Multiplier Hopf algebras imbedded in locally compact quantum groups

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Abstract
Let \((A, \Delta)\) be a locally compact quantum group and \((A_0, \Delta_0)\) a regular multiplier Hopf algebra. We show that if \((A_0, \Delta_0)\) can in some sense be imbedded in \((A, \Delta)\), then \(A_0\) will inherit some of the analytic structure of \(A\). Under certain conditions on the imbedding, we will be able to conclude that \((A_0, \Delta_0)\) is actually an algebraic quantum group with a full analytic structure. The techniques used to show this, can be applied to obtain the analytic structure of a \(*\)-algebraic quantum group \textit{in a purely algebraic fashion}. Moreover, the reason that this analytic structure exists at all, is that the one-parameter groups, such as the modular group and the scaling group, are diagonalizable. In particular, we will show that necessarily the scaling constant \(\mu\) of a \(*\)-algebraic quantum group equals 1. This solves an open problem posed in [8].

Introduction
In [14], the second author introduced \textit{multiplier Hopf algebras}, generalizing the notion of a Hopf algebra to the case where the underlying algebra is not necessarily unital. In [15], he considered those multiplier Hopf algebras that allow a non-zero left invariant functional. It turned out that these objects, termed \textit{algebraic quantum groups}, possess a very rich structure, allowing for example a duality theory. These objects seemed to form an algebraic model of locally compact quantum groups, which at the time had no proper definition.

In [8], Kustermans showed that a \(*\)-algebraic quantum group (with a positivity condition on the left invariant functional) naturally gives rise to a \(C^*\)-\textit{algebraic quantum group}, which by then had been defined in a preliminary way by Woronowicz, Masuda and Nakagami. Kustermans showed however that there was one discrepancy with the proposed definition, in that the invariance of the scaling group with respect to the left Haar weight was only relative.

These investigations culminated in the by now accepted definition of a \textit{locally compact quantum group} by Kustermans and Vaes, as laid down in [6]. This definition was (up to the relative invariance of the scaling group) equivalent with the one proposed by Woronowicz, Masuda and Nakagami, but the set of axioms was smaller and simpler. These axioms were very much inspired by those of \(*\)-algebraic quantum groups, but introducing analysis made it much harder to show that they were sufficiently powerful to carry a theory of locally compact quantum groups with the desired properties.

\textit{In this article}, we examine a converse of the problem studied in [8] and [4]. Namely, instead of starting with a \(*\)-algebraic quantum group and imbedding it into a locally compact quantum group, we start with an imbedding of a general regular multiplier Hopf algebra in a locally compact quantum group, and look whether the multiplier Hopf algebra inherits some structural properties.

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The study of this problem led us to an enhanced structure theory for ∗-algebraic quantum groups. For example, the analytic structure of these objects is a consequence of the fact that all the actions at hand are diagonalizable. This has as a nice corollary that the scaling constant of a ∗-algebraic quantum group is necessarily 1. It is odd that ∗-algebraic quantum groups, which provided a motivation for allowing relative invariance of the scaling group under the Haar weight, turn out to have proper invariance after all.

The paper is organized as follows. In the first part, we introduce the definitions of the objects at play and introduce notations.

In the second part we investigate the following problem: if a multiplier Hopf algebra $A_0$ can be imbedded in a locally compact quantum group, does this give us information about the multiplier Hopf algebra? Firstly, we must specify what we mean by ‘imbedded in’: $A_0$ has to be a subalgebra of the locally compact quantum group, and the respective comultiplications $\Delta_0$ and $\Delta$ have to satisfy formulas of the form $\Delta_0(a)(1 \otimes b) = \Delta(a)(1 \otimes b)$ for $a, b$ in $A_0$. Secondly, we must specify whether we imbed $A_0$ in the von Neumann algebra $M$ or in the C∗-algebra $A$ associated to the locally compact quantum group. Already in the first situation, the objects of $A_0$ will behave nicely with respect to analyticity of the various one-parameter groups. But only in the second case can we conclude, under a mild extra condition, that $A_0$ is invariant under these one-parameter groups. Moreover, $A_0$ will then automatically have the structure of an algebraic quantum group.

In the third part we apply the techniques of the previous section to obtain structural properties of ∗-algebraic quantum groups. We want to stress that this section is entirely of an algebraic nature. For example, we prove in a purely algebraic fashion the existence of a positive right invariant functional on the ∗-algebraic quantum group. Up to now, some involved analysis was necessary to arrive at this.

In the fourth part we consider some special cases. We also look at a concrete example. Since examples of non-compact locally compact groups and ∗-algebraic quantum groups are still hard to find, we have to limit ourselves to some already well-known ones. We will see what can be said about the discrete quantum group $U_q(su(2))$ in this context.

Some of the motivation for this paper comes from [9], where similar questions are investigated in the commutative and co-commutative case. For example, it is shown that the function space $C_0(G)$ of a locally compact group contains a dense multiplier Hopf ∗-algebra, if and only if $G$ contains a compact open subgroup. The multiplier Hopf ∗-algebra will be the space spanned by translates of regular (=polynomial) functions on this compact group.

1 Preliminaries

In this article, we use the concepts of a regular multiplier Hopf (∗)-algebra, a (∗)-algebraic quantum group, a (reduced) C∗-algebraic quantum group and a von Neumann-algebraic quantum group, as introduced respectively in [14], [15], [6] and [7] (see also [18]). Since these objects stem from quite different backgrounds, we will give a brief overview of their definitions.

Regular multiplier Hopf (∗)-algebras

We recall the notion of the multiplier algebra of an algebra. Let $A$ be a non-degenerate algebra (over the field $\mathbb{C}$), with or without a unit. The non-degeneracy condition means that if $ab = 0$ for all $b \in A$, or $ba = 0$ for all $b \in A$, then $a = 0$. As a set, the multiplier algebra $M(A)$ of $A$ consists of couples $(\lambda, \rho)$, where $\lambda$ and $\rho$ are linear maps $A \to A$, obeying the following law:

$$a\lambda(b) = \rho(a)b, \quad \text{for all } a, b \in A.$$
In practice, we write \( m \) for \((\lambda, \rho)\), and denote \( \lambda(a) \) by \( ma \) and \( \rho(a) \) by \( am \). Then the above law is simply an associativity condition. With the obvious multiplication by composition of maps, \( M(A) \) becomes an algebra, called the multiplier algebra of \( A \). Moreover, if \( A \) is a \( * \)-algebra, \( M(A) \) also carries a \( * \)-operation: for \( m \in M(A) \) and \( a \in A \), we define \( m^* \) by \( m^*a = (a^m)^* \) and \( amo^* = (mao^*)^* \). Note that, when \( A \) is actually a \( C^* \)-algebra, one may like to impose some extra continuity conditions on the above linear maps, but actually, the continuity comes for free, as an application of the closed graph theorem shows.

There is a natural map \( A \to M(A) \), letting an element \( a \) correspond with left and right multiplication by it. Because of non-degeneracy, this \( (\cdot, \cdot) \)-algebra morphism will be an injection. In this way, non-degeneracy compensates the possible lack of a unit. Note that, when \( A \) is unital, \( M(A) \) is equal to \( A \).

Let \( B \) be another non-degenerate \( (\cdot, \cdot) \)-algebra. In our paper, a morphism between \( A \) and \( B \) is a non-degenerate \( (\cdot, \cdot) \)-algebra homomorphism \( f : A \to M(B) \). In general, the non-degeneracy of a map \( f \) means that \( f(A)B = B \) and \( Bf(A) = B \). If \( A \) and \( B \) are both \( C^* \)-algebras, non-degeneracy means that \( f(A)B \) is norm-dense in \( B \). In any case, if \( f \) is a morphism from \( A \) to \( B \), then \( f \) can be extended to a unital \( (\cdot, \cdot) \)-algebra morphism from \( M(A) \) to \( M(B) \). A proper morphism between \( A \) and \( B \) is a morphism \( f \) such that \( f(A) \subset B \).

We can now state the definition of a regular multiplier Hopf \( (\cdot, \cdot) \)-algebra. It is the appropriate generalization of a Hopf \( (\cdot, \cdot) \)-algebra to the case where the underlying algebra need not be unital. A regular multiplier Hopf \( (\cdot, \cdot) \)-algebra consists of a couple \((A, \Delta)\), with \( A \) a non-degenerate \( (\cdot, \cdot) \)-algebra, and \( \Delta \), the comultiplication, a morphism from \( A \) to \( A \circ A \), where \( \circ \) denotes the algebraic tensor product. Moreover, \((A, \Delta)\) has to satisfy the following conditions:

M.1 \( (\Delta \circ i)\Delta = (i \circ \Delta)\Delta \) (coassociativity).

M.2 The maps

\[
\begin{align*}
T_{\Delta 2} : A \circ A & \to M(A \circ A) : a \circ b \mapsto \Delta(a)(1 \circ b), \\
T_\Delta : A \circ A & \to M(A \circ A) : a \circ b \mapsto (a \circ 1)\Delta(b), \\
T_{\Delta 1} : A \circ A & \to M(A \circ A) : a \circ b \mapsto \Delta(a)(b \circ 1), \\
T_{\Delta 2} : A \circ A & \to M(A \circ A) : a \circ b \mapsto (1 \circ a)\Delta(b)
\end{align*}
\]

all induce linear bijections \( A \circ A \to A \circ A \).

Here, and elsewhere in the text, we will use \( i \) to denote the identity map.

The \( T \)-maps can be used to define a co-unit \( \varepsilon \) (which will be a \( (\cdot, \cdot) \)-homomorphism from \( A \to \mathbb{C} \)) and an antipode \( S \) (which will be a linear anti-morphism, satisfying \( S(S(a^\ast)^* = a \) for all \( a \in A \) when \( A \) carries an involution). Both co-unit and antipode will be unique, and will satisfy the corresponding equations of those defining them in the Hopf algebra case.

\( (\cdot, \cdot) \)-Algebraic quantum groups

Our second object forms an intermediate step between the former, purely algebraic notion of a regular multiplier Hopf algebra, and the analytic set-up of a locally compact quantum group. A \( (\cdot, \cdot) \)-algebraic quantum group is a regular multiplier Hopf \( (\cdot, \cdot) \)-algebra \((A, \Delta)\), for which there exists a non-zero linear functional \( \varphi \) on \( A \), such that

\[
(i \circ \varphi)(\Delta(a)(b \circ 1)) = \varphi(a)b, \quad \text{for all } a, b \in A.
\]

When \( A \) is a \( * \)-algebra, we impose the extra condition of positivity on \( \varphi \): for every \( a \in A \), we have \( \varphi(a^\ast a) \geq 0 \). This extra condition is in fact very restrictive, as we shall see.

We can prove that \( \varphi \) is unique up to multiplication with a scalar. It will be faithful in the following sense: if \( \varphi(ab) = 0 \) for all \( b \in A \), or \( \varphi(ba) = 0 \) for all \( b \in A \), then \( a = 0 \). The functional \( \varphi \) is called the left Haar functional. Then \((A, \Delta)\) will also have a non-zero functional \( \psi \), again unique up to a scalar, such that

\[
(\psi \circ i)(\Delta(a)(1 \circ b)) = \psi(a)b, \quad \text{for all } a, b \in A.
\]
We will call $\psi$ a right Haar functional. If $A$ is a $\sigma$-algebraic quantum group, we can still choose $\psi$ to be positive. We note however, that to arrive at this functional, a detour into analytic landscape (with aid of the GNS-device for $\varphi$) seemed inevitable. The problem is of course that the evident right Haar functional $\psi = \varphi \circ S$ is not necessarily positive. To create the right $\psi$, we needed some analytic machinery, namely the square root of the modular element, or a polar decomposition of the antipode (see [8]). In this paper we show, that it is possible to arrive at the positivity of the right invariant functional by purely algebraic means (see the remark after Theorem 3.5). This means that $\sigma$-algebraic quantum groups are appropriate objects of study for algebraists with a fear of analysis.

Algebraic quantum groups have some really nice features. For example, there exists a unique automorphism $\sigma$ of the algebra $A$, satisfying $\varphi(ab) = \varphi(b\sigma(a))$ for all $a, b \in A$. It is called the modular automorphism, a notion coming from the theory of weights on von Neumann-algebras (see below). There also exists a unique multiplier $\delta$ such that

\[
(\varphi \otimes \iota)(\Delta(a)(1 \otimes b)) = \varphi(a)\delta b,
\]

\[
(\varphi \otimes \iota)((1 \otimes b)\Delta(a)) = \varphi(a)b\delta,
\]

for all $a, b \in A$. It is called the modular element, as it is the non-commutative equivalent of the modular function in the theory of locally compact groups. When $A$ is a $\sigma$-algebraic quantum group, $\delta$ will indeed be a positive element (i.e. $\delta = q^*q$ for some $q \in M(A)$).

There also is a particular number that can be associated with an algebraic quantum group. Since $\varphi \circ S^2$ is a left Haar functional, the unicity of $\varphi$ implies there exists $\mu \in \mathbb{C}$ such that $\varphi(S^2(a)) = \mu \varphi(a)$, for all $a \in \mathbb{C}$. This number $\mu \in \mathbb{C}$ is called the scaling constant of $(A, \Delta)$.

In an early stage, examples of algebraic quantum groups were found where $\mu \neq 1$ (see [15]). However, it remained an open question whether $\sigma$-algebraic quantum groups existed with $\mu \neq 1$. We will show in this paper that in fact $\mu = 1$ for all $\sigma$-algebraic quantum groups (see Theorem 3.4).

$\sigma$-algebraic quantum groups

A reduced $\sigma$-algebraic quantum group is the non-commutative version of the space of continuous complex functions, vanishing at infinity, of a locally compact group. It consists of a couple $(A, \Delta)$, with $A$ a $\sigma$-algebra, encoding the topology of the quantum space, and $\Delta$ a morphism $A \rightarrow A \otimes A$, encoding the group-multiplication in the quantum space. Here $\otimes$ denotes the minimal $\sigma$-algebraic tensor product. Moreover, $(A, \Delta)$ has to satisfy the following conditions, which are the appropriate non-commutative translations of the associativity condition and the cancelation property\(^1\) of a locally compact group:

C.1 $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ (coassociativity).

C.2 $\{(\omega \otimes \iota)\Delta(a) \mid a \in A, \omega \in A^*\}$ and $\{((\iota \otimes \omega)\Delta(a) \mid a \in A, \omega \in A^*\}$ are norm-dense in $A$. Here $A^*$ denotes the space of continuous functionals on $A$.

In contrast with the theory of locally compact groups, it seems unlikely that a satisfactory theory of locally compact quantum groups can be built from simply topological and algebraic axioms. Some analytic structure has to be included. More precisely, we have to assume the existence of invariant weights, which correspond to the notion of Haar measures on a locally compact group. The final axiom reads:

C.3 There exist faithful KMS-weights $\varphi$ and $\psi$ on $A$, such that

1. $\varphi((\omega \otimes \iota)\Delta(a)) = \varphi(a)\omega(1)$ for $a \in A^+$ and $\omega \in A^*_+$,
2. $\psi((\iota \otimes \omega)\Delta(a)) = \psi(a)\omega(1)$ for $a \in A^+$ and $\omega \in A^*_+$.

\(^1\)Actually, the cancelation property says that $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$, but it can be shown that the given (weaker) condition implies this.
Actually, this axiom can be relaxed in a non-trivial manner (see [6]). For an account of the theory of weights on C*-algebras, we refer the reader to [5].

It turns out that \( \varphi \) and \( \psi \) are unique up to multiplication with a positive scalar. They are called respectively a left and right Haar weight. Unlike the case of algebraic quantum groups, both left and right Haar weight must be part of the definition to have a good theory. The most important objects associated with \((A, \Delta)\) are the multiplicative unitaries, which essentially carry all information about \((A, \Delta)\). To introduce them, we first recall the notion of the GNS-representation associated to \( \psi \). Denote the algebra of \( \psi \)-integrable elements by \( \mathcal{M}_\psi \) and the set of square integrable elements by \( \mathcal{N}_\psi \). The GNS-space associated to \( \psi \) is the closure \( \mathcal{H}_\psi \) of the pre-Hilbert space \( \mathcal{N}_\psi \), with scalar product defined by \( \langle a, b \rangle = \psi(b^* a) \) for \( a, b \in \mathcal{N}_\psi \). The injection of \( \mathcal{N}_\psi \) into \( \mathcal{H}_\psi \) will be denoted by \( \Lambda \). We can construct a faithful representation of \( A \) on \( \mathcal{H}_\psi \) via left multiplication. We can do the same for \( \varphi \), obtaining a Hilbert space \( \mathcal{H}_\varphi \) and an injection \( \Lambda \varphi : \mathcal{N}_\varphi \to \mathcal{H}_\varphi \). Moreover, there exists a unitary from \( \mathcal{H}_\varphi \) to \( \mathcal{H}_\psi \), commuting with the action of \( A \). Since the representations are faithful, we can identify both Hilbert spaces, and denote this Hilbert space as \( \mathcal{H} \). We will let elements of \( A \) act directly on \( \mathcal{H} \) as operators (suppressing the representation). Now we can define the multiplicative unitary \( W \), also called the left regular representation: it is the unitary operator on \( \mathcal{H} \otimes \mathcal{H} \), characterized by

\[
(\iota \otimes \omega)(W^*) \Lambda \varphi(x) = \Lambda \varphi((\iota \otimes \omega)(\Delta(x))), \quad x \in \mathcal{N}_\varphi \text{ and } \omega \in B(\mathcal{H})_*.
\]

Here \( B(\mathcal{H}) \) denotes the space of bounded operators on \( \mathcal{H} \) and \( B(\mathcal{H})_* \) its pre-dual. It implements the comultiplication as follows:

\[
\Delta(x) = W^*(1 \otimes x)W, \quad \text{for all } x \in M.
\]

Moreover, the normclosure of the set \( \{(\iota \otimes \omega)(W) \mid \omega \in B(\mathcal{H})_*\} \) will be equal to \( A \).

We can also define the multiplicative unitary \( V \), called the right regular representation: it is the unitary operator on \( \mathcal{H} \otimes \mathcal{H} \), determined by

\[
(\omega \otimes \iota)(V^*) \Lambda \varphi(x) = \Lambda \varphi((\omega \otimes \iota)(\Delta(x))), \quad x \in \mathcal{N}_\varphi \text{ and } \omega \in B(\mathcal{H})_*.
\]

It also implements the comultiplication:

\[
\Delta(x) = V(x \otimes 1)V^*, \quad \text{for all } x \in M.
\]

Again, the normclosure of the set \( \{(\omega \otimes \iota)(V) \mid \omega \in B(\mathcal{H})_*\} \) will be equal to \( A \).

With the aid of a multiplicative unitary, it is possible to construct a dual quantum group \((\hat{A}, \hat{\Delta})\). Just as in the classical case, the bi-dual will be isomorphic to the original quantum group.

The right regular representation can also be used to define the antipode on \((A, \Delta)\). It is the (possibly unbounded) closed linear map \( S \) from \( A \) to \( A \), with a core consisting of elements of the form \((\omega \otimes \iota)(V^*) \), \( \omega \in B(\mathcal{H})_* \), such that

\[
S((\omega \otimes \iota)(V)) = (\omega \otimes \iota)(V^*).
\]

This map has a polar decomposition, consisting of a (point-wise) normcontinuous one-parameter group \( \tau \) on \( A \) (called the scaling group) and a \(^*\)-anti-automorphism \( R \) of \( A \) (called the unitary antipode). Then the antipode equals the map \( R \circ \tau_{-i/2} \), where \( \tau_{-i/2} \) is the analytic continuation of \( \tau \) to the point \(-i/2\).

Since \( \psi \) and \( \varphi \) are KMS-weights, there exist respective normcontinuous one-parameter groups \( \sigma^* \) and \( \sigma \) on \( A \), called the modular one-parameter groups (associated with \( \psi \) and \( \varphi \)). It can be shown that there exists a (possibly unbounded) positive operator \( \delta \) on \( \mathcal{H} \), affiliated with \( A \) (in the sense of Woronowicz), such that \( \psi = \varphi(\delta^{1/2}, \delta^{1/2}) \) for all \( a \in A \) and \( t \in \mathbb{R} \). For
a correct interpretation of this equality, we refer to [12]. We call $\delta$ the modular element of $(A, \Delta)$. In the sequel, we will frequently use the commutation rules between these objects and $\Delta$:

$$\begin{align*}
\Delta \tau_t &= (\tau_t \otimes \tau_t) \Delta \\
\Delta \sigma_t &= (\tau_t \otimes \sigma_t) \Delta \\
\Delta \tau_t' &= (\sigma_t' \otimes \tau_{-t}) \Delta \\
\Delta \sigma_t' &= (\sigma_t \otimes \sigma_{-t}) \Delta.
\end{align*}$$

There also exists $\nu \in \mathbb{R}^+$ such that $\sigma_t(\delta) = \nu^t \delta$. This constant $\nu$ is called the scaling constant. It arises naturally in the framework considered by Kustermans and Vaes. Therefore, it was an important question whether there exist locally compact quantum groups where this constant is not trivially 1. Such quantum groups do indeed exist: an interesting example is the quantum $az + b$-group (see [17]).

**Von Neumann-algebraic quantum groups**

Our final object is a von Neumann-algebraic quantum group. It consists of a couple $(M, \Delta)$, with $M$ a von Neumann-algebra, and $\Delta$ a normal morphism from $M$ to $M \otimes M$, satisfying the coassociativity condition. Here $\otimes$ denotes the ordinary von Neumann-algebraic tensor product. We also assume that there exist faithful, semi-finite normal weights $\varphi$ and $\psi$ on $M$, satisfying

1. $\varphi((\omega \otimes \iota)\Delta(a)) = \varphi(a)\omega(1)$ for $a \in M^+$ and $\omega \in M_{s,+},$
2. $\psi((\iota \otimes \omega)\Delta(a)) = \psi(a)\omega(1)$ for $a \in M^+$ and $\omega \in M_{s,+}.$

The theory then develops parallel to the $C^*$-algebraic case, but with some significant simplifications (see [18]). Remark for example that its definition does not mention the ‘cancelation law’: this law will automatically be fulfilled! However, it can be shown that it contains as much information as a $C^*$-algebraic locally compact quantum group: there is a canonical bijection between $C^*$-algebraic and von Neumann-algebraic quantum groups. When working in the von Neumann-algebraic context, we will use the same notation as in the $C^*$-algebraic case (the antipode will be denoted by $S$, the scaling group by $\tau$, ...). For the theory of weights on von Neumann-algebras, we refer the reader to [11].

## 2 Multiplier Hopf algebras imbedded in locally compact quantum groups

In this section, we fix a $C^*$-algebraic quantum group $(A, \Delta)$ and a regular multiplier Hopf algebra $(A_0, \Delta_0)$. The von Neumann-algebraic quantum group associated to $(A, \Delta)$ will be denoted by $(M, \Delta)$. We will use notations as before, but the structural maps for $A_0$ will be indexed by 0 (whenever this causes no confusion). We also fix a left Haar weight $\varphi$ on $(A, \Delta)$. As a right Haar weight on $(A, \Delta)$, we choose $\psi = \varphi \circ R$. When using the notation $\mathcal{N}_\varphi$, we will always specify whether we mean the square integrable elements in $M$ or in $A$.

**Assumption**: $A_0 \subseteq M$.

This means that $A_0$ is a subalgebra of $M$, not necessarily invariant under the $^*$-involution. We also want to impose a certain compatibility between $\Delta$ and $\Delta_0$, but we have to be careful: $M(A_0)$ bears no natural relation to $M$. For example, denoting by $j$ the inclusion of $A_0$ in $M$, the identity $(j \otimes j) \circ \Delta_0 = \Delta \circ j$ can be meaningless if $j$ has no well-defined extension to $M(A_0)$. We will however assume the following: for all $a, b \in A_0$,

$$\begin{align*}
\Delta_0(a)(1 \otimes b) &= \Delta(a)(1 \otimes b), \\
\Delta_0(a)(b \otimes 1) &= \Delta(a)(b \otimes 1),
\end{align*}$$

\[6\]
\[(a \otimes 1)\Delta_0(b) = (a \otimes 1)\Delta(b),\]
\[(1 \otimes a)\Delta_0(b) = (1 \otimes a)\Delta(b).\]

Remark. This condition is strictly weaker than the condition \((j \otimes j) \circ \Delta_0 = \Delta \circ j\), when it makes sense. For example, the imbedding of \(C(\mathbb{Z})\) in \(C(\mathbb{Z})\) sending \(\delta_i\) to \(\delta_{2i}\) satisfies the former, but not the latter condition.

Our first result shows that the antipode \(S\) of \(M\) restricts to the antipode \(S_0\) of \(A_0\). The hard part consists of showing that \(A_0\) lies in the domain of \(S\). We will need a lemma which is interesting in its own right. It is a kind of cancelation property involving \(M\) and \(\hat{M}'\), the commutant of the dual quantum group \(\hat{M}\).

Lemma 2.1. Suppose \(a \in M\) and \(x \in \hat{M}'\) satisfy \(ax = 0\). Then \(a = 0\) or \(x = 0\).

Proof. Let \(W\) be the left regular representation for \(M\). We recall that \(W \in M \otimes \hat{M}\) (see e.g. [7]). So if \(xa = 0\), then
\[
W^*(1 \otimes xa)W = (1 \otimes x)W^*(1 \otimes a)W
= (1 \otimes x)\Delta(a)
= 0.
\]
Assume \(x \neq 0\). Choose \(\omega \in B(\mathcal{H})_{+,+}\) such that \(\omega(xx^*) = 1\). Then we have \((\iota \otimes \omega(x \cdot x^*))\Delta(aa^*) = 0\). Applying \(\psi\) and using the strong right invariance property, we get \(\psi(aa^*)\omega(xx^*) = \psi(aa^*) = 0\). Since \(\psi\) is faithful, \(a\) must be zero.

\[\square\]

Remark. Applying \(\tilde{J} \cdot \tilde{J}\), we see that also the following is true: if \(a \in M\) and \(x \in \hat{M}\), then \(ax = 0\) implies either \(a = 0\) or \(x = 0\).

We can show now that the antipodes of \(M\) and \(A_0\) coincide.

Proposition 2.2. \(A_0\) lies in the domain of \(S\), and \(S|_{A_0}\) will be the antipode of \((A_0, \Delta_0)\).

Proof. Let \(b\) be an element of \(A_0\). We will show that \(b \in \mathcal{D}(S)\) and \(S(b) = S_0(b)\). We start by choosing some fixed \(a\) in \(A_0\). We can pick \(p_i, q_i\) in \(A_0\) such that
\[
a \otimes b = \sum_{i=1}^n (p_i \otimes 1)\Delta(q_i).
\]
Then
\[
\Delta(a)(1 \otimes S_0(b)) = \sum_{i=1}^n \Delta(p_i)(q_i \otimes 1).
\]
Let \(y\) be \((\omega_{c,d}(a \cdot \iota))(V) = (\psi \otimes \iota)((c^*a \otimes 1)\Delta(d))\), where \(c, d\) are square integrable elements in \(M\) and \(\omega_{c,d} = \langle \Lambda_{\psi}(d), \Lambda_{\psi}(c) \rangle\). Then
\[
by = (\psi \otimes \iota)((c^*a \otimes b)\Delta(d))
= (\psi \otimes \iota)\sum(c^*p_i \otimes 1)\Delta(q_id).
\]
We know that this last expression is in \(\mathcal{M}_{\psi \otimes \iota}\) (where we use the slice-notation) and that
\[
(\psi \otimes \iota)\sum(c^*p_i \otimes 1)\Delta(q_id) \in \mathcal{D}(S),
\]
with
\[
S((\psi \otimes \iota)\sum(c^*p_i \otimes 1)\Delta(q_id)) = (\psi \otimes \iota)\sum \Delta(c^*p_i)(q_id \otimes 1).
\]
So

\[
S(by) = (\psi \otimes \iota) \sum \Delta(c^* p_i)(q_i d \otimes 1) \\
= (\psi \otimes \iota)(\Delta(c^* a)(d \otimes S_0(b))) \\
= S(y)S_0(b).
\]

Denote by \( C \) the linear span of all such \( y \), with \( c \) and \( d \) varying. We show that \( C \) is an ultra-strong*-core for \( S \). First remark that functionals of the form \( \omega_{e,d}(a \cdot) \) have a norm-dense linear span in \( (\mathcal{M}')_* \). Indeed: if \( z \in \mathcal{M}' \) such that \( (a \cdot \Lambda_\omega(d), \Lambda_\omega(c)) = 0 \) for all \( c, d \in \mathcal{N}_\omega \), then \( az = 0 \), hence \( z = 0 \) by the previous lemma. Then, since \( V \in \mathcal{M}' \otimes M \), there exists for every \( \omega \in B(\mathcal{H})_\ast \) a sequence of \( e_n, d_n \in \mathcal{N}_\omega \) such that \( (\omega_{e_n,d_n}(a \cdot) \otimes \iota)(V) \to (\omega \otimes \iota)(V) \) and \( (\omega_{e_n,d_n}(a \cdot) \otimes \iota)(V^*) \to (\omega \otimes \iota)(V^*) \). Since \( \{(\omega \otimes \iota)(V) \mid \omega \in B(\mathcal{H})_\ast\} \) is an ultra-strong*-core for \( S \), the same will be true for \( C \).

By choosing a net \( y_\alpha \) in \( C \) such that \( y_\alpha \to 1 \) and \( S(y_\alpha) \to 1 \) in the ultra-strong*-topology, we can conclude that \( b \in \mathcal{D}(S) \) and \( S(b) = S_0(b) \).

The previous proposition implies that \( A_0 \subset \mathcal{D}(\tau_z) \) for every \( z \in \mathbb{C} \), i.e. every \( a \in A_0 \) is analytic with respect to \( \tau \). Indeed: \( a \in \mathcal{D}(S) \) means that \( a \in \mathcal{D}(\tau_{-i/2}) \). Since \( S(S_{-1}(a)) = a \) for \( a \in A_0 \), we also have that \( a \in \mathcal{D}(S_{-1}) = \mathcal{D}(\tau_{i/2}) \). So \( A_0 \subset \mathcal{D}(\tau_{ni}) \) for every integer \( n \in \mathbb{Z} \).

This again illustrates the lack of analytic structure of a general algebraic quantum group: if its antipode \( S \) satisfies \( S^{2n} = \iota \), but \( S^2 \neq \iota \), then it can not be imbedded in a locally compact quantum group at all. It can not be at all. Such algebraic quantum groups do indeed exist (see e.g. [15]).

We can also use Lemma 2.1 to prove that actually \( A_0 \subset M(A) \). Fix \( a \in A_0 \). Choose \( b \in A_0 \) and \( \omega \in B(\mathcal{H})_\ast \). Then

\[
a \otimes b = \sum (q_i \otimes 1)\Delta(p_i) \\
= \sum (q_i \otimes 1)V(p_i \otimes 1)V^*,
\]

for some \( p_i, q_i \in A_0 \). Multiplying from the right with \( V \) and applying \( \omega \otimes \iota \), we get \( b(\omega(a \cdot) \otimes \iota)(V) \in A \). But as we have shown, the set \( \{(\omega(a \cdot) \otimes \iota)(V) \mid \omega \in B(\mathcal{H})_\ast\} \) is norm-dense in \( A \). Hence \( bA \subseteq A \). Similarly \( Ab \subseteq A \), and thus \( A_0 \subset M(A) \).

As a second important result, we show that \( A_0 \) consists of analytic elements for \( \sigma \). This follows easily from the following proposition, which elucidates the behavior of \( A_0 \) with respect to the one-parameter group \( \kappa \), where \( \kappa_t = \sigma_t \tau_{-t} \). It will be decisive in obtaining some structural properties of *-algebraic quantum groups, as we will show in the third section.

**Proposition 2.3.** \( A_0 \subset \mathcal{D}(\kappa_z) \) for all \( z \in \mathbb{C} \), and \( \kappa_z(A_0) \subset A_0 \). Here \( \kappa_z \) denotes the analytic continuation of the one-parameter group \( \kappa \) to the point \( z \in \mathbb{C} \), and \( \mathcal{D}(\kappa_z) \) denotes its domain.

**Proof.** Let \( b \) be a fixed element of \( A_0 \). Choose a non-zero \( a \in A_0 \), and write

\[
a \otimes b = \sum_{i=1}^n \Delta(p_i)(1 \otimes q_i),
\]

with \( p_i, q_i \in A_0 \). Using the commutation relations between \( \Delta, \tau, \sigma \) and \( \sigma' \), we get that

\[
\kappa_{-t}(a) \otimes \rho_t(b) = \sum \Delta(p_i)(1 \otimes \rho_t(q_i)), \text{ for all } t \in \mathbb{R},
\]
where $\rho_t = \sigma'_t \tau_t$. Choose $c \in A_0$ such that $c\bar{b} \neq 0$, and multiply this equation to the left with $1 \otimes c$ to get

$$\kappa_{-t}(a) \otimes \rho_t(b) = \sum_{i,j} (1 \otimes c) \Delta(p_i)(1 \otimes \rho_t(q_i)).$$

Choose $a_{ij}, b_{ij} \in A_0$ such that

$$(1 \otimes c) \Delta(p_i) = \sum_{j=1}^{m_i} a_{ij} \otimes b_{ij},$$

and let $L$ be the finite-dimensional space spanned by the $a_{ij}$. We see that $\kappa_{-t}(a) \otimes \rho_t(b) \in L \otimes M$, for every $t \in \mathbb{R}$. Since $c\bar{b}(t) = c\bar{b} \neq 0$, and $\rho_t$ is strongly continuous, we get that there exists a $\delta > 0$ such that $c\bar{b}(t) \neq 0$ for all $t$ with $|t| < \delta$. This means $\kappa_{-t}(a) \in L$ for all $|t| < \delta$.

For every $\varepsilon > 0$, let $K_{\varepsilon} = \text{span}\{\kappa_{t}(a) \mid |t| < \varepsilon\}$, and $n_{\varepsilon} = \dim(K_{\varepsilon})$. For small $\varepsilon$, we have $n_{\varepsilon} \in \mathbb{N}$. Choose an $\varepsilon$ where this dimension reaches a minimum. Then $K = K_{\varepsilon} = K_{\varepsilon/2}$ will be a finite-dimensional subspace containing $a$, invariant under $\kappa_t$, for all $t \in \mathbb{R}$.

Now $\kappa$ induces a continuous homomorphism $\tilde{\kappa} : \mathbb{R} \to GL(K)$. It is a well-known fact that such a homomorphism is necessarily analytic. Thus $a \in \mathcal{D}(\kappa_{\varepsilon})$, and $\kappa_{\varepsilon}(a) \in K \subseteq A_0$. This concludes the proof.

$\square$

**Remarks.** (i) We can actually show that $A_0$ is spanned by eigenvectors for $\kappa_t$. For every continuous one-parameter group of isometries on a finite-dimensional Banach space is diagonalizable. Namely, using notation as in the proof, consider a Banach space isomorphism of $K$ to $\mathbb{C}^n$ with the usual Hilbert space structure. Then $\kappa_t$ will be transformed to a uniformly bounded, norm-continuous one-parameter group of operators on $\mathbb{C}^n$. Since $\mathbb{R}$ is amenable, we can choose another Hilbert structure on $\mathbb{C}^n$ such that the one-parameter group will consist of unitaries. Hence the action is diagonalizable.

(ii) The lemma remains true if we replace $\kappa_t$ by $\rho_t = \tau_t \sigma'_t$ or $\sigma_t \sigma'_t$.

**Corollary 2.4.** $A_0$ consists of analytic elements for $\sigma$.

**Proof.** This follows easily from the previous two statements. If $a \in A_0$, we know that $a$ is analytic for $\tau_t$ and $\kappa_t = \sigma_t \tau_{-t}$. If $z \in \mathbb{C}$, then $\tau_z(\kappa_z(a))$ makes sense, since $A_0$ is invariant under $\kappa_z$. Since $\sigma_z$ is the closure of $\tau_z \circ \kappa_z$, we arrive at $a \in \mathcal{D}(\sigma_z)$.

$\square$

As a consequence, $A_0$ is invariant under $\sigma_{ni}$ and $\sigma'_{ni}$, with $n \in \mathbb{Z}$.

**Remark.** We do not know if $A_0$, or even the von Neumann-algebra $N$ generated by it, has to be invariant under the one-parameter groups $\sigma$ and $\tau$. There seems to be an analytic obstruction to be able to conclude this. It is however easy to see that if $N$ is invariant under either $\sigma$, $\tau$ or $\delta^{-1} \cdot \delta^{it}$, then it is invariant under all of them (see e.g. Proposition 2.9).

Next, we impose a stronger condition on $A_0$:

**Assumption:** $A_0 \subseteq A$

We will say then that $A_0$ has a proper imbedding in $A$. Because $A_0$ is now a subspace of the $C^*$-algebra $A$, we can say more about its connection to $\varphi$. We first need a simple lemma, which also appears in some form in [9]:

**Lemma 2.5.** Suppose that $a \in A \cap \mathcal{D}(\sigma_{1/2})$ and $c \in A$ satisfy $ca = a$. Then $a \in \mathcal{N}_{\varphi}$.

**Proof.** Choose $c$ in $A \cap \mathcal{M}_{\varphi}^+$ such that $0 \leq \|c - e^*c\| \leq 1/2$. This is possible because $\mathcal{M}_{\varphi}^+ \cap A$ is normdense in $A^+$. Then

$$\frac{1}{2}a^*a \leq a^*(1 + c - e^*c)a = a^*ca.$$
Proof. We have that \( A \Delta (\cdot) \) can apply the previous lemma to each element of \( (\bigcup_{z \in \mathbb{C}} \sigma_z(A^n)) \) and \( \sigma(z) \in \mathcal{N}_\varphi \cap \mathcal{M}^*_\varphi \) for all \( z \in \mathbb{C} \).

The previous proposition has the interesting corollary that the scaling constant of \( A \) necessarily trivial. We will come back to this fact in the third section, where we apply our techniques to \( * \)-algebraic quantum groups (see Theorem 3.4).

**Corollary 2.7.** The scaling constant \( \nu \) of \( (A, \Delta) \) equals 1.

**Proof.** We have that \( \nu^{-\frac{1}{2} t} \kappa_t \) induces a one-parameter unitary group \( u_t \) on \( \mathcal{H} \), where \( \kappa_t = \sigma_t \circ \tau_{-t} \). As in the proof of lemma 2.3, there is a finite-dimensional subspace \( K \) of \( A_0 \) that is invariant under \( \kappa \). Therefore \( V = \Lambda_\varphi(K) \) is invariant under \( u_t \). This means that there exists a non-zero \( v = \Lambda_\varphi(x) \in V \) such that \( u_t(v) = e^{it\lambda}v \), for some \( \lambda \in \mathbb{R} \). Hence \( \nu^{-\frac{1}{2} t} \kappa_t(x) = e^{it\lambda}x \).

But, since \( \kappa_t \) is a one-parameter group of \( * \)-automorphisms, we get

\[
\|x\| = \|\kappa_t(x)\| = \|e^{it\lambda} \nu^{\frac{1}{2} t} x\| = \nu^{\frac{1}{2} t} \|x\|.
\]

So \( \nu = 1 \). 

From the previous proposition, it follows that \( A_0 \subset \mathcal{N}_\varphi \cap \mathcal{M}^*_\varphi \). Because \( A_0^2 = A_0 \), we also have \( A_0 \subset \mathcal{M}_\varphi \), so every element of \( A_0 \) is integrable with respect to \( \varphi \). However, we cannot conclude that \( (A_0, \Delta_0) \) is an algebraic quantum group, because we do not know if the restriction \( \varphi_0 \) of \( \varphi \) to \( A_0 \) is non-zero. In any case, it will be left invariant: If \( a, b \in A_0 \), then \( \Delta(a)(b \otimes 1) \in \mathcal{M}_{\otimes \varphi} \), and

\[
(\iota \otimes \varphi_0)(\Delta_0(a)(b \otimes 1)) = (\iota \otimes \varphi)(\Delta(a)(b \otimes 1)) = \varphi(a)b = \varphi_0(a)b.
\]

**Assumption:** \( A_0 \subseteq A \) and \( \varphi|_{A_0} \neq 0 \).

The assumption is sufficient to conclude that \( (A_0, \Delta_0) \) is an algebraic quantum group, as we have shown. Remark that the second condition is automatically fulfilled if \( (A_0, \Delta_0) \) is a multiplier Hopf \( * \)-algebra (with the same \( * \)-involution as in \( A \)).

We now show that \( A_0 \) itself possesses an analytic structure, thus generalizing the results in [4].
Proof. Choose a fixed $\delta$ with respect to $A_0$. Then every $a$ in $A_0$ is a left and a right multiplier for $\delta$ such that $a\delta = a\delta_0$ and $\delta a = \delta_0 a$. Moreover, we have that $\delta^2 A_0 = A_0$ and $A_0 \delta^2 = A_0$.

**Proposition 2.8.** Theorem 2.10. 
\[
\phi(\delta^2 b)\delta^2 a = \sum \phi(\delta^2 p_i)q_i \in A_0.
\]

Every term in the right hand side lies in $\mathcal{M}_i$, so applying $\iota \otimes \phi$ to each side, we get
\[
\phi(\delta^2 b)\delta^2 a = \sum \phi(\delta^2 p_i)q_i \in A_0.
\]

Denote by $L$ the finite-dimensional vector space spanned by the $q_i$. Since $t \rightarrow \phi(\delta^2 b)$ is a continuous function and $\phi(b) = 1$, we can choose $t$ small such that $\phi(\delta^2 b) \neq 0$. For such $t$ we have $\delta^2 a \in L$. A similar argument as in Proposition 2.3 let’s us conclude that the linear span of the $\delta^2 a$, with $t \in \mathbb{R}$, is a finite-dimensional subspace of $A_0$. This easily implies that $a$ is a right multiplier of $\delta$ and $\delta a \in A_0$. So every element of $A_0$ lies in the domain of left multiplication with $\delta^2$, and $\delta^2 A_0 = A_0$. Since the space $B_0 = \{a \in A \mid a^* \in A_0\}$ also has the structure of a multiplier Hopf algebra, and $\phi|_{B_0} \neq 0$, we can conclude that $B_0$ consists of right multipliers for $\delta$. So $A_0$ consists of left multipliers for $\delta$, and $A_0 \delta^2 = A_0$.

Choose a fixed $a \in A_0$ with $\phi(a) \neq 0$. Let $b, c$ be elements in $A_0$. We know that (see [6])
\[
\phi((\iota \otimes \{\Lambda_\phi(b), \Lambda_\phi(c^*)\})\Delta(a)) = \phi(a)(\delta^{1/2}\Lambda_\phi(b), \delta^{1/2}\Lambda_\phi(c^*)) = \phi(a)(\Lambda_\phi(b), \Lambda_\phi(\delta c^*)) = \phi(a)\phi_0(c^*b).
\]

On the other hand,
\[
\phi((\iota \otimes \{\Lambda_\phi(b), \Lambda_\phi(c^*)\})\Delta(a)) = (\phi \otimes \phi)((1 \otimes c)\Delta(a)(1 \otimes b)) = (\phi_0 \otimes \phi_0)((1 \otimes c)\Delta_0(a)(1 \otimes b)) = \phi_0(a)\phi_0(c^*b).
\]

Since $\phi_0$ is faithful, $\delta_0 b = \delta b$ and $b\delta_0 = b\delta$ for all $b \in A_0$.

**Remark.** In general (i.e. when $A_0 \subset M(A)$), we do not have to expect nice behavior of $A_0$ with respect to $\delta$. Consider for example the trivial quantum group $C_1$ in $M(A)$.

As we have remarked, the invariance under the one-parameter groups of $A_0$ follows easily.

**Proposition 2.9.** $\tau_z(A_0) = \sigma_z(A_0) = R(A_0) = A_0$, for all $z \in \mathbb{C}$.

**Proof.** We have $\sigma_{2z}(a) = \delta^{-iz}(\sigma_z \sigma_z(a))\delta^iz$. But $\sigma_z \sigma_z$ and $\delta^{-iz} \cdot \delta^iz$ leave $A_0$ invariant. Hence $\sigma_z(A_0) \subseteq A_0$. Then also $\tau_z = (\tau_z \sigma_{-z}) \circ \sigma_z$ leaves $A_0$ invariant. Since $R = S \circ \tau_{i/2}$, we have that $R$ leaves $A_0$ invariant.

Gathering all we have proven so far, we obtain the following theorem:

**Theorem 2.10.** Let $(A, \Delta)$ be a reduced $C^*$-algebraic quantum group with left Haar weight $\phi$. Let $(A_0, \Delta_0)$ be a regular multiplier Hopf algebra in $A$, such that
\[
\Delta_0(a)(1 \otimes b) = \Delta(a)(1 \otimes b),
\]
\[
\Delta_0(a)(b \otimes 1) = \Delta(a)(b \otimes 1),
\]
The comultiplication in $A_0$ will consist of integrable elements for $\varphi$. If $\varphi|_{A_0} \neq 0$, then $(A_0, \Delta_0)$ will be an algebraic quantum group with left Haar functional $\varphi_0 = \varphi|_{A_0}$. Moreover, $A_0$ will consist of analytic elements for the modular automorphism group, the scaling group, the unitary antipode and left and right multiplication with the modular element of $(A, \Delta)$, and $A_0$ will be invariant under all these actions.

As a corollary, we have

**Corollary 2.11.** Let $(A, \Delta)$ be a reduced $C^*$-algebraic quantum group with a dense, properly imbedded regular multiplier Hopf $*$-algebra $(A_0, \Delta_0)$. Then $(A_0, \Delta_0)$ is a $*$-algebraic quantum group, with associated $C^*$-algebraic quantum group $(A, \Delta)$.

**Proof.** From the foregoing, we know that $(A_0, \Delta_0)$ is a $*$-algebraic quantum group with left Haar functional $\varphi_0 = \varphi|_{A_0}$. The only difficult step left to show, is that $A_0$ is actually a core for the GNS-map $\Lambda_\varphi$. The proof of this follows along the lines of Theorem 6.12. of [8].

Let $\Lambda_0$ be the closure of the restriction of $\Lambda_\varphi$ to $A_0$. Choose a bounded net $(e_j)$ in $A_0$ converging strictly to 1. We can replace $e_j$ by $\frac{1}{\sqrt{\pi}} \int \exp(-t^2) \sigma_t(e_j) dt$, since each will be an element of $A_0$ (because $\{\sigma_t(e_j) \mid t \in \mathbb{R}\}$ only spans a finite-dimensional space in $A_0$), and the net will still be bounded, converging strictly to 1. Moreover, now also $\sigma_{i/2}(e_j)$ will be a bounded net, converging strictly to 1.

Let $x$ be an element of $\mathcal{M}_\varphi \cap A$. Then $xe_j \to x$ in norm. Moreover, $\Lambda_\varphi(xe_j) = J \sigma_{i/2}(e_j)^* J \Lambda_\varphi(x)$. Because $\sigma_{i/2}(e_j)$ also converges $*$-strongly to 1, we have $\Lambda_\varphi(xe_j) \to \Lambda_\varphi(x)$.

The corollary follows, since the multiplicative unitary of $A$ and the multiplicative unitary of $A_0$ on $\mathcal{H}_{\varphi} \otimes \mathcal{H}_{\varphi} = \mathcal{H}_{\varphi_0} \otimes \mathcal{H}_{\varphi_0}$ coincide, and their first leg constitute respectively $A$ and the $C^*$-algebraic quantum group associated to $A_0$.

In our last proposition we will say something about the dual of $(A_0, \Delta_0)$ when $A_0 \subset A$ is a regular multiplier Hopf algebra with $\varphi|_{A_0} \neq 0$.

**Proposition 2.12.** Let $(A, \Delta)$ be the dual locally compact quantum group of $(A, \Delta)$, and let $(A_0, \Delta_0)$ be the dual algebraic quantum group$^2$ of $(A_0, \Delta_0)$. Then

$$j : \hat{A}_0 \to \hat{A} : \varphi_0(\cdot,a) \to (\varphi(\cdot,a) \otimes i)(W)$$

is an injective ($*$-)algebra homomorphism, such that

$$(j \otimes j)(\hat{\Delta}_0(\omega_1)(1 \otimes \omega_2)) = \hat{\Delta}(j(\omega_1))(1 \otimes j(\omega_2)),$$

$$(j \otimes j)(\hat{\Delta}_0(\omega_1)(\omega_2 \otimes 1)) = \hat{\Delta}(j(\omega_1))(j(\omega_2) \otimes 1),$$

for all $\omega_1, \omega_2 \in \hat{A}_0$.

**Proof.** Recall that $W$ denotes the multiplicative unitary of the left regular representation. Remark that the expression $$(\varphi(\cdot,a) \otimes i)(W)$$

$^2$The comultiplication in $\hat{A}_0$ is determined by $\hat{\Delta}_0(\omega)(x \otimes y) = \omega(y x)$. This is not the convention followed in [15].
makes sense, since \( \varphi(a) \) can also be written as a sum of elements of the form \( \varphi(b \cdot c) \), with \( b, c \) in \( A_0 \subset A_\varphi \cap A^* \varphi \). It is also easily seen that \( j \) is injective.

We first check that \( j \) preserves the \(*\)-operation, in case \( A_0 \) is a \(*\)-algebraic quantum group. For simplicity, set \( \hat{a} = j(\varphi_0(a)) \) for \( a \in A_0 \). Then \( a \in \mathcal{D}(\Lambda_\varphi) \) and \( \Lambda_\varphi(\hat{a}) = \Lambda_\varphi(a) \), where \( \hat{\varphi} \) denotes the left Haar weight of the dual locally compact quantum group \((\hat{A}, \hat{\Delta})\). We know that \( \Lambda_\varphi(a) \) lies in the domain of the operator \( P^{1/2} J \delta^{-1/2} J \), where \( P \) is the analytic generator for the unitary one-parameter group \( \Lambda_\varphi(x) \rightarrow \Lambda_\varphi(\tau_t(x)) \) and \( J \) is the modular conjugation for \( \varphi \). But \( P^{1/2} J \delta^{-1/2} J = \hat{\nabla}^{1/2} \), where \( \hat{\nabla} \) is the modular operator for \( \hat{\varphi} \). So \( \hat{a}^* \in \mathcal{D}(\Lambda_\varphi) \) and

\[
\Lambda_\varphi(\hat{a}^*) = j(\hat{\nabla}^{1/2} \Lambda_\varphi(\hat{a}))
= j P^{1/2} J \delta^{-1/2} J \Lambda_\varphi(a)
= j \Lambda_\varphi(\tau_{-i/2}(a) \delta^{-1/2})
= j \Lambda_\varphi((\tau_{-i/2}(a) \delta^{-1}) \delta^{1/2})
= j \Lambda_\varphi(\tau_{-i/2}(a) \delta^{-1})
= \Lambda_\varphi(S(a)^* \delta)
= \Lambda_\varphi(S_0(a)^* \delta),
\]

where we freely used the formulas in [7]. Since \( \varphi_0(S_0(a)^* \delta) \) equals \( \varphi_0(a)^* \), we arrive at \( j(\varphi_0(a)^*) = j(\varphi_0(a))^* \).

Now we show that \( j \) is an algebra morphism. Choose \( a, b \in A_0 \). Choose \( p_i, q_i \in A_0 \) such that \( a \otimes b = \sum \Delta_0(p_i)(q_i \otimes 1) \). Then \( \varphi_0(a \cdot \varphi_0(b) = \sum \varphi_0(q_i) \varphi_0(p_i) = \sum \varphi(q_i) \varphi_0(p_i) \). It is enough then to show that \( \sum \varphi(q_i) \varphi(p_i) \) equals \( \varphi(a \cdot \varphi(b) \) in \( \mathcal{L}(A) \). But evaluating this last functional in \( x \), we get \( (\varphi \otimes \varphi)(\Delta(x)(a \otimes b)) \), which equals \( \sum (\varphi \otimes \varphi)(\Delta(x)(q_i \otimes 1)) = \sum \varphi(q_i) \varphi(x p_i) \), so indeed both functionals are equal.

Similarly, \( j \) respects the comultiplication. Namely:

\[
(\Lambda_\varphi \otimes \Lambda_\varphi)(\hat{\Delta}(\hat{a} \otimes 1)) = \Sigma W \Sigma(\Lambda_\varphi(\hat{a}) \otimes \Lambda_\varphi(\hat{b}))
= \Sigma W \Sigma(\Lambda_\varphi(a) \otimes \Lambda_\varphi(b))
= \Sigma(\Lambda_\varphi \otimes \Lambda_\varphi)((S_0^{-1} \otimes \epsilon)(\Delta_0(b))(a \otimes 1)),
\]

with \( \Sigma \) denoting the flip. A simple computation then shows that this expression is the image under \( (\Lambda_\varphi \otimes \Lambda_\varphi) \) of \( (j \otimes j)(\Delta_0(\varphi_0(\cdot a))(\varphi_0(\cdot b) \otimes 1)) \). Thus

\[
\hat{\Delta}(\hat{a})(\hat{b} \otimes 1) = (j \otimes j)(\hat{\Delta}_0(\varphi_0(\cdot a))(\varphi_0(\cdot b) \otimes 1)).
\]

This implies the second equation of the proposition. The first one follows similarly.

\[\square\]

Remark. The previous proposition says, that the dual \( \hat{A}_0 \) will be properly imbedded in \( \hat{A} \) if \( A_0 \) is properly imbedded in \( (A, \Delta) \). This implies that, under the given conditions, also the dual \( \hat{A}_0 \) of \( A_0 \) will have an analytic structure.

### 3 Structure of \(*\)-algebraic quantum groups

We apply the techniques of the above section to obtain some interesting structural properties of \(*\)-algebraic quantum groups. While many of the results follow easily from the previous section, we have decided to give new proofs, using only algebraic machinery. As such, we can give a purely algebraic proof of the existence of a \textit{positive} right invariant functional on a \(*\)-algebraic quantum group.
We fix a*-algebraic quantum group \((A, \Delta)\) with antipode \(S\), positive left Haar functional \(\varphi\), modular automorphism \(\sigma\) and modular element \(\delta\). As a right Haar functional (not assumed to be positive) we take \(\psi = \varphi \circ S\), with modular automorphism \(\sigma^*\). We adapt the proof of Lemma 2.1 to show that \(A\) is spanned by eigenvectors for \(\kappa = \sigma^{-1}S^2\). We need a lemma.

**Lemma 3.1.** If \(b\) is a non-zero element in \(A_0\) and \(n\) is an even integer, then \(b^*(\sigma^n S^{2n})(b) \neq 0\).

*Proof.* Suppose that \(b \in A_0\) and \(n \in 2\mathbb{Z}\) are such that
\[
b^*(\sigma^n S^{2n}(b)) = 0.
\]
Then
\[
b^* \delta^n(\sigma^n S^{2n}(b)) = 0.
\]
Applying \(\sigma^{-n/2} \circ S^{-n}\), we get
\[
(\sigma^{n/2} S^n(b))^* \delta^n(\sigma^{n/2} S^n)(b) = 0.
\]
Since \(\delta\) is self-adjoint, applying \(\varphi\) to the previous equation and using positivity and faithfulness of \(\varphi\), we obtain
\[
\delta^{n/2}(\sigma^{n/2} S^n)(b) = 0,
\]
hence \(b = 0\).

\(\square\)

**Remark.** To be complete, we give an algebraic argument demonstrating the self-adjointness of \(\delta\). Choose \(a\) and \(b\) in \(A_0\), then \(\varphi(a^*a) b^* \delta b = (\varphi \otimes \iota)((1 \otimes b^*) \Delta(a^*a)(1 \otimes b)) = \varphi(a^*a) b^* \delta^* b\). Applying \(\varphi\) and using polarization, we see that \(\varphi(c^* \delta b) = \varphi(c^* \delta^* b)\) for all \(b, c \in A_0\). Finally, using the modular automorphism and the fact that \(A_0^2 = A_0\), we get \(\varphi(\delta b) = \varphi(\delta^* b)\) for all \(b \in A_0\). This implies \(\delta b = \delta^* b\) and \(\delta b = b\delta^*\) for all \(b \in A_0\). So \(\delta = \delta^*\).

**Lemma 3.2.** If \(a \in A\), then the linear span of the \(\kappa^n(a)\), with \(n \in \mathbb{Z}\), is finite-dimensional.

*Proof.* We can follow the proof as in lemma 2.1:

Let \(b\) be a fixed element of \(A\). Choose a non-zero \(a \in A\), and write
\[
a \otimes b = \sum_{i=1}^{n} \Delta(p_i)(1 \otimes q_i),
\]
with \(p_i, q_i \in A\). Then
\[
\kappa^n(a) \otimes \rho^{-n}(b) = \sum \Delta(p_i)(1 \otimes \rho^{-n}(q_i)), \quad \text{for all} \ n \in \mathbb{Z},
\]
where \(\rho = \sigma^* S^2\). Multiply this equation to the left with \(1 \otimes b^*\) to get
\[
\kappa^n(a) \otimes b^* \rho^{-n}(b) = \sum ((1 \otimes b^*) \Delta(p_i))(1 \otimes \rho^{-n}(q_i)).
\]
Choose \(a_{ij}, b_{ij} \in A\) such that
\[
(1 \otimes b^*) \Delta(p_i) = \sum_{j=1}^{m_i} a_{ij} \otimes b_{ij},
\]
and let \(L\) be the finite-dimensional space spanned by the \(a_{ij}\). We see that \(\kappa^n(a) \otimes b^* \rho^{-n}(b) \in L \otimes A\), for every \(n \in \mathbb{Z}\). Using the previous lemma, we can conclude that \(\kappa^{2n}(a) \in L\) for all \(n \in 2\mathbb{Z}\). But this easily implies that the linear span \(K\) of all \(\kappa^n(a)\), with \(n \in \mathbb{Z}\), is a finite-dimensional, \(\kappa\)-invariant linear subspace of \(A\).

\(\square\)
Denote by \((\hat{A}, \hat{\Delta})\) the dual \(*\)-algebraic quantum group of \((A, \Delta)\). We can regard \(\hat{A}\) and \(M(\hat{A})\) as functionals on \(A\). We know from [4] that \(\hat{\delta} = \varepsilon \circ \kappa\) (this is also not so difficult to prove algebraically). Then
\[
\langle \omega \hat{\delta}, x \rangle = \langle \omega \otimes (\varepsilon \kappa), \Delta(x) \rangle = \langle \omega, \kappa(x) \rangle,
\]
for each \(\omega \in \hat{A}\) and \(x \in A\). If \(\omega\) is of the form \(\varphi(a)\), this means \(\varphi(a) \hat{\delta}\) is a scalar multiple of \(\varphi(\kappa^{-1}(a))\). This implies that for \(\omega\) fixed, the linear span of the \(\omega \hat{\delta}^n\) is finite-dimensional. The same is of course true for left multiplication with \(\hat{\delta}\).

By duality, we conclude that for each \(a\) in \(A\), the linear span of the \(\delta^n a\) is a finite-dimensional space \(K\). (We could also prove this along the lines of Proposition 2.8.) Since \(\delta\) is a self-adjoint operator on \(K\), with Hilbert space structure induced by \(\varphi\), we can diagonalize \(\delta\). Hence we arrive at

**Proposition 3.3.** Let \((A, \Delta)\) be a \(*\)-algebraic quantum group. Then \(A\) is spanned by elements which are eigenvectors for left multiplication by \(\delta\).

We can use this to settle an open question (cf. [8]):

**Theorem 3.4.** Let \((A, \Delta)\) be a \(*\)-algebraic quantum group. Then the scaling constant \(\mu\) equals 1.

**Proof.** Choose a non-zero element \(b \in A\) with \(\delta b = \lambda b\), for some \(\lambda \in \mathbb{R}\). Then \(\varphi(bb^* \delta) = \lambda \varphi(bb^*)\). But the left hand side equals \(\mu \varphi(bb^*) = \mu \lambda \varphi(bb^*)\). Since \(\varphi(bb^*) \neq 0\), we arrive at \(\mu = 1\).

Proposition 3.3 can be strengthened:

**Theorem 3.5.** Let \((A, \Delta)\) be a \(*\)-algebraic quantum group. Then \(A\) is spanned by elements which are simultaneously eigenvectors for \(S^2\), \(\sigma\) and \(\sigma'\), and left and right multiplication by \(\delta\). Moreover, the eigenvalues of these actions are all positive.

**Proof.** We know that \(A\) is spanned by eigenvectors for left multiplication with \(\delta\), and the same is easily seen to be true for \(\kappa\) and \(\rho = \sigma' S^2\). But all these actions commute. Hence we can find a basis of \(A\) consisting of simultaneous eigenvectors. Since \(\sigma, \sigma'\) and \(S^2\) can be written as compositions of the maps \(\kappa, \rho\) and left and right multiplication with \(\delta\), the first part of the theorem is proven.

We show that left multiplication with \(\delta\) has positive eigenvalues. This is easily done. Fix \(a \in A_0\). If \(\lambda\) is an eigenvalue, choose an eigenvector \(b\). Consider \(x = \Delta(a)(1 \otimes b)\). Then 
\[
(\varphi \otimes \varphi)(x^* x) \text{ will be a positive number. But this is equal to } \varphi(a^* a) \varphi(b^* \delta b) = \lambda \varphi(a^* a) \varphi(b^* b).
\]
Hence \(\lambda\) must be positive. As before, duality implies that \(\kappa\) and \(\rho\) have positive eigenvalues, hence the same is true of \(\sigma, \sigma'\) and \(S^2\).

This theorem explains why there exists an analytic structure on a \(*\)-algebraic quantum group \((A, \Delta)\): the actions are all diagonal with positive entries! Hence \(\sigma_\tau, \sigma'_\tau, \tau_{\varepsilon}\) and multiplication with \(\delta^iz\) are all well-defined on \(A\).

We can also see that \(\psi = \varphi \circ S\) is already a positive right invariant functional, since \(\psi(a^* a) = \varphi(a^* a) = \varphi((a \delta^{1/2})^* a \delta^{1/2}) = 0\). Here we use that \(\sigma(\delta^{1/2}) = \delta^{1/2}\), which is easily proven using an eigenvector argument.

Finally remark that the extension of \(\varphi\) to \(M\), with \(M\) the von Neumann algebraic quantum group associated with \(A\), is an almost periodic weight, since the modular operator \(\nabla\) implementing \(\sigma\) on \(\mathcal{H}_\varphi\) is diagonizable.
4 Special cases

Compact and discrete quantum groups

Let \((A, \Delta)\) be a discrete locally compact quantum group. Then \(A\) is the \(C^*\)-algebraic direct sum of matrix algebras \(M_{n_\alpha}(\mathbb{C})\). The algebraic direct sum \(\mathcal{A} = \oplus_{\alpha} M_{n_\alpha}(\mathbb{C})\) has the structure of a multiplier Hopf \(*\)-algebra. So it is easy to see that \(\delta\), being a positive element in \(\prod M_{n_\alpha}\), is diagonalizable with respect to \(\mathcal{A}\). Then the same will be true for \(S^2\), the square of the antipode, since in a discrete quantum group we have \(S^2(a) = \delta^{-1/2}a\delta^{1/2}\). Lastly, \(\sigma\) is diagonalizable since \(\sigma = S^2\) in a discrete quantum group.

Suppose now that \((A_0, \Delta_0)\) is a \(*\)-algebraic quantum group, properly imbedded in \((A, \Delta)\). Suppose \(a\) is a non-zero element in \(A_0\) such that \(a \notin \mathcal{A}\). We know that \(A_0\) has local units, so there exists \(e \in A_0\) with \(ae = a\). Then \(e^*a^*ae = a^*a\), and this implies that infinitely many components of \(e^*e\) have norm greater than \(1\). But this is impossible, since \(e^*e \in A\). So \(A_0 \subset \mathcal{A}\).

The same argument implies that \(A_0\) is again a \(*\)-algebraic quantum group of discrete type, since \(A_0\) itself will be an algebraic direct sum of matrix algebras. In particular, \(A_0\) has a co-integral \(h_0\), which will be a grouplike projection in \(\mathcal{A}\). \((A, \Delta)\) is a discrete quantum group properly imbedded in \(A\). Hence \(A_0 \subset \mathcal{A}\) and \((A_0, \Delta_0)\) is a compact \(*\)-algebraic quantum group. The dual \(p\) of the co-integral \(h_0\) of \(\widehat{A_0}\) in \(\mathcal{A}\) will be a grouplike projection in \(\mathcal{A}\). It will be a unit for \(A_0\).

Locally compact groups

Suppose \(G\) is a locally compact group. Let \((A_0, \Delta_0)\) be a regular multiplier Hopf \(*\)-algebra imbedded in \((L^\infty(G), \Delta)\), where \(\Delta\) is the usual comultiplication determined by \(\Delta(f)(g, h) = f(gh)\). Then \(A_0 \subset M(C_0(G)) = C_0(G)\), so \(A_0\) consists of bounded continuous functions on \(G\). Let \(\tilde{A}_0\) be the normclosure of \(A_0\) in \(L^\infty(G)\). Then \(\Delta\) restricts to a \(*\)-algebra morphism \(\tilde{A}_0 \to M(\tilde{A}_0 \otimes \tilde{A}_0)\). Since \(\tilde{A}_0\) is abelian, this induces a locally compact semigroup structure on the spectrum \(\tilde{X}\) of \(\tilde{A}_0\). Since \(S_0\) extends to an \(S\) on \(\tilde{A}_0\), the semigroup will be a locally compact group. But this means that \(X\) has a Haar measure. So \((A_0, \Delta_0)\) is properly imbedded in the \(C^*\)-algebraic quantum group \((C_0(X), \Delta)\), hence is itself a \(*\)-algebraic quantum group.

Remark however that the Haar functional on \(A_0\) can be different from integration with respect to the Haar measure on \(G\). Consider for example the linear span \(A_0\) of the functions \(f_x : t \to e^{itx}\) in \(L^\infty(\mathbb{R})\), with \(x \in \mathbb{R}\). Then \(A_0\) is a compact Hopf \(*\)-algebra, but none of its non-zero elements are integrable with respect to the Lebesgue measure. It is easy to see that the Haar functional \(\varphi_0\) on \(A_0\) is given by \(\varphi_0(f_x) = \delta_{x,0}\), and that the space \(X\) equals \(\mathbb{R}\) with the discrete topology.

The dual case is not so clear: suppose \(G\) is a locally compact group, and \((A_0, \Delta_0)\) is imbedded in \((L^\tau(G), \Delta)\), where \(\Delta\) is determined by \(\Delta(u_g) = u_g \otimes u_g\) on the generators of \(L^\tau(G)\). Will the \(C^*\)-algebraic closure \(\tilde{A}_0\) with the restriction of \(\Delta\) be of the form \((C^*(X), \Delta)\) for some locally compact group \(X\)? This will of course be true if \(A_0\) is properly imbedded in \(C^*_x(G)\), since then we can apply the theory of the second section to conclude that \(\tilde{A}_0\) is a cocommutative \(C^*\)-algebraic quantum group with normdense \(*\)-algebraic quantum group.
hence of the form \((C^*_r(X), \Delta)\).

Let us now look at the results of the third section in the commutative case. Let \(G\) be a locally compact group with a compact open subgroup \(H\). Consider the regular functions on \(H\) - i.e. the functions generated by the matrix-coefficients of finite-dimensional irreducible representations of \(H\). We can see them as functions on \(G\). The linear span of left translates of these functions by elements of \(G\) is denoted by \(P_0(G)\). In [9], it is shown that \(P_0(G)\) forms a dense multiplier Hopf*-algebra inside \((C_0(G), \Delta)\), with the usual comultiplication, and that every commutative *-algebraic quantum group is of this form. In this setting, the only non-trivial object is the modular function \(\delta\). According to our results, it should be diagonalizable. This is easily seen to be true. For example, the characteristic function of \(H\) will be an eigenvector for left multiplication. Indeed: the Haar measure on \(H\) is the restriction of the Haar measure on \(G\). Hence \(\delta|_H\) is the modular function of \(H\). Since \(H\) is compact, \(\delta|_H = 1\). So every regular function on \(H\) is invariant for left multiplication. Then the translates by some element \(g\) of such functions will be eigenvectors with eigenvalue \(\delta(g)\), and the linear span of all such translates equals \(P_0(G)\).

**The case of the quantum groups \(U_q(su(2))\) and \(SU_q(2)\)**

Finally, we consider a particular, non-trivial example of a multiplier Hopf *-algebra \((A_0, \Delta_0)\), imbedded in the multiplier algebra of a discrete *-algebraic quantum group \((\mathcal{A}, \Delta)\). This is not a situation we have discussed, since this multiplier algebra contains unbounded operators (when acting on the Hilbert space closure of \(\mathcal{A}\) by left multiplication). We will see which of our results are still true in this case.

So as \((A_0, \Delta_0)\), we take the quantum enveloping Lie algebra \(U_q(su(2))\), with \(q\) nonzero in \([-1, 1]\). As a *-algebra, it is generated by two elements, \(E\) and \(K\), with \(K\) invertible and self-adjoint, obeying the following commutation relations:

\[
\begin{align*}
EK &= q^{-1}KE \\
[E, E^*] &= \frac{1}{q-q^{-1}}(K^2 - K^{-2}).
\end{align*}
\]

The comultiplication on the generators is given by

\[
\begin{align*}
\Delta_0(K) &= K \otimes K \\
\Delta_0(E) &= E \otimes K + K^{-1} \otimes E.
\end{align*}
\]

To see that this comultiplication is well-defined, it is enough to check that it respects the commutation relations, but this is easily done. The antipode is determined by

\[
\begin{align*}
S_0(K) &= K^{-1} \\
S_0(E) &= -qE \\
S_0(E^*) &= -q^{-1}E^*.
\end{align*}
\]

As our *-algebraic quantum group \((\mathcal{A}, \Delta)\), we take the *-algebraic quantum group \(\widehat{\mathcal{B}}\), where \(\mathcal{B}\) is the compact *-algebraic quantum group associated with \(SU_q(2)\), Woronowicz’ twisted \(SU(2)\)-group. As a *-algebra, \(\mathcal{B}\) is generated by two elements \(a\) and \(b\), such that

\[
\begin{align*}
ab &= qba \\
ab^* &= q^*b^*a \\
[b, b^*] &= 0 \\
a^*a &= 1 - q^{-2}b^*b \\
aa^* &= 1 - b^*b.
\end{align*}
\]

The co-multiplication is given by:

\[
\begin{align*}
\Delta(a) &= a \otimes a - q^{-1}b \otimes b^* \\
\Delta(b) &= a \otimes b + b \otimes a^*.
\end{align*}
\]

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For convenience, we state that the antipode $S$ is given by

\[
\begin{aligned}
S(a) &= a^* \\
S(a^*) &= a \\
S(b) &= -q^{-1}b \\
S(b^*) &= -qb^*.
\end{aligned}
\]

We will not need the concrete description of the left invariant functional, but we need to know the modular group, which we now denote by $\rho$. To be complete, we also provide the scaling group, which we will denote by $\theta$:

\[
\begin{aligned}
\rho_z(a) &= q^{-2iz}a \\
\rho_z(b) &= b \\
\theta_z(a) &= a \\
\theta_z(b) &= q^{-2iz}b.
\end{aligned}
\]

The modular element will of course be trivial, since the quantum group is compact. The easiest way to see that $A_0$ can be imbedded in $M(\hat{\mathcal{B}})$, is by creating a pairing\(^3\) between $\mathcal{B}$ and $A_0$. For, since $\mathcal{B}$ is compact, it is known that $M(\hat{\mathcal{B}})$ can be identified with the vector space of all linear functionals on $\mathcal{B}$. The fact that there is a pairing, implies that the inclusion of $A_0$ in $M(\hat{\mathcal{B}})$ will be a morphism. The concrete pairing is as follows:

\[
\begin{aligned}
\langle K, a \rangle &= q^{-1/2} \\
\langle K, a^* \rangle &= q^{1/2} \\
\langle K, b \rangle &= 0 \\
\langle K, b^* \rangle &= 0 \\
\langle E, a \rangle &= 0 \\
\langle E, a^* \rangle &= 0 \\
\langle E, b \rangle &= 0 \\
\langle E, b^* \rangle &= -q.
\end{aligned}
\]

Since on the dual of a compact algebraic quantum group the modular group $\sigma$ and the scaling group $\tau$ coincide, we find the following behavior of $A_0$:

\[
\begin{aligned}
\sigma_z(K) &= \tau_z(K) = K \\
\sigma_z(E) &= \tau_z(E) = q^{2iz}E.
\end{aligned}
\]

But although there is general invariance under the scaling (and thus the modular) group, we no longer have that $A_0$ is invariant under left multiplication by $\delta^z$, with $\delta$ the modular element of $\hat{\mathcal{B}}$. For this would imply that actually $\delta^z \in A_0$, since $A_0$ is a Hopf algebra. Remark that this $\delta^z$ is easily computable, for it is given as a functional by $\varepsilon \circ \rho_{iz}$, with $\varepsilon$ the co-unit of $\mathcal{B}$. We find that applying $\delta$ is the same as pairing with $K^{-1} = (K^*K)^{-2}$, so uniqueness gives us that $\delta^it = K^{-4it}$. It is clear that this is no element in $A_0$. Remark also, that right or left multiplication with $\delta$ is no longer diagonal. This is easy to see, using that $A_0$ has $\{K^lE^mF^n | l \in \mathbb{Z}, m, n \in \mathbb{N}\}$ as a basis. In fact, since $\text{span}\{K^lE^n\}$ has infinite dimension for any $X \in A_0$, we get that $A_0 \cap \hat{\mathcal{B}} = \{0\}$.

We note that in this example we are in a special situation: $(SU_q(2), \hat{\Delta})$ is the $C^*$-algebraic quantum group generated by $K$, $K^{-1}$ and $E$, in the sense of Woronowicz. Moreover, the multiplier Hopf $^*$-subalgebra is linked by a pairing to a $^*$-algebraic quantum group. This could explain why we still have invariance under $\tau_\sigma$ and $\sigma_\tau$. For example, the same type of behavior occurs with the quantum $ax + b$-group. Remark that in these cases, the corresponding Hopf $^*$-algebra can be viewed as the infinitesimal version of the quantum group. We do not know if it is a general fact that the one-parameter groups descend to the Hopf $^*$-algebra associated with the quantum group, if such an object is present. In any case, the connection between a locally compact quantum group and a Hopf $^*$-algebra representing the quantum group at an infinitesimal level, is at present not well understood in a general framework.

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\(^3\)In this part we follow the algebraic convention $\langle \Delta(b), x \otimes y \rangle = \langle b, xy \rangle$. 

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