On embedding well-separable graphs

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Abstract
Call a simple graph $H$ of order $n$ well-separable, if by deleting a separator set of size $o(n)$ the leftover will have components of size at most $o(n)$. We prove, that bounded degree well-separable spanning subgraphs are easy to embed: for every $\gamma > 0$ and positive integer $\Delta$ there exists an $n_0$ such that if $n > n_0$, $\Delta(H) \leq \Delta$ for a well-separable graph $H$ of order $n$ and $\delta(G) \geq (1 - \frac{1}{2\chi(H)-1}) + \gamma)n$ for a simple graph $G$ of order $n$, then $H \subset G$. We extend our result to graphs with small band-width, too.

1 Notation
In this paper we will consider only simple graphs. We mostly use standard graph theory notation: we denote by $V(G)$ and $E(G)$ the vertex and the edge set of the graph $G$, respectively. $deg_G(x)$ (or $deg(x)$) is the degree of the vertex $x \in V(G)$, $\delta(G)$ is the minimum degree and $\Delta(G)$ is the maximum degree. Denote $deg_G(v, A)$ the number of neighbors of $v$ in the set $A$. We write $N_G(x)$ (or $N(x)$) for the neighborhood of the vertex $x \in V(G)$, hence, $deg_G(x) = |N_G(x)|$. $N_G(U) = \bigcup_{x \in U} N(x)$ for a set $U \subset V(G)$. $N_G(v, A)$ is the set of neighbors of $v$ in $A$. Set $e(G) = |E(G)|$ and $v(G) = |V(G)|$.

If $A$ and $B$ are disjoint subsets of $V(G)$, then we denote by $e(A, B)$ the number of edges with one endpoint in $A$ and the other in $B$. We write $\chi(G)$ for the chromatic number of $G$. If $A$ is a subset of the vertices of $G$, we write $G - A$ for the graph induced by the vertices of $V(G) - A$.

If $G$ has a subgraph isomorphic to $H$, then we write $H \subset G$. In this case we sometimes call $G$ the host graph. We say that $G$ has an $H$–factor if there are $\lfloor \frac{v(G)}{v(H)} \rfloor$ vertex-disjoint copies of $H$ in $G$ (this notion is somewhat different from the common one: we don’t need that $v(G)$ is a multiple of $v(H)$). Throughout the paper we will apply the relation “$\ll$”: $a \ll b$ if $a$ is sufficiently smaller than $b$.

2 Introduction
In this paper we consider a problem in extremal graph theory. Before getting on the subject of our result let us take a short historical tour in the field.

One of the main results of the area is Turán’s Theorem:

*Part of this research was done during the author’s stay at Max-Planck-Institut für Informatik, Saarbrücken, Germany
†Partially supported by the IST Programme of the EU under contract number IST-1999-14186 (ALCOM-FT), and by OTKA T034475.
Theorem 1 (Turán 1941 [17]) If $G$ is a graph on $n$ vertices, and

$$e(G) > \left(1 - \frac{1}{r-1}\right) \frac{n^2}{2},$$

then $K_r \subset G$.

Another milestone in extremal graph theory is the following theorem:

Theorem 2 (Erdős–Stone–Simonovits 1946/1966 [7, 6]) For every graph $H$ and every real $\varepsilon > 0$ there exists an $N = N(H, \varepsilon)$ such that if $G$ is a graph on $n > N$ vertices, and

$$e(G) > \left(1 - \frac{1}{\chi(H) - 1} + \varepsilon\right) \frac{n^2}{2},$$

then $H \subset G$.

The deep result of Hajnal and Szemerédi shows that when we are looking for a $K_r$–factor in a graph, the situation is different.

Theorem 3 (Hajnal–Szemerédi 1969 [8]) If $G$ is a graph of order $n$ and $\delta(G) \geq \left(1 - \frac{1}{r}\right) n$, then $G$ has a $K_r$–factor.

There are two important changes in the formulation of the above result: first, it is not sufficient to bound the number of edges anymore – we need a lower bound on the minimum degree of the host graph. Second, that $1/(r-1)$ changed to $1/r$.

The following results were conjectured by Alon and Yuster [2, 3], and proved by Komlós, Sárközy and Szemerédi:

Theorem 4 (Komlós–Sárközy–Szemerédi 2001 [13]) Part 1: For every graph $H$ there is a constant $K$ such that if $G$ is a graph on $n$ vertices, then

$$\delta(G) > \left(1 - \frac{1}{\chi(H)}\right) n$$

implies that there is a union of vertex disjoint copies of $H$ covering all but at most $K$ vertices of $G$.

Part 2: For every graph $H$ there is a constant $K$ such that if $G$ is a graph on $n$ vertices, then

$$\delta(G) > \left(1 - \frac{1}{\chi(H)}\right) n + K$$

implies that $G$ has an $H$–factor.

These theorems show that the chromatic number is a crucial parameter in classical extremal graph theory. However, it is easy to come up with examples when the maximum degree turns out to be much more important. We give one possible set of examples for this fact. Let $\{H_d\}_{d \geq 2}$ be a family of random bipartite graphs with equal color classes of size $n/2$ that are obtained as the union of $d$ random 1–factors. Let $r$ be an odd positive integer, and consider the graph $G$ of order $n$ having $r$ independent sets of equal size, and all the edges between any two independent sets. By a standard application of the probabilistic method one can prove that for a given $r$ if $d$ is large enough ($d = \text{constant} \cdot r$ is sufficient), then $H_d \not\subset G$. Since $H_d$ is bipartite for every $d$, this proves, that the critical parameter for embedding expanders cannot be the chromatic number. (Although, the chromatic number still has a role, see [5].) One may think, that the main reason of this fact is that $H_d$ is an expander graph with large expansion rate.

We show, that if a graph is ”far from being an expander”, then again, the chromatic number comes into picture. First, let us define what we mean on ”non–expander” graphs.
Definition 1 Let $H$ be a graph of order $n$. We call $H$ well-separable if there is a subset $S \subset V(H)$ of size $o(n)$ such that all components of $H - S$ are of size $o(n)$.

We call $S$ the separator set, and write $C_1, C_2, \ldots, C_t$ for the components of $H - S$. Note, that if $H$ is an expander graph, then it is not well–separable. We will show the following property of well-separable graphs.

Theorem 5 For every $\gamma > 0$, positive integers $\Delta$ and $k$ there exists an $n_0$ such that if $n > n_0$, $\chi(H) \leq k$, $\Delta(H) \leq \Delta$ for a well-separable graph $H$ of order $n$ and $\delta(G) \geq (1 - \frac{1}{2(k-1)} + \gamma)n$ for a simple graph $G$ of order $n$, then $H \subset G$.

Observe, that trees are well–separable graphs. A conjecture of Bollobás [4] (proved by Komlós, Sárközy and Szemerédi [10]) states that trees of bounded degree can be embedded into graphs of minimum degree $(1/2 + \gamma)n$ for $\gamma > 0$. Since every tree is bipartite, this result is a special case of Theorem 5. (Recently Komlós, Sárközy and Szemerédi extended their result for trees of maximum degree as large as $c n \log n$ [14].)

Our proof of Theorem 5 uses the Regularity Lemma of Szemerédi [16] (sometimes called Uniformity Lemma). In the next section we will give a brief survey on this powerful tool, and related results. For more information see e.g., [15, 9]. We will prove Theorem 5 in the fourth section, and then prove a strengthened version of it, too. In the fifth section we will investigate the case of graphs with small band-width.

3 A review of tools for the proof

We introduce some more notation first. The density between disjoint sets $X$ and $Y$ is defined as:

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

We need the following definition to state the Regularity Lemma.

Definition 2 (Regularity condition) Let $\epsilon > 0$. A pair $(A, B)$ of disjoint vertex sets in $G$ is $\epsilon$-regular if for every $X \subset A$ and $Y \subset B$, satisfying $|X| > \epsilon|A|$, $|Y| > \epsilon|B|$ we have

$$|d(X, Y) - d(A, B)| < \epsilon.$$

We will employ the fact that if $(A, B)$ is an $\epsilon$-regular pair as above, and we place constant $\cdot \epsilon |A|$ new vertices into $A$, the resulting pair will remain $\epsilon'$-regular, with a somewhat larger $\epsilon'$ than $\epsilon$, depending on the constant.

An important property of regular pairs is the following:

Fact 6 Let $(A, B)$ be an $\epsilon$–regular pair with density $d$. Then for any $Y \subset B$, $|Y| > \epsilon|B|$, we have

$$|\{x \in A : \deg(x, Y) \leq (d - \epsilon)|Y|\}| \leq \epsilon|A|.$$

We will use the following form of the Regularity Lemma:

Lemma 7 (Degree Form) For every $\epsilon > 0$ there is an $M = M(\epsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set $V$ into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$, and there is a subgraph $G'$ of $G$ with the following properties:
\[ \ell \leq M, \]

\[ |V_0| \leq \varepsilon |V|, \]

- all clusters \( V_i, i \geq 1, \) are of the same size \( m \) (and therefore \( m \leq \left\lfloor \frac{|V|}{r} \right\rfloor < \varepsilon |V|), \)

- \( \text{deg}_{G'}(v) > \text{deg}_{G}(v) - (d + \varepsilon) |V| \) for all \( v \in V, \)

- \( V_i \) is an independent set in \( G' \) for all \( i \geq 1, \)

- all pairs \( (V_i, V_j), 1 \leq i < j \leq \ell, \) are \( \varepsilon \)-regular, each with density either \( 0 \) or at least \( d \) in \( G' \).

Often we call \( V_0 \) the exceptional cluster. In the rest of the paper we assume that \( 0 < \varepsilon \ll d \ll 1. \)

**Definition 3 (Reduced graph)** Apply Lemma 7 to the graph \( G = (V, E) \) with parameters \( \varepsilon \) and \( d, \) and denote the clusters of the resulting partition by \( V_0, V_1, \ldots, V_\ell, V_0 \) being the exceptional cluster. We construct a new graph \( G_r, \) the reduced graph of \( G' \) in the following way: The non-exceptional clusters of \( G' \) are the vertices of the reduced graph (hence \( |V(G_r)| = \ell \)). We connect two vertices of \( G_r \) by an edge if the corresponding two clusters form an \( \varepsilon \)-regular pair with density at least \( d \).

The following corollary is immediate:

**Corollary 8** Let \( G = (V, E) \) be a graph of order \( n \) and \( \delta(G) \geq cn \) for some \( c > 0, \) and let \( G_r \) be the reduced graph of \( G' \) after applying Lemma 7 with parameters \( \varepsilon \) and \( d. \) Then \( \delta(G_r) \geq (c - \theta)\ell, \) where \( \theta = 2\varepsilon + d. \)

A stronger one-sided property of regular pairs is super-regularity:

**Definition 4 (Super-Regularity condition)** Given a graph \( G \) and two disjoint subsets \( A \) and \( B \) of its vertices, the pair \( (A, B) \) is \((\varepsilon, \delta)\)-super-regular, if it is \( \varepsilon \)-regular and furthermore,

\[ \text{deg}(a) > \delta |B|, \text{ for all } a \in A, \]

and

\[ \text{deg}(b) > \delta |A|, \text{ for all } b \in B. \]

Finally, we formulate another important tool of the area:

**Theorem 9 (Blow-up Lemma [11, 12])** Given a graph \( R \) of order \( r \) and positive parameters \( \delta, \Delta, \) there exists a positive \( \varepsilon = \varepsilon(\delta, \Delta, r) \) such that the following holds: Let \( n_1, n_2, \ldots, n_r \) be arbitrary positive integers and let us replace the vertices \( v_1, v_2, \ldots, v_r \) of \( R \) with pairwise disjoint sets \( V_1, V_2, \ldots, V_r \) of sizes \( n_1, n_2, \ldots, n_r \) (blowing up). We construct two graphs on the same vertex set \( V = \bigcup V_i. \) The first graph \( F \) is obtained by replacing each edge \( \{v_i, v_j\} \) of \( R \) with the complete bipartite graph between \( V_i \) and \( V_j. \) A sparser graph \( G \) is constructed by replacing each edge \( \{v_i, v_j\} \) arbitrarily with an \((\varepsilon, \delta)\)-super-regular pair between \( V_i \) and \( V_j. \) If a graph \( H \) with \( \Delta(H) \leq \Delta \) is embeddable into \( F \) then it is already embeddable into \( G. \)

**Remark 1 (Strengthening the Blow-up Lemma [11])** Assume that \( n_i \leq 2n_j \) for every \( 1 \leq i, j \leq r. \) Then we can strengthen the lemma: Given \( c > 0 \) there are positive numbers \( \varepsilon = \varepsilon(\delta, \Delta, r, c) \) and \( \alpha = \alpha(\delta, \Delta, r, c) \) such that the Blow-up Lemma remains true if for every \( i \) there are certain vertices \( x \) to be embedded into \( V_i \) whose images are a priori restricted to certain sets \( T_x \subset V_i \) provided that

(i) each \( T_x \) within a \( V_i \) is of size at least \( c|V_i|, \)

(ii) the number of such restrictions within a \( V_i \) is not more than \( \alpha|V_i|. \)
4 Proof of Theorem 5

The proof goes along the following lines:

(1) Find a special structure in $G$ by the help of the Regularity Lemma and the Hajnal–Szemerédi Theorem (Theorem 3).

(2) Map the vertices of $H$ to clusters of $G$ in such a way that if $\{x, y\} \in E(H)$, then $x$ and $y$ are mapped to neighboring clusters; moreover, these clusters will form an $(\varepsilon, \delta)$–super–regular pair for all, but at most $o(n)$ edges.

(3) Finish the embedding by the help of the Blow-up Lemma.

4.1 Decomposition of $G$

In this subsection we will find a useful decomposition of $G$.

First, we apply the Degree Form of the Regularity Lemma with parameters $\varepsilon$ and $d$, where $0 < \varepsilon << d \ll \gamma < 1$. As a result, we have $\ell + 1$ clusters, $V_0, V_1, \ldots, V_\ell$, where $V_0$ is the exceptional cluster of size at most $\varepsilon n$, and all the others have the same size $m$. We deleted only a small number of edges, and now all the $(V_i, V_j)$ pairs are $\varepsilon$–regular, with density 0 or larger than $d$. By Corollary 8 we will have that $\delta(G_r) \geq (1 - 2\frac{1}{\ell + 1}) + \gamma'\ell$, where $\gamma' = \gamma - d - 2\varepsilon > 0$.

Applying Theorem 3, we have a $K_k$–factor in $G_r$. It is possible, that at most $k - 1$ clusters are left out from this $K_k$–factor -- such clusters are put into $V_0$. It is easy to transform the $\varepsilon$–regular pairs inside this $K_k$–factor into super–regular pairs: given a $\delta$ with $\varepsilon \ll \delta \ll d$ we have to discard at most $\varepsilon m$ vertices from a cluster to make a regular pair $(\varepsilon, \delta)$–super–regular. In a $k$–clique a cluster has $k - 1$ other adjacent clusters in $G_r$. Hence, it is enough to discard at most $(k - 1)\varepsilon m$ vertices from every cluster, and arrive to the desired result. Note, that now the pairs are $\varepsilon'$–regular, with $\varepsilon' < 2\varepsilon$: for simplicity, we will use the letter $\varepsilon$ in the rest of the paper. We will discard the same number of vertices from every non–exceptional cluster, and get, that all the edges of $G_r$ inside the cliques of the $K_k$–factor are $(\varepsilon, \delta)$–super–regular pairs. For simplicity we will still denote the common cluster size by $m$ in $G_r$. The discarded vertices are placed into $V_0$; now $|V_0| \leq (2k - 1)\varepsilon n$.

Our next goal is to distribute the vertices of $V_0$ among the non–exceptional clusters so as to preserve super–regularity within the cliques of the $K_k$–factor. We also require that the resulting clusters should have about the same size.

For a cluster $V_i$ in $G_r$ denote $clq(V_i)$ the set of the clusters of $V_i$’s clique in the $K_k$–factor, but without $V_i$ itself. Hence, $V_i \notin clq(V_i)$, and $|clq(V_i)| = k - 1$ for every $V_i \in V(G_r)$.

Recall, that every cluster in $G_r$ has the same size, $m$. We want to distribute the vertices of $V_0$ evenly among the clusters of $G_r$: we will achieve that $|V_i| - |V_j| < 4k\varepsilon m$ for every $1 \leq i, j \leq \ell$ after placing the vertices of $V_0$ to non–exceptional clusters. Besides, we require that if we put a vertex $v \in V_0$ into $V_i \in V(G_r)$, then $deg(v, V_j) \geq \delta m$ for every $V_j \in clq(V_i)$.

So as to satisfy the above requirement, let us define an auxiliary bipartite graph $F_1 = F_1(V_0, V(G_r), E(F_1))$. That is, the color classes of $F_1$ are $V_0$ and the set of the non–exceptional clusters. We draw a $\{v, V_i\}$ edge for $v \in V_0$ and $V_i \in V(G_r)$ if $deg_G(v, V_j) \geq \delta m$ for every $V_j \in clq(V_i)$.

Set $\gamma'' = k(\gamma - 2(\varepsilon + d))$. The following lemma is crucial in distributing $V_0$.

**Lemma 10** $deg_{F_1}(v) \geq (1/2 + \gamma'')\ell$ for every $v \in V_0$.

**Proof:** Consider an arbitrary vertex $v \in V_0$. Then we can partition the set of $k$–cliques of the $K_k$–factor into $k + 1$ pairwise disjoint sets $A_0, A_1, \ldots, A_k$. A clique $Q$ is in $A_j$ if $v$ has at least $\delta m$ neighbors in exactly $j$ clusters of $Q$. Set $a_j = k|A_j|/\ell$ for every $0 \leq j \leq k$, that is, $a_j$ is the proportion of cliques in $A_j$. Clearly, $\sum_j a_j = 1$. There are at most $\delta n$ edges connecting $v$ to clusters not adjacent to $v$ in $F_1$. Hence, by the minimum degree condition, $1/k\sum_j ja_j \geq \delta(G_r)/\ell - \delta$. Notice, that if there are at most $k - 2$ clusters in a clique in which $v$ has at least $\delta m$ neighbors, then $v$ is not adjacent to any
clusters of that clique in \( F_1 \). There are two possibilities left: \( v \) has one neighbor in a clique in \( F_1 \), or it is connected to all the clusters in \( F_1 \), depending on whether it has large enough degree to \( k - 1 \) or \( k \) clusters of that clique. Putting these together, the solution of the following linear program is a lower bound for \( \text{deg}_{F_1}(v)/\ell \):

\[
\sum_{j=0}^{k} a_j = 1 \quad \text{and} \quad \sum_{j=0}^{k} ja_j - z = k \left( \frac{2k-3}{2} + \gamma - 2(\varepsilon + d) \right)
\]

where \( a_j, z \geq 0 \)

\[
\min \{ \frac{a_{k-1}}{k} + a_k \}
\]

Let \( A \) be the coefficient matrix of the two equalities above, i.e.,

\[
A = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
0 & 1 & 2 & \ldots & k-1 & k & -1
\end{pmatrix}.
\]

Let \( a^T = (a_0, a_1, \ldots, a_k, z) \), \( b^T = (1, k(2k-3)/(2k-2) + \gamma'') \), and \( c^T = (0, 0, \ldots, 0, 1/k, 1, 0) \). Then the dual of the linear program above is:

\[
A^T u \leq c
\]

\[
\max \{ b^T u \}
\]

It is easy to check that \( u_1 = 2 - k \) and \( u_2 = \frac{k-1}{k} \) is a feasible solution (in fact the optimal solution as well), and therefore \( \max b^T u \geq 1/2 + \gamma'' \).

Applying the lemma above it is easy to distribute the vertices of \( V_0 \) evenly, without violating our requirement. For every \( v \in V_0 \) randomly choose a neighboring cluster in \( F_1 \), and put \( v \) into that cluster. Since \( \text{deg}_{F_1}(v) \geq (1/2 + \gamma'')\ell \), with very high probability (use e.g., Chernoff’s bound) no cluster will get more than \( 2|V_0|/\ell \) new vertices from \( V_0 \). Hence, we have that \( ||V_i| - |V_j|| < 4k\varepsilon m \) for every \( 1 \leq i, j \leq \ell \).

### 4.2 Assigning the vertices of \( H \)

In this subsection we will map the vertices of \( H \) to clusters of \( G_r \). We will heavily use the fact that \( H \) is \( k \)-colorable.

Fix an arbitrary \( k \)-coloration of \( H \). For an arbitrary set \( A \), denote \( A^1, A^2, \ldots, A^k \) the color classes determined by this \( k \)-coloration.

Recall, that \( S \) is the separator set of \( H \) and \( C_1, C_2, \ldots, C_t \) are the components of \( H - S \). We will map \( S \) and all the components in \( H - S \) by the randomized procedure below.

**Mapping algorithm**

**Input:** the set \( A \)

- Pick a clique \( Q = \{Q_1, Q_2, \ldots, Q_k\} \) in the cover of \( G_r \) randomly, uniformly.
- Pick a permutation \( \pi \) on \( \{1, 2, \ldots, k\} \) uniformly at random.
- Assign the vertices of \( A^i \) to the cluster \( Q_{\pi(i)} \) for every \( 1 \leq i \leq k \).

Repeating this algorithm for \( S \) and all the components in \( H - S \), we will have, that the number of vertices of \( H \) assigned to a cluster are almost the same: with probability tending to 1, the difference between the number of assigned vertices to a cluster and the cluster size \( m \) will be at most \( o(n) \). This follows easily from a standard application of Chebyshev’s inequality:
Lemma 11  With positive probability the mapping algorithm assigns \( n/\ell \pm \varepsilon m/\ell \) vertices of \( H \) to every cluster of \( G_r \).

Proof: Let \( V_i \) be an arbitrary cluster of \( G_r \). The above mapping algorithm is a randomized procedure, hence, the number of vertices of \( H \) assigned to \( V_i \) is a random variable. Denote this random variable by \( Z \). Let us define \( n \) indicator random variables \( \{Z_i\}_1^n \), where \( Z_i = 1 \) if and only if \( x_i \) (the \( i \)th vertex of \( H \)) is assigned to \( V_i \). By the Blow-up Lemma, it is necessary to map adjacent vertices in \( H \) to adjacent clusters in \( G_r \). For \( x \in V(H) \) let \( \kappa(x) \) denote the cluster to which \( x \) is assigned. After randomly assigning \( S \) and \( C_1, C_2, \ldots, C_t \), we have that if \( \{x, y\} \in E(H) \) and \( x, y \in S \) or \( x, y \in C_{j} \) for some \( 1 \leq j \leq t \), then \( \{\kappa(x), \kappa(y)\} \in E(G_r) \). On the other hand, there is no guarantee that a vertex in \( S \) and a vertex in some component of \( H - S \) are assigned to adjacent clusters, even when they are adjacent in \( H \).

Therefore, we have to reassign a small subset of \( V(H) \). We will see that no vertex which is at distance larger than \( k \) from \( S \) will change its place, and vertices of \( S \) will not be reassigned. Consider an arbitrary component \( C_j \). Set \( B = N(S) \cap C_j \), and \( B_p = B \cap C_j^p \) for every \( 1 \leq p \leq k \). By the algorithm below we will define \( B_k' \), the subset of \( C_j^k \) which will be reassigned.

1. **Step 1.** Set \( B_k' = B_k \), and \( i = 1 \)
2. **Step 2.** Set \( B_{k-i}' = B_{k-i} \cup \bigcup_{p=0}^{i-1} (N(B_{k-p}') \cap C_j^{k-i}) \)
3. **Step 3.** If \( i < k - 1 \), then set \( i \leftarrow i + 1 \), and go back to Step 2.

Informally, when we determine which vertices to reassign from \( C_{k-i} \), we take into account all the neighbors of \( B_p' \) with \( p > k - i \), and \( B_{k-i} \) itself. It is important, that we proceed backwards, that is, we specify the vertices to be reassigned starting from the last, the \( k \)th color class. Note, that the vertices of \( \bigcup_{p=1}^{k} B_p' \) are at distance at most \( k \) from \( S \). Hence, \( |\bigcup_{p=1}^{k} B_p'| < \Delta^k |S| = o(n) \).

Now we have the sets \( \{B_p\} \). First we will find a new cluster for \( B_1' \): Take an arbitrary cluster \( W_1 \) from the set
We have, that if \( \frac{4}{\gamma} \leq k \), then there are several different directed paths of length two from \( \gamma \) vertices in the common neighborhood of \( 2k-2 \) clusters, and this neighborhood is of size at least \( \gamma'\ell \) by the minimum degree condition of \( G \).

By the help of the above reassigning procedure we achieved, that adjacent vertices of \( H \) are assigned to adjacent clusters of \( G_r \). Let us denote the set of vertices of \( H \) assigned to cluster \( V_i \) by \( L_i \) for every \( 1 \leq i \leq \ell \). Our next goal is to make \( |L_i| = |V_i| \).

### 4.3 Achieving \( |V_i| = |L_i| \)

We have, that if \( \{x, y\} \in E(H) \), then \( \{\kappa(x), \kappa(y)\} \in E(G_r) \). Moreover, the \( \{\kappa(x), \kappa(y)\} \) edges are super–regular pairs for all, but at most \( o(n) \) edges in \( E(H) \).

Still, we cannot apply the Blow–up Lemma, since \( |V_i| = |L_i| \) is not necessarily true for every \( 1 \leq i \leq \ell \). What we know for sure is that \( |V_i| - |L_i| < 5\epsilon km \), because these differences were at most \( o(n) \) after the random mapping algorithm of the previous subsection, and distributing the vertices of \( V_0 \) had contribution at most \( 4\epsilon km \) for every \( 1 \leq i \leq \ell \) (we refer to Subsection 4.1), and we relocated \( o(n) \) vertices in the previous subsection.

We will partition the clusters of \( G_r \) into three disjoint sets: \( V_\prec, V_\succ \) and \( V_\sim \). If \( |V_i| < |L_i| \), then \( V_i \in V_\prec \); if \( |V_i| = |L_i| \), then \( V_i \in V_\sim \), and we put \( V_p \) into \( V_\succ \) if \( |V_p| > |L_p| \). Clearly, it is enough to replace at most \( 5\epsilon km \) vertices of \( G \) so as to achieve \( |V_i| = |L_i| \) for every \( 1 \leq i \leq \ell \), while preserving regularity for the edges of \( G_r \). But we need super–regular pairs for the edges of the \( k \)-cliques of the \( K_k \)-factor, hence, a straightforward relocation of some vertices of \( G \) is not helpful. Instead, we will apply an idea similar to what we used for distributing the vertices of \( V_0 \).

First, we define a directed graph \( F_2 \): the vertices of \( F_2 \) are the clusters of \( G_r \), and \( (V_i, V_j) \in E(F_2) \) if \( (V_i, V_p) \in E(G_r) \) for every \( V_p \in clq(V_j) \). We will have that the out–degree of every cluster is at least \((1/2 + \gamma''\ell)\ell \) by considering the linear program of Subsection 4.1. Since \( \delta(G_r) \geq \left(\frac{2k-1}{2m} + \gamma\right)\ell \), it is easy to see that any \( k-1 \) clusters have at least \((1/2 + \gamma\ell)\ell \) common neighbors. That is, the in–degree of \( F_2 \) is at least \((1/2 + \gamma'\ell)\ell \). Therefore, there is a large number – at least \((\gamma' + \gamma'\ell)\ell \) – of directed paths of length at most two between any two clusters in \( F_2 \).

Let \( V_i \in V_\prec \) and \( V_j \in V_\succ \) be arbitrary clusters. If \( (V_j, V_i) \in E(F_2) \), then we can directly place a vertex from \( V_j \) into \( V_i \) which has at least \( d\ell \) neighbors in \( V_i \) for every \( V_\ell \in clq(V_i) \) (and most of the vertices have actually at least \( d\ell \) neighbors, since \( d \) is the lower bound for the density of regular pairs). If there is no such edge, then there are several different directed paths of length two from \( V_j \) to \( V_i \). These paths differ in their "center" cluster. Assume that \( V_p \) is such a cluster, i.e., \( (V_j, V_p) \) and
(V_p, V_i) are edges in F_2. It is useful to choose V_p randomly, uniformly among the possible "center" clusters.

Take any vertex v ∈ V_j which has at least 8m neighbors in V_s for every V_s ∈ clq(V_p), and put it into V_p. Then choose any vertex from V_p which has at least 8m neighbors in V_i for every V_i ∈ clq(V_i), and put it into V_i. As a result, we decreased ||V_j| − |L_j|| and ||V_i| − |L_i||, while ||V_p| − |L_p|| did not change. Now, by the remark after the definition of a regular pair it is clear that if we make all |V_i| = |L_i| this way, we will preserve regularity and super-regularity as well.

4.4 Finishing the proof

Now we are prepared to prove Theorem 5.

We have to check if the conditions of the Blow-up Lemma are satisfied. There are o(n) edges of E(H) which are problematic: those edges having their endpoints in clusters which do not constitute a super-regular pair. Denote the set of these edges by E'. Suppose that z is a vertex which occurs in some edges of E'. It can have neighbors assigned to at most 2k − 2 clusters V_{x_1}, V_{x_2}, . . . , V_{x_{2k−2}}. Since (κ(x), V_{x_1}) is a regular pair for every 1 ≤ i ≤ 2k − 2, there is a set T_x ⊂ κ(x) of size at least (1 − (2k−2)ε)m (by Fact 6 and applying induction), all the vertices of which have at least (d − ε)2k−2m > 8m neighbors in V_{x_i} for every 1 ≤ i ≤ 2k − 2. T_x will be the set to which x is restricted. Since |E'| = o(n), the number of restricted vertices is small enough, and therefore we can apply the strengthened version of the Blow-up Lemma. 

4.5 Strengthening Theorem 5

We begin this subsection with a definition.

Definition 5 Let 0 < α < 1. We call a graph H on n vertices α-separable, if there is a set S ⊂ V(H) of size at most an such that all components of H − S are of size at most αn.

Obviously, given some 0 < α < 1 if H is well-separable and |V(H)| is large enough, then H is α-separable as well. On the other hand, if α is small enough, then we can substitute well-separability by α-separability:

Theorem 12 For every γ > 0, positive integers ∆ and k there exists an n_0 and an α such that if n > n_0, χ(H) ≤ k, ∆(H) ≤ ∆ for an α-separable graph H of order n and δ(G) ≥ (1 − 1/(2k−1)) + γ)n for a simple graph G of order n, then H ⊂ G.

Proof (sketch): We will apply the same method for embedding α-separable graphs. First, we decompose G by the help of the Regularity Lemma and the Hajnal–Szemerédi Theorem. Then distribute the vertices of H among the clusters of G_r, finally, apply the Blow-up Lemma for finishing the embedding. Since S and the components of H − S can be much larger now, we have to be careful at certain points. We will pay attention only to these points.

Given γ, ∆ and k, we can determine α: Proving Lemma 11 for α-separable graphs we will have that Var(Z) ≤ nVar(Z_1) + αn^2, hence, D(Z) ≤ √2αn. Set λ = √2ℓ, and choose α so that

$$\Pr(|Z − n/ℓ| ≥ εm) ≤ \frac{1}{\lambda^2} = \frac{1}{2ε^2}. \quad (1)$$

It is easy to check that if α ≤ ε^2/(4n^3) then λD(Z) ≤ εm and inequality (1) is satisfied.

After the random mapping algorithm we have to reassign some vertices so as to get that adjacent vertices of H are assigned to adjacent clusters of G_r. At this point we may reassign as many as ∆^k|S| ≤ ∆^kαn vertices of H. Our second criteria for α is that ∆^kαn should be less than εm. Other parts of the proof work smoothly not just for well-separable but for α-separable graphs as well.
Therefore, if
\[ \alpha \leq \max\left\{ \frac{\varepsilon^2}{4\ell^3}, \frac{\varepsilon}{7\Delta} \right\} = \frac{\varepsilon^2}{4\ell^3}, \]
then we can embed \( H \) into \( G \). \qed

5 On graphs with small band-width

Another notion, which measures the "non-expansion" of graphs is band-width. Let us denote the band-width of a graph \( G \) by \( bw(G) \). Notice, that there are well–separable graphs with large band-width: consider \( K_{1,n-1} \), the star on \( n \) vertices. Obviously, it is a well–separable graph, on the other hand its band-width is \( n/2 \).

The following is conjectured by Bollobás and Komlós (see e.g., in [15]):

**Conjecture 13 (Bollobás-Komlós)** For every \( \gamma > 0 \) and positive integers \( r \) and \( \Delta \), there is a \( \beta > 0 \) and an \( n_0 \) such that if \( |V(H)| = |V(G)| = n \geq n_0, \chi(H) \leq r, \Delta(H) \leq \Delta, bw(H) < \beta n \) and \( \delta(G) \geq (1 - \frac{1}{r} + \gamma)n \), then \( H \subset G \).

The special case when \( H \) is bipartite was shown by Abbasi [1]. We will give an alternative proof of this by showing that if the band-width is small enough, then the graph is \( \alpha \)-separable for a small enough \( \alpha \).

**Lemma 14** Let \( 0 < \beta < 1 \), and assume that \( H \) is a graph of order \( n \) with \( bw(H) \leq \beta n \). Then \( H \) is a \( \sqrt{\beta} \)-separable graph.

**Proof:** We can decompose \( H \) in the following way: Consider an ordering of the vertices of \( H \) in which no edge connects two vertices which are farther away from each other than \( \beta n \). Divide the ordering into \( m = 1/\beta \) intervals. For simplicity we assume, that \( m \) and \( \sqrt{m} \) are integers and \( n \) is divisible by \( m \). The \( i \)th interval, \( I_i \) will contain the vertices of order \((i-1)\beta n + 1, \ldots, i\beta n \).

We let
\[ S = \bigcup_{i=1}^{\sqrt{m}} I_{i\sqrt{m}}, \]
and for \( 0 \leq j \leq \sqrt{m} - 1 \), set
\[ C_j = \bigcup_{i=1}^{\sqrt{m}-1} I_{j\sqrt{m}+i}. \]

Clearly, \( S \) consists of \( \sqrt{m} \) intervals, each of length \( n/m \), thus \( |S| \leq n/\sqrt{m} \). If \( x \in C_j \) and \( y \in C_k \) for \( j \neq k \), then \((x,y) \notin E(H)\) because \( bw(H) \leq \beta n \). Hence, we have found a simple decomposition of \( H \) which proves that \( H \) is \( \sqrt{\beta} \)-separable. \qed

Unfortunately, if \( \chi(H) \geq 3 \), then our result does not imply Conjecture 13.

References

[1] S. Abbasi (1998), Spanning subgraphs of dense graphs, Ph.D. theses, Department of Computer Science, Rutgers, the State University of New Jersey.

[2] N. Alon and R. Yuster (1992), Almost \( H \)-factors in dense graphs, Graphs and Combinatorics, 8, 95–102.
[3] N. Alon and R. Yuster (1996), $H$-factors in dense graphs, Journal of Combinatorial Theory, Series B, 66, 269–282.

[4] B. Bollobás (1978), Extremal graph theory, Academic Press, London.

[5] B. Csaba (2003), On the Bollobás–Eldridge conjecture for bipartite graphs, submitted for publication.

[6] P. Erdős and M. Simonovits (1966), A limit theorem in graph theory, Studia Sci. Math. Hungar., 1, 51–57.

[7] P. Erdős and A. H. Stone (1946), On the structure of linear graphs, Bulletin of the American Mathematical Society, 52, 1089–1091.

[8] A. Hajnal and E. Szemerédi (1970) Proof of a Conjecture of Erdős, in “Combinatorial Theory and Its Applications, II” (P. Erdős, and V. T. Sós, Eds.), Colloquia Mathematica Societatis János Bolyai, North-Holland, Amsterdam/London.

[9] J. Komlós (1999), The Blow-up Lemma (survey), Combinatorics, Probability and Computing, 8, 161–176.

[10] J. Komlós, G.N. Sárközy and E. Szemerédi (1995), Proof of a packing conjecture of Bollobás, Combinatorics, Probability and Computing, 4, 241–255.

[11] J. Komlós, G.N. Sárközy and E. Szemerédi (1997) Blow-up Lemma, Combinatorica, 17, 109-123.

[12] J. Komlós, G.N. Sárközy and E. Szemerédi (1998), An Algorithmic Version of the Blow-up Lemma, Random Structures and Algorithms, 12, 297-312.

[13] J. Komlós, G.N. Sárközy and E. Szemerédi (2001), Proof of the Alon–Yuster conjecture, Discrete Mathematics, 255–269.

[14] J. Komlós, G.N. Sárközy and E. Szemerédi (2001), Spanning trees in dense graphs, Combinatorics, Probability and Computing, 397–416.

[15] J. Komlós and M. Simonovits (1993), Szemerédi’s Regularity Lemma and its Applications in Graph Theory (survey), Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), 295–352.

[16] E. Szemerédi (1976), Regular Partitions of Graphs, Colloques Internationaux C.N.R.S N° 260 - Problèmes Combinatoires et Théorie des Graphes, Orsay, 399-401.

[17] P. Turán (1941), On an extremal problem in graph theory (in Hungarian), Matematikai és Fizikai Lapok, 48, 436–452.