Upper Bounds on the average eccentricity of Graphs of Girth 6 and \((C_4, C_5)\)-free Graphs

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Abstract

Let \(G\) be a finite, connected graph. The eccentricity of a vertex \(v\) of \(G\) is the distance from \(v\) to a vertex farthest from \(v\). The average eccentricity of \(G\) is the arithmetic mean of the eccentricities of the vertices of \(G\). We show that the average eccentricity of a connected graph \(G\) of girth at least six is at most \(\frac{9}{2} \left\lceil \frac{n^2}{2\delta^2} - 1 \right\rceil + 7\), where \(n\) is the order of \(G\) and \(\delta\) its minimum degree. We construct graphs that show that whenever \(\delta - 1\) is a prime power, then this bound is sharp apart from an additive constant. For graphs containing a vertex of large degree we give an improved bound. We further show that if the girth condition on \(G\) is relaxed to \(G\) having neither a 4-cycle nor a 5-cycle as a subgraph, then similar and only slightly weaker bounds hold.

Keywords: average eccentricity; eccentricity; eccentric mean; total eccentricity index; minimum degree; girth

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1 Introduction

Let \(G\) be a connected graph. The eccentricity \(e(v)\) of a vertex \(v\) is the distance from \(v\) to a vertex farthest from \(v\), i.e., \(e_G(v) = \max_{w \in V(G)} d_G(v, w)\), where \(V(G)\) denotes the vertex set of \(G\) and \(d_G(v, w)\) is the usual distance between \(v\) and \(w\). The average eccentricity \(\text{avec}(G)\) of \(G\) is defined as the arithmetic mean of the eccentricities of its vertices, i.e., \(\text{avec}(G) = \frac{1}{n} \sum_{v \in V(G)} e_G(v)\), where \(n\) is the order of \(G\). The average eccentricity was introduced under the name eccentric mean by Buckley and Harary [3], but it attracted major attention only after its first systematic study in [5]. One of the basic results in this paper determined the maximum average distance of a connected graph of given order:

**Theorem 1.1.** [5] If \(G\) is a connected graph of order \(n\), then

\[
\text{avec}(G) \leq \frac{1}{n} \left[ \frac{3n^2}{4} - \frac{n}{2} \right].
\]
Several bounds on the average eccentricity have been found since. For example for graphs of given order and size \([2, 21]\), and for maximal planar graphs \([2]\). Several relations between the average eccentricity and other graph parameters, for example independence number \([6, 7, 16]\), domination number \([6, 7, 12, 13, 14]\), clique number \([11, 16]\), chromatic number \([22]\), proximity \([18]\) and Wiener index \([10]\) have been explored. Bounds on the average eccentricity of the strong product of graphs were given in \([4]\).

The natural question if the bound in Theorem 1.1 can be improved for graphs whose minimum degree is greater than 1 was answered in the affirmative in \([5]\), where it was shown that if \(G\) is a graph of order \(n\) and minimum degree \(\delta\), then

\[
avec(G) \leq \frac{9n}{4(\delta + 1)} + \frac{15}{4},
\]

and this inequality is best possible apart from a small additive constant. Further results relating the average eccentricity of a graph to its vertex degrees are known. Bounds on the average eccentricity of trees of given order and maximum degree were given in \([10]\). Trees with given degree sequence that minimise or maximise the average eccentricity were determined in \([20]\). For relations between average eccentricity and Randić index see \([17]\). An upper bound on the average eccentricity in terms of order, size and first Zagreb index was given in \([11]\).

It was observed in \([8]\) that the upper bound \((1.1)\) can be improved for triangle-free graphs and for graphs not containing four-cycles. The aim of this paper is to further pursue the idea of improving \((1.1)\) for graphs not containing certain subgraphs. In this paper we give upper bounds on the average eccentricity of graphs of girth at least 6, and of graphs containing neither 4-cycles nor 5-cycles, in terms of order, minimum degree and maximum degree.

The notation we use is as follows. We denote the vertex set and edge set of a graph \(G\) by \(V(G)\) and \(E(G)\), respectively, and \(n(G)\) stands for the order of \(G\), i.e., for the number of vertices of \(G\). By \(\deg_G(v)\) we mean the degree of \(v\), i.e., the number of vertices adjacent to \(v\). The largest of the eccentricities of the vertices of \(G\) is called the diameter of \(G\) and denoted by \(\text{diam}(G)\).

For \(k \in \mathbb{Z}\), we denote the set of vertices at distance exactly \(k\) and at most \(k\) from a vertex \(v\) by \(N_k(v)\) and \(N_{\leq k}(v)\), respectively. If \(uv\) is an edge of \(G\), then \(N_{\leq k}(uv)\) is the set \(N_{\leq k}(u) \cup N_{\leq k}(v)\). The \(k\)-th power of \(G\), denoted by \(G^k\), is the graph with the same vertex set as \(G\) in which two vertices are adjacent if their distance is not more than \(k\).

The line graph of a graph \(G\) is the graph \(L\) whose vertex set is \(E(G)\), with two vertices of \(L\) being adjacent in \(L\) if, as edges of \(G\), they share a vertex.

A matching of \(G\) is a set of edges in which no two edges share a vertex. The vertex set \(V(M)\) of a matching \(M\) is the set of vertices incident with an edge in \(M\). The distance \(d_G(e_1, e_2)\) between two edges \(e_1\) and \(e_2\) is the smallest of the distances between a vertex incident with \(e_1\) and a vertex incident with \(e_2\). (Note that in general this is not equal to the distance in the line graph of \(G\).)
If $M$ is a set of edges, then the distance $d(e, M)$ between an edge $e$ and $M$ is the smallest of the distances between $e$ and the edges in $M$.

If $A \subseteq V(G)$, then we write $G[A]$ for the subgraph of $G$ induced by $A$.

By $C_n$ we mean the cycle on $n$ vertices. We say a graph is $C_k$-free if it does not contain $C_k$ as a (not necessarily induced) subgraph. A graph is $(C_4, C_5)$-free if it contains neither $C_4$ nor $C_5$ as a subgraph. The girth of a graph $G$ is the length of a smallest cycle of $G$.

2 Preliminary results

In this section we present some results which will be needed for the proof of our main theorems.

If $v$ and $w$ are two adjacent vertices of a graph of girth at least 6, then the sets of vertices at distance at most two from $v$ or $w$, respectively, in $G - vw$ are disjoint if $G$ has girth at least 6, hence we have the following well-known result (see for example [1]).

Lemma 2.1 ([1]). Let $G$ be a graph of girth at least 6 and minimum degree $\delta$. If $v$ and $w$ are adjacent vertices of $G$, then

$$|N_{\leq 2}(vw)| \geq 2(\delta^2 - \delta + 1).$$

It was shown in [1] that if we relax the girth condition to $G$ having neither 4-cycles nor 5-cycles (so triangles are permitted), then a slightly weaker bound on $|N_{\leq 2}(u) \cup N_{\leq 2}(w)|$ holds.

Lemma 2.2 ([1]). Let $G$ be a $(C_4, C_5)$-free graph with minimum degree $\delta \geq 3$. If $v$ and $w$ are adjacent vertices of $G$, then

$$|N_{\leq 2}(vw)| \geq \begin{cases} 
2\delta^2 - 5\delta + 5 & \text{if } \delta \text{ is even,} \\
2\delta^2 - 5\delta + 7 & \text{if } \delta \text{ is odd.}
\end{cases}$$

We also require bounds on the number of vertices within distance three of a vertex of large degree.

Lemma 2.3 ([1]). Let $G$ be a graph of girth 6, minimum degree $\delta \geq 3$ and maximum degree $\Delta$. If $v$ is a vertex of degree $\Delta$, then

$$|N_{\leq 3}(v)| \geq \Delta \delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2}.$$

Lemma 2.4 ([1]). Let $G$ be a $(C_4, C_5)$-free graph of minimum degree $\delta \geq 3$ and maximum degree $\Delta$. If $v$ is a vertex of degree $\Delta$, then

$$|N_{\leq 3}(v)| \geq \Delta(\delta - 1) + (\delta - 2)\sqrt{\Delta(\delta - 3)} + \frac{3}{2}.$$
Let $G$ be a connected graph with a weight function $c : V(G) \to \mathbb{R} \geq 0$. Then the eccentricity of $G$ with respect to $c$ is defined by

$$EX_c(G) = \sum_{v \in V(G)} c(v)e_G(v).$$

If the total weight of the vertices of $G$ is strictly greater than 0, we define the average eccentricity of $G$ with respect to $c$ by

$$avec_c(G) = \frac{\sum_{v \in V(G)} c(v)e_G(v)}{\sum_{v \in V(G)} c(v)}.$$

We usually denote the total weight of the vertices of $G$ by $N$. Hence, if $N > 0$, we have $avec_c(G) = \frac{EX_c(G)}{N}$.

**Lemma 2.5** ([5]). Let $G$ be a connected, weighted graph with a weight function $c; V(G) \to \mathbb{R} \geq 0$. Let $N = \sum_{v \in V(G)} c(v)$. If $c(v) \geq 1$ for all $v \in V(G)$, then

$$avec_c(G) \leq \text{avec}(P_{\lceil N \rceil}).$$

### 3 Bounds in terms of order and minimum degree

In this section we present the first two of our main results: upper bounds on the average eccentricity of graphs of girth at least six and of $(C_4, C_5)$-free graphs in terms of order and minimum degree. The basic proof strategy follows that used in [8].

**Theorem 3.1.** Let $G$ be a connected graph of order $n$, minimum degree $\delta \geq 3$ and girth at least 6. Then

$$avec(G) \leq \frac{9}{2} \left\lceil \frac{n}{\delta^*} \right\rceil + 8,$$

where $\delta^* = 2\delta^2 - 2\delta + 2$.

**Proof.** We first find a matching $M$ of $G$ as follows. Choose an arbitrary edge $e_1$ of $G$ and let $M = \{e_1\}$. If there exists an edge $e_2$ of $G$ at distance exactly 5 from $M$, then let $M = \{e_1, e_2\}$. If there exists an edge $e_3$ at distance 5 from $M$ then let $M = \{e_1, e_2, e_3\}$. Repeat this step, i.e., successively add edges at distance 5 from $M$ until, after $k$ steps say, each edge of $G$ is within distance 4 of $M$. Let $M = \{e_1, e_2, \ldots, e_k\}$.

The sets $N_{\leq 2}(e_i)$ are pairwise disjoint for $i = 1, 2, \ldots, k$. For $i = 1, 2, \ldots, k$ let $T(e_i)$ be a spanning tree of $N_{\leq 2}(e_i)$ that contains $e_i$ and preserves the distances to $e_i$. Since the sets $N_{\leq 2}(e_i)$ are pairwise disjoint, the trees $T(e_i)$ are vertex disjoint, so the union $\bigcup_{i=1}^k T(e_i)$ forms a subforest $T_1$ of $G$. It follows from the construction of $M$ that for every $i \in \{2, 3, \ldots, k\}$ there exists an edge
Since $T_2 = T_1 + \{f_2, f_3, \ldots, f_k\}$ is a subtree of $G$. Now every vertex of $G$ is within distance 5 from some vertex of $V(M)$. Hence we can extend $T_2$ to a spanning tree $T$ of $G$ that preserves the distances to a nearest vertex in $V(M)$. Since the average eccentricity of any spanning tree of $G$ is not less than the average eccentricity of $G$, it suffices to show that

$$\text{avec}(T) \leq \frac{9}{2} \left\lfloor \frac{n}{\delta^*} \right\rfloor + 8.$$  \hspace{1cm} (3.1)

For every vertex $u \in V(T)$ let $u_M$ be a vertex in $V(M)$ closest to $u$ in $T$. The tree $T$ can be thought of as a weighted tree, where each vertex has weight exactly 1. Informally speaking, we now move the weight of every vertex to the closest vertex in $V(M)$. More precisely, we define a weight function $c : V(T) \to \mathbb{R}^\geq 0$ by

$$c(v) = |\{u \in V(T) \mid u_M = v\}|.$$  

Since $d_T(x, x_M) \leq 5$ for all $x \in V(G)$, we have

$$|\text{avec}_c(T) - \text{avec}(T)| = \left| \frac{1}{n} \sum_{u \in V(T)} c(u)e_T(u) - \frac{1}{n} \sum_{v \in V(T)} e_T(v) \right|$$

$$= \left| \frac{1}{n} \sum_{v \in V(T)} e_T(v_M) - \frac{1}{n} \sum_{v \in V(T)} e_T(v) \right|$$

$$\leq \frac{1}{n} \sum_{v \in V(T)} |e_T(v_M) - e_T(v)|$$

$$\leq \frac{1}{n} \sum_{v \in V(T)} d_T(v_M, v) \leq 5.$$  \hspace{1cm} (3.2)

Note that $c(u) = 0$ if $u \notin V(M)$ and $\sum_{v \in V(G)} c(v) = n$, where $n$ is the order of $G$. We consider the line graph $L$ of $T$ and define a new weight function $\overline{c}$ on $V(L) = E(T)$ by

$$\overline{c}(uv) = \begin{cases} 
  c(u) + c(v) & \text{if } uv \in M, \\
  0 & \text{if } uv \notin M.
\end{cases}$$

Let $uv \in M$. For each vertex $x \in N_{\leq 2}(uv)$, we have $x_M \in \{u, v\}$. Hence, by Lemma 2.1, it follows that for all $uv \in M$,

$$\overline{c}(uv) = c(u) + c(v) \geq |N_{\leq 2}(uv)| \geq \delta^*.$$  \hspace{1cm} (3.3)

We now bound the difference between $\text{avec}_c(T)$ and $\text{avec}(L)$. If $x$ and $y$ are vertices of $T$, and $e_x, e_y$ are edges of $T$ incident with $x$ and $y$, respectively, then it is easy to prove that $d_T(x, y) \leq d_L(e_x, e_y) + 1$ and consequently $e_T(x) \leq e_L(e_x) + 1$. Since the weight of $c$ is concentrated entirely in the vertices in
\[ V(M), \text{ we have} \]
\[
\sum_{v \in V(T)} c(v)e_T(v) = \sum_{uv \in M} c(u)e_T(u) + c(v)e_T(v)
\]
\[
\leq \sum_{uv \in M} \bar{e}(uv)(e_L(uv) + 1)
\]
\[
= \left( \sum_{uv \in M} \bar{e}(uv)e_L(uv) \right) + n.
\]

Division by \( n \) now yields
\[
\text{avec}_c(T) \leq \text{avec}_c(L) + 1. \tag{3.4}
\]

Now, if the distance \( d_T(e_i, e_j) \) between two matching edges \( e_i, e_j \in M \) equals five, then \( d_L(e_1, e_2) \leq 6 \). By the construction of \( M \), every edge \( e_i \in M \) with \( i > 1 \) is thus adjacent in \( L^6 \) to an edge \( e_j \in M \) with \( j < i \). It follows that \( L^6[M] \) is connected. Moreover, we have for all pairs \( e, f \in M \) that
\[
d_L(e, f) \leq 6d_L[M](e, f).
\]

Now, for every edge \( e \) of \( T \) there exists an edge \( f \in M \) such that
\[
d_L(e, f) \leq 5.
\]
It follows that for every \( f \in M \) we have
\[
e_L(f) \leq 6e_L[M](f) + 5,
\]
and thus
\[
\text{avec}_c(L) \leq 6\text{avec}_c(L^6[M]) + 5. \tag{3.5}
\]

To normalise the weights of the vertices of \( L^6[M] \), we now define the new weight function \( \bar{e}' \) by \( \bar{e}'(e) = \frac{\bar{e}(e)}{\delta^*} \) for all \( e \in M \). Clearly,
\[
\text{avec}_{\bar{e}'}(G) = \sum_{v \in V(T)} \bar{e}'(v)e_L[M] = \sum_{v \in V(T)} \bar{e}(v)e_L[M] = \text{avec}_c(G) \tag{3.6}
\]

Observe that \( \bar{e}'(e) \geq 1 \) for all \( e \in M \) by (3.3) and that \( \sum_{v \in V(T)} \bar{e}'(v) = \frac{n}{\delta^*} \). We thus have by Lemma 2.5
\[
\text{avec}_{\bar{e}'}(L^6[M]) \leq \frac{3}{4} \left\lfloor \frac{n}{\delta^*} \right\rfloor - \frac{1}{2} \tag{3.7}
\]

From (3.2), (3.4), (3.5), (3.6) and (3.7) we obtain
\[
\text{avec}(T) \leq \text{avec}_c(T) + 5
\]
\[
\leq \text{avec}_c(L) + 6
\]
\[
\leq 6 \text{avec}_{\bar{e}'}(L^6[M]) + 11
\]
\[
\leq 6 \left( \frac{3}{4} \left\lfloor \frac{n}{\delta^*} \right\rfloor - \frac{1}{2} \right) + 11
\]
\[
= \frac{9}{2} \left\lfloor \frac{n}{\delta^*} \right\rfloor + 8,
\]

which is (3.1), as desired. \( \square \)
We now show that the bound in Theorem 3.1 is sharp apart from an additive constant whenever $\delta - 1$ is a prime power. This holds even if we restrict ourselves to a subclass of graphs of girth at least six, to $C_4$-free bipartite graphs.

**Theorem 3.2.** Let $\delta \in \mathbb{N}$ such that $\delta - 1$ is a prime power. Then there exists an infinite family of bipartite $C_4$-free graphs $G$ of order $n$ and minimum degree $\delta$ such that

$$\text{avec}(G) \geq \frac{9n}{2\delta^*} - 5,$$

where $\delta^* = 2\delta^2 - 2\delta + 2$.

**Proof.** Given $\delta$, let $q = \delta - 1$. Then $q$ is a prime power. Our construction is based on the graph $H_q$ first constructed by Reimann [12]. Let $GF(q)$ be the finite field of order $q$. Consider the 3-dimensional vector space $GF(q)^3$, i.e., the set of all triples of elements of $GF(q)^3$. For $i = 1, 2$ let $V_i$ be the set of all $i$-dimensional subspaces of $GF(q)^3$. Now $H_q$ is defined as the bipartite graph with partite sets $V_1$ and $V_2$, where two vertices $v_1 \in V_1$ and $v_2 \in V_2$ are adjacent if and only if $v_1$ is a subspace of $v_2$. It is easy to verify that $H_q$ has $2(q^2 + q + 1)$ vertices, has diameter three, is $(q + 1)$-regular, and that $H_q$ does not contain any 4-cycles.

Let $\ell \in \mathbb{N}$ with $\ell$ even, and let $uv$ be an edge of $H_q$. Let $H^{1}$ and $H^{\ell}$ be disjoint copies of $H_q$, and let $H^2, H^3, \ldots, H^{\ell-1}$ be disjoint copies of $H_q - uv$. Let $G_{\delta,\ell}$ be the graph obtained from the union of $H^1, H^2, \ldots, H^\ell$ by adding the edges $u^{(i)}u^{(i+1)}$ for every $t \in \{1, 2, \ldots, \ell - 1\}$ where $u^{(i)}$ and $v^{(i)}$ are the vertices of $H_t$ corresponding to the vertices $u$ and $v$, respectively, of $H_q$. Clearly, $G_{\delta,\ell}$ is bipartite and $C_4$-free, so its girth is at least six. Its minimum degree is $\delta$. Since $\delta = q + 1$, the order $n$ of $G_{\delta,\ell}$ is

$$n = 2\ell(q^2 + q + 1) = 2\ell(\delta^2 - \delta + 1) = \ell\delta^*.$$

In order to bound the average eccentricity of $G_{\delta,\ell}$ from below, choose vertices $u^*$ of $H^1$ and $v^*$ of $H^\ell$ with $d(u^*, v^*) = d(u^\ell, v^*) = 3$. Since $H_q$ has girth at least 6, the distance between $u^{(i)}$ and $v^{(i)}$ in $H^i$ is at least 5 for $i = 2, 3, \ldots, \ell - 1$. It is easy to verify that in fact $\text{diam}(H^i) = 5$ for $i = 2, 3, \ldots, \ell - 1$. Hence $\text{diam}(G_{\delta,\ell}) = d(u^*, v^*) = 6\ell - 5 = \frac{3n}{\delta - 1} - 5$. If $w \in V(H^i)$, then $e(w) = d(w, v^*) \geq d(v^*, v^*) = 6(\ell - i) - 2$ if $i \leq \frac{\ell}{2}$, and $e(w) = d(w, u^*) \geq d(u^\ell, v^*) = 6(\ell - 1) - 2$ if $i > \frac{\ell}{2}$. Hence

$$\text{EX}(G_{\delta,\ell}) = \sum_{i=1}^{\ell/2} \sum_{w \in V(H^i)} e(w) + \sum_{i=\ell/2+1}^{\ell} \sum_{w \in V(H^i)} e(w) \geq \sum_{i=1}^{\ell/2} \delta^* [6(\ell - i) - 2] + \sum_{i=\ell/2+1}^{\ell} \delta^* [6(\ell - 1) - 2] = \delta^* \left( \frac{9}{2} \ell^2 - 5\ell \right).$$

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Since \( n = \ell \delta^* \), division by \( n \) yields that

\[
\avec(G_{\delta, \ell}) \geq \frac{\delta^*(\frac{9\ell^2}{2} - 5\ell)}{\ell \delta^*} = \frac{9\ell}{2\delta^*} - 5 = \frac{9n}{2\delta^*} - 5,
\]

as desired.

If we relax the condition on \( G \) to have girth at least six to \( G \) being \((C_4, C_5)\)-free, we obtain a bound very similar to Theorems 3.1. We omit the proof as it is almost identical to that of Theorems 3.1.

**Theorem 3.3.** Let \( G \) be a connected \((C_4, C_5)\)-free graph of order \( n \) and minimum degree \( \delta \geq 3 \). Then

\[
\avec(G) \leq \frac{9}{2} \left\lfloor \frac{n}{\delta^*} \right\rfloor + 8,
\]

where \( \delta^* = 2\delta^2 - 5\delta + 5 \) if \( \delta \) is even, and \( \delta^* = 2\delta^2 - 5\delta + 7 \) if \( \delta \) is odd.

We do not know if the bound in Theorem 3.3 is sharp. But since \( \lim_{\delta \to \infty} \frac{\delta^*}{\delta} = 1 \), it is clear from Theorem 3.2 that for large \( \delta \) the coefficient of \( n \) in the bound is close to being optimal.

### 4 Bounds in terms of order, minimum degree and maximum degree

We now show that the bound in Theorem 3.3 can be improved if \( G \) contains a vertex of large degree. The proof of this bound follows broadly that of Theorem 3.1, and also borrows ideas from [9], but several modifications and additional arguments are required.

**Theorem 4.1.** Let \( G \) be a connected graph of order \( n \), minimum degree \( \delta \geq 3 \), maximum degree at least \( \Delta \) and girth at least 6. Then

\[
\avec(G) \leq \frac{n - \Delta^*}{2\delta^*} \frac{9n + 3\Delta^*}{n} + 21,
\]

where \( \delta^* = 2(\delta^2 - \delta + 1) \) and \( \Delta^* = \Delta \delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2} \).

**Proof.** Let \( v_1 \) be a vertex of degree \( \Delta \) and let \( e_1 \) be an edge incident with \( v_1 \). We first find a matching \( M \) of \( G \) as follows. Let \( M = \{e_1\} \). If there exists an edge \( e_2 \) with \( d_G(e_1, e_2) = 6 \), add \( e_2 \) to \( M \). Assume that \( M = \{e_1, e_2, \ldots, e_{i-1}\} \). If there exists an edge \( e_i \) satisfying

(i) \( d_G(e_i, e_1) \geq 6 \),
(ii) \( \min\{d_G(e_i, e_j) \mid j = 2, 3, \ldots, i-1\} \geq 5 \),
(iii) we have equality in (i) or (ii) or both,

then add \( e_i \) to \( M \). We repeat this process until, after \( k \) steps say, every edge not in \( M_0 \cup \{e_1\} \) is within distance 5 of \( e_1 \), or within distance 4 of an edge in \( M_0 \). Let \( M = \{e_1, e_2, \ldots, e_k\} \).
The sets $N_{\leq 3}(e_1)$ and $N_{\leq 2}(e_i)$ for $i = 2, 3, \ldots, k$ are pairwise disjoint. Let $T(e_1)$ be a spanning tree of $N_{\leq 3}(e_1)$ that contains $e_1$ and preserves the distances to $e_1$. For $i = 2, 3, \ldots, k$ let $T(e_i)$ be a spanning tree of $N_{\leq 2}(e_i)$ that contains $e_i$ and preserves the distances to $e_i$. Then the trees $T(e_i)$, $i = 1, 2, \ldots, k$, are vertex disjoint, so the union $T(e_1) \cup \bigcup_{i=2}^{k} T(e_i)$ forms a subforest $T_1$ of $G$. It follows from the construction of $M$ that for every $i \in \{2, 3, \ldots, k\}$ there exists an edge $f_i$ in $G$ joining a vertex in $T(e_i)$ to a vertex in $T(e_j)$ for some $j$ with $1 \leq j < i$. Hence $T_2 := T_1 + \{f_2, f_3, \ldots, f_k\}$ is a subtree of $G$. We extend $T_2$ to a spanning tree $T$ of $G$ that preserves the distances to a nearest vertex in $V(M)$. In $T$ every vertex is within distance 6 from some vertex in $V(M)$. Since the average eccentricity of any spanning tree of $G$ is not less than the average eccentricity of $G$, it suffices to show that

$$
\text{avec}(T) \leq \frac{n - \Delta^*}{2\Delta^*} \frac{9n + 3\Delta^*}{n} + 21. 
$$

(4.1)

For every vertex $u \in V(T)$ let $u_M$ be a vertex in $V(M)$ closest to $u$ in $T$. We may assume that $u_M$ is a vertex incident with $e_1$ whenever $u \in V(T(e_i))$. As in the proof of Theorem 3.1 we define a weight function $c : V(T) \to \mathbb{R}^\geq 0$ by

$$
c(v) = |\{u \in V(T) \mid u_M = v\}|.
$$

Now $d_T(x, x_M) \leq 6$ for all $x \in V(G)$. The same arguments as in the proof of Theorem 3.1 show that

$$
\text{avec}(T) \leq \text{avec}_c(T) + 6.
$$

(4.2)

We consider the line graph $L$ of $T$ and define a new weight function $\overline{c}$ on $V(L) = E(T)$ by

$$
\overline{c}(uv) = \begin{cases} 
c(u) + c(v) & \text{if } uv \in M, \\
0 & \text{if } uv \notin M.
\end{cases}
$$

As in the proof of Theorem 3.1 we have

$$
\text{avec}_c(T) \leq \text{avec}_{\overline{c}}(L) + 1.
$$

(4.3)

Let $H$ be the graph obtained from $L^6[M]$ by joining $e_1$ to every edge $e_i \in M$ for which $d_L(e_1, e_i) \leq 7$. Such edges $e_i$ exist since by the construction of $M$ we have $d_T(e_1, e_2) = 6$ and thus $d_L(e_1, e_2) \leq 7$. Essentially the same argument as in the proof of Theorem 3.1 shows that $H$ is connected. Let $e, f \in M$ and let $P$ be a shortest path from $e$ to $f$ in $H$ of length $\ell$ say. First assume that $P$ does not pass through $e_1$. Then each edge of $P$ yields a path in $L$ of length 6, so $P$ yields a path from $e$ to $f$ of length at most $6\ell$. Now assume that $P$ passes through $e_1$. Then each edge on $P$ not incident with $e_1$ yields a path of length at most 6 in $L$, while each edge of $P$ incident with $e_1$ yields a path of length at most 7 in $L$. Since $P$ has at most two edges incident with $e_1$, $P$ yields a path of length at most $6\ell + 2$. Hence

$$
d_L(e, f) \leq 6d_H(e, f) + 2.
$$

(4.4)
Now, for every edge \( f \in E(T) \) there exists an edge \( g \in M \) such that \( d_L(f, g) \leq 6 \). Hence \( e_L(e) \leq 6 \) for every \( e \in M \), and thus

\[
\text{avec}(L) \leq 6\text{avec}(H) + 8. \tag{4.5}
\]

As in [3.3] we have

\[
\overline{\tau}(e_2), \overline{\tau}(e_3), \ldots, \overline{\tau}(e_k) \geq \delta^*. \tag{4.6}
\]

By Lemma 2.3 we have

\[
\overline{\tau}(e_1) \geq |N \leq 3(v_1)| \geq \Delta^*. \tag{4.7}
\]

Since

\[
\sum_{e \in H} \overline{\tau}(e) = \sum_{v \in V(T)} c_H(v) = n,
\]

we have

\[
|M| \leq n \delta^* + 2. \tag{4.8}
\]

We now modify the weight function \( \overline{\tau} \) to obtain a new weight function \( \overline{\tau}' \). We define

\[
\overline{\tau}'(e_i) = \begin{cases} 
\frac{\overline{\tau}(e_i) - \Delta^* + \delta^*}{\delta^*} & \text{if } i = 1, \\
\frac{\overline{\tau}(e_i)}{\delta^*} & \text{if } i \geq 2.
\end{cases}
\]

Clearly, \( \sum_{v \in V(H)} \overline{\tau}'(v) = \frac{n - \Delta^* + \delta^*}{\delta^*} = N^* \). Hence

\[
\text{avec}(H) = \frac{EX_{\overline{\tau}}(H)}{N^*} = \frac{\frac{1}{\delta^*} \left( \sum_{u \in M} \overline{\tau}(u)e_H(u) - (\Delta^* - \delta^*)e_H(e_1) \right)}{N^*} = \frac{EX_{\overline{\tau}}(H) - (\Delta^* - \delta^*)e_H(e_1)}{n - \Delta^* + \delta^*} = \frac{n}{n - \Delta^* + \delta^*} \text{avec}(H) - \frac{\Delta^* - \delta^*}{n - \Delta^* + \delta^*} e_H(e_1). \tag{4.8}
\]

Rearranging yields

\[
\text{avec}(H) = \frac{n - \Delta^* + \delta^*}{n} \text{avec}(H) + \frac{\Delta^* - \delta^*}{n} e_H(e_1). \tag{4.9}
\]

We now bound the two terms on the right hand side of (4.9) separately. Note that \( \overline{\tau}'(e_i) \geq 1 \) for all \( e_i \in M \). Applying Lemma 2.5 we obtain

\[
\text{avec}(H) \leq \text{avec}(P_{N^*}) = \frac{3}{4} \lfloor N^* \rfloor - \frac{1}{2}.
\]

Now \( \lfloor N^* \rfloor = \lfloor \frac{n - \Delta^* + \delta^*}{\delta^*} \rfloor < \frac{n - \Delta^*}{\delta^*} + 2 \). Hence

\[
\text{avec}(H) < \frac{3(n - \Delta^*)}{4\delta^*} + 1. \tag{4.10}
\]

To bound \( e_H(e_1) \) note that \( H \) has order \( |M| \). Now \( |M| = \sum_{e \in M} 1 \leq \sum_{e \in M} \overline{\tau}(e_i) = \frac{n - \Delta^* + \delta^*}{\delta^*} \). Hence

\[
e_H(e_1) \leq |M| - 1 = \frac{n - \Delta^* + \delta^*}{\delta^*} - 1 = \frac{n - \Delta^*}{\delta^*}. \tag{4.11}
\]
From (4.9), (4.10) and (4.11) we get, after some calculations,
\[
\text{avec}(H) < \frac{n - \Delta^* + \delta^* (3(n - \Delta^*) \frac{3(n - \Delta^*)}{4 \delta^*} + 1)}{n} + \frac{\Delta^* - \delta^* n - \Delta^*}{\delta^*} = \frac{n - \Delta^* 3n + \Delta^*}{4 \delta^*} + \frac{3n - 3\Delta^* + 4 \delta^*}{4n} \leq \frac{n - \Delta^* 3n + \Delta^*}{4 \delta^*} + 1. \tag{4.12}
\]

Applying the inequalities (4.2), (4.3), (4.5) and (4.12) we obtain
\[
\text{avec}(T) \leq \text{avec}(T) + 6 \leq \text{avec}(L) + 7 \leq 6 \text{avec}(H) + 15 < \frac{n - \Delta^* 9n + 3\Delta^*}{2 \delta^*} + 21,
\]
as desired.

The following Theorem demonstrates that the bound in Theorem 4.1 is sharp if \(\delta - 1\) is a prime power, except for an additive term \(O(\sqrt{\Delta})\).

**Theorem 4.2** ([4]). Let \(\delta, k \in \mathbb{N}\) such that \(\delta - 1\) is a prime power and \(k \geq 7\). Then there exist a bipartite, \(C_4\)-free graph \(G^{\delta,k}\) of minimum degree \(\delta\), maximum degree \(\Delta = (q^k - 1)(q^k - 1 - 1) - \frac{1}{q}\), where \(q = \delta - 1\), whose order \(n_{\delta,k}\) satisfies
\[
\Delta^* \leq n_{\delta,k} \leq \Delta^* + 2 \sqrt{\Delta(\delta - 2)} + \frac{1}{2}.
\]

We make use of the fact that the graph in Theorem 4.2 has diameter at least 3, which is easy to check from the construction (see [1]). In the proof of Theorem 4.3 we make use of a graph \(G\), which was first described in [1].

**Theorem 4.3.** Let \(\delta \in \mathbb{N}\) be such that \(\delta - 1\) is a prime power. Then there exist infinitely many connected graphs \(G\) of minimum degree \(\delta\) and girth 6 with
\[
\text{avec}(G) > \frac{n - \Delta^* 9n + 3\Delta^*}{2 \delta^*} n - O(\sqrt{\Delta}),
\]
where \(\Delta\) is the maximum degree and \(n\) the order of \(G\).

**Proof.** Let \(q := \delta - 1\), so \(q\) is a prime power. Let \(k, \ell \in \mathbb{N}\) with \(\ell\) even and \(\ell\) sufficiently large. Consider the graph \(G^{\delta,k}\) in Theorem 4.2 and let \(u^1\) be a vertex of degree \(\Delta\), and let \(v^1\) be a vertex at distance three from \(u^1\).

As in the construction of the graph \(G^{\delta,k}\) in Theorem 4.2 let \(H^2, H^3, \ldots, H^\ell\) be isomorphic to \(H_q\), but let \(H^1\) be the graph \(G^{\delta,k}\). Denote the resulting graph by \(G\). It is easy to verify that \(G\) has minimum degree \(\delta\) and maximum degree \(\Delta\), and that its diameter is \(d(u_1, v_\ell) = 6\ell - 3\). For the order \(n\) of \(G\) we have
\[
n = n(G^{\delta,k}) + (\ell - 1)n(H_q) = n_{\delta,k} + (\ell - 1)\delta^*. \tag{4.13}
\]
We now bound the average eccentricity of $G$ from below. For $i \in \{2, 3, \ldots, \ell\}$ let $V(i)$ be the vertex set of $H^i$, and for $i = 1$ let $V(1)$ be a set of $\delta^*$ vertices of $H^1$. Let $x \in V(i)$. If $i \leq \frac{\ell}{2}$, then
\[
eq_G(x) \geq d_G(x, v^\ell) \geq d_G(v^i, v^\ell) = 6(\ell + 1 - i) - 8,
\]
and if $i > \frac{\ell}{2}$, then
\[
eq_G(x) \geq d_G(x, u^1) \geq d_G(u^i, u^1) = 6i - 8.
\]
The $(n_{\delta,k} - \delta^*)$ vertices in $V(H^1) - V(1)$ have eccentricity at least $6\ell - 8$. Hence,
\[
EX(G) = \left( \sum_{i=1}^{\ell/2} + \sum_{i=\ell/2+1}^\ell \right) \sum_{x \in V(i)} e_G(x) + \sum_{x \in V(H^1) - V(1)} e_G(x) \\
\geq \left( \sum_{i=1}^{\ell/2} \delta^*[6(\ell + 1 - i) - 8] \right) + \left( \sum_{i=\ell/2+1}^\ell \delta^*[6i - 8] \right) + (n_{\delta,k} - \delta^*)(6\ell - 8) \\
= \left( \frac{9}{2} \ell^2 - 5\ell \delta^* + (n_{\delta,k} - \delta^*)(6\ell - 8) \right).
\]
Now $\ell = \frac{n - n_{\delta,k}}{\delta^*} + 1$ by (4.13). Substituting this and dividing by $n$ yields, after simplification,
\[
avec(G) \geq \frac{n - n_{\delta,k}}{2\delta^*} \frac{9n + 3n_{\delta,k}}{n} - 2 + \frac{3\delta^*}{2n} \\
\geq \frac{n - n_{\delta,k}}{2\delta^*} \frac{9n + 3n_{\delta,k}}{n} - 2.
\]
Now let $\varepsilon = n_{\delta,k} - \Delta^*$. Replacing $n_{\delta,k}$ by $\Delta^* + \varepsilon$ in the above lower bound, we obtain
\[
avec(G) > \frac{n - \Delta^* - \varepsilon}{2\delta^*} \frac{9n + 3\Delta^* + 3\varepsilon}{n} - 2 \\
= \frac{n - \Delta^*}{2\delta^*} \frac{9n + 3\Delta^*}{n} - \frac{\varepsilon}{2\delta^*} (6n + 6\Delta^* + 3\varepsilon) - 2.
\]
Since $6n + 6\Delta^* + 3\varepsilon \leq 12n$, and since $0 \leq \varepsilon \leq 2\sqrt{\Delta(\delta - 2)} + \frac{1}{2}$ by Theorem 4.2 we have, for constant $\delta$ and large $n$ and $\Delta$,
\[
avec(G) > \frac{n - \Delta^*}{2\delta^*} \frac{9n + 3\Delta^*}{n} - O(\sqrt{\Delta}),
\]
as desired.

Theorem 4.1 generalises Theorem 3.1 in the sense that it implies (by setting $\Delta = \delta$) a bound that differs from Theorem 3.1 only by having a weaker additive constant.

As in the previous section, a bound slightly weaker than that in Theorem 4.1 holds for all $(C_4, C_5)$-free graphs. We omit the proof, which is very similar to the proof of Theorem 4.1.

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Theorem 4.4. Let $G$ be a connected $(C_4, C_5)$-free graph of order $n$, minimum degree $\delta \geq 3$ and maximum degree $\Delta$. Then

$$\text{avec}(G) \leq \frac{n - \Delta^o}{2\delta^o} \frac{9n + 3\Delta^o}{n} + 21,$$

where $\delta^o = 2\delta^2 - 5\delta + 5$ if $\delta$ is even, $\delta^o = 2\delta^2 - 5\delta + 7$ if $\delta$ is odd, and $\Delta^o = \Delta(\delta - 1) + (\delta - 2)\sqrt{\Delta(\delta - 3)} + \frac{3}{2}$.

We do not know if the bound in Theorem 4.4 is sharp.

References

[1] A.S.A. Alochukwu, P. Dankelmann, Upper bounds on the diameter and radius of graphs of girth 6 and $(C_4, C_5)$-free graphs. (submitted).

[2] P. Ali, P. Dankelmann, M.J. Morgan, S. Mukwembi, T. Vetrík, The average eccentricity, spanning trees of plane graphs, size and order. *Utilitas Math.* 107 (2018), 37-49.

[3] F. Buckley, F. Harary, *Distance in Graphs*. Addison-Wesley, Redwood City, California (1990).

[4] R.M. Casablanca and P. Dankelmann, Distance and eccentric sequences to bound the Wiener index, Hosoya polynomial and the average eccentricity in the strong product of graphs. *Discrete Appl. Math.* 263 (2019), 105-117.

[5] P. Dankelmann, W. Goddard, C.S. Swart, The average eccentricity of a graph and its subgraphs. *Util. Math.* 41 (2004), 41-51.

[6] P. Dankelmann and S. Mukwembi, Upper bounds on the average eccentricity. *Discrete Appl. Math.* 167 (2014), 72-79.

[7] P. Dankelmann, F.J. Osaye, Average eccentricity, $k$-packing and $k$-domination in graphs. *Discrete Math.* 342 vol. 5 (2019), 1261-1274. https://doi.org/10.1016/j.disc.2019.01.004

[8] P. Dankelmann, S. Mukwembi, F.J. Osaye, B.G. Rodrigues, Upper bounds on the average eccentricity of $K_3$-free and $C_4$-free graphs. Discrete Applied Mathematics 270 (2019), 106-114.

[9] P. Dankelmann, F.J. Osaye Average eccentricity, minimum degree and maximum degree in graphs. (submitted)

[10] H. Darabi, Y. Alizadeh, S. Klavzar, K.C. Das, On the relation between Wiener index and eccentricity of a graph. (Manuscript 2018)

[11] K.C. Das, A.D. Maden, I.N. Cangül, A.S. Çevik, On average eccentricity of graphs. Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 87 no. 1 (2017), 23-30.
[12] Z. Du, A. Ilić, On AGX conjectures regarding average eccentricity. MATCH Commun. Math. Comput. Chem. 69 (2013), 597-609.

[13] Z. Du, A. Ilić, A proof of the conjecture regarding the sum of the domination number and average eccentricity. Discrete Appl. Math. 201 (2016), 105-113.

[14] Z. Du, Further results regarding the sum of domination number and average eccentricity. Applied Mathematics and Computation 294 (2017), 299-309.

[15] C. He, S. Li, J. Tu, Edge-grafting transformations on the average eccentricity of graphs and their applications. Discrete Appl. Math. 238 (2018), 95-105.

[16] A. Ilić, On the extremal properties of the average eccentricity. Computers and Mathematics with Applications 64 no. 9 (2012), 2877-2885.

[17] M. Liang, B. Liu, A proof of two conjectures on the Randić index and the average eccentricity. Discrete Math. 312 (2012), 2446-2449.

[18] B. Ma, B. Wu, W. Zhang, Proximity and average eccentricity of a graph. Inform. Process. Lett. 112 (2012), 392-395.

[19] Reiman, I., Über ein Problem von K. Zarankiewicz. Acta Math. Hungar. 9, 269-273 (1958).

[20] H. Smith, L.A. Székely, H. Wang, Eccentricity sum in trees. Discrete Appl. Math. 207 (2016), 120-131.

[21] Y. Tang, B. Zhou, On average eccentricity. MATCH Commun. Math. Comput. Chem. 67 (2012), 405-423.

[22] Y. Tang, D.B. West, Lower bounds for eccentricity-based parameters of graphs. Manuscript (2019). [http://faculty.math.illinois.edu/~west/pubs/avgecc.pdf]