Matching colored points with rectangles

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Abstract. Let $S$ be a point set in the plane such that each of its elements is colored either red or blue. A matching of $S$ with rectangles is any set of pairwise-disjoint axis-aligned rectangles such that each rectangle contains exactly two points of $S$. Such a matching is monochromatic if every rectangle contains points of the same color, and is bichromatic if every rectangle contains points of different colors. In this paper we study the following two problems:

1. Find a maximum monochromatic matching of $S$ with rectangles.
2. Find a maximum bichromatic matching of $S$ with rectangles.

For each problem we provide a polynomial-time approximation algorithm that constructs a matching with at least $1/4$ of the number of rectangles of an optimal matching. We show that the first problem is NP-hard even if either the matching rectangles are restricted to axis-aligned segments or $S$ is in general position, that is, no two points of $S$ share the same $x$ or $y$ coordinate. We further show that the second problem is also NP-hard, even if $S$ is in general position. These NP-hardness results follow by showing that deciding the existence of a perfect matching is NP-complete in each case. The approximation results are based on a relation of our problem with the problem of finding a maximum independent set in a family of axis-aligned rectangles. With this paper we extend previous ones on matching one-colored points with rectangles and squares, and matching two-colored points with segments. Furthermore, using our techniques, we prove that it is NP-complete to decide a perfect matching with rectangles in the case where all points have the same color, solving an open problem of Bereg, Mutsanas, and Wolff [CGTA (2009)].

1 Introduction

Matching points in the plane with geometric objects consists in, given an input point set $S$ and a class $C$ of geometric objects, finding a collection $M \subseteq C$ such that each element of $M$ contains exactly two points of $S$ and every point of $S$ lies in at most one element of $M$. This kind of geometric matching problem

\footnote{This research has been partially supported by grant CONICYT, FONDECYT/Iniciación 11110069 (Chile).}
was introduced by Ábrego et al. [1], calling a geometric matching strong if the
geometric objects are disjoint, and perfect if every point of \( S \) belongs to some
element of \( M \). They studied the existence and properties of matchings for point
sets in the plane when \( C \) is the set of axis-aligned squares, or the family of disks.

Bereg et al. [6] continued the study of this class of problems. They proved by
a constructive proof that if \( C \) is the class of axis-aligned rectangles, then every
point set of \( n \) points in the plane admits a strong matching that matches at least
\( 2\lfloor n/3 \rfloor \) of the points; and left open the computational complexity of finding
such a maximum strong matching. They assume that there can be points with
the same \( x \) or \( y \) coordinate, condition that makes the optimization problem hard.
In the case in which \( C \) is the class of axis-aligned squares, they proved that it is
NP-hard to decide whether a given point set admits a perfect strong matching.

In the setting of colored points, it is well known that every two-colored point
set in the plane such that no three points are collinear, consisting of \( n \) red points
and \( n \) blue points, admits a perfect strong matching with straight segments,
where each segment connects points of different colors [21]. Dumitrescu and
Steiger [13] introduced the study of strong straight segment matchings of two-
colored point sets in the case where each segment must match points of the
same color. The current results are due to Dumitrescu and Kaye [12]: Every two-
colored point set \( S \) of \( n \) points admits a strong straight segment matching that
matches at least \( \frac{6}{7}n - O(1) \) of the points, which can be found in \( O(n^2) \) time; and
there exist \( n \)-point sets such that every strong matching with straight segments
matches at most \( \frac{94}{95}n + O(1) \) points. The computational complexity of deciding
if a given two-colored point set admits a perfect strong matching with straight
segments connecting points of the same color, is still an open problem [13].

Let \( S = R \cup B \) be a set of \( n \) points in the plane such that each element of \( S \) is
colored either red or blue, where \( R \) denotes the set of the points colored red and
\( B \) the set of the points colored blue. A strong matching of \( S \) is monochromatic if
all matching objects cover points of the same color. Likewise, a strong matching
of \( S \) is bichromatic if all matching objects cover points of different colors.

As an extension of the above problems, we study both monochromatic and
bichromatic strong matchings of \( S \) with axis-aligned rectangles. For the monochro-
matic case it is trivial to build examples in which no matching rectangle exists
and examples in which a perfect strong matching exists. For the bichromatic
case, there always exists at least one matching rectangle (i.e. match the red point
and the blue point such that their minimum enclosing rectangle has minimum
area among all combinations of a red point and a blue point) and similar as the
the monochromatic case, one can build examples in which exactly one matching
rectangle exists and examples in which a perfect strong matching exists. Then,
we focus our attention in the following optimization problems:

**Maximum Monochromatic Rectangle Matching (MMRM)** problem: *Find
a monochromatic strong matching of \( S \) with the maximum number of rectangles.*

**Maximum Bichromatic Rectangle Matching (MBRM)** problem: *Find a
bichromatic strong matching of \( S \) with the maximum number of rectangles.*
Results. For each problem we provide a polynomial-time approximation algorithm that constructs a matching with at least 1/4 of the number of rectangles of an optimal matching. In the approximation algorithms we consider that the elements of \( S \) are not necessarily in general position. We say that \( S \) is in \textit{general position} if no two elements of \( S \) share the same \( x \) or \( y \) coordinate. We further use the direct relation of the problems with the \textsc{Maximum Independent Set of Rectangles} problem, which is to find a maximum subset of pairwise disjoint rectangles in a given family of rectangles. We complement the approximation results by showing that the \textsc{MMRM} problem is \textsc{NP}-hard, even if either the matching rectangles are restricted to axis-aligned segments or the points are in general position. We further show that the \textsc{MBRM} problem is also \textsc{NP}-hard, even if the points are in general position. Furthermore, we are able to prove that if all elements of \( S \) have the same color, then the \textsc{MMRM} problem keeps \textsc{NP}-hard, solving an open question of Bereg et al. \cite{Bereg2016}. These \textsc{NP}-hardness results follow by showing that deciding the existence of a perfect matching is \textsc{NP}-complete in each case.

2 Preliminaries

For every point \( p \) of \( S \), let \( x(p), y(p), \) and \( c(p) \) denote the \( x \)-coordinate, the \( y \)-coordinate, and the color of \( p \), respectively. Given two points \( a \) and \( b \) of the plane with \( x(a) \leq x(b) \), let \( D(a,b) \) denote the rectangle which has the segment connecting \( a \) and \( b \) as diagonal, which is in fact the minimum enclosing axis-aligned rectangle of \( a \) and \( b \). If \( a \) and \( b \) are horizontally or vertically aligned, we say that \( D(a,b) \) is a segment, otherwise we say that \( D(a,b) \) is a box. We say that \( D(a,b) \) is red if both \( a \) and \( b \) are colored red. Otherwise, if both \( a \) and \( b \) are colored blue, we say that \( D(a,b) \) is blue. Given \( S \), consider the following two families of axis-aligned rectangles:

\[
\mathcal{R}(S) := \{ D(p,q) \mid p,q \in S; c(p) = c(q); \text{ and } D(p,q) \cap S = \{p,q\} \}
\]

\[
\overline{\mathcal{R}}(S) := \{ D(p,q) \mid p,q \in S; c(p) \neq c(q); \text{ and } D(p,q) \cap S = \{p,q\} \}
\]

Observe that the \textsc{MMRM} problem is equivalent to finding a maximum subset of \( \mathcal{R}(S) \) of independent rectangles. Two rectangles are independent if they are disjoint. Similarly, the \textsc{MBRM} problem is equivalent to finding a maximum subset of \( \overline{\mathcal{R}}(S) \) of independent rectangles. The \textsc{Maximum Independent Set of Rectangles (MISR)} problem is a classical \textsc{NP}-hard problem in computational geometry and combinatorics, and is to find a maximum subset of independent rectangles in a given set of axis-aligned rectangles \cite{Bereg2016,Bereg2016a,Bereg2016b}. The general MISR problem admits a polynomial-time approximation algorithm, which with high probability produces and independent set of rectangles with at least \( \Omega\left(\frac{1}{\log \log m}\right) \) times the number of rectangles in an optimal solution, being \( m \) the number of rectangles in the input \cite{Bereg2016}. There also exist deterministic polynomial-time \( \Omega\left(\frac{1}{\log \log m}\right) \)-approximation algorithms for the MISR problem \cite{Bereg2016b}. Finding a constant-approximation algorithm, or a \textsc{PTAS}, is still an intriguing open question. As we will show later, our two matching problems, being special cases
of the MISR problem, are also NP-hard and we give a polynomial-time 1/4-
approximation algorithm for each of them.

There exists polynomial-time exact algorithms, constant-approximation al-
gorithms, and PTAS’s for special cases of the MISR problem, according to the
intersection graph of the rectangles. The intersection graph is the undirected
type of the input as vertices, and two rectangles are adja-
cent if they are not independent. For any set $H$ of rectangles, let $G(H)$ denote the
intersection graph of $H$. Given two rectangles $R_1$ and $R_2$, we say that $R_2$ pierces
$R_1$ if into the $x$-axis the orthogonal projection of $R_1$ contains the orthogonal
projection of $R_2$, and into the $y$-axis the orthogonal projection of $R_2$ contains
the orthogonal projection of $R_1$. We say that two intersecting rectangles pierce if
one of them pierces the other one (see Figure 1a) [3,22,25]. Independently, Agar-
wal and Mustafa [3] and Lewin-Eytan et al. [22] showed that if all rectangles
are pairwise-piercing then the MISR problem can be solved in polynomial time
since in this case the intersection graph $G$ of the rectangles is perfect. Using a
classical result of Grötschel et al. [16], a maximum independent set of a perfect
graph can be computed in polynomial time. Agarwal and Mustafa [3] general-
ized this fact, claiming that the spanning subgraph of the intersection graph,
with only the edges corresponding to the piercing intersections, is also perfect.
We will use these results on pairwise-piercing rectangles in our approximation
algorithms. If $q$ is the clique number of the intersection graph, there exists a
$(1/4q)$-approximation [3,22]. In our two problems, we can build examples in
which the size of the optimal solution is either big or small, and independently
of that, the clique number $q$ is either big or small. Then, applying this result
does not always give a good approximation.

According to the nature of the rectangles in our families $R(S)$ and $\overline{R}(S)$, two
rectangles can have one of four types of intersection: (1) a piercing intersection
in which the two rectangles pierce (see Figure 1a); (2) a corner intersection in
which each rectangle contains exactly one of the corners of the other one and
these corners are not elements of $S$ (see Figure 1b); (3) a point intersection where
the intersection of the rectangles is precisely an element of $S$ (see Figure 1c); and
(4) a side intersection which is the complement of the above three intersection
types (see Figure 1d).

Let $G := G(R(S))$. Observe that if we consider the spanning subgraph $G'$
of $G$ with edge set the edges corresponding to the piercing intersections, and
compute in polynomial time the maximum independent set for $G'$ [3,22], then
we will obtain a set $H \subseteq R(S)$ of pairwise non-piercing rectangles. In that
case the set $H$ (after a slight perturbation that maintains the same intersection
graph) is a set of pseudo-disks and the PTAS of Chan and Har-Peled [11], for
approximating the maximum independent set in a family of pseudo-disks, can
be applied in $H$ to obtain an independent set $H' \subseteq H \subseteq R(S)$. Unfortunately,
we are unable to compare $|H'|$ with the optimal value of the MISR problem
for $R(S)$. The same arguments apply for $\overline{R}(S)$. On the other hand, there exits
PTAS's for the MISR problem when the rectangles have unit height [10], and
bounded aspect ratio [9,14].
Soto and Telha [25] studied the following problem to model cross-free matchings in two-directional orthogonal ray graphs (2-dorgs): Given both a point set $A_1$ and a point set $A_2$, find a maximum set of independent rectangles over all rectangles having an element of $A_1$ as bottom-left corner and an element of $A_2$ as top-right corner. For $A_1 := R$ and $A_2 := B$, where $S = R \cup B$, this problem is equivalent to the MISR problem over the rectangles $H \subseteq \mathcal{R}(S)$ that have a red point as bottom-left corner and a blue point as top-right corner. The authors solved this problem in polynomial time with the next observations: the rectangles of $H$ have only two types of intersections, piercing and corner, and $H$ can be reduced to a small one $H' \subseteq H$ whose intersection graph is perfect since the elements of $H'$ are pairwise piercing, and a maximum independent set in $H'$ is a maximum independent set in $H$. They proved them by using an LP-relaxation approach. By using simpler combinatorial arguments, we generalize and prove these observations to obtain our approximation algorithms.

3 Approximation algorithms

Given a point set $P$ in the plane, we say that $\mathcal{H}$ is a set of rectangles on $P$ if every element of $\mathcal{H}$ is of the form $D(a, b)$, where $a, b \in P$ and $D(a, b)$ contains exactly the points $a$ and $b$ of $P$. We say that the set $\mathcal{H}$ is complete if for every pair of elements $D(a, b)$ and $D(a', b')$ of $\mathcal{H}$ that have a corner intersection, the other two rectangles of the form $D(p, q)$ having a piercing intersection, where $p \in \{a, a'\}$ and $q \in \{b, b'\}$, also belong to $\mathcal{H}$ (see Figure 2a). Let $G_{p,c}(\mathcal{H})$ denote the spanning subgraph of $G(\mathcal{H})$ with edge set the edges that correspond to the piercing and the corner intersections.

Lemma 1. Let $P$ be a point set and $\mathcal{H}$ be any complete set of rectangles on $P$. Let $D(a, b)$ and $D(a', b')$ be two elements of $\mathcal{H}$ such that $D(a, b)$ and $D(a', b')$ have a corner intersection. A maximum independent set in $G_{p,c}(\mathcal{H} \setminus \{D(a, b)\})$ is a maximum independent set in $G_{p,c}(\mathcal{H})$.
Proof. Let \( I \) denote a maximum independent set of \( G_{p,c}(\mathcal{H}) \). Assume w.l.o.g. that \( x(a') < x(a) \leq x(b') < x(b) \) and \( y(a) < y(a') \leq y(b) < y(b') \) (see Figure 2a). We claim that either \((I \setminus \{D(a,b)\}) \cup \{D(a,b')\}\) or \((I \setminus \{D(a,b)\}) \cup \{D(a',b')\}\) is an independent set, which implies the result. Indeed, if \((I \setminus \{D(a,b)\}) \cup \{D(a,b')\}\) is an independent set, then we are done. Otherwise, at least one of the next two cases is satisfied: (1) there is a rectangle of \( I \setminus \{D(a,b)\} \) that has a corner intersection with both \( D(a',b') \) and \( D(a,b') \) (see Figure 2b); and (2) there is a rectangle of \( I \setminus \{D(a,b)\} \) that has a piercing intersection with both \( D(a',b') \) and \( D(a,b') \) (see Figure 2c). In both cases \( D(a',b') \) is independent from any rectangle in \( I \setminus \{D(a,b)\} \). Hence, \((I \setminus \{D(a,b)\}) \cup \{D(a',b')\}\) is an independent set. \( \square \)

**Lemma 2.** Let \( P \) be a point set and \( \mathcal{H} \) be any complete set of rectangles on \( P \). A maximum independent set in \( G_{p,c}(\mathcal{H}) \) can be found in polynomial time.

**Proof.** Using Lemma 1, the set \( \mathcal{H} \) can be reduced in polynomial time to the set \( \mathcal{H}' \subseteq \mathcal{H} \) such that all edges of the graph \( G_{p,c}(\mathcal{H}') \) correspond to piercing intersections, and a maximum independent set in \( G_{p,c}(\mathcal{H}') \) is a maximum independent set in \( G_{p,c}(\mathcal{H}) \). The former one can be found in polynomial time since \( G_{p,c}(\mathcal{H}') \) is a perfect graph, precisely a comparability graph [3,22,25].

### 3.1 Approximation for the MMRM problem

Let \( S = R \cup B \) be a colored point set in the plane. Let \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) be the next two families of rectangles of \( \mathcal{R}(S) \) (see Figure 3):
Fig. 3: The families $\mathcal{R}_1$ and $\mathcal{R}_2$.

**Lemma 3.** There exists a polynomial-time $(1/2)$-approximation algorithm for the maximum independent set of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively.

**Proof.** Consider the family $\mathcal{R}_1$, the arguments for the family $\mathcal{R}_2$ are analogous. Let $\text{OPT}_1$ denote the size of a maximum independent set in $\mathcal{R}_1$. Observe that a blue and a red rectangle in $\mathcal{R}_1$ can have only a piercing intersection, that two rectangles of the same color cannot have a side intersection, and that $G_{p,c}(\mathcal{R}_1)$ is a complete set of rectangles on $S$. Let $H$ be a maximum independent set of $G_{p,c}(\mathcal{R}_1)$ which can be found in polynomial time by Lemma 2. Note that in $H$ every blue rectangle is independent from every red rectangle, and rectangles of the same color can have point intersections only. Further observe that the graph $G(H)$ is acyclic and thus 2-colorable. Such a 2-coloring of $G(H)$ can be found in polynomial time and gives an independent set $I$ of $H$ with at least $|H|/2$ rectangles, which is an independent set in $\mathcal{R}_1$. The set $I$ is the approximation and satisfies $\text{OPT}_1 \leq |H| \leq 2|I|$. The result thus follows. □

**Theorem 1.** There exists a polynomial-time $(1/4)$-approximation algorithm for the MMRM problem.

**Proof.** Let $\text{OPT}$ denote the size of a maximum independent set in $\mathcal{R}(S)$, and $\text{OPT}_1$ and $\text{OPT}_2$ denote the sizes of the maximum independent sets in $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Let $I_1$ be a $(1/2)$-approximation for the maximum independent set in $\mathcal{R}_1$ and $I_2$ be a $(1/2)$-approximation for the maximum independent set in $\mathcal{R}_2$ (Lemma 3). The approximation for the MMRM problem is to return the set with maximum elements between $I_1$ and $I_2$. Since $\text{OPT} \leq \text{OPT}_1 + \text{OPT}_2 \leq 2|I_1| + 2|I_2| \leq 4 \max\{|I_1|, |I_2|\}$, the result follows. □
3.2 Approximation for the MMRM problem

Let $S = R \cup B$ be a colored point set in the plane. Let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3,$ and $\mathcal{R}_4$ be the next four families of rectangles of $\mathcal{R}(S)$:

- $\mathcal{R}_1$ contains the rectangles with a blue point in the bottom-left corner.
- $\mathcal{R}_2$ contains the rectangles with a red point in the bottom-left corner.
- $\mathcal{R}_3$ contains the rectangles with a blue point in the bottom-right corner.
- $\mathcal{R}_4$ contains the rectangles with a red point in the bottom-right corner.

Each of the above four families are complete sets of rectangles on $S$, where every two rectangles have either a corner or a piercing intersection. Then the maximum independent set in each family can be found in polynomial time (Lemma 2).

These observations imply the next result:

Theorem 2. There exists a polynomial-time (1/4)-approximation algorithm for the MBRM problem.

4 Hardness

In this section we prove that the MMRM problem and the MBRM problem are NP-hard even if further conditions are assumed. To this end we consider the next decision problems:

**Perfect Monochromatic Rectangle Matching (PMRM)** problem: *Is there a perfect monochromatic strong matching of $S$ with rectangles?*

**Perfect Bichromatic Rectangle Matching (PBRM)** problem: *Is there a perfect bichromatic strong matching of $S$ with rectangles?*

Proving that the PMRM problem and the PBRM problem are NP-complete, even on certain additional conditions, implies that the MMRM problem and the MBRM problem are NP-hard under the same conditions.

In our proofs we use a reduction from the Planar 1-in-3 SAT problem which is NP-complete [23]. The input of the Planar 1-in-3 SAT problem is a Boolean formula in 3-CNF whose associated graph is planar, and the formula is accepted if and only if there exists an assignment to its variables such that in each clause exactly one literal is satisfied [23]. Given any planar 3-SAT formula, our main idea is to construct a point set $S = S_1 \cup S_2$, such that: the elements of $S_2$ force to match certain pairs of points in $S_1$ and those pairs can only be matched with (axis-aligned) segments, there always exists a perfect matching with segments for $S_2$ independently of $S_1$, and there exists a perfect matching with segments for $S_1$ independently of $S_2$ if and only the formula is accepted.

The above method can be applied in the construction of Kratochvíl and Nešetřil [20] that proves that finding a maximum independent set in a family of axis-aligned segments is NP-hard. Indeed, we can put the elements of $S_1$ at the endpoints of the segments $T$ of their construction, by first modelling the parallel overlapping segments by segments sharing an endpoint. Then the elements of $S_2$
are added in a way that every two elements of $S_1$ can be matched if and only if they are endpoints of the same segment in $T$. This approach would give us a prove that our optimization problems are NP-hard, but not that our perfect matching decision problems are NP-complete which are stronger results. On the other hand, our hardness proofs give and alternative NP-hardness proof for the problem of finding a maximum independent set in a family of axis-aligned segments [20].

**Theorem 3.** The PMRM problem is NP-complete, even if we restrict the matching rectangles to segments.

**Proof.** Given a combinatorial matching of $S$, certifying that such a matching is monochromatic, strong, and perfect can be done in polynomial time. Then the PMRM problem is in NP. We prove now that the PMRM problem is NP-hard.

Let $\varphi$ be a planar 3-SAT formula. The (planar) graph associated with $\varphi$ can be represented in the plane as in Figure 4, where all variables lie on an horizontal line, and all clauses are represented by non-intersecting three-legged combs [19]. Using this embedding, which can be constructed in a grid of polynomial size [19], we construct a set $S$ of red and blue integer-coordinate points in a polynomial-size grid, such that there exists a perfect monochromatic strong matching with (axis-aligned) segments in $S$ if and only if $\varphi$ is accepted.

![Fig. 4: Planar representation of $\varphi = \left(v_1 \lor v_2 \lor v_3\right) \land \left(v_2 \lor v_3 \lor v_4\right) \land \left(v_1 \lor v_2 \lor v_4\right) \land \left(v_1 \lor v_3 \lor v_5\right) \land \left(v_1 \lor v_4 \lor v_5\right) \land \left(v_2 \lor v_3 \lor v_6\right) \land \left(v_4 \lor v_5 \lor v_6\right)$](image_url)

For an overview of our construction of $S$, refer to Figure 5. We use variable gadgets (the dark-shaded rectangles called variable rectangles) and clause gadgets (the light-shaded orthogonal polygon representing the three-legged comb).

**Variable gadgets:** For each variable $v$, its rectangle $Q_v$ has height 4 and width $6 \cdot d(v)$, where $d(v)$ is the number of clauses in which $v$ appears. We assume that each variable appears in every clause at most once. Along the boundary of $Q_v$, starting from a vertex, we put blue points so that every two successive points are at distance 2 from each other. We number consecutively in clockwise order these $4 + 6 \cdot d(v)$ points, starting from the top-left vertex of $Q_v$ which is numbered 1.

**Clause gadgets:** Let $C$ be a clause with variables $u$, $v$, and $w$, appearing in this order from left to right in the embedding of $\varphi$. Assume w.l.o.g. that the gadget of $C$ is above the horizontal line through the variables. Every leg of the gadget of $C$ overlaps the rectangles of its corresponding variable (denoted $x$) in a rectangle $Q_{x,C}$ of height 1 and width 2, so that the midpoint of the top side of $Q_{x,C}$ is a
blue point in the boundary of $Q_x$. The overlapping satisfies that such a midpoint is numbered with an even number if and only if $x$ appears positive in $C$. We further put three blue points equally spaced in the bottom side of $Q_x, C$, and other 9 blue points in the boundary of the gadget, as shown in Figure 5.

Forcing a convenient matching of the blue points: We add red points (a polynomial number of them) in such a way that any two blue points $a$ and $b$ can be matched if and only if $D(a, b)$ is a segment of any dotted line and does not contain any other colored point than $a$ and $b$ (see Figure 5). This can be done as follows: Since blue points have all integer coordinates, we can scale the blue point set (multiplying by 2) so that every element has even $x$- and $y$-coordinates. Then, we put a red point over every point of at least one odd coordinate that is not over any dotted line. We finally scale again the points, the blue and the red ones, and make a copy of the scaled red points and move it one unit downwards.

Reduction: Observe that in each variable $v$, the blue points along the boundary of $Q_v$ can be matched independently of the other points, and that they have two perfect strong matchings: the 1-matching that matches the $i$th point with the $(i+1)$th point for all odd $i$; and the 0-matching that matches the $i$th point with the $(i+1)$th one for all even $i$. In each clause $C$ in which $v$ appears, each of these two matchings forces a maximum strong matching on the blue points in the leg of the gadget of $C$ that overlaps $Q_v$, until reaching the points in the union of the three legs. We consider that variable $v = 1$ if we use the 1-matching, and consider $v = 0$ if the 0-matching is used. Let $C$ be a clause with variables $u$, $v$, and $w$; and draw perfect strong matchings on the blue points of the boundaries of $Q_u$, $Q_v$, and $Q_w$, respectively, giving values to $u$, $v$, and $w$. Notice that if exactly one among $u$, $v$, and $w$ makes $C$ positive, then the strong matching forced in the blue points of the gadget of $C$ is perfect (see Figure 6 and Figure 7). Otherwise, if none or at least two among $u$, $v$, and $w$ make $C$ positive, then the strong matching forced on the blue points of the gadget of $C$ is not perfect since at least 2 blue points are unmatched (see Figure 8 and Figure 9). Finally, note that the red points admit a perfect strong matching with segments such that no segment contains a blue point. Therefore, we can ensure that the 3-SAT formula

\[ C = (u \lor \overline{v} \lor w) \]

Fig. 5: The variable gadgets and the clause gadgets. In the figure, each variable $u$, $v$, $w$ might participate in other clauses.
ϕ can be accepted if and only if the point set $S$ admits a perfect strong matching with segments.

$$C = (u \lor v \lor w)$$

Fig. 6: If $u = 1$, $v = 1$, and $w = 0$, then only $u$ makes $C$ positive and there exists a perfect strong matching on the blue points.

$$C = (u \lor v \lor w)$$

Fig. 7: If $u = 0$, $v = 0$, and $w = 0$, then only $v$ makes $C$ positive and there exists a perfect strong matching on the blue points.

Suppose now that the two-colored point set $S$ is in general position. In what follows we show that the PMRM problem remains NP-complete under this assumption. To this end we first perturb the two-colored point set of the construction of the proof of Theorem 3 so that no two points share the same $x$- or $y$-coordinate, and second show that two points of $S$ can be matched in the perturbed point set if and only if they can be matched in the original one.

Alliez et al. [5] proposed the transformation that replaces each point $p = (x, y)$ by the point $\lambda(p) := ((1 + \varepsilon)x + \varepsilon^2y, \varepsilon^3x + y)$ for some small enough $\varepsilon > 0$, with the aim of removing the degeneracies in a point set for computing the Delaunay triangulation under the $L_\infty$ metric. Although this transformation can be used for our purpose, by using the fact that the points in the proof of Theorem 3 belong to a grid $[0..N]^2$, where $N$ is polynomially-bounded, we use the simpler transformation $\lambda(p) := ((1+\varepsilon)x+\varepsilon y, \varepsilon x+(1+\varepsilon)y)$ for $\varepsilon = 1/(2N+1)$,
Fig. 8: If $u = 0$, $v = 1$, and $w = 0$, then no variable makes $C$ positive and there does not exist any perfect strong matching on the blue points.

Fig. 9: If $u = 1$, $v = 0$, and $w = 1$, then two variables make $C$ positive and there does not exist any perfect strong matching on the blue points.

which is linear in $\varepsilon$. Both transformations change the relative positions of the initial points in the manner showed in Figure 10. Some useful properties of our transformation, stated in the next lemma, were not stated by Alliez et al. [5].

Fig. 10: Perturbation of the point set to put $S$ in general position.

**Lemma 4.** Let $N$ be a natural number and $P \subseteq [0..N]^2$. The function $\lambda : P \rightarrow \mathbb{Q}^2$ such that

$$
\lambda(p) = \left( x(p) + \frac{x(p) + y(p)}{2N+1}, y(p) + \frac{x(p) + y(p)}{2N+1} \right)
$$
satisfies the next properties:

(a) $\lambda$ is injective and the point set $\lambda(P) := \{\lambda(p) : p \in P\}$ is in general position.

(b) For every two distinct points $a, b \in P$ such that $x(a) = x(b)$ or $y(a) = y(b)$,
we have that $D(a, b) \cap P = \{a, b\}$ if and only if $D(\lambda(a), \lambda(b)) \cap \lambda(P) = \{\lambda(a), \lambda(b)\}$.

(c) For every three distinct points $a, b, c \in P$ such that $x(a) \neq x(b)$ and $y(a) \neq y(b)$,
we have that $c$ belongs to the interior of $D(a, b)$ if and only if $\lambda(c)$
belongs to the interior of $D(\lambda(a), \lambda(b))$.

Proof. Properties (a-c) are a consequence of $0 \leq \frac{x(p)+y(p)}{2N+1} \leq \frac{2N}{2N+1} < 1$. □

Theorem 4. The PMRM problem remains NP-complete on point sets in general position.

Proof. Let $S$ be the colored point set generated in the reduction of the proof
of Theorem 3. Let $N$ be a polynomially-bounded natural number such that
$S \subset [0..N]^2$, and let $S' := \lambda(S)$, where $\lambda$ is the function of Lemma 4. Consider
the next observations:

(a) If $a, b \in S$ are red points that can be matched in $S$ because $x(a) = x(b)$ and
$y(b) = y(a) - 1$, then $\lambda(a)$ and $\lambda(b)$ can also be matched in $S'$ (Property (b)
of Lemma 4).

(b) If $a, b \in S$ are blue points that can be matched in $S$, then we have that
either $x(a) = x(b)$ or $y(a) = y(b)$, which implies that $\lambda(a)$ and $\lambda(b)$ can also
be matched in $S'$ by Property (b) of Lemma 4.

(c) If $a, b \in S$ are blue points that cannot be matched in $S$ because $D(a, b)$ is
a segment containing a red point $c \in S$, then neither $\lambda(a)$ and $\lambda(b)$ can be
matched in $S'$ (Property (b) of Lemma 4).

(d) If $a, b \in S$ are blue points that cannot be matched in $S$ because $D(a, b)$ is a
box containing a point $c \in S$ in the interior, then neither $\lambda(a)$ and $\lambda(b)$ can
be matched in $S'$ since the box $D(\lambda(a), \lambda(b))$ contains $\lambda(c)$ (Property (c)
of Lemma 4).

The above observations imply that there exists a perfect strong rectangle matching
in $S$ if and only if it exists in $S'$. The result thus follows since $S'$ is in general
position by Property (a) of Lemma 4. □

Combining the construction of Theorem 3 with the perturbation of Lemma 4,
we can prove that the PMRM problem is also NP-complete when all points have
the same color, and that the PBRM problem is also NP-complete.

Lemma 5. Let $M_1 := \{(0, 0), (5, 0), (5, 5), (0, 5)\}$ and $M_2 := \{(1, 3), (2, 2), (2, 3),$
$(2, 4), (3, 1), (3, 2), (3, 3), (4, 2)\}$ be two point sets. The point set $M_1 \cup M_2$ has a
perfect strong matching with rectangles, and for every proper subset $M'_1 \subset M_1$
the point set $M'_1 \cup M_2$ does not have any perfect strong matching with rectangles.

Proof. The proof is straightforward (see Figure 11a, Figure 11b and Figure 11c).
Theorem 5. The PMRM problem remains NP-complete if all elements of $S$ have the same color.

Proof. Let $R_0$ and $B_0$ be the sets of the red points and the blue points, respectively, in the proof of Theorem 3. Let $Q$ be a set of (artificial) green points to block the forbidden matching rectangles in $B_0$, that is, for every two points $p, q \in B_0$ we have that $D(p, q)$ does not contain elements of $B_0 \cup Q$ other than $p$ and $q$ if and only if $D(p, q)$ is a matching rectangle in $R_0 \cup B_0$. In other words, $p, q$ can be matched in $R_0 \cup B_0$ if and only if they can be matched in $B_0 \cup Q$. The point set $S_1 := B_0 \cup Q$ belongs to the grid $[0..N]^2$, where $N$ is polynomially-bounded, and is not in general position. Let $S_2 := \lambda(S_1)$, where $\lambda$ is the function of the Lemma 4. We now replace each green point $g$ of $Q$ by a translated and stretched copy $S_g$ of the set $M_1 \cup M_2$ of Lemma 5, with all elements colored blue (see Figure 11a). Let $S := B_0 \cup (\bigcup_{g \in Q} S_g)$. Putting the elements of $S_g$ close enough one another for every $g$, we can guarantee that if we want to obtain a perfect strong matching in $S$ then we must have by Lemma 5 a perfect strong matching in each $S_g$ in particular (see Figure 11b). Therefore, the set $S_g$ acts as the green point $g$ blocking the forbidden matching rectangles in $B_0$. The construction of $S$ starts from the planar 3-SAT formula $\varphi$ of the proof of Theorem 3 and using all the above arguments we can claim that there exists a perfect strong matching in $S$ if and only if the formula $\varphi$ is accepted. Hence, the PMRM problem with input points of the same color is NP-complete since there exists a polynomial-time reduction from the PLANAR 1-IN-3 SAT problem. □
Theorem 6. The PBRM problem is NP-complete, even if the point set $S$ is in general position.

Proof. Let $R_0$ and $B_0$ be the sets of the red points and the blue points, respectively, in the proof of Theorem 5. Change to color red elements of $B_0$, to obtain the colored point set $S_0$, so that for every segment matching two blue points in $R_0 \cup B_0$ exactly one of the matched points is changed to color red (see Figure 12a). For every point $p \in B_0$, let $p'$ denote the corresponding point in $S_0$, and vice versa. Let $Q$ be a set of (artificial) green points to block the forbidden matching rectangles in $S_0$, that is, for every two points $p', q' \in S_0$ we have that $D(p', q')$ does not contain elements of $S_0 \cup Q$ other than $p'$ and $q'$ if and only if $D(p, q)$ is a matching rectangle in $R_0 \cup B_0$. The point set $S_1 := S_0 \cup Q$ belongs to the grid $[0 .. N]^2$, where $N$ is polynomially-bounded, and is not in general position. Let $S_2 := \lambda(S_1)$, where $\lambda$ is the function of the Lemma 4. We now replace each green point $g$ of $Q$ by the set $S_g$ of eight red and blue points in general position (see Figure 12b). Let $S := S_0 \cup (\bigcup_{g \in Q} S_g)$. Putting the elements of $S_g$ close enough one another for every $g$, we can guarantee that $S$ is also in general position and that for every $g$ the points of $S_g$ appear together in both the left-to-right and the top-down order of $S$. This last condition ensures that if we want to obtain a perfect strong matching in $S$ then we must have a perfect strong matching for each $S_g$ in particular (see Figure 12c and Figure 12d) because for all $g$ every red point of $S_g$ cannot be matched with any blue point not in $S_g$. Therefore, the set $S_g$ acts as the green point $g$ blocking the forbidden matching rectangles in $S_0$. Hence, the PBRM problem is NP-complete, even on points in general position. \qed

5 Discussion

We have proved that finding a maximum strong matching of a two-colored point set, with either rectangles containing points from the same color or rectangles containing points of different colors, is NP-hard and provide a $(1/4)$-approximation for each case. Our approximation algorithms provide a $(1/4)$-approximation for the problem of finding a maximum strong rectangle matching of points of the same color, studied by Bereg et al. [9]. However, the approximation ratio is smaller than $2/3$, the one given by Bereg et al. We leave as open to find a better constant approximation algorithm for our problems, or a PTAS. On the other hand, finding a constant-approximation algorithm for the general case of the MAXIMUM INDEPENDENT SET OF RECTANGLES problem is still an intriguing open question.

References

1. B. M. Ábrego, E. M. Arkin, S. Fernández-Merchant, F. Hurtado, M. Kano, J. S. B. Mitchell, and J. Urrutia. Matching points with squares. *Discrete & Computational Geometry*, 41(1):77–95, 2009.
\[ C = (u \lor \neg v \lor w) \]

\[ u = 1 \]
\[ v = 1 \]
\[ w = 0 \]

Fig. 12: Proof of Theorem 6. (a) Changing the colors of the blue points in the gadgets of the proof of Theorem \[ \text{[3]} \] (b) The eight points (close enough one another) that replace each green point. (c) One of the only two ways to match the points corresponding to a green point in order to obtain a perfect matching. (d) The other way.

2. A. Adamaszek and A. Wiese. Approximation schemes for maximum weight independent set of rectangles. CoRR, abs/1307.1774, 2013.
3. P. K. Agarwal and N. H. Mustafa. Independent set of intersection graphs of convex objects in 2d. Computational Geometry, 34(2):83 – 95, 2006.
4. P. K. Agarwal, M. J. van Kreveld, and S. Suri. Label placement by maximum independent set in rectangles. Computational Geometry, 11(3-4):209–218, 1998.
5. P. Alliez, O. Devillers, and J. Snoeyink. Removing degeneracies by perturbing the problem or the world. Technical Report 3316, INRIA, 1997.
6. S. Bereg, N. Mutsanas, and A. Wolff. Matching points with rectangles and squares. Comput. Geom. Theory Appl., 42(2):93–108, 2009.
7. P. Chalermsook. Coloring and maximum independent set of rectangles. In Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, volume 6845 of LNCS, pages 123–134. Springer Berlin Heidelberg, 2011.
8. P. Chalermsook and J. Chuzhoy. Maximum independent set of rectangles. In Proc. of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’09, pages 892–901, Philadelphia, USA, 2009.
9. T. M. Chan. Polynomial-time approximation schemes for packing and piercing fat objects. J. Algorithms, 46(2):178–189, 2003.
10. T. M. Chan. A note on maximum independent sets in rectangle intersection graphs. Information Processing Letters, 89(1):19 – 23, 2004.
11. T. M. Chan and S. Har-Peled. Approximation algorithms for maximum independent set of pseudo-disks. In Proceedings of the Twenty-fifth Annual Symposium on Computational Geometry, SCG ’09, pages 333–340, 2009.
12. A. Dumitrescu and R. Kaye. Matching colored points in the plane: Some new results. *Computational Geometry*, 19(1):69 – 85, 2001.
13. A Dumitrescu and W. L. Steiger. On a matching problem in the plane. *Discrete Mathematics*, 211:183–195, 2000.
14. T. Erlebach, K. Jansen, and E. Seidel. Polynomial-time approximation schemes for geometric intersection graphs. *SIAM J. Comput.*, 34(6):1302–1323, 2005.
15. R. J. Fowler, M. Paterson, and S. L. Tanimoto. Optimal packing and covering in the plane are NP-complete. *Inf. Process. Lett.*, 12(3):133–137, 1981.
16. M. Grötschel, L. Lovász, and A. Schrijver. Polynomial algorithms for perfect graphs. In *Topics on Perfect Graphs*, volume 88, pages 325 – 356. 1984.
17. H. Imai and T. Asano. Finding the connected components and a maximum clique of an intersection graph of rectangles in the plane. *J. Alg.*, 4(4):310–323, 1983.
18. S. Khanna, S. Muthukrishnan, and M. Paterson. On approximating rectangle tiling and packing. In *ACM-SIAM Symposium on Discrete Algorithms*, pages 384–393, 1998.
19. D. E. Knuth and A. Raghunathan. The problem of compatible representatives. *SIAM J. Discret. Math.*, 5(3):422–427, 1992.
20. J. Kratochvíl and J. Nešetřil. Independent set and clique problems in intersection-defined classes of graphs. *Comm. Math. Univ. Carolinae*, 31(1):85–93, 1990.
21. L. C. Larson. *Problem-solving through problems*. Problem books in mathematics. Springer, 1990.
22. L. Lewin-Eytan, J. Naor, and A. Orda. Admission control in networks with advance reservations. *Algorithmica*, 40(4):293–304, 2004.
23. W. Mulzer and G. Rote. Minimum-weight triangulation is NP-hard. *J. ACM*, 55(2), 2008.
24. C. S. Rim and K. Nakajima. On rectangle intersection and overlap graphs. *IEEE Transactions on Circuits and Systems*, 42(9):549–553, 1995.
25. J. A. Soto and C. Telha. Jump number of two-directional orthogonal ray graphs. In *Integer Programming and Combinatorial Optimization*, volume 6655 of *LNCS*, pages 389–403. Springer Berlin Heidelberg, 2011.