Kinetic equations & Boltzmann equation

Step 1-4 are same as that for Grad's moment system.

Project the convection part:

Project the time and space derivative:

Multiply velocity:

Grad's expansion:

Orthogonal polynomial expansion:

Project the time derivatives:

Gro's Moment Method

Procedure of deriving Grad's moment system (collision term is neglected):

1. Orthogonal polynomial expansion:
   \[
   f(t, x, \xi) = \sum_{\alpha \in \mathbb{N}^d} a_{\alpha} f_{\alpha}(t, x, \xi),
   \]
   where \( f_{\alpha}(t, x, \xi) \) is a set of orthogonal basis.

2. Orthogonal projection:
   \[
   p_{\alpha} = \left\{ f_{\alpha}(t, x, \xi) \mid \alpha \in \mathbb{N}^d \right\}, \quad p_{\alpha} = \{ f_{\alpha}(t, x, \xi) \mid |\alpha| \leq M \}, \quad p_{\alpha} = \left\{ f_{\alpha}(t, x, \xi) \mid \alpha \in \mathbb{N}^d \right\},
   \]

3. Grad's expansion:
   \[
   p_{\alpha}(t, x, \xi) = \sum_{\alpha \in \mathbb{N}^d} a_{\alpha} f_{\alpha}(t, x, \xi).
   \]

4. Calculate time and space derivative:
   \[
   \frac{\partial p_{\alpha}}{\partial t} = \sum_{\alpha \in \mathbb{N}^d} a_{\alpha} \frac{\partial f_{\alpha}}{\partial t}.
   \]

5. Project the time derivatives:
   \[
   \frac{\partial p_{\alpha}}{\partial t} = \sum_{\alpha \in \mathbb{N}^d} a_{\alpha} \frac{\partial f_{\alpha}}{\partial t}.
   \]

6. Multiply velocity:
   \[
   \xi \cdot \nabla p_{\alpha} = \sum_{\alpha \in \mathbb{N}^d} a_{\alpha} \xi \cdot \nabla f_{\alpha}.
   \]

7. Project the convection part:
   \[
   p_{\alpha} \xi \cdot \nabla p_{\alpha} = \sum_{\alpha \in \mathbb{N}^d} a_{\alpha} \xi \cdot \nabla f_{\alpha}.
   \]

8. Matching the coefficients of basis functions:
   \[
   g_{\alpha}^{(1)} = \sum_{\alpha \in \mathbb{N}^d} a_{\alpha} = 0, \quad |\alpha| \leq M.
   \]

   Quasi-linear form:
   \[
   \frac{\partial g_{\alpha}}{\partial t} + A_{\alpha} g_{\alpha} = 0.
   \]

Grad's Moment System

Hyperbolicity of Grad's moment system

Hyperbolicity of Grad's 13 and 20 moment system

First-order PDEs

\[
\frac{\partial U}{\partial t} + A(U) \cdot \frac{\partial U}{\partial x} = 0, \quad U \in \mathbb{R}^n
\]

is called hyperbolic in \( \mathbb{R}^n \) if \( A(U) \) is real diagonalizable for any \( U \in \mathbb{R}^n \).

Grad's moment system (

Grad's moment system is NOT hyperbolic.

Diagram of Grad's moment system

Diagram the procedure of deriving Grad's moment system as following:

Time derivative

Convection term

Even around the equilibrium, Grad's moment system is NOT hyperbolic.

Model Reduction of Kinetic Equations by Operator Projection

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Conclusion & References

Grad's moment system is not hyperbolic even around the Maxwellian.

A globally hyperbolic regularization for Grad's moment system is proposed. The problem that loss of hyperbolicity is essentially fixed.

Based on the regularization, a general framework to derive globally hyperbolic system from kinetic equations is proposed.

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