A Tree-Loop Duality Relation at Two Loops and Beyond

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References:
Catani, Gleisberg, Krauss, Rodrigo, Winter  JHEP 0809 (2008) 065
Bierenbaum, Catani, Draggiotis, Rodrigo  arXiv:1007.0194 [hep-ph]
**MOTIVATION:**

IR-singularities between virtual and real contributions cancel after integration → the **duality relation** provides an alternative method to existing ones that make use of this fact. It follows the idea:

Try to express the loop integrals as tree-level integrals which are of the same nature as real-radiation integrals

Since then all integrations are of the same tree-level phase-space type, one hopes for a more efficient implementation into a Monte Carlo program

This is currently under investigation for one-loop integrals

The question for this talk is:
If the answer to the above is: „yes, the method is efficient“, is there a way to extend it to higher loop orders?

A one-loop formula in this line of thinking: The **Feynman tree theorem**
The Feynman Tree Theorem

\[ G_F(q) \equiv \frac{1}{q^2 + i0}, \quad G_A(q) \equiv \frac{1}{q^2 - i0 q_0} \]

Use:

\[ \frac{1}{x \pm i0} = \text{PV} \left( \frac{1}{x} \right) \mp i\pi \delta(x) \]

\[ G_A(q) \equiv G_F(q) + \tilde{\delta}(q), \quad G_R(q) \equiv G_F(q) + \tilde{\delta}(-q) \]

\[ \tilde{\delta}(q) \equiv 2\pi i \theta(q_0) \delta(q^2) = 2\pi i \delta_+(q^2) \]
Feynman Tree Theorem:

\[ 0 = L_A^{(N)}(p_1, p_2, \ldots, p_N) = \int_q \prod_{i=1}^N G_A(q_i) = \int_q \prod_{i=1}^N \left[ G_F(q_i) + \tilde{\delta}(q_i) \right] \]

\[ = L^{(N)}(p_1, p_2, \ldots, p_N) + L_{1-cut}^{(N)}(p_1, p_2, \ldots, p_N) + \cdots + L_{N-cut}^{(N)}(p_1, p_2, \ldots, p_N) \]

\[ L_{m-cut}^{(N)}(p_1, p_2, \ldots, p_N) = \int_q \left\{ \tilde{\delta}(q_1) \cdots \tilde{\delta}(q_m) G(q_{m+1}) \cdots G(q_N) + \text{uneq. perms.} \right\} \]

Feynman Tree Theorem:

\[ L^{(N)}(p_1, p_2, \ldots, p_N) = - \left[ L_{1-cut}^{(N)}(p_1, p_2, \ldots, p_N) + \cdots + L_{N-cut}^{(N)}(p_1, p_2, \ldots, p_N) \right] \]
The goal is:
Search for a loop-tree duality relation where the amount of cuts = number of loops (unlike FTT)

For one loop that means:

One single cut is enough!
Towards a **Duality Theorem**: One Loop

Take the integral directly over the residues

\[
L^{(N)}(p_1, p_2, \ldots, p_N) = -2\pi i \int \sum [\text{Res}_{i-th\ pole} \ G_F(q_i)] \left[ \prod_{j=1 \atop j \neq i}^{N} G_F(q_j) \right] \text{Res}_{i-th\ pole} \left[ \frac{1}{q_i^2 + i\eta} \right] = \int dq_0 \ \delta_+(q_i^2) \left[ \prod_{j=1 \atop j \neq i}^{N} \frac{1}{q_j^2 + i\eta} \right]
\]

\[
\prod_{j=1 \atop j \neq i}^{N} G_F(q_j) = \prod_{j=1 \atop j \neq i}^{N} \frac{1}{q_j^2 + i\eta}
\]

\[
\eta \text{ is a future-like vector, } \eta_{\mu} = (\eta_0, \eta), \quad \eta_0 \geq 0, \quad \eta^2 = \eta_\mu \eta^\mu \geq 0
\]

**Duality Theorem at one loop**

\[
L^{(N)}(p_1, p_2, \ldots, p_N) = -\sum \int q \ \delta(q_i) \left[ \prod_{j=1 \atop j \neq i}^{N} G_D(q_i, q_j) \right]
\]

\[
G_D(q_i, q_j) := \frac{1}{q_j^2 - i\eta(q_j - q_i)}
\]
Duality theorem:

\[
\frac{1}{(q + p_1)^2 - i0 \eta p_1} - \frac{1}{(q - p_1)^2 + i0 \eta p_1} = - \sum_{i=1}^{N} \frac{1}{(q + p_i)^2 - i0 \eta p_i}
\]
At one-loop order:

- Only single cuts
- The i0-preservation of the dual propagator depends on external momenta only → no branch cuts

\[ G_D(q_i, q_j) := \frac{1}{q_j^2 - i0 \eta(q_j - q_i)} \]

Can we obtain a similar formula at higher loops?

- # cuts = # loops
- integration-momentum independent i0-prescription

At higher loop orders there is more than one integration momentum which we will use to group the diagrams into parts

We start by constructing formulae similar to the ones used so far, but for whole sets of inner momenta
In analogy to single propagators, define for any set of (internal) momenta $\alpha_k$:

$$G_{F(A,R)}(\alpha_k) = \prod_{i \in \alpha_k} G_{F(A,R)}(q_i)$$

$$G_D(\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(q_i) \prod_{j \in \alpha_k, j \neq i} G_D(q_i; q_j)$$

$$G_D(\alpha_k) = \tilde{\delta}(q_i) \quad \text{for} \quad \alpha_k = \{i\}$$

$$G_D(q_i, q_j) := \frac{1}{q_j^2 - i0 \eta(q_j - q_i)}$$

NOTE:

- If the momenta depend on different integration momenta: integration-momentum dependence in i0-prescription
- If the momenta depend on the same integration momentum: i0-prescription depends on external momenta only

Hence, we will naturally try to group higher order diagrams into parts depending on the same integration momenta
\[ G_F(-\alpha_k) = G_F(\alpha_k) \]
\[ G_A(-\alpha_k) = G_R(\alpha_k) \]

\[ G_D(-\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(-q_i) \prod_{j \in \alpha_k, j \neq i} G_D(-q_i; -q_j) \]

Change direction of momentum flow \( q_i \rightarrow -q_i \) for all momenta \( q_i \in \alpha_k \)

This will become necessary, starting from two loops
Relation between dual propagator and Feynman propagator:

\[ \tilde{\delta}(q_i) \ G_D(q_i, q_j) = \tilde{\delta}(q_i) \left( G_F(q_j) + \tilde{\theta}(q_j - q_i) \ \tilde{\delta}(q_j) \right) \]

with \( \tilde{\theta}(q) = \theta(\eta q) \)

For ANY set of (internal) momenta, one finds:

**Main Equation I**

\[ G_A(\alpha_k) = G_F(\alpha_k) + G_D(\alpha_k) \]

Non-trivial relation relying on cancellation of theta-functions
"Multiplication formula": How to express $G_D$ in terms of subsets $\beta_N \equiv \alpha_1 \cup \ldots \cup \alpha_N$

$$G_D(\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_N) = G_A(\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_N) - G_F(\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_N)$$

$$G_D(\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_N) = \prod_{i=1}^{N} G_A(\alpha_i) - \prod_{i=1}^{N} G_F(\alpha_i)$$

$$= \prod_{i=1}^{N} \left[ G_F(\alpha_i) + G_D(\alpha_i) \right] - \prod_{i=1}^{N} G_F(\alpha_i)$$

$$= \sum_{\beta_N^{(1)} \cup \beta_N^{(2)} = \beta_N} \prod_{i_1 \in \beta_N^{(1)}} G_D(\alpha_{i_1}) \prod_{i_2 \in \beta_N^{(2)}} G_F(\alpha_{i_2})$$

Partition of $\beta_N$ into exactly two sets $\beta_N^{(1)}$ and $\beta_N^{(2)}$, with elements $\alpha_i$, including the case $\beta_N^{(1)} = \beta_N$ and $\beta_N^{(2)} = \emptyset$ (There is no term with only $G_F$)

For example:

$$G_D(\alpha_1 \cup \alpha_2) = G_D(\alpha_1) G_D(\alpha_2) + G_D(\alpha_1) G_F(\alpha_2) + G_F(\alpha_1) G_D(\alpha_2)$$
The one-loop case revisited:

\[ 0 = \int_{\ell_1} G_A(\alpha_1) = \int_{\ell_1} [G_F(\alpha_1) + G_D(\alpha_1)] \]

Where \( \alpha_1 \) is the set of all inner lines of the one-loop diagram

Solve for the „Feynman“-part

\[ L^{(1)}(p_1, p_2, \ldots, p_N) = - \int_{\ell_1} G_D(\alpha_1) = - \sum_{i \in \alpha_1} \tilde{\delta} (q_i) \prod_{j \in \alpha_1, j \neq i} G_D(q_i; q_j) \]

Original one-loop result
Using the multiplication formula for the set $\alpha_1$
where the elements are given by all single propagators $q_i$

$$\alpha_1 = q_1 \cup \ldots \cup q_N$$

we reproduce the FTT at one loop

$$L^{(1)}(p_1, p_2, \ldots, p_N) = - \int_{\ell_1} G_D(\alpha_1)$$

$$L^{(1)}(p_1, p_2, \ldots, p_N) = - \sum_{\alpha_1^{(1)} \cup \alpha_1^{(2)} = \alpha_1} \int_{\ell_1} \prod_{i_1 \in \alpha_1^{(1)}} \tilde{\delta}(q_{i_1}) \prod_{i_2 \in \alpha_1^{(2)}} G_F(q_{i_2})$$
How can we use this to find a formula for higher order loops with the required properties?

Use Equation I:

The following statement is correct for ANY set of internal momenta depending on a common integration momentum:

\[
0 = \int_{\ell_1} G_A(\alpha_1) = \int_{\ell_1} [G_F(\alpha_1) + G_D(\alpha_1)]
\]

Hence for any set of momenta \( \alpha_1 \cup ... \cup \alpha_N \) depending on the same integration momentum:

\[
\int_{\ell_i} G_F(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) = -\int_{\ell_i} G_D(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N)
\]

Application of the duality theorem!
**Two (and higher) loops:** Find the correct subsets and use Equation II

Group lines with the same integration momentum:  

| Subset | Expression |
|--------|------------|
| $\alpha_1$ | $\{0, 1, \ldots, r\}$ |
| $\alpha_2$ | $\{r+1, \ldots, l\}$ |
| $\alpha_3$ | $\{l+1, \ldots, N\}$ |

The „Loop Lines“
\[ L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{\ell_1} \int_{\ell_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) \]

Apply the duality theorem to the first loop

\[ L^{(2)}(p_1, p_2, \ldots, p_N) = -\int_{\ell_1} \int_{\ell_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2) \]

Use multiplication formula

\[ = -\int_{\ell_1} \int_{\ell_2} \left\{ G_D(\alpha_1) G_D(\alpha_3) + G_D(\alpha_1) G_F(\alpha_3) + G_F(\alpha_1) G_D(\alpha_3) \right\} G_F(\alpha_2) \]

Change direction of momentum-flow for one momentum in this term:

\[ \alpha_1 \rightarrow -\alpha_1 \]
\[ L^{(2)}(p_1, p_2, \ldots, p_N) = - \int_{\ell_1} \int_{\ell_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2) \]

\[ = \int_{\ell_1} \int_{\ell_2} \left\{ -G_D(\alpha_1) G_F(\alpha_2) G_D(\alpha_3) + G_D(\alpha_1) G_D(\alpha_2 \cup \alpha_3) + G_D(\alpha_3) G_D(-\alpha_1 \cup \alpha_2) \right\} \]

Formula with only double-cuts but integration momentum dependent i0-prescription

\[ = \int_{\ell_1} \int_{\ell_2} \left\{ G_D(\alpha_1) G_D(\alpha_2) G_F(\alpha_3) + G_D(-\alpha_1) G_F(\alpha_2) G_D(\alpha_3) + G^*(\alpha_1) G_D(\alpha_2) G_D(\alpha_3) \right\} \]

Contains triple-cuts but has integration-momentum-free i0-prescription

\[ G^*(\alpha_k) \equiv G_F(\alpha_k) + G_D(\alpha_k) + G_D(-\alpha_k) \]

This can also be expressed as:

\[ G^*(\alpha_k) = G_A(\alpha_k) + G_R(\alpha_k) - G_F(\alpha_k) \]
Using again the multiplication formula: **Feynman Tree Theorem at two loops**

\[
L^{(2)}(p_1, \ldots, p_N) = \sum_{\alpha_k^{(1)} \cup \alpha_k^{(2)} = \alpha_k \atop k \in \{1,2,3\}} \int \ell_1 \int \ell_2 \left\{ G_F(\alpha_1) \prod_{i_1 \in \alpha_2^{(1)}} \tilde{\delta}(q_{i_1}) \prod_{i_2 \in \alpha_3^{(1)}} \tilde{\delta}(q_{i_2}) \prod_{i_3 \in \alpha_2^{(2)} \cup \alpha_3^{(2)}} G_F(q_{i_3}) \right. \\
+ G_F(\alpha_2) \prod_{i_1 \in \alpha_1^{(1)}} \tilde{\delta}(-q_{i_1}) \prod_{i_2 \in \alpha_3^{(1)}} \tilde{\delta}(q_{i_2}) \prod_{i_3 \in \alpha_2^{(2)} \cup \alpha_3^{(2)}} G_F(q_{i_3}) \\
+ G_F(\alpha_3) \prod_{i_1 \in \alpha_1^{(1)}} \tilde{\delta}(q_{i_1}) \prod_{i_2 \in \alpha_2^{(1)}} \tilde{\delta}(q_{i_2}) \prod_{i_3 \in \alpha_2^{(2)} \cup \alpha_3^{(2)}} G_F(q_{i_3}) \\
+ \left( \prod_{i_1 \in \alpha_1^{(1)}} \tilde{\delta}(q_{i_1}) + \prod_{i_1 \in \alpha_1^{(1)}} \tilde{\delta}(-q_{i_1}) \right) \prod_{i_2 \in \alpha_2^{(1)}} \tilde{\delta}(q_{i_2}) \prod_{i_3 \in \alpha_3^{(1)}} \tilde{\delta}(q_{i_3}) \prod_{i_4 \in \alpha_2^{(2)} \cup \alpha_3^{(2)}} G_F(q_{i_4}) \right\},
\]
Three loops:

\[ L_{(a),(b),(c)}^{(3)}(p_1, p_2, \ldots, p_N) = \int_{\ell_1} \int_{\ell_2} \int_{\ell_3} G_D(\alpha_1 \cup \alpha_2) G_D(\alpha_3 \cup \alpha_4) G_F(\beta) \]

(a): \( \beta = \emptyset \)  \hspace{1cm} (b): \( \beta = \alpha_5 \)  \hspace{1cm} (c): \( \beta = \alpha_5 \cup \alpha_6 \)

\[ G_D(\alpha_1 \cup \alpha_2) = G_D(\alpha_1) G_D(\alpha_2) + G_D(\alpha_1) G_F(\alpha_2) + G_F(\alpha_1) G_D(\alpha_2) \]

\[ \int_{\ell_1} \int_{\ell_2} \int_{\ell_3} G_D(\alpha_1) G_F(\alpha_2) G_F(\alpha_3) G_D(\alpha_4) \rightarrow - \int_{\ell_1} \int_{\ell_2} \int_{\ell_3} G_D(\alpha_1) G_D(\alpha_2 \cup \alpha_3) G_D(\alpha_4) \]
\[ L^{(3)}_{(a),(b),(c)}(p_1, p_2, \ldots, p_N) = \int_{\ell_1} \int_{\ell_2} \int_{\ell_3} G_D(\alpha_1 \cup \alpha_2) G_D(\alpha_3 \cup \alpha_4) G_F(\beta) \]

\[ = \int_{\ell_1} \int_{\ell_2} \int_{\ell_3} \left\{ G_D(\alpha_2, \alpha_3, \alpha_4) G_F(\alpha_1) + G_D(\alpha_1, \alpha_3, \alpha_4) G_F(\alpha_2) + G_D(\alpha_1, \alpha_2, \alpha_4) G_F(\alpha_3) 
+ G_D(\alpha_1, \alpha_2, \alpha_3) G_F(\alpha_4) + G_D(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \right\} G_F(\beta) \]

\[ - G_D(\alpha_1, \alpha_3) G_D(\alpha_2 \cup -\alpha_4 \cup \beta) - G_D(\alpha_1, \alpha_4) G_D(\alpha_2 \cup \alpha_3 \cup \beta) \]

\[ - G_D(\alpha_2, \alpha_3) G_D(-\alpha_1 \cup -\alpha_4 \cup \beta) - G_D(\alpha_2, \alpha_4) G_D(-\alpha_1 \cup \alpha_3 \cup \beta) \} \]

\[ G_D(\alpha_1, ..., \alpha_N) = \prod_{i=1}^{N} G_D(\alpha_i) \]
Expressed in loop lines (int.-mom.-free i0-prescription):

Cuts, ranging from the number of loops $\rightarrow$ number of loop lines

True for any diagram!
\[
L^{(3)}_{\text{Mercedes}}(p_1, p_2, \ldots, p_N)
= \int_{\ell_1} \int_{\ell_2} \int_{\ell_3} \left\{ -G_D(\alpha_1, \alpha_2, \alpha_3)G_F(\alpha_4, \alpha_5, \alpha_6) + G_D(\alpha_3 \cup \alpha_4 \cup \alpha_5)G_D(\alpha_1, \alpha_2)G_F(\alpha_6) \\
+ G_D(-\alpha_1 \cup \alpha_4 \cup \alpha_6)G_D(\alpha_2, \alpha_3)G_F(\alpha_5) + G_D(-\alpha_2 \cup \alpha_5 \cup -\alpha_6)G_D(\alpha_1, \alpha_3)G_F(\alpha_4) \\
+ G_D(\alpha_1)[G_D(\alpha_3 \cup \alpha_4)G_D(\alpha_5)G_F(\alpha_2 \cup \alpha_6) - G_D(\alpha_2 \cup \alpha_3 \cup \alpha_4 \cup \alpha_6)G_D(\alpha_5) \\
- G_D(\alpha_3 \cup \alpha_4)G_D(-\alpha_2 \cup \alpha_5 \cup -\alpha_6)] \\
+ G_D(\alpha_2)[G_D(-\alpha_1 \cup \alpha_6)G_D(\alpha_4)G_F(\alpha_3 \cup \alpha_5) - G_D(\alpha_1 \cup \alpha_3 \cup \alpha_5 \cup -\alpha_6)G_D(\alpha_4) \\
- G_D(-\alpha_1 \cup \alpha_6)G_D(\alpha_3 \cup \alpha_4 \cup \alpha_5)] \\
+ G_D(\alpha_3)[G_D(-\alpha_2 \cup \alpha_5)G_D(-\alpha_6)G_F(\alpha_1 \cup \alpha_4) - G_D(-\alpha_1 \cup -\alpha_2 \cup \alpha_4 \cup \alpha_5)G_D(-\alpha_6) \\
- G_D(-\alpha_2 \cup \alpha_5)G_D(-\alpha_1 \cup \alpha_4 \cup \alpha_6)] \right\}
\]
Conclusions and Outlook:

The efficiency of the method at one loop has to be investigated and is under investigation!

We constructed a loop-tree duality relation which is easily extendable to higher loop orders, either in the form of

→ n cuts for a n-loop diagram, where the propagators still can involve branch cuts
→ n up to m cuts for a n-loop diagram, with m=#(loop lines), no branch cuts