Research Article
Simpson’s Integral Inequalities for Twice Differentiable Convex Functions

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Received 30 March 2020; Accepted 25 May 2020; Published 27 June 2020

Guest Editor: Praveen Agarwal

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Integral inequality is an interesting mathematical model due to its wide and significant applications in mathematical analysis and fractional calculus. In the present research article, we obtain new inequalities of Simpson’s integral type based on the \(\phi\)-convex and \(\phi\)-quasiconvex functions in the second derivative sense. In the last sections, some applications on special functions are provided and shown via two figures to demonstrate the explanation of the readers.

1. Introduction

Integral inequality is a modern model of approximation theory that describes the growth rate of competing mathematical analysis. This model is also used in various fields such as ordinary differential equations [1–5] and fractional calculus [6–17].

Among the several known inequalities, the most simple is Simpson’s type, which has been successfully applied in several models of ordinary differential equations [18–29] and fractional differential equations [30–32]. Simpson’s integral inequality is as follows: for any four times continuously differentiable function \(F: [\xi_1, \xi_2] \rightarrow \mathbb{R}\) on \((\xi_1, \xi_2)\), Simpson’s integral inequality is defined as follows:

\[
\frac{1}{3} \left[ F(\xi_1) + 2F(\xi_2) + 2F\left(\frac{\xi_1 + \xi_2}{2}\right) \right] - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x)dx \leq \frac{1}{2880\|F^{(4)}\|_{\infty}} (\xi_2 - \xi_1)^4,
\]

where \(\|F^{(4)}\|_{\infty} = \sup_{x \in [\xi_1, \xi_2]} |F^{(4)}(x)| < \infty\).

If the function \(F\) is neither four times differentiable nor is the fourth derivative \(F^{(4)}\) bounded on \((\xi_1, \xi_2)\), then we cannot apply the classical Simpson quadrature formula.

The following literature results obtained by Alomari et al. [18] and Sarikaya et al. [23] become a special case in our findings in Sections 2 and 3.

\[
\text{Lemma 1 (see [18]). Let } F: \mathbb{R} \rightarrow \mathbb{R} \text{ be twice differentiable function on } \mathbb{R} \text{ with } F'' \in L_1[\xi_1, \xi_2], \text{ then we have}
\]

\[
\frac{F(\xi_1) + F(\xi_2)}{2} - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x)dx = \frac{(\xi_2 - \xi_1)^2}{2} \int_0^1 t(1-t)F''(t\xi_1 + (1-t)\xi_2)dt.
\]

\[
\text{Lemma 2 (see [23]). Let } F: \mathbb{R} \rightarrow \mathbb{R} \text{ be twice differentiable function on } \mathbb{R} \text{ such that } F'' \in L_1[\xi_1, \xi_2], \text{ where } \xi_1, \xi_2 \in \mathbb{R}
\]

with \(\xi_1 < \xi_2\), then we have
\[
\frac{1}{6} \left[ F(x) + 4F \left( \frac{\xi_1 + \xi_2}{2} \right) + F(\xi_2) \right] - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x) dx
\]

\[= (\xi_2 - \xi_1)^2 \int_0^1 h(t) F''(t\xi_2 + (1-t)\xi_1) dt,
\]

where
\[
h(t) = \begin{cases} 
\frac{t}{2} \left( 1 - \frac{1}{3} t \right), & \text{if } t \in \left[ 0, \frac{1}{2} \right), \\
\left( 1 - t \right) \left( \frac{t}{2} - \frac{1}{3} \right), & \text{if } t \in \left[ \frac{1}{2}, 1 \right].
\end{cases}
\]

Example 5. Let \( F(x) = x^2 \), then \( F \) is convex and \( \varphi \) convex with \( \varphi (y_1, y_2) = 2y_1 + y_2 \); indeed,

\[F(t\xi_1 + (1-t)\xi_2) = (t\xi_1 + (1-t)\xi_2)^2 \leq \xi_1^2 + t\xi_2^2 + 2t(1-t)\xi_1\xi_2 \leq \xi_1^2 + t\xi_2^2 + t(1-t)(\xi_1^2 + \xi_2^2) \leq \xi_1^2 + t(\xi_1^2 + \xi_2^2)
\]

Remark 1. (i) It is easy to see the definition that every \( \varphi \)-convex function is \( \varphi \)-quasiconvex. (ii) If we take \( \varphi (\xi_1, \xi_2) = \xi_1 - \xi_2 \) in Definition 1, then the definitions of \( \varphi \)-convex and \( \varphi \)-quasiconvex are reduced to the definition of convex function and quasiconvex function, respectively.

Next, we will give examples for the above definitions.

Example 1. Let \( F(x) = x^2 \), then \( F \) is convex and \( \varphi \)-convex with \( \varphi (y_1, y_2) = 2y_1 + y_2 \); indeed,

\[F(t\xi_1 + (1-t)\xi_2) = (t\xi_1 + (1-t)\xi_2)^2 \leq \xi_1^2 + t\xi_2^2 + 2t(1-t)\xi_1\xi_2 \leq \xi_1^2 + t\xi_2^2 + t(1-t)(\xi_1^2 + \xi_2^2) \leq \xi_1^2 + t(\xi_1^2 + \xi_2^2)
\]

Example 2. Let \( F(x) = x^3 \), then \( F \) is not convex but is \( \varphi \)-convex with \( \varphi (y_1, y_2) = 3y_2^2(y_1 - y_2) + 3y_2(y_1 - y_2)^2 + (y_1 - y_2)^3 \); indeed,

\[F(t\xi_1 + (1-t)\xi_2) = (t\xi_1 + (1-t)\xi_2)^3 = (\xi_2 + t(\xi_1 - \xi_2))^3 \leq \xi_1^3 + t(\xi_1 - \xi_2)^3 \]

Example 3. Let \( F: [\xi_1, \xi_2] \to R \), \( 0 < \xi_1 < \xi_2 \), with \( F(x) = 1/x^2 \). We observe that \( F \) is convex on \( [\xi_1, \xi_2] \) and therefore \( \varphi \)-quasiconvex with \( \varphi (y_1, y_2) = y_1 - y_2 \).

Example 4. Let \( F: [\xi_1, \xi_2] \to R \), \( 0 < \xi_1 < \xi_2 \), with \( F(x) = 2/x^3 \). We observe that \( F \) is convex on \( [\xi_1, \xi_2] \) and therefore \( \varphi \)-quasiconvex with \( \varphi (y_1, y_2) = y_1 - y_2 \).

Example 5. Let \( F: [\xi_1, \xi_2] \to R \), \( 0 < \xi_1 < \xi_2 \), with \( F(x) = 2 \). We obviously see that \( F \) is \( \varphi \)-quasiconvex with \( \varphi (y_1, y_2) = y_1 - y_2 \).

The essential object of this study is to establish new Simpson’s integral inequalities for the \( \varphi \)-convex and \( \varphi \)-quasiconvex functions in the second derivative sense at certain powers.

2. Simpson’s Inequality for \( \varphi \)-Convex

In this section, we give a new refinement of Simpson integral inequality for twice differentiable functions.

Theorem 1. Let \( F: [\xi_1, \xi_2] \to R \) be a twice differentiable function on \( [\xi_1, \xi_2] \) such that \( F'' \in L_1[\xi_1, \xi_2] \), where \( \xi_1, \xi_2 \in \mathbb{R} \) with \( \xi_1 < \xi_2 \). If \( |F''| \) is \( \varphi \)-convex on \( [\xi_1, \xi_2] \), then we have

\[\frac{1}{6} \left[ \left| F(\xi_1) + 4F \left( \frac{\xi_1 + \xi_2}{2} \right) + F(\xi_2) \right| - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x) dx \right] \leq \frac{(\xi_2 - \xi_1)^2}{81} \left[ |F''(\xi_1)| + \frac{1}{2} \varphi (|F''(\xi_1)|, |F''(\xi_2)|) \right].
\]
Proof. By making the use of Lemma 2 and the \( \varphi \)-convexity of \( |\mathcal{F}''| \), we find that

\[
\frac{1}{6} \left[ \mathcal{F}(\xi_1) + 4\mathcal{F}\left(\frac{\xi_1 + \xi_2}{2}\right) + \mathcal{F}(\xi_2) \right] - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \mathcal{F}(\xi) d\xi \\
\leq (\xi_2 - \xi_1)^2 \int_0^1 |k(t)||\mathcal{F}''(t\xi_2 + (1 - t)\xi_1)| \, dt
\]

\[
\leq (\xi_2 - \xi_1)^2 \int_0^{1/2} \frac{t}{2} \left( \frac{1}{3} - t \right) \left( |\mathcal{F}''(\xi_1)| + t \varphi(|\mathcal{F}''(\xi_1)|, |\mathcal{F}''(\xi_2)|) \right) \, dt
\]

\[
+ (\xi_2 - \xi_1)^2 \int_{1/2}^1 (1 - t) \left( \frac{t}{2} - \frac{1}{3} \right) \left( |\mathcal{F}''(\xi_1)| + t \varphi(|\mathcal{F}''(\xi_1)|, |\mathcal{F}''(\xi_2)|) \right) \, dt
\]

\[
= q(\xi_2 - \xi_1)^2 (\tau_1 + \tau_2),
\]

where

\[
\tau_1 := \int_0^{1/2} \frac{t}{2} \left( \frac{1}{3} - t \right) \left( |\mathcal{F}''(\xi_1)| + t \varphi(|\mathcal{F}''(\xi_1)|, |\mathcal{F}''(\xi_2)|) \right) \, dt
\]

\[
= \int_0^{1/2} \frac{t}{2} \left( \frac{1}{3} - t \right) |\mathcal{F}''(\xi_1)| \, dt + \int_0^{1/2} \frac{t}{2} \left( \frac{1}{3} - t \right) t \varphi(|\mathcal{F}''(\xi_1)|, |\mathcal{F}''(\xi_2)|) \, dt
\]

\[
= |\mathcal{F}''(\xi_1)| \int_0^{1/2} \frac{t}{2} \left( \frac{1}{3} - t \right) \, dt + \varphi(|\mathcal{F}''(\xi_1)|, |\mathcal{F}''(\xi_2)|) \int_0^{1/2} \frac{t}{2} \left( \frac{1}{3} - t \right) \, dt
\]

\[
= \frac{1}{16} |\mathcal{F}''(\xi_1)| + \varphi(|\mathcal{F}''(\xi_1)|, |\mathcal{F}''(\xi_2)|).
\]

\[
\tau_2 := \int_{1/2}^1 (1 - t) \left( \frac{t}{2} - \frac{1}{3} \right) \left( |\mathcal{F}''(\xi_1)| + t \varphi(|\mathcal{F}''(\xi_1)|, |\mathcal{F}''(\xi_2)|) \right) \, dt
\]

\[
= \int_{1/2}^1 (1 - t) \left( \frac{t}{2} - \frac{1}{3} \right) |\mathcal{F}''(\xi_1)| \, dt + \int_{1/2}^1 (1 - t) \left( \frac{t}{2} - \frac{1}{3} \right) t \varphi(|\mathcal{F}''(\xi_1)|, |\mathcal{F}''(\xi_2)|) \, dt
\]

\[
= |\mathcal{F}''(\xi_1)| \int_{1/2}^1 (1 - t) \left( \frac{t}{2} - \frac{1}{3} \right) \, dt + \varphi(|\mathcal{F}''(\xi_1)|, |\mathcal{F}''(\xi_2)|) \int_{1/2}^1 (1 - t) \left( \frac{t}{2} - \frac{1}{3} \right) \, dt
\]

\[
= \frac{1}{16} |\mathcal{F}''(\xi_1)| + \frac{133}{31104} \varphi(|\mathcal{F}''(\xi_1)|, |\mathcal{F}''(\xi_2)|).
\]

A simple rearrangement gives us the proof. \( \square \)
Corollary 1. Theorem 1 with \( F(\xi_1) = F((\xi_1 + \xi_2)/2) = F(\xi_2) \) gives the following new inequality:

\[
\left| \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x) \, dx - F\left( \frac{\xi_1 + \xi_2}{2} \right) \right| \\
\leq \frac{(\xi_2 - \xi_1)^2}{81} \left[ \left| F''(\xi_1) \right| + \frac{1}{2} \phi\left( |F''(\xi_1)|, |F''(\xi_2)| \right) \right].
\]

(12)

Remark 2. Inequality (9) with \( \phi(|F''(\xi_1)|, |F''(\xi_2)|) = |F''(\xi_2)| - |F''(\xi_1)| \) becomes

\[
\left| \frac{1}{6} \left[ F(\xi_1) + 4F\left( \frac{\xi_1 + \xi_2}{2} \right) + F(\xi_2) \right] - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x) \, dx \right| \\
\leq \frac{(\xi_2 - \xi_1)^2}{162} \left[ \left| F''(\xi_1) \right| + |F''(\xi_2)| \right].
\]

(13)

Moreover, inequality (12) with \( \phi(|F''(\xi_1)|, |F''(\xi_2)|) = |F''(\xi_2)| - |F''(\xi_1)| \) becomes

\[
\left| \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x) \, dx - F\left( \frac{\xi_1 + \xi_2}{2} \right) \right| \\
\leq \frac{(\xi_2 - \xi_1)^2}{162} \left[ \left| F''(\xi_1) \right| + |F''(\xi_2)| \right].
\]

(14)

These are both obtained by Sarikaya et al. [23] in Theorem 2.2 and Corollary 2.3, respectively.

Theorem 2. Let \( F: \mathfrak{X} \to \mathbb{R} \) be a twice differentiable function on \( \mathfrak{X} \) such that \( F'' \in L_1[\xi_1, \xi_2] \), where \( \xi_1, \xi_2 \in \mathfrak{X} \) with \( \xi_1 < \xi_2 \). If \( |F''|^q \) is \( \varphi \)-convex on \([\xi_1, \xi_2]\) and \( q \geq 1 \), then we have

\[
\left| \frac{1}{6} \left[ F(\xi_1) + 4F\left( \frac{\xi_1 + \xi_2}{2} \right) + F(\xi_2) \right] - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x) \, dx \right| \\
\leq (\xi_2 - \xi_1)^2 \left( \int_0^{1/2} \frac{f(t)}{2} \left( \frac{1}{3} - t \right) \, dt \right)^{1-1/q} \left( \int_0^{1/2} \frac{f(t)}{2} \left( \frac{1}{3} - t \right) \left| F''(t\xi_2 + (1-t)\xi_1) \right|^q \, dt \right)^{1/q}
\]

(17)

\[
+ (\xi_2 - \xi_1)^2 \left( \int_{1/2}^1 \left( 1 - t \right) \left( \frac{f(t)}{2} - \frac{1}{3} \right) \, dt \right)^{1-1/q} \left( \int_{1/2}^1 \left( 1 - t \right) \left( \frac{f(t)}{2} - \frac{1}{3} \right) \left| F''(t\xi_2 + (1-t)\xi_1) \right|^q \, dt \right)^{1/q}.
\]

By \( \varphi \)-convexity of \( |F''|^q \) for the last two integrals, we have
\[
\int_0^{1/2} \left( \frac{t}{2} \left( \frac{1}{3} - t \right) \right) \left| F''(t \xi_2 + (1-t) \xi_1) \right|^q dt \\
\leq \int_0^{1/2} \left( \frac{t}{2} \left( \frac{1}{3} - t \right) \right) \left[ \left| F''(\xi_1) \right|^q + t \varphi(\left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q) \right] dt \\
= \left| F''(\xi_1) \right|^q \int_0^{1/2} \left( \frac{t}{2} \left( \frac{1}{3} - t \right) \right) dt + \varphi(\left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q) \int_0^{1/2} \left( \frac{t}{2} \left( \frac{1}{3} - t \right) \right) dt \\
= \frac{1}{162} \left| F''(\xi_1) \right|^q + \frac{59}{31104} \varphi(\left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q).
\]

By substituting (18) and (19) into (17), we have

\[
\left\{ \frac{1}{6} \left[ F(\xi_1) + 4F\left( \frac{\xi_1 + \xi_2}{2} \right) + F(\xi_2) \right] - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x) dx \right\} \\
\leq (\xi_2 - \xi_1)^2 \left( \int_0^{1/2} \left( \frac{t}{2} \left( \frac{1}{3} - t \right) \right) dt \right) \left[ \left| F''(\xi_1) \right|^q + \frac{59}{31104} \varphi(\left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q) \right]^{1/q} \\
+ (\xi_2 - \xi_1)^2 \left( \int_{1/2}^1 (1-t) \left( \frac{t}{2} - \frac{1}{3} \right) dt \right) \left[ \left| F''(\xi_1) \right|^q + \frac{133}{31104} \varphi(\left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q) \right]^{1/q} \\
= (\xi_2 - \xi_1)^2 \left( \frac{1}{162} \right) \left[ \left( \frac{1}{162} \left| F''(\xi_1) \right|^q + \frac{59}{31104} \varphi(\left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q) \right) \right]^{1/q} \\
+ \left( \frac{1}{162} \left| F''(\xi_1) \right|^q + \frac{133}{31104} \varphi(\left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q) \right) \right]^{1/q},
\]

where we used the identity

\[
\int_0^{1/2} \left( \frac{t}{2} \left( \frac{1}{3} - t \right) \right) dt = \int_{1/2}^1 (1-t) \left( \frac{t}{2} - \frac{1}{3} \right) dt = \frac{1}{162}
\]

Thus, we are done.

**Corollary 2.** Theorem 2 with \( F(\xi_1) = F((\xi_1 + \xi_2)/2) = F(\xi_2) \) gives the following new inequality:
\[
\frac{1}{|\xi_2 - \xi_1|} \int_{\xi_1}^{\xi_2} F(x)dx - F\left(\frac{\xi_1 + \xi_2}{2}\right)
\leq (\xi_2 - \xi_1)^2 \left(\frac{1}{162}\right)^{1-1/q} \left[\left(\frac{1}{162}\right)|F''(\xi_1)|^q + \frac{59}{31104}\phi|F''(\xi_1)|^q \right]^{1/q} \\
+ \left(\frac{1}{162}\right)|F''(\xi_1)|^q + \frac{133}{31104}\phi|F''(\xi_1)|^q \right]^{1/q}
\]

(22)

Remark 3. Inequality (15) with \(\phi(|F''(\xi_1)|^q, |F''(\xi_2)|^q) = |F''(\xi_2)|^q - |F''(\xi_1)|^q\) becomes

\[
\frac{1}{6} \int_{\xi_1}^{\xi_2} F(x)dx + 4F\left(\frac{\xi_1 + \xi_2}{2}\right) + F(\xi_2) - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x)dx
\leq (\xi_2 - \xi_1)^2 \left(\frac{1}{162}\right)^{1-1/q} \left(\frac{133}{31104}|F''(\xi_1)|^q + \frac{59}{31104}|F''(\xi_2)|^q \right) \right]^{1/q} \\
+ \left(\frac{59}{31104}|F''(\xi_1)|^q + \frac{133}{31104}|F''(\xi_2)|^q \right) \right]^{1/q}.
\]

(23)

Moreover, inequality (22) with \(\phi(|F''(\xi_1)|^q, |F''(\xi_2)|^q) = |F''(\xi_2)|^q - |F''(\xi_1)|^q\) becomes

\[
\frac{1}{6} \int_{\xi_1}^{\xi_2} F(x)dx + 4F\left(\frac{\xi_1 + \xi_2}{2}\right) + F(\xi_2) - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x)dx
\leq (\xi_2 - \xi_1)^2 \left(\frac{1}{162}\right)^{1-1/q} \left(\frac{133}{31104}|F''(\xi_1)|^q + \frac{59}{31104}|F''(\xi_2)|^q \right) \right) \right]^{1/q} \\
+ \left(\frac{59}{31104}|F''(\xi_1)|^q + \frac{133}{31104}|F''(\xi_2)|^q \right) \right]^{1/q}.
\]

(24)

These are both obtained by Sarikaya et al. [23] in Theorem 2.5 and Corollary 2.6, respectively.

Remark 4. Theorem 2 and Corollary 2 with \(q = 1\) become Theorem 1 and Corollary 1, respectively.

3. Simpson’s Inequality for φ-Quasiconvex

Theorem 3. Let \(F: \mathbb{R} \rightarrow \mathbb{R}\) be a twice differentiable function on \(\mathbb{R}\) provided \(F'' \in L_{1}[\xi_1, \xi_2]\), where \(\xi_1, \xi_2 \in \mathbb{R}\) with \(\xi_1 < \xi_2\). If \(|F''|\) is \(\phi\)-quasiconvex on \([\xi_1, \xi_2]\), then we have

\[
\frac{1}{6} \int_{\xi_1}^{\xi_2} F(x)dx + 4F\left(\frac{\xi_1 + \xi_2}{2}\right) + F(\xi_2) - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x)dx
\leq \frac{(\xi_2 - \xi_1)^2}{81} \max\{|F''(\xi_1)|, |F''(\xi_2)|, \phi(|F''(\xi_1)|, |F''(\xi_2)|)\}.
\]

(25)

Proof. By making use of \(\phi\)-quasiconvexity of \(|F''|\) and Lemma 2, we get

\[
\frac{1}{6} \int_{\xi_1}^{\xi_2} F(x)dx + 4F\left(\frac{\xi_1 + \xi_2}{2}\right) + F(\xi_2) - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} F(x)dx
\leq (\xi_2 - \xi_1)^2 \int_0^1 |k(t)||F''(t\xi_2 + (1-t)\xi_1)|dt
\leq (\xi_2 - \xi_1)^2 \int_0^1 \left(\frac{t}{2}\right) \max\{|F''(\xi_1)|, |F''(\xi_2)|, \phi(|F''(\xi_1)|, |F''(\xi_2)|)\} dt
\leq (\xi_2 - \xi_1)^2 \int_0^1 \left(\frac{t}{2} - \frac{1}{3}\right) \max\{|F''(\xi_1)|, |F''(\xi_2)|, \phi(|F''(\xi_1)|, |F''(\xi_2)|)\} dt
\]

(26)
where

\[ \mathcal{T}_1 = \int_0^{1/2} \left( \frac{1}{2} - t \right) \max\{ |F''(\xi_1)|, |F''(\xi_2)|, \phi(|F''(\xi_2)|, |F''(\xi_1)|) \} \, dt \]
\[ = \max\{ |F''(\xi_1)|, |F''(\xi_2)|, \phi(|F''(\xi_2)|, |F''(\xi_1)|) \} \int_0^{1/2} \left( \frac{1}{2} - t \right) \, dt \]
\[ = \max\{ |F''(\xi_1)|, |F''(\xi_2)|, \phi(|F''(\xi_2)|, |F''(\xi_1)|) \} \left[ \frac{1}{13} \int_0^{1/3} \left( \frac{1}{2} - t \right) \, dt \right] \]
\[ = \frac{1}{162} \max\{ |F''(\xi_1)|, |F''(\xi_2)|, \phi(|F''(\xi_2)|, |F''(\xi_1)|) \}. \]  

\[ \mathcal{T}_2 = \int_{1/2}^1 \left( 1 - t \right) \left( t - \frac{1}{3} \right) \max\{ |F''(\xi_1)|, |F''(\xi_2)|, \phi(|F''(\xi_2)|, |F''(\xi_1)|) \} \, dt \]
\[ = \max\{ |F''(\xi_1)|, |F''(\xi_2)|, \phi(|F''(\xi_2)|, |F''(\xi_1)|) \} \int_{1/2}^1 \left( 1 - t \right) \left( t - \frac{1}{3} \right) \, dt \]
\[ = \frac{1}{162} \max\{ |F''(\xi_1)|, |F''(\xi_2)|, \phi(|F''(\xi_2)|, |F''(\xi_1)|) \}. \]  

\[ \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \mathcal{F}(x) \, dx - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \mathcal{F}(x) \, dx \leq \frac{(\xi_2 - \xi_1)^2}{81} \max\{ |F''(\xi_1)|, |F''(\xi_2)|, \phi(|F''(\xi_2)|, |F''(\xi_1)|) \}. \]  

**Corollary 3.** Theorem 3 with \( \mathcal{F}(\xi) = \mathcal{F}(\frac{\xi_1 + \xi_2}{2}) = \mathcal{F}(\xi_2) \) becomes

A simple rearrangement completes the proof.  

**Theorem 4.** Let \( \mathcal{F}: \mathcal{X} \rightarrow \mathcal{R} \) be a twice differentiable function on \( \mathcal{X} \) provided \( F'' \in L_1[\xi_1, \xi_2] \), where \( \xi_1, \xi_2 \in \mathcal{X} \) with \( \xi_1 < \xi_2 \). If \( |F''|^q \) is \( \phi \)-quasiconvex on \( [\xi_1, \xi_2] \) and \( q \geq 1 \), then we have

\[ \frac{1}{6} \left[ \frac{d}{d \xi_1} \mathcal{F}(\xi_1) + 4 \mathcal{F} \left( \frac{\xi_1 + \xi_2}{2} \right) + \mathcal{F}(\xi_2) \right] - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \mathcal{F}(x) \, dx \]
\[ \leq \frac{(\xi_2 - \xi_1)^2}{81} \left( \max\{ |F''(\xi_1)|^q, |F''(\xi_2)|^q, \phi(|F''(\xi_2)|^q, |F''(\xi_1)|^q) \} \right)^{1/q}, \]  

where \( 1/p + 1/q = 1 \).

**Proof.** Let \( q \geq 1 \), then by using Lemma 2, we have

\[ \frac{1}{6} \left[ \frac{d}{d \xi_1} \mathcal{F}(\xi_1) + 4 \mathcal{F} \left( \frac{\xi_1 + \xi_2}{2} \right) + \mathcal{F}(\xi_2) \right] - \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \mathcal{F}(x) \, dx \]
\[ \leq (\xi_2 - \xi_1)^2 \int_0^1 |k(t)| |F''(t\xi_2 + (1-t)\xi_1)| \, dt \]
\[ = (\xi_2 - \xi_1)^2 \int_0^{1/2} \left( \frac{1}{2} - t \right) |F''(t\xi_2 + (1-t)\xi_1)| \, dt \]
\[ + (\xi_2 - \xi_1)^2 \int_{1/2}^1 (1-t) \left( \frac{t}{2} - \frac{1}{3} \right) |F''(t\xi_2 + (1-t)\xi_1)| \, dt. \]
By making the use of the Hölder’s inequality for the above integrals, we have

\[
\left| \frac{1}{6} \left[ F(\xi_1) + 4F\left(\frac{\xi_1 + \xi_2}{2}\right) + F(\xi_2) \right] - \frac{1}{\xi_2 - \xi_1} \int^{\xi_2}_{\xi_1} F(x) \, dx \right|
\]

\[
\leq (\xi_2 - \xi_1)^2 \left( \int_{1/2}^{1} \left( \frac{1}{2} - t \right) \, dt \right)^{1-(1/q)} \left( \int_{1/2}^{1} \left( \frac{1}{2} - t \right) \left| F''(t\xi_2 + (1-t)\xi_1) \right|^q \, dt \right)^{1/q} + (\xi_2 - \xi_1)^2 \left( \int_{1/2}^{1} \left( 1 - t \left( \frac{t}{2} - \frac{1}{3} \right) \right) \, dt \right)^{1-(1/q)} \left( \int_{1/2}^{1} \left( 1 - t \left( \frac{t}{2} - \frac{1}{3} \right) \right) \left| F''(t\xi_2 + (1-t)\xi_1) \right|^q \, dt \right)^{1/q}.
\]

(31)

By \( \varphi \)-quasiconvexity of \( |F''|^q \) for the last two integrals, we have

\[
\int_{1/2}^{1} \left| (1-t) \left( \frac{t}{2} - \frac{1}{3} \right) \right| \left| F''(t\xi_2 + (1-t)\xi_1) \right|^q \, dt
\]

\[
\leq \int_{1/2}^{1} \left( 1 - t \left( \frac{t}{2} - \frac{1}{3} \right) \right) \max \left\{ \left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q + \varphi \left( \left| F''(\xi_2) \right|^q, \left| F''(\xi_1) \right|^q \right) \right\} \, dt
\]

\[
= \max \left\{ \left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q + \varphi \left( \left| F''(\xi_2) \right|^q, \left| F''(\xi_1) \right|^q \right) \right\} \int_{1/2}^{1} \left( 1 - t \left( \frac{t}{2} - \frac{1}{3} \right) \right) \, dt
\]

\[
= \frac{1}{162} \max \left\{ \left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q + \varphi \left( \left| F''(\xi_2) \right|^q, \left| F''(\xi_1) \right|^q \right) \right\}.
\]

(32)

By substituting (32) and (33) into (31), we have

\[
\left| \frac{1}{6} \left[ F(\xi_1) + 4F\left(\frac{\xi_1 + \xi_2}{2}\right) + F(\xi_2) \right] - \frac{1}{\xi_2 - \xi_1} \int^{\xi_2}_{\xi_1} F(x) \, dx \right|
\]

\[
\leq (\xi_2 - \xi_1)^2 \left( \int_{1/2}^{1} \left( \frac{1}{2} - t \right) \, dt \right)^{1-(1/q)} \left( \frac{1}{162} \max \left\{ \left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q + \varphi \left( \left| F''(\xi_2) \right|^q, \left| F''(\xi_1) \right|^q \right) \right\} \right)^{1/q} + (\xi_2 - \xi_1)^2 \left( \int_{1/2}^{1} \left( 1 - t \left( \frac{t}{2} - \frac{1}{3} \right) \right) \, dt \right)^{1-(1/q)} \left( \frac{1}{162} \max \left\{ \left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q + \varphi \left( \left| F''(\xi_2) \right|^q, \left| F''(\xi_1) \right|^q \right) \right\} \right)^{1/q}
\]

\[
= 2(\xi_2 - \xi_1)^2 \left( \frac{1}{162} \right)^{1-(1/q)} \left( \frac{1}{162} \max \left\{ \left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q + \varphi \left( \left| F''(\xi_2) \right|^q, \left| F''(\xi_1) \right|^q \right) \right\} \right)^{1/q}
\]

\[
= \frac{(\xi_2 - \xi_1)^2}{81} \left( \max \left\{ \left| F''(\xi_1) \right|^q, \left| F''(\xi_2) \right|^q + \varphi \left( \left| F''(\xi_2) \right|^q, \left| F''(\xi_1) \right|^q \right) \right\} \right)^{1/q}.
\]

(34)
where we used the following identity
\[
\int_0^{1/2} \left( \frac{1}{2} - t \right) dt = \int_0^{1/2} \left( 1 - t \left( \frac{1}{2} - \frac{1}{2} \right) \right) dt = \frac{1}{162}.
\] (35)

Thus we are done.

**Corollary 4.** Theorem 4 with \( \Phi(\xi_1) = \Phi((\xi_1 + \xi_2)/2) = \Phi(\xi_2) \) becomes
\[
\left( \frac{\xi_1 - \xi_2}{81} \right)^2 \left( \max \left\{ |F''(\xi_1)|^q, |F''(\xi_1)|^q + \varphi\left( |F''(\xi_1)|^q, |F''(\xi_1)|^q \right) \right) \right)^{1/q}.
\] (36)

**Corollary 5.** Theorem 4 with \( \Phi(\xi_1) = \Phi((\xi_1 + \xi_2)/2) = \Phi(\xi_2) \) becomes
\[
\left( \frac{\xi_1 - \xi_2}{81} \right)^2 \left( \max \left\{ |F''(\xi_1)|^q, |F''(\xi_1)|^q + \varphi\left( |F''(\xi_1)|^q, |F''(\xi_1)|^q \right) \right) \right)^{1/q}.
\] (37)

**Remark 5.** Theorem 4 and Corollary 4 with \( q = 1 \) become Theorem 3 and Corollary 3, respectively.

### 4. Applications

Some applications for our findings are presented.

**4.1. Applications to Special Means.** The special means are itemized as follows:

(i) The arithmetic mean:
\[
\mathcal{A} = \mathcal{A}(\xi_1, \xi_2) = \frac{\xi_1 + \xi_2}{2}, \quad \xi_1, \xi_2 \geq 0.
\] (38)

(ii) The harmonic mean:
\[
\mathcal{H} = \mathcal{H}(\xi_1, \xi_2) = \frac{2\xi_1 \xi_2}{\xi_1 + \xi_2}, \quad \xi_1, \xi_2 > 0.
\] (39)

(iii) The logarithmic mean:
\[
\mathcal{L} = \mathcal{L}(\xi_1, \xi_2) = \left\{ \begin{array}{ll}
\frac{\xi_2 - \xi_1}{\ln \xi_2 - \ln \xi_1}; & \text{if } \xi_1 \neq \xi_2, \\
\xi_1; & \text{if } \xi_1 = \xi_2,
\end{array} \right.
\] (40)

for \( \xi_1, \xi_2 > 0 \).

(iv) The \( p \)-logarithmic mean:
\[
\mathcal{L}_p = \mathcal{L}_p(\xi_1, \xi_2) = \left\{ \begin{array}{ll}
\left( \frac{\xi_2^{p-1} - \xi_1^{p-1}}{(p-1)(\xi_2 - \xi_1)} \right)^{1/p}; & \text{if } \xi_1 \neq \xi_2, \\
\xi_1; & \text{if } \xi_1 = \xi_2,
\end{array} \right.
\] (41)

for \( p \in \mathbb{R} \setminus \{-1, 0\}; \xi_1, \xi_2 > 0 \).

We know that \( \mathcal{L}_p \) is a monotonic nondecreasing function over \( p \in \mathbb{R} \) with \( \mathcal{L}_{-1} = \mathcal{A} \). In particular, we can say that \( \mathcal{A} \leq \mathcal{L} \leq \mathcal{L}_0 \).

Now, using our findings in Section 2, we conclude the following new inequalities.

**Proposition 1.** Let \( \xi_1, \xi_2 \in \mathbb{R} \) with \( 0 < \xi_1 < \xi_2 \). Then, we have
\[
|\frac{1}{3} \alpha(\xi_1, \xi_2) + \frac{2}{3} \alpha(\xi_1, \xi_2) - \mathcal{L}_0^0(\xi_1, \xi_2)| \leq \left( \frac{\xi_1 - \xi_2}{27} \right)^2 \left[ 8 \xi_1^3 + \xi_2^3 \right].
\] (42)

**Proof.** The assertion follows from Theorem 1 with \( \Phi(x) = x^4/20, x \in [\xi_1, \xi_2] \) and a simple computation, where \( |\Phi'\prime'\prime| \) is \( \varphi \)-convex function with \( \varphi(x, y) = 2x + y \) (see Example 1).

**Proposition 2.** Let \( \xi_1, \xi_2 \in \mathbb{R}, 0 < \xi_1 < \xi_2 \). Then, we have
\[
|\frac{1}{3} \alpha(\xi_1, \xi_2) + \frac{2}{3} \alpha(\xi_1, \xi_2) - \mathcal{L}_0^0(\xi_1, \xi_2)| \leq \frac{10}{81} \left( \frac{\xi_2^3 - \xi_1^3}{8} \right)^2 \left[ 2\xi_1^3 + \xi_2^3 - \xi_2^3 \right].
\] (43)

**Proof.** The assertion follows from Theorem 1 and a simple computation applied to \( \Phi(x) = x^4/20, x \in [\xi_1, \xi_2] \), where \( |\Phi'\prime'\prime| \) is \( \varphi \)-convex function with \( \varphi(x, y) = 3y^2(x - y) + 3y(x - y)^2 + (x - y)^3 \) (see Example 2).

The following proposition is a particular case of Corollary 11 in [34] when \( \lambda = 1/3 \) (see Remark 12 in [34]).
Proposition 3. Let \( \xi_1, \xi_2 \in \mathbb{R}, 0 < \xi_1 < \xi_2 \). Then, we have
\[
\left| \frac{1}{3} \mathcal{A}^2(\xi_1, \xi_2) + \frac{2}{3} \mathcal{A}^2(\xi_1, \xi_2) - \mathcal{L}^2(\xi_1, \xi_2) \right| \leq \frac{2(\xi_2 - \xi_1)^2}{81}.
\]
(44)

Proof. The assertion follows from Theorem 3 and a simple computation applied to \( \mathcal{F}(x) = x^2, x \in [\xi_1, \xi_2] \), where \( |\mathcal{F}''(x)| = 2 \) is \( \varphi \)-quasiconvex function with \( \varphi(x, y) = x - y \) (see Example 5).

Proposition 4. Let \( \xi_1, \xi_2 \in \mathbb{R}, 0 < \xi_1 < \xi_2 \). Then, for all \( q > 1 \), we have
\[
\left| \frac{1}{3} \mathcal{H}^{-1}(\xi_1, \xi_2) + \frac{2}{3} \mathcal{H}^{-1}(\xi_1, \xi_2) - \mathcal{L}^{-1}(\xi_1, \xi_2) \right| \leq \frac{(\xi_2 - \xi_1)^2}{81} \max \left\{ \frac{2^q}{\xi_1}, \frac{2^q}{\xi_2} \right\}.
\]
(45)

Proof. The assertion follows from Theorem 4 and a simple computation applied to \( \mathcal{F}(x) = 1/x, x \in [\xi_1, \xi_2] \), where \( |\mathcal{F}''(x)| = 2/\xi^3 \) is \( \varphi \)-quasiconvex function with \( \varphi(x, y) = x - y \) (see Example 4).

4.2. Applications to Simpson’s Formula. Let \( \mathcal{D} \) be a partition of the interval \([\xi_1, \xi_2] \); that is, \( \mathcal{D} : \xi_1 = s_0 < s_1 < \cdots < s_n = \xi_2; h_i = (s_{i+1} - s_i)/2 \) and consider Simpson’s formula:
\[
S(\mathcal{F}, \mathcal{D}) = \sum_{i=1}^{n} \frac{1}{6} \mathcal{F}(s_0) + 4\mathcal{F}(s_i + h_i) + \mathcal{F}(s_{i+1})(s_{i+1} - s_i).
\]
(46)

We know that if \( \mathcal{F} : [\xi_1, \xi_2] \rightarrow \mathbb{R} \) is differentiable such that \( \mathcal{F}^{(1)}(x) \) exists on \([\xi_1, \xi_2] \) and \( K = \max_{x \in [\xi_1, \xi_2]} |\mathcal{F}^{(1)}(x)| \) \( < \infty \). Then, we have
\[
I = \int_{\xi_1}^{\xi_2} \mathcal{F}(s)ds = S(\mathcal{F}, \mathcal{D}) + E_i(\mathcal{F}, \mathcal{L}),
\]
(47)
where the approximation error \( E_i(\mathcal{F}, \mathcal{L}) \) satisfies
\[
|E_i(\mathcal{F}, \mathcal{L})| \leq \frac{K}{90} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^5.
\]
(48)

It is clear that if the function \( \mathcal{F} \) is not four times differentiable or \( \mathcal{F}^{(4)} \) is not bounded on \([\xi_1, \xi_2] \), then (47) cannot be applied.

Theorem 5. Let \( \mathcal{F} : \mathbb{R} \rightarrow \mathbb{R} \) be a twice differentiable function on \( \mathbb{R} \) such that \( \mathcal{F}'' \in L_1[\xi_1, \xi_2] \), where \( \xi_1, \xi_2 \in \mathbb{R} \) with \( \xi_1 < \xi_2 \). If \( |\mathcal{F}''| \) is \( \varphi \)-convex on \([\xi_1, \xi_2] \), then for every division \( \mathcal{D} \) of \([\xi_1, \xi_2] \) we have
\[
\left| E_i(\mathcal{F}, \mathcal{D}) \right| \leq \frac{1}{81} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left| \mathcal{F}''(s_i) + \frac{1}{2} \varphi(\mathcal{F}''(s_i), \mathcal{F}''(s_{i+1})) \right|.
\]
(49)

Proof. By applying Theorem 1 on the subintervals \([s_i, s_{i+1}] \), \( i = 0, 1, 2, \ldots, n-1 \) of the division \( \mathcal{D} \) to get
\[
\left| \frac{1}{6} \mathcal{F}(s_i) + 4\mathcal{F}(s_i + h_i) + \mathcal{F}(s_{i+1})(s_{i+1} - s_i) \right| \leq \frac{(s_{i+1} - s_i)^3}{6} \left| \mathcal{F}''(s_i) + \frac{1}{2} \varphi(\mathcal{F}''(s_i), \mathcal{F}''(s_{i+1})) \right|.
\]
(50)

By summing over \( i \) from 0 to \( n-1 \) and taking into account that \( |\mathcal{F}''| \) is \( \varphi \)-convex to get
\[
\left| S(\mathcal{F}, \mathcal{D}) - \int_{\xi_1}^{\xi_2} \mathcal{F}(s)ds \right| \leq \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left| \mathcal{F}''(s_i) + \frac{1}{2} \varphi(\mathcal{F}''(s_i), \mathcal{F}''(s_{i+1})) \right|.
\]
(51)
which completes our proof.

Corollary 6. Theorem 5 with \( \varphi(x, y) = y - x \) becomes
\[
\left| E_i(\mathcal{F}, \mathcal{D}) \right| \leq \frac{1}{162} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left| \mathcal{F}''(s_i) + \mathcal{F}''(s_{i+1}) \right|.
\]
(52)

Theorem 6. Let \( \mathcal{F} : \mathbb{R} \rightarrow \mathbb{R} \) be a twice differentiable function on \( \mathbb{R} \) such that \( \mathcal{F}'' \in L_1[\xi_1, \xi_2] \), where \( \xi_1, \xi_2 \in \mathbb{R} \) with \( \xi_1 < \xi_2 \). If \( |\mathcal{F}''(q)_q \geq 1 \) is \( \varphi \)-convex on \([\xi_1, \xi_2] \), then for every division \( \mathcal{D} \) of \([\xi_1, \xi_2] \) we have
\[
\left| E_i(\mathcal{F}, \mathcal{D}) \right| \leq \frac{1}{162} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left| \mathcal{F}''(s_i) + \mathcal{F}''(s_{i+1}) \right|.
\]
(53)
where

\[
K^q_p(\mathbf{F}''(s_i), \mathbf{F}''(s_{i+1})) = \left( \frac{1}{162} |\mathbf{F}''(s_i)|^q + \frac{59}{31104} \phi(|\mathbf{F}''(s_i)|^q, |\mathbf{F}''(s_{i+1})|^q) \right)^{1/q} + \left( \frac{1}{162} |\mathbf{F}''(s_{i+1})|^q + \frac{133}{31104} \phi(|\mathbf{F}''(s_i)|^q, |\mathbf{F}''(s_{i+1})|^q) \right)^{1/q}.
\]

(55)

**Proof.** The proof follows from Theorem 2 directly. \(\Box\)

**Proposition 6.** Let \( \mathbf{F}: \mathfrak{F} \subseteq [0, \infty) \rightarrow \mathds{R} \) be a twice differentiable function on \( \mathfrak{F} \) such that \( \mathbf{F}'' \in L_1[\xi_1, \xi_2] \), where \( \xi_1, \xi_2 \in \mathfrak{F} \) with \( \xi_1 < \xi_2 \). If \( |\mathbf{F}''|^q \geq 1 \) is \( \varphi \)-quasiconvex on \( [\xi_1, \xi_2] \), then we have

\[
|E_\xi(\mathbf{F}, \mathcal{D})| \leq \frac{1}{51} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \max\{|\mathbf{F}''(s_i)|^q, |\mathbf{F}''(s_{i+1})|^q|^{1/q} + \phi(|\mathbf{F}''(s_i)|^q, |\mathbf{F}''(s_{i+1})|^q)|. \quad (56)
\]

**Proof.** The proof follows from Theorem 4 directly. \(\Box\)

4.3. **Applications to the Midpoint Formula.** Let \( \mathcal{D} \) be a partition as before. Here we consider the midpoint formula:

\[
M(\mathbf{F}, \mathcal{D}) = \sum_{i=0}^{n-1} (s_{i+1} - s_i) \mathbf{F}(\frac{s_i + s_{i+1}}{2}). \quad (57)
\]

Suppose that the function \( \mathbf{F}: [\xi_1, \xi_2] \rightarrow \mathds{R} \) is differentiable with \( \mathbf{F}''(x) \) existing on \( [\xi_1, \xi_2] \) and \( K = \sup_{x \in [\xi_1, \xi_2]} |\mathbf{F}''(x)| < \infty \), and then, we have

\[
I = \int_{\xi_1}^{\xi_2} \mathbf{F}(s)ds = M(\mathbf{F}, \mathcal{D}) + E_M(\mathbf{F}, \mathcal{D}), \quad (58)
\]

where the approximation error \( E_M(\mathbf{F}, \mathcal{D}) \) satisfies

\[
|E_M(\mathbf{F}, \mathcal{D})| \leq \frac{K}{24} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3. \quad (59)
\]

**Proposition 7.** Let \( \mathbf{F}: \mathfrak{F} \rightarrow \mathds{R} \) be a twice differentiable function on \( \mathfrak{F} \), \( \xi_1, \xi_2 \in \mathfrak{F} \) with \( \xi_1 < \xi_2 \). If \( |\mathbf{F}''|^q \) is \( \varphi \)-convex on \( [\xi_1, \xi_2] \) and \( q \geq 1 \), then for any division \( \mathcal{D} \) of \( [\xi_1, \xi_2] \), we have

\[
|E_M(\mathbf{F}, \mathcal{D})| \quad \leq \quad \frac{1}{81} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left[ |\mathbf{F}''(s_i)|^q + \frac{1}{2} \phi(|\mathbf{F}''(s_i)|^q, |\mathbf{F}''(s_{i+1})|^q) \right]. \quad (60)
\]

**Proof.** By applying Corollary 1 on the subintervals \([s_i, s_{i+1}]\), \( i = 0, 1, \ldots, n-1 \) of the division \( \mathcal{D} \), to get

\[
|E_M(\mathbf{F}, \mathcal{D})| \leq \left( \frac{1}{162} \right)^{1-\frac{1}{q}} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left[ K^q_p(\mathbf{F}''(s_i), \mathbf{F}''(s_{i+1})) \right]. \quad (63)
\]
Proof. By applying Corollary 2 on the subintervals \([s_i, s_{i+1}], (i = 0, 1, \ldots, n - 1)\) of the division \(\mathcal{D}\) to get
\[
\left| (s_{i+1} - s_i) F \left( \frac{s_{i+1} + s_i}{2} \right) - \int_{s_i}^{s_{i+1}} F(s)ds \right|
\]
\[
\leq \left( \frac{1}{162} \right)^{1 - (1/q)} (s_{i+1} - s_i)^3 \left[ K_p^{q}(F''(s_i), F''(s_{i+1})) \right],
\]
where
\[
K_p^{q}(F''(s_i), F''(s_{i+1}))
\]
\[
= \left( \frac{1}{162} \right)^{1 - (1/q)} \max \left[ |F''(s_i)|, |F''(s_{i+1})| + \varphi \left( |F''(s_i)|, |F''(s_{i+1})| \right) \right]^{1/q}.
\]
By summing over \(i\) from 0 to \(n - 1\) to get
\[
\left| E_M(F, \mathcal{D}) \right| \leq \frac{1}{81} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \max \left[ |F''(s_i)|, |F''(s_{i+1})| + \varphi \left( |F''(s_i)|, |F''(s_{i+1})| \right) \right].
\]
which completes our proof. □

Proposition 9. Let \(F: \mathfrak{F} \rightarrow \mathbb{R}\) be a twice differentiable function on \(\mathfrak{F}\), \(\xi_1, \xi_2 \in \mathfrak{F}\) with \(\xi_1 < \xi_2\). If \(|F''|\) is \(\varphi\)-quasiconvex on \([\xi_1, \xi_2]\), then for any division \(\mathcal{D}\) of \([\xi_1, \xi_2]\), we have
\[
|E_M(F, \mathcal{D})| \leq \left( \frac{1}{162} \right)^{1 - (1/q)} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left[ K_p^{q}(F''(s_i), F''(s_{i+1})) \right],
\]
which completes our proof. □

Proposition 10. Let \(F: \mathfrak{F} \rightarrow \mathbb{R}\) be a twice differentiable function on \(\mathfrak{F}\), \(\xi_1, \xi_2 \in \mathfrak{F}\) with \(\xi_1 < \xi_2\). If \(|F''|\) is \(\varphi\)-quasiconvex on \([\xi_1, \xi_2]\) and \(q \geq 1\), then in (30), for every division \(\mathcal{D}\) of \([\xi_1, \xi_2]\), we have
\[
|E_M(F, \mathcal{D})| \leq \frac{1}{81} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \max \left[ |F''(s_i)|^q, |F''(s_{i+1})|^q + \varphi \left( |F''(s_i)|^q, |F''(s_{i+1})|^q \right) \right].
\]
Proof. By applying Corollary 4 on the subintervals \([s_i, s_{i+1}], (i = 0, 1, \ldots, n - 1)\) of the division \(\mathcal{D}\), we get
\[
\left| (s_{i+1} - s_i) F \left( \frac{s_{i+1} + s_i}{2} \right) - \int_{s_i}^{s_{i+1}} F(s)ds \right|
\]
\[
\leq \left( \frac{1}{162} \right)^{1 - (1/q)} (s_{i+1} - s_i)^3 \left[ K_p^{q}(F''(s_i), F''(s_{i+1})) \right],
\]
By summing over \(i\) from 0 to \(n - 1\) to get
\[
\left| E_M(F, \mathcal{D}) \right| \leq \left( \frac{1}{162} \right)^{1 - (1/q)} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \left[ K_p^{q}(F''(s_i), F''(s_{i+1})) \right],
\]
which completes our proof. □
By summing over $i$ from 0 to $n - 1$ to get
\[
|E_M(T, \varphi)| \leq \frac{1}{81} \sum_{i=0}^{n-1} (s_{i+1} - s_i)^3 \max \left\{ |F''(s_i)|^q, |F''(s_{i+1})|^q + \varphi( |F''(s_{i+1})|^q, |F''(s_i)|^q) \right\},
\]
which rearranges to the proof.

\section{5. Illustrative Plots}

Finally, we present two three-dimensional plots to demonstrate the validity of the inequalities (42) and (44) in the case of $\varphi$-convex and $\varphi$-quasiconvex functions, respectively.

From inequality (42), we can define
\[
v(x, y) = \frac{1}{3} \varphi(x^3, y^3) + \frac{2}{3} \varphi(x, y) - L_1^3(x, y)
\]
\[V(x, y) = \frac{q}{27} \left[ \frac{(y-x)^2}{8x^2 + y^2} \right].
\]
(74)

Thus, Figure 1 represents the plot of inequality (42) and $V(x, y) - v(x, y)$.

From inequality (44), we can define
\[
w(x, y) = \frac{1}{3} \varphi(x^2, y^2) + \frac{2}{3} \varphi(x, y) - L_1^4(x, y)
\]
\[W(x, y) = \frac{2(xy-x^2)}{81}.
\]
(75)

Thus, Figure 2 represents the plot of inequality (44) and $W(x, y) - w(x, y)$.

\section{6. Conclusion}

In this study, we have considered Simpson's type integral inequalities for the $\varphi$-convex and $\varphi$-quasiconvex functions in the second derivative sense. Some special cases of our findings are investigated to show the powerfulness of our results. Also, the proposed inequalities can be applied to other mathematical and statistical models, as we have shown in Section 4.
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