Abstract

We discuss the new state $D_{sJ}(2317)$ discovered by BaBar and demonstrate using QCD inequalities that if indeed the $f_0(980)$ and the new $D_{sJ}(2317)$ ($0^+$) are primarily made of four quarks that a new I=0 “$\bar{D}D$ bound state” at a mass smaller than 3660 MeV must exist. Observation of such a state will constitute definitive evidence for four-quark states.

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I. GENERAL PHENOMENOLOGICAL CONSIDERATION.

The new $D_s(2320)$ discovered in BaBar\cite{1} as a narrow $D_s(1970) + \pi^0$ resonance is somewhat puzzling\cite{2, 3, 4, 5}. It has $c\bar{s}$ flavors and natural spin parity $J^P = 0^+, 1^-, 2^+ \ldots$. We assume $J^P = 0^+$ as suggested by the lightness of the state. The only allowed two-body decay is then $D_s(1963) + \pi^0$. $D_s(1963) + \eta$ is $\sim 40$ MeV above 2320 MeV and $D_s(2320) \rightarrow D_s(1963) + \gamma$ is a forbidden 0 $\rightarrow$ 0 electromagnetic transition.

Despite the high $Q$ value in $D_s(2320) \rightarrow D_s(1968) + \pi$ the narrow width of the $S$-wave decay immediately follows if the $I$-spin of $D_s(2320)$ is $I = 0$. The final $D_s + \pi^0$ state has $I = 1$ and the decay rate would naturally be suppressed to be $\Gamma(D_s(2320)) \leq \text{expt. resolution} \approx 10$ MeV. We will henceforth assume $I\{D_s(2320)\} = 0$ which is the case if $D_s(2320)$ is just a $c\bar{s}$ state and/or $c\bar{s}$ with any number of gluons or $\bar{u}u + \bar{d}d$, $I = 0$ light quark pairs.

The puzzling feature is the relative lightness of this state. The simplest assignment of $D_s(2320)$ to be the $^3P$($0^+$), $P$-wave quark model $c\bar{s}$ state conflicts with earlier quark model calculations\cite{6, 7}. These models treat the system as a $q\bar{Q}$ state and accommodate the $^1S$(1963) $D_s$ and the $^3S$(2112) $D_s^*$ states and a pair of $P$-wave states with $j_s = \ell_s + s_s = 3/2$ identified with the narrow $D_{sj}(2573)$ and $D_{si}(2536)$ with likely $J^P = 2^+, 1^+ \text{ spin parities}$. The remaining two $P$-wave states corresponding to $j_s = 1/2$ with $J^P = 1^+0^+$ were predicted to lie above 2400 MeV and hence the $0^+$ state was expected to decay strongly into $DK$ and be very broad. These considerations, on which we elaborate further, make a $P$-wave $c\bar{s}$ interpretation of $D_s(2320)$ difficult. Yet, such an interpretation is deemed to be favored by the recent discovery of the $J^P = 1^+ D_s(2460)$\cite{8, 9}.

We note that for all known $Q\bar{q}$, $q\bar{q}$, and $Q\bar{Q}$ states the gap between the $^3S$ state and the lowest $P$ state exceeds $\approx 320$ MeV, making the $\approx 220$ MeV split between $D_s^*(2112)$ and $D_s(2320)$ clearly stand out as the smallest.

This holds also for the new $0^+$ $c\bar{u}$ state seen in Belle\cite{9} as a broad resonance in the $D\pi$ channel at 2308 MeV (again 300 MeV above the triplet $S$-wave $D^*(2010)$).

If indeed the new $0^+ c\bar{u}$ state is confirmed the lightness of the $D_{sj}(2317)$ is even more puzzling as it is almost degenerate with its non-strange $c\bar{u}$ analog, whereas in all meson/baryon states the penalty for such a $u \rightarrow s$ substitution exceeds $\sim 100$-150 MeV, the constituent quark $s - u$ mass difference.

The large splitting are indeed suggested by the $P$-wave centrifugal barrier:
\[ \frac{2}{\mu} \langle \frac{1}{r^2} \rangle \text{ with } \mu = m_q / 2 \text{ or } \sim m_q, m_Q / 2 \text{ for } \bar{q}Q \text{ or } \bar{q}Q \bar{Q}Q \text{ (since } \langle \frac{1}{r^2} \rangle \geq \frac{1}{<r^2>} \text{ increases with } \mu, \text{ this splitting changes only mildly with } \mu) .

The above suggests considering \( D_s(2320) \) as primarily an \( I = 0 \) four-quark \( c \bar{s}(\bar{u}u + \bar{d}d) / \sqrt{2} \) bound state or the \( I = 0 \) \( (K^+D^0 + K^0D^+) / \sqrt{2} \) bound state.

The idea that certain states can be more readily obtained as four-quark states and considerations of exotic (non-\( \bar{q}q \)) or \( qqq \) states in general date back to the very beginning of quark models and QCD. Baryonium \( (qq) \ldots (\bar{q}\bar{q}) \) states were suggested by considerations of duality \cite{10} and in recent years reappeared as light-heavy \( (QQ')(\bar{q}\bar{q}') \) tetra quarks and/or as molecular bound states of \( DD^* DD^* \) or even \( KK^* \) dominated by pion exchange \cite{11,12,13}.

Since the lightest \( P \)-wave \( s\bar{s} \) state is expected to lie significantly above the \( ^3S(\bar{s}s) \phi(1020) \), the light \( 0^{++}I = 0(980) \) and \( I = 1 a_0(980) \) are unlikely to be just \( s\bar{s} \) state. R. Jaffe suggested \cite{14} that these states arise as four-quark states within bag models, thanks to a favorable pattern of chromomagnetic “hyperfine” attraction. This in turn motivated prediction of striking, strong interactions stable, Hexa \( u^2s^2d^2 \) and Penta–\( \bar{c}s u^2d \) states–which were not found to date.\cite{15,16,17}

In general the paucity of exotic \( \bar{q}q \bar{q}q \) and \( q^3\bar{q}q \) mesons and baryons traces to the lightness of the pseudoscalar pion causing the exotics to be extremely broad and overlapping resonances. The decay of non-exotic \( \bar{q}q \) states say \( \rho \) into a pair of \( \bar{q}q \) pions is suppressed by a \( 1/N_c \) color factor.\cite{18} This is \textbf{not} the case for decay of exotic hadrons which simply fall apart into its constituent non-exotic hadrons with no penalty for creating an extra \( \bar{q}q \) pair.

Thus if the new \( D_s(2320) \) is not a standard \( P \)-wave \( c\bar{s} \) state, it may well be the first genuine exotic four quark \( c\bar{s}(\bar{u}u + \bar{d}d) / \sqrt{2} \) \( \text{ or } \frac{D^0K^+ + D^0K^0}{\sqrt{2}} \) which is narrow.

Our main aim in the following is to point out a likely analogous \( c\bar{s}(\bar{u}u + \bar{d}d) / \sqrt{2} \) or \( D\bar{D} \) state. QCD inequalities then suggest that this new state is lighter than 3660 MeV.

More likely \( D_s \) is the appropriate lowest energy superposition of all three:

\[
D_s(2320) = \alpha |c\bar{s}⟩ + \beta |c\bar{s}(\bar{u}u + \bar{d}d) / \sqrt{2}⟩ + \gamma |\frac{K^+D^0 + K^0D^+}{\sqrt{2}}⟩
\]

(1)

with \( |\beta|^2 + |\gamma|^2 > |\alpha|^2 \) favored by our interpretation.
II. QCD/LATTICE FORMULATION OF THE HADRON SPECTRUM.

The above short summary fails to address the main issue. A reliable theoretical framework for computing the hadronic spectrum in general (and of four-quark states in particular) is lacking.

In principle, one could use a path integral formulation together with an appropriate lattice discretization to numerically evaluate Euclidean $n$-point correlators of local color singlet operators.\[19\]

To find the mass of the lightest hadron in a channel with given quark flavors and Lorentz ($J^P$) quantum numbers, we use the two-point correlator of currents with these quantum numbers:

$$\langle 0|\{\bar{\psi}(x)\Gamma\psi(y)\}\{\bar{\psi}(y)\Gamma\psi(x)\}|0\rangle \equiv F^{(ij)}(x,y)$$

with $\Gamma = 1, \gamma_\mu, \gamma_\nu\gamma_5, \sigma_{\mu\nu}\gamma_5$, the appropriate Dirac matrix. The asymptotic behavior of $F$ as $|x-y| \to \infty$ is controlled by the lowest mass particle—say the $0^+(c\bar{s})$ scalar state of interest when we use $\Gamma = 1$ $i = c, j = s$:

$$F^{(c\bar{s})}(x,y) \big|_{|x-y| \to \infty} \sim |<0|\bar{c}(0)s(0)|D_s^{(0^+)}(2320)>|^2 e^{-m_{c\bar{s}}|x-y|}$$

More generally hadron masses, couplings and scattering amplitudes can be obtained by inverse Laplace transforms of two, and $n$-point Euclidean correlators. The latter can be written after integration over the fermionic fields (which appear in the action in a bilinear form) as path integrals over gauge field configurations $A^a_{\mu}(x)$.

For $F^{(ij)}_\Gamma(x,y)$ we have:

$$F^{(ij)}_\Gamma(x,y) = \int d\mu(A) \text{tr}\{\Gamma S^A_i(x,y)\Gamma S^A_j(y,x)\}$$

with $S^A_i(x,y)$ the propagator of the quark $q_i$, in the background field $A^a_{\mu}(x)$, the trace is over spinor and color indices and $d\mu(A)$ is the (positive!) measure obtained after integrating the fermions to yield the determinant factor

$$d\mu(A) = D(A_{\mu}^{(a)}(x)) e^{-S_{YM}(A_{\mu}(x))} \prod_{i=1}^{n_F} \det(D_A + m_i)$$

and $S_{YM} = \int d^4x L_{YM} = \int d^4x (\vec{E}^2 + \vec{B}^2)$ is the Euclidean Yang-Mills action.

Unfortunately, the computational complexity of generating many gauge backgrounds and evaluating for each the fermionic propagator and, in particular, the fermionic determinant prevents carrying out these calculations.
reliably at present. Note that since the state of interest is likely to contain extra $q\bar{q}$ pairs \{(\bar{u}u + \bar{d}d)/\sqrt{2}\} the "quenched" approximation in which $\det(D_{A} + m_{i}) \to 1$ and quark loops are omitted may be inappropriate.

To bring out such components in the $D_{s}(2320)$ state we can use instead of the correlators of quark bilinears products of quark bilinear such as:

$$F_{s\bar{c},\bar{u}u}(x, y) \sim \langle 0 | [\bar{c}(x)\gamma_{5}s(x) \cdot (\bar{u}(x)\gamma_{5}u(x))]^{+} \cdot \bar{c}(y)\gamma_{5}s(y) \cdot (\bar{u}(y)\gamma_{5}u(s)) |0 \rangle$$ (6)

or

$$F_{s\bar{u},c\bar{s}}(x, y) \sim \langle 0 | [\bar{c}(x)\gamma_{5}u(x) \cdot (\bar{u}(x)\gamma_{5}s(x))]^{+} \cdot [\bar{c}(y)\gamma_{5}c(y)](\bar{u}(y)\gamma_{5}(y)) |0 \rangle$$ (7)

After integrating the fermionic fields and using $\gamma_{5}S_{A}(x,y)\gamma_{5} = (S_{A}(y,x))^{+}$ with the + indicating the adjoint color/spin matrix (6) and (7) yields the following path integrals:

$$F_{s\bar{c}\bar{u}u}(x, y) = \int d\mu(A) \operatorname{tr}((S_{c}^{A}(x,y)^{+}S_{s}^{A}(x,y))) \operatorname{tr}((S_{u}^{A}(x,y)^{+}S_{u}^{A}(x,y)))$$
$$+ \int d\mu(A) \operatorname{tr}(S_{c}^{A}(x,y)^{+}S_{s}^{A}(x,y)) \operatorname{tr}(S_{u}^{A}(x,y)\gamma_{5}) \cdot \operatorname{tr}(S_{u}^{A}(y,y)\gamma_{5})$$ (8)

or

$$F_{s\bar{u},c\bar{s}}(s, y) = \int d\mu(A) \operatorname{tr}(S_{c}^{A}(x,y)^{+}S_{u}^{A}(x,y)) \operatorname{tr}(S_{u}^{A}(x,y)^{+}S_{s}^{A}(x,y))$$
$$+ \int d\mu(A) \operatorname{tr}(S_{c}^{A}(x,y)^{+}S_{s}^{A}(x,y)) \operatorname{tr}(S_{u}^{A}(x,x)\gamma_{5}) \cdot \operatorname{tr}(S_{u}^{A}(y,y)\gamma_{5})$$ (9)

Note that (6) and (8) suggest viewing $D_{s}(2320)$ as a "$D_{s}\eta$" composite whereas (7) and (9) emphasize the more relevant "$DK$" component. Also instead of $\bar{c}(x)\gamma_{5}s(x)$ and $\bar{u}(x)\gamma_{5}u(x)$ we could have used any $\bar{c}(x)\Gamma_{s}(x)$, $\bar{u}(x)\Gamma_{u}(x)$ as for any $\Gamma$ the $s$ channel quantum number are scalar. (In the above expression a $\gamma^{\mu} \cdot \gamma_{\mu}$ or $\sigma^{\mu\nu}\sigma_{\mu\nu}$ etc. contraction is implicit and $\bar{c}(x)\Gamma_{s}(x)$ indicates a contraction over the color indices.) For calculations done with infinite precision, it does not matter if we start with (6), (7) or Eq. (2) with $i, j = c, s$, the lowest energy state in all cases is the same $D_{s}(2320)$.

In practice and for motivating the different inequalities there are clear differences.
In passing we note that a specific member among the three components of $D_s(2320)$ indicated in Eq. (1) can be enhanced if we probe the physical $D_s$ at different distance scales.

Thus at short distances the pure $c\bar{s}$ component may prevail, at intermediate distances the four-quark single bag state is likely to dominate and at larger distances yet the two mesons $D\bar{K}$ bound state may be appropriate.

Finally we note that in the case of the $I = 1$ analog of our $D_s(2320)$ state say $\bar{c}s\bar{ud}, \bar{c}s\frac{\bar{u}u-\bar{d}d}{\sqrt{2}}, \bar{c}s\bar{du}$ where all the quarks necessarily propagate from $x$ to $y$ and only the first “flavor connected” terms appears in Eq. (8) and (9).

For the case of interest with identical $\bar{u}u$ (or $\bar{d}d$) annihilation is possible. We have the additional second “flavor disconnected” terms in (8), (9) in which $\bar{u}u$ annihilate into intermediate gluons and the quark line emerging from $x$ (or $y$) loops back to $x$ (or $y$). The role of such terms in general and its effect on the inequalities we suggest in particular will be discussed at length later. It plays a crucial role in the pseudoscalar $I = 0$ light quark system where the axial anomaly coupled to non-perturbative QCD effects [20] explains the absence of a “ninth” Goldstone boson.

III. MOLECULAR STATE VIA SIMPLE POTENTIAL MODELS.

Let us assume that $D_s(2320)$ can indeed be viewed as a $(D^0K^++D^+K^0)/\sqrt{2}$ non-relativistic bound state. This may be reasonable due to the relatively small binding B.E. $\approx 40$ MeV. There will also be a $D_s\eta^0$ component but being $\approx 200$ MeV heavier will be neglected in sections 3, 4.

We can use a non-relativistic Schrödinger equation with a potential generated by $\rho, \omega$ and the two-meson exchange box diagram (see Fig. 1):

At the present we do not know $g_{\rho\bar{D}D}$, $g_{\omega\bar{D}D}$ and $g_{D^*\pi}$ to better than $\approx 40\%$.

It is unrealistic to consider $D$ and $K$ as point-like when the range of the potential $1/m_v \approx 1/3 - 1/4$ Fermi, i.e. the distance between the mesons when $V(r)$ is appreciably, is smaller than the size of each hadron. Thus we need to include “Form-Factors” $F_v^D(q^2)$ and $F_{\pi\bar{D}D^*}(q^2)$ and $F_v^K(q^2)$, $F_{\pi\bar{K}K^*}(q^2)$ which are poorly known. Thus any conclusion as the existence of a $D\bar{K}$ bound state with $O(40$ MeV) binding is very tentative.

The same applies to the near threshold $\bar{K}K$ bound states [tentatively identified with $f_0(980)$ and $a_0(980)$] and even more so to putative $\bar{D}D$ $0^+$ bound states.
The main point of the present paper is that certain inequalities between the masses of the lowest states in the $\bar{K}K$, $\bar{D}D$ and $DK + KD$ channel can be derived irrespective of all the above ambiguities.

The mass inequality alluded to is:

$$2m_0^{(0^+)}(KD) \geq m_0^{(0^+)}(D\bar{D}) + m_0^{(0^+)}(\bar{K}K)$$  \hspace{1cm} (10)

We will also attempt to derive directly from the path integral expressions (8), (9), the analog inequalities for quadriquarks:

$$2m_0^{(0^+)}\left\{\bar{c}s\frac{(u\bar{u} + d\bar{d})}{\sqrt{2}}\right\} \geq m_0^{(0^+)}\left\{\bar{c}c\frac{\bar{u}u + \bar{d}d}{\sqrt{2}}\right\} + m_0^{(0^+)}\left\{\bar{s}s\frac{\bar{u}u + \bar{d}d}{\sqrt{2}}\right\}$$  \hspace{1cm} (11)

The left-hand side of (10) and (11) is to be identified with the new $D_s(2320)$ the second mass on the right is taken to be that of $f^{(0)}(980)$. Eqs. (10) or (11) then predict the existence of a $0^+$ $D\bar{D}$ bound state (or $c\bar{c}uu + dd$ exotic) at a mass lower than 3660:

$$m^{(0^+)}(D\bar{D}) \leq 2m_{D_s(2320)} - m_{f_0}(980) \sim 3660$$  \hspace{1cm} (12)

In the limit when both $D$ and $K$ are considered to be a “heavy-light” $Q\bar{q}'$ system the forces between $Q\bar{q}'$ and $Q'q$ due to $\pi, \rho, \omega, \ldots$ etc. exchanges
coupling to the light degrees of freedom become universal and independent of the heavy quark flavors.

The attractive potentials **guarantee** that for sufficiently heavy $Q$ and $Q'$ the mesons $Qq'$ and $Q'\bar{q}$ will bind. This persists for the genuine baryonium-like states $Qq'Q'\bar{q}$. The same results are obtained when we view the last problem as binding of a quadri-quark. In the $m_Q, m_{Q'} \to \infty$ limit, the $QQ'$ will bind into a $\bar{3}$ with Coulombic binding

\[ \approx \frac{\alpha_s^2 m_Q m'_{Q}}{2 m_Q + m'_{Q}} \gg \Lambda_{\text{QCD}} . \] (13)

The remaining two light anti-quarks will then necessarily form a combined “baryon-like” state in which the compact \{QQ'\} behaves as a yet heavier “effective $\bar{Q}$” with a mass equal to $m_Q + m_{Q'}$.

In the case discussed here with $Q\bar{Q}'$ (rather than $QQ'$) heavy quarks the $Q\bar{Q}'$ will indeed bind (with twice the binding of $QQ'$) into a compact quarkonium state which is a color singlet. The lowest physical state will then be $(Q\bar{Q}') + (q\bar{q})$, i.e. a quarkonium and a light quark meson. QCD suggests that the residual “van der Waals” multi-gluon exchange interactions between the two color singlet hadrons is attractive. However due to the small size and “rigidity” of the quarkonium state the color polarizability of the latter is likely to be rather small, suppressing the strength of this attraction. This in turn makes the actual binding of a light pseudoscalar and the heavy quarkonium unlikely.

For the case of $D_s(2320)$, $D + K \approx 2360$ is lighter than $\eta + D_s$. However, for $D\bar{D}$ system, $2m_D$ is indeed higher than $m_\eta + m_{\eta_c}$. The $D\bar{D}$ “state” will thus be unstable with respect to re-arrangement and decay to $\eta + \eta_c$.

**IV. POTENTIAL MODEL DERIVATIONS OF THE INEQUALITY.**

For a first simple motivation of the desired inequality we assume that the non-relativistic potentials in the $K\bar{K}$, $D\bar{K}$ and $\bar{D}D$ channels are the same.\[19, 21\]

The lowest eigenvalues of the following three Hamiltonians:

\[
\begin{align*}
  h_{11} &= \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}^2}{2m_1} + V(r) h_{22} = \frac{\vec{p}_2^2}{2m_2} + \frac{\vec{p}^2}{2m_2} + V(r) h_{12} = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}^2}{2m_2} + V(r) \\
\end{align*}
\] (14)
with $\vec{p}, \vec{r}$ the relative moment and distance between the particles in the respective center mass systems, and $m_1 = m_K$, $m_2 = m_D$; are given by the Schrödinger equations:

$$h_{11} |\psi_{11}^{(0)}\rangle = \epsilon_{11}^{(0)} |\psi_{11}^{(0)}\rangle h_{22} |\psi_{12}^{(0)}\rangle = \epsilon_{22}^{(0)} |\psi_{22}^{(0)}\rangle h_{12} |\psi_{12}^{(0)}\rangle = \epsilon_{12}^{(0)} |\psi_{12}^{(0)}\rangle$$ (15)

$|\psi_{11}^{(0)}\rangle, |\psi_{22}^{(0)}\rangle |\psi_{12}^{(0)}\rangle$ being the ground state of the respective Hamiltonian. Eq. (14) implies the operator relation

$$2h_{12} = h_{11} + h_{12}$$ (16)

Taking the expectation value in $\psi_{12}^{0}$:

$$2\langle \psi_{12}^{(0)} | h_{12} | \psi_{12}^{(0)} \rangle \equiv \epsilon_{12}^{(0)} = \langle \psi_{12}^{(0)} | h_{11} | \psi_{12}^{(0)} \rangle + \langle \psi_{12}^{0} | h_{22} | \psi_{12}^{(0)} \rangle$$ (17)

The variational principle implies that the two expectation values on the right-hand side of Eq. (17) exceed the energies of the respective ground states.

This then yields

$$2\epsilon_{12}^{(0)} \geq \epsilon_{11}^{(0)} + \epsilon_{22}^{(0)}$$ (18)

After adding the rest masses ($2m_1 + 2m_2$) to both sides we finally obtain the desired mass inequality

$$2m_{12}^{(0)} \geq m_{11}^{(0)} + m_{22}^{(0)}$$ (19)

i.e. Eq. (10) above.

The operator relation (17) can be projected on any partial $\ell$-wave subspace. Hence the mass inequalities hold for each wave separately. The inequalities also apply when ground state energies are replaced by the sum of energies of the ground state and the first $n$ radial excitations for any $n$. [19]

Considering the $D$ (and the $K$) mesons as $\bar{Q}q$ mesons is clearly an (extreme) idealization and hence the universality of potentials in the $D\bar{D}$, $K\bar{K}$, and $K\bar{D}$ channels is a rather crude approximation. Indeed the coupling $g_{K\bar{K}/\rho\omega}$ and $g_{K\bar{K}^*\pi}$ will all differ from the corresponding $g_{D\bar{D}/\rho\omega}$ and $g_{D\bar{D}^*\pi}$, as will the “form factors” associated with coupling to the $D$ and $K$ of different sizes.

Nonetheless the same mass inequalities (eq. (10)) continue to hold subject to much more general and rather weak assumptions!
The only requirement is that the interactions in the $D\bar{D}$, $K\bar{K}$ and $\bar{K}D$ channels can be viewed (in either momentum or configuration space) as convolutions or “scalar products” of a generalized “vector” associated with couplings to the $K$ line and another vector associated in the same way with the propagating $D$ line. The $\rho/\omega$ exchanges and the more complex interactions due, say, to the pion box diagram of Fig. 1 can be viewed as some generalized “metric” used for this “scalar product”. We only need that this metric be positive, viz., attractive potentials. This is closely related to positive potentials used by E. Lieb in [22].

We note that the “scalar product” form holds also for the full propagation of the $\bar{K}D$ system and with the various exchanges iterated any number of times.

More generally the amplitude for propagating a, say, $\bar{K}D$ system configuration at $t = 0$ to the same configuration at some imaginary time $t = iT$ can be expressed in terms of a path integral.

The asymptotic behavior of this joint say $\bar{K}D$ propagation $\sim \text{tr}\{e^{-TH}\}$ is then dominated by the lightest, bound or threshold, state in this channel and is proportional to $e^{m^{(0)}_{KD}T} T \approx \text{tr}\{e^{-TH}\}$. Likewise the asymptotic behavior of the $D\bar{D}$ and $K\bar{K}$ joint propagations are $e^{-m^{(0)}_{KK}t}$ and $e^{-m^{(0)}_{DD}T}$ respectively.

When there are no interactions the $K, D$ independently propagate from the initial ($t = 0$) to final ($t = T$) configurations. When the interaction is switched on the propagation is modulated by the interaction potential at all intermediate times: $t_1, t_2, \ldots t_n (= \Delta n) \ldots, t_N = N \Delta = T$. (Eventually the $\Delta \rightarrow 0$ $N \rightarrow \infty$ limit is taken). Thus the joint propagation has again the form of a scalar product of $N$ component vectors with the potentials $V(\vec{r}_1(t_i) - \vec{r}_2(t_i))$ playing the role of the positive metric. [19, 22]

Hence

$$P\{\bar{D}D(0 \rightarrow T)\} \approx \tilde{V}_D(T) \ast V_D(T)$$
$$P\{K\bar{K}(0 \rightarrow T)\} \approx \tilde{V}_K(T) \ast V_K(T)$$
$$P\{\bar{D}K(0 \rightarrow T)\} \approx \tilde{V}_D(T) \ast V_K(T)$$

(20)

and the Schwarz inequality implying:

$$\{P_{DD}(T)\} \{P_{KK}(T)\} \geq (P_{KD}(T))^2$$

(21)

yields in the $T \rightarrow \infty$ limit the desired inequality.
V. MASS INEQUALITIES FOR FOUR-QUARK STATES FROM QCD.

We indicated in Section II how the QCD spectrum can be obtained from Euclidean correlators, which in turn can be evaluated via path integrals. A fundamental property underlying the probabilistic, Monte Carlo, approach to generating background gauge configuration—is the positivity of the measure $d\mu(A)$ in Eq. (5) which can be proven in vectorial theories.

The same feature underlies derivation of (QCD) inequalities between Euclidean correlators or products of such correlators. These inequalities when applied in the $|x-y| \to \infty$ limit (where the asymptotic behavior (see Eq. (3)) is dominated by the lightest hadron or hadronic system in the relevant channel) then leads to mass inequalities of the form $m_H > m_{H'}$ or $m_A + m_B \leq 2m_C$ respectively.

The conjugation property

$$\gamma_5 S_A(x, y) \gamma_5 = S_A^+(y, x)$$

allows us to write the pseudoscalar correlator:

$$F_{0^-}^{ud} \equiv \langle 0 | \bar{u}(x) \gamma_5 d(x) \rangle_{+}^{\dagger} \bar{u}(y) \gamma_5 d(y) | 0 \rangle = \int d\mu(A) \text{ tr}\{\gamma_5 S_A^{(u)}(x, y) \gamma_5 S^{(d)}(y, x)\}$$

as

$$F_{ud}^{0^-}(x - y) = \int d\mu(A) \text{ tr}\{S_{(A)}^{(u)}(x, y) + S_{(A)}^{(d)}(y, x)\}$$

In the limit of exact isospin symmetry $m_u^{(0)} = m_d^{(0)}$ and $\alpha_{em} = 0$, the propagators of the $u$ and $d$ quarks in any background configuration of gauge fields $A^{(a)}_{\mu}(x)$ are the same and the integral becomes a sum of squares:

$$F_{ud}^{0^-}(x - y) = \int d\mu(A) \text{ tr}\{(S_A^{(u)}(x, y)) + S_{(A)}^{(d)}(x, y)\}$$

Up to overall constants it is larger than two-point correlators of (light) quark bilinears [23] involving $\Gamma$ matrices other than $\gamma_5$. Using the asymptotic behavior we infer that the pion is the lightest meson:

$$m_\pi \leq m_\rho, m_{A_1}, m_\sigma, m_{f_0}, \ldots$$

Next we compare pseudoscalar propagators involving mixed flavors, [24] e.g.
\begin{align*}
\langle 0 | (\bar{b}(x) \gamma_5 c(x))^+ (\bar{b}(y) \gamma_5 c(y)) | 0 \rangle & \equiv F_{b\bar{c}}^{(0^-)}(x-y) \\
\text{with the corresponding flavor diagonal correlated } F_{b\bar{b}}^{(0^-)}(x-y) \text{ and } F_{c\bar{c}}^{(0^-)}(x-y). \text{ Using the above path integral representation we have:}
\end{align*}

\begin{align*}
F_{b\bar{b}}^{0^-} &= \int d\mu(A) S_{b}^{+(A)}(x,y) S_{b}^{A}(x,y) \\
F_{c\bar{c}}^{0^-} &= \int d\mu(A) S_{c}^{+(A)}(x,y) S_{c}^{A}(x,y) \\
F_{b\bar{c}}^{0^-} &= \int d\mu(A) S_{b}^{+(A)}(x,y) S_{c}^{A}(x,y)
\end{align*}

The Schwarz inequality then yields

\begin{equation}
F_{b\bar{b}}^{(0^-)}(x-y) \cdot F_{c\bar{c}}^{(0^-)}(x-y) \geq |F_{b\bar{c}}^{(0^-)}(x-y)|^2
\end{equation}

and using the asymptotic behavior of the three \( F \) functions as \( |x-y| \to \infty \) the interflavor mass inequalities:

\begin{equation}
2m_{b\bar{c}}^{0^-} \geq m_{c\bar{c}}^{0^-} + m_{b\bar{b}}^{0^-} \equiv m_{\eta_c} + m_{\eta_b}
\end{equation}

In principle such inequalities might be expected for any quark flavors

\begin{equation}
2m_{i\bar{i}}^{0^-} \geq m_{ii}^{0^-} + m_{jj}^{0^-}
\end{equation}

These inequalities are reminiscent of the meson meson N.R. bound state mass inequalities discussed in the previous section. For the case of heavy quark flavors such as \( i = b, j = c \) where a non-relativistic \( Q_i, Q_j \) bound state picture may be appropriate, there is indeed a very suggestive connection: All QCD interactions in the \( 1S \) pseudoscalar channel including the hyperfine chromomagnetic interactions are purely attractive. Such potentials correspond to the positive definite interaction kernel \([22]\) used above.

Unfortunately when light flavors, \( u, d \) \( I = 0 \) states are involved, the above arguments are flawed. The current \( \bar{q}_i(x) \Gamma q_i(x) \) creates from the vacuum a quark and an identical (anti)-quark. After the fermionic (path) integration we have not only the flavor connected parts

\begin{equation}
\int d\mu(A) \text{ tr}\{ \Gamma S_A^i(y, x) \} \text{ tr}\{ \Gamma S_A^i(y, x) \}
\end{equation}

considered above but also the flavor disconnected part:
\[
\int d\mu(A) \, \text{tr}(\Gamma S^i_A(xx)) \cdot \text{tr}(\Gamma S^i_A(yy))
\] (33)

Each of these terms in (33) traces over single propagators starting at \(x\) (or \(y\)) and “looping back” to the initial point.

These contributions seem to be related to \(\bar{q}q\) annihilation into multigluon “glueball” color singlet states. The notion that such “hairpin” annihilation diagrams can be neglected, namely the “Zweig rule”, dates back to the invention of the quark model.

This neglect is certainly justified by asymptotic freedom in the case of heavy quarks and by the fact that many meson nonets are “ideal” with small mixing between the \(\bar{s}s\) and \(\frac{uu + \bar{d}d}{\sqrt{2}}\) that such annihilations would generate.

We will next proceed to deriving QCD mass inequalities for four-quark correlators adopting first the “Zweig rule” i.e. by neglecting flavor disconnected contributions, namely the \(\text{tr}\{D_u^A(xx)\Gamma\} \cdot \text{tr}\{D_u^A(yy)\Gamma\}\) term in (8) and in the analogous expressions for \(F_{\bar{c}c u\bar{u}}(x,y)\) and \(F_{\bar{s}u u\bar{u}}(x,y)\).

We use the correlators of products of quark bilinears appearing in (7) instead of (8) which is more appropriate if we view \(D_s(2320)\) as a \(DK\) bound state since \(c(x)\gamma_5\bar{u}(x)u(x)\gamma_5\bar{s}(x)\) creates, when acting on the vacuum “localized” \(D^0\) and \(K^+\) states.

The relevant Schwarz type inequality here is:

\[
\int d\mu(A) \, \text{tr}(S^+_{s(A)}(x,0)S_{u(A)}(x,0)) \cdot \text{tr}(S^+_{s(A)}(x,0)S_{s(A)}(x,0)) \geq \left| \int d\mu(A) \, \text{tr}(S^+_{c(A)}(x,0)S_{u(A)}(x,0)) \cdot \text{tr}(S^+_{u(A)}(x,0)S_{s(A)}(x,0)) \right|^2
\] (34)

The bound on \(m(D^0_s(2320))\) in terms of \(m(f_0(980))\) and the mass of a new \((c\bar{u})(\bar{c}u), I = 0\) state would then ensue.

There are no obvious flavor disconnected contributions when we start from (7) or from correlators of any of the following scalar products of quark bilinears: \((\bar{u}(x)\Gamma Q(x)) \cdot (\bar{Q}'(x)\Gamma u(x))\). However such terms and in particular \(\bar{u}(x)\gamma_5u(x)\), yielding the \(\text{tr}(D^A_u(x,x)\gamma_5) \cdot \text{tr}(D^A_u(yy)\gamma_5)\) “disconnected” terms in the path integral, do arise if we “Fierz transform” any of the above \(\Gamma_{\alpha} \cdot \Gamma^\alpha\). This reflect the fact that the channels \(D_s\eta^0\) and \(\eta_c\eta_0\) do couple to the \(D_s(2320) 0^+\) state and to the new \(cc'(\bar{u}u + \bar{d}d) 0^+\) state even if the latter are mainly \(KD\) (or \(D\bar{D}\)) bound states.
Such disconnected contributions can be avoided if we consider instead the $I = 1 \ (c\bar{s} \ u\bar{d}) \ (c\bar{c} \ u\bar{d})$ and $(s\bar{s} \ u\bar{d} \ 0^+)$ states with the last being the $a_0(980)$ $K\bar{K}$ threshold state.

However these states can decay into final states involving the very light pion viz., $D_s\pi^+ \ \eta_c\pi^+$ and the observed $\eta\pi$ respectively, with appreciable $Q$ values. This will make them relatively wide and hard to identify. (The fact that $a_0(980)$ still has a relatively small $\Gamma \approx 50 \text{ MeV}$ width is somewhat encouraging.)

Coming back to the $I = 0$ states of interest we need to address the issue of having flavor disconnected terms in general and those involving $\Gamma = \gamma_5$ in particular.
VI. FLAVOR DISCONNECTED CONTRIBUTIONS IN PSEUDOSCALAR CHANNELS.

The following section is largely independent of the rest of this paper. We present it here to preface the discussion of disconnected contributions to the inequalities between correlators of various products of quark bilinears.

Let us compare the Euclidean pseudoscalar correlators with light quarks in the $I = 1$ and $I = 0$ channel, used to compute the $\pi, \eta$ masses, respectively

$$\int d\mu(A) \text{tr}\{S_q^{+(A)}(x,y)S_q^{(A)}(x,y)\} \equiv F^{\pi}(x - y) \sim \epsilon^{-m_\pi|y-x|} e^{-m_\pi|x-y|} \quad (35)$$

$$\int d\mu(A)\left[ \text{tr}\{(S_q^{+(A)}(x,y)S_q^{(A)}(x,y))\} + \text{tr}(S_q^{A}(x,x)\gamma_5) \cdot \text{tr}(S_q^{A}(y,y)\gamma_5)\right]$$

$$\equiv F^{\eta}(x - y) \sim \epsilon^{-m_\eta|x-y|} e^{-m_\eta|x-y|} \quad (36)$$

For simplicity we assume only $SU(2)$ flavor with light and equal mass up and down quarks:

$$m_u^{(0)} = m_d^{(0)} = m_q \quad (37)$$

and further take $\alpha_{em} = 0$. In this limit $I$-spin is exact and the $\eta$ particle–all decays of which are electromagnetic and/or $I$ spin violating–becomes stable. We know experimentally and from QCD or soft pion / current algebra that in the limit $m_q^{0} \to 0$, $m_\pi \to 0$ like:

$$m_\pi^2 f_\pi^2 \simeq m_q^0 \langle \bar{q}q \rangle \quad \text{or} \quad m_\pi^2 \simeq m_q^0 \text{ “}\Lambda_{\text{QCD}}\text{”} \quad (38)$$

On the other hand $\eta$ in $SU(2)$ or $\eta'$ in the real world is massive. The mechanism of generating its mass involves the $U(1)_A$ QCD anomaly and potentially non-perturbative (instanton) effects. The important point for the present discussion is that $m_\eta$ (or $m_{\eta'}$) unlike $m_\pi$ does not depend on $m_q^0$ and stays finite when $m_q^0 \to 0$:

$$m_\eta \approx \text{ “}\Lambda_{\text{QCD}}\text{”} \quad (39)$$

and the ratio

$$m_\eta/m_\pi \approx \sqrt{\frac{\text{ “}\Lambda_{\text{QCD}}\text{”}}{m_q}} \quad (40)$$
FIG. 2: (a) The propagators in the background gauge field appearing in the first, flavor-connected, part of the path integral. (b) The second, flavor disconnected part in the path integral. (c) The flavor disconnected part redrawn so as to bring out the fact that quark lines rather than pure glue propagate most of the way between the points $x$ and $y$ when $|x - y| \gg \Lambda_{\text{QCD}}^{-1}$.

becomes arbitrarily large.

Hence in Eqs. 35, 36 above the contribution of the flavor disconnected $\text{tr}(S_A\gamma_5)\text{tr}(S_A\gamma_5)$ should almost completely cancel the first positive definite $\text{tr}(S^+S)$ over a large range of distance scales:

$$\frac{1}{\Lambda_{\text{QCD}}} = m^{-1}_{q'} \leq (x - y) \leq m^{-1}_\pi \approx \frac{1}{(\Lambda_{\text{QCD}}m_0^0)^{1/2}}$$

If we schematically represent the connected and disconnected parts in Fig. 2 such a cancellation seems implausible. Clearly $S^A_q(x, y)$ which appear in the first term “knows” via

$$S^A_q(x, y) \equiv \langle x| \frac{1}{iD_A + m_0^0}|y\rangle$$

of both $x$ and $y$ and $m_0^0$. On the other hand, $S^A_q(x, x)$ and $S^A_q(y, y)$ “know” each only $x$ or $y$. In an ordinary Feynman diagram sense, there are gluon lines connecting the two loops. The propagation of the latter is independent of the light quark masses. Rather the pure gluonic states propagating between
two small loops around $x$ and $y$ would tend to make the contribution of the second flavor disconnected term $\sim e^{m_{GB}^{(0^-)}|x-y|}$. Since the lightest $0^-$ glueball is massive, $m_{GB}^{(0^-)} \sim 2 \text{ GeV} \gg m_{\pi}$, this second term could not cancel the first positive term $\sim e^{-m_{\pi}|x-y|}$ over the interval in Eq. (41) above.

The resolution of this paradox is very simple. The pure gluonic state hardly propagates over any distance. Rather due to the lightness of the quarks the dominant fermionic paths making most of the contribution to the propagation and path integral in (9) are very extended and go most of the way from $x$ to $y$ (when $|x-y| \gg \Lambda_{QCD}^{-1}$) as indicated in Fig. 2c. Only a tiny gap remains to be bridged over by the gluons.

Paranthetically we note that the actual pattern of fermionic paths may be different from what is suggested by Fig. 2. Thus in the strong coupling limit large E, B fields are excited and there are in general few “channels” along which the fermions can propagate with minimal hindrance. The absolute square appearing in $F_{\pi}(x-y)$ ($\text{tr} S^+(x,y)S(x,y)$) makes it favorable to have also the returning anti-quark line follow the same channel. This then would suggest that the worldlines of the quark and antiquark stay at a finite distance (the width of the above “channels” $\sim \Lambda_{QCD}^{-1}$) from each other and are “confined”. It is amusing to go further [25, 26] and gain a qualitative understanding of SXSB in this limit.

The vacuum “channels” exist independently of the probing at $x$ and $y$ and the $q_i\bar{q}_j$ propagation from $x$ to $y$. These channels random walk with a finite $(\Lambda_{QCD}^{-1})^{-1}$ step and often bend “backwards”. Such bends appear at a given time slice as $\bar{q}q$ pairs admixed inside the propagating pion. This then suggests that a simple explanation of Eq. (38) follows from a more careful consideration of these $\bar{q}q$ paths in the pion. Similar paths with one small break appear in the case of the $\eta$ in the disconnected parts. This may naturally cancel the long-range $e^{-m_{\pi}|x-y|}$ contribution.

VII. QCD MASS INEQUALITIES FOR FOUR-QUARK STATES.

The disconnected “hairpin” diagrams play a crucial role for the $I = 0$, $0^-$ light pseudoscalar channels and drastically decrease the binding therein. As we will next argue the effect of such light quark annihilations on the quadri-quark mass inequalities may enhance the bindings in the $Q\bar{u}\bar{Q}u$ channels. Hence the contribution of the second term in (9) is likely to be positive and
FIG. 3: (a) The first, no-annihilation term. (b) The second term with the light $q - \bar{q}$ quarks annihilating. (c) The second term redrawn to emphasize the short length over which pure glue propagates instead of light quarks, and the connection to Harari-Rosner duality diagrams with $\rho$, $\omega$ exchanges.

The derivation of the inequalities goes through.

To illustrate this point we redraw in Fig. 3b the flavor disconnected terms i.e., the second part in (9) in a manner similar to that used in Fig. 2a. Specifically, we let the light quarks propagate over most of the Euclidean distance $|\vec{x} - \vec{y}|$ with a short gap bridged by gluons. This would be justified if the four-quark state $Q\bar{Q}'q\bar{q}$ indeed is lighter than the state of $\bar{Q}Q'$ plus glue propagating in the middle part of Fig. 3b. [Note that the lightest such glueball state is a $0^{++}$]

The diagram depicted in Fig. 3c is just the classical Harari-Rosner “Duality diagram” with a relatively light $\bar{q}q$ bound state exchanged in the $t$-channel of the meson-meson scattering. (The fact that the four-quark lines eventually merge at $x$ (and at $y$) implies that we project out only the $J^P = 0^+$ state of interest. It does not affect the following argument pertaining to the sign of this contribution.) The lightness is reflected via the narrow gap between the quark and anti-quark propagating in the $t$-channel. In a string model description this separation is proportional to the mass of the $\bar{q}q$ meson. These $\omega, \rho, \sigma$ etc. exchanges were indeed considered in our $D\bar{K}, K\bar{K}$, and
potential model picture of Section III, and are well known to provide attractive forces (particularly in the $\bar{M}M$ diagonal channels). Indeed this underlies the argument for positive interaction kernels used above in the first, heuristic derivation of the desired mass inequality. For QCD inequalities between four-quark states, we need positive “disconnected” (second) part in (7b) just like the “connected” first part. We cannot demonstrate this separately for each background field in the QCD path integral $\Phi$. The following arguments are, however, suggestive.

First, we compare the joint propagation of four quarks from $x$ to $y$ with the separate independent propagation of the two $\bar{q}q$ pairs. We argue that with annihilation neglected

$$\langle 0|\{\bar{c}(x)\gamma_5 u(x)\} \cdot \{\bar{c}(x)\gamma_5 u(x)\}^+ \cdot \{\bar{c}(y)\gamma_5 u(y)\} \cdot \{\bar{c}(y)\gamma_5 u(y)\}|0\rangle \equiv F_{\bar{D}D}(x, y) \geq |\langle 0|\{\bar{c}(x)\gamma_5 u(x)\}^+ \cdot \{\bar{c}(y)\gamma_5 u(y)\}|0\rangle|^2 \equiv (F_{\bar{D}}(x, y))^2 \quad (43)$$

and also motivate

$$\langle 0|\{\bar{c}(x)\gamma_5 c(x)\bar{u}(x)\gamma_5 u(x)\}^+ \cdot \{\bar{c}(y)\gamma_5 c(y)\} \cdot \bar{u}(y)\gamma_5 u(y)|0\rangle \equiv F_{\eta,c}(x, y) \geq \langle 0|\{\bar{c}(x)\gamma_5 c(x)^+\bar{c}\gamma_5(y)|0\rangle \cdot \langle 0|\{\bar{u}(x)\gamma_5 u(y)\}^+\bar{u}(y)\gamma_5 u(y)\rangle \equiv F_{\eta}(x, y) \cdot F_{\eta}(x, y) \quad (45)$$

In the $|x - y| \to \infty$ limit these may suggest that bound $c\bar{u} - \bar{c}u$ or $c\bar{c} - u\bar{u}$ states exist.

$$m^{0+}(c\bar{u}, \bar{c}u) \leq 2m^{0+}(c\bar{u}) \quad (47)$$

$$m^{0+}(c\bar{c}, u\bar{u}) \leq m^{0+}(c\bar{c}) + m^{0+}u\bar{u}) \quad (48)$$

With disconnected terms omitted:

$$F_{\bar{D}D}(x, y) \equiv \int d\mu(A) \ \text{tr}\{S_c^+(A)(x, y)) S_u^+(A)(x, y))\} \text{tr}\{S_u^+(A)(x, y)S_c^+(A)(x, y))\} \quad (49)$$

$$F_{\bar{D}}(x, y) = \int d\mu(A) \ \text{tr} \ S_c^+(A)(x, y)S_u^+(A)(x, y) \quad (50)$$
and

\[
F_{\eta c}(x, y) = \int d\mu(A) \, \text{tr}(S_c^{(a)}(x, y) \, S_c^{(a)}(x, y))
\]

\[
F_{\eta c}(x, y) = \int d\mu(A) \, \text{tr} \{ S_c^{(a)}(x, y) \, S_c^{(a)}(x, y) \} S_c^{(a)}(x, y)
\]

(51)

\[
F_{\eta c}(x, y) = \text{tr}(S_u^+(x, y) S_u(x, y))
\]

(52)

The inequalities

\[
F_{DD}(x, y) \geq |F_0(x, y)|^2
\]

(53)

and

\[
F_{\eta c \eta}(x, y) \geq F_{\eta c}(x, y) F_{\eta}(x, y)
\]

(54)

then follow either as a Schwarz inequality or from more heuristic considerations in the case of equation 54 (see footnote).

These correlator inequalities imply the mass inequalities (47, 48). A more careful analysis [19, 27] suggests that the latter do not imply that \(\bar{D}D\) and/or \(\eta_c\eta\) bound states exist. Rather it implies an attractive \(S\)-wave scattering length.\(^1\)

What is the impact of turning on the flavor disconnected contributions on the mass inequalities? One effect is the enhanced \(\eta\) mass (relative to \(\pi^0\)). This however leaves attractive interactions between hadrons in general and between the \(\eta\) and \(D_s\) and \(\eta\) and \(\eta_c\) in particular. The interaction in the last case involves only gluonic exchanges as there are no common light quark to annihilate and/or exchange. Such two (or more) gluon exchange “Casimir-Polder” type interactions between objects of similar chromo-electric magnetic properties (as any two hadrons invariably are) tend to be always attractive.

The transitions to states including \(\eta, \eta'\) appears to be the main obstruction to both the derivations of the inequalities (which follow if we consider only the \(\bar{D}D, \bar{K}K\), and \(\bar{K}D\) channels or the corresponding four quark correlators).

1 The second inequality holds if \(|S_{i u}^i(x, y)|^2\) and \(|S_{i s}^i(x, y)|^2\) are positively correlated; namely for those \(i\) values; i.e. \(A_{i u}^i(x)\) background field configurations, where \(|S_{i u}^i(x, y)|^2\) is larger/smaller than the average also \(|S_{i s}^i(x, y)|\) tends to be large/small. Indeed those configurations which do (not) impede light quark \((u)\) propagation are likely not to impede that of the heavy quark \((c)\). The heuristic statement becomes much clearer when we have quarks of the same mass and indeed we expect the Casimir polder type attraction to be maximal between objects with similar ratios \(\alpha_M/\alpha_E\) of the magnetic and electric polarizability.
The very same transition may impede the discovery of the predicted new $\bar{D}D$ bound state as the latter may have a broad decay width into $\eta_c + \eta$. Indeed heavier $Q, Q'$ quarks in the $Q\bar{Q}q\bar{q}$ state considered make such a transition more likely. Thus the $D_s(2320)$ bound by $\approx 40$ MeV in the $D\bar{K}$ channel the $D_s\eta$ threshold is yet $\approx 200$ MeV higher. However the new $D\bar{D}$ state which our inequality predicts is $\approx 100$ MeV below the $\bar{D}D$ threshold and 80 MeV above the $\eta_c\eta$ threshold.

The relatively narrow width ($\sim 50$ MeV) of $f_0(980)$ into $\pi\pi$ states (where annihilation of slightly heavier $\bar{s}s$ is involved and the similar width of $a_0(980)$ into $\eta\pi^0$ (where no $\bar{s}s$ annihilation are needed) suggests that also the decay width of the new $\bar{D}D(2660)$ $0^+$ state into $\eta_c\eta$ may not be prohibitively large.

The above discussion suggests attractive interactions in both $\bar{D}D$ and $\eta_c\eta$ channels. Hence even if the $\eta_c\eta$ channel does not bind on its own it generates some threshold enhancement. The mixing of the $\eta_c\eta$ and $\bar{D}D$ channel may in turn further lower the mass of the state of interest.

All the above considerations then suggest that the new $\bar{D}D$ state predicted by our QCD inequalities – while definitely broader than the narrow $D_s(2320)$ state may still manifest in high statistics experiments. Before concluding we add few more comments:

(i) Obviously $B_s$ states analogous to the $D_s(2320)$ exist. The $\bar{B}B$ analog of our predicted $\bar{D}D$ state lies even higher above the $\eta\eta_b$ threshold. The small chromo-electric polarizability of $\eta_b$ will strongly quench the “Casimir-Polder” residual interactions between $\eta_b$ and $\eta$.

(ii) An issue omitted in the above in the role played by annihilation of the “heavy” $\bar{s}s$ pairs in $f_0(980)$ which again is assumed to be an $\bar{s}s(\bar{u}u + \bar{d}d)$ $0^{++}$, $I = 0$ state. The relative small $\Gamma \sim 50$ MeV of $f_0(980) \to \pi\pi$ $S$-wave decay despite the huge Q value indicates that to a large extent the “Zweig rule” is operative and the annihilation of $\bar{s}s$ pairs is suppressed.

(iii) It has been suggested that the new BaBar relatively light $D_s(0^+)$ state could be interpreted as the “parity doublet partner” of $D_s (0^-)$ state. Parity doubling is a vestige of a world where the $S\chi SB$ transition has not occurred and the $Q_5$ symmetry generated via $I_5^\pm = \int d^3x \bar{u}(x)\gamma_5\gamma_0d(x)$; or $\bar{d}(x)\gamma_5\gamma_0u(x)$ and $I_5^3$ is linearly realized via parity doublets. Superficially this seems to be an altogether different interpretation than the states $Q\bar{Q}'\bar{q}q$ with one extra pair of light quarks considered here. This need not be the case. Parity doubling, i.e. degeneracy of $S$ and $P$-wave states is extremely difficult to understand in a simple N.R. quark model due to the prohibitive cost of $P$-
wave $q\bar{q}$ excitations (a point which we belabored in the first section). Indeed unbroken chiral symmetry does not hold for most composite models. We can incorporate this feature by assuming that hadrons contain an “infinite” number of $\bar{q}q$ pairs. In this case $I^{(+)5}_5$ when acting on a fermion, generates an extra ($\bar{u}d$) pair which blends into the “coherent” state with many pairs producing the parity doublet without any energy penalty.

The introduction of a light $\bar{q}q$ pair into the $c\bar{s}$ (or $Q'\bar{Q}$) $0^-$ ground state was motivated above as a way to achieve a lower energy realization of a $0^+$ state than via a simple $P$-wave excitation. This can be viewed as a modest, first step towards the infinitely many light quark anti-quark pairs implicit when ideal parity doubling holds.

The spectroscopy of the $c\bar{c}$ system is well understood and ALL the four $P$-wave states are accounted for. Finding below $D\bar{D}$ threshold the new $c\bar{c}u\bar{u}$ $I=0$ $0^{++}$ state that QCD inequalities suggest (if the four quark interpretation of $D_s(2320)$ and $f_0(980)$ is correct) would constitute compelling evidence for four quark states.

At the present time the discovery of one of the missing $1^+ c\bar{s}$ state at 2460 MeV is deemed to favor interpreting $D_s(2320)$ as a $P$-wave excitation of two quarks. Clearly such an interpretation will be bolstered if the missing heavier and broad $0(+) c\bar{s}$ state and the new $c\bar{c}$ state discussed here will not be discovered.

We note that the KM favoured $b \to c\bar{c}s$ constituting $\sim 20\%$ of all B decays may offer good hunting grounds for the new particle which predominantly decays into $\eta_c\eta$.

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