Asymptotical stability analysis of conformable fractional systems

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ABSTRACT
In this paper, we analyses the asymptotical stability of the system in the form $T_\alpha y(\tau) = A y(\tau) + f(\tau, y(\tau))$ with the initial value $y(\tau_0) = y_0$. With the help of the Grönwall’s Inequality and function analysis, we have proved asymptotical stability of solution for the conformable fractional system. Two examples are included to apply the results.

1. Introduction
Fractional differential systems have gained considerable popularity due to its important applications in physics and engineering [1–8] etc. In recent years, several types of fractional definitions are given, such as Riemann–Liouville, Grunwald–Letnikov and Caputo’s fractional definition and so on. However, there are some disadvantages about Riemann–Liouville and Caputo fractional derivative, such as all of them do not satisfy the following rules,

\[ D_\alpha^\alpha (gf) = fD_\alpha^\alpha g + gD_\alpha^\alpha f, \]
\[ D_\alpha^\alpha \left( \frac{g}{f} \right) = \frac{fD_\alpha^\alpha g - gD_\alpha^\alpha f}{f^2}. \]

Over the past few decades, a simple definition called conformable fractional derivative was proposed in [9]. For more results about conformable fractional derivative, we refer the reader to [10–18]. This derivative seems to be more natural, and it coincides with the classical definition of the first derivative. In 2015, Thabet Abdeljawad proceeded on to develop the definition, some basic concepts about conformable fractional derivative such as chain rule, Grönwall’s Inequality, exponential functions and Lyapunov inequality were studied in [19–22]. In addition, the Laplace transform was introduced to solve the linear differential systems [23].

In order to solve the conformable fractional equations, more and more methods have been proposed, such as invariant subspace method [24], the new extended direct algebraic method [25], the first integral method [26], modified Kudryashov method [27], the analytical method [28] and stochastic method [29], thanks to these methods, the exact solutions are formally established for many systems. Although so many methods have been presented, there are still a large number of systems cannot be solved, hence, the numerical simulations method is proposed, the results are proved to be very accurate [30].

On the other hand, more and more conformable fractional models have been established, such as conformable fractional dynamic cobweb model [31], conformable time-fractional schrödinger model [32], conformable fractional Biswas–Milovic model [33]. The stability of the differential system is also attracted for researchers, that is because the stable system is very important in our life. Recently, stability problems of nonlinear fractional systems have been extensively investigated by many authors [34–36]. In addition, Abdourazek Souahi et al. studied the stability of conformable fractional-order nonlinear systems by using Lyapunov function [37]. However, to the best of the authors’ knowledge, few contributions addressing the asymptotical stability for the conformable fractional system have been reported in the literature, which motivates us to carry out this work.

It is well-known that the Lyapunov function is difficult to obtain for stability analysis of uncertain nonlinear systems. The purpose of this paper is to present more convenient methods to analyse the asymptotical stability of the conformable fractional system. The main contributions of this paper are as follows: (1) By using the Grönwall’s Inequality and function analysis, the asymptotical stability results of a class of conformable fractional system are established, (2) To overcome the difficulty of finding suitable Lyapunov function, the asymptotical stability of the system is studied by the limit method.

The rest of this paper is organized as follows. In Section 2, we introduce some Definitions and the necessary Lemmas. In Section 3, we given our main...
The trivial solution of Equation (5) is called to be
\[
\text{Definition 2.3 (Fractional Exponential Stability [23]):}
\]
\[
\varepsilon
\]
for all \( \varepsilon > 0 \) such that the solution of Equation (5) satisfies
\[
\text{where } 0 < \alpha \leq 1.
\]

\[
\text{Lemma 2.1 ([23]): Assume that } y: [0, \infty) \to \mathbb{R} \text{ such that } y'(\tau) \text{ is continuous. Then the following equation holds}
\]
\[
T_\alpha T_\alpha y(\tau) = y(\tau), 0 < \alpha \leq 1.
\]

\[
\text{Lemma 2.2 ([23]):}
\]
\[
T_\alpha (au + bv) = a T_\alpha u + b T_\alpha v,
\]
\[
T_\alpha (uv) = v T_\alpha u + u T_\alpha v,
\]
\[
T_\alpha \left( \begin{array}{c} u \\ v \end{array} \right) = \frac{v T_\alpha u - u T_\alpha v}{\sqrt{v^2}}.
\]

\[
\text{Lemma 2.3 (Grönwall’s Inequality):}
\]
\[
f(\tau) \leq \lambda + \int_a^\tau f(s) g(s) \, ds, \tau \in [a, b],
\]
then
\[
f(\tau) \leq \lambda e^{\int_a^\tau g(\tau) \, d\tau}, \tau \in [a, b].
\]

\[
\text{Lemma 2.4 ([23]):}
\]
\[
T_\alpha y(\tau) = A y(\tau) + f(\tau, y(\tau)), y(\tau_0) = y_0,
\]
has the solution
\[
T_\alpha y(\tau) = y(\tau) + f(\tau, y(\tau)), y(\tau_0) = y_0,
\]
\[
y(\tau) = y(0) \exp \left( A (\tau - \tau_0)^\alpha \right) + \int_{\tau_0}^\tau \exp \left( A (\tau - \tau_0)^\alpha \right) \times \exp \left( -A (\tau - \tau_0)^\alpha \right) f(s, y(s)) (s - \tau_0)^{1-\alpha} \, ds.
\]

\[
\text{Lemma 2.5 ([34]):}
\]
\[
\exp(At) \in \mathbb{R}^{-w},
\]
where \( \lambda \) is eigenvalue of the real matrix \( A \in \mathbb{R}^{n \times n} \) and \( w = -\max(\text{Re}(A)). \)

\[
\text{Lemma 2.6:}
\]
\[
\exp(At + Bt) = \exp(At) \exp(Bt),
\]
where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times n} \) are real matrices.
3. Main result

In this section, we will pay attention to the following conformable fractional differential system. The main purpose of this section is to analysis the asymptotical stability of the system.

\[ T_\alpha y(\tau) = Ay(\tau) + f(\tau, y(\tau)), y(\tau_0) = y_0, \]

where \( 0 < \alpha \leq 1 \), \( A \in \mathbb{R}^{n \times n} \) is a constant matrix, \( y(\tau) \), \( f(\tau, y(\tau)) \) are column vectors and \( f(\tau, 0) = 0 \).

**Theorem 3.1** ([37]): Let \( x = 0 \) be an equilibrium point of the system (14), and Lyapunov function \( V(\tau, y(\tau)) \) is continuous. If there exist positive constants \( c_1, c_2, c_3 \) satisfying the following conditions:

\[ c_1 \| y \|^2 \leq V(\tau, y) \leq c_2 \| y \|^2, \]

\[ T_\alpha c V(\tau, y) \leq -c_3 \| y \|^2, \]

then the origin of system (14) is fractional exponentially stable.

**Theorem 3.2**: Let \( \hat{y}(\tau) = [y_1(\tau), y_2(\tau), \ldots, y_n(\tau)] \) and \( P \) be positive symmetric matrix, then there exist \( \lambda_1 > 0 \) and \( \lambda_n > 0 \) satisfying the following inequality for arbitrary \( y(\tau) \).

\[ \lambda_1 y^T(\tau)y(\tau) \leq y^T(\tau)Py(\tau) \leq \lambda_n y^T(\tau)y(\tau). \]

**Proof**: \( P \) is positive symmetric matrix implies that there exists orthogonal matrix \( Q \) (or \( Q^T Q = I \)) satisfying

\[ Q^T P Q = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}, \]

where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are eigenvalues of \( P \) and \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \). Let \( y(\tau) = Qz(\tau) \), then

\[ \hat{y}(\tau)^T Py(\tau) = (Qz(\tau))^T P Q z(\tau) = z^T(\tau) PQ z(\tau) = \lambda_{1} z_{1}^{2}(\tau) + \cdots + \lambda_{n} z_{n}^{2}(\tau). \]

It is obvious that

\[ \lambda_{1} z_{1}^{2}(\tau) + \cdots + \lambda_{n} z_{n}^{2}(\tau) \leq \lambda_{1} z_{1}^{2}(\tau) + \cdots + \lambda_{n} z_{n}^{2}(\tau) \leq \| y \|, \]

then we have

\[ \lambda_{1} z_{1}^{2}(\tau) z(\tau) \leq \lambda_{1} z_{1}^{2}(\tau) + \cdots + \lambda_{n} z_{n}^{2}(\tau) \leq \| y \| \| z(\tau) \|. \]

From Definition 2.5 and (23), we have

\[ T_\alpha V(\tau) = (T_\alpha y^T(\tau)) Py(\tau) + y^T(\tau) PT_\alpha y(\tau) \]

\[ = (T_\alpha y(\tau))^T Py(\tau) + y^T(\tau) PT_\alpha y(\tau) \]

\[ = [Ay(\tau) + f(\tau, y(\tau))]^T Py(\tau) + y^T(\tau) [Ay(\tau) + f(\tau, y(\tau))]|P|y(\tau) \]

\[ = [y^T(\tau) A^T + f^T(\tau, y(\tau))] Py(\tau) + y^T(\tau) [Ay(\tau) + f(\tau, y(\tau))]|P|y(\tau) \]

\[ = y^T(\tau) [A^T P + PA] y(\tau) + 2f^T(\tau, y(\tau)) Py(\tau). \]

By Theorem 3.1, it is easy to verify that the origin of system (14) is fractional exponentially stable, the proof is completed.

**Theorem 3.3**: For \( 0 < \alpha \leq 1 \), if the function \( f(\tau, y(\tau)) \) is Lipschitz continuous, \( L \) is Lipschitz constant. Assume that the following assumption is satisfied: There exists a positive symmetric matrix \( P \) and positive constant \( \varepsilon \) such that the following inequalities hold

\[ A^T P + PA + \varepsilon I < 0, \]

\[ L < \frac{\varepsilon}{2\lambda_{\max}(P)}. \]

Then the origin of system (14) is fractional exponentially stable.

**Proof**: Choose a Lyapunov function \( V(\tau) = y^T(\tau) Py(\tau) \), it is obvious that the condition (15) holds.

From Theorem 3.2 and Lemma 2.2, we can conclude that

\[ T_\alpha V(\tau) = (T_\alpha y^T(\tau)) Py(\tau) + y^T(\tau) PT_\alpha y(\tau) \]

\[ = (T_\alpha y(\tau))^T Py(\tau) + y^T(\tau) PT_\alpha y(\tau) \]

\[ = [Ay(\tau) + f(\tau, y(\tau))]^T Py(\tau) + y^T(\tau) [Ay(\tau) + f(\tau, y(\tau))]|P|y(\tau) \]

\[ = [y^T(\tau) A^T + f^T(\tau, y(\tau))] Py(\tau) + y^T(\tau) [Ay(\tau) + f(\tau, y(\tau))]|P|y(\tau) \]

\[ = y^T(\tau) [A^T P + PA] y(\tau) + 2f^T(\tau, y(\tau)) Py(\tau). \]

\[ \leq \varepsilon \| y(\tau) \|^2 + 2L \| y(\tau) \|^2 \leq \varepsilon \| y(\tau) \|^2 + 2L \lambda_{\max}(P) \| y(\tau) \|^2, \]
**Proof:** With the help of Lemma 2.4, the solution of system (14) is obtained

\[
y(\tau) = y(0) \exp \left( \frac{A(\tau - \tau_0)^\alpha}{\alpha} \right) + \int_{\tau_0}^{\tau} \exp \left( \frac{A(\tau - \tau_0)^\alpha}{\alpha} \right) \times e^{-A(s - \tau_0)^\alpha} f(s, y(s)) (s - \tau_0)^{\alpha - 1} ds,
\]

Thus,

\[
||y(\tau)|| \leq ||y(0)|| \exp \left( \frac{A(\tau - \tau_0)^\alpha}{\alpha} \right) + \int_{\tau_0}^{\tau} \exp \left( \frac{A(\tau - \tau_0)^\alpha - (s - \tau_0)^\alpha}{\alpha} \right) \times |f(s, y(s))|(s - \tau_0)^{\alpha - 1} ds.
\]

(28)

According to Lemma 2.5, there exists a constant \( M > 0 \) such that

\[
||\exp \left( \frac{A(\tau - \tau_0)^\alpha}{\alpha} \right)|| \leq Me^{-\alpha (\tau - \tau_0)^\alpha}, \quad ||\exp \left( \frac{A(\tau - \tau_0)^\alpha - (s - \tau_0)^\alpha}{\alpha} \right)|| \leq Me^{-\alpha (\tau - \tau_0)^\alpha - (s - \tau_0)^\alpha},
\]

where \( w = -\max\{\text{Re}\lambda(\mathcal{A})\} \).

Combining (29), (30), and (31), we have

\[
||y(\tau)|| \leq ||y(0)|| Me^{-\alpha (\tau - \tau_0)^\alpha} + \int_{\tau_0}^{\tau} Me^{-\alpha (\tau - \tau_0)^\alpha - (s - \tau_0)^\alpha} \times |f(s, y(s))|(s - \tau_0)^{\alpha - 1} ds,
\]

(32)

the condition \( \lim_{\tau \to +\infty} ||y(\tau)|| = 0 \) implies that there exists \( \delta > 0 \) satisfying the following inequality

\[
||f(\tau, y(\tau))|| \leq \frac{1}{M} ||y(\tau)||, \quad \text{as} \ ||y(\tau)|| < \delta.
\]

(33)

Substituting (33) into (32), we have

\[
||y(\tau)|| \leq ||y(0)|| Me^{-\alpha (\tau - \tau_0)^\alpha} + \int_{\tau_0}^{\tau} Me^{-\alpha (\tau - \tau_0)^\alpha - (s - \tau_0)^\alpha} \times |f(s, y(s))|(s - \tau_0)^{\alpha - 1} ds,
\]

(34)

Multiplying both sides by \( e^{\alpha (\tau - \tau_0)^\alpha} \), we have

\[
e^{\alpha (\tau - \tau_0)^\alpha} ||y(\tau)|| \leq ||y(0)|| M + \int_{\tau_0}^{\tau} e^{\alpha (\tau - \tau_0)^\alpha - (s - \tau_0)^\alpha} \times |f(s, y(s))|(s - \tau_0)^{\alpha - 1} ds.
\]

(35)

From Lemma 2.3, the following inequality holds

\[
e^{\alpha (\tau - \tau_0)^\alpha} ||y(\tau)|| \leq ||y(0)|| M e^{\alpha (\tau - \tau_0)^\alpha} e_{\gamma}(\tau - \tau_0)^\alpha ds,
\]

(36)

inequality (36) implies that

\[
||y(\tau)|| \leq ||y(0)|| Me^{\alpha (\tau - \tau_0)^\alpha} e_{\gamma}(\tau - \tau_0)^\alpha,
\]

(37)

thus, one can obtain

\[
\lim_{\tau \to +\infty} ||y(\tau)|| = 0.
\]

(38)

Therefore, the origin of system (14) is asymptotically stable, the proof is completed.

\begin{center}
\textbf{4. Numerical results}
\end{center}

In this section, two examples will be provided to demonstrate the effectiveness of the proposed results.

**Example 4.1:** Consider the following conformable fractional differential system:

\[
T_\alpha y(\tau) = Ay(\tau) + f(\tau, y(\tau)), y(\tau_0) = y_0,
\]

where \( 0 < \alpha \leq 1, y(\tau) = (y_1(\tau), y_2(\tau))^T, A = [-3, \frac{1}{2}], f(\tau, y(\tau)) = (\sin y_1(\tau), \sin y_2(\tau))^T \).

It is obvious that \( y(\tau) \) is Lipschitz continuous with \( L = 1 \), let \( P = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \epsilon = 1.1, A^T P + P A + \epsilon I = \begin{bmatrix} -1.9 & 1 \\ 1 & -0.9 \end{bmatrix} < 0, \lambda_{\max}(P) = \frac{1}{2}, \) and \( L < \frac{\epsilon}{\lambda_{\max}(P)} \). By using
Theorem 3.3, it is easy to obtain that the trivial solution of system (39) is fractional exponentially stable.

**Example 4.2:** Consider the following conformable fractional differential system:

\[
T_\alpha y(\tau) = Ay(\tau) + f(\tau, y(\tau)), y(\tau_0) = y_0, \quad (40)
\]

where \(0 < \alpha \leq 1\), \(A = \begin{bmatrix} -10 & 10 & 0 \\ -20 & 0 & 0 \\ 0 & -2.5 & 0 \end{bmatrix}\), \(f(\tau, y(\tau)) = \begin{bmatrix} -10y_1y_3 \\ 4y_1^2 \end{bmatrix}\).

\[
\lim_{y(\tau) \to 0} \frac{|f(\tau, y(\tau))|}{|y(\tau)|} = \lim_{y(\tau) \to 0} \frac{\sqrt{(-10y_1y_3)^2 + (4y_1^2)^2}}{\sqrt{y_1^2 + y_2^2 + y_3^2}} \leq \lim_{y(\tau) \to 0} \frac{\sqrt{(-10y_1y_3)^2 + (4y_1^2)^2}}{\sqrt{y_1^2}} = \lim_{y(\tau) \to 0} \sqrt{(-10y_3)^2 + (4y_1)^2} = 0. \quad (41)
\]

Obviously, \(\text{Re} \lambda(A) < 0\). Therefore, by Theorem 3.4, it is clear that the trivial solution of system (40) is asymptotically stable.

5. Conclusions

This paper investigates the problem of asymptotical stability of a class of conformable fractional system. By using the Grönwall’s Inequality and function analysis, we have proved asymptotical stability of solution for the conformable fractional system. Two examples are given to show the validity of the proposed method. In the future, we will consider the limit cycle of the conformable fractional systems.

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