THE SELBERG TRACE FORMULA FOR HECKE OPERATORS ON COCOMPACT KLEINIAN GROUPS

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ABSTRACT. We compute the Selberg trace formula for Hecke operators (also called the trace formula for modular correspondences) in the context of cocompact Kleinian groups with finite-dimensional unitary representations. We give some applications to the distribution of Hecke eigenvalues, and give an analogue of Huber’s theorem.

1. INTRODUCTION

1.1. Motivation. The Selberg trace formula has been well studied for cofinite Kleinian and Fuchsian groups. In the Fuchsian case: [Sel56, Roe66, VKF73, Hej76, Hej83, Ven82, Bus92, Iwa02]; in the Kleinian case: [EGM98, Fri05a, Fri05b]. For the most part (excepting [Hej76, Hej83]) the references above consider the standard Selberg trace formula which entails two objects: a discrete group $\Gamma$, and possibly a finite-dimensional unitary representation $\chi$. A third object to consider is a Hecke operator.

The Selberg trace formula (and the underlying Selberg spectral theory) for Hecke operators (also known as the Selberg trace formula for modular correspondences) is far from a trivial extension of the standard Selberg trace formula. By considering the abundance of Hecke operators for the Modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$, Lindenstrauss and Venkatesh ([LV07], [Gol06, Chap. 4]) gave a new proof for the existence of infinitely many even Maaß forms in $L^2(\mathbb{H} \setminus \Gamma)$.

Strömbergsson ([Str01]) made extensive use of the trace formula for Hecke operators to solve a problem in the well known spectral correspondence for quaternion groups—the Jacquet-Langlands correspondence ([Hej85]). More specifically, Strömbergsson completely determined the image of the Jacquet-Langlands correspondence. In fact, in [Str98], he extended the Selberg trace formula for modular correspondences from the cocompact Fuchsian case ([Hej76]) to the cofinite Fuchsian case.

The possibility for a trace formula for Hecke operators started with Selberg ([Sel89], pp. 444–446, 460–462, 504–505]. In [Hej76] Hejhal gave an explicit trace formula for modular correspondences for cocompact Fuchsian groups with finite-dimensional unitary representations and integer

\footnote{Some authors seem to refer to the subject as The Selberg trace formula for modular correspondences and some as The Selberg trace formula for Hecke operators. For our purposes these are interchangeable. To be perfectly precise, we are not computing the trace of a Hecke operator, but rather the trace of $\mathcal{M}h(\Delta)$, where $\mathcal{M}$ is a Hecke operator, $\Delta$ is the Laplacian, and $h$ is a holomorphic function with a certain decay rate.}

\footnote{The old proof is Selberg’s original method: The determinant of the scattering matrix is shown to have order that is sufficiently small.}

\footnote{Their proof works in much more generality.}

\footnote{The bijective correspondence between the nontrivial automorphic forms on the multiplicative group of a division quaternion algebra and certain cusp forms on $\text{GL}(2)$.}
weight $k \geq 0$ forms. Akiyama and Tanigawa [AT90] proved the Selberg trace formula for modular correspondences for cofinite Fuchsian groups and gave explicit evaluations of the trace formula for the case of the congruence group $\Gamma_0(p)$ ($p$ prime). Hoffmann ([Hof94]) approached the trace formula for Hecke operator, from the representation-theoretic point of view.

Strömbergsson ([Str98]) gave a very detailed derivation of the Selberg trace formula for modular correspondences in the context of weight-zero forms (functions) and trivial unitary representations on cofinite Fuchsian groups. He also gave explicit formulas for the case of $\Gamma = \Gamma_0(N)$ ($N$ is square-free).

In this work, we derive the Selberg trace formula for Hecke operators for cocompact Kleinian groups with finite-dimensional unitary representations. In order for the formula to apply, the cocompact group $\Gamma$ needs to have its commensurator $\text{Comm}(\Gamma)$ (in $\text{PSL}(2, \mathbb{C})$) strictly larger than itself. For each $\alpha \in \text{Comm}(\Gamma) \setminus \Gamma$, we obtain a Hecke operator $M$. When $\Gamma$ is arithmetic, $[\text{Comm}(\Gamma) : \Gamma] = \infty$; in fact $\text{Comm}(\Gamma)$ is a dense subgroup of $\text{PSL}(2, \mathbb{C})$ ([MR03, p. 271, Exer. 8.4.5]).

1.2. Main results. In this section we state our main results, and briefly describe the notation and preliminary material needed to state them (for more details see [2]).

Let $\Gamma < \text{PSL}(2, \mathbb{C})$ be a cocompact Kleinian group acting on hyperbolic three-space $\mathbb{H}^3$. Let $V$ be a finite-dimensional complex inner product space with inner-product $\langle \cdot , \cdot \rangle_V$, and let $\text{Rep}(\Gamma, V)$ denote the set of finite-dimensional unitary representations of $\Gamma$ in $V$. Let $\mathcal{F} \subset \Gamma$ be a (compact) fundamental domain for the action of $\Gamma$ in $\mathbb{H}^3$.

Let $\chi \in \text{Rep}(\Gamma, V)$. The Hilbert space of $\chi$–automorphic functions is the set of measurable functions

$$\mathcal{H}(\Gamma, \chi) \equiv \{ f : \mathbb{H}^3 \rightarrow V \mid f(\gamma P) = \chi(\gamma)f(P) \ \forall \gamma \in \Gamma, P \in \mathbb{H}^3, \text{ and } \langle f, f \rangle \equiv \int_{\mathcal{F}} \langle f(P), f(P) \rangle_V \ dv(P) < \infty \}.$$ 

Finally, let $\Delta = \Delta(\Gamma, \chi)$ be the corresponding positive self-adjoint Laplace-Beltrami operator on $\mathcal{H}(\Gamma, \chi)$.

The commensurator subgroup $\text{Comm}(\Gamma)$ is the set of all $\alpha \in \text{PSL}(2, \mathbb{C})$ such that both $[\Gamma : \Gamma \cap \alpha^{-1}\Gamma \alpha] < \infty$ and $[\alpha^{-1}\Gamma \alpha : \Gamma \cap \alpha^{-1}\Gamma \alpha] < \infty$.

Throughout this paper we will consider elements $\alpha \in \text{Comm}(\Gamma)$ that satisfy:

1. the element $\alpha \in \text{Comm}(\Gamma) \setminus \Gamma$;
2. the unitary representation $\chi \in \text{Rep}(\Gamma, V)$ has a single-valued extension from $\Gamma$ to the set $\Gamma \alpha^\Gamma$, and satisfies the following properties:
   (a) $\chi(g_1 \alpha g_2) = \chi(g_1)\chi(\alpha)\chi(g_2)$ for $g_1, g_2 \in \Gamma$;
   (b) $\chi(\alpha)$ is an invertible linear map on the inner product space $V$.

Associated to the element $\alpha$ is the Hecke operator $M : \mathcal{H}(\Gamma, \chi) \rightarrow \mathcal{H}(\Gamma, \chi)$. The Hecke operator commutes with $\Delta$.

Let $\mathcal{D}$ be an indexing set for the set of eigenvalues of $\Delta$. For each eigenvalue $\lambda = \lambda_m$ ($m \in \mathcal{D}$), let $A_\lambda \equiv A(\lambda, \Gamma, \chi)$ denote the subspace of $\mathcal{H}(\Gamma, \chi)$ spanned by $\{ \lambda_m \mid \lambda_m = \lambda \}$. The subspaces $A_\lambda$ are invariant under $M$. Let $\lambda_{m_1} \cdots \lambda_{m_{k_\lambda}}$ generate $A_\lambda$ ($\lambda$ has multiplicity $k_\lambda$). Since $A_\lambda$ is invariant under $M$, the eigenvalues of $M|_{A_\lambda}$ can be listed as $\omega_{m_1} \cdots \omega_{m_{k_\lambda}}$. Decomposing $\mathcal{H}(\Gamma, \chi)$ into invariant subspaces $A_\lambda$, for each $m \in \mathcal{D}$, we pair $\lambda_m$ and $\omega_m$. 
We next state our main result, the Selberg trace formula for Hecke operators. The various notations are described very briefly following the theorem; for more details see [EGM98].

Our main result is the following:

**Theorem.** (Selberg trace formula) Let \( \Gamma \) be a cocompact Kleinian group. Let \( \alpha \in \text{Comm}(\Gamma) \setminus \Gamma \), and let \( \chi \) be a representation satisfying Assumption [2.1]. Let \( h \) be a holomorphic function on \( \{ s \in \mathbb{C} \mid |\text{Im}(s)| < 2 + \delta \} \) for some \( \delta > 0 \), satisfying \( h(1 + z^2) = O(1 + |z|^2)^{3/2 - \epsilon} \) as \( |z| \to \infty \), and let

\[
g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} h(1 + t^2)e^{-itx} \, dt.
\]

Then

\[
\sum_{m \in \mathcal{D}} h(\lambda_m)\omega_m = \sum_{\{R\}\text{ell}} \frac{\text{tr}_V(\chi(R^{-1})^* g(0) \log N(T_0))}{|E(R)| |(\text{tr}(R))^2 - 4|} + \sum_{\{T\}\text{lox}} \frac{\text{tr}_V(\chi(T^{-1})^* g(\log N(T)))}{|E(T)| |a(T) - a(T)^{-1}|^2} \log N(T_0)
\]

Here, \( \{\lambda_m\}_{m \in \mathcal{D}} \) are the eigenvalues of \( \Delta \) counted with multiplicity, and \( \omega_m \) are the eigenvalues of \( \mathcal{M} \) (the Hecke operator associated to \( \alpha \)) with the convention of Equation [2.6]. The summation with respect to \( \{R\}_{\text{ell}} \) extends over the finitely many \( \Gamma \)-conjugacy classes of elliptic elements \( R \in \Gamma \alpha^{-1}\Gamma \), and for such a class, \( N(T_0) \) is the minimal norm of a hyperbolic or loxodromic element of the centralizer \( C(R) \subset \Gamma ; E(R) \) is the maximal finite subgroup contained in \( C(R) \) (see Lemma [3.5] for more details). The summation with respect to \( \{T\}_{\text{lox}} \) extends over the \( \Gamma \)-conjugacy classes of hyperbolic or loxodromic elements of \( \Gamma \alpha^{-1}\Gamma \), \( T_0 \) denotes a primitive hyperbolic or loxodromic element of minimal norm in \( C(T) \subset \Gamma ; T \) is conjugate in \( \text{PSL}(2, \mathbb{C}) \) to the transformation described by the diagonal matrix with diagonal entries \( a(T), a(T)^{-1} \) with \( |a(T)| > 1 \), and \( N(T) = |a(T)|^2 ; E(T) \) is the finite cyclic elliptic subgroup of \( C(T) \) (see Lemma [3.3] for more details). The sum over elliptic elements is finite, and all other sums converge absolutely.

Please keep in mind that the conjugacy classes \( \{R\}_{\text{ell}} \) and \( \{T\}_{\text{lox}} \), the operator \( \mathcal{M} \), and the eigenvalues \( \omega_m \) all depend on \( \alpha \in \text{Comm}(\Gamma) \setminus \Gamma \).

Our first application of the trace formula is to study the distribution of the Hecke eigenvalues.

Define the **elliptic number** of \( \Gamma \) with respect to \( \alpha, E_\alpha \), by the finite sum

\[
E_\alpha^\Gamma = \sum_{\{R\}_{\text{ell}}} \frac{\log N(T_0)}{|E(R)| |(\text{tr}(R))^2 - 4|}.
\]

Here the sum is over **elliptic conjugacy classes** of the set \( \Gamma \alpha^{-1} \Gamma \), where conjugacy is defined with respect to the group \( \Gamma \). The notations above are defined in [3.5].

We have

**Theorem.** Let \( \Gamma \) be a cocompact Kleinian group with \( \alpha \in \text{Comm}(\Gamma) \setminus \Gamma \). Then

\[
\sum_{m \in \mathcal{D}} \omega_m e^{-\lambda_m t} = \frac{E_\alpha^\Gamma}{\sqrt{4\pi t}} + O(\sqrt{t}) \quad \text{as} \ t \to 0^+.
\]

\[5\] Note that there is no clear relationship between \( T \in \Gamma \alpha^{-1} \Gamma \) and \( T_0 \in \Gamma \).

\[6\] Please note that there is a typographical error in the loxodromic and non cuspidal elliptic terms in [EGM98] Theorem 6.5.1; both terms are missing a factor of \( \frac{1}{2\pi} \).
Here $E_\Gamma^\alpha$ is the elliptic number of $\Gamma$ with respect to $\alpha$, $\{\lambda_m\}_{m \in D}$ are the eigenvalues of $\Delta$ counted with multiplicity, and $\omega_m$ are the eigenvalues of $M$ (the Hecke operator associated to $\alpha$) with the convention of Equation 2.6.

If the set $\Gamma\alpha^{-1}\Gamma$ contains no elliptic elements we have

**Theorem.** Let $\Gamma$ be a cocompact Kleinian group with $\alpha \in \text{Comm}(\Gamma) \setminus \Gamma$. Suppose that the set $\Gamma\alpha^{-1}\Gamma$ contains no elliptic elements. Then

$$\sum_{m \in D} \omega_m e^{-\lambda_m t} = O \left( t^{-1/2} \exp(-c/t) \right) \quad \text{as } t \to 0^+.$$  

Hence

$$\lim_{t \to 0^+} \sum_{m \in D} \omega_m e^{-\lambda_m t} = 0.$$

Here $\{\lambda_m\}_{m \in D}$ are the eigenvalues of $\Delta$ counted with multiplicity, and $\omega_m$ are the eigenvalues of $M$ (the Hecke operator associated to $\alpha$) with the convention of Equation 2.6.

Finally, we give an analogue of Huber’s theorem.

Following [EGM98], we define the length spectrum of $\Gamma\alpha^{-1}\Gamma$. For loxodromic $T_j \in \Gamma\alpha^{-1}\Gamma$ set $\mu_j = \log N(T_j)$. The length spectrum of $\Gamma\alpha^{-1}\Gamma$ is defined to be

$$L_\Gamma^\alpha = \left( \mu_j, \sum_{\{T_j\}_{1 \leq j} \text{log } N(T) = \mu_j} \frac{\log N(T_0)}{|E(T)| |a(T) - a(T)|^2} \right)_{j \geq 1}.$$  

In the two-dimensional case the length spectrum simply comprises the lengths of closed geodesics. Here we really need the complex lengths $a(T)$ and the order of the elliptic, finite subgroup of $C(T)$, $|E(T)|$.

We define the eigenvalue spectra of $\Delta$ and $M$ ($M$ is defined from $\alpha \in \text{Comm}(\Gamma) \setminus \Gamma$) by

$$S_\Gamma^\alpha = (\lambda_j, \omega(\lambda_j))_{j \in D^*}.$$  

Here the symbol $D^*$ means that we do not count with multiplicity, and $\omega(\lambda)$ is the trace of $M$ on the invariant subspace generated by all eigenfunctions (of $\Delta$) with eigenvalue $\lambda$.

**Theorem.** Let $\Gamma$, $\Gamma'$ be cocompact Kleinian groups with $\alpha \in \text{Comm}(\Gamma) \setminus \Gamma$, $\alpha' \in \text{Comm}(\Gamma') \setminus \Gamma'$. Then the following hold:

1. Suppose that $S_\Gamma^\alpha$ and $S_{\Gamma'}^{\alpha'}$ agree up to at most finitely many terms. Then

$$E_\Gamma^\alpha = E_{\Gamma'}^{\alpha'},$$  

$$S_\Gamma^\alpha = S_{\Gamma'}^{\alpha'},$$  

$$L_\Gamma^\alpha = L_{\Gamma'}^{\alpha'}.$$  

2. Suppose that $L_\Gamma^\alpha$ and $L_{\Gamma'}^{\alpha'}$ agree up to at most finitely many terms. Then

$$E_\Gamma^\alpha = E_{\Gamma'}^{\alpha'},$$  

$$S_\Gamma^\alpha = S_{\Gamma'}^{\alpha'},$$  

$$L_\Gamma^\alpha = L_{\Gamma'}^{\alpha'}.$$
By applying the above Theorem to the case of a single fixed group with two different elements \( \alpha, \alpha' \in \text{Comm}(\Gamma) \setminus \Gamma \), we obtain:

**Corollary.** Let \( \Gamma \) be a cocompact Kleinian groups with \( \alpha, \alpha' \in \text{Comm}(\Gamma) \setminus \Gamma \). Let \( \omega(\lambda_m) = \omega'(\lambda_m) \) for all but at most finitely many \( m \in D^* \). Then \( \omega(\lambda_m) = \omega'(\lambda_m) \) for all \( m \in D^* \).

## 2. Preliminaries

In this section we review the basic notions needed to evaluate the Selberg trace formula for modular correspondences. See [Hej76, Chapter 5] for the analogous Fuchsian case.

### 2.1. Cocompact Kleinian groups

Let \( \Gamma < \text{PSL}(2, \mathbb{C}) \) be a cocompact Kleinian group acting on hyperbolic three-space \( \mathbb{H}^3 \). Let \( V \) be a finite-dimensional complex inner product space with inner-product \( \langle \cdot, \cdot \rangle_V \), and let \( \text{Rep}(\Gamma, V) \) denote the set of finite-dimensional unitary representations of \( \Gamma \) in \( V \). Let \( \mathcal{F} \subset \Gamma \) be a (compact) fundamental domain for the action of \( \Gamma \) in \( \mathbb{H}^3 \).

Let \( \chi \in \text{Rep}(\Gamma, V) \). The Hilbert space of \( \chi \)-automorphic functions is the set of measurable functions

\[
\mathcal{H}(\Gamma, \chi) = \{ f : \mathbb{H}^3 \to V \mid f(\gamma P) = \chi(\gamma)f(P) \ \forall \gamma \in \Gamma, P \in \mathbb{H}^3, \ \text{and} \ \langle f, f \rangle = \int_{\mathcal{F}} \langle f(P), f(P) \rangle_V \, dv(P) < \infty \}.
\]

Finally, let \( \Delta = \Delta(\Gamma, \chi) \) be the corresponding positive self-adjoint Laplace-Beltrami operator on \( \mathcal{H}(\Gamma, \chi) \).

### 2.2. Hecke operators

In this section we define the Hecke operators (or modular correspondences). For more details see [Hej76, Chap. 5] for the cocompact Fuchsian case, [Str98] for the cofinite Fuchsian case.

Let \( \Gamma \) be a cocompact Kleinian group. The commensurator subgroup \( \text{Comm}(\Gamma) \) is the set of all \( \alpha \in \text{PSL}(2, \mathbb{C}) \) such that both \( |\Gamma : \Gamma \cap \alpha^{-1} \Gamma \alpha| < \infty \) and \( |\alpha^{-1} \Gamma \alpha : \Gamma \cap \alpha^{-1} \Gamma \alpha| < \infty \).

Let \( \chi \in \text{Rep}(\Gamma, V) \). By definition, \( \text{Comm}(\Gamma) \subset \Gamma \). However, in order to define non-trivial Hecke operators, we will need to start with an element \( \alpha \in \text{Comm}(\Gamma) \) that lies outside of \( \Gamma \). We will also need the unitary representation \( \chi \) to extend from \( \Gamma \) to the set \( \Gamma \alpha \Gamma \). More precisely:

**Assumption 2.1.** Throughout this paper we assume:

1. The element \( \alpha \in \text{Comm}(\Gamma) \setminus \Gamma \);
2. The unitary representation \( \chi \in \text{Rep}(\Gamma, V) \) has a single-valued extension from \( \Gamma \) to the set \( \Gamma \alpha \Gamma \), and satisfies the following properties:
   - (a) \( \chi(g_1 \alpha g_2) = \chi(g_1)\chi(\alpha)\chi(g_2) \) for \( g_1, g_2 \in \Gamma \);
   - (b) \( \chi(\alpha) \) is an invertible linear map on the inner product space \( V \).

See [Hej76, Pages 463-4] for the analogous Fuchsian case.

Let \( d = \left[ \Gamma : \Gamma \cap \alpha^{-1} \Gamma \alpha \right] \). It follows, by conjugation, that \( \left[ \alpha^{-1} \Gamma \alpha : \Gamma \cap \alpha^{-1} \Gamma \alpha \right] = d \).

\(^7\)There is much overlap between the Fuchsian case and our case, the Kleinian case. We will often cite the Fuchsian case, as a reference when proofs in the two cases are identical.
Let the symbol \( \sqcup \) denote disjoint union. Then

\[
\Gamma = \bigsqcup_{i=1}^{d} (\Gamma \cap \alpha^{-1}\Gamma) e_i \quad \text{iff} \quad \Gamma \alpha \Gamma = \bigsqcup_{i=1}^{d} \Gamma \alpha_i e_i.
\]

Setting \( \alpha_i = \alpha e_i \) \((i = 1 \ldots d)\), we define the operator \( M : \mathcal{H}(\Gamma, \chi) \mapsto \mathcal{H}(\Gamma, \chi) \) by

\[
(Mf)(P) = \sum_{i=1}^{d} \chi(\alpha_i)^* f(\alpha_i P).
\]

The operator \( M \) does not depend on the above choice of coset representatives, and it is a bounded linear operator on \( \mathcal{H}(\Gamma, \chi) \) ([Hej76, p. 467] or [Str98]).

The adjoint of \( M \) can be constructed as follows: Since \( \alpha \in \text{Comm}(\Gamma) \), and \( \text{Comm}(\Gamma) \) is a group; \( \alpha^{-1} \) also satisfies Assumption 2.1. As before, we can write

\[
\Gamma \alpha^{-1} \Gamma = e \bigsqcup_{i=1}^{d} \Gamma \beta_i.
\]

It follows that \( e = d \) ([Hej76 p. 469]). Define \( M^* : \mathcal{H}(\Gamma, \chi) \mapsto \mathcal{H}(\Gamma, \chi) \) by

\[
(M^* f)(z) = \sum_{i=1}^{d} \chi(\beta_i^{-1}) f(\beta_i P).
\]

For every \( f, g \in \mathcal{H}(\Gamma, \chi) \) we have ([Hej76 p. 470])

\[
\langle Mf, g \rangle = \langle f, M^* g \rangle.
\]

2.3. Interplay between \( \Delta \) and \( M \). For \( P = z + rj, \ P' = z' + r'j \in \mathbb{H}^3 \) set

\[
\delta(P, P') \equiv \frac{|z - z'|^2 + r^2 + r'^2}{2rr'}.
\]

It follows that \( \delta(P, P') = \cosh(d(P, P')) \), where \( d \) denotes the hyperbolic distance in \( \mathbb{H}^3 \). Next, for \( k \in S([1, \infty)) \) a Schwartz-class function, define the point-pair invariant \( K(P, Q) \equiv k(\delta(P, Q)) \). Note that for all \( \gamma \in \text{PSL}(2, \mathbb{C}) \) we have \( K(\gamma P, \gamma Q) = K(P, Q) \). Define the Poincaré series

\[
K_1(P, Q) \equiv \sum_{T \in \Gamma} \chi(T) K(P, TQ).
\]

The series above converges absolutely and uniformly on compact subsets of \( \mathbb{H}^3 \times \mathbb{H}^3 \), and is the kernel of a bounded operator \( K : \mathcal{H}(\Gamma, \chi) \mapsto \mathcal{H}(\Gamma, \chi) \). The Selberg theory allows us to diagonalize \( K \) with respect to a basis of eigenfunctions of \( \Delta \).

We first need the following spectral expansion: Each \( f \in \mathcal{H}(\Gamma, \chi) \) has an expansion of the form

\[
f(P) = \sum_{m \in D} \langle f, e_m \rangle e_m(P).
\]

Here the sum converges in the Hilbert space \( \mathcal{H}(\Gamma, \chi) \), and \( D \) is an indexing set of the eigenfunctions \( e_m \) of \( \Delta \) with corresponding eigenvalues \( \lambda_m \).

Now, let \( h \) be the Selberg–Harish-Chandra transform of \( k \); explicitly, for \( \lambda = 1 - s^2 \in \mathbb{C} \),

\[
h(\lambda) = h(1 - s^2) \equiv \frac{\pi}{s} \int_1^{\infty} k \left( \frac{1}{2} \left( t + \frac{1}{t} \right) \right) (t^s - t^{-s}) \left( \frac{t - \frac{1}{t}}{t} \right) \frac{dt}{t}.
\]
and let $g$ be the Fourier transform of $h$:

\[(2.4)\]

\[g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} h(1 + t^2)e^{-itx} \, dt.\]

For $v, w \in V$ let $v \otimes w$ be a linear operator in $V$ defined by $v \otimes w(x) = \langle x, w \rangle v$. An immediate application of the spectral expansion (2.2) and the Selberg–Harish-Chandra transform (2.3) gives us (see [Fri05a, Fri05b] or [EGM98, Equation 6.4.10, page 278]):

**Lemma 2.2.** Let $k \in S$ and $h : \mathbb{C} \to \mathbb{C}$ be the Selberg–Harish-Chandra Transform of $k$. Then with $K_\Gamma$ defined above, we have

\[(2.5)\]

\[K_\Gamma(P, Q) = \sum_{m \in D} h(\lambda_m) e_m(P) \otimes e_m(Q).\]

The sum converges absolutely and uniformly on compact subsets of $\mathbb{H}^3 \times \mathbb{H}^3$.

Since the Laplace operator $\Delta$ commutes with the action of $\text{PSL}(2, \mathbb{C})$ on $\mathbb{H}^3$, it follows that $\Delta$ commutes with $\mathcal{M}$. From Selberg’s original ideas ([Sel56]), the operator $\mathcal{K} = h(\Delta)$, and thus $\mathcal{M}$ must also commute with $\mathcal{K}$. A direct proof of this fact is given in [Hej76, p. 473]:

**Lemma 2.3.** Let $\mathcal{K}$ be the linear operator defined from the kernel function $K_\Gamma(P, Q)$. Then, on $\mathcal{H}(\Gamma, \chi)$

\[\mathcal{K}\mathcal{M} = \mathcal{M}\mathcal{K},\]

and on $\mathcal{H}(\Gamma, \chi) \cap C^2(\mathbb{H}^3)$

\[\mathcal{M} \Delta = \Delta \mathcal{M}.\]

For each eigenvalue $\lambda = \lambda_m$ ($m \in D$), let $A_\lambda \equiv A(\lambda, \Gamma, \chi)$ denote the subspace of $\mathcal{H}(\Gamma, \chi)$ spanned by $\{\lambda_m \mid \lambda_m = \lambda\}$. Since $\Delta$ commutes with $\mathcal{M}$, it follows that the subspaces $A_\lambda$ are invariant under $\mathcal{M}$. Let $m_1 \ldots m_{k_\lambda} \in D$ generate $A_\lambda$ ($\lambda$ has multiplicity $k_\lambda$). Since $A_\lambda$ is invariant under $\mathcal{M}$, the eigenvalues of $\mathcal{M}|_{A_\lambda}$ can be listed as $\omega_{m_1} \ldots \omega_{m_{k_\lambda}}$. Next, we define

\[(2.6)\]

\[\omega(\lambda) \equiv \text{tr } \mathcal{M}|_{A_\lambda} = \sum_{j=1}^{k_\lambda} \omega_{m_j}.\]

Decomposing $\mathcal{H}(\Gamma, \chi)$ into invariant subspaces $A_\lambda$, for each $m \in D$ we pair $\lambda_m$ and $\omega_m$. Since $\mathcal{M}$ is a bounded operator, we conclude that the $\omega_m$ ($m \in D$) are uniformly bounded.

3. The Selberg Trace Formula for Hecke Operators

The standard Selberg trace formula is an explicit evaluation of both the integral and spectral of the operator $\mathcal{K}$. In our case of interest we evaluate the trace of $\mathcal{K}\mathcal{M}$. 
3.1. **The kernel of operator** $\mathcal{K}\mathcal{M}$. The first step is to construct a Poincaré series for the kernel of $\mathcal{K}\mathcal{M}$.

For all $f \in \mathcal{H}(\Gamma, \chi)$, by Lemma 2.3 and by definition,

$$\mathcal{K}\mathcal{M}f(P) = \mathcal{M}\mathcal{K}f(P) = \int_{\mathcal{F}} \mathcal{M}_{P}\mathcal{K}_{\Gamma}(P, Q)f(Q)\, dv(Q).$$

Here $\mathcal{M}_{P}\mathcal{K}_{\Gamma}(P, Q)$ is the action of $\mathcal{M}$, treating $\mathcal{K}_{\Gamma}(P, Q)$ as a function of $P$. Set

$$(3.1) \quad \mathcal{K}_{\Gamma}(P, Q) \equiv \sum_{T \in \Gamma} \chi(T^{-1})^{*} K(P, TQ),$$

then follows ([Hej76, p. 476]) that $K_{\Gamma}(P, Q) = \mathcal{M}_{P}\mathcal{K}_{\Gamma}(P, Q)$. Hence:

**Lemma 3.1.** For all $f \in \mathcal{H}(\Gamma, \chi)$,

$$\mathcal{K}\mathcal{M}f(P) = \mathcal{M}\mathcal{K}f(P) = \int_{\mathcal{F}} \mathcal{K}_{\Gamma}(P, Q)f(Q)\, dv(Q).$$

Upon applying $\mathcal{M}$ to (2.5), we obtain

$$(3.2) \quad \mathcal{K}_{\Gamma}(P, Q) = \sum_{m \in \mathcal{D}} h(\lambda_{m}) (\mathcal{M}e_{m}(P)) \otimes e_{m}(Q).$$

The Selberg trace formula will result from a careful evaluation of

$$\text{tr}(\mathcal{K}\mathcal{M}) \equiv \int_{\mathcal{F}} \text{tr}_{V} \mathcal{K}_{\Gamma}(P, P)\, dv(P).$$

3.2. **Spectral trace.** Since $\mathcal{K}_{\Gamma}(P, Q)$ is a continuous function on the compact set $\mathcal{F} \times \mathcal{F}$, the integral

$$\int_{\mathcal{F}} \text{tr}_{V} \mathcal{K}_{\Gamma}(P, P)\, dv(P)$$

converges absolutely. Since $\mathcal{M}$ is bounded, the derivation of the standard Selberg trace formula for cofinite Kleinian groups [EGM98 §6.5] implies that

$$(3.3) \quad \int_{\mathcal{F}} \text{tr}_{V} \mathcal{K}_{\Gamma}(P, P)\, dv(P) = \sum_{m \in \mathcal{D}} h(\lambda_{m})\omega_{m}.$$

3.3. **Integral trace.** Selberg’s clever method of evaluating the integral (3.3) entails decomposing the Poincaré series $\mathcal{K}_{\Gamma}(P, Q)$ into conjugacy classes of various types of isometries (the types being: elliptic, parabolic, hyperbolic, . . .). A similar method works here, though conjugacy classes must be defined in the context of the set $\Gamma\alpha^{-1}\Gamma$. One also must deal with centralizer subgroups of various isometries, which lead to complications not found in the standard Selberg trace formula derivations.

For $T \in \Gamma\alpha^{-1}\Gamma$ set

$$\{T\} \equiv \{T\}_{\Gamma} \equiv \{\sigma^{-1}T\sigma \mid \sigma \in \Gamma\},$$

and set

$$\mathcal{C}(T) \equiv \mathcal{C}_{\Gamma}(T) \equiv \{\sigma \in \Gamma \mid \sigma T = T\sigma\}.$$
It now follows ([Hej76, pp. 480–482]) that

\[
\text{tr}(KM) = \sum_T \text{tr}_V (\chi(T^{-1})^*) \int_{\mathcal{F}(\mathcal{C}(T))} K(P, TP) \, dv(P).
\]

Here the sum is over all conjugacy classes of $\Gamma \alpha^{-1} \Gamma$ with respect to $\Gamma$; $\mathcal{F}(\mathcal{C}(T))$ is a fundamental domain for the subgroup $\mathcal{C}(T)$.

Since the integral in (3.4) is invariant when $T$ is conjugated within $\text{PSL}(2, \mathbb{C})$, it suffices to choose a conjugate of $T$ which has the simplest form to allow one to explicitly evaluate the following integral: For $T \in \Gamma \alpha^{-1} \Gamma$ set

\[
I(T) \equiv I_{\Gamma}(T) = \int_{\mathcal{F}(\mathcal{C}(T))} K(P, TP) \, dv(P).
\]

There are four cases to consider:

1. $T = \text{id}$ (this case does not occur by Assumption 2.1)
2. $T$ is loxodromic
3. $T$ is elliptic
4. $T$ is parabolic.

In fact, we must first compute the structure of each centralizer subgroup $\mathcal{C}(T)$.

See [EGM98, chap. 1] for details on the action of $\text{PSL}(2, \mathbb{C})$ on $\mathbb{H}^3$, and for the definitions of loxodromic, parabolic, . . .

3.4. Loxodromic case. Suppose that $T \in \Gamma \alpha^{-1} \Gamma$ is loxodromic. We can then assume (replacing $T$ by its unique conjugate within $\text{PSL}(2, \mathbb{C})$) that $T$ has the form

\[
T = \begin{pmatrix} a(T) & 0 \\ 0 & a(T)^{-1} \end{pmatrix}
\]

such that $a(T) \in \mathbb{C}$ has $|a(T)| > 1$. Let $N(T)$ denote the norm of $T$, defined by

\[
N(T) \equiv |a(T)|^2.
\]

For $z + rj \in \mathbb{H}^3$

\[
T(z + rj) = a(T)^2 z + N(T)rj.
\]

The element $T$ fixes two points: $0$ and $j\infty \in \hat{\mathbb{C}}$. It follows from elementary calculations that any $\sigma \in \mathcal{C}(T) \subset \Gamma$ must be of the form

\[
\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}.
\]

Since $\Gamma$ is discrete and cocompact, it follows ([EGM98, chap. 5]) that $\mathcal{C}(T)$ has at least one element of minimal norm. Choose one, say $T_0$. The other minimal norm elements are obtained by multiplying by the elliptic elements of the finite cyclic elliptic subgroup $\mathcal{E}(T)$ of order $m(T)$, generated by an element $E_{T_0}$. We now have

\[
\mathcal{C}(T) = \langle T_0 \rangle \times \mathcal{E}(T).
\]

Here $\langle T_0 \rangle = \{ T_0^n \mid n \in \mathbb{Z} \}$. The elliptic element $E_{T_0}$ is conjugate in $\text{PSL}(2, \mathbb{C})$ to an element of the form

\[
\begin{pmatrix} \zeta(T_0) & 0 \\ 0 & \zeta(T_0)^{-1} \end{pmatrix},
\]
where here $\zeta(T_0)$ is a primitive $2m(T)$-th root of unity.

Note that (by definition) if $T \not\in \Gamma, T \not\in C(T)$. It seems conceivable then that $C(T)$ could be a finite group. However, in the next lemma we will show that $N(T_0) > 1$ which implies that $C(T)$ contains an infinite cyclic loxodromic subgroup.

**Lemma 3.2.** Let $T \in \Gamma^{-1}\Gamma$. Then $C(T)$ contains a loxodromic element $T_0$ of minimal norm $N(T_0) > 1$.

**Proof.** Let $k \in \mathcal{S}[1, \infty)$ be a smooth, positive bump function with $k \equiv 1$ on $[1, 2]$, (zero everywhere else) and let $\chi$ be the trivial representation. The right-hand side of (3.3) converges absolutely, and by positivity the left side does too. Hence (3.4) and (3.5) imply that

$$\int_{\mathcal{F}(C(T))} K(P, T P) \, dv(P) < \infty.$$  

(3.7)

For $\varphi \in \mathbb{R}$ set

$$R(\varphi) = \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix}.$$

$R(\varphi)$ is a rotation on $\mathbb{H}^3$ that fixes $0, \infty j \in \mathbb{C}$. For $z + rj \in \mathbb{H}^3$,

$$R(\varphi)(z + rj) = e^{i\varphi} z + rj.$$

Next, suppose there are not loxodromic elements in $C(T)$, that is, that $C(T) = \langle R(\varphi) \rangle$, where $\varphi \equiv \frac{2\pi}{m}$, and $m = |R(\varphi)| = |C(T)|$. A fundamental domain for $C(T)$ is given by (see [EGM98, p. 193])

$$\mathcal{F}_1 = \left\{ \rho e^{i\varphi} + rj \mid r > 0, \rho \geq 0, 0 \leq \varphi \leq \frac{2\pi}{m} \right\}.$$

We claim that

$$\int_{\mathcal{F}_1} K(P, T P) \, dv(P) = \infty.$$  

(3.8)

To see this, first note that using (3.7) and (2.1), we can simplify $K(P, T P)$. We obtain

$$\int_{\mathcal{F}_1} K(P, T P) \, dv(P) = \int_{\mathcal{F}_1} k \left( \frac{|a(t)|^2 - 1|^2|z|^2 + (N(T)^2 + 1)^2}{2N(T)r^2} \right) \frac{dx \, dy \, dr}{r^3}.$$  

Applying the elementary substitution $x \mapsto rx, y \mapsto ry$: the integral becomes

$$\int_{\mathcal{F}_1} k \left( \frac{|a(t)|^2 - 1|^2|z|^2 + (N(T)^2 + 1)^2}{2N(T)} \right) \frac{dx \, dy \, dr}{r} = \int_0^\infty \frac{dr}{r} \int_{z(\mathcal{F}_1)} k \left( \frac{|a(t)|^2 - 1|^2|z|^2 + (N(T)^2 + 1)^2}{2N(T)} \right) \, dx \, dy = \infty.$$

Here $z(\mathcal{F}_1)$ is the standard projection of $\mathcal{F}_1 \subset \mathbb{H}^3$ onto $\mathbb{C}$. Hence $C(T)$ must contain at least one loxodromic element, and since $\Gamma$ is discrete, $C(T)$ must contain a minimally normed element. □
An element $R$ in $\Gamma$ is called \textit{primitive} if there does not exist $S \in \Gamma$ with $S^m = R$ for some positive integer $m$. Since the $T_0$ from Lemma 3.2 has minimal norm in $\mathcal{C}(T)$, it is necessarily primitive.

Now that we see that the structure of $\mathcal{C}(T)$ is identical to the case where $T \in \Gamma$ ([EGM98, p. 193]), hence we can evaluate the integral (3.5), obtaining:

\textbf{Lemma 3.3.} Let $T \in \text{Comm}(\Gamma)$ be loxodromic (hyperbolic elements are considered loxodromic). Let 
\[ \mathcal{C}(T) = \langle T_0 \rangle \times \mathcal{E}(T) \]
with $T_0$ loxodromic and primitive, and $\mathcal{E}(T)$ a finite cyclic subgroup of elliptic elements. Then 
\[ \int_{\mathcal{F}(\mathcal{C}(T))} K(P, TP) \, d\nu(P) = \frac{g(\log N(T)) \log N(T_0)}{|\mathcal{E}(T)| |a(T) - a(T)^{-1}|^2}. \]
Here, $g$ comes from (2.4) and (2.3).

3.5. \textbf{Elliptic case.} Suppose that $R \in \Gamma \alpha^{-1} \Gamma$ is elliptic. We can then assume (replacing $R$ by its unique conjugate within $\text{PSL}(2, \mathbb{C})$) that $R$ has the form 
\[ R = R(\varphi) = \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix} \quad (\varphi \in \mathbb{R}). \]

$R(\varphi)$ is a rotation on $\mathbb{H}^3$ that fixes $0, \infty j \in \hat{\mathbb{C}}$. For $z + rj \in \mathbb{H}^3$, 
\[ R(\varphi)(z + rj) = e^{i\varphi} z + rj. \]

An elementary calculation shows that 
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{C}) \]
commutes with $R(\varphi)$ iff one of the two conditions are satisfied:

1. $b = c = 0$
2. $\exp \frac{i\varphi}{2} = \pm i$ and $a = d = 0$.

In both cases, a similar argument to Lemma 3.2 (see [EGM98] pp. 193–194]) shows that $\mathcal{C}(R)$ contains loxodromic elements. In particular $\mathcal{C}(R)$ contains a loxodromic element of minimal norm $T_0$. Following [EGM98] pp. 193–197] and generalizing to $R \in \text{Comm}(\Gamma)$ we have ([EGM98 Thm. 5.2.1])\footnote{Note that there is \textit{no} clear relationship between $T$ and $T_0$ other than the fact that they have the same fixed points and commute. It seems reasonable to conjecture that $T = T_0^{k/n}$ for $k, n \in \mathbb{N}$.}

\textbf{Lemma 3.4.} Let $R \in \text{Comm}(\Gamma)$ be elliptic. Then there exists a primitive elliptic element $R_0 \in \mathcal{C}(R) \subset \Gamma$ (it is possible that $R_0 = \text{id}$), an element $T_0 \in \mathcal{C}(R)$ loxodromic, of minimal norm in $\mathcal{C}(R)$ so that one of the following cases holds:

1. Either all the elliptic elements of $\mathcal{C}(R)$ are contained in $\mathcal{E}(R) \equiv \langle R_0 \rangle$, and 
\[ \mathcal{C}(R) = \langle T_0 \rangle \times \mathcal{E}(R). \]

\footnote{We modified their hypothesis to allow $R$ to be in $\text{Comm}(\Gamma)$ instead of in $\Gamma$. We are justified by the proof of Lemma 3.2}
(2) Or $R$ is elliptic of order 2, and there exists an elliptic element $S \in C(R)$ so that
\[ E(R) = \langle R_0 \rangle \cup \langle R_0 \rangle S \]
and
\[ C(R) = \langle T_0 \rangle \times E(R). \]

See [EGM98, pp. 191-198] for more details.

Now that we see that the structure of $C(T)$ is identical to the case where $T \in \Gamma$ ([EGM98, p. 193]), we can evaluate the integral (3.5), obtaining:

**Lemma 3.5.** Let $R \in \text{Comm}(\Gamma)$ be elliptic. Let $C(R) = \langle T_0 \rangle \times E(R)$ as in Lemma 3.4. Then
\[ \int_{\mathcal{F}(C(R))} K(P, RP) \, dv(P) = \frac{g(0) \log N(T_0)}{|E(R)| |(tr(R))^2 - 4|}. \]

Here, $g$ comes from (2.4) and (2.3).

We will also need the following:

**Lemma 3.6.** Let $R \in \text{Comm}(\Gamma)$ be elliptic. Then there are only finitely many elliptic conjugacy classes of $\Gamma \alpha^{-1} \Gamma$, where conjugacy is defined with respect to $\Gamma$.

**Proof.** Let $k \in S[1, \infty)$ be non-negative and let $\chi$ be the trivial representation. By (3.4) and (3.3),
\[ \sum_{\{R\}_{\text{ell}}} \int_{\mathcal{F}(C(R))} K(P, RP) \, dv(P) < \infty. \]

Hence by Lemma 3.5
\[ g(0) \sum_{\{R\}_{\text{ell}}} \frac{\log N(T_0)}{|E(R)| |(tr(R))^2 - 4|} < \infty. \]

Now, since $N(T_0) \geq \delta > 1$ (independent of $T_0$) and since $-2 < tr(R) < 2$, in order for there to be infinitely many conjugacy classes, it is necessary for $|E(R)|$ to be unbounded as $R$ goes through all conjugacy classes. But by Lemma 3.4 that would imply the order of each $R_0$ is unbounded; a contradiction since cofinite Kleinian groups have only finitely many conjugacy classes of elliptic elements (and two elements in the same conjugacy class have the same order; see [EGM98]). \[ \square \]

### 3.6. Parabolic case.

Suppose that $T \in \text{Comm}(\Gamma)$ is parabolic. We will soon see that this case can not occur when $\Gamma$ is assumed to be cocompact. That is: $\text{Comm}(\Gamma)$ contains no parabolic elements when $\Gamma$ is cocompact. To see this, suppose that $T$ is already conjugated into the simple form
\[ T = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}. \]

Then, an elementary calculation shows that
\[ S \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{C}) \]
Lemma 3.2 shows that this cannot happen. Hence there are no parabolic elements in $\Gamma$. But, since $\Gamma$ is cocompact, it contains no parabolic elements. So $C(T) = \text{id}$. An argument similar to Lemma 3.2 shows that this cannot happen. Hence there are no parabolic elements in $\text{Comm}(\Gamma) \setminus \Gamma$.

### 3.7. The Selberg trace formula.

By putting together (3.4), Lemma 3.3, Lemma 3.5, Assumption 2.1, and the results of (3.6) and by applying standard approximation techniques in order to allow more general growth conditions on the function $k$ (and hence on $h$ and $g$) (see [Hej76], pp. 32–34) for the details, we obtain:

**Theorem 3.7.** (Selberg trace formula) Let $\Gamma$ be a cocompact Kleinian group. Let $\alpha \in \text{Comm}(\Gamma) \setminus \Gamma$, and let $\chi$ be a representation satisfying Assumption 2.1. Let $h$ be a holomorphic function on $\{ s \in \mathbb{C} \mid |\text{Im}(s)| < 2 + \delta \}$ for some $\delta > 0$, satisfying $h(1 + z^2) = O(1 + |z|^{3/2 - \epsilon})$ as $|z| \to \infty$, and let

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} h(1 + t^2) e^{-itx} \, dt.$$

Then

$$\sum_{m \in \mathcal{D}} h(\lambda_m) \omega_m = \sum_{\{R\}_{\text{ell}}} \frac{\text{tr}_V \left( \chi(R^{-1}) \right) g(0) \log N(T_0)}{|\mathcal{E}(R)| |(\text{tr}(R))^2 - 4|} + \sum_{\{T\}_{\text{lox}}} \frac{\text{tr}_V \left( \chi(T^{-1}) \right) g(\log N(T))}{|\mathcal{E}(T)| |a(T) - a(T)^{-1}|^2} \log N(T_0).$$

Here, $\{\lambda_m\}_{m \in \mathcal{D}}$ are the eigenvalues of $\Delta$ counted with multiplicity, and $\omega_m$ are the eigenvalues of $M$ (the Hecke operator associated to $\alpha$) with the convention of Equation 2.6. The summation with respect to $\{R\}_{\text{ell}}$ extends over the finitely many $\Gamma$–conjugacy classes of elliptic elements $R \in \Gamma \alpha^{-1} \Gamma$, and for such a class, $N(T_0)$ is the minimal norm of a hyperbolic or loxodromic element of the centralizer $C(R) \subset \Gamma$; $\mathcal{E}(R)$ is the maximal finite subgroup contained in $C(R)$ (see Lemma 3.5 for more details). The summation with respect to $\{T\}_{\text{lox}}$ extends over the $\Gamma$–conjugacy classes of hyperbolic or loxodromic elements of $\Gamma \alpha^{-1} \Gamma$, $T_0$ denotes a primitive hyperbolic or loxodromic element of minimal norm in $C(T) \subset \Gamma$; the element $T$ is conjugate in $\text{PSL}(2, \mathbb{C})$ to the transformation described by the diagonal matrix with diagonal entries $a(T), a(T)^{-1}$ with $|a(T)| > 1$, and $N(T) = |a(T)|^2$; $\mathcal{E}(T)$ is the finite cyclic elliptic subgroup of $C(T)$ (see Lemma 3.3 for more details). The sum over elliptic elements is finite, and all other sums converge absolutely.

Please keep in mind that the conjugacy classes $\{R\}_{\text{ell}}$ and $\{T\}_{\text{lox}}$, the operator $M$, and the eigenvalues $\omega_m$ all depend on $\alpha \in \text{Comm}(\Gamma) \setminus \Gamma$.

### 3.8. Self-Adjoint Hecke Operators.

Throughout this work we were under Assumption 2.1. By imposing stronger conditions on $\alpha$ and $\chi$ we can ensure that $M$ is self-adjoint.

**Assumption 3.8.** In addition to Assumption 2.1

1. $\Gamma \alpha \Gamma = \Gamma \alpha^{-1} \Gamma$
2. $\chi(\alpha^{-1}) = \chi(\alpha)^*$.

Assumption 3.8 implies that $M$ is self-adjoint. Since $M$ commutes with $\Delta$ we can choose a simultaneous diagonalization $\{e_m\}_{m \in \mathcal{D}}$ so that

$$\Delta e_m = \lambda_m e_m \quad \text{and} \quad Me_m = \omega_m e_m.$$

12 Note that there is no clear relationship between $T \in \Gamma \alpha^{-1} \Gamma$ and $T_0 \in \Gamma$.
13 Please note that there is a typographical error in the loxodromic and non cuspidal elliptic terms in [EGM98] Theorem 6.5.1; both terms are missing a factor of $\frac{1}{\pi}$. 
4. Application of the Trace Formula to the distribution $\omega_m$.

In the Selberg trace formula (Theorem 3.7), the eigenvalues $\lambda_m$ of $\Delta$ do not depend on $M$. The distribution of these eigenvalues has been well studied [Hub59, Sel56, Hej76, EGM98, Ven82].

More specifically, in the case of cocompact Kleinian groups we have

**Theorem 4.1.** [EGM98, p. 308] Let $\Gamma$ be a cocompact Kleinian group. Then

$$
\sum_{m \in D} e^{-\lambda_m t} \sim \frac{\text{vol}(\Gamma)}{8\pi^{3/2}} t^{-3/2} \quad \text{as} \quad t \to 0^+.
$$

Here $\text{vol}(\Gamma)$ is the volume of a fundamental domain for the action of $\Gamma$ in $\mathbb{H}^3$.

For $\Gamma$ a cocompact Kleinian group with $\alpha \in \text{Comm}(\Gamma) \setminus \Gamma$, define $E_{\alpha}$ the elliptic number of $\Gamma$ with respect to $\alpha$ by

$$
E_{\Gamma}^\alpha \equiv \sum_{\{R\}} \log N(T_0) \frac{|E(R)|}{|E(R)|^2 - 4} \quad \text{(see §3.5 for the definitions of the above notation)}.
$$

Our result is as follows:

**Theorem 4.2.** Let $\Gamma$ be a cocompact Kleinian group with $\alpha \in \text{Comm}(\Gamma) \setminus \Gamma$. Then

$$
\sum_{m \in D} \omega_m e^{-\lambda_m t} = \frac{E_{\Gamma}^\alpha}{\sqrt{4\pi t}} + O(\sqrt{t}) \quad \text{as} \quad t \to 0^+.
$$

Here $E_{\Gamma}^\alpha$ is the elliptic number of $\Gamma$ with respect to $\alpha$, $\{\lambda_m\}_{m \in D}$ are the eigenvalues of $\Delta$ counted with multiplicity, and $\omega_m$ are the eigenvalues of $M$ (the Hecke operator associated to $\alpha$) with the convention of Equation 2.6.

**Proof.** It follows from Lemma 4.3(below) that the loxodromic sum in (4.2) is bounded by $O\left(t^{-1/2} \exp\left(-c/t\right)\right)$ as $t \to 0^+$. Hence

$$
\sum_{m \in D} \omega_m e^{-t\lambda_m} = \frac{\exp(-t)}{\sqrt{4\pi t}} \sum_{\{R\}} \log N(T_0) \frac{|E(R)|}{|E(R)|^2 - 4} + O\left(t^{-1/2} \exp\left(-c/t\right)\right) = \frac{E_{\Gamma}^\alpha}{\sqrt{4\pi t}} + O(\sqrt{t}) \quad \text{as} \quad t \to 0^+.
$$

**Lemma 4.3.** Let $\Gamma$ be a cocompact Kleinian group with $\alpha \in \text{Comm}(\Gamma) \setminus \Gamma$. Then there exists a constant $c_0 > 1$ so that $N(T) > c_0$ for all conjugacy class $\{T\} \in \{T\}_{\text{lox}}$.

**Proof.** Recall that $\{T\}_{\text{lox}}$ is the set of conjugacy classes of loxodromic elements of $\Gamma \alpha^{-1} \Gamma$, where conjugacy is taken with respect to the group $\Gamma$. An application of the Selberg trace formula, for the pair of functions

$$
h(z) = \exp(-zt), \quad g(r) = \frac{\exp(-t)}{\sqrt{4\pi t}} \exp\left(-\frac{r^2}{4t}\right),
$$


yields
\[ (4.2) \sum_{m \in D} \omega_m e^{-t \lambda_m} = \]
\[ \frac{\exp(-t)}{\sqrt{4\pi t}} \sum_{(R)_{\text{al}}} |E(R)| |(\text{tr}(R))^2 - 4| \frac{\exp(-t)}{\sqrt{4\pi t}} \sum_{(T)_{\text{lo}} x} |E(T)| |a(T) - a(T)^{-1}|^2 \log N(T_0). \]

For \( t > 0 \) the left hand side of (4.2) converges absolutely. Fix \( t = 1 \). Now, the sum over loxodromic terms is comprised of positive terms. Hence each term of the form
\[ \exp \left( - \frac{(\log N(T))^2}{4} \right) \log N(T_0) \]

must be bounded by a constant independent of \( T \) and \( T_0 \). Since \( \Gamma \) is a discrete group, there exists a constant \( d_0 > 1 \) so that \( N(T_0) > d_0 \) for all loxodromic \( T_0 \in \Gamma \). Hence \( \log N(T_0) \) is bounded above zero, uniformly (since \( T_0 \in \Gamma \). It thus follows that \( |a(T) - a(T)^{-1}| \) must be uniformly bounded above zero. In other words, there is a constant \( c_0 > 1 \), independent of \( T \), with \( |a(T)| > c_0 \). Noting that \( N(T) = |a(T)|^2 \), the Lemma follows.

Theorem 4.2 immediately implies:

**Theorem 4.4.** Let \( \Gamma \) be a cocompact Kleinian group with \( \alpha \in \text{Comm}(\Gamma) \setminus \Gamma \). Suppose that the set \( \Gamma \alpha^{-1} \Gamma \) contains no elliptic elements. Then
\[ \sum_{m \in D} \omega_m e^{-\lambda_m t} = O \left( t^{-1/2} \exp(-c/t) \right) \text{ as } t \to 0^+. \]

Hence
\[ \lim_{t \to 0^+} \sum_{m \in D} \omega_m e^{-\lambda_m t} = 0. \]

Here \( \{\lambda_m\}_{m \in D} \) are the eigenvalues of \( \Delta \) counted with multiplicity, and \( \omega_m \) are the eigenvalues of \( \mathcal{M} \) (the Hecke operator associated to \( \alpha \)) with the convention of Equation 2.6.

### 5. Analogues of Huber’s Theorem for Hecke Operators

Following [EGM98, p. 202], we define the length spectrum of \( \Gamma \alpha^{-1} \Gamma \). For loxodromic \( T_j \in \Gamma \alpha^{-1} \Gamma \) set \( \mu_j = \log N(T_j) \). The length spectrum of \( \Gamma \alpha^{-1} \Gamma \) is defined to be (see [3] for the notation)
\[ L_\Gamma^\alpha = \left( \mu_j, \sum_{(T)_{\text{lo}} x \mu_j} \log N(T_0) |E(T)| |a(T) - a(T)^{-1}|^2 \right)_{j \geq 1}. \]

In the two-dimensional case the length spectrum is simply comprises the lengths of closed geodesics. Here we really need the complex lengths \( a(T) \) and the order of the elliptic, finite subgroup of \( \mathcal{C}(T) \), \( |E(T)| \).

We define the eigenvalue spectra of \( \Delta \) and \( \mathcal{M} \) (\( \mathcal{M} \) is defined from \( \alpha \in \text{Comm}(\Gamma) \setminus \Gamma \)) by
\[ S_\Gamma^\alpha = (\lambda_j, \omega(\lambda_j))_{j \in D^*}; \]
here the symbol $D^*$ means that we do not count with multiplicity: recall that $\omega(\lambda)$ is the trace of $M$ on the invariant subspace generated by all eigenfunctions (of $\Delta$) with eigenvalue $\lambda$, so multiplicity is already encoded into $\omega(\lambda)$.

**Theorem 5.1.** Let $\Gamma, \Gamma'$ be cocompact Kleinian groups with $\alpha \in \text{Comm}(\Gamma) \setminus \Gamma$, $\alpha' \in \text{Comm}(\Gamma') \setminus \Gamma'$. Then the following hold:

1. Suppose that $S^\alpha_\Gamma$ and $S^\alpha'_{\Gamma'}$ agree up to at most finitely many terms. Then
   
   $$E^\alpha_\Gamma = E^\alpha'_{\Gamma'},$$
   $$S^\alpha_\Gamma = S^\alpha'_{\Gamma'},$$
   $$L^\alpha_\Gamma = L^\alpha'_{\Gamma'}.$$  

2. Suppose that $L^\alpha_\Gamma$ and $L^\alpha'_{\Gamma'}$ agree up to at most finitely many terms. Then
   
   $$E^\alpha_\Gamma = E^\alpha'_{\Gamma'},$$
   $$S^\alpha_\Gamma = S^\alpha'_{\Gamma'},$$
   $$L^\alpha_\Gamma = L^\alpha'_{\Gamma'}.$$  

**Proof.** Upon applying the Selberg trace formula (with trivial representations) to the pair of functions,

$$h(w) = \frac{1}{s^2 + w - 1} - \frac{1}{B^2 + w - 1},$$

$$g(x) = \frac{1}{2s}e^{-sx|x|} - \frac{1}{2B}e^{-B|x|},$$

where $1 < \text{Re}(s) < \text{Re}(B)$, we obtain

$$\frac{1}{2s} \sum_{\{T\}_{\text{int}}} \log N(T_0) m(T)|a(T) - a(T)|^{-1} \sum_{\{T\}_{\text{int}}} \frac{\log N(T_0)}{m(T)|a(T) - a(T)|^{-1}} N(T)^{-2} N(T)^{-B} = \sum_{n \in D^*} \omega(\lambda_n) \left( \frac{1}{s^2 - s^2_n} - \frac{1}{B^2 - s^2_n} \right) - \left( \frac{1}{2s} - \frac{1}{2B} \right) \sum_{\{R\}_{\text{int}}} \frac{\log N(T_0)}{|E(R)||\text{tr}(R)|^2 - 4}.$$  

Here $s^2_n = 1 - \lambda_n$ and $s_n$ is chosen to lie in $\{z \in \mathbb{C} \mid \text{Im} z \geq 0, \text{Re} z \geq 0 \}$. The Theorem follows by comparing asymptotics. Simply apply the proof of the [EGM98, Theorem 3.3, p. 203] replacing [EGM98, Equation 5.2.35, p. 198] with our (5.1). Note that our symbol $w(\lambda_n)$ is the analogue of the multiplicity of $\lambda_n$ as considered in [EGM98]. □

By applying the above Theorem to the case of a single fixed group with two different elements $\alpha, \alpha' \in \text{Comm}(\Gamma) \setminus \Gamma$, we obtain:

**Corollary 5.2.** Let $\Gamma$ be a cocompact Kleinian groups with $\alpha, \alpha' \in \text{Comm}(\Gamma) \setminus \Gamma$. Let $\omega(\lambda_m) = \omega'(\lambda_m)$ for all but at most finitely many $m \in D^*$. Then $\omega(\lambda_m) = \omega'(\lambda_m)$ for all $m \in D^*$. 

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