CYLINDRICAL ESTIMATES FOR MEAN CURVATURE FLOW IN HYPERBOLIC SPACES

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Abstract. We consider the mean curvature flow of a closed hypersurface in hyperbolic space. Under a suitable pinching assumption on the initial data, we prove a priori estimate on the principal curvatures which implies that the asymptotic profile near a singularity is either strictly convex or cylindrical. This result generalizes the estimates obtained in the previous works of Huisken, Sinestrari and Nguyen on the mean curvature flow of hypersurfaces in Euclidean spaces and in the spheres.

1. Introduction. Let $X_0 : M^n \to \mathbb{H}^{n+1}(K)$ be a smooth immersion of an $n$-dimensional hypersurface into the hyperbolic space. Then the mean curvature flow is defined by the partial differential equation:

$$\begin{cases}
\frac{\partial}{\partial t} X(p, t) = -H(p, t)\nu(p, t), & p \in M, \ t \geq 0, \\
X(p, 0) = X_0(p),
\end{cases}$$

where $H(p, t)$ is the mean curvature.

The Ricci flow as well as the mean curvature flow have attracted the attention of many mathematicians, see [5, 6, 8, 9, 10, 11, 12, 13, 14, 15]. In fact, these classifications can be viewed as a start in proving generalised sphere theorems or connected sum theorems. For example, in [9], Huisken and Sinestrari proved,

**Theorem A.** Any smooth closed $n$-dimensional two-convex immersed hypersurface $X_0 : M^n \to \mathbb{R}^{n+1}$ with $n \geq 3$ is diffeomorphic either to $\mathbb{S}^n$ or to a finite connected sum of $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

Then, Brendle and Huisken proved a similar result when $n = 2$, in [1],

**Theorem B.** Let $M_0$ be a closed and embedded surfaces in $\mathbb{R}^3$ with positive mean curvature. Then there exists a mean curvature flow with surgeries starting from $M_0$ which terminates after finitely many steps.

The idea is that by relaxing the convexity condition to a flexible condition, in particular a curvature condition that is close under connected sum, one can obtain a connected sum theorem. Another example is positive scalar curvature for three-manifolds in [3].

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**Theorem C.** Let $M^3$ be a 3-dimensional compact Riemannian manifold with positive scalar curvature. Then $M^3$ is diffeomorphic to $S^3$ or a finite connected sum of $S^2 \times S^1$ and $S^1 \times S^2$.

The proofs of both these theorems use the mean curvature flow and the Ricci flow respectively. They generalized the following theorems in [5, 4], respectively:

**Theorem D.** Any smooth closed $n$-dimensional convex immersed surface $X_0 : M^n \rightarrow \mathbb{R}^{n+1}$ with $n \geq 2$ which flows by mean curvature flow, converges in finite time to a round point.

**Theorem E.** Let $M^3$ be a compact 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature, then $M^3$ also admits a metric of constant positive curvature.

As it is pointed out in [16], the ideas of both proofs are to analyse the formation of singularities, and prove convexity estimates or Hamilton-Ivey estimates. Such estimates show that as we approach a singularity, we enter the boundary cases to the sphere theorems above.

In this paper, we will study the mean curvature flow in the hyperbolic space $\mathbb{H}^{n+1}(K)$ ($K < 0$), under the following assumption:

$$|A|^2 < \frac{1}{n-2}H^2 + 4K.$$ 

Note that when the above curvature pinching condition holds, we have that $H > c > 0$ for some constant $c = \max\{\sqrt{-4K(n-2)}, \sqrt{-2K(n-2)}\}$. If the first inequality holds, we say the hypersurface is quadratically bounded. If the second condition holds, it is called mean convexity. If both inequalities hold we say that the surface is mean convex quadratically bounded. This is related to a rigid curvature cone, introduced by Huisken in [7].

**Theorem F.** Let $n \geq 3$ and $S^{n+1}(K)$ be a spherical space form of sectional curvature $K$ ($K > 0$). Let $M_0$ be a compact hypersurface without boundary which is smoothly immersed in $N$ and supposed that we have on $M_0$

$$|A|^2 < \frac{1}{n-1}H^2 + 2K.$$ 

Then either the mean curvature flow exists for finite time $t \in [0, T)$ and $M_t$ converges to a round point as $t \rightarrow T$ or exists for finite time and converges to a great sphere.

We consider a family of closed hypersurfaces $M_t$ evolving by mean curvature flow in a Riemannian manifold. We are interested in the singular behaviour of the flow. More precisely, we want to show that under the quadratic condition we can classify the blow up limits, without the hypothesis of mean convexity.

When the ambient space is Euclidean, Huisken and Sinestrari [9] showed that the above property holds provided the evolving hypersurface is 2-convex. i.e.,

$$\lambda_1 + \lambda_2 > 0,$$

where $\lambda_1$ and $\lambda_2$ are the two smallest principal curvatures. This result is obtained by means of some priori estimates on suitable functions of the principal curvatures, called cylindrical estimates. Recently, Nguyen [16] has considered mean curvature flow in the sphere and proved cylindrical estimates under a pinching assumption which is related to 2-convexity, but is less restrictive in the regions with small curvature:
Theorem G. Let $M_0$ be a given smooth, closed hypersurface in $S^{n+1}(K)(K > 0)$ which satisfies
\[ |A|^2 < \frac{1}{n-2} H^2 + 4K, \quad H > 0. \]
Then there exists $C_\eta$ such that
\[ |\lambda_1| \leq \eta H \Rightarrow |\lambda_i - \lambda_j|^2 \leq c(n)\eta H^2 + C_\eta, \quad \forall i, j \geq 2 \]
holds for all $\eta > 0$.

Latter, Giuseppe Pipoli and Carlo Sinestrari [17] proved cylindrical estimates for the mean curvature flow when the ambient space is the projective space over either the complex field $C$ or the algebra of quaternions $H$ under the assumptions as in [16].

In this paper, we proved cylindrical estimates for the mean curvature flow in hyperbolic spaces. Our main result is:

Theorem 1.1. Let $n \geq 4$ and let $M_0$ be a closed real hypersurface of $H^{n+1}(K)$ ($K < 0$) which satisfies
\[ |A|^2 < \frac{1}{n-2} H^2 + 4K. \] (1.1)
Then, condition (1.1) holds on $M_t$ for any $0 \leq t \leq T_{\text{max}}$. Moreover, for any $\eta > 0$ there exists a constant $C_\eta$ that depends only on $\eta$ and $M_0$ such that
\[ |\lambda_1| \leq \eta H \Rightarrow |\lambda_i - \lambda_j|^2 \leq c(n)\eta H^2 + C_\eta, \quad \forall i, j \geq 2, \]
everywhere on $M_t$, for a constant $c(n)$ that depends only on the dimension.

As in [9, 16], using the cylindrical estimates, it is possible to prove a pointwise bound on the gradient of the second fundamental form.

Theorem 1.2. Under the hypotheses of the previous theorem, there is $C > 0$ depending only on $M_0$, such that
\[ |\nabla A|^2 \leq C(|A|^4 + 1) \] (1.2)
everywhere on $M_t$, for any $t \leq T_{\text{max}}$.

The interests of the above results lie in possible applications to the construction of a generalized curvature flow after singularities by means of a surgery procedure. In fact, the estimates of this paper and the ones of [16, 17], together with the argument of Section 8 in [2] imply that surgeries similar to one of [9] should exist also for the hypersurface in $H^{n+1}(K)$ satisfying the dimension restrictions and the pinching assumptions stated here. This would imply that any such hypersurfaces is diffeomorphic to a sphere $S^n$ or a finite connected sum of $S^{n-1} \times S^1$.

2. Preliminaries. In this section, we will give some facts about the quadratically bounded hypersurfaces and introduce some notations. For convenience, we will use $H^{n+1}$ to denote the hyperbolic space of sectional curvature $K(K < 0)$ in the rest of this article.

Here we state several well-known equations for submanifolds of $H^{n+1}(K)$.

Proposition 1. Let $M^n \subseteq H^{n+1}(K)$ be a hypersurface of the hyperbolic space. Then we have the following equations:
\[ R_{ijkl} = K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk}, \]
\[ R_{ik} = (n-1)K\delta_{ij} + h_{kj}h_{ij} - h_{ik}h_{jk}, \]
We also have the following consequence of the Codazzi inequality.

**Lemma A.** For any hypersurface $M_0 \subseteq \mathbb{H}^{n+1}(K)$, we have that
\[
|\nabla A|^2 \geq \frac{3}{n+2} |\nabla H|^2. \tag{2.1}
\]

The evolution equations of the main geometric quantities associated with hypersurface evolving by mean curvature flow in hyperbolic space have been derived in [6]. We recall here all the evolution equations which will be using in this paper. For convenience, we may use $\partial_t$ to denote $\frac{\partial}{\partial t}$.

**Lemma B.** Let $X_0 : M^n \to \mathbb{H}^{n+1}(K)$ be a compact hypersurface in $\mathbb{H}^{n+1}(K)$. Then for the mean curvature flow, we have the following evolution equations:
\[
\begin{align*}
\partial_t X &= -H \nu, \\
\partial_t g_{ij} &= -2H h_{ij}, \\
\partial_t h_{ij} &= \Delta h_{ij} - 2H h_{ik} h_{kj} + |A|^2 h_{ij} + 2K H g_{ij} - nK h_{ij}, \\
\partial_t H &= \Delta H + H(|A|^2 + nK), \\
\partial_t |A|^2 &= \Delta |A|^2 - 2|\nabla A|^2 + |A|^2(2|A|^2 + nK) - 4nK \left( |A|^2 - \frac{H^2}{n} \right), \\
\partial_t |H|^2 &= \Delta |H|^2 - 2|\nabla H|^2 + H^2(|A|^2 + nK), \\
\partial_t \Gamma^i_{jk} &= -g^{il}(\nabla_j (H h_{kl}) + \nabla_k (H h_{lj}) - \nabla_l (H h_{jk})) = A \ast \nabla A. \tag{2.8}
\end{align*}
\]

In particular, using the evolution equation of the Christoffel symbol, together with the evolution equation of the second fundamental form, we can drive the evolution equation of the norm of the derivative of the second fundamental form as in [4],
\[
\frac{\partial}{\partial t} |\nabla A|^2 - \Delta |\nabla A|^2 \leq -2|\nabla^2 A|^2 + c_n |A|^2 |\nabla A|^2 + d_n |A||\nabla A|^2. \tag{2.9}
\]

3. **Curvature cones.** In this section, we will prove the curvature pinching condition (1.1) is preserved during the mean curvature flow.

If we assume that the quadratic curvature inequality is strict, and let $\alpha_0 = \frac{1}{n-2}$ and $\beta_0 = 4$, that is,
\[|A|^2 < \alpha_0 H^2 + \beta_0 K,\]
then there exists an $\varepsilon > 0$ such that the following inequality holds everywhere on $M_0 \subseteq \mathbb{H}^{n+1}$:
\[|A|^2 \leq \frac{1}{n-2+\varepsilon} H^2 + 2(2+\varepsilon)K.\]

Let $\beta_n = 4(1+\varepsilon)$ and $\alpha_n = \frac{1}{n-2+\varepsilon}$, then we have the following theorem,

**Theorem 3.1.** Consider the curvature inequality,
\[|A|^2 < \alpha_n H^2 + \beta_n K,\]
which holds everywhere on $M_0$. This inequality is preserved on $M_t$ by the mean curvature flow for every time $0 \leq t \leq T_{\max} < \infty$, when the solution exists.
Proof. Let
\[ Q = |A|^2 - \alpha_n H^2 - \beta_n K, \]
and we get \( Q < 0 \) on \( M_0 \). If the above pinching condition is not preserved, then we will get \( Q = 0 \) at some \( t_0 \). On the one hand, we have the evolution equation,
\[
\partial_t(|A|^2 - \alpha H^2) = \Delta(|A|^2 - \alpha H^2) - 2(\nabla |A|^2 - \alpha |\nabla H|^2) + 2\beta K(|A|^2 + nK) \\
+ 2(|A|^2 - \alpha H^2 - \beta K)(|A|^2 + nK) - 4nK\left(\frac{|A|^2 - H^2}{n}\right),
\]
where we use the fact that \( \alpha = \frac{2}{2n-2} \). On the other hand, we get the following inequality at \( t_0 \),
\[
\beta K(|A|^2 + nK) - 2nK\left(\frac{|A|^2 - \frac{H^2}{n}}{n}\right) \\
= \beta K|A|^2 - nK|A|^2 + (2 - \alpha n)KH^2 = (\beta - n)K|A|^2 + \left(2 - \frac{n}{n-2}\right)KH^2 \\
< \frac{4-n}{n-2}KH^2 + \frac{n-4}{n-2}KH^2 = 0.
\]
Since \( n \geq 4 \), and by applying (2.1) and the parabolic maximum principle, we get a contradiction. Thus we prove the theorem. \( \square \)

Here, we introduce some relations between the quadratic bounds and the principal curvatures in [16].

**Lemma A.** We have the following equations:
\[
|A|^2 - \frac{1}{n-1}H^2 = -2\lambda_1 \lambda_2 + \left(\lambda_1 + \lambda_2 - \frac{H}{n-1}\right)^2 + \sum_{l=3}^{n} \left(\lambda_l - \frac{H}{n-1}\right)
\]
and
\[
|A|^2 - \frac{1}{n-2}H^2 \\
= -2(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) + \left(\lambda_1 + \lambda_2 + \lambda_3 - \frac{H}{n-2}\right)^2 + \sum_{l=4}^{n} \left(\lambda_l - \frac{H}{n-2}\right)
\]

4. **Cylindrical estimates.** In this section, we will prove our main Theorem. That is, we will show that at a singular point if the submanifold does not become spherical, then it becomes cylindrical.

Let \( \overline{M} \) be a Riemannian manifold, the following important property in analysis of singularities of the mean curvature flow for convex hypersurfaces in the convexity estimates was proved in [9, 17].

**Theorem 4.1.** Let \( F_0 : M \to \overline{M} \) be a smooth closed hypersurface immersion with non-negative mean curvature. Then for any \( \eta > 0 \), there is a \( M_\eta \) such that
\[ \lambda_1 \geq -\eta H - M_\eta, \]
everywhere on \( M \times [0, T_{max}) \).

To apply the iteration argument to get our estimates, we will need the following theorem to bound the non-linearity \( Z \) appearing in Simons identity,
\[
\Delta |A|^2 = 2<h_{ij}, \nabla_i \nabla_j H> + 2|\nabla A|^2 + 2Z,
\]
where \( Z = Htr(A^3) - |A|^4 + nK(|A|^2 - \frac{1}{n}H^2). \)
Theorem 4.2. If $M_0$ satisfies the mean convex quadratic bound (1.1), then there exists a constant $\gamma_1$ depending only on $n$ and $M_0$, such that for any $\eta > 0$ there exists a $K_\eta$ such that

$$Z \geq \gamma_1 H^2 \left(|A|^2 - \frac{H^2}{n-1} - \eta H^2\right) - K_\eta(H^3 + H^2 + H + 1).$$

Proof. This proof is a modification in [16]. First, we have

$$|A|^2 - \frac{H^2}{n-1} = \frac{1}{n-1} \left( \sum_{1<i<j} (\lambda_i - \lambda_j)^2 + \lambda_1 \lambda_2 (n \lambda_1 - 2H) \right).$$

We can assume that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. By using the fact that $\lambda_1 + \lambda_2 \geq -\eta H + C_\eta$, $Z$ may be written as

$$Z = \sum_{j=2}^{n} (\lambda_1 \lambda_j + K)(\lambda_i - \lambda_j)^2 + \sum_{1<i<j} (\lambda_i \lambda_j + K)(\lambda_i - \lambda_j)^2.$$

The second term may be estimated as follows

$$\sum_{1<i<j} (\lambda_i \lambda_j + K)(\lambda_i - \lambda_j)^2$$

$$= \sum_{1<i<j} (\lambda_i \lambda_j + \lambda_i \lambda_k + \lambda_j \lambda_k + 2K)(\lambda_i - \lambda_j)^2 - \sum_{1<i<j} (\lambda_k (\lambda_i + \lambda_j) + K)(\lambda_i - \lambda_j)^2.$$

Now we choose $\lambda_k = \lambda_1 < 0$. Otherwise if $\lambda_1 > 0$, the second form is convex and this estimate has been proven for convex surface in [6]. Hence combining with (3.2), we have the estimate

$$-2 \sum_{1<i<j} (\lambda_k (\lambda_i + \lambda_j) + K)(\lambda_i - \lambda_j)^2$$

$$\geq -C \lambda_1 (\lambda_1 + \lambda_2) \sum_{1<i<j} (\lambda_i - \lambda_j)^2 - CK \sum_{1<i<j} (\lambda_i - \lambda_j)^2$$

$$\geq \lambda_1 C_1 K \left( \frac{H^2}{n-2} + 4K \right) - C_2 K \left( \frac{H^2}{n-2} + 4K \right).$$

Combining with (3.2), we can also get

$$\sum_{1<i<j} (\lambda_i \lambda_j + \lambda_i \lambda_k + \lambda_j \lambda_k + 2K)(\lambda_i - \lambda_j)^2$$

$$\geq \left( 4K + \frac{H^2}{n-2} - |A|^2 \right) \sum_{1<i<j} (\lambda_i - \lambda_j)^2$$

$$\geq \left( 4K + \frac{H^2}{n-2} - |A|^2 \right) \left[ (n-1) \left( \frac{H^2}{n-1} + |A|^2 \right) - \lambda_1 (n \lambda_1 - 2H) \right].$$

Noting that

$$\max\{\lambda_i^2, \lambda_j^2\} \leq |A|^2 \leq \frac{H^2}{n-2} + 4K \leq \left( \frac{1}{\sqrt{n-2}} H + 2\sqrt{|K|} \right)^2$$

and $\lambda_n \geq 0, H \geq 0$, hence we get

$$\lambda_n \leq \frac{1}{\sqrt{n-2}} H + 2\sqrt{|K|}. $$
Similar as \( \lambda_1 \leq 0 \) and \( H \geq 0 \), we have
\[
-\lambda_1 \leq \frac{1}{\sqrt{n-2}} H + 2\sqrt{|K|}.
\]
Collecting the remaining terms with a coefficient of \( \lambda_1 \) we have
\[
\lambda_1 \left( \sum_{j=2}^{n} \lambda_j (\lambda_1 - \lambda_j)^2 + \left( 4K + \frac{H^2}{n-2} - |A|^2 \right) (2H - n\lambda_1) \right).
\]
We can use \( \lambda_1 \geq -\eta H - C_0 \) to estimate the first factor. To estimate the second factor in the brackets, we have
\[
\sum_{j=2}^{n} \lambda_j (\lambda_1 - \lambda_j)^2 + \gamma H^2 (2H - n\lambda_1)
\]
\[
\leq C^H \left( \frac{H^2}{n-1} + 4K \right) + \gamma H^2 (2H + \frac{1}{\sqrt{n-2}} H + 2\sqrt{|K|}) \leq C_1 H^3 + C_2 H^2 + C_3 H,
\]
as \( (\lambda_1 - \lambda_j)^2 \leq C |A|^2, \lambda_j \leq \lambda_n \leq \frac{1}{\sqrt{n-2}} H + 2\sqrt{|K|} \) and \( -\lambda_1 \leq \frac{1}{\sqrt{n-2}} H + 2\sqrt{|K|} \). \( \square \)

Now we derive the cylindrical estimates.

**Theorem 4.3.** Let \( M_t \) be a smooth solution of mean curvature flow with \( n \geq 4 \) and \( M_0 \) satisfies \( |A|^2 < \frac{1}{n-2} H^2 + 4K \). Then, for any \( \eta > 0 \) there exists a constant \( C_\eta = C_\eta(n) \) such that
\[
|A|^2 - \frac{1}{n-1} H^2 \leq \eta H^2 + C_\eta
\]
on \( M_t \) for any \( t \in (0, T_{\text{max}}) \).

The above theorem gives a simple corollary.

**Corollary 1.** Let \( M_t \) be a smooth solution of the mean curvature flow with \( n \geq 4 \) and \( M_0 \) satisfying \( |A|^2 < \frac{1}{n-2} H^2 + 4K \). Then, for any \( \eta > 0 \) there exists constants \( c = c(n) \) and \( C_\eta = C_\eta(n) \) such that
\[
|\lambda_1| \leq \eta H \Rightarrow (\lambda_i - \lambda_j)^2 \geq c\eta H^2 + C_\eta.
\]

**Proof.** Let \( M_t \) be a smooth solution of mean curvature flow with \( n \geq 3 \), and \( M_0 \) satisfying \( |A|^2 < \frac{1}{n-2} H^2 + 4K \). Hence we have the estimate
\[
|A|^2 - \frac{1}{n-1} H^2 = \frac{1}{n-1} \left( \sum_{1 < i < j} (\lambda_i - \lambda_j)^2 + \lambda_1 (n\lambda_1 - H) \right).
\]
Then by Theorem 4.3 we get that for any \( \eta > 0 \) there exists a constant \( C_\eta = C_\eta(n) \) such that
\[
1|A|^2 - \frac{1}{n-1} H^2 \leq \eta H^2 + C_\eta.
\]
The bound \( |\lambda_1| \leq \eta H \) then gives the desired estimate. \( \square \)

Using the similar strategy and estimates of [16], we will consider the following function:
\[
f_{\sigma, \eta} = \frac{|A|^2 - \left( \frac{1}{n-1} + \eta \right) H^2}{(aH^2 + b)^{1-\sigma}} = \frac{|A|^2 - \left( \frac{1}{n-1} + \eta \right) H^2}{W^{1-\sigma}}
\]
where \( a = \alpha_n - \left( \frac{1}{n-1} + \eta \right) \) and \( b = \beta_n K \).
Lemma A. For all $0 \leq \sigma \leq \sigma_1$, where $\sigma_1$ only depends on $M_0$, we have
\[
\frac{\partial}{\partial t} f_{\sigma,n} \leq \triangle f_{\sigma,n} + \frac{4a(1-\sigma)H}{W}(\nabla f_{\sigma,n} \nabla H) - \frac{\varepsilon}{12}W^{\sigma-1}|\nabla H|^2 + 2|A|^2 f_{\sigma,n}.
\] (4.1)

Proof. As the first step, we have the evolution equation for $f_0 = f_{0,n}$,
\[
\frac{\partial}{\partial t} f_0 = \frac{1}{aH^2 + b} \left[ \triangle \left( |A|^2 - \left( \frac{1}{n-1} + \eta \right) H^2 \right) - f_0 \triangle (aH^2) \right]
- \frac{2}{aH^2 + b} \left[ \nabla A|^2 - \left( \frac{1}{n-1} + \eta + a f_0 \right) |\nabla H|^2 \right]
+ \frac{2f_0}{aH^2 + b} (b|A|^2 - 2anKH^2 - bnK).
\]

Next, we get
\[
\nabla_i f_0 = \frac{1}{aH^2 + b} \left( \nabla_i \left( |A|^2 - \left( \frac{1}{n-1} + \eta \right) H^2 \right) - f_0 \nabla_i (aH^2) \right),
\]
and
\[
\triangle f_0 = \frac{1}{aH^2 + b} \left( \triangle \left( |A|^2 - \left( \frac{1}{n-1} + \eta \right) H^2 \right) - f_0 \triangle (aH^2) \right) - \frac{4aH}{aH^2 + b} \langle \nabla_i H, \nabla_i f_0 \rangle.
\]

We can choose suitable $a, b$ to ensure $f_0 < 1$. Hence, we get that
\[
\frac{\partial}{\partial t} f_0 \leq \triangle f_0 + \frac{4aH}{aH^2 + b} (\nabla_i H, \nabla_i f_0) - \frac{2}{aH^2 + b} \left( \nabla A|^2 - \left( \frac{1}{n-1} + \eta + a \right) |\nabla H|^2 \right)
+ \frac{2f_0}{aH^2 + b} (b|A|^2 - 2anKH^2 - bnK).
\]

Since we have that $n \geq 3$ and $\frac{3}{n+2} - \left( a + \left( \frac{1}{n-1} + \eta \right) \right) \geq \frac{\varepsilon}{16}$, then we get
\[
\frac{\partial}{\partial t} f_0 \leq \triangle f_0 + \frac{4aH}{aH^2 + b} (\nabla_i H, \nabla_i f_0) - \frac{\varepsilon}{8(aH^2 + b)} |\nabla H|^2
+ \frac{2f_0}{aH^2 + b} (b|A|^2 - 2anKH^2 - bnK).
\]

Noting that we have a negative coefficient in front of the $|\nabla H|^2$ term. This will allow us to use the iteration method, without using a bound on $|h_{jk}\nabla_i H - H\nabla_i h_{jk}|^2$.

Now we compute the evolution equation of the term $W^\sigma = (aH^2 + b)^\sigma$. First, we have
\[
\triangle W^\sigma = \sigma (W^{\sigma-1} 2aH \nabla_i H)
= 2\sigma aHW^{\sigma-1} \Delta H + 4\sigma (\sigma - 1) a^2 H^2 W^{\sigma-2} |\nabla H|^2 + 2a\sigma W^{\sigma-2} |\nabla H|^2.
\]

Hence, we obtain
\[
\partial_t W^\sigma = \triangle W^\sigma - 4\sigma (\sigma - 1) a^2 H^2 W^{\sigma-2} |\nabla H|^2 - 2a\sigma W^{\sigma-2} |\nabla H|^2 + 2\sigma aHW^{\sigma-1} (|A|^2 + nK).
\]
This gives us the following inequality
\[
\partial_t f_{\sigma,n} = \partial_t f_0 W^\sigma
\leq \triangle f_{\sigma,n} - 2(\nabla_i f_0, \nabla_i W^\sigma) + 4aHW^{\sigma-1} (\nabla_i f_0, \nabla_i H)
- \frac{\varepsilon}{12}W^{\sigma-1}|\nabla H|^2 - 4\sigma (\sigma - 1) a^2 H^2 W^{\sigma-2} f_0 |\nabla H|^2 - 2a\sigma W^{\sigma-2} f_0 |\nabla H|^2
- 2f_0 W^{\sigma-1} (b|A|^2 - 2anKH^2 - bnK) + 2a\sigma H^2 W^{\sigma-1} (|A|^2 + nK) f_0.
\]
The coefficient of $|\nabla H|^2$ given by

$$-4\sigma(\sigma - 1)a^2H^2W^{\sigma - 2}f_0 - 2a\sigma W^{\sigma - 2}f_0 = -\frac{2f_0}{W}(a\sigma + 2a^2\sigma(1 - \sigma)\frac{H^2}{W})$$

is negative if $\sigma \in [0, 1]$. We also have

$$(2aH^2 + b)nKf_0W^{\sigma - 1} + 2a\sigma H^2W^{\sigma - 1}nK - 2f_0W^{\sigma - 1}b|A|^2 \leq 0.$$ 

and

$$2a\sigma H^2W^{\sigma - 1}|A|^2f_0 = 2\sigma|A|^2f_{\sigma \eta}aH^2 \leq 2\sigma|A|^2f_{\sigma \eta}.$$ 

Therefore, we obtain

$$\partial_t f_{\sigma \eta} \leq \triangle f_{\sigma \eta} - 2(\nabla_i f_0, \nabla_i W^\sigma) + 2\sigma|A|^2f_{\sigma \eta} - \frac{\varepsilon}{12}W^{\sigma - 1}|\nabla H|^2 + 4aHW^{\sigma - 1}(\nabla_i f_0, \nabla_i H).$$

Using the above inequality, we can derive the evolution inequality for the $L^p$ norm of $f_+ = (f_{\sigma \eta})_+$, where $(f_{\sigma \eta})_+ = \max\{f_{\sigma \eta}, 0\}$.

**Lemma B.** For $\sigma, \eta \in (0, 1)$, we have the following evolution inequality:

$$\partial_t \int_{M_t} f_+^p + p(p - 1) \int_{M_t} f_+^{p-2}\nabla f_+^2 + \frac{\varepsilon}{12}p \int_{M_t} W^{\sigma - 1}|\nabla H|^2 f_+^{p-1}$$

$$\leq 4pa(1 - \sigma) \int_{M_t} \frac{H}{W}||\nabla H||\nabla f_+|f_+^{p-1} + \int_{M_t} 2\sigma p|A|^2 f_+^p,$$

for some $p \geq c_2$.

**Proof.** By directly computing, we get

$$\partial_t \int_{M_t} f_+^p d\mu = p \int_{M_t} (f_+^{p-1}\partial_t f_+ - H^2 f_+^p). \quad (4.2)$$

Using the evolution equation above for $f_{\sigma \eta}$, we have

$$\frac{\partial}{\partial t} f_{\sigma \eta} \leq \triangle f_{\sigma \eta} + \frac{4a(1 - \sigma)H}{W}(\nabla f_{\sigma \eta}, \nabla H) - \frac{\varepsilon}{12}W^{\sigma - 1}|\nabla H|^2 + 2\sigma|A|^2 f_{\sigma \eta}.$$ 

Inserting the above inequality into (4.2), we obtain

$$\partial_t \int_{M_t} f_+^p + p(p - 1) \int_{M_t} f_+^{p-2}\nabla f_+^2 + \frac{\varepsilon}{12}p \int_{M_t} W^{\sigma - 1}|\nabla H|^2 f_+^{p-1}$$

$$\leq 4pa(1 - \sigma) \int_{M_t} \frac{H}{W}||\nabla H||\nabla f_+|f_+^{p-1} + \int_{M_t} 2\sigma p|A|^2 f_+^p.$$ 

In order to use the good negative gradient type terms, we need a Poincaré type inequality. For convenience, we define

$$h_{ij}^\eta := h_{ij} - \left(\frac{1}{n - 1} + \eta\right)g_{ij}.$$ 

**Lemma C.** There exists $c_3 = c_3(M_0)$ and $C = C(n)$ such that

$$\left(\frac{2}{\eta c_3} - p\sigma\right) \int_{M_t} WF_+^p \leq \frac{p + 1}{\beta} \int_{M_t} f_+^{p-2}|\nabla f_+|^2 + (1 + \beta p) \int_{M_t} \frac{f_+^p}{W}|\nabla H|^2 + C|M_0|$$

for $\beta > 0$, $p \geq p_1$ and $\sigma \leq \sigma_1$. 
Proof. From [16], we have

\[ \frac{2}{\eta c} \int_{M_t} W f_+^p \leq \frac{p + 1}{\beta} \int_{M_t} f_+^{p-2} |\nabla f|^2 + (1 + \beta p) \int_{M_t} \frac{f_+^p}{W} |\nabla H|^2 + C \int_{M_t} \frac{H^3}{W} f_+^p. \]

If we choose \( p \geq p_1, \sigma \leq \sigma_1, \text{and let} \)

\[ q = \frac{2 + \sigma}{1 + \beta}, \quad q' = 2\sigma p, \quad B = (pq\sigma)^{\frac{1}{q}}, \]

then we get

\[ \frac{H^3}{W} \leq C' \sqrt{W} \leq \sqrt{W^{-\sigma p}} B V^1 + \frac{C'\sqrt{W^{1+\sigma p}}}{B} \leq \sqrt{W^{1-\sigma p}} \left( \frac{B^3 \sqrt{W^{\eta(1+\sigma p)}}}{q} + 1 + \frac{1}{q} \left( \frac{C'}{B} \right)^{\frac{1}{q}} \right) \]

\[ \leq \rho \sigma W + |K| \sqrt{W^{-\sigma p}} \leq \rho \sigma W + |K| W^{-\sigma p}. \]

Hence we obtain

\[ \int \frac{H^3}{W} f_+^p \leq \rho \sigma \int W f_+^p d\mu + |K| \int W^{-\sigma p} f_+^p. \]

Using the fact that \( (\frac{1}{W})^p \leq 1, \) we arrive at

\[ \int \frac{H^3}{W} f_+^p \leq \rho \sigma \int W f_+^p d\mu + |K| |M|. \]

Therefore, we get

\[ \left( \frac{2}{\eta c_3} - \rho \sigma \right) \int_{M_t} W f_+^p \leq \frac{p + 1}{\beta} \int_{M_t} f_+^{p-2} |\nabla f|^2 + (1 + \beta p) \int_{M_t} \frac{f_+^p}{W} |\nabla H|^2 + C |M_0|. \]

Using the above Poincaré type inequality, we get the following estimate.

**Lemma D.** There exist constants \( c_4 \) and \( c_5 \) such that for any \( p \geq c_4, 0 < \sigma < \frac{c_5}{\sqrt{p}} \) and some constant \( K_2 > 0, \) we have

\[ \frac{d}{dt} \int_{M_t} (f_{\sigma, \eta})_+^p \leq K_2 |M_0|. \]

Proof. Let us consider the following inequality

\[ \partial_t \int_{M_t} f_+^p + p(p - 1) \int_{M_t} f_+^{p-2} |\nabla f_+|^2 + \frac{\varepsilon}{12} p \int_{M_t} W^{-\sigma - 1} |\nabla H|^2 f_+^{p-1} \]

\[ \leq 4p(1 - \sigma) \int_{M_t} \frac{H}{W} |\nabla H| |\nabla f_+| f_+^{p-1} + \int_{M_t} 2\sigma p |A|^2 f_+^p. \]

The term

\[ 4p(1 - \sigma) \int_{M_t} \frac{H}{W} |\nabla H| |\nabla f_+| f_+^{p-1} \]

may be split up. Using Young’s inequality, we get

\[ 4p(1 - \sigma) \int_{M_t} \frac{H}{W} |\nabla H| |\nabla f_{\sigma, \eta}| f_{\sigma, \eta}^{p-1} \]

\[ \leq p \int_{M_t} \frac{1}{W} \left( \frac{2}{\beta} f_{\sigma, \eta}^{p-2} |H^2 | f_{\sigma, \eta} | f_{\sigma, \eta} | \nabla H |^2 + 2 \beta f_{\sigma, \eta}^{p-2} |f_{\sigma, \eta}| |\nabla H|^2 \right) \]

\[ \leq \frac{2p}{\beta} \int_{M_t} f_{\sigma, \eta}^{p-2} |\nabla f_{\sigma, \eta}|^2 + 2p \beta \int_{M_t} \frac{f_{\sigma, \eta}^{p-1}}{W^{1-\sigma}} |\nabla H|^2. \]
And we also have
\[
\varepsilon \int_{M_t} W^\sigma \left| \nabla H \right|^2 f_+^{p-1} \leq \frac{p}{c_1} \int_{M_t} \frac{\left| \nabla H \right|^2}{W} f_+^p \leq \frac{p}{c_1} \int_{M_t} \frac{f_+^{p-1}}{W^{1-\sigma}} \left| \nabla H \right|^2.
\]
For term
\[
\int_{M_t} \frac{2\sigma p}{\beta} |A|^2 f_+^p,
\]
we have a constant \( c_6 = c_6(n) \) such that
\[
\int_{M_t} \frac{2\sigma p}{\beta} |A|^2 f_+^p \leq 2c_6 \int_{M_t} W f_+^p.
\]
These together with Lemma C, we get
\[
\int_{M_t} 2\sigma p |A|^2 f_+^p \leq 2c_6 \int_{M_t} W f_+^p.
\]

Thus, we obtain
\[
\partial_t \int_{M_t} f_+^p \leq \left( -p(p-1) + \frac{2p}{\beta} + 2c_6 \frac{1}{\left( \frac{2}{\eta^3} - \rho \sigma \right)} p + 1 \right) \int_{M_t} f_+^{p-2} \left| \nabla f_+ \right|^2
\]
\[
+ \left( -\frac{p}{c_1} + 2c_1 \rho \beta + 2c_6 \frac{1}{\left( \frac{2}{\eta^3} - \rho \sigma \right)} (1 + \beta p) \right) \int_{M_t} W^{\sigma - 1} \left| \nabla H \right|^2 f_+^{p-1}
\]
\[
+ 2c_6 \frac{1}{\left( \frac{2}{\eta^3} - \rho \sigma \right)} C |M_0|.
\]
Let
\[
p \geq c_7, \quad \sigma \leq \frac{c_5}{\sqrt{p}}, \quad \beta = \frac{1}{100c_1}
\]
where \( c_5 \) and \( c_7 \) are constants which are related to the coefficients of the terms:
\[
\int_{M_t} W^{\sigma - 1} \left| \nabla H \right|^2 f_+^{p-1}, \int_{M_t} f_+^{p-2} \left| \nabla f_+ \right|^2.
\]
When \( p \geq c_7 \) and \( \sigma \leq \frac{c_5}{\sqrt{p}} \), the coefficients of the above terms are negative.

Therefore, we get
\[
\frac{d}{dt} \int_{M_t} (f_+)^p d\mu \leq K_2 |M_0|.
\]

**Proof of Theorem 4.3.** Using the above result, we get the desired estimate by Stampacchia iteration. Then from the \( L^p \) estimate of the previous theorems we can derive a uniform bound on the supremum of the function \((f_+)^p\). We leave out the details as they are entirely analogous to similar theorems in [5, 6, 7].

5. Derivative estimates for the curvature. In this section, we will drive pointwise derivative estimates for the curvature in the mean curvature flow of the surfaces satisfying (1.1). The estimates here depend only on the maximum of the curvature at a point and not on the maximum of the curvature over the entire submanifold. The proof depends on the following gradient estimate:

\[ |\nabla A|^2 \geq \frac{3}{n+2} |\nabla H|^2, \]

which is obtained from Codazzi equation. In the following, if \( n \geq 4 \), the the following constant is positive:

\[ \kappa_n = \frac{3}{n+2} - \frac{1}{n-1} > 0. \]

In fact, this only requires \( n \geq 3 \).

**Theorem 5.1.** Let \( M_t, (t \in [0,T_{\text{max}})) \) be a closed \( (n \geq 4) \)-dimensional quadratically bounded solution to the mean curvature flow in \( \mathbb{H}^{n+1}(K) \). Then there exist constants \( \gamma_2 = \gamma_2(n, M_0) \) and \( \gamma_3 = \gamma_3(n, M_0) \) such that the flow satisfies the uniform estimate

\[ |\nabla A|^2 \leq \gamma_2 |A|^4 + \gamma_3 \]

for all \( t \in [0,T) \).

**Proof.** This proof use similar strategy and estimates in both [16, 17]. Observe that \( \frac{3}{n+2} > \frac{1}{n-1} \) provided \( n \geq 3 \). Let

\[ \kappa_n = \frac{3}{n+2} - \frac{1}{n-1}. \]

By the cylindrical estimate, there exists a \( C_0 \) such that

\[ \left( \frac{1}{n-1} + \eta \right) H^2 - |A|^2 + C_0 \geq 0. \]

Let

\[ g_1 = \left( \frac{1}{n-1} + \eta \right) H^2 - |A|^2 + 2C_0, \]

and

\[ g_2 = \frac{3}{n+2} H^2 - |A|^2 + 2C_0. \]

Then we have \( g_1 > g_2 \geq C_0 \) and \( g_i - 2C_0 = 2(g_i - C_0) - g_i \geq g_i \) for \( i = 1, 2 \). Using the evolution equations for \( |A|^2 \) and \( H^2 \), we have that

\[
\frac{1}{\partial_t} g_1 - \Delta g_1 = -2 \left( \left( \frac{1}{n-1} + \eta \right) |\nabla H|^2 - |\nabla A|^2 \right) + 2|A|^2 (g_1 - 2C_0) + 4nK \left( |A|^2 - \frac{H^2}{n} \right)
\geq 2 \left( 1 - \frac{n+2}{3} \left( \frac{1}{n-1} + \eta \right) \right) |\nabla A|^2 - 2|A|^2 g_1 = 2\kappa_n \frac{n+3}{2} |\nabla A|^2 - 2|A|^2 g_1.
\]

Similarly, we have

\[
\frac{\partial}{\partial_t} g_2 - \Delta g_2 = -2 \left( \frac{2}{n+3} |\nabla H|^2 - |\nabla A|^2 \right) + 2|A|^2 (g_2 - 2C_0) + 4nK \left( |A|^2 - \frac{H^2}{n} \right) \geq -2|A|^2 g_2.
\]
Notice that $|A|^2 \geq \frac{1}{n} H^2 \geq c > 0$ and the lower bounds of the mean curvature are preserved by the flow, there exists a constant $c'$ depending only on the initial data $M_0$ such that $|A|^2 \leq c'|A|^4$. Furthermore, applying this to the evolution equation (2.9), we obtain the evolution equation for the derivative of the second fundamental form,

$$\frac{\partial}{\partial t} |\nabla A|^2 - \Delta |\nabla A|^2 \leq -2|\nabla^2 A|^2 + c'_n |\nabla A|^2 |\nabla^2 A|^2 - c_n |A|^2 |\nabla A|^2$$

for a constant $c_n$ depending only on $n$. The remainder of the proof is then entirely analogous to the proof of the Theorem 6.1 in [9].

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