BOSONIZATION OF A TOPOLOGICAL COSET MODEL AND NON-CRITICAL STRING THEORY

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Abstract

We analyze the relation between a topological coset model based on super $SL(2,R)/U(1)$ coset and non-critical string theory by using free field realization. We show that the twisted $N = 2$ algebra of the coset model can be naturally transformed into that of non-critical string. The screening operators of the coset models can be identified either with those of the minimal matters or with the cosmological constant operator. We also find that another screening operator, which is intrinsic in our approach, becomes the BRST nontrivial state of ghost number 0 (generator of the ground ring for $c = 1$ gravity).
The relation between non-critical strings and topological field theories is the subject of current interest. It has long been suggested that the latter theories describe the unbroken phase of gravity [1], but their precise relation has not been clear.

It has been known that the twisting of $N = 2$ superconformal field theory gives rise to topological theory [1, 2]. This suggests that any non-critical string theories may have hidden $N = 2$ superconformal symmetry. Indeed, several authors have observed that the BRST current and the antighost field $b(z)$ generate an algebra that is quite similar but apparently not identical to the $N = 2$ superconformal algebra [3]. It turns out that the BRST current can be modified by total derivative terms so that the antighost and the physical BRST current exactly generate a topologically twisted $N = 2$ superconformal algebra [4, 5]. This does not identify, however, the structure of the models with $N = 2$ symmetry.

Recently, rather nontrivial correspondence between super $SL(2, R)/U(1)$ coset model [6] and $c = 1$ string has been analyzed through twisted $N = 2$ structure. Mukhi and Vafa [7] have revealed an amazing correspondence between these two models for $k = 3$. In this letter, we discuss the relation of these models and the generalization of the correspondence to the minimal models coupled to gravity by means of the free field realization. We find that there is another interesting correspondence for $k = 1$.

Super $SL(2, R)/U(1)$ model is described by the bosonic coset model of $SL(2, R) \times U(1)/U(1)$ [6]. For a representation of $SL(2, R)_k$, we use the following free field realization [6]:

\[
\begin{align*}
J^+ &= \frac{1}{\sqrt{k-2}} \left[ \beta \gamma^2 - \sqrt{2(k-2)} \gamma \partial \phi_3 + k \partial \gamma \right], \\
J^3 &= \beta \gamma - \sqrt{\frac{k-2}{2}} \partial \phi_3, \\
J^- &= \sqrt{k-2} \beta,
\end{align*}
\]

(1)

where the basic fields $(\beta, \gamma)$ with the dimension $(1, 0)$ and $\phi_3$ satisfy

\[
\beta(z)\gamma(w) \sim \frac{1}{z - w}, \quad \phi_3(z)\phi_3(w) \sim -\ln(z - w).
\]

(2)
We use fermions $\bar{\psi}$ and $\psi$ to represent a $U(1)$ element. Then $N = 2$ algebra can be realized by means of these fields as follows [3]:

$$T = \frac{1}{k-2} \left[ \frac{1}{2} (J^+, J^-) - (J^3)^2 \right] - \frac{1}{2} \bar{\psi} \partial \psi - \frac{1}{2} \psi \partial \bar{\psi} + \frac{1}{k-2} (J^3 - \psi \bar{\psi})^2$$

$$G^+ = \frac{1}{\sqrt{k-2}} \psi J^+ = \frac{1}{k-2} \psi \left[ \beta \gamma - \sqrt{\frac{k-2}{2}} \partial \phi_3 - \psi \bar{\psi} \right]^2,$$

$$G^- = \frac{1}{\sqrt{k-2}} \bar{\psi} J^- = \bar{\psi} \beta,$$

$$J = \frac{1}{k-2} (k \psi \bar{\psi} - 2 J^3). \quad (3)$$

The screening operators of $SL(2, R)$ are given by

$$V_- = \beta e^{\sqrt{2(\pm)} \phi_3}, \quad V_+ = \beta^{k-2} e^{\sqrt{2(\mp)} \phi_3}. \quad (4)$$

We would like to identify $G^-$ with the antighost $b$ of the non-critical string. For this purpose, it turns out to be convenient to bosonize $(\beta, \gamma)$ and $(\bar{\psi}, \psi)$ fields as

$$\beta = e^{\phi_1 + i \phi_2}, \quad \gamma = -i \partial \phi_2 e^{-\phi_1 - i \phi_2},$$

$$\bar{\psi} = e^{-i \sigma}, \quad \psi = e^{i \sigma}, \quad (5)$$

where fields $\phi_1, \phi_2$ and $\sigma$ are normalized to have the ordinary correlations, e. g. $\phi_1(z) \phi_1(w) \sim -\ln(z - w)$. Notice that this is a little different from the usual bosonization rule [10].

We should also note that we have an additional screening operator $S(z)$ in this particular bosonization:

$$S(z) = e^{i \phi_2(z)}. \quad (6)$$

In terms of the free bosons $\phi_1, \phi_2, \phi_3$ and $\sigma$, the generators of $N = 2$ elements are written as

$$T = -\frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 + \frac{1}{2} \partial^2 \phi_1 - \frac{i}{2} \partial^2 \phi_2$$
\[ -\frac{1}{2} (\partial \phi_3)^2 + \frac{1}{\sqrt{2(k-2)}} \partial^2 \phi_3 - \frac{1}{2} (\partial \sigma)^2 \]
\[ + \frac{1}{k-2} \left( \partial \phi_1 - \frac{\sqrt{k-2}}{2} \partial \phi_3 - i \partial \sigma \right)^2, \]
\[ G^+ = \frac{1}{k-2} \left[ \frac{i(k-2)}{2} \partial \phi_1 \partial \phi_2 - (k-1)(\partial \phi_2)^2 \right. \]
\[ + i \sqrt{2(k-2)} \partial \phi_2 \partial \phi_3 - i(k-1) \partial^2 \phi_2 \left. \right] e^{i\sigma - \phi_1 - \phi_2}, \]
\[ G^- = e^{-i\sigma + \phi_1 + i\phi_2}, \]
\[ J = \frac{1}{k-2} \left[ k i \partial \sigma - 2 \left( \partial \phi_1 - \frac{\sqrt{k-2}}{2} \partial \phi_3 \right) \right]. \tag{7} \]

In these expressions, we have three characteristic combinations of bosons. The first is the one lying in the direction of \( N = 2 \) \( U(1) \) charge, which we call \( u \):
\[ iu = i \frac{k}{k-2} \sigma - \frac{2}{\sqrt{k(k-2)}} \phi_1 + \frac{\sqrt{2}}{k} \phi_3. \tag{8} \]

The second represents the direction eliminated by the coset construction:
\[ \sqrt{\frac{k-2}{2}} \rho \equiv -i\sigma + \phi_1 - \sqrt{\frac{k-2}{2}} \phi_3. \tag{9} \]

The third one is in the direction of the screening charge of \( SL(2,R) \):
\[ \sqrt{\frac{2}{k-2}} \phi \equiv \phi_1 + i\phi_2 + \sqrt{\frac{2}{k-2}} \phi_3. \tag{10} \]

We finally define the remaining combination out of four bosons in such a way that it is orthogonal to the other three combinations:
\[ i \sqrt{\frac{2}{k-2}} \tau = \sqrt{\frac{k-2}{k}} \phi_1 + \sqrt{\frac{k}{k-2}} \phi_2 + \frac{\sqrt{2}}{k} \phi_3. \tag{11} \]

The generators of \( N = 2 \) algebra can be expressed by these bosons as
\[ T = -\frac{1}{2} (\partial u)^2 - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial \tau)^2 + \frac{k-1}{\sqrt{2(k-2)}} \partial^2 \phi - \frac{\sqrt{k}}{2} i \partial^2 \tau, \]
\[ G^- = e^{-\sqrt{\frac{k}{2(k-2)}} iu + \sqrt{\frac{k}{2}} \tau}, \]
\[ J = \sqrt{\frac{k}{k-2}} i \partial u. \tag{12} \]

1 The factors of \( i \) are introduced in the above field redefinitions in such a way that all fields have the ordinary correlations. We define the field \( \phi_2 \) anti-hermitian not to mix the hermiticity of the fields.
We are now going to twist this system to topological conformal models \([2]\) by defining the energy-momentum tensor as \(T \rightarrow T + \frac{1}{2} \partial J\). By defining BRST charge \(Q_B = \int \frac{dz}{2\pi i} G^+(z)\), we can find that energy-momentum tensor is BRST trivial: \(T(z) = \{Q_B, G^-(z)\}\).

The twisted energy-momentum tensor of this model is given by

\[
T = -\frac{1}{2} (\partial u)^2 - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial \tau)^2 + \frac{k-1}{\sqrt{2(k-2)}} \partial^2 \phi - i \frac{k}{2} \partial^2 \tau + i \sqrt{\frac{k}{2(k-2)}} \partial^2 u.
\]  

To make a connection with \(c \leq 1\) gravity, let us make our final field redefinition. Since \(G^-\) is given by a simple expression, it is natural to fermionize this as

\[
b \equiv G^- = e^{-\sqrt{\frac{k-2}{k}}iu + \sqrt{\frac{k-2}{k}}i\tau}, \quad c \equiv e^{\sqrt{\frac{k-2}{k}}iu - \sqrt{\frac{k-2}{k}}i\tau}.
\]

Defining the other orthogonal combination of fields \(u\) and \(\tau\) as

\[
X \equiv \sqrt{\frac{2}{k}} u + \sqrt{\frac{k-2}{k}} \tau,
\]

we finally get the elements of the topological algebra of non-critical string:

\[
T = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial X)^2 + \frac{k-1}{\sqrt{2(k-2)}} \partial^2 \phi - \frac{k-3}{\sqrt{2(k-2)}} i \partial^2 X + T_{bc},
\]

\[
G^- = b,
\]

\[
G^+ = j_{BRST}(z) - \frac{1}{k-2} \left[ \frac{k-4}{2} \partial^2 c + i \sqrt{2(k-2)} \partial (c \partial X) \right],
\]

where \(j_{BRST}\) is the ordinary BRST current and \(T_{bc}\) is the energy-momentum tensor of ghosts with conformal weight \((2, -1)\):

\[
T_{bc} = -2b \partial c - \partial bc
\]

\[
= -\frac{1}{2} \left( \sqrt{\frac{k-2}{k}} \partial u - \sqrt{\frac{2}{k}} \partial \tau \right)^2 + \frac{3i}{2} \left( \sqrt{\frac{k-2}{k}} \partial^2 u - \sqrt{\frac{2}{k}} \partial^2 \tau \right).
\]

When \(k-2 = \frac{q}{p}\), the energy-momentum tensor in (16) is that of gravity coupled to \((p, q)\) minimal matter. We see in particular that \(k = 3\) corresponds to \(c = 1\) string. The BRST charge obtained from \(G^+\) in (16) is the usual one, and the cohomology in the boson Fock space is the same \([11, 12, 13]\). For the central charge \(c < 1\), however, we have to make further reduction using Felder cohomology \([15]\).

\(^2\)If one takes the coset representations, it may be different because of the presence of null states \([14]\).
The screening operators (4) of $SL(2, R)$ is given by

$$V_- = e^{\sqrt{\frac{2}{k-2}} \phi}, \quad V_+ = e^{\sqrt{2(k-2)} \phi}. \tag{18}$$

We see that $V_-$ can be identified with the cosmological constant operator. In this identification of matter and Liouville fields, we lack the information on the origin of screening operators for the matter field. This problem is not present, however, for $k = 3$ since screening operators are not required for $c = 1$ matter. We have thus made an explicit connection of super $SL(2, R)_k/U(1)$ at $k = 3$ and $c = 1$ gravity. These results confirm the observation of Ref. [7, 16].

We should note that here is an additional structure which differs from the analysis of Ref. [7] in this correspondence. Namely we have another screening operator (6) which arises from our particular bosonization (5) of $(\beta, \gamma)$ system. The operator can be written in the fields of $c \leq 1$ string as

$$S(z) = b e^{\sqrt{\frac{2}{k-2}} i (X+i\phi)}. \tag{19}$$

When $k = 3$, the zero form version of this operator is given by

$$x(z) = \left( cb + \frac{i}{\sqrt{2}} (\partial X - i\partial \phi) \right) e^{\frac{i}{\sqrt{2}} (X+i\phi)}, \tag{20}$$

which is exactly the state known as the Lian-Zuckerman state of ghost number 0 [11, 12, 13]. Other elementary state $y(z)$ can be obtained by $y(z) = J'_- x(z)$, with $J'_- \equiv \oint \frac{dz}{2\pi i} \exp[-i\sqrt{2}X(z)].$ As is pointed out by Witten [13], these form the ground ring of $c = 1$ string.

It is quite interesting to note that there is another identification of Liouville and matter fields. When $k-2 = -\frac{2}{p}$, the role of $X$ and $\phi$ interchanges: We can identify $\phi$ as $(p,q)$ matter and $X$ as Liouville field. The screening operators (4) or (18) of $SL(2, R)$ then become those for matter field in the $(p,q)$ minimal models:

$$V_- = e^{-i\sqrt{2pq/X}}, \quad V_+ = e^{i\sqrt{2pq/X}}. \tag{21}$$

In this case, $k = 1$ is critical in the sense that the screening operators (21) become generators of $SU(2)$ in the matter sector. With $X \leftrightarrow \phi$, another screening operator (19)
can be identified with BRST-nontrivial state of gravity coupled to minimal matter and again gives one of the generators of ground ring for $c = 1$ gravity.

In this identification, the origin of the cosmological constant operator is not clear. We have to consider the perturbation by the cosmological constant. The operator corresponding to the cosmological constant

$$ce^{i\sqrt{\frac{k-2}{2(k-2)}}X},$$

is expressed by $N = 2$ fields as

$$e^{i\sqrt{\frac{k-2}{2(k-2)}}iX},$$

which is the chiral primary field of conformal weight

$$h = \frac{1}{2} \frac{k}{k-2} = \frac{c}{6}. \tag{24}$$

This state seems to have a special meaning because it is the chiral field of the highest conformal weight, at least for the unitary series [17]. (We do not know much about the non-unitary series.) In other words, the perturbation by the cosmological constant can be identified with that by the chiral primary field of conformal weight $h = c/6$. It is quite interesting to note that this is precisely the integrable perturbation for the unitary minimal model [2].

Finally let us point out that we could have defined “$c = 1$ fields” at the outset by

$$\dot{\phi} = \frac{k-1}{2\sqrt{(k-2)}}\phi - \frac{k-3}{2\sqrt{(k-2)}}iX,$$

$$i\dot{X} = \frac{k-3}{2\sqrt{(k-2)}}\phi - \frac{k-1}{2\sqrt{(k-2)}}iX. \tag{25}$$

The energy-momentum tensor as well as the BRST operator obtained from (16) are then precisely those for $c = 1$ matter coupled to gravity irrespective of the level $k$. The cosmological constant operator (18) now takes the form

$$V_- = \exp \left[ \frac{k-1}{\sqrt{2}(k-2)} \dot{\phi} - \frac{k-3}{\sqrt{2}(k-2)}i\dot{X} \right]. \tag{26}$$
The special value $k = 3$ is singled out by the fact that this precisely gives the correct operator for $c = 1$ gravity.

In conclusion, we have shown that the topological algebra of twisted super $SL(2, R)_k/U(1)$ model can be converted to that of gravity coupled to matter field with suitable redefinition of the fields. BRST nontrivial state appears as a screening operator under this redefinition. It seems interesting to consider further correspondence between these models through free field realization.

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