A theorem for the closed–form evaluation of the first generalized
Stieltjes constant at rational arguments

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Abstract

The Stieltjes constants, also known as generalized Euler’s constants, are of fundamental and long–
standing importance in modern analysis, number theory, theory of special functions and other disci-
plines. Recently , it was conjectured that the first generalized Stieltjes constant at rational argument
γ1(k/n), where k and n are positive integers such that k < n, may be always expressed by means of
the Euler’s constant γ, the first Stieltjes constant γ1, the logarithm of the Γ–function at rational argu-
ment(s) and some relatively simple, perhaps even elementary, function. This conjecture was based
on the evaluation of γ1(1/2), γ1(1/3), γ1(2/3), γ1(1/4), γ1(3/4), γ1(1/6), γ1(5/6), which could be
expressed in this way. In this short article we complete this previous study by deriving an elegant
theorem which allows to evaluate the first generalized Stieltjes constant at any rational argument.
Besides, several related summation formulæ involving the first generalized Stieltjes constant and the
Digamma function are also derived and discussed. Finally , it is also shown that similar theorems
may be also derived for higher Stieltjes constants; in particular, for the second Stieltjes constant this
theorem is provided in an explicite form.

Key words: Stieltjes constants, generalized Euler’s constants, Special constants, Number theory, Zeta
function, Gamma function, Rational arguments, logarithmic integrals, Malmsten’s integrals,
logarithmic series, theory of functions of a complex variable, orthogonal expansions.

I. Introduction and notations

I.1. Introduction

The ζ–functions are one of more important special functions in modern analysis and theory of
functions. The most known and frequently encountered of ζ–functions are Riemann and Hurwitz
ζ–functions. They are classically introduced as following series

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \zeta(s, v) = \sum_{n=0}^{\infty} \frac{1}{(n + v)^s}, \quad v \neq 0, -1, -2, \ldots \]

convergent for \( \mathrm{Re} \, s > 1 \), and may be extended to other domains of \( s \) by the principle of analytic
continuation. It is well known that \( \zeta(s) \) and \( \zeta(s, v) \) are meromorphic on the entire complex \( s \)–plane

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and that their only pole is a simple pole at $s = 1$ with residue 1. They can be, therefore, expanded in the Laurent series in a neighbourhood of $s = 1$ in the following way
\[
\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n (s-1)^n}{n!} \gamma_n, \quad s \neq 1.
\] (1)

and
\[
\zeta(s, v) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n (s-1)^n}{n!} \gamma_n(v), \quad s \neq 1.
\] (2)

respectively. Coefficients $\gamma_n$ appearing in the regular part of expansion (1) are called Stieltjes constants, while those appearing in the regular part of (2), $\gamma_n(v)$, are called generalized Stieltjes constants. It is obvious that $\gamma_n(1) = \gamma_n$ because $\zeta(s, 1) = \zeta(s)$.

The study of these coefficients is an interesting subject and may be traced back to the works of Thomas Stieltjes and Charles Hermite [1, vol. I, letter 71 and following]. In 1885, first Stieltjes, and then Hermite, proved that
\[
\gamma_n = \lim_{m \to \infty} \left\{ \frac{1}{m} \sum_{k=1}^{m} \frac{\ln^nk}{k} - \frac{\ln^{n+1}m}{n+1} \right\}, \quad n = 0, 1, 2, \ldots
\] (3)

Later, this formula was also obtained or simply stated in works of Johan Jensen [38], [40], Jørgen Gram [28], Godfrey Hardy [32], Srinivasa Ramanujan [2] and many others. From (3), it is visible that $\gamma_0$ is the Euler’s constant $\gamma$. However, the study of other Stieltjes constants revealed to be much more difficult and, at the same time, interesting. In 1895, Franel [25], by using contour integration techniques, first, rediscovered the first Jensen’s formula for $\zeta(s)$ [the first expression in (51)], and then, showed that
\[
\gamma_n = \frac{1}{2} \delta_{n,0} + \frac{1}{i} \int_{0}^{\infty} \frac{dx}{e^{2\pi x} - 1} \left\{ \frac{\ln^n(1 - ix)}{1 - ix} - \frac{\ln^n(1 + ix)}{1 + ix} \right\}, \quad n = 0, 1, 2, \ldots
\] (4)

The same integral formula was rediscovered in 1985 by Ainsworth and Howell who also provided a very detailed proof of it [6]. By the same line of reasoning, one can also deduce several similar formulæ
\[
\gamma_n = -\frac{\pi}{2(n+1)} \int_{-\infty}^{+\infty} \frac{\ln^{n+1}(1 + ix)}{\cosh^2\pi x} \, dx \quad n = 0, 1, 2, \ldots
\] (5)

\footnote{See the related priority dispute between Jensen, Kluyver and Franel [25], [40], [39]. Besides, we corrected the Franel’s formula from [25] which was not valid for $n = 0$ [this correction comes from (8) and (9) here after].}

\footnote{The proof is analogous to that given for the formula (8) here after, except that the Hermite representation should be replaced by the third and second Jensen’s formulæ for $\zeta(s)$ (51) respectively.}
and

\[
\begin{align*}
\gamma_1 &= -\left[\gamma - \frac{\ln 2}{2}\right] \ln 2 + i \int_0^\infty \frac{dx}{e^{\pi x} + 1} \left\{ \frac{\ln(1 - ix)}{1 - ix} - \frac{\ln(1 + ix)}{1 + ix} \right\} \\
\gamma_2 &= -\left[2\gamma_1 + \gamma \ln 2 - \frac{\ln^2 2}{3}\right] \ln 2 + i \int_0^\infty \frac{dx}{e^{\pi x} + 1} \left\{ \frac{\ln^2(1 - ix)}{1 - ix} - \frac{\ln^2(1 + ix)}{1 + ix} \right\} \\
\gamma_3 &= -\left[3\gamma_2 + 3\gamma_1 \ln 2 + \gamma \ln^2 2 - \frac{\ln^3 2}{4}\right] \ln 2 + i \int_0^\infty \frac{dx}{e^{\pi x} + 1} \left\{ \frac{\ln^3(1 - ix)}{1 - ix} - \frac{\ln^3(1 + ix)}{1 + ix} \right\}
\end{align*}
\]

first of which is particularly simple. Other important results concerning the Stieltjes constants were obtained by studying series implying integer part of functions. In 1910, Giovanni Vacca discovered a new series for the Euler’s constant

\[
\gamma = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \lfloor \log_2 k \rfloor
\]

for which he gave a simple and elegant geometrical proof [61]. Glaisher, being puzzled by such an unusual demonstration, proposed an arithmetical proof of the same result [27]. Two years later, Hardy, by another analytical method, not only reproved the Vacca’s result (6), but also extended it to the first Stieltjes constant

\[
\gamma_1 = \frac{\ln 2}{2} \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \lfloor \log_2 k \rfloor \cdot (2 \log_2 k - \lfloor \log_2 2k \rfloor)
\]

see [32]. Similar expressions for higher Stieltjes constants were later given by Klu¨yver [41] (Hardy, however, already mentioned this possibility in [32]). Apart from \( \gamma_0 \), no closed–form expressions are known for \( \gamma_n \); however, there are works devoted to their estimations [43], [9], [45], [37], and to the behaviour of their sign [11], [53]. In particular, Briggs in 1955 [11] demonstrated that there are infinitely many changes of sign for them (later, Mitrovi´c [53] extended some results of Briggs). Gram [28], Liang & Todd [47], Ainsworth & Howell [6] and Kreminski [42] discussed some aspects related to the numerical computation of Stieltjes constants.\(^3\)

As regards generalized Stieltjes constants, they are much less studied than the usual Stieltjes constants. In 1927, Wilton [64], showed that they can be given by an asymptotic representation of the same kind as (3)

\[
\gamma_n(v) = \lim_{m \to \infty} \left\{ \sum_{k=1}^{m} \frac{\ln^n(k + v)}{k} - \frac{\ln^{n+1}(m + v)}{n + 1} \right\}, \quad n = 0, 1, 2, \ldots \quad v \neq 0, -1, -2, \ldots
\]

which was also derived by Berndt in 1972, Berndt, being inspired by the similar Lammel’s proof for the Riemann \( \zeta \)–function [43], also proved this formula in [9]. Similarly to the Franel’s method of the

\(^3\)In the latter work, author also considered aspects related to the generalized Stieltjes constants.
derivation of (4), one may derive the following integral representation for the \( n \)th generalized Stieltjes constant

\[
\gamma_n(v) = \left[ \frac{1}{2v} - \frac{\ln v}{n+1} \right] \ln^n v - i \int_0^\infty \frac{dx}{e^{2\pi x} - 1} \left\{ \frac{\ln^p(v - ix)}{v - ix} - \frac{\ln^p(v + ix)}{v + ix} \right\}, \quad n = 0, 1, 2, \ldots
\]

(8)

Re \( v > 0 \).\(^4\) A variant of this formula was also obtained by Mark Coffey \[15\], \[19\] (by the way, he also gave an equivalent of the formula (5) for the generalized Stieltjes constant). From both latter formulæ, it follows that \( \gamma_0(v) = -\Psi(v) \). Take, for instance, (8) and put \( n = 0 \). Then, the latter equation takes the form

\[
\gamma_0(v) = \frac{1}{2v} - \ln v + 2 \int_0^\infty \frac{x dx}{(e^{2\pi x} - 1)(v^2 + x^2)} = -\Psi(v), \quad n = 0, 1, 2, \ldots
\]

(9)

where the last integral was first calculated by Legendre\(^5\). The demonstration of the same result from formula (7) may be found, for example, in \[54\]. For rational \( v \), the 0th Stieltjes constant may be, therefore, expressed by means of the Euler’s constant \( \gamma \) and a finite combination of elementary functions (thanks to the Gauss’ Digamma theorem \[26\], \[8, vol. I, p. 19, §1.7.3\]). However, things are much more complicated for higher generalized Stieltjes constants; currently, no closed–form expressions are known for them and little is known as to their arithmetical properties. Basic properties, such as the multiplication theorem

\[
\sum_{l=0}^{n-1} \gamma_p \left( v + \frac{l}{n} \right) = (-1)^p n \left[ \frac{\ln n}{p+1} - \Psi(nv) \right] \ln^p n + n \sum_{r=0}^{p-1} (-1)^r C_p^r \gamma_{p-r}(nv) \cdot \ln^r n,
\]

\( n = 2, 3, 4, \ldots \), where \( C_p^r \) denotes the binomial coefficient \( C_p^r = \frac{p!}{r!(p-r)!} \), and the recurrent relationship

\[
\gamma_p(v + 1) = \gamma_p(v) - \frac{\ln^p v}{v}, \quad p = 1, 2, 3, \ldots \quad v \neq 0, -1, -2, \ldots
\]

(10)

may be both straightforwardly derived from those for the Hurwitz \( \zeta \)–function, see e.g. \[10, exercise n^6 64, Eq. (63)–(65)\].\(^6\) In attempt to obtain other properties, several summation relations involving single and double infinite series were quite recently obtained in \[13\], \[14\]. Also, many important aspects regarding the Stieltjes constants were considered by Donal Conon in works \[19\], \[18\], \[17\].

Let now focus our attention on the first generalized Stieltjes constant. The most strong and pertinent results in the field of its closed–form evaluation is the formula for the difference between the

\[^4\text{Proof.} \text{Consider the well-known Hermite representation for } \zeta(s,v), \text{ see} \ [35, p. 66], [48, p. 106], [8, vol. I, p. 26, Eq. 1.10(7)]. \text{Recall first that } 2(v^2 + x^2)^{-s/2} \sin[s \arctg(x/v)] = -i(s - vix)^{-s} - (v + ix)^{-s}, \text{ and then, expand } \frac{1}{v^{s+1}} + (s - 1)^{-1}v^{1-s} \text{ into the Laurent series about } s = 1. \text{ Performing the term–by–term comparison of the derived expansion with the Laurent series (2) yields (8). QED}

\[^5\text{And not by Binet as stated in} \ [8, vol. I, p. 18, Eq. 1.7.2(27)], \text{ see} \ [59, vol. II, p. 190] \text{ and} \ [10, exercise n^4 40, eq. (55)].

\[^6\text{As regards the multiplication theorem, see e.g.} \ [18, Eq. (6.6)] \text{ or} \ [10, exercise n^6 64]. \text{ We can also find its particular case for } v = 1/n \text{ in} \ [16, p. 1830, eq. (3.28)].}
first generalized Stieltjes constant at rational argument and its reflected version

$$\gamma_1 \left( \frac{m}{n} \right) - \gamma_1 \left( 1 - \frac{m}{n} \right) = 2\pi \sum_{l=1}^{n-1} \sin \frac{2\pi ml}{n} \cdot \ln \Gamma \left( \frac{l}{n} \right) - \pi \left( \gamma + \ln 2\pi n \right) \csc \frac{m\pi}{n}$$  \hspace{1cm} (11)

In the literature devoted to Stieltjes constants this result is usually attributed to Almkvist and Meurman who obtained it by deriving the functional equation for $\zeta(s,v)$, equation (25), with respect to $s$ at rational $v$, see e.g. [4], [7, p. 261], [12,9], [52, eq. (6)]. However, it was comparatively recently that we discovered that this formula, albeit in a slightly different form, was obtained by Carl Malmsten already in 1846 [50, p. 20 & 38]. In particular, he showed that

$$\sum_{l=0}^{\infty} \left\{ \ln \left( \frac{(2l+1)n - m}{2l+1n + m} \right) - \ln \left( \frac{(2l+1)n + m}{2l+1n - m} \right) \right\} =$$

$$= \begin{cases} -\frac{\pi(\gamma + \ln \pi)}{2n} \tan \frac{\pi m}{2n} - \frac{\pi}{n} \sum_{l=1}^{n-1} (-1)^{l-1} \sin \frac{\pi ml}{n} \cdot \ln \left\{ \frac{\Gamma \left( \frac{n + l}{2n} \right)}{\Gamma \left( \frac{n - l}{2n} \right)} \right\}, & \text{if } m + n \text{ is odd,} \\
-\frac{\pi(\gamma + \ln \pi)}{2n} \tan \frac{\pi m}{2n} - \frac{\pi}{n} \sum_{l=1}^{n-1} (-1)^{l-1} \sin \frac{\pi ml}{n} \cdot \ln \left\{ \frac{\Gamma \left( \frac{n - l}{n} \right)}{\Gamma \left( \frac{n + l}{n} \right)} \right\}, & \text{if } m + n \text{ is even,} \end{cases}$$

where $m$ and $n$ are integers such that $m < n$, see [50, p. 20, Eq. (55)]. It is visible that the left part of this equality contains the difference of two first–order derivatives of $\zeta(s,v)$ at $s \rightarrow 1$ and $v = \frac{1}{2} + \frac{m}{2n}$. Putting $2m - n$ instead of $m$ and using the Laurent series expansion (2) yields, after some simplifications, formula (11). A somewhat different and more simple way to get (11) is to directly apply the Mittag–Leffler theorem to one of the Malmsten’s integrals at rational points; we developed such a method in our preceding study [10, exercises n° 63 and 67].

Recently, Coffey derived several interesting representations for the generalized Stieltjes constants and for their differences [16]. From one of these representations, one may conjecture that in some cases (author gave only two examples of such cases [16, p. 1821, Eqs. (3.33)–(3.34)]), not only the $\Gamma$–function at rational argument (which is more or less predictable from the preceding formula), but also the second–order derivative of the Hurwitz $\zeta$–function could be related, in some way, to the first generalized Stieltjes constant. However, these preliminary findings do not permit to precisely identify their roles in the general problem of the closed–form evaluation of the first Stieltjes constant at any rational argument (the problem which we come to solve here).

Very recently, it has been conjectured that similarly to the Digamma theorem for $\gamma_0(v)$, the first generalized Stieltjes constant $\gamma_1(v)$ at rational $v$ may be expressed by means of the Euler’s constant $\gamma$, the first Stieltjes constant $\gamma_1$, the logarithm of the $\Gamma$–function and some “relatively simple” function, that is to say

$$\gamma_1 \left( \frac{r}{m} \right) = f(l, m, \gamma) + \sum_{l=1}^{m-1} \alpha_l(r, m) \cdot \ln \Gamma \left( \frac{l}{m} \right) + \gamma_1, \hspace{1cm} r = 1, 2, \ldots , m - 1,$$  \hspace{1cm} (12)

\text{Unfortunately, this Malmsten’s work contains a huge quantity of misprints in formulae. We already corrected many of them in our preceding work [10, Section 2.1]. As regards the above–referenced Malmsten’s original equation (55), case $m + n$ even, note that } \Gamma \left( \frac{n+1}{2} \right) \text{ should be replaced by } \Gamma \left( \frac{n+1}{2} \right). \text{ Formula (56) also has an error: } \Gamma \left( \frac{n+1}{2} \right) \text{ should be replaced by } \Gamma \left( \frac{n+1}{2} \right).
where \( a_l(r, m) \) are real coefficients and \( f(l, m, \gamma) \) is some “relatively simple” function, see [10, exercise no 64]. For seven rational values of \( v \) in the range \((0, 1)\), namely for \( \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4} \) and \( \frac{5}{6} \), it has been shown in [10, exercise no 64] that this “relatively simple” function is elementary. In this short manuscript, we extend these precedent researches by providing a theorem which allows to evaluate the first generalized Stieltjes constant at any rational argument in a closed–form and by precisely identifying this “relatively simple” function. The latter consists of elementary functions (containing the Euler’s constant \( \gamma \) as well) and of the reflected sum of two second–order derivatives of the Hurwitz \( \zeta \)–function at zero \( \zeta''(0, p) + \zeta''(0, 1 - p) \), number \( p \) being rational in the range \((0, 1)\). Curiously enough, the derived theorem represents also the finite Fourier series for the first generalized Stieltjes constant, so that classic Fourier analysis tools may be used at their full strength. With the help of the latter, we derive several summation formulæ including summation with trigonometric functions and summation with square. Obviously, the same method can be applied to other discrete functions allowing similar representations. In particular, its application to the little–known Malmsten’s representation for the \( \Psi \)–function yields several beautiful summation formulæ for the Digamma function which are derived in appendices.

I.2. Notations

Throughout the manuscript, following abbreviated notations are used: \( \gamma = 0.5772156649 \ldots \) for the Euler’s constant, \( \gamma_n \) for the \( n \)th Stieltjes constant, \( \gamma_n(p) \) for the \( n \)th generalized Stieltjes constant at point \( p \), \( \lfloor x \rfloor \) for the integer part of \( x \), \( \text{tg} \) for the tangent of \( z \), \( \text{ctg} \) for the cotangent of \( z \), \( \text{ch} \) for the hyperbolic cosine of \( z \), \( \text{sh} \) for the hyperbolic sine of \( z \), \( \text{th} \) for the hyperbolic tangent of \( z \), \( \text{cch} \) for the hyperbolic cotangent of \( z \).\(^6\) In order to avoid any confusion between compositional inverse and multiplicative inverse, inverse trigonometric and hyperbolic functions are denoted as \( \arccos, \arcsin, \arctg, \ldots \) and not as \( \cos^{-1}, \sin^{-1}, \tg^{-1}, \ldots \). Writings \( \Gamma(z), \Psi(z), \zeta(s) \) and \( \zeta(s, v) \) denote respectively the \( \Gamma \)–function, the \( \Psi \)–function (or Digamma function), the Riemann \( \zeta \)–function and the Hurwitz \( \zeta \)–function. When referring to the derivatives of the the Hurwitz \( \zeta \)–function, we always refers to the derivative with respect to its first argument \( s \) (unless otherwise specified). \( \text{Re} \) and \( \text{Im} \) denote, respectively, real and imaginary parts of \( z \). Natural numbers are defined in a traditional way as a set of positive integers, which is denoted by \( \mathbb{N} \). Kronecker symbol of arguments \( l \) and \( k \) is denoted by \( \delta_{l,k} \). Letter \( i \) is never used as index and is \( \sqrt{-1} \). By Malmsten’s integral we mean any integral of the form

\[
\int_{0}^{\infty} \frac{R(\text{sh} px, \text{ch} px) \cdot \ln x}{R(\text{sh} x, \text{ch} x)} \, dx
\]

where \( R \) denotes a rational function and the parameter \( p \) is such that the convergence is guaranteed. Moreover, if \( p \) is rational, then by an appropriate change of variable, the above Malmsten’s integral

\(^8\)Further to remarks we received after the publication of [10], we note that similar closed–form expressions for \( \gamma(1/2), \gamma(1/4), \gamma(3/4) \) and \( \gamma(1/3) \) were also obtained in [17, pp. 17–18].

\(^9\)Most of these notations come from Latin, e.g “ch” stands for cosinus hyperbolicus, “sh” stands for sinus hyperbolicus, etc.
may be reduced to the following ln ln–integrals

\[
\int_0^\infty R(\text{sh } px, \text{ch } px) \cdot \ln x \frac{dx}{R(\text{sh } x, \text{ch } x)} = \begin{cases} 
\int_1^\infty P(u) \frac{\ln \ln u}{Q(u)} du + \ldots \\
\int_1^\infty \frac{P(u)}{Q(u)} \ln \ln u du + \ldots \\
\int_0^\infty \frac{R(y)}{S(y)} \ln \frac{1}{y} dy + \ldots 
\end{cases}
\]

where \(P(u), Q(u), R(y)\) and \(S(y)\) are polynomials in \(u\) and \(y\) respectively, see [10] for more details. Other notations are standard.

II. Evaluation of the first generalized Stieltjes constant at rational argument

II.1. Generalized Stieltjes constants and their relationships to Malmsten's integrals

The formula (11) provides a closed–form expression for the difference of two first Stieltjes constants at rational arguments. It should be therefore interesting to investigate if there could be some expressions containing other combinations of Stieltjes constants. In our previous work [10], we already demonstrated that some Malmsten’s integrals are connected with the first generalized Stieltjes constants. This connection was quite fruitful and permitted not only to prove by another method the known relationship (11), but also to evaluate the first generalized Stieltjes constant \(\gamma_1(p)\) at \(p = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\), etc. by means of elementary functions, the Euler’s constant \(\gamma\), the first Stieltjes constant \(\gamma_1\) and the \(\Gamma\)–function, see for more details [10, exercise 64]. Taking into account that aforementioned manuscript was quite long, many results and theorems were given as exercises with hints and without rigorous proofs. Below, we provide several useful proofs and unpublished results (given as lemmas and corollaries) showing that Malmsten’s integrals of the first and second orders may be expressed by means of the first generalized Stieltjes constants. This connection between Malmsten’s integrals and Stieltjes constants is crucial and plays the central role in the proof of the main theorem of this manuscript.

**Lemma 1.** For any \(|\text{Re } p| < 1\) and \(\text{Re } a > -1\),

\[
\int_0^\infty \frac{x^{a-1}(\text{ch } px - 1)}{\text{sh } x} dx = \frac{\Gamma(a)}{2^a} \left\{ \zeta\left(a, \frac{1}{2} - \frac{p}{2}\right) + \zeta\left(a, \frac{1}{2} + \frac{p}{2}\right) - 2(2^a - 1)\zeta(a) \right\}
\]  

**Proof.** From elementary analysis it is well-known that \(\text{sh}^{-1}x\), for \(\text{Re } x > 0\), may be represented by the following geometric series

\[
\frac{1}{\text{sh } x} = 2 \sum_{n=0}^{\infty} e^{-(2n+1)x}, \quad \text{Re } x > 0.
\]

This series, being uniformly convergent, can be integrated term–by–term. Hence

\[
\int_0^\infty \frac{x^{a-1}(\text{ch } px - 1)}{\text{sh } x} dx = \sum_{n=0}^{\infty} \int_0^\infty x^{a-1} \left\{ e^{-(2n+1-p)x} + e^{-(2n+1+p)x} - 2e^{-(2n+1)x} \right\} dx
\]

\[
= \Gamma(a) \sum_{n=0}^{\infty} \left\{ \frac{1}{(2n+1-p)^a} + \frac{1}{(2n+1+p)^a} - \frac{2}{(2n+1)^a} \right\}
\]

\[
= \frac{\Gamma(a)}{2^a} \left\{ \zeta\left(a, \frac{1}{2} - \frac{p}{2}\right) + \zeta\left(a, \frac{1}{2} + \frac{p}{2}\right) - 2\zeta\left(a, \frac{1}{2}\right) \right\},
\]
where the integral on the left converges if $|\text{Re } p| < 1$ and $\text{Re } a > -1$. In order to obtain (13), it suffices to notice that $\zeta(a, \frac{1}{2}) = (2^a - 1) \zeta(a)$. QED

**Corollary 1.** For any $p$ lying in the strip $|\text{Re } p| < 1$, we always have

$$\int_0^\infty \frac{(\text{ch } px - 1) \ln x}{\text{sh } x} \, dx = (\gamma + \ln 2) \cdot \left\{ \Psi\left(\frac{1}{2} + \frac{p}{2}\right) + \ln 2 - \frac{\pi}{2} \tan \frac{\pi p}{2} \right\}$$

$$+ \gamma^2 + \gamma_1 - \frac{1}{2} \gamma_1 \left(\frac{1}{2} + \frac{p}{2}\right) - \frac{1}{2} \gamma_1 \left(\frac{1}{2} - \frac{p}{2}\right).$$

(14)

This result is straightforwardly obtained from Lemma 1 by differentiating (13) with respect to $a$, and then by making $a \to 1$. In order to evaluate the limit in the right–hand side, we make use of Laurent series (1) and (2).

Another Malmsten’s integral of the first order which also contains Stieltjes constants appear in the next Lemma.

**Lemma 2.** For any $|\text{Re } p| < 1$ and $\text{Re } a > -1$,

$$\int_0^\infty \frac{x^{a-1} \text{sh } px}{\text{ch } x} \, dx = \frac{\Gamma(a)}{2^a} \left\{ \zeta\left(a, \frac{1}{2} + \frac{p}{2}\right) - \zeta\left(a, \frac{1}{2} - \frac{p}{2}\right) \right\}$$

$$- 2^1 - a \zeta\left(a, \frac{1}{4} + \frac{p}{4}\right) + 2^1 - a \zeta\left(a, \frac{1}{4} - \frac{p}{4}\right) \right\\

**Proof.** Analogous to that of Lemma 1.

**Corollary 2.** For any $|\text{Re } p| < 1$,

$$\int_0^\infty \frac{\text{sh } px \cdot \ln x}{\text{ch } x} \, dx = \frac{1}{2} \left\{ \pi (\gamma + \ln 2) \tan \frac{\pi p}{2} - (\gamma + 2 \ln 2) \left[ \Psi\left(\frac{1}{4} + \frac{p}{4}\right) - \Psi\left(\frac{1}{4} - \frac{p}{4}\right) \right] \right\\

+ \gamma_1 \left(\frac{1}{2} - \frac{p}{2}\right) - \gamma_1 \left(\frac{1}{2} + \frac{p}{2}\right) - \gamma_1 \left(\frac{1}{4} - \frac{p}{4}\right) + \gamma_1 \left(\frac{1}{4} + \frac{p}{4}\right) \right\\

This result can be shown in the same way as that in Corollary 1.

By the same line of reasoning, one may also prove that following logarithmic integrals may be
expressed in terms of first generalized Stieltjes constants.

\[
\int_0^\infty \frac{\text{sh} p x \cdot \ln x}{\text{sh} x} \, dx = -\frac{1}{2} \left\{ \pi (\gamma + \ln 2) \tg \frac{\pi p}{2} + \gamma_1 \left( \frac{1}{2} - \frac{p}{2} \right) - \gamma_1 \left( \frac{1}{2} + \frac{p}{2} \right) \right\}
\]

\[
\int_0^\infty \frac{\text{ch} p x \cdot \ln x}{\text{ch} x} \, dx = \frac{1}{2} \left\{ \gamma_1 \left( \frac{1}{2} + \frac{p}{2} \right) + \gamma_1 \left( \frac{1}{2} - \frac{p}{2} \right) - \gamma_1 \left( \frac{1}{4} + \frac{p}{4} \right) - \gamma_1 \left( \frac{1}{4} - \frac{p}{4} \right) \right\} - \frac{1}{2} \ln^2 2
\]

\[
+ \ln 2 \cdot \Psi \left( \frac{1}{2} + \frac{p}{2} \right) + \frac{\pi}{2} (\gamma + \ln 2) \tg \frac{\pi p}{2} - \frac{\pi}{2} (\gamma + 2 \ln 2) \ctg \left( \frac{\pi p}{2} \right)
\]

\[
\int_0^\infty \frac{\text{sh}^2 p x \cdot \ln x}{\text{sh}^2 x} \, dx = \frac{1}{2} \left\{ \ln \pi - \ln \sin \pi p + \pi \left[ \gamma_1 (p) - \gamma_1 (1 - p) \right] - (\gamma + \ln 2) \left( 1 - \pi p \ctg \pi p \right) \right\}
\]

\[
\int_0^\infty \frac{\text{ch} p x \cdot \ln x}{\text{ch}^2 x} \, dx = \frac{p}{2} \left\{ \gamma_1 \left( \frac{p}{2} \right) - \gamma_1 \left( 1 - \frac{p}{2} \right) - \gamma_1 \left( \frac{p}{4} \right) + \gamma_1 \left( 1 - \frac{p}{4} \right) \right\}
\]

\[
- \frac{\pi p}{2} \left\{ (\gamma + 2 \ln 2) \csc \frac{\pi p}{2} + \ln 2 \cdot \ctg \frac{\pi p}{2} \right\} + \ln \tg \frac{\pi p}{4}
\]

where parameter \( p \) should be such that \(|\text{Re} \, p| < 1\) in the first three integrals and \(|\text{Re} \, p| < 2\) in the fourth one. Interestingly, higher Malmsten’s integrals seem to not contain higher Stieltjes constants, but rather other \( \zeta \)-function related constants. For instance, the evaluation of the third–order Malmsten’s integral by the same method yields:

\[
\int_0^\infty \frac{\text{sh}^3 p x \cdot \ln x}{\text{sh}^3 x} \, dx = \frac{1}{4} \left\{ 3 \zeta’ \left( -1, \frac{1}{2} + \frac{p}{2} \right) - 3 \zeta’ \left( -1, \frac{1}{2} - \frac{p}{2} \right) - \zeta’ \left( -1, \frac{1}{2} + \frac{3p}{2} \right) + \zeta’ \left( -1, \frac{1}{2} - \frac{3p}{2} \right) \right\}
\]

\[
+ \frac{3(1 - p^2)}{16} \left\{ \gamma_1 \left( \frac{1}{2} + \frac{p}{2} \right) - \gamma_1 \left( \frac{1}{2} - \frac{p}{2} \right) \right\} - \frac{1 - 9p^2}{16} \left\{ \gamma_1 \left( \frac{1}{2} + \frac{3p}{2} \right) - \gamma_1 \left( \frac{1}{2} - \frac{3p}{2} \right) \right\}
\]

\[
+ \frac{\pi (\gamma + \ln 2)}{16} \left\{ 3(p^2 - 1) \tg \frac{\pi p}{2} - (9p^2 - 1) \tg \frac{3\pi p}{2} \right\} - \frac{3p}{4} \ln \left( 4 \cos^2 \frac{\pi p}{4} - 3 \right)
\]

in the strip \(|\text{Re} \, p| < 1\). In contrast, the evaluation of Malmsten’s integrals containing higher powers of the logarithm in the numerator of the integrand\(^{10}\) leads precisely to higher Stieltjes constants. In fact, differentiating twice (13) with respect to \( a \), and then making \( a \to 1 \), yields

\[
\int_0^\infty \frac{(\text{ch} p x - 1) \ln^2 x}{\text{sh} x} \, dx = -\frac{\ln 2}{3} \left( \pi^2 + 2 \ln^2 2 \right) - \left( \gamma + \ln 2 \right)^2 + \frac{\pi^2}{6} \cdot \left\{ \Psi \left( \frac{1}{2} + \frac{p}{2} \right) - \frac{\pi}{2} \tg \frac{\pi p}{2} \right\}
\]

\[
- \frac{\gamma_2}{6} \left( \pi^2 + 6 \ln^2 2 \right) - \gamma^3 - 2\gamma_1 (\gamma - \ln 2) + (\gamma + \ln 2) \cdot \left\{ \gamma_1 \left( \frac{1}{2} + \frac{p}{2} \right) + \gamma_1 \left( \frac{1}{2} - \frac{p}{2} \right) \right\}
\]

\[
- \gamma_2 \cdot \frac{1}{2} \left\{ \gamma_2 \left( \frac{1}{2} + \frac{p}{2} \right) + \gamma_2 \left( \frac{1}{2} - \frac{p}{2} \right) \right\}
\]

\[(17)\]

\(^{10}\)We propose to call such integrals \textit{generalized Malmsten’s integrals}. 

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where \( |\text{Re } p| < 1 \). More generally, the same integral containing \( \ln^nx \) instead of \( \ln^2x \) will lead to the \( n \)th Stieltjes constants.

II.2. Malmsten’s series and Hurwitz’s reflection formula

We now show that the integral form Lemma 1 may be also evaluated via a trigonometric series.

**Lemma 3.** For any \(-1 < p < 1\) and \( |\text{Re } a| < 1 \),

\[
\int_0^\infty \frac{x^{a-1}(\text{ch } px - 1)}{\text{sh } x} \, dx = \pi^n \sec \frac{\pi a}{2} \sum_{n=1}^\infty (-1)^n \cos \frac{p\pi n - 1}{n^{1-a}}
\]  

(18)

**Proof.** The Mittag–Leffler theorem is a fundamental theorem in the theory of functions of a complex variable and allows to expand meromorphic functions into a series accordingly to its poles.\(^{11}\) Application of this theorem to the meromorphic function \( (\text{ch } pz - 1)/\text{sh } z \), \( p \in (-1, +1) \), having only first–order poles at \( z = \pi ni, n \in \mathbb{Z} \), with residue \((-1)^n(\cos \pi pn - 1)\), leads to the following expansion

\[
\frac{\text{ch } pz - 1}{\text{sh } z} = 2\pi \sum_{n=1}^\infty (-1)^n \frac{\cos p\pi n - 1}{z^2 + \pi^2n^2}, \quad z \in \mathbb{C}, \quad z \neq \pi ni, \quad n \in \mathbb{Z}.
\]

which is uniformly convergent on the entire complex \( z \)-plane except discs \(|z - \pi in| < \epsilon, n \in \mathbb{Z}\), of arbitrary small radius \( \epsilon \). Therefore

\[
\int_0^\infty \frac{x^{a-1}(\text{ch } px - 1)}{\text{sh } x} \, dx = 2\pi \sum_{n=1}^\infty (-1)^n(\cos p\pi n - 1) \int_0^\infty \frac{x^a}{x^2 + \pi^2n^2} \, dx = \pi^n \sec \frac{\pi a}{2} \sum_{n=1}^\infty (-1)^n \frac{\cos p\pi n - 1}{n^{1-a}}
\]

(19)

which holds only for \(-1 < p < 1\) and \( |\text{Re } a| < 1 \) (the elementary integral in the middle, whose evaluation is due to Euler, is convergent only in the strip \(|\text{Re } a| < 1\), see e.g. [62, p. 126, n° 880], [21, p. 197, n° 856.2], [3, p. 256, n° 6.1.17], [29, p. 67, n° 587], [48, p. 51]). However, the above equality can be analytically continued for other values of \( a \): the integral is the analytic continuation of the sum for \( \text{Re } a \geq 1 \), while the sum analytically continues the integral for \( \text{Re } a \leq -1 \). We obviously have to expect trouble with the right-hand part at \( a = \pm 1, \pm 3, \pm 5, \ldots \) because of the secant. Since when \( a = -1, -3, -5, \ldots \) the sum in the right-hand side converges, these points are poles of the first order for the analytic continuation of integral (19). In contrast, for \( a = 1, 3, 5, \ldots, \) the integral on the left remains bounded, and thus, these points are removable singularities for the right-hand side of (19). In other words, formally \( \sum(-1)^n(\cos p\pi n - 1)n^{a-1}, n \geq 1 \), must vanish identically for any odd positive \( a \) (exactly as \( \eta(1 - a) \), the result which has been derived by Euler, see e.g. [22, p. 85]). These matters are treated in detail in the next Corollary. QED

**Corollary 3.** For \( 0 < p < 1 \) and \( \text{Re } a < 1 \),

\[
\begin{align*}
\sum_{n=1}^\infty \cos \frac{2\pi pn}{n^{1-a}} &= \Gamma(a)(2\pi)^{-a} \cos \frac{\pi a}{2} \left\{ \zeta(a, p) + \zeta(a, 1 - p) \right\} \\
\sum_{n=1}^\infty \sin \frac{2\pi pn}{n^{1-a}} &= \Gamma(a)(2\pi)^{-a} \sin \frac{\pi a}{2} \left\{ \zeta(a, p) - \zeta(a, 1 - p) \right\}
\end{align*}
\]

(20a,b)

\(^{11}\)For more details, see [51], [62, pp. 147–148, n° 994–1002], [24, Chap. V, § 27, n° 27.10-2], [58, Chap. VII, p. 175], [48].
Proof. In view of the fact the alternating \( \zeta \)-function \( \eta(s) \) may be reduced to the Riemann \( \zeta \)-function and by making use of the well–known reflection formula for the Riemann \( \zeta \)-function\(^{12}\)

\[ \zeta(1 - s) = 2\zeta(s)\Gamma(s)(2\pi)^{-s} \cos \frac{s}{2} \pi s, \]

we may continue (19) as follows

\[
\pi^a \sec \frac{\pi a}{2} \sum_{n=1}^{\infty} (-1)^n \frac{\cos p \pi n}{n^{1-a}} = \pi^a \sec \frac{\pi a}{2} \left\{ \sum_{n=1}^{\infty} (-1)^n \frac{\cos p \pi n}{n^{1-a}} - (2^a - 1)\zeta(1 - a) \right\} \\
= \pi^a \sec \frac{\pi a}{2} \sum_{n=1}^{\infty} (-1)^n \frac{\cos p \pi n}{n^{1-a}} - 2(2 - 1^a)\Gamma(a)\zeta(a)
\]

Comparing the latter expression to the result of Lemma 1 gives

\[
\sum_{n=1}^{\infty} (-1)^n \frac{\cos p \pi n}{n^{1-a}} = \Gamma(a)(2\pi)^{-a} \cos \frac{\pi a}{2} \left\{ \zeta(\frac{a}{2} + \frac{p}{2}) + \zeta(\frac{a}{2} - \frac{p}{2}) \right\}
\]

Writing in this expression \( 2p - 1 \) instead of \( p \) yields immediately (20a). Now, by partially differentiating (20a) with respect to \( p \) and by remarking that \( a\Gamma(a) = \Gamma(a + 1) \), and then, by writing \( a \) instead of \( a + 1 \), we arrive at (20b). Note also that both sums \( (20a,b) \) may be analytically continued to other domains of \( a \) by means of expressions in corresponding right parts. QED

Nowadays, the results \( (20a,b) \) seem to be not particularly well–known (for instance, advanced calculators such as Wolfram Alpha Pro expresses both series in terms of polylogarithms). Notwithstanding, equation (20b) can be found in an old Malmsten’s work published as early as 1849 [50, p. 17, eq. (48)], and (20a) is a straightforward consequence of (20b).\(^{13}\)

Corollary 4. If we notice that

\[
\Gamma(a) = \frac{\pi}{\sin \pi a \cdot \Gamma(1 - a)} = \frac{\pi}{2 \sin \frac{1}{2} \pi a \cdot \cos \frac{1}{2} \pi a \cdot \Gamma(1 - a)}
\]

then, the sum of (20a) with (20b) leads to the well–known formula

\[
\zeta(a, p) = \frac{2\Gamma(1 - a)}{(2\pi)^{-a}} \left[ \sin \frac{\pi a}{2} \sum_{n=1}^{\infty} \frac{\cos 2\pi pn}{n^{1-a}} + \cos \frac{\pi a}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi pn}{n^{1-a}} \right], \quad 0 < p \leq 1, \quad \Re a < 1
\]

which is usually attributed to Adolf Hurwitz who derived it in 1881, see [36, p. 93], [63, p. 269], [48, p. 107], [60, p. 37], [9, p. 156], [8, vol. I, p. 26, Eq. 1.10(6)].\(^{15}\)

Remark It is quite rarely emphasized that the latter representation coincide with the trigonometric Fourier series for \( \zeta(a, p) \). Remark ing this permits to immediately derive several integral formulæ.

\(^{12}\)There is an interesting relationship related to this famous relationship. It was first proposed by Leonhard Euler in 1749 in [22], who derived it by the method of mathematical induction. Then, it was independently rediscovered and rigorously proved by Bernhard Riemann in 1859 [55], [20, p. 861]. Very similar reflection formulæ were obtained by Carl Malmsten in 1842 [49] and in 1846 [50], as well as by Oscar Schlömilch in 1849 [56], [57], [36], [33, p. 23], [10]. By the way, Malmsten, unlike Riemann, remarked that the proved formulæ is analogous to that already conjectured by Euler.

\(^{13}\)Moreover, Malmsten derived his reflection formulæ (see footnote 12) precisely from this equality.

\(^{14}\)Hurwitz derived all his results for the function \( f(s, a) \) which is related to the modern Hurwitz \( \zeta \)-function as \( f(s, a) \equiv f_w(s, a) = m^{-s}\zeta(s, a/m) \), see [36, p. 89].

\(^{15}\)There is a slight error in this formulæ in the latter reference: it remains valid not only for \( \Re a < 0 \), but also for \( \Re a < 1 \).
whose demonstration by other means is more difficult

\[
\begin{align*}
\int_0^1 \zeta(a, p) \, dp &= 0 \\
\int_0^1 \zeta(a, p) \cos 2\pi n \, dp &= \Gamma(1 - a)(2\pi n)^{a-1} \sin \frac{\pi a}{2} \quad \text{Re} \, a < 1 \\
\int_0^1 \zeta(a, p) \sin 2\pi n \, dp &= \Gamma(1 - a)(2\pi n)^{a-1} \cos \frac{\pi a}{2}
\end{align*}
\]

for \( n = 1, 2, 3, \ldots \). Furthermore, in virtue of the Parseval’s theorem, we also have

\[
\int_0^1 \zeta^2(a, p) \, dp = 2\Gamma^2(1 - a)(2\pi)^{2a-2} \zeta(2 - 2a), \quad \text{Re} \, a < 1, \quad a \neq \frac{1}{2}
\]  

(22)

Differentiating this formula with respect to \( a \) and then setting \( a = 0 \), yields:

\[
2 \int_0^1 \frac{1}{2 - p} \cdot \left( \ln \Gamma(p) - \frac{1}{2} \ln 2\pi \right) \, dp = \frac{\gamma + \ln 2\pi}{6} - \frac{\zeta'(2)}{\pi^2}
\]

Whence, accounting for the well–known result\(^{16}\)

\[
\int_0^1 \ln \Gamma(x) \, dx = \frac{1}{2} \ln 2\pi
\]

we obtain

\[
\int_0^1 x \ln \Gamma(x) \, dx = \frac{\zeta'(2)}{2\pi^2} - \frac{\gamma - 2 \ln 2\pi}{12}
\]

Integration by parts of the latter expression leads to the antiderivatives of \( \ln \Gamma(x) \) which are currently not well–studied yet.

**Corollary 5.** In (21), the index \( n \) may be represented as \( n = mk + l \), where for each \( k = 0, 1, 2, \ldots, \infty \), the index \( l \) runs over \([1, 2, \ldots, m]\) and where \( m \) is some positive integer. Then, (21) may be written in the form:

\[
\zeta(a, p) = \frac{2\Gamma(1 - a)}{(2\pi)^{1-a}} \left[ \sin \frac{\pi a}{2} \sum_{l=1}^{m} \sum_{k=0}^{\infty} \frac{\cos 2\pi p (mk + l)}{(mk + l)^{1-a}} + \cos \frac{\pi a}{2} \sum_{l=1}^{m} \sum_{k=0}^{\infty} \frac{\sin 2\pi p (mk + l)}{(mk + l)^{1-a}} \right]
\]  

(23)

Now, let \( p \) be a rational part of \( m \), i.e. \( p = r/m \), where \( r \) and \( m \) are positive integers such that \( r \leq m \). Then

\[
\cos \left[ 2\pi p (mk + l) \right] = \cos \left( 2\pi rl/m \right), \quad \text{and similarly for the sine. Hence, for positive rational} \, p \, \text{not greater than}
\]

---

\(^{16}\)This result is straightforward from a similar Fourier series expansion for the logarithm of the \( \Gamma \)–function, see e.g. [8, vol. I, pp. 23–24, §1.9.1]. This expansion, attributed erroneously to Ernst Kummer, was first derived by Malmsten and colleagues from the Uppsala University in 1842. We discuss this interesting historical question in details in [10, Sect. 2.2, Fig. 2 and exercise no\(^{20}\)].
1, equation (23) takes the form

\[
\zeta\left(\frac{1}{m}, \frac{r}{m}\right) = \frac{2\Gamma(1-a)}{(2\pi m)^{1-a}} \left[ \sin \frac{\pi a}{2} \sum_{l=1}^{m} \cos \left( \frac{2\pi rl}{m} \right) \sum_{k=0}^{\infty} \frac{1}{(mk+1)^{1-a}} + \cos \frac{\pi a}{2} \sum_{l=1}^{m} \sin \left( \frac{2\pi rl}{m} \right) \sum_{k=0}^{\infty} \frac{1}{(mk+1)^{1-a}} \right]
\]

\[
= 2\Gamma(1-a) \left[ \sin \frac{\pi a}{2} \sum_{l=1}^{m} \cos \left( \frac{2\pi rl}{m} \right) \cdot \zeta\left(1-a, \frac{1}{m}\right) + \cos \frac{\pi a}{2} \sum_{l=1}^{m} \sin \left( \frac{2\pi rl}{m} \right) \cdot \zeta\left(1-a, \frac{1}{m}\right) \right]
\]

\[
= 2\Gamma(1-a) \sum_{l=1}^{m} \left( \frac{2\pi rl}{m} + \frac{\pi a}{2} \right) \cdot \zeta\left(1-a, \frac{1}{m}\right), \quad r = 1, 2, \ldots, m.
\]

This equality holds in the entire complex \(a\)-plane for any positive integer \(m \geq 2\). Furthermore, by putting in the latter formula \(1-a\) instead of \(a\), it may be rewritten as

\[
\zeta\left(1-a, \frac{r}{m}\right) = 2\Gamma(1-a) \sum_{l=1}^{m} \cos \left( \frac{2\pi rl}{m} \right) \cdot \zeta\left(a, \frac{1}{m}\right), \quad r = 1, 2, \ldots, m.
\]

In the case \(r = m\), above formula reduces to the reflection formula for the Riemann \(\zeta\)-function (simply use the multiplication theorem for the Hurwitz \(\zeta\)-function, see e.g. [10]). Formulas (24) and (25) are known as functional equations for the Hurwitz \(\zeta\)-function and were both obtained by Hurwitz in the same article [36, p. 93] in 1881. By the way, the above demonstration also shows that they can be elementary derived from Malmsten’s results (20a,b) obtained as early as 1840ies.

II.3. Closed–form evaluation of the first generalized Stieltjes constant at rational argument and some related results

We now state the main result of this manuscript allowing to evaluate in a closed–form the first generalized Stieltjes constant at any rational argument.

**Theorem** The first generalized Stieltjes constant of any rational argument in the range \((0, 1)\) may be expressed in a closed form via a finite combination of logarithms of the \(\Gamma\)-function, of second–order derivatives of the Hurwitz \(\zeta\)-function at zero, of the Euler’s constant \(\gamma\), of the first Stieltjes constant \(\gamma_1\) and of elementary functions:

\[
\gamma_1\left(\frac{r}{m}\right) = \gamma_1 - \gamma \ln 2m - \ln^2 2 - \ln 2 \cdot \ln \pi m - \frac{\pi}{2} (\gamma + \ln 2\pi m) \cdot \text{ctg} \frac{\pi r}{m}
\]

\[
- \left(\frac{-1}{4}\right)^{r} \left[ 1 - (-1)^{m+1} \right] \cdot (3 \ln 2 + 2 \ln \pi) \ln 2 - \pi \ln \pi \cdot \csc \frac{\pi r}{m} \cdot \sin \left( \frac{\pi r}{m} \left[ \frac{m+1}{2} \right] \right) \cdot \sin \left( \frac{\pi r}{m} \left[ \frac{m-1}{2} \right] \right)
\]

\[
+ 2(\gamma + \ln 2\pi m) \cdot \sum_{l=1}^{\lfloor \frac{r}{m} \rfloor} \cos \frac{2\pi rl}{m} \cdot \ln \pi + \pi l \cdot \sum_{l=1}^{\lfloor \frac{r}{m} \rfloor} \sin \frac{2\pi rl}{m} \cdot \ln \pi + 2\pi \sum_{l=1}^{\lfloor \frac{r}{m} \rfloor} \sin \frac{2\pi rl}{m} \cdot \ln \Gamma\left(\frac{1}{l}\right)
\]

\[
+ \sum_{l=1}^{\lfloor \frac{r}{m} \rfloor} \cos \frac{2\pi rl}{m} \cdot \left\{ \zeta''\left(0, \frac{1}{m}\right) + \zeta''\left(0, 1 - \frac{1}{m}\right) \right\}
\]

This elegant formula holds for any \(r = 1, 2, 3, \ldots, m - 1\), where \(m\) is positive integer greater than \(1\). The Stieltjes constants for other “periods” may be obtained from the recurrent relationship:

\[
\gamma_1(v + 1) = \gamma_1(v) - \frac{\ln v}{v}, \quad v \neq 0.
\]
see, e.g. [10, exercise n° 64, Eq. (64)]. The above theorem is an equivalent of the Gauss' Digamma theorem for the 0th Stieltjes constant $\gamma_0(r/m) = -\Psi(r/m)$. Two alternative forms of the same theorem are given in equations (38) and (41).

**Proof of the Theorem** Consider the integral (13). Put $2p − 1$ instead of $p$ and denote the resulting integral via $J_a(p)$:

$$J_a(p) \equiv \int_0^\infty \frac{x^{a-1} \sin[(2p-1)x]}{x} \, dx = \frac{\Gamma(a)}{2^a} \left\{ \zeta(a, p) + \zeta(a, 1 - p) - 2(2^a - 1)\zeta(a) \right\}, \quad 0 < \text{Re} \, p < 1$$

Let now $p$ be rational $p = r/m$, where $r$ and $m$ are positive integers such that $r < m$. Then, the precedent equation becomes

$$J_a\left(\frac{r}{m}\right) = \frac{\Gamma(a)}{2^a} \left\{ \zeta\left(a, \frac{r}{m}\right) + \zeta\left(a, 1 - \frac{r}{m}\right) - 2(2^a - 1)\zeta(a) \right\}$$

The sum of first two terms in curly brackets may be evaluated via the Hurwitz’ reflection formula (24):

$$\zeta\left(a, \frac{r}{m}\right) + \zeta\left(a, 1 - \frac{r}{m}\right) = \frac{2\Gamma(1-a)}{(2\pi m)^{1-a}} \sum_{l=1}^{m} \left[ \sin\left(\frac{2\pi rl}{m} + \frac{\pi a}{2}\right) + \sin\left(\frac{2\pi (m-r)l}{m} + \frac{\pi a}{2}\right) \right]$$

$$\times \zeta\left(1 - a, \frac{1}{m}\right) = \frac{4\Gamma(1-a)}{(2\pi m)^{1-a}} \sin\frac{\pi a}{2} \sum_{l=1}^{m} \cos\left(\frac{2\pi rl}{m}\right) \cdot \zeta\left(1 - a, \frac{1}{m}\right)$$

Thus, by noticing that $\Gamma(a)\Gamma(1-a) = \frac{1}{2}\pi \csc \frac{\pi a}{2} \cdot \sec \frac{1}{2}\pi a$, the integral $J_a(r/m)$ takes the form:

$$J_a\left(\frac{r}{m}\right) = \frac{2\pi}{2^a(2\pi m)^{1-a}} \sec \frac{\pi a}{2} \sum_{l=1}^{m} \sum_{l_1=1}^{m} \cos\left(\frac{2\pi rl}{m}\right) \cdot \zeta\left(1 - a, \frac{1}{m}\right) - \frac{2\Gamma(a)(2^a - 1)\zeta(a)}{2^a}$$

which is third expression for the integral $J_a$, other two expressions being given by (13) and (18). Let now study each term of the right part, denoted for brevity $f_1$, $f_2$ and $f_3$ respectively, in a neighbourhood of $a = 1$. The first and the third terms have poles of the first order at this point, while the second term $f_2$ is analytic at $a = 1$. Thus, in a neighbourhood of $a = 1$, terms $f_1$ and $f_3$ may be expanded in the Laurent series as follows

$$f_1 = -\frac{2}{a - 1} - 2\ln\pi m - \left(\frac{\pi^2}{12} - \ln^2\pi m\right) \cdot (a - 1) + O(a - 1)^2$$

and

$$f_3 = \frac{1}{a - 1} + \ln 2 + \left(\frac{\pi^2}{12} - \frac{\ln2^2}{2} - \frac{\gamma_1^2}{2} - \gamma_1\right) \cdot (a - 1) + O(a - 1)^2$$

while $f_2$ may be represented by the following Taylor series

$$f_2 = \sum_{l=1}^{m} \cos\left(\frac{2\pi rl}{m}\right) \cdot \zeta\left(0, \frac{1}{m}\right) - (a - 1) \sum_{l=1}^{m} \cos\left(\frac{2\pi rl}{m}\right) \cdot \zeta'\left(0, \frac{1}{m}\right) + \frac{(a - 1)^2}{2} \sum_{l=1}^{m} \cos\left(\frac{2\pi rl}{m}\right) \cdot \zeta''\left(0, \frac{1}{m}\right) \ln\Gamma\left(\frac{1}{m}\right)$$

$$+ O(a - 1)^3 = -\frac{1}{2} - (a - 1) \sum_{l=1}^{m} \cos\left(\frac{2\pi rl}{m}\right) \ln\Gamma\left(\frac{1}{m}\right) + \frac{(a - 1)^2}{2} \sum_{l=1}^{m} \cos\left(\frac{2\pi rl}{m}\right) \cdot \zeta''\left(0, \frac{1}{m}\right) + O(a - 1)^3$$

(33)
because
\[
\begin{align*}
\sum_{l=1}^{m} \cos \left( \frac{2\pi rl}{m} \right) &= 0, \quad r = 1, 2, 3, \ldots, m - 1 \\
\sum_{l=1}^{m} l \cdot \cos \left( \frac{2\pi rl}{m} \right) &= \frac{m}{2}, \quad r = 1, 2, 3, \ldots, m - 1
\end{align*}
\]

In the final analysis, the substitution of (31), (32) and (33) into (30), yields the following representation for the integral \( J_a \left( \frac{r}{m} \right) \) in a neighbourhood of \( a = 1 \):

\[
J_a \left( \frac{r}{m} \right) = \ln \frac{\pi m}{2} + 2A_m(r) + (a - 1) \cdot \left[ -B_m(r) + 2A_m(r) \ln \pi m - \frac{\pi^2}{24} + \frac{\ln^2 \pi m}{2} + \frac{\gamma^2}{2} + \frac{\ln^2 2}{2} + \gamma_1 \right] \\
+ O(a - 1)^2
\]

where
\[
\begin{align*}
A_m(r) &\equiv \sum_{l=1}^{m} \cos \left( \frac{2\pi rl}{m} \right) \cdot \ln \Gamma \left( \frac{l}{m} \right) \\
B_m(r) &\equiv \sum_{l=1}^{m} \cos \left( \frac{2\pi rl}{m} \right) \cdot \zeta'' \left( 0, \frac{l}{m} \right)
\end{align*}
\]

Now, if we look at the integral \( J_a \left( \frac{r}{m} \right) \) defined in (29), we see that it is uniformly convergent and regular near \( a = 1 \) (see appendix C), and hence, may be expanded in the Taylor series about \( a = 1 \):

\[
J_a \left( \frac{r}{m} \right) = J_1 \left( \frac{r}{m} \right) + (a - 1) \left. \frac{\partial J_a \left( \frac{r}{m} \right)}{\partial a} \right|_{a=1} + O(a - 1)^2
\]

Equating right–hand sides of (34) and (35), and then, searching for terms with same powers of \((a - 1)^2\), gives

\[
\begin{align*}
\int_{0}^{\infty} \frac{\text{ch} [(2p - 1)x] - 1}{\text{sh} x} \, dx &= \ln \frac{\pi m}{2} + 2A_m(r) \\
\int_{0}^{\infty} \frac{(\text{ch} [(2p - 1)x] - 1) \ln x}{\text{sh} x} \, dx &= -B_m(r) + 2A_m(r) \ln \pi m - \frac{\pi^2}{24} + \frac{\ln^2 \pi m}{2} + \frac{\gamma^2}{2} + \frac{\ln^2 2}{2} + \gamma_1
\end{align*}
\]

where \( p \equiv r/m \). The sum \( A_m(r) \) may be reduced either to elementary functions (if using the reflection formula for the logarithm of the \( \Gamma \)–function) or to the \( \Psi \)–function and the Euler’s constant \( \gamma \) (see appendix B). We, for the purpose of brevity, prefer to use the latter representation for \( A_m(r) \). Thus, by using (58), the first of the above integrals may be calculated as

\[
\int_{0}^{\infty} \frac{\text{ch} [(2p - 1)x] - 1}{\text{sh} x} \, dx = -\gamma - 2 \ln 2 - \frac{\pi}{2} \text{ctg} \frac{\pi r}{m} - \Psi \left( \frac{r}{m} \right), \quad p \equiv \frac{r}{m}
\]
while the second one reduces to
\[
\int_0^\infty \frac{\cosh[(2p-1)x] - 1}{\sinh x} \ln x \, dx = - \sum_{l=1}^m \cos \frac{2\pi rl}{m} \cdot \zeta''(0, \frac{1}{m}) - \left[ \gamma + \ln 2 + \frac{\pi}{2} \cot \frac{\pi r}{m} + \Psi\left(\frac{r}{m}\right) \right] \ln \pi m
\]
\[
- \frac{\pi^2}{24} - \frac{\ln^2 \pi m}{2} + \frac{\gamma^2}{2} + \frac{\ln^2 2}{2} + \gamma_1 = - \sum_{l=1}^m \cos \frac{2\pi rl}{m} \cdot \zeta''(0, \frac{1}{m}) - \left[ \gamma + \ln 2 + \frac{\pi}{2} \cot \frac{\pi r}{m} + \Psi\left(\frac{r}{m}\right) \right] \ln \pi m
\]
\[
- \frac{\ln^2 \pi m}{2} + \frac{\ln^2 2}{2} - \frac{\ln 2 \pi}{2} \pi m = \frac{r}{m}
\]

(36)

where at the final stage we separate the last term in the sum \(B_m(r)\) whose value is known
\[
\zeta''(0, 1) = \zeta''(0) = \gamma_1 + \frac{\gamma^2}{2} - \frac{\pi^2}{24} - \frac{\ln 2 \pi}{2}
\]

But the integral (36) was also evaluated in (14) by means of first generalized Stieltjes constants. Hence, the comparison of (14) to (36) yields
\[
\gamma_1 \left(\frac{r}{m}\right) + \gamma_1 \left(1 - \frac{r}{m}\right) = 2 \sum_{l=1}^m \cos \frac{2\pi rl}{m} \cdot \zeta''(0, \frac{1}{m}) + 2(\gamma + \ln 2 \pi m) \cdot \left\{ \Psi\left(\frac{r}{m}\right) + \frac{\pi}{2} \cot \frac{\pi r}{m} \right\}
\]
\[
+ 2(\gamma + \ln 2 \pi m) \ln 2 \pi m + \ln^2 2 \pi m - \ln^2 2 \pi + 2 \gamma_1
\]

for each \(r = 1, 2, \ldots, m - 1\). Adding this to (11) and simplifying the result finally gives
\[
\gamma_1 \left(\frac{r}{m}\right) = \gamma_1 + \gamma^2 + \gamma \ln 2 \pi m + \ln 2 \pi \cdot \ln m + \frac{1}{2} \ln^2 m + (\gamma + \ln 2 \pi m) \cdot \Psi\left(\frac{r}{m}\right)
\]
\[
+ \pi \left( \sum_{l=1}^m \sin \frac{2\pi rl}{m} \cdot \ln \Gamma\left(\frac{l}{m}\right) + \sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \zeta''(0, \frac{1}{m}) \right)
\]

(38)

This is the most simple form of the theorem which we are stating here and can be used as is. Notwithstanding, we may also notice that each of two sums from the right-hand side may be further simplified. Since each pair of terms which occupy symmetrical positions relatively to the center (except for \(l = m/2\) when \(m\) is even) may be grouped together, the last sum may be reduced to
\[
\sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \zeta''(0, \frac{1}{m}) =
\]
\[
= \begin{cases} 
\sum_{l=1}^{\lfloor(m-1)/2\rfloor} \cos \frac{2\pi rl}{m} \cdot \left\{ \zeta''(0, \frac{1}{m}) + \zeta''\left(0, 1 - \frac{l}{m}\right) \right\}, & \text{if } m \text{ is odd} \\
\sum_{l=1}^{\lfloor(m-1)/2\rfloor} \cos \frac{2\pi rl}{m} \cdot \left\{ \zeta''(0, \frac{1}{m}) + \zeta''\left(0, 1 - \frac{l}{m}\right) \right\} + (-1)^r \zeta''\left(0, \frac{1}{2}\right), & \text{if } m \text{ is even}
\end{cases}
\]
\[
= \sum_{l=1}^{\lfloor(m-1)/2\rfloor} \cos \frac{2\pi rl}{m} \cdot \left\{ \zeta''(0, \frac{1}{m}) + \zeta''\left(0, 1 - \frac{l}{m}\right) \right\} - \frac{(-1)^r}{4} \left[ 1 - (-1)^{m+1} \right] \cdot (3 \ln 2 + 2 \ln \pi) \ln 2
\]

(39)
because \( \zeta''(0, \frac{1}{2}) = -\frac{3}{2} \ln^2 2 - \ln \pi \ln 2 \), see e.g. [10, exercise no. 24]. Similarly,

\[
\sum_{l=1}^{m-1} \sin \frac{2\pi rl}{m} \cdot \ln \Gamma \left( \frac{l}{m} \right) = \sum_{l=1}^{m-1} \sin \frac{2\pi rl}{m} \cdot \left\{ \ln \Gamma \left( \frac{l}{m} \right) - \ln \Gamma \left( 1 - \frac{l}{m} \right) \right\} - \ln \pi \cdot \csc \frac{\pi r}{m} \cdot \sin \left( \frac{\pi r}{m} \left[ \frac{m+1}{2} \right] \right) \cdot \sin \left( \frac{\pi r}{m} \left[ \frac{m-1}{2} \right] \right)
\]

because for natural \( n \)

\[
\sum_{l=1}^{n} \sin(lx) = \csc \frac{x}{2} \cdot \sin \frac{nx}{2} \cdot \sin \left[ \frac{x}{2} (n+1) \right]
\]

see e.g. [29, no. 58, p. 12]. Thus, by using (39) and (40), as well as the Gauss’ Digamma theorem, equation (38) reduces to (26). QED.

In some cases, it may be more advantageous to have the complete finite Fourier series form. For this aim, it suffices to take again (38) and to use the Malmsten’s representation for the \( \Psi \)–function, see appendix B, formulæ (58)–(59). This yields the following expression

\[
\gamma_1 \left( \frac{r}{m} \right) = \gamma_1 - \gamma \ln 2\pi m - \ln 2\pi \cdot \ln m - \frac{1}{2} \ln^2 m - 2\pi
\]

\[
+ \pi \sum_{l=1}^{m-1} \sin \frac{2\pi rl}{m} \cdot \left\{ \ln \Gamma \left( \frac{l}{m} \right) + \frac{l(\gamma + \ln 2\pi m)}{m} \right\}
\]

\[
+ \sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \left\{ \zeta'' \left( 0, \frac{l}{m} \right) - 2(\gamma + \ln 2\pi m) \ln \Gamma \left( \frac{l}{m} \right) \right\}
\]

(41)

where \( r = 1, 2, 3, \ldots, m - 1 \), and \( m \) is positive integer greater than 1.

**Corollary 6.** For the first generalized Stieltjes constant at rational argument take place following summation formulæ

\[
\begin{align*}
\sum_{r=1}^{m-1} \gamma_1 \left( \frac{r}{m} \right) \cdot \cos \frac{2\pi rk}{m} &= -\gamma_1 + m(\gamma + \ln 2\pi m) \cdot \ln \left( 2 \sin \frac{k\pi}{m} \right) + \frac{m}{2} \left\{ \zeta'' \left( 0, \frac{k}{m} \right) + \zeta'' \left( 0, 1 - \frac{k}{m} \right) \right\} \\
\sum_{r=1}^{m-1} \gamma_1 \left( \frac{r}{m} \right) \cdot \sin \frac{2\pi rk}{m} &= \frac{\pi}{2}(\gamma + \ln 2\pi m)(2k - m) - \frac{\pi m}{2} \left\{ \ln \pi - \ln \sin \frac{\pi k}{m} \right\} + m \pi \ln \Gamma \left( \frac{k}{m} \right)
\end{align*}
\]

(42a,b)

for \( k = 1, 2, 3, \ldots, m - 1 \), where \( m \) is natural greater than 1.

**Proof.** Formula (41) represents the finite Fourier series of the kind (54). Comparing (41) to (54), we
immediately identify

\[
\begin{align*}
    a_m(0) &= \gamma_1 - \gamma \ln 2\pi m - \ln 2\pi \cdot \ln m - \frac{1}{2} \ln^2 m - \ln^2 2\pi, \\
    a_m(l) &= \zeta''(0, \frac{l}{m}) - 2(\gamma + \ln 2\pi m) \ln \Gamma\left(\frac{l}{m}\right), \quad l = 1, 2, 3, \ldots, m - 1 \\
    b_m(l) &= \pi \left\{ \ln \Gamma\left(\frac{l}{m}\right) + \frac{l(\gamma + \ln 2\pi m)}{m} \right\}, \quad l = 1, 2, 3, \ldots, m - 1
\end{align*}
\]

Thus, in virtue of (55), for any \(k = 1, 2, 3, \ldots, m - 1\),

\[
\begin{align*}
    \sum_{r=1}^{m-1} \gamma_1 \left(\frac{r}{m}\right) \cdot \cos \frac{2\pi rk}{m} &= -\gamma_1 + \gamma \ln 2\pi m + \ln 2\pi \cdot \ln m + \frac{1}{2} \ln^2 m + \ln^2 2\pi - \sum_{l=1}^{m-1} \zeta''\left(0, \frac{l}{m}\right) \\
    &+ 2(\gamma + \ln 2\pi m) \sum_{l=1}^{m-1} \ln \Gamma\left(\frac{l}{m}\right) - m(\gamma + \ln 2\pi m) \left[ \ln \Gamma\left(\frac{k}{m}\right) + \ln \Gamma\left(\frac{1 - k}{m}\right) \right] \\
    &+ \frac{\ln \pi - \ln \sin \frac{\pi k}{m}}{\frac{\pi}{2}(m-1) \ln 2\pi - \frac{1}{2} \ln m} \\
    &+ \frac{m}{2} \left\{ \zeta''\left(0, \frac{k}{m}\right) + \zeta''\left(0, 1 - \frac{k}{m}\right) \right\}
\end{align*}
\]

where we respectively used the multiplication theorem for the Hurwitz \(\zeta\)–function, see e.g. [10, exercise no 64], the Gauss’ multiplication theorem and the reflection formula for the logarithm of the \(\Gamma\)–function. Analogously, by (56), we deduce

\[
\begin{align*}
    \sum_{r=1}^{m-1} \gamma_1 \left(\frac{r}{m}\right) \cdot \sin \frac{2\pi rk}{m} &= \frac{\pi m}{2} \left\{ \ln \Gamma\left(\frac{k}{m}\right) - \ln \Gamma\left(\frac{1 - k}{m}\right) + \gamma + \ln 2\pi m \frac{k - (m - k)}{m} \right\} \\
    &= \frac{\pi}{2} (\gamma + \ln 2\pi m)(2k - m) - \frac{\pi m}{2} \left\{ \ln \pi - \ln \sin \frac{\pi k}{m} \right\} + m\pi \ln \Gamma\left(\frac{k}{m}\right)
\end{align*}
\]

which holds for \(k = 1, 2, 3, \ldots, m - 1\).\(^{17}\) \textbf{QED.}

**Corollary 7.** The Parseval’s theorem for the first generalized Stieltjes constant at rational argument has the
following form
\[\sum_{r=1}^{m-1} \gamma_1 \left( \frac{r}{m} \right) = (m-1) \gamma_1^2 - m \gamma_1 (2 \gamma + \ln m) \ln m + C_m + \frac{m}{2} \sum_{l=1}^{m-1} \zeta''' \left( 0, \frac{1}{m} \right) \cdot \left\{ \zeta'' \left( 0, \frac{1}{m} \right) + \zeta'' \left( 0, 1 - \frac{1}{m} \right) \right\} \]

\[-m (\gamma + \ln 2 \pi m) \sum_{l=1}^{m-1} \ln \Gamma \left( \frac{1}{m} \right) \cdot \left\{ \zeta''' \left( 0, \frac{1}{m} \right) + \zeta'' \left( 0, 1 - \frac{1}{m} \right) \right\} + m (\gamma + \ln 2 \pi m) \sum_{l=1}^{m-1} \zeta'' \left( 0, \frac{1}{m} \right) \cdot \ln \sin \frac{\pi l}{m} \]

\[+ m \pi^2 \sum_{l=1}^{m-1} \ln^2 \Gamma \left( \frac{1}{m} \right) + 2 \pi^2 (\gamma + \ln 2 \pi m) \sum_{l=1}^{m-1} \ln \Gamma \left( \frac{1}{m} \right) - \frac{m}{2} \big[ 4 (\gamma + \ln 2 \pi m)^2 - \pi^2 \big] \sum_{l=1}^{m-1} \ln \Gamma \left( \frac{1}{m} \right) \cdot \ln \sin \frac{\pi l}{m} \]

which can be also written (simplified as)
\[\sum_{r=1}^{m-1} \gamma_1 \left( \frac{r}{m} \right) = (m-1) \gamma_1^2 - m \gamma_1 (2 \gamma + \ln m) \ln m + C_m + \frac{m}{2} (1 - (-1)^m) \bigg\{ \left( \frac{3}{2} \ln^2 2 + \ln \pi \cdot \ln 2 \right) \]

\[\times \left[ \left( \frac{3}{2} \ln^2 2 + (\gamma + \ln 4 \pi m) \ln \pi \right) + \frac{\pi^2 \ln^2 \pi}{4} - \ln \frac{2}{\pi} (3 \ln 2 + 2 \ln \pi) (\gamma + \ln 2 \pi m) \ln \pi + \frac{\pi^2}{2} (\gamma + \ln 2 \pi m) \ln \pi \bigg] \]

\[+ \frac{m}{2} \ln \pi \cdot \ln m \cdot (\gamma + \ln 2 \pi m) \ln (4 \pi m^2) - \frac{m}{2} \bigg\{ 2 (\gamma + \ln 2 \pi m)^2 - \frac{\pi^2}{2} \bigg\} \cdot (1 - m) \ln 2 + \ln m \ln \pi \]

\[+ \frac{m}{2} [4 (\gamma + \ln 2 \pi m)^2 - \pi^2] \cdot \sum_{l=1}^{\frac{1}{2}(m-1)} \ln^2 \sin \frac{\pi l}{m} + \frac{m}{2} \cdot \sum_{l=1}^{\frac{1}{2}(m-1)} \left\{ \zeta''' \left( 0, \frac{1}{m} \right) + \zeta'' \left( 0, 1 - \frac{1}{m} \right) \right\}^2 \]

\[+ 2m (\gamma + \ln 2 \pi m) \cdot \sum_{l=1}^{\frac{1}{2}(m-1)} \left\{ \zeta''' \left( 0, \frac{1}{m} \right) + \zeta'' \left( 0, 1 - \frac{1}{m} \right) \right\} \cdot \ln \sin \frac{\pi l}{m} + m \pi^2 \cdot \sum_{l=1}^{\frac{1}{2}(m-1)} \left\{ \ln^2 \Gamma \left( \frac{1}{m} \right) + \ln^2 \Gamma \left( 1 - \frac{1}{m} \right) \right\} \]

\[+ 2 \pi^2 (\gamma + \ln 2 \pi m) \sum_{l=1}^{\frac{1}{2}(m-1)} l \cdot \left\{ \ln \Gamma \left( \frac{1}{m} \right) - \ln \Gamma \left( 1 - \frac{1}{m} \right) \right\} + 2 m \pi^2 \ln \Gamma \left( \gamma + \ln 2 \pi m \right) \cdot \sum_{l=1}^{\frac{1}{2}(m-1)} \ln \Gamma \left( 1 - \frac{1}{m} \right) \]

where, for the purpose of brevity, by \( C_m \) we designated an elementary function depending on \( m \) and containing the Euler's constant \( \gamma \)

\[C_m \equiv -m (m-1) \ln^2 2 - m (m-1) (2 \ln m + 2 \gamma + 3 \ln \pi) \ln^3 2 - m (m-1) (3 \ln^2 \pi + 4 \gamma \ln \pi + \gamma^2 + \frac{5}{2} \pi^2 + \frac{1}{6m} \pi^2) \ln^2 2 \]

\[-m \left[ (m - \frac{2}{3}) \ln \pi - 3 \gamma \right] \ln^2 m \cdot \ln m + 2 m \cdot \ln 2 + 2 m \left[ (1 - m) \ln^2 \pi - (m - \frac{3}{2}) \gamma \ln \pi \right] \ln m \cdot \ln 2 \]

\[+ \frac{\pi^2}{2} \left[ ((6 \pi^2 + 24 \gamma^2) m + 4 \pi^2 (1 - m^2)) \ln m - 4 (m-1) \left( 3 m \ln^3 \pi + 6 m \gamma \ln^2 \pi + \gamma^2 (m+1) \right) \right] \ln 2 + \frac{1}{4} m \ln^4 m + m (\gamma + \frac{1}{2} \ln \pi) \ln^3 m + \frac{1}{12} \left[ 6 m \ln^2 \pi \right. \]

\[+ 18 \gamma m \ln \pi + \pi^2 m^2 + (12 \gamma^2 + 3 \pi^2) m + 2 \pi^2 \ln^2 m + \frac{1}{12} \left[ 12 m \gamma \ln^2 \pi \right. \]

\[+ ((12 \gamma^2 + 9 \pi^2) m + 4 \pi^2 (1 - m^2)) \ln m + 2 \pi^2 (2 + m^2) \gamma \ln m \]

\[- \frac{1}{12} (m-1) \left[ 2 \pi^2 (4 m + 1) \ln^2 \pi + 4 \gamma^2 (m + 1) \ln \pi - \pi^2 \gamma^2 (m - 2) \right] \]

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and where \( m \) is natural greater than 1.

**Proof.** Inserting Fourier series coefficients (43) into (57) and proceeding analogously to (63)–(64), yields, after several pages of careful calculations and simplifications, the above result. The unique formula that should be used in addition to those employed in derivations (63)–(64) is

\[
\sum_{l=1}^{m-1} l \cdot \ln \frac{\pi l}{m} = m \sum_{l=1}^{m-1} \ln \frac{\pi l}{m} = m \prod_{l=1}^{m-1} \sin \frac{\pi l}{m} = m \left[ (1 - m) \ln 2 + \ln m \right] \quad \frac{m}{2}
\]

Also, the fact that the reflected sum \( \zeta''(0, l/m) + \zeta''(0, 1 - l/m) \), as well as the function \( \ln \sin (\pi l/m) \), are both invariant with respect to a change of summation’s index \( l \to m - l \) greatly helps when simplifying formula (46). Note that in the second variant of the Parseval’s theorem (that which contains truncated sums), the term in big curly brackets vanishes when \( m \) is odd; this is because when \( m \) is even, the number of terms in each sum is odd, and the term corresponding to the factor \( l = m/2 \) does not have its pair.

**Remark** From the above formulæ, we see that the sum of \( \zeta''(0, p) \) with its reflected version \( \zeta''(0, 1 - p) \), at positive rational \( p \) less than 1, plays the fundamental role in the evaluation of the first generalized Stieltjes constant at rational argument. In other words, the transcendence of the latter is mainly defined by the expression \( \zeta''(0, p) + \zeta''(0, 1 - p) \) at rational \( p \).\(^{18}\) We do not know which is the transcendence of such a sum, but it is not unreasonable to expect that it is lower than that of solely \( \zeta''(0, p) \). Furthermore, in our previous work \([10]\), we demonstrated that this sum has several comparatively simple integral and series representations; below, we briefly present some of them. In exercises no 20–21, we dealt with integral \( \Phi(\varphi) \), which we, unfortunately, could not reduce to elementary functions (despite of its simple and naive appearance). Written in terms of this integral, the above sum reads\(^{19}\)

\[
\zeta''(0, p) + \zeta''(0, 1 - p) = \pi \cot 2\pi p \cdot \left\{ 2 \ln \Gamma(p) + \ln \sin \pi p + (2p - 1) \ln 2\pi - \ln \pi \right\}
- 2 \ln 2\pi \cdot \ln(2 \sin \pi p) + \left\{ \int_{0}^{\infty} \frac{e^{-x} \ln x \, dx}{\cosh x - \cos 2\pi p} \right. \\
\left. + \int_{1}^{\infty} \frac{\ln x \, dx}{x(x^2 - 2x \cos 2\pi p + 1)} \right\}
\]

Other representations for \( \zeta''(0, p) + \zeta''(0, 1 - p) \) may also involve integrals

\[
\int_{0}^{\infty} \frac{e^{-2x} \ln x \, dx}{\cosh x - \cos 2\pi p} , \quad \int_{0}^{\infty} \frac{\ln x \, dx}{x^2(x^2 - 2x \cos 2\pi p + 1)} , \quad \int_{0}^{\infty} \frac{\ln \left[(2p - 1)x\right] - 1 \ln x \, dx}{x \sinh x}
\]

\(^{18}\)By neglecting, in such a context, the transcendence of the logarithm of the \( \Gamma \)-function.

\(^{19}\)Put in \([10, \text{Eq. 49}] \varphi = \pi(2p - 1)\).
which may be easily deduced from the Hermite representation for the Hurwitz \( \zeta \)-function, see e.g. [8, vol. I, p. 26, Eq. 1.10(7)]. Besides, attempts to obtain Jensen’s like formulæ, see [40], [48, p. 103], could probably bring some new ideas. In fact, the last integral in (47), after a change of variable \( x \to 1/x \), may be also written as

\[
\int_0^\infty \frac{\ln x}{x(x^2 - 2x \cos 2\pi p + 1)} \, dx = \lim_{s \to 1} \left\{ \frac{\partial}{\partial s} \int_0^\infty \frac{x \cdot \ln^{s-\frac{1}{2}}}{x^2 - 2x \cos 2\pi p + 1} \, dx \right\}
\]

The latter integral appears in old Malmsten’s works [49, pp. 20–25] and [50, p. 12]. In particular, from equations (31) [49, p. 21] and (29) [50, p. 12] it follows that

\[
\int_0^\infty \frac{x \cdot \ln^{s-\frac{1}{2}}}{x^2 - 2x \cos 2\pi p + 1} \, dx = -\frac{\Gamma(s)}{2 \sin 2\pi p} \int_0^\infty \frac{\sin \left[ \pi (2p - 1)x \right]}{\sinh \pi x} \cdot \cos(s \arctan x) \left( \frac{1}{x^2 + 1} \right)^{s/2} \, dx
\]

Integrals in the right-hand side are very similar to Jensen’s formulæ for \( \zeta(s) \) derived between 1893 and 1895 in [39] and [40]. Taking into account that these references are hard to find and that the same formulæ were later reprinted with misprints\(^{20}\), we find it useful to reproduce them here as well

\[
\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + 2 \int_0^{\pi/2} \frac{(\cos \theta)^{s-2} \sin \theta}{e^{2\pi \theta} - 1} \, d\theta = \frac{1}{s-1} + \frac{1}{2} + 2 \int_0^\infty \frac{\sin(s \arctan x) \, dx}{(e^{2\pi x} - 1) \left( x^2 + 1 \right)^{s/2}}
\]

\[
\zeta(s) = \frac{2s-1}{s-1} + i 2^{s-1} \int_0^\infty \frac{dx}{e^{\pi x} + 1} \cdot \frac{(1 + ix)^s - (1 - ix)^s}{(x^2 + 1)^s} = \frac{2s-1}{s-1} - 2^{s-1} \int_0^\infty \frac{\sin(s \arctan x) \, dx}{(e^{\pi x} + 1) \left( x^2 + 1 \right)^{s/2}}
\]

\[
\zeta(s) = \frac{\pi}{2(s-1)} \text{Li}_s \left( \frac{1}{1 + i} \right) = \frac{\pi}{2(s-1)} \int_0^\infty \frac{\cos \left[ (s-1) \arctan x \right]}{(x^2 + 1)^{(s-1)/2}} \cdot \frac{dx}{\sqrt{x} \cdot \sqrt{x^2 + 1}}
\]

\[s \in \mathbb{C}, s \neq 1,\] where final simplifications were later done by Lindelöf, [48, p. 103], who also gave details of their derivation.\(^{21}\) Application of contour integration methods to integrals (50) seems quite

\[^{20}\text{In the well-known monograph [8, vol. I], in formula (13) on p. 33, } \left( e^{2\pi i t} + 1 \right)^{-t} \text{ should be replaced by } \left( e^{2\pi i t} + 1 \right)^{-1}.\]

\[^{21}\text{Jensen, in [40], did not provide proofs for these formulæ; he only stated that he had } \text{“found them in his notes”}, \text{ and added that they can be easily derived by the Cauchy’s residue theorem. By the way, the first of these three formulæ was also obtained by Franel [25, 40, 39].}\]
attractive as well (especially if \( p \) is rational). Consider, for instance, the last integral in (50) at \( s = 1 \). This integral, if \( p \) is rational, may be reduced to elementary functions with the help of integral \( n^o \)-3-a from [10, Section 4] (the latter being evaluated by the method of contour integration). Indeed, by \( n^o \)-3-a, we have for \( p \equiv m/n \), numbers \( m \) and \( n \) being integers such that \( m < n \),

\[
\int_{-\infty}^{+\infty} \frac{\text{sh} \left[ \pi (2p-1)x \right]}{\text{sh} \pi x} \cdot \frac{dx}{1 \pm ix} = 2 \int_{0}^{\infty} \frac{\text{sh} \left[ \pi (2p-1)x \right]}{\text{sh} \pi x} \cdot \frac{dx}{x^2 + 1}
\]

\[
= \lim_{a \to 1} \left\{ \frac{\partial}{\partial a} \int_{0}^{\infty} \frac{\text{sh} \left[ \pi (2p-1)x \right]}{\text{sh} \pi x} \cdot \ln(x^2 + a^2) \, dx \right\} = \frac{1}{n} \sum_{l=1}^{2n-1} \sin \left( \frac{2\pi ml}{n} \right) \cdot \Psi \left( \frac{l+1}{2n} \right)
\]

\[
= \frac{\pi (2m-n)}{n} \cos \frac{2\pi m}{n} - 2 \sin \frac{2\pi m}{n} \cdot \ln \left( \frac{2 \sin \frac{\pi m}{n}}{n} \right),
\]

where we performed the final simplification with the help of first two formulae from (60). However, the evaluation of the derivative of the same integral with respect to \( p \) at \( s = 1 \) in a closed form faces much more difficulties. Choosing the sign \( "-" \) in the aforementioned integral gives the branch point at \(-i\). Hence, integration over the contour \( C \) consisting of the line along the real axis from \(-R\) to \(+R\) and the semicircle \( C_R \) of the non–integer radius \( R \) above the real axis, yields, in virtue of the Cauchy’s residue theorem, the following equality

\[
\oint_{C} \frac{\text{sh} \left[ \pi (2p-1)z \right]}{\text{sh} \pi z} \cdot \frac{\ln(1 - iz)}{1 - iz} \, dz = \int_{-R}^{+R} \ldots dx + \int_{C_R} \ldots dz
\]

\[
= 2\pi i \sum_{n=1}^{[R]} \text{Res} \left[ \frac{\text{sh} \left[ \pi (2p-1)z \right]}{\text{sh} \pi z} \cdot \frac{\ln(1 - iz)}{1 - iz} \right] = -2 \sum_{n=1}^{[R]} (-1)^n \sin \left( \pi n (2p-1) \right) \cdot \ln(n+1)
\]

where, in the first line, we omitted integrands for brevity. Now, as \( R \to \infty \) the integral taken along the semicircle \( C_R \) approaches zero. Hence, letting \( R \to \infty \) yields:

\[
\int_{-\infty}^{+\infty} \frac{\text{sh} \left[ \pi (2p-1)x \right]}{\text{sh} \pi x} \cdot \frac{\ln(1 - ix)}{1 - ix} \, dx = -2 \sum_{n=1}^{\infty} (-1)^n \sin \left( \pi n (2p-1) \right) \cdot \ln(n+1)
\]

(52)

which holds for any \( p \in (0,1) \), since we never used its rationality. The latter property could be, for example, used if integrating around an infinitely long rectangle of a prescribed breadth constrained to the parameter \( p \), but the branch point at \( \pm i \) is really annoying.

The sum \( \zeta''(0, p) + \zeta''(0, 1 - p) \) may be also reduced to an important logarithmico–trigonometric series

\[
\zeta''(0, p) + \zeta''(0, 1 - p) = -2(\gamma + \ln 2\pi) \ln(2 \sin \pi p) + 2 \sum_{n=1}^{\infty} \frac{\cos(2\pi p n \cdot \ln n)}{n}
\]

see [10, exercise no. 22]. This series, unlike the similar sine–series, is not known to be reducible to any elementary or “classical” function of analysis. However, it appeared in a number of previous works,
including works of Legendre, Lerch and Landau, see e.g. [46], [44], [30], and by the way, it also arises when simplifying the right part of (52).

As we previously showed in [10, exercise no. 22], the sum \( \zeta''(0, p) + \zeta''(0, 1 - p) \) may be also written in terms of the antiderivatives of the first generalized Stieltjes constant \( \Gamma_1(p) \)

\[
\zeta''(0, p) + \zeta''(0, 1 - p) = -(3 \ln 2 + 2 \ln \pi) \ln 2 - 4 \Gamma_1(1/2) + 2 \Gamma_1(p) + 2 \Gamma_1(1 - p)
\]

By the way, the latter formula, inserted into (26), gives an equation which is in some way analogous the Malmsten’s representation for the Digamma function (58) [in the sense that for rational arguments it provides a connection between the function and its derivative].

Finally, note that most of above representations remain valid everywhere in the strip \( 0 < \text{Re } p < 1 \), so it is not impossible that for rational \( p \) they could be further simplified or reduced to more convenient forms.

III. Extensions of the theorem to the second and higher Stieltjes constants

It can be reasonably expected that similar theorems could be derived for the second and higher Stieltjes constants. Such a demonstration could be carried out again with the help of \( f_a(p) \) and integral (14) where \( \ln x \) is replaced with \( \ln^nx \) [see below how integral (17) is used for the determination of the second Stieltjes constant]. As regards the equation for the difference of generalized Stieltjes constants, which is also necessary, it is simply sufficient to note that from (2) and (24) it follows that

\[
\gamma_n \left( \frac{r}{m} \right) - \gamma_n \left( 1 - \frac{r}{m} \right) = (-1)^n \lim_{a \to 1} \left\{ \zeta^{(n)} \left( a, \frac{r}{m} \right) - \zeta^{(n)} \left( a, 1 - \frac{r}{m} \right) \right\}
\]

\[
= 4(-1)^n \lim_{a \to 1} \frac{\partial^n}{\partial a^n} \left[ \frac{\Gamma(1 - a)}{(2\pi m)^{1-a}} \cos \frac{\pi a}{2} \cdot \sum_{i=1}^{m-1} \sin \frac{2\pi r l}{m} \cdot \zeta \left( 1 - a, \frac{l}{m} \right) \right], \quad n = 1, 2, 3, \ldots
\]

where \( r \) and \( m \) are positive integers such that \( r < m \). In particular, for the second generalized Stieltjes constant, the latter formula takes the form\(^{22}\)

\[
\gamma_2 \left( \frac{r}{m} \right) - \gamma_2 \left( 1 - \frac{r}{m} \right) = 2\pi \sum_{i=1}^{m-1} \sin \frac{2\pi r l}{m} \cdot \zeta'' \left( 0, \frac{l}{m} \right) - 4\pi(\gamma + \ln 2\pi m) \sum_{i=1}^{m-1} \sin \frac{2\pi r l}{m} \cdot \ln \Gamma \left( \frac{l}{m} \right)
\]

\[
+ \pi \left[ \frac{\pi^2}{12} + (\gamma + \ln 2\pi m)^2 \right] \cot \frac{\pi r}{m}
\]

(53)

In order to obtain a formula for \( \gamma_2(r/m) \), we take again the expansion (34) and write down its terms up to \( O(a-1)^3 \). Hence

\[
\int_0^\infty \frac{(\cosh [(2p-1)x] - 1) \ln^2 x}{\sinh x} \, dx = \frac{2}{3} \mathcal{C}_m(r) - 2B_m(r) \ln \pi m + \left\{ 2 \ln^2 \pi m + \frac{\pi^2}{6} \right\} A_m(r) - 2\gamma_1(\gamma - \ln 2)
\]

\[
+ \frac{2}{3} \zeta(3) - \frac{2}{3} \gamma^3 - \gamma_2 + \left( \gamma^2 - \frac{\pi^2}{6} \right) \ln 2 + \frac{\pi^2}{12} \ln \pi m + \ln \pi \cdot \ln m \cdot \ln \pi m + \frac{1}{3} \left\{ \ln^3 \pi + \ln^3 m - \ln^3 2 \right\}
\]

\(^{22}\)This formula appears also in an unpublished work sent to us by Donal Connolly.
Hurwitz zeta–function at zero at rational points

ζ

A. Some results from the theory of finite Fourier series

expressions for higher which is the analog of (38) for the second generalized Stieltjes constant. Obviously, this formula

series, which are essentially variants or particular cases of a same formula, finite Fourier series may take quite
different forms and expressions. For instance, in engineering sciences, one usually deals with the following

where

p \equiv r/m and

C_m(r) \equiv \sum_{l=1}^{m} \cos \frac{2\pi rl}{m} \cdot \zeta'' \left(0, \frac{1}{m}\right)

Comparing the latter integral to (17), and then using (37), we obtain

\gamma_2 \left(\frac{r}{m}\right) + \gamma_2 \left(1 - \frac{r}{m}\right) = 2\gamma_2 + \frac{4}{3} \sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \zeta'' \left(0, \frac{1}{m}\right) - 4(\gamma + \ln 2\pi m) \sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \zeta'' \left(0, \frac{1}{m}\right)

\gamma_m (r) = \gamma_2 - \gamma_2 \ln m + \frac{2}{3} \sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \zeta'' \left(0, \frac{1}{m}\right) - 2(\gamma + \ln 2\pi m) \sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \zeta'' \left(0, \frac{1}{m}\right)

Adding the latter equality to (53) finally yields

\gamma_2 \left(\frac{r}{m}\right) + \gamma_2 \left(1 - \frac{r}{m}\right) = 2\gamma_2 + \frac{4}{3} \sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \zeta'' \left(0, \frac{1}{m}\right) - 4(\gamma + \ln 2\pi m) \sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \zeta'' \left(0, \frac{1}{m}\right)

+ \frac{2}{3} \sum_{l=1}^{m-1} \sin \frac{2\pi rl}{m} \cdot \zeta'' \left(0, \frac{1}{m}\right) - 2\pi(\gamma + \ln 2\pi m) \sum_{l=1}^{m-1} \sin \frac{2\pi rl}{m} \cdot \ln \Gamma \left(\frac{1}{m}\right) - \gamma^3

- \left(\gamma + \ln 2\pi m\right)^2 \cdot \zeta' \left(\frac{r}{m}\right) + \frac{\pi^3}{12} \cdot \ln \frac{\pi r}{m} - \gamma^2 \ln \left(4\pi^2 m^3\right) + \frac{\pi^2}{12} (\gamma + \ln m)

\gamma \left(\ln^2 2\pi + 4 \ln m \cdot \ln 2\pi + 2 \ln^2 m\right) - \left(\ln^2 2\pi + 2 \ln 2\pi \cdot \ln m + \frac{2}{3} \ln^2 m\right) \ln m

which is the analog of (38) for the second generalized Stieltjes constant. Obviously, this formula
can be further simplified or reduced to the complete finite Fourier series form if necessary. Thus,
expressions for higher \gamma_n (r/m) are expected to be quite long and to contain higher derivatives of the
Hurwitz zeta–function at zero at rational points zeta(n)(0,1/m) whose properties are little studied.

A. Some results from the theory of finite Fourier series

Finite Fourier series are well-known and widely used in discrete mathematics, numerical analysis, engineering
sciences (especially in signal and image processing) and in a lot of related disciplines. Unlike usual Fourier
series, which are essentially variants or particular cases of a same formula, finite Fourier series may take quite
different forms and expressions. For instance, in engineering sciences, one usually deals with the following
2m–points Fourier series

f_m(r) = \frac{a_m(0)}{2} + \sum_{l=1}^{m-1} \left( a_m(l) \cos \frac{\pi rl}{m} + b_m(l) \sin \frac{\pi rl}{m} \right) + (-1)^l \frac{a_m(m)}{2}, \quad r = 0, 1, 2, \ldots, 2m-1, \quad m \in \mathbb{N},

With the help of orthogonality relations, one may determine the coefficients in this expansion:

\begin{align*}
a_m(k) &= \frac{1}{m} \sum_{r=1}^{2m-1} f_m(r) \cos \frac{\pi rk}{m}, \quad k = 0, 1, 2, \ldots, m \\
b_m(k) &= \frac{1}{m} \sum_{r=1}^{2m-1} f_m(r) \sin \frac{\pi rk}{m}, \quad k = 1, 2, 3, \ldots, m-1
\end{align*}
as well as derive the Parseval’s theorem
\[
\frac{1}{m} \sum_{r=1}^{2m-1} f_m^2(r) = \frac{a_m^2(0)}{2} + \sum_{l=1}^{m-1} \left( a_m^2(l) + b_m^2(l) \right) + \frac{a_m^2(m)}{2},
\]
see for more details [31, Chapter 6].

In contrast, in our researches, we encounter the following \((m-1)\)-points finite Fourier series
\[
f_m(r) = a_m(0) + \sum_{l=1}^{m-1} \left( a_m(l) \cos \frac{2\pi rl}{m} + b_m(l) \sin \frac{2\pi rl}{m} \right), \quad r = 1, 2, 3, \ldots, m-1, \quad m \in \mathbb{N},
\] (54)
for which inversion formulae and Parseval’s theorem are quite different. Let, first, derive the inversion formulae for the coefficients of this series. Multiplying both sides by \(\cos(2\pi rk/m)\), where \(k = 1, 2, 3, \ldots, m-1\), and summing over \(r \in [1, m-1]\), gives
\[
\sum_{r=1}^{m-1} f_m(r) \cos \frac{2\pi r k}{m} = \sum_{r=1}^{m-1} \left[ a_m(0) + \sum_{l=1}^{m-1} \left( a_m(l) \cos \frac{2\pi rl}{m} + \sum_{l=1}^{m-1} b_m(l) \sin \frac{2\pi rl}{m} \right) \cos \frac{2\pi rk}{m} \right] = a_m(0) - \sum_{l=1}^{m-1} a_m(l) + \frac{m}{2} \left\{ a_m(k) + a_m(m-k) \right\}
\]
Similarly, multiplying both sides of (54) by \(\sin(2\pi rk/m)\), where \(k = 1, 2, 3, \ldots, m-1\), and summing over \(r \in [1, m-1]\), yields
\[
\sum_{r=1}^{m-1} f_m(r) \sin \frac{2\pi r k}{m} = \sum_{r=1}^{m-1} \left[ a_m(0) + \sum_{l=1}^{m-1} \left( a_m(l) \cos \frac{2\pi rl}{m} + \sum_{l=1}^{m-1} b_m(l) \sin \frac{2\pi rl}{m} \right) \sin \frac{2\pi rk}{m} \right] = m \left\{ b_m(k) - b_m(m-k) \right\}
\]
Finally, Parseval’s equality for the finite series (54) reads:
\[
\sum_{r=1}^{m-1} f_m^2(r) = \left( \sum_{r=1}^{m-1} a_m(0) + \sum_{l=1}^{m-1} a_m(l) \cos \frac{2\pi rl}{m} + \sum_{l=1}^{m-1} b_m(l) \sin \frac{2\pi rl}{m} \right)^2 = \sum_{r=1}^{m-1} a_m^2(0) + 2a_m(0) \sum_{l=1}^{m-1} a_m(l) \sum_{l=1}^{m-1} b_m^2(l) + \frac{m}{2} \sum_{r=1}^{m-1} \sum_{l=1}^{m-1} a_m(l) a_m(m-l) \sum_{l=1}^{m-1} b_m^2(l) b_m(m-l) - \frac{\text{m}(\delta_{m-k}-\delta_{m-k})}{2m(m-1)}
\]
Note that in above formulae the performance of the sums’ permutation is permitted since all series are finite.
B. Malmsten’s finite Fourier series representation for the $\Psi$–function and some related summations

In 1842 Carl Malmsten found an interesting representation for the $\Psi$–function of a rational argument:

$$
\Psi\left(\frac{r}{m}\right) = -\gamma - \ln 2\pi m - \frac{\pi}{2} \cotg \frac{\pi r}{m} - 2 \sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \ln \Gamma\left(\frac{l}{m}\right), \quad r = 1, 2, \ldots, m-1.
$$

(58)

where $m$ is natural, see [49, p. 57, Eq. (70)], [10, Eq. (23)]. Malmsten did not notice that this formula may be further simplified and reduced to the Gauss’ Digamma theorem. However, on the other hand, if we recall that

$$
\sum_{l=1}^{m-1} l \cdot \sin \frac{2\pi rl}{m} = -\frac{m}{2} \cotg \frac{\pi r}{m}, \quad r = 1, 2, \ldots, m-1.
$$

we may rewrite (58) as the complete finite–length Fourier series for the $\Psi$–function at rational argument:

$$
\Psi\left(\frac{r}{m}\right) = -\gamma - \ln 2\pi m + \frac{\pi}{m} \sum_{l=1}^{m-1} \sin \frac{2\pi rl}{m} l - 2 \sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \ln \Gamma\left(\frac{l}{m}\right), \quad r = 1, 2, \ldots, m-1.
$$

(59)

This expression may be in some cases more suitable and more convenient to use than the usual Gauss’ Digamma theorem. For instance, from (59), thanks to semi–orthogonality properties of circular functions, we straightforwardly deduce following summation formulae

$$
\begin{align*}
\sum_{r=1}^{m-1} \Psi\left(\frac{r}{m}\right) \cdot \cos \frac{2\pi rk}{m} &= m \ln \left(2 \sin \frac{k\pi}{m}\right) + \gamma, \quad k = 1, 2, \ldots, m-1 \\
\sum_{r=1}^{m-1} \Psi\left(\frac{r}{m}\right) \cdot \sin \frac{2\pi rk}{m} &= \frac{\pi}{2} (2k - m), \quad k = 1, 2, \ldots, m-1 \\
\sum_{r=1}^{m-1} \Psi^2\left(\frac{r}{m}\right) &= (m - 1)\gamma^2 + m(2\gamma + \ln 4m) \ln m - m(m - 1) \ln^2 2 + \frac{\pi^2 (m^2 - 3m + 2)}{12} + 2m \sum_{l=1}^{m-1} \ln^2 \sin \frac{\pi l}{m}
\end{align*}
$$

(60)

Indeed, inserting expressions for coefficients $a_m(0) = -\gamma - \ln 2\pi m$, $a_m(1) = -2\ln \Gamma(1/m)$ and $b_m(l) = \pi l/m$ into (55), we have for the first sum:

$$
\sum_{r=1}^{m-1} \Psi\left(\frac{r}{m}\right) \cdot \cos \frac{2\pi rk}{m} = \gamma + \ln 2\pi m - m \left[ \ln \Gamma\left(\frac{k}{m}\right) + \ln \Gamma\left(1 - \frac{k}{m}\right) \right] + 2 \sum_{l=1}^{m-1} \ln \Gamma\left(\frac{l}{m}\right) = \gamma + m \ln \left(2 \sin \frac{k\pi}{m}\right)
$$

(61)

where the final simplification is performed with the help of the reflection formula and the Gauss’ multiplication theorem for the logarithm of the $\Gamma$–function. Analogously, using (56) yields for the second sum:

$$
\sum_{r=1}^{m-1} \Psi\left(\frac{r}{m}\right) \cdot \sin \frac{2\pi rk}{m} = \frac{m}{2} \left[ \pi k - \pi (m - k) \right] = \frac{\pi}{2} (2k - m)
$$

(62)

Finally, by (57) we derive the Parseval’s theorem for the $\Psi$–function of a discrete argument takes the following form

$$
\sum_{r=1}^{m-1} \Psi^2\left(\frac{r}{m}\right) = (m - 1)(\gamma + \ln 2\pi m)^2 - 4(\gamma + \ln 2\pi m) \sum_{l=1}^{m-1} \ln \Gamma\left(\frac{l}{m}\right) - 4 \sum_{l=1}^{m-1} \ln \Gamma\left(\frac{l}{m}\right) + 2m \sum_{l=1}^{m-1} \ln^2 \Gamma\left(\frac{l}{m}\right) - \frac{\pi^2 (m^2 - 3m + 2)}{12} + 2m \sum_{l=1}^{m-1} \ln^2 \sin \frac{\pi l}{m}
$$

(63)
where the first sum from the second line, thanks to the symmetry of \( \ln \sin \pi l/m \) about \( l = m/2 \) and to the fact that \( \ln \sin \pi l/m = 0 \) for \( l = m/2 \), could be simplified as follows

\[
\sum_{l=1}^{m-1} \ln \Gamma \left( \frac{1}{m} \right) \cdot \left[ \ln \Gamma \left( \frac{1}{m} \right) + \ln \Gamma \left( 1 - \frac{1}{m} \right) \right] = \sum_{l=1}^{m-1} \ln \Gamma \left( \frac{1}{m} \right) \cdot \left[ \ln \pi - \ln \sin \frac{\pi l}{m} \right] = \frac{\ln \pi}{2} \left( (m-1) \ln 2\pi - \ln m \right)
\]

\[
- \sum_{l=1}^{m-1} \ln \Gamma \left( \frac{1}{m} \right) \cdot \ln \sin \frac{\pi l}{m} = \frac{\ln \pi}{2} \left( (m-1) \ln 2\pi - \ln m \right) - \sum_{l=1}^{(m-1)/2} \left[ \ln \pi - \ln \sin \frac{\pi l}{m} \right] \ln \sin \frac{\pi l}{m} = \frac{\ln \pi}{2} \left( (m-1) \ln 2\pi - \ln m \right)
\]

\[
- \left[ \frac{1}{\sin r} \right] + \left[ \frac{1}{\sin r} \right] = \frac{\ln \pi}{2} \left( (m-1) \ln 2\pi - \ln m \right) + \sum_{l=1}^{(m-1)/2} \ln^2 \sin \frac{\pi l}{m}
\]

because

\[
\sum_{l=1}^{m-1} \ln \sin \frac{\pi l}{m} = \ln \prod_{l=1}^{m-1} \sin \frac{\pi l}{m} = \frac{1}{2} \ln 2 + \frac{1}{2} \ln m
\]

and where

\[
\sum_{l=1}^{m-1} l^2 = \frac{m(m-1)(2m-1)}{6} \quad \text{and} \quad \sum_{l=1}^{m-1} l = \frac{m(m-1)}{2}
\]

respectively, which completes the evaluation of the third formula in (60).

In like manner, we may also derive similar summation formulae for the Hurwitz \( \zeta \)-function. Rewriting Hurwitz' functional equation (24) in the form analogous to (54)

\[
\zeta \left( \frac{am}{m} \right) = m^{a-1} \zeta (a) + \frac{2\Gamma (1-a)}{(2\pi m)^{1-a}} \left[ \sin \frac{\pi a}{2} \sum_{l=1}^{m-1} \cos \frac{2\pi rl}{m} \cdot \zeta \left( 1 - a, \frac{l}{m} \right) - \cos \frac{\pi a}{2} \sum_{l=1}^{m-1} \sin \frac{2\pi rl}{m} \cdot \zeta \left( 1 - a, \frac{l}{m} \right) \right]
\]

yields

\[
\begin{align*}
\sum_{r=1}^{m-1} \zeta \left( \frac{am}{m} \right) & \cdot \cos \frac{2\pi rk}{m} = \frac{m\Gamma (1-a)}{(2\pi m)^{1-a}} \sin \frac{\pi a}{2} \cdot \left\{ \zeta \left( 1 - a, \frac{k}{m} \right) + \zeta \left( 1 - a, 1 - \frac{k}{m} \right) \right\} - \zeta (a) \\
\sum_{r=1}^{m-1} \zeta \left( \frac{am}{m} \right) & \cdot \sin \frac{2\pi rk}{m} = \frac{m\Gamma (1-a)}{(2\pi m)^{1-a}} \cos \frac{\pi a}{2} \cdot \left\{ \zeta \left( 1 - a, \frac{k}{m} \right) - \zeta \left( 1 - a, 1 - \frac{k}{m} \right) \right\} \\
\sum_{r=1}^{m-1} \zeta^2 \left( \frac{am}{m} \right) & = \left( m^{2a-1} - 1 \right) \zeta^2 (a) + \frac{2m\Gamma^2 (1-a)}{(2\pi m)^{2-2a}} \sum_{l=1}^{m-1} \zeta \left( 1 - a, 1 - \frac{l}{m} \right) \cdot \zeta \left( 1 - a, \frac{l}{m} \right)
\end{align*}
\]

which hold for any \( r = 1, 2, 3, \ldots, m-1 \) and \( k = 1, 2, 3, \ldots, m-1 \), where \( m \) is positive integer.

C. On the complex differentiability of the integral \( I_a (rl/m) \) in a neighbourhood of \( a = 1 \)

In order to show that \( I_a (rl/m) \) is regular in a neighbourhood of \( a = 1 \), it suffices to ascertain that in the region of uniform convergence, the complex–valued function \( x^{a-1}, x \in \mathbb{R}^+, a \in \mathbb{C} \), is differentiable with respect to \( a \). The necessary and sufficient condition that a complex–valued function be differentiable is that its both real and imaginary parts be differentiable, and that the Cauchy–Riemann equations\(^{23}\) hold. Denoting for brevity \( a \equiv \text{Re} \ a \) and \( \beta \equiv \text{Im} \ a \), we separate the real part from the imaginary one of \( x^{a-1} \) as follows

\[
x^{a-1} = x^{a-1} e^{i\beta} = x^{a-1} e^{i(\beta + \ln x)} = x^{a-1} \cos (\beta \ln x) + i x^{a-1} \sin (\beta \ln x)
\]

\(^{23}\)For historical reasons, some author prefer to call these equations D’Alambert–Euler equations [51].
Verification of the first equation yields
\[
\begin{align*}
\frac{\partial u}{\partial \alpha} &= x^{\alpha-1} \ln x \cdot \cos(\beta \ln x) \\
\frac{\partial v}{\partial \beta} &= x^{\alpha-1} \cos(\beta \ln x) \ln x
\end{align*}
\]
⇒ \( \frac{\partial u}{\partial \alpha} = \frac{\partial v}{\partial \beta} \)

while the second one gives
\[
\begin{align*}
\frac{\partial u}{\partial \beta} &= -x^{\alpha-1} \sin(\beta \ln x) \ln x \\
\frac{\partial v}{\partial \alpha} &= x^{\alpha-1} \ln x \cdot \sin(\beta \ln x)
\end{align*}
\]
⇒ \( \frac{\partial u}{\partial \beta} = -\frac{\partial v}{\partial \alpha} \)

Thus, \( x^{\alpha-1}, x \in \mathbb{R}^+ \), is an entire function on the complex \( \alpha \)-plane, which, together with the uniform convergence, plenty guarantees the existence of the Taylor series for \( f_\alpha(r/m) \) near \( \alpha = 1 \). For a deeper study of regular (or holomorphic) and analytic functions, please refer to these classical complex analysis monographs: [23], [51], [58], [12], [34], [5].

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