ON GENERAL MULTILINEAR SQUARE FUNCTION WITH NON-SMOOTH KERNELS

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Abstract. In this paper, we obtain some boundedness of multilinear square functions $T$ with non-smooth kernels, which extend some known results significantly. The corresponding multilinear maximal square function $T^*$ was also introduced and weighted strong and weak type estimates for $T^*$ were given.

1. Introduction

It is well-known that the multilinear Calderón-Zygmund operators were introduced and first studied by Coifman and Meyer [3, 4, 5], and later by Grafakos and Torres [12, 13]. The study of this subject was recently enjoyed a resurgence of renewed interest and activity. In particular, the study of multilinear singular integral operators with non-standard kernels have recently received pretty much attention.

Before we state some known results, we begin by giving some definitions and notations. For any $v \in (0, \infty)$, a locally integrable function $K_v(x, y_1, \ldots, y_m)$ defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ satisfies the integral condition of $C - Z$ type, if there are some positive constants $\gamma, A$, and $B > 1$, such that

\begin{equation}
(\int_0^\infty |K_v(x, y_1, \cdots, y_m)|^2 \frac{dv}{v})^{\frac{1}{2}} \leq \frac{A}{(\sum_{j=1}^m |x - y_j|)^{mn}},
\end{equation}

\begin{equation}
(\int_0^\infty |K_v(z, y_1, \cdots, y_m) - K_v(x, y_1, \cdots, y_m)|^2 \frac{dv}{v})^{\frac{1}{2}} \leq \frac{A|z - x|^{\gamma}}{(\sum_{j=1}^m |x - y_j|)^{mn+\gamma}},
\end{equation}

whenever $|z - x| \leq \frac{1}{B} \max_{j=1}^m |x - y_j|$; and

\begin{equation}
(\int_0^\infty |K_v(x, y_1, \ldots, y_i, \ldots, y_m) - K_v(x, y_1, \ldots, y'_i, \ldots, y_m)|^2 \frac{dv}{v})^{\frac{1}{2}} \leq \frac{A|y_j - y'_j|^{\gamma}}{(\sum_{j=1}^m |x - y_j|)^{mn+\gamma}},
\end{equation}

whenever $|y - y'| \leq \frac{1}{B} \max_{j=1}^m |x - y_j|$.

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Xue and Yan [20] introduced a multilinear square function $T$ which is defined by
\begin{equation}
T(\vec{f})(x) = \left( \int_0^\infty \left[ \int_{[\mathbb{R}^n]^m} K_v(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) dy_1, \ldots, dy_m \right]^{2 \frac{1}{p}} \frac{dv}{v} \right)^{\frac{1}{2}},
\end{equation}
for any $\vec{f} = (f_1, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)$ and all $x \notin \bigcap_{j=1}^m \text{supp} f_j$. We assume that for some $1 \leq q_1, \ldots, q_m < \infty$ and $0 < q < \infty$, $T$ can be extended to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to $L^q$, where $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$.

Xue and Yan [20] obtain the following results.

**Theorem A** ([20]) Let $T$ be a multilinear square function with the kernel satisfying the integral condition of $C - Z$ type. Then $T$ can be extended to a bounded operator from $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n)$ to $L^{1/m, \infty}(\mathbb{R}^n)$.

**Theorem B** ([20]) Let $T$ be a multilinear square function with the kernel satisfying the integral condition of $C - Z$ type. Let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 \leq p_1, \ldots, p_m < \infty$ and assume that $\vec{w}$ satisfies $A_{\vec{p}}$ condition, then the following results hold:

1. If there is no $p_i = 1$, then $\|T(\vec{f})\|_{L^p(\vec{w})} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}$

2. If there is a $p_i = 1$, then $\|T(\vec{f})\|_{L^{p, \infty}(\vec{w})} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}$

Recently, many mathematicians are concerned to remove or replace the smoothness condition on the kernel [1, 9, 19, 15, 8, 18, 17, 21]. It is natural to ask under the non-smooth condition, does Theorem A and Theorem B still hold or not? In this paper we can give a positive answer: we can extend Theorem A and Theorem B to non-smooth case. Moreover, we introduce the multilinear maximal square function $T^*$ and weighted strong and weak type estimates for $T^*$ are also given.

To begin with, we first recall a class of integral operators $\{A_t\}_{t>0}$, that plays the role of an approximation to the identity modifying the original definition in [?] to extend it to a more general scenario. We assume that the operators $A_t$ are associated with kernels $a_t(x, y)$ in the sense that

$$A_t f(x) = \int_{\mathbb{R}^n} a_t(x, y) f(y) dy$$

for every function $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and the kernels $a_t(x, y)$ satisfy the following size conditions

\begin{equation}
|a_t(x, y)| \leq h_t(x, y) := t^{-n/s} h \left( \frac{|x - y|}{t^{1/s}} \right),
\end{equation}

where $s$ is a positive fixed constant and $h$ is a positive, bounded, decreasing function satisfying

\begin{equation}
\lim_{r \to 0} r^{n+\eta} h(r^s) = 0
\end{equation}

for some $\eta > 0$. These conditions imply that for some $C > 0$ and all $0 < \eta \leq \eta'$, the kernels $a_t(x, y)$ satisfy

$$|a_t(x, y)| \leq C t^{-n/s} (1 + t^{-1/s} |x - y|)^{-n-\eta'}.$$
Assumption (H1) Assume that for each $i = 1, \cdots, m$ there exist operators $\{A^{(i)}_t\}_{t > 0}$ with kernels $a^{(i)}_t(x, y)$ that satisfy condition (1.5) and (1.6) with constants $s$ and $\eta$ and that for every $j = 0, 1, 2, \cdots, m$, there exist kernels $K^{(i)}_{t, v}$ such that

$$< T(f_1, \cdots, A^{(i)}_t f_i, \cdots, f_m), g >$$

$$(1.7)$$

$$= \int_{\mathbb{R}^n} \left( \int_0^\infty \left| \int_{\mathbb{R}^m} K^{(i)}_{t, v}(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) dy \right|^2 \frac{dv}{v} \right)^{1/2} g(x) dx,$$

for all $f_1, \cdots, f_m, g$ in $S$ with $\cap_{k=1}^m \text{supp} f_k \cap \text{supp} \phi = \phi$. There exists a function $\phi \in C(\mathbb{R})$ with $\text{supp} \phi \subset [-1, 1]$ and a constant $\varepsilon > 0$ so that for every $j = 0, 1, \cdots, m$ and every $i = 1, 2, \cdots, m$, we have

$$A \leq \left( \int_0^\infty \left| K_t(x, y_1, \cdots, y_m) - K^{(i)}_{t, v}(x, y_1, \cdots, y_m) \right|^2 \frac{dv}{v} \right)^{1/2}$$

$$(1.8)$$

$$\leq \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}} \sum_{k=1, k \neq i}^m \phi \left( \frac{|y_i - y_k|}{t^{1/s}} \right) + \frac{A t^{\varepsilon/s}}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\varepsilon}}$$

whenever $t^{1/s} \leq |x - y_i|/2$.

Assumption (H2) Assume that there exist operators $\{A_t\}_{t > 0}$ with kernels $a_t(x, y)$ that satisfy condition (1.5) and (1.6) with constants $s$ and $\eta$, and there exist kernels $K^{(0)}_{t, v}(x, y_1, \cdots, y_m)$ such that for all $x, y_1, \cdots, y_m \in \mathbb{R}^n$ and $t > 0$ the representation is valid

$$(1.9)$$

$$K^{(0)}_{t, v}(x, y_1, \cdots, y_m) = \int_{\mathbb{R}^n} K_v(z, y_1, \cdots, y_m) a_t(x, z) dz.$$

Assume also that there exist a function $\phi \in C(\mathbb{R})$ and $\phi \subset [-1, 1]$ and a constant $\varepsilon > 0$ such that

$$(1.10)$$

$$\left( \int_0^\infty \left| K^{(0)}_t(x, y_1, \cdots, y_m) \right|^2 \frac{dv}{v} \right)^{1/2} \leq \frac{A}{(|x - y_k|)^{mn}}$$

whenever $2t^{1/s} \leq \min_{1 \leq j \leq m} |x - y_j|$ and

$$(1.11)$$

$$\left( \int_0^\infty \left| K_v(x, y_1, \cdots, y_m) - K^{(0)}_t(x, y_1, \cdots, y_m) \right|^2 \frac{dv}{v} \right)^{1/2}$$

$$\leq \frac{A}{(|x - y_k|)^{mn}} \sum_{k=1, k \neq j}^m \phi \left( \frac{|x - y_k|}{t^{1/s}} \right) + \frac{A t^{\varepsilon/s}}{(|x - y_k|)^{mn+\varepsilon}}$$

for some $A > 0$, whenever $2t^{1/s} \leq \max_{1 \leq j \leq m} |x - y_j|$.
Assumption (H3) Assume that there exist operators \( \{A_t\}_{t>0} \) with kernels \( a_t(x,y) \) that satisfy condition (1.5) and (1.6) with constants \( s \) and \( \eta \) and there exist kernels \( K^{(0)}_{l,v} \) such that (1.9) holds. Also assume that there exist positive constant \( A \) and \( \varepsilon \) such that,

\[
\left( \int_0^\infty \left| K^{(0)}_{l,v}(x,y_1,\ldots,y_m) - K^{(0)}_{l,v}(x',y_1,\ldots,y_m) \right|^2 \frac{dv}{v} \right)^{1/2} \leq \frac{At^\varepsilon/s}{(\sum_{k=1}^m |x-y_k|)^{mn+\varepsilon}},
\]

whenever \( 2t^{1/s} \leq \min_{1\leq j\leq m} |x-y_j| \) and \( 2|x-x'| \leq t^{1/s} \).

Kernels \( K_v \) that satisfying (1.1), (1.7) and (1.8) with parameters \( m, A, s, \eta, \varepsilon \) are called generalized square function kernels, and their collection is denoted by \( m-GSFK(A, s, \eta, \varepsilon) \).

We say that \( T \) is of class \( m - GSFO(A, s, \eta, \varepsilon) \) if \( T \) has an associated kernel \( K_v \) in \( m-GSFK(A, s, \eta, \varepsilon) \).

2. Endpoint estimate for \( T \)

In this section we prove the endpoint estimate for multilinear generalized square function.

Theorem 2.1. Let \( T \) be a multilinear operator in \( m-GSFO(A, s, \eta, \varepsilon) \). Then \( T \) can be extended to a bounded operator from \( L^1(\mathbb{R}^n) \times \ldots \times L^1(\mathbb{R}^n) \) into \( L^{1/m,\infty}(\mathbb{R}^n) \) with bound

\[
\|T\|_{L^1(\mathbb{R}^n) \times \ldots \times L^1(\mathbb{R}^n) \rightarrow L^{1/m,\infty}(\mathbb{R}^n)} \leq C_{n,m}(A + \|T\|_{L^1(\mathbb{R}^n) \rightarrow L^{1/m,\infty}(\mathbb{R}^n)}).
\]

Proof of Theorem 2.1. For simplicity, we write \( \|T\| = \|T\|_{L^1(\mathbb{R}^n) \rightarrow L^{1/m,\infty}(\mathbb{R}^n)} \). Let \( \tilde{f} = (f_1, \ldots, f_m) \). By homogeneity, we may assume that each \( f_i \in L^1(\mathbb{R}^n) \) and \( \|f_i\|_{L^1(\mathbb{R}^n)} = 1 \).

It suffices to prove that for any \( \lambda > 0 \),

\[
\{|x \in \mathbb{R}^n : |T(\tilde{f})(x)| > 2^m \lambda\} \leq C(A + \|T\|)^{1/m} \lambda^{-1/m}.
\]

For \( j = 1, 2, \ldots, m \), we consider the Calderón–Zygmund decomposition of each function \( f_j \) at level \( (\alpha \lambda)^{1/m} \), where \( \alpha \) is a positive constant to be determined later. Then, each \( f_j \) has the decomposition

\[
f_j = g_j + b_j = g_j + \sum_{k \in I_j} b_{j,k}
\]

with \( I_j \) being some index set, such that

1. \( |g_j(x)| \leq C(\alpha \lambda)^{1/m} \) for all \( x \in \mathbb{R}^n \);
2. there exists a sequence of mutually disjoint cubes \( \{Q_{j,k}\}_{k \in I_j} \) such that \( supp b_{j,k} \subset Q_{j,k} \) and \( \sum_{k \in I_j} |Q_{j,k}| \leq C(\alpha \lambda)^{-1/m} \); and
3. \( \|b_{j,k}\|_{L^1(\mathbb{R}^n)} \leq C(\alpha \lambda)^{1/m} |Q_{j,k}| \).

It follows from (ii) and (iii) that

\[
\|b_j\|_{L^1(\mathbb{R}^n)} \leq C
\]

and hence

\[
\|g_j\|_{L^1(\mathbb{R}^n)} \leq \|f_j\|_{L^1(\mathbb{R}^n)} + \|b_j\|_{L^1(\mathbb{R}^n)} \leq 1 + C.
\]

Further,

\[
\|g_j\|_{L^p(\mathbb{R}^n)} \leq (1 + C)^{1/p_j} \left| C(\alpha \lambda)^{1/m} \right|^{(p_j - 1)/p_j} = C(\alpha \lambda)^{1/(mp_j^*)}.
\]
Let
\[ E_\lambda^{(1)} := \{ x \in \mathbb{R}^n : |T(g_1, g_2, \ldots, g_m)(x)| > \lambda \}; \]
\[ E_\lambda^{(2)} := \{ x \in \mathbb{R}^n : |T(b_1, g_2, \ldots, g_m)(x)| > \lambda \}; \]
\[ E_\lambda^{(3)} := \{ x \in \mathbb{R}^n : |T(g_1, b_2, g_3, \ldots, g_m)(x)| > \lambda \}; \]
\[ \vdots \]
\[ E_\lambda^{(2^m)} := \{ x \in \mathbb{R}^n : |T(b_1, b_2, \ldots, b_m)(x)| > \lambda \}. \]

That is, for \( 1 \leq i \leq 2^m \),
\[ E_\lambda^{(i)} = \{ x \in \mathbb{R}^n : |T(h_1, h_2, \ldots, h_m)(x)| > \lambda \}, \quad h_j \in \{g_j, b_j\} \]
\begin{equation}
\text{and all the sets } E_\lambda^{(i)} \text{ are distinct. Thus,}
\| \{ x \in \mathbb{R}^n : |T(\tilde{f})(x)| > 2^m \lambda \} \| \leq \sum_{i=1}^{2^m} |E_\lambda^{(i)}|,
\end{equation}
\begin{equation}
\text{and we only need to prove that for } 1 \leq i \leq 2^m \text{ and } \lambda > 0,
|E_\lambda^{(i)}| \leq C(A + \|T\|)^{1/m} \lambda^{-1/m}.
\end{equation}

First, we estimate \( |E_\lambda^{(1)}| \). By the \( L^p_1(\mathbb{R}^n) \times \cdots \times L^p_m(\mathbb{R}^n) \to L^{p, \infty}(\mathbb{R}^n) \) boundedness of \( T \) and Chebychev's inequality, one gets that
\begin{equation}
|E_\lambda^{(1)}| \leq \left( \frac{\|T\| \prod_{j=1}^m \|g_j\|_{L^p_j(\mathbb{R}^n)}}{\lambda} \right)^p \leq C \left( \|T\| \left( \frac{\alpha \lambda^{1/m} \sum_{j=1}^m \|g_j\|}{\lambda} \right)^{p} \right)^p = C \|T\|^p \alpha^{p - \frac{1}{m}} \lambda^{\frac{-1}{m}}.
\end{equation}

Now we estimate \( |E_\lambda^{(i)}| \) for \( 2 \leq i \leq 2^m \). Suppose that for some \( 1 \leq \ell \leq m \) we have \( \ell \) bad functions and \( m - \ell \) good functions appearing in \( T(h_1, \ldots, h_m) \), where \( h_j \in \{g_j, b_j\} \). It suffices to prove that
\begin{equation}
|E_\lambda^{(i)}| \leq C \lambda^{-\frac{1}{m}} \left[ \|T\|^p \alpha^{p - \frac{1}{m}} + \alpha^{-\frac{1}{m}} + A \alpha^{1 - \frac{1}{m}} \right].
\end{equation}

Indeed, once we have (2.6), then combining with (2.5) and selecting \( \alpha = (\|T\| + A)^{-1} \), we get (2.4).

Now we prove (2.6). For each cube \( Q_{j,k} \) obtained in the Calderón-Zygmund decomposition in (ii), we denote by \( Q_{j,k}^* \) the cube with the same center as \( Q_{j,k} \) but \( 5\sqrt{n} \) times the side length of that of \( Q_{j,k} \). By (iii), we have
\[ \left| \bigcup_{j=1}^m \bigcup_{k \in I_j} Q_{j,k}^* \right| \leq \sum_{j=1}^m \sum_{k \in I_j} |Q_{j,k}^*| \leq Cm(5\sqrt{n})^n \alpha^{-1/m}. \]
Thus, to get (2.6), it suffices to prove that
\[
(2.7) \quad \left| \left\{ x \notin \bigcup_{j=1}^{m} \bigcup_{k \in I_j} Q_{j,k}^* : |T(h_1, \ldots, h_m)(x)| > \lambda \right\} \right| \leq C \lambda^{-\frac{1}{m}} \left[ \|T\|_{L^p}^{\alpha_p - \frac{1}{m}} + \alpha^{-\frac{1}{m}} + A\alpha^{1 - \frac{1}{m}} \right], \quad h_j \in \{g_j, b_j\}.
\]

Without loss of generality, we may assume that
\[
h_j = \begin{cases} b_j, & j = 1, \ldots, \ell; \\ g_j, & j = \ell + 1, \ldots, m. \end{cases}
\]

Fix \( x \notin \bigcup_{j=1}^{m} \bigcup_{k \in I_j} Q_{j,k}^* \). Then, by the Calderón-Zygmund decomposition, we write
\[
(2.8) \quad T(b_1, \ldots, b_{\ell}, g_{\ell+1}, \ldots, g_m)(x) = \sum_{k_1, k_2, \ldots, k_{\ell}} T(b_{1,k_1}, \ldots, b_{\ell,k_{\ell}}, g_{\ell+1}, \ldots, g_m)(x) \quad = \sum_{j=1}^{\ell} \left( \sum_{k_1, \ldots, k_{j-1}, k_j \in \Theta_j} \sum_{k_{j+1}, \ldots, k_{\ell}} T(b_1, \ldots, b_{\ell,k_{\ell}}, g_{\ell+1}, \ldots, g_m)(x) \right) \quad =: \sum_{j=1}^{\ell} T^{(j)}(b_1, \ldots, b_{\ell}, g_{\ell+1}, \ldots, g_m)(x),
\]

where, for \( 1 \leq j \leq \ell \), we recall that \( \text{supp} \ b_{j,k_j} \subset Q_{j,k_j} \) and define
\[
\Theta_j := \{ k_j \in I_j : \ell(Q_{i,k_i}) > \ell(Q_{j,k_j}) \text{ for } 1 \leq i \leq j - 1, \quad \ell(Q_{j,k_j}) \leq \ell(Q_{i,k_i}) \text{ for } j + 1 \leq i \leq \ell \}.
\]

Notice that \( \Theta_j \) is given by collecting all the indices \( k_j \in I_j \) such that the side length of \( Q_{j,k_j} \) is the first smallest among those of \( \{Q_{1,k_1}, \ldots, Q_{\ell,k_{\ell}}\} \). Therefore, the proof of (2.7) can be reduced to the following estimate:
\[
(2.9) \quad \left| \left\{ x \notin \bigcup_{i=1}^{\ell} \bigcup_{k \in I_i} Q_{i,k}^* : |T^{(j)}(b_1, \ldots, b_{\ell}, g_{\ell+1}, \ldots, g_m)(x)| > \lambda \right\} \right| \leq C \lambda^{-\frac{1}{m}} \left[ \|T\|_{L^p}^{\alpha_p - \frac{1}{m}} + \alpha^{-\frac{1}{m}} + A\alpha^{1 - \frac{1}{m}} \right], \quad 1 \leq j \leq \ell.
\]

Without loss of generality, we may consider only the case \( j = 1 \) in (2.9). Choose \( t_{1,k_1} = (\sqrt{n} \ell(Q_{1,k_1}))^s \), where \( s \) is the constant appearing in (1.5). Write
\[
T^{(1)}(b_1, \ldots, b_{\ell}, g_{\ell+1}, \ldots, g_m)(x) = \sum_{k_2, \ldots, k_\ell} \sum_{k_1 \in \Theta_1} T(b_1, b_{2,k_2}, \ldots, b_{\ell,k_{\ell}}, g_{\ell+1}, \ldots, g_m)(x) = \sum_{k_2, \ldots, k_\ell} \sum_{k_1 \in \Theta_1} T(b_1, b_{2,k_2}, \ldots, b_{\ell,k_{\ell}}, g_{\ell+1}, \ldots, g_m)(x)
\]

\[
= T^{(1)}(b_1, b_{2,k_2}, \ldots, b_{\ell,k_{\ell}}, g_{\ell+1}, \ldots, g_m)(x) - A_{t_{1,k_1}}^{(1)}(b_1, b_{2,k_2}, \ldots, b_{\ell,k_{\ell}}, g_{\ell+1}, \ldots, g_m)(x)
\]

where \( A_{t_{1,k_1}}^{(1)}(b_1, b_{2,k_2}, \ldots, b_{\ell,k_{\ell}}, g_{\ell+1}, \ldots, g_m)(x) \) is the function appearing in (1.5).
\[
\sum \sum T(A_{1,k_1}^{(1)} b_1, k_1, b_2, k_2, \ldots, b_{\ell,k_{\ell}}, g_{\ell+1}, \ldots, g_m)(x) =: T^{(1,1)}(b_1, \ldots, b_{\ell}, g_{\ell+1}, \ldots, g_m)(x) + T^{(1,2)}(b_1, \ldots, b_{\ell}, g_{\ell+1}, \ldots, g_m)(x).
\]

Consider first the operator \(T^{(1,1)}\). We get
\[
\begin{align*}
|T^{(1,1)}(b_1, \ldots, b_{\ell}, g_{\ell+1}, \ldots, g_m)(x)| & \leq \sum \sum \int_0^\infty \int_{\mathbb{R}^n} |K_v(x, y_1, \ldots, y_m) - K_{t, k_1}^{(1)}(x, y_1, \ldots, y_m)| \\
& \times \prod_{j=1}^{\ell} |b_{j,k_j}(y_j)| \prod_{i=\ell+1}^m |g_i(y_i)| d\vec{y} \left(\frac{2}{v}\right)^{\frac{1}{2}} \\
& \leq A \sum \sum \int_{\mathbb{R}^n} \left(\int_0^\infty \left|K_v(x, y_1, \ldots, y_m) - K_{t, k_1}^{(1)}(x, y_1, \ldots, y_m)\right|^2 dv\right)^{\frac{1}{2}} \\
& \times \prod_{j=1}^{\ell} |b_{j,k_j}(y_j)| \prod_{i=\ell+1}^m |g_i(y_i)| d\vec{y} \\
& \leq A \sum \sum \int_{\mathbb{R}^n} \left(\sum_{i=1}^m |x - y_i|\right)^m \\
& \times \prod_{j=1}^{\ell} |b_{j,k_j}(y_j)| \prod_{i=\ell+1}^m |g_i(y_i)| d\vec{y} \\
& + A \sum_{u=2}^m \sum \sum \int_{\mathbb{R}^n} \phi \left(\frac{|y_1 - y_u|}{(t_1, k_1)^{1/s}}\right) \\
& \times \prod_{j=1}^{\ell} |b_{j,k_j}(y_j)| \prod_{i=\ell+1}^m |g_i(y_i)| d\vec{y} \\
& =: Y_1(x) + Y_2(x),
\end{align*}
\]

where we have used Assumption (H1) since \(|x - y_1| \geq 2t_{1,k_1}^{1/s}\). For \(Y_1(x)\), by the definitions of \(t_{1,k_1}\) and \(\Theta_1\), we see that
\[
Y_1(x) \leq A \int_{\mathbb{R}^n} \prod_{j=1}^{\ell} \sum_{k_j \in I_j} \left(\frac{\ell(Q_{j,k_j})}{|x - y_j|} \frac{\epsilon/m}{|x - y_j|^n}\right) \\
\times \prod_{i=\ell+1}^m \left(\frac{\ell(Q_{1,k_1})}{(\ell(Q_{1,k_1}) + |x - y_1|)^n} \frac{\epsilon/m}{\ell(Q_{1,k_1}) + |x - y_1|}\right) d\vec{y}.
\]

Therefore we obtain that for \(\ell + 1 \leq i \leq m\),
\[
\int_{\mathbb{R}^n} \left(\frac{\ell(Q_{1,k_1})}{(\ell(Q_{1,k_1}) + |x - y_1|)^n} \frac{\epsilon/m}{\ell(Q_{1,k_1}) + |x - y_1|}\right) g_i(y_i) dy_i \leq CM(g_i)(x) \leq C(\alpha\lambda)^{1/m}.
\]
Recall that we have assumed that \( x \notin \bigcup_{i=1}^{m} \bigcup_{k \in I_i} Q^*_{i,k} \). Consequently, \( x \notin Q^*_j \) for each \( 1 \leq j \leq \ell \). For such an \( x \) and all \( y \in Q^*_j \), it is easy to see that

\[
\frac{4}{5} |x - c_{j,k}| \leq |x - y| \leq \frac{4}{3} |x - c_{j,k}|,
\]

where \( c_{j,k} \) denotes the center of the cube \( Q^*_j \). Furthermore, we have

\[
\int_{\mathbb{R}^n} \sum_{k_j \in I_j} \left[ \frac{\ell(Q_{j,k})}{|x-y_j|} \right] \frac{\varepsilon}{m} |b_{j,k_j}(y_j)| dy_j
\]

\[
\leq C \sum_{k_j \in I_j} \left[ \frac{\ell(Q_{j,k})}{\frac{4}{5}|x-c_{j,k}|} \right] \left\| b_{j,k} \right\|_{L^1(\mathbb{R}^n)} \frac{\varepsilon}{m} |Q_{j,k}| \frac{1}{|x-c_{j,k}|^n}.
\]

Define the function

\[
M_{j,1/m}(x) := \sum_{k_j \in I_j} \left[ \frac{\ell(Q_{j,k})}{\frac{4}{5}|x-c_{j,k}|} \right] \left\| b_{j,k} \right\|_{L^1(\mathbb{R}^n)} \frac{\varepsilon}{m} |Q_{j,k}| \frac{1}{|x-c_{j,k}|^n}.
\]

Then, we have proved that

\[
Y_1(x) \leq C A \alpha \lambda \prod_{j=1}^{\ell} M_{j,1/m}(x).
\]

Next, we shall use the following estimate in [18, p. 240]:

\[
\int_{x \notin \bigcup_{i=1}^{m} \bigcup_{k \in I_i} Q^*_i} M_{j,1/m}(x) \, dx \leq C \sum_{k_j \in I_j} |Q_{j,k}|.
\]

By this and Hölder’s inequality, we get

\[
\left\{ x \notin \bigcup_{i=1}^{\ell} \bigcup_{k \in I_i} Q^*_i : |Y_1(x)| > \lambda/4 \right\} \leq \left\{ x \notin \bigcup_{i=1}^{\ell} \bigcup_{k \in I_i} Q^*_i : A \alpha \prod_{j=1}^{\ell} M_{j,1/m}(x) > C \right\}
\]

\[
\leq C \int_{x \notin \bigcup_{i=1}^{m} \bigcup_{k \in I_i} Q^*_i} \left( A \alpha \prod_{j=1}^{\ell} M_{j,1/m}(x) \right)^{1/\ell} \, dx
\]

\[
\leq C \left( A \alpha \prod_{j=1}^{\ell} \int_{x \notin \bigcup_{i=1}^{m} \bigcup_{k \in I_i} Q^*_i} M_{j,1/m}(x) \, dx \right)^{1/\ell}
\]
where the last step is by (iii) and the fact $A \alpha < 1$.

Now we consider $Y_2(x)$. Indeed, the estimate of $Y_2$ is exactly the term $T_2^{(11)}$ in [7, p. 2100], which provides us the following estimate:

$$Y_2(x) \leq CA(\alpha \lambda)^{\ell/m} \sum_{k=\ell+1}^{m} \left( \prod_{j=2}^{\ell} J_{j,m^{-1}}(x) \left( \prod_{i \neq k, \ell+1 \leq i \leq m} M(g_i)(x) \right) \right) \times \left( M(g_k)(x) + (\alpha \lambda)^{1/m} J_{1,m^{-1}}(x) \right),$$

where for any $\epsilon > 0$ and $1 \leq j \leq m$ the function $J_{j,\epsilon}(x)$ is given by

$$J_{j,\epsilon}(x) = \sum_{k_j \in I_j} \left[ \ell(Q_{j,k_j}) \right]^{n+\epsilon} + \left[ \ell(Q_{j,k_j}) + |x - c_{j,k_j}| \right]^{n+\epsilon}$$

and $M$ is usual the Hardy-Littlewood maximal operator.

It is known from [7, (2.6)] that for any $p \in (n/(n+\epsilon), \infty)$,

$$\|J_{j,\epsilon}\|_{L^p(\mathbb{R}^n)} \leq C \left( \sum_{k_j \in I_j} |Q_{j,k_j}| \right)^{1/p}.$$

By (iii), we further have that

$$\|J_{j,\epsilon}\|_{L^2(\mathbb{R}^n)} \leq C \left( \sum_{k_j \in I_j} |Q_{j,k_j}| \right)^{1/2} \leq C(\alpha \lambda)^{-1/(2m)}.$$

By the $L^2$-boundedness of $M$, property (i) and (2.2), we have that for all $1 \leq i \leq m$,

$$\|M(g_i)\|_{L^2(\mathbb{R}^n)} \leq C\|g_i\|_{L^2(\mathbb{R}^n)} \leq C(\alpha \lambda)^{1/(2m)} \|g_i\|_{L^1(\mathbb{R}^n)}^{1/2} \leq C(\alpha \lambda)^{1/(2m)}.$$

From (2.15) and (2.16), it follows that

$$\leq \int_{\mathbb{R}^n} \left( \frac{Y_2(x)}{\lambda/4} \right)^{2} \frac{dx}{m-1} \leq C \left( \frac{A(\alpha \lambda)^{\ell/m}}{\lambda} \right)^{2} \sum_{k=\ell+1}^{m} \int_{\mathbb{R}^n} \left( \prod_{j=2}^{\ell} J_{j,m^{-1}}(x) \right)^{2} \frac{dx}{m-1}.$$
Thus, to obtain (2.10), we use (2.19) (2.17)

\[ \int_{\mathbb{R}^n} [F_1(x) \cdots F_{m-1}(x)]^{\frac{2}{m-1}} \, dx \leq \prod_{j=1}^{m-1} \left( \int_{\mathbb{R}^n} [F_j(x)]^2 \, dx \right)^{\frac{1}{m-1}}. \]

This inequality holds with “=” when \( m = 2 \), and it follows from Hölder’s inequality when \( m \geq 3 \). Therefore, by (2.17), we have

(2.18) \[
\left\{ x \in \bigcup_{i=1}^{\ell} \bigcup_{k \in I_i} Q_{i,k}^* : |Y_2(x)| > \lambda/4 \right\}
\leq C \left( \frac{A(\alpha \lambda)^{\ell/m}}{\lambda} \right)^{\frac{2}{m-1}} \left( \sum_{k=\ell+1}^{m} \prod_{j=2}^{\ell} \left\| J_{j, m-1} \right\|^2_{L^2(\mathbb{R}^n)} \right)^{-\frac{1}{m-1}} \times \left( \prod_{i \neq k, \ell+1 \leq i \leq m} \left\| M(g_i) \right\|^2_{L^2(\mathbb{R}^n)} \right) \times \left( \left\| M(g_k) \right\|_{L^2(\mathbb{R}^n)} + (\alpha \lambda)^{1/m} \left\| J_{1, m-1} \right\|_{L^2(\mathbb{R}^n)} \right)^{\frac{2}{m-1}} \leq C \left( \frac{A(\alpha \lambda)^{\ell/m}}{\lambda} \right)^{\frac{2}{m-1}} \left( (\alpha \lambda)^{-\frac{1}{2m}} \cdot (\alpha \lambda)^{-\frac{m-\ell-1}{2m}} \cdot (\alpha \lambda)^{-\frac{1}{2m}} \left\| M(g_k) \right\|^2_{L^2(\mathbb{R}^n)} \right)^{\frac{2}{m-1}} = C(A\alpha)^{\frac{2}{m-1}}(\alpha \lambda)^{-1/m},
\]

which is bounded by \( C\alpha(\alpha \lambda)^{-1/m} \) since \( A\alpha < 1 \) and \( \frac{2}{m-1} \geq 1 \).

The relations (2.13) and (2.18) together yield that

\[
\left\{ x \in \bigcup_{i=1}^{\ell} \bigcup_{k \in I_i} Q_{i,k}^* : |T^{(1,1)}(b_1, \ldots, b_{\ell}, g_{\ell+1}, \ldots, g_m)(x)| > \lambda/2 \right\} \leq C(\lambda \alpha)^{-\frac{1}{m}} [1 + A\alpha].
\]

Thus, to obtain (2.9), it remains to prove

(2.19) \[
\left\{ x \in \bigcup_{i=1}^{\ell} \bigcup_{k \in I_i} Q_{i,k}^* : |T^{(1,2)}(b_1, \ldots, b_{\ell}, g_{\ell+1}, \ldots, g_m)(x)| > \lambda/2 \right\} \leq C(\alpha \lambda)^{-1/m}.
\]

To handle the rest of the proof we use (2.14), Chebychev’s inequality and the \( L^{q_1} \times \cdots \times L^{q_m} \to L^q \) boundedness of \( T \) to get the desired result. This concludes the proof of the Theorem.
3. Weighted estimates for $T$

In this section, we will study the multiple-weighted normal inequalities and weak-type estimates. Our main results in this section can be stated as follows.

**Theorem 3.1.** Let $T$ be a multilinear operator in $m - GSFO(A, s, \eta, \varepsilon)$ with a kernel satisfies Assumption (H2). Let $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$ for $1 \leq p_1, \ldots, p_m < \infty$ and $\omega \in A_p$ with $p \geq 1$. Then the following hold:

1. If there is no $p_i = 1$, then
   \[ \|T(\vec{f})\|_{L^p(\omega)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega)} \]

2. If there is a $p_i = 1$, then
   \[ \|T(\vec{f})\|_{L^{p_i, \infty}(\omega)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega)} \]

The proof of Theorem 3.1 is similar to [16, Cor. 4.2], which is based on the following lemmas.

**Lemma 3.2.** Let $0 < p, \delta < \infty$ and $\omega$ be any Mackenhoupt $A_\infty$ weight. Then there exists a constant $C$ independent of $f$ such that the inequality

\[ \int_{\mathbb{R}^n} (M_{\delta}f(x))^p \omega(x) dx \leq C \int_{\mathbb{R}^n} (M_{\# \delta}f(x))^p \omega(x) dx \]

holds for any function such that the left-hand side is finite.

Moreover, if $\phi : (0, \infty) \to (0, \infty)$ be a doubling, that is, $\phi(2a) \leq C\phi(a)$ for $a > 0$. Then, there exists a constant $C$ depending upon the $A_\infty$ condition of $\omega$ and doubling condition of $\phi$ such that

\[ \sup_{\lambda > 0} \phi(\lambda)\omega(\{y \in \mathbb{R}^n : M_\delta f(y) > \lambda\}) \leq C \sup_{\lambda > 0} \phi(\lambda)\omega(\{y \in \mathbb{R}^n : M_{\# \delta} f(y) > \lambda\}) \]

for any function such that the left-hand side is finite.

Similar as [20, Lemma 4.2], we can easily get

**Lemma 3.3.** Let $T$ be a multilinear operator in $m - GSFO(A, s, \eta, \varepsilon)$. Suppose $\text{supp } f_i \subset B(0, R)$, then there is a constant $C < \infty$ such that for $|x| > 2R$,

\[ T(\vec{f})(x) \leq C \prod_{j=1}^m Mf_j(x). \]

**Lemma 3.4.** Let $0 < \delta < 1/m$ and let $T$ be a multilinear operator in $m - GSFO(A, s, \eta, \varepsilon)$ with a kernel satisfies Assumption (H2). Then there exists a constant $C$ such that for any bounded and compact supported function $f_i, i = 1, \ldots, m$,

\[ M_\delta^{\#} T\vec{f}(x) \leq C \prod_{j=1}^m Mf_j(x). \]

**Proof.** Fix a point $x \in \mathbb{R}^n$ and a cube $Q$ containing $x$. For $0 < \delta < 1/m$, we need to show there exists a constant $c_Q$ such that

\[ \left( \frac{1}{|Q|} \int_Q |T(\vec{f})(z) - c_Q\delta| dz \right)^{1/\delta} \leq C \prod_{j=1}^m Mf_j(x). \]
Let \( c_{Q,t} = \sum_{\vec{\alpha}, \vec{\alpha} \neq \vec{0}} \int_{\mathbb{R}^m} K_t(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) \, dy \), where \( \vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \) with \( \alpha_i = 0 \) or \( \infty \) and let \( c_Q = \left( \frac{\int_{\mathbb{R}^n} |c_{Q,t}|^2 \, dt}{t} \right)^{1/2} \).

\[
\left( \frac{1}{|Q|} \int_Q |T(f)(z) - c_Q| \, dz \right)^{1/\delta} 
\leq C \left( \frac{1}{|Q|} \int_Q \left( \int_0^{\infty} \left| \int_{\mathbb{R}^m} K_t(z, \vec{y}) \prod_{j=1}^m f_j(y_j) \, dy \right| \, dt \right)^{\delta/2} \, dz \right)^{1/\delta} 
+ C \sum_{\vec{\alpha}, \vec{\alpha} \neq \vec{0}} \left( \frac{1}{|Q|} \int_Q \left( \int_0^{\infty} \left| \int_{\mathbb{R}^m} (K_t(z, \vec{y}) - K_t(x, \vec{y})) \prod_{j=1}^m f_j^{\alpha_j}(y_j) \, dy \right| \, dt \right)^{\delta/2} \, dz \right)^{1/\delta} 
:= I_0^\varepsilon + C \sum_{\vec{\alpha} \neq \vec{0}} I_{\vec{\alpha}}.
\]

By using Kolmogorov inequality and Theorem 2.1, we get \( I_0^\varepsilon \leq C \prod_{j=1}^m M f_j(x) \).

Now, we turn our attention to the integral

\[
\int_{\mathbb{R}^m} \left( \int_0^{\infty} |K_v(z, \vec{y}) - K_v(x, \vec{y})|^2 \, dv \right)^{1/2} \prod_{j=1}^m |f_j^{\alpha_j}(y_j)| \, dy,
\]

which can be decomposed as

\[
\int_{\mathbb{R}^m} \left( \int_0^{\infty} |K_v(z, \vec{y}) - K_v(x, \vec{y})|^2 \, dv \right)^{1/2} \prod_{j=1}^m |f_j^{\alpha_j}(y_j)| \, dy 
\leq \int_{\mathbb{R}^m} \left( \int_0^{\infty} |K_v(z, \vec{y}) - K_v(0)(z, \vec{y})|^2 \, dv \right)^{1/2} \prod_{j=1}^m |f_j^{\alpha_j}(y_j)| \, dy 
+ \int_{\mathbb{R}^m} \left( \int_0^{\infty} |K_v(0)(z, \vec{y}) - K_v(x, \vec{y})|^2 \, dv \right)^{1/2} \prod_{j=1}^m |f_j^{\alpha_j}(y_j)| \, dy 
:= I_1 + I_2.
\]

By (1.11) in Assumption (H2) we obtain that

\[
I_1 \leq C \prod_{j \in \{j_1, \ldots, j_l\}} \int_{Q^*} |f_j(y_j)| \, dy_j \left( \int_{(Q^*)^m \setminus \{z - y_j\}^{mn+\varepsilon}} \text{ref} \prod_{j \notin \{j_1, \ldots, j_l\}} |f_j(y_j)| \, dy_j \right) 
+ \int_{(Q^*)^m \setminus \{z - y_j\}^{mn}} \frac{\prod_{j \notin \{j_1, \ldots, j_l\}} |f_j(y_j)| \, dy_j}{\prod_{j \notin \{j_1, \ldots, j_l\}} |z - y_j|^{mn}} 
\leq C \prod_{j \in \{j_1, \ldots, j_l\}} \int_{Q^*} |f_j(y_j)| \, dy_j \left( \sum_{k=1}^{2^n+1} \frac{|Q^*|^{\varepsilon/n}}{(2k|Q^*|^{1/n})^{mn+\varepsilon}} \int_{(2k+1)Q^* \setminus \{z - y_j\}^{mn}} \prod_{j \notin \{j_1, \ldots, j_l\}} |f_j(y_j)| \, dy_j \right) 
+ \sum_{k=1}^{2^n+1} \frac{1}{(2k|Q^*|^{1/n})^{mn}} \int_{(2k+1)Q^* \setminus (2kQ^*)^{m-l}} \prod_{j \notin \{j_1, \ldots, j_l\}} |f_j(y_j)| \, dy_j \right).
\]
\[ \leq C \prod_{j=1}^{m} M f_j(x). \]

\( I_2 \) can be estimated in a similar way, which is also dominated by \( C \prod_{j=1}^{m} M f_j(x) \). Then, by Minkowski’s inequality and above estimates, we get

\[ I_2 \leq C \left( \frac{1}{|Q|} \int_{Q} \left( \int_{\mathbb{R}^m} \left( \int_{0}^{\infty} |K_t(z, \bar{y}) - K_t(x, \bar{y})|^2 \frac{dt}{t} \right)^{1/2} \prod_{j=1}^{m} |f_j^{\alpha_j}(y_j)| d\bar{y} \right)^{\delta} d\bar{z} \right) \]

\[ \leq C \prod_{j=1}^{m} M f_j(x). \]

This concludes the proof. \( \square \)

4. Weighted estimates for \( T^* \)

In this section, we will study the multilinear maximal square function \( T^* \)

\[ T^*(\vec{f})(x) = \left( \int_{0}^{\infty} \sup_{\delta > 0} \left| \int_{\sum_{i=1}^{m} |x - y_i|^2 > \delta^2} K_v(x, y_1, \ldots, y_m) \prod_{j=1}^{m} f_j(y_j) d\bar{y} \right|^2 \frac{d\bar{v}}{v} \right)^{1/2}. \]

It should be pointed out that there is another kind of multilinear maximal square function \( T^{**} \) given by

\[ T^{**}(\vec{f})(x) = \sup_{\delta > 0} \left( \int_{0}^{\infty} \left| \int_{\sum_{i=1}^{m} |x - y_i|^2 > \delta^2} K_v(x, y_1, \ldots, y_m) \prod_{j=1}^{m} f_j(y_j) d\bar{y} \right|^2 \frac{d\bar{v}}{v} \right)^{1/2}. \]

It is obvious that \( T^{**}(\vec{f})(x) \leq T^*(\vec{f})(x) \). Thus, it is more meaningful to give some estimates for operator \( T^* \). In this following, we establish some multiple-weighted normal inequalities and weak-type estimates for \( T^* \).

**Theorem 4.1.** Let \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} \) and \( \omega \in A_p \) with \( p \geq 1 \). Let \( T \) be a multilinear operator in \( m-GSFO(A, s, \eta, \varepsilon) \) with a kernel satisfies Assumptions (H2) and (H3). Then the following inequality holds:

(i) If \( 1 < p_1, \ldots, p_m < \infty \), then

\[ \|T^* \vec{f}\|_{L^p(\omega)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega)}. \]

(ii) If \( 1 \leq p_1, \ldots, p_m < \infty \), then

\[ \|T^* \vec{f}\|_{L^p(\infty)(\omega)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega)}. \]

To prove Theorem 4.1 we need some lemmas.
Lemma 4.2. Let $T$ be a multilinear operator in $m-$GSFO($A, s, \eta, \varepsilon$) with a kernel satisfies Assumptions (H2) and (H3). For any $\eta > 0$, there is a constant $C < \infty$ depending on $\eta$ such that for all $\tilde{f}$ in any product of $L^{q_j}(\mathbb{R}^n)$ spaces, with $1 \leq q_j < \infty$, then the following inequality hold for all $x \in \mathbb{R}^n$

$$T^*(\tilde{f})(x) \leq C \left( M_\eta(T(\tilde{f}))(x) + \prod_{j=1}^{m} M(f_j)(x) \right).$$

Proof. For a fixed point $x$ and $\delta > 0$ we denote by $U_\delta(x) = \{\tilde{y} : \sum_{i=1}^{m} |x - y_i|^2 < \delta^2\}$ and $V_\delta(x) = \{\tilde{y} : \inf_{1 \leq j \leq m} |y_j - x| \geq \delta\}$. It is clear that

$$|T^*(\tilde{f})(x)| \leq \left( \int_0^\infty \sup_{\delta > 0} \left| \int_{(U_\delta(x) \cup V_\delta(x))^c} K_v(x, \tilde{y}) \prod_{i=1}^{m} f_i(y_i) dy_i \right| \left( \frac{dv}{v} \right)^{\frac{1}{2}} \right).$$

By using the size condition and Minkowski’s inequality, we get

$$\left( \int_0^\infty \left| \int_{(U_\delta(x) \cup V_\delta(x))^c} K_v(x, \tilde{y}) \prod_{i=1}^{m} f_i(y_i) dy_i \right| \left( \frac{dv}{v} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq C \left( \int_{(U_\delta(x) \cup V_\delta(x))^c} \left( \int_0^\infty |K_v(z, \tilde{y})|^2 \left( \frac{dv}{v} \right) \right)^{\frac{1}{2}} \prod_{i=1}^{m} |f_i(y_i)| dy_i \right).$$

$$\leq C \left( \int_{(U_\delta(x) \cup V_\delta(x))^c} \left( \frac{A}{\delta + \sum_{i=1}^{m} |y_i - x|^m} \prod_{i=1}^{m} |f_i(y_i)| dy_i \right) \right)^{\frac{1}{2}}.$$

(4.1)

We are ready to estimate the second term. Fix $\delta > 0$ and let $B(x, \delta/2)$ be the ball of center $x$ and radius $\delta/2$. Set $\tilde{f}_0 = (f_1 \chi_{B(z, \delta)}, \cdots, f_m \chi_{B(z, \delta)})$, for any $z \in B(x, \delta/2)$, we have

$$\tilde{T}_\delta(\tilde{f})(z) := \left( \int_0^\infty \left| \int_{V_\delta(z)} K_v(z, \tilde{y}) \prod_{i=1}^{m} f_i(y_i) dy_i \right|^2 \left( \frac{dv}{v} \right) \right)^{\frac{1}{2}}$$
In order to estimate \( \tilde{T}(\tilde{f})(z) \) in (4.1), we deduce that

\[
\left( \int_{0}^{\infty} \left| \int_{\beta} K_{v}(x, y) \prod_{i=1}^{m} f_{i}(y_{i}) dy_{i} \right|^{2} \frac{dv}{v} \right)^{\frac{1}{2}} \leq C \prod_{j=1}^{m} M(f_{j})(x).
\]

where \( t = (\delta/4)^{s} \) and \( s \) is a constant in (1.6). We decompose it into

\[
\left| \tilde{T}_{\delta}(\tilde{f})(x) - \tilde{T}_{\delta}(\tilde{f})(z) \right| \\
\leq \left| \tilde{T}_{\delta}(\tilde{f})(z) - \tilde{T}_{\delta}(\tilde{f})(z) \right| + \left| \tilde{T}_{\delta}(x) - \tilde{T}_{\delta}(\tilde{f})(z) \right| + \left| \tilde{T}_{\delta}(\tilde{f})(x) - \tilde{T}_{\delta}(\tilde{f})(x) \right| \\
= I + II + III.
\]

For the first term we use Assumption (H2). First note that \(|z - y| \geq 2|y - z| \geq 2|y - z| \). By the facts that \( z \in B(x, \delta/2) \) and \(|y_{j} - z| \geq \delta, \delta/2 \leq |y_{j} - x| \leq 2|y_{j} - z| \), we obtain

\[
\frac{|y_{j} - z|}{2} \leq |y - x| \leq 2|y_{j} - z|.
\]

By (1.11) in Assumption (H2) we obtain that

\[
\left| \tilde{T}_{\delta}(\tilde{f})(z) - \tilde{T}_{\delta}(\tilde{f})(z) \right| \\
= \left| \left( \int_{0}^{\infty} \left| \int_{\beta} K_{v}(y, \tilde{y}) m \prod_{i=1}^{m} f_{i}(y_{i}) dy_{i} \right|^{2} \frac{dv}{v} \right)^{\frac{1}{2}} \right| \\
- \left| \left( \int_{0}^{\infty} \left| \int_{\beta} K_{v}(y, \tilde{y}) m \prod_{i=1}^{m} f_{i}(y_{i}) dy_{i} \right|^{2} \frac{dv}{v} \right)^{\frac{1}{2}} \right| \\
\leq \left( \int_{0}^{\infty} \left| \int_{\beta} K_{v}(y, \tilde{y}) m \prod_{i=1}^{m} f_{i}(y_{i}) dy_{i} \right|^{2} \frac{dv}{v} \right)^{\frac{1}{2}}
\]
\[ \leq \left( \int_{V_{\delta}(z)} \left( \int_{0}^{\infty} \left| K_{v}(z, \bar{y}) - K_{t,v}^{(0)}(z, \bar{y}) \right| \prod_{i=1}^{m} \left| f_{i}(y_{i}) \right| dy_{i} \right) \left( \frac{dv}{v} \right) \right)^{\frac{1}{2}} \]

\[ \leq \int_{V_{\delta}(z)} \left( \int_{0}^{\infty} \left| K_{v}(z, \bar{y}) - K_{t,v}^{(0)}(z, \bar{y}) \right| \prod_{i=1}^{m} \left| f_{i}(y_{i}) \right| dy_{i} \right)^{\frac{1}{2}} \]

\[ \leq \int_{V_{\delta}(z)} \delta^{\epsilon} \prod_{i=1}^{m} \left| f_{i}(y_{i}) \right| dy_{i} \]

\[ \leq \prod_{j=1}^{m} M(f_{j})(x). \]

Similarly we get \( III \leq \prod_{j=1}^{m} M(f_{j})(x). \)

We now turn to the second term \( II. \)

\[ |T_{\delta}(\bar{f})(x) - T_{\delta}(\bar{f})(z)| \]

\[ \leq C \left| \left( \int_{0}^{\infty} \left( \int_{V_{\delta}(z)} \int_{\mathbb{R}^{n}} a_{t}(x, y) K_{v}(y, \bar{y}) dy \prod_{i=1}^{m} f_{i}(y_{i}) dy_{i} \right) \left( \frac{dv}{v} \right) \right)^{\frac{1}{2}} \right| \]

\[ + C \left| \left( \int_{0}^{\infty} \left( \int_{V_{\delta}(z)} \int_{\mathbb{R}^{n}} a_{t}(x, y) K_{v}(y, \bar{y}) dy \prod_{i=1}^{m} f_{i}(y_{i}) dy_{i} \right) \left( \frac{dv}{v} \right) \right)^{\frac{1}{2}} \right| \]

\[ - \left( \int_{0}^{\infty} \left( \int_{V_{\delta}(z)} \int_{\mathbb{R}^{n}} a_{t}(x, y) K_{v}(y, \bar{y}) dy \prod_{i=1}^{m} f_{i}(y_{i}) dy_{i} \right) \left( \frac{dv}{v} \right) \right)^{\frac{1}{2}} \]

\[ = II_{1} + II_{2}. \]

As in [8], since \( z \in B(x, \delta/2) \), we use the fact \( V_{\delta z}/2(x) \subset V_{\delta}(z) \) and \( V_{\delta}(x) \setminus V_{\delta}(z) \subset V_{\delta}(x) \setminus V_{\delta z}/2(x) \subset (\mathbb{R}^{n})^{m} \setminus (U_{\delta}(x) \cup V_{3\delta z}/2(x)) \). By (1.10) in Assumption (H2), we obtain

\[ II_{1} \leq C \int_{(\mathbb{R}^{n})^{m} \setminus (U_{\delta}(x) \cup V_{3\delta z}/2(x))} \left( \int_{0}^{\infty} \left| K_{v}(y, \bar{y}) a_{t}(z, y) dy \right|^{2} \left( \frac{dv}{v} \right) \right)^{\frac{1}{2}} \prod_{i=1}^{m} |f_{i}(y_{i})| dy_{i} \]

\[ = \int_{(\mathbb{R}^{n})^{m} \setminus (U_{\delta}(x) \cup V_{3\delta z}/2(x))} \left( \int_{0}^{\infty} \left| K_{t,v}^{(0)}(z, \bar{y}) \right|^{2} \left( \frac{dv}{v} \right) \right)^{\frac{1}{2}} \prod_{i=1}^{m} |f_{i}(y_{i})| dy_{i} \]

\[ \leq C \int_{(\mathbb{R}^{n})^{m} \setminus (U_{\delta}(x) \cup V_{3\delta z}/2(x))} \delta + \sum_{i=1}^{m} \left| y_{i} - x \right|^{m} \prod_{i=1}^{m} |f_{i}(y_{i})| dy_{i} \]

\[ \leq C \prod_{j=1}^{m} M(f_{j})(x). \]
For $II_2$, we use (1.12) in Assumption (H3), and a similar argument as in term $I$ to deduce that for $z \in B(x, \delta/2)$ we have

$$II_2 = \left| \left( \int_0^\infty \left| \int_{V_\delta(z)} K_{I,m}^{(0)}(x, y) \prod_{i=1}^m f_i(y_i) dy_i \left| \frac{2}{dv} \right| \right)^{\frac{1}{2}} \right|$$

$$- \left( \int_0^\infty \left| \int_{V_\delta(z)} \prod_{i=1}^m f_i(y_i) dy_i \right| \frac{2}{dv} \right)^{\frac{1}{2}}$$

$$\leq C \left( \int_0^\infty \left| \int_{V_\delta(z)} (K_{I,m}^{(0)}(x, y) - K_{I,m}^{(0)}(x, \tilde{y})) \prod_{i=1}^m f_i(y_i) dy_i \right| \frac{2}{dv} \right)^{\frac{1}{2}}$$

$$\leq C \left| \int_{V_\delta(z)} \left( \int_0^\infty |K_{I,m}^{(0)}(x, y) - K_{I,m}^{(0)}(x, \tilde{y})| \prod_{i=1}^m f_i(y_i) dy_i \right) \frac{2}{dv} \right|$$

$$\leq C \prod_{j=1}^m M(f_j)(x).$$

Then we obtain

$$II \leq C \prod_{j=1}^m M(f_j)(x).$$

Thus, for any $z \in B(x, \delta/2)$, we have

$$\tilde{T}_\delta(f)(x) \leq C \prod_{j=1}^m M(f_j)(x) + |T(f)(z)| + |T(f_0)(z)|.$$

Fix $0 < \eta < 1/m$. Raising the above inequality to the power $\eta$, integrating over $z \in B(x, \delta/2)$, and dividing by $B$ we obtain

$$\left| \tilde{T}_\delta(f)(x) \right|^{\eta} \leq C \left( \prod_{j=1}^m M(f_j)(x) \right)^{\eta} + M \left( T(f)^{\eta} \right)(x) + \frac{1}{|B|} \int_B |T(f_0)(z)|^{\eta} dz.$$

The left part is the same as in [8]. Then we finish the proof of this Lemma.

\[Q.E.D.\]

**Lemma 4.3.** ([6]) If $\omega \in A_p$ and $p > 1$, then $M$ maps from $L^p(\omega)$ to $L^p(\omega)$.

**Lemma 4.4.** ([2]) If $\omega \in A_p$ and $p \geq 1$, then $M$ maps from $L^{p,\infty}(\omega)$ to $L^{p,\infty}(\omega)$.

**Proof of Theorem 4.1.**
Proof. We choose a positive number $\eta$ such that $\eta < p$, then Theorem 4.1(ii) follows by using Lemma 4.2, Lemma 4.4, Theorem 3.1 and the Hölder inequality for weak spaces (see [11], p.15).

$$||T^*(\vec{f})||_{L^p,\infty}(\omega) \leq C \left( ||M_\eta (T(\vec{f}))||_{L^p,\infty}(\omega) + \prod_{j=1}^m ||M(f_j)||_{L^p,\infty}(\omega) \right)$$

$$= C \left( ||M(||T(\vec{f})||^{\eta})||_{L^p,\infty}^{\frac{1}{\eta}}(\omega) + \prod_{j=1}^m ||M(f_j)||_{L^p,\infty}(\omega) \right)$$

$$\leq C \left( |||T(\vec{f})||^{\eta}||_{L^p,\infty}(\omega) + \prod_{j=1}^m ||f_j||_{L^p(\omega)} \right)$$

$$\leq C \left( ||T(\vec{f})||_{L^p,\infty}(\omega) + \prod_{j=1}^m ||f_j||_{L^p(\omega)} \right)$$

$$\leq C \prod_{j=1}^m ||f_j||_{L^p(\omega)}.$$  

The proof of (i) is similar. \qed

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