EXACT SIMULATION OF THE EXTREMA OF STABLE PROCESSES

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Abstract. We exhibit an exact simulation algorithm for the supremum of a stable process over a finite time interval using dominated coupling from the past (DCFTP). We establish a novel perpetuity equation for the supremum (via the representation of the concave majorants of Lévy processes [PUB12]) and apply it to construct a Markov chain in the DCFTP algorithm. We prove that the number of steps taken backwards in time before the coalescence is detected is finite.

1. Introduction

This paper describes an algorithm for generating exact samples of the extrema of a stable process (see Algorithm 1 below) based on dominated coupling from the past (DCFTP), a coupling method for exact simulation from an invariant distribution of a Markov chain on an ordered state space (cf. [KM00] and the references therein). The chain in Algorithm 1 is based on a novel characterisation for the law of the supremum of a stable process at a fixed time in Theorem 1. Perpetuity (1.1) is established via the stochastic representation for concave majorants of Lévy processes [PUB12] and the scaling property of stable laws (see Section 2 below for the proof of Theorem 1).

Theorem 1. Let \( Y = (Y_t)_{t \in [0,\infty)} \) be a stable process with the stability and positivity parameters \( \alpha \) and \( \rho \), respectively (see Appendix A). Define \( Y_1 = \sup_{s \in [0,1]} Y_s \) and let \((B,U,V,S,Y_1)\) be a random vector with independent components, where \( U,V \) are uniform on \((0,1)\), \( B \) is Bernoulli with parameter \( 1-\rho \) and \( S \) has the law of \( Y_1 \) conditioned on being positive. Then the following equality in law holds:

\[
Y_1 \stackrel{d}{=} \Lambda \left( U^{\frac{1}{\alpha}} Y_1 + (1-U)^{\frac{1}{\alpha}} S \right),
\]

where \( \Lambda = 1 + B(V^{\frac{1}{\rho}} - 1) \). Furthermore, the law of \( Y_1 \) is the unique solution to (1.1).

The universality of stable processes makes them ubiquitous in probability theory and many areas of statistics and natural and social sciences (see the monograph [UZ99] and the references therein). The problem of efficient simulation of stable random variables in the context of statistics was addressed in [DJ14]. Among the path properties, the running supremum \( \bar{Y}_t = \sup_{s \in [0,t]} Y_s \) is of special interest (cf. [SV08, BDP08, Kuz11, Don08]) as it arises in application areas such as optimal stopping, the prediction of the ultimate supremum and risk theory (cf. [BDP11, SV08]).

In general, one has no access to the density, distribution or even characteristic function of \( Y_1 \), making a rejection sampling algorithm (see [Dev86, Sec. II.3]) for \( Y_1 \) difficult to construct. More precisely, if \( Y \) has no positive jumps, the strong Markov property and the fact that \( Y \) does not jump over positive levels imply that \( \bar{Y}_1 \) has the same law as \( Y_1 \) conditioned on being positive [Mic13]. In all

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other cases, the law of $\overline{Y}_1$ is not accessible in closed form and the information about it in the literature is obtained via analytical methods based on the Wiener-Hopf factorisation. If $Y$ has no negative jumps, [BDP08] gives an alternating series expression for the density, while [Kuz11, Don08] give a double series representation for a dense class of parameters. The coefficients in these representations are complicated and it is not immediately clear how one could use them to design a simulation algorithm. Moreover, in the general case, when $\alpha$ is rational the series representation is proved to be convergent for finitely many $\rho$ only [Kuz13]. Our simulation algorithm is based on purely probabilistic methods (it may be regarded as a generalization of the exact simulation algorithm for Vervaat perpetuities in [FH10]) and as such covers the entire class of stable processes.

1.1. Exact Simulation Algorithm. The perpetuity in (1.1) above gives rise to an update function $x \mapsto \phi(x, \Theta)$ of a Markov chain on $(0, \infty)$, where the components of the random vector $\Theta$ are the random variables in Theorem 1 (see (3.1) below for the precise definition of $\phi$). The invariant distribution (i.e. invariant probability measure as defined in [MT09, p. 229]) for the chain $X' = \{X'_n\}_{n \in \mathbb{Z}}$, defined by $X'_n = \phi(X'_{n-1}, \Theta_{n-1})$ with $\{\Theta_n\}_{n \in \mathbb{Z}}$ a sequence of independent copies of $\Theta$, equals that of $\overline{Y}_1$. However, since $x \mapsto \phi(x, \Theta)$ is strictly increasing in $x$ with probability one, the coalescence does not occur, making $X'$ unusable for DCFTP purposes. Fortunately, the structure of the perpetuity in (1.1) is such that the update function $\phi$ can be modified to $x \mapsto \psi(x, \Theta)$, which is constant on a subinterval in $(0, \infty)$ with positive probability and globally non-decreasing. The definition of $\psi$, given in Lemma 1 below, was inspired by [FH10] where such a modification was applied to Vervaat perpetuities. The construction requires an addition of a single independent uniform random variable to the vector $\Theta$ and yields a Markov chain $X = \{X_n\}_{n \in \mathbb{Z}}$ on $(0, \infty)$ via $X_n = \psi(X_{n-1}, \Theta_{n-1})$, where $\{\Theta_n\}_{n \in \mathbb{Z}}$ are independent copies of $\Theta$. The invariant distribution of $X$ equals that of $\overline{Y}_1$ and the coalescence occurs at every step with positive probability. The former follows from Theorem 1 and the fact that the chains $X$ and $X'$ have the same transition probabilities (see Lemma 1 below) and the latter is a consequence of the structure of $\psi$.

Our aim is to sample $X_0$, whose law equals that of $\overline{Y}_1$. By construction of $\psi$ it follows that $\psi(x, \Theta) = \psi(a(\Theta), \Theta)$ for any $x \in (0, a(\Theta)]$, where $\theta \mapsto a(\theta)$ is a positive deterministic function explicitly given in (3.3) of Lemma 1 below. The coalescence for $X$ occurs every time the inequality $X_n \leq a(\Theta_n)$ is satisfied, since, if $-\sigma$ is such a time, then $X_{-\sigma+1} = \psi(a(\Theta_{-\sigma}), \Theta_{-\sigma})$ disregards the value $X_{-\sigma}$ and hence the entire trajectory of $X$ prior to time $-\sigma + 1$.

The task now is to detect whether the event $\{X_n \leq a(\Theta_n)\}$ occurred without knowing the value of $X_n$ (if we had access to $X_n$ for any $n \in \mathbb{Z}$, we would have a sample from the law of $\overline{Y}_1$). DCFTP [KM00] suggests to look for a process $D = \{D_n\}_{n \in \mathbb{Z}}$ satisfying $D_n \geq X_n$ for all $n \in \mathbb{Z}$, which can be simulated backwards in time (starting at 0) together with the i.i.d. sequence $\{\Theta_n\}_{n \in \mathbb{Z}}$.

It is possible to define such a process $D$, which turns out to be stationary but non-Markovian, by “unwinding” the recursion for $X$ backwards in time and bounding the terms (see (3.8) in Sec. 3).

Algorithm 1. Exact sampling from the law of $\overline{Y}_1$

1: Starting at 0, sample $\{(D_n, \Theta_n)\}_{n \in \mathbb{Z}}$ backwards in time until $-\sigma = \sup\{n \leq 0 : D_n \leq a(\Theta_n)\}$
2: Put $X_{-\sigma+1} = \psi(a(\Theta_{-\sigma}), \Theta_{-\sigma})$
3: Compute recursively $X_n = \psi(X_{n-1}, \Theta_{n-1})$ for $n = -\sigma + 2, \ldots, 0$
4: return $X_0$
The backward simulation of \( \{(D_n, \Theta_n)\}_{n \in \mathbb{Z}} \) in step 1 of Algorithm 1 is discussed in Section 4 below. It relies on two ingredients: (A) the simulation of the indicators of independent events with summable probabilities and (B) the simulation of a random walk with negative drift and its future supremum. By the Borel-Cantelli lemma, only finitely many indicators in (A) are non-zero.

A simple and efficient algorithm for the simulation of the entire sequence is given in Section 4.1 below. The algorithm for (B) has been developed in [BS11, Sec. 4]. For completeness, in Section 4.2 below we present the algorithm from [BS11, Sec. 4] applied to the specific random walk that arises in Section 4.1 below. The algorithm for (B) has been developed in [BS11, Sec. 4]. For completeness, in Section 4.2 below we present the algorithm from [BS11, Sec. 4] applied to the specific random walk that arises in Section 4.1 below.

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Theorem 2. The random time \( \sigma \) in Algorithm 1 is finite a.s. Moreover, \( \mathbb{E}[\sigma | \Xi_0] < \infty \) a.s.

In [FH10, Thm 5.1] the authors provide a sharp estimate on \( \mathbb{E}[\sigma] \) for an analogous algorithm in the context of Vervaat perpetuities. Their analysis is based on the fact that their dominating process \( D \) is a birth-death Markov chain and is hence time-reversible with skip-free increments and an explicit invariant distribution (shifted geometric). In the context of Theorem 2, the dominating process \( D \) is non-Markovian, its increments have heavy tails and the multi-dimensional Markov chain \( \Xi \) used to bound \( D \) has a non-explicit invariant probability measure \( \pi \) (which also has heavy tails). Moreover, the law of the time-reversal of \( \Xi \) (with respect to \( \pi \)) is very different from that of \( \Xi \). The key step in the proof of Theorem 2 is provided by [Rev84, Thm 8.1.1], which allows us to conclude that the time-reversed chain has a Harris recurrent modification. However, a quantitative bound on the expected number of steps taken backwards in time in Algorithm 1 remains an open problem.

1.2. Related literature. Exact simulation algorithms for various instances of a general perpetuity equation \( X = A_0 X + A_1 \) (with \( (A_0, A_1) \) and \( X \) independent) have been developed in the literature. [FH10] studies the case \( A_0 = A_1 \geq 0, \mathbb{E}[A_0] < 1 \), specialising to the Vervaat perpetuity for \( A_0 = U^{1/\beta} \) with \( U \) uniform on \( (0,1) \) and \( \beta \in (0,\infty) \), see also [CH17, Dev01]. A sped up version of a DCFTP algorithm [Dev01] in the case \( \beta = 1 \) (i.e. when \( X \) follows the Dickman distribution) is given in [DF10].

In [DJ11], the authors develop the double CFTP algorithm in the case \( A_0 = V \) and \( A_1 = (1-V)Z \), where \( V \) takes value in \( [0,1] \) (and has a computable density) and \( Z \) is independent of \( V \) with support in an interval \( [0,c] \) for some \( c < \infty \). This structure appears similar to perpetuity (2.1) of [0].

1Algorithm 1 was implemented in Julia, see the GitHub repository [GCMUB18] for the code. Free parameters \( (d, \delta, \gamma) \) (see page 5 below) were fixed at \( (\frac{\alpha}{\beta}, \frac{\alpha}{\beta}, \alpha^2) \). On a laptop with Intel@Celeron(R) CPU N2840 @ 2.16GHz×2, Algorithm 1 output approximately 100 samples per second with slight variations for varying \( (\alpha, \rho) \). The numerical performance and optimal choice of \( (d, \delta, \gamma) \) (as a function of \( (\alpha, \rho) \)) are left for future work.
Proposition 1 below, where \( A_0 = U^{1/\alpha} \) and \( A_1 = (1 - U)^{1/\alpha} \) \( \max \{ Y_1, 0 \} \) with \( Y_1 \) an \( \alpha \)-stable random variable independent of the uniform \( U \). Proposition 1 provides a key step in the proof of Theorem 1 above, which in turn is the cornerstone of Algorithm 1. The upper bound \( c \) on the support of \( Z \) in [DJ11] is inversely proportional to the coalescence probability of the chain in the double CFTP algorithm, making its direct application to perpetuity (2.1) impossible, since \( \max \{ Y_1, 0 \} \) not only has infinite support but also a heavy tail. Moreover, even if we could construct a stochastic (rather than constant) upper bound on the relevant support, this bound would by necessity still have to have a heavy tail making the coalescence in a generalisation of the algorithm in [DJ11] unlikely.

In [BS11], the authors allow \( A_1 \) to have a heavy tail, but assume that \( A_0 \) and \( A_1 \) are independent, a requirement clearly violated by perpetuities (1.1) and (2.1) in the present paper. Moreover, a certain domination condition [BS11, Eq. (2) in Assumption (B)] for the density of \( \text{certain domination condition} \) \( \alpha, \rho \) is stipulated, which plays an important role in the coalescence probability but may be hard to establish for the density of a stable law conditioned on being positive.

The problem of the exact simulation of the first passage event of a spectrally positive stable process (resp. a Lévy process with infinite activity and finite variation) is addressed in [Chi18] (resp. [Chi12]). Theorem 1 easily implies the asymptotic behaviour at infinity of the distribution function of \( \overline{Y}_1 \) stated in [Ber96, Prop. VIII.1.4, p. 221]. Excluding the spectrally negative case, perpetuity (1.1) and the Grincevičius-Grey theorem [BDM16, Thm. 2.4.3] yield \( \lim_{x \to \infty} 2\mathbb{P}(Y_1 U^{1/\alpha} > x) / \mathbb{P}(Y_1 > x) = 1 \). By Breiman’s lemma [BDM16, Lem. B.5.1] we have \( \lim_{x \to \infty} 2\mathbb{P}(Y_1 U^{1/\alpha} > x) / \mathbb{P}(Y_1 > x) = 1 \), implying \( \lim_{x \to \infty} \mathbb{P}(Y_1 > x) / x^{-\alpha} = \Gamma(\alpha) \sin(\pi \alpha \rho) / \pi \) via the classical tail behaviour of the stable law [UZ99, Sec. 4.3].

2. Stochastic Perpetuities

Let \( Y \) be a stable process with stability and positivity parameters \( \alpha \) and \( \rho \), respectively (see Appendix A below for definition). Since \( Y_0 = 0 \) and the scaling property yield \( \overline{Y}_t = \sup_{s \in [0,t]} Y_s \overset{d}{=} t^{1/\alpha} \overline{Y}_1 \) for all \( t \in [0, \infty) \), we may restrict our attention to \( \overline{Y}_1 \). Let \( S(\alpha, \rho) \) and \( \overline{S}(\alpha, \rho) \) denote the laws of \( Y_1 \) and \( \overline{Y}_1 \), respectively. Since \( \mathbb{P}(Y_t > 0) = \rho \) for any \( t > 0 \), the extreme cases \( \rho \in \{0, 1\} \) are excluded from our analysis as they correspond to \( Y \) having monotone paths. Let \( U(0,1) \) denote the uniform law on \( (0,1) \) and define \( x^+ = \max \{ x, 0 \} \) for any real number \( x \in \mathbb{R} \).

**Proposition 1.** Let \( (\overline{Y}_1, Z, U) \sim \overline{S}(\alpha, \rho) \times S(\alpha, \rho) \times U(0,1) \). Then the law of \( \overline{Y}_1 \) is the unique solution of the following perpetuity:

\[
\overline{Y}_1 \overset{d}{=} U^+ \overline{Y}_1 + (1 - U)^+ Z^+.
\]  

The proof of Proposition 1 exploits the fact that the supremum of a function lies on its concave majorant, at the end of all (if any) faces with positive slope. The idea is as follows: fix a sample path of \( Y \) and pick a random face of its concave majorant above an independent uniform point in \( [0,1] \). The length of the chosen face is distributed as \( V \sim U(0,1) \) and its height is distributed as the increment of a stable process over a time interval of duration \( V \). Moreover, after removing this
face (together with the path underneath it) the remainder of the concave majorant behaves like a concave majorant of a stable process over the time interval $[0, 1 - V]$, see \cite{PUB12}. This recursive relation and the scaling property of $Y$ will yield the perpetuity in (2.1).

**Proof.** For any $a < b$, the concave majorant of a function $f : [a, b] \to \mathbb{R}$ is defined as the smallest concave function $c : [a, b] \to \mathbb{R}$, such that $c(t) \geq f(t)$ for every $t \in [a, b]$. Following the classical result for the complete description of a concave majorant of random walks, \cite{PUB12} describes the continuous time analogue of these results for Lévy processes (\cite{PUB12} is phrased in terms of the convex minorant, but through a change of sign their results cover the concave majorant).

A stick-breaking process $\{\ell_n\}_{n \geq 1}$ on $[0, 1]$ is defined recursively as follows:

$$\ell_n = V_n (1 - L_{n-1}), \quad n \geq 1,$$

where $L_{n-1} = \ell_1 + \cdots + \ell_{n-1}$, $L_0 = 0$ and $\{V_n\}_{n \geq 1}$ is a sequence of i.i.d. random variables with law $U(0, 1)$ (independent of $Y$). Let $C = (C_t)^{t \in [0, 1]}$ be the concave majorant of the Lévy process $Y$. Let $(d_n - g_n, C_d_n - C_g_n)_{n \geq 1}$ be the lengths and heights of the faces of $C$ picked at random, uniformly on lengths and without replacement ($g_n$ and $d_n$ denote the beginning and end times for the $n$-th face). \cite{PUB12}, Thm. 1 asserts the equality in law

$$(d_n - g_n, C_d_n - C_g_n)_{n \geq 1} \overset{d}{=} (\ell_n, Y_{L_n} - Y_{L_{n-1}})_{n \geq 1}.$$

The concave majorant $(C_t)^{t \in [0, 1]}$ is piecewise linear, with the corresponding slopes forming a non-increasing piecewise constant function in $t$. Hence $\bar{Y}_1$ is always contained in the image of the function $C$. Moreover, the supremum equals the sum of all the positive heights of $C$:

$$\bar{Y}_1 = \sum_{n=1}^\infty (C_d_n - C_g_n)^+ \overset{d}{=} \sum_{n=1}^\infty (Y_{L_n} - Y_{L_{n-1}})^+.$$

Conditional on $\{L_n\}_{n \geq 1}$, the random variables $\{Y_{L_n} - Y_{L_{n-1}}\}_{n \geq 1}$ are independent and have the same distribution as the respective $Y_n$. Hence, for an independent i.i.d. sequence $\{Z_n\}_{n \geq 1}$ with law $S(\alpha, \rho)$ we have

$$(\ell_n, Y_{Z_n} - Y_{Z_{n-1}})_{n \geq 1} \overset{d}{=} \left(\ell_n, \ell_n^{1/\alpha} Z_n\right)_{n \geq 1},$$

implying

$$\bar{Y}_1 \overset{d}{=} \sum_{n=1}^\infty (Y_{L_n} - Y_{L_{n-1}})^+ \overset{d}{=} \sum_{n=1}^\infty \ell_n^{1/\alpha} Z_n^+.$$

It is well-known that $\left\{\frac{\ell_n}{1 - \ell_n}\right\}_{n \geq 2}$ is a stick-breaking process on $[0, 1]$, independent of $\ell_1 \sim U(0, 1)$ (and $\{Z_n\}_{n \geq 1}$). Hence by (2.2) we find the equality in law

$$\bar{Y}_1 \overset{d}{=} \sum_{n=2}^\infty \frac{\ell_n}{1 - \ell_1}^{1/\alpha} Z_n^+,$$

which, together with (2.2), implies the perpetuity

$$\bar{Y}_1 \overset{d}{=} \frac{1}{\ell_1^{1/\alpha}} Z_1^+ + (1 - \ell_1)^{1/\alpha} \bar{Y}_1.$$

Finally, the uniqueness of solution follows from \cite{BDM16}, Thm 2.1.3]. \qed
Let $S^+(\alpha, \rho)$ denote the law of $Y_1$ conditioned on being positive. For any $n, m \in \mathbb{Z}$ define
\begin{equation}
Z^n = \{ k \in \mathbb{Z} : k < n \}, \quad Z^n_m = Z^n \setminus Z^m.
\end{equation}

**Proof of Theorem 2.** Note that the random variable $Z^+$ in Proposition 1 behaves like the product of a Bernoulli random variable and a stable random variable conditioned on being positive, i.e., if $B \sim \text{Ber}(\rho)$ and $S \sim S^+(\alpha, \rho)$ are independent, then $Z^+ \overset{d}{=} BS$. Since $\mathbb{P}(Z^+ = 0) = 1 - \rho > 0$, the idea behind the proof of Theorem 1 is to iterate perpetuity \((2.1)\) backwards in time until the first time we observe $Z^+ > 0$.

More precisely, by Proposition 1 and Kolmogorov’s consistency theorem we can construct a stationary Markov chain $\{(U_n, Z_n, \zeta_n)\}_{n \in \mathbb{Z}^1}$ with invariant law $U(0, 1) \times S(\alpha, \rho) \times \overline{S}(\alpha, \rho)$, where $\{(U_n, Z_n)\}_{n \in \mathbb{Z}^1}$ is an i.i.d. sequence with law $U(0, 1) \times S(\alpha, \rho)$ and
\begin{equation}
\zeta_{n+1} = U_n^\alpha Z_n^+ + (1 - U_n)^{\frac{1}{\alpha}} \zeta_n, \quad n \in \mathbb{Z}^0.
\end{equation}
Define $V_0 = 1$ and $V_n = \prod_{m \in \mathbb{Z}_n^0} (1 - U_m)$ for $n \in \mathbb{Z}^0$. Then the following equality holds
\begin{equation}
\zeta_0 = \sum_{m \in \mathbb{Z}_n^0} (U_m V_{m+1})^{\frac{1}{\alpha}} Z_m^+ + V_n^{\frac{1}{\alpha}} \zeta_n \quad \text{for all } n \in \mathbb{Z}^0.
\end{equation}

Let $\tau = \sup \{ n \in \mathbb{Z}^0 : Z_n > 0 \}$ (with convention $\sup \emptyset = -\infty$) be the last time we see a positive value in the sequence $\{Z_n\}_{n \in \mathbb{Z}^0}$. Substituting $n = \tau$ in equation \((2.4)\), we get
\begin{equation}
\zeta_0 = V_{\tau+1}^{\frac{1}{\alpha}} \left( (1 - U_\tau)^{\frac{1}{\alpha}} \zeta_\tau + U_{\tau+1}^{\frac{1}{\alpha}} Z_\tau \right).
\end{equation}
This equality of course yields the same equality in law. It will hence imply the perpetuity in \((1.1)\), if we prove that the random variables involved have the desired laws and independence structure.

The events $\{Z_n > 0, n \in \mathbb{Z}^0\}$ are independent with probability $\rho$, making $\tau$ a geometric random variable on $\mathbb{Z}^0$ with parameter $\rho$. By construction, the coordinates of the vector $(U_n, Z_n, \zeta_n)$ are independent for any $n \in \mathbb{Z}^0$. Hence we have $(U_\tau, Z_\tau, \zeta_\tau) \sim U(0, 1) \times S^+(\alpha, \rho) \times \overline{S}(\alpha, \rho)$. Moreover, $(U_\tau, Z_\tau, \zeta_\tau)$ is independent of $(\tau, V_{\tau+1})$. Hence \((2.5)\) will imply the perpetuity in the theorem if we prove that $\Lambda$ has the same law as $V_{\tau+1}$. Put differently, as $\tau$ and $U_0$ are independent, it is sufficient to prove the following equality in law
\begin{equation}
V_{\tau+1} \overset{d}{=} 1_{\tau = -1} + 1_{\tau \neq -1} U_0^{\frac{1}{\rho}} = 1 + 1_{\tau \neq -1} \left( U_0^{\frac{1}{\rho}} - 1 \right).
\end{equation}
Since $- \log (1 - U_1) \sim \text{Exp}(1)$ is exponential with mean one, $- \log (V_n)$ is gamma distributed with density $x \mapsto x^{-n-1} e^{-x}/(-n-1)!$ for any $n \in \mathbb{Z}^0$. Hence, on the event $\{\tau \neq -1\}$, the density of the conditional law $- \log (V_{\tau+1}) | \tau$ is given by $x \mapsto x^{-\tau-2} e^{-x}/(-\tau-2)!$. Thus, the conditional law $- \log (V_{\tau+1}) | \{\tau \neq -1\}$ is exponential with density
\begin{equation}
x \mapsto \frac{1}{1 - \rho} \sum_{k=2}^{\infty} \rho (1 - \rho)^{k-1} \frac{x^{k-2}}{(k-2)!} e^{-x} = \rho e^{-\rho x}, \quad x > 0.
\end{equation}
Since $- \log (V_{\tau+1})$ takes the value 0 when $\tau = -1$, which happens with probability $\rho$, and is otherwise exponential with mean $1/\rho$, the distributional identity in \((2.6)\) follows.

The uniqueness of solution for perpetuity \((1.1)\) follows from in [BDM10, Thm 2.1.3]. \(\square\)
3. The Markov chain X and the dominating process D in Algorithm

Let \( A = (0, \infty) \times (0,1) \times (0,1) \times (0,1) \) and define the function \( \phi : (0, \infty) \times A \to (0, \infty) \) by

\[
(3.1) \quad \phi(x, \theta) = \lambda^\frac{1}{\alpha} \left( u^\frac{1}{\alpha} x + (1 - u)^\frac{1}{\alpha} s \right), \quad x \in (0, \infty), \quad \theta = (s, u, w, \lambda) \in A.
\]

Note that the map \( x \mapsto \phi(x, \theta) \) is increasing and linear in \( x \) for all \( \theta \in A \) and does not depend on \( w \). Let \( W \sim U(0,1) \) be independent of random variables \( S, U, \) and \( \Lambda \) defined in Theorem

Then, by Theorem [1] we have \( \zeta \overset{d}{=} \phi(\zeta, \Theta) \), where \( \zeta \sim \mathcal{S}(\alpha, \rho) \) is independent of \( \Theta = (S, U, W, \Lambda) \).

Hence a Markov chain with the update function \( \phi \) has the correct invariant law but does not allow for coalescence: if for any \( x, y \in (0, \infty) \) we have \( \phi(x, \Theta) = \phi(y, \Theta) \), by (3.1) it follows \( x = y \). But the structure of \( \phi \) and the additional randomness in \( W \) allow us to modify the update function \( x \mapsto \phi(x, \theta) \) so that coalescence can be achieved, while keeping the law of the chain unchanged.

**Lemma 1.** Define the functions \( \psi : (0, \infty) \times A \to (0, \infty) \) and \( a : A \to (0, \infty) \) by the formulae

\[
(3.2) \quad \psi(x, \theta) = 1_{\{a(\theta) \geq x\}} w^{\frac{1}{\alpha}} (1 - u)^{\frac{1}{\alpha}} s + 1_{\{a(\theta) < x\}} \lambda^\frac{1}{\alpha} \left( u^\frac{1}{\alpha} x + (1 - u)^\frac{1}{\alpha} s \right),
\]

\[
(3.3) \quad a(\theta) = \left( \lambda^\frac{1}{\alpha} - 1 \right) \left( \frac{1 - u}{u} \right)^\frac{1}{\alpha} s.
\]

The map \( x \mapsto \psi(x, \theta) \) is non-decreasing in \( x \) for all \( \theta \in A \). Moreover, for \( \zeta \) and \( \Theta \) as in the paragraph above, we have \( \phi(x, \Theta) \overset{d}{=} \psi(x, \Theta) \) for all \( x > 0 \) and \( \mathcal{S}(\alpha, \rho) \) is the unique solution of the distributional equation \( \zeta \overset{d}{=} \psi(\zeta, \Theta) \).

**Proof.** The function \( \psi \) takes constant value of \( w^{\frac{1}{\alpha}} (1 - u)^{\frac{1}{\alpha}} s \) for \( x \in (0, a(\theta)] \) and increases linearly on the interval \( (a(\theta), \infty) \) with the right limit satisfying \( \lim_{x \downarrow a(\theta)} \psi(x, \theta) = (1 - u)^{\frac{1}{\alpha}} s > \psi(a(\theta), \theta) \).

Hence the desired monotonicity follows.

We now prove that \( \phi(x, \Theta) \overset{d}{=} \psi(x, \Theta) \) for all \( x > 0 \), i.e the transition probabilities for the update functions \( \phi \) and \( \psi \) coincide. Pick \( x > 0 \) and note that \( \{ \phi(x, \Theta) = \psi(x, \Theta) \} \subset \{ a(\Theta) < x \} \). Thus, for any \( y > 0 \) we have \( \mathbb{P}(\phi(x, \Theta) \leq y, a(\Theta) < x) = \mathbb{P}(\psi(x, \Theta) \leq y, a(\Theta) < x) \). Define

\[
v(u, s) = \left( \frac{(1 - u)^{\frac{1}{\alpha}} s}{u^\frac{1}{\alpha} x + (1 - u)^\frac{1}{\alpha} s} \right)^{\alpha \rho} \in (0, 1),
\]

and note that \( \{ a(\Theta) \geq x \} = \{ \Lambda^\rho \leq v(U, S) \} \). On this event, the definition of \( \Lambda \) in Theorem [1] implies the inequality \( \Lambda < 1 \), in which case \( \Lambda^\rho \) is uniform on \( (0, 1) \). Hence the conditional law of \( \Lambda \), given \( (U, S) \) and \( \{ a(\Theta) \geq x \} \), is uniform on the interval \( (0, v(U, S)) \). Moreover, the conditional law of \( v(U, S) W \), given \( (U, S) \) and on \( \{ a(\Theta) \geq x \} \), is also uniform on \( (0, v(U, S)) \). Hence for any \( y > 0 \) the following equalities hold:

\[
\mathbb{P}(\phi(x, \Theta) \leq y, a(\Theta) \geq x | U, S) = \mathbb{P} \left( \Lambda^\rho \leq \left( \frac{y}{u^\frac{1}{\alpha} x + (1 - U)^\frac{1}{\alpha} S} \right)^{\alpha \rho}, a(\Theta) \geq x \mid U, S \right)
\]

\[
= \mathbb{P} \left( v(U, S) W \leq \left( \frac{y}{u^\frac{1}{\alpha} x + (1 - U)^\frac{1}{\alpha} S} \right)^{\alpha \rho}, a(\Theta) \geq x \mid U, S \right)
\]

\[
= \mathbb{P} \left( W^{\frac{1}{\alpha}} (1 - U)^\frac{1}{\alpha} S \leq y, a(\Theta) \geq x \mid U, S \right)
\]

\[
= \mathbb{P}(\psi(x, \Theta) \leq y, a(\Theta) \geq x | U, S).
\]
Taking expectations in this identity yields the unconditional equality \( \mathbb{P}(\phi(x, \Theta) \leq y, a(\Theta) \geq x) = \mathbb{P}(\psi(x, \Theta) \leq y, a(\Theta) \geq x) \). Hence we get \( \mathbb{P}(\phi(x, \Theta) \leq y) = \mathbb{P}(\psi(x, \Theta) \leq y) \) for all \( y > 0 \), implying the equality in law \( \phi(x, \Theta) \overset{d}{=} \psi(x, \Theta) \) for arbitrary \( x > 0 \).

Pick \( y > 0 \). Since \( \Theta \) and \( \zeta \) are independent, by Theorem 1 we have

\[
\mathbb{P}(\zeta \leq y) = \mathbb{P}(\phi(\zeta, \Theta) \leq y) = \int_{(0,\infty)} \mathbb{P}(\phi(x, \Theta) \leq y) \mathbb{P}(\zeta \leq dx)
= \int_{(0,\infty)} \mathbb{P}(\psi(x, \Theta) \leq y) \mathbb{P}(\zeta \leq dx) = \mathbb{P}(\psi(\zeta, \Theta) \leq y),
\]

implying \( \zeta \overset{d}{=} \psi(\zeta, \Theta) \). Moreover, if there exists some \( \zeta' \) (independent of \( \Theta \)) satisfying \( \zeta' \overset{d}{=} \psi(\zeta', \Theta) \), this calculation implies the equality \( \zeta' \overset{d}{=} \phi(\zeta', \Theta) \). By Theorem 1 we get \( \zeta' \overset{d}{=} \zeta \), as claimed. \( \square \)

By Lemma 1 and Kolmogorov’s consistency theorem, there exists a probability space supporting a sequence \( \{\Theta_n\}_{n \in \mathbb{Z}} \) of independent copies of \( \Theta \) and a stationary Markov chain \( \{X_n\}_{n \in \mathbb{Z}} \), satisfying \( X_{n+1} = \psi(X_n, \Theta_n) \) for all \( n \in \mathbb{Z} \). In the remainder of the paper, \( \{(X_n, \Theta_n)\}_{n \in \mathbb{Z}} \) denotes the corresponding Markov chain on \((0, \infty) \times A\). In order to detect coalescence in Algorithm 1 we now construct a dominating process \( \{D_n\}_{n \in \mathbb{Z}} \).

With this in mind, fix constants \( \delta \) and \( d \) satisfying \( 0 < \delta < d < \frac{1}{\alpha \rho} \). Let \( I^n_k = 1_{\{S_k > e^{\delta(n-1-k)}\}} \) for all \( n \in \mathbb{Z} \) and \( k \in \mathbb{Z}^n \) (see (2.3) above), where \( S_k \sim S^+(\alpha, \rho) \) is the first component of \( \Theta_k \) (see the first paragraph of Section 3). Fix \( \gamma > 0 \) such that \( \mathbb{E}S_1^\gamma < \infty \) (see (A.2)). Markov’s inequality implies

\[
p(m) = \mathbb{P}\left(S_1 \leq e^{\delta m}\right) \geq 1 - e^{-\delta \gamma m} \mathbb{E}S_1^\gamma, \quad m \geq 0,
\]

and hence \( \sum_{m=0}^{\infty}(1-p(m)) < \infty \). Since \( \{S_k\}_{k \in \mathbb{Z}} \) are independent, the Borel-Cantelli lemma ensures that, for a fixed \( n \in \mathbb{Z} \), the events \( \{S_k > e^{\delta(n-1-k)}\} = \{I^n_k = 1\} \) occur for only finitely many \( k \in \mathbb{Z}^n \) a.s. Let \( \chi_n \) be the smallest time beyond which the indicators \( I^n_k \) are all zero:

\[
\chi_n = (n-1) \wedge \inf\{k \in \mathbb{Z}^n : I^n_k = 1\},
\]

with convention \( \inf\emptyset = \infty \). Note that \( -\infty < \chi_n \leq n-1 \) holds a.s. for all \( n \in \mathbb{Z} \). Since the integers are countable, we have \( n-1 \geq \chi_n > -\infty \) for all \( n \in \mathbb{Z} \) a.s.

Define the i.i.d. sequence \( \{F_n\}_{n \in \mathbb{Z}} \) by \( F_n = d + \frac{1}{\alpha} \log(\Lambda_n U_n) \), where \( U_n \) and \( \Lambda_n \) are the second and fourth components of \( \Theta_n \), respectively (see the first paragraph of Section 3). Note that \( d - F_n \) has the same law as a geometric sum of exponential random variables and is hence exponentially distributed with mean \( \mathbb{E}[d - F_n] = \frac{1}{\alpha \rho} \). Let \( C = \{C_n\}_{n \in \mathbb{Z}} \) be a random walk defined by \( C_0 = 0 \) and

\[
C_{n+1} = C_n - F_n, \quad n \in \mathbb{Z}.
\]

Recall definition (2.3) and let \( R = \{R_n\}_{n \in \mathbb{Z}} \) be the reflected process of the walk \( \{C_n\}_{n \in \mathbb{Z}} \), that is

\[
R_n = \sup_{k \in \mathbb{Z}^{n+1}} C_k - C_n, \quad n \in \mathbb{Z}.
\]
For any \( n \in \mathbb{Z} \), define the following random variables

\[
\begin{align*}
D_n &= \exp (R_n) \left( \frac{e^{(1-d)(\chi_n-n)}}{1-e^{1-d}} + \sum_{k \in \mathbb{Z}_{\chi_n}^n} e^{-(n-1-k)d} S_k \left( 1 - U_k \right)^{1/\alpha} \right), \\
D'_n &= \exp (R_n) \left( \frac{1}{1-e^{1-d}} + D''_n \right), \quad \text{where} \quad D''_n = \sum_{k \in \mathbb{Z}^n} e^{-(n-1-k)d} S_k.
\end{align*}
\]

The sum in (3.8) is taken to be zero if \( \mathbb{Z}_{\chi_n}^n = \emptyset \), i.e. if \( \chi_n = n \). Note that the series in \( D''_n \) is absolutely convergent by the Borel-Cantelli lemma, but \( D'_n \) cannot be simulated directly as it depends on an infinite sum. Finally, define the random element \( \Xi_n = (\Theta_n, R_n, D'_n) \) for any \( n \in \mathbb{Z} \).

**Lemma 2.**

(a) \( X_n \leq D_n \leq D'_n \) for all \( n \in \mathbb{Z} \) a.s.

(b) The processes \( R = \{R_n\}_{n \in \mathbb{Z}} \) and \( \Xi = \{\Xi_n\}_{n \in \mathbb{Z}} \) are Markov, stationary and \( \varphi \)-irreducible (see definition [MT09, p. 82]) with respect to the respective invariant distributions.

**Proof.** (a) Since \( EF_1 < 0 \), by the strong law of large numbers we have \( C_n \rightarrow -\infty \) a.s. as \( n \rightarrow \infty \). Hence \( R_n < \infty \) for all \( n \in \mathbb{Z} \) a.s. and a direct termwise comparison yields \( D'_n \geq D_n \) for all \( n \in \mathbb{Z} \). It remains to prove that \( X_n \leq D_n \) for all \( n \in \mathbb{Z} \).

Recall that the function \( \theta \mapsto a(\theta) \) is defined in (3.3). Let \( \tau_n = \sup \{k \in \mathbb{Z}^n : X_k \leq a(\Theta_k)\} \) (with convention \( \sup \emptyset = -\infty \)) be the last time the coalescence occurred before \( n \in \mathbb{Z} \). If \( \tau_n > -\infty \), the value \( X_{1+\tau_n} \) does not depend on \( X_{\tau_n} \), and neither do the values of the chain taken at subsequent times. In particular,

\[
X_n = \psi (X_{n-1}, \Theta_{n-1}) = \psi \left( \cdots \psi \left( W^{\frac{1}{\alpha}}_{\tau_n} \left( 1 - U_{\tau_n} \right)^{\frac{1}{\alpha}} S_{\tau_n}, \Theta_{\tau_n+1} \right), \cdots, \Theta_{n-1} \right).
\]

In general, by (3.2) and (3.3), \( X_n \) can be expressed as

\[
X_n = \sum_{k \in \mathbb{Z}^n_{\tau_n+1}} \exp \left( \frac{1}{\alpha} \sum_{j \in \mathbb{Z}^n_{k+1}} \log (\Lambda_j U_j) \right) \Lambda_k^{\frac{1}{\alpha}} \left( 1 - U_k \right)^{\frac{1}{\alpha}} S_k \\
+ 1_{\{\tau_n > -\infty\}} \exp \left( \frac{1}{\alpha} \sum_{j \in \mathbb{Z}^n_{\tau_n+1}} \log (\Lambda_j U_j) \right) W^{\frac{1}{\alpha}}_{\tau_n} \left( 1 - U_{\tau_n} \right)^{\frac{1}{\alpha}} S_{\tau_n},
\]

where sums over empty sets in (3.10) are defined to be equal to zero and, if \( \tau_n = -\infty \), we define \( \mathbb{Z}^n_{\tau_n+1} = \mathbb{Z}^n \). A termwise comparison then yields

\[
\begin{align*}
X_n &\leq \sum_{k \in \mathbb{Z}^n} e^{C_{k+1} - C_n - (n-1-k)d} (1 - U_k)^{\frac{1}{\alpha}} S_k \\
&\leq e^{K_n} \sum_{k \in \mathbb{Z}^n} e^{-(n-1-k)d} (1 - U_k)^{\frac{1}{\alpha}} S_k \quad \text{for all} \quad n \in \mathbb{Z} \text{ a.s.}
\end{align*}
\]
Recall that $S_k (1 - I_k^n) \leq e^{(n-1-k)} (1 - I_k^n)$ for all $k \in \mathbb{Z}^n$. Since $I_k^n = 0$ for $k < \chi_n$, we get

$$
\sum_{k \in \mathbb{Z}^n} e^{-(n-1-k)d} (1 - U_k) \frac{\mathbf{1}}{d} S_k \leq \sum_{k \in \mathbb{Z}^n} e^{-(n-1-k)(d-\delta)} (1 - U_k) \frac{\mathbf{1}}{d} + \sum_{k \in \mathbb{Z}^n} e^{-(n-1-k)d} (1 - U_k) \frac{\mathbf{1}}{d} S_k
$$

(3.12)

The inequalities in (3.11)–(3.12) and the definition in (3.8) imply $C_{k,d} (1 - U_k) \frac{\mathbf{1}}{d} S_k$ together with the fact that $(\Sigma, \mathbb{Z})$ is Harris recurrent on a complement of a null set. Put differently, for any starting point, the process $(\Theta_n, \mathbb{Z}^{n+1})$ follows from

$$
R_n = \max \left\{ \sup_{k \in \mathbb{Z}^n} C_k - C_n, 0 \right\} = \max \left\{ R_{n-1} + F_{n-1}, 0 \right\}.
$$

By (3.9) we have $D''_{n+1} = S_{n+1} + e^{-d} D''_{n+1}$. Hence $(R_n, D''_n)$ is a function of $\Xi_{n-1} = (\Theta_{n-1}, R_{n-1}, D''_{n-1})$ (recall that $S_{n-1}$ is the first component of the random vector $\Theta_{n-1}$). Since the random elements $\Xi_{n-1}$ and $\Theta_{n}$ are independent, the process $(\Xi_{n})_{n \in \mathbb{Z}}$ is Markov.

The vector $\Xi_n = (\Theta_n, R_n, D''_n)$ is in a bijective correspondence with $(\Theta_n, R_n, D''_n)$. Since $\{\Theta_n\}_{n \in \mathbb{Z}}$ are i.i.d., the following equality in law holds

$$
(R_{n+1}, D''_{n+1}) = \left( \sup_{j \in \mathbb{Z}^1} \sum_{k \in \mathbb{Z}^1} F_{n+k}, \sum_{k \in \mathbb{Z}^1} e^{kd} S_{n+k} \right) \quad \text{d} \quad \left( \sup_{j \in \mathbb{Z}^1} \sum_{k \in \mathbb{Z}^1} F_k, \sum_{k \in \mathbb{Z}^1} e^{kd} S_k \right),
$$

implying the stationarity of $(\{\Theta_n, R_n, D''_n\}_{n \in \mathbb{Z}})$ and hence of $R$ and $\Xi$.

The process $R$ can jump to 0 in a single step and has positive jumps of size at most $1/(\alpha \rho) - d$, both with positive probability. Hence it will hit any subinterval of its state space $[0, \infty)$ from any starting point in a finite number of steps with positive probability, making it $\varphi$-irreducible [MT09, p. 82] with respect to its invariant law.

Since $\Theta_n$ is independent of $(R_n, D''_n)$, the $\varphi$-irreducibility of $\{\Xi_n\}_{n \in \mathbb{Z}}$ follows if, starting from an arbitrary point, we can prove that the process $(\{R_n, D''_n\}_{n \in \mathbb{Z}}$ hits any rectangle in the product $[0, \infty) \times (0, \infty)$ with positive probability. Since we already know that $R$ hits intervals and has (arbitrarily) small positive jumps with positive probability, the independence of $\{D''_n\}_{n \in \mathbb{Z}}$ and $R$, together with the fact that $D''_n$ has a positive density, imply the final statement of the lemma. □

Proof of Theorem 2: By Lemma 2(ii), $\Xi$ is $\pi$-irreducible, where $\pi$ denotes the invariant law of $\Xi$. Hence, by [MT09, Prop. 10.1.1], $\Xi$ is recurrent, meaning that the expected number of visits of the chain $\Xi$ to any set charged by $\pi$ is infinite for all starting points. By [MT09, Thm 9.0.1], the chain $\Xi$ is Harris recurrent on a complement of a $\pi$-null set. Put differently, for any starting point, the number of visits $\Xi$ makes to any set charged by $\pi$ is infinite almost surely.

Consider the Markov chain $\Psi = \{\Psi_n\}_{n \in \mathbb{N}}$, where $\mathbb{N} = \{0, 1, \ldots\}$ and $\Psi_n = \Xi_{-n}$. In the language of [Rev84], $\Psi$ is a chain dual to $\Xi$ with respect to $\pi$. In particular, the invariant law of $\Psi$ equals $\pi$. Since $\Xi$ is Harris recurrent on a state space with a countably generated $\sigma$-algebra, [Rev84, Thm 8.1.1] implies that there exists a modification of $\Psi$ (again denoted by $\Psi$) that is also Harris recurrent. Since $\mathbb{P} (a (\Theta_{-n}) \geq D'_{-n}) > 0$ for any $n \in \mathbb{N}$, it follows that the $\Psi$-stopping time $\sigma'$ =
inf \{n > 0 : a(\Theta_{n}) \geq D_{n}\} is finite almost surely. Moreover, by [MT09], Thm 11.1.4 we have 
\[ E[\sigma' | \Psi_0] < \infty \] almost surely.

Recall that \( \sigma = \inf \{n > 0 : a(\Theta_{n}) \geq D_{n}\} \) is the number of steps taken backwards in time in Algorithm 1. By Lemma 2(i) we have \( \sigma \leq \sigma' \). Since, by definition \( \Psi_0 = \Xi_0 \), the theorem follows. \( \square \)

4. Backward Simulation of \( \{(D_n, \Theta_n)\}_{n \in \mathbb{Z}} \)

The key step in Algorithm 1 consists of the simulation of the process \( \{(D_n, \Theta_n)\}_{n \in \mathbb{Z}} \) backwards in time until the random time \( -\sigma = \sup \{n \in \mathbb{Z}^1 : a(\Theta_n) \geq D_n\} \) (see (2.3) and (3.3) for the definitions of \( \mathbb{Z}^1 \) and \( a(\theta) \), respectively). Recall that \( \{\Theta_n\}_{n \in \mathbb{Z}} \) is an i.i.d. sequence with \( \Theta_n = (S_n, U_n, W_n, \Lambda_n) \) having independent coordinates, where \( S_n, U_n, \Lambda_n \) are distributed as in Theorem 1 and \( W_n \sim U(0,1) \).

At time \( n \in \mathbb{Z} \), the dominating process \( D \) in (3.8) depends on three components: the sequence \( (\chi_n, \{S_k\}_{k \in \mathbb{Z}^0_n}) \), the all-time maximum \( \sup_{k \in \mathbb{Z}^{n+1}} \{C_k\} \) and \( C_n \) (via the reflected process \( R, \) see (3.6) - (3.7)) and the uniform random variables \( \{U_k\}_{k \in \mathbb{Z}_0} \). The time \( \chi_n \) in (3.5) is the last time before \( n \) the random variables \( \{S_k\}_{k \in \mathbb{Z}^0_n} \) exceed a certain adaptive exponential bound. Algorithm 3 for sampling \( (\chi_n, \{S_k\}_{k \in \mathbb{Z}^0_n}) \) is given in Section 4.1 below. A sample for \( (R_n, C_n) \) requires the joint forward simulation of the dual random walk \( -C \) and its ultimate maximum. This problem was solved in [BS11]. The algorithm in [BS11], stated for completeness as Algorithm 7 of Section 4.2 below for the random walk \( C \) in (3.6), requires the simulation of the random walk under the exponential change of measure. Since the increments of \( C \) are shifted negative exponential random variables under the original measure, they remain in the same class under the exponential change of measure, making the simulation in Algorithm 7 simple. Finally, having simulated \( (R, C) \) backwards in time, we need to recover the random variables \( \Lambda_k \) and \( U_k \), conditional on the values of increments \( F_k = d + (1/\alpha) \log(U_n \Lambda_n) \) we have observed. Algorithm 8 in Section 4.3 below describes this step.

**Algorithm 2.** Backward simulation of \( (\sigma, \{(D_n, \Theta_n)\}_{n \in \mathbb{Z}_0^\sigma}) \)

1. Sample \( \chi_{-1} \) and \( \{S_k\}_{k \in \mathbb{Z}_0^\chi_{-1}} \) \( \triangleright \) Use Algorithm 3
2. Sample \( \{(R_k, C_k, \Lambda_k, U_k)\}_{k \in \mathbb{Z}_0^{\chi_{-1}}} \) for some \( N_{-1} \leq \chi_{-1} \) \( \triangleright \) Use Algorithms 7 & 8
3. Bundle up \( \{\Theta_k\}_{k \in \mathbb{Z}_0^\chi_{-1}} \) and compute \( D_{-1} \)
4. Put \( n := -1 \)
5. **while** \( D_n > a(\Theta_n) \) **do**
6. Put \( n := n - 1 \)
7. Sample \( \chi_n \) and \( \{S_k\}_{k \in \mathbb{Z}_0^\chi_{n+1}} \) conditional on \( (\chi_{n+1}, \{S_k\}_{k \in \mathbb{Z}_0^\chi_{n+1}}) \) \( \triangleright \) Use Algorithm 3
8. Sample \( \{(R_k, C_k, \Lambda_k, U_k)\}_{k \in \mathbb{Z}_0^\chi_{n+1}} \) for some \( N_n \leq \chi_n \) \( \triangleright \) Use Algorithms 7 & 8
9. Bundle up \( \{\Theta_k\}_{k \in \mathbb{Z}_0^\chi_{n+1}} \), and compute \( D_n \)
10. **end while**
11. Put \( \sigma = -n \)
12. **return** \( (\sigma, \{\Theta_k\}_{k \in \mathbb{Z}_0^\sigma}) \)

The number of steps \( N_{-1} \) (resp. \( N_n \)) in line 2 (resp. 8) of Algorithm 2 is random since Algorithm 7 which outputs the all-time maximum of the random walk, may need more values of the random walk than required to recover the previous value of the dominating process \( D_{-1} \) (resp. \( D_n \)).

\[ \text{In the notation of Section 4.2 below, the integers } N_n \text{ take the form } \Delta(\tau_m). \]
time of Algorithm 3 is random but has moments of all orders (see Lemma 3 in Section 4.1 below). Algorithm 3 executes a loop of length equal to the number of steps in the random walk $C$ the algorithm is applied two, with each step sampling one Poisson and one Beta random variables (see Section 4.2 below). Hence both Algorithms 3 and 3 are fast. Algorithm 3 of [BS11] (see Section 4.2 below) runs sequentially rejection sampling Algorithms 3, 3 and 3. Each of these algorithms has a finite expected running time, which is easy to quantify in terms of the increments of the walk $C$.

4.1. Simulation of $\left(\chi_n, \{S_k\}_{k \in \mathbb{Z}^d_n}\right)$. Let $N = \{0, 1, \ldots\}$ and consider independent Bernoulli random variables $\{J_n\}_{n \in \mathbb{N}}$ with parameters $p_n = P(J_n = 0), n \in \mathbb{N},$ satisfying $\sum_{n \in \mathbb{N}} (1 - p_n) < \infty.$ By the Borel-Cantelli Lemma the random time $\tau = \sup \{n \geq 0 : J_n = 1\}$ (with convention $\sup \emptyset = -\infty$) satisfies $\tau \in \mathbb{N}$ a.s. Clearly, $J_n = 0$ for all $n \geq \tau$, and $\{\tau < n\} = \bigcap_{k=n}^\infty \{J_k = 0\}$ implies $P(\tau < n) = \prod_{k=n}^\infty p_k$. If there exists $n^* \in \mathbb{N}$ such that for all $n \geq n^*$ we have a positive computable lower bound $q_n \leq \prod_{k=n}^\infty p_k$, then we can simulate $(\tau, \{J_k\}_{k \in \{0, \ldots, \tau\}})$ as follows.

Define the auxiliary function $F : (0, 1) \times (0, 1) \rightarrow \{0, 1\} \times (0, 1)$ by the formula

$$F(u, p) = \begin{cases} (0, \frac{u}{p}) & \text{if } u \leq p, \\ (1, \frac{u-p}{1-p}) & \text{if } u > p. \end{cases}$$

The following observation is simple but crucial: for any $p \in (0, 1)$ and $U \sim U(0, 1)$, the components of the vector $(J, V) = F(U, p)$ are independent, $J$ is Bernoulli with $P(J = 0) = p$ and $V \sim U(0, 1)$.

Sample $\{J_n\}_{n \in \{0, \ldots, n^*-1\}}$ and an independent $U(n^*) \sim U(0, 1)$. Let $(J_{n^*}, U(n^*+1)) = F(U(n^*), p_{n^*})$. Hence $J_{n^*}$ has the correct distribution and is independent of $U(n^*+1) \sim U(0, 1)$. Thus, $J_{n^*}$ is independent of $F(U(n^*+1), p_{n^*+1}) = (J_{n^*+1}, U(n^*+2))$. Define recursively $(J_n, U(n+1)) = F(U(n), p_n)$ for $n \geq n^* + 2$ and note that the sequence $\{J_n\}_{n \in \mathbb{N}}$ of Bernoulli random variables is i.i.d. Moreover, the sequence $\{U(n)\}_{n \geq n^*}$ detects the value of $\tau$ since $\{U(n) \leq q_n\} \subseteq \{U(n) \leq \prod_{k=n}^\infty p_k\} = \{\tau < n\}$.

Algorithm 3. Simulation of $(\tau, \{J_k\}_{k \in \{0, \ldots, \tau\}})$

1: Sample $J_0, \ldots, J_{n^*-1}$ and put $n := n^*-1$
2: Sample $U \sim U(0, 1)$
3: loop
4: \hspace{1em} if $U > p_n$ then
5: \hspace{2em} Put $J_n := 1$ and update $U := U - p_n$
6: \hspace{1em} else if $U \leq q_n$ then
7: \hspace{2em} Compute $\tau$ from $J_0, \ldots, J_{n-1}$ and exit loop
8: \hspace{1em} else
9: \hspace{2em} Put $J_n := 0$ and update $U := U - p_n$
10: \hspace{1em} end if
11: end loop
12: return $(\tau, \{J_k\}_{k \in \{0, \ldots, \tau\}}$)

Algorithm 3 samples a single uniform random variable and performs a binary search. Its running time $\zeta = \inf \{n \geq n^* : U(n) \leq q_n\}$ (with convention $\inf \emptyset = \infty$) has the following properties.
Lemma 3.  (a) If \( \lim_{n \to \infty} q_n = 1 \) then \( \gamma < \infty \) a.s.
(b) If \( \sum_{n=n^*}^{\infty} (1 - q_n) < \infty \) then \( \mathbb{E} \xi < \infty \).
(c) If \( \sum_{n=n^*}^{\infty} (1 - q_n) e^{tn} < \infty \) for some \( t > 0 \), then \( \mathbb{E} e^{tc} < \infty \).

Proof. For all \( n \geq n^* \) we have \( \{ \varsigma \leq n \} \supseteq \{ U^{(n)} \leq q_n \} \), then \( \mathbb{P}(\varsigma > n) \leq \mathbb{P}(U^{(n)} > q_n) = 1 - q_n \).
Hence \( \mathbb{P}(\varsigma = \infty) = \lim_{n \to \infty} \mathbb{P}(\varsigma > n) \leq \lim_{n \to \infty} (1 - q_n) = 0 \) and (a) follows. Similarly, \( \mathbb{E} \xi = \sum_{n=0}^{\infty} \mathbb{P}(\varsigma > n) \leq n^* + \sum_{n=n^*}^{\infty} (1 - q_n) < \infty \) and (b) follows. Note that \( (e^t - 1) \sum_{m=0}^{n-1} e^{tm} = e^{tn} - 1 \).
Exchanging the order of summation in the third equality of the following estimate implies (c):

\[
\mathbb{E} e^{tc} = \sum_{n=0}^{\infty} \mathbb{P}(\varsigma = n) e^{tn} = \sum_{n=0}^{\infty} \mathbb{P}(\varsigma = n) \left( 1 + (e^t - 1) \sum_{m=0}^{n-1} e^{tm} \right)
\]
\[
= 1 + (e^t - 1) \sum_{m=0}^{\infty} e^{tm} \mathbb{P}(\varsigma > m) \leq e^{tn^*} + (e^t - 1) \sum_{n=n^*}^{\infty} (1 - q_n) e^{tn} < \infty.
\]

In Algorithm 2 we are required to sample \( \left( \chi_0, \{ S_k \}_{k \in \mathbb{Z}_{0}^{\chi_0}} \right) \), and then, iteratively for \( n \in \mathbb{Z}^0, \chi_n \) and the remaining \( \{ S_k \}_{k \in \mathbb{Z}_{\chi_n}^{n+1}} \), given the known values \( \left( \chi_{n+1}, \{ S_k \}_{k \in \mathbb{Z}_{\chi_n}^{n+1}} \right) \). To apply Algorithm 3 we need a computable lower bound on the product of probabilities \( p(m) = \mathbb{P}(S_1 \leq e^{\delta m}), m \in \mathbb{N} \). Recall the exponential lower bound on \( p(m) \) in (3.4) and let \( m^* = \max \left\{ 0, \left\lfloor \frac{1}{\delta \gamma} \log \mathbb{E} S_1^\gamma \right\rfloor + 1 \right\} \) (here \( \lfloor x \rfloor = \sup \{ n \in \mathbb{Z} : n \leq x \} \) for any \( x \in \mathbb{R} \)). Note that for any \( m \geq m^* \) we have \( e^{-\delta m} \mathbb{E} S_1^\gamma < 1 \) and may hence define \( \overline{p}(m) = \exp \left( -\frac{e^{-\delta \gamma} \mathbb{E} S_1^\gamma}{1 - e^{-\delta \gamma} \mathbb{E} S_1^\gamma} \right) \in (0, 1) \). The inequality in (3.4) implies

\[
\prod_{j=m}^{\infty} p(j) \geq \prod_{j=m}^{\infty} \left( 1 - e^{-\delta \gamma} \mathbb{E} S_1^\gamma \right) = \exp \left( \sum_{j=m}^{\infty} \log \left( 1 - e^{-\delta \gamma} \mathbb{E} S_1^\gamma \right) \right)
\]
\[
= \exp \left( -\sum_{j=m}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} e^{-\delta \gamma j k} \left( \mathbb{E} S_1^\gamma \right)^k \right) \geq \exp \left( -\sum_{k=1}^{\infty} \frac{e^{-\delta \gamma m k} \left( \mathbb{E} S_1^\gamma \right)^k}{1 - e^{-\delta \gamma k}} \right) \geq \overline{p}(m).
\]
Since for any \( k \in \mathbb{Z}^0_{\chi_0} \) we have \( \mathbb{P}(I_k^0 = 0) = \mathbb{P}(S_k \leq e^{-(k+1)\delta}) = p(-(k + 1)) \), Algorithm 4 can be applied (with \( n^* = m^* \)) to sample the sequence \( \{ I_k^0 \}_{k \in \mathbb{Z}_{\chi_0}^0} \). Moreover, for \( m \in \mathbb{N} \) we get

\[
\overline{p}(m^* + m) \geq \exp \left( -re^{-\delta m} \right) \geq 1 - re^{-\delta m}, \quad \text{where} \quad r = \frac{e^{-\delta m} \mathbb{E} S_1^\gamma}{(1 - e^{-\delta}) (1 - e^{-\delta m} \mathbb{E} S_1^\gamma)} > 0.
\]
Hence, for any \( t \in (0, \delta \gamma) \), Lemma 3(c) implies that the running time \( \varsigma \) satisfies \( \mathbb{E}[e^{\varsigma t}] < \infty \) and therefore possesses moments of all orders. Having obtained \( \left( \chi_0, \{ I_k^0 \}_{k \in \mathbb{Z}_{\chi_0}^0} \right) \), for \( k \in \mathbb{Z}_{\chi_0}^0 \), we sample \( S_k \) as \( S^+(\alpha, \rho) \) conditional on \( S_k \leq e^{-(k + 1)\delta} \) (if \( I_k^0 = 0 \)) or \( S_k > e^{-(k + 1)\delta} \) (if \( I_k^0 = 1 \)), yielding a sample of \( \left( \chi_n, \{ S_k \}_{k \in \mathbb{Z}_{\chi_n}^{n+1}} \right) \).

Assume now that we have already sampled \( \left( \chi_{n+1}, \{ S_k \}_{k \in \mathbb{Z}_{\chi_{n+1}}^{n+1}} \right) \). The adaptive exponential bounds in the indicators \( I_k^{n+1} \) and \( I_k^n \) are different (see Figure 4.1) and the relevant probabilities take the form

\[
p'(m) = \mathbb{P} \left( S_1 \leq e^{\delta m} \bigg| S_1 \leq e^{\delta (m+1)} \right), \quad m \in \mathbb{N}.
\]
Since $\{S_1 \leq e^{\delta m}\} \subset \{S_1 \leq e^{\delta (m+1)}\}$, the inequality $p'(m) \geq p(m)$ holds for any $m \in \mathbb{N}$. Thus
\[
\prod_{j=m}^{\infty} p'(j) \geq \prod_{j=m}^{\infty} p(j) \geq \bar{p}(m)
\]
and Algorithm 3 can be applied with $n^* = \max\{m^*, n - \chi_{n+1}\}$. The same argument as above shows that the running time $\zeta$ has moments of all orders.

\[\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure41.png}
\caption{The adaptive exponential bounds $k \mapsto e^\delta(n-k-1)$ for $n \in \{0, -1, -2\}$ and the corresponding stable random variables conditioned to be positive.}
\end{figure}\]

4.2. Simulation of the Random Walk and Its Reflected Process from [BS11]. In this section we present an overview of the algorithm in [BS11] for the joint simulation of $(C, R)$ defined in (3.6)-(3.7). We refer to [BS11] and [EG00] for the proofs (the latter paper contains the simulation algorithm for the ultimate maximum of a random walk with negative drift and provides a basis for the simulation algorithm in [BS11]).

We describe the algorithm for $d' = \alpha p d$ (see (3.4)), $C' = \alpha p C$ and $R' = \alpha p R$. For ease of exposition we drop the apostrophe, overriding the notation from the rest of the paper. Hence in the present section it is assumed that $d \in (0, 1)$, $\{C_n\}_{n \in \mathbb{Z}}$ is a random walk satisfying $C_{n+1} = C_n - F_n$ for all $n \in \mathbb{Z}$ and $C_0 = 0$, where the i.i.d. increments $\{F_n\}_{n \in \mathbb{Z}}$ are distributed according to $d - F_n \sim \text{Exp}\,(1)$. We stress that this notation only applies in the present section. Let $\eta = \eta(d)$ be the unique positive root of $\psi_d(\eta) = 0$, where $\psi_d(t) = \log(\mathbb{E}^e^{t F_0}) = dt - \log(1 + t)$. Note that $\psi_d'(\eta) = d - \frac{1}{1+\eta} > 0$ and $\eta = 1 - W_{-1}(\frac{-de^{-d}}{d})$, where $W_{-1}$ is the secondary branch of the Lambert W function. Since $\mathbb{E}[\exp(\eta F_n)] = 1$ for all $n \in \mathbb{Z}$, the process $\{\exp(\eta C_n)\}_{n \in \mathbb{Z}}$ is a positive backward martingale started at one, thus inducing a probability measure $\mathbb{P}^n$ on $\sigma$-algebras $\sigma(C_k; k \in \mathbb{Z}_n)$, $n \in \mathbb{Z}$, by the formula $\mathbb{P}^n(A) = \mathbb{E}[1_A e^{\eta C_n}]$ where $A \in \sigma(C_k; k \in \mathbb{Z}_n)$. Under $\mathbb{P}^n$, the process $C$ remains a random walk with i.i.d. increments satisfying $\frac{1}{1+\eta}(d - F_n) \sim \text{Exp}(1)$. Hence $\mathbb{E}^n[C_{-1}] = \psi_d'(\eta) > 0$, implying $\lim_{n \to -\infty} C_n = \infty \mathbb{P}^n$-a.s. by the strong law of large numbers.

For any $k \in \mathbb{Z}$ define (with convention $\sup\emptyset = -\infty$)
\[
T^k_x = \begin{cases} 
\sup \{n \in \mathbb{Z}^k : C_n - C_k > x\} & \text{if } x > 0, \\
\sup \{n \in \mathbb{Z}^k : C_n - C_k < x\} & \text{if } x < 0.
\end{cases}
\]
For ease of notation we let \( T_x = T^0_x \). Let \( E \) be an independent exponential random variable with mean one. Then, for \( x > 0 \), we have \( \mathbb{P}(R_0 > x) = \mathbb{P}^\eta(L_{\eta-1}E > x) \), where \( L_x = \inf \{ y \geq 0 : C_T > x \} \) is the right inverse of \( x \mapsto C_T \), see e.g. [EG00]. Hence for \( x \in (0, x') \), where \( x' \leq \infty \), sampling \( 1_{\{R_0 > x\}} = 1_{\{T_x > -\infty\}} \) conditional on \( 1_{\{R_0 \leq x\}} = 1_{\{T_s = -\infty\}} \), in finite time amounts to sampling \( E \) and \( C_{-1}, \ldots, C_{T_{\eta-1}E} \) under \( \mathbb{P}^\eta \), see Algorithm 4 below.

**Algorithm 4. Simulation of \( 1_{\{R_0 > x\}} \) conditional on \( \{R_0 \leq x'\} \)**

**Require:** \( \infty \geq x' > x > 0 \)

1: loop
2: Sample \( E \sim \text{Exp}(1) \)
3: Sample \( C_0 = 0, C_{-1}, \ldots, C_{T_{\eta-1}E} \) under \( \mathbb{P}^\eta \)
4: Compute \( L_{\eta-1}E \)
5: if \( L_{\eta-1}E \leq x' \) then \( \triangleright \) Accept sample  
6: \( \text{return} \, 1_{\{L_{\eta-1}E > x\}} \)
7: end if
8: end loop

This algorithm repeats independent experiments with probability of success \( \mathbb{P}^\eta(L_{\eta-1}E \leq x') > 0 \). The expected running time of each iteration in the loop is bounded above by \( (\eta^{-1} + d)/\psi_d(\eta) \), see [EG00, Eq. (2.3)]. Hence the expected running time of Algorithm 3 is finite.

In Algorithm 7 below we need to sample the path of the random walk \( \{C_k\}_{k \in \mathbb{Z}_{T_x}^1} \) conditioned on the event \( \{R_0 \in (x, x')\} \), where \( 0 < x < x' \leq \infty \). By a rejection sampling method under \( \mathbb{P}^\eta \) and Algorithm 4 (see [BS11, Lemma 3]), this can be achieved as follows.

**Algorithm 5. Simulation of \( C_0, \ldots, C_{T_x} \) conditional on \( \{T_x > -\infty = T_{x'}\} \)**

**Require:** \( \infty \geq x' > x > 0 \)

1: loop
2: Sample \( C_0 = 0, C_{-1}, \ldots, C_{T_x} \) under \( \mathbb{P}^\eta \)
3: Given \( C_{T_x} \), sample independent \( 1_{\{R_0 \leq x'-C_{T_x}\}} \) and \( U \sim U(0, 1) \) \( \triangleright \) Use Algorithm 4  
4: if \( U \leq \exp(-\eta C_{T_x}) \) and \( 1_{\{R_0 \leq x'-C_{T_x}\}} = 1 \) then \( \triangleright \) Accept sample  
5: \( \text{return} \, \{C_n\}_{n \in \mathbb{Z}_{T_x}^1} \)
6: end if
7: end loop

Since \( L_x \leq x \), we have \( \mathbb{P}(R_0 \leq z) \geq \mathbb{P}(\eta^{-1}E \leq z) = 1 - \exp(-z\eta) \) for all \( z \geq 0 \). Since the overshoot \( C_{T_x} - x \) is in the interval \((0, d)\), the expected running time of Algorithm 5 (i.e. one over the acceptance probability) is smaller than \( \exp(\eta(x + d))/(1 - \exp(-\eta(x' - x - d))) \) if \( x' > x + d \).

In Algorithm 7 we also need to simulate the path of the walk reaching a negative level \(-x\), while staying below a given positive level forever. Algorithm 4 achieves this (see [BS11, Lemma 3]). Its expected running time is bounded above by \( 1/((1 - \exp(-\eta(x' + x)))\mathbb{P}(T_{T_x} < T_{x'})) < \infty \).

**Algorithm 6. Simulation of \( C_0, \ldots, C_{T_{-x}} \) conditional on \( \{T_{x'} = -\infty\} \)**

**Require:** \( x \in (0, \infty) \) & \( x' \in (0, \infty] \)
We now give a brief overview of the algorithm in [BS11] for the simulation of \( \{(C_n, R_n)\}_{n \in \mathbb{Z}} \). Pick \( \kappa > \max\{\log(2)/(3\eta), 1\} \) (see assumption in [BS11, Prop. 3]). [BS11] constructs sequences \( \Delta = \{\Delta(k)\}_{k \geq 0} \) and \( \tau = \{\tau_k\}_{k \geq 0} \) of decreasing negative and increasing positive times, respectively:

1. at the start of each iteration of the algorithm we are given
   \[
   \left( \{\tau_k\}_{k \in \{0, \ldots, m\}}, \{\Delta(k)\}_{k \in \{0, \ldots, m\}}, \{C_n\}_{n \in \mathbb{Z}^1_{\Delta(m)}}, \{R_n\}_{n \in \mathbb{Z}^1_{\Delta(m)}} \right),
   \]
2. at each iteration we sample
   \[
   \left( \tau_{m+1}, \{\Delta(k)\}_{k \in \{m+1, \ldots, m\}}, \{C_n\}_{n \in \mathbb{Z}^1_{\Delta(m+1)}}, \{R_n\}_{n \in \mathbb{Z}^1_{\Delta(m+1)}} \right).
   \]

Note that at the \( m \)-th iteration we have \( \Delta(\tau_m) - \Delta(\tau_{m-1}) \) more values of the walk than of the reflected process. More precisely, the algorithm starts by setting \( \Delta_0 = 0 \) and repeats the following steps: given \( \{\tau_k\}_{k \in \{0, \ldots, m\}} \) and \( \{\Delta_k\}_{k \in \{0, \ldots, m\}} \), then put \( \Delta(\tau_m + 1) = \max \{\Delta(k) \mid k \in \{0, \ldots, m\}\} \). Next, if \( \Delta(k) \) is the last known value of \( \Delta \) and if \( R_{\Delta(k)} > \kappa \), then put \( \Delta(k + 1) = T_{\kappa} \Delta(k) \) and \( \Delta(k + 2) = T_{-2\kappa} \Delta(k) \). If instead \( R_{\Delta(k)} \leq \kappa \) then put \( \tau_{m+1} = k \). Repeat the previous two steps until we can compute \( \tau_{m+1} \), that is, until \( R_{\Delta(\tau_m)} \leq \kappa \). After computing \( \tau_{m+1} \) go back and repeat. By construction (see Proposition 3 in [BS11]) we have

\[
\sup_{n \in \mathbb{Z}^{\Delta(\tau_m) + 1}} \{C_n\} \leq C_{\Delta(\tau_m) - \kappa}, \quad \text{implying} \quad R_n = \max_{k \in \mathbb{Z}^{\Delta(\tau_m) + 1}} \{C_k\} - C_n, \quad n \in \mathbb{Z}^{\Delta(\tau_m) - \kappa}.
\]

Hence, we may compute \( R_n, n \in \mathbb{Z}^1_{\Delta(\tau_m)} \), from the simulated values \( \tau_m, \Delta(\tau_m + 1), \Delta(\tau_m), \{C_n\}_{n \in \mathbb{Z}^1_{\Delta(\tau_m)}} \).

**Algorithm 7. Simulation of the random walk and its reflected process**

**Require:** \( \kappa \geq \max \{\frac{\log(2)}{3\eta}, 1\} \), \( d \in (0, 1) \), and \( m \geq 1 \)

1. Put \( x := \infty \) and \( t := C_0 := \Delta(0) := \tau_0 := 0 \) \( \triangleright \) \( x \) is an upper bound for the tail of the r.w.
2. for \( k \in \{1, \ldots, m\} \) do
3. 3. Put \( t := \tau_{k-1} \)
4. 4. loop
5. 5. Sample \( C_{\Delta(t) - 1}, \ldots, C_{\Delta(t) + 1} \) conditioned on \( \{R_{\Delta(t)} < x\} \) \( \triangleright \) Use Algorithm 6
6. 6. Put \( t := t + 1 \)
7. 7. Sample \( 1 \{R_{\Delta(t)} > \kappa\} \) given \( R_{\Delta(t)} < x - C_{\Delta(t)} \) \( \triangleright \) Use Algorithm 4
8. 8. if \( 1 \{R_{\Delta(t)} > \kappa\} = 1 \) then
9. 9. Sample \( C_{\Delta(t) - 1}, \ldots, C_{\Delta(t) + 1} \) from \( \mathbb{P}^\eta \) \( \triangleright \) Use Algorithm 5
10. 10. Put \( t := t + 1 \)
11. 11. else
12. Put $\tau_k := t$ and exit loop
13. end if
14. end loop
15. end for
16. Compute $\{R_n\}_{n \in \mathbb{Z}^{1}_{\tau_m-1}}$
17. return $\left(\{\tau_k\}_{k \in \{0,\ldots,\tau_m\}}, \{\Delta (k)\}_{k \in \{0,\ldots,\tau_m\}}, \{C_n\}_{n \in \mathbb{Z}^{1}_{\tau_m}}, \{R_n\}_{n \in \mathbb{Z}^{1}_{\tau_m-1}}\right)$

4.3. Sampling $(U_n, \Lambda_n)$ given $F_n$. Algorithm 2 requires the knowledge of $\{(U_n, \Lambda_n)\}_{n \in \mathbb{Z}^{0}}$, given the increments $\{F_n\}_{n \in \mathbb{Z}^{0}}$ of the random walk $C$. Since $\log(U_n\Lambda_n) = \alpha(F_n - d)$ for all $n \in \mathbb{Z}$, by independence, we may restrict attention to $n = 1$. It follows from (2.6) above that $\Lambda_1 \sim \prod_{i=2}^{T} U_i$ for an independent geometric random variable $T$ with parameter $\rho$ on the positive integers (if $T = 1$ the right-hand side is defined to equal one). Hence, by independence, we have $U_1\Lambda_1 \sim \prod_{i=1}^{T} U_i$.

By (2.7), $-\log \Lambda_1$ conditioned on being positive is exponential with mean $1/\rho$. Hence for any $n \geq 1$ and $y > 0$ we obtain

$$\mathbb{P} \left[ T = n \left| -\sum_{i=1}^{T} \log(U_i) = y \right. \right] = \frac{\rho (1-\rho)^{n-1} y^{n-1} e^{-y}}{\rho^{n-1} y^{n-1} e^{-y}} = \frac{((1-\rho) y)^{n-1} e^{-(1-\rho)y}}{(n-1)!}.$$ 

Thus the conditional law of $T - 1$ given $\sum_{i=1}^{T} \log(U_i) = -y$ is Poisson with mean $(1-\rho) y$. If $T = 1$, then $\log(U_1) = y$ and $\Lambda_1 = 1$. If $T > 1$, then for $x \in (0, y)$ we get

$$\mathbb{P} \left[ \log(U_1) \in dx \bigg| T = n, -\sum_{i=1}^{T} \log(U_i) = y \right] = \frac{e^{-x} (y-x)^{n-2} e^{-(y-x)}}{(n-2)!} \frac{g^{n-1} e^{-y}}{(n-1)!} dx = (n-1) \frac{(y-x)^{n-2} g^{n-1}}{y^{n-1}} dx.$$ 

Hence, conditional on $T = n$ and $\log(\prod_{i=1}^{T} U_i) = -y$, the law of $\frac{1}{y} \log(U_1)$ is Beta$(1, n-1)$ (understood as the Dirac measure $\delta_1$ when $n = 1$). Finally we set $\Lambda_1 = \exp(\alpha(F_1 - d))/U_1$.

**Algorithm 8.** Simulation of $\{(U_k, \Lambda_k)\}_{k \in \mathbb{Z}_n^m}$ given $\{F_k\}_{k \in \mathbb{Z}_n^m}$

**Require:** $m, n \in \mathbb{Z}$ and $m < n$

1: for $k \in \mathbb{Z}_n^m$ do
2: Sample $T - 1 \sim \text{Poisson}(-\alpha(F_k - d) (1-\rho))$
3: Sample $L \sim \text{Beta}(1, T - 1)$
4: Let $U_k := \exp(L \alpha(F_k - d))$ and $\Lambda_k := \exp((1-L) \alpha(F_k - d))$
5: end for
6: return $\{(U_k, \Lambda_k)\}_{k \in \mathbb{Z}_n^m}$

**APPENDIX A. SAMPLING THE MARGINALS OF STABLE PROCESSES**

A Lévy process $Y = (Y_t)_{t \in [0, \infty)}$ in $\mathbb{R}$ is strictly stable with index $\alpha \in (0, 2]$ if for any constant $c \geq 0$ the processes $(Y_{ct})_{t \in [0, \infty)}$ and $(c^{1/\alpha}Y_t)_{t \in [0, \infty)}$ have the same law. For brevity, we call $Y$ a stable process. Sampling the increments of $Y$ hence reduces to sampling $Y_1$. Using Zolotarev’s (C) form [UZ92], up to a scaling constant the law of $Y_1$ is parametrised by $(\alpha, \beta) \in (0, 2] \times [-1, 1]$ via

$$\mathbb{E}e^{itY_1} = e^{-|t|^\alpha \theta_{\alpha \theta \beta}(t)}, \text{ where } t \in \mathbb{R}, \theta = \beta \left(1_{\alpha \leq 1} + \frac{\alpha - 2}{\alpha}1_{\alpha > 1}\right),$$

(A.1)
If $\alpha > S$, Sec 4.4]). For alternative ways of sampling from the laws $\rho$, where $E (\rho)$ implies that the stable law is uniquely determined by $\alpha$ and its positivity parameter $\rho = P (Y_1 > 0)$. If $\alpha > 1$, the pair $(\alpha, \rho) \in (0, 2] \times [0, 1]$ must satisfy $\rho \in \left[1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right]$, since $\theta \in \left[1 - \frac{2}{\alpha}, \frac{2}{\alpha} - 1\right]$. Let $S (\alpha, \rho)$ and $S^+ (\alpha, \rho)$ denote the laws of $Y_1$ and $Y_1$ conditioned on being positive, respectively. As $\rho, \alpha \rho \in [0, 1]$ and the Mellin transform determines the law uniquely, (A.2) implies that $(Z' / Z'')^\rho$ follows $S^+ (\alpha, \rho)$, where $Z' \sim S (\alpha \rho, 1)$ and $Z'' \sim S (\rho, 1)$ are independent. Since $P'B + P'' (1 - B)$ follows $S (\alpha, \rho)$, where $P' \sim S^+ (\alpha, \rho)$, $P' \sim S^+ (\alpha, 1 - \rho)$ and $B \sim Ber (\rho)$ are independent, we need only be able to simulate a positive stable random variable with law $S (\alpha, 1)$ for any $\alpha \in (0, 1]$. If $\alpha = 1$, then by (A.1), $Y_1$ is a constant equal to one. If $\alpha \in (0, 1)$, Kanter’s factorisation states

$$
\left(\sin (\alpha \pi U)^\alpha \sin ((1 - \alpha) \pi U)^{1 - \alpha} / \sin (\pi U)\right)^{\frac{1}{\alpha}} E^{1 - \frac{1}{\alpha}} \sim S (\alpha, 1),
$$

where $E$ is exponential with mean one, independent of $U$, which is uniform on $(0, 1)$ (see [UZ99, Sec 4.4]). For alternative ways of sampling from the laws $S (\alpha, \rho)$ and $S^+ (\alpha, \rho)$ we refer to [DJ14].

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