Knowledge on Treelike Spaces

Konstantinos Georgatos
Dipartimento di Informatica e Sistemistica
Università di Roma “La Sapienza”
Via Salaria 113
I-0019 Roma
Italy
e-mail: geo@dis.uniroma1.it.

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Abstract
This paper presents a bimodal logic for reasoning about knowledge during knowledge acquisition. One of the modalities represents (effort during) non-deterministic time and the other represents knowledge. The semantics of this logic are tree-like spaces which are a generalization of semantics used for modeling branching time and historical necessity. A finite system of axiom schemes is shown to be canonically complete for the aforementioned spaces. A characterization of the satisfaction relation implies the small model property and decidability for this system.

1 Introduction

The notion of possible world dominates the literature in modal logic, via Kripke models, as well as in any logic dealing with the epistemic state of a reasoner. The heart of this popularity lies in the identification of an intentional state through common properties of extensional objects. Apart from genuine problems such as logical omniscience this representation suffers from, it is limited in a static description of the reasoner’s epistemic state. The “logic of knowing” is not only embodied in the representation of knowledge but also in the way knowledge is acquired. We do not refer to temporal properties but rather to methodology (though both can be intertwined).

Recently a family of logics was introduced (MP92, Geo94a, Geo93, DMP) with the intention to fill this void. It succeeds in doing so by attaching familiar mathematical structures such as spaces of subsets, topologies and complete lattices of subsets corresponding to a natural knowledge acquisition. This paper extends this work by introducing a bimodal logic belonging to the same family of logics and establishes a correspondence between a particular epistemic pro-
cess of knowledge acquisition with a space of subsets forming a tree (treelike space).

In our framework the view of a reasoner will be represented by a set of possible worlds. Each of these worlds represents an alternative state compatible with the reasoner’s knowledge of actual state. This treatment of knowledge agrees with the traditional one ([Hin62], [HM84], [PR85], [CM86]) expressed in a variety of contexts (artificial intelligence, distributed processes, economics, etc).

We are interested in formulating a basic logical framework for reasoning about a resource-conscious acquiring of knowledge. Such a framework can be applied to many settings such as the ones involving time, computation, physical experiments or observations. In these settings an (discrete or continuous) increase of information available to us takes place and results in an increase of our knowledge. How could this simple idea be embodied in the aforementioned semantical framework? An increase of knowledge can be represented with a restriction of the knower’s view, i.e. of the equivalence class of the alternative worlds. This restriction is nondeterministic (we do not know what kind of additional information will be available to us, if at all) but not arbitrary: it will always contain the actual state of the knower, i.e. it is a neighborhood restriction of the actual state. In this way, set-theoretic considerations come in.

A discrete version of our epistemic framework can arise in scientific experiments or tests. We acquire knowledge by “a step-by-step” process, each step being an experiment or test. The outcome of such an experiment or test is unknown to us beforehand, but after being known it restricts our attention to a smaller set of possibilities. A sequence of experiments, tests, or actions comprises a strategy of knowledge acquisition. This model is in many respects similar to Hintikka’s “oracle” (see [Hin86]). In Hintikka’s model the “inquirer” asks a series of questions $Q_1, Q_2, ..., Q_n, ...$ to an external information source, called “oracle” (can be thought as a knowledge base). The oracle answers yes or no and the inquirer increases his or her knowledge by this piece of additional evidence. At any point of this process the inquirer follows a branch of a tree determined by the possible answers to his or her series of questions. Such an interrogative model is recognized by Gadamer ([Gad75]) as an important part of the epistemic process. Consider the following example:

Example: Suppose that our view, the set of possible worlds, is $\{q_1, q_2, q_3, q_4\}$ and our query consists of two questions $Q_1, Q_2$, in that order. The answer to $Q_1$ is yes in $q_1, q_2$ and no in $q_3, q_4$. The answer to $Q_2$ is yes in $q_1, q_2, q_3$ and no in $q_4$. Then the possible sequences of knowledge states comprise a tree of subsets as shown in Figure 1. The space of subsets labeling the nodes of the tree will be called a treelike space.

The above example shows a transition from the symbolic description of the epistemic process to a description in spatial terms. Instead of going down a proof tree, the one which entails the desired formulae, we intersect nodes of a tree labeled by subsets of a space. This transition is direct; it enables us to think in geometric terms.
Now consider the following example:

**Example:** Suppose that a machine emits a stream of binary digits representing the output of a recursive function $f$. After time $t_1$ the machine emitted the stream 111. The only information we have about the function being computed at this time on the basis of this (finite) observation is that $f(1) = f(2) = f(3) = 1$.

As far as our knowledge concerns, $f$ is indistinguishable from the constant function 1, where $1(n) = 1$ for all $n$. After some additional time $t_2$, i.e. spending more time and resources, 0 might appear and thus we could be able to distinguish $f$ from 1. In any case, each binary stream will be an initial segment of $f$ and this initial segment is a neighborhood of $f$. In this way, we can acquire better knowledge of the function the machine computes. The space of finite binary streams is a structure which models computation. The sets of binary streams under the initial segment ordering is an example of a treelike space.

The above example shows how the same epistemic process appears during observations of programs. Here possible worlds correspond to (total) computations and our view to observations. We can apply the same spatial reasoning to programs through the following correspondence:

- Knowledge states = Sets = Observations
- Possible worlds = Points = Computations.

Therefore a common idea lies behind the knowledge-theoretic, spatial and computational framework. The connection between the last two is not new. Here is how this epistemic framework ties with previous work on establishing links between spatial reasoning and reasoning about programs.
We use two modalities $K$ for knowledge and $\Box$ for effort, i.e. spending of resources. Consider the formula

$$A \rightarrow \Diamond KA,$$

where $A$ is an atomic predicate and $\Diamond$ is the dual of the $\Box$, i.e. $\Diamond \equiv \neg \Box \neg$. It will be clear after the presentation of semantics in Section 2.1 that if the above formula is valid, then the set which $A$ represents is an open set of the topology generated by the subsets of the treelike space as a basis. Under the reading of $\Diamond$ as “possible” and $K$ as “is known”, the above formula says that

“if $A$ is true then it is possible for $A$ to be known”,

i.e. $A$ is affirmative. Vickers defines similarly an affirmative assertion in [Vic89]

“an assertion is affirmative iff it is true precisely in the circumstances when it can be affirmed.”

Affirmative and refutative assertions are closed under infinite disjunctions and conjunctions, respectively. Smyth in [Smy83] observed first these properties in semi-decidable properties. Semi-decidable properties are those properties whose truth set is r.e. and are a particular kind of affirmative assertions. In fact, changing our power of affirming or computing we get another class of properties with a similar knowledge-theoretic character. For example, using polynomial algorithms affirmative assertions become polynomially semi-decidable, i.e. NP properties. If an object has this property then it is possible to know it with a polynomial algorithm even though it is not true we know it now.

Our approach has an independent theoretical interest. A new family of Kripke frames, called subset frames, arises. These are Kripke frames which are equivalent to sets of subsets. In particular, we have identified those which are equivalent to (complete) lattices of subsets and topologies (see [Geo93]). In this paper, we shall identify those which correspond to the above interrogative model, called treelike spaces. Treelike spaces have a particular interest; they correspond to an indeterminist’s theory of time called Ockhamism (see [Pri67]), which gives rise to branching time. We refer the reader to section 2.1 for a detailed discussion.

A family of logics for knowledge and time is studied in [HV89] and various complexity results are established. However, the framework of the above logics is restricted to distributed systems and their interpretation differs significantly from ours.

Interpreting the knowledge modal operator as a universal quantifier we present a novel way of understanding the meaning of quantifiers in varying (ordered) domains (see section 2.2 for a relevant discussion). This is one of the main difficulties in formulating a meaningful first-order system for modal logic (see [Fit93] for a discussion).

The language and semantics of our logical framework is presented in Section 2. In the same section, we present two systems which belong to the same family of logics, studied in [MP92], [Geo93] and [Geo94a]. In Section 3, we
present an axiomatization, called $\text{MPT}$, for our semantics and we prove completeness, small model property, and decidability.

A preliminary version of this paper has appeared in [Geo94b].

## 2 Two Systems: MP and MP*$^*$

### 2.1 Language and Semantics

We follow the notation of [MP92].

We construct a bimodal propositional modal logic. Formally, we start with a countable set $\mathcal{A}$ of atomic formulae, then the language $L$ is the least set such that $\mathcal{A} \subseteq L$ and closed under the following rules:

\[
\begin{align*}
\phi, \psi & \in L \\
\phi \land \psi & \in L \\
\neg \phi, 2\phi, K\phi & \in L
\end{align*}
\]

We abbreviate, as usual, $\phi \land \neg \phi$ with $\bot$ and $\neg \bot$ with $\top$.

The language $L$ can be interpreted inside any spatial context as follows.

**Definition 1** Let $X$ be a set and $\mathcal{O}$ a subset of the powerset of $X$, i.e. $\mathcal{O} \subseteq \mathcal{P}(X)$ such that $X \in \mathcal{O}$. We call the pair $\langle X, \mathcal{O} \rangle$ a subset space. A model is a triple $\langle X, \mathcal{O}, i \rangle$, where $\langle X, \mathcal{O} \rangle$ is a subset space and $i$ a map from $\mathcal{A}$ to $\mathcal{P}(X)$ with $i(\top) = X$ and $i(\bot) = \emptyset$ called initial interpretation.

We denote the set $\{(x, U) : U \in \mathcal{O}, \text{ and } x \in U\} \subseteq X \times \mathcal{O}$ with $X \times \mathcal{O}$. For each $U \in \mathcal{O}$ let $\downarrow U$ be the lower closed set generated by $U$ in the partial order $(\mathcal{O}, \subseteq)$, i.e. the set $\{V : V \in \mathcal{O} \text{ and } V \subseteq U\}$.

**Definition 2** The satisfaction relation $\models_M$, where $M$ is the model $\langle X, \mathcal{O}, i \rangle$, is a subset of $(X \times \mathcal{O}) \times L$ defined recursively by (we write $x, U \models_M \phi$ instead of $(x, U, \phi) \in \models_M$)

\[
\begin{align*}
x, U \models_M A & \quad \text{iff } x \in i(A), \text{ where } A \in \mathcal{A} \\
x, U \models_M \phi \land \psi & \quad \text{if } x, U \models_M \phi \text{ and } x, U \models_M \psi \\
x, U \models_M \neg \phi & \quad \text{if } x, U \not\models_M \phi \\
x, U \models_M K\phi & \quad \text{if for all } y \in U, \ y, U \models_M \phi \\
x, U \models_M \Box\phi & \quad \text{if for all } V \in \downarrow U \text{ such that } x \in V, \ x, V \models_M \phi
\end{align*}
\]

If $x, U \models_M \phi$, for all $(x, U)$ belonging to $X \times \mathcal{O}$, then $\phi$ is valid in $M$, denoted by $M \models \phi$.

The case for atomic formulae shows that we deal with analytic sentences, i.e. sentences which do not change their truth value. If a formula $\Box\phi$ does not contain $K$ then it has the same interpretation as $\phi$. This has also the consequence that the universal substitution rule does not hold. Thus, time does not affect the semantic value of sentences but rather the knowledge we have of them. This
difference makes the □ modality not collapsing to a temporal modality but being closer to necessity.

We abbreviate \( \neg \Box \neg \phi \) and \( \neg \text{K} \neg \phi \) with \( \Diamond \phi \) and \( \text{L} \phi \) respectively. We have that

\[
x, U \models_M \text{L} \phi \quad \text{if there exists } y \in U \text{ such that } y, U \models_M \phi
\]

\[
x, U \models_M \Diamond \phi \quad \text{if there exists } V \in \mathcal{O} \text{ such that } V \subseteq U, \ x \in V, \text{ and } x, V \models_M \phi.
\]

**Definition 3** A *treelike space* is a subset space \( \langle X, \mathcal{O} \rangle \) where for all \( U, V \in \mathcal{O} \), either \( U \subseteq V \), or \( V \subseteq U \), or \( U \cap V = \emptyset \). A model induced by a tree space will be called a *treelike model*.

It is clear that in the countable case the set of subsets of a treelike space forms a tree under the subset ordering.

**Example:** Let

\[
X = \{ f \mid f \text{ recursive } \}.
\]

Now, let

\[
[a_1, a_2, \ldots, a_n] = \{ f \mid f(k) = a_k, \ \text{for } k = 1, 2, \ldots, n \} \subseteq X,
\]

where \( a_1, a_2, \ldots, a_n \) are natural numbers, and

\[
\mathcal{O} = \{ [a_1, a_2, \ldots, a_n] \mid n = 1, 2, \ldots \} \cup \{ X \}.
\]

Then it is easily verified, using definition 2.1, that \( \langle X, \mathcal{O} \rangle \) is a treelike space.

Now let 1 be a predicate with

\[
i(1) = \{ f \mid \text{there exists } n \text{ such that for all } m > n, f(n) = 1 \}.
\]

Then the formula

\[
\Box \text{L} 1
\]

which translates to “it will never be known that 0 appears infinitely often”, is valid in the treelike model \( \langle X, \mathcal{O}, i \rangle \). This comes with no surprise, since the knowledge of “infinitely often” requires an infinite amount of resources. This formula is an example of a refutative assertion (see introduction).

Treelike spaces get their name from treelike frames (see [Pri67]). A *treelike frame* is a pair \( \langle T, \prec \rangle \), where \( T \) is a nonempty set and \( \prec \) is a transitive ordering on \( T \) such that if \( t_1 < t \) and \( t_2 < t \) then either \( t_1 = t \) or \( t_1 < t_2 \) or \( t_2 < t_1 \). Treelike frames have appeared as semantics for the Ockhamist’s concept of non-deterministic time and been used for treating historical necessity and conditionals (see [Tho84] and [VF81]). The validity on these frames is called *Ockhamist validity*. A treelike space is a special form of a treelike frame where the temporal instants of the frame are labeled by subsets of a space and whenever instants are incomparable the respective subsets are disjoint. It can be easily seen that the ordering among subsets is a treelike frame. The similarities do not end here. Let \( \langle T, \prec \rangle \) be a treelike frame and, for each \( t \in T \), \( B_t \) the set of
maximal linear ordered subsets of $T$ containing $t$, i.e. the branches intersecting $t$. Then $\{B_t\}_{t \in T}$ is a treelike space. The difference lies on the interpretation of atomic formulae. We interpret atomic formulae on branches while an Ockhamist assignment interprets atomic formulae on temporal instances. This bring up another dimension of our logic. Our logic is not conservative over a logic which interprets $\Box$ as $F$ (the “future” modality) for if $\phi$ contains no occurrences of $K$ then $\Box \phi$ is valid in a treelike space exactly when $\phi$ is. We adopt the indeterminist’s view of necessity (knowledge). Although $\phi$ may be true in our world, $K \phi$ may be false. This is because there is no special world in our view which deserves to be called actual. Setting apart Ockhamist validity, treelike spaces are more general than treelike frames (and their derivative $T \times W$ frames) due to the fact that we do not assume an overall temporal ordering. In this sense treelike spaces are closer to a more general structure, first introduced by Kamp and subsequently called Kamp frames, where worlds do not participate in the same temporal structure (for definition and discussion see [Tho84]). In fact, it is easily seen that treelike spaces are equivalent to Ockhamist frames introduced by Zanardo in [Zan85] for the completeness of strong Ockhamist validity. At any rate, our work seems to have more than superficial links with work in historical necessity and questions such as what the connections between the two notions of validity are should be the subject of a more systematic investigation.

2.2 MP and MP$^*$

We saw that the semantics of the bimodal language is interpreted in any pair $\langle X, \mathcal{O} \rangle$. What happens when we allow $\mathcal{O}$ to be any class of sets of subsets? If $\mathcal{O}$ is an arbitrary set of subsets then the system MP is complete for such subset spaces. The axiom system MP consists of axiom schemes 1 through 10 and rules of Table 1 (see page 9) and appeared first in [MP92].

The following was proved in [MP92].

**Theorem 4** The axioms and rules of MP are sound and complete with respect to subset spaces.

If $\mathcal{O}$ is a complete lattice under set-theoretic union and intersection then the system MP$^*$ is canonically complete for this class of subset spaces. The axiom system MP$^*$ consists of the axiom schemes and rules of MP plus the following two additional axiom schemes:

$$\Diamond \Box \phi \rightarrow \Box \Diamond \phi$$

and

$$\Diamond (K \phi \land \psi) \land L \Diamond (K \phi \land \chi) \rightarrow \Diamond (K \Diamond \phi \land \Diamond \psi \land L \Diamond \chi).$$

The first axiom is a well-known formula which characterizes incestual frames, i.e. if two points $\beta$ and $\gamma$ in a frame can be accessed by a common point $\alpha$ then there is a point $\delta$ which can be accessed by both $\beta$ and $\gamma$. The second characterizes union.

The following was proved in [Geo93].

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Theorem 5 The axioms and rules of MP* are sound and canonically complete with respect to subset spaces, which are complete lattices.

The proof of the above theorem was later shortened and improved through an elegant embedding of S4 (and therefore intuitionistic logic via the Gödel translation) by Dabrowski, Moss and Parikh in [DMP]. This translation reveals that truth in intuitionistic logic coincides with “possibility of knowing” in our system. It also reveals a connection with another line of work, that of Fischer Servi. In [FS80] and [FS84] the semantics and syntax of the family *-IC of intuitionistic modal logics is studied. This family is is naturally embedded via the Gödel translation to the family (S4-*) of bimodal logics, where S4 is always one of the coordinates (like in our case). However, the semantics called double model structures (birelational modal frames) deviate from our space theoretic framework; a fact that declares itself on the presence of different connecting axioms, i.e. axioms involving both modalities.

3 The system MPT

We add the axioms 11 and 12 to form the system MPT for the purpose of axiomatizing treelike spaces.

A word about the axioms (most of the following facts can be found in any introductory book about modal logic, e.g. [Che80] or [Gol87].) Axiom 8 expresses the fact that the truth of atomic formulae is independent of the choice of subset and depends only on the choice of point. Axioms 8 through 10 and Axioms 11 through 13 are used to axiomatize the normal modal logics S4 and S5 respectively. The former group of axioms expresses the fact that the passage from one subset to its restriction is done in a constructive way, as actually happens in an increase of information or a spending of resources (the classical interpretation of necessity in intuitionistic logic is axiomatized in the same way). The latter group is generally used for axiomatizing logics of knowledge.

Axiom 11 expresses the fact that if a formula holds in arbitrary subsets is going to hold as well in the ones which are neighborhoods of a point. The converse of this axiom is not sound.

Axiom 11 is a well-known axiom which characterizes reflexive, transitive and connected frames, i.e. if two points β and γ in a frame can be accessed by a common point α then either β accesses γ or γ accesses β (or both).

Soundness of Axioms 8 through 10 has already been established for arbitrary subset spaces (see [MP92]). The soundness of Axiom 11 is easy to see, since the subset frame (see [Geo83]), i.e. the birelational modal frame, of a tree model is connected.

Proposition 6 The axiom 11 is sound.

Proof. We shall show soundness for the equivalent formula

$$\Box K\phi \land \Diamond L(\psi \land \Box \phi) \rightarrow L(\Diamond \psi \land \Box \phi).$$
**Axioms**

1. All propositional tautologies
2. \((A \rightarrow \Box A) \land (\neg A \rightarrow \Box \neg A)\), for \(A \in A\)
3. \(\Box (\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)\)
4. \(\Box \phi \rightarrow \phi\)
5. \(\Box \phi \rightarrow \Box \Box \phi\)
6. \(K(\phi \rightarrow \psi) \rightarrow (K \phi \rightarrow K \psi)\)
7. \(K \phi \rightarrow \phi\)
8. \(K \phi \rightarrow KK \phi\)
9. \(\phi \rightarrow KL \phi\)
10. \(K \Box \phi \rightarrow \Box K \phi\)
11. \(\Box (\Box \phi \rightarrow \psi) \lor \Box (\Box \psi \rightarrow \phi)\)
12. \(\Box K \phi \land K(\Box \phi \rightarrow \Box \psi) \rightarrow \Box K(\Box \phi \rightarrow \Box \psi)\)

**Rules**

\[
\begin{align*}
\frac{\phi \rightarrow \psi, \phi}{\psi} & \text{ MP} \\
\frac{\phi}{K \phi} & \text{ K-Necessitation} \\
\frac{\phi}{\Box \phi} & \text{ \Box-Necessitation}
\end{align*}
\]

Table 1: Axioms and Rules of MPT.

Let \(x, U \models \Box K \phi \land \Diamond (\psi \land \Box \phi)\). Then there exists \(V \subseteq U\) such that \(x, V \models \Diamond (\psi \land \Box \phi)\). This implies that there exists \(y \in V\) such that \(y, V \models \psi \land \Box \phi\). Now, observe that \(y, U \models \Box \phi\). For, if \(W \subseteq U\) and \(y \in W\) then there are two cases. Either \(W \subseteq V\) and \(y, W \models \phi\), since \(y, U \models \Box \phi\), and we are done, or \(W \subseteq U\) and \(W \not\subseteq V\) so we have \(V \subseteq W \subseteq U\), since the subsets containing \(y\) are linearly ordered. In this case, we have \(x \in W\), since \(x \in V\). By our assumption \(x, U \models \Box K \phi\), we have \(x, W \models K \phi\). So \(y, W \models \phi\). Now, \(y \in U\) and \(y, U \models \Box \phi\) imply together \(y, U \models \Diamond \psi \land \Box \phi\).

Note that Axiom 10 follows from Axiom 12 (substitute \(\phi\) with \(\top\)). Axiom 11
has a particular interest; if we replace $K$ with the universal quantifier it becomes the well-known Barcan formula
\[
\forall x \Box \phi(x) \rightarrow \Box \forall x \phi(x).
\]

Our system (and therefore $\text{MP}$ and $\text{MP}^*$, since this formula belongs to their axiomatization) can be thought as a propositional analogue of a first order modal system interpreted over varying restricting domains (see [Fit93]).

### 3.1 Completeness

Our proof of completeness is based on a construction of a treelike model which is (strongly) equivalent to each generated canonical submodel of the canonical model of $\text{MPT}$.

The canonical model of $\text{MPT}$ is the structure
\[
\mathcal{C} = (S, \{\circlearrowleft, \rightarrow\}, v),
\]
where
- $S = \{s \subseteq L | s \text{ is } \text{MPT}-\text{maximal consistent}\}$,
- $s \circlearrowleft t$ iff $\{\phi \in L | \Box \phi \in s\} \subseteq t$,
- $s \rightarrow t$ iff $\{\phi \in L | K \phi \in s\} \subseteq t$,
- $v(A) = \{s \in S | A \in s\}$,

along with the usual satisfaction relation (defined inductively):
\[
\begin{align*}
&s \models_{c} A \quad \text{iff} \quad s \in v(A) \\
&s \models_{c} \neg \phi \quad \text{iff} \quad s \not \models_{c} \phi \\
&s \models_{c} \phi \land \psi \quad \text{iff} \quad s \models_{c} \phi \text{ and } s \models_{c} \psi \\
&s \models_{c} \Box \phi \quad \text{iff} \quad \text{for all } t \in S, s \rightarrow t \text{ implies } t \models_{c} \phi \\
&s \models_{c} K \phi \quad \text{iff} \quad \text{for all } t \in S, s \rightarrow t \text{ implies } t \models_{c} \phi.
\end{align*}
\]

We write $\mathcal{C} \models \phi$, if $s \models_{c} \phi$ for all $s \in S$.

A canonical model exists for all consistent bimodal systems with the normal axiom scheme for each modality (as $\text{MPT}$). We have the following well known theorems (see [Che80], or [Gol87]).

**Theorem 7 (Truth Theorem)** For all $s \in S$ and $\phi \in L$,
\[
\models_{c} \phi \quad \text{iff} \quad \phi \in s.
\]

**Theorem 8 (Completeness Theorem)** For all $\phi \in L$,
\[
\mathcal{C} \models \phi \quad \text{iff} \quad \vdash_{\text{MPT}} \phi.
\]

We shall now prove some properties of $\mathcal{C}$. 

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Proposition 9  

a. The canonical frame is reflexive, transitive and connected with respect to the relation $\rightarrow$.

b. The relation $\rightarrow$ is an equivalence relation.

c. For all $s, s', t \in S$, if $s \overset{\phi}{\rightarrow} s' \overset{I}{\rightarrow} t$ then there exists $t' \in S$ such that $s \overset{I}{\rightarrow} t' \overset{\phi}{\rightarrow} t$.

d. For all $s, s' \in S$, if $s \overset{I}{\rightarrow} s'$ and $s \overset{\phi}{\rightarrow} s'$ then $s = s'$.

e. The relation $\overset{\phi}{\rightarrow}$ is antisymmetric.

Proof. For Part a, Axioms 3 through 5 and Axiom 11 characterize reflexive, transitive and connected frames (these axioms comprise the system $S4.3$).

For Part b, $K$ is axiomatized with the $S5$ axioms.

Part c is an immediate consequence of Axiom 10.

To show Part d, let $I_{s,s'} = \{ t \mid s \overset{\phi}{\rightarrow} t \overset{\phi}{\rightarrow} s' \}$, for all pairs $(s, s')$ such that $s \overset{I}{\rightarrow} s'$ and $s \overset{\phi}{\rightarrow} s'$.

We shall prove by induction on the complexity of $\phi$ that, for all pairs $(s, s')$ such that $s \overset{I}{\rightarrow} s'$ and $s \overset{\phi}{\rightarrow} s'$, $\phi$ belongs to some $t \in I_{s,s'}$ if and only if $\phi$ belongs to $s$. This shows that $\bigcup I_{s,s'} \subseteq s$. Further, we have $I_{s,s'} = \{ s \}$, since $s \in I_{s,s'}$. Therefore $s = s'$.

If $\phi$ is an atomic formula $A$ and $A \in t$, for some $t \in I_{s,s'}$, then $\Box A \in s$. Therefore, by axioms 2, $\Box A \in s$. Hence, $A \in s$.

The cases of negation and conjunction are straightforward.

If $\phi = \Box \psi$, let $\Box \psi \in t$, for some $t \in I_{s,s'}$. In particular, $\psi \in t$ and by induction hypothesis, $\psi \in s$. Suppose, towards a contradiction, that $\Box \neg \psi \in s$. Then there exists $r \in S$ such that $s \overset{\Box}{\rightarrow} r$ and $\neg \psi \in r$. Since the frame is connected, $s \overset{\phi}{\rightarrow} s'$ and $s \overset{\phi}{\rightarrow} r$ imply that either $s' \overset{\phi}{\rightarrow} r$ or $r \overset{\phi}{\rightarrow} s'$. If $s' \overset{\phi}{\rightarrow} r$ then $t \overset{\phi}{\rightarrow} r$ which is contradiction, since $\Box \psi \in t$ and $\neg \psi \in r$. If $r \overset{\phi}{\rightarrow} s'$ then, by induction hypothesis, $\neg \psi \in s$ which is a contradiction, since $\psi \in s$ and $s$ is consistent. Hence $\Box \psi \in s$.

If $\phi = K \psi$, let $K \psi \in t$ for some $t \in I_{s,s'}$. Suppose, towards a contradiction, that $L \neg \psi \in s$. Then there exists $r \in S$ such that $s \overset{L}{\rightarrow} r$ and $\neg \psi \in r$. We have $s' \overset{L}{\rightarrow} r$, since $s' \overset{L}{\rightarrow} s$. Since $t \overset{\phi}{\rightarrow} s'$, there exists, by Part 3, $r' \in S$ such that $t \overset{\phi}{\rightarrow} r' \overset{\phi}{\rightarrow} r$. We have $\psi \in r'$, since $K \psi \in t$. Since $s \overset{\phi}{\rightarrow} t \overset{\phi}{\rightarrow} r'$, there exists, by Part 3, $r'' \in S$ such that $s \overset{L}{\rightarrow} r'' \overset{\phi}{\rightarrow} r'$. Notice that $r'' \overset{\phi}{\rightarrow} r' \overset{\phi}{\rightarrow} r$ and $r'' \overset{L}{\rightarrow} r$, and so $r'', r' \in I_{r',r''}$. By our previous assumption, we have $\neg \psi \in r$ and $\psi \in r'$. By induction hypothesis on $I_{r',r''}$, both $\neg \psi$ and $\psi$ should belong to $r''$ which is a contradiction to its consistency.

For Part e we shall prove by induction on the structure of $\phi$ that, for all $s, t \in S$ such that $s \overset{\phi}{\rightarrow} t \overset{\phi}{\rightarrow} s$, $s \in s$ if and only if $\phi \in t$. 

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The cases of atomic formula, negation, conjunction and $\Box$ are straightforward. We shall show the $\phi = K\psi$ step. Let $K\psi \in s$, and suppose $L\neg \psi \in t$ towards a contradiction. Then there exists $r \in S$ such that $t \overset{L}{\rightarrow} r$ and $\neg \psi \in r$. Since $s \overset{\Box}{\rightarrow} t \overset{L}{\rightarrow} r$, there exists $p \in S$ such that $s \overset{L}{\rightarrow} p \overset{\Box}{\rightarrow} r$. Also, $\psi \in p$, since $K\psi \in S$. Now, since $t \overset{L}{\rightarrow} s \overset{L}{\rightarrow} p$ there exists $r' \in S$ such that $t \overset{L}{\rightarrow} r' \overset{\Box}{\rightarrow} p$. This implies $r' \overset{\Box}{\rightarrow} p \overset{\Box}{\rightarrow} r$ and $r \overset{L}{\rightarrow} r'$. Therefore, by Part $[3]$, $r = r'$. Thus we have $r \overset{L}{\rightarrow} p \overset{\Box}{\rightarrow} r$ with $\neg \psi \in r$ and $\psi \in p$ which is a contradiction to the induction hypothesis.

The canonical model is not a (model corresponding to a) treelike model. A counterexample will appear later on (see Figure [3]). However, by defining a number of equivalence relations, we shall be able to construct a treelike model equivalent to each generated part of the canonical model.

For all $t \in S$, let $[t] = \{s \in S \mid s \overset{L}{\rightarrow} t\}$, i.e. the equivalence class under $\overset{L}{\rightarrow}$, where $t$ belongs. Let $C_K = \{[t] \mid t \in S\}$. We define the following relation on $C_K$.

$$[t_1] \leq [t_2] \quad \text{iff} \quad \text{there exist } s_1, s_2 \in S \text{ such that } s_1 \in [t_1], s_2 \in [t_2] \text{ and } s_2 \overset{\Box}{\rightarrow} s_1.$$  

**Proposition 10** The relation $\leq$ is a partial order.

**Proof.** Since $t \overset{\Box}{\rightarrow} t$, we have $[t] \leq [t]$ and reflexivity follows.

For antisymmetry, let $[t_1] \leq [t_2]$ and $[t_2] \leq [t_1]$ for some $t_1, t_2 \in S$. Then there exist $s_1, s_2, s_1', s_2' \in S$ such that $s_1, s_1' \in [t_1], s_2, s_2' \in [t_2]$, $s_2 \overset{\Box}{\rightarrow} s_1$ and $s_1' \overset{\Box}{\rightarrow} s_2'$. Since $s_2 \overset{L}{\rightarrow} s_1 \overset{L}{\rightarrow} s_1'$, there exists $s_2'' \in S$ such that $s_2 \overset{L}{\rightarrow} s_2'' \overset{\Box}{\rightarrow} s_1$. So we have $s_2'' \overset{\Box}{\rightarrow} s_1' \overset{\Box}{\rightarrow} s_2'$ and $s_2'' \overset{L}{\rightarrow} s_2$ which implies, by Proposition $[3]$, $s_2'' = s_2'$. Therefore $s_1' = s_2'$, by $\overset{\Box}{\rightarrow}$'s antisymmetry. Hence $[t_1] = [s_1'] = [s_2'] = [t_2]$.

For transitivity, let $[t_3] \leq [t_2] \leq [t_1]$ for some $t_1, t_2, t_3 \in S$. Then there exist $s_1 \in [t_1], s_2, s_2' \in [t_2]$ and $s_3 \in [t_3]$ such that $s_1 \overset{\Box}{\rightarrow} s_2$ and $s_2' \overset{\Box}{\rightarrow} s_3$. Since $s_1 \overset{L}{\rightarrow} s_2 \overset{L}{\rightarrow} s_2'$, there exists $s_1' \in S$ such that $s_1 \overset{L}{\rightarrow} s_1' \overset{\Box}{\rightarrow} s_2'$. So $s_1' \overset{\Box}{\rightarrow} s_3$, and therefore $[t_3] = [s_3] \leq [s_1'] = [t_1]$.

A subset $X$ of $S$, the domain of the canonical model $C$, is called $K\Box$-closed whenever

$$s \overset{\Box}{\rightarrow} t \quad \text{or} \quad s \overset{L}{\rightarrow} t, \quad \text{then} \quad t \in X.$$  

The intersection of $K\Box$-closed sets is still $K\Box$-closed, therefore we can define the smallest $K\Box$-closed containing $t$, for all $t \in S$. We shall denote this set by $S^t$. Fix $t_0 \in S$. We define the model

$$C^{t_0} = \left(S^{t_0} \overset{\Box}{\rightarrow} |S^{t_0} \times S^{t_0}| \overset{L}{\rightarrow} |S^{t_0} \times S^{t_0}|, v^{t_0}\right),$$

where $\overset{\Box}{\rightarrow} |S^{t_0} \times S^{t_0}|$, $\overset{L}{\rightarrow} |S^{t_0} \times S^{t_0}|$ and $v^{t_0}$ are the restrictions of $\overset{\Box}{\rightarrow}$, $\overset{L}{\rightarrow}$ and $v$ to $S^{t_0} \times S^{t_0}$ and $S^{t_0}$ respectively. We shall call this model the submodel of $C$ generated by $t_0$.  

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Observe that if we restrict the partial order \( \leq \) to \( C^{t_0} \) then \([t_0]\) is the greatest element under \( \leq \).

For each generated submodel of the canonical model, we shall construct a treelike model which is equivalent to it.

For each \( s \in S^{t_0} \), let

\[
[s] = \{ t \in [t_0] \mid \text{there exists } t' \in [s] \text{ such that } t \overset{\supset}{\rightarrow} t' \}.
\]

Notice that \([s] \subseteq [t_0]\).

For each \( s \in S^{t_0} \), let

\[
[s] = \{ t \in [t_0] \mid \text{there exists } t' \in [s] \text{ such that } t \overset{\supset}{\rightarrow} t' \}.
\]

For each \( s \in S^{t_0} \), let

\[
t_1 \sim_s t_2 \text{ iff for all } [s] \subseteq [s'], t_1 \in [s'] \text{ iff } t_2 \in [s'].
\]

**Proposition 11** For all \( s \in S^{t_0} \), the relation \( \sim_s \) is an equivalence relation.

**Proof.** This is because \( \sim_s \) inherits the properties of \( \overset{\supset}{\rightarrow} \).

We denote the equivalence class of \( t \) under \( \sim_s \) with \([t]_s\). We have \([t]_s \subseteq [s] \subseteq [t_0]\).

Let \( \langle X, \mathcal{O}^{t_0} \rangle \) be the subset space where

\[
X = \{ t \mid t \in [t_0] \}
\]

and

\[
\mathcal{O}^{t_0} = \{ [t]_s \mid t \in [s] \text{ and } s \in S^{t_0} \}.
\]

It is clear that \( \mathcal{O}^{t_0} \subseteq \mathcal{P}(X) \).

**Lemma 12** If \([s_1] \leq [s_2] \) and \( t \in [s_1] \cap [s_2] \) then \([t]_{s_1} \subseteq [t]_{s_2}\).

**Proof.** Immediate from the definition of \( \sim_s \).

To elaborate the above process, we present the following simple example. *Example:* A part of the canonical model appears in Figure 2. (Horizontal and downward arrows correspond to \( \overset{\supset}{\rightarrow} \) and \( \overset{\supset}{\rightarrow} \), respectively.) We would like to make subsets of a treelike space correspond to equivalence classes under \( \overset{\supset}{\rightarrow} \). Canonical model worlds related with \( \overset{\supset}{\rightarrow} \) will be represented by a single point. However, this model is not a treelike model: \{\( r_1, t_1 \)\} and \{\( r_3, s_1, t_1 \)\} should make two distinct points. To remedy that, we “trace back” each equivalence class under \( \overset{\supset}{\rightarrow} \) to the uppermost one. For instance, \([t_1] = \{t_1, t_2\}\) is traced back to \([r_1] = \{r_1, r_2, r_3, r_4\}\). The latter forms \([t_1]\). Next, we split \([t_1]\) into equivalence classes under \( \overset{\supset}{\rightarrow} \), i.e. \([r_1]_{t_1} = \{r_1, r_2\}\) and \([r_3]_{t_1} = \{r_3, r_4\}\), since \( r_1 \sim_{t_1} r_2 \) and \( r_3 \sim_{t_1} r_4 \). Finally, we replace \([t_1]\) with as many copies as these equivalence classes (see Figure 3). The infinite case is taken care of by Lemma 3. The resulting space (of Figure 3) is a treelike space. Note that we could have replaced this procedure by one that employs maximal branches but we find the present one simpler.
Figure 2: A generated submodel of the canonical model.

**Proposition 13** The subset space $\langle X, O_{t_0} \rangle$ is a treelike space.

**Proof.** Suppose $[t_1]_{s_1} \cap [t_2]_{s_2} \neq \emptyset$. Let $t \in [t_1]_{s_1} \cap [t_2]_{s_2}$. We have either $s_1 \xrightarrow{3} s_2$ or $s_2 \xrightarrow{3} s_1$, since $t \xrightarrow{3} s'_1$, $t \xrightarrow{3} s'_2$, for some $s'_1 \in [s_1]$ and $s'_2 \in [s_2]$, and the canonical frame is connected. The former implies $[s_1] \leq [s_2]$. Thus, by Lemma 12, $[t]_{s_1} = [t]_{s_1} \subseteq [t]_{s_2} = [t]_{s_2}$. Similarly, the latter implies $[t]_{s_2} \subseteq [t]_{s_1}$.

Let $\langle X, O_{t_0}, i \rangle$ be the treelike model where $X$ and $O_{t_0}$ are as above, and $i(A) = v_{t_0}(A)$ where $v_{t_0}$ is the initial interpretation restricted on $C_{t_0}$.

An element of $X \times S_{t_0}$ can have more than one representation. In order to prove the semantical equivalence we are opting for, we shall choose a canonical representation. So, given a pair $(t, [t']_s) \in X \times O_{t_0}$, its canonical representation is $(t, [t]_s)$ where $s$ is such that $t \xrightarrow{s} s' \xrightarrow{s} s'$. Its existence is assured by the definition of $[t']_s$ and uniqueness by Proposition 9(d). From now on, we shall use the canonical representation wherever is possible.

**Lemma 14** Let $t \in [t_0]$ and $s \in S_{t_0}$ such that $t \xrightarrow{s} s$. Then for all $s' \in [s]$ there exists $t' \in [t_0]$ such that $t' \xrightarrow{s'} s'$ and $t \xrightarrow{s} t'$, i.e. $t' \in [t]_s$.

**Proof.** Let

$$\{t_i\}_{i \in I}$$

be the linear order of all members of $S_{t_0}$ under $\xrightarrow{s}$ such that $t \xrightarrow{s} t_i \xrightarrow{s} s$. 

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Now, let
\[ T' = \{ \Diamond \psi \mid \psi \in s' \} \cup \{ \chi \mid K \chi \in t \} \cup \{ \Diamond \omega \mid K \omega \in t_i, \text{ for some } i \in I \} \cup \{ \Box \phi \mid \Box K \phi \in t \text{ and } \Box \phi \in s' \}. \]

\( T' \) is consistent. For if not, then there would be \( \psi, \omega_1, \omega_2, \ldots, \omega_n, \chi, \phi \) as above with \( i_1, i_2, \ldots, i_n \in I \) and \( i_1 \leq i_2 \leq \ldots \leq i_n \) such that
\[ \not\vdash_{\text{MPT}} \Diamond \psi \land \bigwedge_{k=1}^{n} \Diamond \omega_k \land \chi \rightarrow \Diamond \neg \phi. \]

Thus
\[ \not\vdash_{\text{MPT}} K \left( \Diamond \psi \land \bigwedge_{k=1}^{n} \Diamond \omega_k \land \chi \rightarrow \Diamond \neg \phi \right). \]

We shall prove that the negation of the above formula belongs to \( t \) and reach a contradiction. Since \( \psi \land \Box \phi \in s' \), we have \( L(\psi \land \Box \phi) \in s \). Hence
\[ \Diamond L(\psi \land \Box \phi) \in t_{i_1}. \]

Observe that \( \Box K \phi \in t_{i_1} \), so, by applying axiom 12, we have
\[ L(\Diamond \psi \land \Box \phi) \in t_{i_1}. \]

Since \( K \omega_1 \in t_{i_1} \), we have
\[ L(\Diamond \psi \land \omega_1 \land \Box \phi) \in t_{i_1}. \]
Also, $\square K\phi \in t_{i_2}$ and
$$\lozenge L(\lozenge (\psi \land \omega_1 \land \square \phi)) \in t_{i_2}.$$ So, by axiom 12,
$$L(\lozenge (\lozenge (\psi \land \omega_1) \land \square \phi)) \in t_{i_2}.$$ Since $K\omega_2 \in t_{i_2}$, we have
$$L(\lozenge (\psi \land \omega_2 \land \lozenge \omega_1 \land \square \phi)) \in t_{i_2}.$$ Also, $\square K\phi \in t_{i_3}$ and
$$\lozenge L(\lozenge (\psi \land \omega_2 \land \lozenge \omega_1 \land \square \phi)) \in t_{i_3}.$$ So, by axiom 12,
$$L(\lozenge (\lozenge (\psi \land \omega_2 \land \lozenge \omega_1) \land \square \phi)) \in t_{i_3},$$ i.e.
$$L(\lozenge (\psi \land \lozenge \omega_2 \land \lozenge \omega_1 \land \square \phi)) \in t_{i_3}.$$ Arguing this way and by repeated applications of axiom 12 we have
$$L \left( \lozenge \psi \land \bigwedge_{k=1}^{n} \lozenge \omega_k \land \square \phi \right) \in t.$$ Since $K\chi \in t$, we have
$$L \left( \lozenge \psi \land \bigwedge_{k=1}^{n} \lozenge \omega_k \land \chi \land \square \phi \right) \in t$$ which is the negation of the formula that MPT proves. Therefore $T'$ is consistent. Let $t'$ be a maximal extension of $T'$.

We shall show that $t'$ is the required theory of the lemma. We begin by showing that if $t' \models r' \models s'$ then $r' \models t_i$, for some $i \in I$, i.e. $t \in [r']$. So suppose that $t' \models r' \models s'$. If $r' = s'$ we are done. If not, let
$$R = \{ \psi \mid \square \psi \in t \} \cup \{ L\chi \mid \chi \in r' \}.$$ $R$ is consistent. For if not, then there would be $\psi$ and $\chi$ as above such that
$$\vdash_{\text{MPT}} \psi \rightarrow \neg L\chi.$$ Since $r' \models s'$ and $r' \neq s'$, there exists $\chi' \in r'$ such that $\square \neg \chi' \in s'$. Let $\phi = \chi \land \chi'$. Observe that $\square \neg (\chi \land \chi') \in s'$, i.e. $\square \neg \phi \in s'$, and $\phi \in r'$. Further,
$$\vdash_{\text{MPT}} \psi \rightarrow \neg L(\chi \land \chi'),$$ i.e.
$$\vdash_{\text{MPT}} \psi \rightarrow \neg L\phi,$$
and therefore,
\[ \vdash_{\text{MPT}} \Box \psi \to \Box K \neg \phi. \]

Now, we have \( \Box K \neg \phi \in t \) and \( \Box \neg \phi \in s' \), since \( \Box \psi \in t \). By definition of \( T' \) above, we have \( \Box \neg \phi \in T' \), and therefore \( \Box \neg \phi \in t' \) (\( t' \) is an extension of \( T' \)). In this case, \( \neg \phi \in r' \) which is a contradiction. Therefore \( R \) is consistent. So a maximal extension \( r \) of \( R \) has the property \( t \overset{\diamond}{\rightarrow} r \overset{\dashv}{\rightarrow} s \). Hence \( r = t_i \), for some \( i \in I \).

We must now prove that \( t' \in [t_i] \), for all \( i \in I \). Let
\[ T'_i = \{ \psi \mid \square \psi \in t' \} \cup \{ \omega \mid K \omega \in t_i \}. \]

\( T'_i \) is consistent. If not, then
\[ \vdash_{\text{MPT}} \psi \to \neg \omega, \]
for some \( \phi \) and \( \omega \) as above, which implies
\[ \vdash_{\text{MPT}} \Box \psi \to \Box \neg \omega, \]
i.e.
\[ \vdash_{\text{MPT}} \Box \psi \to \neg \Box \omega. \]
So \( \neg \Box \omega \in t' \), since \( \Box \psi \in t' \). But, by definition, \( \Box \omega \in T' \subseteq t' \) which is a contradiction. Therefore a maximal extension \( t'_i \) of \( T'_i \) is such that \( t' \overset{\diamond}{\rightarrow} t'_i \overset{\dashv}{\rightarrow} t_i \).

Hence \( t' \in [t_i] \).

Combining the above proofs we have \( t \sim_s t' \).

We now have the following theorem.

**Theorem 15** For all \( s \in S^{t_0} \) and \( t \in X \) such that \( t \overset{\diamond}{\rightarrow} s \),
\[ \phi \in s \iff t, [t]_s \models \phi. \]

**Proof.** By induction on the structure of \( \phi \). For an atomic formula \( A \), we have that \( t \in i(A) \) if and only if \( s \in i(A) = v^0(A) \), i.e. \( A \in s \), because of Axiom 2 and \( t \overset{\diamond}{\rightarrow} s \).

Negation and conjunction are straightforward.

Suppose \( \phi = \Box \psi \). Let \( \Box \psi \in s \) and \( t, [t]_s \models \Box \neg \psi \), for some \( s \) and \( t \) as in the theorem’s statement. This implies that there exists \( s' \in S^{t_0} \) such that \( [t]_{s'} \subseteq [t]_s \), \( t \overset{\diamond}{\rightarrow} s' \) and \( t, [t]_{s'} \models \neg \psi \). By induction hypothesis, \( \neg \psi \in s' \). We have now that \( t \overset{\diamond}{\rightarrow} s \) and \( t \overset{\diamond}{\rightarrow} s' \) which, by connectivity, implies either \( s' \overset{\dashv}{\rightarrow} s \) or \( s \overset{\diamond}{\rightarrow} s' \). In the former case, we have \( [s] \leq [s'] \) and hence, by Lemma 12, \( [t]_s \subseteq [t]_{s'} \). So \( [t]_s = [t]_{s'} \). Therefore \( s = s' \), by Proposition 3(3), which is a contradiction to our hypothesis (\( \phi \in s \)). In the latter case, we have \( s \overset{\diamond}{\rightarrow} s' \) which again contradicts our hypothesis (\( \phi \in s \)).

For the other direction, suppose that \( t, [t]_s \models \Box \psi \) and \( \Box \neg \psi \in s \) for some \( s \) and \( t \) as above. Then there exists \( s' \in S^{t_0} \) such that \( s \overset{\diamond}{\rightarrow} s' \) and \( \neg \psi \in s' \). Thus,
$t, [t]_s \models \neg \psi$ by induction hypothesis. Moreover $[t]_{s'} \subseteq [t]_s$ by Lemma \ref{lemma12}, which is a contradiction.

If $\phi = K\psi$, let $K\psi \in s$ and suppose $t, [t]_s \models L \neg \psi$, for some $s$ and $t$ as in the theorem's statement, towards a contradiction. Then there exists $t' \in [t]_s$ such that $t', [t]_s \models \neg \psi$, i.e. $t', [t']_s \models \neg \psi$, for some $s' \in S^{t_0}$ such that $t' \overset{\omega}{\rightarrow} s'$ and $s' \models s$, which is a contradiction.

For the other direction, suppose that $t, [t]_s \models K\psi$ and $L \neg \psi \in s$, for some $s$ and $t$ as above. Then there exist $s' \in S^{t_0}$ such that $s \overset{L}{\rightarrow} s'$ and $\neg \psi \in s'$. By Lemma \ref{lemma14}, there exists $t' \in [t]_s$ such that $t' \overset{\omega}{\rightarrow} s'$. Then we have $t', [t']_s \models \neg \psi$ by induction hypothesis. Therefore $t, [t]_s \models \neg K\psi$ which is a contradiction.

Combining now Proposition \ref{prop13} and Theorem \ref{thm15} we have the following

**Corollary 16** The system MPT is complete with respect to treelike spaces.

### 3.2 Decidability

For each treelike model and formula $\phi$, we shall construct an equivalent finite subset space of bounded size with respect to the complexity of $\phi$. This is a kind of “semantic” filtration, based on geometric properties of treelike models, using a technique first introduced in [Geo94a].

In the following we assume that $\langle X, O \rangle$ is a treelike space. Our aim is to find a partition of $O$, where a given formula $\phi$ “retains its truth value” for each point throughout a member of this partition. It turns out that there exists a finite partition of this kind.

First we need some definitions. (Note that the following hold, although we refer to a treelike space $O$, for an arbitrary family of subsets of $X$.)

**Definition 17** Given a finite family $F = \{U_1, \ldots, U_n\} \subseteq \mathcal{P}(X)$, i.e. of subsets of $X$, we define the remainder of (the principal ideal in $(O, \subseteq)$ generated by) $U_k$ by

$$\text{Rem}^F_{U_k} = \downarrow U_k - \bigcup_{U_i \not\subseteq U_k} \downarrow U_i,$$

where $\downarrow U_k = \{V \in O \mid V \subseteq U_k\}$. Note that $\text{Rem}^F_{U_k} \subseteq O$ (but not necessarily $U_k \in O$).

**Proposition 18** In a finite family $F = \{U_1, \ldots, U_n\} \subseteq \mathcal{P}(X)$ closed under intersection, we have

$$\text{Rem}^F_{U_i} = \downarrow U_i - \bigcup_{U_j \subset U_i} \downarrow U_j,$$

for $i = 1, \ldots, n$. 

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We denote \( \bigcup_{U_i \in F} \downarrow U_i \) with \( \downarrow F \).

**Proposition 19** If \( F = \{ U_1, \ldots, U_n \} \) is a finite family of subsets of \( X \) closed under intersection then

a. \( \text{Rem}^F U_i \cap \text{Rem}^F U_j = \emptyset \), for \( i \neq j \),

b. \( \bigcup_{i=1}^n \text{Rem}^F U_i = \downarrow F \), i.e. \( \{ \text{Rem}^F U_i \}_{i=1}^n \) is a partition of \( \downarrow F \). From now on we shall call a finite family of subsets \( F \) closed under intersection a finite partition (of \( \downarrow F \)),

c. if \( V_1, V_2 \in \mathcal{O}, V_1 \in \text{Rem}^F U_i \) and \( V_1 \subseteq V_2 \subseteq U_i \) then \( V_2 \in \text{Rem}^F U_i \), i.e. \( \text{Rem}^F U_i \) is convex,

d. if \( \{ V_j \}_{j \in J} \subseteq \text{Rem}^F U_i \) then \( \bigcup_{j \in J} U_j \subseteq U_i \).

**Proof.** Parts a, c and d are immediate from the definition. For Part b suppose that \( V \in \downarrow F \) then \( V \in \text{Rem}^F \bigcap_{V \in U_i} U_i \). Every partition of a set induces an equivalence relation on this set. The members of the partition comprise the equivalence classes. We denote the equivalence relation induced by \( F \) by \( \sim_F \).

**Definition 20** Given a set of subsets \( G \), we define the relation \( \sim'_G \) on \( \mathcal{O} \) with \( V_1 \sim'_G V_2 \) if and only if \( V_1 \subseteq U \iff V_2 \subseteq U \) for all \( U \in G \).

We have the following

**Proposition 21** The relation \( \sim'_G \) is an equivalence.

**Proposition 22** Given a finite partition \( F \), we have \( \sim'_F \sim_F \) i.e. the remainders of \( F \) are the equivalence classes of \( \sim'_F \).

**Proof.** Suppose \( V_1 \sim'_F V_2 \) then \( V_1 \) and \( V_2 \) belong to \( \text{Rem}^F U \) where

\[
U = \bigcap \{ U' \mid V_1, V_2 \subseteq U, \ U' \in F \}.
\]

For the opposite direction, suppose \( V_1, V_2 \in \text{Rem}^F U \) and there exists \( U' \in F \) such that \( V_1 \subseteq U' \) while \( V_2 \not\subseteq U' \). Then we have \( V_1 \subseteq U' \cap U, U' \cap U \in F \) and \( U' \cap U \subseteq U \) i.e. \( V_1 \not\in \text{Rem}^F U \).
**Proposition 23** If $\mathcal{G}$ is a finite set of subsets of $X$ then $\text{Cl}(\mathcal{G})$, its closure under intersection, is a finite partition for $\downarrow \mathcal{G}$.

The last proposition enables us to give yet another characterization of remainders: every family of points in a complete lattice closed under arbitrary joins comprises a closure system, i.e. a set of fixed points of a closure operator of the lattice (cf. [GHK + 80]). Here the lattice is the powerset of $X$. If we restrict ourselves to a finite number of fixed points then we just ask for a finite set of subsets closed under intersection i.e. Proposition 23. Thus a closure operator in the lattice of the powerset of $X$ induces an equivalence relation to any family of subsets of $X$. Two subsets are equivalent if they have the same closure, and the equivalence classes of this relation are just the remainders of the subsets which are fixed points of the closure operator.

We now introduce the notion of stability corresponding to what we mean by “a formula retains its truth value on a set of subsets”.

**Definition 24** Let $\mathcal{G} \subseteq \mathcal{O}$ then $\mathcal{G}$ is stable for $\phi$, if for all $x$, either $x, V \models \phi$ for all $V \in \mathcal{G}$, or $x, V \models \neg \phi$ for all $V \in \mathcal{G}$.

**Proposition 25** Let $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{O}$ then

a. if $\mathcal{G}_1 \subseteq \mathcal{G}_2$ and $\mathcal{G}_2$ is stable for $\phi$ then $\mathcal{G}_1$ is stable for $\phi$, and

b. if $\mathcal{G}_1$ is stable for $\phi$ and $\mathcal{G}$ is stable for $\chi$ then $\mathcal{G}_1 \cap \mathcal{G}_2$ is stable for $\phi \land \chi$.

**Proof.** Part a is easy to see while Part b is a corollary of Part a. □

**Definition 26** A finite partition $\mathcal{F} = \{U_1, \ldots, U_n\}$ is called a stable partition for $\phi$, if $\text{Rem}_\mathcal{F} U_i$ is stable for $\phi$, for all $U_i \in \mathcal{F}$.

**Proposition 27** If $\mathcal{F} = \{U_1, \ldots, U_n\}$ is a stable partition for $\phi$, so is

$$\mathcal{F}' = \text{Cl}(\{U_0, U_1, \ldots, U_n\})$$

where $U_0 \in \downarrow \mathcal{F}$.

**Proof.** Let $V \in \mathcal{F}'$, then there exists $U_i \in \mathcal{F}$ such that $\text{Rem}^{\mathcal{F}'} V \subseteq \text{Rem}^{\mathcal{F}} U_i$ (e.g. $U_i = \bigcap\{U_i | U_i \in \mathcal{F}, V \subseteq U_i\}$), i.e. $\mathcal{F}'$ is a refinement of $\mathcal{F}$. But $\text{Rem}^{\mathcal{F}} U_i$ is stable for $\phi$ and so is $\text{Rem}^{\mathcal{F}'} V$ by Proposition 23. □

The above proposition says that a finite stable partition for a treelike space $\mathcal{O}$ remains stable if we “refine” it.

The following is the main theorem of this section. It says that for each formula $\phi$ we can find a stable partition for $\phi$ which is essentially a refinement of the stable partition corresponding to the subformulae of $\phi$.

**Theorem 28 (Partition Theorem)** Let $\mathcal{M} = \langle X, \mathcal{O}, i \rangle$ be a treelike model. Then there exists a family $\{\mathcal{F}^\psi\}_{\psi \in \mathcal{L}}$ of finite stable partitions such that if $\phi$ is a subformula of $\psi$ then $\mathcal{F}^\phi \subseteq \mathcal{F}^\psi$ and $\mathcal{F}^\psi$ is a finite stable partition for $\psi$. 

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Proof. By induction on the structure of the formula $\psi$. In each step we refine the partition of the induction hypothesis. For each $U \in F^\psi$, let $U^\psi = \{ x \in U : x, U \models \psi \}$. This set determines completely the satisfaction of $\psi$ on $\text{Rem}^F U$ whenever $F^\psi$ is stable.

- If $\psi = A$ is an atomic formula then $F^A = \{ X \} = \{ i(\top) \}$, since $O$ is stable for all atomic formulae. We have $X^A = i(A)$.

- If $\psi = \neg \phi$ then let $F^\psi = F^\phi$, since the statement of the theorem is symmetric with respect to negation. We also have $U^\psi = (X - U^\phi) \cap U$, for all $U \in F^\psi$.

- If $\psi = \chi \land \phi$, let $F^\psi = \text{Cl}(F^\chi \cup F^\phi)$. Observe that $F^\chi \cup F^\phi \subseteq F^{\chi \land \phi}$. Now, $F^\psi$ is a stable partition for $\chi \land \phi$ containing $X$, since it is a refinement of both $F^\chi$ and $F^\phi$. Thus, $F^\psi$ is a finite stable partition for $\psi$ containing $X$.

- Suppose $\psi = K\phi$. Then, by induction hypothesis, there exists a finite stable partition $F^\phi = \{ U_1, \ldots, U_n \}$ for $\phi$ containing $X$.

  Now, if $V \in \text{Rem}^F U_i \cap U_i^\phi$, for some $i \in \{1, \ldots, n\}$, then $x, V \models \phi$, for all $x \in V$, by definition of $U_i^\phi$. Hence $x, V \models \phi^\phi$, for all $x \in V$.

  On the other hand, if $V \in \text{Rem}^F U_i - \text{rem}^F U_i$ then there exists $x \in V$ such that $x, V \models \neg \phi$ (otherwise $V \subseteq U_i^\phi$). Thus we have $x, V \models \neg \phi^\phi$, for all $x \in V$. Hence $\text{Rem}^F U_i \cap \text{rem}^F U_i$ and $\text{Rem}^F U_i - \text{rem}^F U_i$ are stable for $K\phi$. Thus the set

  \[ F = \{ \text{Rem}^F U_i \cap \text{rem}^F U_i \} \cup \{ \text{rem}^F U_j - \text{rem}^F U_j \cap \text{rem}^F U_j \} \]

  is a partition of $O$ and its members are stable for $K\phi$. Let

  \[ F^{K\phi} = \text{Cl}(F^\phi \cup U_i^\phi). \]

  We have that $F^{K\phi}$ is a finite set of opens and $F^\phi \subseteq F^{K\phi}$. Thus $F^{K\phi}$ is finite and contains $X$. We have only to prove that $F^{K\phi}$ is a stable partition for $K\phi$, i.e. every remainder of an open in $F^{K\phi}$ is stable for $K\phi$. But for that, observe that $F^{K\phi}$ is a refinement of $F$. Therefore $F^{K\phi}$ is a finite stable partition for $K\phi$, using Proposition 23.[4]

  Now, if $U \in F^\psi$ then either $U^{K\phi} = U$ or $U^{K\phi} = \emptyset$.

- Suppose $\psi = \diamond \phi$. Then, let

  \[ F^{\diamond \phi} = F^\phi, \]

  where $F^\phi$ is a finite stable partition for $\phi$ by induction hypothesis.
We shall show that \( F^\phi \) is also a finite stable splitting for \( \diamond \phi \). Pick \( U \in F^\phi \) and \( x \in U \). If \( x, V \models \neg \phi \), for all \( V \subseteq U \) such that \( x \in V \), we are done, since \( x, V \models \neg \diamond \phi \). If \( x, V \models \phi \), for some \( V \in \text{Rem}^{F^\phi} U \), then \( x, W \models \phi \), for all \( W \in \text{Rem}^{F^\phi} U \), since \( F^\phi \) is stable for \( \phi \). Therefore \( x, W \models \diamond \phi \) for all \( W \in \text{Rem}^{F^\phi} U \). If \( x, V \models \phi \), for some \( V \subseteq U \) such that \( x \in V \), we are done, since \( x, V \models \neg \diamond \phi \).

The following corollary is “folklore”.

**Corollary 29** The formula \( \Box \Box \phi \rightarrow \Box \diamond \phi \) is sound in treelike spaces.

**Proof.** Let \( x, U \models \Box \Box \phi \) in some model \( \langle X, \mathcal{O}, i \rangle \).

By the Partition theorem, there exists a finite stable partition \( F \) for \( \phi \). Further, there is a \( V \in F \) which is “the least” in the following sense: if \( W, W' \in \mathcal{O} \) contain \( x \), \( W \in \text{Rem}^F V \), and \( W' \subseteq W \) then we will also have \( W' \in \text{Rem}^F V \).

The existence of such a set \( V \) is assured by the fact that \( F \) is finite, the members of the partition which \( F \) induces are convex, and the set of subsets in \( \mathcal{O} \) which contain \( x \) is linearly ordered. Moreover, \( \text{Rem}^F V \) contains at least one subset which contains \( x \), say \( W \).

Now, we have either \( U \subseteq V \) or \( V \subseteq U \). In the former case, we have \( U \in \text{Rem}^F V \). Hence \( x, U \models \Box \phi \) as \( \text{Rem}^F V \) is stable for \( \phi \). In the latter case, \( x, W \models \diamond \phi \), since \( W \subseteq V \subseteq U \). Thus we have \( x, W \models \Box \phi \) for the same reasons as above. Hence \( x, U \models \Box \diamond \phi \).

A finite partition does not have a treelike form. Therefore we cannot perform a filtration in a direct manner. First, we shall consider no partition member (remainder) that contains no subset belonging to the initial treelike space. Next, we shall impose a relation \( \leq \) among the remaining members (Definition 3.2).

Two remainders will be related just in case they contain subsets with common elements. This relation is not a partial order. However, it respects the initial treelike ordering (Lemma 32 through 35). Finally, using a number of equivalence relations based on \( \leq \), one for each member of the partition, we shall construct a treelike model equivalent to the initial one (Propositions 38 and 39). Moreover, the underlying space of this model will contain a finite number of subsets.

By the Partition theorem, given a treelike model \( \langle X, \mathcal{O}, i \rangle \) and a formula \( \phi \), there exists a finite partition \( F^\phi \) on \( \mathcal{O} \) stable for \( \phi \). For each \( U \in F^\phi \), let

\[
\overline{U} = \bigcup \text{Rem}^{F^\phi} U
\]

and

\[
\overline{F^\phi} = \{ U \mid U \in F^\phi \text{ and } \overline{U} \neq \emptyset \}.
\]

We have the following
Lemma 30 If \( U_1, U_2 \in \bar{F}^\phi \) with \( \overline{U_1} \subset \overline{U_2} \), and \( V_1, V_2 \in \mathcal{O} \) with \( V_1 \in \text{Rem}^{F^\phi} U_1 \), \( V_2 \in \text{Rem}^{F^\phi} U_2 \) and \( V_1 \cap V_2 \neq \emptyset \), then \( V_1 \subset V_2 \).

Proof. Since \( V_1 \cap V_2 \neq \emptyset \), then, by connectedness, we have either \( V_1 \subseteq V_2 \) or \( V_2 \subseteq V_1 \). If \( V_1 \subseteq V_2 \), then \( V_2 \subset V_2 \) since they belong to distinct equivalence classes. If \( V_2 \subset V_1 \) then we have \( V_2 \subseteq V_1 \subseteq \overline{U_1} \subseteq \overline{U_2} \). Hence \( V_1 \in \text{Rem}^{F^\phi} U_2 \), by Proposition 19(c).

Definition 31 Let \( \prec \) be the following relation on \( \bar{F}^\phi \)

\[ U_1 \prec U_2 \iff \overline{U_1} \cap \overline{U_2} \neq \emptyset, \text{ and for all } x, V_1, V_2 \text{ such that } x \in \overline{U_1} \cap \overline{U_2}, V_1 \in \text{Rem}^{F^\phi} U_1 \text{ with } x \in V_1, \text{ and } V_2 \in \text{Rem}^{F^\phi} U_2 \text{ with } x \in V_2, V_1 \subset V_2. \]

Clearly, we cannot have \( U_1 \prec U_2 \) and \( U_2 \prec U_1 \). Let \( U_1 \leq U_2 \), if either \( U_1 = U_2 \) or \( U_1 \prec U_2 \).

The following lemma allows us to weaken the conditions of the definition of \( \prec \).

Lemma 32 Let \( U_1, U_2 \in \bar{F}^\phi \) with \( U_1 \neq U_2 \). If there exist \( x \in \overline{U_1} \cap \overline{U_2} \) and \( V_1 \in \text{Rem}^{F^\phi} U_1 \), \( V_2 \in \text{Rem}^{F^\phi} U_2 \) with \( x \in V_1 \cap V_2 \) such that \( V_1 \subset V_2 \), then \( U_1 \prec U_2 \).

Proof. Suppose, towards a contradiction, that for \( y \in \overline{U_1} \cap \overline{U_2} \) there exist \( W_1 \in \text{Rem}^{F^\phi} U_1 \) and \( W_2 \in \text{Rem}^{F^\phi} U_2 \) such that \( y \in W_1 \cap W_2 \) and \( W_2 \subseteq W_1 \). By our hypothesis, \( U_1 \neq U_2 \) and \( V_1 \subset V_2 \), and so we have \( U_2 \not\subset U_1 \). This implies that \( \text{Rem}^{F^\phi} U_2 \cap \downarrow U_1 = \emptyset \). Therefore \( W_2 \notin \text{Rem}^{F^\phi} U_2 \) which is a contradiction. Thus \( W_1 \subset W_2 \). Hence \( U_1 \prec U_2 \).

Lemma 33 Let \( U_1, U_2 \in \bar{F}^\phi \). If \( \overline{U_1} \cap \overline{U_2} \neq \emptyset \) then \( U_1 \leq U_2 \) or \( U_2 \leq U_1 \).

Proof. Suppose that \( U_1 \neq U_2 \) and let \( x \in \overline{U_1} \cap \overline{U_2} \). Let \( V_1 \in \text{Rem}^{F^\phi} U_1 \) and \( V_2 \in \text{Rem}^{F^\phi} U_2 \) such that \( x \in V_1 \cap V_2 \). Since \( \mathcal{O} \) is a treelike space, we have either \( V_1 \subseteq V_2 \) or \( V_2 \subseteq V_1 \). Suppose that the former holds. Since \( U_1 \neq U_2 \), we have \( V_1 \subset V_2 \). By Lemma 32, \( U_1 \prec U_2 \). Similarly, if \( V_2 \subseteq V_1 \) then \( U_2 \prec U_1 \).

Lemma 34 \( \leq \) is reflexive and antisymmetric.

Proof. Reflexivity is straightforward. For antisymmetry, suppose that \( \overline{U_1} \cap \overline{U_2} \neq \emptyset \), \( U_1 \leq U_2 \) and \( U_2 \leq U_1 \). If \( U_1 \neq U_2 \) then we have \( U_1 \prec U_2 \) and \( U_2 \prec U_1 \), which is a contradiction.

Instead of transitivity, we have the following property of \( \leq \):

Lemma 35 Let \( U_1, U_2, U_3 \in \bar{F}^\phi \). If \( U_1 \leq U_2 \), \( U_2 \leq U_3 \) and \( \overline{U_1} \cap \overline{U_2} \cap \overline{U_3} \neq \emptyset \) then \( U_1 \leq U_3 \).
Proof. If either \( U_1 = U_2 \) or \( U_2 = U_3 \) we are done, so suppose that \( U_1 < U_2 \) and \( U_2 < U_3 \). Let \( x \in U_1 \cap U_2 \cap U_3 \), \( V_1 \in \text{Rem}^{F^\circ} U_1 \) and \( V_3 \in \text{Rem}^{F^\circ} U_3 \) such that \( x \in V_1 \) and \( x \in V_3 \). Since \( x \in U_2 \), there exists \( V_2 \in \text{Rem}^{F^\circ} U_2 \) such that \( x \in V_2 \). Also, we have \( V_1 \subset V_2 \subset V_3 \), since \( U_1 < U_2 \) and \( U_2 < U_3 \). So, by Lemma 32, \( U_1 < U_3 \).

Since \( F^\circ \subset F^\phi \), \( F^\phi \) is finite. Let \( F^\phi = \{U_1, U_2, \ldots, U_n\} \), for some \( n \). Now, let \( \sim_i \) be the following equivalence relation on \( U_i \)

\[
x \sim_i y \iff \text{for all } U_j, j \in \{1, 2, \ldots, n\} \text{ such that } U_i \subseteq U_j,
\]

\[
x \in U_j \iff y \in U_j.
\]

We denote the equivalence of \( x \) under \( \sim_i \) with \([x]_i\). Observe that the number of equivalence classes is finite, since it depends only on the number of members of the partition.

Lemma 36 Let \( U_k, U_l \in F^\phi \), \( k, l \in \{1, 2, \ldots, n\} \), with \( U_k \cap U_l \neq \emptyset \). Then

a. if \( U_k \leq U_l \) then \([x]_k \subseteq [x]_l\), for all \( x \in U_k \cap U_l \), and

b. if \([x]_k \subset [x]_l\), for some \( x \in U_k \cap U_l \), then \( U_k < U_l \).

Proof. For Part 1 if \( U_k = U_l \) then we are done. Suppose \( U_k < U_l \) and let \( z \in [x]_k \). Let \( U_m \in F^\phi \), \( m \in \{1, 2, \ldots, n\} \), be such that \( U_l \leq U_m \). If \( x \in U_m \) then \( x \in U_k \cap U_l \cap U_m \). So, by Lemma 33, \( U_k \leq U_m \). Since \( z \in U_m \), since \( x \sim_i z \). For the other direction, suppose \( z \in U_m \). Then we have \( z \in U_l \), since \( U_k \leq U_l \) and \( x \sim_i z \). So \( z \in U_k \cap U_l \cap U_m \). Hence, by Lemma 33, \( U_k \leq U_m \). Also, \( x \in U_m \), since \( x \sim_i z \). Therefore \( z \in [x]_l \).

For Part 2 we have either \( U_k < U_l \) or \( U_l \leq U_k \), since \( U_k \cap U_l \neq \emptyset \). Suppose the latter towards a contradiction. Then, by Part 1 and Lemma 33, we have \([x]_l \subseteq [x]_k\) which is a contradiction to our hypothesis.

Lemma 37 Let \( U_k, U_l \in F^\phi \), \( k, l \in \{1, 2, \ldots, n\} \), with \( U_k \cap U_l \neq \emptyset \). If \( U_k < U_l \) then \([x]_k \subset [x]_l\), for all \( x \in U_k \cap U_l \).

Proof. By Lemma 33(2), we have \([x]_k \subseteq [x]_l\). Suppose \([x]_k = [x]_l\). Let \( V \in \text{Rem}^{F^\circ} U_i \) such that \( x \in V \). We have \( V \subseteq [x]_l \). Thus \( V \subseteq [x]_l \subseteq U_l \). So, for each \( y \in V \), there exists \( V_y \in \text{Rem}^{F^\circ} U_j \) such that \( V_y \subset V \). But then \( V = \bigcup_{y \in V} V_y \in \text{Rem}^{F^\circ} U_j \) which is a contradiction, since \( U_i \neq U_j \).

Now, let

\[
\overline{F^\phi} = \{ [x]_i \mid x \in U_i, \ i \in \{1, 2, \ldots, n\} \}.
\]

Proposition 38 The subset space \( (X, \overline{F^\phi}) \) is a treelike space.
Proof. First notice that \( X \in \mathcal{F}_0 \), since \( X \in \mathcal{F}_0 \). Thus \( X = U_{i_0} \), for some \( i_0 \in \{1,2,\ldots,n\} \). Moreover, \( x \sim_{i_0} y \), for all \( x, y \in X \). Hence \( X = [x]_{i_0} \in \mathcal{F}_0 \).

Now, let \([x]_i \cap [y]_j \neq \emptyset\), for some \( x, y \in U_i \) and \( y \in U_j \). Let \( z \in [x]_i \cap [y]_j \). We have \([x]_i \subseteq [z]_i \) and \([y]_j \subseteq [z]_j \). Further, \( z \in U_i \) and \( z \in U_j \), i.e. \( U_i \cap U_j \neq \emptyset \). So, by Lemma 33, we have either \( U_i \leq U_j \) or \( U_j \leq U_i \). By Lemma 33(a), we have either \([z]_i \subseteq [z]_j \) or \([z]_j \subseteq [z]_i \), respectively. Therefore either \([x]_i \subseteq [y]_j \) or \([y]_j \subseteq [x]_i \).

Let \( \mathcal{M} = \langle X, [\mathcal{F}_0], \mathcal{I} \rangle \) be the treelike model where \( \mathcal{I}(A) = \{ [x]_i \mid x \in i(A) \} \).

**Proposition 39** For all \( x \in X, V \in \mathcal{O} \) and \( \psi \in \mathcal{L} \) such that \( \psi \) is a subformula of \( \phi \), if \( V \in \text{Rem}_\psi U_i \), for some \( i \in \{1,2,\ldots,n\} \), then

\[
x, V \models_{\mathcal{M}} \psi \quad \text{iff} \quad x, [x]_i \models_{\mathcal{M}} \psi.
\]

Proof. By induction on the complexity of \( \phi \). The only interesting case is that of \( \phi = \Box \psi \). Suppose \( x, [x]_i \models \Box \neg \psi \) but \( x, V \models_{\mathcal{M}} \psi \), for some \( V \in \text{Rem}_\psi U_i \).

The latter implies that there is \( j \in \{1,2,\ldots,n\} \) such that \( x, [x]_j \models \neg \psi \) and \([x]_j \subseteq [x]_i \). We have \( x, [x]_j \models \psi \), by \( x, V \models_{\mathcal{M}} \psi \) and induction hypothesis. Hence \( [x]_j \subseteq [x]_i \). By Lemma 33(b), we have \( U_j < U_i \). By induction hypothesis, we have \( x, V' \models_{\mathcal{M}} \neg \psi \), for all \( V' \in \text{Rem}_\psi U_j \) such that \( x \in V' \). Also, we have \( V' \models \psi \), since \( U_j < U_i \). Hence \( x, V \models_{\mathcal{M}} \neg \psi \), a contradiction.

Now, suppose \( x, [x]_i \models \Box \psi \) but \( x, V \models_{\mathcal{M}} \neg \psi \), for some \( V \in \text{Rem}_\psi U_i \). So there exists \( V' \in \text{Rem}_\psi U_j \) such that \( x \in V', V' \subseteq V \), and \( x, V' \models_{\mathcal{M}} \neg \psi \). Also, \( x, [x]_j \models \psi \) so, by induction hypothesis, \( x, V \models_{\mathcal{M}} \psi \). The latter implies \( U_i \neq U_j \), since \( \text{Rem}_\psi U_i \) is stable for \( \psi \). Therefore we have \( U_j < U_i \). Hence, by Lemma 33, \([x]_j \subseteq [x]_i \). Thus \( x, [x]_j \models \neg \psi \), by induction hypothesis. Hence \( x, [x]_i \models \Box \neg \psi \), a contradiction to our hypothesis.

Constructing the above model is not adequate for generating a finite model, since there may still be an infinite number of points. It turns out that we only need a finite number of them.

Let \( \mathcal{M} = \langle X, \mathcal{O}, i \rangle \) be a treelike model, and define an equivalence relation \( \sim \) on \( X \) by \( x \sim y \) iff

1. for all \( U \in \mathcal{O} \), \( x \in U \) iff \( y \in U \), and

2. for all atomic \( A \), \( x \in i(A) \) iff \( y \in i(A) \).

Further, denote by \( x^* \) the equivalence class of \( x \), and let \( X^* = \{ x^* \mid x \in X \} \). For every \( U \in \mathcal{O} \), let \( U^* = \{ x^* \mid x \in U \} \), then \( \mathcal{O}^* = \{ U^* \mid U \in \mathcal{O} \} \) is a treelike space on \( X^* \). Define a map \( i^* \) from the atomic formulae to the powerset of \( X^* \) by \( i^*(A) = \{ x^* \mid x \in i(A) \} \). The entire model \( \mathcal{M} \) lifts to the model \( \mathcal{M}^* = \langle X^*, \mathcal{O}^*, i^* \rangle \) in a well-defined way.
Lemma 40 For all $x$, $U$, and $\phi$,

$$x, U \models_{\mathcal{M}} \phi \iff x^*, U^* \models_{\mathcal{M}^*} \phi .$$

Proof. By induction on $\phi$. ■

Theorem 41 If $\phi$ is satisfied in any treelike space then $\phi$ is satisfied in a finite treelike space.

Proof. Let $\mathcal{M} = \langle X, \mathcal{O}, i \rangle$ be such that, for some $x \in U \in \mathcal{O}$, $x, U \models_{\mathcal{M}} \phi$. Let $\mathcal{F}^\phi$ be a finite stable partition (by Theorem 28) for $\phi$ and its subformulae with respect to $\mathcal{M}$. By Proposition 39, $x, U \models_{\mathcal{N}} \phi$, where $\mathcal{N} = \langle X, \mathcal{F}, i \rangle$. We may assume that $\mathcal{F}$ is a treelike space, and we may also assume that the overall language has only the (finitely many) atomic symbols which occur in $\phi$. Then the relation $\sim$ has only finitely many classes. So the model $\mathcal{N}^*$ is finite. Finally, by Lemma 40, $x^*, U^* \models_{\mathcal{N}^*} \phi$. ■

Observe that the finite treelike space is a quotient of the initial one under two equivalences. The one equivalence is on the elements of the treelike space and the number of equivalence classes is a function of the complexity of $\phi$. The other equivalence is on the points of the treelike space and the number of equivalence classes is a function of the atomic formulae appearing in $\phi$. So the overall size of the (finite) treelike space is bounded by a function of the complexity of $\phi$. Thus if we want to test if a given formula is invalid we have a finite number of finite treelike spaces where we have to test its validity. Thus we have the following

Theorem 42 The theory of treelike spaces is decidable.

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