ABSTRACT. Actual research concerning, in particular, the occurrence of “gap-solitons” bifurcating from the continuous spectrum confirms that this part of Bifurcation Theory that started around 40 years ago flourishes. In this lecture we review the origins of “Bifurcation from the continuous spectrum” with regard to the achievements of Jürgen Scheurle and sketch how the early results dealing with the bifurcation of singular solutions have prepared the ground for present and further developments.

1. Introduction. When Hans-Peter Kruse asked me some months ago to participate in the “Festkolloquium” honoring Jürgen Scheurle he also suggested to talk about bifurcation, in particular with regard to achievements due to Jürgen in this area. Indeed, Dynamical Systems and Bifurcation Theory have been the mathematical ties which connected Jürgen and me for nearly 40 years. About 5 years ago, on the occasion of my 65th birthday my group in Cologne organized a symposium and - for good reasons - they asked Jürgen to contribute as one of the main speakers. In 2012 Jürgen showed up with the courtly and charming title “Dynamische Systeme: von Henri Poincaré bis zu Tassilo Küpper”. Now it seems to be my turn, and as the organizers had arranged that Ferdinand Verhulst would cover the area of Dynamical Systems I will be happy to address Bifurcation Theory. Although the notion is quite old going back to Poincaré the formation of a systematic theory only started around 50 years ago – in fact at a time when both Jürgen and I entered mathematical research. We both belonged to groups active in this area – Klaus Kirchgässner in Stuttgart and Norman W. Bazley in Cologne, and we both acknowledge gratefully that our teachers opened the path into a field which at that time was quite new and exciting to us. Both sharing a background in Numerical Analysis we took the chance to enter bifurcation theory, and as it happened we were able to contribute independently in a new section of bifurcation theory related to “bifurcation of singular solutions” and “bifurcation from the continuous spectrum”. In my lecture I will focus on that part of bifurcation theory tracing the footprints Jürgen has marked in this unknown territory followed by a review of some further developments.

2. Classification of Bifurcation Theory. Systematically, Bifurcation Theory relies on some kind of reduction of a parameter-dependent system to a lower dimensional one preserving the essential behaviour. More precisely let us consider
the following setting with a problem of the form

\[ F(u, \lambda) = 0, \]

where \( u \) represents a state vector and where all parameters are collected in some vector \( \lambda \in \mathbb{R}^m \). Here \( F \) denotes a smooth map

\[ F : X \times \mathbb{R}^m \to Y \]

for Banach spaces \( X \) and \( Y \). Bifurcation theory deals with existence and stability of solutions near a special solution \((\bar{u}, \bar{\lambda})\). To set up a classification define

\[ L := \frac{\partial F}{\partial u}(\bar{u}, \bar{\lambda}). \]

In an abstract functional analytic setting then the following alternative holds:

(I) Continuation Case: “\( L^{-1} \) exists and \( L^{-1} : Y \to X \) is bounded.” Implicit Function Theorem implies that all solutions near \((\bar{u}, \bar{\lambda})\) can be parametrized as a function of \( \lambda \); i.e. \( u = u(\lambda) \) locally near \( \bar{\lambda} \).

(II) Bifurcation Case: “\( L \) does not have a bounded inverse.” This case splits into two subcases:

(A) The inverse operator does not exist.

(B) The inverse operator \( L^{-1} \) exists but is unbounded.

In case (I) there is no bifurcation at all; the problem is reduced to continuation of the special solution \((\bar{u}, \bar{\lambda})\); hence a question of “path following”.

In case (II) (A) there is a nontrivial null space \( N \) of \( L \) such that \( n := \text{dim}(N) > 0 \). The standard approach in bifurcation theory relies on a reduction to a reduced (locally equivalent) problem on \( N \times \mathbb{R}^m \) called the bifurcation equation; common procedures are the Lyapunov-Schmidt reduction or the center manifold approach. In typical applications \( n \) is small, even \( n = 1 \) in many situations allowing a straightforward analysis of the reduced problem. The abstract setting of the problem is not restricted to static problems but can also be used to cover dynamical systems, for example to cover Hopf bifurcation. Higher dimensions of \( N \) are often due to symmetries or degeneracies and require more sophisticated tools to treat the situation. For a detailed presentation we refer to [12, 13]. During the past decades this situation has been studied in detail.

The case \( n = \infty \) refers to a special situation [14] and might be considered as a bridge to the case (II)(B) which typically comes along with continuous spectrum of \( L \). With regard to bifurcation the key difficulty is due to the fact that there is no null space of the linearized problem to start with, hence a reduction procedure is not obvious at all. Typically that situation arises in connection with differential equations on unbounded domains related to nonlinear Schrödinger equations defined on the whole space or problems in fluid dynamics defined on an infinite strip.

While cases (I) and (II) (A) attracted a lot of attention the problems related to case (II)(B) appeared in the mid seventies of the last century as a new challenge and a new “branch” in Bifurcation Theory roughly described as “bifurcation from the continuous spectrum”. There are more general concepts of bifurcation in use, for example based on the changes of the topological characteristics of the set of solutions but for smooth problems the above alternative provides a good structure.

3. **Singular Bifurcation Problems.** Jürgen and I had several meetings, for example 1978 at the Oberwolfach workshop organized by Albrecht/Collatz/
Kirchgässner [44]. The essential meeting took place in the fall of 1979 at a conference organized by Amann/Bazley/Kirchgässner [2] at the “Zentrum für Interdisziplinäre Forschung (ZIF)” in Bielefeld. There are two lasting memories to that meeting.

Besides the fruitful mathematical exchange, I very well recall another strong feature of the Stuttgart group around Klaus Kirchgässner when I joined the well trained team on their early morning runs through the hilly outskirts of the Teutoburger Woods. It is widely acknowledged that the familiar atmosphere created by Klaus Kirchgässner in his group aiming far beyond mathematical excellence by including all kinds of activities such as sports, music, poetry etc. substantially contributed to the high standard of mathematical research as well.

In a particular way the lecture “Bifurcation of bounded solutions of semilinear elliptic equations in a strip” [20] attracted my attention as it dealt with problems closely related to my own research [25].

I had just achieved my “Habilitation” on “Singular Bifurcation Problems” the summer term before so I was curiously listening to the joint lecture of Klaus Kirchgässner and Jürgen on “Bifurcation of bounded solutions of semilinear elliptic equations in a strip” dealing with bifurcation of singular solutions; later on this topic was exploited in a series of papers [19, 20, 21, 22].

While the early footprints of Jürgen [38, 39, 40, 41, 42, 43, 44, 45, 46, 47] belong to the well-cultivated terrain of classical bifurcation analysis dealing in particular with specific iteration schemes he now seemed to enter a new area related to bifurcation of singular solutions.

Let me briefly sketch the essential features of that approach.

Kirchgässner and Scheurle [20, 21, 22] considered the nonlinear elliptic boundary value problem:

\[ \Delta u + \lambda u + f(u, u_x, u_y) = 0 \]
\[ u(0, y) = u(1, y) = 0 \]

on the infinite strip \( \Omega = [0, 1] \times \mathbb{R} \) together with Dirichlet boundary conditions on the finite border. Due to the background of the problem in fluid dynamics they focus on bounded solutions. All nonlinear terms of higher order are collected in the function \( f \).

Hence, \( u = 0 \) is a (trivial) solution for all \( \lambda \in \mathbb{R} \), and the spectrum of the linearization of the problem about \( u = 0 \) considered in \( L^2(\Omega) \) is purely continuous, ranging from \( \pi^2 \) to \( \infty \). In that sense problem (1)-(2) fits into the abstract setting of case (II)(B) for \( \lambda \in [\pi^2, \infty) \).

The key approach pursued by Kirchgässner and Scheurle [20] is motivated by the geometry of the domain given by an infinite strip; accordingly the elliptic operator is split in such a way that an evolution problem arises:

\[ u_{yy} = -u_{xx} - \lambda u - f(u, u_x, u_y). \]

Proceeding that way the spectrum of the linearized problem becomes purely discrete, hence classical bifurcation theory is applicable, and working in appropriate spaces Kirchgässner and Scheurle succeed to prove:

- Bifurcation of periodic/bounded solutions
- The existence of “singular” solutions, bifurcating at \( \pi^2 \).

The proof is based on a sophisticated spectral analysis. A detailed description is beyond the scope of this paper. Once the results have been established, it becomes obvious that they can be nicely illustrated in the simple one-dimensional case which
allows a direct analysis using phase plane techniques. The following examples show different cases illustrating the occurrence of singular solutions in simple situations:

\[ u'' + \lambda u + f(u, u') = 0 \]  

(4)

**Example 1.** \( f(u, u') = -u^3 \) Then for \( \lambda < 0 \) there are no nontrivial bounded solutions; for \( \lambda > 0 \) the phase portrait looks as follows:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{phaseportrait_larger_lambda.png}
\caption{Phase portrait for \( \lambda > 0 \)}
\end{figure}

Hence:
- All bounded solutions are periodic (except a singular solution).
- The singular solution bifurcates at \( \lambda = 0 \).
- All bounded solutions shrink to 0 as \( \lambda \to 0 \).

Here the singular solution occurs as a heteroclinic (front-like) solution (see Fig.1 and Fig.2).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{heteroclinic_sing_solut.png}
\caption{Heteroclinic orbit (singular solution)}
\end{figure}

**Example 2.** \( f(u, u') = u^3 \) For \( \lambda > 0 \) all bounded solutions are periodic; there are no singular solutions.

For \( \lambda < 0 \) there exists a singular solution given as a homoclinic (pulse-like) solution.
Again, the singular solution shrinks to 0 (in the sup-norm) as $\lambda \to 0$.

The singular solution is pulse-like and belongs to $L^2(\mathbb{R})$ but there is not always bifurcation with regard to the $L^2$-norm. It turns out that bifurcation depends on the “growth” of the nonlinearity, a fact unknown in classical bifurcation theory where even in the case of non-present nonlinearities formally at least “vertical” bifurcation can be stated (see [30]). In the absence of nonlinear terms the vertical branch is generated by multiples of eigenvectors.

When Jürgen and I set our feet into this area, it was into still unknown territory. It seems that Jürgen soon switched to Dynamical Systems, while I stayed with bifurcation for a while, although soon I also realized that the methods of Dynamical Systems could be useful to understand bifurcation from the continuous spectrum. For the rest of the paper I will give a brief review of some achievements in this area.

Partial results related to nonlinear Schrödinger, Hartree-Fock equations, Sine-Gordon equations, problems involving convolution operators [8] or differential equations on unbounded domains had been obtained. The first results had been obtained by a direct approach using a variety of methods including Dynamical Systems techniques, variational methods, use of lower and upper solutions, perturbation techniques but there was no systematic approach dealing with bifurcation from the continuous spectrum. With regard to the physical background the focus was on square integrable solutions. Hence we will also consider a Hilbert space setting further on.

The first results already showed significant differences when compared with standard bifurcation results. The role of the nonlinearity appeared in a different light. While in standard bifurcation theory the nonlinear terms did not determine the existence of bifurcating solutions (although of course the direction of the bifurcating branches) the order of the higher order terms gained crucial importance.

4. Bifurcation from the continuous spectrum.

4.1. Variational problems. The first systematic approaches were based on variational methods (see [48, 49, 50, 51, 52, 53]) usually proving first existence and, followed up by an investigation of bifurcation. Typically the results concern bifurcation from the lowest point away from the spectrum but almost simultaneously two papers [3, 4] proved existence of solutions above the continuous spectrum for a class of nonlinear elliptic differential equations on unbounded domains with a rapidly growing nonlinearity. Again, bifurcation at the lowest point of the purely continuous spectrum ranging from 0 to $\infty$ could be proven. At the same time every point of the continuous spectrum was shown to be bifurcation point (see [3]).

Due to the rapidly growing nonlinearity it became possible ([4]) to treat these problems in a subspace with a compact embedding allowing techniques to prove
existence. In terms of the solutions this behaviour corresponds to very rapid decay, hence extremely concentrated pulse-like solutions. Later on these embedding results could be used to establish the existence of sharp pulses for some equations in telecommunication allowing an efficient transformation of signals due to an increase of capacity by a factor 5 because of the concentrated pulses ([23]).

The two kinds of bifurcation at the continuous spectrum, with one leading away from the spectrum and with the other one occurring at any point of the continuous spectrum motivated investigations to understand the origins of these phenomena. While a further characterization of the solutions by variational methods was not at hand Dynamical Systems Methods led the way.

4.2. Dynamical systems techniques. A detailed analysis using dynamical systems techniques clarified the situation observed in [3, 4], by characterizing the solutions by nodal properties as it is known for classical problems where nodal properties of the eigenfunctions of the linearized problem carry over to the bifurcating branches. In fact it could be shown that there are infinitely many branches bifurcating at the lowest point of the continuous spectrum characterized by nodal properties; a cross section in the bifurcation diagram for any fixed value of $\lambda$ shows that the infinitely many solutions from different branches with increasing number of zeros accumulate at the trivial solution, hence there is bifurcation as well although not in form of branches (see [17, 18]).

The nonlinear elliptic equation

$$\Delta u + \lambda u + |u|^\sigma u = 0 \quad (5)$$

in $\mathbb{R}^n$ is considered as a key problem for which existence of infinitely many solutions could be established by variational methods. The relationship between the growth of the nonlinearity represented by $\sigma$ and the dimension $n$ of the underlying space is an important feature. Pohozaev’s identity implies $\sigma < \frac{4}{(n-2)}$ (see [17]) as a necessary criterion for the existence of solutions. Using Dynamical Systems techniques it was possible to prove the existence of infinitely many radialsymmetric solutions satisfying the corresponding ordinary differential equation. As a key tool properties of the limiting systems for $r = 0$ and $r = \infty$ were exploited and shown to carry over when connecting both systems.

In this way it was possible to characterize the solutions by nodal properties using winding numbers and to determine a geometric explanation for the restriction between $n$ and $\sigma$. To obtain that result a transformation and compactification using $\rho = r/(r + 1)$ lead to the system

$$u' = v$$

$$v' = -(n - 1) (1 - \rho) v / \rho - \lambda u - |u|^\sigma u$$

$$\rho' = (1 - \rho)^2. \quad (6)$$

Hence $\rho = 0$ r.s.p. $\rho = 1$ correspond to $r = 0$ r.s.p. $r = \infty$.

The limiting system for $\rho = 1$ is given by

$$u' = v$$

$$v' = -\lambda u - |u|^\sigma u \quad (7)$$

with $u = 0$, $v = 0$ as critical point. It is then shown that forward iterates of the line $W^0 = \{(u, v = 0, \rho = 0) | u \in \mathbb{R}\}$ and backwards iterates of the local center stable manifold $W^{cs,loc}(0, 0, 1)$ under the flow of system (6) intersect. The precise way of intersection can be captured by the concept of a winding number. It that way a
number of zeros can be assigned to the solutions $u$ for the proof of the postulated nodal property.

4.3. Perturbation techniques. We have already noticed that not just the direction of bifurcation but bifurcation itself depends on the nonlinear terms; the number of bifurcating branches varies between 0, 1 and infinity as can be seen for the following examples permitting no (a), exactly one (b) or infinitely many nontrivial solutions (c):

**(a)** $- u'' - u^3 = \lambda u$, $u(0) = 0$ $u \in L^2(0, \infty)$ (8)

**(b)** $- u'' - u^3 = \lambda u$, $u'(0) = 0$ $u \in L^2(0, \infty)$ (9)

**(c)** $- u'' - r(x)u^3 = \lambda u$, $u(0) = 0$ $u \in L^2(0, \infty)$ (10)

$r(x)$ : decaying appropriately

To understand why this happens it seemed natural to try perturbation techniques. It is known that a purely continuous spectrum of a linear operator can be changed into a purely discrete spectrum consisting of simple eigenvalues by adding a small parameter dependent (even compact) perturbation. As the perturbation parameter approaches 0 the eigenvalues accumulate at the lowest point of the spectrum. For the perturbed problem classical bifurcation methods giving bifurcation at simple eigenvalues apply. The limit procedure though involves the consideration of two limit processes simultaneously making things difficult. A first successful approach has been carried out by Chiapinelli and Stuart [5] for a class of problems including nonlocal terms; following these ideas A. Plate [37] has extended these results in her Ph.D thesis for a class of other problems of the form

$$- u'' - r(x)|u|^{\sigma}u = \lambda u$$ (11)

$$u'(0) = 0, \quad u \in L^2(0, \infty)$$ (12)

where $r$ satisfies

(V1) $r(x) \geq 0$, continuous, bounded, $\lim_{x \to \infty} r(x) \to 0$

(V2) $\exists A > 0 \quad \exists t \in (0, 1) : r(x) \geq \frac{A}{(1+t)^{17}}, \quad (0 < \sigma < 2(2-t))$

As perturbation Plate introduces the term $\varepsilon u/(1+x)$ which leads to an additional spectrum consisting of negative eigenvalues approaching 0 for $\varepsilon \to 0$.

Indeed a key difficulty arises on the fact that both limit procedures $\lambda \to 0$ and $\varepsilon \to 0$ are difficult to control simultaneously. Usually there are infinitely many branches for the perturbed problem but depending on the nonlinear terms just a single branch remains or all of them remain or even all of them vanish. So far there are no general criteria known.

4.4. Gap-bifurcation. In the beginning bifurcation of the continuous spectrum (at least in form of branches) had been established at the lowest point of the spectrum. Correspondingly, the examples used for motivation and illustration were characterized by a purely continuous spectrum ranging from 0 to $\infty$. Considering the perturbation approach one notices that the eigenvalues of the perturbed problem accumulated at the lowest point of the spectrum filling it up successively during the limit procedure.

What could be expected if there were gaps in the continuous spectrum as it happens for example for elliptic equations with periodic potentials? Results up to this point motivated the hypothesis that the boundary points of the spectrum could
play a key role. Simple examples support this idea as was shown in [2] involving a multiplication operator

\[ f(t)u(t) - \left| \int_0^1 |u(s)| ds \right|^2 \int_0^1 u(s) ds = \lambda u(t) \quad (u \in L^2(0,1)) \]  

with appropriate choice of the function \( f \) to determine the spectrum.

A first systematic approach to establish bifurcation at end points of the continuous spectrum was obtained in [31, 32] for variational problems by converting the bifurcation within a gap into bifurcation at the lowest point. This could be achieved by using an implicit function theorem to eliminate the infinite-dimensional spaces related to the spectrum below the gap, hence a reduction to an already well known problem. The abstract results were applied to explain gap bifurcation for a class of nonlinear Hill’s equations. In a series of subsequent papers the abstract procedures could be simplified by a direct application of variational principals to establish “gap-bifurcation”; see [16, 53] for a review.

It is known that the endpoints of the continuous spectrum for equations of Hill’s type can be characterized by special eigenvalues of associated eigenvalue problems with periodic, respectively anti-periodic boundary conditions. Based on this characterization it became possible to analyze the bifurcation structure for nonlinear Hill’s equation in much more detail including information [29] concerning

- bifurcation at the boundary points,
- results by Alama and Li [1] and Heinz [15] on the number of branches,
- coexistence of bifurcation from zero and infinity,
- global results.

In particular it was shown that the “branches” exist globally as solutions of the differential equation. Only over the gaps they consist of square-integrable solutions. Hence they “disappear” within the \( L^2 \) -setting when reaching the spectrum and reoccur when reaching the gap again. Further characterizations have been worked out by Mrziglod [35].

Although our interest and the first results concerning gap-bifurcation were mainly stimulated by mathematical reasons there is a strong motivation from a physical point of view to study such problems as they arise in the context of “gap-solitons” for example. Using formal expansions Pelinovsky et al [36] have constructed bifurcating solutions within the gaps for problems with periodic structure. It is interesting to see how this area started about 35 years ago has been developed in recent years. As a review of the current achievements exceed the scope of this contribution we refer to some topical paper (see for example Dohnal et al. [9] or [10]).

4.5. Outlook. The recent development in the construction of gap-solutions mainly based on the solution of coupled mode-equations can be seen as a great progress of the theory. From a mathematical point of view it would be challenging to study how this approach can be generalized to understand the phenomenon of bifurcation from the continuous spectrum. Vague ideas to embed the problem in a distributional sense in appropriate spaces using generalized eigenfunctions and to apply the theory of rigged Hilbert spaces as employed by Chiba [6, 7] appear as appropriate attempts but have not yet led to concrete results.

5. More footprints. It seems that the early contributions concerning the bifurcation of singular solutions and the investigation of bifurcation from the continuous spectrum has been expanded into a fruitful and rewarding area. Near the end of the
century both Jürgen and myself have focused our attention to other areas but Dynamical Systems and Bifurcation Theory have been our mathematical ties for many years. There are more joint footprints to note: fruitful cooperation within various Research Programs such as the DFG-Schwerpunkt “Ergodic Theory, Analysis and Efficient Simulation of Dynamical Systems” (1994-2000) [11], the “Joint Research Program with Chinese Universities (Changchun, Beijing, Nanjing, Wuhan)” (1995-2005) [28], collaboration with Jiangong You from Nanjing as Humboldt Fellow or, last but not least, as a trusted partner in the activity group “Nonlinear Phenomena” with annual meetings in the Mathematical Research Center Oberwolfach for 17 years from 1993 to 2010. These meetings were a wonderful opportunity to discuss new and promising areas of mathematical topics and a rewarding experience for our students to enjoy mathematical research in the spirit of Oberwolfach.

We are looking forward to see new footprints of Jürgen in the future hopefully leading into joyful regions.

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![Figure 4. Prof. Tassilo Küpper and Prof. Jürgen Scheurle](image-url)
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