Abstract. We reduce Boyer-Finley equation to a family of compatible systems of hydrodynamic type, with characteristic speeds expressed in terms of spaces of rational functions. The systems of hydrodynamic type are then solved by the generalized hodograph method, providing solutions of the Boyer-Finley equation including functional parameters.

In this paper we construct solutions of the dispersionless non-linear PDE – the Boyer-Finley equation (self-dual Einstein equation with a Killing vector),

\[ U_{xy} = (e^U)_{tt}, \]  

(1)

via reduction to a family of compatible systems of hydrodynamic type.

This equation was actively studied during last twenty years by many authors; we just mention works [1, 2, 3, 4, 5, 6, 7, 8, 9, 14, 15]. So far the most general scheme of the construction of its solutions was developed in [8, 9]. In these works solutions of the Boyer-Finley equation were derived by averaging an appropriate two-point Baker-Akhiezer function in genus zero which corresponds to
the two-dimensional Toda lattice equations, the underlying Riemann surface deforming according to the Whitham equations. Some particular solutions of the Boyer-Finley equation were constructed in [2, 3, 6, 5]; their relationship to the solutions of [8, 9] remains unclear.

The goal of the present paper is to give an alternative scheme of solving the Boyer-Finley equation. Namely, we consider reductions of this equation to multi-component systems of hydrodynamic type in the spirit of [12, 13], see also [10, 14, 15]; the equations for characteristic speeds of these systems are solved in terms of rational branched coverings. The systems of hydrodynamic type are then solved by the generalized hodograph method [11].

Now we describe a way to find solutions of the Boyer-Finley equation. First we ignore the condition of reality of the function \( U \) and construct complex-valued solutions of equation (1); then we formulate restrictions on the parameters which guarantee the reality of \( U \).

Let us assume \( U \) to be a function of \( L \) variables \( l_1, \ldots, l_L \), where \( l_m(x, y, t) \) ("Riemann invariants") satisfy a pair of systems of hydrodynamic type
\[
\begin{align*}
\partial_x l_m &= V_m(l_k) \partial_l l_m, \\
\partial_y l_m &= W_m(l_k) \partial_l l_m.
\end{align*}
\] (2)

A direct substitution of the function \( U(l_1, \ldots, l_L) \) into the Boyer-Finley equation implies the algebraic relation among the functions \( U, V_m \) and \( W_m \),
\[
V_m W_m = e^U,
\] (3)
along with the following differential equations:
\[
\begin{align*}
\partial_m \partial_n U(V_m W_n + V_n W_m) &= 2 \partial_n \partial_m (e^U), \quad m \neq n, \\
\frac{\partial_m V_n}{V_m - V_n} &= \frac{\partial_m W_n}{W_m - W_n}, \quad m \neq n,
\end{align*}
\] (4, 5)
where \( \partial_m \equiv \partial/\partial l_m \).

Relations (3) allow one to parameterize the functions \( V_m \) and \( W_m \) by a new set of variables \( \varphi_m(l_k) \) as follows:
\[
\begin{align*}
V_m &= \exp \left\{ 2i\varphi_m + \frac{U}{2} \right\}, \\
W_m &= \exp \left\{ -2i\varphi_m + \frac{U}{2} \right\}.
\end{align*}
\] (6)

In terms of these variables, equations (4) and (5) take the form
\[
\begin{align*}
\partial_m \partial_n U &= -\frac{\partial_m U \partial_n U}{2 \sin^2(\varphi_m - \varphi_n)}, \\
\partial_m \varphi_n &= \frac{1}{4} \cot(\varphi_n - \varphi_m) \partial_m U,
\end{align*}
\] (7)
where \( m \neq n \).

A class of solutions of this system is related to the space of rational functions in the following way. Consider a rational function
\[
R(\mu) = \mu + \sum_{k=1}^{N-1} \frac{a_k}{\mu - b_k}, \quad \mu \in \mathbb{CP}^1.
\] (8)

In application to Benney’s hierarchy, functions of this form were first introduced in [16]. The equation
\[
l = R(\mu)
\] (9)
defines an $N$–sheeted covering $L$ of the $l$–sphere. A point $P \in L$ is a pair of complex numbers, $P = (l, \mu)$. We consider the generic case when the function $R(\mu)$ has $2N - 2$ non-coinciding finite critical points, i.e., the equation

$$R'(\mu) = 0$$

has $2N - 2$ distinct roots $\mu_1, \ldots, \mu_{2N-2}$. The corresponding critical values,

$$l_n = R(\mu_n), \quad n = 1, \ldots, 2N - 2,$$

are projections onto the $l$–sphere of the branch points of the covering $L$ (we denote branch points by $P_n = (l_n, \mu_n)$; all of them are simple as a corollary of non-coincidence of $\mu_n$ for different $n$). An additional condition we impose on the function $R(\mu)$ is that all $l_n$ are different. Now, observe that the number of parameters of the rational function (8) is equal to the number of branch points; therefore, we can take $l_1, \ldots, l_{2N-2}$ as local coordinates on the space of rational functions. It was shown in [19] that the critical points $\{\mu_m\}$ of the rational function $R(\mu)$ depend on $\{l_n\}$ in the following way:

$$\frac{\partial \mu_m}{\partial l_n} = \frac{\beta_n}{\mu_n - \mu_m}, \quad \frac{\partial \beta_m}{\partial l_n} = \frac{2\beta_n \beta_m}{(\mu_n - \mu_m)^2}, \quad m \neq n.$$ (11)

These equations appeared also in [12, 13] in the theory of hydrodynamic reductions of Benney’s moment equations. The inverse function $\mu(P) = R^{-1}(P)$ is defined on the covering $L$. As a function of $\{l_n\}$, it satisfies the system of differential equations [19]:

$$\frac{\partial \mu}{\partial l_n} = \frac{\beta_n}{\mu_n - \mu}.$$ (12)

Let us now choose two points $Q_1$ and $Q_2$ on the covering $L$ such that their projections $l(Q_1)$ and $l(Q_2)$ onto the $l$–sphere do not depend on $\{l_n\}$. Then, consider the following function:

$$\gamma(P) = \frac{1}{2\pi i} \log \frac{\mu(P) - \mu(Q_1)}{\mu(P) - \mu(Q_2)},$$ (13)

which maps $L$ onto a cylinder. We shall be interested in the images $\{\gamma_m\}$ of branch points under this map as functions of $\{l_n\}$. In the sequel we denote $\mu(Q_1)$ by $\kappa_1$ and $\mu(Q_2)$ by $\kappa_2$. According to [12], they satisfy the equations

$$\frac{\partial \kappa_j}{\partial l_n} = \frac{\beta_n}{\mu_n - \kappa_j}, \quad j = 1, 2.$$ (14)

From the expression (13) for $\gamma(P_m)$ we have

$$\gamma_m = \frac{1}{2\pi i} \log \frac{\mu_m - \kappa_1}{\mu_m - \kappa_2}.$$ (15)

Differentiation of this relation with respect to $l_n$ using (11), (13) and (14) gives

$$\frac{\partial \gamma_m}{\partial l_n} = \frac{1}{2\pi i} \frac{\beta_n}{\mu_n - \mu_m} \left[ \frac{1}{\mu_m - \kappa_1} - \frac{1}{\mu_m - \kappa_2} \right].$$ (16)

If we now express $\mu_n$ and $\mu_m$ in terms of $\gamma_n$, $\gamma_m$ from (15) and set

$$\alpha_n = -\frac{\beta_n}{4\pi^2} \left[ \frac{1}{\mu_m - \kappa_1} - \frac{1}{\mu_m - \kappa_2} \right]^2,$$ (17)
we arrive at the following system of differential equation for the functions \(\gamma_m(\{l_n\})\):

\[
\frac{\partial \gamma_m}{\partial l_n} = -\pi \alpha_n \left( \cot \pi (\gamma_m - \gamma_n) + \cot \pi \gamma_n \right), \quad m \neq n.
\] (18)

Similarly, the functions \(\alpha_n\) satisfy the following equations:

\[
\frac{\partial \alpha_n}{\partial l_m} = 2\pi^2 \frac{\alpha_n \alpha_m}{\sin^2 \pi (\gamma_m - \gamma_n)} , \quad m \neq n.
\] (19)

It turns out that a simple transformation allows one to construct solutions of system (7) from the set of functions \(\gamma_m\) and \(\alpha_m\). Namely, the system of equations (18), (19) coincides with system (7) if rewritten in terms of the new variables \(U\) and \(\varphi_n\) such that

\[
\frac{\partial U}{\partial l_m} = -4\pi^2 \alpha_m
\] (20)

and

\[
\varphi_n = \pi (\gamma_n + \psi),
\] (21)

where

\[
\frac{\partial \psi}{\partial l_m} = \pi \alpha_m \cot \pi \gamma_m.
\] (22)

The existence of functions \(U\) and \(\psi\) is provided by the compatibility conditions,

\[
\frac{\partial}{\partial l_n} \alpha_m = \frac{\partial}{\partial l_m} \alpha_n
\] (23)

and

\[
\frac{\partial}{\partial l_n} (\alpha_m \cot \pi \gamma_m) = \frac{\partial}{\partial l_m} (\alpha_n \cot \pi \gamma_n)
\] (24)

which follow from (19) and (18).

Ultimately, formulae (6) determine \(\{V_m\}\) and \(\{W_m\}\) as functions of \(\{l_k\}\). In order to obtain a solution of the Boyer-Finley equation (11) we need \(U\) as an explicit function of \(x, y\) and \(t\), that is, we need to solve the system of hydrodynamic type (2). The tool which is usually used for this purpose is the generalized hodograph method [11]. Instead of solving (2), we find a smooth solution \(\Lambda(x, y, t) = (l_1, \ldots, l_L)\) of the following system

\[
\Phi_m(\Lambda) = t + V_m(\Lambda)x + W_m(\Lambda)y,
\] (25)

where the functions \(\{\Phi_m\}\) satisfy the linear system

\[
\frac{\partial_m \Phi_n}{\Phi_m - \Phi_n} = \frac{\partial_m V_n}{V_m - V_n} = \frac{\partial_m W_n}{W_m - W_n} , \quad m \neq n.
\] (26)

To see that an implicit solution (25) for \(\{l_m(x, y, t)\}\) indeed satisfies (2), one needs to differentiate (25) with respect to \(x, y\) and \(t\) [11].

To be able to use this method we need to construct functions \(\Phi_m\), i.e. we need to solve for \(\Phi_m\) the system

\[
\frac{\partial_m \Phi_n}{\Phi_m - \Phi_n} = \frac{\partial_m V_n}{V_m - V_n} , \quad m \neq n.
\] (27)
Observe that for \( m \neq n \)
\[
\frac{\partial_n V_m}{V_m - V_n} = -\frac{\pi^2 \alpha_n}{\sin^2 \pi (\gamma_m - \gamma_n)} ;
\]
(28)
this is a simple corollary of definitions of \( V_m \) and \( W_m \) and equations (20), (7).

Then the following functions satisfy equations (27):
\[
\Phi_m = \pi^2 \oint_l H(l) d\gamma \sin^2 \pi (\gamma - \gamma_m),
\]
(29)
where \( l \) is an arbitrary closed contour on the branched covering \( L \) such that its projection on the \( l \)-plane does not depend on the branch points \( \{l_n\} \) and such that \( P_m \notin l \) for all \( m \); \( H(l) \) is an arbitrary function on \( l \) independent of \( \{l_n\} \). The proof of this fact is a simple calculation using equations (18), (12) and the link (13) between \( \gamma \) and \( \mu \).

Note that in this framework we can fix positions of branch points \( \{l_{L+1}, \ldots, l_{2N-2}\} \) and consider the dependence of all functions on the remaining set of variables \( \{l_1, \ldots, l_L\} \), where \( L \leq 2N - 2 \).

The following theorem summarizes our construction of solutions of the Boyer-Finley equation.

**Theorem 1** Let functions \( \alpha_m(\{l_n\}) \) and \( \gamma_m(\{l_n\}) \), \( m, n = 1, \ldots, L \), \( L \leq 2N - 2 \), be associated with an \( N \)-sheeted branched covering as described above. Define the potentials \( U(\{l_n\}) \) and \( \psi(\{l_n\}) \) to be solutions of the following system of equations:

\[
\frac{\partial U}{\partial l_m} = -4\pi^2 \alpha_m , \quad m = 1, \ldots, L ;
\]
(30)
\[
\frac{\partial \psi}{\partial l_m} = \pi \alpha_m \cot \pi \gamma_m , \quad m = 1, \ldots, L .
\]
(31)

Let the \((x, y, t)\)-dependence of branch points \( l_n \), \( n = 1, \ldots, L \), be governed by the following system of \( L \) equations,
\[
\pi^2 \oint_l \frac{H(l) d\gamma}{\sin^2 \pi (\gamma - \gamma_m)} = t + x V_m + y W_m , \quad m = 1, \ldots, L ,
\]
(32)
where
\[
V_m = e^{2\pi i(\gamma_m + \psi) + U/2} , \quad W_m = e^{-2\pi i(\gamma_m + \psi) + U/2} ;
\]
(33)
l is an arbitrary \( \{l_n\} \)-independent contour on \( L \) such that all \( P_m \notin l \); \( H(l) \) is an arbitrary summable \( \{l_n\} \)-independent function on \( l \).

Then the function \( U(\{l_n(x, y, t)\}) \) satisfies the Boyer-Finley equation (1).

**Remark 1** If an \( N \)-sheeted rational branched covering \( L \) with two marked points \( Q_1 \), \( Q_2 \) is fixed, the solution of the Boyer-Finley equation constructed according to this theorem is defined by

(a) a functional parameter \( H(l) \) and
(b) a number \( L \leq 2N - 2 \), which has a meaning of the number of components \( l_m \) satisfying systems of hydrodynamic type (2) with characteristic speeds (33).

The application of theorem 1 in practice requires calculation of quadratures (30) and (31); besides that, one needs to resolve implicit relations (32) to find the dependence of \( l_m \) on \((x, y, t)\).

So far we were dealing with complex solutions of the Boyer-Finley equation (1); it is easy to formulate conditions on the parameters of our solutions which provide the reality of the function \( U \).
Let us assume the function $R(\gamma)$ to satisfy the “reality condition”

$$
\overline{R(\gamma)} = R(\gamma).
$$

Then the branch covering $L$ is invariant with respect to the antiholomorphic involution $\tau$, which acts on the points $(l, \mu)$ of the covering $L$ as follows:

$$
\tau: (l, \mu) \rightarrow (\overline{l}, \overline{\mu}).
$$

Assume also that both points $Q_1, Q_2$ are invariant with respect to $\tau$, i.e., $\kappa_{1,2} \in \mathbb{R}$. Let also the contour $l$ be invariant with respect to the involution and the function $H(P)$ satisfy the relation $H(P) = -\overline{H(\overline{P})}$. Then one can choose the constants of integration in (30) and (31) such that the solution $U(x,y,t)$ of the Boyer-Finley equation given by theorem 1 is real.

Indeed, the invariance of the covering $L$ with respect to $\tau$ means that all $l_m$ are either real or form conjugate pairs; the same holds for the set $\{\mu_m\}$. The expression (13) for the map $\gamma$ implies

$$
\overline{\gamma(P')} = -\gamma(P),
$$

therefore, all $\gamma_m \equiv \gamma(P_m)$ are either imaginary, $\overline{\gamma_m} = -\gamma_m$, or form anti-conjugate pairs. Applying complex conjugation to both sides of equation (18), we find that $\alpha_m$ are either real ($\alpha_m \in \mathbb{R}$ if $l_m \in \mathbb{R}$) or form conjugate pairs, $\alpha_m = \overline{\alpha_m}$, if $l_m = \overline{l_m}$. This readily implies that one can choose the integration constant in the definition (30) of potential $U$ in such a way that $U$ is a real function of $\{l_m\}$.

For completeness, we should also check that the reality condition does not contradict the solvability of system (32). Assume, for simplicity, that all $l_m$ are real, i.e., all $\gamma_m$ are imaginary and all $\alpha_m$ are real. Then the potential function $\psi$ solving system (31) can be chosen to be imaginary, and both $V_m$ and $W_m$ (33) are real. Together with $H(P') = -\overline{H(P)}$ and (36), it implies that both sides of equations (32) are real. Therefore, system (32) gives $L$ real equations for $L$ real variables $\{l_m\}$, and generically has solutions.

Similar consideration applies when some $l_m$'s form conjugate pairs; in this case the corresponding equations (32) will be conjugate to each other, and the number of real equations will again coincide with the number of real variables.

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