MULTIPLE SOLUTIONS OF AN ELLIPTIC HARDY-SOBOLEV EQUATION WITH CRITICAL EXPONENTS ON COMPACT RIEMANNIAN MANIFOLDS.

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ABSTRACT. On a compact Riemannian manifold, we prove the existence of multiple solutions for an elliptic equation with critical Sobolev growth and critical Hardy potential.

1. Introduction

Let \((M, g)\) be a Riemannian manifold of dimension \(n \geq 3\). For a fixed point \(p\) in \(M\), we define the function \(\rho_p\) on \(M\) as follows

\[ \rho_p(x) = \begin{cases} \text{dist}_g(p, x), & x \in B(p, \delta_g), \\ \delta_g, & x \in M \setminus B(p, \delta_g) \end{cases} \]

where, \(\delta_g\) denotes the injectivity radius of \(M\). Let \(h\) be a continuous functions on \(M\). Consider on \(M \setminus \{p\}\) the following Hardy-Sobolev equation:

\[ (E_h) \quad \Delta_g u - \frac{h}{\rho_p^2(x)} u = |u|^{2^*-2} u, \]

where \(2^* = \frac{2n}{n-2}\) is the Sobolev critical exponent.

In this paper, we are interested in the study of existence of multiple solutions of equation \((E_h)\). When dropping the singular term \(\frac{1}{\rho_p^2(x)}\) from equation \((E_h)\) we fall in the so called Yamabe equation which is very known in the literature and whose origin comes from the study of conformal deformation of the metric to constant scalar curvature. A positive solution \(u\) of the Yamabe equation provides a conformal metric \(g' = u^{\frac{4}{n-2}}g\) with scalar curvature a constant function. Of course, the presence of the critical Sobolev exponent made the resolution of such equation difficult and appealed to a more sophisticated analysis. We can refer the reader to the book [5] for a compendium on this topic. Equation \((E_h)\) can be then seen as a Yamabe type equation of a singular type.
When the function $\rho_p$ is of power $0 < \gamma < 2$, the study of the associated equations is related to the study of conformal deformation to constant scalar curvature of metrics which are smooth only in some geodesic ball $B(p, \delta)$ (see [6]). As the inclusion $H^2_1(M) \subset L(M, \rho_p^{-\gamma})$ (where $H^2_1(M)$ and $L(M, \rho_p^{-\gamma})$ are defined in section 2) is compact for $0 < \gamma < 2$, the study of existence of solutions, in this case, goes as in the case of 'regular' Yamabe equation (see [6]).

However, when $\gamma = 2$, regarding the non compactness of the inclusion $H^2_1(M) \subset L(M, \rho_p^{-2})$, equation $(E_h)$ is also critical in terms of the power $\gamma = 2$ of the function $\rho_p$.

In studying equations $(E_h)$, besides the critical Sobolev exponent $2^*$, the singular term plays a prominent role. As it has been shown in [7], it interferes in the decomposition of the Palais-Smale sequence of the functional energy and then collaborates principally in determining the safe energy level for the compactness of the Palais-Smale sequences.

The singular term interferes also in the regularity of solutions in that only weak solutions can be obtained as contrasted to the case of the 'regular' Yamabe equation where strong solutions can be obtained (see [6]). The author in [6] studied equation $(E_h)$ and proved the existence of at least one solution of minimal energy. In this work, we are interested in the existence of multiple solutions of high energy. The main tool that we employ to achieve our interest is the classical Lusternik-Schnirelmann theory (see for example [1]). We note that multiple solutions of minimal energy can also be obtained.

In [2], the authors proved a multiplicity result for a subcritical regular equation on compact Riemannian manifold. They, used Lusternik-Schnirelmann theory together with some astute constructions. We will follow the authors in [2] and [3] in their global framework. As aforesaid, equation $(E_h)$ is double critical, which leads to further technical difficulties to arise and then a deeper analysis needs to be done.

The paper is organized as follows: in section 2 we introduce some notations and useful results that will be of great use and state the main result. In section 3, a noncompact analysis is done. In section 4, we give an overview of the proof of the main result and then collect ingredients for the proof of the main results. The fourth section is devoted to the proof of the main result.

2. Notations, useful results and statement of the main result.

In this section, we introduce some notations and cite results that are useful in our study.
Throughout the paper, we will denote by $B(a, r)$ a ball of center $a$ and radius $r > 0$, the point $a$ will be specified either in $M$ or in $\mathbb{R}^n$, and $B(r)$ is a ball in $\mathbb{R}^n$ of center $0$ and radius $r > 0$.

Let $q \in M$. Denote by $\exp_q$ the exponential map which defines, for $r > 0$ small, a diffeomorphism from $B(r)$ to $B(q, r)$.

Let $H^1_2(M)$ be the Sobolev space consisting of the completion of $C^\infty(M)$ with respect to the norm

$$||u||_{H^1_2(M)} = \int_M (|\nabla u|^2 + u^2) dv_g.$$ 

$M$ being compact, $H^1_2(M)$ is then embedded in $L_q(M)$ compactly for $q < 2^* = \frac{2n}{n-2}$ and continuously for $q = 2^*$.

Let $K(n, 2)$ denote the best constant in Sobolev inequality that asserts that there exists a constant $B > 0$ such that for any $u \in H^1_2(M)$,

$$||u||^2_{L_{2^*}(M)} \leq K^2(n, 2)||\nabla u||^2_{L_2(M)} + B||u||^2_{L_2(M)}.$$ 

The constant $K(n, 2)$ is defined to be

$$K(n, 2) = \inf_{u \in H^1(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{(\int_{\mathbb{R}^n} |u|^{2^*})^{\frac{n}{2^*}}}.$$ 

It is well known that the extremal functions for the above infimum are the family of functions

$$w_\mu(x) = (n(n-2))^{\frac{n-2}{4}} \left( \frac{\mu}{\mu^2 + |x|^2} \right)^{\frac{n-2}{2}}, \mu > 0.$$ 

These family of functions classifies all positive solutions of the Euclidean equation

$$\Delta u = u^{2^*-1}.$$ 

Denote by $L_2(M, \rho_p^{-2})$ the space of functions $u$ such that $\frac{u^2}{\rho_p}$ is integrable. This space is endowed with norm $||u||^2_{L_2(M, \rho_p^{-2})} = \int_M \frac{|u|^2}{\rho_p^2} dv_g$.

In [6], the author proved the following Hardy inequality: let $(M, g)$ be any compact manifold $M$, for every $\varepsilon > 0$ there exists a positive constant $A(\varepsilon)$ such that for any $u \in H^1_2(M)$,

$$\int_M \frac{u^2}{\rho_p^2} dv_g \leq (K^2(n, -2) + \varepsilon) \int_M |\nabla u|^2 dv_g + A(\varepsilon) \int_M u^2 dv_g,$$

with $K(n, -2)$ being the best constant in the Euclidean Hardy inequality

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq K(n, -2)^2 \int_{\mathbb{R}^n} |\nabla u|^2 dx, u \in C^\infty(\mathbb{R}^n).$$
The constant \( K(n, -2) \) is equal to \( \frac{2}{n-2} \) and is not attained.

If \( u \) is supported in some ball \( B(p, \delta), 0 < \delta < \delta_g \), then there exists positive constant \( K_\delta(n, -2) \)

\[
\int_{B(p, \delta)} \frac{u^2}{\rho_p^2} dv_g \leq K_\delta(n, -2) \int_{B(p, \delta)} |\nabla u|^2 dv_g,
\]

with \( K_\delta(n, -2) \) goes to \( K(n, -2) \) as \( \delta \) goes to 0.

On the Euclidean space \( \mathbb{R}^n \), the author in [9] studied the equation

\[
(Eu_\lambda) \quad \Delta u - \frac{\lambda}{|x|^2} = |u|^{\frac{4}{n-2}} u,
\]

where \( \lambda > 0 \) is a positive constant. She proved in particular that for \( \lambda \geq \frac{(n-2)^2}{4} \) there is no positive solution and for \( 0 < \lambda < K^2(n, -2) = \frac{(n-2)^2}{4} \), all positive solutions are the class of functions

\[
w_{\lambda, \xi}(x) = (n(n-2)a^2_\lambda)^{\frac{4-n}{4}} \left( \frac{\xi^{a_\lambda}|x|^{a_\lambda-1}}{\xi^{2a_\lambda} + |x|^{2a_\lambda}} \right)^{\frac{4-n}{4}}, \xi > 0,
\]

where \( a_\lambda = \sqrt{1 - \lambda K^2(n, -2)} \). Note that for \( \lambda = 0 \), we meet the functions (2.5). Furthermore, if we denote by \( S \) the infimum

\[
S = \inf_{u \in D^{1, 2}, u \neq 0} \frac{\left( \int_{\mathbb{R}^n} |\nabla u|^2 - \lambda |u|^2 \right) dx}{\left( \int_{\mathbb{R}^n} |u|^2 dx \right)^{\frac{n}{2}}},
\]

then, the functions defined by (2.5) are extremal for this infimum, that is

\[
S_\lambda = \frac{\left( \int_{\mathbb{R}^n} |\nabla w_{\lambda, \xi}|^2 - \lambda |w_{\lambda, \xi}|^2 \right) dx}{\left( \int_{\mathbb{R}^n} |w_{\lambda, \xi}|^{2^*} dx \right)^{\frac{n}{2}}}.\]

Moreover,

\[
S_\lambda = \frac{(1 - \lambda K^2(n, -2))^{\frac{n-1}{n}}}{K^2(n, 2)}.
\]

Let \( h \) be a continuous function on \( M \) and \( p \in M \) a fixed point. Let us take \( \lambda = h(p) > 0 \) with \( 1 - h(p)K(n, -2)^2 > 0 \). We denote by \( D^* \) the constant

\[
D^* = \frac{\left( S_{h(p)} \right)^{\frac{n}{2}}}{n} = \frac{(1 - h(p)K^2(n, -2))^{\frac{n-1}{n}}}{nK^2(n, 2)}.
\]

Let

\[
\mu = \inf_{u \in H^1_0(M), u \neq 0} \frac{\int_{M} (|\nabla u|^2 - \frac{h}{\rho_p^2} u^2) dv_g}{\left( \int_{M} |u|^2 dv_g \right)^{\frac{n}{2}}},
\]
In [6], the author proved an existence result for equation \((E_h)\) on compact manifold under the condition
\[
\mu < \frac{1 - h(p)K^2(n, -2)}{K^2(n, 2)} = (nD^*)^{\frac{2}{n}}(1 - h(p)K^2(n, -2))^\frac{1}{n},
\]
provided of course that \(1 - h(p)K^2(n, -2) > 0\). Further in this paper, we will see that this existence result is a simple consequence of a Palais-Smale decomposition result that we established in [7]. Furthermore, we consider the reverse inequality \(\mu > (nD^*)^{\frac{2}{n}}\) and show that effectively multiple solutions exist in this case. In a very precise way, we establish the following result

**Theorem 2.1.** Let \((M, g)\) be a compact Riemannian manifold of dimension \(n\). Suppose that the function \(h\) is smooth, changes sign once and satisfies the following conditions

1. \(h(p) > 0, 0 < 1 - h(p)K(n, -2)^2 < \frac{1}{2}\),
2. \(n = \text{Dim}(M) > \frac{2}{a} + 2, a = \sqrt{1 - h(p)K^2(n, -2)}\),
3. \((A(n, a) + h(p))\text{Scal}_g(p) - 6\frac{\Delta h(p)}{n} > 0\), where \(A(n, a)\) is defined by (4.17).

If \(\mu > (nD^*)^{\frac{2}{n}}\), equation \((E_h)\) admits at least \(\text{Cat}(M)\) \((\text{Cat}(M)\) is the Lusternik-Schnirelmann category of \(M\) defined in section 3\) weak solutions such that each weak solution \(u\) satisfies \(D^* < J_h(u) < D^* + g(\varepsilon)\) and at least one weak solution \(u\) such that \(D^* + g(\varepsilon) < J_h(u)\).

3. compactness of Palais-Smale sequences

Let \(J_h\) be the functional defined on \(H^1_0(M)\) by
\[
J_h(u) = \frac{1}{2} \int_M (|\nabla u|^2 - \frac{h}{\rho_0^2}u^2)dv_g - \frac{1}{2^*} \int_M |u|^{2^*}dv_g.
\]
It is a \(C^2\) functional whose critical points are weak solutions of equation \((E_h)\).

A Palais-Smale sequence \(u_m\) \((\text{P-S in short})\) of \(J_h\) at a level \(d\) is defined to be the sequence that satisfies \(J_h(u_m) \to d\) and \(DJ_h(u_m) \varphi \to 0, \forall \varphi \in H^1_0(M)\).

The functional \(J_h\) is said to satisfy P-S condition et level \(d\) if each P-S sequence at level \(d\) is relatively compact.

In this section, we determine a region of levels where P-S condition is satisfied and then critical points of the function \(J_h\) can be obtained. This can be done by analyzing asymptotically the behavior of the P-S sequences. In a previous paper [7], based on blow-up theory in [4] and
on a result in [8], we asymptotically studied P-S sequences of the functional $J_h$ and established a Struwe-type decomposition formula for P-S sequences of the functional $J_h$. For the seek of clearness we cite this theorem and we refer to [7] for a detailed proof. Let us first introduce

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^n} |u|^{2^*} dx, \text{ and}$$

$$J_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{h(p)}{2} \int_{\mathbb{R}^n} u^2 dx - \frac{1}{2} \int_{\mathbb{R}^n} |u|^{2^*} dx.$$

In [7], we established the following decomposition theorem:

**Theorem 3.1.** Let $(M, g)$ be a compact Riemannian manifold with $\dim(M) = n \geq 3$ and let $h$ be a continuous function on $M$ that on the point $p \in M$, it satisfies $0 < h(p) < \frac{1}{K(n, 2, -2)}$.

Let $u_m$ be a P-S sequence of the functional $J_h$ at level $d$. Then, there exist $k \in \mathbb{N}$, sequences $R_i^m > 0, R_i^m \to 0, \ell \in \mathbb{N}$ sequences $\tau_j^m > 0, \tau_j^m \to 0$, converging sequences $x_m \to x_p \neq p$ in $M$, a solution $u \in H^1(M)$ of $(E_h)$, solutions $v_i \in D^{1,2}(\mathbb{R}^n)$ of $E_{u(p)}$ and nontrivial solutions $\nu_j \in D^{1,2}(\mathbb{R}^n)$ of (2.2) such that up to a subsequence

$$m = u + \sum_{i=1}^{k} (R_i^m)^{\frac{2-n}{2}} \eta_r(\exp_p^{-1}(x))v_i((R_i^m)^{-1} \exp_p^{-1}(x))$$

$$+ \sum_{j=1}^{\ell} (\tau_j^m)^{\frac{2-n}{2}} \eta_r(\exp_{x_m}^{-1}(x))\nu_j((\tau_j^m)^{-1} \exp_{x_m}^{-1}(x)) + \mathcal{W}_m,$$

with $\mathcal{W}_m \to 0$ in $H^1(M)$,

and

$$J_h(u_m) = J_h(u) + \sum_{i=1}^{k} J_\infty(v_i) + \sum_{j=1}^{\ell} J(\nu_j) + o(1).$$

Before we derive some consequences of the above theorem we draw attention to the following important remark

**Remark 3.2.** If $u$ is a changing sign of equation $(E_{u \lambda})$ with $\lambda = h(p)$, then $J(u) > 2D^*$. 

In fact write $u = u^+ + u^-$, where $u^+ = \max(u, 0)$ and $v^- = \min(u, 0)$. We then get

$$
\int_{\mathbb{R}^n} \left( |\nabla u^+|^2 - \frac{h(p)}{|x|^2} u^+ u \right) dx = \int_{\mathbb{R}^n} (\nabla u^+ \cdot \nabla u^+ - \frac{h(p)}{|x|^2} uu^+) dx
$$

(3.11)

$$
= \int_{\mathbb{R}^n} |u^+|^{2^*} - 1 uu^+ dx = \int_{\mathbb{R}^n} |u^+|^{2^*} dx
$$

Since $u^+$ cannot be a 'member' of the family of functions defined by (2.5), then by (3.11) we get

$$
J_\infty(u^+) = \frac{1}{n} \int_{\mathbb{R}^n} \left( |\nabla u^+|^2 - \frac{h(p)}{|x|^2} u^+ u \right) dx
$$

$$
> \frac{1}{n} (S_{h(p)})^{\frac{2}{p}} = D^*,
$$

where $S_{h(p)}$ is defined by (2.7).

By the same way, we get

$$
J_\infty(u^-) = \frac{1}{n} \int_{\mathbb{R}^n} \left( |\nabla u^-|^2 - \frac{h(p)}{|x|^2} u^- u \right) dx > D^*
$$

Thus, we obtain

$$
J_\infty(u) = J_\infty(u^+) + J_\infty(u^-) > 2D^*.
$$

Now, we derive from the above theorem the following corollaries

**Corollary 3.3.** Under conditions

1. $\mu \geq (nD^*)^{\frac{2}{p}}$,
2. $0 < 1 - h(p)K^2(n, -2) \leq \frac{1}{2}$,

every P-S sequence of the functional $J_h$ at level $d$ with $D^* < d < 2D^*$ is relatively compact.

**Proof.** By the above theorem, there exist a critical point $u_0$ of $J_h$, a sequence of solutions $v_i$ of $(Eu_0)$ and sequence of non trivial solutions $\nu_j$ such that up to a subsequence (3.9) and (3.10) hold. Suppose that $v_i \neq 0$ for some $i$, either $v_i$ changes sign or not, it must hold $d > 2D^*$. Thus $v_i = 0 \forall i$. Similarly, if there exists $\nu_j \neq 0$, by condition (1) of the corollary will have also $d > 2D^*$. Therefore, all $\nu_j$ are null and thus $u_m$ converges strongly up to a subsequence in $H^1_1(M)$. \hfill \Box

**Corollary 3.4.** Suppose that $\mu > (nD^*)^{\frac{2}{p}}$. Then, for every P-S sequence $u_m$ of $J_h$ at level $D^*$, there exists a sequence of functions $w_m \in H^1_1(M)$ such that $w_n \to 0$ strongly in $H^1_1(M)$ and

$$
u_m = w_m + \phi_{p,R,m,q}, q > 0,
$$

where $\phi_{p,R,m,q}$ is the function defined by (4.25) with $q = p$ and $\xi = R_m q$. 

Proof. First, the condition \( \mu > \left( \frac{nD^*}{n} \right)^2 \) prevents the existence of any critical point \( u_0 \) of \( J_h \) with \( J_h(u_0) = D^* \). Thus, only one function \( v_i \) can be included in the decomposition expression of the above theorem and since this function cannot change sign, then it takes the form of (4.25) with \( q = p \) and \( \xi = R_m \varepsilon \).

\[ \Box \]

Corollary 3.5. under the following conditions

(1) \( 1 - h(p)K^2(n, -2) > 0 \),
(2) \( \mu < \left( \frac{nD^*}{n} \right)^2 \),

there exists a non trivial critical point of \( J_h \).

Proof. Like in corollary 3.3, by applying theorem 3.1, it is not difficult to see that the P-S condition for the functional \( J_h \) is satisfied for any level \( d \) such that \( 0 < d < D^* \).

Consider \( d = \inf_{\mathcal{N}_h} J_h \), where \( \mathcal{N}_h \) is the Nehari manifold defined by (4.12). By applying the Ekeland variational principle, we can obtain a P-S sequence on \( \mathcal{N}_h \) at level \( d \) which is also a P-S sequence \( u_m \) on \( H^1_0(M) \).

It is clear that \( d \geq \frac{1}{n} \mu \frac{\pi}{n} \). Let \( u \in H^1_0(M) \setminus \{0\} \), then \( \Phi(u)u \in \mathcal{N}_h \), where \( \Phi \) is defined by (4.18), and by homogeneity of

\[ I_h(u) = \frac{\int_M (|\nabla u|^2 - \frac{h}{\rho^*} u^2) \, dv_g}{(\int_M |u|^2 \, dv_g)^{\frac{n}{2}}} \],

since \( \Phi(u)u \in \mathcal{N}_h \), we get that

\[ I_h(u) = I_h(\Phi(u)u) = (nJ_h(\Phi(u)u))^2 \geq (nd)^2 \].

Thus we get \( \mu \geq (nd)^2 \) and hence \( d = \frac{1}{n} \mu \frac{\pi}{n} \). Therefore, condition (2) of the corollary implies that \( d < D^* \) and hence the P-S sequence \( u_m \) converges up to a subsequence strongly in \( H^1_0(M) \) to a critical point of \( J_h \).

\[ \Box \]

4. Construction of solutions

In this section, we construct solutions of \( (E_h) \) as critical points of the functional \( J_h \). In searching critical points of the functional \( J_h \), we just apply the following classical theorem.

Theorem 4.1. Let \( J \) be \( C^1 \) real functional defined on a \( C^{1,1} \) Banach manifold \( N \). If \( J \) is bounded from below on \( N \) and satisfies the P-S condition then it has at least \( \text{Cat}(J^c) \) critical points in \( J^c \).

Moreover, if \( N \) is contractible and \( \text{Cat}(J^c) > 1 \) then there exists at least one critical point \( u \notin J^c \).
$J^c$ in the theorem denotes the sub-level set of the functional $J$

$$J^c = \{ u \in N : J(u) < c \}.$$  
and $\text{Cat}(J^c)$ denotes the Lusternik-Schnirelmann category of the set $J^c$.

We recall that the Lusternik-Schnirelmann category $\text{Cat}_Y(X)$ of a topological space $X$ with respect to a topological space $Y$ with $X \subset Y$ is the least integer $k \leq \infty$ such that there exists an open covering of $U_i$ of $X$ with each $U_i$ contractible in $Y$. If $X = Y$, we put $\text{Cat}_X(X) = \text{Cat}(X)$.

Consider the Nehari manifold $\mathcal{N}_h$ which associated to the functional $J_h$

$$\mathcal{N}_h = \{ u \in H^2_1(M) \setminus \{0\}, D J_h(u). u = 0 \}.$$  
It is well known that this manifold defines a natural constraint set for the functional $J_h$ in the sense that a P-S sequence in $\mathcal{N}_h$ is also a P-S of $J_h$ on $H^2_1(M)$. Moreover, for $u \in H^2_1(M) \setminus \{0\}$, we have $\sup_{t>0}(tu) = t_0 u$ with $t_0 = \left( \frac{\int_M (|\nabla u|^2 - \frac{h}{\rho_p} u^2) dv_g}{\int_M |u|^2 dv_g} \right)^{\frac{n}{n-2}}$ and $t_0 u \in \mathcal{N}_h$.

Note that if the function $h$ changes sign only once and $1 - h(p) K(n, -2)^2 > 0$, then $\int_M (|\nabla u|^2 - \frac{h}{\rho_p} u^2) dv_g > 0$. In fact, let $\delta_0 = \max \delta$ such that $h(x) \geq 0$ on $B(p, \delta)$. Without loss of generality we assume that $h(p) = \max_M h$, by inequality (2.4) we have

$$\int_M (|\nabla u|^2 - \frac{h}{\rho_p} u^2) dv_g \geq \int_M |\nabla u|^2 dv_g - \max_M h \int_{B(p, \delta)} u^2 \rho_p dv_g \geq (1 - h(p) K_\delta(n, -2)^2) \int_M |\nabla u|^2 dv_g.$$  

Since $K_\delta(n, -2) \to K(n, -2)$ as $\delta \to 0$ and $1 - h(p) K(n, -2)^2 > 0$, we get the claim true.

The main difficulty in applying theorem 4.1 above is that the P-S condition for the functional $J_h$ is not satisfied for any level because of the presence of the critical exponent $2^*$ and the critical singular term. However, corollary 3.3 gives level rank for which P-S condition is satisfied and consequently the P-S sequence levels would be restricted to this level rank. We therefore construct a subset of the manifold $\mathcal{N}_h$ on which the P-S condition is satisfied. It seems that the 'test' functions defined by (4.19) play an important role in the construction of such subset.

We can assume by the Nash embedding theorem, without loss of generality, that the Riemannian manifold $M$ is embedded in some Euclidean...
Let \( M_r \) be the set
\[
M_r = \{ x \in \mathbb{R}^N : d(x, M) < r \},
\]
and define the radius of the topological invariance \( r_M \) of \( M \) by
\[
r_M = \sup \{ r > 0 : \text{Cat}(M_r) = \text{Cat}(M) \}.
\]
For \( \varepsilon > 0 \), let \( g(\varepsilon) \) be a positive function such that \( g(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).
Let \( \Sigma_\varepsilon \) be the subset of \( \mathcal{N}_h \) defined by
\[
\Sigma_\varepsilon = \{ u \in \mathcal{N}_h \text{ s.t } D^* < J_h(u) < D^* + g(\varepsilon) \text{ for some } g(\varepsilon) \}.
\]
To prove the main theorem, we construct two continuous maps \( I_\varepsilon : M \to \Sigma_\varepsilon \) and \( \beta : \Sigma_\varepsilon \to M_r \) such that the composition \( \beta \circ I_\varepsilon \) is homotopic to the identity. This leads, by the Lusternik-Schnirelmann properties (see [1] for example) that \( \text{Cat}(\Sigma_\varepsilon) > \text{Cat}(M) \). Thus by applying theorem 4.1 on the set \( \Sigma_\varepsilon \), we obtain at least \( \text{Cat}(M) \) critical points of the functional \( J_h \) in \( \Sigma_\varepsilon \). Finally, we end the proof of the main theorem by proving the existence of another critical point \( u \notin \Sigma_\varepsilon \). This can be done by constructing a contractible set \( P_\varepsilon \) that contains \( I_\varepsilon(M) \) and is contractible in \( \mathcal{N}_h \cap J \mathcal{C}_\varepsilon \) for bounded \( C_\varepsilon \).

First, we have to prove that the set \( \Sigma_\varepsilon \) is not empty. This is achieved in lemma (4.2) below.

Let \( q \in M \) be any point of \( M \) and \( 0 < \delta < \frac{\delta_0}{2} \). Define a cut-off function on \( M \), \( \eta_{q,\delta} \), such that \( 0 \leq \eta_{q,\delta} \leq 1 \), \( \eta_{q,\delta}(x) = 1 \), \( x \in B(q, \delta) \), \( \eta_{q,\delta}(x) = 0 \), \( x \in M \setminus B(q, 2\delta) \) and \( |\nabla \eta_{q,\delta}| \leq C \), for some constant \( C > 0 \).

Put \( \rho_p(x) = r \) and consider on \( M \) the function
\[
\phi_\varepsilon(x) = C(n, a) \eta_{p,\delta} \left( \frac{\varepsilon a^2 a^{-1}}{\varepsilon 2a + y^2a} \right)^{\frac{1}{n-1}},
\]
where
\[
(4.13) \quad C(n, a) = (a^2 n(n - 2))^{\frac{n-2}{4}} \quad \text{and} \quad a = \sqrt{1 - h(p) K(n, 2, -2)^2}
\]
Define the constants
\[
C_1(n, a) = \frac{1}{6} C(n, a)^2 \left( \frac{n-2}{2} \right)^2 w_{n-1} \left[ (a - 1)^2 + 2(1 - a) \frac{a(n - 2) + 2}{an - 2} \right]
\]
\[
(4.14) \quad + \quad (1 + a)^2 \left( \frac{an + 2}{an - 2} \right) \left( a(n - 2) + 2 \right) \left( a(n - 2) - 2 \right)
\]
\[
(4.15) \quad C_2(n, a) = C(n, a)^2 \frac{4a^2 w_{n-1} (n-2)(n-1)}{(a(n-2) - 2)(an - 2)}
\]
(4.16) \[ C_3(n,a) = C(n,a)^2 \frac{w_{n-1}(a(n-2) + 2)n}{6(an - 2)}. \]

Let the constant

\[ A(n,a) = \frac{6(\frac{n-2}{n}C_3(n,a) - C_1(n,a))}{C_2(n,a)} \] (4.17)

Consider the projection \( \Phi : H^2_1(M) \setminus \{0\} \rightarrow \mathcal{N}_h \) defined by

\[ \Phi(u) = \left( \frac{\int_M (|\nabla u|^2 - \frac{\rho}{\rho^2} u^2)dv_g}{(\int_M |u|^{2\gamma}dv_g)^{\frac{\gamma-2}{2}}} \right)^{\frac{\alpha-2}{\gamma}} u \] (4.18)

In the remaining of the paper, \( \alpha(\varepsilon) \) is a function such that \( \alpha(\varepsilon) > 0 \) and \( \alpha(\varepsilon) \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \).

In [7], we have proved the following lemma. For completeness, we review briefly the proof

**Lemma 4.2.** Suppose that

\[ n = \text{dim}(M) > 2 + \frac{2}{a}, \quad \text{and} \]

\[ (A(n,a) + h(p))\text{Scal}_g(p) - 6\frac{\Delta h(p)}{n} > 0. \]

Then, there exists \( g(\varepsilon) > 0 \) with \( g(\varepsilon) \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \) such that

\[ D^* < J_h(\Phi(\phi_\varepsilon)) < D^* + g(\varepsilon), g(\varepsilon) > 0, \forall \varepsilon. \] (4.19)

**Proof.** Define for \( 2a\beta - 1 > \alpha > 0 \)

\[ I_\beta^\alpha = \int_0^{\infty} \frac{r^\alpha}{(1 + r^{2a})^\beta} dr. \]

Then, by direct computations (see [7]) one can get

\[ \int_M |\nabla \phi_\varepsilon|^2dv_g \]

\[ = \int_{\mathbb{R}^n} |\nabla U|^2dx - \text{Scal}_g(p)C_1(n,a)I_n^{(a-n-2)\varepsilon^2} + o(\varepsilon^2) + \alpha(\varepsilon). \] (4.20)

Similarly, by writing

\[ h(x) = h(p) + (\nabla_i h)(p)x_i + (\nabla_{i,j} h)(p)x_ix_j + o(r^2) \]
we obtain

\[ (4.21) \quad \int_M \frac{h(x)}{r^2} \varphi^2_g dv_g \]

\[ = h(p) \int_{\mathbb{R}^N} \frac{U^2}{|x|^2} dx - \left[ Scal_g(p)h(p) - \frac{\Delta h(p)}{n} \right] C_2(n, a) I_n^{(n-2)+1} \varepsilon^2 \]

\[ + o(\varepsilon^2) + o(\varepsilon), \]

For the term term \( \int_M |\varphi\varepsilon|^2 dv_g \), one can obtain

\[ (4.22) \quad \int_M |\varphi\varepsilon|^2 dv_g \]

\[ = \int_{\mathbb{R}^n} |U|^2 dx - Scal_g(p) C_3(n, a) I_n^{(n-2)+1} \varepsilon^2 + o(\varepsilon^2) \]

\[ + \alpha_3(\varepsilon), \]

with \( \lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0. \)

Using the fact that

\[ \int_{\mathbb{R}^n} (|\nabla U|^2 - h(p)\frac{U^2}{|x|^2}) dx \]

\[ = (1 - h(p)K(n, 2, -2)^2) \frac{n-1}{K(n, 2)^2} = (nD^*)^\frac{n}{2}, \]

the expansions (4.20), (4.21) and (4.22) yield

\[ \frac{\int_M |\nabla \varphi\varepsilon|^2 - \frac{h}{r^2} \varphi^2_g dv_g}{(\int_M |\varphi\varepsilon|^2 dv_g)^{\frac{n}{2}}} \]

\[ = (nD^*)^\frac{n}{2} \left( 1 + \frac{1}{\int_{\mathbb{R}^n} |U|^2 dx} \left[ Scal_g(p)(\frac{n-2}{n} C_3(n, a) - C_1(n, a)) \right] \right) \]

\[ + \left( \frac{Scal_g(p)h(p)}{6} - \frac{\Delta h(p)}{n} \right) C_2(n, a)) I_n^{(n-2)+1} \varepsilon^2 + o(\varepsilon^2) + o(\varepsilon) \]

with \( \lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0. \)

Now, writing

\[ nJ_n(\Phi(\varphi\varepsilon)) = \left( \frac{\int_M |\nabla \varphi\varepsilon|^2 - \frac{h}{r^2} \varphi^2_g dv_g}{(\int_M |\varphi\varepsilon|^2 dv_g)^{\frac{n}{2}}} \right)^{\frac{n}{2}} \]

we obtain

\[ J_n(\Phi(\varphi\varepsilon)) \]

\[ = D^*(1 + \frac{n}{2(\int_{\mathbb{R}^n} |U|^2 dx)^{\frac{n}{2}}} \left[ Scal_g(p)(\frac{n-2}{n} C_3(n, a) - C_1(n, a)) \right] \]

\[ + \left( \frac{Scal_g(p)h(p)}{6} - \frac{\Delta h(p)}{n} \right) C_2(n, a)) I_n^{(n-2)+1} \varepsilon^2 + o(\varepsilon^2) + o(\varepsilon) \].
That is
\[ J_h(\Phi(\phi_\varepsilon)) = D^* \left[ 1 + B(n, a) \left( (A(n, a) + h(p)) \text{Scal}_g(p) - 6 \frac{\Delta h(p)}{n} \right) \varepsilon^2 \right] + o(\varepsilon^2) + o(\varepsilon). \]

with
\[ B(n, a) = \frac{n}{12C_2(n, a)(\int_{\mathbb{R}^n} |U|^2 dx)^{\frac{n}{2}}}. \]

Therefore, if
\[ (A(n, a) + h(p)) \text{Scal}_g(p) - 6 \frac{\Delta h(p)}{n} > 0, \]

for \( \varepsilon \) small enough, we get (4.19). \( \square \)

4.1. The map \( I_\varepsilon \). In this subsection, we construct a continuous map \( I_\varepsilon : M \to \Sigma_\varepsilon \). For a fixed point \( q \in M \) put \( r_q(x) = \text{dist}_g(q, x), x \in M \) and let \( \phi_{q,\xi} \) be the function
\[ \phi_{q,\xi}(x) = C(n, a) \eta_{q,\delta} \left( \frac{\xi^a r_q(x)^{a-1}}{\xi^{2a} + r_q(x)^{2a}} \right)^{\frac{n-1}{2}}, \xi > 0, \]
where \( a \) and \( c(n, a) \) are defined by (4.13).

For \( \varepsilon \in (0, 1) \), define the function \( I_\varepsilon : M \to \mathcal{N}_h \) by
\[ I_\varepsilon(q) = \Phi((1 - \varepsilon)\phi_{p,\varepsilon} + \varepsilon \phi_{q,\varepsilon}). \]

Let us prove the following lemma

**Lemma 4.3.** The function \( I_\varepsilon \) is continuous, and under the conditions
\( 1 \) \( a(n - 2) > 2, \]
\( 2 \) \( (A(n, a) + h(p)) \text{Scal}_g(p) - 6 \frac{\Delta h(p)}{n} > 0, \]
\( I(q) \in \Sigma_\varepsilon \) for all \( q \in M \).

**Proof.** By continuity of the projection \( \Phi : H^2_1(M) \setminus \{0\} \to \mathcal{N}_h, \) in order to prove the continuity of the function \( I_\varepsilon(q) \), we need to just prove the continuity of the function \( \phi_{q,\varepsilon} \) with respect to \( q \).

Let \( q_j \) be a sequence of points of \( M \) that converges to \( q \) and prove that
\[ \phi_{q_j,\varepsilon} \to \phi_{q,\varepsilon} \text{ in } H^2_1(M) \text{ as } q_j \to q. \]
Put $A_j = B(q_j, 2\delta) \cap B(q, 2\delta)$. Since $q_j \to q$ there exist $j_0$ such that $A_j \neq \emptyset$ for all $j \geq j_0$. Then, for $q_j$ close to $q$ we have

$$\int_{A_j} |(\phi_{q_j,\varepsilon}(x) - \phi_{q,\varepsilon}(x))|^2dv_g$$

$$= \int_{\exp_q^{-1}(A_j)} |(\phi_{q_j,\varepsilon} - \phi_{q,\varepsilon})(\exp_q(z))|^2 \sqrt{|g_{\exp(z)}|}dz$$

$$= C(n, a)^2 \int_{\exp_q^{-1}(A_j)} \eta_{q,\delta}(\exp_q(z))^2(U_{q_j} - U_q)(\exp_q(z))^2 \sqrt{|g_{\exp(z)}|}dz$$

$$+ \int_{\exp_q^{-1}(A_j)} U_{q_j}^2(\exp_q(z))|\eta_{q,\delta}(\exp_q(z)) - \eta_{q_j,\delta}(\exp_q(z))|^2 \sqrt{|g_{\exp(z)}|}dz$$

$$+ 2\int_{\exp_q^{-1}(A_j)} \eta_{q_j,\delta}U_{q_j,\varepsilon}|(\eta_{q,\delta} - \eta_{q_j,\delta})(\exp_q(z))|$$

$$|U_{q_j} - U_q)(\exp_q(z))|^2 \sqrt{|g_{\exp(z)}|}dz]$$

where

$$U_{q,\varepsilon}(x) = \left(\frac{\varepsilon_0 r_q(x)^{a-1}}{\varepsilon^{2n} + r_q(x)^{2n}}\right)^{2-1}, q \in M.$$ 

Using the fact that $U_{q_j} \to u_q$ and $\eta_{q_j,\varepsilon} \to \eta_{q,\varepsilon}$ pointwise together with the boundedness of $\int_{\exp_q^{-1}(A_j)} U_{q_j}^2(\exp_q(z))\sqrt{|g_{\exp(z)}|}dz$, we get that

$$\int_{A_j} |(\phi_{q_j,\varepsilon}(x) - \phi_{q,\varepsilon}(x))|^2dv_g \to 0.$$ 

Of course, outside the set $A_j$, $\int_{M \setminus A_j} |(\phi_{q_j,\varepsilon}(x) - \phi_{q,\varepsilon}(x))|^2dv_g \to 0$.

Similarly, the same conclusion holds for $\int_M |\nabla \phi_{q_j,\varepsilon}(x) - \nabla \phi_{q,\varepsilon}(x)|^2dv_g$. Now, for the proof of second part of the lemma, we begin with the case for $q = p$. In this case $I_*(p) = \Phi(\phi_{p,\varepsilon})$ and then the conclusion follows by lemma 4.2.

For $q \neq p$, let $\delta > 0$ be small enough such that $B(q, 2\delta) \cap B(p, 2\delta) = \emptyset$. In this way, the functions $\phi_{p,\varepsilon}$ and $\phi_{q,\varepsilon}$ are of disjoint supports. Then, we have

$$\int_M ((1 - \varepsilon)\phi_{p,\varepsilon} + \varepsilon \phi_{q,\varepsilon})^2 - \frac{h}{r_p}(1 - \varepsilon)\phi_{p,\varepsilon} + \varepsilon \phi_{q,\varepsilon})^2dv_g$$

$$= (1 - \varepsilon)^2 \left(\int_M ((\nabla \phi_{p,\varepsilon})^2 - \frac{h}{r_p}\phi_{p,\varepsilon})dv_g\right) + \varepsilon^2 \left(\int_M ((\nabla \phi_{q,\varepsilon})^2 - \frac{h}{r_p}\phi_{q,\varepsilon})dv_g\right)$$

We point out that by considering a normal geodesic coordinate system around the point $q$, the expansion (4.20) remains the same for any point $q$, that is

$$\int_M |\nabla \phi_{q,\varepsilon}|^2dv_g$$

$$= \int_M |\nabla U|^2dx - Scal_g(q)C_1(n, a)\int_{\mathbb{R}^n} |\nabla (n-2) + \varepsilon^2 + o(\varepsilon^2) + \alpha(\varepsilon).$$
Moreover, we have

\[
\int_M \frac{|h(x)|}{r_p^2} \phi_{q,\epsilon}^2 dv_g \leq \sup_M |h(x)| \frac{\delta^{-2}}{4} \int_{B(q,2\delta)} \phi_{q,\epsilon}^2(x)dv_g
\]

\[
+ C(n,a) \left( \frac{\epsilon a \delta a^{-1}}{\epsilon^2 \delta^2} \right)^{n-2} \int_{M \setminus B(q,2\delta)} \frac{1}{r_p^2} dv_g.
\]

The second integral is bounded by the Hardy inequality. For the first integral, as in [7], by considering a geodesic normal coordinate system around the point \(q\), direct calculations give

\[
\int_{B(q,2\delta)} \phi_{q,\epsilon}^2(x)dv_g = C(n,a)^2 w_{n-1}\epsilon^2 \int_0^{\infty} \frac{t^{a(n-1)+1}}{(1+t^2a)^{n-2}}dt + o(\epsilon^2).
\]

with \(w_{n-1}\) is the volume of the unit sphere \(S^{n-1} \subset \mathbb{R}^n\). Since \(a(n-2) > 2\), we get that

\[
\int_M \frac{|h(x)|}{r_p^2} \phi_{q,\epsilon}^2 dv_g = O(\epsilon^2) + o(\epsilon^2).
\]

Hence, we obtain

\[
(\int_M |\nabla((1-\epsilon)\phi_{p,\epsilon} + \epsilon \phi_{q,\epsilon})|^2 - \frac{h}{r_p^2}((1-\epsilon)\phi_{p,\epsilon} + \epsilon \phi_{q,\epsilon})^2)dv_g
\]

\[
= (1-\epsilon)^2 \left( \int_{\mathbb{R}^n} |\nabla U|^2 - \frac{h(p)}{|x|^2} dx + \alpha(n,a)\epsilon^2 \right)
\]

\[
+ \epsilon^2 \int_{\mathbb{R}^n} |\nabla U|^2 dx + o(\epsilon^2) + o(\epsilon^2).
\]

with

\[
\alpha(n,a) = \left( \text{Scal}_g(n,a) - \left( \frac{\text{Scal}_g(p)h(p)}{6} - \frac{\Delta h(p)}{n} \right) C_2(n,a) \right) \frac{a(n-2)+1}{n}
\]

On the other hand, since the functions \(\phi_{p,\epsilon}, \phi_{q,\epsilon}\) are of disjoint supports, we have

\[
(\int_M |(1-\epsilon)\phi_{p,\epsilon} + \epsilon \phi_{q,\epsilon}|^2 dv_g)^{2^*}
\]

\[
= \left( (1-\epsilon)^2 \int_{\mathbb{R}^n} |\phi_{p,\epsilon}|^2 dv_g + \epsilon^2 \int_{\mathbb{R}^n} |\phi_{q,\epsilon}|^2 dv_g \right)^{\frac{2^*}{2}}.
\]

Here again the expansion (4.22) holds true for \(\int_M |\phi_{q,\epsilon}|^2 dv_g\). That is

\[
\int_{\mathbb{R}^n} |\phi_{q,\epsilon}|^2 dv_g = \int_{\mathbb{R}^n} |u|^2 dx - \text{Scal}_g(q) C_3(n,a) I_n^{a(n-2)+1} \epsilon^2 + o(\epsilon^2) + o(\epsilon).
\]

Then

\[
(\int_M |(1-\epsilon)\phi_{p,\epsilon} + \epsilon \phi_{q,\epsilon}|^2 dv_g)^{\frac{2^*}{2}}
\]

\[
= \left( (1-\epsilon)^2 + \epsilon^2 \right)^{\frac{2^*}{2}} \left( \int_{\mathbb{R}^n} |U|^2 dx - \frac{(1-\epsilon)^2}{(1-\epsilon)^2 + \epsilon^2} \left( \text{Scal}_g(p) C_1(n,a) I_n^{a(n-2)+1} \epsilon^2 + o(\epsilon^2) \right) \right)^{\frac{2^*}{2}}.
\]
By using the expansions
\[
\frac{(1-\varepsilon)^2}{(1-\varepsilon)^{2\star} + \varepsilon^{2\star}} = 1 + o(\varepsilon^2)
\]
and
\[
((1-\varepsilon)^{2\star} + \varepsilon^{2\star})^{2\star} = (1-\varepsilon)^2(1 + (\varepsilon^{-1} - 1)^2)^{2\star} = (1-\varepsilon)^2\left(1 + \frac{2}{2\star}(\frac{\varepsilon^{2\star}}{1-\varepsilon})^{2\star} + o((\frac{\varepsilon}{1-\varepsilon})^{2\star})\right),
\]
and remark that $2^\star > 2$, we get
\[
\left(\int_M |(1-\varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon}|^2 dv_g\right)^{2\star}
= ((1-\varepsilon)^2(\int_{\mathbb{R}^n} |U|^2 dx)^{2\star}((1-\varepsilon)^{2\star} + \varepsilon^{2\star})^{2\star} + o(\varepsilon^2) + o(\varepsilon))
\]
Thus, we get the development of
\[
(nJ((1-\varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon}))^{2\star} = 
\left(\int_M \left(\left|\nabla((1-\varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon})\right|^2 - \frac{4\pi}{\varepsilon^2}(1-\varepsilon)^{2\star}(\varepsilon\phi_{q,\varepsilon})^2\right) dv_g \right) \left(\int_M |(1-\varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon}|^2 dv_g\right)^{2\star}
\]
as
\[
\left(\frac{nJ((1-\varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon}))^{2\star}}{nJ((1-\varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon}))^{2\star}}\right)^{2\star} = 
\left(\frac{\int_{\mathbb{R}^n} (|\nabla U|^2 - h(p)\frac{U^2}{\varepsilon^2}) dx}{\int_{\mathbb{R}^n} |U|^2 dx}\right)^{2\star} + \frac{1}{\int_{\mathbb{R}^n} |U|^2 dx}\left[\frac{\varepsilon^2}{(1-\varepsilon)^2} \int_{\mathbb{R}^n} |\nabla U|^2 dx + \alpha(\varepsilon) + o(\varepsilon)\right].
\]
Thus, by definition of the constant $D^*$, we get
\[
(nJ((1-\varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon}))^{2\star} < (nD^*)^{2\star} \left(\left(\frac{\varepsilon^2}{(1-\varepsilon)^2} \int_{\mathbb{R}^n} |\nabla U|^2 dx + \alpha(\varepsilon)\right)\right].
\]
Hence,
\[
J(\Phi((1-\varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon})) = D^* + h(\varepsilon),
\]
with
\[
h(\varepsilon) = \frac{\varepsilon^2}{2\int_{\mathbb{R}^n} |U|^2 dx} \left(\left(\frac{\varepsilon^2}{(1-\varepsilon)^2} \int_{\mathbb{R}^n} |\nabla U|^2 dx + o(\varepsilon^2)\right)\right].
\]
Note that under condition (2) of the lemma, $h(\varepsilon) > 0$. Therefore, we obtain

$$D^* < J_h(\Phi((1 - \varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon})) < D^* + g(\varepsilon),$$

with $g$ is any function such that $g(\varepsilon) > h(\varepsilon)$ and $g(\varepsilon) \to 0$. □

4.2. The map $\beta : \Sigma_\varepsilon \to M_{R^*_M}$. In this subsection, we define a map $\beta : \Sigma_\varepsilon \to M_{R^*_M}$. For this aim, we introduce the barycenter function $\beta : \mathcal{N}_h \to \mathbb{R}^n$ defined by

$$\beta(u) = \frac{\int_M (x + q - p)|u|^2^* dv_g}{\int_M |u|^2^* dv_g}.$$ 

The function $\beta$ is well defined as $u \neq 0$ for all $u \in \mathcal{N}_h$ and the manifold $M$ is embedded in some $\mathbb{R}^N$.

Now, we prove some properties of the function $\beta$ through the following lemmas

**Lemma 4.4.** We have

$$\lim_{\varepsilon \to 0} \beta(I_\varepsilon(q)) = q$$

**Proof.** We begin with case where $q = p$. By homogeneity of the function $\beta$, we have

$$\beta(I_\varepsilon(p)) = \beta(\phi_{p,\varepsilon}) = \frac{\int_M x|\phi_{p,\varepsilon}|^2^* dv_g}{\int_M |\phi_{p,\varepsilon}|^2^* dv_g}.$$ 

Then

$$|\beta(I_\varepsilon(p)) - p| \leq \frac{\int_M |x - p||\phi_{p,\varepsilon}|^2^* dv_g}{\int_M |\phi_{p,\varepsilon}|^2^* dv_g}.$$ 

For the numerator, we have

$$\int_M |x - p||\phi_{p,\varepsilon}|^2^* dv_g = C(n, a) \int_M r_p \delta r_p(x) \left( \frac{\varepsilon^a r_p(x)^{a-1}}{\varepsilon^2 a + r_p(x)^{2a}} \right) \frac{n}{2}^{a-1} dv_g.$$ 

We repeat the same calculation as in [7], we get

$$\int_M |x - p||\phi_{p,\varepsilon}|^2^* dv_g = \varepsilon \int_{\mathbb{R}^n} |U|^{2^*} dx - Scal_g(p) C_3(n, a) I_n^{a(n-2)+1} \varepsilon^2 + o(\varepsilon^2) + o(\varepsilon^2).$$

For the denominator, we have already

$$\int_M |\phi_{p,\varepsilon}|^2^* dv_g = \int_{\mathbb{R}^n} |U|^{2^*} dx - Scal_g(p) C_3(n, a) I_n^{a(n-2)+1} \varepsilon^2 + o(\varepsilon^2) + o(\varepsilon^2).$$
By letting $\varepsilon \to 0$, we get the desired equality.

Now, for $q \neq p$, we choose $\delta$ small enough so that $B(q, 2\delta) \cap B(p, 2\delta) = \emptyset$. In this situation, the functions $\phi_{p,\varepsilon}$ and $\phi_{q,\varepsilon}$ have disjoint supports. Then, similarly as above, we have

$$|\beta(I_{\varepsilon}(q)) - q| \leq \frac{\int_M |x - p||(1 - \varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon}|^2dv_g}{\int_M |(1 - \varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon}|^2dv_g}.$$ 

Since the functions $\phi_{p,\varepsilon}$ and $\phi_{q,\varepsilon}$ have disjoint supports, we have

$$\int_M |x - q||\phi_{q,\varepsilon}|^2dv_g$$

$$= (1 - \varepsilon)^2\int_M |x - p||\phi_{p,\varepsilon}|^2dv_g + \varepsilon^2\int_M |x - p||\phi_{q,\varepsilon}|^2dv_g$$

$$\leq (1 - \varepsilon)^2\int_M |x - p||\phi_{p,\varepsilon}|^2dv_g + \varepsilon^2\int_M |x - p||\phi_{q,\varepsilon}|^2dv_g$$

$$+ \varepsilon^2\int_M |x - q||\phi_{q,\varepsilon}|^2dv_g + \varepsilon^2|q - p|\int_M |\phi_{q,\varepsilon}|^2dv_g$$

Like before, we have

$$\int_M |\phi_{q,\varepsilon}|^2dv_g = \int_{\mathbb{R}^n} |U|^2dx - Scal_g(q)C_3(n, a)I_n^{a(n-2)+1}\varepsilon^2 + o(\varepsilon^2), + \alpha(\varepsilon).$$

and

$$\int_M |x - q||\phi_{q,\varepsilon}|^2dv_g = \varepsilon\int_{\mathbb{R}^n} |U|^2dx - Scal_g(q)C_3(n, a)I_n^{a(n-2)+1}\varepsilon^3$$

$$+ \varepsilon o(\varepsilon^2) + \alpha(\varepsilon).$$

Hence, by using the expansion

$$\int_M |(1 - \varepsilon)\phi_{p,\varepsilon} + \varepsilon\phi_{q,\varepsilon}|^2dv_g$$

$$= ((1 - \varepsilon)^2 + \varepsilon^2)\int_{\mathbb{R}^n} |U|^2dx - (1 - \varepsilon)^2Scal_g(p)C_1(n, a)I_n^{a(n-2)+1}\varepsilon^2$$

$$+ o(\varepsilon^2) + \alpha(\varepsilon),$$

we get

$$|\beta(I_{\varepsilon}(q)) - q| \to 0 \text{ as } \varepsilon \to 0.$$

\[\square\]

**Lemma 4.5.** For any $\eta \in (0, 1)$ and for every $u \in \Sigma_{\varepsilon}$, there exists a point $q \in M$ such that

$$\int_{B(q, \frac{\varepsilon}{3})} |u|^2dv_g > (1 - \eta)(S_{h(p)})^{\frac{n}{2}}.$$
Proof. Suppose by contradiction that there exist \( \eta \in (0, 1) \), a sequence \( \varepsilon_m \to 0 \) as \( m \to \infty \) and a sequence \( u_m \in \Sigma_{\varepsilon_m} \) such that for all \( q \in M \)

\[
\int_{B(q, \frac{\varepsilon_m}{4^q})} |u_m|^{2^*} dv_g \leq (1 - \eta)(S_{h(p)})^{\frac{2}{2^*}}.
\]

By the Ekland variational principle, we can assume that \( D_{N_h} J_h(u_m) \to 0 \) as \( m \to \infty \). Since \( D^* < J_h(u_m) < D^* + g(\varepsilon_m) \), for some \( g(\varepsilon_m) > 0 \) and \( g(\varepsilon_m) \to 0 \) as \( m \to 0 \) and since the manifold \( N_h \) defines a natural constraint for the functional \( J_h \) (see [1]), we can assume that \( u_m \) is a P-S sequence of \( J_h \) at level \( D^* \). Thus by corollary 3.4, there exists a sequence of reals \( R_m \to 0 \) as \( m \to \infty \) and a sequence \( w_m \in H^2_1(M) \) that converges strongly to 0 in \( H^2_1(M) \) such that

\[
u_m = \phi_{p, R_m} + w_m.
\]

Hence, by applying the inequality

\[(a + b)^{2^*} \leq a^{2^*} + b^{2^*} + 2^*a^{2^* - 1}b + 2^*ab^{2^* - 1}, a \geq 0, b \geq 0,
\]

and by using the fact that \( w_m \to 0 \) strongly in \( H^2_1(M) \), we obtain

\[
\int_{B(q, \frac{\varepsilon_m}{4^q})} |\phi_{p, R_m}|^{2^*} dv_g \leq (1 - \eta)(S_{h(p)})^{\frac{2}{2^*}}.
\]

Put \( \varepsilon^*_m = R_m \). Then, \( \varepsilon^*_m \to 0 \) as \( m \to \infty \). Thus, by using the expansion (4.22), we have

\[
\int_M |\phi_{p, \varepsilon^*_m}|^{2^*} dv_g = \int_{\mathbb{R}^n} |U|^{2^*} dx - Scal_g(p)C_3(n, a)I^{a(n-2)+1}(\varepsilon^*_m)^2 + o((\varepsilon^*_m)^2) + \alpha(\varepsilon^*_m),
\]

As the function \( U \) is a positive solution of \( (Eu_\lambda) \), we get

\[
\int_M |\phi_{\varepsilon^*_m, p}|^{2^*} dv_g = (nD^*)^{\frac{2}{2^*}} - Scal_g(p)C_3(n, a)I^{a(n-2)+1}(\varepsilon^*_m)^2 + o((\varepsilon^*_m)^2) + \alpha(\varepsilon^*_m)
\]

Recall that the function \( U \) is supported in \( B(p, \delta) \), then by choosing \( \delta \) small, we obtain by

\[
(S_{h(p)})^{\frac{2}{2^*}} - Scal_g(p)C_3(n, a)I^{a(n-2)+1}(\varepsilon^*_m)^2 + o((\varepsilon^*_m)^2) + \alpha(\varepsilon^*_m) \leq (1 - \eta)(S_{h(p)})^{\frac{2}{2^*}}.
\]

Hence, by letting \( m \to \infty \), we get the contradiction:

\[
(S_{h(p)})^{\frac{2}{2^*}} \leq (1 - \eta)(S_{h(p)})^{\frac{2}{2^*}}.
\]

\[
\Box
\]

Lemma 4.6. For \( \varepsilon \) small, \( \beta(u) \in M \), for every function \( u \in \Sigma_{\varepsilon} \).
Proof. It suffices to prove that for every \( u \in \Sigma_\varepsilon \),
\[
|\beta(u) - p| \leq r_M.
\]
Let \( u \in \Sigma_\varepsilon \), by lemma 4.5, we get that for any \( \eta \in (0, 1) \)
\[
\frac{\int_{B(p, r_M)} |u|^2 \, dv_g}{\int_M |u|^2 \, dv_g} > \frac{(1 - \eta)(S_{h(p)})^{\frac{2}{n}}}{n(D^* + g(\varepsilon))} = \frac{(1 - \eta)(S_{h(p)})^{\frac{2}{n}}}{(S_{h(p)})^{\frac{2}{n}} + ng(\varepsilon)}
\]
Then, we obtain
\[
|\beta(u) - p| = \frac{\int_M (x - p)|u|^2 \, dv_g}{\int_M |u|^2 \, dv_g}
\leq \frac{\int_{B(p, r_M)} (x - p)|u|^2 \, dv_g}{\int_M |u|^2 \, dv_g} + \frac{\int_{M \setminus B(p, r_M)} |x - p||u|^2 \, dv_g \, r_M}{2}
\leq \frac{r_M}{2} + D(M)(1 - \frac{\int_{B(p, r_M)} |u|^2 \, dv_g}{\int_M |u|^2 \, dv_g})
\leq \frac{r_M}{2} + D(M)(1 - \frac{(1 - \eta)(S_{h(p)})^{\frac{2}{n}}}{(S_{h(p)})^{\frac{2}{n}} + ng(\varepsilon)})
\]
where \( D(M) \) is the diameter of \( M \). Thus, in order to get the conclusion, it suffices to choose \( \eta \) and \( \varepsilon \) small enough so that
\[
D(M)(1 - \frac{(1 - \eta)(S_{h(p)})^{\frac{2}{n}}}{(S_{h(p)})^{\frac{2}{n}} + ng(\varepsilon)}) \leq \frac{r_M}{2}.
\]
\[\square\]

5. PROOF OF THE MAIN RESULT

Proof. By Lemmas 4.3 and 4.6 the maps \( I_\varepsilon : M \to \Sigma_\varepsilon \) and \( \beta : \Sigma_\varepsilon \to M_{r_M} \) are well defined. Moreover, by lemma 4.4 the composition \( \beta \circ I_\varepsilon : M \to M_{r_M} \) is well defined and is homotopic to the identity. Thus, by the properties of Lusternik-Schnirelmann category, \( \text{Cat} \Sigma_\varepsilon \geq \text{Cat}(M) \).
Since the Palais-Smale conditions are satisfied in the set \( \Sigma_\varepsilon \), by theorem 4.1 there are at least \( \text{cat}(M) \) critical points of the functional \( J_h \).
It remains, to achieve the proof of the theorem, to prove that there exists another critical point \( u \) with \( J_h(u) > D^* + g(\varepsilon) \). For this task, following [2], we construct a set \( P_\varepsilon \) which is contractible in \( N_h \cap J_h^c \).
Let \( V \in D^{1,2}(\mathbb{R}^n) \) be any function and define on the manifold \( M \) the function
\[
V_\varepsilon(x) = \eta_{p, \delta}(\exp_p^{-1}(x))V(\varepsilon^{-1}\exp_p^{-1}(x)), x \in B(p, \delta).
\]
Put $\varphi_{\varepsilon} = (1 - \varepsilon) \phi_{p,\varepsilon} + \varepsilon \phi_{q,\varepsilon}$ and define the set

$$\Omega_{\varepsilon} = \{(1 - t) \varphi_{\varepsilon} + t V_{\varepsilon}, t \in [0, 1]\}$$

Consider $P_{\varepsilon}$, the projection of $\Omega_{\varepsilon}$ on the Nehari manifold $N_h$

$$P_{\varepsilon} = \{\Phi(\omega_{\varepsilon}), \omega_{\varepsilon} \in \Omega_{\varepsilon}\}$$

We notice immediately that $I_{\varepsilon}(M) \subset P_{\varepsilon}$, $P_{\varepsilon}$ is compact and contractible in $N_h$. Then, put

$$c_{\varepsilon} = \sup_{u_{\varepsilon} \in P_{\varepsilon}} J_h(u).$$

We need to prove that $c_{\varepsilon}$ is bounded with respect to $\varepsilon$. For this aim, for $u \in \Omega_{\varepsilon}$ write

$$J_h(\Phi(u)) = \frac{1}{n} \left( \frac{\int_M (|\nabla u|^2 - \frac{4}{p} u^2) dv_g}{\left(\int_M |u|^{2^*} dv_g\right)^{\frac{2}{2^*}}} \right)^{\frac{n}{2}}.$$

We have

$$\int_M |\nabla u_{\varepsilon}|^2 dv_g$$

$$= t^2 \int_M |\nabla V_{\varepsilon}|^2 dv_g + (1 - t)^2 \int_M |\nabla \varphi_{\varepsilon}|^2 dv_g + 2t(1-t) \int_M \nabla V_{\varepsilon} \cdot \nabla \varphi_{\varepsilon} dv_g$$

$$\leq \int_{\mathbb{R}^n} |\nabla V|^2 dx + 2 \int_{\mathbb{R}^n} |\nabla U|^2 dx + 2 \left( \int_{\mathbb{R}^n} |\nabla V|^2 \int_{\mathbb{R}^n} |\nabla U|^2 dx \right)^{\frac{1}{2}} + K_1.$$ (5.26)

Also, we have

$$\left| \int_M \frac{h}{\rho_p^2} \varphi_{\varepsilon}^2 dv_g \right|$$

$$\leq t^2 \int_M \frac{h}{\rho_p^2} V_{\varepsilon}^2 dv_g + (1 - t)^2 \int_M \frac{h}{\rho_p^2} \varphi_{\varepsilon}^2 dv_g + 2t(1-t) \sup_M |h| K(n, -2, \delta)$$

$$\left( \left( \int_M |V_{\varepsilon}|^2 dv_g \int_M |\nabla \varphi_{\varepsilon}|^2 dv_g \right)^{\frac{1}{2}} + \left( \int_M |\varphi_{\varepsilon}|^2 dv_g \int_M |\nabla V_{\varepsilon}|^2 dv_g \right)^{\frac{1}{2}} \right)$$

$$\leq h(p) \left( \int_{\mathbb{R}^n} |V|^2 dx + \int_{\mathbb{R}^n} |\nabla U|^2 dx \right) + C\left( \left( \int_{\mathbb{R}^n} |V|^2 dx \int_{\mathbb{R}^n} |\nabla U|^2 dx \right)^{\frac{1}{2}} \right)$$

$$+ \left( \int_{\mathbb{R}^n} |U|^2 dx \int_{\mathbb{R}^n} |\nabla V|^2 dx \right)^{\frac{1}{2}} + K_2.$$ (5.27)

Moreover, there exists $\varepsilon_o$ and $\varepsilon_1$ such that

$$\int_M |\varphi_{\varepsilon}|^2 dv_g \geq \int_{\mathbb{R}^n} |U|^2 dx - Scal_g(p) C_3(n, a) I_n^{a(n-2) + 1} \varepsilon_o^2 > 0,$$
\[
\int_M |V_\varepsilon|^2 \, dv_g \geq \int_{\mathbb{R}^n} |V|^2 \, dx - \text{Scal}_g(p) C_3(n, a) I_n^{a(n-2)+1} \varepsilon_1^2 > 0.
\]

Then, since \(V_\varepsilon\) and \(\varphi_\varepsilon\) are positive, we get
\[
\int_M |u_\varepsilon|^{2^*} \, dv_g \\
\geq \max(t^{2^*} \int_M |V_\varepsilon|^2 \, dv_g, (1-t)^{2^*} \int_M |\varphi_\varepsilon|^2 \, dv_g) \\
\geq \frac{1}{2^*} \min(\int_{\mathbb{R}^n} |U|^{2^*} \, dx - K_3, \int_{\mathbb{R}^n} |V|^{2^*} \, dx - K_4),
\]
where
\[
K_3 = \text{Scal}_g(p) C_3(n, a) I_n^{a(n-2)+1} \varepsilon_0^2 \quad \text{and} \quad K_4 = \text{Scal}_g(p) C_3(n, a) I_n^{a(n-2)+1} \varepsilon_1^2.
\]
which gives together with estimates (5.26) and (5.27) the thesis. □

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