A LOCAL RELATIVE TRACE FORMULA FOR $F^\times \backslash SL_2(F)$

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Abstract. In this note, we derive explicitly the local relative trace formula for the symmetric space $F^\times \backslash SL_2(F)$ at the level of Lie algebras, where $F$ is a $p$-adic field of residue characteristic greater than two and characteristic zero. This is perhaps one of the simplest non-trivial analogs of the trace formula, and also a motivating example for the author’s work (in preparation) on the relative trace formula.

1. Introduction

1.1. Preamble. The relative trace formula is a tool for studying harmonic analysis on reductive symmetric spaces in the same way that Arthur’s trace formula is a tool for studying harmonic analysis on reductive groups. It was introduced as a variant of the Kuznetsov trace formula (see [4]) and then studied further by Jacquet, Lai, and Rallis in the article [6]. Ever since, it has generated a large amount of work, some of which is surveyed in [5], but it is still far from completely developed. The goal of this article is to provide an explicit motivating example for this extension of the local trace formula to symmetric spaces at the level of Lie algebras.

More precisely, the local trace formula of Arthur (see [1]) has already been used by Waldspurger in [8] to derive a local trace formula for a Lie algebra $g$. This formula has the shape

$$J(f_1, f_2) = J(\hat{f}_1, \hat{f}_2)$$

where $J(\phi_1, \phi_2)$ is an expression written in terms of certain (orbital) integrals of test functions $\phi_1$ and $\phi_2$ that are locally constant and compactly supported on $g$; both sides of this identity have the same general form. This formula provides a technically simpler analog of Arthur’s local trace formula that nevertheless exhibits much of its structure. For example, one encounters weighted orbital integrals on $g$ and essentially the same weights. In addition, it is also a powerful tool in harmonic analysis, and already in the original article, [8], Waldspurger applies this identity to extend earlier work of Harish-Chandra. For these reasons, one would very much like to generalize this identity from reductive groups to reductive symmetric spaces.

The case of $F^\times \backslash SL_2(F)$ is interesting for two reasons: first, it provides one of the simplest concrete analogs of the local trace formula, having only a handful of terms, each with fairly transparent structure; second, it provides a template for the more general relative Lie algebra trace formula, in which one can already see the need for a second truncation procedure, which supplements Arthur’s techniques. One might reasonably expect valuable insight into the development of the relative trace formula on the group from this procedure.

We will work over a $p$-adic field $F$ of residue characteristic greater than two and characteristic zero. In order to state this special case, we need some notation. Let $\mathcal{O}$ be the ring of integers of $F$, let $\nu_\varpi$ be a valuation of $F$, and fix a uniformizing element $\varpi$. Let $\mathcal{K}$ denote a set of representatives of the fourth-power classes of $F^\times$, which we assume to contain 1, and let
$K' := K \cap (F^x)^2$. Set
\[
t_\gamma := \left\{ \left( \begin{array}{cc} b & b\gamma \\ b & b \end{array} \right) : b \in F \right\}
\]
and $T_\gamma = Z_G(t_\gamma)$, the centralizer of $t_\gamma$ in $G$. Let $A_\gamma$ be the $F$-split part of the torus $T_\gamma$. Set
\[
t_\gamma := \left\{ \left( \begin{array}{cc} h & -h \\ h & -h \end{array} \right) : h \in F \right\}
\]
and $h_\epsilon := \left\{ \left( \begin{array}{cc} h & -h \\ h & -h \end{array} \right) : \nu_\omega(h) \geq \nu_\omega(\epsilon) \right\}$
for a new truncation parameter in $F^x$ that we call $\epsilon$. The complement of $h_\epsilon$ in $h$ will be written $h_c$. We will also write
\[
\tilde{f}(X) := \int_K \hat{f}(k^{-1}Xk) \, dk
\]
for a Haar measure $dk$ on $K$, normalized so that $K$ has measure one.

Here is the formula we will calculate:

**Proposition 1.1.** Let $f$ be a locally constant, compactly supported function on the Lie algebra $\mathfrak{sl}_2$ of trace zero matrices. Then there exists an $\epsilon \in F^x$, depending on $f$, for which there is an integral identity,
\[
\sum_{\gamma \in K-K'} \frac{1}{|\mu_4(F)|} \int_{t_\gamma} |\gamma b| \int_G f \left( g^{-1} \left( \begin{array}{cc} b & b\gamma \\ b & b \end{array} \right) g \right) \, dg \, db \\
+ \frac{1}{2} \int_{t_1} |b| \int_{A_1 \setminus G} f \left( g^{-1} \left( \begin{array}{cc} b & b \\ b & b \end{array} \right) g \right) \left( 1 - 2H_B(g) + 2H_{\hat{B}}(g) \right) \, dg \, db \\
= \int_{F^x} \tilde{f}(0 \ 2n) (1 + \nu_\omega(n) - \nu_\omega(\epsilon)) \, dn \quad + \quad \frac{1}{1 - |\omega|} \int_{H \setminus G} \hat{f}(g^{-1}Xg) \, dg \, dX
\]
for measures $dg$ defined in terms of the Haar measures on $G$ and $A_1$, normalized so that $K$ and $A_1 \cap K$ respectively have measure one.

This is the analog of the local trace formula for the symmetric space $F^x \setminus SL_2(F)$ at the level of Lie algebras. Notice that there appears on the right hand side an integral over the truncated set $h_c$. This is because that particular integral would not converge if taken over the entire Lie algebra $h$. Instead, we have had to remove the divergent part, and absorb it into the weight factor in the first term on the right hand side.

Because the set of fixed points of the involution $\theta$ is so simple in this case, it is not very difficult to truncate the set $h$ correctly, so that it combines nicely into a weighted nilpotent integral. In higher rank groups, however, this becomes significantly more demanding, although the author has succeeded in carrying out this procedure for $GL(n,F) \setminus Sp(2n,F)$, a family of symmetric spaces in which this difficulty is perhaps most fully manifest.

**1.2. Outline.** In section 2 we set the stage for this formula more explicitly. In section 3 we pin down the first truncation procedure, which requires the choice of a compactly supported and locally constant function on the symmetric space. In section 4 and section 5 we develop one side of the trace formula and then the other. But we would like to draw attention to section 5, in which one sees for the first time a second truncation procedure which remedies a form of divergence that does not appear in the Lie algebra analog of the original local trace formula. Finally, in section 6 we will combine these developments into a pair of identities, one for the symmetric space $F^x \setminus SL_2(F)$ and another that follows from the local relative trace formula on the symmetric space $\{1\} \setminus F^x$. 
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2. **Context**

The $F$-points $G$ of the algebraic group $G := SL_2$ can be written in the following way:

$$SL_2(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\} \subset GL_2(F).$$

The involution

$$\theta := \text{Inn} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on $GL_2(F)$ preserves $SL_2(F)$ and is defined over $\mathcal{O}$. As a result, it preserves the maximal compact subgroup $K := G(\mathcal{O})$ and acts on the quotient $G/K$. The fixed points of $\theta$ in $G$ will be written $H$. For this involution, $H$ is the set of diagonal matrices with determinant 1:

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \subset SL_2(F).$$

The abelian group $H$ is an $F$-split maximal torus of $SL_2(F)$.

Let $g = sl_2$ be the Lie algebra of $SL_2(F)$ and let $h$ be the Lie algebra of $H$, so that $h$ is a subalgebra of $g$. The Killing form provides an orthogonal decomposition of $g$:

$$g = h \oplus h^\perp$$

where $h^\perp$ is a vector subspace of $g$ that is not a subalgebra. We can express this vector space explicitly in the following way:

$$h^\perp = \left\{ \begin{pmatrix} b \\ c \end{pmatrix} \right\} \subset SL_2(F).$$

Note that this set identifies in a natural way with the tangent space at the identity coset of the symmetric space $H\backslash G$.

Let $f$ be a locally constant, compactly supported function on the Lie algebra $g$, and let $C_c^\infty(g)$ denote the set of such functions. The starting point for the relative Waldspurger trace formula is this variant of the Plancherel identity:

$$\int_{h} \hat{f}(X) dX = \int_{h^\perp} f(X) dX.$$

Because the Fourier transform commutes with the coadjoint action of $G$ on $C_c^\infty(g)$, we can adjust this identity so that each side is a function on $G$:

$$\int_{h} \hat{f}(g^{-1}Xg) dX = \int_{h^\perp} f(g^{-1}Xg) dX.$$

The adjoint action of $H$ preserves $h$ and the Killing form, which means that it necessarily preserves $h^\perp$ as well. We can therefore interpret each side of this identity as a function on the noncompact space $H\backslash G$. After introducing a compactly supported function $\phi(g)$ to guarantee convergence, we can integrate both sides of this identity over $H\backslash G$ to obtain

$$\int_{H\backslash G} \phi(g) \int_{h} \hat{f}(g^{-1}Xg) dX dg = \int_{H\backslash G} \phi(g) \int_{h^\perp} f(g^{-1}Xg) dX dg.$$
The bulk of the development of the trace formula consists in expressing both sides of this identity in terms of possibly weighted orbital integrals.

We proceed in three stages. First, we will choose the function $\phi(g)$ so that it correctly extends Arthur’s truncation procedure. Second, we will develop the right-hand side of this identity, using techniques of Arthur and Harish-Chandra. Third, we will compute the left-hand side explicitly, by breaking up the integral over $\mathfrak{h}$ according to a second truncation procedure.

3. The Choice of $\phi$

Let $A$ denote the torus
$$\left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 - b^2 = 1 \right\}.$$ Because the characteristic polynomial of any element in this torus splits over $F$, this is an $F$-split torus of dimension 1. Because $G$ has rank 1, this torus is also maximal. Fix a Borel subgroup $B$ that contains $A$. Then
$$X_*(A)_{\text{dom}} \cong \mathbb{Z}_{\geq 0}$$
where $X_*(A)_{\text{dom}}$ denotes the set of coweights of $A$ that are positive with respect to the Borel subgroup $B$. In this context, we can define a map given by the Cartan decomposition:
$$\text{Cartan} : G \rightarrow K \backslash G/K \cong X_*(A)_{\text{dom}} \cong \mathbb{Z}_{\geq 0}.$$ This map is proper, and so the preimage of the union of a finite number of positive integers is always a compact set.

We choose a positive integer $\mu$, which is called the truncation parameter. Following Arthur’s truncation procedure for his local trace formula, we associate to this number a truncation function
$$\bar{\omega}(g, \mu) := \begin{cases} 1 & \text{Cartan } \theta(g)^{-1} g \leq \mu \\ 0 & \text{otherwise}. \end{cases}$$ A technical point is that the function
$$\tau : g \mapsto \theta(g)^{-1} g$$
sends $H \backslash G$ into $G$ and is a closed immersion (as in [7]), which implies that $\bar{\omega}$ is compactly supported and locally constant as a function on the symmetric space $H \backslash G$. It is this function that we use to truncate our trace formula: $\phi(g) = \bar{\omega}(g, \mu)$.

4. The $\theta$-split Side

4.1. The Weyl Integration formula for $\mathfrak{h}^\perp$. In this subsection, we provide an integration formula that replaces an integral over $\mathfrak{h}^\perp$ with an iterated integral: the first over the $H$-orbits of elements in $\mathfrak{h}^\perp$ and the second over a chosen set of representatives of these orbits.

Almost every element of $\mathfrak{h}^\perp$ can be written in the following form:
$$\begin{pmatrix} a \gamma \\ a \end{pmatrix}.$$ By conjugating this matrix by elements of $H$, we can adjust $\gamma$ so that it belongs to a chosen set of representatives $\mathcal{K}$ for the fourth-power-classes of $F^\times$. In other words, off a set of measure 0, every element of $\mathfrak{h}^\perp$ is $H$-conjugate to a unique element in
$$t_\gamma := \left\{ \begin{pmatrix} a \gamma \\ a \end{pmatrix} : a \in F \right\}.$$
for some $\gamma \in \mathcal{K}$. Let $\mu_4(F)$ be the set of fourth roots of unity in $F$. Then the integral over $\mathfrak{h}^\perp$ is

$$\int_{\mathfrak{h}^\perp} f(g^{-1}Xg)\,dX = \int_{F^2} f \left( g^{-1}\begin{pmatrix} c & b \\ b a^{-2} & g \end{pmatrix} \right) \,db\,dc$$

$$= \sum_{\gamma \in \mathcal{K}} \frac{1}{|\mu_4(F)|} \int_{F^2} |\gamma b| f( g^{-1}\begin{pmatrix} b a^2 & \gamma b \\ a & a^{-1} \end{pmatrix}^{-1} \begin{pmatrix} b & \gamma b \\ 1 & 1 \end{pmatrix} g) \,\frac{da}{|a|} \,db$$

$$= \sum_{\gamma \in \mathcal{K}} \frac{1}{|\mu_4(F)|} \int_{F^2} |\gamma b| f \left( g^{-1}\begin{pmatrix} c & b \gamma g \\ b & a \end{pmatrix} \right) \,\frac{da}{|a|} \,db$$

where we allow the integrand to be undefined on a set of measure 0. This can be written in the following way:

$$\int_{\mathfrak{h}^\perp} f(g^{-1}Xg)\,dX = \sum_{\gamma \in \mathcal{K}} \frac{1 - |\omega|}{|\mu_4(F)|} \int_{\lambda_\gamma} |\gamma b| \int_{H} f \left( g^{-1}h^{-1}\begin{pmatrix} b & \gamma b \\ 1 & 1 \end{pmatrix} \right) \,dh \,db$$

where

$$dh = \frac{1}{1 - |\omega|} \,\frac{da}{da}$$

has been normalized so that $H \cap K$ has measure one.

Because this is a special case of a generalization of the Weyl integration formula to reductive symmetric spaces (see [3]), we will refer to this identity as the Weyl integration formula.

4.2. The General Expansion. For the choice of truncation function made in section 3, we will consider the expression

$$\int_{H\backslash G} \bar{\omega}(g, \mu) \int_{\mathfrak{h}^\perp} f(g^{-1}Xg)\,dX \,dg$$

which we call the $\theta$-split side. Our goal will be to develop this expression into a sum of (orbital) integrals, each integrated over certain Cartan subalgebras of $\mathfrak{g}$ that contain only elements in $\mathfrak{h}^\perp$.

Expanding the integral over $\mathfrak{h}^\perp$ with the Weyl integration formula, introducing the truncation function $\bar{\omega}(g, \mu)$, and integrating this expression over $H\backslash G$ yields another expression for the $\theta$-split side:

$$\sum_{\gamma \in \mathcal{K}} \frac{1 - |\omega|}{|\mu_4(F)|} \int_{\lambda_\gamma} |\gamma b| \int_{G} f \left( g^{-1}\begin{pmatrix} b & \gamma \mu \\ 1 & 1 \end{pmatrix} \right) \bar{\omega}(g, \mu) \,dg \,db$$

$$= \sum_{\gamma \in \mathcal{K}} \frac{1 - |\omega|}{|\mu_4(F)|} \int_{\lambda_\gamma} |\gamma b| \int_{A_\gamma} f \left( g^{-1}\begin{pmatrix} b & \gamma \mu \\ 1 & 1 \end{pmatrix} \right) \int_{A_\gamma} \bar{\omega}(ag, \mu) \,da \,dg \,db$$

where $A_\gamma$ is the maximal $F$-split subtorus of $T_\gamma := Z_G(t_\gamma)$.

Definition 4.1. The inner integral

$$\omega_\gamma(g, \mu) := \int_{A_\gamma} \bar{\omega}(ag, \mu) \,da = \text{meas}_{A_\gamma}\{a : \text{Cartan } \tau(ag) \leq \mu\}$$

is called the weight factor (associated to $\gamma$).
Notice that when $\gamma$ is not a square, the torus $T_\gamma$ contains a discrete set of $F$-split elements, and so $A_\gamma$ is trivial. In this case, the weight factor associated to $\gamma$ is simply $\bar{\omega}(g, \mu)$. When $\gamma$ is a square, however, these weight factors are slightly more complicated.

Suppose that $\gamma$ belongs to $K' = K \cap (F^\times)^2$. We start by relating $A_\gamma$ to the maximal $F$-split torus $A = A_1$, where 1 represents the trivial fourth-power-class of $F^\times$. If $\sqrt{\gamma}$ represents an element of $K$ whose square maps to the same element as $\gamma$ in $F^\times/(F^\times)^4$, then we can set

$$r_\gamma := \begin{pmatrix} 1+\sqrt{\gamma} & \sqrt{\gamma} \\ \sqrt{\gamma} & 1+\sqrt{\gamma} \end{pmatrix}$$

so that

$$r_\gamma^{-1}A_\gamma r_\gamma = A$$

and

$$\tau(r_\gamma) = r_\gamma^{-1}\theta(r_\gamma) = \begin{pmatrix} 1+\sqrt{\gamma} & \sqrt{\gamma} \\ \sqrt{\gamma} & 1+\sqrt{\gamma} \end{pmatrix} \in A$$

which is sent to the element $\pm \nu(\sqrt{\gamma})$ by the map $A \to X\ast(A)_{dom} \cong \mathbb{Z}_+$. We consider the following part of the $\theta$-split side:

$$\sum_{\gamma \in K'} \frac{1 - |\omega|}{|\mu_4(F)|} \int_{A'\cap G} g^{-1} \left( \begin{smallmatrix} b \\ \gamma b \end{smallmatrix} \right) \omega_\gamma(g, \mu) dg db.$$

The elements $r_\gamma$ allow us to change variables so that this expression becomes

$$\frac{1 - |\omega|}{|\mu_4(F)|} \int_{A'\cap G} f \left( \begin{smallmatrix} b \\ \gamma b \end{smallmatrix} \right) \left( \sum_{\gamma \in K'} \omega_\gamma(r_\gamma g, \mu) \right) dg db$$

(we have also changed variables by the map $b \mapsto b\gamma^{-\frac{1}{2}}$). The inner sum acts as a new weight factor, and we may write it in the following way:

$$\sum_{\gamma \in K'} \omega_\gamma(r_\gamma g, \mu) := \sum_{\gamma \in K'} \int_{A_\gamma} \bar{\omega}(a \gamma g, \mu) da = \sum_{\gamma \in K'} \int_A \bar{\omega}(r_\gamma ag, \mu) da.$$

Our goal for the remainder of this section will be to compute this sum explicitly.

### 4.3. Computation of the Weight Factor

Recall that $\bar{\omega}(r_\gamma ag, \mu)$ is defined by the inequality Cartan $\tau(r_\gamma ag) \leq \mu$. This condition can be clarified by appealing to a special case of Arthur’s key geometric lemma, provided that $\mu$ is bigger than some integer that depends on $g$.

We first require some notation. Let $g = tnk$ be the Iwasawa decomposition of $g$, where $t$ is in $A \cong F^\times$, $n$ is in the unipotent radical $N \cong F$ of $B$, and $k$ is in $K$. Recall the function $H_B$ on $G$, which is defined to be $H_B(g) := \nu(\omega(t))$. We will also set $d(g) := \max \{|H_B(g)|, |H_B(g)|\}$. 

Lemma 4.2. There is a constant $c$ such that when $\mu > c d(g)$, the inequality
\[ \text{Cartan } \tau(r,ag) \leq \mu \]
is equivalent to
\[ -2H_B(g) - \mu \leq \nu_\varpi(\sqrt{\gamma}) + 2\nu_a \leq \mu - 2H_B(g). \]

Proof. First write
\[ \text{Cartan } \tau(r,ag) = \text{Cartan } (\theta(n)^{-1}\tau(r) a^2 t^2 n). \]
This is easier to compute if we temporarily change basis so that $A$ is the set of diagonal matrices. Write
\[ \tau(r) a^2 = \begin{pmatrix} s \\ s^{-1} \end{pmatrix} \text{ and } t^2 = \begin{pmatrix} r \\ r^{-1} \end{pmatrix} \]
and suppose that $\nu_\varpi(s)$ is positive. Then
\[
\text{Cartan } (\theta(n)^{-1}\tau(r) a^2 t^2 n) = \text{Cartan } \begin{pmatrix} 1 \\ -n \\ 1 \end{pmatrix} \begin{pmatrix} sr \\ (sr)^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ n \\ 1 \end{pmatrix} \\
= \text{Cartan } \begin{pmatrix} sr \\ -srn \\ (sr)^{-1} - srn^2 \end{pmatrix} \\
= -\min \{ \nu_\varpi(sr), \nu_\varpi(srn), \nu_\varpi((sr)^{-1} - srn^2) \}. \]

Because there exists a constant $c$ for which $\min \{ \nu_\varpi(r), \nu_\varpi(rn^2) \} > -c d(g)$, if we ensure that $\mu > c d(g)$, then we can simplify this condition to a simpler inequality:
\[ -\nu_\varpi((sr)^{-1}) = -\min \{ \nu_\varpi(sr), \nu_\varpi(srn), \nu_\varpi((sr)^{-1} - srn^2) \} \leq \mu. \]

With the original notation, the condition $\text{Cartan } \tau(r,ag) \leq \mu$ may therefore be written
\[ \nu_\varpi(\sqrt{\gamma}) + 2\nu_a \leq \mu - 2H_B(g) \]
in this case.

Now suppose that $\nu_\varpi(s)$ is negative. We may carry out the same computation, using the Iwasawa decomposition $g = tnk$ for $n$ belonging to $\tilde{N}$, the opposite unipotent radical of $N$. In this case, $\nu_\varpi(r) = H_B(g)$ rather than $H_B(g)$. Computing in a similar fashion,
\[
\text{Cartan } (\theta(n)^{-1}\tau(r) a^2 t^2 n) = \text{Cartan } \begin{pmatrix} 1 \\ -n \\ 1 \end{pmatrix} \begin{pmatrix} sr \\ (sr)^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ n \\ 1 \end{pmatrix} \\
= \text{Cartan } \begin{pmatrix} sr - n^2(sr)^{-1} \\ n(sr)^{-1} \\ (sr)^{-1} \end{pmatrix} \\
= -\min \{ \nu_\varpi((sr)^{-1}), \nu_\varpi(n(sr)^{-1}), \nu_\varpi(sr - n^2(sr)^{-1}) \}. \]

Because $\max \{ \nu_\varpi(r), \nu_\varpi(rn^2) \} < c d(g)$ for some $c$, which we may assume to equal the previous constant $c$, this expression is strictly less than $\mu$ precisely when
\[ -2H_B(g) - \mu \leq \nu_\varpi(\sqrt{\gamma}) + 2\nu_a \]
provided that $\mu > c d(g)$. Combining these inequalities completes the proof of the lemma. \qed

In other words, this lemma implies that if $A$ is equipped with the Haar measure that assigns $A(\mathcal{O}) = A \cap K$ a measure of one, then
\[ \int_A \bar{\omega}(r,ag,\mu) \, da = \# \{ \nu \in \mathbb{Z} : -2H_B(g) - \mu \leq \nu_\varpi(\sqrt{\gamma}) + 2\nu \leq \mu - 2H_B(g) \} \]
when $\mu$ is sufficiently large in a sense that depends on $g$. We would like to sum this right hand side over the elements of $K'$.

Because $F$ is a $p$-adic field, we can assume that $K'$ has been chosen to contain $|\mu_4(F)|$ elements, one half of them having valuation zero and the other half having valuation two. Then

$$
\sum_{\gamma \in K'} \omega_\gamma(r_\gamma g, \mu) = \frac{|\mu_4(F)|}{2} \left\{ \nu : -2H_B(g) - \mu \leq 2\nu \leq \mu - 2H_B(g) \right\} \\
+ \frac{|\mu_4(F)|}{2} \left\{ \nu : -2H_B(g) - \mu \leq 1 + 2\nu \leq \mu - 2H_B(g) \right\} \\
= \frac{|\mu_4(F)|}{2} (1 + 2\mu - 2H_B(g) + 2H_B(g)).
$$

This expression is only valid when $\mu$ is sufficiently large, but we may nevertheless use it to form an approximation for the $\theta$-split side.

**Definition 4.3.** Set $J_-(f, \mu)$ equal to the expression

$$
\frac{1 - |\varpi|}{2} \int_{\mathbb{V}} |b| \int_{A_1 \setminus G} f \left( g^{-1} \left( \begin{array}{c} b \\ b\gamma \end{array} \right) g \right) (1 + 2\mu - 2H_B(g) + 2H_B(g)) \, dg \, db \\
+ \sum_{\gamma \in K - K'} \frac{1 - |\varpi|}{|\mu_4(F)|} \int_{\mathbb{V}} |b| \int_{G} f \left( g^{-1} \left( \begin{array}{c} b \\ b\gamma \end{array} \right) g \right) \, db.
$$

We prove next that the difference between this approximation and the original expression tends to 0 as $\mu$ approaches infinity, so that this is the “best fit” polynomial in $\mu$ to the $\theta$-split side near $\mu = \infty$.

**Proposition 4.4.** For any locally constant, compactly supported function $f$ on the Lie algebra $\mathfrak{g}$,

$$
\lim_{\mu \to \infty} (\theta\text{-split side} - J_-(f, \mu)) = 0.
$$

**Proof.** When $\gamma$ is not a square, the magnitude of the difference

$$
\left| b \right| \int_{A_1 \setminus G} f \left( g^{-1} \left( \begin{array}{c} b \\ b\gamma \end{array} \right) g \right) (1 - \omega_\gamma(g, \mu)) \, dg \, db \leq \left| b \right| \int_{A_1 \setminus G} f \left( g^{-1} \left( \begin{array}{c} b \\ b\gamma \end{array} \right) g \right) \, dg \, db
$$

which is a (normalized) orbital integral. This bound is locally integrable and compactly supported on the Cartan subalgebra $\mathfrak{t}_\gamma$ by results of Harish-Chandra, given for $p$-adic reductive Lie algebras in [2]. This absolutely summable bound does not depend on $\mu$.

When $\gamma$ is a square, one first observes that when $\mu > c d(g)$,

$$
\omega_\gamma(g, \mu) = \frac{|\mu_4(F)|}{2} (1 + 2\mu - 2H_B(g) + 2H_B(g))
$$

and so we can bound the magnitude of the difference

$$
\left| b \right| \int_{A_1 \setminus G} f \left( g^{-1} \left( \begin{array}{c} b \\ b\gamma \end{array} \right) g \right) \left( \frac{|\mu_4(F)|}{2} (1 + 2\mu - 2H_B(g) + 2H_B(g)) - \omega_\gamma(g, \mu) \right) \, dg \, db
$$

by a weighted (and normalized) orbital integral that is absolutely summable by a general theorem of Waldspurger in [3]:

$$
\left| b \right| \int_{A_1 \setminus G} f \left( g^{-1} \left( \begin{array}{c} b \\ b\gamma \end{array} \right) g \right) \left( \frac{|\mu_4(F)|}{2} (1 + (2c + 4)d(g)) \right) \, dg \, db.
$$
This expression is also independent of \( \mu \).

That both these differences are bounded by absolutely summable functions that do not depend on \( \mu \) means that we can use the Lebesgue dominated convergence theorem to study their behavior as \( \mu \) tends to infinity. Specifically, the limit of the difference between the \( \theta \)-split side and this approximation is a linear combination of integrals of

\[
|b| \int_{A^\times \backslash G} f \left( g^{-1} \begin{pmatrix} b & b\gamma \\ -b & b \end{pmatrix} g \right) \lim_{\mu \to \infty} \left( \omega_\gamma(g, \mu) - \frac{|\mu_4(F)|}{2} (1 + 2\mu - 2H_B(g) + 2H_B^\ast(g)) \right) \, dg \, db = 0
\]
or

\[
|b| \int_{A^\times \backslash G} f \left( g^{-1} \begin{pmatrix} b & b\gamma \\ -b & b \end{pmatrix} g \right) \lim_{\mu \to \infty} (1 - \omega_\gamma(g, \mu)) \, dg \, db = 0
\]

which implies that the entire limit equals 0.

\[ \square \]

5. The \( \theta \)-fixed Side

We now consider the left-hand side of the trace formula,

\[
\int_{H \backslash G} \bar{\omega}(g, \mu) \int_{\mathfrak{h}} \hat{f}(g^{-1}Xg) \, dX \, dg = \int_{\mathfrak{h}} \int_{H \backslash G} \hat{f}(g^{-1}Xg) \bar{\omega}(g, \mu) \, dg \, dX
\]

which we call the \( \theta \)-fixed side. Unfortunately, this side is not amenable to Arthur’s techniques: the inner integral would have to be replaced by an unweighted orbital integral which is not locally integrable on \( \mathfrak{h} \) near the origin. But in the case of \( SL_2(F) \), the rank one symplectic group, it is simple enough that we may just compute an expansion directly. This computation does not, however, generalize to higher rank symplectic groups in a straightforward way.

First, we divide \( \mathfrak{h} \) into two sets, which depend on the choice of a second truncation parameter \( \epsilon \in F^\times \).

\[
\mathfrak{h}_\epsilon := \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} : |x| \leq |\epsilon| \right\} \quad \text{and} \quad \mathfrak{h}_\epsilon^c := \{ X \in \mathfrak{h} : X \notin \mathfrak{h}_\epsilon \}.
\]

The \( \theta \)-fixed side can then be broken into two terms:

\[
\int_{\mathfrak{h}_\epsilon} \int_{H \backslash G} \hat{f}(g^{-1}Xg) \bar{\omega}(g, \mu) \, dg \, dX + \int_{\mathfrak{h}_\epsilon^c} \int_{H \backslash G} \hat{f}(g^{-1}Xg) \bar{\omega}(g, \mu) \, dg \, dX.
\]

For the second term, \( \bar{\omega}(g, \mu) \) itself will act as the weight factor. For the first term, we must derive an ad hoc germ expansion.

Because \( H \) is a maximal torus in \( G = SL_2(F) \), we can apply the Iwasawa decomposition to write \( G = HN'K \), where \( N' \) is the unipotent radical of a Borel subgroup \( B' \) that contains \( H \). If we set

\[
\hat{f}(X) := \int_{K} \hat{f}(k^{-1}Xk) \, dX
\]

then the first term can be written

\[
\int_{\mathfrak{h}_\epsilon} \int_{N'} \hat{f}(n^{-1}Xn) \bar{\omega}(n, \mu) \, dn \, dX
\]

because \( \bar{\omega} \) is right \( K \)-invariant and left \( H \)-invariant. In coordinates, this equals

\[
\int_{\{ h \in F : |h| \leq |\epsilon| \}} \int_{F} \hat{f} \left( \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} h & \ast \\ -h & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right) \bar{\omega}(n, \mu) \, dn \, dh
\]
where the measures $dn$ and $dh$ are both the standard additive measures on $F$. This integral can be simplified to

$$\int_{\{h \in F : |h| \leq |\epsilon|\}} \int_F \tilde{f} \left( h \begin{pmatrix} 2hn \\ -h \end{pmatrix} \right) \tilde{\omega} (n, \mu) \ dn \ dh$$

or

$$\int_{\{h \in F : |h| \leq |\epsilon|\}} \int_F \tilde{f} \left( h \begin{pmatrix} 2n \\ -h \end{pmatrix} \right) \tilde{\omega} \left( \left( \frac{n}{|h|} \right), \mu \right) \ \frac{dn}{|h|} \ dh.$$

The key point is that because $\tilde{f}$ is compactly supported, we can choose $|\epsilon|$ so small that $\tilde{f} \left( h \begin{pmatrix} 2n \\ -h \end{pmatrix} \right) = \tilde{f} \left( 0 \begin{pmatrix} 2n \\ 0 \end{pmatrix} \right)$ for all $n$. We fix such an $\epsilon$, which establishes the domains of the terms on the $\theta$-fixed side.

Fubini’s theorem allows us to reverse the order of integration to express the first term on the $\theta$-fixed side as

$$\int_F \tilde{f} \left( 0 \begin{pmatrix} 2n \\ 0 \end{pmatrix} \right) \int_{\{h \in F : |h| \leq |\epsilon|\}} \omega \left( \left( \frac{n}{|h|} \right), \mu \right) \ \frac{dh}{|h|} \ dn.$$

This expression is an integral over the Lie algebra $n$ of the unipotent radical $N$ with weight

$$\int_{\{h \in F : |h| \leq |\epsilon|\}} \omega \left( \left( \frac{n}{|h|} \right), \mu \right) \ \frac{dh}{|h|} = \int_{\{h \in F : |n| \cdot |\omega| \cdot |h| \leq |\epsilon|\}} \frac{dh}{|h|} = (1 - |\omega|) (\nu_\omega(n) + \mu - \nu_\omega(\epsilon) + 1)$$

when $\mu$ is bigger than the valuation of $\epsilon$ minus the valuation of $n$ (otherwise, this expression equals 0). This asymptotic simplification for the weight factor provides an approximation for the $\theta$-fixed side.

**Definition 5.1.** Set $J_+(\hat{f}, \mu) = J_+(\hat{f}, \mu, \epsilon)$ equal to

$$(1 - |\omega|) \int_F \tilde{f} \left( 0 \begin{pmatrix} 2n \\ 0 \end{pmatrix} \right) (\nu_\omega(n) + \mu - \nu_\omega(\epsilon) + 1) \ dn + \int_{H \backslash G} \hat{f}(g^{-1}Xg) \ dg \ dX.$$

This expression is defined for all $\epsilon \in F^\times$, but the local trace formula will only hold when $\epsilon$ is taken very small in a sense that depends on $f$.

Again, we claim that the difference between this approximation and the original expression tends to 0 as $\mu$ approaches infinity. This will imply that the polynomial approximations we have derived for the $\theta$-split side and the $\theta$-fixed side are equal, which is the statement of a working version of the relative trace formula for $F^\times \backslash SL_2(F)$.

**Proposition 5.2.** For any compactly supported, locally constant function on $g$,

$$\lim_{\mu \to \infty} \left( \theta \text{-fixed side} - J_+(\hat{f}, \mu) \right) = 0.$$

**Proof.** When $\mu$ exceeds the valuation of $\epsilon$ minus the valuation of $n$,

$$\int_{\{h \in F : |h| \leq |\epsilon|\}} \tilde{\omega} \left( \left( \frac{n}{|h|} \right), \mu \right) \ dh = \nu_\omega(n) + \mu - \nu_\omega(\epsilon) + 1$$
and so the magnitude of the difference
\[
\tilde{f} \left( \begin{array}{cc} 0 & 2n \\ 0 & 0 \end{array} \right) \left( (\nu_\omega(n) + \mu - \nu_\omega(\epsilon) + 1) - \int_{\{h \in F : |h| \leq |\epsilon|\}} \bar{\omega} \left( \left( \begin{array}{cc} 1 & n \\ 2n & 1 \end{array} \right), \mu \right) dh \right)
\]
is bounded by a constant multiple of
\[
\left| \tilde{f} \left( \begin{array}{cc} 0 & 2n \\ 0 & 0 \end{array} \right) \right|
\]
which is absolutely summable on \( F \). Analogously,
\[
\int_{H \backslash G} \tilde{f}(g^{-1}Xg) \ dg - \int_{H \backslash G} \tilde{f}(g^{-1}Xg) \bar{\omega}(g, \mu) \ dg \leq \int_{H \backslash G} \left| \tilde{f}(g^{-1}Xg) \right| \ dg
\]
which is absolutely summable on \( b_0^c \), by results of Harish-Chandra in [2].

Because these differences are bounded by absolutely summable functions that do not depend on \( \mu \), we can apply the Lebesgue dominated convergence theorem to understand the behavior of this difference as \( \mu \) tends to infinity. Explicitly, the limit of the difference between the first term on the \( \theta \)-fixed side and the first term in its approximation \( J_+(\tilde{f}, \mu) \) is
\[
(1 - |\varpi|) \int_F \tilde{f} \left( \begin{array}{cc} 0 & 2n \\ 0 & 0 \end{array} \right) \lim_{\mu \to \infty} \left( (\nu_\omega(n) + \mu - \nu_\omega(\epsilon) + 1) - \int_{\{h \in F : |h| \leq |\epsilon|\}} \bar{\omega} \left( \left( \begin{array}{cc} 1 & n \\ 2n & 1 \end{array} \right), \mu \right) dh \right) \ dn
\]
which equals zero. Analogously, the difference between the second term on the \( \theta \)-fixed side and the second term in its approximation \( J_+(\tilde{f}, \mu) \) also tends to 0. These limits, taken together, imply the limit of the proposition. \( \square \)

### 6. A Relative Trace Formula for \( F^x \backslash SL_2(F) \)

**Theorem 6.1.** For any locally constant, compactly supported function \( f \) on the Lie algebra \( g = sl_2 \), there is an \( \epsilon \) that depends on \( f \) for which there is an integral identity
\[
\frac{1 - |\varpi|}{2} \int_{t_1} |b| \int_{A \backslash G} f \left( b^{-1} \begin{array}{cc} g & b \\ b & 1 \end{array} \right) g \left( (1 + 2\mu - 2H_B(g) + 2\bar{H}_B(g)) \right) \ dg \ db + \sum_{\gamma \in \mathcal{K} \backslash \mathcal{K}'} \frac{1 - |\varpi|}{|\mu_4(F)|} \int_{t_1} |b\gamma| \int_{G} f \left( b^{-1} \begin{array}{cc} g & b\gamma \\ \gamma & 1 \end{array} \right) g \ dg \ db
\]
\[
= (1 - |\varpi|) \int_F \tilde{f} \left( \begin{array}{cc} 0 & 2n \\ 0 & 0 \end{array} \right) \left( (\nu_\omega(n) + \mu - \nu_\omega(\epsilon) + 1) \right) \ dn + \int_{h_0^c} \int_{H \backslash G} \tilde{f}(g^{-1}Xg) \ dg \ dX.
\]

**Proof.** This identity may be stated succinctly as \( J_-(f, \mu) = J_+(\tilde{f}, \mu) \). We know that the \( \theta \)-fixed side equals the \( \theta \)-split side. By subtracting the limits
\[
\lim_{\mu \to \infty} \text{\( \theta \)-fixed side} - J_+(\tilde{f}, \mu) = 0
\]
\[
\lim_{\mu \to \infty} \text{\( \theta \)-split side} - J_-(f, \mu) = 0
\]
one establishes this limit:
\[
\lim_{\mu \to \infty} J_+(\tilde{f}, \mu) - J_-(f, \mu) = 0.
\]
But both \( J_+(\tilde{f}, \mu) \) and \( J_-(f, \mu) \) are polynomials in \( \mu \), so this can only happen if
\[
J_-(f, \mu) = J_+(\tilde{f}, \mu)
\]
for all $\mu$, which is the stated identity. □

This proposition gives the full trace formula for the symmetric space $F^x \backslash SL_2(F)$, and one of the simplest non-trivial analogs of the local trace formula. If we regard this expression as an identity of polynomials in $\mu$, then we may identify coefficients. This yields two formulas:

$$\int_{t_1} |b| \int_{A_1 \backslash G} f \left( g^{-1} \left( b \ b^\gamma \right) g \right) \, dg \, db = \int_{F} \hat{f} \left( \begin{pmatrix} 0 & 2n \\ 0 & 0 \end{pmatrix} \right) \, dn$$

and

$$\sum_{\gamma \in \mathbb{K} - \mathbb{K}'} \frac{1}{|\mu_4(F)|} \int_{t_1} |b| \int_{A_1 \backslash G} f \left( g^{-1} \left( b \ b^\gamma \right) g \right) \, dg \, db$$

$$+ \frac{1}{2} \int_{t_1} |b| \int_{A_1 \backslash G} f \left( g^{-1} \left( b \ b^\gamma \right) g \right) (1 - 2H_B(g) + 2\overline{H}_B(g)) \, dg \, db$$

$$= \int_{F} \hat{f} \left( \begin{pmatrix} 0 & 2n \\ 0 & 0 \end{pmatrix} \right) (1 + \nu_\infty(n) - \nu_\infty(\epsilon)) \, dn + \frac{1}{1 - |\varpi|} \int_{H \backslash G} \hat{f} \left( g^{-1} X g \right) \, dg \, dX.$$

The first of these formulas can be derived from the Plancherel formula on $t_1$, applied to the function

$$\hat{f}(B') \left( x \right) := \int_{F} \hat{f} \left( \begin{pmatrix} x & 2n \\ -x & 0 \end{pmatrix} \right) \, dn.$$

The second identity is roughly equivalent to the full trace formula for $F^x \backslash SL_2(F)$, and is the integral identity of proposition 1.1.

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