A coprime action version of a solubility criterion of Deskins

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Abstract
Let \( A \) and \( G \) be finite groups of relatively prime orders and suppose that \( A \) acts on \( G \) via automorphisms. We demonstrate that if \( G \) has a maximal \( A \)-invariant subgroup \( M \) that is nilpotent and the Sylow 2-subgroup of \( M \) has class at most 2, then \( G \) is soluble. This result extends, in the context of coprime action, a solubility criterion given by W.E. Deskins.

Keywords Soluble groups · Maximal subgroups · Coprime action · Group action on groups

Mathematics Subject Classification 20D20 · 20D15

1 Introduction
In [2], in the course of a study of the lattice of subinvariant subgroups in a finite group, W.E. Deskins provided an interesting solubility criterion concerning maximal subgroups: When a finite group \( G \) contains a maximal subgroup \( M \) that is nilpotent of class less than 3, then \( G \) is soluble. This result is similar to a theorem of B. Huppert, which originally appeared in [6], except in the case in which \( M \) has a Sylow 2-subgroup of class 2. The criterion of Deskins was also in line with a theorem announced by Thompson: If a finite group \( G \) has a maximal subgroup that is nilpotent of odd order, then \( G \) is soluble. The crucial tool of Deskins’s work, which allowed him to extend the nilpotence class to 2 instead of 1 (abelian), was the First Theorem of Grün (see
for instance [5, IV.3.4]), which is an application of the transfer theory into a Sylow subgroup. Precisely, Grün’s Theorem was used so as to obtain the existence of a normal complement to the maximal subgroup $M$.

In this paper we study such results in the context in which a finite group $A$ with $(|A|, |G|) = 1$ acts on $G$. We ask whether the existence of a maximal $A$-invariant subgroup in $G$ (which needs not be a maximal subgroup) satisfying the same conditions as in Deskins’s theorem must imply the solubility of $G$. We give an affirmative answer.

**Theorem** Let $G$ and $A$ be finite groups of coprime orders and assume that $A$ acts on $G$ by automorphisms. If $G$ has a maximal $A$-invariant subgroup that is nilpotent with a Sylow 2-subgroup of class less than 3, then $G$ is soluble.

At first sight, there seems only to be a subtle difference from Deskins’s theorem, but there exists a great distinction between our development and Deskins’s approach. It is not possible to use Grün’s Theorem in the setting of a coprime action, and instead, we appeal to the Classification of the Finite Simple Groups. We point out that the authors have already obtained a coprime action version of the Thompson’s aforementioned result [1, Theorem B]. This is not done by employing the Classification, but by transferring into the setting of coprime action results like the Glauberman–Thompson criterion for $p$-nilpotence. In fact, this result will be used in the proof of our theorem.

We denote by $\pi(G)$ the set of primes dividing the order of a group $G$. The rest of the notation is standard and all groups are supposed to be finite.

2 Preliminaries

We start with an elementary observation that is needed for the inductive arguments.

**Lemma 2.1** Let $P$ be a finite $p$-group of class 2. If $A \trianglelefteq P$, then the class of $A$ and $P/A$ is less than or equal to 2.

We require the following theorem of Wielandt.

**Theorem 2.2** (IV.7.3, [5]) Let $H$ be a Hall $\pi$-subgroup of a group $G$ which is not a Sylow subgroup of $G$. Suppose that for every $p \in \pi$ and for every Sylow $p$-subgroup $H_p$ of $H$, we have $N_G(H_p) = H$. Then $H$ has a normal $\pi$-complement in $G$.

We also recall the Thompson subgroup. If $p$ is prime and $P$ is a $p$-group, the Thompson subgroup $J(P)$ is the subgroup generated by all abelian subgroups of $P$ of maximal order. It is immediate that $J(P)$ and $Z(J(P))$ are characteristic in $P$, and hence, these subgroups are left invariant by every automorphism acting on $P$, so in particular, by every group acting coprimely on $P$. As we said in the Introduction, in order to prove our result we need to use the celebrated Glauberman–Thompson $p$-nilpotence criterion.

**Theorem 2.3** (Theorem 8.3.1, [4]) Let $P$ be a Sylow $p$-subgroup of a finite group $G$, where $p$ is an odd prime. If $N_G(Z(J(P)))$ is $p$-nilpotent, then $G$ is $p$-nilpotent.
As mentioned in the Introduction, we appeal to the Classification of the Finite Simple Groups. Precisely, we need to determine all non-abelian simple finite groups whose Sylow 2-subgroups are self-normalising as well as all those simple groups whose Sylow 2-subgroups have nilpotence class at most 2. Such groups have been classified by Kondrat’ev [7] and by Gilman and Gorenstein [3], respectively, so we can gather the list of those simple groups satisfying both conditions in the next result.

**Theorem 2.4** Let \( G \) be a finite non-abelian simple group and \( P \) a Sylow 2-subgroup of \( G \). If \( N_G(P) = P \) and \( P \) has class at most 2, then \( G \cong \text{PSL}(2, q) \), where \( q \equiv 7, 9 \pmod{16} \).

**Proof** This is a consequence of combining the main result of [7] and Theorems 7.1 and 7.4 of [4]. \( \square \)

We will also need to know the structure of the Sylow normalisers in \( \text{PSL}(2, q) \), especially for odd primes.

**Lemma 2.5** Let \( G = \text{PSL}(2, q) \), where \( q \) is a power of prime \( p \) and \( d = (2, q + 1) \). Let \( r \in \pi(G) \) and \( R \in \text{Syl}_r(G) \).

(1) If \( r = p \), then \( N_G(R) = R \rtimes C_{q-1} \) is a dihedral group;
(2) If \( 2 \neq r \mid \frac{q+1}{d} \), then \( N_G(R) = C_{\frac{q+1}{d}} \rtimes C_2 \);
(3) If \( 2 \neq r \mid \frac{q-1}{d} \), then \( N_G(R) = C_{\frac{q-1}{d}} \rtimes C_2 \);
(4) Assume \( p \neq r = 2 \).
   (4.1) If \( q \equiv \pm 1 (\pmod{8}) \), then \( N_G(R) = R \);
   (4.2) If \( q \equiv \pm 3 (\pmod{8}) \), then \( N_G(R) = (C_2 \times C_2) \rtimes C_3 \).

**Proof** This follows from [5, Theorem 2.8.27]. \( \square \)

### 3 Proof of the Theorem

**Proof** We study a minimal counter-example. Suppose then that \( G \) is a minimal counter-example to the theorem and let \( M \) be the nilpotent maximal \( A \)-invariant subgroup of \( G \) with a Sylow 2-subgroup of class less than 3. We divide the proof into the following steps.

**Step 1** We can assume that \( M \) is a Hall subgroup of \( G \) and that \( M \) does not contain any \( A \)-invariant normal subgroup of \( G \).

If \( M \) contains a non-trivial \( A \)-invariant normal subgroup \( N \) of \( G \), then by taking into account Lemma 2.1, \( G/N \) satisfies the hypotheses of the theorem, so \( G/N \) is soluble by minimality, and consequently, \( G \) is soluble for \( N \) being nilpotent. Henceforth, it can be assumed \( M \) does not contain any \( A \)-invariant normal subgroup of \( G \).

Suppose that there exists a prime \( p \in \pi(M) \) such that the Sylow \( p \)-subgroup of \( M \) is not a Sylow \( p \)-subgroup of \( G \). Then by elementary coprime action properties there exists an \( A \)-invariant Sylow \( p \)-subgroup \( G_p \) of \( G \) and an \( A \)-invariant Sylow \( p \)-subgroup \( M_p \) of \( M \) with \( M_p < G_p \). Since \( M \) is nilpotent, we have \( M < N_G(M_p) \). \( \square \)
Also, \( N_G(M_p) \) is \( A \)-invariant. By the maximality of \( M \) we get \( N_G(M_p) = G \), that is \( M_p \leq G \), a contradiction with the above paragraph. This shows that \( M_p = G_p \), or equivalently, \( M \) is a Hall subgroup of \( G \).

**Step 2** We can assume that \( M \) is a Sylow 2-subgroup of \( G \).

Suppose that \( M \) is not a Sylow subgroup of \( G \). For every prime \( p \in \pi(M) \) we take \( P \) an \( A \)-invariant Sylow \( p \)-subgroup of \( M \). Then \( M \leq N_G(P) \) and by maximality of \( M \) and Step 1, it follows that \( N_G(P) = M \). Thus, we can apply Theorem 2.2, so there exists a normal complement \( K \) of \( M \) in \( G \). Clearly, \( K \) is \( A \)-invariant. Now let us consider the action of \( MA \) on \( K \). Since the orders of \( MA \) and \( K \) are coprime, we get that \( K \) has a \( MA \)-invariant Sylow \( q \)-subgroup \( Q \). Therefore \( MQ \leq G \) is \( A \)-invariant, and by the maximality of \( M \), we have \( G = MQ \). However, \( Q \) and \( G/Q \) are soluble, so we deduce that \( G \) is soluble as well, a contradiction. This shows that \( M \) is a Sylow \( p \)-subgroup of \( G \) for some prime \( p \).

Next we prove that \( p = 2 \). Assume that \( p \neq 2 \). Let \( J = J(M) \), the Thompson’s subgroup of \( M \), and \( Z = Z(J) \). Note that \( Z \) and \( N_G(Z) \) are \( A \)-invariant by the observation made before Theorem 2.3. Since by Step 1, \( Z \) is not normal in \( G \), we have \( M \leq N_G(Z) < G \). By the maximality of \( M \), we get \( M = N_G(Z) \), so in particular it is a \( p \)-subgroup. Then \( G \) is \( p \)-nilpotent by Theorem 2.3, that is, \( G \) has a normal \( p \)-complement, \( L \), which is obviously \( A \)-invariant too. This means that \( G = ML \) with \( M \cap L = 1 \). The rest of the proof of this step consists in proving that \( L \) is a \( q \)-group for some prime \( q \). Indeed, take \( Q \) an \( A \)-invariant Sylow \( q \)-subgroup of \( L \) for some prime \( q \). The Frattini argument gives \( G = N_G(Q)L \). Now, the Schur–Zassenhaus Theorem assures that \( N_L(Q) \) has complements in \( N_G(Q) \) that are conjugate in \( N_G(Q) \). Since \( A \) acts on the set of complements, Glauberman’s Lemma (for instance [8, Theorem 6.2.2]) implies that there exists an \( A \)-invariant complement \( X \) of \( N_L(Q) \) in \( N_G(Q) \). As a result, \( G = XN_L(Q)L = XL \), so \( X \) is an \( A \)-invariant complement of \( L \) in \( G \). Again by Glauberman’s Lemma, we know that the \( A \)-invariant complements of \( L \) are conjugate in the fixed point subgroup \( C_G(A) \), so in particular, \( X = M^c \) for some \( c \in C_G(A) \). We conclude that \( X \) is a maximal \( A \)-invariant subgroup of \( G \). However, \( X \) normalizes \( Q \) and by maximality of \( X \), we get \( G = XQ \). This forces \( L = Q \), as wanted. As a consequence, \( G \) is soluble by Burnside \( p^aq^b \) Theorem, a contradiction. Hence \( p = 2 \) and \( M \) is a Sylow 2-subgroup of \( G \).

**Step 3** We can assume that \( M \) has nilpotence class 2.

Suppose on the contrary, that the class of \( M \) is not 2, so by hypothesis \( M \) is abelian. As \( N_G(M) = M \) by the maximality of \( M \), we have \( M \leq Z(N_G(M)) \). We can apply then Burnside normal \( p \)-complement Theorem for \( p = 2 \) (for instance [5, 2.2.6]), and we conclude that \( G \) has a normal 2-complement. Now Feit–Thompson Theorem implies that \( G \) is soluble, a contradiction.

**Step 4** Final contradiction.

Let \( N \) be a minimal \( A \)-invariant normal subgroup of \( G \). We can assume that \( N \) is not soluble; otherwise by Step 1, \( N \) is not contained in \( M \), and by maximality we obtain \( NM = G \). As a consequence, \( G \) would be soluble and the proof is finished. Therefore, we can write \( N = S_1 \times \cdots \times S_n \) where \( S_i \) are isomorphic non-abelian simple groups (possibly \( n = 1 \)). Put \( S = S_1, B = N_A(S) \) and let \( T \) be a
transversal of $B$ in $A$. Now, as $M$ is self-normalising in $G$ for being maximal, then $M \cap S$ is self-normalising in $S$ and it has class at most 2 by Lemma 2.1. Then by applying Theorem 2.4, we obtain $S \cong \text{PSL}(2, q)$ with $q \equiv 7, 9 \pmod{16}$. We distinguish separately these two cases. If $q \equiv 9 \pmod{16}$, with $q > 9$, then we can certainly choose an odd prime $r \mid (q - 1)/2$ and $R$ to be a $B$-invariant Sylow $r$-subgroup of $S$. By Lemma 2.5(3), we know that $|N_S(R)| = q + 1$, so $N_S(R)$ has odd index in $S$ and contains properly a Sylow 2-subgroup of $S$. Analogously, if $q \equiv 7 \pmod{16}$, with $q > 7$, there exists an odd prime $r \mid (q + 1)/2$ and we take $R$ to be a $B$-invariant Sylow $r$-subgroup of $S$. Again by Lemma 2.5(2), we know that $|N_S(R)| = (q - 1)$, so $N_S(R)$ has odd index in $S$ and hence, it contains properly a Sylow 2-subgroup of $S$. In both cases, we put $R_0 = \prod_{t \in T} R^t$, which is an $A$-invariant Sylow $r$-subgroup of $N$ because $A$ acts transitively on the $S_t$. We deduce that $|N : N_N(R_0)| = |S : N_S(R)|^n$ is odd too. Now, by the Frattini argument, $G = NN_G(R_0)$ and thus, $|G : N_G(R_0)| = |N : N_N(R_0)|$. We conclude that $N_G(R_0)$ properly contains an $A$-invariant Sylow 2-subgroup of $G$, contradicting the maximality of $M$.

Finally, suppose that $S \cong \text{PSL}(2, 9)$ or $\text{PSL}(2, 7)$. Both groups contain $\{2, 3\}$-Hall subgroups, which are isomorphic to the symmetric group $S_4$. We remark that these subgroups are not all conjugate in $S$. If this were the case, then Glauberman’s Lemma would provide an $A$-invariant $\{2, 3\}$-subgroup, against the maximality of $M$. But this is not the case and we give the following alternative argument. The Sylow 2-subgroups of $S$ are dihedral groups of order 8. Now, $M \cap N$ is an $A$-invariant Sylow 2-subgroup of $N$, which is the direct product of $n$ copies of such a $B$-invariant dihedral group, say $D$, of $S$. Let $K$ be the cyclic group of order 4 of $D$, which is also $B$-invariant for being characteristic, and let $K_0 = \prod_{t \in T} K^t$. It is easily seen that $K_0$ is $A$-invariant because $A$ is acting transitively on the factors. Moreover, since $K$ is characteristic in $D$, then $K_0$ is characteristic in $M \cap N$, so $K_0 \trianglelefteq M$, that is, $M \leq N_G(K_0)$. On the other hand, in both cases $S \cong \text{PSL}(2, 9)$ or $\text{PSL}(2, 7)$, we have that $K$ is normalised by an element of order 3 lying in $S$, so the same occurs with $K_0$ and $N$. We conclude that $N_G(K_0)$ is an $A$-invariant subgroup that contains properly $M$. Again this contradicts the maximality of $M$. 

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