A Swapping Lemma for Switched Systems

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Abstract: Robust stability of a class of linear time-invariant systems affected by piecewise constant parameters with dwell-time constraints is considered. In contrast to other approaches the proposed result relies on separation techniques within the framework of integral quadratic constraints and is based on a novel version of the swapping lemma. In particular, our result allows to take additional knowledge on the variation of the parameter into account. The obtained conditions are expressed as infinite-dimensional LMIs which can be solved e.g. by using sum-of-squares relaxation methods. We illustrate the proposed approach with a numerical example.

Keywords: Switched systems, integral-quadratic-constraints, robust stability, clock-dependent conditions, linear matrix inequalities

1. INTRODUCTION

In the past years considerable attention has been devoted to the analysis and control of linear parameter-varying (LPV) systems which allow to model a wide range of real-world phenomena (Mohammadpour and Scherer (2012); Briat (2015)). Recently, robust stability results were proposed for LPV systems affected by perturbations of smoothly time-varying (Köroğlu and Scherer (2006)), piecewise constant (Allerhand and Shaked (2010); Briat (2015); Xiang (2016)) as well as piecewise differentiable parameters (Briot (2017)). These results are usually based on robust control techniques involving parameter-dependent Lyapunov functions (Allerhand and Shaked (2010); Briat (2015, 2017)), multiplier theory (Sideris and Tchernychev (2000)) or integral quadratic constraints (IQCs) (Petzer (2017); Megretsky and Rantzer (1997); Köroğlu and Scherer (2006)).

In this paper we consider stability analysis of a feedback interconnection involving a linear system $M$ and an uncertainty $\Delta$ as depicted in Fig. 1. The studied class of uncertainties allows us to model an LPV system that is affected by perturbations of piecewise constant parameters. The main contribution of the present paper is a novel robust stability result for such systems and its derivation. The latter is based on introducing an additional clock parameter and is related to the ones of Allerhand and Shaked (2010); Briat (2015, 2017). However, in contrast to those approaches we base our results on a variation of the so-called swapping lemma (see Köroğlu and Scherer (2006); Teng (1991)) and on separation techniques within the general framework of IQCs involving dynamic multipliers. The application of the latter is motivated by the fact that they pose an attractive alternative to parameter dependent Lyapunov functions for constant parametric uncertainties since they offer substantial extra flexibility and even permit to handle nonlinearities systematically (Veenman et al. (2016)). In particular, our result allows to take extra knowledge on the jump-height of the involved piecewise constant parameter at the discontinuities into account. The latter is formalized via the notion of region-of-variation (ROV) inspired from Sideris and Tchernychev (2000) and Köroğlu and Scherer (2006).

The present paper is structured as follows. After a short paragraph on notation, we introduce the considered feedback interconnection as well as the involved region-of-variation and present a novel version of the swapping lemma. Based on the latter, a suitable system extension and a version of an IQC stability theorem, we derive a novel robust stability criterion for systems affected by piecewise constant parametric uncertainties. We illustrate the benefits of our approach by means of a numerical example. Some technical proofs are moved to the appendix.

Notation. For $o \in \{<, \leq, =, >\}$ we use $C_o := \{z \in \mathbb{C} \mid \text{Re}(z) \circ 0\}$, $R_o := C_o \cap \mathbb{R}$, $C^\infty_o := C_o \cup \{\infty\}$ and $\mathbb{R}_c^\infty := \mathbb{R}_c \cup \{\infty\}$. $N$ (or $\mathbb{N}$) denotes the set of positive (nonnegative) integers and $S^n_o$ denotes the set of symmetric real $n \times n$ matrices. For $o \in \{<, \leq, =, >\}$ and a function $F : X \to S^n_o$ we write $F \circ o$ iff $F(x) \circ 0$ for all $x \in X$. $C^\infty_o = (L^\infty_2)$ are the spaces of (locally) square integrable functions from $\mathbb{R}_o$ to $\mathbb{R}^n$ and $L^\infty_2 (L^\infty_2)$ are the spaces of (locally) square summable $\mathbb{R}^n$-valued sequences on $\mathbb{N}_0$. Further, we set $L^\infty_0 \subset \{x \in L^\infty_2 \mid x(0) = 0\}$ as well as $L^\infty_0 \subset \{x \in L^\infty_2 \mid x(0) = 0\}$.

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\[ C^\infty_o \]
For any \( t > 0 \), we let \( f(t^-):=\lim_{s\to t^-}f(s) \) denote the limit from below once it is well defined. For notational simplicity we further set \( f(t^-):=f(t) \). Finally, objects that can be inferred by symmetry or are not relevant are indicated by “\( \bullet \)”. 

2. ROBUST STABILITY ANALYSIS 

2.1 System Description 

For matrices \( A_M \in \mathbb{R}^{m \times m} \), \( B_M \in \mathbb{R}^{m \times q} \), \( C_M \in \mathbb{R}^{q \times m} \), \( D_M \in \mathbb{R}^{q \times q} \) with \( \text{eig}(A_M) \subset C_\subset \), we consider the LTI system described by the operator 

\[
M : \mathcal{L}_2^q \rightarrow \mathcal{L}_2^q, \quad w \mapsto \int_0^z C_M e^{A_M(-s)} B_M w(s) \, ds + D_M w. 
\]

Moreover, for a number \( T \in \mathbb{R}_+ \) and a compact and star-shaped set \( \mathcal{R} \subset \mathbb{R}^2 \), i.e., \( x \in \mathcal{R} \) implies \( \tau x \in \mathcal{R} \) for all \( \tau \in [0,1] \), we consider an uncertainty defined by the map 

\[
\delta = \begin{cases} \Delta, & \text{if } \delta(t) = \delta(s) \text{ and } (\delta(Tk), \partial_k \delta) \in \mathcal{R} \text{ for all } t, s \in [Tk, Tk + 1) \text{ and all } \delta \in \mathcal{R} \end{cases} 
\]

for matrices \( A, B, C, D \) with \( \text{eig}(A) \subset C_\subset \). 

2.2 Swapping Lemma 

Motivated by Teng (1991) and Köröglu and Scherer (2006) we present a version of the swapping lemma for piecewise constant parameters with dwell-time \( T \). Its proof is given in the appendix. 

Lemma 3. For matrices \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times q} \), \( C \in \mathbb{R}^{p \times n} \), \( \Delta \) and \( \psi \) as in (1b) with \( \Delta \in \mathcal{R} \). Then 

\[
\Delta \psi - \psi \Delta_q = \psi_{cd} \psi_{dd} \Delta \psi_{dc}
\]

where 

\[
\psi_{cd} : \mathcal{L}_2^q \rightarrow \ell_0^\infty, \quad \psi_{dc}(w)(k) = \int_0^{Tk} e^{A(T_{k+1} - s)} B w(s) \, ds 
\]

\[
\psi_{dd} : \ell_0^\infty \rightarrow \ell_0^\infty, \quad \psi_{dd}(u)(k) = \sum_{l=1}^k e^{A(T_{k+1} - l)} u(l), 
\]

\[
\psi_{cd} : \ell_0^\infty \rightarrow \mathcal{L}_2^q, \quad \psi_{cd}(u)(t) = C e^{A(T_{k+1} - t)} u(k) 
\]

for all \( t \in [Tk, Tk + 1) \), \( k \in \mathbb{N}_0 \) and 

\[
\Delta : \ell_0^\infty \rightarrow \ell_0^\infty, \quad \Delta(u)(k) = \partial_k u(k).
\]

The corresponding result from Teng (1991), Köröglu and Scherer (2006) involves continuous LTI filters \( \psi_B \) and \( \psi_C \) with realizations \((A, B, I, 0)\) and \((A, I, C, 0)\), respectively. The mappings \( \psi_{dd}, \psi_{dc} \) and \( \psi_{cd} \) are related to those via concatenation with operators as well-known in signal processing. Indeed, with the sampling operator \( S : \mathcal{L}_2^q \rightarrow \ell_0^\infty, S(u)(k) = u(Tk) \), we include \( \psi_{cd} = S \psi_B \) and \( \psi_{dd} = \psi_B L \) with 

\[
I : \ell_0^\infty \rightarrow \mathcal{L}_2^q, \quad u \mapsto \sum_{k=0}^{\infty}(\delta - Tk) u(k)
\]

involving a so-called dirac-comb.

Remark 4. 

- Since \( \mathcal{R} \) is bounded all appearing operators \( \Delta, \Delta, \psi, \psi_{dd}, \psi_{cd}, \psi_{dc} \) are easily seen to be linear, bounded and causal. 
- It is crucial that \( \delta \) commutes with \( A, B, C \) and \( D \). By adjusting \( \psi \), i.e., choosing \( A, B, C \) and \( D \) to be block-diagonal, the swapping lemma can be
1. Introduction and Notations

2. The Switched System Model

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5. Conclusion
elapsd since the last potential discontinuity occurred in the parameter \( \delta \) and which resets its value to zero after a new discontinuity. Interestingly, this yields the following non-trivial KYP-like result for switched systems defined by (2) and (1) based on a hybrid dynamic multiplier.

**Theorem 7.** Let \( \Sigma \) be as in (7) and suppose that there exist \( \varepsilon > 0 \) and a continuously differentiable function \( X : [0, T] \to \mathbb{S}^{m+2n} \) satisfying

\[
(*)^* \begin{pmatrix} (X_F')_0 & 0 \\ \delta \end{pmatrix} \geq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},
\]

where \( X_F := \begin{pmatrix} X \ 0 \end{pmatrix}, X := \frac{1}{T} \left( X(0) \ 0 \right) - X(T) \) and

\[
\begin{pmatrix} A & B & \cdots & A & B \\ B & \cdots & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \leq -\varepsilon I,
\]

(10)

Then \( \Sigma (M, w, \varepsilon) \leq -\varepsilon \|w\|^2 \) for all \( w \in \mathbb{L}_2 \times \ell_{0,2} \).

A proof relying on dissipativity arguments is given in the appendix. The used storage function involves the clock parameter \( \theta \) as well as a continuous and a hybrid filter, i.e. a filter with continuous as well as discrete dynamics.

**Remark 8.** In contrast to Briat (2015, 2017) and due to the applied separation techniques, the LMI (10) is only affected by the clock parameter varying in \([0, T]\) and not by any additional parameter in the uncertainty set \( \mathcal{R} \).

Additionally, note that (10) is composed of a continuous-time KYP-LMI for the weighted system \( \text{col}(\psi M, \psi) \) and a discrete-time KYP-LMI which are coupled through \( \mathcal{P} \) and \( X \). We use both types of LMIs in order to handle the flow and the jump behavior of the interconnection (3).

In summary, by applying Theorems 5, 7 and Lemma 6, we obtain the following robust stability result.

**Corollary 9.** Let \( \psi = \begin{pmatrix} \psi_M \ 0 \end{pmatrix} \), with eig\((A) \in \mathbb{C}_-\) be given. Then the interconnection (2) is stable for all \( \varepsilon > 0 \), a continuous \( P : [0, T] \to \mathbb{S}^{2(p+n)} \) and a continuously differentiable \( X : [0, T] \to \mathbb{S}^{m+2n} \) satisfying (8) and (10).

**Proof.** It remains to show that (a) in Theorem 7 is satisfied. To this end, we observe that the left upper block in (8) and the (2, 2) block in (10) the inequalities

\[
\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \leq -\varepsilon I,
\]

for all \((\delta, 0) \in \mathcal{R}\) with \( P := (*)^* \begin{pmatrix} \begin{pmatrix} \delta^2 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \delta^2 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \) This leads to \( \det(I - D_M \delta) \neq 0 \) for all \((\delta, 0) \in \mathcal{R}\) and proves the claim.

**Remark 10.** For \( T \to \infty \) and due to cancellation of rows and columns, we end up with the conditions for constant parametric uncertainties with dynamic multipliers (cf. e.g. Balakrishnan (2002); Scherer and Köse (2012)).

- A common choice for \( \psi \) are the stable “basis” filters with transfer matrices

\[
\psi(s) := \begin{pmatrix} I_q \left( \frac{s + \alpha}{\nu} \right) I_q \cdots I_q \left( \frac{s + \alpha}{\nu} \right)^T \nu I_q \end{pmatrix}
\]

for some “length” \( \nu \in \mathbb{N}_0 \) and some fixed pole \( \alpha \in \mathbb{C}_- \). These admit nice approximation properties as illustrated in Pinkus (1985) and Scherer (2015).

- In Körögh and Scherer (2006) various relaxations for the LMI (8) are elaborated on. These are ranging from convex hull relaxations for polytopic ROVs \( \mathcal{R} \) to Pólya and sum-of-squares relaxations for more general regions.

- Feasibility of the LMIs (10), (8) is an infinite dimensional problem and thus not numerically tractable in general. However, asymptotic exact sufficient conditions can be obtained e.g. by using the sum-of-squares approach (Parrilo (2000); Scherer and Hol (2006)) similarly as performed in (Briot, 2017, Section 3.4). The resulting finite dimensional semidefinite program can then be solved via standard semidefinite programming solvers such as SeDuMi by Sturm (2001).

Let us finally observe that the computational burden of the feasibility problem in Corollary 9 can be drastically reduced, e.g., for rectangular ROVs \( \mathcal{R} \). To this end we structure the matrix \( P \) as

\[
P = \begin{pmatrix} R_1 & 0 & S_1 \\ 0 & R_2 & 0 \\ S_1^T & 0 & Q_1 \\ 0 & 0 & Q_2 \end{pmatrix}
\]

and use the following natural realization

\[
\begin{pmatrix} \psi G \end{pmatrix} = \begin{pmatrix} A & BC_M \ 0 & B \\ 0 & A_M \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}
\]

with \( G = \begin{pmatrix} A_M & B_M \end{pmatrix} \) and \( \psi = \begin{pmatrix} A & B \end{pmatrix} \). Then Corollary 9 specializes to the following result.

**Corollary 11.** Let \( \psi = \begin{pmatrix} A \ B \end{pmatrix} \) with eig\((A) \in \mathbb{C}_-\) and \( G = \begin{pmatrix} A_M & B_M \end{pmatrix} \) be given. Further, let \( A_M, B_M \) be as in Corollary 9 and set \( \tilde{C}_J := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tilde{D}_J := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). Then the interconnection (2) is stable for all \( \varepsilon > 0, P_2 \in \mathbb{S}^{2n}, \text{a continuous } P_1 : [0, T] \to \mathbb{S}^{2p} \) and a continuous differentiable \( X : [0, T] \to \mathbb{S}^{m+2n} \) satisfying

\[
\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \leq -\varepsilon I,
\]

(12a)

\[
\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \leq -\varepsilon I,
\]

(12b)

Then (12b) reveals even more clearly the continuous and discrete-time KYP structure as addressed in Remark 8. Note that requiring feasibility of (12c) instead of (8) is in general conservative for non-rectangular ROVs.
As an example let us consider interconnection (2) with
\[
\begin{pmatrix}
A_M & B_M \\
C_M & D_M
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 \\
-2 & -1 & 1
\end{pmatrix},
\]
the rectangular ROV \(\mathcal{R}(\beta, \gamma) = [0, \beta] \times [-\gamma, \gamma]\) with parameters \(\beta, \gamma \in \mathbb{R}_+\), and filters (11) with \(\alpha = -5\) and to be chosen lengths \(\nu \in \mathbb{N}_0\). By using Corollary 11, we can now compute for each dwell-time \(T \in \mathbb{R}_+\), each \(\gamma \in \mathbb{R}_+\) and each length \(\nu \in \mathbb{N}_0\) some lower bounds \(\beta_0\) on the stability margin
\[
\max\{\beta \in \mathbb{R} \mid (2) \text{ is stable}\} \in D_T(\mathcal{R}(\beta, \gamma))]
\]
In order to compute such bounds, we use sum-of-squares relaxations (Scherer and Hol (2006)) involving polynomials of degree 4 and \(\varepsilon = 0.01\). The resulting finite dimensional problems are solved with Yalmip (Löfberg (2004)) and SeDuMi (Sturm (2001)).
Illustrated on the left hand side of Fig. 4 are the lower bounds \(\beta_0\) as a function of the dwell-time \(T \in [0, 2]\) for \(\gamma = 1\) obtained from Corollary 11 for the lengths \(\nu \in \{1, 2, 3, 4\}\), and the ones obtained from a special case of (Briat, 2017, Theorem 3). Note that, due to the definition of \(\mathcal{R}, \gamma = 1\) translates into the absence of additional information on the jump-heights. We can observe that (Briat, 2017, Theorem 3) yields superior results for dwell-times \(T < 0.7\) while our approach yields less conservative results for larger dwell-times and for the lengths \(\nu \in \{2, 3, 4\}\). Note that dynamic multipliers are expected to be highly beneficial for constant parametric uncertainties as e.g. pointed out in Poola and Tikku (1995) and that those multipliers are incorporated in our results. Hence an increase of the lower bounds of the stability margin for uncertainties with larger dwell-times makes sense intuitively.

Deicted on the right hand side of Fig. 4 are the lower bounds \(\beta_0\) as a function of the dwell-time \(T \in [0, 1]\) for the length \(\nu = 2\) obtained from Corollary 11 for restricted jump-heights, i.e., \(\gamma \in \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}\). We can observe that our result yields, as expected, less conservative lower bounds when additionally available information on the variation of the uncertainty is used.

4. CONCLUSIONS

In this paper we apply IQC separation techniques and non-trivial extensions of a classical swapping lemma and the KYP lemma in order to derive a novel robust stability result for LTI systems against piecewise constant parametric uncertainties. In particular, our result allows taking additional knowledge on the correlations between the values and the jump-heights of the considered uncertainties into account. The proposed approach might pave the way for systematically covering more general uncertainties within the IQC framework. We conclude with an application of our results by means of a numerical example.
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**Appendix A. TECHNICAL PROOFS**

**Proof of Lemma 3.** Let $w \in \mathcal{L}_2$ be arbitrary and let us abbreviate $h_{lk} := \int_0^T e^{A(Tk-s)} B w(s) \, ds$ for all $k, l \in \mathbb{N}_0$.
Then $\psi_{dc} w(l) = h_{lk}$ for all $l \in \mathbb{N}_0$ and $\psi_{dd} \Delta \psi_{dc} w(k)$ equals

\[
\sum_{l=1}^k e^{A(Tk-l)} \delta h_{lk} = \sum_{l=1}^k [\delta(Tl) - \delta(T(l-1))] h_{lk}
\]

\[
= \delta(Tk) h_{kk} - \delta(0) h_{1k} + \sum_{l=1}^{k-1} \delta(Tl) h_{lk} - \sum_{l=2}^k \delta(T(l-1)) h_{lk}
\]

\[
= \delta(Tk) h_{kk} - \delta(0) h_{1k} - \sum_{l=1}^{k-1} \delta(Tl) [h_{l+1,k} - h_{lk}]
\]

\[
= \delta(Tk) h_{kk} - \sum_{l=0}^{k-1} \delta(Tl) \int_{Tk}^{T(l+1)} e^{A(Tk-s)} B w(s) \, ds
\]

\[
= \delta(Tk) h_{kk} - \int_0^{Tk} e^{A(Tk-s)} B \delta(s) w(s) \, ds
\]

for all $k \in \mathbb{N}_0$, where the last equation is a consequence of $\delta$ being constant on $[Tk, T(T(l+1))$ for all $l \in \mathbb{N}_0$. Let now $k \in \mathbb{N}_0$ and $t \in [Tk, T(k+1))$ be arbitrary. Then we can conclude that $\psi_{dd} \Delta \psi_{dc} w(t)$ equals

\[
C e^{A(t-Tk)} \left( \delta(Tk) h_{kk} - \int_0^{Tk} e^{A(Tk-s)} B \delta(s) w(s) \, ds \right)
\]

\[
= \delta(Tk) \int_0^{Tk} C e^{A(t-s)} B w(s) \, ds - \int_0^{Tk} C e^{A(t-s)} B \delta(s) w(s) \, ds
\]

\[
\pm \int_0^t C e^{A(t-s)} B \delta(Tk) w(s) \, ds
\]

\[
= \delta(Tk) \int_0^t C e^{A(t-s)} B w(s) \, ds - \int_0^t C e^{A(t-s)} B \delta(s) w(s) \, ds
\]

\[
= \delta(t) \int_0^t C e^{A(t-s)} B w(s) \, ds - \int_0^t C e^{A(t-s)} B \delta(s) w(s) \, ds
\]

\[
= \delta(t) \psi(w)(t) - \psi(\delta w)(t),
\]

which yields the claim. 

**Proof of Theorem 7.** Let $w = \text{col}(w_1, w_2) \in \mathcal{L}_2 \times \ell_{0,2} \in \mathcal{L}_2 \times \ell_{0,2}$ be arbitrary and let us begin with several preliminary observations. For $x(0) = 0$ let us define

\[
\left( \begin{array}{c}
\dot{x}(t) \\
z(t)
\end{array} \right) = \left( \begin{array}{c}
A_M & B_M \\
C_M & D_M
\end{array} \right) \left( \begin{array}{c}
x(t) \\
w_1(t)
\end{array} \right) \text{ for all } t \in \mathbb{R}_+.
\]

To infer $z = M_x w = M w_1$ by definition of $M$ and $M_x$. Further, let us define the stable filters

\[
\left( \begin{array}{c}
\xi_1(t) \\
\omega_1(t)
\end{array} \right) = \left( \begin{array}{c}
A B \\
C D
\end{array} \right) \left( \begin{array}{c}
\xi_2(t) \\
\omega_2(t)
\end{array} \right), \quad \left( \begin{array}{c}
\psi_1(t) \\
\omega_1(t)
\end{array} \right) = \left( \begin{array}{c}
A B \\
C D
\end{array} \right) \left( \begin{array}{c}
\psi_2(t) \\
\omega_2(t)
\end{array} \right)
\]

for all $t \in \mathbb{R}_+$ with zero initial conditions $\xi_1(0) = \omega_1(0) = 0$, which are known to satisfy

\[
\psi_1 w_1 = z_1 \quad \text{and} \quad \psi M w_1 = z_2.
\]

In particular, note that

\[
\psi_{dc} M w_1(k) = \psi_{dd} z(k) = \xi_2(Tk) \quad \text{for all } k \in \mathbb{N}_0.
\]

Moreover, observe that

\[
\xi_1(t) = \int_0^t e^{A(t-s)} B w_1(s) \, ds
\]

\[
= \int_0^{Tk} e^{A(t-s)} B w_1(s) \, ds + \int_{Tk}^t e^{A(t-s)} B w_1(s) \, ds
\]

\[
= e^{A(t-Tk)} \xi_1(Tk) + \int_0^t e^{A(t-s)} B w_1(s) \, ds
\]

for all $k \in \mathbb{N}_0$ and all $t \in \mathbb{R}_+$. In order to construct the remaining terms appearing in (9) we need two additional filters. The first one is the discrete filter $\xi_2(0) = 0$, $\xi_3(k+1) = e^{A T} \xi_3(k) + w_2(k+1)$ for all $k \in \mathbb{N}_0$ which satisfies

\[
\psi_{dd} \psi_2 w_2(k) = \xi_3(k) \quad \text{for all } k \in \mathbb{N}_0.
\]

The second one is a resetting filter (or filter with additional impulsive behavior)

\[
\left( \begin{array}{c}
\hat{\xi}_1(t) \\
\hat{\omega}_1(t)
\end{array} \right) = \left( \begin{array}{c}
A B \\
C D
\end{array} \right) \left( \begin{array}{c}
\hat{\xi}_2(t) \\
\hat{\omega}_2(t)
\end{array} \right), \quad \hat{\xi}_1(Tk) = \xi_3(k) + \xi_1(Tk)
\]

for all $t \in [Tk, T(k+1))$ and all $k \in \mathbb{N}_0$. The latter is chosen such that

\[
\psi_{dd} \psi_{dd} \psi_2 w_2 + \psi w_1 = \hat{z}_1.
\]

Indeed, let $k \in \mathbb{N}_0$ and $t \in [Tk, T(k+1))$ be arbitrary. Then

\[
\hat{z}_1 = \psi_{dd} \psi_{dd} \psi_2 w_2 + \psi w_1
\]
\[ z_1(t) = C e^{A(t-T_k)} \xi_3(k) + C e^{A(t-T_k)} \xi_1(T_k) + C \int_{T_k}^{t} e^{A(t-s)} B w_1(s) \, ds + D w_1(t) \]

\[ \xi_1(T(k+1)^-) = e^{A_T} \xi_3(k) + e^{A_T} \xi_1(T_k) + \int_{T_k}^{T(k+1)} e^{A(T(k+1)-s)} B w_1(s) \, ds \]

\[ \xi_1(T(k+1)^-) = e^{A_T} \xi_3(k) + \xi_1(T(k+1)) \]

\[ \xi_1(T(k+1)^-) = \xi_1(T(k+1)) - w_2(k+1) \]

for all \( k \in \mathbb{N}_0 \). With those filters at hand we now set

\[ \tilde{x} := \text{col}(\tilde{x}_2, \tilde{x}_1, x) \]

as well as \( \nu := \tilde{x}^* X(\theta) \tilde{x} \)

We have now everything prepared and can come to the main part of the proof. To this end, let \( k \in \mathbb{N}_0 \) as well as \( t \in [T_k, T(k+1)] \) be arbitrary. Then observe that by \( w_2(0) = 0, \tilde{x}(0) = 0 \) and by construction we have

\[ \dot{x}(t) = (A B) \begin{pmatrix} \tilde{x}(t) \\ w_1(t) \end{pmatrix} \]

\[ \tilde{x}(T^-) = (A_J B_J) \begin{pmatrix} \tilde{x}(T^-) \\ w_2(k) \end{pmatrix} \]

This yields in particular

\[ \nu(T(k+1)^-) - \nu(T_k) = \int_{T_k}^{T(k+1)} \nu(s) \, ds \]

\[ = \int_{T_k}^{T(k+1)} (\bullet)^* \begin{pmatrix} X(\theta(s)) & X(\theta(s))' \\ X(\theta(s)) & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{x}(s) \\ w_1(s) \end{pmatrix} \, ds \]

and

\[ \nu(T_k) - \nu(T^-) = (\bullet)^* X(0) \tilde{x}(T_k) - (\bullet)^* X(T) \tilde{x}(T^-) \]

\[ = (\bullet)^* \begin{pmatrix} X(0) & 0 \\ 0 & -X(T) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{x}(T_k) \\ w_2(k) \end{pmatrix} \, ds \]

Combining both and using (10) yields

\[ \nu(T(k+1)^-) - \nu(T^-) \]

\[ = \int_{T_k}^{T(k+1)} (\bullet)^* \begin{pmatrix} X_f(\theta(s)) & X_f(\theta(s))' \\ 0 & X_f(\theta(s))' \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{x}(s) \\ w_1(s) \end{pmatrix} \, ds \]

\[ \leq - \int_{T_k}^{T(k+1)} (\bullet)^* P (C D C_J D_J) \begin{pmatrix} \tilde{x}(s) \\ w_1(s) \\ \tilde{z}_1(s) \\ \tilde{z}_2(s) \end{pmatrix} \begin{pmatrix} \tilde{x}(s) \\ w_1(s) \\ \tilde{z}_1(s) \\ \tilde{z}_2(s) \end{pmatrix} \, ds \]

\[ \leq - \frac{\varepsilon}{T} \| w_2(k) \|^2 \, ds \]

\[ \leq - \int_{T_k}^{T(k+1)} (\bullet)^* P \begin{pmatrix} \tilde{z}_2(s) \\ \tilde{z}_1(s) \\ \tilde{z}_1(s) \\ \tilde{z}_2(s) \end{pmatrix} \begin{pmatrix} \tilde{z}_2(s) \\ \tilde{z}_1(s) \\ \tilde{z}_1(s) \\ \tilde{z}_2(s) \end{pmatrix} \, ds \]

Due to stability of \( A_M, A \) and \( w \in L_2 \times L_0 \) as well as \( \tilde{x}(0) = 0 \) we can then infer by summation