Statistical properties of Pauli matrices going through noisy channels

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Abstract

We study the statistical properties of the triplet \((\sigma_x, \sigma_y, \sigma_z)\) of Pauli matrices going through a sequence of noisy channels, modeled by the repetition of a general, trace-preserving, completely positive map. We show a non-commutative central limit theorem for the distribution of this triplet, which shows up a 3-dimensional Brownian motion in the limit with a non-trivial covariance matrix. We also prove a large deviation principle associated to this convergence, with an explicit rate function depending on the stationary state of the noisy channel.

1 Introduction

In quantum information theory one of the most important question is to understand and to control the way a quantum bit is modified when transmitted through a quantum channel. It is well-known that realistic transmission channels are not perfect and that they distort the quantum bit they transmit. This transformation of the quantum state is represented by the action of a completely positive map. These are the so-called noisy channels.

The purpose of this article is to study the action of the repetition of a general completely positive map on basic observables. Physically, this model can be thought of as the sequence of transformations of small identical

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pieces of noisy channels on a qubit. It can also be thought of as a discrete approximation of the more realistic model of a quantum bit going through a semigroup of completely positive maps (a Lindblad semigroup).

As basic observables, we consider the triplet \((\sigma_x, \sigma_y, \sigma_z)\) of Pauli matrices. Under the repeated action of the completely positive map, they behave as a 3-dimensional quantum random walk. The aim of this article is to study the statistical properties of this quantum random walk.

Indeed, for any initial density matrix \(\rho_{in}\), we study the statistical properties of the empirical average of the Pauli matrices in the successive states \(\Phi^n(\rho_{in}), n \geq 0\) where \(\Phi\) is some completely positive and trace-preserving map describing our quantum channel. Quantum Bernoulli random walks studied by Biane in [1] corresponds to the case where \(\Phi\) is the identity map. Biane [1] proved an invariance principle for this quantum random walk when \(\rho_{in} = \frac{1}{2}I\).

This article is organized as follows. In section two we describe the physical and mathematical setup. In section three we establish a functional central limit theorem for the empirical average of the quantum random walk associated to the Pauli matrices generalizing Biane’s result [1]. This central limit theorem involves a 3-dimensional Brownian motion in the limit, whose covariance matrix is non-trivial and depends explicitly on the stationary state of the noisy channel. In section four, we apply our central limit theorem to some explicit cases, in particular to the King-Ruskai-Szarék-Werner representation of completely positive and trace-preserving maps in \(M_2(\mathbb{C})\). This allows us to compute the limit Brownian motion for the most well-know quantum channels: the depolarizing channel, the phase-damping channel, the amplitude-damping channel. Finally, in the last section, a large deviation principle for the empirical average is proved.

2 Model and notations

Let \(M_2(\mathbb{C})\) be the set of \(2 \times 2\) matrices with complex coefficients. The set of \(2 \times 2\) self-adjoint matrices forms a four dimensional real vector subspace of \(M_2(\mathbb{C})\). A convenient basis \(B\) is given by the following matrices

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

where \(\sigma_x, \sigma_y, \sigma_z\) are the traditional Pauli matrices, they satisfy the commutation relations: \([\sigma_x, \sigma_y] = 2i\sigma_z\), and those obtained by cyclic permutations
of $\sigma_x$, $\sigma_y$, $\sigma_z$. A state on $M_2(\mathbb{C})$ is given by a density matrix (i.e. a positive semi-definite matrix with trace one) which we will suppose to be of the form

$$\rho = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & 1 - \alpha \end{pmatrix}$$

where $0 \leq \alpha \leq 1$ and $|\beta|^2 \leq \alpha(1-\alpha)$. The noise coming from interactions between the qubit states and the environment is represented by the action of a completely positive and trace-preserving map $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$.

Let $M_1$, $M_2$, $\ldots$, $M_k$, $\ldots$ be infinitely many copies of $M_2(\mathbb{C})$. For each given state $\rho$, we consider the algebra

$$\mathcal{M}_\rho = M_1 \otimes M_2 \otimes \ldots \otimes M_k \otimes \ldots$$

where the product is taken in the sense of $W^*$-algebra with respect to the product state

$$\omega = \rho \otimes \Phi(\rho) \otimes \Phi^2(\rho) \otimes \ldots \otimes \Phi^k(\rho) \otimes \ldots.$$ 

Our main hypothesis is the following. We assume that for any state $\rho$, the sequence $\Phi^n(\rho)$ converges to a stationary state $\rho_\infty$, which we write as

$$\rho_\infty = \begin{pmatrix} \alpha_\infty & \beta_\infty \\ \bar{\beta}_\infty & 1 - \alpha_\infty \end{pmatrix}$$

where $0 \leq \alpha_\infty \leq 1$ and $|\beta_\infty|^2 \leq \alpha_\infty(1-\alpha_\infty)$.

Put

$$v_1 = 2 \text{Re}(\beta_\infty), v_2 = -2 \text{Im}(\beta_\infty), v_3 = 2\alpha_\infty - 1.$$ 

For every $k \geq 1$, we define

$$x_k = I \otimes \ldots \otimes I \otimes (\sigma_x - v_1 I) \otimes I \otimes \ldots$$

$$y_k = I \otimes \ldots \otimes I \otimes (\sigma_y - v_2 I) \otimes I \otimes \ldots$$

$$z_k = I \otimes \ldots \otimes I \otimes (\sigma_z - v_3 I) \otimes I \otimes \ldots$$

where each $(\sigma_i - v_i I)$ appears on the $k^{th}$ place.

For every $n \geq 1$, put

$$X_n = \sum_{k=1}^{n} x_k, \quad Y_n = \sum_{k=1}^{n} y_k, \quad Z_n = \sum_{k=1}^{n} z_k$$
with initial conditions
\[ X_0 = Y_0 = Z_0 = 0. \]

The integer part of a real \( t \) is denoted by \([t]\). To each process we associate a continuous time normalized process denoted by
\[
X_t^{(n)} = n^{-1/2}X_{[nt]}, \quad Y_t^{(n)} = n^{-1/2}Y_{[nt]}, \quad Z_t^{(n)} = n^{-1/2}Z_{[nt]}.
\]

\section{A central limit theorem}

The aim of our article is to study the asymptotical properties of the quantum process \((X_t^{(n)}, Y_t^{(n)}, Z_t^{(n)})\) when \(n\) goes to infinity. This process being truly non-commutative, there is no hope to obtain an asymptotic behaviour in the classical sense.

For any polynomial \(P = P(X_1, X_2, \ldots, X_m)\) of \(m\) variables, we denote by \(\hat{P}\) the totally symmetrized polynomial of \(P\) obtained by symmetrizing each monomial in the following way:
\[
X_{i_1}X_{i_2} \cdots X_{i_k} \longrightarrow \frac{1}{k!} \sum_{\sigma \in S_k} X_{i_{\sigma(1)}} \cdots X_{i_{\sigma(k)}},
\]
where \(S_k\) is the group of permutations of \(\{1, \ldots, k\}\).

\textbf{Theorem 3.1.} Assume that
\[
(A) \quad \Phi^n(\rho) = \rho_\infty + o\left(\frac{1}{\sqrt{n}}\right).
\]

Then, for any polynomial \(P\) of \(3m\) variables, for any \((t_1, \ldots, t_m)\) such that \(0 \leq t_1 < t_2 < \ldots < t_m\), the following convergence holds:
\[
\lim_{n \to +\infty} w \left[ \hat{P}(X_t^{(n)}, Y_t^{(n)}, Z_t^{(n)}, \ldots, X_{t_m}^{(n)}, Y_{t_m}^{(n)}, Z_{t_m}^{(n)}) \right] = E \left[ P(B_{t_1}^{(1)}, B_{t_2}^{(2)}, B_{t_1}^{(3)}, \ldots, B_{t_m}^{(1)}, B_{t_m}^{(2)}, B_{t_m}^{(3)}) \right]
\]
where \((B_{t}^{(1)}, B_{t}^{(2)}, B_{t}^{(3)})_{t \geq 0}\) is a three-dimensional centered Brownian motion with covariance matrix \(C_t\), where
\[
C = \begin{pmatrix}
1 - v_1^2 & -v_1 v_2 & -v_1 v_3 \\
-v_1 v_2 & 1 - v_2^2 & -v_2 v_3 \\
-v_1 v_3 & -v_2 v_3 & 1 - v_3^2
\end{pmatrix}.
\]
Remark:

Theorem 3.1 has to be compared with the quantum central limit theorem obtained in [5] and [9]. In our case, the state under which the convergence holds does not need to be an infinite tensor product of states. We also give here a functional version of the central limit theorem. Finally, in [5] (see Remark 3 p.131), the limit is described as a so-called quasi-free state in quantum mechanics. We prove in Theorem 3.1 that the limit is real Gaussian for the class of totally symmetrized polynomials.

Proof:

Let \( m \geq 1 \) and \((t_0, t_1, \ldots, t_m)\) such that \( t_0 = 0 < t_1 < t_2 < \ldots < t_m \). The polynomial \( P(X_{t_1}^{(n)}, Y_{t_1}^{(n)}, Z_{t_1}^{(n)}, \ldots, X_{t_m}^{(n)}, Y_{t_m}^{(n)}, Z_{t_m}^{(n)}) \) can be rewritten as a polynomial function \( Q \) of the increments: \( X_{t_1}^{(n)} - X_{t_0}^{(n)}, Y_{t_1}^{(n)} - Y_{t_0}^{(n)}, Z_{t_1}^{(n)} - Z_{t_0}^{(n)}, \ldots, X_{t_m}^{(n)} - X_{t_{m-1}}^{(n)}, Y_{t_m}^{(n)} - Y_{t_{m-1}}^{(n)}, Z_{t_m}^{(n)} - Z_{t_{m-1}}^{(n)} \).

A monomial of \( Q \) is a product of the form \( A_{i_1} \ldots A_{i_k} \) for some distinct \( i_1, \ldots, i_k \) in \( \{1, \ldots, m\} \) where \( A_i \) is a product depending only on the increments \( X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)}, Y_{t_i}^{(n)} - Y_{t_{i-1}}^{(n)}, Z_{t_i}^{(n)} - Z_{t_{i-1}}^{(n)} \). Since the \( A_i \)'s are commuting variables, the totally symmetrized polynomial of the monomial \( A_{i_1} \ldots A_{i_k} \) is equal to the product \( \widehat{A}_{i_1} \ldots \widehat{A}_{i_k} \). Consequently, it is enough to prove the theorem for any polynomial \( A_i \).

Let \( i \geq 1 \) fixed, for every \( \nu_1, \nu_2, \nu_3 \in \mathbb{R} \), we begin by determining the asymptotic distribution of the linear combination

\[
(\nu_1^2 + \nu_2^2 + \nu_3^2)^{-1/2} \left( \nu_1 (X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)}) + \nu_2 (Y_{t_i}^{(n)} - Y_{t_{i-1}}^{(n)}) + \nu_3 (Z_{t_i}^{(n)} - Z_{t_{i-1}}^{(n)}) \right) \tag{1}
\]

which can be rewritten as

\[
\frac{1}{\sqrt{n}} \sum_{k=[m_{i-1}]+1}^{[m_i]} \left( \frac{\nu_1 x_k + \nu_2 y_k + \nu_3 z_k}{\sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}} \right).
\]

Consider the matrix

\[
A = \frac{1}{\sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}} \begin{pmatrix}
\nu_1 (\sigma_x - v_1 I) + \nu_2 (\sigma_y - v_2 I) + \nu_3 (\sigma_z - v_3 I) \\
-\nu_1 v_1 - \nu_2 v_2 + \nu_3 (1 - v_3) & \nu_1 - i \nu_2 \\
\nu_1 + i \nu_2 & -\nu_1 v_1 - \nu_2 v_2 - \nu_3 (1 + v_3)
\end{pmatrix}
\]

which we denote by

\[
\begin{pmatrix}
a_1 & a_3 \\
\bar{a}_3 & a_2
\end{pmatrix},
\]
\(a_1, a_2 \in \mathbb{R}, a_3 \in \mathbb{C}\).

From assumption (A) we can write, for every \(n \geq 0\)
\[
\Phi^n(\rho) = \begin{pmatrix}
\alpha_\infty + \phi_n(1) & \beta_\infty + \phi_n(2) \\
\bar{\beta}_\infty + \phi_n(3) & 1 - \alpha_\infty + \phi_n(4)
\end{pmatrix}
\]
where each sequence \(\phi_n(i)\) satisfies: \(\phi_n(i) = o(1/\sqrt{n})\).

Let \(k \geq 1\), the expectation and the variance of \(A\) in the state \(\Phi^k(\rho)\) are respectively equal to
\[
\text{Trace}(A \Phi^k(\rho))
\]
and
\[
\text{Trace}(A^2 \Phi^k(\rho)) - \text{Trace}(A \Phi^k(\rho))^2.
\]

If both following conditions are satisfied:
\[
\sum_{k=[nt_i]}^{[nt_i+1]} \text{Trace}(A \Phi^k(\rho)) = o(\sqrt{n}) \tag{2}
\]
and
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=[nt_{i-1}]+1}^{[nt_{i-1}]} [\text{Trace}(A^2 \Phi^k(\rho)) - \text{Trace}(A \Phi^k(\rho))^2] = a(t_i - t_{i-1}) \tag{3}
\]
then (see Theorem 2.8.42 in \[3\]) the asymptotic distribution of (1) is the Normal distribution \(\mathcal{N}(0, a(t_i - t_{i-1}))\), \(a > 0\).

Let us first prove (2). For every \(k \geq 1\), a simple computation gives
\[
\text{Trace}(A \Phi^k(\rho)) = [a_1 \alpha_\infty + a_3 \bar{\beta}_\infty + \bar{a}_3 \beta_\infty + a_2 (1 - \alpha_\infty) + o(1/\sqrt{n})] = o(1/\sqrt{n}),
\]
hence
\[
\sum_{k=[nt_i]}^{[nt_{i-1}]+1} \text{Trace}(A \Phi^k(\rho)) = \sum_{k=[nt_{i-1}]+1}^{[nt_i]} o(1/\sqrt{n}) = o(\sqrt{n}).
\]
This gives (2).

Let us prove (3). Note that the sequence \((\text{Trace}(A \Phi^a(\rho)))_n\) converges to 0, as \(n\) tends to infinity. As a consequence, it is enough to prove that
\[
\frac{1}{n} \sum_{k=[nt_{i-1}]+1}^{[nt_i]} \text{Trace}(A^2 \Phi^k(\rho))
\]
converges to a strictly positive constant. A straightforward computation gives

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=[nt_{i-1}]+1}^{[nt_{i}]} \text{Trace}(A^2 \Phi^k(\rho)) = a_1^2 \alpha_\infty + a_2^2 (1 - \alpha_\infty) + |a_3|^2 + (a_1 + a_2)(a_3 \beta_\infty + \bar{a}_3 \beta_\infty)
\]

\[
= \frac{(t_i - t_{i-1})}{\nu_1 + \nu_2 + \nu_3} \left[ \nu_1^2 (1 - \nu_1^2) + \nu_2^2 (1 - \nu_2^2) + \nu_3^2 (1 - \nu_3^2) - 2 \nu_1 \nu_2 \nu_3 (1 - \nu_1 \nu_2 \nu_3) \right].
\]

This means that, for every \(\nu_1, \nu_2, \nu_3 \in \mathbb{R}\), for any \(p \geq 1\), the expectation

\[
w \left[ \left( \nu_1 (X^{(n)}_{t_i} - X^{(n)}_{t_{i-1}}) + \nu_2 (Y^{(n)}_{t_i} - Y^{(n)}_{t_{i-1}}) + \nu_3 (Z^{(n)}_{t_i} - Z^{(n)}_{t_{i-1}}) \right)^p \right]
\]

converges to

\[
\mathbb{E} \left[ \left( \nu_1 (B^{(1)}_{t_i} - B^{(1)}_{t_{i-1}}) + \nu_2 (B^{(2)}_{t_i} - B^{(2)}_{t_{i-1}}) + \nu_3 (B^{(3)}_{t_i} - B^{(3)}_{t_{i-1}}) \right)^p \right],
\]

where \((B^{(1)}_t, B^{(2)}_t, B^{(3)}_t)\) is a 3-dimensional Brownian motion with the announced covariance matrix.

The polynomial

\[
\left( \nu_1 (X^{(n)}_{t_i} - X^{(n)}_{t_{i-1}}) + \nu_2 (Y^{(n)}_{t_i} - Y^{(n)}_{t_{i-1}}) + \nu_3 (Z^{(n)}_{t_i} - Z^{(n)}_{t_{i-1}}) \right)^p
\]

can be developed as the sum

\[
\sum_{0 \leq p_1 + p_2 + p_3 \leq p} \nu_1^{p_1} \nu_2^{p_2} \nu_3^{p_3} \sum_{\mathcal{P}} S_1 S_2 \ldots S_p
\]

where the summation in the last sum runs over all partitions \(\mathcal{P} = \{A, B, C\}\) of \{1, \ldots, p\} such that \(|A| = p_1, |B| = p_2, |C| = p - p_1 - p_2\), with the convention:

\[
S_j = \begin{cases} 
X^{(n)}_{t_i} - X^{(n)}_{t_{i-1}} & \text{if } j \in A \\
Y^{(n)}_{t_i} - Y^{(n)}_{t_{i-1}} & \text{if } j \in B \\
Z^{(n)}_{t_i} - Z^{(n)}_{t_{i-1}} & \text{if } j \in C.
\end{cases}
\]

The expectation under \(w\) of the above expression converges to the corresponding expression involving the expectation \(\mathbb{E} \left[ \cdot \right]\) of the Brownian motion \((B^{(1)}_t, B^{(2)}_t, B^{(3)}_t)\). As this holds for any \(\nu_1, \nu_2, \nu_3 \in \mathbb{R}\), we deduce that
\[ \sum P S_1 S_2 \ldots S_p \] converges to the corresponding expectation for the Brownian motion.

We can conclude the proof by noticing that \( \hat{A}_i \) can be written, modulo multiplication by a constant, as \( \sum P S_1 S_2 \ldots S_p \) for some \( p \).

Let us discuss the class of polynomials for which Theorem 3.1 holds. In the particular case when the map \( \Phi \) is the identity map and \( \rho = 1/2I \) (in that case \( v_i = 0 \) for \( i = 1, 2, 3 \) and \( C = I \)), Biane [1] proved the convergence of the expectations in Theorem 3.1 for any polynomial in \( 3m \) non-commuting variables. It is a natural question to ask whether our result holds for any polynomial \( P \) instead of \( \hat{P} \), or at least for a larger class.

Let us give an example of a polynomial for which the convergence in our setting does not hold. Take \( P(X,Y) = XY \). From Theorem 3.1, the expectation under the state \( \omega \) of \( X^{(n)} tY^{(n)} t \) converges as \( n \to +\infty \) to 

\[ 2E[B^{(1)} tB^{(2)} t] . \]

Since we have the following commutation relations

\[ [(\sigma_x - v_1 I), (\sigma_y - v_2 I)] = 2i\sigma_z, \quad [(\sigma_y - v_2 I), (\sigma_z - v_3 I)] = 2i\sigma_x \]

and

\[ [(\sigma_z - v_3 I), (\sigma_x - v_1 I)] = 2i\sigma_y, \]

we deduce that

\[ [X^{(n)} t, Y^{(n)} t] = 2in^{-1/2} Z^{(n)} t + 2itv_3 I, \quad [Y^{(n)} t, Z^{(n)} t] = 2in^{-1/2} X^{(n)} t + 2itv_1 I \]

and

\[ [Z^{(n)} t, X^{(n)} t] = 2in^{-1/2} Y^{(n)} t + 2itv_2 I. \]

Then the expectation under the state \( \omega \) of

\[ X^{(n)} tY^{(n)} t = \frac{1}{2} [\hat{P}(X,Y) + [X^{(n)} t, Y^{(n)} t]] \]

converges to \( E[B^{(1)} tB^{(2)} t] + itv_3 \neq E[B^{(1)} tB^{(2)} t] \), if \( v_3 \) is non zero.

Furthermore, by considering the polynomial \( P(X,Y) = XY^3 + Y^3X \), it is possible to show that the convergence in Theorem 3.1 can not be enlarged.
to the class of symmetric polynomials. Straightforward computations gives
that \( P(X, Y) \) can be rewritten as
\[
\widehat{XY^3} + YX^3 + \frac{3}{4}[X, Y](Y^2 - X^2) + \frac{1}{2}(Y[X, Y]Y - X[X, Y]X) + \frac{1}{4}(Y^2 - X^2)[X, Y]
\]
so the expectation \( w[P(X_t^{(n)}, Y_t^{(n)})] \) converges as \( n \) tends to \(+\infty\) to
\[
\mathbb{E}[P(B_t^{(1)}, B_t^{(2)})] + 3iv_3t(v_1^2 - v_2^2)
\]
which is not equal to \( \mathbb{E}[P(B_t^{(1)}, B_t^{(2)})] \) if \( v_3 \neq 0 \) and \( |v_1| \neq |v_2| \).

In the following corollary we give a condition under which the convergence in Theorem 3.1 holds for any polynomial in \( 3m \) non-commuting variables.

**Corollaire 3.1.** In the case when \( \rho_\infty \) is equal to \( \frac{1}{2}I \), the convergence holds for any polynomial \( P \) in \( 3m \) non-commuting variables, i.e. for every \( t_1 < t_2 < \ldots < t_m \), the following convergence holds:
\[
\lim_{n \to +\infty} w[P(X_t^{(n)}, Y_t^{(n)}, Z_t^{(n)}, \ldots, X_t^{(n)}, Y_t^{(n)}, Z_t^{(n)})] = \mathbb{E}[P(B_t^{(1)}, B_t^{(2)}, B_t^{(3)}, \ldots, B_t^{(1)}, B_t^{(2)}, B_t^{(3)})]
\]
where \( (B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t \geq 0} \) is a three-dimensional centered Brownian motion with covariance matrix \( tI_3 \).

**Proof:**
We consider the polynomials of the form \( S = \frac{1}{N} \sum_{\mathcal{P}} S_1S_2\ldots S_{p_1+p_2+p_3} \) where the summation is done over all partitions \( \mathcal{P} = \{A, B, C\} \) of the set \( \{1, \ldots, p_1 + p_2 + p_3\} \) such that \( |A| = p_1, |B| = p_2, |C| = p_3 \), with the convention:
\[
S_j = \begin{cases} 
X_t^{(n)} - X_{t_{i-1}}^{(n)} & \text{if } j \in A \\
Y_t^{(n)} - Y_{t_{i-1}}^{(n)} & \text{if } j \in B \\
Z_t^{(n)} - Z_{t_{i-1}}^{(n)} & \text{if } j \in C
\end{cases}
\]
and \( N \) is the number of terms in the sum.

From Theorem 3.1 the expectation under the state \( w \) of \( S \) converges to
\[
\mathbb{E}\left[ \prod_{j=1}^3 (B_t^{(j)} - B_{t_{i-1}}^{(j)})^{p_j} \right].
\]
Using the commutation relations (3) with the \( v_i \)'s being all equal to zero, monomials of \( S \) differ of each other by \( n^{-1/2} \) times a polynomial of total degree less or equal to \((p_1 + p_2 + p_3) - 1\). It is easy to conclude by induction. \( \triangle \)
4 Examples

4.1 King-Ruskai-Szarek-Werner’s representation

The set of $2 \times 2$ self-adjoint matrices forms a four dimensional real vector subspace of $M_2(\mathbb{C})$. A convenient basis of this space is given by $\mathcal{B} = \{I, \sigma_x, \sigma_y, \sigma_z\}$. Each state $\rho$ on $M_2(\mathbb{C})$ can then be written as

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + z & x - iy \\ x + iy & 1 - z \end{pmatrix}$$

where $x, y, z$ are reals such that $x^2 + y^2 + z^2 \leq 1$. Equivalently, in the basis $\mathcal{B}$,

$$\rho = \frac{1}{2}(I + x \sigma_x + y \sigma_y + z \sigma_z)$$

with $x, y, z$ defined above. Thus, the set of density matrices can be identified with the unit ball in $\mathbb{R}^3$. The pure states, that is, the ones for which $x^2 + y^2 + z^2 = 1$, constitutes the Bloch sphere.

The noise coming from interactions between the qubit states and the environment is represented by the action of a completely positive and trace-preserving map $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$. Kraus and Choi [2, 7, 8] gave an abstract representation of these particular maps in terms of Kraus operators: There exists at most four matrices $L_i$ such that for any density matrix $\rho$,

$$\Phi(\rho) = \sum_{1 \leq i \leq 4} L_i^* \rho L_i$$

with $\sum_i L_i L_i^* = I$. The matrices $L_i$ are usually called the Kraus operators of $\Phi$. This representation is unique up to a unitary transformation. Recently, King, Ruskai et al [10, 6] obtained a precise characterization of completely positive and trace-preserving maps from $M_2(\mathbb{C})$ as follows. The map $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ being linear and preserving the trace, it can be represented as a unique $4 \times 4$-matrix in the basis $\mathcal{B}$ given by

$$\begin{pmatrix} 1 & 0 \\ t & T \end{pmatrix}$$

with $0 = (0, 0, 0)$, $t \in \mathbb{R}^3$ and $T$ a real $3 \times 3$-matrix. King, Ruskai et al [10, 6]
proved that via changes of basis, this matrix can be reduced to

\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
t_1 & \lambda_1 & 0 & 0 \\
t_2 & 0 & \lambda_2 & 0 \\
t_3 & 0 & 0 & \lambda_3 \\
\end{pmatrix}
\]  

(5)

Necessary and sufficient conditions under which the map \( \Phi \) with reduced matrix \( T \) for which \(|t_3| + |\lambda_3| \leq 1\) is completely positive are (see [6])

\[
(\lambda_1 + \lambda_2)^2 \leq (1 + \lambda_3)^2 - t_3^2 - (t_1^2 + t_2^2) \left( \frac{1 + \lambda_3 \pm t_3}{1 - \lambda_3 \pm t_3} \right) \leq (1 + \lambda_3)^2 - t_3^2 \quad (6)
\]

\[
(\lambda_1 - \lambda_2)^2 \leq (1 - \lambda_3)^2 - t_3^2 - (t_1^2 + t_2^2) \left( \frac{1 - \lambda_3 \pm t_3}{1 + \lambda_3 \pm t_3} \right) \leq (1 - \lambda_3)^2 - t_3^2 \quad (7)
\]

\[
[1 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - (t_1^2 + t_2^2 + t_3^2)]^2 \\
\geq 4 \left[ \lambda_1^2(t_1^2 + \lambda_2^2) + \lambda_2^2(t_2^2 + \lambda_3^2) + \lambda_3^2(t_3^2 + \lambda_1^2) - 2\lambda_1\lambda_2\lambda_3 \right]. \quad (8)
\]

We now apply Theorem 3.1 in this setting. Let \( \Phi \) be a completely positive and trace preserving map with matrix \( T \) given in (5), with coefficients \( t_i, \lambda_i, i = 1, 2, 3 \) satisfying conditions (6), (7) and (8). Moreover, we assume that \(|\lambda_i| < 1, i = 1, 2, 3 \). For every \( n \geq 0 \),

\[
\Phi^n(\rho) = \frac{1}{2} \begin{pmatrix}
1 + \phi_n(3) & \phi_n(1) - i \phi_n(2) \\
\phi_n(1) + i \phi_n(2) & 1 - \phi_n(3) \\
\end{pmatrix}
\]

where the sequences \((\phi_n(i))_{n \geq 0}, i = 1, 2, 3\) satisfy the induction relations:

\[
\phi_n(i) = \lambda_i \phi_{n-1}(i) + t_i.
\]

with initial conditions \( \phi_0(1) = x, \phi_0(2) = y \) and \( \phi_0(3) = z \). Explicit formulae can easily be obtained. We get, for every \( n \geq 0 \),

\[
\phi_n(1) = \left( x - \frac{t_1}{1 - \lambda_1} \right) \lambda_1^n + \frac{t_1}{1 - \lambda_1}
\]

\[
\phi_n(2) = \left( y - \frac{t_2}{1 - \lambda_2} \right) \lambda_2^n + \frac{t_2}{1 - \lambda_2}
\]

\[
\phi_n(3) = \left( z - \frac{t_3}{1 - \lambda_3} \right) \lambda_3^n + \frac{t_3}{1 - \lambda_3}.
\]
Hence, for any state $\rho$, for any $n \geq 1$,

$$\Phi_n(\rho) = \rho_\infty + o(|\lambda|_{max}^n)$$

where $|\lambda|_{max} = \max_{i=1,2,3} |\lambda_i|$ and

$$\rho_\infty = \begin{pmatrix} \alpha_\infty & \beta_\infty \\ \beta_\infty & 1 - \alpha_\infty \end{pmatrix}$$

with $\alpha_\infty = \frac{1}{2} \left(1 + \frac{t_3}{1 - \lambda_3}\right)$ and $\beta_\infty = \frac{1}{2} \left(\frac{t_1}{1 - \lambda_1} - i \frac{t_2}{1 - \lambda_2}\right)$. Theorem 3.1 applies with $v_i = \frac{t_i}{1 - \lambda_i}$, $i = 1, 2, 3$.

We now give some examples of well-known quantum channels. For each of them we give their Kraus operators, their corresponding matrix $T$ in the King-Ruskai-Szarek-Werner’s representation, as well as the vector $v = (v_1, v_2, v_3)$ and the covariance matrix $C$ obtained in Theorem 3.1. It is worth noticing that if $\Phi$ is a unital map, i.e. such that $\Phi(I) = I$, then the covariance matrix $C$ is equal to the identity matrix $I_3$.

1. **The depolarizing channel**:

Kraus operators: for some $0 \leq p \leq 1$,

$$L_1 = \sqrt{1 - p} I, L_2 = \sqrt{\frac{p}{3}} \sigma_x, L_3 = \sqrt{\frac{p}{3}} \sigma_y, L_4 = \sqrt{\frac{p}{3}} \sigma_z.$$

King-Ruskai-Szarek-Werner’s representation:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{4p}{3} & 0 & 0 \\ 0 & 0 & 1 - \frac{4p}{3} & 0 \\ 0 & 0 & 0 & 1 - \frac{4p}{3} \end{pmatrix}$$

The vector $v$ is the null vector and the covariance matrix $C$ in this case is given by the identity matrix $I_3$.

2. **Phase-damping channel**:
Kraus operators: for some $0 \leq p \leq 1$,

$$L_1 = \sqrt{1-p} I, \quad L_2 = \sqrt{p} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-p & 0 \\ 0 & 0 & 1-p \end{pmatrix}, \quad L_3 = \sqrt{p} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

King-Ruskai-Szarek-Werner’s representation:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-p & 0 & 0 \\ 0 & 0 & 1-p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The vector $v$ is the null vector and the covariance matrix $C$ in this case is given by $I_3$.

3. **Amplitude-damping channel:**

Kraus operators: for some $0 \leq p \leq 1$,

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

King-Ruskai-Szarek-Werner’s representation:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & 0 & 0 \\ 0 & 0 & \sqrt{1-p} & 0 \\ t & 0 & 0 & 1-p \end{pmatrix}$$

The vector $v$ is equal to $(0, 0, 1)$. The covariance matrix in this case is given by

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4. **Trigonometric parameterization:**

Consider the particular Kraus operators

$$L_1 = \left[ \cos\left(\frac{v}{2}\right) \cos\left(\frac{u}{2}\right) \right] I + \left[ \sin\left(\frac{v}{2}\right) \sin\left(\frac{u}{2}\right) \right] \sigma_z$$
and

\[ L_2 = \left[ \sin \left( \frac{u}{2} \right) \cos \left( \frac{u}{2} \right) \right] \sigma_x - i \left[ \cos \left( \frac{u}{2} \right) \sin \left( \frac{u}{2} \right) \right] \sigma_y. \]

King-Ruskai-Szarek-Werner's representation:

\[ T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos u & 0 & 0 \\
0 & 0 & \cos v & 0 \\
\sin u \sin v & 0 & 0 & \cos u \cos v
\end{pmatrix} \]

The vector \( v \) is equal to \((0, 0, \frac{\sin u \sin v}{1 - \cos u \cos v})\). The covariance matrix in this case is given by

\[ C = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 - v_3^2
\end{pmatrix} \]

with \( v_3 = \frac{\sin u \sin v}{1 - \cos u \cos v} \).

### 4.2 CP map associated to a Markov chain

With every Markov chain with two states and transition matrix given by

\[ P = \begin{pmatrix}
p & 1 - p \\
q & 1 - q
\end{pmatrix}, \quad p, q \in (0, 1) \]

is associated a completely positive and trace preserving map, denoted by \( \Phi \), with the Kraus operators:

\[ L_1 = \begin{pmatrix}
\sqrt{p} & \sqrt{1 - p} \\
0 & 0
\end{pmatrix} = \frac{\sqrt{p}}{2} (I + \sigma_z) + \frac{\sqrt{1 - p}}{2} (\sigma_x + i\sigma_y) \]

and

\[ L_2 = \begin{pmatrix}
0 & 0 \\
\sqrt{q} & \sqrt{1 - q}
\end{pmatrix} = \frac{\sqrt{1 - q}}{2} (I - \sigma_z) + \frac{\sqrt{q}}{2} (\sigma_x - i\sigma_y). \]

Let \( \rho \) be the density matrix

\[ \frac{1}{2} \begin{pmatrix}
1 + z & x - iy \\
x + iy & 1 - z
\end{pmatrix} \]
where \(x, y, z\) are reals such that \(x^2 + y^2 + z^2 \leq 1\). The map \(\Phi\) transforms the density matrix \(\rho\) into a new one given by

\[
\Phi(\rho) = L_1^* \rho L_1 + L_2^* \rho L_2.
\]

By induction, for every \(n \geq 0\),

\[
\Phi^n(\rho) = \begin{pmatrix} p_n & r_n \\ r_n & 1 - p_n \end{pmatrix}
\]

where the sequences \((p_n)_{n \geq 0}\), and \((r_n)_{n \geq 0}\) satisfy the recurrence relations: for every \(n \geq 1\),

\[
 p_n = p_{n-1}(p - q) + q
\]

and

\[
 r_n = \sqrt{q(1 - q) + p_{n-1}(\sqrt{p(1 - p)} - \sqrt{q(1 - q)})}
\]

with the initial condition \(p_0 = (1 + z)/2\). Assumption (A) is then clearly satisfied with

\[
\rho_\infty = \frac{1}{1 + q - p} \begin{pmatrix} q & \beta \\ \beta & 1 - p \end{pmatrix}
\]

where \(\beta = \left[q\sqrt{p(1 - p)} + (1 - p)\sqrt{q(1 - q)}\right]\). Then, applying Theorem 3.1, if \(P\) is a polynomial of \(3m\) non-commuting variables, for every \(0 < t_1 < t_2 < \ldots < t_m\), the following convergence holds

\[
\lim_{n \to +\infty} w \left[\hat{P}(X_{t_1}^{(n)}, Y_{t_1}^{(n)}, Z_{t_1}^{(n)}, \ldots, X_{t_m}^{(n)}, Y_{t_m}^{(n)}, Z_{t_m}^{(n)})\right] = \mathbb{E}\left[P(B_{t_1}^{(1)}, B_{t_1}^{(2)}, B_{t_1}^{(3)}, \ldots, B_{t_m}^{(1)}, B_{t_m}^{(2)}, B_{t_m}^{(3)})\right]
\]

where \((B_{t_1}^{(1)}, B_{t_1}^{(2)}, B_{t_1}^{(3)})_{t \geq 0}\) is a three-dimensional centered Brownian motion with Covariance matrix \(Ct\) where

\[
C = \begin{pmatrix} 1 - v_1^2 & 0 & -v_1 v_2 \\ 0 & 1 & 0 \\ -v_1 v_2 & 0 & 1 - v_2^2 \end{pmatrix}
\]

with

\[
v_1 = \frac{2}{1 + q - p}[q\sqrt{p(1 - p)} + (1 - p)\sqrt{q(1 - q)}]
\]

and

\[
v_2 = \frac{p + q - 1}{1 + q - p}.
\]
5 Large deviation principle

Let $\Gamma$ be a Polish space endowed with the Borel $\sigma$-field $\mathcal{B}(\Gamma)$. A good rate function is a lower semi-continuous function $\Lambda^* : \Gamma \to [0, \infty]$ with compact level sets $\{ x ; \Lambda^*(x) \leq \alpha \}, \alpha \in [0, \infty]$. Let $v = (v_n)_n \uparrow \infty$ be an increasing sequence of positive reals. A sequence of random variables $(Y_n)_n$ with values in $\Gamma$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to satisfy a Large Deviation Principle (LDP) with speed $v = (v_n)_n$ and the good rate function $\Lambda^*$ if for every Borel set $B \in \mathcal{B}(\Gamma)$,

$$- \inf_{x \in B} \Lambda^*(x) \leq \liminf_n \frac{1}{v_n} \log \mathbb{P}(Y_n \in B) \leq \limsup_n \frac{1}{v_n} \log \mathbb{P}(Y_n \in B) \leq - \inf_{x \in \bar{B}} \Lambda^*(x).$$

For every $k \geq 1$, we define

$$\bar{x}_k = I \otimes \ldots \otimes I \otimes \sigma_x \otimes I \otimes \ldots$$

$$\bar{y}_k = I \otimes \ldots \otimes I \otimes \sigma_y \otimes I \otimes \ldots$$

$$\bar{z}_k = I \otimes \ldots \otimes I \otimes \sigma_z \otimes I \otimes \ldots$$

where each $\sigma_j$ appears on the $k^{th}$ place.

For every $n \geq 1$, we consider the processes

$$X_n = \sum_{k=1}^n \bar{x}_k, \quad Y_n = \sum_{k=1}^n \bar{y}_k, \quad Z_n = \sum_{k=1}^n \bar{z}_k$$

with initial conditions

$$X_0 = Y_0 = Z_0 = 0.$$

To each vector $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$, we associate the euclidean norm $||\nu|| = \sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}$ and $\langle \cdot, \cdot \rangle$ the corresponding inner product.

**Theorem 5.1.** Let $\Phi$ be a completely positive and trace-preserving map for which there exists a state

$$\rho_{\infty} = \begin{pmatrix} \alpha_{\infty} & \beta_{\infty} \\ \beta_{\infty} & 1 - \alpha_{\infty} \end{pmatrix}$$

such that for any given state $\rho$,

$$\Phi^n(\rho) = \rho_{\infty} + o(1).$$
For every $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3^*$, the sequence 
\[
\left( \frac{\nu_1 \bar{X}_n + \nu_2 \bar{Y}_n + \nu_3 \bar{Z}_n}{n} \right)_{n \geq 1}
\]
satisfies a LDP with speed $n$ and the good rate function 
\[
I(x) = \begin{cases} 
\frac{1}{2} \left[ \left( 1 + \frac{x}{||\nu||} \right) \log \left( \frac{||\nu|+x}{||\nu|+(\nu,\nu)} \right) 
+ \left( 1 - \frac{x}{||\nu||} \right) \log \left( \frac{||\nu|-x}{||\nu|-\langle \nu,\nu \rangle} \right) \right] & \text{if } |x| < ||\nu||. \\
+\infty & \text{otherwise.}
\end{cases}
\]
where $v_1 = 2 \Re(\beta_\infty), v_2 = -2 \Im(\beta_\infty), v_3 = 2\alpha_\infty - 1$.

Proof:

The matrix 
\[
B := \nu_1 \sigma_x + \nu_2 \sigma_y + \nu_3 \sigma_z = \begin{pmatrix} v_3 & v_1 - i v_2 \\ v_1 + i v_2 & -v_3 \end{pmatrix}
\]
has two distinct eigenvalues $\pm ||\nu||$.

For every $n \geq 0$, we can write 
\[
\Phi^n(\rho) = \begin{pmatrix} \alpha_\infty + \phi_n(1) & \beta_\infty + \phi_n(2) \\ \beta_\infty + \phi_n(3) & 1 - \alpha_\infty + \phi_n(4) \end{pmatrix}
\]
where the four sequences $(\phi_n(i))_{n \geq 0}$ satisfy $\phi_n(i) = o(1)$.

For any $k \geq 1$, the expectation of $B$ in the state $\Phi^k(\rho)$ is equal to 
\[
\text{Trace}(B \Phi^k(\rho)) = \langle \nu, v \rangle + \varepsilon_k,
\]
with $\varepsilon_n = o(1)$. As a consequence, the distribution of $B$ is 
\[
p_k(||\nu||) = \frac{1}{2} \left[ 1 + \frac{1}{||\nu||} \langle \nu, v \rangle + \varepsilon_k \right] = 1 - p_k(-||\nu||).
\]

Using the fact that $\nu_1 \bar{X}_n + \nu_2 \bar{Y}_n + \nu_3 \bar{Z}_n$ is the sum of $n$ commuting matrices, we get that 
\[
\frac{1}{n} \log w \left( \exp t(\nu_1 X_n + \nu_2 Y_n + \nu_3 Z_n) \right)
= \frac{1}{n} \sum_{k=1}^{n} \log \left( e^{||\nu||t} p_k(||\nu||) + e^{-||\nu||t} (1 - p_k(||\nu||)) \right)
\]

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Since \( \varepsilon_n = o(1) \), we obtain that
\[
\lim_{n \to +\infty} \frac{1}{n} \log w \left( \exp t(\nu_1 X_n + \nu_2 Y_n + \nu_3 Z_n) \right)
= \log \left( \cosh (||\nu||t) + \frac{\langle \nu, v \rangle}{||\nu||} \sinh (||\nu||t) \right)
= \log \left( \cosh (||\nu||t) \right) + \log \left( 1 + \frac{\langle \nu, v \rangle}{||\nu||} \tanh (||\nu||t) \right).
\]

We denote by \( \Lambda(t) \) this function of \( t \).

For every \( t \in \mathbb{R} \), the function \( \Lambda \) is finite and differentiable on \( \mathbb{R} \), then, by Gärtner-Ellis Theorem (see [4]), the LDP holds with the good rate function
\[
I(x) = \sup_{t \in \mathbb{R}} \{ tx - \Lambda(t) \}.
\]

A simple computation leads to the rate function given in the theorem.

\( \triangle \)

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