SUBDIFFERENTIAL FORMULAE FOR THE SUPREMUM OF AN ARBITRARY FAMILY OF FUNCTIONS

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Abstract. This work provides calculus for the Fréchet and limiting subdifferential of the pointwise supremum given by an arbitrary family of lower semicontinuous functions. We start our study showing fuzzy results about the Fréchet subdifferential of the supremum function. Posteriorly, we study in finite- and infinite-dimensional settings the limiting subdifferential of the supremum function. Finally, we apply our results to the study of the convex subdifferential; here we recover general formulae for the subdifferential of an arbitrary family of convex functions.

Key words. variational analysis and optimization, supremum functions, calculus rules, subdifferentials.

AMS subject classifications. 49J52, 49J53, 49Q10

1. Introduction. Many mathematical models concern the study of a constraint minimization problem represented by

\[
\text{minimize } g \quad \text{subject to} \\
f_t(x) \leq 0, \quad \text{for all } t \in T \text{ and } x \in X,
\]

where \( T \) is an index set and the function \( g \) and \( f_t \) are defined in some space \( X \). In these applications the (possibly nonsmooth) pointwise supremum \( f := \sup f_t \) plays a crucial role in solving this optimization problem, because the constraint \( f_t(x) \leq 0 \) for all \( t \in T \) can be recast as one single inequality constraint passing to the supremum function \( f := \sup_T f_t \). For that reason, understanding the subdifferential of the function \( f \) is decisive in computing necessary optimality conditions. Problem (1) has been widely studied when the index set \( T \) is finite, and nowadays these results are available in numerous monographs of optimization and variational analysis (see for instance [2,3,6,7,25–27,36]).

When the set \( T \) is infinite (1) is understood to be a problem of infinite programming, and when the space \( X \) is finite-dimensional the more precise terminology of semi-infinite programming appears due to the finite-dimensionality of the variable \( x \in X \) and the infinitude of \( T \). These classes of problems have been studied over the last sixty years by many researchers for the reason that several models in science can be represented as a constraint of the state or the control of a system during a period of time or in a region of the space. Within this framework, a classical assumption is the compactness of the set \( T \) together with some hypothesis about the continuity of the function \((t,x) \to f_t(x)\) and its gradient; in this context the set of active indices \( T(x) := \{ t \in T : f_t(x) = f(x) \} \) performs an important part in the study (see, e.g., [24]).

More recent papers have studied the convex subdifferential of the supremum function when \( T \) is an arbitrary index set and \( \{ f_t : t \in T \} \) is an arbitrary family of (possibly non-smooth) convex functions (see, for example, [8,12–14,23,37] and the reference therein). Due to the possible emptiness of the set of active indices at a given point \( x \), the authors have considered the \( \varepsilon \)-active index set \( T_\varepsilon(x) := \{ t \in T : f_t(x) \geq f(x) - \varepsilon \} \). In these works researchers have successfully calculated the convex subdifferential of...
the supremum function without any qualification about the data functions \( f_t \)'s, using the set of \( \varepsilon \)-active indices, the \( \varepsilon \)-subdifferential of the data and the normal cone of the domain of the function \( f \), all of which are well-known concepts in convex analysis.

When the data functions \( \{ f_t \}_{t \in T} \) are non-convex and non-smooth, but uniformly locally Lipschitz at point \( \bar{x} \), which means, there are constants \( k, \varepsilon > 0 \) such that

\[
|f_t(x) - f_t(y)| \leq k \|x - y\|, \forall x \in B(\bar{x}, \varepsilon), \forall t \in T, \tag{2}
\]

we can refer to the classical result about the upper-estimate of the Clarke subdifferential of the function \( f \) at the point \( \bar{x} \) (see [6, Theorem 2.8.2]). It is important to recall that in this result the set \( T \) is compact and the function \( t \rightarrow f_t(x) \) is upper-semi continuous for each \( x \in B(\bar{x}, \varepsilon) \). Recently, in [28] (see also [29]) the authors studied the limiting subdifferential of the function \( f \) at \( \bar{x} \); they assumed that \( T \) is an arbitrary index and the functions \( \{ f_t \}_{t \in T} \) satisfy (2). They provided new upper-estimates and improvements of the mentioned result relative to the Clarke subdifferential. Using these calculus rules they derived optimality conditions for infinite and semi-infinite programming.

However, as far as we know, the literature does not provide an upper-estimate for the subdifferential of an arbitrary family of functions \( \{ f_t : t \in T \} \). This observation motivates our research to derive general upper estimations for the subdifferential of the supremum function under an arbitrary index set \( T \) and without the uniform locally Lipschitz condition. The aim of this work is to extend the results of [28] and give general formulae for the subdifferential of the supremum function, in order to apply them to derive necessary optimality conditions for general problems in the framework of infinite programming. The main motivation for considering an arbitrary family of functions comes from the fact that indicators of sets are commonly used in variational analysis to study constraints and set-valued maps related with optimization problems (for example, stability of optimization problems and differentiability of set-valued maps) and they cannot, at least directly, be assumed to be locally Lipschitz. Furthermore, this approach allows us to also study the convex case, and recover general formulae in the convex case, which in particular shows a unifying approach to the study of the subdifferential of the supremum function. For the sake of brevity, we will confine ourselves to extending the results of [28], keeping in mind our applications for a future work.

The rest of the paper is organized as follows: In Section 2 we summarize the notation that we use in this paper, which is classical in variation analysis. In Subsection 3.1 we establish basic properties about the Fréchet subdifferential. We begin Subsection 3.2 giving the definition of robust infimum (see Definition 3.3), this notion fits perfectly with our purpose. It can be understood as a bridge, which allows us to express the subgradient of the supremum function as robust minimum of perturbed functions, when the family \( \{ f_t : t \in T \} \) is an increasing family of functions. Nevertheless, the increasing property of the functions can be obtained considering the max functions over all finite sets of \( T \) (see Theorem 3.8). In Section 4, where the main results are established, we study the limiting subdifferential, this section is divided into two subsections. First, we consider a finite-dimensional space; in this framework we establish a technical result (see Lemma 4.1), which can be applied to several results, but for simplicity we choose only one setting (see Theorem 4.2), where we provide a convex upper-estimation of the subdifferential. Second, we consider an infinite-dimensional Asplund space. This subsection starts with a result concerning a fuzzy calculus rule for the normal cone of an intersection of an arbitrary family.
of sets (see Theorem 4.5). Later, we use the definition of sequential normal epi-
compactness together with some results of separable reduction to get Theorem 4.8;
this gives as a consequence a generalization of [28, Theorem 3.2] (see Theorem 4.9),
for non-necessarily uniformly Lipschitz functions. Finally, in Section 5 we apply our
results to the convex subdifferential, that is, when the functions \( f_t \) are convex. In
this section we get new results and also we recover the general formula of Hantoute-
López-Zálinescu [14, Theorem 4].

2. Notation. Throughout the paper and unless we stipulate to the contrary, we
adopt the following notation, \((X, \| \cdot \|)\) will be an Asplund space (i.e., every separable
subspace of \( X \) has separable dual) and \( X^* \) its topological dual, with its norm denoted
by \( \| \cdot \|_\ast \). The bilinear form \( \langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R} \) is given by
\( \langle x^*, x \rangle := x^*(x) \). The weak* topology on \( X^* \) is denoted by \( w(X^*, X) \) (w*,
for short). The set of all convex, balanced and closed neighborhoods of a point \( x \) with respect to the topology \( \tau \) is
denoted by \( \mathcal{N}_\tau(x) \) (\( \mathcal{N}_\tau \) for short). We will write \( \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\} \) and we adopt
the conventions \( 1/\infty = 0 \), \( 0 \cdot \infty = 0 = 0 \cdot (-\infty) \) and \( \infty + (-\infty) = (-\infty) + \infty = \infty \).

The closed unit ball in \( X \) and \( X^* \) are denoted by \( \mathbb{B} \) and \( \mathbb{B}^* \) respectively. For a
point \( x \in X \) (resp. \( x^* \in X^* \)) and a number \( r \geq 0 \) we set \( \mathbb{B}(x, r) := x + r\mathbb{B} \) (resp.
\( \mathbb{B}^*(x^*, r) = x^* + r\mathbb{B}^* \)). For a function \( f : X \to \overline{\mathbb{R}} \) the set \( \mathbb{B}(x, f, r) \) is defined as
the set of all \( x' \in \mathbb{B}(x, r) \) such that \( |f(x) - f(x')| \leq r \). The symbol \( x' \xrightarrow{\tau} x \) means
\( x' \to x \) and \( f(x') \to f(x) \); we avoid some misunderstandings about the topology \( \tau \)
considered in the last convergence using the notation \( x' \xrightarrow{\tau} x \) which emphasizes that
the convergence \( x' \to x \) is with respect to the topology \( \tau \).

We denote by \( \text{int}(A), \overline{A}, \text{co}(A) \) and \( \text{co}(A) \), the interior, the closure, the convex hull and the closed convex hull of \( A \), respectively. The affine subspace generated by
\( A \) is denoted by \( \text{aff}(A) \). The polar set and annihilator of \( A \) are defined by
\[
\mathcal{A}^\circ := \{ x^* \in X^* \mid \langle x^*, x \rangle \leq 1, \forall x \in A \},
\]
\[
\mathcal{A}^\perp := \{ x^* \in X^* \mid \langle x^*, x \rangle = 0, \forall x \in A \},
\]
respectively. The indicator function of \( A \) is defined as \( \delta_A(x) := 0 \), if \( x \in A \) and
\( \delta_A(x) = +\infty \), if \( x \notin A \).

Let \( f : X \to \overline{\mathbb{R}} \) be a lower semicontinuous (lsc) function finite at \( x \). Then
\[
\hat{\partial} f(x) := \{ x^* \in X^* \mid \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\|h\|} \geq 0 \},
\]
is called the Fréchet (or regular) subdifferential of \( f \) at \( x \).

The limiting (or Mordukhovich, or basic) subdifferential and the singular subdi-
fferential can be defined as
\[
\partial f(x) := \{ w^* \cdot \lim x^*_n : x^*_n \in \hat{\partial} f(x_n), \text{ and } x_n \xrightarrow{\tau} x \},
\]
\[
\partial^\infty f(x) := \{ w^* \cdot \lim \lambda_n x^*_n : x^*_n \in \hat{\partial} f(x_n), \text{ } x_n \xrightarrow{\tau} x \text{ and } \lambda_n \to 0^+ \},
\]
respectively (see, e.g., [2, 3, 25, 27] for more details).

If \( |f(x)| = +\infty \), we set \( \hat{\partial} f(x) := \emptyset \) for any of the previous subdifferentials. It is
important to recall that when \( f \) is convex proper and lsc all of these subdifferentials
coincide with the classical subdifferential of convex analysis
\[
\partial f(x) := \{ x^* \in X^* : \langle x^*, y - x \rangle \leq f(y) - f(x), \forall y \in X \}.
\]
For any set $A$, the Fréchet (or Regular) and the limiting (or Mordukhovich, or basic) normal cone of $A$ at $x$ are given by $\bar{N}(x, A) = \partial A(x)$ and $N(x, A) = \partial A(x)$, respectively.

Consider a set $T$ and a family of functions $\{f_t\}_{t \in T} \subseteq \mathbb{R}^T$, we define the supremum function $f : X \to \mathbb{R}$ by

$$f(x) := \sup_{t \in T} f_t(x), \ \forall x \in X$$

The symbol $\mathcal{P}_T(T)$ denotes the set of all $F \subseteq T$ such that $F$ is finite. For $F \in \mathcal{P}_T(T)$ we denote $f_F(x) := \max_{s \in F} f_s(x)$.

Following the notation of [28], $\mathbb{R}^T$ is defined as the space of all multipliers $\lambda = (\lambda_t)$ and $\mathbb{R}^\mathbb{R}$ denotes the set of all $\lambda \in \mathbb{R}^T$ such that $\lambda_t \neq 0$ for finitely many $t \in T$; by the symbol $\# \lambda$ we denote the cardinal number of $\sup \lambda$. The generalized simplex on $T$ is the set $\Delta(T) := \{ \lambda \in \mathbb{R}^T : (\lambda_t) \geq 0 \text{ and } \sum_{t \in T} \lambda_t = 1 \}$. For a point $\bar{x}$ and $\varepsilon \geq 0$, the set of $\varepsilon$-active indices at $\bar{x}$ is denoted by $T_\varepsilon(\{f_t\}_{t \in T}, \bar{x}) := \{ t \in T : f(\bar{x}) \leq f_t(\bar{x}) + \varepsilon \}$ ($T_\varepsilon(\bar{x})$ for short), meanwhile the set of all $\varepsilon$-active sets at $\bar{x}$ is denoted by $\hat{T}_\varepsilon(\{f_t\}_{t \in T}, \bar{x}) := \{ F \in \mathcal{P}_T(T) : f(\bar{x}) \leq f_F(\bar{x}) + \varepsilon \}$ ($\hat{T}_\varepsilon(\bar{x})$ for short) and finally, we define

$$\Delta(T, \{f_t\}_{t \in T}, \bar{x}, \varepsilon) := \left\{ (\lambda_t) \in \mathbb{R}^T : \begin{array}{l}
\lambda_t \geq 0 \text{ for all } t \in T,
\lambda_t \leq \varepsilon, \forall t \in T \setminus T_\varepsilon(\bar{x})
\text{and } |\sum_{t \in T} \lambda_t - 1| \leq \varepsilon
\end{array} \right\}$$

($\Delta(T, \bar{x}, \varepsilon)$ for short). When $T$ is a directed set ordered by $\preceq$, which means $(T, \preceq)$ is an ordered set and for every $t_1, t_2 \in T$ there exists $t_3 \in T$ such that $t_1 \preceq t_3$ and $t_2 \preceq t_3$, we say that the family of functions is increasing provided that for all $t_1, t_2 \in T$

$$t_1 \preceq t_2 \implies f_{t_1}(x) \leq f_{t_2}(x), \ \forall x \in X.$$

3. Subdifferential of supremum function. In this section we establish some fuzzy calculus rules for the Fréchet subdifferential of the supremum function. First we start subsection 3.1 recalling some basic properties of this subdifferential. Posteriorly, we use the aforementioned properties to get fuzzy calculus rules for the supremum function of an arbitrary family of lower-semicontinuous functions.

3.1. Basic properties of the Fréchet subdifferential. This section is devoted to stipulating some simple properties of the Fréchet subdifferentials. First, let us recall the following relation between the subdifferential and the normal cone to the epigraph of the function; a point $x^*$ belongs to $\partial f(x)$ if and only if $(x^*, -1) \in \bar{N}((x, f(x)), epi f)$.

Now we write the next result, which is useful to understand Fréchet normal vectors to the epigraph of a function in terms of subgradients in the Fréchet subdifferential, this result is well-known and we refer to [3, 20, 25, 27, 31, 34] for the proof.

**Proposition 3.1.** Let $f : X \to \mathbb{R}$ be a proper lsc function and consider a point $(x^*, 0) \in \bar{N}(epi f, (x, f(x)))$. Hence for any $\varepsilon > 0$ there are points $y \in X$ and $(y^*, \lambda) \in N(epi f, (y, f(y)))$ such that $\lambda \in (-\varepsilon, 0)$, $\|y - x\| \leq \varepsilon$, $|f(y) - f(x)| < \varepsilon$ and $y^* \in x^* + \varepsilon\mathbb{R}^*$. Next, we give some basic properties of the Fréchet subdifferentials. The first four properties are classically in the literature, the final one can be proved using [35, Theorem 3.1] by rewriting a Fréchet subgradient satisfying an optimization problem as in [28, Equation (3.8)]. Nevertheless, we provide a proof for completeness.
Proposition 3.2. The Fréchet subdifferential satisfies the following properties:

P(i) Consider an lsc function \( f : X \to \mathbb{R} \) and \( x^* \in \partial f(\bar{x}) \). Then, for every \( \varepsilon > 0 \) there exists \( \gamma > 0 \) such that the function

\[
x \to f(x) - \langle x^*, x - \bar{x} \rangle + \varepsilon \|x - \bar{x}\| + \delta_{\mathbb{B}(\bar{x}, \gamma)}
\]

attains its minimum at \( \bar{x} \).

P(ii) (Calculus estimation) For every \( \varepsilon > 0 \), any point \( x \in X \) and every finite-dimensional subspace \( L \) of \( X \), we have

\[
\hat{\partial}_{\mathbb{B}(x, \varepsilon) \cap L} (x') \subseteq L^\perp, \forall x' \in \text{int} \mathbb{B}(x, \varepsilon).
\]

P(iii) (Enhanced Fuzzy Sum Rule) Consider an lsc function \( g \) and a convex Lipschitz function \( f \) with \( f(x) \in \mathbb{R} \), there are sequences \( (x_n, \alpha_n)_{n \in \mathbb{N}} \) such that \( x_n^* \in \partial f(x_n) \), \( x_n \xrightarrow{J} x_0 \), \( x_n^* \xrightarrow{\| \|} x_0^* \) with \( -x_0^* \in \partial g(x_0) \).

P(iv) (Fuzzy Sum Rule) Consider a finite family of lsc functions \( f_j : X \to \mathbb{R} \) with \( j \in J \) and \( x^* \in \partial (\sum_{j \in J} f_j)(x) \). Then, there are nets \( (x_{\alpha,j}, x^*_{\alpha,j})_{\alpha \in D} \) such that \( x^*_{\alpha,j} \in \partial f_j(x_{\alpha,j}) \), \( x_{\alpha,j} \xrightarrow{J} x \) and \( \sum_{j \in J} x^*_{\alpha,j} \xrightarrow{\| \|} x^* \).

P(v) For every finite family of lsc functions \( f_j : X \to \mathbb{R} \) with \( j \in J \) we have that for all \( x \in X \)

\[
\hat{\partial} f_j(x) \subseteq \bigcap_{\varepsilon > 0} \text{cl} \mathbb{W}^{\varepsilon}\left\{ \sum_{\varepsilon > 0} \lambda \hat{\partial} f_j(x_j) : x_j \in \mathbb{B}(x, f_j, \varepsilon), \lambda \in \Delta(J, x, \varepsilon) \text{ and } \# \lambda \leq \dim(X) + 1 \right\}.
\]

Proof. Items P(i) and P(ii) follow from definition. Item P(iii) is the well-known Enhanced Fuzzy Sum Rule (see, e.g., \([7, 20, 25, 42, 43]\)). Item P(iv) is an equivalence of the Enhanced Fuzzy Sum Rule (see, e.g., \([21]\)). Finally, we must prove Item P(v); to complete this task, it is enough to consider the pointwise maximum of two functions \( g := \max\{f_1, f_2\} \). Let \( x^* \in \partial g(x), \varepsilon \in (0, 1), V \in N_0(w^*) \), so by Item P(i) there exist \( \gamma \in (0, \varepsilon) \) such that the function

\[
y \to g(y) - \langle x^*, x - \bar{x} \rangle + \varepsilon \|y - x\| + \delta_{\mathbb{B}(x, \gamma)}(y)
\]

attains its minimum at \( x \). Hence, assuming that \( \gamma > 0 \) is small enough, one can suppose that

\[
f_i(u) > f_i(x) - \varepsilon, \text{ for all } u \in \mathbb{B}(x, \gamma), i = 1, 2.
\]

Now consider the function

\[
X \times \mathbb{R}^2 \ni (w, \alpha_1, \alpha_2) \to m(\alpha_1, \alpha_2) + \delta_{\text{epi} f_1}(w, \alpha_1) + \delta_{\text{epi} f_2}(w, \alpha_2) - \phi(w) + \delta_{\text{epi} \mathbb{B}(x, \gamma)}(w),
\]

where \( m(\alpha_1, \alpha_2) := \max\{\alpha_1, \alpha_2\} \) and \( \phi(y) := \langle x^*, x - \bar{x} \rangle - \varepsilon \|y - x\| \). This function has a local minimum at the point \( (x_1, f_1(x), f_2(x)) \), so by Item P(iv) we can choose

(i) \( (\alpha_1, \alpha_2) \in \mathbb{R}^2 \) with \( f_i(x) - \alpha_i \leq \gamma/2 \) and \( (q_1, q_2) \in \hat{\partial} m(\alpha_1, \alpha_2) = \{(p_1, p_2) \in \Delta(1, 2) : p_1 = 0 \text{ if } \alpha_i < m(\alpha_1, \alpha_2)\} \).

(ii) \( (w_1, \beta_1) \in \mathbb{B}(x, f_1(x), \gamma/2) \) and \( (w^*_1, \lambda_1) \in \hat{\partial} m f_1(w_1, \beta_1) \) such that \( w^*_1 + w^*_2 \in x^* + V + V, |q_1 + \lambda_1| < \gamma/2 \) and \( |q_2 + \lambda_2| < \gamma/2 \). Consequently, by (5) and Item (ii) we have that \( (w_1, f_i(w_1)) \in \mathbb{B}(x, f_i(x), \varepsilon) \) by classical argumentation we have that \( (w^*_1, \lambda_1) \in \hat{\partial} m f_1(w_1, f_i(w_1)) \) and \( \lambda_i \leq 0 \) (see, e.g., \([7, 20, 25, 27]\)). Now, we
check that \((-\lambda_1, -\lambda_2) \in \Delta(\{1, 2\}, \varepsilon),\) indeed \(|\lambda_1 + \lambda_2 - 1| = |\lambda_1 + \lambda_2 - q_1 + q_2| \leq \varepsilon;\) moreover if \(f_i(x) < g(x)\) (for small enough \(\varepsilon\)) we can assume (by Item (i)) that \(\alpha_i < m(\alpha_1, \alpha_2),\) so \(q_i = 0\) and consequently \(|\lambda_i| \leq \varepsilon.\) Now, if \(\lambda_i^* \neq 0\) for \(i = 1, 2,\) we define \(x_i^* := -\lambda_i^{-1} w_i \in \partial f(w_i);\) otherwise if there exists some \(\lambda_i = 0,\) then one can approximate this element using Proposition 3.1. Therefore, we have proved that

\[ \partial f_J(x) \subseteq \bigcap_{\varepsilon > 0} \{ \sum_{i=1}^k \lambda_i \partial f_J(x_i) : x_i \in B(x, f_J, \varepsilon), \lambda \in \Delta(J, x, \varepsilon) \}. \]

Now assume that \(X\) is finite-dimensional. Consider \(x^* = \sum_{i=1}^k \lambda_i x_i^*\) for some \(k > \dim(X) + 1\) with \(\lambda_i > 0, x_i^* \in \partial f_J(x_i), x_i \in B(x, f_i, \varepsilon)\) and \(\lambda \in \Delta(J, x, \varepsilon).\) Hence, \(\{x_i^*, 1\}_{i=1}^k \subseteq X \times \mathbb{R}\) must be linearly dependent in \(X \times \mathbb{R},\) and there are numbers \((\alpha_i)_{i=1}^k \subseteq \mathbb{R}\) not all equal to zero such that \(\sum_{i=1}^k \alpha_i x_i^* = 0\) and \(\sum \alpha_i = 0.\) Now consider

\[ (6) \quad \beta := \min \left\{ \frac{\lambda}{|\alpha_i|} : i \in I^+ \cup I^- \right\}, \text{ where } I^+ := \{ i : \alpha_i > 0 \} \text{ and } I^- := \{ i : \alpha_i < 0 \}. \]

Then,

1) If \(\beta = \frac{\lambda_{i_0}}{\alpha_{i_0}}\) for some \(i_0 \in I^+\), we notice that

\[ x^* = \sum_{i=1}^k (\lambda_i - \beta \alpha_i) x_i^* = \sum_{i=1}^k (\lambda_i - \beta \alpha_i) x_i^*, \]

moreover \(|\sum_{i=1}^k (\lambda_i - \beta \alpha_i) - 1| = |\sum_{i=1}^k (\lambda_i - 1)| \leq \varepsilon\) and for all \(t_i \notin T_\varepsilon(x)\)

2.1) If \(i \in I^+, 0 \leq \lambda_i - \beta \alpha_i \leq \lambda_i \leq \varepsilon.\)

2) If \(\beta = \frac{\lambda_{i_0}}{\alpha_{i_0}}\) for some \(i_0 \in I^-\), we notice that

\[ x^* = \sum_{i=1}^k (\lambda_i + \beta \alpha_i) x_i^* = \sum_{i=1}^k (\lambda_i + \beta \alpha_i) x_i^*, \]

moreover \(|\sum_{i=1}^k (\lambda_i + \beta \alpha_i) - 1| = |\sum_{i=1}^k (\lambda_i + 1)| \leq \varepsilon\) and for all \(t_i \notin T_\varepsilon(x)\)

2.1) If \(i \in I^-, 0 \leq \lambda_i + \beta \alpha_i \leq \lambda_i \leq \varepsilon.\)

Therefore,

\[ x^* \in \{ \sum_{t \in J} \lambda_t \partial f_t(x_t) : x_t \in B(x, f_t, 2\varepsilon), (\lambda_t) \in \Delta(J, x, 2\varepsilon) \text{ and } \#(\lambda_t) \leq k - 1 \}. \]

Repeating the processes (if \(k - 1 > \dim(X) + 1\)) one gets that

\[ x^* \in \{ \sum_{t \in J} \lambda_t \partial f_t(x_t) : x_t \in B(x, f_t, 2^p \varepsilon), (\lambda_t) \in \Delta(J, x, 2^p \varepsilon) \text{ and } \#(\lambda_t) \leq \dim(X) + 1 \}, \]

with \(p = \#J - \dim(X) - 1.\) \(\square\)
3.2. Fuzzy calculus rules for the subdifferential of the supremum function. In this section $T$ will be an arbitrary index set and $f_t : X \to \mathbb{R}$ will be a family of lsc functions. We recall that $f$ is defined as the supremum function of the family (3).

The next definition is an adaptation of the notion of the robust infimum or the decoupled infimum used in subdifferential theory to get fuzzy calculus rules (see, e.g., [3,19,25,27,36,37]).

Definition 3.3 (robust infimum). We will say that the family $\{f_t : t \in T\}$ has a robust infimum on $B \subseteq X$ provided that

$$\inf_{x \in B} f(x) = \sup_{t \in T} \inf_{x \in B} f_t(x).$$

In addition, if there exists some $\bar{x} \in B$ such that $\sup_{t \in T} \inf_{x \in B} f_t(x) = f(\bar{x})$, then we will say that $\{f_t : t \in T\}$ has a robust minimum on $B \subseteq X$. Finally, we say that the family $\{f_t : t \in T\}$ has a robust local minimum at $\bar{x}$ if $\{f_t : t \in T\}$ has a robust minimum on some neighborhood $B$ of $\bar{x}$.

The next lemma shows a sufficient condition for the existence of a robust minimum. We recall that a function $g : X \to \mathbb{R}$, where $(X, \tau)$ is a topological space, is called $\tau$-incompact provided that for every $\alpha \in \mathbb{R}$ the sublevel set $\{x \in X : g(x) \leq \alpha\}$ is $\tau$-compact.

Lemma 3.4. [Sufficient condition for robust minimum] Let $X$ be a Banach space and $B \subseteq X$. Suppose that $\{f_t : t \in T\}$ is an increasing family of $\tau$-lsc, $B$ is $\tau$-closed and there exists some $t_0$ such that $f_{t_0}$ is $\tau$-incompact on $B$, with $\tau$ some topology coarser (weaker or smaller) than the norm topology. Then the family $\{f_t : t \in T\}$ has a robust minimum on $B$.

Proof. [37, Lemma 3.5]

It is worth mentioning that in the above result the interchange between minimax in (7) is given without any convex-concave assumptions as in classical results (see, e.g., [3,4,11,40,41,44]). This follows from the fact that in our result these assumptions are replaced by the increasing property of the family of functions.

Remark 3.5. it has not escaped our notice that the hypothesis of infcompactness of some $f$ is necessary, even if the supremum function $f$ is incompact. Indeed, consider $f_n(x) = n^2 x^2 - x^4$, then it is easy to see that $f_n \leq f_{n+1}$ and $f = \delta_{\{0\}}$; moreover $\inf_{\mathbb{R}} f_n = -\infty$ and $\inf_{\mathbb{R}} f = 0$.

The next results give us a necessary condition for the existence of robust minimum in terms of an approximate Fermat’s rule. More precisely, we have the following results

Proposition 3.6. Let $\{f_t : t \in T\}$ be an increasing family of lsc functions. If $\{f_t : t \in T\}$ has a robust local minimum at $\bar{x}$, then

$$0 \in \bigcap_{\varepsilon > 0} \overline{c_1}^{\mathbb{R}} \left\{ \partial f_t(x) : x \in \mathbb{B}(\bar{x}, f_t, \varepsilon), t \in T_{\varepsilon}(\bar{x}) \right\}.$$

Proof. Assume that $\{f_t : t \in T\}$ has a robust minimum at $\bar{x}$ on $B := \mathbb{B}(\bar{x}, \eta)$. Pick $\varepsilon \in (0,1)$ and $\gamma \in (0, \min(\eta/2, \varepsilon/2))$, since $\bar{x}$ is a robust minimum there exists some $t \in T$ such that $\inf_B f_t \geq f(\bar{x}) - \gamma^2 \geq f_t(\bar{x}) - \gamma^2$, so $|f_t(\bar{x}) - f(\bar{x})| \leq \gamma^2$ and $\bar{x}$ is a $\gamma^2$-minimum of $f_t + \delta_B$. Hence, by Ekeland’s Variational Principle (see, e.g., [3]) there
exists $x_\gamma \in \mathbb{B}(\bar{x}, \gamma)$ such that $|f_t(x_\gamma) - f_t(\bar{x})| \leq \gamma^2$ and $x_\gamma$ is a minimum of the function $f_t(\cdot) + \delta_{\mathbb{B}}(\cdot) + \gamma \| \cdot - x_\gamma \|$, which implies that $f_t(\cdot) + \gamma \| \cdot - x_\gamma \|$ attains a local minimum at $x_\gamma$. By Proposition 3.2 Item P(iii) there exist sequences $(x_n, x_n^*) \in X \times X^*$ such that $x_n^* \in \partial f_n(x_n)$, $x_n \xrightarrow{\mathcal{H}} x_\gamma$, $x_n^* \rightarrow x^* \in \mathbb{B}^*$. Then, take $n \in \mathbb{N}$ such that $|f_t(x_n) - f_t(x_\gamma)| \leq \gamma$, $\|x_n - x_\gamma\| \leq \gamma$ and $0 \in \partial f_t(x_n) + 2\gamma \mathbb{B}^*$. Therefore, $x_n \in \mathbb{B}(\bar{x}, f_t, \varepsilon)$, $|f_t(x_n) - f_t(x_\gamma)| \leq \varepsilon$, $|f_t(x_n)| \leq \varepsilon$ and $0 \in \partial f_t(x_n) + \varepsilon \mathbb{B}^*$; to that end $0 \in \bigcup \{\partial f_t(x) : x \in \mathbb{B}(\bar{x}, f_t, \varepsilon), t \in T_\varepsilon(\bar{x})\} + \varepsilon \mathbb{B}^*$. \hfill \Box

Now, we notice that, in particular, Lemma 3.4 shows that every minimum over a closed bounded set in a finite-dimensional space is necessarily a robust local minimum. This fact, together with the representation of Item P(i), helps us to understand the subgradients in terms of the definition of a robust local minimum. Also in an infinite-dimensional space, this compactness property can be forced using the $w^*$-topology. Consequently, we use Proposition 3.6 to give an upper-estimation of the subdifferential of the supremum function of an increasing family of functions.

**PROPOSITION 3.7.** Let $\{f_t : t \in T\}$ be an increasing family of lsc functions. Then for all $\bar{x} \in X$

$$\partial f(\bar{x}) \subseteq \bigcup_{\varepsilon > 0} \text{cl} w^* \bigg\{ \partial f_t(x) : x \in \mathbb{B}(\bar{x}, f_t(\bar{x}), \varepsilon), t \in T_\varepsilon(\bar{x}) \bigg\}. $$

*Proof.* Fix $x^* \in \partial f(\bar{x})$, $V \in \mathcal{N}_0(w^*)$, $\varepsilon > 0$ and $L$ a finite-dimensional subspace of $X$ such that $L^\perp \subseteq V$, so by Item P(i) there exist a ball $B := \mathbb{B}(\bar{x}, \eta)$ such that the function $\tilde{f} := f - \langle x^*, \cdot - \bar{x} \rangle + \varepsilon \| \cdot - \bar{x} \| + \delta_{L^\perp B}$ attains its minimum at $\bar{x}$.

Hence, consider the family of functions $\tilde{f}_t := f_t - \langle x^*, \cdot - \bar{x} \rangle + \varepsilon \| \cdot - \bar{x} \| + \delta_{L^\perp B}$. It is easy to see that the family is increasing, $f = \sup_T f_t$ and there exists some $t \in T$ such that $\tilde{f}_t$ is infcompact. Whence, Lemma 3.4 shows that the family $\{\tilde{f}_t : t \in T\}$ has a robust local minimum at $\bar{x}$, and Proposition 3.6 implies

$$0 \in \bigcap_{\gamma > 0} \text{cl} w^* \bigg\{ \tilde{f}_t(x) : x \in \mathbb{B}(\bar{x}, \tilde{f}_t, \gamma), t \in T_\gamma(\{\tilde{f}_t : t \in T\}) \bigg\}. $$

Now take $\nu \in (0, \min\{\varepsilon/3, \eta/3\})$ small enough such that $|\phi(w) - \phi(\bar{x})| \leq \varepsilon/3$ for all $w \in \mathbb{B}(\bar{x}, \nu)$, so by (10) there exist $t \in T_\nu(\{\tilde{f}_t : t \in T\})$, $x \in \mathbb{B}(\bar{x}, \tilde{f}_t, \nu)$ and $w^* \in \partial \tilde{f}_t(x) = \partial f(\bar{x} + \phi + \delta_{B_{L^\perp B}})(x)$ such that $w^* \in x^* + V$. This implies that $x \in \mathbb{B}(\bar{x}, f_t, \nu + \varepsilon/3)$ and $t \in T_{\nu+\varepsilon/3}(\{f_t : t \in T\})$.

Now applying Proposition 3.2 Items P(ii) and P(iv) to $\tilde{f}_t$ we get the existence of points $u \in X$ and $u^* \in X^*$ such that $u^* \in \partial f_t(u)$, $u \in \mathbb{B}(x, f_t, \nu)$ and $u^* \in w^* + L^\perp + V = w^* + V$. Therefore $t \in T_\nu(\{f_t : t \in T\})$, $u \in \mathbb{B}(\bar{x}, f_t, \varepsilon)$ and $x^* \in u^* + V + V$. \hfill \Box

Now we present a fuzzy calculus rule for a not necessarily increasing family of functions; we bypass this assumption using the family of finite sets of the index set $T$, which is always ordered by inclusion.

**THEOREM 3.8.** Let $\{f_t : t \in T\}$ be an arbitrary family of lsc functions. Then for every $\bar{x} \in X$

$$\partial f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl} w^* \bigg\{ \bigcup_{F \in T_\varepsilon(\bar{x})} \bigg\{ \bigg\{ \sum_{i \in F} \lambda_i \partial f_i(x_i) : \lambda \in \Delta(F, x', \gamma) \text{ and } \#\lambda \leq \dim(X) + 1 \bigg\} \bigg\} \bigg\}$$

where $x_i \in \mathbb{B}(x', f_i, \gamma)$.\hfill \Box
Proof. Consider the set \( T := \mathcal{P}_f(T) \), ordered by \( F_1 \leq F_2 \) if and only if \( F_1 \subseteq F_2 \), and the family of functions \( \{ f_F : F \in \tilde{T} \} \) (recall that \( f_F = \max_{x \in F} f_x \)), then it is easy to see that the family \( \{ f_F : F \in \tilde{T} \} \) is an increasing family of functions and \( \sup_{F \in \tilde{T}} F = f \). Let \( x^* \in \partial f(\bar{x}) \), thus by Proposition 3.7

\[
x^* \in \bigcap_{\varepsilon > 0} \text{cl} w^* \left\{ \bigcup \{ \partial f_F(x') : x' \in B(\bar{x}, f_F, \varepsilon), F \in \tilde{T}_\varepsilon(\bar{x}) \} \right\}.
\]

Now, if \( w^* \in \partial f_F(x') \) for some \( x' \in B(\bar{x}, f_F, \varepsilon) \) and \( F \in \tilde{T}_\varepsilon(\bar{x}) \), we get \( x' \in B(\bar{x}, \varepsilon) \) and \( F' \in \tilde{T}_\varepsilon(\bar{x}) \), so using Proposition 3.2 Item P(v) we get

\[
w^* \in \bigcap_{\gamma > 0} \text{cl} w^* \left\{ \sum \lambda_t \partial f_{t_i}(x_{t_i}) : x_{t_i} \in B(x', f_t, \gamma), \lambda \in \Delta(F, x', \gamma) \right\},
\]

then (11) holds.

Here, it is important to compare the above result with \([28, \text{Theorem 3.1 part ii}]\). In the mentioned result, only uniform Lipschitz continuous data was considered. Here, we extend this fuzzy calculus to arbitrary lsc data functions. Since the comparison between both results involves some technical estimations, we prefer to write this as a corollary.

**Corollary 3.9.** **Under the hypothesis of Theorem 3.8 assume that the data function** \( f_t \) **is uniformly locally Lipschitz at** \( \bar{x} \). **Then, for each** \( x^* \in \partial f(\bar{x}) \), \( V \in N_0(w^*) \) **and** \( \varepsilon > 0 \) **there exist** \( \lambda \in \Delta(T_\varepsilon(\bar{x})) \) **and** \( x_t \in B(\bar{x}, \varepsilon) \) **for all** \( t \in T_\varepsilon(\bar{x}) \) **such that**

\[
x^* \in \sum_{t \in T_\varepsilon(\bar{x})} \lambda_t \partial f_t(x_t) + V
\]

**Proof.** Consider \( K \) as the constant of uniform Lipschitz continuity. Pick \( x^* \in \partial f(\bar{x}) \), and by Theorem 3.8 we have that

\[
x^* \in \sum_{t \in F} \lambda_t \partial f_t(x_t) + V
\]

for some \( F \in T_\varepsilon(\bar{x}) \), a point \( x' \in B(\bar{x}, f_F, \varepsilon) \), points \( x_t \in B(x', f_t, \gamma) \), and \( \lambda \in \Delta(F, x', \gamma) \), we can assume that \( \gamma \cdot \# F \leq \varepsilon \). First \( \| x_t - \bar{x} \| \leq ||x_t - x'| + ||\bar{x} - x'|| \leq \varepsilon + \gamma \).

Second \( F_\varepsilon(x') \subseteq T_\varepsilon(K+3)(\bar{x}) \), this is because

\[
\begin{align*}
f_t(\bar{x}) \geq f_t(x') - \varepsilon K & \geq f_F(x') - \varepsilon (K + 1) \geq f_F(\bar{x}) - \varepsilon (K + 2) \\
& \geq f(\bar{x}) - \varepsilon (K + 3).
\end{align*}
\]

Then, let us define \( \tilde{\lambda} : T \to \mathbb{R} \) by

\[
\tilde{\lambda}_t := \begin{cases} \sum_{t \in F_{\gamma}(x')} \lambda_t, & \text{if } t \in F_{\gamma}(x'), \\ 0, & \text{otherwise}. \end{cases}
\]

It is easy to see that \( \tilde{\lambda} \in \Delta(T_{\varepsilon(K+3)}(\bar{x})) \). Furthermore, we claim that

\[
x^* \in \sum_{t \in T} \tilde{\lambda}_t \partial f_t(x_t) + 3K\varepsilon B + V.
\]
Indeed, by (13) there are \( x^*_i \in \partial f_i(x_i) \) and \( v^* \in V \) such that \( x^* = \sum \lambda_i x^*_i + v^* \), then
\[
\left\| \sum_{t \in T} \lambda_t x^*_t - \sum_{t \in T} \lambda_t x^*_t \right\| = \left\| \sum_{t \in F(x')} (\lambda_t - \lambda_t) x^*_t + \sum_{t \in F(x')} \lambda_t x^*_t \right\|
\leq \left\| \sum_{t \in F(x')} \lambda_t - 1 \right\| K + K \varepsilon \leq \left\| \sum_{t \in F} \lambda_t - 1 \right\| K + 2K \varepsilon \leq 3K \varepsilon.
\]
Consequently, (14). Finally, taking \( \varepsilon \) small enough we have that (14) implies (13). \( \square \)

4. Limiting subdifferential of pointwise supremum. This section is divided into two subsections. The first one concerns the study of the notion of the limiting subdifferential in finite-dimensional Banach spaces. This setting is obviously motivated by the theory of semi-infinite programming; in this scenario we can obtain a better estimation of the limiting sequences obtained in Theorem 3.8. This result is given in Lemma 4.1; using this technical lemma, we focus on the particular case when the set \( T \) is a subset of a compact metric space (see Theorem 4.2). The second one corresponds to the infinite-dimensional setting; this subsection begins with a result concerning a fuzzy intersection rule for the normal cone of an arbitrary intersection of sets (see Theorem 4.5), which generalizes [30, Theorem 5.2]. Later the main result of this subsection is given in Theorem 4.8, where we explore the definition of sequential normal epi-compactness (see, e.g., [25]) and with this we extend [28, Theorem 3.2] (see Theorem 4.9).

4.1. Finite-dimensional spaces. In this subsection \( \hat{\partial}, \partial \) and \( \partial^\infty \) mean the Fréchet subdifferential, the limiting subdifferential and the singular limiting subdifferential, respectively.

**Lemma 4.1.** Consider \( \gamma_k \to 0^+ \) and \( x^* = \partial f(x) \) and \( y^* \in \partial^\infty f(x) \). Then there are sequences \( \eta_k \to 0^+ \), \( \{i_{n,k}\} = F_k \in \mathcal{P}_1(T) \), \( \{t_{n,k}\} = F_{n,k} \in \mathcal{P}_1(T) \) with \( \#F_k \leq \dim(X) + 1 \), \( \#F_{n,k} \leq \dim(X) + 1 \), \( x_{n,k} \to x \), \( y_{n,k} \to x \), \( x_{i,k} \to x \), \( y_{i,k} \to x \), \( \lambda_{i,k} \in \Delta(F_k, x_k^*, \gamma_k) \), \( \lambda_{n,k} \in \Delta(F_{n,k}, x_k^*, \gamma_k) \) such that:

i) \( x^* = \lim_{i \to \infty} \sum_{i \in F_k} \lambda_{i,k} \cdot x_{i,k}^* \), \( y^* = \lim_{i \to \infty} \sum_{i \in F_k} \lambda_{i,k} \cdot y_{i,k}^* \),

ii) \( \lim_{i \to \infty} f_{n,k}(x_k^*) = f(x) \), \( \lim_{i \to \infty} f_{n,k}(y_k^*) = f(x) \),

iii) \( \lim \left\{ f_{n,k}(x_{i,k}) - f_{n,k}(x_k^*) \right\} = 0 \) and \( \lim \left\{ f_{n,k}(y_{i,k}) - f_{n,k}(y_k^*) \right\} = 0 \) for all \( i \).

Moreover (by passing to a subsequence) one of the following conditions holds.

(A) There exists \( n_1 \in \mathbb{N} \) with \( n_1 \leq \dim(X) + 1 \) such that \( \lambda_{i,k} \xrightarrow{k \to \infty} \lambda_i > 0 \), \( x_{i,k}^* \xrightarrow{k \to \infty} x_i^* \), \( \lim f_{n,k}(x_{i,k}) = f(x) \) for \( i \leq n_1 \) and \( \lambda_{i,k} \xrightarrow{k \to \infty} \lambda_i > 0 \), \( \lambda_{i,k} \cdot x_{i,k}^* \xrightarrow{k \to \infty} x_i^* \) for \( n_1 < i \leq n \),

and \( x^* = \sum_{i=1}^{n_1} \lambda_i x_i^* + \sum_{i>n_1} x_i^* \), or

(B) There are \( \nu_k \to 0 \) such that \( \nu_k \cdot \lambda_{i,k} \cdot x_{i,k}^* \xrightarrow{k \to \infty} x_i^* \) and \( \sum_{i=1}^{n_1} x_i^* = 0 \) with not all \( x_i^* \) equal to zero.

and (up to a subsequence) one of the following conditions holds.

(A∞) There exists \( n_2 \in \mathbb{N} \) with \( n_2 \leq \dim(X) + 1 \) such that \( \lambda_{i,k} \xrightarrow{k \to \infty} \lambda_i > 0 \), \( y_{i,k}^* \xrightarrow{k \to \infty} y_i^* \), \( \lim f_{n,k}(y_{i,k}) = f(x) \) for \( i \leq n_2 \) and \( \lambda_{i,k} \xrightarrow{k \to \infty} \lambda_i > 0 \), \( \lambda_{i,k} \cdot y_{i,k}^* \xrightarrow{k \to \infty} y_i^* \) for \( n_2 < i \leq n \),
and \( y^* = \sum_{i=1}^{n_1} \lambda_i^\infty y_i^* + \sum_{i=n_2}^{n} y_i^* \), or

\[ (B^\infty) \quad \text{There are } \nu_k \rightarrow 0 \text{ such that } \nu_k \cdot \eta_k \cdot \lambda_i^\infty \cdot y_i^* \xrightarrow{k \rightarrow \infty} y_i^* \text{ and } \sum_{i=1}^{n_1} y_i^* = 0 \text{ with not all } x_i^* \text{ equal to zero.} \]

**Proof.** Define \( N := \dim(X) + 1 \) and consider \( x^* \in \partial f(x) \) (\( y^* \in \partial^\infty f(x) \), resp.), so (by definition) there exist \( x_k \xrightarrow{f} x \) and \( x_k^* \in \partial f(x_k) \) (\( y_k \xrightarrow{f} y \) and \( y_k^* \in \partial f(y_k) \), resp.) such that \( x_k^* \rightarrow x^* \) (\( y_k^* \rightarrow y^* \), resp.). Whence, by Theorem 3.8, there exist \( x_k^* \in \mathbb{B}(x_k, \gamma_k) \) and \( F_k = \{ t_i,k \}_{k=1}^{N} \subseteq T \), with \( |f_{F_k}(x'_k) - f(x'_k)| \leq \gamma_k \) along with elements \( x_{t_i,k} \in \mathbb{B}(x'_k, t_i,k, \gamma_k) \) and \( z_k^* = \sum_{i=1}^{N} \lambda_{t_i,k} x_{t_i,k}^* \) with \( ||z_k^* - x_k^*|| \leq \gamma_k \), \((\lambda_{t_i,k}) \in \Delta(F_k, x'_k, \gamma_k)\) and \( x_{t_i,k}^* \in \partial f_{t_i,k}(x_{t_i,k}). \) Hence, \( x^* = \lim_{k \rightarrow \infty} \sum_{i \in F_k} \lambda_{t_i,k} \cdot x_{t_i,k}^* \) for all \( i \in 1, \ldots, N \).

On the one hand if \( \sup\{||x_{t_i,k}|| : i = 1, \ldots, N ; k \in \mathbb{N}\} < +\infty \) (up to a subsequence) we can assume that \( \lambda_{t_i,k} \rightarrow \lambda_i \) with \( \lambda_i \in \Delta\{1, \ldots, N\} \) and (relabeling it if necessary) we may assume that \( \lambda_k \neq 0 \) for all \( i = 1, \ldots, n_1 \) and \( \lambda_k = 0 \) for all \( i = n_1 + 1, \ldots, N \).

On the other hand if \( \sup\{||x_{t_i,k}|| : i = 1, \ldots, N ; k \in \mathbb{N}\} = +\infty \) (by passing to a subsequence) we can assume that \( \lambda_{t_i,k} \rightarrow \lambda_i \) for all \( i = 1, \ldots, n_1 \) and \( \lambda_{t_i,k} x_{t_i,k}^* \rightarrow x_i^* \) for all \( i = n_1 + 1, \ldots, N \), therefore \( x^* = \sum_{i=1}^{n_1} \lambda_i x_i^* + \sum_{i=n_1}^{n} x_i^* \). Next, we claim that \( \lim_{k \rightarrow \infty} f_{t_i,k}(x_k) = f(x) \) for all \( i = 1, \ldots, n_1 \). Indeed, define \( \gamma := \min\{\lambda_i/2 : i = 1, \ldots, n_1\} \), then for all \( k \) (large enough) such that \( \gamma_k \leq \gamma \) and \( \lambda_k > \gamma \) (recall \( t_{k,i} \in \Delta(F_k, x'_k, \gamma_k) \)) we have that

\[ f_{t_i,k}(x_k) + \gamma_k \geq \max_{s \in F_k} f_s(x'_k) \geq f_{t_i,k}(x'_k), \]

so, taking the limits we obtain that

\[ \lim_{k \rightarrow \infty} f_{t_i,k}(x_k) \geq \lim_{k \rightarrow \infty} \max_{s \in F_k} f_s(x'_k) = f(x) \geq \lim_{k \rightarrow \infty} f_{t_i,k}(x'_k), \]

which implies the desired conclusion.

Now we are going to apply the above result to a framework, where the functions \( f_i \) represent a control in a region. We assume that \( T \) is contained in a metric space and \( \mathbf{T} \) is compact. For this reason we introduce the following definitions.

A family of lsc functions \( \{f_t : t \in \mathbf{T}\} \) is said to be **continuously subdifferentiable at** \( x \) with respect to \( \partial \) provided that for every sequence \( T \times X \times [0, +\infty) \ni (t_n, x_n, \lambda_n) \rightarrow (t, x, \lambda) \in T \times X \times [0, +\infty) \) and points \( w_n^* \in \partial f_{t_n}(x_n) \) with \( \lambda_n w_n^* \rightarrow w^* \) one has

\[ w^* \in \lambda \partial f_t(x) := \left\{ \begin{array}{ll} \lambda \partial f_t(x) & \text{if } \lambda > 0, \\ \partial f_t(x) & \text{if } \lambda = 0, \end{array} \right. \]
To our knowledge, the next definition was introduced in [32], where the authors studied generalized notions of differentiation for parameter-dependent set valued maps and mappings. For a point \( x \in X \) and \( t \in \mathcal{T} \setminus T \) we define the extended subdifferential and the extended singular subdifferential at \( (t, x) \) as

\[
\partial f(t)(x) := \left\{ x^* \in X^* : \exists \xi_k \in T, t_k \to t, x_k \to x, x_k^* \in \partial f(t_k)(x_k) \right. \\
s.t. \ f(t_k)(x_k) \to f(x), \ \text{and} \ x_k^* \to x^* \right\}
\]

\[
\partial^\infty f(t)(x) := \left\{ x^* \in X^* : \exists \xi_k \in T, t_k \to t, \eta_k \to 0^+, x_k \to x, x_k^* \in \partial f(t_k)(x_k) \right. \\
s.t. \ \sup \ f(t_k)(x_k) \leq f(x), \ \text{and} \ \eta_k x_k^* \to x^* \right\},
\]

respectively. Finally, we denote the extended active index set at \( x \) by \( \mathcal{T}(x) = T(x) \cup (\mathcal{T} \setminus T) \).

**Theorem 4.2.** Consider a family of lsc functions \( \{f_t : t \in T\} \) where \( T \) is a subset of a metric space and \( \mathcal{T} \) is compact. Assume that the following conditions hold at a point \( \bar{x} \)

(a) For every \( i \in T \), \( \mathop{\lim\sup}_{(t, x) \to (i, x)} f_t(x) \leq f_t(\bar{x}) \).

(b) The family is \( \{f_t : t \in T\} \) continuously subdifferentiable at \( \bar{x} \).

(c) The set \( \mathop{\text{co}} \left( \bigcup_{t \in \mathcal{T}} \partial^\infty f_t(\bar{x}) \right) \) does not contain lines.

Then

\[
\partial f(\bar{x}) \subseteq \mathop{\text{co}} \left( \bigcup_{t \in T(\bar{x})} \partial f_t(\bar{x}) \right) + \mathop{\text{co}} \left( \bigcup_{t \in T} \partial^\infty f_t(\bar{x}) \right), \ \text{and}
\]

\[
\partial^\infty f(\bar{x}) \subseteq \mathop{\text{co}} \left( \bigcup_{t \in \mathcal{T}} \partial^\infty f_t(\bar{x}) \right).
\]

**Proof.** Consider \( x^* \in \partial f(\bar{x}) \). Now, using the notation of Lemma 4.1 and by the compactness of \( \mathcal{T} \) we can assume that \( t_{k, i} \to t_i \in \mathcal{T} \). Moreover, Item (c) contradicts Lemma 4.1 Items (B) and (B\(^\infty\)), which means, Lemma 4.1 Items (A) and (A\(^\infty\)) must hold. Hence we can write \( x^* = \sum_{i=1}^{n_1} \lambda_i x_i^* + \sum_{i \geq n_1} x_i^* \).

- If \( i \leq n_1 \) and \( t_i \in T \): By assumption Item (a) and Lemma 4.1 Item (A) necessarily \( f(\bar{x}) = f_{t_i}(\bar{x}) \), i.e., \( t \in T(\bar{x}) \). Also, Item (b) implies \( x_i^* \in \partial f_{t_i}(\bar{x}) \).
- If \( i \leq n_1 \) and \( t_i \in \mathcal{T} \setminus T \): By Lemma 4.1 Item (A) we get that \( x_i^* \in \partial f_{t_i}(\bar{x}) \).
- If \( i > n_1 \) and \( t_i \in T \): By assumption Item (b) we get \( x_i^* \in \partial f_{t_i}(\bar{x}) \).
- If \( i > n_1 \) and \( t_i \in \mathcal{T} \setminus T \): By Lemma 4.1 Item (A) implies that \( x_i^* \in \partial f_{t_i}(\bar{x}) \).

This completes the first part. The case \( y^* \in \partial^\infty f(\bar{x}) \) follows similar arguments so we omit the proof.

It is important to mention that similar results have been shown in the literature; we refer to [6,29,32] for some examples. In the above result we did not go for the greater stage of generality, and we established the result only to show one possible application of Lemma 4.1.

**Remark 4.3.** It has not escaped our notice that the convex envelope appears in Theorem 4.2 due to the fact that at the moment of taking the convergent subsequence in the index \( t_{k,i} \to t_i \) we cannot ensure, in a general framework, that there could exist two limit points \( t_i = t_j \) for \( i \neq j \). Nevertheless, the reader can force this condition imposing some assumptions over the index set, the simplest example is when the index set is finite.
Now let us finish this subsection with an example which shows an application of Theorem 4.2 for a countable number of functions.

**Example 4.4.** Consider $T = \mathbb{N}$ and the sequence of functions

$$f_n(x, y) = \begin{cases} \frac{n x^2}{n^{-1}} \log(|y| + 1) - \frac{1}{n} & \text{if } x \geq 0, \\ \frac{n}{n+1} \log(|y| + 1) - \frac{1}{n} & \text{if } x < 0. \end{cases}$$

Here, it is worth noting that all functions $f_n$ are locally Lipschitz continuous, but they are not uniformly Lipschitz continuous, so the results of [28] cannot be applied. Nevertheless, we can apply Theorem 4.2. Indeed, after some calculus, we get that

$$\partial f_n(0, 0) = \{0\} \times \left[-\frac{n}{n-1}, \frac{n}{n-1}\right],$$

$$\partial^\infty f_n(0, 0) = \{(0, 0)\}.$$

We compute the function

$$f(x, y) = \log(|y| + 1) + \delta_{(-\infty, 0]}(x) = \begin{cases} +\infty & \text{if } x > 0, \\ \log(|y| + 1) & \text{if } x \leq 0, \end{cases}.$$  

Then, $\partial f(0, 0) = [0, +\infty) \times [-1, 1]$ and $\partial^\infty f(0, 0) = [0, +\infty) \times \{(0, 0)\}$. In order to apply Theorem 4.2 we notice that $\mathbb{N}$ is a subset of the compact space $\mathbb{N}_{\infty} := \mathbb{N} \cup \{\infty\}$ with the metric $d(a, b) = \frac{|a - b|}{a+b}$. Straightforwardly the assumptions Items (a) and (b) of Theorem 4.2 are satisfied, furthermore, $\mathbb{N}(0, 0) = \emptyset$.

Now, we calculate $\partial^\infty f_n(0, 0)$ and $\partial^\infty f(0, 0)$. First we notice that

$$\hat{\partial} f_n(x, y) \subseteq [0, +\infty) \times \left[-\frac{n}{n-1}, \frac{n}{n-1}\right].$$

Then $\partial^\infty f_n(0, 0) = [0, +\infty) \times [-1, 1]$ and $\partial^\infty f_n(0, 0) = [0, +\infty) \times \{0\}$. In particular, assumption Item (c) of Theorem 4.2 holds. Then, Theorem 4.2 gives us

$$\partial f_n(0, 0) = \text{co} \left( \partial^\infty f_n(0, 0) \right) + \text{co} \left( \bigcup_{n \in \mathbb{N}_{\infty}} \partial^\infty f_n(0, 0) \right) = [0, +\infty) \times [-1, 1],$$

$$\partial^\infty f_n(0, 0) = \text{co} \left( \bigcup_{n \in \mathbb{N}_{\infty}} \partial^\infty f_n(0, 0) \right) = [0, +\infty) \times \{0\},$$

which are exact estimations of the limiting and singular subdifferential of the function $f$ at $(0, 0)$.

**4.2. Infinite-dimensional spaces.** In this section we study the limiting subdifferential of the supremum function in an arbitrary Asplund space $X$.

The first result of this Subsection generalizes the Fuzzy Intersection Rule for Fréchet Normals to Countable Intersections of Cones established in [30, Theorem 5.2].

**Theorem 4.5.** Let $\{A_t\}_{t \in T}$ be an arbitrary family of closed subsets of $X$ and $A := \bigcap_{t \in T} A_t$. Then given $\bar{x} \in X$, $x^* \in \bar{N}(A, \bar{x})$, $\varepsilon > 0$ and $V \in \mathcal{N}(0(w^*))$ there are $F \in \mathcal{P}_T(T)$, $w_\varepsilon \in \mathbb{B}(\varepsilon, \bar{x})$ and $w_t^* \in \bar{N}(A_t, w^*_t)$ such that

$$x^* \in \bigcup_{t \in F} w_t^* + V.$$  

(15)
Consequently, if \( \{ A_t \}_{t \in T} \) is a family of closed cones \( \hat{N}(A_t, w_t) \subseteq N(A_t, 0) \) for all \( t \in T \) and
\[
\hat{N}(A, \bar{x}) \subseteq \text{cl}^w \left\{ \sum_{t \in F} w^*_t \right\} \quad \text{where} \quad w^*_t \in N(A_t, 0) \quad \text{and} \quad t \in F \in \mathcal{P}_T(T).
\]

**Proof.** The first part corresponds to a straightforward application of Theorem 3.8. Now if one considers a closed cone \( K \subseteq X \) and \( u \in K \) one has that
\[
\hat{N}(K, u) \subseteq \hat{N}(K, n^{-1}u), \quad \forall n \in \mathbb{N}.
\]
Therefore \( \hat{N}(A_t, u) \subseteq N(A_t, 0) \) for every \( t \in T \) and \( u \in A_t \), consequently (15) implies (16).

**Remark 4.6.** It important to notice that the results of [8] cannot be applied to derive the above formulæ, since imposing uniform Lipschitz continuity of an indicator function of the set \( \Lambda \) at a point \( \bar{x} \) is equivalent to assume that the point \( \bar{x} \) is an interior point of \( \Lambda \), which give us a trivial conclusion.

The next definition is the notion of sequential normal epi-compactness (SNEC) of functions defined for the limiting subdifferential (see, e.g., [25, Definition 1.116 and Corollary 2.39]).

**Definition 4.7.** A real extended valued function \( f \) finite at \( x \) is said to be SNEC at \( x \) if for any sequences \( (\lambda_k, x_k, x^*_k) \in [0, +\infty) \times X \times X^* \) satisfying \( \lambda_k \to 0 \), \( x_k \xrightarrow{\Delta} x \), \( x^*_k \in \partial f(x_k) \) and \( \lambda_k x^*_k \rightharpoonup 0 \) one has \( \| \lambda_k x^*_k \| \to 0 \). A family of functions \( \{ f_t \}_{t \in T} \) is said to be SNEC on a neighborhood of a point \( \bar{x} \) if there exists a neighborhood \( U \) of \( \bar{x} \) such that for all \( x \in U \) all but one of these are SNEC at \( x \).

We say that the family of functions \( \{ f_t : t \in T \} \) satisfy the limiting condition on a neighborhood of a point \( \bar{x} \) if there exists a neighborhood \( U \) of \( \bar{x} \) such that for all all \( x \in U \) in \( F \in \mathcal{P}_T(T) \)
\[
w^*_t \in \partial f_t(x), \quad t \in F \quad \text{and} \quad \sum_{t \in F} w^*_t = 0 \quad \text{implies} \quad w^*_t = 0, \quad \text{for all} \quad t \in F.
\]

It is worth mentioning that the SNEC property is immediately satisfied if the space \( X \) is finite-dimensional. Moreover, the family of functions \( \{ f_t \}_{t \in T} \) is SNEC and satisfies the limiting condition on a neighborhood of a point \( \bar{x} \), provided that the functions are locally Lipschitz (not necessarily uniform) on a neighborhood \( U \) of \( \bar{x} \).

The next theorem corresponds to the main result of this paper; in this result we give an upper-estimation of the subdifferential of the supremum function only using the above definitions, without the assumption of uniformly locally Lipschitz continuity.

**Theorem 4.8.** Consider a family of lsc functions \( \{ f_t : t \in T \} \). If the family \( \{ f_t : t \in T \} \) is SNEC and satisfy the limiting condition (17) on a neighborhood of \( \bar{x} \). Then
\[
\partial f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^w \left( S(\bar{x}, \varepsilon) \right), \quad \text{and} \quad \partial^\infty f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^w \left( [0, \varepsilon] \cdot S(\bar{x}, \varepsilon) \right),
\]

Where
\[
S(\bar{x}, \varepsilon) := \left\{ \sum_{t \in F} \lambda_t \circ \partial f_t(x') : \right\}
\begin{align*}
F \in \mathcal{P}_T(T), x' \in B(\bar{x}, \varepsilon), \\
|f_F(x') - f(\bar{x})| \leq \varepsilon, \lambda \in \Delta(F) \\
\text{and} \quad f_t(x') = f_t(x') \quad \text{for all} \quad t' \in \text{supp} \Lambda
\end{align*}
\]
and
\[
\lambda \circ \partial f_t(x) := \begin{cases} 
\lambda \partial f_t(x), & \text{if } \lambda > 0, \\
\partial^\infty f_t(x), & \text{if } \lambda = 0.
\end{cases}
\]

**Proof.** Consider \( \varepsilon > 0 \) and \( V \in \mathcal{N}_0(w^*) \). Pick \( x^* \in \partial f(\bar{x}) \) (\( y^* \in \partial^\infty f(\bar{x}) \), resp.). Hence, there exist sequences \( x_j \xrightarrow{\varepsilon} \bar{x} \) and \( x_j^* \xrightarrow{\nu_j} y^* \)(\( \nu_j \to 0^+ \) and \( \nu_j x_j^* \xrightarrow{\nu_j} y^* \), resp.) with \( x_j^* \in \partial f(x_j) \). Now, take \( j_0 \in \mathbb{N} \) such that \( x^* \in x_j^* + V \) (\( x^* \in \nu_j x_j^* + V \) and \( \nu_j \leq \varepsilon \), resp.) and \( x_{j_0} \in \mathbb{B}(\bar{x}, f, \varepsilon) \). Hence, by Theorem 3.8 there exist some \( F \in \mathcal{T}_A(x_{j_0}) \) and \( x' \in \mathbb{B}(x_{j_0}, f, \varepsilon) \) such that \( x_{j_0}^* = w^* + v^* \) with
\[
w^* \in \bigcap_{\gamma > 0} \text{cl}^\circ \left\{ \sum_{t \in T} \lambda_t \partial f_t(x_t) : x_t \in \mathbb{B}(x', f_t, \gamma), \ (\lambda_t) \in \Delta(F, x', \gamma) \right\},
\]
and \( v^* \in V \). One gets \( x' \in \mathbb{B}(\bar{x}, 2\varepsilon) \) and \( |f_F(x') - f(\bar{x})| \leq 3\varepsilon \). Now, we show that
\[
w^* \in S(\bar{x}, 3\varepsilon)
\]

For this purpose let us introduce the following notation; by the symbol \( S(X \times X^*) \) we understand the family of set \( U \times Y \) where \( U \) and \( Y \) are (norm-) separable closed linear subspaces of \( X \) and \( X^* \), a set \( \mathcal{A} \subseteq S(X \times X^*) \) is called a rich family if (i) for every \( U \times Y \in \mathcal{S}(X \times X^*) \), there exists \( V \times Z \in \mathcal{A} \) such that \( U \subseteq V \) and \( Y \subseteq Z \), and (ii) \( \bigcap_{\mathcal{U} \subseteq \mathcal{A}} \mathcal{U} \) is a projection satisfying that
\[
P^*_\mu(X^*) = Y, \quad P^*_\mu(0) = V \quad \text{and} \quad P^*_\mu(X^{**}) = \nabla^{w}(X^{**}, X^*)
\]
Hence, consider \( v^*_k \in Y \) such that \( v^*_k \xrightarrow{w^*} v^* \) and \( v^* = 0 \) on \( V \), so \( v^* = 0 \) on \( \nabla^{w}(X^{**}, X^*) \). Moreover, because \( v^*_k \in Y \) and \( P^*_\mu \) is a projection onto \( Y \) one has \( P^*_\mu(v^*_k) = v^*_k \), then \( \langle v^*, x - P^*_\mu(x) \rangle = \lim \langle v^*_k, x - P^*_\mu(x) \rangle = \lim \langle P^*_\mu(v^*_k), x - P^*_\mu(x) \rangle = \lim \langle v^*_k, P^*_\mu(x) - P^*_\mu(x) \rangle = 0 \) for every \( x \in X \), which implies (using that \( \langle v^*, P^*_\mu(x) \rangle = 0 \) \( \langle v^*, x \rangle = 0 \).

Now, we choose a decreasing sequence of positive numbers \( \gamma_n \searrow 0^+ \), consider \( V_1 \times Y_1 \in \mathcal{A} \) containing \( \langle x', w^* \rangle \), let \( \{e(1, i)\}_{i \in \mathbb{N}} \) be a dense set in \( \mathbb{B} \cap V_1 \) and define
\[
W(1, p) := \{ y^* \in X^* : |\langle y^*, e(1, i) \rangle| \leq \gamma_p, \text{ for all } i = 1, ..., p \}.
\]

Whence for all \( p \geq 1 \) and \( t \in F \) we can pick points \( x_t(1, p) \in \mathbb{B}(x', f_t, \gamma_p) \), subgradients \( x_t^*(1, p) \in \partial f_t(x_t(1, p)) \), \( \lambda(1, p) \in \Delta(F, x', \gamma_p) \) and \( v(1, p)^* \in W(1, p) \) such that \( w^* = \sum \lambda_t(1, p)x_t^*(1, p) + v^*(1, p) \).

Now assume that we have selected \( V_n \times Y_n \in \mathcal{A} \) containing all \( V_k \times Y_k \) for \( k \leq n \), families of points \( \{e(n, i)\}_{i \in \mathbb{N}} \) dense in \( \mathbb{B} \cap V_n \), which contains all previous \( \{e(k, i)\}_{i \in \mathbb{N}} \) for \( k \leq n \), points \( x_t(i, p) \in \mathbb{B}(x', f_t, \gamma_p) \), subgradients \( x_t^*(i, p) \in \partial f_t(x_t(i, p)) \), \( \lambda(i, p) \in \Delta(F, x', \gamma_p) \) and \( v(i, p)^* \in W(i, p) \) such that
\[
w^* = \sum \lambda_t(i, p)x_t^*(i, p) + v^*(i, p), \text{ for } i \leq n \text{ and } p \geq 1.
\]

Then, take \( V_{n+1} \times Y_{n+1} \in \mathcal{A} \) such that \( V_n \times Y_n \subseteq V_{n+1} \times Y_{n+1}, x_t(i, p) \in V_{n+1}, x_t(i, p) \in Y_{n+1} \) for all \( t \in F, i \leq n, p \in \mathbb{N} \), consider \( \{e(n + 1, i)\}_{i \in \mathbb{N}} \) a dense set in \( B \cap V_{n+1} \), and define
\[
W(n + 1, p) := \{ y^* \in X^* : |\langle y^*, e(k, i) \rangle| \leq \gamma_p, \text{ for all } k = 1, ..., n + 1 \text{ and } i = 1, ..., p \}.
\]
Then for all \( p \geq 1 \) and \( t \in F \) we can pick points \( x_t(n+1, p) \in \mathbb{B}(x', f_t, \gamma_p) \), subgradients \( x_t^n(n+1, p) \in \partial_f(x_t(n+1, p)) \), \( \lambda(n+1, p) \in \Delta(F, x', \gamma_p) \) and \( v(n+1, p)^* \in W(n+1, p) \) such that \( w^* = \sum \lambda_t(n+1, p)x_t^n(n+1, p) + v^*(n+1, p) \).

Now we define \( \bigcup_{n \in \mathbb{N}} \bigcup_{t \in \mathbb{N}} Y_n := V \times Y \in A \), \( x_t(n) := x_t(n, n) \), \( x_t^* := x_t^*(n, n) \), \( \lambda_t(n) := \lambda_t(n, n) \), \( v_t(n) := v(n, n) \). Then, by our construction \( x_t(n) \xrightarrow{t} x' \).

Since \( \lambda_t(n) \in \Delta(F, x', \gamma_n) \) we can assume that \( \lambda_t(n) \xrightarrow{n \to \infty} \lambda_t \in [0, 1] \) for every \( t \in F \), and \( \sum_{t \in F} \lambda_t = 1 \); moreover \( f_t(x') = f_P(x') \) for every \( t \in \text{supp} \lambda \).

Then, on the one hand if (there exist some subsequence such that) \( \lambda_t(n)x_t^*(n) \) is bounded for all \( t \in F \), in this case we can assume that

- If \( t \in \text{supp} \lambda \), \( \lambda_t(n)x_t^*(n) \) converge to some \( \lambda_t x_t^* \) with \( x_t^* \in \partial f_t(x') \).
- If \( t \notin \text{supp} \lambda \), \( \lambda_t(n)x_t^*(n) \) converge to some \( x_t^* \in \partial^\infty f_t(x') \).
- \( v^*(k) \xrightarrow{t} v^* \).

Furthermore, \( v^* \) is zero on \( V \). Indeed, the set \( \{ e(i,j) \}_{i,j} \) is dense in \( V \), then for every \( n \geq \max\{i, j\} \) we have that \( \| (v^*(n), e(i,j)) \| \leq \gamma_n \) (recall \( v^*(n) \in W(n, n) \)), so taking the limits \( (v^*, e(i,j)) = 0 \) for every \( i, j \), therefore \( v^* \) is zero on \( V \). Thus, by the property of \( A \) necessarily \( v^* \) is zero on the whole \( X \), hence using (21) we have that (20) holds.

On the other hand, if there exists some \( t \in F \) such that \( \| \lambda_t(n) \cdot x_t^*(n) \| \to +\infty \), we define \( \eta_n := (\max_{i \in F} \{ \| \lambda_t(n)x_t^*(n) \|, \| v^*(n) \| \})^{-1} \). We have \( \eta_n w^* \to 0 \) and (by passing to a subsequence) \( \eta_n \lambda_t(n)x_t^*(n) \xrightarrow{w^*} w^* \in \partial^\infty f(x') \); and by a similar argument as in the first case \( \eta_n v^*(n) \to 0 \), so \( \sum_{t \in F} w_t^* = 0 \). Moreover, by the limiting condition (17) we have \( w_t^* = 0 \). Finally, since all the functions but one of \( f_t \)’s are SNEC at \( x' \) we have \( \eta_n \lambda_t(n)x_t^*(n) \) converge in norm topology to zero, which is a contradiction.

Therefore \( x^* \in S(x, 3\varepsilon) + V + V (x^* \in [0, \varepsilon]S(x, 3\varepsilon) + V + V, \) resp., and by the arbitrariness of \( V \) and \( \varepsilon > 0 \) we conclude (18).

The next result gives us a simplification of the main formulae in Theorem 4.8 under the additional assumption that the data is Lipschitz continuous. The case when the data is uniformly Lipschitz continuous was proved in [28, Theorem 3.2].

**Theorem 4.9.** Let \( \{ f_t : t \in T \} \) be a family of locally Lipschitz functions on a neighborhood of a point \( \bar{x} \in \text{dom} f \). Then

\[
\text{(22) } \partial f(\bar{x}) \subseteq \bigcup_{\varepsilon > 0} \text{cl} w^* \left( S(\bar{x}, \varepsilon) \right), \text{ and } \partial^\infty f(\bar{x}) \subseteq \bigcup_{\varepsilon > 0} \text{cl} w^* \left( [0, \varepsilon] : S(\bar{x}, \varepsilon) \right),
\]

where \( S(\bar{x}, \varepsilon) \) was defined in (19). In addition, if the family is uniformly locally Lipschitz at \( \tilde{x} \), then

\[
\text{(23) } \partial f(\tilde{x}) \subseteq \bigcup_{\varepsilon > 0} \text{cl} w^* \left\{ \sum_{t \in F} \lambda_t \partial f_t(x'): F \in \mathcal{P}(T_r(\tilde{x})), x' \in \mathbb{B}(\tilde{x}, \varepsilon), \lambda \in \Delta(F) \text{ and } f_t(x') = f_P(x') \text{ for all } t \in F \right\},
\]

**Proof.** Consider \( V \in N_0(w^*), \varepsilon > 0 \), a finite-dimensional subspace \( L \ni \tilde{x} \) such that \( L^\perp \subseteq V \) and \( x^* \in \partial f(\tilde{x}) \) (respectively, \( y^* \in \partial^\infty f(\tilde{x}) \)), let \( P : X \to L \) be a continuous linear projection and define \( W = (P^*)^{-1}(V) \). Hence, \( x^*_t \in \partial f_t(x) \) (respectively, \( y^*_t \in \partial^\infty f_t(x) \)). Hence, we apply Theorem 4.8 and we conclude the existence of some \( F \in T_r(\tilde{x}), x' \in \mathbb{B}(\tilde{x}, \varepsilon), \lambda \in \Delta(F) \) such that \( x^*_t \in \sum_{t \in F} \lambda_t \partial(f_t)_t(x') + W \) and
\((f_t)_t(x') = (f_t)_t(x')\) for all \(t', t'' \in \text{supp } \lambda\), then

\[ P^*(x^*_t) = \sum_{t \in F} \lambda_t \delta_t f_t(x') + \sum_{t \in F} \lambda_t \delta_t f_t(x') + L^+ + V, \]

where the last equality follows from the sum rule for Lipschitz functions (see [17, 18, 25]). Therefore \(x^* = P(x^*_t) + x^* - P(x^*_t) \in \sum_{t \in F} \lambda_t \delta_t f_t(x') + V\), which implies \(x^* \in S(x, \varepsilon) + V\). Similarly, for \(y^*_t \in \partial f_t(x)\) one concludes that \(y^* \in [0, \varepsilon] \cdot S(x, \varepsilon) + V\), and from the arbitrariness of \(\varepsilon > 0\) and \(V \in N_0(w^*)\) we conclude the proof of (22).

Finally to prove (23) we notice that if the functions are uniformly locally Lipschitz at \(\bar{x}\) with constant \(K\), then assuming that \(\varepsilon > 0\) is small enough, we have that for any \(t \in T, x \in \mathbb{B}(\bar{x}, \varepsilon)\) and \(|f_t(x) - f(\bar{x})| \leq \varepsilon\) we also have \(f_t(\bar{x}) \geq f(\bar{x}) - (K + 1)\varepsilon\), which means \(t \in T_{(K+1)/\varepsilon}(\bar{x})\).

The next example shows an application of the above results with a family which is not uniformly locally Lipschitz. This example is important because, on the one hand, it provides an exact upper-estimation of the supremum function of a family of functions which are not uniformly locally Lipschitz, and, on the other hand it gives us a nonconvex upper-estimation.

**Example 4.10.** Consider \(T = (0,1)\) and the family of functions \(f_t : \mathbb{R}^2 \to \mathbb{R}\) given by

\[ f_t(x, y) = tx^2 - \frac{|y| + 1}{t}. \]

Here, it is important to notice that all the functions are Lipschitz continuous, but not uniformly Lipschitz continuous, so the results of [28] cannot be applied. Nevertheless, we can apply Theorem 4.9. Indeed, first the supremum function is given by \(f(x, y) = x^2 - |y| - 1\). The limiting subdifferential of \(f\) at \((\bar{x}, \bar{y}) = (0,0)\) is \(\partial f(0,0) = \{0\} \times \{-1,1\}\) and the value of \(f\) at this point is \(f(0,0) = -1\). Now, we compute the limiting subdifferential of \(f\) at \((\bar{x}, \bar{y})\) using Theorem 4.9. Pick \(z^*\) in the right-hand side of (22), then there exist \(\varepsilon_n \to 0^+, F_n \in \mathcal{P}_t(T), (x_n, y_n) \in \varepsilon_n \mathbb{B}, \) and \(\lambda_n \in \Delta(F_n)\) such that \(|f_{t_n}(x, y_n) - f(0,0)| \leq \varepsilon_n, f_{t_n}(x_n, y_n) = f_{t_n}(x_n, y_n)\) for all \(t \in F_n\) and \(z^*_n \in \sum_{t \in F_n} \lambda_n \partial f_{t_n}(x_n, y_n) + \varepsilon_n \mathbb{B}^*\). Now the equation

\[ tx^2_n - \frac{|y_n| + 1}{t} = sx^2_n - \frac{|y_n| + 1}{s} \]

implies \(t = s\), and consequently \(F_n = \{t_n\}\).

Now, using the inequality \(|f_{t_n}(x, y_n) - f(0,0)| = |f_{t_n}(x, y_n) + 1| \leq \varepsilon_n\) one gets \(t_n \to 1\). Therefore, \(z^*_n \in \{(2t_n x_n^2, \frac{1}{t_n}), (2t_n x_n^2, -\frac{1}{t_n})\} + \varepsilon_n \mathbb{B}^*\) with \(t_n \to 1, x_n \to 0\) and \(\varepsilon \to 0, \) consequently \(z^* \in \{0\} \times \{-1,1\}\).

In order to derive a more precise estimation of the subdifferential of the supremum function in [28, Definition 3.4], the authors introduced the definition of *equicontinuous subdifferentiability*. This notion involves some *uniform continuity* of the subdifferentials of the data functions \(f_t\)’s for points close to the active index set.

**Definition 4.11.** Let \(f_t : X \to \mathbb{R} \cup \{\infty\}\) be a family of lsc functions indexed by \(t \in T\). The family is called *equicontinuously subdifferentiable at \(\bar{x} \in X\) if for any weak*-neighborhood \(V\) of the origin in \(X^*\) there is some \(\varepsilon > 0\) such that

\[ \partial f_t(x) \subseteq \partial f_t(\bar{x}) + V, \text{ for all } t \in T_\varepsilon(\bar{x}) \text{ and all } x \in \mathbb{B}(\bar{x}, \varepsilon). \]
Although this definition is precisely for the framework of [28], our formulae involves the singular subdifferential of the nominal data for points close to the point of interest, due to the possible lack of Lipschitz continuity of our data. For that reason we introduce the following definition, which is satisfied trivially when the nominal data is Lipschitz continuous.

**Definition 4.12.** Let \( f_t : X \to \mathbb{R} \cup \{0\} \) be a family of lsc functions indexed by \( t \in T \). The family is called singular equicontinuously subdifferentiable at \( \bar{x} \in X \) if for any weak*-neighborhood \( V \) of the origin in \( X^* \) there is some \( \varepsilon > 0 \) such that

\[
\partial^\infty f_t(x) \subseteq \partial^\infty f_t(\bar{x}) + V, \quad \text{for all } t \in T \text{ and all } x \in B(\bar{x}, \varepsilon).
\]

Finally, we say that the family of functions \( \{f_t : t \in T\} \) is total equicontinuously subdifferentiable at \( \bar{x} \in X \) if \( \{f_t : t \in T\} \) is equicontinuously subdifferentiable and singular equicontinuously subdifferentiable at \( \bar{x} \in X \).

Using the notion of total equicontinuously subdifferentiable we have the following tighter formulae, which represents an extension of [28, Proposition 3.5].

**Theorem 4.13.** In the setting of Theorem 4.8 assume that the family of functions \( \{f_t\}_{t \in T} \) is total equicontinuously subdifferentiable at \( \bar{x} \) and

\[
\lim_{x \to \bar{x}} \sup_{t \in T} |f_t(x) - f_t(\bar{x})| = 0.
\]

Then

\[
\partial f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^w \left\{ \sum_{t \in T} \lambda_t \circ \partial f_t(\bar{x}) : \begin{array}{l}
\lambda \in \Delta(T) \text{ and } \\
\text{supp } \lambda \subseteq T_\varepsilon(\bar{x}) \end{array} \right\}
\]

\[
\partial^\infty f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^w \left\{ [0, \varepsilon] \cdot \left( \sum_{t \in T} \lambda_t \circ \partial f_t(\bar{x}) : \begin{array}{l}
\lambda \in \Delta(T) \text{ and } \\
\text{supp } \lambda \subseteq T_\varepsilon(\bar{x}) \end{array} \right) \right\}.
\]

**Proof.** Consider \( x^* \in \partial f(\bar{x}), \varepsilon > 0 \) and \( V \) a weak*-neighborhood of the origin. First, by (24) and (25) we can take \( \gamma_1 > 0 \) such that for all \( x \in B(\bar{x}, \gamma_1) \)

\[
\partial f_t(x) \subseteq \partial f_t(\bar{x}) + V, \quad \text{for all } t \in T_{\gamma_1}(\bar{x}) \quad \text{and} \quad \partial^\infty f_t(x) \subseteq \partial^\infty f_t(\bar{x}) + V \quad \text{for all } t \in T.
\]

Second, by (26) we can take \( \gamma_2 > 0 \) such that

\[
|f_t(x) - f_t(\bar{x})| \leq \gamma_1/2, \quad \forall t \in T, \quad \forall x \in B(\bar{x}, \gamma_2).
\]

Now, by Theorem 4.8 we have that for \( \gamma = \min\{\gamma_1/2, \gamma_2, \varepsilon/2\} \)

\[
x^* \in \mathcal{S}(\bar{x}, \gamma) + V.
\]

Whence, there exists \( F \in \mathcal{P}_t(T) \), \( \lambda \in \Delta(F) \) and \( x' \in B(\bar{x}, \gamma) \) such that \( |f_F(x') - f(\bar{x})| \leq \gamma \) and \( f_F(x') = f_t(x') \) for all \( t \in \text{supp } \lambda \) and

\[
x^* \in \sum_{t \in F} \lambda_t \circ \partial f_t(x') + V.
\]

Hence, by (31) we have that for all \( t \in \text{supp } \lambda \)

\[
f(\bar{x}) \leq f_F(x') + \gamma = f_t(x') + \gamma \leq f_t(\bar{x}) + \gamma_1/2 + \gamma \leq f_t(\bar{x}) + \gamma_1,
\]
which means that \( t \in T_{\gamma_1}(\bar{x}) \) and consequently \( \text{supp} \lambda \subseteq T_{\gamma_1}(\bar{x}) \). Now, by (29), (30), and (32) we have

\[
x^* + \sum_{t \in T} \lambda_t \cdot \partial f_t(x^*) + \sum_{\lambda_t = 0} \partial \lambda^\infty f_t(x^*) + V \leq \sum_{\lambda_t > 0} \lambda_t \cdot \partial f_t(\bar{x}) + \sum_{\lambda_t = 0} \partial \lambda^\infty f_t(\bar{x}) + V + V + V \leq \sum_{t \in T} \left\{ \lambda_t \cdot \partial f_t(\bar{x}) : \lambda \in \Delta(T) \text{ and } \text{supp} \lambda \subseteq T_e(\bar{x}) \right\} + V + V + V.
\]

Finally, from the arbitrariness of \( \varepsilon \) and \( V \) we conclude (27). The proof of (28) is similar, so we omit the proof.

5. The convex subdifferential. This section is devoted to giving formulae for the convex subdifferential. Due to the closure of the graph of the convex subdifferential under bounded nets with respect to the \( \| \cdot \| \times w^* \)-topology in \( X \times X^* \), we can obtain a similar result to Theorem 4.8 by changing the SNEC assumption for a similar one using nets instead of sequences. For this purpose, it is better to express the limiting condition of Theorem 4.8 in terms of the normal cone of the domain of each function \( f_t \), more precisely, we recall that for any lsc convex function \( h \), the normal cone to the domain of \( h \) at a point \( x \) is given by

\[
N_{\text{dom } h}(x) := \{ x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \forall y \in \text{dom } h \}.
\]

Using this notation we establish the following result.

**Theorem 5.1.** Let \( \{ f_t : t \in T \} \) be a family of proper convex lsc functions satisfying the following assumptions: There exists a neighborhood \( U \) of \( \bar{x} \) such that

a) For all \( x \in U \), all but one of the functions \( \{ f_t : t \in T \} \) and every net \((\lambda_t, x_t, x_t^*) \in [0, +\infty) \times X \times X^* \) satisfying \( \lambda_t \to 0 \), \( x_t \xrightarrow{\Delta} x \), \( x_t^* \in \partial f(x_t) \) and \( \lambda_t x_t^* \xrightarrow{\ast} 0 \) one has \( \| \lambda_t x_t^* \| \to 0 \).

b) For all \( x \in U \) and all \( F \in \mathcal{P}_1(T) \)

\[
w_t^* \in N_{\text{dom } f_t}(x), \ t \in F \quad \text{and} \quad \sum_{t \in F} w_t^* = 0 \quad \text{implies} \quad w_t^* = 0, \ \text{for all} \ t \in F.
\]

Then

\[
\partial f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{co} \left\{ \bigcup_{t \in F_1} \partial f_t(x^*) \right\} + \sum_{t \in F_2} N_{\text{dom } f_t}(x^*)
\]

and the union is over all \( F_1, F_2 \in \mathcal{P}_1(T) \) and \( x' \in B(\bar{x}, \varepsilon) \) such that \( |f_t(x') - f(\bar{x})| \leq \varepsilon \) and \( f_t(x') = f_{t_0}(x') \) for all \( t' \in F_1 \). Moreover, the equality holds, whenever the function \( f \) is continuous at some point, or the space \( X \) is finite-dimensional.

**Proof.** Since the proof of (33) relies on similar arguments as Theorem 4.8 (but without the use of techniques of separable reduction) we prefer to omit the proof.
Now, any point in the right-hand side of (33) is the limit of a net \( w^* \), which has the form of \( w^* = \sum \lambda_v(t)v_v(t) + \sum w^*_v(t) \) with \( v_v \in \partial f_1(x_v) \), \( w^*_v(t) \in N_{\text{dom } f_1}(x_v) \), \( \sum \lambda_v(t) = 1 \) and \( \lambda_v(t) \geq 0 \), then one gets for every \( y \in X \)

\[
\langle w^*_v, y - \bar{x} \rangle \leq f(y) - f(x) + |f_t(x') - f(\bar{x})| + \langle w^*_v, x_v - \bar{x} \rangle.
\]

Therefore, we can conclude the equality in (33) whenever the \( \lim \langle w^*_v, x_v - \bar{x} \rangle = 0 \), and this holds in particular when the function \( f \) is continuous at some point, or the space \( X \) is finite-dimensional, because in these cases the net \( \{w^*_v\} \) is bounded. \( \Box \)

The following results have the intention of establishing formulae without any qualification condition. This is possible by reducing the analysis to subspaces with nice properties for the family of functions. For that reason we denoted by \( \mathcal{F}_x \) the set of all finite-dimensional affine subspaces containing \( x \). This class of sets allows us to give formulae in any (Hausdorff) locally convex topological vector space (lcs for short). It is useful to recall some simple facts about lcs available in pioneer books such as [5, 39]: The topology on every lcs \( X \) is generated by a family of seminorms \( \{\rho_i : i \in I\} \), which will be always assumed to be up-directed, i.e., for every two points \( i_1, i_2 \in I \) there exists \( i_3 \in I \) such that \( \rho_{i_3}(x) \geq \max\{\rho_{i_1}(x), \rho_{i_2}(x)\} \) for all \( x \in X \). For a point \( \bar{x} \) in \( X \), \( r \geq 0 \) and a seminorm \( \rho \) we define \( B_\rho(\bar{x}, r) := \{x \in X : \rho(x - \bar{x}) \leq r\} \). In the (topological) dual of \( X \), denoted by \( X^* \), some examples of topologies are the \( w^* \)-topology denoted by \( w(X^*, X) \) (\( w^* \), for short), which is the topology generated by the pointwise convergence, and the strong topology denoted by \( \beta(X^*, X) \) (\( \beta \), for short), which is the topology generated by the uniform convergence on bounded sets. For a set \( A \subseteq X^* \), the symbol \( \beta \)-seq-\( A \) denotes the set of points which are the limit, with respect to the \( \beta \)-topology, of some sequence lying in \( A \). Finally, for a function \( g : X \to \mathbb{R} \), \( \text{co} \, g \) denotes the convex lsc envelope of \( g \). For more details about the theory of convex analysis in lcs we refer to [22, 33, 44].

Now, let us establish the first general formula without any qualification condition.

**Theorem 5.2.** Let \( X \) be an lcs, let \( I \) be a family of seminorms which generate the topology on \( X \). Consider a family of proper convex lsc functions \( \{f_t : t \in T\} \). Then, for all \( \bar{x} \in X \)

\[
(34) \quad \partial f(\bar{x}) = \bigcap_{\varepsilon > 0, \rho \in I} \beta \text{-seq-}\, \text{cl} \, A_{\varepsilon, L, \rho}(\bar{x}),
\]

where

\[
A_{\varepsilon, L, \rho}(\bar{x}) := \bigcup_{\varepsilon > 0, \rho \in I} \left\{ \text{co} \left( \bigcup_{t \in F_1} \partial f_{t, L}(x') \right) + \sum_{t \in F_2} N_{\text{dom } f_t \cap L}(x') \right\}.
\]

Where \( f_{t, L} := f_t + \delta_{\text{aff(dom } f \cap L)} \) and the union is over all \( x' \in B_\rho(\bar{x}, \varepsilon) \cap L \) and \( F_1, F_2 \in \mathcal{P}(T) \) such that \( f_t(x') = f_{t_1 \cup t_2}(x') \) for all \( t \in F_1 \) and \( |f_t(x') - f(\bar{x})| \leq \varepsilon \).

**Proof.** W.l.o.g. we may assume that \( \bar{x} = 0 \). Consider \( \varepsilon > 0, \rho \in \mathcal{F}_x \), and \( \rho \) a seminorm on \( X \), also we can assume that \( \rho \) is a norm on \( L \), because \( A_{\varepsilon, L, \rho}(0) \subseteq A_{\varepsilon, L, \rho}(0) \), for any \( \rho \geq \rho \). Consider \( W := \text{aff(dom } f \cap L) \), let us show that

\[
(35) \quad \partial(f + \delta_W)(0) \subseteq \beta \text{-seq-}\, \text{cl} \, A_{\varepsilon, L, \rho}(0).
\]

Indeed, take \( x^* \in \partial(f + \delta_W)(0) \) and let \( P : X \to (W, \rho) \) be a continuous linear projection. Hence, \( x^*_{||W} \) (the restriction of \( x^* \) to \( W \)) belongs to \( \partial f_{||W}(0) \). The finite-dimensionality of \( W \) gives us the continuity of \( f_{||W} \) at some point (see [38]), so the
family \( (f_t)_{|W} \) satisfies the hypotheses of Theorem 5.1. Whence, there exists a sequence \( w_n^* \rightarrow x^* \) where

\[
w_n^* \in \operatorname{co} \left( \bigcup_{t \in F_{1,n}} \partial (f_t)_{|W}(x_n^*) \right) + \sum_{t \in F_{2,n}} N_{\text{dom}(f_t)}(x_n^*)
\]

with \( F_{1,n}, F_{2,n} \in \mathcal{P}_\mathcal{I}(T), x_n^* \in B(0,\varepsilon) \cap W \) such that \( |f_t(x_n^*) - f(\bar{x})| \leq \varepsilon \) and \( f_t(x_n^*) = \max_{F_{1,n} \cup F_{2,n}} f_t(x') \) for all \( t \in F_{1,n} \).

Now we define \( x_n^* := P^*(w_n^*) + x^* - P^*(x_n^*) \), it follows that \( x_n^* \in A_{\varepsilon,L,\rho}(0) \).

Moreover, considering \( V := P^{-1}(B_W) \), where \( B_W \) is the unit ball in \( W \), we get

\[
\sigma_V(x^* - y_n^*, v) = \sup_{v \in V} \langle x^* - y_n^*, v \rangle = \sup_{v \in V} \langle P^*(w_n^*) - P^*(x_n^*), v \rangle = \sup_{h \in B_W} \langle z_n^* - x_n^*, h \rangle \rightarrow 0.
\]

Which concludes (35), then using that

\[
\partial f(0) = \bigcap_{L \in \mathcal{F}_0} \partial (f + \delta_{\text{aff}(\text{dom } f \cap L)})(0) \subseteq \bigcap_{\varepsilon > 0, \rho \in \mathcal{I}, L \in \mathcal{F}_0} \beta-\text{seq-} \text{cl } A_{\varepsilon,L,\rho}(0),
\]

we get the first inclusion in (34).

Now, pick \( x^* \in \bigcap_{\varepsilon > 0, \rho \in \mathcal{I}, L \in \mathcal{F}_0} \beta-\text{seq-} \text{cl } A_{\varepsilon,L,\rho}(0) \) and \( y \in \text{dom } f \). Then, take a sequence \( \varepsilon_n \rightarrow 0 \) and pick \( L \in \mathcal{F}_0 \) which contains \( y \) and consider \( \rho \in \mathcal{I} \) such that \( \rho \) is a norm on \( L \) and \( \rho(x_n) \rightarrow 0 \) implies \( |x^*,x| \rightarrow 0 \). Hence, there exist sequences \( F_{1,n}, F_{2,n} \in \mathcal{P}_\mathcal{I}(T), x_n \in B(0,\varepsilon_n) \cap L \) and \( w_n^* \in X^* \) such that \( w_n^* \overset{\beta}{\rightarrow} x^* \),

\[
w_n^* \in \operatorname{co} \left( \bigcup_{t \in F_{1,n}} \partial f_t \right) + \sum_{t \in F_{2,n}} N_{\text{dom } f_t}(x_n)
\]

and \( |f_t(x_n) - f(0)| \leq \varepsilon_n, f_t(x_n) = \max_{F_{1,n} \cup F_{2,n}} f_t(x_n) \) for all \( t \in F_{1,n} \), which implies

\[
\langle w_n^*,y-x_n^* \rangle \leq f(y) - f(0) + \varepsilon_n.
\]

We claim that \( \langle w_n^*,y-x_n^* \rangle \rightarrow \langle x^*,y \rangle \). Indeed, because \( \rho \) is a norm in \( L, x_n \in L \) and \( \rho(x_n) \rightarrow 0 \) necessarily \( x_n \rightarrow 0 \) with respect to the topology on \( X \). Hence, the set \( B := \{ y-x_n : n \in \mathbb{N} \} \) is bounded, so

\[
|\langle w_n^*,y-x_n^* \rangle - \langle x^*,y \rangle | = |\langle w_n^*-x^*,y-x_n^* \rangle - \langle x^*,x_n^* \rangle |
\]

\[
\leq \sigma_B(w_n^*-x^*) + |\langle x^*,x_n^* \rangle | \rightarrow 0.
\]

Finally, taking \( n \rightarrow \infty \) in (36) it yields \( \langle x^*,y-x \rangle \leq f(y) - f(0) \), which concludes the proof due to the arbitrariness of \( y \in \text{dom } f \).

The final goal of this paper is to give an alternative proof of [8, Corollary 6], which, as far as we know, appears to be the most general extension of [14, Theorem 4]. Before presenting this proof we need the following lemma. This result is interesting by itself, since it allows us to understand the subdifferential of any function in terms of the subdifferential of another function.
Lemma 5.3. Let $X$ be an lcs, let $h, g : X \to \mathbb{R}$ be two convex lsc proper functions and let $D \subseteq \text{dom } h$ be a convex subset such that

$$h(x) = g(x) \text{ for all } x \in D.$$ 

Then for every $\bar{x} \in X$

\begin{equation}
\partial(h + \delta_D)(\bar{x}) = \bigcap_{L \in \mathcal{F}_0} \left\{ \text{co} \left\{ S_L(\bar{x}) \right\} + N_{D \cap L}(\bar{x}) \right\},
\end{equation}

where $S_L(\bar{x}) := \limsup \partial(g + \delta_{\text{aff}(D \cap L)})(x^*)$, the limsup is understood to be the set of all $x^* \in X^*$, which are the limit (in the $\beta$-topology) of some sequence $x_n^* \in \partial(g + \delta_{\text{aff}(D \cap L)})(x_n)$ with $x_n \in \text{ri}_L(D)$, $x_n \xrightarrow{\delta} \bar{x}$ and $|\langle x_n^*, x_n - \bar{x} \rangle| \to 0$. Here, $\text{ri}_L(D)$ denotes the interior of $D \cap L$ with respect to $\text{aff}(D \cap L)$.

Proof. W.l.o.g. we may assume that $\bar{x} = 0$. First we notice that

\begin{equation}
\partial(h + \delta_D)(0) = \bigcap_{L \in \mathcal{L}_0} \partial(h + \delta_{D \cap L})(0) = \bigcap_{L \in \mathcal{L}_0} \partial(h + \delta_{\text{cl}(D \cap L)})(0).
\end{equation}

Indeed, the first inequality is straightforward and the second follows from the fact that $\partial(h + \delta_{D \cap L})(0) = \partial(h + \delta_{\text{cl}(D \cap L)})(0)$ thanks to the accessibility lemma (see, e.g., [1]). Now, fix $L \in \mathcal{F}_0$, define $W = \text{aff}(L \cap D)$ and consider a continuous linear projection $P : X \to W$. We claim that

\begin{equation}
\partial(h + \delta_{\text{cl}(D \cap L)})(0) \subseteq \text{co} \left\{ S_L(0) \right\} + N_{\text{dom } f \cap D \cap L}(0).
\end{equation}

Indeed, take $x^* \in \partial(h + \delta_{\text{cl}(D \cap L)})(0)$, using the same finite-dimensional representation as in the proof of Theorem 5.2, one gets the existence of a point $y^* \in \partial(h + \delta_{\text{cl}(D \cap L)})(0)$ and $z^* \in W^*$ such that $x^* = P^*(y^*) + z^*$. Then, by the finite-dimensionality of $W$ $\text{ri}_{\text{aff}(D \cap L)}$ is not empty and consequently $\partial(h + \delta_{\text{cl}(D \cap L)})(0)$ contains a line, which is not possible due to the continuity of $\partial(h + \delta_{\text{cl}(D \cap L)})(0)$. Therefore, $\partial(h + \delta_{\text{cl}(D \cap L)})(0)$ is fixed by virtue of Carathéodory’s Theorem.

Now, $\partial(h + \delta_{\text{cl}(D \cap L)})(0)$ is the interior of $\partial(h + \delta_{\text{cl}(D \cap L)})(0)$, because $u_{n,i} \in \text{ri}_L(D)$. Furthermore, $h(x') = g(x')$ for every $x' \in \text{ri}_L(D)$, which implies that $u_{n,i}^* \in \partial g_L(u_{n,i})$.

Moreover, the vectors $\alpha_{n,i} u_{n,i}^*$ are bounded (to prove this fact, one can argue by contradiction following the proof of Theorem 4.8, and then one shows that $N_{\text{dom } h + \delta_{\text{cl}(D \cap L)}}(0)$ contains a line, which is not possible due to the continuity of $\partial(h + \delta_{\text{cl}(D \cap L)})(0)$. Hence, we may assume that $\alpha_{n,i} u_{n,i}^*$ converges and $\alpha_{n,i} \xrightarrow{n \to \infty} \alpha_i$. More precisely, on the one hand for each index $i$ such that $\alpha_i = 0$, one has that $\alpha_{n,i} u_{n,i}^* \xrightarrow{i \in \alpha_i \neq 0} v_i^*$ and $v_i^* \in N_{\text{dom } f_L}(0)$. Indeed, for every $y \in \text{dom } h_L$

\begin{equation}
\langle u_{n,i}^*, y \rangle - 0 = \lim \langle \alpha_{n,i} u_{n,i}^*, y - u_{n,i} \rangle + \lim \langle \alpha_{n,i} u_{n,i}^*, u_{n,i} - 0 \rangle
\leq \lim \langle \alpha_{n,i} h(y) - h(u_{n,i}) + \langle \alpha_{n,i} u_{n,i}^*, u_{n,i} - 0 \rangle = 0.
\end{equation}

On the other hand, we have that for every index $i$ such that $\alpha_i \neq 0$, $u_{n,i}^* \xrightarrow{i \in \alpha_i \neq 0} v_i^*$ and $|\langle u_{n,i}^*, u_{n,i} \rangle| \to 0$, then using that $u_{n,i}^* \in \partial g_L(u_{n,i})$ we get $g(u_{n,i}) \xrightarrow{i \in \alpha_i = 0} g(0)$. Therefore,

$$y^* = \sum_{i : \alpha_i \neq 0} \alpha_i v_i^* + \sum_{i : \alpha_i = 0} v_i^* + \theta^*.$$
with \( v_i^* \in \limsup \partial f_{\omega_t}(u_{n,i}) \) and \( q^* := \sum_{\{i|a_i=0\}} v_i^* + \theta^* \in N_{dom f_{\omega_t}}(0) \).

Now define \( w_i^* := P^*(v_i^*), \lambda^* := z^* + P^*(q^*), w^* := \sum_{\{i|a_i\neq 0\}} \alpha_i w_i^*, w_{n,i} = P^*(u_{n,i}^*), \)

it follows that \( w_{n,i}^* \to w_i^*, |w_{n,i}^*, u_{n,i}| \to 0 \) and \( w_{n,i}^* \in \partial (g + \delta_{w_{n,i}})(u_{n,i}), u_{n,i} \in \text{ri}_L(\text{dom} h), g(u_{n,i}) \to g(0), \lambda^* \in N_{dom h\cap L}(0) \) and \( x^* = w^* + \lambda^* \), which concludes the proof of (39). Then, using (38) and (39) we conclude the first inclusion in (37).

To prove the opposite inclusion, consider \( x^* \) in the right-hand side of (37) and \( y \in D \), and consider \( L \) as the subspace generated by \( y \). Then, there are \( \alpha_i \geq 0 \) (with \( \sum_i \alpha_i = 1 \)), \( x_{n,i}^* \in \partial (g + \delta_{\text{aff}(D \cap L)})(x_{n,i}) \) and \( x_{n,i} \in \text{ri}_L(D) \) such that \( x_{n,i} \to 0 \), \( x_{n,i} \to y_{i,n}, |x_{n,i,n}| \to 0 \) and \( x^* = \sum_i \alpha_i y_{i,n}^* + \lambda^* \). Moreover, because \( x_{n} \in \text{ri}_L(D) \) and \( h = g \) in \( D \), we get \( \partial (g + \delta_{\text{aff}(D \cap L)})(x_{n}) = \partial (h + \delta_{\text{aff}(D \cap L)})(x_{n}) \). Then,

\[
\langle x^*, y \rangle = \sum_i \alpha_i y_{i,n}^* + \lambda^*, y \leq \sum_i \alpha_i \lim_n (x_{n,i}^*, y - x_{n,i}) + \lim_n \langle x_{n,i}, x_{n,i} \rangle \\
\leq \sum_i \alpha_i \lim_n (h(y) - h(x_{n,i})) = h(y) - h(0).
\]

From the arbitrariness of \( y \) we conclude that \( x^* \in \partial (h + \delta_D)(0) \), which concludes the proof of (37).

**Theorem 5.4.** Let \( X \) be an lcs and let \( \{f_t : t \in T \} \) be an arbitrary family of functions and let \( D \subseteq \text{dom} \text{co} f \) be a convex set such that

\[
\text{co}(f + \delta_D)(x) = \sup_{t \in T} \text{co} f_t(x) \text{ for all } x \in D.
\]

Then for all \( \bar{x} \in X \)

\[
\partial(f + \delta_D)(\bar{x}) = \bigcap_{\varepsilon > 0} \text{co} \left( \bigcup_{t \in T_{\varepsilon}} \partial f_t(\bar{x}) \right) + N_{D \cap L}(\bar{x}).
\]  

**Proof.** W.l.o.g we can assume that \( \bar{x} = 0 \). Because the inclusion \( \supseteq \) is direct, we focus on the opposite one. To prove this inclusion, we can assume that \( \partial(f + \delta_D)(0) \neq \emptyset \), in particular \( (f + \delta_D)(x) = \text{co}(f + \delta_D)(x) \). First, we denote by \( h = \text{co}(f + \delta_D), g_t := \text{co} f_t \) and \( g = \sup_{t \in T} g_t \), then we apply Lemma 5.3 and we get

\[
\partial(f + \delta_D)(0) \subseteq \partial h(0) = \bigcap_{L \in F_0, \varepsilon > 0} \left\{ \text{co} \{S_L(0)\} + N_{D \cap L}(0) \right\}.
\]

We claim that for every \( L \in F_0, \varepsilon > 0 \) and \( U \in N_0(w^*) \)

\[
S_L(0) \subseteq \text{co} \left( \bigcup_{t \in T_{\varepsilon}(0)} \partial f_t(0) \right) + N_{D \cap L}(0) + U + U,
\]

where \( S_L(0) \) was defined in Lemma 5.3. Indeed, consider \( x^* \in S_L(0) \), then by definition there exist sequences \( y_n \in \text{ri}_{\text{aff}(D \cap L)}(D) \) and \( y_n^* \in \partial (g + \delta_{\text{aff}(D \cap L)})(y_n) \) such that \( y_n^* \to x^*, |\langle y_n^*, y_n \rangle | \to 0 \) and \( \langle g(y_n) - g(0) \rangle \to 0 \).

Now, the restriction of each \( y_n^* \) to \( W := \text{aff}(D \cap L) \) belongs to \( \partial g_{W}(y_n) \) and \( y_n \in \text{ri}_{W}(\text{dom} g_{W}) \). Since the function \( g_{W} \) is locally bounded at \( y_n \) we can find a constant \( M_n \) and a closed convex neighborhood \( V_n \) of zero (relative to \( W \)) such that \( g_t(x) \leq g_t(y_n) + M_n - g_t(y_n), \forall x \in y_n + V_n \).
Consequently, by [44, Corollary 2.2.12]

\[ |g_t(x) - g(x')| \leq 3M_{t,n}\rho_{V_n}(x - y), \forall x, x' \in y_n + \frac{1}{2}V_n, \]

where \( M_{t,n} := M_n - g_t(y) \) and \( \rho_{V_n} \) is the Minkowski’s functional associated to \( V_n \), that is, \( \rho_{V_n}(u) := \inf\{s > 0 : u \in sV_n\} \). In particular, each function \((g_t)_{|W} \) is Lipschitz continuous on \( \frac{1}{2}V_n \), it allows us to apply Theorem 4.9 and by a diagonal argument we yield that there exists a sequence of sets \( F_n \in \mathcal{P}_T(T) \), and there are sequences of vectors \( x_n \in W, x^*_n \in \partial(g_t)_{|w}(x_n) \) together with scalars \((\lambda_t(n)) \in \Delta(F_n)\) such that \( x_n \to 0, |g_{F_n}(x_n) - g(0)| \to 0 \) and \( g_t(x_n) = g_{F_n}(x_n) \) for all \( t \in F_n \) and \( x^*_n = \sum_{t \in F_n} \lambda_t(n)x^*_t(n) \to x^*_w \). From the fact that the dimension of \( W \) is finite, we can assume that \#\( F_n \leq \operatorname{dim}(W) + 1 \). Hence, necessarily the points \( x^*_t(n) \) are uniformly bounded in \( W \), otherwise \( N_{\operatorname{dom} f_{|W}(0)} \) contains a line, which is not possible due to \( \text{ri}_{\operatorname{aff}(\mathcal{L} \cap \operatorname{dom} g)}(\operatorname{dom} g_{|W}) \neq \emptyset \) (it can be seen using similar arguments as those given in the proof of Theorem 5.2). Then, we can assume that there exists \( F \in \mathcal{P}_T(T), x \in W, x^*_t \in \partial(g)_{|w}(x) \) and \((\lambda_t) \in \Delta(F)\) such that \( \max_{t \in F} |\langle x^*_t, x \rangle| \leq \varepsilon/5, |g_t(x) - g(0)| \leq \varepsilon/5, g_t(x) = f_{F}(x) \) for all \( t \in F \) and

\[ x^*_w = \sum_{t \in F} \lambda_t x^*_t + (P^*)^{-1}(U), \]

where \( P \) is a continuous projection from \( X \) to \( W \). Then,

\[ x^*_w = \sum_{t \in F} \lambda_t w^*_t + x^* - P^*(y^*_w) + U, \]

here \( w^*_t := P^*(x^*_t) \) and \( w^*_t \in \partial(g_t + \delta_W)(x) \). Furthermore, for all \( t \in F \)

\[ f_t(0) + 2\varepsilon/5 \geq g_t(0) + 2\varepsilon/5 \geq g_t(0) + |\langle x_t, x \rangle| + \varepsilon/5 \geq g_t(x) + \varepsilon/5 \geq g(0) = f(0), \]

Now by Hirriat-Hurruty-Phelps’ formula [15, Theorem 2.1]

\[ \partial(g_t + \delta_W)(x) \subseteq \partial_{\varepsilon/5} g_t(x) + W^⊥ + U, \]

which implies the existence of some point \( \tilde{w}^*_t \in \partial_{\varepsilon/5} g_t(x) \) such that

\[ w^*_t \in \tilde{w}^*_t + W^⊥ + U. \]

Now, let us show that \( \tilde{w}^*_t \in \partial_{\varepsilon} f_t(0) \). Indeed, consider \( z \in X \), then

\[ \langle \tilde{w}^*_t, z \rangle = \langle \tilde{w}^*_t, z - x \rangle + |\langle w^*_t, x \rangle| \leq g_t(z) - g_t(x) + \varepsilon/5 + \varepsilon/5 \leq g_t(z) - g_t(0) + 3\varepsilon/5 \leq f_t(x) - f_t(0) - g_t(0) + 3\varepsilon/5 \leq f_t(z) - f_t(0) + \varepsilon \text{ (by (44)).} \]

Now, according to (43)–(45) we get (42) and from the arbitrariness of \( \varepsilon > 0 \) and \( U \) we conclude that

\[ S_L(0) + N_{D \cap L}(0) \subset \bigcap_{\varepsilon > 0} \text{cl} w^* \left( \text{co} \left( \bigcup_{t \in T(x)} \partial_{\varepsilon} f_t(x) \right) + N_{D \cap L}(x) \right). \]

Finally, using (41) and (46) we conclude the desired inclusion in (40).

**Remark 5.5.** It is worth mentioning that Theorem 5.4 represents a slight extension of [8, Corollary 6], because in this result the authors have assumed that the data functions \( f_t's \) are convex and proper.
6. Conclusions. In this paper, we have provided general formulae for the supremum function of an arbitrary family of lsc functions.

In Section 3, we provided general fuzzy calculus rules in terms of the Fréchet subdifferential. Our approach follows from establishing these fuzzy calculus rules for an increasing family of functions (see Proposition 3.7), where the key tool is the introduction of robust infimum. Later, in Theorem 3.8, we used the power set ordered by inclusion to get general fuzzy calculus rules of an arbitrary family of functions, without any qualification condition, as far as we know this approach is novel.

In Section 4 we established the main results of the paper, where we replaced the Lipschitz continuous assumption of the data by some limiting condition in terms of the singular subdifferentials (see Item (c) and (17)). It has not escape our notice that these kind of conditions are becoming more popular in providing subdifferential calculus rules (see, e.g., [2,3,16–18,25,26,36]). This section was divided into Subsection 4.1 and Subsection 4.2, which focused attention on finite-dimensional and infinite-dimensional settings respectively. In both subsections we gave formulae for the subdifferential of the supremum function under different conditions. Here, It is worth comparing Theorem 4.2 and Theorem 4.8. The main difference between these two results is that the first one is a convex upper-estimate, and the second one corresponds to a non-convex upper-estimate (as we showed in Example 4.10). This difference can be explained, because Theorem 4.2 uses a limiting condition only at the point of interest (see, Item (c)), but Theorem 4.8 uses the information of the subdifferential at a neighborhood of the point of interest (see (17)).

Finally, in Section 5 we shown that our approach can be used to get new formulae for the convex subdifferential, with and without qualification conditions, of the supremum function (see Theorem 5.1 and Theorem 5.2), and also, it allows us to recover [8, Corollary 6] using Theorem 4.8 (see Theorem 5.4), which in particular shows a unifying approach to the study of the subdifferential of the supremum function.

REFERENCES

[1] J. M. Borwein and R. Goebel. Notions of relative interior in Banach spaces. J. Math. Sci. (N. Y.), 115(4):2542–2553, 2003. Optimization and related topics, 1.
[2] J. M. Borwein, B. S. Mordukhovich, and Y. Shao. On the equivalence of some basic principles in variational analysis. J. Math. Anal. Appl., 229(1):228–257, 1999.
[3] J. M. Borwein and Q. J. Zhu. Techniques of variational analysis. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 20. Springer-Verlag, New York, 2005.
[4] J. M. Borwein and D. Zhuang. On Fan’s minimax theorem. Math. Programming, 34(2):232–234, 1986.
[5] N. Bourbaki. Topological vector spaces. Chapters 1–5. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1987. Translated from the French by H. G. Eggleston and S. Madan.
[6] F. H. Clarke. Optimization and nonsmooth analysis, volume 5 of Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 1990.
[7] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski. Nonsmooth Analysis and Control Theory. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 1998.
[8] R. Correa, A. Hantoute, and M. A. López. Towards supremum-sum subdifferential calculus free of qualification conditions. SIAM J. Optim., 26(4):2219–2234, 2016.
[9] M. Cúth and M. Fabian. Rich families and projectional skeletons in Asplund WCG spaces. J. Math. Anal. Appl., 448(2):1618–1632, 2017.
[10] M. Fabian and A. D. Ioffe. Separable reductions and rich families in the theory of Fréchet subdifferentials. J. Convex Anal., 23(3):631–648, 2016.
[11] K. Fan. Minimax theorems. Proc. Nat. Acad. Sci. U. S. A., 39:42–47, 1953.
[12] A. Hantoute. Subdifferential set of the supremum of lower semi-continuous convex functions and the conical hull intersection property. *Top.*, 14(2):355–374, 2006.

[13] A. Hantoute and M. A. López. A complete characterization of the subdifferential set of the supremum of an arbitrary family of convex functions. *J. Convex Anal.*, 15(4):831–858, 2008.

[14] A. Hantoute, M. A. López, and C. Zălinescu. Subdifferential calculus in convex analysis: a unifying approach via pointwise supremum functions. *SIAM J. Optim.*, 19(2):863–882, 2008.

[15] J.-B. Hiriart-Urruty and R. R. Phelps. Subdifferential calculus using $\varepsilon$-subdifferentials. *J. Funct. Anal.*, 118(1):154–166, 1993.

[16] A. D. Ioffe. Approximate subdifferentials and applications. I. The finite-dimensional theory. *Trans. Amer. Math. Soc.*, 281(1):389–416, 1984.

[17] A. D. Ioffe. Approximate subdifferentials and applications. II. *Mathematika*, 33(1):111–128, 1986.

[18] A. D. Ioffe. Approximate subdifferentials and applications. III. The metric theory. *Mathematika*, 36(1):1–38, 1989.

[19] A. D. Ioffe. On the theory of subdifferentials. *Adv. Nonlinear Anal.*, 1(1):47–120, 2012.

[20] B. S. Mordukhovich. Variational analysis and generalized differentiation. I, volume 330 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006. Basic theory.

[21] B. S. Mordukhovich. Variational analysis and generalized differentiation. II, volume 331 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006. Applications.

[22] B. S. Mordukhovich. Variational Analysis and Applications, volume 8. Springer, Cham, 2018.

[23] J.-P. Penot. *Calculus without derivatives*, volume 266 of *Graduate Texts in Mathematics*. Springer, New York, 2013.

[24] H. V. Ngai and M. Théra. A fuzzy necessary optimality condition for non-Lipschitz optimization in Asplund spaces. *SIAM J. Optim.*, 12(3):656–668, 2002.

[25] T. T. A. Nghia. A nondegenerate fuzzy optimality condition for constrained optimization problems without qualification conditions. *Nonlinear Anal.*, 75(18):6379–6390, 2012.

[26] J.-P. Penot. *Calculus without derivatives*, volume 266 of *Graduate Texts in Mathematics*. Springer, New York, 2013.

[27] B. S. Mordukhovich and T. T. A. Nghia. Subdifferentials of nonconvex supremum functions and their applications to semi-infinite and infinite programs with Lipschitzian data. *SIAM J. Optim.*, 23(1):406–431, 2013.

[28] B. S. Mordukhovich and T. T. A. Nghia. Nonsmooth cone-constrained optimization with applications to semi-infinite programming. *Math. Oper. Res.*, 39(2):301–324, 2014.

[29] B. S. Mordukhovich and H. M. Phan. Tangential extremal principles for finite and infinite systems of sets, I: basic theory. *Math. Program.*, 136(1, Ser. B):3–30, 2012.

[30] B. S. Mordukhovich and Y. H. Shao. Nonsmooth sequential analysis in Asplund spaces. *Trans. Amer. Math. Soc.*, 348(4):1295–1280, 1996.

[31] P.-J. Laurent. *Approximation et optimisation*. Hermann, Paris, 1972. Collection Enseignement des Sciences, No. 13.

[32] A. Stefănescu. A general min-max theorem. *Optimization*, 16(4):497–504, 1985.
[42] L. Thibault. Sequential convex subdifferential calculus and sequential Lagrange multipliers. 
*SIAM J. Control Optim.*, 35(4):1434–1444, 1997.

[43] L. Thibault. Limiting convex subdifferential calculus with applications to integration and 
maximal monotonicity of subdifferential. In *Constructive, experimental, and nonlinear 
analysis (Limoges, 1999)*, volume 27 of *CMS Conf. Proc.*, pages 279–289. Amer. Math. 
Soc., Providence, RI, 2000.

[44] C. Zălinescu. *Convex analysis in general vector spaces*. World Scientific Publishing Co., Inc., 
River Edge, NJ, 2002.