BLOW-UP CRITERION FOR THE 3D NON-RESISTIVE COMpressible MAGnetoHYDRODYNAMIC EQUATIONS

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Abstract. In this paper, we prove a blow-up criterion in terms of the magnetic field $H$ and the mass density $\rho$ for the classical (or strong) solutions to the 3D compressible isentropic MHD equations with zero magnetic diffusion and initial vacuum. More precisely, we show that the $L^\infty$ norms of $(H, \rho)$ control the possible blow-up (see [25],[30]) for classical (or strong) solutions, which means that if a solution of the compressible isentropic non-resistive MHD equations is initially smooth and loses its regularity at some later time, then the formation of singularity must be caused by losing the bound of the $L^\infty$ norm of $H$ or $\rho$ as the critical time approaches. Our criterion is similar to [25],[29] for the 3D compressible isentropic Navier-stokes equations.

1. Introduction

Magnetohydrodynamics is that part of the mechanics of continuous media which studies the motion of electrically conducting media in the presence of a magnetic field. The dynamic motion of fluid and magnetic field interact strongly on each other, so the hydrodynamic and electrodynamic effects are coupled. In 3D space, the compressible isentropic MHD equations in a domain $\Omega$ of $\mathbb{R}^3$ can be written as

\[
\begin{align*}
H_t - \text{rot}(u \times H) &= -\text{rot}\left(\frac{1}{\sigma}\text{rot}H\right), \\
\text{div}H &= 0, \\
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \text{div}\tau + \text{rot}H \times H.
\end{align*}
\]

In this system, $x \in \Omega$ is the spatial coordinate; $t \geq 0$ is the time; $H = (H^1, H^2, H^3)$ is the magnetic field; $0 < \sigma \leq \infty$ is the electric conductivity coefficient; $\rho$ is the mass density; $u = (u^1, u^2, u^3) \in \Omega$ is the velocity of fluids; $P$ is the pressure law satisfying

\[ P = A\rho^\gamma, \quad \gamma > 1, \]

where $A$ is a positive constant and $\gamma$ is the adiabatic index; $\tau$ is the stress tensor given by

\[ \tau = 2\mu D(u) + \lambda \text{div}u I, \quad D(u) = \frac{\nabla u + (\nabla u)^T}{2}, \]

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where $D(u)$ is the deformation tensor, $\mathbb{I}_3$ is the $3 \times 3$ unit matrix, $\mu$ is the shear viscosity coefficient, $\lambda$ is the bulk viscosity coefficient, $\mu$ and $\lambda$ are both real constants,

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0,$$

(1.4)

which ensures the ellipticity of the Lamé operator (see [19]). Although the electric field $E$ doesn’t appear in system (1.5), it is indeed induced according to a relation

$$E = \frac{1}{\sigma} \text{rot} H - u \times H$$

by moving the conductive flow in the magnetic field.

The MHD system (1.1) describes the macroscopic behavior of electrically conducting compressible (isentropic) fluids in a magnetic field. It is reasonable to assume that there is no magnetic diffusion (i.e. $\sigma = +\infty$) when the conducting fluid considered is of a very high conductivity, which occur frequently in many cosmical and geophysical problems. Then we need to consider the following system:

$$\begin{cases}
H_t - \text{rot}(u \times H) = 0, \\
\text{div} H = 0, \\
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \text{div} T + \text{rot} H \times H,
\end{cases}$$

(1.5)

which is the so called viscous and non-resistant MHD equations (see [5] [11] [13] [19] [20]).

The aim of this paper is to give a blow-up criterion of classical (or strong) solutions to system (1.5) in a bounded, smooth domain $\Omega \subset \mathbb{R}^3$ with the initial-boundary condition:

$$(H, \rho, u)|_{t=0} = (H_0(x), \rho_0(x), u_0(x)), \quad x \in \Omega; \quad u|_{\partial\Omega} = 0.$$  

(1.6)

Throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:

$${D^k}_r = \{ f \in L^1_{\text{loc}}(\Omega) : |f|_{D^k} = |\nabla^k f|_{L^r} < +\infty \}, \quad D^k = {D^k}_2,$$

$${D^k}_0 = \{ f \in L^6(\Omega) : |f|_{D^k} = |\nabla f|_{L^2} < \infty \text{ and } f|_{\partial\Omega} = 0 \}, \quad ||(f, g)||_X = ||f||_X + ||g||_X,$$

$$||f||_{W^{m,r}} = ||f||_{W^{m,r}(\Omega)}, \quad ||f||_s = ||f||_{H^s(\Omega)}, \quad ||f||_p = ||f||_{L^p(\Omega)};$$

$$|f|_{D^k} = ||f||_{D^k(\Omega)}, \quad |f|_{D^k} = ||f||_{D^k(\Omega)};$$

where $f = (f^1, f^2, f^3)^T \in \mathbb{R}^3$ or $f \in \mathbb{R}$, $g = (g^1, g^2, g^3)^T \in \mathbb{R}^3$ or $g \in \mathbb{R}$, $X$ is some Sobolev space, $A = (a_{ij})_{3 \times 3}$ and $B = (b_{ij})_{3 \times 3}$ are both $3 \times 3$ matrixes. A detailed study of homogeneous Sobolev space may be found in [12].
As has been observed in [10], which proved the existence of unique local strongs and classical solutions with initial vacuum, in order to make sure that the IBVP (1.5)-(1.6) with vacuum is well-posed, the lack of a positive lower bound of the initial mass density \( \rho_0 \) should be compensated with some initial layer compatibility condition on the initial data \((H_0, \rho_0, u_0, P_0)\). For classical solutions, it can be shown as

**Theorem 1.1.** Let constant \( q \in (3, 6] \). If \((H_0, \rho_0, u_0, P_0)\) satisfies
\[
(H_0, \rho_0, P_0) \in H^2 \cap W^{2,q}, \quad \rho_0 \geq 0, \quad u_0 \in D^1_0 \cap D^2, 
\]
and the compatibility condition
\[
Lu_0 + \nabla P_0 - \text{rot}H_0 \times H_0 = \sqrt{\rho_0}g_1 
\]
for some \( g_1 \in L^2 \) and
\[
Lu_0 = -\mu \Delta u_0 - (\lambda + \mu) \nabla \text{div}u_0, 
\]
then there exists a small time \( T_* \) and a unique solution \((H, \rho, u, P)\) to IBVP (1.5)-(1.6) satisfying
\[
(H, \rho, P) \in C([0, T_*]; H^2 \cap W^{2,q}), \quad u \in C([0, T_*]; D^1_0 \cap D^2) \cap L^{3q}(0, T_*; D^3), \quad u_t \in L^2([0, T_*]; D^1_0), \quad t^{\frac{1}{2}}u_t \in L^\infty([0, T_*]; D^3), 
\]
\[
\sqrt{\rho}u_{tt} \in L^2([0, T_*]; D^1_0), \quad t^{\frac{1}{2}}u_{tt} \in L^\infty([0, T_*]; D^3), 
\]
where \( p_0 \) is a constant satisfying \( 1 \leq p_0 \leq \frac{4q}{5q-6} \in (1, 2) \). Moreover, if \( 0 < T < +\infty \) is the maximum time for our classical solution, then
\[
\lim_{T \to T} \sup \int_0^T |D(u)|_{L^\infty(\Omega)} \, dt = \infty. 
\]
want to know the mechanism of break down and the structure of possible singularities? The similar question has been studied for the incompressible Euler equation by Beale-Kato-Majda (BKM) in their pioneering work [3], which showed that the $L^\infty$-bound of vorticity rot $u$ must blow up. Later, Ponce [24] rephrased the BKM-criterion in terms of the deformation tensor $D(u)$. The same result as [24] has been proved by Huang-Li-Xin [16] for compressible isentropic Navier-Stokes equations, which can be shown: if $0 < T < +\infty$ is the maximum time for strong solution, then

$$\limsup_{T \to T} \int_0^T |D(u)|_{L^\infty(\Omega)} dt = \infty. \quad (1.11)$$

Recently, the similar blow-up criterions as (1.11) have been obtained for the 3D compressible isentropic MHD equations in Xu-Zhang [32] which can be shown:

$$\limsup_{T \to T} \int_0^T |\nabla u|_{L^\infty(\Omega)} dt = \infty. \quad (1.12)$$

However, in Zhu [33], the same blow-up criterion as (1.11) for classical (or strong) solutions to IBVP (1.5)-(1.6) has been proved via a subtle estimate for the magnetic field $H$. Some similar results also can be seen in Chen-Liu [22] or Lu-Du-Yao [23]. For the strong solutions to 3D compressible isentropic Navier-stokes equations, Sun-Wang-Zhang [28] proved

$$\limsup_{T \to T} |\rho|_{L^\infty([0,T];L^\infty(\Omega))} = \infty,$$

under the physical assumption (1.4) and $\lambda < 7\mu$.

Our main result in the following theorem shows that the $L^\infty$ norms of the magnetic field $H$ and the mass density $\rho$ control the possible blow-up (see [25][30]) for classical (or strong) solutions, which means that if a solution of the compressible MHD equations is initially regular and loses its regularity at some later time, then the formation of singularity must be caused by losing the bound of the $L^\infty$ norm of $H$ or $\rho$ as the critical time approaches.

**Theorem 1.3.** Let the viscosity coefficients $(\mu, \lambda)$ satisfy

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0, \quad 3\lambda < (29 - \alpha)\mu, \quad (1.13)$$

for any sufficiently small $\alpha > 0$, and $(H_0, \rho_0, u_0, P_0)$ satisfy (1.7)-(1.8). If $(H, \rho, u, P)$ is a classical solution to IBVP (1.5)-(1.6) obtained in Theorem 1.1 and $0 < T < \infty$ is the maximal time of its existence, then

$$\limsup_{T \to T} (|\rho|_{L^\infty([0,T];L^\infty(\Omega))} + |H|_{L^\infty([0,T];L^\infty(\Omega))}) = \infty. \quad (1.14)$$

Moreover, our criterion also holds for the strong solutions obtained in [10].

**Remark 1.1.** We introduce the main ideas of our proof for Theorem 1.3, some of which are inspired by the arguments used in [16][28][29][33].

1) We improve the methods used in [28] to obtain the estimate (3.5) under the assumption (1.13). In order to prove (3.5), the restriction $\lambda < 7\mu$ plays an key role in the analysis shown in [28], and actually, it is only used to get the upper bound of $\int_\Omega \rho|u(t)|^r dx$ for some $r > 3$. However, Wen-Zhu [29] obtain the upper bound of $\int_\Omega \rho|u(t)|^r dx$ under the assumption $3\lambda < 29\mu$, which as a byproduct extends the conclusions obtained in [28]. Compared with [29], we need to deal with the magnetic term appearing in the momentum...
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equations, and due to the initial vacuum, we obtain the upper bound of \( \int_\Omega \rho|u(t)|^r \, dx \) for \( r \in (3, 7/2) \) under the assumption (1.13) for any sufficiently small \( \alpha > 0 \).

In order to get a restriction of \( \mu \) and \( \lambda \) as better as possible, the crucial ingredient to relax the additional restrictions to \( 3\lambda < (29 - \alpha)\mu \) has been observed that

\[
\|\nabla u\|^2 = |u|^2 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + |\nabla |u||^2, \tag{1.15}
\]

for \( |u| > 0 \), and thus

\[
\int_{\Omega \cap |u| > 0} |u|^{r-2} |\nabla |u||^2 \, dx \geq (1 + \phi(\varepsilon_0, \varepsilon_1, r)) \int_{\Omega \cap |u| > 0} |u|^{r-2} |\nabla |u||^2 \, dx, \tag{1.16}
\]

if

\[
\int_{\Omega \cap |u| > 0} |u| \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 \, dx \geq \phi(\varepsilon_0, \varepsilon_1, r) \int_{\Omega \cap |u| > 0} |u|^{r-2} |\nabla |u||^2 \, dx. \tag{1.17}
\]

for some positive function \( \phi(\varepsilon_0, \varepsilon_1, r) \) near \( r = 3 \). The details can be seen in Lemma 3.2.

II) If \( |\rho(0, T; L^\infty(\Omega)) \leq \mu \) and \( |H| L^\infty([0, T]; L^\infty(\Omega)) \) are bounded, we can obtain a high integrability of velocity \( u \), which can be used to control the nonlinear term (See Lemmas 3.2, 3.3). The argument used in [23] is introduced to control the upper bound of \( |\nabla u|_2 \), and a important observation has been shown in Lemma 3.3 that

\[
\begin{cases}
\mathbb{B} = H_t \otimes H + H \otimes H_t \\
= (H^i H^k \partial_k u^i + H^i H^k \partial_k u^i - H^i H^j \partial_k u^j)_{(ij)} - \text{div}(H \otimes H \otimes u), \\
\mathbb{C} = H : H_t = H : (H \cdot \nabla u - u \cdot \nabla H - H \text{div}u) \\
= (H : \nabla u \cdot H - \frac{1}{2}|H|_2^2 \text{div}u) - \frac{1}{2} \text{div}(u|H|_2^2),
\end{cases}
\tag{1.18}
\]

from which, we successfully avoid the difficulty coming from the strong coupling between the magnetic field and velocity when the magnetic diffusion vanishes.

The next difficulty is to control the mass density \( \rho \) and the magnetic field \( H \), both which satisfy hyperbolic equations. To do this, we need to make sure that the velocity \( u \) is bounded in \( L^1([0, T]; D_4^1(\Omega)) \). On the other hand, in order to prove \( u \in L^1([0, T]; D_4^1(\Omega)) \), we have to obtain some priori bounds for \( \nabla \rho \) and \( \nabla H \). Furthermore, the magnetic term in the momentum equation will bring extra difficulty to us. However, via using the argument from [17] and the structure of the magnetic equations, in Lemma 3.3 we show that

\[
\Lambda = \int_\Omega \hat{u} \cdot \left[ \text{div}(H \otimes H - \frac{1}{2}|H|_2^2 I_3) + \text{div}(\text{div}(H \otimes H - \frac{1}{2}|H|_2^2 I_3) \otimes u) \right] \, dx
= \int_\Omega \partial_k u^k H^i H^j \partial^j u^i \, dx + \int_\Omega \left( -\frac{1}{2} \partial_j u^j |H^k|^2 \partial_i u^i \right) \, dx.
\tag{1.19}
\]

Then we get the cancelation to the derivatives of \( \nabla \rho \) and \( \nabla H \) during our computation, which brings us the desired result. Finally via introducing the method used in [23], we will easily show that the solution \( (H, \rho, u, P) \) is indeed a classical one in \((0, T) \times \Omega\).

The rest of this paper is organized as follows. In Section 2, we give some important lemmas which will be used frequently in our proof. In Section 3, we give the proof for the blow-up criterion (1.14) for the classical solutions obtained in Section 3. Firstly in Sections 3.1-3.2, via assuming that the opposite of (1.14) holds, we show that the solution
in \([0, T] \times \Omega\) has the regularity that the strong solution has to satisfy obtained in \([10]\).

Then finally in Section 3.3, based on the estimates shown in Sections 3.1-3.2, we improve
the regularity of \((H, \rho, u, P)\) to make sure that it is also a classical one in \([0, T] \times \Omega\), which
contradicts our assumption.

\[\text{Lemma 2.3.}\]

In this section, we give some important Lemmas which will be used frequently in our
proof. The first one is important in the derivation of our estimate for the higher order
terms of the velocity \(u\) (see Lemma 3.10), which can be seen in the Remark 1 of \([4]\).

\[\text{Lemma 2.1.}\]

If \(h(t, x) \in L^2(0, T; L^2)\), then there exists a sequence \(s_k\) such that
\[s_k \to 0, \quad \text{and} \quad s_k|h(s_k, x)|^2_2 \to 0, \quad \text{as} \quad k \to \infty.\]

The following one is some Sobolev inequalities obtained from the well-known Gagliardo-
Nirenberg inequality:

\[\text{Lemma 2.2.}\]

For \(l \in (3, \infty)\), there exists some generic constant \(C > 0\) that may depend
on \(l\) such that for \(f \in D_0^1(\Omega), g \in D_0^1 \cap D^2(\Omega)\), and \(h \in W^{l,1}(\Omega)\), we have
\[|f|_6 \leq C|f|_{D_0^1}, \quad |g|_\infty \leq C|g|_{D_0^1 \cap D^2}, \quad |h|_\infty \leq C\|h\|_{W^{l,1}}. \quad (2.1)\]

Next we consider the following boundary value problem for the Lamé operator \(L\):
\[
\begin{cases}
\mu \Delta U + (\mu + \lambda)\nabla \div U = F, & \text{in } \Omega, \\
U(t, x) = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where \(U = (U^1, U^2, U^3), F = (F^1, F^2, F^3)\). It is well known that under the assumption
(1.4), (2.2) is a strongly elliptic system. If \(F \in W^{-1,2}(\Omega)\), then there exists a unique weak
solution \(U \in D_0^1(\Omega)\). We begin with recalling various estimates for this system in \(L^q(\Omega)\)
spaces, which can be seen in \([1]\).

\[\text{Lemma 2.3.}\]

Let \(l \in (1, +\infty)\) and \(u\) be a solution of (2.2). There exists a constant \(C\)
depending only on \(\lambda, \mu, p\) and \(\Omega\) such that the following estimates hold:
(1) if \(F \in L^1(\Omega)\), then we have
\[\|U\|_{W^{2,1}} \leq C\|F\|_l; \quad (2.3)\]
(2) if \(F \in W^{-1,1}(\Omega)\) (i.e., \(F = \div f\) with \(f = (f_{ij})_{3 \times 3}, f_{ij} \in L^1(\Omega)\)), then we have
\[\|U\|_{W^{1,1}} \leq C\|f\|_l; \quad (2.4)\]
(3) if \(F = \div f\) with \(f_{ij} = \partial_k h_{ij}^k\) and \(h_{ij}^k \in W^{1,1}_0(\Omega)\) for \(i, j, k = 1, 2, 3\), then we have
\[\|U\|_l \leq C\|h\|_l. \quad (2.5)\]

Moreover, we need an endpoint estimate for \(L\) in the case \(l = \infty\). Let \(BMO(\Omega)\) stand
for the John-Nirenbergs space of bounded mean oscillation whose norm is defined by:
\[\|F\|_{BMO(\Omega)} = \|f\|_{L^2(\Omega)} + |f|_{BMO}; \quad (2.6)\]
with
\[
\begin{align*}
[f]_{B_{\infty}} & = \sup_{x \in \Omega, r \in (0,d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f(x)| \, dy, \\
J_{\Omega_r(x)} & = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y) \, dy,
\end{align*}
\] (2.7)

where \( \Omega_r(x) = B_r(x) \cap \Omega \), \( B_r(x) \) is the ball with center \( x \) and radius \( r \) and \( d \) is the diameter of \( \Omega \). \( |\Omega_r(x)| \) denotes the Lebesgue measure of \( \Omega_r(x) \). Note that
\[
[f]_{B_{\infty}} \leq 2 |f|_{\infty}. 
\] (2.8)

**Lemma 2.4.** If \( F = \text{div} f \) with \( f = (f_{ij})_{i,j} \in L^\infty(\Omega) \cap L^2(\Omega) \), then \( \nabla U \in \text{BMO}(\Omega) \) and there exists a constant \( C \) depending only on \( \lambda, \mu \) and \( \Omega \) such that
\[
|\nabla U|_{\text{BMO}} \leq C (|f|_{\infty} + |f|_2).
\] (2.9)

Due to \( \Omega \) is a bounded domain with smooth boundary, the estimate (2.9) can be found in [1] for a more general setting.

In the next lemma, we will give a variant of the Brezis-Wagner inequality [6], which also can be seen in [28].

**Lemma 2.5.** [6] Let \( \Omega \) be a bounded Lipschitz domain and \( f \in W^{1,l}(\Omega) \) with \( l \in (3, \infty) \). There exists a constant \( C \) depending on \( l \) and the Lipschitz property of \( \Omega \) such that
\[
|f|_{L^\infty(\Omega)} \leq C (1 + (|f|_{\infty} + |f|_2) \ln(e + |\nabla f|_l)).
\] (2.10)

From the conclusions obtained in Lemmas 2.4-2.5 under the assumptions shown in these both lemmas, we quickly deduce that
\[
|f|_{L^\infty(\Omega)} \leq C (1 + (|f|_{\infty} + |f|_2) \ln(e + |\nabla f|_l)),
\] (2.11)

which plays an important role in our proof (see Lemma 3.7).

Finally, for \((H, u) \in C^2(\Omega)\), there are some formulas based on \( \text{div} H = 0 \):

**Lemma 2.6.** Let \((H, \rho, u, P)\) be the unique classical solution obtained in Theorem 1.1 to IBVP (1.5)–(1.6) in \([0, T) \times \Omega\), then we have
\[
\begin{align*}
\text{rot}(u \times H) & = (H \cdot \nabla)u - (u \cdot \nabla)H - H \text{div}, \\
\text{rot} H \times H & = \text{div}(H \otimes H - \frac{1}{2}|H|^2 I_3) = -\frac{1}{2} \nabla |H|^2 + H \cdot \nabla H.
\end{align*}
\] (2.12)

**Proof.** It follows immediately from the following equality:
\[
\begin{align*}
\text{rot}a & = \frac{1}{2} \nabla (|a|^2) - a \cdot \nabla a, \\
\nabla \times (a \times b) & = (b \cdot \nabla) a - (a \cdot \nabla) b + (\text{div} b) a - (\text{div} a) b,
\end{align*}
\]

based on the fact that \( \text{div} H = 0 \). \qed
3. Blow-up criterion (1.14) for classical solutions

Now we prove (1.14). Let \((H, \rho, u, P)\) be the unique classical solution obtained in Theorem 1.1 to IBVP (1.5)–(1.6) in \([0, T) \times \Omega\). Due to \(P = A\rho^\gamma\), we quickly know that \(P\) satisfies

\[
P_t + u \cdot \nabla P + \gamma P \text{div} u = 0, \quad P_0 \in H^2 \cap W^{2,q}.
\] (3.1)

We first give the standard energy estimate that

**Lemma 3.1.**

\[
|\sqrt{\rho}u(t)|^2_2 + |H(t)|^2_2 + |P(t)|_1 + \int_0^T |\nabla u(t)|^2_2 dt \leq C, \quad 0 \leq t < T,
\]

where \(C\) only depends on \(C_0\) and \(T\) (any \(T \in (0, \overline{T}]\)).

**Proof.** We first show that

\[
\frac{d}{dt} \int_\Omega \left( \frac{1}{2} \rho |u|^2 + \frac{P}{\gamma - 1} + \frac{1}{2} H^2 \right) dx + \int_\Omega (\mu |\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2) dx = 0. \tag{3.2}
\]

Actually, (3.2) is classical, which can be shown by multiplying (1.5) by \(u\), (1.5) by \(\frac{|u|^2}{2}\) and (1.5) by \(H\), then summing them together and integrating the resulting equation over \(\Omega\) by parts, where we have used the fact

\[
\int_\Omega \text{rot} H \times H \cdot \text{u} dx = \int_\Omega -\text{rot}(u \times H) \cdot H dx. \tag{3.3}
\]

Next we assume that the opposite of (1.14) holds, i.e.,

\[
\limsup_{T \to \overline{T}} \left( |\rho|_{L^\infty([0, T]; L^\infty(\Omega))} + |\rho|_{L^\infty([0, T]; L^\infty(\Omega))} \right) = C_0 < \infty. \tag{3.4}
\]

Then in Sections 3.1-3.2, we need to show some regularity estimate on \([0, \overline{T}]\) for our classical solution \((H, \rho, u, P)\), which is the same as that of the strong solution obtained in [10]. Finally, based on the estimate obtained in Sections 3.1-3.2, in Section 3.3, we will improve the regularity of \((H, \rho, u, P)\) to make sure it is a classical one on \([0, \overline{T}] \times \Omega\) via the same method used in [33].

3.1. The lower order estimate for \(|u|_{L^\infty([0, \overline{T}]; D^1(\Omega))}\).

Now we improve the energy estimate obtained in Lemma 3.1.

**Lemma 3.2.** If \(\lambda < \frac{2\gamma - \alpha}{3} \mu\) for any sufficiently small \(\alpha > 0\), then there exists \(r \in \left(3, \frac{7}{2}\right)\) such that

\[
\int_\Omega \rho|u(t)|^r dx \leq C, \quad 0 \leq t < T, \tag{3.5}
\]

where \(C\) only depends on \(C_0\) and \(T\) (any \(T \in (0, \overline{T}]\)).
Proof. Firstly, multiplying (1.5) by \( r|u|^{r-2}u \) \((r \geq 3)\) and integrating the resulting equation over \( \Omega \) by parts, then we have

\[
\frac{d}{dt} \int_\Omega \rho |u|^r \, dx + \int_\Omega H_r \, dx = -r(r-2)(\mu + \lambda) \int_\Omega \text{div} |u|^{r-3} u \cdot \nabla |u| \, dx
\]

\[
+ \int_\Omega r \text{div} (|u|^{r-2}u) \, dx - \int_\Omega r (H \otimes H - \frac{1}{2} |H|^2 I_3) : \nabla (|u|^{r-2}u) \, dx,
\]

where

\[
H_r = r |u|^{r-2} (\mu |\nabla u|^2 + (\mu + \lambda) \text{div} |u|^2 + \mu (r-2) |\nabla |u|^2|).
\]

For any given \( \epsilon_1 \in (0, 1) \) and \( \epsilon_0 \in (0, \frac{1}{4}) \), we define a nonnegative function which will be determined in Step 2 as follows

\[
\phi(\epsilon_0, \epsilon_1, r) = \begin{cases} \frac{\mu \epsilon_1 (r-1)}{3 \left( -\frac{\epsilon_0(r-\epsilon_0)}{3} - \lambda + \frac{\epsilon_0(\mu + \lambda)}{4(r-1)} \right)}, & \text{if } \frac{r^2(\mu + \lambda)}{4(r-1)} - \frac{(4 - \epsilon_0) \mu}{3} - \lambda > 0, \\ 0, & \text{otherwise.} \end{cases}
\]

**Step 1:** we assume that

\[
\int_{\Omega \cap |u| > 0} |u| \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 \, dx > \phi(\epsilon_0, \epsilon_1, r) \int_{\Omega \cap |u| > 0} |u|^{r-2} |\nabla |u|^2| \, dx. \tag{3.7}
\]

A direct calculation gives for \(|u| > 0\):

\[
|\nabla u|^2 = |u|^2 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + |\nabla |u|^2|, \tag{3.8}
\]

which plays a important role in the proof. By (3.6) and the Cauchy’s inequality, we have

\[
\frac{d}{dt} \int_\Omega \rho |u|^r \, dx + \int_{\Omega \cap |u| > 0} H_r \, dx
\]

\[
= -r(r-2)(\mu + \lambda) \int_{\Omega \cap |u| > 0} \text{div} |u|^{r-2} |u|^{\frac{r-4}{2}} u \cdot \nabla |u| \, dx
\]

\[
+ \int_\Omega r \text{div} (|u|^{r-2}u) \, dx - \int_\Omega r (H \otimes H - \frac{1}{2} |H|^2 I_3) : \nabla (|u|^{r-2}u) \, dx \tag{3.9}
\]

\[
\leq r(\mu + \lambda) \int_{\Omega \cap |u| > 0} |u|^{r-2} |\text{div} u|^2 \, dx + \frac{r(r-2)^2(\mu + \lambda)}{4} \int_{\Omega \cap |u| > 0} |u|^{r-2} |\nabla |u|^2| \, dx
\]

\[
+ \int_\Omega r \text{div} (|u|^{r-2}u) \, dx - \int_\Omega r (H \otimes H - \frac{1}{2} |H|^2 I_3) : \nabla (|u|^{r-2}u) \, dx.
\]
Via Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we have

\[ J_1 \leq C \int_{\Omega} P|u|^{r-2} \nabla u \, dx \]
\[ \leq \left( \int_{\Omega} |u|^{r-2} \nabla u^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |u|^{r-2} P \, dx \right)^{\frac{1}{2}} \]
\[ \leq C \left( \int_{\Omega} |u|^{r-2} \nabla u^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (|u|^{r-2})^{\frac{6r}{2(r-1)}} \, dx \right)^{\frac{2(r-2)}{12r}} \left( \int_{\Omega} P^{\frac{6r}{4r+4}} \, dx \right)^{\frac{4r+4}{2r}} \]
\[ \leq C \left( \int_{\Omega} |u|^{r-2} \nabla u^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (\nabla |u|^2)^2 \, dx \right)^{\frac{(r-2)}{2r}} \]
\[ \leq \frac{1}{2} \mu r \epsilon_0 \int_{\Omega} |u|^{r-2} \nabla u^2 \, dx + C(\mu, r, \epsilon_0), \tag{3.10} \]

\[ J_2 \leq C \int_{\Omega} |H|^2 |u|^{r-2} \nabla u \, dx \]
\[ \leq \left( \int_{\Omega} |u|^{r-2} \nabla u^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |u|^{r-2} H^2 \, dx \right)^{\frac{1}{2}} \]
\[ \leq C \left( \int_{\Omega} |u|^{r-2} \nabla u^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (|u|^{r-2})^{\frac{6r}{2(r-1)}} \, dx \right)^{\frac{2(r-2)}{12r}} \left( \int_{\Omega} |H|^2 \frac{6r}{2(r+2)} \, dx \right)^{\frac{4r+4}{12r}} \]
\[ \leq C \left( \int_{\Omega} |u|^{r-2} \nabla u^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (|u|^2)^6 \, dx \right)^{\frac{2(r-2)}{12r}} \]
\[ \leq C \left( \int_{\Omega} |u|^{r-2} \nabla u^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (\nabla |u|^2)^2 \, dx \right)^{\frac{(r-2)}{2r}} \]
\[ \leq \frac{1}{2} \mu r \epsilon_0 \int_{\Omega} |u|^{r-2} \nabla u^2 \, dx + C(\mu, r, \epsilon_0), \]

where \( \epsilon_0 \in (0, \frac{1}{4}) \) is independent of \( r \). Then combining (3.8)-(3.10), we quickly have

\[ \frac{d}{dt} \int_{\Omega} \rho |u|^r \, dx + \int_{\Omega \cap \{ |u| > 0 \}} \mu r (1 - \epsilon_0) |u|^{r-2} |\nabla |u||^2 \, dx \]
\[ + \int_{\Omega \cap \{ |u| > 0 \}} \mu r (1 - \epsilon_0) |u|^r \left| \nabla \left( \frac{|u|}{|u|} \right) \right|^2 \, dx + \int_{\Omega \cap \{ |u| > 0 \}} \mu r (r - 2) |\nabla |u||^2 \, dx \quad \tag{3.11} \]
\[ \leq \frac{r(r-2)(\mu + \lambda)}{4} \int_{\Omega \cap \{ |u| > 0 \}} |u|^{r-2} |\nabla |u||^2 \, dx. \]

So according to (3.7) and (3.11), we obtain that

\[ \frac{d}{dt} \int_{\Omega} \rho |u|^r \, dx + rf(\epsilon_0, \epsilon_1, \epsilon_2, r) \int_{\Omega \cap \{ |u| > 0 \}} |u|^{r-2} |\nabla |u||^2 \, dx \leq C, \quad \tag{3.12} \]

where

\[ f(\epsilon_0, \epsilon_1, \epsilon_2, r) = \mu (1 - \epsilon_2) \phi(\epsilon_0, \epsilon_1, r) + \mu (r - 1 - \epsilon_0) - \frac{(r-2)(\mu + \lambda)}{4}. \quad \tag{3.13} \]
Subcase 1: If $3 \notin \{ r \mid r^2(\mu + \lambda) - \frac{(4 - \epsilon_0)\mu}{3} - \lambda > 0 \}$, i.e., $(5 - 8\epsilon_0)\mu < 3\lambda$, it is easy to get

\[
[3, +\infty) \in \{ r \mid r^2(\mu + \lambda) - \frac{(4 - \epsilon_0)\mu}{3} - \lambda > 0 \}.
\]

Therefore, we have

\[
\phi(\epsilon_0, \epsilon_1, r) = \frac{\mu(\epsilon_1 - 1) + \epsilon_0}{3\left(\frac{(4 - \epsilon_0)\mu}{3} - \lambda + \frac{r^2(\lambda + \mu)}{4(r - 1)}\right)}, \tag{3.14}
\]

for any $r \in [3, \infty)$. Substituting (3.14) into (3.13), for $r \in [3, \infty)$, we have

\[
f(\epsilon_0, \epsilon_1, \epsilon_2, r) = \frac{\mu^2(\epsilon_1 - 1 - \epsilon_2)(r - 1)}{3\left(\frac{(4 - \epsilon_0)\mu}{3} - \lambda + \frac{r^2(\lambda + \mu)}{4(r - 1)}\right)} + \mu(r - 1 - \epsilon_0) - \frac{(r - 2)^2(\mu + \lambda)}{4}.
\] \tag{3.15}

For $(\epsilon_1, \epsilon_2, r) = (1, 0, 3)$, we have

\[
f(\epsilon_0, 1, 0, 3) = \frac{16\mu^2}{3\lambda - (5 - 8\epsilon_0)\mu} + \frac{7\mu - \lambda}{4} = -C_1(\lambda - a_1\mu)(\lambda - a_2\mu), \tag{3.16}
\]

then according to $\frac{5 - 8\epsilon_0}{3}\mu < \lambda$, we have $C_1 = \frac{3}{4(3\lambda - (5 - 8\epsilon_0)\mu)} > 0$ and

\[
a_1(\epsilon_0) = \frac{1}{3}\left(13 - 4\epsilon_0 + 4\sqrt{\epsilon_0^2 + 4\epsilon_0 + 16}\right),
\]

\[
a_2(\epsilon_0) = \frac{1}{3}\left(13 - 4\epsilon_0 - 4\sqrt{\epsilon_0^2 + 4\epsilon_0 + 16}\right) < 0. \tag{3.17}
\]

Then if we want to make sure that $f(\epsilon_0, 1, 0, 3) > 0$, we have to assume that

\[
\frac{(5 - 8\epsilon_0)\mu}{3} < \lambda < a_1(\epsilon_0)\mu. \tag{3.18}
\]

It is obviously that $a_1(0) = \frac{28}{3}$ and $a_1'(\epsilon_0) < 0$ for $\epsilon_0 \in (0, \frac{1}{7})$. Due to the continuity of $a_1(\epsilon_0)$ for $\epsilon_0 \in (0, \frac{1}{7})$, we have for any sufficiently $\alpha > 0$, there exists $\epsilon_0 \in (0, \frac{1}{7})$ such that $a_1 = \frac{28}{3}$.

Subcase 2: If $3 \notin \{ r \mid r^2(\mu + \lambda) - \frac{(4 - \epsilon_0)\mu}{3} - \lambda > 0 \}$, i.e., $(5 - 8\epsilon_0)\mu \geq 3\lambda$. In this case, for $r \in (3, \frac{7}{2})$, it is easy to get

\[
r \left[ \mu(\epsilon_1 - 1) + \mu(r - 1 - \epsilon_0) - \frac{(r - 2)^2(\mu + \lambda)}{4} \right] > 3 \left( \frac{7}{4}\mu - \frac{9(\mu + \lambda)}{16} \right) = 3 \left( \frac{19\mu}{16} - \frac{9\lambda}{16} \right) \geq 3 \left( \frac{19\mu}{16} - \frac{3(5 - 8\epsilon_0)\mu}{16} \right) > \frac{1}{4}\mu, \tag{3.20}
\]

which, together with (3.12), implies that

\[
\frac{d}{dt} \int_{\Omega} \rho u^r dx + \frac{1}{4}\mu \int_{\Omega \cap \{|u| > 0\}} |u|^{-2} |\nabla u|^2 dx \leq C, \quad \text{for} \quad r \in (3, \frac{7}{2}). \tag{3.21}
\]
**Step 2:** we assume that
\[
\int_{\Omega \cap |u| > 0} |u|^r \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 \, dx \leq \phi(\epsilon_0, \epsilon_1, r) \int_{\Omega \cap |u| > 0} |u|^{r-2} |\nabla u|^2 \, dx. \tag{3.22}
\]
A direct calculation gives for \(|u| > 0\),
\[
\text{div} u = |u| \text{div} \left( \frac{u}{|u|} \right) + \frac{u \cdot \nabla |u|}{|u|}. \tag{3.23}
\]
Then combining (3.23) and (3.29)-(3.11), we quickly have
\[
\frac{d}{dt} \int_{\Omega} \rho |u|^r \, dx + \int_{\Omega \cap \{|u| > 0\}} \mu r (1 - \epsilon_0) |u|^{r-2} |\nabla u|^2 \, dx
\]
\[
+ \int_{\Omega \cap \{|u| > 0\}} r(\lambda + \mu) |u|^{r-2} |\text{div} u|^2 \, dx + \int_{\Omega \cap \{|u| > 0\}} \mu r (r-2) |u|^{r-2} |\nabla u|^2 \, dx \tag{3.24}
\]
\[
= -r (r-2)(\mu + \lambda) \int_{\Omega \cap \{|u| > 0\}} \left( |u|^{r-2} u \cdot \nabla |u| \text{div} \left( \frac{u}{|u|} \right) + |u|^{r-4} |u \cdot \nabla |u||^2 \right) \, dx.
\]
This gives
\[
\frac{d}{dt} \int_{\Omega} \rho |u|^r \, dx + \int_{\Omega \cap \{|u| > 0\}} r (1 - \epsilon_0) |u|^{r-4} G \, dx \leq C, \tag{3.25}
\]
where
\[
G = \mu (1 - \epsilon_0) |u|^2 |\nabla u|^2 + (\mu + \lambda) |u|^2 |\text{div} u|^2 + \mu (r-2) |u|^2 |\nabla u|^2
\]
\[
+ (r-2)(\mu + \lambda) |u|^2 u \cdot \nabla |u| \text{div} \left( \frac{u}{|u|} \right) + (r-2)(\mu + \lambda) u \cdot \nabla |u|. \tag{3.26}
\]
Now we consider how to make sure that \(G \geq 0\).
\[
G = \mu (1 - \epsilon_0) |u|^2 \left( |u|^2 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + |\nabla |u|^2 | \right) + (\mu + \lambda) |u|^2 \left( |u| \text{div} \left( \frac{u}{|u|} \right) + \frac{u \cdot \nabla |u|}{|u|} \right)^2
\]
\[
+ \mu (r-2) |u|^2 |\nabla u|^2 + (r-2)(\mu + \lambda) |u|^2 u \cdot \nabla |u| \text{div} \left( \frac{u}{|u|} \right)
\]
\[
+ (r-2)(\mu + \lambda) u \cdot \nabla |u|^2
\]
\[
= \mu (1 - \epsilon_0) |u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + \mu (r-1 - \epsilon_0) |\nabla |u|^2 |^2 + (r-1)(\mu + \lambda) u \cdot \nabla |u|^2
\]
\[
+ r(\mu + \lambda) |u|^2 u \cdot \nabla |u| \text{div} \left( \frac{u}{|u|} \right) + (\mu + \lambda) |u|^4 \left( \text{div} \left( \frac{u}{|u|} \right) \right)^2
\]
\[
= \mu (1 - \epsilon_0) |u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + \mu (r-1 - \epsilon_0) |u|^2 |\nabla u|^2
\]
\[
+ (r-1)(\mu + \lambda) \left( u \cdot \nabla |u| + \frac{r}{2(r-1)} |u|^2 \left( \text{div} \frac{u}{|u|} \right) \right)^2
\]
\[
+ (\mu + \lambda) |u|^4 \left( \text{div} \frac{u}{|u|} \right)^2 - \frac{r^2(\mu + \lambda)}{4(r-1)} |u|^4 \left( \text{div} \frac{u}{|u|} \right)^2,
\]
which, combining with the fact
\[
|\text{div}(\frac{u}{|u|})|^2 \leq 3|\nabla(\frac{u}{|u|})|^2,
\]
implies that
\[
G \geq \mu(1 - \epsilon_0)|u|^4|\nabla(\frac{u}{|u|})|^2 + \mu(r - 1 - \epsilon_0)|u|^2|\nabla|u|^2
\]
\[
+ \left(\mu + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)}\right)|u|^4\left(\text{div}(\frac{u}{|u|})\right)^2
\]
\[
\geq \frac{\mu(1 - \epsilon_0)}{3}(1 - \epsilon_0)|u|^4\left|\text{div}(\frac{u}{|u|})\right|^2 + \mu(r - 1 - \epsilon_0)|u|^2|\nabla|u|^2
\]
\[
+ \left(\mu + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)}\right)|u|^4\left(\text{div}(\frac{u}{|u|})\right)^2
\]
\[
\geq \mu(r - 1 - \epsilon_0)|u|^2|\nabla|u|^2 + \left(\frac{4 - \epsilon_0}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)}\right)|u|^4\left(\text{div}(\frac{u}{|u|})\right)^2.
\]
(3.28)

Thus
\[
\int_{\Omega \cap \{|u| > 0\}} r|u|^{-4}Gdx
\]
\[
\geq 3r\left(\frac{4 - \epsilon_0}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)}\right)\int_{\Omega \cap \{|u| > 0\}} |u|^r\left(\text{div}(\frac{u}{|u|})\right)^2 dx
\]
\[
+ \mu r(r - 1 - \epsilon_0)\int_{\Omega \cap \{|u| > 0\}} |u|^{-2}|\nabla|u|^2 dx
\]
\[
\geq 3r\left(\frac{4 - \epsilon_0}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)}\right)\phi(\epsilon_0, \epsilon_1, r)\int_{\Omega \cap \{|u| > 0\}} |u|^{-2}|\nabla|u|^2 dx
\]
\[
+ \mu r(r - 1 - \epsilon_0)\int_{\Omega \cap \{|u| > 0\}} |u|^{-2}|\nabla|u|^2 dx
\]
\[
\geq g(\epsilon_0, \epsilon_1, r)\int_{\Omega \cap \{|u| > 0\}} |u|^{-2}|\nabla|u|^2 dx,
\]
where
\[
g(\epsilon_0, \epsilon_1, r) = \left[3r\left(\frac{4 - \epsilon_0}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)}\right)\phi(\epsilon_0, \epsilon_1, r) + \mu r(r - 1 - \epsilon_0)\right].
\]
(3.30)

where we need that \(\epsilon_0\) is sufficiently small such that \(\epsilon_0 < (r - 1)\epsilon_1\). Then combining (3.25) and (3.29)-(3.30), we quickly have
\[
\frac{d}{dt}\int_{\Omega} \rho|u|^r dx \leq C, \quad \text{for} \quad r \in (3, 7/2).
\]
(3.31)

So combining (3.19)-(3.21) and (3.31) for Step:1 and Step:2, we conclude that if \(3\lambda < (29 - \alpha)\mu\) for any sufficiently \(\alpha > 0\), there exist some constants \(C > 0\) such that
\[
\frac{d}{dt}\int_{\Omega} \rho|u|^r dx \leq C, \quad \text{for} \quad r \in (3, 7/2).
\]
(3.32)
Now for each \( t \in [0, T) \), we denote \( v(t, x) = L^{-1} \text{div} A \) and
\[
A = PI_3 + H \otimes H - \frac{1}{2} |H|^2 I_3,
\]
that is, \( v \) is the solution of
\[
\begin{align*}
\mu \Delta v + (\lambda + \mu) \nabla \text{div} v &= \text{div} A \quad \text{in } \Omega, \\
v(t, x) &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
Due to Lemma 2.3, for any \( l \in (1, +\infty) \), there exists a constant \( C \) independent of \( t \) such that
\[
\begin{align*}
|\nabla v(t)|_l &\leq C (|\rho(t)|_l + |H(t)|_l), \\
|\nabla^2 v(t)|_l &\leq C (|\nabla \rho(t)|_l + |\nabla H(t)|_l).
\end{align*}
\]
Now let us introduce an important quantity:
\[
w = u - v.
\]
In [28], it has been observed that this quantity \( w \) possesses more regularity information than \( u \) under the assumption that the density is upper bounded for compressible isentropic Navier-Stokes equations. However, for compressible isentropic MHD equations, if we also assume that the magnetic field \( H \) is upper bounded, we have

**Lemma 3.3.**
\[
|\nabla w(t)|_2^2 + \int_0^T (|\nabla^2 w|_2^2 + |\sqrt{\rho} w_t|_2^2)dt \leq C, \quad 0 \leq t < T,
\]
where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T] \)).

**Proof.** Firstly, via the continuity equation, we find that \( w \) satisfies
\[
\begin{align*}
\rho w_t - \mu \Delta w - (\lambda + \mu) \nabla \text{div} w &= \rho F, \\
w(t, x) &= 0 \quad \text{on } [0, T) \times \partial \Omega, \quad w(0, x) = w_0(x), \quad \text{ in } \Omega,
\end{align*}
\]
with \( w_0(x) = u_0(x) + v_0(x) \) and
\[
F = -u \cdot \nabla u - L^{-1} \text{div} A_t
= -u \cdot \nabla u - L^{-1} \text{div}(Pu) + (\gamma - 1)L^{-1} \nabla (P \text{div} u)
+ L^{-1} \text{div}(H_t \otimes H + H \otimes H_t) - L^{-1} \nabla (H_t \cdot H_t) = \sum_{i=1}^{5} L_i.
\]

Multiplying the equations in (3.35) by \( w_t \) and integrating the resulting equation over \( \Omega \), via Holder’s inequality, we have
\[
\frac{d}{dt} \int_{\Omega} (\mu |\nabla w|^2 + (\lambda + \mu)|\text{div} w|^2)dx + \int_{\Omega} \rho |w_t|^2 dx
= \int_{\Omega} \rho F \cdot w_t dx \leq C |\sqrt{\rho} F|_2^2 + \frac{1}{2} \int_{\Omega} \rho |w_t|^2 dx,
\]
(3.36)
According to Lemmas 3.1-3.2, we obtain
\[ \frac{d}{dt} \int_{\Omega} (\mu|\nabla w|^2 + (\lambda + \mu)|\text{div}w|^2) dx + \int_{\Omega} \rho|w_t|^2 dx \leq C \sum_{i=1}^{5} |\sqrt{\rho}L_i|^2. \] (3.37)

Next we need to consider the terms $|\sqrt{\rho}L_i|$ for $i = 1, 2, \ldots, 5$. From Lemma 3.2 and (3.34), it follows that
\[ |\sqrt{\rho}L_1| = | - \sqrt{\rho} u \cdot \nabla u| \leq C|\sqrt{\rho} u| \leq C|\sqrt{\rho} u|_2 \leq C, \] (3.38)
\[ |\sqrt{\rho}L_2| = | - (\gamma - 1)\sqrt{\rho}L^{-1} \nabla (P\text{div}u)|_2 \leq C|\sqrt{\rho}L_3|L^{-1} \nabla (P\text{div}u)|_2 \leq C|\sqrt{\rho}L_3|L^{-1} \nabla (P\text{div}u)|_2 \leq C|\sqrt{\rho}L_4|_2 \leq C|\sqrt{\rho}L_5|_2 \],
where we have used the interpolation inequality
\[ |f|_q \leq C(\epsilon)|f|_2 + \epsilon|\nabla f|_2, \quad 2 \leq q < 6. \]

According to Lemmas 3.1, 3.2, we obtain
\[ |\sqrt{\rho}L_2| = | - \sqrt{\rho}L^{-1} \nabla (P\text{div}u)|_2 \leq C|\sqrt{\rho}L_3|L^{-1} \nabla (P\text{div}u)|_2 \] (3.39)

Now we consider the term $\mathbb{B} = (b^{(i,j)}_{(x \times 3)}) = H \otimes H + H \otimes H_t$, due to Lemma 2.6.
\[ H_t = H \cdot \nabla u - u \cdot \nabla H - H \text{div}u, \] (3.40)
we immediately have
\[ b^{(i,j)} = H^i (H^k \partial_k u^j - u^k \partial_k H^i - H^i \partial_k u^k) + H^j (H^k \partial_k u^i - u^k \partial_k H^j - H^j \partial_k u^k) = H^i H^k \partial_k u^j + H^j H^k \partial_k u^i - H^i H^j \partial_k u^k - \partial_k (H^i H^j u^k), \] (3.41)
which means that
\[ \mathbb{B} = (H^i H^k \partial_k u^j + H^j H^k \partial_k u^i - H^i H^j \partial_k u^k)_{(x \times 3)} - \text{div}((H \otimes H) \otimes u) = \mathbb{B}_1 + \mathbb{B}_2. \] (3.42)

Then we easily have
\[ |\sqrt{\rho}L_4| = |\sqrt{\rho}L^{-1} \text{div}(H_t \otimes H + H \otimes H_t)|_2 \leq C|\sqrt{\rho}L_3|L^{-1} \text{div}\mathbb{B}_1|_2 + C|\sqrt{\rho}L^{-1} \text{div}\mathbb{B}_2|_2 \leq C|\nabla u|_2, \] (3.43)
Similarly, we consider the term $\mathbb{C} = H \cdot H_t$, similarly, due to (3.40), we obtain
\[ \mathbb{C} = H \cdot (H \cdot \nabla u - u \cdot \nabla H - H \text{div}u) \]
\[ = (H \cdot \nabla u \cdot H - \frac{1}{2}|H|^2 \text{div}u) - \frac{1}{2} \text{div}(u|H|^2) = \mathbb{C}_1 + \mathbb{C}_2, \] (3.44)
which, according to the Poincare inequality, implies that
\[
|\sqrt{\rho} L_5|_2 = |\sqrt{\rho} L^{-1} \nabla (H \cdot H_t)|_2 \\
= |\sqrt{\rho} L^{-1} \nabla C_1|_2 + |\sqrt{\rho} L^{-1} \nabla C_2|_2 \\
\leq C |\sqrt{\rho} L^{-1} \nabla C_1|_6 + |\sqrt{\rho} L^{-1} \nabla \text{div}(u |H|^2)|_2 \\
\leq C |\nabla L_1^{-1} \nabla C_1|_2 + C |\nabla u|_2 \leq C |\nabla u|_2. \tag{3.45}
\]

Combining (3.38)-(3.45), we have
\[
|\sqrt{\rho} F|_2^2 \leq \epsilon |\nabla^2 w|_2^2 + C(\epsilon)(1 + |\nabla w|_2^2 + |\nabla u|_2^2). \tag{3.46}
\]
Then from Lemma 2.3 and (3.36), we have
\[
|\nabla^2 w|_2^2 \leq C(|\rho w_t|_2^2 + |\rho F|_2^2) \leq C(|\sqrt{\rho} w_t|_2^2 + |\sqrt{\rho} F|_2^2), \tag{3.47}
\]
which implies, by taking \( \epsilon = \frac{1}{3C} \) in (3.46), that
\[
|\sqrt{\rho} F|_2^2 \leq \frac{1}{2}|\sqrt{\rho} w_t|_2^2 + C(\epsilon)(1 + |\nabla w|_2^2 + |\nabla u|_2^2). \tag{3.48}
\]
Substituting (3.47) into (3.37), from Gronwall’s inequality, the desired conclusions can be obtained.

Finally, according to the estimates obtained in (3.31), Lemmas 3.2 3.3 and (3.47), we deduce that

**Lemma 3.4.**
\[
|\nabla u(t)|_2^2 + \int_0^T |\nabla u|_2^2 dt \leq C, \quad 0 \leq t < T,
\]
where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T] \)).

### 3.2. The higher order estimate for \( |u|_{L^\infty([0,T]:D^2(\Omega))} \)

In this section, will give high order regularity estimates for \( w \). This is possible if the initial data \((H_0, \rho_0, u_0, P_0)\) satisfies the compatibility condition (1.3). First for a function or vector field (or even a \(3 \times 3\) matrix) \( f(t,x) \), the material derivative \( f \) is defined by:
\[
\dot{f} = f_t + u \cdot \nabla f = f_t + \text{div}(fu) - f \text{div} u.
\]

**Lemma 3.5 (Lower order estimate of the velocity \( u \)).**
\[
|w(t)|_{D^2}^2 + |\sqrt{\rho} \dot{u}(t)|_2^2 + \int_0^T |\dot{u}|_{D^1}^2 dt \leq C, \quad 0 \leq t \leq T,
\]
where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T] \)).
Proof. We will follow an idea due to Hoff [17]. Applying \( \dot{u} \frac{\partial}{\partial t} + \text{div}(u \cdot) \) to (1.5) and integrating by parts give

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\dot{u}|^2 dx
= - \int_{\Omega} \left( \dot{u} \cdot (\nabla P_t + \text{div}(\nabla P \otimes u)) + \dot{u} \cdot (\Delta u_t + \text{div}(\Delta u \otimes u)) \right) dx
+ (\lambda + \mu) \int_{\Omega} \dot{u} \cdot (\nabla \text{div} u_t + \text{div}(\nabla \text{div} u \otimes u)) dx
+ \int_{\Omega} \dot{u} \cdot \left( \text{div}(H \otimes H - \frac{1}{2} |H|^2 I_3) + \text{div}(\nabla \text{div} u \otimes u) \right) dx
\equiv : \sum_{i=6}^{8} L_i + \Lambda.
\]

According to Lemmas 3.1-3.4, Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce that

\[
L_6 = - \int_{\Omega} \left( \dot{u} \cdot (\nabla P_t + \text{div}(\nabla P \otimes u)) \right) dx
= \int_{\Omega} \left( \partial_j \dot{u}^j P_t + \partial_k \dot{u}^j \partial_j P u^k \right) dx
= \int_{\Omega} \left( - \partial_j \dot{u}^j u^k \partial_k P - \gamma P \text{div} u \partial_j \dot{u}^j + \partial_k \dot{u}^j \partial_j P u^k \right) dx
= \int_{\Omega} \left( - \gamma P \text{div} u \partial_j \dot{u}^j + P \partial_k (\partial_j \dot{u}^j u^k) - P \partial_j (\partial_k \dot{u}^j u^k) \right) dx
\leq C |\nabla \dot{u}|_2 |\nabla u|_2 \leq \epsilon |\nabla \dot{u}|_2^2 + C |\nabla u|_2^4,
\]

\[
L_7 = \mu \int_{\Omega} \left( \dot{u} \cdot (\Delta u_t + \text{div}(\Delta u \otimes u)) \right) dx
= - \int_{\Omega} \mu \left( \partial_i \dot{u}^i \partial_i u^j + \Delta u^j u \cdot \nabla \dot{u}^j \right) dx
= - \int_{\Omega} \mu \left( |\nabla \dot{u}|^2 - \partial_i \dot{u}^i u^k \partial_k \partial_i u^j - \partial_i \dot{u}^i \partial_i u^k \partial_k u^j + \Delta u^j u \cdot \nabla \dot{u}^j \right) dx
= - \int_{\Omega} \mu \left( |\nabla \dot{u}|^2 - \partial_i \dot{u}^i \partial_k \partial_i u^k - \partial_i \dot{u}^i \partial_i u^k \partial_k u^j - \partial_i \dot{u}^i \partial_i u^k \partial_k \dot{u}^j \right) dx
\leq - \frac{\mu}{2} |\nabla \dot{u}|_2^2 + C |\nabla u|_4^4,
\]

and similarly, we have

\[
L_8 = (\lambda + \mu) \int_{\Omega} \left( \dot{u} \cdot (\nabla \text{div} u_t + \text{div}(\nabla \text{div} u \otimes u)) \right) dx \leq - \frac{\mu + \lambda}{2} |\nabla \dot{u}|_2^2 + C |\nabla u|_4^4.
\]

Next we begin to consider the magnetic term \( \Lambda \)

\[
\Lambda = \int_{\Omega} \dot{u} \cdot \left( \text{div}(H \otimes H - \frac{1}{2} |H|^2 I_3) + \text{div}(\nabla \text{div} u \otimes u) \right) dx = \sum_{j=1}^{4} \Lambda_j.
\]
Via the magnetic equations \[\text{(1.5)}\] and integrating by parts, we obtain that

\[
\Lambda_1 = \int_\Omega \hat{u} \cdot \text{div}(H \otimes H)_t \, dx = \int_\Omega (H \otimes H)_t \cdot \nabla \hat{u} \, dx
= \int_\Omega (H \otimes H_t + H_t \otimes H) \cdot \nabla \hat{u} \, dx
= \int_\Omega H \otimes (H \cdot \nabla u - u \cdot \nabla H - H \text{div} u) \cdot \nabla \hat{u} \, dx
+ \int_\Omega (H \cdot \nabla u - u \cdot \nabla H - H \text{div} u) \otimes H \cdot \nabla \hat{u} \, dx
= \int_\Omega H \otimes (H \cdot \nabla u - H \text{div} u) \cdot \nabla \hat{u} \, dx
+ \int_\Omega (\text{div} (u \cdot \nabla) - (u \cdot \nabla H) \otimes H) \cdot \nabla \hat{u} \, dx
= \Lambda_{11} + \Lambda_{12} + \Lambda_{13},
\]

\[
\Lambda_2 = \int_\Omega \hat{u} \cdot \text{div}\left(-\frac{1}{2}|H|^2 I_3\right)_t \, dx = \int_\Omega \left(-\frac{1}{2}|H|^2 I_3\right)_t \cdot \nabla \hat{u} \, dx
= \int_\Omega -(H \cdot H_t I_3) \cdot \nabla \hat{u} \, dx
= \int_\Omega -(H \cdot (H \cdot \nabla u - u \cdot \nabla H - H \text{div} u) I_3) \cdot \nabla \hat{u} \, dx
= \int_\Omega -(H \cdot (H \cdot \nabla u - H \text{div} u) I_3) \cdot \nabla \hat{u} \, dx + \int_\Omega (H \cdot (u \cdot \nabla H) I_3) \cdot \nabla \hat{u} \, dx
= \Lambda_{21} + \Lambda_{22}.
\]

\[
\Lambda_3 = \int_\Omega \hat{u} \cdot \text{div}(\text{div}(H \otimes H) \otimes u) \, dx = \int_\Omega \text{div}(H \otimes H) \otimes u \cdot \nabla \hat{u} \, dx
= \int_\Omega (H \cdot \nabla H) \otimes u \cdot \nabla \hat{u} \, dx = \int_\Omega H^k \partial_k H^i \partial_j \hat{u}^i \, dx
= -\int_\Omega H^k H^i \partial_k u^j \partial_j \hat{u}^i \, dx - \int_\Omega H^k H^i u^j \partial_k \hat{u}^i \, dx = \Lambda_{31} + \Lambda_{32},
\]

\[
\Lambda_4 = \int_\Omega \hat{u} \cdot \text{div}\left(\text{div}\left(-\frac{1}{2}|H|^2 I_3\right) \otimes u\right) \, dx
= \int_\Omega \text{div}\left(-\frac{1}{2}|H|^2 I_3\right) \otimes u \cdot \nabla \hat{u} \, dx = \int_\Omega -H^k \partial_i H^k u^j \partial_j \hat{u}^i \, dx
= \frac{1}{2} \int_\Omega |H|^2 \partial_i u^j \partial_j \hat{u}^i \, dx + \frac{1}{2} \int_\Omega |H|^2 u^j \partial_i \partial_j \hat{u}^i \, dx = \Lambda_{41} + \Lambda_{42}.
\]

Now we observe that

\[
\Lambda_{13} + \Lambda_{32} = \int_\Omega \left(\text{div} (H \otimes (u \cdot \nabla H) - (u \cdot \nabla H) \otimes H)\right) \cdot \nabla \hat{u} \, dx
- \int_\Omega H^k H^i u^j \partial_k \hat{u}^i \, dx
= \int_\Omega \left(-u^k H^j \partial_k H^i \partial_j \hat{u}^i - u^k H^j \partial_k H^i \partial_j \hat{u}^i - H^k H^i u^j \partial_k \hat{u}^i\right) \, dx
\]
Due to the definition of which, together with (3.49)-(3.54), implies that with the zero boundary condition. From Lemma 2.3, we have

\[ C \]

(3.58) that

\[ \int_{\Omega} \partial_t w H_j H_j \partial_j \hat{u}^i \, dx, \]

\[ \Lambda_2 + \Lambda_{42} = \int_{\Omega} \left( H \cdot (u \cdot \nabla H) I_3 \right) : \nabla \hat{u} \, dx + \frac{1}{2} \int_{\Omega} |H|^2 u^j \partial_i \partial_j \hat{u}^i \, dx \] (3.53)

\[ = \int_{\Omega} \left( \frac{1}{2} |H|^2 u^j \partial_i \partial_j \hat{u}^i + H^k u^j \partial_k \hat{u} \right) \, dx \]

\[ = \int_{\Omega} \left( \frac{1}{2} |H|^2 u^j \partial_i \partial_j \hat{u}^i - \frac{1}{2} u^j |H|^2 \partial_i \partial_j \hat{u} + \frac{1}{2} \partial_j u^j |H|^2 \partial_i \hat{u} \right) \, dx \]

\[ = \int_{\Omega} \left( - \frac{1}{2} \partial_j u^j |H|^2 \partial_i \hat{u} \right) \, dx, \]

which implies that

\[ \Lambda \leq C |H|_\infty^2 |\nabla u|_2 |\nabla \hat{u}|_2 \leq \varepsilon |\nabla \hat{u}|_2^2 + C(\varepsilon) |\nabla u|_2^2. \] (3.54)

Due to the definition of \( w \), we know that \( w \) satisfies

\[ \mu \Delta w + (\lambda + \mu) \nabla \text{div} w = \rho \hat{u} \quad \text{in} \quad \Omega, \] (3.55)

with the zero boundary condition. From Lemma 2.3, we have

\[ |w|_{D^2} \leq C |\rho \hat{u}|_2 \leq C |\rho \hat{u}|_2, \] (3.56)

which, together with (3.49)-(3.54), implies that

\[ \frac{1}{2} \int_{0}^{T} \rho |\hat{u}|^2 \, dx + |\hat{u}|_{D^1}^2 \leq C |\nabla u|_4^4 + C \]

\[ \leq C |\nabla u|_2 |\nabla u|_6^2 \leq C |\nabla u|_6^2 (|\nabla w|_6 + |\nabla v|_6) \] (3.57)

\[ \leq C |\nabla u|_6^2 (1 + |\nabla^2 w|_2) \leq C |\nabla u|_6^2 (1 + |\sqrt{\rho} \hat{u}|_2). \]

Then from Gronwall’s inequality, we have

\[ \int_{0}^{T} \rho |\hat{u}|^2(t) \, dx + \int_{0}^{T} |\hat{u}|_{D^1}^2 \leq C, \quad 0 \leq t \leq T. \] (3.58)

According to Lemmas 3.2, 3.5 and using the equations (3.55) again, we deduce from (3.58) that

**Lemma 3.6.**

\[ |\nabla w(t)|_{L^2([0,T];L^\infty(\Omega))} + |\nabla^2 w(t)|_{L^2([0,T];L^4(\Omega))} \leq C, \quad 0 \leq t < T, \]

where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0,T] \)).

Finally, the following lemma gives bounds of \( |\nabla \rho|_q \), \( |\nabla H|_q \) and \( |\nabla^2 u|_q \).
Lemma 3.7.

\[ \| (\rho, H, P)(t) \|_{W^{1,r}} + \| (\rho_t, H_t, P_t)(t) \|_r \leq C, \quad 0 \leq t < T, \quad (3.59) \]

where \( r \in [2, q] \), \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T) \)).

**Proof.** In the following estimates we will use

\[ |\nabla^2 v|_q \leq C (|\nabla \rho|_q + |\nabla H|_q); \]
\[ |\nabla v|_\infty \leq C (1 + |\nabla v|_{BMO(\Omega)} \ln(e + |\nabla^2 v|_q)) \]
\[ \leq C (1 + (|\rho|_{L^2 \cap L^\infty} + |H|_{L^2 \cap L^\infty}) \ln(e + |\nabla \rho|_q + |\nabla H|_q)) \]
\[ \leq C (1 + \ln(e + |\nabla \rho|_q + |\nabla H|_q)). \quad (3.60) \]

Firstly, applying \( \nabla \) to (1.5), multiplying the resulting equations by \( q |\nabla \rho|^{q-2} \nabla \rho \), we have

\[ (|\nabla \rho|^q)_t + \text{div}(|\nabla \rho|^q u) + (q - 1)|\nabla \rho|^q \text{div} u \]
\[ = -q |\nabla \rho|^{q-2}(\nabla \rho)^\top D(u)(\nabla \rho) - q|\nabla \rho|^{q-2}\nabla \rho \cdot \nabla \text{div} u. \quad (3.61) \]

Then integrating (3.61) over \( \Omega \), we immediately obtain

\[ \frac{d}{dt} |\nabla \rho|^q_q \leq C |D(u)|_\infty |\nabla \rho|^q_q + C |\nabla^2 u|_q |\nabla \rho|^{q-1}_q \]
\[ \leq C (|\nabla w|_\infty + |\nabla v|_\infty) |\rho|^q_q + C (|\nabla^2 w|_q + |\nabla^2 v|_q) |\nabla \rho|^{q-1}_q. \quad (3.62) \]

Secondly, applying \( \nabla \) to (1.5), multiplying the resulting equations by \( q|\nabla H|^{q-2} \nabla H \), we have

\[ (|\nabla H|^2)_t - qA : \nabla H|\nabla|^{q-2} + qB : \nabla H|\nabla H|^{q-2} + qC : \nabla H|\nabla H|^{q-2} = 0, \quad (3.63) \]

where

\[ A = \nabla(H \cdot \nabla u) = (\partial_j H \cdot \nabla u^i)_{(ij)} + (H \cdot \nabla \partial_j u^i)_{(ij)}, \]
\[ B = \nabla(u \cdot \nabla H) = (\partial_j u \cdot \nabla H^i)_{(ij)} + (u \cdot \nabla \partial_j H^i)_{(ij)}, \]
\[ C = \nabla(H \text{div} u) = \nabla H \text{div} u + H \otimes \nabla \text{div} u, \quad (3.64) \]

Then integrating (3.63) over \( \Omega \), due to

\[ \int_{\Omega} A : \nabla H|\nabla H|^{q-2} dx \leq C |\nabla u|_\infty |\nabla H|^q_q + C |H|_\infty |\nabla H|^{q-1}_q |u|_{D^2,q}, \quad (3.65) \]
\[
\int_{\Omega} B : \nabla H |\nabla H|^q \, dx \\
= \int_{\Omega} \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} \partial_j u^k \partial_k \partial_j H^i |\nabla H|^q \, dx + \int_{\Omega} \sum_{j=1}^{3} \sum_{i=1}^{3} u^k \partial_k \partial_j H^i |\nabla H|^q \, dx
\]

\[
= C|\nabla u|_{\infty} |\nabla H|^q + \frac{1}{2} \int_{\Omega} \sum_{k=1}^{3} u^k \left( \sum_{j,i} \partial_k |\nabla H|^2 |\nabla H|^q \right) \, dx
\]

\[
= C|\nabla u|_{\infty} |\nabla H|^q + \frac{1}{q} \int_{\Omega} \sum_{k=1}^{3} u^k \partial_k |\nabla H|^q \, dx \leq C|\nabla u|_{\infty} |\nabla H|^q,
\]

\[
\int_{\Omega} C : \nabla H |\nabla H|^q \, dx \leq C|\nabla u|_{\infty} |\nabla H|^q + C|H|_{\infty} |\nabla H|_{\theta} |u|_{D^2,q},
\]

(3.66)

we quickly obtain the following estimate:

\[
\frac{d}{dt} |\nabla H|^q_{\theta} \leq C((|\nabla u|_{\infty} + 1)|\nabla H|^q_{\theta} + C|u|_{D^2,q}|\nabla H|^{q-1}_{\theta}
\]

\[
\leq C(|\nabla w|_{\infty} + |\nabla v|_{\infty})|\nabla H|^q_{\theta} + C(|\nabla^2 w|_{q} + |\nabla^2 v|_{q})|\nabla H|^{q-1}_{\theta}.
\]

(3.67)

Then from (3.60), (3.62), (3.67) and Gronwall’s inequality, we immediately have

\[
\frac{d}{dt} (|\nabla \rho|^q_{\theta} + |\nabla H|^q_{\theta})
\]

\[
\leq C(1 + |\nabla w|_{\infty} + |\nabla v|_{\infty})(|\nabla \rho|^q_{\theta} + |\nabla H|^q_{\theta}) + C(|\nabla^2 w|_{q} (|\nabla \rho|^{q-1}_{\theta} + |\nabla H|^{q-1}_{\theta})
\]

\[
\leq C(1 + ||\nabla w||_{W^{1,q}} + \ln(e + |\nabla \rho|_{\theta} + |\nabla H|_{\theta}))(|\nabla \rho|^q_{\theta} + |\nabla H|^q_{\theta})
\]

\[
+ C(|\nabla^2 w|_{q} (|\nabla \rho|^{q-1}_{\theta} + |\nabla H|^{q-1}_{\theta}).
\]

Via (3.68) and notations:

\[
f = e + |\nabla \rho|_{\theta} + |\nabla H|_{\theta}, \quad g = 1 + ||\nabla w||_{W^{1,q}},
\]

we quickly have

\[
f_t \leq Cgf + Cf \ln f + Cg,
\]

which, together with Lemma 3.6 and Gronwall’s inequality, implies that

\[
\ln f(t) \leq C, \quad 0 \leq t < T.
\]

Then we have obtained the desired estimate for $|\nabla \rho|_{\theta} + |\nabla H|_{\theta}$. And the upper bound of $|\nabla \rho|_{r} + |\nabla H|_{r}$ can be deduced via the Holder inequality.

Finally, the estimates for $\rho_t$ and $H_t$ can be obtained easily via the following relation:

\[
\begin{cases}
H_t = H \cdot \nabla u - u \cdot \nabla H - H \text{div}u, \\
\rho_t = -u \cdot \nabla \rho - \rho \text{div}u, \quad P_t = -u \cdot \nabla P - \gamma P \text{div}u.
\end{cases}
\]

(3.69)

According to the estimates obtained in Sections 3.1-3.2, we deduce that
According to Lemmas 3.2 and 3.7, for \( r \in \Omega \) we easily deduce that

\[
|u(t)|_{D^2} + |\sqrt{\rho}u(t)|_2 + \int_0^T \left( |u_t|^2_{D^1} + |u|^2_{D^2} \right) dt \leq C, \quad 0 \leq t < T,
\]

where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, \overline{T}) \)).

**Proof.** Via the momentum equations \( (1.5)_4, (3.31) \) and Lemma 2.3, we have

\[
|u|_{D^2} \leq (|w|_{D^2} + |v|_{D^2}) \leq C(|w|_{D^2} + |\nabla P|_t + |\nabla H|_t),
\]

which, together with Lemma 3.7, implies that

\[
|u(t)|_{D^2} + \int_0^T |u|^2_{D^2} dt \leq C, \quad 0 \leq t \leq T.
\]

According to Lemmas 3.2 and 3.7 for \( r \in (3, 7/2) \), we quickly have

\[
|\sqrt{\rho}u|_2 \leq C(|\sqrt{\rho}u|_2 + |\sqrt{\rho}u \cdot \nabla u|_2) \leq C(1 + |\rho^{1/2} u_r| |\nabla u|_{L^2}^2) \leq C.
\]

Similarly, we have

\[
\int_0^T |u_t|^2_{D^1} dt \leq C \int_0^T (|u_t|^2_{D^1} + |u \cdot \nabla u|^2_{D^1}) dt \leq C.
\]

3.3. Improved regularity.

In this section, we will get some higher order regularity of \((H, \rho, u, P)\) to make sure that this solution is a classical one in \([0, \overline{T}]\). Based on the estimates obtained in the above sections, in truth, we have already proved that \( \int_0^T |\nabla u|^2_{L^2} ds \leq C \).

**Lemma 3.9 (Higher order estimate).**

\[
|(H, \rho, P)(t)|^2_{D^2} + \|(H_t, \rho_t, P_t)(t)\|^2_1 + \int_0^T \left( |u_t|^2_{D^3} + (H_t, \rho_t, P_t)|^2_2 \right) dt \leq C, \quad 0 \leq t < T,
\]

where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, \overline{T}) \)).

**Proof.** Via \( (1.5)_4 \) and Lemmas 2.3, 3.1–3.8, we show that

\[
|u|_{D^3} \leq C(|\rho u_t|_{D^1} + |\rho u \cdot \nabla u|_{D^1} + |\nabla P|_{D^1} + |\text{rot} H \times H|_{D^1}) \leq C(1 + |u_t|_{D^1} + |P|_{D^2} + |H|_{D^2}).
\]

(3.70)

Firstly, applying \( \nabla^2 \) to \( (1.5)_3 \), and multiplying the result equation by \( 2 \nabla^2 \rho \), integrating over \( \Omega \) we easily deduce that

\[
\frac{d}{dt} |\rho|^2_{D^2} \leq C|\nabla u|_\infty |\rho|^2_{D^2} + C|\rho|_\infty |u|_{D^3} |\rho|_{D^2} + |\nabla \rho|_3 |\nabla^2 \rho|_2 \|\nabla^2 u\|_1,
\]

(3.71)

which, together with (3.70),

\[
\frac{d}{dt} |\rho|_{D^2} \leq C(|\nabla u|_\infty + 1)(1 + |\rho|_{D^2} + |P|_{D^2} + |H|_{D^2}) + C|\nabla u_t|^2_{D^2}.
\]

(3.72)
And similarly, we have
\[
\begin{align*}
\frac{d}{dt}|H|_{D^2} &\leq C(|\nabla u|_\infty + 1)(1 + |P|_{D^2} + |H|_{D^2}) + C|\nabla u_t|^2, \\
\frac{d}{dt}|P|_{D^2} &\leq C(|\nabla u|_\infty + 1)(1 + |P|_{D^2} + |H|_{D^2}) + C|\nabla u_t|^2.
\end{align*}
\tag{3.73}
\]

So combining (3.72) and (3.73), we quickly have
\[
\frac{d}{dt}(|\rho|_{D^2} + |H|_{D^2} + |P|_{D^2}) \leq C(1 + |\nabla u|_\infty)(|\rho|_{D^2} + |H|_{D^2} + |P|_{D^2}) + C(1 + |\nabla u_t|^2).
\tag{3.74}
\]

Then via Gronwall’s inequality and (3.74), we obtain
\[
|\rho|_{D^2} + |H|_{D^2} + |P|_{D^2} + \int_0^t |u(s)|_{D^3}^2 \, dt \leq C, \quad 0 \leq t \leq T.
\]

Finally, due to the relation (3.69), we immediately get the desired conclusions. \( \square \)

Now we will give some estimates for the higher order terms of the velocity \( u \) in the following three Lemmas.

**Lemma 3.10 (Higher order estimate of the velocity \( u \)).**

\[
t|u_t(t)|_{D^3}^2 + t|u(t)|_{D^3}^2 + \int_0^T t(|u_t|^2_{D^2} + |\sqrt{\rho} u_{tt}|^2) \, ds \leq C, \quad 0 \leq t \leq T,
\]

where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T] \)).

**Proof.** Firstly, differentiating (1.3) with respect to \( t \), we have
\[
\rho u_{tt} + Lu_t = -\nabla P_t - \rho_t u_t - (\rho v \cdot \nabla v)_t + (\text{rot} H \times H)_t.
\tag{3.75}
\]

then multiplying (3.75) by \( u_{tt} \) and integrating over \( \Omega \), we have
\[
\int_{\Omega} \rho|u_{tt}|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \mu |\nabla u_t|^2 + (\lambda + \mu)(\text{div} u_t)^2 \right) \, dx \\
= \int_{\Omega} \left( (\nabla P_t - (\rho u \cdot \nabla u)_t - \rho_t u_t + (\text{rot} H \times H)_t) \cdot u_{tt} \right) \, dx = \frac{d}{dt} \Phi_1(t) + \Phi_2(t),
\tag{3.76}
\]

where
\[
\Phi_1(t) = \int_{\Omega} \left( P_t \text{div} u_t - \rho_t (u \cdot \nabla u) \cdot u_t - \frac{1}{2} \rho_t |u_t|^2 + (\text{rot} H \times H)_t \cdot u_t \right) \, dx \equiv \sum_{i=9}^{12} L_i,
\]
\[
\Phi_2(t) = \int_{\Omega} \left( -P_{tt} \text{div} u_t - \rho (u \cdot \nabla u)_t \cdot u_{tt} + \rho (u \cdot \nabla u) \cdot u_{tt} \right) \, dx \\
+ \int_{\Omega} \left( \rho_t (u \cdot \nabla u)_t \cdot u_t + \frac{1}{2} \rho_{tt} |u_t|^2 - (\text{rot} H \times H)_{tt} \cdot u_t \right) \, dx \equiv \sum_{i=13}^{18} L_i.
\]
According to Lemmas 3.1-3.9, Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce that

\[ L_9 = \int_\Omega P_t \nabla u_t dx \leq C |P_t|_2 |\nabla u_t|_2 \leq C |\nabla u_t|_2, \]

\[ L_{10} = -\int_\Omega \rho u \nabla (u \cdot \nabla u \cdot u_t) dx \]

\[ \leq C \int_\Omega (|u|_2^2 |\nabla u_t| + |u|_2^2 |\nabla u|_2 |\nabla u_t|) dx \]

\[ \leq C |u_t|_6 |\nabla u_t|_3 |u_t|_6 + C |u_t|_6^2 |\nabla u_t|_2 |u_t|_6 + C |u|_6^2 |\nabla u_t|_2 \leq C |\nabla u_t|_2, \quad (3.77) \]

\[ L_{11} = -\int_\Omega \rho u \cdot \nabla |u_t|^2 dx \leq C |\rho|_{\infty}^\frac{1}{2} |\nabla u_t|_2 |\sqrt{\rho} u_t|_2 \leq C |\nabla u_t|_2, \]

\[ L_{12} = \int_\Omega H \cdot H_t dx - \int_\Omega (H \cdot \nabla u_t \cdot H_t + H_t \cdot \nabla u_t \cdot H) dx \]

\[ \leq C |\nabla u_t|_2 |H_t|_2 |H|_\infty \leq C |\nabla u_t|_2, \]

which implies that

\[ \Phi_1(t) \leq \frac{\mu}{10} |\nabla u_t(t)|_2^2 + C, \quad 0 \leq t \leq T. \quad (3.78) \]

Let us denote

\[ \Phi^*(t) = \frac{1}{2} \int_\Omega \mu |\nabla u_t|^2 + (\lambda + \mu)(\nabla u_t)^2 dx - \Lambda_3(t), \]

then from (3.78), for \( 0 \leq t \leq T \), we quickly have

\[ C |\nabla u_t|_2^2 - C \leq \Phi^*(t) \leq C |\nabla u_t|_2^2 + C. \quad (3.79) \]

Similarly, according to Lemmas 3.1-3.9, Holder’s inequality and Gagliardo-Nirenberg inequality, for \( 0 < t \leq T \), we deduce that

\[ L_{13} \leq C |P_t|_2 |\nabla u_t|_2, \quad L_{14} \leq |\rho|_{\infty}^\frac{1}{2} |\sqrt{\rho} u_t|_2 (|u|_\infty |\nabla u_t|_2 + |\nabla u|_3 |\nabla u_t|_2), \]

\[ L_{15} \leq C |\rho u_t|_2 |\nabla u_t|_2 |\nabla u|_\infty, \]

\[ L_{16} \leq C |\rho t|_2 |u_t|_6 |\nabla u_t|_2 + C |u|_\infty |u_t|_6 |\nabla u_t|_2 |\rho_t|_3, \quad (3.80) \]

\[ L_{17} \leq C |\rho|_3 |\nabla u_t|_2 |u|_\infty |u_t|_6 + C |\rho|_{\infty}^\frac{1}{2} |\sqrt{\rho} u_t|_3 |u_t|_6 |\nabla u_t|_2, \]

where we have used the facts \( \rho_t = -\nabla (\rho u) \), and

\[ L_{18} = -\int_\Omega (\text{rot} H \times H)_{tt} \cdot u_t dx = \int_\Omega \left( H \otimes H - \frac{1}{2} |H|^2 I_3 \right)_{tt} : \nabla u_t dx \]

\[ \leq C |\nabla u_t|_2 |H_t|_4^2 + C |\nabla u_t|_2 |H_{tt}|_2 |H|_\infty. \]

(3.81)

Combining (3.80)-(3.81), from Young’s inequality, we have

\[ \Phi_2(t) \leq \frac{1}{2} |\sqrt{\rho u_t(t)}|_2^2 + C (1 + |\nabla u_t|_2^2) |\nabla u_t|_2^2 + C (|P_t|_2^2 + |\rho u_t|_2^2 + |H_{tt}|_2^2). \quad (3.82) \]
Then multiplying (3.76) by $t$ and integrating the resulting inequality over $(\tau, t)$ ($\tau \in (0, t)$), from (3.79) and (3.82), we have

$$\int_\tau^t s|\sqrt{\rho}u_t(s)|^2 ds + t|u_t(t)|^2_2 \leq \tau|u_t(\tau)|^2_2 + C \int_\tau^t s(1 + |\nabla u_t(s)|^2_2)ds \leq C$$

Then integrating the resulting inequality over $(\tau, t)$ ($\tau \in (0, t)$), from (3.79) and (3.82), we have

$$\int_\tau^t s|\sqrt{\rho}u_t(s)|^2 ds + t|u_t(t)|^2_2 \leq \tau|u_t(\tau)|^2_2 + C \int_\tau^t (1 + |\nabla u_t(s)|^2_2)ds \leq C$$

for $\tau \leq t \leq T$. From Lemma 3.5, we have $\nabla u_t \in L^2([0, T]; L^2)$, then according to Lemma 2.1, there exists a sequence $s_k$ such that

$$s_k \to 0, \text{ and } s_k|\nabla u_t(s_k)|^2_2 \to 0, \text{ as } k \to \infty.$$

Therefore, letting $\tau = s_k \to 0$ in (3.83), from Gronwall’s inequality, we have

$$\int_0^t s|\sqrt{\rho}u_t(s)|^2 ds + t|u_t(t)|^2_2 \leq C \exp \left( \int_0^T (1 + |\nabla u_t(s)|^2_2)ds \right) \leq C.$$

From (3.70), (3.83), Lemmas 2.3 and 3.1-3.9, we immediately have

$$t|u(t)|^2_{D^3} + \int_0^t s|u_t|^2_{D^2} ds \leq C(t|u_t(t)|_{D^1_0} + 1) + C \int_0^t s(1 + |\sqrt{\rho}u_t|_2^2) ds \leq C.$$

\[\square\]

**Lemma 3.11 (Higher order estimate of the velocity $u$).**

$$|H, \rho, P(t)|_{D^{2,q}} + t|H_t, \rho_t, P_t(t)|_{D^{1,q}} + \int_0^T |u_0|_{D^{3,q}}^p dt \leq C, \quad 0 < t \leq T,$$

where $C$ only depends on $C_0$ and $T$ (any $T \in (0, \mathcal{T})$).

**Proof.** From Lemmas 2.3 and 3.1-3.10 we easily obtain

$$|u|_{D^{3,q}} \leq C(|\rho u_t + \rho \cdot \nabla u|_{D^{3,q}} + |\text{rot}H \times H|_{D^{1,q}} + |P|_{D^{2,q}})$$

$$\leq C(|u_t|_\infty + |\nabla u_t|_q + |u|_{D^{2,q}} + |H|_{D^{2,q}} + |P|_{D^{2,q}}).$$

Due to the Sobolev inequality, Poincare inequality and Young’s inequality, we have

$$|u_t|_\infty \leq C|u_t|_q^{\frac{1}{q}} ||u_t||_{W^{1,q}}^{\frac{q}{q}} \leq C|\nabla u_t|_2 + C|\nabla u_t|_q,$$

then we have

$$|u(t)|_{D^{3,q}} \leq C(|\nabla u_t|_2 + |\nabla u_t|_q + |u|_{D^{2,q}} + |H|_{D^{2,q}} + |P|_{D^{2,q}}).$$
According to Lemmas 3.8, 3.10, let \( F = |\nabla u_t|_2 + |\nabla u_t|_q + |u|_{D^2,q} \) and we have

\[
\Pi = \int_0^t F^{p_0} \, ds = \int_0^t (|\nabla u_t|_2 + |\nabla u_t|_q + |u|_{D^2,q})^{p_0} \, ds
\]

\[
\leq C + C \int_0^t (|u|_{D^2,q}^{p_0} + |\nabla u_t|_q^{p_0} + |u|_{D^2,q}^{p_0}) \, ds
\]

\[
\leq C + C \int_0^t |\nabla u_t|_2^{\frac{p_0}{2q}} |\nabla u_t|_6^{\frac{p_0(3q-6)}{2q}} \, ds
\]

\[
\leq C + C \int_0^t s^{\frac{p_0}{2q}} (s |\nabla u_t|_2^{\frac{p_0(q-6)}{4q}} (s |u_t|_{D^2})^{\frac{p_0(3q-6)}{4q}} \, ds
\]

\[
\leq C + C \left( \sup_{[0,T]} s |\nabla u_t|_2^{\frac{p_0(q-6)}{4q}} \int_0^t s^{\frac{p_0}{2q}} (s |u_t|_{D^2})^{\frac{p_0(3q-6)}{4q}} \, ds \right)
\]

\[
\leq C + C \left( \left( \int_0^t s^{-\frac{2p_0q}{4q-3p_0(3q-6)}} |u_t|_{D^2}^{\frac{4q-3p_0(3q-6)}{4q}} \right) \left( \int_0^t s |u_t|_{D^2}^{p_0(3q-6)} \, ds \right) \right) \leq C.
\]  

Then, applying \( \nabla^2 \) to (3.8), and multiplying the resulting equation by \( q \nabla^2 \rho |\nabla^2 \rho|^{-2} \), integrating over \( \Omega \) we easily deduce that

\[
\frac{d}{dt}|\rho|_{D^2,q}^q \leq C|\nabla u|_\infty |\rho|_{D^2,q}^q + C|\rho|_\infty |\nabla u|_{D^2,q} |\rho|_{D^2,q}^{q-1} + |\nabla \rho|_\infty |u|_{D^2,q} |\rho|_{D^2,q}^{q-1},
\]

which, together with (3.70),

\[
\frac{d}{dt}|\rho|_{D^2,q} \leq C(|\nabla u|_\infty + 1 + F)(1 + |\rho|_{D^2,q} + |P|_{D^2,q} + |H|_{D^2,q}) + CF.
\]  

And similarly, we have

\[
\left\{ \begin{array}{l}
\frac{d}{dt}|H|_{D^2,q} \leq C(|\nabla u|_\infty + F + 1)(1 + |\rho|_{D^2,q} + |P|_{D^2,q} + |H|_{D^2,q}) + F,
\frac{d}{dt}|P|_{D^2,q} \leq C(|\nabla u|_\infty + F + 1)(1 + |\rho|_{D^2,q} + |P|_{D^2,q} + |H|_{D^2,q}) + F.
\end{array} \right.
\]

(3.88)

So we combining (3.87)–(3.88), we quickly have

\[
\frac{d}{dt}(|\rho|_{D^2,q} + |H|_{D^2,q} + |P|_{D^2,q}) \leq C(1 + |\nabla u|_\infty + F)(1 + |\rho|_{D^2,q} + |P|_{D^2,q} + |H|_{D^2,q}) + C(1 + F).
\]

(3.89)

Then via Gronwall’s inequality, (3.85) and (3.89), we obtain

\[
|\rho|_{D^2,q} + |H|_{D^2,q} + |P|_{D^2,q} + \int_0^t |u(s)|_{D^2,q}^{p_0} \, ds \leq C, \quad 0 \leq t \leq T.
\]

Finally, due to relation (3.69), we immediately get the desired conclusions.

\[\square\]

Finally, we have

**Lemma 3.12 (Higher order estimate of the velocity u).**

\[
t^2|u(t)|_{D^3,q} + t^2|u_t(t)|_{D^2,q}^2 + t^2|\nabla u_t(t)|_2^2 + \int_0^T t^2|u_{tt}(t)|_{D^3,q}^2 \, dt \leq C, \quad 0 < t \leq T,
\]
where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T] \)).

This lemma can be easily proved via the method used in Lemma 3.10, here we omit it. And this will be enough to extend the classical solutions of \((H, \rho, u, P)\) beyond \( t \geq T \).

In truth, in view of the estimates obtained in Lemmas 3.1-3.11 we quickly know that the functions \((H, \rho, u, P)|_{t=T} = \lim_{t \to T}(H, \rho, u, P)\) satisfies the conditions imposed on the initial data \((1.7) - (1.8)\). Therefore, we can take \((H, \rho, u, P)|_{t=T}\) as the initial data and apply the local existence Theorem 1.1 to extend our local classical solution beyond \( t \geq T \). This contradicts the assumption on \( T \).

References

[1] P. Acquistapace, On BMO regularity for linear elliptic systems, *Ann. Mat. Pura Appl.* 161 (1992) 231-269.

[2] S. Agmon, A. Dougis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, *Comm. Pure Appl. Math.* 12 (1959) 623-727.

[3] T. Beale, T. Kato and A. Majda, Remarks on the breakdown of smooth solutions for the 3D Euler equation, *Comm. Math. Phys.* 94 (1984) 61-66.

[4] J. L., Boldrini, M. A., Rojas-Medar, E., Fernández-Cara, Semi-Galerkin approximation and strong solutions to the equations of the nonhomogeneous asymmetric fluids, *J.Math.Pures.Appl.* 82 (2003) 1499-1525.

[5] J.A. Bittencourt, *Fundamentals of Plasma Physics* 3rd, Spinger-Verlag, New York, 2004.

[6] H. Brezis and S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, *Comm. Partial Differential Equations* 5 (1980) 771-789.

[7] Y. Cho, H. J. Choe, and H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, *J. Math. Pure. Appl.* 83 (2004) 243-275.

[8] Y. Cho and H. Kim, Existence results for viscous polytropic fluids with vacuum, *J.Differ.Equations* 228 (2006) 377-411.

[9] Y. Cho and H. Kim, On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities, *Manu.Math* 120 (2006) 91-129.

[10] J. Fan and W. Yu, Strong solutions to the magnetohydrodynamic equations with vacuum, *Nonl. Anal.* 10 (2009) 392-409.

[11] J.P. Freidberg, *Ideal magnetohydrodynamic theory of magnetic fusion systems*, Rev. Modern Phys. 54 (3) (1982) The American Physical Society.

[12] G. P. Galdi, *An introduction to the Mathematical Theory of the Navier-Stokes equations*, Springer, New York, 1994.
[13] D.A. Gurnett and A. Bhattacharjee, *Introduction to Plasma Physics*, Cambridge Univ. Press, Cambridge, 2005.

[14] X.D. Huang, J. Li and Z.P. Xin, Global Well-posedness of classical solutions with large oscillations and vacuum, *Comm. Pure. Appl. Math* **65** (2012) 0549-0585.

[15] X.D. Huang, J. Li and Z.P. Xin, Global Well-posedness of classical solutions with vacuum on bounded domains, (2012) Preprint.

[16] X.D. Huang, J. Li and Z.P. Xin, Blow-up criterion for the compressible flows with vacuum states, *Comm. Math. Phys* **301** (2010) 23-35.

[17] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differential Equations*, **120** (1995), 215-254.

[18] S. Kawashima and M. Okada, Smooth global solutions for the one-dimensional equations in magnetohydrodynamics, *Acad. Ser. A Math. Sci* **58** (1982) 384-387.

[19] A.F. Kulikovskiy and G.A. Lyubimov, *Magnetohydrodynamics*, Addison-Wesley, Reading, MA, 1965.

[20] L.D. Landau, E.M. Lifshitz and L.P. Pitaevskii, *Electrodynamics of Continuous Media*, 2nd, Butterworth-Heinemann, London, 1999.

[21] H. Li, X. Xu and J. Zhang, Global classical solutions to the 3D compressible magnetohydrodynamic equations with large oscillations and vacuum, *Siam. J. Math. Anal* **45** (2013) 1356-1387.

[22] M. Liu and S. Liu, Blow-up criterion for 3D viscous-resistive compressible magnetohydrodynamic equations, *Manu.Math* **36** (2012) 1145-1156.

[23] Lu M., Du Y. and Yao Z. A., Blowup criterion for compressible MHD equations, *J. Math. Anal. Appl.* **379** (2011) 425-438.

[24] Ponce, G. Remarks on a paper: Remarks on the breakdown of smooth solutions for the 3D Euler equations, *Comm. Math. Phys* **98** (1985) 349-353.

[25] O. Rozanova, Blow-up of smooth solutions to the barotropic compressible magnetohydrodynamic equations with finite mass and energy, *Proceedings of Symposia in Applied Mathematics* (2008).

[26] O. Rozanova, Blow-up of smooth highly decreasing at infinity solutions to the compressible Navier-Stokes Equations, *J. Differ.Euqations* **245** (2008) 1762-1774.

[27] E. M. Stein, *Singular integrals and Differentiability properties of Functions*, Princeton Univ. Press, Princeton NJ, 1970.

[28] Y. Sun, C. Wang and Z. Zhang, A Beale-Kato-Majda blow-up criterion to the compressible Navier-Stokes Equation, *J. Math.Pure.Appl* **95** (2011) 36-47.

[29] H. Wen and C. Zhu, Blow-up criterions of strong solutions to 3D compressible Navier-Stokes equations with vacuum, *Advances in Mathematics* **248** (2013) 534-572.
[30] Z. P. Xin, Blow-up of smooth solutions to the compressible Navier-Stokes Equations with Compact Density, *Commun. Pure. Appl. Math* 51 (1998) 0229-0240.

[31] Z. P. Xin and W. Yan, On blow-up of classical solutions to the compressible Navier-Stokes Equations, *Commun. Math. Phys.* 321 (2013) 529-541.

[32] X. Xu and J. Zhang, A blow-up criterion for the 3D non-resistive compressible magnetohydrodynamic Equations with initial vacuum, *Nonl. Anal.* 12 (2011) 3442-3451.

[33] S. Zhu, On classical solutions of the compressible Magnetohydrodynamic equations with vacuum, (2014) Unpublished results.

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