Nilpotent elements control the structure of a module

David Ssevviiri
Department of Mathematics
Makerere University, P.O BOX 7062, Kampala Uganda
E-mail: ssevviiri@cns.mak.ac.ug, ssevviirid@yahoo.com

Abstract
A relationship between nilpotency and primeness in a module is investigated. Reduced modules are expressed as sums of prime modules. It is shown that presence of nilpotent module elements inhibits a module from possessing good structural properties. A general form is given of an example used in literature to distinguish: 1) completely prime modules from prime modules, 2) classical prime modules from classical completely prime modules, and 3) a module which satisfies the complete radical formula from one which is neither 2-primal nor satisfies the radical formula.

Keywords: Semisimple module; Reduced module; Nil module; Köthe conjecture; Completely prime module; Prime module; and Reduced ring.

MSC 2010 Mathematics Subject Classification: 16D70, 16D60, 16S90

1 Introduction
Primeness and nilpotency are closely related and well studied notions for rings. We give instances that highlight this relationship. In a commutative ring, the set of all nilpotent elements coincides with the intersection of all its prime ideals - henceforth called the prime radical. A popular class of rings, called 2-primal rings (first defined in [8] and later studied in [23, 26, 27, 28] among others), is defined by requiring that in a not necessarily commutative ring, the set of all nilpotent elements coincides with the prime radical. In an arbitrary ring, Levitzki showed that the set of all strongly nilpotent elements coincides with the intersection of all prime ideals, [29 Theorem 2.6]. The upper nil radical of a ring which is the sum of all its nil ideals is the intersection of all its strongly prime (s-prime for short) ideals, see [35 p. 173] and [36 Proposition 2.6.7]. Every completely prime ring has no nonzero nilpotent elements and every prime ring has no nonzero nilpotent ideals. A ring is semisimple (and hence a direct sum of prime rings) if and only if it is left (right) artinian and semiprime. A ring $R$ is semiprime (i.e., $R$ has no nonzero nilpotent ideals) if and only if $R$ is isomorphic to a subdirect sum of prime rings, [30 Theorem 4.27]. A unital ring $R$ is reduced (i.e., has no nonzero nilpotent elements) if and only if $R$ is a subdirect sum of domains (completely prime rings), [12 Example 3.8.16]. The upper nil
radical of a ring $R$ is zero if and only if $R$ is a subdirect sum of strongly prime rings. Every ring which is a left essential extension of a reduced ring is a subdirect sum of rings which are essential extensions of domains, [7]. A semiprime ring $R$ has index $\leq n$ if and only if $R$ is a subdirect sum of prime rings of index $\leq n$, [2].

There is some effort to understand a relationship between primeness and nilpotency in modules. Modules that satisfy the radical formula (see [3, 19, 24, 32, 33, 37] among others) were studied for this purpose. Behboodi in [5] and [6] defined the Baer lower nil radical for modules and sought its equivalence with the prime radical and the classical prime radical respectively. In [14] and its corrigendum, Groenewald and I, showed that in a uniserial module over a commutative ring, the set of all strongly nilpotent elements of a module coincides with the classical prime radical of that module. Part of the aim of this paper is to continue with this study that establishes a relationship between nilpotency and primeness in a module. We obtain structural theorems which relate reduced modules with prime modules, see Theorems 3.2, 3.3, 4.1 & 4.6. It has been shown that absence of nilpotent module elements allows a module to behave “nicely” by admitting certain structural properties, see paragraph after Question 6.1. The third major object of the paper is that, a general example has been formulated for which a particular case was used in literature to distinguish several kinds of phenomena as we elaborate later.

In Section 2, we give conditions under which a module is nil. A link between nil left ideals and nil submodules is established leading to a possibility of using modules to prove Köthe conjecture in the negative. Köthe conjecture which has existed since 1930 states that, the sum of two nil left ideals is nil, see [36, Theorem 2.6.35], [21, p. 174] and [38] for details about this conjecture.

In Section 3, we give equivalent formulations for a module to be torsion-free and then for it to be reduced. We define co-reduced modules and show that every reduced module over a commutative ring is co-reduced. It is proved that if $M$ is an injective module over a commutative noetherian ring $R$ such that its indecomposable submodules are prime $R$-modules, then $M$ is a reduced module which is isomorphic to a direct sum of prime modules. We introduce $R(M)$ the largest submodule of $M$ which is reduced as a module and show that for a $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$, $R(\mathbb{Q}/\mathbb{Z}) \cong \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ for all prime integers $p$. This provides a general framework through which an already known result can be deduced, i.e., a $\mathbb{Z}$-module $\mathbb{Z}/(p_1^{k_1} \times \cdots \times p_n^{k_n})\mathbb{Z}$ is reduced if and only if $k_1 = k_2 = \cdots = k_n = 1$. This in part answers Question 5.1 that was posed in [39].

Section 4 is devoted towards getting conditions under which a reduced module is semisimple and hence a direct sum of prime modules. We show that if $M$ is a reduced artinian $R$-module such that either $R$ or $R/\text{ann}_R(M)$ is embeddable in $M$, then every $R/\text{ann}_R(M)$-module is semisimple. As corollaries, we get: 1) a faithful artinian completely prime
module is semisimple; 2) an artinian, reduced and faithful free module is semisimple; 3) an artinian, reduced, faithful and finitely generated module over a commutative ring is semisimple; 4) a faithful holonomic module is reduced if and only if it is semisimple and rigid; 5) an artinian $R$-algebra $M$ is a reduced $R/\text{ann}_R(M)$-module if and only if $M$ is a semisimple rigid $R/\text{ann}_R(M)$-module; and 6) if $(R, \mathfrak{m})$ is a Gorenstein artinian local ring, such that the injective envelope $E_R(R/\mathfrak{m})$ of $R/\mathfrak{m}$ is a reduced $R$-module, then $E_R(R/\mathfrak{m})$ is a semisimple $R$-module. It is also proved that, if $R$ is a left artinian ring and the zero submodule of an $R$-module $M$ satisfies the complete radical formula, then whenever $M$ is reduced, it follows that it is also semisimple. As a consequence, a module over a commutative artinian ring is reduced if and only if it is semisimple. This retrieves a well known result: a commutative artinian ring is reduced if and only if it is semisimple.

In Section 5, examples are given to delimit and delineate the theory. For instance, we give what we call the Golden Example. It serves several purposes. It is an example of a prime nil module. It is a general form of the example which was used in [10] to distinguish a prime module from a completely prime module, used in [13] to distinguish a classical prime module from a classical completely prime module and an example used in [11] of a module which satisfies the complete radical formula but it is neither 2-primal nor satisfies the radical formula. It is an example which shows another advantage (in addition to those already known and given in [10]) that completely prime modules have over prime modules. If $M$ is a completely prime $R$-module, then $R/\text{ann}_R(M)$ is embeddable in $M$ and if $M$ is in addition artinian, then $M$ as an $R/\text{ann}_R(M)$-module is semisimple. It is the same example that we have used to show that it is impossible to write a 2-primal module defined in [16] as one for which the prime radical coincides with a submodule generated by the nilpotent elements. This answers in negative Question 8.2.1 posed in [12].

In Section 6, the last section, we give more effects of absence of nilpotent elements on the primeness of a module. In particular, it is shown that a prime module without nilpotent elements is $s$-prime, $l$-prime and completely prime. If $M$ is a faithful $R$-module without nilpotent elements, then $R$ has no nonzero nil ideals, has no locally nilpotent ideals and it has no nonzero nilpotent ideals.

A ring $R$ is reduced if it has no nonzero nilpotent elements. So, a ring with a nonzero nilpotent element is not reduced. This was the motivation in [14] for defining a nilpotent element of a module. Let $R$ be a ring. An $R$-module $M$ is reduced (see [1, 22, 34]) if for all $a \in R$ and $m \in M$, $a^2m = 0$ implies that $aRm = 0$. Reduced modules were called completely semiprime modules in [13]. As for rings, we say that an $R$-module is not reduced if it has a nonzero nilpotent element.

**Definition 1.1.** An element $m$ of an $R$-module $M$ is nilpotent if either $m = 0$ or there exists a positive integer $k$ and an element $a \in R$ such that $a^km = 0$ and $aRm \neq 0$. In this case, $a$ is called the nilpotentiser of $m$ and the least such $k$ is called the degree of
nilpotency of \( m \) with respect to \( a \).

**Definition 1.2.** An \( R \)-module \( M \) is nilpotent if for each \( 0 \neq m \in M \), there exists\(^1\) \( a(m) \in R \) and a fixed \( k \in \mathbb{Z}^+ \) independent of \( m \) such that \( aRm \neq 0 \) and \( a^k m = 0 \).

Throughout this paper, unless stated otherwise, \( R \) will denote a unital associative ring and \( M \) will be a unital left \( R \)-module. We write \( 1 \) for the unity of a ring \( R \). By \( R\text{-Mod} \), \( M_n(\mathbb{R}) \) and \( \text{End}_R(M) \) we shall respectively mean the category of all \( R \)-modules, the ring of all \( n \times n \) matrices over \( R \) and the endomorphism ring of \( M \) over \( R \). \( \text{ann}_R(M) \) (resp. \( \text{ann}_R(m) \)) will denote the ideal \( \{ r \in R : rM = 0 \} \) (resp. the left ideal \( \{ r \in R : rm = 0 \} \)) of \( R \). For an \( R \)-module \( M \) and \( r \in R \), \( (0 : M) \) will denote the set \( \{ m \in M : rm = 0 \} \) of \( M \).

2 Nil modules

**Definition 2.1.** A module is nil if all its elements are nilpotent. It is clear that a nilpotent module is nil. If the ring \( \text{End}_R(M) \) or \( R \) is nil\(^2\) and for all nonzero elements \( m \in M \), \( \text{ann}_R(m) \subsetneq R \); then \( M \) is nil. It is also easy to see that, if the ring \( \text{End}_R(M) \) or \( R \) is nilpotent and for all nonzero elements \( m \in M \), \( \text{ann}_R(m) \subsetneq R \); then \( M \) is nilpotent. Fischer [11, Theorem 1.5] gave a condition for the ring \( \text{End}_R(M) \) to be nilpotent.

A nil module need not be nilpotent. Let \( T_n \) be a ring of all \( n \times n \) matrices over a division ring where \( n \geq 2 \). Each module \( T_nT_n \) is nilpotent. Consider the direct sum \( A := \bigoplus_{n=2}^{\infty} T_n \) which has an ascending chain

\[
T_2 \subset T_2 \oplus T_3 \subset \cdots \subset \bigoplus_{n=2}^{k} T_n \subset \cdots
\]

of submodules such that \( A \) is the union \( A = \bigcup_{n=2}^{\infty} (\bigoplus_{n=2}^{k} T_n) \) of the members of the chain. \( A \) is nil \( A \)-module which is not nilpotent.

A left ideal \( I \) of a ring \( R \) is dense if given any \( 0 \neq r_1 \in R \) and \( r_2 \in R \), there exists \( r \in R \) such that \( rr_1 \neq 0 \) and \( rr_2 \in I \). An ideal \( I \) of a ring \( R \) is essential if for all nonzero ideals \( J \) of \( R \), \( I \cap J \neq 0 \). Every dense ideal is an essential ideal.

**Proposition 1.** If \( R \) is a ring with a left ideal which is both nil and dense, then the module \( _RR \) is nil.

**Proof:** Let \( J \) be a nil and dense left ideal of \( R \). Suppose \( 0 \neq m \in R \). \( 1 \in R \) and by definition of dense left ideals, there exists \( r \in R \) such that \( rm \neq 0 \) and \( r1 = r \in J \). So, \( rRm \neq 0 \). Since \( J \) is nil, \( r \) is nilpotent and hence \( r^k m = 0 \) for some \( k \in \mathbb{Z}^+ \).

---

\(^1\)\( a(m) \) means that \( a \) is dependent on \( m \).

\(^2\)Note that, a nil ring cannot be unital - this is an exception to the general rings used in this paper.
The envelope of a submodule $N$ of an $R$-module $M$ is the set

$$E_M(N) := \{rm : r^k m \in N, r \in R, m \in M, k \in \mathbb{Z}^+\}.$$ 

This set was used to define modules that satisfy the radical formula, see [3, 32, 33, 37] among others. In the context of modules that satisfy the radical formula, $E_M(0)$ was considered to be the module analogue of the set $N(R)$ of all nilpotent elements of a ring $R$. Since $E_M(0)$ is not a submodule, we write the submodule of $M$ generated by $E_M(0)$ as $\langle E_M(0) \rangle$.

**Proposition 2.** For any $R$-module $M$:

1. $\langle E_M(0) \rangle \subseteq \langle N(M) \rangle$;
2. if $E_M(0) = M$, then $N(M) = M$ and hence $M$ is nil;
3. if $m \in N(M)$, then there exists $r \in R$ such that $rm \in E_M(0)$.

**Proof:** If $0 \neq m \in E_M(0)$, then $m = rn$ and $r^k n = 0$ for some $r \in R$, $k \in \mathbb{Z}^+$ and $n \in M$. This shows that $n \in N(M)$. So, $m = rn \in \langle N(M) \rangle$ and $E_M(0) \subseteq \langle N(M) \rangle$. Hence, $\langle E_M(0) \rangle \subseteq \langle N(M) \rangle$. 2 follows immediately from 1 and 3 is trivial.

Let $R$ be a commutative ring. If every nonzero endomorphism $f_r$ of an $R$-module $M$ which is given by $f_r(m) = rm$ for some $r \in R$ and $m \in M$ is both nilpotent and surjective, then $M = E_M(0)$ and $M$ is nil. For if $f_r$ is both nilpotent and surjective, then for all $n \in M$, there exists $m \in M$ such that $n = rm$. Since $f_r$ is nilpotent, there exists a positive integer $k$ such that $r^k m = 0$. It follows that $n \in E_M(0)$ and hence $M = E_M(0)$. Nil modules obtained this way are secondary modules. A module $M$ is secondary [25] if $M \neq 0$ and for each $r \in R$, the endomorphism $m \mapsto rm$ of $M$ is either surjective or nilpotent.

On the other hand, it is impossible for a nilpotent endomorphism $f_r(m) = rm$ of $M$ to be injective and hence it cannot be an isomorphism. Suppose that $f_r(m) = rm \neq 0$ and $f_r$ is nilpotent, i.e., $r^k m = 0$ for some $k \in \mathbb{Z}^+$. Define $m_1 = rm$. Then $f_r(m_1) = rm_1 = r^2 m$. Take $m_2 = rm_1$, we get $f_r(m_2) = rm_2 = r^2 m_1 = r^3 m$, continuing this way, we get $f_r(m_{k-1}) = r^k m = 0$. If injective implies that $m_{k-2} = 0$ so that $f(m_{k-2}) = 0$. Continuing with this process leads to $m_1 = rm = 0$ which is a contradiction.

We here repeat Question 8.2.2 posed in [42].

**Question 2.1 (Module analogue of Köthe conjecture).** Is the sum of nil submodules nil?

The importance of Question 2.1 lies in the fact that it can be used to solve Köthe conjecture in the negative. For if $I_1$ and $I_2$ are nil left ideals of a ring $R$ and $M$ is an $R$-module,
then the submodules $I_1m$ and $I_2m$ of $M$ are also nil. If the sum $I_1m + I_2m = (I_1 + I_2)m$ is not a nil submodule, then it follows that the sum $I_1 + I_2$ is not a nil left ideal of $R$ and hence Köthe conjecture would be false.

It is tempting for one to think that may be hence Köthe conjecture would be false.

**Example 2.** Let $x = 3$. Reduced modules

**Example 3.** Let $I$ is not a nil submodule, then it follows that the sum $I_1 + I_2$ is not a nil left ideal of $R$ and hence Köthe conjecture would be false.

We get equality whenever $M$ is nil.

**Proof:** If $m_i \in \mathcal{N}(M)$, then there exists $a_i \in R$ and $k_i \in \mathbb{Z}^+$ such that $a_i^{k_i}m_i = 0$ and $a_iRm_i \neq 0$. This implies that $m_i \in (0 :_M a_i^{k_i})$ and hence $\mathcal{N}(M) \subseteq \sum_{i \in J \subseteq I} (0 :_M a_i^{k_i})$. To see why $i \in J \subseteq I$, if $m \in M$ has several nilpotentisers, then we take only one leading to $J \subseteq I$. If $M$ is nil, $\mathcal{N}(M) = M$ and it follows that $M = \sum_{i \in J \subseteq I} (0 :_M a_i^{k_i})$.

**Example 1.** Let $R := \mathbb{F}_2[x]/(x^3)$. The module $_R R$ has all its elements nilpotent except $x^2$. It is easy to see that $\mathcal{N}(R R) \subseteq \sum_{i \in I} (0 : R a_i^{k_i}) = R$.

**Example 2.** If $R := M_n(D)$ the matrix ring of order $n$ over a division ring $D$, then the module $_R R$ is nil. So, $R = \mathcal{N}(R R) = \sum_{i \in J \subseteq I} (0 : R a_i^{k_i})$.

**Example 3.** Let $M := M_n(\mathbb{Z}/k\mathbb{Z})$ where $k$ is an integer be the ring of all $n \times n$ matrices defined over the ring $\mathbb{Z}/k\mathbb{Z}$,

$$N := \left\{(m_{ij}) \in M : \sum_{j=1}^{n} m_{ij} = 0 \pmod{k} \forall i \in \{1, 2, \ldots, n\}\right\} \text{ and } R := M_n(\mathbb{Z}).$$

$N$ is a nil $R$-module and $N = \mathcal{N}(N) = \sum_{i \in J \subseteq I} (0 : N a_i^{k_i})$.

### 3 Reduced modules

An $R$-module $M$ is **completely prime** [17, Definition 2.1] if for all elements $a \in R$ and $m \in M$, $am = 0$ implies that either $aM = 0$ or $m = 0$. An $R$-module $M$ is **rigid** [9] if for all $m \in M$, $a \in R$ and a positive integer $k$, $a^km = 0$ implies that $am = 0$. For modules
over commutative rings, rigid modules are indistinguishable from reduced modules.

For a given ring $R$, it is easy to see that

$$\{\text{torsion-free } R\text{-modules}\} \subset \{\text{completely prime } R\text{-modules}\} \subset \{\text{reduced } R\text{-modules}\} \subset \{\text{rigid } R\text{-modules}\}.$$

In Proposition 4, we give equivalent statements in terms of reduced modules and completely prime modules for a module to be torsion-free.

**Proposition 4.** For an $R$-module $M$, the following statements are equivalent:

1. $M$ is torsion-free;
2. for all nonzero elements $m \in M$, $\text{ann}_R(m) = 0$;
3. $M$ is a reduced module and $\text{ann}_R(Rm) = 0$ for all nonzero elements $m \in M$;
4. $M$ is a completely prime module and $\text{ann}_R(M) = 0$.

**Corollary 3.1.** A completely prime module is faithful if and only if it is torsion-free.

**Proposition 5.** If $R$ is a commutative ring and $M$ is an $R$-module, then the following statements are equivalent:

1. $M$ is a reduced $R$-module;
2. for every nonzero $m \in M$, the $R$-module $R/\text{ann}_R(m)$ is reduced;
3. for every nonzero $m \in M$, the cyclic $R$-module $Rm$ is reduced;
4. every minimal submodule of $M$ is reduced;
5. every nonzero endomorphism $f$ of $M$, of the form $f_a(m) = am$, where $a \in R$ and $m \in M$ is not nilpotent;
6. for every nonzero $m \in M$, $\text{ann}_R(m)$ is a semiprime ideal of $R$;
7. for every $r \in R$ and $k \in \mathbb{Z}^+$, $\text{Ker} f_r = \text{Ker} f_r^k$, where $f_r(m) = rm$ with $m \in M$;
8. $(0 :_M r) = (0 :_M r^k)$ for all $r \in R$ and $k \in \mathbb{Z}^+$;
9. $E_M(0) = 0$;
10. $\mathcal{N}(M) = 0$.

Proposition 5 makes it possible to dualise reduced modules defined over commutative rings.
Definition 3.1. Let $R$ be a commutative ring. An $R$-module $M$ is co-reduced if for any endomorphism $f_r(m) = rm$ of $M$ with $r \in R$,

$$\text{Co-Ker } f_r = \text{Co-Ker } f_r^k.$$ 

Suppose $M$ is a module over a commutative ring and for every $r \in R$, $\text{Ker } f_r$ (resp. $\text{Im } f_r$) is a maximal (resp. minimal) submodule of $M$ where $f_r : M \to M$ is the endomorphism $f_r(m) = rm$, then $M$ is reduced (resp. co-reduced). Note that $\text{Ker } f_r \subseteq \text{Ker } f_r^2 \subseteq \text{Ker } f_r^3 \subseteq \cdots$ (resp. $\text{Im } f_r \supseteq \text{Im } f_r^2 \supseteq \text{Im } f_r^3 \supseteq \cdots$) is an ascending chain (resp. descending chain) and is constant when $\text{Ker } f_r$ (resp. $\text{Im } f_r$) is a maximal (resp. minimal) submodule of $M$.

Proposition 6. If a module which is defined over a commutative ring is reduced and has a finite number of elements, then it is co-reduced.

Proof: Let $f_r : M \to M$ be given by $f_r(m) = rm$. Then for a positive integer $k$, $f_r^k(m) = r^k m$. By the first Isomorphism Theorem, $M/(0 :_M r) \cong rM$ and $M/(0 :_M r^k) \cong r^k M$. If $M$ is reduced, by Proposition [5] $(0 :_M r) = (0 :_M r^k)$. This implies that $rM \cong r^k M$. However, $r^k M \subseteq rM$ and $M$ has finite order. It must follow that $r^k M = rM$ and hence $M$ is co-reduced.

Theorem 3.2. If $M$ is an injective module defined over a commutative noetherian ring $R$ such that its indecomposable submodules are prime $R$-modules, then $M$ is a reduced module which is isomorphic to a direct sum of prime modules.

Proof: Any injective module $M$ over a commutative noetherian ring $R$ is isomorphic to $\bigoplus_i \mathcal{E}_R(R/P_i)$ where $P_i$ are prime ideals of $R$ and $\mathcal{E}_R(R/P_i)$ is the injective envelope of the $R$-module $R/P_i$. If each indecomposable component $\mathcal{E}_R(R/P_i)$ of $M$ is a prime $R$-module, then it is also reduced. However, a direct sum of reduced modules is reduced. This shows that $M$ is a reduced module which is isomorphic to a direct sum of prime modules.

Example 4. $\mathbb{Q}/\mathbb{Z}$ is an injective module over $\mathbb{Z}$ (a commutative noetherian ring). However it is not reduced. $\frac{1}{4}$ is a nilpotent element in the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. $2^2 \times \frac{1}{4} \in \mathbb{Z}$ but $2 \times \frac{1}{4} \notin \mathbb{Z}$. We observe that 

$$\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}(p^\infty) \cong \bigoplus_p \mathcal{E}_\mathbb{Z}(\mathbb{Z}/p\mathbb{Z})$$

since by [20] Example 3.36] every injective envelope of a $p$-group is a prufer $p$-group $\mathbb{Z}(p^\infty)$. The indecomposable components $\mathcal{E}_\mathbb{Z}(\mathbb{Z}/p\mathbb{Z})$ are not prime $\mathbb{Z}$-modules.

Example 5. $\mathbb{Q}$ is an injective module over $\mathbb{Z}$ (a commutative noetherian ring). $\mathbb{Q} = \mathcal{E}_\mathbb{Z}(\mathbb{Z})$ which is a prime $\mathbb{Z}$-module and hence a reduced $\mathbb{Z}$-module.
For an $R$-module $M$, let

$$
\mathcal{R}(M) := \{ m \in M : (a^km = 0) \Rightarrow (aRm = 0) \text{ for } a \in R \& k \in \mathbb{Z}^+ \}.
$$

$\mathcal{R}(M)$ is the collection of all non-nilpotent elements of the $R$-module $M$ together with the zero element. A module $M$ is reduced if and only if $\mathcal{R}(M) = M$ and $M$ is nil if and only if $\mathcal{R}(M) = 0$. If $M$ is a module over a commutative ring $R$, then the set $\mathcal{R}(M)$ is a submodule of $M$. For if $m_1, m_2 \in \mathcal{R}(M)$ and there exists $a \in R$ such that $a^k(m_1 + m_2) = 0$ but $aR(m_1 + m_2) \neq 0$. Then either $aRm_1 \neq 0$ or $aRm_2 \neq 0$. If $aRm_1 \neq 0$ and $aRm_2 = 0$. Then $a^km_1 \neq 0$ for all $t \in \mathbb{Z}^+$ since $m_1 \in \mathcal{R}(M)$ and $a^km_2 = 0$. It follows that $a^k(m_1 + m_2) \neq 0$ which is a contradiction. Similarly, if $aRm_1 = 0$ and $aRm_2 \neq 0$, then $a^k(m_1 + m_2) \neq 0$ leading to a contradiction. Now suppose that $aRm_1 \neq 0$ and $aRm_2 \neq 0$. It follows that $a^km_1 \neq 0$ and $a^km_2 \neq 0$ for all $t \in \mathbb{Z}^+$ since $m_1, m_2 \in \mathcal{R}(M)$. This implies that the submodules $a^kRm_1$ and $a^kRm_2$ of $M$ are both nonzero and hence their sum $a^kR(m_1 + m_2)$ is also nonzero. However, $a^k(m_1 + m_2) = 0$ implies that $a^kR(m_1 + m_2) = 0$ which is also a contradiction. This shows that if $m_1, m_2 \in \mathcal{R}(M)$, then $m_1 + m_2 \in \mathcal{R}(M)$. If $m \in \mathcal{R}(M)$ and $r \in R$ such that $a^krm = 0$ for some $k \in \mathbb{Z}^+$ and $a \in R$ but $aR(rm) \neq 0$, then $(ar)Rm \neq 0$. By definition of $\mathcal{R}(M)$, $(ar)^km \neq 0$ for all $t \in \mathbb{Z}^+$ so that $a^krm \neq 0$ which is a contradiction. This shows that $r \in R$ and $m \in \mathcal{R}(M)$ implies that $rm \in \mathcal{R}(M)$.

For modules over commutative rings, $\mathcal{R}(M)$ is the largest submodule of $M$ which is reduced as a module.

**Theorem 3.3.** As $\mathbb{Z}$-modules,

$$
\mathcal{R}(\mathbb{Q}/\mathbb{Z}) \cong \bigoplus_p \mathbb{Z}/p\mathbb{Z}
$$

for all prime integers $p$. Hence, $\mathcal{R}(\mathbb{Q}/\mathbb{Z})$ is a reduced $\mathbb{Z}$-module which is isomorphic to a direct sum of prime $\mathbb{Z}$-modules.

**Proof:**

$$
\mathcal{R}(\mathbb{Q}/\mathbb{Z}) = \mathcal{R}(\bigoplus_p \mathbb{Z}(p^{\infty})) = \bigoplus_p \left( \frac{1}{p^{\infty}} \mathbb{Z} \right) / \mathbb{Z} \cong \bigoplus_p \mathbb{Z}/p\mathbb{Z},
$$

where $\left( \frac{1}{p^{\infty}} \mathbb{Z} \right) / \mathbb{Z}$ is the cyclic subgroup of $\mathbb{Z}(p^{\infty})$ with $p$ elements; it contains those elements of $\mathbb{Z}(p^{\infty})$ whose order divides $p$ and corresponds to the set of $p$-th roots of unity. Each $\mathbb{Z}$-module $\mathbb{Z}/p\mathbb{Z}$ is a simple $\mathbb{Z}$-module and hence a prime $\mathbb{Z}$-module. \hfill \blacksquare

**Corollary 3.4.** The $\mathbb{Z}$-module $\mathbb{Z}/(p_1 \times \cdots \times p_k)\mathbb{Z}$ is reduced whenever the prime integers $p_i$ are all distinct.

**Proof:**

$$
\mathbb{Z}/(p_1 \times \cdots \times p_k)\mathbb{Z} \cong \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k\mathbb{Z}
$$

which is a submodule of a reduced $\mathbb{Z}$-module $\mathcal{R}(\mathbb{Q}/\mathbb{Z}) \cong \bigoplus_p \mathbb{Z}/p\mathbb{Z}$. \hfill \blacksquare

Theorem 3.3 partly answers in affirmative Question 5.1 in [39] which asks whether it is possible to obtain a general framework through which results like Corollary 3.4 which
were obtained in [39] can be retrieved.

Unlike prime rings which are closed under essential extension, prime modules need not be closed under essential extension. The \( \mathbb{Z} \)-module \( \mathbb{Z}/p\mathbb{Z} \) is a prime \( \mathbb{Z} \)-module. However, its essential extension \( \mathcal{E}_\mathbb{Z}(\mathbb{Z}/p\mathbb{Z}) \) which is the prufer \( p \)-group \( \mathbb{Z}(p^\infty) \) is not a prime \( \mathbb{Z} \)-module.

Note that 4 and 5 in Table II were done in [43].

### 4 When is a reduced module semisimple?

It is known that a semisimple module over a commutative ring is reduced. In this section, we investigate conditions when the converse to this holds for modules over arbitrary rings.

**Theorem 4.1.** Let \( R \) be a ring and \( M \) an \( R \)-module such that either the ring \( R \) or the ring \( R/\text{ann}_R(M) \) is embeddable in the module \( M \). If \( M \) is artinian and reduced, then every \( R/\text{ann}_R(M) \)-module is semisimple.

**Proof:** If there exists a monomorphism from \( R \) into \( M \), then \( R \) is isomorphic to a submodule of \( M \). Since \( M \) is artinian, so is \( R \) and hence \( R/\text{ann}_R(M) \) is also artinian. On the other hand, if \( R/\text{ann}_R(M) \) is embeddable in an artinian module \( M \), then \( R/\text{ann}_R(M) \) is also artinian. It is easy to prove that if \( M \) is a reduced \( R \)-module, then \( R/\text{ann}_R(M) \) is a reduced ring. However, a reduced artinian ring is semisimple. Hence, \( R/\text{ann}_R(M) \) is a semisimple ring. So, every \( R/\text{ann}_R(M) \)-module is semisimple.

**Proposition 7.** In the \( R \)-modules \( M \) given in 1-2, \( R \) is embeddable in \( M \); and in the \( R \)-modules \( M \) given in 3-4, \( R/\text{ann}_R(M) \) is embeddable in \( M \).

1. Free \( R \)-modules \( M \),
2. $M = \oplus_i P$ where $P$ is a generator module for $R$-$\text{Mod}$,

3. finitely generated modules $M$ over commutative rings $R$,

4. completely prime $R$-modules $M$.

**Proof:**

1. It is enough to remember that if $M$ is a free $R$-module, then $M$ is isomorphic to the $R$-module $R^n$ with $n$ copies of $R$ for some positive integer $n$.

2. A module $P$ is a generator module for $R$-$\text{Mod}$ if and only if $R$ is a direct summand of a direct sum $M = \oplus_i P$, see [20, Theorem 18.8].

3. Let $M$ be a finitely generated module over a commutative ring $R$ with generators $\{m_1, m_2, \cdots, m_n\}$. Define an $R$-homomorphism $f : R/\text{ann}_R(M) \to M$ by $f(\bar{r}) = (\bar{r}m_1, \bar{r}m_2, \cdots, \bar{r}m_n)$. $f$ is a monomorphism. If $f(\bar{r}) = 0$, then $\bar{r}m_i = 0$ for all $i \in \{1, 2, \cdots, n\}$ and $\bar{r}M = \bar{r}\sum_{i=1}^n Rm_i = \sum_{i=1}^n R\bar{r}m_i = 0$. Thus, $\bar{r} \in \text{ann}_R(M)$.

4. Let $f : R/\text{ann}_R(M) \to M$ be defined by $f(\bar{r}) = rm$ where $\bar{r} = r + \text{ann}_R(M)$. Suppose that $f(\bar{r}) = 0$, then $rm = 0$. Since $M$ is completely prime, $m = 0$ or $rM = 0$. If $m = 0$, Ker $f = R/\text{ann}_R(M)$ and $f$ is the zero homomorphism. If $rM = 0$, $r \in \text{ann}_R(M)$ and $\bar{r} = 0$. So, $f$ is injective. In both cases, $R/\text{ann}_R(M)$ is embeddable in $M$.

In Proposition 7, we have given a desirable property that completely prime modules possess but prime modules do not have; a yet another justification for completely prime modules. See [40, Section 2] for other advantages completely prime modules have over prime modules. In a prime $R$-module $M$, the ring $R/\text{ann}_R(M)$ need not be embeddable in $M$, see Example 6(2 & 4) and Remark 5.1(3).

**Corollary 4.2.** The following statements are true:

1. A faithful artinian completely prime module is semisimple.

2. An artinian, reduced and faithful free module is semisimple.

3. An artinian, reduced, faithful and finitely generated module over a commutative ring is semisimple.

4. A reduced, artinian progenerator module is semisimple.
Proof: It is enough for one to see that a completely prime module is reduced, a progenerator module is a generator module, and by [20, Remark 18.9(B)] any generator module is faithful. The rest follows from Theorem 4.1 and Proposition 7.

Let $A_n$ be the $n$th Weyl algebra. A finitely generated $A_n$-module is called holonomic if it is zero, or it has dimension $n$.

Corollary 4.3. A faithful holonomic module is reduced if and only if it is semisimple and rigid.

Proof: By [10, Theorem 2.2], a holonomic module $R^M$ is artinian. By definition, a holonomic module is a free module and hence by Proposition 7, $R$ is embeddable in $M$. Applying Theorem 4.1 shows that $M$ is a semisimple $R$-module. The remaining part is easy since a reduced module is always rigid. Conversely, if $M$ is semisimple and rigid as an $R$-module, then by [9, Corollary 2.30] it is reduced.

Corollary 4.4. An artinian $R$-algebra $M$ is a reduced $R/\text{ann}_R(M)$-module if and only if $M$ is a semisimple rigid $R/\text{ann}_R(M)$-module.

Proof: An artinian $R$-algebra is an artinian finitely generated module defined over a commutative ring $R$. By Proposition 7, $R/(0 : M)$ is embeddable in $M$. The rest of the proof is mutatis mutandis the one given in Corollary 4.3 above.

Corollary 4.5. Let $(R, m)$ be a Gorenstein artinian local ring. If the injective envelope $\mathcal{E}_R(R/m)$ of $R/m$ is a reduced $R$-module, then $\mathcal{E}_R(R/m)$ is a semisimple $R$-module.

Proof: By [11, Example 3.2], $\mathcal{E}_R(R/m)$ is a finitely generated faithful $R$-module in which $R$ embeds. Since artinian modules are closed under taking injective envelopes, $\mathcal{E}_R(R/m)$ is an artinian module. Thus, $\mathcal{E}_R(R/m)$ is an artinian reduced faithful module in which $R$ embeds. By Theorem 4.1, $\mathcal{E}_R(R/m)$ is a semisimple $R$-module.

A zero submodule of an $R$-module $M$ is said to satisfy the complete radical formula (resp. satisfy the radical formula) if $\langle E_M(0) \rangle = \beta_{co}(M)$ (resp. $\langle E_M(0) \rangle = \beta(M)$) where $\beta_{co}(M)$ (resp. $\beta(M)$) is the completely prime radical of $M$, i.e., the intersection of all completely prime submodules of $M$ (resp. the prime radical of $M$, i.e., the intersection of all prime submodules of $M$).

Theorem 4.6. If $R$ is a left artinian ring, and the zero submodule of an $R$-module $M$ satisfies the complete radical formula; then $M$ is semisimple whenever it is reduced.

Proof: Since $R$ is artinian and therefore $R/J(R)$ is also artinian, by [21, Exercise 4.16], to show that $M$ is semisimple, it is enough to show that $J(R)M = 0$ where $J(R)$ is the Jacobson radical of the ring $R$. Suppose that $\langle E_M(0) \rangle = \beta_{co}(M)$. If $M$ is reduced
\( N(M) = 0 \) and \( \langle E_M(0) \rangle = 0 \) by Proposition 2. It follows that \( \beta_{co}(M) = 0 \). Since \( \beta(R)M \subseteq \beta_{co}(R)M \subseteq \beta_{co}(M) \) (see [17, Lemma 5.4]), where \( \beta_{co}(R) \) (resp. \( \beta(R) \)) is the completely prime radical (resp. prime radical) of \( R \), we have \( \beta(R)M = 0 \). Since \( R \) is left artinian, \( \beta(R) = J(R) \) and hence \( J(R)M = 0 \) as required.

**Corollary 4.7.** Any module over a commutative artinian ring is reduced if and only if it is semisimple.

**Proof:** A semisimple module over a commutative artinian ring is reduced. Any commutative artinian ring \( R \) satisfies the radical formula (see [24]) and hence the zero submodule of an \( R \)-module satisfies the radical formula. However, for modules over commutative rings, there is no distinction between modules that satisfy the radical formula and modules that satisfy the complete radical formula. The rest follows from Theorem 4.6.

**Corollary 4.8.** A commutative artinian ring is reduced if and only if it is semisimple.

**Proof:** For any commutative ring \( R \), \( E_R(0) = \beta(R) \), i.e., the zero ideal of \( R \) satisfies the radical formula. The rest follows from Corollary 4.7.

## 5 Examples

1. If \( R := M_n(D) \) where \( D \) is a division ring, then \( R \) is semisimple and nil. This shows that in a module over a noncommutative ring, existence of nilpotent elements does not in general hinder semisimplicity. This is contrary to what happens for modules over commutative rings.

2. If \( \mathbb{Z} \) is the ring of integers, then \( \mathbb{Z} \) is reduced but not semisimple.

3. Let \( Q := \begin{array}{c}
\bullet \\
\bullet \\
\end{array} \) be a quiver. Its path algebra \( kQ \) has as basis \( \{e_1, e_2, a\} \). Let \( M \) be a 2-dimensional \( kQ \)-module with basis \( \{x, y\} \) and the action of \( kQ \) on \( M \) be given by

|     |  x  |  y  |
|-----|-----|-----|
| \( e_1 \) |  x  | 0   |
| \( e_2 \) | 0   |  y  |
| \( a   \) | 0   |  x  |

\( M \) has only one simple submodule which is generated by \( x \). So, \( M \) is not semisimple.

If \( m_1 = \mu x + \lambda y, m_2 = \mu x, m_3 = \lambda y \in M \) and \( \alpha = pe_1 + qe_2 + ra \in kQ \), then \((ra)^2m_1 = (ra)^2m_3 = 0\) but \((ra)m_1 \neq 0\) and \((ra)m_3 \neq 0\). This shows that \( m_1 \) and \( m_3 \) are nilpotent elements of \( M \). However, \( m_2 \) is not nilpotent and hence \( M \) is not reduced.

4. If the action in Example 3 above is changed to
then we get a simple module and hence a semisimple module which is reduced.

**Example 6 (The Golden Example).** Let \( M := M_n(\mathbb{Z}/k\mathbb{Z}) \) where \( k \) is an integer be the ring of all \( n \times n \) matrices defined over the ring \( \mathbb{Z}/k\mathbb{Z} \),

\[
N := \left\{ (m_{ij}) \in M : \sum_{j=1}^{n} m_{ij} = 0 \pmod{k} \forall i \in \{1, 2, \cdots, n\} \right\} \quad \text{and} \quad R := M_n(\mathbb{Z}).
\]

1. \( M \) is an \( R \)-module with order \( k^{n^2} \) and \( N \) is a minimal submodule of \( M \) with order \( k^n \);
2. As an \( R \)-module, \( N \) is simple. Hence, it is prime with \( \text{ann}_R(N) = M_n(k\mathbb{Z}) \);
3. \( N \) is not a completely prime module, i.e., there exists a nonzero element \( m \in N \) such that \( \text{ann}_R(N) \nsubseteq \text{ann}_R(m) \);
4. The homomorphism \( f : R/\text{ann}_R(N) \to N \) given by \( f(\bar{r}) = rn \) where \( \bar{r} = r + \text{ann}_R(N) \) and \( n \in N \) is not injective;
5. \( N \) is a nil \( R \)-module;
6. \( 0 = \beta(M) \nsubseteq \beta_{co}(M) = E_M(0) = \mathcal{N}(M) = M. \)

**Remark 5.1.** We call Example 6 the Golden Example because it is used for many purposes; we list them below:

1. It is an example of a nil module.
2. It generalizes an example of a prime module which is not completely prime given in [10, Example 1.3] and of a classical completely prime module which is not classical prime given in [13, Example 3.1]. In these two cases, \( k \) and \( n \) were taken to be \( k = n = 2 \).
3. It shows that in a prime \( R \)-module \( M \), the ring \( R/\text{ann}_R(M) \) need not embed in \( M \). More concretely, take \( k = 2 \) in Example 6. This gives another advantage completely prime modules have over prime modules.
4. It shows that a prime module can be nil. We show in Section 6 that \( s \)-prime, \( l \)-prime and completely prime modules cannot be nil.
5. It is an example of a module that satisfies the complete radical formula but it is neither 2-primal nor satisfies the radical formula; see a particular case used in [41].

6. Since \( \beta(M) \neq \beta_{co}(M) \) but \( \beta_{co}(M) = \mathcal{N}(M) \), it gives a negative answer to Question 8.2.1 posed in [42] as to whether we can succeed in writing 2-primal modules in terms of nilpotent elements as is the case for 2-primal rings.

Example 7. Let \( A \) be a ring which is not necessarily unital. Define \( A^1 := \{(a, n) : a \in A, n \in \mathbb{Z}\} \) with component-wise addition and multiplication given by \((a, n)(b, m) = (ab + am + nb, nm)\). Then \( A^1 \) is a ring (called the Dorroh extension of \( A \)) with unity \((0, 1)\). If \( A \) is such that \((A, +)\) has no torsion elements and \( A \) as a ring has at least one nonzero nilpotent element, then the submodule \( B := \{(0, n) : n \in \mathbb{Z}\} \) of the \( A^1 \)-module \( A^1 \) is nil. For if \( a \) is a nonzero nilpotent element of \( A \), then \((a, 0)(0, n) = (an, 0) \neq (0, 0)\) for all \( 0 \neq n \in \mathbb{Z} \) since \((A, +)\) has no torsion elements. This shows that \((a, 0)A(0, n) \neq 0\). However, since \( a^k = 0 \) for some positive integer \( k \), we have \((a, 0)^k(0, n) = (a^k n, 0) = (0, 0)\) which shows that \( B \) is nil.

Example 8. Let \( R \) be a commutative ring and \( S := R[x, y] \), the polynomial ring over \( R \) in the indeterminates \( x \) and \( y \). The regular module \( _S S \) is reduced. However, a finite dimensional \( S \)-module \( R[x, y]/\langle x^4, xy^2, x^3y, y^4 \rangle \) has some nilpotent elements. The generators of \( R[x, y]/\langle x^4, xy^2, x^3y, y^4 \rangle \) are \( \{1, x, x^2, x^3, y, y^2, y^3, xy, x^2y\} \) all nilpotent except \( \{y^3, x^2y, x^3\} \). This information is encoded in a combinatorial object given in Figure 1.

Note that the non-nilpotent module elements \( \{y^3, x^2y, x^3\} \) have been circled and occur at the “sharp points of the stairs” in Figure 1. In general, they can be determined easily by just drawing such combinatorial objects. We also remark that the same circled elements have connections to Hilbert schemes. These connections are outlined in [31, Chapter 7].

Example 9. For any positive integer \( n \) and any prime integer \( p \), \( M_n(\mathbb{Z}/p\mathbb{Z}) \) is a reduced \( \mathbb{Z} \)-module and the \( \mathbb{Z} \)-module \( M_n(\mathbb{Z}/p^k\mathbb{Z}) \) for any positive integer \( k \) greater than 1 contains nilpotent elements.
6 Effect of nilpotents on primeness

Andrunakievich in [1] defined $l$-prime and $s$-prime modules which were further studied in [15] and [18] respectively. We give the effect nilpotent elements have on $l$-prime and $s$-prime modules. Let $L(R)$ and $U(R)$ denote the Levitzki radical and upper nil radical respectively of a ring $R$. A module $M$ is said to be $l$-prime (resp. $s$-prime) if it is prime and $L(R/\text{ann}_R(M)) = 0$ (resp. if it is prime and $U(R/\text{ann}_R(M)) = 0$), see [15] (resp. [18]) for the other equivalent definitions.

Proposition 8. Let $M$ be an $R$-module. If any one of the following is true, then $N(M) \neq 0$, i.e., $M$ contains a nilpotent element.

1. $U(R/\text{ann}_R(M)) \neq 0$,
2. $L(R/\text{ann}_R(M)) \neq 0$,
3. $\beta(R/\text{ann}_R(M)) \neq 0$.

Proof: Suppose $U(R/\text{ann}_R(M)) \neq 0$. Then there exists $0 \neq I/\text{ann}_R(M) \lhd R/\text{ann}_R(M)$ such that $I/\text{ann}_R(M)$ is nil. Then for all $\bar{0} \neq \bar{r} \in I/\text{ann}_R(M)$, there exists a positive integer $k$ such that $\bar{r}^k = \bar{0}$. This is equivalent to saying that, for all $r \in I \setminus \text{ann}_R(M)$, there exists a positive integer $k$ such that $r^k \in \text{ann}_R(M)$. So, $rM \neq 0$ and $r^kM = 0$. This implies that there exists a nonzero element $m$ in $M$ such that $r^km = 0$ and $rm \neq 0$ so that $rRm \neq 0$. This shows that $m \in N(M)$. Since $\beta \subseteq L \subseteq U$, if either $L(R/\text{ann}_R(M)) \neq 0$ or $\beta(R/\text{ann}_R(M)) \neq 0$, it follows that, $U(R/\text{ann}_R(M)) \neq 0$, a case which is already proved.

In Corollary 6.1 below, we retrieve an already known result.

Corollary 6.1. If $M$ is a prime and reduced $R$-module, then

1. $M$ is $s$-prime,
2. $M$ is $l$-prime.

Proof: If $M$ is reduced, $N(M) = 0$. By Proposition 8

$$U(R/\text{ann}_R(M)) = L(R/\text{ann}_R(M)) = 0.$$ 

Since $M$ is in addition prime, it follows from [18, Corollary 2.1] (resp. [15, Proposition 2.2]) that $M$ is $s$-prime (resp. $l$-prime).

This result is already known, because from [17, Theorem 2.10 & Remark 2.11] and [16, Corollary 2.4] a prime and reduced module is completely prime. It was shown in [16, Propositions 3.1 & 3.2] and [15, Proposition 2.7] that a completely prime module is $s$-prime and an $s$-prime module is $l$-prime respectively.
Corollary 6.2 shows that “reduced” in a faithful $R$-module carries over in some sense to the ring $R$.

**Corollary 6.2.** If $M$ is a faithful and reduced $R$-module, then

1. $\mathcal{U}(R) = 0$, i.e., $R$ has no nonzero nil ideals;

2. $\mathcal{L}(R) = 0$, i.e., $R$ has no nonzero locally nilpotent ideals;

3. $\beta(R) = 0$, i.e., $R$ has no nonzero nilpotent ideals.

**Corollary 6.3.** A prime module which is not $l$-prime, not $s$-prime or not completely prime, contains nonzero nilpotent elements.

It is impossible to write a nil module as a sum of completely prime modules. Hence, it is impossible to have a nil module defined over a commutative ring written as a sum of prime modules. If it were possible, the completely prime modules making up the sum would be nil which is a contradiction - completely prime modules are always reduced. It is important also to note that in a module over a commutative ring, prime modules are completely prime. So, existence of nilpotents inhibits some structure; the structure of having a sum of completely prime modules.

**Corollary 6.4.** For any module $M$, $\mathcal{N}(M) = 0$ implies that

$$\mathcal{U}(M) = \mathcal{L}(M) = \beta(M) = 0.$$  

**Proof:** If $\mathcal{N}(M) = 0$, then $M$ is reduced and $\beta_{\text{co}}(M) = 0$. This leads to the desired result since $\beta(M) \subseteq \mathcal{L}(M) \subseteq \mathcal{U}(M) \subseteq \beta_{\text{co}}(M)$. 

For rings, the sum of all nil ideals of a ring coincides with the intersection of all s-prime ideals of that ring. We get the following question.

**Question 6.1.** [14, Question 8.2.3] How does the upper nil radical of a module $M$ which is given as the intersection of all s-prime submodules of $M$ compare with the sum of all nil submodules of $M$?

In conclusion, nilpotent elements of a module control its structure. They inhibit semisimplicity for modules defined over commutative rings. They do not allow a module to be any of the following: torsion-free, completely prime, $l$-prime and $s$-prime. They determine in addition to some other conditions whether a module can be written as a direct sum of prime submodules or not. In a situation where they do not appear, i.e., when the module is reduced, the module behaves nicely; for instance, it behaves as though it is defined over a commutative ring, i.e., it has the insertion of factor property, it is 2-primal and it is symmetric, see [16, Theorems 2.2 & 2.3].
Acknowledgement

I wish to thank professors Ken A. Brown and Michael Wemyss both of University of Glasgow, and Professor Balazs Szendroi of University of Oxford for the hospitality during my visit to the UK and for the discussions that greatly improved this work. This research was supported by LMS and Sida bilateral programme (2015-2020) with Makerere University; project 316: Capacity building in Mathematics and its Applications.

References

[1] A. Andrunakievich and Ju M. Rjabuhin, Special modules and special radicals, *Soviet Math. Dokl.*, 3 (1962), 1790–1793. Russian original: *Dokl. Akad. Nauk SSSR.*, 147 (1962), 1274–1277.

[2] E. P. Armendariz, On Semiprime rings of bounded index, *Proc. Amer. Math. Soc.*, 85(2), (1982), 146–148.

[3] A. Azizi, Radical formula and prime submodules, *J. Algebra*, 307(1), (2007), 454–460.

[4] M. Baser and N. Agayev, On reduced and semicommutative modules, *Turkish J. Math.*, 30, (2006), 285–291.

[5] M. Behboodi, On the prime radical and Baer’s lower nilradical of modules. *Acta Math. Hungar.*, 122, (2008), 293–306.

[6] M. Behboodi, A generalization of Baer’s lower nilradical for modules. *J. Algebra Appl.*, 6, (2007), 337–353.

[7] K. I. Beidar, Y. Fong and E. R. Puczyłowski, On essential extensions of reduced rings and domains, *Arch. Math.*, 83, (2004), 344–352.

[8] G. F. Birkenmeier, H. E. Heatherly and E. K. Lee, Completely prime ideals and associated radicals, in Proc. Biennial Ohio State-Denison Conf., 1992, eds. S. K. Jain and S. T. Rizvi (World Scientific, Singapore, 1993), pp. 102–129.

[9] A. M. Buhphang, S. Halicioglu, A. Harmanci, K. Hera Singh, H. Y. Kose, M. B. Rege, On rigid modules, *East-West J. Math.*, 15(1), (2013), 70–84.

[10] S. C. Coutinho, *A primer of algebraic D-modules*, Cambridge University Press, 1995.

[11] J. W. Fisher, Nil subrings of Endomorphism rings of modules, *Proc. Amer. Math. Soc.*, 34(1), (1972), 75–78.
[12] B. J. Gardner and R. Wiegandt, *Radical theory of rings*, New York: Marcel Dekker, 2004.

[13] N. J. Groenewald and D. Ssevviiri, Classical completely prime submodules, *Hacet. J. Math. Stat.*, **45**(3), (2016), 717–729.

[14] N. J. Groenewald and D. Ssevviiri, Generalization of nilpotency of ring elements to module elements, *Comm. Algebra*, **42**(2), (2014), 571–577.

[15] N. J. Groenewald and D. Ssevviiri, On the Levitzki radical of modules, *Int. Elect. J. Algebra*, **15**, (2014), 77–89.

[16] N. J. Groenewald and D. Ssevviiri, 2-primal modules, *J. Algebra Appl.*, **12**(5), (2013), 1250226, 12 pages.

[17] N. J. Groenewald and D. Ssevviiri, Completely prime modules, *Int. Elect. J. Algebra*, **13**, (2013), 1–14.

[18] N. J. Groenewald and D. Ssevviiri, Köthe’s upper nil radical for modules, *Acta Math. Hungar.*, **138**(4), (2013), 295–306.

[19] J. Jenkins and P. F. Smith, On the prime radical of a module over a commutative ring. *Comm. Algebra*, **20**, (1992), 3593–3602.

[20] T. Y. Lam, *Lecturers on modules and rings*, Springer-Verlag, New York, Inc, 1999.

[21] T. Y. Lam, *A first course in noncommutative rings*, Springer-Verlag, New York, Inc, 1991.

[22] T. K. Lee and Y. Zhou, Reduced modules, *Rings, modules, algebras and abelian groups*, pp. 365–377, Lecturer notes in pure and applied math. **236**, Marcel Dekker, New York, 2004.

[23] Y. Lee, Questions on 2-primal rings, *Comm. Algebra*, **26**(2), (1998), 595–600.

[24] K. H. Leung and H. S. Man, On commutative noetherian rings which satisfy the radical formula, *Glasg Math. J.*, **39**, (1997), 285–293.

[25] I. G. Macdonald, Secondary representations of modules over commutative rings, *Symposia Mathematica*, Vol XI, Academic Press, London, (1973), 23–43.

[26] G. Marks, A taxonomy of 2-primal rings, *J. Algebra*, **266**(2), (2003), 494-520.

[27] G. Marks, On 2-primal Ore-extension, *Comm. Algebra*, **29**(5), (2001), 2113–2123.

[28] G. Marks, Skew polynomial rings over 2-primal rings, *Comm. Algebra*, **27**(9), (1999), 4411–4423.
[29] J. C. McConnell and J. C. Robson, Noncommutative Noetherian rings. American Mathematical Society, 1987.

[30] N. H. McCoy, The theory of rings, Bronx, New York, 1973.

[31] H. Nakajima, Lectures on Hilbert Schemes of points on surfaces, Vol 18, American Mathematical Society, 1962.

[32] A. Nikseresht and A. Azizi, On radical formula in modules, Glasg. Math. J., 53(3), (2011), 657–668.

[33] D. Pusat-Yilmaz and P. F. Smith, Modules which satisfy the radical formula, Acta Math. Hungar., 95, (2002), 155–167.

[34] M. B. Rege and A. M. Buhphang, On reduced modules and rings, Int. Elect. J. Algebra, 3, (2008), 58–74.

[35] L. H. Rowen, Graduate Algebra: Noncommutative view, American Mathematical Society, 2008.

[36] L. H. Rowen, Ring theory, Vol 1, Academic Press, Inc. San Diego, 1988.

[37] H. Sharif, Y. Sharifi and S. Namazi, Rings satisfying the radical formula, Acta Math. Hungar., 71, (1996), 103–108.

[38] A. Smoktunowicz, On some results related to Köthe’s conjecture, Serdica Math. J. 27, (2001), 159-170.

[39] D. Ssevviiri, Effect of nilpotency on semisimplicity and cohomology of the $\mathbb{Z}$-module $\mathbb{Z}/(p_1^{k_1} \times \cdots \times p_n^{k_n})\mathbb{Z}$, arXiv:1705.02528, 2017.

[40] D. Ssevviiri, On completely prime modules, Int. Elect. J. Algebra, 19, (2016), 77–90.

[41] D. Ssevviiri, A complete radical formula and 2-primal modules, arXiv:1612.03021, 2016.

[42] D. Ssevviiri, A contribution to the theory of prime modules, PhD thesis, Nelson Mandela Metropolitan University, 2013.

[43] D. Ssevviiri, Characterization of non-nilpotent elements of the $\mathbb{Z}$-module $\mathbb{Z}/(p_1^{k_1} \times \cdots \times p_n^{k_n})$, Int. J. Algebra, 7, (2013), 699–702.

[44] K. Yamagishii, Embedding of Noetherian rings into faithful modules, J. Math. Kyoto Univ., 23(3), (1983), 461–466.