On smoothness of connection of spatial curve cyclographic projection segments

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Abstract. The correlation between smoothness of connection of segments of a spatial curve and the respective cyclographic projection is considered. The sufficient condition for smoothness $C^{k-1}_a$ at the point of connection of segments of cyclographic projection of a spatial curve described by a polynomial spline of order $k+1$ is formulated and verified. It has been demonstrated that despite the derivative of order $k+1$ of the spline function describing the spatial composite curve along with all the higher-order derivatives not being continuous, it is possible to connect segments of the cyclographic projection with smoothness $C^r_a$, where $r = k + i$, $i = 0, 1, 2, \ldots$. Achieving smoothness $C^{k-1}_a$, $C^r_a$ through the formulated condition of sufficiency is demonstrated on numeric examples. The obtained results of the study can serve as the basis of development of effective algorithms of cyclographic formation of geometric objects applied in geometric optics, road surface form design, and optimal cutting tool trajectory design in pocket machining.

1. Introduction
There is a known spline approach to composite curve construction that allows for construction of a curve out of individual smooth parametric arc segments smoothly connected with each other [1-7]. A curve acquired through the spline approach is referred to as a “spline curve”, keeping in mind that a “spline” is a function defining the mathematical model of a composite curve. A number of theoretical and practical applications feature spline curves as results of interpolation of a discrete set of points with subsequent smoothing. One of the key conditions in spline curve formation is seamless connection of spline segments. The required smoothness $C^k$ of connection is defined by the order of the polynomial modelling each particular segment and the whole spline curve [3,7]. It forms the computational algorithm for coefficients of each polynomial defining the respective segment. In many cases the task of segmental formation of a spline curve serves as the basis for subsequent formation of its images in various representations and transformations often applied in solutions to the problems of geometric optics [8,9], formation of road surface forms [10-11], formation of families of equidistant trajectories in pocket machining in machine-building [12-14]. Solution of the required tasks requires the information on smoothness of connection of cyclographic images of the initial spline curve. The objective of the present paper is to examine the variation of the order of smoothness of connection of spline curve segments in its cyclographic mapping on a plane.

2. Problem Definition
The apparatus of cyclographic mapping of space $E_3$ on plane $z=0$ establishes a bijective correspondence between the multitude of points of space and the multitude of directed circles on plane.
In this mapping the image of a spatial curve is a certain flat curve – an envelope of a one-parameter multitude of directed circles. These circles represent the bases of straight cones, while the vertices of these cones are on the initial curve. Obviously, cyclographic mapping of a composite (segmented) initial curve results in a composite (segmented) image. Spline curves with respective cyclographic images find wide application in solutions to various theoretical and practical problems of geometric optics, road surface form modeling, design of movement of cutting tools in machine-building, etc. The initial curve and its cyclographic projection constitute the basic elements of cyclographic formation of geometric objects in the mentioned applications, hence the problem of the research: to determine the correspondence between smoothness of connection of spatial spline curve segments and smoothness of connection of its cyclographic projection.

3. Theory
Let us consider a spatial spline curve $a$ (Fig. 1) that consists of segments of splines of a certain type (Hermite, Bezier, B-spline, etc.). Spline segments are connected with smoothness $C^k$, $k \in \mathbb{N}$, which means that any point $P_i$ of connection of a pair of segments $a_n$ and $a_{n+1}$ complies with the condition of equality of continuous derivatives up to the order $k$ inclusive of the polynomial vector functions $\overrightarrow{P}_n(t) = \overrightarrow{P}_{n+1}(t), \ t \in [0,1]$ defining these segments. Let us construct a cyclographic projection $a_{\alpha}$ (Fig.1) of a spline curve $a$ consisting of cyclographic images of the respective segments of the initial spline.

Vector functions $\overrightarrow{P}_n(t)$ and $\overrightarrow{P}_{n+1}(t)$ of two respective segments $a_n$ and $a_{n+1}$ can be represented in projections in the following form:

$$\overrightarrow{P}_n(t) = \{x_n(t), y_n(t), z_n(t)\}, \quad \overrightarrow{P}_{n+1}(t) = \{x_{n+1}(t), y_{n+1}(t), z_{n+1}(t)\}. \quad (1)$$

The equations for the cyclographic $\alpha$-projection of the curve are known [15] and they can be represented as follows:

$$x_{\alpha(n+1)}(t) = x(t) + z(t) \frac{-x(t)\cdot y(t) + x(t)\cdot z(t)}{\delta},$$

$$y_{\alpha(n+1)}(t) = y(t) + z(t) \frac{-y(t)\cdot z(t) + y(t)\cdot x(t)}{\delta},$$

$$\delta = (x(t))^2 + (y(t))^2, \quad (2)$$

where $x(t) = \frac{dx(t)}{dt}, \ y(t) = \frac{dy(t)}{dt}, \ z(t) = \frac{dz(t)}{dt}$. Let us represent the equations of cyclographic projections $a_{\alpha(n)}$ and $a_{\alpha(n+1)}$ in the following form:

$$\overrightarrow{P}_{\alpha(n)}(t) = \{x_{\alpha(n)}(t), y_{\alpha(n)}(t)\}, \quad \overrightarrow{P}_{\alpha(n+1)}(t) = \{x_{\alpha(n+1)}(t), y_{\alpha(n+1)}(t)\}, \quad (3)$$

where $x_{\alpha(n)}(t) = F_{\alpha(n)}(x_n(t), y_n(t), z_n(t), x_n'(t), y_n'(t), z_n'(t))$,

$$y_{\alpha(n)}(t) = F_{\alpha(n)}(x_n(t), y_n(t), z_n(t), y_n'(t), z_n'(t), x_n'(t)), \quad (4)$$

$$x_{\alpha(n+1)}(t) = F_{\alpha(n+1)}(x_{n+1}(t), y_{n+1}(t), z_{n+1}(t), x_{n+1}'(t), y_{n+1}'(t), z_{n+1}'(t)),$$

$$y_{\alpha(n+1)}(t) = F_{\alpha(n+1)}(x_{n+1}(t), y_{n+1}(t), z_{n+1}(t), y_{n+1}'(t), z_{n+1}'(t), x_{n+1}'(t)), \quad (5)$$

and primes designate the corresponding derivatives with respect to parameter $t$ of the initial curve $a$. Each of the functions $F_{\alpha(n)}, F_{\alpha(n+1)}, F_{\alpha(n+1)}, F_{\alpha(n+1)}$, as it follows from (2), constitutes a combination of arithmetic operations with coordinate functions of parameter $t$ stated in brackets.
If the point of connection of the initial segments $a_n$ and $a_{n+1}$ has smoothness $C^0$ and $C^1$, then the corresponding point of connection of respective cyclographic projections $a_{\alpha(n)}$ and $a_{\alpha(n+1)}$ has smoothness $C^0_\alpha$. Indeed, connection with smoothness $C^0$ and $C^1$ of the initial segments is provided by conditions
\begin{align*}
  x_n(t=1) &= x_{n+1}(t=0); \\
  y_n(t=1) &= y_{n+1}(t=0); \\
  z_n(t=1) &= z_{n+1}(t=0); \\
  x'_n(t=1) &= x'_{n+1}(t=0); \\
  y'_n(t=1) &= y'_{n+1}(t=0); \\
  z'_n(t=1) &= z'_{n+1}(t=0).
\end{align*}
These conditions, based on (4) and (5), result in equations
\begin{align*}
  x_{\alpha(n)}(t=1) &= x_{\alpha(n+1)}(t=0); \\
  y_{\alpha(n)}(t=1) &= y_{\alpha(n+1)}(t=0),
\end{align*}
providing smoothness $C^0_\alpha$ of connection of cyclographic segments $a_{\alpha(n)}$ and $a_{\alpha(n+1)}$.

Obviously, by subsequently introducing additional conditions
\begin{align*}
  x''_n(t=1) &= x''_{n+1}(t=0); \\
  y''_n(t=1) &= y''_{n+1}(t=0); \\
  z''_n(t=1) &= z''_{n+1}(t=0), \\
  x'''_n(t=1) &= x'''_{n+1}(t=0); \\
  y'''_n(t=1) &= y'''_{n+1}(t=0); \\
  z'''_n(t=1) &= z'''_{n+1}(t=0),
\end{align*}
to the conditions (6), we achieve respective orders of smoothness $C^1_\alpha, C^2_\alpha, \ldots, C^k_\alpha$ at the point of connection of cyclographic segments $a_{\alpha(n)}$ and $a_{\alpha(n+1)}$.

The above allows us to come up with the following statement: subsequent fulfillment of equations of continuous derivatives of polynomial vector functions describing segments of order up to $k$ inclusive at the point of connection of spline curve segments is sufficient in order to achieve smoothness $C^k_\alpha$ at the point of connection of the respective cyclographic images.

4. Results of the experiment

Let us demonstrate the above statement with the following example. The coordinates of points of the designed spline curve are given: $P_0 = (0;0;15)$, $P_1 = (30;3;13)$, $P_2 = (70;-5;15)$, $P_3 = (100;5;12)$. Need
to construct a three-segment Hermite spline curve \( a_i \) of the third order through the given points with smoothness \( C^2 \) at connection points \( P_1 \) and \( P_2 \). It is then required to determine whether smoothness \( C^2 \) is inherited by the respective points of cyclographic projections of segments of the spline \( a_i \).

Let us express the equations of segments of spline curve \( a_i \) in the following form:

\[
\begin{align*}
P_{01}(t) &= \overline{A}_0 t^3 + \overline{A}_1 t^2 + \overline{A}_2 t + \overline{A}_3, \\
P_{12}(t) &= \overline{B}_0 t^3 + \overline{B}_1 t^2 + \overline{B}_2 t + \overline{B}_3, \\
P_{23}(t) &= \overline{C}_0 t^3 + \overline{C}_1 t^2 + \overline{C}_2 t + \overline{C}_3,
\end{align*}
\]

where \( 0 \leq t \leq 1 \). There are twelve unknown coefficients in the system of equations (9). Therefore, in order to acquire the first six equations, let us substitute the boundary values of parameter \( t \) into the equations (9):

\[
\begin{align*}
P_{01}(t = 0) &= \overline{A}_0 = \overline{P}_0; & P_{01}(t = 1) &= \overline{A}_1 + \overline{A}_2 + \overline{A}_3 = \overline{P}_1; \\
P_{12}(t = 0) &= \overline{B}_0 = \overline{P}_2; & P_{12}(t = 1) &= \overline{B}_1 + \overline{B}_2 + \overline{B}_3 = \overline{P}_2; \\
P_{23}(t = 0) &= \overline{C}_0 = \overline{P}_3; & P_{23}(t = 1) &= \overline{C}_1 + \overline{C}_2 + \overline{C}_3 = \overline{P}_3.
\end{align*}
\]

Let us express the first and second derivatives of vector functions of segments of spline curve \( a_i \) in connection points \( P_1 \) and \( P_2 \):

\[
\begin{align*}
P'_{01}(t = 1) &= 3\overline{A}_1 + 2\overline{A}_2 + \overline{A}_3; & P'_{12}(t = 0) &= \overline{B}_1; \\
P''_{01}(t = 1) &= 6\overline{A}_2 + 2\overline{A}_3; & P''_{12}(t = 0) &= \overline{C}_1; \\
P'_{12}(t = 1) &= 6\overline{B}_1 + 2\overline{B}_2; & P''_{12}(t = 0) &= \overline{C}_1; \\
P''_{12}(t = 1) &= 6\overline{B}_2 + 2\overline{B}_3.
\end{align*}
\]

The following equations are acquired through the condition of smoothness \( C^2 \) at the points of connection of segments of spline curve \( a_i \):

\[
\begin{align*}
P_{01}(t = 1) = P_{12}(t = 0) &\rightarrow 3\overline{A}_1 + 2\overline{A}_2 + \overline{A}_3 = \overline{B}_1; \\
P'_{12}(t = 1) = P''_{01}(t = 0) &\rightarrow 6\overline{A}_2 + 2\overline{A}_3 = 2\overline{B}_1; \\
P''_{12}(t = 1) = P''_{12}(t = 0) &\rightarrow 6\overline{B}_2 + 2\overline{B}_3 = 2\overline{C}_1.
\end{align*}
\]

The acquired systems (10) and (11) have ten equations and twelve unknown values. Let us appoint two additional boundary conditions – zero second derivatives in endpoints of spline curve \( a_i \):

\[
\begin{align*}
P''_{01}(t = 0) &= \overline{A}_2 = 0; \\
P''_{23}(t = 1) &= 6\overline{C}_3 + 2\overline{C}_2 = 0.
\end{align*}
\]

As a result, the equations (10), (11), and (12) combined constitute a system if twelve linear equations. Their simultaneous solution results in equations of segments \( P_{01}, P_{12}, \) and \( P_{23} \) of the spline curve \( a_i \):

\[
\begin{align*}
P_{01}(t) &= \{x_{(1)}, y_{(1)}, z_{(1)}\}, & P_{12}(t) &= \{x_{(2)}, y_{(2)}, z_{(2)}\}, & P_{23}(t) &= \{x_{(3)}, y_{(3)}, z_{(3)}\}; \\
P_{01} &= \begin{bmatrix} 3.33 & -4.13 & 1.4 & 0 \\ 0 & 0 & 0 & 0 \\ 26.7 & 7.13 & -3.4 & 0 \\ 0 & 0 & 15 & 0 \end{bmatrix};
\end{align*}
\]
\[ \bar{P}_{12} = \begin{bmatrix} t^3, t^2, t, 1 \end{bmatrix} \]

\[ \begin{bmatrix} -6.7 & 9.7 & -3 & 0 \\ 10 & -12.4 & 4.2 & 0 \\ 36.7 & -52.6 & 0.8 & 0 \\ 30 & 3 & 13 & 0 \end{bmatrix} ; \]

\[ \begin{bmatrix} 3.3 & -5.53 & 1.6 & 0 \\ -10 & 16.6 & -4.8 & 0 \\ 36.7 & -1.07 & 0.2 & 0 \\ 70 & -5 & 15 & 0 \end{bmatrix} ; \]

where \( 0 \leq t \leq 1 \).

Substitution of the acquired coordinate functions of vector equations of segments and their first derivatives into the equations of cyclographic projections (2) results in the equations of the cyclographic images of the segments of the spline curve \( a_5 \):

\[ x_{at(1)}(t) = R_1 - K_1 L M_1 + N_1 \sqrt{M_1^2 + N_1^2 - L_1^2} \]
\[ y_{at(1)}(t) = Q_1 + K_1 -L N_1 + M_1 \sqrt{M_1^2 + N_1^2 - L_1^2} , \]

where \( R_1 = 3.33 \cdot t^5 + 26.7 \cdot t ; \) \( Q_1 = -4.13 \cdot t^3 + 7.13 \cdot t ; \) \( M_1 = 9.99 \cdot t^2 + 26.7 ; \) \( L_1 = 4.2 t^2 - 3.4 ; \) \( N_1 = -12.4 \cdot t^2 + 7.13 ; \) \( K_1 = 1.4t^3 + 15 - 3.4t . \)

\[ x_{at(2)}(t) = R_2 - K_2 L M_2 + N_2 \sqrt{M_2^2 + N_2^2 - L_2^2} \]
\[ y_{at(2)}(t) = Q_2 + K_2 -L N_2 + M_2 \sqrt{M_2^2 + N_2^2 - L_2^2} , \]

where \( R_2 = -6.7t^5 + 10r^2 + 36.7t + 30 ; \) \( Q_2 = 9.7t^4 - 12.4t^2 - 5.27t + 3 ; \) \( M_2 = -20t^2 + 20t + 36.7 ; \) \( L_2 = -9t^5 + 8.4t + 0.8 ; \) \( N_2 = 29t^2 - 24.8t - 5.3 ; \) \( K_2 = 3t^3 + 4.2t^2 + 0.8t + 13 . \)

\[ x_{at(3)}(t) = R_3 - K_3 L M_3 + N_3 \sqrt{M_3^2 + N_3^2 - L_3^2} \]
\[ y_{at(3)}(t) = Q_3 + K_3 -L N_3 + M_3 \sqrt{M_3^2 + N_3^2 - L_3^2} , \]

where \( R_3 = 3.3t^5 - 10r^2 + 36.7t + 70 ; \) \( Q_3 = -5.53t^3 + 16.6r^2 - 1.07t - 5 ; \) \( M_3 = 9.99t^2 - 20t + 36.7 ; \) \( L_3 = 4.8t^2 - 9.6t + 0.2 ; \) \( N_3 = -16.6t^2 + 33.2t - 1.07 ; \) \( K_3 = 1.6t^3 - 4.8t^2 + 0.2t + 15 . \)

Having determined the first and second derivatives of the equations of cyclographic projections of the segments, let us test their equality in points \( P_{at(1)} \) and \( P_{at(2)} \) of connection:

\[ x_{at(1)}(t = 1) = 41.57; \quad x_{at(1)}(t = 0) = 41.57; \quad y_{at(1)}(t = 1) = -5.06; \quad y_{at(1)}(t = 0) = -5.06; \]
\[ x_{at(2)}(t = 1) = 27.22; \quad x_{at(2)}(t = 0) = 27.22; \quad y_{at(2)}(t = 1) = -0.64; \quad y_{at(2)}(t = 0) = -0.64; \]
\[ x_{at(3)}(t = 1) = 20.3; \quad x_{at(3)}(t = 0) = 15.1; \quad y_{at(3)}(t = 1) = -15.4; \quad y_{at(3)}(t = 0) = 53.3; \]
\[ x_{at(1)}(t = 1) = -86.2; \quad x_{at(1)}(t = 0) = 68.05; \quad y_{at(1)}(t = 1) = -9.7; \quad y_{at(1)}(t = 0) = -27.4. \]

As we can see from the numeric results, the values of the second derivatives of coordinate functions of the cyclographic projection segments in connection points \( P_{at(1)} \) and \( P_{at(2)} \) are not equal. This follows from the fact that the equations of these segments have pairwise unequal third derivatives of
coordinate functions of the segments of the spline curve \( a \) at the points \( P_1 \) and \( P_2 \). Indeed, the third derivatives of coordinate functions have the following values in the connection points \( P_1 \) and \( P_2 \):

\[
\begin{align*}
&x_{i1}''(t=1) = 20; \quad y_{i1}''(t=1) = -40; \quad z_{i1}''(t=1) = -33.2; \\
&y_{i2}''(t=1) = -24.8; \quad y_{i2}''(t=0) = 58; \quad y_{i3}''(t=0) = -33.2; \\
z_{i0}''(t=1) = 8.4; \quad z_{i2}''(t=0) = -18; \quad z_{i3}''(t=1) = 9.6.
\end{align*}
\]

Due to inequality of the third derivatives of the coordinate functions of the initial spline curve \( a \), the segments of the cyclographic projection are not connected with smoothness \( C^2_a \), as stated previously. The sufficient condition formulated in the statement guarantees the required smoothness of connection \( C_{a+1}^{k} \) of the cyclographic projections of segments. It follows from the bijective correspondence of prototypes \( a \) and \( a_{n+1} \) and their respective cyclographic images \( a_{a(n)} \) and \( a_{a(n+1)} \), as well as sufficiency of the condition of smoothness \( C^k \) of segments \( a \) and \( a_{n+1} \).

This brings up the question: is it possible, given the same initial conditions (smoothness \( C^k \) of connection of the initial segments), to connect the cyclographic projections of the segments with smoothness \( C^k \)? Obviously, in order to do achieve the order smoothness \( C^k_a \), it is required to introduce one additional condition into the conditions (8) — the condition of equality of derivatives of order \( k+1 \):

\[
x_{n}^{(k+1)}(t=1) = x_{n+1}^{(k+1)}(t=0); \quad y_{n}^{(k+1)}(t=1) = y_{n+1}^{(k+1)}(t=0); \quad z_{n}^{(k+1)}(t=1) = z_{n+1}^{(k+1)}(t=0).
\]

Is the condition (13) at all realizable? Let us keep in mind that a spline is a set of polynomials of order \( k+1 \) with equal continuous derivatives of order up to \( k \) in points of connection [7]. For example, continuity of a cubic spline in connection point equals 2. Therefore, there has to be a continuous second derivative, which provides continuity of curvature of segments in connection point. The \((k+1)\)th derivative of a polynomial of order \( k+1 \) constitutes a certain constant. Through to the condition (13) we establish equality of derivatives of order up to \( k+1 \) inclusive of coordinate functions at the point of connection of two segments of a spline curve. As a result, by fulfilling the conditions (6), (8), (13), with regard for (7) and the equations

\[
\begin{align*}
&y_{a(a)}(t=1) = x_{a(a+1)}(t=0); \quad y_{a(a)}'(t=1) = y_{a(a+1)}'(t=0), \\
&y_{a(a)}''(t=1) = y_{a(a+1)}''(t=0), \quad y_{a(a)}'''(t=1) = y_{a(a+1)}'''(t=0),
\end{align*}
\]

we achieve smoothness \( C^k_a \) at the point of connection of the cyclographic segments \( a_{a(n)} \) and \( a_{a(n+1)} \).

Let us confirm the above on the example of the considered Hermite spline of the third order \( a \). Given the same initial conditions, it is now required to construct a spline curve \( a \) of the third order with the additional condition of equality of the third derivatives in connection points and to determine whether the cyclographic projections of the segments of the initial spline curve are connected with smoothness \( C^2_a \).

The algorithm for finding the unknown coefficients of the spline is basically identical to the one considered in the example above with the exception that it is required to introduce the condition of equality of the third derivatives at the points of connection. In order to do that, let us substitute the system of equations (12) with the following:

\[
\begin{align*}
&\overline{F}_{01}''(t=1) = \overline{F}_{12}''(t=0); \\
&\overline{F}_{12}''(t=1) = \overline{F}_{23}''(t=0);
\end{align*}
\]

where \( \overline{F}_{01}''(t=1) = 6A_1, \overline{F}_{12}''(t=0) = 6B_1, \overline{F}_{12}''(t=1) = 6B_1, \overline{F}_{23}''(t=0) = 6C_1 \).
Simultaneous consideration of the systems of equations (10), (11), and (14) results in the following equations of the initial spline:

\[ \bar{P}_{01}(t) = \{x(0), y(0), z(0)\}, \quad \bar{P}_{12}(t) = \{x(1), y(1), z(1)\}, \quad \bar{P}_{23}(t) = \{x(3), y(3), z(3)\}; \]

\[
\begin{bmatrix}
-3.33 & 4.83 & -1.5 & 0 \\
15 & -20 & 6.5 & 0 \\
18.3 & 18.2 & -7 & 0 \\
0 & 0 & 15 & 0
\end{bmatrix};
\]

\[
\begin{bmatrix}
-3.3 & 4.8 & -1.5 & 0 \\
5 & -5.5 & 2 & 0 \\
38.3 & -7.33 & 1.5 & 0 \\
30 & 3 & 13 & 0
\end{bmatrix};
\]

\[
\begin{bmatrix}
-5 & 9 & -2.5 & 0 \\
38.3 & -3.83 & 1 & 0 \\
70 & -5 & 15 & 0
\end{bmatrix}.
\]

where \(0 \leq t \leq 1\).

The equations of the cyclographic \(\alpha\)-projections of the segments of the spline \(a_i\) are of the following form:

\[
x_{a(t)} = R_i - L_iM_i + N_i\sqrt{M_i^2 + N_i^2 - L_i^2} \\
y_{a(t)} = Q_i + K_i - L_iN_i + M_i\sqrt{M_i^2 + N_i^2 - L_i^2},
\]

where now \(R_i = -3.33t^3 + 15t^2 + 18.3t; \quad Q_i = 4.83t^3 - 20t^2 + 18.2t; \quad M_i = -9.99t^2 + 30t + 18.3; \quad L_i = -4.5t^2 + 13t - 7; \quad N_i = 14.5t^2 - 40t + 18.2; \quad K_i = -1.5t^3 + 6.5t^2 - 7t + 15.\)

\[
x_{a(2)} = R_2 + L_2M_2 + N_2\sqrt{M_2^2 + N_2^2 - L_2^2} \\
y_{a(2)} = Q_2 + K_2 + L_2N_2 + M_2\sqrt{M_2^2 + N_2^2 - L_2^2},
\]

where now \(R_2 = -3.3t^3 + 5t^2 + 38.3t + 30; \quad Q_2 = 4.8t^3 - 5.5t^2 - 7.3t + 3; \quad M_2 = -9.99t^2 + 10t + 38.3; \quad L_2 = -4.5t^2 + 4t + 1.5; \quad N_2 = 14.5t^2 - 11t - 7.33; \quad K_2 = -1.5t^3 + 2t^2 + 1.5t + 13.\)

\[
x_{a(3)} = R_3 - L_3M_3 + N_3\sqrt{M_3^2 + N_3^2 - L_3^2} \\
y_{a(3)} = Q_3 + K_3 - L_3N_3 + M_3\sqrt{M_3^2 + N_3^2 - L_3^2},
\]

where now \(R_3 = -3.3t^3 - 5t^2 + 38.3t + 70; \quad Q_3 = 4.83t^3 + 9t^2 - 3.83t - 5; \quad M_3 = -9.99t^2 - 10t + 38.3; \quad L_3 = -4.5t^2 - 5t + 1; \quad N_3 = 14.5t^2 + 18t - 3.83; \quad K_3 = -1.5t^3 - 2.5t^2 - 5t + 1.\)

Let us evaluate the derivatives at the junctions of the cyclographic projections of the segments of the spline curve \(a_i\) and check their values for equality:
\[ x^{(1)}_{a(1)}(t = 1) = 40.33; \quad x^{(2)}_{a(0)}(t = 0) = 40.33; \quad y^{(1)}_{a(1)}(t = 1) = -6.12; \quad y^{(2)}_{a(2)}(t = 0) = -6.12; \]
\[ x^{(1)}_{a(2)}(t = 1) = 33.66; \quad x^{(2)}_{a(0)}(t = 0) = 33.66; \quad y^{(1)}_{a(1)}(t = 1) = -2.49; \quad y^{(2)}_{a(2)}(t = 0) = -2.49; \]
\[ x^{(1)}_{a(0)}(t = 1) = 4.42; \quad x^{(2)}_{a(0)}(t = 0) = 4.42; \quad y^{(1)}_{a(1)}(t = 1) = -6.36; \quad y^{(2)}_{a(2)}(t = 0) = -6.36; \]
\[ x^{(1)}_{a(2)}(t = 1) = -21.1; \quad x^{(2)}_{a(0)}(t = 0) = -21.1; \quad y^{(1)}_{a(1)}(t = 1) = 12.29; \quad y^{(2)}_{a(2)}(t = 0) = 12.29. \]

The result of evaluation shows smoothness \( C^2 \) of the cyclographic segments at the points of connection. Even though the third derivatives at the points of connection of the segments of the spline curve \( a_3 \) are equal, the respective cyclographic projections can be connected with smoothness \( C^r \), where \( r > 2, r \in N \). The results of the computational experiment conducted in computer algebra system are depicted on figure 2.

**Figure 2.** The results of computational experiment on determination of smoothness of connection of segments of cyclographic projection of a spatial spline curve

5. Consideration of the results

Smoothness \( C^r \) of connection of the initial spline curve segments is accompanied by equality of values of geometric invariants of curvilinear segments [16] at the point of their connection. These geometric invariants are based on equality of derivatives

\[ \bar{P}^{(k)}_n(t = 1) = \bar{P}^{(k)}_{n+1}(t = 0), \quad k = 0, 1, 2, \ldots, \tag{15} \]

where \( k \) represents the order of derivative of polynomial vector-functions describing the connected segments. For example, values \( k = 0 \) and \( k = 1 \) result in equality of values of speed

\[ \left| \frac{d\bar{S}}{dt} \right| = \left| \bar{P}'(t = 1) \right| = \left| \bar{P}'_{n+1}(t = 0) \right| = \left| \frac{dS_{n+1}}{dt} \right| = \left| \bar{P}'_{n+1} \right| \]

in the connection point. Values \( k = 0, 1, 2 \) result in equality of speeds of points, equality of curvatures

\[ k_3(t = 1) = \left[ \frac{\bar{P}'_n(t = 1), \bar{P}'_{n+1}(t = 1)}{\bar{P}'_n(t = 1)} \right] = \left[ \frac{\bar{P}'_{n+1}(t = 0), \bar{P}'_{n+1}(t = 0)}{\bar{P}'_{n+1}(t = 0)} \right] = k_{n+1}(t = 0) \]

and equality of respective accelerations (in magnitude and direction)

\[ \bar{W}_n(t = 1) = \frac{d\bar{V}_n}{dt}(t = 1) = \bar{P}''_n(t = 1) = \bar{P}''_{n+1}(t = 0) = \frac{d\bar{V}_{n+1}}{dt}(t = 0) = \bar{W}_{n+1}(t = 0). \]

It quite obviously follows from these examples that equality of values of geometric invariants at the spline curve segments connection point does not result in equality of the respective derivatives (15) in
such point. If the condition required for smoothness $C^k$ is not achieved, this automatically affects smoothness $C^k_a$.

6. Conclusion
The results of computational experiments have confirmed that achieving the stated sufficient condition indeed guarantees the required smoothness $C^{k-1}_a$ of connection of cyclographic projections of spatial spline curve segments connected with smoothness $C^k$. If the order of a segmented polynomial of the initial spline curve equals $k + 1$, then its continuity equals $k$, which guarantees existence of continuous derivatives of orders up to $k$ inclusive, while the derivative of order $k+1$ exists and is constant. This condition is sufficient in order to connect segments of cyclographic projection of the initial spline curve with smoothness $C^k_a$.

Coordination of smoothness $C^k$ and $C^k_a$ serves as the basis in development of effective algorithms of cyclographic formation of geometric objects applied in solutions of a variety of tasks [17] in various practical fields including:
- geometric optics (formation of elements of the triad of curves and the triad of surfaces with optical reflective properties);
- road surface form design (formation of composite ruled surfaces of roadway through smooth connection of ruled surface segments);
- cutting tool trajectory design (formation of composite trajectories for pocket machining of machine-building products through smooth connection of curvilinear segments).

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