Branch-and-Bound with Barrier: 
Dominance and Suboptimality Detection for 
DD-Based Branch-and-Bound

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The branch-and-bound algorithm based on decision diagrams introduced by Bergman et al. in 2016 is a framework for solving discrete optimization problems with a dynamic programming formulation. It works by compiling a series of bounded-width decision diagrams that can provide lower and upper bounds for any given subproblem. Eventually, every part of the search space will be either explored or pruned by the algorithm, thus proving optimality. This paper presents new ingredients to speed up the search by exploiting the structure of dynamic programming models. The key idea is to prevent the repeated exploration of nodes corresponding to the same dynamic programming states by storing and querying thresholds in a data structure called the Barrier. These thresholds are based on dominance relations between partial solutions previously found. They can be further strengthened by integrating the filtering techniques introduced by Gillard et al. in 2021. Computational experiments show that the pruning brought by the Barrier allows to significantly reduce the number of nodes expanded by the algorithm. This results in more benchmark instances of difficult optimization problems being solved in less time while using narrower decision diagrams.

**Key words**: discrete optimization; decision diagrams; branch-and-bound

**History**:  

1. Introduction
Discrete optimization with decision diagrams (DDs) is a recent framework for solving constraint optimization problems that can be modeled with dynamic programming (DP). Apart from offering new modeling perspectives, this technique can exploit the compactness of DP formulations within an adapted branch-and-bound (B&B) algorithm introduced by Bergman et al. (2016). In addition to performing the search in the DP state space, DD-based B&B has its own way of deriving lower and upper bounds. Both are obtained by
compiling bounded-width DDs that will either under- or over-approximate the objective for a given subproblem. These approximate DDs are respectively called restricted \cite{Bergman2014} and relaxed \cite{Andersen2007, Bergman2014} DDs.

As opposed to classical B&B for \textit{mixed-integer programming} (MIP), the DD-based B&B does not split the search space into disjoint parts. The reason is that in this framework, DDs are based on a DP formulation of a discrete optimization problem and such kind of model typically contains many overlapping subproblems. As a result, the B&B may explore some subproblems multiple times. In addition, the approximate DDs compiled during the algorithm can also repeat a lot of the work done previously because they cover overlapping parts of the search space. A first attempt to address this problem was made in \cite{Coppé2022} by changing the ordering of the nodes in the B&B. With a breadth-first ordering, it is certain that at most one B&B node corresponding to each given subproblem will be explored. However, this approach sacrifices the benefits of best-first search – always exploring the most promising node to try to improve the incumbent solution and tighten the bounds at the same time.

In this paper, the same problem is tackled while allowing a best-first ordering of the B&B nodes. In the same fashion that a closed list prevents the re-expansion of nodes in shortest-path algorithms, we propose to maintain a data structure called the Barrier that stores a threshold for each exact node reached by relaxed DDs during the B&B. To exploit the information contained in relaxed DDs to the fullest extent, we introduce the concept of dominance and pruning thresholds. The former detects dominance relations between partial solutions while the latter accounts for nodes filtered by the rough upper bounds and local bounds introduced by \cite{Gillard2021}, as these thresholds are then propagated in a bottom-up fashion. With the help of the Barrier, partial paths that are guaranteed to be either dominated or suboptimal can thus be pruned during the compilation of following approximate DDs.

In the same line of research, \cite{Rudich2022} were able to speed up the search by reusing previous work when the relaxed DDs are compiled by separation. However, they focus on warm-starting the compilation of relaxed DDs with a subset of a previously compiled DD whereas our contribution aims to avoid the repeated exploration of entire parts of the search space. These are thus transversal improvements that may be combined.
The paper begins with a brief literature review in Section 2 and a summary of DD-based B&B in Section 3, starting from the DD representation of a discrete optimization problem to the latest improvements of the algorithm. This general introduction is followed by a discussion of the caveats that this paper tries to address in Section 4. Next, Section 5 presents the dominance and pruning thresholds, as well as their integration to the B&B and the compilation of approximate DDs. Section 6 then discusses the limitations of the techniques introduced in the paper. Finally, experimental results on three different discrete optimization problems are reported and discussed in Section 7 before concluding.

2. Previous Work

Decision diagrams (DDs) as used today in the field of discrete optimization originate from compact encodings of Boolean functions, first known as binary decision programs (Lee 1959) and later as binary decision diagrams (BDDs) (Akers 1978, Bryant 1986). Multi-valued decision diagrams (MDDs) were then suggested by Kam and Brayton (1990) as an extension of BDDs to variables and functions taking values from discrete sets. These different variants of DDs were successfully applied in different domains such as formal verification (Hu 1995), model checking (Clarke et al. 1994), computer-aided design (Minato 1995) and optimization (Lai et al. 1994, Hachtel and Somenzi 1997, Becker et al. 2005, Hadžić and Hooker 2006, 2007). More recently, Andersen et al. (2007) introduced relaxed DDs that act as a constraint store for constraint programming (CP) solvers. These DDs can represent a superset of the feasible variable assignments with a bounded width to balance the computational costs and the filtering strength. Relaxed DDs were then adapted by Bergman et al. (2014a) as a mean of deriving upper bounds for discrete optimization problems that can be modeled with dynamic programming (Hooker 2013). They were followed closely by their lower bounding counterparts called restricted DDs (Bergman et al. 2014b). Based on these two ingredients, a complete B&B algorithm was presented in (Bergman et al. 2016), providing a novel general framework for discrete optimization. Several improvements to the algorithm have been suggested since: Gillard et al. (2021a) proposed additional bounding procedures to enhance the pruning of the B&B. When relaxed DDs are compiled by separation, Rudich et al. (2022) showed that intermediate DDs could be stored and reused to initialize the following compilations. Another promising research direction is to discover variable orderings that yield approximate DDs with better bounds. Cappart et al. (2022)
designed a reinforcement learning approach to perform this task while Karahalios and van Hoeve (2022) experimented with a portfolio of orderings. For a complete overview of the latest applications of DDs in the field of discrete optimization, we refer the reader to the survey by Castro et al. (2022).

3. Background

This section provides an overview of DD-based discrete optimization. It starts with a reminder on how discrete optimization problems are modeled with dynamic programming (DP). Next, the translation of a DP formulation to DDs is explained. In particular, restricted and relaxed DDs are defined as well as the top-down compilation algorithm used to obtain them. The section ends with a description of the DD-based B&B algorithm built off lower and upper bounds derived from restricted and relaxed DDs respectively.

3.1. Dynamic Programming Formulation

A discrete optimization problem $\mathcal{P}$ is defined by $\max \{ f(x) \mid x \in D \land C(x) \}$, where $x = \langle x_0, \ldots, x_{n-1} \rangle$ is a vector of variables taking values in their respective domains $x_j \in D_j$ with $D = D_0 \times \ldots \times D_{n-1}$, $C$ is a predicate that can represent any set of constraints that solutions need to satisfy and $f$ is the objective function to be maximized. To formulate $\mathcal{P}$ with DP, the following elements are needed:

- the control variables $x_j \in D_j$ with $j \in \{0, \ldots, n-1\}$.
- a state space $S$ partitioned into $n+1$ sets $S_0, \ldots, S_n$ that correspond to distinct stages of the DP model. In particular, $S_j$ contains all states having $j$ variables assigned. Several special states are also defined: the root $\hat{r}$, the terminal $\hat{t}$ and the infeasible state $\hat{0}$.
- the transition functions $t_j : S_j \times D_j \rightarrow S_{j+1}$ for $j = 0, \ldots, n-1$ that specify how states of consecutive stages are connected.
- the transition value functions $h_j : S_j \times D_j \rightarrow \mathbb{R}$ for $j = 0, \ldots, n-1$ that compute the contribution of a transition to the objective.
- an optional root value $v_r$ to account for constant terms in the objective.

If those elements can be defined for $\mathcal{P}$, then finding the optimal solution is equivalent to solving:

$$\begin{align*}
\text{maximize} & \quad f(x) = v_r + \sum_{j=0}^{n-1} h_j(s^j, x_j) \\
\text{subject to} & \quad s^{j+1} = t_j(s^j, x_j), \text{ for all } x_j \in D_j, j = 0, \ldots, n-1 \\
& \quad s^j \in S_j, j = 0, \ldots, n \text{ and } C(x).
\end{align*}$$

(1)
3.2. Decision Diagrams

In the context of discrete optimization, a decision diagram (DD) is a graphical representation of the set of solutions to a given problem \( \mathcal{P} \). DDs are well suited to encode and manipulate compact formulations such as DP models, due to their ability to preserve the uniqueness of overlapping subproblems. In mathematical terms, a decision diagram \( \mathcal{B} = (U, A, \sigma, l, v) \) is a layered directed acyclic graph consisting of a set of nodes \( U \) that are connected by a set of arcs \( A \). Each node is mapped to a DP state by the function \( \sigma \).

The set of nodes \( U \) can be partitioned into layers \( L_0, \ldots, L_n \) corresponding to the successive stages of the DP model, each containing one node for each distinct state. Therefore, arcs \( a = (u_j \rightarrow u_{j+1}) \) connect nodes of consecutive layers \( u_j \in L_j, u_{j+1} \in L_{j+1} \) and represent the transition between states \( \sigma(u_j) \) and \( \sigma(u_{j+1}) \). The label \( l(a) = d \) of an arc encodes the decision that assigns the value \( d \in D_j \) to variable \( x_j \). The value \( v(a) \) of the arc stores the transition value. Both the first and last layer – \( L_0 \) and \( L_n \) – contain a single node, respectively the root \( r \) and the terminal node \( t \). Each \( r \rightarrow t \) path \( p = (a_0, \ldots, a_{n-1}) \) that traverses the DD from top to bottom represents a solution to \( \mathcal{P} \). Its objective value is given by the accumulation of the arc values along the path: \( v(p) = v_r + \sum_{j=0}^{n-1} v(a_j) \). Finally, the set of all solutions appearing in the DD is denoted \( \text{Sol}(\mathcal{B}) \) and \( \mathcal{B} \) is said exact if \( \text{Sol}(\mathcal{B}) = \text{Sol}(\mathcal{P}) \).

Algorithm 1 details the top-down compilation of DDs: starting with the first layer containing only a root node, the DD is developed by applying every possible transition to each node of the last completed layer. At line 10, the width of layer \( j \) is computed and compared against a parameter \( W \) called the maximum width. In order to obtain an exact DD, one can simply set \( W = \infty \). For combinatorial optimization problems however, the size of the corresponding exact DD can grow exponentially, making it intractable to compute. Thankfully, DDs of smaller maximum width can also be compiled. Restricted and relaxed DDs are both capable of maintaining the width of all layers under a given maximum width. The former one yields lower bounds and feasible solutions while the latter can be used to derive upper bounds.

Restricted DDs At line 11, restriction is synonymous with removing surplus nodes. As detailed by Algorithm 2, the least promising nodes of the layer are selected according to a heuristic and simply dropped before resuming the compilation as normal. As a result, some solutions will not appear in restricted DDs. Nonetheless, all remaining solutions are feasible. For a restricted DD \( \mathcal{B} \), we thus have \( \text{Sol}(\mathcal{B}) \subseteq \text{Sol}(\mathcal{P}) \).
Algorithm 1 Compilation of DD rooted at node $u_r$ with maximum width $W$.

1: $i \leftarrow$ index of the layer containing $u_r$
2: $L_i \leftarrow \{u_r\}$
3: for $j = i$ to $n - 1$ do
4:   $\text{pruned} \leftarrow \emptyset$
5:   if $j > i$ then // pruning for the root done in Algorithm 3
6:     for all $u \in L_j$ do
7:       if $\text{Barrier}.\text{contains}(\sigma(u))$ and $v(u) \leq \theta(\text{Barrier}.\text{get}(\sigma(u)))$ then
8:         $\text{pruned} \leftarrow \text{pruned} \cup \{u\}$
9:   $L_j' \leftarrow L_j \setminus \text{pruned}$
10:  if $|L_j'| > W$ then
11:     restrict or relax the layer to get $W$ nodes with Algorithm 2
12:    $L_{j+1} \leftarrow \emptyset$
13:  for all $u \in L_j'$ do
14:    if $\sigma(u) \leq \bar{v}$ then // rough upper bound pruning
15:      continue
16:  for all $d \in D_j$ do
17:    create node $u'$ with state $\sigma(u') = t_j(\sigma(u), d)$
18:    create arc $a = (u \xrightarrow{d} u')$ with $v(a) = h_j(\sigma(u), d)$ and $l(a) = d$
19:    add $u'$ to $L_{j+1}$ and add $a$ to the DD

Algorithm 2 Restriction or relaxation of layer $L_j'$ with maximum width $W$.

1: while $|L_j'| > W$ do
2:   $\mathcal{M} \leftarrow$ select nodes from $L_j'$
3:   $L_j' \leftarrow L_j' \setminus \{\mathcal{M}\}$
4:   create node $\mu$ with state $\sigma(\mu) = \oplus(\mathcal{M})$ and add it to $L_j'$ // for relaxation only
5:   for all $u \in \mathcal{M}$ and arc $a = (u' \xrightarrow{d} u)$ incident to $u$ do
6:     replace $a$ by $a' = (u' \xrightarrow{d} \mu)$ and set $v(a') = \Gamma_{\mathcal{M}}(v(a), u)$

Relaxed DDs As opposed to restricted DDs that delete a part of the solutions, relaxed DDs will never remove feasible solutions but they might introduce infeasible ones. This is achieved by merging nodes together and thus redirecting arcs pointing to different nodes.
to a single meta-node, as performed by lines 4 to 5 of Algorithm 2. It requires defining problem-specific merging operators to merge the corresponding DP states. If $\mathcal{M}$ is the set of states corresponding to the nodes to merge, the operator $\oplus(\mathcal{M})$ gives the state of the merged node. The resulting state should encompass all merged states and preserve their outgoing transitions. A second operator denoted $\Gamma_{\mathcal{M}}$ can be specified to adjust the value of the arcs incident to the merged node. As the merging operator gives an approximate representation of all merged states, it can also introduce infeasible transitions. In addition, merging nodes $u_1, u_2$ allows to combine any $r \rightsquigarrow u_1$ path with any $u_2 \rightsquigarrow t$ path and vice-versa. For any relaxed DD $\overline{B}$ with a valid relaxation operator, we can thus write $\text{Sol}(\overline{B}) \supseteq \text{Sol}(P)$. In relaxed DDs, we distinguish exact nodes from relaxed nodes. A node $u$ in a relaxed DD $\overline{B}$ rooted at node $u_r$ is said exact if all $u_r \rightsquigarrow u$ paths in $\overline{B}$ lead to the same state $\sigma(u)$. All nodes that do not meet this criteria are called relaxed.

**Example 1.** Figure 1(a) shows the exact DD for a dummy problem with 4 variables and binary decisions. The optimal solution is highlighted in blue and has a value of 28. One possible restricted DD of width 3 for this problem is pictured on Figure 1(b). Despite having far fewer nodes and arcs than the exact DD, the lower bound found is the optimal
solution of the problem. With the relaxed DD of Figure 1(c), an upper bound of 29 is found. Exact and relaxed nodes are respectively shown in green and red.

3.3. Branch-and-Bound

The two ingredients presented in Section 3.2 allow to build a DD-only branch-and-bound (B&B) algorithm, as introduced by Bergman et al. (2016). This algorithm is summarized by Algorithm 3. Starting with the root node, line 4 loops over the set of open nodes contained in the Fringe. For each of them, a restricted DD $B$ is compiled at line 12. It may provide a solution with value $v^*(B)$ improving the current best, in which case the incumbent is updated at lines 13 to 15. If $B$ is exact, then this subproblem can be considered as fully explored. Otherwise, a relaxed DD $\overline{B}$ must be developed at line 17. The role of this relaxed DD is twofold. First, it allows to identify the set of subproblems that need to be further explored so that all parts of the search space are covered. This set of subproblems is called an exact cutset (EC) of $B$ and it is defined as a set of exact nodes such that any path from the root of $\overline{B}$ to $t$ must cross at least one of them. Second, $\overline{B}$ allows to compute an upper bound $\overline{v}(u)$ for each node $u \in EC(\overline{B})$, which is used to discard nodes that are guaranteed to lead to solutions of value worse or equal to $\underline{v}$. This check is performed at line 21 before adding a node to the Fringe and is repeated at line 6 when a node is selected for exploration. The upper bound $\overline{v}(u)$ can be either $v^*(\overline{B})$ the value of the best solution in $\overline{B}$, the rough upper bound or the local bound attached to $u$ – more details on that in Section 3.4. This process is repeated until the Fringe is emptied.

The most commonly used EC is the last exact layer (LEL), which is the deepest layer in $\overline{B}$ that contains only exact nodes. Another option is the frontier cutset (FC) defined as:

$$FC(\overline{B}) = \{u \in \overline{B} \mid u \text{ is exact } \land \exists a = (u \xrightarrow{d} u') \text{ such that } u' \text{ is relaxed}\}. \quad (2)$$

Example 2. Back to our example problem defined by Figure 1(a), the DDs represented in Figure 1(b) and (c) could be respectively the first restricted and relaxed DDs compiled during the B&B. After developing the relaxed DD, an EC must be computed and added to the Fringe. On Figure 1(c), nodes $a_2, a_3, b_1, b_2$ and $c_1$ constitute the FC and are highlighted with bold green circles.
Algorithm 3 The DD-based branch-and-bound algorithm.

1: $\text{Fringe} \leftarrow \{r\}$
2: $\text{Barrier} \leftarrow \emptyset$
3: $v \leftarrow -\infty$ // value of the incumbent solution
4: while Fringe is not empty do
5: $u \leftarrow$ best node from Fringe, remove it from Fringe
6: if $\tau(u) \leq v$ then
7: continue
8: if $\text{Barrier.contains}(\sigma(u))$ then
9: $(\theta, \text{explored}) \leftarrow \text{Barrier.get}(\sigma(u))$
10: if $v(u) < \theta$ or $(v(u) = \theta \land \text{explored})$ then
11: continue
12: $\mathcal{B} \leftarrow \text{Restricted}(u)$ // compile restricted DD with Algorithm 1
13: if $v^*(\mathcal{B}) > v$ then // update incumbent
14: $v \leftarrow v^*(\mathcal{B})$
15: $x \leftarrow x^*(\mathcal{B})$
16: if $\mathcal{B}$ is not exact then
17: $\mathcal{B} \leftarrow \text{Relaxed}(u)$ // compile relaxed DD with Algorithm 1
18: compute local bounds with Algorithm 4 applied to $\mathcal{B}$
19: update $\text{Barrier}$ with Algorithm 5 applied to $\mathcal{B}$
20: for all $u' \in \text{ExactCutset}(\mathcal{B})$ do
21: if $\tau(u') > v$ then
22: add $u'$ to Fringe
23: return $(x, v)$

3.4. Additional Filtering Techniques

Gillard et al. (2021a) introduced two additional bounding mechanisms. The first one is the rough upper bound (RUB) that aims at identifying suboptimal nodes as early as possible during the compilation of approximate DDs. It does so by computing a cheap problem-specific upper bound $\tau_{\text{rub}}(u)$ for every node at line 14 of Algorithm 1, which can help pruning many nodes before even generating their successors. This upper bound is defined as $\tau_{\text{rub}}(u) = v(u) + \text{rub}(\sigma(u))$, where $v(u)$ is the value of node $u$ and is equal to the value
Algorithm 4 Computation of the local bound $\overline{v}_{\text{locb}}(u)$ of every node $u$ in a relaxed DD $\overline{B}$.

1: $i \leftarrow$ index of the root layer of $\overline{B}$
2: $(L_i, \ldots, L_n) \leftarrow \text{Layers}(\overline{B})$
3: $v_\uparrow(u) \leftarrow -\infty$ for each node $u \in \overline{B}$
4: $v_\uparrow(t) \leftarrow 0$
5: $\text{mark}(t) \leftarrow \text{true}$
6: for $j = n$ down to $i$ do
7: \hspace{1em} for all $u \in L_j$ do
8: \hspace{2em} if $\text{mark}(u)$ then
9: \hspace{3em} $\overline{v}_{\text{locb}}(u) \leftarrow v_u^*(\overline{B}) + v_\uparrow(u)$
10: \hspace{2em} for all arc $a = (u', u)$ incident to $u$ do
11: \hspace{3em} $v_\uparrow(u') \leftarrow \max\{v_\uparrow(u'), v_\uparrow(u) + v(a)\}$
12: \hspace{2em} $\text{mark}(u') \leftarrow \text{true}$

of the longest $r \rightsquigarrow u$ path known at any point of the execution of the B&B. The second term, denoted $rub(\sigma(u))$, contains the actual upper bound formula that can be specified for each state of a given optimization problem.

In the original B&B algorithm, every node of the EC retrieved from the relaxed DD developed at line 17 of Algorithm 3 is assigned the same upper bound, given by $v^*(\overline{B})$. The local bounds (LocBs) presented by Gillard et al. (2021a) refine this reasoning by computing for each node $u$ the value of the longest path in $\overline{B}$ that actually crosses $u$. Before giving a formal definition of the LocBs, we establish two additional elements of notation. First, we denote by $v^*_u(\overline{B})$ the value of the longest path to a node $u$ within a relaxed DD $\overline{B}$. Similarly, the value of the longest path connecting $u$ and $u'$ in $\overline{B}$ will be referred to as $v^*_{u \rightsquigarrow u'}(\overline{B})$.

**Definition 1 (Local Bound).** Given $\overline{B}$ a relaxed DD for problem $\mathcal{P}$ and a node $u \in \overline{B}$, the local bound $\overline{v}_{\text{locb}}(u)$ of $u$ is the value of the longest path in $\overline{B}$ that crosses $u$, if it exists:

$$\overline{v}_{\text{locb}}(u) = \begin{cases} v^*_u(\overline{B}) + v^*_{u \rightsquigarrow t}(\overline{B}), & \text{if } (u \rightsquigarrow t) \in \overline{B}, \\ -\infty, & \text{otherwise}. \end{cases}$$

LocBs can be computed efficiently by performing a bottom-up traversal of $\overline{B}$, as formalized by Algorithm 4. In the algorithm, the $\text{mark}$ flags are propagated to all nodes that have at least one path reaching the terminal node $t$. In parallel, the value $v^*_{u \rightsquigarrow t}(\overline{B})$ of
each marked node $u$ is computed by accumulating the arc values traversed upwards. Initial estimates for $v_{u\rightarrow t}^{*}(\mathcal{E})$ are denoted $v_{\uparrow}(u)$. At the terminal node, $v_{\uparrow}(t)$ is set to zero at line 4. For all other nodes, the value $v_{\uparrow}$ is updated by their successors at line 11. The local bound computation stated by Equation (3) is then performed at line 9.

Both the RUB and the LocB are incorporated in $v(u)$ the upper bound attached to each node $u$ and used to prune nodes at lines 6 and 21 of Algorithm 3. To maximize the pruning potential of this upper bound, the minimum upper bound obtained is always kept.

**Example 3.** In Figure 1(c), below the value of the longest path shown in blue next to each node $u$, we add the LocB in red. Given the incumbent solution computed in Figure 1(b), the LocBs allow to discard nodes $b_1, b_2$ and $c_1$ from the FC since for each of them $v_{\text{locb}}(u) \leq 28$.

### 3.5. Illustrative Example with the 0–1 Knapsack Problem

In this section, we provide a simple but complete modeling of the Knapsack problem within the framework presented in Section 3 to illustrate how all the components are formulated.

Given a set of items $N = \{0, \ldots, n-1\}$ with weights $W = \{w_0, \ldots, w_{n-1}\}$ and values $V = \{v_0, \ldots, v_{n-1}\}$, the goal is to choose a set of items that maximize the total value while keeping the total weight under a given capacity $C$. In the DP formulation of this problem, we associate a control variable $x_j \in \{0, 1\}$ to each item $j$ that decides whether it is added to the knapsack. States are uniquely identified by the remaining capacity of the knapsack. The root state is thus $\hat{r} = C$ and the transition functions are given by:

$$t_j(s^j, x_j) = \begin{cases} 
  s^j - x_j w_j, & \text{if } s^j \geq x_j w_j, \\
  \hat{0}, & \text{otherwise}.
\end{cases}$$

It means that the weight of item $j$ is subtracted from the capacity when the decision is to include it in the knapsack. If the remaining capacity does not allow to select item $j$, the transition is redirected to the infeasible state $\hat{0}$. Similarly, the value of item $j$ is added to the objective with the transition value functions $h_j(s^j, x_j) = x_j v_j$. The root value is $v_r = 0$ as there is no constant in the objective function.

For the relaxation, remember that the operator $\oplus$ must compute a state such that all transitions originating from states in $\mathcal{M}$ are preserved. In this case, it suffices to keep the maximum remaining capacity among the states in $\mathcal{M}$: $\oplus(\mathcal{M}) = \max \mathcal{M}$. For this simple formulation, the operator $\Gamma_M$ can be defined as the identity function since there is no need
to modify the arc values. Finally, we can define a RUB for this problem. A naive choice is to simply add all remaining items to the knapsack, disregarding the capacity constraint. This can be written as \( \text{rub}(s_j) = \sum_{k=j}^{n-1} v_k \). Of course, a better choice could be to use the LP bound introduced by Dantzig (1957).

4. Caveats of DD-based Branch-and-Bound

As explained in Section 3, DD-based B&B enables solving discrete optimization problems by taking advantage of the compactness of DP models. Nevertheless, a few observations suggest that all the properties of this type of model have not yet been exploited. First, we point out that branching does not split the search space into disjoint parts. Indeed, the very nature of DP models is the ability to solve a large problem by recursively dividing it into smaller overlapping subproblems. To give a better intuition of why this might cause the algorithm to waste computational effort, we propose to look at DD-based B&B as a classical shortest-path algorithm performed on the graph induced by a DP model – in case of directed acyclic graphs, the shortest-path and longest-path problems are equivalent.

Even if Algorithm 3 does not necessarily enqueue the direct successors of a node and that it features many pruning mechanisms, it essentially performs a best-first search without a closed list. In shortest-path algorithms like A* (Hart et al. 1968), nodes are added to the closed list as soon as they are expanded. This list is then used to check if a given node was already explored before expanding it again, or before adding it to the set of open nodes. As DD-based B&B does not accumulate any information except lower and upper bounds, nothing prevents the algorithm from adding multiple nodes with the same DP state to the Fringe nor expanding several of them at different stages of the search.

Our second observation follows from the first but concerns the lower level of DD-based B&B: the approximate DDs developed to obtain bounds and identify an EC can overlap a lot. Suppose we would compute exact DDs for the two subproblems contained in the FC of Figure 1(c) respectively rooted at \( a_2 \) and \( a_3 \). As one can observe on Figure 1(a), they respectively contain 11 and 10 nodes, among which 8 nodes appear in both DDs. Yet, at most one of them contains the optimal path leading to any of these nodes. Thus, if these two DDs are compiled successively within the B&B, most of the work done for the first DD is repeated to compile the second one while many transitions actually yield a value worse or equal than the one obtained before. In general, the structure of DP state transition systems causes such a scenario to occur very frequently.
5. Branch-and-Bound with Barrier

In this section, we explain how the caveats mentioned in Section 4 can be addressed. Our idea is to augment DD-based B&B with some form of closed list. This principle cannot be directly applied, however, as nodes are expanded at two different levels: the B&B queue and the compilation of approximate DDs. In the first case, the optimal value of a node selected for exploration in the B&B is already known if a best-first order is used. On the other hand, during the compilation of approximate DDs, many nodes are explored with no guarantee that their optimal value was found. A classical closed list thus cannot be maintained. Still, we can save a threshold value that should be improved by future transitions reaching the same nodes in order for them to bring any novelty in the search. In particular, we can compute dominance thresholds that leverage the information contained up to the EC of relaxed DDs. When nodes are pruned using RUBs or LocBs, these thresholds must be adjusted to preserve the optimality of the algorithm. We thus define pruning thresholds and show how to combine them with dominance thresholds.

This section first defines dominance thresholds and explains how they can be computed for all exact nodes visited during the compilation of relaxed DDs. Then, it is shown how pruning thresholds are derived and propagated in case of RUB and LocB pruning. Finally, we describe how the B&B and the compilation of approximate DDs can be modified to benefit from the Barrier – the data structure containing all previously computed thresholds.

5.1. Dominance Thresholds

For all exact nodes traversed during the compilation of a relaxed DD, there is no guarantee that their optimal value was obtained. A classical closed list blocking all future expansions of these nodes thus cannot be used. Still, we can attach a threshold to each exact node $u$ that gives the value needed to improve any of the paths traversing node $u$. Further transitions to node $u$ will thus be considered only if they are able to exceed the threshold, preventing the repetition of some of the work done before.

We now discuss how one can compute the highest threshold $\theta(u)$ for an exact node $u$ of a relaxed DD $\overline{B}$ while preserving the optimality guarantees of the algorithm. A first idea for the threshold values is to simply use $v^*_u(\overline{B})$. If a future transition reaching a node $u$ does not improve this value, it can safely be discarded as it is dominated by the first path discovered. This is already a nice observation, but it does not exploit the information encoded by paths traversing $u$ and reaching other exact nodes of the relaxed DD. What if
Figure 2  
(a) A relaxed DD rooted at node $a_2$. Nodes are annotated with the best value obtained $v_u^*(\mathcal{B})$ in blue and their dominance threshold $\theta_d(u)$ in black, where applicable. (b) Compilation of a relaxed DD rooted at node $a_3$, given the dominance thresholds computed in (a).

all transitions originating from $u$ are dominated by other known paths before reaching any node of the EC? Actually, we can compute for each exact node $u$ the hypothetical value that future paths reaching $u$ should surpass in order to improve the longest path of any node in the EC of $\mathcal{B}$ reachable from $u$. We call this value the dominance threshold of $u$.

**Definition 2 (Dominance threshold).** Given $\mathcal{B}$ a relaxed DD for problem $\mathcal{P}$ and an exact node $u \in \mathcal{B}$. We define $\theta_d(u)$ the dominance threshold of $u$ as the minimum value that $v(u)$ needs to exceed in order to improve the value of any node in a given EC of $\mathcal{B}$:

$$\theta_d(u) = \begin{cases} v_u^*(\mathcal{B}), & \text{if } u \in \text{ExactCutset}(\mathcal{B}), \\ \min_{u' \in \text{ExactCutset}(\mathcal{B}) \setminus \{u\}} v_u^*(\mathcal{B}) - v_{u \rightarrow u'}(\mathcal{B}), & \text{otherwise.} \end{cases} \quad (4)$$

which can also be defined recursively:

$$\theta_d(u) = \begin{cases} v_u^*(\mathcal{B}), & \text{if } u \in \text{ExactCutset}(\mathcal{B}), \\ \min_{a = (u \rightarrow u') \in \mathcal{B}} \theta_d(u') - v(a), & \text{otherwise.} \end{cases} \quad (5)$$

**Example 4.** For the same problem as in Example 1, Figure 2(a) shows a relaxed DD compiled from node $a_2$. The FC consists of the nodes $b_2, b_3, c_3, d_3$. For each of them, we set $\theta(u) = v_u^*(\mathcal{B})$ and compute the dominance thresholds of all other exact nodes by applying Equation (5). For node $c_4$, we obtain $\theta_d(c_4) = \theta_d(d_3) - v(c_4 \rightarrow d_3) = 24 - 6 = 18$. 
Algorithm 5 Computation of the threshold $\theta(u)$ of every exact node $u$ in a relaxed DD $\mathcal{B}$ and update of the Barrier.

1: $i \leftarrow$ index of the root layer of $\mathcal{B}$
2: $(L_1, \ldots, L_n) \leftarrow \text{Layers}(\mathcal{B})$
3: $\theta(u) \leftarrow \infty$ for all $u \in \mathcal{B}$
4: for $j = n$ down to $i$ do
5:   for all $u \in L_j$ do
6:     if $\text{Barrier}.\text{contains}(\sigma(u))$ and $v^*_u(\mathcal{B}) \leq \theta(\text{Barrier}.\text{get}(\sigma(u)))$ then
7:       $\theta(u) \leftarrow \theta(\text{Barrier}.\text{get}(\sigma(u)))$
8:     else
9:       if $v_{\text{rub}}(u) \leq v$ then // rough upper bound pruning
10:          $\theta(u) \leftarrow v^*_u(\mathcal{B}) + v - v_{\text{rub}}(u)$
11:     else if $u \in \text{ExactCutset}(\mathcal{B})$ then
12:       if $v_{\text{locb}}(u) \leq v$ then // local bound pruning
13:          $\theta(u) \leftarrow \min\{\theta(u), v^*_u(\mathcal{B}) + v - v_{\text{locb}}(u)\}$
14:     else
15:       $\theta(u) \leftarrow v^*_u(\mathcal{B})$
16:     if $u$ is exact then
17:       $\text{explored} \leftarrow$ false if $u \in \text{ExactCutset}(\mathcal{B})$ else true
18:       $\text{Barrier}.\text{insertOrReplace}(\sigma(u), (\theta(u), \text{explored}))$
19:   for all arc $a = (u' \xrightarrow{d} u)$ incident to $u$ do
20:     $\theta(u') \leftarrow \min\{\theta(u'), \theta(u) - v(a)\}$

Algorithm 5 describes how the thresholds can be computed by performing a single bottom-up pass on a relaxed DD. The thresholds are denoted as $\theta$ since the dominance thresholds are coupled with the pruning thresholds covered in Section 5.2. In the most simple version, the thresholds are initialized at line 3 with a default value of $\infty$ for all nodes, except for nodes of the EC for which we set $\theta(u)$ to $v^*_u(\mathcal{B})$ at line 15. Then, the values are propagated upwards with lines 19 and 20 by applying Equation (5). The threshold computed for each exact node is saved at line 18 in a data structure called the Barrier for later use. At line 6 there is an additional check with the Barrier: if it already contains a threshold larger than $v^*_u(\mathcal{B})$, then $\theta(u)$ is initialized with the stored value and this known
threshold will be used for propagation instead. Note that Algorithm 5 can be applied to any type of EC, but an important detail is that thresholds should not be inserted in the Barrier for exact nodes that are below the EC to allow for their later exploration. As the FC contains all the deepest exact nodes, the algorithm can be applied as is. For the LEL however, a check must be added at line 16. In the next section, we explain how to deal with relaxed DDs with nodes pruned using RUBs or LocBs.

5.2. Pruning Thresholds

When nodes are pruned either with RUBs or LocBs, they must still be taken into account in the computation of the threshold values. Indeed, even if node $u$ has been pruned during the compilation of a relaxed DD either with its RUB at line 14 of Algorithm 1 or with its LocB before adding it to the Fringe at line 21 of Algorithm 3, there could exist a different $r \leadsto u$ path improving the value of $u$ and that would avoid being pruned. And on the other hand, saving a threshold value for pruned nodes could allow to filter paths that are guaranteed to be pruned again. This is why we define a second type of threshold called pruning thresholds. These thresholds must be propagated upwards just like dominance thresholds and combined with them to allow the discovery of paths that would avoid the pruning performed by either RUBs or LocBs in – not necessarily direct – successor nodes. This is also true for relaxed nodes, because the LocBs computed in relaxed DDs are conditioned by the RUB pruning that can happen in further layers.

**Definition 3 (Pruning threshold).** Given $\mathcal{B}$ a relaxed DD for problem $P$, a node $u \in \mathcal{B}$ and an upper bound $v(u)$ for $u$ – that can be either a RUB or a LocB – and the value of the incumbent solution $v$. Let us define $S(u) = \{u\} \cup \{u' \mid (u \leadsto u') \in \mathcal{B}\}$ as the set of nodes reachable from $u$ in $\mathcal{B}$, including itself. We define $\theta_p(u)$ the pruning threshold of $u$ as the minimum value that $v(u)$ needs to exceed in order to avoid the pruning of at least one node in $S(u)$:

$$\theta_p(u) = \min_{u' \in S(u)} \theta_{locp}(u') - v^*(u' \leadsto u)(\mathcal{B})$$

with $\theta_{locp}(u)$ the local pruning threshold that only considers $v(u)$ the upper bound of $u$:

$$\theta_{locp}(u) = \begin{cases} v^*(\mathcal{B}) + v - v(u), & \text{if } v(u) \leq v, \\ \infty, & \text{otherwise.} \end{cases}$$

Pruning thresholds can be defined equivalently in a recursive manner:

$$\theta_p(u) = \min \left\{ \theta_{locp}(u), \min_{a=(u \leadsto u') \in \mathcal{B}} \theta_p(u') - v(a) \right\}.$$
Figure 3 (a) A relaxed DD rooted at node \( a_2 \). Nodes are annotated with their best value obtained \( v_u(B) \) in blue, their LocB \( \bar{v}_{locb}(u) \) in red and their threshold \( \theta(u) \) in black, where applicable. (b) Compilation of a relaxed DD rooted at node \( a_3 \), given the thresholds computed in (a).

In the end, both types of thresholds are combined in a single threshold \( \theta(u) = \min\{\theta_d(u), \theta_p(u)\} \) that allows to prune nodes that do not meet any of the two criteria necessary to bring novelty in the search.

**Example 5.** We consider the same scenario as in Example 4 of a relaxed DD compiled from node \( u_2 \). This time however, we perform LocB pruning for every node of the FC. Also, we suppose that \( d_3 \) was filtered with RUB pruning thanks to the incumbent solution found in Figure 1(b) with value \( v = 28 \), as shown on Figure 3(a). Therefore, we have \( \theta(d_3) = v^*_d(B) + v - \bar{v}_{rub}(d_3) = 24 + 28 - 26 = 26 \). This value is propagated directly to \( c_4 \): \( \theta(c_4) = \theta(d_3) - v(c_4 \rightarrow d_3) = 26 - 6 = 20 \) which gives us stronger threshold. For nodes of the FC, we first compare their LocB with the incumbent solution and notice that they are actually all pruned. However, we must take into account the pruning thresholds propagated from below. For \( c_3 \), we have \( \theta(c_3) = \min\{v^*_c(B) + v - \bar{v}_{locb}(c_3), \theta(d_3) - v(c_3 \rightarrow d_3)\} = \min\{13 + 28 - 22, 26 - 11\} = 15 \).

In Algorithm 5, pruning thresholds are computed at line 10 for nodes pruned with their RUB during the top-down compilation. For nodes of the EC, the pruning threshold is set at line 13 if the LocB computed is smaller than the value of the incumbent solution. It assumes that the LocBs are already computed but in practice, they can be computed...
simultaneously. The propagation phase at line 20 then combines the dominance thresholds and the pruning thresholds by always keeping the minimum threshold. Note that at line 13, the threshold for nodes of the EC that are pruned with their LocB is computed by taking the minimum with its current value, this is to take into account pruning thresholds from relaxed nodes that could have been propagated from below, as specified by Equation (8).

The pruning thresholds presented in this section create a synergy between the dominance thresholds presented in Section 5.1 and the pruning mechanisms introduced by Gillard et al. (2021a). When nodes are pruned with their RUB or their LocB, they will never be re-explored unless a new path obtains a value that could allow them to exceed the value of the incumbent solution, according to the upper bound previously computed, or to do so in one of their successors. Moreover, during the bottom-up traversal of relaxed DDs done to compute LocBs, it is possible to detect "dead ends": nodes with no feasible outgoing transition. As a result, the bottom-up propagation of the thresholds permits to detect both suboptimality and infeasibility earlier in the future compilations of approximate DDs, in addition to dominance relations discussed before. In terms of complexity, computing the thresholds requires a single traversal of the relaxed DDs compiled so it has the same complexity as Algorithm 1 and Algorithm 4 that respectively handle the top-down compilation and the computation of the LocBs.

5.3. Filtering the Search Using the Barrier

Now that we have presented how thresholds can be computed for all exact nodes of relaxed DDs, let us describe in detail how these values are exploited in the B&B. There are two places where the Barrier might be very useful. First, in the top-down compilation of approximate DDs, as described by Algorithm 1. At line 7, the Barrier is queried and whenever a node $u$ does not improve on the threshold $\theta(u)$, it is added to the pruned set. This set of pruned nodes is used to define $L'_j$ at line 9, a clone of the $j$-th layer from which the pruned nodes have been removed. In the rest of the algorithm, the pruned layer $L'_j$ is employed instead of $L_j$ to prevent generating any outgoing transition from the pruned nodes. Nodes from the pruned set are kept in the original layer $L_j$ so that the threshold computations of Algorithm 5 can be performed correctly.

Furthermore, the Barrier can be used when selecting a node for exploration in the B&B loop. At line 8 of Algorithm 3 the Barrier is queried and may contain an entry with a threshold value as well as an explored flag. The node is ignored if its value is less than the
threshold or if it is equal but marked as already explored. Otherwise, the node must be explored and the Barrier is updated with the \textit{explored} flag turned to true. This explains why the filtering is ignored for the first layer in the DD compilation, see line \ref{line:ex} of Algorithm \ref{algo:ex}. When the thresholds are first inserted in the Barrier at line \ref{line:th} of Algorithm \ref{algo:th}, the \textit{explored} flag is set to \textit{true} for all nodes except those of the EC. For a given threshold value $\theta(u)$, the \textit{explored} flag thus helps to distinguish two types of nodes. First, the nodes that have been added to the Fringe from the EC of a relaxed DD $B$ with $\theta(u) = v_u^*(B)$ and that need to be further explored. And second, all exact nodes above the EC of any relaxed DD $B$ in which the threshold $\theta(u)$ was obtained. In this second case, they can be considered as explored since all their outgoing paths were either pruned or cross a node in the EC of $B$.

\textbf{Example 6.} Given the threshold values computed in Figure 2(a) for the relaxed DD rooted at node $a_2$, let us illustrate the compilation of a relaxed DD rooted at node $a_3$. For each transition showed in Figure 2(b), we compare the value obtained with the threshold stored in the Barrier. Transitions leading to nodes $b_3$ and $c_4$ are successfully pruned thanks to the information stored from the previous DD compilation. As a result, an exact DD is compiled from node $a_3$ of width only 2. Now if pruning thresholds are used as in Figure 3(a), stronger thresholds are computed. In this case, the compilation of a relaxed DD rooted at node $a_3$ is stopped by the Barrier at every node visited before, as pictured by Figure 3(b).

6. Limitations

The thresholds presented in Section 5 offer new pruning perspectives that will be shown to be very impactful in Section 7. Yet, they exhibit two main limitations that are discussed in this section.

6.1. Memory Consumption

In order to apply B&B with Barrier, extra information must be stored in memory. Some effort is done to reduce memory consumption by deleting thresholds as soon as they are no longer required by the algorithm. A sufficient condition to remove a threshold is when it concerns a node that belongs to a layer above the \textit{first active layer}.

\textbf{Definition 4 (First active layer).} Given $B$ the exact DD for problem $\mathcal{P}$ with layers $L_0, \ldots, L_n$. The \textit{first active layer} of the B&B specified by Algorithm \ref{algo:th} is defined as the lowest index $j$ such that a node of layer $L_j$ is in the Fringe or is currently selected for exploration at line \ref{line:ex}.
Thresholds related to nodes above the first active layer can be safely removed from the Barrier as there is no way of reaching them again. If memory consumption was nevertheless an issue, a simple solution would be to delete an arbitrary subset of the thresholds stored in the Barrier. This does not compromise the optimality guarantees of the algorithm since the only effect of thresholds is to prevent re-exploration of nodes. However, removing some thresholds decreases the pruning perspectives of the algorithm. Some memory will therefore be saved at the cost of speed. One could even imagine using eviction rules based on an activity measure of the thresholds in the Barrier. Note that all the algorithms presented in the paper are written as if the Barrier could remove some thresholds along the way.

An argument that could be held against the use of the Barrier in the context of DD-based B&B is that in the worst case, it still might need to accommodate as many nodes as there are nodes in the exact DD encoding the problem being solved. While this argument is true, we would like to point out that the aforementioned explosive worst case memory requirement already plagued the original B&B algorithm as it foresees nothing to bound the size of Fringe. Furthermore, we would like to emphasize that maintaining the Barrier is no more costly than maintaining the Fringe and much less costly than memorizing an actual instantiation of the exact DD. Indeed, the maintenance of both the Barrier and the Fringe is $O(|U|)$ – where $U$ is the set of nodes in the state space – whereas the requirement to store an actual instantiation of the exact DD in memory is $O(|U| + |A|)$ where $A$ is the set of arcs connecting the nodes in $U$. It is also worth mentioning that in that case, $|A|$ dominates $|U|$ since by definition of the domains and transition relations, the size of $A$ is bounded by the product of the domain sizes $\prod_{j=0}^{n-1}|D_j|$.

### 6.2. Variables Orderings

Many discrete optimization problems have an imposed variable ordering in their natural DP model. For instances, DP models for sequencing problems usually make decisions about the $j$-th position of the sequence at the $j$-th stage of the DP model (Cire and Van Hoeve 2013). For DP models that allow it, however, it has been shown that variable orderings can yield exact DDs of reduced size as well as approximate DDs with tighter bounds (Bergman et al. 2012, Cappart et al. 2022, Karahalios and van Hoeve 2022). When DDs are used within a B&B algorithm, variable orderings thus constitute an addititonal heuristic that can speed up the search. They can be separated in two categories: static and dynamic variable orderings. The former refers to a single variable ordering used for all DD compilations.
during the B&B. The latter denotes a heuristic that dynamically decides which variable to branch on when generating the next layer of a DD, based on the states contained in the current layer. Since the techniques introduced in this paper are solely based on the overlapping structure of the DP models, a dynamic variable ordering would most likely compromise much of the expected pruning. Indeed, it seems unlikely that many states would overlap in DDs compiled with different variable orderings, although ultimately it depends on the modeling of each specific problem.

7. Computational Experiments

This section presents the results of the computational experiments that were conducted to evaluate the impact of the additional pruning techniques presented in this paper. We performed experiments on three different well-known discrete optimization problems: the Traveling Salesman Problem with Time Windows (TSPTW), the Pigment Sequencing Problem (PSP) – also known as the Discrete Lot Sizing and Scheduling Problem (DLSP) in the literature – and the Single-Row Facility Layout Problem (SRFLP). We describe below the experimental setting used for each problem.

TSPTW The solvers were tested on a classical set of benchmark instances introduced in the following papers (Ascheuer 1996, Dumas et al. 1995, Gendreau et al. 1998, Langevin et al. 1993, Ohlmann and Thomas 2007, Pesant et al. 1998, Potvin and Bengio 1996). A dynamic width was used, where the maximum width for layers at depth $j$ is given by $n \times (j+1) \times \alpha$ with $n$ the number of variables in the instance and $\alpha$ a multiplying parameter for which several values were used in the experiments.

PSP A set of randomly generated instances was created with the number of items in $\{5, 7, 10\}$, the number of periods in $\{50, 100, 150, 200\}$ and the density in $\{0.9, 0.95, 1\}$. The density is computed as the number of demands over the number of time periods. An additional parameter $\rho \in \{0.001, 0.01, 0.1\}$ was introduced to control the proportion between the stocking costs and the changeover costs. The costs are then sampled uniformly in $[\rho \cdot 5000, \rho \cdot 15000]$ and $[(1-\rho) \cdot 5000, (1-\rho) \cdot 15000]$ respectively. To generate the demands, item type and time period pairs were selected uniformly among all possible values and added to the instance as long as it remained feasible. Five instances were generated for each combination of the parameters mentioned. A fixed width was used but which is also proportional to the number of variables: $n \times \alpha$. To better illustrate the efficiency of our approach, the MIP model denoted PIG-A-3 in Pochet and Wolsey (2006) was tested on the same instances.
SRFLP The experiments consist of a compilation of benchmark instances from (Amaral 2006, 2008, 2009, Anjos et al. 2005, Anjos and Vannelli 2008, Anjos and Yen 2009, Duff et al. 1989, Heragu and Kusiak 1991, Hungerländer and Rendl 2013, Nugent et al. 1968, Obata 1981, Sarker 1989, Simmons 1969, Yu and Sarker 2003) for the SRFLP as well as the Single-Row Equidistant Facility Layout Problem and the Minimum Linear Arrangement Problem – two of its subproblems. We limited our experiments to instances with fewer than 50 departments. The same width strategy as for the PSP was applied. The DD approach is compared with the MIP model introduced by Amaral (2009).

The models used to encode these problems within the DD framework are described in the Appendix. For all the experiments, the best results were obtained with a LEL cutset for the classical B&B and with a FC for B&B with Barrier. It is worth mentioning that the baseline approach already uses duplicate state detection in the Fringe. Concerning the heuristics of the approach, the node with the highest upper bound is selected first in the B&B loop. Moreover, nodes with the smallest longest path value are respectively deleted or merged in restricted and relaxed DDs compilations. For each problem, we tested multiple maximum widths by varying the parameter $\alpha \in \{1, 10, 100\}$. The DD approach was implemented in Rust, based on DDO (Gillard et al. 2021b) which is a generic library for DD-based optimization. The source code and all benchmark instances are available at https://github.com/vcoppe/ddo-barrier. The MIP models were solved with Gurobi 9.5.2 (Gurobi Optimization, LLC 2022) with the default settings. For all problems, the solvers were given 1800 seconds to solve each instance to optimality, using a single thread.

7.1. Impact of the Barrier

Figures 4 and 5 show the results of these experiments respectively in terms of computation time and number of DD nodes expanded during the search. This measure of nodes expanded accounts for all nodes expanded during top-down compilations of restricted and relaxed DDs. Each graph presents the total number of instances solved under any given time or nodes expanded. For each of the studied problems and whatever the maximum width used, B&B with Barrier is able to solve significantly more instances than the classical B&B within the time limit. This speedup is directly linked to a reduction of the number of DD nodes expanded, as can be seen Figure 5. This confirms our intuition that a lot of work is unnecessarily repeated by the classical B&B and shows that the pruning techniques introduced in this paper help neutralize much of this problem. Moreover, if we look at the
performance achieved using different maximum widths for the TSPTW and the PSP, one can notice that for the classical B&B, increasing the width helps solving more instances. Indeed, larger DDs allow stronger bounds to be derived and instances to be closed more quickly, as long as they are not too expensive to compile. Yet, the opposite observation can be made about B&B with Barrier: using smaller maximum widths results in solving more instances for the TSPTW and the PSP. This perfectly captures the double benefit of the Barrier. The strength of dominance and pruning thresholds allow to discard many transitions during the compilation of approximate DDs and avoid repeating previous work. As a result, narrower DDs can explore and prune the search space as fast while being much cheaper to compute, and sometimes identify uninteresting parts earlier in the search. Again, this observation about Figure 4 can be verified on Figure 5 where B&B with Barrier
manages to solve significantly more instances within a given maximum number of nodes when using narrower DDs. Another benefit of B&B with Barrier is that it fully exploits the potential of FCs: as all non-improving transitions are blocked by the Barrier, it is only natural to use the deepest possible exact cutset. The same cannot be said in the case of classical B&B because frontier cutsets usually contain many nodes, some of which are the parents of others, and this only exacerbates the caveats mentioned in Section 4.

For the SRFLP, the difference of performance obtained by varying the maximum width is not as significant as for the other two problems. This is probably because the problem is unconstrained and the relaxation is not very tight. As a result, the thresholds computed with small width DDs may not be strong enough to avoid exploring nodes that are explored by DDs of larger widths. As a matter of fact, slightly fewer instances are solved using narrow DDs for a same number of DD nodes expanded. In this case, wider DDs may derive better bounds and thresholds and perform slightly better. Still, using a Barrier with DDs of any width helps closing more instances than classical B&B.

To put these results into perspective, we compare them on Figure 4 to those obtained with MIP models for the PSP and the SRFLP. For the PSP, B&B with Barrier with $\alpha = 1$ could solve 454 of the generated instances while Gurobi solved 345 of them. On the other hand, classical B&B with $\alpha = 100$ solved only 311 instances. The addition of the Barrier thus helped improve DD-based B&B by a large margin and allowed to outperform Gurobi on this set of instances. For the SRLFP, classical B&B and B&B with Barrier with $\alpha = 100$ were able to solve respectively 12 and 4 instances less than Gurobi within the given time budget. Therefore, even if the best results are achieved with the MIP model, using the Barrier closed much of the gap that separates the two techniques.

7.2. Memory Analysis

The performance improvements discussed in Section 7.1 do not come completely for free, as the Barrier must store all the thresholds computed during the B&B algorithm. It is thus important to study the impact of this technique on the memory consumption of the algorithm. For every instance solved by each algorithm, the peak amount of memory used during the execution was recorded. Figure 6 shows the number of instances solved using a given maximum amount of memory. It appears that the memory consumption of classical B&B and B&B with Barrier are of the same order of magnitude. Actually, for the TSPTW and the PSP, as the instances get harder, B&B with Barrier even starts solving
more instances with the same peak amount of memory. Thus, even in terms of memory consumption, the cost of maintaining the Barrier seems to be compensated by its pruning effect which causes DDs to be smaller and sparser.

8. Conclusion

In this paper, we first discussed how the DD-based B&B algorithm tends to repeatedly explore overlapping parts of the search space. Then, we introduced dominance and pruning thresholds with the intention of overcoming this limitation. The former propagate dominance relations between partial solutions obtained within approximate DDs while the latter allow the approach to be combined with the pruning performed by LocBs and RUBs. Both types of thresholds are used in a single pruning mechanism that is able to discard many previously explored nodes during future DD compilations. Finally, we presented experimental results that clearly show the impact of the techniques introduced: B&B with Barrier vastly outperforms the classical B&B algorithm in terms of instances solved on the three discrete optimization problems studied in the paper and compares well to state-of-the-art MIP models. Furthermore, the experiments showed that the memory consumption induced by the Barrier was either acceptable or even overcompensated by its pruning effect.

DD-based B&B is known to parallelize very well and it is one of its main advantages compared to other approaches. The improvements described in this paper are based on a shared data structure that avoids successive DD compilations from being completely blind to previously done work. Therefore, concurrent DD compilations are not fully independent anymore and require some synchronization to access and update the Barrier. As future
work, we thus aspire to optimize the pruning capacities of the Barrier while minimizing contention in the data structure.

Appendix. Models Used in the Experimental Study

The three discrete optimization problems for which we present models in this appendix are minimization problems. The DD-based optimization framework remains applicable but the terminology must be adapted. For instance, the optimal solution of a minimization problem is given by the shortest path in the corresponding exact DD. Moreover, restricted and relaxed DDs respectively provide upper and lower bounds. Similarly, rough upper bounds are replaced with rough lower bounds (RLBs).

A. TSPTW

The TSPTW is a variant of the well-known Traveling Salesman Problem (TSP) where the cities are replaced by a set of customers \( N = \{0, \ldots, n-1\} \) that must each be visited during a given time window \( TW = (e_i, l_i) \). The first customer is a dummy customer that represents the depot where the salesman must begin and end its tour. As in the classical TSP, we are given a symmetrical distance matrix \( D \) that contains in each entry \((i,j)\) the distance \( D_{ij} \) separating customers \( i \) and \( j \). In addition to the time windows controlling the earliest and latest time when the salesman can visit the customers, the horizon \( H \) limits the time at which the salesman must return to the depot.

A.1. DP Model

The DP model presented here extends the one introduced by Held and Karp (1962) for the TSP, which successively decides the next customer to visit and defines states with the set of customers that still must be visited along with the current position of the salesman. For the TSPTW, this state representation is extended to a tuple \((\text{position}, \text{time}, \text{must visit}, \text{might visit})\). As merging of states having the salesman located at different positions will occur, \( \text{position} \) is a set of possible locations at which the salesman might be. Moreover, \( \text{must visit} \) and \( \text{might visit} \) are disjoint sets of customers that must or might be visited in order to close the tour even in case of relaxed states. Finally, states also contain the time at which the salesman reached the current position. We now describe the full DP model:

- Control variables: \( x_j \in N \) with \( j \in N \) decides which customer is visited in \( j\)-th position.
- State space: \( S = \{ (\text{position}, \text{time}, \text{must visit}, \text{might visit}) | \text{position, must visit, might visit} \subseteq N, \text{must visit} \cap \text{might visit} = \emptyset, 0 \leq \text{time} \leq H \} \). The root state is \( *r = (\{0\}, 0, N, \emptyset) \) and the terminal states are all states \( \{ (\{0\}, \text{time}, \emptyset, \text{might visit}) \} \) with \( 0 \leq \text{time} \leq H \).
- Transition functions:

\[
t_j(s^j, x_j) = \begin{cases} 
(t^\text{pos}_j(s^j, x_j), t^\text{time}_j(s^j, x_j), t^\text{must}_j(s^j, x_j), t^\text{might}_j(s^j, x_j)), & \text{if } x_j \in s^j, \text{must visit} \\
(t^\text{pos}_j(s^j, x_j), t^\text{time}_j(s^j, x_j), t^\text{must}_j(s^j, x_j), t^\text{might}_j(s^j, x_j)), & \text{if } x_j \in s^j, \text{might visit} \\
0, & \text{otherwise.}
\end{cases}
\]
where
\begin{align*}
t^{\text{pos}}_{ij}(s^i, x_j) &= \{x_j\} \\
t^{\text{time}}_{ij}(s^i, x_j) &= \max\{e_{x_j}, s^i.\text{time} + \min_{i \in s^i.\text{position}} D_{ix_j}\} \\
t^{\text{must}}_{ij}(s^i, x_j) &= s^i.\text{must}_\text{visit} \setminus \{x_j\} \\
t^{\text{might}}_{ij}(s^i, x_j) &= s^i.\text{might}_\text{visit} \setminus \{x_j\}
\end{align*}

In Equation (9), transitions are only allowed if they respect the time window constraint \(l_{x_j}\). Since the salesman can be at multiple different positions in relaxed states, the minimum distance is used to compute the arrival time. The second condition in Equation (9) ensures that no customers are selected from \(\text{might}_\text{visit}\) when there remains enough stops only for the customers in \(\text{must}_\text{visit}\).

- Transition value functions: \(h_{ij}(s^i, x_j) = \min_{i \in s^i.\text{position}} D_{ix_j}\).
- Root value: \(v_r = 0\).

### A.2. Relaxation

The merging operator is defined as follows:

\[ \oplus (M) = (\oplus_{\text{pos}}(M), \oplus_{\text{time}}(M), \oplus_{\text{must}}(M), \oplus_{\text{might}}(M)) \]

where
\begin{align*}
\oplus_{\text{pos}}(M) &= \bigcup_{s \in M} s.\text{position} \\
\oplus_{\text{time}}(M) &= \min_{s \in M} s.\text{time} \\
\oplus_{\text{must}}(M) &= \bigcap_{s \in M} s.\text{must}_\text{visit} \\
\oplus_{\text{might}}(M) &= \left( \bigcup_{s \in M} s.\text{must}_\text{visit} \cup s.\text{might}_\text{visit} \right) \setminus \left( \bigcap_{s \in M} s.\text{must}_\text{visit} \right)
\end{align*}

These operators ensure that all transitions are preserved and that the transition values are not increased.

Note that the \(\text{must}_\text{visit}\) and \(\text{might}_\text{visit}\) sets are disjoint so that \(\text{might}_\text{visit}\) contains customers that must or might be visited in some states, but no customers that must be visited in all states. The relaxed transition value operator is simply the identity function \(\Gamma_M(v, u) = v\).

### A.3. Rough Lower Bound

The RLB shown in Equation (11) computes an estimate of the remaining distance to cover in order for the salesman to finish his tour and return to the depot. For all customers \(i\) in \(\text{must}_\text{visit}\), the distance of the cheapest edge incident to \(i\) is added. Then, \(n - j - |\text{must}_\text{visit}|\) customers from \(\text{might}_\text{visit}\) must be visited in order to complete the tour. Since this is a lower bound computation, we create a vector \(C\) containing the cheapest edge incident to each customer in \(\text{might}_\text{visit}\) and select the \(n - j - |\text{must}_\text{visit}|\) shortest ones. We denote by \(X_{<,i}\) the \(i\)-th smallest element from a vector \(X\).

\[
\text{rlb}(s^i) = \sum_{i \in s^i.\text{must}_\text{visit}} \text{cheapest}_i + \sum_{i=1}^{n-j-|\text{must}_\text{visit}|} C_{<,i}
\]

A check is also added to detect states where it is impossible to transition independently to each of the customers from \(\text{must}_\text{visit}\) and to at least \(n - j - |\text{must}_\text{visit}|\) customers from \(\text{might}_\text{visit}\), given the current time and the time windows of the remaining customers. In this case, and if the estimate given by Equation (11) prevents the salesman from returning to the depot in time, a RLB of \(-\infty\) is returned to discard the corresponding states.
B. PSP

The PSP is a single-machine planning problem where one has to find a production schedule that satisfies a set of demands at minimal cost. Formally, there is a set of items types $I = \{0, \ldots, n-1\}$ that are associated with a stocking cost $S_i$. When the machine switches the production from item type $i$ to $j$, it incurs a changeover cost $C_{ij}$. The planning spans for a given time horizon $H$. For each time period $0 \leq p < H$ of this horizon, $Q^i_p \in \{0,1\}$ indicates whether an item of type $i$ must be delivered. When an item $i$ is produced at time period $p_1$ and delivered at period $p_2$, stocking the item costs $S_i(p_2 - p_1)$.

To give a better understanding of the problem, we hereby recall the MIP model denoted PIG-A-1 in (Pochet and Wolsey 2006). Variables $x^i_p \in \{0,1\}$ decide whether an item of type $i$ is produced at time period $p$. On the other hand, variables $y^i_p \in \{0,1\}$ decide whether the machine is ready to produce an item of type $i$ at time period $p$. Indeed, the machine can be idle at certain periods. Variables $q^i_p \in \mathbb{N}_0$ accumulate the quantity of items of type $i$ stored at period $p$. Finally, variables $\chi^{i,j}_p \in \{0,1\}$ capture a changeover between item types $i$ and $j$ at period $p$.

$$\min \sum_{i \in I} \sum_{p=0}^{H-1} S_i q^i_p + \sum_{i,j \in I} \sum_{p=0}^{H-1} C_{ij} \chi^{i,j}_p$$

$$q^i_{p-1} + x^i_p = q^i_p + Q^i_p \quad \forall i \in I, 0 \leq p < H \quad (12)$$

$$q^i_{p-1} = 0 \quad \forall i \in I \quad (13)$$

$$x^i_p \leq y^i_p \quad \forall i \in I, 0 \leq p < H \quad (14)$$

$$\sum_{i \in I} y^i_p = 1 \quad \forall 0 \leq p < H \quad (15)$$

$$\chi^{i,j}_p \geq y^i_{p-1} + y^j_p - 1 \quad \forall i,j \in I, 0 < p < H \quad (16)$$

Equation (13) models the stocking of the items and consumes them when needed, with the quantities initialized by Equation (14). Then, Equation (15) ensures that the machine is in the correct mode to produce an item and Equation (16) allows only one mode for each time period. Finally, Equation (17) sets the correct value for the changeover variables.

B.1. DP Model

The DP model introduced in (Gillard and Schaus 2022) is extended to allow relaxation as well as to cover instances where the machine can be idle at some time periods. It proceeds by taking decisions starting from the end of the planning horizon and working backwards to avoid expanding any infeasible states. For clarity, variables $x_j$ decide the type of item produced at period $j$. This implies that the reverse variable ordering $x_{H-1}, \ldots, x_0$ is used. Before pursuing the modeling of the PSP in the DD framework, let us define two additional matrices. The first one gives for each time period $0 \leq p \leq H$ and item type $i$, the previous demand of the same item type:

$$P^i_p = \begin{cases} -1, & \text{if } p = 0, \\ p - 1, & \text{if } Q^i_{p-1} > 0 \\ P^i_{p-1}, & \text{otherwise.} \end{cases}$$

The second matrix contains the remaining quantity to produce for each item type $i$ at every time period $0 \leq p < H$:

$$R^i_p = \begin{cases} Q^i_0, & \text{if } p = 0, \\ R^i_{p-1} + Q^i_p, & \text{otherwise.} \end{cases}$$

Equation (17) ensures that the machine is in the correct mode to produce an item and Equation (16) allows only one mode for each time period. Finally, Equation (17) sets the correct value for the changeover variables.
States \( \langle \text{item}, (\text{previous}_0, \ldots, \text{previous}_{n-1}) \rangle \) where \text{item} is the item type produced at the next time period and \text{previous}_i\) represents the time period of the previous demand to satisfy for item type \text{i}. We pose \( N' = N \cup \{ \bot \} \) where \( \bot \) is a dummy item type used to represent periods where the machine is idle.

- Control variables: \( x_j \in N' \) with \( 0 \leq j < H \) decides which item type is produced at time period \( j \).
- State space: \( S = \{ \langle \text{item}, (\text{previous}_0, \ldots, \text{previous}_{n-1}) \rangle \mid \text{item} \in N', \forall i \in N, 0 \leq \text{previous}_i < H \} \). The root state is \( \hat{r} = \langle \bot, (P^0_H, \ldots, P_H^{n-1}) \rangle \) and terminal states are of the form \( \langle \text{item}, (-1, \ldots, -1) \rangle \) with \text{item} \( \in N' \).

- Transition functions:

\[
t_j(s^l, x_j) = \begin{cases} 
\langle t^\text{item}_j(s^l, x_j), t^\text{prev}_j(s^l, x_j) \rangle, & \text{if } x_j \neq \bot \text{ and } s^l.\text{previous}_{x_j} \geq 0, \\
0, & \text{otherwise}.
\end{cases}
\]

where

\[
t^\text{item}_j(s^l, x_j) = \begin{cases} 
x_j, & \text{if } x_j \neq \bot \\
s^l.\text{item}, & \text{otherwise}.
\end{cases}
\]

\[
t^\text{prev}_j(s^l, x_j) = \begin{cases} 
(s^l.\text{previous}_0, \ldots, P^x_j.s^l.\text{previous}_{x_j}, \ldots, s^l.\text{previous}_{n-1}), & \text{if } x_j \neq \bot \\
s^l.\text{previous}, & \text{otherwise}.
\end{cases}
\]

The second condition in Equation (20) ensures that idle periods can only be scheduled when the total remaining demand is smaller than the number of remaining time periods.

- Transition value functions:

\[
h_j(s^l, x_j) = \begin{cases} 
C_{x_j.s^l.\text{item}}, & \text{if } x_j \neq \bot \text{ and } s^l.\text{item} \neq \bot \\
0, & \text{otherwise}.
\end{cases}
\]

- Root value: \( v_r = 0 \).

### B.2. Relaxation

The merging operator is defined as follows:

\[
\oplus (\mathcal{M}) = \langle \bot, (\min_{s \in \mathcal{M}} s.\text{previous}_0, \ldots, \min_{s \in \mathcal{M}} s.\text{previous}_{n-1}) \rangle.
\]

As merged states might disagree on the item type produced before, it is reset to \( \bot \). For each item type, the minimum time period at which the previous demand occurs is computed, meaning that all demands satisfied by a least one state are considered satisfied in the merged state. As for the TSPTW, the relaxed transition value operator is the identity function \( \Gamma_{\mathcal{M}}(v, u) = v \).

### B.3. Rough Lower Bound

When the changeover costs are ignored, the PSP falls under the Wagner-Whitin conditions (Pochet and Wolsey 2006) that allow to compute the optimal stocking cost. Conversely, if the stocking costs and the delivery constraints are omitted, the PSP can be reduced to the TSP. Therefore, a valid lower bound on the total changeover cost to produce a remaining set of items is to take the total weight of a Minimum Spanning Tree computed on the graph of changeover costs limited to item types that still need to be produced. The optimal weight for all these spanning trees can be precomputed because the number of items is usually small. As there is no overlap between the two lower bounds described, the RLB for the PSP can sum their individual contributions to obtain a stronger lower bound.
C. SRFLP

The SRFLP is an ordering problem that aims to place a set of departments \( N = \{0, \ldots, n-1\} \) on a line. Each department \( i \in N \) has a positive length \( L_i \), and the connection between each pair of departments \( i, j \in N \) is described by a positive traffic intensity \( C_{ij} \). The goal is to find a bijection \( \pi : N \rightarrow N \) that maps each department to a position on the line, while minimizing the total distance covered in the facility:

\[
SRFLP(\pi) = \sum_{k=0}^{n-1} L_k \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} C_{ij} \left( t_j^{\text{must}}(s^j, x_j), t_j^{\text{might}}(s^j, x_j), t_j^{\text{cut}}(s^j, x_j) \right), \quad \text{if } x_j \in s^j.\text{must,place},
\]

\[
0, \quad \text{if } x_j \in s^j.\text{might,place and } |\text{must,place}| < n-j,
\]

\[
\sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} C_{ij} \frac{L_i + L_j}{2}.
\]

The second term of Equation (22) is a constant that does not depend on the ordering \( \pi \), which is usually denoted \( K \). The modeling used in this paper is similar to the one presented in (Coppé et al. 2022) except that merging is included. The following definitions were thus adapted to cover states where each department must, might or might not be added to the ordering, similarly to the TSPTW modeling explained in Section A.

C.1. DP Model

The DP model works as follows: departments are added one by one on the line from left to right. The states are defined by tuples \((\text{must,place}, \text{might,place}, \text{cut})\) where \text{must,place} and \text{might,place} are the set of departments that respectively must and might be placed on the line. The last element of the state, called \text{cut}, is a vector that accumulates the cut value of each department. The cut value of a department is the total traffic intensity incident to this department, coming from placed departments. Formally, the DP model is defined as follows:

- Control variables: \( x_j \in N \) with \( j \in N \) decides which department is placed at position \( j \) on the line.
- State space: \( S = \{ (\text{must,place}, \text{might,place}, \text{cut}) \mid \text{must,place, might,place} \subseteq N, \text{must,place} \cap \text{might,place} = \emptyset \} \). The root state is \( \hat{r} = (N, \emptyset, (0, \ldots, 0)) \) and terminal states \( (\text{must,place}, \text{might,place}, \text{cut}) \) must only verify \( \text{must,place} = \emptyset \).
- Transition functions:

\[
t_j(s^j, x_j) = \begin{cases} t_j^{\text{must}}(s^j, x_j), t_j^{\text{might}}(s^j, x_j), t_j^{\text{cut}}(s^j, x_j), & \text{if } x_j \in s^j.\text{must,place}, \\ 0, & \text{if } x_j \in s^j.\text{might,place and } |\text{must,place}| < n-j, \end{cases}
\]

where

\[
t_j^{\text{must}}(s^j, x_j) = s^j.\text{must,place} \setminus \{x_j\}
\]

\[
t_j^{\text{might}}(s^j, x_j) = s^j.\text{might,place} \setminus \{x_j\}
\]

\[
t_j^{\text{cut}}(s^j, x_j) = (\text{cut}_0, \ldots, \text{cut}_{n-1})
\]

with \( \text{cut}_i = \begin{cases} s^j.\text{cut}[i] + C_{x_j}, & \text{if } x_j \in (s^j.\text{must,place} \cup s^j.\text{might,place}) \setminus \{x_j\}, \\ 0, & \text{otherwise}. \end{cases} \)

The second condition in Equation (23) ensures that no departments can be added from \text{might,place} when there remains space only for the departments in \text{must,place}.

- Transition value functions: the contribution of place department \( x_j \) at position \( j \) is its length \( L_{x_j} \), multiplied by the sum of the cut values of departments that will be located to its right. These departments are contained in \text{must,place} and, in case the state is relaxed, in \text{might,place}. However, there might be more departments in \text{might,place} than needed to complete the ordering. Selecting the \( n-j-|\text{must,place}| \)
smallest cut values among departments of might_place will result in a lower bound on the value obtained by placing any subset of departments in might_place. For a transition from state s^j with decision x_j, we also define cut_might = \{ s^j.cut[i] | i ∈ s^j.might_place \ \{x_j\} \} as the vector of cut values of departments in s^j.might_place. Then, the transition value functions are:

\[ h_j(s^j, x_j) = L_{s^j} \left( \sum_{i \in s^j.might\_place \setminus \{x_j\}} s^j.cut[i] + \sum_{i=1}^{n-j-\|s^j.must\_place\|} cut\_might_{<,i} \right). \]

- Root value: v_r = K.

C.2. Relaxation

The merging operator is defined as follows:

\[ \oplus(M) = (\oplus_{must}(M), \oplus_{might}(M), \oplus_{cut}(M)) \]

where

\[ \oplus_{must}(M) = \bigcap_{i \in M} s.must\_place \]
\[ \oplus_{might}(M) = \bigcup_{s \in M} s.must\_place \cup s.might\_place \setminus (\bigcap_{i \in M} s.must\_place) \]
\[ \oplus_{cut}(M) = \{cut_0, \ldots, cut_{n-1}\} \]

with \( cut_i = \min_{s \in M} \left\{ s.cut[i], \text{ if } i \in s.must\_place \cup s.might\_place, \infty, \text{ otherwise.} \right\} \)

The must_place and might_place sets are aggregated just as in the TSPTW relaxation of Section A.2.

Concerning the cut values, \( \oplus_{cut} \) simply keeps for each department the minimum cut value among the states that still must or might place the department considered. Once again, the relaxed transition value operator is the identity function \( \Gamma_M(v_u) = v \).

C.3. Rough Lower Bound

The RLB used for the SRFLP is divided into separate bounds: the first computes the optimal ordering of the remaining departments with respect to their cut values only, while the second estimates the arrangement cost of the remaining departments, ignoring the departments that have already been placed. The first-generation bound introduced by Simmons (1969) allows to compute the ordering that minimizes the contribution of cut values with respect to departments placed on the left. In that case, the optimal arrangement is given by ordering the departments by decreasing cut-to-length ratios. The only obstacle to using this lower bound is that for a state s^j all departments from s^j.must\_place will be included in the arrangement but only \( n_{might} = n - j - n_{must} \) from might\_place, with \( n_{must} = |s^j.must\_place| \). As the optimal subset of departments to select from s^j.might\_place is unknown, we create \( n_{might} \) artificial departments with the \( n_{might} \) shortest lengths and smallest cut values among the departments in s^j.might\_place. A vector of ratios that contain tuples \( \langle r, c, l \rangle \) is created for departments in s^j.must\_place and s^j.might\_place, where \( r = \frac{c}{l} \), c is a cut value and l is a length. As stated by Equation (27), the k-th shortest length is combined with the \( (n_{might} - k + 1) \)-th smallest cut value as to minimize the lower bound formula given by Equation (30).

\[ length_{might} = \{ L_k | k ∈ s^j.might\_place \} \]
\[ cut_{might} = \{ L_k | k ∈ s^j.might\_place \} \]
\[ ratio_{might} = \left\{ \frac{cut_{might}^k<n_{might}+1}{length_{<,k}}, cut_{might}^k<n_{might}+1, length_{<,k}^k \} | k = 1, \ldots, n_{might} \right\} \]
\[
\text{ratio}^{\text{must}} = \left\langle \frac{s_j^i, \text{cut}(k)}{L_k}, s_j^i, \text{cut}(k), L_k \right| k \in s_j^i, \text{must} \right\rangle \quad (28)
\]
\[
\text{ratio} = \text{concat}(\text{ratio}^{\text{must}}, \text{ratio}^{\text{might}}) \quad (29)
\]

The concatenation of both vectors of ratios is then used to compute the lower bound as in \cite{Simmons1969}.

\[
\text{LB}_{\text{cut}}(s_j^i) = \sum_{k=1}^{n-j} \text{ratio}^{>}, k, \text{cut} \sum_{l=1}^{k-1} \text{ratio}^{>}, l, \text{length} \quad (30)
\]

The second part of the RLB is a lower bound on the arrangement cost of \(n^\text{must} + n^\text{might}\) departments from \(s_j^i, \text{must} \) and \(s_j^i, \text{might} \).\footnote{Simmons, 1969}

First, a subset of traffic intensities is selected: all those connecting two departments in \(s_j^i, \text{must} \) and then the \(n^\text{must} + n^\text{might}\) lowest of those connecting one department from \(s_j^i, \text{must} \) to one in \(s_j^i, \text{might} \) and the \(\frac{n^\text{might}(n^\text{might} - 1)}{2}\) lowest of those connecting two departments in \(s_j^i, \text{might} \). Second, the \(n^\text{might}\) shortest department lengths from \(s_j^i, \text{might} \) are combined with those from \(s_j^i, \text{must} \).

\[
\text{traffic}^{\text{must}, \text{must}} = \{C_{kl} \mid k, l \in s_j^i, \text{must} \}\quad (31)
\]
\[
\text{traffic}^{\text{must}, \text{might}} = \{C_{kl} \mid k \in s_j^i, \text{must} \}, l \in s_j^i, \text{might} \}\quad (32)
\]
\[
\text{traffic}^{\text{might}, \text{might}} = \{C_{kl} \mid k, l \in s_j^i, \text{might} \}\quad (33)
\]
\[
\text{traffic} = \text{concat} \left( \begin{array}{c}
\text{traffic}^{\text{must}, \text{must}}, \\
\text{traffic}^{\text{must}, \text{might}}[1 \ldots n^\text{must} n^\text{might}], \\
\text{traffic}^{\text{might}, \text{might}}[1 \ldots \frac{n^\text{might}(n^\text{might} - 1)}{2}] \end{array} \right) \quad (34)
\]
\[
\text{length}^{\text{must}} = \{L_k \mid k \in s_j^i, \text{must} \}\quad (35)
\]
\[
\text{length} = \text{concat}(\text{length}^{\text{must}}, \text{length}^{\text{might}}[1 \ldots n^\text{might}]) \quad (36)
\]

Given the \text{traffic} and \text{length} vectors, a lower bound on the arrangement cost can then be computed by multiplying the traffic intensities by an optimistic distance. In Equation 37, the lowest traffic intensity is multiplied by the \(n - j - 2\) shortest department lengths. Then, the next two lowest traffic intensities are multiplied by the \(n - j - 3\) shortest department lengths, and so on. With \(T_k = \frac{k(k+1)}{2}\) the \(k\)-th triangular number, we write:

\[
\text{LB}_{\text{edge}}(s_j^i) = \sum_{k=1}^{n-j} \sum_{l=T_{k-1}+1}^{T_k} \text{cut}^{<}, l, \sum_{m=1}^{n-j-k-l} \text{length}^{<}, m. \quad (37)
\]

In the end, the RLB for the SRFLP is given by: \(\text{rlb}(s_j^i) = \text{LB}_{\text{cut}}(s_j^i) + \text{LB}_{\text{edge}}(s_j^i)\).

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