1 Introduction

It is the purpose of this paper to provide an exposition of a cohomological deformation theory for braided monoidal categories, an exposition and proof of a very general theorem of the form “all quantum invariants are Vassiliev invariants,” and to relate to the existence of the Vassiliev invariants obtained via the theorem to the vanishing of obstructions in the deformation theory.

Our deformation theory for braided monoidal categories will be based on a deformation theory for monoidal functors. Remarkably, the deformation theory of monoidal functors, unlike the deformation theory of monoidal categories developed in [3] shares many of the properties of the Gerstenhaber [8, 9, 10] deformation theory of algebras, though the relationship between the two theories is unclear. In this present paper, the deformation theory of monoidal functors is worked out only in the case where the monoidal structure of the source and target categories are undeformed.

Previous theorems of the “all quantum invariants are Vassiliev” type have involved definitions of “quantum invariants” either in terms of the well-known knot polynomial of Jones [14], HOMFLY [12], or Kauffman [16], as in the case of Birman and Lin [2] and Stanford [21], or more generally in terms of constructions tied explicitly to a (simple) Lie algebra, as in the case of Piunikhin [19]. Still, the notion of “quantum invariant” used here is not the most general conceivable, so some preliminaries are in order. The most general reasonable notion of “quantum invariant” of (framed) knots and links is what will here be called a “functorial invariant”, that is, an $R$-valued invariant ($R$ some commutative ring) which arises by choosing an object in an $R$-linear tortile (a.k.a. ribbon) category (with $\text{End}(I) = R$, where $I$ is the monoidal identity object), and considering the image of all framed links under the functor induced by the coherence theorem of Shum [20]. If we wish to deal with unframed links in this context, we must find a normalization procedure. (One will always be available if the chosen object is simple.)

It is not the case that all functorial invariants are Vassiliev invariants: the “counting invariants” associated to finite groups are functorial (cf. Freyd and Yetter [6] and Yetter [24]), but as shown by Altschuler [1] are not generally Vassiliev. Indeed, even the functorial invariants usually shown to “be” Vassiliev invariants are not themselves Vassiliev invariants, but rather become a sequence of Vassiliev invariants when a suitable change of variable by substitution of a power series is applied. Our theorem will be stated in terms of those functorial invariants which arise from tortile (ribbon) deformations of a rigid symmetric monoidal category.

Our setting is more general than that used for previous results of this type, and, we have reason to conjecture, may be the most general setting in which a theorem of this type can be proved. We will point to examples of knot invariants already existing in the literature to show the greater

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2 This paper has been submitted to the proceedings of the Workshop on Higher Dimensional Category Theory and Mathematical Physics, Northwestern University, March 27-28, 1997
3 It has been recently brought to the author’s attention that some aspects of this work have been discovered independently by Davydov [3a]. Davydov’s work does not however deal with “higher-order infinitesimal deformations”, and thus the full parallel with Gerstenhaber’s deformation theory is unrealized.
generality. More importantly, by proving the result in a purely categorical setting, we are able to relate the Vassiliev invariants thus obtained to a new cohomology theory which both classifies and provides the obstruction theory for deformations of braidings on tensor categories. Once the relationship between our deformation theory and the Gerstenhaber deformation theory of algebras is clearer, it is reasonable to think that we will gain an improved understanding of the relation between quantum groups and Vassiliev theory.

We begin with a discussion of the categorical machinery necessary for the statement and proof of the theorem.

2 Tortile Categories and Deformations of Tensor Categories and Functors

The setting in which our theorem is proved involves a class of categories with structure which includes both the categories of representations of quantum groups and various categories of tangles. To make this explicit:

Definition 2.1 A monoidal category $\mathcal{C}$ is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $I$, together with natural isomorphisms $\alpha : \otimes(\otimes \times 1_{\mathcal{C}}) \Rightarrow \otimes(1_{\mathcal{C}} \times \otimes)$, $\rho : \otimes I \Rightarrow 1_{\mathcal{C}}$, and $\lambda : I \otimes \Rightarrow 1_{\mathcal{C}}$, satisfying the pentagon, triangle, and bigon ($\rho I = \lambda I$) coherence conditions (cf. [17]).

A tensor category over $K$, for $K$ some field (or commutative ring) is a monoidal abelian category linear over $K$, with all $- \otimes X$ and $X \otimes -$ exact. Similarly, a semigroupal category is a category equipped with only $\otimes$ and $\alpha$ satisfying the usual pentagon.

Definition 2.2 A (strong) monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between two monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $F$ between the underlying categories, equipped with a natural isomorphism

$$\tilde{F} : F(- \otimes -) \to F(-) \otimes F(-)$$

and an isomorphism $F_0 : F(I) \to I$, satisfying the hexagon and two squares.
Figure 1: Coherence conditions for a monoidal functor
A pair of a functor $F$ and a natural isomorphism $\tilde{F}$ (without the isomorphism $F_0$ as above will be called a semigroupal functor.

**Definition 2.3** A monoidal natural transformation is a natural transformation $\phi : F \Rightarrow G$ between monoidal functors which satisfies

$$\tilde{G}_{A,B}(\phi_{A \otimes B}) = \phi_A \otimes \phi_B(\tilde{F}_{A,B})$$

and $F_0 = G_0(\phi_I)$. A semigroupal natural transformation between semigroupal functors is defined similarly.

We will be concerned with monoidal categories with additional structure.

**Definition 2.4** A braided monoidal category is a monoidal category equipped with a monoidal natural isomorphism $\sigma : \otimes \Rightarrow \otimes(tw)$, where $tw : C \times C \to C \times C$ is the “twist functor” ($tw(f, g) = (g, f)$). And satisfying

![Diagram](image-url)

Figure 2: The hexagon

A braided monoidal category is a symmetric monoidal category if the components of $\sigma$ satisfy $\sigma_{B,A}(\sigma_{A,B}) = 1_{A \otimes B}$ for all objects $A$ and $B$.

**Definition 2.5** A right dual to an object $X$ in a monoidal category $C$ is an object $X^*$ equipped with maps $\epsilon : X \otimes X^* \to I$ and $\eta : I \to X^* \otimes X$ such that the compositions

$$X \xrightarrow{\rho^{-1}} X \otimes I \xrightarrow{X \otimes \eta} X \otimes (X^* \otimes X) \xrightarrow{\alpha^{-1}} (X \otimes X^*) \otimes X \xrightarrow{\epsilon \otimes X} I \otimes X \xrightarrow{\lambda} X$$

and

$$X^* \xrightarrow{\lambda^{-1}} I \otimes X^* \xrightarrow{\eta \otimes X^*} (X^* \otimes X) \otimes X^* \xrightarrow{\Delta} X^* \otimes (X \otimes X^*) \xrightarrow{\gamma \otimes X^*} X^* \otimes I \xrightarrow{\rho} X^*$$

are both identity maps.

A left dual $^*X$ is defined similarly, but with the objects placed on opposite sides of the monoidal product.
This type of duality is an abstraction from the sort of duality which exists in categories of finite dimensional vector spaces. It is not hard to show that the canonical isomorphism from the second dual of a vector space to the space generalizes to give canonical isomorphisms \( k : (X^*)^* \to X \) and \( \kappa : (X)^* \to X \). In general, however, there are not necessarily any maps from \( X^{**} \) or \( **X \) to \( X \) (cf. \[7\`). In cases where every object admits a right (resp. left) dual, it is easy to show that a choice of right (resp. left) dual for every object extends to a contravariant functor, whose application to maps will be denoted \( f^* \) (resp. \(*f\)), and that the canonical maps noted above become natural isomorphisms between the compositions of these functors and the identity functor. Likewise, it is easy to show that \((A \otimes B)^*\) is canonically isomorphic to \(B^* \otimes A^*\), and similarly for left duals.

In the case of a braided monoidal category every right dual is also a left dual, in general in a non-canonical way (cf. \[7\`). In symmetric monoidal categories, we return to the familiar: right duals are canonically left duals. For non-symmetric braided monoidal categories, the structure will be canonical only in the presence of additional structure. As we will be concerned only with the case of categories in which all objects admit duals, we make

**Definition 2.6** A braided monoidal category \( C \) is tortile (or ribbon) if all objects admit right duals, and it is equipped with a natural transformation \( \theta : 1_C \Rightarrow 1_C \), which satisfies

\[
\theta_{A \otimes B} = \sigma_{B,A} (\sigma_{A,B} (\theta_A \otimes \theta_B))
\]

and

\[
\theta_{A^*} = \theta_A^*
\]

**Definition 2.7** A symmetric monoidal category \( C \) is rigid if all objects admit (right) duals.

For details of the canonicity of the correspondence between right and left duals in the tortile case see \[25\].

The importance of tortile categories for knot theory is provided by the extremely beautiful coherence theorem of Shum \[20\]:

**Theorem 2.8** The tortile category freely generated by a single object \( X \) is monoidally equivalent to the category of framed tangles \( \mathcal{FT} \).

To be precise \( \mathcal{FT} \) is the category whose objects are framed, oriented (i.e. signed) finite sets of points in \((0, 1)^2\), and whose arrows are framed oriented compact 1-submanifolds of \([0, 1]^3\) whose intersection with the boundary is normal and consists of finite sets of points lying in \((0, 1)^2 \times \{0\}\) (the source) and \((0, 1)^2 \times \{1\}\) (the target), modulo ambient isotopy rel boundary. Composition and \( \otimes \) are defined by pasting on the third and second coordinates respectively and rescaling. Associativity and unit transformations are given by the obvious ascending submanifolds, while the braiding is given by ascending submanifolds which pass one in front of the other, and duality is given on objects by mirror-imaging in the second coordinate, and reversing all signs, with half curves-of-rotation as structure maps (cf. \[25\]).

This theorem gives rise to the following notion:

**Definition 2.9** A functorial invariant of framed links \( P_X \) is an \( \text{End}(I) \)-valued invariant of framed links, where \( I \) is the unit object in a tortile category \( C \), obtained by choosing an object \( X \) in \( C \), and considering the restriction of the functor.
where $F$ is the free tortile category on one object generator, $G$ is the equivalence functor from Shum’s Coherence Theorem, and $F_X$ is the functor induced by freeness which maps the generator to $X$.

Numerous link invariants discovered since the mid 1980’s are functorial invariants, including the Jones polynomial, and Zariski dense sets of values for the HOMFLY and Kauffman polynomials (the values “arising from quantum groups”), as is Conway’s normalization of the classical Alexander polynomial.

The standard algebraic construction of these functorial invariants begins with the deformation of a universal enveloping algebra for a (simple) Lie algebra to produce a “quantum group”, followed by the construction of the (tortile) category of its representations.

In [25] the author gave an exposition of a theorem of Deligne which dealt with direct deformation of a tensor category (starting with a Tannakian category). We will be concerned with the deformation theory of tensor categories and functors and its application to Vassiliev theory, so we review the necessary definitions:

**Definition 2.10** An infinitesimal deformation of a $K$-linear tensor category $C$ over an Artinian local $K$-algebra $R$ is an $R$-linear tensor category $\tilde{C}$ with the same objects as $C$, but with $\text{Hom}_{\tilde{C}}(a, b) = \text{Hom}_C(a, b) \otimes_K R$, and composition extended by bilinearity, and for which the structure map(s) $\alpha$, $\rho$, and $\lambda$, and if applicable $\eta$ and $\epsilon$) reduce mod $m$ to the structure maps for $C$, where $m$ is the maximal ideal of $R$. A deformation over $K[\epsilon]/\langle \epsilon^{n+1} \rangle$ is an $n$th order deformation.

Similarly an $m$-adic deformation of $C$ over an $m$-adically complete local $K$-algebra $R$ is an $R$-linear tensor category $\tilde{C}$ with the same objects as $C$, but with $\text{Hom}_{\tilde{C}}(a, b) = \text{Hom}_C(a, b) \hat{\otimes}_K R$, and composition extended by bilinearity and continuity, and for which the structure map(s) $\alpha$, $\rho$, and $\lambda$, and if applicable $\eta$ and $\epsilon$) reduce mod $m$ to the structure maps for $C$, where $m$ is the maximal ideal of $R$. (Here $\hat{\otimes}_K$ is the $m$-adic completion of the ordinary tensor product.) An $m$-adic deformation over $K[[x]]$ is a formal series deformation.

Two deformations (in any of the above senses) are equivalent if there exists a monoidal or semigroupal functor, whose underlying functor is the identity, and whose structure maps reduce mod $m$ to identity maps from one to the other. The trivial deformation of $C$ is the deformation whose structure maps are those of $C$. More generally, a deformation is trivial if it is equivalent to a trivial deformation.

Finally, if $K = \mathbb{C}$ (or $\mathbb{R}$), and all hom-spaces in $C$ are finite dimensional, a finite deformation of $C$ is a $K$-linear tensor category with the same objects and maps as $C$, but with structure maps given by the structure maps of a formal series deformation evaluated at $x = \xi$ for some $\xi \in K$ such that the formal series defining all of the structure maps converge at $\xi$.

Similarly, we can define deformations of monoidal and semigroupal functors targeted at tensor categories by

**Definition 2.11** Given a $K$-linear tensor category $D$. An infinitesimal deformation of a monoidal (resp. semigroupal) functor $F : C \to D$ over an Artinian local $K$-algebra $R$ is a monoidal (resp. semigroupal) functor $F' : C \to \tilde{D}$ where the $R$-linear tensor category $\tilde{D}$ is an infinitesimal deformation of $D$ over $R$ and for which the structure maps $\tilde{F}'$ and $F'_0$ (as well as those for $D$) reduce
mod \(m\) to the structure maps for \(F\) (and \(D\)), where \(m\) is the maximal ideal of \(R\). A deformation over \(K[[\varepsilon]]/\langle \varepsilon^{n+1} \rangle\) is an \(n\)th order deformation.

Similarly an \(m\)-adic deformation of \(F\) over an \(m\)-adically complete local \(K\)-algebra \(R\) is a monoidal or semigroupal functor targetted at the \(R\)-linear tensor category \(\tilde{D}\) is an \(m\)-adic deformation of \(D\) and for which the structure maps reduce mod \(m\) to the structure maps for \(F\) and \(D\), where \(m\) is the maximal ideal of \(R\). (Here \(\hat{\otimes}_K\) is the \(m\)-adic completion of the ordinary tensor product.)

An \(m\)-adic deformation over \(K[[x]]\) is the \(m\)-adic completion of a formal series deformation.

Two deformations (in any of the above senses) are equivalent if there exists a monoidal or semigroupal natural isomorphism between them which reduces mod \(m\) to identity maps. The trivial deformation of \(F\) is the deformation whose structure maps are those of \(F\). More generally, a deformation is trivial if it is equivalent to the trivial deformation.

A deformation in any of the above senses will be called purely functorial whenever the target category \(\tilde{D}\) is the trivial deformation of \(D\).

In what follows, the category or functor we deform will generally have a more restrictive structure than its deformations. Thus, we will speak of a braided deformation or a tortile deformation of a rigid symmetric category or a semigroupal deformation of a monoidal functor, meaning one in which only the specified coherence conditions are satisfied.

Deligne’s theorem [25] then says

**Theorem 2.12** Any braided deformation of a Tannakian category \(\mathcal{C}\) admits a unique tortile structure, and is thus a tortile deformation.

In fact the proof given in [25] of this result will carry a more general result once it is observed that the use of intertwining matrices is unnecessary, and that it suffices to note that the map in question is the sum of the identity map with \(m\)-multiples of maps in \(\mathcal{C}\).

**Theorem 2.13** Any braided monoidal deformation of a rigid symmetric \(K\)-linear tensor category \(\mathcal{C}\) for \(K\) any field (char \(k \neq 2\)) admits a unique tortile structure, and is thus a tortile deformation.

These theorems are quite helpful, since they show that even for topological applications where the full tortile structure is required, it is sufficient to understand the structure of braided deformations.

We will also have cause to consider the notion of a reduction of a deformation.

**Definition 2.14** If \(\mathcal{C}\) is a \(K\)-linear semigroupal category, \(R\) an Artinian local ring (resp. an \(m\)-adically complete local ring), \(\tilde{C}\) a deformation (\(m\)-adic deformation) of \(\mathcal{C}\) over \(R\), and \(J\) an ideal of \(R\), then the reduction mod \(J\) of \(\tilde{C}\), denoted \(\tilde{C}/J\), is the category whose hom-spaces are \(\text{Hom}_{\tilde{C}}(X,Y) \otimes_R R/J\) with the obvious composition induced by that on \(\tilde{C}\), and structure maps obtained as the tensor product of the structure maps of \(\tilde{C}\) with \(1 \in R/J\).

**Definition 2.15** If \(\mathcal{C}\) is a \(K\)-linear semigroupal category, \(F : \mathcal{D} \to \mathcal{C}\) a semigroupal functor, \(R\) an Artinian local ring (resp. an \(m\)-adically complete local ring), \(F' : \mathcal{D} \to \tilde{C}\) a deformation (resp. \(m\)-adic deformation) of \(F\) over \(R\), and \(J\) an ideal of \(R\), then the reduction mod \(J\) of \(F'\), denoted \(F'/J\), is the monoidal functor targetted at \(\tilde{C}/J\) and structure maps obtained as the tensor product of the structure maps of \(\tilde{C}\) with \(1 \in R/J\).

We will then need the following:
Lemma 2.16 If \( \tilde{C} \) is an \( n \)th order deformation of a \( K \)-linear semigroupal category \( C \) (resp. a formal series deformation) (perhaps with additional structure, e.g., monoidal, braided or tortile), then for \( k < n \) (resp. for any \( k \in \mathbb{N} \)), \( \tilde{C}/ < \varepsilon^{k+1} > \) is a \( k \)th order deformation of \( C \) with the same additional structure. Similarly if \( F' \) is an \( n \)th order deformation of a semigroupal functor targeted at \( C \) (resp. a formal series deformation), then for any \( k < n \) (resp. any \( k \in \mathbb{N} \)), \( F'/ < \varepsilon^{k+1} > \) is a \( k \)th order deformation of \( F \). If, moreover, \( F' \) is purely functorial, so is \( F'/ < \varepsilon^{k+1} > \).

proof: It suffices to observe that the hom-spaces of the reduction are those of a \( k \)th order deformation, and that coherence conditions which hold before reduction mod \( J \) will still hold after reduction.

Until recently the notion of a deformation of a tensor category was a definition without any attendant theorems, examples coming primarily from categories of representations of a deformed Hopf algebra (quantum group). In \[3\] an exposition was given of the deformation of tensor categories without braiding in terms of a cohomology theory for tensor categories. This theory is somewhat unsatisfactory in that it is unclear whether the obstructions are of a purely cohomological nature.

For our purposes, however, it is much more interesting to consider the deformation theory of tensor functors rather than tensor categories. On the one hand, the obstruction theory will prove to be purely cohomological, and on the other, it provides the correct notion of deformation for braidings thanks to the following results of Joyal and Street \[15\]:

Definition 2.17 A multiplication on a monoidal category \( C \) is a (strong) monoidal functor \((\Phi : C \times C \to C, \tilde{\Phi}, \Phi_0)\) (usually denoted \( \Phi \) by abuse of notation) together with monoidal natural isomorphisms \( r : \Phi(\text{Id}_C, \text{I}) \Rightarrow \text{Id}_C \) and \( l : \Phi(\text{I}, \text{Id}_C) \Rightarrow \text{Id}_C \)

Theorem 2.18 In a monoidal category \( C \), a family of arrows

\[
\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A
\]

is a braiding if and only if the following define a multiplication \( \Phi \) on \( C \): \( \Phi = \otimes, \Phi_0 = \rho^{-1}_I, r = \rho, l = \lambda \) and

\[\tilde{\Phi}_{(A,A'),(B,B')} = \left[(1 \otimes \sigma) \otimes 1\right] : (A \otimes A') \otimes (B \otimes B') \longrightarrow (A \otimes B) \otimes (A' \otimes B')\]

Here, and in what follows, the notation \( [\ ] \) applied to a map in a monoidal category indicates pre- and post- composition by the unique map given by Mac Lane’s coherence theorem \[17\] as a composite of “prolongations” of the structural transformations \( \alpha, \rho, \) and \( \lambda \) so that the source and targets will be as specified. We will also apply this notation to sequences of maps, in which case, Mac Lane coherence maps may be inserted between each pair to provide a well-defined composition, and to both single maps and sequences of maps in settings where a monoidal functor is involved. In this latter setting the coherence map(s) may also involve structural transformations from the monoidal functor (see Epstein \[3\] for the theorems which justify use of this notation in the case of monoidal functors).

A converse theorem also due to Joyal and Street states

Theorem 2.19 For any multiplication \( \Phi \) on a monoidal category \( C \), a braiding \( \sigma \) for \( C \) is defined by

\[
\sigma_{A,B} = l \otimes r(\tilde{\Phi}^{-1}(\Phi(\rho^{-1}_I, \lambda^{-1})(\Phi(\lambda, \rho)(\tilde{\Phi}((r^{-1} \otimes l^{-1}))))))
\]

The multiplication obtained from \( \sigma \) via the previous theorem is isomorphic (an the obvious sense) to \( \Phi \), and if \( \tau \) is any braiding on \( C \), and \( \Phi \) is the multiplication obtained from \( \tau \) via the previous theorem, then \( \sigma = \tau \).
As noted above, the importance of these theorems for us lies in the fact that they reduce the question of studying braidings to the study of monoidal functors, and it is from this point of view that we will examine the deformation theory of braided tensor categories.

3 The Deformation Theory of Semigroupal and Monoidal Functors and Braided Monoidal Categories

We now turn our attention to the actual structure of deformations of braided monoidal categories. As we have stated above, we proceed by the indirect route of developing a deformation theory for monoidal functors, then applying it to classify deformations of a multiplication on a tensor category.

As in [3] our starting point is the observation that the coherence condition for the structure at hand is formally a cohomological condition written multiplicatively. As was the case in [3] for the structure maps of a deformation of a tensor category, if we consider the structure maps for a purely functorial first order deformation of a semigroupal functor, we find that the main coherence condition is equivalent (in the presence of the coherence for the map being deformed) to a familiar cohomological condition, or rather, to what would be a familiar condition were it not for the intervening $\lceil \rceil$’s.

In the present case, if $\tilde{F}' = \tilde{F} + F^{(1)}\epsilon$ for $F^{(1)}$ a natural transformation from $F(- \otimes -)$ to $F(-) \otimes F(-)$ and $\epsilon^2 = 0$, then the hexagon on $\tilde{F}'$ is equivalent to the condition that
\[
\lceil F^{(1)}_{A \otimes B, C} \rceil + \lceil F^{(1)}_{A, B} \otimes F(C) \rceil = \lceil F(A) \otimes F^{(1)}_{B, C} \rceil
\]
recalling that here $\lceil \rceil$ denotes “padding” with coherence maps for $F$ and the semigroupal structures, as is permitted by the results of [3]. Of course, solving for $0$ gives us the condition that $\delta(F^{(1)}) = 0$, where $\delta$ is the familiar coboundary for the bar resolution, but dressed up with $\lceil \rceil$’s on its terms, and using $\otimes$ in place of multiplication.

We now wish to describe a setting in which this “coboundary” is in fact the coboundary in a cochain complex associated to the semigroupal functor $F$. As in [3] let $n \otimes$ (resp. $\otimes^n$) denote the completely left (resp. right) parenthesized $n$-fold iterated semigroupal product. It should then be observed that all of the terms of the formula above are natural transformations from $F^{(3 \otimes)}$ to $\otimes^3(F^3)$. It is thus reasonable to make

Definition 3.1 The deformation complex of a semigroupal functor targetted at a $K$-linear tensor category $\mathcal{C}$ is the $K$-cochain complex $(X^\bullet(F), \delta)$ for which

$X^n(F) = \text{Nat}(F(n \otimes), \otimes^n(F^n))$

with coboundary $\delta$ given by the usual bar-resolution formula with $\otimes$ as multiplication, and $\lceil \rceil$ applied to all terms. Denote the cohomology of the complex by $H^\bullet(F)$.

It follows from the usual argument (and the bilinearity of the composition of natural transformations and $\otimes$ and the coherence conditions for $F$ and $\otimes$) that $\delta^2 = 0$. It is also easy to see that a semigroupal natural isomorphism which reduces to the identity mod $\epsilon$ between two first order deformations is precisely a 1-cochain which cobounds the difference between the two 2-cocycles which name the deformations. Thus we have

Theorem 3.2 There is a natural 1-1 correspondence between the first order purely functorial deformations of a semigroupal functor $F$ and the 2-cocycles of the deformation complex of $F$. Moreover, the semigroupal natural isomorphism classes of first order purely functorial deformations of $F$ are in natural 1-1 correspondence with $H^2(F)$.
Thus, $F$ admits non-trivial purely functorial deformations precisely when $H^2(F) \neq 0$. Because this rigidity is limited to purely functorial deformations, we will say a semigroupal functor $F$ is \textit{purely rigid} whenever $H^2(F) = 0$.

We can now consider the question of extending an $n^{th}$ order deformation to an $(n+1)^{st}$ order deformation, that is, suppose we have an $n^{th}$ order deformation

$$F_n = F + F^{(1)}\epsilon + \ldots + F^{(n)}\epsilon^n$$

where $\epsilon^{n+1} = 0$, and we wish to find an $n+1^{st}$ order deformation

$$F_n = F + F^{(1)}\epsilon + \ldots + F^{(n+1)}\epsilon^{n+1}$$

(for $\epsilon^{n+2} = 0 \neq \epsilon^{n+1}$) by adding another term.

It is easy to see that the required condition on $F^{(n+1)}$ is that

$$\delta(F^{(n+1)}) = \sum_{l+m=n+1,1 \leq l,m \leq n} [F_{A,B}^{(m)} \otimes F(C)(F_{A \otimes B,C}^{(l)})] - \sum_{p+q=n+1,1 \leq p,q \leq n} [F(A) \otimes F_{B,C}^{(q)}(F_{A,B \otimes C}^{(p)})]$$

Thus this right-hand side may be viewed as the obstruction to extending the $n^{th}$ order deformation to an $(n+1)^{st}$ order deformation. As in [8], we will see that this obstruction is always a 3-cocycle, and thus the obstruction to extending a purely functorial deformation by one order may be viewed as a cohomology class in $H^3(F)$.

In fact, the cochain complex associated to a semigroupal functor shares many of the properties of the Hochschild complex of an associative algebra $A$ with coefficients in $A$ which were described by Gerstenhaber [8, 9, 10]. In particular, we have two products defined on cochains:

$$- \cup - : X^n(F) \times X^m(F) \to X^{n+m}(F)$$

given by

$$G \cup H_{A_1,..,A_{n+m}} = [F(A_1) \otimes \ldots \otimes F(A_n) \otimes H_{A_{n+1},..,A_{n+m}}(G_{A_1,..,A_n} \otimes F(A_{n+1}) \otimes \ldots \otimes F(A_{n+m}))]$$

and

$$\langle -,- \rangle : X^n(F) \times X^m(F) \to X^{n+m-1}(F)$$

given by

$$\langle G,H \rangle_{A_1,..,A_{n+m-1}} = \sum (-1)^{ml} [F(A_1) \otimes \ldots \otimes F(A_l) \otimes H_{A_{l+1},..,A_{l+n}} \otimes F(A_{l+n+1}) \otimes \ldots \otimes F(A_{n+m-1})]$$

$$(G_{A_1,..,A_{l+1}\otimes \ldots \otimes A_{l+n},A_{l+n+1},..,A_{n+m-1}})$$

\textbf{Proposition 3.3} This latter product comes from a “pre-Lie system” in the terminology of Gerstenhaber [8] given by

$$\langle G,H \rangle_{A_1,..,A_{n+m-1}}^{(i)} = [F(A_1) \otimes \ldots \otimes F(A_i) \otimes H_{A_{i+1},..,A_{i+n}} \otimes F(A_{i+n+1}) \otimes \ldots \otimes F(A_{n+m-1})]$$

$$(G_{A_1,..,A_{i+1}\otimes \ldots \otimes A_{i+n},A_{i+n+1},..,A_{n+m-1}})$$

where $X^n(F)$ has degree $n - 1$.  

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proof: First, note that the ambiguities of parenthesization in the semigroupal products in this definition are rendered irrelevant by the ⌈⌉ on each term by virtue of the coherence theorem for semigroupal functors.

It is obvious that the product is given by a sum of these terms, so the content of the proposition is really that the ⟨−, −⟩′s satisfy the definition of a pre-Lie system, that is for \( G \in X^m(F) \), \( H \in X^n(F) \), and \( K \in X^p(F) \) that

\[
\langle\langle G, H \rangle^{(i)}, K \rangle^{(j)} = \begin{cases} 
\langle\langle G, K \rangle^{(j)}, H \rangle^{(i+p-1)} & \text{if } 0 \leq j \leq i - 1 \\
\langle G, \langle H, K \rangle^{(j-i)} \rangle^{(i)} & \text{if } i \leq j \leq n
\end{cases}
\]

(recall that a k-chain has degree \( k - 1 \)).

This is a simple computational check. One must remember that naturality will allow one to commute the prolongations of \( K \) and \( H \) in verifying the first case.

We can then write the obstruction equation as

\[
\delta(F^{(n+1)}) = \sum_{i=1}^{n} \langle F^{(i)}, F^{n-i+1} \rangle
\]

In particular, we have a

**Proposition 3.4** For any \( n \)th order purely functorial deformation of a semigroupal functor \( F \) targetted at a \( K \)-linear tensor category, the obstruction to extension to an \((n+1)\)st order deformation is a 3-cocycle.

proof: This follows immediately from the proof in Gerstenhaber [9] of §5, Proposition 3, since that proof depends only on the fact that the operation comes from a pre-Lie system.

In particular, if \( H^3(F) = 0 \), any first order purely functorial deformation of \( F \) can be extended to an \( n \)th order deformation for any \( n \), and thus to a formal series deformation.

The case of a monoidal functor is somewhat less clean than for semigroupal functors. Here we will only consider the case of those purely functorial deformations for which the unit isomorphism \( F_0 \) is deformed trivially, that is, is given by \( F_0 \otimes 1 \), where 1 is the unit of the ring \( R \) over which the deformation takes place. We call such deformations proper.

It appears to be possible to remove this restriction in the case of monoidal functors which are faithful and monic on objects (or equivalent to such a functor), but we will not pursue this further in this paper.

A brief consideration of the unit coherence conditions for monoidal functors shows that a proper \( n \)th order deformation of a monoidal functor must be given by a semigroupal deformation

\[
\tilde{F}' = \tilde{F}' + F^{(1)} \epsilon + \ldots + F^{(n)} \epsilon^n
\]

for which \( F^{(k)}_{A,I} = 0 \) and \( F^{(k)}_{I,A} = 0 \) for all \( k \) and all objects \( A \).

This leads to the following

**Definition 3.5** The proper deformation complex of a monoidal functor targetted at a \( K \)-linear tensor category is the \( K \)-cochain complex \((C^\bullet(F), \delta)\) for which

\[
C^n(F) = \{ \phi_{A_1, \ldots, A_n} | A_i = I \vdash \phi_{A_1, \ldots, A_n} = 0 \} \subset \text{Nat}(F^n \otimes, \otimes^n(F^n))
\]

with coboundary \( \delta \) given by the restriction of the coboundary on the deformation complex of \( F \). We denote the cohomology of this complex by \( \text{H}^\bullet(F) \).
The following proposition ensures that this definition makes sense and that the proper deformation complex has the same type of structures as the deformation complex:

**Proposition 3.6** If \( G \) and \( H \) are proper cochains, then so are \( \delta(G) \), \( G \cup H \), and \( \langle G, H \rangle^{(i)} \).

**proof:** For the latter two, observe that any \( I \) will give a factor in the expression for the product in which \( I \) occurs either as an index of \( G \) or as an index of \( H \), and thus the product will be 0. For the first, note that all terms of \( \delta(G) \) except two will be of the form \( [G] \) for an instance of \( G \) with \( I \) as one of the indices. The remaining two terms will be of the same form (though without \( I \) indices), of opposite signs, and will differ only in which indices are tensored by \( I \). Thus by the presence of the \( [\ ] \) they will be the same and will cancel.

Notice in particular that the obstructions to extending a proper deformation will always be proper cochains.

It then follows immediately that we have results corresponding to Theorem 3.2 and Proposition 3.4.

**Theorem 3.7** There is a natural 1-1 correspondence between the first order proper deformations of a monoidal functor \( F \) and the 2-cocycles of the proper deformation complex of \( F \). Moreover, the monoidal natural isomorphism classes of first order proper deformations of \( F \) are in natural 1-1 correspondence with \( H^2(F) \).

**Proposition 3.8** For any \( n \)th order proper deformation of a monoidal functor \( F \) targetted at a \( K \)-linear tensor category, the obstruction to extension to an \( n + 1 \)st order deformation is a 3-cocycle.

We may now speak of a monoidal functor as properly rigid if \( H^2(F) = 0 \), and observe that if \( H^3(F) = 0 \) then any first order proper deformation can be extended to a formal series deformation.

Now, consider the case of a multiplication on a tensor category \( C \). The requirement that there exist right and left unit transformations for the multiplication imposes an additional condition on the admissible deformations. However, these also are readily understood in terms of the cohomology theory.

In particular, notice that if \( F \) and \( G \) are a composable pair of monoidal (or semigroupal) functors, a deformation of either leads to an induced deformation of the composite (provided the appropriate target categories are \( K \)-linear tensor categories). More precisely, if \( \tilde{F} \) is a deformation of \( F \), then \( \tilde{F}(G) \) is a deformation of \( F(G) \), while if \( \tilde{G} \) is a deformation of \( G \), and both the source and target of \( F \) are \( K \)-linear tensor categories, then \( \tilde{F}_{\text{trivial}}(\tilde{G}_1) \) is a deformation of \( F(G) \), where \( \tilde{F}_{\text{trivial}} \) denotes the trivial deformation of \( F \) and \( \tilde{G}_1 \) denotes the \( R \)-linear extension of \( \tilde{G} : C \rightarrow \tilde{D} \) to \( \tilde{C} \). In terms of induced deformations we then have

**Theorem 3.9** If \( \Phi \) is a multiplication on a tensor category \( C \), and \( \Phi' \) is a proper deformation of \( \Phi \) as a monoidal functor then \( \Phi' \) is a multiplication on \( \tilde{C} \) precisely when the induced deformations \( \Phi'(-, I) \) and \( \Phi'(I, -) \) are trivial.

**proof:** If the induced deformations are trivial, we can compose the trivializing natural isomorphism with the \( R \)-bilinear extension of \( r \) (resp. \( l \)) to obtain the deformed \( r \) (resp. \( l \)). Conversely, if there are deformed \( r \) and \( l \), their composition with the \( R \)-bilinear extension of the inverse to the undeformed \( r \) and \( l \) give the respective trivializations.
4 Vassiliev Invariants for Framed Links

It is usual to discuss Vassiliev theory in terms of unframed oriented knots and links (cf. [2], [21], [23]). We will, however, remain in the setting most natural for functorial invariants (and incidentally most closely connected to 3- and 4-manifold topology), that of framed links.

Following Goryunov [11] one can regard framed knots (and links) not as ordinary knots (and links) with additional structure, but as equivalence classes of mappings from open annular neighborhoods \( U \) of \( S^1 \) (or disjoint unions of such) into \( \mathbb{R}^3 \). Observe that any such mapping \( g : U \to \mathbb{R}^3 \) induces a mapping \( Tg \) from \( i^*(T\mathbb{R}^2) \) to \( T\mathbb{R}^3 \). (Here \( i \) is the inclusion of \( S^1 \) or a disjoint union of \( S^1 \)’s into \( U \).)

For ease in the link setting, we need to specify a countable family of disjoint circles: say \( S_n = \{ z \mid |z - 3n| = 1 \} \subset \mathbb{C} \), then consider disjoint annular neighborhoods of \( S_1 \cup S_2 \cup \ldots \cup S_k \) when dealing with \( k \)-component links.

**Definition 4.1** Two mappings \( g_i : U_i \to \mathbb{R}^3 \) \( i = 1, 2 \) are equivalent if the \( U_i \)’s are annular neighborhoods of the same \( S_1 \cup \ldots \cup S_k \) and the mappings \( Tg_i : i^*(T\mathbb{R}^2) \to T\mathbb{R}^3 \) coincide.

The space of (possibly singular) framed links is the space of all \( C^\infty \) mappings of \( U \)’s modulo this equivalence relation. We denote it by \( \Omega_f \). The subspace of equivalence classes of mapping of neighborhoods of \( S_1 \cup \ldots \cup S_k \) is the space of (possibly singular) framed links of \( k \) components, and will be denoted \( \Omega_f(k) \). Now consider the subspace \( \mathcal{O}_f \) of all (equivalence classes of) mappings such that \( Tg \) is an embedding.

**Definition 4.2** A (non-singular) framed link is a connected component of \( \mathcal{O}_f \). A connected component of \( \mathcal{O}_f(k) = \mathcal{O}_f \cap \Omega_f(k) \) is a (non-singular) framed link of \( k \) components.

First, note that we will often drop the adjective “non-singular” to match the usual usage in knot theory (as in earlier sections). Second, the designation of these maps as non-singular implicitly identifies the element of \( \Omega_f \setminus \mathcal{O}_f \) as singular. We denote the discriminant locus \( \Omega_f \setminus \mathcal{O}_f \) by \( \Sigma_f \). As observed in Goryunov [11] the discriminant is the union of two hypersurfaces, one on which the framing degenerates, denoted \( \Sigma'_f \), and one on which the disjoint union of circles is not embedded, denoted \( \Sigma''_f \).

The minimal degenerations of each type are illustrated in Figure 3. We denote the subspace of \( \Sigma_f \) in which there are exactly \( n \) degenerations of either type by \( \Sigma_{f,n} \), and its intersection with \( \Omega_f(k) \) by \( \Sigma_{f,n}(k) \).

As in Goryunov [11], we co-orient the finite codimensional strata of \( \Sigma_f \) by the local prescriptions given in Figure 3, and give a Vassiliev type prescription for the extension of invariants of framed knots to singular framed knots as in Figure 4.

**Definition 4.3** A Vassiliev invariant of links (resp. links of \( k \) components) is a locally constant function on \( \mathcal{O}_f \) (resp. \( \mathcal{O}_f(k) \)) whose extension according to the prescription of Figure 4 vanishes on \( \Sigma_{f,n+1} \) (resp. \( \Sigma_{f,n+1}(k) \)) for some \( n \), in which case the invariant is said to be of type \( \leq n \). The invariant is of type \( N \) when \( N \) is the minimal such \( n \).

5 Deformations and Vassiliev Invariants

We are now in a position to state our main theorem:
Figure 3: Coörienting the finite codimensional strata

\[ V\left( \begin{array}{c}
\begin{array}{cc}
\vdots
\end{array}
\end{array}\right) - V\left( \begin{array}{c}
\begin{array}{cc}
\vdots
\end{array}
\end{array}\right) = V\left( \begin{array}{c}
\begin{array}{cc}
\vdots
\end{array}
\end{array}\right) \]

\[ V\left( \begin{array}{c}
\begin{array}{cc}
\vdots
\end{array}
\end{array}\right) - V\left( \begin{array}{c}
\begin{array}{cc}
\vdots
\end{array}
\end{array}\right) = V\left( \begin{array}{c}
\begin{array}{cc}
\vdots
\end{array}
\end{array}\right) \]

Figure 4: Vassiliev type extension formulae

**Theorem 5.1** Let \( \mathcal{C} \) be any \( K \)-linear rigid symmetric monoidal category, and let \( \tilde{\mathcal{C}} \) be any \( n \)-th order tortile deformation of \( \mathcal{C} \). For any object \( X \) of \( \tilde{\mathcal{C}} \), let \( V_X \) denote the functor from \( FT \) to \( \tilde{\mathcal{C}} \) induced by Shum’s Coherence Theorem. Then \( V_X \) restricted to \( \text{End}(I) \), regarded as the set of framed links, is a \( K[\epsilon]/\langle \epsilon^{n+1}\rangle \)-valued Vassiliev invariant of type \( \leq n \), and is, moreover, multiplicative under disjoint union.

From this will follow, almost as a corollary

**Theorem 5.2** Let \( \mathcal{C} \) be any \( K \)-linear rigid symmetric monoidal category, and let \( \tilde{\mathcal{C}} \) be any \( n \)-th order tortile deformation of \( \mathcal{C} \) (resp. formal series deformation of \( \mathcal{C} \)). For any object \( X \) of \( \tilde{\mathcal{C}} \), let \( V_X \) denote the functor from \( FT \) to \( \tilde{\mathcal{C}} \) induced by Shum’s Coherence Theorem, and let \( V_{X,k} \) denote the \( K \)-valued framed link invariant which assigns to any link the coefficient of \( \epsilon^k \), for \( k = 0, \ldots, n \), (resp. for \( k \in \mathbb{N} \)). Then \( V_{X,k} \) is a Vassiliev invariant of type \( \leq k \).

The proof of Theorem 5.1 is quite simple and similar to previous proofs of similar results:

The key is to observe that the bilinearity of composition in \( \tilde{\mathcal{C}} \) allows us to use the Vassiliev prescription to extend the functor \( V_X \) from \( FT \) to a larger category of singular framed tangles, \( \bar{FT} \), whose maps are isotopy classes of framed tangles with finitely many degeneracies of either of the two basic types.
Consider a singular framed link with \(n + 1\) degeneracies (of either type). Now, we can represent the framed link as a composition of singular framed tangles, each of which has at most one degeneracy, crossing, framing twist, or extremum. Now the value of the extended functor on such a tangle with a degeneracy of the first type (framing degeneracy) is a monoidal product of identity maps with \(\theta_X - \theta^{-1}_X\) (or its dual), while the value on such a tangle with a degeneracy of the second type is a monoidal product of identity maps with \(\sigma_{X,X} - \sigma^{-1}_{X,X}\).

Now, observe that

\[
\theta_X - \theta^{-1}_X \in \text{Hom}_C(X, X) \otimes \epsilon >
\]

and

\[
\sigma_{X,X} - \sigma^{-1}_{X,X} \in \text{Hom}_C(X \otimes X, X \otimes X) \otimes \epsilon >.
\]

It follows from the bilinearity of composition in \(\hat{C}\) that the composite representing the framed link as an element of \(\text{End}(I)\) lies in \(\text{End}_C \otimes \epsilon^{n+1} = 0\), thus showing \(V_X\) to be Vassiliev of type \(\leq n\). Multiplicativity follows from functoriality.

Theorem 5.2 follows from the Theorem 5.1, Lemma 2.16, and the following lemma, the proof of which is a trivial exercise:

**Lemma 5.3** If \(V\) is an \(A\)-valued Vassiliev invariant, and \(f: A \to B\) is a linear map (\(A\) and \(B\) here are abelian groups) then \(f(V)\) is a \(B\)-valued Vassiliev invariant.

On general principles these results are more general than previous results of this form: our initial category can be any \(K\)-linear rigid symmetric monoidal category for any field \(K\), not just the category of representations of a (semi)simple Lie algebra. In particular, this means that the invariants obtained from any representation of a \(q\)-deformed universal enveloping algebra for a super-Lie algebra (cf. [18]) give rise to sequences of Vassiliev invariants for links of any number of components when we set \(q = e^h\) (or any other formal series for that matter) and consider the coefficients of \(h^n\). (The observation about using other series, of course applies in the ordinary Lie algebra case as well.) Likewise the coefficients of any \(h^n\) in invariants arising from a representation of a multiparameter quantum group (cf. [4], [13], [22]) and any instantiation of the deformation parameters by power series will be Vassiliev invariants (even in the case of quasi-Hopf deformations). In fact, by repeating the arguments given above for quotients of \(K[[x]]\) by nilpotent ideals in the case of \(K[[h_1, \ldots, h_n]]\), one can show that the coefficients of any monomial in \(h_1, \ldots, h_n\) in the invariant associated to a representation of a multiparameter quantum group and any instantiation of the \(i^{th}\) deformation parameter by a power series in \(h_i\) are Vassiliev invariants.

### 6 Questions Raised

The results contained herein are only partial. In order to fully relate Vassiliev theory to the deformation theory of braided monoidal categories, it will be necessary to give a good accounting of that deformation theory in the non-proper case. The difficulties here are two-fold. First is the removal of the restriction that the unit isomorphism not be deformed, and second is the removal of the purely functorial restriction.

We thus ask:

1. Is there a good cohomological deformation theory for purely functorial (but not necessarily proper) deformations of monoidal functors?
2. Can the deformation theory of this work be combined in a satisfactory way with the deformation theory of monoidal categories given in [3] to provide a deformation theory for (not necessarily purely functorial) deformations of monoidal functors?

3. Even a satisfactory answer to question 2. will not suffice to give a complete deformation theory for braided monoidal categories, since in 2. only the target is deformed, while a deformation of the structure maps of the underlying category, induce deformations of the structure maps of both the source and the target of the multiplication. Thus, we ask: Is there a satisfactory cohomological deformation theory for unrestricted deformations of multiplications on monoidal categories?

Likewise, the construction given here of Vassiliev invariants from deformations of monoidal categories leads to a series of questions regarding the converse problem: which Vassiliev invariants arise from functorial invariants?

Chief among these are various precise formulations of that basic question:

1. Given a $K$-valued Vassiliev invariant $v_n$ of type $n$ of framed knots, when do there exist Vassiliev invariants $v_0, ..., v_{n-1}$ of types $0, ..., n-1$ respectively, such that the $K[\eta]/<\eta^{n+1}>$-valued invariant

$$v_0 + v_1 \eta + ... + v_n \eta^n$$

is the restriction to framed knots of an invariant defined by an object in some tortile deformation of a $K$-linear rigid symmetric monoidal category?

2. Given a $K$-valued Vassiliev invariant $v$ of type $n$ of $m$-component links,

(a) When does $v$ extend to a Vassiliev invariant of links as defined above?

(b) When do there exist Vassiliev invariants of lower orders so that the generating function as in question 1. is the invariant defined by an object in some tortile deformation of a $K$-linear rigid symmetric monoidal category?

3. Given a Vassiliev invariant of links as described above, when can it be completed as in questions 1. and 2. by adding lower order Vassiliev invariants to give a functorial invariant?

A partial result toward answers to these three questions can be read off from the results above, namely, a necessary condition for a Vassiliev invariant of framed links to arise as a series coefficient in a tortile deformation of a symmetric tensor category:

**Theorem 6.1** If $v$ is a Vassiliev invariant of (framed) links arising as the coefficient of $\epsilon^n$ in an $n^{th}$ order tortile deformation of a rigid symmetric tensor category, then there exist Vassiliev invariants $v_k$ of (framed) links $k = 1, ..., n$ (with $v_n = v$) such that

$$v(K \coprod L) = v_0(K)v_n(L) + v_1(K)v_{n-1}(L) + ... + v_n(K)v_0(L)$$

and each $v_i$ is of type $\leq i + 1$, where $\coprod$ denotes separated union of links.

In terms of Vassiliev invariants for links of a fixed number of components, we can similarly see
Theorem 6.2 If \( v \) is a Vassiliev invariant of (framed) links of \( N \)-components arising as the coefficient of \( \epsilon^n \) in a tortile deformation of a rigid symmetric tensor category, then for all \( M = 1, \ldots, N \) and \( i = 0, \ldots, n \) there exist Vassiliev invariants \( v_{M,i} \) of \( M \) component (framed) links of type \( \leq i \) such that for all \( K \coprod L \) separated unions of links, with \( K \) having \( M \) components and \( L \) having \( N - M \) components, we have

\[
v(K \coprod L) = v_{M,0}(K)v_{N-M,n}(L) + v_{M,1}(K)v_{N-M,n-1}(L) + \ldots + v_{M,n}(K)v_{N-M,0}(L)
\]

Finally, there are questions of deeper relationships between the cohomological deformation theories of of monoidal categories and of (bi)algebras and Vassiliev theory:

1. What is the relationship between the deformation theory of braided monoidal categories and the original cohomological formulation of Vassiliev invariants [23]?

2. What relationship is there between the Vassiliev invariants arising from the tortile deformations of a particular symmetric monoidal category and the structure of the category itself?

3. What is the relationship between the deformation theory of a (bi)algebra \( A \) and that of monoidal functors associated to the algebra (in particular, the forgetful functor

\[
U : \text{Rep}(A) \to K - \text{vect},
\]

and for (quasi-)triangular \( A \) the tensor product as a monoidal functor with structural transformations as induced by the braiding or symmetry as in Theorem 2.18?
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