Self-consistency of the Two-Point energy Measurement protocol

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A thermally isolated quantum system undergoes unitary evolution by exchanging work with an external work source. The Two-Point energy Measurement (TPM) protocol defines the work exchanged with the system by performing ideal energy measurements on the system before, and after, the unitary evolution. However, the ideal energy measurements used in the TPM protocol ultimately result from a unitary interaction between the system of interest and a measurement apparatus, with such unitary interactions themselves generally requiring an exchange of work with an external source. For the TPM protocol to be self-consistent, we must be able to perform the TPM protocol on the compound of system plus apparatus, thus revealing the total work distribution, such that when ignoring the apparatus degrees of freedom, we recover the original TPM work distribution for the system of interest. In the present manuscript, we show that such self-consistency is satisfied so long as the apparatus is initially prepared in an energy eigenstate. Moreover, we demonstrate that if the apparatus Hamiltonian is equivalent to the “pointer observable”, then: (i) the total work distribution will satisfy the “average” first law of thermodynamics for all system states and system-only unitary processes; and (ii) the total work distribution will be identical to the system-only work distribution, for all system states and system-only unitary processes, if and only if the unitary interaction between system and apparatus does not exchange any work.

I. INTRODUCTION

The definition of work for quantum systems is one of the most contentious issues in quantum thermodynamics, and continues to be a subject of heated debate [1–12]. The paradigmatic scenario is the work done on a thermally isolated system: a system which is only mechanically manipulated, by means of inducing time-dependence on its Hamiltonian, and thus evolves unitarily. Such mechanical manipulation generally results in an exchange of work with an external work source. In the limiting case where the system starts and ends in a classical mixture of energy eigenstates, in any given realization the work done on the system is well defined, and is the difference in energy eigenvalues. By performing ideal energy measurements before, and after, the unitary evolution, one can therefore observe which particular value of work obtains in any given realization without disturbing the system. Furthermore, the average work done, given by the observed probability distribution over work, will be equivalent to the difference in average energies evaluated before, and after, the unitary evolution; the “average” first law of thermodynamics is satisfied. The Two-Point energy Measurement (TPM) protocol extends this procedure for determining the work distribution, namely, performing ideal energy measurements before and after the unitary evolution, to general unitary processes and general states [13, 14]. However, in general if the initial state does not commute with the Hamiltonian, the average first law will be violated; the average work obtained by the TPM protocol will not coincide with the difference in average energies. Indeed, as shown in Ref. [15], no measurement procedure exists which simultaneously recovers the work distribution for systems in a classical mixture of energy eigenstates, and recovers the average work as the difference in average energies, for all states and unitary processes.

That the TPM protocol cannot always satisfy the average first law ultimately rests on one of the central maxims of quantum measurement theory: no information without disturbance [16]. To be sure, ideal measurements are the least disturbing measurements available [17, 18], but only so far as there are some states that are undisturbed by such measurements. This perceived failure of the TPM definition has led to alternative formulations of work, such as defining work as the change in average energy (or change in non-equilibrium free energy for non-unitary processes) simpliciter [19–21], and the Margenau-Hill method and related approaches using quasi-probability distributions [22–26].

Of course, there is another issue raised by the TPM protocol, or indeed any method which uses measurement as part of the definition for work: can such a method be self-consistent? The ideal energy measurements used in the TPM protocol must ultimately result from a physical interaction between the system and a measurement apparatus. The quantum theory of measurement allows for the measurement of any observable to be physically modeled as a normal measurement scheme, which involves a unitary interaction between the system and a measurement apparatus which is initially prepared in a fixed pure state – a condition which is possible to satisfy in principle, thermodynamic limitations on preparing pure states notwithstanding [27–30] – followed by measurement of the apparatus by a sharp pointer observable [31]. Normal measurement schemes have been used to “indirectly” measure work [5, 32]. Of course, such unitary interactions between the system and apparatus themselves result from mechanically manipulating this compound system, and hence are generally accompanied with an exchange of work with an external work source. If the TPM definition of work is valid for the system of interest, therefore, it stands to reason that it is valid for the compound of system plus apparatus, by performing energy measurements.

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on both system and apparatus, before and after the total unitary evolution of the compound system, we thus obtain the total work distribution. We shall say that a given measurement scheme for the TPM protocol is self-consistent if the marginal work distribution for the system, obtained when ignoring the apparatus degrees of freedom, is identical to the original system-only TPM work distribution, for all system states and system-only unitary processes. In the present manuscript, we show that such self-consistency is always achieved if the apparatus is initially prepared in an energy eigenstate.

Interestingly, if we further restrict the measurement scheme such that the apparatus Hamiltonian is equivalent to the pointer observable used to measure the apparatus, then the total work distribution will always satisfy the average first law; the average total work will be the difference in average energy given the total unitary evolution, for all system states and system-only unitary processes (i.e., excluding the apparatus state, and the unitary interaction between system and apparatus, which are fixed by the chosen measurement scheme). This is a consequence of the strong repeatability of ideal energy measurements [18], which implies that given the unitary interaction between system and apparatus, followed by measurement of the apparatus by the pointer observable, “directly” performing an ideal energy measurement on the system is superfluous. Of course, this should not be taken as a refutation of [15], since the initial state of the apparatus is always fixed, and commutes with the Hamiltonian by construction. But this observation does illustrate that it is possible for the TPM work distribution to satisfy the average first law for a large class of initial states that do not commute with the Hamiltonian.

Finally, in the case where the apparatus is initially prepared in an energy eigenstate, and the apparatus Hamiltonian is equivalent to the pointer observable, we show that the total work distribution will be identical with the system-only work distribution, for all system states and system only unitary processes, if and only if the subspace of the apparatus which is involved during the measurement process corresponds with a single degenerate subspace of the apparatus Hamiltonian. This condition is further shown to be equivalent to the statement that the unitary interactions between system and apparatus do not result in any exchange of work with the external work source.

II. TPM PROTOCOL

We consider systems with a separable Hilbert space $\mathcal{H}$, with $\mathcal{L}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$, $\mathcal{T}(\mathcal{H}) := \{A \in \mathcal{L}(\mathcal{H}) : |\text{tr}[A]| < \infty\} \subseteq \mathcal{L}(\mathcal{H})$ the class of finite-trace operators, and $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ the space of positive unit-trace operators (states), respectively. Moreover, we shall assume that the system is thermally isolated, with a bounded, time-dependent Hamiltonian $H(t) = H + H_I(t)$. Here, $H$ is the system’s “bare” Hamiltonian, describing it when it is fully isolated, i.e., isolated both thermally and mechanically. We assume this Hamiltonian to have a discrete spectrum, and may thus write it as

$$H = \sum_m \epsilon_m P_m. \quad (1)$$

Here, $\epsilon_m$ are energy eigenvalues, and $P_m \geq \mathbb{O}$ the corresponding spectral projections such that $P_m P_n = \delta_{m,n} P_m$ and $\sum_m P_m = \mathbb{I}$. By the spectral theorem, the bare Hamiltonian $H$ is associated with a discrete, sharp observable $P := \{P_m\}$, where $m$ is the measurement outcomes which, given a state preparation $\rho \in \mathcal{S}(\mathcal{H})$, are observed with the probability $\text{tr}[P_m \rho] \quad [33]$.

The time-dependence of $H(t)$ is entirely due to the term $H_I(t)$, which results from thermally coupling the system with an external work source. If we assume that the system is only coupled with the work source for times $t \in (t_0, t_1)$, such that $H_I(t) = \mathbb{O}$ for all $t \leq t_0$ and $t \geq t_1$, then the system’s time evolution due to its interaction with the work source will be described by the unitary operator $V := \mathcal{T} \exp(-i \int_{t_0}^{t_1} dt H(t))$, where we note that throughout this manuscript we use $\hbar = 1 \quad [23]$.

The TPM protocol, for revealing the distribution of work due to the interaction between the system and the work source, is thus given by the following sequence of operations:

(i) At time $t = t_0$, perform an ideal measurement of the bare Hamiltonian on the system, which is initially in an arbitrary state $\rho$. Given that outcome $m$ is observed, the system will be prepared in the (unnormalized) state $P_m \rho P_m$. \quad (2)

(ii) Between time $t_0$ and $t_1$, let the system evolve unitarily, given its interaction with the external work source. The system will thus be prepared in the (unnormalized) state $VP_m \rho P_m V^\dagger$. \quad (3)

(iii) At time $t = t_1$, perform an ideal measurement of the bare Hamiltonian on the system. Given that outcome $n$ is observed, the system will be prepared in the (unnormalized) state $P_n V P_m \rho P_m V^\dagger P_n$. \quad (4)

The sequence of energy measurement outcomes $x := (m, n)$ thus corresponds with the work done $w(x) := \epsilon_n - \epsilon_m$, and its probability is given by the Born rule as the trace of the final unnormalized state Eq. (4), which reads

$$p^V_{\rho}(x) := \text{tr}[P_m V^\dagger P_n V \rho]. \quad (5)$$

Therefore the probability distribution for the work done, $w$, given the initial state $\rho$ and unitary operator $V$, is

$$p^V_{\rho}(w) := \sum_x \delta(w - w(x)) p^V_{\rho}(x), \quad (6)$$

where $\delta$ is the Dirac delta function.
where $\delta(a - b) = 1$ if $a = b$, and is zero otherwise. The average work can thus be computed to be
\[
\langle w \rangle^V := \sum_w p^V_w (w) w = \sum_x p^V_x (x) w(x),
\]

\[
= \text{tr}[(V^\dagger HV - H)\rho],
\]

\[
\equiv \text{tr}[(V^\dagger HV - H) T_{A^L}(\rho)],
\]

(7)

where $T^L_{A^L}(\cdot) := \sum_m P_m (\cdot) P_m$ is the Lüders channel for the bare Hamiltonian $H$. Given that for any $A \in \mathcal{L}(\mathcal{H})$, $T^L_{A^L}(A) = A$ if and only if $[H, A] = 0$ [34], it follows that $(\langle w \rangle^V)_\rho = \text{tr}[(V^\dagger HV - H)\rho]$ for all $V$ (for all $\rho$) only if $[H, \rho] = 0$ ([$H, V^\dagger HV = 0$]). In other words, the average first law cannot be satisfied for all states and all unitary processes.

A. Introducing the measurement apparatus in the TPM protocol

As shown above, the TPM protocol relies on performing ideal energy measurements on the system of interest both before, and after, the unitary evolution $V$. Such measurements are physically realized by an appropriate interaction between the system of interest and a measurement apparatus. The quantum theory of measurement allows all measurements on the system of interest to be modeled as a normal measurement scheme [31, 35]. Here, the system of interest first interacts with a measurement apparatus, initially prepared in a fixed pure state, by an appropriate unitary operator. Subsequently, the apparatus is measured by an appropriate pointer observable, and the measurement outcome observed indicates that the corresponding outcome has been observed for the desired system observable.

Since two energy measurements are performed on the system during the TPM protocol, we can generally consider the apparatus to be composed of two parts, one of which interacts with the system at time $t = t_0$, and the other at time $t = t_1$. As such, for the ideal energy measurement performed at time $t_1$, we may mathematically describe the normal measurement scheme by the tuple $(\mathcal{H}^{(i)}_A, |\xi^{(i)}\rangle, U^{(i)}, Z^{(i)})$, where $\mathcal{H}^{(i)}_A$ is the Hilbert space for the apparatus used, which is initially prepared in the pure state $|\xi^{(i)}\rangle; Z^{(i)} := \{Z^{(i)}_m\}$ is a sharp pointer observable, which has the same outcomes as the system observable $P := \{P_m\}$; and $U^{(i)}$ is a joint unitary operator on the compound Hilbert space $\mathcal{H} \otimes \mathcal{H}^{(i)}_A$. This normal measurement scheme will realize an ideal measurement of the bare Hamiltonian on the system of interest if, for all $T \in \mathcal{T}(\mathcal{H})$ and $m$, we have

\[
\text{tr}_{\mathcal{H}^{(i)}_A}[(1 \otimes Z^{(i)}_m) U^{(i)} (T \otimes P[\xi^{(i)}]) U^{(i)\dagger}] = P_m T P_m,
\]

(8)

where $P[\xi^{(i)}] \equiv |\xi^{(i)}\rangle \langle \xi^{(i)}|$ is a projection on the unit vector $|\xi^{(i)}\rangle \in \mathcal{H}^{(i)}_A$, and $\text{tr}_{\mathcal{H}^{(i)}_A} : \mathcal{T}(\mathcal{H} \otimes \mathcal{H}^{(i)}_A) \to \mathcal{T}(\mathcal{H})$ is the partial trace over the apparatus, defined as $\text{tr}[(A \otimes 1)T] = \text{tr}[Atr_{\mathcal{H}^{(i)}_A}[T]]$ for all $A \in \mathcal{L}(\mathcal{H})$ and $T \in \mathcal{T}(\mathcal{H} \otimes \mathcal{H}^{(i)}_A)$ [36, 37]. It is simple to verify that in order for the unitary $U^{(i)}$ to satisfy Eq. (8), it must satisfy

\[
U^{(i)}(|\psi\rangle \otimes |\xi^{(i)}\rangle) = \sum_m P_m |\psi\rangle \otimes |\phi^i_m\rangle \equiv |\psi\rangle \otimes \sum_m P_m |\phi^i_m\rangle.
\]

(9)

for all $|\psi\rangle \in \mathcal{H}$, where $|\phi^i_m\rangle$ are eigenstates of the projection operators $Z^{(i)}_m$, i.e., $Z^{(i)}_m |\phi^i_m\rangle = \delta_{m,n} |\phi^i_m\rangle$ [38].

The TPM protocol can now be performed as follows:

(i) At time $t = t_0$, bring the system, initially prepared in an arbitrary state $\rho$, in contact with the apparatus $\mathcal{H}(0)^A$. The state of the compound system is thus $\rho \otimes P[\xi^{(0)}]$. Subsequently let the system interact with the apparatus by the unitary operator $U^{(0)}$, which prepares the state

\[
U^{(0)}(\rho \otimes P[\xi^{(0)}]) U^{(0)\dagger} = \sum_{m,m'} P_m \rho P_{m'} \otimes |\phi^0_m\rangle \langle \phi^0_{m'}|.
\]

(10)

Finally, perform a measurement of the apparatus by the pointer observable $Z^{(0)}$. Given that outcome $m$ is observed, the system will be prepared in the (unnormalized) state

\[
P_m \rho P_m.
\]

(11)

(ii) Between time $t_0$ and $t_1$, let the system evolve unitarily, given its interaction with the external work source. The system will thus be prepared in the (unnormalized) state

\[
V P_m \rho P_m V^\dagger.
\]

(12)

(iii) At time $t = t_1$, bring the system in contact with the apparatus $\mathcal{H}(1)^A$. The (unnormalized) state of the compound system is thus $V P_m \rho P_m V^\dagger \otimes P[\xi^{(1)}]$. Subsequently let the system interact with the apparatus by the unitary operator $U^{(1)}$, which prepares the (unnormalized) state

\[
U^{(1)}(V P_m \rho P_m V^\dagger \otimes P[\xi^{(1)}]) U^{(1)\dagger} = \sum_{n,n'} P_n V P_m \rho P_m V^\dagger P_{n'} \otimes |\phi^{(1)}_n\rangle \langle \phi^{(1)}_{n'}|.
\]

(13)

Finally, perform a measurement of the apparatus by the pointer observable $Z^{(1)}$. Given that outcome $n$ is observed, the system will be prepared in the (unnormalized) state

\[
P_n V P_m \rho P_m V^\dagger P_n.
\]

(14)

It is evident that the measurement scheme described above is identical to the original TPM protocol involving “direct” measurements on the system.
III. CONSISTENTLY APPLYING THE TPM PROTOCOL TO BOTH SYSTEM AND APPARATUS

The measurement scheme introduced in Sec. II A does not make any assumptions regarding the Hamiltonian of the apparatus, nor the time it takes for the unitary operators $U^{(i)}$ to be generated; indeed, these were assumed to be implemented instantaneously. However, for the unitary operator $U^{(i)}$ on $\mathcal{H} \otimes \mathcal{H}^{(i)}$ to be physical, it must also result from mechanically manipulating the Hamiltonian of this composite system, and thus requires an interaction with an external work source for a finite duration [39]. Let us therefore write the total time-dependent Hamiltonian as $H_{\text{tot}}(t) = H_{\text{tot}} + H(t) + H^{(0)}(t) + H^{(1)}(t)$, where $H_{\text{tot}} = H + H_A^0 + H_A^{(1)}$ is the additive, total bare Hamiltonian of system plus apparatus, and $H(t)$ is the system-only interaction Hamiltonian introduced in Sec. II. We shall denote the bare Hamiltonian of each apparatus in the spectral form as

$$H_A^{(i)} = \sum_{\mu} \lambda^{(i)}(\mu) Q^{(i)}_{\mu},$$

where $\lambda^{(i)}(\mu)$ are energy eigenvalues and $Q^{(i)}_{\mu}$ the spectral projections. The interaction Hamiltonian for the composite system $\mathcal{H} \otimes \mathcal{H}^{(i)}$, due to coupling with an external work source, is denoted $H_{\text{int}}^{(i)}(t)$. Moreover, $H_{\text{int}}^{(i)}(t) = 0$ for all $t < t_0$ and $t > t_0$, and similarly $H_{\text{int}}^{(i)}(t) = 0$ for all $t < t_1$ and $t > t_1$, where $t_0 < t_0 < t_1 < t_1$. In other words, the interaction Hamiltonian $H_{\text{int}}^{(i)}(t)$ is nonvanishing only for a finite duration before the system undergoes its isolated unitary evolution $V$, and similarly $H_{\text{int}}^{(i)}(t)$ is nonvanishing only for a finite duration after the system undergoes its isolated unitary evolution $V$. Therefore, by choosing the interaction Hamiltonians $H_{\text{int}}^{(i)}(t)$ appropriately so that $\mathcal{T} \exp \left(-i \int_{t_0}^{t_1} dt H_{\text{tot}}(t) \right) = U^{(0)}$ and $\mathcal{T} \exp \left(-i \int_{t_0}^{t_1} dt H_{\text{tot}}(t) \right) = U^{(1)}$, the total unitary operator which describes the compound system’s evolution during the extended period $t \in (t_0, t_1)$ will be

$$V_{\text{tot}} := \mathcal{T} \exp \left(-i \int_{t_0}^{t_1} dt H_{\text{tot}}(t) \right),$$

$$= U^{(1)} (V \otimes e^{-i \theta_5 H_A^{(0)}(0)} \otimes e^{-i \theta_1 H_A^{(1)}(0)}) U^{(0)},$$

$$= e^{-i \theta_0 H_A^{(0)}(0)} U^{(1)} V U^{(0)} e^{-i \theta_1 H_A^{(1)}(0)}. \quad (16)$$

Here, $e^{-i \theta_0 H_A^{(0)}}$, where $\theta_0 = t'_1 - t_0$ and $\theta_1 = t_1 - t'_0$, describes the contribution to the total unitary evolution from the bare Hamiltonian of apparatus $\mathcal{H}_A^{(i)}$, i.e., for the time period where the interaction Hamiltonian $H_{\text{int}}^{(i)}(t)$ vanishes. Note that the final line of Eq. (16) is obtained because the unitary operators $e^{-i \theta_0 H_A^{(0)}}$ and $e^{-i \theta_1 H_A^{(1)}}$ commute with $U^{(1)}$ and $U^{(0)}$, respectively, since they act on different Hilbert spaces.

Now we may perform the TPM protocol on the total compound system so as to determine the total work distribution given the total unitary operator in Eq. (16). For this to be consistent with the original TPM protocol on the system alone, however, we require that when averaging out the energy measurements performed on the apparatus, we must obtain the probability distribution given in Eq. (5), for all system states $\rho \in \mathcal{S}(\mathcal{H})$ and system-only unitary operators $V$. In order for this to be satisfied, we demand that $|\xi^{(i)}(\mu)\rangle$ be an eigenstate of the apparatus Hamiltonian $H_A^{(i)}$, with eigenvalue $\lambda^{(i)}(\mu)$. This will ensure that the initial ideal energy measurement of the apparatus will not disturb it, so that the unitary interaction between system and apparatus by the unitary operators $U^{(i)}$ will result in the same state transformation as discussed in Sec. II A.

Let us first note that, given the assumption that the apparatus is initially prepared in an energy eigenstate, and using Eq. (9) and Eq. (16), we can show that for all $|\psi\rangle \in \mathcal{H}$,

$$V_{\text{tot}}(|\psi\rangle \otimes |\xi\rangle) = e^{-i \theta_0 \lambda^{(i)}(\mu)} \sum_{n,n'} P_{n}^{(0)} P_{n'}^{(0)} |e^{-i \theta_1 H_A^{(0)}} |\phi_n^{(0)}\rangle \otimes |\phi_{n'}^{(1)}\rangle,$$ \quad (17)

where $|\xi\rangle := |\xi^{(0)}\rangle \otimes |\xi^{(1)}\rangle \in \mathcal{H}_A := \mathcal{H}_A^{(0)} \otimes \mathcal{H}_A^{(1)}$ is the initial state of the total apparatus. Here, we have used the fact that $|\xi^{(1)}\rangle$ is an energy eigenstate with eigenvalue $\lambda^{(i)}(\mu)$ to infer that the component of $V_{\text{tot}}$ given by $e^{-i \theta_0 H_A^{(0)}}$ only induces a constant phase factor $e^{-i \theta_1 \lambda^{(i)}(\mu)}$, which is not physically observable. Using this, we may now examine the extended TPM protocol, which will be as follows:

(i) At time $t = t'_0$, perform an ideal energy measurement on the total compound system $\mathcal{H} \otimes \mathcal{H}_A$, initially prepared in the state $\rho \otimes |\xi\rangle$. Since the apparatus $\mathcal{H}_A^{(i)}$ is initially prepared in an eigenstate of the apparatus Hamiltonian $H_A^{(i)}$, with energy eigenvalue $\lambda^{(i)}(\mu)$, only the outcomes $(m, 0, 0)$ are observed with non-zero probability, which result in the compound system being prepared in the (unnormalized) state

$$P_{m}^{(0)} P_{n}^{(0)} Q_{n}^{(0)} P|\xi^{(0)}\rangle \otimes Q_{n}^{(1)} P|\xi^{(1)}\rangle Q_{n}^{(1)} = P_{m}^{(0)} P_{n}^{(0)} \otimes P|\xi^{(1)}\rangle.$$

(ii) Between time $t'_0$ and $t'_1$, let the system evolve according to the total unitary operator $V_{\text{tot}}$ defined in Eq. (16) and Eq. (17). This prepares the (unnormalized) state

$$V_{\text{tot}}(P_{m}^{(0)} P_{n}^{(0)} \otimes P|\xi^{(0)}\rangle \otimes P|\xi^{(1)}\rangle V_{\text{tot}}^\dagger$$

$$= \sum_{m, n'} P_{m}^{(0)} P_{n'}^{(0)} V P_{n'}^{(0)} \otimes P|\phi_n^{(0)}\rangle \otimes |\phi_{n'}^{(1)}\rangle.$$ \quad (19)

where $\tilde{P}|\phi_n^{(0)}\rangle := e^{-i \theta_0 H_A^{(0)}} P|\phi_n^{(0)}\rangle e^{i \theta_1 H_A^{(1)}}$. 


(iii) At time $t = t'$, perform an ideal energy measurement on the total compound system $\mathcal{H} \otimes \mathcal{H}_A$. Given the outcomes $(n, \mu, \nu)$, this prepares the (unnormalized) state

$$P_m V_m \rho_m V_m^\dagger = \sum_{n, \mu} P_m \rho_m \sigma_m^0 \Pi_{m,0}^\dagger \Pi_{m,0} = \sum_{n, \mu} P_m \rho_m \Pi_{m,0}^\dagger \Pi_{m,0} |\phi_m^0⟩ \langle \phi_m^0|.$$

The full sequence of measurement outcomes is thus $X := (x, (m, n), (0, \nu))$, where $x := (m, n)$ is the sequence of outcomes for the system, while $(0, \mu)$ and $(0, \nu)$ are the sequences of outcomes for apparatus $\mathcal{H}_A^{(0)}$ and $\mathcal{H}_A^{(1)}$, respectively. The sequence $X$ corresponds with the total work done $\mathcal{W}(X) := w(x) + w_m(0) + w_n(0)$, where $w(x) := \epsilon_n - \epsilon_m$ is the contribution to the total work from the system, while $w_m(0)$ := $\lambda_m^0 - \lambda_0^0$ is the contribution to the total work from apparatus $\mathcal{H}_A^{(1)}$. The probability of observing sequence $X$, meanwhile, is given by the trace of the final unnormalized state Eq. (20), which is

$$p^X_{ρ,ξ}(x) := p^X_{ρ}(x)\text{tr}[Q_m^0 \rho_m |\phi_m^0⟩ \langle \phi_m^0|] = p^X_{ρ}(x),$$

where we recall that $p^X_{ρ}(x)$ is defined in Eq. (5). Note that here, we have used the fact that $Q_m^0$ is a spectral projection of $H_m^{(0)}$ to infer that $\text{tr}[Q_m^0 \rho_m |\phi_m^0⟩ \langle \phi_m^0|] = \text{tr}[Q_m^0 e^{-iθ_m H_m^{(0)}} |\phi_m^0⟩ \langle \phi_m^0|] = \text{tr}[Q_m^0 \rho_m |\phi_m^0⟩ \langle \phi_m^0|].$

Given that $\sum_{\mu} Q_m^0 = \sum_{\mu} Q_n^0 = 1$, the marginal probability distribution for the system-only work will read as

$$\sum_{\mu, \nu} p^X_{ρ,ξ}(X) = p^X_{ρ}(x)\text{tr}[P_m |\phi_m^0⟩ \langle \phi_m^0|] = p^X_{ρ}(x),$$

and so the extended TPM protocol on the compound of system plus apparatus is self-consistent.

A. Satisfying the average first law for the total work

Since the apparatus is initially prepared in an energy eigenstate, the average total work, for the total unitary process discussed in the previous section, clearly satisfies

$$\langle \mathcal{W} \rangle^X_{ρ,ξ} := \sum_X p^X_{ρ,ξ}(X) \mathcal{W}(X),$$

$$= \text{tr}[(V_{tot}^+ H_{tot} V_{tot} - H_{tot})\mathcal{L}_M(ρ) \otimes P[ξ]].$$

To see this, simply compare with Eq. (7). As before, if $ρ$ does not commute with the Hamiltonian $H$, the total work is not guaranteed to satisfy the average first law, i.e., it is possible for some $ρ$ and $V$ to have $\langle \mathcal{W} \rangle^X_{ρ,ξ} \neq \text{tr}[(V_{tot}^+ H_{tot} V_{tot} - H_{tot})ρ \otimes P[ξ]]$ (note that both the apparatus state $|ξ⟩$, and the contribution to $V_{tot}$ from the system-apparatus coupling, i.e., the unitaries $U_l^{(l)}$, are always fixed). However, as we shall show below, if additionally the apparatus Hamiltonian $H_A^{(i)}$ is equivalent to the pointer observable $Z_A^{(i)}$, i.e., if we have

$$H_A^{(i)} = \sum_m \lambda_m^{(i)} Z_m^{(i)},$$

the average first law is guaranteed to be satisfied for the total work.

Let us re-examine the TPM protocol on the compound of system plus apparatus once more in detail, this time assuming that Eq. (24) holds:

(i) At time $t = t'$, perform an ideal energy measurement on the total compound system $\mathcal{H} \otimes \mathcal{H}_A$, initially prepared in the state $ρ \otimes P[ξ]$. As before, we assume that $|ξ^0⟩$ is an eigenstate of $H_A^{(i)}$ with eigenvalue $λ_0^{(i)}$, and hence only the outcomes $(m, 0, 0)$ are observed with non-zero probability, resulting in the compound system to be prepared in the (unnormalized) state

$$P_m \rho_m \otimes Z_m^0 P[ξ^0] Z_0^0 \otimes Z_m^0 P[ξ^0] Z_0^0 = P_m \rho_m \otimes P[ξ^0] \otimes P[ξ^0].$$

(ii) Between time $t'_0$ and $t'$, let the system evolve according to the total unitary operator $V_{tot}$ defined in Eq. (16) and Eq. (17). This prepares the (unnormalized) state

$$V_{tot}(P_m \rho_m \otimes P[ξ^0] \otimes P[ξ^0]) V_{tot}^+ = \sum_{n, n'} P_n V_m \rho_m V_m^+ P_m \otimes P[ξ^0] \otimes P[ξ^0].$$

Note that since $|φ_m^0⟩$ are eigenstates of the projection operators $Z_m^0$, which clearly commute with the Hamiltonian, it follows that $e^{-iθ_m H_m^{(0)}} |φ_m^0⟩ \langle φ_m^0| = P[ξ^0] |φ_m^0⟩ \langle φ_m^0|.$

(iii) At time $t = t'$, perform an ideal energy measurement on the total compound system $\mathcal{H} \otimes \mathcal{H}_A$. Given outcomes $(n, n', n''$, $)$, this prepares the (unnormalized) state

$$P_n V_m \rho_m V_m^+ P_n \otimes Z_n^0 P[ξ^0] Z_n^0 \otimes Z_n^0 P[ξ^0] Z_n^0 = \delta_{m, m'} \delta_{n, n'} P_n V_m \rho_m V_m^+ P_n \otimes P[ξ^0] \otimes P[ξ^0].$$

Equation (27) implies that the only sequences of energy measurement outcomes that are observed with non-zero probability are $X := (x, (0, m), (0, n))$, where we recall that $x := (m, n)$. In other words, the energy transitions of the apparatus fully determine the energy transitions of the system, and vice versa. As such, let us remove some of the
redundancy and write \( X := ((0, 0), x) \), where \((0, 0)\) denotes the energy measurement outcomes on the apparatus at time \( t_0' \), and \( x = (m, n) \) denotes both the energy measurement outcomes on the apparatus at time \( t_1' \), as well as the sequence of energy measurement outcomes on the system at times \( t_0, t_1 \). The total work done given the sequence \( X \) is thus \( W(X) := w(x) + w^{(0)}_\lambda(m) + w^{(1)}_\lambda(n) \), where \( w^{(i)}_\lambda(m) := \lambda^{(i)}_m - \lambda^{(i)}_0 \), with the probability \( p^{V_{\rho,\xi}}_\rho(X) = p^{V}_\rho(x) \). (28)

Note that this is equivalent to Eq. (21) when we replace \( Q^{(i)}_\rho \) with \( Z^{(i)}_m \), which gives \( \text{tr}[Z^{(i)}_m P[\rho^{(i)}]] = 1 \).

As shown in Appendix (A), the average total work reads as
\[
\langle W \rangle^{V_{\rho,\xi}} := \sum_X p^{V_{\rho,\xi}}_\rho(X)W(X),
\]
\[
= \text{tr}[(V^{\dagger}_\rho H_{\text{tot}}V_{\rho} - H_{\text{tot}})(\rho \otimes P[\xi])],
\]
for all \( \rho \in \mathcal{S}(\mathcal{H}) \) and \( V \). As such, we see that so long as the apparatus is initially prepared in an energy eigenstate, and Eq. (24) is satisfied, then not only will the TPM protocol for the total work be self-consistent, but the total work will always satisfy the average first law, even for initial system states \( \rho \) that do not commute with the Hamiltonian. As a final observation, note that Eq. (23) must be equivalent to Eq. (29) when Eq. (24) is satisfied. Consequently, the following equality holds:
\[
\text{tr}[(V^\dagger_\rho H_{\text{tot}}V_{\rho} - H_{\text{tot}})(\rho \otimes P[\xi])]
\]
\[
= \text{tr}[(V^\dagger_\rho H_{\text{tot}}V_{\rho} - H_{\text{tot}})(\mathcal{I}_{\text{tot}}(\rho) \otimes P[\xi])]
\]
(30)
for all \( \rho \in \mathcal{S}(\mathcal{H}) \). This is a consequence of the strong repeatability of ideal energy measurements [18], which implies that directly performing ideal energy measurements on the system is redundant; the structure of the unitary operators \( U^{(i)} \), together with the fact that we measure the apparatus by the pointer observable \( Z^{(i)} \), ensures that the system automatically undergoes an ideal energy measurement.

B. Necessary and sufficient conditions for the total work distribution to be equal to the system-only work distribution

If the apparatus is initially prepared in an energy eigenstate, and the apparatus Hamiltonian is equivalent to the pointer observable, the probability distribution for the total work \( \mathcal{W} \), given an initial total state \( \rho \otimes P[\xi] \) and total unitary operator \( V_{\rho,\xi} \), is
\[
p^{V_{\rho,\xi}}_\rho(\mathcal{W}) := \sum_X \delta(\mathcal{W} - W(X))p^{V_{\rho,\xi}}_\rho(X),
\]
\[
= \sum_X \delta(\mathcal{W} - (w(x) + w^{(0)}_\lambda(m) + w^{(1)}_\lambda(n)))p^{V}_\rho(x),
\]
(31)
where we have used Eq. (28), together with the definition \( w^{(i)}_\lambda(m) := \lambda^{(i)}_m - \lambda^{(i)}_0 \).

It is simple to see that, in general, the total work probability distribution Eq. (31) is different from the system-only work distribution Eq. (6). In order for these distributions to be the same, for all system states \( \rho \) and system-only unitary operators \( V \), we must have \( w^{(i)}_\lambda(x) = 0 \) for all \( x \) such that \( p^{V}_\rho(x) > 0 \) for some \( \rho \) and \( V \). This ensures that for all \( \rho \) and \( V \),
\[
\sum_x \delta(w - w(x))p^{V}_\rho(x) = \sum_x \delta(w - W(X))p^{V}_\rho(x).
\]
(32)

Recall that \( w^{(i)}_\lambda(m) := \lambda^{(i)}_m - \lambda^{(i)}_0 \), where \( \lambda^{(i)}_0 \) is a fixed energy eigenvalue, and that \( p^{V}_\rho(x) := \text{tr}[P\rho V\rho T\rho P\rho V\rho] \). Consequently, the condition \( w^{(i)}_\lambda(x) = 0 \) for all \( x \) such that \( p^{V}_\rho(x) > 0 \) for some \( \rho \) and \( V \) is equivalent to the condition \( \lambda^{(i)}_m = \lambda^{(i)}_0 \) for all \( m \) such that \( P_m > 0 \), i.e., if only a single degenerate energy subspace of the apparatus is involved during the measurement process. Interestingly, we shall see that this condition is equivalent to the statement that the unitary operator \( U^{(i)} \) does not result in any exchanged work with the external work source.

We define by \( \Gamma^{(i)}_{\xi} : \mathcal{L}(\mathcal{H} \otimes \mathcal{H}^{(i)}_{A}) \rightarrow \mathcal{L}(\mathcal{H}) \) the restriction map for \( |\xi^{(i)}\rangle \), defined by the identity
\[
\text{tr}[\Gamma^{(i)}_{\xi}(B)T] = \text{tr}[B(T \otimes P[\xi^{(i)}])]
\]
(33)
for all \( B \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}^{(i)}_{A}) \) and \( T \in \mathcal{L}(\mathcal{H}) \) [40]. Therefore, defining \( H^{(i)} = H + H^{(i)}_{A} \), and recalling that the unitary operator \( U^{(i)} \) always satisfies Eq. (9), we have
\[
\Gamma^{(i)}_{\xi} \left(U^{(i)}H_{\text{tot}}^{(i)}U^{(i)} - H_{\text{tot}}^{(i)}\right) = \sum_m w^{(i)}_\lambda(m)P_m.
\]
(34)
For a detailed proof, refer to Appendix (B). The right hand side of Eq. (34) vanishes if for each \( m \), either \( P_m = 0 \), or \( w^{(i)}_\lambda(m) = 0 \). Consequently, \( w^{(i)}_\lambda(m) = 0 \) for all \( m \) such that \( P_m > 0 \) is necessary and sufficient for the left hand side of Eq. (34) to vanish. But this implies that \( \text{tr}[(U^{(i)}H^{(i)}_{\text{tot}}U^{(i)} - H^{(i)}_{\text{tot}})(\rho \otimes P[\xi^{(i)}])] = 0 \) for all \( \rho \in \mathcal{S}(\mathcal{H}) \); given that the apparatus is in the state \( |\xi^{(i)}\rangle \), then irrespective of what state the system is prepared in, the unitary operator \( U^{(i)} \) will not result in any exchange of work. We refer to this as \( U^{(i)} \) effectively conserving the total energy, which is a weaker condition than full energy conservation, i.e., \( [H^{(i)}_{\text{tot}}, U^{(i)}] = 0 \), which implies that \( \text{tr}[(U^{(i)}H^{(i)}_{\text{tot}}U^{(i)} - H^{(i)}_{\text{tot}})] = 0 \) for all \( \rho \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}^{(i)}_{A}) \).

We note that while a fully degenerate apparatus Hamiltonian, \( H^{(i)}_{A} = \lambda^{(i)}_0 \mathbb{1} \), or a fully energy conserving unitary, \( [H^{(i)}_{\text{tot}}, U^{(i)}] = 0 \), are sufficient conditions for the total work distribution Eq. (31) to equal the system-only work distribution Eq. (6), they are not necessary.

To illustrate the first point, consider the system Hamiltonian \( H = \epsilon_1 P_1 + \epsilon_2 P_2 \), where \( P_1, P_2 > 0 \). However, this is equivalent to \( H = \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3 \) such that \( P_3 = 0 \). Therefore, the ideal measurement of \( H \) can be realized by the normal measurement scheme \( (\mathcal{H}^{(i)}_{A}, \xi, U, Z) \),
with the three-valued pointer observable $Z := \{Z_1, Z_2, Z_3\}$, $Z_m > 0$, and the unitary operator $U$ which satisfies

$$U(|\psi \rangle \otimes |\xi \rangle) = \sum_{m=1}^{3} P_m |\psi \rangle \otimes |\phi_m \rangle,$$

for all $|\psi \rangle \in \mathcal{H}$, where $|\phi_m \rangle$ are eigenstates of $Z_m$. Note that the term for $m = 3$ vanishes, since $P_3|\psi \rangle = O|\psi \rangle = 0$ for all $|\psi \rangle$; the apparatus is never taken to the state $|\phi_3 \rangle$.

Let the apparatus have the Hamiltonian $H_A = \lambda(Z_1 + Z_2) + \lambda'Z_3$, where $\lambda \neq \lambda'$, so that $H_A$ is not fully degenerate. Notwithstanding, if $|\xi \rangle$ is in the support of $Z_1 + Z_2$, we still have $w_A(m) = 0$ for $m = 1, 2$, i.e., for all $m$ corresponding to $P_m > 0$. As stated previously, it is only necessary that a single degenerate energy subspace of the apparatus be “involved” during the measurement process; for all measurement outcomes that are observed, the state of the apparatus starts and ends in the projective subspace of $Z_1 + Z_2$.

To illustrate that full energy conservation by the unitary is also not necessary, consider the single case where $\mathcal{H} \simeq C^2$, with orthonormal basis $\{|0\rangle, |1\rangle\}$, and Hamiltonian $H = \epsilon|1\rangle\langle 1|$, $\epsilon > 0$. A normal measurement scheme for an ideal measurement of $H$ can be given as $(\mathcal{H}_A, |0\rangle, U, Z)$, where $\mathcal{H}_A \simeq C^2$, $Z := \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, and

$$U : \{ |m, 0 \rangle \mapsto |m, m \rangle, |m, 1 \rangle \mapsto |m \oplus_2 1, m \rangle \},$$

where $m = 0, 1$ and $\oplus_2$ denotes addition modulo 2. Note that only the transformation $|m, 0 \rangle \mapsto |m, m \rangle$ is ever utilized, since the apparatus is initially prepared in state $|0\rangle$. If the apparatus Hamiltonian is fully degenerate, $H_A = \lambda I$, then $U$ will effectively conserve the total energy; given $H_{\text{tot}} = H + H_A$, then for any $|\psi \rangle = \alpha|0\rangle + \beta|1\rangle$, we have $\langle \psi, 0|U|H_{\text{tot}}|U|\psi, 0 \rangle = |\beta|^2 \epsilon + \lambda = \langle \psi, 0|H_{\text{tot}}|\psi, 0 \rangle$. However, $[U, H_{\text{tot}}] \neq 0$, since $U|1, 1 \rangle = |0, 1 \rangle$, and hence $\langle 1, 1|U|H_{\text{tot}}|U|1, 1 \rangle = \lambda \neq \langle 1, 1|H_{\text{tot}}|1, 1 \rangle = \epsilon + \lambda$.

### IV. CONCLUSIONS

A definition for work which relies on measurements is self-consistent if it can account for the contribution to work by the measurement process itself, at least in principle. More precisely, for self-consistency we demand that the marginal of the total work distribution for system and measurement apparatus, obtained by ignoring the apparatus degrees of freedom, recovers the original work distribution for the system alone. In the case of the Two-Point energy Measurement (TPM) protocol, we have shown that this is possible so long as the measurement apparatus is initially prepared in an energy eigenstate. Furthermore, if the apparatus Hamiltonian is chosen to be equivalent to the pointer observable, then the total work distribution will always satisfy the average first law; the average total work will equal the change in average energy given the total unitary evolution. This is shown to be a consequence of the repeatability of ideal energy measurements, which implies that directly performing energy measurements on the system is redundant. Finally, we have shown that the total work distribution will be identical to the system-only work distribution if and only if the measurement process does not exchange any work. Extending the present framework of analysis to other definitions of work remain as open questions for further research.

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[1] A. E. Allahverdyan and T. M. Nieuwenhuizen, Phys. Rev. E 71, 066102 (2005).
[2] P. Tárnok, E. Lutz, and P. Hänggi, Phys. Rev. E 75, 050102 (2007).
[3] P. Skrzypczyk, A. J. Short, and S. Popescu, Nat. Commun. 5, 4185 (2014).
[4] G. Watanabe, B. P. Venkatesh, and P. Tárnok, Phys. Rev. E 89, 052116 (2014).
[5] A. J. Roncaglia, F. Cerisola, and J. P. Paz, Phys. Rev. Lett. 113, 250601 (2014).
[6] N. Y. Halpern, A. J. P. Garner, O. C. O. Dahlsten, and V. Vedral, New J. Phys. 17, 095003 (2015).
[7] J. Gemmer and J. Anders, New J. Phys. 17, 085006 (2015).
[8] R. Gallego, J. Eisert, and H. Wilming, New J. Phys. 18, 103017 (2016).
[9] M. Hayashi and H. Tajima, Phys. Rev. A 95, 032132 (2017).
[10] P. Faist and R. Renner, Phys. Rev. X 8, 021011 (2018).
[11] W. Niedenzu, M. Huber, and E. Boukobza, Quantum 3, 195 (2019).
[12] A. Sone, Y.-X. Liu, and P. Cappellaro, Phys. Rev. Lett. 125, 060602 (2020).
[13] M. Esposito, U. Harbola, and S. Mukamel, Rev. Mod. Phys. 81, 1665 (2009).
[14] M. Campisi, P. Hänggi, and P. Tárnok, Rev. Mod. Phys. 83, 771 (2011).
[15] M. Perarnau-Llobet, E. Bäumer, K. V. Hovhannisyan, M. Huber, and A. Acín, Phys. Rev. Lett. 118, 070601 (2017).
[16] P. Busch, in Quantum Reality, Relativ. Causality, Closing Epistemic Circ. (Springer, Dordrecht, 2009) pp. 229–256.
[17] P. J. Lahti, P. Busch, and P. Mittelstaedt, J. Math. Phys. 32, 2770 (1991).
[18] P. Busch, M. Grabowski, and P. J. Lahti, Found. Phys. 25, 1239 (1995).
Appendix A: Proof of Eq. (29)

Let us introduce the operation (completely positive and trace non-increasing map) $J_{x'} : \mathcal{T}(\mathcal{H} \otimes \mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H} \otimes \mathcal{H}_A)$, defined as

$$J_{x'}(\cdot) := (\mathbb{1} \otimes Z_x) V_{tot} (\mathbb{1} \otimes Z_{x'}) (\cdot) (\mathbb{1} \otimes Z_x) V_{tot}^\dagger (\mathbb{1} \otimes Z_x).$$

(A1)

Here, $x := (m,n)$ and $x' := (m',n')$, so that $Z_x := Z_m^{(0)} \otimes Z_n^{(1)}$ and $Z_{x'} := Z_{m'}^{(0)} \otimes Z_{n'}^{(1)}$, and $V_{tot}$ is defined in Eq. (16).

First, let us show that the state transformation given the TP M protocol can be fully described by the operation $J_{x',x}$. Denoting $(0,0) \equiv 0$, we find that for any $T \in \mathcal{T}(\mathcal{H})$, $P[\xi] := P[\xi^{(0)}] \otimes P[\xi^{(1)}]$, and $x$, the following:

$$J_{0,x}(T \otimes P[\xi]) = (\mathbb{1} \otimes Z_x) V_{tot} (\mathbb{1} \otimes Z_0) (T \otimes P[\xi]) (\mathbb{1} \otimes Z_0) V_{tot}^\dagger (\mathbb{1} \otimes Z_x),$$

$$= (\mathbb{1} \otimes Z_x) V_{tot} (T \otimes P[\xi]) V_{tot}^\dagger (\mathbb{1} \otimes Z_x),$$

$$= \sum_{m',m''} (\mathbb{1} \otimes Z_x) \left( P_{m'} V_{m'} P_{m''} V_{m''}^\dagger \otimes \phi_{m'}^{(0)}(\phi_{m''}^{(0)})^\dagger \otimes \phi_{m'}^{(1)}(\phi_{m''}^{(1)})^\dagger \right) (\mathbb{1} \otimes Z_x),$$

$$= P_{m'} V_{m'} P_{m''} V_{m''}^\dagger P_m \otimes P[\phi_m^{(0)}] \otimes P[\phi_m^{(1)}].$$

(A2)

In the second line, we have used the fact that $|\xi\rangle$ is an eigenstate of $Z_0 = Z_0^{(0)} \otimes Z_0^{(1)}$. In the third line, we have used Eq. (16) and Eq. (17). In the final line, we have used the fact that $|\phi_m^{(i)}\rangle$ are eigenstates of $Z_m^{(i)}$. Similarly, it is easy to show that $J_{x',x}(T \otimes P[\xi]) = 0$ for all $x' \neq (0,0)$.

Recall that, given the sequence $X := (0,x)$, the TPM work done is

$$W(X) := w(x) + w_x^{(0)}(m) + w_x^{(1)}(n),$$

$$= (\epsilon_n + \lambda_n^{(0)} + \lambda_n^{(1)}) - (\epsilon_x + \lambda_x^{(0)} + \lambda_x^{(1)}).$$

(A3)

Using Eq. (A2), recalling that $H_{tot} = H + H_{A_0} + H_{A_1}$, and that $|\xi\rangle$ and $|\phi_m^{(i)}\rangle$ are eigenstates of $H_{A_0}^{(i)} + H_{A_1}^{(i)}$, we may verify that

$$\text{tr}[H_{tot} J_{0,x}(\rho \otimes P[\xi])] = (\epsilon_n + \lambda_n^{(0)} + \lambda_n^{(1)}) \text{tr}[J_{0,x}(\rho \otimes P[\xi])],$$

$$\text{tr}[J_{0,x}(H_{tot} (\rho \otimes P[\xi]))] = (\epsilon_x + \lambda_x^{(0)} + \lambda_x^{(1)}) \text{tr}[J_{0,x}(\rho \otimes P[\xi])].$$

(A4)
Consequently, we may express Eq. (A3) as
\[
\mathcal{W}(X) = \frac{\text{tr}[H_{\text{tot}}J_{0,x}(\rho \otimes P[\xi])]}{\text{tr}[J_{0,x}(\rho \otimes P[\xi])]},
\]
Recalling that \(p_{V,\xi}^{\text{tot}}(X) = \text{tr}[J_{0,x}(\rho \otimes P[\xi])]\), we may therefore write the average work as
\[
\langle \mathcal{W} \rangle_{V,\xi}^{\text{tot}} := \sum_X p_{V,\xi}^{\text{tot}}(X) \mathcal{W}(X),
\]

\[
= \sum_{x',x} \text{tr}[H_{\text{tot}}J_{x',x}(\rho \otimes P[\xi])] - \sum_{x',x} \text{tr}[J_{x',x}(H_{\text{tot}}\rho \otimes P[\xi])].
\]

(A6)

Noting that \(\sum_{x',x} J_{x',x} \) is a trace-preserving operation, it follows that
\[
\sum_{x',x} \text{tr}[J_{x',x}(H_{\text{tot}}\rho \otimes P[\xi])] = \text{tr}[H_{\text{tot}}(\rho \otimes P[\xi])].
\]

(A7)

Similarly, noting that \(\sum_{x'} Z_{x'} H_{\text{tot}} Z_{x'} = H_{\text{tot}}\), while \(\sum_x Z_x P[\xi] Z_x = P[\xi]\), we have
\[
\sum_{x',x} \text{tr}[H_{\text{tot}}J_{x',x}(\rho \otimes P[\xi])] = \text{tr}[H_{\text{tot}} V_{\text{tot}}(\rho \otimes P[\xi]) V_{\text{tot}}^\dagger].
\]

(A8)

Therefore, the average total work reads
\[
\langle \mathcal{W} \rangle_{V}^{\text{tot}} = \text{tr}[(V_{\text{tot}}^\dagger H_{\text{tot}} V_{\text{tot}} - H_{\text{tot}})(\rho \otimes P[\xi])]
\]

(A9)

for all \(\rho \in S(\mathcal{H})\) and \(V\).

Appendix B: Proof of Eq. (34)

Let \(H = \sum_m \epsilon_m P_m\) be the Hamiltonian of system \(\mathcal{H}\), and \(H_A = \sum_m \lambda_m Z_m\), the Hamiltonian of system \(\mathcal{H}_A\), such that \(H_{\text{tot}} = H + H_A\) is the total, additive Hamiltonian of the compound system \(\mathcal{H} \otimes \mathcal{H}_A\). Moreover, let \(|\xi\rangle \in \mathcal{H}_A\) be a unit vector which is an eigenstate of \(H_A\) with eigenvalue \(\lambda_0\). Finally, let \(U\) be a unitary operator on \(\mathcal{H} \otimes \mathcal{H}_A\) such that, for all \(|\psi\rangle \in \mathcal{H}\),
\[
|U(\psi \otimes \xi)\rangle = \sum_m |P_m \psi \otimes \phi_m\rangle,
\]

(B1)

where \(|\psi \otimes \xi\rangle \equiv |\psi\rangle \otimes |\xi\rangle\), \(|P_m \psi\rangle \equiv |P_m \psi\rangle\), and \(|\phi_m\rangle\) are eigenstates of \(H_A\) with eigenvalue \(\lambda_m\). It follows that for all \(|\psi\rangle \in \mathcal{H}\), we have
\[
\langle \psi \otimes \xi| H_{\text{tot}}|\psi \otimes \xi\rangle = \langle \psi| H|\psi\rangle \langle \xi| \xi\rangle + \langle \psi| \psi\rangle \langle \xi| H_A|\xi\rangle,
\]

\[
= \langle \psi| H|\psi\rangle + \langle \psi| \psi\rangle \lambda_0,
\]

\[
= \langle \psi| \left(H + \lambda_0 \mathbb{1}\right)|\psi\rangle.
\]

(B2)

In the first line we use the additivity of \(H_{\text{tot}}\), in the second line we use the fact that \(\langle \xi| \xi\rangle = 1\) and \(\langle \xi| H_A|\xi\rangle = \lambda_0\), and in the final line we use the fact that \(\langle \psi| \psi\rangle \lambda_0 = \langle \psi| \lambda_0 \mathbb{1}|\psi\rangle\). Similarly, for all \(|\psi\rangle \in \mathcal{H}\) we have
\[
\langle \psi \otimes \xi| U^\dagger H_{\text{tot}} U|\psi \otimes \xi\rangle = \langle U(\psi \otimes \xi)| H_{\text{tot}}|U(\psi \otimes \xi)\rangle,
\]

\[
= \sum_{m,n} \langle P_m \psi \otimes \phi_m| H_{\text{tot}}|P_n \psi \otimes \phi_n\rangle,
\]

\[
= \sum_{m,n} \langle \psi| P_m H P_n |\psi\rangle \langle \phi_m| \phi_n\rangle + \langle \psi| P_m P_n |\psi\rangle \langle \phi_m| H_A|\phi_n\rangle,
\]

\[
= \sum_m \langle \psi| P_m H P_m |\psi\rangle + \langle \psi| P_m |\psi\rangle \lambda_m,
\]

\[
= \langle \psi| \left(H + \sum_m \lambda_m P_m\right)|\psi\rangle.
\]

(B3)
In the first line we use the definition of the adjoint of a unitary operator $U$, in the second line we use Eq. (B1), in the third line we use the additivity of $H_{\text{tot}}$ and the fact that the projection operators $P_m$ are self-adjoint, in the fourth line we use $\langle \phi_m | \phi_n \rangle = \delta_{m,n}$ together with $P_m P_n = \delta_{m,n} P_m$ and $\langle \phi_m | H_{\lambda} | \phi_m \rangle = \lambda_m$, and in the final line we use the fact that $\sum_m P_m H P_m = H$.

Combining Eq. (B2) with Eq. (B3) implies that, for all $|\psi\rangle \in \mathcal{H}$, we have

$$
\langle \psi \otimes \xi | (U^\dagger H_{\text{tot}} U - H_{\text{tot}}) | \psi \otimes \xi \rangle = \langle \psi | \left( \sum_m \lambda_m P_m - \lambda_0 \mathbb{1} \right) | \psi \rangle,
$$

and

$$
= \langle \psi | \left( \sum_m (\lambda_m - \lambda_0) P_m \right) | \psi \rangle,
$$

where in the final line we use the fact that $\sum_m P_m = \mathbb{1}$. From the above equation, it follows that

$$
\Gamma_{\xi}(U^\dagger H_{\text{tot}} U - H_{\text{tot}}) = \sum_m (\lambda_m - \lambda_0) P_m,
$$

where $\Gamma_{\xi} : \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_A) \to \mathcal{L}(\mathcal{H})$ is the restriction map for $|\xi\rangle \in \mathcal{H}_A$ defined as $\text{tr}[\Gamma_{\xi}(B)T] = \text{tr}[B(T \otimes P[\xi])]$ for all $B \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_A)$ and $T \in \mathcal{T}(\mathcal{H})$. 
