D2-branes with magnetic flux in the presence of RR fields

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Abstract

D2-branes are studied in the context of Born-Infeld theory as a source of the 3-form RR gauge potential. Considering the static case with only a radial magnetic field it is shown that a locally stable hemispherical deformation of the brane exists which minimises the energy locally. Since the D2-brane carries also the charge of D0-branes, and the RR spacetime potential is unbounded from below, these can tunnel to condense on the D2-brane. The corresponding instanton-like configuration and the tunneling rate are derived and discussed.

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1 Introduction

The low energy dynamics of D-branes described by Born-Infeld theory on the worldvolume of the brane is a topic of intense investigation and has led to useful insights into how string theory is interwoven with electromagnetic phenomena. Such investigations are useful in domains where gravitational (closed string) effects can be ignored in the leading approximation. The dynamics of these branes changes drastically in the presence of external (spacetime) forces, such as those of RR fields contained in type II superstring theories, since the D-brane carries the appropriate RR charges and so couples to the potentials. In the following we will be concerned with these interactions in the particular case of D2-branes.

Born-Infeld theory by itself (i.e. without RR fields) has been shown [1] to imply string-like brane excitations which owing to their charge and tension can be identified in the appropriate limit as fundamental strings. Such a string may be looked at as a collapsed brane or the original brane as one with a dissolved

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fundamental string [2]. The conserved (axion) charge of these strings along (say) $x^1$ results from the electric component $B_{01}$ of the NS B-field contained in the Born-Infeld action, and their tension is of order 1 (as distinct from the $1/(\text{string coupling } g)$ behaviour of the tension of D-branes). Considering Born-Infeld theory of the D2-brane in the static limit and with only the electric component of the $U(1)$ gauge field in its world volume, these strings are globally stable. However, on application of the RR field, i.e. with minimal coupling of the brane to the RR potential, the Born-Infeld string has been shown to be only locally stable and can tunnel to or expand into a D2-brane [2, 3], which is also described as the polarisation of a system of fundamental strings into a higher dimensional brane. This tunneling has been considered in detail in ref. [3].

It is natural to extend such investigations to the magnetic counterpart or rather to the fully electromagnetic formulation which, of course, introduces complications as soon as Lorentz boosts or deviations from a static case are required. Such investigations have been carried out recently in various directions [4, 5, 6, 7, 8, 9, 10, 11], and our objective here is to extend some considerations of refs. [2] and [4] with particular reference to brane-antibrane pairs as in ref. [1] and the case of dielectric branes [10, 11].

Since the D2-brane couples not only to the three-form RR potential but in the presence of magnetic flux also to the one-form potential of D0-branes, the magnetic case is very different from that of the purely electric case. Static torus-like brane configurations in the presence of only a pure magnetic field have been derived in ref. [4] as well as locally stable spherical configurations which can be related to the dielectric D-branes of ref. [10]. The presence of the magnetic flux provides these theories locally with energy minima at nontrivial expanded configurations analogous to the separation of charges in a dielectric medium. The related stabilisation of branes (i.e. prevention of their collapse to trivial or point-like pure tension configurations) by the presence of magnetic flux was pointed out earlier in ref. [12] in the context of WZW models.

In the following we remain in the context of the model of refs. [2] and [4] and show that a locally stable hemispherical brane configuration can be shown to exist for a pure magnetic field in the world volume. We demonstrate this explicitly by considering fluctuations around the brane. The considerations are analogous to those of branes and their antibranes of ref. [1] whose stability was studied in detail in ref. [13]. We then consider the Euclidean time pseudoparticle brane configuration and its relation to the classical brane solution. Having found these, we calculate the transition rate of the 2-brane through the RR potential barrier and interpret the result as a set of D0-branes condensing on the D2-brane as suggested some time ago [12]. Irrespective of the physical significance of the result, we consider the explicit calculations which the model permits, to be very instructive for comparison with other cases such as those of refs. [1] and [13], and in providing hints on what one may expect in higher dimensional cases. Some specific calculations, such as the comparison with the spherical D2-brane of ref. [4, 10], the evaluation of the fluctuation determinant and the Lorentz transformation of the action are shifted into appendices.
The Born-Infeld action describing a $D2$-brane as the source of the 3-form RR gauge potential $A_{\mu \nu \rho}$ and its coupling to this RR 3-form gauge potential in type $IIA$ superstring theory is

$$I = -T_2 \int d^3 \xi \sqrt{-\det \left( \partial_\alpha X^\mu \partial_\beta X_\mu + 2\pi \alpha' F_{\alpha \beta} \right) + \frac{1}{3!} \epsilon^{\alpha \beta \gamma} A_{\mu \nu \rho} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho} \right], \quad (1)$$

where $T_2 = 1/4\pi^2 g$ is the $p = 2$ volume tension of the $D2$–brane obtained from

$$T_p = \frac{2\pi}{(2\pi l_s)^{p+1} g} \quad (2)$$

where we set $l_s = 1$ and $\lambda = 2\pi l_s^2 = 2\pi \alpha' = 2\pi$. Here $\mu_p = T_p$ is the RR charge of the brane under the $(p + 1)$–form RR potential. We follow ref. [2] but include also a magnetic field. With $\alpha' = 1$, we have

$$X^0 = t, \ X^1 = z, \ X^2 = R(t, z) \cos \theta, \ X^3 = R(t, z) \sin \theta, \ others = const \quad (3)$$

and

$$F_{\theta z} = \partial_\theta A_z - \partial_z A_\theta \quad (4)$$

and

$$H = dA, \ H_{0123} = h \quad (5)$$

and the target space metric with signature $-1, +1, +1, +1 \cdots$, and $\alpha, \beta = t, z, \theta, E_z = 2\pi F_{t z}, E_\theta = 2\pi F_{t \theta}, B = 2\pi F_{\theta z}$:

$$\left( \begin{array}{ccc} \partial_\alpha X^\mu \partial_\beta X_\mu + 2\pi \alpha' F_{\alpha \beta} \\
-1 + \dot{\hat{R}}^2 & \dot{\hat{R}} \dot{\hat{R}}' + E_z & E_\theta \\
-E_z + \dot{\hat{R}} \dot{\hat{R}} & 1 + \dot{\hat{R}}^2 & B \\
-E_\theta & -B & \dot{\hat{R}}^2 \end{array} \right). \quad (6)$$

Then

$$I = \int dt dz d\theta \mathcal{L}(R, A_0, A_z, A_\theta) \quad (7)$$

where for $E_\theta = 0$

$$\mathcal{L} = \frac{1}{4\pi^2 g} \left\{ -\sqrt{R^2(1 - \dot{\hat{R}}^2 + \dot{\hat{R}}^2 - E_z^2)} + B^2(1 - \dot{\hat{R}}^2) + \frac{h}{2} \dot{\hat{R}}^2 \right\}. \quad (8)$$

The four resulting Euler-Lagrange equations reduce to two constraints and one equation of motion but imply also certain conditions and therefore have to be considered carefully. The equations are respectively:
1. For $R$ we obtain the equation of motion

$$-\frac{\partial}{\partial t}\left(\frac{R^2 \dot{R} + \dot{R}B^2}{\sqrt{R^2(1 - \dot{R}^2 + \dot{R}^2 - E_z^2) + B^2(1 - \dot{R}^2)}}\right) + \frac{\partial}{\partial z}\left(\frac{R'R}{\sqrt{R^2(1 - \dot{R}^2 + \dot{R}^2 - E_z^2) + B^2(1 - \dot{R}^2)}}\right) - \frac{R(1 - \dot{R}^2 + \dot{R}^2 - E_z^2)}{\sqrt{R^2(1 - \dot{R}^2 + \dot{R}^2 - E_z^2) + B^2(1 - \dot{R}^2)}} + hR = 0. \quad (9)$$

2. For $A_0$ we obtain the equation equivalent to the Gauss law in Maxwell theory, i.e.

$$\frac{\partial D_E}{\partial z} = 0, \quad D_E \equiv \frac{R^2E_z}{\sqrt{R^2(1 - \dot{R}^2 + \dot{R}^2 - E_z^2) + B^2(1 - \dot{R}^2)}}. \quad (10)$$

3. For $A_\theta$ we obtain the equation

$$\frac{\partial (2\pi D)}{\partial z} = 0, \quad 2\pi D \equiv \frac{2\pi B(1 - \dot{R}^2)}{\sqrt{R^2(1 - \dot{R}^2 + \dot{R}^2 - E_z^2) + B^2(1 - \dot{R}^2)}}. \quad (11)$$

4. Finally for $A_z$ we obtain the equation

$$\frac{\partial}{\partial t}(2\pi D_E) + \frac{\partial}{\partial \theta}(2\pi D) = 0. \quad (12)$$

We observe that for the purely electric case considered in ref. [2] the second and the fourth equations imply that the electric quantity $D_E$ is independent of both $z$ and $t$ and is therefore a constant. In the purely magnetic case, which we concentrate on here, we conclude from the third and fourth equations that the magnetic quantity $D$ does not possess an explicit $z$-dependence, nor an explicit $\theta$-dependence, but can depend explicitly on $t$. In addition - and this is a vital point - $D$ is a functional of $R$ which is a function of $z$ and $\theta$ and possibly of $t$.

In the static and purely magnetic case the two remaining equations are

$$\frac{\partial}{\partial z}\left(\frac{R'R}{\sqrt{R^2(1 + \dot{R}^2) + B^2}}\right) - \frac{R(1 + R^2)}{\sqrt{R^2(1 + \dot{R}^2 + B^2)}} + hR = 0. \quad (13)$$

and

$$D = \frac{2\pi B}{\sqrt{R^2(1 + R^2 + B^2)}} \quad (14)$$

Setting

$$P^2 = 1/(1 - D^2)$$

and solving the latter equation for $B$ we obtain

$$B = PD\sqrt{1 + \dot{R}^2} \quad (15)$$
and

\[ R^2(1 + R'^2) + B^2 = R^2(1 + R^2)P^2 \]  \hspace{1cm} (16)

This equation allows us to express the quantity \( P^2 \) and so \( D \) in terms of \( R \) and \( B^2 \). Using this, eq. (13) leads to

\[ \frac{\partial}{\partial z} \left( \frac{RR'}{P\sqrt{1 + R'^2}} \right) - \frac{1}{P} \sqrt{1 + R'^2} + hR = 0. \]  \hspace{1cm} (17)

With further manipulations this equation can be converted into

\[ \frac{d}{dz} \left( \frac{R}{P\sqrt{1 + R'^2}} - \frac{hR^2}{2} \right) = 0, \]

\[ \frac{R}{P\sqrt{1 + R'^2}} - \frac{hR^2}{2} = C, \]  \hspace{1cm} (18)

where \( C \) is a constant. Eq. (18) is also contained in the work of ref. [4], and can also be obtained from a Legendre transformed Lagrangian density

\[ \mathcal{L}_B := -\mathcal{L} + B \frac{\partial \mathcal{L}}{\partial B}, \quad D = -4\pi^2 g \frac{\partial \mathcal{L}}{\partial B} \]  \hspace{1cm} (19)

where

\[ \mathcal{L}_B(R) = \frac{1}{4\pi^2 g} \left\{ \frac{R\sqrt{1 + R'^2}}{P} - \frac{hR^2}{2} \right\} \]  \hspace{1cm} (20)

We comment on other derivations later.

We now define the Hamiltonian density \( \mathcal{H} \) by

\[ \mathcal{H} = P_R \dot{R} + P_{A_z} \dot{A}_z + P_{A_\theta} \dot{A}_\theta - \mathcal{L}, \]  \hspace{1cm} (21)

where \( P_R, P_{A_z}, P_{A_\theta} \) are the conjugate momenta of \( R, A_z, A_\theta \), i.e. \( P = \partial \mathcal{L}/\partial \dot{R} \), etc. Then

\[ \mathcal{H} = \frac{1}{4\pi^2 g} \left\{ \frac{R^2 + B^2 + R^2R'^2}{\sqrt{(1 - R^2)(R^2 + B^2) + R^2(R^2 - E_z^2)}} - \frac{h}{2} R^2 \right\} + \frac{1}{g} D_E A_0' \]  \hspace{1cm} (22)

This expression is to be supplemented by the two constraint equations (10) and (11) for \( E_z \) and \( B \).

The magnetic flux in the world volume is for \( \dot{R} = 0 \) given by

\[ N = \frac{1}{2\pi} \int d\theta dz F_{\theta z} = \frac{1}{2\pi} \int dz \frac{R^2 D \sqrt{1 + R'^2}}{\sqrt{D_E^2 + R^2(1 - D^2)}}. \]  \hspace{1cm} (23)

This integral of the worldvolume 2-form field strength is in general not finite. Thus to obtain a finite expression for this charge, one has to close the membrane configuration as already discussed in ref. [14]. This is essentially the charge associated with the worldvolume vector potential. The latter is a 1-form which couples to a D0-brane.
The Hamiltonian $H$ now becomes in going to the static case $(\dot{R} = 0)$

$$H = \frac{1}{4\pi^2 g} \int dzd\theta \left[ (R^2 + D^2_E) \sqrt{\frac{1 + R'^2}{(D^2_E + R^2 - R^2 D^2)}} - \frac{\hbar R^2}{2} \right].$$  

(24)

In performing the variation

$$\frac{d}{dz} \left( \frac{\delta H}{\delta R'} \right) - \frac{\delta H}{\delta R} = 0$$

one has to remember that the expression $D$ is a functional of $R$ as expressed by eq. (16). Considering the static magnetic case this means that before the variation of the Hamiltonian

$$H = \frac{1}{4\pi^2 g} \int dzd\theta \left[ P R \sqrt{1 + R'^2} - \frac{\hbar R^2}{2} \right]$$

(25)

is performed, one has to replace $P$ with the help of expression (16) in terms of $B$. Finally again replacing the expression with $B$ by that with $P$ one arrives at the same equation which was obtained earlier from the Lagrangian. The Hamiltonian will be needed later for the calculation of the energy of our minimum energy configuration. In the above we considered the case of a cylindrical geometry. For comparison we provide in Appendix A the main formulae (for flux and energy) of the spherical case considered in refs. [4, 10].

3 Pure magnetic case

We consider the static and purely magnetic case. The total magnetic flux is nonzero which means that we have objects dissolved in the D2-brane carrying magnetic charge. Being the singularities of the magnetic field, these objects must be pointlike, hence they are D0-branes. In ref. [4] it was shown that the number of D0-branes can in this case be simply related to the magnetic flux. The Hamiltonian $H$ is given by eq. (24). We know that if the D2-brane is coupled to magnetic flux it also carries the charge of the D0-brane [15]. Thus it is sensible to consider the limit of the spherical configuration shrinking to a point. The worldvolume magnetic flux is given by

$$N = \frac{PD}{2\pi} \int dzR \sqrt{1 + R^2} = \frac{PDS}{(2\pi)^2}, \quad S = \int dzd\theta R \sqrt{1 + R'^2},$$

(26)

where $S$ is the area of the world volume which follows simply from the fieldless static metric contained in eq. (3). One can see from this relation that if $D$ were kept constant in a variation, a nonvanishing variation of $S$ would have to be compensated by a nonvanishing variation of $N$, and thus the number of D0-branes would not be fixed unless compensated appropriately from a reservoir of D0-charge around the D2-brane. From the relation (26) we obtain

$$D^2 = \frac{(NT_0)^2}{(ST_2)^2 + (NT_0)^2}, \quad NT_0 = ST_2 DP,$$

(27)
so that for $S$ shrinking to zero $D \to 1$ and the energy can be presented as the mass of $N$ D0-branes, i.e.

$$E_{R\to 0} = T_0 N. \quad (28)$$

One can also express the energy like eq. (27), as observed earlier in ref. [4]:

$$H = \sqrt{(ST_2)^2 + (NT_0)^2} - \frac{h}{4\pi g} \int dz R^2. \quad (29)$$

Varying this expression (and so the surface area $S$ with respect to $R$) one again obtains the same equation of motion as with the other methods for $D = D(R)$. The equation of motion, eq. (18), can be rewritten to give

$$R' = \pm \frac{h}{hR^2 + 2C} \sqrt{(R_+^2 - R^2)(R^2 - R^{-2})} \quad (30)$$

with

$$R_+^2 R_-^2 = \frac{4C^2}{h^2}, \quad R_+^2 + R_-^2 = \frac{4}{P^2 h^2} (1 - CP^2 h). \quad (31)$$

We are looking for nonperiodic, finite energy solutions. Consequently we set $R_-=0$ yielding for the integration constant $C=0$. Then eq. (30) simplifies to

$$R' = \pm \frac{\sqrt{R_+^2 - R^2}}{R}, \quad (32)$$

where $R_+ = \frac{2}{P h}$ and the configuration $R$ is geometrically a radius. The solutions of this equation are the configurations $R$ with respectively positive ($z_0 > z$) and negative ($z_0 < z$) derivatives as in eq. (32), i.e.

$$R(z) = -\int_{z_0 - R_+}^{z} \frac{(z - z_0)dz}{\sqrt{R_+^2 - (z - z_0)^2}} = \sqrt{R_+^2 - (z - z_0)^2},$$

$$R(z) = +\int_{z}^{z_0 + R_+} \frac{(z - z_0)dz}{\sqrt{R_+^2 - (z - z_0)^2}} = \sqrt{R_+^2 - (z - z_0)^2} \quad (33)$$

with the enveloping sphere

$$R^2 + (z - z_0)^2 = R_+^2. \quad (34)$$

This sphere is thus the envelope of the pair of spherical shells with $z_0 - R_+ \leq z \leq z_0, z_0 \leq z \leq z_0 + R_+$, or the pair of circles of radius $R$ at positions

$$z = z_0 \pm \sqrt{R_+^2 - R^2} = \pm \int_{R_+}^{R} \frac{RdR}{\sqrt{R_+^2 - R^2}}. \quad (35)$$

Thus $z(R)$ is double valued. The two possible signs define the two hemispherical configurations on the enveloping sphere given by eq. (34) and as indicated in Fig.1. These two hemispherical configurations defined by respectively positive or negative $R'$ (observe that $R$ originally defined as a radius is always positive, the
variable angle $\theta$ being understood) can be looked at as a brane and its antibrane analogous to the appearance of branes and their antibranes in ref.\[1\]. In fact, that the sphere (34) represents an unstable brane-antibrane pair can be seen by differentiating (34) with respect to $z$ and re-inserting the equation, which gives

$$\frac{dR}{dz} = \frac{z - z_0}{\sqrt{R^2 + (z - z_0)^2}}. \quad (36)$$

We see that this is an odd function which reverses its sign on passing through $z_0$. This is one of the characteristic properties of the configuration called a bounce [10]. (Like a periodic instanton it can be loosely looked at as an instanton-antiinstanton pair, the instanton (or antiinstanton) being a monotonically increasing (or decreasing) function of its argument contrary to the behaviour of the combination). The sphere is the limiting form of a spheroidal bulge like that discussed in the electric case in ref. \[4\]; hence the behaviour of $R'$ here is also that in this limit, i. e. that of a function with the shape typical of an odd first excited state wave function. Thus as expected in ref. [1] for the D3-brane model considered there and demonstrated for this in ref. [13], the brane-antibrane pair is unstable, which means that small fluctuations in its neighbourhood possess a negative mode, the tachyon (how the brane-antibrane system with magnetic flux $N = 2$ is related to tachyon condensation is described for instance in ref. [15]).

The appearance of these brane-antibrane pairs is a consequence of the two possible signs of the derivative in eq. (32), and so of two possible solutions, and these in turn are a consequence of the square-root form of the Born-Infeld Lagrangian density. Thus Born-Infeld theory leads very naturally to brane-antibrane configurations through a linkage of the solutions or continuation of the one solution to the other with the opposite sign of its derivative as in the cases considered in refs. [1] and [2].

For the energy $E$ of the solution with $R\sqrt{1 + R'^2} = R_+$ and $z_0 - R_+ \leq z \leq z_0$ as integration domain one finds from $H$

$$E = \frac{NT_0}{2D} - \left(\frac{4\pi R_+^3}{3}\right)\frac{hT_2}{2}. \quad (37)$$

With eq.(27) the first part can be rewritten so that

$$E = \frac{1}{2}\sqrt{(NT_0)^2 + (ST_2)^2} - \left(\frac{4\pi R_+^3}{3}\right)\frac{hT_2}{2} \quad (38)$$

(With the factor 1/2 the volume part is the same as that in the electric case of ref. \[4\]). This result is very physical with the first part representing the square root of the sum of the squares of the masses of the D2-brane and the N D0-branes, and the second the energy stored in the volume. The negative sign of the latter indicates that the (hemi)spherical configuration of radius $R_+$ minimises the energy locally as will be shown below. We observe from eqs.(24) and (25) that the potential is unbounded below for large values of $R$. Thus tunneling of the locally stable hemispherical configuration is possible, and one can calculate the appropriate transition rate. This will be examined in the following.
4 Small fluctuations and the pre-exponential factor

We now consider the second variation of $H$ around the positive derivative solution (33) and demonstrate that it is positive definite under small fluctuations. This means that any small deviation from this solution of (33) increases the energy. Consequently we have a minimum of the energy functional. This minimum is not a global minimum because the potential is unbounded below as observed earlier; thus it is only a local minimum. Next we determine the fluctuation operator describing the behaviour of the second variation of the energy functional in the vicinity of this classical solution. Born-Infeld theory is a covariant theory; therefore the existence of the static finite energy solution implies the existence of the instanton-type pseudoparticle solution which is really a bounce. In calculating the tunneling transition rate in the next section, we use the semiclassical approximation around this instanton-type solution. The argument of the exponential of the semiclassical amplitude is the pseudoparticle action, and the pre-exponential factor is, as usual, the determinant of the fluctuation operator of the pseudoparticle solution. From Lorentz invariance it follows that the fluctuation operators of both the static and the pseudoparticle solutions must coincide up to a possible sign factor, so that it suffices to calculate the determinant in the static case.

The second variation of $H$ is given by

$$\delta^2 H = \frac{1}{2} \int dz \left[ \frac{\delta^2 H}{\delta R'^2} (\delta R')^2 + 2 \frac{\delta^2 H}{\delta R\delta R'} \delta R \delta R' + \frac{\delta^2 H}{\delta R^2} (\delta R)^2 \right]$$

(39)

This is evaluated in Appendix B and the result is

$$\delta^2 H = \frac{1}{2} \int dz \delta R \hat{M} \delta R = \frac{1}{2} \int \frac{dR}{R} \delta R \hat{M} \delta R$$

(40)

where the fluctuation operator $\hat{M}$ is

$$\hat{M} = -\frac{1}{2\pi gP} \frac{1}{R'} \frac{d}{dz} \frac{RR'^2(1 + D^2 R'^2)}{(1 + R'^2)^{3/2}} \frac{d}{dz} \frac{1}{R'}.$$  

(41)

The derivative $R'$ is positive for one hemisphere as explained earlier, and negative for the other. Thus one of these minimises the energy locally, the other does not. We present $\hat{M}$ as a product because this has the advantage that its determinant can be easily calculated. This is our next step.

The determinant of the fluctuation operator $\hat{M}$ must be normalized. As a normalization point we choose $D = 0 (P = 1)$ which is the case of a vanishing magnetic field. We denote the corresponding classical solution by $R_0$ and the corresponding fluctuation operator by $\hat{M}_0$. The normalized determinant is then

$$\det_n \hat{M} = \frac{\det \hat{M}}{\det \hat{M}_0}.$$  

(42)

We use the following two properties of determinants:

$$\det(AB) = \det(BA), \quad \frac{\det(TA)}{\det(TB)} = \frac{\det A}{\det B}.$$  

(43)
The expression for the normalized determinant then simplifies and we have
\[
\det_n \hat{M} = \frac{\det[R(1 + D^2 R'^2)(1 + R'^2)^{-3/2}/P]}{\det[R_0(1 + R'^2_0)^{-3/2}]}.
\] (44)

For our solution with integration constant \( C = 0 \) we have
\[
\frac{R(1 + R'^2)^{-3/2}}{P} = \frac{\hbar^3 R^4}{8} P^2, \quad (1 + D^2 R'^2) = 1 + D^2 R'^2
\] (45)
resulting in
\[
\det_n \hat{M} = \frac{\det(R^4 + R^2 P^2 D^2 R'^2_+)}{\det R_0^4}.
\] (46)

The expression \( R^4 + R^2 P^2 D^2 R'^2_+ \) is a c-number and its determinant yields an integral:
\[
\det(R^4 + R^2 P^2 D^2 R'^2_+) = \exp[\text{Tr}[\ln(R^4 + R^2 P^2 D^2 R'^2_+)]]
\]
\[
= \exp\left[\int_{-R_+}^0 \frac{dz}{R_+} \ln(R^4 + R^2 P^2 D^2 R'^2_+)\right].
\] (47)

The evaluation of the integral is straightforward and yields
\[
\int_{-R_+}^0 \frac{dz}{R_+} \ln(R^4 + R^2 P^2 D^2 R'^2_+) = -4 + 2 \ln(2R^2) + \ln(P^2 - 1) + P \ln\left(\frac{P + 1}{P - 1}\right)
\] (48)
which gives for the determinant
\[
\det_n \hat{M} = \frac{P^2 - 1}{P^4} \left(\frac{P + 1}{P - 1}\right)^P = \frac{D^2}{P^2} \left(\frac{P + 1}{P - 1}\right)^P.
\] (49)

One may note that the expressions (48) and (49) are finite for \( P \to 1 \).

5 The tunneling amplitude

To describe the tunneling by the instanton-type pseudoparticle solution in Euclidean time, we make a Wick rotation and set \( t \to i\tau \). The way to find this pseudoparticle solution in a case like the one here was already proposed in [2]. One performs an exchange \( \tau \leftrightarrow z \), so that having the static solution one obtains the pseudoparticle one. However, the Lagrangian of eq.(8) does not exhibit an explicit symmetry under this exchange. This might seem strange but has a simple explanation if we include in our considerations the polar component \( E_\theta \) of the electric field which has therefore been given explicitly in eq.(6). We now perform a right angle rotation in the \((z, \tau)\) plane, i.e.
\[
z \to z' = \tau, \quad \tau \to \tau' = -z
\] (50)
under which electromagnetic components transform as follows:
\[
E_z \to E_z
\] (51)
and
\[ E_\theta \rightarrow B, \ B \rightarrow -E_\theta, \] (52)
where \( B \) is as before the magnetic field. The above transformations show that in the purely magnetic case the action does not have the explicit symmetry under the exchange \( \tau \leftrightarrow z \). On the other hand the transformations (52) are electromagnetic duality transformations [17] leaving the equations of motion invariant [17, 18]. Therefore the Lorentz transformations yield for the pseudoparticle the same equation (26). We verify this explicitly in Appendix C. Also one can make the backward Lorentz rotation and obtain the pseudoparticle solution in the original frame. The shape of the pseudoparticle is that of a sphere which, however, is not invariant under this rotation and instead becomes an ellipsoid. It is instructive to obtain the same ellipsoid directly from the Euler-Lagrange eq. (9). We set \( E_z = 0, R' = 0 \), so that in this nonstatic case
\[ B^2 = \frac{D^2 R^2}{1 - R^2 - D^2} \]
and obtain for the time dependent solution the equation
\[ -\frac{d}{dt} \left( \frac{R \dot{R}}{\sqrt{1 - \dot{R}^2 - D^2}} \right) - \sqrt{1 - \dot{R}^2 - D^2} + hR = 0 \] (53)
Going to Euclidean time with \( t = i\tau \) and using the fact that the equation does not contain \( \tau \) explicitly, allows us to convert it into the following first order differential equation
\[ \frac{R}{P^2 \sqrt{1 - D^2 + (\frac{dR}{d\tau})^2}} = \frac{h^2}{2} R^2 = C, \] (54)
where \( C \) is a constant. For the discussion below it is more transparent to consider first the general case with \( C \neq 0 \). Then
\[ \frac{dR}{d\tau} = \pm \frac{h}{2P(C + \frac{h}{2} R^2)} \sqrt{(R^2_+ - R^2)(R^2 - R^2_+)} \] (55)
where
\[ \frac{R^2_+}{P^2 h^2} = \frac{1}{1 - C h P^2} \pm \sqrt{1 - 2 C h P^2} \] (56)
For large \( R \) the energy given by the Hamiltonian \( H \) of eq.(23) decreases without limit. Thus the motion of the pseudoparticle in Euclidean time starts with zero “velocity” \( dR/d\tau \) at \( R = R_+ \) and bounces back from the potential wall at point \( R_+ \) at the time-symmetric point \( \tau = \tau_0 \) until it again reaches \( R = R_+ \). At the time-symmetric point the reversal of the velocity implies a switch from one sign of the square root to the other. Now we again choose the integration constant \( C \) to be zero as in eq. (52). We thus have \( R_+ = 0 \) and obtain
\[ P^2 \left( \frac{dR}{d\tau} \right)^2 + 1 = \frac{R^2_+}{R^2}, \] (57)
From this we obtain the solution

$$P^2 \left( \sqrt{R^2 - R_+^2} - \sqrt{R^2 - R_-^2} \right)^2 + (\tau - \tau_0)^2 = 0, \quad (58)$$

where $R_-$ is the bouncing or turning point at Euclidean time $\tau_0$. However, in evaluating the action of the bounce and hence the tunneling rate it is more convenient to integrate with respect to $R$ by setting $d\tau = dR/\dot{R}$ and using the derivative relation (57), i.e. also

$$PR\sqrt{1 + \left( \frac{dR}{d\tau} \right)^2} = \sqrt{R_+^2 + D^2P^2R^2}. \quad (59)$$

The pseudoparticle action $I_E$ is now obtained from eq.(7), i.e.

$$I_E = -T_2 \int dz \int d\tau \left\{ -PR\sqrt{1 + \left( \frac{dR}{d\tau} \right)^2} + \frac{\hbar}{2} R^2 \right\} \quad (60)$$

The orbit of the bounce is that from $R_+$ to $R_-$ and back, so that

$$I_E = -2\pi PT_2 \int dz \int_{R_+^2}^{R_-^2} dR^2 \left\{ \sqrt{\frac{R_+^2 + D^2P^2R^2}{R_+^2 - R^2}} - \frac{\hbar}{2} \frac{R^2}{\sqrt{R_+^2 - R^2}} \right\} \quad (61)$$

Evaluating this by integrating over the compact length $L$ of a torus we obtain

$$I_E = -4\pi PT_2LR_+^2F_1(1, -1/2, 3/2; -D^2P^2) + \left( \frac{4\pi R_+^3}{3} \right) L\hbar T_2 \quad (62)$$

Using eq.(27) the action $I_E$ can be rewritten as

$$I_E = -L \frac{NT_0}{D} F_1(1, -1/2, 3/2; -D^2P^2) + \left( \frac{4\pi R_+^3}{3} \right) L\hbar T_2 \quad (63)$$

Since action corresponds to energy × length, this result is, as expected, similar to the energy (38) of the sphere but takes into account the ellipsoidal deformation. In computing the decay or tunneling rate we have to subtract from the action of the pseudoparticle the action of the initial state which means here the square-root part of the action. For the argument of the hypergeometric function we also have

$$D^2P^2 = \left( \frac{\pi N}{R_+^2} \right)^2. \quad (64)$$

We observe that for no magnetic field and so charge, or no D0-branes, the radius $R_+$ is zero, that is, there is no dielectric effect.

With this the semiclassical approximation of the tunneling amplitude $\Gamma$ is given by

$$\Gamma = (det\hat{M})^{-\frac{1}{2}} \exp(-I_E). \quad (65)$$
Inserting the appropriate quantities, the final expression for the tunneling rate is seen to be

$$\Gamma = \frac{P}{D} \left( \frac{P - 1}{P + 1} \right)^{p/2} \exp \left( - \frac{4\pi R^3}{3} P h T_2 \right)$$

$$= \frac{P}{D} \left( \frac{P - 1}{P + 1} \right)^{p/2} \exp \left( - \frac{8L}{3\pi g P^2 h^2} \right).$$

(66)

In the limit of $D \to 0, P \to 1$, there is no electromagnetic field (cf. (12) and (13)). But we still have a static spherical solution which is the excitation of the original brane under the influence of the RR field. In the opposite limit of $D \to 1, P \to \infty$ of a strong magnetic field, it is better to express the tunneling amplitude via the D0-brane number. From eqs.(26) and (27) it follows that

$$P^2 = \left( \frac{4}{\pi h^2 N} \right)^2 \left[ 1 + O \left( \frac{1}{P^2} \right) \right], \quad \pi h^2 N \ll 1.$$  

(67)

Thus the tunneling rate becomes

$$\Gamma \approx \frac{4}{\pi h^2 N} \exp \left( - \frac{\pi L h^2 N^2}{6g} \right).$$

(68)

6 The purely temporal case

We consider here the cylindrical D2-brane with magnetic field in a purely time-dependent case and demonstrate an analogy with the static, purely electric case [2, 3]. This permits the consideration of quantum classical transitions in the magnetic case to be taken over from the electric case.

We start from our original D2-brane action

$$S_{D2}[R, A_\theta, A_z] = -T_2 \int dt d\theta dz \left[ \sqrt{R^2(1 + R^2 - \dot{R}^2) + B^2(1 - \dot{R}^2)} - \frac{h}{2} R^2 \right]$$

(69)

with $-\pi \leq \theta \leq \pi$ and $-L/2 \leq z \leq L/2$. Varying the action (69) yields the following equations of motion:

$$\frac{\partial}{\partial z} D = \frac{\partial}{\partial \theta} D = 0,$$

$$\frac{\partial}{\partial t} \sqrt{R^2(1 + R^2 - \dot{R}^2) + B^2(1 - \dot{R}^2)} - \frac{\partial}{\partial z} \sqrt{R^2(1 + R^2 - \dot{R}^2) + B^2(1 - \dot{R}^2)}$$

$$+ \frac{R(1 + R^2 - \dot{R}^2)}{\sqrt{R^2(1 + R^2 - \dot{R}^2) + B^2(1 - \dot{R}^2)}} - hR = 0,$$

(70)

where now with $B = 2\pi F_{\theta z}$

$$D = \frac{B(1 - \dot{R}^2)}{\sqrt{R^2(1 + R^2 - \dot{R}^2) + B^2(1 - \dot{R}^2)}}.$$  

(71)
Since the equations \( \partial_z D = \partial_\theta D = 0 \) have already been solved in the static case \((R = R(z))\), we now consider the solution of \( \partial_z D = \partial_\theta D = 0 \) in the purely temporal case, i.e. for \( R = R(t) \).

In this case \( F_{\theta z} = F_{\theta z}(t) \) could be an arbitrary function of time. Since, however, we are considering the purely magnetic case, it is appropriate to assume again \( F_{\theta z} = \text{const} \), because if not, the time-dependent magnetic field usually generates an electric field. Thus, we choose

\[
F_{\theta z} = \frac{N}{L},
\]

where

\[
N = \frac{1}{2\pi} \int d\theta dz F_{\theta z}
\]

is the total number of D0-brane particles or quantised flux through the cylindrical surface of length \( L \).

Using (72) and \( R' = 0 \) the last of eqs. (70) reduces to

\[
\frac{\partial}{\partial t} \left( \sqrt{R^2 + \xi^2} \right) + R \sqrt{\frac{1 - \dot{R}^2}{R^2 + \xi^2} - \frac{h}{2} R^2} = 0
\]

where

\[
\xi = \frac{N\lambda}{L}, \quad \lambda = 2\pi.
\]

One can also show that eq.(74) is obtained directly by varying the action

\[
\tilde{S}_{D_2}[R] = -2\pi LT_2 \int dt \left[ \sqrt{(R^2 + \xi^2)(1 - \dot{R}^2)} - \frac{h}{2} R^2 \right].
\]

In general, we cannot insert a classical solution into the action before varying it, however here this is permissible for the constant solution (72), which can also be shown by varying action (77). Before solving (74) it is helpful to consider the potential \( V_{D_2}(R) \) which can be read off from (73),

\[
V_{D_2}(R) = 2\pi LT_2 \left( \sqrt{R^2 + \xi^2} - \frac{h}{2} R^2 \right).
\]

It is interesting to reexpress the potential as

\[
V_{D_2}(R) = \sqrt{(ST_2)^2 + (NT_0)^2} - h\sqrt{V}T_2
\]

where \( S = 2\pi RL \) and \( V = \pi R^2 L \). In fact, \( S \) and \( V \) are respectively surface area and volume of the cylindrical D2-brane in flat spacetime. Hence, the potential consists of two terms, i.e. the surface energy of the D2-brane with \( N \) D0-branes dissolved in it and the volume energy.

The shape of the potential \( V_{D_2}(R) \) is as follows. If \( \xi h > 1 \), \( V_{D_2}(R) \) is a monotonically decreasing function and \( R = 0 \) becomes a point of instability. If \( \xi h < 1 \), \( V_{D_2}(R) \) has a local minimum at \( R = 0 \) and a global maximum at \( R \equiv R_* = \sqrt{1/h^2 - \xi^2} \). Thus we have quantum tunneling in this case. Here we
confine ourselves to the latter case ($\xi h < 1$). We summarize several particular values computed from the potential:

\[ V_{D_2}(R = 0) = 2\pi L T_2 \xi \equiv NT_0, \quad (79) \]
\[ V_{D_2}(R = R_*) = \frac{\pi L T_2}{h}(1 + h^2 \xi^2), \]
\[ V''_{D_2}(R = 0) = \frac{1}{\xi} - h > 0, \]
\[ V''_{D_2}(R = R_*) = -h(1 - h^2 \xi^2) < 0. \]

We solve eq.(74) for $\xi h < 1$. Since we have tunneling in this case, it is more convenient to go to Euclidean time by introducing $\tau = -it$. Then eq.(74) becomes

\[ -\frac{d}{d\tau} \left( \dot{R} \sqrt{\frac{R^2 + \xi^2}{1 + R^2}} + R \sqrt{\frac{1 + \dot{R}^2}{R^2 + \xi^2}} - hR = 0 \right) \quad (80) \]

where a dot denotes differentiation with respect to $\tau$. In fact, eq.(80) can be derived by varying the Euclidean version of action (76)

\[ I_{D_2}^Euc = 2\pi L T_2 \int d\tau \left( \sqrt{(R^2 + \xi^2)(1 + \dot{R}^2)} - \frac{h}{2} R^2 \right). \quad (81) \]

One can show that eq.(80) can be converted into the following first order form

\[ \sqrt{\frac{R^2 + \xi^2}{1 + R^2}} - \frac{h}{2} R^2 = C, \quad (82) \]

where $C$ is an integration constant. After some manipulations one can reexpress eq.(82) in the following way:

\[ \dot{R} = \frac{h}{2C + hR^2} \sqrt{(R_+^2 - R^2)(R^2 - R_-^2)} \quad (83) \]

where

\[ R_+^2 + R_-^2 = \frac{4(1 - Ch)}{h^2}, \quad (84) \]
\[ R_+^2 R_-^2 = \frac{4(C^2 - \xi^2)}{h^2}. \]

Comparing eq.(84) with corresponding equations in refs.[2, 3], one can see that eq.(83) is exactly the same as that of the purely electric case there if the electric displacement $D$ there is identified with $\xi$. Thus the general periodic instanton solution of (84) and its classical Euclidean action can be read off directly from ref.[3].

Here we consider only the vacuum solution ($R_+ = 0$, $C = \xi$), which is determined by

\[ \sqrt{R_+^2 - R^2} + \frac{2\xi}{hR_+} \ln \frac{R_+ + \sqrt{R_+^2 - R^2}}{R} = -h(\tau_0 - \tau) \quad (85) \]
where \( R_+ = 2\sqrt{1 - \xi h/h} \) and the corresponding Euclidean action is

\[
I_d = N T_0 \int d\tau + L \left( \frac{4\pi}{3} R_+^3 \right) \frac{hT_2}{2}.
\]

(86)

Here the first term is the contribution of \( N \) D0-branes and the second term is the contribution of the D2-brane.

Finally we show that our formulation allows \( D \) to be time-dependent but \( N \) to be fixed. Using (71) and (82) \( D \) is expressed in Euclidean space as

\[
D = \sqrt{1 + \dot{R}^2 + \frac{\xi^2}{R^2 + \xi^2}} = \frac{\xi}{C + \frac{h}{2} R^2}.
\]

(87)

Since \( R \) is dependent on time, \( D \) should also depend on time. Using (71) and (73) \( N \) can be generally expressed in Euclidean space as

\[
N = \frac{D}{2\pi \lambda} \int d\theta dz R \sqrt{1 + R'^2 + \dot{R}^2} \sqrt{(1 + \dot{R}^2)(1 + \dot{R}^2 - D^2)}.
\]

(88)

Thus if we consider \( R = R(\tau) \), \( N \) reduces to

\[
N = \frac{L}{\lambda} \frac{D R}{\sqrt{1 + \dot{R}^2 - D^2}}
\]

(89)

Inserting (87) into (89) we obtain \( N = L \xi / \lambda \), which is our original definition of \( N \).

One can also calculate the quantum-classical transition in this case using the periodic instanton or sphaleron solutions. The criterion for a first-order transition can be read off directly from ref. [3] as

\[
\hbar \frac{N \lambda}{L} < \frac{1}{2}.
\]

(90)

Thus the number of D0-branes as well as the RR-potential are involved in the criterion.

7 Concluding remarks

In the above we have considered D2-branes in the presence of spacetime RR fields in the context of a model with world volume cylindrical symmetry, and we have found locally stable hemispherical deformations of the brane, the complementary hemispherical configurations being unstable. We have also demonstrated that these two configurations together comprise an enveloping sphere representing a brane-antibrane pair which in view of its associated Euclidean time bounce configuration is unstable. This configuration is analogous to the brane-antibrane configuration constructed in ref. [1] where the presence of the bounce was anticipated. The stability/instability of the associated field configurations was investigated in detail in ref. [13], where it was pointed out in particular that
the brane-antibrane configurations (constructed from the combination of a stable brane and an unstable antibrane) are again unstable. We have derived explicitly the operator of small fluctuations about such configurations from which the local stability or instability of such configurations in the sense of minimising the energy locally may be deduced. We then calculated the corresponding transition rate for the decay of such a locally stable brane configuration through the hump of the RR potential. It might be somewhat easier to repeat the same steps in the case of a pure electric field aligned with the RR field in view of the explicit symmetry of the action under the exchange of time and space coordinates. More interesting is the consideration of the polar component of the electric field together with the magnetic field. In this case the action has a nice symmetry as discussed in section 5. Also, having both electric and magnetic fields in this case, one might expect the existence of both strings as well as D0-branes. Another interesting direction of extension is to consider the same phenomena with D3-branes. The D3-brane is selfdual and physical quantities in different regions of the energy and/or coupling constant can have the same analytic expression [10]. But in all these cases the states are unstable because the strong RR field makes the potential unbounded from below. One might try to consider D0 and spherical D2-branes applied to a radially decreasing RR field at infinity to avoid this instability. The proper principle, however, is to somehow take into account gravity, which might be a good candidate to stabilize the system as has been discussed, for instance in ref. [8].

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Appendix A

Here we summarise a few points in relation to refs. [4, 10] in which a spherical D2-brane was considered instead of the cylindrical one considered here. In a simplified way ref. [10] has

\[ X^0 = t, \quad X^1 = r(t) \sin \theta \cos \phi, \quad X^2 = r(t) \sin \theta \sin \phi, \quad X^3 = r(t) \cos \theta, \quad \text{others} = \text{const.} \]  

(Dirichlet) \hspace{1cm} (A1)

so that with \( X^0 = t \)

\[ ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) + \sum_{i=4}^9 (dX_i)^2. \]

Now one takes as world volume coordinates \( \xi_\alpha \) of the D2-brane (denoted by indices \( \alpha, \beta, \cdots \)) the variables \( t, \theta, \phi \). Thus \( r \), or the function \( r(t) \), originally the third of the three polar coordinates, acts as a scalar excitation of the brane. Then \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \) and \( H = dA, H_{0123} = h, \) and as in [4] we take the background RR four-form field strength to be \( H_{0123} = h = \text{const} \equiv h \epsilon_{123} \). This is a field strength aligned with the 123 subspace of spacetime, i.e. orthogonal to the 4 \cdots 9 part. The target space metric is thus changed from Minkowsky to \( S_2(r) \times R^7 \). Then the Born-Infeld action integral becomes

\[ I = \int d^3 \xi L(r, A_\theta, A_\phi) \] \hspace{1cm} (A2)

where

\[ L = -T_2 \left\{ \left[ -(-1 + r^2)(r^4 \sin^2 \theta + 4\pi^2 F_{\theta\phi}^2) \right. \right. \]

\[ \begin{align*}
-4\pi^2 r^2 ((\partial_\theta A_\phi)^2 + \sin^2 \theta (\partial_\phi A_\theta)^2) \right]^{1/2} \\
-\frac{hr^3 \sin \theta}{3} \end{align*} \] \hspace{1cm} (A3)

The expression \( F_{\theta\phi} = \frac{N}{2} \sin \theta \) (apart from its normalisation) is not a choice; rather it is dictated by the Euler-Lagrange equations derived from \( L \) for \( F_{\theta\phi} \) in the present case. Consider for simplicity the static case and ignore the Wess-Zumino contribution. Set

\[ \frac{F_{\theta\phi}}{\sqrt{r^4 \sin^2 \theta + 4\pi^2 (\partial_\theta A_\phi - \partial_\phi A_\theta)^2}} \equiv D_s \]

Then we obtain the equations

\[ \frac{\partial D_s}{\partial \theta} = 0, \quad \frac{\partial D_s}{\partial \phi} = 0 \]

so that \( D_s \) is independent of \( \theta \) and \( \phi \), but, of course, depends on \( r \). Now we can solve the former equation for \( F_{\theta\phi} \) and obtain

\[ F_{\theta\phi} = \frac{D_s r^2 \sin \theta}{\sqrt{1 - D_s^2 r^4}} \] \hspace{1cm} (A4)
This relation corresponds exactly to the relation one obtains in the cylindrical case, there with \( \sin \theta \) replaced by \( R \sqrt{1 + R'^2} \). Since
\[
2\pi N = \int d\phi d\theta F_{\theta\phi}
\]
the flux, obtained by integrating over the closed surface of the 2-sphere is quantised. The number \( N \) is identified with the number of \( D0 \)-branes. From here on many of the considerations leading to the minimised energy parallel those in our considerations above and therefore will not be given here. We cite only the expression for the potential
\[
V_{D2} = 4\pi T_2 \left\{ \sqrt{r^4 + \pi^2 N^2} - \frac{hr^3}{3} \right\} = \sqrt{(ST_2)^2 + (NT_0)^2} - 4\pi T_2 \frac{hr^3}{3}. \tag{A5}
\]

**Appendix B**

We present here the main steps involved in the determination of the operator of small fluctuations since the nontrivial procedure can be useful in other analogous considerations. Our starting point is the Hamiltonian
\[
H = \frac{1}{2\pi g} \int dz \left( PR\sqrt{1 + R^2} - \frac{h}{2} R^2 \right) \tag{B1}
\]
where \( P = 1/\sqrt{1 - D^2} \), \( D = D(R) \), and for convenience we set \( p(D) \equiv PR\sqrt{1 + R^2} = \sqrt{R^2(1 + R^2) + B^2} \equiv p(B) \). Thus, since \( D = D(R) \), we first replace in the Hamiltonian \( p(D) \) by \( p(B) \) and then perform the variation. Then after each variation we can return to expressions in terms of \( D \). Proceeding in this way the first variation yields
\[
\frac{\delta H}{\delta R} = \frac{R(1 + R^2)}{2\pi gp(B)} - \frac{hR}{2\pi g}, \quad \frac{\delta H}{\delta R'} = \frac{R^2 R'}{2\pi gp(B)}. \tag{B2}
\]
From
\[
\frac{\delta H}{\delta R} - \frac{d}{dz} \frac{\delta H}{\delta R'} = 0 \tag{B3}
\]
we obtain the equation of motion which when integrated and with integration constant chosen equal to zero implies the relation
\[
\frac{R}{P\sqrt{1 + R^2}} - \frac{h}{2} R^2 = 0. \tag{B4}
\]
It is at the configuration given by this equation that the second variation of \( H \) is to be evaluated. First we obtain
\[
\frac{\delta^2 H}{\delta R^2} = \frac{1}{2\pi g} \left( \frac{1 + R'^2}{p(B)} - \frac{R^2(1 + R'^2)^2}{p(B)^3} - h \right)
\]
\[
= \frac{1}{2\pi g} \left[ \frac{D^2 \sqrt{1 + R^2}}{P} - h \right] \tag{B5}
\]
and similarly (omitting now the intermediate step)

\[
\frac{\delta^2 H}{\delta R'^2} = \frac{1}{2\pi g} \frac{R(1 + D^2 R'^2)}{P(1 + R'^2)^{3/2}}, \quad \frac{\delta^2 H}{\delta R \delta R'} = \frac{1}{2\pi g} \frac{R'(D^2 + 1)}{P \sqrt{1 + R'^2}} = \frac{\delta^2 H}{\delta R \delta R'}
\]

(B6)

These expressions are inserted into the second variation and give

\[
\delta^2 H = \frac{1}{2\pi g} \frac{1}{2} \int dz \left[ \frac{\delta^2 H}{\delta R'^2} (\delta R')^2 + 2 \frac{\delta^2 H}{\delta R \delta R'} \delta R \delta R' + \frac{\delta^2 H}{\delta R^2} (\delta R)^2 \right]
\]

\[
= \frac{1}{4\pi g P} \int dz [A] + \frac{D^2}{4\pi g P} \int dz [B]
\]

(B7)

where

\[
A = \frac{R}{(1 + R'^2)^{3/2}} \left( \frac{d \delta R}{dz} \right)^2 + 2 \frac{R'}{\sqrt{1 + R'^2}} \delta R \frac{d \delta R}{dz} - h P (\delta R)^2,
\]

\[
B = \frac{RR'^2}{(1 + R'^2)^{3/2}} \left( \frac{d \delta R}{dz} \right)^2 + 2 \frac{R'}{\sqrt{1 + R'^2}} \delta R \frac{d \delta R}{dz} + \frac{\sqrt{1 + R'^2}}{R} (\delta R)^2
\]

(B8)

By writing

\[
\frac{d}{dz} \delta R = R' \left( \frac{d}{dz} \frac{\delta R}{R'} - \delta R \frac{d}{dz} \frac{1}{R'} \right)
\]

(B9)

one can arrive after some manipulations at the expression

\[
\left( \frac{d \delta R}{dz} \right)^2 = R'^2 \left( \frac{d}{dz} \frac{\delta R}{R'} \right)^2 + R'' \frac{d}{dz} (\delta R)^2
\]

(B10)

We insert this expression into the expressions for \(A\) and \(B\). Considering first the quantity \(A\) and ignoring total derivatives, we can rewrite this as

\[
A = \frac{RR'^2}{(1 + R'^2)^{3/2}} \left( \frac{d}{dz} \frac{\delta R}{R'} \right)^2 + V (\delta R)^2,
\]

(B11)

where

\[
V = - \frac{1}{R'} \frac{d}{dz} \frac{RR''}{(1 + R'^2)^{3/2}} - \frac{d}{dz} \frac{R'}{\sqrt{1 + R'^2}} - h P
\]

(B12)

One can show that for the solutions of eq. (B4) \(V = 0\). Considering now the quantity \(B\) and ignoring total derivatives, we can rewrite this as

\[
B = \frac{RR'^4}{(1 + R'^2)^{3/2}} \left( \frac{d}{dz} \frac{\delta R}{R'} \right)^2 + U (\delta R)^2,
\]

(B13)

where

\[
U = - \frac{1}{R'} \frac{d}{dz} \frac{RR'^2 R''}{(1 + R'^2)^{3/2}} - \frac{d}{dz} \frac{R'}{\sqrt{1 + R'^2}} + \frac{\sqrt{1 + R'^2}}{R}.
\]

(B14)

One can show that for the solutions of eq. (B4) \(U = 0\). Thus finally we are left with (replacing \(dz\) by \(dR/R'\))

\[
\delta^2 H = \frac{1}{4\pi g P} \int \frac{dR}{R'} \frac{RR'^2(1 + D^2 R'^2)}{(1 + R'^2)^{3/2}} \left( \frac{d}{dz} \frac{\delta R}{R'} \right)^2.
\]

(B15)
Thus this remaining term is positive definite for positive derivative $R'$, and negative for negative $R'$. Thus for solutions $R$ with $R'$ positive the energy is minimised and for those with $R'$ negative it is maximised. We can now write the second variation
\[ \delta^2 H = \frac{1}{2} \int dz \delta R \hat{M} \delta R \]  
where the fluctuation operator $\hat{M}$ is
\[ \hat{M} = -\frac{1}{2\pi g} \frac{1}{P R' d \tau' dz} \frac{1}{\left(1 + R^2\right)^{3/2}} \frac{d}{dz} \left[ R R' \right] (B17) \]

### Appendix C

The action $I$ is given by eq. (7) with the Lagrangian (8) and in this the magnetic field $B$ is given by eq. (11). Then we make a Wick rotation by setting $t = i\tau$. The resulting Euclidean action is
\[ I_E = \int d\tau dz d\theta L_E \]  
where (dots now refer to Euclidean time)
\[ L_E = \frac{1}{4\pi^2 g} \left\{ -\sqrt{R^2 (1 + \dot{R}^2 + R'^2) + B^2 (1 + \dot{R}^2) + \frac{\hbar}{2} R^2} \right\}. \] (C2)

with magnetic field $B$ given by
\[ B^2 = \frac{D^2 R^2}{1 + R^2} \frac{1 + \dot{R}^2 + \ddot{R}}{1 + R^2 - D^2} \] (C3)

The induced metric is
\[ g_{\alpha\beta} = \begin{pmatrix} 1 + \ddot{R}^2 & \dot{R} R' & 0 \\ \dot{R} R' & 1 + R'^2 & 0 \\ 0 & 0 & R^2 \end{pmatrix} \] (C4)

and the field tensor (cf. eq. (3))
\[ 2\pi F_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & -B & 0 \end{pmatrix} \] (C5)

Now we make the right angle rotation to
\[ z \rightarrow \tilde{z} = \tau, \quad \tau \rightarrow \tilde{\tau} = -z \] (C6)

so that (omitting tildes on $R$ for simplicity)
\[ R \rightarrow R, \quad R' \rightarrow -\dot{R}, \quad \dot{R} \rightarrow R'. \] (C7)

The magnetic field $\tilde{B}$ becomes
\[ \tilde{B}^2 = \frac{D^2 R^2}{1 + R^2} \frac{1 + \dot{R}^2 + \ddot{R}}{1 + R^2 - D^2} \] (C8)
and the induced metric $\tilde{g}$

\[
\tilde{g}_{\alpha\beta} = \begin{pmatrix}
1 + \ddot{R}^2 & -\dot{R}R' & 0 \\
-\dot{R}R' & 1 + R'^2 & 0 \\
0 & 0 & R^2
\end{pmatrix}
\] (C9)

and the field tensor

\[
\tilde{F}_{\alpha\beta} = \begin{pmatrix}
0 & 0 & \tilde{B} \\
0 & 0 & 0 \\
-\tilde{B} & 0 & 0
\end{pmatrix}.
\] (C10)

These expressions together yield the Lagrangian

\[
\tilde{\mathcal{L}}_E = \frac{1}{4\pi^2 g} \left\{ -\sqrt{R^2(1 + \dot{R}^2 + h^2/2R^2)} \right\}.
\] (C11)

and the action

\[
\tilde{I}_E = \int d\tilde{\tau} d\tilde{z} d\theta \tilde{\mathcal{L}}_E
\] (C12)

Since the Lagrangian is a Lorentz scalar the result could also have been written down directly from (C2). For the instanton solution $R' = 0$, and the Lagrangian assumes the following form which demonstrates its equivalence with that of the static solution:

\[
\tilde{\mathcal{L}}_E = \frac{1}{4\pi^2 g} \left\{ -PR\sqrt{1 + \dot{R}^2 + h^2/2R^2} \right\}.
\] (C13)

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Fig. 1 The two circular shells and the enveloping sphere
