Construction of unshielded singular solutions of the harmonic field equations

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Abstract

Singular solutions of the harmonic Einstein evolution equation are constructed which are related to spatially global and time-local solutions for a certain class of quasilinear hyperbolic systems of second order. The constructed singularities of curvature invariants occur generically and are accessible by g.a.p. curves. The singularities are not strongly censored, and for strongly asymptotically predictable space-times, they are located in the causal past of the future null infinity, and are, hence, not shielded by a black hole. This is an alternative construction of singularities, which may be applied to other hyperbolic equations such as the Euler equation (cf. [3] for a different construction method- both of our constructions are fundamentally different from supercritical blow-up constructions in the Katz-Pavlovic model or singular solution constructions for heat-flow maps in specific dimensions).

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1 Harmonic Einstein equations, unshielded singularities and cosmic censorship

In this section we consider the mathematical and physical background of unshielded or 'naked' singularities, which are, according to one attempt of definition, singularities located in the causal past of the future null infinity. The concept of a 'singularity' in classical gravity is elusive as the extensions of different proposes for this concept seems to be either too extensive or too narrow for different reasonable purposes. Even the reasonable concept of shielded singularities just mentioned is a bit narrow in the sense that it is usually defined relative to strongly asymptotically predictable space-times. Due to this situation, it seems to be easier to prove existence results of singularities than to exclude a certain type of singularities in a broad variety of senses. In order to prove a convincing singularity existence result it can be sufficient to choose a rather strong concept of a singularity such as the blow up of a curvature invariant along a curve of finite generalised affine parameter length. Our considerations here are motivated by a certain structure of the Einstein field equation, but similar constructions can be done for a certain class of quasilinear hyperbolic equations of second order as well. The field equations determine the coefficients
The line element
\[ ds^2 = \sum_{i,j=0}^{n} g_{ij} dx_i dx_j \]
of a semi-Riemannian manifold \( M \), where the zero component refers to time. The usual assumption is that there are three spatial dimensions, i.e., \( n = 3 \), but since the Kaluza-Klein paper there have always been hypotheses with \( n > 3 \) where the Lorentz metric can be generalized to arbitrary dimension straightforwardly and no specific Lorentz-group structure is needed here. It seems reasonable to be not specific with respect to dimension and just assume \( n \geq 3 \).

Depending on the nature of singularities considered they are in general located on the boundary \( \partial M \) of a manifold \( M \) with respect to some topology which has to be defined according to the purposes of the investigation. Such boundaries can be very bizarre and may have counterintuitive properties. Since our intention in this paper is to construct singularities related to curvature blow-ups we may use a rather strong topology. Note that depending on the topology we can include or must exclude (parts of) the boundary from the manifold itself. Especially, if we want basic invariants such as dimension to be well-defined. We better work with \( C^p \)-manifolds for \( p \geq 1 \), maybe with exceptions for specific very restricted sets. For, otherwise, we may run into problems concerning the invariance of domain and so on We shall consider a rather strong topology. The harmonic field equations involve second order derivatives of the metric components such that there is no classical solution of these hyperbolic equations in the function space \( C^1 \setminus C^2 \) of these metric components. In our construction there is only one point of space-time in the latter space. This point will be on the boundary of a classical solution. The solution of the harmonic field equation assume data \( g_{0ij} \in C^{1,\delta} (\mathbb{R}^n, \mathbb{R}) \), \( 1 \leq i, j \leq n \), which are smooth in the complement of the origin and in \( H^1 \). Related assumptions are made for the first order time derivatives, i.e., \( g_{0ij} \in C^{1,\delta} (\mathbb{R}^n, \mathbb{R}) \), \( 1 \leq i, j \leq n \), and the first order spatial derivatives, i.e., \( g_{0ij,k} \in C^{0,\delta} (\mathbb{R}^n, \mathbb{R}) \), \( 1 \leq i, j \leq n \) of the metric component functions. Note that the curvature invariants involve second order derivatives of the metric tensor, and can, hence, may bow up for some metric components \( g_{ij} \) with \( g_{ij} \in C^{1,\delta} \). In order to prove time-local existence it is usually assumed there is a Cauchy surface \( C \) and a local time neighborhood of \( C \) such that the metric components have a uniform Lorentzian signature. More precisely \( g_{ij} \) satisfy a harmonic field equation on \((-\epsilon, \epsilon) \times \mathbb{R}^n\) with respect to harmonic coordinates \((t, x)\) and where \( t \in (-\epsilon, \epsilon) \) for some small \( \epsilon > 0 \). This allows us to subsume the field equations (in this region) under a standard class of quasilinear hyperbolic equations. The spatially global assumption may be weakened to a spatially local assumption (as is most likely the case with hyperbolic equations), but a detailed treatment of this generalisation would involve too much technicalities and may obscure the main idea to be communicated here.
Therefore we are a bit generous with respect to the Cauchy surface assumption. Note that the loss of well-posedness of the field equations (beyond local-time well-posedness) may be due to singularities or to the loss of a given Lorentzian signature as time passes by. In the following we sometimes use Einstein summation, and use the more classical notation with explicit symbols of sums if we want to emphasize some structure of equations. Notation of ordinary partial derivatives with respect to the variable $x^i$ is either denoted by a subscript $\frac{\partial}{\partial x^i}$ or by $\frac{\partial}{\partial x^i}$. It is well-known that the field equations can be subsumed under a certain class of quasilinear hyperbolic systems of second order -which were seemingly first studied systematically by Hilbert and Courant. This subsumption is used in [2], but the result obtained on the abstract level is not strong enough for our purposes. For this reason we stick with the special field equations, where we can use special features. In the physical context, as long as considerations of higher dimensions seem to be of a speculative type, it seems appropriate to consider the classical field equations in classical space with spatial dimension three and then remark that the result can be generalized (if needed). It is in space-time dimension $3 + 1$, where calculations of the field equations produced predictions which were confirmed by experiment. So we think of $n = 4$ in general, and keep the treatment general as this costs us nothing. Recall that the signature of the metric $g_{ij}$ is the number of positive eigenvalues of the matrix $(g_{ij})$, i.e., the spatial dimension $n$ in our case, where the index zero is reserved for the time dimension. For a Lorentz metric on $\mathbb{R} \times \mathbb{R}^n$ we may consider the field equations as a first order quasi-linear hyperbolic system with harmonic coordinates for $g_{ij}, h_{ij} = \frac{\partial g_{ij}}{\partial t}$ of the form

\[
\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= h_{ij} \\
\frac{\partial g_{ij,k}}{\partial t} &= \frac{\partial h_{ij}}{\partial x^k} \\
\frac{\partial h_{ij}}{\partial t} &= -g_{00} \left( 2g^{0k} \frac{\partial h_{ij}}{\partial x^k} + g^{km} \frac{\partial h_{ij,k}}{\partial x^m} - 2H_{ij} \right),
\end{aligned}
\]  

with data $g_{ij}(t_0, \cdot)$ and $h_{ij}(t_0, \cdot)$ at some time $t_0$, and where

\[
H_{ij} \equiv H_{ij} \left( g_{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \right) = g^{\alpha\beta} g_{k\gamma} \Gamma^k_{i\beta} \Gamma^\gamma_{j\alpha} + \frac{1}{2} \left( \frac{\partial g_{ij}}{\partial x^\alpha} \Gamma^\alpha + g_{ij} \Gamma^\rho_{\alpha\beta} g^{\alpha\eta} g^{\beta\sigma} \frac{\partial g_{\eta\sigma}}{\partial x^\rho} + g_{ij} \Gamma^\rho_{\alpha\beta} g^{\alpha\eta} g^{\beta\sigma} \frac{\partial g_{\eta\sigma}}{\partial x^\rho} \right).
\]  

Furthermore, we use the convention

\[
\Gamma^i = g^{\alpha\beta} \Gamma^i_{\alpha\beta},
\]

where we recall that the Christoffel symbols are defined to be

\[
\Gamma^i_{\alpha\beta} = \frac{1}{2} g^{ij} \left( g_{\rho\alpha,\beta} + g_{\rho\beta,\alpha} - g_{\alpha\beta,\rho} \right).
\]  

This system is equivalent to the vacuum field equations

\[
R^h_{\mu\nu} = 0
\]

of course, where the upper script $h$ indicates that the Ricci tensor $R_{\mu\nu}$ is written in harmonic coordinates. The coordinates are called harmonic because, usually,
the (vacuum) Einstein equations are written in coordinates where they take the form

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \]  

which contains an additional 'potential' term \( \frac{1}{2} g_{\mu\nu} R \). Here, recall that the Ricci tensor is given by

\[ R_{\mu\nu} = \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} - \frac{\partial \Gamma^\alpha_{\mu\alpha}}{\partial x^\nu} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\alpha}, \]  

and that the scalar curvature is given by

\[ R = g^{\mu\nu} R_{\mu\nu}. \]  

Note that \( R \) is a scalar function where the evaluation of a scalar at a point \( p \in M \) is denoted by \( R_p \). Historically, it was a major step to find this additional term (saving covariance), and \( G_{\mu\nu} \) is called the Einstein tensor. Maybe it should be called the Einstein-Hilbert tensor, because it is quite possible that Hilbert was the first in November 1915 who wrote it on a blackboard in Göttingen in a derivation via variational calculus \(^1\). In this article we construct singular solutions for the vacuum equations. We note that our method can be applied to extended equations of the form

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu},\]  

where \( \Lambda \) is a cosmological constant, and \( T_{\mu\nu} \) is the energy momentum tensor. Such generalisations depend on conditions on the additional terms, of course. For example a positive cosmological constant \( \Lambda \) has a damping effect in the region where the metric tensor satisfies a Lorentz condition (and can be written in harmonic form). However, generalisations of the following results are possible for positive and negative cosmological constants. Actually sufficient conditions for the construction are described in the last section in the context of quasi-linear hyperbolic equation systems of second order, and the consequences for the Einstein field equations can be drawn as long as the field equations can be subsumed under that systems equipped with a set of conditions for (some type) of local conditions (cf. last section). Up to the 'material' term \( T_{\mu\nu} \) the field equations look locally like ordinary wave equations of course (easily to solve), but globally these simple equations are glued together which makes them non-linear and difficult to solve. Accordingly, most of the research concerns specific solutions to the field equations, while general research is more in the context of hyperbolic systems of second order which are investigated by more general methods. Results are then applied to the field equations without using their special structure. For example, in \(^2\) it is observed that the field equations for a Lorentz metric can be subsumed by hyperbolic systems of second order of the form

\[ a_{00} \frac{\partial^2 \psi}{\partial t^2} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 \psi}{\partial x^i \partial x^j} + \sum_{i=1}^{n} (a_{0i} + a_{ii}) \frac{\partial^2 \psi}{\partial t \partial x_i} + b, \]  

\(^1\)Note that four pages of Hilbert’s corresponding publication are missing in the archive of the academy in Berlin while the variational principle is given in the correct form, and it seems very unlikely that Hilbert did or could not not derive the equations in (7) from the variational principles - probably it is a paragraph in the missing pages.
where $\psi = (\psi_1, \ldots, \psi_n)^T$ is a $n$-vector-valued function of time $t \in [0, T]$ and spatial variables $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, and $a_{ij}, 1 \leq i, j \leq n$ is a collection of $n \times n$-matrix valued functions of suppressed arguments of $t, x, \psi, \frac{\partial \psi}{\partial t}, \nabla \psi$, and $b$ is a $n$-vector-valued function of the same arguments (the latter sentence is a citation of [2], p. 274 essentially). Local triviality and global complexity are characteristics of the Einstein field equation as their derivation principles are simple (cf. the equivalence principle) while second order tensors like the Ricci tensor (which contain the global information) can have a rich structure, so rich, that general investigations of the field equations work with further assumptions, for example with the assumption of asymptotically predictable space-times. The difficulty of defining certain concepts such as 'singularity', 'black holes', 'weak cosmic censorship' is related to that richness such that these concepts are defined relative to such classes of space-time (such as the mentioned class of strongly asymptotically predictable space-times). We recall a related class of concepts which lead us to a concept of black holes, unshielded (naked) singularities, and a concept of weak censorship. We refer the reader to [1, 4, 5, 6, 7] for a more detailed discussion of these notions. First we introduce a class of space-times for which black holes are well-defined.

**Definition 1.1.** A strongly asymptotically predictable space-time is an asymptotically flat space-time $(M, g)$ such that there exists an open region $U$ in the conformal space-time extension $(\tilde{M}, \tilde{g})$ such that

i) $U \supset M \cap J^- (I^+)$,

ii) $(U, \tilde{g})$ is globally hyperbolic.

In this context of asymptotically predictable space-times black holes can be defined without reference to elusive concept of a singularity. We have

**Definition 1.2.** The region $B \subseteq M$ is called a black hole of a strongly asymptotically predictable space-time $(M, g)$, if it is the complement of the causal part $J^-$ of the future null infinity $I^+$, i.e., $B = M \setminus J^- (I^+)$. 

**Definition 1.3.** The boundary $H^+ := M \cap \partial J^- (I^+)$ is the event horizon of a black hole.

**Definition 1.4.** A singularity of space-time is called naked if it is located in the causal past of null infinity $J^- (I^+)$

The weak cosmic censorship conjecture maintains that there is no naked singularity. This concept is due to Hawking and Penrose, of course. It seems to be a rather involved concept, but it is a certain way of making precise Penrose’s early statement of 1969 (citation):

"does there exist a 'cosmic censor' who forbids the appearance of naked singularities closing each one in an absolute event horizon?"

The strong cosmic censorship hypothesis for a metric Lorentzian manifold $(M, g_{ij})$ is often defined by strong global hyperbolicity, i.e., the requirement that there is a Cauchy surface $S$ such that

$$M = D^+ (S) \cup D^- (S),$$

(12)
where $D^+(S)$ (resp. $D^-(S)$) are arcwise connected components separated by $S$ and represent the causal future and the causal past relative to the Cauchy surface $S$ defined by causal curves. The elusive nature of the concept of singularities then led to a weak interpretation of the concept of a singularity in terms of geodesically incompleteness. Next we discuss some notions of the vague concept of singularities and different attempts to make it precise. This is important in order to understand the role of the Hawking-Penrose theorem, the conjectures of weak and strong cosmic censorship, and the results and arguments of this paper, which can be read as comments on these theorems and claims.

Defining singularities by geodesic incompleteness is a well-motivated approach because it seems that other definitions are far too narrow or far too wide. However, we may use a strong definition of singularity and prove its existence for a generic set of Lorentz metrics satisfying the Einstein evolution equation in order to disprove weak cosmic censorship statements which are based on weaker (wider or more extensive) notions of singularities. First let is recall the relevant notions. Note that the line element $ds^2 = g_{ij}dx^i dx^j$ defines a metric $g: TM \times TM \to \mathbb{R}$, which is a bilinear form, and where $TM$ denotes the tangential bundle on $M$.

**Definition 1.5.** The generalized affine parameter length of a curve $\gamma: [0,c) \to M$ with respect to a frame $E_s = (e_i(s))_{0 \leq i \leq n}, 0 \leq s \leq c$ of a family of basis vectors in $\mathbb{R}^n$ (abbreviated by g.a.p.) is given by

$$l_E(\gamma) = \int_0^c \left( \sum_{i=0}^{n} g\left(\gamma(s), e_i(s)\right) \right) ds$$

**Definition 1.6.** A curve $\gamma: [0,c) \to M$ is incomplete if it has finite g.a.p. with respect to some frame, and it is inextensible if there is no limit in $M$ of $\gamma(s)$ as $s$ approaches $c$. Furthermore a space-time is called incomplete if it contains an incomplete inextensible curve.

Singularities may be approached by future directed incomplete inextensible curves, i.e. curves with nonnegative Lorentz-metric at all points of the curve, and by past-directed inextensible curves, i.e., curves with nonpositive Lorentz-metric at all points of the curve. Future-directed curves starting at a point $p \in M$ are denoted by $I^+(p)$ and past-directed curves are noted by $I^-(p)$. Null-curves or light-curves starting at a point $p$ are curves with zero value of the Lorentz-metric at all points of the curve considered, and are located at the boundary of $I^+(p)$, where it is a matter of taste to include or exclude the boundary of $I^+(p)$ in the definition of $I^+(p)$ (we included this boundary here). The set of singularities related to a Lorentz manifold $(M, g_{ij})$ which are endpoints of inextensible curves in $I^+(p)$ starting from some point $p \in M$ are denoted by $M^+$, and the set of singularities which are endpoints of inextensible curves in $I^-(p)$ starting from some point $p \in M$ are denoted by $M^-$. For a curve $\gamma: [0,c) \to M$ with positive $c \in \mathbb{R} \cup \{\infty\}$ we define

$$I^+(\gamma) := \cup_{q \in I^+(0,c)} I^+(q), \ I^-(\gamma) := \cup_{q \in I^-(0,c)} I^-(q).$$

**Definition 1.7.** A space-time is geodesically incomplete if it contains a geodesic curve which is incomplete.
The Hawking Penrose theorem states this is not our main concern here, we refer to [1] for the precise discussion of its assumptions. We have

**Theorem 1.8. (Hawking Penrose Theorem)** Assume that a time oriented space-time \((M,g_{ij})\) satisfies the conditions

i) \(R_{ij}X^iX^j \geq 0\) for any non-spacelike vector \(X^i\).

ii) The timelike and null generic conditions are satisfied.

iii) There are no closed timelike curves.

iv) One of the three condition holds
   iv(a) There exists a trapped surface.
   iv(b) There exists an achronal set without edges.
   iv(c) There exists a point \(p \in M\) such that for each future directed null-
geodesic through \(p\) the expansion becomes negative.

Then the manifold \((M,g_{ij})\) contains at least one incomplete timelike or null geodesic.

Generic occurrence of singularities for the field equations, where ‘generic’ is to be understood as ‘generic relative to physically reasonable space-times’, is one feature of the field equations which may be successfully expressed by theorem 1.8. Another proposed feature of Hawking and Penrose is that singularities are (at least weakly) censored, i.e. shielded by black holes or it is even not possible to reach the singularity from any point in space-time by an incomplete curve of finite g.a.p. length. It is in this respect that our result provides contradictory evidence. Let us first recall the principles of strong and weak cosmic censorships.

**Definition 1.9. (Strong cosmic censorship).** A singularity point \(p \in M^+\) (resp. \(p \in M^-\)) of a space-time \((M,g)\) is strongly censored if for every \(q \in M\) we have \([p] \not\in I^+(q)\) (resp. \(I^-(q)\)). Accordingly, a space-time is said to have strongly censored singularities if all singularities in \(M^+\) and \(M^-\) are strongly censored.

**Definition 1.10. (Weak cosmic censorship).** A singularity point \(p \in M\) of a space-time \((M,g)\) is weakly censored if it is not in the causal past of the future null infinity. Accordingly, a space-time is said to have strongly censored singularities if for all Cauchy surfaces of \(M\) all singularities in \(M^+\) and \(M^-\) are strongly censored.

The reason for the characterization of singularities by geodesically incomplete curves is that other characterizations turn out to be too narrow or too extensive, but counterexamples concerning the cosmic censorship hypotheses can be

**Definition 1.11.** We say that \(p \in M^+\) is a strong scalar curvature singularity if there is a incomplete g.a.p. finite curve \(\gamma : [0,c) \to M\) such that

\[
\lim_{s \uparrow c} \gamma(s) = p, \quad (16)
\]

and

\[
\forall \epsilon > 0 \forall C > 0 \exists s \in [c - \epsilon, c) : |R_{\gamma(s)}| \geq C. \quad (17)
\]

We also say that the scalar curvature invariant blows up at \(p\).
Note that curvature tensor invariants involve second order spatial derivatives of the metric component functions \(g_{ij}\). For this reason there are metric functions which have Hölder continuous first order derivatives and where curvature invariants blows up. There are even metric functions \(g_{ij}\) which have Hölder continuous first order derivative and are smooth in the complement of the origin \((0,0)\). First we give an example

\[
\left( g_{ij} \right)_{1 \leq i, j \leq 2} = \begin{pmatrix} 1 + \phi_\delta(x^1) f(x^1) & 0 \\ 0 & 1 \end{pmatrix},
\]

where \(f\) is a univariate function on the field of real number \(\mathbb{R}\) of the form \(z \rightarrow f(z) = z^3 \cos \left( \frac{1}{z} \right)\) for \(z \neq 0\) and \(f(0) = 0\), and \(\phi_\delta \in C^\infty\) is a function with support \((\delta, \delta)\) and with \(\phi_\delta(0) = 1\) (as known for partitions of unity). Evaluating the derivatives of \(f\) you observe that the second derivative is discontinuous and even blows up at \(z = 0\). It follows that second order derivatives of the metric \(g_{ij}\) with respect to the spatial variable \(x^1\) blow up at the origin. As some first order derivatives of the Christoffel symbols in the definition of the Ricci tensor \(R_{ij}\) in \(\mathbb{R}^3\) contain such non-vanishing second order spatial derivatives of the metric \(g_{ij}\), a simple calculation shows that the non-vanishing second order derivative terms do not cancel, and as \(g_{ij}\) is positive definite and bounded with a bounded inverse \(g^{ij}\) the scalar curvature \(\tilde{R} = g^{ij}R_{ij}\) blows up at the origin and is smooth in the complement of the origin. Such phenomena are consistent with constraint equations on the Cauchy surface as we shall observe later.

The latter example is rather generic. We have

**Proposition 1.13.** Let \(C^{1,\delta} \equiv C^{1,\delta}(\mathbb{R}^n)\) the space of differentiable functions with Hölder continuous first order derivatives of Hölder exponent \(\delta \in (0,1)\),

\[
\left| f(x) \right|_{1,\delta} := \sum_{0 \leq |\alpha| \leq 1} \sup_{y \in \mathbb{R}^n} \left| D_\alpha^n f(y) \right| + \sup_{y \neq z , \ y, z \in \mathbb{R}^n} \frac{\left| D_\alpha^n f(z) - D_\alpha^n f(y) \right|}{|z - y|^{\delta}},
\]

For fixed time \(t = t_0 \in \mathbb{R}\) we consider the metric functions \(g_{ij}^{t_0} \equiv g_{ij}(t_0,.)\), \(1 \leq i, j \leq n\) for \(0 \leq i, j \leq n\). Then in any neighborhood of \(g_{ij}^{t_0} \in C^2 \cap H^1\) in \((C^{1,\delta}, ||, ||_{1,\delta})\) there is a function \(\tilde{g}_{ij}^{t_0} \in C^{1,\delta} \cap H^1\) such that typical curvature invariants of \(\tilde{g}_{ij}^{t_0}\) (such as the scalar curvature) blow up at the origin and are well-defined in the complement of the origin.

**Proof.** The scalar curvature satisfies

\[
\tilde{R} = g^{\alpha\nu} R_{\nu\mu} = g^{\alpha\nu} \left( \frac{\partial \Gamma^\alpha_{\nu\mu}}{\partial x^\alpha} - \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} - \Gamma^\gamma_{\mu\beta} \Gamma^\beta_{\nu\alpha} - \Gamma^\gamma_{\nu\beta} \Gamma^\beta_{\mu\alpha} \right),
\]

For Example 1.12. Note that we have some freedom of choice in lower dimensional subspaces (subspace dimension \(\leq 2\)) for the spatial part of the metric \(g_{ij}\), \(1 \leq i, j \leq 3\) and still solve the field equations. The weak singularities at the origin are located at the boundary of the equation and are not part of the classical solution constructed. For example, consider a metric of spatial dimension \(n = 2\) which depends only on one spatial variable, say \(x^1\), and is constant with respect to time such that for \(1 \leq i, j \leq 2\) for fixed time we have

\[
\left( g_{ij} \right)_{1 \leq i, j \leq 2} = \begin{pmatrix} 1 + \phi_\delta(x^1) f(x^1) & 0 \\ 0 & 1 \end{pmatrix},
\]
where the last three terms in

\[ \Gamma^i_{\alpha\beta,j} = \frac{1}{2} g^{ij} \left( g_{\rho\alpha,\beta} + g_{\rho\beta,\alpha} - g_{\alpha\beta,\rho} \right) \]

\[ + \frac{1}{2} g^{ij} \left( g_{\rho\alpha,\beta,j} + g_{\rho\beta,\alpha,j} - g_{\alpha\beta,\rho,j} \right) \]  

(21)

contain second order derivatives of the metric. First, consider the univariate function \( f : \mathbb{R} \to \mathbb{R} \) with

\[ f(z) = \begin{cases} 
  z^3 \cos \left( \frac{1}{z} \right), & \text{if } z \neq 0 \\
  0, & \text{if } z = 0. 
\end{cases} \]  

(22)

The first derivative is

\[ f'(z) = \begin{cases} 
  3z^2 \cos \left( \frac{1}{z} \right) + z \sin \left( \frac{1}{z} \right), & \text{if } z \neq 0 \\
  0, & \text{if } z = 0. 
\end{cases} \]  

(23)

Note that for \( z \neq 0 \)

\[ \frac{f'(z) - f'(0)}{z - 0} = 3z \cos \left( \frac{1}{z} \right) + \sin \left( \frac{1}{z} \right) \]  

(24)

such that the limit does not exists at \( z = 0 \). The second derivative is

\[ f''(z) = \begin{cases} 
  6z \cos \left( \frac{1}{z} \right) - 3 \sin \left( \frac{1}{z} \right) + \sin \left( \frac{1}{z} \right) - \frac{1}{z} \cos \left( \frac{1}{z} \right), & \text{if } z \neq 0 \\
  \text{undefined if } z = 0. 
\end{cases} \]  

(25)

Now consider a \( C^\infty \)-function \( \phi_{\delta,\epsilon} : \mathbb{R} \to \mathbb{R} \) with support in \((-\epsilon, \epsilon)\) and such that \( \phi_{\delta,\epsilon}(z) = 1 \) for \( z \in (-\delta, \delta) \). Such functions are well known from the context of partitions of unity. Then in any neighborhood (with respect to the normed function space stated) of a metric \( g_{ij}(t_0,.) \) (evaluated at some time \( t_0 \)) we find a metric \( \tilde{g}_{ij}(t_0,.) \) such that for some \( \delta > 0 \) the function

\[ x \to \tilde{g}_{ij}(t_0, x) \]  

(26)

where

\[ \tilde{g}_{11} = g_{11} + \delta \phi_{\delta,2\delta}(x^1) f(x^1) \]  

(27)

stays in the neighborhood and such that the scalar curvature blows up at the origin and is well-defined in the complement of the origin.

The upshot of the following considerations is as follows. In the following section we state the main theorem time local solutions of the Einstein field equations with strong singularities in the sense of definition 1.11. This result gives a counter example to various interpretations of the strong or weak cosmic censorship hypotheses. Note that the time-local existence theorem is not subsumed by \[2\]. As a consequence there is also a counter example of the weak cosmic censorship. In the last section we prove the theorem.
2 Construction of a class of unshielded singular solutions of the harmonic field equation

Next we are concerned with the main theorem which asserts the existence of unshielded (naked) singular solutions of the harmonic field equations. We construct a spatially global and time-local solution for $C^{1,\delta}$ data on a Cauchy surface which are smooth in the complement of the origin and in the Sobolev space $H^1$. The initial data functions have a certain Lorentz signature on the Cauchy surface, and we assume that the same signature is given in a local time-neighborhood of the Cauchy surface. As we have bounded continuous metric components, a uniform Lorentz condition for the initial data $g_{0ij}$, $1 \leq i, j \leq n$ is obviously sufficient in order implement this assumption. We call such a Cauchy surface a local Cauchy surface. This is just a local hyperbolicity assumption. Hence, we assume that at some time there is a local Cauchy surface such that the harmonic field equations can be subsumed by a class of quasilinear hyperbolic equations of second order. There is a local existence theory for such type of equations (cf. [2]), which proves the existence of a time-local and unique solution up to a small time horizon $T > 0$ for regular data, where the components of the metric are assumed to be in $H^s$ for some appropriate exponent such as $s > \frac{n}{2} + 1$ (and related assumptions for the first order spatial and time derivatives). However, this local existence result cannot be applied in our situation, because we have only $C^{1,\delta}$ regularity at one point and second order derivatives may be not integrable. There are two possible reasons for time locality of existence theorems. One possible reason is that a solution ‘develops’ singularities after finite time. The other reason is that the metric does not satisfy a Lorentz condition after some time, and then cannot be subsumed under the type of quasilinear hyperbolic systems for which the local existence results are proved. However, if a strict Lorentz condition is satisfied uniformly on a Cauchy surface $C$ at time $t = t_0$, then it holds in a neighborhood for time $t \in (t_0 - \rho, t_0 + \rho)$ for some $\rho > 0$. The harmonic form of the Einstein field equations confirm time symmetry as a natural property of the theory. For this reason of time symmetry (which we find in many hyperbolic equations of mathematical physics) we can build in the singularity (here a curvature singularity) into the initial data on the Cauchy surface and then show that there exists a time-local solution. This local solution can be extended locally to past time and to future time. An alternative approach is to start with smooth data on the Cauchy surface, and then show that a singularity can develop at the tip of a cone. We considered such a construction for the vorticity form of the incompressible Euler equation in [3]. Note that the alternative construction considered here may be applied to the Euler equation as well.

In the following we use the term ‘classical solution of a differential equation’ on a certain domain in the usual sense that a solution function satisfies the differential equation pointwise, and where the (partial) derivatives exist in the classical Weierstrass sense. We have

Theorem 2.1. For data functions

$$g_{ij}(0,\cdot) = g_{0ij} : \mathbb{R}^n \to \mathbb{R}, \quad g_{ij,t}(0,\cdot) : \mathbb{R}^n \to \mathbb{R},$$

$$g_{ij,k}(0,\cdot) : \mathbb{R}^n \to \mathbb{R}, \quad 1 \leq i, j \leq n,$$

(28)
where
\[ g_{0ij}(0,.) \in C^\infty(\mathbb{R}^n \setminus \{(0,0)\}) \cap C^{1,\delta}(\mathbb{R}^n) \cap H^1 \]
\[ h_{ij0} \equiv g_{ij,t}(0,.) \in C^\infty(\mathbb{R}^n \setminus \{(0,0)\}) \cap C^{1,\delta}(\mathbb{R}^n) \cap H^1 \]
\[ h_{ij0} \equiv g_{ij,t}(0,.) \], \quad \xi_{ij,k} \equiv g_{ij,k}(0,.) \in C^\infty(\mathbb{R}^n \setminus \{(0,0)\}) \cap C^{0,\delta}(\mathbb{R}^n) \cap H^1 \]

there is a time-local solution to the Cauchy problem

\[
\begin{aligned}
\partial g_{ij} / \partial t &= h_{ij} \\
\partial g_{ij,k} / \partial t &= \partial h_{ij} / \partial x^k \\
\partial h_{ij} / \partial t &= -g_{00} \left( 2g^{0k} \partial h_{ij} / \partial x^k + g^{km} \partial g_{ij,k} / \partial x^m - 2H_{ij} \right) \\
g_{ij}(0,.) &= g_{0ij}, \quad g_{ij,t}(0,.) = h_{0ij}, \quad \xi_{ij,k}(0,.) = \xi_{0ij,k}.
\end{aligned}
\]

on a time horizon \([0,T]\) for some \(T > 0\) which is classical in the complement of the origin. Here, the harmonic term is given by

\[
H_{ij} \equiv H_{ij} \left( g_{\alpha\beta} \partial g_{\alpha\beta} / \partial x^\alpha \right) = g^{\alpha\beta} g_{ij} \Gamma^c_{ij} \Gamma^\alpha_{c} \\
+ \frac{1}{2} \left( \partial g_{ij} / \partial x^\alpha \Gamma^\alpha_{ij} + g_{ij} \Gamma^\rho_{ij} g^{\alpha\eta} g^{\beta\sigma} \partial g_{\alpha\eta} / \partial x^\rho + g_{ij} \Gamma^\rho_{ij} g^{\alpha\eta} g^{\beta\sigma} \partial g_{\alpha\eta} / \partial x^\rho \right).
\]

Moreover, for a dense set of data \(g_{0ij}, 1 \leq i,j \leq n\) in the functions space \((C^{1,\delta} \cap H^1, ||.||_{1,\delta})\) the scalar curvature blows up at the origin \((t,x) = 0\). We shall observe below that the metric can be chosen such that it satisfies the usual constraints on the Cauchy surface outside the origin (which is part of the boundary of the Lorentzian of the manifold, not part of the manifold itself), and that a Cauchy problem with such data is well-defined.

We have to mention that any solution of the Einstein field equation has to satisfy some constraint equations. These are the Hamiltonian constraint equation and the momentum constraint equation. However, this is no obstacle for our construction, as we explain in the proof of the next corollary based on the proof of the main theorem. As the constructed metric solutions \(g_{ij}\) with singular scalar curvature are bounded on the domain of well-posedness, it is clear that there are causal curves of finite g.a.p. length (in the domain where the solutions exists) which reach the point of singular scalar curvature at the origin. We get

Corollary 2.2. Theorem 2.1 implies that the weak and strong cosmic censorship in the sense of definition 1.9 and definition 1.10 are violated for a dense set of constrained data and the corresponding spatially global and time-local solutions \(g_{ij}\) described in theorem 2.1.

Proof. It has to be shown that the constraint equations on the Cauchy surface can be satisfied for the viscosity limit of the constructed local solution for all points in the complement of the origin such that the Einstein field equations are satisfied. First note that in coordinates around the Cauchy surface with line element

\[ ds^2 = -dt^2 + g_{ij} dx^i dx^j, \]
and with
\[ h_{ij} = \frac{1}{2} g_{ij,t} \]  
we have (for the vacuum equation) the Hamilton constraint equation
\[ R - h_{ij} h^{ij} + (h_{k}^{i})^2 = 0 \]  
and the momentum constraint equation
\[ D_{j} h_{ij}^{i} - D_{i} h_{ij}^{j} = 0, \]
where \( D \) denotes the connection of \( g_{ij} \). Shifting the Cauchy surface to small positive time \( t_0 > 0 \) for the convolute functions (with the Gaussian \( G_{\nu} \) defined below and evaluated at \( t_0 > 0 \))
\[ h^{(\nu)}_{ij} := h_{ij} * G_{\nu}, \quad R^{(\nu)} := R * G_{\nu} \]  
we can determine data \( h_{ij}^{(\nu)} \) which solve the 'viscous' Hamilton constraint equation
\[ R^{(\nu)} - h^{(\nu)}_{ij} h^{(\nu),ij} + (h^{(\nu)}_{i})^2 = 0 \]  
and the 'viscous' momentum constraint equation
\[ D_{j} h^{(\nu)}_{ij}^{i} - D_{i} h^{(\nu)}_{ij}^{j} = 0, \]  
The local solution of the Cauchy problem can then be solved locally for data as in Theorem \( 2.1 \) according to the construction below such that the viscous constraint equations are satisfied. From the construction of the local solution below it follows that in the viscous limit \( \nu \downarrow 0 \) the original constraint equations in (34) and in (35) are satisfied in the complement of the origin. \( \square \)

3 Proof of theorem \( 2.1 \)

Since the initial data are of weaker regularity at one point we do not claim uniqueness but construct a time-local solution (branch). We construct local solutions via viscosity limits \( \nu \downarrow 0 \) of local solutions of the extended system (\( \nu > 0 \) a positive 'viscosity' parameter)
\[
\begin{align*}
\frac{\partial g_{ij}^{(\nu)}}{\partial t} &= \nu \Delta g_{ij}^{(\nu)} + h_{ij}^{(\nu)}, \\
\frac{\partial g_{ij}^{(\nu)}}{\partial t} &= \nu \Delta g_{ij,k} + h_{ij}^{(\nu)}_{k}, \\
\frac{\partial h_{ij}^{(\nu)}}{\partial t} &= \nu \Delta h_{ij}^{(\nu)}, \\
-g_{00}^{(\nu)} \left( 2g_{(\nu)0k} \frac{\partial h_{ij}^{(\nu)}}{\partial x^k} + g_{(\nu)km} \frac{\partial h_{ij}^{(\nu)}}{\partial x^m} - 2H_{ij}^{(\nu)} \right), \\
g_{ij}^{(\nu)}(0,.) &= g_{0i}^{(\nu)}, \quad g_{ij}^{(\nu)}(0,.) = h_{ij}^{(\nu)}, \quad g_{ij,k}^{(\nu)}(0,.) = g_{ij,k}^{(\nu)}. \end{align*}
\]
For a function $f : (0, \rho) \times \mathbb{R}^n \to \mathbb{R}$ on some time horizon $\rho > 0$ we abbreviate

$$f \ast G_{\nu} = \int_0^\rho \int_{\mathbb{R}^n} f(s,y)G_{\nu}(\cdot, s, \cdot - y)dyds$$

(40)

for the convolution with the Gaussian $G_{\nu}$, and for a function $f_0 : \mathbb{R}^n \to \mathbb{R}$ we write

$$f_0 \ast_{sp} G_{\nu} = \int_{\mathbb{R}^n} f_0(y)G_{\nu}(\cdot, \cdot - y)dy$$

(41)

for the convolution with the Gaussian restricted to the spatial variables.

We define an iterative scheme $g_{ij}^{(v)_l}, g_{ij,k}^{(v)_l}, h_{ij}^{(v)_l}, 1 \leq i,j,k \leq n, l \geq 0$ with integration index $l \geq 0$. We initialize the scheme with

$$g_{ij}^{(v)_0} = g_{0ij} \ast G_{\nu}, g_{ij,k}^{(v)_0} = g_{0ij,k} \ast G_{\nu}, h_{ij}^{(v)_0} = h_{0ij} \ast G_{\nu},$$

(42)

where $G_{\nu}$ is the fundamental solution of

$$G_{\nu,t} - \nu \Delta G_{\nu} = 0.$$  

(43)

In order to simplify the local contraction result we define the scheme with a delay of one step. For this reason we have to extend the initialisation, where we introduce

$$g_{ij}^{(v)_{-1}} = g_{0ij} \ast G_{\nu}, g_{ij,k}^{(v)_{-1}} = g_{0ij,k} \ast G_{\nu}, h_{ij}^{(v)_{-1}} = h_{0ij} \ast G_{\nu}.$$  

(44)

Note that the upper script $-1$ does not refer to the inverse but is an iteration index, which is needed for formal reasons of initialization of the delayed scheme.

As the functions $g_{ij}^{(v)_p}, g_{ij,k}^{(v)_p}, h_{ij}^{(v)_p}, 1 \leq i,j,k \leq n$ are given for $p \in \{-1, 0\}$, for $l \geq 1$ the functions $g_{ij}^{(v)_l}, g_{ij,k}^{(v)_l}, h_{ij}^{(v)_l}, 1 \leq i,j,k \leq n$ are defined as solutions of the equation

$$\begin{equation}
\begin{cases}
\frac{\partial g_{ij}^{(v)_l}}{\partial t} = \nu \Delta g_{ij}^{(v)_l} + h_{ij}^{(v)_{l-1}} \\
\frac{\partial g_{ij,k}^{(v)_l}}{\partial t} = \nu \Delta g_{ij,k}^{(v)_l} + \frac{\partial h_{ij}^{(v)_{l-1}}}{\partial x_k} \\
\frac{\partial h_{ij}^{(v)_l}}{\partial t} = \nu \Delta h_{ij}^{(v)_l} \\
- g_{00}^{(v)_l - 2} \left( 2g_{ij}^{(v)_l - 2,0k} \frac{\partial h_{ij}^{(v)_{l-1}}}{\partial x_k} + g_{ij}^{(v)_l - 2,km} \frac{\partial g_{ij}^{(v)_{l-2}}}{\partial x_m} - 2H_{ij}^{(v)_{l-1}} \right) \\
g_{ij}^{(v)_l}(0,\cdot) = g_{0ij}, g_{ij,k}^{(v)_l}(0,\cdot) = h_{0ij}, g_{ij}^{(v)_l}(0,\cdot) = g_{0ij}.
\end{cases}
\end{equation}$$

(45)

where

$$H_{ij}^{(v)_{l-1}} = H_{ij}^{(v)_{l-1}} \left( g_{ij}^{(v)_{l-2}}, \frac{\partial g_{ij}^{(v)_{l-2}}}{\partial x^\gamma} \right)$$

$$= g_{ij}^{(v)_{l-2,\alpha\beta}} \frac{\partial g_{ij}^{(v)_{l-2}}}{\partial x^\gamma} \Gamma_{ij}^{(v)_{l-2,\delta \epsilon}} 1_{\gamma \delta \epsilon} g_{ij}^{(v)_{l-2,\alpha\beta}}$$

$$+ \frac{1}{2} \left( \frac{\partial g_{ij}^{(v)_{l-2}}}{\partial x^\gamma} \Gamma_{ij}^{(v)_{l-2,\alpha\beta}} g_{ij}^{(v)_{l-2,\alpha\beta}} - g_{ij}^{(v)_{l-2,\alpha\beta}} \frac{\partial g_{ij}^{(v)_{l-2}}}{\partial x^\gamma} \right).$$

(46)
Here,
\[ \Gamma^{(\nu)l-2,i} = g^{(\nu)l-2,\alpha\beta} \Gamma^{(\nu)l-2,i} \alpha\beta, \]
and
\[ \Gamma^{(\nu)l-2,i} = \frac{1}{2} g^{(\nu)l-2,i\rho} \left( g^{(\nu)l-2}_{\mu\alpha,\beta} + g^{(\nu)l-2}_{\rho\beta,\alpha} - g^{(\nu)l-2} \right). \]

We have the representation
\[ g_{ij}^{(\nu)l} = g_{0ij} \ast_{sp} G_{\nu} + h_{ij}^{(\nu)l-1} \ast G_{\nu}, \]
\[ g_{ij,k}^{(\nu)l} = g_{0ij,k} \ast_{sp} G_{\nu} + \frac{\delta h_{ij}^{(\nu)l-1}}{\delta x^k} \ast G_{\nu}, \]
\[ h_{ij}^{(\nu)l} = h_{0ij} \ast_{sp} G_{\nu} \]
\[ -g_{00}^{(\nu)l-2} \left( 2g^{(\nu)l-2,ok} \frac{\partial h_{ij}^{(\nu)l-1}}{\partial x^k} + g^{(\nu)l-2,km} \frac{\partial g_{ij}^{(\nu)l-2}}{\partial x^m} - 2H_{ij}^{(\nu)l-1} \right) \ast G_{\nu}. \]

The first crucial step is the convergence of the function \( h_{ij}^{(\nu)l} \) as \( l \uparrow \infty \). We obtain this convergence by a local contraction result in an appropriate Banach space. In order to prepare this result, we use the convolution rule and write the second equation in (49) in the form
\[ g_{ij,k}^{(\nu)l} = g_{0ij,k} \ast_{sp} G_{\nu} + h_{ij}^{(\nu)l-1} \ast G_{\nu,k}, \]
avoiding the consideration of the convergence of spatial derivatives of \( h_{ij} \) using later the fact that spatial first order derivatives of the Gaussian \( G_{\nu,k} \) are locally integrable. Furthermore this shows that the core of the proof is to have convergence of the \( h_{ij}^{(\nu)l} \) (as \( l \uparrow \infty \)) determined by the third equation. We again use the convolution rule in order to avoid first order spatial derivatives of \( h_{ij}^{(\nu)l} \) and second order spatial derivatives of the metric \( g_{ij}^{(\nu)l} \) in the convoluted term. We get for \( l \geq 1 \)
\[ h_{ij}^{(\nu)l} = h_{0ij} \ast_{sp} G_{\nu} - \left( g_{00}^{(\nu)l-2} g^{(\nu)l-2,ok} h_{ij}^{(\nu)l-1} \right) \ast G_{\nu,k} \]
\[ + \left( g_{00}^{(\nu)l-2} g^{(\nu)l-2,ok} \right)_{,k} h_{ij}^{(\nu)l-1} \ast G_{\nu} + \left( g_{00}^{(\nu)l-2} g^{(\nu)l-2,km} g_{ij,k}^{(\nu)l-2} \right) \ast G_{\nu,m} \]
\[ - \left( g_{00}^{(\nu)l-2} g^{(\nu)l-2,km} \right)_{,m} \left( g_{ij,k}^{(\nu)l-2} \right) \ast G_{\nu} - 2H_{ij}^{(\nu)l-1} \ast G_{\nu}. \]

Note that by the use of the convolution rule all terms involve only first derivatives of the metric \( g_{ij} \). Using (51) we can compute the functions \( h_{ij}^{(\nu)l} \), \( 1 \leq i,j \leq n \). Next we consider the series \( h_{ij}^{(\nu)l} \), \( l \geq 1 \), \( 1 \leq i,j \leq n \). In order to prove convergence in a strong function space we consider for \( l \geq 2 \) the functional series
\[ h_{ij}^{(\nu)l} = h_{ij}^{(\nu)1} + \sum_{m=2}^{l} \delta h_{ij}^{(\nu)m}, \]
where
\[ \delta h_{ij}^{(\nu)m} = h_{ij}^{(\nu)m} - h_{ij}^{(\nu)m-1}. \]
We have
\[
\delta h_{ij}^{(ν)|l} = -\left( g_{i0}^{(ν)|l-1} 2g^{(ν)|l-1,0k} h_{ij}^{(ν)|l-1} \right) \ast G_{ν,k} \\
+ \left( g_{i0}^{(ν)|l-1} 2g^{(ν)|l-1,0k} h_{ij}^{(ν)|l-1} \right) \ast G_{ν} + \left( g_{i0}^{(ν)|l-1,km} g_{ij,k}^{(ν)|l-1} \right) \ast G_{ν,m} \\
- \left( g_{i0}^{(ν)|l-1,km} g_{ij,k}^{(ν)|l-1} \right) \ast G_{ν,m} - 2H_{ij}^{(ν)|l-1} \ast G_{ν} \\
+ \left( g_{i0}^{(ν)|l-2} 2g^{(ν)|l-1,0k} h_{ij}^{(ν)|l-2} \right) \ast G_{ν,k} \\
- \left( g_{i0}^{(ν)|l-2} 2g^{(ν)|l-1,0k} h_{ij}^{(ν)|l-2} \right) \ast G_{ν} - \left( g_{i0}^{(ν)|l-1,km} g_{ij,k}^{(ν)|l-2} \right) \ast G_{ν,m} \\
+ \left( g_{i0}^{(ν)|l-2,km} g_{ij,k}^{(ν)|l-2} \right) \ast G_{ν,m} + 2H_{ij}^{(ν)|l-2} \ast G_{ν}.
\]

We have essentially \( n(n-1)/2 \) metric functional increments \( δg_{i,j}^{(ν)|l} \), \( n^2(n-1)/2 \) metric functional increments \( δg_{i,j}^{(ν)|l} \) and \( n(n-1)/2 \) increments \( δh_{i,j}^{(ν)|l} \). Accordingly, we define
\[
G_{i}^{(ν)} = (δg_{i,j}^{(ν)|l})_{1≤i≤j≤n}, \quad G_{i,l}^{(ν)} = (δg_{i,j}^{(ν)|l})_{1≤i≤j≤n, 1≤k≤n}
\]
and
\[
\mathcal{K}_{i,l}^{(ν)} = (δh_{i,j}^{(ν)|l})_{1≤i≤j≤n}
\]

Hence with \( d = n(n+1)/2 + n^2(n+1)/2 + n(n+1)/2 \) and as we have a delay in the iteration above the functional
\[
F : \left[ C_{H^{2,ν}}^{m,l} (\overline{D_T}) \right]^{2d} \rightarrow \left[ C_{H^{2,ν}}^{m,l} (\overline{D_T}) \right]^{2d}
\]
\[
\left( G_{i}^{(ν)} , G_{i,l}^{(ν)} , G_{i-1,l}^{(ν)} , G_{i-1}^{(ν)} , \mathcal{K}_{i-1,l}^{(ν)} , \mathcal{K}_{i-1}^{(ν)} \right)^T
\]
\[
= F \left( G_{i-1}^{(ν)} , G_{i-1,l}^{(ν)} , G_{i-2,l}^{(ν)} , G_{i-2}^{(ν)} , \mathcal{K}_{i-2,l}^{(ν)} , \mathcal{K}_{i-2}^{(ν)} \right)
\]

with the function space \( C_{H^{2,ν}}^{m,l} (\overline{D_T}) \) which is defined as
\[
C_{H^{2,ν}}^{m,l} (\overline{D_T}) = \{ f \in C_{o}^{m,l} (\overline{D_T}) \mid \forall t \in [0,T] : f(t, \cdot) \in H^{2} \}
\]

along with the functions space \( C_{H^{2,ν}}^{m,l} (\overline{D_T}) \) in (75) below.

Note that by the use of the the convolution rule we have obtained that representations of the increments have only first order derivatives of the metric functions \( g^{(ν)|p} \) for some \( p ≥ -1 \). Some of the related Gaussian \( G_{ν} \)-terms then get first order derivatives, where these Gaussians have local standard \( L^1 \)-estimates in the time interval \((0, T)\) (open at 0) and on a ball around a fixed spatial argument \( x \) (cf. also the proof of Lemma 3.2 below. Similarly, for the \( h_{i,j}^{(ν)|l} \)-terms. Outside the ball around a fixed argument \( x \) we surely have \( L^1 \)-estimates of the Gaussian such that we can apply Young inequalities in order to get contraction of \( F \) on same time interval, i.e., we have
Lemma 3.1. There is a time horizon $T > 0$ such that the map $F$ is a contraction on the function space $\left[ C^{1,2}_0(D_T) \right]^{2d}$

The latter contraction result leads to the pointwise limit

$$g^{(\nu)}_{ij} = \lim_{l \to \infty} g^{(\nu)}_{ij,l} \in C^{1,2}((0,T],[0,\nu]),$$

where

$$g^{(\nu)}_{ij}(0,.) \in C^{1,\delta}([\mathbb{R}^n]) \cap H^1.$$  \hspace{1cm} (60)

We denote

$$h^{(\nu)}_{ij} = \lim_{l \to \infty} h^{(\nu)}_{ij,l}$$

for all $1 \leq i, j \leq n$. Accordingly we write for all $1 \leq i, j \leq n$

$$H^{(\nu)}_{ij} = \lim_{l \to \infty} H^{(\nu)}_{ij,l}$$

where we recall that we have

$$g^{(\nu)}_{ij}(0,.) = g^{(\nu)}_{ij,0}, \quad g^{(\nu)}_{ij}(0,.) = h^{(\nu)}_{ij,0}, \quad g^{(\nu)}_{ij,k}(0,.) = g^{(\nu)}_{0ij,k}$$

for the initial data (which do not depend on the iteration index $l \geq -1$. We observe

Lemma 3.2. The contraction constant $c \in (0,1)$ of Lemma 3.1 can be chosen independently of the viscosity constant $\nu > 0$.

Proof. We have to show that there is an $L^1((0,T) \times [\mathbb{R}^n])$ upper bound of Gaussian $G_{\nu}$ and its first order derivatives $G_{\nu,k}, 1 \leq k \leq n$ which are independent of the viscosity $\nu > 0$. First, for $\Delta x = x - y$ and $\Delta s = \nu \Delta t$ we have for the essential factor of the Gaussian for some $C > 0$ and $\delta > 0$ (where $z = \Delta_x$)

$$\frac{1}{\sqrt{\Delta s}} \exp \left( -\frac{\Delta x^2}{\Delta s} \right) = \left( \frac{\Delta s^2}{\Delta x} \right)^{\gamma - \delta} \frac{1}{\Delta s^\gamma |\Delta x|^{n+2\delta}} \exp \left( -\frac{\Delta x^2}{\Delta s} \right)$$

\hspace{1cm} (64)

$$\leq \frac{1}{\Delta s^\gamma |\Delta x|^{n+2\delta}} \sup_{z \in [\mathbb{R}]} (z^2)^{\gamma - \delta} \exp (-z^2) = \frac{C}{\Delta s^\gamma |\Delta x|^{n+2\delta}}.$$  \hspace{1cm} (65)

Similarly, for the first order spatial derivatives of the (essential factor of the) Gaussian we have for $\delta \in \left( \frac{3}{4}, 1 \right)$ in a ball $B_R(x)$ of radius $R > 0$ the estimate

$$\left( \frac{1}{\sqrt{\Delta s}} \exp \left( -\frac{\Delta x^2}{\Delta s} \right) \right)_{i,j} \leq \frac{C}{\Delta s^\gamma |\Delta x|^{n+2\delta}}.$$  \hspace{1cm} (66)

Hence for $T > 0$ and some $c' > 0$ we have

$$\int_0^T \int_{B_R(x)} \frac{d\Delta t d\Delta x}{\Delta s^\gamma |\Delta x|^{n+1+2\delta}} \leq \frac{c'}{\nu^\delta} |T|^{1-\delta} R^{2\delta-1}.$$  \hspace{1cm} (67)

This means that for $R = \nu^2$ we have $R^{2\delta-1} = \nu^{4\delta-2}$ and $\delta \in \left( \frac{3}{4}, 1 \right)$ we have

$$\frac{c'}{\nu^\delta} |T|^{1-\delta} \nu^{4\delta-2} = c' |T|^{1-\delta} \nu^{3\delta-2} \downarrow 0$$

\hspace{1cm} (68)
as $\nu \downarrow 0$. Furthermore, for the complement $\mathbb{R}^n \setminus B_R(x)$ with $R = \nu^2$ we have for some finite constant $C > 0$ the estimate

$$\left| \int_{|\Delta x| \geq \nu^2} (-\Delta)^{\frac{3}{2}} \frac{e^{-\frac{|x|^2}{2\nu^2}}}{\nu^3} \exp \left( -\frac{|x|^2}{2\nu^2} \right) dy \right| \leq \frac{1}{\sqrt{\nu^2}} \exp \left( -\frac{|x|^2}{2\nu^2} \right) \right|_{\nu^2}$$

$$\leq \left| \frac{1}{\sqrt{\nu^2}} \exp \left( -\frac{|x|^2}{2\nu^2} \right) \right|_{\nu^2}$$

$$\leq \frac{1}{\sqrt{\nu^2}} \left| \frac{1}{\sqrt{\nu^2}} \exp \left( -\frac{|x|^2}{2\nu^2} \right) \right|_{\nu^2}$$

$$\leq \frac{C}{\nu^2}$$

It follows that we have a uniform bound

$$\sup_{\nu > 0} \left( \| G_{\nu} \|_{L^1(0,T)} + \sum_{k=1}^n |G_{\nu,k} \|_{L^1(0,T) \times \mathbb{R}^n} \right) \leq C$$

for some $C > 0$.

Next we choose a sequence $\nu_l$, $l \geq 1$ with $\lim_{l \to \infty} \nu_l = 0$, and consider the functional series $g_{ij}^{(\nu)_l}$, $l \leq 2$, where we consider the representations

$$g_{ij}^{(\nu)_l} = g_{ij}^{(\nu)_0} + \sum_{p=2}^l \delta g_{ij}^{(\nu)_p}.$$  

(70)

The componentwise differentiation (up to second order) of the limit $g_{ij} := \lim_{l \to \infty} g_{ij}^{(\nu)_l}$ of this functional series is a delicate matter. However, we proceed as follows. First we consider functions on the domain $D_T = [0, T] \times [-\frac{\pi}{2}, 0^+]$.

$$g_{ij}^{(\nu)_l}(\cdot, \tan((\cdot)) : D_T \to \mathbb{R}, \quad g_{ij}^{(\nu)_l}(\cdot, \tan((\cdot)) : D_T \to \mathbb{R}$$

(71)

for all 'spatial indices' $1 \leq i, j, k, l \leq n$. Here we denote $\tan(x) = (\tan(x^1), \ldots, \tan(x^n))^T$.

Since $g_{ij}^{(\nu)_l}(t, \cdot) \in H^2 \cap C^2$ for all $t \in (0, T]$ for all these functions we have

$$\forall t \in [0, T] : \quad g_{ij}^{(\nu)_l}(t, \tan(\frac{\pi}{2})) = 0$$

$$\forall t \in [0, T] : \quad g_{ij}^{(\nu)_l}(t, \tan(\frac{\pi}{2})) = 0$$

(72)

Hence we have periodic extensions

$$d_{ij}^{(\nu)_l} : D_T \to \mathbb{R}, \quad d_{ij}^{(\nu)_l}(t, \tan((\cdot)) : D_T \to \mathbb{R}, \quad d_{ij}^{(\nu)_l}(t, \tan(\frac{\pi}{2})) = 0$$

(73)
Note that we can recover the viscosity limit \( g_{ij} (\nu \downarrow 0) \) and its derivatives up to second order from the limits of the functions \( d_{ij}^{\nu,l} \) and \( e_{ij,k}^{\nu,l} \) respectively. We denote the standard closure of the domain \( D_T \) by \( \overline{D_T} \). We are interested in the strong convergence of the increments

\[
\begin{align*}
&d_{ij}^{\nu,l} := d_{ij}^{\nu,l} - d_{ij}^{\nu,l}(0,\cdot), \quad e_{ij,k}^{\nu,l} := e_{ij,k}^{\nu,l} - e_{ij,k}^{\nu,l}(0,\cdot) \\
&f_{ij,k}^{\nu,l} := f_{ij,k}^{\nu,l} - f_{ij,k}^{\nu,l}(0,\cdot).
\end{align*}
\]  

(74)

These increments are located in the appropriate function space \( (m=1 \text{ and } l=2) \)

\[
C_{0,l}^{m,l}(\overline{D_T}) := \{ f : \overline{D_T} \rightarrow \mathbb{R} | f \in C_{0,l}^{m,l} \text{ and } \forall x f(0,x) = 0 \}.
\]  

(75)

For functional series in this function space we may use the following classical result

**Theorem 3.3.** Consider a functional series \( F_m := \sum_{n=1}^{m} f_n, \ m \geq 1 \) with \( f_n \in C_{0,l}^{0,1}(\overline{D_T}) \). Assume that \( F_m(c), \ m \geq 1 \) converges for fixed \( c \in D_T \), and assume that the first order spatial derivative functional series \( F_{m,1} := \sum_{n=1}^{m} f_{n,1}^{\nu,l}, \ m \geq 1 \) converges uniformly to a function \( F = \lim_{m \uparrow \infty} \sum_{n=1}^{m} f_n, \) and such that for all \( (t,x) \in \overline{D_T} \)

\[
F_{i,t}(t,x) = \lim_{m \uparrow \infty} \sum_{n=1}^{m} f_{n,i}(t,x) \text{ holds.}
\]  

(76)

**Lemma 3.4.** There is a sequence \( \nu_l, l \geq 1 \) with \( \lim_{l \uparrow \infty} \nu_l = 0 \) such that the limits of the functional increments in \((73)\) are in the function space \((74)\), i.e.,

\[
d^{0} := \lim_{l \uparrow \infty} d_{ij}^{\nu_l,1}, \quad e^{0} := \lim_{l \uparrow \infty} e_{ij,k}^{\nu_l,1}, \quad f^{0} := \lim_{l \uparrow \infty} f_{ij,k}^{\nu_l,1} \in C_{0,l}^{0,1}(\overline{D_T})
\]  

(77)

Hence,

\[
g_{ij} = \lim_{l \uparrow \infty} g_{ij}^{(\nu_l),l} \in C^{0,2}(\overline{D_T}), \quad g_{ij,k} = \lim_{l \uparrow \infty} g_{ij,k}^{(\nu_l),l} \in C^{0,1}(\overline{D_T}),
\]  

(78)

and, hence, for the first order time derivative, we get

\[
h_{ij} = \lim_{l \uparrow \infty} h_{ij}^{(\nu_l),l} \in C^{0,1}(\overline{D_T}).
\]  

(79)

Plugging \( g_{ij}^{(\nu_l)} \) into the harmonic field equation and using the local contraction result we observe that the limit \( g_{ij}, g_{ij,k}, h_{ij} \) in \((75)\) and in \((76)\) satisfies the harmonic field equation. This may also be observed as follows: abbreviating

\[
f = (f_1, \cdots, f_M) = \left( \begin{pmatrix} g_{0ij}^{(\nu)} \end{pmatrix}_{1 \leq i,j \leq n}, \begin{pmatrix} g_{0ij,k}^{(\nu)} \end{pmatrix}_{1 \leq i,j,k \leq n}, \begin{pmatrix} h_{0ij}^{(\nu)} \end{pmatrix}_{1 \leq i,j \leq n} \right)^T,
\]  

(80)

and

\[
\nu^{(\nu)} = \begin{pmatrix} g_{ij}^{(\nu)} \end{pmatrix}_{1 \leq i,j \leq n}, \begin{pmatrix} g_{ij,k}^{(\nu)} \end{pmatrix}_{1 \leq i,j,k \leq n}, \begin{pmatrix} h_{ij}^{(\nu)} \end{pmatrix}_{1 \leq i,j \leq n} \right)^T,
\]  

(81)
and
\[
F(\nu) = \left( F_1 \left( \left( h_{ij}^{(\nu)} \right)_{1 \leq i, j \leq n} \right), F_2 \left( \left( h_{ij,k}^{(\nu)} \right)_{1 \leq i, j, k \leq n} \right), F_3 \left( \left( v^{(\nu)} \right) \right) \right)^T,
\]

(82)
where
\[
F_1 \left( \left( h_{ij}^{(\nu)} \right)_{1 \leq i, j \leq n} \right) = \left( h_{ij}^{(\nu)} \right)^T_{1 \leq i, j \leq n},
\]

(83)
\[
F_2 \left( \left( h_{ij,k}^{(\nu)} \right)_{1 \leq i, j, k \leq n} \right) = \left( h_{ij,k}^{(\nu)} \right)^T_{1 \leq i, j, k \leq n},
\]

(84)
and
\[
F_3 \left( v^{(\nu)} \right) = \left( -\nu \partial v^{(\nu)}_0 - \frac{2\nu}{2^\nu} \partial x^k + \partial g^{(\nu)} \partial x^m - 2H^{(\nu)} \right)^T_{1 \leq i, j \leq n},
\]

(85)
with the obvious identifications. The equation in (39) may then be abbreviated as
\[
v^{\nu}_{\nu,t} - \nu \Delta v^{\nu} = F(v^{\nu}), \quad v^{\nu}(0,.) = f
\]

(86)
where in the limit \( \nu \downarrow 0 \) the function \( v := v^0 := \lim_{\nu \downarrow 0} v^{\nu} \) satisfies the equation
\[
v_{\nu,t} = F(v).
\]

(87)
For \( v^{\nu,f} := v^{\nu} - f \) we have \( v^{\nu,f}(0,.) \equiv 0 \) and
\[
v^{\nu,f}_{\nu,t} - \nu \Delta v^{\nu,f} = \nu \Delta f + F(v^{\nu,f} + f)
\]

(88)
This leads to the representation
\[
v^{\nu,f} = \nu \Delta f \ast G_\nu + F(v^{\nu,f} + f) \ast G_\nu,
\]

(89)
where the convolution is understood componentwise. The latter statement has a classical interpretation only for smoothed initial data \( f \ast G_\nu \). However, we can rewrite in terms of first order derivative, i.e., we have
\[
v^{\nu,f} = \nu \sum_{i=1}^n f_i \ast G_{\nu,i} + F(v^{\nu,f} + f) \ast G_\nu,
\]

(90)
where the derivative \( .,i \) is understood componentwise of course, i.e.,
\[
\text{div } f = (f_{1,i}, \ldots, f_{n,i})^T.
\]

(91)
Hence, we have
\[
v^f = \lim_{\nu \downarrow 0} v^{\nu,f} = \lim_{\nu \downarrow 0} \nu \sum_{i=1}^n f_i \ast G_{\nu,i} + \lim_{\nu \downarrow 0} F(v^{\nu,f} + f) \ast G_\nu,
\]

(92)
According to our analysis of the Gaussian above the first term on the rightside of the latter equation cancels, and, using continuity of \( F \), we observe that the components of \( v^{\nu,f} \) converge in the function space \( L^2 \) as \( \nu \downarrow 0 \). Considering the time derivative of the last term on the right side of equation (92) for fixed \( \nu > 0 \) and then going to the limit \( \nu \downarrow 0 \) we observe that the limit function \( v^f \) satisfies
\[
v^f_{\nu,t} = F(v^{\nu,f} + f),
\]

(93)
which is equivalent of (87).
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