Proof of two divisibility properties of binomial coefficients conjectured by Z.-W. Sun

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Abstract
For all positive integers \( n \), we prove the following divisibility properties:
\[
(2n + 3) \binom{2n}{n} \bigg| \binom{6n}{3n} \binom{3n}{n} \quad \text{and} \quad (10n + 3) \binom{3n}{n} \bigg| 21 \binom{15n}{5n} \binom{5n}{n}.
\]
This confirms two recent conjectures of Z.-W. Sun. Some similar divisibility properties are given. Moreover, we show that, for all positive integers \( m \) and \( n \), the product \( am^{\binom{am+bn-1}{am}} \binom{an+bn}{an} \) is divisible by \( m + n \). In fact, the latter result can be further generalized to the \( q \)-binomial coefficients and \( q \)-integers case, which generalizes the positivity of \( q \)-Catalan numbers. We also propose several related conjectures.

Keywords: congruences, binomial coefficients, \( p \)-adic order, \( q \)-Catalan numbers, reciprocal and unimodal polynomials

1 Introduction
In [18, 19], Z.-W. Sun proved some divisibility properties of binomial coefficients, such as
\[
(2n + 1) \binom{2n}{n} \bigg| \binom{6n}{3n} \binom{3n}{n},
\]
\[
(10n + 1) \binom{3n}{n} \bigg| \binom{15n}{5n} \binom{5n-1}{n-1}.
\]
Some similar divisibility results were later obtained by Guo [10] and Guo and Krattenthaler [11]. A generalization of (1.1) was recently given by Sepanski [15]. It is worth mentioning that Bober [6] has completely described when ratios of factorial products of the form
\[
\frac{(a_1 n)! \cdots (a_k n)!}{(b_1 n)! \cdots (b_{k+1} n)!}
\]
with \(a_1 + \cdots + a_k = b_1 + \cdots + b_{k+1}\) are always integers.

Let
\[
S_n = \frac{(3n)!}{2(2n+1)(2n)!}, \quad \text{and} \quad t_n = \frac{(15n)!}{(10n+1)(3n)!}.
\]

In this paper we first prove the following two results conjectured by Z.-W. Sun [18, 19].

**Theorem 1.1** (see [18, Conjecture 3(i)]) Let \(n\) be a positive integer. Then
\[
3S_n \equiv 0 \pmod{2n+3}. \tag{1.3}
\]

**Theorem 1.2** [19, Conjecture 1.3] Let \(n\) be a positive integer. Then
\[
21t_n \equiv 0 \pmod{10n+3}. \tag{1.8}
\]

We shall also give more congruences for \(S_n\) and \(t_n\) as follows.

**Theorem 1.3** Let \(n\) be a positive integer. Then
\[
\begin{align*}
105S_n &\equiv 0 \pmod{2n+5}, \tag{1.4} \\
315S_n &\equiv 0 \pmod{2n+7}, \tag{1.5} \\
6435S_n &\equiv 0 \pmod{2n+9}, \tag{1.6} \\
3003t_n &\equiv 0 \pmod{2n+1}, \tag{1.7} \\
88179t_n &\equiv 0 \pmod{10n+7}, \tag{1.8} \\
43263t_n &\equiv 0 \pmod{10n+9}. \tag{1.9}
\end{align*}
\]

Let \(Z\) denote the set of integers. Another result in this paper is the following.

**Theorem 1.4** Let \(a, b, m, n\) be positive integers. Then
\[
\frac{abm}{(a+b)(m+n)} \binom{am + bn}{am} \binom{an + bn}{an} = \frac{abm}{m+n} \binom{am + bn - 1}{am} \binom{an + bn}{an} \in Z. \tag{1.10}
\]

Letting \(a = b = 1\) in (1.10), we get the following result, of which a combinatorial interpretation was given by Gessel [9, Section 7].

**Corollary 1.5** Let \(m, n\) be positive integers. Then
\[
\frac{m}{2(m+n)} \binom{2m}{m} \binom{2n}{n} \in Z. \tag{1.11}
\]

In the next section, we give some lemmas. The proofs of Theorems 1.1–1.3 will be given in Sections 3–5 respectively. A proof of the \(q\)-analogue of Theorem 1.4 will be given in Section 6. We close our paper with some further remarks and open problems in Section 7.
2 Some lemmas

For the $p$-adic order of $n!$, there is a known formula

$$\text{ord}_p n! = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor,$$

(2.1)

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding $x$. In this section, we give some results on the floor function $\lfloor x \rfloor$.

Lemma 2.1 For any real number $x$, we have

$$\lfloor 6x \rfloor + \lfloor x \rfloor \geq \lfloor 3x \rfloor + 2 \lfloor 2x \rfloor,$$

(2.2)

$$\lfloor 15x \rfloor + \lfloor 2x \rfloor \geq \lfloor 10x \rfloor + \lfloor 4x \rfloor + \lfloor 3x \rfloor.$$

(2.3)

Proof. See [6, Theorem 1.1] and one of the 52 sporadic step functions given in [6, Table 2, line# 32]. □

Lemma 2.2 Let $m$ and $n$ be two positive integers such that $m|2n + 3$ and $m \geq 5$. Then

$$\left\lfloor \frac{6n}{m} \right\rfloor + \left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{3n}{m} \right\rfloor + 2 \left\lfloor \frac{2n}{m} \right\rfloor + 1.$$  

(2.4)

Proof. Let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of $x$. Then (2.4) is equivalent to

$$\{ \frac{6n}{m} \} + \{ \frac{n}{m} \} = \{ \frac{3n}{m} \} + 2 \{ \frac{2n}{m} \} - 1.$$  

(2.5)

Now suppose that $m|2n + 3$ and $m \geq 5$. We have

$$\{ \frac{2n}{m} \} = \frac{m - 3}{m} > \frac{1}{3}, \quad \text{and} \quad \left\lfloor \frac{2n}{m} \right\rfloor = \frac{2n + 3}{m} - 1 \equiv 0 \pmod{2}.$$  

It follows that

$$\{ \frac{6n}{m} \} = \begin{cases} \frac{2m - 9}{m}, & \text{if } m = 5, 7, \\ \frac{m - 9}{m}, & \text{if } m \geq 9, \end{cases}$$

$$\{ \frac{n}{m} \} = \frac{m - 3}{2m},$$

$$\{ \frac{3n}{m} \} = \begin{cases} \frac{3m - 9}{2m}, & \text{if } m = 5, 7, \\ \frac{m - 9}{2m}, & \text{if } m \geq 9. \end{cases}$$

Therefore, the identity (2.5) is true for any positive integer $m \geq 5$. □
Lemma 2.3 Let $m$ and $n$ be two positive integers such that $m|10n + 3$ and $m \geq 9$. Then

$$\left\lfloor \frac{15n}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor = \left\lfloor \frac{10n}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor + 1. \tag{2.6}$$

Proof. It is easy to see that (2.6) is equivalent to

$$\left\lfloor \frac{15n}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor = \left\lfloor \frac{10n}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor - 1. \tag{2.7}$$

Now suppose that $m|10n + 3$ and $m \geq 9$. We have

$$\left\lfloor \frac{10n}{m} \right\rfloor = \frac{m - 3}{m} \geq \frac{2}{3}, \quad \text{and} \quad A := \left\lfloor \frac{10n}{m} \right\rfloor = \frac{10n + 3}{m} - 1 \equiv 0, 2, 6, 8 \pmod{10}.$$

It is easy to check that

$$\left\lfloor \frac{15n}{m} \right\rfloor = \frac{m - 9}{2m},$$

$$\left\lfloor \frac{2n}{m} \right\rfloor, \left\lfloor \frac{4n}{m} \right\rfloor, \left\lfloor \frac{3n}{m} \right\rfloor = \begin{cases} \left( \frac{2m - 6}{10m}, \frac{4m - 12}{10m}, \frac{3m - 9}{10m} \right), & \text{if } A \equiv 0 \pmod{10}, \\ \left( \frac{6m - 6}{10m}, \frac{2m - 12}{10m}, \frac{9m - 9}{10m} \right), & \text{if } A \equiv 2 \pmod{10}, \\ \left( \frac{4m - 6}{10m}, \frac{8m - 12}{10m}, \frac{m - 9}{10m} \right), & \text{if } A \equiv 6 \pmod{10}, \\ \left( \frac{8m - 6}{10m}, \frac{6m - 12}{10m}, \frac{7m - 9}{10m} \right), & \text{if } A \equiv 8 \pmod{10}, \end{cases}$$

and so the identity (2.7) holds. \qed

3 Proofs of Theorem 1.1

First Proof. Let $\gcd(a, b)$ denote the greatest common divisor of two integers $a$ and $b$. For any positive integer $n$, since $\gcd(2n + 3, 4n + 2) = 1$, to prove Theorem 1.1, it is enough to show that

$$(2n + 3) \left\lfloor \frac{3 \binom{6n}{3m} \binom{3n}{n}}{\binom{2n}{n}} \right\rfloor. \tag{3.1}$$

By (2.1), for any odd prime $p$, the $p$-adic order of

$$\frac{\binom{6n}{3m} \binom{3n}{n}}{\binom{2n}{n}} = \frac{(2n + 2)!(6n)!(n)!}{(2n + 3)!(3n)!(2n)!^2}.$$
is given by
\[
\sum_{i=1}^{\infty} \left( \left\lfloor \frac{2n + 2}{p^i} \right\rfloor + \left\lfloor \frac{6n}{p^i} \right\rfloor + \left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{2n + 3}{p^i} \right\rfloor - \left\lfloor \frac{3n}{p^i} \right\rfloor - 2 \left\lfloor \frac{2n}{p^i} \right\rfloor \right). \tag{3.2}
\]
Note that
\[
\left\lfloor \frac{2n + 2}{p^i} \right\rfloor - \left\lfloor \frac{2n + 3}{p^i} \right\rfloor = \begin{cases} 
-1, & \text{if } p^i \mid 2n + 3, \\
0, & \text{otherwise.}
\end{cases}
\]
By Lemmas 2.1 and 2.2, for \( p \geq 5 \), the summation (3.2) is clearly greater than or equal to 0. For \( p = 3 \), we have (3.2) \( \geq -1 \) because if the positive integer \( i \) satisfies \( 3^i \mid 2n + 3 \) and \( 3^i < 5 \) then we must have \( i = 1 \). This proves that
\[
\frac{3 \binom{6n}{3n}}{(2n + 3) \binom{3n}{n}}
\]
is always an integer. Hence (3.1) holds.

**Second Proof** (provided by T. Amdeberhan and V.H. Moll). Replacing \( n \) by \( n + 1 \) in (1.1), we see that (after some rearrangement)
\[
\frac{\binom{6n+3}{3n}}{2(2n+3) \binom{3n+2}{n+1}} = \frac{6(6n+5)(6n+1)S_n}{(n+1)(2n+3)} \in \mathbb{Z}.
\]
Hence, \((2n + 3)\binom{6n+5}{3n+1}(6n + 1)S_n\). Since \( \gcd(2n + 3, 2) = \gcd(2n + 3, 6n + 5) = \gcd(2n + 3, 6n + 1) = 1 \), we must have \((2n + 3)\mid 3S_n\).

**Remark.** Z.-W. Sun [18, Conjecture 3(i)] also conjectured that \( S_n \) is odd if and only if \( n \) is a power of 2. After reading a previous version of this paper, Quan-Hui Yang told me that it is easy to show that \( \text{ord}_2((6n)!n!/(3n)!(2n)!^2) \) equals the number of 1’s in the binary expansion of \( n \) by noticing
\[
\text{ord}_2(6n)! = 3n + \text{ord}_2(3n)!, \quad \text{ord}_2(2n)! = n + \text{ord}_2n!,
\]
and using Legendre’s theorem. T. Amdeberhan and V.H. Moll also pointed out this.

## 4 Proof of Theorem 1.2

For any positive integer \( n \), since \( \gcd(10n + 3, 10n + 1) = 1 \), to prove Theorem 1.2, it is enough to show that
\[
(10n + 3) \left\lfloor \frac{21 \binom{15n}{5n-1}}{(3n)} \right\rfloor. \tag{4.1}
\]
Furthermore, since \(\gcd(10n+3, 5) = 1\) and \(\binom{5n}{n} = 5\binom{n-1}{n-1}\), one sees that (4.1) is equivalent to

\[
(10n + 3) \left\lfloor \frac{21 \binom{15n}{5n}}{\binom{3n}{n}} \right\rfloor.
\]  

(4.2)

By (2.1), for any odd prime \(p\), the \(p\)-adic order of

\[
\frac{\binom{15n}{5n} \binom{3n}{n}}{(10n + 3) \binom{3n}{n}} = \frac{(10n + 2)! (15n)! (2n)!}{(10n + 3)! (10n)! (4n)! (3n)!}
\]

is given by

\[
\sum_{i=1}^{\infty} \left( \left\lfloor \frac{10n + 2}{p^i} \right\rfloor + \left\lfloor \frac{15n}{p^i} \right\rfloor + \left\lfloor \frac{2n}{p^i} \right\rfloor - \left\lfloor \frac{10n + 3}{p^i} \right\rfloor - \left\lfloor \frac{4n}{p^i} \right\rfloor - \left\lfloor \frac{3n}{p^i} \right\rfloor \right).
\]  

(4.3)

Note that

\[
\left\lfloor \frac{10n + 2}{p^i} \right\rfloor - \left\lfloor \frac{10n + 3}{p^i} \right\rfloor = \begin{cases} -1, & \text{if } p^i | 10n + 3, \\ 0, & \text{otherwise}. \end{cases}
\]

By Lemmas 2.1 and 2.3, for \(p \geq 11\), or \(p = 5\), the summation (4.3) is clearly greater than or equal to 0. For \(p = 3, 7\), we have (4.3) \(\geq -1\) because there is at most one index \(i \geq 1\) satisfying \(p^i | 10n + 3\) and \(p^i < 9\) in this case. This proves that

\[
\frac{21 \binom{15n}{5n}}{(10n + 3) \binom{3n}{n}}
\]

is always an integer. Namely, (4.2) is true.

## 5 Proof of Theorem 1.3

**Lemma 5.1** Let \(m\) and \(n\) be two positive integers. Then

\[
\left\lfloor \frac{6n}{m} \right\rfloor + \left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{3n}{m} \right\rfloor + 2 \left\lfloor \frac{2n}{m} \right\rfloor + 1,
\]

(5.1)

if \(m|2n + 5\) and \(m \geq 9\), or \(m|2n + 7\) and \(m \geq 11\), or \(m|2n + 9\) and \(m \geq 15\).

**Proof.** The proof is similar to that of Lemma 2.2. We only consider the case when \(m|2n + 5\) and \(m \geq 9\). In this case, we have

\[
\left\lfloor \frac{2n}{m} \right\rfloor = \frac{m - 5}{m} > \frac{1}{3}, \quad \text{and} \quad \left\lfloor \frac{2n}{m} \right\rfloor = \frac{2n + 5}{m} - 1 \equiv 0 \pmod{2}.
\]
It follows that
\[
\begin{align*}
\left\{ \frac{6n}{m} \right\} &= \begin{cases} 
\frac{2m - 15}{m}, & \text{if } m = 9, 11, 13, \\
\frac{m - 15}{m}, & \text{if } m \geq 15,
\end{cases} \\
\left\{ \frac{n}{m} \right\} &= \frac{m - 5}{2m} , \\
\left\{ \frac{3n}{m} \right\} &= \begin{cases} 
\frac{3m - 15}{2m}, & \text{if } m = 9, 11, 13, \\
\frac{m - 15}{2m}, & \text{if } m \geq 15,
\end{cases}
\end{align*}
\]
and so
\[
\left\{ \frac{6n}{m} \right\} + \left\{ \frac{n}{m} \right\} = \left\{ \frac{3n}{m} \right\} + 2 \left\{ \frac{2n}{m} \right\} - 1.
\]
This proves (5.1).

**Lemma 5.2** Let \(m\) and \(n\) be two positive integers. Then
\[
\left\lfloor \frac{15n}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor = \left\lfloor \frac{10n}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor + 1, \tag{5.2}
\]
if \(m|2n + 1\) and \(m \geq 15\), or \(m|10n + 7\) and \(m \geq 21\), or \(m|10n + 9\) and \(m \geq 27\).

**Proof.** The proof is similar to that of Lemma 2.3. We only consider the case when \(m|10n + 9\) and \(m \geq 27\). In this case, we have
\[
\left\{ \frac{10n}{m} \right\} = \frac{m - 9}{m} \geq \frac{2}{3}, \quad \text{and} \quad A := \left\lfloor \frac{10n}{m} \right\rfloor = \frac{10n + 9}{m} - 1 \equiv 0, 2, 6, 8 \pmod{10}.
\]
It follows that
\[
\left\{ \frac{15n}{m} \right\} = \frac{m - 27}{2m},
\]
and so
\[
\left( \left\{ \frac{2n}{m} \right\}, \left\{ \frac{4n}{m} \right\}, \left\{ \frac{3n}{m} \right\} \right) = \begin{cases} 
\left( \frac{2m - 18}{10m}, \frac{4m - 36}{10m}, \frac{3m - 27}{10m} \right), & \text{if } A \equiv 0 \pmod{10}, \\
\left( \frac{6m - 18}{10m}, \frac{2m - 36}{10m}, \frac{9m - 27}{10m} \right), & \text{if } A \equiv 2 \pmod{10}, \\
\left( \frac{4m - 18}{10m}, \frac{8m - 36}{10m}, \frac{m - 27}{10m} \right), & \text{if } A \equiv 6 \pmod{10}, \\
\left( \frac{8m - 18}{10m}, \frac{6m - 36}{10m}, \frac{7m - 27}{10m} \right), & \text{if } A \equiv 8 \pmod{10}.
\end{cases}
\]
Hence,
\[
\left\{ \frac{15n}{m} \right\} + \left\{ \frac{2n}{m} \right\} = \left\{ \frac{10n}{m} \right\} + \left\{ \frac{4n}{m} \right\} + \left\{ \frac{3n}{m} \right\} - 1,
\]
which means that (5.2) holds.

Proof of Theorem 1.3. Since the proofs of the congruences (1.4)–(1.9) are similar in view of Lemmas 5.1 and 5.2, we only give proofs of (1.5) and (1.9). Noticing that \(\gcd(2n + 1, 2n + 7) = 1\) or 3, to prove (1.5), it suffices to show that
\[
(2n + 7) \left| \frac{105(6n)(3n)}{(2n)_n} \right.
\]
(5.3)
Let
\[
X_n := \frac{(6n)(3n)}{(2n + 7)(2n)_n} = \frac{(2n + 6)!(6n)!(n)!}{(2n + 7)!(3n)!(2n)!}.
\]
By (2.1), for any odd prime \(p\), we have
\[
\text{ord}_p X_n = \sum_{i=1}^{\infty} \left( \left\lfloor \frac{2n + 6}{p^i} \right\rfloor + \left\lfloor \frac{6n}{p^i} \right\rfloor + \left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{2n + 7}{p^i} \right\rfloor - \left\lfloor \frac{3n}{p^i} \right\rfloor - 2 \left\lfloor \frac{2n}{p^i} \right\rfloor \right).
\]
Note that (5.1) is also true for \(m = 3\) and \(n \equiv 1 \pmod{3}\), and
\[
\left\lfloor \frac{2n + 6}{p^i} \right\rfloor - \left\lfloor \frac{2n + 7}{p^i} \right\rfloor = \begin{cases} -1, & \text{if } p^i \div|2n + 7, \\ 0, & \text{otherwise}. \end{cases}
\]
By Lemmas 2.1 and 5.1, we obtain
\[
\begin{cases} \text{ord}_p X_n \geq 0, & \text{if } p \geq 11, \\ \text{ord}_p X_n \geq -1, & \text{if } p = 3, 5, 7. \end{cases}
\]
This proves (5.3).

Similarly, since \(\gcd(10n + 9, 10n + 1) = \gcd(10n + 9, 5) = 1\), the congruence (1.9) is equivalent to
\[
(10n + 9) \left| \frac{43263(15n)(5n)}{(10n)_n(3n)_n} \right.
\]
(5.4)
Let
\[
Y_n := \frac{(15n)(5n)}{(10n + 9)(3n)_n} = \frac{(10n + 8)!(15n)!(2n)!}{(10n + 9)!(10n)!(4n)!(3n)!}.
\]
Then, for any odd prime \( p \), \( \text{ord}_p Y_n \) is given by

\[
\sum_{i=1}^{\infty} \left( \left\lfloor \frac{10n + 8}{p^i} \right\rfloor + \left\lfloor \frac{15n + 1}{p^i} \right\rfloor + \left\lfloor \frac{2n}{p^i} \right\rfloor - \left\lfloor \frac{10n + 9}{p^{i+1}} \right\rfloor - \left\lfloor \frac{10n}{p^{i+1}} \right\rfloor - \left\lfloor \frac{4n}{p^{i+1}} \right\rfloor - \left\lfloor \frac{3n}{p^{i+1}} \right\rfloor \right).
\]

Note that (5.2) also holds for \( m = 7, 13, 17 \) and any positive integer \( n \) such that \( m|10n+9 \).

Similarly as before, we have

\[
\begin{cases}
\text{ord}_p Y_n \geq 0, & \text{if } p = 5, 7, 13, 17, \text{ or } p \geq 29, \\
\text{ord}_p Y_n \geq -1, & \text{if } p = 11, 19, 23, \\
\text{ord}_p Y_n \geq -2, & \text{if } p = 3.
\end{cases}
\]

Observing that \( 43263 = 3^2 \cdot 11 \cdot 19 \cdot 23 \), we complete the proof of (5.4).

6 A \( q \)-analogue of Theorem 1.4

Recall that the \( q \)-binomial coefficients are defined by

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{(1 - q^n)(1 - q^{n-1})\cdots(1 - q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}, \quad \text{if } 0 \leq k \leq n, \\
0, \quad \text{otherwise}.
\]

We begin with the announced strengthening of Theorem 1.4.

**Theorem 6.1** Let \( a, b, m, n \geq 1 \). Then

\[
\frac{1 - q^{\text{gcd}(am,m+n)}}{1 - q^{m+n}} \left[ \begin{array}{c} am + bm - 1 \\ am \end{array} \right]_q \left[ \begin{array}{c} an + bn \\
\end{array} \right]_q
\]

is a polynomial in \( q \) with non-negative integer coefficients.

**Corollary 6.2** Let \( a, b, m, n \geq 1 \). Then

\[
\frac{1 - q^{am}}{1 - q^{m+n}} \left[ \begin{array}{c} am + bm - 1 \\ am \end{array} \right]_q \left[ \begin{array}{c} an + bn \\
\end{array} \right]_q
\]

is a polynomial in \( q \) with non-negative integer coefficients.

It is easily seen that Theorem 1.4 can be obtained upon letting \( q \to 1 \) in Corollary 6.2. Moreover, when \( a = b = m = 1 \), the numbers (6.2) reduce to the \( q \)-Catalan numbers

\[
C_n(q) = \frac{1 - q}{1 - q^{2n+1}} \left[ \begin{array}{c} 2n \\ n \end{array} \right]_q.
\]

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It is well known that the \( q \)-Catalan numbers \( C_n(q) \) are polynomials with non-negative integer coefficients (see [2, 3, 5, 7]). There are many different \( q \)-analogues of the Catalan numbers (see F¨urlinger and Hofbauer [7]). For the so-called \( q, t \)-Catalan numbers, see [8, 13, 12].

Recall that a polynomial \( P(q) = \sum_{i=0}^{d} p_i q^i \) in \( q \) of degree \( d \) is called \textit{reciprocal} if \( p_i = p_{d-i} \) for all \( i \), and that it is called \textit{unimodal} if there is an integer \( r \) with \( 0 \leq r \leq d \) and \( 0 \leq p_0 \leq \cdots \leq p_r \geq \cdots \geq p_d \geq 0 \). An elementary but crucial property of reciprocal and unimodal polynomials is the following.

\textbf{Lemma 6.3} If \( A(q) \) and \( B(q) \) are reciprocal and unimodal polynomials, then so is their product \( A(q)B(q) \).

Lemma 6.3 is well known and its proof can be found, e.g., in [1] or [16, Proposition 1].

Similarly to the proof of [11, Theorem 3.1], the following lemma plays an important role in the proof of Theorem 6.1. It is a slight generalization of [14, Proposition 10.1.(iii)], which extracts the essentials out of Andrews [4, Proof of Theorem 2].

\textbf{Lemma 6.4} Let \( P(q) \) be a reciprocal and unimodal polynomial and \( m \) and \( n \) positive integers with \( m \leq n \). Furthermore, assume that \( A(q) = \frac{1-q^n}{1-q^m} P(q) \) is a polynomial in \( q \). Then \( A(q) \) has non-negative coefficients.

\textit{Proof.} See [11, Lemma 5.1]. \hfill \Box

\textit{Proof of Theorem 6.1.} It is well known that the \( q \)-binomial coefficients are reciprocal and unimodal polynomials in \( q \) (cf. [17, Ex. 7.75.d]), and by Lemma 6.3, so is the product of two \( q \)-binomial coefficients. In view of Lemma 6.4, for proving Theorem 6.1 it is enough to show that the expression (6.1) is a polynomial in \( q \). We shall accomplish this by a count of cyclotomic polynomials.

Recall the well-known fact that

\[ q^n - 1 = \prod_{d|n} \Phi_d(q), \]

where \( \Phi_d(q) \) denotes the \( d \)-th cyclotomic polynomial in \( q \). Consequently,

\[ \frac{1-q^{\gcd(am,m+n)}}{1-q^{m+n}} \left[ am + bm - 1 \right]_{q^a} \left[ an + bn \right]_{q^a} = \prod_{d=2}^{\min\{am+bm-1, an+bn\}} \Phi_d(q)^{e_d}, \]

with

\[ e_d = \chi(d \mid \gcd(am, m+n)) - \chi(d \mid m+n) + \left\lfloor \frac{am + bm - 1}{d} \right\rfloor + \left\lfloor \frac{an + bn}{d} \right\rfloor - \left\lfloor \frac{am}{d} \right\rfloor - \left\lfloor \frac{bm - 1}{d} \right\rfloor - \left\lfloor \frac{an}{d} \right\rfloor - \left\lfloor \frac{bn}{d} \right\rfloor, \tag{6.3} \]
where $\chi(S) = 1$ if $S$ is true and $\chi(S) = 0$ otherwise. This is clearly non-negative, unless $d | m + n$ and $d \nmid \gcd(am, m + n)$.

So, let us assume that $d | m + n$ and $d \nmid \gcd(am, m + n)$, which means that $d \nmid am$ and therefore

$$\left\lfloor \frac{am + bm - 1}{d} \right\rfloor + \left\lfloor \frac{an + bn}{d} \right\rfloor = \frac{(a + b)(m + n)}{d} - 1,$$

$$\left\lfloor \frac{am}{d} \right\rfloor + \left\lfloor \frac{an}{d} \right\rfloor = \frac{a(m + n)}{d} - 1,$$

$$\left\lfloor \frac{bm - 1}{d} \right\rfloor + \left\lfloor \frac{bn}{d} \right\rfloor = \frac{b(m + n)}{d} - 1,$$

and so $e_d = 0$ is also non-negative in this case. This completes the proof of polynomiality of (6.1). □

Proof of Corollary 6.2. This follows immediately from Theorem 6.1 and the fact that $\gcd(am, m + n) | am$. □

7 Concluding remarks and open problems

On January 2, 2014 T. Amdeberhan and V.H. Moll (personal communication) found the following generalization of Theorem 1.1, which was soon proved by Q.-H. Yang [21] and C. Krattenthaler.

Conjecture 7.1 Let $a, b$ and $n$ be positive integers with $a > b$. Then

$$(2bn + 1)(2bn + 3) \left( \begin{array}{c} 2bn \\ bm \end{array} \right) \left| \frac{3(a - b)(3a - b)}{an} \left( \begin{array}{c} an \\ bn \end{array} \right) \right..$$

Let $[m!] = (1 - q) \cdots (1 - q^m)$. By a result of Warnaar and Zudilin [20, Proposition 3], one sees that, for any positive integer $n$, the polynomial

$$\frac{[6n]!}{[3n]![2n]!^2}$$

has non-negative integer coefficients. Similarly as before, we can prove the following generalization of congruences (1.3)–(1.5).

Theorem 7.2 Let $n$ be a positive integer. Then all of

$$\frac{(1 - q)[6n]![n]!}{(1 - q^{2n+1})[3n]![2n]!^2}, \quad \frac{(1 - q^3)[6n]![n]!}{(1 - q^{2n+3})[3n]![2n]!^2}, \quad \frac{(1 - q)(1 - q^3)[6n]![n]!}{(1 - q^{2n+1})(1 - q^{2n+3})[3n]![2n]!^2},$$

$$\frac{(1 - q^2)(1 - q^6)(1 - q^7)[6n]![n]!}{(1 - q^{2n+3})(1 - q^{2n+5})(1 - q^{2n+7})[3n]![2n]!^2} \quad (n \geq 2),$$

$$\frac{(1 - q^3)^2(1 - q^6)(1 - q^7)[6n]![n]!}{(1 - q^{2n+1})(1 - q^{2n+3})(1 - q^{2n+5})(1 - q^{2n+7})[3n]![2n]!^2} \quad (n \geq 2),$$

are polynomials in $q$. 

THE ELECTRONIC JOURNAL OF COMBINATORICS 21(2) (2014), #P2.54

11
We have the following two related conjectures.

**Conjecture 7.3** All the polynomials in Theorem 7.2 have non-negative integer coefficients.

**Conjecture 7.4** Let $n \geq 2$. Then the polynomial $\frac{[6n]! [2n]!}{[3n]! [2n]!^2}$ is unimodal.

It is obvious that the polynomial $\frac{[6n]! [2n]!}{[3n]! [2n]!^2}$ is reciprocal. If Conjecture 7.4 is true, then, applying Lemma 6.3, we conclude that the first two polynomials in Theorem 7.2 have non-negative integer coefficients.

It was conjectured by Warnaar and Zudilin (see [20, Conjecture 1]) that

\[
\frac{[15n]! [2n]!}{[10n]! [4n]! [3n]!}
\]

has non-negative integer coefficients. Similarly, we have the following generalization of Theorem 1.2.

**Theorem 7.5** Let $n$ be a positive integer. Then both

\[
\frac{(1 - q)[15n]! [2n]!}{(1 - q^{10n+1})[10n]! [4n]! [3n]!}, \quad \text{and} \quad \frac{(1 - q^3)(1 - q^7)[15n]! [2n]!}{(1 - q)(1 - q^{10n+3})[10n]! [4n]! [3n]!}
\]

are polynomials in $q$.

We end the paper with the following conjecture, strengthening the above theorem.

**Conjecture 7.6** The two polynomials in Theorem 7.5 have non-negative integer coefficients.

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