Stable intersection of middle-$\alpha$ Cantor sets

M. Pourbarat*

Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, G.C., Evin, Tehran 19839, Iran

Abstract

In the present paper, we introduce a pair of middle Cantor sets namely $(C_{\alpha}, C_{\beta})$ having stable intersection, while the product of their thickness is smaller than one. Furthermore, the arithmetic difference $C_{\alpha} - \lambda C_{\beta}$ contains at least one interval for each nonzero number $\lambda$.

Keywords: Middle-$\alpha$ Cantor sets, stable intersection, thickness, Palis conjecture, arithmetic differences.

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1 Introduction

Regular Cantor sets appear in dynamical systems when hyperbolic sets intersect stable and unstable manifolds of its points. The study of metrical and topological properties of intersection of regular Cantor sets emerges naturally in the theory of homoclinic bifurcations. Meanwhile, stable intersections between Cantor sets which come from stable

*E-mail: m-pourbarat@sbu.ac.ir
and unstable foliations of a horseshoe, provide examples of open sets of nonhyperbolic
diffeomorphisms after unfolding a homoclinic tangency. In these cases, the open set of
diffeomorphisms presenting persistent tangencies between the stable and unstable folia-
tions of the horseshoe stably has positive lower density at the initial parameter of the
bifurcation, in parametrized families (see [M1] and [PT]).

Regular Cantor sets appear in number theory too, related to diophantine approxima-
tions. Many Cantor sets given by combinatorial conditions on the continued fraction of
real numbers emerge in this situation. In the study of the classical Markov and Lagrange
spectra related to them, we usually deal with the arithmetic difference of two regular
Cantor sets (see [M2]). On the other hand, the intersection of regular Cantor sets can
be interpreted by their arithmetic difference as

\[ K - K' := \{ x - y \mid x \in K, y \in K' \} = \{ t \in \mathbb{R} \mid K \cap (K' + t) \neq \emptyset \}. \]

The arithmetic difference of regular Cantor sets can be so complicate even for the
simplest possible family of pairs of Cantor sets (see [MO] and [P3]). For instance, con-
jecture of Palis remains still open in the affine case (see [P1], [P2] and [PT]). Also, there
exist regular Cantor sets \( K \) and \( K' \), such that \( K - K' \) has positive Lebesgue measure,
but does not contain any interval (see [S]). Having stable intersection of regular Cantor
sets, clearly implies the existence of an interval contained in their arithmetic difference.

Before stating our main result, we pose some notations. A Cantor set \( K \) is regular
or dynamically defined if:

i) there are disjoint compact intervals \( K_1, K_2, \ldots, K_r \) such that \( K \subset K_1 \cup \cdots \cup K_r \)
and the boundary of each \( K_i \) is contained in \( K \),
ii) there is a $C^{1+\epsilon}$ expanding map $\psi$ defined in a neighborhood of set $K_1 \cup K_2 \cup \cdots \cup K_r$ such that $\psi(K_i)$ is the convex hull of a finite union of some intervals $K_j$ satisfying:

ii.1 For each $i$, $1 \leq i \leq r$ and $n$ sufficiently big, $\psi^n(K \cap K_i) = K$,

ii.2 $K = \bigcap_{n=0}^{\infty} \psi^{-n}(K_1 \cup K_2 \cup \cdots \cup K_r)$.

The set $\{K_1, K_2, \cdots, K_r\}$ is, by definition, a Markov partition for $K$, and the set $D := \bigcup_{i=1}^{r} K_i$ is the Markov domain of $K$.

A regular Cantor set is affine if $D\psi$ be constant on every interval $K_i$. The simplest kind of affine Cantor sets are middle-$\alpha$ Cantor sets that generalizes in the most natural way, the usual ternary Cantor set which corresponds to $p=3$ in below definition.

**Definition.** Let $p > 2$ and $\alpha := 1 - \frac{2}{p}$. Then middle-$\alpha$ Cantor set can be written as

$$C_\alpha := \left\{ x \in \mathbb{R} \mid x = (1 - \frac{1}{p}) \sum_{i=0}^{\infty} \frac{a_i}{p^i}, \ a_i \in \{0, 1\} \right\},$$

that is a regular Cantor set with the Markov partition $\{K_1, K_2\}$ and expanding map

$$\phi(x) := \begin{cases} px & x \in K_1 := [0, \frac{1}{p}] \\ px - p + 1 & x \in K_2 := [1 - \frac{1}{p}, 1] \end{cases}.$$  

We say that the Cantor set $K$ is close on the topology $C^{1+\epsilon}$ to a Cantor set $\tilde{K}$ with the Markov partition $\{\tilde{K}_1, \tilde{K}_2, \cdots, \tilde{K}_s\}$ defined by expanding map $\tilde{\psi}$ if and only if $r = s$, the extremes of $K_i$ are near the corresponding extremes of $\tilde{K}_i$, $i = 1, 2, ..., r$ and supposing $\psi \in C^{1+\epsilon}$ with Holder constant $C$, we must have $\tilde{\psi} \in C^{1+\epsilon}$ with Holder constant $\tilde{C}$ such that $(\tilde{C}, \tilde{\epsilon})$ is near $(C, \epsilon)$ and $\tilde{\psi}$ is close to $\psi$ in the $C^1$ topology.

**Definition.** Regular Cantor sets $K$ and $K'$ have *stable intersection* if for any pair of regular Cantor sets $(\tilde{K}, \tilde{K}')$ near $(K, K')$, we have $\tilde{K} \cap \tilde{K}' \neq \emptyset$.  

3
Besides the Hausdorff dimension, there is another fractal invariant namely thickness introduced by Newhouse, that plays a relevant role in determining stable intersection of regular Cantor sets (see [N]). Such thickness condition was generalized by Moreira in [M1] as follows:

**Definition.** Take $U$ be a bounded gap of Cantor set $K$ and $L_U, R_U$ be the intervals at its left and its right, respectively, that separate it from the closest larger gaps.

$$K: \quad (\quad L_U \quad U \quad R_U \quad )$$

Let $\tau_R(U) := \frac{|R_U|}{|U|}$ and $\tau_L(U) := \frac{|L_U|}{|U|}$. The right thickness $\tau_R$ and left thickness $\tau_L$ are

$$\tau_R(K) := \inf_U \tau_R(U) \quad , \quad \tau_L(K) := \inf_U \tau_L(U)$$

and the Newhouse thickness $\tau(K)$ is the minimum of lateral thicknesses $\tau_R(K)$ and $\tau_L(K)$.

One of the special characteristics of above definition is the existence of an open and dense subset in $C^{1+\epsilon}$ topology of regular Cantor sets whose elements have these lateral thicknesses varying continuously. But, Newhouse thickness is continuous in $C^{1+\epsilon}$ topology of all regular Cantor sets.

Now we state some fundamental results on stable intersection of regular Cantor sets:

I) If $HD(K) + HD(K') < 1$, therefore no translations of $K$ and $K'$ have stable intersection (see [PT]),

II) If $\tau(K) \cdot \tau(K') > 1$ and $K$ is linked to $K'$, then $(K, K')$ have stable intersection (see [N] and [PT]),

III) If $HD(K) + HD(K') > 1$ and $K$ is linked to $K'$, then $(K, K')$ have generically
stable intersection (see [MY]).

Note that, \( K \) is linked to \( K' \) means closure of each gap of \( K \) does not contain \( K' \).

Moreira introduced affine Cantor sets with more than 2 expanding maps and small lateral thicknesses that having stable intersection. He showed that lateral thicknesses change continuously at affine Cantor sets defined by two expanding maps. Moreover, for affine Cantor sets defined by two affine expanding maps:

- If \( \tau_R(K) \cdot \tau_L(K') > 1, \tau_L(K) \cdot \tau_R(K') > 1 \) and \( K \) is linked to \( K' \), then \( (K, K') \) have stable intersection (see [M1]).

In [HMP], we constructed a pair of affine Cantor sets with the simplest possible combinatorics which have stable intersection, while \( \tau_R(K) \cdot \tau_L(K') < 1 \). These considerations motivate the following problem, that will be discussed in this work:

**Problem 1.** Does there exist a non empty open set in the space of affine Cantor sets defined by two expanding maps contained in the region

\[
\left\{ (K, K') \mid \tau_R(K) \cdot \tau_L(K') < 1 \text{ and } \tau_L(K) \cdot \tau_R(K') < 1 \right\}
\]

such that their elements have stable intersection?

Indeed, we consider a pair of special middle-\( \alpha \) Cantor sets that gives an affirmative solution to this problem. The main challenge is to construct a recurrent compact set of relative configurations, since, by the proposition in Subsection 2.3 of [MY], any relative configuration contained in a recurrent compact set is a configuration of stable intersection. The rest of this paper is outlined as follows:

In Section 2, we present the necessary definitions and state the recurrence condition
on relative configurations of [MY] which implies stable intersection of pairs of regular Cantor sets.

In Section 3, we translate this condition to the setting of affine Cantor sets, which gives a recurrent condition on a simpler space of relative configurations.

Constructing a recurrent compact set of the relative configurations of middle-α Cantor sets, under special conditions, is the main subject of Section 4.

2 Basic definitions

We will use notations similar to those of [MY] which are restated here.

Regular Cantor sets can also be defined as follows:

Let \( A \) be a finite alphabet, \( B \) a subset of \( A^2 \), and \( \Sigma \) the subshift of finite type \( A^\mathbb{Z} \) with allowed transitions \( B \). We will always assume that \( \Sigma \) is topologically mixing and every letter in \( A \) occurs in \( \Sigma \).

An expanding map of type \( \Sigma \) is a map \( g \) with the following properties:

i) the domain of \( g \) is a disjoint union \( \bigcup_B I(a,b) \), where for each \( (a,b) \), \( I(a,b) \) is a compact subinterval of \( I(a) := [0,1] \times \{a\} \),

ii) for each \( (a,b) \in B \), the restriction of \( g \) to \( I(a,b) \) is a smooth diffeomorphism onto \( I(b) \) satisfying \( |Dg(t)| > 1 \) for all \( t \).

The regular Cantor set associated to \( g \) is the maximal invariant set

\[
K := \bigcap_{n \geq 0} g^{-n}(\bigcup_B I(a,b)).
\]

These two definitions are equivalent. On one hand, we may, in the first definition, take
\[ I(i) := I_i \] for each \( i \leq r \), and, for each pair \( i, j \) such that \( \psi(I_i) \supset I_j \), take \( I(i, j) = I_i \cap \psi^{-1}(I_j) \). Conversely, in the second definition, we can consider an abstract line containing all intervals \( I(a) \) as subintervals, and \( \{ I(a, b) \mid (a, b) \in B \} \) as the Markov partition.

Also, a regular Cantor set \( K \) is affine if \( Dg \) be constant on every \( I(a, b) \).

Let \( \Sigma^- := \{(\theta_n)_{n \leq 0} \mid (\theta_i, \theta_{i+1}) \in B \text{ for } i < 0 \} \). We equip \( \Sigma^- \) with the following ultrametric distance: for \( \bar{\theta} \neq \tilde{\theta} \in \Sigma^- \), set
\[
d(\theta, \tilde{\theta}) := \begin{cases} 1 & \theta_0 \neq \tilde{\theta}_0, \\ |I(\theta \land \tilde{\theta})| & \text{otherwise}, \end{cases}
\]
where \( \theta \land \tilde{\theta} := (\theta_{-n}, \ldots, \theta_0) \) if \( \tilde{\theta}_j = \theta_j \) for \( 0 \leq j \leq n \) and \( \tilde{\theta}_{-n-1} \neq \theta_{-n-1} \).

Suppose that \( \theta \in \Sigma^- \) and \( n \in \mathbb{N} \), let \( \theta^n := (\theta_{-n}, \ldots, \theta_0) \) and \( B(\theta^n) \) be the affine map from \( I(\theta^n) \) onto \( I(\theta_0) \) which the diffeomorphism \( k_{\theta}^n := B(\theta^n) \circ f_{\theta^n}^{-1} \) is orientation preserving. Then, for each \( \theta \in \Sigma^- \), there is a smooth diffeomorphism \( k_{\theta} \) such that \( k_{\theta}^n \) converges to \( k_{\theta} \) in \( \text{Diff}^r_+(I(\theta_0)) \), for any \( r \in (1, +\infty) \), uniformly in \( \theta \).

Next, we define renormalization operators. For \( (a, b) \in B \), let
\[
f_{a,b} := [g|_{I(a,b)}]^{-1},
\]
this is a contracting diffeomorphism from \( I(b) \) onto \( I(a,b) \). If \( \underline{a} := (a_0, a_1, \ldots, a_n) \) is a word of \( \Sigma \), then we put
\[
f_{\underline{a}} := f_{a_0, a_1} \circ \cdots \circ f_{a_{n-1}, a_n},
\]
this is a contracting diffeomorphism from \( I(a_0) \) onto a subinterval of \( I(a_0) \), that we denote by \( I(\underline{a}) \). Also let \( F_{\theta} \) be the affine map from \( I(\theta_0) \) onto \( I(\theta_0, \theta_0) \) with the same orientation of \( f_{\theta_{-1}, \theta_0} \).
Consider $A := \{(\theta, A) \mid \theta \in \Sigma^- \text{ and } A : \ I(\theta_0) \rightarrow \mathbb{R} \text{ is an affine embedding map}\}$. The renormalization operators $T_{\theta_1, \theta_0} : A \rightarrow A$ are defined by

$$T_{\theta_1, \theta_0}(\theta, A) := (\theta \theta_1, A \circ F^{\theta_1}), \quad (\theta_0, \theta_1) \in \mathcal{B}.$$ 

For two sets of data $(A, B, \Sigma, g)$ and $(A', B', \Sigma', g')$ defining regular Cantor sets $K$ and $K'$, denote $C$ be the quotient of $A \times A$ by the diagonal action on the left of affine group.

A non empty compact set $L$ in $C$ is recurrent if for every $u \in L$ and $\ell, \ell' \geq 0$ with $\ell + \ell' > 0$, when $(\theta, A)$ and $(\theta', A')$ represents $u$, then there exist words $a = (a_0, \cdot \cdot \cdot, a_\ell)$ and $a' = (a'_0, \cdot \cdot \cdot, a'_{\ell'})$ in $\Sigma$ and $\Sigma'$, respectively, with $a_0 = \theta_0$ and $a'_0 = \theta'_0$, such that $(T_a(\theta, A), T_{a'}(\theta', A')) = v$ belongs to $L^c := \text{int} L$. The following proposition has been proved in [MY].

**Proposition 1.** Any relative configuration (of limit geometries) contained in a recurrent compact set is stably intersecting.

In the end of this section, we suppose that $S := \Sigma^- \times \Sigma'^- \times \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. One can see that the fibers of the quotient map $C \rightarrow S$ are one-dimensional and have a canonical affine structure. Moreover, this bundle map is trivializable. We choose an explicit trivialization $C \cong S \times \mathbb{R}$ in order to have a coordinate in each fiber.

### 3 Transfer of renormalization operators on the space $S \times \mathbb{R}$

Assume that $K$ is an affine Cantor set together with the Markov partition $\{I(n, m)\}_{m,n \in A}$ and an expanding map

$$\Phi \mid_{I(n,m)} (x) := p(n,m) \cdot x + q(n,m),$$

$$p(n,m) := p(n,m) + q(n,m),$$

$$q(n,m) := p(n,m) - q(n,m).$$
where $A := \{1, 2, 3, \cdots, N\}$. Let $I(n) := [a^1_n, a^2_n]$ and $I(n, m) := [a^1_{n,m}, a^2_{n,m}]$, then for each $(n, m) \in B$, we have:

$$\Phi(I(n, m)) = I(m).$$

In the case $P(\theta_0, \theta_1) > 0$, map $F^{\theta_1}$ is

$$F^{\theta_1} : I(\theta_1) \rightarrow I(\theta_0, \theta_1),$$

$$F^{\theta_1}(x) = \frac{a^2_{\theta_0, \theta_1} - a^1_{\theta_0, \theta_1}}{a^{1}_{\theta_1} - a^{1}_{\theta_1}} (x - a^1_{\theta_1}) + a^1_{\theta_0, \theta_1}.$$

Also in the opposite orientation, map $F^{\theta_1}$ is

$$F^{\theta_1}(x) = \frac{a^2_{\theta_0, \theta_1} - a^1_{\theta_0, \theta_1}}{a^{1}_{\theta_1} - a^{1}_{\theta_1}} (x - a^1_{\theta_1}) + a^2_{\theta_0, \theta_1}.$$

Therefore, in both cases, we obtain

$$F^{\theta_1}(x) = \frac{1}{p(\theta_0, \theta_1)} x - \frac{q(\theta_0, \theta_1)}{p(\theta_0, \theta_1)}.$$

To continue, we construct the homeomorphism between $S \times \mathbb{R}$ and $C$, then we transfer all renormalization operators to $S \times \mathbb{R}$.

**Theorem 1.** The map

$$L : C \rightarrow S \times \mathbb{R}$$

$$[(\varrho, ax + b), (\varrho', a'x + b')] \mapsto (\varrho', \varrho', a', \frac{b' - b}{a})$$

is a homeomorphism between the space of relative configurations $C$ and $S \times \mathbb{R}$.

**Proof.** $L$ is well defined:

Let

$$[(\varrho, ax + b), (\varrho', a'x + b')] = [(\varrho, a_1x + b_1), (\varrho', a'_1x + b'_1)].$$
Then there exist $c, d \in \mathbb{R}$ such that
\[ a_1 x + b_1 = c(a x + b) + d, \quad a'_1 x + b'_1 = c(a' x + b') + d, \]
therefore,
\[ \frac{a'}{a} = \frac{a'_1}{a_1}, \quad \frac{b' - b}{a} = \frac{b'_1 - b_1}{a_1}. \]
$L$ is onto:
\[ \forall (\theta, \theta', s, t) \in S \times \mathbb{R}, \quad L\left(\left[(\theta, x), (\theta', sx + t)\right]\right) = (\theta, \theta', s, t). \]
Also, $L$ is one to one:
Suppose that
\[ L\left(\left[(\theta, x), (\theta', sx + t)\right]\right) = L\left(\left[(\theta, x), (\theta', s'x + t')\right]\right), \]
then $s = s'$ and $t = t'$.
From the structure of $L$, we see that $L$ and $L^{-1}$ are continuous. □

Now we transfer the renormalization operators of relative configurations $C$ to the space $S \times \mathbb{R}$. To do this, let $((\theta_1, \theta_0), (\theta'_1, \theta'_0)) \in B \times B'$, then we obtain
\[
(\theta, \theta', s, t) \xrightarrow{L^{-1}} [(\theta, x), (\theta', sx + t)] \xrightarrow{\left(T_{\theta_1, \theta_0, \theta'_1, \theta'_0}\right)} \\
\left(\theta, \theta_1, \frac{1}{P(\theta_0, \theta_1)} x - \frac{q(\theta_0, \theta_1)}{P(\theta_0, \theta_1)}, (\theta'_1, \theta'_0)\right) \xrightarrow{L} \\
\left(\theta, \theta_1, \theta'_1, \frac{p(\theta_0, \theta_1)}{P(\theta'_0, \theta'_1)} s, p(\theta_0, \theta_1) t - \frac{q(\theta'_0, \theta'_1)}{P(\theta'_0, \theta'_1)} p(\theta_0, \theta_1) s + q(\theta_0, \theta_1)\right).
\]

We denote $L \circ \left(T_{\theta_1, \theta_0, \theta'_1, \theta'_0}\right) \circ L^{-1}$ by $T((\theta_1, \theta_0), (\theta'_1, \theta'_0))$ and for the sake of comfort we select those which are in these kinds:
\[
T_{(\theta_1, \theta_0), id} : (\theta, \theta', s, t) \longrightarrow (\theta \theta_1, \theta'_1, p_{\theta_1, \theta_0} s, p_{\theta_1, \theta_0} t + q_{\theta_1, \theta_0}).
\]

\[ T_{(id, \theta'_1, \theta'_0)} : (q, q', s, t) \rightarrow (q, \theta'^j_1, s \frac{s}{p'_{\theta'_1, \theta'_0}}, t - q'_{\theta_1, \theta_0} \frac{s}{p'_{\theta'_1, \theta'_0}}). \]

4 Construction of recurrent sets

In this section, we introduce a pair of middle-\(\alpha\) Cantor sets that have a recurrent compact set in the relative configurations, while \(\tau(K) \cdot \tau(K') < 1\).

**Theorem 2.** Suppose that \(K\) and \(K'\) are two homogenous Cantor sets with the convex hull \([0, 1]\) and expanding maps \(\phi\) and \(\phi'\) as

\[
\begin{align*}
K : & \quad \frac{1}{p} & \quad \frac{1}{p} & \quad \frac{1}{q} & \quad \frac{1}{q} \\
K' : & \quad \frac{1}{p} & \quad \frac{1}{p} & \quad \frac{1}{q} & \quad \frac{1}{q}
\end{align*}
\]

\[
\phi(x) := \begin{cases} 
px & x \in [0, \frac{1}{p}] \\
px - p + 1 & x \in [1 - \frac{1}{p}, 1]
\end{cases} \quad \phi'(x) := \begin{cases} 
qx & x \in [0, \frac{1}{q}] \\
qx - q + 1 & x \in [1 - \frac{1}{q}, 1]
\end{cases},
\]

where

\[
p := \gamma^{40} := (1.0321)^{40} = 3.538923071..., \]

\[
q := \gamma^{31} := (1.0321)^{31} = 2.663024240....
\]

Then pair \((K, K')\) have stable intersection, while their thickness product is smaller than one.

**Proof.** At first, we see that

\[
\tau(K) \cdot \tau(K') = \frac{1}{p} \cdot \frac{1}{q} = \frac{1}{(p - 2)(q - 2)} = 0.980062299...
\]

As explained at the end of Section 3, transferred renormalization operators can be considered

\[
\mathbb{R}^* \times \mathbb{R} \rightarrow \mathbb{R}^* \times \mathbb{R}
\]
\[
(s,t) \xrightarrow{T_0} (ps, pt) \quad * \quad (s,t) \xrightarrow{T_1} (ps, pt - p + 1)
\]
\[
(s,t) \xrightarrow{T_0'} (\frac{s}{q}, t) \quad (s,t) \xrightarrow{T_1'} (\frac{s}{q}, t + \frac{q-1}{q}s)
\]
Let \(s_1 := \frac{1}{1 - \frac{2}{q} + 2(1 - \frac{1}{q})s^{39}}\), \(s_2 := \frac{1 - \frac{2}{q} - 2(1 - \frac{1}{q})s^{39}}{q} \) and

\[
\Delta := \{(s,t) \mid \gamma^{-1}s_2 \leq s \leq \gamma s_1, \quad -s + (1 - \frac{1}{q})\frac{s}{q^{39}} \leq t \leq 1 - (1 - \frac{1}{q})\frac{s}{q^{39}} \}.
\]

Note that, since \(\tau_1 \cdot \tau_2 < 1\) and \((p - 2)(q - 2) < \gamma\), we have \(\gamma^{-1}s_2 < s_1 < s_2 < \gamma s_1\).

Observe that the numerical approximations are

\[
\gamma^{-1}s_2 \approx 1.1220161, \quad s_1 \approx 1.1349444, \quad s_2 \approx 1.1580329, \quad \gamma s_1 \approx 1.1713761.
\]

Take

\[
\mathcal{L} := \Delta \setminus \Delta_1 \cup \Delta_2,
\]

with

\[
\Delta_1 := \{(s,t) \mid L^1 : t + (1 - \frac{1}{q})(1 + \frac{1}{q^{39}})s > 1, \quad L^2 : t + (1 - \frac{1}{q})\frac{s}{q^{39}} > \frac{1}{p} \},
\]

\[
\Delta_2 := \{(s,t) \mid L^3 : t + (\frac{1}{q} - (1 - \frac{1}{q})\frac{1}{q^{39}})s < 1 - \frac{1}{p} \}
\]

\[
\Delta_1 := \{(s,t) \mid L^4 : t + (1 - (1 - \frac{1}{q})\frac{1}{q^{39}})s < 1 - \frac{1}{p} \}, \quad L^5 : t + (\frac{1}{q} - (1 - \frac{1}{q})\frac{1}{q^{39}})s < 0
\]

\[
\Delta_2 := \{(s,t) \mid L^6 : t + (1 - \frac{1}{q})(1 + \frac{1}{q^{39}})s > \frac{1}{p} \}.
\]

We are going to show that compact set \(\mathcal{L}\), as displayed in Figure 1, is a recurrent set for the operators (\(*\)).

Every vertical lines \(s = s_0\) pass over itself with suitable compositions of the operators (\(*\)) since \(p^{31} = q^{40}\). Therefore we can transfer the operators (\(*\)) on these lines by

**Proposition 2.** Let \(s \in \mathbb{R}\) and \(\{a_k\}_{k=0}^{30}, \{b_k\}_{k=0}^{30}\) be two finite sequences of numbers 0 and 1, then the maps
Figure 1:

\[ T_s(t) := p^{31}t + a_s, \quad * * \]
\[ a_s := -(p-1)p^{30}\left( \sum_{k=0}^{30} \frac{a_k}{p^k} - \frac{p(q-1)}{q(p-1)}s \sum_{k=0}^{39} \frac{b_k}{q^k} \right) \]

are return maps to the vertical line \( s \).

**Proof.** Suppose that \( \{b_k\}_{k=0}^{\infty} \) and \( \{a_k\}_{k=0}^{\infty} \) are two arbitrary sequences of numbers 0 and 1. For every \( a_k \) and \( b_k \), we can consider the operators of \( * \) in below form

\[ T_{a_k}(s,t) := (ps, pt - (p-1)a_k) \quad , \quad T_{b_k}(s,t) := (\frac{s}{q}, t + (\frac{q-1}{q})b_k s) \]

Let \( m, n \in \mathbb{N} \), then we obtain

\[ T_{a_{m-1}} \circ \ldots \circ T_{a_0}(s,t) = (p^m s, p^m t - (p-1) \sum_{k=0}^{m-1} a_k p^{m-1-k}), \]
\[ T_{b_{n-1}} \circ \ldots \circ T_{b_0}(s,t) = \left( \frac{s}{q^n}, t + \frac{q}{q^n}(q-1) \sum_{k=0}^{n-1} b_k q^{n-1-k} \right). \]

To prove the relations (i) and (ii), we use induction. Case \( m = n = 1 \) is valid. Suppose that assertion satisfies for cases \( i \) and \( j \), then we have:

\[ T_{a_i} \circ T_{a_{i-1}} \circ \ldots \circ T_{a_0}(s,t) = (p^{i+1} s, p^{i+1} t - (p-1) \sum_{k=0}^{i-1} a_k p^{i-k} - (p-1)a_i) \]
\[ = (p^{i+1} s, p^{i+1} t - (p-1) \sum_{k=0}^{i} a_k p^{i-k} ), \]
and we see that the relations (i) and (ii) hold for cases $i+1$ and $j+1$.

Replace $m=31$ and $n=40$ in the relations (i) and (ii), then we obtain

$$T_{b_{39}} \circ \ldots \circ T_{b_{30}} \circ T_{a_{30}} \circ \ldots \circ T_{a_{0}} (s, t) = \left( s, p^{31}t - (p - 1) \sum_{k=0}^{30} a_k p^{30-k} + \frac{p^{31}s}{q} (q - 1) \sum_{k=0}^{39} b_k q^{39-k} \right) = \left( s, p^{31}t - (p - 1)p^{30} \left( \sum_{k=0}^{30} a_k p^{k} - \frac{p(q-1)}{q(p-1)} s \sum_{k=0}^{39} b_k q^{k} \right) \right).$$

This completes the proof of proposition. □

If $\{a_{ik}\}_{k=0}^{30}$ and $\{b_{ik}\}_{k=0}^{39}$ be two arbitrary finite sequences of numbers $0,1$ and

$$s^* := \frac{p(q-1)}{q(p-1)} s,$$

then all of the return maps (or operators) are

$$T_s(t) = p^{31}t + a_s,$$

$$a_s := -(p - 1)p^{30} \left( \sum_{k=0}^{30} a_k p^{k} - s^* \sum_{k=0}^{39} b_k q^{k} \right) \quad a_{ik}, b_{ik} = 0 \text{ or } 1.$$

Here we deal with $2^{31+40}$ squares of length $p^{-31} = \gamma^{-1240} = 0.0 \ldots 0966\ldots$, which project under angle $\theta := \cot^{-1} s$ and determine this position, see Figure 2.

**Definition.** The set $R \subset \mathbb{R}$ is a recurrent if for every element of $R$ there exist suitable composites of maps (**), so transfers that element to $R^o$.

Let $I_s^\pm := [a_1, b_1] \cup [a_2^\pm, b_2^\pm] \cup [a_3, b_3]$ with

$$a_1 := -\frac{q(p-1)}{p(q-1)}s^* + (1 - \frac{1}{p}) \frac{s^*}{q^{39}}, \quad b_1 := \frac{1}{p} - (1 - \frac{1}{p})(1 + \frac{1}{q^{39}})s^*,$$

$$a_2^- := -\frac{p-1}{p(q-1)}s^* + (1 - \frac{1}{p}) \frac{s^*}{q^{39}}, \quad b_2^- := 1 - (1 - \frac{1}{p})(1 + \frac{1}{q^{39}})s^*,$$

$$a_2^+ := 1 - \frac{1}{p} - \frac{q(p-1)}{p(q-1)}s^* + (1 - \frac{1}{p}) \frac{s^*}{q^{39}}, \quad b_2^+ := \frac{1}{p} - (1 - \frac{1}{p}) \frac{s^*}{q^{39}},$$

$$a_3 := 1 - \frac{1}{p} - \frac{p-1}{p(q-1)}s^* + (1 - \frac{1}{p}) \frac{s^*}{q^{39}}, \quad b_3 := 1 - (1 - \frac{1}{p}) \frac{s^*}{q^{39}}.$$

$$I_s^\pm : \begin{array}{cccc}
  a_1 & b_1 & a_2^\pm & b_2^\pm & a_3 & b_3
\end{array}$$
Lemma 1. For every $s_1 \leq s \leq \frac{q(p-1)}{p(q-1)}$, the set $I_s^-$ and for every $\frac{q(p-1)}{p(q-1)} < s \leq s_2$, the set $I_s^+$ is a recurrent set for maps (**).

Proof. If $s_1 \leq s \leq s_2$, then

\[
\delta_1 := 0.987915116... = \frac{q-1}{(p-1)(q-2)+2(p-1)(q-1)q^{3q}} < s^s < s^{s^s} < \frac{(p-2)(q-1)}{(p-1)-2(p-1)(q-1)q^{3q}} = 1.00801257... =: \delta_2.
\]

Fix $s = \cotg \theta$ and relinquish of notation $\pm$ on our calculations.

At first, we show that for every point in interval

\[
J_s := \left[0 - \frac{q(p-1)}{p(q-1)}(1 - \frac{1}{q} + \frac{1}{q^2})s^s, \frac{1}{p^{2q}} - \frac{q(p-1)}{p(q-1)}(1 - \frac{1}{q})s^s\right],
\]

there exist suitable maps of (**) that send that point to $I_s^o$. We remind that $J_s$ is
projection of the union of these 14 squares

\[ C_{ij} := [0, \frac{1}{pq}] \times [0, \frac{1}{pq}] + (c_i, c_j), \quad 1 \leq i, j \leq 4 \]

with

\[(c_i, c_j) \in \{c_i\}_{i=1}^4 \times \{c_j\}_{j=1}^4 := \left\{ (0, (1 - \frac{1}{p})\frac{1}{pq}, (1 - \frac{1}{p})\frac{1}{pq}), \frac{1}{p}\frac{1}{pq} + \frac{1}{pq} \right\} \times \left\{ (0, (1 - \frac{1}{q})\frac{1}{pq}, (1 - \frac{1}{q})\frac{1}{pq}), \frac{1}{q}\frac{1}{pq} + \frac{1}{pq} \right\}, \]

except the squares \( C_{41} \) and \( C_{14} \). Let \( C \) be one of these squares, \( T \) be its correspondent operator and \( \Pi_\theta \) be the projection map on \( \mathbb{R}^2 \). Set \( K(C) := \Pi_\theta(C) - T^{-1}(I_s^\circ) \) contains 4 components if \( s \in (s_1, s_2) \) and 2 components if \( s = s_1 \) or \( s = s_2 \).

We show that intervals \( \Pi_\theta(C) \) overlap each other and each point of \( K(C) \) goes to \( I_s^\circ \) under other operators of (**) Suppose that \( T_{ij}(t) = p^{31}t + a_{ij} \) and \( T_{i'j'}(t) = p^{31}t + a_{i'j'} \) are the corresponding operators to the squares \( C_{ij} \) and \( C_{i'j'} \), respectively. Therefore we have \( T_{i'j'}(t) = T_{ij}(t) - (a_{ij} - a_{i'j'}) \) and we see that on special conditions point \( t \) goes to \( I_s^\circ \) under the operator \( T_{i'j'} \), in fact,

i) If

\[
\max \left\{ \frac{-1}{p} + \frac{(p-1)(q-2)}{pq(q-1)}s^* + 2\left(1 - \frac{1}{p}\right)\frac{s^*}{pq}, \quad 1 - \frac{2}{p} - \frac{(p-1)}{pq(q-1)}s^* + 2\left(1 - \frac{1}{pq}\right)\frac{s^*}{pq} \right\} \\
= \max \left\{ a_2^* - b_1 < a_{ij} - a_{i'j'} < \frac{1}{p} + \frac{p-1}{pq(q-1)}s^* - 2\left(1 - \frac{1}{p}\right)\frac{s^*}{pq} = b_1 - a_1, \right. \\
\]

then all points of last 3 components of \( K(C_{ij}) \) will be sent to \( I_s^\circ \) by the operator \( T_{i'j'} \) and right side of \( \Pi_\theta(C_{ij}) \) intersects left side of \( \Pi_\theta(C_{i'j'}) \),

ii) If

\[
1 - b_1 = 1 - \frac{1}{p} + (1 - \frac{1}{p})(1 + \frac{1}{pq})s^* < a_{ij} - a_{i'j'} \\
< 1 + \frac{q(p-1)}{p(q-1)}s^* - 2\left(1 - \frac{1}{p}\right)\frac{s^*}{pq} = b_3 - a_1, \\
\]

16
then the last component of $K(C_{ij})$ will be sent to $I^s_\circ$ by the operator $T_{ij'}$ and right side of $\Pi_\theta(C_{ij})$ intersects left side of $\Pi_\theta(C_{ij'})$.

Similar result is valid for negative cases.

We need numeral values of right and left sides of above inequality, indeed,

$$\begin{align*}
\max a_2 \pm b_1 &= 0.008670035..., \quad \min b_1 - a_1 = 0.708758125..., \\
\max 1 - b_1 &= 1.440604766..., \quad \min b_3 - a_1 = 2.134944413....
\end{align*}$$

We split interval $J_\circ$ in such cases:

**Case 1.** $t \in K(C_{31})$, we consider the operators $T_{41}$ and $T_{42}$ as

$$\begin{align*}
T_{41}(t) &= p^{31}t - (p - 1)p^{30}(\frac{s^*_{q^{38}}}{q} - \frac{s^*_{q^{38}}}{q}) = p^{31}t + \frac{p-1}{p}(q^2 + q)s^*, \\
T_{42}(t) &= p^{31}t - (p - 1)p^{30}(\frac{1}{p^{30}} - \frac{s^*_{q^{38}}}{q^{38}} - \frac{s^*_{q^{38}}}{q^{38}}) = p^{31}t + \frac{p-1}{p}(-p + q^2 + q)s^*).
\end{align*}$$

On the other hand, the corresponding operator $C_{31}$ is

$$T_{31}(t) = p^{31}t - (p - 1)p^{30}(\frac{s^*_{q^{38}}}{q^{38}}) = p^{31}t + \frac{p-1}{p}(q^2)s^*$$

and we obtain

$$\begin{align*}
-1.925836829 < a_{31} - a_{41} &= \frac{p-1}{p}(-q)s^* < -1.887440068, \\
+0.613086242 < a_{31} - a_{42} &= \frac{p-1}{p}(p - qs^*) < +0.651483003,
\end{align*}$$

therefore the element $t$ will be sent to $I^s_\circ$ by using the operators $T_{41}$ or $T_{42}$, and interval $\Pi_\theta(C_{31})$ connects $\Pi_\theta(C_{41})$ to $\Pi_\theta(C_{42})$.

**Case 2.** $t \in K(C_{42})$, here we need the operator

$$T_{32}(t) = p^{31}t - (p - 1)p^{30}(\frac{1}{p^{30}} - \frac{s^*_{q^{38}}}{q^{38}}) = p^{31}t + \frac{p-1}{p}(-p + q^2s^*)$$

and we obtain

$$a_{42} - a_{32} = \frac{p-1}{p}(q)s^* = a_{41} - a_{31},$$

therefore first 3 components of $K(C_{42})$ map to $I^s_\circ$ by the operator $T_{31}$ since
and the last component of $K(C_{42})$ maps to $I_s$ by using the operator $T_{32}$ and interval $\Pi_\theta(C_{42})$ connects $\Pi_\theta(C_{31})$ to $\Pi_\theta(C_{32})$.

**Case 3.** $t \in K(C_{32})$, in this case, we consider the operator

$$T_{21}(t) = p^{31}t - (p-1)p^{30}(-s^*) = p^{31}t + \frac{p-1}{p}(q)s^*$$

and we have:

$$+ 0.599935514 < a_{32} - a_{21} = \frac{p-1}{p}(-p + (q^2 - q)s^*) < +0.663790257,$$

therefore the operators $T_{12}$ or $T_{21}$ send the point $t$ to $I_s$ and interval $\Pi_\theta(C_{32})$ connects $\Pi(C_{42})$ to $\Pi(21)$.

**Case 4.** $t \in K(C_{21}) \cup K(C_{22}) \cup K(C_{11})$, when $t \in K(C_{21})$, we have $T_{11}(t) = p^{31}t$ and we see that numbers $a_{21} - a_{32}$ and $a_{21} - a_{11}$ satisfy $(i)$ and $(ii)$ conditions above, this says that point $t$ goes to $I_s$ under the operators $T_{32}$ or $T_{11}$ and interval $\Pi_\theta(C_{21})$ connects $\Pi(C_{32})$ to $\Pi(C_{11})$. Elements $K(C_{22})$ and $K(C_{11})$ behave like (2) and (3) cases above.

**Case 5.** $t \in K(C_{12}) \cup K(C_{43})$, in this case, we obtain

$$a_{22} = \frac{p-1}{p}(-p + qs^*), \quad a_{12} = \frac{p-1}{p}(-p),$$

$$a_{43} = \frac{p-1}{p}(-p^2 + (q^2 + q)s^*), \quad a_{33} = \frac{p-1}{p}(-p^2 + q^2 s^*),$$

therefore,

$$a_{12} - a_{22} = \frac{p-1}{p}(-q)s^* = a_{33} - a_{43},$$

$$-0.608256623 < a_{12} - a_{43} = \frac{p-1}{p}(p^2 - p - (q^2 + q)s^*) < -0.467608362,$$

as above discussion, the point $t$ goes to set $I_s$ and the projections cover together.

**Case 6.** Other cases, here we use relation

$$T_{ij}(t) = T_{i(j-2)}\left(t - \left(\frac{p-1}{p}\right)\frac{1}{p^{29}}\right)$$
that sends the point $t$ to $I_s^\circ$ since point $t - \left(\frac{p-1}{p}\right)\frac{1}{p^{29}}$ comes back to above cases.

Till now, we have shown that points of interval $J_s$ will be sent to $I_s^\circ$.

Let $D$ be the union of these 14 squares and $D_1$ be all the squares in $[0, \frac{1}{p^{28}}] \times [0, \frac{1}{q^{37}}]$, except two corner squares $[0, \frac{1}{p^{28}}] \times [\frac{1}{q^{40}}, \frac{1}{q^{37}}]$ and $[\frac{1}{p^{28}} - \frac{1}{p^{31}}, \frac{1}{p^{28}}] \times [0, \frac{1}{q^{37}}]$. We are going to pave all the squares in $[0, \frac{1}{p}] \times [0, \frac{1}{q}]$ by set $D_1$, of course, except two corner squares

$$[0, \frac{1}{p^{28}}] \times \left[\frac{1}{q} - \frac{1}{q^{40}}, \frac{1}{q}\right], \quad \left[\frac{1}{p} - \frac{1}{p^{28}}, \frac{1}{p}\right] \times [0, \frac{1}{q^{40}}].$$

We represent this set to $\mathcal{D}$ and we construct it by induction.

Let $D_n$ be the set of all the squares in $[0, \frac{1}{p^{i+1}}] \times [0, \frac{1}{q^{j+1}}]$, except two corner squares

$$[0, \frac{1}{p^{i+1}}] \times \left[\frac{1}{q^{j+1}} - \frac{1}{q^{40}}, \frac{1}{q^{j+1}}\right], \quad \left[\frac{1}{p^{i+1}} - \frac{1}{p^{i+1}}, \frac{1}{p^{i+1}}\right] \times [0, \frac{1}{q^{40}}],$$

for a pair $(i, j)$ with condition $1 \leq i \leq 28$, $1 \leq j \leq 37$ and also $D_n$ supports that every line with slope $\theta = \theta_s$, that passes among itself, has a conflict with a copy of $D_1$. Before defining set $D_{n+1}$, as shown in Figure 3, put

$$E_n := D_n + \left(1 - \frac{1}{p}, \frac{1}{q^{40}}\right), \quad F_n := D_n + \left(0, 1 - \frac{1}{q}\right)$$
I) If \( \Pi_\theta(D_n) \cup \Pi_\theta(E_n) \) be connected, then we let \( D_{n+1} \) be the set of all the squares in
\[
[0, \frac{1}{p^2}] \times [0, \frac{1}{q^2}], \text{ except } [0, \frac{1}{p^2}] \times [\frac{1}{q^2}, \frac{1}{q^2}], \frac{1}{q^2}] \text{ and } [\frac{1}{p^2}, \frac{1}{p^2}, \frac{1}{q^2}] \times [0, \frac{1}{q^2}]
\]
otherwise,

II) If \( \Pi_\theta(D_n) \cup \Pi_\theta(F_n) \) be connected, then we let \( D_{n+1} \) be the set of all the squares in
\[
[0, \frac{1}{p^2}] \times [0, \frac{1}{q^2}], \text{ except } [0, \frac{1}{p^2}] \times [\frac{1}{q^2}, \frac{1}{q^2}] \text{ and } [\frac{1}{p^2}, \frac{1}{p^2}, \frac{1}{q^2}] \times [0, \frac{1}{q^2}].
\]

Condition (I) is equivalent with the relation
\[
(1 - \frac{1}{p^2}) - (1 - \frac{1}{q^2}) \frac{q(p-1)s^*}{p(q-1)} < \frac{1}{p^2} - (1 - \frac{1}{q^2}) \frac{q(p-1)s^*}{p(q-1)} \Rightarrow \frac{p-2}{s} < \frac{q^3(q-1)}{q^5(q-1)(q-2)} + \frac{2(p-1)p^i}{(p-1)(q-1)} s^* \leq \frac{p^i}{q^i} < \frac{q^3(q-1)}{q^5(q-1)(q-2)} + \frac{2(p-1)p^i}{(p-1)(q-1)} s^*. \tag{1}
\]

Condition (II) is equivalent with the relation
\[
0 - (1 - \frac{1}{q^2}) \frac{q(p-1)s^*}{p(q-1)} < \frac{1}{p^2} - (1 - \frac{1}{q^2}) \frac{q(p-1)s^*}{p(q-1)} \Rightarrow \frac{p-1}{s} < \frac{q^3(q-1)}{q^5(q-1)(q-2)} + \frac{2(p-1)p^i}{(p-1)(q-1)} s^*. \tag{2}
\]

At first, we show that assertion is valid for \( n = 1 \). We can put \( i = 28 \) and \( j = 37 \) in the relation (2) and then we obtain that interval \( \Pi_\theta(D + (0, (1 - \frac{1}{p^2}, 0)) \) intersects interval \( \Pi_\theta(D) \). This matter is similar for sets \( \Pi_\theta(D + ((1 - \frac{1}{p^2}, 0, (1 - \frac{1}{p^2}, 0)) \). The relation (1) is not valid here, and we had to use their operators.

Two squares that occur on points
\[
(c_1, c_4) = \left((1 - \frac{1}{p})(\frac{1}{p^{29}} + \frac{1}{p^{30}}), 0\right), \quad (c_8, c_5) := \left((1 - \frac{1}{p})(\frac{1}{p^{28}} + \frac{1}{p^{30}}), (1 - \frac{1}{q})(\frac{1}{q^{37}} + \frac{1}{q^{38}} + \frac{1}{q^{39}})\right)
\]
have the following operators
\[
T_{14}(t) = p^{31}t + \frac{p-1}{p}(-p^2 - p),
\]
\[
T_{85}(t) = p^{31}t + \frac{p-1}{p}(-p^3 + (q^3 + q^2 + q)s^*)
\]

20
and we get
\[ + 0.025457501 < a_{85} - a_{14} = \frac{p-1}{p} \left( -p^3 + p^2 + p + (q^3 + q^2 + q)s^* \right) < 0.438403979. \]

Therefore the projection of two squares \( C_{84} \) and \( C_{14} \) fill each other’s middle gaps, also the square \( C_{84} \) set upper than \( C_{14} \) under projection \( \Pi_\theta \), this expresses that boundary points fall in set \( J_s \) or \( J_s + \Pi_\theta \left( (1 - \frac{1}{p}) \frac{1}{p-1}, (1 - \frac{1}{q}) \frac{1}{q-1} \right) \).

Suppose that assertion is valid for \( n \) and we check property of \( D_{n+1} \).
Since \( (p-2)(q-2) > 1 \), the relations (1) and (2) do not happen together, therefore the process stops when
\[
\frac{q-1}{(p-1)(q-2)s^*} - \frac{2(q-1)p^i}{(q-2)q^{39}} - 1 < \frac{p^i}{q^i} - 1 < \frac{(p-2)(q-1)}{(p-1)s^*} + \frac{2(q-1)p^i}{q^{39}} - 1.
\]

In fact, we show that always
\[
|\frac{p^i}{q^i} - 1| = \max \left\{ \frac{(p-2)(q-1)}{(p-1)s^*} + \frac{2(q-1)p^i}{q^{39}} - 1, \frac{2(q-1)p^i}{(q-2)q^{39}} - 1 \right\}.
\]

To prove the relation (3), we divide \( 1 \leq i \leq 27 \) in the following cases:

1) If \( i = 27 \), then \( \frac{p^i}{q^{39}} = \frac{q}{p^4} \) and we obtain
\[
\min \left\{ \left| \frac{q^{27}}{q^3} - 1 \right| \mid 1 \leq j \leq 37 \right\} = \min \left\{ \left| \frac{q^{27}}{q^3} - 1 \right|, \left| \frac{q^{27}}{q^3} - 1 \right| \right\}
\]
\[
= \min \{ |\gamma^{26} - 1|, |\gamma^{-5} - 1| \} = 0.146131269.\]

On the other hand,
\[
\frac{2(q-1)p^{27}}{q^{39}} = 2\frac{q-1}{p^4} = 0.056470184, \quad \frac{2(q-1)p^{27}}{(q-2)q^{39}} = 0.085170618\ldots,
\]
therefore the relation (3) happens anytime.

2) If \( i = 26 \), then \( \frac{p^{26}}{q^{39}} = \frac{q}{p^4} \) and by
\[
\min \left\{ \frac{p^{26}}{q^3} - 1 \mid 1 \leq j \leq 37 \right\} = \min \left\{ \frac{p^{26}}{q^3} - 1, \frac{p^{26}}{q^3} - 1 \right\}
\]
\[
= \min \{ |\gamma^{27} - 1|, |\gamma^{-14} - 1| \} = 0.357467488\ldots.
\]

21
the relation (3) is valid.

3) If \( i \leq 25 \), then maximum value in the right side of the relation (3) happens when \( i = 25 \), thus,

\[
\frac{2(q - 1)p^i}{q^{39}} = 0.004508966... \quad , \quad \frac{2(q - 1)p^i}{(q - 2)q^{39}} = 0.006800605....
\]

We show that \( |\frac{p^i}{q^j} - 1| > 0.028738306, \) where \( 1 \leq j \leq 37 \). Since \( \frac{p^i}{q^j} = \gamma^{40i-31j} \) and \( 40i - 31j \neq 0 \), value

\[
\min\left\{ \left| \frac{p^i}{q^j} - 1 \right| \mid 1 \leq j \leq 37 \ , \ 1 \leq i \leq 25 \right\}
\]

happens when \( 40i - 31j = \pm 1 \), (for example \( i = 7 \) and \( j = 9 \)). Also, we have:

\[
\min\left\{ |\gamma - 1| , |\gamma^{-1} - 1| \right\} = 0.031101637....
\]

Therefore \( D_{n+1} \) always exists since sets \( E_n \) and \( F_n \) are copies of \( D_n \) and the relation (3) is valid. This process continues until we construct \( \mathcal{D} \), here is \( i = 1 \) and \( j = 1 \) since \( (p - 2)(q - 2) > 1 \). Therefore, every line \( L_\theta \) that passes among the convex hull \( \mathcal{D} \), had to meet at least one copy of set \( D_1 \) and always projected point will be sent to \( I_\theta \) by using one of the operators \((**))\), like case (6) above. Also, sets \( \mathcal{D} + (1 - \frac{1}{p}, 0) \), \( \mathcal{D} + (0, 1 - \frac{1}{q}) \) and \( \mathcal{D} + (1 - \frac{1}{p}, 1 - \frac{1}{q}) \) behave like set \( \mathcal{D} \), as case (6) above.

For \( s = s_1 \), \( I_s \) is connected interval \((-s, 1)\), but our images are not the points which have been obtained from the intersection of lines \( L^3, L^1 \) and \( L^6, L^5 \) yet. The same situation holds for \( s = s_2 \).

Here the proof is finished. \( \square \)
Lemma 2. If $\gamma^{-1}s_2 \leq s \leq s_1$ or $s_2 \leq s \leq \gamma s_1$, then interval $I_s = (-s, 1)$ is a recurrent set for the maps (**).

Proof. It is enough to show that every line with slope $\theta = \cot^{-1} s$ conflicts with one of the squares in $[0, 1] \times [0, 1]$. Like the above lemma, we do this by induction.

Let $D_1 := C_{11}$ and suppose that $D_n$ be the set of all the squares in $[0, \frac{1}{p^{i+1}}] \times [0, \frac{1}{q^{j+1}}]$, for a pair of $i,j$ and every line $L_\theta$ intersects one of the squares $D_n$. Let $E_n, F_n$ be like Lemma 1 and

I) Let $D_{n+1}$ be the set of all the squares in $[0, \frac{1}{p^i}] \times [0, \frac{1}{q^j}]$ if $\Pi_\theta(D_n) \cup \Pi_\theta(E_n)$ be connected otherwise,

II) Let $D_{n+1}$ be the set of all the squares in $[0, \frac{1}{p^{i+1}}] \times [0, \frac{1}{q^{j+1}}]$ if $\Pi_\theta(D_n) \cup \Pi_\theta(F_n)$ be connected.

If we take small sentences in the relations (1) and (2) of previous lemma and replace $s^* := \frac{p(q-1)}{q(p-1)}s$, then the process stops when

$$\frac{q}{p(q-2)s} < \frac{p^i}{q^j} < \frac{q}{p^i}(p-2).$$

We need to show that the relation $s_1 < \frac{p^i}{q^j}s < s_2$ does not happen any time:

1) If $\gamma^{-1}s_2 \leq s \leq s_1$, then

$$p^i > q^j \implies 1 < \gamma < \frac{p^i}{q^j} \implies \gamma^{-1}s_2 < s < \frac{p^i}{q^j}s,$$

$$p^i < q^j \implies \frac{p^i}{q^j} < \gamma^{-1} < 1 \implies \frac{p^i}{q^j}s < s^{-1}s_1 < s_1.$$

2) If $s_2 \leq s \leq \gamma s_1$, then

$$p^i > q^j \implies 1 < \gamma < \frac{p^i}{q^j} \implies s_2 < \frac{p^i}{q^j}s < s_1 < s_1,$$

$$p^i < q^j \implies \frac{p^i}{q^j} < \gamma^{-1} < 1 \implies \frac{p^i}{q^j}s < s_1 < s_1.$$

Thus, construction continues till $i = j = 0$. This completes the proof of the lemma. □
Now we show that $\mathcal{L}$ is a recurrent set for the operators ($\ast$).

**Case 1.** If $s_1 \leq s \leq s_2$, then we use Lemma 1.

**Case 2.** If $\gamma^{-1}s_2 < s < s_1$ or $s_2 < s < \gamma s_1$, then we use Lemma 2 and the operators

$$
\pi_2 \circ T_0^{40} \circ T_1^{31}(t) = p^{31}t - p^{31} + 1, \quad \pi_2 \circ T_1^{40} \circ T_0^{31}(t) = p^{31}t + (p^{31} - 1)s
$$

of course, if it is needed.

**Case 3.** If $s = \gamma^{-1}s_2$ or $s = \gamma s_1$, then

$$
\min \left\{ \frac{p^i}{q^j} \mid p^i > q^j \right\} = \frac{p^7}{q^9} = \gamma, \quad \max \left\{ \frac{p^i}{q^j} \mid p^i < q^j \right\} = \frac{p^{24}}{q^{21}} = \gamma^{-1},
$$

therefore point $(\gamma^{-1}s_2, t)$ passes on line $s = \frac{p^7}{q^9} \cdot \gamma^{-1}s_2 = s_2$ and it had to fall in set $s_2 \times (-s_2, 1)$ since satisfies condition of the Lemma 2. By using the operators $T_0^{40} \circ T_1^{31}$ and $T_1^{40} \circ T_0^{31}$ and Lemma 1, we transfer point $(s, t)$ to $\mathcal{L}$.

The same situation is valid for point $(\gamma s_1, t) \in \mathcal{L}$.

This completes the proof of the theorem. \(\square\)

Replacing $K$ and $K'$ to the forms of $C_\alpha$ and $C_\beta$, respectively and also appropriate selection of the recurrent set in Theorem 2 yields below proposition.

**Proposition 3.** Set $C_\alpha - \lambda C_\beta$ contains at least one interval for each $\lambda \in \mathbb{R}^\ast$.

We close this paper by posing a question which inspired by Theorem 2 and Palis conjecture.

**Open Problem 1.** Does there exist an open and dense subset in the mysterious region

$$
\Omega := \left\{ (C_\alpha, C_\beta) \mid HD(C_\alpha) + HD(C_\beta) > 1 \text{ and } \tau(C_\alpha) \cdot \tau(C_\beta) < 1 \right\},
$$

such that their elements have stable intersection?
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