Improved actions, the perfect action, and scaling by perturbation theory in Wilson's renormalization group: the two dimensional $O(N)$-invariant non linear $\sigma$-model in the hierarchical approximation

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Abstract

We propose a method using perturbation theory in the running coupling constant and the idea of scaling to determine improved actions for lattice field theories combining Wilson’s renormalization group with Symanzik’s improvement program. The method is based on the analysis of a single renormalization group transformation. We test it on the hierarchical $O(N)$ invariant $\sigma$ model in two dimensions.
1 Introduction

Improved actions for lattice field theories were proposed by Symanzik \cite{S} to reduce systematic errors due to a finite lattice spacing. The improvement parameters are calculated by bare perturbation theory. Improved actions also naturally arise in the block spin renormalization group approach of Wilson \cite{WK}. There they come in form of a renormalized trajectory of lattice actions completely free of lattice artefacts. One way to approach the renormalized trajectory starting from any reasonable ansatz is to iterate renormalization group transformations. A different way due to Hasenfratz and Niedermayer \cite{HN} for asymptotically free theories consists of a classical treatment of a single step. It gives the asymptotic form of the renormalized trajectory called perfect action. We propose to approximate systematically the renormalized trajectory in the spirit of Symanzik by a scaling analysis of a single renormalization group transformation in terms of perturbation theory in the running coupling constant.

Consider the renormalization group flow of an asymptotically free lattice field $\phi$ on a unit lattice with action $S(\phi)$ generated by a renormalization group transformation $R$. Imagine a one dimensional renormalized trajectory $S^{RT}(\phi, f)$ parametrized by the value of a running coupling $f$. A block spin transformation will do nothing to an action on this renormalized trajectory but change the value of the running coupling $f$ into $f'$. The whole dynamics of the renormalization group is then encoded in this change of the running coupling. ($f$ can for instance be marginally relevant, which means that $f' = f + \Delta \beta(f)$ with $\Delta \beta(f) = \beta_2 f^2 + O(f^3)$ where $\beta_2 > 0$.) An action on the renormalized trajectory therefore satisfies the discrete Callan-Symanzik type equation $RS^{RT}(\phi, f') = S^{RT}(\phi, f)$ and is said to scale. Recall that we formulate everything in terms of transformations on unit lattices. (The reader who prefers to see explicit dependence on lattice spacings is invited to rewrite the equations using dimensional analysis.)

In the domain where the running coupling is small, perturbation theory can be applied to compute $S^{RT}(\phi, f)$ (at least in a small field region) as a solution to the Callan-Symanzik type renormalization group equation. The strategy is as follows. Consider a lattice action of some general form $S(\phi) = \sum_a P_a(f)O_a(\phi)$. Here $P_a(f)$ denote polynomials in the running coupling containing free improvement parameters. In the renormalization group terminology the action can contain irrelevant terms. We then compute a block spin transformation to a given order $s$ of perturbation theory in the running coupling $f$ to obtain $RS(\phi, f) = R^{(s)}S^{(s)}(\phi, f) + O(f^{s+1})$. From the result we determine the effective running coupling $f'$ as a function of $f$, invert this relation, and express the effective action in terms of $f'$. We can then compare the original action as a function of $f$ with the effective action as a function of $f'$. The
idea is then to demand that the action be invariant up to corrections of order $s+1$ to
determine the form of the action and fix the values of the improvement parameters.
Imagine that we have succeeded in doing so. The result is the exact renormalized
trajectory $S^{(s)}(\phi, f)$ up to order $s$ in the running coupling. In other words we have
removed all scaling violations to a given order in the running coupling.

In practice the improvement procedure is performed step by step in perturba-
tion theory. To get started we can take an unimproved action and see what kind
of terms are generated in a first or second order calculation. Then the action is ex-
tended by the corresponding terms with couplings depending appropriately on the
running coupling with undetermined coefficients. Eventually this procedure has to
be iterated until no further terms appear. Fortunately a finite order calculation can
only produce a finite number of terms. Then the coefficients are fixed as described
above completing the first improvement step.

In a pilot study we have worked out this improvement program for the hierar-
chical two dimensional $O(N)$-invariant $\sigma$-model. The existence of a renormalized
trajectory in this model and asymptotic freedom (for $N > 1$) have been proved rig-
orously by Gawedzki and Kupiainen [GK]. We have no doubts that our action can
be proved to be an excellent approximation to the true renormalized trajectory in a
small field region using [GK] kind of bounds. As expected the leading term of the
renormalized trajectory can be obtained by a perfect action calculation à la [HN].
We take this perfect action as the starting point of the improvement program. The
hierarchical $\sigma$-model itself is of marginal interest. However, it is an optimal labo-
rary to test renormalization group ideas. The implementation of our program to
the full $O(N)$-invariant $\sigma$-model is under investigation.

Let us mention that the improvement also works when dealing with more than
one coupling to be renormalized. The renormalization group is then reduced to
a transformation of a set of running couplings, the Callan-Symanzik type equation
being exactly the one above. It is conceivable that the improvement program admits
a nonperturbative formulation in terms of polymer representations.

2 The hierarchical $O(N)$ model

In the hierarchical (local approximation) renormalization group for general $N$ com-
ponent models we are left to deal with functions $Z(\phi)$ of a single variable $\phi$ with
values in $\mathbb{R}^N$. For $O(N)$ models one assumes further that $Z(M\phi) = Z(\phi)$ for all
$M \in O(N)$, requiring $Z(\phi)$ to be a function of $|\phi|$. We consider the hierarchical
renormalization group transformation in two dimensions given by (the case $L = \sqrt{2}$
of [GK]

\[ RZ(\psi) = \mathcal{N} \int d\mu_\gamma(\zeta)Z(\psi + \zeta)^2. \tag{1} \]

It preserves \( O(N) \)-invariance. \( \gamma \) is a real positive number and

\[ d\mu_\gamma(\zeta) = (2\pi\gamma)^{-\frac{N}{2}}d^N\zeta \exp\left(-\frac{\zeta^2}{2\gamma}\right). \tag{2} \]

the corresponding Gaussian measure on \( \mathbb{R}^N \). \( \mathcal{N} \) is an optional normalization constant. Concerning background material on hierarchical models see [GK] and references therein. In the following we will write

\[ Z(\phi) = \exp(-V(\phi)). \tag{3} \]

A sufficiently general form of \( V(\phi) \) for \( \sigma \) models close to the renormalized trajectory proves to be

\[ V(\phi) = \sum_{a \geq 2} P_a(f)(|\phi| - \frac{1}{f})^a \tag{4} \]

with \( P_2(f) = \frac{1}{4\gamma} + O(f^2) \) and \( P_a(f) = O(f^a) \) for \( a > 2 \). The running coupling is given by the inverse radius. In this setup \( \phi \) is not restricted to take values on the sphere with radius \( \frac{1}{f} \), the reason being that our recursion does not preserve this condition. The normalization constant \( \mathcal{N} \) is conveniently chosen such that \( V(\phi) = 0 \) for \( |\phi| = \frac{1}{f} \). We adopt this renormalization condition. When computing single renormalization group transformations we will speak of the previous action as the bare action and of the outcome as the effective or renormalized action.

### 3 Perfect action

To calculate the perfect action [HN] for the hierarchical \( O(N) \) model we write the potential in the form

\[ V(\phi) = r^2\mathcal{V}\left(\frac{|\phi|}{r} - 1\right) \tag{5} \]

with \( r = \frac{1}{f} \). The hierarchical transformation then takes the form

\[ \exp\left(-r^2\mathcal{V}\left(\frac{1}{r}||\psi|| - 1\right)\right) = \mathcal{N}\left(\frac{2\pi\gamma}{r^2}\right)^{-\frac{N}{2}}\int d^N\xi \exp\left(-\frac{r^2\xi^2}{2\gamma^2} - 2r^2\mathcal{V}(\frac{1}{r}||\psi + \xi|| - 1)\right). \tag{6} \]
The fluctuation field has been rescaled by $r$. It turns out in perturbation theory that the effective radius is

$$r' = r - \frac{\gamma}{2} (N - 1) \frac{1}{r} + O\left(\frac{1}{r^2}\right).$$  \hspace{1cm} (7)

In the limit when $r$ and hence also $r'$ is sent to infinity the fluctuation integral can be evaluated by the saddlepoint method. Rescaling also the block spin $\psi$ by $r$ the equation for the perfect action becomes

$$V_{RT}(|\psi| - 1) = \inf_{\xi \in \mathbb{R}^N} \left( \frac{1}{2\gamma} \xi^2 + 2V_{RT}(|\psi + \xi| - 1) \right).$$  \hspace{1cm} (8)

This equation can be solved by the ansatz $c_2(|\phi| - r)^2$ with a single parameter $c_2$. The solution is $c_2 = \frac{1}{4\gamma}$. The perfect action approximation for the renormalized trajectory in this model is then given by

$$V_{RT}(\phi) = \frac{r^2}{4\gamma} \left( \frac{1}{r} |\phi| - 1 \right)^2,$$  \hspace{1cm} (9)

The right way to think of this formula is as a line of fixed points parametrized by $r$ of the classical renormalization group transformation. Note that unlike [HN] the action is not just multiplied by $r^2$. We have tested this approximation numerically as will be explained below.

4 Perturbation theory

The perturbation expansion for the $O(N)$ model can be computed to high orders using computer algebra. Let us explain the method in a second order calculation for the perfect action. As bare potential we take

$$V(\phi) = \frac{r^2}{4\gamma} \left( \frac{1}{r} |\phi| - 1 \right)^2.$$ \hspace{1cm} (10)

The effective potential will be $O(N)$-invariant. Without loss of generality we can therefore take the block spin to be given by $\psi = (r + \Psi)\hat{e}$ with $\hat{e}$ an $N$ component unit vector, say $(0, \ldots, 0, 1)^T$. The shift of $\Psi$ serves to place us into the minimum of the bare potential. We then decompose orthogonally the fluctuation field into $\zeta = \sigma\hat{e} + \pi$ with respect to the direction of $\psi$. The one component variable $\sigma$ is the radial fluctuation field while the $N - 1$ component variable $\pi$ is the tangential
fluctuation field. The bare potential is expanded in powers of the coupling $f = \frac{1}{r}$. Up to second order it is given by

$$V(\psi + \zeta) = \frac{1}{4\gamma} \psi^2 + \frac{1}{2\gamma} \psi \sigma + \frac{1}{4\gamma} \sigma^2 + \frac{f}{4\gamma} \psi \pi^2 + \frac{f}{4\gamma} \sigma \pi^2 - \frac{f^2}{4\gamma} \psi^2 \pi^2 - \frac{f^2}{2\gamma} \psi \sigma \pi^2 - \frac{f^2}{2\gamma} \sigma^2 \pi^2 + \frac{f^2}{16\gamma} (\pi^2)^2 + O(f^3).$$

(11)

One immediately observes a linear and a quadratic term in $\sigma$ which cannot be treated as perturbations. The mass term changes the $\sigma$ covariance to $\frac{1}{2\gamma}$. The source term is taken into account by a shift of the radial variable $\sigma = \xi - \frac{1}{2} \Psi$. The renormalization group transformation then becomes

$$\exp(-V'(\Psi)) = \exp\left(-\frac{1}{4\gamma} \Psi^2\right) A' \int d\mu \frac{1}{2\gamma}(\xi) \int d\mu \frac{1}{2\gamma}(\pi) \exp(-2V(\Psi, \xi, \pi))$$

(12)

in terms of $\xi$ and $\pi$. The bare potential takes the form

$$V(\Psi, \xi, \pi) = \frac{f}{8\gamma} \Psi \pi^2 + \frac{f}{4\gamma} \xi \pi^2 - \frac{f^2}{16\gamma} \Psi^2 \pi^2 - \frac{f^2}{4\gamma} \Psi \xi \pi^2 - \frac{f^2}{4\gamma} \xi \pi^2 + \frac{f^2}{16\gamma} (\pi^2)^2 + O(f^3).$$

(13)

At this point perturbation theory is applicable. Although the potential is non-polynomial to begin with only finitely many terms show up at finite order with a leading trilinear vertex. Note that the perfect action is recovered when fluctuations are completely neglected. The perturbation expansion is straightforward using the Gaussian correlations

$$\int d\mu \frac{1}{2\gamma}(\xi) \xi^{2n} = \left(\frac{\gamma}{2}\right)^n \prod_{m=0}^{n-1} (2m + 1)$$

(14)

and

$$\int d\mu \frac{1}{2\gamma}(\pi^2)^n = \gamma^n \prod_{m=0}^{n-1} (2m + N - 1).$$

(15)

Computing the fluctuation integral to second order perturbation theory we obtain an effective potential of the form

$$V'(\Psi) = \left(\frac{1}{4\gamma} - \frac{3}{16} (N - 1) f^2\right) \Psi^2 + \frac{1}{4} (N - 1) f \Psi + O(f^3).$$

(16)
We then determine the value \( \delta r \) of \( \Psi \) at which the effective potential attains its minimum and substitute \( \Psi = \Phi + \delta r \). The change of \( r \) is due to the linear term in \( \Psi \). The meaning of this variable is \( \Phi = |\psi| - r' \) with \( r' = r + \delta r \) the renormalized radius. To second order perturbation theory in \( f \) the change of the radius is \( \delta r = -\frac{\gamma}{2} (N-1)f + O(f^3) \). From this we find a renormalized coupling \( f' = \frac{1}{r'} \) of the form

\[
f' = f + \frac{\gamma}{2} (N-1)f^3 + O(f^5)\tag{17}\]

(The vanishing of the \( f^4 \) term of the \( \Delta \beta \) function follows from a fourth order calculation.) In particular we have confirmed that the model is perturbatively asymptotically free for \( N > 1 \). That is, when perturbation theory applies we find a flow where the coupling slowly grows and the radius slowly shrinks. The effective potential becomes

\[
V'(\Phi) = \left( \frac{1}{4\gamma} - \frac{3}{16} (N-1)f^2 \right) \Phi^2 + O(f^3)\tag{18}
\]

in terms of \( \Phi = |\psi| - r' \) and \( f' \). We also see that this action remains invariant in the sense of scaling to first order. Scaling violation shows up in a second order flow of the overall prefactor. In a zeroth improvement step they can be compensated for by changing the bare action into

\[
V(\phi) = \left( \frac{1}{4\gamma} + c_2^{(2)} f^2 \right) \left( |\phi| - \frac{1}{f} \right)^2.\tag{19}
\]

The correct value of the improvement parameter is \( c_2^{(2)} = \frac{3}{8} (N-1) \). The resulting action can then be seen to scale even to second order.

To appreciate this result consider for instance the polynomial bare potential

\[
V(\phi) = \frac{r^2}{\gamma} \left( \frac{1}{r^2} \phi^2 - 1 \right)^2.\tag{20}
\]

Perturbation theory reveals that its effective action contains among other terms a cubic one at first order in the effective coupling. Therefore this action does not scale even to first order.

## 5 Improved action

Let us now also remove the scaling violations of second order. As a bare potential we take the second order improved one which we write in the form

\[
V^{(2)}(\phi) = P_2^{(2)}(f) \left( |\phi| - \frac{1}{f} \right)^2\tag{21}
\]
with \( P_2^{(2)}(f) = \left[ \frac{1}{4\gamma} + \frac{3}{8}(N - 1)f^2 \right] \). The effective potential computed to third order perturbation theory is given by

\[
V^{(2)}(\Phi) = \left( \frac{1}{4\gamma} - \frac{3}{8}(N - 1)f^2 \right)\Phi^2 + \frac{7}{48}(N - 1)f^2\Phi^3 + O(f^4)
\]  

(22)

in terms of \( f' \). What is new is a cubic term in \( \Phi = |\phi| - r' \). As it will be generated anyway to third order it is natural to include it to this order in the bare action. In other words let us make the ansatz

\[
V^{(3)}(\phi) = P_2^{(3)}(f) \left( |\phi| - \frac{1}{f} \right)^2 + P_3^{(3)}(f) \left( |\phi| - \frac{1}{f} \right)^3
\]  

(23)

with \( P_2^{(3)}(f) = P_2^{(2)}(f) + c_2^{(3)}f^3 \) and \( P_3^{(3)}(f) = c_3^{(3)}(N - 1)f^3 \). Here we have anticipated a possible cubic correction to \( P_2(f) \) which in fact turns out not to exist. The ansatz contains two new improvement parameters. Computing again the effective potential to third order, the correct values of these are seen to be \( c_2^{(3)} = 0 \) and \( c_3^{(3)} = \frac{7}{36} \). This action reproduces itself up to fourth order corrections. The improvement scheme can now be iterated. Suppose that we have found the potential

\[
V^{(s)}(\phi) = \sum_{a=2}^{s} P_a^{(s)}(f) \left( |\phi| - \frac{1}{f} \right)^a
\]  

(24)

which scales up to order \( s \). It’s image under a renormalization group transformation evaluated in perturbation theory to order \( s + 1 \) is of the form

\[
V^{(s+1)}(\phi) = \sum_{a=2}^{s+1} P_a^{(s+1)}(f') \left( |\phi| - \frac{1}{f} \right)^a
\]  

(25)

containing polynomials \( P_a^{(s+1)}(f) = \left[ P_a^{(s)}(f) + c_a^{(s+1)}f^{s+1} \right] \) for \( a \leq s \) and \( P_a^{(s+1)}(f) = c_a^{(s+1)}f^{s+1} \). This general form again is reproduced to this order. The corresponding effective potential to order \( s + 1 \) is given by

\[
V^{(s+1)}(\Phi) = \sum_{a=2}^{s+1} P_a^{(s+1)}(f')\Phi^a + O(f'^{s+2})
\]  

(26)

with \( P_a^{(s+1)}(f) = \left[ P_a^{(s)}(f') + c_a^{(s+1)}f'^{s+1} \right] \) for \( a \leq s \) and \( P_a^{(s+1)}(f) = c_a^{(s+1)}f'^{s+1} \). The effective coefficients \( c_a^{(s+1)} \) depend linearly on their bare counterparts \( c_a^{(s+1)} \). (To order \( s + 1 \) they have no other choice.) In order that there be no scaling violation to order \( s + 1 \) the polynomials \( P_a^{(s+1)}(x) \) and \( P_a^{(s+1)}(x) \) have to be equal.
From this we obtain a system of linear equations for the coefficients $c_a^{(s+1)}$. This system turns out to have a unique solution. To order $f^6$ it is given by

\begin{align}
P_2^{(6)}(f) &= \frac{1}{4\gamma} - \frac{3}{8}(N-1)f^2 + \frac{29\gamma(N-1)}{56} + \frac{17\gamma(N-1)^2}{12} - \\
&\quad \left(\frac{571\gamma^2(N-1)^3}{84} + \frac{327167\gamma^2(N-1)^2}{70560} + \frac{10305\gamma^2(N-1)}{3472}\right)f^6,
\end{align}

\begin{align}
P_3^{(6)}(f) &= \frac{7}{36}(N-1)f^3 - \frac{239\gamma(N-1)}{360} + \frac{361\gamma(N-1)^2}{336} - \\
&\quad \left(\frac{1479\gamma(N-1)}{1736} + \frac{37753\gamma(N-1)^2}{35280}\right)f^5,
\end{align}

\begin{align}
P_4^{(6)}(f) &= -\frac{15}{112}(N-1)f^4 + \frac{1479\gamma(N-1)}{1736} + \frac{37753\gamma(N-1)^2}{35280} - \\
&\quad \left(\frac{18\gamma(N-1)^3}{48} - \frac{11(N-1)^2 + 32(N-1)}{r^3}\right)f^6,
\end{align}

\begin{align}
P_5^{(6)}(f) &= \frac{31}{300}(N-1)f^5,
\end{align}

\begin{align}
P_6^{(6)}(f) &= -\frac{21}{248}(N-1)f^6.
\end{align}

(27)

We observe that the power series for $P_a(f)$ contains only even (odd) powers of $f$ when $a$ is even (odd). Furthermore we observe that the signs of the coefficients alternate. The complete series is not expected to converge due to instanton singularities. It would however be very interesting to apply the machinery of resummation methods to a high order approximation to the renormalized trajectory. What remains to be said to order $f^6$ is the recursion formula

\begin{align}
f' &= f + \frac{\gamma(N-1)}{2}f^3 - \frac{\gamma^2(N-1)^2}{12}f^5 + O(f^7),
\end{align}

(28)

for the running coupling. For the sake of completeness we also include the recursion

\begin{align}
r' &= r - \frac{\gamma}{2}(N-1)\frac{1}{r} + \frac{\gamma^2}{12}(3(N-1)^2 + N-1)\frac{1}{r^3} - \\
&\quad \frac{\gamma^3}{48}(18(N-1)^3 - 11(N-1)^2 + 32(N-1))\frac{1}{r^5} + O(f^7)
\end{align}

(29)

for the effective radius.

6 Numerical results

Fortunately we are not left alone with perturbation theory. Hierarchical renormalization group transformations can also be computed numerically. Therefore we are
able to test our perturbative results and find the limits of their validity. Going back to the point of departure we write the transformation in the form

$$\exp (-\mathcal{V}(|\psi|)) = \mathcal{N} \int d^N \xi \exp \left(-\frac{1}{2\gamma} \xi^2 - 2\mathcal{V}(|\psi + \xi|)\right).$$  \hspace{1cm} (30)$$

with $\mathcal{V}(|\phi|) = V(\phi)$. The fluctuation integral is most conveniently done in $N$ dimensional polar coordinates given by $|\xi| = R$ and $\psi \cdot \xi = |\psi| R \cos \theta$. The independent polar angles are immediately integrated over, changing just the (irrelevant) normalization. The result is

$$\exp (-\mathcal{V}(|\psi|)) = \mathcal{N}' \int_0^\infty dR \int_0^\pi d\theta w(R, \theta) B \left(\sqrt{|\psi|^2 + 2|\psi| R \cos \theta + R^2}\right)$$  \hspace{1cm} (31)$$

with a weight function

$$w(R, \theta) := R^{N-1} \sin^{N-2} \theta \exp \left(-\frac{1}{2\gamma} R^2\right)$$  \hspace{1cm} (32)$$

and a Boltzmann factor (squared by the block volume two)

$$B(\varphi) := \exp (-2\mathcal{V}(\varphi)).$$  \hspace{1cm} (33)$$

$\varphi$ denotes a positive real valued single component field (the modulus of $\psi$). In a numerical evaluation the range of the $R$-integration can be restricted to $[0, R_{\text{max}}]$ with $R_{\text{max}}$ sufficiently large. The contribution of $(R_{\text{max}}, \infty)$ is both supressed by the exponential decay of the weight function $w(R, \theta)$ and the decay of the Boltzmann factor. Furthermore it is sufficient to follow the flow of $B(\varphi)$ in an interval $\varphi \in [\varphi_{\text{min}}, \varphi_{\text{max}}]$ with numerically negligible corrections. Let us emphasize that the interval should include a large field region not covered by perturbation theory in order to test our improved potential. We parametrize $\mathcal{V}(\varphi)$ by a set of $N_{\varphi}$ values $V_i = V(\varphi_i)$ at equidistant points $\varphi_i \in [\varphi_{\text{min}}, \varphi_{\text{max}}]$. Intermediate values are computed by cubic-spline interpolation. The integral (31) is performed by standard numerical methods yielding a set of effective values $V'_i$ from which we compute parameters $r'$ and $C'_a$ by a least square fit using the ansatz

$$V'(\varphi) = \sum_{a=2}^s C'_a (\varphi - r')^a.$$  \hspace{1cm} (34)$$

The renormalized trajectory can be computed by iterative renormalization group steps. (All numerical results refer to parameter values $\gamma = 1, N = 3, \varphi_{\text{min}} =$
We start with an unimproved bare potential $V^{(0)}(|\phi|) = 1/(4\gamma) (|\phi| - r)^2$ and iterate renormalization group transformations. After each step the resulting effective potential comes closer to the renormalized trajectory giving successively better approximations (see figure 1). We took the values of the couplings $C'_a(f') =: P'^{RT}_a(f')$ after 10 iterations as estimates of the true renormalized couplings at the radius $r' = 1/f'$. By varying the bare radius $r$ we computed the renormalized trajectory for a set of renormalized radii $r' \in [0, 100]$. The results were then compared with the coefficients $P^{(6)}_a(f')$ of the improved perfect action approximation $V^{(6)}$. As expected one finds excellent agreement (fig. 2) for large $r = 1/f$. The relative deviation of the perturbative coefficients from their nonperturbative values is less than $10^{-3}$ for $r > 30$ and remains smaller than two percent for $r > 5$. The improvement compared to the unimproved perfect action is especially visible in the large field region (fig. 3).

Around $r = 5$ the deviation increases dramatically. This breakdown of perturbation theory in the crossover region from the ultraviolet to the infrared regime is illustrated in fig. 3. The renormalized trajectory connects the UV-fixpoint with the IR-fixpoint. The IR-fixpoint potential is a quadratic single well whereas the UV-fixpoint potential has a double well shape. Along the renormalized trajectory the radius $r$ and the depth of this well shrink simultaneously. At some (pseudo) critical point $r_c \approx 2.04$ (see fig. 5) the radius and the depth vanish and remain zero all the way to the IR fixed point. Obviously the coupling $f = 1/r$ diverges at this point and looses its meaning as expansion parameter. Though highly interesting in its own we do not intend to investigate the infrared behavior in terms of the flow in this paper. Let us only mention that it is conceivable that one could perform a change of coordinates to another expansion parameter in the crossover region. A natural candidate is the mass parameter in the potential since the double well becomes very flat before it turns over.

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![Graph](image_url)

Figure 1: Flow of the coupling $C_2(f)$ towards it’s renormalized value $P_2^{RT}(f)$ under consecutive applications of the RG transformation. The calculation was performed with a bare radius of 50. The other couplings $C_a(f)$ with $2 < a \leq 6$ behave similar.
Figure 2: Comparison of the perfect renormalized coupling $P^{RT}_2(f)$ with it’s improved and unimproved approximations $P^{(6)}_2(f), P^{(0)}_2(f)$

Figure 3: Large field behavior of the effective potential.
Figure 4: Flow of the effective potential. The plot shows $V'(|\psi|)$ after $n=0,3,6,9$ and 30 RG steps starting with the unimproved perfect action approximation $V^{(0)}(|\psi|) = 1/(4\gamma)(|\psi| - r)^2$ at $r = 5$ and ending in the HT fixpoint $V_{HT}(|\psi|) = 1/(4\gamma)|\psi|^2$.

Figure 5: Flow of the radius in the IR. At $r_c \approx 2.04$ the effective radius vanishes with $r' \approx |r - r_c|^{0.5}$. 
