A UNIVERSAL TROPICAL JACOBIAN OVER $M_{g}^{\text{trop}}$

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Abstract. We introduce and study polystable divisors on a tropical curve, which are the tropical analogue of polystable torsion-free rank-1 sheaves on a nodal curve. We construct a universal tropical Jacobian over the moduli space of tropical curves of genus $g$. This space parametrizes equivalence classes of tropical curves of genus $g$ together with a $\mu$-polystable divisor, and can be seen as a tropical counterpart of Caporaso universal Picard scheme. We describe polyhedral decompositions of the Jacobian of a tropical curve via polystable divisors, relating them with other known polyhedral decompositions.

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1. Introduction

1.1. Background. In the last few years, tropical geometry has uncovered deep connections with algebraic geometry, in particular when applied to the moduli theory of algebraic curves. This interplay became evident in paper [5], which contains a beautiful interpretation of the moduli space of tropical curves as the skeleton of the Berkovich analytification of the moduli space of stable curves. In this sense, tropical geometry can be seen as a confluence point between algebraic and analytic geometry. The result motivated the construction and the analysis of tropical analogues of other interesting moduli spaces in algebraic geometry (see [1], [7], [13], [14], [23], [28], [28], [29], just to make few examples). This new point of view could also provide a better understanding of the boundary of a given compactification.

Sometimes, the construction of a tropical moduli space can also help to handle the combinatorial issues of a given problem in algebraic geometry, or even suggest its solution. For example, a “tropically inspired” resolution of the universal Abel map was recently given in [17] and [2] (though the tropical setting does not appear explicitly in [17]).
A fundamental problem in algebraic geometry is how to attain a compactified universal Picard variety over the moduli space of stable curves. There are essentially two proper compactifications, Caporaso universal Picard scheme \( \mathcal{P}_{d,g} \), constructed in [9], and later generalized in [26] by Pandharipande for higher ranks, and the universal Picard stack \( \mathcal{J}_{\mu,g} \), constructed in [20] by Melo building on the work of Esteves [16]. The space \( \mathcal{P}_{d,g} \) lives over the moduli space of stable curve, in stark contrast with the space \( \mathcal{J}_{\mu,g} \), that naturally lives only over the moduli of pointed curves, as it is necessary to have enough sections to define its objects. On the other hand, the second one is a fine moduli stack, while the first one is not.

A natural problem in tropical geometry is the construction of a universal tropical Jacobian over the moduli space \( \mathcal{M}_{\text{trop}}^g \) of tropical curve (as usual, in the category of generalized cone complexes). This problem has attracted a lot of interest in the last years, due to the centrality of its algebro-geometric counterpart. To our knowledge, the first construction of a universal tropical Jacobian (as a topological space) for tropical curves of a fixed combinatorial type was carried out in [19]. The first and third author recently introduced a universal tropical Jacobian in [1]. This space is the tropical counterpart of the Picard stack \( \mathcal{J}_{\mu,g} \). As expected, the construction holds over the moduli space of pointed tropical curves, and could be reproduced over \( \mathcal{M}_{\text{trop}}^g \) only in the nondegenerate case. This paper is dedicated to the construction of a universal tropical Jacobian over \( \mathcal{M}_{\text{trop}}^g \) as a generalized cone complex, and it is a natural completion of the results in [1].

1.2. Outline of the results. The objects parametrized in Caporaso space \( \mathcal{P}_{d,g} \) are stably balanced line bundle on quasistable curves. Later, Pandharipande re-interpreted \( \mathcal{P}_{d,g} \) as the space parametrizing polystable torsion-free rank-1 sheaves on stable curves. We introduce their tropical analogues, that are \( \mu \)-polystable (pseudo-)divisors on graphs and tropical curves. Here, \( \mu \) denotes a polarization.

The key ingredient to construct a moduli space via polystable divisors is that every divisor on a tropical curve is equivalent to a \( \mu \)-polystable divisor. Moreover, two equivalent \( \mu \)-polystable divisors have the same combinatorial type (see Theorem 5.9). Our proof relies on purely combinatorial arguments. The same result is also proved in [15, Proposition 4.4] using a more geometric approach. We use this result to construct a generalized cone complex \( \mathcal{P}_{\text{trop}}^{\mu,g} \) over \( \mathcal{M}_{\text{trop}}^g \). Each cone parametrizes equivalence classes of polystable divisors on tropical curves with fixed combinatorial type. We need a subtle analysis of how the cones can be glued in a nice way as prescribed by specializations of graphs (see Example 6.11).

**Theorem (6.14 and 6.15).** The generalized cone complex \( \mathcal{P}_{\text{trop}}^{\mu,g} \) has dimension \( 4g-3 \) and it is connected in codimension 1. The natural forgetful map \( \pi_{\text{trop}}: \mathcal{P}_{\mu,g}^{\text{trop}} \rightarrow \mathcal{M}_{\text{trop}}^g \) is a map of generalized cone complexes, and we have:

\[
(\pi_{\text{trop}})^{-1}[X] \cong J(X)/\text{Aut}(X),
\]

for every stable tropical curve \( X \) of genus \( g \). (Here, \( J(X) \) is the tropical Jacobian.)

The space \( \mathcal{P}_{\mu,g}^{\text{trop}} \) parametrizes equivalence classes of pairs \( (X,D) \), where \( X \) is a stable tropical curve of genus \( g \) and \( D \) is a \( \mu \)-polystable divisor on \( X \).

We point out that, looking back to algebraic geometry, there are other natural candidates for a universal tropical Jacobian over \( \mathcal{M}_{\text{trop}}^g \) (see Section 4.1). None of them are suitable for our purposes. In fact, one could try to define a universal object parametrizing all \( \mu \)-semistable divisors on tropical curves. The problem is
that the corresponding topological space is not a cone complex. One could try to solve this issue by considering just simple $\mu$-semistable divisors. In this case, even though the corresponding topological space is a generalized cone complex, its points are not in one to one correspondence with linear equivalence classes of divisors on tropical curves.

We also construct polyhedral decompositions of the Jacobian $J(X)$ of a tropical curve $X$. Polyhedral decompositions of the tropical Jacobian are closely related to toroidal compactifications of the Jacobian of a curve (see \cite{25}, \cite{24}, \cite{15}). An interesting decomposition is studied in \cite{11} in degree $g$ through break divisors. A similar analysis is done in \cite{1} for all degrees through quasistable divisors on tropical curves. The strategy is to define a polyhedral complex $P_{\mu}^\text{trop}(X)$ by means of $\mu$-polystable divisors on $X$. This is done by gluing polytopes along faces as prescribed by specializations. Then we compare $P_{\mu}^\text{trop}(X)$ with $J(X)$:

**Theorem 6.6.** Given a tropical curve $X$, there is a homeomorphism $P_{\mu}^\text{trop}(X) \rightarrow J(X)$.

It is interesting to observe that, if $p_0$ is a point of $X$ and $J_{p_{0,i}}^\text{trop}(X)$ is the polyhedral complex of $(p_0, \mu)$-quasistable divisors on $X$ (also homeomorphic to $J(X)$), then we have a refinement map of polyhedral complexes $J_{p_{0,i}}^\text{trop}(X) \rightarrow P_{\mu}^\text{trop}(X)$ (see Proposition 6.7). The same polyhedral decompositions of the tropical Jacobian by means of polystable divisors and the study of their relationship with quasistable divisors also appear in \cite{15} Proposition 5.8 and Corollary 5.10.

A tool to compare quasistable (or, more generally, semistable) divisors and polystable divisors is Proposition 5.4. There, we show that for every $\mu$-semistable pseudo-divisor on a graph $\Gamma$, there is a unique minimal $\mu$-polystable pseudo-divisor on $\Gamma$ that specializes to it. This distinguished $\mu$-polystable pseudo-divisor is obtained via an iterative procedure reminding of the construction of the graded of a torsion-free rank-1 sheaf on a curve by means of Jordan-Holder filtrations (see \cite{16} Section 1.3 and \cite{15} Proposition 3.7). Proposition 5.4 can be viewed as a discrete analogue of the property of a geometric invariant theory quotient stating that each semistable orbit contains a unique polystable orbit in its closure.

In Section 6.3 we discuss natural stratifications (in the sense of \cite{12}, Definition 1.4.2) of universal Picard moduli spaces.

About ten days before submitting this paper to ArXiv, we were made aware of the paper \cite{15} by Christ, Payne, and Shen. They also consider polystable divisors on tropical curves to describe polyhedral decompositions of the tropical Jacobian, and relate them to Mumford models of compactified Jacobians. Some of the present results appear in \cite{15}, although the methods and purpose of the two papers are somewhat different. We made an effort to highlight the common results.

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2. Preliminaries

2.1. Posets. A poset (partially ordered set) is a pair $(S, \leq)$ where $S$ is a set and $\leq$ is a partial order on $S$. A chain in $S$ is a sequence $x_0 < x_1 < \ldots < x_n$. We call
n the length of the chain. We say that $S$ is ranked (of length $n$) if the maximal chains have all the same length $n$. The maximal length of chains in $S$ is precisely the Krull dimension of $S$ as a topological space. Hence, if $S$ is ranked, then it is of pure dimension. If $S$ is a ranked poset of length $n$, then the codimension of $x \in S$ is $n - \dim_S(x)$, i.e., the length of all maximal chains starting from $x$. We say that $S$ is graded if it has a function $\text{rk}: S \to \mathbb{N}$, called rank function, such that $\text{rk}(x) = \text{rk}(y) + 1$ whenever $y < x$ and there is no element $z \in S$ such that $y < z < x$. Every ranked poset is graded with rank function given by the dimension.

Let $S$ be a ranked poset. We say that $S$ is connected in codimension one if for every maximal elements $y, y' \in S$ there are two sequences of elements $x_1, \ldots, x_n \in S$ and $y_0, \ldots, y_n \in S$ such that

1. $x_i$ has codimension 1 for every $i = 1, \ldots, n$.
2. $y_i$ is maximal for every $i = 0, \ldots, n$.
3. $y_0 = y$ and $y_n = y'$.
4. $x_{i+1} < y_i$ for every $i = 0, \ldots, n - 1$ and $x_i < y_i$ for every $i = 1, \ldots, n$.

We call these sequences a path in codimension 1 from $y$ to $y'$.

2.2. Cones and polyhedra. We briefly introduce cones and polyhedra. We will adopt the terminology of [1, Section 3.2]. Given a finite set $S \subset \mathbb{R}^n$ we define

$$\text{cone}(S) := \left\{ \sum_{s \in S} \lambda_s s | \lambda_s \in \mathbb{R}_{\geq 0} \right\}.$$

A subset $\sigma \subset \mathbb{R}^n$ is called a polyhedral cone if $\sigma = \text{cone}(S)$ for some finite set $S \subset \mathbb{R}^n$. If there is a subset $S \subset \mathbb{Z}^n$ with $\sigma = \text{cone}(S)$ then $\sigma$ is called rational. Throughout, cone will mean rational polyhedral cone. A face of a cone $\sigma$ is the intersection of $\sigma$ with some linear subspace $H \subset \mathbb{R}^n$ of codimension one such that $\sigma$ is contained in one of the closed half-spaces determined by $H$. A generalized cone complex is the colimit (as topological space) of a finite diagram of cones with face morphisms (i.e., morphisms of cones taking faces to faces). We refer to [5, Section 2] for the more details on cone complexes.

A polyhedron $P \subset \mathbb{R}^n$ is an intersection of a finite number of half-spaces of $\mathbb{R}^n$. A face of a polyhedron $P$ is the intersection of $P$ and a hyperplane $H$ such that $P$ is contained in a closed half-space determined by $H$. A polyhedral complex is the colimit (as topological space) of a finite poset of polyhedra with face morphisms (i.e., morphisms of polyhedra taking faces to faces).

2.3. Graphs. Let $\Gamma$ be a graph. We denote by $V(\Gamma)$ the set of vertices and by $E(\Gamma)$ the set of edges of $\Gamma$. We also denote by $b_0(\Gamma)$ and $b_1(\Gamma)$ its first and second Betti numbers, i.e., $b_0(\Gamma)$ is the number of connected components and

$$b_1(\Gamma) := |E(\Gamma)| - |V(\Gamma)| + b_0(\Gamma).$$

Sometimes, we will refer to $b_1(\Gamma)$ as the genus of the graph. We let $\text{Aut}(\Gamma)$ be the group of automorphisms of $\Gamma$.

Given a subset $V \subset V(\Gamma)$, we let $E(V) \subset E(\Gamma)$ be the subset of edges of $\Gamma$ connecting vertices in $V$. Given disjoint subsets $V, W \subset V(\Gamma)$ of $V(\Gamma)$ we define the set $E(V, W) \subset E(\Gamma)$ of the edges that connect a vertex in $V$ to a vertex in $W$. We set $\delta_{V, V} := |E(V, V)|$. More generally, given subsets $V, W \subset V(\Gamma)$, we define $E(V, W) := E(V \setminus W, W \setminus V)$. 
Given a subset $\mathcal{E} \subset E(\Gamma)$ and a vertex $v$ of $\Gamma$, we define the \textit{valence} of $v$ in $\mathcal{E}$, denoted by $\text{val}_\mathcal{E}(v)$, as the number of edges in $\mathcal{E}$ incident to $v$ (with loops counting twice). In the case that $\mathcal{E} = E(\Gamma)$, we simply write $\text{val}(v)$ and call it the \textit{valence} of $v$. For every subset $V \subset V(\Gamma)$, we set $\text{val}_\mathcal{E}(V) := \sum_{v \in V} \text{val}_\mathcal{E}(v)$.

The graph $\Gamma$ is \textit{circular} if it is connected and its vertices have all valence 2. A \textit{cycle} on $\Gamma$ is a circular subgraph of $\Gamma$.

Consider a subset $\mathcal{E} \subset E(\Gamma)$. We let $\Gamma / \mathcal{E}$ and $\Gamma_\mathcal{E}$ be the graphs obtained by the contraction of edges in $\mathcal{E}$ and by the removal of edges in $\mathcal{E}$, respectively. There is a natural surjection $V(\Gamma) / \mathcal{E} \to V(\Gamma / \mathcal{E})$ and a natural identification $E(\Gamma / \mathcal{E}) = E(\Gamma) \setminus \mathcal{E}$. Moreover, we have $V(\Gamma_\mathcal{E}) = V(\Gamma)$ and $E(\Gamma_\mathcal{E}) = E(\Gamma) \setminus \mathcal{E}$. The subset $\mathcal{E}$ is called \textit{non-disconnecting} if $\Gamma_\mathcal{E}$ is connected, otherwise it is \textit{disconnecting}.

We let $\Gamma^\mathcal{E}$ be the graph obtained from $\Gamma$ by inserting a vertex, called \textit{exceptional}, in the interior of any edge $e \in \mathcal{E}$. We denote by $v_e$ the new vertex inside $e$. Thus, for every edge $e \in \mathcal{E}$, we get exactly two edges $e_1, e_2$ of $\Gamma^\mathcal{E}$ incident to $v_e$. We say that $e_1, e_2$ are the edges \textit{over} $e$, and that $e$ is the edge \textit{under} $e_1$ (or $e_2$). A \textit{refinement} of $\Gamma$ is a graph obtained by iterating the operation taking $\Gamma$ to $\Gamma^\mathcal{E}$.

A graph $\Gamma$ \textit{specializes} to a graph $\Gamma'$, and we write $\iota : \Gamma \to \Gamma'$, if there is a subset $\mathcal{E} \subset E(\Gamma)$ such that $\Gamma'$ is isomorphic to $\Gamma / \mathcal{E}$. Then a specialization $\iota : \Gamma \to \Gamma'$ comes equipped with a surjective map $\iota^V : V(\Gamma) \to V(\Gamma')$ and an injective map $\iota^E : E(\Gamma') \to E(\Gamma)$. We simply write $\iota = \iota^V$ and see $E(\Gamma')$ as a subset of $E(\Gamma)$ via $\iota^E$.

Sometimes we consider an orientation on $\Gamma$. In this case, for every edge $e \in E(\Gamma)$, we denote by $s(e), t(e) \in V(\Gamma)$ the source and the target of an (oriented) edge $e \in E(\Gamma)$, respectively. For every subset $\mathcal{E} \subset E(\Gamma)$ and every edge $e \in E(\Gamma)$, we let $e^\ast, e^\dagger$ the edges of $\Gamma^\mathcal{E}$ over $e$, with $e^\ast$ incident to $s(e)$ and $e^\dagger$ incident to $t(e)$, respectively. Moreover, given an oriented cycle $\gamma$ on $\Gamma$, we define

\begin{equation}
\gamma(e) = \begin{cases} 
0 & \text{if } e \text{ is not a edge of } \gamma \\
1 & \text{if the orientations on } \gamma \text{ and } \Gamma \text{ coincide on } e \\
-1 & \text{otherwise.}
\end{cases}
\end{equation}

A (vertex) \textit{weighted graph} is a graph $\Gamma$ together with a function $w_\Gamma : V(\Gamma) \to \mathbb{Z}_{\geq 0}$, called \textit{weight function}. A weighted graph $\Gamma$ is \textit{stable} if $\text{val}(v) + 2w(v) \geq 3$ for every vertex $v \in V(\Gamma)$. The genus of $\Gamma$ is $g(\Gamma) := \sum_{v \in V(\Gamma)} w_\Gamma(v) + b_1(\Gamma)$. A \textit{specialization} $\iota : \Gamma \to \Gamma'$ of weighted graphs $\Gamma$ and $\Gamma'$ is a graph specialization such that $w_{\Gamma'}(v') = w_\Gamma(\iota^{-1}(v'))$ for every $v' \in V(\Gamma')$.

2.4. Tropical curves. A \textit{tropical curve} is a metric space $X$ such that there exists a weighted graph $\Gamma$ and a function $\ell : \Gamma \to \mathbb{R}^{E(\Gamma)}$, called \textit{length function}, so that $X$ is obtained by gluing segments $[0, \ell(e)] \subset \mathbb{R}$ for every $e \in E(\Gamma)$ at their end vertices, as prescribed by the combinatorial data of the graph. We call $(\Gamma, \ell)$ (or simply $\Gamma$ when the function $\ell$ is clear), a \textit{model} of the tropical curve $X$. We will identify isometric tropical curves, so a tropical curve could admit different models. For every edge $e \in E(\Gamma)$, we let $e^0 \subset X$ be the interior of the corresponding segment of $X$. For points $p, q \in X$, we denote by $\overline{pq}$ the segment contained in $e$ with endpoints $p$ and $q$. If an orientation is chosen on $\Gamma$, we denote by $\overrightarrow{pq}$ the oriented segment.

Let $X$ be a tropical curve. The \textit{genus} of $X$ is the genus of one of its underlying weighted graphs. The \textit{valence} of a point $p \in X$ is the valence of $p$ as a vertex of any model containing $p$ as a vertex. Note that $X$ inherits a well-defined weight
function \(w_X : X \rightarrow \mathbb{Z}_{\geq 0}\) from any one of its models, where \(w_X(p) = 0\) is not a vertex of the model. We say that \(X\) is stable if \(\delta_{X,p} + 2w_X(p) \geq 3\) for every point \(p \in X\) such that \(\delta_{X,p} \leq 1\). The stable model of \(X\) is the model \(\Gamma\) such that

\[V(\Gamma) = \{p \in X; \text{ either } \text{val}_X(p) \neq 2, \text{ or } \text{val}_X(p) = 2 \text{ and } w_X(p) \neq 0\}.
\]

A polarization of degree \(d\) on \(X\) is a function \(\mu : X \rightarrow \mathbb{R}\) such that \(\mu(p) = 0\) for all, but finitely many \(p \in X\), and \(\sum_{p \in X} \mu(p) = d\). Note that when \(X\) has model \(\Gamma\) and \(\mu\) is a polarization on \(\Gamma\), then there is a natural induced polarization on \(X\). Conversely, all polarizations on \(X\) come from polarizations on models of \(X\).

Given a polarization \(\mu\) on \(X\), the \(\mu\)-model of \(X\) is the model whose vertices are the vertices of its stable model and the points \(p\) of \(X\) such that \(\mu(p) \neq 0\).

A (tropical) subcurve \(X\) is a tropical curve \(Z\) admitting an injection \(Z \subset X\) that is an isometry over each connected component of \(Z\). If \(\Gamma\) is a model of \(X\) and \(Z\) is a subcurve of \(X\), then there exists a minimal refinement \(\Gamma'\) of \(\Gamma\) such that \(Z\) is induced by a subgraph \(\Gamma'_Z\) of \(\Gamma'\). We define

\[\delta_{X,Z} := \sum_{v \in V(\Gamma'_Z) \setminus E(\Gamma'_Z)} \text{val}_{E(\Gamma'_Z)}(v).
\]

Given a subset \(V \subset V(\Gamma)\), for a model \(\Gamma\) of \(X\), we define \(X_V\) as the subcurve of \(X\) with model \(\Gamma(V)\) and length function induced by the length function of \(X\).

3. Divisors on Graphs and Tropical Curves

3.1. Divisors on Graphs. Let \(\Gamma\) be a (weighted) graph. A divisor \(D\) on \(\Gamma\) is a function \(D : V(\Gamma) \rightarrow \mathbb{Z}\). The degree of \(D\) is the integer \(\deg D := \sum_{v \in V(\Gamma)} D(v)\).

Let \(D\) be a divisor on \(\Gamma\). Given a subset \(\mathcal{E} \subset E(\Gamma)\), we define the divisor \(D_\mathcal{E}\) on \(\Gamma_\mathcal{E}\) as \(D_\mathcal{E}(v) := D(v)\), for every \(v \in V(\Gamma_\mathcal{E}) = V(\Gamma)\). If \(\iota : \Gamma \rightarrow \Gamma'\) is a specialization of graphs, we define the divisor \(\iota_*D\) on \(\Gamma'\) such that \(\iota_*D(v') = \sum_{v \in \iota^{-1}(v')} D(v)\).

A pseudo-divisor on \(\Gamma\) is a pair \((\mathcal{E}, D)\) where \(\mathcal{E} \subset E(\Gamma)\) and \(D\) is a divisor on \(\Gamma_\mathcal{E}\) such that \(D(v) = -1\) for every exceptional vertex \(v \in V(\Gamma_\mathcal{E})\). If \(\mathcal{E} = \emptyset\), then \((\mathcal{E}, D)\) is just a divisor of \(\Gamma\). We denote by \(\text{Aut}(\Gamma, \mathcal{E}, D)\) the subgroup of \(\text{Aut}(\Gamma)\) made of maps sending \((\mathcal{E}, D)\) to itself. More generally, two triples \((\Gamma, E, D)\) and \((\Gamma', E', D')\) are isomorphic if there is a graph isomorphism from \(\Gamma\) to \(\Gamma'\) mapping \((\mathcal{E}, D)\) to \((\mathcal{E}', D')\).

If \(\iota : \Gamma \rightarrow \Gamma'\) is a specialization of graphs and \((\mathcal{E}, D)\) is a pseudo-divisor on \(\Gamma\), we define \(\iota_*((\mathcal{E}, D))\) as the pseudo-divisor on \(\Gamma'\) given by \((\mathcal{E} \cap E(\Gamma'), (\iota \circ \iota)_*(D))\), where \(\iota : \Gamma \rightarrow \Gamma'\) is the specialization induced by \(\iota\).

Given an integer \(d\), a degree-\(d\) polarization on \(\Gamma\) is a function \(\mu : V(\Gamma) \rightarrow \mathbb{R}\) such that \(\sum_{v \in V(\Gamma)} \mu(v) = d\). If \(\mu\) is a polarization on \(\Gamma\), then we set \(\mu(V) := \sum_{v \in V} \mu(v)\) for every subset \(V \subset V(\Gamma)\). For every specialization of graphs \(\iota : \Gamma \rightarrow \Gamma'\), there is an induced degree-\(d\) polarization \(\iota_*\mu\) on \(\Gamma'\) defined as \(\iota_*\mu(v') := \sum_{v \in \iota^{-1}(v')} \mu(v)\).

A universal genus-\(g\) polarization (of degree \(d\)) is the datum of a polarization \(\mu_\Gamma\) for every genus-\(g\) stable weighted graph \(\Gamma\) such that \(\mu_\Gamma = \iota_*\mu_\Gamma\) for every specialization \(\Gamma \rightarrow \Gamma'\). The canonical genus-\(g\) universal polarization of degree \(d\) is

\[\mu_\Gamma(v) = \frac{d(2g_\Gamma(v) - 2 + \text{val}(v))}{2g - 2}.
\]

Let \(\mu\) be a polarization of degree \(d\) on \(\Gamma\). If \(\mathcal{E} \subset E(\Gamma)\) is a subset of edges, then \(\mu_\mathcal{E}(v) := \mu(v) + \frac{1}{2} \text{val}_\mathcal{E}(v)\) defines a polarization \(\mu_\mathcal{E}\) on \(\Gamma_\mathcal{E}\) of degree \(d + |\mathcal{E}|\). Given
a subdivision $\Gamma^\mathcal{E}$ of $\Gamma$ for some $\mathcal{E} \subset E(\Gamma)$, there is an induced degree-$d$ polarization $\mu^\mathcal{E}$ on $\Gamma^\mathcal{E}$ such that $\mu^\mathcal{E}(v) = \mu(v)$ if $v \in V(\Gamma)$, and $\mu^\mathcal{E}(v) = 0$ otherwise.

Let $\Gamma$ be a connected graph. Let $\mu$ be a degree-$d$ polarization on $\Gamma$ and $D$ a degree-$d$ divisor on $\Gamma$. For every subset $V \subset V(\Gamma)$, we set

$$\beta_D(V) = \deg(D|_V) - \mu(V) + \frac{\delta_{V, \Gamma}}{2}. \quad (2)$$

By [1] Lemma 4.1, for subsets $V, W \subset V(\Gamma)$ we have:

$$\beta_D(V \cup W) + \beta_D(V \cap W) = \beta_D(V) + \beta_D(W) - |E(V, W)|. \quad (2)$$

The divisor $D$ is $\mu$-semistable (respectively, $\mu$-stable) on $\Gamma$ if $\beta_D(V) \geq 0$ (respectively, $\beta_D(V) > 0$) for every subset $V \subset V(\Gamma)$ (respectively, for every non-empty proper subset $V \subsetneq V(\Gamma)$). Given a vertex $v_0 \in V(\Gamma)$, the divisor $D$ is $(v_0, \mu)$-quasistable if $\beta_D(V) \geq 0$ for every proper subset $V \subsetneq V(\Gamma)$, with strict inequality if $v_0 \in V$.

A polarization $\mu$ is said to be nondegenerate if every $\mu$-semistable divisor is actually $\mu$-stable (see [12] for a more explicit characterization). A universal polarization is said to be nondegenerate if it is nondegenerate for each graph.

We need to extend the stability conditions to the case of non-connected graphs. The notion of $\mu$-semistability naturally extends for divisors on non-connected graphs.

**Remark 3.1.** Let $\Gamma$ be a graph. Let $\mu$ be a degree-$d$ polarization on $\Gamma$ and $D$ a $\mu$-semistable divisor of degree-$d$ on $\Gamma$. If $\Gamma'$ is a connected component of $\Gamma$, then

$$\beta_D(V(\Gamma')) + \beta_D(V(\Gamma')^c) = 0.$$ 

Hence $\beta_D(V(\Gamma')) = 0$. In particular, $\mu(\Gamma') = \deg(D|_{\Gamma'})$, which is an integer, so $\mu|_{\Gamma'}$ is a polarization on $\Gamma'$.

If $\Gamma$ is a non-connected graph and $\mu$ is a degree-$d$ polarization on $\Gamma$, we say that a divisor $D$ of degree $d$ on $\Gamma$ is $\mu$-stable if it is $\mu$-semistable and $D|_{\Gamma'}$ is $\mu|_{\Gamma'}$-stable for every connected component $\Gamma'$ of $\Gamma$.

More generally, let $(\mathcal{E}, D)$ be a pseudo-divisor on $\Gamma$ and let $\mu_\mathcal{E}$ be the induced polarization on $\Gamma_\mathcal{E}$. For every subset $V \subset V(\Gamma_\mathcal{E}) = V(\Gamma)$, we set

$$\beta_{\mathcal{E}, D}(V) = \deg(D|_V) - \mu_\mathcal{E}(V) + \frac{\delta_{V, \Gamma}}{2}. \quad (3)$$

Notice that $\beta_{\mathcal{E}, D}(V) = \beta_D(V)$ if $\mathcal{E} = \emptyset$.

For subsets $\mathcal{E}, \mathcal{E}' \subset E(\Gamma)$ such that $\mathcal{E}' \subset \mathcal{E}$, we can consider the pseudo-divisor $(\mathcal{E}', D')$ on $\Gamma_{\mathcal{E} \setminus \mathcal{E}'}$, where $D' = D_{\mathcal{E} \setminus \mathcal{E}'}$. We have $D_\mathcal{E} = (D_{\mathcal{E} \setminus \mathcal{E}'})_{\mathcal{E}'}$, and hence

$$\beta_{\mathcal{E}, D_{\mathcal{E} \setminus \mathcal{E}'}}(V) = \beta_{\mathcal{E}', D}(V). \quad (3)$$

Taking $\mathcal{E}' = \emptyset$ in Equation (3), we get an analogue of Equation (2) for pseudo-divisors: for every subsets $V, W \subset V(\Gamma_\mathcal{E}) = V(\Gamma)$,

$$\beta_{\mathcal{E}, D}(V \cup W) + \beta_{\mathcal{E}, D}(V \cap W) = \beta_{\mathcal{E}, D}(V) + \beta_{\mathcal{E}, D}(W) - |E(V, W) \setminus \mathcal{E}|. \quad (4)$$

We could have defined $\beta_{\mathcal{E}, D}(V)$ in terms of the refined graph $\Gamma^\mathcal{E}$. Indeed, for every subset $V \subset V(\Gamma_\mathcal{E})$, we have that $\beta_{\mathcal{E}, D}(V) = \beta_D(\widetilde{V})$, where $\widetilde{V} \subset V(\Gamma^\mathcal{E})$ is:

$$\widetilde{V} := V \cup \{v_e; e \in \mathcal{E} \cap (E(V) \cup E(V^c))\}.$$ 

Note that for every subset $V \subset V(\Gamma^\mathcal{E})$,

$$\beta_D(V) \geq \beta_D(\widetilde{V} \cap V(\Gamma)) \quad (5)$$

A UNIVERSAL TROPICAL JACOBIAN OVER $\overline{M}_{g,n}^{\text{top}}$
with equality if and only if $V = \overline{V \cap V(\Gamma)}$.

We say that a pseudo-divisor $(E, D)$ is simple if $E$ is non-disconnecting. We say that $(E, D)$ is $\mu$-semistable if $\beta_{E, D}(V) \geq 0$ for every subset $V \subset V(\Gamma)$. One can see that $(E, D)$ is semistable if and only if $D_E$ is $\mu_E$-semistable on $\Gamma$, or, equivalently, if and only if $D$ is $\mu^E$-semistable on $\Gamma^E$ (see [1 Proposition 4.6]).

Given a graph $\Gamma$, we define the poset $\mathcal{PD}(\Gamma)$ of pseudo-divisors on $\Gamma$ with partial order $(E, D) \geq (E', D')$ if $(E, D)$ specializes to $(E', D')$. We also define the category $\mathcal{PD}_g$ whose objects are triples $(\Gamma, E, D)$, where $\Gamma$ is a stable weighted graph of genus $g$ and $(E, D)$ is a pseudo-divisor on $\Gamma$, and the morphisms are given by specializations. We let $\mathcal{PD}_g$ be the poset $\mathcal{PD}_g/\sim$ where $(\Gamma, E, D) \sim (\Gamma', E', D')$ if they are isomorphic.

**Remark 3.2.** An easy adaptation of [1 Proposition 4.6] implies that semistability is preserved under graph specialization.

Given a graph $\Gamma$, a degree-$d$ polarization $\mu$ and a vertex $v_0 \in V(\Gamma)$, we define $SSD_\mu(\Gamma)$ and $QD_{v_0, \mu}(\Gamma)$ as the subposets of $\mathcal{PD}(\Gamma)$ of $\mu$-semistable and $(v_0, \mu)$-quasistable pseudo-divisors, respectively.

We define the rank map $\text{rk}: \mathcal{PD}(\Gamma) \to \mathbb{N}$ as
\[
\text{rk}(E, D) = |E| - b_0(\Gamma_E) + b_1(\Gamma).
\]
Note that $\text{rk}(E, D) \leq b_1(\Gamma)$. Also, if $(E, D) \geq (E', D')$ then $\text{rk}(E, D) \geq \text{rk}(E', D')$.

**Lemma 3.3.** Let $\Gamma$ be a graph. Let $(E_1, D_1), (E_2, D_2)$ be pseudo-divisors on $\Gamma$ such that $(E_1, D_1) \geq (E_2, D_2)$. For each subset $E \subset E(\Gamma)$ such that $E_2 \subset E \subset E_1$, there exists a unique pseudo-divisor $(E, D)$ such that
\[
(E_1, D_1) \geq (E, D) \geq (E_2, D_2).
\]
In particular if $(E_1, D_1)$ is $\mu$-semistable for some polarization $\mu$ on $\Gamma$, then $(E, D)$ is $\mu$-semistable.

**Proof.** The condition $(E_1, D_1) \geq (E_2, D_2)$ gives rise to a graph specialization $\iota: \Gamma^{E_1} \to \Gamma^{E_2}$. Note that $\iota$ factors through unique specializations $\iota_1: \Gamma^{E_1} \to \Gamma^E$ and $\Gamma^E \to \Gamma^{E_2}$. It is enough to take the divisor $D = \iota_1(D_1)$ on $\Gamma^E$. The last sentence follows from Remark 3.2. $\square$

**Lemma 3.4.** Let $\Gamma$ be a graph and $\mu$ be a degree-$d$ polarization on $\Gamma$. Let $\iota: (E_1, D_1) \to (E_2, D_2)$ be a specialization of degree-$d$ pseudo-divisors on $\Gamma$. For each $V \subset V(\Gamma)$, we have $\beta_{E_1, D_1}(V) \leq \beta_{E_2, D_2}(V)$.

**Proof.** We compute:
\[
\beta_{E_1, D_1}(V) = \deg(D_1|_V) - \mu_{E_1}(V) + \frac{\delta_{\Gamma^{E_1}, V}}{2}
\]
\[
= \deg(D_1|_V) - \left(\mu_{E_2}(V) + \frac{\text{val}_{E_1 \setminus E_2}(V)}{2}\right) + \frac{\delta_{\Gamma^{E_1}, V} - \text{val}_{E_1 \setminus E_2}(V)}{2}
\]
\[
= \beta_{E_2, D_2}(V) + \deg(D_1|_V) - \deg(D_2|_V) - \text{val}_{E_1 \setminus E_2}(V).
\]
However, $D_2(v) \geq D_1(v) - \text{val}_{E_1 \setminus E_2}(v)$ for every $v \in V$. The result follows. $\square$
Lemma 3.5. Let \( \iota : \Gamma_1 \to \Gamma_2 \) be a specialization of graphs and let \((\mathcal{E}_1, D_1)\) be a pseudo-divisor on \(\Gamma_1\). Define \((\mathcal{E}_2, D_2) = \iota_* (\mathcal{E}_1, D_1)\). For every \(V \subset V(\Gamma_2)\), we have
\[
\beta_{\mathcal{E}_2, D_2}(V) = \beta_{\mathcal{E}_1, D_1}(\iota^{-1}(V)).
\]

Proof. Note that \(\mathcal{E}_2 = \mathcal{E}_1 \cap E(\Gamma_2)\) and
\[
\deg(D_2|_V) = \deg(D_1|_{\iota^{-1}(V)}) - |(\mathcal{E}_1 \setminus E(\Gamma_2)) \cap E(\iota^{-1}(V))|,
\]
\[
\iota_* (\mu|_{\mathcal{E}_2}(V)) = \mu|_{\mathcal{E}_1}(\iota^{-1}(V)) - |(\mathcal{E}_1 \setminus E(\Gamma_2)) \cap E(\iota^{-1}(V))|,
\]
\[
\delta_{\Gamma_2, V} = \delta_{\Gamma_1, \iota^{-1}(V)}.
\]
(In the formula we consider \(E(\Gamma_2)\) as a subset of \(E(\Gamma_1)\).) The result follows. \(\square\)

3.2. Divisors on a tropical curve. Let \(X\) be a tropical curve. A divisor on \(X\) is a map \(D : X \to \mathbb{Z}\) such that \(D(p) \neq 0\) for finitely many points \(p \in X\). The degree of a divisor \(D\) on \(X\) is the integer \(\deg D := \sum_{p \in X} D(p)\). We say that \(D\) is effective if \(D(p) \geq 0\) for every \(p \in X\). The set \(\Div(X)\) of divisors on \(X\) is an Abelian group.

A rational function on \(X\) is a continuous, piece-wise linear function \(f : X \to \mathbb{R}\) with integer slopes. A principal divisor on \(X\) is a divisor of type
\[
\Div_X(f) := \sum_{p \in X} \ord_p(f) p \in \Div(X),
\]
where \(f\) is a rational function on \(X\) and \(\ord_p(f)\) is the sum of the incoming slopes of \(f\) at \(p\). A principal divisor has degree zero. The set \(\Prin(X)\) of principal divisors on \(X\) is a subgroup of \(\Div(X)\).

Given divisors \(D_1, D_2\) on \(X\), we say that \(D_1\) and \(D_2\) are equivalent if \(D_1 - D_2\) is a principal divisor. The Picard group \(\Pic(X)\) of \(X\) is defined as
\[
\Pic(X) := \Div(X) / \Prin(X).
\]
The degree-d Picard group of \(X\) is \(\Pic^d(X) := \Div^d(X) / \Prin(X)\).

The Jacobian of \(X\) is the real torus defined as:
\[
J(X) = \Omega(X) / H_1(X, \mathbb{Z})
\]
where \(\Omega(X)\) is the space of harmonic 1-forms on \(X\) (we refer to [6, Section 3] and [22, Section 6] for more details). Recall that there is a canonical isomorphism between \(\Pic^0(X)\) and \(J(X)\).

3.3. Unitary divisors on tropical curves. In this paper we will restrict our attention to a special type of divisors on a tropical curve, called unitary divisors. These divisors will be enough later to define a universal tropical Jacobian.

Definition 3.6. Let \(X\) be a tropical curve and \(\Gamma\) a model of \(X\). A divisor \(D\) on \(X\) is a \(\Gamma\)-unitary divisor (or simply unitary divisor if the model of \(X\) is clear) if for every \(e \in E(\Gamma)\) we have \(D(p) = 0\) for each point \(p \in e^0\), except for at most one point \(p_{D,e} \in e^0\) for which \(D(p_{D,e}) = -1\).

Throughout the paper, we will use the notation \(p_{D,e}\) to denote the point of \(e^0\) (if it exists) such that \(D(p_{D,e}) = -1\).

Let us define the combinatorial type of a unitary divisor \(D\) on \(X\). First, we let \(\mathcal{E} = \{ e \in E(\Gamma); \exists p \in e^0 \text{ such that } D(p) = -1 \}\).
Then, we define the divisor $D$ on $\Gamma^\varepsilon$ such that $D(v) = D(v)$ for every $v \in V(\Gamma)$ and $D(v_e) = -1$ for every exceptional vertex $v_e \in V(\Gamma^\varepsilon)$. In this way, we obtain a pseudo-divisor $(\mathcal{E}, D)$ on $\Gamma$, called the combinatorial type of $\mathcal{D}$.

Let $X$ be a tropical curve and $\mu$ be a degree $d$ polarization on $X$. From now on, we let $\Gamma$ be the $\mu$-model of $X$. For a divisor $\mathcal{D}$ on $X$ and a subcurve $Z \subset X$, we let

$$\beta_D(Z) = \deg(D|_Z) - \mu(Z) + \frac{\delta_{X, Z}}{2}.$$  

We say that $D$ is $\mu$-semistable if $\beta_D(Z) \geq 0$ for every $Z \subset X$. By [1, Proposition 5.3], if $D$ is a unitary $\mu$-semistable divisor, then its combinatorial type $(\mathcal{E}, D)$ is a $\mu$-semistable pseudo-divisor on $\Gamma$.

By the first and second paragraph of the proof of [1, Proposition 5.3] and Equation (5), for every $V \subset V(\Gamma^\varepsilon)$ we have:

(6) $\beta_D(X_V) = \beta_D(V) \geq \beta_{\mathcal{E}, D}(V \cap V(\Gamma))$, with equality iff $V = V(\Gamma^\varepsilon) 

\begin{align*}
\text{and } \beta_D(Z) \geq \beta_D(Z_\cap V(\Gamma^\varepsilon)), & \\
\text{for every subcurve } Z \subset X.  
\end{align*}$

Consider two degree-$d$ divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ on $X$ of combinatorial types $(\mathcal{E}_1, D_1)$ and $(\mathcal{E}_1, D_2)$. We will often have to decide whether or not they are equivalent. This motivates the following definitions.

**Definition 3.7.** Let $X$ be a tropical curve with a fixed model $\Gamma$. A difference divisor on $X$ is a divisor $\mathcal{D}$ which can be written as $\mathcal{D}_1 - \mathcal{D}_2$, for $\mathcal{D}_1$ and $\mathcal{D}_2$ unitary divisors on $X$ of the same degree. Equivalently, a difference divisor is a degree-$0$ divisor of type

$$\mathcal{D} = \mathcal{D}_0 + \sum_{e \in \mathcal{E}_1} p_e - \sum_{e \in \mathcal{E}_2} q_e,$$

where $\mathcal{D}_0$ is supported on $V(\Gamma)$, the sets $\mathcal{E}_1, \mathcal{E}_2$ are subsets of $\mathcal{E}(\Gamma)$, and $p_e, q_e \in e^\circ$.

An important case is that in which the divisors defining a difference divisor have the same combinatorial type.

**Definition 3.8.** An $\mathcal{E}$-divisor is the difference divisor of two unitary divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ with the same combinatorial type $(\mathcal{E}, D)$. Equivalently, a unitary divisor is a divisor of type

$$\mathcal{D} = \sum_{e \in \mathcal{E}} (p_e - q_e),$$

where $p_e, q_e \in e^\circ$ are, possibly equal, points in the interior of $e$.

We are interested in properties of principal $\mathcal{E}$-divisors. Before, we need a lemma.

**Lemma 3.9.** Let $X$ be a tropical curve with model $\Gamma$. Let $\mathcal{D}$ be a principal difference divisor on $X$, and write $\mathcal{D} = \text{Div}(f)$, for some rational function $f$ on $X$. Let $Z$ be the subcurve of $X$ where $f$ attains its minimum and set $V = V(\Gamma) \cap Z$. Then $V$ is nonempty and no connected component of $Z$ is contained in the interior of any edge of $\Gamma$. Moreover, $X_V \subset Z$ and $X_{V^c} \subset Z^c$. 
Proof. We argue that no connected component of $Z^c$ is contained in the interior of any edge of $\Gamma$. Assume, by contradiction, that $Z^c$ has such a connected component. Let $Y$ be the locus where $f$ attains its maximum on this component. Then

$$1 \geq \deg(D|_Y) = \deg(\text{Div}(f)|_Y) \geq 2,$$

where the inequality in the left-hand side follows because $D$ is a difference divisor. This is a contradiction. The same argument proves that $Z^c$ contains no edges $e$ whose vertices are in $Z$.

Similarly, no connected component of $Z$ is contained in the interior of any edge. Indeed, if $Y$ is a connected component contained in the interior of some edge, then

$$-1 \leq \deg(D|_Y) = \deg(\text{Div}(f)|_Y) \leq -2,$$

which is a contradiction.

Therefore, $V$ is nonempty. Moreover, if $e \in E(\Gamma)$ is an edge connecting vertices $v$ and $w$ in $V$, then $v, w \in Z$. If $e^0 \cap Z^c \neq \emptyset$, then $Z^c$ would have a component in the interior of $e$ or it would contain all of $w$, which is not possible by the first part of the proof. This proves that $X_V \subset Z$. Arguing similarly, we have that $X_{V^c} \subset Z^c$. □

Let $X$ be a tropical curve with a model $\Gamma$. Let $E$ be a subset of $E(\Gamma)$ and define $\iota: \Gamma \to \Gamma' := \Gamma/(E(\Gamma) \setminus E)$. For each $w \in V(\Gamma')$, define $V_w := \iota^{-1}(w)$.

**Lemma 3.10.** Let $X$ be a tropical curve with model $\Gamma$. Let $D = \sum_{e \in E}(p_e - q_e)$ be a principal $E$-divisor on $X$, and write $D = \text{Div}(f)$, for a rational function $f$ on $X$. For every $e \in E$ with $p_e \neq q_e$, consider the orientation on $e$ induced by $q_e p_e$. Then:

1. the slope of $f$ is 0 everywhere, except on the segments $\overline{q_e p_e}$, where it is 1.
2. for every edge $e \in E(\Gamma)$, we have $\ell(\overline{q_e p_e}) = f(t(e)) - f(s(e))$.
3. If $e_1$ and $e_2$ are edges connecting $V_{w_1}$ and $V_{w_2}$, with $w_1 \neq w_2$, then $\ell(\overline{q_{e_1} p_{e_1}}) = \ell(\overline{q_{e_2} p_{e_2}})$ and $\iota(t(e_1)) = \iota(t(e_2))$.
4. If $e$ is an edge connecting $V_w$ with itself, then $p_e = q_e$.
5. If $\Gamma_E$ is connected, then $D = 0$.

**Proof.** Let $Z$ be the subcurve of $X$ where $f$ attains its minimum. Then, $Z \cap Z^c$ consists of points $q_e \in e^0$ such that $D(q_e) = -1$, with $e$ running through a subset of $E$. Define $V = Z \cap V(\Gamma)$. By Lemma 3.9, we have that $V \neq \emptyset$ and $X_V \subset Z$. Moreover, $E(V, V^c) \subset E$. In particular, $V = \bigcup V_{w_i}$, and $p_e = q_e$ for every edge $e \in E$, such that $e$ connects each $V_{w_i}$, with itself.

Since the slope of $f$ is zero on $Z$ and $D(q_e) = -1$, for each $e \in E(V, V^c)$, the slope of $f$ on the segment $\overline{q_e p_e}$ must be 1. Since $D(p_e) = 1$, we have that $f$ has slope 0 on $e \setminus \overline{q_e p_e}$.

Consider the subcurve $X_{V^c}$. We know that $f$ has slope 0 on $e \setminus \overline{p_e q_e}$, for every $e \in E(V, V^c)$, hence $\text{Div}(f)|_{X_{V^c}} = \text{Div}(f|_{X_{V^c}})$. Therefore, $D|_{X_{V^c}} = \text{Div}(f|_{X_{V^c}})$, which is a principal $(E \cap E(V^c))$-unitary divisor on $X_{V^c}$. So item (1) follows by induction. Note that item (2) readily follows from item (1).

To prove item (3), let $p: X \to Y$ be the contraction of all subcurves $X_{V_{w_i}}$, for $w \in V(\Gamma')$ (note that the graph underlying a model of $Y$ is obtained contracting $\Gamma'$ at its loops). By items (1) and (2), the function $f$ is constant on each $X_{V_{w_i}}$, and so $f$ induces a rational function $\tilde{f}$ on $Y$. Moreover, for every edge $e$ connecting vertices $w_1$ and $w_2$ of $\Gamma'$,

$$\ell(\overline{q_e p_e}) = \tilde{f}(w_2) - \tilde{f}(w_1)$$

(assuming that $w_2 = t(e)$). This proves item (3).
Items (4) and (5) are consequences of the previous items. \( \square \)

4. Special polytopes and cones

We define certain polytopes and cones that will play an important role later in defining some polyhedral decomposition of the tropical Jacobian, and a universal tropical Jacobian.

Let \( X \) be the lengths of the edges of \( E \). Note that if \( (P, D) \) the linear quotient linear map \( K \) called the Abel map in combinatorial type \( (E, D) \). Consider the subspace \( L \) generated by \( u \) \( V \) when \( X \). We see that \( 3 \) is non-disconnecting, then \( 3 \) or, equivalently, \( 3 \) is as in Figure 1. Let \( p_0 \) be a point in \( X \). There exists a linear map \( K_{E,D}(X) \to \Omega(X)^\vee \), whose composition with the quotient \( \Omega(X)^\vee \to J(X) \) gives rise to a map

\[
K_{E,D}(X) \to J(X),
\]

called the Abel map. The Abel map \( K_{E,D}(X) \to J(X) \) takes each divisor \( D \in K_{E,D}(X) \) to the equivalence class of \( D - dp_0 \) and it is continuous (see \( 4.1 \) and the proof of \( 5.10 \) for more details).

Now, we define another interesting polytope associated to a pseudo-divisor \( (E, D) \). Consider the subspace \( L_E \subset \mathbb{R}^E \) generated by the vectors

\[
u_V = \sum_{e \in E} u_e - \sum_{e \in E} u_e \quad \text{when } V \text{ runs through all subsets of } V(\Gamma) \text{ such that } E(V, V^c) \subset E \text{ (recall that } u_e \text{ are the vectors of the canonical basis of } \mathbb{R}^E \text{). Equivalently, } L_E \text{ is given by the equations}
\]

\[
\sum_{e \in \gamma \subset E} \gamma(e)x_e = 0,
\]

where \( \gamma \) runs through all cycles of \( \Gamma \) that intersect \( E \) (recall Equation \( 1 \) and that \( x_e \) are the coordinates of \( \mathbb{R}^E \)). Note that if \( \Gamma_E \) is connected, then \( L_E = 0 \). We also note that \( \dim(L_E) = b_0(\Gamma_E) - b_0(\Gamma) \), hence \( \dim(P_{E,D}) = \rk(\mathcal{E}) \).

Given a pseudo-divisor \( (E, D) \) on the model \( \Gamma \) of the tropical curve \( X \), we consider the linear quotient linear map \( T_E: \mathbb{R}^E \to \mathbb{R}^E/L_E \). We define the polytope

\[
P_{E,D}(X) = T_E(K_{E,D}(X)).
\]

Note that if \( (E, D) \) is simple, i.e., if \( E \) is non-disconnecting, then

\[
P_{E,D}(X) = K_{E,D}(X).
\]

**Example 4.1.** Let \( X \) be a tropical curve with model \( \Gamma \) as in Figure 1. Let \( \ell_1, \ell_2, \ell_3 \) be the lengths of the edges of \( X \). Let \( (E, D) \) be a pseudo-divisor such that \( E = E(\Gamma) \). We have \( K_{E,D}(X) = [0, \ell_1] \times [0, \ell_2] \times [0, \ell_3] \subset \mathbb{R}^E = \mathbb{R}^3 \). The subspace \( L_E \subset \mathbb{R}^3 \) is generated by \( u_{e_1} + u_{e_2} + u_{e_3} \) or, equivalently, \( L_E \) is given by the equations \( x_{e_1} = x_{e_2} = x_{e_3} \). We see that \( P_{E,D}(X) \subset \mathbb{R}^E/L_E \) is a hexagon as in Figure 1. Two
unitary divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ with combinatorial type $(\mathcal{E}, \mathcal{D})$ are equivalent if and only if $\mathcal{D}_1 - \mathcal{D}_2 = \sum_{e \in E(\Gamma)} (p_e - q_e)$, with $p_e, q_e \in \mathbb{C}$ such that:

- all the $q_e$ lie in the same connected component after separating $X$ at all points $p_e$ in the edge $e$;
- there is $r \in \mathbb{R}$ such that $\ell(q_e p_e) = r$ for every $e \in E(\Gamma)$.

These conditions hold if and only if $u_{\mathcal{D}_1} - u_{\mathcal{D}_2} \in L_{\mathcal{E}}$, where $u_{\mathcal{D}_1}, u_{\mathcal{D}_2}$ are the point of $K_{\mathcal{E}, \mathcal{D}}$ corresponding to $\mathcal{D}_1, \mathcal{D}_2$.

Figure 1. The model a tropical curve

**Lemma 4.2.** Let $X$ be a tropical curve with oriented model $\Gamma$. Let $\mathcal{D} = \sum_{e \in \mathcal{E}} (p_e - q_e)$ be an $\mathcal{E}$-divisor on $X$, for some $\mathcal{E} \subset E(\Gamma)$, and $u_{\mathcal{D}}$ be the vector in $\mathbb{R}^{\mathcal{E}}$ given by

$$u_{\mathcal{D}} = (\varepsilon(e) \ell(q_e p_e))_{e \in \mathcal{E}}$$

where $\varepsilon(e)$ is 1 if $e$ and $q_e p_e$ have the same orientation, and $-1$ otherwise. Then $u_{\mathcal{D}} \in L_{\mathcal{E}}$ if and only if $\mathcal{D}$ is principal.

**Proof.** If $\mathcal{D}$ is principal, write $\mathcal{D} = \text{Div}(f)$, for some rational function $f$ on $X$. Let $\gamma$ be a cycle of $\Gamma$. Then, by Lemma 3.10 we have

$$\sum_{e \in \gamma \cap \mathcal{E}} \gamma(e) \ell(q_e p_e) = \sum_{e \in \gamma} \gamma(e) \ell(q_e p_e) = \pm \sum_{e \in \gamma} \gamma(e)(f(t(e)) - f(s(e))) = 0.$$

Conversely, if $u_{\mathcal{D}} \in L_{\mathcal{E}}$, we define a rational function $f$ on $X$ as follows. Let $g : X \to \mathbb{R}$ be the function such that $g(p) = 1$, if $p \in q_e p_e$, and $g(p) = 0$, otherwise. Choose a vertex $v \in V(\Gamma)$. For each point $p \in X$, let $\eta_p$ be a path from $v$ to $p$. Then $\int_{\eta_p} \eta_p(t)g(t)dt$ does not depend on the path $\eta_p$ by Equation (9). We set $f(p) = \int_{\eta_p} \eta_p(t)g(t)dt$, for every $p \in X$. Then $f$ is a rational function on $X$ and $\text{Div}(f) = \sum_{e \in \mathcal{E}} (p_e - q_e) = \mathcal{D}$, so $\mathcal{D}$ is principal. \qed

**Proposition 4.3.** The interior $P_{\mathcal{E}, \mathcal{D}}(X)$ parametrizes equivalence classes of unitary divisors $\mathcal{D}$ in $X$ with combinatorial type $(\mathcal{E}, \mathcal{D})$.

**Proof.** If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two equivalent divisors with combinatorial type $(\mathcal{E}, \mathcal{D})$, we have that $\mathcal{D}_1 - \mathcal{D}_2$ is a principal $\mathcal{E}$-divisor. By Lemma 3.2 we get $u_{\mathcal{D}_1 - \mathcal{D}_2} \in L_{\mathcal{E}}$. Let $u_{\mathcal{D}_1}, u_{\mathcal{D}_2} \in K_{\mathcal{E}, \mathcal{D}}(X)$ be the points associated to $\mathcal{D}_1$ and $\mathcal{D}_2$. Note that $u_{\mathcal{D}_1} - u_{\mathcal{D}_2} = u_{\mathcal{D}_1 - \mathcal{D}_2}$, then $u_{\mathcal{D}_1} - u_{\mathcal{D}_2} \in L_{\mathcal{E}}$. We deduce that $\mathcal{D}_1$ and $\mathcal{D}_2$ correspond to the same point in $P_{\mathcal{E}, \mathcal{D}}(X)$.

Conversely, if $\mathcal{D}_1$ and $\mathcal{D}_2$ correspond to the same point in $P_{\mathcal{E}, \mathcal{D}}(X)$, then $u_{\mathcal{D}_1 - \mathcal{D}_2} = u_{\mathcal{D}_1} - u_{\mathcal{D}_2} \in L_{\mathcal{E}}$. Hence, by Lemma 4.2 we have that $\mathcal{D}_1$ and $\mathcal{D}_2$ are equivalent. \qed

Let $X$ be a tropical curve with model $\Gamma$. If $(\mathcal{E}, \mathcal{D}) \geq (\mathcal{E}', \mathcal{D}')$ are two pseudo-divisors on $\Gamma$, there is a natural face-inclusion of polytopes $K_{\mathcal{E}', \mathcal{D}'}(X) \subset K_{\mathcal{E}, \mathcal{D}}(X)$ induced by the inclusion $\mathbb{R}^{\mathcal{E}'} \subset \mathbb{R}^{\mathcal{E}}$. By Equation (9), $L_{\mathcal{E}'} = L_{\mathcal{E}} \cap \mathbb{R}^{\mathcal{E}'}$. Hence, there is a natural inclusion

\begin{equation}
P_{\mathcal{E}', \mathcal{D}'}(X) \subset P_{\mathcal{E}, \mathcal{D}}(X).
\end{equation}

Note that this inclusion is not necessarily a face-inclusion.
By Proposition 4.3, the Abel map $K_{E,D}(X) \to J(X)$, defined in Equation (8), factors through a continuous map

$$P_{E,D}(X) \to J(X).$$

So far, we constructed polytopes that parametrize equivalence classes of unitary divisors on a single tropical curve. We can construct a more universal parameter space in the following way.

Let $\Gamma$ be a graph and $E$ a subset of $E(\Gamma)$. For each subset $V \subset V(\Gamma)$ such that $E(V, V^c) \subset E$, define $\alpha : E(\Gamma) \rightarrow \{0, \pm 1\}$ as $\alpha_V(e) = 1$ (respectively, $-1$), if $e$ is incident to $V$ (respectively, $V^c$) and lies over an edge in $E(V, V^c)$. Otherwise, we define $\alpha_V(e) = 0$. We define the vector in $\mathbb{R}^{E(\Gamma)}$:

$$w_V = \sum_{e \in E(\Gamma)} \alpha_V(e)u_e.$$

(Recall that $u_e$ are the vectors of the canonical basis of $\mathbb{R}^E$.) We let $L_E \subset \mathbb{R}^{E(\Gamma)}$ be the subspace generated by all vectors $w_V$, as $V$ runs through all subsets of $V(\Gamma)$ such that $E(V, V^c) \subset E$. Also, we denote by

$$T_E : \mathbb{R}^{E(\Gamma)} \to \mathbb{R}^{E(\Gamma)} / L_E$$

the linear quotient map. We have two natural maps

$$f_E : \mathbb{R}^{E(\Gamma)} \to \mathbb{R}^{E(\Gamma)} \quad \text{and} \quad g_E : \mathbb{R}^{E(\Gamma)} \to \mathbb{R}^E$$

defined as $f_E(u_e) = u_e$ where $e$ is the (unique) edge under $e'$, and $g_E(u_e) = u_e$ if $e' = e^*$, otherwise $g_E(u_e) = 0$. In other words, the map $f_E$ takes a vector $(x_{e'})_{e' \in E(\Gamma)}$ to $(y_e)_{e \in E(\Gamma)}$, while $g_E$ takes $(x_{e'})$ to $(z_e)_{e \in E}$, where

$$y_e = \sum_{e' \text{ over } e} x_{e'} \quad \text{and} \quad z_e = x_{e^*}.$$

One can see that we have an isomorphism:

$$(f_E, g_E) : \mathbb{R}^{E(\Gamma)} \to \mathbb{R}^{E(\Gamma)} \times \mathbb{R}^E.$$

We denote by $\tau(\Gamma, E, D)$ the image cone of $\mathbb{R}^{E(\Gamma)}_{\geq 0}$ under $(f_E, g_E)$:

$$\tau(\Gamma, E, D) := (f_E, g_E)(\mathbb{R}^{E(\Gamma)}_{\geq 0}) \subset \mathbb{R}^{E(\Gamma)} \times \mathbb{R}^E.$$

**Remark 4.4.** We note that the cone $\tau(\Gamma, E, D)$ can be described as

$$\tau(\Gamma, E, D) = \{((x_e)_{e \in E(\Gamma)}, (z_e)_{e \in E}) ; x_e \geq z_e \geq 0 \text{ for } e \in E \text{ and } x_e \geq 0 \text{ for } e \in E(\Gamma)\}.$$  

(Recall that $x_e$ are the coordinates of $\mathbb{R}^E$.) Hence the image of the second projection $pr : \tau(\Gamma, E, D) \to \mathbb{R}^{E(\Gamma)}$ is contained in $\mathbb{R}^{E(\Gamma)}_{\geq 0}$, and for every $(x_e)_{e \in E(\Gamma)} \in \mathbb{R}^{E(\Gamma)}_{\geq 0}$,

$$pr^{-1}((x_e)_{e \in E(\Gamma)}) = \{((x_e)_{e \in E(\Gamma)}, (z_e)_{e \in E}) ; x_e \geq z_e \geq 0\}.$$  

So, if $X$ is a tropical curve identified with the point $(\ell(e))_{e \in E(\Gamma)} \in \mathbb{R}^{E(\Gamma)}_{> 0}$, where $\Gamma$ and $\ell$ are a model and the length function of $X$ (recall that $\mathbb{R}^{E(\Gamma)}_{> 0}$ parametrizes tropical curves $X$ with model $\Gamma$), we may rephrase Equation (15) as:

$$pr^{-1}(X) = \{X\} \times K_{E,D}(X).$$
Lemma 4.5. Let $\Gamma$ be a graph and $\mathcal{E}$ a subset of $E(\Gamma)$. We have that $(f_{\mathcal{E}}, g_{\mathcal{E}})(\mathcal{L}_{\mathcal{E}}) = \{0\} \times L_{\mathcal{E}}$. In particular, the induced map:

$$
(f_{\mathcal{E}}, g_{\mathcal{E}}): \mathbb{R}_{\geq 0}^{E(\mathcal{E})}/\mathcal{L}_{\mathcal{E}} \rightarrow \mathbb{R}_{\geq 0}^{E(\Gamma)} \times \mathbb{R}^E/L_{\mathcal{E}}
$$

is an isomorphism.

Proof. For every subset $V \subset V(\Gamma)$ such that $E(V, V^c) \subset \mathcal{E}$, we have $(f_{\mathcal{E}}, g_{\mathcal{E}})(w_V) = (0, u_V)$, by definition of $w_V$ and $u_V$. The result follows.

For a graph $\Gamma$ and a pseudo-divisor $(\mathcal{E}, D)$ on $\Gamma$, we define the cone:

$$
\sigma_{(\Gamma, \mathcal{E}, D)} = \mathcal{T}_{\mathcal{E}}(\mathbb{R}_{\geq 0}^{E(\mathcal{E}^\mathcal{F})}) \subset \mathbb{R}_{\geq 0}^{E(\mathcal{E})}/\mathcal{L}_{\mathcal{E}}.
$$

By Lemma 4.5 we have a natural (projection) map $\pi_{\mathcal{E}}: \sigma_{(\Gamma, \mathcal{E}, D)} \rightarrow \mathbb{R}_{\geq 0}^{E(\Gamma)}$. The following diagram is commutative:

\begin{equation}
\begin{array}{ccc}
\mathbb{R}_{\geq 0}^{E(\mathcal{E}^\mathcal{F})} & \xrightarrow{\mathcal{T}_{\mathcal{E}}} & \sigma_{(\Gamma, \mathcal{E}, D)} \\
(f_{\mathcal{E}}, g_{\mathcal{E}}) \downarrow & & \downarrow \pi_{\mathcal{E}} \\
\tau_{(\Gamma, \mathcal{E}, D)} & \xrightarrow{(\text{Id}, \mathcal{T}_{\mathcal{E}})} & (\text{Id}, \mathcal{T}_{\mathcal{E}})(\tau_{(\Gamma, \mathcal{E}, D)}) \\
\downarrow \mathcal{T}_{\mathcal{E}} & & \downarrow \mathcal{T}_{\mathcal{E}} \\
\mathbb{R}_{\geq 0}^{E(\Gamma)} & \xrightarrow{\mathcal{P}_{\mathcal{E}}} & \mathbb{R}_{\geq 0}^{E(\Gamma)}
\end{array}
\end{equation}

where $\mathcal{P}_{\mathcal{E}}: \mathbb{R}_{\geq 0}^{E(\Gamma)} \times \mathbb{R}^E/L_{\mathcal{E}} \rightarrow \mathbb{R}_{\geq 0}^{E(\Gamma)}$ is the projection on the second factor. Since $\mathbb{R}_{\geq 0}^{E(\mathcal{E}^\mathcal{F})}$ is isomorphic to $\tau_{(\Gamma, \mathcal{E}, D)}$ (via $(f_{\mathcal{E}}, g_{\mathcal{E}})$), it follows that $\sigma_{(\Gamma, \mathcal{E}, D)}$ is isomorphic to $(\text{Id}, \mathcal{T}_{\mathcal{E}})(\tau_{(\Gamma, \mathcal{E}, D)})$.

Proposition 4.6. For every tropical curve $X \in \mathbb{R}_{\geq 0}^{E(\Gamma)}$, there is an isomorphism of polytopes $\pi_{\mathcal{E}}^{-1}(X) \cong \mathcal{P}_{E,D}(X)$. In particular, the open cone $\sigma_{(\Gamma, \mathcal{E}, D)}$ parametrizes equivalence classes of pairs $(X, D)$, where $X$ is a tropical curve with model $\Gamma$ and $D$ is a unitary divisor with combinatorial type $(\mathcal{E}, D)$, and two pairs $(X_1, D_1)$ and $(X_2, D_2)$ are equivalent if $X_1 = X_2$ and $D_1$ and $D_2$ are linearly equivalent.

Proof. Consider Diagram (17). Since the map $(f_{\mathcal{E}}, g_{\mathcal{E}})$ is an isomorphism, we have that $\pi_{\mathcal{E}}^{-1}(X)$ and $\mathcal{P}_{\mathcal{E}}^{-1}(X)$ are isomorphic as polytopes. By Equation (16) we have $\mathcal{P}_{\mathcal{E}}^{-1}(X) = \{X\} \times \mathcal{K}_{\mathcal{E}, D}(X)$, and therefore:

$$
\mathcal{P}_{\mathcal{E}}^{-1}(X) = (\text{Id}, \mathcal{T}_{\mathcal{E}})(\{X\} \times \mathcal{K}_{\mathcal{E}, D}(X)) = \{X\} \times \mathcal{P}_{E,D}(X).
$$

Hence, $\pi_{\mathcal{E}}^{-1}(X)$ is isomorphic to $\mathcal{P}_{E,D}(X)$, as required.

The last statement follows from the first statement and from Proposition 4.5. □

Note that given a specialization of triples $(\Gamma_1, \mathcal{E}_1, D_1) \geq (\Gamma_2, \mathcal{E}_2, D_2)$, then $\mathcal{L}_{\mathcal{E}_2} = \mathcal{L}_{\mathcal{E}_1} \cap \mathbb{R}_{\geq 0}^{E(\mathcal{E}_2^\mathcal{F})}$, and hence we have a natural inclusion

$$
\sigma_{(\Gamma_2, \mathcal{E}_2, D_2)} \subset \sigma_{(\Gamma_1, \mathcal{E}_1, D_1)}.
$$

Proposition 4.7. Let $\iota: \Gamma_1 \rightarrow \Gamma_2$ be a specialization of graphs. Let $(\mathcal{E}_1, D_1)$ be a pseudo-divisor on $\Gamma_1$, and set $(\mathcal{E}_2, D_2) = \iota_*(\mathcal{E}_1, D_1)$. Then $\sigma_{(\Gamma_2, \mathcal{E}_2, D_2)}$ is a face of $\sigma_{(\Gamma_1, \mathcal{E}_1, D_1)}$. 


Proof. We know that $\mathbb{R}^{E(\Gamma, 2)}_{\geq 0}$ is naturally a face of $\mathbb{R}^{E(\Gamma, 1)}_{\geq 0}$ and there is an inclusion $\sigma(\Gamma_2, \xi_2, D_2) \subset \sigma(\Gamma_1, \xi_1, D_1)$ by Equation (18). Let $E \subset E(\Gamma_1)$ be the set of edges contracted by $\iota$. Let $H$ be the hyperplane in $\mathbb{R}^{E(\Gamma, 1)}_{\geq 0}$ defined as $\sum_{e} x_e = 0$, where the sum runs through all edges in $E(\Gamma, 1)$ that lie over an edge in $E$. Note that $E_{\xi_1} \subset H$ because, if $e_1$ and $e_2$ are the two edges lying over some edge $e' \in E \cap E_1$, then the vectors $u_{e_1}$ and $u_{e_2}$ appears with opposite signs in $wv$, for every $V$ (see Equation (13)). Since $H \cap \mathbb{R}^{E(\Gamma, 1)}_{\geq 0} = \mathbb{R}^{E(\Gamma, 2)}_{\geq 0}$, and $\mathbb{R}^{E(\Gamma, 2)}_{\geq 0}$ is contained in a single half-space defined by $H$, it follows that

$$(H/L_{E_1}) \cap \sigma(\Gamma_1, \xi_1, D_1) = \sigma(\Gamma_2, \xi_2, D_2)$$

and $\sigma(\Gamma_2, \xi_2, D_2)$ is contained in a single semi-space defined by $H/L_{E_1}$. Therefore, $\sigma(\Gamma_2, \xi_2, D_2)$ is a face of $\sigma(\Gamma_1, \xi_1, D_1)$, as required. □

4.1. Universal tropical Jacobians: a first attempt. As mentioned in the introduction, the category of generalized cone complexes is the right category to construct tropical moduli space. We will construct two spaces over $M^g$ which come close to being universal tropical Jacobians. Throughout, we denote by $M^g$ the moduli space of tropical curves. For more details on $M^g$, we refer to [21, Section 2], [3] Sections 2.1 and 3.2, [10] Section 3], [11] Section 3], [9] Section 4].

First of all, we define the generalized cone complex

$$J^{\text{ps}, \text{trop}}_{\mu, g} = \lim_{\rightarrow} \sigma(\Gamma, \xi, D)$$

where $\Gamma$ is a stable weighted graph of genus $g$ and $(\xi, D)$ is a simple $\mu$-semistable pseudo-divisor on $\Gamma$. Since $(\xi, D)$ is simple, it follows that $\sigma(\Gamma, \xi, D) = \mathbb{R}^{E(\xi)}_{\geq 0}$ and each specialization of $(\Gamma, \xi, D)$ is also simple. Therefore $J^{\text{ps}, \text{trop}}_{\mu, g}$ is indeed a generalized cone complex. The space $J^{\text{ps}, \text{trop}}_{\mu, g}$ parametrizes equivalence classes of pairs $(X, D)$, where $X$ is a stable tropical curve of genus $g$ and $D$ is a simple $\mu$-semistable divisor on $X$. The problem with $J^{\text{ps}, \text{trop}}_{\mu, g}$ is that it is possible to have two simple $\mu$-semistable divisors $D_1$ and $D_2$, with different combinatorial type, that are linearly equivalent. In a way, the cone complex $J^{\text{ps}, \text{trop}}_{\mu, g}$ has too many points. In Section 6.3 we will see that the algebro-geometric counterpart of $J^{\text{ps}, \text{trop}}_{\mu, g}$ is a stack over $\overline{M}_g$, which is not separated.

The second space we can construct is the topological space

$$J^{\text{trop}}_{\mu, g} = \lim_{\rightarrow} \sigma(\Gamma, \xi, D)$$

where $\Gamma$ is a stable weighted graph of genus $g$ and $(\xi, D)$ is a $\mu$-semistable pseudo-divisor on $\Gamma$. By Equation (18) we have that $\sigma(\Gamma, \xi, D) \subset \sigma(\Gamma, \xi, D)$ when $(\Gamma, \xi, D) \geq (\Gamma', \xi', D')$, hence the limit above is well defined. However, as we will see in Proposition 6.3, the topological space $J^{\text{trop}}_{\mu, g}$ is not a generalized cone complex, in the sense that some inclusions $\sigma(\Gamma, \xi, D) \subset \sigma(\Gamma, \xi, D)$ are not face morphisms.

This motivates the search for a better-behaved limit, which is given by the notion of $\mu$-polystable divisors, that we will introduce in the next section. It is worth to observe that, as a topological spaces, the universal tropical Jacobian we will construct in Section 6 is homeomorphic to $J^{\text{trop}}_{\mu, g}$. 

5. Polystable divisors

We now introduce the key definition of polystable divisor on graphs and tropical curves. We will prove that in the equivalence class of a divisor on a tropical curve there is a polystable representative. It turns out that two equivalent polystable divisors have the same combinatorial type and their difference is an $\mathcal{E}$-divisor. This property will be crucial to construct a universal tropical Jacobian over $M_g^{\text{trop}}$.

5.1. Polystability on graphs. We begin with the definition of polystability for a pseudo-divisor on a graph.

**Definition 5.1.** We say that a pseudo-divisor $(\mathcal{E}, D)$ on a graph $\Gamma$ is $\mu$-polystable if $\beta_{\mathcal{E}, D}(V) \geq 0$ for every subset $V \subset V(\Gamma)$, with strict inequality if $E(V, V^c) \not\subset \mathcal{E}$ (that is, if $E$ is not the set of vertices of a union of connected components of $\Gamma_\mathcal{E}$). Equivalently, $(\mathcal{E}, D)$ is $\mu$-polystable if $D_\mathcal{E}$ is $\mu_\mathcal{E}$-stable on $\Gamma_\mathcal{E}$.

Note that every $\mu$-polystable pseudo-divisor is $\mu$-semistable. Moreover, every $\mu$-stable divisor is a $\mu$-polystable pseudo-divisor (with $\mathcal{E} = \emptyset$). The following result tells us that polystability is well behaved under contractions of graphs.

**Lemma 5.2.** If $\iota : \Gamma_1 \to \Gamma_2$ is a specialization of graphs and $(\mathcal{E}_1, D_1)$ is a $\mu$-polystable pseudo-divisor, then $\iota_*(\mathcal{E}_1, D_1)$ is $\iota_*(\mu)$-polystable.

**Proof.** The proof follows directly by the definition and Lemma 3.3. \hfill $\Box$

Given a graph $\Gamma$ and a degree-$d$ polarization $\mu$ on $\Gamma$, we define $\mathcal{PSD}_\mu(\Gamma)$ as the subposets of $\mathcal{PD}(\Gamma)$ consisting of $\mu$-polystable pseudo-divisors on $\Gamma$. We have natural inclusions $(v_0 \in V(\Gamma))$ is any vertex of $\Gamma)$:

$$Q_{D_{\mu,g}}(\Gamma) \subset SSD_\mu(\Gamma) \quad \text{and} \quad \mathcal{P}_{\mu}(\Gamma) \subset SSD_\mu(\Gamma).$$

For a nondegenerate polarization $\mu$, all the above inclusions are equalities.

If $\mu$ is a genus-$g$ universal polarization, we also define $\mathcal{P}_{\mu,g}$ as the category whose objects are triples $(\Gamma, \mathcal{E}, D)$ where $\Gamma$ is a genus-$g$ stable weighted graph and $(\mathcal{E}, D)$ is a $\mu$-polystable pseudo-divisor on $\Gamma$, and morphisms are given by specializations. As usual, we let $\mathcal{P}_{\mu,g}$ be the poset $\mathcal{P}_{\mu,g}/\sim$, where the relation $\sim$ is isomorphism.

**Proposition 5.3.** Let $\Gamma$ be a graph and $\mu$ a degree-$d$ polarization on $\Gamma$. Let $(\mathcal{E}, D)$ be a $\mu$-semistable pseudo-divisor on $\Gamma$. Assume that $V \subset V(\Gamma)$ is a subset with $\beta_{\mathcal{E}, D}(V) = 0$ and $E(V, V^c) \not\subset \mathcal{E}$. Then, there exists a unique $\mu$-semistable pseudo-divisor $(\mathcal{E} \cup E(V, V^c), D_1)$ on $\Gamma$ such that $(\mathcal{E} \cup E(V, V^c), D_1) > (\mathcal{E}, D)$.

**Proof.** It is sufficient to consider the case where $\mathcal{E} = \emptyset$. Indeed if $\mathcal{E} \neq \emptyset$, we can just replace $\Gamma$, $\mu$ and $D$ with $\Gamma_\mathcal{E}$, $\mu_\mathcal{E}$ and $D_\mathcal{E}$ (see also Equation 3).

From now on, assume that $\mathcal{E} = \emptyset$. Let $D_1$ be the divisor on $\Gamma_{E(V, V^c)}$ defined as

$$D_1(v) = \begin{cases} D(v) + \text{val}_{E(V, V^c)}(v) & \text{if } v \in V, \\ D(v) & \text{if } v \notin V, \\ -1 & \text{if } v \text{ is exceptional.} \end{cases}$$

(19)

It is clear that $(E(V, V^c), D_1) > (\emptyset, D)$.

We now argue that $(E(V, V^c), D_1)$ is $\mu$-semistable. Let $W$ be a subset of $V(\Gamma_{E(V, V^c)}) = V(\Gamma)$. Define $W_1 = W \cap V$ and $W_2 = W \cap V^c$. Note that there are no edges in
Therefore, by the definition of polystability, we get
\[ \beta_{E(V, V'), D_1}(W) = \beta_{E(V, V'), D_1}(W_1) + \beta_{E(V, V'), D_1}(W_2). \]

We have \( \beta_{E(V, V'), D_1}(W_1) = \beta_D(W_1) \geq 0 \) and
\[
\beta_{E(V, V'), D_1}(W_2) = \beta_D(W_2) - |E(V, W_2)| = \beta_D(V \cup W_2) - \beta_D(V) = \beta_D(V \cup W_2) \geq 0.
\]

By Equation (20), this readily implies that \( (E(V, V'), D_1) \) is \( \mu \)-semistable.

To prove uniqueness, just note that if \( (E(V, V'), D') \) is another pseudo-divisor with \( (E(V, V'), D') > (0, D) \), then \( \deg(D'|_V) < \deg(D_1|_V) \), and hence
\[ \beta_{E(V, V'), D'}(V) < \beta_{E(V, V'), D_1}(V) = 0, \]
which means that \( (E(V, V'), D') \) is not \( \mu \)-semistable.

Let \( \Gamma \) be a graph. By definition, if \( (\mathcal{E}, D) \) is a \( \mu \)-semistable pseudo-divisor on \( \Gamma \) such that \( \beta_{E, D}(V) > 0 \) for every subset \( V \subseteq V(\Gamma) \) such that \( E(V, V') \not\subseteq \mathcal{E} \), then \( (\mathcal{E}, D) \) is \( \mu \)-polystable. Therefore, by Proposition 5.3 starting from a \( \mu \)-semistable pseudo-divisor \( (\mathcal{E}, D) \), one can construct a sequence of specializations
\[ (\mathcal{E}_k, D_k) > (\mathcal{E}_{k-1}, D_{k-1}) > \ldots > (\mathcal{E}_0, D_0) = (\mathcal{E}, D) \]
of \( \mu \)-semistable divisors, where \( (\mathcal{E}_k, D_k) \) is \( \mu \)-polystable and \( (\mathcal{E}_j, D_j) \) is not \( \mu \)-polystable for \( j < k \). In the next result, we show that the polystable divisor \( (\mathcal{E}_k, D_k) \) is uniquely determined.

**Proposition 5.4.** Let \( \Gamma \) be a graph and \( \mu \) a degree- \( d \) polarization on \( \Gamma \). Let \( (\mathcal{E}, D) \) be a \( \mu \)-semistable pseudo-divisor. Then there exists a unique minimal \( \mu \)-polystable pseudo-divisor \( (\mathcal{E}', D') \) on \( \Gamma \) such that \( (\mathcal{E}', D') \geq (\mathcal{E}, D) \).

**Proof.** If \( (\mathcal{E}, D) \) is \( \mu \)-polystable, just take \( (\mathcal{E}', D') = (\mathcal{E}, D) \). Otherwise, there is a subset \( V \subseteq V(\Gamma) \) such that \( \beta_{E, D}(V) = 0 \) and \( E(V, V') \not\subseteq \mathcal{E} \). Applying Proposition 5.3 to \( (\mathcal{E}, D) \) and \( V \) we obtain a unique \( \mu \)-semistable pseudo-divisor \( (\mathcal{E} \cup E(V, V'), D_1) \) such that \( (\mathcal{E} \cup E(V, V'), D_1) > (\mathcal{E}, D) \).

Assume that \( (\mathcal{E}, D) \) is a \( \mu \)-polystable divisor such that \( (\mathcal{E}, D) > (\mathcal{E}, D) \). Let us prove that \( (\mathcal{E}, D) \geq (\mathcal{E} \cup E(V, V'), D_1) \). First, we prove that \( E(V, V') \subseteq \mathcal{E} \). Indeed, by Lemma 3.3 we have \( \beta_{E, D}(V) \leq \beta_{E, D}(V) = 0 \), hence \( \beta_{E, D}(V) = 0 \).

Therefore, by the definition of polystability, we get \( E(V, V') \subseteq \mathcal{E} \). By Lemma 5.3 there is a unique \( \mu \)-semistable pseudo-divisor \( (\mathcal{E} \cup E(V, V'), D) \) such that \( (\mathcal{E}, D) \geq (\mathcal{E} \cup E(V, V'), D) > (\mathcal{E}, D) \). By Proposition 5.3 we have that \( D = D_1 \). This proves that \( (\mathcal{E}, D) \geq (\mathcal{E}, D) \).

Following the same argument, if
\[ (\mathcal{E}_k, D_k) > (\mathcal{E}_{k-1}, D_{k-1}) > \ldots > (\mathcal{E}_0, D_0) = (\mathcal{E}, D) \]
is a sequence as in Equation (21), then \( (\mathcal{E}, D) \geq (\mathcal{E}_k, D_k) \). This implies that \( (\mathcal{E}_k, D_k) \) is a minimal \( \mu \)-polystable pseudo-divisor on \( \Gamma \) such that \( (\mathcal{E}_k, D_k) > (\mathcal{E}, D) \), and that it is the unique one satisfying these properties.

As a consequence of Proposition 5.4 we obtain an order preserving map:
\[ \text{pol} : SSD_\mu(\Gamma) \rightarrow PSD_\mu(\Gamma) \]
taking $(\mathcal{E}, D)$ to $\text{pol}(\mathcal{E}, D) := (\mathcal{E}', D')$, where $(\mathcal{E}', D')$ is the minimal $\mu$-polystable pseudo-divisor such that $(\mathcal{E}', D') \geq (\mathcal{E}, D)$. Note that the map $\text{pol}$ is a section of the natural inclusion map $\mathcal{PSD}_\mu(\Gamma) \to \mathcal{SSD}_\mu(\Gamma)$. Moreover, we have

$$\text{rk}(\text{pol}(\mathcal{E}, D)) \geq \text{rk}(\mathcal{E}, D),$$

because $\text{pol}(\mathcal{E}, D) \geq (\mathcal{E}, D)$.

**Lemma 5.5.** Let $(\mathcal{E}_1, D_1)$ and $(\mathcal{E}_2, D_2)$ be two $\mu$-polystable divisors on a graph $\Gamma$ such that $(\mathcal{E}_1, D_1) \geq (\mathcal{E}_2, D_2)$. Then $\text{rk}(\mathcal{E}_1, D_1) > \text{rk}(\mathcal{E}_2, D_2)$.

**Proof.** By Equation (3), we can assume that $\mathcal{E}_2 = \emptyset$, and in this case $D_2$ is stable. Also, it is sufficient to prove the statement for a connected component of $\Gamma$.

Assume that $\text{rk}(\mathcal{E}_1, D_1) = \text{rk}(\emptyset, D_2) = 0$. Consider the specialization \(\iota: \Gamma \to \Gamma' = \Gamma/(E(\Gamma) \setminus E)\).

Since $\text{rk}(\mathcal{E}_1, D_1) = 0$, the graph $\Gamma'$ is a tree. By Lemma 5.2 the pseudo-divisors $\iota_*(\mathcal{E}_1, D_1)$ and $(\iota_*(\emptyset, D_2)$ are polystable. So we can assume that $\Gamma = \Gamma'$ is a tree and $\mathcal{E} = E(\Gamma)$.

The specialization $(\mathcal{E}_1, D_1) \to (\emptyset, D_2)$ is induced by a specialization $\iota: \Gamma E_1 \to \Gamma$. We define an orientation on $\Gamma$ such that $\iota(e) = (e)$ for every $e \in E(\Gamma)$ (recall that $v_e$ is the exceptional vertex of $\Gamma E_1$ contained in $e$). Since $\Gamma$ is a tree, this orientation is acyclic, so it has a sink $v_0$. We get that $D_2(v_0) = D_1(v_0) - \text{val}_E(\Gamma)(v_0)$, then

$$\beta_{D_2}(v_0) = D_2(v_0) - \mu(v_0) + \frac{\text{val}_E(\Gamma)(v_0)}{2}$$

$$= D_1(v_0) - \mu(v_0) - \frac{\text{val}_E(\Gamma)(v_0)}{2}$$

$$= D_1(v_0) - \mu_E(\Gamma)(v_0)$$

$$= \beta_{E_1, D_1}(v_0) = 0,$$

which is a contradiction.  \(\square\)

**Proposition 5.6.** Let $\Gamma$ be a graph, $\mu$ polarization on $\Gamma$ and $v_0$ a vertex of $\Gamma$. If $(\mathcal{E}, D)$ is a $\mu$-polystable pseudo-divisor, then there is a $(v_0, \mu)$-quasistable pseudo-divisor $(\mathcal{E}', D')$ such that $\text{pol}(\mathcal{E}', D') = (\mathcal{E}, D)$ and $\text{rk}(\mathcal{E}', D') = \text{rk}(\mathcal{E}, D)$.

**Proof.** Consider the specialization $\iota: \Gamma \to \Gamma' = \Gamma/(E(\Gamma) \setminus E)$.

Let $T \subset E$ be a spanning tree of $\Gamma'$. We know, by Equation (3), that $(\mathcal{E}, D)$ is $\mu$-polystable if and only if $(T, D_{\mathcal{E} \setminus T})$ is a $\mu_{\mathcal{E} \setminus T}$-polystable pseudo-divisor on $\Gamma_{\mathcal{E} \setminus T}$. Hence, we can assume that $\mathcal{E} = T$. In this case, $\Gamma'$ is a tree. We will view $\iota(v_0)$ as the root of $\Gamma'$. This gives rise to an orientation $s, t: E \to V(\Gamma)$ (pointing away from the root). Moreover, $\text{rk}(\mathcal{E}, D) = 0$.

We define the pseudo-divisor $(\emptyset, D')$ on $\Gamma$ as

$$D'(v) = \begin{cases} D(v) - 1 & \text{if } v = t(e) \text{ for some } e \in \mathcal{E} \\ D(v) & \text{otherwise.} \end{cases}$$

We have that $\text{rk}(\emptyset, D') = 0 = \text{rk}(\mathcal{E}, D)$ and $(\mathcal{E}, D) \geq (\emptyset, D')$. By Lemma 5.5 all that is left is to prove is that $D'$ is $(v_0, \mu)$-quasistable. Since $D'$ is the specialization of a $\mu$-polystable pseudo-divisor, it is $\mu$-semistable. Hence, there remains to prove that $\beta_{D'}(V) > 0$ for every proper subset $V \subsetneq V(\Gamma)$ such that $v_0 \in V$. 


Let $V \subset V(\Gamma)$. For each vertex $w \in V(\Gamma')$, let $V_w \subset V$ be the subset of vertices $v \in V$ such that $\iota(v) = w$. Moreover, let $w_0 := \iota(v_0)$ and, for each $w \in V(\Gamma')$ with $w \neq w_0$, let $e_w$ be the unique edge in $\mathcal{E}$ such that $\iota(t(e)) = w$. By Equation (2):

$$\beta_{\mathcal{D}'}(V) = \sum_{w \in V(\Gamma')} \beta_{\mathcal{D}'}(V_w) - |\mathcal{E}_V|,$$

where $\mathcal{E}_V$ is the set:

$$\mathcal{E}_V = \{ e \in \mathcal{E}; e \in E(V_{\iota(s(e))}, V_{\iota(t(e))}) \}.$$

However,

$$\beta_{\mathcal{D}'}(V_w) = \beta_{\mathcal{E}, \mathcal{D}}(V_w) + |\mathcal{E}_w|,$$

where $\mathcal{E}_w$ is the set

$$\mathcal{E}_w = \{ e \in \mathcal{E}; \iota(s(e)) = w \text{ and } e \in E(V_w, V_{\iota(e)}) \}.$$

Note that the $\mathcal{E}_w$ are pairwise disjoint. Then

$$\beta_{\mathcal{D}'}(V) = \sum_{w \in V(\Gamma')} \beta_{\mathcal{E}, \mathcal{D}}(V_w) - |\mathcal{E}_V| + \prod_{w \in V(\Gamma')} |\mathcal{E}_w|.$$

On the other hand, we have $\mathcal{E}_V \subset \bigsqcup \mathcal{E}_w$.

Assume now that $\beta_{\mathcal{D}'}(V) = 0$, for some $V \subset V(\Gamma)$. Then $\beta_{\mathcal{E}, \mathcal{D}}(V_w) = 0$, for every $w \in V(\Gamma')$, and $\mathcal{E}_V = \bigsqcup \mathcal{E}_w$. In this case, since $(\mathcal{E}, \mathcal{D})$ is $\mu$-polystable, we get that either $V_w = \iota^{-1}(w)$ or $V_w = \emptyset$. Let

$$W := \{ w \in V(\Gamma'); V_w = \iota^{-1}(w) \}.$$

Note that $w_0 \in W$, because $v_0 \in V$ and hence $v_0 \in V_{w_0} \neq \emptyset$. We claim that $W$ is equal to $V(\Gamma')$. Indeed, if this is not the case, then there is an edge $e \in \mathcal{E}$, such that $\iota(s(e)) \in W$ and $\iota(t(e)) \notin W$. In other words, $V_{\iota(s(e))} = \iota^{-1}(\iota(s(e)))$ and $V_{\iota(t(e))} = \emptyset$. This means that $e \in \mathcal{E}_{\iota(s(e))} \setminus \mathcal{E}_V$, hence $\mathcal{E}_V \neq \bigsqcup \mathcal{E}_w$, a contradiction. Then $W$ is equal to $V(\Gamma')$, and hence $V$ is equal to $V(\Gamma)$, and we are done. □

**Example 5.7.** Consider the graph $\Gamma$ in Figure 1. Let $\mu$ be the polarization on $\Gamma$ of degree $-1$ given by $\mu(v) = -1/2$ for every $v \in V(\Gamma)$. Figure 2 illustrates the poset $\mathcal{PSD}_\mu(\Gamma)$ of $\mu$-polystable pseudo-divisors of degree $-1$ on $\Gamma$. If $v_0$ is one of the two vertices of $\Gamma$, then the intersection $\mathcal{PSD}_\mu(\Gamma) \cap \mathcal{QD}_{0, \mu}(\Gamma)$ consists of the 3 pseudo-divisors in Figure 2 of type $(\mathcal{E}, \mathcal{D})$, with $\mathcal{E} \neq E(\Gamma)$. The intersection $\mathcal{QD}_{0, \mu}(\Gamma) \setminus \mathcal{PSD}_\mu(\Gamma)$ consists of the pseudo-divisors $(\mathcal{E}', \mathcal{D}')$, with $|\mathcal{E}'| \in \{0, 1, 2\}$ and $\mathcal{D}'(v_0) = 1$. For any one of them, $\text{pol}(\mathcal{E}', \mathcal{D}')$ is the polystable pseudo-divisor $(E(\Gamma), \mathcal{D})$ on the left in Figure 2. Note that $\text{rk}(\mathcal{E}', \mathcal{D}') = \text{rk}(E(\Gamma), \mathcal{D})$ if $|\mathcal{E}'| = 2$.

![Figure 2](image_url)

Figure 2. The poset of polystable pseudo-divisors.
5.2. Polystability on tropical curves. We consider polystability for divisors on tropical curves. Let $X$ be a tropical curve, $\mu$ a polarization on $X$ and $\Gamma$ a $\mu$-model of $X$.

**Definition 5.8.** A divisor $D$ on $X$ is $\mu$-polystable if there is a $\mu$-polystable pseudo-divisor $(\mathcal{E}, D)$ on $\Gamma$ such that $D \in K_{\mu}^2, D(X)$.

The goal of this section is to prove the following theorem (see also [15, Proposition 4.4] for the same statement).

**Theorem 5.9.** Let $X$ be a tropical curve and $\mu$ a degree-$d$ polarization on $X$. The following properties hold.

1. Every degree-$d$ divisor on $X$ is equivalent to a $\mu$-polystable divisor.

2. Two equivalent $\mu$-polystable divisors have the same combinatorial type.

**Remark 5.10.** Although a $\mu$-polystable divisor on a tropical curve is not unique in its equivalence class, Theorem 5.9 tells us that the difference of equivalent polystable divisors is well behaved. In fact, let $D_1$ and $D_2$ be two equivalent $\mu$-polystable divisors. By item (2) of Theorem 5.9, the divisors $D_1$ and $D_2$ have the same combinatorial type $(\mathcal{E}, D)$, hence $D_1 - D_2$ is a principal divisor of type:

$$D_1 - D_2 = \sum_{e \in \mathcal{E}} (p_e - q_e),$$

where $p_e, q_e \in e^0$ are, possibly equal, points in the interior of $e$. In other words, $D_1 - D_2$ is a principal $\mathcal{E}$-divisor. Recall that we described some important properties of $\mathcal{E}$-divisors in Lemmas 3.10 and 4.2. Later on, this will be crucial to construct a universal tropical Jacobian over the moduli space of tropical curves.

Before proving Theorem 5.9, we shall give an example to explain how to convert a divisor into a polystable divisor.

**Example 5.11.** Let $X$ be a tropical curve $X$ as in Example 4.1. Let $X$ be the polarization on $X$ such that $\mu(v) = -1/2$ for every $v \in V(\Gamma)$. Consider the divisor $D = v_0 - v_1 - p$, where $V(\Gamma) = \{v_0, v_1\}$ and $p \in e^0$ for some $e \in E(\Gamma)$. Note that $D$ is $v_0$-quasistable, but not polystable. In Figure 3 we illustrate how to convert $D$ into a polystable divisor.

![Figure 3. Converting a quasistable divisor into a polystable divisor.](image)

**Proof of item (1) of Theorem 5.9.** By [1] Theorem 5.6, every divisor on $X$ is equivalent to a $\mu$-semistable unitary divisor. So it suffices to show that every $\mu$-semistable unitary divisor $D$ on $X$ is equivalent to a $\mu$-polystable divisor $X$.

Let $\Gamma$ be the $\mu$-model of $X$. Let $(\mathcal{E}, D)$ be the combinatorial type of $D$. If $(\mathcal{E}, D)$ is $\mu$-polystable, there is nothing to do. Otherwise there exists $V \subset V(\Gamma)$ such that $E(V, V^c) \not\subset \mathcal{E}$ and $\beta_{\mathcal{E}, D}(V) = 0$. We can choose $V$ such that $E(V, V^c)$ is a minimal disconnecting subset. We apply Proposition 5.3 to get a $\mu$-semistable pseudo-divisor $(\mathcal{E} \cup E(V, V^c), D_1)$ such that $(\mathcal{E} \cup E(V, V^c), D_1) > (\mathcal{E}, D)$.
We claim that there exists a divisor \( D_1 \) on \( X \) with combinatorial type \((E \cup E(V, V^c), D_1)\) that is equivalent to \( D \). Indeed, for each \( e \in E(V, V^c) \) let \( v_e, w_e \) be the vertices incident to \( e \) such that \( v_e \in V \) and \( w_e \in V^c \). Also, for each \( e \in E \cap E(V, V^c) \), let \( p_e \in e^\circ \) be the point of \( e^\circ \) such that \( D(p_e) = -1 \), while, if \( e \in E(V, V^c) \setminus E \), let \( p_e := v_e \). Let \( \ell \) be the length function on \( X \) and define:

\[
r = \min\{\ell(p_e, w_e); e \in E(V, V^c)\}.
\]

For \( e \in E(V, V^c) \), let \( q_e \in \overline{p_e w_e} \) be the point such that \( \ell(p_e, q_e) = \frac{r}{2} \). Let \( f \) be the rational function with slope 1 on \( \overline{q_e p_e} \) and 0 everywhere else. Note that

\[
\text{Div}(f)(p) = \begin{cases} \text{val}_{E(V \setminus E)}(v_e) & \text{if } p = v_e \text{ for some } e \in E(V, V^c) \setminus E, \\ 1 & \text{if } p = p_e \text{ for some } e \in E(V, V^c) \cap E, \\ -1 & \text{if } p = q_e \text{ for some } e \in E(V, V^c), \\ 0 & \text{otherwise}. \end{cases}
\]

Define \( D_1 = D + \text{Div}(f) \). Then, comparing to Equation (10), the combinatorial type of \( D_1 \) is \((E \cup E(V, V^c), D_1)\), proving the claim.

Repeating this process and recalling Equation (24) one can prove that there exists a \( \mu \)-polystable divisor \( D_k \) equivalent to \( D \).

Before proving item (2) of Theorem 5.9 we need a lemma.

**Lemma 12.** Let \( X \) be a tropical curve. Let \( D \) be a \( \mu \)-polystable divisor on \( X \) of combinatorial type \((\mathcal{E}, D)\). Let \( Z \) be a tropical subcurve of \( X \). Consider \( V = V' \cap V(\Gamma) \) and, for every \( e \in E(V, V^c) \), let \( v_e \) be the endpoint of \( e \) with \( v_e \in V \). If \( V \neq \emptyset \) and \( \beta_D(Z) = 0 \), then \( E(V, V^c) \subset E \) and \( \overline{v_e p_{D, e}} \subset Z \) for every \( e \in E(V, V^c) \).

**Proof.** If \( Z = Z_1 \cup Z_2 \), then \( \beta_D(Z) = \beta_D(Z_1) + \beta_D(Z_2) \), and the result for \( Z \) follows from the result for \( Z_1 \) and \( Z_2 \). Therefore, we can assume that \( Z \) is connected. By Equation (7), we have

\[
0 = \beta_D(Z) \geq \beta_{\mathcal{E}, D}(V).
\]

Since \((\mathcal{E}, D)\) is \( \mu \)-polystable, we have that \( \beta_{\mathcal{E}, D}(V) = 0 \) and so, by the definition of polystability, we deduce that \( E(V, V^c) \subset E \).

Let us prove that \( \overline{v_e p_{D, e}} \subset Z \) for every \( e \in E(V, V^c) \). Since equality holds in Equation (23), we have that \( Z' \cap V(\Gamma^c) = Z \cap V(\Gamma) \) by Equations (4) and (7). In particular, \( p_{D, e} \in Z \) for every \( e \in E \cap E(V, V^c) \). Hence,

\[
\beta_D(Z \cup \overline{v_e p_{D, e}}) = \deg(D|_{Z \cup \overline{v_e p_{D, e}}}) - \mu(Z \cup \overline{v_e p_{D, e}}) + \frac{\delta_{X, Z \cup \overline{v_e p_{D, e}}}}{2} \\
= \deg(D|_Z) - \mu(Z) + \frac{\delta_{X, Z}}{2} - \frac{\delta_{X, Z \cup \overline{v_e p_{D, e}}}}{2} \\
= \beta_D(Z) - \frac{\delta_{X, Z} - \delta_{X, Z \cup \overline{v_e p_{D, e}}}}{2}.
\]

Since \( \beta_D(Z) = 0 \) and \( \beta_D(Z \cup \overline{v_e p_{D, e}}) \geq 0 \), we get

\[
\delta_{X, Z} \leq \delta_{X, Z \cup \overline{v_e p_{D, e}}}.
\]

This can only happen if \( \overline{v_e p_{D, e}} \subset Z \) (recall that \( v_e, p_{D, e} \in Z \)), and in this case, equality holds in (24). \( \square \)
Proof of item (2) of Theorem 7.9. Let $\Gamma$ be the $\mu$-model of $X$. Let $D$ and $D'$ be equivalent $\mu$-polystable divisors on $X$ of combinatorial type $(\mathcal{E}, D)$ and $(\mathcal{E}', D')$. We proceed by induction on the number of connected components of $\Gamma$. Write $D' = D + \text{Div}(f)$, with $f$ a rational function on $X$. Let $Z \subset X$ be the set of points where $f$ attains its minimum. Define $V = V(\Gamma) \cap Z$. By Lemma 3.9, we have $X_V \subset Z$, hence in particular $\delta_{X,Z} = |E(V, V')|$. For every $e \in E(V, V')$, let $v_e, w_e$ be the endpoints of $e$, with $v_e \in V$ and $w_e \in V'$, so that $v_e \in Z$ and $w_e \notin Z$. Also, define $p_e$ such that $Z \cap e = \overline{v_e p_e}$ (by Lemma 3.9 such a point $p_e$ exists). We let $k_e > 0$ be the slope of $f$ going out of $p_e$ in the direction of $e$. Then

$$
\beta_{D'}(Z) - \beta_D(Z) = \deg_D(Z) - \deg_D(Z) = \sum_{e \in E(V, V')} k_e.
$$

Since a polystable divisor is semistable, we get $k_e = 1$ for every $e \in E(V, V')$, hence $\beta_D(Z) = 0$ and $\beta_{D'}(Z) = |E(V, V')|$. By Lemma 5.12, we obtain that

$$
(25) \quad E(V, V') \subset \mathcal{E} \cap \mathcal{E}'.
$$

and $p_{D,e} \in Z$ for every $e \in E(V)$. Similarly, if $Z_0$ is some sufficiently small neighborhood of $Z$, then $\beta_D(Z_0) = 0$, hence $p_{D',e} \in Z_0$, so that $p_{D,e} \neq p_{D',e}$. Moreover, since $f$ is constant on $Z$, it is constant on a neighborhood of $X_V$. Therefore $D|_{X_V} = D'|_{X_V}$, and hence,

$$
(26) \quad (\mathcal{E}, D)|_{\Gamma(V)} = (\mathcal{E}', D')|_{\Gamma(V)},
$$

where $\Gamma(V) = (V, E(V))$ is the subgraph of $\Gamma$ induced by $V$.

If $f$ is constant, then $Z = X$, and we are done. So, assume that $f$ is not constant, and hence $Z \neq X$.

We claim that the slope of $f$ is 1 over $\overline{p_{D,e} p_{D',e}}$ and 0 over $\overline{v_e p_{D,e} \cup p_{D',e} w_e}$, for every $e \in E(V, V')$. Indeed for every $e \in E(V, V')$, the point $p_{D,e}$ (respectively, $p_{D',e}$) is the unique point of $e^0$ over which $D$ (respectively, $D'$) is different from 0. Since the slope of $f$ changes from 0 to 1 at the point $p_e$ of $e^c$, which is different from $p_{D',e}$, we get $D(p_e) \neq 0$, and hence $p_e = p_{D,e}$. Thus $f$ has slope zero on $\overline{v_e p_e}$ and 1 on $\overline{p_{D,e} w_e}$, for some $q_e \in e \cap Z^c$. The condition $D' - D = \text{Div}(f)$ implies $q_e = p_{D',e}$, and forces $f$ to have slope zero over $\overline{q_e w_e}$ and hence, over the whole set $\overline{v_e p_e} \cup \overline{q_e w_e}$, proving the claim.

Now take $X_{V'}$. Consider the rational function $\tilde{f} := f|_{X_{V'}}$ and the polarization $\tilde{\mu} := \mu_{E(V)}|_{X_{V'}}$ on $X_{V'}$. Define the divisors on $X_{V'}$:

$$
\tilde{D} := D|_{X_{V'}} \quad \text{and} \quad \tilde{D}' := D'|_{X_{V'}}.
$$

Since $f$ has slope 0 on $\overline{p_{D',e} w_e}$ for every $e \in E(V)$, we deduce that:

$$
\text{Div}(\tilde{f}) = \text{Div}(f)|_{X_{V'}} = \tilde{D}' - \tilde{D}.
$$

Moreover, $\tilde{D}$ and $\tilde{D}'$ have combinatorial type $(\mathcal{E}, D)|_{\Gamma(V')} = (\mathcal{E}', D')|_{\Gamma(V')}$, which are $\tilde{\mu}$-polystable pseudo-divisors, by Equation (26). By inductive hypothesis,

$$
(27) \quad (\mathcal{E}, D)|_{\Gamma(V')} = (\mathcal{E}', D')|_{\Gamma(V')}.
$$

Combining Equations (25), (26) and (27), we get $(\mathcal{E}, D) = (\mathcal{E}', D')$, as required. \qed
6. A universal tropical Jacobian over $M_g^{\text{trop}}$

6.1. A polyhedral decomposition of the tropical Jacobian. In this section, we will construct a polyhedral decomposition of the Jacobian of a tropical curve by means of $\mu$-polystable divisors, and we will compare it with known decompositions.

In this section, we denote by $X$ a tropical curve with a polarization $\mu$ and $\mu$-model $\Gamma$. We already defined the polytopes $P_{E,D}(X)$ for a pseudo-divisor $(E, D)$ on $\Gamma$. Recall that if $(E', D') \geq (E, D)$, then there are natural inclusions $K_{E,D}(X) \subset K_{E',D'}(X)$ and $P_{E,D}(X) \subset P_{E',D'}(X)$ (recall Equation \((\text{i})\)). We will prove that these polytopes glue nicely when we consider $\mu$-polystable pseudo-divisors.

**Proposition 6.1.** Let $(E, D)$ be a $\mu$-semistable pseudo-divisor on $\Gamma$. If we let $(E', D') = \text{pol}(E, D)$, then

$$P_{E,D}(X) \subset P_{E',D'}(X).$$

**Proof.** If $(E, D)$ is $\mu$-polystable, there is nothing to do. Otherwise, there is $V \subset V(\Gamma)$ such that $E_{\text{sp}}(V) = 0$ and $E(V, V^c) \not\subset E$. Define $E_1 = E \cup E(V, V^c)$ and let $(E_1, D_1)$ be the pseudo-divisor as in Proposition \((\text{i})\). Thus $P_{E,D}(X) \subset P_{E_1,D_1}(X)$.

We claim that $P_{E,D}^0(X) \subset P_{E_1,D_1}^0(X)$. Let $D$ be a divisor on $\Gamma$ with combinatorial type $(E, D)$. Recall that $p_{D,e}$ is the point in $e^0$ such that $D(p_{D,e}) = -1$. For $e \in E(V, V^c)$, let $p_{D,e} := v_e$. Recall that $D$ corresponds to a point in $K_{E,D}^0(X)$ and hence to a point in $P_{E,D}^0(X)$ (and vice-versa, every point in $P_{E,D}^0(X)$ corresponds to a divisor $D$ of this form). For each $e \in E(V, V^c)$, let $v_{e_1}, w_e \in V(\Gamma)$ be the vertices incident to $e$ with $v_{e_1} \in V$ and $w_e \in V^c$. Let $Z$ be the subcurve of $X$:

$$Z = X_{V^c} \cup \bigcup_{e \in E(V, V^c) \cap \Sigma} \overline{v_e p_{D,e}}.$$

As in the proof of Theorem \((\text{c})\) item (1), take $r = \min(\ell(p_{D,e}, w_e); e \in E(V, V^c))$. Consider the rational function $f$ with slope 0 everywhere and slope 1 on $\overline{p_{D,e}, q_e}$, where $q_e$ is the point in $p_{D,e}, w_e$ such that $\ell(p_{D,e}, q_e) = r/2$. Hence $D + \text{Div}(f)$ has combinatorial type $(E_1, D_1)$ (recall Equation \((\text{c})\). This means that the points in $K_{E_1,D_1}^0(X)$ corresponding to $D$ and $D + \text{Div}(f)$ have the same image in $P_{E_1,D_1}(X)$ (recall Proposition \((\text{c})\)). However, the point associated to $D + \text{Div}(f)$ lies in the interior $K_{E_1,D_1}^0(X)$, and hence in the interior $P_{E_1,D_1}^0(X)$. Hence we get an inclusion $P_{E,D}^0(X) \subset P_{E_1,D_1}^0(X)$, proving the claim.

Using Equation \((\text{c})\), we can iterate the argument and obtain the result. \qed

**Proposition 6.2.** If $(E, D)$ and $(E', D')$ are $\mu$-polystable pseudo-divisors on $\Gamma$ such that $(E', D') \geq (E, D)$, then $P_{E,D}(X)$ is a face of $P_{E',D'}(X)$. Conversely, every face of $P_{E',D'}(X)$ arises in this way.

**Proof.** Let $(E', D')$ be a $\mu$-polystable pseudo-divisor on $\Gamma$. First, note that every face of $P_{E',D'}(X)$ is the image of a face of $K_{E',D'}(X)$, hence it is of the form $K_{E_0, D_0}(X)$ for some $\mu$-semistable pseudo-divisor $(E_0, D_0)$ (recall Remark \((\text{c})\)). By Proposition \((\text{c})\),

$$P_{E_0,D_0}^0 \subset P_{\text{pol}(E_0,D_0)}^0 \subset P_{E',D'},$$

hence $P_{E_0,D_0} = P_{\text{pol}(E_0,D_0)}$. This proves the second statement.

Second, let $(E, D)$ be a $\mu$-polystable pseudo-divisor on $\Gamma$ with $(E', D') \geq (E, D)$. Hence $P_{E,D}(X)$ is contained in a minimal face $P_{E'',D''}(X)$ (see Equation \((\text{c})\)), where $(E'', D'')$ is $\mu$-polystable and $(E'', D'') \geq (E', D')$. Thus $P_{E,D}(X) \cap P_{E'',D''}(X) \neq \emptyset$. 


By Proposition 4.3 and Theorem 5.3 we have \((\mathcal{E}^\prime, D^\prime) = (\mathcal{E}, D')\), and we are done.

\[\square\]

**Proposition 6.3.** Let \(\Gamma\) be a graph and \(\mu\) a polarization on \(\Gamma\). The poset \(\mathcal{PSD}_{\mu}(\Gamma)\) is ranked of dimension \(b_1(\Gamma)\) and is connected in codimension 1.

**Proof.** We begin noting that the maximal elements \((\mathcal{E}, D) \in \mathcal{PSD}_{\mu}(\Gamma)\) are the ones that satisfy \(\text{rk}(\mathcal{E}, D) = b_1(\Gamma)\) (recall that \(\text{rk}(\mathcal{E}, D) \leq b_1(\Gamma)\)).

By Lemma 5.5 if \(\text{rk}(\mathcal{E}, D) = b_1(\Gamma)\), then \((\mathcal{E}, D)\) is maximal. Conversely, if \((\mathcal{E}, D)\) is a \(\mu\)-polystable pseudo-divisor, then, by Proposition 5.0 there is a \((v_0, \mu)\)-quasistable pseudo-divisor \((\mathcal{E}', D')\) such that \(\text{pol}(\mathcal{E}', D') = (\mathcal{E}, D)\). By [1] Proposition there exists a \((v_0, \mu)\)-quasistable pseudo-divisor \((\mathcal{E}_0, D_0)\) with \(\text{rk}(\mathcal{E}_0, D_0) = b_1(\Gamma)\) and \((\mathcal{E}_0, D_0) \geq (\mathcal{E}', D')\). However, this means that \(\text{pol}(\mathcal{E}_0, D_0) \geq (\mathcal{E}', D')\), hence \(\text{pol}(\mathcal{E}_0, D_0) \geq \text{pol}(\mathcal{E}', D') = (\mathcal{E}, D)\). But \(\text{rk}(\text{pol}(\mathcal{E}_0, D_0)) = b_1(\Gamma)\), which means that \(\text{rk}(\text{pol}(\mathcal{E}_0, D_0)) = b_1(\Gamma)\). Therefore, every \(\mu\)-polystable divisor \((\mathcal{E}, D)\) is less or equal than a \(\mu\)-polystable divisor with rank \(b_1(\Gamma)\).

Now, every maximal chain in \(\mathcal{PSD}_{\mu}(\Gamma)\) ends in a maximal element \((\mathcal{E}, D)\). By Proposition 6.2 the maximal chains ending in \((\mathcal{E}, D)\) correspond precisely the maximal chains of faces of \(P_{\mathcal{E}, D}(X)\), which all have length \(\dim P_{\mathcal{E}, D}(X) = \text{rk}(\mathcal{E}, D) = b_1(\Gamma)\). This proves that \(\mathcal{PSD}_{\mu}(\Gamma)\) is ranked of dimension \(b_1(\Gamma)\).

We now prove that \(\mathcal{PSD}_{\mu}(\Gamma)\) is connected in codimension 1. Let \((\mathcal{E}_1, D_1)\) and \((\mathcal{E}_2, D_2)\) be two \(\mu\)-polystable pseudo-divisors on \(\Gamma\) with rank \(b_1(\Gamma)\). By Proposition 5.6 we can consider \((v_0, \mu)\)-quasistable divisors \((\mathcal{E}_i', D_i')\) on \(\Gamma\) such that \(\text{pol}(\mathcal{E}_i', D_i') = (\mathcal{E}_i, D_i)\) and \(\text{rk}(\mathcal{E}_i', D_i') = g_i\) for \(i = 1, 2\). We know by [1] Proposition 4.13 that \(\mathcal{QD}_{v_0, \mu}(\Gamma)\) is connected in codimension 1. Hence there exists a path in codimension 1 in \(\mathcal{QD}_{v_0, \mu}(\Gamma)\) connecting \((\mathcal{E}_i', D_i')\) with \((\mathcal{E}_j', D_j')\). Applying the map \(\text{pol}\) to the whole sequence, and recalling Equation (22) and the fact that \(\text{pol}\) is order-preserving, we get a sequence in codimension 1 in \(\mathcal{PSD}_{\mu}(\Gamma)\) connecting \((\mathcal{E}_1, D_1)\) with \((\mathcal{E}_2, D_2)\).

\[\square\]

**Theorem 6.4.** The poset \(\mathcal{PSD}_{\mu, g}\) is ranked of dimension \(4g - 3\) and connected in codimension 1.

**Proof.** The proof is essentially the same as in [1] Theorem 4.15, combining Proposition 6.3 with the same results for the poset of genus-\(g\) stable weighted graphs in [8] Theorem 3.2.5 and [11] Fact 4.12.

\[\square\]

**Definition 6.5.** Let \(X\) be a tropical curve with a polarization \(\mu\) and \(\mu\)-model \(\Gamma\). The \(\mu\)-polystable Jacobian of \(X\) is the polyhedral complex

\[P_{\mu}^{\text{trop}}(X) = \lim_{\rightarrow} P_{\mathcal{E}, D}(X)\]

where the limit is taken over the poset \(\mathcal{PSD}_{\mu}(\Gamma)\). In particular, we have

\[P_{\mu}^{\text{trop}}(X) = \bigsqcup_{(\mathcal{E}, D) \in \mathcal{PSD}_{\mu}(\Gamma)} P_{\mathcal{E}, D}(X).\]

We have the following theorem giving a polyhedral decomposition of the tropical Jacobian (see also [15] Proposition 5.8).

**Theorem 6.6.** Given a tropical curve \(X\), there is a homeomorphism \(P_{\mu}^{\text{trop}}(X) \to J(X)\).
Proof. Since $P_{trop}^\mu(X)$ is compact and $J(X)$ is Hausdorff it is sufficient to construct a continuous bijective map $\alpha: P_{trop}^\mu(X) \to J(X)$. Fix a point $p_0 \in X$, and let $\alpha$ be the map taking a $\mu$-polystable divisor $D$ on $\Gamma$ to the class of the divisor $D - dp_0$. It is a bijection by Theorem 5.9. It is continuous, because each map $\alpha|_{PE,D(X)}: PE,D(X) \to J(X)$ is equal to the continuous map of Equation (12).

Recall that, given a tropical curve $X$ and a point $p_0 \in X$, we define the Jacobian of $X$ with respect to $(p_0, \mu)$ as the polyhedral complex:

\begin{equation}
J_{trop}^{p_0,\mu}(X) = \lim_{\rightarrow} K_{E,D}(X),
\end{equation}

where $(E,D)$ runs through all $(p_0, \mu)$-quasistable divisors (see [1, Definition 5.7]).

We have the following proposition, see also [15, Corollary 5.10].

**Proposition 6.7.** Let $X$ be a tropical curve and $p_0$ be a point of $X$. We have a refinement map of polyhedral complexes $J_{trop}^{p_0,\mu}(X) \to P_{trop}^\mu(X)$.

**Proof.** A $(p_0, \mu)$-quasistable divisor $(E,D)$ is simple by [1 Proposition 4.6]. Thus Equation (28) implies that $K_{E,D}(X) = P_{E,D}(X)$. By Proposition 5.4 and Equation (11), we have inclusions $P_{E,D}(X) \subset P_{pol(E,D)}(X)$. Hence we obtain a refinement map $J_{p_0,\mu}(X) \to P_{trop}^\mu(X)$ of polyhedral complexes. □

**Example 6.8.** Consider the tropical curve of Example 4.1 and the polarization $\mu$ of Example 5.7. In Figure 4 we draw a picture of the Jacobian $P_{trop}^\mu(X)$ with its natural polyhedral decomposition. This is a hexagon (with suitable identifications). One can check that $J_{trop}^{p_0,\mu}(X)$ is as in [1 Figure 4], and it is clear that we have a refinement map $J_{trop}^{p_0,\mu}(X) \to P_{trop}^\mu(X)$.

![Figure 4. The Jacobian $P_{trop}^\mu(X)$.](image-url)
6.2. A universal tropical Jacobian. We are in a position to introduce a universal tropical Jacobian over $M_g^{\text{trop}}$. The idea is to glue together all the cones $\sigma(\Gamma, \mathcal{E}, D)$ for all graphs $\Gamma$ of genus $g$ and all $\mu$-polystable pseudo-divisors $(\mathcal{E}, D)$ on $\Gamma$. First of all, we have the following analogue of Propositions 6.1 and 6.2 for the cones $\sigma(\Gamma, \mathcal{E}, D)$.

**Proposition 6.9.** Let $\Gamma$ be a graph and $(\mathcal{E}, D)$ be a $\mu$-semistable pseudo-divisor on $\Gamma$. The following properties hold.

1. If we let $(\mathcal{E}', D') = \text{pol}(\mathcal{E}, D)$, then $\sigma^0(\Gamma, \mathcal{E}, D) \subset \sigma^0(\Gamma, \mathcal{E}', D')$.

2. Given a $\mu$-polystable pseudo-divisor $(\mathcal{E}', D')$ such that $(\mathcal{E}, D) \geq (\mathcal{E}', D')$, we have that $\sigma(\Gamma, \mathcal{E}, D)$ is a face of $\sigma(\Gamma, \mathcal{E}', D')$ if and only if $(\mathcal{E}, D)$ is $\mu$-polystable.

**Proof.** Just use the results in Propositions 6.1 and 6.2 together with the fact that $\pi^{-1}_X(X) \cong P_{\mathcal{E}, D}(X)$ from Proposition 4.6.

**Proposition 6.10.** Let $(\mathcal{E}_1, D_1)$ and $(\mathcal{E}_2, D_2)$ be $\mu$-polystable divisors on the graphs $\Gamma_1$ and $\Gamma_2$. If $(\Gamma_1, \mathcal{E}_1, D_1) \geq (\Gamma_2, \mathcal{E}_2, D_2)$, then there exists a natural face morphism $\sigma(\Gamma_1, \mathcal{E}_1, D_1) \rightarrow \sigma(\Gamma_2, \mathcal{E}_2, D_2)$. Conversely, every face of $\sigma(\Gamma_1, \mathcal{E}_1, D_1)$ arises in this way.

**Proof.** Assume that $(\Gamma_1, \mathcal{E}_1, D_1) \geq (\Gamma_2, \mathcal{E}_2, D_2)$. Let $\iota: \Gamma_1 \rightarrow \Gamma_2$ be the induced specialization and set $(\mathcal{E}', D') = \iota_*(\mathcal{E}_1, D_1)$. We have that

$$(\Gamma_1, \mathcal{E}_1, D_1) \geq (\Gamma_2, \mathcal{E}', D') \geq (\Gamma_2, \mathcal{E}_2, D_2).$$

By Proposition 4.7, the cone $\sigma(\Gamma_1, \mathcal{E}_1, D_1)$ is a face of $\sigma(\Gamma_2, \mathcal{E}_1, D_1)$. By Lemma 5.2, we have that $(\mathcal{E}', D')$ is $\mu$-polystable. Then, by Lemma 5.3, the cone $\sigma(\Gamma_2, \mathcal{E}_1, D_1)$ is a face of $\sigma(\Gamma_2, \mathcal{E}', D')$, and consequently a face of $\sigma(\Gamma_1, \mathcal{E}_1, D_1)$.

Conversely, the faces of $\sigma(\Gamma_1, \mathcal{E}_1, D_1)$ must be images of faces of $\mathbb{R}_0^{E(\Gamma)}$ and hence, by Remark 5.2, they are of the form $\sigma(\Gamma_2, \mathcal{E}_1, D_2)$, where $(\mathcal{E}_2, D_2)$ is a $\mu$-semistable pseudo-divisor on $\Gamma_2$ such that $(\Gamma_2, \mathcal{E}_2, D_2) \geq (\Gamma_1, \mathcal{E}_1, D_1)$. As before, denote by $(\mathcal{E}', D') := \iota_*(\mathcal{E}_1, D_1)$, where $\iota: \Gamma_1 \rightarrow \Gamma_2$ is the induced specialization. Since $(\mathcal{E}', D') \geq (\mathcal{E}_2, D_2)$, we have $(\mathcal{E}', D') \geq \text{pol}(\mathcal{E}_2, D_2)$ by Proposition 5.4. Hence $\sigma(\Gamma_2, \mathcal{E}_1, D_1)$ is a face of $\sigma(\Gamma_2, \mathcal{E}_2, D_2)$ by the first part of the proof. It follows from Proposition 6.9 that $\sigma^0(\Gamma_2, \mathcal{E}_2, D_2) \subset \sigma^0(\Gamma_2, \text{pol}(\mathcal{E}_2, D_2))$, which implies that $\sigma(\Gamma_1, \mathcal{E}_1, D_1) = \sigma(\Gamma_2, \text{pol}(\mathcal{E}_2, D_2))$. Hence, every face of $\sigma(\Gamma_1, \mathcal{E}_1, D_1)$ comes from a $\mu$-polystable pseudo-divisor.

**Example 6.11.** There is a subtle aspect about Proposition 6.10 that Karl Christ pointed out to us and that we want to illustrate in this example. Consider the tropical curve $X$ and its model $\Gamma$ as in Figure 4. Let $(\mathcal{E}, D)$ be a pseudo-divisor on $\Gamma$, where $\mathcal{E} = E(\Gamma)$. The cone $\sigma(\Gamma, \mathcal{E}, D)$ is equal to $\mathbb{R}^4_{>0}/L$, where $L$ is the linear subspace generated by the vector $(1, 1, -1, -1)$ (the coordinates are $(x, y, z, w)$). Identifying $\mathbb{R}^4/L$ with $\mathbb{R}^3$, where $\mathbb{R}^4 = \mathbb{R}^3 + \mathbb{R}^3 - \mathbb{R}^3$, we can think of $\sigma(\Gamma, \mathcal{E}, D)$ as the cone generated by the vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ and $(1, 1, -1)$. There are 4 possible specializations:

$$x = z = 0, \quad y = w = 0, \quad x = w = 0, \quad z = y = 0,$$

which correspond to the faces

$$(0, 1, 0), (1, 1, -1), \quad (1, 0, 0), (0, 0, 1), \quad (0, 1, 0), (0, 0, 1), \quad (1, 0, 0), (1, 1, -1).$$

Note that we cannot consider the specializations $x = 0, y = 0, z = 0, w = 0, x = y = 0, z = w = 0$, as the combinatorial type we get after contraction is not polystable. The potential issue is that two equivalent polystable divisors might not
remain equivalent upon contraction. This, though, does not happen. In fact, fix a parameter \( t > 0 \) and consider a family of tropical curves \( \Gamma(t) \) as in Figure 5 with lengths \( \ell_1(t) > 0 \) and \( \ell_2 > 0 \), and assume that \( \lim_{t \to 0} \ell_1(t) = 0 \). Consider, for each \( t > 0 \), two equivalent polystable divisors \( D_1(t) \) and \( D_2(t) \) with combinatorial type as in Figure 5. Assume that \( D_i(t) \) is given by the pair \((x_i(t), y_i(t))\), where \( x_i(t), y_i(t) \) are the distances to the vertex \( v_0 \). These two divisor are equivalent, if, and only if, \( x_1(t) - x_2(t) = y_1(t) - y_2(t) \). However, \( |x_1(t) - x_2(t)| < \ell_1(t) \), hence \( \lim_{t \to 0}(y_1(t) - y_2(t)) = 0 \). We deduce that, if the limits \( \lim_{t \to 0} D_1(t) \) and \( \lim_{t \to 0} D_2(t) \) exist, then they must be the same. In a way, when an edge is contracted, one has to do it “continuously”: the points on the other edge are forced to come together.

**Figure 5.** A tropical curve

**Definition 6.12.** Let \( \mu \) be a genus-\( g \) universal polarization of degree \( d \). The universal tropical Jacobian \( P_{\mu, g}^{\text{trop}} \) over \( M_{g, \mu}^{\text{trop}} \) is the generalized cone complex:

\[
P_{\mu, g}^{\text{trop}} = \lim_{\Gamma} \sigma(\Gamma, E, D) = \bigoplus_{[\Gamma, E, D]} \sigma^0(\Gamma, E, D) / \text{Aut}(\Gamma, E, D)
\]

where the limit is taken over all triples \((\Gamma, E, D)\) running through all objects in the category \( \text{PSD}_{g, \mu} \) and the union is taken over all equivalence classes \([\Gamma, E, D]\) in \( \text{PSD}_{g, \mu} \). If \( \mu \) is the canonical genus-\( g \) universal polarization of degree \( d \), we simply write:

\[
P_{d, g}^{\text{trop}} := P_{\mu, g}^{\text{trop}}.
\]

**Remark 6.13.** Recall that, by [18] Section 5], there exists a unique genus-\( g \) universal polarization of degree-\( d \), which is the canonical polarization. Maintaining the label \( \mu \), however, enables to generalize in an easy way to other moduli spaces, which do admit other universal polarizations.

**Proposition 6.14.** The generalized cone complex \( P_{\mu, g}^{\text{trop}} \) parametrizes equivalence classes \((X, D)\), where \( X \) is a stable tropical curve of genus \( g \) and \( D \) is a \( \mu \)-polystable divisor on \( X \), and two pairs \((X_1, D_1)\) and \((X_2, D_2)\) are equivalent if there exists an isomorphism \( \iota: X_1 \to X_2 \) such that \( \iota_*(D_1) \) is linearly equivalent to \( D_2 \).

**Proof.** By Proposition 4.6 each open cone \( \sigma^0(\Gamma, E, D) \) parametrizes equivalence classes of pairs \((X, D)\), where \( X \) is a tropical curve with model \( \Gamma \) and \( D \) is a unitary divisor on \( X \) with combinatorial type \((E, D)\). Two pairs \((X_1, D_1)\) and \((X_2, D_2)\) are equivalent if \( X_1 = X_2 \) and \( D_1 \) and \( D_2 \) are linearly equivalent.

Let \( X \) be a stable tropical curve of genus \( g \). The stable model \( \Gamma \) of \( X \) is a genus-\( g \) stable weighted graph. By definition, if \( D \) is a \( \mu \)-polystable divisor of \( X \), then \( D \) is unitary and has combinatorial type equal to some \( \mu \)-polystable pseudo-divisor \((E, D)\) on \( \Gamma \). Therefore \((X, D)\) corresponds to a point in \( \mathbb{R}^{E(\Gamma)} \), and hence to a point in \( \sigma_0(\Gamma, E, D) \). Then the pair \((X, D)\) corresponds to a point in \( P_{\mu, g}^{\text{trop}} \). If \((X, D)\) and \((X', D')\) are in the same equivalence class, then there is an isomorphism
\( \iota: X \to X' \) such that \( \iota_*(D) \) is linearly equivalent to \( D' \). Consider the three points \( p_1, p_2, p_3 \) in \( \mathbb{R}^{E(\Gamma^\text{trop})}_{>0} \) corresponding to \( (X, D), (X', \iota_*(D)), (X', D') \). The points \( p_1 \) and \( p_2 \) will get identified in \( \mathbb{R}^{E(\Gamma^\text{trop})}_{>0}/\text{Aut}(\Gamma, \mathcal{E}, D) \), while \( p_2 \) and \( p_3 \) will identify in \( \mathbb{R}^{E(\Gamma^\text{trop})}_{>0}/\mathcal{E} \). Hence, \( p_1 \) and \( p_3 \) will correspond to the same point in

\[
\sigma^0_{\Gamma, \mathcal{E}, D}/\text{Aut}(\Gamma, \mathcal{E}, D) = (\mathbb{R}^{E(\Gamma^\text{trop})}_{>0}/\mathcal{E})/\text{Aut}(\Gamma, \mathcal{E}, D) \subset P^{\text{trop}}_{\mu,g}.
\]

On the other hand if \( (X, D) \) and \( (X', D') \) corresponds to the same point in \( P^{\text{trop}}_{\mu,g} \) contained in some cell \( \sigma^0_{\Gamma, \mathcal{E}, D}/\text{Aut}(\Gamma, \mathcal{E}, D) \), then there is an isomorphism \( \iota: \Gamma_X \to \Gamma_{X'} \) such that \( \iota_*(\mathcal{E}, D) = (\mathcal{E}', D') \). Moreover, it follows from Lemma 4.6 that the metrics of \( X \) and \( X' \) are equal, hence \( \iota \) induces an isomorphism of metric graphs \( \iota: X \to X' \). This means that \( (X', D') \) and \( (X', \iota_*(D)) \) are the same point in \( \sigma^0_{\Gamma, \mathcal{E}, D} \). By Proposition 6.10 the divisors \( D' \) and \( \iota_*(D) \) are linearly equivalent. Hence \( (X, D) \) and \( (X', D') \) are equivalent, as required.

Conversely, given a triple \( (\Gamma, \mathcal{E}, D) \), every point in \( \mathbb{R}^{E(\Gamma^\text{trop})}_{>0} \) corresponds to a pair \( (X, D) \). By a similar argument as above, we see that if two such points are identified in \( \sigma^0_{\Gamma, \mathcal{E}, D}/\text{Aut}(\Gamma, \mathcal{E}, D) \), then the pairs are in the same equivalence class. \( \square \)

**Theorem 6.15.** The generalized cone complex \( P^{\text{trop}}_{\mu,g} \) has dimension \( 4g - 3 \) and it is connected in codimension 1. The natural forgetful map \( \pi^{\text{trop}}: P^{\text{trop}}_{\mu,g} \to M^{\text{trop}}_g \) is a map of generalized cone complexes, and we have:

\[
(\pi^{\text{trop}})^{-1}[X] \cong J(X)/\text{Aut}(X),
\]

for every stable weighted tropical curve \( X \) of genus \( g \).

**Proof.** The fact that \( P^{\text{trop}}_{\mu,g} \) has pure dimension \( 4g - 3 \) and is connected in codimension 1 follows from Theorem 6.3.

For each cone \( \sigma_{(\Gamma, \mathcal{E}, D)} \), the map \( \pi^{\text{trop}} \) induces a map \( \sigma_{(\Gamma, \mathcal{E}, D)} \to M^{\text{trop}}_g \) that factors through a chain of maps

\[
\sigma_{(\Gamma, \mathcal{E}, D)} = T_{\mathcal{E}}(\mathbb{R}^{E(\Gamma^\text{trop})}_{>0}) \to \mathbb{R}^{E(\Gamma)}_{>0} \to M^{\text{trop}}_g,
\]

where the first one is the map defined in Lemma 1.3 composed with the projection on the first factor, and the second one is the natural map. Hence \( \pi^{\text{trop}} \) is a morphism of generalized cone complexes.

For every stable weighted tropical curve \( X \) of genus \( g \), there is a natural map \( h: P^{\text{trop}}_\mu(X) \to P^{\text{trop}}_{\mu,g} \) and we have \( (\pi^{\text{trop}})^{-1}([X]) = \text{Im}(h) \). Moreover, \( h(D) = h(D') \) if and only if there exists an automorphism \( \alpha: X \to X \) such that \( \alpha_*(D) = D' \), which implies that \( \text{Im}(h) \cong J(X)/\text{Aut}(X) \). \( \square \)

### 6.3. Final comments.

To wrap up this paper, let us make a few observations on stratifications of some universal compactified Jacobians over the moduli space of stable curves. They all are essentially consequences of \[12\] Propositions 3.4.1, 3.4.2.

Let \( I \) be a torsion-free rank-1 sheaf on a nodal curve \( X \). The **combinatorial type** of the pair \( (X, I) \) is the triple \( (\Gamma, \mathcal{E}, D) \), where \( \Gamma \) is the usual dual graph of the nodal curve \( X \), the set \( \mathcal{E} \) is the subset \( \mathcal{E} \subset E(\Gamma) \) of the edges corresponding to the nodes over which \( I \) fails to be invertible, and \( D \) is the divisor on \( \Gamma^\mathcal{E} \) such that \( D(v) = -1 \) if \( v \in V(\Gamma^\mathcal{E}) \) is exceptional, while \( D(v) = \deg(I|_{X_v}) \) if \( v \in V(\Gamma) \) is not exceptional (here \( X_v \) is the component of \( X \) corresponding to \( v \)).
As we saw in the introduction, the compactified Picard scheme $\overline{P}_{d,g}$ over $\overline{M}_g$ parametrizes isomorphism classes of stably balanced line bundles on quasistable curves or, equivalently, pairs $(X, I)$, where $X$ is a stable curve of genus $g$ and $I$ is a $\mu$-polystable torsion-free rank-1 sheaf on $X$ of degree $d$ (see [9] and [26]). For every graph $\Gamma$ and every pseudo-divisor $(E, D)$ on $\Gamma$, we let $P_{\Gamma,E,D} \subset \overline{P}_{d,g}$ be the subscheme where $(X, I)$ has combinatorial type isomorphic to $(\Gamma, E, D)$. Then we have a stratification:

$$\overline{P}_{d,g} = \bigsqcup_{(\Gamma, E, D) \in PSD_{d,g}} P_{\Gamma,E,D},$$

There also is the Jacobian $\overline{J}_{d,g}$ over $\overline{M}_g$ introduced in [3], [16] and [20]. This is the Deligne-Mumford stack parametrizing isomorphism classes of pairs $(X, I)$ where $X$ is a stable curve of genus $g$ and $I$ is a simple torsion-free rank-1 sheaf of degree $d$ on $X$. We have a stratification:

$$\overline{J}_{d,g} = \bigsqcup_{(\Gamma, E, D)} J_{\Gamma,E,D},$$

where $(\Gamma, E, D)$ runs through all stable weighted graphs $\Gamma$ of genus $g$ and simple pseudo-divisors $(E, D)$ on $\Gamma$. Recall that $\overline{J}_{d,g}$ is neither separated nor of finite type over $\overline{M}_g$.

Finally, we have the compactified Jacobian $\overline{J}^{ss}_{\mu,g}$. This is the Deligne-Mumford stack over $\overline{M}_g$ parametrizing pairs $(X, I)$ where $X$ is a stable curve of genus $g$ and $I$ is a $\mu$-semistable simple torsion-free rank-1 sheaf on $X$ (see [16] and [20]). We have a stratification:

$$\overline{J}^{ss}_{\mu,g} = \bigsqcup_{(\Gamma, E, D)} J_{\Gamma,E,D},$$

where $(\Gamma, E, D)$ runs through stable weighted graphs $\Gamma$ of genus $g$ and simple $\mu$-semistable pseudo-divisors $(E, D)$ on $\Gamma$. Recall that $\overline{J}^{ss}_{\mu,g}$ is not separated over $\overline{M}_g$. In the above formulas, $J_{\Gamma,E,D}$ is the locus parametrizing pairs $(X, I)$ whose combinatorial type is isomorphic to $(\Gamma, E, D)$.

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