EXTENDED SUPERCONFORMAL ALGEBRAS
FROM CLASSICAL AND QUANTUM
HAMILTONIAN REDUCTION

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ABSTRACT

The classification of extended superconformal algebras of the Knizhnik-Bershadsky
type with $W$-algebra like composite operators occurring in the commutation relations,
but with generators only of conformal dimension $1$, $\frac{3}{2}$ and 2, has recently
been reconsidered by various authors from various points of view. We argue that
a particularly natural classification seems to arise on the basis of hamiltonian re-
duction of affine Lie superalgebras with even subalgebras $G \oplus sl(2)$. Based on
generic formulas for the Poisson bracket structure of the classical Gel’fand-Dickey
algebra, we introduce similar generic formulations for all these algebras at the quan-
tum level. Similarly, we rewrite the free field (Feigin-Fuchs) representations of all
these algebras by using the BRST formalism and the free field realization of the
affine Lie superalgebra. Again we emphasize the unifying aspects partly based on
previously known results, partly on our own work when everything is formulated
in the language of reduced affine Lie superalgebras. We also discuss the screening
operators of these algebras. Although completely explicit results for all of those are
known only in a limited number of cases, enough may be said about the general
structure, to in fact provide what appears to be the complete pattern of singular
vectors in the free field realisation.

1. Introduction

The role of extended superconformal algebras in formulations of such related sub-
jects as string theory, topological field theory and conformal field theory has been
remarkable. This has been true both as far as the world sheet supersymmetry and

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the target space geometry is concerned. It seems not unlikely that new and unforeseen applications will turn up. Thus it appears to be of general interest to have these extended superconformal algebras reasonably classified.

One must distinguish several different ways in which the supersymmetries have been implemented. The first way studied considered only what may be termed extended superconformal Lie (or perhaps, super-Lie) algebras: those for which commutators (anticommutators) of generators are themselves members of the algebra, i.e., the expression for the commutators involve only a linear combination of the generators themselves. This class of algebras appears to have been adequately classified for some time. For these algebras it seems reasonable to restrict the number, \( N \), of supersymmetries to be less than 4. For example, the linear \( o(N) \) supersymmetric extension of the Virasoro algebra, which was introduced by Ademollo et al. [1], was realized as the superconformal transformation on extended superspace with internal \( o(N) \) symmetry. However, this formulation leads to negative conformal weight generators and the absence of a central extension unless \( N \leq 4 \) [1], [2].

In a slightly different approach one regards linear \( N = 2, 4 \) extended superconformal algebras as a symmetry of supersymmetric non-linear sigma models on group manifolds or on coset spaces such as hermitian symmetric spaces [3] or Wolf spaces [4].

Knizhnik and Bershadsky [5], however, have introduced the non-linear extension of superconformal algebras with \( u(N) \) and \( so(N) \) affine symmetries. The precise form of the algebra follows from closure and associativity of the operator product expansions (the OPE method). This non-linearity means that these extended superconformal algebras are closely related to Zamolodchikov’s \( W \)-algebras [6]. \( W \)-algebras are known to play a very useful role in the classification of chiral algebras in conformal field theories, so this non-linear extension deserves further study. The representation theory has been studied in the case of \( so(N) \)-extended algebras [7] and in special cases such as the so-called doubly extended \( N = 4 \) algebra [4, 8] for which a fair amount by now is known [9].

For actual applications in conformal field theory or string theory, it must be expected that the extended superconformal algebra will be a sub-algebra of an even larger, perhaps \( W \)-extended (or super \( W \)-extended) chiral algebra. However, it seems of some interest to first address the classification problem for the extended superconformal algebras themselves, especially since that problem appears so much simpler. The classification presented here has independently been found by Bowcock and by Fradkin and Linetsky [10, 11]. Our emphasis here is on the unifying aspect provided by the idea of hamiltonian reduction of Lie superalgebras.

As is by now well known, there is a beautiful way of obtaining \( W \)-algebras from the hamiltonian reduction [12, 13, 14, 15] of suitable affine Lie algebras by considering the constraints on the phase space of currents using a gauge fixing procedure, (see also the recent review by Bouwknegt and Schoutens [16]). Recently various types of further extensions of the \( W \)-algebras [17, 18, 19, 20, 21] including their supersym-
metric generalization [22, 23, 24] have been considered, also by using the hamiltonian reduction technique.

There is a simple rule for finding at least the conformal dimension of the \( W \)-generators in any one particular case. In fact the labelling of the \( W \)-algebra as proposed by the review of Bouwknegt and Schoutens [16] makes use of that directly. The reduction is characterized by a particular \( sl(2) \) subalgebra of the original Lie algebra. The generators of the original Lie algebra occur as spin multiplets of that \( sl(2) \) subalgebra with spins \( \{ s_i \} \), say. Then the conformal dimensions of the generators of the ensuing \( W \)-algebra are simply the set of numbers, \( \{ s_i + 1 \} \), the constant 1, being the conformal dimension of the WZNW currents of the unconstrained affine lie-algebra.

In our case we want to consider the set of extended superconformal algebras, which may be characterized as follows: (i) There is an energy-momentum tensor of conformal dimension 2, (ii) there is a number of supercurrents, \( G_\gamma \), primary conformal fields of dimension \( \frac{3}{2} \) and primary affine fields of some affine algebra \( \hat{G} \), transforming according to some completely reducible representation of \( G \), and where the label \( \gamma \), is a weight label of that representation, or indeed, as we shall see, an odd root of the Lie superalgebra; (iii) there are the generators of the affine Lie algebra \( \hat{G} \), with conformal dimension 1; (iv) all other generators are composites of the above.

It follows that within the framework of hamiltonian reduction of Lie superalgebras which we are going to adopt, we need consider situations where a certain \( sl(2) \) subalgebra is embedded so that the generators of the subalgebra, \( G \), have spin 0. That immediately implies that we are looking for situations where there is a subalgebra of the form \( G \oplus sl(2) \). The generators of the \( sl(2) \) itself have spin 1 and indeed give rise as usual to the energy momentum tensor after reduction. We see that we are looking for situations where the remaining generators have spin \( \frac{1}{2} \).

It is easy from the known theory of Lie superalgebras [25] to look up the ones having even subalgebras of the form \( G \oplus sl(2) \). The result is given in table (1). Previously we [26] have considered these in some detail after two of us considered the classical reduced algebras [27]. The well known superconformal algebras occur as special cases. Thus \( N = 1 \) corresponds to \( osp(1|2) \) and \( N = 2 \) to \( osp(2|2) \) as is very well known. For \( N = 4 \), there are several cases with \( D(2|1) \) and \( A(1|1) \) forming extremes between which \( D(2|1; \alpha) \) interpolates. It is amusing and perhaps significant to notice that this classification corresponds to that of the reduced holonomy groups of non-symmetric Riemannian manifolds [28]. In nearly all of the cases in table (1) the odd generators do indeed carry spin \( \frac{1}{2} \) of the \( sl(2) \) subalgebra. There is only one exception: in the case of \( B(1|n) \) the spin is 1, so that the supercurrents acquire conformal dimension 2 after reduction. Hence this kind of algebra falls outside the scope of what we consider here, but may deserve separate investigation. In the appendix we provide details on the root systems of these algebras.

We expect that these extended superconformal algebras might correspond to non-linear sigma models with rich geometrical structures such as quaternionic or octo-
Table 1: Lie Superalgebras with an even subalgebra $A_1$

| $g$  | $g_0$  | $G$                  | $G$ (alternatively) |
|------|--------|----------------------|---------------------|
| $A(n|1)$ | $sl(n+1,2)$ | $A_n \oplus A_1 \oplus u(1)$ | $sl(n+1) \oplus u(1)$ |
| $B(n|1)$ | $osp(2n+1|2)$ | $B_n \oplus A_1$ | $so(2n+1)$ |
| $D(n|1)$ | $osp(2n|2)$ | $D_n \oplus A_1$ | $so(2n)$ |
| $D(2|n)$ | $osp(4|2n)$ | $A_1 \oplus A_1 \oplus C_n$ | $sp(2n) \oplus sl(2)$ |
| $D(2|1;\alpha)$ | $A_1 \oplus A_1 \oplus A_1$ | $A_1 \oplus A_1$ | $sl(2) \oplus sl(2)$ |
| $F(4)$ | $B_3 \oplus A_1$ | $B_3$ | $so(7)$ (spin(7)) |
| $G(3)$ | $G_2 \oplus A_1$ | $G_2$ | $G_2$ |
| $B(1|n)$ | $osp(3|2n)$ | $A_1 \oplus C_n$ | $C_n \oplus sp(2n)$ |

nionic structures. In the similar classification based mostly on the OPE method the role of an underlying Lie superalgebra was also hinted at in various ways. Our treatment presented here, however, provides a complete account also of the free field realizations.

To avoid confusion, is should be emphasized that the affine Lie superalgebras we are going to consider differ from the supersymmetrization of WZNW models giving rise to super affine Lie algebras. However, it may be of interest to consider the two ideas put together, as has recently been done. The results have a slightly different perspective and seems of particular value when further $W$ type extensions need be considered. Thus one considers the embedding, not of $sl(2)$, but of the “supersymmetric extension”, $osp(1|2)$. But that does not seem necessary for our purpose of understanding the extended superalgebras per se, without further extensions.

The purpose of the present paper is to present the hamiltonian reduction in the classical and quantum cases and to provide a unifying account of all these extended superconformal algebras. Similarly we study the quantum algebras further using free field representations, emphasizing again the unifying aspects offered by the formalism, so that generic formulas may be written down. For practical calculations the free field representations are useful for the computation of correlation functions once the pattern of singular vectors have been understood. To that end we analyze the structure of screening operators.

This paper is organized as follows: In sect. 2, we first review some basic properties of affine Lie superalgebras, and then we discuss the classical hamiltonian reduction for an affine Lie superalgebra $\hat{g}$ associated with a Lie superalgebra $g$ which has the even subalgebra $G \oplus sl(2)$, and we derive the classical $G$ extended superconformal Gel’fand-Dickey algebra. In sect. 3 we use the BRST gauge fixing procedure and the Wakimoto realization of the affine Lie superalgebra to derive the free field realization of the extended superconformal algebra at the level of BRST charges and energy momentum tensors. In sect 4, we explain how the classical results for the algebras generalize in
all cases and we complete the free field construction with generic formulas pertaining to all cases. In sect. 5, we investigate the structure of screening operators and the null field structure of degenerate representations of the $G$ extended superconformal algebras.

2. The Classical Hamiltonian Reduction

2.1 Lie superalgebras

We start with explaining some definitions of basic classical Lie superalgebras and their affine extensions. Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ be a rank $n$ basic classical Lie superalgebra with an even subalgebra $\mathfrak{g}_0$ and an odd subspace $\mathfrak{g}_1$. $\Delta = \Delta^0 \cup \Delta^1$ is the set of roots of $\mathfrak{g}$, where $\Delta^0$ ($\Delta^1$) is the set of even (odd) roots. Denote the set of positive even (odd) roots by $\Delta^+_0$ ($\Delta^+_1$). The superalgebra $\mathfrak{g}$ has a canonical basis $\{E_\alpha, e_\gamma, h^i\}$ ($\alpha \in \Delta^0$, $\gamma \in \Delta^1$, $i = 1, \ldots, n$), which satisfies (anti-)commutation relations

\[ [E_\alpha, E_\beta] = \begin{cases} N_{\alpha, \beta} E_{\alpha + \beta}, & \text{for } \alpha, \beta, \alpha + \beta \in \Delta^0, \\ 2\alpha \cdot h \over \alpha^2, & \text{for } \alpha + \beta = 0, \end{cases} \]

\[ \{e_\gamma, e_{\gamma'}\} = N_{\gamma, \gamma'} E_{\gamma + \gamma'}, \text{ for } \gamma, \gamma' \in \Delta^1, \gamma + \gamma' \in \Delta^0, \]

\[ \{e_\gamma, e_{-\gamma}\} = \gamma \cdot h, \text{ for } \gamma \in \Delta^1_+, \]

\[ [e_\gamma, E_\alpha] = N_{\gamma, \alpha} e_{\gamma + \alpha}, \text{ for } \alpha \in \Delta^0, \gamma + \alpha \in \Delta^1, \]

\[ [h^i, E_\alpha] = \alpha^i E_\alpha, \ [h^i, e_\gamma] = \gamma^i e_\gamma. \] (1)

The even subalgebra $\mathfrak{g}_0$ is generated by $\{E_\alpha, h^i\}$. The odd subspace $\mathfrak{g}_1$ is spanned by $\{e_\gamma\}$. $\mathfrak{g}_0$ acts on $\mathfrak{g}_1$ as a faithful, completely reducible representation. The Killing form $\langle \ , \rangle$ on $\mathfrak{g}$ is defined by

\[ \langle E_\alpha, E_\beta \rangle = \frac{2}{\alpha^2} \delta_{\alpha + \beta, 0}, \quad \langle e_\gamma, e_{-\gamma} \rangle = -(e_{-\gamma}, e_{\gamma'}) = \delta_{\gamma, \gamma'}, \quad \langle h^i, h^j \rangle = \delta_{ij}, \] (2)

for $\alpha, \beta \in \Delta^0$, $\gamma, \gamma' \in \Delta^1_+$, $i, j = 1, \ldots, n$.

An affine Lie superalgebra $\hat{\mathfrak{g}}$ at level $k$ is the (untwisted) central extension of $\mathfrak{g}$ and consists of the elements of the form $(X(z), x_0)$, where $X(z)$ is a $\mathfrak{g}$-valued Laurent polynomial of $z \in \mathbb{C}$ and $x_0$ is a number $\pi \mathbb{Z}$. The commutation relation for two elements $(X(z), x_0)$ and $(Y(z), y_0)$ is given by

\[ [ (X(z), x_0), (Y(z), y_0) ] = ( [ X(z), Y(z) ] , \ k \oint_{2\pi i} \frac{dz}{2\pi i} (X(z), \partial Y(z)) ). \] (3)

The dual space $\hat{\mathfrak{g}}^*$ of $\hat{\mathfrak{g}}$ is generated by the current $(J(z), a_0)$. Using the Killing form $\langle \ , \rangle$ on $\mathfrak{g}$, we may identify $\hat{\mathfrak{g}}^*$ with $\hat{\mathfrak{g}}$. The inner product $\langle \ , \rangle$ of $(J(z), a_0) \in \hat{\mathfrak{g}}^*$ and $(X(z), x_0) \in \hat{\mathfrak{g}}$ is given by

\[ \langle (J, a_0), (X, x_0) \rangle = \oint_{2\pi i} \frac{dz}{2\pi i} (J(z), X(z)) + a_0 x_0. \] (4)
One defines the coadjoint action $\text{ad}^*$ of $\hat{g}$ on $\hat{g}^*$ by
\[
\langle \text{ad}^*(X, x_0), (J, a_0) \rangle = -\langle (J, a_0), [ (X, x_0), (Y, y_0) ] \rangle.
\] (5)

Using Eq.(3), one gets
\[
\text{ad}^*(X, x_0)(J(z), a_0) = [ [X(z), J(z)] + ka_0 \partial X(z), 0 ].
\] (6)

Namely, the coadjoint action simply takes the form of an infinitesimal gauge transformation of the current $J(z)$. Denote this gauge transformation with the gauge parameter $\Lambda(z)$ as $\delta\Lambda$:
\[
\delta\Lambda J(z) = [ \Lambda(z), J(z) ] + ka_0 \partial \Lambda(z),
\] (7)

In the following we take the number $a_0$ to be $1$. In terms of the canonical basis
\[
J(z) = \sum_{\alpha \in \Delta^0} \frac{\alpha^2}{2} J_\alpha(z) E_\alpha + \sum_{\gamma \in \Delta^1} j_\gamma(z) e_\gamma + \sum_{i=1}^{n} H^i(z) h^i.
\] (8)

and
\[
\Lambda(z) = \sum_{\alpha \in \Delta^0} \varepsilon_\alpha(z) E_\alpha + \sum_{\gamma \in \Delta^1} \xi_\gamma(z) e_\gamma + \sum_{i=1}^{n} \varepsilon^i(z) h^i,
\] (9)

we can express the gauge transformation Eq.(7) in terms of components:
\[
\delta\Lambda J_\alpha = \sum_{\beta, \alpha - \beta \in \Delta^0} \frac{(\alpha - \beta)^2}{\alpha^2} N_{\alpha - \beta \varepsilon_\beta J_{\alpha - \beta} - 2(\alpha \cdot H) \varepsilon_\alpha + \frac{2k}{\alpha^2} \partial \varepsilon_\alpha + (\alpha \cdot \varepsilon) J_\alpha}
\]
\[
+ \sum_{\gamma, \alpha - \gamma \in \Delta^1} \frac{2}{\alpha^2} N_{\gamma \alpha - \gamma \xi_\gamma j_{\alpha - \gamma}},
\]
\[
\delta\Lambda j_\gamma = \sum_{\alpha \in \Delta^0} N_{\alpha \gamma - a - \alpha} \varepsilon_\alpha j_{\gamma - a} + (\gamma \cdot \varepsilon) j_\gamma + \sum_{\alpha \in \Delta^0} \frac{\alpha^2}{2} N_{\gamma - a \alpha \xi_\gamma - a} J_\alpha - \xi_\gamma \gamma \cdot H + k \partial \xi_\gamma,
\]
\[
\delta\Lambda H^i = \sum_{\alpha \in \Delta^0} \alpha^i \varepsilon_\alpha J_{-a} + \sum_{\gamma \in \Delta^1} \gamma^i(\xi_\gamma j_{-\gamma} + \xi_{-\gamma} j_\gamma) + k \partial \varepsilon^i.
\] (10)

Writing the gauge transformations $\delta\Lambda$ as
\[
\delta\Lambda = \oint \frac{dz}{2\pi i} (\Lambda(z), J(z))
\]
\[
= \oint \frac{dz}{2\pi i} \sum_{\alpha \in \Delta^0} \varepsilon_\alpha J_{-a} + \sum_{\gamma \in \Delta^1} (j_\gamma \xi_{-\gamma} + \xi_{-\gamma} j_\gamma) + \sum_{i=1}^{n} \varepsilon^i H^i,
\] (11)
one can introduce a canonical Poisson structure on the dual space $\hat{g}^*$. This Poisson structure is conveniently summarized in the form of “the operator product expansions”: Poisson brackets between modes are obtained from these exactly as commutators from OPE’s in conformal field theory. The result is:

$$J_\alpha(z)J_\beta(w) = \begin{cases} \frac{N_{\alpha,\beta}J_{\alpha+\beta}(w)}{\alpha^2} + \cdots, & \text{for } \alpha, \beta, \alpha + \beta \in \Delta^0, \\ \frac{2\alpha H(w)}{\alpha^2} + \cdots, & \text{for } \alpha + \beta = 0, \end{cases}$$

$$j_{\pm \gamma}(z)j_{\pm \gamma'}(w) = \frac{\pm N_{\pm \gamma, \pm \gamma'}J_{\pm(\gamma+\gamma')}(w)}{z-w} + \cdots, \text{ for } \gamma, \gamma' \in \Delta^1, \gamma + \gamma' \in \Delta^0,$$

$$j_{\pm \gamma}(z)j_{\mp \gamma}(w) = \frac{-N_{\pm \gamma, \mp \gamma'}J_{\mp(\gamma-\gamma')}(w)}{z-w} + \cdots, \text{ for } \gamma, \gamma' \in \Delta^1, \gamma - \gamma' \in \Delta^0,$$

$$j_{\gamma}(z)j_{-\gamma}(w) = -\frac{k}{(z-w)^2} - \gamma \cdot H(w) + \cdots, \text{ for } \gamma \in \Delta^1_+,$$

$$J_\alpha(z)J_{\gamma}(w) = \frac{-N_{-\alpha,\gamma \mp \alpha}J_{\gamma \mp \alpha}(w)}{z-w} + \cdots, \text{ for } \gamma, \alpha \in \Delta^1, \alpha \in \Delta^0,$$

$$H^i(z)J_\alpha(w) = \frac{\alpha^i J_\alpha(w)}{z-w} + \cdots, \text{ for } \gamma, \alpha \in \Delta^1, \alpha \in \Delta^0,$$

$$H^i(z)H^j(w) = \frac{k\delta^{ij}}{(z-w)^2} + \cdots. \quad (12)$$

Here we have used some identities for the structure constants coming from the Jacobi identities:

$$\frac{2N_{\alpha,\beta}}{(\alpha + \beta)^2} = \frac{2N_{\beta,-\alpha - \beta}}{\alpha^2} = \frac{2N_{-\alpha,-\beta,\alpha}}{\beta^2}, \text{ for } \alpha, \beta \in \Delta^0,$$

$$\frac{2N_{\gamma,\gamma'}}{(\gamma + \gamma')^2} = N_{\gamma',-\gamma - \gamma'} = -N_{-\gamma,-\gamma',\gamma}, \text{ for } \gamma, \gamma' \in \Delta^1_+,$$

$$\frac{2N_{\gamma,-\gamma'}}{(\gamma - \gamma')^2} = N_{-\gamma',-\gamma + \gamma'} = N_{-\gamma + \gamma',\gamma}, \text{ for } \gamma, \gamma' \in \Delta^1_+,$$

$$\frac{2N_{-\gamma,-\gamma'}}{(\gamma + \gamma')^2} = -N_{-\gamma',\gamma + \gamma'} = N_{\gamma + \gamma',-\gamma}, \text{ for } \gamma, \gamma' \in \Delta^1_+. \quad (13)$$

In the following, as argued in the introduction, we will study the class of Lie superalgebras, with even algebras of the form $G \oplus A_1$. Using Kac’s notation, these algebras are classified as follows $A(n|1)$ ($n \geq 1$), $A(1,0) = C(2)$, $B(n|1)$ ($n \geq 0$), $B(1|n)$ ($n \geq 1$), $D(n|1)$ ($n \geq 2$), $D(2|n)$ ($n \geq 1$), $D(2|1; \alpha)$, $F(4)$ and $G(3)$ (see table [1]). The embedding of $A_1$ in $g$ carried by $g_1$ has spin $\frac{1}{2}$, except for $B(1|n)$. In the case of $B(1|n)$ the embedding has spin 1. In the spin $\frac{1}{2}$ embedding case, which is the only case we will be studying in this paper for the reasons given in the introduction, the odd subspace $g_1$ belongs to the spin $\frac{1}{2}$ representation with respect to the even
subalgebra $A_1$ and therefore splits into two parts:

$$g_1 = (g_1)^{+\frac{1}{2}} \oplus (g_1)^{-\frac{1}{2}}.$$  \hspace{1cm} (14)

Likewise, the odd root space $\Delta^1$ may be divided into two parts

$$\Delta^1 = \Delta^1_{+\frac{1}{2}} \cup \Delta^1_{-\frac{1}{2}},$$  \hspace{1cm} (15)

where $(g_1)^{\pm\frac{1}{2}}$ is spanned by the generators whose roots belong to $\Delta^1_{\pm\frac{1}{2}}$. More explicitly, the sets $\Delta^1_{\pm\frac{1}{2}}$ consist of the roots $\gamma \in \Delta^1$ satisfying

$$\frac{\gamma \cdot \theta}{\theta^2} = \pm \frac{1}{2},$$  \hspace{1cm} (16)

where $\theta$ is the simple root of $A_1$. By choosing appropriate simple roots for $g$, we can take the root space $\Delta^1_{+\frac{1}{2}}$ as the space of odd positive roots;

$$\Delta^1_+ = \Delta^1_{+\frac{1}{2}},$$  \hspace{1cm} (17)

and then

$$\Delta^1_{+\frac{1}{2}} = \Delta^1_{-\frac{1}{2}} + \theta,$$  \hspace{1cm} (18)

holds. We note that each odd space $(g_1)^{\pm\frac{1}{2}}$ belongs to a fundamental representation of $G$ of dimension $|\Delta^1_{\pm\frac{1}{2}}|$, this being the number of positive odd roots. These will be related to the supercurrents as discussed below. The corresponding representation of $G$ (possibly of the form $G^{(1)} \oplus G^{(2)}$) has weights simply equal to

$$\gamma_\perp(G^{(i)}) \equiv P_{G^{(i)}}(\gamma)$$

namely the (positive in this case) odd root projected onto the root space of $G^{(i)}$. If we denote the representation matrices in the canonical basis by

$$t^\alpha_{\gamma,\gamma'}, t^i_{\gamma,\gamma'}$$

then we have (using the Jacobi identities, see also below) for $\gamma \in \Delta^1_+$

$$t^\alpha_{\gamma,\gamma'} = -N_{-\alpha, -\gamma} \delta_{\gamma + \alpha, \gamma'}$$

$$t^i_{\gamma,\gamma'} = (\gamma_\perp)^i \delta_{\gamma,\gamma'}$$

$$tr(t^i t^j) = c_F \delta_{ij}$$

$$tr(t^\alpha t^\beta) = c_F \frac{2}{\alpha^2} \delta_{\alpha + \beta, 0}$$

$$c_F = \sum_{\gamma \in \Delta^1_+} \frac{\gamma_\perp^2}{r(G)}$$  \hspace{1cm} (19)
Table 2: Properties of Lie superalgebras giving rise to ESA’s

| g       | sdim  | rank | $h^\vee$ | $|\Delta^1_+|$ | $c_F^{(1)}$ | $c_F^{(2)}$ |
|---------|-------|------|----------|----------------|-------------|-------------|
| $A(n|1)$ | $n^2 - 2n$ ($-2$ for $n = 1$) | $n + 2$ | $n - 1$ | $2(n + 1)$ | 2 | $\frac{(n-1)^2}{2(n+1)}$ |
| $B(n|1)$ | $(n - 1)(2n - 1)$ | $n + 1$ | $3 - 2n$ | $2n + 1$ | -2 | -2 |
| $D(n|1)$ | $(n - 1)(2n - 3)$ | $n + 1$ | $4 - 2n$ | $2n$ | -2 | -2 |
| $D(2n)$ | $(n - 2)(2n - 3)$ | $n + 2$ | $2n - 2$ | $4n$ | 4 | $-2n$ |
| $D(2|1; \alpha)$ | 1 | 3 | 0 | 4 | $-2\gamma$ | $-2(1 - \gamma)$ |
| $F(4)$ | 8 | 4 | $-3$ | 8 | -2 | -2 |
| $G(3)$ | 3 | 3 | 2 | 7 | 2 | -2 |

where

$r(G) = \text{rank}(G)$.

Table 2 gives several parameters relevant for the different Lie superalgebras we are considering. One may show that

$$t^\alpha_{\gamma,\gamma'} = -t^\alpha_{\bar{\theta}-\gamma',\bar{\theta}-\gamma}.$$

The Casimir operator in that representation takes the form

$$C_{\gamma,\gamma'} = (\sum_{\alpha \in \Delta^0(G)} \frac{\alpha^2}{2} t^\alpha t^{-\alpha} + \sum_{i=1}^{r(G)} t^i t^i)_{\gamma,\gamma'}$$  \hspace{1cm} (20)

Then it is convenient to introduce the currents corresponding to $\tilde{G}$ in the following basis

$$J_{\gamma,\gamma'}(z) \equiv \sum_{\alpha \in \Delta^0(G)} \frac{\alpha^2}{2} t^\alpha_{\gamma,\gamma'} J_{-\alpha}(z) + \sum_{i=1}^{r(G)} t_{\gamma,\gamma'}^i H_i(z)$$  \hspace{1cm} (21)

Notice that for given $\gamma, \gamma'$, $J_{\gamma,\gamma'}$ contains at most a single term, which may be written in several ways, for example as:

$$J_{\gamma,\gamma'}(z) = \begin{cases} N_{\gamma,\gamma'} J_{-\gamma,\gamma'}(z) & \text{for } \gamma - \gamma' \in \Delta^0(G) \\
\gamma \cdot H(\gamma) & \text{for } \gamma = \gamma' \\
0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (22)

It follows that the root system of the even subalgebra of $G$ is expressed in the form of $\gamma - \gamma'$, where $\gamma$ and $\gamma'$ are some positive odd roots. This fact turns out
to be essential for the study of the structure of general $G$ extended superconformal algebras.

2.2 Classical hamiltonian reduction

We now consider the hamiltonian reduction of the phase space of the currents $\hat{g}^*$ associated with any one of the set of Lie superalgebras $g$ with an even subalgebra $G \oplus A_1$. We want to impose the constraint on the even subalgebra $A_1$ while keeping the affine $G$ algebra symmetry. We start from the phase space $\mathcal{F}$ of the currents with the constraint

$$J_{-\theta}(z) = 1.$$  

(23)

Denote by $\mathcal{N}$ the gauge group which preserves the constraint Eq.(23). Putting $-\theta$ for $\alpha$ in the first formula of eqs. Eq.(10), we get

$$\delta_{\Lambda}J_{-\theta} = \frac{2(\theta \cdot H)}{\theta^2} \varepsilon_{-\theta} + \frac{2k}{\theta^2} \partial \varepsilon_{-\theta} - (\theta \cdot \varepsilon)J_{-\theta} + \sum_{\gamma \in \Delta_1^+} \frac{2}{\theta^2} N_{-\gamma, -\theta + \gamma} \xi_{-\gamma} j_{-\theta + \gamma}.$$  

(24)

Here we use the relation Eq.(18) and $N_{\alpha, -\theta - \alpha} = 0$ for $\alpha \in \Delta(G)$. Thus $\mathcal{N}$ is characterized by the algebra conditions

$$\varepsilon_{-\theta} = \theta \cdot \varepsilon = 0, \quad \xi_{-\gamma} = 0, \quad \text{for } \gamma \in \Delta_1^+.$$  

(25)

So the Lie algebra $\mathfrak{n}$ of the gauge group $\mathcal{N}$ is equal to $\hat{G} \oplus \mathfrak{n}_1$, where $\hat{G}$ is the affine Lie algebra corresponding to the even subalgebra $G$ and $\mathfrak{n}_1$ is generated by the elements:

$$\Lambda(z) = \varepsilon_\theta(z) E_\theta + \sum_{\gamma \in \Delta_1^+} \xi_{\gamma}(z)e_{\gamma}.$$  

(26)

Let $\mathcal{N}_1$ be the gauge group corresponding to $\mathfrak{n}_1$. The reduced phase space $\mathcal{M}$ is defined as the quotient space $\mathcal{M} = \mathcal{F}/\mathcal{N}_1$. By the standard gauge fixing procedure, one chooses one or the other gauge slice as a representative of the reduced phase space. In the so-called “Drinfeld-Sokolov (DS)"-gauge the generators of the classical extended algebra are rather easily read off. It is defined by [12]:

$$J_\theta(z) = T(z), \quad \theta \cdot H(z) = 0, \quad j_\gamma(z) = G_\gamma(z), \quad j_{-\gamma}(z) = 0,$$  

(27)

for $\gamma \in \Delta_1^+$. And the generic gauge transformation projected on the DS gauge slice becomes the transformation corresponding to the extended superconformal algebra [12, 13, 14, 17, 18, 19, 20, 21, 22, 23] in our case. Hence the final step is to consider the generic gauge transformation preserving the gauge condition Eq.(27). In particular

$$\delta_{\Lambda}J_{-\theta} = -(\theta \cdot \varepsilon) + \frac{2k}{\theta^2} \partial \varepsilon_{-\theta},$$

$$\delta_{\Lambda}(\theta \cdot H) = \theta^2 \varepsilon_{\theta} - \theta^2 \varepsilon_{-\theta} T + k \partial (\theta \cdot \varepsilon) + \sum_{\gamma \in \Delta_1^+} \theta \cdot \gamma \xi_{-\gamma} G_\gamma,$$
\[ \delta \lambda j_{-\gamma} = N_{-\theta, \theta - \gamma} \varepsilon_{-\theta} G_{\theta - \gamma} + \sum_{\gamma' \in \Delta^1_+} N_{\gamma, \gamma'} \xi_{-\gamma} J_{\gamma^0 - \gamma} \]
\[ + N_{\theta - \gamma, \gamma} \xi_{\theta - \gamma} + \xi_{-\gamma} (\gamma \cdot H) + k \partial \xi_{-\gamma}, \]
(28)

for \( \gamma \in \Delta^1_+ \). By solving the conditions which preserve the DS-gauge \( \delta J_{-\theta} = 0 \), \( \delta (\theta \cdot H) = 0 \) and \( \delta j_{-\gamma} = 0 \), we can express the parameters \( \varepsilon_{-\theta}, \theta \cdot \varepsilon \) and \( \xi_{\gamma} (\gamma \in \Delta^1_+) \) in terms of the other parameters, \( \varepsilon_{-\theta}, \xi_{-\gamma} \) (\( \gamma \in \Delta^1_+ \)), \( \varepsilon_{\alpha} \) and \( \alpha \cdot \varepsilon \) (\( \alpha \in \Delta(G) \)):

\[ \theta \cdot \varepsilon = \frac{2k}{\theta^2} \partial \varepsilon_{-\theta}, \]
\[ \varepsilon_{-\theta} = \varepsilon_{-\theta} T - \frac{2k^2}{(\theta^2)^2} \partial^2 \varepsilon_{-\theta} - \frac{1}{2} \sum_{\gamma \in \Delta^1_+} \xi_{-\gamma} G_{\gamma}, \]
\[ \xi_{\theta - \gamma} = \frac{-1}{N_{\theta - \gamma, \gamma}} \{ N_{-\theta, \theta - \gamma} G_{\theta - \gamma} \varepsilon_{-\theta} + \sum_{\gamma' \in \Delta^1_+} N_{\gamma, \gamma'} \xi_{-\gamma} J_{\gamma^0 - \gamma} + \xi_{-\gamma} (\gamma \cdot H) + k \partial \xi_{-\gamma} \}, \]
(29)

In the DS gauge, the gauge transformation of \( J_{\theta}(z) = T(z), J_{\beta}(z), \beta \cdot H(z) \) and \( G_{\gamma}(z) \) may then be worked out in terms of these. We refer to our papers [26] for the straightforward but slightly messy details.

We then convert this result into formal “operator product expansions”. We denote the gauge transformation in the DS-gauge as

\[ \delta = \oint \frac{dz}{2\pi i} (A(z) , J_{DS}(z)), \]
(30)

where

\[ J_{DS}(z) = \frac{\theta^2}{2} T(z) E_0 + \sum_{\gamma \in \Delta^1_+} G_{\gamma}(z) e_\gamma + \sum_{\alpha \in \Delta(G)} \frac{\alpha^2}{2} J_{\alpha}(z) E_\alpha + \sum_{i=1}^{\text{rank}(G)} H^i h^i, \]
(31)

This conveniently generates the Poisson bracket structure on the reduced phase space. Here we take \( h^i \) (\( i = 1, \ldots, \text{rank}(G) \)) as the generators of the Cartan subalgebra of \( G \). We also introduce the rescalings \( T = \frac{\theta^2}{2k} T \) and \( G_{\gamma} = \sqrt{\frac{N_{\gamma, \gamma}}{k}} G_{\gamma} \) for \( \gamma \in \Delta^1_+ \). We further use several identities for the structure constants, like the Jacobi identities Eq.(13), as well as

\[ \frac{N_{\theta, \theta - \gamma}}{N_{\gamma, \gamma - \theta}} = - \frac{N_{\gamma', \gamma - \theta}}{N_{\gamma', \theta - \gamma}}, \]
(32)

for \( \gamma, \gamma' \in \Delta^1_+ \), as may be checked using explicit matrix representations of \( g \) [14, 32] in table [I]. Similarly one may derive

\[ N_{\gamma, \theta - \gamma} = - N_{\gamma, \theta, - \gamma} = \frac{\theta^2}{2} \]
(33)

from the fact that \( g = G \oplus A_1 \). Using all these properties one may achieve a major simplification of the complicated, generic expression we have previously presented [24].
To this end it is very useful to make use of the affine currents of $G$ in the basis of Eq. (21). The result finally is:

\[ \tilde{T}(z)\tilde{T}(w) = \frac{-6k}{\theta^2} \frac{z-w}{(z-w)^4} + \frac{2\tilde{T}(w)}{(z-w)^2} + \frac{\partial \tilde{T}(w)}{z-w} + \cdots, \]

\[ \tilde{G}_\gamma(w) \tilde{G}_\gamma(w) = \frac{2\tilde{G}_\gamma(w)}{(z-w)^2} + \frac{\partial \tilde{G}_\gamma(w)}{z-w} - \frac{1}{k} \sum_{\gamma' \in \Delta_1^+} J_{\gamma',\gamma} G_{\gamma'}(w), \]

\[ J_{\beta}(z)\tilde{G}_\gamma(w) = \frac{-N_{-\delta, -\gamma} \tilde{G}_{\delta + \gamma}(w)}{z-w} + \cdots, \quad \text{and} \quad H^i(z)\tilde{G}_\gamma(w) = \frac{\gamma^i \tilde{G}_\gamma(w)}{z-w} + \cdots, \]

\[ \tilde{G}_{\theta - \gamma'}(z)\tilde{G}_\gamma(w) = \frac{2k\delta_{\gamma',\gamma}}{z-w} - \frac{2J_{\gamma',\gamma}(w)}{(z-w)^2} + \frac{\delta_{\gamma,\gamma'} \theta^2}{z-w} \tilde{T} - \partial J_{\gamma,\gamma'}(w) + \frac{1}{k} (J^2)_{\gamma,\gamma'}(w), \quad \text{Eq. (34)} \]

Here we imply the notation:

\[ (J^2)_{\gamma,\gamma'} \equiv \sum_{\gamma'' \in \Delta_1^+} J_{\gamma,\gamma''} J_{\gamma'',\gamma} \quad \text{Eq. (35)} \]

Notice that this expression is nonvanishing only if either $\gamma - \gamma' \in \Delta(G)$, $\gamma = \gamma'$ or $\gamma + \gamma' = \theta$. At this point, the supercurrents $\tilde{G}_\gamma$ are not primary fields with respect to $\tilde{T}(z)$. However, if we define the total energy-momentum tensor $T_{ESA}$ by adding the Sugawara form $T_{Sugawara}$ of the $\hat{G}$ affine Lie algebra:

\[ T_{ESA} = \tilde{T} + T_{Sugawara} \]

where

\[ T_{Sugawara} = \frac{1}{2k} \left\{ \sum_{\alpha \in \Delta(G)} \frac{\alpha^2}{2} J_{\alpha} J_{-\alpha} + \sum_{i=1}^{\text{rank}(G)} H^i H^i \right\} \]

\[ = \frac{1}{2kc_F} tr(J^2), \quad \text{Eq. (36)} \]

we may check that the supercurrents have conformal weight 3/2 with respect to $T_{ESA}$. The result is the classical or Gel’fand-Dickey extended superconformal algebra. In sect. 4 we shall give the explicit form generalized to the arbitrary quantum case. The classical value of the central charge $c_{ESA}$ is $-12k/\theta^2$. Similarly, the classical central extension of the surviving affine Lie algebra $\bar{G}$ (or $\bar{G} = \oplus_i \bar{G}^{(1-i)}$) is given by

\[ K_{cl}^{(i)} = \frac{2k}{\alpha_L^2}, \]

where $\alpha_L$ denotes the “long root” so that this definition is the one pertaining to a normalization where the longest root has length squared equal to 2. The relative minus sign between these expressions is significant. In fact the metrics in the root space of the $A_1$ to be reduced and that of $G$ have opposite signs in all cases except for $D(2|n)$, cf. Appendix A. Thus in most cases $c_{ESA} \geq 0 \iff K_{cl} \geq 0$. This property turns out to hold in the quantum case as well. In the case of $D(2|n)$,
\( \hat{G} \) splits into two commuting algebras having opposite sign central extensions [10].

3. The Quantum Hamiltonian Reduction

3.1 BRST formalism

In this section we discuss the quantum hamiltonian reduction [14] for an affine Lie superalgebra \( \hat{g} \) at level \( k \), generated by \( J_\alpha(z) \ (\alpha \in \Delta^0) \), \( j_\gamma(z) \ (\gamma \in \Delta^1) \) and \( H^i(z) \ (i = 1, \ldots, n) \). Let \( T_{WZNW}(z) \) be the energy-momentum tensor of an affine Lie superalgebra \( \hat{g} \), defined by the Sugawara form:

\[
T_{WZNW} = \frac{1}{2(k + h')} \left( : \sum_{\alpha \in \Delta^0} \frac{\alpha^2}{2} (J_\alpha J_{-\alpha} + J_{-\alpha} J_\alpha) + \sum_{i=1}^n H^i H^i \right) + \sum_{\gamma \in \Delta^1} (j_\gamma j_{-\gamma} - j_{-\gamma} j_\gamma) : \tag{37}
\]

where \( h' \) is the dual Coxeter number of \( g \) and \( : : \) denotes the normal ordering. In order to impose the constraint for the currents at the quantum level, we have to “improve” the energy-momentum tensor by a contribution from the Cartan currents \( H^i(z) \):

\[
T_{\text{improved}}(z) = T_{WZNW}(z) - \mu \cdot \partial H(z). \tag{38}
\]

Here \( \mu \) is an \( n \)-vector. With respect to the improved energy-momentum tensor \( T_{\text{improved}}(z) \), the currents corresponding to the roots \( \alpha \) have conformal weights \( 1 + \mu \cdot \alpha \).

We are concerned with a class of Lie superalgebras whose even subalgebras are \( G \oplus A_1 \), where \( G \) is a semisimple Lie algebra. In the previous section we have considered the constraint \( J_{-\theta}(z) = 1 \) but the currents have conformal dimension one, so in order for the constraint to make sense, \( J_{-\theta}(z) \) must have improved conformal dimension, 0, whereas the currents of \( \hat{G} \) should continue to have conformal dimension 1. This means that the vector \( \mu \) should satisfy

\[
\mu \cdot \theta = 1, \quad \mu \cdot \alpha = 0, \quad \text{for} \ \alpha \in \Delta(G). \tag{39}
\]

These equations Eq.(39) determine the vector \( \mu \) uniquely:

\[
\mu = \frac{\theta}{\theta^2}. \tag{40}
\]

From Eq.(11), we find that the fermionic currents \( j_\gamma(z) \ (j_{-\gamma}(z)) \) for the positive roots \( \gamma \in \Delta^1 \) have conformal weight \( 3/2 \) (1/2). The current \( J_\theta(z) \) has conformal weight 2.

Now we use the BRST-gauge fixing procedure. In the previous section we took the Drinfeld-Sokolov gauge Eq.(27) and derived the Poisson bracket structure for the currents. In order to study the representation of the algebra, it is very useful to
consider the free field representation. This is realized in the classical case by taking
the “diagonal” gauge

\[ J_\theta(z) = 0, \quad \theta \cdot H(z) = a_0 \partial \phi(z), \]
\[ j_{-\gamma}(z) = \sqrt{N_{-\gamma,-\theta+\gamma}} \chi_\gamma(z), \quad \text{for } \gamma \in \Delta_1. \tag{41} \]

in Eq.(8). In the quantum case we similarly introduce a free boson \( \phi(z) \), coupled
to the world sheet curvature, and free fermions \( \chi_\gamma(z) \), satisfying operator product

\[ \chi_\gamma(z) \chi_{\theta-\gamma'}(w) = \frac{\delta_{\gamma,\gamma'}}{z-w} + \cdots, \quad \text{for } \gamma, \gamma' \in \Delta_1. \tag{42} \]

This is consistent with the OPE’s of \( \hat{g} \) together with the constraint. The role of
the free fermions is to act as auxiliary fields that convert the constraints into first class
ones.

We further introduce fermionic ghosts \( (b_\theta(z), c_\theta(z)) \) with conformal weights \((0,1)\)
and bosonic ghosts \( (\tilde{b}_\gamma(z), \tilde{c}_\gamma(z)) \) of weight \((\frac{1}{2}, \frac{1}{2})\) for \( \gamma \in \Delta_1 \). The BRST current
\( J_{BRST}(z) \) is defined as \[ J_{BRST}(z) = c_\theta(J_\theta - 1) + \sum_{\gamma \in \Delta_1} \tilde{c}_\gamma(j_{-\gamma} - \sqrt{N_{-\gamma,-\theta+\gamma}} \chi_\gamma) + \frac{1}{2} \sum_{\gamma \in \Delta_1} N_{\gamma,\theta-\gamma} : \tilde{c}_\gamma \tilde{c}_{\theta-\gamma} b_\theta :. \tag{43} \]

We can easily show that the BRST charge \( Q_{BRST} = \oint \frac{dz}{2\pi i} J_{BRST}(z) \) satisfies the nilpo-
tency condition \( Q_{BRST}^2 = 0 \). The total energy-momentum tensor is expressed as

\[ T_{total}(z) = T_{improved}(z) + T_{\chi} + T_{\text{ghost}}(z), \tag{44} \]

where

\[ T_{\text{ghost}}(z) = : (\partial b_\theta) c_\theta + \frac{1}{2} \sum_{\gamma \in \Delta_1} (\tilde{b}_\gamma \partial \tilde{c}_\gamma - (\partial \tilde{b}_\gamma) \tilde{c}_\gamma) :, \]
\[ T_{\chi}(z) = -\frac{1}{2} \sum_{\gamma \in \Delta_1} : \chi_{\theta-\gamma} \partial \chi_\gamma :. \tag{45} \]

The central charge \( c \) of the total system is computed to be

\[ c_{total} = c_{WZNW} - 12k\mu^2 + \frac{1}{2} |\Delta_1| - 2 - |\Delta_1|, \tag{46} \]

(\text{where } |\Delta_1| \text{ is still the number of positive odd roots). The last two terms are con-
tributions from ghost fields. The central charge } c_{WZNW} \text{ of the WZNW models on the Lie supersupergroup at level } k \text{ is given by the formula:}

\[ c_{WZNW} = \frac{k \text{ sdimg}}{k + h^c}, \tag{47} \]
Table 3: Central charges for $G$ extended superconformal algebras

| $G$          | $c_{\text{total}}$                                      |
|--------------|--------------------------------------------------------|
| $A(n|1)$     | $[6k^2 + k(n^2 + 3n - 9) - n^2 - 2n + 3]/[k + n - 1]$  |
| $B(n|1)$     | $[(k + 1)(4n^2 + 4n - 15 - 6k)]/[2(k + 3 - 2n)]$        |
| $D(n|1)$     | $[(k + 1)(4n^2 - 16 - 6k)]/[2(k + 4 - 2n)]$             |
| $D(2|n)$     | $[6k^2 + k(2n^2 + 3n - 8)] + 4 - 4n^2]/[k + 2n - 2]$   |
| $D(2|1; \alpha)$ | $-3 - 6k$                                      |
| $F(4)$      | $2(-2k^2 + 7k + 9)]/[k - 3]$                          |
| $G(3)$      | $[9k^2 + 13k - 22]/[2(k + 2)]$                         |

where the super dimension $\text{sdim}_g$ of a Lie superalgebra $g$ is defined as $\text{dim}_g - \text{dim}_{\bar{g}}$.

The list for the corresponding central charges is shown in table 3. The results here are in complete agreement with previous calculations using a variety of different techniques. Thus, the results for $A(n|1)$, $B(n|1)$, $D(n|1)$ were obtained by Knizhnik and Bershadsky [5]; the one for $D(2|n)$ and $G(3)$ agrees with Bowcock and Fradkin and Linetsky [10], and the result for $F(4)$ with that of Fradkin and Linetsky [11]. Our treatment here, however, provides a unifying framework.

In the classical limit $k \to \infty$, the expression Eq.(53) becomes $-12k/\theta^2$, which agrees with the result (34) obtained in the previous section.

3.2 The free field representation

We consider the free field representations of $G$ extended superconformal algebra based on the Wakimoto construction [32] of the affine Lie superalgebra $\hat{g}$ [36]. Let us introduce bosonic ghosts $(\beta_\alpha(z), \gamma_\alpha(z))$ for even positive roots $\alpha \in \Delta_0^+$ with conformal dimensions $(1,0)$, fermionic ghosts $(\eta_\gamma(z), \xi_\gamma(z))$ for odd positive roots $\gamma \in \Delta_1^+$ with (unimproved) conformal dimensions $(1,0)$ and $n$ free bosons $\phi(z) = (\phi_1(z), \ldots, \phi_n(z))$ coupled to the world-sheet curvature, satisfying the operator product expansions:

$$
\beta_\alpha(z)\gamma_{\alpha'}(w) = \frac{\delta_{\alpha,\alpha'}}{z-w} + \cdots, \quad \eta_\gamma(z)\xi_{\gamma'}(w) = \frac{\delta_{\gamma,\gamma'}}{z-w} + \cdots,
$$

$$
\phi_1(z)\phi_j(w) = -\delta_{ij}\ln(z-w) + \cdots. \quad \text{(48)}
$$

Using these free fields the energy-momentum tensor is expressed as

$$
T_{WZW}(z) = \sum_{\alpha \in \Delta_0^+} :\beta_\alpha \partial \gamma_\alpha : - \sum_{\gamma \in \Delta_1^+} :\eta_\gamma \partial \xi_\gamma : - \frac{1}{2} : (\partial \phi)^2 : - \frac{i \rho \cdot \partial^2 \phi}{\alpha_+}, \quad \text{(49)}
$$

where $\alpha_+ = \sqrt{k + h^\vee}$ and $\rho = \rho_0 - \rho_1$. $\rho_0$ ($\rho_1$) is half the sum of positive even (odd)
roots:
\[
\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta^0_+} \alpha, \quad \rho_1 = \frac{1}{2} \sum_{\gamma \in \Delta^1_+} \gamma. \tag{50}
\]

The Cartan part currents \( \hat{H}^i(z) \) are given by
\[
\hat{H}^i(z) = -i \alpha^i \partial \phi^i + \sum_{\alpha \in \Delta^0_+} \alpha : \gamma \alpha : + \sum_{\gamma \in \Delta^1_+} \gamma : \xi \eta :. \tag{51}
\]

The currents indicated by a “hat” will differ by the final full ones by certain fermionic contributions to be described. After improvement of the energy momentum tensor, the ghost systems achieve conformal dimensions, \( (0,1) \) for \( (\beta_\theta, \gamma_\theta) \) and \( (\frac{1}{2}, \frac{1}{2}) \) for \( (\eta_\gamma, \xi_\gamma) \). In the BRST gauge fixing procedure it is natural to assume that the multiplets \( (\beta_\theta, \gamma_\theta, b_\theta, c_\theta) \) and \( (\eta_\gamma, \xi_\gamma, b_\gamma, c_\gamma) \) for \( \gamma \in \Delta^1_+ \) form Kugo-Ojima quartets \(^{37}\), and that the corresponding energy-momentum tensor is BRST-exact. In the case of the \( sl(N) \) affine Lie algebra, this has been proven both by homological techniques \(^{38}\), and by a more direct method \(^{39}\). This ansatz enables us to write the total energy-momentum tensor in the form:
\[
T_{\text{total}}(z) = T_{\text{ESA}}(z) + \{Q_{\text{BRST}}, *\}, \tag{52}
\]

where \( T_{\text{ESA}} \) is the energy-momentum tensor of \( G \) extended superconformal algebra:
\[
T_{\text{ESA}}(z) = -\frac{1}{2} : (\partial \varphi)^2 : -i \left( \frac{\rho}{\alpha_+} - \alpha_+ \mu \right) \cdot \partial^2 \varphi
+ \sum_{\alpha \in \Delta^0_+(G)} : \beta \partial \gamma \alpha : + \frac{1}{2} \sum_{\gamma \in \Delta^1_+} : (\partial \chi \gamma) \chi_{\theta - \gamma} :. \tag{53}
\]

The central charge of \( T_{\text{ESA}} \) is given by the formula
\[
c = n - 12 \left( \frac{\rho}{\alpha_+} - \alpha_+ \mu \right)^2 + 2 |\Delta_+(G)| + \frac{1}{2} |\Delta^1_+|. \tag{54}
\]

This may be shown to be equal to \( c_{\text{total}} \) Eq.\(^{10}\).

In the case that \( \hat{G} \) is expressed as a direct sum of simple affine Lie algebras \( \oplus_i \hat{G}_i \), the relation between the level \( k \) of the affine Lie superalgebra \( \hat{g} \) and that, \( K_i \), of the \( i \)'th affine Lie algebra \( \hat{G}_i \), is given by considering the decomposition of the vector \( \rho/\alpha_+ - \alpha_+ \mu \) into the roots of the even subalgebras \( \oplus_i G_i \oplus A_1 \):
\[
\frac{\rho}{\alpha_+} - \alpha_+ \mu = \sum_i \frac{\rho_{G_i}}{\alpha_+} + \left( \frac{1}{2} - \frac{|\Delta^1_+|}{|\Delta^0_+|} \right) \frac{\alpha_+}{\theta^2} \theta. \tag{55}
\]

Here \( \rho_{G_i} \) is half the sum of positive roots of the subalgebra \( G_i \). The above decomposition means that
\[
k + h^\vee = \frac{\alpha^2_+}{2} (K_i + H_i^\vee), \tag{56}
\]
where $H_i^\gamma$ is the dual Coxeter number of the even subalgebra $G_i$ in a notation appropriate to the long root being normalized to length squared, 2, whereas $\alpha_{Li}$ is an actual long long root of the subalgebra $G_i$. Notice that the central extensions of the affine sub algebras receive additional contributions from the free fermions, see next section.

The $n$ bosons $\varphi(z) = (\varphi_1(z), \ldots, \varphi_n(z))$ coupled to the world sheet curvature, can be divided into two classes, due to the decomposition of the Cartan subalgebra $h = H_G \oplus H_{A_1}$, where $H_G$ and $H_{A_1}$ are the Cartan subalgebras of the even subalgebras $G$ and $A_1$, respectively. A boson $\theta \cdot \varphi$ in the $\theta$ direction of the root space of the even subalgebra $A_1$, commutes with the bosons lying along the root space of the even subalgebra $G$. The remaining $n-1$ free bosons are used for the free field representation of the $\hat{G}$ affine Lie algebra, combined with $(\beta_\alpha, \gamma_\alpha)$-systems ($\alpha \in \Delta_+(G)$). If we define a Feigin-Fuchs boson $\phi$:

$$\phi(z) = \frac{\theta \cdot \varphi(z)}{\sqrt{\theta^2}},$$

the energy-momentum tensor Eq.(53) becomes

$$T_{ESA}(z) = T_\phi(z) + T_G(z) + T_\chi(z),$$

where

$$T_\phi = -\frac{1}{2}(\partial \phi)^2 - iQ \partial^2 \phi, \quad Q = \sqrt{\theta^2(\frac{2 - |\Delta_1^+|}{2\alpha_+} - \frac{2\alpha_+}{\theta^2})}.$$  

$T_G(z)$ is the Sugawara energy-momentum tensor of the affine Lie algebra $\hat{G}$.

4. Generic Free Field Representations and Explicit Form of all Algebras

Having provided a free field representation for the energy momentum tensor, we still have to do the same for the affine $G$-currents and in particular for the $|\Delta_+^1|$ supercurrents, $G_\gamma$. We have at our disposal already the free (Feigin Fuchs) scalar, $\phi(z)$ and the $|\Delta_+^1|$ free fermions, $\chi_\gamma(z)$. We then build the affine currents $J_\alpha(z), H_i(z)$ as

$$J_\alpha(z) = \hat{J}_\alpha(z) + J^f_\alpha(z)$$

$$H_i(z) = \hat{H}_i(z) + H^f_i(z)$$

where the fermionic parts are given by

$$J^f_\alpha(z) = \frac{1}{2} \sum_{\gamma, \gamma' \in \Delta_+^1} t^\alpha_{\gamma', \gamma} : \chi_\gamma(z) \chi_{\theta-\gamma'}(z) :$$

$$H^f_i(z) = \frac{1}{2} \sum_{\gamma \in \Delta_+^1} (\gamma_i)_i : \chi_\gamma(z) \chi_{\theta-\gamma}(z) :$$

The “hatted” currents commute with the fermionic ones but are also affine $G$ currents with levels such that the total match up to what is required. More precisely, for $G = \oplus_i G^{(i)}$, the central extensions of the affine subalgebras are denoted by $K^i_{\text{tot}}$ in a
Table 4: Central charges for \( G \) extended superconformal algebras, in terms of the level \( K_{\text{tot}} \) of the full affine algebra.

| \( g \) | \( k \) | \( C_{\text{total}} \) |
|--------|--------|------------------|
| \( A(n|1) \) | \( K_{\text{tot}} + 1 \) | \( \frac{6(K_{\text{tot}})^2 + K_{\text{tot}}(n^2 + 3n + 3) + n}{K_{\text{tot}} + n} \) |
| \( B(n|1) \) | \( -K_{\text{tot}} - 1 \) | \( \frac{K_{\text{tot}}(6K_{\text{tot}} + 4n^2 + 4n - 9)}{2(K_{\text{tot}} + 2n - 2)} \) |
| \( D(n|1) \) | \( -K_{\text{tot}} - 1 \) | \( \frac{2(K_{\text{tot}} + 2n - 3)}{K_{\text{tot}}(K_{\text{tot}} + 4n^2 + 10)} \) |
| \( D(2|n) \) | \( 2K_{\text{tot}}^{1} \) | \( \frac{2(K_{\text{tot}} + n - 1)}{12(K_{\text{tot}})^2 + K_{\text{tot}}(2n^2 + 3n + 8) + 2 - 2n^2} \) |
| \( D(2|1; \alpha) \) | \( -\gamma(K_{\text{tot}}^{1} + 1) \) | \( -3 + 6\gamma(K_{\text{tot}}^{1} + 1) \) |
| \( F(4) \) | \( -K_{\text{tot}} - 1 \) | \( \frac{2(2K_{\text{tot}})^2 + 11K_{\text{tot}}}{K_{\text{tot}} + 4} \) |
| \( G(3) \) | \( K_{\text{tot}} + 1 \) | \( \frac{9(K_{\text{tot}})^2 + 31K_{\text{tot}}}{2(K_{\text{tot}} + 3)} \) |

notation pertaining to the length squared of the maximal root being 2. We then have

\[
K_{i}^{\text{tot}} = K_{i} + K_{i}^{f}
\]

where \( K_{i} \) is the contribution from the hat algebra tabulated in table (3). These may be treated by the standard Wakimoto like free field realisations as above.

So far we have used the parameter, \( k \), the central extension of the Lie superalgebra given by the “hat’ed” currents, for expressing everything. It may be convenient for various purposes and comparisons to express the central charges as functions of the central extensions of the \( \hat{G} \) algebra in a standard normalization pertaining to the longest root having length squared equal to 2. This we do in table 4. For \( D(2|n) \) we have written the central charge in terms of the level of the affine \( sp(2n) \), and for \( D(2|1; \alpha) \) we have written it in terms of the level of \( su(2)^{+} \), one of two commuting \( su(2) \)’s. The parameter, \( \gamma \) (not to be confused with the odd roots!) in the tables is related to the continuous parameter, \( \alpha \), occurring in \( D(2|1; \alpha) \) as

\[
\gamma \equiv \frac{\alpha}{1 + \alpha}.
\]

Introducing the previously defined combinations Eq.(21), we find for the supercurrents in all cases the following remarkably simple general expression

\[
G_{\gamma}(z) = \sum_{\gamma' \in \Delta^{+}_{1}} \left\{ \delta_{\gamma,\gamma'} \left[ \frac{(k + \theta^{2}/2)}{\alpha_{+}} \partial + \frac{i}{\sqrt{2}} \sqrt{\frac{\theta^{2}}{2}} \partial \phi(z) \right] \right. \\
- \frac{1}{\alpha_{+}} \sum_{i} \left[ j^{(i)}_{\gamma,\gamma'}(z) + c_{2}^{(i)} J_{\gamma,\gamma'}^{(i)}(z) \right] \chi_{\gamma'}(z)
\]

(62)
Table 5: Constants occurring in the free field representations.

| \( \mathfrak{g} \) | \( \theta^2/2 \) | \( c_1^{(1)} \) | \( c_2^{(1)} \) | \( k + h^\vee \) | \( \frac{\alpha^2}{2}(K + H^\vee)^{(1)} \) | \( \frac{\alpha^2}{2}(K + H^\vee)^{(2)} \) |
|-----------------|--------------|-------------|-------------|-------------|----------------|----------------|
| \( A(n|1) \)    | -1           | 1           | \frac{4(n+1)}{(n-1)^2} | \( k + n - 1 \) | \( K + n + 1 \) |                     |
| \( B(n|1) \)    | -2           | 0           | \( k - 2n + 3 \) | \( -(K + 2n - 1) \) |                     |                     |
| \( D(n|1) \)    | -2           | 0           | \( k - 2n + 4 \) | \( -(K + 2n - 2) \) |                     |                     |
| \( D(2|n) \)    | -1           | 1           | 0            | \( 2(K^{(1)} + n + 1) \) | \( -(K^{(2)} + 2) \) |                     |
| \( D(2|1; \alpha) \) | -1           | \frac{2}{3} | \( k \) | \( -\gamma(K^{(1)} + 2) \) | \( -(1 - \gamma)(K^{(2)} + 2) \) |                     |
| \( F(4) \)      | -\frac{4}{3} | \frac{\alpha}{2} | \( k - 3 \) | \( -(K + 5) \) |                     |                     |
| \( G(3) \)      | -\frac{4}{3} | \frac{\alpha}{2} | \( k + 2 \) | \( K + 4 \) |                     |                     |

The only difference between the various different algebras is that the constants, \( c_1, c_2 \) take different values. These are provided in table (5). In the cases \( A(n|1), D(2|n) \) and \( D(2,1; \alpha) \) the surviving affine algebra, \( G \), is nonsimple: \( G = G^{(1)} \oplus G^{(2)} \), and the constants similarly break up in two. These are listed separately. It is worth while noticing that these constants are independent of the affine levels, expressed variously in terms of the affine Lie super algebra level, \( k \), or in terms of the ordinary affine \( G^{(i)} \) levels.

The above exceedingly compact form of the generic free field realisation specializes to (much more complicated looking) results in the special cases of \( \mathfrak{osp}(N|2) \) and \( A(n|1), n > 1 \) [10], \( A(1|1) \) [11], [8], and \( D(2|1; \alpha) \) [8], as well as our results for \( D(2|n) \) and \( D(2|1; \alpha) \) [26]. The results for \( F(4) \) and \( G(3) \) are new.

The full quantum algebra may now be constructed directly from the free field expressions. Indeed the constants are partly determined by demanding closure. Most of the OPE’s of the quantum ESA are rather trivial. Thus the OPE’s between the
energy momentum tensor and the other generators merely indicate that the conformal dimensions of the affine currents is 1, whereas that of the supercurrents is $3/2$. Finally the energy momentum tensor with itself requires knowledge of the central charges already provided in table (3). Similarly the OPE’s with the affine currents partly just indicate that the energy momentum tensor is an affine scalar whereas the supercurrents are primaries transforming according to suitable representations as already discussed at length. The affine currents with themselves generate the central extensions of the affine algebras as we have also discussed.

There remains the highly non trivial OPE’s between the supercurrents themselves. A glance at the several partly covered examples already presented in the literature will convince anyone that previous formulations were unsatisfactory. Now however, we are able to provide a very simple general expression:

$$G_{\theta,\gamma'}(z)G_{\gamma}(w) = \frac{f_1(k)}{k + \hbar^\vee (z - w)^3} - \frac{1}{k + \hbar^\vee} \sum_i f_2^{(i)}(k) \left( \frac{2}{(z - w)^2} + \frac{\partial J_{\gamma,\gamma'}^{(i)}(w)}{z - w} \right)$$

$$- \frac{2}{\theta^2} \delta_{\gamma,\gamma'} \frac{T(w) - T_G(w)}{z - w} + \frac{1}{k + \hbar^\vee} \frac{(J^2)_{\gamma,\gamma'}(w)}{z - w}$$

(63)

Here

$$T_G(w) \equiv \frac{1}{2(k + \hbar^\vee)} \left\{ \sum_{\alpha \in \Delta^0(G)} \frac{\alpha^2}{2} J_{-\alpha}(w) + \sum_i H_i H_i(w) \right\}$$

is similar to the sugawara energy momentum tensor, but in fact carries the incorrect normalization, namely the normalization that would be needed for the “hat” part whereas here we are only dealing with the full generators. Also, the trace part of $(J^2)_{\gamma,\gamma'}(w)$ is of a similar form but (in general) with yet another normalization. In each concrete case one may wish to combine these two contributions. The great similarity between this expression and the classical one given previously (and corresponding to the limit $k \to \infty$) is only obtained when the composite operator $J^2$ is “normal ordered” according to the symmetrized prescription:

$$: A(z)B(z) : \equiv \frac{1}{2} \oint_z \frac{dw \ A(z)B(w) + A(w)B(z)}{z - w}.$$  

It should be emphasized that the striking simplicity of these expressions is perhaps slightly deceptive. This point is particularly relevant for the cases where the algebra $G$ is non-simple. In these cases, namely, there is no à priori natural relative normalization between the representation matrices and indeed between the roots of the two commuting algebras, had it not been for the fact that the structure arose from a Lie superalgebra reduction. But the fact that these commuting algebras are embedded in a simple Lie algebra implies a precise choice for that relative normalization. It is only with that very choice, which allows us to introduce the currents $J_{\gamma,\gamma'}$ in a particular normalization, that the simplicity obtains. So even though the
final expressions appear superficially to make no reference to the Lie super algebra, in fact that underlying structure remains of importance.

5. Degenerate Representation of $G$ Extended Superconformal Algebras

The free field realizations, which we have summarized for all extended superconformal algebras above, can give rise to efficient calculational methods. To begin with, however, they give rise to very highly reducible representations. Thus, in order to discuss representation theory based on them we must introduce appropriate screening charges and understand how reducible representations can be viewed as cohomologies of those. We here take a number of steps in this direction.

The Fock space of the $G$ extended superconformal algebras is a tensor product of ones for $|\Delta^1_+|$ fermions $\chi_\gamma$, free fields for the affine Lie algebra $\hat{G}$, and a free boson $\phi(z)$ coupled to the world sheet curvature.

The free field representations of affine Lie algebras $\hat{G}$ follows from our discussion above[32]. Let $\bar{\alpha}_i (i = 1,\ldots,r)$ be simple roots of the even subalgebra $G$. $\bar{\lambda}_i (i = 1,\ldots,r)$ the fundamental weights of $G$ satisfying $2\bar{\lambda}_i \cdot \bar{\alpha}_j = \delta_{ij}$. $\Phi_{\bar{\lambda}}(z)$ is a primary field of the affine Lie algebra $\hat{G}$ at level $K$, with weight $\bar{\lambda}$ in the highest weight module with highest weight $\bar{\Lambda}$. In the free field representation, this field can be expressed as $p_{\bar{\lambda}}(z)e^{i\bar{\lambda} \cdot \phi(z)}$, where $p_{\bar{\lambda}}(z)$ is a polynomial consisting of terms, $\gamma_{\alpha_1} \cdots \gamma_{\alpha_k}(z)$ ($\alpha_i \in \Delta_+(G)$) such that $\bar{\lambda} = -\bar{\Lambda} + \alpha_1 + \cdots + \alpha_k$. (Note that in the present prescription, the vertex operator $e^{i\bar{\lambda} \cdot \phi(z)}$ represents the lowest weight state $\Phi_{\bar{\Lambda}}(z)$.)

Denote the total Fock space as $F_{\chi,\bar{\Lambda},p} = F^x_{\chi} \otimes F^G_{\bar{\Lambda}} \otimes F^\phi_p$, where $F^x$ is a fermionic Fock space constructed from $\chi_\gamma$ ($\gamma \in \Delta^1_+$). $F^G_{\bar{\Lambda}}$ is a Fock space of the algebra $\hat{G}$ built on a primary field $\Phi_{\bar{\lambda}}(z)$. $F^\phi_p$ is a Fock space built on a vertex operator $V_p(z) = e^{iv\sqrt{\theta}\phi(z)}$. The dual spaces $(F^G_{\bar{\Lambda}})^*$ and $(F^\phi_p)^*$ are isomorphic to $F^G_{-2\rho_G - \bar{\Lambda}}$ and $F^\phi_{-Q - p}$, respectively.

A primary field of a $G$ extended superconformal algebra is expressed as the products of three fields:

$$V_{\gamma_1,\ldots,\gamma_{\lambda},p}(z) = \chi_{\gamma_1}(z) \cdots \chi_{\gamma_{\lambda}}(z)\Phi_{\bar{\lambda}}(z)e^{ip\sqrt{\theta}\phi(z)}$$

(64)

where $\gamma_i$ are positive odd roots. The conformal weight of Eq.(64) is given by

$$\Delta = \frac{l}{2} + \frac{\bar{\Lambda}(\bar{\Lambda} + 2\rho_G)}{2\alpha_+^2} + \frac{1}{2}(p^2 + Qp)\theta^2.$$  

(65)

5.1 Screening operators

In order to study the representation of the algebra using free fields, we must specify the screening operators which commute with the generators of the extended superconformal algebra. We consider screening operators which correspond to the simple
roots of the Lie superalgebra $g$. These screening operators are BRST-equivalent to those of the affine Lie superalgebra $\hat{g}$ \[^{[14]}\]. In the present choice of the simple root system of the Lie superalgebra in table \[^{[1]}\] the simple roots of the even subalgebra $G$ are a subset of those of $g$ (see Appendix A). Thus, first of all we shall get the screening operators corresponding to the simple roots of the affine Lie algebra $\hat{G}$. Since the remaining simple roots are odd, they will correspond to fermionic type screening operators. As discussed for example by Kato and Matsuda\[^{[2]}\] fermionic screening operators may only occur in single contour integrals without producing ambiguous short distance behaviours. Hence it is convenient to introduce yet another screening operator, which characterizes the $A_1$ even subalgebra corresponding to the root $\theta$, even though that root does not belong to our choice of simple roots.

5.1.1. Affine screening operators

First we can take the standard screening operators $S_{\dot{\alpha}_i}(z)$ of the affine Lie algebra $\hat{G}$ as those of $G$ extended superconformal algebra:

$$S_{\dot{\alpha}_i}(z) = s_{\dot{\alpha}_i}(z) e^{\frac{i\gamma_\alpha z}{\alpha_+}} ,$$

(66)

where $s_{\dot{\alpha}_i}$ consists of terms like $\beta_{\dot{\alpha}_i}$ and $\gamma_{\alpha_1} \cdots \gamma_{\alpha_k} \beta_{\alpha_1+\cdots+\alpha_k+\dot{\alpha}_i}$ with $\alpha_1, \ldots, \alpha_k \in \Delta_+(G)$. These screening operators are used for the characterization of singular vectors in the Fock modules of the affine Lie algebra $\hat{G}$.

5.1.2. Fermionic screening operators

Next, we consider the screening operators, which corresponds to the odd simple roots. In the case of $A(n|1)$ there are two of those, in all other cases there is only one, cf. Appendix A. We have a fermion for every (negative) odd root (our labeling pertains to the positive odd roots), corresponding to the fundamental representation of the even subalgebra $G$ of dimensions $|\Delta_1^+|$. Denote the highest weight of these representations as $\bar{\Lambda}^*$. As we have seen, the odd roots themselves (suitably projected to the root subspaces of the appropriate even subalgebras) may be taken as the weights. In order that the screening operator commutes with the $\hat{G}$ currents, it should be a $\hat{G}$ singlet operator i.e. the operator product expansion with the $\hat{G}$ currents should be regular. These observations lead to fermionic screening operators of the following type:

$$S_f(z) = \sum_\gamma \chi_\gamma \phi_{\theta-\gamma} \exp(-\frac{i\sqrt{\theta^2} \phi(z)}{2\alpha_+}),$$

(67)

where $\gamma$ runs over the roots corresponding to the weights of the representation (with the highest weight $\bar{\Lambda}^*$).

The case of $A(n|1)$, where the algebra $G$ is $sl(n+1) \oplus u(1)$, is slightly more complicated than the other cases due to the fact that the representation of the fermions (or the supercurrents) is reducible: Roots $\gamma$ and $\theta - \gamma$ belong to different irreducible representations. Correspondingly we get two odd roots and two fermionic screening
operators by restricting the sum above to weights of one or the other irreducible representation. Further, it may be convenient to explicitly bosonize the (“hat part” of the) $u(1)$ algebra. We refer to our paper[26] for more details.

One may verify that these fermionic screening operators indeed either commute with or produce total derivatives in OPE's with the generators. For the supercurrents this holds by virtue of the Knizhnik-Zamolodchikov equations [43].

This treatment generalizes previous ones in special cases[10, 11, 44, 26] and again provides a unified account.

5.1.3. Screening operators corresponding to the $\theta$-direction

Finally we need a screening operator to characterize the $\theta$ direction. Denote this screening operator by $S_\theta(z)$. Based on several worked out examples we expect that this operator takes the form:

$$S_\theta(z) = s_\theta(z)\exp\left(\frac{2i\alpha_+\phi(z)}{\sqrt{\theta^2}}\right),$$

(68)

where $s_\theta(z)$ is a $G$ singlet operator containing (among many others) a term $\prod_{\gamma \in \Delta^+_1} \chi_\gamma$, and hence having conformal dimension $|\Delta^+_1|/2$. The requirement that $S_\theta(z)$ has conformal dimension 1, then fixes the “momentum” in the vertex operator part. This assumption is justified in part by the following list of results:

When we consider the $osp(N|2)$ case, for $N = 1$ and 2, the screening operators are the well known ones of $N = 1$ and 2 minimal models, and they are of this form: In fact they may be given as $S_\theta = \psi_1 \exp(-\alpha_+\phi(z))$ and $S_\theta = (\psi_1 \psi_2 - \frac{1}{K} J_{12})\exp(-\alpha_+\phi(z))$, respectively.

For the $N = 3$ case, this kind of screening operator has been found by Miki [44].

In the case of $N = 4$ $sl(2)$ superconformal algebra this screening operator has been obtained by Matsuda [11], and it is again of the above form. Also we have provided the explicit form for the so-called doubly extended $N = 4$ algebra[4, 11] corresponding to the case of $D(2|1;\alpha)$ which generalizes the two $N = 4$ cases corresponding to affine $so(4)$ and $su(2)$.

One must expect that this procedure can be generalized to any $N$, but so far we have not found completely general expressions. From the cases studied it seems clear that we must expect all kinds of terms with pairs of fermions from $\prod_{\gamma \in \Delta^+_1} \chi_\gamma$ being replaced by appropriate affine current operators, in such a way that the combined group theory quantum numbers work out correctly.

In the following we assume the existence of this kind of screening operator for any $G$. Even though we do not have the explicit form of $s_\theta(z)$ in all cases, the knowledge of the quantum numbers suffices for the determination of singular vectors as we shall see.

5.2. Structure of null fields

Based on the above observations on the screening operators, we discuss the structure of singular vectors of $G$ extended superconformal algebras. We consider the Neveu-Schwarz sector for simplicity.
Firstly we consider the singular fields corresponding to the affine Lie algebra \( \hat{G} \). These are given by the following screened vertex operators:

\[
\Psi_{\beta_1, \ldots, \beta_m}(z) = \oint du_1 \cdots du_m S_{\beta_1}(u_1) \cdots S_{\beta_m}(u_m) \Phi_{\bar{\Lambda}}(z),
\]

where \( \beta_a (a = 1, \ldots, m) \) are simple roots of the algebra \( G \); i.e. \( \beta_a = \bar{\alpha}_{i_a} \) for some \( i_a = 1, \ldots, r \). The contours of integrations are taken for example as in the prescription by Kato and Matsuda [42]. The “mass-shell-condition” that the above contour integral represents a singular field is derived either by considerations of pole singularities [42], or equivalently by noticing that the quantum numbers of the contour integral are the same as those of the field \( \Phi_{\bar{\Lambda}}(z) \) whereas, if the result is a descendant it must be of a primary pertaining to the weight \( -\sum_{a=1}^{m} \beta_a \).

It is then a simple matter to work out the difference in conformal dimension of these two fields, using Eq.(65) and demand that it be an integer:

\[
\frac{1}{\alpha_+^2} \sum_{a<b} \beta_a \cdot \beta_b - \frac{1}{\alpha_+^2} \sum_{a=1}^{m} \beta_a \cdot \bar{\Lambda} = -M, \tag{70}
\]

with a positive integer \( M \). If \( \sum_{a=1}^{m} \beta_a = n'\bar{\alpha} \) and \( M = nn' \) for a positive root \( \bar{\alpha} \in \Delta_+(G) \) and positive integers \( n \) and \( n' \), we get the Kac-Kazhdan formula [45]:

\[
(\bar{\Lambda} + \rho_G) \cdot \bar{\alpha} = +n\alpha_+^2 + n'\bar{\alpha}_+^2. \tag{71}
\]

This formula and its dual, which is obtained by replacing \( \bar{\Lambda} \) by \(-2\rho_G - \bar{\Lambda}\), characterize the singular vectors of the Fock module of the affine Lie algebra \( \hat{G} \).

The fermionic singular vectors are given by considering a screened vertex operator of the form:

\[
\Psi^{\bar{\Lambda}, \rho}(z) = \oint du S_f(u) V^{\bar{\Lambda}}(z), \tag{72}
\]

where \( V^{\bar{\Lambda}}(z) = \Phi_{\bar{\Lambda}} \exp(i p \sqrt{\theta^2} \phi(z)) \). Again the contour integral has the quantum numbers of \( V^{\bar{\Lambda}}(z) \), but (if singular) must be a descendant of a primary of the Fock module \( F^{G}_{\bar{\Lambda} + \bar{\Lambda}_*} \otimes F^{\phi}_{p - \frac{1}{\alpha_+}} \) since \( S_f \) carries momentum \(-1/2\alpha_+\). Equivalently the non-zero existence of the above contour integral requires the condition:

\[
\frac{\bar{\Lambda} \cdot \bar{\Lambda}^*}{\alpha_+^2} - \frac{p\theta^2}{2\alpha_+} = -M, \tag{73}
\]

where \( M \) is a positive integer. In this case \( \Psi^{\bar{\Lambda}, \rho}(z) \) is a singular vector at level \( M - \frac{1}{2} \).
The null fields in the $\theta$ direction can be obtained from the screened vertex operators:
\[
\Psi^p_r(z) = \oint du_1 \cdots du_r S_\theta(u_1) \cdots S_\theta(u_r) V_p(z),
\] (74)
where $V_p(z) = e^{i p v \sqrt{\theta^2 \phi(z)}}$. The on-shell condition becomes
\[
\frac{r(r-1)}{2} \left( \frac{2\alpha_+}{\theta^2} \right)^2 \theta^2 + 2 r p \alpha_+ = -M.
\] (75)
with a positive integer $M$. Writing $M$ as $\frac{r(s-|\Delta_1^1|+2)}{2}$, we find that $p$ is given by
\[
p_{r,-s} = \frac{r-1}{\theta^2} \alpha_+ - \frac{s - |\Delta_1^1| + 2}{4\alpha_+}.
\] (76)
In this case $\Psi^p_r(z)$ is a null field in the Fock module $F^\phi_{-Q_{pr,s}}$ at level $\frac{r s}{2}$, where
\[
p_{r,s} = \frac{1-r}{\theta^2} \alpha_+ + \frac{s + |\Delta_1^1| - 2}{4\alpha_+}.
\] (77)
The precise range for $s$ cannot be identified unambiguously until the multiplicities of zeros of the Kac determinant (or something equivalent) has been dealt with.

The formulas Eq.(71), Eq.(73) and Eq.(77) characterize the whole singular vector structure of the $G$ extended superconformal algebras.

6. Conclusions and Discussion

In the present paper we have studied $G$ extended superconformal algebras from the viewpoint of classical and quantum hamiltonian reductions of an affine Lie superalgebra $\hat{g}$, with even Lie subalgebras $\hat{G} \oplus \hat{sl}(2)$. That framework seems to provide an attractive classification scheme for extended superconformal algebras of the type with generators consisting of just the energy-momentum tensor, the (dimension $3/2$) supercurrents and some affine currents, the commutation relations, however, being allowed to contain quadratic composites of these generators.

We have derived very compact generic expressions for all these algebras and their free field realizations in the classical case, and we have demonstrated that these expressions also hold in the quantum case modulo simple “renormalizations” of certain constants. We have provided all these constants, thereby completing previous works with partial results [3, 14, 18, 23, 26, 40, 41, 44].

We have introduced a set of screening operators for all these $G$ extended superconformal algebras. Using the null field construction, we have identified the primary fields corresponding to degenerate representations of these algebras.

For future work, it would be of interest to work out in detail the cohomologies of the BRST operators we have constructed in order to understand the origin of the free field realizations more completely from that point of view. In general the quantum hamiltonian reduction which we have employed works “algorithmically” only
at the level of the energy momentum tensor. It would be very useful to have a better understanding of how the remaining generators are built in terms of free fields as well. The fact that we have found expressions of such generic nature for the supercurrents, may perhaps yield some clues towards such an understanding.

For the screening operators our understanding seems to be enough for constructing singular vectors. Thus it would seem possible to find the Kac-determinants in all cases. For detailed calculation of correlation functions there still remains the task of constructing the completely explicit version of one of the screening charges, which is so far known only in a finite number of cases. With this knowledge it is in principle possible to calculate correlation functions and characters for these models. They will then be expressed as products of those of $\hat{G}$ affine Lie algebras, Virasoro minimal models and free fermions.

Compared to the linearly extended superconformal algebras, a geometrical interpretation of the present non-linear algebras is unclear. It seems an interesting problem to try to find a way of interpreting these non-linear symmetries in terms of non-linear $\sigma$-models on non-symmetric Riemannian manifolds. This problem is important in order to clarify the geometrical meaning of the $W$-algebras.

The present construction of the extended superconformal algebra is based on the hamiltonian reduction of the affine Lie superalgebras. It is well understood that in the case of $W$-algebras associated with simple Lie algebras, the hamiltonian reduction \cite{12, 13, 14, 15} provides a connection to various integrable systems such as Toda field theory \cite{16} and the generalized KdV hierarchy \cite{17}. In the present case it is natural to expect the super Liouville model coupled to Wess-Zumino-Novikov-Witten (WZNW) models or the KdV hierarchy coupled to affine Lie algebras to arise. In the bosonic case, this kind of integrable system has been partially studied \cite{20}.

One might further generalize the hamiltonian reduction procedure to any Lie superalgebra. This would then give rise to a super $W$-algebra coupled to WZNW models. Interesting work in this direction has been undertaken in particular in the classical case \cite{30}.

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Appendix. Root Systems of Lie Superalgebras

In this appendix, we describe the root systems of the Lie superalgebras with the even subalgebra $G \oplus A_1$ as given in table 1 (with the exception of $B(1|n)$, cf. the discussion in the text). $\theta$ is the simple root of $A_1$. We use the orthonormal basis $e_i$ ($i \geq 1$) with positive metric and $\delta_j$ ($j \geq 1$) with negative metric:

$$e_i \cdot e_j = \delta_{ij}, \quad \delta_i \cdot \delta_j = -\delta_{ij}, \quad e_i \cdot \delta_j = 0. \quad (78)$$

1. $A(n|1)$ ($n \geq 1$), (rank $n + 2$, the dual Coxeter number $h^\vee = n - 1$)

Simple roots:
\[\alpha_1 = \delta_1 - e_1, \quad \alpha_i = e_{i-1} - e_i, \quad (i = 2, \ldots, n + 1), \quad \alpha_{n+2} = e_{n+1} - \delta_2.\]

Positive even roots: \(e_i - e_j, \quad (1 \leq i < j \leq n + 1), \quad \theta = \delta_1 - \delta_2.\)

Positive odd roots: \(\delta_1 - e_j, \quad e_j - \delta_2, \quad (j = 1, \ldots, n + 1).\)

2. \(B(n|1)\) (\(n \geq 0\)), \((\text{rank } n + 1, h^\vee = 3 - 2n)\)

Simple roots:
\[\alpha_1 = e_1 - \delta_1, \quad \alpha_{i+1} = \delta_i - \delta_{i+1}, \quad (i = 1, \ldots, n - 1), \quad \alpha_{n+1} = \delta_n.\]

Positive even roots: \(\theta = 2e_1, \quad \delta_i \pm \delta_j, \quad (1 \leq i < j \leq n), \quad \delta_i, \quad (i = 1, \ldots, n)\)

Positive odd roots: \(e_1, \quad e_1 \pm \delta_j, \quad (j = 1, \ldots, n).\)

3. \(D(n|1)\) (\(n \geq 2\)), \((\text{rank } n + 1, h^\vee = 4 - 2n)\)

Simple roots:
\[\alpha_1 = e_1 - \delta_1, \quad \alpha_{i+1} = \delta_i - \delta_{i+1}, \quad (i = 1, \ldots, n - 1), \quad \alpha_{n+1} = \delta_{n-1} + \delta_n.\]

Positive even roots: \(\theta = 2e_1, \quad \delta_i \pm \delta_j, \quad (1 \leq i < j \leq n).\)

Positive odd roots: \(e_1 \pm \delta_j, \quad (j = 1, \ldots, n).\)

4. \(D(2|n)\) (\(n \geq 1\)) \((\text{rank } n + 2, h^\vee = 2n - 2)\)

Simple roots:
\[\alpha_1 = -\delta_2 - \delta_1, \quad \alpha_2 = \delta_1 - e_1, \quad \alpha_{i+2} = e_i - e_{i+1}, \quad (i = 1, \ldots, n - 1), \quad \alpha_{n+2} = 2e_n.\]

Positive even roots: \(-\delta_2 - \delta_1, \quad \theta = \delta_1 - \delta_2, \quad e_i + e_j, \quad (1 \leq i < j \leq n), \quad e_i - e_j, \quad (1 \leq i < j \leq n).\)

Positive odd roots: \(\delta_1 \pm e_j, -\delta_2 \pm e_j, \quad (j = 1, \ldots, n).\)

5. \(D(2|1; \alpha)\) (\(\alpha \neq 0, -1, \infty\)), \((\text{rank } 3, h^\vee = 0)\)

Simple roots:
\[\alpha_1 = \frac{1}{2}(\sqrt{3}e_1 + \delta_1 + \delta_2 + \delta_3), \quad \alpha_2 = -\delta_1, \quad \alpha_3 = \delta_1 - \delta_2, \quad \alpha_4 = \delta_2 - \delta_3.\]

Positive even roots: \(\alpha_2, \quad \alpha_3, \quad \theta = 2\alpha_1 + \alpha_2 + \alpha_3.\)

Positive odd roots: \(\alpha_1, \quad \alpha_1 + \alpha_2, \quad \alpha_1 + \alpha_3, \quad \alpha_1 + \alpha_2 + \alpha_3.\)

6. \(F(4)\) \((\text{rank}: 4, h^\vee = -3)\)

Simple roots:
\[\alpha_1 = \frac{1}{2}(\sqrt{3}e_1 + \delta_1 + \delta_2 + \delta_3), \quad \alpha_2 = -\delta_1, \quad \alpha_3 = \delta_1 - \delta_2, \quad \alpha_4 = \delta_2 - \delta_3.\]

Positive even roots: \(\alpha_2, \quad \alpha_3, \quad \alpha_4, \quad \alpha_2 + \alpha_3, \quad \alpha_3 + \alpha_4, \quad 2\alpha_2 + \alpha_3, \quad \alpha_2 + \alpha_3 + \alpha_4, \quad 2\alpha_2 + 2\alpha_3 + \alpha_4, \quad \theta = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4.\)

Positive odd roots: \(\alpha_1, \quad \alpha_1 + \alpha_2, \quad \alpha_1 + \alpha_2 + \alpha_3, \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \quad \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4.\)

7. \(G(3)\) \((\text{rank}: 3, h^\vee = 2)\)

Simple roots:
\[\alpha_1 = \sqrt{2}\delta_1 + \frac{2e_1 - e_2 - e_3}{3}, \quad \alpha_2 = -\frac{e_1 + 2e_2 - e_3}{3}, \quad \alpha_3 = -e_2 + e_3.\]

Positive even roots: \(\alpha_2, \quad \alpha_3, \quad \alpha_2 + \alpha_3, \quad 2\alpha_2 + \alpha_3, \quad 3\alpha_2 + \alpha_3, \quad 3\alpha_2 + 2\alpha_3, \quad \theta = 2\alpha_1 + 4\alpha_2 + 2\alpha_3.\)
Positive odd roots: \( \alpha_1 \), \( \alpha_1 + \alpha_2 \), \( \alpha_1 + \alpha_2 + \alpha_3 \), \( \alpha_1 + 2\alpha_2 + \alpha_3 \), \( \alpha_1 + 3\alpha_2 + \alpha_3 \), \( \alpha_1 + 3\alpha_2 + 2\alpha_3 \), \( \alpha_1 + 4\alpha_2 + 2\alpha_3 \).

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