An unbiased estimate for the mean of a \(\{0, 1\}\) random variable with relative error distribution independent of the mean

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Abstract

Say \(X_1, X_2, \ldots\) are independent identically distributed Bernoulli random variables with mean \(p\). This paper builds a new estimate \(\hat{p}\) of \(p\) that has the property that the relative error, \(\hat{p}/p - 1\), of the estimate does not depend in any way on the value of \(p\). This allows the construction of exact confidence intervals for \(p\) of any desired level without needing any sort of limit or approximation. In addition, \(\hat{p}\) is unbiased. For \(\epsilon\) and \(\delta\) in \((0, 1)\), to obtain an estimate where \(P(|\hat{p}/p - 1| > \epsilon) \leq \delta\), the new algorithm takes on average at most \(2\epsilon^{-2}p^{-1}\ln(2\delta^{-1})(1 - (14/3)\epsilon)^{-1}\) samples. It is also shown that any such algorithm that applies whenever \(p \leq 1/2\) requires at least \((1/5)\epsilon^{-2}(1+2\epsilon)(1-\delta)\ln((2-\delta)\delta^{-1})p^{-1}\) samples. The same algorithm can also be applied to estimate the mean of any random variable that falls in \([0, 1]\).

1 Introduction

Say \(X_1, X_2, X_3, \ldots\) are independent, identically distributed (iid) Bernoulli random variables with mean \(p\). Write \(X_i \sim \text{Bern}(p)\) to denote \(P(X_i = 1) = p\) and \(P(X_i = 0) = 1 - p\). The purpose of this work is to present a new algorithm for estimating \(p\) with \(\hat{p}\) so that the relative error \(\hat{p}/p - 1\) has a known distribution that does not depend on the value of \(p\). In other words, with this algorithm it is possible to compute \(P(a \leq \hat{p}/p - 1 \leq b)\) exactly for any \(a \leq 0 \leq b\), without needing any kind of approximation or limiting behavior.

Many randomized approximation algorithms operate by reducing the problem to finding \(p\) for an iid stream of Bernoulli random variables. For
example, approximating the permanent of a matrix with positive entries \[3\],
the number of solutions to a disjunctive normal form expression \[4\], the
volume of a convex body \[5\], and counting the lattice points inside a polytope
can all be put into this framework. In general, anywhere an acceptance
rejection method is used to build an approximation algorithm, our method
applies.

For these types of algorithms, the cost is usually dominated by the num-
ber of Bern(\(p\)) draws that are needed, and so the focus is on minimizing the
expected number of such draws needed.

**Definition 1.** Suppose \(A\) is a function of \(X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)\) and auxil-
ary randomness (represented by \(U \sim \text{Unif}([0, 1])\)) that outputs \(\hat{p}\). Let \(T\) be
a stopping time with respect to the natural filtration so that the value of \(\hat{p}\)
only depends on \(U\) and \(X_1, \ldots, X_T\). Then call \(T\) the running time of the
algorithm.

The simplest algorithm for estimating \(p\) just fixes \(T = k\), and sets
\[
\hat{p}_k = \frac{X_1 + X_2 + \cdots + X_k}{k}.
\]

In this case \(\hat{p}_k\) has a binomial distribution with parameters \(k\) and \(p\). The
standard deviation of \(\hat{p}_k\) is \(\sqrt{p(1 - p)/k}\). Therefore, to get an estimate which
is close to \(p\) in the sense of having small relative error, \(k\) should be of the form
\(C/p\) (for some constant \(C\)) so that the standard deviation is \(p\sqrt{(1 - p)/C}\)
and so roughly proportional to \(p\). From the Central Limit Theorem, roughly
\(2e^{-2}\ln(2/\delta)/p\) samples are necessary to get \(\hat{p}_k/p \in [1 - \epsilon, 1 + \epsilon]\) for \(\epsilon \in (0, 1)\).
(See Section 3 for a more detailed form of this argument.) But \(p\) is unknown
at the beginning of the algorithm!

Dagum, Karp, Luby and Ross \[2\] dealt with this circularity problem with
their stopping rule algorithm. In this context of Bern(\(p\)) random variables,
their algorithm can be written as follows.

Fix \((\epsilon, \delta)\) with \(\epsilon \in (0, 1)\) and \(\delta > 0\). Let \(T\) be the smallest integer
such that \(X_1 + \cdots + X_T \geq 1 + (1 + \epsilon)4(\epsilon - 2)\ln(2/\delta)e^{-2}\). Then \(\hat{p}_{\text{DKLR}} = (X_1 + \cdots + X_T)/T\).

Call this method DKLR. They showed the following result for their
estimate (Stopping Rule Theorem of \[2\]).

\[
\mathbb{P}(1 - \epsilon \leq \hat{p}_{\text{DKLR}}/p \leq 1 + \epsilon) > 1 - \delta,
\]   (1)

and \(\mathbb{E}[T] \leq [1 + (1 + \epsilon)4(\epsilon - 2)\ln(2/\delta)e^{-2}]/p\).
They also showed that any such \((\epsilon, \delta)\) approximation algorithm that applies to all \(p \in [0, 1/2]\) (Lemma 7.5 of [2]) must satisfy

\[
E[T] \geq (4e^2)^{-1}(1 - \delta)(1 - \epsilon)^2(1 - p)e^{-2}\ln(\delta^{-1}(2 - \delta)).
\]

The factor of \(4(e - 2) = 2.873\ldots\) in the running time of DKLR is somewhat artificial. As mentioned earlier, a heuristic Central Limit Theorem argument (see Section 3) indicates that the correct factor in the running time should be 2 (this is the same 2 in the denominator of the exponential in the standard normal density).

Our algorithm is similar to DKLR, but with a continuous modification that yields several desirable benefits. The DKLR estimate \(\frac{X_1 + \cdots + X_T}{T}\) is a fixed integer divided by a negative binomial random variable. In the algorithm proposed here, the estimate is a fixed integer divided by a Gamma random variable. Since Gamma random variables are scalable, the relative error of the estimate does not depend on the value of \(p\).

This allows a much tighter analysis of the error, since the value of \(p\) is no longer an issue. In particular, the algorithm attains (to first order) the \(2\epsilon - 2p - 1\) running time that is likely the best possible. The new algorithm is called the Gamma Bernoulli approximation scheme (GBAS).

**Theorem 1.** The GBAS method of Section 2, for any integer \(k \geq 2\), outputs an estimate \(\hat{p}\) using \(T\) samples where \(E[T] = k/p\), \(E[\hat{p}] = p\), and the density of \(\hat{p}/p - 1\) is

\[
\frac{(k - 1)^k}{(k - 1)!} \frac{\exp(-(k - 1)/(s + 1))}{(s + 1)^{k+1}} \text{ for } s \geq -1.
\]

Suppose \(\epsilon \in (0, 3/4)\), \(\delta \in (0, 1)\), and

\[
k = \lceil 2\epsilon^{-2}p^{-1}\ln(2\delta^{-1})(1 - (14/3)\epsilon)^{-1}\rceil,
\]

then \(\mathbb{P}(-\epsilon \leq (\hat{p}/p) - 1 \leq \epsilon) > 1 - \delta\).

The lower bound of [2] for random variables in \([0, 1]\) can be improved for \([0, 1]\) random variables. The following theorem is proved in Section 3.

**Theorem 2.** For \(\epsilon > 0\) and \(\delta \in (0, 1)\) any algorithm that returns \(\hat{p}\) for \(p \in [0, 1/2]\) satisfying \(\mathbb{P}(-\epsilon \leq (\hat{p}/p) - 1 \leq \epsilon) > 1 - \delta\) must have

\[
E[T] \geq (1/5)e^{-2}(1 + 2\epsilon)(1 - \delta)\ln((2 - \delta)\delta^{-1})p^{-1}.
\]

As \(\epsilon\) and \(\delta\) go to 0, the ratio between the upper and lower bounds converges to 10 for these results. From Central Limit Theorem considerations, it is likely that the upper bound constant of 2 is the correct one (see Section 3).
2 The GBAS Algorithm

The algorithm is based upon properties of a one dimensional Poisson point process. Write $\text{Exp}(\lambda)$ for the exponential distribution with rate $\lambda$ and mean $1/\lambda$. So $A \sim \text{Exp}(\lambda)$ has density $f_A(t) = \lambda \exp(-\lambda t) \cdot 1(t \geq 0)$. Here $1(\text{expression})$ denotes the indicator function that evaluates to 1 if the expression is true and is 0 otherwise.

Let $A_1, A_2, \ldots$ be iid $\text{Exp}(\lambda)$ random variables. Set $T_i = A_1 + \cdots + A_i$. Then $P = \{T_i\}_{i=1}^{\infty}$ is a one dimensional Poisson point process of rate $\lambda$.

The sum of exponential random variables is well known to be a Gamma distributed random variable. (It is also called the Erlang distribution.) For all $i$, the distribution of $T_i$ is Gamma with shape and rate parameters $i$ and $\lambda$. The density of this random variable is

$$f_{T_i}(t) = (i-1)!^{-1} \lambda^i t^{i-1} \exp(-t\lambda)1(t \geq 0),$$

and write $T_i \sim \text{Gamma}(i, \lambda)$.

The key property used by the algorithm is thinning where each point in $P$ is retained independently with probability $p$. The result is a new Poisson point process $P'$ which has rate $\lambda p$. (See for instance [6, p. 320].)

The intuition is as follows. For a Poisson point process of rate $\lambda$, the chance that a point in $P$ lies in an interval $[t, t+h]$ is approximately $\lambda h$, while the chance that a point in $P'$ lies in interval $[t, t+h] = \lambda ph$ since points are only retained with probability $p$. Hence the new rate is $\lambda p$.

For completeness the next lemma verifies this fact directly by establishing that the distribution of the minimum point in $P'$ is $\text{Exp}(\lambda p)$.

**Lemma 1.** Let $G \sim \text{Geo}(p)$ so for $g \in \{1, 2, \ldots\}$, $\mathbb{P}(G = g) = (1-p)^{g-1}p$. Let $A_1, A_2, \ldots \sim A$ where $A \sim \text{Exp}(\lambda)$. Then

$$A_1 + A_2 + \cdots + A_G \sim \text{Exp}(\lambda p).$$

**Proof.** $G$ has moment generating function $M_G(t) = \mathbb{E}[\exp(-tG)] = pe^t/(1-(1-p)e^t)$ when $t < -\ln(1-p)$. The moment generating function of $A$ is $M_A(t) = \mathbb{E}[\exp(-tA)] = \lambda(\lambda - t)^{-1}$ when $t < \lambda$. The moment generating function of $A_1 + \cdots + A_G$ is the composition

$$M_G(M_A(t)) = \frac{p\lambda(\lambda - t)^{-1}}{1 - (1-p)\lambda(\lambda - t)^{-1}} = \frac{p\lambda}{p\lambda - t},$$

when $t < p\lambda$, and so $A_1 + \cdots + A_G \sim \text{Exp}(\lambda p)$.\hfill\QED
Another useful fact is that exponential distributions (and so Gamma distributions) scale easily.

**Lemma 2.** Let $X \sim \text{Gamma}(a, b)$. Then for $\beta \in \mathbb{R}$, $\beta X \sim \text{Gamma}(a, \beta^{-1}b)$.

**Proof.** The moment generating function of $X$ is $M_X(t) = [b/(b-t)]^a$ for $t < b$, so that of $\beta X$ is

$$
E[\exp(-t\beta X)] = M_X(\beta t) = [b/(b-\beta t)]^a = [\beta^{-1}b/(\beta^{-1}b-t)]^a
$$

exactly the moment generating function of a $\text{Gamma}(a, \beta^{-1}b)$. □

Together these results give the GBAS approach.

| GBAS | Input: $k$ |
|------|-------------|
| 1)   | $S \leftarrow 0, R \leftarrow 0.$ |
| 2)   | Repeat |
| 3)   | $X \leftarrow \text{Bern}(p), A \leftarrow \text{Exp}(1)$ |
| 4)   | $S \leftarrow S + X, R \leftarrow R + A$ |
| 5)   | Until $S = k$ |
| 6)   | $\hat{p} \leftarrow (k - 1)/R$ |

**Lemma 3.** The output $\hat{p}$ of GBAS satisfies

$$
\frac{p}{\hat{p}} \sim \text{Gamma}(k, k-1),
$$

and $E[\hat{p}] = p$. The number of Bern($p$) calls $T$ in the algorithm satisfies $E[T] = k/p$. The relative error $(\hat{p}/p) - 1$ has density

$$
f(s) = \frac{(k-1)^k}{(k-1)!} \frac{\exp(-(k-1)/(s+1))}{(s+1)^{k+1}} \quad \text{for } s \geq -1.
$$

**Proof.** From Lemma[1] the distribution of $R$ is Gamma($k, p$). From Lemma[2] the distribution of $p/\hat{p} = pR/(k-1)$ is Gamma($k, k-1$). Hence $E[\hat{p}] = E[p/X]$ where $X \sim \text{Gamma}(k, k-1)$. Now

$$
E[1/X] = \int_0^\infty \frac{1}{s(k-1)!} \frac{(k-1)^k}{s^{k-1}} \exp(-(k-1)s) \, ds
$$

$$
= \frac{(k-1)^k}{(k-1)!} \int_0^\infty \frac{(k-2)!}{(k-1)^{k-1}} \frac{1}{s^k} \exp(-(k-1)s) \, ds
$$

$$
= \frac{(k-1)^k}{(k-1)!} \, \frac{(k-2)!}{(k-1)^{k-1}} = \frac{k-1}{k-1} = 1,
$$

5
so \( \mathbb{E}[\hat{p}] = \mathbb{E}[p/X] = p \).

Since \( T \), the number of \textit{Bern}(\( p \)) drawn by the algorithm, is the sum of \( k \) geometric random variables (each with mean \( 1/p \)), \( T \) has mean \( k/p \).

The density of \( (\hat{p}/p) - 1 \) follows from the fact that \( p/\hat{p} \) has a \textbf{Gamma}(\( k, k-1 \)) distribution.

Note that for given \( k \) this probability can be computed exactly in \( k \) steps using the incomplete gamma function. Hence for a given error bound and accuracy requirement, it is possible to exactly find the minimum \( k \) using less work than flipping \( k/p \) coins.

To show Theorem 1, bounds on the tail of a Gamma random variable are needed. Chernoff bounds \cite{1} are the simplest way to bound the tails.

\textbf{Fact 1 (Chernoff bounds).} Let \( X_1, X_2, \ldots \) be iid random variables with finite mean and finite moment generating function for \( t \in [a, b] \), where \( a \leq 0 \leq b \). Let \( \gamma \in (0, \infty) \), and \( h(\gamma) = \mathbb{E}[\exp(tX)]/\exp(t\gamma \mathbb{E}[X]) \). Then

\[
\mathbb{P}(X \geq \gamma \mathbb{E}[X]) \leq h(\gamma) \quad \text{for all } t \in [0, b] \text{ and } \gamma \geq 1.
\]

\[
\mathbb{P}(X \leq \gamma \mathbb{E}[X]) \leq h(\gamma) \quad \text{for all } t \in [a, 0] \text{ and } \gamma \leq 1.
\]

\textbf{Lemma 4.} For \( X \sim \textbf{Gamma}(k, k-1) \), let \( g(\gamma) = \gamma/\exp(\gamma - 1) \). Then

\[
\mathbb{P}(X \geq \gamma \mathbb{E}[X]) \leq g(\gamma)^k \quad \text{for all } \gamma \geq 1
\]

\[
\mathbb{P}(X \leq \gamma \mathbb{E}[X]) \leq g(\gamma)^k \quad \text{for all } \gamma \leq 1.
\]

\textit{Proof.} For \( X \sim \textbf{Gamma}(k, k-1) \), \( \mathbb{E}[X] = k/(k-1) \) and the moment generating function is \( \mathbb{E}[\exp(tX)] = (1-t/(k-1))^{-k} \) when \( t < k - 1 \). Letting \( \alpha = t/(k-1) \), that makes \( h(\gamma) \) from the Chernoff bound

\[
h(\gamma) = \frac{(1 - \alpha)^{-k}}{\exp(\alpha k \gamma)}.
\]

Letting \( \alpha = 1 - 1/\gamma \) minimizes the right hand side, making it

\[
[\gamma/\exp(\gamma - 1)]^k.
\]

\textbf{Lemma 5.} For \( \gamma \in [0, 2] \),

\[
\frac{\gamma}{\exp(\gamma - 1)} \leq \exp(-1/2)(\gamma - 1)^2 + (1/3)(\gamma - 1)^3
\]
Proof. Let $\beta = \gamma - 1$, then the goal is to show (after taking the natural logarithm of both sides)

$$\ln(1 + \beta) - \beta \leq -(1/2)\beta^2 + (1/3)\beta^3.$$ 

The Taylor series expansion gives

$$\ln(1 + \beta) - \beta = -(1/2)\beta^2 + (1/3)\beta^3 - (1/4)\beta^4 + \cdots.$$ 

When $\beta \in [-1, 0)$, all the terms on the right hand side are negative, so truncating gives the result. When $\beta \in [0, 1]$, the terms alternate and are decreasing in absolute value, so truncation again gives the desired result. □ 

Lemma 6. For $\epsilon \in (0, 3/14)$, when

$$k \geq 2\epsilon^{-2}(1 - (14/3)\epsilon)^{-1}\ln(2\delta^{-1}),$$

$$P(|\hat{p}/p - 1| > \epsilon) < \delta.$$

Proof. Let $X \sim \text{Gamma}(k, k - 1)$. Then $(\hat{p}/p) - 1 \sim (1/X) - 1$, so

$$P(|\hat{p}/p - 1| > \epsilon) = P(|(1/X) - 1| > \epsilon)$$

$$= P(-\epsilon > (1/X) - 1) + P((1/X) - 1 > \epsilon)$$

$$= P((1 - \epsilon)^{-1} < X) + P(X < (1 + \epsilon)^{-1})$$

$$= P(X > \gamma_1 E[X]) + P(X < \gamma_2 E[X]),$$

$$\leq [\gamma_1/\exp(\gamma_1 - 1)] + [\gamma_2/\exp(\gamma_2 - 1)].$$

where $\gamma_1 = [k/(k-1)](1-\epsilon)^{-1}$ and $\gamma_2 = [k/(k-1)](1+\epsilon)^{-1}$. Note $x/\exp(x-1)$ is an increasing function when $x < 1$, and a decreasing function when $x > 1$. For $k \geq \epsilon^{-2}$,

$$1 \leq \frac{k}{k-1} \leq \frac{1}{1-\epsilon^2}.$$ 

Hence

$$P(|\hat{p}/p - 1| > \epsilon) \leq [\gamma_1'/\exp(\gamma_1'-1)] + [\gamma_2'/\exp(\gamma_2'-1)]$$

where

$$\gamma_1 > \gamma_1' = \frac{1}{1-\epsilon}, \quad \gamma_2 < \gamma_2' = \frac{1}{1-\epsilon^2} \cdot \frac{1}{1+\epsilon}.$$

This means

$$\gamma_1'-1 = \frac{\epsilon}{1-\epsilon}, \quad \gamma_2'-1 = -\frac{\epsilon(1-\epsilon-\epsilon^2)}{(1-\epsilon^2)(1+\epsilon)}.$$
Using the bound from the previous lemma
\[
\frac{\gamma_2'}{\exp(\gamma_2' - 1)} \leq \exp\left(-\frac{1}{2} \frac{\epsilon^2}{1 - \epsilon^2} + \frac{1}{3} \frac{\epsilon^3}{(1 - \epsilon)^3}\right) \leq \exp\left(-\frac{1}{2} \frac{\epsilon^2}{2} + \frac{7}{3} \epsilon^3\right)
\]

Now turn to \(\gamma_1'\):
\[
\frac{\gamma_1'}{\exp(\gamma_1' - 1)} \leq \exp\left(-\frac{1}{2} \frac{\epsilon^2(1 - \epsilon - \epsilon^2)^2}{(1 + \epsilon)^2} + \frac{1}{3} \frac{\epsilon^3(1 - \epsilon - \epsilon^2)^3}{(1 + \epsilon)^3}\right)
\]
By clearing the denominator it is possible (if tedious) to verify that
\[-\frac{1}{2} \frac{\epsilon^2(1 - \epsilon - \epsilon^2)^2}{(1 + \epsilon)^2} + \frac{1}{3} \frac{\epsilon^3(1 - \epsilon - \epsilon^2)^3}{(1 + \epsilon)^3} \leq -\frac{1}{2} \frac{\epsilon^2}{2} + \frac{7}{3} \epsilon^3
\]
Hence
\[
\frac{\gamma_1'}{\exp(\gamma_1' - 1)} \leq \exp\left(-\frac{1}{2} \frac{\epsilon^2}{2} + \frac{7}{3} \epsilon^3\right).
\]
Combining the tail bounds for \(\gamma_1'\) and \(\gamma_2'\) gives the result.

3 Lower bound on running time

The new algorithm intentionally introduces random smoothing to make the estimate easier to analyze. For a fixed number of flips, a sufficient statistic for the mean of a Bernoulli random variable is the number of times the coin came up heads. Call this number \(N\).

For \(k\) flips of the coin, \(N\) will be a binomial random variable with parameters \(k\) and \(p\). Then \(\hat{p}_k = N/k\) is the unbiased estimate of \(p\). By the Central Limit Theorem, \(\hat{p}_k\) will be approximately normally distributed with mean \(p\) and standard deviation \(\sqrt{p(1-p)/k}\). Therefore (for small \(p\), \(\hat{p}_k/p\) will be approximately normal with mean 1 and standard deviation \(1/\sqrt{pk}\). Let \(Z\) denote such a normal. Then well known bounds on the tails of the normal distribution give
\[
\frac{\exp(-\epsilon^2 pk/2)}{\sqrt{2\pi}} \left(\frac{1}{\epsilon^2 pk} - \frac{1}{(\epsilon^2 pk)^3}\right) \leq \mathbb{P}(Z > 1 + \epsilon) \leq \frac{\exp(-\epsilon^2 pk/2)}{\sqrt{2\pi}} \left(\frac{1}{\epsilon^2 pk}\right)
\]

Therefore, to get \(\mathbb{P}(Z > 1 + \epsilon) < \delta/2\) requires about \(2\epsilon^{-2}p^{-1}\ln(2\delta^{-1})\) samples. A bound on the lower tail may be found in a similar fashion. Since only about this many samples are required by the algorithm of Section 2, the constant of 2 in front is most likely the best possible.
To actually prove a lower bound, follow the approach of [2] that uses Wald’s sequential probability ratio test. Consider the problem of testing hypothesis \( H_0 : p = p_0 \) versus \( H_1 : p = p_1 \), where \( p_1 = p_0/(1 + \epsilon)^2 \). Suppose there is an approximating scheme that approximates \( p \) within a factor of \( 1 + \epsilon \) with chance at least \( 1 - \delta/2 \) for all \( p \in [p_1, p_0] \) using \( T \) flips of the coin. Then take the estimate \( \hat{p} \) and accept \( H_0 \) (reject \( H_0 \)) if \( \hat{p} \geq p_1(1 + \epsilon) \) and accept \( H_1 \) (reject \( H_1 \)) if \( \hat{p} \leq p_1(1 + \epsilon) \).

Then let \( \alpha \) be the chance that \( H_0 \) is rejected even though it is true, and \( \beta \) be the chance that \( H_1 \) is accepted even though it is false. From the properties of the approximation scheme, \( \alpha \) and \( \beta \) are both at most \( \delta/2 \).

Wald presented the sequential probability ratio test for testing \( H_0 \) versus \( H_1 \), and showed that it minimized the expected number of coin flips among all tests with the type I and II error probabilities \( \alpha \) and \( \beta \) [7]. This result was formulated as Corollary 7.2 in [2].

**Fact 2** (Corollary 7.2 of [2]). If \( T \) is the stopping time of any test of \( H_0 \) versus \( H_1 \) with error probabilities \( \alpha \) and \( \beta \) such that \( \alpha + \beta = \delta \), then

\[
\mathbb{E}[T | H_0] \geq -(1 - \delta)\omega_0^{-1} \ln((2 - \delta)\delta^{-1}).
\]

where \( \omega_0 = \mathbb{E}[\ln(f_1(X)/f_0(X))] \) with \( X \sim \text{Bern}(p_0) \), \( f_0(x) = p_0 \mathbf{1}(x = 1) + (1 - p_0) \mathbf{1}(x = 0) \), and \( f_1(x) = p_1 \mathbf{1}(x = 1) + (1 - p_1) \mathbf{1}(x = 0) \).

This gives the following lemma for \( \text{Bern}(p) \) random variables.

**Lemma 7.** Fix \( \epsilon > 0 \) and \( \delta \in (0, 1) \). Let \( T \) be the stopping time of any \((1 + \epsilon, \delta/2)\) approximation scheme that applies to \( X_i \sim \text{Bern}(p) \) for all \( p \in [0, 1] \). Then

\[
\mathbb{E}[T] \geq (1/5)\epsilon^{-2}(1 + 2\epsilon)(1 - \delta)\ln((2 - \delta)\delta^{-1})p^{-1}.
\]

**Proof.** As noted above, using the approximation scheme with \( \epsilon \) and \( \delta/2 \) to test if \( p_0 = p \) or \( p_1 = p_0/(1 + \epsilon)^2 \) gives \( \alpha \leq \delta/2 \) and \( \beta \leq \delta/2 \). Here

\[
\omega_0 = p_0[\ln(p_1/p_0)] + (1 - p_0)[\ln((1 - p_1)/(1 - p_0))]
= p_0[\ln(p_1/p_0) + (1/p_0 - 1)\ln((1 - p_1)/(1 - p_0))]
= p_0\ln\left[\frac{p_1(1 - p_1)^{1/p_0 - 1}}{p_0(1 - p_0)^{1/p_0 - 1}}\right].
\]

Consider a function of the form \( g(x) = x(1 - x)^{1/c - 1} \) where \( c \) is a constant. Then \( g(x) > 0 \) for \( x \in (0, 1) \), and \( g'(x) = g(x)x^{-1} + (1/c - 1)g(x)(1 - x)^{-1} \), which gives

\[g'(x) > 0 \iff x^{-1} + (1/c - 1)(1 - x)^{-1} \iff x < c.\]
Hence for all $p_0 > p_1$, $\ln(p_1 (1 - p_1)^{1/p_0 - 1})$ is strictly increasing in $p_1$. Setting $p_1 = p_0$ gives $\omega_0 = 0$, so $\omega_0 < 0$ for $0 < p_1 < p_0 \leq 1$.

Using $\alpha + \beta \leq \delta$ and $\omega_0 < 0$ in Fact 2 gives

$$\mathbb{E}[T] \geq -\omega_0^{-1}(1 - \delta) \ln((2 - \delta)\delta^{-1}).$$

Since $\ln(1+x) = x - x^2/2 + \cdots$ is alternating and decreasing in magnitude for $x \in (0, 1)$:

$$\ln\left(\frac{p_1}{p_0}\right) = \ln\left(\frac{1}{(1 + \epsilon)^2}\right) = -2 \ln(1 + \epsilon) \geq -2\epsilon.$$ 

Also, since $1 - (1 + \epsilon)^{-2} = (2\epsilon + \epsilon^2)/(1 + \epsilon)^2$.

$$\left(\frac{1}{p_0} - 1\right) \ln\left(\frac{1 - p_1}{1 - p_0}\right) = \left(\frac{1 - p_0}{p_0}\right) \ln\left(\frac{1 - p_0(1 + \epsilon)^{-2}}{1 - p_0}\right)$$

$$= \left(\frac{1 - p_0}{p_0}\right) \ln\left(\frac{1 + p_0(1 - (1 + \epsilon)^{-2})}{1 - p_0}\right)$$

$$= \left(\frac{1 - p_0}{p_0}\right) \left[\left(\frac{p_0(1 - (1 + \epsilon)^{-2})}{1 - p_0}\right) - \frac{1}{2} \left(\frac{p_0(1 - (1 + \epsilon)^{-2})}{1 - p_0}\right)^2\right]$$

$$\geq \frac{2\epsilon + \epsilon^2}{(1 + \epsilon)^2} - \frac{1}{2} \left[\frac{2\epsilon + \epsilon^2}{(1 + \epsilon)^2}\right]^2 \cdot \frac{p_0}{1 - p_0}.$$ 

For $p_0 \leq 1/2$, $p_0/(1 - p_0) \leq 1$ and the last factor of the second term can be removed. Putting the bounds on the terms of $\omega_0$ together,

$$\omega_0 \geq p_0 \left[\frac{-2\epsilon + \frac{2\epsilon + \epsilon^2}{(1 + \epsilon)^2} - \frac{1}{2} \left(\frac{2\epsilon + \epsilon^2}{(1 + \epsilon)^2}\right)^2}{(1 + \epsilon)^4}\right]$$

$$= p_0 \frac{-5\epsilon^2(1 + 2\epsilon + (3/2)\epsilon^2 + (2/5)\epsilon^3)}{(1 + \epsilon)^4}$$

$$\geq -p_0 5\epsilon^2/(1 + 2\epsilon).$$ 

The last inequality follows from the fact that for $\epsilon > 0$,

$$(1 + 2\epsilon)(1 + 2\epsilon + (3/2)\epsilon^2 + (2/5)\epsilon^3) \leq (1 + \epsilon)^4.$$ 

\qed
4 Extension to $[0, 1]$ random variables

A well known trick allows extension of the algorithm to $[0, 1]$ random variables with mean $\mu$, rather than just Bernoulli’s.

**Lemma 8.** Let $W$ be a $[0, 1]$ random variable with mean $\mu$. Then for $U \sim \text{Unif}(0, 1)$, $\mathbb{P}(U \leq W) = \mu$.

**Proof.** For $U \sim \text{Unif}(0, 1)$ and $W \in [0, 1]$,

$$
\mathbb{P}(U \leq W) = \int_{w=0}^{1} \mathbb{P}(U \leq w) \, dF(w) = \int_{w=0}^{1} w \, dF(w) = \mathbb{E}[W].
$$

Therefore the algorithm of Section 2 can be applied to any $[0, 1]$ random variable at the cost of one uniform on $[0, 1]$ per draw of the random variable.

5 Conclusions

A new algorithm for estimating the mean of $[0, 1]$ variables is given with the remarkable property that the relative error in the estimate has a distribution independent of the quantity to be estimated. The estimate is unbiased. To obtain an estimate which has absolute relative error $\epsilon$ with probability at least $1 - \delta$ requires at most $2\epsilon^{-2}(1 - (14/3)\epsilon)^{-1}p^{-1} \ln(2\delta^{-1})$ samples. The factor of 2 is an improvement over the factor of $4(e - 2)$ in [2]. Informal Central Limit Theorem arguments indicate that this factor of 2 in the running time is the best possible. The provable lower bound on the constant is improved from the $(1/4)e^{-2} \approx 0.0338$ of [2] to $1/5$ for $\{0, 1\}$ random variables.

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