HARDY-LITTLEWOOD CONSTANTS EMBEDDED INTO INFINITE PRODUCTS OVER ALL POSITIVE INTEGERS

RICHARD J. MATHAR

ABSTRACT. A group of infinite products over low-order rational polynomials evaluated at the sequence of prime numbers is loosely called the Hardy-Littlewood constants. In this manuscript we look at them as factors embedded in a super-product over primes, semiprimes, 3-almost primes etc. Numerical tables are derived by transformation into series over $k$-almost prime zeta functions. Alternative product representations in a basis of $k$-almost prime products associated with Euler's formula for the Riemann zeta function are also pointed out.

1. Appetizer

As pointed out by J. Vos Post [16], infinite products over rational polynomials evaluated at integers $n$ can be reordered as the integers are individually classified as $k$-almost primes [4][3, §1.2],

$$\prod_{n=2}^{\infty} \frac{n^2 - 1}{n^2 + 1} = \prod_{k=1}^{\infty} \prod_{\Omega(n) = k} \frac{n^2 - 1}{n^2 + 1}. $$

This classification uses

Definition 1. (Big-Omega, number of prime factors counted with multiplicity)

$$\Omega(n) \equiv \sum_j e_j$$

where $n = 2^{e_1}3^{e_2}5^{e_3}7^{e_4}11^{e_5} \cdots$ is the prime factorization of $n$.

One approach to numerical practise is the logarithm of the individual terms, [10, 1.513.1]

$$\log \prod_{n=2,\Omega(n) = k} \frac{n^s - 1}{n^s + 1} = \sum_{n=2,\Omega(n) = k} \log \frac{n^s - 1}{n^s + 1} = -\sum_{n=2,\Omega(n) = k} \log \frac{1}{1 - 1/n^s} $$

$$= -2 \sum_{l=1}^{\infty} \frac{1}{2l-1} \sum_{n=2,\Omega(n) = k} \frac{1}{n^{s(2l-1)}} = -2 \sum_{l=1}^{\infty} \frac{1}{2l-1} P_k(s(2l-1)),$$

supposed we know how to compute the $k$-almost prime zeta functions of

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Definition 2. (\textit{$k$-almost-prime Zeta Function,}\ [11])

(4) \[ P_k(s) \equiv \sum_{\Omega(n)=k} \frac{1}{n^s}. \]

Examples of this factorization of (1) are gathered in Table\ 1. The particular case of $k = 1$, product over the primes, results in rational numbers if $s$ is even (chapter 7 of Ramanujan’s first notebook),

(5) \[ \prod_p \frac{p^s - 1}{p^s + 1} = \prod_p \frac{1 - p^{-s}}{1 + p^{-s}} = \frac{\zeta(2s)}{\zeta(s)} = \frac{2B_{2s}}{(2s)B_2}, \]

with $B_s$ the Bernoulli numbers, $\zeta(.)$ Riemann’s zeta function.

Proof. This follows from Euler’s formula

(6) \[ \frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right), \]

the sign-switched sibling [9, 11]

(7) \[ \prod_p \frac{1}{1 + p^{-s}} = \prod_p \frac{1 - p^{-s}}{(1 - p^{-s})(1 + p^{-s})} = \prod_p \frac{1 - p^{-s}}{1 - p^{-2s}} = \frac{\zeta(2s)}{\zeta(s)}, \]

and (again Euler’s) [11, 23.2.16]

(8) \[ \zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|, \quad n = 1, 2, 3, \ldots. \]
Remark 1. The numerical approach to infinite products of rational polynomials as \([7]\) is factorization of numerator and denominator over \(C\) with root sets \(\alpha_i\) and \(\beta_i\), use of the limit of \(\Gamma\)-function ratios \([1\ (6.1.47)]\)—which ought evaluate to unity in the sense \(\sum (\alpha_i - \beta_i) = 0\), easily verified by subtraction of penultimate coefficients related by the law of Vieta—to end up with a product over \(\Gamma\)-ratios \([8\ (\S\ 1.3)]\):

\[
\prod_{n=1}^{\infty} \frac{\prod_i (n - \alpha_i)}{\prod_i (n - \beta_i)} = \frac{\prod_i \Gamma(1 - \beta_i)}{\prod_i \Gamma(1 - \alpha_i)} \lim_{n \to \infty} \prod_i \frac{\Gamma(n - \alpha_i)}{\Gamma(n - \beta_i)} = \frac{\prod_i \Gamma(1 - \beta_i)}{\prod_i \Gamma(1 - \alpha_i)}.
\]

For lucky roots of the polynomials, further simplification may be possible by use of the functional equations of the Gamma-function \([8\ (\S\ 1.2)]\)\([7]\).

2. Almost-Prime Zeta Functions of the 2nd Kind

Formulas like \([5]\) and \([7]\) indicate that the products defined in \([1]\) might not represent the atomic constituents of this arithmetic. A vague hope to establish some useful arithmetic basis—plus a glimpse at Euler’s formula \([6]\)—proposes

**Definition 3.** \((k\text{-almost prime zeta functions of the 2nd kind})\)

\[
1/\zeta_k(s) \equiv \prod_{n \geq 2, \ell(n) = k} \left(1 - \frac{1}{n^s}\right), \quad \zeta_1(s) = \zeta(s), \quad \Re s > 1.
\]

With reference to \([4]\) there are logarithms, the logarithmic derivative,

\[
\log \zeta_k(s) = \sum_{j=1}^{\infty} \frac{1}{j} P_k(js), \quad \frac{\zeta'(s)}{\zeta_k(s)} = \sum_{j=1}^{\infty} P'_k(js),
\]

and their Möbius inversion

\[
P_k(s) = \sum_{j=1}^{\infty} \frac{\mu(j)}{j} \log \zeta_k(js).
\]

This definition factorizes each of the constants of \([1]\),

\[
\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^s}\right) = \prod_{k=1}^{\infty} \frac{1}{\zeta_k(s)}.
\]

The unrestricted host product over all integers is evaluated with \([9]\),

\[
\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^s}\right) = \prod_{n=1}^{\infty} \frac{(n+1)^s - 1}{(n+1)^s} = \frac{1}{\prod_{l=1}^{s-1} \Gamma(2 - e^{2\pi il/s})}
\]

\[
\begin{align*}
1/2, & \quad s = 2, \\
0.89396597366290\ldots, & \quad s = 3, \\
0.919019477593744\ldots, & \quad s = 4, \\
0.96325651757559\ldots, & \quad s = 5, \\
0.982684277742192\ldots, & \quad s = 6, \\
0.991654953472834\ldots, & \quad s = 7, \\
0.995923315077783\ldots, & \quad s = 8, \\
0.997991715347709\ldots, & \quad s = 9, \\
0.999005442480989\ldots, & \quad s = 10.
\end{align*}
\]
where $\tau \equiv \pi/\sqrt{2}$.

Cyclotomic factorizations (avoiding divergent expressions, i.e., using only polynomial orders of 2 or higher) generalize [7] if $x = 1/n$ is set to visualize the structure:

| $k$ | $s$ | $\zeta_k(s)$ |
|-----|-----|--------------|
| 1   | 2   | 1.64439406684822643647241516664602518218949901206798437735... |
| 1   | 3   | 1.202056031595942853997381615114499907486829234049888192... |
| 1   | 4   | 1.08232323371113819151600369654116790274750951918726907682... |
| 1   | 5   | 1.036927551433692633136548645703416805709809195012198191747... |
| 2   | 1   | 1.0173430619844419371451579297909205279018174900328535651842... |
| 2   | 7   | 1.0083492738319262863979754984979675959983650565238706417... |
| 2   | 8   | 1.00407735619794439378685238506524652589607990649580020329... |
| 3   | 2   | 1.1541354291311922127531364676082653062021377019769166316101... |
| 3   | 3   | 1.02423061182696151581575509852679023950490215454774... |
| 3   | 4   | 1.0501517289991717978274016968522912941640476627857960524772... |
| 3   | 5   | 1.001137144907153269444743321059478627202283056156805839703... |
| 3   | 6   | 1.000268773319796626045829352019049130925353448777256227247... |
| 4   | 2   | 1.00006493711920871039924752459791612134264254357082696815... |
| 4   | 3   | 1.000158887627043741809695219326910562197188495738139234951... |
| 4   | 4   | 1.0000391759943508120128800108357232107129793809390341917... |
| 4   | 5   | 1.0000097060551954054769844769158843631069856641087764961... |
| 4   | 6   | 1.00000242127132784709472971654363631286102379240142530861... |
| 4   | 8   | 1.0000006068080358708589512628099422482514999324370824... |
| 5   | 2   | 1.0394324293040944419806920524167341067921270674614826029... |
| 5   | 3   | 1.0305612573636271593427319083437227566674738470640544373... |
| 5   | 4   | 1.003145079830586145685270439706663042500303824169155161... |
| 5   | 5   | 1.000355736020549219327661940592084981748430962203403243... |
| 5   | 6   | 1.00004524129169197709855106142415587292132319822741524992... |
| 5   | 7   | 1.00000570389042425132873140525671868968753054036047135029... |
| 5   | 8   | 1.00000260681399460978137801483720108089957265667040951... |
| 5   | 9   | 1.000000765166478505645787735130050216875881798866985821... |
| 5   | 10  | 1.0000000094786093685641739015981695687606286480202523638... |
| 5   | 11  | 1.0000000111778200443203922814899184474717980169845755406... |
| 5   | 12  | 1.00000000014655193378639781888790100132018989904578221764... |
| 6   | 2   | 1.010069695181975191078741060439035427867568810378713067067... |
| 6   | 3   | 1.00384011077312192420561416488116578237502799964152453503... |
| 6   | 4   | 1.00019679948504124660855296980733877871995656022576263542... |
| 6   | 5   | 1.00001121056796202645792471438751692254495180390266097... |
| 6   | 6   | 1.000000656486763670825403540412473976972447884634538215... |
| 6   | 7   | 1.000000039640201863338482299569864455986578036971179818... |
| 6   | 8   | 1.0000000024245420677635138784543831778216839998486273473... |
| 6   | 9   | 1.0000000014944664045752269172969167215376338782896827053... |
| 6   | 10  | 1.000000000925645528057890635386834145659663528928231940... |
| 6   | 11  | 1.0000000000575104536414294815860508388520719031157305... |
| 6   | 12  | 1.00000000000003580369489719342307308503503504446132847... |

Table 2. $\zeta_k$ defined in [10]. The first block is the familiar Riemann zeta function [11] Table 23.2. As $s \to \infty$, $\zeta_k \sim 1 + 2^{-ks}$.
From $1 - x^4 = (1 - x^2)(1 + x^2)$ we generate

$$\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{1}{n^4} \right) = \prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{1}{n^2} \right) \left( 1 + \frac{1}{n^2} \right).$$

Division leads to

$$\prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{1}{n^2} \right) = \frac{\zeta_k(2)}{\zeta_k(4)},$$

and in general to

$$\prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{1}{n^s} \right) = \frac{\zeta_k(s)}{\zeta_k(2s)}.$$ 

In that sense, the $\zeta_k$ keep up to the promise to generate the rational polynomials of Section 1.

From $1 - x^6 = (1 - x^3)(1 + x^3) = (1 - x^3)(1 + x^2 + x^4)$ we generate

$$\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{1}{n^2} + \frac{1}{n^4} \right) = \frac{\zeta_k(2)}{\zeta_k(6)},$$

and generally from $1 - x^{2s} = (1 - x^2)(1 + x^2 + x^4 + \cdots + x^{2s-2})$

$$\prod_{n \geq 2, \Omega(n) = k} \left( 1 + \sum_{j=1}^{s} \frac{1}{n^{2j}} \right) = \frac{\zeta_k(2)}{\zeta_k(2s + 2)},$$

The three different ways of grouping factors on the right-hand side of $(1 - x^8) = (1 - x^2)(1 + x^2)(1 + x^4)$ establish in addition

$$\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{1}{n^2} + \frac{1}{n^4} - \frac{1}{n^6} \right) = \frac{\zeta_k(4)}{\zeta_k(2) \zeta_k(8)},$$

a special case of

$$\prod_{n \geq 2, \Omega(n) = k} \left( 1 + \sum_{j=1}^{s} \left( -\frac{1}{n^{2j}} \right)^j \right) = \begin{cases} \frac{\zeta_k(4)}{\zeta_k(2) \zeta_k(2s + 2)} & \text{s odd;} \\ \frac{\zeta_k(4) \zeta_k(2s + 2)}{\zeta_k(2) \zeta_k(4s + 4)} & \text{s even.} \end{cases}$$

From $1 - x^9 = (1 - x^3)(1 + x^3 + x^6)$ by division through $1 - x^3$,

$$\prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{1}{n^3} + \frac{1}{n^{6j}} \right) = \frac{\zeta_k(3j)}{\zeta_k(9j)}.$$

Other formats are products of these in disguise. The elementary examples are

$$\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{1 \pm 2n^s}{n^{2s}} \right) = \prod_{n \geq 2, \Omega(n) = k} \left( 1 \pm \frac{1}{n^s} \right)^2,$$

$$\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{n^{s+1} \pm n^s \mp 1}{n^{2s+1}} \right) = \prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{1}{n^s} \right) \left( 1 \pm \frac{1}{n^{s+1}} \right),$$

to be continued in Appendix A.
Table 3. Artin’s constants defined in (26). Where the $k$-column is empty, the value is $A(r)$, else $A_k(r)$.

| $r$ | $k$ | $A(r)$ | $A_k(r)$ |
|-----|-----|--------|----------|
| 1   | 1   | 0.296675134743591034570155020219142864864831519178947890816... |     |
| 1   | 2   | 0.37359813661228805472850434641641511162924860650042094... |     |
| 1   | 3   | 0.8390421542448660076842611119415412549283071666868273300... |     |
| 1   | 4   | 0.95875211643573092771470255789286162569490448502359901592... |     |
| 1   | 5   | 0.98962886716642766550432283745792430805755758953926549448... |     |
| 2   | 2   | 0.67391736337635754166440897932263443856475981231267176792... |     |
| 2   | 3   | 0.69750135849636590328467035082092294073153946214515395354... |     |
| 2   | 4   | 0.96993232501525316214920207789129575961145794796906088006... |     |
| 2   | 5   | 0.9965899274802421274341959046329894929910103911783820658... |     |
| 3   | 3   | 0.85654444885345217442616794813595388216657280031765214035... |     |
| 3   | 4   | 0.9935215897105054066074926924141642940111507867781566018... |     |
| 3   | 5   | 0.9995275856535535637452493248364530863560321267404987... |     |
| 4   | 4   | 0.9298384739543468522831846953455348944998030548225365236... |     |
| 4   | 5   | 0.931265184160004348923720555067682558423743587801059016... |     |
| 4   | 6   | 0.99859105600757375458715021357813127784820993206862159036... |     |
| 5   | 5   | 0.96632127636693029160389479360257259743836587571173... |     |
| 5   | 6   | 0.96668868565775127403283729300162642141238221819397907... |     |
| 5   | 7   | 0.9994527313937636005026101681219660928715403131567977... |     |
| 5   | 8   | 0.999952053519372944786356394802140716661867177761801... |     |

3. Artin’s Constant

Definition 4. (Artin’s Constants of order $r$)

\[ A(r) = \prod_{n=2}^{\infty} \left(1 - \frac{1}{n^r(n-1)}\right) = \prod_{k=1}^{\infty} A_k(r); \quad A_k(r) = \prod_{n=2}^{\infty} A_{\Omega(n)=k}(r) \left(1 - \frac{1}{n^r(n-1)}\right). \]

\[ \log A(r) = \sum_{n=2}^{\infty} \log \left(1 - \frac{1}{n^r(n-1)}\right) = -\sum_{n=2}^{\infty} \sum_{s=1}^{\infty} \frac{1}{sn^r(s+1)^s} \]
\[ = -\sum_{n=2}^{\infty} \sum_{s=1}^{\infty} \sum_{l=0}^{\infty} \frac{(-s)^l}{l! sn^r(s+1)^s l} = -\sum_{s=1}^{\infty} \frac{1}{s!} \left[\zeta(r+1)s + l - 1\right] \]
\[ = \sum_{s=2}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{s-rj-1}{j-1}\right) [1 - \zeta(s)]. \]

This demonstrates how

\[ \log A_k(r) = -\sum_{s=2}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{s-rj-1}{j-1}\right) P_k(s) \]
is derived; the actual numerical evaluation of $A^{(r)}$ is done easier via \[29\]. Occasionally these Γ-functions simplify:

\[A^{(1)} = -\frac{\sin(\pi \phi)}{\pi}, \quad \phi = \frac{\sqrt{5} + 1}{2}.\]

**Definition 5. (Decremented Generalized Lucas Sequences)**

\[a_{r,s} \equiv s \sum_{j=1}^{s/(r+1)} \frac{1}{j} \left( s - jr - 1 \right).\]

The attribute “decremented” stresses that $1 + a_{r,s}$ are the reference values that one would find, for example, in the Online Encyclopedia of Integer Sequences [15].

These integer sequences provide the notational shortcut

\[\log A^{(r)} = -\sum_{s=2}^{\infty} \frac{a_{r,s}}{s} P_k(s),\]

and have recurrences

\[a_{r,s} = 0, \quad s \leq 1; \quad a_{r,s} = 2a_{r,s-1} - a_{r,s-2} + a_{r,s-r-1} - a_{r,s-r-2}, \quad r \geq 1\]

and generating functions

\[\sum_{s=0}^{\infty} a_{r,s} x^s = \frac{x^{1+r}(1 + r - rx)}{(1 - x)(1 - x - x^{1+r})} = -r - \frac{1}{1 - x} + \frac{1 + r - rx}{1 - x - x^{1+r}}.\]

Values for $s \geq 2$ are

\[a_{1,s} = 2, 3, 6, 10, 17, 28, 46, 75, 122, 198, 321, 520, 842, 1363, 2206, 3570, \ldots\]
\[a_{2,s} = 0, 3, 4, 5, 9, 14, 20, 30, 45, 66, 97, 143, 210, 308, 452, 663, 972, 1425, 2089, \ldots\]
\[a_{3,s} = 0, 0, 4, 5, 6, 7, 12, 18, 25, 33, 46, 65, 91, 125, 172, 238, 330, 456, 629, 868, \ldots\]

The argument of the product \[26\] admits an exponential product expansion \[12, 13\]

\[1 - \frac{1}{n^r(n - 1)} = \prod_{j=1}^{\infty} \left( 1 - \frac{1}{n^r} \right)^{\gamma^{(A)}_{r,j}}.\]

From the Laurent expansion

\[\frac{1}{n^r(n - 1)} = \sum_{j=0}^{\infty} \frac{1}{n^{r+1+j}},\]

we see that this is (up to a sign flip) the inverse Euler transform of the all-1 sequence padded with a short string of initial zeros \[6, 2\].

\[\gamma_{1,j}^{(A)} = 0, 1, 1, 1, 2, 2, 4, 5, 8, 11, 18, 25, 40, 58, 90, 135, 210, 316, 492, \ldots\]
\[\gamma_{2,j}^{(A)} = 0, 0, 1, 1, 1, 1, 2, 2, 3, 4, 6, 7, 11, 14, 20, 27, 39, 52, 75, 102, 145, \ldots\]
\[\gamma_{3,j}^{(A)} = 0, 0, 0, 1, 1, 1, 1, 2, 2, 3, 3, 5, 6, 8, 10, 14, 17, 24, 30, 41, 53, \ldots\]
Whereas (28) expands $A_k^{(r)}$ in the $P_k$ basis, (34) generates them in $\zeta_k$ basis:

$$A_k^{(r)} = \prod_{\Omega(n) = k} \prod_{j} \left(1 - \frac{1}{n^j}\right)^{\gamma_{r,j}^{(A)}} = \prod_{j} \zeta_k(j)^{-\gamma_{r,j}^{(A)}}.$$  

The same result could be obtained by plugging (12) into the right hand side of (31), then exponentiation, which reveals the Möbius pair

$$\frac{1}{j} \sum_{l|j} \mu(l) a_{r,j/l} = \gamma_{r,j}^{(A)}; \quad a_{r,s} = \sum_{l|s} \gamma_{r,l}^{(A)}.$$

**Remark 2.** This connection between the two types of coefficients does not depend on the mediation by $P$ nor on summation over $n$. Supposed any $\gamma_j$ are defined in

$$X = \prod_{j} \left(1 - \frac{1}{n^j}\right)^{\gamma_j},$$

which implies

$$\log X = \sum_{j} \log \left(1 - \frac{1}{n^j}\right)^{\gamma_j} = \sum_{j} \gamma_j \log \left(1 - \frac{1}{n^j}\right) = - \sum_{j} \gamma_j \sum_{s \geq 1} \frac{1}{jn^{sj}}.$$  

Let furthermore $g_k$ be defined via

$$\log X = - \sum_{k} g_k \frac{1}{kn^k},$$

then—by comparison of coefficients of equal powers $k = sj$ of $n$—

$$\frac{g_k}{k} = \sum_{j|k} \frac{\gamma_j}{k/j}; \quad g_k = \sum_{j|k} j \gamma_j; \quad \gamma_j = \frac{1}{j} \sum_{l|j} \mu(l) g_{j/l}.$$  

### 4. Twin Prime Constants

**Definition 6.** (Twin Prime Constants of order $r$)

$$T^{(r)} = \prod_{n=3}^{\infty} \left(1 - \frac{1}{(n-1)^r}\right) = \prod_{k=1}^{\infty} T_k^{(r)}; \quad T_k^{(r)} = \prod_{n=3, \Omega(n)=k}^{\infty} \left(1 - \frac{1}{(n-1)^r}\right).$$

$$\log T^{(r)} = \sum_{n=3}^{\infty} \log \left(1 - \frac{1}{(n-1)^r}\right) = - \sum_{n=3}^{\infty} \sum_{s=1}^{\infty} \frac{1}{s(n-1)^{rs}}$$

$$= - \sum_{n=3}^{\infty} \sum_{s=1}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^l}{s n^{rs+l}} = - \sum_{n=3}^{\infty} \sum_{s=1}^{\infty} \sum_{l=0}^{\infty} \frac{(rs)_l}{s! n^{rs+l}}$$

$$= - \sum_{s=1}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(r s + l)}{\Gamma(r s + l)} \left[\zeta(r s + l) - 1 - \frac{1}{2^{rs+l}}\right]$$

$$= - \sum_{s=r}^{\infty} \sum_{j=1}^{[s/r]} \frac{1}{j} \left(\frac{s}{r j} - 1\right) \left[\zeta(s) - 1 - \frac{1}{2^s}\right] = - \sum_{n=2}^{\infty} \sum_{s=1}^{\infty} \frac{1}{s n^{rs}} = \sum_{s=1}^{\infty} \frac{1}{s} [1 - \zeta(rs)].$$
This is a sum rule for the zeta function. The actual values of $T^{(r)}$ duplicate those of (55) because

$$T^{(r)} = \prod_{n=2}^{\infty} \left( 1 - \frac{1}{n^r} \right).$$

Repeating (46) yields

$$\log T^{(r)}_k = - \sum_{s=1}^{\infty} \sum_{j=1}^{\lfloor s/r \rfloor} \frac{1}{j} \left( \frac{s-1}{r_j-1} \right) \times \left\{ \begin{array}{ll} P_k(s), & k > 1 \\ P(s) - \frac{1}{2^r}, & k = 1 \end{array} \right..$$

One can rewrite this as

$$\log T^{(r)}_k = - \sum_{s=1}^{\infty} \frac{r}{s} t_{r,s} \times \left\{ \begin{array}{ll} P_k(s), & k > 1 \\ P(s) - \frac{1}{2^r}, & k = 1 \end{array} \right..$$

by introducing integer sequences $t_{r,s}$ via

**Definition 7.** (Binomial transform of aerated all-1 sequences)

$$t_{r,s} = \sum_{j=1}^{\lfloor s/r \rfloor} \frac{s}{r_j} \left( \frac{s-1}{r_j-1} \right).$$

Generating functions are

$$\sum_{s=1}^{\infty} t_{r,s} x^s = \left\{ \begin{array}{ll} \frac{x^2}{(1-x)(1-2x)}, & r = 2, \\ \frac{x^3}{(1-x)(1-2x)(1-x+x^2)}, & r = 3, \\ \frac{x^6}{(1-x)(1-2x)(1-x+2x^2)}, & r = 4, \\ \frac{x^6}{(1-x)(1-2x)(1-3x+2x^2)}, & r = 5, \\ \frac{x^6}{(1-x)(1-2x)(1-4x+3x^2)}, & r = 6, \\ \frac{x^7}{(1-x)(1-2x)(1-5x+2x^2+3x^3+x^4)}, & r = 7. \end{array} \right.$$

(52) $t_{2,s} = 0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383 \ldots$

(53) $t_{3,s} = 0, 0, 1, 4, 10, 21, 42, 84, 169, 340, 682, 1365, 2730, 5460, 10921 \ldots$

(54) $t_{4,s} = 0, 0, 0, 1, 5, 15, 35, 71, 135, 255, 495, 991, 2015, 4095, 8255, 16511 \ldots$

The exponents $\gamma^{(T)}_{r,s}$ of the inverse Euler transformation are defined as

$$1 - \frac{1}{(n-1)^r} = \prod_{j=1}^{\infty} \left( 1 - \frac{1}{n^j} \right)^{\gamma^{(T)}_{r,s,j}}, \quad T^{(r)}_k = \prod_{j \geq 2} \zeta_k^{\gamma^{(T)}_{r,s,j}}.$$

Examples at indices $j \geq 2$ are

(56) $\gamma^{(T)}_{2,j} = 1, 2, 3, 6, 9, 18, 30, 56, 99, 186, 335, 630, 1161, 2182, 4080, \ldots,$

(57) $\gamma^{(T)}_{3,j} = 0, 1, 3, 10, 18, 30, 56, 99, 186, 335, 630, 1161, 2182, 4080, \ldots,$

(58) $\gamma^{(T)}_{4,j} = 0, 0, 1, 4, 10, 20, 35, 60, 100, 180, 325, 620, 1160, 2200, 4110, \ldots,$

(59) $\gamma^{(T)}_{5,j} = 0, 0, 0, 1, 5, 15, 35, 70, 126, 215, 355, 605, 1065, 2002, 3855, \ldots.$
Table 4. Constants defined in (45).

| r | k | \( T_k^{(r)} \) |
|---|---|----------------|
| 2 | 1 | 0.660161815846869573927812110014555778432623360284734313194 ... |
| 2 | 2 | 0.80450826142474200347760975580484025884291623584411373425 ... |
| 2 | 3 | 0.95057282297004930144625580414573414733401747433181150 ... |
| 2 | 4 | 0.9892051323193015395807816797711957195936925403036958416322 ... |
| 3 | 1 | 0.8553921020033986085509619238173369142779617326433761891909 ... |
| 3 | 2 | 0.9507543513576159108562851848128299370568198257735656138328 ... |
| 3 | 3 | 0.995747203093411378754068730666734766129225361751865362705 ... |
| 3 | 4 | 0.99954971609067455096627510606570048701416230463141527581767 ... |
| 4 | 1 | 0.932472878050225154245474117724776691267802933315976922689 ... |
| 4 | 2 | 0.9856013155080764416552558374557003044523755820395912681851 ... |
| 4 | 3 | 0.99848471180207066093145392549433667590055380610960553390 ... |
| 4 | 4 | 0.9997512687670946226180604671247861013554958292485688974 ... |
| 5 | 1 | 0.9676641641449273298648548828455406702060373504935596363790 ... |
| 5 | 2 | 0.995514067093758631219139726981275567920402258163509688 ... |
| 5 | 3 | 0.9993294788727187501921136192631982385467156034977564808 ... |
| 5 | 4 | 0.999984095207649504306461010984203017854835139390019305 ... |

\[ \gamma_{2,j}^{(T)} \] and \( \gamma_{3,j}^{(T)} \) differ only by one at \( j = 2, 3 \) and 6, caused by (60).

Equivalent to (40) we have

\[ \frac{T_k^{(2)}}{T_k^{(3)}} = \frac{\zeta_k(6)}{\zeta_k(2)\zeta_k(3)}. \]

Equation (61) we have

\[ \frac{r}{j} \sum_{l \mid j} \mu(l) t_{r,j/l} = \gamma_{r,j}^{(T)} ; \quad t_{r,s} = \frac{1}{r} \sum_{l \mid s} t_{r,l}. \]

5. Quadratic Class Number

A sign flip in a denominator of (26) provides

**Definition 8.** (Quadratic Class numbers of order \( r \))

\[ Q^{(r)} \equiv \prod_{n=2}^{\infty} \left( 1 - \frac{1}{n^r(n+1)} \right) = \prod_{k=1}^{\infty} Q_k^{(r)} ; \quad Q_k^{(r)} \equiv \prod_{\Omega(n)=k}^{\infty} \left( 1 - \frac{1}{n^r(n+1)} \right). \]

The special value

\[ Q^{(1)} = -\frac{2 \sin(\pi \phi)}{\pi} = 2 A^{(1)} , \quad \phi = \frac{\sqrt{5} + 1}{2} , \]

is found with (45). Essentially duplicating the calculation in (27) we have

\[ \log Q^{(r)} = \sum_{n=2}^{\infty} \log \left( 1 - \frac{1}{n^r(n+1)} \right) = -\sum_{n=2}^{\infty} \sum_{s=1}^{\infty} \frac{1}{sn^s(n+1)^s} \]

\[ = \sum_{s=2}^{\infty} \sum_{j=1}^{\lfloor s/(1+r) \rfloor} \frac{(-1)^{s-rj-1}}{j} \left[ s - rj - 1 \right] \left[ 1 - \zeta(s) \right]. \]
Table 5. Constants defined in (62). Where the $k$-column is empty, the value is $Q^{(r)}$, else $Q_{k}^{(r)}$.

| $r$ | $k$ | $Q^{(r)}$, $Q_{k}^{(r)}$ |
|-----|-----|---------------------|
| 1   | 0.9933502694871820691403100404382857297296630838357895781633 | |
| 1   | 0.9744220096165927366033503266372101885643417098404944... | |
| 1   | 0.884490615792645569516256302139631981971519790670026883... | |
| 1   | 0.9647847436676197914413891183776203727210690154050544279... | |
| 1   | 0.990034423011674270451032420897059042914873419205301830... | |
| 2   | 0.86146502800903007271207874138497482256486581138963373814... | |
| 2   | 0.88151383972517077692839818229322784712986925720867633670... | |
| 2   | 0.980376289243855939864938619579944425382856840532312748237... | |
| 2   | 0.9972353990884350663951799297032224301445660441648743356... | |
| 2   | 0.99963476242161345108738573415958178438661947563013927774... | |
| 3   | 0.94358856975055819450398681669287975738498422884215535276... | |
| 3   | 0.9477326214367537935925137576418961303363163231741385282... | |
| 3   | 0.995928002823211103055653495384975519658530911655791754... | |
| 3   | 0.999714159404675699601581021439803474511163319960541... | |
| 3   | 0.999981370624803353143270623749614063667365657803958130... | |
| 4   | 0.9748949133590183455579606732396279842729547820066491224441... | |
| 4   | 0.9758241530476682416790114365947998319716474212920649014... | |
| 4   | 0.999800321328525656810967111070833869672365579632939061... | |
| 4   | 0.9999681720442870527054885740878252304234422728579249674... | |
| 4   | 0.999998498076734056792367569736016787447939575489593508... | |
| 5   | 0.9882966010836655618833547056524518156929576975297400237... | |
| 5   | 0.988504979741249608751106623851186664400985083275346188120... | |
| 5   | 0.9978348176640731283537327557402867111772830353712345808... | |
| 5   | 0.99999625085490627373262839133715607260086508728852979934... | |
| 5   | 0.9999993808696971405345913863150793233794604460092491194... | |

Table 5 is calculated via

$$\log Q_{k}^{(r)} = -\sum_{s=2}^{\infty} \frac{|s/(r+1)|}{s} \sum_{j=1}^{\infty} \frac{(-1)^{s-(r+1)p}}{j} \left( \frac{s-r-1}{j-1} \right) P_{k}(s) = -\sum_{s=2}^{\infty} \frac{1}{s} q_{r,s} P_{k}(s).$$

The previous line introduces auxiliary integer sequences $q_{r,s}$ equivalent to (60) with

**Definition 9.** (Binomial transforms of aerated alternating-1 sequences)

$$q_{r,s} \equiv s \sum_{j=1}^{\lfloor s/(r+1) \rfloor} \frac{(-1)^{j-s(r+1)}}{j} \left( \frac{s-jr-1}{j-1} \right).$$

Lists at $s \geq 2$ are

- $q_{1,s} = 2, -3, 6, -10, 17, -28, 46, -75, 122, -198, 321, -520, 842, -1363, 2206, -3570\ldots$
- $q_{2,s} = 0, 3, -4, 5, -3, 0, 4, -6, 5, 0, -7, 13, -14, 8, 4, -17, 24, -19, 1, 24, -44, 46, -23, -20\ldots$
- $q_{3,s} = 0, 0, 4, -5, 6, -7, 12, -18, 25, -33, 46, -65, 91, -125, 172, -238, 330, -456, 629\ldots$
- $q_{4,s} = 0, 0, 0, 5, -6, 7, -8, 9, -5, 0, 6, -13, 21, -25, 24, -17, 3, 19, -45, 70, -88, 92, -74, 30, \ldots$
Recurrences and generating function are sign-flipped variants of \( \text{(32)} \) and \( \text{(33)} \),
\[
q_{r,s} = 0, \quad s \leq 1; \quad q_{r,s} = -2q_{r,s-1} - q_{r,s-2} + q_{r,s-r-1} + q_{r,s-r-2}, \quad r \geq 1,
\]
\[
\sum_{s=0}^{\infty} q_{r,s} x^s = \frac{x^{1+r}(1+r+rx)}{(1+x)(1+x-x^{1+r})} = -r - \frac{1}{1+x} + \frac{1+r+rx}{1+x-x^{1+r}}.
\]

The zeta-expansion exponents \( \gamma_{\gamma_{r,j}}^{(Q)} \) are defined to satisfy
\[
1 - \frac{1}{n^{r}(1+n)} = \prod_{j=1}^{\infty} \left( 1 - \frac{1}{n^j} \right)^{\gamma_{\gamma_{r,j}}^{(Q)}} \quad \text{with} \quad Q = \sum_{s=1}^{\infty} \sum_{j=1}^{\infty} \mu(l) q_{r,j+l} = \gamma_{\gamma_{r,j}}^{(Q)}; \quad q_{r,s} = \sum_{l_s} \gamma_{\gamma_{r,j}}^{(Q)}.
\]

6. FELLER-TÖRNIER

**Definition 10.** (Feller-Tornier Constants)
\[
F^{(r)} = \prod_{n=2}^{\infty} \left( 1 - \frac{2}{n^r} \right) = \prod_{k=1}^{\infty} F_k^{(r)}; \quad F_k^{(r)} = \prod_{n=2}^{\infty} \left( 1 - \frac{2}{n^r} \right).
\]

Closed form expressions generated from the \( n \) roots of \( (n+1)^r - 2 \) via \( \text{(9)} \) are:
\[
F^{(2)} = -\frac{\sin(\sqrt{2}\pi)}{\sqrt{2}\pi},
\]
\[
F^{(4)} = -\frac{\sinh(\sqrt{2}\pi) \sin(\sqrt{2}\pi)}{\pi^2 \sqrt{2}},
\]
\[
F^{(6)} = -\frac{\sin(2^{1/6}) \left[ \cosh^2(2^{-5/6}\sqrt{3}) - \cos^2(2^{-5/6}) \right]}{\pi^3 \sqrt{2}}.
\]

Table \( \text{X} \) is produced accumulating
\[
\log F_k^{(r)} = \sum_{n=2, \Omega(n)=k} \log \left( 1 - \frac{2}{n^r} \right) = \sum_{j=1}^{\infty} \sum_{j=1}^{\infty} \frac{2j}{j} P_k(rj).
\]

The zeta-expansion exponents \( \gamma_{\gamma_{r,j}}^{(F)} \) are
\[
1 - \frac{2}{n^r} = \prod_{j=1}^{\infty} \left( 1 - \frac{1}{n^j} \right)^{\gamma_{\gamma_{r,j}}^{(F)}} \quad \text{with} \quad F_k^{(r)} = \prod_{j} \zeta(j)^{-\gamma_{\gamma_{r,j}}^{(F)}}.
Table 6. Constants defined in (75). Where the \(k\)-column is empty, the value is \(F(r)\), else \(F_k(r)\).

| \(r\) | \(k\) | \(F(r)\) | \(F_k(r)\) |
|---|---|---|---|
| 2 | 1 | 0.2169542943774633693686403963437596599913299714241452950 \ldots |
| 2 | 2 | 0.3226409893924467057953169254823706657095057965832709961 \ldots |
| 2 | 3 | 0.746598592930283504509198848531702307441861924359721333 \ldots |
| 2 | 4 | 0.925267218004050491882121205770653605835184213438593596030 \ldots |
| 2 | 4 | 0.98014142381968561660209779704982568698612662988320829609 \ldots |
| 3 | 1 | 0.67689273700988819361023732672438921279767839745978845273 \ldots |
| 3 | 2 | 0.95281131498858970286646083562413457907297149293392766096 \ldots |
| 3 | 3 | 0.993911463538908983115880607314708743157809479259098501054 \ldots |
| 3 | 4 | 0.99923235438433096529327826907215356983612425541770296004 \ldots |
| 4 | 1 | 0.8497329913847187662650537036291604389282010424286164649 \ldots |
| 4 | 2 | 0.99902875865057552192815157916326949139155672507312329848 \ldots |
| 4 | 3 | 0.9993712185968720447212785129866869154145994816088393076 \ldots |
| 4 | 4 | 0.9999606410241055795100062330383249199360879799036126331 \ldots |
| 5 | 1 | 0.92688085771065686532556950479909232524104194213179034 \ldots |
| 5 | 2 | 0.9200591929568505755219281515791632694913915567250731232984 \ldots |
| 5 | 3 | 0.99772614956085550822520064832476818851830182381404991433 \ldots |
| 5 | 4 | 0.99998846129017559729606214395367084797186701036126206135 \ldots |

(79) in conjunction with (12) reveals

\[
\gamma^{(F)}_{r,j} = \begin{cases} 
0, & r \nmid j, \\
\frac{1}{r} \sum_{d \mid (j/r)} 2^d \mu \left( \frac{j}{rd} \right), & r \nmid j,
\end{cases}
\]

so values in the range \(j \geq 2\) start as

\[
\gamma^{(F)}_{2,j} = 2, 0, 1, 0, 2, 0, 3, 0, 6, 0, 9, 0, 18, 0, 30, 0, 56, 0, 99, 0, 186, 0, 335, \ldots
\]

\[
\gamma^{(F)}_{3,j} = 0, 2, 0, 0, 1, 0, 2, 0, 3, 0, 6, 0, 9, 0, 18, 0, 30, 0, 56, 0, \ldots
\]

The others are obvious: the count of zero fillers—factors effectively dropping out in (80)—grows simply as \(r - 1\).

7. Hardy-Littlewood

The Hardy-Littlewood constants, in the narrow sense, are the cases \(k = 1\) in
Definition 11.

\[(84)\]  
\[C^{(3)} = \prod_{n=4}^{\infty} \left( 1 - \frac{3n - 1}{(n-1)^3} \right) = \frac{2}{9}; \quad C_k^{(3)} = \prod_{n \geq 4, \Omega(n) = k} \left( 1 - \frac{3n - 1}{(n-1)^3} \right);\]

\[(85)\]  
\[C^{(4)} = \prod_{n=5}^{\infty} \left( 1 - \frac{6n^2 - 4n + 1}{(n-1)^4} \right) = \frac{3}{32}; \quad C_k^{(4)} = \prod_{n \geq 5, \Omega(n) = k} \left( 1 - \frac{6n^2 - 4n + 1}{(n-1)^4} \right);\]

\[(86)\]  
\[C^{(r)} = \prod_{n>r} \frac{n^{r-1}(n-r)}{(n-1)^r} = \frac{(r-1)!}{r^{r-1}}; \quad C_k^{(r)} = \prod_{n>r, \Omega(n) = k} \frac{n^{r-1}(n-r)}{(n-1)^r}, \quad r \geq 3.\]

\[T^{(2)} = C^{(2)}\] might be linked in here. As before, we turn to the logarithm and define associate integer expansion coefficients \(c_s^{(r)}\) from their Laurent expansion,

\[(87)\]  
\[\log C_k^{(r)} = \sum_{n>r, \Omega(n) = k} \log \frac{n^{r-1}(n-r)}{(n-1)^r} = \sum_{n>r, \Omega(n) = k} \log \frac{1 - r/n}{(1 - 1/n)^r} = \sum_{n>r, \Omega(n) = k} \left[ \log \left( 1 - \frac{r}{n} \right) - r \log \left( 1 - \frac{1}{n} \right) \right] = \sum_{n>r, \Omega(n) = k} \left[ -\sum_{s \geq 1} \frac{r^s}{s} + \sum_{s \geq 1} \frac{1}{sn^s} \right],\]

which can be summarized after interchange of summations as

\[(88)\]  
\[\log C_k^{(r)} = -\sum_{n>r, \Omega(n) = k} \sum_{s=2}^{\infty} \frac{1}{s} \frac{c_s^{(r)}}{n^s}.\]

in a simple common format:

\[(89)\]  
\[c_s^{(r)} = \begin{cases} 
0, & s < 2; \\
r^s - r, & s \geq 2.
\end{cases}\]

Plugging \(\Box\) into the right hand side of \(88\), we compensate for the fact that the lower limits \(n > r\) discard some \(n\) for the prime \((k = 1)\) and semiprime \((k = 2)\) cases:

\[(90)\]  
\[\log C_k^{(r)} = -\sum_{s \geq 2} \frac{1}{s} c_s^{(r)} \times \left\{ \begin{array}{ll}
P_k(s) - \frac{1}{s} & , \quad r = 5, 6, \quad k = 1; \\
P_k(s) - \frac{1}{s} & , \quad r = 3, 4, \quad k = 1; \\
P_k(s) - \frac{1}{s} & , \quad r = 6, 7, 8, \quad k = 2; \\
P_k(s) - \frac{1}{s} & , \quad r = 4, 5, \quad k = 2; \\
0 & , \quad r = 3, \quad k = 2; \\
P_k(s) & , \quad 3 \leq r < 8, \quad k = 3.
\end{array} \right\}
\]

Remark 3. The convergence of these series is slow, given that \(c_s^{(r)}\) grows roughly \(\sim r^s\) and the terms right from the brace fall roughly \(\sim r^{-s}\). The standard acceleration technique is to split the products \(86\) into \(\prod_{n>r} = \prod_{r \leq n \leq M} \prod_{n>M}\) with some free integer \(M\) of the order of some tenths of \(r\), to calculate the first of these two products explicitly, and to subtract all of the inverse powers of the \(k\)-almost primes below \(M\) on the right hand side of \(90\), so the term to the right of the brace falls off roughly \(\sim M^{-s}\) \([13]\).
Table 7. Hardy-Littlewood constants from equations \([84]–[86]\).

| \(k\) | \(C_k^{(3)}\) | \(C_k^{(4)}\) | \(C_k^{(5)}\) |
|---|---|---|---|
| 1 | 0.635166354604271207206696591272522417342065687332372450899\ldots | 0.3074948787583270931233544860710768530221785199950663928298\ldots | 0.186614297358358396659248794188337840073944945698307980114\ldots |
| 2 | 0.424234558470737325218539671836177441479432573726566654172\ldots | 0.4616917583647737302320552441835623105041873484187592372\ldots | 0.29804202048775428882828545138051778848665100325386176\ldots |
| 3 | 0.8619782171154063977603892881783630108022267215309958070\ldots | 0.723165327592227885742644081506537901793428021559355399358\ldots | 0.547976628430869696044385054920027204371593584787789976\ldots |
| 4 | 0.967013385529829002970609867736752117366812663687575395\ldots | 0.933085922756286271428677500179124619333526568552797663388\ldots | 0.8874238454202366623916084035101114754091961065986923636\ldots |
| 5 | 0.99198654248377761368258943704065646126247353044705005169\ldots | 0.98381478527489679025462251437698529467124102983780408\ldots | 0.97278732807392409260448518740376836949079797225729870114\ldots |

Exponents \(\gamma^{(C)}_{r,j}\) are defined with the aim to decompose \([80]\):

\[
\frac{n^{r-1}(n-r)}{(n-1)^r} = \prod_{j \geq 1} \left(1 - \frac{1}{n^j}\right)^{\gamma^{(C)}_{r,j}},
\]

related to \(c_s^{(r)}\) via a Möbius transform as described in \([44]\, [13]\):

\[
\gamma_{3,j} = 3, 8, 18, 48, 116, 312, 810, 2184, 5880, 16104, 44220, 122640, 341484, \ldots,
\gamma_{4,j} = 6, 20, 60, 204, 670, 2340, 8160, 29120, 104754, 381300, 1397740, 5162220, \ldots,
\gamma_{5,j} = 10, 40, 150, 624, 2580, 11160, 48750, 217000, 976248, 4438920, 20343700, \ldots
\]

Eq. \([91]\) rephrases the constants \([84]–[86]\) as

\[
C_k^{(r)} = \prod_{j \geq 2} \lambda_k(r, j)^{-\gamma^{(C)}_{r,j}},
\]
where \( \lambda_k \) takes into account that the products may have been defined without the first one to three primes or semiprimes,

\[
\lambda_k(r, j) = \prod_{\Omega(n)=k, n \geq r} \frac{1}{1-1/n^j} = \zeta_k(j) \times \begin{cases} 
(1 - \frac{1}{2^r})(1 - \frac{1}{3^r})(1 - \frac{1}{5^r}), & r = 5, k = 1; \\
(1 - \frac{1}{3^r})(1 - \frac{1}{3^r}), & r = 3, 4, k = 1; \\
(1 - \frac{1}{2^r}), & r = 4, 5, k = 2; \\
1, & r = 3, k = 2; \\
1, & r = 3, 4, 5, k \geq 3.
\end{cases}
\]

8. Summary

The familiar Hardy-Littlewood, Artin’s, Feller-Tornier and similar constants are infinite products over the prime numbers. We have generalized these to products over \( k \)-almost primes, and provide tables for low ranks and small \( k \). The products over all values of \( k \) are infinite “host” products, easily evaluated as a multi-gamma functions associated with the roots of the defining rational polynomial.

**Appendix A. Hybrids**

Hybrids are products and ratios of the constants discussed above, which provide access to other forms of products. One example of such a reduction is, see (18),

\[
\prod_{n \geq 2, \Omega(n)=k} \left( 1 - \frac{3n^s+2}{n^{2s}} \right) = \prod_{n \geq 2, \Omega(n)=k} \left( 1 - \frac{2}{n^s} \right) \left( 1 + \frac{1}{n^s} \right) = F_k^r \left( \zeta_k(s) \middle\frac{2s}{\zeta_k(2s)} \right).
\]

An—obviously incomplete—sample of these is:

\[
\prod_{n \geq 2, \Omega(n)=k} \left( 1 + \frac{2n^s - 1}{(n^s - 1)^2} \right) = \zeta_k^2(s),
\]

\[
\prod_{n \geq 2, \Omega(n)=k} \left( 1 + \frac{n^{s+l} + n^s - 1}{n^{2s+l} - n^{s+l} - n^s + 1} \right) = \zeta_k(s+l)\zeta_k(s),
\]

\[
\prod_{n \geq 2, \Omega(n)=k} \left( 1 - \frac{1}{n^s + n^{s-1} + \cdots + n^2 + n + 1} \right) = \zeta_k(s+1) \frac{\zeta_k(s)}{\zeta_k(s)},
\]

\[
\prod_{n \geq 2, \Omega(n)=k} \left( 1 - \frac{n^l - 1}{n^{s+l} - 1} \right) = \zeta_k(s+l) \frac{\zeta_k(s)}{\zeta_k(s)},
\]

\[
\prod_{n \geq 2, \Omega(n)=k} \left( 1 - \frac{2n^s - 1}{n^{2s}} \right) = \frac{1}{\zeta_k^2(s)}, \quad \prod_{n \geq 2, \Omega(n)=k} \left( 1 - \frac{n^{s+l} + n^s - 1}{n^{2s+l}} \right) = \frac{1}{\zeta_k(s+l)\zeta_k(s)},
\]

\[
\prod_{n \geq 2, \Omega(n)=k} \left( 1 + \frac{1}{n(n^{s-1} + n^{s-2} + \cdots + n + 1)} \right) = \frac{\zeta_k(s)}{\zeta_k(s+1)},
\]

\[
\prod_{n \geq 2, \Omega(n)=k} \left( 1 - \frac{1}{n^s + 1} \right) = \frac{\zeta_k(2s)}{\zeta_k(s)}.
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 - \frac{n^{(l-1)s} - 1}{n^s - 1}\right) = \frac{\zeta_k(ls)}{\zeta_k(s)},
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 + \frac{1}{n^s(n-1) - 1}\right) = \frac{1}{A^{(s)}_k},
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 - \frac{1}{n^{s+2} - n^{s+1} - n + 1}\right) = \zeta_k(s + 1)A^{(s)}_k,
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 + \frac{2n + 1}{n^3 - 2n - 1}\right) = \frac{\zeta_k(2)}{A^{(1)}_k},
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 + \frac{1}{n^{s+1} - n^s - 1}\right) = \frac{1}{\zeta_k(s + 1)A^{(s)}_k},
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 - \frac{2n + 1}{n^s}\right) = \frac{A^{(1)}_k}{\zeta_k(2)}, \quad \prod_{n \geq 2, \Omega(n) = k} \left(1 - \frac{2n^s + n^{s-1} + n^{s-2} + \cdots + n + 1}{n^{2s+1}}\right) = \frac{A^{(s)}_k}{\zeta_k(s + 1)},
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 + \frac{n - 2}{n^{s+1} - n^s - n + 1}\right) = \zeta_k(s)A^{(s)}_k,
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 - \frac{n - 2}{n^{s+1} - n^s - 1}\right) = \frac{1}{\zeta_k(s)A^{(s)}_k},
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 - \frac{n^s + n^{s-1} + \cdots + n + 1}{n^{2s}}\right) = \frac{A^{(s)}_k}{\zeta_k(s)},
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 - \frac{n^{s-1} + n^{s-2} + \cdots + n^2 + 2n + 1}{n^{s+1}}\right) = \frac{A^{(1)}_k}{\zeta_k(s)},
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 + \frac{1}{n^s(n+1) - 1}\right) = \frac{1}{Q^{(s)}_k},
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 + \frac{1}{n^{s+2} + n^{s+1} - n - 1}\right) = \zeta_k(s + 1)Q^{(s)}_k,
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 + \frac{2n + 1}{n^3 - 2n + 1}\right) = \frac{\zeta_k(2)}{Q^{(1)}_k},
\]

\[
\prod_{n \geq 2, \Omega(n) = k} \left(1 - \frac{1}{n(n^{s+1} + n^s - 1)}\right) = \frac{1}{\zeta_k(s + 1)Q^{(s)}_k},
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{2n - 1}{n^2} \right) = \frac{Q_k^{(1)}}{\zeta_k(2)}, \quad \prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{2n^{s+1} + n^s - 1}{n^{2s+1}(n + 1)} \right) = \frac{Q_k^{(s)}}{\zeta_k(s + 1)},
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{n}{n^{s+1} + n^s - n - 1} \right) = \zeta_k(s)Q_k^{(s)},
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{n}{n^{s+1} + n^s - 1} \right) = \frac{1}{\zeta_k(s)Q_k^{(s)}},
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{n^2 + n - 1}{n^4} \right) = \frac{Q_k^{(2)}}{\zeta_k(2)}, \quad \prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{n^{s+1} + 2n^s - 1}{n^{2s}(n + 1)} \right) = \frac{Q_k^{(s)}}{\zeta_k(s)},
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{2}{n^s - 2} \right) = \frac{1}{F_k^{(s)}},
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{1}{n^s - 1} \right) = \zeta_k(s)F_k^{(s)}, \quad \prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{2n^l - 1}{n^{s+l} - 1} \right) = \zeta_k(s+l)F_k^{(s)},
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{1}{n^s - 2} \right) = \frac{1}{\zeta_k(s)F_k^{(s)}}, \quad \prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{2n^l - 1}{n^{l(s)} - 2} \right) = \frac{1}{\zeta_k(s+l)F_k^{(s)}},
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{3n^s - 2}{n^{2s}} \right) = \frac{F_k^{(s)}}{\zeta_k(s)},
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{n^l - 1}{n^{l(s+1)} - n^s - 1} \right) = \frac{A_k^{(s+l)}}{A_k^{(s)}},
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{n^l - 1}{n^{s+l} + n^s + l - 1} \right) = \frac{A_k^{(s)}}{A_k^{(s+l)}},
\]
\[
\prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{2n^{s+1} - 1}{n^{2s}(n^2 - 1)} \right) = \frac{A_k^{(s)}}{Q_k^{(s)}},
\]

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(129) \[ \prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{2}{n^{s+2} - n^s - n + 1} \right) = \frac{A_k^{(s)}}{Q_k^{(s)}}, \]

(130) \[ \prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{2}{n^{s+2} - n^s - n - 1} \right) = \frac{Q_k^{(s)}}{A_k^{(s)}}, \]

(131) \[ \prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{2n^{s+1} - n^s - 2}{n^{2s}(n - 1)} \right) = A_k^{(s)} F_k^{(s)}, \]

(132) \[ \prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{3n^{s+1} - 2n^s - 2}{n^{2s+1}(n - 1)} \right) = A_k^{(s)} F_k^{(s+1)}, \]

(133) \[ \prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{2n - 3}{n^{s+1} - n^s - 2n + 2} \right) = \frac{A_k^{(s)}}{F_k^{(s)}}, \]

(134) \[ \prod_{n \geq 2, \Omega(n) = k} \left( 1 + \frac{n + 2}{n^{s+2} + n^{s+1} - 2n - 2} \right) = \frac{Q_k^{(s)}}{F_k^{(s+1)}}, \]

(135) \[ \prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{n + 2}{n(n^{s+1} + n^s - 1)} \right) = \frac{F_k^{(s+1)}}{Q_k^{(s)}}, \]

(136) \[ \prod_{n \geq 2, \Omega(n) = k} \left( 1 - \frac{2n^l + 2}{n^{s+l} - 2} \right) = \frac{F_k^{(s)}}{F_k^{(s+l)}}. \]

Examples which involve \( C_k^{(r)} \) have been left out for aesthetic reasons, as the dependence of the lower limit \( n \) on \( r \) in (86) leads to convoluted subcase notation, like

(137) \[ \prod_{n \geq 5, \Omega(n) = k} \left( 1 + \frac{3}{n(n - 4)} \right) = \begin{cases} \frac{(27/16)C_k^{(3)}}{C_k^{(3)}}, & k = 2, \\ \frac{C_k^{(3)}}{C_k^{(3)}}, & k \neq 2, \end{cases} \]

which notices that the semiprime \( n = 4 \) contributes to \( C_k^{(3)} \) with a factor 16/27 if \( k = 2 \), but not otherwise.

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URL: http://www.strw.leidenuniv.nl/~mathar
E-mail address: mathar@strw.leidenuniv.nl

Leiden Observatory, Leiden University, P.O. Box 9513, 2300 RA Leiden, The Netherlands