On the Flatness of Immediate Observation Petri Nets *

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Abstract. In a previous paper \cite{12} we introduced immediate observation (IO) Petri nets, a class of interest in the study of population protocols (a model of distributed computation), and enzymatic chemical networks. We showed that many problems for this class are \textsc{PSPACE}-complete, including parameterized problems asking whether an infinite set of Petri nets with the same underlying net but different initial markings satisfy a given property. The proofs of \textsc{PSPACE} inclusion did not provide explicit algorithms, leaving open the question of practical verification procedures. In the first part of this paper we show that IO Petri nets are globally flat, thus allowing their safety properties to be checked by efficient symbolic model checking tools using acceleration techniques, like FAST \cite{3}.

In the second part, we extend IO nets in two natural ways: by lifting the restriction on the number of so-called "observed" places, and by lifting the restriction on so-called "destination" places. The first extension proves to be essentially equivalent to the IO model. The second extension however is much more expressive and is no longer globally flat, but we show that its parametrized reachability, coverability and liveness problems remain decidable in \textsc{PSPACE}. Additionally, we observe that the pre-image computation for this second extension is locally flat, which allows application of tools like FAST to the reachability problem. This second class captures in a simple way the core reason that the IO models allow verification in \textsc{PSPACE}, and we believe it is of independent theoretical interest.

Keywords: Petri Nets · Reachability Analysis · Parameterized Verification · Flattability · Model Checking

1 Introduction

We continue the study started in \cite{12} of the theory of immediate observation Petri nets, a class of Petri nets having applications to the study of such distributed computation models as population protocols and chemical reaction networks.

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Immediate observation Petri nets (IO nets) follow the definition of immediate observation population protocols introduced by Angluin et al. in their seminal paper on the expressive power of population protocols [1]. In an IO net each transition is defined by three places: the source place, the destination place, and the observed place. The transition can move one token from the source place to the destination place, provided that the observed place is not empty (if the observed place coincides with the source place, there should be two tokens, and one of them is moved). In our previous paper [12], we studied the complexity of verification problems for IO nets. We showed that the standard verification problems for IO nets, like reachability, are in \textit{PSPACE}. Moreover, these results also apply when the set of initial or final markings is specified as a \textit{cube}, i.e. a set of markings obtained by specifying a lower bound and an upper (possibly infinite) bound on the token count for each place. While these \textit{PSPACE} bounds are much lower than the bounds for verification problems in general Petri nets, our results still suggest high cost of verification in the worst case, and leave open the question whether verification can be made practical in a typical case.

In the first part of this paper we strengthen our results on the complexity of verification for IO nets by demonstrating that IO nets are globally flat. This means that there is a single sequence of transitions $t_1\ldots t_k$ such that for all markings $M' \xrightarrow{t_1} M$, we have $M' \xrightarrow{t_1\ldots t_k} M$ for some $j_1,\ldots, j_k \geq 0$. Global flatness of IO nets allows application of existing symbolic model checking tools like FAST [3], LASH [5] and TREX [2]. The tools track semilinear sets obtained by iterated application of transitions (acceleration) or fixed combinations of transitions, and include heuristics that improve convergence speed for models that are far from the worst case.

We then turn our attention to the possible extensions of the class of IO nets. The definition of IO nets is originally rooted in the study of population protocols with binary interaction. For this reason, the definition includes some requirements that are implicitly expected in the original context but become just a technical restriction in the case of Petri nets.

First we lift the requirement that each transition has exactly one observed place. This restriction is purely technical and doesn’t affect the model behaviour. We define immediate multiple observation (IMO) nets in a way that is a straightforward generalisation of IO nets. We observe that such nets have the same behaviour with a minor technical change if we consider precise bounds. In particular, the same proof as for IO nets shows that IMO nets are globally flat.

Then we consider a more significant generalisation which captures the underlying reason why IO nets have \textit{PSPACE} verification complexity. One of the properties of IO nets is the fact that every transition only consumes tokens from one place, making it simpler to fire the same transition repeatedly. The idea of this new generalisation, called branching immediate multiple observation (BIMO) nets, is to preserve this but lift the other restrictions, both on the observations and on the destinations. The BIMO class is a generalisation of another class of Petri nets where transitions cannot remove tokens from more than one
place at a time: basic parallel processes introduced in [6]. It is shown in [6] that reachability for basic parallel processes (BPP) is \textsc{NP}-complete.

The BIMO generalisation also has a simple description in terms natural for Petri nets. One way to describe IO nets is to say that for every transition \( t \), \( |\*t| = |\*t^*| = 2 \) and \( |\*t - \*t^*| = |\*t^* - \*t| = 1 \) where the difference of two multisets is a pointwise maximum of zero and the difference. In a similar way, BPP nets are described by the single condition \( |\*t| = 1 \). Our new class of BIMO nets, described by the condition \( |\*t - \*t^*| = 1 \), generalizes both of these classes.

As a generalisation of IO nets, reachability, coverability, liveness and liveness for BIMO nets are \textsc{PSPACE}-hard. However, a BIMO net can have a non-semilinear reachability relation, and therefore some BIMO nets are not globally flat. Nevertheless, we prove that reachability, coverability, liveness and similar problems are still \textsc{PSPACE}-complete for BIMO nets. Furthermore, the cube-to-cube reachability problem is still in \textsc{PSPACE} and the pre-image of a cube is a finite union of cubes of exponential norm. We also prove that the pre-image computation in BIMO nets is locally flat, which implies that tools like \textsc{FAST} can still be applied to reachability and coverability problems in BIMO nets.

Finally, the last section discusses some model checking implications of the global and local flatness of IO and IMO nets.

For space reasons, all missing proofs and some technical details are relegated to the appendix.

2 Preliminaries and Definitions

**Multisets.** A multiset on a finite set \( E \) is a mapping \( C: E \to \mathbb{N} \), i.e. for any \( e \in E \), \( C(e) \) denotes the number of occurrences of element \( e \) in \( C \). Let \( \{e_1, \ldots, e_n\} \) denote the multiset \( C \) such that \( C(e) = |\{j \mid e_j = e\}| \). Operations on \( \mathbb{N} \) like addition or comparison are extended to multisets by defining them component-wise on each element of \( E \). Subtraction is allowed in the following way: if \( C, D \) are multisets on set \( E \) then for all \( e \in E \), \( (C - D)(e) = \max(C(e) - D(e), 0) \). We define \( |C| \) \( \overset{\text{def}}{=} \sum_{e \in E} C(e) \) the sum of the occurrences of each element in \( C \). Given a total order \( e_1 \prec e_2 \prec \cdots \prec e_n \) on \( E \), a multiset \( C \) can be equivalently represented by the vector \((C(e_1), \ldots, C(e_n)) \in \mathbb{N}^n\).

**Place/transition Petri nets with weighted arcs.** A Petri net \( N \) is a triple \((P, T, F)\) consisting of a finite set of places \( P \), a finite set of transitions \( T \) and a flow function \( F: (P \times T) \cup (T \times P) \to \mathbb{N} \).

A marking \( M \) is a multiset on \( P \), and we say that a marking \( M \) puts \( M(p) \) tokens in place \( p \) of \( P \). The size of \( M \), denoted by \( |M| \), is the total number of tokens in \( M \). The preset \( \*t \) and postset \( \*t^* \) of a transition \( t \) are the multisets on \( P \) given by \( \*t(p) = F(p, t) \) and \( \*t^*(p) = F(t, p) \). A transition \( t \) is enabled at a marking \( M \) if \( \*t \leq M \), i.e. \( \*t \) is component-wise smaller or equal to \( M \). If \( t \) is enabled then it can be fired, leading to a new marking \( M' = M - \*t + \*t^* \). We note this \( M \xrightarrow{t} M' \).
Reachability and coverability. Given \( \sigma = t_1 \ldots t_n \) we write \( M \xrightarrow{\sigma} M_n \) when \( M \xrightarrow{t_1} M_1 \xrightarrow{t_2} \ldots \xrightarrow{t_n} M_n \), and call \( \sigma \) a firing sequence. We write \( M' \xrightarrow{\sigma} M'' \) if \( M' \xrightarrow{\sigma} M'' \) for some \( \sigma \in T^* \), and say that \( M'' \) is reachable from \( M' \). A marking \( M \) covers another marking \( M' \), written \( M \geq M' \) if \( M(p) \geq M'(p) \) for all places \( p \). A marking \( M \) is coverable from \( M' \) if there exists a marking \( M'' \) such that \( M' \xrightarrow{\sigma} M'' \geq M \).

We call reachability relation the set of pairs of markings \((M, M')\) such that \( M \xrightarrow{\sigma} M' \), and we denote it \( \xrightarrow{\sigma} \). The sets of predecessors and successors of a set \( M \) of markings of \( N \) are defined as follows: \( \text{pre}^*(M) \overset{\text{def}}{=} \{ M' \mid \exists M \in \mathcal{M} . M' \xrightarrow{\sigma} M \} \), and \( \text{post}^*(M) \overset{\text{def}}{=} \{ M \mid \exists M' \in \mathcal{M} . M' \xrightarrow{\sigma} M \} \).

2.1 Flat Petri nets

Here we define the global flatness following [16]. The systems called flat (non-globally) in [16] we call locally post*-flat.

**Definition 1.** Let \( N = (P, T, F) \) be a Petri net.

- \( N \) is globally flat if there exist transition words \( w_1, w_2, \ldots, w_k \in T^* \) such that for every \((M', M)\), \( M' \xrightarrow{\sigma} M \) if and only \( M' \xrightarrow{w_1^{j_1} \ldots w_k^{j_k}} M \) for some \( j_1, \ldots, j_k \geq 0 \).
- \( N \) is locally \( \text{pre}^* \)-flat (respectively locally post*-flat) if for every \( M \) (respectively \( M' \)) there exist transition words \( w_1, w_2, \ldots, w_k \in T^* \) such that for every \( M' \) (respectively \( M \)) , \( M' \xrightarrow{\sigma} M \) if and only \( M' \xrightarrow{w_1^{j_1} \ldots w_k^{j_k}} M \) for some \( j_1, \ldots, j_k \geq 0 \).

We introduce a new measure of the length of firing sequences.

**Definition 2.** Let \( N \) be a Petri net, and let \( \sigma \) be a firing sequence. Let \((k_1, \ldots, k_m)\) be the unique tuple of positive natural numbers such that \( \sigma = t_1^{k_1} t_2^{k_2} \ldots t_m^{k_m} \) and \( t_i \neq t_{i+1} \) for every \( i = 1, \ldots, m - 1 \). We say that \( \sigma \) has accelerated length \( m \), and let \( |\sigma|_a \) denote the accelerated length of \( \sigma \).

2.2 Counting constraints

We recall the formalism of counting sets, which are possibly infinite sets of markings, and their finite representation called counting constraints. The notations and definitions are the same as in our previous article [12].

**Counting sets and counting constraints.** Let \( N = (P, T, F) \) be a Petri net. A set of markings \( \mathcal{C} \) is a cube if there exist mappings \( L : Q \to \mathbb{N} \) and \( U : Q \to \mathbb{N} \) such that marking \( M \) is in \( \mathcal{C} \) if and only if \( L \leq M \leq U \). Mappings \( L \) and \( U \) are called the upper bound and lower bound of \( \mathcal{C} \) respectively, and we call \((L, U)\) the representation of \( \mathcal{C} \). A cube represented by \((L, U)\) is denoted \([L, U]\). A counting constraint \( \Gamma \) is a finite set of representations of cubes \{\((L_1, U_1), \ldots, (L_k, U_k)\)\}. 

Counting constraint $\Gamma$ represents the set $[\Gamma] \coloneqq (L_1, U_1) \cup \ldots \cup (L_k, U_k)$. A set of markings $\mathcal{S}$ represented by a counting constraint $\Gamma$ is called a counting set.

Notice that a cube has a unique representation; we allow ourselves to use $\mathcal{C}$ for both the counting set and the counting constraint. However there can be more than one counting constraint for one counting set. For example $\{(1, 3), (2, 4)\}$ and $\{(1, 4)\}$ define the same counting set, the cube $[1, 4]$.

**Measures of counting constraints.** The $l$-norm and the $u$-norm of a cube $\mathcal{C} = (L, U)$ are defined by:

$$||\mathcal{C}||_l = \sum_{p \in P} L(p) \quad ||\mathcal{C}||_u = \sum_{p \in P} U(p) \quad \text{(and 0 if } U(p) = \infty \text{ for all } p).$$

The $l$-norm (respectively $u$-norm) of a counting constraint $\Gamma = \llbracket \mathcal{C}_1, \ldots, \mathcal{C}_k \rrbracket$ is the maximum $l$-norm (respectively $u$-norm) of a cube in $\Gamma$, i.e.:

$$||\Gamma||_l \coloneqq \max_{i \in \{1, m\}} \{||\mathcal{C}_i||_l\} \quad ||\Gamma||_u \coloneqq \max_{i \in \{1, m\}} \{||\mathcal{C}_i||_u\}.$$

The $l$-norm (respectively $u$-norm) of a counting set $\mathcal{S}$ is the smallest $l$-norm (respectively $u$-norm) of a counting constraint representing $\mathcal{S}$.

Proposition 5 of [10] rewritten below shows that a boolean combination of counting sets is still a counting set and gives bound on the norms of counting constraints representing these combinations.

**Proposition 1.** Let $\Gamma_1, \Gamma_2$ be counting constraints.

- There exists a counting constraint $\Gamma$ with $[\Gamma] = [\Gamma_1] \cup [\Gamma_2]$ such that $||\Gamma||_u \leq \max\{||\Gamma_1||_u, ||\Gamma_2||_u\}$ and $||\Gamma||_l \leq \max\{||\Gamma_1||_l, ||\Gamma_2||_l\}$.
- There exists a counting constraint $\Gamma$ with $[\Gamma] = [\Gamma_1] \cap [\Gamma_2]$ such that $||\Gamma||_u \leq ||\Gamma_1||_u + ||\Gamma_2||_u$ and $||\Gamma||_l \leq ||\Gamma_1||_l + ||\Gamma_2||_l$.
- There exists a counting constraint $\Gamma$ with $[\Gamma] = \mathbb{N}^n \setminus [\Gamma_1]$ such that $||\Gamma||_u \leq n||\Gamma_1||_l$ and $||\Gamma||_l \leq n||\Gamma_1||_u + n.$

3 IO nets are globally flat

In this section, we show that the class of immediate observation Petri nets is globally flat. First we define this class and recall previous results.

3.1 IO nets

**Definition 3.** A transition $t$ of a Petri net is an immediate observation transition if there are three places $p_s, p_d, p_o$, not necessarily distinct, such that $\mathbf{t} = \{p_s, p_o\}$ and $\mathbf{t}^* = \{p_d, p_o\}$. We call $p_s, p_d, p_o$ the source, destination, and observed places of $t$, respectively. A Petri net is an immediate observation net if and only if all its transitions are immediate observation transitions.
Example 1. Figure 1 illustrates an IO net taken from the literature on population protocols [1]. The places of the net are represented by circles, the transitions by squares and the flow function by weighted arcs. Let $M_0$ be a marking with tokens only in $p_1$. Then place $p_3$ is reachable from $M_0$ if and only if $M_0(p_1) \geq 3$.

We define the parameterized versions of the classic problems of reachability, coverability and liveness.

- Cube-reachability: Given a net $N$ and cubes $C, C'$ of $\mathbb{N}$, decide if there are markings $M \in C$ and $M' \in C'$ such that $M$ is reachable from $M'$.
- Cube-coverability: Given a net $N$ and cubes $C, C'$ of $\mathbb{N}$, decide if there are markings $M \in C$ and $M' \in C'$ such that $M$ is coverable from $M'$.
- Cube-liveness: Given a net $N$ and a cube $C$ of $\mathbb{N}$, decide whether every marking of $C$ is live. Recall that a marking $M_0$ is live if for every marking $M$ reachable from $M_0$ and for every transition $t$ of $N$, some marking reachable from $M$ enables $t$.

In our previous paper [12], we showed that these problems are PSPACE-complete for IO nets. The PSPACE-hardness proofs used a simulation of bounded tape Turing machines by IO nets. The PSPACE inclusion proofs derived from good properties of the IO reachability relation, and in particular that the pre-image $\text{pre}^*$ and post-image $\text{post}^*$ of a counting set is a counting set.

**Theorem 1 (Theorem 6, [12]).** Let $N$ be an IO net with place count $n$, and let $S$ be a counting set. Then $\text{pre}^*(S)$ is a counting set and we can bound its norm by

$$\|\text{pre}^*(S)\|_u \leq \|S\|_u \text{ and } \|\text{pre}^*(S)\|_l \leq \|S\|_l + n^3.$$ 

The same holds for $\text{post}^*$ by using the net with reversed transitions.

Notice that since a counting set is a finite union of cubes, the problems over counting sets instead of cubes are also PSPACE-complete.

### 3.2 Bunch matrices

We rephrase Section 6 of our previous paper [12] in terms of bunch matrices, which we define below. The detailed explanation of the rephrasing can be found in the appendix.
In the following, for each IO net \( N = (P, T, F) \) of place count \( |P| = n \), we choose an arbitrary ordering \( p_1, \ldots, p_n \) of the places of \( P \).

**Theorem 2.** For each IO net \( N = (P, T, F) \) of place count \( n \), there exists a set \( B \) of \( n \times n \) matrices called bunch matrices and such that each bunch matrix \( B \) has an associated value called the accelerated length \( |B|_a \) of \( B \). The following properties hold:

(i) For each two markings \( M' \) and \( M \), \( M' \xrightarrow{\sigma} M \) iff there exists a bunch matrix \( B \) such that
\[
M'(p_i) = \sum_j B_{i,j} \\
M(p_i) = \sum_j B_{j,i}
\]
for every \( p_i \in P \). We call source marking and target marking of \( B \) the markings \( M' \) and \( M \) respectively. Moreover, there exists a firing sequence \( \sigma \) of accelerated length \( |\sigma|_a = |B|_a \) such that \( M' \xrightarrow{\sigma} M \).

(ii) Let \( B \) be a bunch matrix such that \( B_{i,j} > n \) for some indices \( i, j \). Let \( B' \) be the matrix equal to \( B \) everywhere except on \( i, j \) where \( B'_{i,j} = n \). Matrix \( B' \) is also a bunch matrix, and \( |B'|_a \leq |B|_a \).

(iii) Let \( B \) be a bunch matrix, and let \( B' \) be the matrix equal to \( B \) everywhere except on one index \( i, j \) where \( B_{i,j} \geq 1 \) and \( B'_{i,j} > B_{i,j} \). Matrix \( B' \) is also a bunch matrix, and \( |B'|_a \leq |B|_a \).

(iv) For \( B \) a bunch matrix, \( |B|_a \leq (n^2 \times \max_{i,j} B_{i,j} + 1)^n \).

The underlying idea is that since there is no token creation or destruction in IO nets, we can de-anonymize the tokens and consider their trajectories through a firing sequence. Informally, a de-anonymization of a firing sequence \( M' \xrightarrow{\sigma} M \) is summarized by a bunch matrix \( B \), where the number \( B_{i,j} \) represents the number of tokens that go from \( p_i \) in \( M' \) to \( p_j \) in \( M \).

**Example 2.** For example, consider the firing sequence \( t_3 t_1 t_3 t_4 \) from marking \((3, 0, 1)\) to marking \((0, 0, 4)\) in the IO net of Figure 1. In Figure 2 we de-anonymize the firing sequence into a possible combination of trajectories of its tokens. The corresponding bunch matrix is \[
\begin{pmatrix}
0 & 0 & 3 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] as there are 3 tokens that
go from $p_1$ to $p_3$ and 1 token that goes from $p_3$ to $p_3$. Note that in general de-anonymization and bunch matrices are not uniquely defined with regards to a firing sequence.

### 3.3 IO nets are globally flat

Theorem 2 allows us to bound the accelerated length of firing sequences, and the global flatness of IO nets is a consequence of this.

**Lemma 1.** Let $N = (P, T, F)$ be an IO net of place count $n$. For every pair of markings $(M', M)$ of $N$ such that $M' \xrightarrow{\sigma} M$, there exists a firing sequence $\sigma$ of accelerated length $|\sigma|_a \leq (n^3 + 1)^n$ such that $M' \xrightarrow{\sigma} M$.

This immediately gives us our main theorem.

**Theorem 3.** Immediate observation nets are globally flat.

**Proof.** All firing sequences of accelerated length at most $(n^3 + 1)^n$ can be expressed in the language

$$\left( \prod_{t \in T} t^* \right)^{(n^3+1)^n}$$

where $\prod_{t \in T} t^*$ is the concatenation of the starred iterations of the transitions of $N$. This language realizes the global flatness of net $N$ and it is the concatenation of $|T|(n^3 + 1)^n$ words of accelerated length 1.

A consequence of IO nets being globally flat is that their reachability relation is semilinear [13]. Using the result on bunch matrices, we exhibit an exponential size semilinear set that realizes this.

**Theorem 4.** Let $N$ be an IO net of place count $n$. The reachability relation of $N$ is semilinear, and there exists a semilinear set realizing this of size at most exponential in $n$.

**Proof (Sketch).** We prove the result by exhibiting a set of bases $B_n$ and periods such that $B$ is the semi-linear set $\bigcup_{b \in B_n} L(b, P_b)$ where $P_b$ is the set of periods of a base $b \in B_n$ and $L(b, P_b)$ is the linear set of base $b$ and period set $P_b$:

- The set of bases $B_n$ is the bunch matrices with $B_{i,j} \leq n$ for each pair $(i, j)$ of indices.
- For $b \in B_n$, the periods of $P_b$ are the subclass of unit matrices $P_b = \{ e_{i,j} \mid b_{i,j} = n \}$ where $e_{i,j}$ is the matrix that has a 1 in index $i, j$ and 0 elsewhere. Notice that for $b \in \{0, \ldots, n-1\}^{n^2}$, the period sets $P_b$ are empty.

The proof that the set of bunch matrices is indeed this semilinear set is in the appendix. Then the reachability relation $\rightarrow$ of $N$ is a semilinear set as the projection onto its lines and columns of the set of bunch matrices.

In the following sections we extend the immediate observation formalism, first by lifting the restriction on the number of observed tokens and then by also lifting the restriction on the number of destination tokens.
4 Extension: Immediate Multiple Observation

We consider a first natural extension of IO nets called immediate multiple observation nets, in which a token may observe multiple tokens instead of just the one.

**Definition 4.** A transition $t$ of a Petri net is an immediate multiple observation (IMO) transition if there are places $p_s, p_d, p_{o_1}, p_{o_2}, \ldots, p_{o_k}$ for some $k \in \mathbb{N}$, not necessarily distinct, such that $t = \{p_s, p_{o_1}, \ldots, p_{o_k}\}$ and $t^* = \{p_d, p_{o_1}, \ldots, p_{o_k}\}$. We call $p_s$ and $p_d$ the source and destination places of $t$, and $p_{o_1}, \ldots, p_{o_k}$ the observed places of $t$. A Petri net is an immediate multiple observation net if and only if all its transitions are IMO transitions.

We let $m_o = \max_{t \in T} \{\min(\bullet t(p), \bullet t^*(p)) \mid p \in \bullet t \cap \bullet t^*\}$ denote the maximal observation degree of $N$, that is the maximal number of tokens needed in an observed place to fire $t$, for any $t \in T$.

Notice that IMO nets are conservative Petri nets, like IO nets.

![Fig. 3: An IMO net.](image)

**Example 3.** Figure 3 is an IMO net such that $p_2$ is reachable from a marking $M_0$ with only tokens in $p_1$ if and only if $M_0(p_1) \geq 3$. In contrast to the IO net of Figure 1 this “threshold check” can be realized in only one transition.

We present a a result for IMO nets analog to Theorem 2 for IO nets.

**Theorem 5.** For each IMO net $N$ of place count $n$ and maximal observation degree $m_o$, there exists a set $B$ of $n \times n$ matrices called bunch matrices and such that each bunch matrix $B$ has an associated value called the accelerated length $|B|_a$ of $B$. The following properties hold:

(i) For each two markings $M'$ and $M$, $M' \xrightarrow{\sigma} M$ iff there exists a bunch matrix $B$ such that

$$M'(p_i) = \Sigma_j B_{i,j}$$

$$M(p_i) = \Sigma_j B_{j,i}$$

for every $p_i \in P$. We call source marking and target marking of $B$ the markings $M'$ and $M$ respectively. Moreover, there exists a firing sequence $\sigma$ of accelerated length $|\sigma|_a = |B|_a$ such that $M' \xrightarrow{\sigma} M$. 


(ii) Let $B$ be a bunch matrix such that $B_{i,j} > n$ for some indices $i,j$. Let $B'$ be the matrix equal to $B$ everywhere except on $i,j$ where $B'_{i,j} = n \cdot m_o$. Matrix $B'$ is also a bunch matrix, and $|B'|_a \leq |B|_a$.

(iii) Let $B$ be a bunch matrix, and let $B'$ be the matrix equal to $B$ everywhere except on one index $i,j$ where $B_{i,j} \geq 1$ and $B'_{i,j} > B_{i,j}$. Matrix $B'$ is also a bunch matrix, and $|B'|_a \leq |B|_a$.

(iv) For $B$ a bunch matrix, $|B|_a \leq (n^2 \cdot \max_{i,j} B_{i,j} + 1) ^ n$.

The idea is that we can reuse the exact same proofs and formalisms as for IO nets except that instead of making sure there is at least one token in each observed place, we make sure there are at least $m_o$. The proof details are in the appendix.

From this result it follows that we can prove the same results for IMO as we could for IO in our previous paper [12]: the backward and forward reachability sets of a counting set are counting sets, and the cube-reachability, cube-coverability and cube-liveness problems are PSPACE-complete.

**Theorem 6.** Let $N$ be an IMO net with place count $n$ and maximal observation degree $m_o$, and let $S$ be a counting set. Then $pre^*(S)$ is a counting set and we can bound its norm by

$$\|pre^*(S)\|_u \leq \|S\|_u \text{ and } \|pre^*(S)\|_l \leq \|S\|_l + m_o \cdot n^3.$$  

The same holds for $post^*$ by using the net with reversed transitions.

**Theorem 7.** The cube-reachability, cube-coverability and cube-liveness problems for immediate multiple observation nets are PSPACE-complete.

The proof details are in the appendix, but they differ very little from the IO proofs.

Like for IO nets, we can now also prove that IMO nets are globally flat via a bound on the accelerated length of the firing sequences.

**Lemma 2.** Let $N = (P, T, F)$ be an IMO net of place count $n$ and maximal observation degree $m_o$. For every pair of markings $(M', M)$ of $N$ such that $M' \not\rightarrow M$, there exists a firing sequence $\sigma$ of accelerated length $|\sigma|_a \leq (n^3 \cdot m_o + 1)^n$ such that $M' \not\rightarrow M$.

And this yields the result:

**Theorem 8.** Immediate multiple observation nets are globally flat.

5 Extension: Branching Immediate Observation

We now consider another extension of IO nets called branching immediate multiple observation nets, in which a token may observe multiple tokens and there may be more than one destination place. This new extension is not globally flat
anymore, but it does continue to be verifiable in \textbf{PSPACE}, and the backward (but not forward) reachability set of a counting set is a counting set. In addition, BIMO nets are locally pre*-flat, thus allowing the use of symbolic model checking tools like FAST. In a sense, this model captures the restriction on general Petri nets that makes IO (and IMO) \textbf{PSPACE} verifiable.

\textbf{Definition 5.} A transition $t$ of a Petri net is an branching immediate multiple observation (BIMO) transition if $|\bullet t - t^*| \leq 1$. We remind the reader that we have defined multiset subtraction as the component-wise maximum between the multiset subtraction and 0. A Petri net is a branching immediate multiple observation net if and only if all its transitions are BIMO transitions.

We let $m_o = \max \{\min(\bullet t(p), t^*(p)) \mid t \in T, p \in \bullet t \cap t^*\}$ denote the maximal observation degree of $N$, that is the maximal number of tokens needed in an observed place to fire $t$, for any $t \in T$.

We let $o_d = \max \{\max(\bullet t(p) - \bullet t^*(p), 0) \mid t \in T, p \in t^*\}$ denote the maximal output degree of $N$, that is the maximal number of tokens added to a destination place for any $t \in T$.

The interpretation of BIMO transitions in terms of observation is as follows:

- either $t$ is of the form $\bullet t = \langle p_s, p_{o1}, \ldots, p_{ok} \rangle$ and $t^* = \langle p_{d1}, \ldots, p_{dl}, p_{o1}, \ldots, p_{ok} \rangle$,
- or $t$ is of the form $\bullet t = \langle p_{o1}, \ldots, p_{ok} \rangle$ and $t^* = \langle p_{d1}, \ldots, p_{dl}, p_{o1}, \ldots, p_{ok} \rangle$,

with $p_s$ the source place, $p_{d1}, \ldots, p_{dl}$ the destination places, and $p_{o1}, \ldots, p_{ok}$ the observed places of $t$, all of them not necessarily distinct. In other words, each BIMO transition is such that there is at most one source place. Notice that contrary to IO and IMO transitions, a BIMO transition can destroy tokens (via transitions $t$ with $t^* = \langle p_{o1}, \ldots, p_{ok} \rangle$) or create tokens (via transitions $t$ with $|t^*| \geq |t|$). Thus BIMO nets are \textit{not} conservative Petri nets, unlike IMO nets.

\textbf{Remark 1.} One can also define \textit{merging immediate multiple observation} (MIMO) nets. It is easy to see that reversing all the transitions in a BIMO net yields a MIMO net, and vice versa. Our results for BIMO nets immediately imply similar result for MIMO nets with changes implied by this reversal, e.g. MIMO nets are locally post*-flat but not locally pre*-flat.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{bimo_net_example.png}
\caption{A BIMO net example}
\end{figure}
Example 4. The Petri net of Figure 4 represents a client server interaction. If the server $S$ observes a client $C$, it creates a worker $W$. Worker $W$ creates a response $R$ and terminates, and a client $C$ “leaves” after having observed a response. The saved response expires after some time. This net is a BIMO net.

We start by showing that BIMO nets are not globally flat.

Proposition 2. Branching immediate multiple observation nets are not globally flat.

Proof. The 3-dimensional VASS of Lemma 2.8 of [14], with states $p, q$, counters $c_1, c_2, c_3$ and transitions $p \xrightarrow{0.1,-1} p, q \xrightarrow{0.0,0} q, q \xrightarrow{1.0,0} p$, has a non-semilinear reachability set. This VASS can be rewritten as an equivalent BIMO net using standard VASS to Petri net techniques, as illustrated in Figure 5. Therefore the reachability relation of this BIMO net is not semilinear. If a Petri net is globally flat then it has a semi-linear reachability set (see Corollary 3 of [7]), and so this example shows that arbitrary BIMO nets are not globally flat.

![Fig. 5: A non-flat BIMO net.](image)

The proof of Proposition 2 shows that even if we do not allow destruction of tokens, an arbitrary BIMO net is still not globally flat. In the rest of this section, we describe a technical procedure of firing sequence transformation which then allows us to prove results for BIMO nets.

5.1 Colored Executions

The general idea is to keep track of three possible place states: white, black and red. Tokens in a white place will be preserved or further split later along the execution. A black place contains some tokens that will be destroyed later. A red place contains a lot of tokens.

For the rest of this section, we fix a BIMO net $N = (P, T, F)$ of place count $n$, and maximal observation degree $m_o$. Additionally, for a transition $t$, we call $q$ a source place of $t$ if $t^*(q) < t(t(q)$ and a destination place of $t$ if $t^*(q) < t^*(q)$.
Definition 6. A colored marking $C$ is a marking with additional data: each place is declared black, red, or white.

A colored step is a pair of colored markings $C', C$ such that there exists a transition $t \in T$ for which the following conditions hold:

(i) Each destination place of $t$ may only change its color from black in $C'$ to red in $C$. The source place of $t$ may only change its color to white in $C$, but only if no destination place is white in $C'$ (and therefore in $C$). The remaining places must keep the same color in $C'$ and $C$.

(ii) If $t$ has a source place and it is white in $C'$, at least one of the destination places of $t$ must also be white in $C'$, and therefore also in $C$ (in particular, there must be at least one destination place).

(iii) Transition $t$ can be fired $l \geq 0$ times to obtain $C$ from $C'$ (ignoring the colors of both markings).

(iv) All the observation requirements of $t$ are satisfied in $C'$, even if the two markings coincide (in particular in the case $l = 0$).

Definition 7. Let $N$ be a BIMO net. A colored execution $\rho$ with target size $m$ is a sequence $C_1 C_2 \ldots C_k$ of colored markings of $N$ with the following properties:

- Any two adjacent colored markings $C_i, C_{i+1}$ form a colored step.
- All the places of $C_k$ are white and $C_k$ has $m$ tokens, i.e. $|C_k| = m$.

We call $k$ the length of the colored execution, and denote it $|\rho|$.

Example 5. We consider two firing sequences $\sigma_1, \sigma_2$ on the net of Figure 4. The first sequence $\sigma_1$ starts with one token in $S$ and three tokens in $C$, as shown on Figure 4 and $\sigma_1 = t_2^{20} t_3^{15} t_1^{50} t_3^{15} t_4^{104} t_1^{20} t_4^{80} t_3^{104}$. The second sequence $\sigma_2$ starts with one token in $S$ and one token in $C$, and $\sigma_2 = t_2^{20} t_3 t_4 t_2 t_4$. The corresponding
sequences of markings are represented in the tables of Figure 6, where the successive columns are the successive markings, and each line of a column contains the number of tokens in the state $S, C, W$ or $R$ corresponding to the line.

We build colored executions from these firing sequences by assigning colors to the places of the markings of $\sigma_1, \sigma_2$. Observe that in both sequences, place $C$ must be black until the last firing of $t_3$ and place $R$ must be black until the last firing of $t_4$, as $t_3$ and $t_4$ have no destination places. As $t_2$ is only fired before $t_4$, $W$ must be black until the last firing of $t_2$. On the other hand, $S$ is never the source place of a transition and therefore to be white at the end it must be white throughout. Following these observations, we obtain the colored executions $\rho_1$ and $\rho_2$ represented in Figure 7.

The process for obtaining a coloured execution can be generalised.

**Lemma 3.** Let $M', M$ be markings of $N$. For each firing sequence $\sigma$ such that $M' \xrightarrow{\sigma} M$, there is a colored execution $\rho$ with the same initial and final marking (with some colors assigned to each place) such that $|\rho| = |\sigma| + 1$.

For each colored execution $\rho$ there is a firing sequence $\sigma$ from the initial marking to the final marking (ignoring colors) of $\rho$ with accelerated length $|\sigma| \leq |\rho| - 1$.

The rest of the section formalizes our intention to have red places contain enough tokens that we can ignore their exact count. We give some intuition before specifying the technical details. Firstly, each red place should have more than the total final amount of tokens $m$, and if a red place is observed, it should have more than $m_o$ tokens. We also need to avoid accidentally spending too many tokens of a red place before the moment it changes from red to white. This will be achieved by skipping the transitions where all the destinations are red. If there are transitions from red places that transfer many tokens to black places, then these are made red as well. The fewer black places there are, the fewer tokens are needed to turn the remaining black places red. And so the token threshold for a place to be red becomes smaller as the count $B$ of black places diminishes.

**Definition 8.** Given a target size $m$ and the maximal observation degree $m_o$, we call $m_* = \max(m, m_o)$ the instance size. In general, this is the same as $m$ and it is only used to avoid special cases with a large observation degree and very small target size.

We call a marking red-balanced for target size $m$ if all the red places have at least $\max(m_* n^B, m_* n^{B+1} - b - w)$ tokens, where $B$ is the number of black places, and $b$ and $w$ is the number of tokens in the black and white places, correspondingly.

We call a colored marking red-black-balanced for a target size $m$, if each red place has at least $m_* n^{B+1} - b - w$ tokens and each black place has less than $m_* n^B$ tokens.

Notice that a colored marking with only white places is red-black-balanced.
Lemma 4. A red-black-balanced marking is also red-balanced and each red place has strictly more than $m_B n^B$ tokens. Each red-balanced marking can be made red-black-balanced by making some black places red.

Lemma 5. Let $\rho$ be a colored execution of initial marking $C_0$ with target size $m$. Let $\tilde{C}_0$ be a colored marking such that the pair $(C_0, \tilde{C}_0)$ satisfies the following conditions:

- Each place of $\tilde{C}_0$ has the same color as in $C_0$ or is red instead of black.
- Each white place has the same number of tokens as in $C_0$.
- Each black place of $\tilde{C}_0$ has at least as many tokens as in $C_0$.
- $\tilde{C}_0$ is red-black-balanced for target size $m$.

Then there exists a colored execution $\tilde{\rho}$ with initial marking $\tilde{C}_0$ and the same final marking as $\rho$. Moreover, all the colored markings of $\tilde{\rho}$ are red-black-balanced for target size $m$.

Additionally, one can require that all the colored markings of $\tilde{\rho}$ have unique combinations of the set of red places, the set of black places, and the token counts in black and white places (token counts in the red places are ignored).

Proof (Construction). First we prove the claim without the uniqueness condition. This is proven using induction over the length $|\rho| \geq 1$. If there is only one colored marking then all places are white, therefore $C_0 = \tilde{C}_0$ and we can take $\tilde{\rho} \overset{\text{def}}{=} \rho$.

Otherwise, $|\rho| > 1$. Consider the first colored step $C_0, C_1$ in $\rho$ such that $C_0 \xrightarrow{t^l} C_1$ in the net $N$ for some $t \in T$ and $l \geq 0$. We construct a colored marking $\tilde{C}_1$ such that $\tilde{C}_0, \tilde{C}_1$ is a colored step and the pair $(C_1, \tilde{C}_1)$ verifies the conditions of the Lemma. We apply the induction hypothesis to $(C_1, \tilde{C}_1)$ and $\rho$ truncated of its first colored step (thus with initial marking $C_1$). The sequence of markings $\tilde{C}_0 \tilde{C}_1 \ldots$ provides us our desired colored execution $\tilde{\rho}$.

As $\tilde{C}_0$ is red-black-balanced, each place contains either at least as many tokens as in $C_0$ or at least $m_B$ tokens, so all observation conditions of $t$ are satisfied at $\tilde{C}_0$. To construct $\tilde{C}_1$, we fire $t$ from $\tilde{C}_0$ a certain number of times, in the process changing the color of any black place with $m_B n^B$ or more tokens to red. This change decreases $B$ and the threshold $m_B n^B$, allowing us to change multiple black places to red. Then, if the source place of $t$ changes its color to white between $C_0$ and $C_1$, we also make this place white in $\tilde{C}_1$. The other places of $\tilde{C}_1$ are colored as in $\tilde{C}_0$. It remains to specify how many times we fire $t$.

If any of the source or destination places of $t$ are white in $C_1$ (if they are white in $C_0$, they stay white in $C_1$), then the constraint of having the same number of tokens in all white places in $\tilde{C}_1$ and $C_1$ defines how many times we need to fire $t$. Otherwise, transition $t$ is fired at most $l$ times, stopping early if all the destination places become red. Note that this may mean zero times if all the destination places are already red.

To satisfy the uniqueness condition of the constructed colored execution, we remove the steps between two repetitions of the same combination. The
combinations may differ in the number of tokens in red places: we use the same
construction as above to adapt the subsequent part of the execution. The proof
that the above construction produces a colored execution $\hat{\rho}$ satisfying all the
requirements of the lemma can be found in the appendix.

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
S & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
C & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 0 & 0 \\
\hline
W & 0 & 20 & 15 & 50 & 50 & 64 & 53 & 53 & 0 & 0 \\
\hline
R & 0 & 0 & 5 & 5 & 5 & 5 & 16 & 16 & 69 & 69 \\
\hline
\end{tabular}
\end{center}

Fig. 8: A colored execution with red-black-balanced markings.

\textit{Example 6.} Let us construct red-black-balanced colored executions from the
colored executions $\rho_1$ and $\rho_2$ of Example 5. In both executions, the final marking
is of size $m = 1$, and the maximum observation degree of the net is $m_o = 1$, so
$m_* = 1$. Therefore the upper bounds for the number of token in a black place
are $m_*n^3 = 1 \times 4^3 = 64$ when there are 3 black places, 16 for 2 black places, and
4 for 1 black place.

In the first execution $\rho_1$, we do not change anything until place $W$ gets 100
tokens. We fire $t_1$ fewer times at this step and only put 64 tokens into $W$ and
make it red. At the next step $R$ gets a lot of tokens; we fire $t_2$ fewer times and
only put 16 tokens there, and make $R$ red. We completely skip $t_4$ on the next
step, as $R$ is not changing to white yet. When the token counts in $W$ and $R$
become 0, we make the corresponding place white.

The second execution $\rho_2$ is already red-black-balanced.

\section{5.2 Reachability and Local Flatness in BIMO nets}

Lemma 5 allows us to bound the accelerated length of firing sequences, and
the local pre'-flatness of BIMO nets is a consequence of this. It also allows
us to bound the size of the markings along firing sequences, and this implies
the \textit{PSPACE} complexity of the reachability problem for BIMO nets. With these
results, we prove that the pre-image of a cube is a counting set of bounded norm,
and that many counting set problems such as cube-reachability, cube-coverability
and cube-liveness are solvable in \textit{PSPACE}.

For the rest of this section, we fix a BIMO net $N = (P, T, F)$ of place count
$n$, maximal observation degree $m_o$ and maximal output degree $o_d$.

\textbf{Lemma 6.} Let $M', M$ be two markings of $N$, and let $|M'| = m', |M| = m,
m_* = \max(m, m_o)$. If there exists a firing sequence $\sigma$ from $M'$ to $M$, then there
exists a firing sequence $\sigma'$ from $M'$ to $M$ of accelerated length at most

$$(m_*n^n + m + 2)^n$$
and such that the intermediate markings of $\sigma'$ are of size at most

$$(m' + o_d n (m + m \cdot n^n) (m \cdot n^n + m + 2)^n) (o_d n)^n.$$  

Note that both the accelerated length and intermediate token counts are counted by a polynomial in $m, m'$ and $m_o$ with coefficients exponential in $n$ and degree itself polynomial in $n$. The size of the accelerated length and intermediate token counts is therefore exponential in the size of the input $(M', M, N)$.

This Lemma implies important results for BIMO nets.

**Theorem 9.** BIMO nets are locally $pre^*$-flat.

**Proof.** Let $N = (P, T, F)$ be a BIMO net, and $M$ be a marking of size $M$. We use essentially the same proof as in Theorem 3 all the firing sequences of length at most $(m \cdot n^n + m + 2)^n$ that reach $M$ can be expressed by a language that concatenates $(m \cdot n^n + m + 2)^n$ sequences of all the transitions of $T$, starred. This language depends on $m$ and thus it realizes the local $pre^*$-flatness of BIMO, rather than its global flatness.

**Remark 2.** Note that local $pre^*$-flatness allows us to apply symbolic execution tools to verify reachability. To verify that $M' \rightarrow M$ such tools compute either $pre^*(M)$ or $post^*(M')$, and require local $pre^*$-flatness or local $post^*$-flatness, correspondingly. In the $post^*$ case, we use the tool to find out whether $M' \rightarrow M$ in the reverse net obtained by reversing the direction of each transition, which is a MIMO net (see Remark 1).

**Theorem 10.** The reachability problem for BIMO nets is $PSPACE$-complete.

**Proof.** This problem is $PSPACE$-hard because it is $PSPACE$-hard for IO nets, which are a subclass of BIMO nets.

Let $N$ be a BIMO net of place count $n$, maximal observation degree $m_o$ and maximal output degree $o_d$. If there is a firing sequence between two markings of size $m'$ and $m$, then by Lemma 6 there is a firing sequence with intermediate markings of size exponential in the size of the input. As $NPSPACE = PSPACE$ by Savitch’s theorem, it is enough to say that this firing sequence can be guessed in polynomial space.

**Lemma 7.** Let $M$ be a marking of size $m$. The pre-image $pre^*(M)$ is a counting set and its norm is bound by

$$\|pre^*(M)\|_u \leq n \times m \cdot n^n \text{ and } \|pre^*(M)\|_l \leq n \times m \cdot n^n.$$  

In the proof we show that the token counts above $m \cdot n^n$ are not distinguishable from the point of view of reachability of $M,$ and then use Lemma 6.

**Example 7.** We consider the colored execution $\rho_2$ of Example 5. The final marking $M$ of $\rho_2$ is the marking with exactly one token in $S$. Using Lemma 5 we
show that the pre-image of $M$ is the set of markings with exactly one token in $S$.

By observation of the net, we know that the number of tokens in $S$ does not change along any firing sequence, so markings of the pre-image of $M$ must have exactly one token in $S$. Lemma 5 allows us to take as initial marking $\tilde{M}$ any marking with more tokens in the black places than the original initial marking $M' = \{S, C\}$, and construct a colored execution from $\tilde{M}$ to $M$. This yields a firing sequence to $M$ from any $\tilde{M}$ of the form $\pre^*(\{S, C\})$. Finally, for markings of the form $\pre^*(\{S, C\})$, we can apply the same reasoning to the suffix of $\rho_2$ starting from the fifth marking of $\rho_2$. Finally, for markings of the form $\pre^*(\{S, C\})$, we can apply the same reasoning to the suffix of $\rho_2$ starting from the sixth marking of $\rho_2$.

**Lemma 8.** Let $C$ be a cube. The pre-image $\pre^*(C)$ is a counting set and its norm is bound by

$$\|\pre^*(C)\|_u \leq n \times \max(\|C\|_l + \|C\|_u, m)n^n$$

and

$$\|\pre^*(C)\|_l \leq n \times \max(\|C\|_l + \|C\|_u, m)n^n.$$  

In the proof, we add token-destroying transitions to the net in order to reduce the problem to single-marking reachability.

We observe that the following result from the proof of Theorem 4.50 of [11] is applicable to BIMO nets. The proof is almost identical to the original proof, and is supplied in the appendix for the sake of completeness.

**Theorem 11.** Let $S_1$ and $S_2$ be two functions that take as arguments a BIMO net $N$ and a finite list of counting constraints $X$, and return counting sets $S_1(N, X)$ and $S_2(N, X)$ respectively.

Assume that $S_1(N, X)$ and $S_2(N, X)$ have norms at most exponential in the size of $(N, X)$, as well as PSPACE-decidable membership (given input $(M, N, X)$, decide whether $M \in S_1(N, X)$).

Then the same is true about the counting sets $S_1(N, X) \cap S_2(N, X)$, $S_1(N, X) \cup S_2(N, X)$, $S_1(N, X)$, $\pre^*(S_1(N, X))$. Furthermore the emptiness of the aforementioned sets is decidable in PSPACE, given input $(N, X)$.

Theorem 11 allows us to verify a wide range of properties for BIMO nets in PSPACE. For example, the results for cubes that could be solved in IO are still solvable for BIMO.

**Theorem 12.** The cube-reachability, cube-coverability and cube-liveness problem for branching immediate multiple observation nets are PSPACE-complete.

**Proof (Reachability).** The PSPACE-hardness derives from the PSPACE-hardness for IO nets, which are a subclass of BIMO nets. Let $N$ be a BIMO net, and $C', C$ two cubes. Cube $C'$ can reach $C$ if and only if $C' \cap \pre^*(C)$ is non empty. By Theorem 11 this is solvable in PSPACE in the size of $N$ and $X \equiv C', C$.

The rest of the proof for coverability and liveness is in the appendix.
Remark 3. These results for BIMO nets entail a second proof that IMO and IO nets are verifiable in PSPACE, as IMO and IO nets are a subclass of BIMO nets.

6 Implication for Model Checking

Another consequence of the global flatness of IMO nets and Theorem 11 for BIMO nets is the decidability of some model checking problems for these nets. The problems require to check whether a given marking or all markings from a given set satisfy a formula in CTL style with cubes as atomic formulas. We use the logic UB, the Unified system of Branching time, introduced in [4]. Note that UB is a fragment of (and inspiration to) Computational Tree Logic (CTL). We follow the notation of [9].

Logic Syntax and Semantics. Given a Petri net $N$, UB formulas are inductively defined using the constant true, connectives $\neg$, $\land$, and $\lor$, and operators $EF$, $EG$ and $E(t)$ for each transition $t$. We extend UB to the logic $UB + Cube$ with additional atomic formulas of the form $Cube_C$ for each cube $C$ definable on $N$. We interpret the formulas of $UB + Cube$ on markings of Petri net $N$. $Cube_C$ is satisfied on markings belonging to $C$. $E(t)\varphi$ is satisfied if $t$ is enabled and firing it leads to a marking satisfying $\varphi$. $EF\varphi$ is satisfied if there exists a firing sequence to a marking satisfying $\varphi$, and $EG\varphi$ is satisfied if there exists an infinite firing sequence such that all markings along it satisfy $\varphi$. The constant true and connectives $\neg$, $\land$, and $\lor$ are interpreted in the usual way.

The logics $EF$ (respectively $EG$) are obtained from UB by removing the operator $EG$ (respectively $EF$). The logics $EF + Cube$ and $EG + Cube$ are obtained in the same way as the logic $UB + Cube$.

By Corollary 4 of [7] and by Theorem 4.4 and Section 5.2 of [8], global flatness implies that model checking $EF$ and $EF + Cube$ formulas is decidable for IO nets and IMO nets for each single initial marking and for each counting sets of initial markings. Actually, it also implies decidability of model checking $EF + Presb$ formulas, defined in the same way but with arbitrary semilinear atomic predicates, for semilinear sets of initial markings.

Theorem 13. The model checking problem for $EF + Cube$ formulas is decidable for BIMO nets and counting sets of initial markings.

As BIMO nets are a superclass of BPP nets, Theorem 4.3 from [9] implies undecidability of the model checking problem for $EG$ and BIMO nets with a single initial marking. Therefore, by extension, UB and CTL model checking are also undecidable for BIMO nets. Model checking with a single initial marking is clearly decidable for IO nets and IMO nets because there is only a finite set of reachable markings. However, if we allow a counting set of initial markings, $EG$ model checking stays undecidable for IO.

Theorem 14. The model checking problem for $EG + Cube$ formulas is undecidable for IO nets and counting sets of initial markings.
7 Conclusion

We have shown that immediate observation Petri nets are globally flat, allowing the use of existing efficient verification tools. We have also shown that allowing multiple observation does not change the properties of IO nets; this also implies that allowing multiple observation for IO population protocols changes neither the expressive power nor the verification complexity.

We have also studied branching immediate multiple observation nets, which are simultaneously a generalisation of IO (and IMO) nets, and of the Basic Parallel Processes model. The class of BIMO nets significantly extends the expressive power of both IO nets and BPP nets, bringing together process creation and (restricted) cross-process interaction via a simple and natural definition. While such an extension does not preserve global flatness, we have proven that local flatness is still preserved, many verification problems are still in \textit{PSPACE}, and some model checking problems are still decidable.

While the post-image of a single marking for a BIMO nets is not always semilinear, the preimage of a counting set of markings is always a counting set. Moreover, as long as the preimage of a markings is finite, there is an exponential upper bound on the number of marking in the preimage. This suggests the study of related questions, such as the properties of the preimage of a semilinear set, and the bounds on the size of the post-image of a marking when it is finite.

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A Appendix: Pruning and bunch matrices

In this section we recall the setting used for the Pruning Theorem in our previous article [12], and show how to reformulate it into Theorem 2.

A.1 Setting

Since the transitions of IO nets do not create or destroy tokens, we can give tokens identities. Given a firing sequence, each token of the initial marking follows a trajectory, or sequence of steps, through the places of the net until it reaches the final marking of the sequence.

Definition 9. A trajectory of a IO net $N$ is a sequence $\tau = p_1 \ldots p_k$ of places. We denote $\tau(i)$ the $i$-th place of $\tau$. The $i$-th step of $\tau$ is the pair $\tau(i)\tau(i+1)$ of adjacent places.

A history is a multiset of trajectories of the same length. The length of a history is the common length of its trajectories. Given a history $H$ of length $h$ and index $1 \leq i \leq h$, the $i$-th marking of $H$, denoted $M_H^i$, is defined as follows: for every place $p$, $M_H^i(p)$ is the number of trajectories $\tau \in H$ such that $\tau(i) = p$. The markings $M_H^0$ and $M_H^h$ are called the initial and final markings of $H$.

A history $H$ of length $h \geq 1$ is realizable in an IO net $N$ if there exist transitions of $N$ $t_1, \ldots, t_{h-1}$ and numbers $k_1, \ldots, k_{h-1} \geq 0$ such that $M_H^0 \xrightarrow{t_1^{k_1}} M_H^1 \xrightarrow{t_2^{k_2}} \cdots \xrightarrow{t_{h-1}^{k_{h-1}}} M_H^h$, where for every transition $t$ we define $M \xrightarrow{t} M'$ iff $M = M'$.

A step $\tau(i)\tau(i+1)$ of a trajectory $\tau$ is horizontal if $\tau(i) = \tau(i+1)$, and non-horizontal otherwise. A history $H$ of length $h$ is well-structured if for every $1 \leq i \leq h - 1$ one of the two following conditions hold:

- For every trajectory $\tau \in H$, the $i$-th step of $\tau$ is horizontal.
For every two trajectories $\tau_1, \tau_2 \in H$, if the $i$-th steps of $\tau_1$ and $\tau_2$ are non-horizontal, then they are equal.

Remark 4. Notice that a history of length 1 is always realizable. Notice that there may be more than one realizable history corresponding to a firing sequence in an IO net, because the firing sequence does not keep track of which token goes where, while the history does.

We then have the following result.

**Lemma 9.** Let $N$ be an IO net. Then $M \xrightarrow{\ast} M'$ iff there exists a well-structured history realizable in $N$ with $M$ and $M'$ as initial and final markings.

We now proceed to give a syntactic characterization of the well-structured realizable histories.

**Definition 10.** $H$ is compatible with $N$ if for every trajectory $\tau$ of $H$ and for every non-horizontal step $\tau(i) \tau(i + 1)$ of $\tau$, the net $N$ contains a transition $(\tau(i), p_o) \xrightarrow{} (\tau(i + 1), p_o)$ for some place $p_o$ and $H$ contains a trajectory $\tau'$ with $\tau'(i) = \tau'(i + 1) = p_o$.

**Lemma 10.** Let $N$ be an IO net. A well-structured history is realizable in $N$ iff it is compatible with $N$.

We introduce bunches of trajectories.

**Definition 11.** A bunch is a multiset of trajectories with the same length and the same initial and final place.

Every well-structured realizable history containing a bunch of size larger than $n$ can be “pruned”, meaning that the bunch can be replaced by a smaller one, while keeping the history well-structured and realizable.

**Lemma 11 (Pruning Lemma).** Let $N$ be an IO net of place count $n$. Let $H$ be a well-structured history realizable in $N$ containing a bunch $B \subseteq H$ of size larger than $n$. There exists a nonempty bunch $B'$ of size at most $n$ with the same initial and final places as $B$, such that the history $H' \overset{\text{def}}{=} H - B + B'$ (where $+$, $-$ denote multiset addition and subtraction) is also well-structured and realizable in $N$.

Multiple applications of this Pruning Lemma lead to the Pruning Theorem which intuitively states that if $M$ is coverable from a marking $M''$, then it is also coverable from a “small” marking $S'' \leq M''$, where “small” means $|S''| \leq |M| + n^3$.

**Theorem 15 (Pruning Theorem).** Let $N = (P, T, F)$ be an IO net of place count $n$, let $M$ be a marking of $N$, and let $M'' \xrightarrow{\ast} M'$ be a firing sequence of $N$ such that $M' \geq M$. There exist markings $S''$ and $S'$ such that

\[
M'' \xrightarrow{\ast} M' \geq M \\
\geq \geq \\
S'' \xrightarrow{\ast} S' \geq M
\]

and $|S''| \leq |M| + n^3$. 
A.2 Reformulation

We link the length of a well-structured realizable history and the length of its corresponding executions.

**Lemma 12.** Let $N$ be an IO net, and let $H$ be a well-structured realizable history of length $h$. Every firing sequence $\sigma$ such that $H$ is realizable by $\sigma$ is of accelerated length $|\sigma|_a \leq h$.

**Proof.** The definition of a realizable history requires that there exist transitions $t_1, \ldots, t_{h-1}$ and numbers $k_1, \ldots, k_{h-1} \geq 0$ such that $M_H^1 \xrightarrow{t_1^{k_1}} \cdots \xrightarrow{t_{h-1}^{k_{h-1}}} M_H^h$. As the transitions may not be distinct, and the $k_i$ can be equal to 0, this yields a firing sequence from $M_H^1$ to $M_H^h$ of accelerated length at most $h$.

To ease the reformulation, we first extract a small lemma that was a part of a longer proof in [12]. The Pruning Lemma expresses that we can reduce, or prune, a “large” bunch to a “small” one. The following Boosting Lemma expresses that any non-empty bunch can be augmented, or boosted, to an arbitrary larger size.

**Lemma 13 (Boosting Lemma).** Let $N$ be an IO net. Let $H$ be a well-structured history realizable in $N$ containing a bunch $B \subseteq H$ of size at least 1, with initial and final place $p$ and $p'$, respectively. Then for every natural $k$ there exists a well-structured and realizable history $H' \supset H$ such that the multiset $H' - H$ contains exactly $k$ trajectories and they all have initial and final place $p$ and $p'$.

**Proof.** Note that $H$ is well-structured and realizable, and therefore compatible with $N$. Let $\tau$ be a trajectory of bunch $B$. Consider the history $H' = H + k \times \{\tau\}$. Observe that $H'$ is also well-structured (we have only added non-horizontal steps already present in $H$) and compatible with $N$ (for the same reason). At the same time, $H' - H$ contains exactly $k$ copies of $\tau$ which indeed goes from $p$ to $p'$.

We can now prove Theorem 2.

**Theorem 2.** For each IO net $N = (P, T, F)$ of place count $n$, there exists a set $B$ of $n \times n$ matrices called bunch matrices and such that each bunch matrix $B$ has an associated value called the accelerated length $|B|_a$ of $B$. The following properties hold:

(i) For each two markings $M'$ and $M$, $M' \xrightarrow{\sigma} M$ iff there exists a bunch matrix $B$ such that

\[
M'(p_i) = \Sigma_j B_{i,j} \\
M(p_i) = \Sigma_j B_{j,i}
\]

for every $p_i \in P$. We call source marking and target marking of $B$ the markings $M'$ and $M$ respectively. Moreover, there exists a firing sequence $\sigma$ of accelerated length $|\sigma|_a = |B|_a$ such that $M' \xrightarrow{\sigma} M$. 

Let $B$ be a bunch matrix such that $B_{i,j} > n$ for some indices $i, j$. Let $B'$ be the matrix equal to $B$ everywhere except on $i, j$ where $B'_{i,j} = n$. Matrix $B'$ is also a bunch matrix, and $|B'|_a \leq |B|_a$.

Let $B$ be a bunch matrix, and let $B'$ be the matrix equal to $B$ everywhere except on one index $i, j$ where $B_{i,j} \geq 1$ and $B'_{i,j} > B_{i,j}$. Matrix $B'$ is also a bunch matrix, and $|B'|_a \leq |B|_a$.

For $B$ a bunch matrix, $|B|_a \leq (n^2 \times \max_{i,j} B_{i,j} + 1)^n$.

Proof. We start by constructing $B$. For each realizable well-structured history $H$ we define the bunch matrix $B_H$ such that $(B_H)_{i,j}$ is the size of the maximal bunch with the initial place $p_i$ and the final place $p_j$. We define the accelerated length $|B|_a$ of a bunch matrix $B$ to be the minimal length of a history $H$ such that $B = B_H$. Now $B$ is the set of matrices $B_H$ for all realizable well-structured histories $H$.

Let us now prove the required properties of the constructed bunch matrices.

(i) Observe that for any history $H$

$$M'(p_i) = \sum_j (B_H)_{i,j}$$
$$M(p_i) = \sum_j (B_H)_{j,i}$$

for every $p_i \in P$ iff the initial marking of $H$ is $M'$ and the final marking of $H$ is $M$. If $M' \rightarrow M$, we take an arbitrary realizable well-structured history $H$ with these initial and final markings, and take $B_H \in B$. If there is a bunch matrix $B_H \in B$, we take the the corresponding history $H$ and obtain the necessary firing sequence $\sigma$ from $H'$ by Lemma 12. Note that we can choose the history of the minimal length for this bunch matrix, i.e. $|B_H|_a$.

(ii) This is an immediate implication of the pruning lemma (lemma 11). Indeed, consider $B = B_H$, where length of $H$ is equal to $|B|_a$. We can prune the bunch from $p_i$ to $p_j$ in $H$, obtaining a well-structured and realizable history $H'$ with length at most $|B|_a$. The corresponding bunch matrix $B' = B_{H'}$ is a bunch matrix and has accelerated length not larger than the length $H'$, in particular, $|B'|_a \leq |B|_a$.

(iii) This property is proven in the same way, except by using the Boosting Lemma instead of the Pruning Lemma.

(iv) Consider the realizable well-structured history $H$ such that $B = B_H$ and the length of $H$ is equal to $|B|_a$. All the intermediate markings of $H$ are unique, as the length of $H$ is minimal and cycles can be removed. But the number of trajectories in $H$ is at most $n \times \max_{i,j} B_{i,j}$, and the number of markings with at most that number of tokens is less than $(n \times \max_{i,j} B_{i,j} + 1)^n$ as each of $n$ places can have from 0 to $n \times \max_{i,j} B_{i,j}$ tokens. This proves the bound on the length of $H$, and, therefore, on the accelerated length of $B$.

This Theorem yields the following result on accelerated length of firing sequences.

**Lemma 1.** Let $N = (P,T,F)$ be an IO net of place count $n$. For every pair of markings $(M', M)$ of $N$ such that $M' \rightarrow M$, there exists a firing sequence $\sigma$ of accelerated length $|\sigma|_a \leq (n^3 + 1)^n$ such that $M' \rightarrow M$. 
Proof. Let $N = (P, T, F)$ be an IO net of place count $n$, and let markings $M'$, $M$ be such that $M' \rightarrow M$. By Theorem 4(i) there exists a bunch matrix $B$ of source and target markings $M'$ and $M$. By Theorem 4(ii) there exists a bunch matrix $B'$ such that for every $i, j$, $B_{i,j}' = \min(B_{i,j}, n)$ and $|B'|_a \leq |B|_a$. The biggest element of $B'$ is $B_{i,j}' \leq n$, so by Theorem 4(iv) $|B'|_a \leq (n^3 + 1)^n$. By Theorem 4(iii) bunch matrix $B$ has at most the same accelerated length $|B|_a = |B'|_a$ (as $|B|_a \leq |B'|_a \leq |B|_a$, they are equal). Finally, by Theorem 4(i) there exists a firing sequence $\sigma$ of accelerated length $|\sigma|_a = |B|_a \leq (n^3 + 1)^n$ such that $M' \rightarrow M$ and this concludes our proof.

Theorem 4. Let $N$ be an IO net of place count $n$. The reachability relation of $N$ is semilinear, and there exists a semilinear set realizing this of size at most exponential in $n$.

Proof. We prove the result by exhibiting a set of bases $B_n$ and periods such that $B$ is the semi-linear set $\bigcup_{b \in B_n} L(b, P_b)$ where $P_b$ is the set of periods of a base $b \in B_n$ and $L(b, P_b)$ is the linear set of base $b$ and period set $P_b$:

- The set of bases $B_n$ is the bunch matrices with $B_{i,j} \leq n$ for each pair $(i, j)$ of indices.
- For $b \in B_n$, the periods of $P_b$ are the subclass of unit matrices $P_b = \{e_{i,j} \mid b_{i,j} = n\}$ where $e_{i,j}$ is the matrix that has a $1$ in index $i, j$ and $0$ elsewhere. Notice that for $b \in \{0, \ldots, n - 1\}^n$, the period sets $P_b$ are empty.

Let $B \in L(b, P_l)$ for some $b \in B_n$. If $b \in B_n$, then it is in $B$ by definition. Otherwise, there exists $b \in B_n \setminus \{0, \ldots, n - 1\}^n$ with $P_b = \{p_1, \ldots, p_k\}$ for some $k \geq 1$ and coefficients $\lambda_1, \ldots, \lambda_k \in \mathbb{N}$ such that $B = b + \lambda_1 p_1 + \ldots + \lambda_k p_k$. Since bases are bunch matrices by definition of $B_n$, by Theorem 4(iii) applied to each pair of indices $(i, j)$ such that $b_{i,j} = n \geq 1$, $B$ is a bunch matrix. So $\bigcup_{b \in B_n} L(b, P_b) \subseteq B$.

Now consider $B \in B$. By Theorem 4(iii) there exists a bunch matrix $B'$ such that for every $i, j$, $B'_{i,j} = \min(B_{i,j}, n)$. Matrix $B'$ is in $B_n$. Let $I$ be the set of pair of indices $(i, j)$ such that $B'_{i,j} = n$. Then $B$ is in $L(B', P_{B'})$:

$$B = B' + \sum_{i \in I} (B_i - B'_i)p_i.$$

Thus the reachability relation $\rightarrow$ of $N$ is a semilinear set as the projection onto its lines and columns of the set of bunch matrices.

B Appendix: Results on IMO nets

In this section we demonstrate that the proofs for the IO nets are applicable to IMO nets with only minor modifications.

We use the same definitions of histories, well-structured histories and realizability, as well as bunches. We observe that the connection between realizable histories and firing sequences still holds, as the definition of realizability requires existence of enabled transitions.
Definition 12. \( H \) is compatible with an IMO net \( N \) if for every trajectory \( \tau \) of \( H \) and for every non-horizontal step \( \tau(i)\tau(i+1) \) of \( \tau \), the net \( N \) contains a transition \( (\tau(i),p_0_1,\ldots,p_0_n) \rightarrow (\tau(i+1),p_0_1,\ldots,p_0_n) \) for some places \( p_0_1,\ldots,p_0_n \) and \( H \) contains a submultiset of trajectories \( \{\tau'_1,\ldots,\tau'_k\} \) with \( \tau'_j(i) = \tau'_j(i+1) = p_{o_j} \) for all \( 1 \leq l \leq k \).

Lemma 14. Let \( N \) be an IMO net. A well-structured history is realizable in \( N \) iff it is compatible with \( N \).

Proof. Let a well-structured history \( H \) be realizable in \( N \). Consider an arbitrary non-horizontal step \( \tau(i)\tau(i+1) = p_sp_d \) in some trajectory of this history. All the non-horizontal steps at the corresponding position in \( H \) are equal by well-structuredness, and realizability implies that there is an enabled transition with source place \( p_s \) and destination place \( p_d \) at marking \( M_H^l \) in \( N \). This transition can be applied as many times as there are equal steps at the corresponding position in \( H \). Therefore the observed places of this transition are marked (with the corresponding multiplicities) both before and after iterating this transition, which corresponds to \( H \) containing trajectories with the steps \( p_{o_j}p_{o_j} \) at the corresponding position. As this holds for each non-horizontal step in \( H \), \( H \) is compatible with \( N \).

Now assume that \( H \) is compatible with \( N \). If some position in \( H \) contains only horizontal steps, we can use zero iterations of an arbitrary transition. If a position contains some number of (equal) non-horizontal steps \( p_sp_d \), it also contains horizontal steps \( p_{o_j}p_{o_j} \) (with necessary multiplicities) such that \( (p_s,p_0_1,\ldots,p_0_n) \rightarrow (p_d,p_0_1,\ldots,p_0_n) \) is a transition in \( N \). All the other steps at the corresponding position are horizontal. Therefore we can iterate the transition \( (p_s,p_0_1,\ldots,p_0_n) \rightarrow (p_d,p_0_1,\ldots,p_0_n) \) to obtain the next marking.

We can now reformulate the Pruning Lemma.

Lemma 15 (Pruning Lemma for IMO nets). Let \( N \) be an IMO net of place count \( n \) and maximal observation degree \( m_o \). Let \( H \) be a well-structured history realizable in \( N \) containing a bunch \( B \subseteq H \) of size larger than \( n \times m_o \).

There exists a nonempty bunch \( B' \) of size at most \( n \times m_o \) with the same initial and final places as \( B \), such that the history \( H' \equiv H - B + B' \) (where \( +, - \) denote multiset addition and subtraction) is also well-structured and realizable in \( N \).

Proof. Let \( P_B \) be a set of all places visited by at least one trajectory in the bunch \( B \). For every \( p \in P_B \) let \( f(p) \) and \( l(p) \) be the earliest and the latest moment in time when this place has been used by any of the trajectories (the first and the last occurrence can be in different trajectories).

Let \( \tau_p \in P_B \) be a trajectory that first goes to \( p \) by the moment \( f(p) \), then waits there until \( l(p) \), then goes from \( p \) to the final place. To go to and from \( p \) it uses fragments of trajectories of \( B \).

We will take \( B' = \{\tau_p \mid p \in P_B \} \) and prove that replacing \( B \) with \( m_o \times B' \) in \( H \) does not violate the requirements for being a well-structured history realizable
First let us check the well-structuring condition. Note that we build \( \tau_p \) by taking fragments of existing trajectories and using them at the exact same moments in time, and by adding some horizontal fragments. Therefore, the set of non-horizontal steps in \( B' \) is a subset (if we ignore multiplicity) of the set of non-horizontal steps in \( B \), and the replacement operation cannot increase the set of non-horizontal steps occurring in \( H \).

Now let us check compatibility with \( N \). Consider any non-horizontal step in \( H' \) in any trajectory at position \((i, i+1)\). By construction, the same step at the same position is also present in \( H \). History \( H \) is realizable in \( N \) and thus by Lemma 14 it is compatible with \( N \), so \( H \) contains the enabling horizontal steps \( p_{oj} \) in some trajectories at that position \((i, i+1)\). For each of these horizontal steps there are two cases: either that step \( p_{oj} \) was provided by a bunch being pruned, or by a bunch not affected by pruning. In the first case, note that the place \( p_{oj} \) of this horizontal step must be first observed no later than \( j \), and last observed not earlier than \( j+1 \). This implies \( f(p_{oj}) \leq i < i+1 \leq l(p_{oj}) \). As \( H' \) contains \( m_o \) horizontal steps \( p_{oj} \) for all positions between \( f(p_{oj}) \) and \( l(p_{oj}) \), in particular it contains them at position \((i, i+1)\). In the second case the same horizontal steps are present in \( H' \) as a part of the same trajectories.

So \( H' \) is well-structured and compatible with \( N \), and thus by Lemma 14 realizable in \( N \).

Theorem 5 and the rest of its consequences for IMO nets are proved analogously to Theorem 2 and the rest of its consequences for IO nets.

### Appendix: Results on BIMO nets

#### Lemma 3

Let \( M', M \) be markings of \( N \). For each firing sequence \( \sigma \) such that \( M' \xrightarrow{\sigma} M \), there is a colored execution \( \rho \) with the same initial and final marking (with some colors assigned to each place) such that \( |\rho| = |\sigma| + 1 \).

For each colored execution \( \rho \) there is a firing sequence \( \sigma \) from the initial marking to the final marking (ignoring colors) of \( \rho \) with accelerated length \( |\sigma|_a \leq |\rho| - 1 \).

**Proof.** Let \( \sigma = t_1t_2\ldots t_k \) be a firing sequence such that \( M_1 \xrightarrow{t_1} M_2 \xrightarrow{t_2} \ldots \xrightarrow{t_k} M_{k+1} \). We build a colored execution \( \rho \) from \( \sigma \). Let us set the colored markings of our execution as \( C_i = M_i \) for \( 1 \leq i \leq k+1 \) and with all places white unless assigned the color black in the following procedure:

1. The source places of transitions \( t \) with no destination places are set to black in each \( C_i \) such that \( t \) is fired after \( M_i \), i.e. \( t = t_j \) for some \( i \leq j \leq k \).
2. The source places of transitions \( t \) with only black destination places are set to black in each \( C_i \) such that \( t \) is fired after \( M_i \), i.e. \( t = t_j \) for some \( i \leq j \leq k \).
As this procedure increases the number of black places without decreasing it, it terminates. The only change of color for a place in the resulting sequence of colored markings $\rho$ is from black to white, and by construction at that moment all destination places are also black thus satisfying Definition [11]. At the same time, any transition with white source place must have a white destination place, thus satisfying Definition [12], as otherwise the source place would be changed to black. Finally, the last colored marking $C_{k+1}$ has only white places, as any transition with no destination places must have been fired before. Therefore we obtain a colored execution.

It remains to prove the other direction. Given a colored execution we can just use the iterated transitions that exist by the definition of colored steps.

**Lemma 16.** The number of tokens in white places cannot decrease along the markings of a colored execution. In particular, the number of tokens in the white places of a given marking never exceeds the size of the final marking.

**Proof.** By Definition [12] if a transition removes a token from a white place, it must add a token to some other white place.

**Lemma 4.** A red-black-balanced marking is also red-balanced and each red place has strictly more than $m^* n^B$ tokens. Each red-balanced marking can be made red-black-balanced by making some black places red.

**Proof.** Consider a red-black-balanced marking. Lemma [10] says that $w \leq m$, therefore we have $w \leq W m$ (and $W = 0$ implies $w = 0$). Observe that the existence of a red place implies $B + W \leq n - 1$. We obtain:

\[
m_* n^{B+1} - b - w \geq m_* n^{B+1} - B(m_* n^B - 1) - W m \\
\geq m_* n^{B+1} - (n - 1)m_* n^B = m_* n^B(n - n + 1) = m_* n^B.
\]

Now let $C$ be a red-balanced marking of target size $m$. We prove the lemma by induction over the number $B$ of black places. If $B = 0$, there are no black places, and the marking is already red-black-balanced.

Consider a marking that is not red-black-balanced. Make the black place with the maximum amount of tokens red. By assumptions, it has enough tokens to satisfy the new lower bound for red places (given the reduction of the number of black places). All the other red places already had enough tokens. Therefore we obtain a red-balanced marking with one fewer black place and we can apply the induction hypothesis.

**Lemma 5.** Let $\rho$ be a colored execution of initial marking $C_0$ with target size $m$. Let $\hat{C}_0$ be a colored marking such that the pair $(C_0, \hat{C}_0)$ satisfies the following conditions:

- Each place of $\hat{C}_0$ has the same color as in $C_0$ or is red instead of black.
- Each white place has the same number of tokens as in $C_0$.
- Each black place of $\hat{C}_0$ has at least as many tokens as in $C_0$.
- $\hat{C}_0$ is red-black-balanced for target size $m$. 

Then there exists a colored execution $\tilde{\rho}$ with initial marking $\tilde{C}_0$ and the same final marking as $\rho$. Moreover, all the colored markings of $\tilde{\rho}$ are red-black-balanced for target size $m$.

Additionally, one can require that all the colored markings of $\tilde{\rho}$ have unique combinations of the set of red places, the set of black places, and the token counts in black and white places (token counts in the red places are ignored).

Proof. First we prove the claim without the uniqueness condition. This is proven using induction over the length $|\rho| \geq 1$. If there is only one colored marking then all places are white, therefore $C_0 = \tilde{C}_0$ and we can take $\tilde{\rho} \equiv \rho$.

Otherwise, $|\rho| > 1$. Consider the first colored step $C_0, C_1$ in $\rho$ such that $C_0 \xrightarrow{t} C_1$ in the net $N$ for some $t \in T$ and $l \geq 0$. We construct a colored marking $\tilde{C}_1$ such that $\tilde{C}_0, \tilde{C}_1$ is a colored step and the pair $(\tilde{C}_1, \tilde{C}_1)$ verifies the conditions of the Lemma. We apply the induction hypothesis to $(\tilde{C}_1, \tilde{C}_1)$ and $\rho$ truncated of its first colored step (thus with initial marking $\tilde{C}_1$). The sequence of markings $\tilde{C}_0 \tilde{C}_1 \ldots$ provides us our desired colored execution $\tilde{\rho}$.

Construction of $\tilde{C}_1$. To construct $\tilde{C}_1$, we fire $t$ from $\tilde{C}_0$ a certain number of times, in the process changing the color of any black place with $m_\ast n^B$ or more tokens to red. This change decreases $B$ and the threshold $m_\ast n^B$, allowing us to change multiple black places to red. Then, if the source place of $t$ changes its color to white between $\tilde{C}_0$ and $C_1$, we also make this place white in $\tilde{C}_1$. The other places of $\tilde{C}_1$ are colored as in $\tilde{C}_0$. It remains to specify how many times we fire $t$.

If any of the source or destination places of $t$ are white in $C_1$ (if they are white in $C_0$, they stay white in $C_1$), then the constraint of having the same number of tokens in all white places in $\tilde{C}_1$ and $C_1$ defines how many times we need to fire $t$. Otherwise, transition $t$ is fired at most $l$ times, stopping early if all the destination places become red. Note that this may mean zero times if all the destination places are already red.

Now we prove that the definition of $\tilde{C}_1$ is correct and the constructed $\tilde{C}_1$ does indeed satisfy the conditions of the Lemma with regards to $C_1$. As $\tilde{C}_0$ is red-black-balanced, each place contains either at least as many tokens as in $C_0$ (if the place is white or black) or more than $m_\ast$ tokens (if the place is red), so all observation conditions of $t$ are satisfied at $\tilde{C}_0$. For greater clarity, we introduce the following notation: for $C$ a colored configuration, we note $B(C), b(C), W(C), w(C)$ the count of black places, the number of tokens in black places, the count of white places and the number of tokens in white places of $C$ respectively. We split the problem into cases depending on $t$ in $\rho$.

Case 1: transition $t$ has a source place $p_\ast$ which changes its color to white in $C_1$. Note that in this case no destination place of $t$ can be white by Definition[16][16]. By Lemma[16][16] since $p_\ast$ turns white, it has at most $m$ tokens in $C_1$. Since $C_0 \xrightarrow{t} C_1$, we have $C_0(p_\ast) \geq C_1(p_\ast)$. If $p_\ast$ is black in $\tilde{C}_0$, then it has at least as many tokens as in $C_0$, and if $p_\ast$ is red, then $C_0(p_\ast) > m_\ast \geq m$. In any case, $\tilde{C}_0(p_\ast)$ is at least
As each firing of $t$ removes exactly one token, we can fire $t$ as many times as necessary to have $\tilde{C}_1(p_s) = C_1(p_s)$.

Now let us show that $(C_1, \tilde{C}_1)$ satisfy the conditions of the lemma. Observe that $\tilde{C}_0$ is red-black-balanced and therefore red-balanced by Lemma 4. Our firings of $t$ change the color of $p_s$ to white (which keeps the marking red-balanced), and possibly increase the number of tokens in some places. By Lemma 4 we can change the color of some places from black to red and obtain a red-black-balanced colored marking $\tilde{C}_1$.

It remains to show that all the black places in $\tilde{C}_1$ have at least as many tokens as in $C_1$. If $p_s$ is black in $\tilde{C}_0$, then $\tilde{C}_0(p_s) \geq C_0(p_s)$ and we fire $t$ at least $l$ times. Thus we add at least $C_0(p_s) - C_1(p_s)$ tokens. If $p_s$ is red in $\tilde{C}_0$, all the destination places are red in $\tilde{C}_1$, and we do not need to compare their token counts with $C_1$ (and the other black places do not change their token counts).

Indeed, initially red place $p_s$ contains at least $m_s n^{B(C_0)+1} - b(\tilde{C}_0) - w(\tilde{C}_0)$ tokens in $\tilde{C}_0$, and in the end white place $p_s$ contains at most $m$ tokens in $\tilde{C}_1$. Therefore $t$ must be fired at least $m_s n^{B(C_0)+1} - b(\tilde{C}_0) - w(\tilde{C}_0) - m$ times, and so each destination place receives at least as many tokens. Consider a black destination place $p_d$ with $\tilde{C}_0(p_d)$ tokens. All the non-red places except $p_d$ together contain at most $m_s n^{B(C_0)} \times (n - 2)$ tokens. Therefore

$$\tilde{C}_1(p_d) \geq m_s n^{B(C_0)+1} - b(\tilde{C}_0) - w(\tilde{C}_0) - m + \tilde{C}_0(p_d)$$

$$\geq m_s n^{B(C_0)+1} - m_s n^{B(C_0)} \times (n - 2) - \tilde{C}_0(p_d) - m + \tilde{C}_0(p_d)$$

$$= m_s n^{B(C_0)+1} - m_s n^{B(C_0)} \times (n - 2) - m$$

$$\geq 2m_s n^{B(C_0)} - m > m_s n^{B(C_0)}.$$

Place $p_d$ has too many tokens to be black in a red-black-balanced marking and must be changed to red.

**Case 2: $C_1$ has the same set of white places as $C_0$, and one of these white places is the source place of $t$ or a destination place of $t$.** Note that if the source place of $t$ is white, then some destination place must also be white. Therefore the lower bounds for red places cannot increase.

In this case we fire $t$ exactly $l$ times from $\tilde{C}_0$ to obtain $\tilde{C}_1$, then make some black places red if needed as described in Lemma 4. We can show that it is possible to construct $\tilde{C}_0 = \tilde{C}_0^{(0)} \overset{t}{\to} \tilde{C}_0^{(1)} \overset{t}{\to} \ldots \overset{t}{\to} \tilde{C}_0^{(l)} = \tilde{C}_1$ without running out of tokens in the source place $p_s$. As before, we apply Lemma 4 after each step to keep the markings red-black-balanced. If $p_s$ is white or black in $\tilde{C}_0$, it has at least as many tokens as in $C_0$ and we are done. It remains to consider the situation when $p_s$ is red. We observe that by the assumption of the case some output place is white in $\tilde{C}_0$. We note $C$ the marking obtained by firing $t$ from $\tilde{C}_0^{(j)}$ before recoloring anything. As firing $t$ increases the token count in at least one white place, we have $w(C) \geq w(\tilde{C}_0^{(j)}) + 1$, and, as in Case 1,

$$\tilde{C}_0^{(j)}(p_s) \geq m_s n^{B(C_0^{(j)}+1)} - w(\tilde{C}_0^{(j)}) - b(\tilde{C}_0^{(j)}).$$
At the same time we know that \( \tilde{C}_0^{(j)}(p_s) \geq m_s n^{B(\tilde{C}_0^{(j)})} + 1 \) by Lemma \([4]\). Firing \( t \) decreases the token count in \( p_s \) by 1 and cannot decrease the count of tokens in black places. This combined with the above reasoning yields the following inequalities:

\[
C(p_s) \geq m_s n^{B(\tilde{C}_0^{(j)})} = m_s n^{B(C)},
\]

and

\[
C(p_s) \geq m_s n^{B(C)+1} - w(C) - b(C).
\]

Therefore \( C \) is red-balanced. To obtain \( \tilde{C}_0^{(j+1)} \) we change some black places to red according to Lemma \([4]\), which yields a red-black-balanced marking. In particular, it has strictly more than \( m_s \) tokens which is enough to fire \( t \) again.

We see that we can fire transition \( t \) the necessary number of times from \( C_0 \), and obtain a red-black-balanced marking. The token counts in all places change in the same way between \( C_0 \) and \( C_1 \) and between \( C_0 \) and \( C_1 \).

**Case 3: the source and destination places of \( t \) are all black or red in \( C_1 \)** (and therefore in \( C_0 \) and \( \tilde{C}_0 \)). We construct a sequence \( \tilde{C}_0 = \tilde{C}_0^{(0)} \rightarrow \tilde{C}_0^{(1)} \rightarrow \ldots \rightarrow \tilde{C}_0^{(\ell')} = \tilde{C}_1 \) step by step, changing the color of black places with too many tokens to red as in Lemma \([4]\). We stop either after \( \ell \) steps or when all the destination places become red, whichever happens first. As long as we keep firing, there is at least one black destination place, so the lower bound on the token count in the red places not concerned with the transition cannot increase.

We prove by induction that \( \tilde{C}_0^{(j)} \) is red-black-balanced for \( 0 \leq j \leq \ell' \). It is true for \( j = 0 \), suppose it is true for some \( 0 \leq j \leq \ell' - 1 \). Since \( j < \ell' \), there is a black destination place in \( \tilde{C}_0^{(j)} \).

Suppose the source place \( p_s \) is black in \( \tilde{C}_0^{(j)} \). Then firing \( t \) preserves the red-balanced property as the number of tokens in black places does not decrease. Now we can apply Lemma \([4]\) and obtain a red-black-balanced colored marking \( \tilde{C}_0^{(j+1)} \).

Otherwise, source place \( p_s \) is red and the number of tokens in black places increases when we fire \( t \). Marking \( \tilde{C}_0^{(j)} \) is red-black-balanced, so

\[
\tilde{C}_0^{(j)}(p_s) \geq m_s n^{B(\tilde{C}_0^{(j)})} + 1 \quad \text{by Lemma} \quad \tilde{C}_0^{(j)}(p_s) \geq m_s n^{B(\tilde{C}_0^{(j)})} + 1.
\]

We note \( C \) the marking obtained by firing \( t \) from \( \tilde{C}_0^{(j)} \) before recoloring anything. Firing \( t \) decreases the token count in \( p_s \) by 1 and increases the black token count by at least 1. This combined with the above reasoning yields the following inequalities:

\[
C(p_s) \geq m_s n^{B(\tilde{C}_0^{(j)})} + 1 - w(\tilde{C}_0^{(j)}) - b(\tilde{C}_0^{(j)}),
\]

and

\[
C(p_s) \geq m_s n^{B(C_0^{(j)})} \quad \text{since we have not recolored anything and since the white places are not affected by the firing of} \quad t, \quad B(C_0^{(j)}) = B(C) \quad \text{and} \quad w(C_0^{(j)}) = w(C_0^{(j)}) = m_s n^{B(C)}.
\]
w(C), and so C is red-balanced. Now we can apply Lemma 4 and obtain a red-black-balanced colored marking Ĉ_{0_0}^{(j+1)}.

Thus we can keep firing t and obtain red-black-balanced markings. If we fire the transition t times, the changes in the token count between Ĉ_{0_0} and Ĉ_{1_1} are the same as between C_0 and C_1. Otherwise, p_s loses less tokens, and all the destination places become red so the exact number of tokens there does not matter.

Now we prove that we can construct a colored execution that does not visit two colored markings with the same set of red places, same set of black places, and same token counts in black and white places. Let Ĉ_{i_0} be the one constructed above. Let there be two Ĉ_i, Ĉ_j with i < j that have the same combination of the set of red places, the set of black places, and the token counts in black and white places. Then the above construction allows as to transform the colored execution with initial marking Ĉ_j and the same final marking as p into a colored execution of the same length with the same final marking but with the initial marking Ĉ_i. Together with Ĉ_0, . . . , Ĉ_{i-1}, this yields a shorter colored execution fulfilling all the conditions. We repeat this reasoning to eliminate all repeating combinations from the colored execution, thus concluding the proof.

Lemma 6. Let M', M be two markings of N, and let |M'| = m', |M| = m, m* = max(m, m_0). If there exists a firing sequence σ from M' to M, then there exists a firing sequence σ' from M' to M of accelerated length at most

\[(m_0 \cdot n^m + m + 2)^n\]

and such that the intermediate markings of σ' are of size at most

\[(m' + o_d n (m + m_0 \cdot n^m)) (m_0 \cdot n^m + m + 2)^n (o_d n)^n].

Proof. By Lemma 3 we can construct a colored execution ρ leading from M' to M (with some colors assigned to each place). By Lemma 4 we can make the initial colored marking red-black-balanced for target size m. We apply Lemma 5 to obtain a colored execution with only red-black-balanced markings, where each marking has a unique combination of the color sets and token counts in non-red places.

Note that such a combination is described by specifying for each of the n places either that it is black and adding a token count between 0 and m_0 \cdot n^m - 1, or that it is white and adding a token count between 0 and m, or that it is red. Therefore, since ρ contains each of these combinations at most once, its length is at most (m_0 \cdot n^m + m + 2)^n. The firing sequence with bounded accelerated length is then given by applying Lemma 3 to ρ'.

Now let us consider how many tokens can be created by a red-black-balanced colored step (C_0, t, C_1) for C_0, C_1 two colored markings and t ∈ T a transition, depending on the colors of t’s source and destination places.

If t has a white source place in C_0 or a white destination place in C_1, then it can be fired at most m times in a row while adding at most o_d \cdot n tokens each time.
If \( t \) has a black source place in \( C_0 \) or a black destination place in \( C_1 \), then it can be fired at most \( m \cdot n \) times in a row (after which a black source place is exhausted, and a black destination place becomes red) while adding at most \( o_d \cdot n \) tokens each time.

If \( t \) has a red source place in \( C_0 \) and only red destination places in \( C_1 \), then either the source place does not become white in \( C_1 \), in which case it is fired zero times, or it does become white and then \( t \) is fired at most as many times as its source place contains tokens. It contains at most the total number of tokens, and so firing \( t \) multiple times creates at most the total number of tokens times \( o_d \cdot n \) tokens. This can happen at most \( n \) times as each such firing sequence corresponds to an increase in the number of white places.

Each firing step happens as many times as there are colored steps in \( \tilde{\rho} \), thus yielding the desired bound.

**Lemma 7.** Let \( M \) be a marking of size \( m \). The pre-image \( \text{pre}^*(M) \) is a counting set and its norm is bound by

\[
\|\text{pre}^*(M)\|_u \leq n \times m \cdot n \quad \text{and} \quad \|\text{pre}^*(M)\|_l \leq n \times m \cdot n .
\]

**Proof.** We show that the token counts above \( m \cdot n \) are not distinguishable from the point of view of reachability of \( M \).

Let \( M' \) be some marking in \( \text{pre}^*(M) \) and let \( \sigma \) be a firing sequence from \( M' \) to \( M \). By Lemma 3, we can construct a colored execution \( \rho \) leading from \( M' \) to \( M \) with added colors. Lemma 3 applied to the initial marking of \( \rho \) changes its colors so that it is red-black-balanced for target size \( m \). We apply Lemma 3 to \( \rho \) and its own red-black-balanced initial marking to obtain a colored execution \( \tilde{\rho} \) with red-black-balanced markings from a colored marking \( C' \) such that \( \forall p, C'(p) = M'(p) \) to a colored marking \( C \) such that \( \forall p, C(p) = M(p) \).

Since \( C' \) is red-black-balanced, the places with \( m \cdot n \) or more tokens are red. Lemma 3 can be applied to \( \tilde{\rho} \) and to any colored marking \( C \) equal to \( C' \) except on red places, where \( C \) contains an arbitrary amount of tokens above the redlower bound for target size \( m \). This provides a colored execution to \( C \) from \( C' \), and so by Lemma 3 there exists a firing sequence from this new marking stripped of its colors to \( M \), thus proving that it belongs to \( \text{pre}^*(M) \).

Thus \( \text{pre}^*(M) \) is a counting set as a union of cubes with bounds on places \( p \) of the form \( a \leq p \leq a \) for \( a \in \{0, \ldots, m \cdot n - 1\} \) and \( m \cdot n \leq p \leq \infty \), of which there are a finite number.

**Lemma 8.** Let \( C \) be a cube. The pre-image \( \text{pre}^*(C) \) is a counting set and its norm is bound by

\[
\|\text{pre}^*(C)\|_u \leq n \times \max(\|C\|_l + \|C\|_u, m_o) n^a
\]

and

\[
\|\text{pre}^*(C)\|_l \leq n \times \max(\|C\|_l + \|C\|_u, m_o) n^a .
\]

**Proof.** Every cube \( C \) can be decomposed into a finite union of what we call in this proof simple cubes: cubes with bounds on each place \( p \) of the form \( a_p \leq p \leq a_p \) and \( b_p \leq p \leq \infty \) for some \( a_p, b_p \in \mathbb{N} \). The lower bounds \( L \) of these simple
cubes are bound by the addition of all the highest bounds used to define \( \mathcal{C} \), i.e.
\[ |L| \leq \|\mathcal{C}\|_u + \|\mathcal{C}\|_l. \]

Let \( \mathcal{C} \) be a cube. Assume that \( \mathcal{C} \) is a simple cube as described above. For each place \( p \) with upper bound \( \infty \), we add a transition \( t \) with preset \( \bullet t = \{p\} \) and postset \( t^* = \emptyset \). It is easy to see that for every marking \( M' \), the modified net \( N' \)
has a firing sequence from \( M' \) to the lower bound \( L \) of \( \mathcal{C} \) iff the original net \( N \)
has a firing sequence from \( M' \) to some marking in \( \mathcal{C} \). It then remains to apply
Lemma 7 to \( L \) and \( N' \), which has the same place count, maximal observation
degree and maximal output degree as \( N \). If \( \mathcal{C} \) is actually a union of simple cubes,
then the bounds still apply as the norm of a union is the maximum of the norms.

**Theorem 11.** Let \( S_1 \) and \( S_2 \) be two functions that take as arguments a BIMO
net \( N \) and a finite list of counting constraints \( X \), and return counting sets
\( S_1(N, X) \) and \( S_2(N, X) \) respectively.

Assume that \( S_1(N, X) \) and \( S_2(N, X) \) have norms at most exponential in the
size of \( (N, X) \), as well as \( \text{PSPACE} \)-decidable membership (given input \( (M, N, X) \),
decide whether \( M \in S_1(N, X) \)).

Then the same is true about the counting sets \( S_1(N, X) \cap S_2(N, X), S_1(N, X) \cup
S_2(N, X), S_1(N, X), \text{pre}^*(S_1(N, X)). \) Furthermore the emptiness of the aforementioned sets is decidable in \( \text{PSPACE} \), given input \( (N, X) \).

**Proof.** The exponential bounds for the norms follow immediately from the norms
of set-theoretical combinations of counting constraints in Proposition 1 and
Lemma 8. The membership complexity for union, intersection and complement
is easy to see. It remains to demonstrate that the complexity of membership in
\( \text{pre}^*(S_1(N, X)) \) can be decided in \( \text{PSPACE} \).

By Savitch’s Theorem, \( \text{NPSPACE} = \text{PSPACE} \), so it is sufficient to provide
a nondeterministic algorithm. Given \( (M', N, X) \), we want to decide whether \( M' \in
\text{pre}^*(S_1(N, X)) \). As in the proof of Lemma 8 it is enough to check that \( M' \)
can reach a marking of size \( \|S_1(N, X)\|_u + \|S_1(N, X)\|_l \) in \( S_1(N, X) \) in a modified net
with a few extra (destroying) transitions. The algorithm guesses a firing sequence
in this modified net starting at \( M' \), step by step, guessing each time a marking
of size bound by Lemma 6 with \( m' = \|M'\| \) and \( m = \|S_1(N, X)\|_u + \|S_1(N, X)\|_l \),
and checking after each step if the reached configuration is in \( S_1(N, X) \).

At every moment we store descriptions of two configurations, the current
one and the next one, which can be done in polynomial space as they are of size
exponential in the size of the input \( (M', S_1(N, X), N) \), where \( S_1(N, X) \) also has
size exponential in the input.

If a counting set is of exponential norm, then it contains \( \text{PSPACE} \) describable
markings and thus checking emptiness is done by simply guessing such a marking
and checking whether it is in the set.

**Theorem 12.** The cube-reachability, cube-coverability and cube-liveness problem
for branching immediate multiple observation nets are \( \text{PSPACE} \)-complete.

**Proof.** These problems are \( \text{PSPACE} \)-hard because they are \( \text{PSPACE} \)-hard for IO
nets, which are a subclass of BIMO nets.

For the following problems, let \( N \) be a BIMO net, and \( \mathcal{C}', \mathcal{C} \) two cubes.
Coverability. Cube $C'$ covers cube $C$ if and only if the upward closure of $C$ is reachable from $C'$. The upward closure of $C$ is still a counting set (we just replace all upper bounds by $\infty$) of norms at most the norms of $C$. By the above paragraph on reachability, this problem is solvable $\text{PSPACE}$.

Liveness. Let $t$ be a transition of $N$. The set $\text{En}(t)$ of markings that enable $t$ contains the markings that put at least one token in the source place (if there is one) and at least the multiset of necessary tokens in the observed places. Clearly, $\text{En}(t)$ is a cube. Then $\text{pre}^*(\text{En}(t))$ is the set of markings $M$ from which one cannot execute transition $t$ anymore by any firing sequence starting in $M$. So the set $\mathcal{L}$ of live markings of $N$ is given by

$$\mathcal{L} = \text{pre}^* \left( \bigcup_{t \in T} \text{pre}^*(\text{En}(t)) \right)$$

Deciding whether $C \subseteq \mathcal{L}$ is equivalent to deciding whether $C \cap \overline{\mathcal{L}} = \emptyset$ holds. By Theorem 11 this can be solved in $\text{PSPACE}$ in the size of the input, i.e. net $N$ and cube $C$.

D Appendix: Model Checking

Theorem 13. The model checking problem for $\text{EF} + \text{Cube}$ formulas is decidable for BIMO nets and counting sets of initial markings.

Proof. Let $N$ be a BIMO net. We consider the problem of whether a given markings $M$ of $N$ satisfy a given formula $\varphi$. We define the size of a formula as the length of the binary encoding of the symbols of the formula and the bounds of the cubes in the formula. By induction over the formula size, we can see that a $\text{EF} + \text{Cube}$ formula of size $s$ represents a counting set of norm exponential in $s$. This set contains at most an exponential number of cubes, because each cube is fully described by $2^n$ numbers of exponential size (or explicitly infinite). As each such cube is representable in polynomial space, the entire counting set can be represented in exponential space. It is easy to see that recursion over subformulas allows us to compute this counting set in exponential space. It remains to verify whether the set difference of the initial set of markings and the set of markings satisfying the formula is an empty set, i.e. contains no cubes.

Theorem 14. The model checking problem for $\text{EG} + \text{Cube}$ formulas is undecidable for IO nets and counting sets of initial markings.

Proof. We just outline how to simulate a BPP net using an IO net and a $\text{EG} + \text{Cube}$ condition and then send the reader to the proof of undecidability for BPP nets in Theorem 4.3 from [9].

Let $N = (P, T, F)$ be a BPP net. We construct an IO net $N' = (P', T', F')$, and we initialize $P'$ with the places $P$. We add an extra place to $P'$ for "reserve" tokens, $p_{\bot}$. For each transition $t$ with source place $p_s$ and destination places
If $t$ has no destination places, we add a transition $p_s \rightarrow p_{\bot}$. Otherwise, we add the following transitions: $p_s \rightarrow p_{\text{source}}(t), p_{\text{source}}(t) \rightarrow p_{d_1}(t), (p_{\bot}, p_{d_1}(t)) \rightarrow (p_{d_j}(t), p_{d_{j-1}}(t))$, and $p_{d_j}(t) \rightarrow p_{d_j}$ for $j$ from 2 to $k$.

We observe that the following global conditions ensure that the permitted executions are simulating a BPP execution:

- No transition places for two different transitions can be marked at the same time.
- No transition place can contain more than one token.
- The place $p_{\text{source}}(t)$ can only be marked if no other transition place is marked.
- If a place $p_{d_j}(t)$ is marked, either all places $p_{d_{j'}}(t)$ for $j' < j$ are marked, or all places $p_{d_{j'}}(t)$ for $j' > j$ are marked.

Indeed, once we start modelling a BPP transition, we cannot model a different non-destruction transition in parallel, and we are forced to perform all the steps in order. A firing sequence $M' \xrightarrow{\gamma} M$ in $N$ can be simulated by a firing sequence $L' \xrightarrow{\gamma} L$ in $N'$ such that $M' = L' + \{ k'p_{\bot} \}$ and $M = L + \{ kp_{\bot} \}$ for some $k', k \geq 0$. 