H-space structure on pointed mapping spaces

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Abstract We investigate the existence of an $H$-space structure on the function space, $F_*(X,Y,\ast)$, of based maps in the component of the trivial map between two pointed connected CW-complexes $X$ and $Y$. For that, we introduce the notion of $H(n)$-space and prove that we have an $H$-space structure on $F_*(X,Y,\ast)$ if $Y$ is an $H(n)$-space and $X$ is of Lusternik-Schnirelmann category less than or equal to $n$. When we consider the rational homotopy type of nilpotent finite type CW-complexes, the existence of an $H(n)$-space structure can be easily detected on the minimal model and coincides with the differential length considered by Y. Kotani. When $X$ is finite, using the Haefliger model for function spaces, we can prove that the rational cohomology of $F_*(X,Y,\ast)$ is free commutative if the rational cup length of $X$ is strictly less than the differential length of $Y$, generalizing a recent result of Y. Kotani.

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1 Introduction

Let $X$ and $Y$ be pointed connected CW-complexes. We study the occurrence of an $H$-space structure on the function space, $F_*(X,Y,\ast)$, of based maps in the component of the trivial map. Of course when $X$ is a co-$H$-space or $Y$ is an $H$-space this mapping space is an $H$-space. Here, we are considering weaker conditions, both on $X$ and $Y$, which guarantee the existence of an $H$-space structure on the function space. In Definition 3 we introduce the notion of $H(n)$-space designed for this purpose and prove:

Proposition 1 Let $Y$ be an $H(n)$-space and $X$ be a space of Lusternik-Schnirelmann category less than or equal to $n$. Then the space $F_*(X,Y,\ast)$ is an $H$-space.
The existence of an $H(n)$-structure and the Lusternik-Schnirelmann category (LS-category in short) are hard to determine. We first study some properties of $H(n)$-spaces and give some examples. Concerning the second hypothesis, we are interested in replacing $\text{cat}(X) \leq n$ by an upper bound on an approximation of the LS-category (see [5, Chapter 2]). We succeed in Proposition 7 with an hypothesis on the dimension of $X$ but the most interesting replacement is obtained in the rational setting which constitutes the second part of this paper.

We use Sullivan minimal models for which we refer to [6]. We recall here that each finite type nilpotent CW-complex $X$ has a unique minimal model $(\wedge V, d)$ that characterises all the rational homotopy type of $X$. We first prove that the existence of an $H(n)$-structure on a rational space $X_0$ can be easily detected from its minimal model. It corresponds to a valuation of the differential of this model, introduced by Y. Kotani in [11]:

The differential $d$ of the minimal model $(\wedge V, d)$ can be written as $d = d_1 + d_2 + \cdots$ where $d_i$ increases the word length by $i$. The differential length of $(\wedge V, d)$, denoted $dl(X)$, is the least integer $n$ such that $d_{n-1}$ is non zero.

As a minimal model of $X$ is defined up to isomorphism, the differential length is a rational homotopy type invariant of $X$, see [11, Theorem 1.1]. Proposition 8 establishes a relation between $dl(X)$ and the existence of an $H(n)$-structure on the rationalisation of $X$.

Finally, recall that the rational cup-length $\text{cup}_0(X)$ of $X$ is the maximal length of a nonzero product in $H^{>0}(X; \mathbb{Q})$. In [11], by using this cup-length and the invariant $dl(Y)$, Y. Kotani gives a necessary and sufficient condition for the rational cohomology of $\mathcal{F}_*(X, Y, *)$ to be free commutative when $X$ is a rational formal space and when the dimension of $X$ is less than the connectivity of $Y$. We show here that a large part of the Kotani criterium remains valid, without hypothesis of formality and dimension. We prove:

**Theorem 2** Let $X$ and $Y$ be nilpotent finite type CW-complexes, with $X$ finite.

1. The cohomology algebra $H^*(\mathcal{F}_*(X, Y, *; \mathbb{Q}))$ is free commutative if $\text{cup}_0(X) < dl(Y)$.

2. If $\dim(X) \leq \text{conn}(Y)$, then the cohomology algebra $H^*(\mathcal{F}_*(X, Y, *; \mathbb{Q}))$ is free commutative if, and only if, $\text{cup}_0(X) < dl(Y)$.

As an application, we describe in Theorem 12 the Postnikov tower of the rationalisation of $\mathcal{F}_*(X, Y, *)$ where $X$ is a finite nilpotent space and $Y$ a finite type CW-complex whose connectivity is greater than the dimension of $X$. 
Our description implies the solvability of the rational Pontrjagin algebra of $\Omega(\mathcal{F}_n(X,Y, *))$.

Section 2 contains the topological setting and the proof of Proposition 1. The link with rational models is done in Section 3. Our proof of Theorem 2 uses the Haefliger model for mapping spaces. In order to be self-contained, we recall briefly Haefliger’s construction in Section 4. The proof of Theorem 2 is contained in Section 5. Finally, Section 6 is devoted to the description of the Postnikov tower.

In this text, all spaces are supposed of the homotopy type of connected pointed CW-complexes and we will use cdga for commutative differential graded algebra. A quasi-isomorphism is a morphism of cdga’s which induces an isomorphism in cohomology.

2 Structure of $H(n)$-space

First we recall the construction of Ganea fibrations, $p^X_n : G_n(X) \to X$.

- Let $F_0(X) \overset{i_0} \to G_0(X) \overset{p_0^X} \to X$ denote the path fibration on $X$, $\Omega X \to PX \to X$.
- Suppose a fibration $F_n(X) \overset{i_n} \to G_n(X) \overset{p_n^X} \to X$ has been constructed. We extend $p^X_n$ to a map $q_n : G_n(X) \cup C(F_n(X)) \to X$, defined on the mapping cone of $i_n$, by setting $q_n(x) = p^X_n(x)$ for $x \in G_n(X)$ and $q_n([y,t]) = *$ for $[y,t] \in C(F_n(X))$.
- Now convert $q_n$ into a fibration $p^X_{n+1} : G_{n+1}(X) \to X$.

This construction is functorial and the space $G_n(X)$ has the homotopy type of the $n^{th}$-classifying space of Milnor [12]. We quote also from [8] that the direct limit $G_{\infty}(X)$ of the maps $G_n(X) \to G_{n+1}(X)$ has the homotopy type of $X$. As spaces are pointed, one has two canonical applications $i^*_n : G_n(X) \to G_n(X \times X)$ and $i^*_n : G_n(X) \to G_n(X \times X)$ obtained from maps $X \to X \times X$ defined respectively by $x \mapsto (x,*)$ and $x \mapsto (*,x)$.

Definition 3 A space $X$ is an $H(n)$-space if there exists a map $\mu_n : G_n(X \times X) \to X$ such that $\mu_n \circ i^*_n = \mu_n \circ i^*_n = p^X_n : G_n(X) \to X$.

Directly from the definition, we see that an $H(\infty)$-space is an $H$-space and that any space is an $H(1)$-space. Recall also that any co-$H$-space is of LS-category 1. Then, Proposition 1 contains the trivial cases of a co-$H$-space $X$ and of an $H$-space $Y$. 
Proof of Proposition [1] From the hypothesis, we have a section \( \sigma: X \to G_n(X) \) of the Ganea fibration \( p_n^X \) and a map \( \mu_n: G_n(Y \times Y) \to Y \) extending the Ganea fibration \( p_n^Y \), as in Definition 3. If \( f \) and \( g \) are elements of \( \mathcal{F}_n(X,Y,\ast) \), we set

\[
f \bullet g = \mu_n \circ G_n(f \times g) \circ G_n(\Delta_X) \circ \sigma,
\]

where \( \Delta_X \) denotes the diagonal map of \( X \). One checks easily that \( f \bullet \ast \simeq \ast \bullet f \simeq f \).

In the rest of this section, we are interested in the existence of \( H(n) \)-structures on a given space. For the detection of an \( H(n) \)-space structure, one may replace the Ganea fibrations \( p_n^X \) by any functorial construction of fibrations \( \hat{p}_n: \hat{G}_n(X) \to X \) such that one has a functorial commutative diagram,

\[
\begin{array}{ccc}
\hat{G}_n(X) & \to & G_n(X) \\
\downarrow \hat{p}_n & & \downarrow p_n^X \\
X & \to & G_n(X) \\
\end{array}
\]

Such maps \( \hat{p}_n \) are called fibrations à la Ganea in [13] and substitutes to Ganea fibrations here. Moreover, as we are interested in product spaces, the following filtration of the space \( G_\infty(X) \times G_\infty(Y) \) plays an important role:

\[
(G(X) \times G(Y))_n = \bigcup_{i+j=n} G_i(X) \times G_j(Y).
\]

In [10], N. Iwase proved the existence of a commutative diagram

\[
\begin{array}{ccc}
(G(X) \times G(Y))_n & \to & G_n(X \times Y) \\
\downarrow \bigcup (p_i^X \times p_j^Y) & & \downarrow p_n^{X \times Y} \\
X \times Y & \to & G_n(X) \times G_n(Y) \\
\end{array}
\]

and used it to settle a counter-example to the Ganea conjecture. Therefore, in Definition 3, we are allowed to replace the Ganea space \( G_n(X \times X) \) by \( (G(X) \times G(X))_n \). Moreover, if \( \hat{p}_n: \hat{G}_n(X) \to X \) are substitutes to Ganea fibrations as above, we may also replace \( G_n(X \times X) \) by

\[
(\hat{G}(X) \times \hat{G}(Y))_n = \bigcup_{i+j=n} \hat{G}_i(X) \times \hat{G}_j(Y).
\]

We will use this possibility in the rational setting.

In the case \( n = 2 \), we have a cofibration sequence,

\[
\Sigma(G_1(X) \wedge G_1(X)) \xrightarrow{Wh} G_1(X) \vee G_1(X) \to G_1(X) \times G_1(X),
\]
coming from the Arkowitz generalisation of a Whitehead bracket, \([2]\). Therefore, the existence of an \(H(2)\)-structure on a space \(X\) is equivalent to the triviality of \((p_1^X \vee p_1^X) \circ \text{Wh}\). As the loop \(\Omega p_1^X\) of the Ganea fibration \(p_1^X : G_1(X) \to X\) admits a section, we get the following necessary condition:

– if there is an \(H(2)\)-structure on \(X\), then the homotopy Lie algebra of \(X\) is abelian, i.e. all Whitehead products vanish.

**Example 4** In the case \(X\) is a sphere \(S^n\), the existence of an \(H(2)\) structure on \(S^n\) implies \(n = 1, 3\) or \(7\), \([1]\). Therefore, only the spheres which are already \(H\)-spaces endow a structure of \(H(2)\) space. One can also observe that, in general, if a space \(X\) is both of category \(n\) and an \(H(2n)\)-space, then it is an \(H\)-space. The law is given by

\[
\sigma : X \times X \xrightarrow{\sigma} G_{2n}(X \times X) \xrightarrow{\mu_{2n}} X,
\]

where the existence of the section \(\sigma\) to \(p_{2n}^{X \times X}\) comes from \(\text{cat}(X \times X) \leq 2 \text{cat}(X)\).

**Example 5** If we restrict to spaces whose loop space is a product of spheres or of loop spaces on a sphere, the previous necessary condition becomes a criterion. For instance, it is proved in \([3]\) that all Whitehead products are zero in the complex projective 3-space. This implies that \(\mathbb{C}P^3\) is an \(H(2)\)-space. (Observe that \(\mathbb{C}P^3\) is not an \(H\)-space.) From \([3]\), we know also that the homotopy Lie algebra of \(\mathbb{C}P^2\) is not abelian. Therefore \(\mathbb{C}P^2\) is not an \(H(2)\)-space.

The following example shows that we can find \(H(n)\)-spaces, for any \(n > 1\).

**Example 6** Denote by \(\varphi_r : K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2r)\) the map corresponding to the class \(x^r \in H^{2r}(K(\mathbb{Z}, 2); \mathbb{Z})\), where \(x\) is the generator of \(H^2(K(\mathbb{Z}, 2); \mathbb{Z})\). Let \(E\) be the homotopy fibre of \(\varphi_r\). We prove below that \(E\) is an \(H(r - 1)\)-space.

First we derive, from the homotopy long exact sequence associated to the map \(\varphi_r\), that \(\Omega E\) has the homotopy type of \(S^1 \times K(\mathbb{Z}, 2r - 2)\). Therefore, the only obstruction to extend \(G_{r-1}(E) \vee G_{r-1}(E) \to E\) to \((G(E) \times G(E))_{r-1} = \bigcup_{i+j=r-1} G_i(E) \times G_j(E)\) lies in

\[
\text{Hom}(H_{2r}, ((G(E) \times G(E))_{r-1}; \mathbb{Z}), \pi_{2r-2}(E)).
\]

If \(A\) and \(B\) are CW-complexes, we denote by \(A \sim_n B\) the fact that \(A\) and \(B\) have the same \(n\)-skeleton. If we look at the Ganea total spaces and fibres, we get:

\[
\Sigma \Omega E \sim_2 S^2 \vee S^{2r-1} \vee S^{2r}, \quad F_1(E) = \Omega E \ast \Omega E \sim_2 S^3 \vee S^{2r} \vee S^{2r},
\]

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and more generally, \( F_s(E) \sim_{2r} S^{2s+1} \), for any \( s, \ 2 \leq s \leq r - 1 \). Observe also that \( H_{2r}(F_2(E); \mathbb{Z}) \to H_{2r}(G_1(E); \mathbb{Z}) \) is onto. (As we have only spherical classes in this degree, this comes from the homotopy long exact sequence.) As a conclusion, we have no cell in degree \( 2r \) in \( (G(E) \times G(E))_{r-1} \) and \( E \) is an \( H(r-1) \)-space.

We end this section with a reduction to a more computable invariant than the LS-category. Consider \( \rho^X_n: X \to G_n[X] \) the homotopy cofibre of the Ganea fibration \( p^X_n \). Recall that, by definition, \( \text{wcat}_G(X) \leq n \) if the map \( \rho^X_n \) is homotopically trivial. Observe that we always have \( \text{wcat}_G(X) \leq \text{cat}(X) \), see [E Section 2.6] for more details on this invariant.

**Proposition 7** Let \( X \) be a CW-complex of dimension \( k \) and \( Y \) be a CW-complex \((c - 1)\)-connected with \( k \leq c - 1 \). If \( Y \) is an \( H(n) \)-space such that \( \text{wcat}_G(X) \leq n \), then \( F_*(X,Y,* ) \) is an \( H \)-space.

**Proof** Let \( f \) and \( g \) be elements of \( F_*(X,Y,*) \). Denote by \( \tilde{i}^X_n: \tilde{F}_n(X) \to X \) the homotopy fibre of \( \rho^X_n: X \to G_n[X] \). This construction is functorial and the map \( (f,g): X \to Y \times Y \) induces a map \( \tilde{F}_n(f,g): \tilde{F}_n(X) \to \tilde{F}_n(Y \times Y) \) such that \( \tilde{i}^Y_Y \circ \tilde{F}_n(f,g) = (f,g) \circ \tilde{i}^X_n \).

By hypothesis, we have a homotopy section \( \tilde{\sigma}: X \to \tilde{F}_n(X) \) of \( \tilde{i}^X_n \). Therefore, one gets a map \( X \to \tilde{F}_n(Y \times Y) \) as \( \tilde{F}_n(f,g) \circ \tilde{\sigma} \).

Recall now that, if \( A \to B \to C \) is a cofibration with \( A \) \((a - 1)\)-connected and \( C \) \((c - 1)\)-connected, then the canonical map \( A \to F \) in the homotopy fibre of \( B \to C \) is an \((a + c - 2)\)-equivalence. We apply it in the following situation:

\[
\begin{array}{ccc}
G_n(Y \times Y) & \xrightarrow{p_n^Y \times Y} & Y \times Y \\
\downarrow j_n^Y \times Y & & \downarrow \tilde{i}^Y_Y \times Y \\
\tilde{F}_n(Y \times Y) & \xrightarrow{\tilde{i}^Y_Y} & G_n[Y \times Y]
\end{array}
\]

The space \( G_n(Y \times Y) \) is \((c - 1)\)-connected and \( G_n[Y \times Y] \) is \( c \)-connected. Therefore the map \( j_n^Y \times Y \) is \((2c - 1)\)-connected. From the hypothesis, we get \( k \leq c - 1 < 2c - 1 \) and the map \( j_n^Y \times Y \) induces a bijection

\[ [X,G_n(Y \times Y)] \xrightarrow{\sim} [X,\tilde{F}_n(Y \times Y)]. \]

Denote by \( g_n: X \to G_n(Y \times Y) \) the unique lifting of \( \tilde{F}_n(f,g) \circ \tilde{\sigma} \). The composition \( g \circ f \) is defined as \( \mu_n \circ g_n \) where \( \mu_n \) is the \( H(n) \)-structure on \( Y \).

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If we set \( g = \ast \), then \( \tilde{F}_n(f, g) \) is obtained as the composite of \( \tilde{F}_n(f) \) with the map \( \tilde{F}_n(Y) \to \tilde{F}_n(Y \times Y) \) induced by \( y \mapsto (y, \ast) \). As before, one has an isomorphism
\[
[X, G_n(Y)] \cong [X, \tilde{F}_n(Y)].
\]
A chase in the following diagram shows that \( f \ast \ast = f \) as expected,
\[
\begin{array}{ccc}
G_n(Y) & \to & G_n(Y \times Y) \\
\downarrow & & \downarrow \\
\tilde{F}_n(X) & \to & \tilde{F}_n(Y \times Y) \\
\downarrow & & \downarrow \\
X & \to & Y.
\end{array}
\]

3 Rational characterisation of \( H(n) \)-spaces

Define \( m_H(X) \) as the greatest integer \( n \) such that \( X \) admits an \( H(n) \)-structure and denote by \( X_0 \) the rationalisation of a nilpotent finite type CW-complex \( X \). Recall that \( dl(X) \) is the valuation of the differential of the minimal model of \( X \), already defined in the introduction.

**Proposition 8** Let \( X \) be a nilpotent finite type CW-complex of rationalisation \( X_0 \). Then we have:
\[
m_H(X_0) + 1 = dl(X).
\]

**Proof** Let \( (\wedge V, d) \) be the minimal model of \( X \). Recall from [7] that a model of the Ganea fibration \( p_n^X \) is given by the following composition,
\[
(\wedge V, d) \to (\wedge V/ \wedge >n V, \bar{d}) \to (\wedge V/ \wedge >n V, \bar{d}) \oplus S,
\]
where the first map is the natural projection and the second one the canonical injection together with \( S \cdot S = S \cdot V = 0 \) and \( d(S) = 0 \). As the first map is functorial and the second one admits a left inverse over \( (\wedge V, d) \), we may use the realisation of \( (\wedge V, d) \to (\wedge V/ \wedge >n V, \bar{d}) \) as substitute for the Ganea fibration.

Suppose \( dl(X) = r \). We consider the cdga \( (\wedge V', d') \otimes (\wedge V'', d'')/I_r \) where \( (\wedge V', d') \) and \( (\wedge V'', d'') \) are copies of \( (\wedge V, d) \) and where \( I_r \) is the ideal \( I_r = \oplus_{i+j \geq r} \wedge^i V' \otimes \wedge^j V'' \). Observe that this cdga has a zero differential and that the morphism
\[
\varphi : (\wedge V, d) \to (\wedge V', d') \otimes (\wedge V'', d'')/I_r
\]
defined by \( \varphi(v) = v' + v'' \) satisfies \( \varphi(dv) = 0 \). Therefore \( \varphi \) is a morphism of cdga’s and its realisation induces an \( H(n) \)-structure on the rationalisation \( X_0 \). That shows: \( m_H(X_0) + 1 \geq \text{dl}(X) \).

Suppose now that \( m_H(X_0) + 1 > \text{dl}(X) = r \). By hypothesis, we have a morphism of cdga’s \( \varphi: (\wedge V, d) \to (\wedge V', d') \otimes (\wedge V'', d'')/I_{r+1} \).

By construction, in this quotient, a cocycle of wedge degree \( r \) cannot be a coboundary. Since the composition of \( \varphi \) with the projection on the two factors is the natural projection, we have \( \varphi(v) - v' - v'' \in \wedge^+ V' \otimes \wedge^+ V'' \). Now let \( v \in V \), of lowest degree with \( d_r(v) \neq 0 \). From \( d_r(v) = \sum_{i_1, i_2, \ldots, i_r} c_{i_1 i_2 \ldots i_r} v_{i_1} v_{i_2} \cdots v_{i_r} \), we get

\[
\varphi(dv) = \sum_{i_1, i_2, \ldots, i_r} c_{i_1 i_2 \ldots i_r} (v'_{i_1} + v''_{i_1}) \cdot (v'_{i_2} + v''_{i_2}) \cdots (v'_{i_r} + v''_{i_r}).
\]

This expression cannot be a coboundary and the equation \( d\varphi(x) = \varphi(dx) \) is impossible. We get a contradiction, therefore one has \( m_H(X_0) + 1 = \text{dl}(X) \). \( \square \)

4 The Haefliger model

Let \( X \) and \( Y \) be finite type nilpotent CW-complexes with \( X \) of finite dimension. Let \( (\wedge V, d) \) be the minimal model of \( Y \) and \( (A, d_A) \) be a finite dimensional model for \( X \), which means that \( (A, d_A) \) is a finite dimensional cdga equipped with a quasi-isomorphism \( \psi \) from the minimal model of \( X \) into \( (A, d_A) \). Denote by \( A^\vee \) the dual vector space of \( A \), graded by

\[
(A^\vee)^{-n} = \text{Hom}(A^n, \mathbb{Q}).
\]

We set \( A^+ = \bigoplus_{i=1}^\infty A^i \), and we fix an homogeneous basis \( (a_1, \ldots, a_p) \) of \( A^+ \). The dual basis \( (a^s)^{1 \leq s \leq p} \) is a basis of \( B = (A^+)^\vee \) defined by \( \langle a^s; a_t \rangle = \delta_{st} \).

We construct now a morphism of algebras

\[
\varphi: \wedge V \to A \otimes \wedge(B \otimes V)
\]

by

\[
\varphi(v) = \sum_{s=1}^p a_s \otimes (a^s \otimes v).
\]

In \( [9] \) Haefliger proves that there is a unique differential \( D \) on \( \wedge(B \otimes V) \) such that \( \varphi \) is a morphism of cdga’s, i.e. \( (d_A \otimes D) \circ \varphi = \varphi \circ d \).
In general, the cdga \((\land(B \otimes V), D)\) is not positively graded. Denote by \(D_0: B \otimes V \to B \otimes V\) the linear part of the differential \(D\). We define a cdga \((\land Z, D)\) by constructing \(Z\) as the quotient of \(B \otimes V\) by \(\oplus_{j \leq 0}(B \otimes V)^j\) and their image by \(D_0\). Haefliger proves:

**Theorem 9** [9] The commutative differential graded algebra \((\land Z, D)\) is a model of the mapping space \(\mathcal{F}_*(X, Y, \ast)\).

## 5 Proof of Theorem 2

**Proof** We start with an explicit description of the Haefliger model, keeping the notation of Section 4. The cdga \((\land V, d)\) is a minimal model of \(Y\) and we choose for \(V\) a basis \((v^k)\), indexed by a well-ordered set and satisfying \(d(v^k) \in \land(v^r)_{r < k}\) for all \(k\). As homogeneous basis \(\langle a_s \rangle_{1 \leq s \leq p}\) of \(A\), we choose elements \(h_i, e_j\) and \(b_j\) such that:

- the elements \(h_i\) are cocycles and their classes \([h_i]\) form a linear basis of the reduced cohomology of \(A\);
- the elements \(e_j\) form a linear basis of a supplement of the vector space of cocycles in \(A\), and \(b_j = d_A(e_j)\).

We denote by \(h^i, e^j\) and \(b^j\) the corresponding elements of the basis of \(B = (A^+)^Y\). By developing \(D_0(\sum a^s \otimes (a^s \otimes v)) = 0\), we get a direct description of the linear part \(D_0\) of the differential \(D\) of the Haefliger model:

\[
D_0(b^j \otimes v) = -(-1)^{|b^j|} e^j \otimes v \quad \text{and} \quad D_0(h^i \otimes v) = 0, \quad \text{for each} \ v \in V.
\]

A linear basis of the graded vector space \(Z\) is therefore given by the elements:

\[
\left\{ \begin{array}{ll}
    b^j \otimes v_k, & \text{with} \ |b^j \otimes v_k| \geq 1, \\
    e^j \otimes v_k, & \text{with} \ |e^j \otimes v_k| \geq 2, \\
    h^i \otimes v_k, & \text{with} \ |h^i \otimes v_k| \geq 1.
\end{array} \right.
\]

Now, from \(\varphi(dv) = (D - D_0)\varphi(v)\) and \(d(v) = \sum c_{i_1 i_2 \ldots i_r} v_{i_1} v_{i_2} \ldots v_{i_r}\), we deduce:

\[
(D - D_0)(a^s \otimes v) = \pm \sum c_{i_1 i_2 \ldots i_r} \sum_{a_{i_1}, a_{i_2}, \ldots, a_{i_r}} (a^s; a_{i_1} a_{i_2} \cdots a_{i_r}) (a_{i_1} \otimes v_{i_1}) \cdots (a_{i_r} \otimes v_{i_r})
\]

where, as usual, the sign \(\pm\) is entirely determined by a strict application of the Koszul rule for a permutation of graded objects.

Suppose first that \(\text{cup}_0(X) < \text{dl}(Y)\).
We prove, by induction on $k$, that in $(\wedge Z, D)$ the ideal $I_k$ generated by the elements
\[
\begin{cases}
  b^j \otimes v_s, & s \leq k, \quad \text{with degree at least 1}, \\
  e^j \otimes v_s, & s \leq k, \quad \text{with degree at least 2},
\end{cases}
\]
is a differential ideal and that the elements $h^i \otimes v_s$, with $s \leq k$ and $|h^i \otimes v_s| \geq 1$, are cocycles in the quotient $((\wedge Z)/I_k, D)$. Note that this ideal is acyclic as shown by the formula given for $D_0$. Therefore the quotient map $\rho: (\wedge Z, D) \to ((\wedge Z)/I_k, D)$ is a quasi-isomorphism. The induction will prove that the differential is zero in the quotient, which is the first assertion of Theorem 2.

Begin with the induction. One has $dv_1 = 0$ which implies $(D - D_0)(a^s \otimes v_1) = 0$. Therefore, we deduce $D(b^j \otimes v_1) = -(-1)^{|b^j|} e^j \otimes v_1$ and $D(h^i \otimes v_1) = 0$. That proves the assertion for $k = 1$.

We suppose now that the induction step is true for the integer $k$. Taking the quotient by the ideal $I_k$ gives a quasi-isomorphism
\[
\rho: (\wedge Z, D) \to (\wedge T, D) := ((\wedge Z)/I_k, D).
\]
As the elements $b^j \otimes v_s$ and $e^j \otimes v_s$, $s \leq k$, have disappeared and as $	ext{cup}_0(X) < \text{dl}(Y)$, we have $\rho \circ \varphi(dv_{k+1}) = 0$. Therefore $D(b^j \otimes v_{k+1}) = -(-1)^{|b^j|} e^j \otimes v_{k+1}$ and $D(h^i \otimes v_{k+1}) = 0$. The induction is thus proved.

We consider now the case $\text{cup}_0(X) \geq \text{dl}(Y)$ in the case $\text{dim}(X) \leq \text{conn}(Y)$.

We choose first in the lowest possible degree $q$ an element $y \in V$ that satisfies $dy = d_r y + \cdots$ with $d_r (y) \neq 0$ and $r \leq \text{cup}_0(X)$. As above we can kill all the elements $e^j \otimes v$ and $b^j \otimes v$ with $|v| < q$ and keep a quasi-isomorphism $\rho: (\wedge Z, D) \to (\wedge T, D) := (\wedge Z/I_{q-1}, D)$.

Next we choose cocycles, $h_1, h_2, \cdots, h_m$, such that the class $[\omega]$, associated to the product $\omega = h_1 \cdot h_2 \cdot \cdots h_m$, is not zero. We choose $m \geq r$ and suppose that $\omega$ is in the highest degree for such a product. Let $\omega'$ be an element in $A^*$ such that $\langle \omega'; \omega \rangle = 1$. Then, by the Haefliger formula, the differential $D$ is zero in $\wedge T$ in degrees strictly less than $|\omega' \otimes y|$. Observe that $|\omega' \otimes y| \geq 2$ and that the $D_r$ part of the differential $D(\omega' \otimes y)$ is a nonzero sum. This proves that the cohomology is not free.

\textbf{Example 10} Let $X$ be a space with $\text{cup}_0(X) = 1$, which means that all products are zero in the reduced rational cohomology of $X$. Then, for any nilpotent finite type CW-complex $Y$, the rational cohomology $H^*(\mathcal{F}_*(X, Y, *); \mathbb{Q})$ is a free commutative graded algebra. For instance, this is the case for the (nonformal) space $X = S^3 \vee S^3 \cup_\omega e^8$, where the cell $e^8$ is attached along a sum of triple Whitehead products.
**Example 11** When the dimension of $X$ is greater than the connectivity of $Y$, the degrees of the elements have some importance. The cohomology can be commutative free even if $\text{cup}_0(X) \geq \text{dl}(Y)$. For instance, consider $X = S^5 \times S^{11}$ and $Y = S^8$. One has $\text{cup}_0(X) = \text{dl}(Y) = 2$ and the function space $\mathcal{F}_s(X,Y,\ast)$ is a rational $H$-space with the rational homotopy type of $K(\mathbb{Q}, 3) \times K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 10)$, as a direct computation with the Haefliger model shows.

6 Rationalisation of $\mathcal{F}_s(X,Y,\ast)$ when $\text{dim}(X) \leq \text{conn}(Y)$

Let $X$ be a finite nilpotent space with rational LS-category equal to $m - 1$ and let $Y$ be a finite type nilpotent CW-complex whose connectivity $c$ is greater than the dimension of $X$. We set $r = \text{dl}(Y)$ and denote by $s$ the maximal integer such that $m/r^s \geq 1$, i.e. $s$ is the integral part of $\log_r m$.

**Theorem 12** There is a sequence of rational fibrations $K_k \to F_k \to F_{k-1}$, for $k = 1, \ldots, s$, with $F_0 = \ast$, $F_s$ is the rationalisation of $\mathcal{F}_s(X,Y,\ast)$ and each space $K_k$ is a product of Eilenberg-MacLane spaces. In particular, the rational loop space homology of $\mathcal{F}_s(X,Y,\ast)$ is solvable with solvable index less than or equal to $s$.

**Proof** By a result of Cornea [4], the space $X$ admits a finite dimensional model $A$ such that $m$ is the maximal length of a nonzero product of elements of positive degree. We denote by $(\wedge, d)$ the minimal model of $Y$.

We consider the ideals $I_k = A^{>m/r^k}$, and the short exact sequences of cdga’s

$$I_k/I_{k-1} \to A/I_{k-1} \to A/I_k.$$  

These short exact sequences realise into cofibrations $T_k \to T_{k-1} \to Z_k$ and the sequences

$$\wedge((A^+/I_k)^{\wedge} \otimes V), D) \to (\wedge((A^+/I_{k-1})^{\wedge} \otimes V), D) \to (\wedge((I_k/I_{k-1})^{\wedge} \otimes V), D)$$

are relative Sullivan models for the fibrations

$$\mathcal{F}_s(Z_k,Y,\ast) \to \mathcal{F}_s(T_{k-1},Y,\ast) \to \mathcal{F}_s(T_k,Y,\ast).$$

Now since the cup length of the space $Z_k$ is strictly less than $r$, the function spaces $\mathcal{F}_s(Z_k,Y,\ast)$ are rational $H$-spaces, and this proves Theorem 12. □
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