Hyperbolically embedded virtually free subgroups of relatively hyperbolic groups

Yoshifumi Matsuda∗†, Shin-ichi Oguni‡, Saeko Yamagata§

Abstract

We show that if a group is not virtually cyclic and is hyperbolic relative to a family of proper subgroups, then it has a hyperbolically embedded subgroup which contains a finitely generated non-abelian free group as a finite index subgroup.

Keywords: relatively hyperbolic groups; hyperbolically embedded subgroups; strongly relatively undistorted subgroups; almost malnormal subgroups.

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1 Introduction

The notion of relatively hyperbolic groups was introduced in [6] and has been studied by many authors (see for example [2], [4], [5] and [13]). In this paper we consider relatively hyperbolic groups in accordance with a definition due to D. Osin [13, Definition 2.35]. Note that for certain cases (e.g. for finitely generated groups), this definition has several equivalent formulations (see for example [7, Sections 3 and 5]).

In [14], D. Osin introduced the notion of hyperbolically embedded subgroups of a relatively hyperbolic group.

Definition 1.1. ([14, Definition 1.4]) Let $G$ be a group which is hyperbolic relative to a family $\mathbb{K}$ of subgroups. A subgroup $H$ of $G$ is said to be hyperbolically embedded into $G$ relative to $\mathbb{K}$ if $G$ is hyperbolic relative to $\mathbb{K} \cup \{H\}$.

∗Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914 Japan, ymatsuda@ms.u-tokyo.ac.jp
†Supported by the Global COE Program at Graduate School of Mathematical Sciences, the University of Tokyo, and Grant-in-Aid for Scientific Researches for Young Scientists (B) (No. 22740034), Japan Society of Promotion of Science.
‡Department of Mathematics, Faculty of Science, Ehime University, 2-5 Bunkyo-cho, Matsuyama, Ehime, 790-8577 Japan, oguni@math.sci.ehime-u.ac.jp
§Faculty of Education and Human Sciences, Yokohama National University, 240-8501 Yokohama, Japan, yamagata@ynu.ac.jp
If $G$ is infinite and hyperbolic relative to a family $\mathbb{K}$ of proper subgroups, then there exists a virtually infinite cyclic subgroup of $G$ which is hyperbolically embedded into $G$ relative to $\mathbb{K}$ (see [14, Corollaries 1.7 and 4.5]). Also if $G$ is torsion-free, hyperbolic and not cyclic, then it contains a free subgroup of rank two which is quasiconvex and malnormal in $G$, that is, hyperbolically embedded into $G$ relative to the empty family $\emptyset$ (see [9, Theorem C] and [2, Theorem 7.11]). In this paper we show the following (see also Theorem 5.1).

**Theorem 1.2.** Suppose that a group $G$ is not virtually cyclic and is hyperbolic relative to a family $\mathbb{K}$ of proper subgroups. Then there exists a finitely generated and virtually non-abelian free subgroup of $G$ which is hyperbolically embedded into $G$ relative to $\mathbb{K}$. Moreover if $G$ is torsion-free, then it contains a free subgroup of rank two which is hyperbolically embedded into $G$ relative to $\mathbb{K}$.

We refer to [12, Theorems 1.4 and 1.5] for applications of this theorem to the study of convergence actions of groups. We also refer to [11, Theorem 6.3] for another application.

**Remark 1.3.** After the first version of this paper appeared, the notion of a hyperbolically embedded subgroup was further generalized in [3]. It turns out that we can prove a stronger version of Theorem 1.2 by using the argument in the proof of [3, Theorem 6.14 (c)] (see [12, Appendix B] for details). In what follows, hyperbolically embedded subgroups which we consider are those in the sense of [14].

In Section 2 we recall the fact that hyperbolically embedded subgroups of a relatively hyperbolic group are characterized as strongly relatively undistorted and almost malnormal subgroups. Strongly relatively undistorted free subgroups of rank two of a relatively hyperbolic group are found in Section 3. In Section 4 we construct almost malnormal subgroups of a virtually free group with additional properties. Theorem 1.2 is proved in Section 5.

### 2 Characterization of hyperbolically embedded subgroups

The strategy of our proof of Theorem 1.2 is based on Osin’s characterization of hyperbolically embedded subgroups of relatively hyperbolic groups stated below.

To state the characterization, we begin by introducing several definitions. Let $G$ be a group. For a family $\mathbb{K}$ of subgroups of $G$, we put $\mathbb{K} = \bigcup_{K \in \mathbb{K}} K \setminus \{1\}$. A subset $X$ of $G$ is called a relative generating set of $G$ with respect to $\mathbb{K}$ if $G$ is generated by $X \cup \mathbb{K}$. The group $G$ is said to be finitely generated relative to $\mathbb{K}$ if there exists a finite relative generating set of $G$ with respect to $\mathbb{K}$. When $Z$ is a (possibly infinite) generating set of $G$, we denote by $\Gamma(G, Z)$ the Cayley graph of $G$ with respect to $Z$ and by $d_Z$ the word metric with respect to $Z$. 


**Definition 2.1.** Let $G$ be a group which is finitely generated relative to a family $\mathbb{K}$ of subgroups. A subgroup $H$ of $G$ is said to be strongly undistorted relative to $\mathbb{K}$ in $G$ if $H$ is generated by some finite subset $Y$ and for some finite relative generating set $X$ of $G$ with respect to $\mathbb{K}$, the natural map $(H,d_Y) \to (G,d_{X\cup K})$ is a quasi-isometric embedding.

**Definition 2.2.** Let $G$ be a group and $H$ a subgroup of $G$. The subgroup $H$ is said to be malnormal (resp. almost malnormal) in $G$ if for every element $g$ of $G \setminus H$, the intersection $H \cap gHg^{-1}$ is trivial (resp. finite).

**Theorem 2.3.** ([14, Theorem 1.5]) Let $G$ be a group which is hyperbolic relative to a family $\mathbb{K}$ of subgroups. Then a subgroup $H$ of $G$ is hyperbolically embedded into $G$ relative to $\mathbb{K}$ if and only if $H$ is strongly undistorted relative to $\mathbb{K}$ and almost malnormal in $G$.

For finitely generated relatively hyperbolic groups, we have the following characterization of strongly relatively undistorted subgroups.

**Definition 2.4.** ([13, Definitions 4.9 and 4.11]) Let $G$ be a group with a finite family $\mathbb{K}$ of subgroups. Suppose that $G$ is generated by a finite set $X$. A subgroup $H$ of $G$ is said to be quasiconvex relative to $\mathbb{K}$ in $G$ if there exists a constant $\sigma \geq 0$ satisfying the following: if $h_1$ and $h_2$ are elements of $H$ and $p$ is a geodesic from $h_1$ to $h_2$ in $\Gamma(G,X \cup K)$, then for every vertex $v$ on $p$, there exists an element $h$ of $H$ such that $d_X(v,h) \leq \sigma$. A subgroup $H$ of $G$ is said to be strongly quasiconvex relative to $\mathbb{K}$ in $G$ if $H$ is quasiconvex relative to $\mathbb{K}$ in $G$ and for every element $K$ of $\mathbb{K}$ and every element $g$ of $G$, the intersection $H \cap gKg^{-1}$ is finite.

**Theorem 2.5.** ([13, Theorem 4.13]) Let $G$ be a group which is hyperbolic relative to a finite family $\mathbb{K}$ of subgroups. Suppose that $G$ is finitely generated. Then a subgroup $H$ of $G$ is strongly undistorted relative to $\mathbb{K}$ if and only if $H$ is strongly quasiconvex relative to $\mathbb{K}$.

### 3 Strongly relatively undistorted free subgroups

When a group $G$ is hyperbolic relative to a family $\mathbb{K}$ of subgroups, a subgroup of $G$ is said to be parabolic with respect to $\mathbb{K}$ if it is conjugate to a subgroup of some element of $\mathbb{K}$. The main purpose of this section is to show the following.

**Proposition 3.1.** Let $G$ be a group which is hyperbolic relative to a family $\mathbb{K}$ of proper subgroups and $\Gamma$ a subgroup of $G$ which is neither virtually cyclic nor parabolic with respect to $\mathbb{K}$. If $\Gamma$ contains an element of infinite order, then it contains a free subgroup $F$ of rank two which is strongly undistorted relative to $\mathbb{K}$ in $G$.

Proposition 3.1 yields the following corollary.

**Corollary 3.2.** Let $G$ be a group which is not virtually cyclic and is hyperbolic relative to a family $\mathbb{K}$ of proper subgroups. Then $G$ contains a free subgroup $F$ of rank two which is strongly undistorted relative to $\mathbb{K}$ in $G$.
Proof of Corollary 3.2 using Proposition 3.1. It follows from [14, Corollary 4.5] that the group $G$ contains an element of infinite order. Hence the assertion follows from Proposition 3.1.

For the proof of Proposition 3.1 we prepare several lemmas.

When a group $G$ is hyperbolic relative to a family $\mathcal{K}$ of proper subgroups, an element $g$ of $G$ is said to be parabolic with respect to $\mathcal{K}$ if it is conjugate to an element of a subgroup of $G$ which belongs to $\mathcal{K}$. Otherwise $g$ is said to be hyperbolic with respect to $\mathcal{K}$.

**Lemma 3.3.** Let $G$ be a group which is hyperbolic relative to a family $\mathcal{K}$ of proper subgroups and $\Gamma$ a subgroup of $G$ which is neither virtually cyclic nor parabolic with respect to $\mathcal{K}$. Suppose that either $G$ is countable and $\mathcal{K}$ is finite or $\Gamma$ contains an element of infinite order. Then there exists an element $h$ of $\Gamma$ which is of infinite order and hyperbolic with respect to $\mathcal{K}$.

**Proof.** First suppose that $G$ is countable and $\mathcal{K}$ is finite. Then we can consider a geometrically finite convergence action of $G$ on a compact metrizable space such that the set of all maximal parabolic subgroups of the action is equal to the collection of all conjugates of elements of $\mathcal{K}$ which are infinite (see for example [7, Definition 3.1]). Since $\Gamma$ is neither virtually cyclic nor parabolic with respect to $\mathcal{K}$, the restriction of this action to $\Gamma$ is a non-elementary convergence action. Hence $\Gamma$ contains an element $h$ which is loxodromic with respect to this action (see [16, Theorem 2T]).

Next suppose that $\Gamma$ contains an element $h$ of infinite order. We only have to consider the case where $h$ belongs to the conjugate $gKg^{-1}$ for some element $K$ of $\mathcal{K}$ and some element $g$ of $G$. Since $\Gamma$ is not parabolic with respect to $\mathcal{K}$, we can take an element $\gamma$ of $\Gamma \setminus gKg^{-1}$. By [14, Lemma 4.4], there exists an integer $n$ such that the element $\gamma h^n$ of $\Gamma$ is of infinite order and hyperbolic with respect to $\mathcal{K}$. 

**Lemma 3.4.** ([14, Theorem 4.3 and Corollary 1.7]) Let $G$ be a group which is hyperbolic relative to a family $\mathcal{K}$ of subgroups and $h$ be an element of $G$ which is of infinite order and hyperbolic with respect to $\mathcal{K}$. Then there exists a unique subgroup $E(h)$ of $G$ such that $E(h)$ is virtually cyclic, contains $h$ and maximal among such subgroups of $G$. Moreover $E(h)$ is hyperbolically embedded into $G$ relative to $\mathcal{K}$.

The following lemma is shown by a similar argument in the proof of [10, Corollary 1.12].

**Lemma 3.5.** Let $G$ be a group generated by a finite set $X$ and hyperbolic relative to a finite family $\mathcal{K}$ of subgroups. Let a subgroup $H$ of $G$ be hyperbolically embedded into $G$ relative to $\mathcal{K}$. We denote the union $\mathcal{K} \cup \{H\}$ by $\mathcal{H}$. Suppose that a subgroup $Q$ of $G$ is strongly quasiconvex relative to $\mathcal{H}$ in $G$. Then there exists a constant $C(Q,H) \geq 0$ with the following property: for every subgroup $R$ of $H$ such that

(a) $Q \cap H \subset R$;
(b) $d_X(1, r) \geq C(Q, H)$ for every element $r$ of $R \setminus Q$;
(c) the subgroup $R$ is quasiconvex relative to $K$ in $G$,
the natural homomorphism $Q \ast_{Q\cap R} R \to G$ is injective and its image $(Q \cup R)$ is strongly quasiconvex relative to $K$ in $G$.

Proof. By [10, Theorem 5.12], there exists a constant $C(Q, H) \geq 0$ with the following property: for every subgroup $R$ of $H$ satisfying above conditions (a) and (b), the natural homomorphism $Q \ast_{Q\cap R} R \to G$ is injective, its image $(Q \cup R)$ is quasiconvex relative to $H$ in $G$ and for every element $g$ of $G$ and every element $H'$ of $H$, the intersection $(Q \cup R) \cap gH'g^{-1}$ is either finite or conjugate to $R$ in $\langle Q \cup R \rangle$.

Now we suppose that $R$ satisfies above condition (c) and show that $(Q \cup R)$ is strongly quasiconvex relative to $K$ in $G$.

First we show that $(Q \cup R)$ is quasiconvex relative to $K$ in $G$. By [10, Theorem 1.1 (2)], it suffices to show that for every element $g$ of $G$, the intersection $(Q \cup R) \cap gHg^{-1}$ is quasiconvex relative to $K$ in $G$. Every finite subgroup of $G$ is automatically quasiconvex relative to $K$ in $G$. Since $R$ is quasiconvex relative to $K$ in $G$, it is well-known that every conjugate of $R$ is also quasiconvex relative to $K$ in $G$. Thus the subgroup $(Q \cup R)$ is quasiconvex relative to $K$ in $G$.

Next we show that for every element $g$ of $G$ and every element $K$ of $K$, the intersection $(Q \cup R) \cap gKg^{-1}$ is finite. We have only to consider the case where there exists an element $s$ of $(Q \cup R)$ such that $(Q \cup R) \cap gKg^{-1}$ is equal to $sRs^{-1}$. Then $(Q \cup R) \cap gKg^{-1}$ is contained in $s(s^{-1}gK(s^{-1}g)^{-1} \cap H)s^{-1}$. Since $H$ is hyperbolically embedded into $G$ relative to $K$, the intersection $s^{-1}gK(s^{-1}g)^{-1} \cap H$ is finite. Hence $(Q \cup R) \cap gKg^{-1}$ is also finite.

Thus $(Q \cup R)$ is strongly quasiconvex relative to $K$ in $G$.

By using the above lemmas, we prove the following, which implies Proposition 3.1 for finitely generated groups.

Lemma 3.6. Let $G$ be a group which is hyperbolic relative to a finite family $\mathbb{K}$ of proper subgroups and $\Gamma$ a subgroup of $G$ which is neither virtually cyclic nor parabolic with respect to $\mathbb{K}$. Suppose that $G$ is finitely generated. Then $\Gamma$ contains a free subgroup $F$ of rank two which is strongly quasiconvex relative to $\mathbb{K}$ in $G$.

Proof. By Lemma 3.3 there exists an element $h$ of $\Gamma$ which is of infinite order and hyperbolic with respect to $\mathbb{K}$. We denote by $H$ a subgroup $E(h)$ of $G$ given by Lemma 3.4. We put $\mathbb{H} = \mathbb{K} \cup \{H\}$. Then $\mathbb{H}$ consists of proper subgroups of $G$ and $G$ is hyperbolic relative to $\mathbb{H}$. Since $H$ is virtually infinite cyclic, the subgroup $\Gamma$ is not parabolic with respect to $\mathbb{H}$.

Hence it follows from Lemma 3.3 that there exists an element $q$ of $\Gamma$ which is of infinite order and hyperbolic with respect to $\mathbb{H}$. We denote by $Q$ the infinite cyclic subgroup of $\Gamma$ generated by $q$. By Lemma 3.4 and Theorem 2.6 the subgroup $Q$ is strongly quasiconvex relative to $\mathbb{H}$ in $G$. Since $Q$ is torsion-free, this implies that the intersection $Q \cap H$ is trivial.
Let $X$ be a finite generating set of $G$ and $C(Q, H) \geq 0$ a constant given by Lemma 3.5. Since $h$ is of infinite order, there exists a positive integer $k$ such that $d_X(1, h^{kn}) \geq C(Q, H)$ for every integer $n \in \mathbb{Z} \setminus \{0\}$. We denote by $R$ the infinite cyclic subgroup of $\Gamma$ generated by $h^k$. Since $R$ is a finite index subgroup of $H$, it is strongly quasiconvex relative to $K$ in $G$ by Theorem 2.5. Hence $R$ satisfies conditions (a), (b) and (c) in Lemma 3.5. By Lemma 3.5 the subgroup $\langle Q \cup R \rangle$ of $\Gamma$ is a free group of rank two which is strongly quasiconvex relative to $K$ in $G$.

For the proof of the general case, we need the following lemma which is obtained from a specialization of [13, Theorem 2.44] together with [13, Proposition 2.49].

**Lemma 3.7.** Let $G$ be a group which is hyperbolic relative to a family $K$ of proper subgroups and $X$ a finite relative generating set of $G$ with respect to $K$. Then there exists a finite subfamily $K_0 = \{K_1, \ldots, K_m\}$ of $K$ such that $G$ splits as the free product

\[ G = G_0 \ast (\ast_{K \in K \setminus K_0} K), \]

where $G_0$ is the subgroup of $G$ which is generated by $K_1, \ldots, K_m$ and $X$. Moreover there exist a finitely generated subgroup $Q$ of $G_0$ and a family $L = \{L_1, \ldots, L_m\}$ of subgroups of $Q$ satisfying the following:

(i) the finite relative generating set $X$ is contained in $Q$ and for every $i \in \{1, \ldots, m\}$, the subgroup $L_i$ is contained in $K_i$;

(ii) the group $G_0$ is isomorphic to the fundamental group of the graph of groups $\mathcal{G}$ drawn in Figure 1;

(iii) the subgroup $Q$ is hyperbolic relative to $L$, the set $X$ is a relative generating set of $Q$ with respect to $L$ and the natural map $(Q, d_{X \cup L}) \to (G, d_{X \cup K})$ is an isometric embedding, where we put $L = \bigcup_{L \in L} L \setminus \{1\}$.

**Lemma 3.8.** In the setting of Lemma 3.7 we have the following:

(1) if no elements of $K$ contain $X$, then $L$ consists of proper subgroups of $Q$.

(2) if a subgroup of $Q$ is strongly quasiconvex relative to $L$ in $Q$, then it is strongly undistorted relative to $K$ in $G$.
(3) if a subgroup of $Q$ is hyperbolically embedded into $Q$ relative to $\mathbb{L}$, then it is hyperbolically embedded into $G$ relative to $\mathbb{K}$.

Proof. (1) This follows from condition (i) in Lemma 3.7.
(2) This follows from Theorem 2.3 and condition (iii) in Lemma 3.7.
(3) Suppose that a subgroup $V$ of $Q$ is hyperbolically embedded into $Q$ relative to $\mathbb{L}$. Then $V$ is strongly quasiconvex relative to $\mathbb{L}$ in $Q$ by Theorems 2.3 and 2.5. By assertion (2), the subgroup $V$ is strongly undistorted relative to $\mathbb{K}$ in $G$.

We claim that $V$ is almost malnormal in $G$. Indeed, since $V$ is hyperbolically embedded into $Q$ relative to $\mathbb{L}$, it is almost malnormal in $Q$. Hence it suffices to show that if $g$ is an element of $G \setminus Q$, then the intersection $V \cap gVg^{-1}$ is finite. First suppose that $g$ belongs to $G \setminus G_0$. Since $G_0$ is a free factor of $G$, the intersection $G_0 \cap gG_0g^{-1}$ is trivial. Hence the intersection $V \cap gVg^{-1}$ is also trivial. Next suppose that $g$ belongs to $G_0 \setminus Q$. We denote by $T$ the Bass-Serre covering tree of the graph of groups $\mathcal{G}$. Then the group $Q$ is the stabilizer group of a vertex $v$ of $T$ and we have $gv \neq v$. Since the intersection $Q \cap gQg^{-1}$ fixes both $v$ and $gv$, it fixes an edge of $T$. Hence $Q \cap gQg^{-1}$ is parabolic with respect to $\mathbb{L}$. Since every element of $\mathbb{L}$ is contained in an element of $\mathbb{K}$, the intersection $Q \cap gQg^{-1}$ is parabolic with respect to $\mathbb{K}$. Since the subgroup $V$ is strongly quasiconvex relative to $\mathbb{K}$ in $G$, the intersection $V \cap (Q \cap gQg^{-1})$ is finite. Hence the intersection $V \cap gVg^{-1}$ is also finite. □

Proof of Proposition 3.7 Since $\Gamma$ contains an element of infinite order, it follows from Lemma 3.3 that there exists an element $h$ of $\Gamma$ which is of infinite order and hyperbolic with respect to $\mathbb{K}$. Let $E(h)$ be a subgroup of $G$ given by Lemma 3.4. Since $\Gamma$ is not virtually cyclic, we can take an element $\gamma$ of $\Gamma \setminus E(h)$. By Lemma 3.4 every subgroup of $G$ that contains $\{h, \gamma\}$ is not virtually cyclic. We take a finite relative generating set $X$ of $G$ with respect to $\mathbb{K}$ which contains $\{h, \gamma\}$. Note that since $h$ is hyperbolic with respect to $\mathbb{K}$, no elements of $\mathbb{K}$ contain $X$.

Let $Q$ and $\mathbb{L}$ be given by Lemma 3.7. By Lemma 3.8 (1), the family $\mathbb{L}$ consists of proper subgroups of $Q$. Since $\Gamma \cap Q$ contains $\{h, \gamma\}$ and each element of $\mathbb{L}$ is contained in some element of $\mathbb{K}$, the subgroup $\Gamma \cap Q$ is neither virtually cyclic nor parabolic with respect to $\mathbb{L}$.

Since $Q$ is finitely generated, it follows from Lemma 3.6 that $\Gamma \cap Q$ contains a free subgroup $F$ of rank two which is strongly quasiconvex relative to $\mathbb{L}$ in $Q$. By Lemma 3.8 (2), the subgroup $F$ is strongly undistorted relative to $\mathbb{K}$ in $G$. □

Remark 3.9. We give an alternative proof of Corollary 3.2.

Let $F'$ be a free group of rank two with a free basis $Y'$ and $X$ a finite relative generating set of $G$ with respect to $\mathbb{K}$. By [11, Theorem 1.1], there exists a quotient group $G'$ of $G$ and an embedding $\iota: F' \to G'$ such that $G'$ is hyperbolic relative to $\{\psi(K)\}_{K \in \mathbb{K}} \cup \{\iota(F')\}$, where $\psi: G \to G'$ denotes the natural projection. Since $\iota(F')$ is a hyperbolic group, it follows from [13, Theorem 2.40] that $G'$ is hyperbolic relative to $\{\psi(K)\}_{K \in \mathbb{K}}$. Hence $\iota(F')$ is hyperbolically
embedded into $G'$ relative to $\{\psi(K)\}_{K \in \mathcal{K}}$. By Theorem 2.3, the natural map $(F', d_{Y'}) \to (G', d_{\psi(X) \cup K'})$ is a quasi-isometric embedding, where $\mathcal{K}'$ denotes $\bigcup_{K \in \mathcal{K}} \psi(K) \setminus \{1\}$. We take a subset $Y'$ of $G$ such that $Y$ consists of two elements and $\psi(Y)$ is equal to $\iota(Y')$. We denote by $F$ the subgroup of $G$ which is generated by $Y$. Then $F$ is a free group of rank two. We can confirm that the natural map $(F, d_Y) \to (G, d_X \cup K)$ is a quasi-isometric embedding. This finishes the proof of Corollary 3.2.

4 Almost malnormal subgroups of virtually free groups

In this section, we show the following (compare with [9, Theorem 5.16] for the case of non-abelian free groups of finite rank), which is necessary for the proof of Theorem 1.2.

**Theorem 4.1.** Let $M$ be a finitely generated and virtually non-abelian free group and let $\{M_l \mid l \in \{1, \ldots, n\}\}$ be a finite family of finitely generated subgroups of $M$ of infinite index. Then there exists a proper subgroup $V$ of $M$ satisfying the following:

(i) the subgroup $V$ is finitely generated, virtually non-abelian free, and almost malnormal in $M$;

(ii) for every $l \in \{1, \ldots, n\}$ and every element $m$ of $M$, the intersection $mVm^{-1} \cap M_l$ is finite.

For the proof of Theorem 4.1 we prepare two lemmas.

**Lemma 4.2.** If a proper subgroup $H$ of an infinite group $G$ is almost malnormal in $G$, then $H$ is an infinite index subgroup of $G$.

**Proof.** Assume that $H$ is a finite index subgroup of $G$. Then for every element $g$ of $G$, the intersection $H \cap gHg^{-1}$ is a finite index subgroup of $G$ and hence it is infinite. This contradicts the assumption that $H$ is almost malnormal in $G$. \hfill \square

**Lemma 4.3.** Let $F$ be a non-abelian free group of finite rank and let $\{H_l \mid l \in \{1, \ldots, n\}\}$ be a finite family of finitely generated subgroups of $F$ of infinite index. Let $U$ be a finite subgroup of $\text{Out}(F)$. We denote by $\pi: \text{Aut}(F) \to \text{Out}(F)$ the quotient map and put $A = \pi^{-1}(U)$. Then there exists a proper subgroup $H$ of $F$ satisfying the following:

(i) the subgroup $H$ is a free subgroup of rank two and is malnormal in $F$;

(ii) for every $l \in \{1, \ldots, n\}$ and every element $a$ of $A$, the intersection $H_l \cap a(H)$ is trivial;

(iii) for every element $a$ of $A$, either $a(H)$ is equal to $H$ or the intersection $H \cap a(H)$ is trivial.
Proof. We put $U = \{u_i \mid i \in \{1, \ldots, m\}\}$ and choose an element $a_i$ of $\pi^{-1}(u_i)$ for each $i \in \{1, \ldots, m\}$. We denote by $\mathcal{M}$ the collection of all proper subgroups $H'$ of $F$ satisfying the following:

- the subgroup $H'$ is a free subgroup of rank two and is malnormal in $F$;
- for every $i \in \{1, \ldots, m\}$, every $l \in \{1, \ldots, n\}$ and every element $f$ of $F$, the intersection $a_i^{-1}(H_l) \cap fH'f^{-1}$ is trivial.

By [9, Theorem 5.16], the collection $\mathcal{M}$ is not empty. We remark that every element of $\mathcal{M}$ satisfies conditions (i) and (ii) in Lemma [13].

For each $i \in \{1, \ldots, m\}$ and each element $H'$ of $\mathcal{M}$, we put as follows:

$$\mathfrak{K}_i = \{K \subseteq H' \mid K \neq \{1\} \text{ and } K = H' \cap f a_i(H') f^{-1} \text{ for some } f \in F\};$$

$$I_1(H') = \{i \in \{1, \ldots, m\} \mid \mathfrak{K}_i = \emptyset\};$$

$$I_2(H') = \{i \in \{1, \ldots, m\} \mid \mathfrak{K}_i = \{H'\}\};$$

$$I_3(H') = \{i \in \{1, \ldots, m\} \mid \mathfrak{K}_i \neq \emptyset \text{ and } \mathfrak{K}_i \neq \{H'\}\}.$$ 

Since every finitely generated subgroup of $F$ is quasiconvex in $F$ (see [15, Section 2]), the subgroup $a_i(H')$ as well as $H'$ is quasiconvex and malnormal in $F$. By [2, Theorem 7.11], the group $F$ is hyperbolic relative to $\{a_i(H')\}$. Since $H'$ is quasiconvex in $F$, it follows from [10, Theorem 1.1 (1)] that $H'$ is quasiconvex relative to $\{a_i(H')\}$ in $F$. By [7, Theorem 9.1], the collection $\mathfrak{K}_i$ has a finite set of representatives of $H'$-conjugacy classes $\mathfrak{K}_i = \{K_{i,j} \mid j \in \{1, \ldots, n_i\}\}$ and $H'$ is hyperbolic relative to $\mathfrak{K}_i$. For every $j \in \{1, \ldots, n_i\}$ the subgroup $K_{i,j}$ is finitely generated and malnormal in $H'$ (see [13, Propositions 2.29 and 2.36]).

For the proof of the lemma, it suffices to show that there exists an element $H$ of $\mathcal{M}$ such that $I_3(H)$ is empty. Indeed if $H$ is such an element of $\mathcal{M}$, then for every $a \in A$, either both $H \cap a(H)$ and $H \cap a^{-1}(H)$ are equal to $H$ or the intersection $H \cap a(H)$ is trivial. If the former occurs, then $a(H)$ is equal to $H$. Hence the subgroup $H$ has the desired properties.

Let $H'$ be an element of $\mathcal{M}$. We have only to consider the case where the set $I_3(H')$ is not empty. Then for every $i \in I_3(H')$ and every $j \in \{1, \ldots, n_i\}$, the group $K_{i,j}$ is a proper malnormal subgroup of $H'$ and hence it is of infinite index in $H'$ by Lemma [4.3]. By [9, Theorem 5.16], there exists a proper subgroup $H''$ of $H'$ satisfying the following:

- $H''$ is a free subgroup of rank two and is malnormal in $H'$;
- for every $i \in I_3(H')$, every $j \in \{1, \ldots, n_i\}$ and every element $h'$ of $H'$, the intersection $K_{i,j} \cap h'H''h'^{-1}$ is trivial.

Since $H'$ belongs to $\mathcal{M}$, the subgroup $H''$ also belongs to $\mathcal{M}$.

We claim that if $i \in \{1, \ldots, m\}$ belongs to $I_3(H')$, then for every element $f$ of $F$, the intersection $H'' \cap a_i(fH'f^{-1})$ is trivial. Indeed, the intersection $H' \cap a_i(fH'f^{-1})$ is either trivial or conjugate to $K_{i,j}$ in $H'$ for some $j \in \{1, \ldots, n_i\}$. If the former occurs, the claim obviously holds. If the latter occurs, the intersection $H'' \cap a_i(fH'f^{-1})$ is conjugate to a subgroup of $K_{i,j} \cap h'H''h'^{-1}$ for some $j \in \{1, \ldots, n_i\}$. This completes the proof.
\{1, \ldots, n_i\} and some element $h'$ of $H'$. By the choice of $H''$, the intersection $H'' \cap a_i(fH''f^{-1})$ is trivial.

The above claim implies that $I_3(H')$ is contained in $I_1(H'')$. Since $H''$ is a subgroup of $H'$, the set $I_1(H'')$ is also contained in $I_1(H'')$. Hence the union $I_1(H') \cup I_3(H'')$ is a proper subset of the union $I_1(H') \cup I_3(H'')$ if $I_3(H'')$ is not empty. By repeating this procedure if necessary, we can find an element $H$ of $\mathcal{M}$ such that $I_3(H)$ is empty.

Now we are ready to prove Theorem 4.1. Given a subgroup $H$ of a group $G$, we put

$$V_G(H) = \{g \in G \mid H \cap gHg^{-1} \text{ is of finite index both in } H \text{ and } gHg^{-1}\}.$$

**Proof of Theorem 4.1.** It follows from the assumption that $M$ has a finite index normal subgroup $F$ which is a non-abelian free group of finite rank. The action of $M$ on $F$ by conjugations induces a homomorphism from $M$ to $\text{Out}(F)$. We denote the image of this homomorphism by $U$. Since $F$ is a finite index subgroup of $M$, $U$ is a finite subgroup of $\text{Out}(F)$. For each $l \in \{1, \ldots, n\}$, we put $H_l = M_l \cap F$. Since $M_l$ is finitely generated and $F$ is a finite index subgroup of $M$, this implies that $H_l$ is also finitely generated. Since the subgroup $M_l$ is of infinite index in $M$ and the subgroup $F$ is of finite index in $M$, the subgroup $H_l$ is of infinite index in $F$ and of finite index in $M_l$.

Therefore we can take a subgroup $H$ of $F$ given by Lemma 4.3. We put $V = V_{M_l}(H)$. By [8, Theorem 1.6], the group $H$ is a finite index subgroup of $V$. Hence $V$ is a finitely generated and virtually non-abelian free group. By the definition of $U$ and condition (iii) in Lemma 4.3, for every element $m$ of $M$, either $mHm^{-1}$ is equal to $H$ or the intersection $H \cap mHm^{-1}$ is trivial. Hence for every element $m$ of $M \setminus V$, the intersection $H \cap mHm^{-1}$ is trivial. Since $H$ is a finite index subgroup of $V$, this implies that $V$ is almost malnormal in $M$. By condition (ii) in Lemma 4.3, for every $l \in \{1, \ldots, n\}$ and every element $m$ of $M$, the intersection $H_l \cap mHm^{-1}$ is trivial and hence the intersection $M_l \cap mVm^{-1}$ is finite.

### 5 Proof of Theorem 1.2

We prove Theorem 1.2. As the case of Corollary 3.2, it suffices to show the following in view of [14, Corollary 4.5].

**Theorem 5.1.** Let $G$ be a group which is hyperbolic relative to a family $\mathcal{K}$ of proper subgroups and $\Gamma$ a subgroup of $G$ which is neither virtually cyclic nor parabolic with respect to $\mathcal{K}$. If $\Gamma$ contains an element of infinite order, then there exists a finitely generated and virtually non-abelian free subgroup $V$ of $G$ which is hyperbolically embedded into $G$ relative to $\mathcal{K}$ and contains $V \cap \Gamma$ as a finite index subgroup. Moreover if $G$ is torsion-free, then $\Gamma$ contains a free subgroup of rank two which is hyperbolically embedded into $G$ relative to $\mathcal{K}$.
This generalizes a result due to I. Kapovich [9, Theorem C] for torsion-free hyperbolic groups. For the proof of Theorem 5.4 we show the following lemma.

**Lemma 5.2.** Let $G$ be a group which is hyperbolic relative to a finite family $\mathcal{K}$ of proper subgroups and $\Gamma$ a subgroup of $G$ which is neither virtually cyclic nor parabolic with respect to $\mathcal{K}$. Suppose that $G$ is finitely generated. Then there exists a finitely generated and virtually non-abelian free subgroup $V$ of $G$ which is hyperbolically embedded into $G$ relative to $\mathcal{K}$ and contains $V \cap \Gamma$ as a finite index subgroup. Moreover if $G$ is torsion-free, then $\Gamma$ contains a free subgroup of rank two which is hyperbolically embedded into $G$ relative to $\mathcal{K}$.

**Proof.** By Lemma 3.4, the group $\Gamma$ contains a free subgroup $F$ of rank two which is strongly quasiconvex relative to $\mathcal{K}$ in $G$. We put $M = V_G(F)$. Since $F$ is strongly quasiconvex relative to $\mathcal{K}$ in $G$, it follows from [8, Theorem 1.6] that $F$ is a finite index subgroup of $M$. Hence $M$ is strongly quasiconvex relative to $\mathcal{K}$ in $G$ and we have $V_G(M) = M$. By [8, Corollary 8.7], there exists only finitely many double cosets $MgM$ in $G$ such that $gMg^{-1}$ is not equal to $M$ and the intersection $M \cap gMg^{-1}$ is infinite. We denote the collection of such double cosets by $\{MgM \mid l \in \{1, \ldots, n\}\}$. For each $l \in \{1, \ldots, n\}$, we put $M_l = M \cap g_lMg_l^{-1}$.

We claim that for each $l \in \{1, \ldots, n\}$, $M_l$ is of infinite index in both $M$ and $g_lMg_l^{-1}$. Indeed, assume that this does not hold. Since $g_l$ does not belong to $M$ and $V_G(M)$ is equal to $M$, the subgroup $M_l$ is of infinite index in either $M$ or $g_lMg_l^{-1}$. By replacing $g_l$ by its inverse if necessary, we may assume that the subgroup $M_l$ is of finite index in $M$ and of infinite index in $g_lMg_l^{-1}$. It follows from [9, Lemma 6.6] that for every positive integer $k$, the subgroup $M \cap g_l^kMg_l^{-k}$ is of finite index in $M$ and of infinite index in $g_lMg_l^{-1}$. Then for every positive integer $p$, the subgroup $\bigcap_{k=1}^{p} M \cap g_l^kMg_l^{-k}$ is of finite index in $M$ and hence the intersection $\bigcap_{k=1}^{p} M \cap g_l^kMg_l^{-k}$ is equal to $M_l$. For each $l \in \{1, \ldots, n\}$, we put $M_l = M \cap g_lMg_l^{-1}$.

We claim that for each $l \in \{1, \ldots, n\}$, the subgroup $M_l$ is finitely generated. Indeed, since $M_l$ is strongly quasiconvex relative to $\mathcal{K}$ in $G$, it follows from Theorem 2.5 that the conjugate $g_lMg_l^{-1}$ is also strongly quasiconvex relative to $\mathcal{K}$ in $G$. Hence $M_l$ is strongly quasiconvex relative to $\mathcal{K}$ in $G$ (see for example [7, Theorem 9.8] and [13, Theorem 4.18]). By Theorem 2.5, the subgroup $M_l$ is finitely generated.

Hence it follows from Theorem 4.4 that there exists a finitely generated and virtually non-abelian free subgroup $V$ of $M$ such that $V$ is almost malnormal in $M$ and $mVm^{-1} \cap M_l$ is finite for every $l \in \{1, \ldots, n\}$ and every element $m$ of $M$. Since $M$ contains a finitely generated free subgroup of finite index and $V$ is a finitely generated subgroup of $M$, the subgroup $V$ is undistorted in $M$, that is, it is strongly undistorted relative to the empty family $\emptyset$ in $M$. Since $M$ is strongly undistorted relative to $\mathcal{K}$ in $G$ by Theorem 2.5, the subgroup $V$ is strongly undistorted relative to $\mathcal{K}$ in $G$.

We claim that $V$ is almost malnormal in $G$. Indeed, assume that $V$ is not
almost malnormal in $G$. Then there exists an element $g$ of $G \setminus V$ such that the intersection $V \cap gVg^{-1}$ is infinite. In particular the intersection $M \cap gMg^{-1}$ is also infinite. Since $V$ is almost malnormal in $M$, the element $g$ belongs to $G \setminus M$. Then for some $l \in \{1, \ldots, n\}$ and some elements $m_1$ and $m_2$ of $M$, the element $g$ is equal to $m_1 g m_2$. Therefore the intersection $V \cap (m_1 g m_2) V (m_1 g m_2)^{-1}$ is infinite and hence the intersection $m_1^{-1} V m_1 \cap g_1 M g_1^{-1}$ is also infinite. This contradicts the condition that for every $l \in \{1, \ldots, n\}$ and every element $m$ of $M$, the intersection $m V m^{-1} \cap M$ is finite.

Thus the subgroup $V$ is strongly undistorted relative to $K$ and almost malnormal in $G$. By Theorem 2.3, the subgroup $V$ is hyperbolically embedded into $G$ relative to $K$.

For the case where $G$ is torsion-free, we can take a desired subgroup $V$ of $\Gamma$ by applying [9, Theorem 5.16] to $F$ instead of applying Theorem 4.1 to $M$ in the above argument.

**Proof of Theorem 5.1**. The proof is done in the same way as the proof of Proposition 3.1 by using Lemma 5.2 and Lemma 3.8 (3) instead of Lemma 3.6 and Lemma 3.8 (2), respectively.

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