Notes on noncommutative supersymmetric gauge theory on the fuzzy supersphere

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Abstract

In these notes we review Klimčík’s construction of noncommutative gauge theory on the fuzzy supersphere. This theory has an exact SUSY gauge symmetry with a finite number of degrees of freedom and thus in principle it is amenable to the methods of matrix models and Monte Carlo numerical simulations. We also write down in this article a novel fuzzy supersymmetric scalar action on the fuzzy supersphere.

The differential calculus on the fuzzy sphere is 3−dimensional and as a consequence a spin 1 vector field $\vec{C}$ is intrinsically 3−dimensional. Each component $C_i, i = 1, 2, 3$, is an element of some matrix algebra Mat$_N$. Thus $U(1)$ symmetry will be implemented by $U(N)$ transformations. On the fuzzy sphere $S^2_N$ it is not possible to split the vector field $\vec{C}$ in a gauge-covariant fashion into a tangent 2-dimensional gauge field and a normal scalar fluctuation. Thus in order to reduce the number of independent components from 3 to 2 we impose the gauge-covariant condition

$$\frac{1}{2}(x_i C_i + C_i x_i) + \frac{C_i^2}{\sqrt{N^2-1}} = 0.$$  (1)

$x_i = L_i/\sqrt{L_i^2}$ (where $L_i$ are the generators of $SU(2)$ in the irreducible representation $\frac{N-1}{2}$ of the group) are the matrix coordinates on fuzzy $S^2_N$. The action on the fuzzy sphere $S^2_N$ is given by

$$S_N[C] = \frac{1}{4Ng^2} Tr F_{ij}^2 - \frac{1}{2Ng^2} \epsilon_{ijk} Tr_L \left[ \frac{1}{2} F_{ij} C_k - \frac{i}{6} [C_i, C_j] C_k \right].$$  (2)

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$F_{ij}$ is the curvature on the fuzzy sphere, viz $F_{ij} = i[Y_i, Y_j] + \epsilon_{ijk} Y_k$ where the covariant derivatives $Y_i$ are defined by $Y_i = L_i + C_i$. The action by analogy with (2) reads

$$\delta V = \text{Str} \vartriangle F \ast F + \beta \text{Str}(A \ast F - \frac{1}{3} A \ast A \ast A).$$

The first term is similar to the usual Yang-Mills action whereas the second term is a (real-valued) Chern-Simons-like contribution. $\alpha$ and $\beta$ are two real parameters. The Hodge triangle $\vartriangle$ is defined as the identity map between one-forms and two-forms and thus $\vartriangle F$ should be considered as a one-form. This action will have the correct continuum limit provided we impose the following supersymmetric- and gauge-covariant conditions on the supergauge field $A$. The first condition is the supersymmetric analogue of (1) defined by

$$[D_+, A_-] - [D_-, A_+] + \frac{1}{4} \{D_0, W\} + [A_+, A_-] + \frac{1}{4} W^2 = 0.$$  

We will also impose the following supersymmetric constraints

$$b_+ = b_- = c_+ = c_- = c_3 = 0.$$  

These constraints will reduce the number of independent components of $A$ and $F$ from 8 to 2. $D_{\pm,0}$ are the generators of $OSP(2,2)$ in the complement of $OSP(2,1)$. The generators of $OSP(2,1)$ are denoted by $R_i, V_\pm$. The expressions of the curvatures $F_\pm$, $f$ and $c_i, b_\pm$ in terms of the gauge fields $A_\pm, W$ and $C_i, B_\pm$ are given by

$$F_\pm = 2[X_\pm, Y_\pm] \pm 2[X_\pm, Y_3] + [X_0, Z_\pm] + 2X_\pm, \quad f = 4\{Z_+, X_-\} - 4\{Z_-, X_+\} + 2X_0$$
$$c_\pm = \mp 2\{X_\pm, X_\pm\} - Y_\pm, \quad c_3 = 2\{X_+, X_-\} - Y_3, \quad b_\pm = [X_0, X_\pm] - Z_\pm.$$  

The supercovariant derivatives $X_\pm, X_0, Y_i$ and $Z_\pm$ are defined by $X_\pm = D_\pm + A_\pm$, $X_0 = D_0 + W, Y_i = R_i + C_i$, and $Z_\pm = V_\pm + B_\pm$.

In the rest of much of these notes we will go into the detail of the above construction following [1,2]. Motivated by this construction we introduce in section 4 a novel fuzzy supersymmetric scalar action on the fuzzy supersphere.

For other constructions of fuzzy supersymmetric gauge models see [5]. The work [6] is a numerical study of the type of supersymmetry which is involved in IKKT models [7,8] so it is not the same as fuzzy SUSY. This paper is organized as follows
1 The continuum supersphere

1.1 The Lie algebras $osp(2, 1)$ and $osp(2, 2)$

We start with the $osp(2, 1)$ Lie algebra. It consists of three even generators $R_+$ and $R_-$ and two odd generators $V_+$ with commutation and anticommutation relations

$$[R_+, R_-] = 2R_3, \ [R_3, R_\pm] = \pm R_\pm$$

and

$$[R_3, V_\pm] = \pm \frac{1}{2} V_\pm, \ [R_\pm, V_\pm] = 0, \ [R_\pm, V_\mp] = V_\pm.$$

The following notation is also useful $R_\pm = \Lambda_1 \pm i\Lambda_2, \ R_3 = \Lambda_3, \ V_+ = \Lambda_4$ and $V_- = \Lambda_5$. The above commutation and anticommutation relations take now the following forms respectively

$$[\Lambda_i, \Lambda_j] = i\epsilon_{ijk}\Lambda_k, \ [\Lambda_i, \Lambda_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}\Lambda_\beta, \ \{\Lambda_\alpha, \Lambda_\beta\} = \frac{1}{2}(C\sigma_i)_{\alpha\beta}\Lambda_i.$$

The most important point is that $V_\pm$ transform as an $SU(2)$ spinor.
Let us also introduce $osp(2,2)$. We add two more odd generators $D_\pm$ and one even generator $D_0$ with the commutation and anticommutation relations

\[ [R_3, D_\pm] = \pm \frac{1}{2} D_\pm, \quad [R_\pm, D_\pm] = 0, \quad [R_\mp, D_\mp] = D_\mp. \tag{11} \]

\[ \{D_\pm, D_\pm\} = \mp \frac{1}{2} R_\pm, \quad \{D_\pm, D_\mp\} = \frac{1}{2} R_3. \tag{12} \]

\[ \{D_\pm, V_\pm\} = 0, \quad \{D_\pm, V_\mp\} = \pm \frac{1}{4} D_0. \tag{13} \]

and

\[ [D_0, R_i] = 0, \quad [D_0, V_\pm] = D_\pm, \quad [D_0, D_\pm] = V_\pm. \tag{14} \]

Again we denote $D_+ = \Lambda_6$, $D_- = \Lambda_7$ and $D_0 = \Lambda_8$. Then the commutation and anticommutation relations (11) and (12) take the forms respectively

\[ [\Lambda_i, \Lambda_\alpha] = \frac{1}{2} (\sigma_i)_{\beta\alpha} \Lambda_\beta, \quad \{\Lambda_\alpha, \Lambda_\beta\} = -\frac{1}{2} (C \sigma_i)_{\alpha\beta} \Lambda_i. \tag{15} \]

Here $\alpha, \beta = 6, 7$. Note that $D_\pm$ transform also as an $SU(2)$ spinor. Equation (8) and (11) can be put in the form

\[ [\Lambda_i, \Lambda_\alpha] = \frac{1}{2} (\tilde{\sigma}_i)_{\beta\alpha} \Lambda_\beta \tag{16} \]

Equation (9), (12) and (13) can be put together in the form

\[ \{\Lambda_\alpha, \Lambda_\beta\} = \frac{1}{2} (\tilde{C} \tilde{\sigma}_i)_{\alpha\beta} \Lambda_i + \frac{1}{4} (\tilde{\epsilon} \tilde{C})_{\alpha\beta} \Lambda_8. \tag{17} \]

Equation (14) takes the form

\[ [\Lambda_8, \Lambda_i] = 0, \quad [\Lambda_8, \Lambda_\alpha] = \tilde{\epsilon}_{\alpha\beta} \Lambda_\beta. \tag{18} \]

Here ( in the last three equations ) $\alpha, \beta = 4, 5, 6, 7$ and

\[
\tilde{\sigma}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}, \quad \tilde{\epsilon} = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}.
\tag{19}
\]

Also

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\tag{20}
\]
1.2 The supersphere

The supersphere $S^{(3,2)}$ (with ordinary $S^3$ as its even part) is given by the points $\psi \in \mathbb{C}^{(2,1)}$ which satisfy

$$|\psi|^2 = 1. \quad (21)$$

We have $\psi = (z, \theta) = (z_1, z_2, \theta)$ and $\bar{\psi} = (\bar{z}, \bar{\theta}) = (z_1^+, z_2^+, \bar{\theta})$ and the norm is given by

$$|\psi|^2 \equiv \bar{\psi}\psi = \bar{z}z + \theta\bar{\theta} = |z_1|^2 + |z_2|^2 + \theta\bar{\theta}. \quad (22)$$

In above $z_1, z_2$ are complex variables and $\theta, \bar{\theta}$ are Grassmann numbers. The group manifold of $osp(2, 1)$ is $S^{(3,2)}$ in the same way that the group manifold of $su(2)$ is $S^3$. Furthermore the supersphere $S^{(2,2)}$ is an adjoint orbit of $OSP(2, 1)$ in the same way that the sphere $S^2$ is an adjoint orbit of $SU(2)$. In other words we must consider the supersymmetric Hopf fibration $S^1 \to S^{(3,2)} \to S^{(2,2)}$ by analogy with the Hopf fibration $S^1 \to S^3 \to S^2$. We define thus the coordinates functions on $S^{(2,2)}$ by the following functions on $S^{3,2}$

$$\omega_a(\psi) = \bar{\psi}\Lambda_a^{(2)}\psi, \quad a = 1, ..., 5. \quad (23)$$

A point on $S^{(2,2)}$ is given by the supervector $\omega = (\omega_1, ..., \omega_5)$. $\Lambda_a^{(2)}$ are the generators of $OSP(2, 1)$ in the 3–dimensional fundamental representation characterized by the superspin $j = \frac{1}{2}$. It consists of the $SU(2)$ irreducible representations $\frac{1}{2}$ and 0. The generators are given explicitly by

$$\Lambda_i^{(2)} = \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda_4^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_5^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (24)$$

Remark that under $\psi \to \psi h, \quad h = \exp(i\gamma)$ with $\gamma$ real numbers we have $\omega_a \to \omega_a$. Thus the points $\psi h$ on $S^{(3,2)}$ correspond to the same point $\omega$ on $S^{(2,2)}$. This shows that $OSP(2, 1)$ is a principal $U(1)$ bundle over the coset space $OSP(2, 1)/U(1)$ which is diffeomorphic to the sphere $S^{(2,2)}$. We can compute the explicit expressions

$$\omega_1 = \frac{1}{2} \bar{z}\sigma_i z, \quad \omega_4 = -\frac{1}{2}(z_1^+ \theta + z_2 \bar{\theta}), \quad \omega_5 = \frac{1}{2}(-z_2^+ \theta + z_1 \bar{\theta}). \quad (25)$$

By using these equations we can immediately compute

$$\omega_i^2 + C_{\alpha\beta}\omega_\alpha\omega_\beta = \frac{1}{4}. \quad (26)$$

This is the defining equation of the supersphere $S^{(2,2)}$. We define the grade adjoint $++$ by $z_i^{++} = z_i^+, \quad \theta^{++} = \bar{\theta}$ and $\bar{\theta}^{++} = -\theta$ and by the requirement that $(AB)^{++} = (-1)^{d_AD_B} A^{++}$ where $d_A$ and $d_B$ are the degrees of $A$ and $B$ respectively. For an even object the degree is equal 0 while for an odd object the degree is equal to 1. Hence we have the reality conditions

$$\omega_i^{++} \equiv \omega_i^+ = \omega_i, \quad \omega_\alpha^{++} = -C_{\alpha\beta}\omega_\beta, \quad \alpha, \beta = 4, 5. \quad (27)$$
The action of the group $OSP(2, 2)$ on $S^{(2,2)}$ preserves (26) and (27) but it is not the same as the adjoint action of the group $OSP(2, 1)$. This is because the Lie algebra $osp(2, 1)$ is not invariant under the action of the generators $\Lambda_6, \Lambda_7$ and $\Lambda_8$ of $OSP(2, 2)$. Let us define the $OSP(2, 2)$ coordinates functions

$$\Omega_a(\psi) = \bar{\psi} \Lambda_a^{(\frac{1}{2})} \psi, \ a = 1, \ldots, 8.$$  

They define an $OSP(2, 2)$ vector. We will have the following extra generators (in addition to (24)) in the 3–dimensional fundamental representation $j = \frac{1}{2}$ of $OSP(2, 2)$

$$\Lambda_8^{(\frac{1}{2})} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \Lambda_6^{(\frac{1}{2})} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \Lambda_7^{(\frac{1}{2})} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$  

Explicitly we have

$$\Omega_8 = 2 - \bar{z}z, \Omega_6 = \frac{1}{2}(z^+_1 \theta - z_2 \bar{\theta}), \Omega_7 = \frac{1}{2}(z^+_2 \theta + z_1 \bar{\theta}) \quad (30)$$

$$\Omega_{8+}^+ \equiv \Omega_8, \Omega_{\alpha+} = C_{\alpha\beta} \Omega_\beta, \alpha, \beta = 6, 7.$$  

We can immediately compute

$$- \frac{1}{4} \Omega_8^2 + \tilde{C}_{\alpha\beta} \Omega_\alpha \Omega_\beta = - \frac{1}{4}, \alpha, \beta = 6, 7.$$  

By adding (26) (with the substitutions $\omega_i \longrightarrow \Omega_i$ and $\omega_\alpha \longrightarrow \Omega_\alpha$, $\alpha = 4, 5$) and (32) we get the $OSP(2, 2)$ Casimir

$$\Omega_i^2 - \frac{1}{4} \Omega_8^2 + \tilde{C}_{\alpha\beta} \Omega_\alpha \Omega_\beta = 0.$$  

Let us also compute the following

$$\omega_3 \omega_4 = (\frac{1}{2} |z_1|^2 - \frac{1}{2} |z_2|^2) \omega_4 = - \frac{1}{4}(|z_1|^2 z^+_2 \theta + |z_1|^2 z_2 \bar{\theta} - |z_2|^2 z^+_1 \theta - |z_2|^2 z_1 \bar{\theta}) \quad (34)$$

$$(\omega_1 + i \omega_2) \omega_5 = (z^+_1 z_2) \omega_5 = \frac{1}{2}(|z_1|^2 z_2 \bar{\theta} - |z_2|^2 z^+_1 \theta). \quad (35)$$

Hence by using also $\sqrt{\omega_i^2} = \frac{1}{2} \bar{z}z$ we obtain

$$\Omega_6 = - \frac{1}{\sqrt{\omega_i^2}}(\omega_3 \omega_4 + (\omega_1 + i \omega_2) \omega_5). \quad (36)$$

By using $\Omega_6^{++} = \Omega_7, \omega_4^{++} = -\omega_5, \omega_5^{++} = \omega_4, \omega_6^{++} = \omega_i$ and $(i)^{++} = -i$ we obatin

$$\Omega_7 = \frac{1}{\sqrt{\omega_i^2}}(\omega_3 \omega_5 - (\omega_1 - i \omega_2) \omega_4). \quad (37)$$

Finally by using again $\sqrt{\omega_i^2} = \frac{1}{2} \bar{z}z$ we obtain

$$\Omega_8 = 2 - 2 \sqrt{\omega_i^2}.$$  

(38)
1.3 Laplacians

Define \( n_i = 2Rw_i, n_\alpha = 2R\omega_\alpha \). Then

\[
n_i^2 + C_{\alpha\beta}n_\alpha n_\beta = R^2. \tag{39}
\]

The delta function \( \delta(n_i^2 + C_{\alpha\beta}n_\alpha n_\beta - R^2) \) will have an expansion of the general form \( \delta(n_i^2 + C_{\alpha\beta}n_\alpha n_\beta - R^2) = \delta(n_i^2 - R^2) + C_{\alpha\beta}n_\alpha n_\beta X \) where \( X \) is given by

\[
X = \frac{1}{2} \left[ \frac{d}{dn_5 dn_4} \delta(n_i^2 + C_{\alpha\beta}n_\alpha n_\beta - R^2) \right]_{n_4 = n_5 = 0} = \frac{d\delta(n_i^2 - R^2)}{dn_i^2}. \tag{40}
\]

Thus

\[
\delta(n_i^2 + C_{\alpha\beta}n_\alpha n_\beta - R^2) = \frac{1}{2R} \delta(r - R) + \frac{n_4 n_5}{2rR} \frac{d\delta(r - R)}{dr}, \quad n_i^2 = r^2. \tag{41}
\]

In above we have assumed that \( r \geq 0 \). A scalar superfield \( \Phi \) on \( S^{(2,2)} \) has an expansion of the form (with \( \alpha, \beta = 4, 5 \))

\[
\Phi = \phi_0 + C_{\alpha\beta}\psi_\alpha n_\beta + \phi_1 C_{\alpha\beta}n_\alpha n_\beta. \tag{42}
\]

\( \phi_0 \equiv \phi_0(n_i) \) and \( \phi_1 \equiv \phi_1(n_i) \) are scalar functions while \( \psi \) is a Majorana spinor field with two components Grassman functions \( \psi_\alpha \equiv \psi_\alpha(n_i) \). In the terminology of supersymmetry in 4–dimensional Minkowski spacetime the field \( \phi_0 \) is the \( D \)–term of the superfield \( \Phi \) while the field \( \phi_1 \) is the \( F \)–term of the superfield. The integral of this superfield over the supersphere is defined by (with \( d\Omega \) denoting the solid angle)

\[
I(\Phi) = \int r^2 dr d\Omega dn_4 dn_5 \delta(n_i^2 + C_{\alpha\beta}n_\alpha n_\beta - R^2) \Phi. \tag{43}
\]

Since the volume form and the delta function are invariant under the \( OSP(2,1) \) action the integral should be invariant under the susy action on \( \Phi \). A straightforward calculation (using also \( \int dn_4 = \int dn_5 = 0 \) and \( \int dn_4 dn_5 n_4 n_5 = -1 \)) we obtain

\[
I(\Phi) = \int d\Omega \left[ \frac{d}{dr} \left( \frac{r}{2R} \phi_0 - R\phi_1 \right) \right]_{r=R}. \tag{44}
\]

Thus as in the case of supersymmetry in 4 dimensions the integral depends only on the \( D \)– and \( F \)–terms of the superfield. The \( OSP(2,1) \) and \( OSP(2,2) \) Laplacians (by inspection of equations (23), (32) and (33)) are given respectively by the equations

\[
\begin{align*}
K_{2,1} &= \Lambda_i^2 + C_{\alpha\beta}\Lambda_\alpha\Lambda_\beta \\
K_{2,2} &= \Lambda_i^2 - \frac{1}{4}\Lambda_8^2 + \tilde{C}_{\alpha\beta}\Lambda_\alpha\Lambda_\beta. \tag{45}
\end{align*}
\]

The Laplacian on \( S^{(2,2)} \) is given by

\[
\Delta = K_{2,1} - K_{2,2} = \frac{1}{4}\Lambda_8^2 + \Lambda_6\Lambda_7 - \Lambda_7\Lambda_6. \tag{46}
\]

The action for the scalar superfield \( \Phi \) is given by

\[
S = I(\Phi^{++}\Delta\Phi) = \int r^2 dr d\Omega dn_4 dn_5 \delta(n_i^2 + C_{\alpha\beta}n_\alpha n_\beta - R^2) \Phi^{++}\Delta\Phi. \tag{47}
\]
1.4 Scalar action

In the calculation of the above action we need the $D$– and $F$–terms of the superfield $\Phi^{++}\Delta\Phi$. Because of the constraint the superfield $\Phi$ can be rewritten in the form $\Phi = \phi_2 + C_{\alpha\beta}\psi_\alpha n_\beta$ where $\phi_2 = \phi_0 + \phi_1(R^2 - r^2)$. Hence the $D$–term of the superfield $\Phi^{++}\Delta\Phi$ is

$$[\Phi^{++}\Delta\Phi]_0 = \phi_2^{++}\Delta\phi_2 = \phi_2\Delta\phi_2.$$  \hspace{1cm} (48)

The $F$–term will be extracted from

$$[\Phi^{++}\Delta\Phi]_1 = (C_{\alpha\beta}\psi_\alpha n_\beta)^{++}\Delta(C_{\alpha\beta}\psi_\alpha n_\beta) = (C_{\alpha\beta}\psi_\alpha n_\beta)\Delta(C_{\alpha\beta}\psi_\alpha n_\beta).$$  \hspace{1cm} (49)

In above we have assumed that the superfield $\Phi$ is real and hence $\Phi^{++} = \Phi$ or equivalently $\phi_0^{++} = \phi_0^+ = \phi_0$, $\phi_1^{++} = \phi_1^+ = \phi_1$ and $\psi_\alpha^{++} = -C_{\alpha\beta}\psi_\beta$. We have also assumed that cross terms are linear in $n_\alpha$ which we will show.

The $D$–term: First we calculate the $D$–component. The action of the generators $\Lambda_6 = D_+$ and $\Lambda_7 = D_-$ on $\phi_0$ is defined by

$$\Lambda_6\phi_0 \equiv (D_+ n_i)\partial_i\phi_0 = -\frac{1}{2}[(\sigma_1)_{66}n_6 + (\sigma_1)_{76}n_7] \partial_i\phi_0 = -\frac{1}{2}[n_6\partial_3 + n_7\partial_\pm]\phi_0$$

and

$$\Lambda_7\phi_0 \equiv (D_- n_i)\partial_i\phi_0 = -\frac{1}{2}[(\sigma_1)_{67}n_6 + (\sigma_1)_{77}n_7] \partial_i\phi_0 = -\frac{1}{2}[n_6\partial_- - n_7\partial_3]\phi_0.$$  \hspace{1cm} (50)

These equations are consistent with the commutation relations $[\Lambda_\alpha, \Lambda_\beta] = -\frac{1}{2}(\sigma_1)_{\alpha\beta}\Lambda_\gamma$ where $\alpha, \beta = 6, 7$. Let us also say that the operators $D_\pm$ correspond to the generators $D_\pm$ in the adjoint representation of $OSP(2, 2)$. Furthermore $\partial_\pm = \partial_1 \pm i\partial_2$ and $n_6$, $n_7$ are given by $n_6 = 2R\Omega_6$, $n_7 = 2R\Omega_7$ and hence we must have from (33) and (37) the expressions

$$n_6 = -\frac{1}{r}(n_3n_4 + (n_1 + in_2)n_5), \quad n_7 = \frac{1}{r}(n_3n_5 - (n_1 - in_2)n_4).$$  \hspace{1cm} (51)

Remark that $\Lambda_6\phi_0$ and $\Lambda_7\phi_0$ are odd and hence a second action of $\Lambda_6$ and $\Lambda_7$ will involve anticommutation relations instead of commutation relations. We have

$$\Lambda_7\Lambda_6\phi_0 = -\frac{1}{2}\left[\Lambda_7(n_6\partial_3\phi_0) + \Lambda_7(n_7\partial_+\phi_0)\right]$$

$$\Lambda_6\Lambda_7\phi_0 = -\frac{1}{2}\left[\Lambda_6(n_6\partial_-\phi_0) - \Lambda_6(n_7\partial_3\phi_0)\right]$$

$$\hspace{1cm} = -\frac{1}{2}\left[\Lambda_6(n_6\partial_-\phi_0) - n_6(\Lambda_6\partial_3\phi_0) + (\Lambda_7n_7)\partial_+\phi_0 - n_7(\Lambda_7\partial_+\phi_0)\right]$$

$$\Lambda_6\Lambda_7\phi_0 = -\frac{1}{2}\left[\Lambda_6(n_6\partial_-\phi_0) - n_6(\Lambda_6\partial_3\phi_0) - (\Lambda_6n_7)\partial_3\phi_0 + n_7(\Lambda_6\partial_3\phi_0)\right]$$

$$\hspace{1cm} = -\frac{1}{2}\left[(\Lambda_6n_6)\partial_\phi_0 - n_6(\Lambda_6\partial_3\phi_0) - (\Lambda_6n_7)\partial_3\phi_0 + n_7(\Lambda_6\partial_3\phi_0)\right]$$  \hspace{1cm} (52)

\footnote{Take the case of the ordinary generators of $SU(2)$ denoted here by $\Lambda_i = R_i$. We know that $\Lambda_i\phi_0 \equiv (R_i\phi_0)(\bar{n}) = -i\epsilon_{ijk}n_j\partial_k\phi_0$. This can be put in the form $\Lambda_i\phi_0 = (R_i n_j)\partial_j\phi_0$.}
The quantities \( (\Lambda_6 \partial_3 \phi_0) \) and \( (\Lambda_6 \partial_\pm \phi_0) \) will be given by similar expressions to (50). From the anticommutation relations \( \{ D_\pm, D_\pm \} = \mp \frac{1}{2} R_\pm \) and \( \{ D_\pm, D_+ \} = \frac{1}{2} R_3 \) we have

\[
\Lambda_6 n_6 = D_+ n_6 = -\frac{1}{2} n_+ , \quad \Lambda_6 n_7 = D_+ n_7 = \frac{1}{2} n_3 , \quad n_+ = n_1 + i n_2
\]

\[
\Lambda_7 n_6 = D_- n_6 = \frac{1}{2} n_3 , \quad \Lambda_7 n_7 = D_- n_7 = +\frac{1}{2} n_- , \quad n_- = n_1 - i n_2 .
\]

(53)

Note here that the odd coordinates associated with \( \Lambda_6,7 \) will always be denoted by \( n_{6,7} \) although we will denote sometimes the operators \( \Lambda_6 \) and \( \Lambda_7 \) by \( D_+ \) and \( D_- \) respectively. So \( n_+ \) and \( n_- \) are always bosonic coordinates associated with \( \Lambda_+ = \Lambda_1 + i \Lambda_2 \) and \( \Lambda_- = \Lambda_1 - i \Lambda_2 \). We compute (with \( \mathcal{L}_3 = i(n_1 \partial_2 - n_2 \partial_1) \))

\[
\Lambda_7 \Lambda_6 \phi_0 = -\frac{1}{4} \left[ (n_1 \partial_1 + \mathcal{L}_3) \phi_0 - n_6 n_7 \partial^2 \phi_0 \right]
\]

\[
\Lambda_6 \Lambda_7 \phi_0 = -\frac{1}{4} \left[ (n_1 \partial_1 + \mathcal{L}_3) \phi_0 + n_6 n_7 \partial^2 \phi_0 \right].
\]

(54)

Thus

\[
(\Lambda_6 \Lambda_7 - \Lambda_7 \Lambda_6)(\phi_0) = \frac{1}{2} [n_1 \partial_1 \phi_0 - n_6 n_7 \partial^2 \phi_0] .
\]

(55)

Similarly \( \Lambda_8(\phi_0) = D_0(n_i) \partial_i \phi_0 = 0 \) since \( [\Lambda_8, R_i] = [D_0, R_i] = 0 \). Hence

\[
\phi_0 \Delta(\phi_0) = \frac{1}{2} \left[ n_1 \partial_1 \phi_0 - n_6 n_7 \partial^2 \phi_0 \right] = \frac{1}{2} \phi_0 \left[ r \partial_r \phi_0 + \frac{R^2 - r^2}{2} \partial^2 \phi_0 \right] = \frac{1}{2} \phi_0 \left[ \frac{R^2 - r^2}{2} \partial_r^2 + \frac{R^2}{r^2} \partial_r + \frac{1}{2} \frac{R^2}{r^2} - 1 \mathcal{L}_a^2 \right] \phi_0 .
\]

(56)

In above we have used the results \( n_6 n_7 = -n_4 n_5 = -\frac{R^2 - r^2}{2} \) and \( \partial^2 = \partial_r^2 + \frac{2}{r} \partial_r + \frac{\mathcal{L}_a^2}{r^2} \). Finally we get

\[
\frac{d}{d r} \left[ \frac{r}{2 R} \phi_0 \Delta \phi_0 \right]_{r=R} = \frac{R}{4} \left[ \left( \frac{d \phi_0}{d r} \right)^2 \right]_{r=R} - \frac{1}{4 R} \phi_0 \mathcal{L}_a^2 \phi_0 .
\]

(57)

The corresponding action is

\[
I_0 = \int d \Omega \left[ \frac{r}{2 R} \phi_0 \Delta \phi_0 \right]_{r=R} = \frac{R}{4} \int d \Omega \left( \frac{d \phi_0}{d r} \right)^2 + \frac{1}{4 R} \int d \Omega (\mathcal{L}_a \phi_0)^2 .
\]

(58)

The full action coming from the D–term is obtained from above by replacing \( \phi_0 \) with \( \phi_2 \). We get

\[
I_D = \int d \Omega \left[ \frac{r}{2 R} \phi_2 \Delta \phi_2 \right]_{r=R} = \frac{R}{4} \int d \Omega \left( \frac{d \phi_2}{d r} \right)^2 + \frac{1}{4 R} \int d \Omega (\mathcal{L}_a \phi_2)^2
\]

\[
= \int d \Omega \left[ \frac{r}{2 R} \phi_2 \Delta \phi_2 \right]_{r=R} = \frac{R}{4} \int d \Omega \left( \frac{d \phi_2}{d r} - 2 R \phi_1 \right)^2 + \frac{1}{4 R} \int d \Omega (\mathcal{L}_a \phi_0)^2 .
\]

(59)
The \( F \)-term: Now we have (with \( D_+ = \Lambda_7, D_- = \Lambda_6 \) and \( \alpha = 4, 5 \))

\[
\Lambda_6(\psi_\alpha) = D_+(n_i)\partial_i \psi_\alpha = \frac{1}{2}(D_+n_+)\partial_- \psi_\alpha + \frac{1}{2}(D_-n_-)\partial_+ \psi_\alpha + D_+(n_3)\partial_3 \psi_\alpha
\]

\[
= -\frac{1}{2}n_7\partial_+ \psi_\alpha - \frac{1}{2}n_6\partial_3 \psi_\alpha
\]

\[
\Lambda_7(\psi_\alpha) = D_-(n_i)\partial_i \psi_\alpha = \frac{1}{2}(D_-n_-)\partial_- \psi_\alpha + \frac{1}{2}(D_-n_-)\partial_+ \psi_\alpha + D_-\partial_3 \psi_\alpha
\]

\[
= -\frac{1}{2}n_6\partial_- \psi_\alpha + \frac{1}{2}n_7\partial_3 \psi_\alpha.
\] (60)

In above we have also used the fact (which we can check from the commutation relations \([D_+, R_\pm] = -D_\pm, [D_\pm, R_\pm] = 0 \) and \([D_\pm, R_3] = \pm \frac{1}{2}D_\pm\) ) that

\[
D_-(n_+) = -n_6, D_+(n_-) = -n_7, D_+(n_+) = D_-(n_-) = 0, D_+(n_3) = -\frac{1}{2}n_6, D_-(n_3) = \frac{1}{2}n_7\] (61)

We will also need (from the anticommutation relations \(\{D_\pm, V_\pm\} = 0\) and \(\{D_\pm, V_\mp\} = \pm \frac{1}{2}D_0\)) with \(D_0 = \Lambda_8\) the actions

\[
D_+(n_5) = \frac{1}{4}n_8, D_+(n_4) = 0, D_-(n_5) = 0, D_-(n_4) = -\frac{1}{4}n_8.
\] (62)

The even coordinate \(n_8\) is defined by \(n_8 = 2R\Omega_8 = 2(2R - r)\). Next

\[
\Lambda_6(C_{\alpha\beta}\psi_\alpha n_\beta) = \Lambda_6(\psi_4n_5 - \psi_5n_4) = \Lambda_6(\psi_4)n_5 - \psi_4 D_+(n_5) - \Lambda_6(\psi_5)n_4 + \psi_5 D_+(n_4)
\]

\[
= \Lambda_6(\psi_4)n_5 - \frac{1}{4}\psi_4 n_8 - \Lambda_6(\psi_5)n_4
\]

\[
= -\frac{1}{4}\psi_4 n_8 + \frac{1}{2}n_7\partial_+ \psi_4 + \frac{1}{2}n_6 n_5\partial_3 \psi_4 - \frac{1}{2}n_7\partial_+ \psi_5 - \frac{1}{2}n_6 n_4\partial_3 \psi_5
\]

\[
\Lambda_7(C_{\alpha\beta}\psi_\alpha n_\beta) = \Lambda_7(\psi_4n_5 - \psi_5n_4)
\]

\[
= \Lambda_7(\psi_4)n_5 - \psi_4 D_-(n_5) - \Lambda_7(\psi_5)n_4 + \psi_5 D_-(n_4)
\]

\[
= \Lambda_7(\psi_4)n_5 - \Lambda_7(\psi_5)n_4 - \frac{1}{4}\psi_5 n_8
\]

\[
= -\frac{1}{4}\psi_5 n_8 + \frac{1}{2}n_6 n_5\partial_- \psi_4 - \frac{1}{2}n_7 n_5\partial_3 \psi_4 - \frac{1}{2}n_6 n_4\partial_- \psi_5 + \frac{1}{2}n_7 n_4\partial_3 \psi_5.
\] (63)

By using now \(n_7 n_5 = \frac{1}{r}n_-n_4 n_5, n_7 n_4 = n_6 n_5 = \frac{1}{r}n_3 n_4 n_5\) and \(n_6 n_4 = \frac{1}{r}n_+ n_4 n_5\) we obtain

\[
\Lambda_6(C_{\alpha\beta}\psi_\alpha n_\beta) = -\frac{1}{4}\psi_4 n_8 - \frac{1}{2r}(n_-\partial_+ \psi_4 + n_3\partial_3 \psi_4 - n_3\partial_+ \psi_5 + n_+\partial_3 \psi_5)n_4 n_5
\]

\[
\Lambda_7(C_{\alpha\beta}\psi_\alpha n_\beta) = -\frac{1}{4}\psi_5 n_8 - \frac{1}{2r}(n_3\partial_- \psi_4 - n_-\partial_3 \psi_4 + n_4\partial_- \psi_5 + n_3\partial_3 \psi_5)n_4 n_5.
\] (64)

We need now to compute the following (using also \(D_\pm(n_8) = -n_4, 5\))

\[
\Lambda_6(\psi_\alpha n_8) = \frac{1}{2}D_+(n_+ n_8) - \frac{1}{2}D_+(n_- n_8) + \frac{1}{2}D_+(n_3)\partial_3 \psi_\alpha n_8 - \psi_\alpha D_+(n_8)
\]

\[
= -\frac{1}{2}n_7\partial_+ \psi_\alpha n_8 - \frac{1}{2}n_6\partial_3 \psi_\alpha n_8 + \psi_\alpha n_4
\]

\[
\Lambda_7(\psi_\alpha n_8) = \frac{1}{2}D_-(n_+ n_8) + \frac{1}{2}D_-(n_- n_8) + \frac{1}{2}D_-(n_3)\partial_3 \psi_\alpha n_8 - \psi_\alpha D_-(n_8)
\]

\[
= -\frac{1}{2}n_6\partial_- \psi_\alpha n_8 + \frac{1}{2}n_7\partial_3 \psi_\alpha n_8 + \psi_\alpha n_5.
\] (65)
Next step is to compute the following action

\[
\Lambda_7\left(\frac{1}{2r}(n_-\partial_+\psi_4 + n_3\partial_3\psi_4 - n_3\partial_+\psi_5 + n_+\partial_3\psi_5)n_4n_5\right) = -\frac{1}{8r}n_5n_8\left[n_-\partial_+\psi_4 + n_3\partial_3\psi_4 - n_3\partial_+\psi_5 + n_+\partial_3\psi_5\right]
\]
\[
\Lambda_6\left(\frac{1}{2r}(n_3\partial_-\psi_4 - n_-\partial_3\psi_4 + n_+\partial_-\psi_5 + n_3\partial_3\psi_5)n_4n_5\right) = -\frac{1}{8r}n_4n_8(n_3\partial_-\psi_4 - n_-\partial_3\psi_4 + n_+\partial_-\psi_5 + n_3\partial_3\psi_5).
\]

This action corresponds to \(\Lambda_{6,7}\) acting on the factor \(n_4n_5\). The action of \(\Lambda_{6,7}\) on the other terms leads to products which involve \(n_4n_5\) and \(n_{6,7}\) and hence they are zero by (51).

We now compute in a straightforward way

\[
n_5\Lambda_6\Lambda_7(C_{\alpha\beta}\psi_\alpha n_\beta) = \frac{n_4n_5n_8}{8r}\left[(n_-\partial_+ - n_+\partial_-)\psi_5 - (n_3\partial_- - n_-\partial_3)\psi_4 - \frac{2r}{n_8}\psi_5\right]
\]
\[
n_4\Lambda_6\Lambda_7(C_{\alpha\beta}\psi_\alpha n_\beta) = \frac{n_4n_5n_8}{8r}(n_3\partial_+ - n_+\partial_3)\psi_5.
\] (67)

\[
n_4\Lambda_7\Lambda_6(C_{\alpha\beta}\psi_\alpha n_\beta) = \frac{n_4n_5n_8}{8r}\left[(-n_+\partial_- + n_-\partial_+)\psi_4 - (n_3\partial_+ - n_+\partial_3)\psi_5 + \frac{2r}{n_8}\psi_4\right]
\]
\[
n_5\Lambda_7\Lambda_6(C_{\alpha\beta}\psi_\alpha n_\beta) = \frac{n_4n_5n_8}{8r}(n_3\partial_- - n_-\partial_3)\psi_4.
\] (68)

So ( with \(L_3 = \frac{1}{2}(n_+\partial_- - n_-\partial_+), L_\pm = L_1 \pm iL_2 = \mp(n_\pm\partial_3 - n_3\partial_\pm)\) and \(\psi = (\psi_1, \psi_2)\) )

\[
(C_{\alpha\beta}\psi_\alpha n_\beta)(\Lambda_6\Lambda_7 - \Lambda_7\Lambda_6)(C_{\alpha\beta}\psi_\alpha n_\beta) = \frac{n_4n_5n_8}{4r}\left[\psi_4L_-\psi_4 - \psi_4L_3\psi_5 - \psi_5L_3\psi_4 - \psi_5L_+\psi_5 - \frac{r}{n_8}\psi_4\psi_5 + \frac{r}{n_8}\psi_5\psi_4\right]
\]
\[
= -\frac{C_{\alpha\beta}n_\alpha n_\beta n_8}{8r}\psi^T(\sigma_\alpha L_\alpha + \frac{r}{n_8})(C\psi).
\] (69)

The full result is then

\[
(C_{\alpha\beta}\psi_\alpha n_\beta)\Delta(C_{\alpha\beta}\psi_\alpha n_\beta) = -\frac{C_{\alpha\beta}n_\alpha n_\beta n_8}{8r}\psi^T(\sigma_\alpha L_\alpha + 2\frac{r}{n_8})(C\psi).
\] (70)

The contribution of this \(F\)–term to the action is given by

\[
I_F = -R\int d\Omega\left[-\frac{n_8}{8r}\psi^T(\sigma_\alpha L_\alpha + 2r\frac{r}{n_8})(C\psi)\right]|_{r=R} = \frac{R}{4}\int d\Omega\psi^T(\sigma_\alpha L_\alpha + 1)(C\psi).
\] (71)

The total action

\[
I = \frac{R}{4}\int d\Omega\left\langle\frac{d\phi_0}{dr} - 2R\phi_1\right\rangle^2 + \frac{1}{4R}\int d\Omega(L_\alpha\phi_0)^2 + \frac{R}{4}\int d\Omega\psi^T(\sigma_\alpha L_\alpha + 1)(C\psi).
\] (72)
2 The fuzzy supersphere

We consider the irreducible representation with $OSP(2, 1)$ superspin equal $L$. This representation consists of the direct sum of the $SU(2)$ representations with spins $L$ and $L - \frac{1}{2}$. Let $\mathcal{L}(L, L)$ be the space of linear operators from the corresponding representation space into itself. The action of the superalgebra $OSP(2, 2)$ on $\mathcal{L}(L, L)$ is described by the operators

$$ R_i = \begin{pmatrix} R_i^{(L)} & 0 \\ 0 & R_i^{(L - \frac{1}{2})} \end{pmatrix}, \quad V_{\alpha} = \begin{pmatrix} 0 & V_{\alpha}^{(L, L - \frac{1}{2})} \\ V_{\alpha}^{(L - \frac{1}{2}, L)} & 0 \end{pmatrix} $$

$$ D_0 = \begin{pmatrix} 2L & 0 \\ 0 & 2L + 1 \end{pmatrix}, \quad D_\alpha = \begin{pmatrix} 0 & -V_{\alpha}^{(L, L - \frac{1}{2})} \\ V_{\alpha}^{(L - \frac{1}{2}, L)} & 0 \end{pmatrix}. \quad (73) $$

The dimension of the first block of $R_i$ and $D_0$ is $(2L + 1) \times (2L + 1)$ while the dimension of the second block is $(2L) \times (2L)$. The upper and lower off-diagonal blocks are therefore rectangular matrices with dimensions $(2L + 1) \times (2L)$ and $(2L) \times (2L + 1)$ respectively. In the above equation the definitions of $R_i^{(l)}$ are the usual ones, i.e ( with $R_i^{(l)} = R_i^{(l)} \pm iR_2^{(l)}$ and $l = L, L - \frac{1}{2}$ )

$$ (R_{\pm}^{(l)})_{l_3 \pm 1, l_3} = \sqrt{(l \mp l_3)(l \pm l_3 + 1)}, \quad (R_{3}^{(l)})_{l_3, l_3} = l_3, \quad (74) $$

whereas $V_{\alpha}^{(L, L - \frac{1}{2})}$ and $V_{\alpha}^{(L - \frac{1}{2}, L)}$ are given by

$$ (V_{\pm}^{(L, L - \frac{1}{2})})_{l_3 \pm \frac{1}{2}, L - \frac{1}{2} l_3} = -\frac{1}{2} \sqrt{L \pm l_3 + \frac{1}{2}}, \quad (V_{\pm}^{(L - \frac{1}{2}, L)})_{L - \frac{1}{2}, l_3 \pm \frac{1}{2}, l_3} = \pm \frac{1}{2} \sqrt{L \mp l_3}. \quad (75) $$

We will also denote the operators given in (73) by $\Lambda_i^{(L)} \equiv R_i$, $i = 1, 2, 3$, $\Lambda_\alpha^{(L)} \equiv V_{\pm}$, $D_\pm \equiv D_0$. For $L = \frac{1}{2}$ we get the 3-dimensional fundamental representation of $OSP(2, 2)$ given in (24) and (29).

The above irreducible representation with superspin $L$ is characterized by the value of the $OSP(2, 1)$ Casimir operator $K_{2,1} = R_i^2 + C_{\alpha\beta}V_{\alpha}V_{\beta}$ which is equal $L(L + \frac{1}{2})$ in this representation, viz

$$ K_{2,1} = R_i^2 + C_{\alpha\beta}V_{\alpha}V_{\beta} = L(L + \frac{1}{2}). \quad (76) $$

The above operators (73) give also a non-typical irreducible representation of $OSP(2, 2)$ characterized by the value of the $OSP(2, 2)$ Casimir operator

$$ K_{2,2} = R_i^2 + C_{\alpha\beta}V_{\alpha}V_{\beta} - C_{\alpha\beta}D_\alpha D_\beta - \frac{1}{4}D_0^2 = 0. \quad (77) $$

This means in particular two things, 1) this representation ( as opposed to typical ones of $OSP(2, 2)$ ) is irreducible with respect to the $OSP(2, 1)$ subgroup and 2) the $OSP(2, 2)$ generators $D_\alpha$ and $D_0$ can be realized nonlinearly in terms of the $OSP(2, 1)$ generators.
The space $\mathcal{L}(L, L)$ is isomorphic to the algebra of supermatrices $Mat(2L + 1, 2L)$. The dimension of the Hilbert space on which this algebra acts is $N = (2L + 1) + (2L) = 4L + 1$. The coordinates operators on the fuzzy supersphere are defined by $\hat{n}_i = 2R\hat{\Omega}_i$, $\hat{n}_{4,5} = 2R\hat{\Omega}_{4,5}$ where

$$
\hat{\Omega}_i = \frac{R_i}{2\sqrt{L(L + \frac{1}{2})}}, \quad \hat{\Omega}_4 = \frac{V_+}{2\sqrt{L(L + \frac{1}{2})}}, \quad \hat{\Omega}_5 = \frac{V_-}{2\sqrt{2L(L + \frac{1}{2})}}.
$$

(78)

The remaining coordinates operators $\hat{n}_{6,7,8} = 2R\hat{\Omega}_{6,7,8}$ are similarly defined by

$$
\hat{\Omega}_8 = \frac{D_0}{2\sqrt{L(L + \frac{1}{2})}}, \quad \hat{\Omega}_6 = \frac{D_+}{2\sqrt{L(L + \frac{1}{2})}}, \quad \hat{\Omega}_7 = \frac{D_-}{2\sqrt{L(L + \frac{1}{2})}}.
$$

(79)

These coordinates operators satisfy the commutation and anticommutation relations (with $a, b, c = 1, ..., 8$)

$$
[\hat{n}_a, \hat{n}_b] = \hat{n}_a\hat{n}_b - (-1)^{d_{ab}d_{ac}}\hat{n}_b\hat{n}_a = \frac{iR}{\sqrt{L(L + \frac{1}{2})}} f_{abc}\hat{n}_c.
$$

(80)

The definition of the structure constants $f_{abc}$ is obvious from (10) and (15). These coordinates operators must also satisfy the constraints

$$
\hat{n}_i^2 + C_{\alpha\beta}\hat{n}_\alpha\hat{n}_\beta = R^2.
$$

(81)

$$
\hat{n}_i^2 + \tilde{C}_{\alpha\beta}\hat{n}_\alpha\hat{n}_\beta - \frac{1}{4}\hat{n}_8^2 = 0.
$$

(82)

The continuum limit is defined by $L \to \infty$ in which $\hat{n}_a \to n_a$ and $\hat{\Omega}_a \to \Omega_a$. To see this more explicitly we notice that under the adjoint action of $OSP(2, 1)$ the algebra $Mat(2L + 1, 2L)$ decomposes as

$$
Mat(2L + 1, 2L) \equiv L \otimes L = 0 \oplus \frac{1}{2} \oplus 1 \oplus ... \oplus 2L - \frac{1}{2} \oplus 2L.
$$

(83)

The dimension of this space is $N^2$ and a generic element is a polynomial in $\hat{n}_{i,4,5}$. Recall that $\hat{n}_{6,7,8}$ can be realized nonlinearly in terms of the $\hat{n}_{i,4,5}$. Among these polynomials we can define the matrix superspherical harmonics. A given $N \times N$ supermatrix can be expanded in terms of these superspherical harmonics. In the continuum limit $Mat(2L + 1, L)$ approaches the algebra of superfunctions on the supersphere. In particular the matrix superspherical harmonics go to the ordinary superspherical harmonics which are the eigensuperfunctions of the Casimir operator $R_i^2 + C_{\alpha\beta} \hat{\nu}_\alpha \hat{\nu}_\beta$ and $R_3$.

A very important remark is to note that elements of $Mat(2L + 1, 2L)$ (in other words superfields) can be even or odd if $Mat(2L + 1, 2L)$ is defined over a graded commutative algebra $\mathbf{P}$ instead of the field of complex numbers. In this case we will denote this algebra by $Mat(2L + 1, 2L; \mathbf{P})$. In the fuzzy case we have the definitions $\mathcal{R}_i \Phi = [R_i, \Phi]$ and $\mathcal{V}_\alpha \Phi_{\text{odd}} =$
\{V_\alpha, \Phi_{\text{odd}}\}, \{V_\alpha, \Phi_{\text{even}}\} = [V_\alpha, \Phi_{\text{even}}]\) where \(\Phi\) is any element of \(\text{Mat}(2L + 1, 2L; P)\), \(\Phi_{\text{odd}}\) is an odd element of \(\text{Mat}(2L + 1, 2L; P)\) and \(\Phi_{\text{even}}\) is an even element of \(\text{Mat}(2L + 1, 2L; P)\). Strictly speaking the fuzzy supersphere is identified with the even elements of \(\text{Mat}(2L + 1, 2L; P)\) while the odd elements will be crucial in constructing gauge theories. The inner product on \(\text{Mat}(2L + 1, 2L; P)\) is defined by

\[
(\Phi_1, \Phi_2) \equiv \text{STr} \Phi_1^{++} \Phi_2.
\]  

This satisfies \(\text{STr}(1_N) = 1\) and \(\text{STr}[X,Y] = 0\).

A general supermatrix \(\Phi \in \text{Mat}(2L + 1, 2L; P)\) and its graded involution \(\Phi^{++}\) are given by

\[
\Phi = \begin{pmatrix} \phi_R & \psi_R \\ \psi_L & \phi_L \end{pmatrix}, \quad \Phi^{++} = \begin{pmatrix} \phi_R^{++} & \mp \psi_L^{++} \\ \pm \psi_R^{++} & \phi_L^{++} \end{pmatrix}.
\]  

\(\phi_R\) and \(\phi_L\) are \((2L + 1) \times (2L + 1)\) and \((2L) \times (2L)\) matrices while \(\psi_R\) and \(\psi_L\) are respectively \((2L + 1) \times (2L)\) and \((2L) \times (2L + 1)\) matrices. In \(\Phi^{++}\) the upper signs refer to the case when \(\Phi\) is an even superfield (in which case the off-diagonal blocks are fermionic and the diagonal blocks are bosonic), while the lower signs refer to the case when \(\Phi\) is an odd (in which case the off-diagonal blocks are bosonic and the diagonal blocks are fermionic).

The Laplacian on the fuzzy supersphere is given by

\[
\Delta = K_{2,1} - K_{2,2} = \frac{1}{4} D_0^2 + D_6 D_7 - D_7 D_6.
\]

The definition of \(D_{0,6,7}\) are obvious by analogy with \(R_i\) and \(V_{4,5}\) given above. The fuzzy supersphere of size \(N = 4L + 1\) is by definition the spectral triple consisting of 1) the algebra of supermatrices \(\text{Mat}(2L + 1, 2L)\) together with 2) the representation space of the superspin \(\tilde{L}\) of \(\text{OSP}(2,1)\) with inner product given by the supertrace \(\text{STr}\) and graded involution given by \(++\) and 3) the Laplacian \(\Delta\) which is the most important ingredient. The Laplacian fixes the metric aspects of the space uniquely while the algebra alone will only give topology.

### 3 Gauge theory

#### 3.1 Klimcik differential complex

The Laplacian on the fuzzy supersphere depends only on the \(\text{OSP}(2,2)\) generators \(D_{\pm,0}\) in the adjoint representation. This means in particular that the \(\text{OSP}(2,2)\) generators in the directions \(D_{\pm}\) are the supersymmetric covariant derivatives on the fuzzy supersphere while the \(\text{OSP}(2,2)\) generators \(V_{\pm}\) are the supersymmetry generators. A gauge field on the supersphere is a superspin 1/2 multiplet composed of 3 superfields \(A_{\pm}\) and \(W\) in the directions \(D_{\pm}\) and \(D_0\) respectively. These superfields \(A_{\pm}\) and \(W\) transform under \(\text{OSP}(2,1)\) in the same way as
$D_\pm$ and $D_0$. The supercovariant derivatives (as opposed to the covariant derivatives in the non-supersymmetric case) are thus

$$X_\pm = D_\pm + A_\pm, \quad X_0 = D_0 + W.$$  \hfill (87)

In order to construct gauge theory on the fuzzy supersphere we must in fact start from an $OSP(2,2)$ supervector. Thus we need to add a superspin 1 multiplet composed of 5 more superfields $C_i$ and $B_\pm$ (which transform under $OSP(2,1)$ in the same way as $V_\pm$ and $R_i$) with supercovariant derivatives

$$Y_i = R_i + C_i, \quad Z_\pm = V_\pm + B_\pm.$$  \hfill (88)

In the following we will construct explicitly the action principle of the $OSP(2,2)$ vector gauge superfield $(A_\pm, W, C_i, B_\pm)$. In the case of the fuzzy supersphere this action principle will be a supermatrix model. We will also need to write down constraints which must be satisfied by these superfields in order to have the correct number of degrees of freedom on the fuzzy supersphere.

The differential complex over the fuzzy supersphere is defined by

$$\Psi^j_N = \oplus_{j=0}^3 \Psi^j_N,$$  \hfill (89)

The elements of $\Psi^j_N$ are the $j$-forms. We have the following identifications

$$\Psi^0_N = \Psi^3_N = Mat(2L + 1,2L), \quad \Psi^1_N = \Psi^2_N = \otimes_{i=1}^8 Mat(2L + 1,2L)_i.$$  \hfill (90)

We must clearly have $Mat(2L + 1,2L)_i = Mat(2L + 1,2L)$. A zero-form is thus an element $\Phi$ of the algebra $\Psi^0_N = Mat(2L + 1,2L)$ while a one-form is an element of $\Psi^1_N$ of the form

$$A = (A_\pm, W, C_i, B_\pm).$$  \hfill (91)

The 5 superfields $C_i, B_\pm$ transform as a superspin 1 multiplet under $OSP(2,1)$ while the remaining 3 superfields $A_\pm$ and $W$ will transform as a superspin 1/2 under $OSP(2,1)$. All these superfields are elements of the algebra $Mat(2L + 1,2L)$. We can also write zero-forms and one-forms as $3N \times 3N$ supermatrices of the form

$$M_1 = r_+ \otimes C_- + r_- \otimes C_+ + 2r_3 \otimes C_3 + 2v_+ \otimes B_- - 2v_- \otimes B_+ - 2d_+ \otimes A_- + 2d_- \otimes A_+ - \frac{1}{2} d_0 \otimes W,$$

$$M_0 = 1_3 \otimes \Phi.$$  \hfill (92)

In above $\Lambda_i^{(1)} \equiv r_i, \ i = 1, 2, 3(\pm, 3)$, $\Lambda_\alpha^{(1)} = v_\alpha, \ \alpha = 4(+), 5(-), \ \Lambda_\alpha^{(2)} = d_\alpha, \ \alpha = 6(+) , 7(-)$ and $\Lambda_8^{(1)} \equiv d_0$ are the supermatrices of the 3-dimensional superspin 1/2 fundamental representation of $OSP(2,2)$ corresponding to the generators $R_i, V_\pm, D_\pm$ and $D_0$ respectively. We will also use the notation $\Lambda_i \equiv R_i, \ i = 1, 2, 3(\pm, 3), \ \Lambda_\alpha = V_\alpha, \ \alpha = 4(+), 5(-), \ \Lambda_\alpha = D_\alpha, \ \alpha = 6(+) , 7(-)$ and $\Lambda_8 \equiv D_0$. \hfill \end{proof}
The exterior derivatives
\[ \delta \text{ad}_O \Psi \]
where
\[ M = \eta \equiv O. \]

Let us introduce the quadratic Casimirs
\[ C_G = r_+ \otimes R_- + r_- \otimes R_+ + 2r_3 \otimes R_3 + 2v_+ \otimes V_- - 2v_- \otimes V_+ - 2d_+ \otimes D_- + 2d_- \otimes D_+ \]
\[ C_H = r_+ \otimes R_- + r_- \otimes R_+ + 2r_3 \otimes R_3 + 2v_+ \otimes V_- - 2v_- \otimes V_+ \]
\[ C = C_G - C_H = -2d_+ \otimes D_- + 2d_- \otimes D_+ - \frac{1}{2} d_0 \otimes D_0. \] (94)

The exterior derivative We introduce a coboundary operator \( \delta : \Psi^i \to \Psi^{i+1} \) defined on 0-forms by
\[ \delta \Phi = \left( r_+ \otimes R_- + r_- \otimes R_+ + 2r_3 \otimes R_3 + 2v_+ \otimes V_- - 2v_- \otimes V_+ \right. \]
\[ \left. - 2d_+ \otimes D_- + 2d_- \otimes D_+ - \frac{1}{2} d_0 \otimes D_0 \right) \Phi. \] (95)

The exterior derivative on one-forms is on the other hand given by
\[ \delta M_1 = \delta^G M_1 - \delta^H M_1^H. \] (96)

\( M_1^H \) is the orthogonal projection of \( M_1 \) from \( G \otimes \text{Mat}(2L + 1, 2L) \) into \( H \otimes \text{Mat}(2L + 1, 2L) \) where \( G = \text{osp}(2, 2) \) and \( H = \text{osp}(2, 1) \). In other words
\[ M_1^H = r_+ \otimes C_- + r_- \otimes C_+ + 2r_3 \otimes C_3 + 2v_+ \otimes B_- - 2v_- \otimes B_+. \] (97)

The exterior derivatives \( \delta^G \) and \( \delta^H \) are defined by (with the notation \( M_1 \equiv h_A \otimes C_A \) and \( \text{ad}_O \equiv O = [O, \cdot] \))
\[ \delta^G M_1 = 2(-1)^{h_A} \hat{\eta}_{\Lambda_a} \text{ad}_a(\{\cdot\}) \text{ad}_A + \frac{1}{2} d_G M_1 \]
\[ \delta^H M_1 = 2(-1)^{h_A} \bar{\eta}_{\Lambda_a} \text{ad}_a(\{\cdot\}) \text{ad}_A + \frac{1}{2} d_H M_1. \] (98)

\( \hat{\eta}, \bar{\eta} \) stand for the block diagonal matrices \( \hat{\eta} = 2(1_3, \tilde{C}, -1/4), \bar{\eta} = 2(1_3, C) \) and \( a, b = 1, \ldots, 8 \) in \( \delta^G M_1 \) and \( a, b = 1, \ldots, 5 \) in \( \delta^H M_1 \). The Dynkin numbers \( d_G, d_H \) are defined by
\[ STr XY = \frac{4L^2}{d_G} STr_G(X_1Y_1), \quad STr XY = \frac{4L^2}{d_H} STr_H(X_1Y_1). \] (99)
where $STr_G$, $STr_H$ are the supertraces in the adjoint representations of $G$ and $H$ respectively. We choose $X = Y = R_i$ so $XY = R_i^2 = \text{diag}((R_i^L)^2,(R_i^{L-\frac{1}{2}})^2)$ for $H$ and $XY = R_i^2 = \text{diag}((R_i^L)^2,(R_i^{L-\frac{1}{2}})^2,(R_i^{L-\frac{3}{2}})^2,(R_i^{L-1})^2)$ for $G$. The supermatrices $X_1$ and $Y_1$ correspond to the representation $L = 1$. Using the property $STr \Phi = Tr_{2L+1} \Phi_R - Tr_{2L} \Phi_L$ of $STr$ we compute for $G$ that $STr R_i^2 = 6L^2$ and hence $d_G = 6$ while for $H$ we compute $STr R_i^2 = 3L(L + \frac{1}{2})$ and hence $d_H = 6 + 3/L$.

We have explicitly

$$\delta M_1 = \delta^G M_1^\perp + (\delta^G - \delta^H) M_1^H, \quad M_1^\perp = M_1 - M_1^H. \quad (100)$$

We can immediately compute

$$2\delta^G(d_\pm \otimes A_\mp) = 4d_\mp \otimes R_\pm A_\pm \pm 4d_\pm \otimes R_3 A_\mp + 2d_0 \otimes V_\pm A_\mp - 2v_\pm \otimes D_0 A_\mp$$

$$- 4r_\pm \otimes D_\mp A_\mp \mp 4r_3 \otimes D_\pm A_\mp + d_G d_\pm \otimes A_\mp, \quad (101)$$

and

$$\frac{1}{2} \delta^G(d_0 \otimes W) = - 2d_+ \otimes V_- W + 2d_- \otimes V_+ W + 2v_+ \otimes D_- W - 2v_- \otimes D_+ W$$

$$+ \frac{1}{4} d_G d_0 \otimes W. \quad (102)$$

Hence

$$\delta^G M_1^\perp = - 4r_- \otimes D_+ A_+ + 4r_+ \otimes D_- A_- + 4r_3 \otimes (D_+ A_- + D_- A_+)$$

$$- 2v_- \otimes (D_0 A_+ - D_+ W) + 2v_+ \otimes (D_0 A_- - D_- W)$$

$$- d_- \otimes (4R_+ A_+ + 4R_3 A_+ + 2V_+ W - d_G A_+)$$

$$+ d_+ \otimes (4R_- A_+ - 4R_3 A_- + 2V_- W - d_G A_-)$$

$$+ d_0 \otimes (- 2V_+ A_- + 2V_- A_+ - \frac{1}{4} d_G W). \quad (103)$$

Furthermore (with the notation $M_1^H = h_A \otimes C_A$)

$$(\delta^G - \delta^H) M_1^H = (-1)^{h_A} (-4 add_+ \otimes D_- + 4 add_- \otimes D_+ - add_0 \otimes D_0)(h_A \otimes C_A)$$

$$+ \frac{1}{2} (d_G - d_H) M_1^H. \quad (104)$$

We compute

$$2(-4 add_+ \otimes D_- + 4 add_- \otimes D_+ - add_0 \otimes D_0)(r_i \otimes C_i) = 4d_+ \otimes (D_- C_3 - D_+ C_-)$$

$$+ 4d_- \otimes (D_+ C_3 + D_- C_+), \quad (105)$$

and

$$- 2(-4 add_+ \otimes D_- + 4 add_- \otimes D_+ + add_0 \otimes D_0)(v_+ \otimes B_- - v_- \otimes B_+)$$

$$= - 2d_+ \otimes D_0 B_- + 2d_- \otimes D_0 B_+ + 2d_0 \otimes (D_+ B_- - D_- B_+). \quad (106)$$
Thus

\[(\delta^G - \delta^H)M_1^H = \frac{1}{2} (d_G - d_H)(r_+ \otimes C_- + r_- \otimes C_+ + 2r_3 \otimes C_3 + 2v_+ \otimes B_- - 2v_- \otimes B_+)\]

\[+ 2d_+ ( -D_0B_- + 2D_-C_3 - 2D_+C_-) - 2d_- ( -D_0B_+ - 2D_+C_3 - 2D_-C_+)\]

\[+ 2d_0 (D_+B_+ - D_-B_+).\]

(107)

The final result is (with \(d_G = 4, \ d_H = 6\))

\[\delta A_\pm = 2A_\pm + 2D_\pm C_\pm - 2\mathcal{R}_\pm A_\mp \mp 2D_\pm C_3 \mp 2\mathcal{R}_3 A_\pm + D_0 B_\pm - \mathcal{V}_\pm W\]

\[\delta W = 2W + 4\mathcal{V}_+ A_- + 4\mathcal{V}_- A_+ + 4D_- B_+ - 4D_+ B_-\]

\[\delta B_\pm = -B_\pm + D_0 A_\pm - D_\pm W\]

\[\delta C_3 = -C_3 + 2D_+ A_- + 2D_- A_+\]

\[\delta C_\pm = -C_\pm \mp 4D_\pm A_\pm.\]

(108)

Lastly the action of the coboundary operator on two-forms a t hree-forms is given by the obvious definitions (with \(M_2 = h_A \otimes C_A\))

\[\delta M_2 = -\frac{1}{2} 13 \otimes \mathcal{H}_A c_A\]

\[= D_+ a_- - D_- a_+ + \frac{1}{4} D_0 w - \mathcal{V}_+ b_- + \mathcal{V}_- b_+ - \frac{1}{2} \mathcal{R}_+ c_- - \frac{1}{2} \mathcal{R}_- c_+ - \mathcal{R}_3 c_3.\]

(109)

\[\delta M_3 = \delta (13 \otimes \phi) = 0.\]

(110)

The product \(*\) The associative product \(*\) between the forms is a map \(* : \Psi^i_N \otimes \Psi^j_N \rightarrow \Psi^{i+j}_N\) defined for \(i = 1\) by (with \(h_A \otimes X_A\) standing for one-forms, two-forms and three-forms)

\[(13 \otimes \Phi) \ast (h_A \otimes X_A) = h_A \otimes \Phi X_A.\]

(111)

For \(i = 2\) we have

\[(h_A \otimes C_A) \ast (13 \otimes \Phi) = h_A \otimes C_A \Phi, \ (h_A \otimes C_A) \ast (13 \otimes \phi) = 0,\]

and

\[(h_A \otimes C_A) \ast (h_A' \otimes C_A') = (h_A \otimes C_A) \ast_G (h_A' \otimes C_A') - (h_A \otimes C_A) \ast_H (h_A' \otimes C_A'),\]

(113)

where

\[(h_A \otimes C_A) \ast_{G,H} (h_A' \otimes C_A') = 2(-1)^{Ch'} ad(h_A)h_B' \otimes C_A C_B'.\]

(114)
The indices $A$ and $B$ run over the superalgebra $H$ for $*_H$ whereas for $*_G$ they run over the superalgebra $G$. Explicitly we have

$$(h_A \otimes C_A) * (h_A' \otimes C_A') = 2(r_i \otimes C_i + v_+ \otimes B_- - v_- \otimes B_+)_G(-2d_+ \otimes A'_- + 2d_- \otimes A'_+ - \frac{1}{2}d_0 \otimes W') + (-2d_+ \otimes A_- + 2d_- \otimes A_+ - \frac{1}{2}d_0 \otimes W)_G(h_A' \otimes C_A').$$

The first line is computed to be given by

First line \(=\) \(2d_+ \otimes (B_- W' + 2C_- A'_+ - 2C_3 A'_+ - 2d_- \otimes (B_+ W' + 2C_+ A'_+ + 2C_3 A'_+))\)

= \(2d_0 \otimes (B_- A'_+ - B_+ A'_-).

The second line is computed to be given by

Second line \(=\) \(2d_+ \otimes (-WB'_- + 2A_- C'_3 - 2A_+ C'_3) + 2d_- \otimes (WB'_+ + 2A_+ C'_3 + 2A_- C'_3 - 2d_0 \otimes (A_- B'_+ - A_+ B'_-) + 2r_+ \otimes A_- A'_- - 4r_- \otimes A_+ A'_- + 4r_3 \otimes (A_- A'_+ + A_+ A'_-))\)

\(=\) \(2v_+ \otimes (A_- W' - WA'_-) + 2v_- \otimes (A_+ W' - WA'_+).

Thus we obtain

\[\begin{align*}
A_\pm A'_\pm &= \pm 2A_\pm C'_\pm \mp 2C_3 A'_\pm + WB'_\pm - B_\pm W' + 2A_\pm C'_\pm - 2C_3 A'_\pm \\
W \ast W' &= 4A_- B'_+ + 4B_+ A'_- - 4A_+ B'_- - 4B_- A'_+ \\
C_\pm C'_\pm &= \mp 4A_\pm A'_\pm \\
C_3 C'_3 &= 2A_- A'_+ + 2A_+ A'_- \\
B_\pm B_\pm &= WA'_\pm - A_\pm W'.
\end{align*}\]

Also for \(i = 2\) we have

\[\begin{align*}
(h_A \otimes C_A) * (h_A' \otimes c_A') &= -\frac{1}{2}(-1)^{i}STr(h_A h_B') \otimes C_A c_B'.
\end{align*}\]

By using the identities $STrr_i r_j = \frac{1}{2} \delta_{ij}$, $STrv_v = \mp \frac{1}{2}$, $STrd_v = \pm \frac{1}{2}$ and $STrd_0 = -2$ (all other supertraces are zero) we obtain immediately the results

\[\begin{align*}
(h_A \otimes C_A) * (h_A' \otimes c_A') &= -\frac{1}{2}C_+ c_- - \frac{1}{2}C_- c'_+ - C_3 c'_3 - A_+ a'_- - A_- a'_+ + \frac{1}{4}Ww' \\
&+ B_- b'_+ - B_+ b'_-.
\end{align*}\]

For $i = 3$ we have the two non-vanishing products $\((h_A \otimes c_A) * (1_3 \otimes \Phi)\)$ and $\((h_A \otimes c_A) * (h_A' \otimes C_A')\)$ with obvious definitions by analogy with the products $\((1_3 \otimes \Phi) * (h_A \otimes c_A)\)$ and $\((h_A' \otimes C_A') * (h_A \otimes c_A)\)$. In particular the product of two-forms with one-forms is given as above with reversed order of small and capital letters. For $i = 4$ we have one non-zero product given by $\((1_3 \otimes \phi) * (1_3 \otimes \Phi)\)$ while the rest are zero.

This coboundary operator is nilpotent, i.e it satisfies $\delta^2 = 0$. The product $*$ is compatible with $\delta$ so that the Leibniz rule is respected. Thus we must have $\delta(X^i * Y^j) = \delta X^i * Y^j + (-1)^i X^i * \delta Y^j$. 

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3.2 Gauge action

We consider one-forms \( A = (A_\pm, W, C_i, B_\pm) \) satisfying the reality condition \( A^{++} = A \). Thus we must have \( A^{++}_\pm = \pm A_\mp, W^{++} = W, C^{++}_i = C_i \) and \( B^{++}_\pm = \mp B_\mp \). We define the curvature by

\[
F = \delta A + A^* A = (F_\pm, f, c_\pm).
\]

We can immediately compute

\[
F_\pm = 2[X_\mp, Y_\pm] \pm 2[X_\pm, Y_3] + [X_0, Z_\pm] + 2X_\pm
\]

\[
f = 4\{Z_+, X_-\} - 4\{Z_-, X_+\} + 2X_0
\]

\[
c_\pm = \mp 2\{X_\pm, X_\pm\} - Y_\pm
\]

\[
c_3 = 2\{X_+, X_-\} - Y_3
\]

\[
b_\pm = [X_0, X_\pm] - Z_\pm.
\]

Recall that \( X_\pm = D_\pm + A_\pm, X_0 = D_0 + W, Y_i = R_i + C_i \) and \( Z_\pm = V_\pm + B_\pm \). We define the supersymmetric noncommutative \( U(1) \) gauge action by

\[
S_L[A] = \alpha \text{Str} \circ F \circ F + \beta \text{Str}(A^* \delta A + \frac{2}{3} A^* A \circ A^* A)
\]

\[
= \alpha \text{Str} \circ F \circ F + \beta \text{Str}(A^* F - \frac{1}{3} A^* A \circ A^* A).
\]

The first term is similar to the usual Yang-Mills action whereas the second term is a (real-valued) Chern-Simons-like contribution. \( \alpha \) and \( \beta \) are two real parameters. The Hodge triangle \( \circ \) is defined as the identity map between \( \Psi_1^N \) and \( \Psi_2^N \) and thus \( \circ F \) should be considered as a one-form. Explicitly we have

\[
\circ F \circ F = b_- b_+ - b_+ b_- - c_1^2 + F_+ F_- - F_- F_+ + \frac{1}{4} f^2
\]

\[
A \circ A = B_- b_+ - B_+ b_- - C_i c_i + A_+ F_- - A_- F_+ + \frac{1}{4} W f,
\]

and

\[
A \circ A \circ A = B_- [W, A_+] - B_+ [W, A_-] - 2C_+ A_- A_- + 2C_- A_+ A_+ - 2C_3 \{A_+, A_-\}
\]

\[
+ \ A_+ (2[C_3, A_-] - 2[C_-, A_+]) + [W, B_-]) - A_- (- 2[C_3, A_+] - 2[C_+, A_-] + [W, B_+])
\]

\[
+ \ W \{A_-, B_+\} - W \{A_+, B_-\}.
\]

We need first to show gauge invariance of the above action. The invariance of the Yang-Mills term is obvious whereas the invariance of the Chern-Simons-like term requires some work in order to be established. Towards this end we will need to rewrite the Chern-Simons-like term in a completely covariant fashion.
First by thinking about $C_G$ as a one-form we can show after a long calculation that we must have

$$STrC_G \ast F = STr\left(-2D_+A_- + 2D_-A_+ + V_-B_+ - V_+B_- - R_iC_i - \frac{1}{2}D_0W\right)$$

$$+ \frac{1}{2}STrW\left\{D_-, B_+\} - \{D_+, B_-\} - \{V_-, A_+\} + \{V_+, A_-\}\right)$$

$$- STrA_+\left([D_-, C_3] - [D_+, C_-] - \frac{1}{2}[D_0, B_-] + [R_-, A_+] - [R_3, A_-] - \frac{1}{2}[W, V_-]\right)$$

$$- STrA_-\left([D_-, C_+] + [D_+, C_3] + \frac{1}{2}[D_0, B_+] - [R_+, A_-] - [R_3, A_+] + \frac{1}{2}[W, V_+]\right)$$

$$- STrC_+\{D_-, A_-\} + STrC_-\{D_+, A_+\} - \frac{1}{2}STrB_+([D_0, A_-] - [D_-, W])$$

$$+ \frac{1}{2}STrB_-([D_0, A_+] - [D_+, W]) - STrC_3\left\{D_-, A_+\} + \{D_+, A_-\}\right)\right). \quad (126)$$

In above we have used the results

$$STrD_\pm F_\pm = STr\left(\pm R_\mp C_\pm \pm 2D_\pm A_\pm \pm R_3C_3 \pm D_\pm A_\pm - V_\mp B_\pm \pm \frac{1}{4}D_0W\right)$$

$$+ 2A_\pm [D_\mp, C_\pm] \pm 2A_\pm [D_\mp, C_3] - W\{D_\mp, B_\pm\}$$

$$STrV_\mp b_\pm = STr\left(- D_\mp A_\pm \mp \frac{1}{4}D_0W - V_\mp B_\pm - W\{V_\mp, A_\pm\}\right)$$

$$\frac{1}{4}STrD_0 f = STr\left(D_+A_- - D_-A_+ + V_-B_+ - V_+B_- + \frac{1}{2}D_0W - A_-[D_0, B_+] + A_+[D_0, B_-]\right)$$

$$STrR_\mp c_\pm = STr\left(\mp 4D_\mp A_\pm - R_\mp C_\pm \pm 2A_\pm [R_\mp, A_\pm]\right)$$

$$STrR_3c_3 = STr\left(D_+A_- - D_-A_+ - R_3C_3 - A_-[R_3, A_+] - A_+[R_3, A_-]\right). \quad (127)$$
More calculation yields

\[
STrC_G \ast F = STr\left(-D_+ A_- + D_- A_+ - V_- B_+ + V_+ B_- - R_i C_i + \frac{1}{2} D_0 W\right)
\]
\[
+ \frac{1}{2} STrW\left(\frac{1}{4} f - \frac{1}{2} W - \{A_-, B_+\} + \{B_-, A_+\}\right)
\]
\[
- STrA_+\left(-\frac{1}{2} F_- + A_- - [A_-, C_3] + [A_+, C_-] + \frac{1}{2}[W, B_-]\right)
\]
\[
- STrA_-\left(\frac{1}{2} F_- - A_+ - [A_-, C_3] + [A_+, C_-] - \frac{1}{2}[W, B_+]\right)
\]
\[
- STrC_+\left(\frac{1}{4} c_+ + \frac{1}{4} c_- - \frac{1}{2}(A_-, A_-)\right) - STrC_-\left(\frac{1}{4} c_+ + \frac{1}{4} c_- + \frac{1}{2}\{A_+, A_+\}\right)
\]
\[
- \frac{1}{2} STrB_+\left(b_+ + B_+ - [W, A_-]\right) + \frac{1}{2} STrB_-\left(b_+ + B_- - [W, A_+]\right)
\]
\[
- STrC_3\left(\frac{1}{2} c_3 + \frac{1}{2} c_3 - \{A_-, A_+\}\right). \tag{128}
\]

Equivalently

\[
STrC_G \ast F = \frac{1}{2} STrA \ast F - \frac{1}{2} STrA \ast A - \frac{1}{2} STr\left((C_H + A_H) \ast \lhd(C_H + A_H) - C_H \ast \lhd C_H\right)
\]
\[
- STr\left((C_\perp + A_\perp) \ast \lhd(C_\perp + A_\perp) - C_\perp \ast \lhd C_\perp\right). \tag{129}
\]

In above \(C_\perp = C = C_G - C_H\), \(A_H\) is the projection of \(A\) in the directions along the generators of \(H = osp(2, 1)\) and \(A_\perp\) is the corresponding orthogonal part. Explicitly we have

\[
STr(C_H + A_H) \ast \lhd(C_H + A_H) = STr\left(-Y_i^2 - Z_+ Z_- + Z_- Z_+\right)
\]
\[
STr(C_\perp + A_\perp) \ast \lhd(C_\perp + A_\perp) = STr\left(\frac{1}{4} X_0^2 + X_+ X_- - X_- X_+\right). \tag{130}
\]

The supersymmetric noncommutative \(U(1)\) gauge action becomes

\[
S_L[A] = \alpha STr \diamond F \ast F + \frac{\beta}{3} STr\left(2(C_H + A_H) \ast F + 2(C_\perp + A_\perp) \ast F + (C_H + A_H) \ast \lhd (C_H + A_H)\right)
\]
\[
+ 2(C_\perp + A_\perp) \ast \lhd(C_\perp + A_\perp) - C_H \ast \lhd C_H - 2C_\perp \ast \lhd C_\perp). \tag{131}
\]

This establishes gauge invariance of the system under

\[
C_G + A \longrightarrow U \ast (C_G + A) \ast U^{++}. \tag{132}
\]

\(U\) is a zero-form with \(U^{++} = U^+\) and hence this transformation law means

\[
X_{\pm,0} \longrightarrow UX_{\pm,0} U^+, \ Y_i \longrightarrow UY_i U^+, \ Z_\pm \longrightarrow UZ_\pm U^+. \tag{133}
\]
The next step is to notice that the system as it stands contains too many degrees of freedom and hence we must impose some extra constraints in order to reduce the number of independent components of $A$ and $F$ from 8 to 2 since we are in two dimensions. We impose

$$(\delta A + A * A)_H = 0, \quad \Leftrightarrow \ b_+ = b_- = c_+ = c_- = c_3 = 0,$$  \hspace{1cm} (134)$$

and

$$(C_\bot + A_\bot) * \ll (C_\bot + A_\bot) - C_\bot * \ll C_\bot = 0.$$ \hspace{1cm} (135)$$

Both constraints are obviously gauge covariant.

These two constraints as well as the action (131) are invariant under all $OSP(2,1)$ supersymmetry transformations. Indeed the two quantities $(C_H + A_H) * \ll (C_H + A_H)$ and $(C_\bot + A_\bot) * \ll (C_\bot + A_\bot)$ are separately invariant under $OSP(2,1)$ which is the reason behind the invariance of the second constraint and the 4th and 5th terms of the action under $OSP(2,1)$. Furthermore a generic one-form and a generic two-form will always decompose under $OSP(2,1)$ into a direct sum of a superpin 1/2 multiplet and a superspin 1 multiplet. For example for the one-form $A$ and for the two-form $F$ the components $A_+, A_-, W$ and $F_+, F_-, f$ form $OSP(2,1)$ multiplets with superspin 1/2 while the other five components $B_+, B_-, C_i$ of $A$ and $b_+, b_-, c_i$ of $F$ form multiplets with superspin 1. This is the reason why the first constraint is $OSP(2,1)$ invariant. The invariance of the rest of the action under $OSP(2,1)$ is obvious since it is covariant under full $OSP(2,2)$.

### 3.3 The continuum limit

The constraint (135) reads explicitly

$$[D_+, A_-] - [D_-, A_+] + \frac{1}{4}(D_0, W) + [A_+, A_-] + \frac{1}{4}W^2 = 0.$$  \hspace{1cm} (136)$$

In the continuum limit this becomes

$$n_6 A_- - n_7 A_+ + \frac{1}{4}n_8 W = 0.$$ \hspace{1cm} (137)$$

After some calculation we get the solution (by using $\omega_6 = \frac{1}{2}(\bar{z}_1 \theta - z_2 \bar{\theta})$, $\omega_7 = \frac{1}{2}(z_1 \bar{\theta} + \bar{z}_2 \theta)$, $\omega_8 = 2 - \bar{z}z$ and $\bar{z}z + \theta \bar{\theta} = 1$)

$$W = \bar{z}z(A_+ \bar{\theta} + A_\bot \bar{\theta}) = \bar{A}_+ \bar{\theta} + A_\bot \bar{\theta},$$ \hspace{1cm} (138)$$

where

$$A_+ = \frac{1}{2}(A + \frac{z_1^+}{\bar{z}_2} \bar{A}), \quad A_- = -\frac{1}{2}(\bar{A} + \frac{\bar{z}_1}{z_2} A).$$ \hspace{1cm} (139)$$
The constraint (134) leads to the equations
\[
B_\pm = [D_0, A_\pm] - [D_\pm, W],
\]
\[
C_\pm = \mp 4\{D_\pm, A_\pm\}
\]
\[
C_3 = 2\{D_+, A_-\} + 2\{D_-, A_+\}. \tag{140}
\]

We need now to compute the action (with \(L \longrightarrow \infty\))
\[
S_L[A] = \alpha STr(F_+ F_- - F_- F_+ + \frac{1}{4} f^2) + \beta STr(A_+ F_- - A_- F_+ + \frac{1}{4} W f). \tag{141}
\]

Explicitly we have
\[
F_\pm = [D_0, [D_0, A_\pm]] + 2A_\pm - [D_0, [D_\pm, W]] - [V_\pm, W] \mp 12[D_\pm, \{D_\pm, A_\pm\}] \pm 12[D_\pm, A_\mp]
\]
\[
f = 2W + 4\{D_+, [D_+, W]\} - 4\{D_-, [D_+, W]\} + 4\{V_+, A_-\} - 4\{D_+, [D_0, A_-]\}
- 4\{V_-, A_+\} + 4\{D_-, [D_0, A_+]\}. \tag{142}
\]

Because of the constraint (137) we have only two independent superfields. We will work in the local coordinates \(t = z_1/z_2, \bar{t} = z_1^*/z_2^*, \ b = -\theta/z_2\) and \(\bar{b} = -\bar{\theta}/z_2^*\). We introduce the parametrization
\[
A_+ = \frac{1}{2}(A - \bar{t}\bar{A}), \quad A_- = -\frac{1}{2}(\bar{A} + tA), \quad W = \bar{b}\bar{A} - bA. \tag{143}
\]

In terms of \(t, \bar{t}\) and \(b, \bar{b}\) the supersymmetric covariant derivatives \(D, \bar{D}\) and the supersymmetric charges \(Q, \bar{Q}\) are given respectively by
\[
D = \partial_b + b\partial_t, \quad D = \partial_b + \bar{b}\partial_{\bar{t}}, \tag{144}
\]
and
\[
Q = \partial_b - b\partial_t, \quad \bar{Q} = \partial_b - \bar{b}\partial_{\bar{t}}. \tag{145}
\]

In terms of \(t, \bar{t}\) and \(b, \bar{b}\) the \(OSP(2, 2)\) generators are given by
\[
R_+ = -\partial_t - \bar{t}^2\partial_{\bar{t}} - \bar{b}\partial_b, \quad R_- = \partial_t + t^2\partial_{\bar{t}} + tb\partial_b, \quad R_3 = \bar{t} \partial_t - t \partial_{\bar{t}} + \frac{1}{2} b \partial_b - \frac{1}{2} \bar{b} \partial_{\bar{b}}
\]
\[
V_+ = \frac{1}{2}(Q + \bar{t}\bar{Q}), \quad V_- = \frac{1}{2}(\bar{Q} - tQ), \tag{146}
\]
and
\[
D_0 = \bar{b}\partial_b - b\partial_{\bar{b}}, \quad D_+ = \frac{1}{2}(D - \bar{t}\bar{D}), \quad D_- = -\frac{1}{2}(\bar{D} + tD). \tag{147}
\]

Let us compute
\[
(D_0^2 + 2)A_+ = \frac{1}{2} \bar{b}\bar{D}A + \frac{1}{2} bDA + \bar{b}bD\bar{A} - \bar{t} \bar{b} D\bar{A} - \frac{1}{2} \bar{t}\bar{D}\bar{A} - \frac{1}{2} b\bar{D}A + A - \bar{t}\bar{A}. \tag{148}
\]
\((\mathcal{D}_0 \mathcal{D}_+ + \mathbb{V}_+)(W) = -\frac{1}{2} \bar{b}b(\bar{t}D\bar{D}A + \bar{D}DA) + \frac{3}{2} \bar{b}b(D^2 \bar{A} + \bar{t}D^2 A) + \frac{1}{2}(\bar{t}\bar{A} - A) + (\bar{t}b - \frac{1}{2} \bar{b})\bar{D}A + \bar{D}A\)

\[+ \ (-\bar{b} + \frac{1}{2} \bar{t}b)\bar{D}A + \frac{1}{2}(bDA - \bar{b}\bar{D}A). \] (149)

In above we have used the identities \(\mathcal{D}^2 = \partial_t, \ \bar{\mathcal{D}}^2 = \partial_t, \ \mathcal{D}\bar{\mathcal{D}} = \partial_t\partial_b, \ \bar{\mathcal{D}}\mathcal{D} = \partial_t\partial_b\) and \(\{\mathcal{D}, \bar{\mathcal{D}}\} = 0\).

We can also compute

\[-12\mathcal{D}_- (\mathcal{D}_+ A_+) = 3\bar{D}DA_+ - 3\bar{t}tD\bar{D}A_+ + 3t\bar{D}^2 A_+ - 3\bar{t}\bar{D}^2 A_+ - 3\bar{b}\bar{D}A_+ \]

\[= -\frac{3}{2} \bar{t}\bar{D}DA - \frac{3}{2} \bar{t}tD\bar{D}A - \frac{3}{2} \bar{t}t\bar{D}^2 A + \frac{3}{2} tD^2 A + \frac{3}{2} \bar{D}DA - \frac{3}{2} \bar{t}\bar{D}^2 A \]

\[= \frac{3}{2} \bar{b}D\bar{A} + \frac{3}{2} \bar{t}tD\bar{D}A + \frac{3}{2} \bar{b}D\bar{A} + \frac{3}{2} \bar{t}\bar{A} - \frac{3}{2} \bar{b}\bar{D}A - \frac{3}{2} \bar{t}\bar{b}\bar{D}A. \] (150)

Also

\[12D_+^2 A_- = 3D^2 A_- + 3t\bar{D}^2 A_- + 3\bar{b}D\bar{A}_- \]

\[= -\frac{3}{2} D^2 A - \frac{3}{2} tD^2 A - \frac{3}{2} A - \frac{3}{2} t\bar{D}^2 A - \frac{3}{2} \bar{t}t\bar{D}^2 A - \frac{3}{2} \bar{b}D\bar{A} - \frac{3}{2} \bar{t}\bar{b}\bar{D}A. \] (151)

We get immediately (with \(y = 1 + \bar{t}t + \bar{b}b, \ \omega = \bar{D}A + D\bar{A}\) and \(2\omega_6 = -\bar{t}b + \bar{b}b\) )

\[F_+ = -\frac{3}{2} y \left( D^2 A + \bar{t}D^2 A + \bar{t}D\bar{D}A - \bar{D}DA + b\omega \right) + (\bar{b} - \bar{t}b)\omega \]

\[= -\frac{3}{2} \left( D(y\omega) + \bar{t}D(y\omega) \right) - 4\omega_6\omega. \] (152)

Since \(F_+^+ = F_\omega \) we must have (by using also \(D^+ = -\bar{D}, \ \bar{D}^+ = D, \ \bar{A}^+ = A, \ A^+ = -\bar{A}\) and \(\omega^+ = \omega, \ 2\omega^+_6 = 2\omega_7 = -\bar{t}b - b\) )

\[F_- = -\frac{3}{2} \left( \bar{D}(y\omega) - t\bar{D}(y\omega) \right) - 4\omega_7\omega. \] (153)

Also

\[4\mathcal{D}_+ \mathcal{D}_- W - 4\mathcal{D}_- \mathcal{D}_+ W = 2(1 + \bar{t}t)\bar{D}DW - bDW - \bar{b}\bar{D}W \]

\[= 2y(\bar{b}D\bar{D}A - b\bar{D}DA) - 2(1 + \bar{t}t)\omega - W - \bar{b}b\omega. \] (154)

\[4\mathbb{V}_+ A_- - 4\mathbb{V}_- A_+ = 2y(bD^2 A + \bar{b}\bar{D}^2 A) - (1 + \bar{t}t)\omega - W. \] (155)

\[-4\mathcal{D}_+ \mathcal{D}_0 A_- + 4\mathcal{D}_- \mathcal{D}_0 A_+ = (1 + \bar{t}t)(\mathcal{D}\mathcal{D}_0 A - \bar{D}\mathcal{D}_0 A) + b\mathcal{D}_0 A + \bar{b}\mathcal{D}_0 \bar{A} \]

\[= y(\bar{b}\mathcal{D}\mathcal{D}A - b\bar{D}DA + bD^2 A + \bar{b}\bar{D}^2 A - \omega). \] (156)

Hence

\[f = 3y(\bar{b}\mathcal{D}\mathcal{D}A - b\bar{D}DA + bD^2 A + \bar{b}\bar{D}^2 A - 2\omega) + 2(y + \bar{b}b)\omega \]

\[= 3(b\mathcal{D}(y\omega) + \bar{b}\mathcal{D}(y\omega)) - 4(y + \bar{b}b)\omega. \] (157)
We can immediately compute the Yang-Mills action

\[ S_{YM}[A] = \alpha STr(F_+ F_ - F_- F_ + \frac{1}{4} f^2) \]

\[ = \alpha STr\left(\frac{9}{2} y\mathcal{D}(y\omega)\mathcal{D}(y\omega) + 4y^2\omega^2\right) \]

\[ = \frac{\alpha}{2\pi i} \int \frac{d\bar{t}dt db d\bar{b}db}{y}\left(\frac{9}{2} y\mathcal{D}(y\omega)\mathcal{D}(y\omega) + 4y^2\omega^2\right). \tag{158} \]

In the last line we have also converted the supertrace into a superintegral. Similarly the Chern-Simons action becomes

\[ S_{CS}[A] = \beta STr(A_+ F_ - A_- F_ + \frac{1}{4} W f) \]

\[ = \beta STr\left(-\frac{3}{4} y\left( A\mathcal{D}(y\omega) + \bar{A}\mathcal{D}(y\omega) \right) \right) \]

\[ = \frac{\beta}{2\pi i} \int \frac{d\bar{t}dt db d\bar{b}db}{y}\left(-\frac{3}{4} y\left( A\mathcal{D}(y\omega) + \bar{A}\mathcal{D}(y\omega) \right) \right). \tag{159} \]

The last step is to rewrite the above actions in terms of components of the superfields \( A \) and \( \bar{A} \). Introduce

\[ iA = \zeta + bv + \frac{1}{2} b^w + iu + b\bar{b})(\frac{\eta}{1 + \bar{t}t} + \partial_t\bar{\zeta}) \]

\[ i\bar{A} = -\bar{\zeta} + b\bar{v} - \frac{1}{2} b^w - iu + b\bar{b})(\frac{\eta}{1 + \bar{t}t} + \partial_t\zeta). \tag{160} \]

\( w \) and \( u \) are real bosonic fields while \( v \) is a complex bosonic field. Clearly \( w^{++} = w, u^{++} = u, v^{++} = \bar{v} \). The fermionic fields \( \zeta \) and \( \eta \) are such that \( \zeta^{++} = -\bar{\zeta}, \bar{\zeta}^{++} = \zeta, \eta^{++} = \bar{\eta}, \bar{\eta}^{++} = -\eta \).

We can immediately compute

\[ iy\omega = iu + b\bar{\eta} - b\bar{v} - b\bar{b} + \bar{b} \left(1 + \bar{t}t\right)(\partial_t v - \partial_t \bar{v}) + \frac{i}{1 + \bar{t}t} u. \tag{161} \]

The Kahler term can now be put in the form (with \( \int dbd\bar{b} = -1, \int db = 0, \int d\bar{b} = 0 \))

\[ \frac{1}{2\pi i} \int \frac{d\bar{t}dt db d\bar{b}db}{y}\left(\frac{9\alpha}{2} y\mathcal{D}(y\omega)\mathcal{D}(y\omega) \right) = \frac{1}{2\pi i} \int d\bar{t}dt\left(\frac{9}{2}\alpha \right) \left[-(1 + \bar{t}t)^2(\partial_t v - \partial_t \bar{v})^2 + \partial_t u\partial_t v + \frac{u^2}{(1 + \bar{t}t)^2} + \bar{\eta}\partial_t \bar{\eta} - \partial_t \bar{v}\bar{\eta} - 2iu(\partial_t v - \partial_t \bar{v})\right]. \tag{162} \]

The superpotential takes the form

\[ \frac{1}{2\pi i} \int \frac{d\bar{t}dt db d\bar{b}db}{y}(4\alpha y^2\omega^2) = \frac{1}{2\pi i} \int d\bar{t}dt\left(\frac{2\bar{\eta}}{1 + \bar{t}t} - \frac{u^2}{(1 + \bar{t}t)^2} + 2iu(\partial_t v - \partial_t \bar{v})\right). \tag{163} \]

Similarly

\[ \frac{1}{2\pi i} \int \frac{d\bar{t}dt db d\bar{b}db}{y}\left( A\mathcal{D}(y\omega) + \bar{A}\mathcal{D}(y\omega) \right) = \frac{1}{2\pi i} \int d\bar{t}dt\left(\frac{3}{4} \beta \right) \left(\frac{2\bar{\eta}}{1 + \bar{t}t} - \frac{u^2}{(1 + \bar{t}t)^2} + 2iu(\partial_t v - \partial_t \bar{v})\right). \tag{164} \]
In order to cancel the coupling between the fields $u$ and $v$ we choose $\beta = -\frac{2}{3}\alpha$. This will also cancel the mass term of the $u$ field. We obtain finally (with $S_L[A] = S_{YM}[A] + S_{CS}[A]$ and $L\to\infty$)

$$
S_L[A] = \frac{1}{2\pi i} \int d\bar{t} dt \left( \frac{9}{2\alpha} \right) \left( - (1+i\bar{t})^2 (\partial_i v - \partial_{\bar{t}} v)^2 + \partial_i u \partial_{\bar{t}} u + \eta \partial_i \eta + \bar{\eta} \partial_{\bar{t}} \bar{\eta} + \frac{2\bar{\eta} \eta}{1+i\bar{t}} \right). \quad (165)
$$

### 4 Concluding remarks: A new fuzzy SUSY scalar action

The next natural step is to take the action (3) with the corresponding constraints (4) and (5) and write the whole thing in terms of the components of $X_{\pm,0}, Y_i, Z_{\pm}$ thus reducing the supertrace $STr$ to an ordinary trace $Tr$. The fermionic fields should then be integrated out before we can attempt any numerical investigation. This complicated exercise will not be pursued here.

A possibly much simpler supersymmetric action than the above pure gauge action is given by the following fuzzy supersymmetric scalar action. We introduce the superscalar fields $\Phi_H$ and $\Phi_\perp$ defined by the expressions

$$
\Phi_H = (C_H + A_H) \star (C_H + A_H) = -Y_i^2 - Z_+ Z_- + Z_- Z_+ \\
\Phi_\perp = (C_\perp + A_\perp) \star (C_\perp + A_\perp) = \frac{1}{4} X_0^2 + X_+ X_- - X_- X_+.
$$

The action we write (without any extra constraints and with full supersymmetry) is

$$
S_L[\Phi] = STr(a_\perp \Phi_\perp + b_\perp \Phi_\perp^2 + c_\perp \Phi_\perp^3 + d_\perp \Phi_\perp^4 + ...) \\
+ STr(a_H \Phi_H + b_H \Phi_H^2 + c_H \Phi_H^3 + d_H \Phi_H^4 + ...).
$$

This action is supersymmetric for the same reason that (3) is supersymmetric. It is gauge covariant since $\Phi_H$ and $\Phi_\perp$ are gauge covariant fields. The parameters $a, b, c, d, ...$ are the coupling constants of the model. The partition function thus reads

$$
Z_L[a, b, c, d, ...] = \int dX_\pm dX_0 dZ_\pm dY_i e^{-S_L[\Phi]}.
$$

We conclude this article by introducing the matrix components of $X_{\pm,0}, Y_i, Z_\pm$ in the following way. The superfields $X_0 = D_0 + W$ and $Y_i = R_i + C_i$ are real scalar superfields so that they are even elements of the superalgebra $Mat(2L+1,2L)$ while $X_\pm = D_\pm + A_\pm$ and $Z_\pm = V_\pm + B_\pm$ are odd elements of $Mat(2L+1,2L)$ (we think of them as real spinor superfields). This means that instead of considering the algebra $Mat(2L+1,2L)$ over the field of complex numbers $C$ we consider it over a graded commutative algebra $P$. Then the one-forms are actually elements of the space

$$
\Psi_N^1(P) = G_0 \otimes Mat(2L+1,2L; P)_0 \oplus G_1 \otimes Mat(2L+1,2L; P)_1.
$$

27
$G_0$ and $G_1$ are the even and odd parts of the super-Lie algebra $G = osp(2, 2)$ while $Mat(2L + 1, 2; P)_{0,1}$ are the subspaces of $Mat(2L + 1, 2L; P)$ with even and odd grading respectively with respect to the gradings of $Mat(2L + 1, 2L)$ and $P$. $\Psi^1_N(P)$ is isomorphic to the space of one-forms $\Psi^1_N$ we had constructed previously.

Let us introduce the $(2L + 1) \times (2L + 1)$ fermionic matrices $\psi_{\pm R}, \chi_{\pm R}$, the $2L \times 2L$ fermionic matrices $\psi_{\pm L}, \chi_{\pm L}$, the $(2L + 1) \times 2L$ bosonic matrices $X_{\pm R}, Z_{\pm R}$ and the $2L \times (2L + 1)$ bosonic matrices $X_{\pm L}, Z_{\pm L}$ as follows

$$X_\pm = \begin{pmatrix} \psi_{\pm R} & X_{\pm R} \\ X_{\pm L} & \psi_{\pm L} \end{pmatrix}, \quad X_+ = X_+^{++} = \begin{pmatrix} \psi_{++ R} & +X_{++ L} \\ -X_{++ R} & \psi_{++ L} \end{pmatrix}. \tag{170}$$

$$Z_\pm = \begin{pmatrix} X_{\pm R} & Z_{\pm R} \\ Z_{\pm L} & \chi_{\pm L} \end{pmatrix}, \quad Z_+ = Z_+^{++} = \begin{pmatrix} X_{++ R} & +Z_{++ L} \\ -Z_{++ R} & \chi_{++ L} \end{pmatrix}. \tag{171}$$

Furthermore we introduce the $(2L + 1) \times (2L + 1)$ bosonic matrices $Y_{iR} = Y_{iR}^{++}, X_{0R} = X_{0R}^{++}$, the $2L \times 2L$ bosonic matrices $Y_{iL} = Y_{iL}^{++}, X_{0L} = X_{0L}^{++}$, the $(2L + 1) \times 2L$ fermionic matrices $\phi_{iR} = -\phi_{iL}^{++}, \phi_{0R} = -\phi_{0L}^{++}$ and the $2L \times (2L + 1)$ fermionic matrices $\phi_{iL} = \phi_{iR}^{++}, \phi_{0L} = \phi_{0R}^{++}$ as follows

$$Y_i = \begin{pmatrix} Y_{iR} & \phi_{iR} \\ \phi_{iL} & Y_{iL} \end{pmatrix}, \quad X_0 = \begin{pmatrix} X_{0R} & \phi_{0R} \\ \phi_{0L} & X_{0L} \end{pmatrix}. \tag{172}$$

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