Spacetime topology from causality

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Abstract

We prove that a globally hyperbolic spacetime with its causality relation is a bicontinuous poset whose interval topology is the manifold topology. This provides an abstract mathematical setting in which one can study causality independent of geometry and differentiable structure.

1 Introduction

It has been known for some time that the topology of spacetime could be characterized purely in terms of causality – but what is causality? In this paper, we prove that the causality relation is much more than a relation – it turns a globally hyperbolic spacetime into what is known as a bicontinuous poset. The order on a bicontinuous poset allows one to define an intrinsic topology called the interval topology. On a globally hyperbolic spacetime, the interval topology is the manifold topology.
The importance of these results and ideas is that they suggest an abstract formulation of causality – a setting where one can study causality independently of geometry and differentiable structure.

## 2 Domains, continuous posets and topology

A poset is a partially ordered set, i.e., a set together with a reflexive, antisymmetric and transitive relation.

**Definition 2.1** Let \( (P, \sqsubseteq) \) be a partially ordered set. A nonempty subset \( S \subseteq P \) is **directed** if \( (\forall x, y \in S)(\exists z \in S) x, y \sqsubseteq z \). The **supremum** of \( S \subseteq P \) is the least of all its upper bounds provided it exists. This is written \( \bigsqcup S \).

These ideas have duals that will be important to us: A nonempty \( S \subseteq P \) is **filtered** if \( (\forall x, y \in S)(\exists z \in S) z \sqsubseteq x, y \). The **infimum** \( \bigwedge S \) of \( S \subseteq P \) is the greatest of all its lower bounds provided it exists.

**Definition 2.2** For a subset \( X \) of a poset \( P \), set

\[
\uparrow X := \{ y \in P : (\exists x \in X) x \sqsubseteq y \} \quad \& \quad \downarrow X := \{ y \in P : (\exists x \in X) y \sqsubseteq x \}.
\]

We write \( \uparrow x = \uparrow \{x\} \) and \( \downarrow x = \downarrow \{x\} \) for elements \( x \in X \).

A partial order allows for the derivation of several intrinsically defined topologies. Here is our first example.

**Definition 2.3** A subset \( U \) of a poset \( P \) is **Scott open** if

1. \( U \) is an upper set: \( x \in U \& x \sqsubseteq y \Rightarrow y \in U \), and
2. \( U \) is inaccessible by directed suprema: For every directed \( S \subseteq P \) with a supremum,

\[ \bigsqcup S \in U \Rightarrow S \cap U \neq \emptyset. \]

The collection of all Scott open sets on \( P \) is called the **Scott topology**.

**Definition 2.4** A dcpo is a poset in which every directed subset has a supremum. The **least element** in a poset, when it exists, is the unique element \( \bot \) with \( \bot \sqsubseteq x \) for all \( x \).

The set of **maximal elements** in a dcpo \( D \) is

\[ \max(D) := \{ x \in D : \uparrow x = \{x\} \}. \]

Each element in a dcpo has a maximal element above it.
Definition 2.5 For elements \( x, y \) of a poset, write \( x \ll y \) iff for all directed sets \( S \) with a supremum,

\[
y \subseteq \bigcup S \Rightarrow (\exists s \in S) \ x \subseteq s.
\]

We set \( \downarrow x = \{ a \in D : a \ll x \} \) and \( \uparrow x = \{ a \in D : x \ll a \} \).

For the symbol \( \ll \), read “approximates.”

Definition 2.6 A basis for a poset \( D \) is a subset \( B \) such that \( B \cap \downarrow x \) contains a directed set with supremum \( x \) for all \( x \in D \). A poset is continuous if it has a basis. A poset is \( \omega \)-continuous if it has a countable basis.

Continuous posets have an important property, they are interpolative.

Proposition 2.7 If \( x \ll y \) in a continuous poset \( P \), then there is \( z \in P \) with \( x \ll z \ll y \).

This enables a clear description of the Scott topology,

Theorem 2.8 The collection \( \{ \uparrow x : x \in D \} \) is a basis for the Scott topology on a continuous poset.

And also helps us give a clear definition of the Lawson topology.

Definition 2.9 The Lawson topology on a continuous poset \( P \) has as a basis all sets of the form \( \uparrow x \uparrow F \), for \( F \subseteq P \) finite.

The next idea, as far as we know, is new.

Definition 2.10 A continuous poset \( P \) is bicontinuous if

- For all \( x, y \in P \), \( x \ll y \) iff for all filtered \( S \subseteq P \) with an infimum,

\[
\bigwedge S \subseteq x \Rightarrow (\exists s \in S) \ s \subseteq y,
\]

and

- For each \( x \in P \), the set \( \uparrow x \) is filtered with infimum \( x \).

Example 2.11 \( \mathbb{R}, \mathbb{Q} \) are bicontinuous.
**Definition 2.12** On a bicontinuous poset $P$, sets of the form

$$(a, b) := \{x \in P : a \ll x \ll b\}$$

form a basis for a topology called the interval topology.

The proof uses interpolation and bicontinuity. A bicontinuous poset $P$ has $\uparrow x \neq \emptyset$ for each $x$, so it is rarely a dcpo. Later we will see that on a bicontinuous poset, the Lawson topology is contained in the interval topology (causal simplicity), the interval topology is Hausdorff (strong causality), and $\leq$ is a closed subset of $P^2$.

**Definition 2.13** A continuous dcpo is a continuous poset which is also a dcpo. A domain is a continuous dcpo.

We now consider an example of a domain that will be used later in proofs.

**Example 2.14** Let $X$ be a locally compact Hausdorff space. Its upper space

$$UX = \{\emptyset \neq K \subseteq X : K \text{ is compact}\}$$

ordered under reverse inclusion

$$A \subseteq B \iff B \subseteq A$$

is a continuous dcpo:

- For directed $S \subseteq UX$, $\bigcup S = \bigcap S$.
- For all $K, L \in UX$, $K \ll L \iff L \subseteq \text{int}(K)$.
- $UX$ is $\omega$-continuous iff $X$ has a countable basis.

It is interesting here that the space $X$ can be recovered from $UX$ in a purely order theoretic manner:

$$X \simeq \text{max}(UX) = \{\{x\} : x \in X\}$$

where max($UX$) carries the relative Scott topology it inherits as a subset of $UX$. Several constructions of this type are known.
3 The causal structure of spacetime

A manifold $\mathcal{M}$ is a locally Euclidean Hausdorff space that is connected and has a countable basis. A connected Hausdorff manifold is paracompact if and only if it has a countable basis. A Lorentz metric on a manifold is a symmetric, nondegenerate tensor field of type $(0,2)$ whose signature is $(-+++)$.

**Definition 3.1** A spacetime is a real four-dimensional smooth manifold $\mathcal{M}$ with a Lorentz metric $g_{ab}$.

Let $(\mathcal{M}, g_{ab})$ be a time orientable spacetime. Let $\Pi^+_{\leq}$ denote the future directed causal curves, and $\Pi^+_{<}$ denote the future directed time-like curves.

**Definition 3.2** For $p \in \mathcal{M}$,

$$I^+(p) := \{q \in \mathcal{M} : (\exists \pi \in \Pi^+_{\leq}) \pi(0) = p, \pi(1) = q\}$$

and

$$J^+(p) := \{q \in \mathcal{M} : (\exists \pi \in \Pi^+_{<}) \pi(0) = p, \pi(1) = q\}$$

Similarly, we define $I^-(p)$ and $J^-(p)$.

We write the relation $J^+$ as

$$p \sqsubseteq q \equiv q \in J^+(p).$$

The following properties from [3] are very useful:

**Proposition 3.3** Let $p, q, r \in \mathcal{M}$. Then

(i) The sets $I^+(p)$ and $I^-(p)$ are open.

(ii) $p \sqsubseteq q$ and $r \in I^+(q) \Rightarrow r \in I^+(p)$

(iii) $q \in I^+(p)$ and $q \sqsubseteq r \Rightarrow r \in I^+(p)$

(iv) $\text{Cl}(I^+(p)) = \text{Cl}(J^+(p))$ and $\text{Cl}(I^-(p)) = \text{Cl}(J^-(p))$.

We always assume the chronology conditions that ensure $(\mathcal{M}, \sqsubseteq)$ is a partially ordered set. We also assume strong causality which can be characterized as follows [5]:

**Theorem 3.4** A spacetime $\mathcal{M}$ is strongly causal iff its Alexandroff topology is Hausdorff if and only if its Alexandroff topology is the manifold topology.

The Alexandroff topology on a spacetime has $\{I^+(p) \cap I^-(q) : p, q \in \mathcal{M}\}$ as a basis [5].
4 Global hyperbolicity

Penrose has called *globally hyperbolic* spacetimes “the physically reasonable spacetimes [7].” In this section, $\mathcal{M}$ denotes a globally hyperbolic spacetime, and we prove that $(\mathcal{M}, \sqsubset)$ is a bicontinuous poset.

**Definition 4.1** A spacetime $\mathcal{M}$ is *globally hyperbolic* if it is strongly causal and if $\uparrow a \cap \downarrow b$ is compact in the manifold topology, for all $a, b \in \mathcal{M}$.

**Lemma 4.2** If $(x_n)$ is a sequence in $\mathcal{M}$ with $x_n \sqsubset x$ for all $n$, then

$$\lim_{n \to \infty} x_n = x \Rightarrow \bigsqcup_{n \geq 1} x_n = x.$$  

**Proof.** Let $x_n \sqsubset y$. By global hyperbolicity, $\mathcal{M}$ is causally simple, so the set $J^-(y)$ is closed. Since $x_n \in J^-(y)$, $x = \lim x_n \in J^-(y)$, and thus $x \sqsubset y$. This proves $x = \bigsqcup x_n$. \(\square\)

**Lemma 4.3** For any $x \in \mathcal{M}$, $I^-(x)$ contains an increasing sequence with supremum $x$.

**Proof.** Because $x \in \text{Cl}(I^-(x)) = J^-(x)$ but $x \not\in I^-(x)$, $x$ is an accumulation point of $I^-(x)$, so for every open set $V$ with $x \in V$, $V \cap I^-(x) \neq \emptyset$. Let $(U_n)$ be a countable basis for $x$, which exists because $\mathcal{M}$ is locally Euclidean. Define a sequence $(x_n)$ by first choosing

$$x_1 \in U_1 \cap I^-(x) \neq \emptyset$$

and then whenever

$$x_n \in U_n \cap I^-(x)$$

we choose

$$x_{n+1} \in (U_n \cap I^+(x_n)) \cap I^-(x) \neq \emptyset.$$  

By definition, $(x_n)$ is increasing, and since $(U_n)$ is a basis for $x$, $\lim x_n = x$. By Lemma 4.2, $\bigsqcup x_n = x$. \(\square\)

**Proposition 4.4** Let $\mathcal{M}$ be a globally hyperbolic spacetime. Then

$$x \ll y \Leftrightarrow y \in I^+(x)$$

for all $x, y \in \mathcal{M}$.
Proof. Let \( y \in I^+(x) \). Let \( y \not\subseteq \bigcup S \) with \( S \) directed. By Prop. 3.3(iii),
\[
y \in I^+(x) \land y \not\subseteq \bigcup S \Rightarrow \bigcup S \in I^+(x)
\]
Since \( I^+(x) \) is manifold open and \( \mathcal{M} \) is locally compact, there is an open set \( V \subseteq \mathcal{M} \) whose closure \( \text{Cl}(V) \) is compact with \( \bigcup S \in V \subseteq \text{Cl}(V) \subseteq I^+(x) \). Then, using approximation on the upper space of \( \mathcal{M} \),
\[
\text{Cl}(V) \ll \{ \bigcup S \} = \bigcap_{s \in S} [s, \bigcup S]
\]
where the intersection on the right is a filtered collection of nonempty compact sets by directedness of \( S \) and global hyperbolicity of \( \mathcal{M} \). Thus, for some \( s \in S \), \( [s, \bigcup S] \subseteq \text{Cl}(V) \subseteq I^+(x) \), which gives \( x \subseteq s \). This proves \( x \ll y \).

Now let \( x \ll y \). By Lemma 4.3 there is an increasing sequence \( (y_n) \) in \( I^-(y) \) with \( y = \bigcup y_n \). Then since \( x \ll y \), there is \( n \) with \( x \subseteq y_n \). By Prop. 3.3(ii),
\[
x \subseteq y_n \land y_n \in I^-(y) \Rightarrow x \in I^-(y)
\]
which is to say that \( y \in I^+(x) \). \( \square \)

Theorem 4.5 If \( \mathcal{M} \) is globally hyperbolic, then \( (\mathcal{M}, \subseteq) \) is a bicontinuous poset with \( \ll = I^+ \) whose interval topology is the manifold topology.

Proof. By combining Lemma 4.3 with Prop. 4.4 \( x \) contains an increasing sequence with supremum \( x \), for each \( x \in \mathcal{M} \). Thus, \( \mathcal{M} \) is a continuous poset. For the bicontinuity, Lemmas 4.2, 4.3 and Prop. 4.4 have “duals” which are obtained by replacing ‘increasing’ by ‘decreasing’, \( I^+ \) by \( I^- \), \( J^- \) by \( J^+ \), etc. For example, the dual of Lemma 4.3 is that \( I^+ \) contains a decreasing sequence with infimum \( x \). Using the duals of these two lemmas, we then give an alternate characterization of \( \ll \) in terms of infima:
\[
x \ll y \equiv (\forall S) \bigwedge S \subseteq x \Rightarrow (\exists s \in S) s \subseteq y
\]
where we quantify over filtered subsets \( S \) of \( \mathcal{M} \). These three facts then imply that \( \uparrow x \) contains a decreasing sequence with inf \( x \). But because \( \ll \) can be phrased in terms of infima, \( \uparrow x \) itself must be filtered with inf \( x \).

Finally, \( \mathcal{M} \) is bicontinuous, so we know it has an interval topology. Because \( \ll = I^+ \), the interval topology is the one generated by the timelike causality relation, which by strong causality is the manifold topology. \( \square \)
Bicontinuity, as we have defined it here, is really quite a special property, and some of the nicest posets in the world are not bicontinuous. For example, the powerset of the naturals $\mathcal{P}\omega$ is not bicontinuous, because we can have $F \ll G$ for $G$ finite, and $F = \bigcap V_n$ where all the $V_n$ are infinite.

## 5 Causal simplicity

It is also worth pointing out before we close, that causal simplicity also has a characterization in order theoretic terms.

**Definition 5.1** A spacetime $\mathcal{M}$ is **causally simple** if $J^+(x)$ and $J^-(x)$ are closed for all $x \in \mathcal{M}$.

**Theorem 5.2** Let $\mathcal{M}$ be a spacetime and $(\mathcal{M}, \sqsubseteq)$ a continuous poset with $\ll = I^+$. The following are equivalent:

(i) $\mathcal{M}$ is causally simple.

(ii) The Lawson topology on $\mathcal{M}$ is a subset of the interval topology on $\mathcal{M}$.

**Proof** (i) $\Rightarrow$ (ii): We want to prove that

$$\{ \uparrow x \cap \uparrow F : x \in \mathcal{M} \& F \subseteq \mathcal{M} \text{ finite} \} \subseteq \text{int}_\mathcal{M}.$$  

By strong causality of $\mathcal{M}$ and $\ll = I^+$, int$_\mathcal{M}$ is the manifold topology, and this is the crucial fact we need as follows. First, $\uparrow x = I^+(x)$ is open in the manifold topology and hence belongs to int$_\mathcal{M}$. Second, $\uparrow x = J^+(x)$ is closed in the manifold topology by causal simplicity, so $\mathcal{M} \setminus \uparrow x$ belongs to int$_\mathcal{M}$. Then int$_\mathcal{M}$ contains the basis for the Lawson topology given above.

(ii) $\Rightarrow$ (i): First, since $(\mathcal{M}, \sqsubseteq)$ is continuous, its Lawson topology is Hausdorff, so int$_\mathcal{M}$ is Hausdorff since it contains the Lawson topology by assumption. Since $\ll = I^+$, int$_\mathcal{M}$ is the Alexandroff topology, so Theorem 3.4 implies $\mathcal{M}$ is strongly causal.

Now, Theorem 3.4 also tells us that int$_\mathcal{M}$ is the manifold topology. Since the manifold topology int$_\mathcal{M}$ contains the Lawson by assumption, and since

$$J^+(x) = \uparrow x \quad \text{and} \quad J^-(x) = \downarrow x$$

are both Lawson closed (the second is Scott closed), each is also closed in the manifold topology, which means $\mathcal{M}$ is causally simple. \(\square\)

Note in the above proof that we have assumed causally simplicity implies strong causality. If we are wrong about this, then (i) above should be replaced with ‘causal simplicity+strong causality’.
6 Conclusion

It seems there is something of a gap in the hierarchy of causality conditions. One goes from global hyperbolicity all the way down to causal simplicity. It might be good to insert a new one in between these two. Some possible candidates are to require $(\mathcal{M}, \sqsubseteq)$ a continuous (bicontinuous) poset. All of these might provide generalizations of global hyperbolicity for which a lot could probably be proved. Bicontinuity, in particular, has the nice consequence that one does not have to explicitly assume strong causality as one does with global hyperbolicity. Is $\mathcal{M}$ bicontinuous iff it is causally simple?

In the forthcoming [4], the fact that globally hyperbolic spacetimes are bicontinuous will enable us to prove that spacetime can be reconstructed from a countable dense set and the causality relation in a purely order theoretic manner using techniques from an area known as domain theory [6].

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