Logarithmic Correction to Newton Potential in
Randall-Sundrum Scenario

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Abstract

Using a fixed-energy amplitude in Randall-Sundrum single brane scenario, we compute the Newton potential on the brane. It is shown that the correction terms to the Newton potential involve a logarithmic factor. Especially, when the distance between two point masses are very small compared to $AdS$ radius, the contribution of KK spectrum becomes dominant compared to the usual inversely square law. This fact may be used to prove the existence of an extra dimension experimentally.

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Recently, much attention is paid to the Randall-Sundrum (RS) brane-world scenario [1,2]. Especially, RS solved the linearized gravitational fluctuation equation and used the result to compute the Newton potential on the 3-brane

\[ V_{N,RS} = G_N \frac{m_1 m_2}{r} \left(1 + \frac{R^2}{r^2}\right) \]

where \( R \) is a radius of \( AdS_5 \). The first term in r.h.s. of Eq.(1) is an usual gravitational potential contributed from zero mode of the gravitational fluctuation equation. The next term is a correction to the Newton potential contributed from the continuum KK spectrum.

The slightly different correction to the Newton potential is again derived in Ref. [3,4] with consideration of the bending effect of the 3-brane and in Ref. [5] by computing the one-loop corrections to the gravitational propagator. In both cases the final Newton potential is different from Eq.(1) by a constant factor as follows;

\[ V_{N,GT} = G_N \frac{m_1 m_2}{r} \left(1 + \frac{2}{3} \frac{R^2}{r^2}\right). \]

In this short letter we argue that the correction to the Newton potential on the brane involves a logarithmic factor. Especially, for the short-range gravity this factor is dominant compared to the usual inversely square law. This means the contribution of KK continuum becomes significant in this limit.

Our starting point is a fixed-energy amplitude derived in Ref. [6,7]:

\[ \hat{G}[R, R : E] = R(\Delta_0 + \Delta_{KK}) \]

where

\[ \Delta_0 = \frac{2}{m^2 R^2} \]
\[ \Delta_{KK} = \frac{1}{mR} \frac{K_0(mR)}{K_1(mR)} \]

and \( K_\mu(z) \) is an usual modified Bessel function, and \( m \) is a mass parameter introduced from the linearized fluctuation equation. It is worthwhile explaining briefly how Eq.(3) is derived at this stage. Firstly, we treated the gravitational fluctuation equation as a pure quantum-mechanical Schrödinger equation and transformed the singular nature of the 3-brane into the
singular behavior of Hamiltonian operator in quantum mechanics. Assuming that the extra dimension is a single copy of $AdS_5$, i.e. $AdS/CFT$ setting, we have shown in Ref. [6,7] that the fixed-energy amplitude, which is a Laplace transform of the gravitational propagator in Euclidean time, is crucially dependent on the boundary condition (BC) at the location of brane, which is parametrized by a single real parameter $\xi$. Setting $\xi = 1/2$, which means the Dirichlet and Neumann BCs are included with equal weight, one can derive Eq.(3).

In this letter we would like to derive a Newton potential localized on the 3-brane from Eq.(3). From Ref. [8] the Newton potential on the brane is approximately

$$\tilde{V}(r) \sim G_N \frac{m_1 m_2}{r} \left[ 1 + \frac{2}{3} \int_{m_0}^{\infty} d m R e^{-m r} |\psi_m(R)|^2 \right], \tag{5}$$

where the first term of Eq.(3) is contributed from $\Delta_0$ in Eq.(4) and next term is contributed from the continuum KK states. Thus, the problem reduces to compute $\psi_m(R)$ from $R \Delta_{KK}$.

Since, in general, the fixed energy amplitude is represented as

$$\hat{G}[x, y; E] = \sum_n \frac{\phi_n(x) \phi_n^*(y)}{m_n^2 + E_n}, \tag{6}$$

for the discrete states, one can derive an eigenstate $\phi_n(x)$ by computing the residue of $\hat{G}[x, y; E]$. For the continuum states, however, we need an appropriate integral representation. For example, let us consider the simple free particle case whose fixed-energy amplitude is

$$\hat{G}_F[x, y; E] = e^{-m|x-y|/m} \tag{7}$$

where $E = m^2/2$. Hence, the problem is how to convert Eq.(4) into the continuum version of Eq.(3). Thus we want to find an integral representation of $\hat{G}_F[x, y; E]$ as follows;

$$\hat{G}_F[x, y; E] = \int dk \frac{\phi_k(x) \phi_k^*(y)}{m_k^2 + k^2}. \tag{8}$$

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1 When Eq.(3) is derived in Ref. [3,4], we have used a translation of the extra dimension as $z = y + R$. Thus, the location of the 3-brane in this new coordinate is $z = R$. That is why we consider $\psi_m(R)$ in Eq.(5).
After assuming that \( \phi_k(x) = \mathcal{N} e^{ikx} \), one can perform the integration \( \mathcal{N} \) in the complex \( k \)-plane and finally find \( |\phi_m(0)|^2 = 1/2\pi \) which makes the normalization constant \( \mathcal{N} \) of the continuum state to be \( \mathcal{N} = 1/\sqrt{2\pi} \). This normalization factor corresponds to a classical picture in which a particle starts from a particular point with all momenta equally likely \([9]\). Note that the normalization constant \( \mathcal{N} \) is independent of the energy parameter \( m \) in the free particle case. However, it is not a generic property of quantum mechanics, \( i.e. \) the normalization is dependent on \( m \) for the general case as will be shown in RS2 case.

In the real calculation for the correction to the Newton potential we do not need an explicit form of the continuum wave function. In fact, what we need is a value of the continuum wave function at a particular point. Thus, for the computation of the Newton potential it is sufficient to know the fixed-energy amplitude at the location of the 3-brane.

Applying the same procedure to \( R\Delta_{KK} \) it is straightforward to derive \( \psi_m(R) \) as follows

\[
|\psi_m(R)|^2 = \frac{1}{2\pi} \left| \frac{H_0^{(1)}(mR)}{H_1^{(1)}(mR)} \right|^2
\]

where \( H_\nu^{(1)}(z) \) is an usual Hankel function. As commented earlier, \( |\psi_m(R)|^2 \) is dependent on the parameter \( m \). When deriving Eq.\( (9) \) we have used a relation of the Hankel function to the modified Bessel function

\[
K_\nu(z) = \frac{i\pi}{2} e^{\frac{i\nu\pi}{2}i} H_\nu^{(1)}(iz).
\]

Inserting Eq.\( (9) \) into Eq.\( (5) \) with \( m_0 = 0 \) because of no mass gap in RS2 picture, the Newton potential becomes

\[
V_N = G_N \frac{m_1 m_2}{r} (1 + \Delta V)
\]

\[
\Delta V = \frac{1}{3\pi} \int_0^\infty du \left| \frac{H_0^{(1)}(u)}{H_1^{(1)}(u)} \right| e^{-\pi u}
\]

where \( u = mR \). It seems to be impossible to get an analytical expression for \( \Delta V \). Since, however, the light mass in KK spectrum should make a dominant contribution to \( \Delta V \), we can use the asymptotic formula
\[ \lim_{u \to 0} H_1^{(1)}(u) \sim -\frac{2i}{\pi} \ln u. \] 

\[ \lim_{u \to 0} H_0^{(1)}(u) \sim \frac{2i}{\pi} \ln u. \]

Then, the modification of the Newton potential \( \Delta V \) reduces to approximately

\[ \Delta V \sim \frac{1}{3\pi} \int_0^\infty du \ln u |e^{-\frac{r}{R}}. \]

Using the integral formula

\[ \int_0^\infty x^{\nu-1} \ln x e^{-\mu x} dx = \frac{1}{\mu^\nu} \Gamma(\nu) [\psi(\nu) - \ln \mu] \] 

\[ \int_0^1 x \ln x e^{-\mu x} dx = \frac{1}{\mu^2} \left[ 1 - e^{-\mu} - \gamma - Ei(-\mu) - \ln \mu \right] \]

where \( \psi(z) \), \( \gamma \), and \( Ei(z) \) are Digamma function, Euler’s constant, and Exponential-Integral function respectively, the r.h.s. of Eq.(13) can be analytically computed, which yields

\[ \Delta V \sim f \left( \frac{r}{R} \right) \frac{R^2}{r^2} \] 

where

\[ f \left( \frac{r}{R} \right) = \frac{2}{3\pi} \left[ -1 + \gamma + \ln \frac{r}{R} + 2e^{-\frac{r}{R}} - 2Ei \left( -\frac{r}{R} \right) \right]. \]

Fig. 1 shows \( r/R \)-dependence of \( f(r/R) \) which indicates that the short-range and long-range behaviors of gravitational potential are completely different from each other. When two point masses on the brane are separated with a great distance, \( i.e. \ r/R \to \infty \), Eq.(16) indicates

\[ \Delta V \sim \frac{2}{3\pi} \frac{R^2}{r^2} \ln \frac{r}{R}. \]

Thus as stressed earlier the modification of the Newton potential involves a logarithmic factor. Although this logarithmic factor is extremely small due to the power term, it can be measured by an appropriate experimental setting. When the distance between the point masses are very small, \( i.e. \ r/R \to 0 \), Eq.(16) indicates

\[ \Delta V \sim -\frac{2}{3\pi} \frac{R^2}{r^2} \ln \frac{r}{R}. \]
which yields very strong attractive force compared to the usual inversely square law. This means the contribution of the KK spectrum is dominant for the short range gravitational potential.

The checking of the logarithmic correction to the Newton potential, especially the short-range behavior of it, by experiment may give a strong evidence for the existence of KK spectrum and as a result, the existence of the extra dimension.

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Figure Captions

Figure 1

Plot of $f(r/R)$. This figure indicates that the short-range and long-behaviors of the gravitational potential are completely different from each other. “RS” and “TG” in this figure stand for Randall-Sundrum and Garriga-Tanaka. Thus, these points represent Eq.(1) and Eq.(2) respectively.
Fig. 1