Conormal Geometry of Maximal Minors

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Let \( A \) be a Noetherian local domain, \( N \) be a finitely generated torsion-free module, and \( M \) a proper submodule that is generically equal to \( N \). Let \( A[N] \) be an arbitrary graded overdomain of \( A \) generated as an \( A \)-algebra by \( N \) placed in degree 1. Let \( A[M] \) be the subalgebra generated by \( M \). Set \( C := \text{Proj}(A[M]) \) and \( r := \dim C \). Form the (closed) subset \( W \) of \( \text{Spec}(A) \) of primes \( p \) where \( A[N]_p \) is not a finitely generated module over \( A[M]_p \), and denote the preimage of \( W \) in \( C \) by \( E \). We prove this: (1) \( \dim E = r - 1 \) if either (a) \( N \) is free and \( A[N] \) is the symmetric algebra, or (b) \( W \) is nonempty and \( A \) is universally catenary, and (2) \( E \) is equidimensional if (a) holds and \( A \) is universally catenary.

Our proof was inspired by some recent work of Gaffney and Massey, which we sketch; they proved (2) when \( A \) is the ring of germs of a complex-analytic variety, and applied it to perfect a characterization of Thom's \( A_f \)-condition in equisingularity theory. From (1), we recover, with new proofs, the usual height inequality for maximal minors and an extension of it obtained by the authors in 1992. From the latter, we recover the authors’ generalization to modules of Böger’s criterion for integral dependence of ideals. Finally, we introduce an application of (1), being made by the second author, to the geometry of the dual variety of a projective variety, and use it to obtain an interesting example where the conclusion of (1) fails and \( A[N] \) is a finitely generated module over \( A[M] \).

1. The Theorem and Applications

(1.1) Introduction. Our main result is the following theorem. Its proof occupies nearly all of the next section.

Let \( A \) be a Noetherian local domain, \( N \) be a finitely generated torsion-free module, and \( M \) a nonzero proper submodule. Set \( X := \text{Spec}(A) \) and \( Y := \text{Supp}(N/M) \). Let \( A[N] \) be an arbitrary graded domain containing \( A \) and generated as an \( A \)-algebra by \( N \) placed in degree 1. Thus \( A[N] \) either is the \textit{Rees algebra} (that is, the quotient of the symmetric algebra by its torsion) or is a quotient of the Rees algebra by a homogeneous prime ideal that intersects

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$N$ in $0$. Let $A[M]$ be the subalgebra generated by $M$. Set $P := \text{Proj}(A[N])$, let $p : P \to X$ denote the structure map, and set $Z := \mathbf{V}(M \cdot A[N])$; so $Z$ is the subscheme of $P$ whose homogeneous ideal is generated by $M$. Set $C := \text{Proj}(A[M])$, let $c : C \to X$ denote the structure map, and set $E := c^{-1}p(Z)$ and $F := c^{-1}Y$. Finally, set $r := \dim P$.

Note that $p(Z) \subset Y$ since, off $Y$, the ideal $M \cdot A[N]$ is irrelevant; so $E \subset F$.

Note that $P$ and $C$ are integral, and that $p : P \to X$ and $c : C \to X$ are surjective, being proper and being dominating as $A \subset A[M] \subset A[N]$. If $Y \neq X$, then generically $M$ and $N$ are equal; whence, by (3.4)(ii) of [13],

$$\dim C = r.$$ (1.1.1)

**Theorem.** Preserve the notation above, and assume $Y \neq X$.

(1) If either (a) $N$ is free, and $A[N]$ is the symmetric algebra, (b) $A[N]$ is not a finitely generated $A[M]$-module, and $A$ is universally catenary, or (c) $Z = p^{-1}p(Z)$ as sets, and $Z$ is nonempty, or (d) $P$ has dimension $r$ at some point of $Z$, then

$$\dim E = r - 1 \text{ and } \dim F = r - 1.$$  

Furthermore, if (a) holds, then $p(Z) = Y$ and $E = F$.

(2) Assume either that $N$ is free, and $A[N]$ is the symmetric algebra, or that $Z = p^{-1}p(Z)$ as sets. If $A$ is universally catenary, then $C$, $P$, and $E$ are biequidimensional.

The theorem has applications in algebra and in geometry, which will be discussed in this section. In short, Part (1) with Hypothesis (a) implies the usual height inequality for maximal minors, because $Y$ is defined by the zeroth Fitting ideal $I$ of $N/M$. The height inequality was given one of its first proofs by Buchsbaum and Rim [3, 3.5] as an application of their theory of multiplicities of submodules of free modules. (See [4, Ch. 2] for a discussion of other proofs.) Following in their footsteps, but assuming that $p(Z)$ is the closed point, the authors recovered (1) with (a) and proved (1) with (b)–(d) in (10.2) and (10.3) of [13]. These four results are recovered in the present article via new proofs; moreover, these proofs are substantially shorter, simpler, and more direct than the old. If $A$ is universally catenary, then (1) can be reduced to the case where $p(Z)$ is the closed point by localizing at a generic point of $p(Z)$; if $A$ is not universally catenary, then (1) appears to be new.

Part (1) with Hypothesis (b) provides a criterion for $N$ to be “integrally dependent” on $M$. As such, (1) with (b) is the main ingredient in the authors’ generalization to modules [13, (10.9)] of B"oger’s criterion, which in turn generalized to ideals not of finite colength Rees’s celebrated characterization of integral dependence by multiplicity. Part (1) with (b) is therefore a main ingredient in the work [6] of Gaffney and the first author, which generalizes to modules Teissier’s principle of specialization of integral dependence, and applies it in equisingularity theory.
Part (2) asserts the pure (graded) codimensionality of the extension $I \cdot A[M]$ of the Fitting ideal $I$ to the Rees algebra $A[M]$, without any assumption on the codimensionality of $I$ itself. (Graded (co)dimension is defined via chains of homogeneous primes, but is equal to the usual notion, defined via chains of arbitrary primes by Theorem 1.5.8 on p. 31 in [2].)

This codimensionality result about $I \cdot A[M]$ is new. It was proved recently by Gaffney and Massey [5, (5.7)], [15, 4.2] when $A$ is the ring of germs of a complex-analytic variety, and their proof inspired ours. They introduced the remarkable idea of expressing $F$ as the union of closed sets, each the exceptional divisor of a suitable blowup, and our corresponding blowup is a stylized version of theirs. They constructed and used germs of complex-analytic curves in a remarkable way, which inspired our work with “paths.” Their proof and ours differ mainly because we need to pay careful attention to the dimension theory, which is so much more delicate for general Noetherian rings than for geometric rings. In particular, we must introduce a certain blowup $B$, dominating $C$, which is unnecessary in their proof.

In the application of the theorem to projective geometry, $X$ is a variety, $Y$ is contained in its singular locus, and $C$ is its conormal variety; the latter is the closure of the locus of pairs $(x, H)$ where $x$ is a simple point and $H$ is a hyperplane tangent to $X$ at $x$. So $F$ is a locus of limit tangent hyperplanes at singular points of $X$. Part (1) of the theorem provides two cases where $F$ has codimension 1: Case (a) $X$ is a singular local complete intersection; Case (b) the normal module $N$ is not integrally dependent on the Jacobian module $M$. (In fact, Case (b) includes Case (a).) Thus we obtain a nontrivial lower bound on the dimension of the dual variety $X'$; see the second author’s paper [19]. Put differently, when $X'$ is small, the conclusion of (1) fails if $Y$ is nonempty, and then $N$ is dependent on $M$.

Part (2) implies that $F$ is equidimensional if $X$ is a local complete intersection. Gaffney and Massey recently proved a similar statement in complex-analytic geometry. They applied it to perfect some work of Gaffney and the first author’s in the equisingularity theory of a family of germs of isolated complete-intersection singularities (ICIS germs), equipped with a function $f$. The final result is a definitive characterization of Thom’s $A_f$-condition in terms of the constancy of numbers of vanishing cycles, or Milnor numbers.

(1.2) A Global Extension. It is straightforward, but tedious, to extend the theorem, obtaining the following corollary, which recovers (10.2) and (10.3) of [13]. A proof will be given in (2.10).

Let $X$ be a Noetherian scheme of finite dimension, $N$ a coherent sheaf, and $M$ a proper coherent subsheaf. Set $Y := \text{Supp}(N/M)$. Let $O_X[N]$ be a graded quasi-coherent algebra generated by $N$ in degree 1, and let $O_X[M]$ be the subalgebra generated by $M$. Set $P := \text{Proj}(O_X[N])$, let $p: P \to X$ denote the structure map, and set $Z := V(M \cdot O_X[N])$. Set $C := \text{Proj}(O_X[M])$, let
\( c : C \to X \) denote the structure map, and set \( E := c^{-1}p(Z) \) and \( F := c^{-1}Y \). Finally, set \( r := \dim P \).

**Corollary.** Preserve the notation above.

(1) If either (a) \( N \) is locally free of constant rank, \( \mathcal{O}_X[N] \) is the symmetric algebra, \( \dim Y < \dim X \), and there exists a point \( y \in Y \) where \( \dim \mathcal{O}_{X,y} = \dim X \), or (b) \( X \) is a closed subscheme of a universally catenary and biequidimensional scheme, \( \dim p^{-1}p(Z) < r \), and \( Z \) meets an \( r \)-dimensional component of \( P \), or (c) \( Z = p^{-1}p(Z) \) as sets, \( Z \) is nonempty, \( \dim Z < r \), and \( X \) is local, or (d) \( \dim \mathcal{O}_{P,z} = r \) for some point \( z \in Z \), and \( \dim p^{-1}p(Z) < r \), then \( \dim C = r \), and \( \dim E = r - 1 \).

Furthermore, if \( N \) is locally free and \( \mathcal{O}_X[N] \) is the symmetric algebra, then \( p(Z) = Y \) and \( E = F \).

(2) Assume either that (a) holds or that \( Z = p^{-1}p(Z) \) as sets, \( \dim Z < r \), and \( P \) is equidimensional. If \( X \) is universally catenary and biequidimensional, then so are \( C \), \( P \), and \( E \).

(1.3) **The Height Inequality.** The usual height inequality is this:

\[
d \leq m - n + 1, \tag{1.3.1}
\]

where \( d \) is the height of any minimal prime of the ideal of maximal minors of an \( n \) by \( m \) matrix with \( n \leq m \) and with entries in an arbitrary Noetherian ring \( A \). The inequality is trivial if \( d = 0 \). Otherwise, as we are now going to see, it results from (1) with (a) of our theorem (and is nearly equivalent to it); compare [13, (10.4)]. Indeed, localizing at the prime and dividing by an arbitrary minimal prime of \( A \), we may assume that \( A \) is a local domain of dimension \( d \).

Let \( M \) be the column space of the matrix, and in the natural way, view \( M \) as a subspace of the free module \( N \) of rank \( n \). Then \( N/M \) is supported precisely at the closed point of Spec(\( A \)). Let \( A[N] \) be the symmetric algebra, \( A[M] \) the subalgebra generated by \( M \). Set \( P := \text{Proj}(A[N]) \) and \( C := \text{Proj}(A[M]) \). Standard dimension theory implies that \( \dim P = d + n - 1 \). Since \( M \) is generated by \( m \) elements, \( C \) is a closed subscheme of \( \mathbb{P}^{m-1}_A \). So \( \dim E \leq m - 1 \) since \( E \) is the closed fiber of \( C \). Hence, (1) with (a) of (1.1) implies the inequality \( d + n - 2 \leq m - 1 \), and so (1.3.1).

The height inequality (1.3.1) can be rewritten in the following form:

\[
m \geq d + n - 1. \tag{1.3.2}
\]

As such, it is a lower bound on the minimal number \( m \) of generators of a proper submodule \( M \) of a free module \( N \) over a Noetherian ring, given in terms of the rank \( n \) of \( N \) and the height \( d \) of any prime minimal in \( \text{Supp}(N/M) \), provided this set is nonempty and nowhere dense.

The lower bound (1.3.2) also holds in this general setup: let \( A \) be a universally catenary Noetherian ring, \( N \) be a finitely generated module, and \( M \)
a proper submodule; let \( A[N] \) be an arbitrary graded \( A \)-algebra generated by \( N \) placed in degree 1, and \( A[M] \) the subalgebra generated by \( M \); let \( W \) be the subset of \( \text{Spec}(A) \) of primes \( \mathfrak{p} \) where \( A[N]_{\mathfrak{p}} \) is not a finitely generated module over \( A[M]_{\mathfrak{p}} \); let \( \mathfrak{p} \) be minimal in \( W \); and let \( \mathfrak{q} \) be a homogeneous prime of \( A[N] \) such that its contraction \( \mathfrak{q}_0 := A \cap \mathfrak{q} \) is strictly contained in \( \mathfrak{p} \) and the localized quotient \( (A[N]/\mathfrak{q})_\mathfrak{p} \) is not a finitely generated module over \( A[M]_\mathfrak{p} \); then (1.3.2) holds with, for \( m \), the minimal number of generators of \( M \), for \( d \), the height of \( \mathfrak{p}/\mathfrak{q}_0 \), and for \( n \), the transcendence degree of \( A[N]/\mathfrak{q} \) over \( A/\mathfrak{q}_0 \). This assertion results similarly from (1) with (b) of (1.1), after localizing at \( \mathfrak{p} \) and replacing \( A[N] \) by \( A[N]/\mathfrak{q} \); see the proof of (1.2) given in (2.10).

(1.4) Integral Dependence. Turned around, the corollary yields the following general criterion for integral dependence in terms of dimensions. Let \( A \) be a universally catenary andequidimensional Noetherian ring, \( N \) a finitely generated module, and \( M \) a nonzero proper submodule. Let \( A[N] \) be any graded algebra generated by \( N \) in degree 1, and \( A[M] \) the subalgebra generated by \( M \). Define the maps \( p: P \to X \) and \( c: C \to X \) and the subschemes \( Z \) and \( E \) as in (1.1); set \( r := \dim P \). Call \( N \) integrally dependent on \( M \) if \( A[N] \) is a finitely generated module over \( A[M] \) (even if \( A[N] \) is not the Rees algebra); it is equivalent to require \( Z \) to be empty, see the middle of (2.1). Then (1) with (b) of (1.2) yields this CRITERION: \( N \) is integrally dependent on \( M \) if (i) \( P \) is equidimensional, if (ii) \( \dim p^{-1}p(Z) < r \), and if (iii) \( \dim E < r - 1 \).

The preceding criterion of ours for modules generalizes the following criterion of Böger’s for ideals [1, p. 208]: in a universally catenary and equidimensional Noetherian local ring \( A \), let \( M \) and \( N \) be nonzero proper ideals with \( M \subset N \); then \( N \) is integrally dependent on \( M \) if (\( \alpha \)) \( N_\mathfrak{p} \) is integrally dependent on \( M_\mathfrak{p} \) for every minimal prime \( \mathfrak{p} \) of \( A/M \), and (\( \beta \)) \( \text{ht}(M) = \ell(M) \) where \( \ell(M) \) is the analytic spread.

Indeed, let \( A[N] \) and \( A[M] \) be the (ordinary) Rees algebras. Then \( C \) and \( P \) are the blowups of \( \text{Spec}(A) \) along \( V(M) \) and \( V(N) \). So \( C, P, \) and \( X \) are equidimensional of dimension \( r \). Hypothesis \( \alpha \) implies that \( p(Z) \) is nowhere dense in \( V(M) \); so

\[
\dim p(Z) < \dim V(M) = r - \text{ht}(M),
\]

and \( \dim p^{-1}p(Z) < r \). If \( \Phi \) denotes the closed fiber of \( C \), then by definition \( \ell(M) := \dim \Phi + 1 \). Hence, standard dimension theory and Hypothesis \( \beta \) yield

\[
\dim E \leq \ell(M) - 1 + \dim p(Z) < r - 1.
\]

Thus all three hypothesis of our criterion hold.

Böger replaced Hypothesis \( \alpha \) by the equality of multiplicities,

\[
e(M_\mathfrak{p}) = e(N_\mathfrak{p}),
\]

but the two versions of the hypothesis are equivalent by a celebrated theorem of Rees’s. The latter was generalized to submodules of a free module by Rees
in 4.1 of [18] and then generalized further independently by Kirby and Rees in 6.5 of [7] and by the authors in (6.7a)(iii) of [13]. Also, Böger assumed that $A$ is quasi-unmixed (or formally equidimensional), but this hypothesis implies that $A$ is universally catenary and equidimensional; see p. 251 and following in [17].

(1.5) Projective Geometry. (See [10] and [19].) Let $X$ be a subvariety (or closed, reduced and irreducible subscheme) of dimension $d$ of the projective $m$-space $\mathbb{P}^m$ over an algebraically closed ground field of arbitrary characteristic.

Let $I$ be the sheaf of ideals, and form the usual right exact sequence,

$$\frac{I}{I^2} \xrightarrow{\delta} \Omega^1_{\mathbb{P}^m}|X \to \Omega^1_X \to 0.$$  

(1.5.1)

Locally $\delta$ carries a function $f$ vanishing on $X$ to its differential $df$. So locally the transpose $\delta^*$ is represented by a usual Jacobian matrix.

Consider the following nested sequence of three torsion-free sheaves:

$$M := \text{Image}(\delta^*) \subset N' := (\text{Image} \delta)^* \subset N := (\frac{I}{I^2})^*.$$ 

where $N'$ and $N$ are the duals. The latter is known as the normal module. The first sheaf $M$ can be viewed locally as the column space of a Jacobian matrix; so $M$ is known as the Jacobian module of $X$.

Let $x \in X$. First, suppose $X$ is smooth at $x$. Then (1.5.1) splits at $x$. Hence all three sheaves are equal and are free of rank $m - d$ at $x$. Next, suppose $X$ is a complete intersection at $x$. Then $\frac{I}{I^2}$ is free at $x$. So since $X$ is reduced, $\delta$ is injective. Hence $N'$ and $N$ are equal and free at $x$. If they are also equal at $x$ to $M$, then $x$ must be a simple point because then $\frac{I}{I^2}$ is free and (1.5.1) splits at $x$. Finally, suppose $X$ is normal at $x$. Then $N'$ and $N$ are equal (but not necessarily free) at $x$ because $\mathcal{O}_{X,x}$ satisfies Serre’s conditions ($S_2$) and ($R_1$).

Let $C(X)$ be the conormal variety: by definition, $C(X)$ is the closure of the set of pairs $(x,H)$ where $x$ is a simple point of $X$ and where $H$ is a hyperplane tangent to $X$ at $x$. Then $\dim C(X) = m - 1$. Furthermore,

$$C(X) = \text{Proj}(\mathcal{O}_X[M])$$

(1.5.2)

where $\mathcal{O}_X[M]$ is the Rees algebra (Gaffney, private comm., May 1990). Indeed, this algebra is sheaf of domains, so $\text{Proj}(\mathcal{O}_X[M])$ is irreducible. There is a natural embedding of the Proj in the product of $\mathbb{P}^m$ and its dual space: this embedding is induced by the global Jacobian map,

$$\mathcal{O}_X^{m+1} \to \mathcal{N}(-1),$$

which arises from the first map $\delta$ in (1.5.1) and the natural inclusion map of $\Omega^1_{\mathbb{P}^m}|X$ into $\mathcal{O}_X^{m+1}(-1)$. Finally, the two sides of (1.5.2) are equal over the smooth locus of $X$ as $M$ is locally the column space of a Jacobian matrix.

Assume that $x$ is an isolated singular point of $X$ (see [19] for a more general discussion). Let $F$ be the fiber of $C(X)$ over $x$. Part (1) of our theorem in
Let \( X' \) be the **dual variety**: by definition, \( X' \) is the image of \( C(X) \) under the second projection. So \( X' \) contains the image of \( F \), which may be identified with \( F \). If \( X' \) is not a hypersurface and if \( \dim F = m - 2 \), then \( X' = F \). If the characteristic is zero, then the dual variety of \( X' \) is equal to \( X \) (see I-4 in [9]). However, the dual variety \( F' \) of \( F \) is a cone in \( \mathbb{P}^m \); its vertex is \( x \), and its base is the dual of \( F \), viewed as a subvariety of the hyperplane of hyperplanes through \( x \). Moreover, since \( x \) is an isolated singular point, if \( F \) is a cone, then the base is smooth, and \( x \) is the only singular point. In sum, we have proved this: In characteristic zero, if the dual variety \( X' \) is not a hypersurface and if, at the isolated singular point \( x \), either \( X \) is a complete intersection or, more generally, the normal module is not integrally dependent on the Jacobian module, then \( X' \) has codimension 2 and \( X \) is a cone over a smooth base.

(1.6) **Example.** The discussion in (1.5) leads to the following construction of an example where the conclusion of Part (1) of the theorem in (1.1) fails and there is nontrivial integral dependence. Over an algebraically closed field \( k \) of any characteristic, let \( G \) be a smooth subvariety of \( \mathbb{P}^{m-1} \) whose dual variety \( G' \) is of dimension at most \( m - 3 \); specific \( G \) will be described below (and more possible \( G \) are described in [8, p. 360] and [9, I-7]). Let \( X \) be the projecting cone over \( G \) with vertex \( x \) in \( \mathbb{P}^m \). Then its dual variety \( X' \) is equal to \( G' \). Hence, by (1.5), at \( x \) the normal module \( N \) must be integrally dependent on the Jacobian module \( M \). However, algebraically this example is trivial if the two modules are equal; this possibility will now be investigated.

Sequence (1.5.1) induces the following short exact sequence:

\[
0 \to M \to N' \to \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X) \to 0.
\]

Since \( x \) is an isolated singular point, the \( \mathcal{E}xt^1 \) is concentrated at \( x \). Moreover, as is well known (see (1.4.3) in [12] for example), it is then equal to the module \( T^1 := T^1(\mathcal{O}_X/k, \mathcal{O}_X) \) of deformation theory. Hence, \( x \) is a rigid singularity if and only if \( M \) and \( N' \) are equal at \( x \). Moreover, \( N' \) and \( N \) are equal if \( G \) is arithmetically normal. For instance, take \( A := \mathbb{P}^a \). Then \( x \) is rigid by a theorem of Thom, Grauert–Kerner, and Schlessinger; see (2.2.8) in [12]. So \( N = M \). Finally, take \( A \) to be a smooth quartic surface in \( \mathbb{P}^3 \), a K3-surface. Then the proof of the latter theorem shows that \( T^1 \neq 0 \); indeed, \( h^1(\mathcal{H}om(\Omega^1_A, \mathcal{O}_A)) = 20 \) and \( h^2(\mathcal{O}_A) = 1 \), whence \( H^1(\widetilde{N}) \neq 0 \) where \( \widetilde{N} \) appears at the end of the proof of
(2.2.6) in [12]. So \( \mathcal{N} \) is integrally dependent on \( \mathcal{M} \), but not equal to it. This is the desired example.

(1.7) Equisingularity Theory. Let \( X \) be a complex-analytic germ at 0 in \( \mathbb{C}^a \times \mathbb{C}^b \). Say \( X : f_1 = 0, \ldots, f_k = 0 \) on a neighborhood of 0 in \( \mathbb{C}^a \times \mathbb{C}^b \), where each \( f_i \) is an analytic function \( f_i(x,y) \) of the two sets of variables, \( x = (x_1, \ldots, x_a) \) and \( y = (y_1, \ldots, y_b) \).

For fixed \( y \), let \( X_y \subset \mathbb{C}^a \) denote the locus of \( x \) such that \( (x,y) \in X \). Let \( Y \) be the locus of \( y \) with \( (0,y) \in X \), assume that \( Y \) contains a neighborhood of 0 in \( \mathbb{C}^b \), and identify \( Y \) with \( 0 \times Y \). View \( Y \) as the parameter space and \( X \) as the total space of the family of \( X_y \). Finally, assume that the \( X_y \) are germs of isolated complete-intersection singularities (ICIS germs) of dimension \( a - k \geq 1 \).

Let \( f \) be a nonconstant analytic function on \( X \) vanishing on \( Y \). Set 
\[
Z_u := f^{-1}u \quad \text{and} \quad Z_{u,y} := Z_u \cap X_y.
\]
Let \( \Sigma(f) \) denote the “critical set,” the union of the singular sets of the various \( Z_u \). Let \( \Sigma_Y(f) \) denote the union of the singular sets of the various \( Z_{u,y} \).

Form the following three conormal varieties: first \( C(X,f) \), the closure of the set of pairs \((w,H)\) where \( w := (x,y) \) is a point of \( X - \Sigma(f) \) and \( H \) is a hyperplane in \( \mathbb{C}^a \times \mathbb{C}^b \) tangent at \( w \) to \( Z_{fw} \); second, \( C(X,f;Y) \), the closure of the set of \((w,H)\) where \( w \) is a point of \( X - Y \) and \( H \) is a hyperplane in \( \mathbb{C}^a \times \mathbb{C}^b \) tangent at \( w := (x,y) \) to \( Z_{fw,y} \); third \( C(Y) \), the set of pairs \((w,H)\) where \( w := (0,y) \) is a point of \( Y \) and \( H \) is a hyperplane containing \( Y \) (in other words, \( C(Y) \) is simply \( Y \times \mathbb{P}^{a-1} \)).

Extend \( f \) over a neighborhood of \( X \) in \( \mathbb{C}^a \times \mathbb{C}^b \) on which \( f_1, \ldots, f_k \) are defined, and denote the extension too by \( f \); the choice of extension is immaterial. Form the following two Jacobian modules on \( X \): first \( \mathcal{N} \), the column space of the Jacobian matrix of the functions \( f_1, \ldots, f_k, f \) with respect to all \( a + b \) variables \( x, y \); second \( \mathcal{M} \), that with respect to \( x \) alone. So \( \mathcal{M} \subset \mathcal{N} \subset \mathcal{E} := \mathcal{O}_X^{k+1} \).

The reasoning in the proof of (1.5.2) yields these identifications:

\[
C(X,f) = \text{Projan}(\mathcal{N}) \quad \text{and} \quad C(X,f;Y) = \text{Projan}(\mathcal{M}).
\]

Finally, denote the preimage in \( C(X,f) \) of \( Y \) by \( F \), and that of 0 by \( C(X,f)_0 \).

Thom’s \( A_f \)-condition at 0 may be put succinctly as the condition that

\[
C(X,f)_0 \subset C(Y).
\]

It is a well-known preliminary condition for the pair \( X, f \) to be topologically trivially along \( Y \) at 0. Recently, it was proved to be equivalent to a weaker condition of topological equisingularity, which involves the constancy of numbers of vanishing cycles, or Milnor numbers.
The Lê-Saito theorem is a celebrated step in this direction, and asserts the following: in the case where \( X \) is all of \( \mathbb{C}^a \times \mathbb{C}^b \) and each \( Z_{0,y} \) has an isolated singularity at 0, if the Milnor number \( \mu(Z_{0,y}) \) is constant in \( y \), then \( A_f \) holds. Lê and Saito proved the theorem using Morse theory, but Teissier reproved it right away using more algebraic-geometric methods.

Following in Teissier’s footsteps, Gaffney, Massey and the first author recently generalized the Lê-Saito theorem as follows: in the setup above, at 0, the germs of \( \Sigma(f) \) and \( Y \) are equal and \( A_f \) holds if and only if, for \( y \) near 0, the germ \( Z_{0,y} \) has an isolated singularity at 0, and both \( \mu(X_y) \) and \( \mu(Z_{0,y}) \) are constant in \( y \). Indeed, \([6, \S 5]\) contains a proof that \( A_f \) implies the constancy, and a proof of a weak converse. The definitive converse is proved in \([5, (5.8)]\); see also \([11, (2.2)]\).

A clean composite sketch will now be made of these proofs, highlighting the use of the complex-analytic version of the theorem in (1.1).

Suppose that, for \( y \) near 0, the germ \( Z_{0,y} \) has an isolated singularity at 0, and let \( e(y) \) denote the Buchsbaum–Rim multiplicity of the restriction \( M|_{X_y} \) in \( E|_{X_y} \) at 0. Theorems of Lê and Greuel and of Buchsbaum and Rim yield

\[
e(y) = \mu(X_y) + \mu(Z_{0,y}).
\]

Since Milnor numbers are upper semicontinuous \([14, \text{bot. p. 126}]\), the two of them are constant in \( y \) if and only if \( e(y) \) is so. Thus we have to prove that, near 0, the germ \( Z_{0,y} \) has an isolated singularity and \( e(y) \) is constant if and only if, at 0, the germs of \( \Sigma(f) \) and \( Y \) are equal and \( A_f \) holds. We’ll prove that each condition holds if and only if, at 0, the germs of \( \Sigma(f) \) and \( Y \) are equal and \( N \) is integrally dependent on \( M \).

Suppose \( A_f \) holds at 0. The gradients of \( f_1, \ldots, f_k, f \) define hyperplanes \( H \) tangent to the \( Z_u \). So, along any path to 0 not lying entirely in \( Y \), each \( H \) approaches a hyperplane that contains \( Y \). Therefore, each of the last \( b \) components of each gradient vanishes at 0 along the curve to order higher than the order of one, or more, of the first \( a \) components. Hence, by the curve criterion, \( N \) is integrally dependent on \( M \) at 0.

Suppose \( \Sigma(f) = Y \) and that \( N \) is integrally dependent on \( M \). Then \( \Sigma_Y(f) = Y \); indeed, \( \Sigma_Y(f) = \text{Supp}(E/M) \), and \( \text{Supp}(E/M) = Y \) as \( E = N \) off \( Y \) and a free module is not dependent on any proper submodule. Hence \( Z_{0,y} \) has an isolated singularity at 0. Let \( \mathcal{I} \) be the zeroth Fitting ideal of \( E/M \). Then \( \mathcal{O}_X/\mathcal{I} \) is determinantal, and hence Cohen–Macaulay by a theorem of Eagon. Moreover, the support of \( \mathcal{O}_X/\mathcal{I} \) is equal to \( Y \); in particular, \( \mathcal{O}_X/\mathcal{I} \) is a finitely generated \( \mathcal{O}_Y \)-module. Consequently, \( \mathcal{O}_X/\mathcal{I} \) is a Cohen–Macaulay \( \mathcal{O}_Y \)-module, and therefore, by the Auslander–Buchsbaum formula, a free \( \mathcal{O}_Y \)-module. Hence, the restriction of \( \mathcal{O}_X/\mathcal{I} \) to \( X_y \) has constant length. Since \( X_y \) is Cohen–Macaulay, it follows from some theorems of Buchsbaum and Rim that this length is equal to \( e(y) \). Thus \( e(y) \) is constant.

Conversely, suppose \( Z_{0,y} \) has an isolated singularity at 0. Then, replacing
X by a smaller representative of its germ, we may assume that $\Sigma_Y(f)$ is finite over $Y$. Suppose $e(y)$ is constant. Then $\Sigma_Y(f) = Y$ because of the upper semi-continuity of the following sum: the sum, over all the points $w$ in the fiber of $\Sigma_Y(f)$ over $y$, of the Buchsbaum–Rim multiplicity of the restriction $M|_{X_y}$ in $E|_{X_y}$ at $w$.

Hence $\text{Supp}(N/M) \subseteq Y$. So, if $W$ denotes the locus where $N$ is not integrally dependent on $M$, then $W \subseteq Y$. In fact, $W \neq Y$ because $A_f$ holds generically on $Y$ by the generic Thom lemma, and because $A_f$ implies dependence. Since $M$ has $a$ generators, $C(X, f; Y)$ embeds in $X \times \mathbb{P}^{a-1}$. Hence, the preimage of $W$ in $C(X, f; Y)$ has dimension at most $a + b - 2$. However, $C(X, f; Y)$ has dimension $a + b$. Therefore, by the complex-analytic version of the criterion of integral dependence discussed in (1.4) above, $N$ is dependent on $M$.

Again suppose that $\Sigma(f) = Y$ and that $N$ is integrally dependent on $M$. Then, as noted above, $\text{Supp}(E/M) = Y$. Hence, by the complex-analytic version of the corollary in (1.2), each component $F'$ of $F$ has dimension $a + b - 1$. Since $N$ is dependent on $M$, the inclusion of $M$ into $N$ induces a finite surjective map,

$$g: C(X, f) \to C(X, f; Y).$$

Hence, $\text{dim } g(F') = a + b - 1$. However, $C(X, f; Y) \subset X \times \mathbb{P}^{a-1}$ as noted above. Hence $g(F') = Y \times \mathbb{P}^{a-1}$. Therefore, $F'$ maps onto $Y$. By the generic Thom lemma, the inclusion $F' \subset C(Y)$ holds generically over $Y$; hence, it holds globally over $Y$. Thus $A_f$ holds, and the proof is complete.

2. Proof of the theorem and corollary

(2.1) Preliminaries. Until the last section (2.10), preserve the notation of (1.1), and assume $Y \neq X$. Form the natural commutative diagram

$$\begin{array}{ccc}
B & \overset{b}{\longrightarrow} & P \\
\downarrow q & & \downarrow p \\
C & \overset{c}{\longrightarrow} & X
\end{array}$$

where $B := \text{Bl}_Z(P)$ is the blowup along $Z$; see [13, (2.1)]. Then $Z \neq P$ since $p(Z) \neq X$ and $p$ is surjective. Hence $b: B \to P$ is proper and surjective, as $P$ is integral. Set $D := b^{-1}Z$.

For each nonzero element $\nu$ of $N$, form the ring of elements of degree 0 and its affine scheme, which is a standard open subscheme of $P$:

$$A[N/\nu] := A[N]_{(\nu)} = A[N]/(1 - \nu)$$
and $P_\nu := \text{Spec}(A[N/\nu]) \subset P$.

Since $\nu$ need not lie in $M$, the corresponding ring and scheme must be defined differently, but compatibly when $\nu \in M$:

$$A[M/\nu] := \text{Image}(A[M] \to A[N] \to A[N/\nu]);$$
$$Q_\nu := \text{Spec}(A[M/\nu]) \subset Q := \text{Spec}(A[M]).$$
The augmentation homomorphism $A[M] \rightarrow A$ defines a section of $Q/X$. Form the corresponding blowups:
\[ \tilde{Q} := \text{Bl}_X Q \] and \[ \tilde{Q}_\nu := \text{Bl}_{X \cap Q_\nu} Q_\nu. \]

The exceptional divisor of $\tilde{Q}$ is $C := \text{Proj}(A[M])$; whence, that of $\tilde{Q}_\nu$ is
\[ E_\nu := \tilde{Q}_\nu \cap C. \]

Since $Q_\nu$ is closed in $Q$, also $E_\nu$ is closed in $C$.

Note that $E_\nu \subset E := c^{-1}p(Z)$. Indeed, work off $p(Z)$, or assume for the moment that $Z$ is empty. Then the homogeneous ideal $M \cdot A[N]$ is irrelevant. So, if $N_k$ denotes the $k$th graded piece of $A[N]$, then $M \cdot N_k = N_{k+1}$ for $k \gg 0$. Hence $A[N]$ is a finitely generated $A[M]$-module. (For use elsewhere, note that this argument is reversible (compare with [13], (2.3)): if $A[N]$ is finitely generated, then $Z$ is empty.) Hence $A[N/\nu]$ is a finitely generated $A[M/\nu]$-module. Now, $M$ generates the unit ideal in $A[N/\nu]$. Therefore $X \cap Q_\nu$ is empty off $p(Z)$; whence, so is $E_\nu$. Thus $E_\nu \subset E$.

Set $Z_\nu := Z \cap P_\nu$. The inclusion $A[M/\nu] \hookrightarrow A[N/\nu]$ induces maps,
\[ P_\nu \rightarrow Q_\nu \] and $q_\nu: B_\nu \rightarrow \tilde{Q}_\nu$ where $B_\nu := \text{Bl}_{Z_\nu}(P_\nu) = b^{-1}P_\nu$.

Set $D_\nu := B_\nu \cap D$. Then the restriction $q_\nu|D_\nu$ is equal to the restriction,
\[ q: D_\nu \rightarrow E_\nu. \]

Hence, if $\nu$ varies so that $Z = \bigcup Z_\nu$, then
\[ q(D) = \bigcup q(D_\nu) \subset \bigcup E_\nu \subset E \subset F \]
(2.1.1)

(2.2)Lemma. If $P$ has dimension $r$ at some $z \in Z_\nu$ for some $\nu$, then
\[ \dim E = \dim F = \dim E_\nu = r - 1. \]
Furthermore, $E_\nu$ is biequidimensional if $A$ is universally catenary.

Indeed, the map $P_\nu \rightarrow Q_\nu$ carries $z$ to a point $x$ of $X \cap Q_\nu$, and $x$ must be the unique closed point since $Z$ is closed in $P$. Also, $E_\nu$ is the exceptional divisor of $\tilde{Q}_\nu := \text{Bl}_{X \cap Q_\nu} Q_\nu$. Hence, $\dim E_\nu = r - 1$ will hold by (3.2)(iii) of [13] if
\[ \dim \mathcal{O}_{Q_\nu,x} = r, \]
(2.2.1)
and this equation will now be established.

Generically, the modules $N$ and $M$ are equal to each other, since $Y \neq X$. So, generically, the algebras $A[N/\nu]$ and $A[M/\nu]$ are equal to each other. Denote their common transcendence degree over $A$ by $f$. Then $f$ is the dimension of the generic fiber of $p: P \rightarrow X$ as $P_\nu$ is an open subset of $P$. Set $d := \dim X$. Then (3.2)(ii) of [13] yields $d + f = r$. 


Let \( m \) be the maximal ideal of \( A[M/\nu] \) representing \( x \), and \( n \) that of \( A[N/\nu] \) representing \( z \). Then \( n \) contracts to \( m \). Also, the residue field extension \( k(m)/k(x) \) is trivial. So, by standard theory [16, (14.C), p. 84],

\[
\operatorname{ht} m \leq d + \operatorname{tr.deg}_A A[M/\nu] - \operatorname{tr.deg}_{k(x)} k(m) = d + f - 0 = r.
\]

By the Hilbert Nullstellensatz, \( k(n)/k(m) \) is algebraic. So, similarly,

\[
\operatorname{ht} n \leq \operatorname{ht} m + \operatorname{tr.deg}_{A[M/\nu]} A[N/\nu] - \operatorname{tr.deg}_{k(m)} k(n) = \operatorname{ht} m + 0 - 0.
\]

Now, \( n = r \) since \( P \) has dimension \( r \) at \( z \). Hence \( \operatorname{ht} m = r \). Thus (2.2.1) holds, and so \( \dim E_\nu = r - 1 \).

Note that \( \dim C > \dim F \), for \( C \) is irreducible and \( C \neq F \) as \( Y \neq X \). So

\[
r = \dim C > \dim F \geq \dim E \geq \dim E_\nu = r - 1
\]

by (1.1.1), by (2.1.1), and by what was just proved. Hence all the dimensions are as asserted.

Finally, suppose \( A \) is universally catenary. Then \( \mathcal{O}_{Q_\nu,x} \) is too. Now, to prove that \( E_\nu \) is biequidimensional, we may replace \( Q_\nu \) by \( \operatorname{Spec} \mathcal{O}_{Q_\nu,x} \). After this replacement, \( \bar{Q}_\nu \) is biequidimensional by (3.8) of [13]. Hence its Cartier divisor \( E_\nu \) is too. The proof is now complete.

(2.3) Proof of (1) in (1.1). We’ll prove that each of the three hypotheses (a)–(c) implies Hypothesis (d), that \( P \) has dimension \( r \) at some \( z \in Z \). Then \( z \in Z_\nu \) for some \( \nu \) because, as \( \nu \) runs through a set of generators of \( N \), the various \( P_\nu \) cover \( P \). Hence (2.2) will yield the dimension assertions.

First, assume (a). Then \( A[N]/(M \cdot A[N]) \) is equal to the symmetric algebra on \( N/M \), and so \( Z = \mathbb{P}(N/M) \). Hence \( p(Z) = Y \), and so \( E = F \). Hence the closed fiber of \( Z \) contains a closed point \( z \) because \( Y \) is nonempty as \( M \neq N \) by hypothesis. Finally, \( P \) has dimension \( r \) at \( z \) by [13, (3.6)].

Second, assume \( A[N] \) is not a finitely generated module over \( A[M] \). Then \( Z \) is nonempty by virtue of part of the argument in (2.1) showing \( E_\nu \subset E \). So \( p(Z) \) contains the closed point of \( X \). Hence the closed fiber of \( Z \) contains a closed point \( z \). Finally, \( P \) has dimension \( r \) at \( z \) by [13, (3.8)].

Third, assume that \( Z = p^{-1}p(Z) \) as sets and that \( Z \) is nonempty. Now, \( P \) has dimension \( r \) at some point \( z \). Then \( p(z) \) is the closed point of \( X \). So \( p(z) \in p(Z) \) since \( Z \) is nonempty and is closed. Hence \( z \in Z \) since \( Z = p^{-1}p(Z) \). The proof is now complete.

(2.4) Paths. Let \( V \) be an \( X \)-scheme. By a path to \( v \in V \) will be meant an \( X \)-map \( \operatorname{Spec}(R) \to V \), where \( R \) is a local overdomain of \( A \), such that the closed point of \( \operatorname{Spec}(R) \) maps to \( v \).

A path to \( w \in C \) is given by a map of graded \( A \)-algebras \( A[M] \to R[t] \) where \( t \) is an indeterminate; so the path is determined by the piece in degree 1 of this map, which is an \( A \)-linear map,

\[
\pi: M \to R.
\]
Such a $\pi$ will be called a parameterized path, or pp for short. Of course, $R\pi(M) = R$.

Let $K$ and $L$ be the fraction fields of $A$ and $R$. Let $A[M] \otimes K \to L[t]$ be a map of graded $K$-algebras, $\rho: M \otimes K \to L$ the piece in degree 1. Suppose $R\rho(M) = Rr$ for some nonzero $r \in L$. Set $u := rt$ and $\pi := (\rho/r)|M$. Then $\pi$ is the piece in degree 1 of the induced map of graded $A$-algebras $A[M] \to R[u]$. The latter defines a map $\text{Spec}(R) \to C$. This map is a path to $w \in C$, where $w$ is the image of closed point of $\text{Spec}(R)$; so $w$ is determined by the composition $M \to R \to k(R)$, where $k(R)$ is the residue field.

Let $\pi: M \to R$ be a pp to $w \in C$, and $\pi_K: M \otimes K \to L$ the extension. Recall that $M \otimes K = N \otimes K$. Suppose $R\pi_K(N) = Rt$ for some $t \in L$. Then, by the discussion above, $\psi := (\pi_K/t)|N$ is a pp to some $z \in P$. This pp lifts to a path to some $u \in B$ since $R\psi(M) = R/t$, and $u \in D$ if $1/t$ lies in the maximal ideal $m_R$. Furthermore, $q(u) = w$; see [13, (2.1)].

For instance, suppose that $A[M]$ is the Rees algebra (or equivalently, that $A[N]$ is the Rees algebra). Then $A[M] \otimes K$ is the symmetric algebra over $K$ on the vector space $M \otimes K$. Hence any $A$-linear map $\rho: M \otimes K \to L$ extends to a map of graded $K$-algebras $A[M] \otimes K \to L[t]$. So, if $R\rho(M) = Rr$, then $\pi := (\rho/r)M$ is a pp to some $w \in C$, and if $R\pi_K(N) = Rt$ where $t \in m_R$, then $w \in q(D)$.

(2.5) Lemma. Let $w \in C$. Then $w \in q(D)$ if there is a pp $\pi: M \to R$ to $w$ where $R$ is a valuation ring and if either (a) $R\pi_K(N) \not= R$, or (b) $A[N]$ is the Rees algebra, and there is a pp $\theta: N \to R$ to a point $z$ of $P$ in $Z$.

Indeed, since $R$ is a valuation ring, then $R\pi_K(N) = Rt$ for some $t \in L$. If (a) holds, then $1/t \in m_R$; whence, $w \in q(D)$ by (2.4).

Suppose (a) fails, but (b) holds. By hypothesis, $\theta: N \to R$ is a pp to a point of $P$ in $Z$; so $R\theta(N) = R$ and $\theta(M) \subset m_R$. Since $R$ is a valuation ring, $R\theta(M) = Rs$ for some $s \in m_R$. If $Rs \neq m_R$, then

$$s = rr'$$

for some $r, r' \in m_R$;

in fact, any $r \in (m_R - Rs)$ works. If $Rs = m_R$, then the displayed equation can be achieved by adjoining a square root $r$ of $s$ to $K$ and then replacing $R$ by a valuation ring of the extension dominating $R$.

The displayed equation implies that $\theta(M) \subset Rs \subset m_Rr$. Set $\pi' := \pi + (\theta/r)$. Then $\pi': M \to R$ is a pp to $w \in C$ by (2.4) because $A[N]$ is the Rees algebra, $R\pi(M) = R$ and $\theta(M) \subset m_Rr$. Finally, (a) holds for $\pi'$. Otherwise, (a) would fail for both $\pi$ and $\pi'$. Then $\theta/r$ would carry $N$ into $R$, and so $\theta(N) \subset Rr \subset m_R$. However, $R\theta(N) = R$ since $\theta$ is a pp. Replace $\pi$ by $\pi'$. Then $w \in q(D)$ by Case (a). The proof is now complete.

(2.6) Lemma. If $\nu$ varies so that $Z = \bigcup Z_{\nu}$, then

$$q(D) = \bigcup E_{\nu}.$$
Indeed, $q(D) \subset \bigcup E_{\nu}$ by (2.1.1). Conversely, given $w \in E_{\nu}$, let $R$ be a valuation ring dominating the local ring of $\tilde{Q}_{\nu}$ at $w$, and form the corresponding ring map $\mu: A[M/\nu] \to R$. Then $R\mu(M/\nu) = Rr$ for some $r \in m_R$ because $E_{\nu}$ is the exceptional divisor of $\tilde{Q}_{\nu} := \text{Bl}_{X \cap Q_{\nu}} Q_{\nu}$. Moreover, $\mu$ induces a map of graded $K$-algebras $A[M] \to R$, whose piece in degree 1 is the composition,

$$\rho: M \otimes K \to (M/\nu) \otimes K \to L.$$ 

Then $R\rho(M) = Rr$. Set $\pi := \rho/r$. Then $\pi$ is a pp to $w \in C$ by (2.4). Moreover, $\mu$ induces a map of graded $K$-algebras $A[M] \otimes K \to L[t]$, whose piece in degree 1 is the composition,

$$\rho: M \otimes K \to (M/\nu) \otimes K \to L.$$ 

Then $R\rho(M) = Rr$. Set $\pi := \rho/r$. Then $\pi$ is a pp to $w \in C$ by (2.4). Moreover, $\mu$ induces a map of graded $K$-algebras $A[M] \to R$, whose piece in degree 1 is the composition,

$$\rho: M \otimes K \to (M/\nu) \otimes K \to L.$$ 

Set $\pi := \rho/r$. Then $\pi$ is a pp to $w \in C$ by (2.4). Moreover, $\mu$ induces a map of graded $K$-algebras $A[M] \otimes K \to L[t]$, whose piece in degree 1 is the composition,

$$\rho: M \otimes K \to (M/\nu) \otimes K \to L.$$ 

Then $R\rho(M) = Rr$. Set $\pi := \rho/r$. Then $\pi$ is a pp to $w \in C$ by (2.4). Moreover, $\mu$ induces a map of graded $K$-algebras $A[M] \to R$, whose piece in degree 1 is the composition,

$$\rho: M \otimes K \to (M/\nu) \otimes K \to L.$$ 

Then $R\rho(M) = Rr$. Set $\pi := \rho/r$. Then $\pi$ is a pp to $w \in C$ by (2.4). Moreover, $\mu$ induces a map of graded $K$-algebras $A[M] \otimes K \to L[t]$, whose piece in degree 1 is the composition,

$$\rho: M \otimes K \to (M/\nu) \otimes K \to L.$$ 

Then $R\rho(M) = Rr$. Set $\pi := \rho/r$. Then $\pi$ is a pp to $w \in C$ by (2.4). Moreover, $\mu$ induces a map of graded $K$-algebras $A[M] \otimes K \to L[t]$, whose piece in degree 1 is the composition,

$$\rho: M \otimes K \to (M/\nu) \otimes K \to L.$$
where $B := \text{Bl}_Z(P)$ is the blowup along $Z$; see [13, (2.1)]. Then $q: B \to C$ is surjective by [13, (2.6)] since $\mathcal{O}_X[M] \subset \mathcal{O}_X[N]$.

Suppose for a moment that $\mathcal{N}$ is locally free of rank $n$ and that $\mathcal{O}_X[N]$ is the symmetric algebra. Then $Z = \mathbf{P}(\mathcal{N}/\mathcal{M})$. Hence $p(Z) = Y$, and so $E = F$. Also, if $\dim Y < \dim X$, then, by [13, (3.6)] applied at each closed point of $Y$ and at each of $X$,

$$\dim p^{-1}Y = \dim Y + n - 1 < \dim X + n - 1 = \dim P =: r.$$  

Hence $\dim p^{-1}p(Z) < r$ under Hypothesis (a) too.

Let $C'$ be a component of $C$. Since $q$ is surjective, $C' = q(B')$ for some component $B'$ of $B$. Then, by [13, (3.2)],

$$\dim C' \leq \dim B' = \dim b(B') \leq \dim P =: r. \quad (2.2.1)$$

Let $E'$ be a component of $E$. Then $\dim E' \leq r - 1$. Otherwise, $E'$ is a component $C'$ of $C$ of dimension $r$ because of (2.2.1). Then $cq(B') \subset p(Z)$, so $b(B') \subset p^{-1}(Z)$. So, if $\dim p^{-1}p(Z) < r$, then there is a contradiction.

Consider Part (1). To prove dim $C = r$ and dim $E = r - 1$, it is enough, by the above, to find one $C'$ of dimension $r$ and one $E'$ of dimension $r - 1$.

We may assume (d) holds, as there is a point $z \in Z$ with dim $\mathcal{O}_{P,z} = r$ also if (a), (b) or (c) holds. Indeed, if (a) holds, then any closed point $z \in Z$ lying over $y \in Y$ will do by [13, (3.6)] applied locally at $y \in X$. If (b) holds, then $Z$ meets an $r$-dimensional component $P'$ of $P$, and any closed point $z \in P' \cap Z$ will do by [13, (3.8)]. Finally, assume (c), and take $z \in P$ such that $\dim \mathcal{O}_{P,z} = r$. Then $p(z)$ is the unique closed point of $X$. So $p(z) \in p(Z)$ since $Z$ is nonempty and is closed. Hence $z \in Z$ since $Z = p^{-1}p(Z)$.

We may localize the setup at $p(z)$. Then $X$ is the spectrum of a local ring, $A$ say. Moreover, the two $\mathcal{O}_X$-algebras are associated to two graded $A$-algebras, $A[N]$ and $A[M]$ say.

Take a component $P'$ of $P$ whose local ring at $z$ has dimension $r$, and give $P'$ its reduced structure. Then $P' = \text{Proj}(A'[N'])$ where $A'[N']$ is a graded domain and a quotient of $A[N]$. Let $M'$ be the image of $M$ in $N'$, and use a prime to indicate the corresponding constructions. Then $Z' = Z \cap P'$; hence (d) continues to hold. Moreover, the $C'$ and $E'$ are closed subsets of $C$ and $E$; hence, the latter have dimensions $r$ and $r - 1$ if the former do.

By hypothesis, $\dim p^{-1}p(Z) < r$; so $p^{-1}p(Z') \neq P'$ and so $p(Z') \neq X'$. Since $A'$ is a domain and $N'$ is torsion free, $N'$ is free on a dense open set, $U$ say, of $X'$. Then $Y' \cap U = p(Z') \cap U$ by the second paragraph of the proof. Hence $Y' \neq X'$. So dim $C' = r$ by (1.1.1), and dim $E' = r - 1$ by (1) with (d) of the theorem. The proof of (1) of the corollary is now complete.

Consider Part (2). To prove it, we may localize at an arbitrary closed point of $X$. Indeed, since $X$ is universally catenary and biequidimensional, $P$ is equidimensional also if (a) holds by (3.2)(ii) of [13], and so $P$ is biequidimensional by (3.8) of [13]. We may also replace $P$ by an arbitrary component
Indeed, each component $C'$ of $C$ corresponds to some $P'$ by the third paragraph above. Let $E'$ be a component of $E$. Then $E'$ lies in some $C'$, which corresponds to some $P'$. If $p^{-1}(Z) = Z$, then any closed point of $P'$ lies in $Z$. If (a) holds, then the whole closed fiber of $P$ lies in $P''$, and so any closed point of $Z$ lies in $P'$. Finally, proceeding as in the proof of Part (1), reduce (2) of the corollary to (2) of the theorem. Thus (2) is proved, and the proof of the corollary is complete.

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