Stabilizer Formalism for Generalized Concatenated Quantum Codes

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Abstract—The concept of generalized concatenated quantum codes (GCQCs) provides a systematic way for constructing good quantum codes from short component codes. We introduce a stabilizer formalism for GCQCs, which is achieved by defining quantum coset codes. This formalism offers a new perspective for GCQCs and enables us to derive a lower bound on the code distance of stabilizer GCQCs from component codes parameters, for both non-degenerate and degenerate component codes. Our formalism also shows how to exploit the error-correcting capacity of component codes to design good GCQCs efficiently.

I. INTRODUCTION

Error-correcting codes are necessary to overcome restrictions in computation and communication due to noise, but developing algorithms for finding 'good' codes is generically an intractable problem and evidently the central question of coding theory. 'Good codes' are special in that they have good trade-off among rate, distance, encoding and decoding costs, thereby reducing requisite space and time resources.

In classical settings, constructing generalized concatenated codes, which incorporate multiple outer codes concatenated with multiple inner codes, is a promising approach for realizing good trade-off among those parameters [1], [2]. Recently, generalized concatenation has been introduced into the quantum scenario, providing a systematic way to construct good quantum codes with short component codes [3], [4].

The stabilizer formalism plays a central role in almost all branches of quantum information science, especially in quantum coding theory. Stabilizer codes, which are quantum analogues of classical linear codes, form the most important class of quantum error-correcting codes (QECCs) [5], [6]. The stabilizer formalism serves not only a role analogous to the classical parity-check matrix, but also takes a role analogous to the classical generator matrix during the decoding and encoding procedures [5], [7]. However, the stabilizer formalism for generalized concatenated quantum codes (GCQCs) has not been investigated in much detail previously, and the understanding of GCQCs is still far from satisfactory compared to their classical counterparts.

In this work we introduce the stabilizer formalism for GCQCs, thereby providing a new perspective for the GCQC framework as well as a powerful and systematic technique for constructing good stabilizer codes. By using our stabilizer formalism, we derive a lower bound on the achievable distance for GCQCs. Moreover, our stabilizer formalism for GCQCs clarifies how to exploit the error-correcting capacity of component codes to improve the performance of the resultant codes efficiently.

II. GENERALIZED CONCATENATED STABILIZER CODES

A qudit is a quantum system modeled by a $q$-dimensional Hilbert space $\mathbb{C}^q$, where $q$ is a prime power. A stabilizer (or additive) quantum code encoding $k$ qudits into an $n$-qudit system, with minimum distance $d$, is denoted by $[[n,k,d]]_q$.

A. Idea of generalized concatenated quantum codes

A concatenated stabilizer code is constructed from two component quantum codes: an outer code $A$ with parameters $[[N,K,D]]_Q$ and an inner code $B$ with parameters $[[n,k,d]]_q$ such that $Q = q^k$. The concatenated code $A \circ B$ is constructed in the following way: for any state $|\phi\rangle = \sum_{j_1=0}^{Q-1} \alpha_{j_1} |j_1\rangle$ of the outer code $A$, replace each basis vector $|j_1\rangle$ (where $j_1 = 0, \ldots, Q-1$ for $l = 1, \ldots, N$) by a basis vector $|\psi_{j_1}\rangle$ of the inner code $B$. This mapping yields

$$|\phi\rangle \mapsto |\tilde{\phi}\rangle = \sum_{j_1=0}^{Q-1} \alpha_{j_1}^* |\psi_{j_1}\rangle \cdots |\psi_{j_N}\rangle,$$

and the resultant code is an $[[nN,kK,D]]_q$ stabilizer code where $D \geq dD$ [5], [8].

For GCQCs, the role of the basis vectors of the inner quantum code is taken on by subcodes of the inner code [3]. In its simplest version (two-level version), a GCQC is also constructed from two quantum codes: an outer code $A_1$ with with parameters $[[N,K_1,D_1]]_Q$, and an inner code $B_1$ with parameters $[[n,k_1,d_1]]_q$, such that the inner code $B_1$ could be

1In the sequel, we usually denote the outer parameters by capital Latin characters and the inner parameters by their small counterparts.
isomorphic, we can, on an abstract level, rewrite the decomposition (2) as
\[ B_1 = \bigoplus_{j=0}^{Q_1-1} B_{2(j)}^{(j)}. \] (2)
and each \( B_{2(j)}^{(j)} \) is an \( \|n, k_j, d_j\|_q \) code with basis vectors \( \{\ket{\psi_{(j)}^{(j)}}\}_j \), \( j = 0, \ldots, Q_1 - 1 \) and \( d_2 \geq d_1 \). Thus we have \( q^{k_1 k_2} = Q_1 \).  

To construct a GCQC, replace each basis state \( \ket{j} \) of the outer code \( A_1 \) with a basis state \( \{\ket{\psi_{(j)}^{(j)}}\} \) of \( B_{2(j)}^{(j)} \). In this way, each basis state \( \ket{j} \) of the outer code is mapped to the subcode \( B_{2(j)}^{(j)} \). Consequently, given a state \( \ket{\phi} = \sum_{j_{(1)} \cdots j_{(N)}} \alpha_{j_{(1)} \cdots j_{(N)}} \ket{j_{(1)} \cdots j_{(N)}} \) of the outer code together with an unencoded basis state \( \ket{i_1 \cdots i_N} \in (\mathbb{C}^{d_2})^\otimes N \), the encoding of a GCQC is given by the following mapping (3):
\[ \ket{\phi} \ket{i_1 \cdots i_N} \rightarrow \sum_{j_{(1)} \cdots j_{(N)}} \alpha_{j_{(1)} \cdots j_{(N)}} \ket{\phi^{(j_{(1)})}} \cdots \ket{\phi^{(j_{(N)})}}. \] (3)

This then gives a GCQC code with parameters \( \|N, \mathcal{K}, \mathcal{D}\|_q \), where \( N = nN, \mathcal{K} = (k_1 - k_2)K_1 + k_3N \), and the minimum distance \( \mathcal{D} \) is to be determined. Note that the basis states \( \ket{i_1 \cdots i_N} \) span a trivial outer code \( \|N, N, 1\|_q \), where \( Q_2 = q^{k_3} \). Therefore, two outer codes and two inner codes are used, which is where the name ‘two-level concatenation’ comes from.

**B. Quantum coset codes**

We adapt the concept of coset codes [11], [12], [10] to the quantum scenario to provide an alternative understanding for stabilizer GCQCs. Coset codes will help to build a systematic interpretation for GCQCs from the viewpoint of the stabilizer formalism.  

We choose any subcode \( B_{2(i)}^{(i)} \) in the decomposition (2) and denote it as \( B_{2} \). Continuing the partitioning process, say
\[ B_i = \bigoplus_{j=0}^{Q_{i+1}-1} B_{2(i)}^{(j)}. \] (4)
for \( i = 2, 3, \ldots, m \), we obtain a chain of subcodes
\[ B_{m+1} \subset B_m \subset \cdots \subset B_3 \subset B_2 \subset B_1, \] (5)
where all subcodes \( B_{2(i)}^{(i)} \) on level \( i \) have parameters \( \|n, k_i, d_i\|_q \). To simplify notation, we use \( B_i \) to denote any of the sub-codes \( B_{2(i)}^{(i)} \) on level \( m + 1 \), all subcodes are one-dimensional subspaces, and we choose \( B_{m+1} = \{|0\rangle\} \).

As the subspaces \( B_{2(i)}^{(i)} \) in the decomposition (3) are all isomorphic, we can, on an abstract level, rewrite the decomposition as a tensor product of a vector space of dimension \( Q_i \), spanned by orthonormal states \( \ket{j} \) corresponding to the indices \( j \) in the decomposition (2), and the subcode \( B_{i+1} \). We denote this situation by
\[ B_i = \|B_i / B_{i+1}\| \otimes B_{i+1}. \] (6)

It turns out that the co-factor \( \|B_i / B_{i+1}\| \) in (3) can be identified with an additive quantum code of dimension
\[ Q_i = \dim \|B_i / B_{i+1}\| = q^{k_i k_{i+1}}. \] (7)

Note that both \( B_{i+1} \) and \( \|B_i / B_{i+1}\| \) are defined with respect to a quantum system with \( n \) qudits. In analogy to coset codes in the context of generalized concatenated codes [1], [9], [10], we call \( B_i / B_{i+1} \) a quantum coset code.

This then directly leads to
\[ B_1 = \|B_1 / B_2\| \otimes \|B_2 / B_3\| \otimes \cdots \otimes \|B_m / B_{m+1}\|. \] (8)
i.e., the quantum code \( B_1 \) is abstractly a tensor product of \( m \) coset codes \( \|B_1 / B_2\|, \|B_2 / B_3\|, \ldots, \|B_m / B_{m+1}\| = B_m \). These \( m \) quantum coset codes will be used as inner codes to be concatenated with \( m \) outer codes \( A_i (i = 1, 2, \ldots, m) \) to form an \( m \)-level concatenated quantum code.

On each level, the basis state \( \ket{j} \in \mathbb{C}^{Q_i} \) of the ‘coordinate space’ of the outer code \( A_i = \|N_i, K_i, D_i\|_q \) is mapped to the basis index \( j \) of the corresponding quantum coset code \( B_i / B_{i+1} \). Hence, the \( i \)th level of concatenation yields the concatenated code
\[ C_i = A_i \circ \|B_i / B_{i+1}\|. \] (9)
The resultant \( m \)-level concatenated code \( C \) is then the abstract tensor product of those \( m \) concatenated codes, i.e.,
\[ C = C_1 \otimes C_2 \otimes \cdots \otimes C_m. \] (10)

**III. STABILIZER FORMALISM FOR GENERALIZED CONCATENATED QUANTUM CODES**

A. Stabilizers for the inner codes

We now develop the stabilizer formalism for GCQCs based on the coset codes \( \|B_i / B_{i+1}\| \). For simplicity we consider the case \( q = 2 \), i.e., all codes \( B_i \)s are qubit stabilizer codes. The extension to larger dimensions \( q \) is straightforward.

For the code \( B_1 = \|n, k_1, d_1\|_q \), let \( S_{B_1} = \{g_1, g_2, \ldots, g_{n-k_1}\} \) denote the set of generators of the stabilizer group. The corresponding sets of logical X- and Z-operators for the \( k_1 \) encoded qubits are denoted by \( \overline{X}_{B_1} = \{\overline{X}_{B_1}, \overline{X}_{B_1^{+}}\} \) and \( \overline{Z}_{B_1} = \{\overline{Z}_{B_1}, \overline{Z}_{B_1^{-}}\} \). Similarly, for the code \( B_i \), we use \( S_{B_i}, \overline{X}_{B_i}, \overline{Z}_{B_i} \) to denote the set of stabilizer generators, the logical X-, and the logical Z-operators, respectively.

Note that \( B_i = \|n, k_i, d_i\|_q \) is a subcode of \( B_1 \), for \( 2 \leq i \leq m + 1 \). Thus \( S_{B_i} \) can be chosen as the union of \( S_{B_{i-1}} \) and a set comprising \( k_1 - k_i \) commuting logical operators of \( B_{i-1} \) which is denoted as \( \hat{S}_{B_i} \). Without loss of generality, we choose \( \hat{S}_{B_i} = \{\overline{Z}_{B_i}, \overline{Z}_{B_i^{+}}\} \). Thus we have
\[ S_{B_i} = S_{B_{i-1}} \cup \hat{S}_{B_i} = \{g_1, g_2, \ldots, g_{n-k_i}, \overline{Z}_{B_i^{+}}, \overline{Z}_{B_i}^{+}, \ldots, \overline{Z}_{B_i^{+}}^{-}\}, \] (11)
\[ \overline{X}_{B_i} = \{\overline{X}_{B_i^{+}}, \overline{X}_{B_i^{+}}, \overline{X}_{B_i^{+}}, \ldots, \overline{X}_{B_i^{+}}^{-}\}, \] (12)
\[ \overline{Z}_{B_i} = \{\overline{Z}_{B_i^{+}}, \overline{Z}_{B_i^{+}}, \overline{Z}_{B_i^{+}}, \ldots, \overline{Z}_{B_i^{+}}^{-}\}. \] (13)

Note that eventually we will arrive at \( B_{m+1} = \{|00\cdots 0\rangle\} \). This logical state \( \{|0\rangle\} \) is the only vector shared by all \( B_i \).

Recall that the code \( \|B_i / B_{i+1}\| \) is an additive quantum code with dimension \( Q_i = q^{k_i k_{i+1}} \). Let \( \hat{S}_{B_i / B_{i+1}} \) denote the
set of generators of its stabilizer group. Defining the set
\[ S_{B_{ii}} = \overline{Z}_{B_{ii}} = \{ Z_{k_{i-1}+1}, Z_{k_{i-1}+2}, \ldots, Z_{k_i} \} \]
we have
\[
S_{B_{ii}} = S_{B_{ii}} \cup \overline{S}_{B_{ii}} = \{ g_1, \ldots, g_{n-k}, \overline{Z}, \overline{Z}_{k_1}, \overline{Z}_{k_1-k_{i-1}}, \ldots, \overline{Z}_{k_i} \}.
\]  
(14)
The logical operators of \([B_i/B_{i+1}]\) are
\[
\overline{Z}_{[B_i/B_{i+1}]} = \{ \overline{Z}_{k_1}, \overline{Z}_{k_1-k_{i-1}}, \ldots, \overline{Z}_{k_i} \}
\]  
and
\[
\overline{X}_{[B_i/B_{i+1}]} = \{ \overline{X}_{k_1}, \overline{X}_{k_1-k_{i-1}}, \ldots, \overline{X}_{k_i} \}.
\]  
(15)
The structure is illustrated in Fig. 1

![Fig. 1. Structure of a quantum coset code obtained by nesting B_i](image)

Note that each \(X^a\) (\(Z^b\)) is a Pauli operator corresponding to the \(i\)th qubit for each block with \(r_i\) qubits. For the concatenation at the \(i\)th level, each basis vector \(|j\rangle\) of the ‘coordinate space’ of \(A_i\) will be mapped to a basis vector \(|b^{(j)}\rangle\) of the coset code \([B_i/B_{i+1}]\). Therefore, in order to import the constraints coming from the stabilizer generators \(S_{A_i}\), we need to replace the Pauli operators \(X^a\) (\(Z^b\)) for each block of \(r_i\) qubits by the corresponding logical operators of \([B_i/B_{i+1}]\), which are given by Eqs. (15) and (16).

For each of the \(N\) blocks in total, this procedure encodes \(r_i\) qubits into \(n\) qubits. For each \(G_j = X^a \otimes Z^b\), \((1 \leq j \leq N)\), the replacement mentioned above yields
\[
\overline{G}_j = (\overline{X}_{k_1}, \overline{X}_{k_1-k_{i-1}}, \ldots, \overline{X}_{k_i}) \cdot (\overline{Z}_{k_1-k_{i-1}}, \overline{Z}_{k_1-k_{i-1}+1}, \ldots, \overline{Z}_{k_i}).
\]  
(17)
Thus each generator \(G \in S_{A_i}\) is mapped to \(\overline{G} = \bigotimes_{j=1}^N \overline{G}_j \in S_{A_i}\), where \(\overline{S}_{A_i}\) denotes the resulting set of generators after the replacement.

For each outer code \(A_i\), denote the set of logical operators by \(L_{A_i}\). Then using a similar replacement as for the stabilizer generators, we obtain a set of logical operators for the \(i\)th level concatenated code, which we denote by \(\overline{L}_a\). We then have the following proposition, which is a direct consequence of Eq. (18).

**Proposition 1.** The set of stabilizer generator \(S_{C}\) for the generalized concatenated quantum code \(C = [N, K, D]_q\) is given by
\[
S_{C} = S_{L} \cup \bigcup_{i=1}^{m} \overline{S}_{A_i},
\]  
(18)
and the set of logical operators \(L_{C}\) is given by
\[
L_{C} = \bigcup_{i=1}^{m} \overline{L}_i.
\]  
(19)
Note that we may multiply any logical operator by an element of the stabilizer without changing its effect on the code.

**Example 2.** Consider \(B_1 = [4, 2, 2]_2\) with stabilizer generators \(S_{B_1} = \{ XXXX, ZZZZ \}\) and logical operators \(\overline{Z}_1 = ZZZ, \overline{X}_1 = X X I, \overline{Z}_2 = X I I, \overline{X}_2 = X X I I\). Then take the subcodes \(B_2 = [4, 1, 2]_2\) with stabilizer generators \(S_{B_2} = \{ \overline{Z}_1 \}\) and \(B_3 = [4, 0, 2]_2\) with stabilizer generators by \(S_{B_3} = \{ \overline{Z}_1, \overline{Z}_2 \}\). Thus the coset code \([B_1/B_2]\) has dimension 2 with logical operators \(\{ \overline{Z}_1, \overline{X}_1 \}\). It will be used as the inner code for the first level of concatenation. Since \(B_3 = [0]_3\), we have

Here we omit the tensor product symbol, i.e., \(X^{n_1}X^{n_2} \cdots X^{n_\ell}\) is to be read as \(X^{n_1} \otimes X^{n_2} \otimes \cdots \otimes X^{n_\ell}\), similarly for \(Z^{n_1}Z^{n_2} \cdots Z^{n_\ell}\).
If the first outer code $a$ is degenerate code, then together with the trivial code $A$, the identity operator on each of the corresponding $n$ qubits obtained by mapping one coordinate sub-block. The resulting GCQC has parameters $C = \{8,3,2\}$.

IV. Parameters of GCQCs

In order to derive the parameters of the GCQCs from our stabilizer formalism, we will use the following lemma. We keep the notation from the previous sections. In addition, for a stabilizer code with stabilizer generators $S$, we denote the normalizer group of $S$ by $N(S)$.

**Lemma 3.** Consider the restriction $\overline{W}_{ij}$ and $\overline{V}_{ij}$ of any two elements $W \in N(S_A)$ and $V \in N(S_A)$ $(1 \leq i \leq j \leq m)$ to sub-block $\bar{r}$ and $\bar{s}$ $(r,s \in \{1,\ldots,N\})$, respectively, each block corresponding to $n$ qubits obtained by mapping one coordinate of the outer code to the $n$ qubits of the inner code. Then the product $W_{ij} \cdot V_{ij}$ has weight at least $d_i$, unless $W_{ij} \cdot V_{ij} = \overline{W}_{ij} = \text{id}$.

**Proof:** Case 1: $i = j$:

$\overline{W}_{ij} \cdot \overline{V}_{ij}$ is composed of the logical operators of $B_i$, whose distance is $d_i$, thus $\overline{W}_{ij} \cdot \overline{V}_{ij}$ has weight at least $d_i$.

Case 2: $i < j$:

$\overline{W}_{ij} \cdot \overline{V}_{ij}$ is composed of the logical operators from $B_i$ and $B_j$. Further, $B_j \subset B_i$ implies $d_j \geq d_i$, thus $\overline{W}_{ij} \cdot \overline{V}_{ij}$ has weight at least $d_i$.

**Theorem 4.** Consider a GCQC $C = [N,K,D]_q$ which is composed of $m$ outer codes $A_i = [N,K_i,D_i]_q$ and $m$ inner codes $[B_i/B_{i+1}]_q$ for $i = 1,2,\ldots,m$, where the code $B_i = [n,k_i,d_i]_q$ is in the sub-code chain $B_{i+1} \subset B_i \subset \cdots \subset B_2 \subset B_1$ and $Q_i = q^{k_i - k_{i+1}}$. Let $A_0$ be the first degenerate code regarding the ordering $A_1 > A_2 > \cdots > A_m$ of the outer codes. Then the parameters of $C$ are given as

1) $N = nN$;

2) $K = \sum_{i=1}^{m} (k_i - k_{i+1})K_i$;

3) $D \geq \min\{d_1,d_2,d_2,\ldots,d_{m-1}d_{m-1},d_m\}$.

Note that if all outer codes are non-degenerate codes, it follows from Eq. (22) that

$D \geq \min\{d_1,d_2,d_2,\ldots,d_{m}d_{m}\}$.

If the first outer code $a$ is degenerate code, then

$D \geq d_1 \min\{D_i\}$.

**Proof:**

1) Eq. (20) is evidently true.

2) For each $A_i = [N,K_i,D_i]_q$, the number of independent generators in $S_{A_i}$ is $r_i(N-K_i)$, which is also the number of independent generators in $S_{A_i}$. The number of independent generators in $S_j$ is equal to $(n-k_i)N$. Therefore, according to Proposition 1 we have

$$K = nN - (n-k_i)N - \sum_{i=1}^{m} r_i(N-K_i) = \sum_{i=1}^{m} (k_i - k_{i+1})K_i,$$

(25)

where $r_i = k_i - k_{i+1}$ for $i = 1,2,\ldots,m$.

3) For a stabilizer code with stabilizer $S$, the minimum distance is the minimum weight of an element in $N(S) \setminus S$. In other words, it is the minimum weight of non-trivial logical operators. We consider different cases how a logical operator of a GCQC can be composed according to Proposition 1 (see Fig. 2).

**Fig. 2.** Constitution of logical operators for a GCQC with all terms defined in the body.

For the $i$th level of concatenation, we know that the distance of $[B_i/B_{i+1}]_q$ is at least $d_i$. As the distance of $A_i$ is $D_i$, according to our replacement strategy, the non-trivial logical operators obtained from $L_i$ and $\overline{S}_{A_i}$ have weight at least $d_i$ on at least $D_i$ sub-blocks of length $n$. Therefore, the minimal weight is at least $d_iD_i$. Multiplying two non-trivial elements $\tilde{t}_i$ and $\tilde{t}_j$ from two different levels $i$ and $j$ with $i < j$, from Lemma 3 the product $\tilde{t}_{ij} = \tilde{t}_i \cdot \tilde{t}_j$ must have weight at least $d_i$ on at least $D_i$ sub-blocks of length $n$. Denoting the weight of an operator $\tilde{t}$ as $\text{wgt}(\tilde{t})$, for any element $\tilde{t} \in L_i$ (see Eq. (19)), we have

$$\text{wgt}(\tilde{t}) \geq \min\{d_1,d_2,d_2,\ldots,d_{m}d_{m}\}.$$

(26)

Next we consider the minimal weight of the elements obtained by multiplying a logical operator $\tilde{t} \in L_i$ by a non-trivial stabilizer element $\overline{G} \in \overline{S}_{C_i}$: First let $\tilde{t} = \overline{G} \cdot \tilde{t}_i$, where $\overline{G} \in S_i$ or $\overline{G} \in \overline{S}_{A_i}$, and $\tilde{t}_i \in L_i$ for $1 \leq i, j \leq m$. Then we analyze $\text{wgt}(\tilde{t})$ based on the following cases (see Fig. 2):

(i) $\overline{G} \in S_i$: $\text{wgt}(\tilde{t}) \geq d_iD_i$ according to Eq. (5).

(ii) $\overline{G} \in \overline{S}_{A_i}$ and $i = j$: $\text{wgt}(\tilde{t}) \geq d_iD_i$.

(iii) $\overline{G} \in \overline{S}_{A_i}$ and $i < j$: $\text{wgt}(\tilde{t}) \geq d_iD_i$ according to Lemma 3.
(iv) $\mathcal{C} \in \mathcal{S}_A$, and $i > j$: $\text{wt}(\mathcal{L}) \geq d_i \times \max\{d_j, \text{wt}(G)\}$ according to Lemma 4. If $A_j$ is a non-degenerate outer code, then $\text{wt}(G) \geq d_j$, thus $\text{wt}(\mathcal{L}) \geq d_i d_j$. If $A_j$ is a degenerate outer code, then there exist at least one non-trivial element $G \in S_A$ such that $\text{wt}(G) < d_j$. Considering $D_i < D_j$ is probably true, thus $\text{wt}(\mathcal{L}) \geq d_i D_j$ is not guaranteed, but $\text{wt}(\mathcal{L}) \geq d_i D_j$ evidently is.

Any non-trivial logical operator can be decomposed as a combination of the four cases discussed above. Now we are ready to get the distance of a GCQC as shown by Eqs. (22), (23), and (24).

Note that a degenerate outer code might also be viewed as a non-degenerate code, but with a smaller distance. As clarified by Theorem 4 and illustrated by the following example, despite the larger minimum distance of the degenerate outer code, the minimum distance of the resulting GCQC is not increased in general.

Example 5. Let $B_1 = \{0, 1\}_Z$ with stabilizer generators $S_{B_1} = \{ZZZZ, ZZII\}$ and logical operators $\{IIII, XXXX\}$. The subcodes $B_1$ and $B_2$ are $B_2 = \{0, 1\}_Z$ with stabilizer generators $S_{B_2} = \{ZI, Z\}$, and $B_1 = \{0, 1\}_Z$ with stabilizer generators $S_{B_1} = \{ZI, Z\}$. Then $[B_1/B_2]$ is a code of dimension 2 with logical operators $\{Z_1, Z_2\}$, while the inner code to be used on the first level of concatenation, $[B_1/B_2]$, is the inner code to be used for the second level of concatenation.

The outer code $A_1 = \{XIII, IXXX\}$ is a degenerate code with stabilizer generators $S_{A_1} = \{XXII, XXXZ\}$, and logical operators $\{IIII, XIIII\}$. Furthermore, let $A_2 = \{XXII, XIIII\}$ be the trivial code with logical operators $\{ZI, Z\}$. According to our replacement strategy, $L = \mathcal{L}_{A_2} = \mathcal{L}_{A_1}$, and $L = \mathcal{L}_{A_2} = \mathcal{L}_{A_1}$. Note that the minimum weight of elements in $L$ is 2, and $\text{wt}(\mathcal{L}) = 2$. Now consider the product $G \cdot \mathcal{T}$ which is obviously a logical operator of the resulting GCQC as well and which plays the same role as $\mathcal{L}$. It is easy to check that multiplication by $G$ reduces the weight of this logical operator from $d_i D_2 = 2$ to $d_i D_2 = 1$, as predicted by Eqs. (22) and (24).

In fact, $A_1$ can also be viewed as a non-degenerate code with parameters $\{XIII, IXXX\}$. This gives the lower bound $\text{wt}(G) \geq D_1$, and therefore $\text{wt}(G \cdot \mathcal{T}) \geq d_i \times \max\{D_j, D_1\} \geq d_i D_1 = 1$, which is consistent with Eq. (23). In summary, the resultant GCQC has parameters $\{XXII, XIIZ\}$, but not $\{XXII, XIXI\}$ as one would expect for non-degenerate codes.

V. Discussion

We have developed the structure of the stabilizer and logical operators of generalized concatenated quantum codes. With the help of quantum coset codes $[B_i/B_{i+1}]$, the resulting code can be considered as an abstract tensor product of codes $C_i$ corresponding to the $i$th level of concatenation. For the code $C_i$, the lower bound on the minimum distance is $d_i D_i$. This lower bound is met only if all the non-identity entries of some logical operator of minimum weight $D_i$ of $A_i$ are mapped onto the logical operators of minimum weight $d_i$ of $[B_i/B_{i+1}]$. In some cases, it is possible to use a clever map to improve the minimum distance of $C_i$ and thereby that of the resulting code.

Example 6. Take both $A_i$ and $B_i$ as $\{0, 1\}_Z$ with stabilizer generator $\{Z\}$ and logical operators $\{Z\}$. Take $B_2 = \{00\}$ as the trivial one-dimensional code. Then $[B_1/B_2]$ is the result of the logical Z- and the logical Z-operator of $[B_1/B_2]$. In other words, we let $\mathcal{Z}_1 = XX, \mathcal{X}_1 = ZI$. Then according to our replacement strategy, we obtain a concatenated code $[4, 1, 2]_Z$ with stabilizer generators $\{ZIIII, IIIZZ, XXXX\}$. Once again, despite the original choice of logical operators for $B_1$, the resulting GCQC is probably true, thus $\text{wt}(\mathcal{L}) = 2$.

This example indicates that the minimum distance of the resulting GCQC might be significantly improved compared to the lower bound when a deliberate nesting strategy is used. That is because such a strategy could be used to optimize the weight distribution for the logical operators of the inner code $B_i$. The stabilizer of the quantum coset codes $[B_i/B_{i+1}]$ depends on this choice, and hence the parameters of the inner codes as well. In combination with suitable chosen outer codes, the error-correcting capacity of component codes could be exploited efficiently and the overall performance might be better.

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