Working Paper: Exact Convex Relaxation for Optimal Power Flow in Tree Networks

Lingwen Gan, Ufuk Topcu, Na Li, Steven Low

Abstract

The optimal power flow problem is non-convex. We modify its voltage constraint to be “slightly” more conservative, and then relax the modified problem to a convex problem in tree networks. A sufficient condition for exact relaxation is provided. This condition can be checked before solving the relaxation, and holds in various test networks, including IEEE 13, 34, 37, 123-bus test networks and a real distribution network with high penetration of distributed generation.

I. INTRODUCTION

The optimal power flow (OPF) problem seeks to minimize a certain objective function, such as power loss and generation cost, subject to physical constraints including Kirchoff’s laws, voltage regulation constraints, and power injection constraints. There has been extensive research on OPF since Carpentier’s first formulation in 1962 [1], and surveys can be found in [2], [3], [4], [5], [6]. OPF is in general non-convex, and a lot of algorithms have been proposed to approximate OPF or relax OPF to convex problems. Relaxation methods have the potential of providing exact OPF solutions, and are the focus of this work. We call a relaxation exact if every solution to the relaxed problem also solves the original problem.

In transmission networks, which are usually mesh networks, semi-definite relaxation (SDR) has been proposed to solve OPF [7], [8], [9], but whether or when SDR is exact can only be checked after solving SDR. In distribution networks, which are usually tree networks, different convex relaxations for OPF have been proposed based on the DistFlow power flow model proposed in [10], [11]. Reference [12] puts forward a second-order-cone relaxation (SOCR) for OPF, and proves that SOCR is exact if there are no lower bounds on power injection. This condition can be checked before solving SOCR. See also [13], [14] for other relaxations for OPF.

The lower bounds on power injection are important for various applications including demand response [15] and Volt/VAR control [16]. Motivated by this shortcoming of [12], references [17], [18] work on other sufficient...
conditions for the exactness of SOCR. Though references [17], [18] bring back the lower bounds on power injection, they remove the upper bounds on voltage. The upper bounds on voltage are important for voltage regulation, especially when there are distributed energy sources increasing the likelihood of over-voltage, i.e., voltage exceeds voltage upper bound.

Contributions of this work include the following. Firstly, we show that SOCR is not always exact, especially in the presence of distributed generation. Furthermore, over-voltage is a cause for inexact SOCR. Secondly, we modify OPF by using a “slightly” more conservative voltage constraint to deal with the over-voltage issue. The modified OPF (MOPF) problem is still non-convex, therefore we relax it to a convex problem called relaxed MOPF (RMOPF). Thirdly, we provide a sufficient condition for the exactness of RMOPF. This condition can be checked before solving RMOPF, and holds in various test networks including IEEE 13, 34, 37, 123-bus test networks [19] and a practical network of Southern California Edison [20] with high penetration of distributed generation. The condition roughly imposes that reverse power flow—power flowing from the loads to the substation—in the network should not be too large. At last, we present our proof technique. While prior works [17], [18] focus on the dual method—prove properties of the dual variables—to derive sufficient conditions for exact relaxation, this work explores the primal problem directly. We show that the results in [17] can be re-established using the technique in this paper.

The rest of the paper is organized as follows. In Section II, we introduce the OPF problem and its relaxation SOCR, as well as a counter example where SOCR is not exact. In Section III, we propose the modified OPF problem MOPF, give a convex relaxation RMOPF for MOPF, provide a sufficient condition for the exactness of RMOPF, and check that this condition holds in various test networks. In Section IV, we present the proof technique, and use this technique to re-establish the results in [17].

II. THE OPTIMAL POWER FLOW PROBLEM

A. The Optimal Power Flow Problem

Consider a tree distribution network\(^1\) that consists of \(N + 1\) buses. Index the substation bus by 0 and the other branch buses by \(i = 1, \ldots, N\). Let \(\mathcal{N} := \{0, \ldots, N\}\) denote the set of all buses. If two buses \(i, j \in \mathcal{N}\) are connected by a line, we denote \(i \sim j\); otherwise, we denote \(i \nottimes j\).

**Definition 1:** The ordered set \((i_0, i_1, \ldots, i_k) \subseteq \mathcal{N}\) where \(k \in \mathbb{N}\) is called a path from bus \(i_0\) to bus \(i_k\), if \(i_t \sim i_{t+1}\) for \(t = 0, \ldots, k - 1\).

There exists a unique path between two arbitrary buses in a tree [21], and we denote the unique path from bus 0 to bus \(i\) by \(\mathcal{P}_i\), for \(i \in \mathcal{N}\). Each transmission line in the network connects two buses, say \(i\) and \(j\). Then, either \(i \in \mathcal{P}_j\) or \(j \in \mathcal{P}_i\), i.e., either bus \(i\) is on the path from bus 0 to bus \(j\), or vice versa. We denote the transmission line by \((i, j)\) if \(i \in \mathcal{P}_j\), and \((j, i)\) otherwise, i.e., we put the bus that is closer to bus 0 in front. Let

\[
\mathcal{L} := \{(i, j) \mid i \sim j \text{ and } i \in \mathcal{P}_j\}
\]

\(^1\)Most distribution networks are tree networks.
denote the set of all transmission lines.

For each bus $i \in \mathcal{N}$, let $p_i$ and $q_i$ denote its real and reactive power injection respectively, let $V_i$ denote its voltage, and define $v_i := |V_i|^2$. Bus 0—the substation—has fixed voltage magnitude, i.e., $v_0$ is a constant. For each transmission line $(i, j) \in \mathcal{L}$, let $r_{ij}$ and $x_{ij}$ denote its resistance and reactance respectively, let $P_{ji}$ and $Q_{ji}$ denote the real and reactive power flow from $j$ to $i$ respectively, let $I_{ij}$ denote the current flowing from $i$ to $j$, and define $\ell_{ij} := |I_{ij}|^2$. Define

$$r := (r_{ij}, (i, j) \in \mathcal{L})^T, \quad p := (p_1, \ldots, p_N)^T,$$
$$x := (x_{ij}, (i, j) \in \mathcal{L})^T, \quad q := (q_1, \ldots, q_N)^T,$$
$$P := (P_{ji}, (i, j) \in \mathcal{L})^T, \quad v := (v_1, \ldots, v_N)^T,$$
$$Q := (Q_{ji}, (i, j) \in \mathcal{L})^T, \quad \ell := (\ell_{ij}, (i, j) \in \mathcal{L})^T,$$

i.e., use the letter without subscript to denote a column vector of the corresponding quantity. For any two vectors $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$ and $b = (b_1, \ldots, b_k) \in \mathbb{R}^k$ where $k \in \{1, 2, \ldots, \}$, define the relations $\succ, \succeq, \prec, \preceq$ as

$$a \succ b \iff a_i > b_i \text{ for } i = 1, \ldots, k,$$
$$a \succeq b \iff a_i \geq b_i \text{ for } i = 1, \ldots, k,$$
$$a \prec b \iff a_i < b_i \text{ for } i = 1, \ldots, k,$$
$$a \preceq b \iff a_i \leq b_i \text{ for } i = 1, \ldots, k.$$

| $\mathcal{N}$ | the set of buses |
| $i \sim j$ | bus $i$ and $j$ are connected |
| $i \not\sim j$ | bus $i$ and $j$ are not connected |
| $\mathcal{P}_i$ | the path from bus 0 to bus $i$ |
| $(i, j)$ | a transmission line, $i \sim j$ and $i \in \mathcal{P}_j$ |
| $\mathcal{L}$ | the set of transmission lines |
| $p_i$ | real power injection at bus $i$ |
| $q_i$ | reactive power injection at bus $i$ |
| $V_i$ | voltage at bus $i$ |
| $v_i$ | $v_i = |V_i|^2$ |
| $r_{ij}$ | resistance of $(i, j)$ |
| $x_{ij}$ | reactance of $(i, j)$ |
| $P_{ji}$ | real power flow from $j$ to $i$ |
| $Q_{ji}$ | reactive power flow from $j$ to $i$ |
| $I_{ij}$ | current flowing from $i$ to $j$ |
| $\ell_{ij}$ | $\ell_{ij} = |I_{ij}|^2$ |

The optimal power flow (OPF) problem is
OPF:

\[
\begin{align*}
\text{min} & \quad \sum_{(i,j) \in \mathcal{L}} r_{ij} \ell_{ij} \\
\text{over} & \quad p, q, P, Q, \ell, v \\
\text{s.t.} & \quad P_{ji} - p = \sum_{(j,k) \in \mathcal{L}} (P_{kj} - r_{jk} \ell_{jk}), (i, j) \in \mathcal{L}; \\
& \quad Q_{ji} - q = \sum_{(j,k) \in \mathcal{L}} (Q_{kj} - x_{jk} \ell_{jk}), (i, j) \in \mathcal{L}; \\
& \quad v_i = v_j - 2(r_{ij} P_{ji} + x_{ij} Q_{ji}) + (r_{ij}^2 + x_{ij}^2) \ell_{ij}, (i, j) \in \mathcal{L}; \\
& \quad \ell_{ij} = \frac{P_{ji}^2 + Q_{ji}^2}{v_j}, (i, j) \in \mathcal{L}; \\
& \quad p \preceq P \preceq \bar{p}, q \preceq Q \preceq \bar{q}.
\end{align*}
\]

where \( r, x, v, \bar{v}, p, \bar{p}, q, \bar{q} \) and \( v_0 \) are given constants.

**Remark 1:** The objective function in (1) is the power loss, and can be generalized to any function of the type

\[
f \left( \sum_{(i,j) \in \mathcal{L}} r_{ij} \ell_{ij} \right) + g(p, q),
\]

where \( f: \mathbb{R} \to \mathbb{R} \) is strictly increasing and \( g: \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) is arbitrary, without changing the results in this paper. For ease of presentation, we work with the objective function in (1).

**Remark 2:** Equations (2)–(5) are due to the underlying physical laws that power flow satisfies [10], [11], and called the DistFlow model in the literature. It is claimed in [22] that once substation voltage \( v_0 \) and power injection \( p \) and \( q \) are specified, then there exists a unique practical \( P, Q, \ell, v \) satisfying the DistFlow model (2)–(5). The control is to find the optimal power injection \( p \) and \( q \), and let the other variables \( P, Q, \ell \) and \( v \) be determined by the DistFlow model (2)–(5).

**Remark 3:** Equation (6) is the voltage regulation constraint. Voltage is usually expressed in terms of the per unit value. For instance, the voltage constraint can be \( 0.95 \text{p.u.} \leq |V_i| \leq 1.05 \text{p.u.} \) [23], which translates into \( 0.9 \leq v_i \leq 1.1 \).

**Remark 4:** Equation (7) are the power injection constraints. In applications including demand response and Volt/VAR control, power injection constraints are usually considered to be box constraints as (7), but results in this paper extend to arbitrary power injection constraints (they do not even need to be convex).

**B. The Second-Order-Cone Relaxation**

OPF is non-convex due to the non-affine equality constraint in (5). An approach (see [12]) is relaxing this constraint to the inequality constraint

\[
\ell_{ij} \geq \frac{P_{ji}^2 + Q_{ji}^2}{v_j}, (i, j) \in \mathcal{L}.
\]
Equation (8) is equivalent to a second-order-cone constraint. Therefore, we call the following convex problem the second-order-cone relaxation (SOCR) of OPF.

**SOCR:**

$$\min \sum_{(i,j) \in \mathcal{L}} r_{ij} \ell_{ij}$$

over $p, q, P, Q, \ell, v$

s.t. (2)–(4), (6)–(8).

If (5) is satisfied at an optimal solution $w^{\text{opt}}$ to SOCR, then $w^{\text{opt}}$ also solves OPF.

**Definition 2:** A relaxation is **exact** if every of its optimal solutions solves the original problem.

When a relaxation is exact, we can solve the original problem by solving the relaxed problem. The original problem is usually non-convex and difficult to solve, while the relaxed problem is usually convex and easy to solve.

In this section, OPF is the non-convex original problem, and SOCR is the convex relaxed problem. SOCR is exact if and only if every of its optimal solutions satisfies the constraint in (5). It is proved in [12] that if there are no lower bounds $p$ and $q$ in (7), then SOCR is exact. In applications including demand response and Volt/VAR control, lower bounds $p$ and $q$ exist, requiring further analysis.

C. **SOCR is Not Always Exact.**

SOCR is not always exact. Furthermore, when it is not exact, the solution to SOCR is physically meaningless. We illustrate this through a 2-bus network in Fig. 1. Bus 0 is the substation and has fixed voltage magnitude $v_0 = 1$.

The branch bus has renewables generating 1p.u. real power, of which $p \in [0, 1]$ will be injected to the grid and the rest $1 - p$ will be curtailed. The reactive power injection $q$ is 0. Line resistance and reactance are $r = x = 0.1$, and the voltage constraint is $0.9 \leq v \leq 1.1$. The objective is to minimize the sum of power loss $r \ell = 0.1 \ell$ and
curtailment $1 - p$. The OPF can be written as

$$\begin{align*}
\min & \quad 0.1\ell + (1 - p) \\
\text{over} & \quad p, q, \ell, v \\
\text{s.t.} & \quad v = 1 + 0.2p + 0.2q - 0.02\ell, \\
& \quad \ell = \frac{p^2 + q^2}{v}, \\
& \quad 0.9 \leq v \leq 1.1, \\
& \quad 0 \leq p \leq 1, \quad q = 0,
\end{align*}$$

and the corresponding SOCR is

$$\begin{align*}
\min & \quad 0.1\ell + (1 - p) \\
\text{over} & \quad p, q, \ell, v \\
\text{s.t.} & \quad v = 1 + 0.2p + 0.2q - 0.02\ell, \\
& \quad \ell = \frac{p^2 + q^2}{v}, \\
& \quad 0.9 \leq v \leq 1.1, \\
& \quad 0 \leq p \leq 1, \quad q = 0,
\end{align*}$$

The solution to (10) is

$$p = 0.7705, \quad q = 0, \quad \ell = 2.7049, \quad v = 1.1,$$

which violates $\ell = (p^2 + q^2)/v$, therefore it is physically meaningless.

There is a tendency to operate at $p = 1$ since curtailment is penalized, but large $p$ leads to over-voltage, i.e., $v > 1.1$. The real optimal solution for (9)—if the power loss term $0.1\ell$ is negligible—is to increase $p$ up to $p^{opt}$ where $v = 1.1$. However, after relaxing (9) to (10), it is possible to increase $p$ beyond $p^{opt}$ to obtain a smaller objective, while allowing $\ell > (p^2 + q^2)/v$ to satisfy $v \leq 1.1$.

To summarize, SOCR is not always exact, and attains a physically meaningless solution when it is not exact. Furthermore, over-voltage is a cause of inexact relaxation.

### III. A Modified OPF Problem

In this section, we modify OPF to deal with the over-voltage issue, by changing the constraint $v \leq \pi$ in (6) to $v^{\text{lin}} \leq \pi$, where $v^{\text{lin}}$ is a linear approximation as well as an upper bound for $v$. We first propose a modified OPF problem MOPF in Section III-A and then compare MOPF with OPF in Section III-B and III-C. Afterwards we provide a convex relaxation RMOPF for MOPF as well as a sufficient condition for the exactness of RMOPF in Section III-D. This condition can be checked before solving RMOPF, and holds in various test networks as shown in Section III-E.

\[2\] We solve SOCR-example using the Matlab toolbox CVX [24].
A. A Modified OPF Problem

We modify OPF to deal with the over-voltage issue, which may cause inexact relaxation (see the example in Section II-C). The way we modify OPF is to replace the constraint \( v \leq \bar{v} \) in (6) by a new constraint \( v^{\text{lin}} \leq \bar{v} \). Here \( v^{\text{lin}} \) is a linear approximation as well as an upper bound for \( v \).

We start with introducing the linear approximations \( P^{\text{lin}}, Q^{\text{lin}} \) and \( v^{\text{lin}} \) to \( P, Q \) and \( v \). They are used to study optimal placement and sizing of shunt capacitors in [10], [11], to minimize power loss and balance load in [25], and to control reactive power injection for voltage regulation in [16]. The linear approximations \( P^{\text{lin}}, Q^{\text{lin}} \) and \( v^{\text{lin}} \) are solution to

\[
P_{ji} - p_j = \sum_{(j,k) \in \mathcal{L}} P_{kj}, \quad (i,j) \in \mathcal{L},
\]

\[
Q_{ji} - q_j = \sum_{(j,k) \in \mathcal{L}} Q_{kj}, \quad (i,j) \in \mathcal{L},
\]

\[
v_i = v_j - 2(r_{ij} P_{ji} + x_{ij} Q_{ji}), \quad (i,j) \in \mathcal{L},
\]

i.e., ignoring the \( \ell \) terms in (2)–(4). The \( \ell \) terms are much smaller than the other terms for a broad class of distribution networks including the ones considered in this paper. Linear approximations \( P^{\text{lin}}, Q^{\text{lin}} \) are valid when \( r_{ij} \ell_{ij} \ll |P_{ji}| \) and \( x_{ij} \ell_{ij} \ll |Q_{ji}| \) for \((i,j) \in \mathcal{L}, \) i.e., power loss is much smaller than power flow. Linear approximation \( v^{\text{lin}} \) is valid when we further have \( r_{ij} |P_{ji}(p,q)| \ll v_j \) and \( x_{ij} |Q_{ji}(p,q)| \ll v_j \) for \((i,j) \in \mathcal{L}, \) which happens to be the requirement of voltage stability [16]. Therefore, \( v^{\text{lin}} \) is valid when power loss is insignificant and voltage is stable. The approximations have closed-form expressions

\[
P^{\text{lin}}_{ji} = \sum_{k,j \in \mathcal{P}_k} p_k, \quad (i,j) \in \mathcal{L},
\]

\[
Q^{\text{lin}}_{ji} = \sum_{k,j \in \mathcal{P}_k} q_k, \quad (i,j) \in \mathcal{L},
\]

\[
v^{\text{lin}}_j = v_0 + \sum_{(k,l) \in \mathcal{P}_j} 2(r_{kl} P^{\text{lin}}_{lk} + x_{kl} Q^{\text{lin}}_{lk}),
\]

\[
\quad j = 1, \ldots, N.
\]

Note that given \( p, q, v_0 \), the quantities \( P^{\text{lin}}, Q^{\text{lin}} \) and \( v^{\text{lin}} \) can be computed without solving the nonlinear power flow equations (2)–(5).

Equations (14) and (15) imply that \( P^{\text{lin}}_{ji} \) and \( Q^{\text{lin}}_{ji} \) are equal to the sum of their subsequent real and reactive power injections respectively for \((i,j) \in \mathcal{L} \). It follows that \( P^{\text{lin}} \) is a linear function of \( p, Q^{\text{lin}} \) is a linear function of \( q, \) and \( v^{\text{lin}} \) is a linear function of \( p,q \). Furthermore, the following lemma shows that \( P^{\text{lin}}, Q^{\text{lin}}, \) and \( v^{\text{lin}} \) are upper bounds on \( P, Q, \) and \( v \).

**Lemma 1:** Let \((p,q,P,Q,\ell,v)\) satisfy (2)–(4) and (8), then

\[
P \preceq P^{\text{lin}}(p), \quad Q \preceq Q^{\text{lin}}_{ji}(q), \quad \text{and} \quad v \preceq v^{\text{lin}}(p,q).
\]

**Proof:** In Appendix A
We replace the voltage constraint $v \leq \overline{v}$ in (6) on $v$ by the voltage constraint $v^{\text{lin}}(p, q) \leq \overline{v}$ in (17) on $v^{\text{lin}}(p, q)$ to obtain the modified OPF (MOPF) problem

\[
\text{MOPF:}
\min \sum_{(i,j) \in \mathcal{L}} r_{ij} \ell_{ij}
\text{over } p, q, P, Q, \ell, v
\text{s.t. \(2) - (5), (7); \)}
\overline{v} \leq v, \ v^{\text{lin}}(p, q) \leq \overline{v}.
\] (17)

\[\text{B. Comparison of MOPF and OPF}\]

Problem MOPF and OPF only differ in the voltage constraint, i.e., the upper and lower bounds on the voltage profile. Consider the hypothetical scenario in Fig. 2. The dashed curves illustrate the voltage upper and lower bounds in OPF. MOPF has the same voltage lower bound as OPF. Since $v \leq v^{\text{lin}}$, the constraint $v^{\text{lin}} \leq \overline{v}$ in MOPF can be interpreted as having a stricter upper bound on the voltage, as illustrated by the dotted curve. Since MOPF has a stricter voltage constraint than OPF, every feasible voltage profile for MOPF is also feasible for OPF, but voltage profiles that are feasible for OPF are not necessarily feasible for MOPF. For instance, in Fig. 2, voltage profile B, shown by a dash dot curve, is feasible for both MOPF and OPF, while voltage profile A, shown by a solid curve, is feasible only for OPF. On the other hand, when power loss is insignificant and voltage is stable—normal operation points satisfy these, $v^{\text{lin}}$ and $v$ are “close,” therefore the two upper bounds in Fig. 2 are “close.”

Fig. 2. Voltage constraints for MOPF and OPF along a path $\mathcal{P}_j$ for some $j \in \mathcal{N}$. Dashed curves illustrate the voltage upper and lower bounds in OPF. MOPF has the same voltage lower bound, but a smaller voltage upper bound illustrated by the dotted curve. Both voltage profiles A and B satisfy the voltage constraint in OPF, but only B satisfies the voltage constraint in MOPF.

We now formalize the intuition from Fig. 2. Let $\mathcal{F}_{\text{OPF}}$ denote the feasible set for OPF, and $\mathcal{F}_{\text{MOPF}}$ denote the
feasible set of MOPF, then \( F_{\text{MOPF}} \subseteq F_{\text{OPF}} \). Define

\[
\varepsilon := \max_{v \in \text{lin}(p, q)} v_i(p, q) - v_i(p, q)
\]

s.t. \( p \preceq p \preceq \bar{p}, \)

\[
q \preceq q \preceq \bar{q},
\]

\( i = 1, \ldots, N \)

as the maximum deviation from \( v_{\text{lin}}(p, q) \) to \( v(p, q) \), then

\[(v_i(p, q) \leq \bar{v}_i - \varepsilon) \Rightarrow (v_{i,\text{lin}}(p, q) \leq v_i) . (18)\]

Consider OPF with a stricter voltage upper bound \( v_i \leq \bar{v}_i - \varepsilon \)

\[\text{OPF-}\varepsilon: \]

\[
\min \sum_{(i,j) \in \mathcal{L}} r_{ij} \ell_{ij}
\]

over \( p, q, P, Q, \ell, v \)

s.t. (2)–(5), (7);

\[
v_i \leq v_i \leq \bar{v}_i - \varepsilon, i = 1, \ldots, N.
\]

It follows from (18) that \( F_{\text{OPF-}\varepsilon} \subseteq F_{\text{MOPF}} \). Hence,

\[F_{\text{OPF-}\varepsilon} \subseteq F_{\text{MOPF}} \subseteq F_{\text{OPF}},\]

i.e., MOPF is “sandwiched” between OPF and OPF with stricter voltage upper bound.

C. Evaluate \( \varepsilon \)

As seen in OPF-\(\varepsilon\), the maximum deviation \( \varepsilon \) from \( v_{\text{lin}}(p, q) \) to \( v(p, q) \) can be considered as the conservatively in voltage upper bound. If \( \bar{v}_i = 0.9, \bar{v}_i = 1.1 \) for all \( i \in \mathcal{N} \) and \( \varepsilon = 0.01 \), then the voltage constraint is \( 0.9 \leq v_i \leq 1.1 \) for OPF and \( 0.9 \leq v_i \leq 1.09 \) for OPF-\(\varepsilon\). For voltage profiles that satisfy the voltage constraint of OPF but violate that of OPF-\(\varepsilon\), we must have \( 1.09 < v_i \leq 1.1 \) for some \( i \in \mathcal{N} \). The voltage \( v_i \) is close to though still below the voltage upper bound in OPF, and such operating points are to be avoided due to safety consideration. The smaller \( \varepsilon \) is, the “closer” is OPF-\(\varepsilon\) to OPF, and the less favorable are the operating points that are feasible for OPF but infeasible for OPF-\(\varepsilon\).

It has long been accepted in the literature [10], [11], [25], [16] that the gap \( \varepsilon \) between \( v_{\text{lin}} \) and \( v \) is negligible for practical networks, and we provide an empirical evidence that \( \varepsilon = 0.0036 \) for the network in Fig. 5. It is a practical distribution network of Southern California Edison (SCE) [20] with high penetration of distributed generation (11.3MW peak load and 6.4MW nameplate distributed generation capacity). Line impedances, peak spot load, and nameplate ratings of shunt capacitors and photo voltaic generators of the network are summarized in Table I. In evaluating \( \varepsilon \), we assume that the loads are drawing peak spot apparent power at power factor 0.97, all
Fig. 3. An industrial feeder of SCE with high distributed generation (photovoltaics) penetration. Bus 1 is the substation and the 6 loads attached to it model the other feeders on this substation.

TABLE I
LINE IMPEDANCES, PEAK SPOT LOAD, AND NAMEPLATE RATINGS OF CAPACITORS AND PV GENERATIONS OF THE NETWORK SHOWN IN FIG. 3.

| Network Data | Load Data | PV Generators |
|--------------|-----------|---------------|
| Line Data    | Load Data | PV Generators |
| From Bus. | To Bus. | R (Ω) | X (Ω) | From Bus. | To Bus. | R (Ω) | X (Ω) | From Bus. | To Bus. | R (Ω) | X (Ω) | Bus No. | Peak MVA | Bus No. | Peak MVA | Bus No. | Nameplate Capacity |
| 1           | 2         | 0.259 | 0.808 | 8       | 41      | 0.107 | 0.031 | 21      | 22      | 0.198 | 0.046 | 1       | 30      | 34      | 0.2     | 13       | 1.5MW     |
| 2           | 13        | 0     | 0     | 8       | 35      | 0.076 | 0.015 | 22      | 23      | 0     | 0     | 11      | 0.67    | 36      | 0.27    | 17       | 0.4MW     |
| 2           | 3         | 0.031 | 0.092 | 8       | 9       | 0.031 | 0.031 | 27      | 31      | 0.046 | 0.015 | 12      | 0.45    | 38      | 0.45    | 19       | 1.5 MW    |
| 3           | 4         | 0.046 | 0.092 | 9       | 10      | 0.015 | 0.015 | 27      | 28      | 0.107 | 0.031 | 14      | 0.89    | 39      | 1.34    | 23       | 1 MW      |
| 3           | 14        | 0.092 | 0.031 | 9       | 42      | 0.153 | 0.046 | 28      | 29      | 0.107 | 0.031 | 16      | 0.07    | 40      | 0.13    | 24       | 2 MW      |
| 3           | 15        | 0.214 | 0.046 | 10      | 11      | 0.107 | 0.076 | 29      | 30      | 0.061 | 0.015 | 18      | 0.67    | 41      | 0.67    | 24       | 2 MW      |
| 3           | 20        | 0.336 | 0.061 | 10      | 46      | 0.229 | 0.122 | 32      | 33      | 0.046 | 0.015 | 21      | 0.45    | 42      | 0.13    |          |           |
| 4           | 5         | 0.107 | 0.183 | 11      | 47      | 0.031 | 0.015 | 33      | 34      | 0.031 | 0     | 22      | 2.23    | 44      | 0.45    |          |           |
| 4           | 26        | 0.081 | 0.015 | 11      | 12      | 0.076 | 0.046 | 35      | 36      | 0.076 | 0.015 | 25      | 0.45    | 45      | 0.45    |          |           |
| 5           | 6         | 0.015 | 0.031 | 15      | 18      | 0.046 | 0.015 | 35      | 37      | 0.076 | 0.046 | 26      | 0.2     | 46      | 0.45    |          |           |
| 6           | 27        | 0.168 | 0.061 | 15      | 16      | 0.107 | 0.015 | 35      | 38      | 0.107 | 0.015 | 28      | 0.13    |          |          |          |           |
| 6           | 7         | 0.031 | 0.046 | 16      | 17      | 0     | 0     | 42      | 43      | 0.061 | 0.015 | 29      | 0.13    | 1000    | 1000    |          |           |
| 7           | 32        | 0.076 | 0.015 | 18      | 19      | 0     | 0     | 43      | 44      | 0.061 | 0.015 | 30      | 0.2     |          |          |          |           |
| 7           | 8         | 0.015 | 0.015 | 20      | 21      | 0.122 | 0.092 | 43      | 45      | 0.061 | 0.015 | 31      | 0.07    |          |          |          |           |
| 8           | 40        | 0.046 | 0.015 | 20      | 25      | 0.214 | 0.046 |          |          |        |        | 32      | 0.13    |          |          |          |           |
| 8           | 39        | 0.244 | 0.046 | 21      | 24      | 0     | 0     |          |          |        |        | 33      | 0.27    |          |          |          |           |

Base Voltage (kV) = 12.35
Base kVA = 1000
Substation Voltage = 12.35
Base Voltage (kV) = 12.35
Base Voltage (kV) = 12.35
shunt capacitors are switched on, and distributed generators are generating real power at their nameplate capacities with 0 reactive power ($p$ equals $\bar{p}$ equals the real power injection in this case, and $q$ equals $\bar{q}$ equals the reactive power injection in this case). We also assume $v_0 = 1$.

The maximum deviation $\varepsilon$ from $v^{\text{lin}}$ to $v$ is considerable only when the power loss is significant with respect to the power flow, or the voltage is unstable. Both cases are to be avoided in practice.

D. Exact Relaxation for MOPF

MOPF is still non-convex due to the non-affine equality constraint in [5], and we relax it to the inequality constraint in [5] to obtain a convex relaxation RMOPF.

RMOPF:

$$\min \sum_{(i,j) \in \mathcal{L}} r_{ij} \ell_{ij}$$

over $p, q, P, Q, \ell, v$

s.t. (2)–(4), (7)–(8), (17).

In this section, MOPF is the non-convex original problem, and RMOPF is the convex relaxed problem. RMOPF is exact if and only if every of its optimal solutions satisfies the constraint in [5]. When RMOPF is exact, we can solve MOPF by solving RMOPF.

The main result of this paper is a sufficient condition for the exactness of RMOPF, which can be checked before solving RMOPF. Recall the closed-form expressions for $P^{\text{lin}}$ and $Q^{\text{lin}}$ in (14)–(15), and define

$$P_{ji}^{\text{rev}} := \max \{P_{ji}^{\text{lin}}, 0\}, \quad Q_{ji}^{\text{rev}} := \max \{Q_{ji}^{\text{lin}}, 0\}$$

for $(i, j) \in \mathcal{L}$ as the nonnegative part of $P_{ji}^{\text{lin}}$ and $Q_{ji}^{\text{lin}}$. We refer to $P^{\text{rev}}$ and $Q^{\text{rev}}$ as reverse power flow hereafter. Define

$$a_{i11} := \prod_{(k,l) \in \mathcal{P}_i} \left( 1 - \frac{2r_{kl} P_{lk}^{\text{rev}}(\bar{p})}{\bar{v}_l} \right),$$

$$a_{i12} := \sum_{(k,l) \in \mathcal{P}_i} \frac{2r_{kl} Q_{lk}^{\text{rev}}(\bar{q})}{\bar{v}_l},$$

$$a_{i21} := \sum_{(k,l) \in \mathcal{P}_i} \frac{2r_{kl} P_{lk}^{\text{rev}}(\bar{p})}{\bar{v}_l},$$

$$a_{i22} := \prod_{(k,l) \in \mathcal{P}_i} \left( 1 - \frac{2r_{kl} Q_{lk}^{\text{rev}}(\bar{q})}{\bar{v}_l} \right)$$

for $i \in \mathcal{N}$, and note that $a_{i_mn}^i$ for $m, n \in \{1, 2\}$ and $i \in \mathcal{N}$ can be computed before solving RMOPF. The sufficient condition for the exactness of RMOPF is

**Theorem 1:** RMOPF is exact if

$$\begin{pmatrix}
  a_{i11} & -a_{i12} \\
  -a_{i21} & a_{i22}
\end{pmatrix}
\begin{pmatrix}
  r_{ij} \\
  x_{ij}
\end{pmatrix} > 0, \forall (i, j) \in \mathcal{L}. \quad (20)$$
Reader needs to read Section IV-A and IV-B before reading the proof of Theorem 1 in Appendix D.

Note that condition (20) can be checked before solving RMOPF. It can be written in a simpler (and easier to comprehend) form

\[
\frac{r_{ij}}{x_{ij}} \in (\bar{b}_i, \bar{b}_i) := \left( \frac{a_{12}^i}{a_{11}^i}, \frac{a_{22}^i}{a_{21}^i} \right), \forall (i,j) \in \mathcal{L}
\]

(21)

when \(a_{11}^i > 0\) and \(a_{22}^i > 0\) for \(i \in \mathcal{N}\). In practice, \(a_{11}^i > 0\) and \(a_{22}^i > 0\) hold for \(i \in \mathcal{N}\). Hence, one can check (21) instead of (20) (but equivalently) to guarantee exact RMOPF.

Note that the interval \((\bar{b}_i, \bar{b}_i)\) in (21) depends on the reverse power flow. When there is zero reverse power flow, \(\bar{b}_i = 0\) and \(\bar{b}_i = \infty\) for \(i \in \mathcal{N}\). The larger the reverse power flow is, the smaller the intervals \((\bar{b}_i, \bar{b}_i)\) are. Define the minimum interval

\[
(b, \bar{b}) := \cap_{i \in \mathcal{N}} (\bar{b}_i, \bar{b}_i)
\]

as the intersection of all intervals in (21). If the minimum interval \((b, \bar{b})\) covers the range of \(\frac{r_{ij}}{x_{ij}}\), then (21) holds, which implies exact RMOPF. According to [26], the \(r/x\) range for practical transmission lines is \([0.1, 10]\), and we will refer to this range as the practical range hereafter.

E. Case Studies

In this section, we show that the minimum interval \((b, \bar{b})\) covers the practical \(r/x\) range \([0.1, 10]\) in various test networks, including IEEE 13, 34, 37, 123-bus distribution test networks [19] and a practical distribution network of SCE [20] with high penetration of distributed generation.

We calculate the minimum interval \((b, \bar{b})\) for each of the IEEE test networks, and summarize the results in Table II. It can be seen that the practical range \([0.1, 10]\) is covered by each of the minimal intervals. In our calculations, we consider the worst case where there is no load, while all shunt capacitors are switched on. This worst case maximizes reverse power flow, and minimizes \((b, \bar{b})\). Since the IEEE test networks are unbalanced three-phase networks, we assume different phases are decoupled, calculate the minimum interval for each phase, and take their intersection as the final minimum interval \((b, \bar{b})\).

| test circuit | \((b, \bar{b})\) |
|--------------|-----------------|
| IEEE 13-bus  | (0.0119, \infty) |
| IEEE 34-bus  | (0.0049, \infty) |
| IEEE 37-bus  | (0, \infty) |
| IEEE 123-bus | (0.0067, \infty) |
| SCE 47-bus   | (0.0366, 15.5739) |

TABLE II
MINIMUM INTERVALS OF DIFFERENT TEST CIRCUITS
There is no distributed generation in each of the four IEEE test networks. Therefore the reverse real power flow $P^{\text{rev}} = 0$, and it follows that $\overline{b}_i = \infty$ for $i \in \mathcal{N}$. This is why $\overline{b} = \infty$ for all four IEEE test networks. Distributed generation potentially introduces reverse real power flow, and shrinks the minimum interval $(\underline{b}, \overline{b})$.

To explore the effect of distributed generation, we revisit the SCE distribution network in Fig. 3 which has five photo voltaic (PV) generators and four shunt capacitor banks generating reverse power flow. Its minimum interval is calculated to be $(0.0366, 15.5739)$, which covers the practical range $[0.1, 10]$. In our calculation, we also consider the worst case where there is no load, while shunt capacitors are all switched on and PVs are all generating real power at their nameplate capacities. This worst case maximizes reverse power flow, and minimizes $(\underline{b}, \overline{b})$.

We are conservative in checking (21). First of all, instead of checking that $(\underline{b}_i, \overline{b}_i)$ covers $r_{ij}/x_{ij}$ for each $(i,j) \in \mathcal{L}$, we check that their intersection $(\underline{b}, \overline{b})$ covers the practical $r/x$ range $[0.1, 10]$ for all practical transmission lines. Secondly, we consider the worst case (which is unlikely to happen) where reverse power flow is maximized and therefore $(\underline{b}, \overline{b})$ is minimized. While $(\underline{b}, \overline{b})$ covers the practical range $[0.1, 10]$ after these two conservative steps, condition (21) is more widely satisfied. For instance, if we use the load data in Table I to calculate the minimum interval of the SCE network in Fig. 3 we obtain $(\underline{b}, \overline{b}) = (0.0245, 1209.7)$, which is much larger than the interval $(0.0366, 15.5739)$ obtained assuming the worst case.

To summarize, condition (21) is satisfied in all test networks. Therefore RMOPF is exact in all test networks.

IV. PROOF OF THE MAIN RESULT

We have given a sufficient condition for the exactness of RMOPF in Theorem 1 and shown that this condition holds in various test networks in Section III-E. In this section, we present the proof technique for Theorem 1 as well as some other theoretical results. We first present the proof technique for Theorem 1 in Section IV-A and IV-B and then use the technique to re-establish the results in prior work [17] in Section IV-C. At last, we study the uniqueness of MOPF solutions in Section IV-D.

Prior works [17], [18] use the duality theory [27, Chapter 5] to derive sufficient conditions for the exactness of SOCR. The idea is to prove that the optimal dual variables corresponding to (8) are always positive. It follows from complementary slackness that the equality in (8) is attained at any optimal solution to SOCR, therefore SOCR is exact. We call this method the dual method.

We explore a different method to derive sufficient conditions for the exactness of RMOPF in this work. Instead of working with the dual variables, we directly work with primal variables $p, q, P, Q, \ell,$ and $v$. The idea is to prove that for any feasible point $w = (p, q, P, Q, \ell, v)$ of RMOPF that violates (5), we can find another feasible point $w' = (p', q', P', Q', \ell', v')$ of RMOPF that has a smaller objective value. It follows that the optimal solution $w^{\text{opt}}$ to RMOPF must satisfy (5), therefore RMOPF is exact. We call this method the primal method.

A. The Primal Construction

The key to primal method lies in constructing a new feasible point $w'$ of RMOPF that has a smaller objective value, given any feasible point $w$ of RMOF that violates (5). For ease of presentation, we present the construction
in a one-line network shown in Fig. 4, and the construction in a general tree network can be found in Appendix D. In the one-line network, we abbreviate \( r_{ij}, x_{ij}, P_{ji}, Q_{ji} \), and \( \ell_{ij} \) by \( r_j, x_j, P_j, Q_j \) and \( \ell_j \) respectively. With this simplified notation, we re-state MOPF and RMOPF as follows.

**MOPF-line:**

\[
\min \sum_{i=1}^{N} r_i \ell_i \\
\text{over } p, q, P, Q, \ell, v \\
\text{s.t. } P_N = p_N, Q_N = q_N; \\
P_{i-1} = P_i - r_i \ell_i + p_{i-1}, \ i = 1, \ldots, N; \\
Q_{i-1} = Q_i - x_i \ell_i + q_{i-1}, \ i = 1, \ldots, N; \\
v_{i-1} = v_i - 2(r_i P_i + x_i Q_i) \\
+ (r_i^2 + x_i^2) \ell_i, \ i = 1, \ldots, N; \\
\ell_i = \frac{P_i^2 + Q_i^2}{v_i}, \ i = 1, \ldots, N; \\
v \preceq v, \\
v^\text{lin}(p, q) \preceq v, \\
p \preceq p \preceq \overline{p}, \ q \preceq q \preceq \overline{q}.
\]

**RMOPF-line:**

\[
\min \sum_{i=1}^{N} r_i \ell_i \\
\text{over } p, q, P, Q, \ell, v \\
\text{s.t. } \text{(22)-(25), (27)-(29);} \\
\ell_i \geq \frac{P_i^2 + Q_i^2}{v_i}, \ i = 1, \ldots, N.
\]

Given any feasible point \( w = (p, q, P, Q, \ell, v) \) of RMOPF-line that violates \( (26) \), let

\[
m := \min \left\{ i \geq 1 \mid \ell_i > \frac{P_i^2 + Q_i^2}{v_i} \right\}
\]
denote the first bus that violates (26), then \( m \in \mathbb{N} \). Pick \( \epsilon > 0 \) satisfying \( \epsilon \leq \frac{P_m^2 + Q_m^2}{v_m} - \ell_m \) (such \( \epsilon \) exists according to the definition of \( m \)), and construct \( w'(\epsilon) := (p', q', P', Q', \ell', v') \) according to Algorithm 1. First of all, Algorithm 1 keeps \( p \) and \( q \) unchanged, so that \( w' \) still satisfies (28)–(29). The main construction is for \( P', Q' \) and \( \ell' \), after which \( v' \) is simply constructed to satisfy (25).

In the construction of \( P', Q' \) and \( \ell' \), we set \( \ell' \) as

\[
\ell'_k = \begin{cases} 
\ell_k & k > m, \\
\ell_k - \epsilon & k = m, \\
\max\{P_k^2, P_k^2\} + \max\{Q_k^2, Q_k^2\} & k < m.
\end{cases}
\]

That is, we do not change \( \ell_k \) for \( k > m \); we reduce \( \ell_m \) to \( \ell'_m = \ell_m - \epsilon \); and we modify \( \ell_k \) for \( k < m \) so that constraint (30) remains satisfied (assuming \( v'_k = v_k \)) after \( P_k \) and \( Q_k \) are changed. The construction of \( P' \) and \( Q' \) is simply to satisfy (22)–(24). Since

\[
\ell'_{m} \Rightarrow P'_{m-1}, Q'_{m-1} \Rightarrow \ell'_{m-1} \Rightarrow P'_{m-2}, Q'_{m-2} \Rightarrow \ldots,
\]

The construction of \( P', Q' \) and \( \ell' \) can be done recursively. Through its construction given in Algorithm 1 \( w'(\epsilon) \) may only violate (27) and (30) of all the constraints in RMOPF.

B. The Proof of Main Result

The first question is whether \( w'(\epsilon) \) is feasible for RMOPF-line. It turns out that, if the power flow \( P_{m-1}, Q_{m-1}, \ldots, P_0, Q_0 \) increases, then not only \( w'(\epsilon) \) is feasible for MOPF-line, but also \( w'(\epsilon) \) has a smaller objective value than \( w \).

**Lemma 2:** The new point \( w'(\epsilon) \) is feasible for RMOPF-line, and has a smaller objective value than \( w \), i.e.,

\[
\sum_{i=1}^{N} r_i \ell'_i < \sum_{i=1}^{N} r_i \ell_i,
\]

provided that

\[
P'_k > P_k, \ Q'_k > Q_k, \ k = 0, \ldots, m - 1.
\]

(31)

**Proof:** In Appendix B.

We illustrate Lemma 2 through Fig. 5. In constructing the new point \( w' \), we do not change \( P_m, Q_m \) but reduce \( \ell_m \),

![Fig. 5. Illustration of (31).](image)

which is proportional to the real and reactive power loss on \((m-1, m)\). Hence, \( P'_{m-1} > P_{m-1} \) and \( Q'_{m-1} > Q_{m-1} \).

August 21, 2012 DRAFT
Algorithm 1 Primal Construction

Input:

\( (p, q, P, Q, \ell, v) \), feasible for RMOPF-line but violates (26):

- \( m \), equals to \( \min \{ i \geq 1 \mid \ell_i > \frac{P_i^2 + Q_i^2}{v_i} \} \);
- \( \epsilon \in \left( 0, \frac{P_m^2 + Q_m^2}{v_m} - \ell_m \right) \).

Output:

\( w'(\epsilon) = (p', q', P', Q', \ell', v'). \)

1) Construct \( p' \) and \( q' \):

- \( p' \leftarrow p \), \( q' \leftarrow q \).

2) Construct \( P' \), \( Q' \) and \( \ell' \):

- for \( k = m + 1, \ldots, N \):
  
  \[ P'_k \leftarrow P_k, \quad Q'_k \leftarrow Q_k, \quad \ell'_k \leftarrow \ell_k; \]

- for \( k = m \):
  
  \[ P'_m \leftarrow P_m, \quad Q'_m \leftarrow Q_m, \quad \ell'_m \leftarrow \ell_m - \epsilon; \]

- for \( k = m - 1, m - 2, \ldots, 1 \),

  \[ P'_k \leftarrow P'_{k+1} - r_{k+1} \ell'_{k+1} + p'_k; \]

  \[ Q'_k \leftarrow Q'_{k+1} - x_{k+1} \ell'_{k+1} + q'_k; \]

  \[ \ell'_k \leftarrow \max \{ P'_k^2, P'_k \} + \max \{ Q'_k^2, Q'_k \}; \]

- for \( k = 0 \):

  \[ P'_0 \leftarrow P'_{k+1} - r_{k+1} \ell'_{k+1} + p'_k; \]

  \[ Q'_0 \leftarrow Q'_{k+1} - x_{k+1} \ell'_{k+1} + q'_k. \]

3) Construct \( v' \):

- \( v'_0 \leftarrow v_0; \)

- for \( k = 1, 2, \ldots, N \),

  \[ v'_k \leftarrow v'_{k-1} + 2(r_k P'_k + x_k Q'_k) - (r_k^2 + x_k^2) \ell'_k. \]

Intuitively, after increasing \( P_{m-1} \) and \( Q_{m-1} \), power flow \( P_k \) and \( Q_k \), for \( k = m - 2, \ldots, 1, 0 \), should also increase. Lemma 2 says that if these power flows indeed increase, then \( w' \) is feasible and “better” than \( w \) (in the sense that \( w' \) has a smaller objective value than \( w \)). It follows that \( w \) cannot be optimal for RMOPF-line. Since \( w \) is chosen as an arbitrary feasible point of RMOPF-line that violates (26), RMOPF-line is exact.
Inequality (31) guarantees the exactness of RMOPF-line, and we now investigate when (31) holds. With the simplified notation for one-line networks, equations (14)–(15) and (19) take the form
\[
P_{\text{lin}}^k = \sum_{n \geq k} p_n, \quad Q_{\text{lin}}^k = \sum_{n \geq k} q_n, \\
P_{\text{rev}}^k = \max \left\{ P_{\text{lin}}^k, 0 \right\}, \quad Q_{\text{rev}}^k = \max \left\{ Q_{\text{lin}}^k, 0 \right\}
\]
for \( k \in \mathcal{N} \). We also have
\[
a_{i11}^1 = \prod_{k=1}^{i-1} \left( 1 - 2r_k \frac{P_{\text{rev}}^k(\bar{p})}{v_k} \right), \quad a_{i12}^1 = \sum_{k=1}^{i-1} 2r_k \frac{Q_{\text{rev}}^k(\bar{q})}{v_k}, \\
a_{i21}^1 = \sum_{k=1}^{i-1} 2x_k P_{\text{rev}}^k(\bar{p}) \frac{v_k}{v}, \quad a_{i22}^1 = \prod_{k=1}^{i-1} \left( 1 - 2x_k Q_{\text{rev}}^k(\bar{q}) \frac{v_k}{v} \right)
\]
for \( i \in \mathcal{N} \).

**Lemma 3:** Inequality (31) holds, if
\[
\begin{pmatrix}
a_{i11}^i & -a_{i12}^i \\
-a_{i21}^i & a_{i22}^i
\end{pmatrix}
\begin{pmatrix}
\rho_i \\
x_i
\end{pmatrix} > 0
\]
(32)
for \( i = 1, \ldots, N \), and \( \epsilon \) is sufficiently small.

**Proof:** In Appendix C. □

Condition (32) is nothing but (20) in the one-line network. We combine Lemma 2 and 3 to get the following theorem.

**Theorem 2:** RMOPF-line is exact if (32) holds for \( i = 1, \ldots, N \).

**Proof:** If an optimal solution \( w^{\text{opt}} \) to RMOPF-line violates (26), then we can construct a new point \( w(\epsilon) \) from \( w^{\text{opt}} \) according to Algorithm 1. If (32) holds for \( i = 1, \ldots, N \), then we can pick sufficiently small \( \epsilon \) such that \( w(\epsilon) \) is feasible for RMOPF-line and has a smaller objective value than \( w^{\text{opt}} \), according to Lemma 2 and 3. This contradicts the assumption that \( w^{\text{opt}} \) is optimal. Hence, any optimal solution \( w^{\text{opt}} \) to RMOPF-line must satisfy (26), therefore RMOPF-line is exact. □

Generalization of Theorem 2 in tree networks is Theorem 1, whose proof is provided in Appendix D.

### C. Connection with Prior Work

In [17], four sufficient conditions for the exactness of a convex relaxation for OPF are given, using the dual method. There are mainly two differences between the convex relaxation in [17] and RMOPF. Firstly, the relaxation in [17] is \( \ell_{ij} \geq (P_{ij}^2 + Q_{ij}^2)/v_i \) with \( P_{ij} \) and \( Q_{ij} \) denoting the power flow from \( i \) to \( j \), while the relaxation in RMOPF is \( \ell_{ij} \geq (P_{ji}^2 + Q_{ji}^2)/v_j \) with \( P_{ji} \) and \( Q_{ji} \) denoting the power flow from \( j \) to \( i \). Secondly, the voltage constraint in [17] is \( v \leq v \); while the voltage constraint in MOPF is \( v^\text{lin} \leq v \). The four sufficient conditions in [17] are

(p1) \( P_{\text{lin}}(\bar{p}) \geq 0, \quad Q_{\text{lin}}(\bar{q}) \geq 0 \) for \( (i,j) \in \mathcal{L} \).

(p2) \( r_{ij}/x_{ij} = r_{kl}/x_{kl} \) for any \( (i,j), (k,l) \in \mathcal{L} \).
We establish similar sufficient conditions for the exactness of RMOPF in the following theorem, using the primal method.

**Theorem 3:** RMOPF is exact if any of the following conditions holds.

(c1) \( P_{\text{rev}}^{ji}(p) = 0, Q_{\text{rev}}^{ji}(q) = 0 \) for \((i,j) \in \mathcal{L}\).

(c2) \( r_{ij}/x_{ij} = r_{kl}/x_{kl} \) for any \((i,j),(k,l) \in \mathcal{L}\), and
\[
\nu_j - 2r_{ij}P_{\text{rev}}^{ji}(\bar{p}) - 2x_{ij}Q_{\text{rev}}^{ji}(\bar{q}) > 0 \quad \text{for} \quad (i,j) \in \mathcal{L}.
\]

(c3) \( r_{ij}/x_{ij} \leq r_{jk}/x_{jk} \) for any \((i,j),(j,k) \in \mathcal{L}\), and
\[
P_{\text{rev}}^{ji}(p) = 0, \nu_j - 2x_{ij}Q_{\text{rev}}^{ji}(\bar{q}) > 0 \quad \text{for} \quad (i,j) \in \mathcal{L}.
\]

(c4) \( r_{ij}/x_{ij} \geq r_{jk}/x_{jk} \) for any \((i,j),(j,k) \in \mathcal{L}\), and
\[
Q_{\text{rev}}^{ji}(\bar{q}) = 0, \nu_j - 2r_{ij}P_{\text{rev}}^{ji}(\bar{p}) > 0 \quad \text{for} \quad (i,j) \in \mathcal{L}.
\]

**Proof:** In Appendix E.

We compare conditions p1-p4 with conditions c1-c4. Firstly, they can all be checked before solving the corresponding relaxation. Secondly, since \( P_{\text{lin}}^{ij} = -P_{\text{lin}}^{ji} \) and \( Q_{\text{lin}}^{ij} = -Q_{\text{lin}}^{ji} \) for all \((i,j) \in \mathcal{L}\), conditions p1 and c1 are identical. Thirdly, \( \nu_j - 2r_{ij}P_{\text{rev}}^{ji}(\bar{p}) - 2x_{ij}Q_{\text{rev}}^{ji}(\bar{q}) > 0 \) always holds in practical networks, for every \((i,j) \in \mathcal{L}\). Hence, conditions p2 and c2 are identical for practical networks. Similarly, conditions p3 and c3 are identical for practical networks, and conditions p4 and c4 are identical for practical networks.

To summarize, we have established sufficient conditions for the exactness of RMOPF using the primal method. These conditions are are identical to the conditions given in [17] in practical networks.

**D. Uniqueness of MOPF Solutions**

We study the uniqueness of MOPF solutions in this section. We assume that RMOPF is exact, which implies a point solves MOPF if and only if it solves RMOPF. Therefore we only need to study the uniqueness of RMOPF solutions. When the objective function of RMOPF is strictly convex, the solution is unique. However, the objective function is often linear—not strictly convex, e.g., the power loss. We show that with arbitrary convex objective functions (not necessarily strictly convex), RMOPF has a unique solution if it is exact.

**Theorem 4:** If RMOPF is exact and its objective function is convex, then RMOPF has a unique solution. Furthermore, MOPF has the same unique solution.

**Proof:** In Appendix F.

**V. Conclusion**

We have proposed a convex relaxation for solving a modified optimal power flow problem in tree networks. A sufficient condition for the exactness of the relaxation is given. This condition can be checked before solving the
relaxation, and holds in various test networks.

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A. Proof of Lemma 1

Since \((p, q, P, Q, \ell, v)\) satisfies (2)–(4) and (8), we have \(\ell_{ij} \geq (P_{ji}^2 + Q_{ji}^2)/v_j \geq 0\) for all \((i, j) \in L\). It follows that

\[
P_{ji} = \sum_{k: j \in P_k} p_k - \sum_{(k,l): j \in P_k} r_{kl} \ell_{kl} \leq \sum_{k: j \in P_k} p_k = P_{ji}^{lin}(p)
\]

for \((i, j) \in L\). Similarly, \(Q_{ji} \leq Q_{ji}^{lin}(q)\) for \((i, j) \in L\).

It follows from (4) that

\[
v_j - v_i = 2(r_{ij}P_{ji} + x_{ij}Q_{ji}) - (r_{ij}^2 + x_{ij}^2)\ell_{ij} \leq 2(r_{ij}P_{ji}^{lin} + x_{ij}Q_{ji}^{lin})
\]

for \((i, j) \in L\). We sum up the inequalities over \(P_j\) to obtain

\[
v_j - v_0 \leq \sum_{(k,l) \in P_j} 2(r_{kl}P_{kl}^{lin} + x_{kl}Q_{kl}^{lin}) = v_j^{lin}(p, q) - v_0,
\]

which implies \(v_j \leq v_j^{lin}(p, q)\), for \(j = 1, \ldots, N\).

B. Proof of Lemma 2

By the construction in Algorithm 1 \(w'\) satisfies (22)–(25) and (28)–(29). To show that \(w'\) is feasible for RMOPF-line, we are left to check that \(w'\) satisfies (27) and (30).

Claim 1: The voltage \(v' > v\) if (31) holds.

Proof: Define \(\Delta P := P' - P\), \(\Delta Q := Q' - Q\) and \(\Delta v := v' - v\). If (31) holds, then \(\Delta P_k > 0\), \(\Delta Q_k > 0\) for \(k = 0, \ldots, m - 1\), and \(\Delta P_k = 0\), \(\Delta Q_k = 0\) for \(k = m, \ldots, N\). It follows from (23)–(25) that

\[
v_i = v_{i-1} + r_i(P_i + P_{i-1} - p_{i-1}) + x_i(Q_i + Q_{i-1} - q_{i-1}),
\]

which implies

\[
\Delta v_i = \Delta v_{i-1} + r_i(\Delta P_i + \Delta P_{i-1}) + x_i(\Delta Q_i + \Delta Q_{i-1}) \geq \Delta v_{i-1},
\]

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for \( i = 1, \ldots, N \). Moreover, the inequality becomes strict when \( i = 1 \) since
\[
\Delta P_0 > 0, \quad \Delta Q_0 > 0.
\]
Hence,
\[
\Delta v_N \geq \ldots \geq \Delta v_2 \geq \Delta v_1 > \Delta v_0 = 0,
\]
which completes the proof of Claim 1.

It follows from Claim 1 that \( v'_i > v_i \geq v \) and
\[
\ell'_i = \frac{\max\{P^{\alpha}_i, P^\alpha\} + \max\{Q^{\alpha}_i, Q^\alpha\}}{v_i} \geq \frac{P^\alpha_k + Q^\alpha_k}{v'_i}
\]
for \( i = 1, \ldots, N \). Hence, \( w' \) satisfies (27) and (30), therefore feasible for RMOPF-line. It has a smaller objective value than \( w \) because
\[
\sum_{i=1}^{N} r_i \ell_i - \sum_{i=1}^{N} r_i \ell'_i = \left( \sum_{i=0}^{N} p'_i - \sum_{i=1}^{N} r_i \ell'_i \right) - \left( \sum_{i=0}^{N} p_i - \sum_{i=1}^{N} r_i \ell_i \right) = P'_0 - P_0 > 0.
\]

C. Proof of Lemma 3

Lemma 4: Given \((a_1, \ldots, a_n), (b_1, \ldots, b_n), (c_1, \ldots, c_n), (d_1, \ldots, d_n)\) satisfying \( 0 \leq a_k < 1, \ b_k \geq 0, \ c_k \geq 0, \ 0 \leq d_k < 1 \) for \( k = 1, \ldots, n \), where \( n \in \{1, 2, \ldots\} \), define matrices
\[
A_k = \begin{pmatrix}
1 - a_k & -b_k \\
-c_k & 1 - d_k
\end{pmatrix}
\]
for \( k = 1, \ldots, n \). Let \( u_{n+1} > 0 \) be a \( 2 \times 1 \) vector, and define
\[
u_k := A_k A_{k+1} \cdots A_n u_{n+1}
\]
for \( k = 1, \ldots, n \). If
\[
\begin{pmatrix}
\prod_{k=1}^{n} (1 - a_k) & -\sum_{k=1}^{n} b_k \\
-\sum_{k=1}^{n} c_k & \prod_{k=1}^{n} (1 - d_k)
\end{pmatrix} u_{n+1} > 0,
\]
then \( u_k > 0 \) for \( k = 1, \ldots, n + 1 \).

Proof: We prove Lemma 4 by induction on \( n \).

(i) When \( n = 1 \), Lemma 4 is trivial.
(ii) Suppose Lemma 4 holds for \( n = 1, \ldots, K \) \((K \geq 1)\). When \( n = K + 1 \), we have
\[
\left( \prod_{k=2}^{n} (1 - a_k) - \sum_{k=2}^{n} b_k \right) u_{n+1} \geq \left( \prod_{k=1}^{n} (1 - a_k) - \sum_{k=1}^{n} b_k \right) u_{n+1} \succ 0.
\]

According to induction hypothesis for \( n = K \), \( u_k \succ 0 \) for \( k = 2, \ldots, n + 1 \). To complete the induction, we are left to show that \( u_1 \succ 0 \). The idea is to construct a new sequence of \( K \) matrices and apply the induction hypothesis for \( n = K \) again. The construction is
\[
a_1' := 1 - (1 - a_1)(1 - a_2), \quad b_1' := b_1 + b_2, \quad c_1' := c_1 + c_2, \quad d_1' := 1 - (1 - d_1)(1 - d_2); \\
a_k' := a_{k+1}, \quad b_k' := b_{k+1}, \quad c_k' := c_{k+1}, \quad d_k' := d_{k+1} \text{ for } k = 2, \ldots, K.
\]
Define matrices
\[
A_k' := \begin{pmatrix} 1 - a_k & -b_k \\ -c_k & 1 - d_k \end{pmatrix}
\]
for \( k = 1, \ldots, K \), and \( u_k' := A_k' \cdots A_K u_{n+1} \) for \( k = 1, \ldots, K \). Apply the induction hypothesis for \( n = K \), and we obtain \( u_1' \succ 0, u_2' \succ 0 \). Then
\[
\begin{align*}
u_1 &= A_1 A_2 A_3 \cdots A_{K+1} u_{n+1} \\
&= A_1 A_2 A_3' \cdots A_K' u_{n+1} \\
&= A_1 A_2 u_2' \preceq A_1' u_2' \succ 0,
\end{align*}
\]
which completes the induction.

According to (i) and (ii), Lemma 4 holds for \( n = 1, 2, \ldots \). 

Recall that we are given a point \( w = (p, q, P, Q, \ell, v) \) that is feasible for RMOPF-line but violates (26), and \( m \) is the first bus that violates (26). Given any \( \epsilon \) satisfying \( 0 < \epsilon \leq \frac{P_m^2 + Q_m^2}{v_m^2} - \ell_m \), we can construct a point \( w'(\epsilon) \).

What we want to prove in Lemma 3 is that if inequality (32) holds and \( \epsilon \) is sufficiently small, then inequality (31) holds.

Define non-negative numbers
\[
\begin{align*}
\pi_k &:= \frac{2r_k P_{k}^{rev}(\bar{p})}{v_k}, & \varpi_k &:= \frac{2r_k Q_{k}^{rev}(\bar{q})}{v_k}, \\
\bar{\pi}_k &:= \frac{2x_k P_{k}^{rev}(\bar{p})}{v_k}, & \bar{\varpi}_k &:= \frac{2x_k Q_{k}^{rev}(\bar{q})}{v_k}
\end{align*}
\]
for \( k = 1, \ldots, N \). Inequality (32) is equivalent to
\[
 a_{11}^i r_i > a_{12}^i x_i \geq 0 \quad \text{and} \quad a_{22}^i x_i > a_{21}^i r_i \geq 0,
\]
which implies \( a_{11}^i > 0 \) and \( a_{22}^i > 0 \), for \( i = 1, \ldots, N \). Since \( a_{11}^i \) and \( a_{22}^i \) are products of \( 1 - \sigma_k \) and \( 1 - \overline{d}_k \) respectively, we have \( \overline{\sigma}_k < 1 \) and \( \overline{d}_k < 1 \) for \( k = 1, \ldots, N - 1 \). For \( x \in \mathbb{R} \), define \( x^+ := \max\{x, 0\} \). Define
\[
 a_k := \frac{2 r_k P_k^+}{v_k}, \quad b_k := \frac{2 r_k Q_k^+}{v_k}, \quad c_k := \frac{2 x_k P_k^+}{v_k}, \quad d_k := \frac{2 x_k Q_k^+}{v_k}
\]
for \( k = 1, \ldots, N \), then we have
\[
 0 \leq a_k \leq \overline{\sigma}_k < 1, \quad 0 \leq b_k \leq \overline{\sigma}_k,
\]
\[
 0 \leq c_k \leq \overline{\sigma}_k, \quad 0 \leq d_k \leq \overline{d}_k < 1
\]
for \( k = 1, \ldots, N - 1 \). It follows from (32) that
\[
 \left( \begin{array}{c}
 \prod_{k=1}^{m-1} (1 - a_k) - \sum_{k=1}^{m-1} b_k \\
 - \sum_{k=1}^{m-1} c_k \prod_{k=1}^{m-1} (1 - d_k)
 \end{array} \right) \left( \begin{array}{c}
 r_m \\
 x_m
 \end{array} \right) \geq 0
\]
Define matrices \( A_k \) according to (33) for \( k = 1, \ldots, N \), and
\[
 u_m := (r_m, x_m)^T \succ 0.
\]
It follows from Lemma 4 that \( A_k \cdots A_{m-1} u_m \succ 0 \) for \( k = 1, \ldots, m \). We will show that this implies
\[
 \left( \begin{array}{c}
 \Delta P_k \\
 \Delta Q_k
 \end{array} \right) = A_{k+1} \cdots A_{m-1} u_m \epsilon + O(\epsilon^2) \succ 0 \quad (34)
\]
for \( k = 0, \ldots, m - 1 \), if \( \epsilon \) is sufficiently small.

We prove (34) by induction on \( k \).

i) When \( k = m - 1 \), it follows from Algorithm 1 that
\[
 \left( \begin{array}{c}
 \Delta P_k \\
 \Delta Q_k
 \end{array} \right) = \left( \begin{array}{c}
 r_m \\
 x_m
 \end{array} \right) \epsilon = u_m \epsilon > 0.
\]
ii) Suppose (34) holds for $k = t$ (1 ≤ $t$ ≤ $m - 1$), we will show that (34) also holds for $k = t - 1$. We have

$$
\Delta \ell_t = \max\{P_t^2, Q_t^2\} \frac{v_t}{v_t} + \max\{P_t^2 - P_t^2, 0\} + \max\{Q_t^2 - Q_t^2, 0\} \frac{v_t}{v_t}
$$

$$
= \max\{2P_t \Delta P_t + O(\epsilon^2), 0\} \frac{v_t}{v_t} + \max\{2Q_t \Delta Q_t + O(\epsilon^2), 0\} \frac{v_t}{v_t}
$$

$$
= \frac{2P_t^+ \Delta P_t}{v_t} + \frac{2Q_t^+ \Delta Q_t}{v_t} + O(\epsilon^2)
$$

when $\epsilon$ is sufficiently small such that $\Delta P_t > 0$ and $\Delta Q_t > 0$. It follows that

$$
\begin{pmatrix}
\Delta P_{t-1} \\
\Delta Q_{t-1}
\end{pmatrix} = \begin{pmatrix}
P_t \\
Q_t
\end{pmatrix} - \begin{pmatrix}
r_t \\
x_t
\end{pmatrix} \Delta \ell_t
$$

$$
= A_t \begin{pmatrix}
P_t \\
Q_t
\end{pmatrix} + O(\epsilon^2)
$$

(35)

$$
= A_t \cdots A_{m-1} u_m \epsilon + O(\epsilon^2).
$$

Since $A_t \cdots A_{m-1} u_m \succ 0$, (34) holds for $k = t - 1$ if $\epsilon$ is sufficiently small.

According to (i) and (ii), (34) holds for $k = 0, \ldots, m - 1$ if $\epsilon$ is sufficiently small, which completes the proof of Lemma 3.

D. Proof of Theorem 7

If a solution $w^{opt} = (p^{opt}, q^{opt}, P^{opt}, Q^{opt}, \ell^{opt}, v^{opt})$ to RMOPF violates (5), then there exists a line $(i, j)$ such that

$$
\ell^{opt}_{ij} > \frac{(P^{opt}_{ji})^2 + (Q^{opt}_{ji})^2}{v^{opt}_{ji}}
$$

and (5) holds on $P_i$. For $\epsilon$ satisfying

$$
0 < \epsilon \leq \frac{(P^{opt}_{ji})^2 + (Q^{opt}_{ji})^2}{v^{opt}_{ji}} - \ell^{opt}_{ij},
$$

we construct a new point $w(\epsilon)$ by

1) setting $p = p^{opt}$, $q = q^{opt}$,

2) setting $\ell_{kl} = \ell^{opt}_{kl}$, $P_{kl} = P^{opt}_{kl}$, $Q_{kl} = Q^{opt}_{kl}$ for $(k, l) \notin P_j$;

3) setting $\ell_{kl}$, $P_{kl}$, $Q_{kl}$ as Algorithm 1 for $(k, l) \in P_j$;

4) setting $v$ to satisfy (4).

Define

$$
P_0 := p_0 + \sum_{(0,k) \in \mathcal{L}} (P_{ko} - r_{0k} \ell_{0k}),
$$

$$
Q_0 := q_0 + \sum_{(0,k) \in \mathcal{L}} (Q_{ko} - x_{0k} \ell_{0k})
$$
as the power flow from the substation to the main grid. Follow Appendix C we can prove that if (20) holds, then the point \( w(\epsilon) \) satisfies
\[
\Delta P_{lk} > 0, \quad \Delta Q_{lk} > 0
\]
for \((k, l) \in P_i\) and \( \Delta P_0 > 0, \Delta Q_0 > 0 \) for sufficiently small \( \epsilon \). Follow Appendix B we can further prove that such \( w(\epsilon) \) is feasible for RMOPF and has a smaller objective value than \( w^{\text{opt}} \), which contradicts with the assumption that \( w^{\text{opt}} \) is optimal for RMOPF. Hence, any optimal solution \( w^{\text{opt}} \) to RMOPF satisfies (5), therefore RMOPF is exact.

E. Proof of Theorem 3

We only give the proof for a one-line network as in Fig. 4. The proof can be generalized to tree networks following Appendix D. With the simplified notation in a one-line network, we re-state Theorem 3 as

**Theorem 5:** RMOPF-line is exact if any of the following conditions holds.

- \( P_{k}^{\text{rev}}(\bar{p}) = 0, Q_{k}^{\text{rev}}(\bar{q}) = 0 \) for \( k = 1, \ldots, N \).
- \( r_m/x_m = r_n/x_n \) for any \( 1 \leq m, n \leq N \), and  
  \( \psi_k - 2r_kP_k^{\text{rev}}(\bar{p}) - 2x_kQ_k^{\text{rev}}(\bar{q}) > 0 \) for \( k = 1, \ldots, N \).
- \( r_m/x_m \leq r_n/x_n \) for \( 1 \leq m \leq n \leq N \), and  
  \( P_k^{\text{rev}}(\bar{p}) = 0, \psi_k - 2x_kQ_k^{\text{rev}}(\bar{q}) > 0 \) for \( k = 1, \ldots, N \).
- \( r_m/x_m \geq r_n/x_n \) for \( 1 \leq m \leq n \leq N \), and  
  \( Q_k^{\text{rev}}(\bar{q}) = 0, \psi_k - 2r_kP_k^{\text{rev}}(\bar{p}) > 0 \) for \( k = 1, \ldots, N \).

We prove Theorem 5 through Claim 2–5.

**Claim 2:** RMOPF-line is exact if \( P_k^{\text{rev}}(\bar{p}) = 0, Q_k^{\text{rev}}(\bar{q}) = 0 \) for \( k = 1, \ldots, N \).

**Proof:** If \( P_k^{\text{rev}}(\bar{p}) = 0 \) and \( Q_k^{\text{rev}}(\bar{q}) = 0 \) for \( k = 1, \ldots, N \), then \( a_{11} = a_{22} = 1, a_{12} = a_{21} = 0 \) for \( i = 1, \ldots, N \). It follows that (32) is satisfied for \( i = 1, \ldots, N \), therefore RMOPF-line is exact according to Theorem 2.

**Claim 3:** RMOPF-line is exact if \( r_m/x_m = r_n/x_n \) for \( 1 \leq m, n \leq N \), and  
  \( \psi_k - 2r_kP_k^{\text{rev}}(\bar{p}) - 2x_kQ_k^{\text{rev}}(\bar{q}) > 0 \) for \( k = 1, \ldots, N \).

**Proof:** Let \( \eta := r_1/x_1 \) denote the \( r/x \) ratio for all transmission lines. We will show that (31) holds for sufficiently small \( \epsilon \). Since
\[
\Delta P_k = - \sum_{n=k+1}^{N} r_n \Delta \ell_n = - \eta \sum_{n=k+1}^{N} x_n \Delta \ell_n = - \eta \Delta Q_k
\]
for \( k = 0, \ldots, N \), it suffices to show \( \Delta Q_k > 0 \) for \( k = 0, \ldots, m-1 \) to prove (31). For brevity, we ignore the second order term \( O(\epsilon^2) \) in the following derivation, but the second order term can be considered following Appendix C.
It follows from (35) that

\[ \Delta Q_{t-1} = -\frac{2\epsilon}{v_t} \Delta P_t + \left(1 - \frac{2\epsilon}{v_t} \right) \Delta Q_t \]

\[ = -\frac{2\epsilon}{v_t} \Delta Q_t + \left(1 - \frac{2\epsilon}{v_t} \right) \Delta Q_t \]

\[ = \frac{1}{v_t} \left( v_t - 2\epsilon \right) \Delta Q_t \]

for \( t = 1, \ldots, m - 1 \). Therefore \( \Delta Q_t \) has the same sign for \( t = 0, \ldots, m - 1 \) since \( v_t - 2\epsilon \geq 0 \) for \( t = 1, \ldots, m - 1 \). By construction, \( \Delta Q_{m-1} = \epsilon > 0 \). Hence, \( \Delta Q_t > 0 \) for \( t = 0, \ldots, m - 1 \). Therefore (31) holds, which implies exact RMOPF-line.

\[ \text{Claim 4: RMOPF-line is exact if } r_m/x_m \leq r_n/x_n \text{ for } 1 \leq m \leq n \leq N, \text{ and } P_k^{\text{rev}}(\bar{p}) - 2x_tQ_k^{\text{rev}}(\bar{q}) > 0 \text{ for } k = 1, \ldots, N. \]

**Proof:** We will show that (31) holds for sufficiently small \( \epsilon \). For brevity, we ignore the second order term \( O(\epsilon^2) \) in the following derivation, but the second order term can be considered following Appendix C.

We have \( 0 \leq P_k^+ \leq P_k^{\text{rev}}(\bar{p}) = 0 \), which implies \( P_k^+ = 0 \), for \( k = 1, \ldots, N \). It follows from (35) that

\[ \left( \begin{array}{c} \Delta P_{t-1} \\ \Delta Q_{t-1} \end{array} \right) = \left( \begin{array}{cc} 1 - \frac{2\epsilon}{v_t} & 0 \\ 0 & 1 - \frac{2\epsilon}{v_t} \end{array} \right) \left( \begin{array}{c} \Delta P_t \\ \Delta Q_t \end{array} \right) \]

for \( t = 1, \ldots, m-1 \). Therefore \( \Delta Q_t \) has the same sign for \( t = 0, \ldots, m-1 \) since \( 1 - 2\epsilon Q_t^+ / v_t = \frac{1}{v_t} (v_t - 2x_tQ_t^{\text{rev}}(\bar{q})) > 0 \) for \( t = 1, \ldots, m - 1 \). By construction, \( \Delta Q_{m-1} = \epsilon > 0 \). Hence, \( \Delta Q_t > 0 \) for \( t = 0, \ldots, m - 1 \). Now we prove that

\[ \frac{\Delta P_t}{\Delta Q_t} \geq \frac{r_m}{x_m} \]

(36)

for \( t = 0, \ldots, m - 1 \) by induction on \( t \).

i) When \( t = m - 1 \), we have

\[ \frac{\Delta P_t}{\Delta Q_t} = \frac{r_m \epsilon}{x_m \epsilon} = \frac{r_m}{x_m} \]

Inequality (36) holds.

ii) Suppose (36) holds for \( t = k \) (1 \( \leq k \leq m - 1 \)), then

\[ \frac{\Delta P_{k-1}}{\Delta Q_{k-1}} = \frac{\Delta P_k - 2x_k Q_k^+ \Delta Q_k}{1 - \frac{2x_k Q_k^+}{v_k}} \]

\[ = \frac{\Delta P_k - 2x_k Q_k^+}{1 - \frac{2x_k Q_k^+}{v_k}} \Delta Q_k \]

\[ \geq \frac{r_m - 2x_k Q_k^+}{1 - \frac{2x_k Q_k^+}{v_k}} = \frac{r_m}{x_m} \]

i.e., (36) also holds for \( t = k - 1 \). Here the first inequality is because (36) holds for \( t = k \) (induction hypothesis); and the second inequality is because \( r_m/x_m \geq r_k/x_k \).
According to (i) and (ii), (36) holds for \( t = 0, \ldots, m - 1 \). It follows that \( \Delta P_t > 0 \) for \( t = 0, \ldots, m - 1 \). We have proven that (31) holds, which implies exact RMOPF-line.

**Claim 5:** RMOPF-line is exact if \( r_m/x_m \leq r_n/x_n \) for \( 1 \leq m \leq n \leq N \), and \( Q^\text{rev}_k(\bar{q}) = 0 \), \( \nu_k - 2r_k P^\text{rev}_k(\bar{q}) > 0 \) for \( k = 1, \ldots, N \).

**Proof:** The proof of Claim 5 is similar to that of Claim 4 and omitted for brevity.

**F. Proof of Theorem 4**

**Claim 6:** RMOPF has a unique solution if it is exact and its objective function is convex.

**Proof:** Let \( f \) denote the convex objective function of RMOPF, let \( \tilde{w} = (\tilde{p}, \tilde{q}, \tilde{P}, \tilde{Q}, \tilde{\ell}, \tilde{v}) \), \( \hat{w} = (\hat{p}, \hat{q}, \hat{P}, \hat{Q}, \hat{\ell}, \hat{v}) \) denote two arbitrary solutions to RMOPF, and define

\[
\tilde{f}^\text{opt} := f(\tilde{w}) = f(\hat{w})
\]

as the optimal value of RMOPF. Since RMOPF is exact,

\[
\tilde{\ell}_{ij}\tilde{v}_j = \tilde{P}_{ji}^2 + \tilde{Q}_{ji}^2, \quad \hat{\ell}_{ij}\hat{v}_j = \hat{P}_{ji}^2 + \hat{Q}_{ji}^2
\]

(37)

for \((i, j) \in \mathcal{L}\). For any \( \theta \in (0, 1) \), define

\[
w(\theta) := \theta \hat{w} + (1 - \theta) \tilde{w}.
\]

The point \( w(\theta) \) is feasible for RMOPF since RMOPF is convex. It is also optimal for RMOPF because

\[
f(w(\theta)) \leq \theta f(\hat{w}) + (1 - \theta) f(\tilde{w}) = \tilde{f}^\text{opt}.
\]

If RMOPF is exact, it follows that

\[
\ell_{ij}(\theta) v_j(\theta) = P_{ji}^2(\theta) + Q_{ji}^2(\theta)
\]

(38)

for \((i, j) \in \mathcal{L}\). Substitute

\[
\ell_{ij}(\theta) = \theta \ell_{ij} + (1 - \theta) \hat{\ell}_{ij}, \quad v_j(\theta) = \theta \hat{v}_j + (1 - \theta) \tilde{v}_j,
\]

\[
P_{ji}(\theta) = \theta \hat{P}_{ji} + (1 - \theta) \tilde{P}_{ji}, \quad Q_{ji}(\theta) = \theta \hat{Q}_{ji} + (1 - \theta) \tilde{Q}_{ji}
\]

into (38) and simplify using (37) to obtain

\[
\hat{v}_j\tilde{\ell}_{ij} + \tilde{v}_j\hat{\ell}_{ij} = 2 \left( \tilde{P}_{ji}\hat{P}_{ji} + \hat{Q}_{ji}\tilde{Q}_{ji} \right)
\]

for \((i, j) \in \mathcal{L}\). It follows that

\[
2 \left( \tilde{P}_{ji}\hat{P}_{ji} + \hat{Q}_{ji}\tilde{Q}_{ji} \right)
\]

\[
= \frac{\hat{v}_j}{\tilde{v}_j} \frac{\tilde{P}_{ji}^2 + \tilde{Q}_{ji}^2}{\tilde{v}_j} + \frac{\tilde{v}_j}{\hat{v}_j} \frac{\hat{P}_{ji}^2 + \hat{Q}_{ji}^2}{\hat{v}_j}
\]

\[
= \frac{\hat{v}_j}{\tilde{v}_j} \frac{\tilde{P}_{ji}^2}{\tilde{v}_j} + \frac{\tilde{v}_j}{\hat{v}_j} \frac{\hat{P}_{ji}^2}{\hat{v}_j} + \frac{\tilde{v}_j}{\hat{v}_j} \frac{\hat{Q}_{ji}^2}{\hat{v}_j} + \frac{\hat{v}_j}{\tilde{v}_j} \frac{\tilde{Q}_{ji}^2}{\tilde{v}_j}
\]

\[
\geq 2\sqrt{\hat{P}_{ji}^2 \tilde{P}_{ji}^2} + 2\sqrt{\hat{Q}_{ji}^2 \tilde{Q}_{ji}^2}
\]

\[
\geq 2 \tilde{P}_{ji}\hat{P}_{ji} + 2 \hat{Q}_{ji}\tilde{Q}_{ji}.
\]
for \((i,j) \in \mathcal{L}\). The two inequalities must both attain equalities. The first inequality attaining equality implies that 
\[
\left| \frac{\hat{P}_{ji}}{\tilde{P}_{ji}} \right| = \frac{\hat{v}_j}{\tilde{v}_j} = \frac{\hat{Q}_{ji}}{\tilde{Q}_{ji}}
\]
for \((i,j) \in \mathcal{L}\). The second inequality attaining equality implies that \(\hat{P}_{ji}\) and \(\tilde{P}_{ji}\) have the same sign, \(\hat{Q}_{ji}\) and \(\tilde{Q}_{ji}\) have the same sign, for \((i,j) \in \mathcal{L}\). It follows that 
\[
\hat{P}_{ji} = \frac{\hat{v}_j}{\tilde{v}_j} = \frac{\hat{Q}_{ji}}{\tilde{Q}_{ji}}
\]
for \((i,j) \in \mathcal{L}\), which further implies 
\[
\frac{\hat{P}_{ji}}{\tilde{P}_{ji}} = \frac{\hat{v}_j}{\tilde{v}_j} = \frac{\hat{Q}_{ji}}{\tilde{Q}_{ji}} = \frac{\hat{\ell}_{ij}}{\tilde{\ell}_{ij}}
\]
for \((i,j) \in \mathcal{L}\). Define 
\[
\eta_j := \frac{\hat{v}_j}{\tilde{v}_j}
\]
for \(j \in \mathcal{N}\), then \(\eta_0 = 1\), and 
\[
\eta_i = \frac{\hat{v}_i}{\tilde{v}_i} = \frac{\hat{v}_j - 2(r_{ij}\hat{P}_{ji} + x_{ij}\hat{Q}_{ji}) + (r_{ij}^2 + x_{ij}^2)\hat{\ell}_{ij}}{\tilde{v}_j - 2(r_{ij}\tilde{P}_{ji} + x_{ij}\tilde{Q}_{ji}) + (r_{ij}^2 + x_{ij}^2)\tilde{\ell}_{ij}} = \eta_j
\]
for \((i,j) \in \mathcal{L}\). It follows that \(\eta_j = 1\) for \(j \in \mathcal{N}\), which implies \(\hat{P} = \tilde{P}, \hat{Q} = \tilde{Q}, \hat{\ell} = \tilde{\ell}\), and \(\hat{v} = \tilde{v}\). It is not difficult to further prove that \(\hat{p} = \tilde{p}\) and \(\hat{q} = \tilde{q}\), which completes the proof of \(\hat{w} = \tilde{w}\). Claim 6 follows from \(\hat{w} = \tilde{w}\).

To complete the proof of Theorem 4, we only need to show that RMOPF and MOPF has the same solution when RMOPF is exact. Any solution \(w'\) to RMOPF is also optimal for MOPF because RMOPF is exact. For any solution \(w_{\text{opt}}\) to MOPF, we have \(f(w_{\text{opt}}) \leq f(w')\). Hence, \(w_{\text{opt}}\) is optimal for RMOPF. We have proved that solutions to MOPF and RMOPF are the same when RMOPF is exact, and Theorem 4 follows.