Intersections of Leray Complexes and Regularity of Monomial Ideals

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Abstract
For a simplicial complex $X$ and a field $K$, let $\tilde{h}_i(X) = \dim \tilde{H}_i(X; K)$. It is shown that if $X, Y$ are complexes on the same vertex set, then for $k \geq 0$

$$\tilde{h}_{k-1}(X \cap Y) \leq \sum_{\sigma \in Y} \sum_{i+j=k} \tilde{h}_{i-1}(X[\sigma]) \cdot \tilde{h}_{j-1}(\text{lk}(Y, \sigma)) .$$

A simplicial complex $X$ is $d$-Leray over $K$, if $\tilde{H}_i(Y; K) = 0$ for all induced subcomplexes $Y \subset X$ and $i \geq d$. Let $L_K(X)$ denote the minimal $d$ such that $X$ is $d$-Leray over $K$. The above theorem implies that if $X, Y$ are simplicial complexes on the same vertex set then

$$L_K(X \cap Y) \leq L_K(X) + L_K(Y) .$$

Reformulating this inequality in commutative algebra terms, we obtain the following result conjectured by Terai: If $I, J$ are square-free monomial ideals in $S = K[x_1, \ldots, x_n]$, then

$$\text{reg}(I + J) \leq \text{reg}(I) + \text{reg}(J) - 1$$

where $\text{reg}(I)$ denotes the Castelnuovo-Mumford regularity of $I$.

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1 Introduction

Let $X$ be a simplicial complex on the vertex set $V$. The induced subcomplex on a subset of vertices $S \subset V$ is $X[S] = \{ \sigma \in X : \sigma \subset S \}$. Let $\emptyset$ be the void complex and let $\{\emptyset\}$ be the empty complex. Any non-void complex contains $\emptyset$ as a unique $(-1)$-dimensional face. The star of a subset $A \subset V$ is $\text{St}(X, A) = \{ \tau \in X : \tau \cup A \in X \}$. The link of $A \subset V$ is $\text{lk}(X, A) = \{ \tau \in \text{St}(X, A) : \tau \cap A = \emptyset \}$. If $A \not\in X$ then $\text{St}(X, A) = \text{lk}(X, A) = \emptyset$.

All homology groups considered below are with coefficients in a fixed field $\mathbb{K}$ and we denote $\tilde{h}_i(X) = \dim_{\mathbb{K}} \tilde{H}_i(X)$. Note that $\tilde{h}_{-1}(\emptyset) = 0 \neq 1 = \tilde{h}_{-1}(\{\emptyset\})$.

Our main result is the following

**Theorem 1.1.** Let $X, Y$ be finite simplicial complexes on the same vertex set. Then for $k \geq 0$

$$\tilde{h}_{k-1}(X \cap Y) \leq \sum_{\sigma \in Y} \sum_{i+j=k} \tilde{h}_{i-1}(X[\sigma]) \cdot \tilde{h}_{j-1}(\text{lk}(Y, \sigma)) .$$  \hspace{1cm} (1)

We next discuss some applications of Theorem 1.1. A simplicial complex $X$ is $d$-Leray over $\mathbb{K}$ if $\tilde{H}_i(Y) = 0$ for all induced subcomplexes $Y \subset X$ and $i \geq d$. Let $L_{\mathbb{K}}(X)$ denote the minimal $d$ such that $X$ is $d$-Leray over $\mathbb{K}$. Note that $L_{\mathbb{K}}(X) = 0$ iff $X$ is a simplex. $L_{\mathbb{K}}(X) \leq 1$ iff $X$ is the clique complex of a chordal graph (see e.g. [11]).

The class $\mathcal{L}_d^d$ of $d$-Leray complexes over $\mathbb{K}$ arises naturally in the context of Helly type theorems [3]. The *Helly number* $h(\mathcal{F})$ of a finite family of sets $\mathcal{F}$ is the minimal positive integer $h$ such that if $\mathcal{K} \subset \mathcal{F}$ satisfies $\bigcap_{K \in \mathcal{K'}} K \neq \emptyset$ for all $\mathcal{K}' \subset \mathcal{K}$ of cardinality $\leq h$, then $\bigcap_{K \in \mathcal{K}} K \neq \emptyset$. The *nerve* $N(\mathcal{K})$ of a family of sets $\mathcal{K}$, is the simplicial complex whose vertex set is $\mathcal{K}$ and whose simplices are all $\mathcal{K}' \subset \mathcal{K}$ such that $\bigcap_{K \in \mathcal{K}'} K \neq \emptyset$. It is easy to see that for any field $\mathbb{K}$

$$h(\mathcal{F}) \leq 1 + L_{\mathbb{K}}(N(\mathcal{F})).$$

For example, if $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$, then by the Nerve Lemma (see e.g. [2]) $N(\mathcal{F})$ is $d$-Leray over $\mathbb{K}$, hence follows Helly’s Theorem: $h(\mathcal{F}) \leq d + 1$. This argument actually proves the Topological Helly Theorem: If $\mathcal{F}$ is a finite family of closed sets in $\mathbb{R}^d$ such that the intersection of any subfamily of $\mathcal{F}$ is either empty or contractible, then $h(\mathcal{F}) \leq d + 1$.

Nerves of families of convex sets however satisfy a stronger combinatorial property called *d-collapsibility* [11], that leads to some of the deeper extensions of Helly’s Theorem. It is of considerable interest to understand which
combinatorial properties of nerves of families of convex sets in $\mathbb{R}^d$ extend to arbitrary $d$-Leray complexes. For some recent work in this direction see [1, 6].

One consequence of Theorem 1.1 is the following

**Theorem 1.2.** Let $X_1, \ldots, X_r$ be simplicial complexes on the same finite vertex set. Then

$$L_{K}(\bigcap_{i=1}^{r} X_i) \leq \sum_{i=1}^{r} L_{K}(X_i) \quad (2)$$

$$L_{K}(\bigcup_{i=1}^{r} X_i) \leq \sum_{i=1}^{r} L_{K}(X_i) + r - 1 \quad . \quad (3)$$

**Example:** Let $V_1, \ldots, V_r$ be disjoint sets of cardinalities $|V_i| = a_i$, and let $V = \bigcup_{i=1}^{r} V_i$. Let $\Delta(A)$ denote the simplex on vertex set $A$, with boundary $\partial \Delta(A) \simeq S^{|A|-2}$. Consider the complexes

$$X_i = \Delta(V_1) * \cdots * \Delta(V_{i-1}) * \partial \Delta(V_{i}) * \Delta(V_{i+1}) * \cdots * \Delta(V_r) \quad . \quad$$

Then

$$\bigcap_{i=1}^{r} X_i = \partial \Delta(V_1) * \cdots * \partial \Delta(V_{r}) \simeq S^{\sum_{i=1}^{r} a_i - r - 1}$$

and

$$\bigcup_{i=1}^{r} X_i = \partial \Delta(V_1 \cup \ldots \cup V_r) \simeq S^{\sum_{i=1}^{r} a_i - 2} \quad .$$

The only non-contractible induced subcomplex of $X_i$ is $\partial \Delta(V_i)$, therefore $L_{K}(X_i) = a_i - 1$. Similar considerations show that $L_{K}(\bigcup_{i=1}^{r} X_i) = \sum_{i=1}^{r} a_i - 1$ and $L_{K}(\bigcap_{i=1}^{r} X_i) = \sum_{i=1}^{r} a_i - r$ , so equality is attained in both (2) and (3).

Theorem 1.2 was first conjectured in a different but equivalent form by Terai [8], in the context of monomial ideals. Let $S = K[x_1, \ldots, x_n]$ and let $M$ be a graded $S$-module. Let $\beta_{ij}(M) = \dim_K \text{Tor}^{i}_{j}(K, M)$ be the graded Betti numbers of $M$. The *regularity* of $M$ is the minimal $\rho = \text{reg}(M)$ such that $\beta_{ij}(M)$ vanish for $j > i + \rho$ (see e.g. [4]).

For a simplicial complex $X$ on $[n] = \{1, \ldots, n\}$ let $I_X$ denote the ideal of $S$ generated by $\{ \prod_{i \in A} x_i : A \not\in X \}$. The following fundamental result of Hochster relates the Betti numbers of $I_X$ to the topology of the induced subcomplexes $X$. 

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Theorem 1.3 (Hochster [5]).

$$\beta_{ij}(I_X) = \sum_{|W|=j} \dim_K \tilde{H}_{j-i-2}(X[W]) .$$  \hspace{1cm} (4)

Hochster’s formula (4) implies that $\text{reg}(I_X) = L_K(X) + 1$. The case $r = 2$ of Theorem 1.2 is therefore equivalent to the following result conjectured by Terai [8].

Theorem 1.4. Let $X$ and $Y$ be simplicial complexes on the same vertex set. Then

$$\text{reg}(I_X + I_Y) = \text{reg}(I_{X\cap Y}) \leq \text{reg}(I_X) + \text{reg}(I_Y) - 1$$
$$\text{reg}(I_X \cap I_Y) = \text{reg}(I_{X\cup Y}) \leq \text{reg}(I_X) + \text{reg}(I_Y) .$$

□

Theorem 1.4 can also be formulated in terms of projective dimension. Let $X^* = \{ \tau \subset [n] : [n] - \tau \not\in X \}$ denote the Alexander dual of $X$. Terai [7] showed that

$$\text{pd}(S/I_X) = \text{reg}(I_{X^*}) .$$ \hspace{1cm} (5)

Using (5) it is straightforward to check that Theorem 1.4 is equivalent to

Theorem 1.5.

$$\text{pd}(I_X \cap I_Y) \leq \text{pd}(I_X) + \text{pd}(I_Y)$$
$$\text{pd}(I_X + I_Y) \leq \text{pd}(I_X) + \text{pd}(I_Y) + 1 .$$

□

In Section 2 we give a spectral sequence for the relative homology group $H_*(Y, X \cap Y)$, which directly implies Theorem 1.1. The proof of Theorem 1.2 is given in Section 3.

2 A Spectral Sequence for $H_*(Y, X \cap Y)$

Let $K$ be a simplicial complex. The subdivision $\text{sd}(K)$ is the order complex of the set of the non-empty simplices of $K$ ordered by inclusion. For $\sigma \in K$ let $D_K(\sigma)$ denote the order complex of the interval $[\sigma, \cdot] = \{ \tau \in K : \tau \supset \sigma \}$.
$D_K(\sigma)$ is called the dual cell of $\sigma$. Let $D_K(\sigma)$ denote the order complex of the interval $(\sigma, \cdot] = \{\tau \in K : \tau \supsetneq \sigma\}$. Note that $D_K(\sigma)$ is isomorphic to $\text{sd}(\text{lk}(K, \sigma))$ via the simplicial map $\tau \to \tau - \sigma$. Since $D_K(\sigma)$ is contractible, it follows that $\text{H}_i(D_K(\sigma), \partial D_K(\sigma)) \cong \tilde{\text{H}}_{i-1}(\text{lk}(K, \sigma))$ for all $i \geq 0$. Write $K(p)$ for the family of $p$-dimensional simplices in $K$. The proof of Theorem 1.1 depends on the following

**Proposition 2.1.** Let $X$ and $Y$ be two complexes on the same vertex set $V$, such that $\dim Y = n$. Then there exists a homology spectral sequence $\{E^r_{p,q}\}$ converging to $\text{H}_*(Y, X \cap Y)$ such that

$$E^1_{p,q} = \bigoplus_{\sigma \in Y(n-p)} \bigoplus_{i+j \geq 0} \tilde{\text{H}}_{i-1}(X[\sigma]) \otimes \tilde{\text{H}}_{j-1}(\text{lk}(Y, \sigma))$$

for $0 \leq p \leq n$, $0 \leq q$, and $E^1_{p,q} = 0$ otherwise.

**Proof:** In the sequel we identify abstract complexes with their geometric realizations. Let $\Delta$ denote the simplex on $V$. For $0 \leq p \leq n$ let

$$K_p = \bigcup_{\sigma \in Y \atop \dim \sigma \geq n-p} \Delta[\sigma] \times D_Y(\sigma) \subset Y \times \text{sd}(Y)$$

and

$$L_p = \bigcup_{\sigma \in Y \atop \dim \sigma \geq n-p} X[\sigma] \times D_Y(\sigma) \subset (X \cap Y) \times \text{sd}(Y)$$

Write $K = K_n$, $L = L_n$. Let

$$\pi : K \to \bigcup_{\sigma \in Y} \Delta[\sigma] = Y$$

denote the projection on the first coordinate. For a point $z \in Y$, let $\tau = \text{supp}(z)$ denote the minimal simplex in $Y$ containing $z$. The fiber $\pi^{-1}(z) = \{z\} \times D_Y(\tau)$ is a cone, hence $\pi$ is a homotopy equivalence. Similarly, the restriction

$$\pi|_L : L \to \bigcup_{\sigma \in Y} X[\sigma] = X \cap Y$$

is a homotopy equivalence. Let $F_p = C_*(K_p, L_p)$ be the group of cellular chains of the pair $(K_p, L_p)$. The filtration $0 \subset F_0 \subset \cdots \subset F_n = C_*(K, L)$
gives rise to a homology spectral sequence \( \{ E^r \} \) converging to \( H_*(K, L) \cong H_*(Y, X \cap Y) \). We compute \( E^1 \) by excision and the Künneth formula:

\[
E^1_{p,q} = H_{p+q}(F_p/F_{p-1}) \cong H_{p+q}(K_p, L_p \cup K_{p-1}) \cong \\
H_{p+q} \left( \bigcup_{\sigma \in Y(n-p)} \Delta[\sigma] \times D_Y(\sigma), \bigcup_{\sigma \in Y(n-p)} X[\sigma] \times D_Y(\sigma) \cup \Delta[\sigma] \times \hat{D}_Y(\sigma) \right) \cong \\
\bigoplus_{\sigma \in Y(n-p)} H_{p+q}(\Delta[\sigma] \times D_Y(\sigma), X[\sigma] \times D_Y(\sigma) \cup \Delta[\sigma] \times \hat{D}_Y(\sigma)) \cong \\
\bigoplus_{\sigma \in Y(n-p)} \bigoplus_{i+j=p+q} H_i(\Delta[\sigma], X[\sigma]) \otimes H_j(D_Y(\sigma), \hat{D}_Y(\sigma)) \cong \\
\bigoplus_{\sigma \in Y(n-p)} \bigoplus_{i+j=p+q} \tilde{h}_{i-1}(X[\sigma]) \otimes \tilde{h}_{j-1}(\text{lk}(Y, \sigma)) .
\]

**Remark:** The derivation of the above spectral sequence may be viewed as a simple application of the method of simplicial resolutions. See Vassiliev’s papers [9, 10] for a description of this technique, and for far reaching applications to plane arrangements and to spaces of Hermitian operators.

**Proof of Theorem 1.1:** By Proposition 2.1

\[
\tilde{h}_{k-1}(X \cap Y) \leq \tilde{h}_{k-1}(Y) + h_k(Y, X \cap Y) \leq \\
\tilde{h}_{k-1}(Y) + \sum_{p+q=k} \dim E^1_{p,q} = \\
\tilde{h}_{k-1}(Y) + \sum_{\emptyset \neq \sigma \subseteq Y} \sum_{\dim \sigma \geq n-k} \tilde{h}_{i-1}(X[\sigma]) \cdot \tilde{h}_{j-1}(\text{lk}(Y, \sigma)) \leq \\
\sum_{\sigma \subseteq Y} \sum_{i+j=k} \tilde{h}_{i-1}(X[\sigma]) \cdot \tilde{h}_{j-1}(\text{lk}(Y, \sigma)) .
\]
3 Intersection of Leray Complexes

We first recall a well-known characterization of \(d\)-Leray complexes. For completeness we include a proof.

**Proposition 3.1.** For a simplicial complex \(X\), the following conditions are equivalent:

(i) \(X\) is \(d\)-Leray over \(\mathbb{K}\).

(ii) \(\tilde{H}_i(lk(X, \sigma)) = 0\) for every \(\sigma \in X\) and \(i \geq d\).

It will be convenient to prove a slightly more general result. Let \(k, m \geq 0\). We say that a simplicial complex \(X\) on \(V\) satisfies condition \(P(k, m)\) if \(\tilde{H}_i(lk(X[A], B)) = 0\) for all \(B \subset A \subset V\) such that \(|A| \geq |V| - k\) and \(|B| \leq m\).

**Claim 3.2.** If \(k \geq 0\) and \(m \geq 1\) then conditions \(P(k, m)\) and \(P(k + 1, m - 1)\) are equivalent.

**Proof:** Suppose \(B \subset A \subset V\) and \(B_1 \subset A_1 \subset V\) satisfy \(B = B_1 \cup \{v\}\), \(A = A_1 \cup \{v\}\) for some \(v \notin A_1\), and let

\[ Z_1 = lk(X[A_1], B_1) \quad \text{and} \quad Z_2 = St(lk(X[A], B_1), v) \]

Then

\[ Z_1 \cup Z_2 = lk(X[A], B_1) \quad \text{and} \quad Z_1 \cap Z_2 = lk(X[A], B) \]

and by Mayer-Vietoris there is an exact sequence

\[ \ldots \rightarrow \tilde{H}_{i+1}(lk(X[A], B_1)) \rightarrow \tilde{H}_i(lk(X[A], B)) \rightarrow \tilde{H}_i(lk(X[A], B_1)) \rightarrow \tilde{H}_i(lk(X[A_1], B_1)) \rightarrow \ldots \quad (6) \]

**P(k, m) \Rightarrow P(k + 1, m - 1):** Suppose \(X\) satisfies \(P(k, m)\) and let \(B_1 \subset A_1 \subset V\) such that \(|V| - |A_1| = k + 1\) and \(|B_1| \leq m - 1\). Choose a \(v \in V - A_1\) and let \(A = A_1 \cup \{v\}\), \(B = B_1 \cup \{v\}\). Let \(i \geq d\), then by the assumption on \(X\), both the second and the fourth terms in (6) vanish. It follows that \(\tilde{H}_i(lk(X[A_1], B_1)) = 0\) as required.

**P(k + 1, m - 1) \Rightarrow P(k, m):** Suppose \(X\) satisfies \(P(k + 1, m - 1)\) and let \(B \subset A \subset V\) such that \(|V| - |A| \leq k\) and \(|B| = m\). Choose a \(v \in B\) and let \(A_1 = A - v\), \(B_1 = B - v\). Let \(i \geq d\), then by the assumption on \(X\), both the first and the third terms in (6) vanish. It follows that \(\tilde{H}_i(lk(X[A], B)) = 0\) as required.
Proof of Proposition 3.1: Let $X$ be a complex on $n$ vertices. Then (i) is equivalent to $P(n, 0)$, while (ii) is equivalent to $P(0, n)$. On the other hand, $P(n, 0)$ and $P(0, n)$ are equivalent by Claim 3.2.

Proof of Theorem 1.2: By induction it suffices to consider the $r = 2$ case. Let $X, Y$ be complexes on $V$ with $L_K(X) = a$, $L_K(Y) = b$, and let $k > a + b$. Then for any $\sigma \in Y$ and for any $i, j$ such that $i + j = k$, either $i > a$ hence $\tilde{h}_{i-1}(X[\sigma]) = 0$, or $j > b$ which by Proposition 3.1 implies that $\tilde{h}_{j-1}(\text{lk}(Y, \sigma)) = 0$. By Theorem 1.1 it then follows that $\tilde{h}_{k-1}(X \cap Y) = 0$. Therefore

$$L_K(X \cap Y) \leq \max_{S \subset V} (L_K(X[S]) + L_K(Y[S])) = L_K(X) + L_K(Y).$$

(7)

Next, let $k \geq L_K(X) + L_K(Y) + 1$. Then by (7) and the Mayer-Vietoris sequence

$$\to \tilde{H}_k(X) \oplus \tilde{H}_k(Y) \to \tilde{H}_k(X \cup Y) \to \tilde{H}_{k-1}(X \cap Y) \to$$

it follows that $\tilde{H}_k(X \cup Y) = 0$. Hence

$$L_K(X \cup Y) \leq \max_{S \subset V} (L_K(X[S]) + L_K(Y[S]) + 1) = L_K(X) + L_K(Y) + 1.$$

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