SUPERFLUID PHASES OF TRIPLET PAIRING

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1. INTRODUCTION

Triplet pairing is exemplified physically in at least two strongly interacting quantum many-fermion systems:

(i) In laboratories on earth: Superfluid phases are observed when liquid $^3$He is cooled to mK temperatures (triplet pairing of spin–1/2 atoms) [1].

(ii) In the quantum fluid interior of a neutron star: It is generally believed that the neutrons pair-condense at temperatures below $\sim 10^9$ K (triplet pairing of spin-1/2 nucleons) [2].

There are important differences in these realizations. In dense neutron matter, $^1S_0$ pairing, which dominates in the inner crustal region of the star, is quenched by the strong short-range repulsion in the nucleon-nucleon ($NN$) interaction. At $NN$ collision energies corresponding to the relevant density regime, the most attractive phase shift is found in the $^3P_2$ channel, indicating that this state will dominate the pairing. The $^3P_0$ phase shift is only weakly attractive, while the $^3P_1$ phase shift is repulsive; these channels are therefore expected to have only minor effects on the problem. The state dependence of the spin-orbit component of the $NN$ interaction is responsible for this pre-eminence of the $^3P_2$ state. A complication is introduced by the tensor component of the interaction, which implies a coupling of the $^3P_2$ and $^3F_2$ channels.

On the other hand, in liquid $^3$He, the atom-atom interaction is spherically symmetric and spin-independent to a very good approximation. Tensor and spin-orbit components are effectively absent. Thus there is no coupling to higher $L$ waves, but
the three $J$ states for $L = 1$, $S = 1$, namely $^3P_2$, $^3P_1$, and $^3P_0$ now enter on an equal footing.

On balance, the neutron matter – in spite of the complication of tensor coupling – is simpler than liquid $^3\text{He}$, since fewer magnetic substates enter the problem. Therefore we focus first on neutron matter as a “training ground” for the more challenging case of liquid $^3\text{He}$.

Quantitative, microscopic prediction of the superfluid phases and phase diagram of systems manifesting triplet pairing (or pairing in orbital angular momentum states beyond $S$-wave), has hitherto eluded the best efforts of theorists. For one thing, the popular Ginzburg-Landau approach is restricted to temperatures near the critical temperature $T_c$. Secondly, iterative procedures, routinely employed to calculate the energy gap in weakly interacting systems with $S$-wave pairing, are afflicted with slow convergence and uncertain accuracy when applied to the many coupled, singular nonlinear integral equations that come into play for pairing in higher angular momentum states. The limitations of standard iterative procedures become most apparent when one attempts to construct the superfluid phase diagram of the system, which is controlled by tiny energy splittings between different solutions of the BCS pairing problem.

These difficulties can now be overcome by means of a recently introduced separation method [3–8] for solving the BCS gap equations associated with pairing states of arbitrary angular-momentum content. In this paper, we implement this method to identify and classify the spectrum of phases of the $^3P_2-^3F_2$ pairing model. Nominal, the model refers to superfluid neutron matter at densities of order twice that found in heavy nuclei. However, the results obtained are universal in the sense that the angular properties of the solutions corresponding to the existing phases are independent of the detailed nature of the system under study, and especially its pairing interaction. Indeed, these results can be derived analytically and depend only on two energy scales, one governing the overall strength of the pairing effect and the other measuring the strength of the coupling between $P$ and $F$ states. There exist the usual solutions involving a single value of the magnetic quantum number $M$ (and its negative) – obviously three in number corresponding to $M = 0, \pm 1, \pm 2$. In addition we find that there exist ten real multicomponent solutions. Five of these have angle-dependent order parameters with nodes (and therefore are of relatively high energy), while the other five are nodeless. In contrast to the case of superfluid $^3\text{He}$, transitions occur between phases with nodeless order parameters. Results for the temperature dependence of the competition between the various phases have been obtained. In principle, and presumably also in practice, the same approach can be used to complete the catalog of superfluid phases of liquid $^3\text{He}$.

2. BCS FORMALISM FOR ANY-CHANNEL PAIRING

Adopting a spinor representation, the $2 \times 2$ gap matrix for the general BCS problem of pairing in a state of triplet spin $(S = 1)$ and triplet isospin $(T = 1)$ has the expansion

$$\Delta_{\alpha\beta}(p) = \sum_{J,L,M} \Delta_L^{JM}(p) (G_{LM}(n))_{\alpha\beta}$$

(1)
in terms of the spin-angle matrices
\[ (G^M_{LM}(n))_{\alpha\beta} = \sum_{M_2M_L} C^1_{LM_2M_L} C^1_{LM} Y_{LM_L}(n). \] (2)

The coupled partial-wave gap components \( \Delta^J_M(p) \) solve the set of gap equations
\[ \Delta^J_M(p) = \sum_{L'J_1M_1} (-1)^{L+L'} \int \int \langle p|V^J_{LL'}|p_1\rangle S^{JMJ_1M_1}_{LL'}(n_1) \times \frac{\tanh(E(p_1)/2T)}{2E(p_1)} \Delta^{J_1M_1}_L(p_1)p_1^2dp_1dn_1. \] (3)

The quasiparticle energy \( E(p) = [\xi^2(p) + D^2(p)]^{1/2} \) is constructed from the gap components \( \Delta^J_M(p) \) through
\[ D^2(p) = \sum_{LJM, L_1J_1M_1} (\Delta^J_M(p))^* \Delta^{J_1M_1}_L(p) S^{JMJ_1M_1}_{LL_1}(n) \] (4)
together with the single-particle spectrum \( \xi(p) \) of the normal Fermi liquid, often parametrized in terms of an effective mass \( M^* \). Angular dependence is introduced into the quasiparticle energy \( E(p) \) via the spin trace
\[ S^{JMJ_1M_1}_{LL_1}(n) = \text{Tr} \left[ (G^M_{LM}(n))^* G^{J_1M_1}_{L_1M_1}(n) \right], \] (5)
which obviously complicates explicit solution of the system of gap equations.

The pairing matrix elements \( \langle p|V^J_{LL'}|p_1\rangle \) are generated by the spin-angle expansion
\[ V(p, p_1) = \sum_{LL',JM} (-1)^{L-L'} \langle p|V^J_{LL'}|p_1\rangle G^M_{LM}(n) (G^M_{LM}(n_1))^* \] (6)
of the totality of vertex diagrams irreducible in the particle-particle channel.

A close inspection reveals that complete solution of this set of equations presents awesome difficulties: note, in particular, the coupling to the generic angular momentum labels \( L_1, J_1, \) and \( M_1 \) quantum numbers via the squared-gap quantity \( D^2(p) \) appearing in the quasiparticle energy \( E(p) \). Practical approximate solution (which might still be extremely accurate) will require a series of justifiable simplifications.

To wit, in dealing with neutron matter, the free-space two-body interaction has the salient feature that the components of the central forces nearly cancel each other, as reflected in the behavior of the experimental \( P \)-scattering phases. It is assumed that this feature is preserved by the effective interaction inside neutron matter. We shall later comment on the veracity of this assumption. If it holds, the pivotal role of the spin-orbit force in promoting the \( ^3P_2 \) pairing channel then implies that contributions to triplet pairing from “nondiagonal” terms with \( L' \neq 1 \) or \( J_1 \neq 2 \) on the right-hand side of the set of gap equations can be evaluated within perturbation theory, in terms of the set of principal gap amplitudes \( \Delta^{2M}_L(p) \), with \( M = 0, \pm 1, \pm 2. \)
In fact, if we choose to invoke time-reversal invariance, the problem may be treated in terms of only three complex functions, namely $\Delta^2_1(p)$ with $M = 0, 1, 2$.

Another simplifying feature is the existence of a small parameter across the range of interesting pairing problems. When dealing with anisotropic gaps as may arise in pairing states beyond the $S$-wave, it is helpful to define the energy gap parameter $\Delta_F$ as the square root of the angle average of $D^2(p)$, evaluated at the Fermi surface. The ratio $d_F = \Delta_F/\epsilon_F$ of the gap to the Fermi energy provides a small parameter for BCS pairing theory. For $^3P_2$ (or $^3P_2-^3F_2$) pairing in neutron matter, a leading order approximation in $d_F$ is good to about one part in $10^5$ or $10^6$. A similar or higher accuracy is to be expected in liquid $^3$He.

3. THE SEPARATION METHOD

The separation approach developed in Refs. [3-8] facilitates essential further simplifications (linked in part to the existence of a (very) small parameter). In this approach, any given pairing matrix element is expressed identically as a separable part, plus a remainder that vanishes when either momentum argument is on the Fermi surface. We write

$$\langle p | V^J_{LL'} | p_1 \rangle = v^J_{LL'} \phi^J_{LL'}(p) \phi^J_{LL'}(p_1) + W^J_{LL'}(p, p_1)$$

and verify that the choices $v^J_{LL'} \equiv \langle p_F | V^J_{LL'} | p_F \rangle$ and $\phi^J_{LL'}(p) \equiv \langle p | V^J_{LL'} | p_F \rangle / v^J_{LL'}$ produce the desired behavior when $p$ or $p_1$ hits $p_F$. Substitution of this identity into the set of gap equations, followed by simple manipulation and argumentation, establishes a decomposition of the form

$$\Delta^J_M(p) = D^J_M \chi^J_{LL'}(p) \quad (|M| = 0, 1, 2)$$

for the general gap component and provides separate equations for the shape factor $\chi^J_{LL'}(p)$ (normalized to unity at $p = p_F$) and the numerical amplitudes $D^J_M$ that determine the angle-dependence of the gap.

The shape factor $\chi^J_{LL'}(p)$ is given by a nonsingular integral equation involving a set of kernels proportional to the residual interaction $W^J_{LL'}(p, p_1)$ (see Ref. [8] for details). The vanishing of $W$ at the Fermi surface removes the logarithmic singularity characteristic of BCS theory, which is banished to the set of equations for the amplitudes $D^J_M$, which are coupled nonlinear equations for a set of numbers.

To very high accuracy, the integral equation for $\chi^J_{LL'}$ is linear and independent of the amplitudes $D^J_M$. For all practical purposes (e.g., appealing to extremely rapid convergence of small-parameter expansions), we are free to make the replacements

$$E(p) \rightarrow |\xi(p)|, \quad \tanh(E(p)/2T) \rightarrow 1$$

in any integral involving the residual interaction $W$ as a factor. The shape factors $\chi^J_{LL'}(p)$ are therefore determined by the interaction $W$ and may be calculated independently of the $D^J_M$ by a matrix inversion routine. The nonlinear and singular aspects of the problem reside entirely in the set of equations for the amplitudes $D^J_M$,.
which may also be solved numerically by standard methods (e.g. Newton-type algorithms). In fact, we have even been able to solve scaled versions of the $D^M_L$ equations *analytically* in interesting cases, as will emerge below.

In view of these crucial simplifications revealed and expedited by the separation transformation, it is seen that

(i) For given $L$, $L'$, and $J$, the $\chi$ function is universal, i.e., independent of magnetic quantum number $M$ and temperature $T$.

(ii) The factorization

$$\Delta^2_M(p) = D^2_M \chi(p) \quad (|M| = 0, 1, 2)$$

may be asserted for the principal gap components, where $\chi(p) \equiv \chi^{21}_{11}(p)$.

(iii) Explication of the phase diagram of dense superfluid neutron matter reduces to determination of the three amplitudes $D^2_M$, since the character of the phase diagram itself is independent of the shape factor $\chi(p)$.

### 4. PERTURBATIVE TREATMENT

Now let us focus on the nondiagonal contributions to the right-hand side of the gap equations (3), where nondiagonal means one or more of the conditions $L' = 1$, $L_1 = 1$, $J_1 = 2$ is not met. Two nondiagonal contributions are found to be of leading importance: The first contains the integral of the product $V^2_{31} S^2_{13} \Delta^2_M$ while the second contains the integral of the product $V^2_{11} S^2_{13} \Delta^2_M$. There is a single small factor in each product, namely $V^1_{31}$ in the first instance and $\Delta^2_M$. All other nondiagonal contributions involve two or more small factors. Retaining the leading nondiagonal pieces and ignoring all the others, we have what is called the $^3P_2 - ^3F_2$ pairing problem, which is generally regarded as a satisfactory model of superfluid neutron matter.

Rapid convergence of the pertinent nondiagonal integrals is instrumental to success of the perturbation approach used to treat the nondiagonal effects. To affirm this behavior, we observe that the dominant contributions to these integrals come from the vicinity of the Fermi surface. Then, if $E(p)$ is significantly larger than the energy gap value (as it will be for large $p$), $E(p)$ and $|\xi(p)|$ will practically coincide, and consequently the angular integration yields zero.

We conclude that in evaluating the nondiagonal contributions it suffices to know the “minor” gap components $\Delta^2_M(p)$ at $p = p_F$. These quantities will in fact be determined in terms of the coefficients $D^2_M$. To excellent approximation, we may retain only the dominant contribution to the right-hand side of the gap equations (3) that diverges like $\ln d_F$ as $\Delta_F \to 0$. We are then led to the desired connection

$$\Delta^2_3(p = p_F) = \eta D^2_1,$$

where $\eta = -\langle p_F | V^2_{13} | p_F \rangle / v_F$ and $v_F \equiv \langle p_F | V^2_{11} | p_F \rangle$. Similar relations may be obtained for other minor gap components, notably $\Delta^0_0$ and $\Delta^1_1$.

What can we say about the parameter $\eta$, which represents the coupling to the “nondiagonal” states? If perturbation theory is to be valid, it should be comfortably small. For pairing matrix corresponding directly to in-vacuum neutron-neutron interaction, $\eta$ depends smoothly on density, varying around 0.3 in the interval
ρ₀ < ρ < 2ρ₀. Medium modification of the spin-orbit force is probably not important. Due to the relativistic origin of this component, it should not be much affected by polarization or correlation corrections. This claim is consistent with empirical analyses of the spin-orbit splitting in finite nuclei.

On the other hand, medium modification of the tensor force may be more significant, especially as one approaches the density at which pion condensation occurs. Thus, the parameter η is somewhat uncertain, although it is still expected to be rather small.

To sketch out the superfluid phase diagram of the system, we need to determine the key amplitudes \( D^{2M}_1 \) (\( M = 0, 1, 2 \)) to leading perturbative order in the coupling η. Exploiting the linear connection (11) between the \( D^{2M}_1 \) amplitudes and the minor-component values \( Δ^2_M(p_F) \), simple manipulations applied to the coupled gap equations (1) at \( p = p_F \) yield

\[
D^{2M}_1 + v_F \sum_{M_1} D^{2M}_{1M_1} \int \int \phi(p) \frac{\tanh(E_0(p)/2T)}{2E_0(p)} S^{2M2M_1}_{11}(n) \chi(p)p^2dpdn = ηv_Fr_M
\]

(12)

with \( φ(p) = \langle p|V_1^2|p_F⟩/v_F, E_0(p) = E(p; η = 0) \), and

\[
r_M = \sum_{M_1} D^{2M}_{1M_1} \int \int \left[S^{2M2M_1}_{31}(n) + S^{2M2M_1}_{13}(n)\right] \frac{\tanh(E_0(p)/2T)}{2E_0(p)} p^2dpdn.
\]

(13)

We restrict the search for solutions of these equations to those with real coefficients \( D^{2M}_1 \), reasoning that solutions with complex \( D \) amplitudes will lie at energies high enough to make them physically irrelevant.

Inserting the explicit form of \( S^{2M2M_1}_{11}(n) \) from Eq. (5) into Eq. (12), we derive a system of three equations for the gap value \( Δ_F \) and the two ratios

\[
λ_1 = D^{21}_1/D^{20}_1 \sqrt{6} \quad \text{and} \quad λ_2 = D^{22}_1/D^{20}_1 \sqrt{6},
\]

(14)

which serve generally to determine the angular dependence of the gap (or alternatively its composition with respect to the magnetic quantum number \( M \)). In a notation and form compatible with the earlier treatment of the pure \( 3P_2 \) problem [4,6], these basic equations read

\[
λ_2 + v_F [λ_2(J_0 + J_3) − λ_1J_1 − J_3] = ηv_Fr_2, \quad (15a)
\]

\[
λ_1 + v_F [−(λ_2 + 1)J_1 + λ_1(J_0 + 4J_5 + 2J_3)/4] = ηv_Fr_1, \quad (15b)
\]

\[
1 + v_F [−(λ_2J_3 + λ_1J_1)/3 + J_5] = ηv_Fr_0, \quad (15c)
\]

with

\[
J_i = \int \int f_i(θ, φ) \phi(p) \frac{\tanh(E_0(p)/2T)}{2E_0(p)} \chi(p) p^2dpdn / 4π,
\]

(16)

\[
f_0 = 1 − 3z^2, \quad f_1 = 3xz/2, \quad f_3 = 3(2x^2 + z^2 − 1)/2, \quad f_5 = (1 + 3z^2)/2, \quad (17)
\]

\[
z = \cos θ, \quad x = \sin θ \cos φ, \quad y = \sin θ \sin φ. \quad (18)
\]
Upon setting \( \eta = 0 \), Eqs. (15a)–(15c) collapse to Eqs. (12)–(14) of Ref. [4], which refer to the case \( \kappa_1 = \kappa_2 = 0 \) and were solved analytically in Refs. [4,6].

If we had not used the separation method, we would have been faced at this point with a system of coupled singular nonlinear integral equations for a set of functions, not numbers. The great advantage gained is transparent.

5. SEARCHING FOR MULTICOMPONENT SOLUTIONS

The basic equations (15a)–(15c) possess the familiar one-component solutions corresponding to definite \(|M|\), i.e., \(|M| = 0, 1, 2\). In addition, there exist multicomponent solutions whose structure and spectrum we seek to establish, via a two-step process. We first note that \( J_5 \), which is the only \( J_i \) integral containing a principal term going like \( \ln(\epsilon_F/\Delta_F) \), is responsible for the gap magnitude \( \Delta_F \). On the other hand, \( J_5 \) is irrelevant to the phase structure.

**Step 1.** Thus, we begin by eliminating \( J_5 \) from the first pair (15a)–(15b) of basic equations and reduce the number of \( J_i \) integrals in each of the pair to two:

\[
(\lambda_2 + 1)[3\lambda_1(\lambda_2 + 1)J_0 - 2(\lambda_1^2 - 2\lambda_2^2 + 6)J_1] = \eta B_1, \tag{19a}
\]

\[
(\lambda_2 + 1)[(\lambda_1^2 - 4\lambda_2)J_1 + \lambda_1(\lambda_2 + 1)J_3] = \eta B_2, \tag{19b}
\]

\[
B_1 = 2\lambda_1(2\lambda_2 + 3)r_2 - 4(\lambda_2^2 - 3)r_1 - 6\lambda_1(\lambda_2 + 2)r_0, \\
B_2 = -\lambda_1r_2 + 4\lambda_2r_1 - 3\lambda_1\lambda_2r_0. \tag{20}
\]

**Step 2.** Assuming \( \lambda_2 \neq 1 \), we perform the rotation

\[
(x, z) = (-t \sin \vartheta + u \cos \vartheta, t \cos \vartheta + u \sin \vartheta) \tag{21}
\]

and choose the angle \( \vartheta \) so as to remove the integral \( J_1 \) from the “sanitized” first pair of basic equations, (19a)–(19b). Specifically, if \( \zeta = \tan \vartheta \) is chosen to obey the algebraic equation

\[
\lambda_1 \zeta^2 - (\lambda_2 - 3)\zeta - \lambda_1 = 0, \tag{22}
\]

then substitution of the transformed integrals \( J_i \) converts the key pair of equations (19a)–(19b) into

\[
(\lambda_2 + 1)[A_1J_0 + A_2J_3] = \eta B_1, \tag{23a}
\]

\[
(\lambda_2 + 1)[A_1J_0 + A_2J_3] = -2\eta B_2, \tag{23b}
\]

with

\[
A_1 = \frac{3}{2}\lambda_1(1 + \lambda_2)(2 - \zeta^2) - \frac{3}{2}(\lambda_1^2 - 2\lambda_2^2 + 6)\zeta, \\
A_2 = -3\lambda_1(1 + \lambda_2)\zeta^2 - (\lambda_1^2 - 2\lambda_2^2 + 6)\zeta. \tag{24}
\]

We observe that the left-hand members of Eqs. (23a) and (23b) are identical! It is in fact just this coincidence that leads to the striking universalities of the pure
Figure 1. Solutions for the coefficient ratios $\lambda_1$ and $\lambda_2$ in the case $\eta \equiv 0$, corresponding to the uncoupled $^3P_2$ pairing problem solved in Refs. [4,6]. The particular (point) solution ($\lambda_1 = 0, \lambda_2 = 3$) is indicated by the solid dot and the degenerate solutions by the solid curves.

$^3P_2$ pairing problem first revealed in Ref. [4]. Of course, in this problem, which corresponds to $\eta = 0$, we have the special circumstance that the right-hand side of each equation vanishes identically, so that Eqs. (23a) and (23b) coincide and yield only the single constraint

$$ (\lambda_2 + 1)[A_1J_0 + A_2J_3] = 0 $$

(25)
on the parameters $\lambda_1$ and $\lambda_2$.

It ensues that the solutions of the pure $^3P_2$ pairing problem display remarkable universalities as expressed in two kinds of degeneracies, independently of temperature, density, and details of the in-medium interaction:

- **Energetic degeneracy.** There exists an upper group of states, degenerate in energy, whose angle-dependent order parameters have nodes, and a lower group with nodeless order parameters. Relative to the absolute pairing energy, the splitting between upper and lower states is small (of order 2% in the neutron-matter problem).
- **Parametric degeneracy.** The multicomponent solutions, which satisfy the spectral condition [4,6]

$$ (\lambda_1^2 + 2 - 2\lambda_2)(\lambda_1^2 - 2\lambda_2^2 - 6\lambda_2) = 0, $$

(26)
Figure 2. Multicomponent solutions of the $^3P_2$ pairing problem, defined by the coefficient ratios $\lambda_1$ and $\lambda_2$. Solutions with nodeless (respectively, node-bearing) order parameters in the case of pure $^3P_2$ pairing are indicated by open (respectively, filled) circles.

display a parametric degeneracy with respect to the parameters $\lambda_1$ and $\lambda_2$. Thus, as seen in Fig. 1 (which was constructed analytically), the multicomponent solutions generally define curves rather than points in the ($\lambda_1, \lambda_2$) plane.

6. BREAKING THE PARAMETRIC DEGENERACY

The strong parametric degeneracy intrinsic to the uncoupled $^3P_2$ pairing problem is lifted in the case of $^3P_2-^3F_2$ pairing, i.e., at $\eta \neq 0$. With the coupling turned on, the true solutions of the problem are represented by a set of isolated points in the ($\lambda_1, \lambda_2$) plane.

From the analytical standpoint, the underlying “mechanism” runs as follows. Upon equating the right-hand members of the double-sanitized pair of basic equations (23a)–(23b) in the small–$|\eta|$ limit ($|\eta|$ infinitesimal), one obtains an additional relation between the parameters $\lambda_1(\eta = 0)$ and $\lambda_2(\eta = 0)$:

$$\lambda_1 r_2 - (\lambda_2 - 3)r_1 - 3\lambda_1 r_0 = 0.$$  

(27)
This relation supplements the spectral condition (26) nontrivially and removes the parametric degeneracy. The system formed by (27) and (26) is solved by applying the same rotation in $x-z$ coordinates as introduced previously. After some algebra, one may then obtain the full set of solutions of the coupled-channel $^3P_2-^3F_2$ pairing problem. A salient feature of these solutions, made explicit via the separation scheme, is their virtually complete independence of the temperature $T$.

We summarize the results of this analysis with a catalog of the possible solutions for $^3P_2-^3F_2$ pairing described in the BCS framework. One simplification, already evident in Fig. 1 but also quite general, is that the relevant pairing energies are independent of the sign of $\lambda_1$, so that we only need to consider $\lambda_1 > 0$. The modified picture in the $(\lambda_1, \lambda_2)$ plane is displayed in Fig. 2.

First of all, there remain the three well-known single-component solutions with $|M| = 0, 1, \text{ or } 2$. Beyond these, the collection of unitary solutions of the $^3P_2-^3F_2$ pairing problem contains ten multicomponent solutions, corresponding to more complicated superfluid phases.

Five of these additional solutions, denoted $O_k$ ($k = 1, \ldots, 5$), have nodeless order parameters and include:

(a) Two two-component solutions $O_{\pm 3}$, identical to those found in the pure $^3P_2$ pairing problem, with $\lambda_1 = 0$ and $\lambda_2 = \pm 3$.

(b) Three three-component solutions:
   - Two of them, $O_1$ and $O_4$, are associated with the upper branch of $\lambda_1^2 - 2\lambda_2^2 - 6\lambda_2 = 0$ and have $\lambda_2 = 3(\sqrt{21} - 4)/5$ and $\lambda_2 = 3$, respectively.
   - The third, $O_2$, is associated with the lower branch of the same equation and has $\lambda_2 = -3(\sqrt{21} + 4)/5$.

The other five solutions, denoted by $X_k$ ($k = 1, \ldots, 5$) do possess nodes.

(a) Two two-component solutions $X_{\pm 1}$, again identical to those found in the pure $^3P_2$ pairing problem, with $\lambda_1 = 0$ and $\lambda_2 = \pm 1$.

(b) Three three-component solutions, $X_2, X_3,$ and $X_4$, associated with the parabola $\lambda_2 = \lambda_1^2/2 + 1$ and having $\lambda_2 = 13 - 2\sqrt{35}$, $\lambda_2 = 3$, and $\lambda_2 = 13 + 2\sqrt{35}$, respectively.

These general features of the spectrum of solutions of the $^3P_2-^3F_2$ problem are expected to persist even if $|\eta|$ is not so small.

7. THE BATTLE BETWEEN PHASES

To complete the phase diagram of superfluid neutron matter, we need the gap values $\Delta_F$ for the various phases. These are found through the third basic equation (15c), which involves $J_5$. To determine which phase wins the competition at a given temperature $T$, we compare the free-energy shifts

$$F_s = -\int_0^g \Delta_F^2(g') \frac{dg'}{(g')^2}$$

due to pairing in the corresponding superfluid states, where $g$ is the relevant coupling constant.
Figure 3. Temperature dependence of the splitting between the phase $O_1$ (or $O_2$) and the phases $O_{\pm 3}$ (or the one-component phase with $M = 0$), in terms of the difference $\delta \Delta^2_F(T)$ between the respective $\Delta^2_F(T)$ values, measured relative to $|\delta \Delta^2_F(T = 0)|$. To set the energy scales involved, the latter quantity is around 2% of $\Delta^2_F$ for $\eta = 0.3$, while the gap parameter $\Delta_F(T = 0)$ itself reaches a maximum value near 0.44 MeV at $k_F \simeq 2.1$ fm$^{-1}$ when the pairing interaction is given by the Argonne $v_{18}$ potential and free normal-state single-particle energies are employed [8].

At low $T$, the only viable contestants are the solutions with nodeless order parameters, since the other solutions lie too high in energy. The separation between the two groups of states simply cannot be bridged if the value of $|\eta|$ remains rather small.

As documented in the last section, the parametric degeneracy of the $^3P_2$ pairing problem in the $\lambda_1 - \lambda_2$ plane, embodied in the spectral relation (26), is completely eradicated when the $\eta$-coupling is switched on. On the other hand, the energetic degeneracy between the different superfluid phases is only partially lifted. Specifically, the spectrum of pairing energies decays into several groups of nearly degenerate states: The $O_1$ and $O_2$ phases form the lowest-energy group, followed by the phases $O_{\pm 3}$ along with the one-component phase with $M = 0$, and so on.

Raising the temperature from $T = 0$ toward the critical temperature $T_c$, the splitting between the two groups lowest in energy shrinks, until, at $T \simeq 0.7 T_c$, their roles are interchanged and transitions occur. This behavior is quantified in Fig. 3. We observe that the analogy with the A-B phase transition in superfluid $^3$He is imperfect. Whereas the order parameter of the A-phase solution has nodes while that of the B-phase does not, the transition in neutron matter takes place between phases with nodeless order parameters.
8. CONCLUSIONS

We close with some details and caveats and examine some prospects for observable implications of our findings.

First, there is the matter of unresolvable fine structure in the phase diagram. Introduction of the $^3P_0$ and $^3P_2$ pairing channels would entail a further lifting of energy degeneracies, but current ignorance of the in-medium interaction precludes reliable estimation of these effects.

Secondly, we point out that the region near the critical temperature $T_c$ may admit subtle phase transitions not uncovered by our analysis. In this regime, strong-coupling corrections [9,10] can no longer be neglected in deciding the contest between phases. Also, the character of the phase diagram may be influenced significantly by external magnetic fields, a fact not considered in our treatment. And further, the phenomenon of fermion condensation [11], occurring as a precursor to pion condensation [12-14], may provide an exotic source of phase transitions in superfluid neutron matter.

In summary, it has been established that the superfluid phase diagram of dense neutron matter can in principle exhibit several triplet superfluid phases (at least 13!). Transitions between the different phases are expected to occur as a young neutron star cools. Since the gap value changes in these transitions, their occurrence may ultimately be detected in the thermal history and/or the rotational dynamics of the star. Such transitions may produce a significant variation of the moment of inertia of the star, or alterations of the distribution of the angular momentum between the crust and the vortex system, resulting in a change of the star’s angular velocity.

The existing analysis of triplet pairing in neutron matter provides a firm foothold for the more ambitious project of a complete characterization of the repertoire of superfluid phases of liquid $^3$He, which is even more complex, promising a richness of phenomena not yet revealed by experiment or theory. Since $^3P_2$, $^3P_1$, and $^3P_0$ states all enter the picture, one will be faced with nine nonlinear equations for the relevant gap components (in contrast to the five encountered in the pure $^3P_2$ case). Nevertheless, the separation method will continue to provide an incisive tool of analysis.

Thus, the procedure applied here may in principle be extended to count all possible solutions of the set of nine BCS equations for the gap function of superfluid $^3$He. We may begin by severing the couplings between $^3P_2$, $^3P_1$, and $^3P_0$ components, which facilitates a reference analytic solution. With this solution at hand, we may restore the channel couplings and solve the resulting set of equations numerically step by step, under increase of the coupling parameters. Attention then turns to (i) verification of the structure of the A-phase predicted by ABM [15,16], (ii) evaluation of the difference between free energies of A and B phases vs. temperature and pressure, and the barrier between the two phases (important for understanding the huge delay of the phase transition between the supercooled A-phase and the B-phase [17–19]), (iii) investigation of strong-coupling corrections beyond the Ginzburg-Landau approximation, and (iv) examination of the phase diagram of liquid $^3$He in an external magnetic field.
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