Four Dimensional Orbit Spaces of Compact Coregular Linear Groups

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Abstract

All four dimensional orbit spaces of compact coregular linear groups have been determined. The results are obtained through the integration of a universal differential equation, that only requires as input the number of elements of an integrity basis of the ideal of polynomial invariants of the linear group. Our results are relevant and lead to universality properties in the physics of spontaneous symmetry breaking at the classical level.

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1 Introduction

In theories in which the ground state of the system is determined by a minimum of a potential $V(x)$, $x \in \mathbb{R}^n$, which is invariant under the transformations of a compact group $G$ acting linearly in $\mathbb{R}^n$, the characterization of the schemes of spontaneous symmetry breaking rests on the determination of the minimum of the potential. Owing to the high number of variables involved and the degeneracy of the minimum along an orbit of $G$, this is generally a difficult problem to solve, even if a polynomial approximation is used for the potential.

The degeneracy associated with the $G$-invariance of the potential can be eliminated by considering the potential as a function in the orbit space $\mathbb{R}^n/G$ (see ref. [1] and references therein). If $V(x)$ is a polynomial or a $C^\infty$-function, this is easily achieved in the following way. Let $\{p_1(x), \ldots, p_q(x)\}$ be a minimal integrity basis (hereafter abbreviated in IB) of the ring $\mathbb{R}^n[x]^G$ of polynomial invariants of $G$. The IB defines an orbit map $p(x)$, mapping $\mathbb{R}^n$ onto a semi-algebraic subset, $p(\mathbb{R}^n)$, of $\mathbb{R}^q$, and $p(\mathbb{R}^n)$ is a faithful image of $\mathbb{R}^n/G$ (see, for instance, [2] or [3] and references therein) whose geometric stratification is strictly related to the isotropy type stratification of the orbit space $\mathbb{R}^n/G$. The invariance properties of the potential allow to write it in the form $\hat{V}(p(x))$ [4, 5]. The function $\hat{V}(p)$, which is defined in the whole of $\mathbb{R}^q$, when restricted to $p(\mathbb{R}^n)$ has the same
regularity properties and the same range as $V(x)$. Thus it can be advantageously used in the determination of the minimum of the potential.

As long as the potential is not specified, each point $p \in p(\mathbb{R}^n)$ can be considered as the representative of a possible ground state of the system. Points lying on the same stratum represent possible ground states of the system and their invariance groups are conjugated subgroups of $G$. The set $p(\mathbb{R}^n)$ yields therefore a suggestive geometric picture of the possible configurations (phases) of the system after spontaneous symmetry breaking.

Often, invariance properties are the only bounds which are imposed to the potential, beyond regularity and stability properties and/or bounds to the degree, when the potential is a polynomial. If the symmetry groups of the potentials of different theories share isomorphic orbit spaces, the potentials have the same formal expression if they are written as functions in orbit space, despite the completely different physical meaning of the variables and parameters involved in their definitions. Thus, the problems of determining the geometric features of the phase space, the location and the stability properties of the minima of the potential, the number of phases and the allowed phase transitions, reduce to identical mathematical problems in all these theories. This remark makes the orbit space approach to a model independent analysis of spontaneous symmetry breaking particularly appealing.

The price to pay in the orbit space approach to the minimization of a potential $V$ is essentially twofold:

1. IB’s are sometimes difficult to determine.

2. The domain of the function $\tilde{V}(p)$ is not the whole of $\mathbb{R}^q$, but reduces to the semi-algebraic subset $p(\mathbb{R}^n)$, which is not trivial to determine.

The difficulties mentioned under items 1 and 2 can however be bypassed. In fact, it has been shown that the polynomial relations defining $p(\mathbb{R}^n)$ and its strata can be determined from the positivity properties of a matrix $\hat{P}(p)$, defined only in terms of the gradients of the elements of an IB. The semialgebraic subset of $\mathbb{R}^q$ where $\hat{P}(p) \geq 0$ and has rank $k$ is the union of all the $k$-dimensional strata. If $G$ is a compact coregular linear group (hereafter abbreviated in CCLG), i.e. if there is no algebraic relation among the elements of an IB, the matrix $\hat{P}(p)$ plays the role of an inverse metric matrix in the interior of $\mathbb{R}^n/G$ and the isomorphism classes of the orbit spaces of all the compact coregular linear groups can be classified in terms of equivalence classes of matrices $\hat{P}(p)$. The matrices $\hat{P}(p)$ have been shown to be solutions of a canonical differential equation in the variables $p_1, \ldots, p_q$, that contains as free parameters only the degrees $d_1, \ldots, d_q$ of the IB. Therefore, they can all be determined, even if the classification of the compact coregular linear groups is not yet complete and/or the explicit form of the elements $p_a(x)$ of the corresponding IB’s is not known.

For $q \leq 3$, all the relevant solutions of the canonical equation have been provided in a preceding paper; they share the following remarkable features that have been conjectured to hold for all values of $q$:

a) Only particular values of the ratios between the degrees are allowed and for each choice of the degrees, there is only a finite number of non equivalent solutions.

b) In convenient IB’s, the coefficients of the polynomials $\hat{P}_{ab}(p)$ are integer numbers.

In this paper, after recalling the basic steps leading to the formulation of the canonical equation, we shall provide all its solutions in the case of four dimensional orbit spaces. From the explicit
form of the solutions it will be evident that the statements listed under items a–b remain true for \( q = 4 \) too.

The plan of the paper is the following. In Section 2 we shall recall some known results concerning
the characterization of orbit spaces (see, for instance, \( [1, 4, 4, 13] \)) and in Sections 3 and 4 we shall
resume the main results of reference \( [15] \). In Section 5 the explicit expressions of the relevant
solutions of the canonical equation for \( q = 4 \) are reported.

2 Orbit Spaces of Compact Linear Groups

Let \( G \) be a compact group of \( n \times n \) matrices acting linearly in the Euclidean space \( \mathbb{R}^n \). We shall
assume, without loss of generality, that \( G \) is a subgroup of \( SO_n(\mathbb{R}) \).

If \( G_x \) denotes the isotropy subgroup of \( G \) at \( x \), the class of the isotropy subgroups at the points
of the orbit \( \Omega \) through \( x \), is the class of subgroups conjugated to \( G_x \) in \( G \) and is called the orbit
type of \( \Omega \). All the points lying on orbits with the same orbit type form an isotropy type stratum of
(the action of \( G \) in) \( \mathbb{R}^n \). Isotropy type strata are in a one-to-one correspondence with orbit types
and the connected components of a stratum are smooth manifolds with the same dimensions.

The orbit space of the action of \( G \) in \( \mathbb{R}^n \) is the quotient space \( \mathbb{R}^n/G \), endowed with the quotient
topology and differentiable structure. The connected components of the image of orbit space of an
isotropy type stratum of \( \mathbb{R}^n \) are smooth manifolds with the same dimensions.

Almost all the images of the orbits of \( G \) lie in a unique stratum \( \Sigma_p \) of \( \mathbb{R}^n/G \), the principal
stratum, which is a connected open dense subset of \( \mathbb{R}^n/G \). The boundary \( \Sigma_p \setminus \Sigma \) of the principal
stratum is the union of disjoint singular strata. All the strata lying on the boundary \( \Sigma \setminus \Sigma \) of a
stratum \( \Sigma \) are open in \( \Sigma \setminus \Sigma \).

The collection of all orbit types is a finite and partially ordered set: the orbit type of a stratum
\( \Sigma \) is contained in the orbit types of the strata lying in its boundary and there is a unique minimum
orbit type, the principal orbit type, corresponding to the principal stratum.

A faithful image of the orbit space \( \mathbb{R}^n/G \) and its stratification can be obtained from classical
invariant theory in the following way. Let \( \{p_1(x), \ldots , p_q(x)\} \) denote an homogeneous IB of the ring
\( \mathbb{R}^n[x]^G \) of all \( G \)-invariant polynomials and \( d_i \) denote the homogeneity degree of \( p_i(x) \).

The orbit map, \( p : \mathbb{R}^n \to \mathbb{R}^q : x \to (p_1(x), \ldots , p_q(x)) \), maps all the points of \( \mathbb{R}^n \) lying on an
orbit of \( G \) onto a unique point of \( \mathbb{R}^q \) and induces a diffeomorphism of \( \mathbb{R}^n/G \) onto a semi-algebraic
q-dimensional connected closed subset \( p(\mathbb{R}^n) \) of \( \mathbb{R}^q \). Like all semialgebraic sets, \( p(\mathbb{R}^n) \) is the disjoint
union of finitely many connected semi-algebraic differentiable varieties \( \{E_i\} \) (the primary strata),
such that the boundary of each \( E_i \) is empty or the union of lower dimensional primary strata and
each primary stratum is open in its closure. The connected components of the isotropy type strata
of \( \mathbb{R}^n/G \) are in a one-to-one correspondence with the primary strata of \( p(\mathbb{R}^n) \); the interior of \( p(\mathbb{R}^n) \)
represents the principal stratum, the boundary hosts the singular strata.

There is some arbitrariness in the choice of the elements of an IB, but their number \( q \) and their
degrees are determined by \( G \) alone. If \( G \) is coregular, as we shall assume in the rest of the paper,
\( p(\mathbb{R}^n) \) is a q-dimensional subset of \( \mathbb{R}^q \).

If \( \{p\} \) and \( \{p'\} \) are distinct IB’s of \( G \), then \( p(\mathbb{R}^n) \) and \( p'(\mathbb{R}^n) \) are isomorphic semi-algebraic
sets, since the IB transformation \( p'(p) \) is a bi-rational mapping.

3 The matrix \( \hat{P}(p) \)

The classification of the orbit spaces of all CCLG’s can be reduced to the classification of the orbit
spaces of the CCLG’s with no fixed points, as we shall do. In fact, if in \( \mathbb{R}^n \) there are linear \( G-\)
invariants, generating the linear subspace $W$ of $\mathbb{R}^n$, then $\mathbb{R}^n/G$ is homeomorphic to $W \times W_\perp/G_\perp$, where the linear group $G_\perp$ denotes the reduction of the linear group $G$ to the orthogonal complement $W_\perp$ of $W$ in $\mathbb{R}^n$. In these cases the degrees of all the polynomial invariants are necessarily $\geq 2$ and the following conventions can be adopted:

$$p_q(x) = \sum_{i=1}^{n} x_i^2.$$  

(1)

$$d_1 \geq d_2 \geq \ldots \geq d_q = 2,$$  

(2)

Hereafter, by an IB we shall always mean a *minimal homogeneous integrity basis* fulfilling eqs. (1) and (2).

A polynomial $\hat{F}(p)$ defines a $G$-invariant polynomial $F(x) = \hat{F}(p(x))$. Vice-versa, for each invariant polynomial $F(x)$, there exist a unique polynomial $\hat{F}(p)$, such that $F(x) = \hat{F}(p(x))$. $\hat{F}(p)$ will be said to be $w$-homogeneous with weight $w$ if $\hat{F}(p(x))$ is homogeneous in $x$ with degree $w$.

The set $p(\mathbb{R}^n)$ can be characterized by means of a $q \times q$ matrix $\hat{P}(p)$, defined in $\mathbb{R}^q$ by the following relations [12, 13]:

$$P_{ab}(x) = \sum_{i=1}^{n} \partial_i p_a(x) \partial_i p_b(x) = \hat{P}_{ab}(p(x)), \quad a, b = 1, \ldots, q.$$  

(3)

The matrix $\hat{P}(p)$ has the following properties:

**P1. Symmetry.** The matrix $\hat{P}(p)$ is symmetric.

**P2. Homogeneity.** The matrix elements $\hat{P}_{ab}(p)$ are real $w$-homogeneous polynomial functions of $p$ and

$$w(\hat{P}_{ab}) = d_a + d_b - 2;$$  

(4)

moreover, owing to eq. (1)

$$\hat{P}_{qa}(p) = \hat{P}_{aq}(p) = 2d_a p_a, \quad a = 1, \ldots, q.$$  

(5)

**P3. Tensor character.** $\hat{P}(p)$ transforms as a rank 2 contravariant tensor under IB transformations (hereafter abbreviated in IBT’s) $p' = f(p)$, where $f(p)$ is a convenient $w$-homogeneous polynomial map, whose Jacobian matrix $J(p)$ is an upper block-triangular matrix with constant diagonal blocks,

$$\hat{P}'(p'(x)) = J(p)\hat{P}(p)J^T(p),$$  

(6)

The determinant of $\hat{P}(p)$ is a relative invariant of the group of IBT’s.

**P4. Positivity.** The set $p(\mathbb{R}^n)$ is the only subset of $\mathbb{R}^q$ where $\hat{P}(p) \geq 0$. In the interior of $p(\mathbb{R}^n)$, the rank of $\hat{P}(p)$ equals $q$; on the boundary, it is lower and almost everywhere equal to $q - 1$. The subset of $\mathbb{R}^q$ where $\hat{P}(p) \geq 0$ and has rank $k$ is the union of all the $k$-dimensional strata [12, 14, 16]. The image $\mathcal{S} = p(S^{n-1})$ of the unit sphere of $\mathbb{R}^n$ is a compact connected $(q - 1)$-dimensional semialgebraic subset of the hyperplane $\Pi = \{ p \in \mathbb{R}^q \mid p_q = 1 \}$, which is the intersection of $p(\mathbb{R}^n)$ with $\Pi$ [14, 15].
P5. Boundary conditions \cite{14,15}. Let \( \mathcal{I}(\hat{\sigma}) \) denote the ideal of all the polynomials in \( p \) vanishing on \( \hat{\sigma} \), a \((q-1)\)-dimensional primary stratum (or a union of primary strata) of \( p(\mathbb{R}^n) \). Then, for all \( f \in \mathcal{I}(\hat{\sigma}) \)

\[
\sum_{1}^{q} b \hat{P}_{ab}(p) \partial_{b} f(p) \in \mathcal{I}(\hat{\sigma}).
\] (7)

Two \( \hat{P} \)-matrices, related by a relation like eq.(6) will be said to be equivalent. The \( q \)-dimensional orbit spaces of the CCLG’s \( G \) and \( G' \), acting in \( \mathbb{R}^n \) and, respectively, \( \mathbb{R}^n' \), will be said to be isomorphic if their \( \hat{P} \) matrices are equivalent; in particular, in case of isomorphic orbit spaces, \( p'(\mathbb{R}^n') = f(p(\mathbb{R}^n)) \), where \( p' = f(p) \) has the form of an IBT.

The identification of the class of all IB’s with the class of coordinate systems makes the interior of \( \mathbb{R}^n/G \) a differentiable manifold and \( \hat{P}(p) \) yields an inverse metric matrix in it. Thus, in order to classify the isomorphism classes of the orbit spaces of all the CCLG’s, it is sufficient to classify the equivalence classes of matrices \( \hat{P}(p) \).

4 The Canonical Equation

The results summarized below have been proved in \cite{14,15}.

If \( \hat{B} \) is the boundary of \( p(\mathbb{R}^n) \), the ideal \( \mathcal{I}(\hat{B}) \), formed by the polynomials in \( p \) vanishing in the whole of \( \hat{B} \), has a unique independent generator \( A(p) \). For \( f(p) = A(p) \), eq.(6) reduces to

\[
\sum_{1}^{q} b \hat{P}_{ab}(p) \partial_{b} A(p) = \lambda^{(A)}(p) A(p), \quad a = 1, \ldots, q,
\] (8)

where \( \lambda^{(A)}(p) \) is a contravariant vector field with \( w \)-homogeneous components.

The vector \( \lambda^{(A)}(p) \) can be reduced to the canonical form \( \lambda^{(A)}(p) = 2 \delta_{aq} w(A) \) in the so-called \( A \)-bases. In an \( A \)-basis, the boundary conditions assume the following canonical form:

\[
\sum_{1}^{q} b \hat{P}_{ab}(p) \partial_{b} A(p) = 2 \delta_{aq} w(A) A(p), \quad a = 1, \ldots, q.
\] (9)

From eq.(8) one deduces that, in every \( A \)-basis, the following facts are true:

i) \( A(p) \) is a factor of \( \det \hat{P}(p) \) and its sign can be chosen to be positive inside \( p(\mathbb{R}^n) \); we shall call it a complete active factor. Consequently:

\[
w(A) \leq w(\det \hat{P}) = 2 \sum_{1}^{q} d_{a} - 2q.
\] (10)

ii) \( A(p)|_{p_{0}=1} \) has a unique local non degenerate maximum lying at \( p_{0} = (0, \ldots , 1) \). This means that \( p_{0} \) is in the interior of \( S \) and that the Hessian matrix \( H(p) \) of \( A(p) \) is negative definite at \( p_{0} \). This allows, in particular, to set a lower bound to the weight of \( A \):

\[
w(A) \geq 2d_{1}.
\] (11)

iii) \( \hat{P}(p_{0}) \) is proportional to \( H(p_{0}) \); it is block diagonal and, in a subclass of \( A \)-bases (standard \( A \)-bases), it is diagonal:

\[
\hat{P}_{ab}(p_{0}) = d_{a} d_{b} \delta_{ab}, \quad a, b = 1, \ldots, q.
\] (12)
Two different standard A-bases are related by an IBT $p'_\alpha = f_\alpha(p)$, $\alpha = 1, \ldots, q - 1$, not involving $p_q$; the corresponding Jacobian matrix is orthogonal at $p_0$.

In eq.(9), $\hat{P}_{ab}(p)$ and $A(p)$ can be considered as unknown polynomials satisfying only the conditions listed under items P1-P3. In this case eq.(9) will be called the canonical equation.

Let us now assume that the only information we have on a CCLG with no fixed points is the number of elements of its IB’s. Then, the $\hat{P}$-matrix of $G$, in a standard A basis, has to be searched among the solutions of the canonical equation (in which the degrees $d_i$ are considered as integer parameters $\geq 2$), satisfying the initial conditions specified in eq.(12). We shall call these solutions allowable $\hat{P}$ matrices.

If the degrees $d_a$ are given explicit numerical values, the polynomials $\hat{P}_{ab}(p)$ and $A(p)$ can be expanded in the $p_a$’s. After elimination of these variables, the canonical equation reduces to a system of algebraic equations in the unknown coefficients of the polynomials. For $q = 3$ the number of variables and equations is already frightening.

If the degrees are maintained as free parameters, as we did, only the dependence on $p_1$ can be rendered explicit in the unknown polynomials involved in the canonical equation. After the elimination of $p_1$, the canonical equation expands into a seemingly awful system of coupled algebraic and differential equations. For $q \leq 4$, a clever use of the initial conditions has allowed to solve it. A complete list of the allowable solutions for $q = 3$ has been published in ref.[15]; a list of the allowable solutions for $q = 4$ will be given in the next section.

The allowable solutions we have found for $q = 3$ and for $q = 4$ share the following relevant features:

a) The canonical equation admits allowable solutions only for particular values of the degrees and for each choice of the degrees there is only a finite number of non equivalent (with respect to IBT’s) allowable solutions.

b) Let $D$ denote the MCD of $d_1, \ldots, d_{q-1}$ and $R$ the following $q \times q$ matrix, defined in $\Pi$:

$$R_{ab}(p_1, \ldots, p_{q-1}) = \hat{P}_{ab}(p_1, \ldots, p_{q-1})/(d_a d_b).$$

(13)

For each allowable solution corresponding to the weights $(d_1, \ldots, d_{q-1}, 2)$, there is tower of allowable solutions corresponding to the weights $(s d_1/D, \ldots, s d_{q-1}/D, 2)$, where $s \in \mathbb{N}$ is a scale parameter. Different solutions belonging to the same tower are not equivalent, but share the same $R$-matrix; as a consequence, their positivity regions, in the hyperplane $\Pi$, are isomorphic.

c) The $A$-basis can be chosen so that all the coefficients of the polynomials $\hat{P}_{ab}(p)$ are integer numbers.

d) All the allowable solutions satisfy the following positivity condition, analogous to the positivity condition stated under item P4: $\hat{P}(p)$ is $\geq 0$ in a unique connected semi-algebraic subset of $\mathbb{R}^q$, whose intersection with $\Pi$ is a $(q - 1)$-dimensional, compact, connected semi-algebraic subset, $\mathcal{S}$, of $\mathbb{R}^{q-1}$.

e) $A(p)$ is a factor of det $\hat{P}(p)$ which vanishes on the whole boundary of the region where $\hat{P}(p) \geq 0$ and there are no other factors of det $\hat{P}(p)$ vanishing in a $(q - 1)$-dimensional semi-algebraic subset of the boundary.
The statement under item b), the fact that \( A(p) \) is a factor of \( \hat{P}(p) \) (see item e)) and the fact that \( \hat{P}(p)|_{p=1} \) is positive semi-definite only in a compact set (see item d)), hold true for all values of \( q [15, 17] \). Even if we have no proof that also the other features of the allowable solutions we have listed are independent of the particular range of values chosen for \( q \), we believe they yield a reasonable support to the following conjecture [14]:

**Conjecture.** For all integer \( q > 1 \) the allowable solutions of the canonical equation fullfil the conditions stated above under items a), c), d) and e).

We have not proved that every allowable solution of the canonical equation yields the \( \hat{P} \)-matrix of a CCLG; it is clear, however, that the \( \hat{P} \)-matrix of every CCLG is in the set of these solutions. Therefore, the selection rules mentioned under item a) are obeyed also by the CCLG’s; moreover, the orbit spaces of all the CCLG’s, whose IB’s are formed by the same number \( q (\leq 4 \) and, if the conjecture holds true, for all \( q \)) of elements with the same degrees \( (d_1, \ldots, d_q) \), can be classified in a finite number of isomorphism classes; for each CCLG, the IB can be chosen so that the \( \hat{P} \)-matrix elements are \( w \)-homogeneous polynomials with integer coefficients.

All the finite coregular linear groups have been classified and the corresponding IB’s determined [18]. All the compact coregular simple real linear Lie groups, and in many cases their IB’s, can be determined from the results obtained by Schwarz for the complex coregular simple linear Lie groups [19]. The \( \hat{P} \)-matrices of the images of the 2-, 3- [1] and 4-dimensional [20] orbit spaces of all these real groups should be found among the allowable solutions of the canonical equation we have found. For some of them the identification is immediate, since there is only one allowable solution with suitable degrees, for the other cases a direct check is needed [20]. At our knowledge, in the literature there is no complete classification of general compact coregular linear Lie groups.

Some of the solutions we have found may correspond also to non-coregular CLG’s. A criterium to recognize such solutions will be presented in a forthcoming paper [20].

### 5 Allowable solutions of the canonical equation for \( q = 4 \)

Below we shall report a (hopefully) complete list of the matrices \( R \) characterizing the distinct towers of allowable solutions of the canonical equation for \( q = 4 \). The representative matrices \( R(p_1, \ldots, p_{q-1}) \) are expressed in a convenient \( A \)-basis, \( A \) being a complete active factor of \( \det \hat{P} \), with weight \( d_A \). The corresponding matrices \( \hat{P}(p_1, \ldots, p_q) \) can be easily recovered from eq.(13) and the \( w \)-homogeneity of \( \hat{P}_{ab}(p) \). We shall also make use of the following definition:

\[
\det R = \rho A. \tag{14}
\]

The towers of allowable solutions will be labelled by an indexed capital latin letter and a set of positive integer parameters \( j_1, j_2, \ldots \). Different labels generally correspond to unequivalent allowable \( \hat{P} \)-matrices. The degrees are expressed as functions of \( s \) and of the \( j_i \)'s. The limitations on the possible values of \( s \) and of the \( j_i \)'s coming from eq.(2) will be understood.

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1The list of allowable solutions given in ref.[21] is incomplete and the labels used to distinguish the different towers do not generally agree with the labels used in the present paper.
Class $A1(j_1,j_2,j_3,j_4)$

$$d_A = 2d_1, \quad d_1 = s(j_1 + j_2)(j_3 + j_4)/4, \quad d_2 = s(j_1 + j_2)/2, \quad d_3 = s;$$

$j_1 \geq j_2$ and $j_1 = j_2$ for odd $d_3$; $j_3 \geq j_4$ and $j_3 = j_4$ for odd $j_1$ or $j_2$;

$$R_{11} = [j_1j_2 + (j_1 - j_2)p_3][j_3j_4(j_1 - p_3)^{j_1/2}(j_2 + p_3)^{j_2/2} + (j_3 - j_4)p_2][j_3(j_1 - p_3)^{j_1/2}(j_2 + p_3)^{j_2/2} - p_2]^{j_3-1} [j_4(j_1 - p_3)^{j_1/2}(j_2 + p_3)^{j_2/2} + p_2]^{j_4-1}(j_1 - p_3)^{j_1/2-1}(j_2 + p_3)^{j_2/2-1},$$

$R_{12} = 0,$

$R_{13} = 0,$

$$R_{22} = [j_1j_2 + (j_1 - j_2)p_3][j_3j_4(j_1 - p_3)^{j_1/2}(j_2 + p_3)^{j_2/2} + (j_3 - j_4)p_2](j_1 - p_3)^{j_1/2-1}(j_2 + p_3)^{j_2/2-1},$$

$R_{23} = 0,$

$R_{33} = j_1j_2 + (j_1 - j_2)p_3,$

$$A = [j_3(j_1 - p_3)^{j_1/2}(j_2 + p_3)^{j_2/2} - p_2]^{j_3}[j_4(j_1 - p_3)^{j_1/2}(j_2 + p_3)^{j_2/2} + p_2]^{j_4} - p_1^2,$$

$$\rho = [j_1j_2 + (j_1 - j_2)p_3]^2[j_3j_4(j_1 - p_3)^{j_1/2}(j_2 + p_3)^{j_2/2} + (j_3 - j_4)p_2](j_1 - p_3)^{j_1/2-1}(j_2 + p_3)^{j_2/2-1}.$$

Class $A2(j_1,j_2,j_3,j_4)$

$$d_A = 2d_1, \quad d_1 = s[(j_1 + 1)(j_2 + j_3)/2 + j_4], \quad d_2 = s(j_1 + 1), \quad d_3 = 2s;$$

$j_3 \geq j_2$, and $j_3 = j_2$ for even $j_1$; $j = (j_1 + 1)(j_2 + j_3)$;

$$R_{11} = [(j + 2j_4)[2j_4 - (j - 2j_4)p_3][j_2j_3(p_3 + j)^{(j_1+1)/2} + (j_2 - j_3)p_2](p_3 + j)^{(j_1-1)/2} - 4j_4^2p_3^2](2j_4 - p_3)^{j_2-1}[j_2(p_3 + j)^{(j_1+1)/2} - p_2]^{j_2-1}[j_3(p_3 + j)^{(j_1+1)/2} + p_2]^{j_3-1},$$

$R_{12} = 0,$

$R_{13} = 0,$

$$R_{22} = (j + 2j_4)[j_2j_3(p_3 + j)^{(j_1+1)/2} + (j_2 - j_3)p_2](p_3 + j)^{(j_1-1)/2},$$

$$R_{23} = 2j_4p_2,$$

$R_{33} = 2jj_4 - (j - 2j_4)p_3,$

$$A = (j + 2j_4)(2j_4 - p_3)^{j_4}[j_2(p_3 + j)^{(j_1+1)/2} - p_2]^{j_2}[j_3(p_3 + j)^{(j_1+1)/2} + p_2]^{j_3} - p_1^2,$$

$$\rho = (j + 2j_4)[2jj_4 - (j - 2j_4)p_3][j_2j_3(p_3 + j)^{(j_1+1)/2} + (j_2 - j_3)p_2](p_3 + j)^{(j_1-1)/2} - 4j_4^2p_3^2.$$
Class $A3(j_1, j_2, j_3)$

\[ d_A = 2d_1 + d_3, \quad d_1 = s(j_1 + 1)(j_2 + j_3)/2, \quad d_2 = s(j_1 + 1), \quad d_3 = 2s; \]

\[ j_2 \geq j_3, \text{ and } j_2 = j_3 \text{ for even } j_1; \quad j = (j_1 + 1)(j_2 + j_3)/2; \]

\[ R_{11} = (j + 1)[j_2j_3(p_3 + j)]^{(j_1+1)/2} + (j_3 - j_2)p_2][j_2(p_3 + j)^{(j_1+1)/2} + p_2]^{j_2-1}[j_3(p_3 + j)^{(j_1+1)/2} - p_2]^{j_3-1}(p_3 + j)^{(j_1-1)/2}, \]

\[ R_{12} = 0, \]

\[ R_{13} = p_1, \]

\[ R_{22} = (j + 1)[j_2j_3(p_3 + j)]^{(j_1+1)/2} + (j_3 - j_2)p_2](p_3 + j)^{(j_1-1)/2}, \]

\[ R_{23} = p_2, \]

\[ R_{33} = j + (1 - j)p_3, \]

\[ A = (1 - p_3)[j_2(p_3 + j)]^{(j_1+1)/2} + p_2]^{j_2}[j_3(p_3 + j)^{(j_1+1)/2} - p_2]^{j_3} - p_4^2, \]

\[ \rho = (j + 1)^2[j_2j_3(p_3 + j)]^{(j_1+1)/2} + (j_3 - j_2)p_2](p_3 + j)^{(j_1-1)/2}. \]

Class $A4(j_1, j_2)$

\[ d_A = 2d_1 + j_1d_2, \quad d_1 = 2s, \quad d_2 = s(j_1 + j_2), \quad d_3 = 2s; \]

\[ R_{11} = (j_1 + j_2)(j_1 + 2j_2)[2j_2(j_1 + j_2) + p_3]^{2j_2-1}, \]

\[ R_{12} = j_2(j_1 + 2j_2)p_2[2j_2(j_1 + j_2) + p_3]^{j_2-1} \]

\[ R_{13} = j_1(j_1 + j_2)p_1, \]

\[ R_{22} = (j_1 + 2j_2)[j_1(j_1 + j_2) - p_3]^{j_1-1}(j_1^2p_1 + [2j_1j_2(j_1 + j_2)^2 + (j_1^2 - 2j_2^2)p_2][2j_2(j_1 + j_2) + p_3]^{j_2-1}, \]

\[ R_{23} = j_1j_2p_2, \]

\[ R_{33} = (j_1 + j_2)[2j_1j_2(j_1 + j_2) + j_1 - j_1j_2p_3], \]

\[ A = ([2j_2(j_1 + j_2) + p_3]^{2j_2} - p_1}[j_1(j_1 + j_2) + j_2][j_1(j_1 + j_2) - p_3]^{j_1}[2j_2]^{j_2} - p_1] - p_2^2, \]

\[ \rho = (j_1 + 2j_2)^2[j_1^2p_1 + [2j_1j_2(j_1 + j_2)^2 + (j_1^2 - 2j_2^2)p_2][2j_2(j_1 + j_2) + p_3]^{j_2-1}. \]
Class $A5(j_1, j_2, j_3)$

\[ d_A = 2d_1 + d_2, \quad d_1 = s(j_1 j_2 + j_3), \quad d_2 = 2s j_2, \quad d_3 = 2s; \]

\[ j = j_1 j_2 + j_2, \quad j' = j_1 j_2 + j_3; \]

\[ R_{11} = (j + j_3) j'(j' j_3 - p_3)^{j_3-1} \{ j_1 jj' j_3 + (j_3^2 - j j_1 j_2) p_3 \} (j j' + p_3) j_2 - 1 + j_3^2 p_2 \} [ j_1 (j j' + p_3)^{j_2} + p_2 ]^{j_1-1}, \]

\[ R_{12} = j_1 j_2 (j + j_3) p_1 (j j' + p_3)^{j_2-1}, \]

\[ R_{13} = -j_2 j_3 p_1, \]

\[ R_{22} = j' (j + j_3) (j j' + p_3)^{j_2-1} [ j_1 (j j' + p_3)^{j_2} + (1 - j_1) p_2 ], \]

\[ R_{23} = j' j_3 p_2, \]

\[ R_{33} = j' [j j' j_3 + (j_3 - j) p_3], \]

\[ A = \{ j_1 j_2 j_3 + (j_3 - j) p_3 \}, \]

\[ \rho = (j + j_3)^2 [ j_1 jj' j_3 - (j j_1 j_2 - j_3^2) p_3 ] (j j' + p_3)^{j_2-1} + j_3^2 p_2 \}.

Class $A6(j_1, j_2, j_3)$

\[ d_A = 2d_1 + d_2, \quad d_1 = s(j_1 + j_2)(j_3 + 1)/2, \quad d_2 = s(j_1 + j_2), \quad d_3 = s; \]

\[ j_1 \geq j_2 \text{ and } j_2 = j_1 \text{ for odd } d_3. \]

\[ R_{11} = (j_3 + 2) [ j_1 j_2 + (j_1 - j_2) p_3 \} (j_1 - p_3)^{j_1-1} \{ j_2 + p_3 \} j_2^{-1} \text{ } [ j_3 + 1 \} (j_1 - p_3)^{j_1} (j_2 + p_3)^{j_2} + p_2 ]^{j_3}, \]

\[ R_{12} = p_1 [ j_1 j_2 + (j_1 - j_2) p_3 \} (j_1 - p_3)^{j_1-1} (j_2 + p_3)^{j_2-1}, \]

\[ R_{13} = 0, \]

\[ R_{22} = [ j_1 j_2 + (j_1 - j_2) p_3 \} (j_1 - p_3)^{j_1-1} (j_2 + p_3)^{j_2} - 1 \} (j_3 + 1 \} (j_1 - p_3)^{j_1-1} (j_2 + p_3)^{j_2} - j_3^2 p_2], \]

\[ R_{23} = 0, \]

\[ R_{33} = j_1 j_2 + (j_1 - j_2) p_3, \]

\[ A = \{ (j_1 - p_3)^{j_1} (j_2 + p_3)^{j_2} - p_2 \} \{ (j_3 + 1 \} (j_1 - p_3)^{j_1} (j_2 + p_3)^{j_2} + p_2 ]^{j_3+1} - p_1^2 \}, \]

\[ \rho = (j_3 + 2) [ j_1 j_2 + (j_1 - j_2) p_3 ]^2 (j_1 - p_3)^{j_1-1} (j_2 + p_3)^{j_2-1}.
Class \( A7(j_1, j_2) \)

\[
d_A = 2d_1 + d_2 + d_3, \quad d_1 = sj_1(j_2 + 1), \quad d_2 = 2sj_1, \quad d_3 = 2s;
\]

\[
j = j_1(j_2 + 2);
\]

\[
R_{11} = (j + 1)(j_2 + 2)(p_3 + j)^{j_1 - 1}[(j_2 + 1)(p_3 + j)^{j_1} + p_2]^{j_2},
\]

\[
R_{12} = (j + 1)p_1(p_3 + j)^{j_1 - 1},
\]

\[
R_{13} = p_1,
\]

\[
R_{22} = (j + 1)(p_3 + j)^{j_1 - 1}[(j_2 + 1)(p_3 + j)^{j_1} - j_2p_2],
\]

\[
R_{23} = p_2,
\]

\[
R_{33} = j + (1 - j)p_3,
\]

\[
A = (1 - p_3)[(p_3 + j)^{j_1} - p_2][((j_2 + 1)(p_3 + j)^{j_1} + p_2)^{j_2 + 1} - p_2^2],
\]

\[
\rho = (j + 1)^2(j_2 + 2)(p_3 + j)^{j_1 - 1}.
\]

Class \( A8(j_1, j_2) \)

\[
d_A = 2d_1 + 2d_2, \quad d_1 = s(j_1 + 1), \quad d_2 = s(j_2 + 1), \quad d_3 = 2s;
\]

\[
R_{11} = (j_1 + j_2 + 2)(j_1 + 1 - p_3)^{j_1},
\]

\[
R_{12} = 0,
\]

\[
R_{13} = -(j_2 + 1)p_1,
\]

\[
R_{22} = (j_1 + j_2 + 2)(j_2 + 1 + p_3)^{j_2},
\]

\[
R_{23} = (j_1 + 1)p_2,
\]

\[
R_{33} = (j_1 + 1)(j_2 + 1) + (j_1 - j_2)p_3,
\]

\[
A = [(j_1 + 1 - p_3)^{j_1 + 1} - p_1^2][(j_2 + 1 + p_3)^{j_2 + 1} - p_2^2],
\]

\[
\rho = (j_1 + j_2 + 2)^2.
\]
Class $A9(j_1)$

\[ d_A = 2d_1 + d_3, \quad d_1 = 2s(j_1 + 1), \quad d_2 = s(j_1 + 1), \quad d_3 = 2s; \]
\[ R_{11} = (2j_1 + 3)(2j_1 + 2 - p_3)^{2j_1+1}, \]
\[ R_{12} = -p_2(2j_1 + 3)(2j_1 + 2 - p_3)^{j_1}, \]
\[ R_{13} = -p_1, \]
\[ R_{22} = (2j_1 + 3)(2j_1 + 2 - p_3)^{j_1}, \]
\[ R_{23} = -p_2, \]
\[ R_{33} = 2j_1 + 2 + (2j_1 + 1)p_3, \]
\[ A = (1 + p_3)[(2j_1 + 2 - p_3)^{j_1+1} + p_1][(2j_1 + 2 - p_3)^{j_1+1} - p_1 - 2p_2^2], \]
\[ \rho = (2j_1 + 3)^2(2j_1 + 2 - p_3)^{j_1}. \]

Class $A10(j_1)$

\[ d_A = 2d_1 + d_2, \quad d_1 = s(j_1 + 1), \quad d_2 = 2s, \quad d_3 = s; \]
\[ R_{11} = (j_1 + 2)(j_1 + 1 + p_2)^{j_1}, \]
\[ R_{12} = p_1, \]
\[ R_{13} = 0, \]
\[ R_{22} = j_1 + 1 - j_1p_2, \]
\[ R_{23} = -p_3(j_1 + 1), \]
\[ R_{33} = j_1 + 2, \]
\[ A = [(j_1 + 1 + p_2)^{j_1+1} - p_2^2](1 - p_2 - p_3^2), \]
\[ \rho = (j_1 + 2)^2. \]
Class $B1(j_1)$

\[ d_A = 2d_1, \quad d_1 = 6s_1, \quad d_2 = 4s, \quad d_3 = 3s; \]
\[ R_{11} = [-p_2^2 + p_2 p_3^2 - 2p_3^2 + 4][-p_2^3 - 3p_2^2 + 6p_2 p_3^2 - p_3^4 - 4p_3^2 + 4]^{j_1 - 1}, \]
\[ R_{12} = 0, \]
\[ R_{13} = 0, \]
\[ R_{22} = -p_2 + p_3^2 + 2, \]
\[ R_{23} = 2p_3, \]
\[ R_{33} = p_2 + 2, \]
\[ A = [-p_2^3 - 3p_2^2 + 6p_2 p_3^2 - p_3^4 - 4p_3^2 + 4] - p_1^2, \]
\[ \rho = -p_2^2 + p_2 p_3^2 - 2p_3^2 + 4. \]

Class $B2$

\[ d_A = 3d_1, \quad d_1 = 4s, \quad d_2 = 3s, \quad d_3 = 3s; \]
\[ R_{11} = -p_1 + p_2^2 + p_3^2 + 2, \]
\[ R_{12} = 2p_2, \]
\[ R_{13} = 2p_3, \]
\[ R_{22} = p_1 + 2, \]
\[ R_{23} = 0, \]
\[ R_{33} = p_1 + 2, \]
\[ A = -p_1^3 - 3p_1^2 + 6p_1 (p_2^2 + p_3^2) - (p_2^2 + p_3^2)^2 - 4(p_2^2 + p_3^2) + 4, \]
\[ \rho = p_1 + 2. \]
Class $B3(j_1, j_2)$

$d_A = 3d_1, \quad d_1 = 2s(j_1 + j_2), \quad d_2 = 3s(j_1 + j_2)/2, \quad d_3 = s$;

$j_1 \geq j_2$ and $j_1 = j_2$ for odd $d_3$;

$R_{11} = (j_1j_2 + (j_1 - j_2)p_3)(j_1 - p_3)^{j_1-1}(j_2 + p_3)^{j_2-1}[-p_1(j_1 - p_3)^{j_1} (j_2 + p_3)^{j_2} + p_2^2 + 2(j_1 - p_3)^{3j_1} (j_2 + p_3)^{3j_2}],$

$R_{12} = 2p_2[j_1j_2 + (j_1 - j_2)p_3](j_1 - p_3)^{2j_1-1}(j_2 + p_3)^{2j_2-1},$

$R_{13} = 0,$

$R_{22} = (j_1j_2 + (j_1 - j_2)p_3)(j_1 - p_3)^{j_1-1}(j_2 + p_3)^{j_2-1}[p_1 + 2(j_1 - p_3)^{2j_1} (j_2 + p_3)^{2j_2}],$

$R_{23} = 0,$

$R_{33} = j_1j_2 + (j_1 - j_2)p_3,$

$A = p_1^3 - 3p_1^2(p_3 + 6j_1)^{2j_1} (j_2 + p_3)^{2j_2} + 6p_1p_2^2(j_1 - p_3)^{j_1} (j_2 + p_3)^{j_2} - p_2^4 - 4p_2^3(j_1 - p_3)^{3j_1} (j_2 + p_3)^{3j_2} + 4(j_1 - p_3)^6(j_2 + p_3)^{6j_2},$

$\rho = [j_1j_2 + (j_1 - j_2)p_3]^2(j_1 - p_3)^{j_1-1}(j_2 + p_3)^{j_2-1}.$

Class $B4(j_1)$

$d_A = 3d_1 + d_3, \quad d_1 = 4sj_1, \quad d_2 = 3sj_1, \quad d_3 = 2s$;

$R_{11} = (6j_1 + 1)(p_3 + 6j_1)^{j_1-1}[-p_1(p_3 + 6j_1)^{j_1} + p_2^3 + 2(p_3 + 6j_1)^{3j_1}],$

$R_{12} = 2(6j_1 + 1)p_2(p_3 + 6j_1)^{2j_1-1},$

$R_{13} = p_1,$

$R_{22} = (6j_1 + 1)(p_3 + 6j_1)^{j_1-1}[p_1 + 2(p_3 + 6j_1)^{2j_1}],$

$R_{23} = p_2,$

$R_{33} = 6j_1 + (1 - 6j_1)p_3,$

$A = (1 - p_3)[-p_1^3 - 3p_1^2(p_3 + 6j_1)^{2j_1} + 6p_1p_2^2(p_3 + 6j_1)^{j_1} - p_2^4 - 4p_2^3(p_3 + 6j_1)^{3j_1} + 4(p_3 + 6j_1)^{6j_1}],$

$\rho = (6j_1 + 1)^2(p_3 + 6j_1)^{j_1-1}.$
Class C1($j_1, j_2$)

\[ d_A = 2d_1, \quad d_1 = 3s(j_1 + 2j_2), \quad d_2 = 6s, \quad d_3 = 4s; \]
\[
R_{11} = \{-p_2^2[(j_1 - 2j_2)p_3 + 4(j_1^2 - j_1j_2 + 2j_2^2)] + 4j_2p_2[(3j_1 - j_2)p_3^2 + 12j_1(j_1 - j_2)p_3 - 16j_2(j_1^2 - j_2^2)] - j_1(j_1p_3^2 + 4(j_1^2 - 2j_1j_2 + 3j_2^2)p_3^2 - 16j_2(3j_1^2 - 7j_1j_2 - j_2^2)p_3^2 + 192j_2(j_1^3 - j_2^3)p_3 - 256j_2^3(j_1 + j_2)^3}\}
\[
p_{12} = 0, \quad R_{13} = 0, \quad R_{22} = \{(-j_1 + 2j_2)p_3 + 4j_2(3j_1 - 2j_2)]p_2 + 2j_1[(j_1 + 3j_2)p_3^2 - 8j_2(j_1 - 2j_2)p_3 + 16j_2^2(j_1 + j_2)]\}
\[
R_{23} = 2(-j_1 + 2j_2)p_2 + j_1(12j_2 - p_3), \quad R_{33} = p_2 + 2(j_1 + j_2)(4j_2 - p_3),
\[
A = \{p_2 + j_1(3p_3 + 4j_1 + 4j_2)\}^{j_1} \{-p_2^2 + 4j_2p_2(3p_3 - 4j_2) - (j_1 + 2j_2)p_3^2 + 4j_2[3(j_1 - j_2)p_3^2 - 12j_1j_2p_3 + 16j_2^2(j_1 + j_2)]\}^{j_2 - 1},
\[
r = -p_2^2[(j_1 - 2j_2)p_3 + 4(j_1^2 - j_1j_2 + 2j_2^2)] + 4j_2p_2[(3j_1 - j_2)p_3^2 + 12j_1(j_1 - j_2)p_3 - 16j_2(j_1^2 - j_2^2)] - j_1(j_1p_3^2 + 4(j_1^2 - 2j_1j_2 + 3j_2^2)p_3^2 - 16j_2(3j_1^2 - 7j_1j_2 - j_2^2)p_3^2 + 192j_2(j_1^3 - j_2^3)p_3 - 256j_2^3(j_1 + j_2)^3].
\]

Class C2($j_1$)

\[ d_A = 2d_1 + d_2, \quad d_1 = 6sj_1, \quad d_2 = 6s, \quad d_3 = 4s; \]
\[
R_{11} = \{(2j_1 + 1)(p_3 - 4j_1)p_2 - 2(j_1 - 1)p_3 - 8j_1(j_1 + 1)]\{-p_2^2 + 4j_1p_2(3p_3 - 4j_1) - (2j_1 + 1)p_3^2 - 12j_1(j_1 - 1)p_3^2 - 48j_1^2p_3 + 64j_1^3(j_1 + 1)\}^{j_1 - 1},
\[
R_{12} = -p_1(4j_1 + p_3), \quad R_{13} = -2p_1,
\[
R_{22} = p_2(2j_1 - 1)p_3 - 4j_1(3 - 2j_1)] + 2(3j_1 + 1)p_3^2 + 16j_1(2j_1 - 1)p_3 + 32j_1^2(j_1 + 1),
\[
R_{23} = 2(2j_1 - 1)p_2 + p_3(12j_1 - p_3), \quad R_{33} = p_2 + 2(j_1 + 1)(4j_1 - p_3),
\[
A = \{p_2 + 3p_3 + 4(j_1 + 1)]\{-p_2^2 + 4j_1p_2(3p_3 - 4j_1) - (2j_1 + 1)p_3^2 - 12j_1(j_1 - 1)p_3^2 - 48j_1^2p_3 + 64j_1^3(j_1 + 1)\}^{j_1 - 1},
\[
r = (2j_1 + 1)(p_3 - 4j_1)p_2 - 2(j_1 - 1)p_3 - 8j_1(j_1 + 1).
Class $C_3(j_1)$

\[ d_A = 2d_1 + 2d_2, \quad d_1 = 3s(j_1 + 1), \quad d_2 = 6s, \quad d_3 = 4s; \]

\[ R_{11} = (j_1 + 3)(4j_1 + 8 + p_3)[(j_1 + 1)(4j_1 + 8 + 3p_3) + p_2]^{j_1}, \]

\[ R_{12} = 2p_1(4 + p_3), \]

\[ R_{13} = 4p_1, \]

\[ R_{22} = p_2[(1 - j_1)p_3 - 4(3j_1 + 1)] + 2(j_1 + 1)[(j_1 + 4)p_3^2 - 8(j_1 - 1)p_3 + 16(j_1 + 2)], \]

\[ R_{23} = 2(1 - j_1)p_2 + (j_1 + 1)p_3(12 - p_3) \]

\[ R_{33} = p_2 + 2(j_1 + 2)(4 - p_3), \]

\[ A = [-p_2^2 + 4p_2(3p_3 - 4) - (j_1 + 3)p_3^2 + 12j_1p_3^2 - 48(j_1 + 1)p_3 + 64(j_1 + 2)][(j_1 + 1)(4j_1 + 8 + 3p_3) + p_2]^{j_1+1} - p_1^2, \]

\[ \rho = (j_1 + 3)(4j_1 + 8 + p_3). \]

Class $C_4$

\[ d_A = 3d_1, \quad d_1 = 6s, \quad d_2 = 4s, \quad d_3 = 3s; \]

\[ R_{11} = p_1(p_2 - 4) + 8(p_2^2 + 2p_2 + 8), \]

\[ R_{12} = 2p_1 - p_2(p_2 - 12), \]

\[ R_{13} = 2p_3(p_2 + 4), \]

\[ R_{22} = p_1 - 4p_2 + 16, \]

\[ R_{23} = 4p_3, \]

\[ R_{33} = 3(p_2 + 8), \]

\[ A = (p_1 + 3p_2 - p_3^2 + 8)[-p_1^2 + 4p_1(3p_2 - 4) - 3p_2^3 - 48p_2 + 128], \]

\[ \rho = 3(p_2 + 8). \]
Class \( C5(j_1, j_2) \)

\[ d_A = 3d_1, \quad d_1 = 3s(j_1 + j_2), \quad d_2 = 2s(j_1 + j_2), \quad d_3 = s; \]

\( j_1 \geq j_2 \) and \( j_1 = j_2 \) for odd \( d_3; \)

\[ R_{11} = [j_1j_2 + (j_1 - j_2)p_3](j_1 - p_3)^{j_1-1}(j_2 + p_3)^{j_2-1}(p_1[p_2 - 4(j_1 - p_3)^{2j_1}(j_2 + p_3)^{2j_2}] + 8p_2^2(j_1 - p_3)^{j_1}(j_2 + p_3)^{j_2} + 16p_2(j_1 - p_3)^{3j_1}(j_2 + p_3)^{3j_2} + 64(j_1 - p_3)^{5j_1}(j_2 + p_3)^{5j_2}), \]

\[ R_{12} = [j_1j_2 + (j_1 - j_2)p_3](j_1 - p_3)^{j_1-1}(j_2 + p_3)^{j_2-1}[2p_1(j_1 - p_3)^{j_1}(j_2 + p_3)^{j_2} - p_2^2 + 12p_2(j_1 - p_3)^{2j_1}(j_2 + p_3)^{2j_2}], \]

\[ R_{13} = 0, \]

\[ R_{22} = [j_1j_2 + (j_1 - j_2)p_3](j_1 - p_3)^{j_1-1}(j_2 + p_3)^{j_2-1}(p_1 - 4p_2(j_1 - p_3)^{j_1}(j_2 + p_3)^{j_2} + 16(j_1 - p_3)^{3j_1}(j_2 + p_3)^{3j_2}], \]

\[ R_{23} = 0, \]

\[ R_{33} = j_1j_2 + (j_1 - j_2)p_3, \]

\[ A = [p_1 + 3p_2(j_1 - p_3)^{j_1}(j_2 + p_3)^{j_2} + 8(j_1 - p_3)^{3j_1}(j_2 + p_3)^{3j_2}][-p_1^2 + 4p_1(j_1 - p_3)^{j_1}(j_2 + p_3)^{j_2}[3p_2 - 4(j_1 - p_3)^{2j_1}(j_2 + p_3)^{2j_2}] - 3p_2^3 - 16(j_1 - p_3)^{4j_1}(j_2 + p_3)^{4j_2}[3p_2 - 8(j_1 - p_3)^{2j_1}(j_2 + p_3)^{2j_2}], \]

\[ \rho = [(j_1j_2 + (j_1 - j_2)p_3)^2(j_1 - p_3)^{j_1-1}(j_2 + p_3)^{j_2-1}. \]

Class \( C6(j_1) \)

\[ d_A = 3d_1 + d_3, \quad d_1 = 6sj_1, \quad d_2 = 4sj_1, \quad d_3 = 2s; \]

\[ R_{11} = (9j_1 + 1)(p_3 + 9j_1)^{j_1-1}[p_1[p_2 - 4(p_3 + 9j_1)^{2j_1} + 8(p_3 + 9j_1)^{j_1}[p_2^2 + 2p_2(p_3 + 9j_1)^{2j_1} + 8(p_3 + 9j_1)^{4j_1}]], \]

\[ R_{12} = (9j_1 + 1)(p_3 + 9j_1)^{j_1-1}[2p_1(p_3 + 9j_1)^{j_1} - p_2[p_2 - 12(p_3 + 9j_1)^{2j_1}]], \]

\[ R_{13} = p_1, \]

\[ R_{22} = (9j_1 + 1)(p_3 + 9j_1)^{j_1-1}[p_1 - 4(p_3 + 9j_1)^{j_1}[p_2 - 4(p_3 + 9j_1)^{2j_1}]]; \]

\[ R_{23} = p_2, \]

\[ R_{33} = 9j_1 + (1 - 9j_1)p_3, \]

\[ A = (1 - p_3)[p_1 + (p_3 + 9j_1)^{j_1}[3p_2 + 8(p_3 + 9j_1)^{2j_1}][-p_1^2 + 4p_1(p_3 + 9j_1)^{j_1}[3p_2 - 4(p_3 + 9j_1)^{2j_1}] - 3p_2^3 - 16(p_3 + 9j_1)^{4j_1}[3p_2 - 8(p_3 + 9j_1)^{2j_1}], \]

\[ \rho = (9j_1 + 1)^2(p_3 + 9j_1)^{j_1-1}. \]
Class $D1(j_1)$

\[ d_A = 2d_1, \quad d_1 = 15s_j_1, \quad d_2 = 10s, \quad d_3 = 6s; \]

\[ R_{11} = [p_2(p_3 - 12) + 11p_3^2 + 24p_3 - 576)(4p_2 - p_3^2 - 8p_3 - 192)[-p_3^3 + 6p_2^2(5p_3 - 24) - 15p_2p_3^2(p_3 - 12) + p_3^4 - 55p_3^4 - 280p_3^2 - 2880p_3^2 + 110592]_1^{j_1 - 1}, \]

\[ R_{12} = 0, \]

\[ R_{13} = 0, \]

\[ R_{22} = 4p_2(p_3 - 3) - (p_3 - 48)(p_3^2 + 4p_3 + 24), \]

\[ R_{23} = 6p_2 - 5p_3(p_3 - 12), \]

\[ R_{33} = p_2 - 2(7p_3 - 48), \]

\[ A = [-p_3^3 + 6p_2^2(5p_3 - 24) - 15p_2p_3^2(p_3 - 12) + p_3^4 - 55p_3^4 - 280p_3^2 - 2880p_3^2 + 110592]_1^{j_1 - 1} - p_1^2, \]

\[ \rho = [p_2(p_3 - 12) + 11p_3^2 + 24p_3 - 576)(4p_2 - p_3^2 - 8p_3 - 192). \]

Class $D2$

\[ d_A = 3d_1, \quad d_1 = 10s, \quad d_2 = 6s, \quad d_3 = 4s; \]

\[ R_{11} = p_1(16p_2 - p_3^2 - 192) - 4p_3^2 + 704p_2^2 + 2p_2(p_3^2 - 156p_3^2 + 5376) + 46p_3^4 + 192p_3^4 - 8064p_3^2 + 294912, \]

\[ R_{12} = 24p_1 - 20p_3^2 + 5p_2(p_3^2 + 192) + 2p_3^2(p_3 - 96), \]

\[ R_{13} = p_3(12p_2 - p_3^2), \]

\[ R_{22} = p_1 - 2(28p_2 - 11p_3^2 - 768), \]

\[ R_{23} = p_3(p_3 + 48), \]

\[ R_{33} = p_2 + 6p_3 + 96, \]

\[ A = -p_3^3 + 3p_2^2(40p_2 - 5p_3^2 - 768) - 3p_3^2(20p_2^2 - 5p_3^2(p_3 + 192) - 10p_2p_3^2(p_3 - 48) + 4p_3^2(p_3 - 20)) + 4p_3^2 - 880p_2^2 - 10p_2^2(p_3^2 - 84p_3^2 + 1792) - 5p_2^2(63p_3^4 + 128p_3^4 - 5952p_3^4 + 147456) + 40p_2p_3^2(p_3^2 + 30p_3^2 - 456p_3^2 - 768p_3 + 18432) - 4(45p_3^7 - 580p_3^6 + 2160p_3^5 - 20160p_3^4 - 184320p_3^3 + 4423680p_3^2 - 113246208), \]

\[ \rho = p_2 + 6p_3 + 96. \]
Class $D3(j_1, j_2)$

$$d_A = 3d_1, \quad d_1 = 5s(j_1 + j_2), \quad d_2 = 3s(j_1 + j_2), \quad d_3 = s;$$

$j_1 \geq j_2$ and $j_2 = j_1$ for odd $d_3$;

$$R_{11} = [j_1 j_2 + (j_1 - j_2)p_3](j_1 - p_3)^{j_1 - 1}(j_2 + p_3)^{j_2 - 1}\{4p_1(j_1 - p_3)^{j_1}(j_2 + p_3)^{j_2}[p_2 - 3(j_1 - p_3)^{3j_1}(j_2 + p_3)^{3j_2}] - [p_2 - 48(j_1 - p_3)^{3j_1}(j_2 + p_3)^{3j_2}][p_2^2 + 4p_2(j_1 - p_3)^{3j_1}(j_2 + p_3)^{3j_2} + 24(j_1 - p_3)^{6j_1}(j_2 + p_3)^{6j_2}]\},$$

$$R_{12} = [j_1 j_2 + (j_1 - j_2)p_3](j_1 - p_3)^{2j_1 - 1}(j_2 + p_3)^{2j_2 - 1}[6p_1(j_1 - p_3)^{j_1}(j_2 + p_3)^{j_2} - 5p_2^2 + 60p_2(j_1 - p_3)^{3j_1}(j_2 + p_3)^{3j_2}],$$

$$R_{13} = 0,$$

$$R_{22} = [j_1 j_2 + (j_1 - j_2)p_3](j_1 - p_3)^{j_1 - 1}(j_2 + p_3)^{j_2 - 1}[p_1 - 14p_2(j_1 - p_3)^{2j_1}(j_2 + p_3)^{2j_2} + 96(j_1 - p_3)^{5j_1}(j_2 + p_3)^{5j_2}],$$

$$R_{23} = 0,$$

$$R_{33} = j_1 j_2 + (j_1 - j_2)p_3,$$

$$A = -p_1^2 + 6p_1^2(j_1 - p_3)^{2j_1}(j_2 + p_3)^{2j_2}[5p_2 - 24(j_1 - p_3)^{3j_1}(j_2 + p_3)^{3j_2}] - 15p_1p_2^2(j_1 - p_3)^{j_1}(j_2 + p_3)^{j_2}[p_2 - 12(j_1 - p_3)^{3j_1}(j_2 + p_3)^{3j_2} + p_2^2 - 55p_2(j_1 - p_3)^{3j_1}(j_2 + p_3)^{3j_2} - 280p_2^2(j_1 - p_3)^{6j_1}(j_2 + p_3)^{6j_2} - 2880p_2^2(j_1 - p_3)^{9j_1}(j_2 + p_3)^{9j_2} + 110592(j_1 - p_3)^{15j_1}(j_2 + p_3)^{15j_2},$$

$$\rho = [j_1 j_2 + (j_1 - j_2)p_3]^2(j_1 - p_3)^{j_1 - 1}(j_2 + p_3)^{j_2 - 1}.$$

Class $D4(j_1)$

$$d_A = 3d_1 + d_3, \quad d_1 = 10sj_1, \quad d_2 = 6sj_1, \quad d_3 = 2s;$$

$$R_{11} = (15j_1 + 1)(p_3 + 15j_1)^{j_1 - 1}\{4p_1(p_3 + 15j_1)^{j_1}[p_2 - 3(p_3 + 15j_1)^{3j_1}] - [p_2 - 48(p_3 + 15j_1)^{3j_1}][p_2^2 + 4p_2(p_3 + 15j_1)^{3j_1} + 24(p_3 + 15j_1)^{6j_1}]\},$$

$$R_{12} = (15j_1 + 1)(p_3 + 15j_1)^{2j_1 - 1}[6p_1(p_3 + 15j_1)^{j_1} - 5p_2^2 + 60p_2(p_3 + 15j_1)^{3j_1}],$$

$$R_{13} = p_1,$$

$$R_{22} = (15j_1 + 1)(p_3 + 15j_1)^{j_1 - 1}[p_1 - 14p_2(p_3 + 15j_1)^{2j_1} + 96(p_3 + 15j_1)^{5j_1}],$$

$$R_{23} = p_2,$$

$$R_{33} = 15j_1 + (1 - 15j_1)p_3,$$

$$A = (1 - p_3)[-p_1^2 + 6p_1^2(p_3 + 15j_1)^{2j_1}][5p_2 - 24(p_3 + 15j_1)^{3j_1}] - 15p_1p_2^2(p_3 + 15j_1)^{j_1}[p_2 - 12(p_3 + 15j_1)^{3j_1} + p_2^2 - 55p_2(p_3 + 15j_1)^{3j_1} - 280p_2^2(p_3 + 15j_1)^{6j_1} - 2880p_2^2(p_3 + 15j_1)^{9j_1} + 110592(p_3 + 15j_1)^{15j_1},$$

$$\rho = (15j_1 + 1)^2(p_3 + 15j_1)^{j_1 - 1}.$$
Class $E_1$

\[ d_A = 4d_1, \quad d_1 = 5s, \quad d_2 = 4s, \quad d_3 = 3s; \]
\[ R_{11} = -4p_1p_3 + 3p_2^2 - 27p_2 + 18(2p_3^2 + 9), \]
\[ R_{12} = -6p_1 + p_3(5p_2 + 54), \]
\[ R_{13} = 2(3p_2 + p_3^2), \]
\[ R_{22} = 3p_2 + 2(4p_3^2 + 27), \]
\[ R_{23} = p_1 + 12p_3, \]
\[ R_{33} = p_2 + 9, \]
\[ A = p_1^4 + 40p_1^3p_3 - 6p_1^2[5p_2^2 + 5p_2(p_3^2 + 9) - 30p_3^3 + 162] + 4p_1p_3[5p_2^2 + 45p_2^2 - 30p_2(p_3^2 - 27) + 4p_3^2(4p_3^2 - 135)] - 3p_2^2 + 45p_2^4 - 5p_2^3(2p_3^2 - 27) - 15p_2^2(p_1^4 + 162p_3^2 + 243) + 1620p_2p_3(p_2^3 + 6) - 4(80p_3^6 + 135p_3^4 + 7290p_3^2 - 19683), \]
\[ \rho = 1. \]

Class $E_2$

\[ d_A = 4d_1, \quad d_1 = 6s, \quad d_2 = 4s, \quad d_3 = 3s; \]
\[ R_{11} = -4p_1 + 5p_2^2 + 5p_3^2 + 12, \]
\[ R_{12} = 2p_2(p_3 + 3), \]
\[ R_{13} = p_2^2 - p_3(p_3 - 6), \]
\[ R_{22} = p_1 + 3p_3 + 6, \]
\[ R_{23} = 3p_2, \]
\[ R_{33} = p_1 - 3p_3 + 6, \]
\[ A = -p_1^4 - 16p_1^3 + 2p_1^2[3p_2^2(p_3 + 5) - p_3^3 + 15p_3^2 - 36] - 12p_1[p_2^4 + 2p_2^2(p_3^2 + 6p_3 - 6) + p_3^2(p_2^2 - 4p_3 - 12)] + p_2^6 - 5p_2^4(2p_3^2 - 18p_3 + 15) + 9p_2^2(p_3^4 + 4p_3^3 - 10p_3^2 + 24p_3 - 24) - 9(p_3 + 2)^2(2p_3^3 - 3p_3^2 + 12p_3 - 12), \]
\[ \rho = 1. \]
Class $E_3$

\[d_A = 4d_1, \quad d_1 = 8s, \quad d_2 = 6s, \quad d_3 = 4s;\]

\[R_{11} = 2p_1(p_2 - 54) + 15p_2^2 + 216p_2p_3 + 324(4p_3^2 + 18p_3 + 81),\]

\[R_{12} = 6p_1p_3 - p_2^2 + 18p_2(p_3 + 12) + 1620p_3,\]

\[R_{13} = 6p_1 - p_2(p_3 - 27) + 54p_3,\]

\[R_{22} = 4[3p_1 - p_2(p_3 + 9) + 27(p_3^2 - 2p_3 + 27)],\]

\[R_{23} = p_1 - 3p_2 - 6p_3^2 + 108p_3,\]

\[R_{33} = p_2 - 12p_3 + 81,\]

\[A = (p_1 + 4p_2 + 36p_3 + 162)[p_1^3 - 6p_1^2(6p_2 + 4p_3^2 - 90p_3 + 243) + 12p_1(2p_2^2(p_3 + 9) - 12p_2(2p_3^2 + 18p_3 - 81) + 3(4p_3^3 + 24p_3^2 - 540p_3^2 + 2916p_3 - 6561)] - 4[p_3^2 + 4p_2^2(2p_3^2 - 944784) + 20p_2(p_3^2 - 324)] + 6p_2^2(11p_3^2 - 729p_3 + 729) + 36p_2(4p_3^4 - 120p_3^2 - 2916p_3 + 6561) + 162(8p_3^5 - 12p_3^4 + 288p_3^3 + 13122p_3 - 59049)],\]

\[\rho = 1.\]

Class $E_4$

\[d_A = 4d_1, \quad d_1 = 12s, \quad d_2 = 8s, \quad d_3 = 6s;\]

\[R_{11} = -6[12p_1p_3 - 21p_2^2 - p_2(11p_3^2 - 864) - 36(17p_3^2 + 972)],\]

\[R_{12} = -6[12p_1 - p_3(21p_2 + p_3^2 + 756)],\]

\[R_{13} = p_2^2 + 180p_2 + 60p_3^2,\]

\[R_{22} = 6(9p_2 + 7p_3^2 + 972),\]

\[R_{23} = p_1 + 180p_3,\]

\[R_{33} = 5p_2 + 324,\]

\[A = p_1^4 + 576p_1^3p_3 - 2p_1^2[p_3^3 + 810p_2^2 + 36p_2(11p_3^2 + 1944) + 6(p_3^4 - 4536p_3^2 + 314928)] + 288p_1p_3[7p_2^3 + p_2^2(3p_3^2 + 648) - 144p_2(2p_3^2 - 243) + 20p_2^2(p_3^2 - 324)] + p_2^6 - 324p_2^3 - 36p_2^2(14p_3^2 - 729) - 12p_2^2(p_3^4 + 3780p_3^2 - 104976) + 6p_2(95p_3^4 - 40824p_3^2 + 944784) - 2160p_2p_3^2(p_3^4 - 1080p_3^2 - 104976) + 36(p_3^8 - 3696p_3^6 - 536544p_3^4 - 13604896p_3^2 + 11019960576),\]

\[\rho = 1.\]
Class \( E_5 \)

\[ d_A = 4d_1, \quad d_1 = 30s, \quad d_2 = 20s, \quad d_3 = 12s; \]

\[ R_{11} = 90p_1(12p_2 + p_3^2 + 2^3 \cdot 3^3 \cdot 5 \cdot p_3 - 2^6 \cdot 3^6 \cdot 5^2) - 36p_2^2(19p_3 - 2^2 \cdot 3^3 \cdot 5 \cdot 47) - p_2(29p_3^2 - 2^4 \cdot 3^4 \cdot 5 \cdot p_3^2 - 2^7 \cdot 3^7 \cdot 5^2 - 2^11 \cdot 3^10 \cdot 5^4) + 45(1119 - p_2^3 - 2^4 \cdot 3^3 \cdot 5^2 - 2^6 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot p_3^2 + 2^13 \cdot 3^11 \cdot 5^4 \cdot p_3 + 2^12 \cdot 3^15 \cdot 5^6), \]

\[ R_{12} = 360p_1(3^2 \cdot 3^3 \cdot 5) - 486p_2^2 - 9p_2(19p_2^2 - 2^3 \cdot 3^4 \cdot 5 \cdot p_3 - 2^7 \cdot 3^7 \cdot 5^3) - p_3(p_3^2 - 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot p_3^2 + 2^3 \cdot 3^7 \cdot 5^3 \cdot 13 \cdot p_3 - 2^7 \cdot 3^10 \cdot 5^4 \cdot 19), \]

\[ R_{13} = 2160p_1 + p_2^2 - 1980p_2(p_3 - 2^3 \cdot 3^3 \cdot 5) - 55p_3^2(p_3 - 2^2 \cdot 3^5 \cdot 5), \]

\[ R_{22} = 540p_1 - 324p_2(p_3 + 3^4 \cdot 5^2) - 19p_3^2 + 2^2 \cdot 3^7 \cdot 5^2 \cdot p_3^2 - 2^3 \cdot 3^8 \cdot 5^2 \cdot 7 \cdot p_3 + 2^7 \cdot 3^12 \cdot 5^5, \]

\[ R_{23} = p_1 - 810p_2 - 495p_3(p_3 - 2^2 \cdot 3^3 \cdot 5^2), \]

\[ R_{33} = 11p_2 - 6750(p_3 - 2^4 \cdot 3^4), \]

\[ A = p_1^4 - 360p_1^3[36p_2 + 5(p_2^2 - 2^3 \cdot 3^3 \cdot 5^2 \cdot p_3 + 2^6 \cdot 3^7 \cdot 5^2)] + 2p_2^2[-p_2^3 + 810p_2^2(13p_3 + 2^3 \cdot 3^3 \cdot 5 \cdot 11) + 15p_2(29p_1^3 - 2^2 \cdot 3^5 \cdot 5^2 \cdot 19p_2^2 - 2^7 \cdot 3^9 \cdot 5^2 \cdot p_3 - 2^10 \cdot 3^12 \cdot 5^4) + p_3^2 + 2^3 \cdot 5^3 \cdot 5^3 \cdot p_3^2 + 2^4 \cdot 3^7 \cdot 5^2 \cdot 5^3 \cdot p_3^2 = 2^6 \cdot 3^11 \cdot 5^7 \cdot p_3^2 + 2^7 \cdot 3^14 \cdot 5^7 \cdot p_3^2 + 2^8 \cdot 3^{13} \cdot 5^6 \cdot p_3 - 2^{12} \cdot 3^{15} \cdot 5^7 \cdot 2p_3^3 \cdot 5 \cdot 36 \cdot 5 \cdot p_3 + 2^3 \cdot 3^{10} \cdot 5^7 \cdot p_3^3], 19(19p_3 - 2^2 \cdot 3^6 \cdot 23 \cdot p_3^2 - 2^5 \cdot 3^8 \cdot 5^2 \cdot p_3 - 2^7 \cdot 3^12 \cdot 5^5) + 2p_3^2(p_3 - 3^3 \cdot 5 \cdot p_3^2 - 2^5 \cdot 3^7 \cdot 5^3 \cdot 37 \cdot p_3^2 + 2^10 \cdot 3^10 \cdot 5^4 \cdot 13 \cdot p_3^2 - 2^{14} \cdot 3^15 \cdot 5^6 \cdot p_3 - 2^{12} \cdot 3^20 \cdot 5 \cdot p_3 + 2^2 \cdot 3^5 \cdot 5 \cdot 11 \cdot 463 \cdot p_3^2 - 2^4 \cdot 3^8 \cdot 5^3 \cdot 11 \cdot 563 \cdot p_3^2 + 2^11 \cdot 3^{12} \cdot 5^5 \cdot 59 \cdot p_3^2 + 2^11 \cdot 3^14 \cdot 5^6 \cdot 41 \cdot p_3^2 - 2^{14} \cdot 3^20 \cdot 5^8 \cdot p_3 - 2^{17} \cdot 3^24 \cdot 5^9)] - 330p_2(p_3 + 3^4 \cdot 5^2 \cdot 17 \cdot p_3^2 - 2^2 \cdot 3^7 \cdot 5^3 \cdot 107 \cdot p_3^4 + 2^5 \cdot 3^10 \cdot p_3 \cdot 2^8 \cdot 3^{15} \cdot 5^6 \cdot 7 \cdot p_3^3 - 2^{12} \cdot 3^{18} \cdot 5^8 \cdot p_3 - 2^{14} \cdot 3^{22} \cdot 5^8) + p_3^2 + 2 \cdot 5^2 \cdot 17 \cdot 23 \cdot p_3^2 - 3^5 \cdot 3^14 \cdot 5^4 \cdot 1913 \cdot p_3^4 + 2^3 \cdot 3^{13} \cdot 5^7 \cdot 7 \cdot p_3^2 - 2^{12} \cdot 3^{18} \cdot 5^8 \cdot p_3^2 - 2^{14} \cdot 3^{22} \cdot 5^8) + 499p_3^2 - 2^8 \cdot 3^{15} \cdot 5^6 \cdot 7 \cdot p_3^3 - 2^{12} \cdot 3^{18} \cdot 5^8 \cdot p_3^3 - 2^{12} \cdot 3^{18} \cdot 5^8 \cdot p_3^3 - 2^{14} \cdot 3^{22} \cdot 5^8) + p_3^2 + 2 \cdot 5^2 \cdot 17 \cdot 23 \cdot p_3^2 - 3^5 \cdot 3^14 \cdot 5^4 \cdot 1913 \cdot p_3^4 + 2^3 \cdot 3^{13} \cdot 5^7 \cdot 7 \cdot p_3^2 - 2^{11} \cdot 3^{14} \cdot 5^6 \cdot 17 \cdot 47 \cdot p_3^2 - 2^{10} \cdot 3^7 \cdot 5^8 \cdot 13 \cdot 23 \cdot p_3^2 - 2^12 \cdot 3^20 \cdot 5 \cdot 107 \cdot 3 \cdot p_3^2 - 2^4 \cdot 5 \cdot 2^7 \cdot p_3^2 - 2^3 \cdot 3^14 \cdot 5^7 \cdot 3 \cdot p_3^2 - 2^{16} \cdot 3^{24} \cdot 5 \cdot 13 \cdot p_3^2 - 2^{19} \cdot 3^{29} \cdot 5^3 \cdot p_3^2 + 2^{24} \cdot 3^36 \cdot 5 \cdot 15, \]

\( \rho = 1 \).

In order to allow the reader to get a glimmering of the shape of possible 4-dimensional orbit spaces, we conclude by giving a graphical representation of the (intersections with the coordinate planes of) the positivity regions \( \mathcal{S} \) of some of the allowable \( \mathcal{P} \)-matrices listed above. Our examples are selected by picking up the solution corresponding to the lowest values of the parameters \( j \) and \( s \) in each tower of allowable solutions, but for the case \( \mathcal{A}(1,2,2) \). The sections \( p_1 = 0, p_2 = 0 \) and \( p_3 = 0 \) of \( \mathcal{S} \) will be represented in three distinct graphs on the same row, with axes \( (p_3, p_2) \), \( (p_3, p_1) \) and, respectively \( (p_2, p_1) \). The unit lengths in the two coordinate axes of a graph, or in the homonymous axes of graphs lying in different rows, may be different, but they have been chosen to be the same in homonymous axes of graphs lying in the same row.
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References

[1] G. Sartori, La Rivista del Nuovo Cimento, 14 (1991) 1-120.
[2] D. Mumford, Geometric invariant theory, Erg. Math., Bd. 34, Springer, Berlin, Heidelberg, New York, 1965.
[3] G.W. Schwarz, Inst. Hautes Etudes Sci. Publ. Math. 51 (1980) 37-135.
[4] H. Whitney, Ann. of Math. 66 (1957) 545-556.
[5] G.E. Bredon, Introduction to Compact Transformation Groups, Academic Press, New York, 1972.
[6] D. Hilbert, Math. Ann. 36 (1890) 473-534.
[7] E. Noether, Math. Ann. 77 (1916) 89-92.
[8] G.W. Schwarz, Topology 14 (1975) 63-68.
[9] Yu. M. Gufan, Soviet Phys. Solid State, 13 (1971) 175-184.
[10] M. V. Jarić, Lecture Notes in Physics 135 (1980) 12-16.
[11] M. V. Jarić, J. Math. Phys. 24 (1983) 917-921.
[12] M. Abud and G. Sartori, Phys. Lett. 104 B (1981) 147-152.
[13] M. Abud and G. Sartori, Ann. Phys. 150 (1983) 307-372.
[14] G. Sartori, Mod. Phys. Lett. A 4 (1989) 91-98.
[15] G. Sartori and V. Talamini, Comm. Math. Phys., 139 (1991) 559-588.
[16] C. Procesi and G. W. Schwarz, Inv. Math. 81 (1985) 539-554.
[17] G. Sartori and V. Talamini, in preparation.
[18] G. C. Shephard and J. A. Todd, Canad. J. Math. 6 (1954) 274-304.
[19] G. W. Schwarz, Invent. Math. 49 (1978) 167-191.
[20] G. Sartori and G. Valente, in preparation.
[21] G. Sartori and V. Talamini, Anales de Física, Monografías, Vol. 1 (1993) 459-462, Edited by M.A. del Olmo, M. Santander and J. Mateos Guilarte (CIEMAT, Madrid).