Axially Symmetric Cosmological Models with Perfect Fluid and Cosmological Constant

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Abstract

Following recent considerations of a non-zero value for the vacuum energy density and the realization that a simple Kantowski-Sachs model might fit the classical tests of cosmology, we study the qualitative behavior of three anisotropic and homogeneous models: Kantowski-Sachs, Bianchi I and Bianchi III universes, with dust and cosmological constant, in order to find out which are physically permitted.

In fact, these models undergo isotropisation, except for the Kantowski-Sachs model \( (\Omega_k > 0) \) with \( \Omega_\Lambda < \Omega_{\Lambda M} \) and for the Bianchi III \( (\Omega_k < 0) \) with \( \Omega_\Lambda < \Omega_{\Lambda M} \), and the observations will not be able to distinguish between these models and the standard model.

If we impose that the Universe should be very much isotropic since the last scattering epoch \( (z \approx 1000) \), meaning that the Universe should have approximately the same Hubble parameter in all directions, we are led to a matter density parameter very close to the unity at the present time.

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Lately, the issue of whether or not there is a non-zero value for the vacuum energy density, or cosmological constant, has been raised quite often. Even taking the Hubble constant to be in the range 60-75 km/s/Mpc it is possible to show [1] that the standard model of flat space with vanishing cosmological constant is ruled out. On the other hand, if the classical tests of cosmology are applied to a simple Kantowski-Sachs metric and the results compared with those obtained for the standard model, the observations will not be able to distinguish between these models if the Hubble parameters along the orthogonal directions are assumed to be approximately equal [2]. Indeed, as Collins and Hawking [3] point out, the number of cosmological solutions which demonstrate exact isotropy well after the big bang origin of the Universe is a small fraction of the set of allowable solutions of the cosmological equations. It is therefore prudent to take seriously the possibility that the Universe is expanding anisotropically. Note also that in [4] it is shown that some shear free anisotropic models display a FLRW-like behaviour. Taking all this into consideration, we discuss the behavior of some homogeneous but anisotropic models with axial symmetry, filled with a pressureless perfect fluid (dust) and a non vanishing cosmological constant. For this, we will restrict our study to the the metric forms

\[ ds^2 = -c^2 dt^2 + a^2(t) dr^2 + b^2(t) \left( \frac{dv^2}{1 - kv^2} + v^2 d\phi^2 \right), \]  

(1)

with the two scale factors \( a(t) \) and \( b(t) \); \( k \) is the curvature index of the 2-dimensional surface \( \frac{dv^2}{1 - kv^2} + v^2 d\phi^2 \) and can take the values +1, 0, −1, giving the following three different metrics: Kantowski-Sachs, Bianchi I, and Bianchi III, respectively.

Einstein equations for the metric (1), for which the matter content is in the form of a perfect fluid and a cosmological term, \( \Lambda \), are then as follows:

\[ 2 \frac{\dot{a}}{a} \frac{\dot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{kc^2}{b^2} = 8\pi G \rho + \Lambda c^2, \]  

(2)

\[ 2 \frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{kc^2}{b^2} = -8\pi G p c^2 + \Lambda c^2, \]  

(3)

\[ \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a} \dot{b}}{a b} = -8\pi G p c^2 + \Lambda c^2, \]  

(4)

where \( \rho \) is the matter density and \( p \) is the (isotropic) pressure of the fluid. Here \( G \) is the Newton’s gravitational constant and \( c \) is the speed of light. If we consider a vanishing pressure (\( p = 0 \)), which is compatible with the present conditions for the Universe, the last two equations take the form

\[ 2 \frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{kc^2}{b^2} = \Lambda c^2, \]  

(5)

\[ \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a} \dot{b}}{a b} = \Lambda c^2, \]  

(6)

and Eq.(5) can easily be integrated to give

\[ \frac{\dot{b}^2}{b^2} = \frac{M_1}{b^3} - \frac{kc^2}{b^2} + \frac{\Lambda c^2}{3}. \]  

(7)

where \( M_1 \) is a constant of integration.

The Hubble constants corresponding to the scales \( a(t) \) and \( b(t) \) are defined by
\[ H_a \equiv \dot{a}/a \quad \text{and} \quad H_b \equiv \dot{b}/b. \]

Using them to introduce the following dimensionless parameters, in analogy with which it is usually done in the Friedmann-Lemaître-Robertson-Walker (FLRW) universes, let us define

\[ \frac{M_1}{b^3 H_b^2} \equiv \Omega_M, \quad (8) \]

\[ \frac{kc^2}{b^2 H_b^2} \equiv \Omega_k, \quad (9) \]

and

\[ \frac{\Lambda c^2}{3H_b^2} \equiv \Omega_\Lambda. \quad (10) \]

The conservation equation (7) can now be rewritten as

\[ \Omega_M - \Omega_k + \Omega_\Lambda = 1. \quad (11) \]

Now defining the dimensionless variable \( y = b/b_0 \) where \( b_0 = b(t_0) \), is the angular scale factor for the present age of the Universe, and using Eq.(11) (taken for \( t = t_0 \)), one may rewrite Eq.(7) as

\[ \dot{y} = \pm H_b b \left[ 1 + \Omega_M \left( \frac{1}{y} - 1 \right) + \Omega_\Lambda \left( y^2 - 1 \right) \right], \quad (12) \]

where the density parameters \( \Omega \) and \( H_b \) with the zero subscript, denote as before these quantities at the present time \( t_0 \). In this way, the number of independent parameters have been reduced. Substituting Eq.(7) into Eq.(2) gives

\[ \dot{a} = \frac{M_\rho - M_1 a + \frac{2}{3} \Lambda c^2 a b^2}{2\sqrt{M_1 b - kc^2 b^2 + \frac{4}{3} c^2 b^4}}, \quad (13) \]

where \( M_\rho \) is a constant proportional to the matter in the Universe,

\[ M_\rho = 8\pi G \rho b^2. \quad (14) \]

Using the procedure above, Eq.(13) can be rewritten in the form

\[ \Omega_\rho - \Omega_M + 2 \Omega_\Lambda = 2 \frac{H_a}{H_b}, \quad (15) \]

where

\[ \Omega_\rho = \frac{M_\rho}{ab^2 H_b^2}. \quad (16) \]

\( \Omega \) is defined as a matter density parameter

\[ \Omega = \frac{8\pi G \rho}{2H_a H_b + H_b^2}, \quad (17) \]

just like in FLRW models, and such that \( \Omega = 1 \) when \( k = 0 \) and \( \Lambda = 0 \), and which is related to \( \Omega_\rho \) by

\[ \Omega = \frac{\Omega_\rho}{1 + 2 \frac{H_a}{H_b}}. \quad (18) \]
Although $\Omega_M$ is not the matter density parameter, it performs the same important role. We emphasize the fact that if for one particular time $H_a = H_b$ and $\Omega_\Lambda = 1$, then, by Eqs.(11), (15) and (18), $3\Omega = \Omega_\rho = \Omega_M = \Omega_k$; and if $0 < \Omega_\Lambda \ll 1$ and $\Omega_M = 1$, then, $\Omega_k = \Omega_\Lambda$ and $\Omega \simeq 1$.

Introducing another dimensionless variable $x = a/a_0$, Eq.(13) takes the form

$$
\dot{x} = H_{b0} \frac{\Omega_{M0}(1 - \frac{x}{y}) + \Omega_{\Lambda0}(-1 + xy^2) + H_{a0}}{y/\sqrt{\Omega_{M0}(\frac{x}{y} - 1)} + \Omega_{\Lambda0}(y^2 - 1) + 1},
$$

(19)

and its number of independent parameters was also reduced, now at the expense of Eq.(15) taken for the present time $t = t_0$.

The behavior of $y(t)$ may be carried out looking for the $y$ values where $\dot{y} = 0$. This analysis was made by Mariano Moles [5] for FLRW models, in great detail.

There are two $\Omega_\Lambda$ values which characterize two zones of distinct behavior of scale factor $b$. Starting with condition $\dot{y} = 0$ one may obtain

$$
\Omega_{\Lambda0} = \frac{\Omega_{M0}(y - 1) - y}{y^3 - y}
$$

(20)

If we consider $\Omega_{\Lambda0} = \Omega_{\Lambda0}(y)$, as a function of $y$, then this function presents a relative maximum and a minimum, that we will denote by $\Omega_{\Lambda c}$ and $\Omega_{\Lambda M}$, respectively. The relative maximum depends on $\Omega_{M0}$ in the following way: For $\Omega_{M0} < 1/2$ we have

$$
\Omega_{\Lambda c} = \frac{3\Omega_{M0}}{2} \left\{ \left[ \frac{(\Omega_{M0} - 1)^2}{\Omega_{M0}^2 - 1 + \frac{1 - \Omega_{M0}}{\Omega_{M0}}} \right]^{1/3} + \frac{1}{\sqrt{(\Omega_{M0} - 1)^2/\Omega_{M0}^2 - 1 + (1 - \Omega_{M0})/\Omega_{M0}}} \right\} - (\Omega_{M0} - 1),
$$

(21)

for $\Omega_{M0} \geq 1/2$ the expression is

$$
\Omega_{\Lambda c} = -3\Omega_{M0} \cos \left( \frac{\theta + 2\pi}{3} \right) - (\Omega_{M0} - 1).
$$

(22)

The relative minimum is done by

$$
\Omega_{\Lambda M} = -3\Omega_{M0} \cos \left( \frac{\theta + 4\pi}{3} \right) - (\Omega_{M0} - 1),
$$

(23)

where $\theta = \cos^{-1} \left( \frac{\Omega_{M0} - 1}{\Omega_{M0}} \right)$. These expressions are limiting zones of the $(\Omega_{\Lambda0}, \Omega_{M0})$ plane, where $\dot{y} = 0$ has three or one solutions (for details see [5, Moles]). The $\Omega_{\Lambda M}$ expression is also defined for $\Omega_{M0} > 1/2$, but it has the meaning of a maximum only for $\Omega_{M0} > 1$. The $\Omega_{\Lambda0}$ less or equal to $\Omega_{\Lambda M}$ imposes the recollapse of scale factor $b$, while greater values produces inflexional behaviors for $b$. The $\Omega_{\Lambda0}$ values greater or equal to $\Omega_{\Lambda c}$ are physically “forbidden” because they don’t reproduce the present Universe (see [5]). Obviously, $\Omega_{\Lambda M} < \Omega_{\Lambda c}$ always.
Although we are considering anisotropic models, the Eq.(12) is exactly the same as the one obtained by \[5\] for the homogeneous and isotropic FLRW models. From Eq.(12) and Eq.(19) one obtains the differential equation

$$\frac{dx}{dy} = \frac{\Omega_M a_0 (1 - \frac{x}{y}) + \Omega_0 (-1 + xy^2)}{\Omega_M (1 - y) + \Omega_0 (y^3 - y) + y},$$

(24)

This equation has to comply with the conditions imposed by Eq.(11) and Eq.(15) evaluated at $t_0$. There are some particular values of the parameter for which this equation has exact solutions. However, for the majority of the values of the parameter, the solution has only been obtained by numerical integration.

Although we are dealing with anisotropic models, we may admit that at a certain moment of time, which we can take as the present time $t_0$, the Hubble parameters along the orthogonal directions may be assumed to be approximately equal, $H_a \simeq H_b$. This hypothesis has been considered in [2, Henriques] for the case of a Kantowski-Sachs (KS) model. From this study was derived the conclusion that the classical tests of cosmology are not at present sufficiently accurate to distinguish between a FLRW model and the KS defined in that paper, with $(H_{a0} \simeq H_{b0})$, except for small values of $b_0$.

The Eq.(24), with $H_{a0} = H_{b0}$, has three distinct possible integrations, one for each $k$ value, (see Figure 1). There are several density values for which the corresponding curves to $k = \pm 1$ models have vertical asymptotes on the right of vertical axis, limiting $y$ values below. So, we see that for the KS model, the scale factor $a(t)$ starts from infinity if $b(t)$ starts from zero. For the Bianchi I model, the scale factors are always proportional or even equal. In this situation we don’t have an anisotropic model; in fact, we can easily prove that this model is isotropic by a properly reparametrization of the coordinates. For the Bianchi III model, the scale factor $b(t)$ never starts from zero, but has an initial value different from zero when $a$ is null. The following plot shows the zones where each model acts (Figure 2).

Taking into account the analysis given in [3], we may get the behavior of $y(t)$, since this dimensionless parameter obeys the same differential equation for these models and for the standard model. Now, going back to Figure 1, one can then determine the $x(t)$ behavior. The plotting below summarizes the several possibilities for the three models: Kantowski-Sachs, Bianchi I and Bianchi III models, respectively.

Figure 1: Scale factors relation, that is, the $x$ $y$ dependence for the three models.
Figure 2: The Kantowski-Sachs model corresponds to the region above the straight line; the Bianchi III model corresponds the region below the straight line; the straight line represents the set of region for the Bianchi I model.

Figure 3: On the left the scale factor for the Kantowski-Sachs model ($\Omega_{k_0} > 0$) when $\Omega_{\Lambda_0} < \Omega_{\Lambda M}$. On the right the scale factor for the Kantowski-Sachs model ($\Omega_{k_0} > 0$) when $\Omega_{\Lambda M} < \Omega_{\Lambda_0} < \Omega_{\Lambda c}$.

The present technology allows us to “see” the epoch of last scattering of radiation at a redshift of about 1000, i.e., we can actually observe the most distant information that the Universe provides. Thus, we observe a great isotropy from the Cosmic Microwave Background Radiation (CMBR) because this radiation possesses a near-perfect black body spectrum \[6\]. The high level of isotropy from this epoch to our days imposes that the two Hubble factors $H_a$ and $H_b$ must remain approximately equals from this epoch to the present. In other words, we must impose a high isotropy level from the last scattering onwards, in our expressions, i.e.,

$$\frac{\Delta H}{H_a} = \frac{H_a - H_b}{H_a},$$

such that $|\frac{\Delta H}{H_a}| \ll 1$. We performed several numerical simulations and concluded that the sum $\Omega_{M_0} + \Omega_{\Lambda_0}$ must be close to the unity from above for Kantowski-Sachs and from below
Figure 4: On the left the scale factor for the Bianchi I model ($\Omega_{k0} = 0$) when $\Omega_{\Lambda 0} < 0$. On the right the scale factor for Bianchi I model ($\Omega_{k0} = 0$) when $\Omega_{\Lambda 0} \geq 0$.

Figure 5: On the left the scale factor for the Bianchi III model ($\Omega_{k0} < 0$) when $\Omega_{\Lambda 0} < \Omega_{\Lambda M}$. On the right the scale factor for the Bianchi III model ($\Omega_{k0} < 0$) when $\Omega_{\Lambda M} < \Omega_{\Lambda 0} < \Omega_{\Lambda c}$.

for Bianchi III models. We summarize in the table below the $\Delta H/H_a$ at $z = 1000$, for Kantowski-Sachs and Bianchi III models, if we suppose $H_{a0} = H_{b0}$.

|       | $\Omega_{M0}$ | $\Omega_{\Lambda 0}$ | $\Omega_0$ | $\Omega_{k0}$ | $\frac{\Delta H}{H_a}$ (%) |
|-------|----------------|----------------------|-------------|---------------|-----------------------------|
| K-S   | $10^{-15}$     | $1$                  | $1 - 4.974 \times 10^{-8}$ | $+7.46 \times 10^{-8}$ | $-0.4 \times 10^{-4}$       |
| B III | $1 - 10^{-8}$  | $9.9 \times 10^{-9}$ | $1 - 9.93 \times 10^{-9}$ | $-10^{-15}$ | $+9.4 \times 10^{-2}$      |
|       | $9.99 \times 10^{-13}$ | $1 - 10^{-12}$ | $9.997 \times 10^{-13}$ | $-10^{-15}$ | $+1.0 \times 10^{-4}$      |

We excluded the situations in that $\Omega_{\Lambda 0}$ is close to unity, because they are not compatible with the $\Omega_0$ values observed today. The two $\Omega_0$ values $1 - 6.67 \times 10^{-8}$ and $1 - 9.93 \times 10^{-8}$ for Kantowski-Sachs and Bianchi III models, respectively, were chosen so they would be both compatible with the estimated value for our Universe matter density and the restriction $\frac{\Delta H}{H_a} \ll 1$ on $z = 1000$. These two values for $\Omega_0$ are within the range used in the plots of right hand side figures (3) and (5), respectively.

\footnote{It is obvious that for Bianchi I model ($\Omega_{M0} + \Omega_{\Lambda 0} = 1$), with our restrictions, we have always $\Delta H/H_a = 0$.}
Conclusion

For the Kantowski-Sachs model, we conclude that if the scale factor $b(t)$ starts from zero, then the scale factor $a(t)$ will start from infinity and decreases afterwards. When $\Omega_0 < \Omega_{\Lambda M}$, $b(t)$ reaches the maximum value recollapsing after that. So, $a(t)$ will reach a relative maximum, when $b(t)$ is maximum, because $x(y)$ has a relative minimum for $y < 1$ (see Figure 1). After that, when $b(t) = 0$, $a(t)$ goes to infinity again. When $\Omega_{\Lambda M} < \Omega_0 < \Omega_{\Lambda c}$, the scale factor $b(t)$ grows indefinitely, giving place to an inflationary scenario. Then, $a(t)$ decreases reaching a minimum value, and growing after that indefinitely, and becoming proportional to $b(t)$. The initial singularity is of a “cigar” type.

For the Bianchi I model ($\Omega_{k_0} = 0$), the scale factors $a(t)$ and $b(t)$ are proportional or even equal. Thus, this model turns out to be an isotropic one. However, when $\Omega_0 < \Omega_{\Lambda M}$, $a(t)$ and $b(t)$ reach the maximum and recollapse after that. And when $\Omega_{\Lambda M} < \Omega_0 < \Omega_{\Lambda c}$, $a(t)$ and $b(t)$ grow indefinitely after an inflection.

For the Bianchi III model ($\Omega_{k_0} < 0$), when $\Omega_0 < \Omega_{\Lambda M}$, $b(t)$ starts from an initial non vanishing value ($b(t = 0) = b_0 > 0$), reaching a maximum and recollapsing after that until reaches the same value for $t = 0$. Also, $a(t)$ has a similar behavior, but starts from zero and recollapses to zero. When $\Omega_{\Lambda M} < \Omega_0 < \Omega_{\Lambda c}$, $b(t)$ starts again from a non vanishing value ($b_0 > 0$), growing indefinitely with an inflection. In this case, $a(t)$ starts from zero and grows indefinitely becoming proportional to $b(t)$. So, the initial singularity is of a “pancake” type.

In conclusion, these models undergo isotropisation, except for the Kantowski-Sachs model ($\Omega_{k_0} > 0$) with $\Omega_0 < \Omega_{\Lambda M}$ and for the Bianchi III ($\Omega_{k_0} < 0$) with $\Omega_0 < \Omega_{\Lambda M}$. If we impose that the Universe should be very much isotropic since the last scattering epoch ($z \approx 1000$), meaning that the Universe should have approximately the same Hubble parameter in all directions, we are led to a matter density parameter very close to the unity at the present time.

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