ON THE DEHN FUNCTIONS OF KÄHLER GROUPS

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Abstract. We address the problem of which functions can arise as Dehn functions of Kähler groups. We explain why there are examples of Kähler groups with linear, quadratic, and exponential Dehn function. We then proceed to show that there is an example of a Kähler group which has Dehn function bounded below by a cubic function and above by \( n^6 \). As a consequence we obtain that for a compact Kähler manifold having non-positive holomorphic bisectional curvature does not imply having quadratic Dehn function.

1. Introduction

A Kähler group is a group which can be realized as fundamental group of a compact Kähler manifold. Kähler groups form an intriguing class of groups. A fundamental problem in the field is Serre’s question of “which” finitely presented groups are Kähler. While on one side there is a variety of constraints on Kähler groups, many of them originating in Hodge theory and, more generally, the theory of harmonic maps on Kähler manifolds, examples have been constructed that show that the class is far from trivial. Filling the space between examples and constraints turns out to be a very hard problem. This is at least in part due to the fact that the range of known concrete examples and construction techniques are limited. For general background on Kähler groups see [2] (and also [11, 3] for more recent results).

Known constructions have shown that Kähler groups can present the following group theoretic properties: they can

- be non-residually finite [47] (see also [16]);
- be nilpotent of class 2 [12, 44];
- admit a classifying space with finite \( k \)-skeleton, but no classifying space with finitely many \( k+1 \)-cells [21] (see also [4, 34, 10]); and
- be non-coherent [31] (also [42, 25]).

On the other side strong constraints on Kähler groups exclude many groups from being Kähler. One of the simplest constraints is that their first Betti number must be even, meaning that for example free abelian groups of odd rank and free groups can not be Kähler. Other constraints include that Kähler groups are one-ended [29], are virtually nilpotent if they are virtually solvable [20] and have quadratically presented Malcev algebra [19].

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A fundamental object of study in asymptotic group theory is the Dehn function of a finitely presented group. This relates to other questions in Group theory, such as solvability of the word problem. In this work we address the following question.

**Question 1.** Which functions can be realised as Dehn functions of Kähler groups?

Note that for nilpotent groups, the fact that the Malcev algebra of a Kähler group is quadratically presented is likely to imply that the Dehn function is quadratic (see e.g. [49] and also [43, Section 6.2]). On the other hand, a straightforward search combining known examples of Kähler groups with classical results on Dehn functions shows that they include groups with linear, quadratic, and exponential Dehn function (see Section 3). Since all three functions occur as Dehn functions of many lattices in semi-simple Lie groups [32, 24, 33], this is may not be too surprising, considering that many of the known examples of Kähler groups arise from such lattices (see e.g. [46]). However, these three functions only cover a small fraction of the functions that can be obtained as Dehn functions of finitely presented groups and they play a very special role among them: hyperbolic groups can be characterised by the property that their Dehn function is linear [28] and there is no non-hyperbolic group with subquadratic Dehn function [28, 6, 39, 40, 48]. Note that Gromov’s result [28] actually only requires weaker assumptions than subquadratic Dehn function to obtain hyperbolicity and Wenger gives a further improvement to this, which he shows to be optimal (see [43, Section 1] for details on these assumptions and [18] for a detailed explanation of Gromov’s proof). In contrast to these results, the set of \( \alpha \geq 2 \) such that there is a group with Dehn function \( n^\alpha \) is Dense in \([2, \infty)\) [7].

Hence, it is natural to ask if Kähler groups can attain any Dehn functions other than linear, quadratic and exponential, and the main part of our work will be dedicated to proving that the answer to this question is positive. This is achieved by showing that an example of a Kähler subgroup of a direct product of three surface groups, which was constructed by Dimca, Papadima and Suciu [21], has Dehn function bounded below by a cubic function and above by \( n^6 \).

**Theorem 1.1.** There is a Kähler group \( G \) with the following properties:

1. \( G \) has Dehn function \( \delta_G \) with \( n^3 \leq \delta_G(n) \leq n^6 \);
2. \( G \) is not coherent;
3. \( G \) is of type \( \mathcal{F}_2 \) and not of type \( \mathcal{F}_3 \).

It turns out that the group \( G \) of Theorem 1.1 can be realised as fundamental group of a smooth projective variety \( X \) with Stein universal cover \( \tilde{X} \) [21]. To obtain Theorem 1.1 we generalise work by Dison on the Dehn function of subgroups of direct products of free groups [23]. While the upper bound is a straightforward consequence of his work, the lower bound requires some new ideas.

Theorem 1.1 has the following interesting consequence:

**Corollary 1.2.** There is a compact Kähler manifold with non-positive holomorphic bi-sectional curvature which does not admit a quadratic isoperimetric function.

**Structure.** In Section 2 we summarize results on Dehn functions that we shall need. In Section 3 we provide examples of Kähler groups with linear, quadratic and exponential Dehn function. In Section 4 we give a lower bound on the distortion of certain elements in...
fibre products of hyperbolic groups. We apply this bound in Section 5 to show that there is a Kähler group with Dehn function bounded below by a cubic function. In Section 6 we give an upper bound on the Dehn function of this group allowing us to prove Theorem 1.1. We finish by listing some open questions arising from our work in Section 7.

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## 2. Dehn functions

Let \( G \) be a finitely presented group and let \( G \cong \langle X \mid R \rangle \) be a finite presentation for \( G \). A word \( w(X) \) of length \( l(w(X)) = m \) in the alphabet \( X \) is an expression of the form \( w(X) = x_1^{\pm 1} \ldots x_m^{\pm 1} \) with \( x_i \in X \) for \( 1 \leq i \leq m \). We call a word \( w(X) \) null-homotopic in \( G \) if it represents the trivial element in \( G \). Every null-homotopic word is freely equal to a word of the form \( \prod_{j=1}^n u_j(X) r_j^{\pm 1}(u_j(X))^{-1} \) for some words \( u_i(X) \) and elements \( r_i \in R \).

The area of a null-homotopic word \( w(X) \) is

\[
\text{Area}(w(X)) = \min \left\{ k \mid w(X) = \prod_{j=1}^k u_j(X) r_j^{\pm 1}(u_j(X))^{-1} \right\}
\]

For non-decreasing functions \( f, g : \mathbb{N} \to \mathbb{R}_{\geq 0} \) (or \( : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \)) we say that \( f \) is asymptotically bounded by \( g \) if there are constants \( C_1, C_2, C_3 \geq 1 \) such that \( f(n) \leq C_1 g(C_2 n) + C_3 \) for all \( n \in \mathbb{N} \) (or \( n \in \mathbb{R} \)). We write \( f \asymp g \) if \( f \) is asymptotically bounded by \( g \). We further say that \( f \) is asymptotically equal (or asymptotically equivalent) to \( g \) and write \( f \asymp g \) if \( f \leq g \leq f \).

For an element \( g \in G \) we denote by \( |g|_G = \text{dist}_{Cay(G,X)}(1,g) \) its distance from the origin in the Cayley graph \( Cay(G,X) \) with respect to the generating set \( X \) of \( G \).

The Dehn function of \( G \) is the function

\[
\delta_G(n) = \max \left\{ \text{Area}(w(X)) \mid w(X) \text{ null-homotopic with } l(w(X)) \leq n \right\}.
\]

We say that a function \( f : \mathbb{N} \to \mathbb{R}_{\geq 0} \) is an isoperimetric function for \( G \) if \( \delta_G(n) \leq f(n) \).

For a subgroup \( N \leq G \) of a finitely generated group \( G \) and generating sets \( X \) of \( G \) and \( Y \) of \( N \), the distortion of \( N \) in \( G \) is the function \( \Delta_G^N(n) = \max \{|g|_H \mid g \in N, |g|_G \leq n \} \).

A priori the definitions of \( | \cdot |_G \) and \( \delta_G \), and \( \Delta_G^n \) depend on a choice of presentation for \( G \) and generating set for \( N \leq G \). However, up to asymptotical equivalence, they are independent of these choices and hence it makes sense to speak of them as properties of a group rather than of a presentation.

We want to summarize a few important properties of Dehn functions which we will require.

**Theorem 2.1.** Let \( G \) be a finitely presented group and \( \delta_G \) be its Dehn function. Then the following hold:

1. \( \delta_G \) is linear if and only if \( G \) is hyperbolic;
2. if \( \delta_G \) is subquadratic (i.e. \( \delta_G(n) \leq n^2 \)) then it is linear;
3. if \( G = \pi_1 X \) for \( X \) a closed non-positively curved Riemann manifold or, more generally, if \( G \) is\( \text{CAT}(0) \), then the Dehn function of \( G \) is at most quadratic. In particular, if \( G \) is not hyperbolic then \( \delta_G(n) \asymp n^2 \);
4. if \( G = G_1 \times G_2 \) is a direct product of two infinite groups then
   \[ \delta_G(n) \asymp \max \{ n^2, \delta_{G_1}(n), \delta_{G_2}(n) \}. \]
Proof. For (1) see [28]. For (2) see [28] and also [30, 18]. For (3) see [11, III.Γ.1.6]. For (4) see [8]. □

3. Linear, quadratic and exponential Dehn functions

Comparing the examples of Kähler groups in the literature to results on Dehn functions of hyperbolic groups, non-positively curved groups and lattices, it is not hard to see that they include examples with linear, quadratic and exponential Dehn function. In this section we want to give an overview of known examples of Kähler groups for which we could determine their Dehn function.

Many of these examples rely on a result by Toledo.

Proposition 3.1 ([46]). Let $G$ be a semi-simple Lie group with associated symmetric space $X$. Assume that $X$ is an irreducible Hermitian symmetric space and that $X$ is neither the one- nor two-dimensional complex unit ball, nor the Siegel upper half plane of genus 2. Then every non-uniform lattice $\Gamma \leq G$ is a Kähler group.

Note that Toledo also shows that non-uniform lattices in $SU(1,1)$ and $SU(2,1)$ are not Kähler, while it is not known if non-uniform lattices in $Sp(4,\mathbb{R})$ are Kähler [46].

Linear Dehn function: By Theorem 2.1(1), classifying Kähler groups with linear Dehn function is equivalent to classifying hyperbolic Kähler groups. Hyperbolic Kähler groups include the fundamental groups $\Gamma_g = \pi_1 S_g$ of closed orientable surfaces $S_g$ of genus $\geq 2$ and cocompact lattices $\Gamma \leq PU(n,1)$, which correspond to compact complex ball quotients $\mathbb{B}^n/\Gamma$ (see [41, Section 2] and also [15, 45] for examples of such lattices). As an immediate consequence we obtain that there are Kähler groups with linear Dehn function.

Quadratic Dehn function: It is easy to obtain Kähler groups with quadratic Dehn function by taking direct products of hyperbolic groups and applying Theorem 2.1(4). In particular, we obtain that $\mathbb{Z}^n$ and $\Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ have quadratic Dehn function for $n \geq 1$, $r \geq 2$, and $g_i \geq 2$. Searching a bit further we can also find examples of Kähler groups with quadratic Dehn function which do not (virtually) decompose as a direct product. They include:

- irreducible non-hyperbolic cocompact lattices in semi-simple Lie groups whose associated symmetric space is Hermitian, since non-compact symmetric spaces are CAT(0) and thus all cocompact lattices have Dehn function bounded above by a quadratic function by Theorem 2.1(3);
- the symplectic groups $Sp(2g,\mathbb{Z})$ which have quadratic Dehn function for $g \geq 5$ by a result of Cohen [17], and are known to be Kähler for $g \geq 3$ by Proposition 3.1;
- Heisenberg groups $H_{2k+1}$ for $k \geq 4$, which are Kähler if and only if $k \geq 4$ ([12, 44] and [14]) and have quadratic Dehn function for $k \geq 2$ [1];
- the compactifications of non-arithmetic lattices constructed in Py’s work [12], since they are non-positively curved and non-hyperbolic (the fundamental group of the cusps maps to $\mathbb{Z}^2$-subgroups in the compactification).

A further interesting class of Kähler groups with quadratic Dehn function are Hilbert modular groups defined by totally real number fields $K/Q$ of degree $n = [K : Q] \geq 3$. More precisely, let $\phi : K \to \mathbb{R}^n$ be the homomorphism defined by the $n$ non-trivial embeddings of $K$ in $\mathbb{R}$. Then $\phi$ defines an embedding $PSL(2, K) \to (PSL(2, \mathbb{R}))^n$ and the image of
a cocompact lattice in $PSL(2, K)$ defines a non-uniform lattice in $(PSL(2, \mathbb{R}))^n$. Such a lattice is called a Hilbert modular group. It is Kähler by [40]. Hilbert modular groups contain $\mathbb{Z}^2$ subgroups and are thus not hyperbolic. In particular, their Dehn function can not be linear. On the other hand they are $\mathbb{Q}$-rank one lattices and thus by work of Drutu [24] have Dehn function bounded above by $n^{2+\epsilon}$ for every $\epsilon > 0$. In fact Drutu’s work shows that the Dehn function of Hilbert modular groups is quadratic (see also [50]).

Note that very recently Leuzinger and Young showed that all irreducible non-uniform lattices in a connected center-free semisimple Lie group without compact factors have quadratic Dehn function [33], confirming a Conjecture of Gromov for this very general class of lattices. Hence, any Kähler groups of this form also have quadratic Dehn function.

**Exponential Dehn function:** By work of Leuzinger and Pittet [32], all irreducible non-uniform lattices in semi-simple Lie groups of $\mathbb{R}$-rank 2 have exponential Dehn function. Hence, to see that there are Kähler groups with exponential Dehn function it suffices to find an example of a Kähler group which can be realised by such a lattice. A class of such examples is given by non-uniform lattices in $SU(2, n)$ for $n \geq 2$, since this is a semi-simple irreducible Lie group of real rank 2 (e.g. [30]) and it follows from Proposition 3.1 that non-uniform lattices in $SU(2, n)$ are Kähler.

The existence of non-uniform lattices in $SU(2, n)$ is well-known. An example is the group $SU(2, n, \mathbb{Z}[i])$, which is a non-uniform lattice by Godement’s compactness criterion [5, 38], since it contains the unipotent element

$$I_{n+2} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 + \frac{i}{2} & \frac{i}{2} & -1 & : & : \\ 0 & -\frac{i}{2} & -\frac{i}{2} & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ : & : \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

where $I_{n+2}$ is the identity matrix in $GL(n + 2, \mathbb{C})$ (for further details see [37, Section 3]).

**Other Dehn functions:** This leaves us with the question if there are any Kähler groups whose Dehn function does not fall into one of these three categories. While it is possible that such groups can be found among lattices in semi-simple Lie groups whose associated symmetric space is Hermitian, we are not aware of any known Kähler example. Indeed, most of the lattices for which the Dehn function is known seem to fall in one of the previous three categories, which might not be too surprising in the light of the results of Leuzinger and Pittet [32], and Leuzinger and Young [33]. One notable exception is the 3-Heisenberg group which has cubic Dehn function [13, Chapter 8], [26]. However, the latter is not Kähler.

The remaining sections will be dedicated to proving that Kähler groups can admit a Dehn function which is not linear, quadratic or exponential, by proving upper and lower bounds on the Dehn function of a concrete example of a Kähler subdirect product of three surface groups constructed by Dimca, Papadima and Suciu [21].
4. Distortion of fibre products of hyperbolic groups

For short exact sequences of groups

\[ 1 \to N_1 \to G_1 \xrightarrow{\phi_1} Q_1 \to 1, \]
\[ 1 \to N_2 \to G_2 \xrightarrow{\phi_2} Q_2 \to 1, \]

their (asymmetric) fibre product \( P \leq G_1 \times G_2 \) is the group \( P = \{(g_1, g_2) \mid \phi_1(g_1) = \phi_2(g_2)\} \).

Recall the well-known 0-1-2-Lemma.

Lemma 4.1 (0-1-2 Lemma). Consider the short exact sequences defined above. If \( Q \) is finitely presented and \( G_1 \) and \( G_2 \) are finitely generated, then the fibre product \( P \leq G_1 \times G_2 \) is finitely generated.

The proof is straight-forward by constructing an explicit finite generating set. We will explain its construction in the special case of a symmetric fibre product, that is, \( G_1 = G_2 \) and \( \phi_1 = \phi_2 \), since we will require it later. The general case is very similar.

Let \( \mathcal{X} \) be a finite generating set for \( G \). Denote by \( \mathcal{X}_\Delta \) the corresponding generating set of \( G_1 \) and by \( \mathcal{X}_\Delta \) the one of the diagonal embedding \( \text{Diag}(G) \hookrightarrow G \times G = G_1 \times G_2 \).

The set \( \mathcal{X}_Q = \phi_1(\mathcal{X}_1) = \phi_2(\mathcal{X}_2) \) is a finite generating set for \( Q \). Since \( Q \) is finitely presented we can find a finite set of relations \( R_Q \) among the elements of \( \mathcal{X}_Q \), such that \( Q \cong \langle \mathcal{X}_Q \mid R_Q \rangle \). Denote by \( R_1 \) the lift of \( R_Q \) to words in \( \mathcal{X}_1 \). Since \( N_1 = \ker \phi_1 \), it follows that \( N_1 = \langle \langle R_1 \rangle \rangle \) is finitely generated as normal subgroup of \( G_1 \). It is now easy to see that a finite generating set of \( P \) is given by \( \mathcal{X}_\Delta \cup R_1 \).

This generating set allows us to give a lower bound for the distortion of \( P \) in \( G \times G \) in terms of the Dehn function \( \delta_Q \) of \( Q \) with respect to the presentation \( Q \cong \langle \mathcal{X}_Q \mid R_Q \rangle \).

Proposition 4.2. Let \( G = \langle \mathcal{X} \mid S \rangle \) be a finitely presented hyperbolic group, let \( Q \) be a finitely presented group and let \( \phi : G \to Q \) be an epimorphism. Let \( Q = \langle \mathcal{X}_Q \mid R_Q \rangle \) be a finite presentation for \( Q \) with \( S \subset R \). Let \( P \leq G \times G \) be the symmetric fibre product of \( \phi \) and let \( h_n = (g_{1,n}, 1) \in P \) be a sequence of elements of the intersection \( P \cap (G_1 \times \{1\}) \) with \( |g_{1,n}|_G \asymp n \). Assume that each \( g_{1,n} \) admits a representative word \( v_n(X) \) with \( |v_n(X)|_{\text{Free}(\mathcal{X})} \asymp n \) such that the null-homotopic word \( v_n(\mathcal{X}_Q) \) in \( Q \) satisfies \( \text{Area}(v_n(\mathcal{X}_Q)) \asymp \delta_Q(n) \). Then \( |h_n|_P \asymp \delta_Q(n) \).

Proof. If \( Q \) is hyperbolic then the conclusion is trivially true, since we have the asymptotic inequalities \( |h_n|_P \geq |g_{1,n}|_G \asymp n \asymp \delta_Q(n) \), so assume that \( Q \) is not hyperbolic.

Let \( G \cong \langle \mathcal{X} \mid S \rangle \) be a finite presentation for \( G \). Choose a finite presentation \( Q = \langle \mathcal{X}_Q \mid R_Q \rangle \) as above. We denote by \( S_1 \subset R_1 \) the subset corresponding to the subset \( S \subset R \).

Since \( h_n \in P \cap (G_1 \times \{1\}) \), the image \( \phi(h_n) = \phi_1(g_{1,n}) \in Q \) represents the trivial word. There is a word \( \omega_n(\mathcal{X}_\Delta, R_1) \) with \( h_n = \omega_n(\mathcal{X}_\Delta, R_1) \) in \( P \). Since \( [G_1 \times \{1\}, \{1\} \times G_2] = \{1\} \), we obtain \( \omega_n(\mathcal{X}_\Delta, R_1) = \omega_n(\mathcal{X}_1, R_1) \cdot \omega_n(\mathcal{X}_2, 1) \). Hence, the word \( \omega_n(\mathcal{X}_2, 1) \) represents the trivial element in \( G_2 \).

It follows that the word \( \omega_n(\mathcal{X}, 1) \), obtained from \( \omega_n(\mathcal{X}_1, R_1) \) by deleting all occurrences of elements of \( R_1 \), represents the trivial word in \( G_1 \). Hence, \( \omega_n(\mathcal{X}, 1) \) is freely equal to a product of finitely many conjugates of elements of \( S_1 \), whose number will be denoted by \( k_2,n \geq 0 \). Since hyperbolic groups have linear Dehn function, we have that for a minimal choice of \( k_2,n \),

\[ l(\omega_n(\mathcal{X}, 1)) \geq \delta_G(\omega_n(\mathcal{X}, 1))_{\text{Free}(\mathcal{X})}) \geq k_2,n \] (4.1)
and thus
\[ l(\omega_n(X_1, \mathcal{R}_1)) = k_{1,n} + l(\omega_n(X_1, 1)) \geq k_1 + k_2, \]
where \( k_{1,n} \geq 0 \) denotes the number of occurrences of elements of \( \mathcal{R}_1 \) in \( \omega_n(X_1, \mathcal{R}_1) \).

However, it follows from the previous paragraph that \( \omega_n(X_1, \mathcal{R}_1) \) is freely equal to a product of \( k_{1,n} + k_2 \) conjugates of elements of \( \mathcal{R}_1 \). Hence, the area of \( \omega_n(X_1, \mathcal{R}_1) \) in \( Q = \langle X_1 \mid \mathcal{R}_1 \rangle \) provides us with a lower bound on \( k_{1,n} + k_2 \) and \( \text{Area}_Q(\omega_n(X_1, \mathcal{R}_1)) \) yields
\[ l(\omega_n(X_1, \mathcal{R}_1)) \geq k_{1,n} + k_2 \geq \text{Area}_Q(\omega_n(X_1, \mathcal{R}_1)). \tag{4.2} \]

The word \( \omega_n(X_1, \mathcal{R}_1) \cdot (v_n(X_1))^{-1} \) is null-homotopic in \( G_1 \). Thus, it can be written as a product of \( k_{3,n} \geq 0 \) conjugates of elements of \( S_1 \), where we choose \( k_{3,n} \) to be minimal with this property. Since hyperbolic groups have linear Dehn function, we deduce
\[ k_{3,n} \leq |\omega_n(X_1, \mathcal{R}_1) \cdot (v_n(X_1))^{-1}|_{\text{Free}(X_1)} \leq |\omega_n(X_1, \mathcal{R}_1)|_{\text{Free}(X_1)} + n. \tag{4.3} \]

Using that \( \delta_Q(n) \leq \text{Area}_Q(\omega_n(X_1, \mathcal{R}_1)) + k_{3,n} \geq \text{Area}_Q(\omega_n(X_1, \mathcal{R}_1)) \times \delta_Q(n) \tag{4.4} \)

Combining inequalities (4.2), (4.3) and (4.4), this yields
\[ \delta_Q(n) \leq \text{Area}_Q(\omega_n(X_1, \mathcal{R}_1)) + k_{3,n} \]
\[ \leq \text{Area}_Q(\omega_n(X_1, \mathcal{R}_1)) + n + |\omega_n(X_1, \mathcal{R}_1)|_{\text{Free}(X_1)} \]
\[ \leq 2 \cdot l(\omega_n(X_1, \mathcal{R}_1)) + n. \tag{4.2} \]

Since \( Q \) is non-hyperbolic, we obtain that \( \frac{\delta_Q(n)}{n} \to \infty \) as \( n \to \infty \), and thus \( l(\omega_n(X_1, \mathcal{R}_1)) \geq \delta_Q(n) \). It follows that \( |h_n|_p \geq \delta_Q(n) \). \( \square \)

A special type of fibre products that we are interested in are the \textit{coabelian subgroups} of a direct product of groups, that is, subgroups \( H \leq G_1 \times \cdots \times G_r \) with \( H = \ker \theta \) for some epimorphism \( \theta : G_1 \times \cdots \times G_r \to \mathbb{Z}^N \).

For a group \( G \) consider an epimorphism \( \phi : G \to Q = \mathbb{Z}^N \) and denote by \( \phi_1 : G_1 \to Q \), \( \phi_2 : G_2 \to Q \) two copies of this epimorphism. Then the coabelian subgroup \( K = \ker (\phi_1 + \phi_2) \leq G_1 \times G_2 \) is the fibre product of the short exact sequences
\[ 1 \to \ker \phi \to G \xrightarrow{\phi} Q \to 1, \]
\[ 1 \to \ker \phi \to G \xrightarrow{-\phi} Q \to 1. \]

We obtain the following consequence of Proposition 4.2

\textbf{Corollary 4.3.} Let \( G \) be a finitely presented hyperbolic group, let \( Q = \mathbb{Z}^N \) for some \( N \geq 0 \) and let \( \phi : G \to Q \) be an epimorphism. Let \( K \) be as defined in the previous paragraph and assume that there is an automorphism \( \nu : G \to G \) such that \( (\phi \circ \nu)(g) = -\phi(g) \) for all \( g \in G \).

Let \( h_n = (g_{1,n}, 1) \in K \) be a sequence of element of the intersection \( K \cap (G_1 \times \{1\}) \), such that \( |g_{1,n}|_Q = n \) and \( g_{1,n} \) has the same properties as the element \( g_{1,n} \) in Proposition 4.3. Then \( |h_n|_K \geq \delta_Q(n) \).
Proof. It is immediate from the existence of the automorphism $\nu$ that $K$ is isomorphic to the symmetric fibre product $P$ of the short exact sequence defined by $\phi : G \to Q$. The automorphism is induced by the automorphism $(\text{id}_G, \nu) : G_1 \times G_2 \to G_1 \times G_2$ of the product. The element $h_n$ is invariant under this automorphism. Hence, Proposition 4.2 implies that $|h_n|_K \asymp |h_n|_P \geq \delta_Q(n)$. \hfill $\square$

5. SUBGROUPS OF DIRECT PRODUCTS OF SURFACE GROUPS

Now consider a surface group $\Gamma_2 = \pi_1S_2$ of genus 2 and fix a presentation $\Gamma_2 = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] \rangle$.

Define an epimorphism $\phi : \Gamma_2 \to \mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$

$$\phi(a_i) \mapsto a, \quad \phi(b_i) \mapsto b$$

and take $r \geq 2$ copies $\phi_i : \Gamma_2^{(i)} \to \mathbb{Z}^2$, $1 \leq i \leq r$, where we denote the generators of $\Gamma_2^{(i)}$ by $a_j^{(i)}, b_j^{(i)}, j = 1, 2$.

As in Section 3 we define coabelian subgroups $K_r = \ker \theta_r$ for $\theta_r = \sum_{i=1}^r \phi_i : \Gamma_2^{(1)} \times \cdots \times \Gamma_2^{(r)} \to \mathbb{Z}^2$.

It is not hard to see that the groups $K_r$ for $r \geq 3$ are in fact explicit realizations of the Kähler subgroups of direct products of surface groups constructed by Dimca, Papadima and Suciu [21] by taking a 2-fold branched cover of genus 2 of an elliptic curve (see [35] for more details and an explicit construction of finite presentations for the $K_r$).

The rest of this section will be concerned with proving the following result:

Theorem 5.1. The Dehn function of the Kähler group $K_3$ satisfies $\delta_{K_3}(n) \geq n^3$.

It is straight-forward to check that

$$\nu : \Gamma_2 \to \Gamma_2$$

$$\phi(a_i) \mapsto (a,b)a_i^{-1}(a,b)^{-1}$$

$$\phi(b_i) \mapsto (a,b)b_i^{-1}(a,b)^{-1}$$

defines an automorphism of $\Gamma_2$ which satisfies $\phi \circ \nu = -\phi$.

Lemma 5.2. Let $Q = \mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$ and let $h_m = \left([\left((a_1^{(1)})^m, (b_1^{(1)})^m\right), 1]\right) \in K_2$, for $m \geq 1$. Then $|h_m|_{K_2} \geq \delta_Q(m)(\asymp m^2)$.

Proof. Since $\phi_1 \left([\left((a_1^{(1)})^m, (b_1^{(1)})^m\right)]ight) = 1$, we have $h_m \in K_2 \cap \left(\Gamma_2^{(1)} \times \{1\}\right)$.

The group $\Gamma_2^{(1)}$ retracts onto the free subgroup $F_2 = \langle a_1^{(1)}, b_1^{(1)} \rangle \leq \Gamma_2^{(1)}$ via the homomorphism defined by

$$\Gamma_2^{(1)} \to F_2$$

$$a_1^{(1)} \mapsto a_1^{(1)}$$

$$b_1^{(1)} \mapsto b_1^{(1)}$$

$$a_2^{(1)}, b_1^{(1)} \mapsto 1.$$

Hence, $F_2 \leq \Gamma_2^{(1)}$ is an undistorted subgroup and in particular we have $|g_m|_{\Gamma_2^{(1)}} \times |g_m|_{F_2} = 4m$, where the last inequality is realised by the sequence of words $v_m(X) = \left([\left((a_1^{(1)})^m, (b_1^{(1)})^m\right)]ight)$. 


with \(|v_m(\mathcal{X})|_{\text{Free}(\mathcal{X})} = 4m\), for \(\mathcal{X} = \{a_1^{(1)}, b_1^{(1)}, a_2^{(1)}, b_2^{(1)}\}\). It is well-known that \(\delta_Q(n) \times n^2\) and that the sequence of words \(w_m(\{a, b\}) = [a^m, b^m]\) satisfies \(\text{Area}_Q(w_m(\{a, b\})) \asymp \delta_Q(m)\).

Replace the presentation for \(Q\) with the presentation
\[
Q \equiv \langle a_1^Q, b_1^Q, a_2^Q, b_2^Q \mid [a_1^Q, b_1^Q], a_1^Q(a_2^Q)^{-1}, b_1^Q(b_2^Q)^{-1}, [a_1^Q, b_1^Q][a_2^Q, b_2^Q] \rangle,
\]
via the identifications \(a_1^Q \to a, b_1^Q \to b\). Under this change of presentation the word \(v_m(\mathcal{X}_Q)\) gets identified with the word \(w_m(\{a, b\})\) under this identification. Thus, \(\text{Area}_Q(v_m(\mathcal{X}_Q)) \asymp \delta_Q(m) \times m^2\).

It follows that the homomorphism \(\phi : G \to Q\) as defined at the beginning of this section, \(\nu\) as defined in \(5.1\) and \(h_m\) satisfy all conditions of Corollary \(4.3\). We obtain that \(|h_m|_{\mathcal{K}_2} \geq \delta_Q(m) \times m^2\). \(\square\)

Remark 5.3. Note that in fact the same argument works for any sequence of elements \(g_m\) which is contained in an undistorted free subgroup \(H \leq \Gamma_2^{(2)}\), has reduced length asymptotically equivalent to \(m\) and can be represented by words \(v_m(\mathcal{X})\) satisfying the conditions of Proposition \(4.2\). Thus the conclusions of Lemma \(5.2\) apply to this more general class of elements in \(\mathcal{K}_2 \cap \langle \Gamma(1) \times \{1\} \rangle\).

The rest of the proof of Theorem \(5.4\) will require a suitable decomposition of \(K_3\) to which we can apply the following consequence of \(23\) Theorem 6.1.

**Theorem 5.4.** Let \(\Lambda = G_1 \ast_H G_2\) be finitely presented with \(H\) a proper subgroup of each \(G_i\), and \(H = \langle \mathcal{B} \rangle, G_1 = \langle \mathcal{A}_1 \rangle\) and \(G_2 = \langle \mathcal{A}_2 \rangle\), where \(\mathcal{B}, \mathcal{A}_1\) and \(\mathcal{A}_2\) are finite generating sets. Let \(\mathcal{P} = \langle \mathcal{A}_1, \mathcal{A}_2 \mid \mathcal{R} \rangle\) be a finite presentation for \(\Lambda\) then there is a constant \(C = C(\mathcal{P}, \mathcal{B}) > 0\) such that the following holds:

Given elements \(h \in H, g_1 \in G_1 \setminus H\) and \(g_2 \in G_2 \setminus H\) with \([g_1, h] = [g_2, h] = 1\), and words \(w = w(\mathcal{A}_1)\) representing \(h\) and \(u_i = u_i(\mathcal{A}_i)\) representing \(g_i, i = 1, 2\), then
\[
\text{Area}_\mathcal{P}([w, (u_1u_2)^n]) \geq C \cdot n \cdot \text{dist}_{\text{Cay}(H, \mathcal{B})}(1, h)
\]

Note that \(23\) Theorem 6.1 shows that for an explicit presentation of \(\Lambda\) of the form \(\langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{B} \mid \mathcal{R} \rangle\) one can choose \(C = 2\). It is straightforward to see that Theorem \(5.4\) follows from this result, using that \(\text{Area}_\mathcal{P}\) and \(\text{dist}_{\text{Cay}(H, \mathcal{B})}\) are equivalent up to multiplicative constants under change of presentation and generating set.

Projection of the subgroup \(K_3 \leq \Gamma_2^{(1)} \times \Gamma_2^{(2)} \times \Gamma_2^{(3)}\) onto the third factor induces a short exact sequence
\[
1 \to K_2 \to K_3 \overset{p_3}{\to} \Gamma_2^{(3)} \to 1
\]
with \(K_2 = \ker \left(\Gamma_2^{(1)} \times \Gamma_2^{(2)} \to \mathbb{Z}^2\right)\). The homomorphism
\[
(1, \nu, \text{id}_{\Gamma_2^{(3)}}) : \Gamma_2^{(3)} \to K_3
\]
provides a splitting of this sequence. Hence, it follows that \(K_3 \cong K_2 \rtimes \Gamma_2^{(3)}\) is a semidirect product.

Consider the decomposition \(\Gamma_2^{(3)} = \text{Free}(a_1^{(3)}, b_1^{(3)}) \ast [a_1^{(3)}, b_1^{(3)}] = \text{Free}(a_2^{(3)}, b_2^{(3)})\) into an amalgamated product over \(\mathbb{Z}\). Its lift under \(p_3\) provides a decomposition of \(K_3\) as amalgamated free product
\[
K_3 \cong \left(K_2 \rtimes \text{Free}(a_1^{(3)}, b_1^{(3)})\right) \ast_H \left(K_2 \rtimes \text{Free}(a_2^{(3)}, b_2^{(3)})\right),
\]
Proof of Theorem 5.1. Let $K$ be a presentation for $G$. Then

$$H = p_3^{-1}(\left[\left(a_1^{(3)}, b_1^{(3)}\right)\right]) = K_2 \times \left[\left(a_1^{(3)}, b_1^{(3)}\right)\right] \cong K_2 \times \left[\left(a_1^{(3)}, b_1^{(3)}\right)\right],$$

where the last identity follows from the fact that

$$\left(1, \nu(a_1^{(3)}), \nu(b_1^{(3)}), \left(a_1^{(3)}, b_1^{(3)}\right)\right) K_2 = \left(1, \left[a_1^{(3)}, b_1^{(3)}\right], b_1^{(3)}\right) K_2$$

as cosets of $K_2 \leq H$.

To simplify notation we will now write $K_2 \times Z$ for $K_2 \times \left[\left(a_1^{(3)}, b_1^{(3)}\right)\right]$.

Lemma 5.5. Let $h \in (K_2 \times Z) \cap \Gamma_2^{(1)} \times \{1\} \times \{1\}$. Then $|h|_{K_2 \times Z} \leq |h|_{K_2}$.

Proof. Let $\mathcal{Y}$ be a generating set for $K_2$ and let $z = \left(1, 1, \left[a_1^{(3)}, b_1^{(3)}\right]\right)$. Then $\mathcal{Y}_z = \mathcal{Y} \cup \{z\}$ is a generating set for $K_2 \times Z$. It is clear that with respect to this generating set we have $|h|_{K_2 \times Z} \leq |h|_{K_2}$.

Conversely, let $\omega(\mathcal{Y}, z)$ be a word of minimal length in $K_2 \times Z$ representing $h$ in $K_2 \times Z$. Then $\omega(\mathcal{Y}, z) = \omega(\mathcal{Y}, 1) \omega(1, z)$ and in particular $\omega(1, z)$ represents the trivial element in $Z$, while $\omega(\mathcal{Y}, 1)$ is a word in $\mathcal{Y}$ representing $h$. Hence, we obtain

$$|h|_{K_2 \times Z} = l(\omega(\mathcal{Y}, z)) = l(\omega(\mathcal{Y}, 1) \omega(1, z)) \geq l(\omega(\mathcal{Y}, 1)) \geq |h|_{K_2}.$$

As a consequence of Lemma 5.5 and Lemma 5.2, we obtain

Corollary 5.6. For $m \geq 1$, the element $h_m = \left[\left(a_1^{(1)} \cdot m, b_2^{(1)} \cdot m\right), 1, 1\right] \in K_2 \times Z$ satisfies $|h_m|_{K_2 \times Z} \geq m^2$.

Proof of Theorem 5.7. Let $G_1 = K_2 \times \text{Free}(a_1^{(3)}, b_1^{(3)})$ and $G_2 = K_2 \times \text{Free}(a_2^{(3)}, b_2^{(3)})$. Extend the set $\left\{\left(1, \nu(a_1^{(3)}), a_1^{(3)}\right) \cdot m, \left(1, (a_1^{(3)})^{-1}\right) \cdot m, \left(b_1^{(1)} \cdot (b_2^{(1)})^{-1}, 1\right) \cdot m\right\} \subset G_1$ to a finite generating set $\mathcal{A}_1$ of $G_1$ and the set $\left\{\left(1, \nu(a_2^{(3)}), a_2^{(3)}\right) \cdot m\right\} \subset G_2$ to a finite generating set $\mathcal{A}_2$ of $G_2$ and choose any finite generating set $\mathcal{B}$ for $H$. Let further $\mathcal{P} = \langle \mathcal{A}_1, \mathcal{A}_2 \mid \mathcal{R} \rangle$ be a finite presentation for $K_3$.

Choose elements represented by words

$$g_1 = u_1(A_1) = \left(1, \nu(a_1^{(3)}), a_1^{(3)}\right) \in \left(K_2 \times \text{Free}(a_1^{(3)}, b_1^{(3)})\right) \setminus H$$

and

$$g_2 = u_2(A_2) = \left(1, \nu(a_2^{(3)}), a_2^{(3)}\right) \in \left(K_2 \times \text{Free}(a_2^{(3)}, b_2^{(3)})\right) \setminus H.$$

Since $h_m \in K_3 \cap \Gamma_2^{(1)}$, we have $[g_1, h_m] = [g_2, h_m] = 1$. The element $h_m$ is represented by the word

$$w_m(A_1) = \left[\left(\left(a_1^{(1)} \cdot m, (a_1^{(3)})^{-1} \cdot m, (b_1^{(1)} \cdot (b_2^{(1)})^{-1}, 1\right) \cdot m\right)\right]$$

of length $l(w_m) = 4m$ with respect to the generating set $A_1$.

Thus, by Theorem 5.4 there is $C = C(\mathcal{P}, \mathcal{B}) > 0$ such that

$$\text{Area}_{\mathcal{P}}\left[\left[w_m, (u_1 u_2)^m\right]\right] \geq C \cdot m \cdot \text{dist}_{\mathcal{C}_{\mathcal{A}}(H, \mathcal{B})}(1, h_m).$$

The word $[w_m, (u_1 u_2)^m]$ has length $l([w_m, (u_1 u_2)^m]) = 12m$ in the alphabet $A_1 \cup A_2$. Hence, it follows from Corollary 5.6 that

$$\delta_{K_3}(m) \geq \text{Area}_{\mathcal{P}}\left[\left[w_m, (u_1 u_2)^m\right]\right] \geq Cm|h_m|_{K_2 \times Z} \geq m^3.$$

This completes the proof of Theorem 5.1. $\Box$
Note that $K_3$ is a Kähler group of type $F_2$, not of type $F_3$ (see [21]), which contains the subgroup $K_2$ of type $F_1$, not of type $F_2$. In particular, $K_3$ is not coherent. Hence, we can summarize the main properties of $K_3$ as follows.

**Theorem 5.7.** The group $K_3$ is a non-coherent Kähler group of type $F_2$, not of type $F_3$. The Dehn function of $K_3$ is bounded below by $\delta_{K_3}(n) \geq n^3$.

As a consequence we obtain:

**Proof of Corollary 1.2** Dimca, Papadima and Suciu obtain the group $K_3$ as fundamental group of the (compact) smooth generic fibre $H$ of a surjective holomorphic map $f : S_2^{(1)} \times S_2^{(2)} \times S_2^{(3)} \to E$ for $E$ an elliptic curve. In particular, $H \subset S_2^{(1)} \times S_2^{(2)} \times S_2^{(3)}$ is an embedded complex submanifold. The manifold $S_2^{(1)} \times S_2^{(2)} \times S_2^{(3)}$ admits a metric with non-positive holomorphic bisectional curvature, since it is a product of closed Riemann surfaces. Since holomorphic bisectional curvature is non-increasing when passing to complex submanifolds [27, Section 4], it follows that $H$ can be endowed with a metric of non-positive holomorphic bisectional curvature. By Theorem 5.7 $K_3$, does not admit a quadratic isoperimetric function and the result follows. \qed

Note that Corollary 1.2 is in contrast to the case of non-positive (Riemannian) sectional curvature, as the latter implies that the fundamental group admits a quadratic isoperimetric function.

### 6. An upper bound

For a group $G$ with finite presentation $G = \langle X \mid R \rangle$, an area-radius pair $(\alpha, \rho)$ is a pair of functions $\alpha : \mathbb{N} \to \mathbb{R}_{\geq 0}$ and $\rho : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that for every null-homotopic word $w(X)$ with $l(w(X)) \leq n$, there is a free equality $w(X) = \prod_{i=1}^{k} u_i(X)^{r_i+1}(u_i(X))^{-1}$ with $k \leq \alpha(n)$ and $\max \{l(u_i(X)) \mid 1 \leq i \leq k\} \leq \rho(n)$. Note that area-radius pairs are independent of the choice of presentation (up to equivalence). For further details on area-radius pairs we refer the reader to [22, Section 3].

An upper bound on the Dehn function of $K_3$ can be obtained as a consequence of the following theorem by Dison:

**Theorem 6.1 ([22, Theorem 11.15]).** For $r \geq 3$, let $G_1, \ldots, G_r$ be finitely presented groups with area-radius pairs $(\alpha_i, \rho_i)$, for $1 \leq i \leq r$, and let $A$ be an abelian group of rank $m = \dim (A \otimes \mathbb{Z} \mathbb{Q})$. Define $\alpha(n) = \max \{n^2, \alpha_i(n), 1 \leq i \leq r\}$, $\rho(n) = \max \{n, \rho_i(n), 1 \leq i \leq r\}$.

If $\phi : G_1 \times \cdots \times G_r \to A$ is a homomorphism such that the restriction of $\phi$ to each factor $G_i$ is surjective, then $\rho^{2m} \alpha$ is an isoperimetric function for $\ker \phi$.

**Lemma 6.2.** Let $G = \langle X \mid R \rangle$ be a finitely presented group and let $\delta_G$ be the Dehn function of $G$. Then there is $C > 0$ such that $(\delta_G, C \cdot \delta_G)$ defines an area-radius pair for $G$.

**Proof.** It is not hard to see that, given a van Kampen diagram of area $k = \text{Area}(w(X))$ for a word $w(X)$, then $w(X)$ is freely equal to a word of the form

$$w(X) = \prod_{j=1}^{k} u_j(X)^{r_j+1}(u_j(X))^{-1}$$

with $l(u_j(X)) \leq C \cdot k$, where $C = \max \{l(r(X)) \mid r \in R\}$. Hence, $(\delta_G, C \delta_G)$ is an area-radius pair for $G$. \qed
As an easy consequence we obtain an upper bound on the Dehn function of the group $K_3$.

**Corollary 6.3.** The Dehn function of $K_3$ satisfies $\delta_{K_3}(n) \leq n^6$.

**Proof.** The group $K_3$ is the kernel of the homomorphism $\theta_3 : \Gamma_2^{(1)} \times \Gamma_2^{(2)} \times \Gamma_2^{(3)} \to \mathbb{Z}^2$ which is surjective on factors. Since surface groups are hyperbolic the groups $\Gamma_2^{(i)}$ have linear Dehn function $\delta_{G_2^{(i)}}(n) = n$. By Lemma 6.2 they admit area-radius pairs $(\alpha_i, \rho_i)$ with $\alpha_i(n) \asymp \rho_i(n) \asymp \delta_{G_2^{(i)}}(n) \asymp n$. Theorem 6.1 implies that $\delta_{K_3}(n) \asymp \rho_4 \cdot \alpha \asymp n^6$. □

Note that Theorem 6.1 was used in a very similar way in [22, Corollary 12.6] to show that subdirect products of $r$ limit groups of type $\mathcal{F}_{r-1}$ have polynomial Dehn function and in [22, Proposition 13.3(2)] to show that the examples in [23] have Dehn function bounded above by $n^6$.

**Proof of Theorem 1.1.** This is now an immediate consequence of Theorem 5.7 and Corollary 6.3. □

### 7. Questions

This work begs intriguing questions and we want to finish by listing some of them. One may first ask whether our result can be extended as follows.

**Question 2.** For every integer $k$, is there $k' \geq k$ and a Kähler group $G$ with Dehn function satisfying $n^k \leq \delta_G(n) \leq n^{k'}$?

Note that the same reasoning as in [22, Proposition 13.3(3)] and Section 6 can be applied to see that the groups constructed in [21], [4] and [34] must all have Dehn function bounded above by $n^6$. Hence, these examples can not be used to obtain a positive answer to this question. However, these arguments do not apply to many of the groups constructed very recently by the first author from maps onto higher-dimensional tori [36]. Thus, in the light of Theorem 1.1 one could try to search for groups with larger Dehn function among these examples.

Recall that the gap between 1 and 2 in Theorem 2.1(2) is the only gap in the isoperimetric spectrum, that is, for every $\alpha \in [2, \infty)$ and $\epsilon > 0$ there is a group $G$ with Dehn function $n^\alpha$ and $|\alpha - \beta| < \epsilon$. It is thus natural to ask if the same is true in the class of Kähler groups.

**Question 3.** What is the isoperimetric spectrum of Dehn functions of Kähler groups? Does it contain any gaps on $[2, \infty)$?

We do not know the exact asymptotic of the Dehn function of the group of Theorem 1.1 and in particular we do not know whether there exists a single point in the isoperimetric spectrum of Dehn functions of Kähler groups that is different from 1 or 2.

We might wonder whether there exist Kähler groups with arbitrary large Dehn function. For instance

**Question 4.** Do all Kähler groups have solvable word problem?

Recall that the word problem is solvable if and only if the Dehn function is a recursive function. At this time even the following question is open.

**Question 5.** Do all Kähler groups admit an exponential isoperimetric function?
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