The Hall effect of dipole chain in one dimensional Bose-Einstein condensation

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We find a breather behavior of the dipole chain, this breather excitation obey fractional statistics, it could be an experimental quantity to detect anyon. A Hall effect of magnetic monopole in a dipole chain of ultracold molecules is also presented, we show that this Hall effect can induce the flip of magnetic dipole chain.

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A. Introduction

Fractional Hall effect occurs for electrons confined to two dimensions in a strong magnetic field\(^1\). The Laughlin wave function has provided a correct description to this quantum Hall liquid\(^2\), and it leads us to new profound understanding to quantum many body system.

The rapidly developed optical method in controlling ultracold cold molecules gas has opened an unique door to the quantum many body systems\(^3\). There has been some proposals to realize the strongly correlated states of fractional Hall type in optical lattice, such as rotating cold atom gases\(^4\), melting a Mott-insulator state in a superlattice potential\(^5\), and so on. While in our paper, a totally different physical system to the realization of quantum Hall state in molecule is proposed based on the dipolar interaction between ultracold molecules.

The Bose-Einstein condensation (BEC) with dipole-dipole interactions has been thoroughly studied\(^6\) since the dipole-dipole interaction was introduced into ultracold molecular gas by Yi and You\(^7\). Yu shows two column of parallel dipole molecules has an inverse square interaction\(^8\), which happened to be described by the well known Calogero-Sutherland Model\(^9\), the solution of this model is just the Laughlin wave function. So far, the quantum Hall states of ultracold molecular clouds with dipole-dipole interaction is still blank to our acknowledge. So we shall show how the Hall effect arise from the ultracold molecular dipole chain, the anyons may also be observed in our physical system.

In this paper, we find a method of controlling the magnetic dipole momentum through quantum Hall effect. Hall effect can induce the flip of magnetic monopole in two column of ultracold molecular dipole chain. The anyon appears as the breathers mode of the dipole chain, they obey fractional statistics.

B. Hall effect of electric dipole chain

The BEC may be achieved in a dilute gas of ultracold polar molecules\(^10\). When the deBroglie wavelength becomes comparable to the inter-molecule spacing, molecules lose their individuality, forming a Bose-Einstein condensate. In this case, a collection of millions of molecules can then be described as a single entity: a coherent field\(^11\). Applying the ‘cloverleaf’ magnetic trap, which has a weak confinement in \(y\) dimension, but very strong along the other two dimensions, the molecule cloud may be squeezed into a cigar-shaped with an aspect ratio of about 15:1\(^12\).

At zero temperature, when the confining field becomes stronger, the cigar shaped molecule cloud would grow thinner, until the molecules sit in a line one by one along the \(y\) direction. Considering one column of ultracold molecules placed on the line \(x = x_2, y \in (-\infty, +\infty), z = 0\), they are polarized by a strong homogeneous electric field \(E\) penetrating through the \(x - y\) plane with an angle \(\theta_0\) to the +\(x\)-axes(Fig. 1). So there is a electric dipole chain at \(x_2\), they stand shoulder to shoulder in a column along \(y\) axes. The nearest neighbor dipole-dipole interaction is repulsive, this keeps the stability of the cigar-like BEC.

Besides the homogeneous electric field penetrating through the \(x - y\) plane, we also set up a homogeneous magnetic field in the \(y\) direction, it goes perpendicular into the \(x - z\) plane.

We consider two column of dipole chains at \(x_1\) and \(x_2\) (Fig. 1). We first study the electric dipole chain at \(x_2\) to investigate macroscopic behavior of the dipole chain.

The electric dipole moment is defined from the electric polarization charge \(\vec{d}_E = q \vec{d}\), \(\vec{d}\) is the vector pointing from \(-q\) to \(+q\) polarization electric charge. When the electric polarization charge \(+q\) is moving in the pane perpendicular to the magnetic field, it feels a Lorentz force,

\[
\vec{f}_e = q \vec{v} \times \vec{B},
\]

\(\vec{v}\) is the speed of the charge. For the electric dipole chain at \(x_2\), its speed is \(v_2 = \frac{dx_2}{dt}\). The Lorentz force’s direction obey the right hand rule. When \(v_2 > 0\), according to Eq. (1), the \(+q\) would move towards the +\(z\)-direction. While the corresponding negative charges \(-q\)
of the dipole are moving toward the $-z$-direction. In the ultracold molecule, the electron cloud moves towards to $-z$-direction. This strengthened the electric dipole momentum, therefore the repulsive interaction between the nearest neighboring electric dipole would increase, then the cigar-like giant particle grows longer. On the contrary, when the dipole chain moves in the $-x$-direction, the dipole chain grows shorter.

The direction of the moving electric dipole is given by the balance between the lorentz force of magnetic field and electric field. Let $\theta_0$ denote the angle between the electric field and $x$-axes, and $\theta$ denotes the direction of the electric dipole momentum, then

$$\tan \theta = E \tan \theta_0 + \frac{vB}{E \cos \theta_0}.$$ \hspace{1cm} (2)

This provide us a new method to control the direction of the electric dipole chain. One need to adjust the ratio between electric field and magnetic field, and the velocity of the dipole chain.

C. Breather modes of Calogero-Sutherland gas

We consider two dipole chains at $x_1$ and $x_2$ (Fig. 1), they are oscillating around the equilibrium point. Then the dipole would grows longer and shorter periodically, it will be shown that this breather mode open a new door to anyon.

The interaction between two parallel electric dipoles is

$$V_{d}(y, y') = \frac{E^2}{4\pi \epsilon_0} \frac{1 - 3 \cos^2 \alpha}{r^3},$$ \hspace{1cm} (3)

with $y$ and $y'$ as their location and $\alpha$ is the angle between $r = y - y'$ and $d$. If the harmonic trap along the $y$-direction is very flat, $\rho_0(y)$ may be approximated by the average density $\tilde{\rho}_0$. When the angle between the electric dipole and the $x-y$ plane is $\theta$, the interaction potential between the two column of dipole chain is

$$V = \frac{2d^2\tilde{\rho}_0 N_0}{2} \left( 1 + \frac{3\pi \cos \theta}{4} \right) \frac{1}{|x_1 - x_2|^2}. \hspace{1cm} (4)$$

where $N_0$ is the molecule number in a condensate, $d$ is the magnitude of the electric dipole moment $d$. In the following, we denote $g = 2d^2\tilde{\rho}_0 (1 + \frac{3\pi \cos \theta}{4})$ as the coupling factor. This interaction between the two dipole chain can be easily controlled through Eq. (2).

At zero temperature, the weakly interacting ultracold molecules of the dipole form one dimensional BEC, they act like a whole to external perturbation. So each dipole chain may be treated as one giant particle, whose position on the $x$-axes is marked by $x_i$. The Hamiltonian of this physical system is

$$H_{CS} = \frac{1}{2} \sum_{i=1}^{2} \left( -\frac{d^2}{dx_i^2} + \omega^2 x_i^2 \right) + \frac{1}{2} \rho_0(\lambda - 1) \left| x_1 - x_2 \right|^2.$$ \hspace{1cm} (5)

here we have set the units $\hbar = c = 1$, $\lambda = \frac{1 - \sqrt{1 - 4g}}{2}$.

Using the operator decomposition method in supersymmetry quantum mechanics, we find the operator of the breather excitation,

$$\hbar j = -i \frac{d}{dx_j} + A, \hspace{0.5cm} A = -i \omega x_j + i \lambda \frac{1}{x_j - x_k}, \hspace{0.5cm} (j \neq k), \hspace{1cm} (6)$$

$A = (A_1, A_2)$ is the vector potential. So it is easy to see the Calogero-Sutherland model for two giant particles describes the same physics as a two dimensional electron gas. The Calogero-Sutherland model \cite{5} reduced to \cite{6,7},

$$H_{CS} = \frac{1}{2} \sum_j h_j^\dagger h_j + \hbar \omega (\lambda + 1),$$ \hspace{1cm} (7)

with the ground state energy $E_0 = \hbar \omega (\lambda + 1)$. Here $h_j^\dagger$ is the generating operator of a kind elementary excitation—the breather excitation. It has an exact ground state wave function obtained from $h_j |\Psi_{0,\lambda}\rangle = 0$,

$$|\Psi_{0,\lambda}(x_1, x_2)\rangle = (x_1 - x_2)^\lambda e^{-\frac{1}{2} \omega^2 (x_1^2 + x_2^2)}. \hspace{1cm} (8)$$

The statistics of the two dipole chain giant particle relies on $\lambda$. For $\lambda = 2k$, the wave function is symmetric when the two giant particles exchange their position, then they behaves like bosons. When $\lambda = 2k + 1$, the wave function is anti-symmetric for the exchange of the two particles. More over, $\lambda$ is fractional number, the two giant particles obey fractional statistics.

The breather excitation mode is also modified by the statistics of the two giant particles. According to the ground state energy, Eq. (7), the normal frequency $\omega$ of the breather is modified by the interaction between the two giant particles,

$$\omega' = \omega(\lambda + 1), \hspace{0.5cm} \lambda = \frac{1 - \sqrt{1 - 8d^2\tilde{\rho}_0(1 + \frac{3\pi \cos \theta}{4})}}{2}. \hspace{1cm} (9)$$
Equation (9), the critical angle of the dipole reads
\[ \cos \theta < \frac{4}{3\pi} \left( \frac{1}{8d^2 \rho_0} - 1 \right). \]  

This excitation spectrum can be probed by measuring the frequencies of collective oscillation in experiment [14].

If \( g < \frac{1}{2} \), the ground state is well defined, according to Eq. (9), the critical angle of the dipole reads
\[ \cos \theta < \frac{4}{3\pi} \left( \frac{1}{8d^2 \rho_0} - 1 \right). \]  

This puts a constrain on the electric and magnetic field. As we know, if \( \lambda \) continuously changed from an odd number to an even number, another kind of excitation—anyon—would appear [15], so this is a good system to detect anyons, the electric field, magnetic field are both macroscopic physical parameter. For a fixed value of \( \lambda \), combing Eq. 2 and Eq. 10, we can find the proper range of magnetic field and electric field. In experiment, it is more convenient to move the electric field in the horizontal plane than to move the dipole chain.

The above discussion only concerns about two giant particles, in fact, it also holds for the more general case of \( N \) giant particles. For \( N \) giant particles, the physical system is governed by the Calogero-Sutherland model
\[ H_{CS} = \frac{1}{2} \sum_{i} \left(-\frac{d^2}{dx_i^2} + \omega^2 x_i^2\right) + \frac{1}{2} \sum_{i \neq j} \frac{\lambda(\lambda-1)}{\mu^2} \mathbf{R}_{ij}^2, \]  

its ground state wave function is of a familiar form of Laughlin wave function. So there are also breathers for the many dipole chain system, they obey fractional statistics.

**D. Hall effect of magnetic dipole chain**

In fact, in our physical system, the magnetic field goes perpendicular to the \( x-z \) plane, so the magnetic dipole chain lies head to tail in the \( y \) direction. But it has been proved that, for two dipole chain polarized in this way, the interaction between the two is zero, no matter it is static or in motion.

As analyzed in the above, when the electric dipole chain is oscillating around the equilibrium point, it induced a breather mode. If the ultracold molecules have magnetic dipole momentum, this dipole chain is a combination of electric dipole pole chain(like centipede along \( y \) axes) and magnetic dipole chain(head to tail along \( y \) axes). While the electric dipole momentum is much larger than the magnetic dipole momentum. So the magnetic momentum interaction in one dipole contributes a small perturbation. Though the magnetic dipole has a small attractive interaction, which is opposite to the repulsive interaction between electric dipoles, it also present the same breather behavior as the electric case.

We shall introduce Dirac’s magnetic monopole [16] to study this self-consistence. A magnetic dipole may be viewed as a monopole-anti-monopole pair, the magnetic momentum is defined by the product of the monopole charge and vector from the anti-monopole to the monopole, \( \mu_m = g \mathbf{d}, g \) is the charge of magnetic monopole, \( \mathbf{d} \) is the vector pointing from anti-magnetic-monopole \(-g\) to positive magnetic monopole \(+g\).

The Maxwell equations for a vacuum without sources possess an interesting symmetry under the exchange of electric field \( E \) and magnetic field \( B \), i.e., \( E \rightarrow B \) and \( B \rightarrow -E \), this symmetry is called the electric-magnetic duality. So for a magnetic monopole with charge \( g \) moving in the plane perpendicular to electric field, it feels a dual Lorentz force,
\[ \mathbf{f}_g = -g \mathbf{v} \times \mathbf{E}, \]  

this duality is self-content during the movement of the dipole chain. The dipole chain is a series of monopole-anti-monopole pair head to tail along \( y \) direction. As the magnetic dipole chain moves in \(+x\) direction, the monopole \(+g\) in the electric field is govern by the dual Lorentz force (10), so there is a transverse force towards \(+y\) direction acted on the magnetic monopole, the anti-monopole \(-g\) would move to \(-y\)-direction correspondingly. This strengthen the cancellation of the monopole and anti-monopole charge in the middle of dipole chain except the charges at the two end points of the giant particle. Monopoles with \(+g\) are focused on the \(+y\) end point, and anti-monopole with \(-g\) are focused on the \(-y\) end point. This is the Hall effect of magnetic monopoles.

There is also a corresponding Hall current of magnetic monopole, the corresponding Hall conductance is determined by the net force include the Lorentz force
\[ \mathbf{f}_g = g(\mathbf{B} - \mathbf{v} \times \mathbf{E}), \]  

\( \mathbf{v} \) is the charge velocity. The balance \( f = 0 \) provides a relation between the magnetic field along y-direction \( E_y \) and the applied magnetic field \( E_z \),
\[ B = \frac{\mathbf{v}^0 E_z}{\mathbf{v}_x}. \]  

When the velocity of the dipole chain exceed the critical velocity \( v > v^0 \), the net force would change its sign, in that case, the magnetic dipole chain would turn to the opposite direction. When the velocity of the giant particle is twice of the critical velocity \( v^c \), there would be an opposite dipole chain with the same strength. This has provide us a new method to obtain a antiparallel dipole
array. This flip of magnetic dipole momentum is induced by the its movement in electric field.

The Dirac quantization condition says that in the presence of magnetic monopoles, the product of electric and magnetic charges must be an integral multiple of $1/2$, i.e., $e\tau = 2g_0/n, (n = 1, 2, 3, \ldots)$. So the dual Hall resistance of magnetic monopole $R_g$ reads

$$R_g = \frac{\hbar}{n} 4g_0^2 = \frac{1}{\nu} 4\nu e_0^2.$$  \hfill (14)

here $\nu$ is the filling factor of Landau level, $g_0 = \frac{1}{2}e$ is the unit of magnetic monopole charge, $e$ is the unit electric charge. The magnetic monopole has been drawing great attention of physicists all the time since Dirac postulated its existence in 1931. There is still no definite evidence to confirm its existence in spite of the great effort physicists have made in the last decades of years.

This Hall effect would provided us a new understanding to magnetic monopole.

In fact, Eq. (14) means the magnetic monopole obey the left handed rule, it leads to self-content results with electric dipole chain. On the contrary, if magnetic monopole obey the right handed rule, when the magnetic dipole is lengthened, the electric dipole chain would be shortened, this two motions are conflict. Which case is correct on earth? it lies in the hand of exact experiment.

**E. conclusion**

In summary, when two column of ultracold molecules of Bose-Einstein condensation are polarized by an orthogonal magnetic field and electric field, they become two dipole chains. When the dipole is oscillating in the transverse section, it induces a breather excitation in the longitudinal direction. The length of dipole chain behaves like a breather, it grows longer and shorter periodically. This breather excitation obey the fractional statistics. This could be an experimental quantity to detect the anyon. By adjusting the speed of the dipole chain, one can observe a magnetic dipole flip induced by Hall effect above a critical velocity.

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