Fermi gas response to non-adiabatic switching of external potential

Y. Adamov B. Mouzykantskii

Department of Physics, University of Warwick, Coventry, CV4 7AL, England

Abstract

We compute analytically the distribution function $P(E)$ for the energy $E$ acquired by a Fermi gas after being subjected to an arbitrary time-dependent external potential (switching event). We relate the distribution function to a solution of a matrix Riemann-Hilbert problem and present explicit formulae for the low order cumulants of $P(E)$. These general results are used to find the distribution of dissipated energy in a biased quantum point contact.

1 Introduction

A rapid switching of an impurity potential in a Fermi gas causes creation of electron-hole pairs which carry away some energy (so called “shake-up” process). This non-trivial many body effect was first studied more than 30 years ago in relation to the X-ray absorption in metals [1, 2]. In this case the perturbing potential appear instantly (i.e. the switching time $T_s$ is of the order of the inverse conduction band width $D$) and the distribution $P(E)$ of energy $E$ carried away by electron hole pairs is a power law $P(E) \sim E^{-\alpha}$ in the threshold region $E \ll \hbar/T_s \sim D$.

It was later found that the same physics is important in the absorption of ions by metallic surfaces [3, 4]. In this case the perturbing potential changes slowly (i.e. $T_s \gg \hbar/D$). The distribution $P(E)$ was found theoretically for arbitrary energies [3, 4] (the result is also quoted in Eq. (7) below) provided that the potential $V(r,t)$ only causes $s$-wave scattering.

Recent advances in fabrication technology have made it possible to subject a Fermi gas to potentials with non-trivial time and space dependence. For example such potentials are used in adiabatic electron pumps (see e.g. [5]). While the distribution of the pumped charge has received a lot of attention recently [5, 6, 7, 8] the closely related distribution of dissipated energy is still unknown. In this case the time-dependent potential $V(r,t)$ changes slowly in time (as in the case of ion absorption) but neither the results nor the methods from [4] are applicable since other (non $s$-wave) scattering channels are important. Finding the distribution $P(E)$ for an arbitrary (slow) time-dependent potential turns out to be a non-trivial problem and some novel ideas are required to solve it.

We study the ideal Fermi gas in the presence of a time-dependent potential $V = V(r,t)$ assuming that the potential is zero in the distant past ($t \to -\infty$) and
becomes time independent in the distant future. We compute the probability $P(E)$ that the gas acquires energy $E$ after this switching process. The results are expressed through the scattering matrix $S(E_F, t) \equiv S(t)$ at the Fermi energy $E_F$ on the instantaneous value of the potential $V(r, t)$. It is more convenient to deal with the characteristic function

$$\chi(\lambda) = \int e^{-iE\lambda} P(E) dE$$

rather than with the probability distribution itself. In the rest of the paper we focus on the case when the scattering matrix changes slowly on the scale of Wigner’s delay time, i.e.

$$S^\dagger \frac{\partial S}{\partial E} S^\dagger \frac{\partial S}{\partial t} \ll 1. \tag{2}$$

Here and everywhere below we use units with $\hbar = 1$. We start by briefly summarising the facts which are already established in the literature (Eqs. (3 – 7) below).

At low energies $E \ll 1/T_s$ the distribution function has the power law asymptote \[1, 9\]

$$P(E) = C(E - \Delta E)^\alpha, \quad \alpha = \frac{\text{tr} \ln S(\infty)}{4\pi^2} - 1, \tag{3}$$

where $\Delta E$ is the difference between the ground state energies of the Fermi gas with and without potential $V(r, \infty)$ \[1, 4\] and is given by

$$\Delta E = \frac{i}{2\pi} \int_{-\infty}^{0} dE \text{tr} \ln S(E, \infty). \tag{4}$$

The average absorbed energy $\langle E \rangle = \int P(E) E dE$ and the probability $P_{00}$ to remain in the ground state are given by

$$\langle E \rangle = -\frac{1}{4\pi} \int dt \text{tr}\{(\partial_t S) S^\dagger (\partial_t S) S^\dagger\}, \tag{5}$$

$$P_{00} = \exp\left\{\frac{1}{4\pi^2} \int_{0}^{\infty} \text{tr} \left(D(\omega) D(-\omega)\right) \omega d\omega\right\} \tag{6}$$

respectively \[1, 1\], where $D(\omega) = \int e^{i\omega t} \ln S(t) dt$. When $V(r, \infty) \neq 0$ the integral in \(6\) diverges logarithmically giving rise to the Anderson orthogonality catastrophe \[1, 2, 4\].

Finally, in the “commutative case” when the scattering matrix $S(t)$ can be diagonalised in a time-independent basis (e.g. when there is only $s$-wave scattering) the problem admits a complete solution and the characteristic function is given by \[4, 3\]

$$\ln \chi(\lambda) = -i\Delta E\lambda + \frac{1}{4\pi^2} \int_{0}^{\infty} |D(\omega)|^2 \omega (1 - e^{-i\omega \lambda}) d\omega. \tag{7}$$

To illustrate the above results and compare them with our findings we consider a simple model of a quantum point contact biased by a rectangular voltage...
pulse $V_b(t) = V_0 \theta(t) \theta(T_s - t)$. After a gauge transformation (see Sec. 4 for details) the potential becomes local in space and the scattering matrix takes the form

$$S(t) = \begin{bmatrix} A e^{i \phi(t)} & -B^* \\ B & A^* e^{-i \phi(t)} \end{bmatrix},$$

(8)

where $A, B$ are transmission and reflection amplitudes and $\phi(t) = \int_{-\infty}^{t} eV_b(\tau) d\tau$ is the Faraday flux.

The results (3–7) are of not much use for the simple model introduced above: since the matrices in (8) do not commute with each other at different times, Eq. (7) is not applicable and the only known result is Eq. (5) for the average energy. This result, however, is just a straightforward combination of the Joule law $\langle E \rangle = \int I(t)V_b(t) dt$ with the Landauer-Büttiker formula [13, 14] for the current across the junction $I = \frac{e^2}{2\pi} |A|^2 V_b$.

In this paper we develop a technique for finding distribution function $P(E)$ which allows us

– to obtain the cumulants expansion of $P(E)$ explicitly and to compute the second and third cumulants for an arbitrary switching process (see Eqs. (24, 25))

– to relate $P(E)$ to a solution of a matrix Riemann-Hilbert (RH) problem (Eq. (29))

– to find the distribution function of energy dissipation in the quantum point contact biased by a long rectangular pulse (see below).

We finish the introduction by illustrating the relation between $P(E)$ and the statistics of transmitted charge. Consider a quantum point contact described above. When the rectangular pulse is long i.e. $T_s \gg 1/(eV_0)$ we solve the RH problem asymptotically to logarithmic accuracy in $n = eV_0 T_s / (2\pi) \gg 1$; this gives

$$\ln \chi(\lambda) = n \ln(|A|^2 e^{-ieV_0 \lambda} + |B|^2) +$$

$$+ \frac{1}{2\pi^2} \ln \left( \frac{2\pi n}{eV_0 \lambda} \right) \ln^2(|A|^2 e^{-ieV_0 \lambda} + |B|^2),$$

(9)

where we assume $eV_0 \lambda \ll n$. Since each electron reaching the contact contributes $eV_0$ to the total dissipated energy if it is transmitted (probability $|A|^2$) and contributes nothing if it is reflected (probability $|B|^2$) we expect at least for long times $T_s$ that characteristic function (8) can be obtained from the distribution of transmitted charge. The latter is known to be binomial [15]

$$B(N, k) = \frac{N!}{k!(N - k)!} |A|^{2k} |B|^{2(N-k)},$$

(10)

where $B(N, k)$ is the probability that out of $N$ incident electrons $k$ are transmitted. The distribution $p(N)$ of the number of incident electrons can be found from the energy dissipation of a completely open contact (i.e. one with $A = 1$). The mean number of electrons transmitted through an open contact is given by $\langle N \rangle = n$, and the distribution $p(N)$ is Gaussian in the vicinity of its maximum (see Sec. 4.1 for details)

$$p(N) \sim e^{-\pi^2 (N - n)^2 / (4 \ln n)} \quad \text{at} \quad n - N \sim 1.$$

(11)
There are therefore two sources of shot noise in a quantum point contact – one is the fluctuations in the number of incident electrons and the other is the fluctuations in the number of reflected electrons. We are now able to establish that these two sources are statistically independent by observing that the characteristic function (9) corresponds (with logarithmic accuracy) to the convolution of \( p(N) \) and \( B(N, k) \):

\[
P(E) = \sum_{N,k} p(N) B(N, k) \delta(E - eV_0 k).
\]

The physical picture of the two statistically independent sources of quantum shot noise was suggested (but not proven) in [16]. Note, that in order to confirm the statistical independence we need to obtain the sub-leading term in the large \( n \) expansion of \( \chi(\lambda) \) (namely the term proportional to \( \ln n \) in Eq (9)) and analyse its dependence on the transmission amplitude \( A \). We are not aware of any other technique capable of obtaining this term for a partially open channel.

In fact, measuring the energy dissipated in a contact in the presence of a rectangular bias pulse can be viewed as a “quantum charge measurement scheme” alternative to (and somewhat conceptually simpler than) using a precessing spin as was suggested in [15]. The detailed comparison of the two measurement schemes will be the subject of future work.

2 Overview of the method

Let \( H(t) = H_0 + V(r, t) \) be the time dependent Hamiltonian of the Fermi gas in the external potential, where \( H_0 \) is the Hamiltonian of the free Fermi gas.

If \( V(r, \infty) \neq 0 \) then there are two contributions to the energy of the gas at \( t \to \infty \). The first is the sum of the energies of the electron-hole pairs and the second originates from the adiabatic shift of the single particle energy levels. If the energy levels in the vicinity of the Fermi surface shift uniformly by \( V(r, \infty) \) then we can adiabatically switch off the potential \( V(r, \infty) \) preserving the energy of existing electron-hole pairs and creating no additional ones. Therefore to account for \( V(r, \infty) \) we should add \(-i\Delta E\lambda \) to the characteristic function \( \chi(\lambda) \). We assume \( V(r, t > T_s) = 0 \) in the rest of the paper.

The evolution operator \( U_t \) which relates the many-body wave function \( |\Psi(t)\rangle \) to the the ground state wave function \( |\Psi_0\rangle \) via \( |\Psi(t)\rangle = U_t|\Psi_0\rangle \) obeys the Schrödinger equation

\[
i \frac{d}{dt} U_t = H(t) U_t.
\]

The characteristic function (11) is given by the average

\[
\chi(\lambda) = \lim_{t \to \infty} \langle \Psi_0 | U_t^{-1} e^{-iH_0 \lambda} U_t e^{iH_0 \lambda} |\Psi_0\rangle.
\]

Usually such averages are evaluated by expanding both \( U \) and \( U^{-1} \) in the powers of \( V(r, t) \) and by evaluating the result using the Keldysh diagram technique (see [3] for application of this method in the commutative case). We follow a different approach suggested in [3] and separate the single-particle scattering problem from the averaging over the Fermi-gas ground state. The scattering problem
consists of finding the electron annihilation operator $\hat{c}_k(t)$ in the Heisenberg representation

$$\hat{c}_k(t) = U_t^{-1} c_k U_t = e^{-i\epsilon_k t} \sum_{k'} \sigma_{kk'}(t) c_{k'},$$

(15)

where $\epsilon_k$ is the energy of the electron with momentum $k$. The matrix $\sigma_{kk'}(t)$ becomes time independent at $t > T_s$ when the potential is switched off. Only this limiting value is needed for computation of the characteristic function (14).

It can be expressed as a Fourier component of the scattering matrix

$$\sigma_{kk'}(\infty) = \frac{\pi^{d/2-1}}{\Gamma(\frac{d}{2}) \nu(E)} \int dt S_{nn'}(E,t) e^{i(\epsilon_k - \epsilon_{k'}) t}$$

(16)

where $\nu(E)$ is the density of states, $d$ is the number of space dimensions, $n = k/|k|$ and $E = (\epsilon_k + \epsilon_{k'})/2$. Eq. (16) is the first term in the expansion in $(S^\dagger \frac{\partial S}{\partial E} S^\dagger)\frac{\partial S}{\partial t}$ and is valid as long as $(\epsilon_k - \epsilon_{k'}) S^\dagger \frac{\partial S}{\partial t} \ll 1$. The detailed calculation is presented in appendix $[A]$.

Using (15), (16), we can compute the limiting value of the Hamiltonian in the Heisenberg representation

$$\hat{H}(t) = U_t^{-1} H_0 U_t = \sum_k \epsilon_k \hat{c}_k \hat{c}_k + \sum_{k,k'} \epsilon_k W_{kk'} \hat{c}_k \hat{c}_{k'},$$

(17)

where

$$W_{kk'} = \frac{\pi^{d/2-1}}{\Gamma(\frac{d}{2}) \nu(E)} \int i S^\dagger(E,t) \frac{\partial S(E,t)}{\partial t} e^{i(\epsilon_k - \epsilon_{k'}) t} dt.$$  

(18)

Now the characteristic function (14) acquires the form of a time independent expectation value and can be evaluated using a linked-cluster expansion (see appendix $[B]$ for details)

$$\ln \chi(\lambda) = \frac{i\lambda}{4\pi} \int dt \text{tr}\{(\partial_r S) S^\dagger (\partial_t S) S^\dagger\} + \mathcal{W}$$

(19)

where $\mathcal{W}$ is the sum of the ring diagrams starting with the second order term

$$\mathcal{W} = -\frac{1}{2} \bigcirc - \frac{1}{3} \bigcirc \cdots$$

(20)

Here the Green’s function is

$$G(p,\Delta \tau) = -ie^{i\tau} \left( \theta(-p) \theta(\Delta \tau) - \theta(p) \theta(-\Delta \tau) \right)$$

(21)

and the vertex is given by $A(p - p', \tau) = M(p - p') \theta(\tau - \theta(p - \lambda)$ where

$$M(p - p') = i \int S^\dagger(t) \frac{\partial S}{\partial t} e^{-i(p - p') t} dt.$$  

(22)

We integrate over $\tau$ and sum over matrix indices of $M(p - p')$ in each vertex and sum over $p$ in each line.
The diagram expansion (20) makes the computation of the $n$-th order cumulant

$$\langle \langle E^n \rangle \rangle = i^n \frac{\partial^n \ln \chi(\lambda)}{\lambda^n} |_{\lambda=0}$$  \hspace{1cm} (23)

relatively straightforward. Indeed, since the $n$-th diagram is proportional to $\lambda^n$ at small $\lambda$ no more than $n$ terms is needed for the $n$-th cumulant. This gives the following results for the second and the third cumulants

$$\langle \langle E^2 \rangle \rangle = \frac{1}{4\pi^2} \int_0^\infty \text{tr} \left( M(p)M(-p) \right) p dp$$  \hspace{1cm} (24)

$$\langle \langle E^3 \rangle \rangle = \frac{1}{(2\pi)^3} \int_0^\infty \min(p_1,p_2) \text{tr} \left( M(p_2 - p_1)[M(-p_2), M(p_1)] \right) d p_1 d p_2 +$$

$$+ \frac{1}{4\pi^2} \int_0^\infty \text{tr} \left( M(p)M(-p) \right) p^2 dp.$$  \hspace{1cm} (25)

Finding the characteristic function $\chi(\lambda)$ is a much more difficult problem which we relate to the RH problem described below. We start by introducing the matrix valued function $S_+(t,y)$ which arbitrary interpolates between 1 at $y = 0$ and $S(t + \lambda)$ at $y = 1$

$$S_+(t,1) = S(t + \lambda), \quad S_+(t,0) = 1$$  \hspace{1cm} (26)

and analogously the matrix valued function $S_-(t,y)$ interpolating between 1 and $S(t)$

$$S_-(t,1) = S(t), \quad S_-(t,0) = 1.$$  \hspace{1cm} (27)

It is also convenient to define the new variables $z = \tau + i t$ and $\bar{z} = \tau - i t$. For a fixed $0 < y < 1$ the RH problem consists of finding two matrix valued functions $f_\pm(z,y)$ that satisfy the following conditions:

(i) $f_+(\bar{z},y)$ is antianalytic when $\text{Re} \bar{z} > 0$

(ii) $f_-(\bar{z},y)$ is antianalytic when $\text{Re} \bar{z} < 0$

(iii) $f_\pm(\infty,y) = 1$

(iv) for real $t$

$$f_+(\tau + 0,y)f_-(\tau - 0,y)^{-1} = Q(t,y),$$  \hspace{1cm} (28)

where $Q = S_+^{-1} S_-$. Once the solution of the RH problem (or approximation to it) is found the characteristic function of dissipated energy is given by the two-dimensional integral

$$\ln \chi(\lambda) = \frac{i}{4\pi} \int \text{tr} \left\{ dS_+ S_+^{-1} \wedge dS_- S_-^{-1} \right\}$$

$$- \frac{i}{4\pi} \int \text{tr} \left\{ Q^{-1} dQ \wedge df^{-1}_- \right\},$$  \hspace{1cm} (29)

where $\wedge$ is the wedge product, the integrals are over the strip ($0 < y < 1$, $-\infty < t < +\infty$) and the orientation is chosen in such a way that $\int dy \wedge dt$ is positive.

The difficult step in the outlined procedure is the solution of the RH problem. We consider two special cases. In the quantum point contact case with a particular $t$-dependence of $S$-matrix the RH problem can be solved asymptotically.
The “commutative” case when the $S$ matrix can be diagonalised in some time-independent basis is discussed below. For pure $s$-wave scattering the $S$ matrix has only one nontrivial eigenvalue $S(t) = e^{-2i\delta_V(t)}$, where $\delta_V(t)$ is the phase shift at the Fermi energy. The addition of more scattering channels is straightforward. Formulating (28, 29), we choose the interpolating functions $S_{\pm}$ in the form

$$
S_-(t, y) = e^{-2iy\delta_V(t)} , \quad S_+ = e^{-2iy\delta_V(t+\lambda)}
$$

which gives $Q = e^{-2iy(\delta_V(t) - \delta_V(t+\lambda))}$. Employing the method from [17] we obtain the solution

$$
\ln f_\pm(\bar{z}, y) = \frac{iy}{\pi} \int dt_1 \frac{\delta_V(t_1) - \delta_V(t_1 + \lambda)}{\bar{z} + i\lambda_1 \pm 0}.
$$

Substituting (31, 30) into (29) gives (7).

Equations (24), (25), (29) and Eq. (9) from the introduction are the main results of the paper.

3 Wess-Zumino action and the Riemann-Hilbert problem

In this section we employ the Wess-Zumino action to find $\mathcal{W}$ given by the diagram series (20). Firstly, we show that the functional derivative of $\mathcal{W}$ with respect to the vertex $A$

$$
J = 2i \frac{\delta \mathcal{W}}{\delta A} = -2iGAG - 2iGAGAG - \cdots
$$

satisfies the equation for the current in the Wess-Zumino model. Observe that the Green’s function $G(p, \tau)$ given by Eq. (21) obeys the same equation as the Green’s function of a massless left-moving particle in one dimension, $p$ being the momentum of the particle and $x$ is the conjugate variable to $p$

$$
(\partial_\tau - \partial_x)G(x - x', \tau - \tau') = -i\delta(x - x')\delta(\tau - \tau').
$$

Note, that in the coordinate representation the vertex $A$ takes the form

$$
A(\tau, x) = iS^\dagger \frac{dS}{dx} (\theta(\tau) - \theta(\tau - \lambda)).
$$

Using (33) we get for the $n$-th order ($n > 2$) of the perturbation expansion (32)

$$
(\partial_\tau - \partial_x)j_n = -i[A, j_{n-1}].
$$

The second order diagram can be presented in the form

$$
-\frac{1}{2} \int \frac{dp}{2\pi} \frac{d\omega}{2\pi} \Pi(p, \omega)A(p, \omega)A(-p, -\omega)
$$

where $\Pi(p, \omega) = \frac{-ip}{2\pi(\omega + p - \delta\text{sign } p)}$ is a polarisation operator. Varying this equation over $A$ we have the following equation for $j_2$

$$
(\partial_\tau - \partial_x)j_2 = -\frac{1}{\pi} \partial_x A.
$$
Combining (33) and (36) we arrive at the implicit equation for \( \mathcal{W} \)

\[
(\partial_\tau - \partial_x) j + i[A,j] = -\frac{1}{\pi} \partial_x A; \quad j = 2i \frac{\delta \mathcal{W}}{\delta A}.
\]  

(37)

This equation is very similar to the equation for currents in the Wess-Zumino model. To exploit this analogy we introduce the functional

\[
\tilde{\mathcal{W}} = \mathcal{W} + \frac{i\lambda}{8\pi} \int \text{tr}(\partial_xSS^{-1})^2 dx = \mathcal{W} - \frac{i}{8\pi} \int A^2(x,\tau) d\tau d\tau
\]

which obeys the implicit equation

\[
(\partial_\tau - \partial_x) \tilde{j} + i[A,\tilde{j}] = -\frac{1}{2\pi} (\partial_\tau + \partial_x) A; \quad \tilde{j} = 2i \frac{\delta \tilde{\mathcal{W}}}{\delta A}.
\]  

(39)

Polyakov and Wiegman (PW) \[18\] solved (39) for the case when \( A \) has the form

\[ A = i(\partial_\tau - \partial_x)gg^{-1}; \quad g|_{\tau\to\infty, x\to\infty} = 1, \]

(40)

where \( g \) takes values in \( SU(N) \). Under these conditions Eq. (39) has the solution

\[
\tilde{\mathcal{W}}(A) = -\frac{i}{8\pi} \int d\tau d\tau \text{tr}\{\partial_\mu gg^{-1}\partial_\mu gg^{-1}\} - \frac{i}{12\pi} \int d\tau d\tau d\tau d\tau \text{tr}\{\partial_\mu gg^{-1}\partial_\nu gg^{-1}\partial_\lambda gg^{-1}\}.
\]  

(41)

In the second term in (41) the integration is over a membrane in the target space with boundary \( g(\tau, x) \). The current is given by \( \tilde{j} = \frac{\delta}{\delta A}(\partial_\tau + \partial_x)gg^{-1} \).

In our case the PW procedure is not applicable because the vertex \( A \) does not admit representation (40). This can be demonstrated by using Eq. (32) for the current and observing that it does not decay at \( x, \tau \to \infty \), in contradiction to the PW prediction. For example, the second term in (32) gives \( j_2 = 2i \int_{[x-x', \tau-\tau', -\infty, -\infty]} A(\tau', x') dx' d\tau' \), which is not zero if \( x \to \infty, \tau \to \infty \) along the line \( x = -\tau \).

Boundary conditions that are more general than (40) were considered in \[19, 20\] using a rather involved method. Instead, we overcome the problem by rotating the contour of integration over the energy \( \epsilon \) in diagram expansion \[21\] (the procedure is clearer in the energy representation) and take \( \epsilon = i\xi \) where \( \xi \) is real and varies from \(-\infty \to \infty \). As the Fourier transform of \( [\theta(\tau) - \theta(\tau - \lambda)] \) is \( \frac{e^{i\epsilon(\tau')\lambda}}{i(\tau'-\tau)} \), we also need to rotate \( \lambda \) to keep the integrals convergent. In this way we obtain \( \chi(\lambda) \) at \( \lambda = -i\alpha \); where \( \alpha \) is real and positive. The characteristic function for a real \( \lambda \) is obtained using analyticity of \( \chi(\lambda) \) in the lower half-plane. The vertex in the coordinate representation \[21\] is \( A(\tau, x) = iS(\chi) \frac{\delta S(x)}{\delta x} \theta(\tau + \alpha) - \theta(\tau) \) and the rotated Green’s function is \( G(x-x', \tau-\tau') = \frac{1}{i(\tau'-\tau)} \frac{1}{i(x-x')} \). This Green’s function satisfies equation

\[ 2\partial_\tau G = \frac{\delta (x-x')}{\delta x} \delta (\tau-\tau') \]

where we denote \( z = \tau + ix \).

Similar arguments as above lead to the equation for the current \( \tilde{j} = 2i \frac{\delta \tilde{\mathcal{W}}}{\delta A} \)

(note that the current now goes to zero as \( z \to \infty \))

\[ 2\partial_x \tilde{j} - [A, \tilde{j}] = \frac{i}{\pi} \partial_z A. \]

(42)
Equation (40) now takes the form
\[ 2\partial_z g g^{-1} = A = iS^\dagger(x) \frac{\partial S(x)}{\partial x} (\theta(\tau + \alpha) - \theta(\tau)) \] (43)
and has the solution with \( g \in GL(N, \mathbb{C}) \) satisfying \( g|_{z \to \infty} = 1 \),
\[ g(\tau, x) = \begin{cases} f(\bar{z}), & \tau < -\alpha, \tau > 0 \\ S^\dagger(x) f(\bar{z}), & -\alpha < \tau < 0. \end{cases} \] (44)
The conditions on the function \( f(\bar{z}) = f(\tau - ix) \) are as follows
(i) \( f \) is antianalytical in the complement of the two vertical (i.e. parallel to \( x \) axis) branch cuts at \( \tau = 0 \) and \( \tau = -\alpha \)
(ii) \( f(\infty) = 1 \)
(iii) Values of \( f \) on the opposite sides of the cuts are related by
\[ f(-0 - \alpha - ix) = S^\dagger(x) f(+0 - \alpha - ix) \]
\[ S^\dagger(x) f(-0 - ix) = f(+0 - ix). \] (45)
Conditions (i)-(iii) describe a matrix Riemann-Hilbert problem. Once the decomposition (43) is found the solution of (42) is given by the Wess-Zumino functional
\[ \tilde{\mathcal{W}}(A) = -\frac{1}{8\pi} \int d\tau \text{tr}\{\partial_\mu gg^{-1} \partial_\mu gg^{-1}\} - \frac{i}{12\pi} \int d\tau \text{dye}^{\mu
u\lambda} \text{tr}\{\partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\lambda gg^{-1}\}. \] (46)
In the second integral \( g \) is continued on the membrane that has \( g(\tau, x) \) as a boundary. We choose the continuation \( g(\tau, x, y) \) with \( 0 < y < 1 \) based on an arbitrary interpolation \( S(x, y) \) between \( S(x) = S(x, 1) \) and \( 1 = S(x, 0) \)
\[ g(\tau, x, y) = \begin{cases} f(\bar{z}, y), & \tau < -\alpha, \tau > 0 \\ S^\dagger(x, y) f(\bar{z}, y), & -\alpha < \tau < 0. \end{cases} \] (47)
The function \( f(\bar{z}, y) \) has the same branch cuts and behavior at infinity as \( f(\bar{z}) \) and obeys (45) at each value of \( y \)
\[ f(-0 - \alpha - ix, y) = S^{-1}(x, y) f(+0 - \alpha - ix, y) \]
\[ S^{-1}(x, y) f(-0 - ix, y) = f(+0 - ix, y). \] (48)
Substituting (47) into (46) and taking into account (38) we get (see appendix C for details)
\[ \mathcal{W} = -\frac{i}{4\pi} \int_{\beta_1} \text{tr}\{dSS^{-1} df(-\alpha + 0 - ix, y) f^{-1}\} - \frac{i}{4\pi} \int_{\beta_2} \text{tr}\{dSS^{-1} df(-0 - ix, y) f^{-1}\} - \frac{\alpha}{4\pi} \int dx \text{tr}\{\partial_x S(x) S^{-1}(x) \partial_x S(x) S^{-1}(x)\}. \] (49)
where \( \beta_1 \) is the plane \( \tau = -\alpha \) and \( \beta_2 \) is the plane \( \tau = 0 \). Note that the last term in (49) exactly cancels the contribution from the first order diagram in (19).
The only remaining step is the analytic continuation of (49) from real positive \( \alpha \) to \( \alpha = i\lambda \). The method of the continuation is illustrated on Fig. 1.
Figure 1: Branch cuts for the continuation of \( f(\bar{z},y) \) to the complex values of \( \alpha \).

Take the complex value \( \alpha = \alpha' + i\alpha'' \) and obtain the continuation of (47) in the form

\[
g(\tau,x,y) = \begin{cases} 
  f(\bar{z},y), & \tau < -\alpha', \tau > 0 \\
  S^{-1}(x + \frac{\alpha''}{\alpha'} \tau, y) f(\bar{z},y), & -\alpha' < \tau < 0 
\end{cases}
\]

(50)

where

\[
  f(-0 - \alpha' - ix, y) = S^{-1}(x - \alpha'', y) f(+0 - \alpha' - ix) \\
  S^{-1}(x, y) f(-0 - ix, y) = f(+0 - ix, y). 
\]

(51)

In the limiting case \( \alpha = i\lambda \) the surfaces \( \tau = -\alpha' \), \( \tau = 0 \) coincide and (49) takes the form (29).

4 Energy dissipation in a quantum point contact

In this section we apply Eq. (29) to a biased quantum point contact. The contact is described by the Hamiltonian

\[
  H = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + U(x), 
\]

(52)

where the potential \( U(x) \) is local in space and therefore the scattering matrix

\[
  S_0 = \begin{bmatrix} A & -B^* \\ B & A^* \end{bmatrix} 
\]

(53)

is almost energy independent. The presence of the bias potential \( V_b(t) \) is taken into account by adding the term \( eV_b(t)\theta(x) \) to the Hamiltonian (52). The gauge transformation \( \psi(x,t) \to \psi(x,t)e^{-i\phi(t)\theta(x)} \) where \( \phi(t) = \int_{-\infty}^{t} eV(\tau)d\tau \) makes the Hamiltonian local in space

\[
  H(t) = \frac{1}{2} \left( -i \frac{\partial}{\partial x} - \delta(x)\phi(t) \right)^2 + U(x) 
\]

(54)

with the scattering matrix given by (53).
In what follows, we consider the rectangular bias pulse with amplitude $V_0$ and duration $T_s$. The flux $\phi(t)$ is given by

$$\phi(t) = \begin{cases} 
0, & t < 0 \\
VeV_0t, & 0 < t < T_s \\
eV_0T_s, & t > T_s.
\end{cases} \quad (55)$$

The RH problem for the scattering matrix (8) can be solved in two limiting cases: when the contact is transparent (i.e. does not reflect electrons) and when the time $T_s$ is large.

### 4.1 Transparent contact

In the case of transparent contact the scattering matrix (8) acquires the form

$$S(t) = \begin{bmatrix} e^{i\phi(t)} & 0 \\
0 & e^{-i\phi(t)} \end{bmatrix}. \quad (56)$$

Since it remains diagonal at all times Eq. (3) can be used, which gives

$$\ln \chi(\lambda) = -\left(\frac{eV_0}{2\pi}\right)^2 \left[(\lambda - T_s)^2 \ln \left(i\frac{\lambda}{T_s} - i\right) + (\lambda + T_s)^2 \ln \left(i\frac{\lambda}{T_s} + i\right) - 2\lambda^2 \ln \left(i\frac{\lambda}{T_s}\right)\right]. \quad (57)$$

Classically all electrons in the energy strip $(E_F, E_F - eV_0)$ that can reach the contact are transmitted. The average number of such electrons is $n = \nu eV_0 v_F T_s = eV_0 T_s / (2\pi)$, where $v_F$ is a Fermi velocity and $\nu = 1/(2\pi v_F)$ is the density of states per unit length. Since each transmitted electron contributes energy $eV_0$ to the total dissipated energy it is convenient to introduce a new rescaled distribution function

$$P_c(k) = eV_0 P(eV_0k) \quad (58)$$

that measures energy in the units of $eV_0$. The new distribution $P_c(k)$ depends only on the parameter $n$. We plot $P_c(k)$ for three different values of $n$ in Fig. 2. For small $n$ the distribution function has a divergence at small $k$,

$$P_c = C_n/k^{1-2n^2} \text{ at } k \ll 1/n, \quad (59)$$

which is the Nozieres and De Dominicis asymptote (3) with the prefactor $C_n$ given by

$$C_n = 2n \sin(2\pi n^2) e^{-3n^2} \Gamma(1 - 2n^2)(2\pi n)^{2n^2 - 1}. \quad (60)$$

In the other limit $n \gg 1$ the distribution looks like Gaussian with mean value $n$ and the standard deviation $\sigma = \frac{1}{2} \sqrt{\ln n}$. The Gaussian approximation is valid only in the region $k - n \sim 1$. The distribution function at $k \gg n$ is determined by the term $\lambda^2 \ln \lambda$ in (57), which leads to the logarithmically divergent second cumulant.
\[
\phi(t) = 2\pi n \quad T \quad t
\]
\[
P_c(k) = \begin{cases} 
0.5 & \text{for } n = 2 \\
0.5 & \text{for } n = 12.2 
\end{cases}
\]

Figure 2: \(P_c(k)\) is the probability that the one-dimensional Fermi gas in the wire acquires the energy \(E = keV_0\) after applying the voltage \(V_0\) during the time \(T_s\). The distribution \(P_c(k)\) is parametrised by a single parameter \(n = eV_0T_s/(2\pi)\), which is the expected number of electrons passed through the contact.

(a) The Faraday flux \(\phi(t)\) as a function of time. The potential switches on at \(t = 0\) and switches off at \(t = T_s\).

(b) \(P_c(k)\) for \(n = 0.4\) shows the Fermi-Edge singularity (the broken curve is the power law asymptote \(P_c = Cn/k^{1-2n^2}\) with \(C_n\) given by Eq. (60).

(c) For \(n = 0.8 > 1/\sqrt{2}\) the behaviour at small \(k\) is also determined by the power law asymptote (broken curve). However, now the charge distribution is not divergent at small \(k\).

(d), (e) \(P_c(k)\) for the larger values of \(n\) crosses over to the Gaussian distribution (shown by the broken curve) with the mean value \(n\) and the standard deviation \(\sigma = \frac{1}{2}\sqrt{\ln n}\).

4.2 Large \(T_s\) asymptote of \(\chi(\lambda)\)

For a semi-transparent contact we can still solve Eq. (28) asymptotically when \(T_s \gg 1/(eV_0)\). The matrix \(Q = S^\dagger(x + \lambda, y)S(x, y)\) has the form

\[
Q = \begin{bmatrix}
|A|^2e^{-i(\phi_+ - \phi_-)} + |B|^2 & -A^*B^*(e^{-i\phi_+} - e^{-i\phi_-}) \\
AB(e^{i\phi_+} - e^{i\phi_-}) & |A|^2e^{i(\phi_+ - \phi_-)} + |B|^2
\end{bmatrix},
\]

where \(\phi_+ = y\phi(x + \lambda), \phi_- = y\phi(x)\). In the limit \(\lambda \ll T_s\) it has the simple time dependence

\[
Q = Q_0 + Q_+ e^{ieV_0x} + Q_- e^{-ieV_0x}
\]

where \(Q_0\) is diagonal and \(Q_-(Q_+)\) are upper (lower) triangular matrices, which are almost time-independent. This suggest the following anzats for \(f_{\pm}\)

\[
\begin{align*}
    f_+ &= f_{0+} + f_{1+} e^{ieV_0\xi} \\
    f_- &= f_{0-} + f_{1-} e^{ieV_0\xi}
\end{align*}
\]

(63)
After substituting (63) into (28) we get the approximate solution

\[ f_+ = (1 + Q_+ Q_c^{-1} e^{-y e V_0 \bar{z}}) f_{0+} \]
\[ f_- = (1 - Q_0^{-1} Q_- e^{y e V_0 \bar{z}}) f_{0-} \]  

(64)  
(65)

where \( f_{0+}, f_{0-} \) are diagonal matrices satisfying the equation

\[ f_{0+} f_{0-}^{-1} = Q_c \]  

(66)

with

\[ Q_c = Q_0 - Q_+ Q_0^{-1} Q_- = \left( \begin{array}{cc} |A|^2 e^{-i(\phi_+ - \phi_-)} + |B|^2 & 0 \\ 0 & \frac{1}{|A|^2 e^{-i(\phi_+ - \phi_-)} + |B|^2} \end{array} \right) \]  

(67)

The solution for \( \ln f_{0-} \) is given by

\[ \ln f_{0-} = \frac{1}{2\pi i} \int \ln \left( \frac{|A|^2 e^{-i(\phi_+ - \phi_-)} + |B|^2}{x' - x + i0} \right) dx' \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right]. \]  

(68)

Substituting \( f_- = A_- f_{0-} \), where \( A_- = 1 - Q_0^{-1} Q_- \), into (29) we get

\[ \mathcal{W} = \frac{i}{4\pi} \int \text{tr} \left\{ dS_+ S_+^{-1} \wedge dS_- S_-^{-1} \right\} - \frac{i}{4\pi} \int d \left\{ Q^{-1} dQ \wedge dA_- A_-^{-1} \right\} - \frac{i}{4\pi} \int d \left\{ Q^{-1} dQ \wedge A_- (d \ln f_{0-}) A_-^{-1} \right\}. \]  

(69)

The first two integrals are proportional to \( T_s \), the last integral is proportional to \( \ln(T_s/\lambda) \). After long but straightforward algebra we finally get (3).

5 Conclusions

We considered the energy dissipation in a Fermi gas subjected to a perturbation with time dependent scattering matrix of a general form. Result (29) relates the dissipated energy distribution function \( P(E) \) to a solution of the matrix Riemann-Hilbert problem (28).

For a quantum point contact biased by a rectangular voltage pulse we computed \( P(E) \) analytically for long pulse durations and analysed the relation between \( P(E) \) and distribution of transmitted charge. We also obtained second and third cumulants of the dissipated energy for a general switching process.

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A One-particle scattering problem

In this section we relate the electron annihilation operator \( \hat{c}_k \) in the Heisenberg representation to the one-particle scattering matrix \( S(E, t) \) and derive Eqs. (15, 16) from the overview.

Solving the equation for the time evolution of \( \hat{c}_k \)

\[
\dot{\hat{c}}_k = -i\epsilon_k \hat{c}_k - i \sum_{k'} V_{kk'}(t) \hat{c}_{k'}
\]  

(70)

perturbatively we get

\[
\hat{c}_k(t) = e^{-i\epsilon_k t} \left( c_k - i \int_{-\infty}^{t} dt' V_{kk'}(t') e^{i(\epsilon_k - \epsilon_{k'}) t'} c_{k'} + i^2 \int_{-\infty}^{t} dt' V_{kk'}(t') e^{i(\epsilon_k - \epsilon_{k'}) t'} \times \int_{-\infty}^{t'} dt'' V_{kk''}(t'') e^{i(\epsilon_k - \epsilon_{k''}) t''} c_{k''} \ldots \right).
\]  

(71)

In the limit \( t \to \infty \) we get

\[
\hat{c}_k(t_\infty) = e^{-i\epsilon_k t_\infty} \sum_{k'} (\delta_{kk'} - i F_{kk'}(\epsilon_k, \epsilon_{k'})) c_{k'}.
\]  

(72)

Here we introduced the function \( F_{kk'}(\epsilon, \epsilon') \) satisfying the integral equation

\[
F_{kk'}(\epsilon, \epsilon') = V_{kk'}(\epsilon - \epsilon') + \sum_{k_1} \int \frac{d\epsilon_1}{2\pi} V_{kk_1}(\epsilon - \epsilon_1) G_{k_1}(\epsilon_1) F_{k_1 k'}(\epsilon_1, \epsilon'),
\]  

(73)

where \( V_{kk'}(\omega) = \int dt V_{kk'}(t) e^{-i\omega t} \) and \( G_k(\epsilon) = \frac{1}{\epsilon - \epsilon_k + i\eta} \) is the retarded Green’s function. To take into account the slowness of \( V_{kk'}(t) \) we introduce the new variables \( E = (\epsilon + \epsilon')/2 \) and \( \omega = \epsilon - \epsilon' \) and notice that \( F \) varies slowly when \( E \) changes but varies fast when \( \omega \) changes. Using Wigner’s representation \( F_{kk'}(E, t) = \int e^{-i\omega t} F_{kk'}(E, \omega) \frac{d\omega}{2\pi} \) we get the approximate solution of (73)

\[
F_{kk'}(E, t) = V_{kk'}(t) + \sum_{k_1} V_{kk_1} G_{k_1}(E) F_{k_1 k'}(E, t).
\]  

(74)

The first order correction to (74) is given by

\[
\delta F = \frac{i}{2} \frac{\partial V}{\partial t} \frac{\partial G}{\partial E} F + \frac{i}{2} V \frac{\partial G}{\partial E} \frac{\partial F}{\partial t} - i \frac{\partial V}{\partial t} \frac{\partial G}{\partial E} F.
\]  

(75)

Eq. (74) is similar to the equation that one gets in quantum scattering theory (see [22, §130]).

Substituting \( F_{kk'} \) given by (74) into (72) and neglecting the terms of the order \( \frac{\partial V}{\partial t} \frac{\partial G}{\partial E} \) we get (16).

B Linked cluster expansion for \( \chi(\lambda) \)

From Eq. (18) we see that the characteristic function (14) is given by the linked cluster expansion

\[
\ln \chi = ln \langle \Psi_0 | e^{-iH(\infty)\lambda} e^{iH_0\lambda} | \Psi_0 \rangle = - \frac{1}{2} - \frac{1}{3} \cdot \cdot \cdot ,
\]  

(76)
where the solid line is the Green’s function \( G_k(\tau_2 - \tau_1) \) and the vertex is \( W_{kk'}(\theta(\tau) - \theta(\tau - \lambda)) \). An integration with respect to fictitious time \( \tau \) is implied in each vertex and summation with respect to \( k \) in each line.

The causal Green’s function is given by the sum

\[
G_k = G^R_k \theta(\epsilon_k) + G^A_k \theta(-\epsilon_k)
\]

where \( G^R(G^A) \) are the retarded and advanced Green’s function respectively

\[
G^R_k = -ie^{-i\epsilon_k(\tau_2 - \tau_1)}\theta(\tau_2 - \tau_1)
\]

and

\[
G^A_k = ie^{-i\epsilon_k(\tau_2 - \tau_1)}\theta(\tau_1 - \tau_2).
\]

In the diagrams of second and higher orders the main contribution in the sum over \( k \) comes from the region near the Fermi surface. For example, in the second order diagram we have

\[
\begin{array}{c}
\text{Causal Green's function is a sum of advanced and retarded ones.}
\end{array}
\]

since the causal Green’s function is a sum of advanced and retarded ones. The diagrams with only advanced or only retarded lines are equal to zero because of incompatible conditions set out by the time \( \theta \)-functions. In the diagrams with both retarded and advanced lines because \( W_{kk'} \) is large only if \( \epsilon_k - \epsilon_{k'} < \frac{1}{T_s} \) all energies are close to each other and therefore close to the Fermi surface. This allows us to substitute \( S(E_F, t) = S(t) \) for \( S(E, t) \) in Eq. (18). We notice that

\[
\int d^d k (2\pi)^d d\tau = \frac{\Gamma(d/2)}{\pi^{d/2-1}} \int d\epsilon_k \frac{2\pi}{d} \int d\nu.
\]

As the Green’s functions do not depend on direction of \( k \) the integration over \( n = k/|k| \) involves only indexes of scattering matrix in each vertex. This integration can be replaced by summation over scattering channels if we use an appropriate basis for the scattering matrix.

The first diagram in (76) requires special attention because the contributions from all the region under the Fermi surface is significant and the energy dependence of \( S(E, t) \) can not be neglected. We notice, however, that this diagram is proportional to the average absorbed energy, because

\[
- \quad = -i\lambda \langle 0 \vert \hat{H}_\infty - H_0 \vert 0 \rangle = -i\lambda \langle E \rangle
\]

The average absorbed energy was computed in [11] and is given by [12]. Redefining the Green’s function and the vertex in the diagram expansion (76) in a more convenient way we get (20).

The integral in (46) with \( g \) given by (47,48) splits into two parts: the surface integral

\[
\Omega(g) = -\frac{1}{\delta\pi} \int \text{tr}(\partial\mu gg^{-1}\partial\mu gg^{-1})d\tau dx
\]

\[
C \quad \text{Integrating of continued } g
\]

The integral in (46) with \( g \) given by (47,48) splits into two parts: the surface integral

\[
\Omega(g) = -\frac{1}{\delta\pi} \int \text{tr}(\partial\mu gg^{-1}\partial\mu gg^{-1})d\tau dx
\]
and the volume integral

\[ \Gamma(g) = -\frac{i}{12\pi} \int e^{\mu\nu\lambda} \text{tr}(\partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\lambda gg^{-1}) d\tau dx dy. \quad (80) \]

Firstly, we will deal with the surface integral. Because of the identity 
\[ \text{tr}(\partial_\mu gg^{-1} \partial_\nu gg^{-1}) = 4 \text{tr}(\partial_\mu gg^{-1} \partial_\nu gg^{-1}) \]
the expression under the integral is zero in the region where \( \tau > 0 \) or \( \tau < -\alpha \) where \( \partial_\tau g = 0 \). Combining

\[ \text{tr}(\partial_\mu (gh)h^{-1} \partial_\mu (gh)h^{-1}) = \text{tr}(\partial_\mu gg^{-1} \partial_\nu gg^{-1}) + \text{tr}(\partial_\mu hh^{-1} \partial_\mu hh^{-1}) + 2 \text{tr}(g^{-1} \partial_\mu gg^{-1} \partial_\mu hh^{-1}) \quad (81) \]

with Eq. (44) we get

\[ \Omega(g) = -\frac{1}{8\pi} \int_{-\alpha}^{0} d\tau \int dx \text{tr}(\partial_x S^{-1}(x)S(x)\partial_x S^{-1}(x)S(x)) - \frac{1}{4\pi} \int_{-\alpha}^{0} d\tau \int dx \text{tr}(S\partial_x S^{-1}\partial_x f(\bar{z})f^{-1}(\bar{z})). \quad (82) \]

Notice, that \( \partial_x f = -i\partial_\tau f \) due to antianalyticity of \( f \).

In Eq. (80) the interior of the sphere is separated into three parts by surfaces \( \tau = -\alpha \) and \( \tau = 0 \). Everywhere except in the middle part where \( 0 < \tau < -\alpha \) the differential form \( \text{tr}(dgg^{-1} \wedge dgg^{-1} \wedge dgg^{-1}) \) is zero because in those regions \( g \) depends on two variables only. In the remaining part we have \( g = S^{-1}(x,y)f(\bar{z},y) \) and the differential form is exact

\[ \text{tr}\{dgg^{-1}\} = -\text{tr}\{(S^{-1}dS)^3\} + \text{tr}\{(df^{-1})^3\} + 3d\text{tr}(dSS^{-1} \wedge df^{-1}) = 3d\text{tr}(dSS^{-1} \wedge df^{-1}). \quad (83) \]

Applying Stokes’ theorem we get

\[ \Gamma(g) = -\frac{i}{4\pi} \int_{-\alpha}^{0} d\tau \int dx \text{tr}(S\partial_x S^{-1}\partial_x f(\bar{z})f^{-1}) - \frac{i}{4\pi} \int_{-\alpha}^{0} d\tau \text{tr}\{dSS^{-1}df(-\alpha + 0 - ix,y)f^{-1}\} - \frac{i}{4\pi} \int_{-\alpha}^{0} d\tau \text{tr}\{dSS^{-1}df(0 - ix,y)f^{-1}\} \quad (84) \]

where \( \beta_1 \) is a surface \( \tau = -\alpha \) and \( \beta_2 \) is a surface \( \tau = 0 \). The first term in (84) cancels the second one from (82) and thus we obtain (49).

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\]
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