DIVERGENCE MEASURES ESTIMATION AND ITS ASYMPOTOTIC NORMALITY THEORY : DISCRETE CASE

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Abstract.

1. Introduction

1.1. Motivations. In this paper, we study the convergence of empirical discrete probability distributions supported on a finite set.

Let throughout the following \( \mathcal{X} = \{c_1, c_2, \ldots, c_r\} \) \( (r \geq 2 \) be a finite countable space. The distributions probability on \( \mathcal{X} \) are finite dimensional vectors \( p \) in

\[
\mathcal{P}(\mathcal{X}) = \left\{ p = (p_c)_{c \in \mathcal{X}} : p_c \geq 0, \forall c \in \mathcal{X} \text{ and } \sum_{c \in \mathcal{X}} p_c = 1 \right\}.
\]

A divergence measure on \( \mathcal{P}(\mathcal{X}) \) is a function

\[
D : (\mathcal{P}(\mathcal{X}))^2 \to \mathbb{R}, (p, q) \mapsto D(p, q)
\]

such that \( D(p, q) = 0 \) for any \( p \) such that \( (p, p) \) in the domain of application of \( D \).

The function \( D \) is not necessarily an application. And if it is, it is not always symmetrical and it does neither have to be a metric. In lack of symmetry, the following more general notation is more appropriate :

\[
D : \mathcal{P}_1(\mathcal{X}) \times \mathcal{P}_2(\mathcal{X}) \to \mathbb{R}, (p, q) \mapsto D(p, q),
\]

where \( \mathcal{P}_1(\mathcal{X}) \) and \( \mathcal{P}_2(\mathcal{X}) \) are two families of distributions probability on \( \mathcal{X} \), not necessarily the same. To better explain our concern, let us introduce some of the most celebrated divergence measures.

We may present the following divergence measure : let \( (p, q) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \) with \( \mathcal{X} = \{c_1, c_2, \ldots, c_r\} \), \( p = (p_1, \ldots, p_r) \) and \( q = (q_1, \ldots, q_r) \) two probabilities distribution on \( \mathcal{X} \).

(1) The \( L_2^2 \)-divergence measure :

\[
D_{L_2^2}(p, q) = \sum_{j=1}^{r} (p_j - q_j)^2.
\]
(2) The family of Renyi’s divergence measures indexed by $\alpha \neq 1, \alpha > 0$, known under the name of Renyi-$\alpha$:

$$D_{R,\alpha}(p, q) = \frac{1}{\alpha - 1} \log \left( \sum_{j=1}^{r} p_j^\alpha q_j^{1-\alpha} \right).$$

(3) The family of Tsallis divergence measures indexed by $\alpha \neq 1, \alpha > 0$, also known under the name of Tsallis-$\alpha$:

$$D_{T,\alpha}(p, q) = \frac{1}{\alpha - 1} \left( \sum_{j=1}^{r} p_j^\alpha q_j^{1-\alpha} - 1 \right).$$

(4) The Kulback-Leibler divergence measure

$$D_{KL}(p, q) = \sum_{j=1}^{r} p_j \log(q_j/p_j).$$

The latter, the Kullback-Leibler measure, may be interpreted as a limit case of both the Renyi’s family and the Tsallis’ one by letting $\alpha \to 1$. As well, for $\alpha$ near 1, the Tsallis family may be seen as derived from $D_{R,\alpha}(p, q)$ based on the first order expansion of the logarithm function in the neighborhood of the unity.

From this small sample of divergence measures, we may give the following remarks.

(a) The $L^2$-divergence measure is both an application and a metric on $\mathcal{P}^2$, where $\mathcal{P}$ is the class of probability measures on $\mathbb{R}^d$ such that

$$\sum_{j} p_j^2 < +\infty.$$ 

(b) For both the Renyi and the Tsallis families, we may have computation problems and lack of symmetry. Let give examples. It is clear from the very form of these divergence measures that we do not have symmetry, unless for the special case where $\alpha = 1/2$. Both families are build on the following functional

$$I_{\alpha}(p, q) = \sum_{j} p_j^\alpha q_j^{1-\alpha}$$

1.2. Previous work and main contributions. Our main contribution may be summarized as follows, for data sampled from one or two unknown random variables, we derive almost sure convergency and central limit theorems for empirical $\phi-$ divergences

1.3. Overview of the paper.
2. Distribution limit for empirical $\phi-$divergence

2.1. Notation and definitions. Before we state the main results we need a few definitions. Define the empirical probability distribution generated by i.i.d. random variables $X_1, \ldots, X_n$ from the distribution probability $p$ as

$$\hat{p}_n = \left(\hat{p}_{cn}\right)_{c \in X}, \text{ where } \hat{p}_{cn} = \frac{1}{n} \sum_{i=1}^{n} 1_{c_j}(X_i) \quad (2.1)$$

and $\hat{q}_m$ is defined in the same way by $Y_1, \ldots, Y_m \sim q$ that is

$$\hat{q}_m = \left(\hat{q}_{cm}\right)_{c \in X}, \text{ where } \hat{q}_{cm} = \frac{1}{m} \sum_{i=1}^{m} 1_{c_j}(Y_i) \quad (2.2)$$

Definition 1. The $\phi$-divergence between the two probability distributions $p$ and $q$ is given by

$$J(p, q) = \sum_{j \in D} \phi(p_j, q_j) \quad (2.3)$$

where $\phi : [0, 1]^2 \to \mathbb{R}$ is a measurable function on which we will make the appropriate conditions.

The results on the functional $J(p, q)$ will lead to those on the particular cases of the Renyi, Tsallis, and Kullback-Leibler measures.

2.2. Main results. Since for a fixed $j \in \{1, \ldots, r\}$, $n\hat{p}_{cj}$ has a binomial distribution with parameters $n$ and success probability $p_j$, therefore

$$E[\hat{p}_{cj}] = p_j \quad \text{and} \quad V(\hat{p}_{cj}) = \frac{p_j(1-p_j)}{n}. \quad (2.4)$$

Furthermore, by the strong law of large numbers we have that $\hat{p}_{cj}$ converges almost surely (and hence in probability) to $p_j$ for every fixed $j \in \{1, \ldots, r\}$.

By the theorem central limit

$$\sqrt{n}(\hat{p}_{cj} - p_j) \sqrt{p_j(1-p_j)} \Rightarrow N(0, 1), \text{ as } n \to +\infty, \quad (2.5)$$

where we use the symbol $\Rightarrow$ to denote convergence in distribution.

Also for a fixed $j \in \{1, \ldots, r\}$, we have

$$\sqrt{m}(\hat{q}_{cj} - q_j) \sqrt{q_j(1-q_j)} \Rightarrow N(0, 1), \text{ as } m \to +\infty. \quad (2.6)$$

More generally since $n\hat{p}_n$ is a sample of size $n$ from a multinomial distribution with probabilities $p$, therefore (see Lo et al. (2016))

$$\sqrt{n}(\hat{p}_n - p) \xrightarrow{p} N_r(0, \Sigma(p)) \text{ as } n \to +\infty$$

where $N_r(0, \Sigma(p))$ is the multinomial covariance matrix given by

$$\Sigma(p) = \begin{bmatrix}
    p_1(1-p_1) & -p_1p_2 & \cdots & -p_1p_r \\
    -p_2p_1 & p_2(1-p_2) & \cdots & -p_2p_r \\
    \vdots & \vdots & \ddots & \vdots \\
    -p,rp_1 & -p,rp_2 & \cdots & p_r(1-p_r)
\end{bmatrix}$$
3. Asymptotic theory for ϕ-divergence measure

3.1. Boundness assumption and notations. Define

\[ D = \{ j \in \{ 1, 2, \cdots, r \} \text{ such that } p_j, q_j \geq \kappa > 0 \} \]

Let

\[ J(p, q) = \sum_{j \in D} \varphi(p_j, q_j) \]

where \( \varphi : [0, 1]^2 \to \mathbb{R} \) is a measurable function having continuous second order partial derivatives defined as follows:

\[ \varphi^{(1)}_1(s, t) = \frac{\partial \varphi}{\partial s}(s, t), \quad \varphi^{(1)}_2(s, t) = \frac{\partial \varphi}{\partial t}(s, t) \]

and

\[ \varphi^{(2)}_1(s, t) = \frac{\partial^2 \varphi}{\partial s^2}(s, t), \quad \varphi^{(2)}_2(s, t) = \frac{\partial^2 \varphi}{\partial t^2}(s, t), \quad \varphi^{(1, 2)}(s, t) = \frac{\partial^2 \varphi}{\partial s \partial t}(s, t). \]

Set

\[ A_{1,p} = \sum_{j \in D} |\varphi^{(1)}_1(p_j, q_j)|, \quad A_{2,q} = \sum_{j \in D} |\varphi^{(1)}_2(p_j, q_j)|, \]

\[ A_{3,q} = \sum_{j \in D} |\varphi^{(1)}_1(q_j, p_j)|, \quad A_{4,p} = \sum_{j \in D} |\varphi^{(1)}_2(q_j, p_j)|. \]

Based on (2.1) and (2.2), we will use the following empirical ϕ-divergences.

\[ J(\hat{p}_n, q) = \sum_{j \in D} \varphi(\hat{p}_n^j, q_j), \quad J(p, \hat{q}_m) = \sum_{j \in D} \varphi(p_j, \hat{q}_m^j), \]

and

\[ J(\hat{p}_n, \hat{q}_m) = \sum_{j \in D} \varphi(\hat{p}_n^j, \hat{q}_m^j). \]

Set

\[ a_n = \sup_{j \in D} |\hat{p}_n^j - p_j|, \quad b_m = \sup_{j \in D} |\hat{q}_m^j - q_j|, \]

and

\[ c_{n,m} = \max(a_n, b_m). \]

3.2. Statements of the main results. The first concerns the almost sure efficiency of the estimators.

**Theorem 1.** Let \( X \) a finite countable space and \((p, q) \in \mathcal{P}(X)^2\), and \( \hat{p}_n \) and \( \hat{q}_m \) be generated by i.i.d. samples \( X_1, \cdots, X_n \sim p \) and \( Y_1, \cdots, Y_m \sim q \). Then the following asymptotic results hold for the empirical ϕ-divergences.

(a) One sample

\[ \limsup_{n \to +\infty} \frac{|J(\hat{p}_n, q) - J(p, q)|}{a_n} \leq A_{1,p}, \quad a.s \]

\[ \limsup_{m \to +\infty} \frac{|J(p, \hat{q}_m) - J(p, q)|}{b_m} \leq A_{2,q}, \quad a.s \]
Divergence measures estimation and its asymptotic normality theory: discrete case

(b) Two samples

\[
\limsup_{(n,m) \to (+\infty, +\infty)} \frac{|J(\hat{p}_n, \hat{q}_m) - J(p, q)|}{c_{n,m}} \leq A_{1,p} + A_{2,q} \quad \text{a.s.} \tag{3.4}
\]

where \(a_n, b_n\) and \(c_{n,m}\) are as in (3.1).

The second concerns the asymptotic normality of the estimators.

\textbf{Theorem 2.} Let

\[
V_{1,p} = \sum_{j \in D} p_j(1 - p_j)(\phi_1^{(1)}(p_j, q_j))^2 \quad \text{and} \quad V_{2,q} = \sum_{j \in D} q_j(1 - q_j)(\phi_2^{(1)}(p_j, q_j))^2.
\]

Under the same assumptions as in theorem 1, the following central limit theorems hold for empirical \(\phi\)-divergences

(a) One sample: as \(n \to +\infty\),

\[
\sqrt{n}(J(\hat{p}_n, q) - J(p, q)) \rightsquigarrow N(0, V_{1,p}), \tag{3.5}
\]

(b) Two samples: as \(n \to +\infty\) and \(m \to +\infty\),

\[
\left(\frac{nm}{mV_{1,p} + nV_{2,q}}\right)^{1/2}(J(\hat{p}_n, \hat{q}_m) - J(p, q)) \rightsquigarrow N(0, 1) \tag{3.7}
\]

\[II - \text{Direct extensions.}\]

Quite a few number of divergence measures are not symmetrical. Among these non-symmetrical measures are some of the most interesting ones. For such measures, estimators of the form \(J(\hat{p}_n, q), J(p, \hat{q}_m)\) and \(J(\hat{p}_n, \hat{q}_m)\) are not equal to \(J(q, \hat{p}_n), J(\hat{q}_m, f)\) and \(J(\hat{q}_m, \hat{p}_n)\) respectively.

In one-sided tests, we have to decide whether the hypothesis \(p = q\), for \(q\) known and fixed, is true based on data from \(p\). In such a case, we may use the statistics one of the statistics \((J(\hat{p}_n, q)\) and \(J(q, \hat{p}_n)\) to perform the tests. We may have information that allows us to prefer one of them. If not, it is better to use both of them, upon the finiteness of both \(J(p, q)\) and \(J(q, p)\), in a symmetrized form as

\[
J_{(s)}(p, q) = \frac{J(p, q) + J(q, p)}{2}. \tag{3.8}
\]

The same situation applies when we face double-side tests, i.e., testing \(p = q\) from data generated by \(p\) et \(q\).

\textbf{Asymptotic a.e. efficiency.}

\textbf{Theorem 3.} Under the same assumptions as in theorem 1, the following hold...
One sample:

\[
\limsup_{n \to +\infty} \frac{|J_{(s)}(\hat{p}_n, q) - J_{(s)}(p, q)|}{a_n} \leq \frac{1}{2} (A_{1,p} + A_{4,p}) \text{ a.e.,}
\]

\[
\limsup_{n \to +\infty} \frac{|J_{(s)}(p, \hat{q}_m) - J_{(s)}(p, q)|}{b_n} \leq \frac{1}{2} (A_{2,q} + A_{3,q}) \text{ a.e.,}
\]

Two samples:

\[
\limsup_{(n,m) \to (+\infty, +\infty)} \frac{|J_{(s)}(\hat{p}_n, \hat{q}_m) - J_{(s)}(p, q)|}{c_{n,m}} \leq \frac{1}{2} (A_{1,p} + A_{2,q} + A_{3,q} + A_{4,p}) \text{ a.e.}
\]

### Asymptotic Normality

Denote

\[
V_{3,q} = \sum_{j \in D} q_j (1 - q_j) (\phi_1^{(1)}(q_j, p_j))^2 \quad \text{and} \quad V_{4,p} = \sum_{j \in D} p_j (1 - p_j) (\phi_2^{(1)}(q_j, p_j))^2.
\]

We have

\[
V_{1,4,p} = V_{1,p} + V_{4,p} \quad \text{and} \quad V_{2,3,q} = V_{2,q} + V_{3,q}.
\]

We have

### Theorem 4

Under the same assumptions as in theorem 1, the following hold

(a) One sample: as \( n \to +\infty \)

\[
\sqrt{\frac{n}{V_{1,4,p}}} \left( J_{(s)}(\hat{p}_n, q) - J_{(s)}(p, q) \right) \Rightarrow \mathcal{N}(0, 1),
\]

\[
\sqrt{\frac{n}{V_{2,3,q}}} \left( J_{(s)}(p, \hat{q}_m) - J_{(s)}(p, q) \right) \Rightarrow \mathcal{N}(0, 1).
\]

(b) Two samples: as \( (n,m) \to (+\infty, +\infty) \)

\[
\left( \frac{nm}{mV_{1,4,p} + nV_{2,3,q}} \right)^{1/2} \left( J_{(s)}(\hat{p}_n, \hat{q}_m) - J_{(s)}(p, q) \right) \Rightarrow \mathcal{N}(0, 1).
\]

**Remark** The proof of these extensions will not be given here, since they are straight consequences of the main results. As well, such considerations will not be made again for particular measures for the same reason.

### 4. Particular Cases

#### 4.1. Renyi and Tsallis families

These two families are expressed through the functional

\[
\mathcal{I}_\alpha(p, q) = \sum_{j \in D} p_j^{\alpha} q_j^{1-\alpha}, \quad \alpha > 0, \quad \alpha \neq 1,
\]

which is of the form of the \( \phi \)-divergence measure with

\[
\phi(x, y) = x^\alpha y^{1-\alpha}, \quad (x, y) \in \{(p_j, q_j), j \in D\}.
\]
A- (a)- The asymptotic behavior of the Tsallis divergence measure.

Denote
\[ A_{T,\alpha,1} = \frac{\alpha}{|\alpha - 1|} \sum_{j \in D} \left( \frac{p_j}{q_j} \right)^{\alpha - 1} \quad \text{and} \quad A_{T,\alpha,2} = \sum_{j \in D} \left( \frac{p_j}{q_j} \right)^{\alpha}. \]

We have

**Corollary 1.** Under the same assumptions as in theorem 1, and for any \( \alpha > 0, \alpha \neq 1 \), the following hold

(a) One sample:
\[ \limsup_{n \to +\infty} \frac{|D_{T,\alpha}(\hat{p}_n, q) - D_{T,\alpha}(p, q)|}{a_n} \leq A_{T,\alpha,1} \quad \text{a.s.} \]
\[ \limsup_{n \to +\infty} \frac{|D_{T,\alpha}(p, \hat{q}_n) - D_{T,\alpha}(p, q)|}{b_n} \leq A_{T,\alpha,2} \quad \text{a.s.} \]

(b) Two samples:
\[ \limsup_{(n,m) \to (+\infty, +\infty)} \frac{|D_{T,\alpha}(\hat{p}_n, \hat{q}_m) - D_{T,\alpha}(p, q)|}{c_{n,m}} \leq A_{T,\alpha,1} + A_{T,\alpha,2} \quad \text{a.s.} \]

Denote
\[
\sigma^2_{T,\alpha,1}(p, q) = \frac{\alpha^2}{(\alpha - 1)^2} \left[ \sum_{j \in D} p_j (p_j/q_j)^{2\alpha - 2} - \left( \sum_{j \in D} p_j (p_j/q_j)^{\alpha - 1} \right)^2 \right]
\]
\[
\sigma^2_{T,\alpha,2}(p, q) = \sum_{j \in D} q_j (p_j/q_j)^{2\alpha} - \left( \sum_{j \in D} q_j (p_j/q_j)^{\alpha} \right)^2.
\]

We have

**Corollary 2.** Under the same assumptions as in theorem 1, and for any \( \alpha > 0, \alpha \neq 1 \), the following hold

\[ \sqrt{n} \left( D_{T,\alpha}(\hat{p}_n, q) - D_{T,\alpha}(p, q) \right) \sim N(0, \sigma^2_{T,\alpha,1}(p, q)) \quad \text{as} \quad n \to +\infty, \]
\[ \sqrt{n} \left( D_{T,\alpha}(p, \hat{q}_n) - D_{T,\alpha}(p, q) \right) \sim N(0, \sigma^2_{T,\alpha,2}(p, q)) \quad \text{as} \quad n \to +\infty, \]
and as \((n, m) \to (+\infty, +\infty),\)
\[ \left( \frac{mn}{n\sigma^2_{T,\alpha,2}(p, q) + m\sigma^2_{T,\alpha,1}(p, q)} \right)^{1/2} \left( D_{T,\alpha}(\hat{p}_n, \hat{q}_m) - D_{T,\alpha}(p, q) \right) \sim N(0, 1). \]

As to the symmetrized form
\[ D_{T,\alpha}^{(s)}(p, q) = \frac{D_{T,\alpha}(p, q) + D_{T,\alpha}(g, f)}{2}. \]
we need the supplementary notations:
\[ A_{T,\alpha,3} = \frac{\alpha}{|\alpha-1|} \sum_{j \in D} (q_j/p_j)^{\alpha-1}, \quad A_{T,\alpha,4} = \sum_{j \in D} (q_j/p_j)^\alpha, \]
\[ \sigma^2_{T,\alpha,3} = \frac{\sigma^2}{(\alpha-1)^2} \left[ \sum_{j \in D} q_j (q_j/p_j)^{2\alpha-2} - \left( \sum_{j \in D} q_j (q_j/p_j)^{\alpha-1} \right)^2 \right] \]
and
\[ \sigma^2_{T,\alpha,4} = \sum_{j \in D} p_j (q_j/p_j)^{2\alpha} - (q_j/p_j)^\alpha \cdot \left( p_j (q_j/p_j)^\alpha \right)^2. \]

We have

**Corollary 3.** Let Assumptions ?? and ?? hold and let (BD) be satisfied. Then for any \( \alpha > 0, \alpha \neq 0, \)
\[ \limsup_{n \to +\infty} \left| \frac{D_{T,\alpha}^{(s)}(\tilde{p}_n, g) - D_{T,\alpha}^{(s)}(p, q)}{a_n} \right| \leq \frac{A_{T,\alpha,1} + A_{T,\alpha,4}}{2} \text{ a.s.,} \]
\[ \limsup_{n \to +\infty} \left| \frac{D_{T,\alpha}^{(s)}(\tilde{f}, \tilde{q}_n) - D_{T,\alpha}^{(s)}(p, q)}{b_n} \right| \leq \frac{A_{T,\alpha,2} + A_{T,\alpha,3}}{2} \text{ a.s.,} \]
and
\[ \limsup_{(n,m) \to (+\infty, +\infty)} \left| \frac{D_{T,\alpha}^{(s)}(\tilde{p}_n, g_m) - D_{T,\alpha}^{(s)}(p, q)}{c_{n,m}} \right| \leq \frac{A_{T,\alpha,1} + A_{T,\alpha,2}}{2} \text{ a.s.} \]

Denote
\[ \sigma^2_{T,\alpha,1;3}(p, q) = \sigma^2_{T,\alpha,1}(p, q) + \sigma^2_{T,\alpha,4}(p, q), \]
\[ \sigma^2_{T,\alpha,2;3}(p, q) = \sigma^2_{T,\alpha,2}(p, q) + \sigma^2_{T,\alpha,3}(p, q). \]

We also have

**Corollary 4.** Let Assumptions ?? and ?? hold and let (BD) be satisfied. Then for any \( \alpha > 0, \alpha \neq 0, \) we have
\[ \sqrt{n} \left( D_{T,\alpha}^{(s)}(\tilde{p}_n, g) - D_{T,\alpha}^{(s)}(p, q) \right) \rightsquigarrow N(0, \sigma^2_{T,\alpha,1;3}(p, q)), \text{ as } n \to +\infty, \]
\[ \sqrt{n} \left( D_{T,\alpha}^{(s)}(\tilde{f}, \tilde{q}_n) - D_{T,\alpha}^{(s)}(p, q) \right) \rightsquigarrow N(0, \sigma^2_{T,\alpha,2;3}(p, q)), \text{ as } n \to +\infty, \]
and as \( (n,m) \to (+\infty, +\infty), \)
\[ \left( \frac{nm}{m \sigma^2_{T,\alpha,1;4}(p, q) + n \sigma^2_{T,\alpha,2;3}(p, q)} \right)^{1/2} \left( D_{T,\alpha}^{(s)}(\tilde{p}_n, g_m) - D_{T,\alpha}^{(s)}(p, q) \right) \rightsquigarrow N(0, 1). \]

**A-(b)-** The asymptotic behavior of the Renyi-\( \alpha \) divergence measure.

The treatment of the asymptotic behavior of the Renyi-\( \alpha, \alpha > 0, \alpha \neq 1 \) is obtained from Part (A) (a) by expansions and by the application of the delta method.

We first remark that
\[ D_{R,\alpha}(p, q) = \frac{1}{\alpha - 1} \log (I_\alpha(p, q)). \]
Corollary 5. Under the same assumptions as in theorem 1, and for any $\alpha > 0$, $\alpha \neq 1$, the following hold

$$
\lim_{n \to +\infty} \frac{|D_{R,\alpha}(\hat{p}_n, g) - D_{R,\alpha}(p, q)|}{a_n} \leq \frac{A_{T,\alpha}}{I_{\alpha}(p, q)} =: A_{R,\alpha,1} \ a.s.,
$$

$$
\lim_{n \to +\infty} \frac{|D_{R,\alpha}(f, \hat{q}_n) - D_{R,\alpha}(p, q)|}{b_n} \leq \frac{A_{T,\alpha}}{I_{\alpha}(p, q)} =: A_{R,\alpha,2} \ a.s.,
$$

and

$$
\lim_{n,m \to +\infty} \frac{|D_{R,\alpha}(\hat{p}_n, g_m) - D_{R,\alpha}(p, q)|}{c_{n,m}} \leq A_{R,\alpha,1} + A_{R,\alpha,2} \ a.s.
$$

Denote

$$
\sigma_{R,\alpha,1}^2(p, q) = \frac{\sigma_{T,\alpha,1}^2(p, q)}{I_{\alpha}^2(p, q)} \quad \text{and} \quad \sigma_{R,\alpha,2}^2(p, q) = \frac{\sigma_{T,\alpha,2}^2(p, q)}{I_{\alpha}^2(p, q)}.
$$

We have

Corollary 6. Let Assumptions ?? and ?? hold and let (BD) be satisfied. Then for any $\alpha > 0$, $\alpha \neq 1$,

$$
\sqrt{n} \left( D_{R,\alpha}(\hat{p}_n, g) - D_{R,\alpha}(p, q) \right) \overset{\text{a.s.}}{\to} N(0, \sigma_{R,\alpha,1}^2(p, q)), \quad \text{as} \ n \to +\infty,
$$

$$
\sqrt{n} \left( D_{R,\alpha}(f, \hat{q}_n) - D_{R,\alpha}(p, q) \right) \overset{\text{a.s.}}{\to} N(0, \sigma_{R,\alpha,2}^2(p, q)), \quad \text{as} \ n \to +\infty,
$$

and as $(n, m) \to (+, +)$

$$
\left( \frac{mn}{n \sigma_{R,\alpha,2}(p, q) + m \sigma_{R,\alpha,1}(p, q)} \right)^{1/2} \left( D_{R,\alpha}(\hat{p}_n, g_m) - D_{R,\alpha}(p, q) \right) \overset{\text{a.s.}}{\to} N(0, 1).
$$

As to the symetrized form

$$
D_{R,\alpha}^{(s)}(p, q) = \frac{D_{R,\alpha}(p, q) - D_{R,\alpha}(g, f)}{2},
$$

we need the supplementary notations:

$$
A_{R,\alpha,3} = \frac{A_{T,\alpha,3}}{I_{\alpha}(p, q)}, \quad \text{and} \quad A_{R,\alpha,4} = \frac{A_{T,\alpha,4}}{I_{\alpha}(p, q)}
$$

$$
\sigma_{R,\alpha,3}(p, q) = \frac{\sigma_{T,\alpha,3}(p, q)}{I_{\alpha}^2(p, q)} \quad \text{and} \quad \sigma_{R,\alpha,4}(p, q) = \frac{\sigma_{T,\alpha,4}(p, q)}{I_{\alpha}^2(p, q)}.
$$

Corollary 7. Let Assumptions ?? and ?? hold and let (BD) be satisfied. Then for any $\alpha > 0$, $\alpha \neq 1$,

$$
\lim_{n \to +\infty} \frac{|D_{R,\alpha}^{(s)}(\hat{p}_n, g) - D_{R,\alpha}^{(s)}(p, q)|}{a_n} \leq (A_{R,\alpha,1} + A_{R,\alpha,3})/2 =: A_{R,\alpha,1}^{(s)} \ a.s.,
$$

$$
\lim_{n \to +\infty} \frac{|D_{R,\alpha}^{(s)}(f, \hat{q}_n) - D_{R,\alpha}^{(s)}(p, q)|}{a_n} \leq (A_{R,\alpha,2} + A_{R,\alpha,3})/2 =: A_{R,\alpha,2}^{(s)},
$$

and

$$
\lim_{(n, m) \to (+, +)} \frac{|D_{R,\alpha}^{(s)}(\hat{p}_n, g_m) - D_{R,\alpha}^{(s)}(p, q)|}{c_{n,m}} \leq A_{R,\alpha,1}^{(s)} + A_{R,\alpha,2}^{(s)}.
$$
Denote
\[ \sigma^2_{R,\alpha,1:4}(p, q) = \sigma^2_{R,\alpha,1}(p, q) + \sigma^2_{R,\alpha,4}(p, q) \]
\[ \sigma^2_{R,\alpha,2:3}(p, q) = \sigma^2_{R,\alpha,2}(p, q) + \sigma^2_{R,\alpha,3}(p, q) \]

We also have

**Corollary 8.** Under the same assumptions as in theorem 1, and for any \( \alpha > 0, \alpha \neq 1 \), the following hold

\[
\sqrt{n} \left( D_{R,\alpha}(\tilde{p}_n, g) - D_{R,\alpha}(p, q) \right) \Rightarrow N(0, \sigma^2_{R,\alpha,1:4}(p, q)), \text{ as } n \to +\infty
\]
\[
\sqrt{n} \left( D_{R,\alpha}(f, \tilde{q}_m) - D_{R,\alpha}(p, q) \right) \Rightarrow N(0, \sigma^2_{R,\alpha,2:3}(p, q)), \text{ as } n \to +\infty
\]

and as \((n, m) \to (+\infty, +\infty)\),

\[
\left( \frac{mn}{n \sigma^2_{R,\alpha,2:3}(p, q) + m \sigma^2_{R,\alpha,1:4}(p, q)} \right)^{1/2} \left( D_{R,\alpha}(\tilde{p}_n, g_m) - D_{R,\alpha}(p, q) \right) \Rightarrow N(0, 1).
\]

**B - Kulback-Leibler Measure**

5. PROOFS

To keep the notation simple, we introduce the two following notations:

\[ \Delta^c_{j,n} = \hat{p}_j - p_j \text{ and } \Delta^c_{j,m} = \hat{q}_j - q_j, \forall j \in D, \]

therefore

\[ a_n = \sup_{j \in D} |\Delta^c_{j,n}|, \quad b_m = \sup_{j \in D} |\Delta^c_{j,m}|, \quad \text{and } c_{n,m} = \max(a_n, b_m). \]

For any \( j \in D \), set

\[ \hat{q}_n(p_j) = \sqrt{n} \Delta^c_{j,n} \text{ and } g_m(q_j) = \sqrt{m} \Delta^c_{j,m} \]

We will use in the following the mean value theorem and the delta method.

For one sample estimation, define

\[ J(\hat{p}_n, q) = \sum_{j \in D} \varphi(\hat{p}_j, q_j) \]

For a fixed \( j \in D \), we have

\[ \varphi(\hat{p}_n, q_j) = \varphi(p_j + \Delta^c_{j,n}, q_j) \]
\[ = \varphi(p_j, q_j) + \Delta^c_{j,n} \varphi_1(1)(p_j + \theta_{1,j} \Delta^c_{j,n}, q_j) \]

by applying the mean value theorem to the function \((,) \mapsto \phi((,), q_j)\) and where \( \theta_{1,j} \)

is some number lying between 0 and 1. In the sequel, any \( \theta_{i,j}, i = 1, 2, \ldots \) satisfies \(|\theta_{i,j}| < 1.\)
By applying again the mean values theorem to the function \( (. ) \mapsto \phi_1(1)(., q_j) \), we have

\[
\phi_1^{(1)}(p_j + \theta_{1,j} \Delta_{p_n}^c, q_j) = \phi_1^{(1)}(p_j, q_j) + \theta_{1,j} \Delta_{p_n}^c \phi_1^{(2)}(p_j + \theta_{2,j} \Delta_{p_n}^c, q_j)
\]

We can write (5.1) as

\[
\varphi(p_n^c, q_j) = \varphi(p_j, q_j) + \Delta_{p_n}^c \phi_1^{(1)}(p_j, q_j) + \theta_{1,j} \Delta_{p_n}^c \phi_1^{(2)}(p_j + \theta_{2,j} \Delta_{p_n}^c, q_j)
\]

Now we have

\[
J(\hat{p}_n, q) - J(p, q) = \sum_{j \in D} \Delta_{p_n}^c \phi_1^{(1)}(p_j, q_j)
\]

\[
+ \sum_{j \in D} \theta_{1,j} \Delta_{p_n}^c \phi_1^{(2)}(p_j + \theta_{2,j} \Delta_{p_n}^c, q_j),
\]

hence

\[
|J(\hat{p}_n, q) - J(p, q)| \leq a_n \sum_{j \in D} \phi_1^{(1)}(p_j, q_j)
\]

\[
+ a_n^2 \sum_{j \in D} \phi_1^{(2)}(p_j + \theta_{2,j} \Delta_{p_n}^c, q_j),
\]

Therefore

\[
\limsup_{n \to \infty} \frac{|J(\hat{p}_n, q) - J(p, q)|}{a_n} \leq A_{1,p} + a_n \sum_{j \in D} \phi_1^{(2)}(p_j + \theta_{2,j} \Delta_{p_n}^c, q_j).
\]

We know that \( A_{1,p} < \infty \) and

\[
\sum_{j \in D} \phi_1^{(2)}(p_j + \theta_{2,j} \Delta_{p_n}^c, q_j) \to \sum_{j \in D} \phi_1^{(2)}(p_j, q_j) < \infty \quad \text{as} \quad n \to \infty.
\]

This proves (3.2).

Formula (3.3) is obtained in a similar way. We only need to adapt the result concerning the first coordinate to the second.

The proof of (3.4) comes by splitting \( \sum_{j \in D} (\phi(p_n^c, q_n^c) - \phi(p_j, q_j)) \), into the following two terms

\[
\sum_{j \in D} (\phi(p_n^c, q_n^c) - \phi(p_j, q_j)) = \sum_{j \in D} (\phi(p_n^c, q_n^c) - \phi(p_j, q_n^c))
\]

\[
+ \sum_{j \in D} (\phi(p_j, q_n^c) - \phi(p_j, q_j))
\]

\[
= I_{n,1} + I_{n,2}
\]
we have
\[ \sqrt{n}(J(\hat{p}_n, q) - J(p, q)) = \sum_{j \in D} \hat{q}_n(p_j)\phi_1^{(1)}(p_j, q_j) + \sqrt{n}R_{2,n} \]
where
\[ R_{2,n} = \sum_{j \in D} \theta_{1,j}(\Delta_{p_n}^{c_j})^2\phi_1^{(2)}(p_j + \theta_{2,j}\Delta_{p_n}^{c_j}, q_j). \]

From (2.5) and for a fixed \( j \in D \), we have
\[ \hat{q}_n(p_j) \overset{d}{\rightarrow} \mathcal{N}(0, p_j(1 - p_j)) \quad \text{as} \quad n \rightarrow +\infty, \]
therefore
\[ \hat{q}_n(p_j)\phi_1^{(1)}(p_j, q_j) \overset{d}{\rightarrow} \mathcal{N} \left(0, p_j(1 - p_j)(\phi_1^{(1)}(p_j, q_j))^2\right), \quad \text{as} \quad n \rightarrow +\infty. \]

Hence
\[ \sum_{j \in D} \hat{q}_n(p_j)\phi_1^{(1)}(p_j, q_j) \overset{d}{\rightarrow} \mathcal{N}(0, V_{1,p}), \quad \text{as} \quad n \rightarrow +\infty, \]
where
\[ V_1 = \sum_{j \in D} p_j(1 - p_j)(\phi_1^{(1)}(p_j, q_j))^2 \]
since \( X_1, X_2, \cdots \) is a sequence of i.i.d. replications of \( X \).

Let show that \( \sqrt{n}R_{2,n} = o_P(1) \). We have
\[ |\sqrt{n}R_{2,n}| \leq \sqrt{n}a_n^2 \sum_{j \in D} \phi_1^{(2)}(p_j + \theta_{2,j}\Delta_{p_n}^{c_j}, q_j) \]

Let show that
\[ \sqrt{n}a_n^2 = o_P(1) \]
By the Bienaymé-Tchebychev inequality, we have, for any \( \epsilon > 0 \) and for any \( j \in \{1, \cdots, r\} \)
\[ \mathbb{P}(\sqrt{n}(|\hat{\rho}_n^{c_j} - p_j|)^2 \geq \epsilon) = \mathbb{P}\left( |\hat{\rho}_n^{c_j} - p_j| \geq \frac{\epsilon}{n^{1/4}} \right) \leq \frac{p_j(1 - p_j)}{\epsilon n^{1/2}}, \]
which implies that \( \sqrt{n}a_n^2 \) converges in probability to 0 as \( n \rightarrow +\infty \).

Finally from (5.3) we have \( \sqrt{n}R_{2,n} \rightarrow_P 0 \) as \( n \rightarrow +\infty \) which implies
\[ \sqrt{n}(J(\hat{p}_n, q) - J(p, q)) \overset{d}{\rightarrow} \mathcal{N}(0, V_{1,p}), \quad \text{as} \quad n \rightarrow +\infty. \]
This ends the proof of (3.5).

The result (3.6) is obtained by a symmetry argument by swapping the role of \( p \) and \( q \).
Now, it remains to prove Formula (3.7) of the theorem. Let us use bi-variate Taylor-Lagrange-Cauchy formula to get,

\[
J(\hat{p}_n, \hat{q}_m) - J(p, q) = \sum_{j \in D} \Delta_{p_n} \phi_1^{(1)}(p_j, q_j) + \sum_{j \in D} \Delta_{q_m} \phi_2^{(1)}(p_j, q_j) \\
+ \frac{1}{2} \sum_{j \in D} \left( (\Delta_{p_n}^j)^2 \phi_1^{(2)} + \Delta_{p_n}^j \Delta_{q_m}^j \phi_{1,2}^{(2)} + (\Delta_{q_m}^j)^2 \phi_2^{(2)} \right) \left( u_n^{e_j}, v_m^{e_j} \right),
\]

where

\[
(u_n^{e_j}, v_m^{e_j}) = (p_j + \theta \Delta_{p_n}^j, q_j + \theta \Delta_{q_m}^j).
\]

Thus we get

\[
J(\hat{p}_n, \hat{q}_m) - J(p, q) = \frac{1}{\sqrt{n}} \sum_{j \in D} \hat{q}_n(p_j) \phi_1^{(1)}(p_j, q_j) + \frac{1}{\sqrt{m}} \sum_{j \in D} q_m(q_j) \phi_2^{(1)}(p_j, q_j) + R_{n,m},
\]

where \( R_{n,m} \) is given by

\[
\frac{1}{2} \sum_{j \in D} \left( (\Delta_{p_n}^j)^2 \phi_1^{(2)} + \Delta_{p_n}^j \Delta_{q_m}^j \phi_{1,2}^{(2)} + (\Delta_{q_m}^j)^2 \phi_2^{(2)} \right) \left( u_n^{e_j}, v_m^{e_j} \right).
\]

But we have

\[
\sum_{j \in D} \hat{q}_n(p_j) \phi_1^{(1)}(p_j, q_j) = N_n(1) + o_P(1)
\]

\[
\sum_{j \in D} q_m(q_j) \phi_2^{(1)}(p_j, q_j) = N_m(2) + o_P(1),
\]

where

\[
N_n(1) \sim \mathcal{N}(0, V_{1,p}) \, \text{ and } \, N_m(2) \sim \mathcal{N}(0, V_{2,q})
\]

with

\[
V_{2,q} = \sum_{j \in D} q_j(1 - q_j)(\phi_2^{(1)}(p_j, q_j))^2
\]

Since \( N_n(1) \) and \( N_m(2) \) are independent, we have

\[
\frac{1}{\sqrt{n}} \sum_{j \in D} \hat{q}_n(p_j) \phi_1^{(1)}(p_j, q_j) + \frac{1}{\sqrt{m}} \sum_{j \in D} q_m(q_j) \phi_2^{(1)}(p_j, q_j) = \mathcal{N} \left( 0, \frac{V_{1,p}}{n} + \frac{V_{2,q}}{m} \right) \\
+ o_P \left( \frac{1}{\sqrt{n}} \right) + o_P \left( \frac{1}{\sqrt{m}} \right).
\]

Therefore, we have

\[
J(\hat{p}_n, \hat{q}_m) - J(p, q) = \mathcal{N} \left( 0, \frac{V_{1,p}}{n} + \frac{V_{2,q}}{m} \right) + o_P \left( \frac{1}{\sqrt{n}} \right) + o_P \left( \frac{1}{\sqrt{m}} \right) + R_{n,m}.
\]
Hence
\[
\frac{1}{\sqrt{\frac{V_1}{n} + \frac{V_2}{m}}} (J(\hat{p}_n, \hat{q}_m) - J(p, q)) = N(0, 1) + o_P \left( \frac{1}{\sqrt{\frac{V_1}{n} + \frac{V_2}{m}}} \right)
\]
\[
+ \frac{1}{\sqrt{\frac{V_1}{n} + \frac{V_2}{m}}} R_{n,m}.
\]

That leads to
\[
\sqrt{\frac{nm}{mV_1 + nV_2}} (J(\hat{p}_n, \hat{q}_m) - J(p, q)) = N(0, 1) + o_P(1) + \sqrt{\frac{nm}{mV_1 + nV_2}} R_{n,m},
\]

since \(m/(mV_1 + nV_2)\) and \(m/(nV_1 + nV_2)\) are bounded, and then
\[
o_P \left( \frac{1}{\sqrt{n}} \right) \left( \frac{1}{\sqrt{V_1/n + V_2/m}} \right) = o_P(1)
\]
and
\[
o_P \left( \frac{1}{\sqrt{m}} \right) \left( \frac{1}{\sqrt{V_1/n + V_2/m}} \right) = o_P(1).
\]

It remains to prove that \(\sqrt{\frac{nm}{mV_1 + nV_2}} R_{n,r} = o_P(1)\). But we have by the continuity assumptions on \(\phi\) and on its partial derivatives and by the uniform converges of \(\Delta^c_{n_j}\) and \(\Delta^c_{m_j}\) to zero, that
\[
\left| \sqrt{\frac{nm}{mV_1 + nV_2}} R_{n,r} \right| \leq
\]
\[
\frac{1}{2} \left( \sqrt{\frac{na^2}{n}} \sum_{j \in D} \phi_1^{(2)}(p_j, q_j) + o(1) \right) \left( \sqrt{\frac{m}{mV_1 + nV_2}} \right)
\]
\[
+ \frac{1}{2} \left( \sqrt{\frac{mb^2}{m}} \sum_{j \in D} \phi_2^{(2)}(p_j, q_j) + o(1) \right) \left( \sqrt{\frac{n}{mV_1 + nV_2}} \right)
\]
\[
+ \frac{1}{2} \left( \sqrt{\frac{na_{n^2}b_m}{n}} \sum_{j \in D} \phi_3^{(2)}(p_j, q_j) + o(1) \right) \left( \sqrt{\frac{n}{mV_1 + nV_2}} \right)
\]

As previously, we have \(\sqrt{na^2_n} = o_P(1)\), \(\sqrt{mb^2_m} = o_P(1)\) and \(\sqrt{na_{n^2}b_m} = o_P(1)\).

From there, the conclusion is immediate.
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