Universal Algebra Applied to Hom-Associative Algebras, and More

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Abstract The purpose of this paper is to discuss the universal algebra theory of hom-algebras. This kind of algebra involves a linear map which twists the usual identities. We focus on hom-associative algebras and hom-Lie algebras for which we review the main results. We discuss the envelopment problem, operads, and the Diamond Lemma; the usual tools have to be adapted to this new situation. Moreover we study Hilbert series for the hom-associative operad and free algebra, and describe them up to total degree equal 8 and 9 respectively.

Introduction

Abstract algebra is a subject that may be investigated on many different levels of maturity. At the most elementary level that still meets the standards of mathematical rigor, the investigator simply postulates some set of axioms (usually in the form of a definition) and then goes on to derive random consequences of these axioms, hopefully topping it off with examples to illustrate the range of possible outcomes for the results that are stated (as there have been some spectacular instances of mathematical theories that died due to having no nontrivial examples where they were...
This level of investigation may produce a nicely whole theory of something, but in the hands of an immature investigator it runs a significant risk of ending up as a random collection of facts that don’t combine to anything greater than themselves; the whole of a good theory should be greater than the sum of its parts.

One way of reaching a higher level can be to investigate matters using the techniques of universal algebra, since these combine looking at concrete examples with the generality of investigating the generic case. Another way is to employ the language of category theory to investigate matters on a level that is even more abstract. Indeed, category theory has become so fashionable that modern presentations of universal algebra may treat it as a mere application of the categorical formalism. This has the advantage of allowing definitions of for example free algebras to be given that do not presuppose a specific construction machinery, but on the other hand it runs the risk of losing itself in the heavens of abstraction, because the difficulties have been postponed rather than taken care of; doing any nontrivial example may bring them all back with a vengeance. Therefore we were glad to see how Yau in [44] would proceed from an abstract categorical definition to concrete constructions of many free algebras of relevance to hom-associative and hom-Lie algebras—glad, but also a bit curious as to why the constructions were not more systematic.

For better or worse, there is probably a simple reason for someone doing _ad hoc_ constructions rather than the standard systematic ones here: even though the systematic constructions are well known within the Formal languages, Logic, and Discrete mathematics communities, they are _not so_ within the Algebra community. Therefore one aim for us in writing this paper has been to bring to the attention of the Algebra community this veritable treasure-trove of methods and techniques that universal algebra and formal languages have to offer. Another aim was of course to find out more about hom-algebras, as what as come so far is only the beginning of the exploration of these.

The first motivation to study nonassociative hom-algebras comes from quasi-deformations of Lie algebras of vector fields, in particular _q_-deformations of Witt and Virasoro algebras [1, 6, 8, 9, 10, 13, 15, 27, 32, 25]. The deformed algebras arising in connection with _σ_-derivation are no longer Lie algebras. It was observed in the pioneering works that in these examples a twisted Jacobi identity holds. Motivated by these examples and their generalisation on the one hand, and the desire to be able to treat within the same framework such well-known generalisations of Lie algebras as the color and Lie superalgebras on the other hand, quasi-Lie algebras and subclasses of quasi-hom-Lie algebras and hom-Lie algebras were introduced by Hartwig, Larsson and Silvestrov in [19, 29, 30, 31].

The hom-associative algebras play the role of associative algebras in the hom-Lie setting. They were introduced by Makhlouf and Silvestrov in [35]. Usual functors between the categories of Lie algebras and associative algebras were extended to hom-setting, see [44] for the construction of the enveloping algebra of a hom-Lie algebra. Likewise, many classical structures as alternative, Jordan, Malcev, graded algebras and _n_-ary algebras of Lie and associative type, were considered in this framework, see [2, 3, 4, 5, 7, 34, 36, 37, 38, 39, 41, 46, 47, 48, 49, 50]. Notice that Hom-algebras over a PROP were defined and studied in [51] and deformations...
of hom-type of the Associative operad from the point of view of the confluence property discussed in [26].

The main feature of all these algebras is that classical identities are twisted by a homomorphism. Pictorially, drawing the multiplication $m$ as a circle and the linear map $\alpha$ as a square, hom-associativity may be written as

$$
\begin{pmatrix}
\circ & \square
\end{pmatrix} =
\begin{pmatrix}
\bullet & \circ & \bullet
\end{pmatrix}
$$

In this paper, we summarize the basics of hom-algebras in the first section. We emphasize on hom-associative and hom-Lie algebras. We show first the paradigmatic example of $q$-deformation of $\mathfrak{sl}_2$ using $\sigma$-derivations, leading to an interesting example of hom-Lie algebra. We provide the general method and some other procedures to construct examples of hom-associative or hom-Lie algebras. We describe the free hom-nonassociative algebra constructed by Yau. It leads to free hom-associative algebra and to the enveloping algebra of a hom-Lie algebra. In Section 2 we recall the basic concepts in universal algebra as signature $\Omega$, $\Omega$-algebra, formal terms, normal form, rewriting system, and quotient algebra. We emphasize on hom-associative algebras and discuss the envelopment problem. Section 3 is devoted to operadic approach. We discuss this concept and universal algebra for operads. We provide a diamond lemma for operads and discuss ambiguities for symmetric operads. Then we focus on hom-associative algebras operad for which attempt to resolve the ambiguities. Likewise we study congruence modulo hom-associativity and Hilbert series in this case. Moreover we study Hilbert series for the hom-associative operad and compute several dozen terms of it exactly using techniques from formal languages (notably regular tree languages).

1 Hom-algebras: definitions, constructions and examples

We summarize in this section the basics about hom-associative algebras and hom-Lie algebras.

The hom-associative identity $\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z)$ is a generalisation of the ordinary associative identity $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. Study of it could be motivated simply by the creed that “one should always generalise”, and in Subsection 3.5 we will briefly consider the view that hom-associativity (in a rather abstract setting) can be considered as homogenisation of ordinary associativity, but historically the hom-associative identity was first suggested by an application; the line of thought went from $\sigma$-derivations, then to hom-Lie algebras, before finally touching upon hom-associative algebras. We sketch the $\sigma$-derivation development in the first subsection below, but the rest of the text does not depend on the material presented there, so the reader who prefers to skip to Subsection 1.2 now should have no problem doing so.
1.1 \( q \)-Deformations and \( \sigma \)-derivations

Let \( A \) be an associative \( \mathbb{K} \)-algebra with unity 1. Let \( \sigma \) be an endomorphism on \( A \). By a twisted derivation or \( \sigma \)-derivation on \( A \), we mean a \( \mathbb{K} \)-linear map \( \Delta : A \rightarrow A \) such that a \( \sigma \)-twisted product rule (Leibniz rule) holds:

\[
\Delta(ab) = \Delta(a)b + \sigma(a)\Delta(b). \tag{2}
\]

The ordinary derivative \((\partial a)(t) = a'(t)\) on the polynomial ring \( A = \mathbb{K}[t] \) is a \( \sigma \)-derivation for \( \sigma = \text{id} \). If on a superalgebra \( A = A_0 \oplus A_1 \) one defines \( \sigma(a) = a \) for \( a \in A_0 \) but \( \sigma(a) = -a \) for \( a \in A_1 \), then (2) precisely captures the parity adjustments of the product rule that derivations in such settings tend to exhibit, and it does so in a manner that unifies the even and odd cases. Returning to the the polynomial ring \( A = \mathbb{K}[t] \), the \( \sigma \)-derivation concept offers a unified framework for various derivation-like operators, perhaps most famously the Jackson \( q \)-derivation operator \((D_qa)(t) = \frac{1}{q^{-1}}(a(q^t) - a(t))\) for some \( q \in \mathbb{K} \), that has the ordinary derivative as the \( q \to 1 \) limit and the product rule \( D_q(ab)(t) = D_q(a)(t)b(t) + a(q^t)D_q(b)(t) \); this is thus a \( \sigma \)-derivation for \( \sigma(a)(t) = a(q^t) \), which acts on the standard basis for \( \mathbb{K}[t] \) as \( \sigma(t^n) = q^n t^n \). (See [24] and references therein.)

The big algebraic insight about derivations is that they form Lie algebras, from which one can go on to universal enveloping algebras and exploit the connections to formal groups and Lie groups. What about twisted derivations, then? A quick calculation will reveal that they do not form a Lie algebra in the usual way, but there can still be a Lie-algebra-like structure on them.

We let \( \mathcal{D}_{\sigma}(A) \) denote the set of \( \sigma \)-derivations on \( A \). As with vector fields in differential geometry, one may define the product of some \( a \in A \) and \( \Delta \in \mathcal{D}_{\sigma}(A) \) to be the \( a \cdot \Delta \in \mathcal{D}_{\sigma}(A) \) defined by \((a \cdot \Delta)(b) = a\Delta(b)\) for all \( b \in A \); hence \( \mathcal{D}_{\sigma}(A) \) can be regarded as a left \( A \)-module. The annihilator \( \text{Ann}(\Delta) \) of some \( \Delta \in \mathcal{D}(A) \) is the set of all \( a \in A \) such that \( a \cdot \Delta = 0 \). By [19, Th. 4], if \( A \) is a commutative unique factorisation domain then \( \mathcal{D}_{\sigma}(A) \) is as a left \( A \)-module free and of rank one, which lets us use the following construction to exhibit a Lie-algebra-like structure on \( \mathcal{D}_{\sigma}(A) \).

**Theorem 1** ([19, Th. 5]). Let \( A \) be a commutative associative \( \mathbb{K} \)-algebra with unit 1 and let \( \sigma : A \rightarrow A \) be an algebra homomorphism other than the identity map. Fix some \( \Delta \in \mathcal{D}_{\sigma}(A) \) such that \( \sigma (\text{Ann}(\Delta)) \subseteq \text{Ann}(\Delta) \). Define a binary operation \([ \cdot , \cdot ]_{\sigma} \) on the left \( A \)-module \( A \cdot \Delta \) by

\[
[a \cdot \Delta, b \cdot \Delta]_{\sigma} = (\sigma(a) \cdot \Delta) \circ (b \cdot \Delta) - (\sigma(b) \cdot \Delta) \circ (a \cdot \Delta) \quad \text{for all } a, b \in A, \tag{3}
\]

where \( \circ \) denotes composition of functions. This operation is well-defined and satisfies the two identities

\[
[a \cdot \Delta, b \cdot \Delta]_{\sigma} = (\sigma(a)\Delta(b) - \sigma(b)\Delta(a)) \cdot \Delta, \tag{4}
\]

\[
[b \cdot \Delta, a \cdot \Delta]_{\sigma} = -[a \cdot \Delta, b \cdot \Delta]_{\sigma} \tag{5}
\]
for all \(a, b \in A\). If there in addition is some \(\delta \in A\) such that
\[
\Delta(\sigma(a)) = \delta \sigma(\Delta(a)) \quad \text{for all } a \in A,
\]
then \([\cdot, \cdot]_\sigma\) satisfies the deformed six-term Jacobi identity
\[
\partial_{a,b,c} \left( [\sigma(a) \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma] + \delta \cdot [a \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma] \right) = 0
\]
for all \(a, b, c \in A\).

The algebra \(A \cdot \Delta\) in the theorem is then a quasi-hom-Lie algebra with, in the notation of [29], \(\alpha(a \cdot \Delta) = \sigma(a) \cdot \Delta\), \(\beta(a \cdot \Delta) = (\delta a) \cdot \Delta\), and \(\omega = -id_{A \cdot \Delta}\). For \(\delta \in \mathbb{K}\), as is the case with \(\Delta = D_q\), (7) further simplifies to the deformed three-term Jacobi identity (12) of a hom-Lie algebra.

As example of how the method in Theorem 1 ties in with more basic deformation approaches, we review the results in [30, 31] concerned with this quasi-deformation scheme when applied to the simple Lie algebra \(\mathfrak{sl}_2(\mathbb{K})\). Recall that the Lie algebra \(\mathfrak{sl}_2(\mathbb{K})\) can be realized as a vector space generated by elements \(H, E\) and \(F\) with the bilinear bracket product defined by the relations
\[
[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.
\]
A basic starting point is the following representation of \(\mathfrak{sl}_2(\mathbb{K})\) in terms of first order differential operators acting on some vector space of functions in a variable \(t\):
\[
E \mapsto \partial, \quad H \mapsto -2t \partial, \quad F \mapsto -t^2 \partial.
\]
To quasi-deform \(\mathfrak{sl}_2(\mathbb{K})\) means that we firstly replace \(\partial\) by some twisted derivation \(\Delta\) in this representation. At our disposal as deformation parameters are now \(A\) (the “algebra of functions”) and the endomorphism \(\sigma\). After computing the bracket on \(A \cdot \Delta\) by Theorem 1 the relations in the quasi-Lie deformation are obtained by pullback.

Let \(A\) be a commutative, associative \(\mathbb{K}\)-algebra with unity 1, let \(t\) be an element of \(A\), and let \(\sigma\) denote a \(\mathbb{K}\)-algebra endomorphism on \(A\). As above, \(\mathcal{D}_\sigma(A)\) denotes the linear space of \(\sigma\)-derivations on \(A\). Choose an element \(\Delta\) of \(\mathcal{D}_\sigma(A)\) and consider the \(\mathbb{K}\)-subspace \(A \cdot \Delta\) of elements on the form \(a \cdot \Delta\) for \(a \in A\). The elements \(e := \Delta\), \(h := -2t \cdot \Delta\), and \(f := -t^2 \cdot \Delta\) span a \(\mathbb{K}\)-linear subspace
\[
\mathcal{S} := \text{LinSpan}_\mathbb{K}\{\Delta, -2t \cdot \Delta, -t^2 \cdot \Delta\} = \text{LinSpan}_\mathbb{K}\{e, h, f\}
\]
of \(A \cdot \Delta\). We restrict the multiplication (4) to \(\mathcal{S}\) without, at this point, assuming closure. Now, \(\Delta(t^2) = \Delta(t \cdot t) = \sigma(t) \Delta(t) + \Delta(t) t = (\sigma(t) + t) \Delta(t)\). Under the natural (see [30]) assumptions \(\sigma(1) = 1\) and \(\Delta(1) = 0\), (4) leads to
\[
[h, f] = 2\sigma(t) \Delta(t) \cdot \Delta, \quad (9a)
\]
\[
[h, e] = 2\Delta(t) \cdot \Delta, \quad (9b)
\]
\[
[e, f] = -\left(\sigma(t) + t\right) \Delta(t) \cdot \Delta. \quad (9c)
\]
hence as long as \( \sigma \) and \( \Delta \), similarly to their untwisted counterparts, yield that the degrees of \( t \) in the expressions on the right hand side remain among those present in the generating set for the \( \mathbb{K} \)-linear subspace \( \mathcal{S} \), it follows that \( \mathcal{S} \) indeed is closed under this bracket.

In the particular case that \( \sigma(t) = qt \) for some \( q \in \mathbb{K} \) and \( \Delta = Dq \), we obtain a family of hom-Lie algebras deforming \( \mathfrak{sl}_2 \), defined with respect to the basis \( \{e, f, h\} \) by the brackets and the linear map \( \alpha \) as follows:

\[
\begin{align*}
[h, f] &= -2qf, & \alpha(f) &= q^2 f, \\
[h, e] &= 2e, & \alpha(e) &= qe, \\
[e, f] &= \frac{1}{2}(1 + q)h, & \alpha(h) &= qh.
\end{align*}
\]

This is a hom-Lie algebra for all \( q \in \mathbb{K} \) but not a Lie algebra unless \( q = 1 \), in which case we recover the classical \( \mathfrak{sl}_2 \).

### 1.2 Hom-algebras: Lie and associative

An ordinary Lie or associative algebra may informally be described as an underlying linear space (often assumed to be a vector space, but we will typically allow it to be a more general module) on which is defined some bilinear map \( m \) called the multiplication (or in the Lie case sometimes the bracket). Depending on what identities this multiplication satisfies, the algebra is classified as being associative, commutative, anticommutative, Lie, etc. A hom-algebra may similarly be described as an underlying linear space on which is defined two maps \( m \) and \( \alpha \). The multiplication \( m \) is again required to be bilinear, whereas \( \alpha \) is merely a linear map from the underlying set to itself. The ‘hom-’ prefix is historically because \( \alpha \) in many examples turn out to be a homomorphism with respect to some operation (not necessarily the \( m \) of the hom-algebra, even though that is certainly not uncommon), but the modern understanding is that \( \alpha \) may be any linear map.

Practically, the point of incorporating some extra map \( \alpha \) in the definition of an algebra is that this can be used to “twist” or “deform” the identities defining a variety of algebras, and thus offer greater opportunities for capturing within an abstract axiomatic framework the many concrete “twisted” or “deformed” algebras that have emerged in recent decades. It was shown in [19] that hom-Lie algebras are closely related to discrete and deformed vector fields and differential calculus and that some \( q \)-deformations of the Witt and the Virasoro algebras have the structure of a hom-Lie algebra. The paradigmatic example (given above) is the \( \mathfrak{sl}_2 \) Lie algebra which deforms to a new nontrivial hom-Lie algebra by means of \( \sigma \)-derivations. Hom-associative algebras are likewise a generalisation of a usual associative algebras. A common recipe for producing the hom-analogue of a classical identity is to insert \( \alpha \) applications wherever some variable is not acted upon by \( m \) as many times as the others.
Definition 2. Let $\mathcal{R}$ be some associative and commutative unital ring. Formally, an $\mathcal{R}$-hom-algebra $\mathcal{A}$ is a triplet $(A, m, \alpha)$, where $A$ is an $\mathcal{R}$-module, $m: A \times A \rightarrow A$ is a bilinear map, and $\alpha: A \rightarrow A$ is a linear map. As usual, the algebra $\mathcal{A}$ and its carrier set $A$ are notationally identified whenever there is no risk of confusion.

The hom-associative identity for $\mathcal{A}$ is the formula
\[ m(\alpha(x), m(y, z)) = m(m(x, y), \alpha(z)) \quad \text{for all } x, y, z \in A. \] (11)

A hom-algebra which satisfies the hom-associative identity is said to be a hom-associative algebra. Similarly, the hom-Jacobi identity for $\mathcal{A}$ is the formula
\[ m(\alpha(x), m(y, z)) + m(\alpha(y), m(z, x)) + m(\alpha(z), m(x, y)) = 0 \quad \text{for all } x, y, z \in A. \] (12)

For a hom-algebra $\mathcal{A}$ to be a hom-Lie algebra, it must satisfy the hom-Jacobi identity and the ordinary anticommutativity (skew-symmetry) identity
\[ m(x, x) = 0 \quad \text{for all } x \in A. \] (13)

A hom-algebra $\mathcal{A}$ is said to be multiplicative if $\alpha$ is an endomorphism of the algebra $(A, m)$, i.e., if
\[ m(\alpha(x), \alpha(y)) = \alpha(m(x, y)) \quad \text{for all } x, y \in A. \] (14)

Now let $\mathcal{A} = (A, m, \alpha)$ and $\mathcal{A}' = (A', m', \alpha')$ be two hom-algebras. A morphism $f: \mathcal{A} \rightarrow \mathcal{A}'$ of hom-algebras is a linear map $f: A \rightarrow A'$ such that
\[ m(f(x), f(y)) = f(m(x, y)) \quad \text{for all } x, y \in A, \] (15)
\[ \alpha(f(x)) = f(\alpha(x)) \quad \text{for all } x \in A. \] (16)

A linear map $f: A \rightarrow A'$ that merely satisfies the first condition (15) is called a weak morphism of hom-algebras.

The concept of weak morphism is somewhat typical of the classical algebra attitude towards hom-algebras: the multiplication $m$ is taken as part of the core structure, whereas the map $\alpha$ is seen more as an add-on. In both universal algebra and the categorical setting, it is instead natural to view $m$ and $\alpha$ as equally important for the hom-algebra concept, even though it is of course also possible to treat weak morphisms (for example with the help of a suitable forgetful functor) within these settings, should weak morphisms turn out to be of interest for the problems at hand. Yau [44] goes one step in the opposite direction and considers hom-algebras as being hom-modules with a multiplication; this makes $\alpha$ part of the core structure whereas $m$ is the add-on.

As usual, the squaring form (13) of the anticommutative identity implies the more traditional
\[ m(x, y) = -m(y, x) \quad \text{for all } x, y \in A \] (17)
in any hom-algebra $\mathcal{A}$. The two are equivalent in an algebra over a field of characteristic $\neq 2$, but (17) implies nothing about $m(x, x)$ in an algebra over a field of
characteristic equal to 2, and for hom-algebras over other rings more intermediate outcomes are possible.

An example of a hom-Lie algebra was given in the previous subsection. A similar example of a hom-associative algebra would be:

**Example 3.** Let \( \{e_1, e_2, e_3\} \) be a basis of a 3-dimensional linear space \( A \) over some field \( \mathbb{K} \). Let \( a, b \in \mathbb{K} \) be arbitrary parameters. The following equalities

\[
m(e_1, e_1) = ae_1, \quad m(e_2, e_2) = ae_2, \\
m(e_1, e_2) = m(e_2, e_1) = ae_2, \quad m(e_2, e_3) = be_3, \\
m(e_1, e_3) = m(e_3, e_1) = be_3, \quad m(e_3, e_2) = m(e_3, e_3) = 0, \\
\alpha(e_1) = ae_1, \quad \alpha(e_2) = ae_2, \quad \alpha(e_3) = be_3,
\]

define the multiplication \( m \) and linear map \( \alpha \) on a hom-associative algebra on \( \mathbb{K}^3 \). This algebra is not associative when \( a \neq b \) and \( b \neq 0 \), since \( m(m(e_1, e_1), e_3) = m(e_1, m(e_1, e_3)) = (a - b)e_3 \).

**Example 4 (Polynomial hom-associative algebra [45]).** Consider the polynomial algebra \( A = \mathbb{K}[x_1, \ldots, x_n] \) in \( n \) variables. Let \( \alpha \) be an algebra endomorphism of \( A \) which is uniquely determined by the \( n \) polynomials \( \alpha(x_i) = \sum \lambda_i x_1^{r_1} \cdots x_n^{r_n} \) for \( 1 \leq i \leq n \). Define \( m \) by

\[
m(f, g) = f(\alpha(x_1), \ldots, \alpha(x_n))g(\alpha(x_1), \ldots, \alpha(x_n))\tag{18}
\]

for \( f, g \in A \). Then \( (A, m, \alpha) \) is a hom-associative algebra. (This example is a special case of Corollary 7.)

**Example 5 ([47]).** Let \( (A, m, \alpha) \) be a hom-associative \( \mathbb{R} \)-algebra. Denote by \( M_n(A) \) the \( \mathbb{R} \)-module of \( n \times n \) matrices with entries in \( A \). Then \( (M_n(A), m', \alpha') \) is also a hom-associative algebra, in which \( \alpha' : M_n(A) \to M_n(A) \) is the map that applies \( \alpha \) to each matrix element and the multiplication \( m' \) is the ordinary matrix multiplication over \( (A, m) \).

The following result states that hom-associative algebra yields another hom-associative algebra when its multiplication and twisting map are twisted by a morphism. The following results work as well for hom-Lie algebras and more generally \( G \)-hom-associative algebras. These constructions introduced in [45] and generalized in [50] were extended to many other algebraic structures.

**Theorem 6.** Let \( \mathcal{A} = (A, m, \alpha) \) be a hom-algebra and \( \beta : \mathcal{A} \to \mathcal{A} \) be a weak morphism. Then \( \mathcal{A}_\beta = (A, m_\beta, \alpha_\beta) \) where \( m_\beta = \beta \circ m \) and \( \alpha_\beta = \beta \circ \alpha \) is also a hom-algebra. Furthermore:

1. If \( \mathcal{A} \) is hom-associative then \( \mathcal{A}_\beta \) is hom-associative.
2. If \( \mathcal{A} \) is hom-Lie then \( \mathcal{A}_\beta \) is hom-Lie.
3. If \( \mathcal{A} \) is multiplicative and \( \beta \) is a morphism then \( \mathcal{A}_\beta \) is multiplicative.
Proof. For the hom-associative and hom-Jacobi identities, it suffices to consider what a typical term in these identities looks like. We have

\[ m_\beta \left( \alpha_\beta(x), m_\beta(y, z) \right) = (\beta \circ m) \left( (\beta \circ \alpha)(x), (\beta \circ m)(y, z) \right) = \]

\[ = \beta \left( m \circ \beta \otimes \beta \left( \alpha(x), m(y, z) \right) \right) = \beta \left( (\beta \circ m)(\alpha(x), m(y, z)) \right) = \]

\[ = (\beta \circ \beta)(m(\alpha(x), m(y, z))) \]

Hence either side of the hom-associative and hom-Jacobi respectively identities for \( A_\beta \) comes out as \( \beta \circ 2 \) of the corresponding side of the corresponding identity for \( A \), and thus these identities for \( A_\beta \) follow directly from their \( A \) counterparts. The anticommutativity identity similarly follows from its counterpart, as does the multiplicative identity via

\[ m_\beta(\alpha_\beta(x), \alpha_\beta(y)) = \beta\left( m(\alpha(x), \beta(\alpha(y))) \right) = \beta^{\circ 2}\left( m(\alpha(x), \alpha(y)) \right) = \]

\[ = \beta^{\circ 2}\left( \alpha(m(x, y)) \right) = \beta\left( \alpha((\beta \circ m)(x, y)) \right) = \alpha_\beta(m_\beta(x, y)) \]

for all \( x, y \in A \).

The \( \alpha = \operatorname{id} \) special case of Theorem 6 yields.

Corollary 7. Let \((A, m)\) be an associative algebra and \( \beta : A \rightarrow A \) be an algebra endomorphism. Then \( A_\beta = (A, m_\beta, \beta) \) where \( m_\beta = \beta \circ m \) is a multiplicative hom-associative algebra.

That result also has the following partial converse.

Corollary 8 ([18]). Let \( A = (A, m, \alpha) \) be a multiplicative hom-algebra in which \( \alpha \) is invertible. Then \( A' = (A, \alpha^{-1} \circ m, \operatorname{id}) \) is a hom-algebra. In particular, any multiplicative hom-associative or hom-Lie algebra where \( \alpha \) is invertible may be regarded as an ordinary associative or Lie respectively algebra, albeit with an awkwardly defined operation.

Proof. Take \( \beta = \alpha^{-1} \) in Theorem 6.

An application of that corollary is the identity

\[ m(x_0, m(x_1, x_2)) = m(m(\alpha^{-1}(x_0), x_1), \alpha(x_2)) \]

which hold in multiplicative hom-associative algebras with invertible \( \alpha \), and generalises to change the “tilt” of longer products. The idea is to rewrite the product in terms of the corresponding associative multiplication \( \tilde{m} = \alpha^{-1} \circ m \), with respect to which \( \alpha \) and \( \alpha^{-1} \) are also algebra homomorphisms, and apply the ordinary associative law to change the “tilt” of the product before converting the result back to the hom-associative product \( m \).
Since many (hom-)Lie algebras of practical interest are finite-dimensional, and
injectivity implies invertibility for linear operators on a finite-dimensional space,
one might expect hom-Lie algebras to be particularly prone to fall under the domain
of that corollary, but the important condition that should not be forgotten is that of
the algebra being multiplicative. For example the $q$-deformed $sl_2$ of (10) is easily
seen to not be multiplicative for general $q$.

An identity that may seem conspicuously missing from Definition 2 is that of
the unit; although they do not make sense in Lie algebras due to contradicting anti-
commutativity, units are certainly a standard feature of associative algebras, so why
has there been no mention of hom-associative unital algebras? The reason is that
they, by the following theorem, constitute a subclass of that of hom-associative al-
gebras which is even more restricted than that of the multiplicative hom-associative
algebras. Unitality of hom-associative algebras were discussed first in [18].

**Theorem 9.** Let $A$ be a hom-associative algebra. If there is some $e \in A$ such that

$$m(e, x) = x = m(x, e) \quad \text{for all } x \in A$$

then

$$m(\alpha(x), y) = m(x, \alpha(y)) = \alpha(m(x, y)) \quad \text{for all } x, y \in A.$$  

**Proof.** For the first equality,

$$m(\alpha(x), y) = m(\alpha(x), m(e, y)) = m(m(x, e), \alpha(y)) = m(x, \alpha(y))$$

by hom-associativity. For the second equality,

$$m(x, \alpha(y)) = m(e, m(x, e), \alpha(y)) = m(e, m(x, y))) = \alpha(m(x, y))$$

by hom-associativity and the first equality.

An identity such as (20) has profound effects on the structure of a hom-associative
algebra. Basically, it means applications of $\alpha$ are not located in any particular position
in a product, but can move around unhindered. At the same time, even a single $\alpha$
somewhere will act as a powerful lubricant that lets the hom-associative identity shuffle around parentheses as easily as the ordinary associative identity. In particular, any product of $n$ algebra elements $x_1, \ldots, x_n$ where at least one is in the image of $\alpha$ will effectively be an associative product; probably not the wanted outcome if one’s aim is to create new structures through deformations of old ones.

On the other hand, $\alpha$ satisfying (20) obviously have some rather special properties. One may for any algebra $A = (A, m)$ define the centroid $\text{Cent}(A)$ of $A$ as
the set of all linear self-maps $\alpha : A \rightarrow A$ satisfying the condition $\alpha(m(x, y)) = m(\alpha(x), y) = m(x, \alpha(y))$ for all $x, y \in A$. Notice that if $\alpha \in \text{Cent}(A)$, then we have $m(\alpha^p(x), \alpha^q(y)) = (\alpha^{p+q} \circ m)(x, y)$ for all $p, q \geq 0$. The construction of hom-algebras using elements of the centroid was initiated in [5] for Lie algebras. We have
Proposition 10. Let \((A, m)\) be an associative algebra and \(\alpha \in \text{Cent}(A)\). Set for \(x, y \in A\)

\[
m_1(x, y) = m(\alpha(x), y),
\]

\[
m_2(x, y) = m(\alpha(x), \alpha(y)).
\]

Then \((A, m_1, \alpha)\) and \((A, m_2, \alpha)\) are hom-associative algebras.

Indeed we have

\[
m_1(\alpha(x), m_1(y, z)) = m(\alpha^2(x), m(\alpha(y), z)) = \alpha(m(\alpha(x), m(\alpha(y), z))) = m(\alpha(x), \alpha(m(\alpha(y), \alpha(z)))).
\]

Remark 11. The definition of unitality which fits with Corollary 7 was introduced in [18] and then used for hom-bialgebra and hom-Hopf algebras in [7].

Let \((A, m, \alpha)\) be a hom-associative algebra. It is said to be unital if there is some \(e \in A\) such that

\[
m(e, x) = \alpha(x) = m(x, e) \quad \text{for all} \ x \in A.
\]

Therefore, similarly to Corollary 7, a unital associative algebra gives rise a unital hom-associative algebra.

1.3 Admissible and enveloping hom-algebras

Two concepts that are of key importance in the theory of ordinary Lie algebras are those of Lie-admissible and enveloping algebras. In the setting of hom-algebras, these concepts are defined as follows, with the classical non-hom concepts arising in the special case \(\alpha = \text{id}\).

Definition 12. Let a hom-algebra \(A = (A, m, \alpha)\) be given. Define \(b(x, y) = m(x, y) - m(y, x)\) to be the commutator (bracket) corresponding to \(m\), and let \(A^-\) be the hom-algebra \((A, b, \alpha)\). The algebra \(A\) is said to be hom-Lie-admissible if the hom-algebra \(A^-\) is hom-Lie.

Now let \(L\) be a hom-Lie algebra. \(A\) is said to be an enveloping algebra for \(L\) if \(L\) is isomorphic to some hom-subalgebra \(B = (B, b_B, \alpha_B)\) of \(A^-\) such that \(B\) generates \(A\).

It was shown in [35, Prop. 1.6] that any hom-associative algebra is hom-Lie-admissible. On one hand, this becomes another method of constructing new hom-Lie algebras, but it is more interesting when wielded to the opposite end of studying a given hom-Lie algebra through a corresponding enveloping algebra. To explain why this is so, we will briefly review the classical theory of ordinary Lie and associative algebras.
On a Lie group, the exponential map $v \mapsto \exp(v)$ allows transitioning from tangent vectors to non-infinitesimal shifts; $\exp(tv)$ is the point where you end up if travelling from the identity point at velocity $v$ for time $t$. Under the interpretation that identifies vectors with invariant vector fields, and vector fields with derivations on the ring of scalar-valued functions ("scalar fields", in the physicist terminology), the exponential map may in fact be defined via the elementary power series formula

$$
\exp(v) = \sum_{n=0}^{\infty} \frac{v^n}{n!}
$$

(where multiplication of vectors is composition of differential operators) and in the Lie group $(\mathbb{R}, +)$ this turns out to be Taylor’s formula: $\exp(t \frac{d}{dx})$ is the shift operator mapping an analytic function $f$ to the shifted variant $x \mapsto f(x+t)$.

When doing the same in a more general Lie group, one must however be careful to note that vector fields need not commute, and that already the degree 2 term of for example $\exp(u+v)$ contains $uv$ and $vu$ terms that need not be equal. The role of the Lie algebra is precisely to keep track of the extent to which vector fields do not commute, so the proper place to do algebra with vector fields to the aim of studying the exponential map must be in an enveloping algebra of the Lie algebra of invariant vector fields on the underlying Lie group.

Conversely, one may start with a Lie algebra $\mathfrak{g}$ and ask oneself what the corresponding Lie group would be like, by studying formal series in the basic vector fields, while keeping in mind that these should satisfy the commutation relations encoded into $\mathfrak{g}$; this leads to the concept of formal groups. An important step towards it is the construction of the (associative) universal enveloping algebra $U(\mathfrak{g})$, which starts with the free associative algebra generated by $\mathfrak{g}$ as a module and imposes upon it the relations that

$$
xy - yx = [x,y] \quad \text{for all } x, y \in \mathfrak{g}, \quad (22)
$$

where on the left hand side we have multiplication in $U(\mathfrak{g})$ but on the right hand side the bracket operation of the Lie algebra $\mathfrak{g}$. More technically, the free associative algebra in question can be constructed as the tensor algebra $T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^\otimes n$ where the product of $x_1 \otimes \cdots \otimes x_m \in \mathfrak{g}^\otimes m$ and $y_1 \otimes \cdots \otimes y_n \in \mathfrak{g}^\otimes n$ is $x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n \in \mathfrak{g}^\otimes (m+n)$. Imposing the commutation relations can then be done by taking the quotient by the two-sided ideal $J(\mathfrak{g})$ in $T(\mathfrak{g})$ that is generated by all $xy - yx - [x,y]$ for $x, y \in \mathfrak{g}$, i.e.,

$$
U(\mathfrak{g}) := T(\mathfrak{g})/J(\mathfrak{g}) = T(\mathfrak{g})/\langle \{ xy - yx - [x,y] | x, y \in \mathfrak{g} \} \rangle.
$$

With this in mind, it is only natural to generalise this construction to the hom-case, and in [44] Yau does so. Since he in the non-associative case cannot take advantage of familiar concepts such as the tensor algebra, this construction will however involve a few steps more than one might be used to from the non-hom setting. Notably, Yau begins with setting up the free hom-algebra $F_{HAs}(\mathfrak{g})$: neither hom-associativity nor ordinary associativity is inherent. Then he goes on to impose hom-associativity by taking a quotient, which results in the free hom-associative algebra $F_{HAs}(\mathfrak{g})$; this is what corresponds to the tensor algebra $T(\mathfrak{g})$. Another quotient imposes also the commutation relations, to finally yield the universal enveloping hom-associative algebra $U_{HLie}(\mathfrak{g})$. 
When reading through the technical details of these constructions, which we shall quote below for the reader’s convenience, they may seem a daring plunge forward into very general algebra, that harnesses advanced combinatorial objects to achieve a clear picture of the algebra. It may be that they are that, but our main point in the next section is that they are also an entirely straightforward application of the basic methods of universal algebra, so there is in fact very little that was novel in these constructions. The reader who has grasped the material in Section 2 will be able to recreate something equivalent to the following (modulo some minor optimisations) from scratch.

For \( n \geq 1 \), let \( T_n \) denote the set of isomorphism classes of plane\(^1\) binary trees with \( n \) leaves and one root. The first \( T_n \) are depicted below.

\[
\begin{align*}
T_1 &= \{1\}, & T_2 &= \left\{ \begin{array}{c}
\\\\\\\end{array} \right\}, & T_3 &= \left\{ \begin{array}{c}
\begin{array}{c}
\\\\\\\end{array}
\end{array} \right\}, \\
T_4 &= \left\{ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \right\}.
\end{align*}
\]

Each dot represents either a leaf, which is always depicted at the top, or an internal vertex. An element in \( T_n \) will be called an \( n \)-tree. The set of nodes (= leaves and internal vertices) in a tree \( \psi \) is denoted by \( N(\psi) \). The node of an \( n \)-tree \( \psi \) that is connected to the root (the lowest point in the \( n \)-tree) will be denoted by \( v_{\text{low}} \). In other words, \( v_{\text{low}} \) is the lowest internal vertex in \( \psi \) if \( n \geq 2 \) and is the only leaf if \( n = 1 \).

Given an \( n \)-tree \( \psi \) and an \( m \)-tree \( \varphi \), their grafting \( \psi \vee \varphi \in T_{n+m} \) is the tree obtained by placing \( \psi \) on the left and \( \varphi \) on the right and joining their roots to form the new lowest internal vertex, which is connected to the new root. Pictorially, we have

\[
\psi \vee \varphi = \begin{array}{c}
\begin{array}{c}
\psi \\
\varphi
\end{array}
\end{array}.
\]

Note that grafting is a nonassociative operation. As we will discuss below, the operation of grafting is for generating the multiplication \( m \) of a free nonassociative algebra.

To handle hom-algebras, we need to introduce weights on plane trees. A weighted \( n \)-tree is a pair \( \tau = (\psi, w) \), in which \( \psi \in T_n \) is an \( n \)-tree and \( w \) is a function from the set of internal vertices of \( \psi \) to the set \( \mathbb{N} \) of non-negative integers. If \( v \) is an internal vertex of \( \psi \), then we call \( w(v) \) the weight of \( v \). The \( n \)-tree \( \psi \) is called the underlying

---

\(^1\) Yau, like many other algebraists, actually uses the term ‘planar’ rather than ‘plane’, but this practice is simply wrong as the two words refer to slightly different graph-theoretical properties: a graph is planar if it can be embedded in a genus 0 surface, but plane if it is given with such an embedding. To speak of a ‘planar tree’ is a tautology, because trees by definition contain no cycles, will therefore have no subdivided \( K_5 \) or \( K_{3,3} \) as subgraph, and thus by Kuratowski’s Theorem be planar. What is of utmost importance here is rather that the trees are given with a (combinatorial) embedding into the plane, since that specifies a local cyclic order on edges incident with a vertex, which is what the isomorphisms spoken of are required to preserve. As rooted trees, the two elements of \( T_3 \) are isomorphic, but as plane rooted trees they are not.
Likewise, the grafting of two weighted trees is defined as above by connecting them to a new root for which the weight is 0. There is also an operation to change the weight; for \( \tau = (\psi, w) \), we define \( \tau[r] = (\psi, w') \) where \( w'(v) = w(v) + r \) and \( w'(v) = w(v) \) for all internal vertices \( v \neq v_{low} \).

Now let an \( R \)-module \( A \) and a linear map \( \alpha: A \rightarrow A \) be given. As a set,

\[
F_{\text{HNAs}}(A) = \bigoplus_{n \geq 1} \bigoplus_{\tau \in T^w_n} A^{\otimes n}.
\]

We write \( A^{\otimes n}_\tau \) for the component in this direct sum that corresponds to the values \( n \) and \( \tau \) of these summation indices. There is a canonical isomorphism \( A^{\otimes n}_\tau \cong A^{\otimes n} \).

For any \( n \geq 1, \tau \in T^w_n \), and \( x_1, \ldots, x_n \in A \), we write \( (x_1 \otimes \cdots \otimes x_n)_\tau \) for the element of \( A^{\otimes n}_\tau \) that corresponds to \( x_1 \otimes \cdots \otimes x_n \in A^{\otimes n} \). The linear map \( \alpha \) is extended to a linear map \( \alpha_F: F_{\text{HNAs}}(A) \rightarrow F_{\text{HNAs}}(A) \) by the rule

\[
\alpha_F((x_1 \otimes \cdots \otimes x_n)_\tau) = (x_1 \otimes \cdots \otimes x_n)_{\tau[1]} \quad \text{for} \quad \tau \notin T_1
\]

and the multiplication \( m_F \) on \( F_{\text{HNAs}}(A) \) is defined by

\[
m_F((x_1 \otimes \cdots \otimes x_n)_\tau, (x_{n+1} \otimes \cdots \otimes x_{n+m})_\sigma) = (x_1 \otimes \cdots \otimes x_{n+m})_{\tau \vee \sigma}
\]

and bilinearity. This \( (F_{\text{HNAs}}(A), m_F, \alpha_F) \) is the free (nonassociative) \( R \)-hom-algebra generated by the hom-module \((A, \alpha)\).

From there, the corresponding free hom-associative algebra is constructed as the quotient

\[
\tilde{F}_{\text{HNAs}}(A) := F_{\text{HNAs}}(A)/J^w
\]

where \( J^w = \bigcup_{n \geq 1} J^n \) and \( J^1 \subseteq J^2 \subseteq \cdots \subseteq J^w \subset F_{\text{HNAs}}(A) \) is an ascending chain of two-sided ideals defined by

\[
J^1 = \left\langle \text{Im}(m_F \circ (m_F \otimes \alpha_F - \alpha_F \otimes m_F)) \right\rangle,
J^{n+1} = \left\langle J^n \cup \alpha_F(J^n) \right\rangle \quad \text{for} \quad n \geq 1.
\]

The universal enveloping algebra of a hom-Lie algebra \((g, b, \alpha)\) is similarly obtained as the quotient

\[
U_{\text{HLie}}(g) := F_{\text{HNAs}}(g)/I^w
\]

where \( I^w \) is the two-sided ideal obtained if one starts with

\[
I^1 = \left\langle \text{Im}(m_F \circ (m_F \otimes \alpha_F - \alpha_F \otimes m_F)) \cup \{m_F(x,y) - m_F(y,x) - b(x,y) \mid x, y \in g\} \right\rangle
\]

and then similarly lets \( I^{n+1} = \left\langle I^n \cup \alpha_F(I^n) \right\rangle \) for \( n \geq 1 \) and \( I^w = \bigcup_{n \geq 1} I^n \). Since \( I^n \subseteq J^n \) for all \( n \geq 1 \), it follows that \( U_{\text{HLie}}(g) \) may alternatively be regarded as a quotient of \( F_{\text{HNAs}}(g) \). This further justifies labelling the hom-associative algebra \( U_{\text{HLie}}(g) \) as...
the hom-analogue for a hom-Lie algebra $\mathfrak{g}$ of the universal enveloping algebra of a Lie algebra.

There is however one important question regarding this $U_{\text{HLie}}$ which has not been answered by the above, and in fact seems to be open in the literature: Is $U_{\text{HLie}}(\mathfrak{g})$ for every hom-Lie algebra $\mathfrak{g}$ an enveloping algebra of $\mathfrak{g}$? It follows from the form of the construction that there is a linear map $j: \mathfrak{g} \rightarrow U_{\text{HLie}}(\mathfrak{g})$ with the properties that

$$j([x,y]) = j(x)j(y) - j(y)j(x)$$
$$j(\alpha(x)) = \alpha(j(x))$$

for all $x,y \in \mathfrak{g}$,

and hence $j$ becomes a morphism of hom-Lie algebras $\mathfrak{g} \rightarrow U_{\text{HLie}}(\mathfrak{g})^-$, but it is entirely unknown whether $j$ is injective. A failure to be injective would obviously render these hom-associative enveloping algebras of hom-Lie algebras less important than the ordinary associative enveloping algebras of ordinary Lie algebras, as they would fail to capture all the information encoded into the hom-Lie algebra.

Another way of phrasing the conjecture that the canonical homomorphism is injective is that the ideal $I_{\infty}$ used to construct $U_{\text{HLie}}(\mathfrak{g})$ does not contain any degree 1 elements; such elements would correspond to linear dependencies in $U_{\text{HLie}}(\mathfrak{g})$ between the images of basis elements in $\mathfrak{g}$. A simple argument for this conjecture would be that such dependencies do not occur in the associative case, and since the hom-associative case has “more degrees of freedom” than the associative case, it shouldn’t happen here either. An argument against it comes from the converse of the Poincaré–Birkhoff–Witt Theorem [43]: If the canonical homomorphism $\mathfrak{g} \rightarrow U(\mathfrak{g})$ is injective, then $\mathfrak{g}$ is a Lie algebra; the ordinary universal enveloping algebra construction only manages to envelop the algebra one starts with if that algebra is a Lie algebra. What can be hoped for is of course that the conditions inherent in $U_{\text{HLie}}$ have precisely those deformations relative to the conditions of $U_{\text{Lie}}$ that makes everything work out for hom-Lie algebras instead, but they could just as well end up going some other way.

To positively resolve the envelopment problem, one would probably have to prove a hom-analogue of the Poincaré–Birkhoff–Witt Theorem. Methods for this—particularly the Diamond Lemma—are available, but the calculations required seem to be rather extensive. To negatively resolve the envelopment problem, it would be sufficient to find one hom-Lie algebra $\mathfrak{g}$ for which the canonical homomorphism $\mathfrak{g} \rightarrow U_{\text{HLie}}(\mathfrak{g})$ is not injective. Yau does show in [44, Th. 2] that $U_{\text{HLie}}(\mathfrak{g})$ satisfies an universal property with respect to hom-associative enveloping algebras, so a hom-Lie algebra $\mathfrak{g}$ which constitutes a counterexample cannot arise as a subalgebra of $\mathfrak{A}^-$ for any hom-associative algebra $\mathfrak{A}$. 

2 Classical universal algebra: free algebras and their quotients

2.1 Discrete free algebras

A basic concept in universal algebra is that of the signature. A signature $\Omega$ is a set of formal symbols, together with a function arity: $\Omega \rightarrow \mathbb{N}$ that gives the arity, or “wanted number of operands”, for each symbol. Symbols with arity 0 are called constants (or said to be nullary), symbols with arity 1 are said to be unary, symbols with arity 2 are said to be binary, symbols with arity 3 are said to be ternary, and so on; one may also speak about a symbol being $n$-ary. A convenient shorthand, used in for example [12], for specifying signatures is as a set of “function prototypes”: symbols of positive arity are followed by a parenthesis containing one comma less than the arity, whereas constants are not followed by a parenthesis. Hence $\Omega$ prototypes: symbols of positive arity are followed by a parenthesis containing one comma less than the arity, whereas constants are not followed by a parenthesis. Hence $\Omega = \{a(), m(), x, y\}$ is the signature of four symbols $a$, $m$, $x$, and $y$, where $a$ is unary, $m$ is binary, and the remaining two are constants. The signature for a hom-algebra is thus $\{a(), m(), \}$, whereas the signature for a unary hom-algebra would be $\{a(), m(), 1\}$: a unit would be an extra constant symbol.

Given a signature $\Omega$, a set $A$ is said to be an $\Omega$-algebra if it for every symbol $x \in \Omega$ comes with a map $f_x: A^{\text{arity}(x)} \rightarrow A$; these maps are the operations of the algebra. Note that no claim is made that the operations fulfill any particular property (beyond matching the respective arities of their symbols), so the $\Omega$-algebra structure is not determined by $A$ unless that set has cardinality 1; therefore one might want to be more formal and say it is $A = (A, \{f_x\}_{x \in \Omega})$ that is the $\Omega$-algebra, but we shall in what follows generally be concerned with only one $\Omega$-algebra structure at a time on each base set.

What the $\Omega$-algebra concept suffices for, despite imposing virtually no structure upon the object in question, is the definition of an $\Omega$-algebra homomorphism: a map $\phi: A \rightarrow B$ is an $\Omega$-algebra homomorphism from $(A, \{f_x\}_{x \in \Omega})$ to $(B, \{g_x\}_{x \in \Omega})$ if

$$\phi(f_x(a_1, \ldots, a_{\text{arity}(x)})) = g_x(\phi(a_1), \ldots, \phi(a_{\text{arity}(x)}))$$

for all $a_1, \ldots, a_{\text{arity}(x)} \in A$ and $x \in \Omega$. (23)

It is easy to verify that these homomorphisms obey the axioms for being the morphisms in the category of $\Omega$-algebras, so that category $\Omega$-algebra is what one gets. One may then define (up to isomorphism) the free $\Omega$-algebra as being the free object in this category, or more technically state that $F_{\Omega}(X)$ together with $i: X \rightarrow F_{\Omega}(X)$ is the free $\Omega$-algebra generated by $X$ if there for every $\Omega$-algebra $A$ and every map $j: X \rightarrow A$ exists a unique $\Omega$-algebra homomorphism $\phi: F_{\Omega}(X) \rightarrow A$ such that $j = \phi \circ i$. An alternative claim to the same effect is that $F_{\Omega}$, interpreted as a functor from Set to $\Omega$-algebra, is left adjoint of the forgetful functor mapping an $\Omega$-algebra to its underlying set.

Although these definitions may seem frightfully abstract, the objects in question are actually rather easy to construct: $F_{\Omega}(X)$ is merely the set $T(\Omega, X)$ of all formal terms in $\Omega \cup X$, where the elements of $X$ are interpreted as symbols of arity 0. Hence
the first few elements of $T(\{a(), m(, )\}, \{x, y\})$ are

$$x, y, a(x), a(y), a(a(x)), a(a(y)), m(x, x), m(x, y), m(y, x), m(y, y), \ldots$$

and the operations $\{f_x\}_{x \in \Omega}$ in the free $\Omega$-algebra $T(\Omega, X)$ merely produce their formal terms counterparts:

$$f_x(t_1, \ldots, \text{arity}(x)) := x(t_1, \ldots, \text{arity}(x))$$

for all $t_1, \ldots, \text{arity}(x) \in T(\Omega, X)$ and $x \in \Omega$.

Conversely, the unique morphism $\phi$ of the universal property turns out to evaluate formal terms in the codomain $\Omega$-algebra, so for any given $j: X \rightarrow B$ it can be defined recursively through

$$\phi(t) = \begin{cases} j(x) & \text{if } t = x \in X, \\ g_x(\phi(t_1), \ldots, \phi(t_n)) & \text{if } t = x(t_1, \ldots, t_n) \text{ where } x \in \Omega \end{cases}$$

for all $t \in T(\Omega, X)$.

### 2.2 Quotient algebras

Completely free algebras might be cute, but most of the time one is rather interested in something with a bit more structure, in the sense that certain identities are known to hold; in an associative algebra, the associativity identity holds, whereas in a hom-associative algebra the hom-associative identity (11) holds. One approach to imposing such properties on one’s algebras is to restrict attention to the subcategory of $\Omega$-algebras which satisfy the wanted identities, and then look at the free object of that subcategory. Another approach is to take a suitable quotient of the free object from the full category.

In general $\Omega$-algebras, the denominator in a quotient is a congruence relation on the numerator, and an $\Omega$-algebra congruence relation is an equivalence relation which is preserved by the operations; $\equiv$ is a congruence relation on $A = (A, \{f_x\}_{x \in \Omega})$ if it is an equivalence relation on $A$ and

$$f_x(a_1, \ldots, a_n) \equiv f_x(b_1, \ldots, b_n)$$

for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, $x \in \Omega$, and $n = \text{arity}(x)$ such that $a_1 \equiv b_1$, $a_2 \equiv b_2$, \ldots, and $a_n \equiv b_n$.

The quotient $\left(\left\{ B, g_x \right\}_{x \in \Omega} \right) := A/\equiv$ then has $B$ equal to the set of $\equiv$-equivalence classes in $A$, and operations defined by
congruence relations are precisely those for which this definition makes sense. Conversely, the relation ≡ defined on some \( \Omega \)-algebra \( A \) by \( a \equiv b \) iff \( \phi(a) = \phi(b) \) will be a congruence relation whenever \( \phi \) is an \( \Omega \)-algebra homomorphism.

It should at this point be observed that defining specific congruence relations to that they respect particular identities is not an entirely straightforward matter; it would for example be wrong to expect a simple formula such as ‘\( b \equiv b' \) iff \( b = m(a(b_1), m(b_2, b_3)) \) and \( b' = m(m(b_1, b_2), a(b_3)) \) for some \( b_1, b_2, b_3 \in F_\Omega(X) \)’ to set up the congruence relation imposing hom-associativity on \( F_\Omega(X) \), as it actually fails even to define an equivalence relation. Instead one considers the family of all congruence relations which fulfill the wanted identities, and picks the smallest of these, which also happens to be the intersection of the entire family; this makes precisely those identifications of elements which would be logical consequences of the given axioms, but nothing more. Thus to construct the free hom-associative \( \{ a(), m(, )\} \)-algebra generated by \( X \), one would let \( \Omega = \{ a(), m(, )\} \) and form \( T(\Omega, X)/\equiv \), where \( \equiv \) is defined by

\[
\begin{align*}
t \equiv t' & \iff t \sim t' \text{ for every congruence relation } \sim \text{ on } T(\Omega, X) \\
& \text{ satisfying } m(a(t_1), m(t_2, t_3)) \sim m(m(t_1, t_2), a(t_3)) \text{ for all } t_1, t_2, t_3 \in T(\Omega, X).
\end{align*}
\]

Another thing that should be observed is that this construction of the free hom-associative algebra is not effective, i.e., one cannot use it to implement the algebra on a computer, nor to reliably carry out calculations with pen and paper. The construction does suggest both an encoding of arbitrary algebra elements—since the algebra elements are equivalence classes, just use any element of a class to represent it—and an implementation of operations—just perform the corresponding operation of \( F_\Omega(X) \) on the equivalence representatives—but it does then not suggest any algorithm for deciding equality. Providing such an algorithm is of course equivalent to solving the word problem for the algebra/congruence relation in question, so there cannot be a universal method which works for arbitrary algebras, but nothing prevents seeking a solution that works a particular algebra, and indeed one should always consider this an important problem to solve for every class of algebras one considers.

One common form of solutions to the word problem is to device a normal form map for the congruence relation \( \equiv \): a map \( N : T(\Omega, X) \to T(\Omega, X) \) such that \( N(t) \equiv t \) for all \( t \in T(\Omega, X) \) and \( t \equiv t' \) iff \( N(t) = N(t') \); this singles out one element from each equivalence class as being the normal form representative of that class, thereby reducing the problem of deciding congruence to that of testing whether the respective normal forms are equal. Normal form maps are often realised as the limit of a system of rewrite rules derived directly from the defining relations; we shall return to this matter in Subsection 3.3.
2.3 Algebras with linear structure

One thing that has so far been glossed over is that e.g. a hom-associative algebra is not just supposed to have a non-associative multiplication $m$ and a homomorphism $a$, it is also supposed to have addition and multiplication by a scalar. The general way to ensure this is of course to extend the signature with operations for these, and then impose the corresponding axioms on the congruence relation used, but a more practical approach is usually to switch to a category where the wanted linear structure is in place from the start. As it turns out the free object in the category of algebras with a linear structure can be constructed as the set of formal linear combinations of elements in the free (without linear structure) algebra, our constructions above remain highly useful.

Let $R$ be an associative and commutative ring with unit. An $\Omega$-algebra $(A, \{f_x\}_{x \in \Omega})$ is $R$-linear if $A$ is an $R$-module and each operation $f_x$ is $R$-multilinear, i.e., it is $R$-linear in each argument. An $\Omega$-algebra homomorphism $\phi : A \rightarrow B$ is an $R$-linear $\Omega$-algebra homomorphism if $A$ and $B$ are $R$-linear $\Omega$-algebras and $\phi$ is an $R$-module homomorphism. An $R$-linear $\Omega$-algebra congruence relation $\equiv$ is an $\Omega$-algebra congruence relation on an $R$-linear $\Omega$-algebra which is preserved also by module operations, i.e., $a_1 \equiv b_1$ and $a_2 \equiv b_2$ implies $ra_1 \equiv rb_1$ (for all $r \in R$) and $a_1 + a_2 \equiv b_1 + b_2$.

The free $R$-linear $\Omega$-algebra generated by a set $X$ can be constructed as the set of all formal linear combinations of elements of $T(\Omega, X)$, i.e., as the free $R$-module with basis $T(\Omega, X)$; we will denote this free algebra by $R\{\Omega, X\}$ (continuing the notation family $R[X], R(X), R(\langle X \rangle)$). The universal property it satisfies is that any function $j : X \rightarrow A$ where $A$ is an $R$-linear $\Omega$-algebra gives rise to a unique $R$-linear $\Omega$-algebra homomorphism $\phi : R\{\Omega, X\} \rightarrow A$ such that $j = \phi \circ i$, where $i$ is the function $X \rightarrow R\{\Omega, X\}$ such that $i(x)$ is $x$, or more precisely the linear combination which has coefficient 1 for the formal term $x$ and coefficient 0 for all other terms.

A consequence of the above is that $R\{\Omega, X\}$ is the free $R$-module with basis $X$, which might be seen as restrictive. There is an alternative concept of free $R$-linear $\Omega$-algebra which is generated by an $R$-module $M$ rather than a set $X$, in which case the above universal property must instead hold for $j$ being an $R$-module homomorphism $M \rightarrow A$; in more categoric terms, this corresponds to the functor producing the free algebra being left adjoint of not the forgetful functor from $R$-linear $\Omega$-algebra to Set, but left adjoint of the forgetful functor from $R$-linear $\Omega$-algebra to $R$-module. It is however quite possible to get to that also by going via $R\{\Omega, X\}$, as all one has to do is take $X = M$ and then consider the quotient by the smallest congruence relation $\equiv$ which has $i(a) + i(b) \equiv i(a + b)$ and $ri(a) \equiv i(ra)$ for all $a, b \in M$ and $r \in R$ (it is useful here to make the function $i : X \rightarrow R\{\Omega, X\}$ figuring in the universal property of $R\{\Omega, X\}$ explicit, as $\equiv$ would otherwise seem a triviality); the result is the free object in the category of $R$-linear $\Omega$-algebras that are equipped with an $R$-module homomorphism $i'$ from $M$, just like the alternative universal property would require.
No doubt some readers may find this construction wasteful—a separate constant symbol for every element of the module $M$, with a host of identities just to make them “remember” this module structure, immediately rendering most of the symbols redundant—and would rather prefer to construct the free $R$-linear $\Omega$-algebra on the $R$-module $M$ by direct sums of appropriate tensor products of $M$ with itself, somehow generalising the tensor algebra construction $T(M) = \bigoplus_{n=0}^{\infty} M^\otimes n$. However, from the perspectives of constructive set theory and effectiveness, such constructions are guilty of the exact same wastefulness; they only manage to sweep it under the proverbial rug that is the definition of the tensor product. As is quite often the case, one ends up doing the same thing either way, although the presentation may obscure the correspondencies between the two approaches.

Another stylistic detail is that of whether the denominator in a quotient should be a congruence relation or an ideal. For $R$-linear $\Omega$-algebras, the equivalence class of 0 turns out to be an ideal, and conversely a congruence relation $\equiv$ is uniquely determined by its equivalence class of 0 since $a \equiv b$ if and only if $a - b \equiv 0$. In our experience, an important advantage of the congruence relation formalism is that it makes the dependency on the signature $\Omega$ more explicit, since it is not uncommon to see authors continue to associate “ideal” and/or related concepts with the definition these have in a more traditional setting; particularly continuing to use ‘two-sided ideal’ and ‘$\langle S \rangle$’ as they would be defined in an $\{m(,)\}$-algebra even though all objects under consideration are really $\{m(,),a(,)\}$-algebras. To be explicit, an ideal $I$ in an $R$-linear $\Omega$-algebra $(A,\{f_x\}_{x \in \Omega})$ is an $R$-submodule of $A$ with the property that

$$f_x(a_1,\ldots,a_{\text{arity}(x)}) \in I \text{ whenever } \{a_1,\ldots,a_{\text{arity}(x)}\} \cap I \neq \emptyset,$$

for all $a_1,\ldots,a_{\text{arity}(x)} \in A$ and $x \in \Omega$.

Note that for constants $x$, the left operand of $\cap$ above is always empty, and thus this condition does not require that (the values of) constants would be in every ideal. It does however imply that unary operations map ideals into themselves, and higher arity operations take values within the ideal as soon as any operand is in the ideal.

### 2.4 Algebra constructions revisited

Modulo some minor details, this universal algebra machinery allows us to reproduce quickly the constructions of free hom-nonassociative algebras, free hom-associative algebras, and universal enveloping hom-associative algebras from Subsection 1.3, as well as various others that [44] treat more cursory. The plane binary trees are simply an alternative encoding of formal terms over the signature $\{m(,\})$; the correspondence of one to the other is arguably not entirely trivial, but well-known, and it is clearly the binary trees that have the weaker link to the algebra. There is perhaps a slight mismatch in that a formal term would encode an actual constant within
each leaf, whereas the binary trees as specified rather take the leaves to mark places where a constant can be inserted, but we shall return to that in the next section.

The weighting added to the trees is a method of encoding also the $\alpha$ operation of a hom-algebra; the unstated idea is that the weight $w(v)$ of a node $v$ specifies how many times $\alpha$ should be applied to the partial result of that node. This is thus why grafting creates new nodes with weight 0—grafting is multiplication, so when the outermost operation was a multiplication, no additional $\alpha$s are to be applied—and why $\alpha$ raises the weight of the root node $v_{\text{low}}$ only. One would like to think of a weighted $n$-tree as a specification of how $n$ elements in a hom-algebra are being composed—for example the term $\alpha^3(m(\alpha(m(\alpha^2(x_1), x_2)), \alpha^4(x_3)))$ would correspond to the weighted 3-tree

![3-tree diagram]

—but there is a catch: weights were supposed to appear only on the internal vertices, not on the leaves, so the above is not strictly a weighted tree as defined in [44]. This choice of disallowing weights on leaves corresponds to the dichotomy in the definition of $\alpha_F$ for $F_{\text{HNAs}}$: as the underlying $\alpha$ on 1-tree terms, but as a shift [1] on $n$-tree terms for $n > 1$. This in turn corresponds to the choice of making $F_{\text{HNAs}}$ a functor from $\mathcal{R}$-hom-module to $\mathcal{R}$-hom-algebra rather than a functor from $\mathcal{R}$-module to $\mathcal{R}$-hom-algebra; the former produces objects that are less free than those of the latter. It is arguably a strength of the universal algebra method that this distinction appears so clearly, and also a strength that it prefers the more general approach.

What one would do in the universal algebra setting to recover the exact same $F_{\text{HNAs}}(A)$ as Yau defined is to impose $a(x) \equiv \alpha(x)$ for all $x \in A$ as conditions upon a congruence relation $\equiv$, and then take the quotient by that. Technically, one would start out with the free $\mathcal{R}$-linear $\Omega$-algebra $R\{\Omega, A\}$ and impose upon it (in addition to $a(x) \equiv \alpha(x)$) the silly-looking congruences

$$rx \equiv (rx), \quad x + y \equiv (x + y) \quad \text{for all } x, y \in A \text{ and } r \in \mathcal{R};$$

the technical point here is that addition and multiplication in the left hand sides refer to the operations in $\mathcal{R}\{\Omega, A\}$, whereas those on the right hand side refer to operations in $A$. What happens is effectively the same as in the set-theoretic construction of tensor product of modules. Similarly, to recover the hom-associative $F_{\text{HAs}}(A)$ one would start out with $\mathcal{R}\{\Omega, A\}$ for $\Omega = \{a(), m(),\}$ and quotient that by the smallest $\mathcal{R}$-linear $\Omega$-algebra congruence relation $\equiv$ satisfying the linearity condition (25) and

$$x y \equiv (x y), \quad (x + y) z \equiv (x z) + (y z) \quad \text{for all } x, y, z \in A \text{ and } r, s \in \mathcal{R};$$

the technical point here is that addition and multiplication in the left hand sides refer to the operations in $\mathcal{R}\{\Omega, A\}$, whereas those on the right hand side refer to operations in $A$. What happens is effectively the same as in the set-theoretic construction of tensor product of modules. Similarly, to recover the hom-associative $F_{\text{HAs}}(A)$ one would start out with $\mathcal{R}\{\Omega, A\}$ for $\Omega = \{a(), m(),\}$ and quotient that by the smallest $\mathcal{R}$-linear $\Omega$-algebra congruence relation $\equiv$ satisfying the linearity condition (25) and
\[ a(x) \equiv \alpha(x) \quad \text{for all } x \in A, \quad (26a) \]
\[ m(a(t_1), m(t_2, t_3)) \equiv m(m(t_1, t_2), a(t_3)) \quad \text{for all } t_1, t_2, t_3 \in T(\Omega, A). \quad (26b) \]

Finally, in order to recover \( U_{\text{HLie}}(g) \) for the hom-Lie algebra \( g = (A, b, \alpha) \), one needs only impose also the condition
\[ m(x, y) - m(y, x) \equiv b(x, y) \quad \text{for all } x, y \in A \quad (26c) \]
on the congruence relation \( \equiv \). What in this step has been noticeably simplified in comparison to the presentation of Subsection 1.3 is that the infinite sequence of alternatingly generating two-sided ideals and applying \( \alpha_f \) has been compressed into just one operation, namely that of forming the generated congruence relation. This has not made the whole thing more effective, but it greatly simplifies reasoning about it.

For the reader approaching the above as it a deformation of the associative universal enveloping algebra of a Lie algebra, it might instead be more natural to impose the conditions in the order (26b) first, (26c) second, and (26a) last. Doing so might also raise the question of why one should stop there, as opposed to imposing some additional condition on \( a \), such as \( a(m(t_1, t_2)) \equiv m(a(t_1), a(t_2)) \)? The reason not to ask for that particular condition is that it forces the resulting hom-algebra to be multiplicative, and it is easily checked that if \( A \) is a multiplicative hom-algebra, then \( A^- \) is multiplicative as well; doing so would immediately destroy all hope of getting an enveloping algebra, unless the hom-Lie algebra one started with was already multiplicative.

For a hom-Lie algebra presented in terms of a basis, such as the \( q \)-deformed \( \mathfrak{sl}_2 \) of (10), it is usually more natural to seek its \( U_{\text{HLie}} \) by starting with only the basis elements as constant symbols. In that example one would instead take \( X = \{e, f, h\} \) and seek a congruence relation on \( \mathbb{K}\{\Omega, X\} \), namely that which satisfies
\[ a(e) \equiv qe, \quad a(f) \equiv q^2f, \quad a(h) \equiv qh, \quad (27a) \]
\[ m(a(t_1), m(t_2, t_3)) \equiv m(m(t_1, t_2), a(t_3)) \quad \text{for all } t_1, t_2, t_3 \in T(\Omega, X). \quad (27b) \]
\[ m(e, f) - m(f, e) \equiv \frac{1}{2}(1 + q)h, \quad m(e, h) - m(h, e) \equiv -2e, \quad m(h, f) - m(f, h) \equiv -2qf. \quad (27c) \]

It suffices to impose hom-associativity for monomial terms (those that can be formed using \( a \), \( m \), and elements of \( X \) only) as anything else is a finite linear combination of such terms.

In these equations, it should be observed that (27a) and (27c) are three discrete conditions each, whereas (27b) imposing hom-associativity is an infinite family of conditions. This is mirrored in (26) by the difference in ranges: in (26a), \( x \) ranges only over elements of \( A \) (i.e., terms that are constants), but in (26b) the variables range over arbitrary terms. Comparing this to presentations of associative algebras on the form \( \mathcal{R}(x, y, z \mid \ldots) \), the discrete conditions are like prescribing a re-
lation between the generators $x$, $y$, and $z$, whereas the infinite family used for hom-associativity is like prescribing a Polynomial Identity for the algebra. In rewriting theory, one would rather say (27a) and (27c) are equations of ground terms whereas (27b) is an equation involving variables (note that this is a different sense of ‘variable’ than in ‘variable’ as generator of $\mathcal{R}(x, y, z)$).

The exact same analysis can be carried out for the hom-dialgebras and diweighted trees of [44, Secs. 5–6]; the main point of deviation is merely that one starts out the signature $\{a(), l(), r()\}$ (because a dialgebra has separate left multiplication $\cdot$ and right multiplication $\cdot'$) rather than the hom-algebra signature $\{a(), m()\}$. The di-weighted tree encoding takes another step away from the canonical formal terms by bundling into the weight the left/right nature of each multiplication with the number of $\alpha$s to apply after it. This is not quite as ad hoc as it may seem, because in non-hom dialgebras the associativity-like axioms have the effect that general products of $n$ elements look like $(\cdots (x_1 \cdot x_2 ) \cdot x_m \cdot (\cdots (x_{n-1} \cdot x_n ) \cdot \cdots )$; the left/right nature of a multiplication is pretty much determined by its position in relation to the switchover factor $x_m$, so there it makes sense to seek a mostly unified encoding of the two. It is however far from clear that the same would be true also for general hom-dialgebras; free hom-associative algebras are certainly far more complicated than free associative algebras.

3 A newer setting: free operads

One awkward point above is that for example the hom-associativity axiom, despite in some sense being just one identity, required an infinite family of equations to be imposed upon the free hom-associative algebra; shouldn’t there be a way of imposing it in just one step? Indeed there is, but it requires broadening one’s view, and to think in terms of operads rather than algebras. A programme for this was outlined in [20].

3.1 What is an operad?

Nowadays, many introductions to the operad concept are available, for example [33, 40, 42]. What is important for us to stress is the analogy with associative algebras: Operators acting on (say) a vector space can be added together, taken scalar multiples of, and composed; any given set of operators will generate an associative algebra under these operations. When viewed as functions, operators are only univariate however, so one might wonder what happens if we instead consider multivariate functions (still mapping some number of elements from a vector space into that same space)? One way of answering that question is that we get an operad.

Composition in operads work as when one uses dots ‘.’ to mark the position of “an argument” in an expression: From the bivariate functions $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$, one
may construct the compositions \( f(g(\cdot, \cdot), \cdot), f(\cdot, g(\cdot, \cdot)), g(f(\cdot, \cdot), \cdot), \) and \( g(\cdot, f(\cdot, \cdot)) \), which are all trivariate. Note in particular that the “variable-based” style of composition that permits forming e.g. the bivariate function \((x, y) \mapsto f(g(x, y), y)\) from \(f\) and \(g\) is not allowed in an operad, because it destroys multilinearity; \(f(x, y) = xy\) is a bilinear map \(\mathbb{R}^2 \to \mathbb{R}\), but \(h(x) = f(x, x) = x^2\) is nonlinear.\(^2\) In an expression that composes several operad elements into one, one is however usually allowed to choose where the various arguments are used: \(g(f(x_1, x_2), x_3), g(f(x_2, x_1), x_3), g(f(x_3, x_1), x_2)\), etc. are all possible as operad elements. This is formalised by postulating a right action of the group \(\Sigma_n\) of permutations of \(\{1, \ldots, n\}\) on those operad elements which take \(n\) arguments; in function notation one would have \(f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) = (f\sigma)(x_1, \ldots, x_n)\).

More formally, an operad \(\mathcal{P}\) is a family \(\{\mathcal{P}(n)\}_{n \in \mathbb{N}}\) of sets, where \(\mathcal{P}(n)\) is “the set of those operad elements which have arity \(n\)”. Alternatively, an operad \(\mathcal{P}\) can be viewed as a set with an arity function, in which case \(\mathcal{P}(n)\) is a shorthand for \(\{ a \in \mathcal{P} \mid \text{arity}(a) = n \}\). Both approaches are (modulo some formal nonsense) equivalent, and we will employ both since some concepts are easier under one approach and others are easier under the other.

Composition can be given the form of composing one element \(a \in \mathcal{P}(m)\) with the \(m\) elements \(b_i \in \mathcal{P}(n_i)\) for \(i = 1, \ldots, m\) (i.e., one for each “argument” of \(a\)) to form \(a \circ b_1 \otimes \cdots \otimes b_m \in \mathcal{P}\left(\sum_{i=1}^m n_i\right)\); note that the “\(\circ\)” and the \(m-1\) “\(\otimes\)” are all part of the same operad composition. (There is a more general concept called PROP where \(b_1 \otimes \cdots \otimes b_m\) would be an actual element, but we won’t go into that here.) Operad composition is associative in the sense that the unparenthesized expression

\[
a \circ b_1 \otimes \cdots \otimes b_l \circ c_1 \otimes \cdots \otimes c_m
\]

is the same whether it is interpreted as

\[
(a \circ b_1 \otimes \cdots \otimes b_l) \circ c_1 \otimes \cdots \otimes c_m
\]

or as

\[
a \circ (b_1 \circ c_1 \otimes \cdots \otimes c_{m_1}) \otimes \cdots \otimes (b_l \circ c_{m_1+\cdots+m_{l-1}+1} \otimes \cdots \otimes c_{m_1+\cdots+m_l})
\]

where \(m = \sum_{i=1}^l m_i\) and \(b_i \in \mathcal{P}(m_i)\) for \(i = 1, \ldots, l\).\(^3\)

Since \(\Sigma_n\) acts on the right of each \(\mathcal{P}(n)\), this action satisfies \((a\sigma)\tau = a(\sigma\tau)\) for all \(a \in \mathcal{P}(n)\) and \(\sigma, \tau \in \Sigma_n\). There is also a condition called equivariance that

\(^2\) It may then seem serendipitous that Cohn [11, p. 127] citing Hall calls an algebraic structure with the variable-based form of composition a clone, since it gets its extra power from being able to “clone” input data, but he explains it as being a contraction of ‘closed set of operations’. In the world of Quantum Mechanics, the well-known ‘No cloning’ theorem forbids that kind of behaviour (essentially because it violates multilinearity), so by sticking to operads we take the narrow road.

\(^3\) As the number of ellipses (\ldots) above indicate, the axioms for operads are somewhat awkward to state, even though they only express familiar properties of multivariate functions. The PROP formalism may therefore be preferable even if one is only interested in an operad setting, since the PROP axioms can be stated without constantly going \('\ldots'\).
Example 14. A permutation is a reason of the actions on it of the symmetric groups. Dropping everything involving permutations above, one instead arrives at the concept of a non-symmetric or non-Σ operad.

Example 13. For every set $A$, there is an operad $\text{Map}_A$ such that $\text{Map}_A(n)$ is the set of all maps $A^n \to A$; in particular, $\text{Map}_A(0)$ may be identified with $A$. For $a \in \text{Map}_A(m)$ and $b_i \in \text{Map}_A(n_i)$ for $i = 1, \ldots, m$, the composition $a \circ b_1 \otimes \cdots \otimes b_m$ is defined by

$$(a \circ b_1 \otimes \cdots \otimes b_m)(x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{m,1}, \ldots, x_{m,n_m}) = a(b_1(x_{1,1}, \ldots, x_{1,n_1}), \ldots, b_m(x_{m,1}, \ldots, x_{m,n_m}))$$

for all $x_{1,1}, \ldots, x_{m,n_m} \in A$. $\text{id} \in \text{Map}_A(1)$ is the identity map on $A$. The permutation action is defined by $(a \circ b_1 \otimes \cdots \otimes b_m)(x_1, \ldots, x_n) = a(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$.

An alternative notation for composition is $\gamma(a, b_1, \ldots, b_m) = a \circ b_1 \otimes \cdots \otimes b_m$; that $\gamma$ is then called the structure map, or structure maps if one requires each map to have a signature on the form $\mathcal{P}(m) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_m) \to \mathcal{P}(\sum_{i=1}^m n_i)$. An alternative composition concept is the $i$th composition $\circ_i$, which satisfies $a \circ_i b = a \circ i d \otimes (i-1) \otimes b \otimes i d \otimes (m-i)$ for $a \in \mathcal{P}(m)$ and $i = 1, \ldots, m$. Note that $i$th composition, despite being a binary operation, is not at all associative in the usual sense and expressions involving it must therefore be explicitly parenthesized; operad associativity does however imply that subexpressions can be regrouped (informally: “parentheses can be moved around”) provided that the position indices are adjusted accordingly.

An operad homomorphism $\phi : \mathcal{P} \to \Omega$ is a map that is compatible with the operad structures of $\mathcal{P}$ and $\Omega$: $\text{arity}_\Omega(\phi(a)) = \text{arity}_\mathcal{P}(a)$, $\phi(a \circ b_1 \otimes \cdots \otimes b_m) = \phi(a) \circ \phi(b_1) \otimes \cdots \otimes \phi(b_m)$, $\phi(a \sigma) = \phi(a) \sigma$, and $\phi(\text{id}_\mathcal{P}) = \text{id}_\Omega$ for all $a \in \mathcal{P}(m)$, $b_i \in \mathcal{P}$ for $i = 1, \ldots, m$, $\sigma \in \Sigma_n$, and $m \in \mathbb{N}$. A suboperad of $\mathcal{P}$ is a subset of $\mathcal{P}$ that is closed under composition, closed under permutation action, and contains the identity element. The operad generated by some $\Omega \subseteq \mathcal{P}$ is the smallest suboperad of $\mathcal{P}$ that contains $\Omega$.

Let $\mathcal{R}$ be an associative and commutative unital ring. An operad $\mathcal{P}$ is said to be $\mathcal{R}$-linear if (i) each $\mathcal{P}(n)$ is an $\mathcal{R}$-module, (ii) every structure map $(a, b_1, \ldots, b_m) \mapsto a \circ b_1 \otimes \cdots \otimes b_m$ is $\mathcal{R}$-linear in each argument separately, and (iii) each action of a permutation is $\mathcal{R}$-linear.

Example 14. The $\text{Map}_A$ operad is in general not $\mathcal{R}$-linear, but if $A$ is an $\mathcal{R}$-module, then the suboperad $\text{End}_A$ where $\text{End}_A(n)$ consists of all $\mathcal{R}$-multilinear maps $A^n \to A$ will be $\mathcal{R}$-linear. $\text{End}_A(0)$ can also be identified with $A$.

The operad concept defined above is sometimes called a symmetric operad, because of the actions on it of the symmetric groups. Dropping everything involving permutations above, one instead arrives at the concept of a nonsymmetric or non-$\Sigma$ operad.
operad. Much of what is done below could just as well be done in the non-$\Sigma$ setting, but we find the symmetric setting to be more akin to classical universal algebra.

### 3.2 Universal algebra for operads

Regarding universal algebra, an interesting thing about operads is that they may serve as generalisations of both the algebra concept and the signature concept. The way that an operad $\mathcal{P}$ may generalise a signature $\Omega$ is that a set $A$ is said to be a $\mathcal{P}$-algebra if it is given with an operad homomorphism $\phi : \mathcal{P} \rightarrow \text{Map}_A$; the operation $f_x$ of some $x \in \mathcal{P}$ is then simply $\phi(x)$. Being an operad-algebra is however a stronger condition than being a signature-algebra, because the map $\phi$ will only be a homomorphism if every identity in $\mathcal{P}$ is also satisfied in $\phi(\mathcal{P})$; this can be used to impose "laws" on algebras, and several elementary operads are defined to precisely this purpose: an algebra is an $\text{Ass}$-algebra iff it is associative, a $\text{Com}$-algebra iff it is commutative, a $\text{Lie}$-algebra iff it is a Lie algebra, a $\text{Leib}$-algebra iff it is a Leibniz algebra, and so on. It is therefore only natural that we will shortly construct an operad $\text{HAss}$ whose algebras are precisely the hom-associative algebras.

Before taking on that problem, we should however give an example of how identities in an operad become laws of its algebras. To that end, consider $\mathbb{N}$ as an operad by making arity$(n) = n$; this uniquely defines the operad structure, since the arity of any particular composition is given by the axioms, and that in turn determines the value since every $\mathbb{N}(n)$ only has one element. What can now be said about an $\mathbb{N}$-algebra $A$ if $f : \mathbb{N} \rightarrow \text{Map}_A$ is the given operad homomorphism? Clearly $f(2) : A^2 \rightarrow A$ is a binary operation. If $\tau \in \Sigma_2$ is the transposition, one furthermore finds that

$$f(2)(x,y) = f(2\tau)(x,y) = (f(2)\tau)(x,y) = f(2)(y,x)$$

for all $x,y \in A$, so $f(2)$ is commutative. Similarly it follows from $2 \circ 1 \otimes 2 = 3 = 2 \circ 2 \otimes 1$ that $f(2)(x,f(2)(y,z)) = f(2)(f(2)(x,y),z)$ for all $x,y,z \in A$, and thus $f(2)$ is associative. Finally one may deduce from $\text{id} = 1 = 2 \circ 0 \otimes 1$ that $f(0)$ is a unit element with respect to $f(2)$, so in summary any $\mathbb{N}$-operad algebra carries an abelian monoid structure. This is almost the same as $\text{Com}$ is supposed to accomplish, so one might ask whether in fact $\text{Com} = \mathbb{N}$, but traditionally $\text{Com}$, $\text{Ass}$, etc. are taken to be the $\mathbb{R}$-linear (for whatever ring $\mathbb{R}$ of scalars is being considered) operads that impose the indicated laws on their algebras. $\text{Com}$ is thus rather characterised by having dim $\text{Com}(n) = 1$ for all $n$, and may if one wishes be constructed as $\mathbb{R} \times \mathbb{N}$.

While specific operads may sometimes be constructed through elementary methods as above, the general approach to constructing an operad that corresponds to a specific set of laws is instead the universal algebraic one, which rather employs the point of view that an operad is a generalisation of an algebra. Obviously any specific $\Omega$-algebra $(A, \{f_x\}_{x \in \Omega})$ gives rise to the operad $\text{Map}_A$, but the operad that more naturally generalises $A$ as an $\Omega$-algebra is the suboperad of $\text{Map}_A$ that is generated by $\{f_x\}_{x \in \Omega}$. Conversely, if $A$ is supposed to be some kind of free algebra,
one may choose to construct it as the constant component of the corresponding free operad.

An equivalence relation \( \equiv \) on an operad \( \mathcal{P} \) is an \textit{operad congruence relation} if:

1. \( a \equiv a' \) implies \( \text{arity}(a) = \text{arity}(a') \),
2. \( a \equiv a' \) and \( b_i \equiv b_i' \) for \( i = 1, \ldots, \text{arity}(a) \) implies \( a \circ b_1 \otimes \cdots \otimes b_{\text{arity}(a)} \equiv a' \circ b_1' \otimes \cdots \otimes b'_{\text{arity}(a)} \), and
3. \( a \equiv a' \) implies \( a\sigma \equiv a'\sigma \) for all \( \sigma \in \Sigma_{\text{arity}(a)} \).

As for algebras, it follows that the quotient \( \mathcal{P}/\equiv \) carries an operad structure, and the canonical map \( \mathcal{P} \to \mathcal{P}/\equiv \) is an operad homomorphism. If additionally \( \mathcal{P} \) is \( \mathcal{R} \)-linear and \( \equiv \) is an \( \mathcal{R} \)-module congruence relation on each \( \mathcal{P}(n) \), then \( \equiv \) is an \( \mathcal{R} \)-linear operad congruence relation and the corresponding \textit{operad ideal} \( \mathcal{I} \) is defined by \( \mathcal{I}(n) = \{ a \in \mathcal{P}(n) | a \equiv 0 \} \) for all \( n \in \mathbb{N} \) (note that each \( \mathcal{P}(n) \) has a separate 0 element). Equivalently, \( \mathcal{I} \subseteq \mathcal{P} \) is an operad ideal if each \( \mathcal{I}(n) \) is a submodule of \( \mathcal{P}(n) \), each \( \mathcal{I}(n) \) is closed under the action of \( \Sigma_n \), and \( a \circ b_1 \otimes \cdots \otimes b_m \in \mathcal{I} \) whenever at least one of \( a, b_1, \ldots, b_m \) is an element of \( \mathcal{I} \).

So far, the operad formalism is very similar to that for algebras, but an important difference occurs when one wishes to impose laws on a congruence. For an algebra, the hom-associativity condition (24) required an infinite family of identities.

The corresponding condition in the operad \( \text{Map}_A \) requires only the single identity \( f_m \circ f_a \otimes f_m \equiv f_m \circ f_m \circ f_a \), as the infinite family is recovered from this using composition on the right: \( f_m \circ f_a \otimes f_m \circ f_1 \otimes f_2 \otimes f_3 \equiv f_m \circ f_m \circ f_a \circ f_1 \otimes f_2 \otimes f_3 \). The \( \text{Ass}, \text{Com}, \text{Lie} \), etc. operads can all be seen to be finitely presented, and the same holds for their free algebras if generated as the arity 0 component of an operad, even though they are not finitely presented within the \( \Omega \)-algebra formalism!

The universal property satisfied by the free operad \( \mathcal{F} \) on \( \Omega \) is that it is given with an arity-preserving map \( i : \Omega \to \mathcal{F} \) such that for every operad \( \mathcal{P} \) and every arity-preserving map \( j : \Omega \to \mathcal{P} \) such that \( j = \phi \circ i \). A practical construction of that free operad is to let \( \mathcal{F}(n) \) be the set of all \( n \)-variable contexts [12, p. 17], but since we’ll anyway need some notation for these, we might as well give an explicit definition based on Polish notation for expressions.

**Definition 15.** A \textit{(left-)Polish term} on the signature \( \Omega \) is a finite word on \( \Omega \cup \{ \square_i \}_{i=1}^\infty \) (where it is presumed that \( \square_i \notin \Omega \) and \( \text{arity}(\square_i) = 0 \) for all \( i \)), which is either \( \square_i \), for some \( i \geq 1 \), or \( x\mu_1 \cdots \mu_n \) where \( x \in \Omega \), \( n = \text{arity}(x) \), and \( \mu_1, \ldots, \mu_n \) are themselves Polish terms on \( \Omega \). A Polish term is an \textit{n-context} if each symbol \( \square_i \) for \( i = 1, \ldots, n \) occurs exactly once and no symbol \( \square_i \) with \( i > n \) occurs at all. For \( \square_1, \ldots, \square_9 \) we will write 1, \ldots, 9 for short. Denote by \( \gamma_\Omega(n) \) the set of all \( n \)-contexts on \( \Omega \).

The action of \( \sigma \in \Sigma_n \) on \( \gamma_\Omega(n) \) is that each \( \square_i \) is replaced by \( \square_{\sigma^{-1}(i)} \). The composition \( \mu \circ v_1 \otimes \cdots \otimes v_n \) is a combined substitution and renumbering: first each \( \square_i \) in \( \mu \) is replaced by the corresponding \( v_i \), then the \( \square_i \)'s in the composite term are renumbered so that the term becomes a context—preserving the differences within each \( v_i \) and giving \( \square_i \)'s from \( v_i \) lower indices than those from \( v_j \) whenever \( i < j \).
For any associative and commutative unital ring $\mathbb{R}$, and for every $n \in \mathbb{N}$, denote by $\mathcal{R}(\Omega)(n)$ the set of all formal $\mathbb{R}$-linear combinations of elements of $\mathcal{Y}_\Omega(n)$. Extend the action of $\sigma \in \Sigma_n$ on $\mathcal{Y}_\Omega(n)$ to $\mathcal{R}(\Omega)(n)$ by linearity. Let $\mathcal{R}(\Omega) = \bigcup_{n \in \mathbb{N}} \mathcal{R}(\Omega)(n)$. Extend the composition on $\mathcal{Y}_\Omega$ to $\mathcal{R}(\Omega)$ by multilinearity. When $\mathcal{Y}_\Omega$ is viewed as a subset of $\mathcal{R}(\Omega)$, its elements are called monomials.

With $\text{id} = 1 = \Box_1$, this makes $\mathcal{Y}_\Omega$ the free operad on $\Omega$ and $\mathcal{R}(\Omega)$ is the free $\mathbb{R}$-linear operad on $\Omega$.

For $\Omega = \{x, a(\cdot), m(\cdot, \cdot)\}$, one may thus find in $\mathcal{Y}_\Omega(0)$ elements such as $x$, $ax$, $mxx$, $amxx$, and $maxx$ which in parenthesized notation would rather have been written as $x$, $a(x)$, $m(x, x)$, $a(m(x, x))$, and $m(a(x), x)$ respectively. In $\mathcal{Y}_\Omega(1)$ we similarly find $1$, $a1$, $aa1$, $m1x$, and $max1x$ which in parenthesized notation could have been written as $\Box_1$, $a(\Box_1)$, $a(a(\Box_1))$, $m(x, \Box_1)$, $m(\Box_1, x)$, and $m(a(x), m(\Box_1, x))$.

In $\mathcal{R}(\Omega)(2)$ there are elements such as $m12 - m21$ and $m12 + m21$ which would be mapped to 0 by any operad homomorphism $f$ to $\text{Map}_A$ for which $f(m)$ is commutative or anticommutative respectively. Finally there is in $\mathcal{R}(\Omega)(3)$ the elements $m1m23 - mm123$ and $ma1m23 - mm12a3$ which have similar roles with respect to associativity and hom-associativity respectively.

A practical problem, which is mostly common to the Polish and the parenthesized notations, is that it can be difficult to grasp the structure of one of these expressions just from a quick glance at the written forms of them; small expressions may be immediately recognised by the trained eye, but larger expressions almost always require a conscious effort to parse. This is unfortunate, as the exact structure is very important when working in a setting this general. The structure can however be made more visible by drawing expressions rather than writing them; informally one depicts an expression using its abstract syntax tree, but those of a more formalistic persuasion may think of these drawings as graph-theoretical objects underlying the trees (in the sense of [12, pp. 15–16]) of these terms. A few examples can be

$$m12 = \begin{bmatrix} \Box_1 \end{bmatrix}, \quad m21 = \begin{bmatrix} \Box_2 \end{bmatrix}, \quad mm312 = \begin{bmatrix} \Box_1 \Box_2 \end{bmatrix},$$

and several more can be found below. A Polish term may even be read as a direct instruction for how to draw these trees: in order to draw $\mu = x\nu_1 \cdots \nu_{\text{arity}(\cdot)}$, first draw a vertex for $x$ as the root, and then draw the subtrees $\nu_i$ through $\nu_{\text{arity}(\cdot)}$ above the $x$ vertex and side by side, letting the order of edges along the top of a vertex show the order of the subexpressions. The “inputs” $\Box_k$ of a context are represented by edges to the top side of the drawing, with $\Box_1$ being leftmost, $\Box_2$ being second to left, and so on.

**Definition 16.** An element of $\mathcal{Y}_\Omega(n)$ is said to be plane if the $\Box$ symbols (if any) occur in ascending order: none to the left of $\Box_1$, only $\Box_1$ to the left of $\Box_2$, and so on. (Equivalently, the drawing procedure described above will not produce any crossing edges.) An element of $\mathcal{R}(\Omega)(n)$ is plane if it is a linear combination of plane elements. An element of $\mathcal{R}(\Omega)(n)$ is planar if it is of the form $a\sigma$ for some
plane $a \in \mathcal{R}\{\Omega}\{1\}$ and $\sigma \in \Sigma_n$. Finally, an ideal in $\mathcal{R}\{\Omega\}$ is said to be planar if it is generated by planar elements.

Elements in a planar ideal need not be planar, but every element in a planar ideal can be written as a sum of planar elements that are themselves in the ideal.

### 3.3 The Diamond Lemma for operads

This and the following sections rely heavily on results and concepts from [21]. We try to always give a reference, where a concept is first used that will not be explained further here, to the exact definition in [21] of that concept.

Let a signature $\Omega$ and an associative and commutative unital ring $\mathcal{R}$ be given. Consider the free $\mathcal{R}$-linear operad $\mathcal{R}\{\Omega\}$ and its suboperad of monomials $\mathcal{Y}_\Omega$. Let $V(i,j)$ be the set of all maps $\mathcal{R}\{\Omega\}(j) \rightarrow \mathcal{R}\{\Omega\}(i)$ that are on the form

$$a \mapsto (\lambda \circ_k (a \circ v_1 \otimes \cdots \otimes v_j)) \sigma$$

where $v_r \in \mathcal{Y}_\Omega(n_r)$ for $r = 1, \ldots, j$, $\lambda \in \mathcal{Y}_\Omega(\ell)$, $\ell \geq k \geq 1$, $\sigma \in \Sigma_i$, and $i = \ell - 1 + n_1 + \cdots + n_j$. The family $V = \bigcup_{i,j \in \mathbb{N}} V(i,j)$ is then a category [21, Def. 6.8], and each $v \in V(i,j)$ is an injection $\mathcal{Y}_\Omega(j) \rightarrow \mathcal{Y}_\Omega(i)$. Also note that with respect to the tree (drawing) forms of monomials, each $v \in V(i,j)$ defines an embedding of $\mu \in \mathcal{Y}_\Omega(j)$ into $v(\mu)$; this will be important for identifying $V$-critical ambiguities.

**Definition 17.** A rewriting system for $\mathcal{R}\{\Omega\}$ is a set $S = \bigcup_{i \in \mathbb{N}} S(i)$ such that $S(i) \subseteq \mathcal{Y}_\Omega(i) \times \mathcal{R}\{\Omega\}(i)$ for all $i \in \mathbb{N}$. The elements of a rewrite system are called (rewrite) rules. The components of a rule $s$ are often denoted $\mu_s$ and $a_s$, meaning $s = (\mu_s, a_s)$ for all rules $s$.

For a given rewriting system $S$, define $T_1(S)(i) = \bigcup_{j \in \mathbb{N}} \{t_{v,s} \}_{v \in V(i,j), s \in S(i)}$, where $t_{v,s}$ is the $\mathcal{R}$-linear map $\mathcal{R}\{\Omega\}(i) \rightarrow \mathcal{R}\{\Omega\}(i)$ which satisfies

$$t_{v,s}(\lambda) = \begin{cases} v(a_s) & \text{if } \lambda = v(\mu_s), \\ \lambda & \text{otherwise}, \end{cases}$$

for all $\lambda \in \mathcal{Y}_\Omega(i)$. (29)

The elements of $T_1(S)(i)$ are called the simple reductions (with respect to $S$) on $\mathcal{R}\{\Omega\}(i)$. For each $i \in \mathbb{N}$, let $T(S)(i)$ be the set of all finite compositions of maps in $T_1(S)(i)$.

Sometimes, a claim that $t_{v,s}(a) = b$ is more conveniently written as $a \xrightarrow{s} b$ (for example when several such claims are being chained, as in $a \xrightarrow{s_1} b \xrightarrow{s_2} c$). When doing that, we may indicate what $v$ is by inserting parentheses into the Polish term on the tail side of the arrow that is being changed by the simple reduction: the outer parenthesis then surrounds the $\mu_s \circ v_1 \otimes \cdots \otimes v_j$ part, whereas inner parentheses surround the various $v_k$ subterms of it, although these inner parentheses are for brevity omitted where $v_k = \text{id}$. See Example 20 for some examples of this.
With respect to $T(S)$, all maps in $V$ are absolutely advanceable [21, Def. 6.1]. The following subsets of $R\{\Omega\}$ are defined in [21, Def. 3.4], but so important that we include the definitions here:

\[
Irr(S)(i) = \{ a \in R\{\Omega\}(i) \mid t(a) = a \text{ for all } t \in T(S)(i) \},
\]

\[
J(S)(i) = \sum_{t \in T(S)(i)} \{ a - t(a) \mid a \in R\{\Omega\}(i) \}
\]

for all $i \in \mathbb{N}$. We write $a \equiv b \pmod{S}$ for $a - b \in J(S)$. An $a \in Irr(S)$ is said to be a \textit{normal form} of $b \in R\{\Omega\}$ if $a \equiv b \pmod{S}$.

$J(S)$ is the operad ideal in $R\{\Omega\}$ that is generated by $\{ \mu_s - a_s \mid s \in S \}$. $Irr(S)$ is what we want to use as model for the quotient $R\{\Omega\}/J(S)$, and we use Theorem 18 below to tell us that it really is. An \textit{ambiguity} [21, Def. 5.9] of $T_i(S)(i)$ is a triplet $(v_1, v_2, v_3)$ such that $v_1(\mu_{s_1}) = \mu = v_2(\mu_{s_2})$. The ambiguity is \textit{plane} if $\mu$ is plane.

\textbf{Theorem 18 (Basic Diamond Lemma for Symmetric Operads).}

If $P(i)$ is a well-founded partial order on $\gamma_i(\Omega)$ such that $a_i \in DSM(\mu_i, P(i))$ for all $i \in \mathbb{N}$, and moreover for all $i, j \in \mathbb{N}$ every $v \in V(i, j)$ is monotone [21, Def. 6.4] with respect to $P(j)$ and $P(i)$, then the following claims are equivalent:

(a) For all $i \in \mathbb{N}$, every ambiguity of $T_i(S)(i)$ is resolvable [21, Def. 5.9].

(a') For all $i \in \mathbb{N}$, every $V$-critical [21, Def. 6.8] ambiguity of $T_i(S)(i)$ is resolvable.

(a'') For all $i \in \mathbb{N}$, every plane $V$-critical ambiguity of $T_i(S)(i)$ is resolvable.

(b) Every element of $R\{\Omega\}$ is persistently [21, Def. 4.1] and uniquely [21, Def. 4.6] reducible, with normal form map $t^S$ [21, Def. 4.6].

(c) Every element of $R\{\Omega\}$ has a unique normal form, i.e., $R\{\Omega\}(i) = J(S)(i) \oplus Irr(S)(i)$ for all $i \in \mathbb{N}$.

\textbf{Proof.} Taking $M(i) = R\{\Omega\}(i)$ and $\gamma(i) = \gamma_i(\Omega)$, this is mostly a combination of Theorem 5.11, Theorem 6.9, and Construction 7.2 of [21]. Theorem 5.11 provides the basic equivalence of (a), (b), and (c). Theorem 6.9 says (a') is sufficient, as resolvability implies resolvability relative to $P$. Construction 7.2 shows the $V$, $P$, and $T_i(S)$ defined above fulfill the conditions of these two theorems.

What remains to show is that (a'') implies (a'). Let $(t_{i_1}, \mu_{i_1}, t_{i_2}, \mu_{i_2})$ be a $V$-critical ambiguity of some $T_i(S)(i)$, and let $\sigma \in \Sigma_i$ be such that $\mu\sigma$ is plane. Then $w: a \mapsto a\sigma$ and $w^{-1}: a \mapsto a\sigma^{-1}$ are both elements of $V(i, i)$, and hence $(t_{i_1}, \mu_{t_{i_1}}, t_{i_2}, \mu_{i_2})$ is an absolute shadow of the plane and $V$-critical ambiguity $(v_{i_1}, \mu, v_{i_2}, \mu_{i_2})$. The latter is resolvable by (a''), so it follows from [21, Lemma 6.2] that the former is resolvable as well.

\textbf{Remark 19.} Theorem 18 may also be viewed as a slightly streamlined version of [23, Cor. 10.26], but that approach is probably overkill for readers uninterested in the PROP setting.

It may be observed that $Irr(S)(i)$ is closed under the action of $\Sigma_i$, regardless of $S$; this is thus a restriction of the applicability of this diamond lemma, as its conditions
can never be fulfilled when $\mathcal{R}\{\Omega\}(i)/\mathcal{B}(S)(i)$ is fixed under a non-identity element of $\Sigma$. All of that is however a consequence of the choice of $V$, and a different choice of $V$ (e.g. excluding the permutation $\sigma$ from (28)) will result in a different (but very similar-looking) diamond lemma, with a different set of critical ambiguities and a different domain of applicability.

For an ambiguity $(\iota_{1, s_{1}}, \mu, \iota_{2, s_{2}})$ to be $V$-critical in this basic diamond lemma, it is necessary that the graph-theoretical embeddings into $\mu$ of $\mu_{s_{1}}$ and $\mu_{s_{2}}$ have at least one vertex in common (otherwise the ambiguity is a montage) and furthermore these two embeddings must cover $\mu$ (otherwise the ambiguity is a proper $V$-shadow). Enumerating the critical ambiguities formed by two given rules $s_{1}$ and $s_{2}$ is thus mostly a matter of listing the ways of superimposing the two trees $\mu_{s_{1}}$ and $\mu_{s_{2}}$.

**Example 20 (Ass operad).** Let $\Omega = \{m(, )\}$. Consider the rewriting system $S = \{s\}$ where $s = (m1m23, mm123)$. Graphically, this rule takes the form

$$
\begin{bmatrix}
\text{[ ]} \\
\text{[ ]}
\end{bmatrix} \rightarrow
\begin{bmatrix}
\text{[ ]} \\
\text{[ ]}
\end{bmatrix}
$$

The (non-unital) **associative operad** $Ass$ over $\mathcal{R}$ can then be defined as the quotient $\mathcal{R}\{\Omega\}/\mathcal{B}(S)$.

One way of partially ordering trees that will be compatible with this rule is to count, separately for each input, the number of times the path from that input to the root enters an $m$ vertex from the right; denote this number for input $i$ of the tree $\mu$ by $h_{i}(\mu)$. Then define $\mu \geq v$ in $P'(n)$ if and only if $h_{i}(\mu) \geq h_{i}(v)$ for all $i = 1, \ldots, n$, and define a partial order $P(n)$ by $\mu \geq v$ in $P(n)$ if and only if $\mu \geq v$ in $P'(n)$ and $\mu \not< v$ in $P'(n)$, i.e., let $P(n)$ be the restriction to a partial order of the quasi-order $P'(n)$. For the left hand side of $s$ above one has $h_{1} = 0$, $h_{2} = 1$, and $h_{3} = 2$ whereas the left hand side has $h_{1} = 0$, $h_{2} = 1$, and $h_{3} = 1$, so $S$ is indeed compatible with $P$. Furthermore $P(n)$ is clearly well-founded; $\sum_{i=1}^{n} h_{i}(\mu)$ is simply the rank of $\mu$ in the poset $(\mathcal{Y}_{\Omega}(n), P(n))$.

The only plane critical ambiguity of $S$ is $(\iota_{1,s}, \mu_{1}m2m34, \iota_{2,s})$, where $v_{1}(\mu) = \mu \circ_{1} m12$ and $v_{2}(\mu) = m12 \circ_{2} \mu$. This is resolved as follows:

$$
\begin{align*}
(m1m2)(m34) &\xrightarrow{s} (m(m12)m34) \xrightarrow{s} \begin{bmatrix}
\text{[ ]} \\
\text{[ ]}
\end{bmatrix} \\
(m1)(m2m34) &\xrightarrow{s} (m1(m23)m4) \xrightarrow{s} \begin{bmatrix}
\text{[ ]} \\
\text{[ ]}
\end{bmatrix}
\end{align*}
$$
Hence the conditions of Theorem 18 are fulfilled, \( R\{\Omega\}(n) = \mathcal{I}(S)(n) \oplus \text{Irr}(S)(n) \) for all \( n \in \mathbb{N} \), and \( \text{Ass}(n) \cong \text{Irr}(S)(n) \) as \( R \)-modules for all \( n \in \mathbb{N} \). Since a monomial \( \mu \) is irreducible iff it does not contain an \( m \) as right child of an \( m \), i.e., iff every right child of an \( m \) is an input, it follows that the only thing that distinguishes two irreducible elements of \( \mathcal{Y}(n) \) is the order of the inputs. On the other hand, every permutation of the inputs gives rise to a distinct irreducible element, so \( \dim \text{Ass}(n) = |\Sigma_n| = n! \) for all \( n \geq 1 \), exactly as one would expect.

For \( n = 0 \) one gets \( \dim \text{Ass}(0) = \dim R\{\Omega\}(0) = |\mathcal{Y}(0)| = 0 \) however, which is perhaps not quite what the textbooks say \( \text{Ass} \) should have. The reason it comes out this way is that we took \( \text{Ass} \) to be the operad for associative algebras, period; had we instead taken it to be the operad for unital associative algebras then \( \dim \text{Ass}(n) = n! \) would have held also for \( n = 0 \). Obviously \( \dim R\{\Omega\}(0) = 0 \) because \( \Omega \) doesn’t contain any constants, but requiring a unit introduces such a constant \( u \). Making that constant behave like a unit requires two additional rules \((\mu u, 1)\) and \((m1u, 1)\) in the rewriting system however, and we felt the resolution of the resulting ambiguities are perhaps better left as exercises.

Another useful exercise is to similarly construct the Leib operad, which merely amounts to replacing the rewriting system \( S \) with \( \mathcal{S}' = \{s'\} \), where \( s = (m1m23, mm123 - mm132) \). Using brackets as notation for the operation in a Leibniz algebra, this rule corresponds to the law that \([x, [y, z]] = [[x, y], z] - [[x, z], y]\). Graphically, \( s' \) takes the form

\[
\begin{bmatrix}
\text{\includegraphics[scale=0.5]{circle}}
\end{bmatrix} \rightarrow \begin{bmatrix}
\text{\includegraphics[scale=0.5]{circle}}
\end{bmatrix} - \begin{bmatrix}
\text{\includegraphics[scale=0.5]{circle}}
\end{bmatrix}
\]

which unlike associativity is not planar, but that makes no difference for the Diamond Lemma machinery. The left hand side of \( s' \) is the same as the left hand side of \( s \), so \( \text{Irr}(\mathcal{S}') = \text{Irr}(\mathcal{S}) \) and both rewriting systems have the same sites of ambiguities. What is different are the resolutions, where the resolution in the Leibniz case is longer since it involves more terms; a compact notation such as the Polish one is highly recommended when reporting the calculations. Still, it is well within the realm of what can be done by hand.

### 3.4 The hom-associative operad

When pursuing the same approach for the hom-associative identity, one of course needs an extra symbol for the unary operation, so \( \Omega = \{m(\cdot), a(\cdot)\} \). Drawing \( m \) as a circle and \( a \) as a square, hom-associativity is then the congruence

\[
\begin{bmatrix}
\text{\includegraphics[scale=0.5]{circle}}
\end{bmatrix} \equiv \begin{bmatrix}
\text{\includegraphics[scale=0.5]{circle}}
\end{bmatrix}
\]

(32)
which can be expressed as a rule $s = (ma1m23, mm12a3)$. It is thus straightforward to define $\mathcal{R}_{\alpha ss} = R\{\Omega\}/J\{\{s\}\}$, but not quite so straightforward to decide whether two elements of $R\{\Omega\}$ are congruent modulo $J\{\{s\}\}$, because $\{s\}$ is not a complete rewriting system; the ambiguity one has to check fails to resolve:

Failed resolutions should not be taken as disasters however; they are in fact opportunities to learn, since what the above demonstrates is that $mm1a2am34 ≡ mm1m23aa4 (\text{mod } \{s\})$ (or as a law: $(xα(y))α(zw) = (x(yz))α(α(w))$ for all $x, y, z, w$), which was probably not apparent from the definition of hom-associativity. Therefore one’s response to this discovery should be to make a new rule out of this new and nontrivial congruence, so that one can use it to better understand congruence modulo hom-associativity.

A problem with this congruence is however that the left and right hand sides are not comparable under the same partial order as worked fine for the associative and Leibniz operads: $(h_0, h_1, h_2, h_3) = (0, 1, 1, 2)$ for the left hand side but $(h_0, h_1, h_2, h_3) = (0, 1, 2, 1)$ for the right hand side; finding a compatible order can be a rather challenging problem for complex congruences. In the case of hom-associativity though, the fact that all inputs are at the same height in the left and right hand sides makes it possible to use something very classical: a lexicographic order. Recursively it may be defined as having $\mu > \nu$ if:

- $\mu = m\mu'\mu''$ and $\nu = m\nu'\nu''$, where $\mu' > \nu'$, or
- $\mu = m\mu'\mu''$ and $\nu = m\nu'\nu''$, where $\mu' = \nu'$ and $\mu'' > \nu''$, or
- $\mu = a\mu'$ and $\nu = a\nu'$, or
- $\mu = a\mu'$ and $\nu = a\nu'$, where $\mu' > \nu'$.

Equivalently, one may define it as the word-lexicographic order on the Polish notation, over the order on letters which has $m < a$ and each $\square_i$ unrelated to all other letters. With this order, it is clear that the congruence (33) should be turned into the rule $(mm1a2am34, mm1m23aa4)$.

In general, the idea to “find all ambiguities, try to resolve them, make new rules out of everything that doesn’t resolve, and repeat until everything resolves” is called the Critical Pairs/Completion (CPC) procedure; its most famous instance is the Buchberger algorithm for computing Gröbner bases. ‘Critical pairs’ corresponds to identifying ambiguities, whereas ‘completion’ is the step of adding new rules; a rewriting system is said to be complete when all ambiguities are resolvable.

In the case at hand, the calculations quickly become extensive, so we make use of a program [22] one of us has written that automates the CPC procedure in the operadic setting (actually, in the more general PROP setting). Running it with (32) as input quickly leads to the discovery of (33) and several more identities:
And so on... When we stopped it, the program had 1 rule (32) of order (number of vertices) 3, 1 rule (33) of order 5, 1 rule (34) of order 7, 2 rules (35,36) of order 8, 1 rule (37) of order 9, 4 rules (38–41) of order 10, 7 rules of order 11, 12 rules of order 12, 19 rules of order 13, and 38 rules of order 14. Besides those 85 ambiguities that had given rise to new rules, 280 had turned out to be resolvable and 22417 had still not been processed; obviously the program wasn’t going to finish anytime soon, and it’s a fair guess that the complete rewriting system it sought to compute is in fact infinite. Certainly (32), (33), (34), and (37) look suspiciously like the beginning of an infinite family of rules, and indeed the expected sequence with one tower of m’s and another tower of a’s continues for as long as we have run the computations.

What is now our next step, when automated deduction has failed to deliver a complete answer? One approach is to try to guess the general pattern for these rules, and from that construct a provably complete rewriting system; we shall return to that problem in a later article. Right here and now, it is however possible to wash out several pieces of hard information even from the incomplete rewriting system presented above.

### 3.5 Hilbert series and formal languages

A useful observation about the hom-associativity axiom (32) is that it is homogeneous in pretty much every sense imaginable: there are the same number of m’s in the left and right hand sides, there are the same number of a’s in the left and right hand sides, and the inputs are all at the same height in the left as in the right hand sides. (The last is not even true for the ordinary associativity rule (30), so from a very abstract symbolic point of view, hom-associativity may actually be regarded as a homogenised form of ordinary associativity.) It is a well-known principle in Gröbner basis calculations that CPC procedures working on homogeneous rewriting systems only generates homogeneous rules and never derives smaller rules from
larger ones; once the procedure has processed all ambiguities up to a particular order, one knows for sure that no more rules of that order remain to be discovered. Hence the ten rules shown above are all there are of order 10 or less, and since no advanceable map of those used for Theorem 18 can reduce the order, it follows that those rules do effectively describe \( \mathcal{H}_{\text{Ass}} \) up to order 10. There is of course nothing special about order 10, so we may state these observations more formally as follows.

**Lemma 21.** Let \( Y_{k,\ell} \) be the subset of \( \mathcal{H}_\Omega \) whose elements contain exactly \( k \) vertices \( a \) and exactly \( \ell \) vertices \( m \); it follows that \( \mathcal{H}_\Omega(n) = \bigcup_{k=0}^{n} Y_{k,n-1} \). Let \( S_{k,\ell} \) be the set of rules the CPC procedure has generated from \( \{ \text{ma1m23, mm12a3} \} \) after processing all ambiguities at sites in \( \bigcup_{j=0}^{\infty} \bigcup_{i=0}^{\infty} Y_{i,j} \) but no ambiguities with sites outside this set. Let \( S = \bigcup_{k,\ell \in \mathbb{N}} S_{k,\ell} \). Then the following holds:

1. \( S_{1,2} = \{ \{ \text{ma1m23, mm12a3} \} \} \).
2. \( S_{k,\ell} \subseteq S_{k+1,\ell} \) and \( S_{k,\ell} \subseteq S_{k,\ell+1} \) for all \( k, \ell \in \mathbb{N} \).
3. \( \text{Irr}(S_{k,\ell}) \supseteq \text{Irr}(S_{k+1,\ell}) \) and \( \text{Irr}(S_{k,\ell}) \supseteq \text{Irr}(S_{k,\ell+1}) \) for all \( k, \ell \in \mathbb{N} \).
4. All ambiguities of \( S \) are resolvable.
5. \( \text{Irr}(S) \cap Y_{k,\ell} = \text{Irr}(S_{k,\ell}) \cap Y_{k,\ell} \).
6. Every element of \( Y_{k,\ell} \) has a unique normal form modulo \( S_{i,j} \), for all \( i \geq k \) and \( j \geq \ell \).

To finish off, we shall apply a bit of formal language theory to compute the beginning of the Hilbert series of \( \mathcal{H}_{\text{Ass}} \). The kind of information encoded in this is, just like the \( \dim \mathcal{A}_{\text{Ass}}(n) = n! \) result mentioned above, basically the numbers of dimensions of the various components of the operad, although in the case of \( \mathcal{H}_{\text{Ass}} \) it is trivial to see that \( \dim \mathcal{H}_{\text{Ass}}(n) = \infty \) for all \( n \geq 0 \) since inserting more \( a \)'s into an expression does not change its arity. Instead one should partition by both \( a \) and \( m \) to get finite-dimensional components. Furthermore there is a rather boring factorial factor which is due to the action of \( \Sigma_n \), so we restrict attention to plane monomials, factor out that factorial, and define the Hilbert series of \( \mathcal{H}_{\text{Ass}} \) to be the formal power series

\[
H(a,m) = \sum_{k,\ell \in \mathbb{N}} \frac{|Y_{k,\ell} \cap \text{Irr}(S)|}{(j+1)!} a^km^j.
\]  

(42)

Note that this is also the Hilbert series of the free hom-associative algebra with one generator on which \( f_a \) acts freely. Indeed, that algebra is preferably constructed as \( \mathcal{R} \{ \Omega' \} (0) / \mathcal{H}(S)(0) \) where \( \Omega' = \{ m(\cdot), a(\cdot), x \} \), and since no rule in \( S \) changes \( x \) in any way, it follows that \( \text{Irr}(S) \subseteq \mathcal{R} \{ \Omega \} \) is in bijective correspondence to \( \text{Irr}(S)(0) \subseteq \mathcal{R} \{ \Omega' \} (0) \)—just put an \( x \) in every input! However, if one prefers the Hilbert series for the free algebra counting \( a \) and \( x \) rather than \( a \) and \( m \), then it should instead be stated as \( xH(a,x) \), since there is always one \( x \) more in an element of \( Y_{\ell\Omega}(0) \) than there are \( m \)'s. Finally, the Hilbert series for the free hom-associative algebra with \( k \) generators \( x_1, \ldots, x_k \) is \( kxH(a,kx) \), since there for every constant symbol (which is what \( x \) becomes the counting variable for) are \( k \) choices of what that symbol should be.

As approximations of \( H(a,m) \), we furthermore define
$$H_{k,\ell}(a,m) = \sum_{i,j \in \mathbb{N}} \frac{|Y_{i,j} \cap \mbox{Irr}(S_{k,\ell})|}{(j+1)!} a^m \quad \text{for all } k, \ell \in \mathbb{N}. \quad (43)$$

By claim 3 of Lemma 21, $H_{ij} \geq H_{k,\ell}$ coefficient by coefficient whenever $i \leq k$ and $j \leq \ell$. By claim 5, the coefficient of $a^m$ in $H(a,m)$ is equal to the coefficient in $H_{k,\ell}(a,m)$ whenever $i \leq k$ and $j \leq \ell$. Therefore, when one wishes to compute the beginning of $H(a,m)$, one may alternatively compute the beginning of $H_{k,\ell}(a,m)$ for sufficiently large $k$ and $\ell$.

To get an initial bound, let us first compute $H_{0,0}(a,m)$. From the basic observation that a plane element of $\mathcal{Y}_\omega$ is either id, $a1 \circ v$ for some plane $v \in \mathcal{Y}_\omega$, or $m12 \circ v_1 \circ v_2$ for some plane $v_1, v_2 \in \mathcal{Y}_\omega$, it follows that the language of all plane elements of $\mathcal{Y}_\omega$ satisfies the equation $L = \{id\} \cup (a1 \circ L) \cup (m12 \circ L \circ L)$, and consequently that $H_{0,0}$ satisfies the functional equation

$$H_{0,0}(a,m) = 1 + aH_{0,0}(a,m) + mH_{0,0}(a,m)^2; \quad (44)$$

the details of this correspondence between combinatorial constructions and functional equations can be found in for example [16, Ch. 1]. Solving that equation symbolically yields

$$H_{0,0}(a,m) = \frac{1 - a - \sqrt{(1-a)^2 - 4m}}{2m} \quad (45)$$

and using Newton’s generalised binomial theorem one can even get a closed form formula for the coefficients:

$$H_{0,0}(a,m) = \frac{1}{2m} \left( 1 - a - \sum_{n=0}^{\infty} \left( \frac{1}{n} \right) ((1-a)^2)^n \right) \left( (-4m)^n \right)$$

$$= - \frac{1}{2m} \sum_{n=1}^{\infty} \left( \frac{1}{n} \right) (1-a)^{2n} (-4m)^n \left( \ell=n-1 \right)$$

$$= 2 \sum_{\ell=0}^{\infty} \left( \frac{1}{\ell+1} \right) (1-a)^{-2\ell-1} (-4m)^\ell$$

$$= \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k+1} \left( \frac{2\ell+1}{2\ell+1} \right) \left( \frac{(-2\ell-1)}{k} \right) (-a)^k (-4m)^\ell$$

$$= \sum_{k,\ell \in \mathbb{N}} \frac{1}{\ell+1} \left( \frac{k+2\ell}{k,\ell,\ell} \right) a^m \ell.$$
For any finite set of rules, it is straightforward to set up a system of equations for the language $L_0$ of plane monomials that are reducible by at least one of these rules; in the case of $S_{1,2}$, one such equation system is

$$L_0 = (a_1 \circ L_0) \cup (m_{12} \circ L_0 \otimes L_1) \cup (m_{12} \circ L_1 \otimes L_0) \cup (m_{12} \circ L_2 \otimes L_3),$$  \hspace{1cm} (46a)

$$L_1 = (m_{12} \circ L_1 \otimes L_1) \cup (a_1 \circ L_1) \cup \{id\},$$  \hspace{1cm} (46b)

$$L_2 = a_1 \circ L_1,$$  \hspace{1cm} (46c)

$$L_3 = m_{12} \circ L_1 \otimes L_1,$$  \hspace{1cm} (46d)

(where we as usual consider operad composition of sets to denote the sets of operad elements that can be produced by applying the composition to elements of the given sets). A more suggestive presentation might however be as the BNF grammar

\[
\langle \text{reducible} \rangle ::= a \langle \text{reducible} \rangle | m \langle \text{reducible} \rangle \langle \text{arbitrary} \rangle | m \langle \text{arbitrary} \rangle \langle \text{reducible} \rangle \\
| m \langle \text{left} \rangle \langle \text{right} \rangle \\
\langle \text{arbitrary} \rangle ::= a \langle \text{arbitrary} \rangle | m \langle \text{arbitrary} \rangle \langle \text{arbitrary} \rangle | \Box_i \\
\langle \text{left} \rangle ::= a \langle \text{arbitrary} \rangle \\
\langle \text{right} \rangle ::= m \langle \text{arbitrary} \rangle \langle \text{arbitrary} \rangle
\]

whose informal interpretation is that a Polish term is $\langle \text{reducible} \rangle$ by $S_{1,2}$ if one of the children of the root node is itself $\langle \text{reducible} \rangle$, or if the root node is an $m$ whose $\langle \text{left} \rangle$ child is an $a$ and whose $\langle \text{right} \rangle$ child is an $m$. This can be trivially extended to larger sets of rules by adding to the formula for $L_0$ one production for each new rule (describing the root of the $m_i$ of that rule) and one new variable (together with its defining equation) for every internal edge in the $m_i$ of the new rule. Hence if also taking (33) into account, the system grows to

$$L_0 = (a_1 \circ L_0) \cup (m_{12} \circ L_0 \otimes L_1) \cup (m_{12} \circ L_1 \otimes L_0) \cup (m_{12} \circ L_2 \otimes L_3) \\
\cup (m_{12} \circ L_2 \otimes L_3) \cup (m_{12} \circ L_4 \otimes L_5),$$

$$L_1 = (m_{12} \circ L_1 \otimes L_1) \cup (a_1 \circ L_1) \cup \{id\},$$

$$L_2 = a_1 \circ L_1,$$  \hspace{1cm} (46b)

$$L_3 = m_{12} \circ L_1 \otimes L_1,$$  \hspace{1cm} (46c)

$$L_4 = m_{12} \circ L_1 \otimes L_1,$$  \hspace{1cm} (46d)

$$L_5 = a_1 \circ L_7,$$  \hspace{1cm} (46e)

$$L_6 = a_1 \circ L_1,$$  \hspace{1cm} (46f)

$$L_7 = m_{12} \circ L_1 \otimes L_1.$$  \hspace{1cm} (46g)

Smaller systems for the same $L_0$ are often possible (and can save work in the next step), but here we are content with observing that a finite system exists.

While the system (46) is of the same general type as the equation that was used to derive (44), it would not be correct to simply convert it in the same way to an equation system for $H_{1,2}$, since there is a qualitative difference: the unions in (46)
are not in general disjoint, for example because \( L_0 \subset L_1 \) and thus \( m12 \circ L_0 \otimes L_0 \subseteq m12 \circ L_0 \otimes L_1, m12 \circ L_1 \otimes L_0 \). This may be possible to overcome through inclusion–exclusion style combinatorics, but we would rather like to attack this issue using tools from formal language theory. In the terminology of [12], an equation system such as (46) defines a nondeterministic finite top-down tree automaton; it is finite because the set of states is \{0,1,2,3\} (finite) and it is the nondeterminism that can cause the unions to be non-disjoint. By the Subset Construction [12, Th. 1.1.9] however, there exists an equivalent deterministic finite bottom-up tree automaton whose states are subsets of the set of top-down states; moreover this bottom-up automaton may be regarded as an \( \Omega \)-algebra \( (A, \{ f_s \}_{s \in \Omega}) \). In the case of (46), this \( \Omega \)-algebra has

\[
A = \{ \{1\}, \{1,2\}, \{1,3\}, \{0,1,3\}, \{0,1,2\} \}
\]

and operations given by the tables

| first operand | \( f_s \) | \( f_m \) when second operand is: |
|---------------|---------|--------------------------------|
| \{1\}         | \{1,2\} | \{1\} \{1,2\} \{1,3\} \{0,1,3\} \{0,1,2\} |
| \{1,2\}       | \{1,2\} | \{1,3\} \{0,1,3\} \{0,1,2\} |
| \{1,3\}       | \{1,2\} | \{1,3\} \{0,1,3\} \{0,1,2\} |
| \{0,1,3\}     | \{0,1,2\} | \{0,1,3\} \{0,1,2\} |
| \{0,1,2\}     | \{0,1,2\} | \{0,1,3\} \{0,1,2\} |

When such an \( \Omega \)-algebra \( (A, \{ f_s \}_{s \in \Omega}) \) is given, the equation system of generating functions takes the form

\[
G_b(a, m) = a \sum_{c \in A} G_c(a, m) + m \sum_{c,d \in A} G_c(a, m)G_d(a, m) + \begin{cases} 1 & \text{if } b = \{1\}, \\ 0 & \text{otherwise} \end{cases}
\]

for all \( b \in A \) \quad (47)

where the extra term for \( b = \{1\} \) is because that is the state that inputs are considered to be in. The generating function for reducible plane monomials is the sum of all \( G_b \) such that \( b \ni 0 \), since 0 was the top-down \( \langle \text{reducible} \rangle \) state, and consequently the generating function for irreducible plane monomials is the sum of all \( G_b \) such that \( b \ni 0 \). Thus we have

\[
H_{1,2}(a, m) = G_{\{1\}}(a, m) + G_{\{1,2\}}(a, m) + G_{\{1,3\}}(a, m),
\]

\[
G_{\{1\}}(a,m) = 1,
\]

\[
G_{\{1,2\}}(a,m) = aH_{1,2}(a,m),
\]

\[
G_{\{1,3\}}(a,m) = mH_{1,2}(a,m)^2 - mG_{\{1,2\}}(a,m)G_{\{1,3\}}(a,m)
\]

where the definition of \( H_{1,2}(a,m) \) was used to shorten the last two right hand sides a bit. Solving as above is still possible, but results in the somewhat messier expression
imposing the hom-associativity identity (32) reduces by 4368 the dimension of the hom-algebra operad is

\[ H_{1,2}(a,m) = \frac{1 - a - am^2 - \sqrt{(1 - a - am^2)^2 + 4(1 - am + a^2m)m}}{2(1 - am + a^2m)m} = \]

\[ = \sum_{k=0}^{\infty} 2 \left( \frac{1}{2} \right) (1 - a - am^2)^{1 - 2k} 4^k (1 - am + a^2 m^2)^k m^k = \cdots \]

which is probably not so important to put on closed form; the interesting quantity is \( H(a,m) \), and the terms in \( H_{1,2} \) which coincide with their counterparts in \( H(a,m) \) can be determined by an ansatz in the equation system already.

**Theorem 22.** The Hilbert series \( H(a,m) \) for the hom-associative operad \( \tilde{\mathfrak{as}} \) satisfies \( H(a,m) = 1 + m + a + 2m^2 + 3am + a^2 + 5m^3 + 9am^2 + 6a^2m + a^3 + 14m^4 + 30am^3 + 26a^2m^2 + 10a^3m + a^4 + 42m^5 + 105am^4 + 110a^2m^3 + 60a^3m^2 + 15a^4m + a^5 + 132m^6 + 378am^5 + 465a^2m^4 + 315a^3m^3 + 120a^4m^2 + 21a^5m + a^6 + 429m^7 + 1386am^6 + 1960a^2m^5 + 1575a^3m^4 + 770a^4m^3 + 217a^5m^2 + 28a^6m + a^7 + 1430m^8 + 5148am^7 + 8232a^2m^6 + 7644a^3m^5 + 4494a^4m^4 + 1680a^5m^3 + 364a^6m^2 + 36a^7m + a^8 + \cdots \). In particular, the difference to the Hilbert series \( H_{0,0}(a,m) \) for the free hom-algebra operad is

\[ H_{0,0}(a,m) - H(a,m) = \]

\[ = am^2 + 4a^2m^2 + 10a^3m^2 + 20a^4m^2 + 35a^5m^2 + 56a^6m^2 + 5am^3 + 30a^2m^3 + 105a^3m^3 + 280a^4m^3 + 630a^5m^3 + 21am^4 + 165a^2m^4 + 735a^3m^4 + 2436a^4m^4 + 84am^5 + 812a^2m^5 + 4368a^3m^5 + 330am^6 + 3780a^2m^6 + 1287am^7 + \cdots \]

**Remark 23.** The interpretation of for example the term \( 4368a^3m^5 \) above is thus that imposing the hom-associativity identity (32) reduces by 4368 the dimension of the space of plane operad elements that can be formed with 3 operations \( \alpha \) and 5 multiplications.

**Proof.** As shown above for \( H_{1,2} \), but taking all of (32)–(36) into account, so that one instead considers \( S_{5,3} \cup S_{4,4} \) and thus gets all terms of total degree \( \leq 8 \).

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