Determination of the order of stochastically linear dynamic systems by using non-parametric estimation of a regression function

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Abstract. In order to determine the accurate order of the model of discrete linear dynamic system, nonparametric algorithm is investigated. The algorithm is based on the application of the rule of selection of significant variables depending on a bandwidth. Selected variables are included in the Nadaraya–Watson regression estimation. The investigated dynamic system was simulated under diverse working conditions. The accuracy of the models estimated with the non-parametric methods is investigated. Computational experiments showed that using the non-parametric algorithms, the model orders can be determined. Because of the large number of cases investigated, the probability of the general validity of this nonparametric algorithm is high.

1. Introduction

The development of algorithm for the identification of linear dynamic systems is a classic problem in the field of mathematical modeling. Experience shows that often the state of the system is known only at discrete points in time. This paper considers the one-dimensional stochastically dynamic system described by a discrete difference equation:

\[ x_t = F(x_{t-1}, \ldots, x_{t-k}, u_t, \xi_t). \] (1)

where F is an unknown functional, \( x_t \) is the output of the system, \( u_t \) is the input of the system, \( \xi_t \) is a noise with zero mean and finite variance, \( t \) is discrete time, \( k \) is the order of the difference equation. The value of \( k \) is limited \( k \leq k_{\text{max}} \), sample \( \{u_i, x_i\}, i = 1, \ldots, s \), where \( s \) is the sample size, the \( u_i, x_i \) are measurements of the input and output of the system at discrete points in time \( t_i \). It is the purpose of the present paper to provide the algorithm of estimation of the order \( k \).

In most papers on parameter estimation the order of the system under study is assumed to be known a priori. Most of the time this information is unknown and has to be determined as part of the identification what described as structural identification. The methods of structural identification can be divided into two groups. The first group methods are based on choosing a subspace of regressors of a linear model of the classical [1–3]. The second group methods are based on choosing the order of a model of a difference linear equation [4–7]. To limit the possible model orders \( k_{\text{max}} \), the following parameter testing methods can be used: Normality test, Determinant ratio test. To determine the most accurate order \( k \) the can be used the Statistical F-test, Independence test and Signal error test. The
overwhelming majority of structural identification methods of determination the parametric structure of a linear dynamic system model is based on enumerated methods, that is necessitates a laborious and lengthy process from a computational point of view. Especially it concerns spaces of large dimensions. It is the purpose of the present paper to development new method for determining the structure of a linear dynamic system, based on the use of non-parametric estimates. [8–10].

2. Problem set-up and algorithm
As previously mentioned, considers the dynamic system described by a discrete difference equation (1). The measurements shall be taken at equal intervals $\Delta t$.

Let’s introduced the following notation

$$z_t = \left( z_1, z_2, \ldots, z_{k+1} \right) = \left( x_{t-k}, \ldots, x_{t-1}, u_t \right). \quad (2)$$

then

$$x_t = f \left( z_t \right). \quad (3)$$

Taking into account the notation (2), the system model (1) can be shown in the following diagram (figure 1). The figure 1 illustrates the model of a dynamic system in discrete time, where the variables $x_{t-1}, \ldots, x_{t-k}$ is the input of the system.

![Figure 1. Dynamic system modeling scheme.](image)

In figure 1 DE is a delay element.

Nonparametric model of a dynamic system is described by the equation [8]:

$$x'_s = - \frac{1}{s} \sum_{i=1}^{s} x_j \cdot H \left( \frac{u_j - u_i}{c_s^u} \right) \prod_{j=1}^{k} \frac{1}{c_s^j} \cdot H \left( \frac{x_{t-j} - x_{t-j}}{c_s^{x/j}} \right), \quad (4)$$

where $H(\cdot)$ is a kernel function and $c_s$ is a bandwidth, satisfying the assumptions [11]:
\[ c_s > 0; \lim_{s \to \infty} c_s = 0; \]
\[ H(c_s^{-1}(t - t_i)) < \infty; \]
\[ c_s^{-1} \int_{\Omega(u)} H(c_s^{-1}(t - t_i)) dx = 1; \]
\[ \lim_{s \to \infty} sc_s = \infty; \]
\[ \lim_{s \to \infty} c_s^{-1} H(c_s^{-1}(t - t_i)) = \delta(t - t_i), \]
\[ \int_{\Omega(u)} H'(u) du = 0 \quad c_s^{-1} \int_{\Omega(u)} uH'(u) du = -1, \]

where \( u \) is an arbitrary variable.

It should be noted that the model (4) can be used only at equal intervals of measurement \( \Delta t \).

The optimal bandwidths \( c_s^{x_1}, c_s^{x_2}, ..., c_s^{x_k} \) are found by minimizing a quadratic error function (6) by using the sliding exam method

\[ R(c_s^{x_1}, c_s^{x_2}, ..., c_s^{x_k}) = \sum_{q=1}^{3} \left( x_s^q - x_t^q \right)^2 = \min_{c_s^{x_1},...,c_s^{x_k}} \sum_{q=1}^{3} x_s^q, q \neq i, \]

where \( i \) is the index in the equation (4).

The Nelder-Mead simplex method for minimization of function (5) is used. This method for multidimensional unconstrained optimization without derivatives is effective to solve minimization problem, where calculation of the minimized function is low speed and the function values are subject to noise. The range of possible values of bandwidths for selecting the initial vertices of a deformable polyhedron is given in the range \( c_s \in [0.01, 4] \). From this region, \( n + k + 1 \) points are arbitrarily chosen, where \( n \) is the number of input variables, \( k \) is the order of the difference equation that form the simplex \( n + k \)-dimensional space.

In the estimate (4) each variable \( x_{s-1}, ..., x_{s-k} \) has a coefficient \( c_s^{x_1}, ..., c_s^{x_k} \). The significance of the output variable \( x_{s-1}, ..., x_{s-k} \) from the right side of the equations (4) to the final value of the estimate depends on

\[ \frac{1}{c_s^{x_j}} H \left( \frac{x_{s-j} - x_{t-j}}{c_s^{x_j}} \right). \]

We can construct the following chain of inequalities:

\[ \text{if } c_s^{x_1} < c_s^{x_2} < ... < c_s^{x_k}, \text{ then } \frac{1}{c_s^{x_1}} > \frac{1}{c_s^{x_2}} > ... > \frac{1}{c_s^{x_k}}. \]

Let’s consider the second component of equation (7). The coefficients \( c_s^{x_1}, ..., c_s^{x_k} \) and the nuclear function \( H(\cdot) \) must satisfy the following property:

\[ \frac{1}{c_s^w} \int_{\Omega(u)} H \left( \frac{w - w_i}{c_s^w} \right) dw = 1, \]
where $w$ is some variable. Based on the condition (9), given that $c_s^{1} = c_s^{2} = ... = c_s^{k}$, we can construct the following sequence of inequalities:

$$
\text{if } c_s^{1} < c_s^{2} < ... < c_s^{k} \text{ then } H\left(\frac{x_{s-1} - x_{i-1}}{c_s^{1}}\right) < H\left(\frac{x_{s-2} - x_{i-k}}{c_s^{2}}\right) < ... < H\left(\frac{x_{t-k} - x_{t-k}}{c_s^{k}}\right).
$$

For example, let $c_s^{1}$ is the bandwidth for $x_{s-1}$, and $c_s^{2}$ for $x_{s-2}$. If $c_s^{1} < c_s^{2}$ then $H\left(\frac{x_{s-1} - x_{i-1}}{c_s^{1}}\right) < H\left(\frac{x_{s-2} - x_{i-k}}{c_s^{2}}\right)$, and the output variable $x_{s-1}$ has a greater influence on the output value $x_s$ than $x_{s-2}$. Graphically, this is expressed as follows (figure 2).

![Figure 2. The dependence of the value of nuclear function $H(\cdot)$ of the bandwidths $c_j$.](image)

In figure 2: $H_1 = H\left(\frac{x_{s-1} - x_{i-1}}{c_s^{1}}\right), H_2 = H\left(\frac{x_{s-2} - x_{i-k}}{c_s^{2}}\right)$.

The algorithm for calculating significant variables $x_{s-j}$ is based on the following scheme. First, the initial value of $k$ is given. The model is constructed by equation (4) and the relative error $W_0$ is calculated:

$$
W = \sqrt{\frac{1}{s} \sum_{i=1}^{s} (x_i - x'_i)^2} \sqrt{\frac{1}{s-1} \sum_{i=1}^{s} (m_i - x_i)^2}
$$

where $m_i$ is an expected value.

For each $i$ - th iteration, the following set of actions is performed:

- For each coefficient $c_s^{1}, ..., c_s^{k}$ the optimal value is found:
  $c_s^{1} = c_s^{2} = ... = c_s^{k} = c_s^k$

- The maximum of all the values obtained is found: $c_{\text{max}}^s$
• The model is constructed by the equation (4). The multiplier 
\[ H \left( \frac{x_{t-j} - x_{t-j}}{c^* s_j} \right) \]
is excluded, taking into account that \( j \) is a number for \( c_{max}^* s_j \).

• A relative error \( W_i \) is calculated.

These actions will be repeated until \( W_i \geq W_{i-1} \).

3. Results

The non-parametric method described before is applied in this paper on a simulated system of third order \((k=3)\) described by a discrete difference equation:

\[ x_t = 0.5 \cdot x_{t-1} + 1.3x_{t-2} - 0.4x_{t-3} - 0.47x_{t-4} + 0.01u_t \]  
(10)

Set the sampling rate:

\[ h = \frac{T}{s} = 0.1, \]

where \( T \) is the system operation time, \( s \) is the sample size.

The additional noise signal \( \xi_i \) is normally distributed \( M(\xi) = 0, D(\xi) < \infty \), where \( D(\xi) \) is varied within the region of \( 0 < D(\xi) < 1 \) in very small steps. The output signal is calculated by:

\[ x_i = x(t_i) + c \cdot \xi_i, \]

where \( x(t_i) \) is the output of the system (disregarding the effect of noise). The constant \( c \) is determined the level (intensity) of the noise. The output of the system is calculated by:

\[ u_i = 0.4 + \sin(0.2 \cdot t_i). \]

Let us set the initial value \( k = 6 \). The simulation results using the equation (4) are presented in figure.

**Figure 3.** The simulation results.

a) the true value \( k \) is unknown, the maximum value \( k_{max} = 6 \) is taken when calculating the model

b) the true value \( k \) is found using the algorithm \((k = 3)\)
The relative error of the first experiment (figure 3a) is: \( W_0 = 0.19 \). The optimal bandwidths are: 
\[
\begin{align*}
 c_1^* &= 0.578 \cdot u_t, \\
 c_2^* &= 0.127 \cdot x_{t-1}, \\
 c_3^* &= 0.192 \cdot x_{t-2}, \\
 c_4^* &= 0.397 \cdot x_{t-3}, \\
 c_5^* &= 1.154 \cdot x_{t-4}, \\
 c_6^* &= 1.789 \cdot x_{t-5}, \\
 c_7^* &= 1.991 \cdot x_{t-6}.
\end{align*}
\]
After completing all steps of the algorithm, the following variables were excluded: \( x_{t-4}, x_{t-5}, x_{t-6} \). Thus, the structure of the dynamic system model is equal to:
\[
 x_0 = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \alpha_3 x_{t-3} + \beta u_t,
\]
what corresponds to the structure of the difference equation of the system (10).

The simulation results using the non-parametric algorithm are shown in figure 3 b). The found order is \( k = 3 \). The relative error of the second experiment is \( W_3 = 0.038 \). Thus, it can be concluded that the use of the algorithm allows to improve the quality of the non-parametric model by determining \( k \), if the information about the order of the difference equation is unknown from a priori information.

Comparison between known methods for determining model order of a linear dynamic system, a non-parametric algorithm and a brute force algorithm has been made. Studies were conducted under noise \( \xi \) conditions. The order \( k \) was determined from a range of 1-10. The results of the comparison are presented in table 1.

| Table 1. Comparison of the efficiency of the algorithms for determining \( k \). |
|-----------------|-----------------|-----------------|
|                 | Allowable noise, % | Calculation time, ms | The ability to determine \( k \) in normal operation |
| Statistical F-test | 13              | 6864             | + |
| Independence test       | 9               | 4988             | + |
| Signal error test        | 8               | 4279             | - |
| Brute force algorithm    | 14              | 3799             | + |
| Non-parametric algorithm | 15              | 2376             | + |

As can be seen, the majority of cases, the non-parametric algorithm is more resistant to noise, and allows to determine the model order for a smaller calculation time, in conditions of normal operation of the dynamic system.

The algorithm has been tested on several equations, at different levels of the noise and the relative errors demonstrates the effectiveness of the algorithm. Some results are presented in table 2.

| Table 2. Identification of the stochastically linear dynamic systems. |
|-----------------|-----------------|-----------------|
|                  | Noise, %        | \( W \)          |
| Dynamic system  | 3%              | 7%              | 15%            |
| \( x_t = 0.9 \cdot x_{t-1} + 0.1 u_t \) | 0.044 0.049 0.08 | |
| \( x_t = 0.45 \cdot x_{t-1} - 0.89 x_{t-2} + 0.015 u_t \) | 0.051 0.067 0.081 | |
| \( x_t = 0.15 \cdot x_{t-1} + 0.27 x_{t-2} + 0.1 u_t \) | 0.049 0.121 0.133 | |
| \( x_t = 0.21 \cdot x_{t-1} + 0.54 x_{t-2} - 0.37 x_{t-3} + 0.2 u_t \) | 0.069 0.122 0.144 | |
| \( x_t = 2.69 \cdot x_{t-1} - 2.46 x_{t-2} + 0.73 x_{t-3} + 0.015 u_t \) | 0.074 0.156 0.213 | |
| \( x_t = 0.5 \cdot x_{t-1} + 1.3 x_{t-2} - 0.4 x_{t-3} - 0.47 x_{t-4} + 0.01 u_t \) | 0.11 0.145 0.18 | |
| \( x_t = 0.9 \cdot x_{t-1} + 0.1 u_t \) | 0.13 0.168 0.21 | |
Table 2 shows the several experiments, where the dynamic systems were described by discrete difference equations (first column). Overall the results indicate that the use of a non-parametric algorithm is effective for solving the problem of determining a discrete linear mathematical model of a stochastically dynamic system from the measured input and output signals. The non-parametric method gives similar good results for a system with noise (with one exception, the relative errors \( W<20\% \)).

**Reference**

[1] Boks D and Dzhenkins G 1974 *Time Series Analysis, Forecast and Management* (Moskow: Mir)

[2] Ivakhnenko A and Muller I 1984 *Self-organization of predictive models* (Kiev: Technica)

[3] Strizhov V and Krymova E 2010 *Methods for choosing regression models* (Moskow: VTS RAN)

[4] Karabutov N 2007 Observed informational portraits and the structural identification task of *SICPRO* 7 89-115

[5] Woodside C 1971 Estimation of the order of linear systems *Automatica* 7 727-33

[6] Unbehauen H and Gohring B 1974 Tests for determining model order in parameter estimation *Automatica* 10 233-44

[7] Van den Boom A and Van der Endenden A 1974 The determination of the orders of process and noise dynamics *Automatica* 10 245-56

[8] Medvedev A 2010 Theory of nonparametric systems *Modeling Bulletin of SibSAU* 30 4-9

[9] Medvedev A 2010 Theory of nonparametric systems. Processes *Bulletin of SibSAU* 29 4-9

[10] Medvedev A 2015 *Fundamentals of the theory of adaptive systems* (Krasnoyarsk: SibSAU)

[11] Nadaraya E 1983 *Nonparametric estimation of probability density and regression curve* (Tbilisi: University Press)