CONTRACTION PROPERTY OF CERTAIN CLASSES OF LOG-M−SUBHARMONIC FUNCTIONS IN THE UNIT BALL

DAVID KALAJ

ABSTRACT. We prove a contraction property of certain classes of smooth functions, whose absolute values of elements are log-hyperharmonic functions in the unit ball, thus extending the results of Kulikov to higher-dimensional space (GAFA (2022)). Moreover, by applying those results we get some new results for harmonic mappings in the complex plane.

CONTENTS

1. Introduction ................................. 1
1.1. Admissible monoid .......................... 4
1.2. Statement of main results ..................... 6
1.3. Extremal set ................................ 6
1.4. Structure of the paper ...................... 7
2. The proof of the main result .................... 7
2.1. Isoperimetric inequality for the hyperbolic ball and the function \( \Upsilon \) .......... 7
3. Weak-type estimate for the \( \mathcal{M} \)−Hardy class \( \mathcal{H}^p \) and the proof of Theorem 1.5 .......... 11
4. Proof of Theorem 1.6 ......................... 13
5. Some applications for the case \( n = 2 \) .......... 14
6. Appendix .................................... 17
Acknowledgments .............................. 22
References ..................................... 22

1. INTRODUCTION

In this paper \( \mathbb{B} = \{ x \in \mathbb{R}^n : |x| < 1 \} \) is the unit ball. Here and in the sequel for \( x = (x_1, \ldots, x_n) \), \( |x| := \sqrt{\sum_{k=1}^{n} x_k^2} \).

Assume that \( \mathcal{M} \) is the group of Möbius transformations of the unit ball onto itself. We introduce the Möbius invariant hyperbolic measure on the unit ball. For \( x \in \mathbb{B} \) we define it as

\[
d\tau(x) = \frac{2^n}{(1 - |x|^2)^n} \frac{dV(x)}{\omega_n},
\]

where \( \omega_n = V(B) \) is the volume of the unit ball.

Key words and phrases. Hyperbolic harmonic functions, isoperimetric inequality.
A mapping \( u \in C^2(\mathbb{B}^n, \mathbb{C}) \) or more generally \( u \in C^2(\mathbb{B}^n, \mathbb{R}^k) \) is said to be hyperbolic harmonic or \( \mathcal{M} \)-harmonic if \( u \) satisfies the hyperbolic Laplace equation

\[
\Delta_h u(x) = (1 - |x|^2)^2 \Delta u(x) + 2(n - 2)(1 - |x|^2) \sum_{i=1}^n x_i \frac{\partial u}{\partial x_i}(x) = 0,
\]

where \( \Delta \) denotes the usual Laplacian in \( \mathbb{R}^n \). We call \( \Delta_h \) the hyperbolic Laplace operator. See Rudin [24] and Stoll [26].

The Poisson kernel for \( \Delta_h \) is defined by

\[
P_h(x, \zeta) = \frac{(1 - |x|^2)^{n-1}}{|x - \zeta|^{2n-2}}, \quad (x, \zeta) \in \mathbb{B} \times \mathbb{S}.
\]

Then for fixed \( \zeta \), \( x \to P_h(x, \zeta) \) is \( \mathcal{M} \)-harmonic function and for a mapping \( f \in L(\mathbb{S}) \), the function

\[
u(x) = P_h[f](x) := \int_{\mathbb{S}} P_h(x, \zeta) f(\zeta) d\sigma(\zeta)
\]

is the Poisson extension of \( f \) and it is \( \mathcal{M} \)-harmonic in \( \mathbb{B} \).

We say that a real function \( u \) is \( \mathcal{M} \)-subharmonic if \( \Delta_h u(x) \geq 0 \). The definition can be extended to the case of upper semicontinuous functions, by using the so-called invariant mean value property [26].

Now we define the Hardy type space.

(1) As in [26, Definition 7.0.1], for \( 0 < p \leq \infty \), we denote by \( S^p \) the Hardy-type space of non-negative \( \mathcal{M} \)-subharmonic functions \( f \) on \( \mathbb{B} \) such that

\[
\|f\|_p^p = \sup_{0 < r < 1} \int_{\mathbb{S}} |f(r \zeta)|^p d\sigma(\zeta) < \infty.
\]

When \( p = \infty \) we put \( \|f\|_\infty = \sup_{x \in \mathbb{B}} f(x) \).

(2) For \( 0 < p < \infty \) we say that a Borel function \( f : \mathbb{B} \to \mathbb{C} \) belongs to the Hardy space \( h^p \) if \( |f| \in S^p \). Then we define \( \|f\|_p := \|f\|_p \). When \( p = \infty \) we put \( \|f\|_\infty = \sup_{x \in \mathbb{B}} |f(x)| \).

Further

\[
(\Delta_h u)(m(x)) = \Delta_h (u \circ m)(x),
\]

for every Möbius transformation \( m \in \mathcal{M} \) of the unit ball onto itself.

For \( n = 2 \) the \( \mathcal{M} \)-harmonic and \( \mathcal{M} \)-subharmonic functions are just harmonic and subharmonic functions.

If \( f \) is \( \mathcal{M} \)-subharmonic, then we have the following Riesz decomposition theorem of Stoll [26, Theorem 9.1.3]:

\[
f(x) = F_{\hat{f}}(x) - \int_{\mathbb{B}} G_h(x, y) d\mu_f(y),
\]

provided that \( f \in S^1 \), where \( F_{\hat{f}}(x) \) is the least \( \mathcal{M} \)-harmonic majorant of \( f \) and \( \mu_f \) is the \( \mathcal{M} \)-Riesz measure of \( f \), and \( G_h(x, y) \) is the Green function of \( \Delta_h \). If \( f \in S^p \), where \( p > 1 \), then \( g(x) = F_{\hat{f}}(x) = P_h[\hat{f}](x) \), where \( \hat{f} \) is the boundary function of \( f \) ([26, Theorem 7.1.1]).
From the formula (1.3), by putting \( u = \text{Id} \), and \( m \in \mathcal{M} \), we get
\[
\Delta_h m = 2(n - 2)(1 - |m|^2)m.
\]
So Möbius transformations are (considered as vectorial functions) hyperbolic harmonic only in the case \( n = 2 \).

By putting \( u(x) = g(|x|) \) and inserting in (1.1) we arrive to the equation
\[
\Delta_h u = (1 - r^2)^n \left( \frac{(2(2 + n)r^2 + (1 + n)(1 - r^2))g'(r)}{r} + (1 - r^2)g''(r) \right),
\]
where \( r = |x| \). The hypergeometric function \( F \), which we use in this paper is defined by
\[
F\left[ a, b, c ; u, v ; t \right] := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{n!(u)_n(v)_n} t^n, \quad \text{for } |t| < 1,
\]
and by the continuation elsewhere. Here \((a)_n\) denotes the shifted factorial, i.e.,
\( (a)_n = a(a+1)\ldots(a+n-1) \) with any real number \( a \).

Then one of solutions to the equation \( \Delta_h \log v = -4b \), is given by
\[
v = \exp \left\{ \frac{b(2 - n)r^2}{(n - 1)n} F\left[ 1, 1, 2 - \frac{n}{2} ; 2, 1 + \frac{n}{2} ; r^2 \right] \right\} (1 - r^2)^{\frac{b}{n-1}}.
\]
Then by setting \( b = (n - 1)^2 \) we obtain
\[
\Phi_n(r) = \exp \left\{ \frac{(n - 1)(2 - n)r^2}{n} F\left[ 1, 1, 2 - \frac{n}{2} ; 2, 1 + \frac{n}{2} ; r^2 \right] \right\} (1 - r^2)^{n-1}.
\]
We will also sometimes write \( \Phi_n(x) \) instead of \( \Phi_n(|x|) \). Note that
\[
E_n (1 - r^2)^{n-1} \leq \Phi_n(r) \leq (1 - r^2)^{n-1}
\]
and inequality is strict for \( n > 2 \) and \( r > 0 \). Here
\[
E_n = \exp \left\{ \frac{(n - 1)(2 - n)}{n} F\left[ 1, 1, 2 - \frac{n}{2} ; 2, 1 + \frac{n}{2} ; 1 \right] \right\}.
\]
If \( n = 2 \), then \( \Phi_n(|x|) = 1 - |x|^2 \) and this coincides with the case treated in [15] by Kulikov.

**Definition 1.1.** For \( 0 < p < \infty \) and \( \alpha > 1 \) we say that a complex smooth function \( f \) in \( \mathbb{B} \) belongs to the \( \mathcal{M} \)-Bergman space \( B^p_{\alpha} \) if
\[
\| f \|_{\alpha,p}^p = c(\alpha) \int_{\mathbb{B}} |f(x)|^p \Phi^\alpha_n(|x|) d\tau(x) < \infty,
\]
where
\[
\frac{1}{c(\alpha)} = \int_{\mathbb{B}} \Phi^\alpha_n(x)(1 - |x|^2)^{-n} dV(x),
\]
and \( \Phi_n \) is a function defined in (1.4) above.
Observe that in view of (1.5),
\[
\frac{1}{c(\alpha)} \geq E_n^\alpha \int_0^1 r^n (1 - r^2)^{\alpha(n-1)-1} dr.
\]

This implies that
\[
\lim_{\alpha \to 1+} c(\alpha) = 0.
\]

1.1. Admissible monoid. We define $\mathcal{E}$ to be the set of real analytic complex functions $g$ in $\mathbb{B}$ such that $f := \log |g|$ is $\mathcal{M}$--subharmonic. If $g(a) = 0$, then we put $f(a) = -\infty$. Let $\mathcal{E}_+ = \{ f \in \mathcal{E} : f \geq 0 \}$. Observe that $\mathcal{E}$ is a monoid where the operation is simply the multiplication of two functions. Observe that $1 = e^0$, so $1 \in \mathcal{E}$. This monoid contains the Abelian group $\mathcal{G} = \{ e^f : \Delta_h f = 0 \}$. Then for $a, b \in \mathcal{E}_+$, $c, d \in \mathcal{E}$, $p \geq 0$ and $\alpha, \beta > 0$ we have

(1) $a \cdot b \in \mathcal{E}_+$, $c \cdot d \in \mathcal{E}$,
(2) $a^p \in \mathcal{E}_+$, $c^p \in \mathcal{E}$,
(3) $\alpha a + \beta b \in \mathcal{E}_+$.

In other words, $\mathcal{E}_+$ is a convex cone and at the same time a monoid.

Previous statements follow from the straightforward calculation of the invariant Laplacian. First of all, we have
\[
\alpha e^f = e^{\log \alpha + f}.
\]
So if $a \in \mathcal{E}_+$ and $\alpha > 0$, then so is $\alpha a$. Further if $a = e^f$ and $b = e^g$, then $c = a + b = e^{\log (e^f + e^g)}$. Thus
\[
\Delta_h \log c = \Delta_h \log (e^f + e^g) = \frac{e^f \Delta_h f + e^g \Delta_h g}{e^f + e^g} + \frac{e^{f+g}(1 - r^2)|\nabla (f + g)|^2}{e^f + e^g},
\]
where
\[
A = \Delta_h g = f, \quad B = \Delta_h f = g.
\]

So if $A, B \geq 0$, then $\Delta_h \log (e^f + e^g) \geq 0$. We also refer to the paper [6] for some similar properties of log-subharmonic functions.

Let us collect some additional features of $\mathcal{M}$--harmonic and $\mathcal{M}$--subharmonic functions.

(1) If log $f$ is $\mathcal{M}$--subharmonic, then $f = e^{\log f}$ is $\mathcal{M}$-subharmonic.
(2) If log $f$ is $\mathcal{M}$--subharmonic, then $p \log f$ is $\mathcal{M}$-subharmonic and so $f^p = e^{p \log f}$ is $\mathcal{M}$-subharmonic.
(3) If $f$ is $\mathcal{M}$--harmonic, then
\[
|f| = \max\{ f(x), -f(x) \}
\]
is $\mathcal{M}$--subharmonic.
(4) If $f$ is $\mathcal{M}$--harmonic, then $f \circ m$ is $\mathcal{M}$--harmonic, for every Möbius transformation $m \in \mathcal{M}$ of the unit ball onto itself.
(5) If $f$ is $\mathcal{M}$--harmonic, then $|f|^p$ is $\mathcal{M}$--subharmonic for $p \geq 1$. 
For the above facts, we refer to the monograph by M. Stoll [26]. Let us illustrate the proof of one of properties: If $u$ is $\mathcal{M}$—subharmonic, then $e^u$ is $\mathcal{M}$—subharmonic. Indeed

$$\Delta_h e^u = (1 - |x|^2)^2 |\nabla u|^2 e^u$$

(1.8)
$$+ e^u((1 - |x|^2)^2 \Delta u(x) + 2(n - 2)(1 - |x|^2) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}(x))$$

$$= (1 - |x|^2)^2 |\nabla u|^2 e^u + e^u \Delta_h u \geq 0.$$

Remark 1.2. Unfortunately, the class of holomorphic functions in $\mathbb{B} \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ for $n \geq 2$ is not contained in the monoid $\mathcal{E}$. To see this, let $n = 2$ and take the holomorphic mapping $f(w, z) = g(z)$ which depends only on $z$. Here $g$ is a holomorphic non-vanishing function. Then straightforward computations give that

$$\Delta_h \log |f(w, z)| = 8(1 - |w|^2 - |z|^2)\Re \left( \frac{z g'(z)}{g(z)} \right), \quad |z| < 1.$$

Hence the function $f$ is $\mathcal{M}$—subharmonic if and only if $g$ is a starlike function, i.e. if $\Re \left( \frac{z g'(z)}{g(z)} \right) \geq 0$ for $|z| < 1$.

**Definition 1.3.** For $0 < p < \infty$ and $\alpha > 1$ we define

1. the $\mathcal{M}$—Hardy monoid $\mathbb{H}^p$ consisting of functions $f$ in $\mathbb{H}^p \cap \mathcal{E}$, having harmonic majorant $F_f \in \mathbb{H}^p$;
2. the $\mathcal{M}$—Bergman monoid $\mathbb{B}^p_\alpha$ consisting of functions $f$ in $\mathbb{B}^p_\alpha \cap \mathcal{E}$ having harmonic majorant $F_f \in \mathbb{B}^p_\alpha$ which for $n \geq 3$ satisfies the additional condition: there is a sequence of bounded $\mathcal{M}$—harmonic functions $F_k$ such that $\|F_f - F_k\|_{\alpha, p} \to 0$ when $k \to \infty$.

We also refer to [27] for the corresponding Bergman space of holomorphic mappings in the space which is different from our space.

Remark 1.4. Note that the additional condition in the previous definition is redundant for smooth subharmonic functions and for $n = 2$. In this case the dilatations $f_\rho(z) = f(\rho z)$ are subharmonic, $\rho = n/(n + 1)$ and converge in $\mathbb{B}^p_\alpha$ norm to $f$. See e.g. [10, Proposition 1.3].

Observe that for $f(x) \equiv 1$ we have $\|f\|_p = \|f\|_{\alpha, q} = 1$ for all $p, q > 0$ and $\alpha > 1$.

An important property of these spaces is that point evaluations are continuous functionals. For this fact see Proposition 6.1.

One of the interesting questions about those spaces is which space is the subset of the other space. We consider the case when the quotient $\mathbb{B}^p_\alpha$ is held constant, in which case we have

$$\mathbb{B}^p_\alpha \subset \mathbb{B}^q_\beta, \quad \frac{p}{\alpha} = \frac{q}{\beta} = r, \quad p < q$$

and $\mathbb{H}^r$ is contained in all these spaces. This follows from our results.

Recently for $n = 2$, it was asked whether these embeddings are actually contractions, that is whether the norm $\|f\|_{\mathbb{B}^\alpha}$ is decreasing in $\alpha$. In the case of Bergman
spaces, this question was asked by Lieb and Solovej [16]. They proved that such contractivity implies their Wehrl-type entropy conjecture for the SU(1, 1) group. In the case of contractions from the Hardy spaces to the Bergman spaces, it was asked by Pavlović in [23] and by Brevig, Ortega-Cerdà, Seip, and Zhao [3] concerning the estimates for analytic functions. In a recent paper [15], Kulikov, confirmed these conjectures, and he proved more general results where the function \( t^r \) is replaced with a general convex or monotone function, respectively.

1.2. Statement of main results. In this paper, we extend all those results to the higher-dimensional space by proving the following theorems.

**Theorem 1.5.** Let \( p > 0 \). \( G : [0, \infty) \to \mathbb{R} \) be an increasing function. Then the maximum value of

\[
\int_{\mathbb{B}} G(|f(x)|^p \Phi_n(x)) d\tau(x)
\]

is attained for \( f(x) \equiv 1 \), subject to the condition that \( f \in h^p \) and \( \|f\|_p = 1 \).

**Theorem 1.6.** Let \( p > 0 \) and \( \alpha > 0 \). Let \( G : [0, \infty) \to \mathbb{R} \) be a convex function. Then the maximum value of

\[
\int_{\mathbb{B}} G(|f(x)|^p \Phi_n(x))^{\alpha} d\tau(x)
\]

is attained for \( f(x) \equiv 1 \), subject to the condition that \( f \in B_{\alpha}^p \) and \( \|f\|_{\alpha,p} = 1 \).

Applying these theorems to the convex and increasing function \( G(t) = t^{s}, s > 1 \), we get that all the embeddings above between Hardy and Bergman monoids are contractions, and we have the following corollary (Note that Theorem 1.5 is used for the second inequality and Theorem 1.6 is used for the first one).

**Corollary 1.7.** For all \( 0 < p < q < \infty \) and \( 1 < \alpha < \beta < \infty \) with \( \frac{p}{\alpha} = \frac{q}{\beta} = r \) for all \( f \in h^r \), we have

\[
\|f\|_{\beta,q} \leq \|f\|_{\alpha,p} \leq \|f\|_{h^r}
\]

with equality for \( f(z) \equiv c \), where \( c \in \mathbb{C} \), or for \( f \) belonging to the extremal set below.

1.3. Extremal set. It is important to mention that the Möbius group acts not only on the measure \( \tau \) but on the spaces \( B_{\alpha}^p \) as well. More precisely, given a function \( f \in B_{\alpha}^p \) and \( m \in \mathcal{M} \), the function

\[
g(x) = f(m(x)) \frac{\Phi_n^{\alpha/p}(|m(x)|)}{\Phi_n^{\alpha/p}(|x|)}
\]

also belongs to the space \( B_{\alpha}^p \) and moreover it has the same norm as \( f \) and the same distribution of the function \( |f(x)|^p \Phi_n^\alpha(x) \) with respect to the measure \( \tau \). We need to check that \( \Delta_h \log g \geq 0 \), if \( \Delta_h \log f \geq 0 \), and this follows from the formula
and straightforward calculations:

\[
\Delta_h \log g(x) = \Delta_h \log(f(m(x))) + \Delta_h \log \Phi^\frac{\alpha}{p}(|m(x)|) \\
+ \Delta_h \log \Phi^\frac{\alpha}{p}(|x|) \\
= \Delta_h \log(f(y))_{y=m(x)} + \Delta_h \log \Phi^\frac{\alpha}{p}(|y|)_{y=m(x)} - \Delta_h \log \Phi^\frac{\alpha}{p}(|y|)_{y=x} \\
\geq 0 + \left(4(n-1)^2 - 4(n-1)^2\right) \frac{\alpha}{p} = 0.
\]

To prove the second statement, we only need to point out the well-known formula for the Jacobian of Möbius transformations of the unit ball onto itself

\[
J_m(x) = \frac{(1 - |m(x)|^2)^n}{(1 - |x|^2)^n}.
\]

See e.g. [26, p. vii].

In particular, when \(f(x) \equiv 1\) we get \(g(x) = \frac{\Phi^\frac{\alpha}{p}(|m(x)|)}{\Phi^\frac{\alpha}{p}(|x|)}\) and the function \(g\) gives us the maximal value in (1.9) and (1.10) for every \(m \in M\).

We believe that our results also can be formulated for the Hardy and Bergman spaces in the upper half-space, by using a conformal mapping from the unit ball or by directly translating our methods.

1.4. **Structure of the paper.** The paper contains 5 more sections. In Section 2 we prove a general monotonicity theorem for the hyperbolic measure of the superlevel sets of log-Möbius subharmonic functions, which is an adaptation of the beautiful method from [21, 15]. Then, in Sections 3 and 4 we deduce from it Theorems 1.5 and 1.6 respectively. Notice that the proof of Theorem 1.6 is even simpler than the proof of the analogous theorem in [15] for the planar case. Finally, in Section 5 we briefly discuss an application of Corollary 1.7 to coefficient estimates for harmonic functions and some important classes of log-subharmonic functions. In the Appendix below are proved two propositions that deal with Hardy and weight-Bergman spaces of \(M\)-subharmonic functions.

2. **The proof of the main result**

We begin with

2.1. **Isoperimetric inequality for the hyperbolic ball and the function \(\Upsilon\).** For a Borel set \(E \subset B\) we recall the definition of the hyperbolic volume

\[
|E|_h = V(E) = \int_E \left(\frac{2}{1 - |x|^2}\right)^n dx.
\]

Moreover the hyperbolic perimeter is defined by

\[
|\partial E|_h = P(E) = \int_{\partial E} \left(\frac{2}{1 - |x|^2}\right)^{n-1} d\mathcal{H}^{n-1}(x).
\]

Assume that \(B_s\) is the ball centered at the origin with the radius \(\tanh \frac{s}{2}\).
The isoperimetric property of hyperbolic ball was established by E. Schmidt [25] see also [2, 11]. He proved that for every Borel set \( E \subset \mathbb{B} \) of finite perimeter \( P(E) \), such that \( V(E) = V(\mathbb{B}_s) \) and \( s > 0 \) we have

\[(2.1)\quad P_s \leq P(E),\]

where \( P_s \) is the perimeter of \( \mathbb{B}_s \) defined by

\[P_s = n\omega_n \sinh^{n-1}(s) = P(\mathbb{B}(0, \tanh s/2)).\]

The volume of \( \mathbb{B}_s \) is given by

\[V_s = v(s) := n\omega_n \int_0^s \sinh^{n-1}(t) dt = V(\mathbb{B}(0, \tanh s/2)).\]

Here \( \omega_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \) is the Euclidean volume of the unit ball. Since \( s \to v = V_s \) is increasing, it has an inverse function \( S(v) = s \). Then define the function \( \Upsilon \) by

\[(2.2)\quad \Upsilon(v) = \frac{v}{P^2 S(v)}\]

and thus by \((2.1)\)

\[(2.3)\quad P(E)^2 / V(E) \geq 1/ \Upsilon(V(E)),\]

with an equality in \((2.3)\) if and only if \( E \) is a ball. In the following remark, we give a connection with the isoperimetric inequality in the Euclidean space and it is given a specific inequality for \( n = 2 \).

**Remark 2.1.** For \( s > 0 \) we have

\[
\frac{V_s}{P_s^{n/(n-1)}} = \frac{n\omega_n \int_0^s \sinh^{n-1}(t) dt}{\omega_n^{n/(n-1)} \sinh^n(s)} = \frac{\psi(s)}{\phi(s)} = \frac{\psi'(t)}{\phi'(t)} = c_0 \frac{1}{\cosh t} \leq c_0
\]

where

\[c_0 = n \frac{1}{\omega_n^{1/2}}.\]

Now

\[P^2 \geq V / \Upsilon(V) \geq cV^{2(n-1)/n}\]

where

\[c = n^2 \omega_n^{2/n}.\]

It can be proved that

\[(2.4)\quad P^n - c^n V^{n-1} = P^n - n^n \omega_n V^{n-1} \geq (n - 1)^n V^n.\]

If \( n = 2 \) then the estimate \((2.4)\) is equivalent to the estimate \((2.3)\) (for the hyperbolic plane of negative Gaussian curvature \(-1\) see [22]). In this case \( \Upsilon(V) = \frac{1}{4\pi^2} \). It seems unlikely that we can give an explicit expression for the function \( \Upsilon \) for a higher-dimensional case, but we don’t need it in the proofs of our results.
Let $f$ be a real analytic complex valued function such that $v = |f|$ is log-$\mathcal{M}$-log-subharmonic function in $\mathbb{B}$ and such that $u(x) = v(x)^a \Phi_n^a(x)$ is bounded and goes to 0 uniformly as $|x| \to 1$. Then the superlevel sets $A_t = \{ x : u(x) > t \}$ for $t > 0$ are compactly embedded into $\mathbb{B}$ and thus have finite hyperbolic measure $\mu(t) = \tau(A_t)$.

In this section, we prove the following theorem which says that a certain function related to this measure is decreasing.

**Theorem 2.2.** Let $\alpha \geq 1$ and $a \geq 0$ and assume that $f$ is a real analytic complex valued function such that $v = |f| : \mathbb{B} \to [0, +\infty)$ is a log-$\mathcal{M}$-subharmonic function. Assume further that the function $u(x) = |f(x)|^a \Phi_n^a(x)$ is bounded and $u(x)$ tends to 0 uniformly as $|x| \to 1$. Then the function

$$g(t) = t \exp \left[ \int_0^{\mu(t)} \gamma \Upsilon(x) dx \right],$$

is decreasing on the interval $(0, t_0)$, where $\gamma = \alpha(n - 1)^2$, $\Upsilon$ is defined in (2.2) and $t_0 = \max_{x \in \mathbb{B}} u(x)$.

If $f(x) \equiv 1$, the function $g$ turns out to be constant and this is an important property of $g$.

The proof of this theorem is mostly based on the methods developed in [21], translated from the Euclidean to the hyperbolic setting, and in [15] translated from the planar case to the higher-dimensional case. We already introduced the hyperbolic hyper-surface area, associated with the measure $\tau$.

**Proof of Theorem 2.2.** We start with the formula

$$\mu(t) = \tau(A_t) = \int_{A_t} \frac{2^n}{(1 - |x|^2)^n} dx = \int_{t}^{\max u} \int_{|u(x)| = \kappa} \frac{2^n}{(1 - |x|^2)^n} d\mathcal{H}^{n-1}(x) d\kappa.$$

Then we get

$$- \mu'(t) = \int_{u=t} |\nabla u|^{-1} \frac{2^n d\mathcal{H}^{n-1}(x)}{(1 - |x|^2)^n}$$

along with the claim that $\{ x : u(x) = t \} = \partial A_t$ and that this set is a smooth hypersurface for almost all $t \in (0, t_0)$. Here $dS = d\mathcal{H}^{n-1}$ is $n - 1$ dimensional Hausdorff measure. Observe that a similar formula has been proved in [15]. These assertions follows the proof of Lemma 3.2 from [21]. We point out that, since $u$ is real analytic, then it is well-known fact from measure theory that the level set $\{ x : u(x) = t \}$ has a zero measure ([20]), and this is equivalent to the fact that the $\mu$ is continuous.

Following the approach from [21] [15], our next step is to apply the Cauchy–Schwarz inequality to the hyperbolic area of $\partial A_t$:

$$|\partial A_t|^2 = \left( \int_{\partial A_t} \frac{2^{n-1} dS}{(1 - |x|^2)^{n-1}} \right)^2 \leq \int_{\partial A_t} |\nabla u|^{-1} \frac{2^n dS}{(1 - |x|^2)^n} \int_{\partial A_t} |\nabla u|^{2n-2} dS.$$
Let \( \nu = \nu(x) \) be the outward unit normal to \( \partial A_t \) at a point \( x \). Note that, \( \nabla u \) is parallel to \( \nu \), but directed in the opposite direction. Thus we have \( |\nabla u| = -\langle \nabla u, \nu \rangle \). Also, we note that since for \( x \in \partial A_t \) we have \( u(x) = t \), we obtain for \( x \in \partial A_t \) that

\[
\frac{|\nabla u(x)|}{t} = \frac{|\nabla u(x)|}{u} = \langle \nabla \log u(x), \nu \rangle.
\]

Now the second integral on the right-hand side of (2.6) can be evaluated by the Gauss’s divergence theorem:

\[
\int_{\partial A_t} \frac{|\nabla u|dS}{(1 - |x|^2)^{n-2}} = -t \int_{A_t} \text{div} \left( \frac{\nabla \log u(x)}{(1 - |x|^2)^{n-2}} \right) dx
\]

\[
= -t \int_{A_t} \frac{1}{(1 - |x|^2)^n} \Delta_h \log u(x) dx.
\]

Now we plug \( u = v(x)\alpha \Phi_n^\alpha(x) \), where \( v(x) = |f(x)| \), and calculate

\[
-t \Delta_h \log(v^\alpha \Phi_n^\alpha) = -(at \Delta_h \log v + t\alpha \Delta_h \log \Phi_n) \leq 0 + 4t\gamma,
\]

where \( \gamma = \alpha(n - 1)^2 \). By using (2.5) and (2.6) we obtain

\[
|\partial A_t|^2 \leq (-\mu'(t)) \int_{\partial A_t} \frac{|\nabla u|2^{n-2}dS}{(1 - |x|^2)^{n-2}}
\]

\[
\leq -2^{n-2} \cdot 4t\gamma \mu'(t)\mu(t)/2^n
\]

\[
= -t\gamma \mu'(t)\mu(t).
\]

So by (2.3), we have

(2.7)

\[
t\gamma \mu'(t)\mu(t) + \frac{\mu(t)}{\Upsilon(\mu(t))} \leq 0
\]

with equality in (2.7) if and only if \( v \) is a constant because in that case \( A_t \) is a ball centered at the origin.

Thus

\[
G(t) = -\int_t^{t_o} \gamma \mu'(t)\Upsilon(\mu(t))dt - \int_t^{t_o} \frac{1}{t} dt = \int_0^{\mu(t)} \gamma \Upsilon(x)dx - \log \left( \frac{t_o}{t} \right)
\]

is non-increasing. In the case \( v \equiv 1 \), \( t_o = 1 \) and \( \mu(t_o) = 0 \). Moreover

\[
g(t) := \exp(G(t)) = t \exp \left[ \int_0^{\mu(t)} \gamma \Upsilon(x)dx \right]
\]

is non-increasing.

\[\square\]

**Remark 2.3.** Note that for the function \( f(x) \equiv 1 \) everywhere in the proof above we have equalities for all values of \( a \) and \( \alpha \).
3. Weak-type estimate for the $M$–Hardy class $h^p$ and the proof of Theorem 1.5

In this section, we are going to prove the following bound for the measure of the so-called superlevel sets of functions from the Hardy spaces. Theorem 1.5 is then an easy matter. In what follows we keep the same notation as in the previous section.

**Theorem 3.1.** Let $f$ be a real analytic complex valued function such that $v = |f|$ is log-$M$-subharmonic, i.e. assume that $f \in h^p$, where $p > 0$ and assume that its $h^p$ norm is 1 and put $u(x) = |f(x)|^p \Phi_n(x)$. Then for all $t \in (0, \infty)$ we have

$$
\mu(t) \leq \mu_1(t),
$$

where

$$
\mu(t) = |\{x : u(x) \geq t\}|_h \quad \text{and} \quad \mu_1(t) = |\{x : \Phi_n(x) \geq t\}|_h.
$$

Note that this theorem extends the corresponding result in [15], where [3, Conjecture 2] is verified. Indeed it is easy to check that

$$
|\{x : \Phi_n(x) \geq t\}|_h = 4\pi \max\{1/t - 1, 0\},
$$

for $n = 2$ which coincides with the corresponding result of Kulikov in [15] after normalization.

**Proof of Theorem 3.1.** Put $t_0 = \max_{x \in B} u(x)$. This number is well-defined since we have $u(x) \to 0$ as $|x| \to 1$ uniformly because of Proposition 6.1 below. The condition $p > 1$ in Proposition 6.1 make no trouble. Indeed, if $p > 0$ and if $f$ is a positive log-$M$–subharmonic function such that $f \in h^p$. Then $g = f^{p/2}$ is positive log-$M$–subharmonic function such that $g \in h^2$. Therefore by Proposition 6.1, $u(x) = |g(x)|^2(1 - |x|^2) \to 0$ as $|x| \to 1$.

In particular, for $t \geq t_0$ the bound (3.1) holds trivially.

Assume that there exists some $0 < t_1 < t_0$ such that $\mu(t_1) > \mu_1(t_1)$. Then $\mu(t_1) = \mu_1(t_1/c)$ for some $c > 1$, because $\lim_{s \to 0} \mu_1(s) = +\infty$. We claim that in that case for all $0 < t < t_1$ we have $\mu(t) \geq \mu_1(t/c)$.

Indeed, by applying the pointwise bound together with $u(x) \to 0$ as $|x| \to 1$, we see that Theorem 2.2 can be applied to $f$ with $a = p$, $\alpha = 1$, and we get that

$$
g(t) = t \exp \left[ \int_0^{\mu(t)} \gamma \Upsilon(x) dx \right],
$$

is decreasing. Since
\[ g(t) = t \exp \left[ \int_0^{t_1} \gamma \Upsilon(x) \, dx \right] \]
\[ = t \exp \left[ \int_0^{t_1/c} \gamma \Upsilon(x) \, dx \right] \]
\[ = c \cdot t/c \exp \left[ \int_0^{t/c} \gamma \Upsilon(x) \, dx \right] \]
\[ < g(t) = t \exp \left[ \int_0^{t} \gamma \Upsilon(x) \, dx \right] \]

which implies that \( \mu(t) \geq \mu_1(t/c) \).

Now we are going to use Proposition 6.2, which implies that \( \| f \|_{pr/r,pr} \to \| f \|_p = 1 \) as \( r \to 1^+ \). Note that we can express the \( B^{pr} \) norms through the function \( \mu(t) \):

\[ \| f \|_{pr/r,pr} = c_r \int_0^{t_1} \mu(t) t^{-1} \, dt, \]

where \( c_r = c(r) \) is defined in (1.6), and it satisfies the relation \( c_r \int_0^1 \mu_1(r) t^{r-1} \, dt = 1 \). We now use the fact that \( c_r \to 0 \) as \( r \to 1 \) (see (1.7)). By the formula (3.2) and above bound we have

\[ \| f \|_{pr/r,pr} \geq c_r \int_0^{t_1} (\mu_1(r/c)) t^{r-1} \, dt = c_r c \int_0^{t_1/c} (\mu_1(s)) s^{r-1} \, ds. \]

On the other hand

\[ 1 = c_r \int_0^1 \mu_1(t) t^{r-1} \, dt \]
\[ = c_r \int_0^{t_1/c} \mu_1(t) t^{r-1} \, dt + c_r \int_{t_1/c}^1 \mu_1(t) t^{r-1} \, dt = P(r) + Q(r). \]

Since \( c_r \to 0 \) as \( r \to 1 \), we have that \( Q(r) \to 0 \) as \( r \to 1 \) because the function we are integrating is bounded. Therefore, \( P(r) \to 1 \) as \( r \to 1 \). On the other hand the right-hand side of (3.3) is at least \( cP(r) \). Therefore \( 1 = \lim_{r \to 1} \| f \|_{pr/r,pr} \geq c \lim_{r \to 1} P(r) = c \) which is a contradiction. Recall that \( c > 1 \). 

\( \square \)

**Proof of Theorem 1.5.** As in (1.5), we can assume that \( \lim_{t \to 0^+} G(t) = 0 \). Then this integral can be expressed through the function \( \mu(t) \) as

\[ \int_0^\infty \mu(t) dG(t). \]

Note that here we used that the function \( \mu(t) \) is continuous, that is the sets \( \{ x \in \mathbb{B} : u(x) = t \} \), \( t > 0 \), have zero measure.

Since \( G \) is increasing, measure \( dG(t) \) is positive. Thus, by (3.1) this integral is at most

\[ \int_0^\infty \mu_1(t) dG(t), \]
which is the value of \( \|f\|_{\alpha,p} = 1 \) for \( f(x) \equiv 1 \).

4. PROOF OF THEOREM 1.6

As in the proof of Theorem 1.5, we restrict ourselves to the case \( \lim_{t \to 0^+} G(t) = 0 \). Let \( \mu(t) = \tau(\{x : u(x) > t\}) \) where \( u(x) = |f(x)|^p(\Phi_n(x))^\alpha \). Applying Theorem 2.2 to \( f \) with \( a = p \), we get that the function
\[
g(t) = t \exp \int_0^t \gamma \Upsilon(x) dx,
\]
is decreasing on \( (0, t_\circ) \) with \( t_\circ = \max_{x \in D} u(x) \). Proposition 6.1 ensures the existence of \( t_\circ \).

For \( f \equiv 1 \), \( g \) is a constant function equal to 1.

Let
\[
\Lambda(u) = \int_0^u \gamma \Upsilon(x) dx
\]
and \( \Theta = \Lambda^{-1} \). Note that \( \Theta \) is increasing. Then
\[
\mu(t) = \Theta \left( \log \frac{g(t)}{t} \right).
\]

We assume that \( \|f\|_{\alpha,p} = 1 \), that is
\[
I_1 = \int_0^{t_\circ} \mu(t) dt = \int_0^{t_\circ} \Theta \left( \log \frac{g(t)}{t} \right) dt = \frac{1}{c(\alpha)}.
\]

Now the integral in (1.10) can be rewritten as
\[
I_2 = \int_0^{t_\circ} \Theta \left( \log \frac{g(t)}{t} \right) G'(t) dt.
\]

Then by Lemma 4.1 below, by taking \( \Phi(s) = \Theta(\log(s)) \) and \( \Psi(t) = G'(t) \), the maximum of \( I_2 \) under \( I_1 = \frac{1}{c(\alpha)} \) is attained for \( g \equiv 1 \).

Lemma 4.1. Assume that \( \Phi, \Psi \) are positive increasing functions and \( g \) positive non-increasing such that
\[
\int_0^{t_\circ} \Phi \left( \frac{g(t)}{t} \right) dt = \int_0^{t_\circ} \Phi \left( \frac{1}{t} \right) dt = c.
\]

Then
\[
\int_0^{t_\circ} \Phi \left( \frac{g(t)}{t} \right) \Psi(t) dt \leq \int_0^{t_\circ} \Phi \left( \frac{1}{t} \right) \Psi(t) dt.
\]

Proof: Choose \( a \in [0, t_\circ] \) such that \( g(t) \geq 1 \) for \( t \leq a \) and \( g(t) \leq 1 \) for \( t \geq a \). Then
\[
\chi(t) := (\Phi(g(t)/t) - \Phi(1/t)) (\Psi(t) - \Psi(a)) \leq 0
\]
for all $t \in [0, t_0]$. By integrating $\chi(t)$ for $t \in (0, t_0)$ we give

$$\int_0^{t_0} \Phi \left( \frac{1}{t} \right) \Psi(a) - \Phi \left( \frac{g(t)}{t} \right) \Psi(a) - \Phi \left( \frac{1}{t} \right) \Psi(t) + \Phi \left( \frac{g(t)}{t} \right) \Psi(t) \, dt$$

$$= \Psi(a) \int_0^{t_0} \left( \Phi \left( \frac{g(t)}{t} \right) - \Phi \left( \frac{1}{t} \right) \right) \, dt$$

$$+ \int_0^{t_0} \Psi(t) \left( \Phi \left( \frac{g(t)}{t} \right) - \Phi \left( \frac{1}{t} \right) \right) \, dt \leq 0.$$

Since

$$\int_0^{t_0} \left( \Phi \left( \frac{g(t)}{t} \right) - \Phi \left( \frac{1}{t} \right) \right) \, dt = 0,$$

the result follows.

\[ \Box \]

5. SOME APPLICATIONS FOR THE CASE $n = 2$

Let $B_{2/p}^2$, $1 < p < 2$ be the space of harmonic functions in the unit disk $\mathbb{D}$ subject to the condition

$$\| f \|_{2/p, 2}^2 := \int_\mathbb{D} |f(z)|^2 (1 - |z|^2)^{2/p-1} \frac{dxdy}{\pi} < \infty$$

and let $h^p$ be the standard Hardy space with the norm

$$\| \| f \| \|_{h^p} := \| \sqrt{|a|^2 + |b|^2} \|_{p}^2.$$

Assume that $f = a + \bar{b}$ is a harmonic function defined in the unit disk, where $a$ and $b$ are holomorphic functions. Then $\log(|a|^2 + |b|^2)$ is subharmonic (see e.g. [12]).

Then we obtain a special case of contraction from the Hardy space to a Bergman space is

$$(h^p, \| \| \| \|) \subset B_{2/p}^2$$

for $1 < p < 2$, which extends a corresponding result in [15] and a classical result [7, 4] and also the recent results [14, 17]. Namely for

$$f(z) = a + \bar{b} = \sum_{n=0}^\infty a_n z^n + \sum_{n=0}^\infty b_n z^n \in B_{2/p}^2,$$

where $b_0 = 0$, we can express its norm as follows

$$\| f \|_{2/p, 2}^2 = \sum_{n=0}^\infty \frac{|a_n|^2 + |b_n|^2}{c_{2/p}(n)} = \left( n + \frac{2}{p} - 1 \right).$$

Thus, for a function $f \in h^p$, from Corollary [1.7] we have

$$\sum_{n=0}^\infty \frac{|a_n|^2 + |b_n|^2}{c_{2/p}(n)} \leq \| f \|^2_{h^p}.$$

Now by [12 Theorem 2.1], for $f \in h^p$, $b_0 = 0$, for $1 < p \leq 2$, we obtain the following inequality

$$\sum_{n=0}^\infty \frac{|a_n|^2 + |b_n|^2}{c_{2/p}(n)} \leq \frac{1}{1 - \frac{1}{\cos \pi/p}} \| f \|^2_{h^p},$$

(5.2)
where
\[ \|f\|_p^p = \int_{\partial D} |f(\zeta)|^p |d\zeta| \frac{1}{2\pi}. \]

By using the result from [18], that the Jacobian of a harmonic diffeomorphism is superharmonic, and the fact that the Jacobian \(J(f,z) = |a'(z)|^2 - |b'(z)|^2\) is real analytic, Corollary 1.7 implies

**Corollary 5.1.** Let \(1 < p < 2\) and \(\alpha > 1\). Assume that \(f\) is a harmonic diffeomorphism of the unit disk \(D\) onto a two-dimensional domain \(\Omega\). Assume further that \(1/J_f(z) \in h_{p/\alpha}\). Then we have
\[
\left( (\alpha - 1) \int_D \frac{1}{J_f^p(z)} (1 - |z|^2)^{\alpha-2} \frac{dx dy}{\pi} \right)^{1/p} \leq \|1/J_f(z)\|_{p/\alpha}.
\]

Since \(\log(|a|^2 + |b|^2)\) is subharmonic, if \(a\) and \(b\) are analytic functions (see property 3 of the admissible monoid), in view of Corollary 1.7, we obtain the following Corollary

**Corollary 5.2.** Assume that \(f = a + \bar{b} : D \to \mathbb{C}\) is a harmonic mapping and that \(\alpha > 1, p > 1\). Then
\[
\left( (\alpha - 1) \int_D (|a|^2 + |b|^2)^{p/2} (1 - |z|^2)^{\alpha-2} \frac{dx dy}{\pi} \right)^{1/p} \leq \|(a|^2 + |b|^2)^{1/2}\|_{p/\alpha}.
\]

**Corollary 5.3.** For \(p > 1\) and for a harmonic function \(f \in h^p\) we have the following isoperimetric type inequality
\[
\|f\|_{B^{2p}} \leq C_p \|f\|_p,
\]
where
\[
C_p = \frac{\sqrt{2} \cos \frac{\pi}{4p}}{(1 - |\cos \frac{\pi}{p}|)^{1/2}}.
\]

Here \(B^{2p}\) is the Bergman space of harmonic mappings belonging to the Lebesgue space \(L^{2p}(D)\).

Observe that for \(p \geq 2\),
\[
C_p = \frac{1}{2} \csc \frac{\pi}{4p}.
\]

The inequality (5.3) for \(p > 1\) being an integer has been proved by the author in [12, Theorem 2.11]. We also refer to a recent improvement of (5.3) in [19] for \(p > 2\), which is based on (5.3) for \(p = 2\) and Jensen’s inequality. For \(p \in (1,2]\),
\[
C_p = \cos \left[ \frac{\pi}{4p} \right] \sec \left[ \frac{\pi}{2p} \right].
\]

The inequality (5.3) for such constant \(C_p\) has been proved for real-valued harmonic functions in [13]. In this case we do not have such a restriction.
Proof of Corollary 5.3. Let \( f(z) = a(z) + \overline{b(z)} \) and assume w.l.g. that \( a(0) = 0 \).
By integrating \([12, \text{eq. } 2.3]\) in the interval \([0, 1]\) we get
\[
\left( \int_{\mathbb{D}} |a + \overline{b}|^2 \frac{dxdy}{\pi} \right)^{\frac{1}{2p}} \leq I := \sqrt{2} \cos \frac{\pi}{4p} \left( \int_{\mathbb{D}} (|a|^2 + |b|^2)^p \frac{dxdy}{\pi} \right)^{\frac{1}{2p}}.
\]
Then by Corollary 5.2, by choosing \( \alpha = 2 \), we get
\[
I \leq J := \sqrt{2} \cos \frac{\pi}{4p} \| |a|^2 + |b|^2 \|_{h_p}.
\]
Now \([12, \text{Theorem } 2.1]\) implies
\[
J \leq \sqrt{2} \cos \frac{\pi}{4p} \left( \frac{\| |a|^2 + |b|^2 \|_{h_p}}{(1 - |\cos \frac{\pi}{p})^{1/2}} \| f \|_p \right).
\]
The result follows. \( \square \)

Assume now that \( f : \mathbb{D} \to \Sigma \subset \mathbb{R}^n \) is a conformal parameterisation of the minimal surface \( \Sigma \). Since \( \log |f_x(z)|^2 = \log \left( |p(z)(1 + |q(z)|^2)\right) \), where \( p \) and \( q \) are holomorphic functions in \( z = x + iy \) (the so-called Enneper-Weierstrass parameters), we have the following corollary.

**Corollary 5.4.** Assume that \( \alpha > 1, p > 1 \) and \( f : \mathbb{D} \to \mathbb{R}^n \) is the Enneper-Weierstrass parameterisation of a minimal surface in \( \mathbb{R}^n \). Then
\[
\left( \alpha - 1 \right) \int_{\mathbb{D}} |f_x(z)|^p (1 - |z|^2)^{\alpha - 2} \frac{dxdy}{\pi} \right)^{1/p} \leq \| f_x \|_p/\alpha.
\]
For \( p = \alpha = 2 \), the above formula is simply the isoperimetric inequality for minimal surfaces.

By plugging \( p = \alpha \) in Corollary 1.7 we have

**Corollary 5.5.** For every smooth superconvex positive function \( f \) defined in the interval \((-1, 1)\) and for \( \alpha > 1 \), we have the sharp inequality
\[
\frac{\Gamma(\alpha)}{\sqrt{\pi^1}} \left[ \alpha - \frac{1}{2} \right] \int_{-1}^{1} \left( 1 - x^2 \right)^{\alpha - \frac{3}{2}} f^\alpha(x) dx \leq \left( \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} dx \right)^{\alpha}.
\]
In particular, for \( \alpha = 2 \) we have
\[
2\pi \int_{-1}^{1} \sqrt{1 - x^2} f^2(x) dx \leq \left( \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} dx \right)^2.
\]
Equalities are attained for \( f \) being a constant function.
Proof. If $f$ is real analytic, then we apply Corollary 1.7 for $n = 2$ and $p = 1$ to the log-subharmonic function $f(x, y) = f(x)$. By Fubini’s theorem we have

\begin{equation}
\int_D |f(z)|^\alpha (1 - |z|^2)^{\alpha - 2}dxdy = \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)^{\alpha - 2} f^\alpha(x)dy
\end{equation}

\begin{align*}
= \int_{-1}^1 \frac{\sqrt{\pi}}{\Gamma(\frac{\alpha}{2})} \frac{\alpha - 3}{\alpha - 2} \Gamma(\alpha - 1) f^\alpha(x)dx.
\end{align*}

On the other hand

\begin{equation}
\int_D |f(z)||dz| = 2 \int_{-1}^1 \frac{f(x)dx}{\sqrt{1 - x^2}}.
\end{equation}

Furthermore $f$ is logarithmically convex (superconvex) if and only if

\begin{equation}
\Delta \log f(x) = -\frac{f'(x)^2}{f(x)^2} + \frac{f''(x)}{f(x)} \geq 0.
\end{equation}

By using (5.7), (5.8) and Corollary 1.7 we obtain (5.5).

If $f$ is not real analytic, then in view of (5.9), $f''$ is positive, and so $f'$ is non-decreasing. In particular there is a segment $[a, b]$ (possibly $[a, b] = \emptyset$) so that the zero set of $f'$ is $N(f') = [a, b]$. Then $f$ is decreasing in $[-1, a]$, increasing in $[b, 1]$ and constant in $[a, b]$. Let $g(x) = 1 + \epsilon(x - a)(x - b)$. Then $g$ is strictly decreasing in $[-1, (a + b)/2]$ and strictly increasing in $[(a + b)/2, 1]$. Moreover

\begin{equation}
\frac{d^2}{dx^2} \log h(x) = \frac{\epsilon (2 - \epsilon (a^2 + b^2 - 2(a + b)x + 2x^2))}{(1 + \epsilon(a - x)(b - x))^2}.
\end{equation}

Hence $h$ is logharmonic for positive and small enough $\epsilon$. Thus $f_\epsilon(x) = f(x) + \epsilon g(x)$ is strictly decreasing in $[-1, 1/2(a + b)]$ and strictly increasing in $[1/2(a + b), 1]$. Moreover $h(x, y) = f_\epsilon(y)$ is also logsubharmonic. In particular the set $\{x : f_\epsilon(x) = c\}$ has zero linear measure and hence $\{(x, y) \in D : h(x) = t\}$ has zero hyperbolic measure. Therefore $\mu(t) = |\{(x, y) : f_\epsilon(x) = t\}|_h$ is continuous. The rest of the proof is the repetition of the proof of real analytic case and this implies that $f_\epsilon$ satisfies (5.5). By letting $\epsilon \to 0^+$ we obtain that $f$ satisfies (5.5). \qed

6. APPENDIX

The point evaluations are continuous functionals in $h^p$ and $B_\alpha^p$. This is the content of the following proposition.

**Proposition 6.1.** Let $p > 1$ and $\alpha > 1$. There are two constants $C_1 = C_1(n)$ and $C_2 = C_2(n, \alpha)$ such that, if $\bar{f}$ is $\mathcal{M}$-subharmonic and belongs to the Hardy space $\mathcal{S}^p$, then

\begin{equation}
|f(x)|^p \Phi_n(x) \leq C_1 \|f\|_p^p
\end{equation}

and if $f \in B_\alpha^p$ then

\begin{equation}
|f(x)|^p \Phi_\alpha^\alpha(x) \leq C_2 \|f\|_{\alpha, p}^p.
\end{equation}
Moreover

1) If \( f \in h^p \) then \( \lim_{|x| \to 1} |f(x)|^p \Phi_n(x) = 0 \), and
2) if \( f \in B_\alpha^p \) then \( \lim_{|x| \to 1} |f(x)|^p \Phi_n(x) = 0 \).

Proof. First of all relation (6.1) follows from (1.5) and [26, Lemma 7.2.1].

To prove (6.2) we proceed similarly. This time we make use of (1.5) and [26, Eq. 10.1.5], which is formulated for \( M \)–harmonic functions, but the same proof can be applied for the \( M \)–subharmonic functions.

We will prove now statement a). Let \( f \) be a positive \( \log -M \)–subharmonic functions \( f \) on the unit ball belonging to the Hardy space \( S^p \).

Let \( F = P_h[\hat{f}] (x) \) be the least harmonic majorant of \( f \) ([26 Theorem 7.1.1]), which is in \( S^p \). Now we define the mapping \( F_r(x) = P_h[f_r](x) \), \( f_r(x) = F(rx) \), \( x \in \mathbb{S} \) and \( 0 < r < 1 \). Then for \( \epsilon > 0 \) there is \( r \) such that

\[
\|F_r - F\|_p \leq \epsilon.
\]

Let us prove (6.3). Since \( \hat{f} \in L^p(\mathbb{S}) \), there is a continuous function \( g \) in \( \mathbb{S} \), such that \( \|g - \hat{f}\|_p < \epsilon/3 \). Let \( G(x) = P_h[g](x) \) and \( G_r(x) = P_h[g_r] \), where \( g_r(x) = G(rx) \), \( x \in \mathbb{S} \), \( 0 < r < 1 \). Then \( G_r - F_r \) is \( M \)–harmonic and \( |G_r - F_r|^p \) is \( \mathcal{M} \)–subharmonic for \( p \geq 1 \). The same hold for \( G - H \) and \( |G - H|^p \). Then by [26 Theorem 5.4.2] we have

\[
\|G - F\|_p = \|P_h[\hat{f} - g]\|_p \leq \|\hat{f} - g\|_p \leq \epsilon/3
\]

and

\[
\|G_r - F_r\|_p = \|P_h[f_r - g_r]\|_p \\
\leq \|f_r - g_r\|_p \\
= \left( \int_{\mathbb{S}} |F(r\eta) - G(r\eta)|^p d\sigma(\eta) \right)^{1/p} \\
\leq \left( \int_{\mathbb{S}} |\hat{f}(\eta) - g(\eta)|^p d\sigma(\eta) \right)^{1/p} = \|\hat{f} - g\| \leq \epsilon/3.
\]

Thus

\[
\|F - F_r\|_p \leq \|F_r - G_r\|_p + \|G_r - G\|_p + \|G - F\|_p \\
\leq \|g - \hat{f}\|_p + \|g_r - g\|_p + \|g - \hat{f}\|_p \\
\leq 2\epsilon/3 + \|g_r - g\|_p.
\]

Now by [26 Corollary 5.3.4], because \( g \) is continuous on \( \mathbb{S} \), there is \( r < 1 \) such that \( \|g_r - g\|_p < \epsilon/3 \). This implies (6.3).

Then we have

\[
|F(x)| \leq |F(x) - F_r(x)| + |F_r(x)|.
\]

From (6.1) and (6.3), we get

\[
|F(x) - F_r(x)|^p \Phi_n(x) \leq C\epsilon^p.
\]
On the other hand
\[ \lim_{|x| \to 1} |F_r(x)|^p \Phi_n(x) = 0. \]
So
\[ \limsup_{|x| \to 1} |F(x)|^p \Phi_n(x) \leq C e^p. \]
Because \( F \) is the harmonic majorant of \( f \), by the previous relation we get
\[ \limsup_{|x| \to 1} |f(x)|^p \Phi_n(x) \leq \limsup_{|x| \to 1} |F(x)| \Phi_n(x) \leq C e^p. \]
Thus
\[ \limsup_{|x| \to 1} |f(x)|^p \Phi_n(x) = 0 \]
as was stated.

The relation b) can be proved similarly. Let \( F = F_f \) be the harmonic majorant of \( f \) and let \( F_k \) be as in Definition 1.3. For \( \epsilon > 0 \) we can choose \( k \) such that
\[ (6.4) \quad \|F - F_k\|_{\alpha,p} \leq \epsilon, \]
where we recall
\[ \|H\|_{\alpha,p} = c(\alpha) \int_B |H(x)|^p \Phi_n^\alpha(|x|) d\tau(x). \]
Then we have
\[ |F(x)| \leq |F(x) - F_k(x)| + |F_k(x)|. \]
From (6.1) and (6.4), we get
\[ |F(x) - F_k(x)|^p \Phi_n^\alpha(x) \leq C e^p. \]
On the other hand
\[ \lim_{|x| \to 1} |F_k(x)|^p \Phi_n^\alpha(x) = 0, \]
because \( F_k \) is bounded. So
\[ \limsup_{|x| \to 1} |F(x)|^p \Phi_n^\alpha(x) \leq C e^p. \]
Because \( F \) is the harmonic majorant of \( f \), by the previous relation we get
\[ \limsup_{|x| \to 1} |f(x)|^p \Phi_n^\alpha(x) \leq \limsup_{|x| \to 1} |F(x)|^p \Phi_n^\alpha(x) \leq C e^p. \]
Thus
\[ \limsup_{|x| \to 1} |f(x)|^p \Phi_n^\alpha(x) = 0 \]
as it was stated. □

**Proposition 6.2.** Let \( p > 1 \) and \( \alpha > 1 \). Then
\[ (6.5) \quad h^p \subset B^{op}_{\alpha}. \]
Furthermore assume that \( f \in h^p \). Then
\[ (6.6) \quad \lim_{\alpha \to 1+} \|f\|_{\alpha,p} = \|f\|_p. \]
Moreover

\[(6.7) \quad \lim_{\alpha \to 1^+} \|f\|_{\alpha,p} = \|f\|_p.\]

**Proof.** By (1.8), \(\Delta_h f \geq 0\), and by corresponding theorem [26 Theorem 5.4.2], we get

\[\|f\|_p = \lim_{r \to 1^-} \int_{\mathbb{S}} |f(r\zeta)|^p d\sigma(\zeta).\]

Then

\[
\|f\|_{\alpha,p}^p = c(\alpha) \int_{\mathbb{S}} |f(x)|^p \Phi_n^{\alpha}(|x|)(1 - |x|^2)^{-n} dV(x) \frac{1}{\omega_n}
\]

\[(6.8) \quad = c(\alpha) \int_0^1 \Psi(r) dr \int_{\mathbb{S}} |f(r\zeta)|^p d\sigma(\zeta)
\leq \|f\|_p^p,
\]

here \(\alpha > 1\), \(\Psi(r) = r^{n-1} \Phi_n^{\alpha}(r)(1 - r^2)^{-n}\) and

\[c(\alpha) \int_0^1 \Psi(r) dr = 1.\]

On the other hand (6.8) implies

\[\limsup_{\alpha \to 1^+} \|f\|_{\alpha,p} \leq \|f\|_p.\]

Furthermore, because

\[\|f\|_p = \lim_{r \to 1^-} \int_{\mathbb{S}} |f(r\zeta)|^p d\sigma(\zeta),\]

for every \(\epsilon > 0\) there exists \(r_1 < 1\) such that for \(r \in (r_1, 1)\) we have

\[\int_{\mathbb{S}} |f(r\zeta)|^p d\sigma(\zeta) \geq \|f\|_p^p - \epsilon.\]

Then

\[
\|f\|_{\alpha,p}^p = c(\alpha) \int_0^1 r^{n-1} \Phi_n^{\alpha}(r)(1 - r^2)^{-n} dr \int_{\mathbb{S}} |f(r\zeta)|^p d\sigma(\zeta)
\geq c(\alpha) \int_{r_1}^1 r^{n-1} \Phi_n^{\alpha}(r)(1 - r^2)^{-n} dr \int_{\mathbb{S}} |f(r\zeta)|^p d\sigma(\zeta)
\geq c(\alpha) \int_{r_1}^1 r^{n-1} \Phi_n^{\alpha}(r)(1 - r^2)^{-n} dr (\|f\|_p^p - \epsilon).
\]

By letting \(\alpha \to 1^+\), and noting that

\[1 > c(\alpha) \int_{r_1}^1 r^{n-1} \Phi_n^{\alpha}(r)(1 - r^2)^{-n} dr \to 1,\]

when \(\alpha \to 1^+\), we get

\[\liminf_{\alpha \to 1^+} \|f\|_{\alpha,p} \geq \|f\|_p^p - \epsilon.\]
and this finishes the proof of (6.6).

Let us prove (6.7). If \( f \) is \( \mathcal{M} \)-log-subharmonic and belongs to the class \( S^p \), \( p > 1 \) then we proceed similarly as in the end of the proof of the previous proposition. Let \( F = P[\hat{f}](x) \) be the least harmonic majorant of \( f \) \([26 \text{ Theorem 7.1.1}]\) which is in \( S^p \). Now by \([6.3]\) for every \( \epsilon > 0 \), there is \( 0 < r_0 < 1 \) such that, if \( F_r(x) = P_h[F_r](x), f_r = F|r_S \), such that if \( r > r_0 \), then

\[
(6.11) \quad \|F - F_r\|_p = \|\hat{f} - f_r\|_p \leq \epsilon.
\]

We use the Hardy-Littlewood type inequality for \( \mathcal{M} \)-harmonic functions in the unit ball by Stoll: \([26 \text{ Theorem 10.6.3}] \) (i.e. \([26 \text{ Corollary 10.6.5}] \))

\[
(6.12) \quad \|g\|_{\beta,\beta p} \leq C(p,\beta,n) \|\hat{g}\|_p.
\]

Observe that \((6.12)\) implies \( h^p \subset B^{op}_\alpha \). Furthermore

\[
\|f\|_{\alpha,\alpha p} \leq \|F\|_{\alpha,\alpha p} \leq \|F - F_r\|_{\alpha,\alpha p} + \|F_r\|_{\alpha,\alpha p}.
\]

From \((6.12)\), applied to \( g = F - F_r \) and \((6.11)\), for \( r > r_1 \) we obtain

\[
\|F - F_r\|_{\alpha,\alpha p} \leq C\epsilon.
\]

If in \((6.8)\) we put \( \alpha p \) instead of \( p \) we get

\[
\|F_r\|_{\alpha,\alpha p} \leq \|f_r\|_{\alpha p}.
\]

Since \( f_r \) is continuous up to the boundary, we have \( \lim_{\alpha \to 1^+} \|f_r\|_{\alpha p} = \|f_r\|_p \). Thus

\[
\limsup_{\alpha \to 1^+} \|f\|_{\alpha,\alpha p} \leq C\epsilon + \|f_r\|_p \leq C\epsilon + \|\hat{f}\|_p.
\]

Observe that because of \([26 \text{ Remark 7.4.4}] \), we have \( \|f\|_p = \|\hat{f}\|_p \). Since \( \epsilon > 0 \) is arbitrary we get

\[
\limsup_{\alpha \to 1^+} \|f\|_{\alpha,\alpha p} \leq \|f\|_p.
\]

The opposite inequality is much easier and follows from Jensen’s inequality. Indeed

\[
\int_S |f(r\zeta)|^{\alpha p} d\sigma(\zeta) \geq \left( \int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^\alpha.
\]

Hence for \( r_1 \) satisfying \((6.9)\) we have

\[
\|f\|^{\alpha p}_{\alpha,\alpha p} = c(\alpha) \int_0^{r_1} r^{n-1} \Phi_n^\alpha(r)(1 - r^2)^{-\frac{n}{2}} dr \int_S |f(r\zeta)|^{\alpha p} d\sigma(\zeta)
\]

\[
\geq c(\alpha) \int_{r_1}^1 r^{n-1} \Phi_n^\alpha(r)(1 - r^2)^{-\frac{n}{2}} dr \left( \int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^\alpha.
\]

By \((6.9)\) and \((6.10)\) we obtain

\[
\liminf_{\alpha \to 1^+} \|f\|^{\alpha p}_{\alpha,\alpha p} \geq (\|f\|_p^p - \epsilon)^\alpha.
\]

This finishes the proof of proposition.
Acknowledgments. I would like to thank the anonymous referee for very helpful comments that had a significant impact on this paper. I would like to thank Dr. P. Melentijević for drawing my attention to the papers \[15\], \[21\]. I am thankful to Prof. Kristian Seip and Dr. Aleksei Kulikov for very helpful discussion.

REFERENCES

[1] L. V. Ahlfors, *Moebius transformations in several dimensions*. (Preobrazovaniya Mebiusa v mnogomernom prostranstve). Transl. from the English. Sovremennaya Matematika. Vvodnye kursy. Moskva: Mir. 112 p. R. 0.90 (1986).

[2] V. Bögelein, F. Duzaar, Frank; C. Scheven, *A sharp quantitative isoperimetric inequality in hyperbolic n-space*. Calc. Var. Partial Differ. Equ. 54, No. 4, 3967-4017 (2015).

[3] O. F. Brevig, J. Ortega-Cerdà, K. Seip, and J. Zhao, *Contractive inequalities for Hardy spaces*, Funct. Approx. Comment. Math., vol. 59, no. 1, pp. 41–56, 2018.

[4] J. Burbea, *Sharp inequalities for holomorphic functions*, Illinois J. Math., vol. 31, no. 2, pp. 248–264, 1987.

[5] T. M. Flett, *On the rate of growth of mean values of holomorphic and harmonic functions*. Proc. Lond. Math. Soc., III. Ser. 20, 749-768 (1970).

[6] P. Graczyk, T. Kemp, Todd; J.-J. Loeb, *Hypercontractivity for log-subharmonic functions*. J. Funct. Anal. 258, No. 6, 1785-1805 (2010).

[7] G. H. Hardy and J. E. Littlewood, *Some new properties of Fourier constants*. Math. Ann., vol. 97, no. 1, pp. 159–209, 1927.

[8] G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals. III*. Math. Z. 34, 403–439 (1931).

[9] W. K. Hayman, P. B. Kennedy, *Subharmonic functions. Vol. I*. London Mathematical Society Monographs. No. 9. London-New York-San Francisco: Academic Press, a subsidiary of Harcourt Brace Jovanovich, Publishers, XVII, 284 p. (1976).

[10] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman Spaces* Graduate Texts in Mathematics. 199. New York, NY: Springer. ix, 286 p. (2000).

[11] I. Izmestiev, *A simple proof of an isoperimetric inequality for Euclidean and hyperbolic cone-surfaces*. Differ. Geom. Appl. vol. 43, pp. 95–101, 2015.

[12] D. Kalaj, *On Riesz type inequalities for harmonic mappings on the unit disk*. Trans. Am. Math. Soc. 372, No. 6, 4031-4051 (2019).

[13] D. Kalaj, David, E. Bajrami, *On some Riesz and Carleman type inequalities for harmonic functions in the unit disk*. Comput. Methods Funct. Theory 18, No. 2, 295–305 (2018).

[14] A. Kulikov, *A contractive Hardy Littlewood inequality*. Bull. London Math. Soc., 53, pp. 740–746, 2020.

[15] A. Kulikov, *Functionals with extrema at reproducing kernels*. arXiv:2203.12349

[16] E.H. Lieb and J.P. Solovej, *Wehrl-type coherent state entropy inequalities for SU(1,1) and its AX+B subgroup, Partial differential equations, spectral theory, and mathematical physics ’ the Ari Laptev anniversary volume*. 301–314, EMS Ser. Congr. Rep., EMS Press, Berlin, 2021.

[17] A. Llinares, *On a conjecture about contractive inequalities for weighted Bergman spaces*. arXiv:2112.09962

[18] V. Manojlović, *Bi-Lipschicity of quasiconformal harmonic mappings in the plane*. Filomat 23, No. 1, 85-89 (2009).

[19] P. Melentijević, *Hollenbeck-Verbitsky conjecture on best constant inequalities for analytic and co-analytic projections*. arXiv:2203.14364

[20] B. S. Mityagin, *The zero set of a real analytic function*. Math Notes 107, 529–530 (2020). https://doi.org/10.1134/S0001434620030189

[21] F. Nicola, P. Tilli, *The Faber-Krahn inequality for the short-time Fourier transform*. Invent. math. (2022). https://doi.org/10.1007/s00222-022-01119-8.
[22] R. Osserman, *The isoperimetric inequality*. Bull. Amer. Math. Soc., 84(6), pp. 1182–1238, 1978.

[23] M. Pavlović, *Function classes on the unit disk*. De Gruyter Studies in Mathematics, vol. 52, De Gruyter, Berlin, 2014.

[24] W. Rudin, *Function theory in the unit ball of Ca. Reprint of the 1980 original*. Classics in Mathematics. Berlin: Springer. xviii, 436 p. (2008).

[25] E. Schmidt, *Über die isoperimetrische Aufgabe im n-dimensionalen Raum konstanter negativer Krümmung. I. Die isoperimetrischen Ungleichungen in der hyperbolischen Ebene und für Rotationskörper im n-dimensionalen hyperbolischen Raum*. Math. Z., 46, pp. 204–230, 1940.

[26] M. Stoll, *Harmonic and subharmonic function theory on the hyperbolic ball*. London Mathematical Society Lecture Note Series 431. Cambridge: Cambridge University Press, xv, 225 p. (2016).

[27] D. Vukotić: *A sharp estimate for $A^p_{\alpha}$ functions in $\mathbb{C}^n$*. Proc. Am. Math. Soc. 117, No. 3, 753-756 (1993).

University of Montenegro, Faculty of Natural Sciences and Mathematics, Cetinski put b.b. 81000 Podgorica, Montenegro

*Email address*: davidk@ucg.ac.me