RIGHT EXACT GROUP COMPLETION AS A TRANSFINITE INVARIANT OF HOMOLOGY EQUIVALENCE

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ABSTRACT. We consider a functor from the category of groups to itself $G \mapsto \mathbb{Z}_\infty G$ that we call right exact $\mathbb{Z}$-completion of a group. It is connected with the pronilpotent completion $\hat{G}$ by the short exact sequence $1 \to \lim^1 M_n G \to \mathbb{Z}_\infty G \to \hat{G} \to 1$, where $M_n G$ is $n$-th Baer invariant of $G$. We prove that $\mathbb{Z}_\infty \pi_1 (X)$ is an invariant of homological equivalence of a space $X$. Moreover, we prove an analogue of Stallings’ theorem: if $G \to G'$ is a 2-connected group homomorphism, then $\mathbb{Z}_\infty G \cong \mathbb{Z}_\infty G'$. We give examples of 3-manifolds $X, Y$ such that $\overline{\pi_1 (X)} \cong \overline{\pi_1 (Y)}$ but $\mathbb{Z}_\infty \pi_1 (X) \neq \mathbb{Z}_\infty \pi_1 (Y)$. We prove that for a finitely generated group $G$ we have $(\mathbb{Z}_\infty G)/\gamma_\omega = \hat{G}$. So the difference between $\hat{G}$ and $\mathbb{Z}_\infty G$ lies in $\gamma_\omega$. This allows us to treat $\mathbb{Z}_\infty \pi_1 (X)$ as a transfinite invariant of $X$. The advantage of our approach is that it can be used not only for 3-manifolds but for arbitrary spaces.

Introduction

The main motivation of this research is the old problem of J. Milnor, of finding a realizable transfinite version of $\mu$-invariants for links [18]. The pronilpotent completion of the link group provides a natural concordance invariant. How one can find a transfinite analog of this invariant, which can differ links with the same group completion up to concordance? We are not able to construct such an invariant for links, but one can extend the problem to the class of all spaces (or 3-manifolds) and change the concordance by homology equivalence (or homology cobordism). The pronilpotent completion of the fundamental group provides a natural homology equivalence invariant. In this paper we consider one of its transfinite extensions.

For a group $G$ we denote by $\gamma_n G$ its lower central series and by $\hat{G}$ the pronilpotent completion

$$\hat{G} = \varinjlim G/\gamma_n G.$$ 

It is well-known that the pronilpotent completion of the fundamental group is an invariant of $\mathbb{Z}$-homological equivalence, i.e. if $X \to Y$ is a $\mathbb{Z}$-homological equivalence of spaces, then $\overline{\pi_1 (X)} \to \overline{\pi_1 (Y)}$ is an isomorphism. In this paper we present a functor from the category of groups to itself $G \mapsto \mathbb{Z}_\infty G$, that we call the right exact $\mathbb{Z}$-completion, which gives a stronger invariant of homological equivalence than the pronilpotent completion. Moreover, we give examples of 3-manifolds $X, Y$, such that $\overline{\pi_1 (X)} \cong \overline{\pi_1 (Y)}$ but $\mathbb{Z}_\infty \pi_1 (X) \neq \mathbb{Z}_\infty \pi_1 (Y)$.

If $X$ is a space we denote by $\mathbb{Z}_\infty X$ its Bousfield-Kan $\mathbb{Z}$-completion [5]. Then by definition we set

$$\mathbb{Z}_\infty G := \overline{\pi_1 (\mathbb{Z}_\infty BG)},$$

where $BG$ is the classifying space. For a group $G$ we denote by $M_n G$ its $n$-th Baer invariant, which is also known as $n$-nilpotent multiplier (see [6], [15], [11] or Section 2 for the definition of Baer invariants). We prove that there is a short exact sequence

$$1 \longrightarrow \lim^1 M_n G \longrightarrow \mathbb{Z}_\infty G \longrightarrow \hat{G} \longrightarrow 1.$$ 

Moreover, for a finitely generated group $G$ this short exact sequence induces isomorphisms

$$\gamma_\omega (\mathbb{Z}_\infty G) \cong \lim^1 M_n G, \quad \mathbb{Z}_\infty G/\gamma_\omega \cong \hat{G},$$

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where $\gamma_\omega$ is the intersection of the lower sentral series (Proposition 3.13). We also prove an analogue of Stallings’ theorem for the right exact $\mathbb{Z}$-completion: a 2-connected homomorphism $G \to G'$ induces an isomorphism $\mathbb{Z}_0 G \cong \mathbb{Z}_0 G'$ (Proposition 3.6).

We also give a description of $\mathbb{Z}_\infty G$ on the language of combinatorial group theory. Namely, if $G = F/R$ is a free presentation of $G$, then there is an exact sequence

$$\hat{R}_F \to \hat{F} \to Z_\infty G \to 1,$$

where $\hat{R}_F = \lim R/\gamma_n(R, F)$ and $\gamma_n(R, F)$ is defined by recursion: $\gamma_1(R, F) = R$ and $\gamma_{n+1}(R, F) = [\gamma_n(R, F), F]$ (see Proposition 3.12). Moreover, we prove the following statement. If $U \to E \to G$ is a short exact sequence of groups such that $E$ is finitely presented and $H_2 E$ is finite, then there is an exact sequence

$$\hat{U}_E \to \hat{E} \to Z_\infty G \to 1$$

(see Proposition 3.10). This statement is the main tool for computation of $Z_\infty G$. In order to compute $Z_\infty G$, one has to find a finitely presented extension $E$ with finite $H_2 E$ and computable $\hat{E}$ and $\hat{U}_E$.

The homotopy type of the Bousfield-Kan $\mathbb{Z}$-completion $Z_\infty X$ is invariant with respect to $\mathbb{Z}$-homology equivalence i.e. if $X \to Y$ is a $\mathbb{Z}$-homology equivalence, then $Z_\infty X \to Z_\infty Y$ is a homotopy equivalence [5]. Therefore $\pi_1(Z_\infty X)$ is invariant with respect to $\mathbb{Z}$-homological equivalence. We prove that

$$\pi_1(Z_\infty X) = Z_\infty \pi_1(X)$$

(Proposition 3.3). Hence $Z_\infty \pi_1(X)$ is an invariant of $\mathbb{Z}$-homological equivalence. If $\pi_1(X)$ is a finitely generated group, we can recover $\overline{\pi_1(X)}$ from $Z_\infty \pi_1(X)$ as follows

$$\overline{\pi_1(X)} = Z_\infty \pi_1(X)/\gamma_\omega.$$ 

In this sense $Z_\infty \pi_1(X)$ is a stronger invariant than $\overline{\pi_1(X)}$.

We construct examples of 3-manifolds that can’t be distinguished by $\hat{\pi}_1$ but can be distinguished by $Z_\infty \pi_1$. Consider the following matrix $a_k = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}$, where $k$ is an odd integer number. The matrix defines the following semidirect product

$$G_k = \mathbb{Z} \rtimes_{a_k} \mathbb{Z}^2,$$

where $\mathbb{Z}$ acts on $\mathbb{Z}^2$ by $a_k$. The construction of mapping torus associated with the homeomorphism on $(S^1)^2$ induced by $a_k$ gives an aspherical 3-manifold $X_k = K(G_k, 1)$ with a fiber sequence

$$(S^1)^2 \to X_k \to S^1$$

and the monodromy action induced by $a_k$.

**Theorem 4.1.** Let $k$ and $l$ be two odd integers. Then the following holds.

1. $\overline{\pi_1(X_k)} \cong \overline{\pi_1(X_l)}$.
2. If $kl \neq 1, 7 \pmod{8}$, then $Z_\infty \pi_1(X_k) \ncong Z_\infty \pi_1(X_l)$.

The condition $kl \neq 1, 7 \pmod{8}$ is connected with the fact that a $2$-adic integer $\alpha \in \mathbb{Z}_2$ is a square if and only if $\alpha \equiv 1 \pmod{8}$. We use the group $Z_\infty \pi_1$ in order to differ 3-manifolds. However, the advantage of this invariant is that it can be used for arbitrary spaces.

In this paper we use the notion of right exact functor in the sense of Keune (see [14], [11]). Let us define it in a general categorial setting. Let $C$ be a category. A couple of epimorphisms $\alpha_0, \alpha_1 : c' \twoheadrightarrow c$ is called split couple of epimorphisms if there exists a morphism $s : c' \to c$ such that $\alpha_0 s = \text{id}_c = \alpha_1 s$. In other words, $s$ is a splitting for both epimorphisms $\alpha_0, \alpha_1$ at the same time. Such couples of epimorphisms occur in the beginning of a simplicial object. A functor

$$\Phi : C \to D$$
is called \textit{right exact} (in the sense of Keune) if it commutes with coequalizers of split couples of epimorphisms. It is proven in [1] that a functor on the category of groups \( \Phi : \text{Gr} \to \text{Gr} \) is right exact if and only if the natural map from its zero derived functor

\[ L_0\Phi \to \Phi \]

is an isomorphism. Moreover, the zero derived functor \( L_0\Phi \) is a right exact functor for any functor \( \Phi \). Hence, taking the zero derived functor is a universal way to make a right exact functor from a functor. We prove that the functor \( G \mapsto \mathbb{Z}_\infty G \) is the zero derived functor of the functor \( G \mapsto \hat{G} \)

\[(G \mapsto \mathbb{Z}_\infty G) = L_0(G \mapsto \hat{G}).\]

In particular, \( G \mapsto \mathbb{Z}_\infty G \) is right exact, while \( G \mapsto \hat{G} \) is not right exact. That is why we call \( \mathbb{Z}_\infty G \) the \textit{right exact} \( \mathbb{Z}_\infty \)-completion of \( G \).

Also we have to mention that the paper [7] played a significant role in developing transfinite methods in homology cobordism of 3-manifolds. In a forthcoming paper [8], Jae Choon Cha and Kent E. Orr have developed a full theory of Milnor invariants for 3-manifolds. Their homology cobordism invariants, indexed on finite and infinite ordinals, include Milnor’s classical link invariants as a special case, arising as invariants of 0-surgery on a link. As one corollary of their theory, they classify all possible transfinite lower central series quotients of the Vogel localized groups of oriented, closed 3-manifolds. Some of their significant examples were initially suggested by our results, and use the same twisted torus bundles. Although these examples prove interesting in the context of their paper and ours, the connections between their methods and ours seem provocative and interesting.

1. \textbf{Right exact functors in the sense of Keune}

Let \( \mathcal{C} \) be a category. A couple of epimorphisms \( \alpha_0, \alpha_1 : c' \rightrightarrows c \) is called \textit{split couple of epimorphisms} if there exists a map \( s : c' \to c \) such that \( \alpha_0 s = \text{id}_c = \alpha_1 s \). In other words, \( s \) is a splitting for both epimorphisms \( \alpha_0, \alpha_1 \) at the same time. A functor

\[ \Phi : \mathcal{C} \to \mathcal{D} \]

is called \textit{right exact} (in the sense of Keune) if for any split couple of morphisms \( \alpha_0, \alpha_1 : c' \rightrightarrows c \) the couple \( (\alpha_0, \alpha_1) \) has the coequalizer in \( \mathcal{C} \), the couple \( (\Phi \alpha_0, \Phi \alpha_1) \) has the coequalizer in \( \mathcal{D} \) and the natural morphism

\[ \text{coeq}(\Phi \alpha_0, \Phi \alpha_1) \to \Phi(\text{coeq}(\alpha_0, \alpha_1)) \]

is an isomorphism.

\textbf{Remark 1.1.} Some authors call a functor right exact if it commutes with finite colimits. We do not use this notion. By a right exact functor we always mean a right exact functor in this weaker sense of Keune.

In this section we present a result of [1] about right exact functors on the category of groups. Namely, we give several equivalent descriptions of right exact functors on the category of groups.

Following Kan [13], [10], we say that a simplicial group \( F_\bullet \) is free if all groups \( F_n \) are free and it is possible to chose bases of these groups so that they are stable under degeneracy maps. A free simplicial resolution is a weak equivalence of simplicial groups \( F_\bullet \to G \), where \( G \) is considered as a constant simplicial group and \( F_\bullet \) is a free simplicial group. The derived functor of a functor \( \Phi : \text{Gr} \to \text{Gr} \) is defined as follows \( L_n\Phi(G) := \pi_n(\Phi(F_\bullet)) \), where \( F_\bullet \to G \) is a free simplicial resolution. Note that there is a natural map from the zero derived functor to the functor itself

\[ L_0\Phi \to \Phi. \]

Let \( G \) be a group. We say that \( U \) is a \( G \)-group if \( U \) is a group together with a right action of \( G \) on \( U \) by automorphisms. In this case we can consider the semidirect product \( G \ltimes U \). Morphisms
of $G$-groups are homomorphisms preserving the action of $G$. The category of $G$-groups is denoted by $G\text{-Gr}$. A normal subgroup $U$ of $G$ will be always considered as a $G$-group with the action by conjugation.

Let $\Phi : \text{Gr} \to \text{Gr}$ be a functor. For a group $G$ we consider a functor from the category of $G$-groups to the category of $\Phi G$-groups

$$\Phi_G : G\text{-Gr} \to \Phi G\text{-Gr}$$

given by

$$\Phi_G U := \text{Ker}(\Phi(G \times U) \to \Phi G).$$

The action of $\Phi G$ on $\Phi_G U$ goes via the map $\Phi G \to \Phi(G \times U)$. Note that there is an isomorphism

$$\Phi(G \times U) = \Phi G \times \Phi_G U.$$ 

Akhtiamov, Ivanov and Pavutnitskiy proved the following statement.

**Theorem 1.2** ([1], Th.1.5). The following statements about a functor $\Phi : \text{Gr} \to \text{Gr}$ are equivalent.

1. $\Phi$ is right exact.
2. The natural map $L_0 \Phi \to \Phi$ is an isomorphism.
3. For a short exact sequence $U \to G \to H$ the sequence

$$\Phi_G U \to \Phi G \to \Phi H \to 1$$

is exact, where the map $\Phi_G U \to \Phi G$ is induced by the map $G \times U \to G, (g, u) \mapsto gu$.
4. For a simplicial group $G_\bullet$, the natural map $\pi_0(\Phi G_\bullet) \to \Phi(\pi_0(G_\bullet))$ is an isomorphism.

**Remark 1.3.** Note that for any functor $\Phi$ the functor $L_0 \Phi$ is right exact. Indeed, the functors $\Phi$ and $L_0 \Phi$ are equal on free groups, and hence $L_0(L_0 \Phi) \cong L_0 \Phi$.

2. **Baer invariants of groups**

It is known that integral homology of a group $H_*G = H_*(G, \mathbb{Z})$ are derived functors of the functor of abelianization [19, Ch. II. §5]. More precisely, if we take a free simplicial resolution $F_\bullet \to G$ then

$$H_{n+1}G = \pi_n((F_\bullet)_{ab}).$$

There is another sequence of functors starting from $H_2G$ which can be defined on the language of derived functors.

We denote by $\gamma_n G$ the lower central series of $G$ and by $\nu_n G$ we denote the quotient $\nu_n G = G/\gamma_{n+1} G$. We say that $\nu_n G$ is the $n$-nilpotenization of $G$. We define the Baer invariant $M_n$ as the first derived functor of the functor of $n$-nilpotenization:

$$M_n G = \pi_1(\nu_n F_\bullet).$$

Derived functors do not depend of the choice of the resolution, so $M_n G$ is an invariant of $G$. The first homotopy group of a simplicial group is abelian. Hence $M_1 G$ is an abelian group. Note that $M_1 G = H_2 G$. If we present the group $G$ as a quotient of a free group $G = F/\mathcal{R}$, then there is the following version of Hopf’s formula for the Baer invariant

$$M_n G = \frac{R \cap \gamma_{n+1} F}{\gamma_{n+1} (R, F)},$$

where $\gamma_{n+1} (R, F) = [\gamma_n (R, F), F]$ and $\gamma_1 (R, F) = R$ (see [6], [15], [11]). Baer invariant $M_n G$ is also known as $n$-nilpotent multiplier which is the reason for the notation.

**Proposition 2.1.** If $G$ is finitely presented, then $M_n G$ is finitely generated for any $n$. 

Proof. Let $G = F/R$ be a free presentation, where $F$ is a finitely generated free group and $R$ is finitely generated as a normal subgroup. Then $F \ltimes R$ is a finitely generated group. Note that
\[ \nu_n(F \ltimes R) = \nu_n F \ltimes R/\gamma_{n+1}(R, F). \]
A subgroup of a finitely generated nilpotent group is finitely generated. Therefore $(R \cap \gamma_{n+1}F)/\gamma_{n+1}(R, F)$ is finitely generated because it is a subgroup of $\nu_n(F \ltimes R)$. □

Ellis proved the following theorem about Baer invariants.

**Theorem 2.2** ([11, Th. 2], [17, Th. 1.74]). If $H_2G$ is a torsion group, then $M_nG$ is a torsion group for any $n$.

**Corollary 2.3.** If $G$ is finitely presented and $H_2G$ is finite, then $M_nG$ is finite for any $n$.

Proof. This follows from Proposition [2.1] and Theorem [2.2] □

### 3. Right exact $\mathbb{Z}$-completion of a group.

For a group $G$ we define its pro-nilpotent completion as the inverse limit of its $n$-nilpotenizations
\[ \hat{G} = \lim_{\leftarrow} \nu_nG, \]
where $\nu_nG = G/\gamma_{n+1}G$ and $\gamma_{\ast}G$ is the lower central series. The functor of pro-nilpotent completion is not right exact. However its zero derived functor is right exact.

Let $R$ be either a subring of $\mathbb{Q}$ or $\mathbb{Z}/n$. For a group $G$ we define its right exact $R$-completion $R_\infty G$ as the fundamental group of the Bousfield-Kan $R$-completion [5] of the classifying space:
\[ R_\infty G := \pi_1(R_\infty BG). \]
The map $BG \to R_\infty BG$ induces a map
\[ G \to R_\infty G. \]
In this paper we will be interested only in the case $R = \mathbb{Z}$.

Further we use the homotopy theory of simplicial groups ([16, Ch. VI], [5, Ch. IV]). In particular, we use the Kan’s loop functor $\mathcal{G}$ from the category of reduced simplicial sets to the category of simplicial groups and the functor of classifying space of a simplicial group $\bar{W}$:
\[ \mathcal{G} : \text{sSets}_{\text{red}} \rightleftarrows \text{sGr} : \bar{W}. \]
We use the following interpretation of the Bousfield-Kan $\mathbb{Z}$-completion of a reduced simplicial set $X$.
\[ Z_\infty X_{\ast} \simeq \bar{W}(\mathcal{G} X_{\ast}) \]
(see [3, Ch. IV, Prop. 4.1]). One of the main properties of the Bousfield-Kan $\mathbb{Z}$-completion $Z_\infty X_{\ast}$ is that its homotopy type is an invariant of homological equivalence i.e. if $X \to Y$ is a $\mathbb{Z}$-homological equivalence, then $Z_\infty X \to Z_\infty Y$ is a homotopy equivalence.

**Proposition 3.1.** The functor of right exact $\mathbb{Z}$-completion of groups is the zero derived functor of the functor of pro-nilpotent completion of groups
\[ (G \mapsto Z_\infty G) = L_0(G \mapsto \hat{G}). \]
In particular, the functor $G \mapsto Z_\infty G$ is right exact. Moreover, for any group $G$ there is a natural short exact sequence
\[ 1 \to \lim_{\leftarrow}^1 M_nG \to Z_\infty G \to \hat{G} \to 1. \]

**Remark 3.2.** The group $\lim_{\leftarrow}^1 M_nG$ is abelian. However, the extension (3.1) is not necessarily central.
Proof. If we take \( X_\ast = \mathcal{W}G \), where \( G \) is considered as a constant simplicial group, we obtain a simplicial classifying space of \( G \). Then \( F_\ast = \mathcal{G}(\mathcal{W}G) \rightarrow G \) is a free simplicial resolution of \( G \). Hence
\[
\mathbb{Z}_\infty G = \pi_1(\mathbb{Z}_\infty BG) = \pi_1(\mathcal{W}F_\ast) = \pi_0(\hat{F}_\ast).
\]
Therefore \( G \rightarrow \mathbb{Z}_\infty G \) is the zero derived functor of the functor of pro-nilpotent completion. The Milnor’s short exact sequence has the following form here
\[
1 \rightarrow \lim^{-1} \pi_1(\nu_n F_\ast) \rightarrow \pi_0(\lim_{n} \nu_n F_\ast) \rightarrow \lim \pi_0(\nu_n F_\ast) \rightarrow 1.
\]
The functor \( \nu_n \) is right exact, and hence, \( \pi_0(\nu_n F_\ast) = \nu_n G \), which gives the required short exact sequence. \( \square \)

**Proposition 3.3.** Let \( X \) be a path-connected pointed space. Then
\[
\pi_1(\mathbb{Z}_\infty X) = \mathbb{Z}_\infty(\pi_1(X)).
\]

**Proof.** We replace \( X \) by a reduced simplicial set \( X_\ast \). Consider the simplicial group \( G_\ast = \mathcal{G}X_\ast \). The fact that \( G \rightarrow \mathbb{Z}_\infty G \) is a right exact functor together with Theorem 1.2 (1)⇒(4) imply
\[
\pi_0(\mathbb{Z}_\infty G_\ast) = \mathbb{Z}_\infty(\pi_1(G_\ast)).
\]
Since \( G_\ast \) consists of free groups, we obtain \( \mathbb{Z}_\infty G_\ast = \hat{G}_\ast \). Therefore
\[
\pi_1(\mathbb{Z}_\infty X_\ast) = \pi_1(\mathcal{W}\hat{G}_\ast) = \pi_0(\hat{G}_\ast) = \mathbb{Z}_\infty \pi_0(G_\ast) = \mathbb{Z}_\infty \pi_1(X_\ast).
\]

**Corollary 3.4.** The group \( \mathbb{Z}_\infty \pi_1(X) \) is an invariant of homological equivalence of spaces.

A group homomorphism \( G \rightarrow G' \) is said to be 2-connected if it induces an isomorphism on the first homology groups \( H_1 G \cong H_1 G' \) and an epimorphism on the second homology groups \( H_2 G \twoheadrightarrow H_2 G' \). Bousfield proved the following lemma.

**Lemma 3.5** ([3] Lemma 6.1]). A homomorphism \( f : G \rightarrow G' \) is 2-connected if and only if there is a \( \mathbb{Z} \)-homological equivalence of path-connected pointed spaces \( \hat{f} : X \rightarrow X' \) such that \( \pi_1(\hat{f}) \) is isomorphic \( f \) (in the category of homomorphisms of groups).

Stallings’ theorem [20] says that any 2-connected homomorphism \( G \rightarrow G' \) induces an isomorphism \( \nu_n G \cong \nu_n G' \) for any \( n \). It follows that it induces an isomorphism of pro-nilpotent completions \( \hat{G} \cong \hat{G}' \).

**Proposition 3.6** (Stallings’ theorem for \( \mathbb{Z}_\infty G \)). Any 2-connected homomorphism \( G \rightarrow G' \) induces an isomorphism
\[
\mathbb{Z}_\infty G \cong \mathbb{Z}_\infty G'.
\]

**Proof.** It follows from Lemma 3.5 and Proposition 3.3 because a \( \mathbb{Z} \)-homological equivalence \( X \rightarrow X' \) induces a homotopy equivalence \( \mathbb{Z}_\infty X \cong \mathbb{Z}_\infty X' \). \( \square \)

**Corollary 3.7.** A 2-connected homomorphism \( G \rightarrow G' \) induces an isomorphism
\[
\lim^{-1} M_n G \cong \lim^{-1} M_n G'.
\]

**Proposition 3.8.** Let \( G \) be a finitely presented group such that \( H_2 G \) is finite. Then the natural morphism \( \mathbb{Z}_\infty G \rightarrow \hat{G} \) is an isomorphism:
\[
\mathbb{Z}_\infty G \cong \hat{G}.
\]

**Proof.** By Corollary 2.3 the Baer invariants \( M_n G \) are finite. The Mittag-Leffler condition implies \( \lim^{-1} M_n G = 0 \). \( \square \)

If \( U \) is a normal subgroup of a group \( G \), then we set \( \hat{U}_G = \lim U/\gamma_n(U,G) \).
Proposition 3.9. Let $U \rightarrow E \rightarrow G$ be a short exact sequence of groups and assume that the map $\mathbb{Z}_\infty E \rightarrow \hat{E}$ is an isomorphism. Then there is an exact sequence

$$\hat{U}_E \rightarrow \hat{E} \rightarrow \mathbb{Z}_\infty G \rightarrow 1.$$ 

Proof. Note that $\nu_n(E \ltimes U) = \nu_n E \ltimes (U | \gamma_{n+1}(U, E))$. Then $E \ltimes U = \hat{E} \ltimes \hat{U}_E$. Since the functor $G \mapsto \mathbb{Z}_\infty G$ is right exact, by Theorem 1.2 we have an exact sequence

$$\mathbb{Z}_\infty E \rightarrow \hat{E} \rightarrow \mathbb{Z}_\infty G \rightarrow 1.$$ 

Since $\mathbb{Z}_\infty E \cong \hat{E}$, the commutative square

$$\begin{array}{ccc}
\mathbb{Z}_\infty E & \rightarrow & \mathbb{Z}_\infty E \\
\downarrow & & \downarrow \\
\hat{U}_E & \rightarrow & \hat{E}
\end{array}$$

implies that the map $\mathbb{Z}_\infty E \rightarrow \hat{E}$ has the same image as $\hat{U}_E \rightarrow \hat{E}$. Therefore we obtain a short exact sequence $\hat{U}_E \rightarrow \hat{E} \rightarrow \mathbb{Z}_\infty G \rightarrow 1$. \qed

Proposition 3.10. Let $U \rightarrow E \rightarrow G$ be a short exact sequence of groups. Assume that $E$ is finitely presented and $H_2 E$ is finite. Then there is an exact sequence

$$\hat{U}_E \rightarrow \hat{E} \rightarrow \mathbb{Z}_\infty G \rightarrow 1.$$ 

Remark 3.11. Proposition 3.10 gives an effective way for computation of $\mathbb{Z}_\infty G$. One has to find an extension $U \rightarrow E \rightarrow G$ with with finitely presented $E$, finite $H_2 E$ and computable $\hat{E}$ and $\hat{U}_E$, and then use the exact sequence.

Proof of Proposition 3.10. This follows from Proposition 3.9 and Proposition 3.8. \qed

If $G = F/R$ is a free presentation of $G$ we set $\bar{R} = \lim_{\leftarrow} (R \cdot \gamma_n F)/\gamma_n F$. Note that $\bar{R}$ is the closure of $R$ in $\hat{F}$ in the limit topology.

Proposition 3.12. If $G = F/R$ is a free presentation of a group $G$, then there is a commutative diagram with exact rows and columns

$$\begin{array}{cccc}
\bar{R}_F & \rightarrow & \hat{F} & \rightarrow & \mathbb{Z}_\infty G & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \bar{R} & \rightarrow & \hat{F} & \rightarrow & \hat{G} & \rightarrow & 1 \\
& & \downarrow & & & & \downarrow & & \downarrow & & \\
& & & & \lim^1 M_n G.
\end{array}$$

Proof. The first horizontal exact sequence follows from Proposition 3.10. The second horizontal exact sequence follows from the exact sequence $(R \cdot \gamma_n F)/\gamma_n F \rightarrow \nu_n F \rightarrow \nu_n G$. The exactness of the vertical left hand sequence follows from the snake lemma. \qed

For a group $G$ we set $\gamma_\omega G = \cap_{n=1}^{\infty} \gamma_n G$ and $\nu_\omega G = G/\gamma_\omega G$.

Proposition 3.13. If $G$ is finitely generated, then the maps $G \rightarrow \mathbb{Z}_\infty G \rightarrow \hat{G}$ induce isomorphisms

$$\nu_n G \cong \nu_n (\mathbb{Z}_\infty G) \cong \nu_n \hat{G}$$

for any $n$, and the maps from Proposition 3.14 define isomorphisms

$$\gamma_\omega (\mathbb{Z}_\infty G) \cong \lim^1 M_n G, \quad \nu_\omega (\mathbb{Z}_\infty G) \cong \hat{G}.$$
Proof. Let $G = F/R$ be a free presentation of $G$, where $F$ is a finitely generated free group. Then we have two exact sequences $\hat{R}_F \to \hat{F} \to Z_{\infty}G \to 1$ and $1 \to R \to \hat{R} \to G \to 1$. It is known [2] that $\nu_nF \cong \nu_n\hat{F}$ and $\nu_nG \cong \nu_n\hat{G}$. The image of $\hat{R}_F$ in $\nu_nF$ is $(R \cdot \gamma_{n+1}F)/\gamma_{n+1}F$. The image of $\hat{R}$ in $\nu_nF$ is the same. Then the exact sequences $\hat{R}_F \to \nu_nF \to \nu_n(Z_{\infty}G) \to 1$ and $\hat{R} \to \nu_nF \to \nu_n\hat{G} \to 1$ imply that the map $Z_{\infty}G \to \hat{G}$ gives rise to an isomorphism $\nu_n(Z_{\infty}G) \cong \nu_n\hat{G}$.

Set $\bar{K} = \text{Ker}(Z_{\infty}G \to \hat{G})$. By Proposition 3.1 we have $\bar{K} \cong \lim_{\leftarrow} M_nG$. Since $\gamma_\omega(\hat{G}) = 1$, we have $\gamma_\omega(Z_{\infty}G) \subseteq \bar{K}$. On the other hand the isomorphism $\nu_n(Z_{\infty}G) \cong \nu_n\hat{G}$ implies that $\bar{K} \subseteq \gamma_{n+1}Z_{\infty}G$. Thus $\bar{K} = \gamma_\omega(Z_{\infty}G)$.

Proposition 3.14. For a nilpotent group $N$ the natural map $N \to Z_{\infty}N$ is an isomorphism $N \cong Z_{\infty}N$.

Proof. Consider a free presentation $N = F/R$. Then $\gamma_kF \subseteq R$ for some $k$. It follows that $\gamma_{n+k}F \subseteq \gamma_n(R, F)$ for any $n$. Therefore the map $M_{n+k}N \to M_nN$ is trivial for any $n$. Hence $\lim_{\leftarrow} M_nN = 0$. □

4. Examples of 3-manifolds with isomorphic $\pi_1$ but non-isomorphic $Z_{\infty}\pi_1$.

For a homeomorphism of a manifold $f : X \to X$ one can consider its mapping torus:

$$X_f = \frac{[0, 1] \times X}{(1, x) \sim (0, f(x))}.$$  

Then $X_f$ is also a manifold such that $\dim(X_f) = \dim(X) + 1$. Moreover, there is a fiber bundle

$$X \to X_f \to S^1$$  

with the monodromy action induced by $f$.

Consider the following matrices

$$a_k = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}$$

where $k$ is an odd integer. They define homeomorphisms $f_k : (S^1)^2 \to (S^1)^2$. Their mapping tori are 3-manifolds which will be denoted by $X_k$. The homotopy long exact sequence of the fibration $(S^1)^2 \to X_k \to S^1$ implies that

$$X_k \cong K(G_k, 1),$$

where $G_k = \mathbb{Z} \ltimes_{a_k} \mathbb{Z}^2$.

Theorem 4.1. Let $k$ and $l$ be two odd integers.

1. Then $\pi_1(X_k) \cong \pi_1(X_l)$.
2. If $kl \not\equiv 1, 7 \pmod{8}$, then $Z_{\infty}\pi_1(X_k) \not\cong Z_{\infty}\pi_1(X_l)$.

In order to prove this theorem, we need to prove several lemmas. Further we will always assume that $k, l$ are odd integers

$$k, l \in 2\mathbb{Z} + 1.$$  

Lemma 4.2. There is an isomorphism $\hat{G}_k \cong \mathbb{Z} \ltimes_{a_k} \mathbb{Z}_2^2$, where $\mathbb{Z}_2$ is the group of 2-adic integers. Moreover, the group $\mathbb{Z} \ltimes_{a_k} (\mathbb{Z}/2^n)^2$ is nilpotent for any $n$.

Proof. It is easy to check that for arbitrary matrix $a \in GL_a(\mathbb{Z})$ we have $\gamma_{n+1}(\mathbb{Z} \ltimes_{a} \mathbb{Z}^s) = 0 \ltimes b^n(\mathbb{Z}^s)$, where $b = a - 1$ and $n \geq 1$. Set $b_k = a_k - 1$. Then $\gamma_{n+1}(G_k) = 0 \ltimes b_k^n(\mathbb{Z}^2)$. By induction we prove that

$$b_k^n = \begin{pmatrix} (-2)^n & 0 \\ (-2)^{n-1}nk & (-2)^n \end{pmatrix}.$$
Hence $b_2^k(\mathbb{Z}^2) \subseteq 2^{n-1}\mathbb{Z}^2$. On the other hand
\begin{equation}
b_k^n \cdot \begin{pmatrix} (-2)^n & -(-2)^{-n}nk \\ 0 & (-2)^n \\ 0 & (-2)^n \end{pmatrix} = \begin{pmatrix} (-2)^n & 0 \\ 0 & (-2)^n \end{pmatrix}.
\end{equation}
This implies that $2^{n^2}\mathbb{Z}^2 \subseteq b_k^n(\mathbb{Z}^2)$. Therefore the filtrations $b_k^n(\mathbb{Z}^2)$ and $2^n\mathbb{Z}^2$ of $\mathbb{Z}^2$ are equivalent. It follows that $\hat{G}_k = \mathbb{Z} \ltimes_{a_k} \mathbb{Z}_2^2$ and $\mathbb{Z} \ltimes_{a_k} \mathbb{Z}/2^n$ is nilpotent for any $n$. □

**Lemma 4.3.** There is an isomorphism $\hat{G}_k \cong \hat{G}_l$.

**Proof.** Generally, if we have two groups $H_1$ and $H_2$ where $f_i \in \text{Aut}(H_i)$, then a homomorphism $\varphi : H_1 \rightarrow H_2$ satisfying $f_1 \varphi = f_2 \varphi$ defines a homomorphism $1 \times \varphi : \mathbb{Z} \ltimes_{f_1} H_1 \rightarrow \mathbb{Z} \ltimes_{f_2} H_2$. We take $\varphi = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ as a matrix in $\text{GL}_2(\mathbb{Z}_2)$. Here we use that odd numbers are invertible in $\mathbb{Z}_2$. A direct computation shows that $a_l \varphi = \varphi a_k$. Therefore $\varphi$ defines an isomorphism $1 \times \varphi : \mathbb{Z} \ltimes_{a_k} \mathbb{Z}_2^2 \rightarrow \mathbb{Z} \ltimes_{a_l} \mathbb{Z}_2^2$. Thus $\hat{G}_k \cong \hat{G}_l$.

We denote by $N$ the free 2-generated nilpotent group of class 2. In other words $N = \nu_2 F(x, y)$. It has the following presentation
\[ N = \langle x, y \mid [x, y, y] = [x, y, x] = 1 \rangle. \]
There exists a unique automorphism of $F(x, y)$ such that $x \mapsto x^{-1}$ and $y \mapsto x^k y^{-1}$. It induces an automorphism
\[ a_k : N \rightarrow N, \quad a_k(x) = x^{-1}, \quad a_k(y) = x^k y^{-1}. \]
It lifts the automorphism $a_k : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$. Note that $a_k([x, y]) = [x, y]$. Define the following group
\[ E_k = \mathbb{Z} \ltimes_{a_k} N. \]
Then $(0, [x, y])$ is in the center of this group. Therefore we have a central extension
\[ 1 \rightarrow \mathbb{Z} \rightarrow E_k \rightarrow G_k \rightarrow 1. \]
It is easy to check that $E_k$ has a following presentation
\[ E_k = \langle a, x, y \mid x^a = x^{-1}, \quad y^a = x^k y^{-1}, \quad [x, y, x] = [x, y, x] = 1 \rangle. \]

**Lemma 4.4.** There is an isomorphism $H_2 E_k \cong \mathbb{Z}/4$, the isomorphism $\mathbb{Z}_\infty E_k \cong \hat{E}_k$ and an exact sequence
\[ \mathbb{Z} \xrightarrow{[x, y]} \hat{E}_k \rightarrow \mathbb{Z}_\infty G_k \rightarrow 1. \]

**Remark 4.5.** The group $E_k$ is “better” than the group $G_k$ because its second homology is finite, and hence, its right exact completion equals to its usual completion. We think about $E_k$ as about a resolution of $G_k$. Recall that, the same type of construction is used in [12], in order to prove that the $H\mathbb{Z}$-length of a free noncyclic group is $\geq \omega + 2$.

**Proof of Lemma 4.4.** The spectral sequence of the extension $N \rightarrow \hat{E}_k \rightarrow \mathbb{Z}$ gives a short exact sequence
\[ 1 \rightarrow (H_2 N) \rightarrow H_2 E_k \rightarrow (H_1 N) \rightarrow 1. \]
There is an isomorphism $H_1 N = \mathbb{Z}^2$, where $\mathbb{Z}$ acts on $\mathbb{Z}^2$ by $a_k$. Hence $(H_1 N)^{\mathbb{Z}} = \mathbb{Z}$ and $H_2 E_k = (H_2 N)^{\mathbb{Z}}$. Hopf’s formula gives an isomorphism $H_2 N = \gamma_3 F / \gamma_4 F$, where $F = F(x, y)$ is the free group. The two elements $[x, y, y], [x, y, x]$ form a basis of $H_2 N \cong \mathbb{Z}^2$. The Hopf’s formula is natural in the following sense: if we have two presentations of two groups $G = F/R$ and $G' = F'/R'$ and a homomorphism $\varphi : G \rightarrow G'$ then any lifting of the homomorphism to the free groups $F \rightarrow F'$ induces the homomorphism $(R \cap [F, F])/[R, F] \rightarrow (R' \cap [F', F'])/[R', F']$ corresponding to $\varphi_* : H_2 G \rightarrow H_2 G'$. Then the action of $\mathbb{Z}$ on $H_2 N$ is the following: $[x, y, x] \mapsto [x^{-1}, x^k y^{-1}, x^{-1}]$, and $[x, y, y] \mapsto$
In order to prove that this construction gives a well-defined group
(4.2)
\[
[x, y, x] \mapsto -[x, y, x], \\
[x, y, y] \mapsto k \cdot [x, y, x] - [x, y, y].
\]
It follows that \(Z\) acts on \(H_2N\) by the matrix \(a_k\). Then \((H_2N)_{\mathbb{Z}}\) is isomorphic to the quotient \(\mathbb{Z}^2/b_k(\mathbb{Z}^2)\), where \(b_k = a_k - 1\). Computing the Smith normal form of the matrix \((\begin{smallmatrix} -2 & k \\ 0 & -2 \end{smallmatrix})\) we obtain the matrix \((\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})\). Therefore \(H_2E_k = (H_2N)_{\mathbb{Z}} = \mathbb{Z}/4\). Proposition 3.3 implies that \(Z_{\infty}E_k \cong \hat{E}_k\). Then the central extension \((4.1)\) together with Proposition 3.10 give the exact sequence \(\mathbb{Z} \to \hat{E} \to Z_{\infty}G_k \to 1\).

Note that \([N, N]\) is the cyclic group generated by \([x, y]\). Hence, there is a following central extension
(4.3)
\[
1 \to \mathbb{Z} \to N \to \mathbb{Z}^2 \to 1.
\]
Any element of \(N\) can be uniquely presented as \(a^s b^t [(a, b)]^u\). Then the product can be defined by the formula
\[
(x^s y^t [x, y]^u)(x^{s'} y^{t'} [x, y]^{u'}) = x^{s+s'} y^{t+t'} [x, y]^{u+u'-ts'}.
\]
This means that the extension \((4.3)\) can be defined by the 2-cocycle \(\alpha : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}\) given by
\[
\alpha((s, t), (s', t')) = -ts'.
\]

Let \(R\) be an associative ring. We denote by \(N \otimes \mathbb{Z}^2\) the group of formal expressions \(x^s y^t [x, y]^u\) where \(s, t, u \in R\), and the multiplication is defined by the formula \((4.1)\). This group can be obtained as the group corresponding to the 2-cocycle \(\alpha_R : R^2 \times R^2 \to R\) given by the same formula \(\alpha_R((s, t), (s', t')) = -ts'\)
\[
1 \to R \to N \otimes R \to R^2 \to 1.
\]
In order to prove that this construction gives a well-defined group \(N \otimes R\) one just need to check that \(\alpha_R\) is a 2-cocycle, which can be done by a direct computation.

Let’s describe the action of \(a_k'\) on \(N\) more explicitly. It is easy to check that
\[
a_k'(y^t) = (x^k y^{-1})^s = x^{kt} y^{-t} [x, y]^{kt(t-1)/2}
\]
and \(a_k'([x, y]) = [x, y]\). It follows that
\[
a_k'(x^s y^t [x, y]^u) = x^{-s+kt} y^{-t} [x, y]^{u+kt(t-1)/2}.
\]
It is easy to see that the same formula defines an automorphism of \(N \otimes \mathbb{Z}_2\):
\[
a_k'_{\mathbb{Z}_2} : N \otimes \mathbb{Z}_2 \to N \otimes \mathbb{Z}_2
\]
because elements of the form \(t(t-1)\) are uniquely divisible by 2 in \(\mathbb{Z}_2\). For simplicity we will denote \(a_k' = a_k'_{\mathbb{Z}_2}\).

**Lemma 4.6.** The obvious map \(E_k \to \mathbb{Z} \rtimes a_k' (N \otimes \mathbb{Z}_2)\) induces an isomorphism
\[
\hat{E}_k \cong \mathbb{Z} \rtimes a_k' (N \otimes \mathbb{Z}_2)
\]
and there are short exact sequences
\[
\begin{array}{c}
0 \to \mathbb{Z} \\
\downarrow \\
\hat{E}_k \\
\downarrow \\
\hat{G}_k
\end{array}
\]
Moreover, the group \(\mathbb{Z} \rtimes a_k' (N \otimes \mathbb{Z}/2^n)\) is nilpotent for any \(n\).
Proof. Set $E = E_k$. Consider a subgroup of $E_n \subseteq N$

$$E_n = \{x^s y^t [x, y]^u \mid s, t \in 2^{n+1}Z, u \in 2^n Z\}.$$  

Since $t \in 2^{n+1}Z$ implies that $t(t-1)/2 \in 2^n Z$, the explicit formula for $a'_k$ gives that

$$a'_k(E_n) \subseteq N.$$  

Therefore $E_n = 0 \lt E_n$ is a normal subgroup of $E$. Then $Z \lt a'_k (N \otimes Z_2) = \lim \ E / E_n$. We also set $G = G_k$ and $G_n = Z \lt a_k 2^{n+1}Z^2$. Note that there is a central extension

$$1 \longrightarrow Z / 2^n \longrightarrow E / E_n \longrightarrow G / G_n \longrightarrow 1.$$  

By Lemma 1.2 $G / G_n$ is nilpotent, and hence, $E / E_n$ is nilpotent. This follows that for any $n$ there exists $m(n)$ such that $\gamma_{m(n)} E \subseteq E_n$. Hence we have the following diagram

$$
\begin{array}{c}
1 \longrightarrow N / (N \cap \gamma_{m(n)} E) \longrightarrow E / \gamma_{m(n)} E \longrightarrow G / \gamma_{m(n)} G \longrightarrow 1 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
1 \longrightarrow Z / 2^n \longrightarrow E / E_n \longrightarrow G / G_n \longrightarrow 1
\end{array}
$$

We want to prove that the central map induce an isomorphism on limits. By Lemma 1.2 the right hand vertical map induces an isomorphism on limits. Then we need to prove that the left hand map induces an isomorphism on limits. It is enough to prove that the filtrations $[N, N] \cap \gamma_n E$ and $([x, y]^{2^n})$ of the cyclic group $[N, N] = \{(x, y)\}$ are equivalent. We already know that $N \cap \gamma_{m(n)} E \subseteq ([x, y]^{2^n})$ because the left hand vertical map is well defined. On the other hand, if we use the presentation $E = (x, y, a \mid x^a = x^{-1}, y^a = x^k y^{-1}, [x, y, y] = [x, y, x] = 1)$, we obtain

$$[x, a, \ldots, a, y] = [x, y]^{(-2)^n} \in [N, N] \cap \gamma_{n+2} E.$$  

Hence $([x, y]^{2^n}) \subseteq [N, N] \cap \gamma_{n+2} E$. It follows that $E = Z \lt a'_k (N \otimes Z_2)$.  

The short exact sequences follow from the the description of $E_k$ and Lemma 1.4.  

Denote by $N$ the quotient of $N \otimes Z_2$ by the subgroup $\{[x, y]^n \mid n \in Z\}$. Then any element of the group $N$ has the form $x^s y^t [x, y]^u$, where $s, t \in Z_2$ and $u \in Z_2 / Z$:

$$N = (N \otimes Z_2) / ([x, y]), \quad N = \{x^s y^t [x, y]^u \mid s, t \in Z_2, u \in Z_2 / Z\}.$$  

The product of elements in $N$ is given by

$$x^s y^t [x, y]^u \cdot x'^s y'^t [x, y]^u' = x^{s+s'} y^{t+t'} [x, y]^{u+u'-\text{ts}},$$  

where $\text{ts}^{-}$ is the image of $ts'$ in $Z_2 / Z$.  

**Corollary 4.7.** There is an isomorphism

$$\mathbb{Z}_\infty G_k \cong Z \lt a'_k N.$$  

Moreover

$$\gamma_\omega(\mathbb{Z}_\infty G_k) \cong \mathbb{Z}_2 / Z, \quad \nu_\omega(\mathbb{Z}_\infty G_k) \cong \hat{G}_k.$$  

**Corollary 4.8.** There is an isomorphism

$$\lim^1 M_n G_k \cong \mathbb{Z}_2 / Z.$$
Proof of the Theorem 4.1. We proved that $\hat{G}_k \cong \hat{G}_l$ in Lemma 4.3.

Then we only need to prove that $kl \not\equiv 1, 7 \pmod{8}$ implies $\mathbb{Z}_\infty G_k \not\equiv \mathbb{Z}_\infty G_l$. Assume the contrary, that there is an isomorphism $\phi : \mathbb{Z}_\infty G_k \to \mathbb{Z}_\infty G_l$. Then it induces an isomorphism of quotients by $\gamma_\omega$ which are equal to the usual completions $\phi' : \hat{G}_k \to \hat{G}_l$. Since $\hat{G}_k \cong \mathbb{Z} \times \mathbb{Z}_2^2$, we have that the set of 3-divisible elements of $\hat{G}_k$ is $0 \times \mathbb{Z}_2^2$. Therefore we obtain an isomorphism $\phi'' : \mathbb{Z}_2^2 \to \mathbb{Z}_2^2$ and an isomorphism $\tilde{\phi} : \mathbb{Z} \to \mathbb{Z} :$

$$
1 \longrightarrow \mathbb{Z}_2^2 \longrightarrow \hat{G}_k \longrightarrow \mathbb{Z} \longrightarrow 1
$$

$$
1 \longrightarrow \mathbb{Z}_2^2 \longrightarrow \hat{G}_l \longrightarrow \mathbb{Z} \longrightarrow 1.
$$

Note that $\tilde{\phi}(1) = \pm 1$. Therefore

$$
\phi'(a) = a^{s_0}x^{s_0}y^{t_0}
$$

for some $s_0, t_0 \in \mathbb{Z}_2$. This implies that

$$
\phi''a_k = a_k^{t_1}\phi''.
$$

Any homomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$ is the multiplication by a 2-adic number. Therefore $\phi''$ is given by an invertible $2 \times 2$-matrix over $\mathbb{Z}_2$:

$$
\phi'' = \begin{pmatrix}
\varphi_{11} & \varphi_{12} \\
\varphi_{21} & \varphi_{22}
\end{pmatrix}, \quad \varphi_{ij} \in \mathbb{Z}_2.
$$

The equation $\phi''a_k = a_k^{t_1}\phi''$ implies

$$
\begin{pmatrix}
-\varphi_{11} & -\varphi_{12} + k\varphi_{11} \\
-\varphi_{21} & -\varphi_{22} + k\varphi_{21}
\end{pmatrix}
= \phi''a_k = a_k^{t_1}\phi'' = \begin{pmatrix}
-\varphi_{11} \pm l\varphi_{21} & -\varphi_{12} \pm l\varphi_{22} \\
-\varphi_{21} & -\varphi_{22}
\end{pmatrix}.
$$

Since $k, l \neq 0$, this implies that $\varphi_{21} = 0$ and $k\varphi_{11} = \pm l\varphi_{22}$. Therefore

$$
\phi'' = \begin{pmatrix}
\alpha & \beta \\
0 & \alpha k
\end{pmatrix},
$$

where $\alpha = \varphi_{22}/k = \pm \varphi_{11}/l$ and $\beta = \varphi_{12}$. Here we use that odd numbers are invertible in $\mathbb{Z}_2$. Since $\phi''$ is invertible, $\alpha$ is also invertible in $\mathbb{Z}_2$. If we present elements of $\mathbb{Z}_\infty G_k = \mathbb{Z} \times \mathcal{N}$ in the form $a^n x^s y^t [x, y]^u$, where $n \in \mathbb{Z}$, $s, t \in \mathbb{Z}_2$ and $u \in \mathbb{Z}_2/\mathbb{Z}$, then this means that

$$
\varphi(x^s) = x^{\alpha ls}[x, y]^{u_0(s)}, \quad \varphi(y^t) = x^{\beta t}y^{\alpha kt}[x, y]^{v_0(t)}
$$

for some $u_0(s), v_0(t) \in \mathbb{Z}_2/\mathbb{Z}$. Therefore

$$
\varphi([x, y]^u) = \varphi([x^u, y]) = [\varphi(x^u), \varphi(y)] = [x^{\alpha lu}, x^\beta y^{\alpha k}] = [x, y]^{\alpha^2 k lu}.
$$

Since $\varphi$ is well defined, $u \in \mathbb{Z}$ implies $\alpha^2 k lu \in \mathbb{Z}$. It follows that there exists $m \in \mathbb{Z}$ such that

$$
\alpha^2 k l = m.
$$

Since $\alpha, k, l$ are invertible in $\mathbb{Z}_2$, $m$ is odd. Then $[x, y]^1/m$ is a well defined element of $\mathcal{N}$ and

$$
\varphi([x, y]^1/m) = [x, y] = 1.
$$

Since $\varphi$ is an isomorphism, we obtain $[x, y]^1/m = 1$. Therefore $1/m \in \mathbb{Z}$ and hence $m = \pm 1$. Thus

$$
\pm kl = \alpha^2.
$$

The element 1 is the only square in $\mathbb{Z}/8$. Hence $\pm kl \equiv 1 \pmod{8}$. This contradicts to our assumption.
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