On all possible static spherically symmetric EYM solitons and black holes \(^\ast\)^\(^\dagger\)^\(^\ddagger\)

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Abstract

We prove local existence and uniqueness of static spherically symmetric solutions of the Einstein-Yang-Mills equations for any action of the rotation group (or SU(2)) by automorphisms of a principal bundle over space-time whose structure group is a compact semisimple Lie group \(G\). These actions are characterized by a vector in the Cartan subalgebra of \(g\) and are called regular if the vector lies in the interior of a Weyl chamber. In the irregular cases (the majority for larger gauge groups) the boundary value problem that results for possible asymptotically flat soliton or black hole solutions is more complicated than in the previously discussed regular cases. In particular, there is no longer a gauge choice possible in general so that the Yang-Mills potential can be given by just real-valued functions. We prove the local existence of regular solutions near the singularities of the system at the center, the black hole horizon, and at infinity, establish the parameters that characterize these local solutions, and discuss the set of possible actions and the numerical methods necessary to search for global solutions. That some special global solutions exist is easily derived from the fact that su(2) is a subalgebra of any compact semisimple Lie algebra. But the set of less trivial global solutions remains to be explored.

1 Introduction

The classical interaction between gravitational and Yang-Mills fields is described by a complicated highly nonlinear field equations which, even when reduced to a system of ordinary differential equations in the static spherically symmetric case, leads to many interesting mathematical and physical problems. Physically these solutions have shown that equilibrium configurations of black holes can be much more complicated than had previously been thought since mass, charge and angular momentum are clearly not enough to characterize them. There is even numerical evidence for the existence of non spherical axisymmetric static black holes \(^1\). Mathematically, an analysis of the solution space of the static spherically symmetric equations requires an interesting combination of geometrical, algebraic, analytic and numerical techniques. The global solutions of SU(2)-EYM equations have been extensively studied analytically \(^{10, 11, 21}\). For a fairly comprehensive summary of the substantial literature on the subject we refer to the review article \(^{22}\). Almost all these investigations, however, have only studied the gauge groups SU(2) and occasionally SU(\(n\)) for \(n > 2\), and only for the most obvious ansatz for a spherically symmetric gauge field.

But spherical symmetry for Yang-Mills fields is more complicated to define than for tensor fields on a manifold

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because there is no unique way to lift an isometry on space-time to the bundle space. The only natural way to define spherical symmetry of a Yang-Mills field is to require that it be invariant under an action of the rotation group by automorphisms on the principal bundle whose structure group is the gauge group $G$. A conjugacy class of such automorphisms is characterized by a generator $\Lambda_0$ which is an element of a Cartan subalgebra $\mathfrak{h}$ of the complexified Lie algebra $\mathfrak{g}$ of $G$. If one restricts consideration to fields which are bounded at the center or, in the presence of black hole fields, to those for which the Yang-Mills-curvature falls off sufficiently fast at infinity then $\Lambda_0$ must be a defining vector of an $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{g}$.

One of these classes of actions of the symmetry group is somewhat distinguished. It corresponds to a principal defining vector in Dynkin’s terminology and we will call it a principal action. Almost all work for larger gauge groups has been done for this case [12, 13, 15, 17]. A slightly bigger class of actions, which we call regular, consists of those for which the defining vector lies in the interior of a Weyl chamber. For those, for example, Brodbeck and Straumann [6,7] proved that all bounded static asymptotically flat solutions are unstable against time dependent perturbations.

Of course, one is interested in global solutions of the boundary value problem that results from demanding boundedness at the singularities of the differential equations at the center or the horizon and at infinity. This global existence has long been established for $G = SU(2)$ by Smoller et al. [20, 21] (see also [3]). It is easy to “imbed” these solutions into the set of solutions of the theory with arbitrary compact semi-simple $G$ since the latter always has $SU(2)$ subgroups. The problem is thus not to prove that global solutions exist but to explore the global solution space and, hopefully, characterize different types of solutions, for example, by their behavior near the center or near infinity.

In [18] we have therefore considered and solved the local existence problem for bounded solutions near the singularities for the regular symmetry group actions. We have identified the set of “initial conditions” that must be given at $r = 0$ or $r = \infty$ to guarantee the local existence and uniqueness of a bounded solution. For arbitrary gauge groups this required an fairly intricate application of the $\mathfrak{sl}(2, \mathbb{C})$ representation theory on the (complexified) Lie algebra $\mathfrak{g}$ of $G$.

The purpose of the present paper is simply to extend these results to the case where the symmetry group action need not be regular. It turns out that the situation is qualitatively not very different. There are still similar algebraic equations that restrict the possible initial data. But the number of functions that will characterize the gauge potential is no longer just the rank of the Lie algebra $\mathfrak{g}$ or of one of its subalgebras. Even the set of reduced field equations can no longer be determined simply from the structure of the Cartan subalgebra (one needs access to all the Lie brackets of $\mathfrak{g}$). Moreover, a simple gauge choice that allows one to describe the potential in terms of rank($\mathfrak{g}$) real functions is no longer available. Complex functions must be allowed which increases substantially the number of parameters to be determined in a numerical solution of the boundary value problem.

In section 2 we review definitions and previous results mostly from [18] and then describe in section 3 how the field equations can be handled computationally. In section 4 we give some examples of the possible symmetry group actions for low dimensional gauge groups. The methods of section 3 do not lend themselves easily to a general existence proof. This needs to be done differently. In section 5 we state and prove some algebraic lemmas that needed in section 3 for the local existence and uniqueness proofs. We conclude by giving a preliminary example of a numerical irregular solution showing that imaginary parts of the functions $w_\alpha(r)$ develop even if the values $w_\alpha(0)$ or $w_\alpha(\infty)$ are all real. We have not yet found a (nontrivial) global asymptotically flat numerical solution.

2 Yang-Mills potentials and field equations

Let $P$ be a principal bundle with a compact semi-simple structure group $G$ over a static spherically symmetric space-time manifold. For simplicity we consider only actions of the group $SU(2)$ by principal bundle automorphisms on $P$ that project onto the action of $SO(3)$ on space-time which defines the spherical sym-
Equivalence classes of these spherically symmetric $G$-bundles are in one-to-one correspondence to conjugacy classes of homomorphisms of the isotropy subgroup, $U(1)$ in this case, into $G$. The latter, in turn, are given by their generator $\Lambda_3$, the image of the basis vector $\tau_3$ of $\mathfrak{su}(2)$ (where $\{\tau_i\}$ is a standard basis with $[\tau_i, \tau_j] = \epsilon_{ijk} \tau_k$), lying in an integral lattice $I$ of a Cartan subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}_0$ of $G$. This vector $\Lambda_3$, when nontrivial, then characterizes up to conjugacy an $\mathfrak{su}(2)$ subalgebra. (See, for example, [4]; we follow the notation of this text and also of [10].)

It is convenient to pass to the complexified Lie algebra $\mathfrak{g}$ of $\mathfrak{g}_0$ and define

\[ \Lambda_0 := 2i\Lambda_3 . \]  

We now regard $\mathfrak{g}_0$ as a compact real form of $\mathfrak{g}$ which defines the conjugation $c$ on $\mathfrak{g}$ (a Lie algebra automorphism satisfying $c \circ c = \mathbb{I}$). Then we can write $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$, $c(X+iY) = X - iY$ if $X,Y \in \mathfrak{g}_0$, and also $\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0$ where $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. Moreover,

\[ \Lambda_0 \in i\mathfrak{h}_0, \quad c(\Lambda_0) = -\Lambda_0 . \]

We will use the following notation from Lie algebra theory (following [10], [8], [4]): $\text{Ad}_{\mathfrak{g}} : \mathfrak{g}_0 \to \mathfrak{g}_0 \forall g \in G$ is the adjoint action of the Lie group $G$ on its (real) Lie algebra $\mathfrak{g}_0$ while $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is defined by $\text{ad}(X)(Y) := [X,Y]$. Define also the centralizer of $X$ in $\mathfrak{g}$ by

\[ \mathfrak{g}^X := \{ Y \in \mathfrak{g} \mid [X,Y] = 0 \} \]

and write $\mathfrak{g}_0^X$ for the corresponding centralizer of the real Lie algebra.

Wang’s theorem [14, 23] on connections that are invariant under actions transitive on the base manifold has been adapted to spherically symmetric space-time manifolds by Brodbeck and Straumann [1]. They show that in a Schwarzschild type coordinate system $(t, r, \theta, \phi)$ and the metric

\[ g = -NS^2dt^2 + N^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \]  

a gauge can always be chosen such that the $\mathfrak{g}_0$-valued Yang-Mills-connection form is locally given by $A = \tilde{A} + \hat{A}$ where

\[ \tilde{A} = N(t,r)S(t,r)A(t,r)dt + B(t,r)dr \]

is an $\text{Ad}(\Lambda_3)$-invariant 1-form (i.e. with values in $\mathfrak{g}_0^{\Lambda_3}$) on the quotient space parametrized by the $r$ and $t$ coordinates and

\[ \hat{A} = \Lambda_1 d\theta + (\Lambda_2 \sin \theta + \Lambda_3 \cos \theta)d\phi . \]

Here $\Lambda_3$ is the constant isotropy generator as above and $\Lambda_1$ and $\Lambda_2$ are functions of $r$ and $t$ that satisfy

\[ [\Lambda_2, \Lambda_3] = \Lambda_1 \quad \text{and} \quad [\Lambda_3, \Lambda_1] = \Lambda_2 . \]

In this paper we will only consider the static magnetic case for which $\Lambda_1$ and $\Lambda_2$ as well as $N$ and $S$ are functions of $r$ only and the ‘electric’ part $\hat{A}$ of the potential vanishes.

With

\[ A_{\pm} := \mp \Lambda_1 - i\Lambda_2 \]  

\[ \text{This ignores some interesting effects due to the fact that } SO(3) \text{ is not simply connected. For an analysis of } SO(3) \text{ actions on } SU(2) \text{ bundles see [2].} \]
equations (2.7) become

\[ [\Lambda_0, \Lambda_\pm] = \pm 2\Lambda_\pm \]  \hspace{1cm} (2.9)

and the \( \Lambda_\pm(\tau) \) satisfy the reality condition

\[ \Lambda_\mp = -c(\Lambda_\pm) \]  \hspace{1cm} (2.10)

With this choice for the gauge potential, the Einstein-Yang-Mills (EYM) equations can be written as

\[ m' = NG + r^{-2}P, \]  \hspace{1cm} (2.11)

\[ S^{-1}S' = 2r^{-1}G, \]  \hspace{1cm} (2.12)

\[ r^2N \Lambda''_+ + 2(m - r^{-1}P)\Lambda'_+ + F = 0, \]  \hspace{1cm} (2.13)

\[ [\Lambda'_+ , \Lambda_-] + [\Lambda'_- , \Lambda_+] = 0 \]  \hspace{1cm} (2.14)

where \( ', \) \( := d/dr \) and

\[ N := 1 - \frac{2m}{r}, \quad G := \frac{1}{2}(\Lambda'_+ , \Lambda'_-), \quad P := -\frac{i}{2} (\hat{F}, \hat{F}), \]  \hspace{1cm} (2.15)

\[ \hat{F} := \frac{i}{2}(\Lambda_0 - [\Lambda_+, \Lambda_-]), \]  \hspace{1cm} (2.16)

\[ F := -i[\hat{F}, \Lambda_+]. \]  \hspace{1cm} (2.17)

Here \( , \) is an invariant inner product on \( g \). It is determined up to a factor on each simple component of a semi-simple \( g \) and induces a norm \( | | \) on (the Euclidean) \( h \) and therefore its dual. We choose these factors so that \( , \) restricts to a negative definite inner product on \( g_0 \).

For several purposes, in particular for numerical solutions, equations (2.11)-(2.13) are best replaced by an equivalent system that regularizes the almost singularity when \( N \) is close to zero. These equations, introduced by Breitenlohner, Forgács and Maison [3] in the \( SU(2) \) case, take the form

\[ \dot{\tau} = rN, \quad \dot{N} = N(K - N) - 2G, \quad \dot{K} = 1 - K^2 + 2G, \quad \dot{S} = (K - N)S \]  \hspace{1cm} (2.18)

\[ \Lambda'_+ = rU_+ \quad \text{and} \quad \dot{U}_+ = -(K - N)U_+ - F/r \]  \hspace{1cm} (2.19)

where

\[ N := \sqrt{N}, \quad S := NS, \quad K := \frac{1}{2N} (1 + N + 2G - 2P/r^2), \quad U_+ := N\Lambda'_+, \quad G := \frac{1}{2} \|U_+\|^2 \]  \hspace{1cm} (2.20)

and the dot denotes a \( \tau \)-derivative. Note that the \( \tau \) variable behaves somewhat like the logarithm of \( r \) near 0 and for \( r \to \infty \) (where \( N \to 1 \)).

For later use, we introduce a non-degenerate Hermitian inner product \( g \) by \( \langle X|Y \rangle := -(c(X), Y) \) for all \( X, Y \) in \( g \). Then \( \langle | \rangle \) restricts to a real positive definite inner product on \( g_0 \). From the invariance properties of \( , \) it follows that \( \langle | \rangle \) satisfies

\[ \langle X|Y \rangle = \langle Y|X \rangle, \quad \langle c(X)|c(Y) \rangle = \langle X|Y \rangle, \quad \text{and} \quad \langle [X,c(Y)]|Z \rangle = \langle X|[Y, Z] \rangle \]

for all \( X, Y, Z \in g \). Treating \( g \) as a \( \mathbb{R} \)-linear space by restricting scalar multiplication to multiplication by reals, we can introduce a positive definite inner product \( \langle \langle | \rangle \rangle : g \times g \to \mathbb{R} \) on \( g \) defined by

\[ \langle \langle X|Y \rangle \rangle := \text{Re}\langle X|Y \rangle \quad \forall X, Y \in g. \]  \hspace{1cm} (2.21)

Let \( \| | \) denote the norm induced on \( g \) by \( \langle \langle | \rangle \rangle \), i.e.

\[ \|X\| = \sqrt{\langle \langle X|X \rangle \rangle} \quad \forall X \in g. \]  \hspace{1cm} (2.22)
From the above properties satisfied by $\langle | \rangle$, it straightforward to verify that $\langle | \rangle$ satisfies
\[
\langle \langle X|Y \rangle \rangle = \langle \langle Y|X \rangle \rangle, \quad \langle \langle c(X)|c(Y) \rangle \rangle = \langle \langle X|Y \rangle \rangle, \quad \text{and} \quad \langle \langle [X,c(Y)]|Z \rangle \rangle = \langle \langle X|[Y,Z] \rangle \rangle
\] (2.23)
for all $X,Y,Z \in \mathfrak{g}$.

Using the norm (2.24), equations (2.13) can be written as
\[
G = \frac{1}{2}||\Lambda^\prime||^2 \quad \text{and} \quad P = \frac{1}{2}||\hat{F}||^2
\] (2.24)
which shows that $G \geq 0$ and $P \geq 0$. Energy density, radial and tangential pressure are then given by
\[
4\pi e = r^{-2}(NG + r^{-2}P), \quad 4\pi p_r = r^{-2}(NG - r^{-2}P), \quad 4\pi p_\theta = r^{-4}P.
\] (2.25)

The main result of this paper is that the EYM equations (2.11)-(2.14) admit local bounded solutions in the neighborhood of the origin $r = 0$, a black hole horizon $r = r_H > 0$, and as $r \to \infty$. To prove this local existence to the EYM equations (2.11)-(2.14), we proceed in three steps.

1. First we prove the existence of local solutions $\{\Lambda_+(r), m(r)\}$ to the equations (2.11) and (2.13).
2. Then we determine which solutions from step 1 satisfy equation (2.14).
3. Finally, equation (2.12) can be integrated for all solutions from step 2 to obtain the metric function $S(r)$.

Our method for carrying out the first step will be to prove that there exists a change of variables so that the field equations (2.11) and (2.13) can be put into a form to which the following (slight generalization of a) theorem by Breitenlohner, Forgacs and Maison \(\bb{3}\) applies.

**Theorem 1.** The system of differential equations
\[
i \frac{du_i}{dt} = \theta^\mu f_i(t,u,v) \quad i = 1, \ldots, m
\]
\[
i \frac{dv_j}{dt} = -h_j(u)v_j + t^\nu g_j(t,u,v) \quad j = 1, \ldots, n
\] (2.26) (2.27)

where $\mu_i$, $\nu_j$ are integers greater than 1, $f_i$ and $g_j$ analytic functions in a neighborhood of $(0,c_0,0) \in \mathbb{R}^{1+m+n}$, and $h_j : \mathbb{R}^m \to \mathbb{R}$ functions, positive in a neighborhood of $c_0 \in \mathbb{R}^m$, has a unique analytic solution $t \mapsto (u_i(t), v_j(t))$ such that
\[
u_i(t) = c_i + O(t^{\mu_i}) \quad \text{and} \quad v_j(t) = O(t^{\nu_j})
\] (2.28)
for $|t| < R$ for some $R > 0$ if $|c - c_0|$ is small enough. Moreover, the solution depends analytically on the parameters $c_i$.

The next lemma shows that if $\{\Lambda_+(r), m(r)\}$ is a solution to the field equations (2.11) and (2.13), then the quantity $[\Lambda'_+, \Lambda_-] + [\Lambda'_-, \Lambda_+]$ satisfies a first order linear differential equation. This unexpected result is what allows us to carry out step 2 and thereby construct local solutions.

**Lemma 1.** If $\{\Lambda_+(r), m(r)\}$ is a solution to the field equations (2.11) and (2.13) then $\gamma(r) := [\Lambda_+(r), \Lambda'_-(r)] + [\Lambda_-(r), \Lambda'_+(r)]$ satisfies the differential equation $\gamma' = -2(r^2 N)^{-1}(m - r^{-1} P) \gamma$.

**Proof.** Differentiating $\gamma$ yields
\[
\gamma' = [\Lambda'_+, \Lambda''_-] + [\Lambda'_-, \Lambda''_+]
\]
\[
= - \frac{2}{r^2 N} \left( m - \frac{1}{r} P \right) \gamma + \frac{i}{r^2 N} ([\Lambda_-, [\hat{F}, \Lambda_+]] + [\Lambda_+, [\Lambda_-, \hat{F}]]), \quad \text{by (2.11) and (2.13)}
\]
while $[\Lambda_-, [\hat{F}, \Lambda_+]] + [\Lambda_+, [\Lambda_-, \hat{F}]] = 0$ by (2.9), (2.16), and the Jacobi identity. Combining the these two results proves the lemma. \(\square\)
3 Solving the field equations computationally

Here we describe without proofs the more practical aspects of solving the field equations. That all these constructions work will follow from the proofs given in sections \( \square \) and \( \square \).

We consider only situations where the EYM field is nonsingular at the center and/or the gravitational and Yang-Mills field fall off rapidly at infinity. From the expressions for the physical quantities (2.25) and (2.24) it then follows that \( \hat{F} \) vanishes there so that by (2.16) \( \{A_0, \Lambda_+, \Lambda_-\} \) form a triple defining an \( \mathfrak{sl}(2) \) (or \( \mathfrak{A}_1 \)) subalgebra of \( \mathfrak{g} \). This restricts the (constant) \( \Lambda_+ \) vector to be a \textit{defining vector} of such a subalgebra which have been classified by Dynkin \( \square \). Alternatively, in the terminology of \( \square \), the set \( \{A_0, \Omega_+, \Omega_-\} \), where we define \( \Omega_\pm \) to be the limiting values of \( \Lambda_\pm \) at \( r = 0 \) or \( \infty \), is called a standard triple defining a nilpotent orbit, \( A_0 \) is the neutral and \( \Omega_+ \) the nilpositive element.

It is known \( \square \) that there is always an automorphism of \( \mathfrak{g} \) that maps \( A_0 \) onto the fundamental Weyl chamber so that its characteristic (or weighted Dynkin diagram) \( \chi = (\chi_1, \ldots, \chi_\ell) := (\alpha_1(\Lambda_0), \ldots, \alpha_\ell(\Lambda_0)) \) with respect to a chosen Cartan subalgebra and its dual basis \( \{\alpha_1, \ldots, \alpha_\ell\} \), satisfies \( \chi_k \in \{0, 1, 2\} \quad \forall \quad k = 1, \ldots, \ell \). We call the symmetry group action and the vector \( \Lambda_0 \) \textit{regular} if \( \Lambda_0 \) lies in the interior of the Weyl chamber or, equivalently, if all \( \chi_k \) are positive. The action and \( A_0 \) are called \textit{principal} if \( \chi_k = 2 \quad \forall \quad k \).

Since for semi-simple gauge groups all constructions are easily decomposed into those of the simple factors a classification need only be done for the simple Lie groups. It turns out that only for the \( \mathfrak{A}_\ell \) (or \( \mathfrak{sl}(\ell + 1) \)) series of the classical Lie algebras and then only for even \( \ell \) there are regular actions other than the principal one. (See \( \square \), Theorem 2.) Moreover, in those cases the gauge field corresponds to one of a direct sum of two lower-dimensional simple Lie algebras of type \( \mathfrak{A}_k \). It is for that reason that we could confine ourselves in \( \square \) to the principal case when studying the local existence problem of solutions for regular symmetry group actions.

In general, given a fixed defining vector \( A_0 \), the Yang-Mills field will be fully determined, in view of (2.11), if \( \Lambda_+ \) is known as a function of \( r \). Condition (2.9) implies that \( \Lambda_+ \) must lie in the vector subspace \( V_2 \) of \( \mathfrak{g} \) where we define, more generally, the eigenspaces of \( \text{ad}(\Lambda_0) \) by

\[
V_n := \{X \in \mathfrak{g} \mid A_0.X = nX\} \quad n \in \mathbb{Z},
\]

Here, and in the following, the adjoint action of \( \text{span}_\mathbb{C}\{A_0, \Omega_+, \Omega_-\} \cong \mathfrak{sl}(2, \mathbb{C}) \) on \( \mathfrak{g} \), is denoted by a dot,

\[
X.Y := \text{ad}(X)(Y) \quad \forall \quad X \in \text{span}_\mathbb{C}\{A_0, \Omega_+, \Omega_-\}, \quad Y \in \mathfrak{g}.
\]

In terms of a Chevalley-Weyl basis (see \( \square \) or \( \square \) for the notation) \( \{h_j := h_{\alpha_j}, e_\alpha, e_{-\alpha} | j = 1, \ldots, \ell; \alpha \in R^+\} \), where \( R^+ \) is the set of positive roots of \( \mathfrak{g} \) with respect to the root basis \( \{\alpha_1, \ldots, \alpha_\ell\} \), we then have

\[
\Lambda_+(r) = \sum_{\alpha \in S_{\Lambda_0}} w_\alpha(r)e_\alpha \in V_2 \quad \text{and} \quad \Lambda_-(r) = \sum_{\alpha \in S_{\Lambda_0}} \bar{w}_\alpha(r)e_{-\alpha} \in V_2 \quad \text{(3.1)}
\]

where

\[
S_{\Lambda_0} := \{\alpha \in R | \alpha(A_0) = 2\} \quad \text{(3.2)}
\]

In the regular case the Stiefel set \( S_{\Lambda_0} \) is necessarily a \( H \)-system \( \square \), i.e. forms the basis of a root system generating a subalgebra \( \mathfrak{g}_{\Lambda_0} \) of \( \mathfrak{g} \) (see \( \square \)). Since for base root vectors \( \alpha, \beta \) also \( [e_\alpha, e_\beta] \neq 0 \) only if \( \beta = -\alpha \) and \( [e_\alpha, e_{-\alpha}] = h_\alpha \) it follows then from (2.16) that \( \hat{F} \) lies in the Cartan subalgebra of \( \mathfrak{g}_{\Lambda_0} \). Substituting (3.1) into (2.14) leads to

\[
w'_\alpha \bar{w}_\beta - w_\alpha \bar{w}'_\beta = 0 \quad \forall \alpha, \beta \in S_{\Lambda_0} \quad \text{(3.3)}
\]

so that all the phases of the complex \( w_\alpha(r) \) are constant.
In the general case, $S_{\Lambda_0}$ will not be linearly independent and the set $\{[e_\alpha, e_{-\beta}] \mid \alpha, \beta \in S_{\Lambda_0}\}$ will not lie in the Cartan subalgebra $h$ of $\mathfrak{g}$. There is no simple gauge transformation now that will allow us to choose the $w_\alpha$ real, although, as follows from Lemma 1, if \eqref{2.14} is satisfied at one regular value of $r$ it will be in a whole interval.

The Yang-Mills field $\hat{F}$ no longer takes all its values in the Cartan subalgebra. It is instead given by

$$\hat{F} = \frac{i}{2} \left( \Lambda_0 - \sum_{\alpha, \beta \in S_{\Lambda_0}} w_\alpha(r) \bar{w_\beta}(r) [e_\alpha, e_{-\beta}] \right).$$

(3.4)

However, the term $F$ appearing in the Yang-Mills equation \eqref{2.13} lies in $V_2$ since $[X, [Y, Z]] \in V_2$ whenever two of the $X, Y, Z$ lie in $V_2$ and one in $V_{-2}$.

For computational purposes \eqref{2.13} can then be written

$$r^2 N w''_\alpha + 2(m - r^{-1} P) w'_\alpha + f_\alpha = 0$$

(3.5)

where

$$f_\alpha := w_\alpha + \frac{1}{2} \sum_{\beta, \gamma, \delta \in S_{\Lambda_0}} \mu_{\alpha \beta \gamma \delta} w_\beta w_\gamma \bar{w_\delta}$$

(3.6)

and

$$P = \frac{1}{8} ||\Lambda_0||^2 - \sum_{\alpha \in S_{\Lambda_0}} |\alpha|^{-2} |w_\alpha|^2 - \frac{1}{4} \sum_{\alpha, \beta, \gamma, \delta \in S_{\Lambda_0}} |\alpha|^{-2} \mu_{\alpha \beta \gamma \delta} w_\alpha w_\beta w_\gamma \bar{w_\delta}.$$

(3.7)

(Here the Jacobi identity and the invariance of the inner product $(\ , \ )$ were used.) Thus the general structure of the Lie algebra $\mathfrak{g}$ enters the equations only via the quantity $\mu_{\alpha \beta \gamma \delta}$ which is defined by

$$[e_\beta, [e_\gamma, e_{-\delta}]] = \sum_{\alpha \in S_{\Lambda_0}} \mu_{\alpha \beta \gamma \delta} e_\alpha \ \forall \ \beta, \gamma, \delta \in S_{\Lambda_0}.$$

(3.8)

To start off the numerical integration of a bounded solution near one of the singular points we need to sum a power series at a point nearby. This is again similar to the method used in the regular case, but we must allow for possibly complex functions $w_\alpha(r)$. This has most of all the unpleasant effect that the values of $w_\alpha$ can take at the center and at infinity form no longer just a finite set with a few signs to be chosen arbitrarily, but an $M$-dimensional real variety in the space $\mathbb{C}^M$ (where $M$ denotes the number of elements in $S_{\Lambda_0}$ and thus the number of functions $w_\alpha$ needed to characterize the gauge potential $\Lambda_+ + \Omega_+$). Since there is no reason to believe that for a global solution on $[0, \infty)$ the values of $\Lambda_+(0)$ and $\Lambda_+(\infty)$ should be the same we can, for example, choose an arbitrary $\Omega_+ = \Lambda_+(\infty)$ which amounts to a global gauge choice. But then the value of $\Lambda_+$ at the center could be any point of this $M$-dimensional real variety so that the coordinates describing the latter need to be given as parameters as well as some of the first few power series coefficients.

Wishing to find a local analytic solution to equations \eqref{2.11} and \eqref{3.5} we expand all quantities in a power series in $r$ near $r = 0$. The lowest order terms $w_{\alpha,0} = w_\alpha(0)$ are constrained by $\hat{F} = 0$ so that

$$\Lambda_0 = [\Omega_+, \Omega_-] \ \text{ and } \ \Omega_- = -c(\Omega_+)$$

or

$$\Lambda_0 := \sum_{i=1}^\ell \lambda_i h_i = \sum_{\alpha, \beta \in S_{\Lambda_0}} w_{\alpha,0} \bar{w}_{\beta,0} [e_\alpha, e_{-\beta}]$$

(3.9)
which gives, if we write \( h_\alpha = \sum_{i=1}^{\ell} h_{\alpha,i} \) and \( R \) denotes the set of all roots of the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \),

\[
\sum_{\alpha \in S_{\Lambda_0}} h_{\alpha,i} |w_{\alpha,0}|^2 = \lambda_i \quad (i = 1, \ldots, \ell) \quad \text{and} \quad \sum_{\alpha,\beta \in S_{\Lambda_0}} w_{\alpha,0} \bar{w}_{\beta,0} [e_\alpha, e_{-\beta}] = 0. \quad (3.11)
\]

We have not yet been able to determine whether equations (3.11) have solutions for all simple Lie algebras and all defining vectors \( \Lambda_0 \). For all low-dimensional examples we have so far considered solutions exist and form an \( M \)-dimensional real variety in \( \mathbb{C}^M \). Moreover, it appears that there always exist vectors \( \Omega_+ \) with only real components with respect to the basis \( \{ e_\alpha \} \).

Suppose now that \( \Omega_+ \) has been chosen. Then equation (2.11) yields the recurrence relation

\[
m_k = \frac{1}{k} \left( P_{k+1} + G_{k-1} - 2 \sum_{i=2}^{k-3} m_{k-i} G_i \right) \quad (3.12)
\]

and (2.13) gives

\[
A(\Lambda_+,k) - k(k-1)\Lambda_+,k = b_k \quad (k = 2, 3, \ldots) \quad (3.13)
\]

where the \( b_k \) are complicated expressions of lower order terms and \( A \) is defined by

\[
A = \frac{1}{2} \text{ad}(\Omega_+) \circ (\text{ad}(\Omega_-) + \text{ad}(\Omega_+) \circ c). \quad (3.14)
\]

This \( A \) is an \( \mathbb{R} \)-linear operator which turns out to be symmetric with respect to the inner product (2.21) and restricts to \( V_2 \). As will be shown in section 5, half of its eigenvalues are zero and the remaining ones are positive integers of the form \( k(k+1) \) for \( k = 1, 2, \ldots \). We will later need the notation

\[
\mathcal{E} := \{ s \in \mathbb{N} \mid s(s+1) \text{ is an eigenvalue of } A \} \quad (3.15)
\]

Moreover, to every eigenvector with nonzero eigenvalue there exists one (its multiple by \( i \)) that has eigenvalue zero.

Explicitly, if we write \( \Omega_+ = \sum_{\alpha=1}^M (u_{\alpha,0} + iv_{\alpha,0}) e_\alpha \), where \( M := |S_{\Lambda_0}| \), then \( A \) is given by the matrix

\[
(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.16)
\]

where

\[
a_{\alpha\beta} = -\sum_{\gamma,\delta} (\mu_{\alpha\delta(\beta\gamma)} u_{\gamma,0} u_{\delta,0} + \mu_{\alpha\delta(\beta\gamma)}^\ast v_{\gamma,0} v_{\delta,0}) \quad (3.17)
\]

\[
b_{\alpha\beta} = \sum_{\gamma,\delta} (\mu_{\alpha\delta(\beta\gamma)} - \mu_{\alpha\gamma(\beta\delta)}) u_{\gamma,0} v_{\delta,0} \quad (3.18)
\]

\[
c_{\alpha\beta} = \sum_{\gamma,\delta} (\mu_{\alpha\gamma(\beta\delta)} - \mu_{\alpha\delta(\beta\gamma)}) u_{\gamma,0} v_{\delta,0} \quad (3.19)
\]

\[
d_{\alpha\beta} = -\sum_{\gamma,\delta} (\mu_{\alpha\gamma(\beta\delta)} u_{\gamma,0} u_{\delta,0} + \mu_{\alpha\gamma(\beta\delta)}^\ast v_{\gamma,0} v_{\delta,0}) \quad (3.20)
\]

Now let \( E \) be the matrix whose columns are eigenvectors of \( A \) such that

\[
AE = \mathcal{E}A \quad (3.21)
\]
where $\Lambda = \text{diag}(0 \ldots 0\lambda_1 \ldots \lambda_M)$ with $\lambda_i \leq \lambda_j$ if $i \leq j$. Then we know (from section 3.3) that if $E$ is of the form $E = \begin{pmatrix} * & e_1 \\ * & e_2 \end{pmatrix}$ then also $\begin{pmatrix} -e_2 & e_1 \\ e_1 & e_2 \end{pmatrix}$ will be a matrix of eigencolumnvectors with now the correct correspondence between the real and imaginary parts.

To generate the powerseries for $u$ and $v$ we now find from the recurrence relation (3.13) for the coefficients of $r^k$ that

$$
\begin{pmatrix} u_k \\ v_k \end{pmatrix} = E^T X_k
$$

(3.22)

where $X_{k,\alpha} = ((E^T)^{-1}b_k)_{\alpha}/(\lambda_\alpha - k(k - 1))$ if $\lambda_\alpha \neq k(k - 1)$ or a new free parameter otherwise.

This shows that the general solution near $r = 0$ is of a similar form as in the regular case (3.18), Theorem 4, namely

$$
\alpha(r) = \alpha_{\alpha,0} + \sum_{\beta=1}^{M} E_{\alpha,\beta} r^{k_\beta + 1} \tilde{w}_\beta(r), \quad \alpha = 1, \ldots, M
$$

(3.23)

where $k_\alpha(k_\alpha + 1)$ is the $\alpha$-th nonzero eigenvalue of $A$ and $\tilde{w}_\alpha(r)$ are analytic functions near $r = 0$. The general solution $\{m(r), \alpha_\alpha(r)|\alpha = 1, \ldots, M\}$ is determined by $M$ real parameters for the initial values $\alpha_\alpha(0)$ as well as another $M$ real parameters that can be arbitrarily chosen in the coefficients $\tilde{w}_\beta(0)$ of $r^{k_\beta + 1}$.

The general power series solution in $r^{-1}$ near infinity is very similar.

For very special choices of the parameters one expects to find solutions of EYM-equations corresponding to gauge groups that are subgroups of $G$. In one simple case this is easy to see.

Since every compact semi-simple (non-Abelian) Lie group $G$ has $SU(2)$ as a subgroup there must be $SU(2)$-solutions embedded among general $G$-solutions for any symmetry group action. There may be several conjugacy classes of such subgroups, but there is a distinguished one related to the homomorphism defined by the action. Let the action and hence $\Lambda_0$ be given and pick any $\Omega_0$, thus selecting a specific $\mathfrak{sl}(2)$-subalgebra generated by the triple $\{\Lambda_0, \Omega_+, \Omega_- = -c(\Omega_+))\}$ among the conjugacy class associated with $\Lambda_0$. If we then let

$$
\Lambda_+(r) = \psi(r)\Omega_+ \quad \text{with } \psi(0) = 1
$$

(3.24)

it follows from (2.14) that $\psi(r) = e^{\gamma_0} u(r)$ for a constant $\gamma_0$ and a real function $u(r)$. The remaining equations (2.11)-(2.13) then become

$$
\begin{align*}
m' &= g_0^2 (Nu'^2 + \frac{1}{2}r^{-2}(1 - u^2)^2), \\
S^{-1}S' &= 2g_0^2 r^{-1} u^2, \\
r^2Nu'' + (2m - g_0^2 r^{-1}(1 - u^2)^2)u' + (1 - u^2)u = 0,
\end{align*}
$$

(3.25)-(3.27)

where $g_0 := \frac{1}{r}||A_0||$. They reduce with

$$
\rho := r/g_0, \quad \mu := m/g_0
$$

(3.28)

to the equations for the $SU(2)$ theory,

$$
\begin{align*}
d\mu/d\rho &= N(du/d\rho)^2 + \frac{1}{2}\rho^{-2}(1 - u^2)^2, \\
S^{-1}dS/d\rho &= 2\rho^{-1}(du/d\rho)^2, \\
\rho^2 Nd^2u/d\rho^2 + (2\mu - \rho^{-1}(1 - u^2)^2)du/d\rho + (1 - u^2)u &= 0,
\end{align*}
$$

(3.29)-(3.31)

where now $N = 1 - 2\mu/\rho$.

Much is known about the solutions of these equations. In particular, it follows, that for any compact semi-simple Lie group $G$ and any action of the symmetry group $SU(2)$ an infinite discrete set of global asymptotically flat (soliton and black hole) solutions exists.
4 The spherically symmetric static EYM models for some small gauge groups

We have not yet found a simple general way to derive the field equations for arbitrary semi-simple \( g \) and arbitrary choice of \( \Lambda_0 \). But in the following tables \( \text{Table 4} \) to \( \text{Table 4} \) we list for lower dimensional Lie groups and the actions of \( SU(2) \) given by their characteristic some basic properties of the system of equations, namely the size of the size \( M = |S_{\Lambda_0}| \) of the Stiefel set which corresponds to the number of complex functions \( w_\alpha(r) \) that describe the gauge potential, the set \( \mathcal{E} \) with the superscript denoting the dimension of the eigenspace if it is greater than one, and the subalgebra to which the equations reduce if \( S_{\Lambda_0} \) is a \( \Pi \)-system ('-' indicates that there is no reduction). We leave out the principal action, which is always regular, as well as the trivial action of the symmetry group.

Just to give an idea of the structure the \( M \)-dimensional real subvariety \( \Sigma \) of \( \mathbb{C}^M \) we list in table \( \text{Table 4} \) the equations that define it for a few cases. (In the regular case for a Lie algebra of rank \( \ell \) the equations would require that \( |w_i| = \text{const} \) for certain fixed constants for all \( i = 1, 2, \ldots, \ell \).)

When one wishes to find global numerical solutions one can, for example, pick a simple choice for \( \Lambda_+ (\infty) \) – it seems possible to choose all the \( w_\alpha (\infty) \) real, and many of them 0. But then the data at \( r = 0 \) must be left arbitrary, i.e. a suitable parametrization of the variety \( \Sigma \) must be introduced. This is not difficult for the low dimensional cases, but not straightforward.

### Table 1: Irregular actions for Lie algebras \( A_2 \) to \( A_4 \) and \( B_2 \) to \( B_4 \)

| \( \chi \) | \( |S_{\Lambda_0}| \) | \( \mathcal{E} \) | reduction | \( \chi \) | \( |S_{\Lambda_0}| \) | \( \mathcal{E} \) | reduction |
|---|---|---|---|---|---|---|---|
| \( A_2 \) (\( SU(3) \)) | \( B_2 \) (\( SO(5) \)) | \( A_3 \) (\( SU(4) \)) | \( B_3 \) (\( SO(7) \)) | \( A_4 \) (\( SU(5) \)) | \( B_4 \) (\( SO(9) \)) |
| 11 | 1 | 1 | \( A_1 \) | 01 | 1 | 1 | \( A_1 \) |
| 20 | 3 | 1^3 | – | 20 | 3 | 1^3 | – |
| \( A_3 \) | \( A_1 \oplus A_1 \) | \( A_1 \oplus A_2 \) | \( B_2 \oplus A_1 \) |
| 101 | 1 | 1 | \( A_1 \) | 010 | 1 | 1 | \( A_1 \) |
| 020 | 4 | 1^4 | – | 101 | 2 | 1^2 | \( A_1 \oplus A_1 \) |
| 202 | 4 | 1^3,6 | – | 220 | 4 | 1,2^2,3 | – |
| | | | | 200 | 5 | 1^2 | – |
| | | | | 020 | 6 | 1^2,2 | – |
| \( A_4 \) | \( B_2 \oplus A_1 \) | | | | | | |
| 1001 | 1 | 1 | \( A_2 \) | 0100 | 1 | 1 | \( A_1 \) |
| 2112 | 3 | 1,2,3 | \( A_3 \oplus A_2 \) | 2101 | 3 | 1^2,3 | \( B_2 \oplus A_1 \) |
| 1111 | 3 | 1^2,2 | \( A_3 \oplus A_2 \) | 1010 | 4 | 1^4 | – |
| 0110 | 4 | 2^4 | – | 0201 | 5 | 1,2^2,3 | – |
| 2002 | 6 | 2^3,3 | – | 2220 | 5 | 1,3^2,5 | – |
| | | | | 0001 | 6 | 1^6 | – |
| | | | | 2200 | 6 | 1,2^4,3 | – |
| | | | | 2000 | 7 | 1^2 | – |
| | | | | 2020 | 8 | 1^2,2^2,3^2 | – |
| | | | | 0002 | 9 | 1^6,2^3 | – |
| | | | | 0200 | 10 | 1^6,2 | – |
Table 2: Irregular actions for the Lie algebras $C_3 (Sp(6))$, $C_4 (Sp(8))$ and $D_4 (SO(8))$

| $\chi$ | $|S_{\Lambda_0}|$ | $\mathcal{E}$ | reduction | $\chi$ | $|S_{\Lambda_0}|$ | $\mathcal{E}$ | reduction |
|--------|-----------------|----------------|-----------|--------|-----------------|----------------|-----------|
| $C_3$  | $(Sp(6))$       |                |           | $C_4$  | $(Sp(8))$       |                |           | $D_4$  | $(SO(8))$       |                |           |
| 100    | 1               | 1              | $A_1$     | 1000   | 1               | 1              | $A_1$     | 1000   | 1               | 1              | $A_1$     |
| 210    | 2               | 1, 3           | $C_2$     | 2100   | 2               | 1, 3           | $C_2$     | 2100   | 2               | 1, 3           | $C_2$     |
| 010    | 3               | $1^3$          |           | 0100   | 3               | $1^3$          |           | 0100   | 3               | $1^3$          |           |
| 020    | 4               | $1, 2^4$       |           | 0200   | 4               | $1, 2^4$       |           | 0200   | 4               | $1, 2^4$       |           |
| 002    | 6               | $1^6$          |           | 0020   | 6               | $1^6$          |           | 0020   | 6               | $1^6$          |           |

Table 3: Irregular actions for the Lie algebras $F_4$ and $G_2$

| $\chi$ | $|S_{\Lambda_0}|$ | $\mathcal{E}$ | reduction | $\chi$ | $|S_{\Lambda_0}|$ | $\mathcal{E}$ | reduction |
|--------|-----------------|----------------|-----------|--------|-----------------|----------------|-----------|
| $F_4$  |                 |                |           | $G_2$  |                 |                |           |
| 1000   | 1               | 1              | $A_1$     | 01     | 1               | 1              | $A_1$     | 1000   | 1               | 1              | $A_1$     |
| 1012   | 3               | 1, 3, 5        | $C_3$     | 10     | 1               | 1              | $A_1$     | 1010   | 5               | $1^3$, 2, 3  |           |
| 1010   | 5               | $1^3$, 2, 3    |           | 0101   | 5               | $1^2$, 2^3    |           | 0101   | 5               | $1^2$, 2^3    |           |
| 1000   | 6               | $1^6$          |           | 1000   | 6               | $1^6$          |           | 1000   | 6               | $1^6$          |           |
| 2001   | 6               | $1, 2^4$, 3    |           | 2001   | 6               | $1, 2^4$, 3    |           | 2001   | 6               | $1, 2^4$, 3    |           |
| 2202   | 6               | $1, 2, 3, 5^2$, 7 |       | 2002   | 9               | $1^6$, 2^3    |           | 2002   | 9               | $1^6$, 2^3    |           |
| 0001   | 7               | $1^7$          |           | 0001   | 7               | $1^7$          |           | 0001   | 7               | $1^7$          |           |
| 2200   | 7               | $1, 3^5$, 5    |           | 2200   | 7               | $1, 3^5$, 5    |           | 2200   | 7               | $1, 3^5$, 5    |           |
| 0002   | 8               | $1, 2^7$       |           | 0002   | 8               | $1, 2^7$       |           | 0002   | 8               | $1, 2^7$       |           |
| 0202   | 8               | $1^3$, 2, 3, 4, 5^4 |          | 0202   | 8               | $1^3$, 2, 3, 4, 5^4 |          | 0202   | 8               | $1^3$, 2, 3, 4, 5^4 |          |
| 0010   | 9               | $1^6$, 2^4    |           | 0010   | 9               | $1^6$, 2^4    |           | 0010   | 9               | $1^6$, 2^4    |           |
| 0200   | 12              | $1^6$, 2^4, 3^4 |          | 0200   | 12              | $1^6$, 2^4, 3^4 |          | 0200   | 12              | $1^6$, 2^4, 3^4 |          |
| 2000   | 14              | $1^6$, 2      |           | 2000   | 14              | $1^6$, 2      |           | 2000   | 14              | $1^6$, 2      |           |
Table 4: The "initial value" surface $\Sigma$ for a few irregular actions for some Lie algebras

| Lie algebra | $\chi$ | $\Sigma$ |
|-------------|--------|---------|
| $A_3$       | 020    | $|w_1|^2 + |w_2|^2 = 1, |w_3| = |w_4| = |w_1|, w_1\bar{w}_3 = -w_2\bar{w}_4,$ $w_1\bar{w}_2 = -w_3\bar{w}_4$ |
| $B_4$       | 1010   | $|w_1|^2 + |w_2|^2 = 1, |w_3| = |w_1|, w_4 = 1, w_2\bar{w}_3 = w_1\bar{w}_2$ |
| $E_6$       | 120001 | $|w_1|^2 + |w_2|^2 = 1, |w_3| = |w_1|, |w_4| = |w_3|, |w_5| = |w_2|, |w_6| = 2, |w_7| = |w_1|, w_1\bar{w}_2 = w_5\bar{w}_7, w_1\bar{w}_3 = -w_4\bar{w}_7, w_1\bar{w}_4 = -w_5\bar{w}_7, w_2\bar{w}_3 = w_2\bar{w}_5, w_2\bar{w}_4 = -w_3\bar{w}_5$ |
| $F_4$       | 1010   | $|w_1|^2 + |w_3|^2 = 3, |w_3|^2 + |w_4|^2 + |w_5|^2 = 4, -|w_2|^2 + 2|w_3|^2 + |w_5|^2 = 3, w_1\bar{w}_3 + w_2\bar{w}_4 = w_4\bar{w}_5$ |
| $G_2$       | 02     | $|w_1|^2 + 2|w_2|^2 + |w_3|^2 = 2, |w_2|^2 + 2|w_3|^2 + |w_4|^2 = 2, w_1\bar{w}_2 + 2w_2\bar{w}_3 + w_3\bar{w}_4 = 0$ |

5 Algebraic Results

In this section we collect all of the algebraic results needed to prove the local existence theorems. We will employ the same notation as in [18] section 6.

Before proceeding, we recall some results from [18].

**Proposition 1.** There exists $M$ highest weight vectors $\xi^1, \xi^2, \ldots, \xi^M$ for the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ that satisfy

(i) the $\xi^j$ have weights $2k_j$ where $j = 1, 2, \ldots, M$ and $1 = k_1 \leq k_2 \leq \cdots \leq k_M$,

(ii) if $V(\xi^j)$ denotes the irreducible submodule of $\mathfrak{g}$ generated by $\xi^j$, then the sum $\sum_{j=1}^M V(\xi^j)$ is direct,

(iii) if $\xi^j = (1/\lambda)\Omega\cdot\xi^j$ then

$$c(\xi^j) = (-1)^j \xi^j_{2k_j - \lambda},$$

(iv) $M = |S_\lambda|$ and the set $\{\xi^j_{k_j - 1} \mid j = 1, 2, \ldots, M\}$ forms a basis for $V_2$ over $\mathbb{C}$.

According to Lemma 1 of [18] the $\mathbb{R}$-linear operator $A: \mathfrak{g} \to \mathfrak{g}$ defined by (3.14) satisfies

$$A(V_2) \subset V_2$$

and hence restricts to an operator on $V_2$ which we denote by $A_2$.

We label the integers $k_j$ from proposition [1] as follows

$$1 = k_{J_1} = k_{J_1 + 1} = \cdots = k_{J_1 + m_1 - 1} < k_{J_2} = k_{J_2 + 1} = \cdots = k_{J_2 + m_2 - 1} \cdots < k_{J_l} = k_{J_l + 1} = \cdots = k_{J_l + m_l - 1},$$

where $J_1 = 1, J_l + m_l = J_{l+1}$ for $l = 1, 2, \ldots, I$ and $J_{I+1} = M - 1$. Then

$$E = \{k_l := k_{J_l} \mid l = 1, 2, \ldots, I\}.$$
The set \( \{ \xi^j_{k_j-1} \mid j = 1, 2, \ldots M \} \) forms a basis over \( \mathbb{C} \) of \( V_2 \) by proposition (iv) while \( I \) is the number of distinct nonzero eigenvalues of \( A_2 \). Therefore the set of vectors \( \{ X^l_s, Y^l_s \mid l = 1, 2, \ldots , I ; s = 0, 1, \ldots , m_l-1 \} \) where

\[
X^l_s := \left\{ \begin{array}{ll}
\xi^j_{k_j-1} & \text{if } k_j \text{ is odd} \\
\frac{i}{\xi^j_{k_j-1}} & \text{if } k_j \text{ is even}
\end{array} \right. \quad \text{and} \quad Y^l_s := iX^l_s,
\]

(5.3)

forms a basis of \( V_2 \) over \( \mathbb{R} \). Then lemma 2 of [18] shows that this basis is an eigenbasis of \( V_2 \) and we have

\[
A_2(X^l_s) = k_l(k_l+1)X^l_s \quad \text{and} \quad A_2(Y^l_s) = 0 \quad \text{for} \quad l = 1, 2, \ldots , I ; s = 0, 1, \ldots , m_l-1 .
\]

(5.4)

An immediate consequence of this result is that \( \text{spec}(A_2) = \{ 0 \} \cup \{ k_j(k_j+1) \mid j = 1, 2, \ldots I \} \) and \( m_j \) is the dimension of the eigenspace corresponding to the eigenvalue \( k_j(k_j+1) \). Note that \( I \) is the number of distinct positive eigenvalues of \( A_2 \).

Define

\[
E^l_0 = \text{span}_R \{ Y^l_s \mid s = 0, 1, \ldots , m_l-1 \} , \quad E^l_+ = \text{span}_R \{ X^l_s \mid s = 0, 1, \ldots , m_l-1 \} ,
\]

(5.5)

and

\[
E_0 = \bigoplus_{l=1}^I E^l_0 , \quad E_+ = \bigoplus_{l=1}^I E^l_+ .
\]

(5.6)

Then

\[
E_0 = \ker (A_2)
\]

(5.7)

and \( E^l_+ \) is the eigenspace of \( A_2 \) corresponding to the eigenvalue \( k_l(k_l+1) \). Moreover, using proposition (iv), it is clear that

\[
V_2 = E_0 \oplus E_+ .
\]

(5.8)

To simplify notation in what follows, we introduce one more quantity

\[
E^l := \bigoplus_{q=0}^l E^l_0 \oplus E^l_+ .
\]

(5.9)

We then have the useful lemma from [18].

**Lemma 2.** If \( X \in V_2 \) then \( X \in E^l \) if and only if \( \Omega^k(X) = 0 \) or \( \Omega^{k+2}(X) = 0 \).

We will also need the the map \( \tilde{\sim} : \mathbb{Z}_{\geq -1} \to \{ 1, 2, \ldots , I \} \) defined by

\[
\tilde{\sim} -1 = \tilde{0} = 1 \quad \text{and} \quad \tilde{s} = \max \{ t \mid k_t \leq s \} \quad \text{if} \quad s > 0 .
\]

(5.10)

It is shown in lemma 5 of [18] that this map satisfies

\[
k_{\tilde{s}} \leq s \quad \text{for every} \quad s \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad k_{\tilde{s}} \leq s < k_{\tilde{s}+1} \quad \text{for every} \quad s \in \{ 0, 1, \ldots k_I - 1 \} .
\]

(5.11)

The last result we will need from [18] is the following lemma.

**Lemma 3.** If \( X \in V_2 \), \( k_{\tilde{p}} + s < k_{\tilde{p}+1} \) (\( s \geq 0 \)), and \( \Omega^{k_{\tilde{p}+s}}(X) = 0 \), then \( \Omega^{k_{\tilde{p}}}(X) = 0 \).
We will also frequently use the following fact

\[(l \pm 1)^* = \tilde{l} \pm 1 \quad \forall \ l \in \mathcal{E} \tag{5.12}\]

**Proposition 2.** If \(X_a \in E^a\), \(Y_b \in E^b\), and \(Z_c \in E^c\) then \([c(X_a), Y_b], Z_c] \in E^{(a+b+c)}\).

**Proof.** Suppose \(X_a \in E^a\), \(Y_b \in E^b\), and \(Z_c \in E^c\). Then

\[\Omega^{k_2}_{+}.c(X_a) = \Omega^{k_2}_{+}.Y_a = \Omega^{k_2}_{+}.Z_c = 0 \tag{5.13}\]

by lemma 3. Now,

\[\Omega^p_{+}.[[c(X_a), Y_b], Z_c] = \sum_{l=0}^{p} \sum_{m=0}^{l} \binom{p}{l} \binom{l}{m} W_{pabclm}\]

where \(W_{pabclm} = [[\Omega^p_{+}.c(X_a), \Omega^{l-m}_{+}.Y_b], \Omega^{m-l}_{+}.Z_c]\). It then follows from (5.13) that \(W_{pabclm} = 0\) if \(m \geq k_2 + 2\) or \(l \geq k_2\). This implies that \(W_{pabclm} = 0\) unless \(p < l + k_2 < m + k_2 < k_3 + k_2 + k_2 + 2\).

But this can never be satisfied if \(p = k_3 + k_2 + k_2\) and so we arrive at \(\Omega^{k_2}_{+}.[[c(X_a), Y_b], Z_c] = 0\). But \(k_3 + k_2 + k_2 \leq a + b + c\) by (5.11) and hence it follows that \(\Omega^{k_2}_{+}.[[c(X_a), Y_b], Z_c] = 0\). But then lemma 3 implies that \(\Omega^{k_2}_{+}.[[c(X_a), Y_b], Z_c] = 0\) and hence \([c(X_a), Y_b], Z_c] \in E^{(a+b+c)}\).

**Proposition 3.** If \(X_a \in E^a\) and \(Y_b \in E^b\) then 

\([c(X_a, c(Y_b)), \Omega_+], [[\Omega_+, X_a], Y_b], [[\Omega_, X_a], Y_b] \in E^{(a+b)}\).

**Proof.** This proposition can be proved using the same techniques as proposition 2.

**Lemma 4.** If \(l \in \mathcal{E}\) and \(Z \in E^l_+ \oplus E^{l-1}_+\) then \(\Omega^{l+1}_{+}.c(Z) = l(l+1)\Omega^{l-1}_{+}Z\).

**Proof.** Since \(Z \in E^l_+ \oplus E^{l-1}_+\) there exist real constants \(a^s_q, b^s_q\) such that

\[Z = \sum_{q=1}^{i-1} \sum_{s=0}^{m_q-1} \left( a^s_q \xi^q_{s+1} + b^s_q \right) X^q_s + \sum_{s=0}^{m_{l-1}} a^s_l X^l_s \tag{5.14}\]

where \(X^q_s = \xi^q_{k_q-1}\) if \(k_q\) is odd and \(X^q_s = \xi^q_{k_q+1}\) if \(k_q\) is even. But

\[\Omega^{l-1}_{+}.X^q_s = 0 \quad \text{for}\ q \leq \tilde{l} - 1 \tag{5.15}\]

by lemma 4, and so we get

\[\Omega^{l-1}_{+}.Z = \sum_{s=0}^{m_{l-1}} a^s_l \Omega^{l-1}_{+}X^l_s \tag{5.16}\]

Now, \(c(X^q_s) = \xi^q_{k_q+1}\) if \(k_q\) is odd and \(c(X^q_s) = \xi^q_{k_q+1}\) if \(k_q\) is even by proposition 2 and so

\[\Omega^{l+1}_{+}.c(X^q_s) = k_q(k_q + 1)X^q_s \tag{5.17}\]

Since \(l \in \mathcal{E}\) implies that \(k_q = l\), it follows easily form (5.14), (5.15), and (5.17) that

\[\Omega^{l+1}_{+}.c(Z) = l(l+1) \sum_{s=0}^{m_{l-1}} a^s_l \Omega^{l-1}_{+}X^l_s \tag{5.18}\]

Comparing (5.16) and (5.18), we see that \(\Omega^{l+1}_{+}.c(Z) = l(l+1)\Omega^{l-1}_{+}Z\).
**Proposition 4.** If \( Z_l \) with \( l = 0, 1, \ldots k \) is a sequence of vectors such that
\[
Z_0 = \Omega_+ , \quad Z_l \in E^l \quad l = 1, 2, \ldots k \quad \text{and} \quad Z_l \in E^l_+ \oplus E^{l-1}_- \quad \text{if} \ l \in E
\]
then for any \( j = 1, \ldots , l \)
\[
\sum_{s=1}^{k-1} [c(Z_{j-s}), Z_s], \Omega_+ ] \in \begin{cases} E^k & \text{if} \ k \notin E \\ E^{k-1} & \text{if} \ k \in E \end{cases}.
\]

**Proof.** Suppose \( Z_l \) is as in the hypotheses of the proposition, then
\[
\Omega^{k_l+2} c(Z_l) = \Omega^{k_l} Z_l = 0.
\] (5.19)

Now,
\[
-\Omega^p \sum_{s=1}^{k-1} [c(Z_{k-s}), \Omega_+ ] \sum_{l=0}^{k-1} \sum_{s=1}^{k-1} \left( \frac{p+1}{l} \right) W_{pkls}
\] (5.20)
where \( W_{pkls} = [\Omega^l_+, c(Z_{k-s}), \Omega^{l+1-l}_+, Z_s] \). From (5.19) we see that \( W_{pkls} = 0 \) if \( l \geq k_{(k-s)} \) or \( p + 1 - l \geq k_s \). Thus
\[
W_{pkls} = 0 \quad \text{unless} \quad p + 1 - l + k_s < k_{s} + k_{(k-s)} + 2.
\] (5.21)

Now, suppose \( p = k \). Then using the fact that \( k_s \leq s \) and \( k_{(k-s)} \leq k - s \), we get from (5.21) that \( W_{pkls} = 0 \) unless \( k + 1 < l + k_s < k + 2 \). Since this is impossible to satisfy \( W_{pkls} = 0 \) for all \( l, s \). Thus the sum (5.21) vanishes, i.e. \( -\Omega^k \sum_{s=1}^{k-1} [c(Z_{k-s}), \Omega_+ ] = 0 \), and we get
\[
\sum_{s=1}^{k-1} [c(Z_{k-s}), \Omega_+ ] \in E^k
\] (5.22)
by lemmas 2 and 3. Suppose further that \( k \in E \) and let \( p = k - 1 \). Then \( a_{pkls} = 0 \) unless \( k < l + k_s < k_s + k_{(k-s)} + 2 \) by (5.22). Now, \( k_s \leq s \) and \( k_{(k-s)} \leq k - s \), so suppose \( k_s < s \) or \( k_{(k-s)} < k - s \). Then \( k_{(k-s)} + k_s < k + s + s = k \) which will make the inequality \( k < l + k_s < k_s + k_{(k-s)} + 2 \) impossible to satisfy. Therefore we see that \( W_{pkls} = 0 \) unless \( k_{(k-s)} = k - s \) and \( k_s = s \) (i.e. \( k - s, s \in E \)). However, if \( k_{(k-s)} = k - s \) and \( k_s = s \), then \( k < l + k_s < k_s + k_{(k-s)} + 2 \) will be satisfied only if \( l + s = 1 \). So \( W_{pkls} = 0 \) unless \( k - s, s \in E \) and \( l + s = 1 \). This allows us to write the sum (5.20) as
\[
-\Omega^{k-1} \sum_{s=1}^{k-1} [c(Z_{k-s}), Z_s], \Omega_+] \\
= \sum_{s=1}^{k-1} \left( \frac{k}{k-s+1} \right)(k-s)(k-s+1)\Omega^{k-s-1} Z_{k-s}, \Omega^{s-1}_+, Z_s] \\
= \sum_{s=1}^{k-1} \left( \frac{k!}{(k-s-1)!(s-1)!} \right) \Omega^{k-s-1} Z_{k-s}, \Omega^{s-1}_+, Z_s].
\]
Assume now that \( k \) is odd. Then we can write the above sum as

\[
-\Omega_k^{-(k-1)} \sum_{s=1}^{k-1} [c(Z_{k-s}), Z_s], \Omega_+ = \sum_{s=1}^{k-1} \frac{k!}{(k-s-1)! (s-1)!} [\Omega_+^{k-s-1} Z_{k-s}, \Omega_+^{s-1} Z_s]
\]

\[
+ \sum_{s=1}^{k-1} \frac{k!}{(k-s-1)! (s-1)!} [\Omega_+^{k-s-1} Z_{k-s}, \Omega_+^{s-1} Z_s]
\]

\[
= \sum_{s=1}^{k-1} \frac{k!}{(k-s-1)! (s-1)!} \{[\Omega_+^{k-s-1} Z_{k-s}, \Omega_+^{s-1} Z_s] + [\Omega_+^{s-1} Z_s, \Omega_+^{k-s-1} Z_{k-s}]\} = 0.
\]

Similar arguments show that \(-\Omega_k^{-(k-1)} \sum_{s=1}^{k-1} [c(Z_{k-s}), Z_s], \Omega_+ = 0\) if \( k \) is even. Therefore \( \sum_{s=1}^{k-1} [c(Z_{j-s}), Z_s], \Omega_+ \in E^{k-1} \) by lemmas \( 2 \) and \( 3 \) and \( (5.12) \).

**Proposition 5.** If \( Z_l \) with \( l = 0, 1, \ldots k \) is a sequence of vectors such that

\[
Z_0 = \Omega_+, \quad Z_l = E_l \quad l = 1, 2, \ldots k \quad \text{and} \quad Z_l \in E_+ \oplus E_l \quad \text{if} \ l \in \mathcal{E}
\]

then, for any \( j = 1, \ldots, l \)

\[
\sum_{s=1}^{k-1} \sum_{j=1}^{s} [c(Z_{j-s}), Z_s], Z_k = \begin{cases} E_k & \text{if} \ k \notin \mathcal{E} \\ E^{k-1} & \text{if} \ k \in \mathcal{E} \end{cases}
\]

**Proof.** Proved using similar arguments as for proposition \( 4 \). \( \square \)

### 6 Local Existence Proofs

For \( q = 1, 2, \ldots J \) let

\[
\begin{align*}
\text{pr}_q^+: V_2 \to E_+^q, & \quad \text{pr}_0^+: V_2 \to E_0^q, & \quad \text{and} \quad \text{pr}_q: V_2 \to E_0^q \oplus E_+^q
\end{align*}
\]

(6.1)

denote the projections determined by the decomposition \( (5.8), (5.9) \) of \( V_2 \).

#### 6.1 Solutions regular at the origin

**Theorem 2.** Fix \( X \in E_+ \) and \( \Omega_+ \in E_+ \) that satisfies \( [\Omega_+, \Omega_-] = \Lambda_0 \) where \( \Omega_- := -c(\Omega_+) \). Then there exist a unique solution \( \{\Lambda_+(r, Y), m(r, Y)\} \) to the system of differential equations \( (2.13) \) and \( (2.14) \) that is analytic in a neighborhood of \( (r, Y) = (0, X) \) in \( \mathbb{R} \times E_+ \) and satisfies \( m = O(\ell^4) \) and

\[
\text{pr}_+^+(\Lambda_+ - \Omega_+) = Y_s r^{s+1} + O(r^{s+2}), \quad \text{pr}_0^+(\Lambda_+) = O(r^{s+2}) \quad \forall s \in \mathcal{E}
\]

where \( \Omega_+ := \Lambda_+(0) \) and \( Y_s := \text{pr}_+^+(Y) \). Moreover, these solutions also satisfy \( P = O(\ell^2) \) and \( G = O(\ell^2) \).

**Proof.** Introduce new variables \( \{u_+^s, u_0^s \mid s \in \mathcal{E}\} \) via

\[
\begin{align*}
u_+^s :& = \text{pr}_+^+(\Lambda_+ - \Omega_+) r^{-s-1} \quad \text{and} \quad u_0^s : = \text{pr}_0^+(\Lambda_+) r^{-s-2}
\end{align*}
\]

(6.2)

where \( \Omega_+ := \Lambda_+(0) \). This allows us to write \( \Lambda_+ \) as

\[
\Lambda_+(r) = \Omega_+ + \sum_{s \in \mathcal{E}} (u_+^s(r) + r u_0^s(r)) r^{s+1}.
\]

(6.3)

But regularity at \( r = 0 \) requires that \( \Omega_+ \) satisfy \( [\Omega_+, \Omega_-] = \Lambda_0 \) where \( \Omega_- := -c(\Omega_+) \). This can be seen easily from \( (2.16), (2.24), (2.25) \) and the requirement that the pressure remains finite at \( r = 0 \).
Lemma 5. For every $s \in \mathcal{E}$ there exists analytic maps $F_1^s : E_+ \to E_0^0 \oplus E_0^4$ and $F_2^s : E_0 \times E_+ \times \mathbb{R} \to E_0^0 \oplus E_0^4$ such that, for $F$ given in (2.17), $pr^s F = -s(s + 1)u_r^s r^{s+1} + r^{s+2} F_1^s(u^r) + r^{s+3} F_2^s(u^0, u^r, r)$ where

$$u^0 := \sum_{a \in \mathcal{E}} u^0_a \quad \text{and} \quad u^r := \sum_{a \in \mathcal{E}} u^r_a.$$  \hfill (6.4)

Proof. Let $u_s = ru^0_s + u^r_s$. Then from (2.17) we find

$$F = \sum_{a \in \mathcal{E}} A_2(u_a) r^{a+1} + \frac{1}{2} \sum_{a,b \in \mathcal{E}} \left([[u_a, c(u_b)], \Omega_+] \right) + \left([[\Omega_-, c(u_a)], u_b] \right) + \left([[\Omega_-, u_a], u_b] \right)) r^{a+b+2}$$

$$+ \frac{1}{2} \sum_{a,b,c \in \mathcal{E}} [[u_a, c(u_b)], u_c] r^{a+b+c+3}.$$  \hfill (6.5)

But from (6.4) we see that

$$A_2(u_a) = a(a + 1)u^r_a.$$  \hfill (6.6)

Also, for $a \in \mathcal{E}$ we have $k_a = a$ by lemma 5.11, and hence

$$\tilde{a} \leq \tilde{b} \iff a \leq b.$$  \hfill (6.7)

Using the two results (2.6) and (6.6), we get

$$pr^s F = -s(s + 1)u^s_r r^{s+1} + \frac{1}{2} \sum_{a,b \in \mathcal{E}} pr^s \left([[u_a, c(u_b)], \Omega_+] \right) + \left([[\Omega_-, c(u_a)], u_b] \right) + \left([[\Omega_-, u_a], u_b] \right)) r^{a+b+2}$$

$$+ \frac{1}{2} \sum_{a,b,c \in \mathcal{E}} pr^s [[u_a, c(u_b)], u_c] r^{a+b+c+3}$$

by propositions 2 and 3. Substituting $u_a = ru^0_a + u^r_a$ into the above expression completes the proof. \hfill \Box

For every $s \in \mathcal{E}$ define

$$v^+_s := u^+_s \quad \text{and} \quad v^0_s := (ru^0_s)'.$$  \hfill (6.8)

Lemma 6. There exists analytic functions $\hat{P} : E_0 \times E_+ \times \mathbb{R} \to \mathbb{R}$ and $\hat{G} : E_0 \times E_0 \times E_+ \times E_+ \times \mathbb{R} \to \mathbb{R}$ such that $P = r^4 \|u^+_r\|^2 + r^5 \hat{P}(u^0, u^r, r)$ and $G = r^2 \|u^+_r\|^2 + r^3 \hat{G}(u^0, v^0, u^r, v^r, r)$ where

$$v^0 := \sum_{a \in \mathcal{E}} v^0_a \quad \text{and} \quad v^r := \sum_{a \in \mathcal{E}} v^r_a.$$  \hfill (6.9)

and $u^0$, $u^r$ are defined by (6.4).

Proof. The existence of the analytic function $\hat{G}$ follows easily from the definition (2.15) of $G$ and equations (6.4) and (6.8).

Similarly from the definition (2.15) of $P$ we have $P = r^4 \|u^+_r\|^2 + r^5 \hat{P}(u^0, u^r, r) + r^5 Q(u^0, u^r, r)$ where $Q$ is a polynomial in $r$, $u^0$ and $u^r$. Now, using (2.23), (3.14), and $A_2(u^+_r) = 2u^+_r$, it is not difficult to show that $\frac{1}{8}\|\Omega_+, c(u^+_r)\| + \|\Omega_-, c(u^+_r)\| = \|u^+_r\|$ and this completes the proof. \hfill \Box
From (5.3)–(5.7) we get $A_2(u_0^+) = s(s+1)u_0^+$ and $A_2(u_0) = 0$ since $u_0^+ \in E_0^+$ and $u_0^0 \in E_0^0$. Using this result, lemma 3 and equations (6.4) and (6.8), the field equations (2.11) and (2.13) can be written as

\[
rv_s' = rv_s', \tag{6.9}
\]

\[
rv_s' = -2(s+1)v_s^+ - \frac{2}{rN} \left( m - \frac{1}{rP} \right) v_s^+ - s(s+1) \left( \frac{1}{N} - 1 \right) u_s^+ + \frac{2(s+1)}{r^2N} \left( m - \frac{1}{rP} \right) u_s^+ - \frac{r}{Npr_s^0F_s^1(u^+, u^+, r)} - \left( \frac{1}{N} - 1 \right) pr_s^0F_s^1(u^+) - pr_s^0F_s^1(u^+), \tag{6.10}
\]

\[
ru_s' = -u_s^0 + v_s^0, \tag{6.11}
\]

\[
rv_s' = -2(s+1)v_s^0 - s(s+1)u_s^0 - \frac{2}{N} \left( m - \frac{1}{rP} \right) v_s^0 \tag{6.12}
\]

\[
\frac{2(s+1)}{rN} \left( m - \frac{1}{rP} \right) v_s^0 - \frac{r}{Npr_s^0F_s^2(u^0, u^+, r)} - \left( \frac{1}{N} - 1 \right) pr_s^0F_s^1(u^+) - pr_s^0F_s^1(u^+).
\]

where $s \in \mathcal{E}$. For every $s \in \mathcal{E}$, introduce two new variables

\[
x_s := -(s+1)u_0^0 - \frac{s+1}{s} v_s^0, \quad y_s := (s+1)u_s^0 + v_s^0,
\]

and define

\[
f_s := -\frac{2}{N} \left( m - \frac{1}{rP} \right) v_s^0 - \frac{2(s+1)}{rN} \left( m - \frac{1}{rP} \right) u_s^0 - \frac{r}{Npr_s^0F_s^0(u^0)} - \left( \frac{1}{N} - 1 \right) pr_s^0F_s^1(u^+). \tag{6.13}
\]

Then equations (6.11) and (6.12) can be written as

\[
rx_s = -(s+2)x_s + \frac{s+1}{s} pr_s^0F_s^1(u^+) + \frac{s+1}{s} f_s, \tag{6.14}
\]

\[
ry_s = -(s+2)y_s - pr_s^0F_s^1(u^+)f_s, \tag{6.15}
\]

for every $s \in \mathcal{E}$. Define $\mu = r^{-3} \left( m - r^3 \|u_1^+\|_2^2 \right)$. Then the mass equation (2.14) becomes

\[
ry' = -3\mu + r \left\{ \hat{P}(u^0, u^+, r) + \hat{G}(u^0, v^0, u^+, v^+, r) - 2\langle |u_1^+| v_1^+ \rangle \right\} - 2r \left( \mu + \|u_1^+\|^2 \right) \left( 2 \|u_1^+\|^2 + rG(u^0, v^0, u^+, v^+, r) \right). \tag{6.15}
\]

For every $s \in \mathcal{E}$, introduce one last change of variables

\[
\hat{v}_s^+ := v_s^+ + \frac{1}{2(s+1)} pr_s^0F_s^1(u^+), \quad \hat{x}_s := x_s - \frac{s+1}{s(s+2)} pr_s^0F_s^1(u^+), \quad \text{and} \quad \hat{y}_s := y_s + \frac{1}{(s+1)} pr_s^0F_s^1(u^+).
\]

Let $\hat{v}^+ := \sum_{s \in \mathcal{E}} v_s^+$, $\hat{x} := \sum_{s \in \mathcal{E}} x_s$, $\hat{y} := \sum_{s \in \mathcal{E}} y_s$ and $\eta(r) := (\hat{x}(r), \hat{y}(r), u^+(r), \hat{v}^+(r), \mu(r), r)$. Fix $X \in E_+$ and let $N_X$ be a neighborhood of $X$ in $E_+$. Define a set $D(N_X, \epsilon) \equiv D(N_X, \epsilon) := E_0 \times E_0 \times N_X \times E^+ \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$. Then using lemmas 5 and 3 and equations (6.9), (6.10), (6.13), (6.14), and (6.15) one can...
show that there exists an $\epsilon > 0$ and analytic maps $U_\epsilon, V_\epsilon : D(N_X, \epsilon) \to E^+_\epsilon$, $X_\epsilon, Y_\epsilon : D(N_X, \epsilon) \to E^0_\epsilon$, and $M : D(N_X, \epsilon) \to \mathbb{R}$ such that for all $s \in \mathcal{E}$

\[ ru^t_s(r) = \rho U_\epsilon(\eta(r)), \quad r\tilde{v}^t_s(r) = -(s + 1) \tilde{\tau}_s(r) + rV_\epsilon(\eta(r)), \quad r\tilde{\tau}_s(r) = -(s + 1) \tilde{\tau}_s(r) + rY_\epsilon(\eta(r)), \]

and $r\tilde{\mu}'(r) = -3\tilde{\mu}(r) + r\tilde{M}(\eta(r))$. This system of differential equations is in the form to which theorem 1 applies. Applying this theorem shows that for fixed $X \in E^+_\epsilon$ there exist a unique solution \{u^+_s(r, Y), \tilde{\tau}_s(r, Y), \tilde{\mu}(r, Y)\} that is analytic in a neighborhood of $(r, Y) = (0, X)$ and that satisfies $u^+_s(r, Y) = Y_\epsilon + O(r), \tilde{\tau}_s(r, Y) = O(r), \tilde{\mu}(r, Y) = O(r)$, and $\mu_s(Y) = O(r)$ where $Y_\epsilon = pr^+_\epsilon(Y)$. From these results it is not hard to verify that $m(r) = O(r^3)$ and $w^0(r) = O(r^0)$. Also, it is clear that $P = O(r^4)$ and $G = O(r^5)$ by lemma 4.

**Theorem 3.** Every solution from theorem 2 satisfies equation (2.14) on a neighborhood of $r = 0$.

**Proof.** Let \{\Lambda_+(r), m(r)\} be a solution of the equations (2.11) and (2.13) on a neighborhood $\mathcal{N}$ of $r = 0$, which we know exists by theorem 2. From theorem 2 it is clear that $\Lambda_+'(0) = 0$, and this implies that $\gamma(0) = 0$ where $\gamma(r)$ is defined in lemma 2. Also, because $m = O(r^3)$ and $P = O(r^4)$ for these solutions we see, by shrinking $\mathcal{N}$ if necessary, that the function $f(r) = -2(r^2 N)^{-1}(m - \frac{1}{2}P)$ is analytic on $\mathcal{N}$. From lemma 2 $\gamma$ satisfies the differential equation $\gamma' = f(r)\gamma$. Solving this equation we find $\gamma(r) = \gamma(0) \exp(\int_0^r f(\tau)d\tau)$ for all $r \in \mathcal{N}$. But $\gamma(0) = 0$, so $\gamma(r) = 0$ for all $r \in \mathcal{N}$.

### 6.2 Asymptotically flat solutions

In proving that local solutions exist near $r = 0$, we were able to “guess” the appropriate transformations needed to bring the field equations (2.11) and (2.13) in to a form for which theorem 1 applies. Near, $r = \infty$ the equations become much more difficult to analyze and guessing the appropriate transformation is no longer possible. Instead, we will show that the field equations (2.11) and (2.13) admit a formal power series solutions about the point $r = \infty$. This formal power series solution will then be used to construct a transformation to bring the equations (2.11) and (2.13) in to a form for which theorem 1 applies.

Let $z = \frac{1}{r}$, and define

\[ \overset{\circ}{f} = \frac{df}{dz} \]

for any function $f$. Then the field equations (2.11) and (2.13) can be written as

\[ z^2 \overset{\circ}{m} + NG + z^2 P = 0, \quad \overset{\circ}{z}^2 N \overset{\circ}{\Lambda}_+ + 2z(1 - 3zm + z^2 P) \overset{\circ}{\Lambda}_+ + \overset{\circ}{f} = 0. \]

Assume a powerseries expansion of the form

\[ \Lambda_+ = \sum_{k=0}^{\infty} \Lambda_{+,k} z^k \quad \text{and} \quad m = \sum_{k=0}^{\infty} m_k z^k. \]

We will define $\Lambda_{-,k} := -c(\Lambda_{+,k})$ and $\Omega_+ := \Lambda_{+,0}$. From the requirement that the total magnetic charge vanish we have that $[\Omega_+, \Omega_-] = \Lambda_0$. Substituting the powerseries (5.18) in the equations (6.16) and (6.17) yields the recurrence equations

\[ m_1 = m_2 = 0, \quad m_k = \frac{1}{k} \left( -G_{k-3} + 2 \sum_{j=0}^{k-4} m_j G_{k-4-j} - P_{k-1} \right) \quad k = 3, 4, 5, \ldots \]

\[ A_2(\Lambda_{+,k}) - k(k + 1)\Lambda_{+,k} = h_k + f_k \quad k = 1, 2, 3, \ldots \]
It is clear from (6.20) that $m$ satisfies

$$\sum_{j=0}^{k} (j+1)(k+1-j) \langle \Lambda_{+,k+1-j} | \Lambda_{+,j+1} \rangle \quad k \geq 0,$$

(6.21)

$$\hat{F}_0 := 0, \quad \hat{F}_k := \frac{1}{2} \sum_{j=1}^{k} \sum_{s=0}^{j} [\Lambda_{-,j-s}, \Lambda_{+,s}], \Lambda_{+,k-j} \quad k \geq 1,$$

(6.22)

$$P_0 = P_1 = 0, \quad P_k := \frac{1}{2} \sum_{j=1}^{k-1} \langle \langle \hat{F}_j | \hat{F}_{k-j} \rangle \rangle \quad k \geq 2,$$

(6.23)

$$h_1 = 0, \quad h_k := 2 \sum_{j=0}^{k-1} (k-j-1) (P_{j-2} - (k-j+1)m_j) \Lambda_{+,k-j-1} \quad k \geq 2,$$

(6.24)

and

$$f_1 = 0, \quad f_k := \frac{1}{2} \left\{ \sum_{j=1}^{k-1} \sum_{s=0}^{j} [\Lambda_{-,j-s}, \Lambda_{+,s}], \Lambda_{+,k-j} + \sum_{s=1}^{k-1} [\Lambda_{-,k-s}, \Lambda_{+,s}], \Omega_+ \right\} \quad k \geq 2.$$  

(6.25)

Note that with these definitions that $\hat{F} = \sum_{k=0}^{\infty} \hat{F}_k z^k$ and $P = \sum_{k=0}^{\infty} P_k z^k$ while $G = \sum_{k=0}^{\infty} G_k z^{k+4}$.

**Theorem 4.** Fix $X \in E_+$ and $m_\infty \in \mathbb{R}$. Then there exists a unique solution $\{\Lambda_{+,k}, m_k\}_{k=0}^{\infty}$ to the recurrence equations (6.19) and (6.20) that satisfies

$$m_0 = m_\infty, \quad m_1 = m_2 = 0 ; \quad pr_{+}^{k} \Lambda_{+,k} = X_k \quad \forall k \in \mathcal{E}$$

(6.26)

and

$$\Lambda_{+,k} \in \begin{cases} E_{+}^k & \text{if } k \notin \mathcal{E} \\ E_{+}^k \oplus E_{-}^{k-1} & \text{if } k \in \mathcal{E} \end{cases}.$$  

(6.27)

where $X_k := pr_{+}^{k} X$.

**Proof.** Fix $X \in E_+$, $m_\infty \in \mathbb{R}$, and let $X_k = pr_{+}^{k} X$ for all $k \in \mathcal{E}$. We will use induction to prove that the recurrence equations (6.19) and (6.20) can be solved. When $k = 1$, the equations (6.19) and (6.20) reduce to $m_1 = 0$ and $A_2(\Lambda_{+,1}) - 2\Lambda_{+,1} = 0$. This can be solved in $E_0 \oplus E_1^k$ by letting $\Lambda_{+,1} = X_1$. Note that since min $\mathcal{E} = 1$, we have $1 = 1$.

We now assume that for $k \leq l$, $\{\Lambda_{+,k}, m_k\}$ is a solution to the recurrence equations (6.19) and (6.20) that satisfies

$$\Lambda_{+,k} \in \begin{cases} E_{+}^k & \text{if } k \notin \mathcal{E} \\ E_{+}^k \oplus E_{-}^{k-1} & \text{if } k \in \mathcal{E} \end{cases}.$$  

It is clear from (6.20) that $m_{l+1}$ is then determined. From (6.21)-(6.27) and propositions [3] and [3] it follows that

$$h_{l+1} + f_{l+1} \in \begin{cases} E_{+}^{l+1} & \text{if } l + 1 \notin \mathcal{E} \\ E_{+}^l & \text{if } l + 1 \in \mathcal{E} \end{cases}.$$  

(6.28)
Suppose $l + 1 \notin \mathcal{E}$. Then $A_2 - (l + 1)(l + 2)\mathbb{I}$ is invertible and
\[\Lambda_{+1} = (A_2 - (l + 1)(l + 2)\mathbb{I})^{-1}(h_{l+1} + f_{l+1}).\]
But then (6.28) implies that $\Lambda_{+1} \in E_{l+1}^f$.
Alternatively, suppose $l + 1 \in \mathcal{E}$. Then $\ker(A_2 - (l + 1)(l + 2)\mathbb{I}) = E_{l+1}^f$ by (6.25) and (5.12). Therefore, (6.28) shows that
\[\Lambda_{+1} = ((A_2 - (l + 1)(l + 2)\mathbb{I}))[E^f]^{-1}(h_{l+1} + f_{l+1}) + X_{l+1}\]
solves (6.29) since $X_{l+1} \in \ker(A_2 - (l + 1)(l + 2)\mathbb{I})$. It also clear from (6.28) and $X_{l+1} \in E_{l+1}^f$ that
\[\Lambda_{+1} \in E_{l+1}^f \oplus E_{l+1}^f.\]
That prove that $\Lambda_{+1} \in E_{l+1}^f$ satisfies the induction hypothesis and so the proof is complete.

**Theorem 5.** Fix $X \in E_+$, $m_\infty > 0$, and $\Omega_+ \in E_+$ that satisfies $[\Omega_+ , \Omega_-] = 0$ where $\Omega_- := -e(\Omega_+)$. Then there exist a unique solution $\{\Lambda_+(r, a, Y), m(r, a, Y)\}$ to the system of differential equations (2.11) and (2.13) that is analytic in the variables $(r^{-1}, a, Y)$ in a neighborhood of $(0, m_\infty, X) \in \mathbb{R}^2 \times E_+$ and satisfies
\[m = a + O(r^{-3})\]
and
\[pr_0^\delta(\Lambda_+ - \Omega_+) = \frac{Y_s}{r^3} + O(r^{-s}), \quad pr_0^\delta(\Lambda_+) = O(r^{-s}), \quad \forall s \in \mathcal{E}\]
where $\Omega_+ := \Lambda_+(0)$ and $Y_s := pr_0^\delta(Y)$.

**Proof.** Fix $X \in E_+$ and let $\Lambda_{+, k} = \Lambda_{+, k}(X, m_\infty)$ and $m_k = m_k(X, m_\infty)$ be solutions to the recurrence equations (6.19) and (6.20) which satisfy $m_0 = m_\infty$, $m_1 = m_2 = 0$, $pr_k^\delta \Lambda_{+, k} = X_k$ for all $k \in \mathcal{E}$, and
\[\Lambda_{+, k} \in \begin{cases} \bigoplus_{q=1}^k E_0^q \oplus E_+^q & \text{if } k \notin \mathcal{E} \\ E_0^k \oplus \bigoplus_{q=1}^{k-1} E_0^q \oplus E_+^q & \text{if } k \in \mathcal{E} \end{cases},\]
where $X_k = pr_k^\delta X$. Define $U := \sum_{k=0}^n \Lambda_{+, k} z^k$ and $M := \sum_{k=0}^n m_k z^k$ and introduce new variables $\phi(z)$ and $\sigma(z)$ via
\[\Lambda_+ = U + z^{n-3} \phi \quad \text{and} \quad m = M + z^{n-3} \sigma,\]
where the integer $n$ is to be chosen later. Define
\[N_p := 1 - 2Mz, \quad \tilde{F}_p := \tilde{F}(\Lambda_0 + [U, c(U)]),\]
\[F_p := -i[\tilde{F}_p, U], \quad P_p := \frac{1}{2}\|Fh_p\|^2, \quad \text{and} \quad G_p := \frac{1}{2}z^3||U||^2.\]
From these definitions it is clear that the quantities $N_p, \tilde{F}_p, F_p, P_p$ and $G_p$ are all polynomial in the variables $X$ and $m_\infty$. Now, because $U$ and $M$ are the first $n$ terms in the powerseries solution to the field equations (6.16) and (6.17) about the point $z = 0$ they satisfy
\[z^2 N_p \tilde{U} + 2z(1 - 3M + z^2 P_p)U + \mathcal{F}_p = z^{n-1}(a_1(X, m_\infty) + a_2(X, m_\infty)g),\]
\[z^2 M + N_p G_p + z^2 P_p = z^n b(X, m_\infty),\]
\[z^2 \mathcal{F}_p + 2z(1 - 3M + z^2 P_p)F_p + 2z^2 P_p G_p + z^2 G_p = z^n \tilde{b}(X, m_\infty),\]
where \( a_1, a_2 : V_2 \times \mathbb{R} \to \mathbb{R} \) and \( b : V_2 \times \mathbb{R} \to \mathbb{R} \) are polynomial in their variables.

From (6.32) and (2.17) it follows that \( \mathcal{F} = \mathcal{F}_p - z^n A(\phi) + z^{-2} \sum_{j=1}^3 \mathcal{F}_{R,j}(\phi, X, m_\infty, z) \) where \( \mathcal{F}_{R,j} : V_2 \times E_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) \( j = 1, 2, 3 \) are analytic maps that satisfy \( \mathcal{F}_{R,j}(\epsilon Y_1, Y_2, x_1, x_2) = \epsilon \mathcal{F}_{R,j}(Y_1, Y_2, x_1, x_2) \) for all \( \epsilon \in \mathbb{R} \). It is also not difficult to see from (6.32) and (2.17) that

\[
G = \frac{z^4}{2} \|U\|^2 + z^4 (\langle U \rangle (n-3)z^{n-4}\phi + z^{-3}\phi)) + \frac{1}{2}((n-3)z^{n-4}\phi + z^{-3}\phi)
\]

and \( P = P_p + z^{n-2} \sum_{j=1}^4 P_{R,j}(\phi, X, m_\infty, z) \) where \( P_{R,j} : V_2 \times E_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) \( j = 1, 2, 3, 4 \) are analytic maps that satisfy \( P_{R,j}(\epsilon Y_1, Y_2, x_1, x_2) = \epsilon P_{R,j}(Y_1, Y_2, x_1, x_2) \) for all \( \epsilon \in \mathbb{R} \). Note also that \( N = N_p - 2z^{n-2}\sigma \).

Let

\[
\omega := \phi \quad \text{and} \quad \theta := z^{-1}\phi.
\]

Using the above results, straightforward calculation shows that there exists analytic maps \( \mathcal{G} : V_2 \times V_2 \times E_+ \times \mathbb{R}^3 \to V_2 \) and \( \mathcal{M} : V_2 \times V_2 \times E_+ \times \mathbb{R}^3 \to \mathbb{R} \) such that equation (6.16) and (6.17) can be written as

\[
N(\omega - 2(n-3)\omega + (n-3)(n-4)\theta + 2(n-3)\theta + 2\omega - A_2(\theta) - z\mathcal{G}(\theta, \omega, X, m_\infty, \sigma, z) = 0.
\]

We can rewrite (6.36) as

\[
z \hat{\omega} = -2(n-2)\omega + (A_2 - (n-3)(n-2)\mathbb{1}) \theta + z \hat{\mathcal{G}}(\theta, \omega, X, m_\infty, \sigma, z)
\]

where

\[
\hat{\mathcal{G}}(Y_1, Y_2, Y_3, x_1, x_2, x_3) = \frac{1}{x_3} \left( \frac{1}{1 - 2(\mathcal{M}(Y_3, x_1) + x_3^{-1}x_2) x_3} - 1 \right) (2(n-3)Y_1 + 2Y_2 - A_2(Y_1)) + \mathcal{G}(Y_1, Y_2, Y_3, x_1, x_2, x_3).
\]

Because \( \mathcal{G} \) is analytic, it is clear that \( \hat{\mathcal{G}} \) is analytic in a neighborhood of \((0, 0, X, m_\infty, 0, 0) \in V_2 \times V_2 \times E_+ \times \mathbb{R}^3 \).

For \( Y \in V_2 \) and \( s \in \mathcal{E} \), define \( Y_+^s := \text{pr}_+^s Y \) and \( Y_0^s := \text{pr}_0^s Y \). Recalling that \( \text{pr}_0^s A_2 = A_2 \text{pr}_0^s = 0 \) and \( \text{pr}_+^s A_2 = A_2 \text{pr}_+^s = s(s + 1)\text{pr}_+^s \) for every \( s \in \mathcal{E} \), we can write (6.36) and (6.37) as

\[
z \tilde{\omega}_+^s = -\theta_+^s + \omega_+^s,
\]

\[
z \tilde{\omega}_0^s = -2(n-2)\omega_+^s + (s(s+1) - (n-3)(n-2))\theta_+^s + z \tilde{\mathcal{G}}_+^s(\theta, \omega, X, m_\infty, \sigma, z),
\]

and

\[
z \tilde{\omega}_0^s = -2(n-2)\omega_+^s - (n-3)(n-2)\theta_0^s + z \tilde{\mathcal{G}}_0^s(\theta, \omega, X, m_\infty, \sigma, z),
\]

for all \( s \in \mathcal{E} \). For every \( s \in \mathcal{E} \), introduce one last change of variables

\[
\zeta_+^s := -\theta_0^s - \omega_0^s, \quad \zeta_0^s := (n-2)\theta_0^s + \omega_0^s,
\]

\[
\eta_+^1 := \frac{1}{2s+1}((s-n+3)\theta_0^s - \omega_0^s), \quad \eta_0^1 := \frac{1}{2s+1}((s+n-2)\theta_0^s + \omega_+^s).
\]
and let \( \zeta_j := \sum_{s \in \mathcal{E}} \zeta_j^s \) and \( \eta^j := \sum_{s \in \mathcal{E}} \eta^j_s \) \( j = 1, 2 \). Using this transformation we see from the above results that (6.33)--(6.41) can be written as

\[
\begin{align*}
\dot{z}_s^j &= -(n-j)z_s^j + z\mathcal{K}_s^j(\zeta^1, \zeta^2, \eta^1, \eta^2, X, \sigma, z) \\
\dot{\eta}_s^j &= -(n-(-1)^j s-j)\eta_s^j + z\mathcal{H}_s^j(\zeta^1, \zeta^2, \eta^1, \eta^2, X, \sigma, z)
\end{align*}
\]

\( \forall s \in \mathcal{E}, j = 1, 2 \), \(6.42\)
ung

\[
\dot{z}_s^j = -(n-j)z_s^j + z\mathcal{K}_s^j(\zeta^1, \zeta^2, \eta^1, \eta^2, X, \sigma, z) \\
\dot{\eta}_s^j = -(n-(-1)^j s-j)\eta_s^j + z\mathcal{H}_s^j(\zeta^1, \zeta^2, \eta^1, \eta^2, X, \sigma, z)
\]

\( \forall s \in \mathcal{E}, j = 1, 2 \), \(6.43\)

where \( \mathcal{K}_s^j \) and \( \mathcal{H}_s^j \) \( j = 1, 2 \) are \( E_0 \) and \( E_+ \) valued maps, respectively, that are analytic in a neighborhood of \((0, 0, X, \sigma, \sigma, 0, 0) \in V_2 \times V_2 \times E_+ \times \mathbb{R}^3 \). The system of differential equations given by (6.34), (6.42), and (6.43) is equivalent to the original system (6.16), (6.17). Moreover, if we choose \( n = \max\{3, 3 + \max \mathcal{E}\} \), then (6.34), (6.42), and (6.43) are in a form to which theorem 1 applies. Applying this theorem shows that there exist a unique solution \( \{\sigma(z, a, Y), \zeta^1_j(z, a, Y), \eta^j_z(z, a, Y)\} \) that is analytic in a neighborhood of \((z, a, Y) = (0, m_\infty, X) \) and that satisfies \( \zeta^1_j(z) = O(z) \), \( \zeta^2_j(z) = O(z) \), and \( \sigma(z) = O(z) \). It then follows from (6.30) that \( \text{pr}_0^+(\Lambda_+ - \Omega_+) = Y_s z^n + O(r^s) \), \( \text{pr}_0^+(\Lambda_+) = O(z^n) \), and \( m = a + O(z^3) \).

**Theorem 6.** Every solution from theorem \( \mathcal{H} \) satisfies equation (2.14) on a neighborhood of \( r^{-1} = 0 \).

**Proof.** Let \( z = 1/r \) and \( \{\Lambda_+(z), m(z)\} \) be a solution of the equations (2.11) and (2.13) on a neighborhood \( \mathcal{N} \) of \( z = 0 \), which we know exists by theorem \( \mathcal{H} \). From lemma \( \mathcal{H} \) it is easy to see that in terms of the \( z \) variable

\[
\gamma(z) = -z^2([\Lambda_+(z), \sigma(z)] + [\Lambda_-(z), \sigma(z)])
\]

and \( \gamma(z) \) satisfies \( \check{\gamma}(z) = f(z)\gamma(z) \) where \( f(z) = 2N^{-1}(m-2zP) \). By theorem \( \mathcal{H} \) and shrinking \( \mathcal{N} \) if necessary, we see that \( f(z) \) is analytic on \( \mathcal{N} \). Therefore we can solve \( \check{\gamma}(z) = f(z)\gamma(z) \) to get \( \gamma(z) = \gamma(0) \exp(\int_0^z f(\tau)d\tau) \) for all \( z \in \mathcal{N} \). But (6.44) shows that \( \gamma(0) = 0 \). Therefore \( \gamma(z) = 0 \) for all \( z \in \mathcal{N} \) and the theorem is proved.

6.3 Regular black hole solutions

**Theorem 7.** Let \( t = r - r_H \) and suppose \( X \in V_2 \) satisfies

\[
\nu(X) := \frac{1}{r_H} - \frac{1}{r} \|\Lambda_0 + [X, c(X)]\|^2 > 0.
\]

Then there exist a unique solution \( \{\Lambda_+(t, Y), N(t, Y)\} \) to the system of differential equations (2.11) and (2.13) that is analytic in a neighborhood of \((0, X) \in \mathbb{R} \times \mathcal{V} \times \mathbb{S} \) and satisfies

\[
N(t) = \nu(Y)t + O(t^2) \quad \text{and} \quad \Lambda_+(t) = Y + O(t) \,.
\]

**Proof.** The proof is exactly the same as the proof of theorem 6 of [15] with \( E_+ \) replaced by \( V_2 \).

**Theorem 8.** Every solution from theorem \( \mathcal{H} \) satisfies equation (2.13) on a neighborhood of \( r = r_H \).

**Proof.** Let \( t = r - r_H \) and suppose \( X \in V_2 \) satisfies \( \nu := r_H^{-1} - r_H^{-1} \|\Lambda_0 + [X, c(X)]\|^2 > 0 \). Then we know by the previous theorem that there exists a solution \( \{\Lambda_+(t), N(t)\} \) to the system of differential equations (2.11) and (2.13) that is analytic in a neighborhood of \( t = 0 \) and satisfies

\[
N(t) = \nu t + O(t^2) \quad \text{and} \quad \Lambda_+(t) = X + O(t) \,.
\]
Figure 1: Gauge group \( SO(5) \) or \( B_2 \): characteristic (20), \( \Lambda_0 = 2h_1 + h_2, (w_\alpha(0)) = (0, 1, 0), \| \Omega_+ \|^2 = 2, \beta = (0.1, -0.35, -0.28) \). The function \( v_2 \) is identically 0, \( u_3 = -u_1 \), and \( v_3 = -v_1 \). The quantity \( L \) is \( \| \Lambda_0 \| \).

For a globally regular solution it should nowhere exceed its value at the center and at infinity.

Let \( f(t) := -2t ((t + r_H)^2 N(t))^{-1} (m(t) - (t + r_H)^{-1} P(t)) \). Then (6.45) shows that \( f(t) \) is analytic in a neighborhood of \( t = 0 \). A short calculation shows that \( f(0) = -1 \), and therefore we can write \( f(t) = -1 + tg(t) \) where \( g(t) \) is analytic in near \( t = 0 \). Consider the differential equation on \( V_2 \),

\[
\frac{d\eta(t)}{dt} = -\eta(t) + tg(t)\eta(t). \tag{6.46}
\]

It has \( \eta(t) = 0 \) as a solution, and therefore this is the unique analytic solution near \( t = 0 \) by theorem 1. But lemma 1 shows that \( \gamma(t) = [\Lambda_+(t), \frac{d\Lambda_+}{dt}(t)] + [\Lambda_-(t), \frac{d\Lambda_-}{dt}(t)] \) also solves (6.46) in an neighborhood of \( t = 0 \). Because \( \gamma(t) \) is analytic near \( t = 0 \), we must have \( \gamma(t) = 0 \) near \( t = 0 \).

7 A numerical example

So far we have not yet found a global numerical solution that has the correct fall off behavior for \( r \to \infty \). The following example is for the gauge group \( SO(5) \) (\( B_2 \)) for the action with characteristic (20), one of the simplest irregular cases that will not reduce. Figure 1 shows a solution near the center with real initial values \( w_\alpha(0) \) which develops nonzero imaginary components \( v_\alpha \). Figure 2 shows a solution for large \( r \). As is apparent the function \( N \) decreases from its value 1 at infinity as \( r \) decreases down to some minimum, but then increases rapidly. Only by very careful tuning of the data at infinity one can possibly avoid this behavior and construct a ‘physical’ solution in which \( N \) always remains between 0 and 1. For a globally bounded solution we have also indications that necessarily \( \| \Lambda_+ \| \leq \| \Omega_+ \| \).

References
Figure 2: Gauge group $SO(5)$ or $B_2$: characteristic (20), $A_0 = 2 \mathbf{h}_1 + \mathbf{h}_2$, $(w_\alpha(\infty)) = (0, 1, 0)$, $\|\Omega_+\|^2 = 2$, $m_\infty = 0.52$, $\alpha = (-8, -2, -1)$. The function $v_2$ is identically 0, $u_3 = -u_1$, and $v_3 = -v_1$. The quantity $L$ is $\|\Lambda_+\|^2$. For a globally regular solution it should nowhere exceed its value at the center and at infinity. Also $N$ should remain between 0 and 1.

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