Bottom Quark Mass Determination from low-$n$ Sum Rules

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We study the uncertainties in the $\overline{\text{MS}}$ bottom quark mass determination using relativistic sum rules to $\mathcal{O}(\alpha_s^2)$. We include charm mass effects and secondary $b\bar{b}$ production and treat the experimental continuum region more conservatively than previous analyses. The PDG treatment of the region between the resonances $\Upsilon(4S)$ and $\Upsilon(5S)$ is reconsidered. Our final result reads: $\bar{m}_b(\bar{m}_b) = (4.20 \pm 0.09)$ GeV.

For the sake of reliable measurements at the current $B$-factory experiments, a precise knowledge of the bottom quark mass $m_b$ will be essential. In particular, the precision on the extraction of the Cabibbo–Kobayashi–Maskawa matrix elements $V_{cb}$ from the data will depend on the uncertainty on the bottom mass. For example, an error of 60 MeV in $m_b$ leads to a 3% uncertainty in $V_{cb}$ from the semileptonic partial width $\Gamma(B \to X_u \ell \nu)$.\textsuperscript{[1]}

In this talk we present the results of a detailed compilation of uncertainties in the $\overline{\text{MS}}$ bottom quark mass.\textsuperscript{[2]} Our analysis is more conservative than an earlier one in Ref. \textsuperscript{[3]} and includes a number of effects that were previously neglected. Our method consists of determining the $b$ mass by fitting the experimental moments of the $b\bar{b}$ cross section in $e^+e^-$ annihilation to their corresponding theoretical expressions. The moments are defined as follows:

$$P_n = \int \frac{ds}{s^{n+1}} R_{b\bar{b}}(s),$$

where $R_{b\bar{b}} = \sigma(e^+e^- \to b\bar{b} + X)/\sigma(e^+e^- \to \mu^+\mu^-)$. The virtual $Z$ contribution is strongly suppressed and neglected.

We use low-$n$ ("relativistic") moments, i.e. $n \leq 4$. Relativistic moments exhibit the nice feature that they are dominated by scales of order $m_b$ and that they can be computed in fixed-order perturbation theory. However, they have the disadvantage that they strongly depend on the badly known experimental continuum region. As for the $b$-mass definition, we adopt the $\overline{\text{MS}}$ mass, since it is an appropriate definition for processes where $b$ quarks are off-shell.

A compilation of the theoretical moments up to $\mathcal{O}(\alpha_s^2)$ was given in \textsuperscript{[3]}. In our work \textsuperscript{[2]} we also included the effects at $\mathcal{O}(\alpha_s^3)$ of the non-zero charm mass and of secondary $b\bar{b}$ production, with the $b\bar{b}$ pair coming from gluon radiation off light quarks. Referring, for simplicity, to $\bar{m}_b(\bar{m}_b)$ the theoretical moments have a simple form:

$$P_n = \frac{1}{(4\bar{m}_b(\bar{m}_b))^n} \left\{ f_n^0 + \left( \frac{\alpha_s(\mu)}{\pi} \right) f_n^{10} \right. \right.$$  

$$\left. + \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left( f_n^{20}(r) - \frac{1}{4} \beta_0 f_n^{10} \ln \left( \frac{\bar{m}_b^2(\bar{m}_b)}{\mu^2} \right) \right) \right.$$  

$$\left. + \frac{\left( \frac{G^2}{4\bar{m}_b(\bar{m}_b)} \right)^2}{(4\bar{m}_b(\bar{m}_b))^2} \left[ g_n^0 + \left( \frac{\alpha_s(\mu)}{\pi} \right) g_n^{10} \right] \right\}. \quad (2)$$

In Eq. (2), $r = m_c/m_b$, $\mu$ is the renormalization scale and we have included the contribution from the dimension four gluon condensate \textsuperscript{[4]}. The general expression of $P_n$ for $\bar{m}_b(\bar{m}_b)$ can be found in Refs. \textsuperscript{[2]}. Results for the coefficients $f_n$ are reported in Table \textsuperscript{[1]}.

We count out that the contribution of secondary $b\bar{b}$ production and charm mass affects only $f_n^{20}$. Such effects turn out to be small, as can be seen comparing the values for $f_n^{20}$ in Table \textsuperscript{[1]} with the ones of Ref. \textsuperscript{[3]}.

Table \textsuperscript{[2]} displays the impact of charm mass corrections in terms of $\Delta f_n = f_n^{20}(r) - f_n^{20}(0)$. The smallness of $c$-mass effects is however strongly related to the use of the $\overline{\text{MS}}$ mass definition which we have adopted. In fact, if we had chosen the pole scheme for the bottom mass, the inclusion of the charm mass would have had a much stronger impact, as shown in Table \textsuperscript{[3]}.

This can be understood from the fact that the finite charm mass represents an infrared cut-off in the loop integrations.
and that the pole-mass definition is much more sensitive to infrared momenta. To evaluate the experimental moments, we consider the region of the resonances $\Upsilon(1S)$–$\Upsilon(6S)$ and the continuum. We compute the moments of a generic resonance $k$ in the narrow width approximation, i.e.

$$\langle P_n \rangle_k = \frac{9\pi \Gamma_k^{e^+e^-}}{\alpha(10 \text{ GeV})m_k^{2n+1}} ,$$

(3)

where $\Gamma_k^{e^+e^-}$ is the partial $e^+e^-$ width for the $k$-th resonance. For the $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$ and $\Upsilon(6S)$ we use the averages for masses and widths quoted in the PDG [5]. For the region between the $\Upsilon(4S)$ and the $\Upsilon(5S)$, i.e. between 10.5 and 10.95 GeV, we do not use the PDG averages, which were based on results from CUSB [6] and CLEO [7] Collaborations. Both experiments observed an enhancement at about 10.7 GeV. While CUSB did not assign the enhancement to any resonance, CLEO fitted it to a $\eta B^*$ resonance with mass $m_{B^*} = 10.684 \pm 0.013$ GeV and $e^+e^-$ width $\Gamma_{B^*}^{e^+e^-} = 0.20 \pm 0.11$ keV. The PDG averages, on the other hand, ignore the $B^*$ results and, therefore, lead to a contribution to $P_n$ that is smaller than the original CUSB and CLEO data. In our analysis we took the averages from the original CUSB and CLEO data, assuming the larger uncertainties from CLEO (See Ref. [2] for more details).

As far as the continuum is concerned, we subdivide it into three parts: 11.1-12.0 GeV (region 1), where possible data may come from CLEO; 12 GeV - $M_Z$ (region 2) and above $M_Z$ (region 3). There is no direct experimental data in the region above 11.1 GeV. Nevertheless the measurements of $R_b$ by LEP I and LEP II agree with the perturbative QCD prediction within 1% at $M_Z$ and 10% in the region between 133 and 207 GeV explored by LEP II. It is therefore not unreasonable to rely on perturbative QCD to estimate the contribution to the experimental moments above the $\Upsilon(6S)$. In the analysis of Ref. [8] the small theoretical errors in the continuum region were inherently taken as the experimental uncertainties. Since this leads to an implicit model-dependence, we adopt a more transparent treatment and take an assigned fraction of the theoretical prediction as the experimental uncertainty of the continuum. In this way the impact of the unknown experimental continuum contribution can be traced more easily. The experimental moments are quoted in Table 1. The uncertain-

### Table 1

| $n$ | 1  | 2  | 3  | 4  |
|-----|----|----|----|----|
| $f_n^0$ | 0.2667 | 0.1143 | 0.0677 | 0.0462 |
| $f_n^{0*}$ | 0.6387 | 0.2774 | 0.1298 | 0.0508 |
| $f_n^1$ | 0.5333 | 0.4571 | 0.4063 | 0.3694 |
| $f_n^{1*}(0)$ | 0.9446 | 0.8113 | 0.5172 | 0.3052 |
| $f_n^2$ | 0.8606 | 1.2700 | 1.1450 | 0.8682 |
| $f_n^{2*}$ | 0.0222 | 0.4762 | 0.8296 | 1.1240 |

### Table 2

| $r$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|-----|-----|-----|-----|-----|-----|
| $\Delta f_1$ | -0.0021 | -0.0078 | -0.0164 | -0.0266 | -0.0382 |
| $\Delta f_2$ | -0.0098 | -0.0092 | -0.0187 | -0.0302 | -0.0430 |
| $\Delta f_3$ | -0.0024 | -0.0101 | -0.0204 | -0.0330 | -0.0466 |
| $\Delta f_4$ | -0.0030 | -0.0109 | -0.0219 | -0.0348 | -0.0491 |
we fit the ratio ing renormalization group equations (method 2); we determine \( \bar{m} \) single-moment fits and subsequently \( \bar{m} \) mods: we fit single moments and get directly normalization scale renormalization group equations, vary the renor-

As in Table 2, but in the pole-mass scheme.

| \( r \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|-------|-----|-----|-----|-----|-----|
| \( \Delta f_1 \) | 0.0809 | 0.1565 | 0.2113 | 0.2656 | 0.3145 |
| \( \Delta f_2 \) | 0.0684 | 0.1267 | 0.1765 | 0.2203 | 0.2593 |
| \( \Delta f_3 \) | 0.0608 | 0.1106 | 0.1531 | 0.1896 | 0.2221 |
| \( \Delta f_4 \) | 0.0545 | 0.0988 | 0.1358 | 0.1676 | 0.1764 |

Table 4
Individual contributions to the experimental moments including uncertainties. In the continuum the displayed uncertainties are the theoretical ones only.

| contribution | \( P_1 \) \( \times 10^3 \) GeV\(^2 \) | \( P_2 \) \( \times 10^5 \) GeV\(^4 \) | \( P_3 \) \( \times 10^7 \) GeV\(^6 \) | \( P_4 \) \( \times 10^9 \) GeV\(^8 \) |
|--------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| \( \Upsilon(1S) \) | 0.766(29) | 0.856(32) | 0.956(36) | 1.068(40) |
| \( \Upsilon(2S) \) | 0.254(16) | 0.252(16) | 0.251(15) | 0.250(15) |
| \( \Upsilon(3S) \) | 0.211(29) | 0.196(27) | 0.183(26) | 0.171(24) |
| \( [\Upsilon(4S) - \Upsilon(3S)] \) | 0.251(95) | 0.218(82) | 0.190(72) | 0.165(62) |
| \( \Upsilon(6S) \) | 0.048(11) | 0.039(9) | 0.032(7) | 0.027(6) |
| 11.1 GeV – 12.0 GeV | 0.418(57) | 0.314(44) | 0.236(34) | 0.178(27) |
| 12.0 GeV – \( M_Z \) | 2.467(26) | 0.886(21) | 0.414(13) | 0.217(8) |
| \( M_Z - \infty \) | 0.047(1) | 0.000(0) | 0.000(0) | 0.000(0) |

For the determination of \( \bar{m}_b(\bar{m}_b) \) and of the corresponding uncertainties, we use four methods: we fit single moments and get directly \( \bar{m}_b(\bar{m}_b) \) (method 1); we determine \( \bar{m}_b(\mu) \) from single-moment fits and subsequently \( \bar{m}_b(\bar{m}_b) \) using renormalization group equations (method 2); we fit the ratio \( P_n/P_{n+1} \) and get \( \bar{m}_b(\bar{m}_b) \) (method 3); we determine \( \bar{m}_b(\mu) \) by fitting \( P_n/P_{n+1} \) and compute \( \bar{m}_b(\bar{m}_b) \) using renormalization group equations (method 4). We employ four-loop renormalization group equations, vary the renor-

and the error corresponding to a 10% variation of the theoretical prediction. The latter error scales roughly linearly, i.e. assuming a 5% (20%) fraction decreases (increases) the error by a factor of two. In order to get the combined errors, in the resonance region we treat half of the errors as un-
correlated (added linearly) and half of the errors as correlated (added quadratically). The errors in the continuum do not have any statistical corre-
lation, hence we add them linearly. Moreover, we add linearly the errors coming from the resonance and from the continuum regions.

We note that the errors yielded by fits of the first two moments \( P_1 \) and \( P_2 \) are rather large. As for the results given by fits of the moment ratios, the fit of \( P_2/P_3 \) using method 3 yields a rather small error of about 50 MeV. However, this result holds only if the same value of \( \mu \) is chosen for both \( P_2 \) and \( P_3 \): a larger error would instead be found using independent values of \( \mu \) for the numerator and denominator of the ratio. Since we believe that \( P_3 \) can be calculated reliably using


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Table 5
Central values and uncertainties for $\bar{m}_b$ ($\bar{m}_b$).

| Method 1 (2) | Method 3 (4) |
|---|---|
| $n$ | 1 | 2 | 3 | 1 | 2 |
| central | 4210 (4214) | 4200 (4205) | 4197 (4200) | 4191 (4195) | 4191 (4191) |
| $\Upsilon(1S)$ | 14 (13) | 12 (12) | 11 (11) | 11 (11) | 9 (9) |
| $\Upsilon(2S)$ | 7 (7) | 6 (6) | 5 (5) | 4 (4) | 3 (3) |
| $\Upsilon(3S)$ | 14 (14) | 10 (10) | 8 (8) | 7 (7) | 3 (3) |
| $4S - 5S$ | 45 (44) | 32 (32) | 22 (22) | 18 (18) | 4 (4) |
| $\Upsilon(6S)$ | 5 (5) | 3 (3) | 2 (2) | 2 (2) | 0 (0) |
| combined | 67 (67) | 50 (50) | 38 (38) | 33 (33) | 15 (15) |
| region $1_{1\%}$ | 27 (27) | 17 (17) | 11 (11) | 7 (7) | 2 (2) |
| region $2_{1\%}$ | 12 (12) | 8 (8) | 4 (4) | 2 (2) | 0 (0) |
| region $3_{1\%}$ | 115 (114) | 33 (33) | 13 (13) | 49 (49) | 29 (29) |
| region $3_{10\%}$ | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) |
| $\delta m_c$ | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) |
| $\delta \alpha_s(M_Z)$ | 17 (18) | 10 (11) | 6 (6) | 3 (3) | 2 (2) |
| $\delta \mu$ | 23 (5) | 16 (14) | 11 (27) | 15 (27) | 3 (50) |
| total | 251 (233) | 127 (125) | 79 (95) | 110 (121) | 51 (99) |

both methods 1 and 2, we adopt the error on $P_3$ as our final estimate of the uncertainty in the MS bottom mass determination. Rounding to units of 10 MeV, we obtain:

$$\bar{m}_b = (4.20 \pm 0.09) \text{ GeV},$$

assuming a 10% error for the experimental continuum regions 2 and 3. Within the error range, our result is in agreement with the estimate of [3]. Our error is nonetheless larger than the 50 MeV of Ref. [3], which is due to the different treatment of the resonance region and to the more conservative choice for the experimental error in the continuum region.

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