THE COMPOSITION SERIES OF IDEALS OF THE PARTIAL-ISOMETRIC CROSSED PRODUCT BY SEMIGROUP OF ENDMORPHISMS

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Abstract. Let \( \Gamma^+ \) be the positive cone in a totally ordered abelian group \( \Gamma \), and \( \alpha \) an action of \( \Gamma^+ \) by extendible endomorphisms of a \( C^* \)-algebra \( A \). Suppose \( I \) is an extendible \( \alpha \)-invariant ideal of \( A \). We prove that the partial-isometric crossed product \( I := I \times_{\alpha}^\text{piso} \Gamma^+ \) embeds naturally as an ideal of \( A \times_{\alpha}^\text{iso} \Gamma^+ \), such that the quotient is the partial-isometric crossed product of the quotient algebra. We claim that this ideal \( I \) together with the kernel of a natural homomorphism \( \phi : A \times_{\alpha}^\text{piso} \Gamma^+ \to A \times_{\alpha}^\text{iso} \Gamma^+ \) gives a composition series of ideals of \( A \times_{\alpha}^\text{piso} \Gamma^+ \) studied by Lindiarni and Raeburn.

1. Introduction

Let \((A, \Gamma^+, \alpha)\) be a dynamical system consisting of the positive cone \( \Gamma^+ \) in a totally ordered abelian group \( \Gamma \), and an action \( \alpha : \Gamma^+ \to \text{End} A \) of \( \Gamma^+ \) by extendible endomorphisms of a \( C^* \)-algebra \( A \). A covariant representation of the system \((A, \Gamma^+, \alpha)\) is defined for which the semigroup of endomorphisms \( \{\alpha_s : s \in \Gamma^+\} \) are implemented by partial isometries, and then the associated partial-isometric crossed product \( C^* \)-algebra \( A \times_{\alpha}^\text{piso} \Gamma^+ \), generated by a universal covariant representation, is characterized by the property that its nondegenerate representations are in a bijective correspondence with covariant representations of the system. This generalizes the covariant isometric representation theory: the theory that uses isometries to represent the semigroup of endomorphisms in a covariant representation of the system. We denoted by \( A \times_{\alpha}^\text{iso} \Gamma^+ \) for the corresponding isometric crossed product.

Suppose \( I \) is an extendible \( \alpha \)-invariant ideal of \( A \), then \( a + I \mapsto \alpha_x(a) + I \) defines an action of \( \Gamma^+ \) by extendible endomorphisms of the quotient algebra \( A/I \). It is well-known that the isometric crossed product \( I \times_{\alpha}^\text{iso} \Gamma^+ \) sits naturally as an ideal in \( A \times_{\alpha}^\text{iso} \Gamma^+ \) such that \((A \times_{\alpha}^\text{iso} \Gamma^+)/I \times_{\alpha}^\text{iso} \Gamma^+) \cong A/I \times_{\alpha}^\text{iso} \Gamma^+ \). We show that this result is valid for the partial-isometric crossed product.

Moreover if \( \phi : A \times_{\alpha}^\text{piso} \Gamma^+ \to A \times_{\alpha}^\text{iso} \Gamma^+ \) is the natural homomorphism given by the canonical universal covariant isometric representation of \((A, \Gamma^+, \alpha)\) in \( A \times_{\alpha}^\text{iso} \Gamma^+ \), then \( \ker \phi \) together with the ideal \( I \times_{\alpha}^\text{piso} \Gamma^+ \) give a composition series of ideals of \( A \times_{\alpha}^\text{iso} \Gamma^+ \), from which we recover the structure theorems of \([6]\). Let us now consider the framework of \([6]\). A system that consists of the \( C^* \)-subalgebra \( A := B_{\Gamma^+} \) of \( \ell^{\infty}(\Gamma^+) \).

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spanned by the functions $1_s$ satisfying
\[1_s(t) = \begin{cases} 1 & \text{if } t \geq s \\ 0 & \text{otherwise} \end{cases}\]
and the action $\tau : \Gamma^+ \to \text{End } B_{\Gamma^+}$ given by the translation on $\ell^\infty(\Gamma^+)$. We choose an extendible $\tau$-invariant ideal $I$ to be the subalgebra $B_{\Gamma^+\infty}$ spanned by $\{1_x - 1_y : x < y \in \Gamma^+\}$. Then the composition series of ideals of $B_{\Gamma^+} \times_{\tau}^{piso} \Gamma^+$, that is given by the two ideals $\ker \phi$ and $B_{\Gamma^+\infty} \times_{\tau}^{piso} \Gamma^+$, produces the large commutative diagram in [6, Theorem 5.6]. This result shows that the commutative diagram in [6, Theorem 5.6] exists for any totally ordered abelian subgroup (not only for subgroups of $\mathbb{R}$), and that we understand clearly where the diagram comes from.

Next, if we consider a specific semigroup $\Gamma^+$ such as the additive semigroup $\mathbb{N}$ in the group of integers $\mathbb{Z}$, then the large commutative diagram gives a clearer information about the ideals structure of $c \times_{\tau}^{piso} \mathbb{N}$. We can identify that the left-hand and top exact sequences in diagram [6, Theorem 5.6] are indeed equivalent to the extension of the algebra $\mathcal{K}(\ell^2(\mathbb{N}, c_0))$ of compact operators on the Hilbert module $\ell^2(\mathbb{N}, c_0)$ by the algebra $\mathcal{K}(\ell^2(\mathbb{N}))$ provided by the algebra $\mathcal{K}(\ell^2(\mathbb{N}, c))$ of compact operators on $\ell^2(\mathbb{N}, c)$. Moreover it is known that $\text{Prim } \mathcal{K}(\ell^2(\mathbb{N}, c)) \simeq \text{Prim } (\mathcal{K}(\ell^2(\mathbb{N})) \otimes c) \simeq \text{Prim } c$ is homeomorphic to $\mathbb{N} \cup \infty$. Together with a knowledge about the primitive ideal space of the Toeplitz $C^*$-algebra generated by the unilateral shift, our theorem on the composition series of ideals of $c \times_{\tau}^{piso} \mathbb{N}$ provides a complete description of the topology on the primitive ideal space of $c \times_{\tau}^{piso} \mathbb{N}$.

We begin with a section containing background material about the partial-isometric crossed product by semigroups of extendible endomorphisms. In Section 3, we prove the existence of a short exact sequence of partial-isometric crossed products, which generalizes [2, Theorem 2.2] of the semigroup $\mathbb{N}$. Then we consider this and the other natural exact sequence described earlier in [4], to get the composition series of ideals in $A \times_{\alpha}^{piso} \Gamma^+$.

We proceed to Section 4 by applying our results in Section 3 to the distinguished system $(B_{\Gamma^+}, \Gamma^+, \tau)$ and the extendible $\tau$-invariant ideal $B_{\Gamma^+\infty}$ of $B_{\Gamma^+}$. It can be seen from our Proposition [4] that the large commutative diagram of [6, Theorem 5.6] remains valid for any subgroup $\Gamma$ of a totally ordered abelian group. Finally in the last section we describe the topology of primitive ideal space of $c \times_{\tau}^{piso} \mathbb{N}$ by using this large diagram.

2. Preliminaries

A bounded operator $V$ on a Hilbert space $H$ is called an isometry if $\|V(h)\| = \|h\|$ for all $h \in H$, which is equivalent to $V^*V = 1$. A bounded operator $V$ on a Hilbert space $H$ is called a partial isometry if it is isometry on $(\ker V)^\perp$. This is equivalent to $VV^*V = V$. If $V$ is a partial isometry then so is the adjoint $V^*$, where as for an isometry $V$, the adjoint $V^*$ may not be an isometry unless $V$ is unitary. Associated to a partial isometry $V$, there are two orthogonal projections $V^*V$ and $VV^*$ on the initial space $(\ker V)^\perp$ and on the range $VH$ respectively. In a $C^*$-algebra $A$, an element $v \in A$ is called an isometry if $v^*v = 1$ and a partial isometry if $vv^*v = v$.

An isometric representation of $\Gamma^+$ on a Hilbert space $H$ is a map $S : \Gamma^+ \to B(H)$ which satisfies $S_x := S(x)$ is an isometry, and $S_{x+y} = S_xS_y$ for all $x, y \in \Gamma^+$. So
an isometric representation of \( \mathbb{N} \) is determined by a single isometry \( S_1 \). Similarly

a partial-isometric representation of \( \Gamma^+ \) on a Hilbert space \( H \) is a map \( V : \Gamma^+ \to B(H) \) which satisfies \( V_x := V(x) \) is a partial isometry, and \( V_{x+y} = V_x V_y \) for all \( x, y \in \Gamma^+ \). Note that the product \( VW \) of two partial isometries \( V \) and \( W \) is a partial

isometry precisely when \( V^*V \) commutes with \( WW^* \). Proposition 2.1). Thus a partial isometry \( V \) is called a power partial isometry if \( V^n \) is a partial isometry for every \( n \in \mathbb{N} \), so a partial-isometric representation of \( \mathbb{N} \) is determined by a single power partial isometry \( V_1 \). If \( V \) is a partial-isometric representation of \( \Gamma^+ \), then every \( V_x V^*_x \) commutes with \( V_t V^*_t \), and so does \( V_x V^*_x \) with \( V_t V^*_t \).

Now we consider a dynamical system \((A, \Gamma^+, \alpha)\) consisting of a \( C^*\)-algebra \( A \), an action \( \alpha \) of \( \Gamma^+ \) by endomorphisms of \( A \) such that \( \alpha_0 = \text{id} \). Because we deal with non unital \( C^*\)-algebras and non unital endomorphisms, we require every endomorphism \( \alpha_x \) to be extendible to a strictly continuous endomorphism \( \overline{\alpha}_x \) on the multiplier algebra \( M(A) \) of \( A \). This happens precisely when there exists an approximate identity \((a\lambda)\) in \( A \) and a projection \( p_{\alpha_x} \in M(A) \) such that \( \alpha_x(a\lambda) \) converges strictly to \( p_{\alpha_x} \) in \( M(A) \).

**Definition 2.1.** A covariant isometric representation of \((A, \Gamma^+, \alpha)\) on a Hilbert space \( H \) is a pair \((\pi, S)\) of a nondegenerate representation \( \pi : A \to B(H) \) and an isometric representation of \( S : \Gamma^+ \to B(H) \) such that \( \pi(\alpha_x(a)) = S_x \pi(a) S_x^* \) for all \( a \in A \) and \( x \in \Gamma^+ \).

An isometric crossed product of \((A, \Gamma^+, \alpha)\) is a triple \((B, j_A, j_{\Gamma^+})\) consisting of a \( C^*\)-algebra \( B \), a canonical covariant isometric representation \((j_A, j_{\Gamma^+})\) in \( M(B) \) which satisfies the following:

(i) for every covariant isometric representation \((\pi, S)\) of \((A, \Gamma^+, \alpha)\) on a Hilbert space \( H \), there exists a nondegenerate representation \( \pi \times S : B \to B(H) \) such that \( (\pi \times S) \circ j_A = \pi \) and \( (\pi \times S) \circ j_{\Gamma^+} = S \); and

(ii) \( B \) is generated by \( j_A(A) \cup j_{\Gamma^+}(\Gamma^+) \), we actually have

\[
B = \overline{\text{span}}\{j_{\Gamma^+}(x)^*j_A(a)j_{\Gamma^+}(y) : x, y \in \Gamma^+, a \in A\}.
\]

Note that a given system \((A, \Gamma^+, \alpha)\) could have a covariant isometric representation \((\pi, S)\) only with \( \pi = 0 \). In this case the isometric crossed product yields no information about the system. If a system admits a non trivial covariant representation, then the isometric crossed product does exist, and it is unique up to isomorphism: if there is such a covariant isometric representation \((t_A, t_{\Gamma^+})\) of \((A, \Gamma^+, \alpha)\) in a \( C^*\)-algebra \( C \), then there is an isomorphism of \( C \) onto \( B \) which takes \((t_A, t_{\Gamma^+})\) into \((j_A, j_{\Gamma^+})\). Thus we write the isometric crossed product \( B \) as \( A \times^\alpha \Gamma^+ \).

The partial-isometric crossed product of \((A, \Gamma^+, \alpha)\) is defined in a similar fashion involving partial-isometries instead of isometries.

**Definition 2.2.** A covariant partial-isometric representation of \((A, \Gamma^+, \alpha)\) on a Hilbert space \( H \) is a pair \((\pi, S)\) of a nondegenerate representation \( \pi : A \to B(H) \) and a partial-isometric representation \( S : \Gamma^+ \to B(H) \) of \( \Gamma^+ \) such that \( \pi(\alpha_x(a)) = S_x \pi(a) S_x^* \) for all \( a \in A \) and \( x \in \Gamma^+ \). See in the Remark 2.3 that this equation implies \( S_x \pi(a) S_x^* = \pi(a) S_x^* S_x \) for \( a \in A \) and \( x \in \Gamma^+ \). Moreover, [4 Lemma 4.2] shows that every \((\pi, S)\) extends to a partial-isometric covariant representation \((\pi, S)\) of \((M(A), \Gamma^+, \overline{\alpha})\), and the partial-isometric covariance is equivalent to \( \pi(\alpha_x(a)) S_x = S_x \pi(a) \) and \( S_x^* S_x = \pi(\overline{\alpha}_x(1)) \) for \( a \in A \) and \( x \in \Gamma^+ \).
A partial-isometric crossed product of \((A, \Gamma^+, \alpha)\) is a triple \((B, j_A, j_{\Gamma^+})\) consisting of a \(C^*\)-algebra \(B\), a canonical covariant partial-isometric representation \((j_A, j_{\Gamma^+})\) in \(M(B)\) which satisfies the following

(i) for every covariant partial-isometric representation \((\pi, S)\) of \((A, \Gamma^+, \alpha)\) on a Hilbert space \(H\), there exists a nondegenerate representation \(\pi \times S : B \rightarrow B(H)\) such that \((\pi \times S) \circ j_A = \pi\) and \((\pi \times S) \circ j_{\Gamma^+} = S\); and

(ii) \(B\) is generated by \(j_A(A) \cup j_{\Gamma^+}(\Gamma^+)\), we actually have

\[
B = \mathbb{M}_{\mathbb{C}} \{ j_{\Gamma^+}(x)^* j_A(a) j_{\Gamma^+}(y) : x, y \in \Gamma^+, a \in A \}.
\]

Unlike the theory of isometric crossed product: every system \((A, \Gamma^+, \alpha)\) admits a non trivial covariant partial-isometric representation \((\pi, S)\) with \(\pi\) faithful \([6, \text{Example 4.6}]\). In fact \([6, \text{Proposition 4.7}]\) shows that a canonical covariant partial-isometric representation \((j_A, j_{\Gamma^+})\) of \((A, \Gamma^+, \alpha)\) exists in the Toeplitz algebra \(T_X\) associated to a discrete product system \(X\) of Hilbert bimodules over \(\Gamma^+\), which (i) and (ii) are fulfilled, and it is universal: if there is such a covariant partial-isometric representation \((t_A, t_{\Gamma^+})\) of \((A, \Gamma^+, \alpha)\) in a \(C^*\)-algebra \(C\) that satisfies (i) and (ii), then there is an isomorphism of \(C\) onto \(B\) which takes \((t_A, t_{\Gamma^+})\) into \((j_A, j_{\Gamma^+})\). Thus we write the partial-isometric crossed product \(B\) as \(A \times_{\alpha}^{\text{piso}} \Gamma^+\).

**Remark 2.3.** Our special thanks go to B. Kwasniewski for showing us the proof arguments in this remark. Assuming \((\pi, S)\) is covariant, then by \(C^*\)-norm equation we have \(\|\pi(a) S_x^* - S_x^* \pi(\alpha_x(a))\| = 0\), therefore \(\pi(a) S_x^* = S_x^* \pi(\alpha_x(a))\) for all \(a \in A\) and \(x \in \Gamma^+\), which means that \(S_x \pi(a) = \pi(\alpha_x(a)) S_x\) for all \(a \in A\) and \(x \in \Gamma^+\). So \(S_x^* S_x \pi(a) = S_x^* \pi(\alpha_x(a)) S_x = (\pi(\alpha_x(a^*)) S_x)^* S_x = (\pi(\alpha_x^* a^*))^* S_x = \pi(a) S_x^* S_x\).

### 3. The short exact sequence of partial-isometric crossed products

**Theorem 3.1.** Suppose that \((A \times_{\alpha}^{\text{piso}} \Gamma^+, i_A, V)\) is the partial-isometric crossed product of a dynamical system \((A, \Gamma^+, \alpha)\), and \(I\) is an extendible \(\alpha\)-invariant ideal of \(A\). Then there is a short exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & I \times_{\alpha}^{\text{piso}} \Gamma^+ & \overset{\mu}{\longrightarrow} & A \times_{\alpha}^{\text{piso}} \Gamma^+ & \overset{\gamma}{\longrightarrow} & A/I \times_{\overline{\alpha}}^{\text{piso}} \Gamma^+ & \longrightarrow & 0,
\end{array}
\]

where \(\mu\) is an isomorphism of \(I \times_{\alpha}^{\text{piso}} \Gamma^+\) onto the ideal

\[
\mathcal{D} := \text{span}\{V_x^* i_A(i)V_y : i \in I, x, y \in \Gamma^+\}\text{ of } A \times_{\alpha}^{\text{piso}} \Gamma^+.
\]

If \(q : A \rightarrow A/I\) is the quotient map, \(i_I, W\) denote the maps \(I \rightarrow I \times_{\alpha}^{\text{piso}} \Gamma^+\), \(W : \Gamma^+ \rightarrow M(I \times_{\alpha}^{\text{piso}} \Gamma^+)\), and similarly for \(i_{A/I}\), \(U\) the maps \(A/I \rightarrow A/I \times_{\overline{\alpha}}^{\text{piso}} \Gamma^+\), then

\[
\mu \circ i_I = i_A|_I, \quad \overline{\pi} \circ W = V \quad \text{and} \quad \gamma \circ i_{A/I} = i_A|_I \circ q, \quad \overline{\gamma} \circ V = U.
\]

**Proof.** We make some minor adjustment to the proof of \([11, \text{Theorem 3.1}]\) for partial isometries. First, to check that \(\mathcal{D}\) is indeed an ideal of \(A \times_{\alpha}^{\text{piso}} \Gamma^+\). Let \(\xi = V_x^* i_A(i)V_y \in \mathcal{D}\). Then \(V_x^* \xi\) is trivially contained in \(\mathcal{D}\), and computations below show that \(i_A(a)\xi\) and \(V_s\xi\) are all in \(\mathcal{D}\) for \(a \in A\) and \(s \in \Gamma^+\):

\[
i_A(a)\xi = i_A(a)V_x^* i_A(i)V_y = (V_x^* i_A(a))V_y = (i_A(\alpha_x(a^*))V_x)^* i_A(i)V_y = V_x^* i_A(\alpha_x(a) i)V_y.
\]
Then $\tilde{\Phi}$ satisfies $\psi$ so $\tilde{\Phi}$ is a covariant representation of $(I,\Gamma,\alpha)$. Let $\gamma : A \to M(I)$ be the homomorphism satisfying $\gamma(a)i = ai$ for $a \in A$ and $i \in I$. Then $i_A^\alpha(\gamma(a)i)$ converges in norm to $i_A(\gamma(a)i)$ in $V \times V$. However

$$i_A(V_s' \gamma(a)i)V_t = V_s' i_A^\alpha(\gamma(a)i)V_t$$

by covariance, it follows that $j_I(e_A)^* V_s' i_A^\alpha(\gamma(a)i)V_t$ converges in norm to $V_s' i_A^\alpha(\gamma(a)i)V_t$. We can similarly show that $V_s' i_A^\alpha(\gamma(a)i)V_t$ converges in norm to $V_s' i_A^\alpha(\gamma(a)i)V_t$. Thus $j_I(e_A) \to 1_{M(D)}$ strictly, and hence $j_I$ is nondegenerate.

We claim that the triple $(D, j_I, S)$ is a partial-isometric crossed product of $(I,\Gamma,\alpha)$. A routine computations show the covariance of $(j_I, S)$ for $(I,\Gamma,\alpha)$. Suppose now $(\pi, T)$ is a covariant representation of $(I,\Gamma,\alpha)$ on a Hilbert space $H$. Let $\rho : A \to M(I) \to B(H)$. Then by extendibility of ideal $I$, that is $\alpha|_I \circ \varphi = \varphi \circ \alpha$, the pair $(\rho, T)$ is a covariant representation of $(A,\Gamma,\alpha)$. The restriction $(\rho \times T)|_D$ to $D$ of $\rho \times T$ is a nondegenerate representation of $D$ which satisfies the requirement $(\rho \times T)|_D \circ j_I = \pi$ and $(\rho \times T)|_D \circ S = T$. Thus the triple $(D, j_I, S)$ is a partial-isometric crossed product for $(I,\Gamma,\alpha)$, and we have the homomorphism $\mu = i_A|_{I \times V}$.

Next we show the exactness. Let $\Phi$ be a nondegenerate representation of $A \times^{piso}_\alpha \Gamma$ with kernel $D$. Since $I \subset \ker \Phi \circ i_A$, we can have a representation $\overline{\Phi}$ of $A/I$, which together with $\overline{\Phi} \circ V$ is a covariant partial-isometric representation of $(A/I,\Gamma^+,\overline{\alpha})$. Then $\Phi \times (\overline{\Phi} \circ V)$ lifts to $\Phi$, and therefore $\ker \gamma \subset \ker \Phi = D$. \hfill \Box

Corollary 3.2. Let $(A,\Gamma^+,\alpha)$ be a dynamical system, and $I$ an extendible $\alpha$-invariant ideal of $A$. Then there is a commutative diagram

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0 \to \ker \phi_I \to I \times^{piso}_\alpha \Gamma^+ \to 0
0 \to \ker \phi_A \to A \times^{piso}_\alpha \Gamma^+ \to 0
0 \to \ker \phi_{A/I} \to A/I \times^{piso}_\alpha \Gamma^+ \to 0
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Diagram for Corollary 3.2
Proof. The three row exact sequences follow from [1], the middle column from Theorem 3.1 and the right column exact sequence from [1]. By inspection on the spanning elements, one can see that \( \mu(\ker \phi_I) \) is an ideal of \( \ker \phi_A \) and \( \mu^{iso} \circ \phi_I = \phi_A \circ \mu \), thus first and second rows commute. Then Snake Lemma gives the commutativity of all rows and columns. \( \square \)

4. THE EXAMPLE

We consider a dynamical system \((B_{\Gamma^+}, \Gamma^+, \tau)\) consisting of a unital \(C^*\)-subalgebra \(B_{\Gamma^+}\) of \(\mathcal{L}(\Gamma^+)\) spanned by the set \(\{1_s : s \in \Gamma^+\}\) of characteristic functions \(1_s\) of \(\{x \in \Gamma^+ : x \geq s\}\), the action \(\tau\) of \(\Gamma^+\) on \(B_{\Gamma^+}\) is given by \(\tau_x(1_s) = 1_{s+x}\). The ideal \(B_{\Gamma^+, \infty} = \text{span}\{1_i - 1_j : i < j \in \Gamma^+\}\) is an extendible \(\tau\)-invariant ideal of \(B_{\Gamma^+}\). Then we want to show in Proposition 4.1 that an application of Corollary 3.2 to the system \((B_{\Gamma^+}, \Gamma^+, \tau)\) gives \([6\), Theorem 5.6].

The crossed product \(B_{\Gamma^+} \times^{iso}_{\tau} \Gamma^+\) is a universal \(C^*\)-algebra generated by the canonical isometric representation \(t\) of \(\Gamma^+\); every isometric representation \(w\) of \(\Gamma^+\) gives a covariant isometric representation \((\pi_w, w)\) of \((B_{\Gamma^+}, \Gamma^+, \tau)\). Suppose \(\{\varepsilon_x : x \in \Gamma^+\}\) is the usual orthonormal basis in \(\mathcal{L}^2(\Gamma^+)\), and let \(T_s(\varepsilon_x) = \varepsilon_{x+s}\) for every \(s \in \Gamma^+\). Then \(s \mapsto T_s\) is an isometric representation of \(\Gamma^+\), and the Toeplitz algebra \(T(\Gamma)\) is the \(C^*\)-subalgebra of \(B(\mathcal{L}^2(\Gamma^+))\) generated by \(\{T_s : s \in \Gamma^+\}\). So there exists a representation \(\mathcal{I} := \pi_T \times \mathcal{T}\) of \((B_{\Gamma^+} \times^{iso}_{\tau} \Gamma^+)\) such that \(\mathcal{I}(T_x) = T_x\) and \(\mathcal{I}(1_s) = T_xT_s^*\) for all \(x \in \Gamma^+\). This representation is faithful by [3, Theorem 2.4]. Thus \(B_{\Gamma^+} \times^{iso}_{\tau} \Gamma^+\) and the Toeplitz algebra \(T(\Gamma) = \pi_T \times \mathcal{T}(B_{\Gamma^+} \times^{iso}_{\tau} \Gamma^+)\) are isomorphic, and the isomorphism takes the ideal \(B_{\Gamma^+, \infty} \times^{iso}_{\tau} \Gamma^+\) of \(B_{\Gamma^+} \times^{iso}_{\tau} \Gamma^+\) onto the commutator ideal \(\mathcal{C}_\Gamma = \text{span}\{T_x(1 - TT^*)T_y^* : x, y \in \Gamma^+\}\) of \(T(\Gamma)\).

Similarly, the crossed product \(B_{\Gamma^+} \times^{piso}_{\tau} \Gamma^+\) has a partial-isometric version of universal property by [6, Proposition 5.1]: every partial-isometric representation \(v\) of \(\Gamma^+\) gives a covariant partial-isometric representation \((\pi_v, v)\) of \((B_{\Gamma^+} \times^{iso}_{\tau} \Gamma^+)\) with \(\pi_v(1_s) = v_xv_x^*\), and then \(B_{\Gamma^+} \times^{piso}_{\tau} \Gamma^+\) is the universal \(C^*\)-algebra generated by the canonical partial-isometric representation \(v\) of \(\Gamma^+\). Now since \(x \mapsto T_x\) and \(x \mapsto T_x^*\) are partial-isometric representations of \(\Gamma^+\) in the Toeplitz algebra \(T(\Gamma)\), there exist (by the universality) a homomorphism \(\varphi_T\) and \(\varphi_{T^*}\) of \(B_{\Gamma^+} \times^{piso}_{\tau} \Gamma^+\) onto \(T(\Gamma)\).

Next consider the algebra \(C(\hat{\Gamma})\) generated by \(\{\lambda_x : x \in \Gamma\}\) of the evaluation maps \(\lambda_x(\xi) = \xi(x)\) on \(\hat{\Gamma}\). Let \(\psi_T\) and \(\psi_{T^*}\) be the homomorphisms of \(T(\Gamma)\) onto \(C(\hat{\Gamma})\) defined by \(\psi_T(T_x) = \lambda_x\) and \(\psi_{T^*}(T_x) = \lambda_{-x}\).

**Proposition 4.1.** [6, Theorem 5.6] Let \(\Gamma^+\) be the positive cone in a totally ordered abelian group \(\Gamma\). Then the following commutative diagram exists:
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(4.1)

\[ 0 \to \ker \varphi_T \cap \ker \varphi_T^* \to \ker \varphi_T^* \xrightarrow{\varphi_T} C_T \to 0 \]

\[ 0 \to \ker \varphi_T \to B_{\Gamma^+} \xrightarrow{\varphi_T} T(\Gamma) \to 0 \]

\[ 0 \to \psi_T \to C(\hat{\Gamma}) \to 0 \]

(4.2)

\[ 0 \to \ker \varphi_{B_{\Gamma^+},\infty} \to B_{\Gamma^+,\infty} \xrightarrow{\varphi_{B_{\Gamma^+},\infty}} B_{\Gamma^+,\infty} \xrightarrow{\gamma_{B_{\Gamma^+,\infty}}} 0 \]

\[ 0 \to \ker \varphi_Q \to Q^{\text{piso}} \xrightarrow{\phi_Q} Q^{\text{iso}} \to 0 \]

where \( \Psi \) maps each generator \( v_x \in B_{\Gamma^+} \times^{\text{piso}} \Gamma^+ \) to \( \delta^*_x \in C^*(\Gamma) \simeq C(\hat{\Gamma}) \).

**Proof.** Apply Corollary 3.2 to the system \((B_{\Gamma^+}, \Gamma^+, \tau)\) and the extendible ideal \(B_{\Gamma^+,\infty}\). Let \(Q^{\text{piso}} := B_{\Gamma^+}/B_{\Gamma^+,\infty} \times^{\text{piso}} \Gamma^+\) and \(Q^{\text{iso}} := B_{\Gamma^+}/B_{\Gamma^+,\infty} \times^{\text{iso}} \Gamma^+\). Then we have:

We claim that exact sequences in this diagram and (4.1) are equivalent. The middle exact sequences of (4.1) and (4.2) are trivially equivalent via the isomorphism \( T : B_{\Gamma^+} \times^{\text{piso}} \Gamma^+ \to T(\Gamma) \). By viewing \( B_{\Gamma^+} \) as the algebra of functions that have limit, the map \( f \in B_{\Gamma^+} \mapsto \lim_{x \in \Gamma^+} f(x) \) induces an isomorphism \( B_{\Gamma^+}/B_{\Gamma^+,\infty} \to \mathbb{C} \), which intertwines the action \( \tau \) and the trivial action \( \text{id} \) on \( \mathbb{C} \). So \((B_{\Gamma^+}/B_{\Gamma^+,\infty}, \Gamma^+, \tilde{\tau}) \simeq (\mathbb{C}, \Gamma^+, \text{id}) \). Moreover, \( \Sigma \) combines with the isomorphism \( h : B_{\Gamma^+}/B_{\Gamma^+,\infty} \times^{\text{iso}} \Gamma^+ \to \mathbb{C} \times^{\text{id}} \Gamma^+ \to C^*(\Gamma) \simeq C(\hat{\Gamma}) \) to identify the right-hand exact sequence equivalently to

\[ 0 \to C_T \to T(\Gamma) \xrightarrow{\psi_T^*} C(\hat{\Gamma}) \to 0. \]

For the bottom sequence, we consider the pair of

\[ \iota_C : z \in \mathbb{C} \mapsto z1_{T(\Gamma)} \] and \[ \iota_{\Gamma^+} : x \in \Gamma^+ \mapsto T^*_x \in T(\Gamma). \]
It is a partial-isometric covariant representation, such that \((\mathcal{T}(\Gamma), i_\mathcal{C}, i_{\Gamma^+})\) is a partial-isometric crossed product of \((\mathbb{C}, \Gamma^+, \text{id})\). So we have an isomorphism

\[
\Upsilon : Q^{\text{piso}} \rightarrow \mathbb{C} \times_{\text{id}} \mathbb{C}^{\text{piso}} \xrightarrow{\rho} \mathcal{T}(\Gamma) \text{ in which } \Upsilon(i_{\Gamma^+}(x)) = T_x^* \text{ for all } x,
\]

and moreover if \((j_Q, u)\) denotes the canonical covariant partial-isometric representation of the system \((Q := B_{\Gamma^+}/B_{\Gamma^+}, \Gamma^+, \tilde{\tau})\) in \(Q^{\text{piso}}\), then \(\Upsilon\) satisfies the equations \(\Upsilon(u_x) = T_x^*\) and \(\Upsilon(j_Q(1_x + B_{\Gamma^+})) = i_\mathcal{C}((\lim_y 1_x(y)) = 1 \text{ for all } x \in \Gamma^+\). To see \(\Upsilon(\ker \phi_Q) = C_\Gamma\), recall from [4, Proposition 2.3] that

\[
\ker \phi_Q := \overline{\text{span}}\{u_x j_Q(a)(1 - u_x^* u_x)u_y : a \in Q, x, y, z \in \Gamma^+\}.
\]

Since \(\Upsilon(u_x^* j_Q(a)(1 - u_x^* u_x)u_y)\) is a scalar multiplication of \(T_x((1 - T_x T_y^* T_x^*)\), therefore \(\Upsilon(\ker \phi_Q) = C_\Gamma\). Consequently the two exact sequences are equivalent:

\[
\begin{array}{ccc}
0 & \rightarrow & \ker \phi_Q \\
\downarrow & & \downarrow \\
\Upsilon & \rightarrow & \mathcal{T}(\Gamma) \\
\downarrow & \rightarrow & \downarrow \\
C_\Gamma & \rightarrow & C(\hat{\Gamma})
\end{array}
\]

For the second column exact sequence, we note that the isomorphism \(j : Q^{\text{piso}} \simeq \mathbb{C} \times_{\text{id}} \mathbb{C}^{\text{piso}} \xrightarrow{\phi} \mathcal{T}(\Gamma)\) satisfies \(j \circ \gamma = \varphi_{T^*}\). This implies

\[
B_{\Gamma^+, \infty} \times_{\tilde{\tau}}^{\text{piso}} \Gamma^+ \simeq \ker(j \circ \gamma) = \ker \varphi_{T^*},
\]

and therefore the second column sequence of diagram [4.1] is equivalent to \(0 \rightarrow \ker \varphi_{T^*} \rightarrow B_{\Gamma^+, \infty} \times_{\tilde{\tau}}^{\text{piso}} \Gamma^+ \rightarrow \mathcal{T}(\Gamma) \rightarrow 0\).

Next we are working for the first row. The homomorphism \(\phi_B_{\Gamma^+}\) in the following diagram

\[
B_{\Gamma^+, \infty} \times_{\tilde{\tau}}^{\text{piso}} \Gamma^+ \xrightarrow{\phi_B_{\Gamma^+}} B_{\Gamma^+} \times_{\tilde{\tau}} \Gamma^+
\]

restricts to the homomorphism \(\phi_{B_{\Gamma^+}, \infty}\) of the ideal \(B_{\Gamma^+, \infty} \times_{\tilde{\tau}}^{\text{piso}} \Gamma^+ \simeq \ker \varphi_{T^*}\) onto \(B_{\Gamma^+, \infty} \times_{\tilde{\tau}}^{\text{iso}} \Gamma^+ \simeq C_\Gamma\). So the homomorphism \(\varphi_{T^*} : \ker \varphi_{T^*} \rightarrow C_\Gamma\) has kernel \(I := \ker \varphi_{T^*} \cap \ker \varphi_{T}^*\), and therefore first row exact sequence of the two diagrams are indeed equivalent.

Finally we show that such \(\Psi\) exists. Consider \(C(\hat{\Gamma}) \simeq C^*(\Gamma) \simeq \mathbb{C} \times_{\text{id}} \Gamma\) is the \(C^*\)-algebra generated by the unitary representation \(x \in \Gamma \mapsto \delta_x \in \mathbb{C} \times_{\text{id}} \Gamma\). Then we have a homomorphism \(\pi_{\delta_x} \times \delta^* : B_{\Gamma^+} \times_{\tilde{\tau}}^{\text{piso}} \Gamma^+ \rightarrow \mathbb{C} \times_{\text{id}} \Gamma\) which satisfies \(\pi_{\delta_x} \times \delta^*(v_x) = \delta_x^*\) for all \(x \in \Gamma^+\), and hence it is surjective. By looking at the spanning elements of \(\ker \varphi_{T}\) and \(\ker \varphi_{T^*}\) we can see that these two ideals are contained in \(\ker(\pi_{\delta_x} \times \delta^*)\), therefore \(\mathcal{J} := \ker \varphi_{T} + \ker \varphi_{T^*}\) must be also in \(\ker(\pi_{\delta_x} \times \delta^*)\). For the other inclusion, let \(\rho\) be a unital representation of \(B_{\Gamma^+} \times_{\tilde{\tau}}^{\text{piso}} \Gamma^+\) on a Hilbert space \(H_\rho\) with ker \(\rho = \mathcal{J}\). Then for \(s \in \Gamma^+\) we have \(\rho((1 - v_s v_s^*) - (1 - u_s^* u_s)) = 0\) because \(1 - v_s v_s^* \in \ker \varphi_{T^*}\), and \(1 - u_s^* u_s \in \ker \varphi_{T}\) belong to \(\mathcal{J}\). So \(0 = \rho(v_s^* v_s - u_s^* u_s)\), which implies that \(\rho(v_s^* v_s) = \rho(v_s v_s^*)\). On the other hand the equation \(\rho((1 - v_s v_s^*) + (1 - v_s^* v_s)) = 0\) gives \(\rho(v_s v_s^*) = I\). Therefore \(\rho(v_s v_s^*) = \rho(v_s v_s^*) = I\), and this means \(\rho(v_s)\) is unitary for every \(s \in \Gamma^+\). Consequently a representation \(\tilde{\rho} : \mathbb{C} \times_{\text{id}} \Gamma \rightarrow B(H_\rho)\) exists, and it
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does not hallucinate.

5. The Primitive Ideals of $c \times_{piso}^{\mathbb{N}}$

Suppose $\Gamma^+$ is now the additive semigroup $\mathbb{N}$. The algebra $B_0$ is conveniently viewed as the $C^*$-algebra $c$ of convergent sequences, the ideal $B_{[\mathbb{N},\infty}$ with $c_0$, and the action $\tau$ of $\mathbb{N}$ on $c$ is generated by the unilateral shift: $\tau_n(x_0,x_1,x_2,\cdots) = (0,x_0,x_1,x_2,\cdots)$. The universal $C^*$-algebra $c \times_{piso}^{\mathbb{N}}$ is generated by a power partial isometry $v := i_{\mathbb{N}}(1)$. The Toeplitz algebra $T(\mathbb{Z})$ is the $C^*$-subalgebra of $B(\ell^2(\mathbb{N}))$ generated by isometries $\{T_n : n \in \mathbb{N}\}$, where $T_n(e_i) = e_{n+i}$ on the set of usual orthonormal basis $\{e_i : i \in \mathbb{N} \cup \{0\}\}$ of $\ell^2(\mathbb{N})$, and the commutator ideal of $T(\mathbb{Z})$ is $K(\ell^2(\mathbb{N}))$. Kernels of $\varphi_T$ and $\varphi_T^*$ are identified in [6, Lemma 6.2] by

$$\ker \varphi_T = \text{span}\{g_{i,j}^m : i,j,m \in \mathbb{N}\}; \quad \ker \varphi_T^* = \text{span}\{f_{i,j}^m : i,j,m \in \mathbb{N}\}$$

where

$$g_{i,j}^m = v_i^*v_mv_m^*(1-v^*v)v_j \quad \text{and} \quad f_{i,j}^m = v_i^*v_mv_m^*(1-vv^*)v_j^*.$$  

Moreover $I := \ker \varphi_T \cap \ker \varphi_T^*$ is an essential ideal in $c \times_{piso}^{\mathbb{N}}$ [6, Lemma 6.8], given by

$$\text{span}\{f_{i,j}^m - f_{i,j}^{m+1} = g_{m-i,m-j}^{m+1} - g_{m-i,m-j}^m : m \in \mathbb{N}, 0 \leq i,j \leq m\}.$$  

The main point of [6, §6] is to show that there exist isomorphisms of $\varphi_T$ and $\ker \varphi_T^*$ onto the algebra

$$A := \{f : \mathbb{N} \rightarrow K(\ell^2(\mathbb{N})) : f(n) \in P_nK(\ell^2(\mathbb{N}))P_n \text{ and } \varepsilon(f) = \lim_{n} f(n) \text{ exists}\},$$

where $P_n := 1 - T_{n+1}T_{n+1}$ is the projection of $\ell^2(\mathbb{N})$ onto the subspace spanned by $\{e_i : i = 0,1,2,\cdots,n\}$, and such that they restrict to isomorphisms of $I$ onto the ideal

$$A_0 := \{f \in A : \lim_{n} f(n) = 0\} \text{ of } A.$$  

We shall show in Proposition [5.1] that $A$ and $A_0$ are related to the algebras of compact operators on the Hilbert $c$-module $\ell^2(\mathbb{N},c)$ and on the closed sub-$c$-module $\ell^2(\mathbb{N},c_0)$. We supply our readers with some basic theory of the $C^*$-algebra of operators on this Hilbert module, and let us begin with recalling the module structure of $\ell^2(\mathbb{N},c)$ and its closed sub-module. The vector space $\ell^2(\mathbb{N},c)$, containing all $c$-valued functions $a : \mathbb{N} \rightarrow c$ such that the series $\sum_{n \in \mathbb{N}} a(n)^*a(n)$ converges in the norm of $c$, forms a Hilbert $c$-module with the module structure defined by $(a \cdot x)(n) = a(n)x$ for $x \in c$, and its $c$-valued inner product given by $\langle a,b \rangle = \sum_{n \in \mathbb{N}} a(n)^*b(n)$. In fact the module $\ell^2(\mathbb{N},c)$ is naturally isomorphic to the Hilbert module $\ell^2(\mathbb{N}) \otimes c$ that arises from the completion of algebraic (vector space) tensor product $\ell^2(\mathbb{N}) \otimes c$ associated to the $c$-valued inner product defined on simple tensor product by $\langle \xi \otimes x, \eta \otimes y \rangle = \langle \xi,\eta \rangle x^*y$ for $\xi,\eta \in \ell^2(\mathbb{N})$ and $x,y \in c$. The isomorphism is implemented by the map $\phi$ that takes $(e_i \otimes x) \in \ell^2(\mathbb{N}) \otimes c$ to the element $\phi(e_i \otimes x) \in \ell^2(\mathbb{N},c)$ which is the function

$$\phi(e_i \otimes x)(n) = \begin{cases} x & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}.$$  

By exactly the same arguments, we see that the two Hilbert $c_0$-modules $\ell^2(\mathbb{N},c_0)$ and $\ell^2(\mathbb{N}) \otimes c_0$ are isomorphic. However since $c_0$ is an ideal of $c$, it follows that the $c_0$-module $\ell^2(\mathbb{N},c_0)$ is a closed sub-$c$-module of $\ell^2(\mathbb{N},c)$,
and respectively $\mathcal{L}(\ell^2(N), c)$ is a closed sub-$c$-module of $\ell^2(N) \otimes c$. Moreover the $c$-module isomorphism $\phi$ restricts to $c_0$-module isomorphism $\mathcal{L}(\ell^2(N), c_0) \cong \ell^2(N) \otimes c_0$.

Next, we consider the $C^*$-algebra $\mathcal{L}(\ell^2(N), c)$ of adjointable operators on $\ell^2(N, c)$, and the ideal $\mathcal{K}(\ell^2(N, c))$ of $\mathcal{L}(\ell^2(N, c))$ spanned by the set $\{\theta_{ab} : a, b \in \ell^2(N, c)\}$ of compact operators on the module $\ell^2(N, c)$. The algebra $\mathcal{K}(\ell^2(N, c))$ is defined by the same arguments, and note that $\mathcal{K}(\ell^2(N, c_0))$ is an ideal of $\mathcal{K}(\ell^2(N, c))$. The isomorphism of two modules $\ell^2(N, c)$ and $\ell^2(N) \otimes c$, implies that $\mathcal{K}(\ell^2(N, c)) \cong \mathcal{K}(\ell^2(N) \otimes c)$, which by the Hilbert module theorem, this is the $C^*$-algebraic tensor product $\mathcal{K}(\ell^2(N)) \otimes c$ of $\mathcal{K}(\ell^2(N))$ and $c$. We shall often use the characteristics functions $\{1_n : n \in N\}$ as generator elements of $c$ and the spanning set $\{\theta_{ij1_n, ej1_n} : i, j, n \in N\}$ of $\mathcal{K}(\ell^2(N, c))$ in our computations.

There is another ingredient that we need to consider to state the Proposition. Suppose $S \in \mathcal{L}(\ell^2(N, c))$ is an operator defined by $S(a)(i) = a(i - 1)$ for $i \geq 1$ and zero otherwise. One can see that $S^2 = 1$, i.e. $S$ is an isometry. Let $p \in \mathcal{L}(\ell^2(N, c))$ be the projection $(p(a))(n) = 1_n a(n)$ for $a \in \ell^2(N, c)$, and similarly $q \in \mathcal{L}(\ell^2(N, c_0))$ be the projection $(q(a))(n) = 1_n a(n)$ for $a \in \ell^2(N, c_0)$. Then the following two partial isometric representations of $N$ in $p \mathcal{L}(\ell^2(N, c))p$ defined by

$$w : n \in N \mapsto pS_n p \quad \text{and} \quad t : n \in N \mapsto pS_n p,$$

induce the representations $\pi_w \times w$ and $\pi_t \times t$ of $c \times^\text{piso}N$ in $p \mathcal{L}(\ell^2(N, c))p$ which satisfy $\pi_w \times w(v_i) = pS_n^* p$ and $\pi_t \times t(v_i) = pS_n p$ for all $i \in N$. These $\pi_w \times w$ and $\pi_t \times t$ are faithful representations [4, Example 4.3].

**Proposition 5.1.** The representations $\pi_w \times w$ and $\pi_t \times t$ map $\ker \phi_T$ and $\ker \phi_{T^*}$-isomorphically onto the full corner $p \mathcal{K}(\ell^2(N, c))p$. Moreover, they restrict to isomorphisms of the ideal $\ker \phi_T \cap \ker \phi_{T^*}$ onto the full corner $q \mathcal{K}(\ell^2(N, c_0))q$.

**Remark 5.2.** It follows from this Proposition that Prim ker $\phi_T$ and Prim ker $\phi_{T^*}$ are both homeomorphic to Prim $c$. In fact, since ker $\phi_{T^*} \cong c_0 \times^\text{piso}N$ by [2 Corollary 3.1], we can therefore deduce that $c_0 \times^\text{piso}N$ is Morita equivalent to $\mathcal{K}(\ell^2(N, c))$. This is a useful fact for our subsequent work on the partial-isometric crossed product of lattice semigroup $N \times N$.

**Proof of Proposition 5.1.** We only have to show that $\pi_t \times t(\ker \phi_{T^*}) = q \mathcal{K}(\ell^2(N, c_0))q$ and $\pi_t \times t(\ker \phi_T \cap \ker \phi_{T^*}) = q \mathcal{K}(\ell^2(N, c_0))q$. The rest of arguments is done in [4, Example 4.3].

Note that the algebra $p \mathcal{K}(\ell^2(N, c))p$ is spanned by $\{p\theta_{ij1_n, ej1_n} : i, j, n \in N\}$. Since $\pi_t \times t(f_{ij}^n) = p\theta_{ij1_n, ej1_n}p$ for every $i, j, n \in N$, therefore $\pi_t \times t(\ker \phi_{T^*}) = p \mathcal{K}(\ell^2(N, c))p$.

Similarly we consider that $\{q\theta_{ij1_n, ej1_n} : i, j \leq n \in N\}$ spans $q \mathcal{K}(\ell^2(N, c_0))q$. We use the equation $\theta_{ij1_n, ej1_n} = \theta_{ij1_n, ej1_n}$ for every $n \in N$, in the computations below, to see that

$$\pi_t \times t(f_{ij}^n - f_{ij}^{n+1}) = p(\theta_{ij1_n, ej1_n} - \theta_{ij1_n+1, ej1_n})p = p(\theta_{ij1_n} - (\theta_{ij1_n+1}, ej1_n))p = p(\theta_{ij1_n, ej1_n})p.$$
To convince that every \( p(\ell^pC_{(n)})p \) belongs to \( qK(\ell^p(N,c_0))q \), we need the embedding \( \iota^K \) of \( qK(\ell^p(N,c_0))q \) in \( pK(\ell^p(N,c))p \) stated in Lemma 5.3. In fact, every element \( p(\ell^pC_{(n)})p \) spans \( \iota^K(qK(\ell^p(N,c_0))q) \), therefore \( \pi_t \times t(\ker \varphi_T \cap \ker \varphi_T) = \iota^K(qK(\ell^p(N,c_0))q) \).

Lemma 5.3. Let \( p \in L(\ell^2(N,c)) \) and \( q \in L(\ell^2(N,c_0)) \) are the projections defined by \( (p(a))(n) = 1, a(n) \) for \( a \in \ell^2(N,c) \), and \( (q(a))(n) = 1, a(n) \) for \( a \in \ell^2(N,c_0) \). Then the full corner \( qK(\ell^2(N,c_0))q \) embeds naturally via \( \iota^K(q\theta^{\epsilon_0}) = p\theta^{\epsilon_0}p \) as an ideal in \( pK(\ell^2(N,c))p \), and there exists a short exact sequence

\[
0 \longrightarrow qK(\ell^2(N,c_0))q \xrightarrow{\iota^K} pK(\ell^2(N,c))p \xrightarrow{q^K} K(\ell^2(N)) \longrightarrow 0,
\]

where \( q^K(p\theta^{\epsilon_0}p) = \theta_{x,y} \) with \( x, y \in \ell^2(N) \) are given by \( x_i = \lim_{n \to \infty} (1, a(i))(n) \) and \( y_i = \lim_{n \to \infty} (1, b(i))(n) \). In particular we have

\[
q^K(p\theta^{\epsilon_0}p) = q^K(\theta^{\epsilon_0}_{p(\ell^2(N,c))p}) = q^K(\theta^{\epsilon_0}_{p(\ell^2(N,c))p}) = q^K(\theta^{\epsilon_0}_{p(\ell^2(N,c))p}) = T_i(1 - TT^*)T_j^* \in K(\ell^2(N)).
\]

Proof. Apply \[2\] Lemma 2.6] for the module \( X := \ell^2(N,c) \) and \( I = c_0 \). In this case we have the submodule \( XI = \ell^2(N,c_0) \). Note that if \( a \in \ell^2(N,c) \), then every sequence \( a(i) \in C \) is convergent in \( C \), and the map \( q : a \mapsto (q(a))(i) = \lim_{n \to \infty} (a(i))(n) \) gives \( 0 \to \ell^2(N,c_0) \to \ell^2(N,c) \to \ell^2(N) \to 0. \) Moreover \[2\] Lemma 2.6] proves that \( \iota^K(\theta_{a,b}^{XI}) = \theta_{a,b}^X \) and \( q^K(\theta_{a,b}^{XI}) = q^K(\theta_{a,b}^{XI}) \) give the exactness of the sequence

\[
0 \longrightarrow K(\ell^2(N,c_0)) \xrightarrow{\iota^K} K(\ell^2(N,c)) \xrightarrow{q^K} K(\ell^2(N)) \longrightarrow 0.
\]

Since \( \iota^K(q\theta^{XI}_{a,b}) = \theta_{a,b}^{XI} \) for every \( a,b \in \ell^2(N,c_0) \), the corner \( qK(\ell^2(N,c_0))q \) is embedded into \( pK(\ell^2(N,c))p \) such that \( q^K \) is defined by \( q^K(p\theta^{XI}_{a,b}p) = q^K(\theta_{p(a),p(b)}) = \theta_{x,y} \) where \( x_i = \lim_{n \to \infty} (1, a(i))(n) \) and \( y_i = \lim_{n \to \infty} (1, b(i))(n) \). Thus we obtain the required exact sequence.

Proposition 5.4. There are isomorphisms \( \Theta : pK(\ell^2(N,c))p \to \ker \varphi_T \) and \( \Theta : pK(\ell^2(N,c))p \to \ker \varphi_T \) defined by \( \Theta(p\theta_{a,b}^{\epsilon_0}) = g_{i,j}^n \) and \( \Theta(\theta_{a,b}^{\epsilon_0}) = f_{i,j}^n \) for all \( i,j,n \in \mathbb{N} \) such that the following commutative diagram has all rows and columns exact:
Proof. We apply Proposition 4.1 to the system \((c, N, \tau)\). Let \(\{v_i : i \in \mathbb{N}\}\) denote the generators of \(c \times \tau^\text{piso} N\), and \(\{\delta_i : i \in \mathbb{Z}\}\) the generator of \(C^*(\mathbb{T})\). Then the homomorphism \(\Psi : c \times \tau^\text{piso} N \to C(\mathbb{T})\) given by Proposition 4.1 satisfies \(\Psi(v_i) = \delta_i^* = (z \mapsto \overline{z}^i) \in C(\mathbb{T})\) for every \(i \in \mathbb{N}\). Moreover \(\Theta = (\pi_w \times w)^{-1}\) and \(\Theta_\ast = (\pi_t \times t)^{-1}\), by Proposition 5.7 and Lemma 5.12. Thus we know from the diagram that the set \(\text{Prim } c \times \tau^\text{piso} N\) is given by the sets \(\text{Prim } K(\mathbb{L}^2(\mathbb{N}, c))\) and \(\text{Prim } T(\mathbb{Z})\). Since

\[
\text{Prim } T(\mathbb{Z}) = \text{Prim } K(\mathbb{L}^2(\mathbb{N})) \cup \text{Prim } C(\mathbb{T}) = \{0\} \cup \mathbb{T},
\]

and \(\text{Prim } K(\mathbb{L}^2(\mathbb{N}, c))\) is homeomorphic to

\[
\text{Prim } c = \text{Prim } c_0 \cup \text{Prim } C \simeq \mathbb{N} \cup \{\infty\},
\]

therefore \(\text{Prim } c \times \tau^\text{piso} N\) consists of a copy of \(\{I_n\}\) of \(\mathbb{N}\) embedded as an open subset, a copy of \(\{J_z\}\) of \(\mathbb{T}\) embedded as a closed subset. We identify these ideals in Proposition 5.7 and Lemma 5.12.

Note for now that \(\ker \varphi_T\) and \(\ker \varphi_T^\ast\) are primitive ideals of \(c \times \tau^\text{piso} N\); the Toeplitz representation \(T\) of \(T(\mathbb{Z})\) on \(\mathbb{L}^2(\mathbb{N})\) is irreducible by [7] Theorem 3.13, and \(\varphi_T\) and \(\varphi_T^\ast\) are surjective homomorphisms of \(c \times \tau^\text{piso} N\) onto \(T(\mathbb{Z})\), so \(T \circ \varphi_T\) and \(T \circ \varphi_T^\ast\)
are irreducible representations of \( c \times \tau piso N \) on \( \ell^2(N) \). Moreover, irreducibility of the representation \( \text{id} \circ \phi^\tau : pK(\ell^2(N,c))p \xrightarrow{q} K(\ell^2(N)) \xrightarrow{\text{id}} B(\ell^2(N)) \) implies that the kernel \( \mathcal{I} = \ker \varphi_T \cap \ker \varphi_{\tau^*} \simeq qK(\ell^2(N,c))q \) of \( \text{id} \circ \phi^\tau \) is a primitive ideal of \( pK(\ell^2(N,c))p \simeq \ker \varphi_T \). Similarly, \( \mathcal{I} \) is a primitive ideal of \( \ker \varphi_{\tau^*} \simeq pK(\ell^2(N,c))p \). Although \( \mathcal{I} \notin \text{Prim} c \times \tau piso N \), the ideal \( \mathcal{I} \) is essential in \( c \times \tau piso N \) by [6, Lemma 6.8], so the space \( \text{Prim} \mathcal{I} \simeq \text{Prim} c_0 \) is dense in \( \text{Prim} c \times \tau piso N \).

Next consider that \( \mathcal{K}(\ell^2(N)) = \overline{\text{span}} \{ e_{ij} : T_i(1 - TT^*)T_j^* : i,j \in N \} \), and recall that there is a natural isomorphism \( \Lambda \) of \( \mathcal{K}(\ell^2(N,c)) \simeq \mathcal{K}(\ell^2(N)) \otimes c \) onto the algebra
\[
\mathcal{C}(N \cup \{ \infty \}, \mathcal{K}(\ell^2(N))) := \{ f : N \to \mathcal{K}(\ell^2(N)) : \lim_n f(n) \text{ exists in } \mathcal{K}(\ell^2(N)) \}
\]
given by \( \Lambda(e_{ij} \otimes 1_k)(n) = 1_k(n)e_{ij} \) for \( i,j,k,n \in N \). Then \( \Lambda(p\mathcal{K}(\ell^2(N,c))p) \subset \mathcal{A} \) because
\[
\left[ \Lambda(p(e_{ij} \otimes 1_m)p) \right](n) = \left[ \Lambda(e_{ij} \otimes 1_{mvi\nu j}) \right](n) = \begin{cases} 
  e_{ij} & \text{if } n \geq m \lor i \lor j \\
  0 & \text{otherwise}
\end{cases} = \pi_n(f_{ij}^m) = \pi_n^*(g_{ij}^m)
\]
Since \( \Lambda = \pi \circ \Theta_* = \pi^* \circ \Theta \), \( \Lambda \) maps the corners \( p\mathcal{K}(\ell^2(N,c))p \) and \( q\mathcal{K}(\ell^2(N,c_0))q \) isomorphically onto the algebra \( \mathcal{A} \) and \( \mathcal{A}_0 \) respectively. Construction of this isomorphism in [6, §6] involves the representations \( \pi_n \) and \( \pi_n^* \), for each \( n \in N \), of \( c \times \tau piso N \) on \( \ell^2(N) \) that are associated to the partial-isometric representations \( k \mapsto P_nT_kP_n \) and \( k \mapsto P_nT_k^*P_n \) respectively, where \( P_n := 1 - T_{n+1}T_{n+1}^* \) is the projection onto \( H_n := \overline{\text{span}} \{ e_i : i = 0, 1, 2, \ldots, n \} \). For every \( a \in \ker \varphi_{\tau^*} \), the sequence \( \{ \pi_n(a) \}_{n \in N} \) is convergent in \( \mathcal{K}(\ell^2(N)) \), and then the map \( a \in \ker \varphi_{\tau^*} \mapsto \pi(a) := \{ \pi_n(a) \}_{n \in N} \in \mathcal{A} \) defines the isomorphism.

These observations suggest that an extension of \( \pi \) should give a representation of \( c \times \tau piso N \) in the algebra \( C_b(N, B(\ell^2(N))) \), and then primitive ideals are the kernels of evaluation maps. But we can consider a smaller algebra which gives more information on the image of \( \pi \). Note that the algebra \( \mathcal{C}(N \cup \{ \infty \}, B(\ell^2(N))) \) is too small to consider, because the sequence \( (P_nT_kP_n)_{n \in N} \) as we see, does not converge to \( T_k \) in the operator norm on \( B(\ell^2(N)) \), but it converges strongly to \( T_k \). Therefore we consider the set \( C_b(N \cup \{ \infty \}, B(\ell^2(N)), s, s) \) of functions \( \xi : N \to B(\ell^2(N)) \) such that \( \lim_n \xi_n \) exists in the \(*\)-strong topology on \( B(\ell^2(N)) \), and which satisfies \( \| \xi \|_\infty := \sup_n \| \xi_n \| < \infty \). By [3, Lemma 2.56], it is a \( C^* \)-algebra with the pointwise operation from \( B(\ell^2(N)) \) and the norm \( \| \cdot \|_\infty \). Then let
\[
\mathcal{B} := \{ f : \mathbb{N} \to B(\ell^2(N)) : \sup_{n \in \mathbb{N}} \| f(n) \|_{B(\ell^2(N))} < \infty, f(n) \in P_nB(\ell^2(N))P_n \text{ and } \lim_{n \to \infty} f(n) \text{ exists in the } *\text{-strong topology on } B(\ell^2(N)) \}.
\]
Note that \( \mathcal{B} \) is a subalgebra of \( C_b(N \cup \{ \infty \}, B(\ell^2(N)), s, s) \) because \( P_nB(\ell^2(N))P_n \simeq B(H_n) \) is closed in \( B(\ell^2(N)) \) for every \( n \in N \), and \( \mathcal{B} \) has an identity \( 1_\mathcal{B} = (P_0, P_1, P_2, \cdots) \).

**Proposition 5.5.** There are faithful representations \( \pi \) and \( \pi^* \) of \( c \times \tau piso N \) in the algebra \( \mathcal{B} \), which defined on each generator \( v_k \in c \times \tau piso N \) by
\[
\pi(v_k)(n) := \pi_n(v_k) = P_nT_kP_n \text{ and } \pi^*(v_k)(n) := \pi_n^*(v_k) = P_nT_k^*P_n \text{ for } n \in N.
\]
These representations $\pi$ and $\pi^*$ are the extension of isomorphisms $\pi : \ker \varphi_T \to A$ and $\pi^* : \ker \varphi_T \to A$ of [6, Theorem 6.1].

Proof. The map $\pi$ is induced by the partial-isometric representation $k \mapsto W_k$ where $W_k(n) = P_nT_kP_n$, and similarly for $\pi^*$ by $k \mapsto S_k$ where $S_k(n) = P_nT_k^*P_n$ for $n \in \mathbb{N}$. These are unital representations: $\pi(1) = \pi(v_0) = (P_0, P_1, P_2, \cdots) = \pi^*(1)$.

By [6, Proposition 5.4], the representation $\pi$ is faithful if and only if for any $r > 0$ and $i < j \in \mathbb{N}$, we have $\xi^r_{i,j} \in \mathcal{B}$ for which

$$\xi^r_{i,j} := (\pi(1) - \pi(v_r)\pi(v_r))((\pi(v_i)\pi(v_i)^* - \pi(v_j)\pi(v_j)^*)$$

is a nonzero element of $\mathcal{B}$. Let $r > 0$ and $i < j \in \mathbb{N}$, then we consider the three cases $0 < r < i < j$, $i < r < j$ and $i < j < r$ separately. If $0 < r < i < j$, then

$$\xi^r_{i,j}(i) = (P_i - \pi_i(v_r)^*\pi_i(v_r))((\pi_i(v_i)\pi_i(v_i)^* - \pi_i(v_j)\pi_i(v_j)^*)$$

$$= (P_i - P_iT_i^*P_i)(P_iP_iT_i^*P_i - P_iP_iT_i^*P_i)$$

and that $[\xi^r_{i,j}(i)](e_i) = (P_i - P_i)(e_i) = e_i$. If $i < j < r$ then similar computations show that $[\xi^r_{i,j}(i)](e_i) = [P_i(P_iT_i^*P_i)](e_i) = e_i$, and for $i < r < j$ we have $[\xi^r_{i,j}(r)](e_r) = (P_r - P_r)(e_r) = e_r$. Thus $\xi^r_{i,j} \neq 0$ in $\mathcal{B}$. The same outline of arguments is valid to show the representation $\pi^*$ is also faithful.}

So we have for every $n \in \mathbb{N}$ the representations $\pi_n = \varepsilon_n \circ \pi$ and $\pi_n^* = \varepsilon_n \circ \pi^*$ of $C_\times \times \pi_0 \mathbb{N}$ on $H_n$, where $\varepsilon_n$ are the evaluation map of $C_\times \times \pi_0 (\mathbb{N} \cup \{\infty\}, B(\ell^2(\mathbb{N}))_{**})$. Hence they are irreducible, indeed every nonzero vector of the subspace $H_n$ of $\ell^2(\mathbb{N})$ is cyclic for $\pi_n^*$: if $(h_0, h_1, \cdots, h_n) \in H_n$ with $h_j \neq 0$ for some $j$, then for every $i \in \{0, 1, 2, \cdots, n\}$, we have

$$(\pi_n^*(g^n_{i,j})(h_0, h_1, \cdots, h_n) = [T_i(1-\tau T^*)T_j^*](h_0, h_1, \cdots, h_n)$$

$$= (0, \cdots, h_j, \cdots, 0),$$

so $\pi_n^*(\frac{1}{h_j} g^n_{i,j})(h) = e_i$, and therefore $H_n = \text{span}\{\pi_n^*(e)h : e \in c \times \pi_0 \mathbb{N}\}$. Same arguments work for $\pi_n$.

Note for every $n \in \mathbb{N}$ that $\pi_n(f^m_{i,j}) = e_{ij} = \pi_n(g^k_{n-i,n-j})$ for all $0 \leq i, j, m, k \leq n$, and similarly $\pi_n^*(f^m_{i,j}) = e_{ij} = \pi_n^*(j^k_{n-i,n-j})$ for all $0 \leq i, j, m, k \leq n$. Thus every $f^m_{i,j} - g^k_{n-i,n-j}$ is contained in $\ker \pi_n$, and similarly $(g^m_{i,j} - j^k_{n-i,n-j}) \in \ker \pi_n^*$. We shall see many more elements of $\ker \pi_n$ as well as $\ker \pi_n^*$ in Proposition 5.7.

But now we recall that for $n \in \mathbb{N}$ the partial-isometric representation $J^n : \mathbb{N} \to B(H_n)$ in [6, §3] defined by $J^n_i(e_r) = \begin{cases} e_{r+1} & \text{if } r + t \in \{0, 1, \cdots, n\} \\ 0 & \text{otherwise,} \end{cases}$ induces the representation $\pi^R_n \times J^n$ of $(c \times \pi_0 \mathbb{N}, v)$ on $H_n$. In fact $\pi^R_n \times J^n = \pi_n$, because for every $k \in \mathbb{N}$ we have $(\pi^R_n \times J^n(v_k))(e_r) = J^n_r(e_r) = P_nT_kP_n(e_r)$ where $r \in \{0, 1, 2, \cdots, n\}$.

The ideal $\ker \pi^R_n = 0$ and $\pi^R_n \times J^n$ appears in the structure of $\mathbb{C} \times \pi_0 \mathbb{N}$ in [6, Lemma 5.7]. To be more precise about it, we need some results in [6, §5] related to the system $(\mathbb{C}^{n+1}, \tau, \mathbb{N})$. The crossed product $\mathbb{C}^{n+1} \times \pi_0 \mathbb{N}$ is the universal $\tau$-algebra generated by a canonical partial-isometric representation $w$ of $\mathbb{N}$ such that $w_r = 0$ for $r \geq n + 1$. Let $q_n : (c \times \pi_0 \mathbb{N}, v) \to (\mathbb{C}^{n+1} \times \pi_0 \mathbb{N}, w)$ be the homomorphism induced by $w$:
\[ N \rightarrow \mathbb{C}^{n+1} \times_{piso} \mathbb{N} \text{, and note that it is surjective. Then Lemma 5.7 of [6] shows that} \]
\[ \ker q_n = \ker (\oplus_{r=0}^{n} \rho^{n}_r \times J_r) = \bigcap_{r=0}^{n} \ker (\rho^{n}_r \times J_r). \]
So by these arguments we obtain the following equation
\[ (5.2) \quad \ker q_n = \bigcap_{r=0}^{n} \ker \pi_r \text{ for every } n \in \mathbb{N}. \]

**Lemma 5.6.** For \( n \in \mathbb{N} \), let \( L_n \) be the ideal of \((c \times_{piso} \mathbb{N}, v)\) generated by \( \{v_r : r \geq n + 1\}\). Then \( L_n = \ker q_n \), and it is isomorphic to
\[ (5.3) \quad \{ \xi \in \pi (c \times_{piso} \mathbb{N}) \subset C_b (\mathbb{N} \cup \{ \infty \}, B (\ell^2 (\mathbb{N}))^{s-s}) : \xi \equiv 0 \text{ on } \{0, 1, 2, \ldots, n\} \}. \]

**Proof.** We have \( L_n \subset \ker q_n \) because \( q_n (v_k) = 0 \) for all \( k \geq n + 1 \). To see \( \ker q_n \subset L_n \), let \( \rho \) be a representation of \( c \times_{piso} \mathbb{N} \) on \( H_\rho \) where \( \ker \rho = L_n \). Since \( \rho (v_t) = 0 \) for every \( t \geq n + 1 \), by the universal property of \( \mathbb{C}^{n+1} \times_{piso} \mathbb{N} \), there exists a representation \( \tilde{\rho} \) of \( \mathbb{C}^{n+1} \times_{piso} \mathbb{N} \) on \( H_\rho \) which satisfies \( \tilde{\rho} \circ q_n = \rho \). Thus \( \ker q_n \subset \ker \rho = L_n \).

Next we show that \( \pi (L_n) \) and \( (5.3) \) are equal. Let \( r \geq n + 1 \), and consider \( \pi (v_r) \) is the sequence \((P_i T_r P_i)_{i \in \mathbb{N}}\). If \( 0 \leq i \leq n \) then \( 0 \leq i + 1 \leq n + 1 \leq r \) and
\[ P_i T_r P_i = (1 - T_{i+1} T_{i+1}^{s}) T_{i+1} T_{r-(i+1)} P_i = 0. \]
So \( \pi (L_n) \) is a subset of \( (5.3) \). For the other inclusion, suppose \( f \in \pi (c \times_{piso} \mathbb{N}) \) in which \( f (i) = 0 \) for all \( 0 \leq i \leq n \). Since \( f = \pi (\xi) \) for some \( \xi \in c \times_{piso} \mathbb{N} \), and \( \pi (\xi) (i) = \pi_i (\xi) = f (i) \) for all \( i \in \mathbb{N} \), we therefore have \( \pi_i (\xi) = f (i) = 0 \) for all \( 0 \leq i \leq n \). Thus \( \xi \in \cap_{n=0}^{n} \ker \pi_i = \ker q_n \), and hence \( f = \pi (\xi) \in \pi (L_n) \).

Let \( \pi_\infty := \lim_n \pi_n \) and \( \pi^* \infty := \lim_n \pi^*_n \) where the limits are taken with respect to the strong topology of \( B (\ell^2 (\mathbb{N})) \). Then \( \pi_\infty \) and \( \pi^*_\infty \) are the irreducible representations \( \varphi_T : v_k \mapsto T_k \) and \( \varphi_{T^*} : v_k \mapsto T_k^* \) of \( c \times_{piso} \mathbb{N} \) on \( H_\infty := \ell^2 (\mathbb{N}) \). Thus by [6] Lemma 6.2 we have
\[ \ker \pi_\infty = \ker \varphi_T = \text{span} \{ g^m_{i,j} := v^*_i v_m v^*_j (1 - v^* v) v_j : i, j, m \in \mathbb{N} \} \]
\[ \ker \pi^*_\infty = \ker \varphi_{T^*} = \text{span} \{ f^m_{i,j} := v^*_i v_m v^*_j (1 - v^* v) v^*_j : i, j, m \in \mathbb{N} \}. \]

For \( n \in \mathbb{N} \), let \( \pi_n \) and \( \pi^*_n \) be the irreducible representations of \( c \times_{piso} \mathbb{N} \) on the subspace \( H_n \) of \( \ell^2 (\mathbb{N}) \), that are induced by the partial-isometric representations \( k \mapsto P_k T_k P_n \) and \( k \mapsto P_k T^*_k P_n \). Let \( L_n \) be the ideal of \((c \times_{piso} \mathbb{N}, v)\) generated by \( \{v_r : r \geq n + 1\}\). Then \( \pi_n \) is the representation
\[ \varepsilon_n \circ \pi : c \times_{piso} \mathbb{N} \rightarrow B \subset C_b (\mathbb{N} \cup \{ \infty \}, B (\ell^2 (\mathbb{N}))^{s-s}) \rightarrow B (H_n), \]
and similarly \( \pi^*_n = \varepsilon_n \circ \pi^* \). So ker \( \pi_n \simeq \ker \varepsilon_n \simeq \ker \pi^*_n \).

**Proposition 5.7.** Let \( n \in \mathbb{N} \), then
\[ (a) \quad \ker \pi_n = \ker \pi^*_n \simeq \ker \varepsilon_n = \{ \xi \in B : \xi (n) = 0 \}; \]
\[ (b) \quad \ker \pi_\infty \simeq \ker \pi^*_\infty = \{ \xi \in B : \ast - \text{strong lim}_n \xi (n) = 0 \}. \]
Furthermore

(c) \( \ker \pi_n = \text{span}\{g_{i,j}^m - f_{n-i,n-j}^k + \eta : 0 \leq i, j, m, k \leq n, \eta \in L_n\} \),
\( \ker \pi_n = \text{span}\{f_{i,j}^m - g_{n-i,n-j}^k + \eta : 0 \leq i, j, m, k \leq n, \eta \in L_n\} \), and
\( \ker \pi_n^* = \ker \pi_n \) for \( n \in \mathbb{N} \), in particular we have \( \ker \pi_0 = \ker \pi_0^* = L_0; \)

(d) \( \ker \pi_n|_{\ker \varphi_T^*} = \text{span}\{(f_{i,j}^m - f_{i,j}^k) + f_{x,y}^k : 0 \leq i, j, m, k \leq n, \) one of \( x, y, z \geq n + 1\} \),
\( \ker \pi_n|_{\ker \varphi_T^*} = \text{span}\{(g_{i,j}^m - g_{i,j}^k) + g_{x,y}^k : 0 \leq i, j, m, k \leq n, \) one of \( x, y, z \geq n + 1\},
\( \Theta_n^{-1}(\ker \pi_n|_{\ker \varphi_T^*}) = \Theta_n^{-1}(\ker \pi_n|_{\ker \varphi_T^*}) \), and
\( \ker \pi_n|_{\ker \varphi_T^*} \simeq \{a \in A : a(n) = 0\} \simeq \ker \pi_n|_{\ker \varphi_T^*}; \)

(e) \( \ker \pi_n|_I = \text{span}\{g_{i,j}^m - g_{i,j}^{m+1} : 0 \leq i, j \leq m \in \mathbb{N}, \) and \( m \neq n\} = \ker \pi_n|_I \)
\( = \text{span}\{f_{i,j}^m - f_{i,j}^{m+1} : 0 \leq i, j \leq m \in \mathbb{N}, \) and \( m \neq n\} \) is isomorphic to the ideal
\( \{a \in A : a(n) = 0\}. \)

Remark 5.8. Note that the representations \( \pi_n|_{\ker \varphi_T^*} \) and \( \pi_n^*|_{\ker \varphi_T^*} \) are equivalent to the evaluation map \( \varepsilon_n : f \in A \mapsto f(n) \in B(H_n) \) of \( A \) on \( H_n \), so we have \( \ker \pi_n|_{\ker \varphi_T^*} \simeq \ker \pi_n|_{\ker \varphi_T^*} \) is isomorphic to \( \{f \in A : f(n) = 0\} \), and \( \ker \pi_n|_I = \ker \pi_n^*|_I \simeq \{f \in A_0 : f(n) = 0\} \); and \( \ker \pi_\infty \simeq \ker \pi_\infty^* \simeq A. \)

Proof of Proposition 5.7. Fix \( n \in \mathbb{N} \). We show for \( \pi_n \), and skip the proof for \( \pi_n^* \) because it contains the same arguments. We clarify firstly that the space
\[ \mathcal{J} := \text{span}\{f_{i,j}^m - g_{n-i,n-j}^k + \eta : 0 \leq i, j, m, k \leq n, \eta \in L_n\} \]
is an ideal of \((c \times \mathbb{N}, v)\) by showing \( v\mathcal{J} \subset \mathcal{J} \) and \( v^*\mathcal{J} \subset \mathcal{J} \). Let \( i = n \), then
\[ vv_kv_k^*(1 - v^*v)v_{n-j} = v(v^*vv_kv_k^*v(1 - v^*v)v_{n-j} = vv_kv_k^*v^*v(1 - v^*v)v_{n-j} = v_{k+1}v_{k+1}^*(v - v^*v)v_{n-j} = 0, \]
therefore \( v(f_{i,j}^m - g_{n-i,n-j}^k + \eta) = vv_mv_nv_m(1 - v^*v)v_{n-j} - vv_kv_k^*v(1 - v^*v)v_{n-j} + v\eta = f_{n+1,j}^m + v\eta \) belongs to \( \mathcal{J} \) because \( f_{n+1,j}^m \in L_n \). If \( 0 \leq i \leq n - 1, \) then \( 1 \leq i + 1 \leq n \) and \( n - i \geq 1 \), and we have
\[ vv_{n-i}^*v_kv_k^* = vv^*v_{n-i}v_{n-i}^*v_{n-i}^*v_kv_k^* \]
\[ = v_{n-i}^*v_{n-i}v_{n-i}^*v_{n-i}^*v_kv_k^* \]
\[ = v_{n-i}^*v_{n-i}v_{n-i}^*v_kv_k^* = v_{n-i}^*v_{n-i}^*v_{n-i}^*v_{n-i}^*v_kv_k^* \]
so \( v(f_{i,j}^m - g_{n-i,n-j}^k + \eta) = f_{i+1,j}^m - g_{n-i,n-j}^k + v\eta \in \mathcal{J}. \)

Now we check for \( v^*\mathcal{J} \), and assume \( i = 0, \) then
\[ v^*[f_{i,j}^m - g_{n-i,n-j}^k + \eta] = v^*[v_mv_m(1 - v^*v)v_{n-j}^*v^*v_{n-i}^*v_kv_k^*(1 - v^*v)v_{n-j}^*v^*v_{n-i}^*v_kv_k^* + v\eta] = 0 - g_{n+1,i,n-j}^k + v^*\eta \in \mathcal{J} \]
because \( g_{n+1,i,n-j}^k \in L_n \). It follows by similar computations for \( 1 \leq i \leq n \) that
\[ v^*[f_{i,j}^m - g_{n-i,n-j}^k + \eta] = f_{i+1,j}^m - g_{n-i,n-j}^k + v^*\eta \in \mathcal{J}. \]

Next we show that \( \mathcal{J} = \ker \pi_n, \) one inclusion \( \mathcal{J} \subset \ker \pi_n \) is clear because \( \pi_n(f_{i,j}^m) = \pi_n(g_{n-i,n-j}^k) = T_i(1 - TT_n^*)^n \) and \( L_n \subset \ker \pi_n. \) For the other inclusion, let \( \sigma : c \times \mathbb{N} \rightarrow B(H_n) \) be a nondegenerate representation with \( \ker \sigma = \mathcal{J}. \) Note that \( B(H_n) = \text{span}\{e_{ij} := T_i(1 - TT_n^*)^n : 0 \leq i, j \leq n\}. \) Since \( \{f_{i,j}^m : 0 \leq i, j, k \leq n\} \) is
a matrix-units for $B(H_\sigma)$, there is a homomorphism $\psi$ of $B(H_\sigma)$ into $B(H_n)$ which satisfies $e_{ij} \mapsto \sigma(f^m_{i,j})$. Therefore $\sigma = \psi \circ \pi_n$, and hence $\ker \pi_n \subset \ker \sigma = \mathcal{J}$.

Using the spanning elements of $\ker \pi_n$ and $\ker \pi_n^*$, and the equation $f^m_{i,j} - g^k_{i,n-j} = -(g^k_{n-i,n-j} - f^m_{n-(i-n),(n-j)})$, we see that they contain each other, therefore $\ker \pi_n = \ker \pi_n^*$ for every $n \in \mathbb{N}$. The ideal $L_0$ is $\ker \pi_0 = \ker \pi_0^*$ because $f^0_{0,0} - g^0_{0,0} = v^*v - vv^* \in L_0$.

For (d), let now $\mathcal{J}$ be $\text{span}\{(f^m_{i,j} - f^k_{i,j}) + f^x_{y} : 0 \leq i, j, m, k \leq n, \text{one of } x, y, z \geq n + 1\}$. Then the same idea of calculations shows that $\mathcal{J}$ is an ideal of $\ker \varphi_{T^*}$, and it is contained in $\ker \pi_n|_{\ker \varphi_{T^*}}$, then for the other inclusion let $\sigma$ be a non-degenerate representation of $\ker \varphi_{T^*}$ such that $\ker \sigma = \mathcal{J}$, get the homomorphism $\psi : B(H_n) \to B(H_\sigma)$ defined by $\psi(e_{ij}) = \sigma(f_{i,j}^n)$, and hence the equation $\psi \circ \pi_n = \sigma$ implies that $\ker \pi_n|_{\ker \varphi_{T^*}} = \mathcal{J}$. By computations on the spanning elements we see that the equation $\Theta^{-1}_n(\ker \pi_n|_{\ker \varphi_{T^*}}) = \Theta^{-1}(\ker \pi_n^*|_{\ker \varphi_{T^*}})$ is hold. The same arguments work for the proof of (e), and we skip this. $\square$.

**Remark 5.9.** The map $n \in \mathbb{N} \cup \{\infty\} \mapsto I_n := \ker \pi_n^* \in \text{Prim}(c \times_{piso}^\pi \mathbb{N})$ parameterizes the open subset $\{P \in \text{Prim}(c \times_{piso}^\pi \mathbb{N}) : \ker \varphi_T \simeq A \not\subseteq P\}$ of $\text{Prim}(c \times_{piso}^\pi \mathbb{N})$ homeomorphic to $\text{Prim} A$. Note that the $\infty$ corresponds to the ideal $\ker \pi_n^* = \ker \varphi_{T^*} \in \text{Prim}(c \times_{piso}^\pi \mathbb{N})$, and it corresponds to $\mathcal{I} = \ker \varphi_{T^*}|_{\ker \varphi_{T^*}} \in \text{Prim} A$.

**Lemma 5.10.**

(i) $\bigcap_{n=0}^m I_n = L_m$ for every $m \in \mathbb{N}$;

(ii) $\bigcap_{n \geq m} I_n = \ker \pi_n^* \cap \ker \pi_\infty$ for every $m \in \mathbb{N}$.

**Proof.** Part (i) follows from (5.2) and Lemma 5.6. For (ii), note that $q_\infty$ is the identity map on $c \times_{piso}^\pi \mathbb{N}$, and that $\bigoplus_{i \in \mathbb{N}} \pi_i = (\bigoplus_{i \in \mathbb{N}} (\pi_i^N \times J^i)) \circ \text{id}$. So $\bigcap_{n \in \mathbb{N}} I_n = \{0\}$ by faithfulness of $\bigoplus_{i \in \mathbb{N}} (\pi^N_{pi} \times J^i)$ [6 Corollary 5.5].

The inclusion $\bigcap_{n \geq m} I_n \subset \ker \pi_\infty^*$ for every $m \in \mathbb{N}$ follows from the next arguments:

$$\ker(\pi_n^*|_{\ker \pi_\infty}) \cong \{f \in A : f(n) = 0 \forall n > m\}$$

$$\subset \{f \in A : \lim_{n \to \infty} f(n) = 0\} = A_0 \cong \ker \pi_\infty^*|_{\ker \pi_\infty} \subset \ker \pi_\infty^* \in \text{Prim} c \times_{piso}^\pi \mathbb{N},$$

so the two ideals $J := \bigcap_{n \geq m} I_n$ and $L := \ker \pi_\infty$ of $c \times_{piso}^\pi \mathbb{N}$ satisfy $J \cap L \subset \ker \pi_\infty^*$, therefore either $J \subset \ker \pi_\infty^*$ or $L \subset \ker \pi_\infty$, but the latter is not possible. To show $J \subset \ker \pi_\infty$, since $\ker \pi_n = \ker \pi_n^*$ for each $n$, we act similarly using the fact that

$$\bigcap_{n \geq m} \ker(\pi_n|_{\ker \pi_n^*}) \cong \{f \in A : f(n) = 0 \forall n > m\} \subset \ker \pi_\infty \subset \text{Prim} c \times_{piso}^\pi \mathbb{N}.$$ 

Therefore, $J \subset \ker \pi_\infty^* \cap \ker \pi_\infty$. Moreover, since $g^0_{0,0} - g^1_{0,0} \neq 0$ which satisfies $\pi_n^*(g^0_{0,0} - g^1_{0,0}) = 0$ for all $n \geq 1$, it follows that $\{0\} \subset \bigcap_{n \geq m} I_n$. $\square$.

**Remark 5.11.** Part (ii) of Lemma 5.10 confirms with the fact that $\mathcal{I}$ is an essential ideal of $c \times_{piso}^\pi \mathbb{N}$ [6 Lemma 6.8].

Next consider for $z \in \mathbb{T}$, the character $\gamma_z \in \hat{\mathbb{Z}} \simeq \mathbb{T}$ defined by $\gamma_z : m \mapsto e^{2\pi im}$. Note that the map $\gamma_z : k \in \mathbb{N} \mapsto \gamma_z(k)$ is a partial-isometric representation of $\mathbb{N}$ in $\mathbb{C} \simeq B(\mathbb{C})$. Consequently for each $z \in \mathbb{T}$, we have a representation $\pi_{\gamma_z} \times \gamma_z$ of $c \times_{piso}^\pi \mathbb{N}$
on $\mathbb{T}$ such that $\pi_{\gamma_z} \times \gamma_z(v_k) = \gamma_z(k) = \varpi^k$ for $k \in \mathbb{N}$, and it is irreducible. Moreover we know that the homomorphism $\Psi : c \times_{\tau}^\text{piso} \mathbb{N} \to C(\mathbb{T})$ is the composition of the Fourier transform $\mathbb{C} \times \text{id} \mathbb{Z} \simeq C^*(\mathbb{Z}) \simeq C(\mathbb{T})$ with $\ell \times \delta^* : c \times_{\tau}^\text{piso} \mathbb{N} \to C \times \text{id} \mathbb{Z}$, in which $\ell : (x_n) \in c \mapsto \lim_n x_n \in \mathbb{C}$ and $\delta$ is the unitary representation of $\mathbb{Z}$ on $\mathbb{C} \times \text{id} \mathbb{Z}$.

\textbf{Lemma 5.12.} For $z \in \mathbb{T}$, the character $\gamma_z : k \mapsto \varpi^k$ in $\hat{\mathbb{Z}} \simeq \mathbb{T}$ gives an irreducible representation $\pi_{\gamma_z} \times \gamma_z$ of $c \times_{\tau}^\text{piso} \mathbb{N}$ on $\mathbb{T}$ such that $\pi_{\gamma_z} \times \gamma_z = \varepsilon_z \circ (\ell \times \delta^*)$. Denote by $J_z$ the primitive ideal $\ker \pi_{\gamma_z} \times \gamma_z$ of $c \times_{\tau}^\text{piso} \mathbb{N}$. Then $\ker \pi_{\infty}$ and $\ker \pi_{\infty}^*$ are contained in $J_z$ for every $z \in \mathbb{T}$. Moreover every ideal $I_n$ for $n \in \mathbb{N}$ is not contained in any $J_z$.

\textit{Proof.} By using the Fourier transform we can view $\mathbb{C} \times \text{id} \mathbb{Z} \simeq C^*(\mathbb{Z})$ as $C(\mathbb{T})$, and it follows that $v_k \in c \times_{\tau}^\text{piso} \mathbb{N}$ is mapped into the function $t \mapsto \varpi^k \in C(\mathbb{T})$.

We know that primitive ideals of $C(\mathbb{T})$ are given by the kernels of evaluation maps $\varepsilon_t(f) = f(t)$ for $t \in \mathbb{T}$, and the character $\gamma_z$ is a partial-isometric representation of $\mathbb{N}$ on $\mathbb{T}$ for $z \in \mathbb{T}$. Then by inspection on the generators, we see that the representation $\pi_{\gamma_z} \times \gamma_z$ of $c \times_{\tau}^\text{piso} \mathbb{N}$ on $\mathbb{T}$ satisfies $\pi_{\gamma_z} \times \gamma_z = \varepsilon_z \circ (\ell \times \delta^*)$. So the primitive ideal $J_z := \ker \pi_{\gamma_z} \times \gamma_z$ of $c \times_{\tau}^\text{piso} \mathbb{N}$ is lifted from the quotient $C(\mathbb{T})/J \simeq C(\mathbb{T})$.

Since $\pi_{\gamma_z} \times \gamma_z(f_{m,i}) = 0 = \pi_{\gamma_z} \times \gamma_z(g_{m,i}^n)$, ker $\pi_\infty = \ker \varphi_T$ and ker $\pi_{\infty}^* = \ker \varphi_T^*$, are contained in $J_z$ for every $z \in \mathbb{T}$. Finally, since $\pi_{\gamma_z} \times \gamma_z(v_{n+1}) = \varpi^{n+1} \neq 0$ for $n \in \mathbb{N}$, $I_n \not\subset J_z$ for any $z \in \mathbb{T}$. \hfill $\square$

\textbf{Theorem 5.13.} The maps $n \in \mathbb{N} \cup \{\infty\} \cup \{\infty^*\} \mapsto I_n$ and $z \in \hat{\mathbb{Z}} \mapsto J_z$ combine to give a bijection of the disjoint union $\mathbb{N} \cup \{\infty\} \cup \{\infty^*\} \cup \mathbb{T}$ onto Prim$(c \times_{\tau}^\text{piso} \mathbb{N})$, where $I_{\infty} := \ker \varphi_T$. Then the hull-kernel closure of a nonempty subset $F$ of $\mathbb{N} \cup \{\infty\} \cup \{\infty^*\} \cup \mathbb{T}$ is given by

(a) the usual closure of $F$ in $\mathbb{T}$ if $F \subset \mathbb{T}$;
(b) $F$ if $F$ is a finite subset of $\mathbb{N}$;
(c) $F \cup \mathbb{T}$ if $F \subset (\{\infty\} \cup \{\infty^*\})$;
(d) $F \cup (\{\infty\} \cup \{\infty^*\} \cup \mathbb{T})$ if $F \neq \mathbb{N}$ is an infinite subset of $\mathbb{N}$;
(e) $\mathbb{N} \cup \{\infty\} \cup \{\infty^*\} \cup \mathbb{T}$ if $\mathbb{N} \subset F$.

\textit{Proof.} The diagram \textbf{5.1} together with Proposition \textbf{5.7} give a bijection map of $\mathbb{N} \cup \{\infty\} \cup \{\infty^*\} \cup \mathbb{T}$ onto Prim$(c \times_{\tau}^\text{piso} \mathbb{N})$.

Lemma \textbf{5.10} (ii) gives the closure of the subset $F$ in (e), and Lemma \textbf{5.10} (iii) gives the closure of the subset $F$ in (d). If $F \subset (\{\infty\} \cup \{\infty^*\})$, then $\overline{F} = F \cup \mathbb{T}$ because $\ker \pi_\infty, \ker \pi_{\infty}^* \subset J_z$ for every $z \in \mathbb{T}$ by Lemma \textbf{5.12}.

To see that $\overline{F} = F$ for a finite subset $F = \{n_1, n_2, \ldots, n_j\}$ of $\mathbb{N}$, we note that if an ideal $P \in \text{Prim}(c \times_{\tau} \mathbb{N})$ satisfies $\bigcap_{i=1}^j I_{n_i} \subset P$, then

- $\bullet$ $P \neq J_z$ for any $z \in \mathbb{T}$ because $v_{n_j+1} \in \bigcap_{i=1}^j I_{n_i}$ but $v_{n_j+1} \not\in J_z$;
- $\bullet$ $P \neq I_\infty, I_{\infty}^*$ because $v_{n_j+1} \in \bigcap_{i=1}^j I_{n_i}$ but $v_{n_j+1} \not\in I_\infty, I_{\infty}^*$;
- $\bullet$ $P \neq I_n$ for $n \not\in F$ because $(g_{0,0}^n - g_{0,0}^{n+1}) \in \bigcap_{i=1}^j I_{n_i}$ but $(g_{0,0}^n - g_{0,0}^{n+1}) \not\in I_n$ for $n \not\in F$.

So it can only be $P = I_j$ for some $j \in F$. Finally the usual closure of $F$ in $\mathbb{T}$ is followed by the fact that the map $z \mapsto J_z$ is a homeomorphism of $\mathbb{T}$ onto the closed set Prim$C(\mathbb{T})$. \hfill $\square$
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REFERENCES

[1] S. Adji, *Invariant ideals of crossed products by semigroups of endomorphisms*, Proc. Conference in Functional Analysis and Global Analysis in Manila, October 1996 (Springer, Singapore 1996), 1–8.

[2] S. Adji and A. Hosseini, *The Partial-Isometric Crossed Products of $c_0$ by the Forward and the Backward Shifts*, Bull. Malays. Math. Sci. Soc. (2) 33(3) (2010), 487–498.

[3] S. Adji, M. Laca, M. Nilsen and I. Raeburn, *Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups*, Proc. Amer. Math. Soc. 122 (1994), no. 4, 1133–1141.

[4] S. Adji, S. Zahmatkesh, *Partial-isometric crossed products by semigroups of endomorphisms as full corners*, J. Aust. Math. Soc. 96 (2014), 145–166.

[5] N. J. Fowler, P. Muhly and I. Raeburn, *Representations of Cuntz-Pimsner Algebras*, Indiana University Math. J. 52 (2003), no. 3, 569–605.

[6] J. Lindiarni and I. Raeburn, *Partial-isometric crossed products by semigroups of endomorphisms*, J. Operator Theory 52 (2004), 61–87.

[7] G.J. Murphy, *Ordered groups and Toeplitz algebras*, J. Operator Theory 18 (1987), 303–326.

[8] I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-Trace $C^*$-Algebras*, Mathematical Surveys and Monographs, 60 (American Mathematical Society, Providence, RI, 1998).

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