A Distributed control framework
for the optimal operation of DC microgrids

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Abstract— In this paper we propose an original distributed control framework for DC microgrids. We first formulate the (optimal) control objectives as an aggregative game suitable for the energy trading market. Then, based on duality, we analyze the equivalent distributed optimal condition for the proposed aggregative game and design a distributed control scheme to solve it. By interconnecting the DC microgrid and the designed distributed control system in a power preserving way, we steer the DC microgrid’s state to the desired optimal equilibrium, satisfying a predefined set of local and coupling constraints. Finally, based on singular perturbation system theory, we analyze the convergence of the closed-loop system. The simulation results show excellent performance of the proposed control framework.

I. INTRODUCTION

As an important part of the (actual and future) energy system, the direct current (DC) microgrids are widely deployed in several applications, such as renewable energy sources, trains, aircraft, ships and charging stations for the more and more popular electric vehicles [2]. To improve the energy dispatch efficiency and trading fairness for the DC microgrids, one of the most effective options is the adoption of a distributed control and optimization framework (DCOF) [3]. Within such a framework, energy trading, control, and optimization processes will operate in a fully distributed way offering power stability, information privacy, plug-and-play capabilities, and market adaptability for large-scale power networks. However, compared with the opponent centralized framework, the DCOF requires to pay more attention to the design of the control objectives and deal with the constraints (especially with the coupling constraints). In general, the control objectives for the DC microgrids focus on system-level requirements, e.g., system stability [4] and convergence rate. However, the optimization objectives might focus also on economic aspects, such as maximizing the profit from selling power, minimizing power costs, and reducing power losses. For the control objectives, there are several results (see for instance [5], [6] and the references therein). We can divide the research on the optimization into two parts: modeling and algorithm design. In the modeling part, convex optimization (e.g. quadratic programming) and non-cooperative games (e.g. aggregative games [7], [8]) are the most commonly used (see e.g. [9] and the references therein for further details). In the algorithm design process, the challenges come from dealing with the following three aspects: local constraints, coupling constraints, and global information [10]. We mainly have three different methods for dealing with the local constraints. The first one is called the penalty method, and it uses a penalty function to embed the local constraints into the objective function [11]. The second method is called the projection method, and it restricts the descent direction within the feasible direction [12]. The last method is the Lagrange multiplier method, which dualizes the local constraints such that the corresponding dual problem does not have local constraints [13]. On the other hand, one of the most effective methods for handling the coupling constraints is the multiplier consensus method [14]. Such a method employs the Lagrange dual method to first deal with the coupling constraints, and then converts the resulting dual problem into an optimization problem suitable for the design of distributed algorithms [14]. For the aggregative information, one of the most common approaches is to design a (faster) estimation system (such as dynamical average consensus algorithms) to estimate the global information in a fully distributed fashion [15].

After modeling the control and optimization objectives and designing the control system, the next step is to connect the control system with the dynamics of the considered DC microgrid. Since the DC microgrid’s dynamics can be shown to be passive, ensuring passivity of the controller as well, implies that, through a suitable interconnection, the closed-loop system is still a passive system. Inspired by such an idea, we design the control system based on the Lagrange dual theory, and prove that the closed-loop system converges to the desired (optimal) equilibrium, maximizing the profit while satisfying both the local and coupling constraints.

Notations: \( \text{col} \{ x, y, \cdots \} = [x^\top, y^\top, \cdots]^\top \), where the notation \( \text{col} \) represents “vector stack”. Without additional explanation, we use \( x \) to denote the “vector stack” of \( x_1, \ldots, x_n \), that is \( x = \text{col} \{ x_i \}_{i \in N} = \text{col} \{ x_1, \cdots, x_n \} \). Let \( |x|_+ = \max \{ 0, x \} \). The notation \( \{x\}_i \) represents the \( i \)-th entry of the vector \( x \). The notations \( 0_1 \) and \( 1_1 \) denote \( n \)-dimension vectors whose entries are 0 and 1. Also, we omit the dimension when it is clear. The notation \( \partial_x \) denotes the sub-gradient with respect to \( x \). The notation \( J_{F,x}(x) \) denotes the Jacobian matrix of the function \( F(x) \) with respect...
to $x$. The symbols with the superscripts "r" and "∗" denote constant references and equilibriums (or Nash equilibriums). The """ character denotes the control states corresponding to the microgrid’s ones.

II. MODEL DESCRIPTION

Following [4] and the references therein, we consider a microgrid consisting of a certain number of distributed generation units (DGUs), equipped with distributed controllers and decision systems. Moreover, we consider that each DGU includes constant impedance and constant current loads, and DGUs are interconnected with each other via distribution power lines. Let the sets $\mathcal{N} \triangleq \{1, \ldots, n\}$ and $\mathcal{E} \triangleq \{1, \ldots, m\}$ denote the DGU and the transmission line index sets, respectively. For the readers’ convenience we refer to [1, Fig. 1] that shows the electric scheme of the DGU $i$ and line $k$ (see also [1, Tab. I] for the description of the used symbols).

According to Kirchhoff’s law, we can write the dynamics of the DC microgrid as follows (see e.g. [4]):

$$
\begin{align*}
L \dot{I} &= -V - RI + u, \\
C \dot{V} &= I + BI_i - Z_L^{-1}V - I_L \\
L_i \dot{I}_i &= -R_i I_i - B^\top V.
\end{align*}
$$

(1)

where $B \in \mathbb{R}^{m \times n}$ is the adjacency matrix associated with the arbitrary oriented graph of the connected and undirected graph $\mathcal{G}(\mathcal{N}, \mathcal{E})$. We assume that each transmission line is under the control of a unique DGU connected to it, and we let $\mathcal{E}_i$ represent the index set of the transmission lines being under the control of the DGU $i \in \mathcal{N}$. Hence, we have

$$
\bigcap_{i \in \mathcal{N}} \mathcal{E}_i = \emptyset, \quad \bigcup_{i \in \mathcal{N}} \mathcal{E}_i = \mathcal{E}.
$$

(2)

Let now define for convenience $x_i \triangleq \text{col}\{I_i, V_i, I_{c,i}\} \in \mathbb{R}^{2+|\mathcal{E}_i|}$, with $I_{c,i} \triangleq \text{col}\{I_{l,k}\}_{k \in \mathcal{E}_i} \in \mathbb{R}^{|\mathcal{E}_i|}$, to denote the state vector of the DGU $i \in \mathcal{N}$. The objective of this paper is to design a distributed control scheme that stabilizes the considered DC microgrid at the desired equilibrium solving a pre-designed game problem. To achieve this goal, we use passivity theory [16] to interconnect the considered microgrid’s dynamics (1) with the distributed control system we design in the following sections.

Before formulating the game problem, we describe the feasible region of operation of the considered microgrid by introducing the following set of coupling constraints:

$$
K \triangleq \left\{ (u, x) \in \mathbb{R}^{m+3n} \mid \begin{array}{l}
I + BI_i - Z_L^{-1}V = I_L \\
R_i I_i + B^\top V = 0 \\
A x - s_A = 0
\end{array} \right\},
$$

(3)

where we omit the definitions of the matrix $A \in \mathbb{R}^{(m+n) \times n}$ and the vector $s_A \in \mathbb{R}^{m+n}$. Moreover, for every DGU $i \in \mathcal{N}$, we introduce the following set of local constraints:

$$
\Omega_i \triangleq \left\{ (u_i, x_i) \in \mathbb{R}^{3+|\mathcal{E}_i|} \mid \begin{array}{l}
V_i + R_i I_i - u_i = 0 \\
V_i^{\min} \leq V_i \leq V_i^{\max} \\
r_{c,i}^{\min} \leq I_{c,i} \leq r_{c,i}^{\max}
\end{array} \right\},
$$

(4)

where the superscripts ‘min’ and ‘max’ represent the minimum and maximum values of the the corresponding state, respectively. Based on (3) and (4), we can then define the following feasible set:

$$
\pi_i (x_{-i}) = \left\{ (u_i, x_i) \in \Omega_i \mid A_x x_i + \sum_{j=1, j \neq i}^n A_j x_j + s_{A_i} = 0 \right\},
$$

(5)

where for all $i \in \mathcal{N}$, $A = \{A_1, \ldots, A_n\}$, $s_{A_i} = \sum_{j=1}^n s_{A_j}$ is constant and satisfy $A = \{A_1, \ldots, A_n\}$, and $x_{-i}$ denotes the stack of all the rivals’ decisions, i.e., $x_{-i} = \text{col}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. We can now introduce the main goal of the paper, which can be formulated as an aggregative game problem, i.e., for all $i \in \mathcal{N}$

$$
\min_{u_i, x_i} \quad f_i (u_i, x_i, x_{-i})
$$

s.t. $\quad (u_i, x_i) \in \pi_i (x_{-i}),$

with

$$
f_{1,i} (u_i, x_i, x_{-i}) = f_{1,i} (u_i, x_i) - f_{2,i} (x_i, x_{-i})
$$

$$
f_{1,i} (u_i, x_i) \triangleq \alpha_i \|u_i - u_i^r\|^2 + \frac{1}{2} \|x_i - x_i^r\|^2_{A_x},
$$

(6)

(7)

$$
f_{2,i} (x_i, x_{-i}) \triangleq (l - p_r s_{p_r}) V_i^r I_i,
$$

and $s_{p_{pr}} = \sum_{i=1}^n I_i$, where $f_{1,i}(u_i, x_i)$ represents the cost associated with the deviation of the $i$-th DGU’s state and input with respect to the corresponding references, while $f_{2,i}(x_i, x_{-i})$ represents the profit of DGU $i$, where $(l - p_r s_{p_r}) > 0$ is the selling price of the generated power $V_i^r I_i$. Moreover, $\alpha_i$ is a positive constant and the matrix $A_{x,i} \triangleq \text{diag} \{\alpha_{I_i}, \alpha_{V_i}, \text{diag} \{\alpha_{I_{c,i}}\}_{k \in \mathcal{E}_i}\}$ is positive definite; the parameters $p_r$ and $l$ are positive constants ensuring that the price of power is always positive, i.e., $(l - p_r s_{p_r}) > 0$ for all the feasible currents $I_1, \ldots, I_n$. According to the constraint (3), we can guarantee such a condition by introducing the following assumption:

**Assumption 1 (Parameter setting)** Let the following condition

$$
0 < l - p_r \sum_{i=1}^n \frac{V_{i}^{\max}}{Z_{L,i} + L_{i}}
$$

(8)

hold for all $i \in \mathcal{N}$.

Note that the increase of the generated currents’ sum $s_{p_{pr}}$ implies a reduction of the power price $l - p_r s_{p_r}$ (and vice versa), as usual in the energy trading market.
III. Problem Analysis

In this section, we analyze the optimality conditions associated with the game problem (6). First, we introduce the definition of the generalized Nash equilibrium (GNE) [10, Definition 1].

Definition 1 (Generalized Nash equilibrium) The point \((u^*, x^*)\) is a GNE for the game (6) if and only if it solves the following problem:

\[
\min_{u_i, x_i} f_i(u_i, x_i, x^*) \quad \text{s.t.} \quad (u_i, x_i) \in \pi_i(x^*).
\]

(9)

for all \(i \in \mathcal{N} \).

Now, before formulating the dual problem of (9), for all \(i \in \mathcal{N} \), we define the following penalty function (distance function):

\[
g_i(x_i) = \rho_i \left( [V_{i}^{\min} - V_{i}]_+ + [V_{i} - V_{i}^{\max}]_+ \right)
\]

\[
= g_{1,i}(v_i) + \sum_{k \in \mathcal{E}_i} \rho_{I_{i},k} \left( [I_{l,k}^{\min} - I_{l,k}]_+ + [I_{l,k} - I_{l,k}^{\max}]_+ \right),
\]

where the positive constants \(\rho_i\) and \(\rho_{I_{i},k}\), \(k \in \mathcal{E}_i\), represent the penalty parameters. According to [11, Lemma 4], there exist positive penalty parameters for the sub-problems in (9) such that its solution and the solution to the corresponding penalized problem coincide. For all \(i \in \mathcal{N} \), we can then define the following Lagrange function

\[
L_i(u_i, x_i, x^*, \gamma_i, \lambda_i) = f_i(u_i, x_i, x^*) + g_i(x_i) + \gamma_i(D_i^\top x_i - u_i)
\]

(10)

where \(D_i = \text{col}\{1, R_i, 0\} \in \mathbb{R}^{2+|\mathcal{E}_i|}\) is a constant vector and \(\gamma_i \in \mathbb{R}\) represents the Lagrange multipliers associated with the constraints \(D_i^\top x_i - u_i = 0\) (see the first equality constraints in (4)). Similarly, \(\lambda_i \in \mathbb{R}^{m+n}\) represents the Lagrange multipliers associated with the coupling constraints in (5). Note also that the penalty function \(g_i(x_i)\) is not differentiable at some points. Hence, we need to introduce the sub-gradient of \(g_i(x_i)\) consisting of the sub-gradients of the penalty functions \(g_{1,i}(v_i)\) and \(g_{2,k}(I_{l,k})\). For all \(i \in \mathcal{N}\), the sub-gradient of \(g_{1,i}(V_i)\) is as follows [17]

\[
\partial g_{1,i}(V_i) \in \begin{cases} 
-\rho_i & \text{if } V_i < V_{i}^{\min}, \\
-\rho_i, 0 & \text{if } V_i = V_{i}^{\min}, \\
0 & \text{if } V_{i}^{\min} < V_i < V_i^{\max}, \\
[0, \rho_i] & \text{if } V_i = V_{i}^{\max}, \\
\rho_i & \text{if } V_i > V_{i}^{\max}.
\end{cases}
\]

The sub-gradient of \(g_{2,k}(I_{l,k})\) can be obtained as in (11), thus we omit it for the sake of simplicity. To guarantee that a GNE exists for the game problem (6), we introduce the following assumption:

Assumption 2 (Non-empty feasible set) The feasible set given by the intersection of \(K\) in (3) and \(\Omega_1, \ldots, \Omega_n\) in (4) is non-empty.

Since the constraints of each sub-problem in (6) are affine, Assumption 2 guarantees that \(\pi_i\) satisfies Slater’s constraint qualification. Therefore, the following KKT conditions are a necessary and sufficient condition for the optimal condition of the problem (6) (refer to [13, Section 5.2.3] for details):

\[
\forall i \in \mathcal{N}, \begin{cases} 
\nabla_{u_i} L_i (u_i^*, x_i^*, x_{-i}^*, \gamma_i^*, \lambda_i^*) = 0, \\
\partial_{x_i} L_i (u_i^*, x_i^*, x_{-i}^*, \gamma_i^*, \lambda_i^*) \geq 0, \\
\nabla_{\gamma_i} L_i (u_i^*, x_i^*, x_{-i}^*, \gamma_i^*, \lambda_i^*) = 0, \\
\nabla_{\gamma_i} L_i (u_i^*, x_i^*, x_{-i}^*, \gamma_i^*, \lambda_i^*) = 0,
\end{cases}
\]

(12)

where \((u_i^*, x_i^*, \gamma_i^*, \lambda_i^*)\) is the saddle-point of the Lagrange function (10). Since the multipliers \(\lambda_1^*, \ldots, \lambda_n^*\) can vary from each other, the solution of the KKT condition (12) may not be unique. To shrink the solution set of the KKT condition (12) to a convex set (or a singleton), such that we can develop a fully distributed algorithm, we need to introduce the following definition [18, Definition 3.2], [19].

Definition 2 (Normalized Nash equilibrium) A GNE \((u^*, x^*)\) is a normalized Nash equilibrium (NNE) associated with the given \(r_1, \ldots, r_n > 0\), if there exist the Lagrange multipliers \(\gamma^*\) and \(\lambda^*\) such that \((u^*, x^*, \gamma^*, \lambda^*)\) solves the KKT condition (12) and satisfies the additional condition

\[
r_1 \lambda_1^* = \cdots = r_n \lambda_n^*.
\]

(13)

Remark 1 The values of the Lagrange multipliers \(\lambda_1^*, \ldots, \lambda_n^*\) concerning the coupling constraints represent the shadow price of all the DGUs. From a trading market point of view, the values of \(r_1, \ldots, r_n\) can be designed by a higher level decision system (for example, the government) in order to model different market scenarios.

For the sake of analysis, let \(A_r \triangleq \text{diag}\{r_i\}_{i \in \mathcal{N}} \otimes \mathbb{I}_{m+n}\), \(L \triangleq G \otimes \mathbb{I}_{m+n}\), where \(L\) represents the Laplacian matrix associated with \(G\). Since \(G\) is undirected and connected, then the condition (13) is equivalent to \(L A_r \lambda^* = 0\). Then, we introduce the following proposition playing a crucial role in the later controller design, as in [7].

Proposition 1 There exist \(u^* \triangleq \text{col}\{u_i^*\}_{i \in \mathcal{N}}\), \(\nu^* \triangleq \text{col}\{\nu_i^*\}_{i \in \mathcal{N}} \in \mathbb{R}^n\) satisfying

\[
\begin{cases} 
-(I + L) \nu^* - L \nu^* + n I^* = 0, \\
\n\nu^* = 0
\end{cases}
\]

(14)

if and only if

\[
\nu_i^* = \cdots = \nu_n^* = \sum_{i=1}^{n} I_i^*.
\]

(15)

where \(I_i^* \in \mathbb{R}\) for all \(i \in \mathcal{N}\).

Proof It holds that

\[
\|\nu^*\| = 0 \iff \nu_i^* = \cdots = \nu_n^*.
\]

(16)

By substituting the second equality of (14) in the first equality and multiplying both sides by \(1_n\), we can obtain
the condition (15). From (15) and rank($L$) = $n - 1$, we
deduce that there exists $\nu^*$ satisfying

$$v^* - nI^* = \left( \sum_{i=1}^{n} I_i^* \right) 1_n - nI^* = L\nu^*. \quad (17)$$

By combining (17) and (16), we obtain the condition (14),
which completes the proof.

According to [18, Proposition 3.2], the NNE associated with
a given $r > 0$ of the problem (6) corresponds to the solution
of the following variation inequality:

$$x^* \in K \cap \Omega, \langle F_r(u^*, x^*), x - x^* \rangle \geq 0, \forall x \in K \cap \Omega, \quad (18)$$

where $\Omega \triangleq \bigcap_{i=1}^{n} \Omega_i$, and the vector function $F_r(u, x)$ is the
pseudo-gradient (refer to [18]) defined as follows

$$F_r(u, x) = \text{col} \{ r_i \nabla_{(u_i, x_i)} f_i(u_i, x_i, \nu, x_i - x) \}_{i \in \mathcal{N}}. \quad (20)$$

To ensure that the variational inequality (18) (as well as the
problem (6)) has a unique NNE $(u^*, x^*)$ for a fixed $r > 0$,
we need to introduce the following assumption:

**Assumption 3 (Bound for parameters)** For all $i \in \mathcal{N}$, the
parameter $r_i$ satisfies the following condition:

$$2r_i \alpha_i + (6 - n) r_i p_i V_i^r - \sum_{i=1}^{n} r_i p_i V_i^r > 0. \quad (19)$$

Under Assumption 3, one can verify that the Jacobian matrix
$J_{F_r}(u, x) > 0$ and thus $F_r(u, x)$ is strict monotone for all
$(u, x) \in K \cap \Omega$ (refer to [20, Theorem 2.3.3]). Therefore,
the variational inequality (18) has a unique solution under
Assumptions 2 and 3. All the parameters in (19) have to
be designed, and thus Assumption 3 is not a strict condition.
Following from the analysis in Proposition 1, we can deduce
that the following constraint

$$\begin{bmatrix}
A_r \{ \text{col} \{ A_i(x_i^* - s_A) \}_{i \in \mathcal{N}} - L A_r \lambda^* - L \theta^* \} = 0, \\
L A_r \lambda^* = 0
\end{bmatrix}$$

is equivalent to the constraint $A x^* - s_A = 0$. Therefore,
by involving the constraint (13) and Proposition 1, we can rewrite
the condition (12) is a distributed form as

$$\begin{cases}
-v_i^* - L_i v_i^* - L_i \nu^* + n I_i^* = 0, \\
L_i \nu^* = 0, \\
r_i \alpha_i(u_i^* - u_i^*) + \gamma_i^* = 0, \\
r_i F_i(x_i^*, v_i^*) + r_i A_i^T \lambda_i^* + \gamma_i^* D_i \in \mathcal{O}, \\
D_i^T x_i^* - u_i^* = 0, \\
r_i((A x_i^* - s_A) - L_i A_r \lambda^* - L_i \theta^*) = 0, \\
L_i A_r \lambda^* = 0,
\end{cases} \quad (20)$$

where for all $i \in \mathcal{N}$, the vector $\theta_i^* \in \mathbb{R}^{m+n}$ denotes
the dual variables associated with the consensus constraint
$L_i A_r \lambda^* = 0$. We use the vector $L_i^T \in \mathbb{R}^n$ and matrix
$L_i \in \mathbb{R}^{(m+n) \times n(m+n)}$ to denote the rows of $L$ and $L$

associated with the DGU $i \in \mathcal{N}$, respectively. Moreover,
$F_i(x_i^*, v_i^*)$ is defined as

$$F_i(x_i^*, v_i^*) = \left[ \begin{array}{c}
\alpha_i(I_i^* - I_i^r) - (l_i p_i V_i^r v_i^* + p_i V_i^r I_i^r) \\
\alpha_i V_i^* - V_i^r + \partial g_1, i(V_i^r) \\
\text{col} \{ (I_i^r - I_i^r) + \partial g_{2, i}(I_i^r) \}_{k \in \mathcal{E}_i} \end{array} \right].$$

**Remark 2** Based on [21, Theorem 3.4], the penalty param-
eters $\rho_i$ and $\rho_{l, k}$ satisfy the following condition

$$\rho_i \geq \alpha_{V_i}(V_i^\max - V_i^r) + \nabla V_i(\lambda_i^*, (A x - s_A)) + \gamma_i^*,$$

for all $i \in \mathcal{N}$ and $k \in \mathcal{E}_i$. Hence, the penalty parameters
should be large enough such that they satisfy (21).

**IV. ALGORITHM DESIGN AND ANALYSIS**

Based on (20), we can now design the distributed con-
troller for each DGU $i \in \mathcal{N}$. By connecting the designed
controller to the DGU $i \in \mathcal{N}$ in a passive way (see e.g. [4]),
we obtain the following closed-loop system:

$$\begin{aligned}
\dot{x}_i & = G_{g, i}(u_i, x_i), \\
\varepsilon \dot{v}_i & = -v_i - L_i v_i - L_i \nu + n I_i, \\
\dot{u}_i & = -\alpha_i u_i + \gamma_i - \epsilon I_i, \\
\dot{\lambda}_i & = -r_i F_i(x_i, v_i) - r_i A_i^T \lambda_i - \gamma_i D_i, \\
\dot{\theta}_i & = L_i A_r \lambda_i, \\
\gamma_i & = -u_i + D_i x_i,
\end{aligned} \quad (22a)$$

where (22a) denotes the dynamics of each DGU $i \in \mathcal{N}$, and
the non-negative constants $\varepsilon$ and $\epsilon$ denote the control system
parameters. For the sake of the later convergence analysis,
let $s_f \triangleq \text{col} \{ v, v \}$ and $s_d \triangleq \text{col} \{ u, \dot{x}, \lambda, \theta, \gamma \}$. Then we can write (22) as:

$$\begin{aligned}
\dot{x} & = G_g(u, x), \\
\varepsilon \dot{s}_f & = G_f(s_f, \dot{I}), \\
\dot{s}_d & = G_d(v, s_d, \dot{I}),
\end{aligned} \quad (23a)$$

where we omit the detailed definitions of the maps $G_g, G_f$
and $G_d$. Note that, in the framework of singular perturbation
system theory [22], (23b) describes the dynamics of the fast
system, while (23a) and (23c) those of the slow system. Let $h(\dot{I}) \triangleq \text{col} \{ h_3(\dot{I}), h_2(\dot{I}) \}$ and $s_h \triangleq s_f - h(\dot{I})$ represent
the solution of the equation $G_f(s_f, \dot{I}) = 0$ and the corre-
sponding boundary layer system state, respectively. We can write
the boundary layer system and reduced-order system as follows:

$$\begin{aligned}
\dot{x} & = G_g(u, x), \\
\varepsilon \dot{s}_h & = G_f(s_h + h, \dot{I}), \\
\dot{s}_d & = G_d(h_v, s_d, \dot{I}),
\end{aligned} \quad (24a)$$

where we abbreviate $h(\dot{I})$ as $h$. 

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Theorem 1 (Convergence analysis) Let Assumptions 2 and 3 hold and the initial state \( x_0 \) satisfy \( x_0 \) such that (22) converges to the largest invariant set \( \Phi_{s,f} \) for all \( \varepsilon \) satisfying \( 0 < \varepsilon < \varepsilon^* \), where

\[
\Phi_{s,f} = \{ (s_f, x, s_d) \mid G_g(u, x) = 0, G_f(s_f, I) = 0, G_d(v, s_d, I) = 0 \}.
\] (25)

Proof Let \( E_b(s_b) \) and \( E_u(x, s_d) \) denote respectively the Lyapunov functions of the boundary layer system (24b) and the reduced-order system (24a), (24c), i.e.,

\[
E_b(s_b) = \sigma_x ||x||^2 + \frac{1}{2} ||v_b||^2 + \frac{1}{2} ||v_b||^2 + \langle v_b, v_b \rangle,
\]

\[
E_u(x, s_d) = \frac{1}{2} \left( \langle \dot{h}, I \rangle^L + \langle \dot{h}, I \rangle^L + \langle \dot{h}, I \rangle^L + \langle \dot{h}, I \rangle^L \right) + \frac{1}{2} \langle x, x^* \rangle^2
\]

\[+ \frac{1}{2} \sigma_x ||s_d - s_d^0||^2,\]

where \( \sigma \) represents the largest singular value of the Laplacian matrix \( L \). Then, we can define the composite Lyapunov function as follows:

\[V(s_b, s_s) = (1 - \varepsilon)E_u(x, s_d) + \varepsilon E_b(s_b).\]

The convergence analysis follows from [22, Theorem 11.3]. For convenience, we define \( s_d = \alpha \{ x, s_d \} \) and

\[G_b(h_v, s_d, I) = \alpha \{ G_g(x, u), G_d(h_v, s_d, I) \}.\]

Since Proposition 1 ensures that \( h_v = \sum_{i=1}^{n} \dot{I}_i \) and it is easy to verify that \( G_d(h_v, s_d, I) \) is a monotone function with respect to \( s_d \), we can deduce that \( J_{G_d,s_d}(h_v, s_d, I) \) is positive semi-definite and

\[\frac{\partial E_u}{\partial s_d} G_d(h_v, s_d, I) \leq -\frac{1}{\varepsilon} G_f(h_v, s_d, I) J_{F_e, f}(u, x) G_f(h_v, s_d, I) \]

\[-\dot{u}^T \dot{I} - ||A_r(\lambda - \lambda^*)||^2_2 - \zeta ||\hat{I} - \hat{I}^*||^2.\]

where \( G_f(h_v, s_d, I) \) denotes the dynamics of \( \dot{I} \), and \( \zeta \) is the coefficient of strong monotonicity of \( F_e \). In addition, we have

\[\frac{\partial E_u}{\partial x} \dot{x} \leq \begin{bmatrix} -R & I & 0 \\ 0 & \hat{B}^{-1} & 0 \\ 0 & -B^T & -R_t \end{bmatrix} x + \dot{u}^T \dot{I}.
\] (27)

Thus we can observe that there exists a positive \( \alpha_1 \) such that

\[\frac{\partial E_u}{\partial s_d} G_b(h_v, s_d, I) \leq -\alpha_1 G_b^2(h_v, s_d, I),\]

where \( G_b(h_v, s_d, I) \) defines \( G_b(h_v, s_d, I) = ||G_f(h_v, s_d, I)|| + \|A_r(\lambda - \lambda^*)\|_L - ||\hat{I} - \hat{I}^*||. \) Next we proceed by taking the time derivative of \( E_b(s_b) \), and based on the fact that \( 1_n v_0 = 0 \) (following from \( 1_n v_0 = 0 \), we have:

\[\frac{\partial E_b}{\partial s_b} s_b = -s_b \begin{bmatrix} 2\sigma + 1 & I + L & L + L^2 \\ L + L^2 & L + L^2 & \alpha_1 \end{bmatrix} s_b, \]

\[\Delta A_n, \]

where \( \alpha_n \) is a constant. Then, one can verify that all the eigenvalues of the matrix \( A_n \) are positive (we omit the proof due to space limitation). Hence, based on the property of the Rayleigh quotient, we have

\[\frac{\partial E_b}{\partial s_b} s_b \leq -\delta_{\min}(A_n) ||s_b||^2 = -\alpha_2 ||s_b||^2.\]

(29)

Now, since the function \( G_b(s_b + h_v, s_s, I) \) is linear with respect to \( s_b + h_v \), we can deduce that there exists a positive constant \( \beta_1 \) such that

\[\frac{\partial E_u}{\partial s_s} [G_b(s_b + h_v, s_s, I) - G_b(h_v, s_s, I)] \leq \beta_1 G_b(s_b + h_v, s_s, I) \]

\[\leq \beta_1 G_b(h_v, s_s, I) ||s_b||.\]

(30)

Finally, one can show that (31) holds. Furthermore, since \( \partial h/\partial s_b \) and \( \partial h/\partial I \) are constant matrices, then we can deduce that there exist two positive constants \( \beta_2 \) and \( \xi \) such that the following inequality holds

\[\left[ \frac{\partial E_u}{\partial s_b} - \frac{\partial E_u}{\partial s_s} \right] G_b(s_b + h_v, s_s, I) \]

\[\leq \beta_2 G_b(h_v, s_s, I) ||s_b|| + \xi ||s_b||.\]

(32)

So far, we have verified all the conditions in [22, Theorem 11.3], i.e., (28), (29), (30) and (32). Hence, we can conclude that if

\[0 < \varepsilon < \varepsilon^* \]

then system (22) converges to the largest invariant set \( \Phi_{s,f} \). 

V. SIMULATIONS

In this section, we assess the performance of the proposed distributed control system (22) in simulation, considering a microgrid with four DGUs in a ring topology. We set the price parameters \( p_t \) and \( t \) as 5 and 0.01, respectively. Also, we select the fast system parameter \( \varepsilon \) equal to 0.01. The parameters of the objective and penalty functions are reported in [1, Tab. II]. We report the parameters of all the DGUs and transmission lines in Tables [1, Tab. III and IV].

We consider that the microgrid initial conditions are within the feasible set and the system remain unperturbed for the first 5 seconds. Then, at the time instant \( t = 5 \) s, each current-type load \( I_{L,i} \) and resistance-type load \( Z_{L,i} \) is decreased by 3 units. We present the results in Fig. 1 and 2, and we observe that the microgrid’s states converge to the equilibrium within a short time after the loads change. Moreover, the new equilibrium satisfies the optimal condition (20).

In general, the simulation results show excellent performance both in terms of optimality and transient response.

VI. CONCLUSIONS

In this paper, we first design a distributed control system to solve the power trading problem (described as an aggregative game) in a DC microgrid. Then, we interconnect the designed control system with the microgrid in a passive way and analyze the convergence of the overall closed-loop system. Although we prove that there exists \( \varepsilon^* \) for the fast system, we do not explicitly provide its exact bound, which is left as a future work.
\[
\frac{\partial E_b}{\partial s_s} - \frac{\partial E_b}{\partial s_b} \frac{\partial h}{\partial s_s} G_s(s_b + h\nu, s_s, I) \leq s_s^\top \sigma I + I + L L \sigma I \left( \frac{\partial h}{\partial s_s} G_s(h\nu, s_s, I) - \frac{\partial h}{\partial I} (p_r V^r \circ s_b) \right).
\] (31)

Fig. 1. (a) The loads’ and decision system’ voltages. (b) The DGU control voltages.

Fig. 2. (a) The DGUs’ and the decision system’s currents. (b) The transmission lines’ and decision system’s currents.

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