THE HURWITZ SUBGROUPS OF $E_6(2)$

EMILIO PIERRO†

Fakultät für Mathematik,
Universität Bielefeld,
33602, Bielefeld,
Germany

Abstract. We prove that the exceptional group $E_6(2)$ is not a Hurwitz group.
In the course of proving this, we complete the classification up to conjugacy of
all Hurwitz subgroups of $E_6(2)$, in particular, those isomorphic to $L_2(8)$ and
$L_3(2)$.

1. Introduction

Hurwitz groups are finite quotients of the triangle group
\[ \Delta := \Delta(2,3,7) = \langle x, y, z \mid x^2 = y^3 = z^7 = xyz = 1 \rangle \]
and interest in them stems from their connection to Riemann surfaces. A result due
to Hurwitz [5] states that if $\Sigma_g$ is a compact Riemann surface of genus $g > 1$ and $G$
is a group of conformal automorphisms of $\Sigma_g$, then the order of $G$ is bounded above
by $84(g - 1)$. Groups which attain this bound are known as Hurwitz groups and for
a finite group $G$ to be a Hurwitz group, this is equivalent to $G$ being a finite quotient
of $\Delta(2,3,7)$. We refer the reader to the most recent and most excellent survey on
Hurwitz groups by Conder [1] and the references therein for more information.

Rather unexpectedly, the study of Hurwitz groups becomes extremely useful for
capturing and classifying conjugacy classes of small non-abelian finite quasi-simple
subgroups of a finite group $G$. The purpose of this paper is then twofold. Our main
result is the following.

Theorem 1. Let $H \leq E_6(2)$ be a Hurwitz group. Then $H$ is isomorphic to $L_2(8)$,
$L_3(2)$, $2^3 L_3(2)$ or $3 D_4(2)$. In particular, $E_6(2)$ is not a Hurwitz group.

In the course of proving the above theorem it becomes necessary to classify, up
to conjugacy, subgroups of $E_6(2)$ of the isomorphism types named in Theorem 1.
These are then listed in Table 2. The majority of the work towards the proof of
the above theorem and Table 2 is due to Kleidman and Wilson [7] which we shall
cite almost continuously. We note, however, that there is a slight error [7, pg.288]
where the authors identify $F_4(2)$ as a Hurwitz group. This is known to be false
(see Table 3) and, following the suggestion of Wilson to the author, the proof of
Theorem 1 does not rely on the main result of 7. We are able to use results from

E-mail address: emilio.pierro@uwa.edu.au.
Date: April 30, 2018.
to prove almost everything, however for certain conjugacy classes of subgroups we rely on GAP \([10]\) to determine the precise structure of their normalisers.

In and of itself, Theorem 1 is a minor result in the study of Hurwitz groups. However, it is a necessary first step towards determining which members of the families of exceptional groups of Lie type of (twisted) rank at least 4 are Hurwitz groups. For exceptional groups of (twisted) rank less than 4, we summarise the known results in Table 1. In addition to those mentioned, the group \(F_4(2)\) is known not to be a Hurwitz group, see for example Table 6. For general \(q\), however, it is not known whether \(F_4(q)\) is a Hurwitz group.

| \(K\) | Hurwitz | Reference |
|------|---------|-----------|
| \(2B_2(2^{2m+1})\) | None | |
| \(2G_2(3^{2m+1})\) | \(m \geq 1\) | Malle [8], Jones [6] |
| \(2F_4(2^{2m+1})\) | \(m \equiv 1\) mod 3 | Malle [9] |
| \(^3D_4(p^n)\) | \(p \neq 3, p^n \neq 4\) | Malle [9] |
| \(G_2(p^n)\) | \(p^n \geq 5\) | Malle [8] |

Table 1. Status of exceptional groups of Lie type as Hurwitz groups.

Our method, which is standard, is outlined in some detail in Section 2. In brief, given a finite group \(G\), it is possible to determine the order of \(\text{Hom}(\Delta, G)\) and then using knowledge of the subgroup structure of \(G\), determine which elements of this set are in fact epimorphisms. In Section 3 we record various necessary preliminary results about the structure of \(E_6(2)\) and in Section 4 we account for the elements of \(\text{Hom}(\Delta, G)\). In the Appendix we record the conjugacy classes and normalisers of Hurwitz subgroups of the groups \(L_6(2), O_{10}^+(2)\) and \(F_4(2)\) which will be necessary for the proof of Theorem 1. We provide these without proof since they can be easily determined in GAP.

We use ATLAS [2] notation throughout. In particular, we use the notation \(nX\) to denote a conjugacy class of elements of order \(n\), ordered alphabetically in decreasing size of their centraliser. If \(nX\) and \(nY\) are two conjugacy classes where elements from \(nX\) are powers of elements of \(nY\), then we write the union of these two classes as \(nXY\). When we consider the normaliser of a subgroup \(H\) in \(E_6(2)\), we shall normally write this as \(N_{E_6}(H)\) for brevity. A similar practice will be used for the subgroups isomorphic to \(F_4(2), O_{10}^+(2), L_6(2)\) and \(^3D_4(2)\) with the obvious abbreviations.

Acknowledgements. The author is grateful to Rob Wilson and Alastair Litterick for many helpful conversations in the preparation of this paper. The author is also grateful to Marston Conder for assistance in the verification of these results.

This work was supported by the SFB 701 and by ARC grant DP140100416.

2. Preliminaries

The techniques we employ are standard and ‘well-known’ to those in the area. We include a description of them here so as to be as self-contained as possible. We begin with some terminology.

Definition 2. Let \(G\) be a finite group and let \(2X, 3X\) and \(7X\) be \(G\)-conjugacy classes of elements. If there exists \(x \in 2X, y \in 3X\) and \(z \in 7X\) such that \(xyz = 1\),
then we call \((x, y, z)\) a Hurwitz triple and \(H = \langle x, y, z \rangle\) a Hurwitz subgroup of type \((2X, 3X, 7X)\), or usually a \((2X, 3X, 7X)\)-subgroup.

In order to count the number of elements of the set
\[
\mathcal{H} = \{(x, y, z) \in G \times G \times G \mid x^2 = y^3 = z^7 = xyz = 1\}
\]
it is necessary to determine the structure constants of \(G\). Note that one can naturally identify elements of \(\mathcal{H}\) with elements of \(\text{Hom}(\Delta, G)\). For this we use the well known structure constant formula due to Frobenius \([3]\) as follows. If \(C_1, C_2\) and \(C_3\) are three not necessarily distinct conjugacy classes in a finite group \(G\), then the order of the set
\[
\mathcal{C} = \{(x, y, z) \in C_1 \times C_2 \times C_3 \mid xyz = 1\}
\]
is given by the following formula
\[
|\mathcal{C}| = \frac{|C_1||C_2||C_3|}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}.
\]
Note that we give the specific form for three conjugacy classes but such a formula exists for any finite number of conjugacy classes. In practice, we shall ‘normalise’ this constant and define
\[
n_G(C_1, C_2, C_3) = \frac{|\mathcal{C}|}{|G|}.
\]

It will also be necessary to determine the structure constants within Hurwitz subgroups of \(G\). If \(H\) is a Hurwitz subgroup of \(G\) of type \((2X, 3X, 7X)\), where \(2X\), \(3X\) and \(7X\) are \(G\)-conjugacy classes, then, by abuse of notation, we denote by \(n_H(2X, 3X, 7X)\) the sum over all combinations of normalised structure constants \(n_H(C_i, C_j, C_k)\) where \(C_i, C_j\) and \(C_k\) are \(H\)-conjugacy classes of elements which fuse to the \(G\)-classes \(2X, 3X\) and \(7X\) respectively. Since ultimately we shall only deal with four isomorphism classes of Hurwitz subgroups of \(G\), this should not cause too much confusion.

Given a Hurwitz triple \((x, y, z)\) in \(G\) of type \((2X, 3X, 7X)\) this clearly generates a unique Hurwitz subgroup \(H = \langle x, y \rangle \leq G\). Letting \(H_i\) for \(i \in I\) denote the \(G\)-conjugacy class representatives of \((2X, 3X, 7X)\)-subgroups, we obtain the following formula
\[
n_G(2X, 3X, 7X) = \sum_{i \in I} \frac{n_{H_i}(2X, 3X, 7X)}{[N_G(H_i) : H_i]}.
\]

3. Auxiliary structural results

In this section we collect a number of relevant results from \([7]\) and prove further results that will be necessary for classifying normalisers in \(E_6(2)\) of Hurwitz subgroups. We begin with some local analysis.
We reproduce from [7] Table 2 the conjugacy classes and normalisers of elements of orders 2, 3 and 7 in $E_6(2)$ for reference.

\[ N_E(2A) \cong 2_{+}^{1+20}: L_6(2) \]
\[ N_E(2B) \cong [2^{24}]: S_6(2) \leq 2^{16}: O^{+}_{10}(2) \]
\[ N_E(2C) \cong [2^{27}]: (L_2(2) \times L_3(2)) \leq 2^{5+9+18}: (L_2(2) \times L_3(2) \times L_3(2)) \]
\[ N_E(3A) \cong L_2(2) \times L_6(2) \]
\[ N_E(3B) \cong (3 \times O^+_6(2)): 2 \leq O^{+}_{10}(2) \]
\[ N_E(3C) \cong 3(3^2: Q_8 \times L_3(4)): S_3 \]
\[ N_E(7AB) \cong (7 \times 3D_4(2)): 3 \]
\[ N_E(7C) \cong (7: 3 \times L_3(2) \times L_3(2)): 2 \leq (L_3(2) \times L_3(2) \times L_3(2)): S_3 \]
\[ N_E(7D) \cong (7^2: 3 \times L_3(2)): 2 \leq (L_3(2) \times L_3(2) \times L_3(2)): S_3 \]

The character table of $E_6(2)$ is known and can be found in GAP. The non-zero normalised $(2,3,7)$-structure constants are then as follows.

\[ n_E(2A, 3A, 7C) = 1/28224 \quad n_E(2C, 3A, 7C) = 11/192 \]
\[ n_E(2B, 3A, 7C) = 1/288 \quad n_E(2C, 3B, 7D) = 43/24 \]
\[ n_E(2B, 3B, 7AB) = 1/6048 \quad n_E(2C, 3C, 7C) = 15/7 \]
\[ n_E(2B, 3B, 7D) = 5/56 \quad n_E(2C, 3C, 7D) = 137/21 \]

An important class of subgroups of $E_6(2)$ are those isomorphic to $L_3(2)^3$ whose existence and uniqueness up to conjugacy follows from [7] Propositions 2.3, 5.4. For the rest of the paper we denote by $T$ the normaliser in $E_6(2)$ of such a subgroup, that is

\[ T \cong (L_3(2) \times L_3(2) \times L_3(2)): S_3. \]

Note that the action of an element of order 3 in $T \setminus T''$ permutes the factors of $T''$ cyclically, whereas an element of order 2 acts as an outer automorphism on each factor of $T''$ and interchanges two of its factors [7] Proposition 2.3. The fusion of elements of $T$ of orders 3 or 7 is given in [7] Lemma 2.6 and [7] Lemma 8.3. To summarise, let $t$ be an element of order 3 or 7 in $T''$.

1. If $t$ projects non-trivially onto one factor of $T''$, then $t \in 3A$ or $7C$.
2. If $t$ projects non-trivially onto two factors of $T''$, then $t \in 3B, 7AB$ or $7D$.
3. If $t$ projects non-trivially onto three factors of $T''$, then $t \in 3C, 7C$ or $7D$.

The following lemma determines the fusion of a subgroup of $T$ isomorphic to $L_3(2)$ using the above information.

**Lemma 3.** Let $H$ be a subgroup of $T$ with $H \cong L_3(2)$. Then $H$ is of one of the following $E_6(2)$-types: $(2A, 3A, 7C), (2B, 3B, 7AB), (2B, 3B, 7D), (2C, 3C, 7C)$ or $(2C, 3C, 7D)$.

**Proof.** Let $H \cong L_3(2) \leq T$. We may immediately reduce to the case that the structure constant corresponding to the type of $H$ is non-zero. If $H$ projects onto one factor of $T''$, then $H$ contains subgroups isomorphic to $7A: 3A$. Since the centraliser of a $2B$-involution does not contain $L_3(2) \times L_3(2)$, $H$ must be of type $(2A, 3A, 7C)$. If $H$ projects onto two factors, then $H$ contains subgroups isomorphic to $7AB: 3B$ or $7D: 3B$ and hence $H$ is of type $(2B, 3B, 7AB), (2B, 3B, 7D)$ or
(2C, 3B, 7D). However, type (2C, 3B, 7D) cannot occur by [7, Proposition 8.4] since the centraliser in $E_6(2)$ of $H$ contains a subgroup of shape $L_3(2)$. Finally, if $H$ projects onto three factors of $H$, then $H$ contains subgroups isomorphic to $7C: 3C$ or $7D: 3C$, hence $H$ must be of type $(2C, 3C, 7C)$ or type $(2C, 3C, 7D)$. \hfill \Box

In the sequel, we shall need to classify many conjugacy classes of subgroups of $E_6(2)$ isomorphic to $L_3(2) \cong L_2(7)$. In particular, the analysis of subgroups isomorphic to $7C: 3C$ will be crucial. We summarise the necessary results in the following lemma.

**Lemma 4.** Let $P \cong 7: 3 \leq L_3(2) \leq E_6(2)$.

1. If elements of order 7 in $P$ fuse to $E_6(2)$-class $7C$, then either
   
   (a) $P \cong 7C: 3A$ and $N_E(7C: 3A) \cong (7: 3 \times L_3(2) \times L_3(2)): 2$ or
   
   (b) $P \cong 7C: 3C$ and $N_E(7C: 3C) \cong 3 \times ((3 \times (7: 3)): 2).

2. If elements of order 7 in $P$ fuse to $E_6(2)$-class $7D$, then either
   
   (a) $P \cong 7D: 3B$ and $N_E(7D: 3B) \cong (7: 3 \times L_3(2)): 2$, or
   
   (b) $P \cong 7D: 3C$ and $N_E(7D: 3C) \cong 7: 6 \times 3$.

**Proof.** Let $P \cong 7: 3$ where $z \in P$ belongs to $7C$ or $7D$ and $y \in P$ has order 3. Let $T'' \cong K_1 \times K_2 \times K_3$ where $K_1 \cong L_3(2)$. Since $P$ is contained in a conjugate of $T''$ we see from Lemma 3 that $P$ is isomorphic to one of the four types given. For the proof of (1), suppose that $z \in 7C$. Without loss of generality we can assume that $z$ projects non-trivially onto only one factor of $T''$, say $K_1$. If $y \in 3A$, then, since $y$ also projects non-trivially onto $K_1$, the structure of $N_E(P)$ is clear. Now suppose that $y \in 3C$ so that $y$ projects non-trivially onto all three factors of $T''$. It is clear that the only elements of $T''$ centralising $P$ are elements of order 3 and $C_E(P)$ is normalised by elements of order 2 but not 3 in $T \setminus T''$. The proof of part (2) is contained in the proof of [7, Proposition 8.4], which completes the proof. \hfill \Box

We now turn to large simple subgroups of $E_6(2)$. The isomorphism classes of simple subgroups of $E_6(2)$ are determined in [7, Section 4]. Those which will be of most importance to us are those isomorphic to $F_4(2)$, $O_{10}^-(2)$, $L_6(2)$ and $3D_4(2)$. From [7, Section 6] we see that subgroups of $E_6(2)$ isomorphic to $F_4(2)$ are all conjugate, as are those isomorphic to $O_{10}^+(2)$ or $L_6(2)$. Subgroups isomorphic to $3D_4(2)$ fall into two conjugacy classes and any $3D_4(2)$ subgroup is contained in a subgroup isomorphic to $F_4(2)$. The fusion of conjugacy classes of $F_4(2)$, $O_{10}^-(2)$ and $L_6(2)$ into $E_6(2)$ are given in Table 3. These can easily be determined by the power maps given in [2] and by GAP. The fusion of conjugacy classes of $3D_4(2)$ into $F_4(2)$ can also be determined. This is necessary to prove the following lemma.

**Lemma 5.** Let $H \cong 3D_4(2) \leq E_6(2)$. If $N_E(H) \cong (7 \times 3D_4(2)): 3 \leq E_6(2)$, then a Hurwitz generating triple of $H$ is of $E_6(2)$-type $(2C, 3C, 7C)$.

**Proof.** Let $H$ be as in the hypothesis and let $(x, y, z)$ be a Hurwitz generating triple for $H$. By [9, Proposition 4], $(x, y, z)$ is of $3D_4(2)$-type $(2B, 3B, 7D)$. The fusion of $3D_4(2)$ into $F_4(2)$ can easily be determined (up to outer automorphism of $F_4(2)$) and from the fusion of $F_4(2)$ into $E_6(2)$ we see that $(x, y, z)$ is of $E_6(2)$-type either $(2C, 3C, 7C)$ or $(2C, 3C, 7D)$.

Let $P \cong 7: 3 \leq 3D_4(2)$. From [2, pg. 89] we see that $P$ does not contain elements from $3D_4(2)$-class $7ABC$, since such elements are normalised only by elements of order 2. If $P$ is contained in a maximal subgroup of $3D_4(2)$ isomorphic
to \((7 \times L_2(8)) : 2\), then elements of order 3 in \(P\) fuse to \(3D_4(2)\)-class 3A, since \(C_D(3B)\) is not divisible by 7. Since all elements of order 3 in \(N_D(7D)\) are \(3D_4(2)\)-conjugate, it follows that \(P \cong 7D : 3A\) is unique up to conjugacy in \(3D_4(2)\).

Since \(N_D(3A) \cong S_3 \times L_2(8)\), these elements fuse to \(F_4(2)\)-class 3A or 3B and hence \(E_6(2)\)-class 3A or 3B. Then, by Lemma 8, subgroups isomorphic to \(P\) are of \(E_6(2)\)-type \(7C: 3A\) or \(7D: 3B\). Since \(P\) is contained in a conjugate of \(C_D(7ABC) \cong 7 \times L_3(2)\), if \(C_E(H) \cong 7\), then \(N_E(P)\) contains \(7 \times (7: 3) \times L_3(2)\) and hence the \(3D_4(2)\)-class \(7D\) fuses to the \(E_6(2)\)-class \(7C\). This completes the proof. □

4. THE HURWITZ SUBGROUPS OF \(E_6(2)\)

We now account for all of the conjugacy classes of Hurwitz subgroups of \(E_6(2)\) and prove that they are as stated in Table 2. The proof is organised into lemmas according to the type of each Hurwitz subgroup.

(2A, 3A, 7C)-subgroups.

**Lemma 6.** There is a unique conjugacy class of \((2A, 3A, 7C)\)-subgroups in \(E_6(2)\). If \(H\) belongs to this class, then \(H \cong L_3(2)\) and \(N_E(H) \cong (L_3(2) \times L_3(2) \times L_3(2)) : 2\).

**Proof.** From the examination of the normalisers of \(p\)-elements [7, Table 2], we see that a factor of \(T''\) must be of type \((2A, 3A, 7C)\). From the discussion of the action of elements in \(T \setminus T''\), it is clear that subgroups projecting onto one factor are conjugate in \(E_6(2)\) and that \(N_E(H) \cong (L_3(2) \times L_3(2)) : 2\). This accounts for the entire \(n_E(2A, 3A, 7C)\) structure constant. □

(2B, 3A, 7C)-subgroups.

**Lemma 7.** There is a unique conjugacy class of \((2B, 3A, 7C)\)-subgroups in \(E_6(2)\). If \(H\) belongs to this class, then \(H \cong L_3(2)\) and \(N_E(H) \cong L_3(2) \times S_4 \times S_4\).

**Proof.** From Table 4 we see that \(E_6(2)\) contains a class of \((2B, 3A, 7C)\)-subgroups with representative \(H \cong L_3(2)\) and \(N_L(H) \cong H \times S_4\). Let \(y \in S\) have order 3 and note that \(y\) belongs to \(E_6(2)\)-class 3A since it belongs to \(L_6(2)\)-class 3B. Then, \(C_E(y)' \cong L_6(2)\) contains \(H \times S_4\). Since \(C_E(y)' \cap S\) is trivial, it follows that \(N_E(H)\) contains \(C \cong H \times S_4 \times S_4\). Let \(P \cong 7C : 3A\). Since the group \(P \times S_4 \times S_4\) is self-normalising in \(E_6(2)\) it follows that \(N_E(H) = C\). Since this accounts for the full structure constant, the proof is complete. □

(2B, 3B, 7AB)-subgroups.

**Lemma 8.** There is a unique conjugacy class of \((2B, 3B, 7AB)\)-subgroups in \(E_6(2)\). If \(H\) belongs to this class, then \(H \cong L_3(2)\) and \(N_E(H) \cong L_3(2) \times G_2(2)\).

**Proof.** The existence and structure follows from [7, Proposition 5.2] and we see immediately that they account for the entire structure constant. □

(2B, 3B, 7D)-subgroups.

**Lemma 9.** There are two conjugacy classes of \((2B, 3B, 7D)\)-subgroups in \(E_6(2)\) with representatives \(H\) as follows:

1. \(H \cong L_3(2)\) and \(N_E(H) \cong (L_3(2) \times L_3(2)) : 2\); or,
2. \(H \cong L_3(2)\) and \(N_E(H) \cong L_3(2) \times S_4\).

**Proof.** The existence and structure follows from [7, Proposition 8.4]. It remains to observe that these classes account for the entire structure constant. □
Lemma 10. There are two conjugacy classes of \((2C,3A,7C)\)-subgroups in \(E_6(2)\) with representatives \(H\) as follows:

1. \(H \cong L_3(2)\) and \(N_E(H) \cong (L_3(2) \times D_8) : 2\) or
2. \(H \cong 2^3L_3(2)\) and \(N_E(H) \cong 2^3L_3(2) \times (2^2 \times 2^2) : S_3\).

Proof. For the proof we use GAP to explicitly construct conjugacy class representatives and their normalisers in \(E_6(2)\). We take as generators for \(E_6(2)\) those given in [4] Section 3.3 since this allows us to easily construct the parabolic and maximal rank subgroups of \(E_6(2)\). From Table 6 we see that there exists \((2C,3A,7C)\)-subgroups of \(E_6(2)\) isomorphic to \(L_3(2)\) and \(2^3L_3(2)\). Let \(P \cong 7C : 3A\) denote the normaliser in \(H\) of a Sylow 7-subgroup and note that \(P\) does not depend on the isomorphism type of \(H\). We then construct the normaliser in \(E_6(2)\) of a 3A-element in \(P\) and check which of the 1113210 \(2C\)-involutions inverting the 3A element generate with \(P\) a subgroup isomorphic to \(H\). Of these, there are 441 distinct subgroups isomorphic to \(L_3(2)\) and 588 distinct subgroups isomorphic to \(2^3L_3(2)\). We then see that under the action of \(N_E(P)\) isomorphic groups belong to the same orbit, hence \(H\) is determined up to conjugacy by its isomorphism type. We are then able to construct the full normaliser in \(E_6(2)\) of each isomorphism type and they are as stated. Finally, since these conjugacy classes account for the entire structure constant, the proof is complete. □

Lemma 11. There are four conjugacy classes of \((2C,3B,7D)\)-subgroups in \(E_6(2)\) with representatives \(H\) as follows:

1. \(H \cong L_3(2)\) and \(N_E(H) \cong (L_3(2) \times D_8) : 2\)
2. \(H \cong 2^3L_3(2)\) and \(N_E(H) \cong 2^3L_3(2) \times S_4\)
3. \(H \cong 2^3L_3(2)\) and \(N_E(H) \cong 2^3L_3(2) \times D_8\) or
4. \(H \cong 2^3L_3(2)\) and \(N_E(H) \cong 2^3L_3(2) \times 2^2\).

Proof. By [4] Proposition 8.4 there exists a unique conjugacy class of subgroups \(H \cong L_3(2)\) in \(E_6(2)\) and \(N_E(H)\) is as given. From Table 6 we see that there exist \((2C,3B,7D)\)-subgroups isomorphic to \(2^3L_3(2)\) and we now determine the conjugacy classes of such subgroups. Let \(H \cong 2^3L_3(2)\) and let \(P \cong 7D : 3B\) denote the normaliser in \(H\) of a Sylow 7-subgroup. As in the proof of the previous lemma, we use GAP to construct conjugacy class representatives and their normalisers in \(E_6(2)\) for each of the remaining classes. There are 128520 \(2C\)-involutions inverting a 3B-element in \(P\). Gathering these together, there are 140 distinct subgroups isomorphic to \(2^3L_3(2)\) generated by \(P\) and any such involution. Under the action of \(N_E(P) \cong (P \times L_3(2)) : 2\) these fall into three orbits of lengths 14, 42 and 84 with normalisers in \(E_6(2)\) as given. Since these four conjugacy classes account for the entire structure constant, the proof is complete. □

Lemma 12. There are four conjugacy classes of \((2C,3C,7C)\)-subgroups in \(E_6(2)\) with representatives \(H\) as follows:

1. \(H \cong 3^2D_4(2)\) and \(N_E(H) \cong (7 \times 3^4D_4(2)) : 3\)
2. \(H \cong L_3(2)\) and \(N_E(H) \cong (3 \times L_3(2)) : 2\)
3. \(H \cong L_2(8)\) and \(N_E(H) \cong S_4 \times (L_2(8) : 3)\), or,
(4) \( H \cong L_2(8) \) and \( N_E(H) \cong 2 \times L_2(8) \).

Proof. From \([7\text{ Proposition 6.11}]\) we see that subgroups isomorphic to \( ^3D_4(2) \) exist and fall into two conjugacy classes distinguished by their normaliser in \( E_6(2) \). It follows from Lemma \([5]\) that if \( H \cong ^3D_4(2) \), then \( N_E(H) \cong (7 \times ^3D_4(2)) : 3 \).

From Table \([3]\) we see that subgroups isomorphic to \( O_{10}^+(2) \) contain a unique conjugacy class of subgroups isomorphic to \( L_3(2) \) of \( E_6(2) \)-type \((2C, 3C, 7C)\). Let \( H \cong L_3(2) \) denote a representative from this class. Since \( H \) is centralised in \( O_{10}^+(2) \) by an element of order 3 which fuses to an element from the \( E_6(2) \)-class \( 3B \), we see that the normaliser in \( E_6(2) \) of \( H \) is equal to \( N_O(H) \cong (3 \times L_3(2)) : 2 \).

Now suppose that \( H \cong L_2(8) \) and note that there exist subgroups of \( O_{10}^+(2) \) satisfying the hypotheses. Consider a dihedral subgroup of order 18 in \( H \). Since elements of order 9 in \( O_{10}^+(2) \) belong to \( E_6(2) \)-class \( 9A \) and \( N_E(9A) \cong 9 \times S_3 \), it follows that \( N_E(H) \leq S_3 \times L_2(8) : 3 \). By also considering the centraliser in \( E_6(2) \) of a dihedral group of shape \( 7C : 3C \), we see that non-trivial elements of \( C_E(H) \) belong to \( E_6(2) \)-classes \( 2B \) or \( 3B \). We see from Table \([5]\) that there exists a conjugacy class of subgroups \( H \cong L_2(8) \) satisfying the hypothesis with \( N_E(H) \cong S_3 \times L_2(8) : 3 \), and all such \( H \) which centralise an element of order 3 are conjugate in \( E_6(2) \). Finally, we are able to find check in GAP that the conjugacy class of subgroups \( H \) in \( O_{10}^+(2) \) with \( N_O(H) \cong 2 \times H \) are unique up to conjugacy in \( E_6(2) \) and \( N_E(H) = N_O(H) \). Since these four conjugacy classes account for the entire structure constant, the proof is complete. \( \square \)

\((2C, 3C, 7D)\)-subgroups.

Lemma 13. There are 66 conjugacy classes of \((2C, 3C, 7D)\)-subgroups in \( E_6(2) \) with representatives \( H \) as follows:

1. \( H \cong ^3D_4(2) \) and \( N_E(H) \cong ^3D_4(2) : 3 \),
2. \( H \cong L_3(2) \) and \( N_E(H) \cong L_3(2) : 2 \),
3. \( H \cong L_2(8) \) and \( N_E(H) \cong S_4 \times (7 \times L_2(8)) : 3 \); or,
4. \( H \cong L_2(8) \) and \( N_E(H) \cong 2 \times (7 \times L_2(8)) : 3 \) of which there are 63 classes.

Proof. From Lemma \([5]\) and \([7\text{ Proposition 6.11}]\) subgroups isomorphic to \( ^3D_4(2) \) exist and are accounted for up to conjugacy. If \( H \cong L_2(8) \), then \( H \) is determined up to conjugacy in \([7\text{ Section 9}]\). Now suppose that \( H \cong L_3(2) \) and note that there exists a conjugacy class of subgroups in \( F_4(2) \) satisfying the hypothesis with \( N_F(H) \cong L_3(2) : 2 \). By Lemma \([4]\) if the centraliser \( C_E(H) \) is non-trivial, then it has order 3. But, from Tables \([3\text{ and 5}]\) and from the structure of \( C_E(3C) \) we see that \( C_E(H) \) does not contain elements of order 3. Hence \( N_E(H) \cong L_3(2) : 2 \). Since the entire structure constant has now been accounted for, the proof is complete. \( \square \)

References

1. M. Conder, An update on Hurwitz groups, Groups Complex. Cryptol. 2 (2010), no. 1, 35–49.
2. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, An Atlas of finite groups, Oxford University Press, Eynsham, 1985.
3. F. G. Frobenius, Über Gruppencharaktere, Sitzungsber. König. Preuss. Acad. Wiss. Berlin (1896), 985–1021.
4. R. B. Howlett, L. J. Rylands, and D. E. Taylor, Matrix generators for exceptional groups of Lie type, J. Symbolic Comput. 31 (2001), no. 4, 429–445. MR 1823074
5. A. Hurwitz, über algebraische Gebilde mit eindeutigen Transformationen in sich, Math. Ann. 41 (1892), no. 3, 403–442. MR 1510753
6. G. A. Jones, Ree groups and Riemann surfaces, J. Algebra 165 (1994), no. 1, 41–62.
Appendix A. Hurwitz subgroups of $E_6(2)$, $F_4(2)$, $O_{10}^+(2)$ and $L_6(2)$

| Type       | $H$                      | $N_E(H)$                          | Contribution | Overgroups of $H$ |
|------------|--------------------------|-----------------------------------|--------------|------------------|
| $(2A,3A,7C)$ | $L_3(2)$                 | $(H \times L_3(2) \times L_3(2)):2$ | 1/28224      | $T$              |
| $(2B,3A,7C)$ | $L_3(2)$                 | $H \times S_4 \times S_4$        | 1/288        | $F_4(2), O_{10}^+(2), L_6(2)$ |
| $(2B,3B,7AB)$ | $L_3(2)$                 | $H \times G_2(2)$               | 1/6048       | $L_6(2), T$      |
| $(2B,3B,7D)$ | $L_3(2)$                 | $(H \times L_3(2)):2$            | 1/168        | $L_6(2)$         |
| $(2C,3A,7C)$ | $L_3(2)$                 | $H \times S_4$                   | 1/12         | $F_4(2)$         |
| $(2C,3B,7D)$ | $L_3(2)$                 | $(H \times D_8):2$               | 1/8          | $F_4(2)$         |
| $(2C,3C,7C)$ | $L_3(2)$                 | $(3 \times H):2$                 | 1/3          | $N_E(3B)$        |
| $(2C,3C,7D)$ | $L_3(2)$                 | $H : 2$                          | 1/7          | $N_E(7A)$        |

Table 2. Conjugacy classes of Hurwitz subgroups of $E_6(2)$.
In the case of (*) there are 63 classes.

| $E_6(2)$  | 2A | 2B | 2C | 3A | 3B | 3C | 7AB | 7C | 7D | 9A | 9B |
|-----------|----|----|----|----|----|----|-----|----|----|----|----|
| $L_6(2)$  | 2A | 2B | 2C | 3B | 3A | 3C | 7AB | 7C | 7D | 9A | 9B |
| $O_{10}^+(2)$ | 2A | 2BC | 2D | 3B | 3AC | 3D | 7A | 7B | 7A | 9AB | 9B |
| $F_4(2)$  | 2A | 2BC | 2D | 3A | 3B | 3C | 7A | 7B | 7A | 9B | 9A |

Table 3. Conjugacy class fusion

Current address: Centre for the Mathematics of Symmetry and Computation, School of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia.
|      | $L_6(2)$-Type | $H$ | $N_L(H)$ | $E_6(2)$-type |
|------|--------------|-----|---------|---------------|
|      | $(2A,3B,7CD)$ | $L_3(2)$ | $L_3(2) \times L_3(2)$ | $(2A,3A,7C)$ |
|      | $(2B,3B,7CD)$ | $L_3(2)$ | $L_3(2) \times S_4$ | $(2B,3A,7C)$ |
|      | $(2B,3C,7AB)$ | $L_3(2)$ | $L_3(2) \times S_3$ | $(2B,3B,7AB)$ |
|      | $(2B,3C,7E)$  | $L_3(2)$ | $L_3(2)$ | $(2B,3B,7D)$ |
|      | $(2C,3A,7E)$  | $L_2(8)$ | $(7 \times L_2(8)) : 3$ | $(2C,3C,7D)$ |
|      | $(2C,3C,7E)$  | $2^3 L_3(2)$ | $2^3 L_3(2)$ | $(2C,3B,7D)$ |

**Table 4.** The Hurwitz subgroups of $L_6(2)$

|      | $O^+_8(2)$-Type | $H$ | $N_O(H)$ | $E_6(2)$-type |
|------|----------------|-----|---------|---------------|
|      | $(2A,3B,7A)$  | $L_3(2)$ | $(L_3(2) \times S_3 \times S_3) : 2$ | $(2A,3A,7C)$ |
|      | $(2B,3B,7A)$  | $L_3(2)$ | $L_3(2) \times S_3 \times S_3$ | $(2B,3A,7C)$ |
|      | $(2C,3B,7A)$  | $L_3(2)$ | $L_3(2) \times 2^4$ | $(2B,3A,7C)$ |
|      | $(2D,3B,7A)$  | $L_3(2)$ | $(L_3(2) \times 2^2) : 2$ | $(2C,3A,7C)$ |
|      |                | $L_3(2)$ | $2^3 L_3(2) \times 2^2$ | |
|      |                | $L_3(2)$ | $2^3 L_3(2) \times S_3$ | |
|      | $(2D,3D,7A)$  | $L_3(2)$ | $(3 \times L_3(2)) : 2$ | $(2C,3C,7C)$ |
|      |                | $L_2(8)$ | $L_2(8) : 3$ | |
|      |                | $L_2(8)$ | $2 \times L_2(8)$ | |
|      |                | $L_2(8)$ | $S_3 \times (L_2(8) : 3$ | |

**Table 5.** The Hurwitz subgroups of $O^+_8(2)$

|      | $F_4(2)$-Type | $H$ | $N_F(H)$ | $E_6(2)$-type |
|------|--------------|-----|---------|---------------|
|      | $(2A,3A,7B)$ | $L_3(2)$ | $(L_3(2) \times L_3(2)) : 2$ | $(2A,3A,7C)$ |
|      | $(2B,3B,7A)$ | $L_3(2)$ | $(L_3(2) \times L_3(2)) : 2$ | $(2B,3B,7D)$ |
|      | $(2C,3A,7B)$ | $L_3(2)$ | $L_3(2) \times S_4$ | $(2B,3A,7C)$ |
|      | $(2C,3B,7A)$ | $L_3(2)$ | $L_3(2) \times S_4$ | $(2B,3B,7D)$ |
|      | $(2D,3A,7B)$ | $L_3(2)$ | $(L_3(2) \times D_8) : 2$ | $(2C,3A,7C)$ |
|      |                | $L_3(2)$ | $2^3 L_3(2) \times D_8$ | |
|      |                | $L_3(2)$ | $2^3 L_3(2) \times S_4$ | |
|      | $(2D,3C,7A)$  | $L_3(2)$ | $L_3(2) : 2$ | $(2C,3C,7D)$ |
|      |                | $L_2(8)$ | $2 \times L_2(8)$ | |
|      | $(2D,3C,7B)$  | $L_3(2)$ | $L_3(2) : 2$ | $(2C,3C,7D)$ |
|      |                | $L_2(8)$ | $2 \times L_2(8)$ | |
|      |                | $L_2(8)$ | $S_3 \times (L_2(8) : 3$ | |
|      |                | $3D_4(2)$ | $3D_4(2) : 3$ | |

**Table 6.** The Hurwitz subgroups of $F_4(2)$