Supersymmetry on the lattice

Alessandra Feo\textsuperscript{a} \textsuperscript{*}

\textsuperscript{a}School of Mathematics, Trinity College, Dublin 2, Ireland

Lattice results in supersymmetry are summarized. Past, present and future perspectives are discussed.

1. INTRODUCTION

Supersymmetry or fermion-boson symmetry is one of the most fascinating topics in field theory. Even if has not yet been observed in Nature, thousands of papers have been written on this subject. Its validity in particle physics follows from the common belief in unification through the feasibility of incorporating quantum gravity. From a theoretical point of view, non-perturbative studies of supersymmetric (SUSY) gauge theories turn out to have remarkably rich properties which are of great physical interest, as have been shown in \cite{1}. For this reason, much effort has been dedicated to formulating a lattice version of SUSY theories, using Wilson fermions, see \cite{2}. The lattice formulation has been succesful to extract non-perturbative dynamics in field theory, specially in QCD, and may be able to provide additional information and confirm the existing analytical calculations. Whether supersymmetry is or not an exact symmetry is a question that must be settle by going beyond perturbation theory.

1.1. SYM theory in the continuum

In the continuum, the action for the \( N = 1 \) SYM theory with an \( SU(N_c) \) gauge group is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu
u}^a(x) F^{a\mu\nu}(x) + \frac{1}{2} \bar{\lambda}^a(x) \gamma_\mu \partial_\mu \lambda^a(x), \tag{1}
\]

where \( \lambda^a \) is a 4-component Majorana spinor and satisfies the Majorana condition \( \bar{\lambda}^a = \lambda^a T C \). The gluon fields are represented by \( A_\mu = -ig A_\mu^a T^a \) and \( F_{\mu\nu} = -ig F_{\mu\nu}^a T^a \). \( D_\mu \lambda^a = \partial_\mu \lambda^a + g \bar{\lambda}^a F_{\mu}^a \lambda^c \) is the covariant derivative in the adjoint representation. The continuum SUSY transformations read

\[
\delta A_\mu(x) = -2g \bar{\lambda}(x) \gamma_\mu \varepsilon, \tag{2}
\]

\[
\delta \lambda(x) = -i \frac{\sigma_{\rho\tau}}{g} F_{\rho\tau}(x) \varepsilon, \quad \delta \bar{\lambda}(x) = i \bar{\varepsilon} \sigma_{\rho\tau} F_{\rho\tau}(x),
\]

where \( \sigma_{\rho\tau} = \frac{i}{2}[\gamma_\rho, \gamma_\tau] \), \( \lambda = \lambda^a T^a \) and \( \varepsilon \) is a global Grassmann parameter with Majorana properties. These transformations relate fermions and bosons, leave the action invariant and commute with the gauge transformations so that the resulting Noether current \( S_\mu(x) \) is gauge invariant. For \( N = 1 \) SYM theory the supercurrent is given by

\[
S_\mu(x) = -F_{\rho\tau}^a(\varepsilon) \sigma_{\rho\tau} \gamma_\mu \lambda^a(x). \tag{3}
\]

Classically, the Noether current is conserved, \( \partial_\mu S_\mu(x) = 0 \), provided the fields satisfy the equations of motion. Furthermore one finds that it fullfills a spin 3/2 constraint: \( \gamma_\mu S_\mu(x) = 0 \).

1.2. Superfields

Supersymmetry in Eq. \cite{3} can be better understood in the language of superfields \cite{4}. In the Wess-Zumino gauge, the action for the \( N = 1 \) SYM theory is

\[
S_{SYM} = \text{Re} \left\{ \int d^4 x d^2 \theta \text{Tr}[W^a W_a] \right\}. \tag{4}
\]

\( W(x, \theta, \bar{\theta})_a = -\frac{1}{4} (\bar{D} \bar{D}) e^{-2V} D_\alpha e^{2V} \) is the spinorial field strength superfield which depends on the four coordinates \( x \) and the anticommuting Weyl spinor variables \( \theta_\alpha, \bar{\theta}_{\bar{\alpha}} \). \( \bar{D}_{\bar{\alpha}} \) and \( D_\alpha \) are the superspace derivatives while \( V = V(x, \theta, \bar{\theta}) \) is the vector superfield which takes the
form [8]

\[ V = \theta\sigma^a \tilde{\theta} A_\mu(x) + i(\theta \dot{\theta}) \tilde{\psi}_w(x) - i(\bar{\theta} \dot{\theta}) \bar{\psi}_w(x) + \frac{1}{2}(\theta \dot{\theta})D(x). \]  

(5)

The particle content can be identified as one bosonic vector field \( A_\mu = -igA^a_\mu T^a \), its SUSY partner, a complex Weyl spinor field \( \psi_w = T^a \bar{\psi}_w^a \), and an auxiliary scalar field \( D \). The action can be expressed in terms of these components as

\[ S_{\text{SYM}} = \int d^4x \left\{-\frac{i}{4}F^{a}_{\mu\nu}\bar{F}^{a\mu\nu} + \frac{1}{2}i\bar{\psi}_w^a\sigma^\mu(D_\mu\bar{\psi}_w^a)^a \right. \\
\left. -\frac{1}{2}i(D_\mu\bar{\psi}_w^a)\bar{\sigma}^\mu\psi_w^a + \frac{1}{2}D^2 \right\}. \]

(6)

In the on-shell case, the auxiliary scalar field \( D \) is eliminated by the equation of motion \( D = 0 \). Introducing one massless Majorana fermion, the gluino, in the adjoint representation \( \lambda = ((\bar{\psi}_w)\omega, (\bar{\psi}_w)^\omega) \), and going to the Euclidean space, we recover Eq. (5).

1.3. SUSY Ward Identities (WIs)

The existence of a renormalized SUSY Ward-type identity

\[ \partial_{\mu}\chi^R_\mu(x) = 2m_R\chi_R(x), \]  

(7)

is generally assumed. It is obtained by performing a variation of the functional integral with respect to a local, smooth transformation, \( \varepsilon = \varepsilon(x) \), and then putting the sources to zero. \( S_R^R \) is the renormalized supercurrent and \( \chi_R = Z\chi \), with \( \chi = \frac{1}{2}F_{\mu\nu}^a\sigma_{\mu\nu}\lambda^a \). \( m_R \) is the renormalized gluino mass. For \( m_R = 0 \) we have supersymmetry while a non-vanishing value of \( m_R \) breaks supersymmetry softly.

1.4. Non-perturbative effects of the SYM dynamics

Because of asymptotic freedom and the corresponding infrared instability, it is reasonable to guess that, as in conventional QCD, there are many non-perturbative phenomena that occur in this theory. For the \( N = 1 \) SYM theory we may expect confinement and spontaneous chiral symmetry breaking. The discrete chiral symmetry is expected to be broken by a non-zero gluino condensate while the confinement is realized by colorless bound states described by the effective action belonging to chiral supermultiplets.

It is generally assumed that supersymmetry is not anomalous (Eq. (6) holds) and only the mass term is responsible for a soft breaking. However, in [3] the question of whether non-perturbative effects may cause a supersymmetry anomaly has been raised. Only a study of the continuum limit of the lattice SUSY WIs can shed light on this question.

1.5. Chiral symmetry breaking

Introducing a non-zero gluino mass term in Eq. (6), \( L_{\text{mass}} = m_\tilde{g}\lambda^\dagger\lambda \), breaks supersymmetry softly (which implies that the non-renormalization theorem and cancellation of divergences are preserved [9]). In the massless case, the global chiral symmetry is \( U(1)_{\lambda} \):

\[ \lambda \rightarrow e^{-i\varphi}\gamma_5\lambda, \quad \lambda \rightarrow \tilde{\lambda} e^{-i\varphi}\gamma_5, \]  

(8)

which is anomalous. This implies that the divergence of the axial current, \( J^5_{\mu} = \lambda\gamma_5\lambda \), is

\[ \partial_{\mu}J^5_{\mu} = \frac{N_c g^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} F^{a}_{\mu\nu} F^{a}_{\rho\sigma}. \]  

(9)

The anomaly leaves a \( Z_{2N_c} \) subgroup of \( U(1)_{\lambda} \) unbroken. Eq. (8) is equivalent to \( \tilde{m}_\tilde{g} \rightarrow \tilde{m}_\tilde{g} e^{-2i\varphi}\gamma_5 \) and \( \Theta_{\text{SYM}} \rightarrow \Theta_{\text{SYM}} - 2N_c\varphi \).

In the SUSY case, \( m_\tilde{g} = 0 \), the \( U(1)_{\lambda} \) symmetry is unbroken if \( \varphi \equiv \frac{\Lambda}{N_c} \) for \( k = 0, 1, \ldots, 2N_c - 1 \). \( Z_{2N_c} \) is expected to be spontaneously broken to \( Z_2 \) by a value of \( \langle \lambda\lambda \rangle \neq 0 \) [10]. The consequence of this spontaneous chiral symmetry breaking is the existence of a first order phase transition at \( m_\tilde{g} = 0 \). That means the existence of \( N_c \) degenerate ground states with different orientations of the gluino condensate \( (k = 0, \ldots, 2N_c - 1) \),

\[ \langle \lambda\lambda \rangle = c\Lambda^4 e^{\frac{2m_\lambda}{N_c}}, \]  

(10)

where \( \Lambda \) is the dynamical scale of the theory which can be calculated on the lattice, for example, while \( c \) is a numerical constant which depends on the renormalization scheme used to compute \( \Lambda \). Eq. (10) shows the dependence on the gauge group. For \( SU(2) \) two degenerate ground
states with opposite sign of the gluino condensate, \((\lambda\lambda) < 0\) and \((\lambda\lambda) > 0\) appear, while for \(SU(3)\) there are three degenerate vacua at \(k = k_c\) (in \([6]\) a fourth state with \((\lambda\lambda) = 0\) is claimed). A first numerical study of SYM theory with a gauge group \(SU(3)\) is in \([6]\).

### 1.6. Magnitude of the gluino condensate

The calculation of the gluino condensate for the \(N = 1\) SYM theory is a puzzle. Two approaches in the literature have been used which give different results for \(c\). One is based on weak-coupling instanton (WCI) calculations \([10]\) and gives \(c = 1\). In the second, calculations based on strong-coupling instanton (SCI) give \(c = 2/((N_c - 1)!/(3N_c - 1))^{1/N_c}\) \([11]\). For a review on SCI see \([12]\). Several suggestions have been put forward to resolve the puzzle, see for example \([13]\). Ref. \([14]\) cast serious doubts on the SCI picture. The introduction of a fourth state with \(k > 1\) \([15]\) can not straightforwardly resolve the disagreement between the SCI and WCI results \([16]\). The KS vacuum can indeed potentially resolve the mismatch at the 1-instanton sector \((k = 1)\) but it fails to do so for the topological sectors with \(k > 1\) \([14]\). Using an instanton liquid picture gives qualitatively similar results and evidence for the gluino condensate \([16]\).

### 1.7. Light hadron spectrum

For the \(N = 1\) SYM theory a low energy effective action has been proposed by Veneziano and Yankielowicz (VY) \([17]\). The action contains all degrees of freedom, gauge invariant and colorless, composite fields: \(F^a_{\mu\nu}, F^a_{\mu\nu} F^a_{\rho\sigma}, \lambda^a\), in analogy to QCD, while \(\chi = \sigma_{\mu\nu} F^a_{\mu\nu}\lambda^a\) is a new type of composite operator formed by a gluino \(\lambda\) and a gauge field \(F\), both in the adjoint representation. These fields can be combined to form the chiral supermultiplet, containing the expression for the anomalies as component fields \([18]\)

\[
S(x, \theta) = \phi(x) + \sqrt{2}\theta \chi(x) + \theta \theta F(x).
\]

\[
\phi = \frac{\beta(2)}{2g}(\psi_w)^\dagger(\psi_w)_\alpha,
\]

is proportional to the gluino bilinear (here \(\psi_w\) denotes a 2-component Weyl spinor), while the other components, which we do not report here (see \([3]\)), contain combinations of gluino-gluino field and gluino-gluon fields. The VY action has the form \([3]\):

\[
\mathcal{L}_{eff} = \frac{1}{\alpha}(S^\dagger S)^{1/3}|D + \gamma((S\log S - S)|_F + h.c.)
\]

Expanding the effective VY action around its minimum, it is found that the low energy spectrum forms a supermultiplet, consisting of a scalar meson \(\tilde{\lambda}^a\lambda^a\) (the \(a - f_0\)), a pseudoscalar meson \(\tilde{\lambda}^a\gamma_5\lambda^a\) (the \(a - \eta^p\)), and a spin-1/2 gluino-glueball particle, the \(\chi\). Glueballs are absent in this formulation.

In the SUSY point these masses are degenerate. The introduction of a \(m_{\tilde{g}} \neq 0\) breaks supersymmetry softly and leads to a splitting of the multiplet. How the spectrum is influenced by the soft SUSY breaking has been studied in \([19]\):

\[
M_{a-\eta^p} = N_c\alpha\Lambda + \frac{40\pi^2|m_{\tilde{g}}|}{3N_c} + \cdots,
\]

\[
M_{a-\chi} = N_c\alpha\Lambda + \frac{48\pi^2|m_{\tilde{g}}|}{3N_c} + \cdots,
\]

\[
M_{a-f_0} = N_c\alpha\Lambda + \frac{56\pi^2|m_{\tilde{g}}|}{3N_c} + \cdots
\]

Unfortunately, the range of applicability of the linear mass formulae is not known because of the unknown magnitude of the constants and of the higher order terms in Eq. \([12]\).

Introducing an extra term in the effective VY action, Farrar et al. (FGS) solved the question of including glueballs in the low energy spectrum \([20]\). For unbroken supersymmetry we expect to see two chiral multiplets (not one), at the bottom of the SYM spectrum. The lighter one contains a \(0^++\)glueball \(\approx F_{\mu\nu} F_{\mu\nu}\), a \(0^-\)glueball \(\varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\) and a gluino-glueball ground state, while the heavier supermultiplet is the VY one. Moreover, in the FGS picture, a non-zero mixing between the \(a - f_0\) and \(0^+\) glueball is possible. Of course, other generalizations beside the FGS picture are conceivable.
2. LATTECE FORMULATION OF \( N = 1 \) SYM

The question of whether it is possible to formulate successfully SUSY theories on the lattice has been addressed in the past by several authors \cite{21,22,23} (the reader is referred to \cite{23} for a successful contruction of a lattice 2d Wess-Zumino model, and some discussion concerning Ref. \cite{21}). It can be seen that a lattice regularized version of a gauge theory is not SUSY since the Poincaré invariance (a sector of the superalgebra) is lost, thus \( \{ Q_\alpha, Q_\beta \} = 2\sigma^{\mu}_{\alpha\beta}P_\mu \). Poincaré invariance is achieved automatically without fine tuning in the continuum limit because operators that violate Poincaré invariance are all irrelevant. Moreover, if the SUSY theory contains scalar fields one can have scalar mass terms that break supersymmetry. Since these operators are relevant, fine tuning is necessary in order to cancel their contributions.

Another problem is the question of how to balance bosonic and fermionic modes, the numbers of which are constrained by the supersymmetry: the naive lattice fermion formulation results in twice as many fermions as in QCD by using different fermion formula-

\[ k \] is the hopping parameter defined as \( k = 1/(2(4 + m_0a)) \), where \( m_0 \) is the bare mass, and the matrix for the gauge field link in the adjoint representation is

\[ V_{ab,\mu} \equiv V_{ab,\mu}[U] = \frac{1}{2} Tr(U_{\mu\nu}^T T_a U_{\nu\mu} T_b). \]

The fermion matrix for the gluino in Eq. (13) is not hermitian but it satisfies \( Q^\dagger = \gamma_5 Q \gamma_5 \). That allows for the definition of the hermitian fermion matrix \( Q = \gamma_5 Q \). The path integral over the Majorana fermions gives the Pfaffian,

\[ \int [d\lambda] e^{-\frac{1}{4} \lambda Q \lambda} = \int [d\lambda] e^{-\frac{1}{4} \lambda^T C Q \lambda} = Pf(M), \]

where \( M = CQ \) is an antisymmetric matrix.

It is easy to see that \( Pf(M) = \pm \sqrt{\text{det}Q} \). In the effective CV action the absolute value of the Pfaffian is taken into account (this may cause the sign problem). The omitted sign can be included by reweighting the expectation values according to the formula,

\[ \langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} \text{sign} Pf(M) \rangle_\text{CV}}{\langle \text{sign} Pf(M) \rangle_\text{CV}}. \]

The spectral flow is a method which checks the value of the sign of the Pfaffian. The experience of the DESY-Münster collaboration shows that below the critical line \( k_c(\beta) \), corresponding to zero gluino mass \( m_\tilde{g} = 0 \), negative Pfaffians practically never appear \cite{21,22,23}. The factor 1/2 in front of \( \log \det Q[U] \) shows that Majorana fermions imply a flavor number \( N_f = 1/2 \). This can be achieved by the hybrid molecular dynamics (HMD) algorithm \cite{25} which is applicable to any number of flavors. The HMD has been checked for the CV action in \( N = 1 \) \( SU(2) \) SYM at small lattices \( 4^3 \times 8 \) \cite{26}. Another method for simulation with non-even numbers of flavors is based on the multibosonic algorithm proposed by Lüscher \cite{27}. A
The fermion determinant is approximated as \[ \lim_{n \to \infty} \text{det}(Q^1 Q)^{N_f} = \frac{1}{\text{det} P_{n_1}^{(1)}(Q^1 Q) \text{det} P_{n_2}^{(2)}(Q^1 Q)}. \]

Unquenched results using TSMB have been reported by the DESY-Münster-Roma collaboration (the first large scale numerical simulation of \( N = 1 \) SYM theory). For \( SU(2) \) see \[31,32,33\], while for \( SU(3) \) see \[34,35\]. Interesting quenched results (which was pioneering work) are in \[34,35\].

In order to check the expected pattern of spontaneous chiral symmetry breaking, let us first write the expression for the renormalized gluino condensate, obtained by additive and multiplicative renormalizations: \( \langle \bar{\lambda} \lambda \rangle_R(\mu) = Z(\mu) \langle \bar{\lambda} \lambda \rangle - b_0(a_\mu) \). In a numerical simulation, a first order phase transition (or cross-over) should show up (on small or modere lattices) as a jump in the expectation value of the gluino condensate at \( k = k_c \). By tuning the hopping parameter \( k \) to \( k_c \), for a fixed \( \beta \) value, one expects to see a two peak structure in the distribution of the gluino condensate. By increasing the volume the tunneling between the two ground states becomes less and less probable and at some point practically impossible. Outside the phase transition region, the observed distribution can be fitted by a single Gaussian but in the transition region a good fit can only be obtained with two Gaussians.

The DESY-Münster collaboration performed the first lattice investigation of the gluino condensate. Results for \( SU(2) \) are reported in Fig. 1 and show that a first order phase transition occurs at \( k_c = 0.1955 \pm 0.0005 \) for \( \beta = 2.3 \). The lattice size used is \( 6^3 \times 12 \). For \( SU(3) \), preliminary results are in \[9\]. Here the pattern is more complicated, \( Z_6 \to Z_2 \). The probability distribution for the gluino condensate was measured for \( k = 0.1950 \) and \( \beta = 5.6 \) on a rather small lattice \( 4^3 \times 8 \), with encouraging results. Of course, a more accurate study by going to larger lattices will clarify the nature of the transition.

SUSY restoration can be also verified by a direct inspection of the low energy mass spectrum \[32,34\]; this is expected to reproduce the SUSY multiplets predicted in \[17,20\]. An accurate analysis of the spectrum is, however, a non-trivial task from the computational point of view and an independent method for checking SUSY restoration would be welcome. Moreover, the mixing between the states has been measured. Numerical simulations show that this mixing is practically zero \[34,35\]. At this conference, new results for the low lying spectrum of \( N = 1 \) SYM, for \( 12^3 \times 24 \) and \( k = 0.1955 \) and for \( 16^3 \times 32 \) and \( k = 0.194 \), are reported by Peetz \[41\].

A study with a large number of colors \( N_c \) and strong coupling, by considering the hopping parameter expansion as a sum over lattice paths (random walks), is \[37\]. The surprising result is that the quenched approximation is exact at this order (and it is also consistent with the quenched results of Donini et al. in \[35\]). Formulæ for the propagators and masses of 2- and 3-gluino states are also presented. The 2-gluino masses do coincide with the results for the meson spectrum in ordinary lattice QCD at strong coupling \[35\]. A preliminary study of a 3-gluino states is reported also in \[38,40\]. This particle does not appear in \[17,20\] even if it contains the same quan-
tum numbers. A numerical analysis of this issue would clarify whether it contributes or not in the mass spectrum. At this conference, a study of the strong coupling expansion for $N = 1$ SYM theory using the Hamiltonian formalism is also presented [51]. It is shown that the theory is effectively described by an $SO(4)$ antiferromagnet.

### 2.2. Domain wall fermions

A very nice innovation is the domain wall fermions (DWF) approach. A new lattice fermion regulator, which improves the lattice formulation for fermions because the zero gluino mass is achieved without fine tuning. Application of DWF in SUSY theories has been explored in [43,44] and also suggested in [45], with a different approach as [46]. First Monte Carlo simulations for $N = 1$ $SU(2)$ SYM with DWF, using the lines of Refs. [43,44] are in [46,47]. DWF were introduced in [18] and were further developed in [38]. For a recent review on DWF for SUSY gauge theories see [49].

There are two unwelcome difficulties in using Wilson fermions as mentioned in the previous section. The first one is the need for fine tuning. The second one is related with the sign of the Pfaffian. DWF are defined by extending the space-time to five dimensions. Also a non-zero five dimensional mass or domain wall height $m_0$, which controls the number of flavors, is present. $L_s$ is the size of the fifth dimension and free boundary conditions for the fermions are implemented. As a result the two chiral components of the Dirac fermion are separated with one chirality bound exponentially on one wall and the other on the opposite wall. For any value of $a$ the two chiralities mix only by an amount that decreases exponentially as $L_s \rightarrow \infty$. For $L_s = \infty$, chiral symmetry is expected to be exact even at finite lattice spacing. So, there is no need for fine tuning. DWF offer for the first time the opportunity to separate the continuum limit, $a \rightarrow 0$, from the chiral limit, $L_s \rightarrow \infty$.

DWF introduce two extra parameters: $L_s$ and $m_0$. These two parameters together with the four dimensional mass $m_f$ control the effective fermion mass $m_{\text{eff}}$. In the free theory one finds [52]

$$m_{\text{eff}} = m_0(2 - m_0)(m_f + (1 - m_0)L_s^4),$$

with $0 < m_0 < 2$. The value of $m_0 = 1$ is optimal because finite $L_s$ effects do not contribute to $m_{\text{eff}}$. In the interactive theory, one would not expect such optimal value, due to the fact that $m_0$ will fluctuate. Then the goal would be to have $L_s$ large enough to have the second term of Eq. (16) small, in order to simulate at reasonably small masses and extrapolate to the chiral limit, $m_f \rightarrow 0$ and $L_s \rightarrow \infty$.

The effective action, which is not reported here (more details in [46]) contains $L_s$ heavy species. When $L_s$ is increased they may introduce bulk effects which must be substracted. This can be done by introducing Pauli-Villars type fields [43]. The fermionic path integral gives the Pfaffian

$$\int [d\lambda] e^{-S_F} = Pf(M_F) = \sqrt{\text{det}(D_F)}$$

which is positive for $m_f > 0$. This is a second advantage of the DWF approach.

Numerical simulations with DWF for $N = 1$ $SU(2)$ SYM are reported in [46,47], using the (HMDR) of [25]. They concern only the study of the gluino condensate, for rather small volumes, $4^4$ and $8^4$. The results for $m_f = 0$ are shown in Fig. 2. As can be seen, $\langle \bar{\lambda} \lambda \rangle (L_s \rightarrow \infty)$ has a non-zero VEV which remains different from zero, even for rather small volumes. These results (and also those in [46]) support the non-zero value for the gluino condensate.

Also the DWF has difficulties in its implementation. Beside the two extra parameters, it may happen that the two chiralities do not decouple,
even for $L_s \to \infty$. In this case the chiral symmetry can not be restored. This may need large values of $L_s$ and for this reason much expensive cost in the simulations. From the computational point of view, SYM is harder to simulate than QCD, using DWF, while with Wilson fermions, SYM is easier to simulate than QCD.

2.3. SUSY WIs on the lattice

Another independent way to study the SUSY limit is by means of the SUSY WIs. On the lattice they contain explicit SUSY breaking terms and the SUSY limit is defined to be the point in parameter space where these breaking terms vanish and the SUSY WIs take their continuum form. These issues have been investigated numerically by the DESY-Münster-Roma collaboration, using Wilson fermions. 

In the continuum, the SUSY WIs are given by Eq. (7) where $S^R_\mu(x) \equiv Z_S S_\mu(x) + Z_T T_\mu(x)$ and $Z_S$ and $Z_T$ are multiplicative renormalization factors. $T_\mu(x) = F_{\mu}^a(x)\gamma_\mu \lambda^a(x)$ is a mixing current with dimension $7/2$ as $S_\mu(x)$.

In the Wilson formulation, supersymmetry is not realized on the lattice. One might still define some lattice SUSY transformations (which reduce to the continuum ones in the limit $a \to 0$). One choice is

$$\delta U_\mu(x) = -agU_\mu(x)\varepsilon(x)\gamma_\mu \lambda(x) - a\varepsilon(x + a\hat{\mu})\gamma_\mu \lambda(x + a\hat{\mu})U_\mu(x),$$

$$\delta \lambda(x) = -\frac{i}{g} \sigma_{\mu\nu} G_{\mu\nu}(x) \varepsilon(x).$$

In the case of a gauge invariant operator insertion $O(y)$, we find for the bare SUSY WIs

$$\langle O(y) \nabla_\mu S_\mu(x) \rangle - 2m_0 \langle O(y) \chi(x) \rangle + \langle \frac{\delta O(y)}{\delta \varepsilon(x)} \rangle \big|_{\varepsilon = 0} \rangle = \langle O(y) X_S(x) \rangle.$$  

Comparing with Eq. (7), beside the presence of a non-zero bare mass term in the action which breaks supersymmetry softly, the rest of the SUSY breaking results in the presence of the $X_S$ term, that appear since the action is not fully SUSY. In order to renormalize the SUSY WIs a possible operator mixing has to be taken into account. In the case of gauge invariant operator insertion, $X_S$ mixes with the following operators of equal or lower dimension $\nabla_\mu S_\mu, \nabla_\mu T_\mu$ and $\chi$. The SUSY WIs can be written as

$$\langle O(y) \nabla_\mu S_\mu(x) \rangle + Z_S Z_T^{-1} \langle O(y) \nabla_\mu T_\mu(x) \rangle = m_R Z_S^{-1} \langle \phi(y) \chi(x) \rangle.$$  

Here the gauge invariant operator $O(y)$ at point $y$ is assumed to be sufficiently far away from $x$ in such a way that contact terms are avoided. This is the on-shell regime. $m_R = m_0 - a^{-1} Z_S$ and $\nabla_\mu$ is the lattice derivative. In numerical simulations, Eq. (20) can be computed at fixed $\beta$ and $k$. Thus, by choosing two elements of the $4 \times 4$ matrices, a system of two equations can be solved for $Z_S Z_T^{-1}$ and $m_R Z_S^{-1}$. One must clearly ensure that these two equations are non-trivial and independent. To do this, a proper definition of the supercurrent operator and the operator insertion $O$ is necessary. In Ref. [33], two different definitions for the supercurrent operator are considered in Eq. (20): the local definition [3], and more involved, the point-split definition, reported also in [53], which differs from the local one in order $O(a)$.

The dependence of $Z_T Z_S^{-1}$ and $m_R Z_S^{-1}$ on $k$, are shown in Fig. 3 and Fig. 4 respectively, with two different operator insertion, $\chi(x)$ and $T_0(x)$, for a lattice size $12^3 \times 24$. Fitting results are considered only for the insertion $\chi(x)$. In Fig. 3, $Z_T Z_S^{-1} = -0.039(7)$, for the point-split current and $Z_T Z_S^{-1} = 0.185(7)$, for the local current [33]. Also, the combination of $Z_T Z_S^{-1}$ shows no dependence on $k$. In fact, the latter is an $O(a)$ effect. An estimate of $Z_T Z_S^{-1}$ at the 1-loop order, using the point split current and $\beta = 2.3$ is $Z_T Z_S^{-1} = Z_T|_{\text{1-loop}} = -0.074$ [44]. Preliminary studies for $Z_T Z_S^{-1}$, using the local current definition, are reported in [33, 55]. Turning to Fig. 4, the expectation is that $m_R Z_S^{-1}$ vanishes linearly when $k \to k_c$. Also, a determination of $k_c$ by performing an extrapolation to zero gluino mass is obtained. The results is $k_c = 0.19750(38)$ for the point split current and $k_c = 0.19647(27)$ for the local one. These values can be compared with the previous determination from the first order phase transition, $k_c = (0.1955 \pm 0.0005)$ [24], see also Fig. 4. A study of the continuum limit of the lattice SUSY WIs, using Wilson fermions, at
the 1-loop order is $\frac{5\gamma_5}{2}$. It is interesting to see whether an analytic calculation of the $Z_S$ and $Z_T$ renormalization factors is in agreement with the numerical ones. Compared with numerical simulations [33], a lattice perturbative calculation of the SUSY WIs is more complicated. In order to do perturbation theory, we have to fix the gauge, which implies that new terms appear in the SUSY WIs: the gauge fixing term, the Faddeev-Popov term and a term coming from the gauge variation of the involved operators [56]. This is also discussed in [57]. Taking into account all these contributions we have [56],

$$
\langle \mathcal{O}_\mu S_\mu(x) \rangle - 2m_0 \langle \mathcal{O}_x(x) \rangle + \langle \frac{\delta \mathcal{O}}{\delta \bar{\epsilon}(x)} |_{\epsilon=0} \rangle - \langle \frac{\delta \mathcal{O}_{GF}}{\delta \bar{\epsilon}(x)} |_{\epsilon=0} \rangle - \langle \frac{\delta \mathcal{O}_{FP}}{\delta \bar{\epsilon}(x)} |_{\epsilon=0} \rangle = \langle \mathcal{O}X_S(x) \rangle.
$$

In [56] the operator insertion studied (at the 1-loop order) is $\mathcal{O} := A_i^a(y) \bar{\lambda}^b(z)$, which is a non-gauge invariant operator (a similar choice of operators as in the numerical case, for example $\chi$, would lead to a 2-loop calculation). In this case, $X_S(x)$ mixes with operators of equal or lower dimension [56], as for the on-shell case, plus $\sum_i Z_i A_i$, where the additional operators $A_i$ correspond to mixing with non-gauge invariant operators. In [12] an on-shell 1-loop perturbative calculation (using an exact expression for $X_S$), gives $Z_T Z_S^{-1} \equiv Z_T|_{1-loop} = -0.074$, using a point-split definition for the supercurrent, while preliminary studies using the local supercurrent, in the off-shell regime [58], give also results of the same order for $Z_T$.

2.4. Ginsparg-Wilson (GW) fermions

The GW relation [59] is defined to be

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D,$$

(21)

where $\gamma_5 D$ is an hermitian lattice Dirac operator. An explicit solution to the GW algebra, free of doubling species is reported in [59]. It exhibits highlight chiralities properties [59] and locality [62]. Either DWF (already discussed) or operators obtained from a renormalization group (RG) blocking of the continuum one, the fixed-point (FP) operator [63], satisfy the GW relation.

Theoretical applications of GW algebra to lattice supersymmetry are reported in [59,60]. For the Wess-Zumino model [58]. In [64], it is suggested that a lattice version of a perturbatively finite theory preserves supersymmetry to all orders in perturbation theory, in the sense that the SUSY breaking terms induced by the failure of the Leibniz rule become irrelevant in the continuum limit. Differences between implementing GW fermions and Wilson fermions are also analyzed. In [65] a conflict between lattice chiral symmetry and the Majorana condition for a Yukawa-type coupling is pointed out. If one adopts GW operators, a precise analysis of supersymmetry and its breaking require a consistent formulation of Majorana fermions.

In Ref. [66] a new formulation of SYM on the lattice with an exact fermionic symmetry is pre-
sented. First, it is considering the model in a fundamental lattice which is called the one-cell model and it is derived the preSUSY transformations. Then, it is extended to the entire lattice. The lattice action has a peculiar form: a translational invariance, not the usual one, and it is called Ichimatsu lattice (similar to a chessboard). At this conference, first non-perturbative results on an Ichimatsu lattice gauge theory is presented \[70\], while in \[71\] the study of the phase transition is reported. Although this is an interesting formulation, one has to note that, because of the staggered fermion action, there is no exact balance between fermionic and bosonic degrees of freedom.

3. EXACT SUPERSYMMETRY ON THE LATTICE

Improving lattice supersymmetry seems to be a difficult task for gauge theories. In fact, most SUSY theories, as for example \(N = 2\) or \(N = 4\) SYM, contain scalar bosons which typically produce SUSY violating relevant operators, which has to be fined tune away. For \(N = 1\), as we have seen, only a fine tuning is needed in order to eliminate the mass term. Unfortunately, because there is no discrete version of supersymmetry which can be implemented to forbid scalar masses and unwanted relevant operators, it is desiderable to construct lattice structures which directly display at least a subset of exact supersymmetry in order to decrease the number of fine tuning to do. In Ref. \[72\] a similar approach is used, for the 2d Wess-Zumino model with extended \(N = 2\) supersymmetry on the lattice. The model is then discretized in a way that preserves exactly a subnet of the continuum SUSY transformations. This is enough to guarantee that the full symmetry is restored without fine tuning in the continuum limit. Also numerical results using the HMC algorithm are presented \[73\], and show the equality between fermions and boson masses and also the verification of the WIs to high precision.

Another nice example of exact lattice supersymmetry is \[74\], where a perfect SUSY action for the 2d and 4d Wess-Zumino model, free case, is presented. The perfect action is achieved in terms of block variable RG transformation, which maps a system from a fine lattice to a coarser lattice in a specific way, that keeps the partition function and all expectation values invariant. It is preserved invariance under a continuous SUSY type of field transformations in a local perfect lattice action, which contains also a remnant chiral symmetry. This perfect formulation also cures the problem related with the Leibniz rule on the lattice \[22\]. In the perfect lattice formulation it is obtained the consistently blocked continuum translation operator. Therefore the algebra with the field variations closes. Ref. \[77\] uses a similar approach.

At this conference a talk by Kaplan \[74\], based on a recent remarkable paper \[75\] has been presented. It is a new method for implementing supersymmetry on a spatial lattice for a variety of SYM theories in 3 + 1 dimensions, including \(N = 4\), in a way that eliminates or reduces the problem of fine tuning. The formalism is presented in the Minkowski space but a generalization to Euclidean space is under way. The motivation is based on a recent work on deconstruction of SUSY theories \[76\]. The spatial lattice is created by “orbifolding”. Starting from a “mother theory”, being a SUSY quantum mechanics (QM) system with extended supersymmetry, a gauge group \(U(kN^d)\), and a global \(R\) symmetry group \(G_R\), the spatial lattice is constructed by orbifolding out a \(Z_N^d\) factor from \(G_R \times U(kN^d)\), which will result in a \(U(k)\) gauge symmetry, living on a \(d\)-dimensional spatial lattice. This is called the “daughter theory”. Taking then the continuum limit of this “daughter theory”, in some point in the classical moduli space of vacua, produces a higher dimensional quantum field theory with the original extended supersymmetry restored (previously broken by the orbifolding procedure), together with Poincaré invariance. The important point is that the lattice generated by orbifolding can retain some of the exact supersymmetries which facilitates the recovery of the remaining ones in the continuum limit. This method also protects the renormalizability of the theory. In \[77\], a SUSY QM model with extended supersymmetry in 0+1 dimensions is considered, and then the types of lattices that
can be obtained via orbifolding are shown.

### 3.1. Other related topics

At this conference, a nice example of a Hamiltonian lattice version for the 2d Wess-Zumino model is presented by Campostrini [77]. Previous results in [78]. Developing numerical simulations techniques using the Hamiltonian approach [79] turns out to be very advantageous (but contains also some disadvantages). Powerful many-body techniques are available: the Green Function Monte Carlo (GFMC) algorithm [80]. Since \( H \) is conserved, it is possible to preserve exactly a 1d SUSY subalgebra of the continuum \( N = 1 \) SUSY algebra. This is an advantage in comparison with the standard lattice formulation and is enough to guarantee the most important properties of SUSY-like paring of positive energy states [77,78]. Moreover, fermions are implemented directly, and need not be integrated out. A disadvantage is that fermions may lead to sign problems, but at least in 1 + 1 dimensions, it can be by-passed. In [77], simulations using GFMC are performed. The algorithm is very efficient in computing the ground state energy \( E_0 \) (because it can distinguish between 0 and \( 10^{-5} \)), and therefore can be used to study the pattern of supersymmetry breaking, which corresponds to \( E_0 > 0 \), while unbroken supersymmetry corresponds to \( E_0 = 0 \). Focusing on \( V(\phi) = \lambda \phi^2 + \lambda_0 \), predictions for the strong coupling and the weak coupling regime are quite different and it is interesting to study both numerically and analytically and the crossover from the strong to the weak coupling [77].

At this conference a related and very interesting method, namely, SUSY Discretized Light-Cone Quantization (SDLQC) has been reported by Trittmann [81]. SDLQC is a discrete, Hamiltonian and manifestly SUSY approach to solving a quantum field theory. Some results (including a review) for the \( N = 1 \) SUSY Chern Simons theory, in 2d and 3d, including correlators and bound states are in [82]. The theory is discretized by imposing periodic boundary conditions on the boson and fermion field in terms of discrete momentum modes, \( k = nP/K \), where \( K \) is a positive integer that determines the resolution (typically of order 10 in the numerical calculations). The continuum limit is reached for \( K \to \infty \). The low energy spectrum mass is determined using the fit \( M^2 = M^2_\infty + b(1/K) \) and approximate BPS states which have non-zero masses are found. In [83], a numerical test of the Maldacena conjecture within the 10 – 15% is found. A schematic comparison of SDLQC and lattice gauge theory is also presented.

Ref. [83] casts a Hamiltonian study of SYM QM, which faces the problem of the introduction of a cut-off that violates supersymmetry and the restoration of supersymmetry when it is taken away. It has been applied to the Wess-Zumino QM and for the SYM QM, in 2d and 4d. There are no sign problems. Also, the complete spectrum, Witten index and identification of SUSY multiplets has been determined [84].

### 4. CONCLUSIONS

A big effort has been made in order to describe supersymmetry on the lattice. Traditional Wilson fermions have been used in realistic computations with nice results. Improved chiral fermions results are starting too. New exciting ideas [75] are now waiting for numerical applications. It has also been shown that most properties of \( N = 1 \) SYM are analogous to non-SUSY YM and QCD and can be tested on the lattice in order to understand a possible transition from SYM to YM. For this reason, placing real-world QCD in a wider variety of theories may help us to better understand it.

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