Wilsonian Effective Field Theory and String Theory*

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Abstract

We argue that deriving an effective field theory from string theory requires a Wilsonian perspective with a physical cutoff. Employing proper time regularization we demonstrate the decoupling of states and contrast this with what happens in dimensional regularization. In particular we point out that even if the cosmological constant (CC) calculated from some classical action at some ultra-violet scale is negative, this does not necessarily imply that the CC calculated at cosmological scales is also negative, and discuss the possible criteria for achieving a positive CC starting with a CC at the string/KK scale which is negative. Obviously this has implications for swampland claims.

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1 Introduction: EFT from string theory - physical cutoffs vs dimreg

String theory is supposed to be an ultraviolet complete theory of quantum gravity. Currently this assertion can be explicitly demonstrated only in the context of perturbation theory around flat space but it is widely expected to be valid more generally. This is because in going from point particles to strings one explicitly introduces (albeit in a Lorentz invariant fashion) a short distance cutoff - namely the string length \( l_s = \sqrt{\alpha'} \) which is a measure of the size of fundamental strings. Equivalently there is an ultra-violet (UV) mass scale governed by the tension of the string \( T_s \equiv M_{(n)}^2 = n/\alpha', \quad n \in \mathbb{Z}^+ \).

In order define the limits of validity of the EFT let us write the ten-dimensional background metric of string theory as,

\[
ds^2 = G_{MN}dX^M dX^N = e^{\phi/2}[e^{-6u(x)}g_{\mu\nu}(x)dx^\mu dx^\nu + e^{2u(x)}\hat{g}_{mn}(y)dy^m dy^n].
\]

(1)

Here \( \hat{g}_{mn}(y) \) is a fiducial metric on the internal space \( X \) with coordinates \( y \) and fiducial volume \( \int_X \hat{g}^{1/3} dy = (2\pi \alpha')^6 \) and \( \phi \) is the dilaton. The volume (modulus) of the internal space is then \( V = e^{6u} \). We have then the relation between the 4D Planck scale and string scale,

\[
T_s \equiv M_s^2 = M_P^2 \frac{e^{\phi/2}}{2V}.
\]

(2)

Here \( M_P^2 = 1/8\pi G \) is the low energy Planck scale. The actual cutoff for a 4D EFT is however below the string scale and is given by the Kaluza-Klein scale

\[
M_{KK}^2 = \frac{M_s^2}{V^{1/3}} = \frac{M_P^2}{2V^{4/3}} e^{\phi/2}.
\]

(3)

We will focus on type IIB compactifications which are the best studied in terms of moduli stabilization, phenomenology and cosmology. The two main scenarios are those of KKLT [1] and LVS [2]. In both the dilaton and complex structure moduli are determined by a set of internal fluxes while the Kaehler moduli (including the volume \( V \)) are determined by a combination of internal fluxes and non-perturbative terms (which depend also on a choice of open string data in particular a gauge group).

Now given a set of internal space data, i.e. choice of Calabi-Yau (CY) manifold, internal fluxes and open string data, one has a 4D EFT with a cutoff essentially given by (3). However it is possible to keep the field content and the interactions of the EFT fixed (say by fixing the CY and the open string data) while varying the fluxes. In the string theory EFT context one can identify this as a Wilsonian UV cutoff for the EFT.

On the other hand suppose we have identified a string model in the landscape of solutions to the string equations, that describes our universe. At a minimum it must contain the standard model spectrum and perhaps the spectrum of a supersymmetric extension of it such as the MSSM. In that case the input for the EFT coming from string theory would be a set of couplings (including the cosmological constant) that would be the initial conditions for the RG evolution of the field theory. These are defined not at some arbitrary adjustable scale but at a physical scale - the

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1For simplicity we will ignore the effect of warping as we may do for an internal space of very large volume.
effective Kaluza-Klein scale. In fact this was pointed out a while ago in [3] where a comparison of the string loop calculation with the Coleman-Weinberg one-loop potential with a physical cutoff was made.

This is in contrast to the situation in a purely field theoretic analysis. In the latter the parameters including the CC are fixed at some scale through experiments done at that scale. The RG is then used to compute the effective couplings for processes at a different scale. For instance one may use the electromagnetic coupling $\alpha$ at very low energies (say measured in Thompson scattering) and then evolve it up to the Z-pole to compute QED corrections to processes at that scale. Similarly with the strong coupling $\alpha_s$ which is usually quoted at the Z-pole, may then be used after RG evolution to compute QCD corrections at a different scale.

However none of this depends on the value of those couplings at the cutoff scale for the standard model coming from the UV completion of the standard model whatever it is. Indeed any such UV completion must give values for those couplings which are in agreement with what has been computed from running the RG up from low energies to the cutoff scale. What we will find is that the evolution between low energy and the KK scale will be very different in regularizations that employ a physical cutoff and in dimensional regularization (dimreg). The basic reason is that the latter involves integration over all scales (up to infinite momenta) and the consequent absence of decoupling, which has to be put in by hand.

The most salient manifestation of this is in the evolution of the CC. Whereas with a physical cutoff one sees that there is no significant evolution up until the lowest mass scale in dimensional regularization this is not the case and results in a logarithmic infrared divergence as the scale becomes arbitrarily small. Given that the initial CC (at the KK scale) is supposed to be finite there is no way to infer a finite CC at cosmological scales. To put it another way if one inputs the measured CC at the latter scale, which is the standard prescription for renormalization in dimreg, then the CC at the UV scale would diverge - but that would be in conflict with the assumed existence of a finite UV theory - such as string theory which has to replace the field theory once one crosses the cutoff value such as the KK scale. In order to correct this one has, as we mentioned above, to put in the decoupling of states by hand.

Furthermore let us suppose that the various no-go theorems which appear to rule out dS solutions to string theory are actually valid. This would however only mean that at the cutoff scale for the field theory (presumably some effective KK scale or perhaps the string scale) there is no dS solution i.e. the CC is either zero or negative or indeed the potential is runaway. However the observed CC cannot be compared directly with this string theory calculation since one needs to take into account the effect of all the low energy fluctuations below the KK/string scale. In dim reg (at least to one loop) this appears to yield only negative contributions and hence would not lift a negative CC to a positive value at cosmological scales. On the contrary with a physical cutoff depending on the details of low energy physics there is a possibility, as we shall see later of lifting the low energy CC to positive values even if the string theory CC is negative.

\(^2\)This author does not believe this to be the case since this would mean that all the explicit constructions of a positive CC at the string scale are invalid for some reason which the advocates of swampland conjectures have yet to demonstrate!

\(^3\)An example of this possibility has already been given in [3].
2 Proper time regularization of the quantum effective action

The quantum theory corresponding to a given classical action $I[\phi]$ is given by the quantum effective action $\Gamma(\phi_c)$ defined (implicitly and formally) by the formula

$$e^{-\Gamma(\phi_c)} = \int [d\phi] e^{-I[\phi] - J_c(\phi - \phi_c)} \bigg|_{J = -\partial \Gamma / \partial \phi_c}. \quad (4)$$

By translating the integration variable $\phi = \phi_c + \phi'$ we have the following expressions,

$$e^{-\Gamma(\phi_c)} = \int [d\phi'] e^{-I[\phi_c + \phi'] - J_c \phi'} \bigg|_{J = -\partial \Gamma / \partial \phi_c} = \int [d\phi'] e^{-I[\phi_c] + \frac{1}{2} \phi', \frac{\partial^2}{\partial \phi'^2} \phi' + I_c[\phi_c, \phi'] + (J + \partial I[\phi_c] / \partial \phi_c) \phi'} \bigg|_{J = -\delta \Gamma / \delta \phi_c}. \quad (5)$$

In the second line above $I_c[\phi_c, \phi']$ contains all powers of $\phi'$ which are higher than quadratic in the expansion of $I[\phi_c + \phi']$, and the third line is the result of doing the Gaussian integral over $\phi'$. Also $K[\phi_c]$ is the kinetic operator in the presence of the background field $\phi_c$ and we’ve used a condensed notation so that for example $J, \phi \equiv \int \sqrt{g} \phi^2 J_i$ etc.

Now typically a QFT cannot be assumed to be ultraviolet complete. Thus the “initial” action $I$ in the above is actually some effective action obtained by integrating out high energy degrees of freedom (above some UV scale $\Lambda$) from some fundamental UV complete theory such as string theory. Thus we should replace $I \to I_{\Lambda}$. Also the above is a formal expression that needs to be regularized if we are to evaluate it in perturbation theory. A convenient way of doing this for our purposes is to introduce the Schwinger proper time regularization$^3$.

$$K_{\Lambda}^{-1}(\phi_c; x, y) = \langle x | \int_{1/\Lambda^2} \infty ds e^{-K_{\Lambda}[\phi_s]} | y >, \ln K_{\Lambda}[\phi_c; x, y] = - < x | \int_{1/\Lambda^2} \infty ds e^{-K_{\Lambda}[\phi_s]} | y >. \quad (6)$$

The quantum effective action should be independent of the regularization scale $\Lambda$. This implies that the “initial” action which one may think of as coming from the low energy limit of some UV complete theory such as string theory is dependent on the scale $\Lambda$, i.e. as we stated before, $I[\phi] \to I_{\Lambda}[\phi]$ where in the latter all couplings (including masses and the CC) are $\Lambda$ dependent. Hence the subscript $\Lambda$ on $K$ in the RHS of the above equations. The regularized definition of the 1PI action is then obtained by making these replacements in (5).

$$e^{-\Gamma(\phi_c)} = e^{-I_{\Lambda}[\phi_c]} e^{-\frac{1}{2} \ln K_{\Lambda}[\phi_c]} e^{-I_{\Lambda}[\phi_c, \phi_c] - \frac{1}{2} \frac{\partial^2}{\partial \phi'^2} \phi' + \frac{1}{2} j \int K_{\Lambda}[\phi_c]^{-1} J | J = \delta I_{\Lambda}[\phi_c] / \delta \phi_c - \delta \Gamma / \delta \phi_c. \quad (7)$$

Note that the second exponential factor is the one-loop determinant and the last two exponential factors give the higher than one-loop contributions. Requiring the independence of $\Gamma$ from the cutoff scale $\Lambda$ i.e. $\Lambda \frac{\partial \Gamma}{\partial \Lambda} = 0$ then gives the set (in general infinte) of RG equations for the couplings in $I_{\Lambda}$.

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$^3$We work in the Euclidean formulation of quantum field theory. This is just a matter of convenience. If one worked in the Lorentzian formulation we would have to include $i\epsilon$ prescriptions and use contour rotation to define integrals over time components of momenta. The expressions for effective actions and RG equations, are of course not affected.

$^3$In general $K$ will of course be a matrix over space-time indices as well as internal indices labelling the different fields as well as their components.
Define the heat kernel\textsuperscript{6}

\[ H(s|x, x') = \langle x | e^{-Ks} | x' \rangle, \quad \langle x | x' \rangle = \frac{1}{\sqrt{g}}\delta^D(x - x'), \quad (8) \]

the formal solution of heat equation \( \partial_s H(s|x, x') = -KH(s|x, x') \). Note that in general \( K \) (and hence \( H \)) is a matrix in internal (field) space as well as in space time. For a scalar field theory \( K = -\Box I + V''(\phi_c) \) where \( V'' \) is the second derivative matrix of the scalar potential. A regularized one-loop effective action is then given by keeping just the first two exponential factors of \( I \),

\[ \Gamma^{(0+1)}[\phi_c] = I_\Lambda[\phi_c] + \frac{1}{2} \text{Tr} \ln K_\Lambda[\phi_c]. \]

The one-loop contribution may be evaluated as,

\[ \Gamma^{(1)} = \frac{1}{2} \text{Tr} \ln K = -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \text{Tr} e^{-K[\phi_c]s} \hspace{1cm} (9) \]

\[ = -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \int d^4x \sqrt{g} \text{tr} H(s|x, x). \]

\[ = -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \int d^4x \sqrt{g} \text{tr} e^{-V''(\phi_c)s} \frac{1}{(4\pi s)^2}. \]

(10)

Here \( V'' \) is the mass matrix and we’ve used the standard result for the heat kernel of the canonical kinetic (derivative) term in flat space and ignored space-time variations of the field (and the metric \( g \)), since in this work we will just focus on the effective potential.

The expression \( I \) gives a well-defined expression for the quantum effective action that is obtained by integrating over all quantum fluctuations down to arbitrarily small energy scales. It is well-defined in that there is no issue with UV divergences since it is expressed in terms of the Wilsonian action at the scale \( \Lambda \) and the effect of integrating over all quantum fluctuations from \( \Lambda \) down to zero. It is assumed that there are no IR divergences or that they can be effectively taken care of even when there are massless particles.

On the other hand one may also define the Wilsonian action at some lower (non-zero) scale \( \mu \) in terms of that at the scale \( \Lambda \) by integrating over the quantum fluctuations between those two scales. For this purpose we may define the propagator and the log of the kinetic matrix by introducing an IR cutoff into the definitions \( I \), i.e.

\[ K^{-1}_{\mu, \Lambda}(\phi_c; x, y) = \langle x | \int_{1/\mu^2}^{1/\Lambda^2} ds \left[ e^{-\hat{K}_\Lambda[\phi_c]s} \right] y \rangle, \quad \ln K_{\mu, \Lambda}[\phi_c; x, y] = \langle x | \int_{1/\mu^2}^{1/\Lambda^2} ds \left[ e^{-\hat{K}_\Lambda[\phi_c]s} \right] y \rangle. \quad (11) \]

Using these in \( I \) then gives the Wilsonian effective action at scale \( \mu \) in terms of that at the scale \( \Lambda \),

\[ e^{-I_\mu(\phi_c)} = e^{-I_\Lambda[\phi_c]} e^{-\frac{1}{2} \text{Tr} \ln K_{\mu, \Lambda}[\phi_c]} e^{-I_{\Lambda}[\phi_c] - \frac{1}{2\pi^2} \int d^4x \left[ e^{-\hat{K}_\Lambda[\phi_c]s} \right] J \cdot [\delta I_\Lambda/\partial \phi_c - \delta I_\mu/\partial \phi_c]. \quad (12) \]

\textsuperscript{6}For our purposes here it is sufficient to work in flat background. The discussion can be easily exted to general curved backgrounds. A useful review of heat kernel methods is \[5\]. For an application for comparing RG running in higher dimensional and 4D supergravity see \[6\].
To one-loop we have (upto derivative terms which we ignore since we are only considering the effective potential below),

$$I_\mu(\phi_c) = I_\Lambda[\phi_c] + \frac{1}{2} \text{Tr} \ln K_{\mu,\Lambda}[\phi_c] + \ldots$$

$$= I_\Lambda[\phi_c] - \frac{1}{2} \int_{1/\Lambda^2}^ {1/\mu^2} \frac{ds}{s} \int d^4x \sqrt{g} \text{tr} e^{-V''(\phi_c)s} \frac{1}{(4\pi s)^2} + \ldots$$

From this we have the one loop beta function equation\(^7\)

$$\mu \frac{d}{d\mu} I_\mu[\phi_c] = \frac{\mu^4}{16\pi^2} \int d^4x \sqrt{g} \text{tr} e^{-V''(\phi_c)/\mu^2}.$$ (14)

3 A toy model of decoupling - Heat kernel vs dim reg

Let us now consider a toy model with low energy degrees of freedom coupled to high energy modes. Let us see how this happens explicitly in the context of heat kernel regularization and then compare it to what one would have done in dimensional regularization.

Let us calculate the \(\beta\)-function for a scalar field theory with two scales. The potential is

$$V(\phi, \Phi) = \Lambda_{cc} + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} M^2 \Phi^2 + \frac{\lambda}{4!} \phi^4 + \frac{\eta}{4} \phi^2 \Phi^2 + \ldots,$$ (15)

where \(M^2 \gg m^2\) and the ellipses stand for higher dimension operators. The coefficients of the operators in the above are taken to be defined at some UV scale \(\Lambda\) which we may be taken to be the KK scale in the string theory context. For the purposes of this discussion we may think of \(\phi\) to be the standard model Higgs with physical (pole) mass \(m_{\text{phys}} = m + O(\lambda, \eta, \ldots)\) and \(\Phi\) a Higgs of some grand unified theory with physical (pole) mass \(M_{\text{phys}} = M + O(\lambda, \eta, \ldots)\). The kinetic operator in the background \(\langle \phi_c, 0 \rangle\) (we set \(\Phi\) to zero in order to focus on the light field beta function) is then \(K_{ij} = \Box \delta_{ij} + V_{ij}(\phi_c, 0), i, j = \phi, \Phi\). For the above potential we have \(V_{\phi\phi}(\phi_c, 0) = m^2 + \frac{\lambda}{2} \phi^2_c, V_{\phi\Phi} = M^2 + \eta \phi^2_c\). Hence from (14) we have

$$\mu \frac{d}{d\mu} \Gamma^{(1)} = \int d^4x \sqrt{g} \frac{\mu^4}{16\pi^2} [e^{-V_{\phi\phi}/\mu^2} + e^{-V_{\phi\Phi}/\mu^2}].$$ (16)

The effective potential at scale \(\mu\) is then given in terms of the action at scale \(\Lambda\) by the relation,

$$V^{(0+1)}(\phi, \Phi; \mu) = \Lambda_{cc}(\mu) + \frac{1}{2} m^2(\mu) \phi^2 + \frac{1}{2} M^2(\mu) \Phi^2 + \frac{\lambda(\mu)}{4!} \phi^4 + \frac{\eta(\mu)}{4} \phi^2 \Phi^2 + \ldots$$

$$= \Lambda_{cc}(\Lambda) + \frac{1}{2} m^2(\Lambda) \phi^2 + \frac{1}{2} M^2(\Lambda) \Phi^2 + \frac{\lambda(\Lambda)}{4!} \phi^4 + \frac{\eta(\Lambda)}{4} \phi^2 \Phi^2 +$$

$$+ \Delta V^{(1)}, \Delta V^{(1)} = -\frac{1}{2} \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s} \text{tr} e^{-V''(\phi, \Phi)s} \frac{1}{(4\pi s)^2}.$$ (17)

Note that in the above we’ve defined the generalized mass matrix \(V'' \equiv \partial^2 V/\partial \phi_i \partial \phi_j, \phi_i = \phi, \Phi\). It is important to stress that there is no issue of subtracting divergent quantities here as is the case

\(^7\)Using this procedure an exact RG equation was derived in [7] and is in effect the RG improved version of the equation below, i.e. with \(V''_{\Lambda}(\phi_c) \to V''_{\mu}(\phi_c)\).
in the usual textbook discussion of renormalization in QFT. Equation (17) relates the (finite) Wilsonian potential at the scale $\Lambda$ to the (finite) Wilsonian potential at the scale $\mu$. The one loop contribution to the relation $V^{(1)}$ is clearly well defined and incorporates the result of integrating out the degrees of freedom between the two scales. It can be represented as a difference of two incomplete Gamma functions.

$$\Delta V^{(1)} = -\frac{1}{2} \text{tr} \left( V''(\phi_c) \right)^2 \int_{V''(\phi_c)/\Lambda^2}^{V''(\phi_c)/\mu^2} dt \frac{1}{(4\pi t)^2} e^{-t},$$

$$= -\frac{1}{32\pi^2} \text{tr} \left( V''(\phi_c) \right)^2 \left[ \Gamma \left( -2, V''(\phi_c)/\Lambda^2 \right) - \Gamma \left( -2, V''(\phi_c)/\mu^2 \right) \right].$$

(18)

$\Gamma(\alpha, x) \equiv \int_x^\infty e^{-t} t^{\alpha-1}$, is the incomplete Gamma function\textsuperscript{8} For $\alpha = -r$, $r \in \mathbb{Z}^+$ this has the expansion

$$\Gamma(-r, x) = \frac{(-1)^r}{r!} \sum_{j=0}^{r-1} \frac{(-1)^j j!}{x^{j+1}} + E_1(x) - e^{-x} \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{nn!}, |\text{arg } x| < \pi.$$  

(19)

In the above $\gamma = .57721...$, is the Euler-Mascheroni constant. The quantum effective potential up to one loop is then given in this procedure of regularization by

$$V^{(0+1)}_{1\text{PI}}(\phi, \Phi) = \Lambda_{cc}(\Lambda) + \frac{1}{2} m^2(\Lambda) \phi^2 + \frac{1}{2} M^2(\Lambda) \Phi^2 + \frac{\lambda(\Lambda)}{4!} \phi^4 + \frac{\eta(\Lambda)}{4} \phi^2 \Phi^2 + \ldots + V^{(1)}(\phi, \Phi; \Lambda)$$

(21)

$$V^{(1)}(\phi, \Phi; \Lambda) = \frac{1}{32\pi^2} \text{tr} \left( V''(\phi, \Phi) \right)^2 \Gamma \left( -2, V''(\phi, \Phi)/\Lambda^2 \right)$$

$$= -\frac{1}{64\pi^2} \text{tr} \left[ e^{-V''/\Lambda^2} (\Lambda^4 - \Lambda^2 V'') - \left( V'' \right)^2 \left( \gamma + \ln \frac{V''}{\Lambda^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{nn!} \left( \frac{V''}{\Lambda^2} \right)^n \right) \right].$$

(22)

The RG equations for the local potential are equivalent to the statement $\Lambda dV_{1\text{PI}}/d\Lambda = 0$. Note again that there is no issue of infinities here. Given the “initial” local potential (or more generally the action) at the scale $\Lambda$ equation (21) gives the effective potential to one loop. Note that up to this point the discussion is quite general and holds for any scalar field theory. In the particular case of our toy model the matrix $V''$ is given by (\ldots).

\textsuperscript{8}See Gradshteyn and Ryzhik, Tables of Integrals Series and Products section 8.35 or Abramowitz and Stegan, Handbook of Mathematical functions, section 6.5.
Although we’ve focussed on a simple scalar field theory to illustrate the effects of decoupling (discussed below), we note that the generalization to include gauge fields and fermions is straightforward - one simply replaces the traces by supertraces.

How does the above calculation compare to the one in dimreg (reviewed in the Appendix). The dimreg calculation effectively implies that all scales are integrated over - so we need to send the cutoff $\Lambda$ to infinity. So in (22) the exponential factor will tend to unity and the inverse powers of $\Lambda$ will tend to zero. As for the divergent terms, the potential at $\Lambda$ (usually called the unrenormalized potential (or action if derivative terms are included) which is really an completely undefined object with infinite couplings!) has to be replaced by the so-called renormalized action (with couplings defined at some arbitrary renormalization scale $\mu$) plus so-called counter terms (which are of course divergent) chosen to cancel the divergent one loop terms in the above. Thus for instance the logarithmic term is rewritten as

\[
(V'')^2 \ln \left( \frac{V''}{\Lambda^2} \right) = (V'')^2 \ln \left( \frac{V''}{\mu^2} \right) + (V'')^2 \ln \left( \frac{\mu^2}{\Lambda^2} \right),
\]

with the second (local) term on the RHS cancelling the counter term in the “classical” action leaving us with just the finite first term. Thus in this procedure the effective potential to one-loop is

\[
V'_{\text{1PI}}(\phi, \Phi) = \Lambda_{cc}(\mu) + \frac{1}{2} m^2(\mu) \phi^2 + \frac{1}{2} M^2(\mu) \Phi^2 + \frac{\lambda(\mu)}{4!} \phi^4 + \frac{\eta(\mu)}{4} \phi^2 \Phi^2 + \ldots + V^{(1)}(\phi, \Phi; \mu)
\]

\[
V^{(1)}(\phi, \Phi; \mu) = \frac{1}{64\pi^2} \text{tr} \left[ \left( \frac{V''}{\mu^2} + \text{scheme dependent finite terms} \right) \right]
\]

in agreement (up to scheme dependence) with the dimreg calculation of the appendix i.e. eqn. (46).

However this interpretation defeats the purpose of having a physical cutoff. The maximum value of the latter is supposed to be finite since beyond that value the theory should be replaced by a more fundamental one which is UV complete as in the case of string theory where necessarily $\Lambda < M_{\text{KK}}$. Sending $\Lambda$ to infinity thus makes no sense and indeed that is why conceptually dimreg is not appropriate for interpreting the EFT of string theory.

### 3.1 The cosmological constant

First consider the evolution of the cosmological constant $\Lambda_{cc}$. Comparing the coefficients of the unit operator (or $\sqrt{g}$) in eqn. we have

\[
\beta_{\Lambda_{cc}} \equiv \mu \frac{d}{d\mu} \Lambda_{cc} = \mu^4 \frac{d}{d\mu} \left[ e^{-\frac{m^2}{\mu^2}} + e^{-\frac{M^2}{\mu^2}} \right]
\]

\[
\simeq \frac{\mu^4}{16\pi^2} \times 2, \quad \mu \gg M \gg m
\]

\[
\simeq \frac{\mu^4}{16\pi^2}, \quad m \ll \mu \ll M
\]

\[
\simeq 0, \quad \mu \ll m \ll M
\]
Here for the purposes of illustration we have made a crude approximation in the last three equations which will be refined in the next subsection.

What do these equations imply for the low energy parameters that should be used to discuss physics at say the standard model scale or below, assuming that the initial values are set by some UV complete theory such as string theory. Given that the latter is the only theory of quantum gravity that can even in principle include the standard model we will focus on it.

In string theory there are two scales - one the string scale $M_s$ is the mass of the lowest string excitations (and defines the tension of the string). This controls the low energy expansion of the string field theory. This is a ten dimensional local field theory with an infinite number of terms with the coefficients of higher dimensional operators controlled by inverse powers of $M_s^2$. The four dimensional theory is obtained by compactifying six of the spatial dimensions, and that introduces another scale - the Kaluza-Klein scale $M_{KK}$, whose inverse sets the length scale of the compactified space. It is the latter which controls the low energy expansion of the four dimensional theory since in proceeding via the ten-dimensional field theory we’ve already assumed that the volume of the extra six dimensions is large and hence for consistency we need to have $M_{KK} < M_s$. We will assume that the theory we are discussing comes from a particular point in the landscape of string theory. This means that the initial data for the RG evolution of the parameters of the field theory are fixed by string theory.

For our toy model this means that the initial potential takes the form (15) with parameters

$$\Lambda_{cc} \rightarrow \Lambda_0 \equiv \Lambda_{cc}(M_{KK}), \ m \rightarrow m_0 \equiv m(M_{KK}), \ M \rightarrow M_0 \equiv M(M_{KK}), \ldots.$$

These are all determined by string theory and the corresponding action may be used to compute physical processes at energies just below the KK scale. The RG equations then determine the effective low energy action that may be used at energy scales well below the KK scale.

We integrate in stages first following the simplest approximations to (23), i.e. (24),(25) and (26). From the first we get

$$\Lambda_{cc}(M) = \Lambda_0 + \frac{1}{16\pi^2} \frac{1}{2} \left( M^4 - M_{KK}^4 \right).$$

From the second we have

$$\Lambda_{cc}(m) = \Lambda_{cc}(M) + \frac{1}{16\pi^2} \frac{1}{4} \left( m^4 - M^4 \right).$$

Finally from the third we have

$$\Lambda_{cc}(\mu \ll m) = \Lambda_{cc}(m)$$

reflecting the fact that the CC hardly evolves between cosmological scales and the lightest (non-zero) physical mass scale. Putting these equations together we can express the cosmological CC in terms of the CC coming from string theory,

$$\Lambda_{cc}(\mu \ll m) = \Lambda_0 + \frac{1}{64\pi^2} \left( m^4 - M^4 \right) + \frac{1}{32\pi^2} \left( M^4 - M_{KK}^4 \right). \quad (27)$$

This is a fairly crude approximation since we ignored the regions where the exponential terms are different from zero or one. Nevertheless this equation nicely illustrates the fine tuning problem
for the CC, at least in a non-supersymmetric theory or a string theory with a SUSY breaking scale above the KK scale. The input CC coming from string theory at the KK scale, namely $\Lambda_0$, has to be such as to account for each mass threshold that is crossed as one goes from this scale down to cosmological scales. The argument of Bousso and Polchinski [8] is that given a sufficiently complicated compactification manifold (a Calabi-Yau space with a large number of cycles for instance with internal fluxes turned on) this is always possible.

Let us now include correction terms coming from the exponentials in (23). In the region $m^2 \ll M^2 < \mu^2 < M_{KK}^2$,

$$\mu \frac{d}{d\mu} \Lambda_{cc} \simeq \frac{\mu^4}{16\pi^2} \left[ 1 + \left( 1 - \frac{M^2}{\mu^2} + \frac{1}{2} \frac{M^4}{\mu^2} + O \left( \frac{M^6}{\mu^2} \right) \right) \right]$$

$$= \frac{1}{16\pi^2} \left[ 2\mu^4 - M^2\mu^2 + \frac{1}{2} M^4 + \ldots \right].$$

Integrating this we get

$$\Lambda_{cc}(m) - \Lambda_{cc}(M_{KK}) \simeq \frac{M^4 - M_{KK}^4}{32\pi^2} + \frac{M_{KK}^2 - M^2}{32\pi^2} M^2 + \frac{1}{64\pi^2} M^4 \ln \left( \frac{M^2}{M_{KK}^2} \right) + \ldots \quad (28)$$

For $m^2 < \mu^2 \ll M^2$ we have

$$\mu \frac{d}{d\mu} \Lambda_{cc} \simeq \frac{\mu^4}{16\pi^2} \left( e^{-m^2/\mu^2} + 0 \right)$$

$$= \frac{\mu^4}{16\pi^2} \left( 1 - \frac{m^2}{\mu^2} + \frac{1}{2} \left( \frac{m^2}{\mu^2} \right)^2 + \ldots \right)$$

Integrating this we get

$$\Lambda_{cc}(m) - \Lambda_{cc}(M) \simeq \frac{m^4 - M^4}{64\pi^2} + \frac{(M^2 - m^2)^2}{32\pi^2} m^2 + \frac{m^4}{64\pi^2} \ln \left( \frac{m^2}{M^2} \right) + \ldots \quad (29)$$

Finally in the region $\mu^2 \ll m^2$, we have

$$\mu \frac{d}{d\mu} \Lambda_{cc} \simeq 0, \Rightarrow \Lambda_{cc}(\mu \ll m) \simeq \Lambda_{cc}(m) \quad (30)$$

Collecting the three expressions we have for the long distance CC the expression

$$\Lambda_{CC}(\mu \ll m) = \Lambda_{CC}(M_{KK}) - \frac{M_{KK}^4 - m^4}{32\pi^2} + \frac{M_{KK}^2 - M^2}{32\pi^2} M^2 + \frac{(M^2 - m^2)^2}{32\pi^2} m^2$$

$$+ \frac{M^4}{64\pi^2} \ln \left( \frac{M^2}{M_{KK}^2} \right) + \frac{m^4}{64\pi^2} \ln \left( \frac{m^2}{M^2} \right) + \ldots \quad (31)$$

Before we go on to discuss the evolution of the other parameters, let us generalize these formulae to a SUSY theory where the supersymmetry is broken below the KK scale. In this case in the formula
for the effective action the trace instruction will be replaced by the supertrace defined (with \( \mathcal{M} \) being the field dependent mass matrix) by

\[
\text{Str} (\mathcal{M}^2)^n = \sum_j (2j + 1)(-1)^j \text{tr} (\mathcal{M}_j^2)^n.
\]

Let us take again a hierarchy of mass scales (corresponding for instance to a superGUT model) with a high scale supermultiplet at a mass scale \( \bar{M} \) and a low scale one at a sale \( \tilde{m} \). Let us also denote the corresponding supermultiplet mass matrices as \( \mathcal{M} \) and \( \mathcal{m} \). Equation (28) is then replaced by

\[
\Lambda_{cc}(\bar{M}) - \Lambda_{cc}(M_{KK}) \simeq \frac{\bar{M}^4 - M_{KK}^4}{32\pi^2} \text{Str}(\mathcal{M}^2)^0 - \frac{\bar{M}^2 - M_{KK}^2}{32\pi^2} \text{Str}\mathcal{M}^2 + \frac{1}{64\pi^2} \text{Str}\mathcal{M}^4 \ln \left( \frac{M^2}{M_{KK}^2} \right) + \ldots,
\]

and eqn. (29) by

\[
\Lambda_{cc}(\tilde{m}) - \Lambda_{cc}(M) \simeq \frac{\tilde{m}^4 - \bar{M}^4}{64\pi^2} \text{Str}(\mathcal{m}^2)^0 - \frac{\tilde{m}^2 - \bar{M}^2}{32\pi^2} \text{Str}\mathcal{m}^2 + \frac{1}{64\pi^2} \text{Str}\mathcal{m}^4 \ln \left( \frac{\bar{m}^2}{M^2} \right) + \ldots
\]

For a theory in which there is an equal number of fermionic and bosonic degrees of freedom and hence even for a broken supersymmetric theory, \( \text{Str}(\mathcal{M}^2)^0 = \text{Str}(\mathcal{m}^2)^0 = 0 \). Also of course (30) is unchanged. Hence we have for the low energy cosmological constant in a SUSY theory,

\[
\Lambda_{cc}(\mu \ll \bar{m}) \simeq \frac{\bar{M}^4 - M_{KK}^4}{32\pi^2} \text{Str}(\mathcal{M}^2)^0 - \frac{\bar{M}^2 - M_{KK}^2}{32\pi^2} \text{Str}\mathcal{M}^2 + \frac{1}{64\pi^2} \text{Str}\mathcal{M}^4 \ln \left( \frac{\bar{m}^2}{M^2} \right) + \ldots
\]

The subscript on the the supertrace instruction implies that it is to be taken over the supermultiplets at that scale. So for example in a superGUT theory the subscript \( \bar{M} \) implies the supertrace over the GUT scale supermultiplets and the subscript \( \tilde{m} \) implies the supertrace over the MSSM supermultiplets. It is assumed also that the splitting within a multiplet is much smaller than \( \bar{M} - \tilde{m} \).

Note that the second and third terms on the RHS of (34) are in fact positive since typically the \( \text{Str}\mathcal{M}^2(m^2) \) is positive. On the other hand the third and fourth terms are negative. The expression shows that whether the quantum correction to the “classical” CC generated by string theory (i.e. the initial condition for the evolution of the CC) does not necessarily have to be positive (see also [4]). Depending on the physics below the KK scale, it may be the case that the final CC (at cosmological scales) can indeed be positive even if the string theory generated CC is negative at the KK scale. For instance one could take the LVS minimum (before the so-called uplift which is less well established) with a negative CC albeit with broken SUSY. In the case of KKLMT however this is somewhat more problematic since the minimum before “uplift” is a SUSY preserving AdS space.

Let us now compare this formula (i.e. (34)) with what is obtained in dimensional regularization which gives (see Appendix eqn. (50))
\[ \Lambda_{cc}(\mu \ll m) = \Lambda_{cc}(M_{KK}) + \frac{1}{64\pi^2} \text{Str} \left( m^4 + M^4 \right) \ln \left( \frac{\mu^2}{M_{KK}^2} \right) \]  

(35)

The decoupling that is manifest in (34) is absent in the above - one needs to put it in by hand. In fact as a consequence it appears that this expression has an infrared divergence as \( \mu \to 0 \)! Now if we did not have an UV complete theory one usually thinks of the cutoff (which is here a physical scale) as a scale which is at the end of the day sent to infinity. \( \Lambda_{cc}(M_{KK}) \) would then be thought of as the “bare” CC which has no physical significance. The only number we have is the measured large distance CC i.e. the LHS of (34) (35). The difference in the RHS’s of two formulae is of no consequence and just means that the counter terms in the two schemes are different. In the first case one would need to subtract a quadratic (in a non-SUSY theory also a quartic) divergence as well as a log divergence whereas in the dim reg case only a log divergence would be subtracted. However once we have a meaningful UV complete theory such as string theory the initial value is not divergent and (for a given point in the landscape) has a well defined value. This means that only a physical cutoff scheme makes sense in this context.

### 3.2 Masses and couplings

Let us now discuss the evolution of the coupling \( \lambda \). Using (16) and comparing the coefficients of \( \phi_4^2 \) (for instance) on both sides we have,

\[
\beta_{\lambda} \equiv \mu \frac{d}{d\mu} \lambda = \frac{3\lambda^2}{16\pi^2} e^{-\frac{m^2}{\mu^2}} + \frac{3\eta^2}{16\pi^2} e^{-\frac{M^2}{\mu^2}}.
\]

(36)

Again we see very clearly from the heat kernel method how states decouple. Thus we have the following expressions for the beta function in three different regimes

\[
\beta_{\lambda} \simeq \frac{3\lambda^2}{16\pi^2} + \frac{\eta^2}{32\pi^2}, \mu \gg M \gg m.
\]

(37)

\[
\simeq \frac{3\lambda^2}{16\pi^2}, M \gg \mu \gg m
\]

(38)

\[
\simeq 0, m \gg \mu.
\]

(39)

This is in contrast to the result in dimensional regularization where the decoupling has to be introduced by hand. In fact if one did the usual calculation in dimreg the answer is just the first line above (37) as we’ve reviewed in the Appendix (see eqn. (49)).

If one ignores the evolution of \( \eta \) these equations can be integrated. Thus we have, starting with (39)

\[
\lambda(m) \simeq \lambda(\mu), \mu \ll m
\]

(40)

\[
\lambda(\mu) = \frac{\lambda(m)}{1 - \frac{3}{16\pi^2} \lambda(m) \ln \frac{\mu}{m}}, \mu \ll M < m \exp \left[ \frac{16\pi^2}{3\lambda(m)} \right]
\]

(41)

\[
\lambda(\mu) = \frac{\eta}{\sqrt{6}} \tan \left( \frac{3}{16\pi^2} \eta \ln \frac{\mu}{M} + \tan^{-1} \sqrt{6\lambda(M)}/\eta \right), M_{KK} \gg \mu > M
\]

(42)
We note in passing that in order to see how the last equation reduces to the second in the limit \( \eta \to 0 \) (as it should) we have to use the Taylor series expansion around infinity for the inverse tangent \( \tan^{-1} x = \frac{x}{2} - \frac{1}{x} + \ldots \). Also in the second equation the last condition is the requirement that the high mass threshold must be crossed before the Landau pole. A similar condition holds for the third equation - namely that \( M_{KK} \) should be less than the pole on the RHS of that equation when the argument of the tangent hits \( \pi/2 \).

Let us consider now the beta function for the light mass,

\[
\beta_{m^2} \equiv \frac{d}{d\mu} m^2 = -\frac{\lambda}{16\pi^2} \mu^2 e^{-\frac{m^2}{\mu^2}} - \frac{\eta}{16\pi^2} \mu^2 e^{-\frac{M^2}{\mu^2}}.
\]

(43)

Then in the three regimes we have

\[
\beta_{m^2} = -\frac{1}{16\pi^2} (\lambda + \eta) \mu^2 + \frac{1}{16\pi^2} (\lambda m^2 + \eta M^2) + O(M^2/\mu^2), \quad \mu \gg M \gg m,
\]

\[
\simeq -\frac{\lambda}{16\pi^2} \mu^2 + \frac{1}{16\pi^2} \lambda m^2, \quad m \ll \mu \ll M,
\]

\[
\simeq 0, \quad \mu \ll m.
\]

On the other hand what happens in dimreg is that one gets the first line of the above set of eqns without the first term since that corresponds to a quadratic divergence (in the usual non-Wilsonian discussion) that is absent in dimreg, i.e. we have

\[
\beta_{m^2} = \frac{1}{16\pi^2} (\lambda m^2 + \eta M^2),
\]

(44)

and as with the dimreg eqn. for \( \lambda \) this is valid at all scales.

But again one would not see the decoupling of the heavy states that is manifest in the second and third lines above. This phenomenon in dimreg for the couplings the masses and the CC is not surprising, since dimereg involves integration over all scales and decoupling has to be introduced by hand\(^9\).

4 Conclusions

The question we’ve addressed in this note is the evolution of the couplings down to long distance scales when the initial (“classical”) action is given by the effective field theory of some UV complete theory such as string theory. This EFT is expected to be valid up to the Kaluza-Klein scale if the UV theory is string theory. We have argued that in this case it makes more sense to use a physical cut-off when defining the Wilsonian action for scales well below the KK scale. The same is the case when obtaining the 1PI action which is supposed to incorporate all quantum fluctuations i.e. to arbitrarily low scales. It is defined by an initial “classical” action at the KK scale and then integrating all quantum fluctuations down to the long wave length limit to get the quantum effective action.

As we argued earlier, while in a purely field theoretic scenario (say with just renormalizable couplings) the issue of an initial action as a meaningful entity does not arise since it is essentially

\(^9\)For a very clear recent discussion of this see 9 section 7.2.3.
a cut off dependent object called the bare or unrenormalized action which in the limit when the cut off is removed goes to infinity. In fact in dimreg the analog of $\int d^D x$ for negative dimension (since the quartic divergence in the CC can only be regularized for negative $D$) is meaningless. Actually the situation is even worse since the procedure needs to be well defined for all real values of $D$ from negative values to $D = 4$. Thus a regularized all orders expression such as (7) which was possible in heat kernel regularization (or more generally in any such physical regulator scheme) makes no sense in dimreg.

Nevertheless dimreg remains by far the easiest way of calculating in perturbation theory beyond one-loop. The issue we have presented is a conceptual one. Once we have an initial “classical” action coming from a UV complete theory such as string theory a physical cutoff gives us a clear way of relating low energy physics to the UV theory. Furthermore this gives an explicit understanding of decoupling and most importantly shows us that depending on the nature of the physics in between the UV cutoff (KK) scale and the cosmological scale, a negative CC at the string/KK scale may still yield the observed positive CC. This would relieve the current tension between the observed positive CC at cosmological scales and the CC from string constructions which is typically negative.

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Appendix: Coleman-Weinberg one-loop potential in dimreg

Calculating with a a momentum space cutoff $\Lambda$ one gets (see for example [10])

$$V = V_0 + \frac{1}{64\pi^2} \text{Str} \mathcal{M}^0 \Lambda^4 \ln \frac{\Lambda^2}{\mu^2} + \frac{1}{32\pi^2} \text{Str} \mathcal{M}^2 \Lambda^2 + \frac{1}{64\pi^2} \text{Str} \mathcal{M}^4 \ln \frac{\mathcal{M}^2}{\Lambda^2} + \ldots$$  \hspace{1cm} (45)

$$\text{Str} \mathcal{M}^{2n} = \sum_j (2J + 1)(-1)^J M_j^{2n}$$

This is the same as our heat kernel regularized expression (21)(22) when $\Lambda^2 \gg |V''[\phi]|$. On the other hand in dimreg only the fourth term is present. For completeness we review the calculation below.

In dimensional regularization one may start with the D dimensional version of the proper time representation eqn. (9) but without the cut-off in the $s$-integral, i.e.

$$\Gamma^{(1)}_D = -\frac{1}{2} \int d^D x \sqrt{g} \int_0^\infty ds \text{tr} e^{-V''(\phi_c)s} \frac{1}{(4\pi s)^{D/2}}$$

While the space-time integral makes no sense for negative or non-integral $D$, the integral is well-defined for $D < 0$. Introducing the arbitrary mass scale $\mu$ and using the integral representation
for the Gamma function $\Gamma(z) = \int_0^\infty e^{-t^z} dt$ and writing $D = 4 - \epsilon$ and expanding in a Laurent series in $\epsilon$ we get (see for example Peskin and Schroeder \[11\] eqns. (11.77,78))

$$\Gamma^{(1)}_{4-\epsilon} = -\frac{1}{2} \int d^4 x \sqrt{g} \text{tr} \left( \frac{V''}{(4\pi)^2} \right)^2 \frac{1}{2} \left( \frac{2}{\epsilon} - \gamma + \ln(4\pi) - \ln \frac{V''}{\mu^2} + \frac{3}{2} + O(\epsilon) \right),$$

where $\mu$ is an arbitrary scale factor. In $\overline{\text{MS}}$ one adds the counter term

$$\delta S = \frac{1}{2} \int d^4 x \sqrt{g} \text{tr} \left( \frac{V''}{(4\pi)^2} \right)^2 \frac{1}{2} \left( \frac{2}{\epsilon} - \gamma + \ln(4\pi) \right),$$

to the original action (with couplings defined at the mass scale $\mu$) so that we have for the one-loop corrected quantum effective action (1PI action to one-loop)

$$\Gamma_{1\text{PI}} \simeq S_{\text{cl}}(\mu) + \lim_{\epsilon \to 0} \left( \delta S + \Gamma^{(1)}_{4-\epsilon} \right) = S_{\text{cl}}(\mu) + \frac{1}{2} \int d^4 x \sqrt{g} \text{tr} \left( \frac{V''}{(4\pi)^2} \right)^2 \frac{1}{2} \left( \ln \frac{V''}{\mu^2} - \frac{3}{2} \right)$$

In our toy model the classical potential is,

$$V(\phi, \Phi; \mu) = \Lambda_{\text{cc}}(\mu) + \frac{1}{2} m^2(\mu) \phi^2 + \frac{1}{2} M^2(\mu) \Phi^2 + \frac{\lambda(\mu)}{4!} \phi^4 + \frac{\eta(\mu)}{4} \phi^2 \Phi^2 + \ldots.$$

The $\beta$-functions are obtained by demanding that $\Gamma_{1\text{PI}}$ is independent of the arbitrary scale $\mu$. Thus we may read off the flow equations for the couplings:

$$\mu \frac{d\Lambda_{\text{cc}}}{d\mu} = \frac{1}{32\pi^2} \left( m^4 + M^4 \right)$$

$$\mu \frac{dm^2}{d\mu} = \frac{1}{16\pi^2} \left( \lambda m^2 + \eta M^2 \right)$$

$$\mu \frac{d\lambda(\mu)}{d\mu} = \frac{3}{16\pi^2} \left( (\lambda^2 + \eta^2) \right)$$

Let us focus on the cosmological constant. Integrating the first equation between $M_{\text{KK}}$ and cosmological scales $\mu \ll m$ we get (after generalizing to include also fermions and gauge bosons)

$$\Lambda_{\text{cc}}(\mu \ll m) = \Lambda_{\text{cc}}(M_{\text{KK}}) + \frac{1}{64\pi^2} \text{Str} \left( m^4 + M^4 \right) \ln \left( \frac{\mu^2}{M_{\text{KK}}^2} \right).$$

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