Polyhedral Deformations of Cone Manifolds

A Aalam

Abstract Two single parameter families of polyhedra $P(\psi)$ are constructed in three dimensional spaces of constant curvature $C(\psi)$. Identification of the faces of the polyhedra via isometries results in cone manifolds $M(\psi)$ which are topologically $S^1 \times S^2$, $S^3$ or singular $S^2$. The singular set of $M(\psi)$ can have self intersections for some values of $\psi$ and can also be the Whitehead link or form other configurations. Curvature varies continuously with $\psi$. At $\psi = 0$ spontaneous surgery occurs and the topological type of $M(\psi)$ changes. This phenomenon is described.

0 Introduction

We study continuous families of cone manifolds $M_\psi$ parametrised by cone angle which begin at cone angle zero with the complement of the Whitehead link in $S^3$. We consider the case of equal cone angles on all singular link components. Increasing cone angles the families trace different paths in Dehn surgery space joined by what we call a Dehn surgery transition point. The cone structures for certain non-zero values of cone angles exist in projective models or in $S^3$.

In one Dehn surgery direction the cone manifolds are for certain cone angles, obtained by surgery on the Whitehead link in $S^3$ resulting in a topologically distinct singular set in $S^2 \times S^1$. As cone angle is increased the topological type of the singular set changes and the hyperbolic cone manifold develops two cusps and becomes $S^3$ at cone angle $\frac{2}{3}\pi$. The topological type of the singular set and the structure of $M_\psi$ remain unaltered as cone angle increases beyond $\frac{2}{3}\pi$ until we reach a cone angle $\omega$ where $M_\psi$ becomes $\mathbb{R}^3$ with topologically the same type of singularity. Increasing cone angle past $\omega$ the singular set reverts back to its pre-$\frac{2}{3}\pi$ cone angle topological type and $M_\psi$ becomes spherical in $S^2 \times S^1$.

At cone angle $\pi$ the underlying polyhedron becomes a lens in $S^3$ from which the cone manifold is obtained by suitable identifications. For cone angles in the interval $[\pi, \zeta]$, $M_\psi$ is spherical and the topological type of its singular set is unchanged but it is now in $S^3$. At cone angle $\zeta$, $M_\psi$ becomes the suspension of a sphere with four cone points. It remains the well understood sphere with four singularities for cone angles larger than $\zeta$.

Investigating the deformation on the other side of Dehn surgery we obtain the Whitehead link in $S^3$ for certain non-zero cone angles. A complete investigation will be carried out later.
Working in hyperbolic or spherical three space of constant curvature it is usual for curvature to be normalised to plus or minus one. While there are good reasons for this convention, this does not allow us to envisage a continuous family of cone structures in which curvature changes from positive to negative with Euclidean space a point in a continuum. cf [1].

Here a Cone Manifold is a PL manifold with a possibly empty codimension two locally flat submanifold called the singular set. In dimensions two and three the singular set consists of isolated points and curves but not graphs respectively. The geometric model is a spherical, Euclidean, or hyperbolic space of constant curvature where the constant is any real number. Points in the complement of the singular set have neighbourhoods homeomorphic to neighbourhoods in the model. Points on the singular set have neighbourhoods homeomorphic to neighbourhoods in the topological space obtained by identifying boundaries of the intersection of two half spaces referred to as a wedge in the model. The homeomorphism takes the singular set to the axis of rotation in the topological space and transition functions are isometries. Orbifolds are represented by discrete structures where cone angles are of the form $\frac{2\pi}{n}, n \in \mathbb{N}$. cf [1].

The topological space obtained by identifying in pairs faces of a polyhedron in a space of constant curvature via isometries is a cone manifold if:

1. No edge is identified with its inverse in the equivalence class induced by the identifications, and the identifications of wedges along faces are cyclic for each equivalence class of edges.

2. The cone angle at each edge, i.e. the sum of the dihedral angles about the edge is $\leq 2\pi$.

3. The neighbourhood of each vertex is a cone on a sphere. If a vertex neighbourhood is, for example, a cone on a torus the topological space is not a manifold.

4. There are either two or no edges emanating from a vertex where cone angle is not $2\pi$. If two edges then the cone angles must be equal and the edges must be lined up. The vertex neighbourhood is obtained by identifying wedge half planes.

cf [1].

It is reasonable to expect that our deformation process holds for any hyperbolic link complement since the polyhedral description of a link complement utilised here is canonical and we can expect to be able to deform this construction by opening cusps in $M_0$ in the way we have described. Our methods can be viewed as part of an approach to describe all compact connected 3-manifolds through deformations by removing singularities of branched singular covers of $S^3$ along universal links thereby providing an approach to the Poincare Conjecture in dimension three. The important phenomenon of self intersecting singular set in a cone manifold occurs in our work. Thus our construction may prove useful in studying this phenomenon.
1 Cone angle interval $[0, \omega]$

**Proposition 1.1** $M_\psi$ is $S^2 \times S^1$ with two singular components and is hyperbolic for cone angles $\psi \in (0, \frac{\pi}{3})$. $M_{\frac{2\pi}{3}}$ is cusped hyperbolic and has a self intersecting singular set. There exists $\omega \in \mathbb{R}$ such that $M_\psi$ is hyperbolic with a self intersecting singular set when $\frac{2\pi}{3} < \psi < \omega$ and $M_\omega$ is $\mathbb{R}^3$.

**Proof** $M_0$ the complement of the Whitehead link in $S^3$ is obtained from a two cusped octahedron $P_0$ with identifications described in the projective Klein model of $\mathbb{H}^3$ in figure 2. We refer the reader to [2], [3] and [4] for details of this construction.

![Figure 1: The Whitehead link](image)

![Figure 2: $P_0$ with identifications](image)

To find out more about $P_\psi$ we view it relative to a coordinate system and inside a reference box in the Klein model as in figure 4. The length, width and
height of the reference box are denoted by $a$, $b$ and $c$ respectively. Opening cusps in $M_0$ we obtain $P_\psi$ for $\psi \in (0, \frac{2}{3}\pi)$ as in figure 3.

Figure 3: opening cusps

From figure 4 we have the following identifications:
Face identifications:

\[ A \leftrightarrow A' \quad \text{i.e.} \quad \triangle(\text{onp}) \leftrightarrow \triangle(\text{jhi}) \]  \hspace{1cm} (1)
\[ C \leftrightarrow C' \quad \text{i.e.} \quad \triangle(\text{sqr}) \leftrightarrow \triangle(\text{klm}) \]  \hspace{1cm} (2)
\[ D \leftrightarrow D' \quad \text{i.e.} \quad \text{hexagon}(\text{inpqsj}) \leftrightarrow \text{hexagon}(\text{hmlrsj}) \]  \hspace{1cm} (3)
\[ B \leftrightarrow B' \quad \text{i.e.} \quad \text{hexagon}(\text{nokmhi}) \leftrightarrow \text{hexagon}(\text{poklrq}) \]  \hspace{1cm} (4)

Edge pairings:

\[ ok \leftrightarrow ok \]  \hspace{1cm} (5)
\[ sj \leftrightarrow sj \]  \hspace{1cm} (6)
\[ on \leftrightarrow op \leftrightarrow ji \leftrightarrow jh \]  \hspace{1cm} (7)
\[ sr \leftrightarrow sq \leftrightarrow km \leftrightarrow kl \]  \hspace{1cm} (8)
\[ np \leftrightarrow ml \leftrightarrow hi \leftrightarrow rq \]  \hspace{1cm} (9)
\[ in \leftrightarrow hm \leftrightarrow lr \leftrightarrow pq \]  \hspace{1cm} (10)

Edge pairings (5) and (6) represent singular components of \( M_\psi \).

We observe:

The dihedral angle between planes incident
in a member of (5) or (6) is the cone angle \( \psi \). \hspace{1cm} (11)

Moreover (7), (8) and (9) are not part of the singular set. Therefore

The dihedral angle between planes incident
in a member of (7), (8) or (9) is \( \pi \). \hspace{1cm} (12)

Information in (10) represents segments \( \beta_1, \beta_2, \beta_3 \) and \( \beta_4 \) which are identified
to give a singular component \( \beta \) of the singular set of \( M_\psi \). Considering (10) with
equal cone angles on all singular components we deduce:

Planes incident in a member of (10) intersect at dihedral angle \( \frac{\psi}{4} \). \hspace{1cm} (13)

We deduce from (10) that \( a = b \). Box coordinates can therefore be normalised
to give \( a = 1 = b \) and \( c \) where \( c \) is measured in the \( z \)-direction. The
reference box is therefore a cube with square base and height \( 2c \) as in figure 4.

Topologically \( M_\psi \) is \( S^1 \times S^2 \) with two unlinked singular components as in
figure 5.

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To find out about the geometry of $M_\psi$ we look at $P_\psi$ inside the Klein model of $\mathbb{H}^3$. Let $R$ be the radius of the Klein ball $B_R$ and let $c$ represent the height of the main box. We will show that $P_\psi$ lives inside $B_R$ and can be specified by $R$ and $c$. We also have

$$\text{curvature of } \mathbb{H}^3 = -\frac{1}{R^2}$$ (14)

The geometry of $M_\psi$ can therefore be described in terms of $\psi$. We now obtain $R$ and $c$ as functions of $\psi$.

Figure 5 depicts $P_\psi$ in the Klein model.

Figure 6: $P_\psi$ in the Klein model
In homogeneous coordinates the equations of planes of $P_\psi$ and their poles are:

| Plane | Equation | Pole |
|-------|----------|------|
| $A$   | $cx + cy + z = c$ | $(cR, cR, R, c)$  
| $B$   | $cx - cy - z = c$ | $(cR, -cR, -R, c)$  
| $C$   | $-cx - cy + z = c$ | $(-cR, -cR, R, c)$  
| $D$   | $-cx + cy - z = c$ | $(-cR, cR, -R, c)$  

Let $v = (v_1, v_2, v_3, v_4), w = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$. Then the hyperbolic bilinear form is given by

$$\langle v, w \rangle_H = v_1 w_1 + v_2 w_2 + v_3 w_3 - v_4 w_4$$

Let $\theta$ be the dihedral angle of intersection between planes $P$ and $Q$ with poles $v$ and $w$ respectively. Then

$$\cos \theta = -\frac{\langle v, w \rangle_H}{\sqrt{\langle v, v \rangle_H \langle w, w \rangle_H}}$$

Combining (10), (13), (15) and (17) we can derive expressions for $\cos \psi$ and $\cos \frac{\psi}{4}$ in terms of $R$ and $c$ to obtain:

$$c^2 = \frac{1 + \cos \psi}{2\cos \frac{\psi}{4} - \cos \psi + 1}$$  \hspace{1cm} (18)

$$R^2 = \frac{1 + \cos \psi}{2\cos \frac{\psi}{4} + \cos \psi - 1}$$  \hspace{1cm} (19)

Substituting $(R, c)$-expressions of $\cos \frac{\psi}{4}$ and $\cos \psi$ in (18) and (19) we can verify these identities.

We note $\omega = 4\cos^{-1}(\frac{2}{\sqrt{3}} \cos(\frac{1}{3}\cos^{-1}(-\frac{3\sqrt{3}}{8}))) \approx 2.311984...$ is the smallest positive value of $\psi$ for which the denominator of (19) is zero. Hence $\omega$ is the smallest positive value of $\psi$ for which $R^2$ is infinite. This combined with (14) implies that curvature is zero and $M_\psi$ is therefore Euclidean when $\psi = \omega$.

Combining (12), (15), (18) and (17) the equation of plane $N$ in figure 6 is:

$$N : -x + y + cz = R^2$$

Let $p = (-1, 1, -c, R)$ denote the pole of $N$ with $\mathbb{H}^3$ embedded in $\mathbb{R}P^3$. We have

$$\langle p, p \rangle = 2 + c^2 - R^2 = \frac{3 - 4\cos^2 \frac{\psi}{4}}{4 - 4\cos \frac{\psi}{4}(2\cos \frac{\psi}{4} - 1)}$$  \hspace{1cm} (21)
Therefore

\[
\langle \mathbf{p}, \mathbf{p} \rangle \begin{cases} 
> 0 & \text{when } 0 < \psi < \frac{2}{3} \pi, \\
= 0 & \text{when } \psi = \frac{2}{3} \pi, \\
< 0 & \text{when } \frac{2}{3} \pi < \psi < \omega.
\end{cases}
\] (22)

Information contained in (22) corresponds to the configurations depicted in figure 7 in the case of \( \mathbb{H}^2 \) embedded in \( \mathbb{R}P^2 \) and is true for \( n \) hyperplanes in \( \mathbb{H}^n \) embedded in \( \mathbb{R}P^n \).

![Diagram](image)

**Figure 7:** (a) \( \langle \mathbf{p}, \mathbf{p} \rangle > 0 \) : lines meet outside \( \mathbb{H}^2 \). (b) \( \langle \mathbf{p}, \mathbf{p} \rangle = 0 \) : lines meet on \( \partial \mathbb{H}^2 \). (c) \( \langle \mathbf{p}, \mathbf{p} \rangle < 0 \) : lines meet inside \( \mathbb{H}^2 \).

Referring to (20), figure 6 and figure 7, when \( \langle \mathbf{p}, \mathbf{p} \rangle > 0 \) there is a plane \( N \) inside \( \mathbb{H}^3 \). When \( \langle \mathbf{p}, \mathbf{p} \rangle = 0 \) the plane \( N \) is the point of intersection of the planes \( A, C \) and \( D \) on \( \partial \mathbb{H}^3 \). When \( \langle \mathbf{p}, \mathbf{p} \rangle < 0 \) the plane \( N \) is the point of intersection of the planes \( A, C \) and \( D \) inside \( \mathbb{H}^3 \).

We now verify that \( P_\psi \subset \mathbb{H}^3 \cup \partial \mathbb{H}^3 \) for \( \psi \in [0, \frac{2}{3} \pi] \). Since \( P_\psi \) is symmetric with respect to the origin it is sufficient to show that the vertices \( (A \cap N \cap C) \) and \( (A \cap N \cap D) \) of figure 6 live inside \( \mathbb{H}^3 \cup \partial \mathbb{H}^3 \). Let

\[
f(\psi) = R^2 - (A \cap N \cap C)^2 = \frac{1}{2}(R^2 - c^2)(2 + c^2 - R^2)
\] (23)

where \( (A \cap N \cap C)^2 \) denotes square of the distance of vertex \( (A \cap N \cap C) \) from the origin. We have \( f > 0 \) for \( \psi \in (0, \frac{2}{3} \pi) \) and \( f = 0 \) when \( \psi = 0 \) or \( \frac{2}{3} \pi \).

Let

\[
g(\psi) = R^2 - (A \cap N \cap D)^2 = \frac{(R^2 - 1)(2 + c^2 - R^2)}{c^2 + 1}
\] (24)
We note $g > 0$ when $\psi \in (0, \frac{2}{3}\pi)$ and $g = 0$ if $\psi = 0$ or $\frac{2}{3}\pi$.

Therefore $P_\psi \subset \mathbb{H}^3 \cup \partial\mathbb{H}^3$ when $\psi \in [0, \frac{2}{3}\pi]$. Hence $M_\psi$ is hyperbolic when $\psi \in [0, \frac{2}{3}\pi]$.

We note from (22) and figure 7 that $M_{\frac{2}{3}\pi}$ is cusped. From figure 8 we observe that $M_{\frac{2}{3}\pi}$ has two cusps, it is topologically $S^3$ with singular set as shown in figure 9 and its cusp neighborhoods are Euclidean turnovers as in figure 10.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{\textit{P}_{\frac{2}{3}\pi} \text{ inside } \mathbb{H}^3 \cup \partial\mathbb{H}^3}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{Singular set of $M_\psi$ when $\psi \in [\frac{2}{3}\pi, \omega)$ and its cusp neighbourhoods when $\psi = \frac{2}{3}\pi$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10.png}
\caption{A Euclidean turnover}
\end{figure}
From (22) and figure 7 we observe that the bounding planes of $P_\psi$ meet inside $H_3$ when $\psi \in \left(\frac{2}{3}\pi, \omega\right)$. Therefore, $P_\psi$ is in $H_3$. Hence $M_\psi$ is hyperbolic when $\psi \in \left(\frac{2}{3}\pi, \omega\right)$. The singular set of $M_\psi$ for $\psi \in \left(\frac{2}{3}\pi, \omega\right)$ is shown in figure 9.

We note from (19) that $R(\omega) = \infty$ hence $M_\omega$ is Euclidean. $P_\omega$ is a tetrahedron with deleted vertices from which $M_\omega$ is obtained using identifications. We observe that $M_\omega$ is $\mathbb{R}^3$. The singular set of $M_\omega$ is shown in figure 11.

Figure 11: Singular set of $M_\omega$

2 Cone angles larger than $\omega$

**Proposition 2.1** $M_\psi$ is spherical when cone angle $\psi$ is larger than $\omega$. There exists $\zeta \in \mathbb{R}$ such that $M_\psi$ is topologically $S^3$ with a self intersecting singular set when $\omega < \psi < \zeta$. $M_\psi$ is the suspension of a sphere with four cone points when $\zeta \leq \psi$.

**Proof** When $\omega < \psi < \pi$ we note $R^2$ the square radius of the Klein model becomes negative so that the model has imaginary radius. This leads us to use the spherical bilinear form for $v = (v_1, v_2, v_3, v_4), w = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$:

$$\langle v, w \rangle_{S^3} = v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4$$

when $\psi \in \left(\omega, \pi\right)$.

This is the usual scalar product on $\mathbb{R}^4$. Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Since $H_3^R = \{x \mid \langle x, x \rangle_H = -R^2\}$ we define $S^3_R^1 = \{x \mid \langle x, x \rangle_{S^3} = R^2\}$. Thus $S^3_R$ is the sphere of radius $R$. The “Klein model” $\mathbb{K}_R^3$ for the sphere of radius $R$ is the hyperplane $\mathbb{K}_R^3 = \{x \mid t = R\}$. In contrast to the hyperbolic case we don’t need to verify that the polytope lies inside the sphere of radius $R$ in $\mathbb{K}_R^3$ since projection from the origin which is not conformal defines a 1-1 correspondence between $\mathbb{K}_R^3$ and the upper hemisphere of $S^3_R$. The metric on $\mathbb{K}_R^3$ is then the pull back of the metric on $S^3_R$. As in the hyperbolic case reflections in planes through the origin and rotation about axes through the origin are both Euclidean and spherical isometries.

If a plane in $\mathbb{K}_R^3$ has equation $\alpha x + \beta y + \gamma z = \delta$ and $t = R$ then in homogeneous coordinates the equation is $\alpha x + \beta y + \gamma z - (\delta / R) t = 0$. In the spherical
case the pole has homogeneous coordinates \((\alpha, \beta, \gamma, -\delta/R)\) whereas in the hyperbolic case the pole has coordinates \((\alpha, \beta, \gamma, \delta/R)\). If a pair of planes with poles \(v\) and \(w\) in the spherical case intersect with dihedral angle \(\theta\) then

\[
\pm \cos \theta = \frac{(v, w)_S}{\sqrt{(v, v)_S (w, w)_S}}
\]

(26)

Using (25), (26) and the spherical version of (15) we obtain

\[
0 \leq R^2 = -\frac{1 + \cos \psi}{2\cos \frac{\psi}{4} - 1 + \cos \psi}
\]

(27)

when \(\omega \leq \psi \leq \zeta\).

When \(\psi \in (\omega, \pi)\) we observe that \(P_\psi\) is a tetrahedron with identification as in figure 12.

Figure 12: \(P_\psi\) inside the reference box

Therefore \(M_\psi\) is \(S^3\) with a self intersecting singular set as in figure 13 with spherical turnover neighbourhoods.

Figure 13: A spherical turnover
Since $R^2(\pi) = 0 = c^2(\pi)$ the three dimensional model has collapsed into a two dimensional disc at cone angle $\pi$. By projection onto $S_1^3 = \{ x \mid x^2 + y^2 + z^2 + t^2 = 1 \}$ the sphere of radius 1 we observe that $P_\pi$ is a lens with angle $\psi/4$ as in figure 14 from which $M_\pi$ is obtained by identifications.

![Figure 14: $P_\pi$ is a lens with angle $\frac{\pi}{4}$](image)

We note $\zeta = 4 \cos^{-1}(\frac{2}{\sqrt{3}} \cos(\frac{1}{3} \cos^{-1}(\frac{-3\sqrt{3}}{8} + \frac{4}{3} \pi))) \approx 5.191298...$ is the second smallest value of $\psi$ for which $R$ is infinite. The equation of $c^2$ is the same as in the hyperbolic case. The model continues to be $S^3$ for $\pi < \psi < \zeta$ therefore $M_\psi$ is spherical with a self intersecting singular set as in figure 15 when $\psi \in (\pi, \zeta)$.

![Figure 15: Singular set of $M_\pi$](image)

As cone angle $\psi$ approaches $\zeta$ the segments $h_1$ and $h_2$ decrease in length towards 0 as shown in figure 16 so that we get the singular set shown in figure 17 when $\psi = \zeta$. 

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Figure 16: Segments $h_1$ and $h_2$ decrease in length as $\psi$ increases towards $\zeta$.

We therefore obtain $M_\psi$ as the suspension of a sphere with four cone points as in figure 17 when $\zeta < \psi$.

Figure 17: $M_\zeta$ is the suspension of a sphere with four cone points.

3 Spontaneous surgery

We now look at how we get spontaneous surgery and the Whitehead link as the singular set. $P_\psi$ and its identifications “after” spontaneous surgery are shown in figure 18. Face pairings are $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$ and $D \leftrightarrow D'$ and edges with the same label are identified.
Figure 18: $P_\psi$ after spontaneous surgery gives $S^3$ with the Whitehead link as its singular set

We note that $M_\psi$ “after” spontaneous surgery is the result of performing $(0,1)$-Dehn surgery on one component of the Whitehead link in $S^3$. Dihedral angles at the $\beta$ edges of $P_\psi$ are now the cone angle $\psi$ and other incidence angles between the planes of $P_\psi$ are as they were prior to spontaneous surgery. Calculations to show the behaviour of $M_\psi$ after surgery remain to be done.

References

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Dr Aalam
PO Box 18810
London SW7 2ZR
UK.
email: aalam@mth.kcl.ac.uk