Three-dimensional analytical discrete-ordinates method for structured illumination

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Abstract

The radiative transport equation is considered in the spatial frequency domain. We extend the analytical discrete-ordinates method to three dimensions by rotating reference frames.

Keywords: radiative transport equation, discrete ordinates, rotated reference frames, structured illumination

1. Introduction

In this paper we consider the three-dimensional radiative transport equation by extending the method of analytical discrete ordinates (ADO) [1, 2, 3] to three dimensions using rotated reference frames. In one dimension, analytical solutions are available in terms of singular eigenfunctions by Case’s method [4, 5]. The method was recently extended to three dimensions [6]. Our three-dimensional ADO relies on the knowledge of the three-dimensional spherical-harmonics method [7, 8].

The discrete-ordinates method first proposed by Wick [7] and Chandrasekhar [8] was further developed to ADO. The method was constructed for the one-dimensional equation, the searchlight problem in three dimensions was considered with ADO in the case of isotropic scattering, for which the scattering phase function is constant [9]. ADO relies on the knowledge of the three-dimensional singular eigenfunction. The discrete-ordinates method was developed for the one-dimensional transport equation by extending the method of analytical solutions are available in terms of singular eigenfunctions by Case’s method [4, 5]. The method was recently extended to three dimensions [6]. Our three-dimensional ADO relies on the knowledge of the three-dimensional spherical-harmonics method [7, 8].

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Here, is the anisotropic factor. We give the source term as
\[ \tilde{g}(\mathbf{s}) \]
We assume that
\[ q \in [0, 2\pi] \]
is the azimuthal angle of \( \mathbf{s} \), and \( L_l^m(\mu) \) are associated Legendre polynomials. The constant \( g \in [0, 1) \) is the anisotropic factor. We give the source term as
\[ g(\mathbf{r}, \mathbf{s}) = I_0 e^{i q_0 \cdot \mathbf{r}} \delta(\mathbf{s} - \mathbf{s}_0), \quad \mathbf{q}_0 \in \mathbb{R}^2. \]
The Fourier transform \( \tilde{g}(\mathbf{q}, \mathbf{s}) \) is given by
\[ \tilde{g}(\mathbf{q}, \mathbf{s}) = \int_{\mathbb{R}^2} e^{-i \mathbf{q} \cdot \mathbf{r}} g(\mathbf{r}, \mathbf{s}) d\mathbf{r} = (2\pi)^2 I_0 \delta(\mathbf{q} - \mathbf{q}_0) \delta(\mathbf{s} - \mathbf{s}_0). \]
Here, \( \mathbf{s} \in \mathbb{S}^2 \) is given by
\[ \mathbf{s} = \begin{pmatrix} \omega \\ \mu \end{pmatrix}, \quad \omega = \frac{\sqrt{1 - \mu^2} \cos \varphi}{\sqrt{1 - \mu^2} \sin \varphi} \]
for \(-1 \leq \mu \leq 1, 0 \leq \varphi < 2\pi\). We define \( \mathbb{S}^2_+ \) as
\[ \mathbb{S}^2_+ = \{ \mathbf{s} \in \mathbb{S}^2; \ 0 < \pm \mu \leq 1, 0 \leq \varphi < 2\pi \}. \]
We assume that \( \mathbf{s}_0 \in \mathbb{S}^2_+ \). Moreover we defined
\[ \Gamma_0^0 = \{ (\mathbf{r}, \mathbf{s}) \in \partial \Omega \times \mathbb{S}^2_+; z = 0 \}, \]
\[ \Gamma_1^L = \{ (\mathbf{r}, \mathbf{s}) \in \partial \Omega \times \mathbb{S}^2_-; z = L \}. \]
We decompose the specific intensity into the ballistic and scattering terms as
\[ I(\mathbf{r}, \mathbf{s}) = I_b(\mathbf{r}, \mathbf{s}) + I_s(\mathbf{r}, \mathbf{s}). \]
The ballistic term \( I_b(\mathbf{r}, \mathbf{s}) \) satisfies
\[ \begin{cases} (\mathbf{s} \cdot \nabla + 1) I_b(\mathbf{r}, \mathbf{s}) = 0 & \text{in } \Omega \times \mathbb{S}^2, \\
I_b(\mathbf{r}, \mathbf{s}) = g(\mathbf{r}, \mathbf{s}) & \text{on } \Gamma_0^0, \\
I_b(\mathbf{r}, \mathbf{s}) = 0 & \text{on } \Gamma_1^L. \end{cases} \]
The scattering term \( I_s(\mathbf{r}, \mathbf{s}) \) satisfies
\[ \begin{cases} (\mathbf{s} \cdot \nabla + 1) I_s(\mathbf{r}, \mathbf{s}) = \omega \int_{\mathbb{S}^2} p(\mathbf{s}, \mathbf{s}') I_s(\mathbf{r}, \mathbf{s}') d\mathbf{s}' + S(\mathbf{r}, \mathbf{s}) & \text{in } \Omega \times \mathbb{S}^2, \\
I_s(\mathbf{r}, \mathbf{s}) = 0 & \text{on } \Gamma_0^0, \\
I_s(\mathbf{r}, \mathbf{s}) = 0 & \text{on } \Gamma_1^L. \end{cases} \]
where
\[ S(\mathbf{r}, \mathbf{s}) = \omega \int_{\mathbb{S}^2} p(\mathbf{s}, \mathbf{s}') I_s(\mathbf{r}, \mathbf{s}') d\mathbf{s}'. \]
We obtain
\[ I_b(\mathbf{r}, \mathbf{s}) = e^{-z/\mu} g \left( \rho - \frac{z}{\mu} \omega, \mathbf{s} \right), \]
and \( I_b = 0 \) for \( \mu < 0 \). Hence,
\[ S(\mathbf{r}, \mathbf{s}) = \omega I_0 \rho(\mathbf{s}, \mathbf{s}_0) e^{-1/\mu} \omega_0 z/\mu_0, \]
where \( \omega_0, \mu_0 \) are components of \( \mathbf{s}_0 \). The Fourier transform \( \tilde{I}_s(\mathbf{q}, z, \mathbf{s}) \) is given by
\[ \tilde{I}_s(\mathbf{q}, z, \mathbf{s}) = \int_{\mathbb{S}^2} e^{-i \mathbf{q} \cdot \mathbf{r}} I_s(\mathbf{r}, \mathbf{s}) d\mathbf{r}, \]
where
\[ q = q \left( \frac{\cos \varphi_q}{\sin \varphi_q} \right). \]
Similarly, we introduce \( q_0, \varphi_{q_0} \) for \( \mathbf{q}_0 \). This \( \tilde{I}_s(\mathbf{q}, z, \mathbf{s}) \) is obtained as the solution to the following equation.
\[ \left\{ \begin{align*}
(\mu \frac{\partial}{\partial z} + 1 + i\omega \cdot \mathbf{q}) \tilde{I}_s &= \omega \int_{\mathbb{S}^2} p(\mathbf{s}, \mathbf{s}') \tilde{I}_s(\mathbf{r}, \mathbf{s}') d\mathbf{s}' \\
&+ \tilde{S}, \quad z \in (0, L), \ \mathbf{s} \in \mathbb{S}^2,
\tilde{I}_s(\mathbf{q}, z, \mathbf{s}) &= 0, \quad z = 0, \ \mathbf{s} \in \mathbb{S}^2_+,
\tilde{I}_s(\mathbf{q}, z, \mathbf{s}) &= 0, \quad z = L, \ \mathbf{s} \in \mathbb{S}^2_-, \quad (2.3)
\end{align*} \]
We note that
\[ \omega \cdot \mathbf{q} = q \sqrt{1 - \mu^2} \cos(\varphi - \varphi_q). \]
We will obtain \( \tilde{I}_s \) by using discrete ordinates:
\[ \tilde{I}_s \approx \tilde{I}_s, \]
where \( \tilde{I}_s \) is the solution to the following equation.
\[ \left\{ \begin{align*}
(\mu \frac{\partial}{\partial z} + 1 + i\omega \cdot \mathbf{q}) \tilde{I}_s &= \omega \sum_{\nu'=1}^{2N} w_{\nu'} \int_0^{2\pi} p(\mathbf{s}_\nu', \mathbf{s}_\nu') \tilde{I}_s(\mathbf{r}, \mathbf{s}_\nu') d\varphi' \\
&+ \tilde{S}, \quad z \in (0, L), \ \mathbf{s}_\nu \in \mathbb{S}^2,
\tilde{I}_s(\mathbf{q}, z, \mathbf{s}) &= 0, \quad z = 0, \ \mathbf{s}_\nu \in \mathbb{S}^2_+,
\tilde{I}_s(\mathbf{q}, z, \mathbf{s}) &= 0, \quad z = L, \ \mathbf{s}_\nu \in \mathbb{S}^2_- \quad (2.6)
\end{align*} \]
Here, we discretize the integral by discrete ordinates with the Gauss-Legendre quadrature of weights \( w_i \) \((i = 1, \ldots, N)\). We use the Golub-Welsch algorithm \cite{19}. We label \( \mu_i \) \((i = 1, \ldots, N)\) such that \( 0 < \mu_1 < \mu_2 < \cdots < \mu_N < 1 \) and \(-1 < \mu_{2N} < \cdots < \mu_{N+1} < 0 \) \((\mu_{N+1} = -\mu_1, \ 1 \leq i \leq N)\). Moreover we introduced
\[
\hat{s}_i = \left( \frac{\omega_i}{\mu_i} \right), \quad \omega_i = \sqrt{\frac{1 - \mu_i^2}{1 - \mu_i^2 \sin^2 \varphi}}
\]
for \( i = 1, 2, \ldots, 2N, \) \( 0 \leq \varphi < 2\pi \). We have
\[
\omega_i \cdot q = q \sqrt{1 - \mu_i^2 \cos (\varphi - \varphi_q)}.
\]
Let us define
\[
\omega_i^0(q, \varphi) = q \sqrt{1 - \mu_i^2 \cos \varphi}.
\]
To solve \eqref{eq:2.7}, we decompose \( \hat{I}_s \) into a particular solution \( \psi_p \) and a general solution \( \psi_g \) as
\[
\hat{I}_s(q, z, \hat{s}_i) = \psi_p(q, z, \hat{s}_i) + \psi_g(q, z, \hat{s}_i). \tag{2.7}
\]
Let \( \chi_{(0,L)}(z) \) be the characteristic function, which is 1 for \( 0 < z < L \) and 0 otherwise. The particular solution \( \psi_p \) satisfies

\[
\left( \mu_i \frac{\partial}{\partial z} + 1 + i \omega_i \cdot q \right) \psi_p(q, z, \hat{s}_i) = \sum_{i'=1}^{2N} w_{i'} \int_0^{2\pi} p(\hat{s}_i, \hat{s}_{i'}) \psi_p(q, z, \hat{s}_{i'}) \, d\varphi' + \tilde{S}(q, z, \hat{s}_i) \chi_{(0,L)}(z)
\]
for \( z \in \mathbb{R}, \mu_i \in [-1, 1], \) and \( \psi_g \) satisfies

\[
\left( \mu_i \frac{\partial}{\partial z} + 1 + i \omega_i \cdot q \right) \psi_g(q, z, \hat{s}_i) = \sum_{i'=1}^{2N} w_{i'} \int_0^{2\pi} p(\hat{s}_i, \hat{s}_{i'}) \psi_g(q, z, \hat{s}_{i'}) \, d\varphi',
\]
\( z \in (0, L), \mu_i \in [-1, 1], \psi_g(q, 0, \hat{s}_i) = -\psi_p(q, 0, \hat{s}_i), \mu_i \in (0, 1], \psi_g(q, L, \hat{s}_i) = -\psi_p(q, L, \hat{s}_i), \mu_i \in [-1, 0). \)

3. Eigenmodes

We begin by defining unit vector \( \hat{k} \) as \cite{11}
\[
\hat{k} = \hat{k}(\nu, q) = \left( \frac{-i \nu q}{\hat{k}_{z}(\nu q)} \right), \quad \hat{k}_z(\nu q) = \sqrt{1 + (\nu q)^2},
\]
where \( \nu \in \mathbb{R} \). We note that \( \hat{k} \cdot \hat{k} = 1 \). We have
\[
\hat{s}_i \cdot \hat{k} = -i \nu q \sqrt{1 - \mu_i^2 \cos (\varphi - \varphi_q)} + \hat{k}_z(\nu q) \mu_i. \tag{3.1}
\]
Let us consider the homogeneous equation below.
\[
\left( \mu_i \frac{\partial}{\partial z} + 1 + i \omega_i \cdot q \right) I(q, z, \hat{s}_i)
= \sum_{i'=1}^{2N} w_{i'} \int_0^{2\pi} p(\hat{s}_i, \hat{s}_{i'}) I(q, z, \hat{s}_{i'}) \, d\varphi'
\]
for \( r \in \mathbb{R}^3, \hat{s}_i \in \mathbb{S}^2 \). Let \( R_{\hat{k}} \) be the operator which rotates the reference frame in such a way that the \( z \)-axis is rotated to the direction of \( \hat{k} \) \cite{21}. We assume the following separated solution.
\[
I(q, z, \hat{s}_i) = R_{\hat{k}(\nu, q)} \Phi^{m}_{\nu}(\hat{s}_i) e^{-\hat{k}_z(\nu q) z / \nu}, \tag{3.2}
\]
where
\[
\Phi^{m}_{\nu}(\hat{s}_i) = \Phi^{m}_{\nu}(\mu_i, \varphi) = \phi^{m}(\nu, \mu_i) \left( 1 - \mu_i^2 \right)^{|m|/2} e^{im\varphi}.
\]
The function \( \phi^{m}(\nu, \mu_i) \) is normalized as
\[
\sum_{i=1}^{2N} w_{i} \phi^{m}(\nu, \mu_i) \left( 1 - \mu_i^2 \right)^{|m|} = 1. \tag{3.3}
\]
The normalization condition \cite{13,33} implies
\[
\frac{1}{2\pi} \sum_{i=1}^{2N} \int_0^{2\pi} \phi^{m}(\nu, \hat{s}_i \cdot \hat{k}) \left[ 1 - (\hat{s}_i \cdot \hat{k})^2 \right] |m| d\varphi = 1.
\]
Note that
\[
R_{\hat{k}} Y_{lm}(\hat{s}) = \sum_{m'=-l}^{l} e^{-im'\varphi_k} d_{m'm}^{l}(\theta) Y_{lm'}(\hat{s}).
\]
Using \( Y_{lm}^{\ast}(\hat{s}) = (-1)^m Y_{-l,-m}(\hat{s}) \), we have
\[
R_{\hat{k}} Y_{lm}^{\ast}(\hat{s}) = (-1)^m \sum_{m'=-l}^{l} e^{-im'\varphi_k} d_{m'm}^{l}(\theta) Y_{lm'}^{\ast}(\hat{s})
= \sum_{m'=-l}^{l} e^{im'\varphi_k} d_{m'm}^{l}(\theta) Y_{lm'}^{\ast}(\hat{s}).
\]
By substitution we have
\[
\left( 1 - \frac{\hat{s}_i \cdot \hat{k}(\nu, q)}{\nu} \right) R_{\hat{k}(\nu, q)} \Phi^{m}_{\nu}(\hat{s}_i)
= \sum_{i'=1}^{2N} w_{i'} \int_0^{2\pi} p(\hat{s}_i, \hat{s}_{i'}) R_{\hat{k}(\nu, q)} \Phi^{m}_{\nu}(\hat{s}_{i'}) \, d\varphi'. \tag{3.4}
\]
Now, we view \cite{33,34} in the reference frame whose \( z \)-axis is rotated to the direction of \( \hat{k} \). We note that \( \mu_i = \hat{s}_i \cdot \hat{z} \), where \( \hat{z} = (0, 0, 1) \). Furthermore, \( p(\hat{s}_i, \hat{s}_{i'}) \) is invariant under rotation because it depends only on \( \hat{s}_i \cdot \hat{s}_{i'} \). Thus we...
can rewrite (3.4) as

\[
(1 - \frac{\mu_i}{\nu}) \Phi^m_n(\mathbf{s}_i) = c_0 \sum_{l' = 0}^{l \text{max}} \sum_{m' = -l'}^{l'} g^{i}_{l'} Y_{l'm'}(\mathbf{s}_i) \\
\times \sum_{\nu' = 1}^{2N} w^{\nu'} \int_0^{2\pi} Y^*_{l'm'}(\mathbf{s}_i') \Phi^m_n(\mathbf{s}_i') \, d\varphi' \tag{3.5}
\]

for \( i = 1, \ldots, 2N \). We have

\[
\sum_{\nu' = 1}^{2N} w^{\nu'} \int_0^{2\pi} Y^*_{l'm'}(\mathbf{s}_i') \Phi^m_n(\mathbf{s}_i') \, d\varphi' = \delta_{mm'}(-1)^m \sqrt{(2l'+1)\pi} g^{i}_{l'}(\nu), \tag{3.6}
\]

where

\[
g^{m}_{l'}(\nu) = (-1)^m \sqrt{\frac{(l - m)!}{(l + m)!}} \sum_{\nu' = 1}^{2N} w^{\nu'} \phi^m(\nu, \mu_i) (1 - \mu^2)^{|m|/2} P^m_l(\mu_i). \]

Noting that \( P_l^{-m}(\mu_i) = (-1)^m[(l - m)!/(l + m)!] P^m_l(\mu_i) \) and \( P^m_l(\mu_i) = (-1)^m(2m - 1)!(2l - 1)!(1 - \mu^2)^{|m|/2} (m \geq 0) \), we obtain

\[
g^{m}_{l'}(\nu) = \frac{(2m - 1)!}{(2m)!} \sqrt{\frac{(2m)!}{2m!}}, \quad g^{m}_{l'}(\nu) = (-1)^m g^{m}_{l'}(\nu),
\]

for \( m \geq 0 \). Equation (3.5) is written as

\[
(1 - \frac{\mu_i}{\nu}) \phi^m(\nu, \mu_i) (1 - \mu^2)^{|m|/2} = \frac{c_0}{2} (-1)^m \sum_{l'=|m|}^{l \text{max}} (2l'+1) \sqrt{\frac{(l'-m)!}{(l'+m)!}} \sum_{\nu' = 1}^{2N} w^{\nu'} P^m_l(\mu_i) g^{i}_{l'}(\nu).
\]

Let us define

\[
p^m_l(\mu) = (-1)^m \sqrt{\frac{(l - m)!}{(l + m)!}} P^m_l(\mu) (1 - \mu^2)^{-|m|/2}.
\]

We have

\[
p^m_l(\mu) = \frac{(2m - 1)!}{(2m)!}, \quad p^m_{l'}(\mu) = (-1)^m p^m_l(\mu),
\]

for \( m \geq 0 \). Moreover from \((l + m + 1)P^{m}_{l+1}(\mu) = (2l + 1)\mu P^{m}_{l}(\mu) - (l + m)P^{m}_{l-1}(\mu)\),

\[
(1 + \frac{l}{l+m})^2 - m^2 P^{m}_{l+1}(\mu) = (2l+1)\mu P^{m}_{l}(\mu) - \sqrt{1 + m^2} P^{m}_{l-1}(\mu)
\]

for \( l \geq |m| + 1 \), and

\[
p^m_{l+1}(\mu) = \sqrt{2|m|} + 1 \mu P^m_{|m|}(\mu).
\]

Polynomials such as \( p^m_l(R^k \mu_i) \) can be evaluated using the above-mentioned recurrence relation. Thus,

\[
(\nu - \mu_i) \phi^m(\nu, \mu_i) = \frac{c_0}{2} \sum_{l'=|m|}^{l \text{max}} (2l'+1) \sqrt{\frac{(l'-m)!}{(l'+m)!}} \sum_{\nu' = 1}^{2N} w^{\nu'} P^m_l(\mu_i) g^{i}_{l'}(\nu).
\]

We note that

\[
g^{m}_{l'}(\nu) = \sum_{i=1}^{2N} w^{\nu} \phi^m(\nu, \mu_i) p^m_l(\mu_i) (1 - \mu^2)^{|m|/2}.
\]

The following recurrence relations are obtained.

\[
\sqrt{(l+1)^2 - l^2} g^m_{l+1}(\nu) + \sqrt{1 - l^2} g^m_{l-1}(\nu) = (2l + 1) \mu g^m_{l}(\nu) - \frac{c_0}{2} \sum_{i=1}^{l \text{max}} (2l+1)(2l'+1) g^{i}_{l'} g^{m}_{l}(\nu)
\]

\[
\times \sum_{i=1}^{2N} w^{\nu'} P^m_l(\mu_i) p^m_{l'}(\mu_i) (1 - \mu^2)^{|m|/2}.
\]

For large \( N \), we can replace the sum over \( \mu_i \) with the integral. Using \( \int_{-1}^1 P^m_l(\mu) P^m_{l'}(\mu) d\mu = 2(l + m)\delta_{l,l'}/(2l + 1)(l - m)! \), the recurrence relations below are approximately obtained.

\[
\sqrt{(l+1)^2 - l^2} g^m_{l+1}(\nu) + \sqrt{1 - l^2} g^m_{l-1}(\nu) \approx \nu h_l g^m_{l}(\nu)
\]

for \( l \geq |m| + 1 \), and

\[
\sqrt{2|m| + 1} g^m_{|m|+1}(\nu) \approx \nu h_m g^m_{|m|}(\nu),
\]

where

\[
h_l = \begin{cases} 
(2l + 1) (1 - \nu^2), & 0 \leq l \leq l_{\text{max}}, \\
2l + 1, & l > l_{\text{max}}.
\end{cases}
\]

Within the approximation of discrete ordinates, \( g^m_{l'}(\nu) \) are equivalent to the normalized Chandrasekhar polynomials \( P^m_l(\mu) \). For numerical calculation, \( g^m_{l'}(\nu) \) can be computed using the recurrence relations. To compute \( g^m_{l'}(\nu) \) for \( \nu > 1 \), we define \( \tilde{g}^m_{l'}(\nu) \). Then we have

\[
\tilde{g}^m_{l-1}(\nu) = \sqrt{(l+1)^2 - l^2} \nu h_l - \sqrt{(l+1)^2 - l^2} g^m_{l}(\nu)
\]

for \( l = |m| + 1, |m| + 2, \ldots \). We can set \( \tilde{g}^m_{|m|}(\nu) = 0 \) for sufficiently large \( l \). Using \( \tilde{g}^m_{|m|}(\nu) \), we obtain

\[
g^m_{|m|+1}(\nu) = \tilde{g}^m_{|m|}(\nu) g^m_{|m|}(\nu), \quad l = |m|, |m| + 1, \ldots.
\]

From (3.7),

\[
\phi^m(\nu, \mu_i) = \frac{c_0}{2} \sum_{\nu' = 1}^{2N} w^{\nu'} \phi^m(\nu, \mu_i) P^m_l(\mu_i) g^{i}_{l'}(\nu),
\]

where

\[
g^m(\nu, \mu_i) = \sum_{l'=|m|}^{l \text{max}} (2l'+1) g^{i}_{l'} P^m_l(\mu_i) g^{m}_{l'}(\nu).
\]
If \( \nu \neq \hat{s}_i \cdot \hat{k} \), we have

\[
\phi^m(\nu, \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q})) = \frac{\varpi \nu g^m(\nu, \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q}))}{2 - \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q})}.
\]

Thus,

\[
\mathcal{R}_{\hat{k}(\nu, \mathbf{q})} \Phi^m_\nu(\hat{s}_i) = \frac{(-1)^m \varpi \nu}{\nu - \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q})} \sum_{l=|m|}^{l_{max}} \sqrt{2l + 1} \pi^{|m|} g^m_l(\nu) \times \left( \mathcal{R}_{\hat{k}(\nu, \mathbf{q})} Y_{lm}(\hat{s}_i) \right).
\]

The above eigenmodes satisfy the orthogonality relation.

**Lemma 3.1.**

\[
\mathcal{N}(\nu, \mathbf{q}) = \frac{\varpi^2 \nu^2}{2} \sum_{i=1}^{2N} w_i \mu_i 
\times \int_0^{2\pi} \left( \mathcal{R}_{\hat{k}(\nu, \mathbf{q})} \Phi^m_\nu(\hat{s}_i) \right) \left( \mathcal{R}_{\hat{k}(\nu', \mathbf{q})} \Phi^m_{\nu'}(\hat{s}_i) \right) \, d\varphi' = \mathcal{N}(\nu, \mathbf{q}) \delta_{\nu \nu'} \delta_{mm'},
\]

where

\[
\mathcal{N}(\nu, \mathbf{q}) = \frac{\varpi^2 \nu^2}{2} \sum_{i=1}^{2N} w_i \mu_i 
\times \int_0^{2\pi} \left( \mathcal{R}_{\hat{k}(\nu, \mathbf{q})} \Phi^m_\nu(\hat{s}_i) \right) \left( \mathcal{R}_{\hat{k}(\nu', \mathbf{q})} \Phi^m_{\nu'}(\hat{s}_i) \right) \, d\varphi'.
\]

**Proof.** We begin by expressing (3.4) as

\[
\left( 1 - \frac{\hat{s}_i \cdot \hat{k}(\nu, \mathbf{q})}{\nu} \right) \mathcal{R}_{\hat{k}(\nu, \mathbf{q})} \Phi^m_\nu(\hat{s}_i)
= \varpi \sum_{l=0}^{l_{max}} \left[ \frac{g^m_l(\nu, \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q}))}{\nu - \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q})} \right] \sum_{l=0}^{l_{max}} \left[ \frac{g^m_l(\nu, \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q}))}{\nu + \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q})} \right] \int_0^{2\pi} Y^*_{lm}(\hat{s}_i, \varphi) \mathcal{R}_{\hat{k}(\nu, \mathbf{q})} \Phi^m_{\nu'}(\hat{s}_i', \varphi') \, d\varphi'.
\]

Note that \( \nu = \nu^m \) depends on \( m \). Let us consider \( \hat{k}_1 = \hat{k}(\nu^m_1, \mathbf{q}) \) and \( \hat{k}_2 = \hat{k}(\nu^m_2, \mathbf{q}) \). We have

\[
\mathcal{R}_{\hat{k}_1} \Phi^m_{\nu_1}(\hat{s}_i) = \left( \mathcal{R}_{\hat{k}_2} \Phi^m_{\nu_2}(\hat{s}_i) \right) \mathcal{R}_{\hat{k}_1} \left( 1 - \frac{\mu_1}{\nu_2} \right) \Phi^m_{\nu_1}(\hat{s}_i)
= \left( \mathcal{R}_{\hat{k}_2} \Phi^m_{\nu_2}(\hat{s}_i) \right) \sum_{l=0}^{l_{max}} \left[ \frac{g^m_l(\nu, \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q}))}{\nu + \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q})} \right] \int_0^{2\pi} Y^*_{lm}(\hat{s}_i, \varphi) \mathcal{R}_{\hat{k}_1} \Phi^m_{\nu_1}(\hat{s}_i', \varphi') \, d\varphi'.
\]

and

\[
\mathcal{R}_{\hat{k}_2} \Phi^m_{\nu_2}(\hat{s}_i) \mathcal{R}_{\hat{k}_1} \left( 1 - \frac{\mu_1}{\nu_2} \right) \Phi^m_{\nu_1}(\hat{s}_i)
= \left( \mathcal{R}_{\hat{k}_2} \Phi^m_{\nu_2}(\hat{s}_i) \right) \varpi \sum_{l=0}^{l_{max}} \left[ \frac{g^m_l(\nu, \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q}))}{\nu - \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q})} \right] \sum_{l=0}^{l_{max}} \left[ \frac{g^m_l(\nu, \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q}))}{\nu + \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q})} \right] \int_0^{2\pi} Y^*_{lm}(\hat{s}_i, \varphi) \mathcal{R}_{\hat{k}_2} \Phi^m_{\nu_2}(\hat{s}_i', \varphi') \, d\varphi'.
\]

By subtraction,

\[
\left( \mathcal{R}_{\hat{k}_2} \Phi^m_{\nu_2}(\hat{s}_i) \right) \mathcal{R}_{\hat{k}_1} \left( 1 - \frac{\mu_1}{\nu_2} \right) \Phi^m_{\nu_1}(\hat{s}_i)
= \left( \mathcal{R}_{\hat{k}_2} \Phi^m_{\nu_2}(\hat{s}_i) \right) \varpi \sum_{l=0}^{l_{max}} \left[ \frac{g^m_l(\nu, \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q}))}{\nu - \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q})} \right] \sum_{l=0}^{l_{max}} \left[ \frac{g^m_l(\nu, \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q}))}{\nu + \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q})} \right] \int_0^{2\pi} Y^*_{lm}(\hat{s}_i, \varphi) \mathcal{R}_{\hat{k}_2} \Phi^m_{\nu_2}(\hat{s}_i', \varphi') \, d\varphi'.
\]

We note that \( \hat{s}_i \cdot \hat{k}(\nu, \mathbf{q}) = \nu \nu_i \cdot \mathbf{q} + \hat{k}_z(\nu q) \mu_i \). This relation is also obtained as follows using \( d^{1}_{m0}(\theta) = \cos \theta \), \( d^{0}_{m0}(\theta) = -(1/\sqrt{2}) \sin \theta \), \( d^{m}_{m0}(\theta) = (-1)^m \sin \theta \), and \( Y_{l - m}(\hat{s}) = (-1)^m Y^*_{lm}(\hat{s}) \):

\[
\mathcal{R}_{\hat{k}(\nu, \mathbf{q})} \Phi^m_\nu(\hat{s}_i) = \sqrt{\frac{4\pi}{3}} \mathcal{R}_{\hat{k}(\nu, \mathbf{q})} Y_{10}(\hat{s}_i)
= \sqrt{\frac{4\pi}{3}} \sum_{m = -1}^{1} e^{-i m \varphi} \frac{d^{m0}_{m0}(\theta)}{\sin \theta} Y_{1m}(\hat{s}_i)
= \frac{1}{\sqrt{1 - \mu^2}} \sin \theta \mathcal{R}_{\hat{k}} \left( \frac{1}{\sqrt{1 - \mu^2}} \cos(\varphi - \varphi_k) + \mu \cos \theta_k \right)
= \hat{k}_z(\nu q) \mu_i - i \nu q \sqrt{\frac{1}{1 - \mu^2}} \cos(\varphi - \varphi_k),
\]

where we used

\[
\cos \theta_k(\nu, \mathbf{q}) = \hat{k}_z(\nu q), \quad \sin \theta_k(\nu, \mathbf{q}) = i |\nu q|,
\]

and

\[
\varphi_k(\nu, \mathbf{q}) = \varphi_k + \frac{\pi}{2} \pm \frac{\pi}{2} \quad (\pm \nu > 0).
\]

Hence by summing both sides of the above two equations by \( \sum_{i=1}^{2N} w_i \int_0^{2\pi} \, d\varphi' \),

\[
\left( \frac{\hat{k}_z(\nu q)}{\nu_2} - \frac{\hat{k}_z(\nu q)}{\nu_1} \right) \sum_{i=1}^{2N} w_i \int_0^{2\pi} \mu_i \left( \mathcal{R}_{\hat{k}_1} \Phi^m_{\nu_1}(\hat{s}_i) \right) \left( \mathcal{R}_{\hat{k}_2} \Phi^m_{\nu_2}(\hat{s}_i) \right) \, d\varphi' = 0.
\]
Suppose \( \nu = \nu_1 = \nu_2 \) but \( m_1 \neq m_2 \). In this case, \( \mathbf{k} = \mathbf{k}_1 = \mathbf{k}_2 \) and
\[
\sum_{i=1}^{2N} w_i \int_{0}^{2\pi} \mu_i \left( \mathbf{R}_k \Phi^m_{\nu_1}(\mathbf{s}_i) \right) \left( \mathbf{R}_k \Phi^m_{\nu_2}(\mathbf{s}_i) \right) d\varphi
= \sum_{i=1}^{2N} w_i \int_{0}^{2\pi} \mu_i \left( \mathbf{R}_k \Phi^m_{\nu_1}(\mathbf{s}_i) \Phi^m_{\nu_2}(\mathbf{s}_i) \right) d\varphi \propto \delta_{m_1,-m_2}.
\]
Noting that \( \nu^{-m} = \nu^m \) and \( \Phi^m_{\nu}(\mathbf{s}_i) = \Phi^m_{-\nu}(\mathbf{s}_i) \), we have
\[
\sum_{i=1}^{2N} w_i \int_{0}^{2\pi} \mu_i \left( \mathbf{R}_k \Phi^m_{\nu_1}(\mathbf{s}_i) \right) \left( \mathbf{R}_k \Phi^m_{\nu_2}(\mathbf{s}_i) \right) d\varphi
\propto \delta_{\nu_1,\nu_2} \delta_{m_1,m_2}.
\]
For \( m = m_1 = m_2 \) and \( \nu = \nu_1 = \nu_2 \), we obtained from \( \text{[3.1]} \). We have
\[
\mathcal{N}(\nu, q) = \sum_{i=1}^{2N} w_i \mu_i \int_{0}^{2\pi} \left[ \mathbf{R}_k(\nu, q) \Phi^m(\mathbf{s}_i) \Phi^m(\mathbf{s}_i) \right] d\varphi
= \left( \frac{\omega_{\nu}}{2} \right)^2 \sum_{i=1}^{N} w_i \mu_i
\times \int_{0}^{2\pi} \left[ \left( g^m(\nu, \nu(m) \mathbf{q}) \mu_i \right)^2 \right]^{[m]} \times \left( 1 - \left( -\nu \mathbf{q} \cdot \mathbf{k} \right)^2 \right)^{[m]} \times \left( 1 - \left( -\nu \mathbf{q} \cdot \mathbf{k} \right) \mu_i \right)^2
\times \left( 1 - \left( -\nu \mathbf{q} \cdot \mathbf{k} \right)^2 \right)^{[m]} \right] d\varphi.
\]
Since the integral over \( \varphi \) is taken from 0 to 2\( \pi \), we can replace \( \mathbf{q} \cdot \mathbf{k} \) in the above integral with \( \omega_0(q, \varphi) \). Finally, we arrive at \( \mathcal{N}(\nu, q) \) in Lemma \( \text{[3.1]} \).

**Lemma 3.2.** \( \mathcal{N}(\nu, q) = -\mathcal{N}(\nu, q) \).

**Proof.** Use \( g^m(-\nu, -\mu_i) = g^m(\nu, \mu_i) \).

Let us put \( \mu_i = 1 \). The resonance \( \nu = \mathbf{k} \cdot \mathbf{s} \) occurs when \( \nu > 1 \). Since \( \nu - \mathbf{k} \cdot \mathbf{s} = \nu - \mathbf{k}^2 \mathbf{q} \) for \( \mu_i = 1 \), there exists \( q_\nu \) such that \( \nu - \mathbf{k}^2 \mathbf{q} = 0 \) for \( \nu > 1 \). Indeed, we have
\[
q_\nu = q = \sqrt{1 - \frac{1}{\nu^2}}.
\]
Let us calculate eigenvalues. Equation \( \text{[3.5]} \) can be rewritten as \( (i = 1, \ldots, 2N, -l_{\text{max}} \leq m \leq l_{\text{max}}) \)
\[
\left( 1 - \frac{\mu_i}{\nu} \right) \phi^m(\nu, \mu_i) = \frac{\omega_0}{2} \sum_{l'=|m|}^{l_{\text{max}}} g^l(2l' + 1)p^m_{l'}(\mu_i)
\times \sum_{i=1}^{2N} w_i \nu_{l'} p^m_{l'}(\mu_i) \phi^m(\nu, \mu_i).
\]
Hence for \( i = 1, \ldots, N \),
\[
\left( 1 - \frac{\mu_i}{\nu} \right) \phi^m(\nu, \mu_i) = \frac{\omega_0}{2} \sum_{l'=|m|}^{l_{\text{max}}} g^l(2l' + 1)p^m_{l'}(\mu_i)
\times \sum_{i=1}^{N} w_i \nu_{l'} p^m_{l'}(\mu_i) \phi^m(\nu, \mu_i).
\]

By adding and subtracting two equations we obtain
\[
\mathbf{U}^m(\nu) = \Phi^m(\nu) + \Phi^m(\nu), \quad \mathbf{V}^m(\nu) = \Phi^m(\nu) - \Phi^m(\nu).
\]

By adding and subtracting two equations we obtain
\[
\mathbf{U}^m(\nu) = \frac{1}{\nu} \mathbf{V}^m(\nu) = \frac{\omega_0}{2} (W^m_+ + W^m_-) \mathbf{U}^m(\nu),
\]
\[
\mathbf{V}^m(\nu) = \frac{1}{\nu} \mathbf{U}^m(\nu) = \frac{\omega_0}{2} (W^m_+ - W^m_-) \mathbf{V}^m(\nu).
\]

Hence we obtain \( \text{[3.4]} \)
\[
E^m_+ E^m_+ \mathbf{U}^m = \frac{1}{\nu^2} \mathbf{V}^m, \quad \text{(3.10)}
\]
where
\[
E^m_+ = \left[ I_N - \frac{\omega_0}{2} (W^m_+ + W^m_-) \right] \mathbf{U}^{-1}.
\]
4. Particular and general solutions

Let us consider the fundamental solution \( G_q(z, \tilde{s}_i; z', \tilde{s}'_i) \) for \( z \in \mathbb{R} \):

\[
\begin{align*}
G_q(z, \tilde{s}_i; z', \tilde{s}'_i) &= \frac{\varepsilon}{4\pi} \sum_{i=1}^{2N} w_i \int_0^{2\pi} G_q(z, \tilde{s}_i; z', \tilde{s}'_i) d\varphi \\
&= \frac{\varepsilon}{4\pi} \sum_{i=1}^{2N} w_i \int_0^{2\pi} G_q(z, \tilde{s}_i; z', \tilde{s}'_i) d\varphi \\
&+ \delta(z - z') \delta_{i,0} \delta(\varphi - \varphi_0)
\end{align*}
\]

for \( z \in \mathbb{R}, 1 \leq i \leq 2N, \varphi \in [0, 2\pi] \). Here, \( \varphi_0 \) is the azimuthal angle of \( \tilde{s}_i \). We obtain (see Appendix A)

\[
G_q(z, \tilde{s}_i; z', \tilde{s}'_i) = \sum_{l=-\infty}^{l_{\max}} \sum_{n=1}^N \frac{w_{ln}}{N(\nu_n, q)} e^{\pm \tilde{k}_l(nu_0 q)(z - z')/\nu_0} \\
\times \left( R_{k_\pm}^{\nu_0} (\tilde{s}_i) \right) \left( R_{k_\pm}^{\nu_0} (\tilde{s}'_i) \right)
\]

where upper (lower) signs are taken for \( z > z' \) (\( z < z' \)) and \( k_{\pm} = k(\pm \nu_0, q) \). Let us define

\[
C(\tau; \zeta, \eta) = \frac{e^{-\tau/\zeta} - e^{-\tau/\eta}}{\zeta - \eta}, \quad S(\tau; \zeta, \eta) = \frac{1 - e^{-\tau/\zeta} - e^{-\tau/\eta}}{\zeta + \eta}.
\]

We note that \( C(-\tau; -\zeta, -\eta) = -C(\tau; \zeta, \eta) \) and

\[
C(\tau; \eta, \eta) = \frac{\tau}{\eta}, \quad S(\tau; -\eta, \eta) = -\frac{\tau}{\eta^2}.
\]

We obtain

\[
\psi_p(q, z, \tilde{s}_i) = \sum_{i=1}^{2N} \int_0^{2\pi} G_q(z, \tilde{s}_i; z', \tilde{s}'_i) S(q, z', \tilde{s}'_i) d\varphi' \\
\times e^{-(1+i\omega_0)z'/\mu_0} dz',
\]

where

\[
\psi_p(q, z, \tilde{s}_i) = 4\pi^2 \omega \rho I_0 \delta(q) - \mu_0 \\
\times \sum_{m=-l_{\max}}^{l_{\max}} \sum_{n=1}^N \frac{\nu_0}{N(\nu_0, q)} \left( R_{k_\pm}^{\nu_0} (\tilde{s}_i) \right) \left( R_{k_\pm}^{\nu_0} (\tilde{s}'_i) \right)
\]

Here,

\[
\sum_{i=1}^{2\pi} \int_0^{2\pi} G_q(z, \tilde{s}_i; z', \tilde{s}'_i) p(\tilde{s}_i, \tilde{s}'_i) d\varphi' \\
\approx \sum_{m=-l_{\max}}^{l_{\max}} \sum_{n=1}^N \frac{1}{N(\nu_n, q)} e^{\pm \tilde{k}_l(nu_0 q)(z - z')/\nu_n} \left( R_{k_\pm}^{\nu_0} (\tilde{s}_i) \right) \left( R_{k_\pm}^{\nu_0} (\tilde{s}'_i) \right)
\]

\[
\times \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} g^l \left( R_{k_\pm}^{\nu_0} (\tilde{s}_i) \right) \left( R_{k_\pm}^{\nu_0} (\tilde{s}'_i) \right)
\]

\[
\times \sum_{l'=1}^{l_{\max}} \sum_{m=-l'}^{l'} g^{l'} \left( R_{k_\pm}^{\nu_0} (\tilde{s}_i) \right) \left( R_{k_\pm}^{\nu_0} (\tilde{s}'_i) \right)
\]

Hence,

\[
\psi_p(q, z, \tilde{s}_i) = 4\pi^2 \omega \rho I_0 \delta(q) - \mu_0 \\
\times \sum_{m=-l_{\max}}^{l_{\max}} \sum_{n=1}^N \frac{\nu_0}{N(\nu_n, q)} \left( R_{k_\pm}^{\nu_0} (\tilde{s}_i) \right) \left( R_{k_\pm}^{\nu_0} (\tilde{s}'_i) \right)
\]

where

\[
\psi_1 = \frac{1}{\nu_n - \tilde{s}_i \cdot k_+} \left( 1 + i\omega_0 \cdot q_0 + \nu_0 \right) \\
\times \sum_{l=0}^{l_{\max}} \sqrt{2l + 1} g^l \left( R_{k_\pm}^{\nu_0} (\tilde{s}_i) \right)
\]

\[
\times \sum_{l'=1}^{l_{\max}} \sqrt{2l' + 1} g^{l'} \left( R_{k_\pm}^{\nu_0} (\tilde{s}_i) \right)
\]
\[
\psi_2 = \sum_{l=\max}^{\max} \sqrt{2l+1} g^l_{\nu_n} (-\nu_n) \left( R_{k_-} Y_{l_m}(\mathbf{s}_i) \right) \times \sum_{l'=\max}^{\max} \sqrt{2l'+1} g^l'_{\nu_n} (-\nu_n) \left( R_{k_-} Y^*_{l'm}(\mathbf{s}_i) \right) 
\]

Hereafter we assume \( \mathbf{s}_0 = \mathbf{z} = t(0,0,1) \).

Then
\[
\psi_{p}(\mathbf{q}, z, \mathbf{s}_i) = 4\pi^2 \varkappa^2 I_0 \delta(\mathbf{q} - \mathbf{q}_0) \sum_{m=-\max}^{\max} \sum_{n=1}^{N} \frac{(-1)^m \nu_n}{N(\nu_n, q_0) k_z(\nu_n q_0)} \times \left( R_{k(\nu_n, \mathbf{q}_0)} \Phi^m_{\nu_n}(\mathbf{s}_i) \right) C \left( z; 1, \frac{\nu_n}{k_z(\nu_n q_0)} \right) \times \sum_{l'=\max}^{\max} \sqrt{2l'+1} g^l'_{\nu_n} (-\nu_n) \left( R_{k_-} Y^*_{l'm}(\mathbf{z}) \right). 
\]

Next we consider the general solution. Using the eigenmodes developed in (33) we can express \( \psi_{g}(\mathbf{q}, z, \mathbf{s}_i) \) as
\[
\psi_{g}(\mathbf{q}, z, \mathbf{s}_i) = 4\pi^2 \varkappa^2 I_0 \delta(\mathbf{q} - \mathbf{q}_0) \sum_{m=-\max}^{\max} \sum_{n=1}^{N} \frac{\nu_n}{N(\nu_n, q_0) k_z(\nu_n q_0)} \times \left[ a^m_n(q_0) R_{k_+} \Phi^m_{\nu_n}(\mathbf{s}_i) e^{-\frac{k_z(\nu_n q_0) z}{\nu_n}} + \right.
\]

Thus,
\[
\psi_{g}(\mathbf{q}, z, \mathbf{s}_i) = 4\pi^2 \varkappa^2 I_0 \delta(\mathbf{q} - \mathbf{q}_0) \sum_{m=-\max}^{\max} \sum_{n=1}^{N} \frac{\nu_n}{N(\nu_n, q_0) k_z(\nu_n q_0)} \times \left[ a^m_n(q_0) e^{-\frac{k_z(\nu_n q_0) z}{\nu_n}} + \right.
\]

The coefficients \( a^m_n(q_0), b^m_n(q_0) \) will be determined from the boundary conditions. Below, we will see that only the case of \( \varphi_{\mathbf{q}_0} = 0 \) is necessary, and \( a^m_n, b^m_n \) depends on \( q_0 \).

We substitute the expressions (33) and (34) for \( \psi_{p}, \psi_{g} \) in the boundary conditions. Then we multiply \( \Phi^m_{\nu_n}(\mathbf{s}_i) \) on both sides of the boundary conditions, sum over \( \mu_i \) (\( i = 1, \ldots, N \) at \( z = 0 \) and \( i = N + 1, \ldots, 2N \) at \( z = L \)), and integrate over \( \varphi \). We define
\[
\psi_{m',m}^{\rho, \pm}(\mathbf{q}_0) = \sum_{i=1}^{N} \int_{0}^{2\pi} \Phi^m_{\nu_n}(\mu_i, \varphi) R_{k_+} \Phi^m_{\pm \nu_n}(\mu_i, \varphi) \frac{d\varphi}{2\pi},
\]

We set \( \varphi_{\mathbf{q}_0} = 0 \) and recall \( \omega_0^1(q_0, \varphi) = q_0 \sqrt{1 - \mu_1^2} \cos \varphi \). The following linear system is obtained with a matrix \( B \) of dimension \( 2N(2l_{\max} + 1) \).
\[
B = \left( \begin{array}{c} \vdots \\ a_n^m \\ \vdots \\ b_n^m \\ \vdots \\ \vdots \\ v^{(1)}_{l_1} \\ \vdots \\ v^{(2)}_{l_2} \\ \vdots \\ \vdots \end{array} \right),
\]

for \( \mu_i > 0 \). Here, \( B \in \mathbb{C}^{2N(2l_{\max} + 1) \times 2N(2l_{\max} + 1)} \) is given by
\[
B = \left( \begin{array}{cccc} \cdots & B^{(11)} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & B^{(12)} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & B^{(21)} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array} \right).
\]
with
\[
\{ B^{(1)} \}_{n'm',nm} = \frac{\pi \nu_n}{\mathcal{N}(\nu_n, q_0)} \Psi_{n',m}(q_0),
\]
\[
\{ B^{(2)} \}_{n'm',nm} = \frac{\pi \nu_n}{\mathcal{N}(\nu_n, q_0)} \nu_{n',m}(q_0),
\]
\[
\{ B^{(21)} \}_{n'm',nm} = \frac{\pi \nu_n}{\mathcal{N}(\nu_n, q_0)} \Psi_{n',m}(q_0),
\]
\[
\{ B^{(22)} \}_{n'm',nm} = \frac{\pi \nu_n}{\mathcal{N}(\nu_n, q_0)} \Psi_{n',m}(q_0).
\]
Moreover,
\[
\{ v^{(1)} \}_{n'm'} = - \sum_{m=-\max}^{\max} \sum_{n=1}^{N} \frac{(-1)^m}{\mathcal{N}(\nu_n, q_0)} \hat{s}_{n,m}(q_0) S \left( L ; 1, \frac{\nu_n}{k_z(\nu_n q_0)} \right)
\]
\[
\times \sum_{l=|m|}^{\max} \sqrt{(2l + 1)\pi g_l^m(\nu_n) R_{\nu_n} Y_{l,m}(z)}.
\]
\[
\{ v^{(2)} \}_{n'm'} = - \sum_{m=-\max}^{\max} \sum_{n=1}^{N} \frac{(-1)^m}{\mathcal{N}(\nu_n, q_0)} \hat{s}_{n,m}(q_0) C \left( L ; 1, \frac{\nu_n}{k_z(\nu_n q_0)} \right)
\]
\[
\times \sum_{l=|m|}^{\max} \sqrt{(2l + 1)\pi g_l^m(\nu_n) R_{\nu_n} Y_{l,m}(z)}.
\]
Using (3.6), we approximately have
\[
\sum_{l'=1}^{2N} w_{l'} \int_{0}^{2\pi} R_{\nu_n} \Phi_{l'}(\hat{s}_{l'}(q_0) Y_{l,m}(\hat{s}_{l'}(q_0)) d\varphi
\]
\[
\approx \sum_{l'=1}^{2N} w_{l'} \int_{0}^{2\pi} \Phi_{l'}(\hat{s}_{l'}(q_0) Y_{l,m}(\hat{s}_{l'}(q_0)) d\varphi
\]
\[
= \delta_{nn'} (-1)^m \sqrt{2l + 1} \pi g_l^m(\nu).)
\]

The function \( F(q,z) \) is given by
\[
F(q,z, \hat{s}_i) = S(q,z, \hat{s}_i)
\]
\[
= \sum_{l'=1}^{2N} w_{l'} \int_{0}^{2\pi} \Psi_{l'}(q, z, \hat{s}_i) \psi(q, z, \hat{s}_i) \ d\varphi'
\]
\[
\approx (2\pi)^2 \sum_{l'=1}^{2N} w_{l'} \int_{0}^{2\pi} \Psi_{l'}(q, z, \hat{s}_i) \psi(q, z, \hat{s}_i) \ d\varphi'
\]
\[
= F_1 + F_2,
\]
where
\[
F_1 = 4\pi^{\alpha - 1} \sum_{l'=1}^{2N} \int_0^{2\pi} \delta(q - q_0) \sum_{m=-\max}^{\max} \sum_{n=1}^{N} \frac{\nu_n}{\mathcal{N}(\nu_n, q_0)} k_z(\nu_n q_0)
\]
\[
\times \sum_{l=|m|}^{\max} \sqrt{(2l + 1)\pi g_l^m(\nu_n) R_{\nu_n} Y_{l,m}(z)}.
\]
\[
F_2 = 4\pi^{\alpha - 1} \sum_{l'=1}^{2N} \int_0^{2\pi} \delta(q - q_0) \sum_{m=-\max}^{\max} \sum_{n=1}^{N} \frac{\nu_n}{\mathcal{N}(\nu_n, q_0)} (-1)^m
\]
\[
= a^m_{\nu_n}(q_0) e^{-k_z(\nu_n q_0)} \left[ \int_0^{2\pi} \Psi_{l'}(q, z, \hat{s}_i) \psi(q, z, \hat{s}_i) \ d\varphi' \right]
\]
\[
= b^m_{\nu_n}(q_0) e^{-k_z(\nu_n q_0)} \left[ \int_0^{2\pi} \Psi_{l'}(q, z, \hat{s}_i) \psi(q, z, \hat{s}_i) \ d\varphi' \right]
\]
\[
= \sum_{l=|m|}^{\max} \sqrt{(2l + 1)\pi g_l^m(\nu_n) R_{\nu_n} Y_{l,m}(z)}.
\]

5. Post-processing procedure

According to the post-processing procedure (23) (23), we substitute the obtained solution \( \hat{s}(q, z, \hat{s}_i) \) to the radiative transport equation below rather than directly calculate the inverse Fourier transform of \( \hat{s}_i \).

From (2.6), we have the following equation for \( \hat{s}_i(q, z, \hat{s}_i) \)
\[
\frac{\partial}{\partial z} \hat{s}_i(q, z, \hat{s}_i) e^{(1+\alpha)(1+\beta)} \equiv \frac{1}{\mu_i} e^{(1+\alpha)(1+\beta)} F(q, z, \hat{s}_i).
\]
We made use of the fact that \( \varphi - \varphi_0 \) can be redefined as \( \varphi \) in the integral over \( \varphi \) on the right-hand side. Hence,

\[
F(q, z, \hat{s}_l) = (2\pi)^2 \omega I_0 p(\hat{s}_l, \hat{z}) e^{-z \delta(q - q_0)} + 4\pi^2 \omega I_0 \delta(q - q_0) \sum_{m=-l_{\text{max}}}^{l_{\text{max}}} \sum_{n=1}^{N} \frac{\nu_n}{N} \frac{\alpha^m(\nu_n, \hat{k}_z(\nu_n, q_0))}{2\pi k_z(\nu_n, q_0)} \times \left[ \frac{A^m(\nu_n, \hat{k}_z(\nu_n, q_0)) C(z; 1, \frac{\nu_n}{k_z(\nu_n, q_0)})}{2\pi k_z(\nu_n, q_0)} \right] + a_n^m(q_0) e^{-k_z(\nu_n, q_0)z/\nu_n} \times \sum_{l=0}^{l_{\text{max}}} g_l^l(2l + 1) N^{-1} g_l^m(\nu_n) \left( R_{\hat{k}_z} Y_{lm}(\hat{s}_l) \right)
\]

Note the relation

\[
(1 - \mu) \alpha^m(\nu_n, \hat{k}_z(\nu_n, q_0)) = \left( 1 - \left( \hat{k}_z(\nu_n, q_0) \right)^2 \right)^{m/2} (2l + 1) N^{-1} g_l^m(\nu_n) \left( R_{\hat{k}_z} Y_{lm}(\hat{s}_l) \right)
\]

and we used

\[
(1 - \mu)^{m/2} (2l + 1) N^{-1} g_l^m(\nu_n) \left( R_{\hat{k}_z} Y_{lm}(\hat{s}_l) \right)
\]

where \( \theta \) is the angle corresponding to \( \mu \). Hence,

\[
\alpha^m(\nu_n, \hat{k}_z(\nu_n, q_0)) = \sum_{l=0}^{l_{\text{max}}} (2l + 1) g_l^l(2l + 1) \left( \frac{1}{2\pi} \right)^{(l - m)!} (l + m)! \nu_n \left( R_{\hat{k}_z} Y_{lm}(\hat{s}_l) \right).
\]

For \( \mu_i > 0 \), we integrate the above equation from 0 to \( z \) and integrate from \( z \) to \( L \) when \( \mu_i < 0 \). That is,

\[
\begin{align*}
I_{s}(q, z, \hat{s}_l) &= \frac{(2\pi)^2 \omega I_0}{1 + i\omega_i \cdot q_0} p(\hat{s}_l, \hat{z}) \times e^{-z S(z; 1, \frac{\mu_i}{1 + i\omega_i \cdot q_0})} \delta(q - q_0) + I_3 + I_4,
\end{align*}
\]

For \( \mu_i > 0 \), we obtain

\[
\begin{align*}
I_{s}(q, z, \hat{s}_l) &= \frac{(2\pi)^2 \omega I_0}{1 + i\omega_i \cdot q_0} p(\hat{s}_l, \hat{z}) \times C(z; 1, \frac{\mu_i}{1 + i\omega_i \cdot q_0}) \delta(q - q_0) + I_1 + I_2,
\end{align*}
\]

where

\[
I_1 = \frac{4\pi^2 \omega I_0}{1 + i\omega_i \cdot q_0} S(z; 1, \frac{\mu_i}{1 + i\omega_i \cdot q_0}) \sum_{l=-l_{\text{max}}}^{l_{\text{max}}} \sum_{n=1}^{N} \frac{\nu_n}{N} \frac{\alpha^m(\nu_n, \hat{k}_z(\nu_n, q_0))}{2\pi k_z(\nu_n, q_0)} \times \left[ \frac{A^m(\nu_n, \hat{k}_z(\nu_n, q_0)) C(z; 1, \frac{\mu_i}{1 + i\omega_i \cdot q_0})}{2\pi k_z(\nu_n, q_0)} \right] \times \sum_{l=0}^{l_{\text{max}}} g_l^l(2l + 1) N^{-1} g_l^m(\nu_n) \left( R_{\hat{k}_z} Y_{lm}(\hat{s}_l) \right),
\]

and

\[
I_2 = \frac{4\pi^2 \omega I_0}{1 + i\omega_i \cdot q_0} \delta(q - q_0) \sum_{l=-l_{\text{max}}}^{l_{\text{max}}} \sum_{n=1}^{N} \frac{\nu_n}{N} \frac{\alpha^m(\nu_n, \hat{k}_z(\nu_n, q_0))}{2\pi k_z(\nu_n, q_0)} \times \left[ \frac{A^m(\nu_n, \hat{k}_z(\nu_n, q_0)) C(z; 1, \frac{\mu_i}{1 + i\omega_i \cdot q_0})}{2\pi k_z(\nu_n, q_0)} \right] \times \sum_{l=0}^{l_{\text{max}}} g_l^l(2l + 1) N^{-1} g_l^m(\nu_n) \left( R_{\hat{k}_z} Y_{lm}(\hat{s}_l) \right).
\]
We obtain
\[
I_3 = \frac{4\pi^2 c^2}{1 + i\omega \cdot \mathbf{q}_0} \delta(\mathbf{q} - \mathbf{q}_0) \sum_{m=-l_{\text{max}}}^{l_{\text{max}}} \sum_{n=1}^{N} \nu_n \frac{\nu_n \hat{k}_z(\nu_n q_0) e^{-\nu_n z} S \left( L - z; 1, \frac{|\mu_i|}{1 + i\omega_i \cdot \mathbf{q}_0} \right)}{2\pi} \left( \nu_n \hat{k}_z(\nu_n q_0) - \nu_n \right)
\]

where
\[
I_4 = \frac{4\pi^2 c^2}{1 + i\omega \cdot \mathbf{q}_0} \delta(\mathbf{q} - \mathbf{q}_0) \sum_{m=-l_{\text{max}}}^{l_{\text{max}}} \sum_{n=1}^{N} \nu_n \frac{\nu_n \hat{k}_z(\nu_n q_0) e^{-\nu_n z} S \left( L - z; 1, \frac{|\mu_i|}{1 + i\omega_i \cdot \mathbf{q}_0} \right)}{2\pi} \left( \nu_n \hat{k}_z(\nu_n q_0) - \nu_n \right)
\]

The first term on the right-hand side of the above equation vanishes. In the half space of \( L \to \infty \), we obtain
\[
J_+ (\rho; \mathbf{q}_0) = \frac{4\pi^2 c^2}{(2\pi)^2} e^{i\mathbf{q}_0 \cdot \rho} \sum_{i=1}^{N} w_i \mu_i
\]

6. Reflectance

We note that
\[
I = 4\pi^2 c^2 \delta(\mathbf{q} - \mathbf{q}_0) \sum_{m=-l_{\text{max}}}^{l_{\text{max}}} \sum_{n=1}^{N} \nu_n R_{k+} Y_{lm}(\hat{s}_i)
\]

and
\[
I = 4\pi^2 c^2 \delta(\mathbf{q} - \mathbf{q}_0) \sum_{m=-l_{\text{max}}}^{l_{\text{max}}} \sum_{n=1}^{N} \nu_n R_{k+} Y_{lm}(\hat{s}_i)
\]

7. Concluding remarks

In one dimension, the equivalence between the discrete-ordinates method and spherical-harmonics method is known \[12\]. In three dimensions, however, the series from the method of rotated reference frames \[11\], which is the spherical harmonics method, diverges because components of \( \mathbf{k} \) become very large even if \( |\mathbf{k} \cdot \hat{\mathbf{z}}| = 1 \) always holds.

Kim and Keller \[13\] and Kim \[12\] established discrete-ordinates with plane-wave decomposition for the three-dimensional radiative transport equation. Their method can be viewed as separation of variables assuming the form
\[
\hat{I}(\mathbf{q}, \mathbf{r}, \hat{s}_i) = V_\lambda(\hat{s}_i, \mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{r}}
\]

The explicit form of the eigenvectors is obtained and we can compute eigenvalues \( \nu \) by diagonalizing a matrix once.

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Appendix A. Fundamental solution

Let us obtain $G(q, z')$ which satisfies Eq. (111). Since $G(q, z') - G(q, z)$ can be determined from the source term. We note that the right hand side can be written as

$$G(q, z') = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{N} A_{m} B_{n} \mathcal{R}_{\kappa}(\nu_{n}, q) \Phi_{m}(\hat{\nu}_{n}(z')) \Phi_{n},$$

where coefficients $A_{m} = A_{m}(q, z)$ will be determined from the source term. We note that the superscript $m$ of $\nu_{n} = \nu_{n}^{m}$ was omitted in the above equation. With the source term, the jump condition is written as

$$\mu_{i} G(q, z') = G(q, z).$$

Hence,

$$\mu_{i} A_{m} = \sum_{n=1}^{N} \mathcal{R}_{\kappa}(\nu_{n}, q) \Phi_{m}(\hat{\nu}_{n}(z)) - \sum_{n=1}^{N} \mathcal{R}_{\kappa}(\nu_{n}, q) \Phi_{m}(\hat{\nu}_{n}(z)),$$

Using the orthogonality relation in Lemma 5.1, we obtain

$$A_{m} = \frac{\nu_{m}(q) \mathcal{R}_{\kappa}(\nu_{m}(q))}{\mathcal{P}_{m}(\hat{\nu}_{m}(z))},$$

$$B_{n} = \frac{-\nu_{n}(q) \mathcal{R}_{\kappa}(\nu_{n}(q))}{\mathcal{P}_{m}(\hat{\nu}_{n}(z))} \Phi_{m}(\hat{\nu}_{n}(z)).$$

Thus we arrive at (122).

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