OPTIMAL STRATEGIES FOR A TIME-DEPENDENT HARVESTING PROBLEM

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ABSTRACT. We focus on an optimal control problem, introduced by Bressan and Shen in [5] as a model for fish harvesting. We consider the time-dependent case and we establish existence and uniqueness of an optimal strategy. We also study a related differential game, and we prove existence of Nash equilibria. From the technical viewpoint, the most relevant point is establishing the uniqueness result. This amounts to prove precise a-priori estimates for solutions of suitable parabolic equations with measure-valued coefficients. All the analysis focuses on one-dimensional fishing domains.

1. Introduction. This paper deals with a model for fish harvesting introduced by Bressan and Shen in [5]. The model involves an optimization problem for a payoff functional representing the profit of fish companies. Differently from [5], we consider the time-dependent case and we prove existence and (local) uniqueness of optimal fishing strategies. We also exhibit first order necessary conditions for optimality and we establish the existence of Nash equilibria in the case of several competing players, i.e. fish companies. We always focus on the case of one-dimensional bounded fishing domains.

Before discussing our results, we go through the main features of the model proposed in [5]. Consider a one-dimensional fishing domain (a confined portion of a river, for instance), modeled by the real interval $[0, R]$, and a time interval $[0, T]$. With $\varphi = \varphi(t, x)$, we denote the fish density at time $t \in [0, T]$ at the point $x \in [0, R]$. When no fishing activity is conducted, the evolution of the fish population is modeled by the parabolic equation.

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\[ \partial_t \varphi = \partial_{xx}^2 \varphi + \varphi f(t,x,\varphi). \]  

(1)

A reasonable choice for the source term \( f \) is the logistic law

\[ f(t,x,\varphi) = \alpha(t,x)(h(t, x) - \varphi), \]  

(2)

where \( h(t, x) \) denotes the maximum fish population that can be supported by the habitat at the point \( x \) and at the time \( t \), and \( \alpha \) is a reproduction rate. Equation (1) is augmented with the initial datum

\[ \varphi(0,x) = \varphi_0(x), \]  

(3)

and with the homogeneous Neumann boundary conditions

\[ \partial_x \varphi(t,0) = \partial_x \varphi(t,R) = 0. \]  

(4)

Denote by \( \mu = \mu(t,x) \) the intensity of the harvesting conducted by a fish company. We modify equation (1) by setting

\[ \partial_t \varphi = \partial_{xx}^2 \varphi + \varphi f(t,x,\varphi) - \varphi \mu. \]  

(5)

To define an optimal control problem, we first introduce the cost functional

\[ \int_0^T \int_0^R c(t,x) \mu(t,x) \, dt \, dx. \]  

(6)

In the above expression, \( c \) is a nonnegative, lower semicontinuous function representing the cost of the fishing effort. One could for instance have a cost \( c \) which is monotone increasing with respect to the distance of the point \( x \) from the fish company hub. Also, the presence of a natural park where no fishing is allowed can be modeled by setting \( c(t,x) = +\infty \) in that region. We can now define the payoff functional by setting

\[ J(\mu) := \int_0^T \int_0^R \varphi(t,x) \mu(t,x) \, dt \, dx - \Psi \left( \int_0^T \int_0^R c(t,x) \mu(t,x) \, dt \, dx \right). \]  

(7)

In the above expression, \( \Psi \) is a nondecreasing, convex function (the simplest possible choice is the identity). The function \( \varphi \) is the solution of the initial-boundary value problem obtained by coupling (5) with (3) and (4). Note, in particular, that \( \varphi \) depends on \( \mu \) and hence the functional \( J \) is nonlinear. In the paper, we address the problem of maximizing the payoff functional \( J \), i.e. of finding an optimal fishing strategy \( \mu \), under the constraints

\[ \mu(t,x) \geq 0, \quad \int_0^T \int_0^R b(t,x) \mu(t,x) \, dt \, dx \leq 1. \]  

(8)

In the above expression, the nonnegative function \( b \) models the maximum amount of harvesting power within the capabilities of the fish company. In practice, it may depend on the number of fishermen and on the size of the fishing boats. We actually search for optimal strategies that are not necessarily functions, but more generally nonnegative Radon measures. This is motivated by the following two considerations.

- From the analytic viewpoint, we remark that the functional \( J \) has only linear growth with respect to \( \mu \); hence, in general, an optimal strategy \( \mu \) does not belong to \( L^1([0,T] \times [0,R]) \). Note that a quadratic harvesting cost such like (see [2, 8, 10, 11, 12])

\[ \int_0^T \int_0^R c(t,x) \mu^2(t,x) \, dt \, dx \]
is entirely natural from the mathematical viewpoint and it would give an optimal strategy \( \mu^{opt} \in L^2([0,T] \times [0,R]) \). However, the linear cost (6) provides a more realistic model.

- From the \textit{modeling} viewpoint, it is reasonable to expect that, when for instance there is a natural park, the optimal strategy concentrates the fishing effort at the park border. An explicit analytic example where the optimal strategy contains atomic parts concentrated at discontinuity points of \( c \) is exhibited in [6].

To conclude the discussion of the model, we point out that cost functional (7) only takes into account the running cost and gain occurring on the given interval \([0,T]\). In particular, the optimal strategy \( \mu \) might lead to overfishing and resource depletion. Owing to this consideration, it could be interesting to consider different cost functionals. For instance, a reasonable choice could be

\[
\tilde{J}(\mu) := \int_0^T \int_0^R \varphi(t,x) \mu(t,x) dt\,dx + \int_0^R \varphi(T,x) dx - \Psi \left( \int_0^T \int_0^R c(t,x) \mu(t,x) dt\,dx \right).
\]

In the previous expression, the second term represents the total fish mass at the time \( t = T \). As another possibility, one could also consider a cost functional like

\[
\tilde{J}(\mu) := \int_0^{+\infty} \int_0^R \omega(t) \varphi(t,x) \mu(t,x) dt\,dx - \Psi \left( \int_0^{+\infty} \int_0^R \omega(t) c(t,x) \mu(t,x) dt\,dx \right),
\]

where \( \omega \) is a weight function that is decaying sufficiently fast at \(+\infty\). In this way, the cost functional takes into account a possibly infinite time horizon, but the decay of the weight function \( \omega \) expresses the fact that one is more interested in the close future than in the far one. Note, however, that in the present work we stick to the model introduced in [5] and we focus on the cost functional (7).

We now briefly discuss some previous results concerning the analysis of this model. In [5] the analysis focuses on the one-dimensional, steady state problem, and (5) reduces to a second order ordinary differential equation. The authors established existence and local uniqueness of optimal strategies, and discussed the related differential game proving existence of Nash equilibria. See also [6] for related results. In [4] Bressan, Coclite and Shen established existence of optimal strategies for the steady case by considering multidimensional fishing domains. Finally, in [9] Coclite and Garavello established existence of optimal strategies in multi-dimensional domains in the time-dependent case.

The main results of the present paper are the following ones.

- We establish existence of an optimal strategy \( \mu \) for the payoff functional \( J \) in (7) subject to the constraints (8), and first order necessary conditions for optimality. Also, we show that the optimal strategy is locally unique, i.e. it is unique in the class of measures with sufficiently small total variation. The uniqueness result is stated as Theorem 5.1. Note that, while the existence proof is the same as in [4, 9], the uniqueness proof is, from the technical viewpoint, the most relevant result of the present paper. More precisely, as in [4, 9], the uniqueness follows from the concavity of the cost functional. However, the proof of the concavity in the time-dependent case is different than the proof in the static case. The main technical differences with the analysis in [4, 9] are discussed in the next paragraph.
By relying on the above local uniqueness result we establish existence of Nash equilibria for a differential game modeling the case where there are several competing fish companies that exploit the same environment; see Theorem 6.2 for the precise result.

The local uniqueness of optimal strategies was established by Bressan and Shen [5] in the steady case. The main novelties of our analysis compared to the one in [5] are the following ones.

• In both cases, the main point of the argument is showing that the functional $J$ is locally concave. This amounts to establish suitable a-priori estimates on the solutions of parabolic equations with measured-valued coefficients similar to (5). However, as mentioned before, in the steady case the parabolic equation (5) reduces to a second order ordinary differential equation: this makes the analysis considerably simpler than the time-dependent case. In particular, in the time-dependent case we establish precise estimates on solutions of parabolic equations with measured-valued coefficients by making extensive use of the Duhamel representation formula.

• The analysis in [5] is based on a technical assumption, i.e. condition [5, Equation (5.15)]. In the time-dependent case we replace [5, Equation (5.15)] with (73), namely with the requirement that the initial fish density distribution is sufficiently close, in the $H^1$ norm, to a constant representing the maximal fish density supported by the environment. Whether or not uniqueness can be obtained without this smallness assumption is, to the best of our knowledge, an open problem.

The exposition is organized as follows. In §2 we list the mathematical symbols, the notation and the main hypotheses of the paper. In §3, we state existence, uniqueness and stability results for the initial-boundary value problem (3), (4), (5) in the case when $\mu$ is a given nonnegative Radon measure. In §4 we deduce existence of an optimal strategy maximizing (7) subject to the constraint (8) and we establish necessary conditions for optimality. In §5 we establish the local uniqueness of the optimal strategy by relying on a technical lemma proved in §7. Finally, in §6 we introduce the differential game and we establish existence of Nash equilibria. The Appendix collects some results concerning the Duhamel representation formula and the fundamental solution of the heat equation.

2. Mathematical symbols, notation and hypotheses. For the reader’s convenience, we collect here the mathematical symbols, the notation, and the main hypotheses used in the paper. With $C(a_1, \ldots, a_k)$ we denote a constant which only depends on the quantities $a_1, \ldots, a_k$: its precise value can vary from occurrence to occurrence. Also, $K$ denotes a universal constant (i.e., a number) and again its precise value can vary from occurrence to occurrence.

General mathematical symbols. Here is a list of the mathematical symbols used in the paper.

• $\mathbb{R}^+$: the interval $[0, +\infty[$.
• $C^0([0, R])$: the space of continuous functions defined on the interval $[0, R]$.
• $H^1([0, R])$: the Sobolev space $W^{1,2}([0, R])$, endowed with the norm
  \[
  \|u\|_{H^1([0, R])} := \sqrt{\|u\|_{L^2([0, R])}^2 + \|\partial_x u\|_{L^2([0, R])}^2}.
  \]

Note that $H^1([0, R])$ compactly embeds into $C^0([0, R])$ and we have the inequality...
\[ \|u\|_{C^0([0,R])} \leq C(R)\|u\|_{H^1([0,R])} \quad \text{for every } u \in H^1([0,R]). \quad (9) \]

- \( H^*(0,R) \): the dual space of \( H^1(0,R) \).
- \( C_c^\infty(\Omega) \): the space of smooth, compactly supported functions defined on the open set \( \Omega \).
- \( M([0,R]) \): the space of (signed) Radon measures on the interval \([0,R]\). We denote by
  \[ \|\mu\|_{M([0,R])} := |\mu|(0,R) \]
  the total variation of the (signed) Radon measure \( \mu \).
- \( M_+(0,R) \): the space of nonnegative Radon measures on the interval \([0,R]\).
- \( \text{a.e. } x, \text{ a.e. } (t,x) \): for \( L^1 \) almost every \( x \), for \( L^2 \) almost every \( (t,x) \). Here \( L^1 \), \( L^2 \) denote the Lebesgue measure on \( \mathbb{R}^1 \) and \( \mathbb{R}^2 \), respectively.

Notation introduced in the present paper. Here is a list of notation.

- \( \alpha_1 \): the Lipschitz constant in (11).
- \( M \): the constant defined by (21).
- \( F \): the constant defined in (24).
- \( \alpha_2 \): the Lipschitz constant in (56).
- \( T \): the length of the time interval where we set our problem.
- \( R \): the length of the space interval where we set our problem.
- \( h \): the function \( h \) in (12) and the constant \( h \) in (54).
- \( h_* \): the constant defined as in (55).

Hypotheses. We list here the main hypotheses.

- \( (H.1) \): the hypothesis introduced at page 869.
- \( (H.2) \): the hypothesis introduced at page 870.
- \( (H.3) \): the hypothesis introduced at page 871.
- \( (H.4) \): the hypothesis introduced at page 871.
- \( (H.5) \): the hypothesis introduced at page 871.
- \( (H.6) \): the hypothesis introduced at page 877.
- \( (H.7) \): the hypothesis introduced at page 879.
- \( (H.8) \): the hypothesis introduced at page 879.

3. A parabolic problem with measure-valued coefficients. In this section we discuss the well-posedness of the following parabolic initial-boundary value problem with a time-dependent, measure-valued coefficient \( \mu \):

\[
\begin{align*}
\partial_t \varphi &= \partial_{xx} \varphi - \varphi \mu + f(\cdot, \cdot, \varphi) \varphi, \\
\partial_x \varphi(t,0) &= \partial_x \varphi(t,R) = 0, \\
\varphi(0,x) &= \varphi_0(x).
\end{align*}
\]

We introduce the following hypotheses on the function \( f \) and on the coefficient \( \mu \).

(\(H.1\)) The function \( f : [0,T] \times [0,R] \times \mathbb{R} \to \mathbb{R} \) is continuous. Also, it is twice continuously differentiable with respect to the variable \( \varphi \), namely the partial derivatives \( \partial_{\varphi} f \) and \( \partial_{\varphi^2} f \) both exist and are continuous. Finally, there are a constant \( \alpha_1 > 0 \) and a continuous, nonnegative function \( h : [0,T] \times [0,R] \to \mathbb{R}^+ \) such that

\[
-\alpha_1 \leq \partial_{\varphi} f(t,x,\varphi) < 0, \quad \text{for all } (t,x,\varphi) \in [0,T] \times [0,R] \times \mathbb{R}, \quad (11)
\]

\[
f(t,x,\varphi) > 0, \quad \text{if and only if } \varphi < h(t,x). \quad (12)
\]
(H.2) The measured valued coefficient $\mu$ satisfies $\mu \in L^\infty([0, T]; \mathcal{M}([0, R]))$, namely it satisfies the following conditions:

i) for a.e. $t \in [0, T]$, we have $\mu_t \in \mathcal{M}^+([0, R])$.

ii) For every Borel set $B \subset [0, R]$, the map $t \mapsto \mu_t(B)$ is $L^1$-measurable.

iii) We have

$$\text{ess sup}_{0 \leq t \leq T} \|\mu_t\|_{\mathcal{M}([0, R])} < +\infty.$$  \hfill (13)

Also, in the following we identify $\mu \in L^\infty([0, T]; \mathcal{M}([0, R]))$ with the measure defined by setting

$$\mu(E) := \int_0^T \int_0^R \mathbb{1}_E(t, x) \, d\mu_t(x) \, dt,$$  \hfill (14)

where $\mathbb{1}_E$ denotes the characteristic function of $E$. Owing to [1, Proposition 2.26], $\mu$ is a Borel measure on $[0, T] \times [0, R]$. Also, by using standard techniques one can establish the following properties.

**Remark 1.** Note that a function $\varphi$ such that

$$\varphi \in L^2([0, T]; H^1([0, R])) \quad \text{and} \quad \partial_t \varphi \in L^2([0, T]; H^1([0, R]))$$  \hfill (15)

is $\mu$-measurable, summable and satisfies

$$\int_{[0, T] \times [0, R]} \varphi(t, x) \, d\mu(t, x) = \int_0^T \int_0^R \varphi(t, x) \, d\mu_t(x) \, dt$$  \hfill (16)

and

$$\left| \int_{[0, T] \times [0, R]} \varphi(t, x) \, d\mu(t, x) \right| \leq C(R) \text{ess sup}_{0 \leq t \leq T} \|\mu_t\|_{\mathcal{M}([0, R])} \|\varphi\|_{L^2([0, T]; H^1([0, R]))}. $$  \hfill (17)

We can now provide the definition of weak solution of (10).

**Definition 3.1.** Assume (H.1) and (H.2). A function $\varphi : [0, T] \times [0, R] \to \mathbb{R}$ is a weak solution of (10) if $\varphi$ satisfies (15) and, for every test function $v \in C_c^\infty(-\infty, T]\times[0, R\} = 0$

$$\int_0^T \int_0^R (-\partial_t \varphi - \partial_x v \partial_x \varphi) \, dx \, dt - \int_0^T \int_0^R v(t, x) \varphi(t, x) \, d\mu_t(x) \, dt$$

$$+ \int_0^T \int_0^R v(t, x) f(t, x, \varphi(t, x)) \, dx \, dt + \int_0^R v(0, x) \varphi_0(x) \, dx = 0.$$  \hfill (18)

Note that the second term in the above expression is well defined owing to Remark 1. We now state a well-posedness result concerning (10).

**Theorem 3.2.** Assume (H.1) and (H.2) and suppose that the initial datum $\varphi_0$ satisfies

$$\varphi_0 \in L^\infty([0, R]), \quad \varphi_0(x) \geq 0 \quad \text{for a.e. } x \in [0, R].$$  \hfill (19)

Then the initial-boundary value problem (10) admits a unique weak solution, in the sense of Definition 3.1. Also, the solution enjoys the following properties: first,

$$0 \leq \varphi(t, x) \leq M, \quad \text{for a.e. } (t, x) \in [0, T]\times[0, R],$$  \hfill (20)

where

$$M := \max\{\|h\|_{L^\infty}, \|\varphi_0\|_{L^\infty}\}.$$  \hfill (21)
Second, we have stability with respect to the initial datum and with respect to the coefficient $\mu$. Namely, if we term $\hat{\varphi}$ the solution of the initial-boundary value problem

$$\begin{cases}
\partial_t \hat{\varphi} = \partial_{xx}^2 \hat{\varphi} - \hat{\varphi} \hat{\mu} + f(\cdot, \cdot, \hat{\varphi}) \hat{\varphi}, \\
\partial_x \hat{\varphi}(t, 0) = \partial_x \hat{\varphi}(t, R) = 0, \\
\hat{\varphi}(0, x) = \hat{\varphi}_0(x).
\end{cases}$$

(22)

then we have

$$\|\varphi(t, \cdot) - \hat{\varphi}(t, \cdot)\|_{L^2([0, R])}^2 + \int_0^t \|\partial_x \varphi(s, \cdot) - \partial_x \hat{\varphi}(s, \cdot)\|_{L^2([0, R])}^2 ds$$

$$\leq C(\alpha_1, M, T, R, F) \left[ \text{ess sup}_{t \in [0, T]} \|\mu_t - \hat{\mu}_t\|_{L^2([0, R])}^2 + \|\varphi_0 - \hat{\varphi}_0\|_{L^2([0, R])}^2 \right]$$

(23)

for every $0 \leq t \leq T$, where

$$F := \max \{ f(t, x, 0) : (t, x) \in [0, T] \times [0, R] \}.$$  

(24)

The proof of the above theorem can be obtained by minor modifications of the theory developed in [3]. A complete proof is contained in the online version of the present paper, available as arXiv:1602.05737.

4. Optimal control problem. Consider the payoff functional

$$J : L^\infty([0, T]; \mathcal{M}_+([0, R])) \to \mathbb{R},$$

$$J(\mu) := \int_0^T \int_0^R \varphi(t, x) d\mu_t(x) dt - \Psi \left( \int_0^T \int_0^R c(t, x) d\mu_t(x) dt \right).$$

(25)

In the above expression, $\varphi$ is the weak solution of the initial-boundary value problem (10), according to Definition 3.1. In this section we discuss both the existence of an optimal strategy $\mu$ and some first order necessary conditions. Note that $\varphi$ depends on $\mu$, and hence the functional $J$ is nonlinear. We assume that the functions $c$ and $\Psi$ satisfy the following hypotheses.

(H.3) The function $c : [0, T] \times [0, R] \to \mathbb{R}^+ \cup \{+\infty\}$ is lower semicontinuous.

Note that hypothesis (H.3) implies that the function $c$ is $\mu$-integrable (in the sense of [1, p.8]) and hence that the second term in (25) is well-defined.

(H.4) The function $\Psi : \mathbb{R} \cup \{+\infty\} \to \mathbb{R}$ is twice continuously differentiable, non-decreasing and convex.

In the following we aim at discussing the existence and uniqueness of an optimal $\mu$ for $J$ under the constraint

$$\int_0^R b(t, x) d\mu_t(x) \leq 1 \quad \text{for a.e. } t \in [0, T].$$

(26)

The function $b$ satisfies the following hypothesis:

(H.5) The function $b : [0, T] \times [0, R] \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and bounded away from 0, namely

$$b(t, x) \geq b_0 > 0, \quad \text{for every } (t, x) \in [0, T] \times [0, R]$$

(27)

for some suitable constant $b_0 > 0$. 

Note that $b$ is $\mu_t$-integrable because it is lower semicontinuous and positive, and hence the integral at the left hand side of (26) is well-defined. Note furthermore that by combining (27) with (26) we get

$$\text{ess sup}_{t \in [0, T]} \|\mu_t\|_{\mathcal{M}([0, R])} \leq \frac{1}{b_0}.$$ (28)

We now focus on the problem

$$\text{maximize } J(\mu), \text{ defined as in (25), among } \mu \in L^\infty([0, T]; \mathcal{M}_+(\mathbb{R})) \text{ satisfying (26).}$$ (29)

The following result establishes the existence of an optimal strategy $\mu^*$ for problem (29).

**Proposition 1.** Assume (H.1)-(H.5). Then the optimization problem (29) admits an optimal solution $\mu^*$.

The proof of Proposition 1 is based on the same argument as the proof of Theorem 1 in [5] (see also [9, Theorem 4.1]) and so we omit it.

The next result provides an expression for the Gateaux derivative of the solution to (10) with respect to the measure $\mu$, and gives a first order necessary condition for $\mu^*$ to be an optimal strategy.

**Proposition 2.** Assume (H.1)-(H.5). Let $\mu^* \in L^\infty([0, T]; \mathcal{M}_+(\mathbb{R}))$ be an optimal strategy for the problem (29) and $\varphi_*$ be the corresponding solution of (10). For every $\nu \in L^\infty([0, T]; \mathcal{M}_+(\mathbb{R}))$, such that $\mu^* + \varepsilon \nu$ satisfies (26) for $\varepsilon > 0$ sufficiently small, the directional derivative of $\varphi_*$ at $\mu^*$, in the direction $\nu - \mu^*$, is given by the function $\varphi_1 \in L^2([0, T]; H^1([0, R]))$ that is a weak solution, in the sense of Definition 3.1, of the initial-boundary value problem

\begin{align*}
\partial_t \varphi_1 &= \partial_{xx} \varphi_1 - \varphi_1 \mu^* - \nu \varphi_* + \partial_x g(\cdot, \cdot, \varphi_*) \varphi_1, \\
\partial_x \varphi_1(t, 0) &= \partial_x \varphi_1(t, R) = 0, \\
\varphi_1(0, x) &= 0.
\end{align*} (30)

Moreover, the following inequality holds:

$$
\int_0^T \int_0^R \varphi_1(t, x)d\mu_t^*(x)dt + \int_0^T \int_0^R \varphi_*(t, x)d\nu_t(x)dt - \Psi' \left( \int_0^T \int_0^R c(t, x)d\mu_t^*(x)dt \right) \int_0^T \int_0^R c(t, x)d\nu_t(x)dt \leq 0.
$$ (31)

The function $g : [0, T] \times [0, R] \times \mathbb{R} \to \mathbb{R}$ at the first line of (30), is defined by setting

$$g(t, x, \varphi) := f(t, x, \varphi).$$ (32)

We explicitly point out that the above necessary conditions are not interesting in the case when the constraint (26) is satisfied as an equality by the optimal strategy $\mu^*$. Indeed, in this case the admissible variation $\nu$ must satisfy

$$
\int_0^R b(t, x)d\nu_t(x) = 0 \quad \text{for a.e. } t \in [0, T],
$$

which, owing to (27), implies that $\nu \equiv 0.$
Proof of Proposition 2. Fix $\nu$ as in the statement of the theorem and, for every real number $\varepsilon > 0$, consider the quantity

$$J(\mu^* + \varepsilon \nu) = \int_0^T \int_0^R \varphi_\varepsilon(t, x) d(\mu^*_\varepsilon + \varepsilon \nu)(x) dt - \Psi \left( \int_0^T \int_0^R c(t, x) d(\mu^*_\varepsilon + \varepsilon \nu)(x) dt \right).$$

In the previous expression, we term $\varphi_\varepsilon$ the solution of the initial-boundary problem (10) in the case $\mu = \mu^* + \varepsilon \nu$. The heuristic idea to establish (30) is to differentiate both $J(\mu^* + \varepsilon \nu)$ and $\varphi_\varepsilon$ with respect to the variable $\varepsilon$. The rigorous proof is organized into the following steps.

**Step 1.** We construct an approximate derivative of $\varphi_*$. For every $\varepsilon > 0$, define

$$\psi_\varepsilon = \frac{\varphi_\varepsilon - \varphi_*}{\varepsilon}. \tag{33}$$

Note that $\psi_\varepsilon$ is the weak solution, in the sense of Definition 3.1, of the initial-boundary value problem

$$\begin{cases}
    \partial_t \psi_\varepsilon = \partial^2_{xx} \psi_\varepsilon - \psi_* \mu^* - \varphi_\varepsilon \nu + G_\varepsilon, \\
    \partial_x \psi_\varepsilon(t, 0) = \partial_x \psi_\varepsilon(t, R) = 0, \\
    \psi_\varepsilon(0, x) = 0.
\end{cases} \tag{34}$$

provided that

$$G_\varepsilon(t, x) = \frac{f(t, x, \varphi_\varepsilon) \varphi_\varepsilon - f(t, x, \varphi_*) \varphi_*}{\varepsilon}. \tag{35}$$

In other words, for every test function $v \in C^\infty([-\infty, T] \times \mathbb{R})$ we have

$$\int_0^T \int_0^R \left( \partial_t v \psi_\varepsilon - \partial_{xx} v \partial_x \psi_\varepsilon \right) dx dt - \int_0^T \int_0^R v \psi_\varepsilon \mu^*_\varepsilon(x) dt$$

$$- \int_0^T \int_0^R v \varphi_\varepsilon d\nu_\varepsilon(x) dt + \int_0^T \int_0^R v G_\varepsilon dx dt = 0. \tag{36}$$

Note that

$$|G_\varepsilon(t, x)| \leq \frac{1}{\varepsilon} |f(t, x, \varphi_\varepsilon) \varphi_\varepsilon - f(t, x, \varphi_*) \varphi_*| + \frac{1}{\varepsilon} |f(t, x, \varphi_\varepsilon) \varphi_* - f(t, x, \varphi_*) \varphi_*|$$

$$\leq |f(t, x, \varphi_\varepsilon)| \frac{|\varphi_\varepsilon - \varphi_*|}{\varepsilon} + |\varphi_*| |\alpha_1| \frac{|\varphi_\varepsilon - \varphi_*|}{\varepsilon} \tag{35}$$

$$\leq \left[ \alpha_1 |\varphi_\varepsilon| + F \right] |\psi_\varepsilon| + \alpha_1 |\varphi_*| |\psi_\varepsilon| \tag{11),(24),(33}$$

$$\leq \left[ \alpha_1 M + F \right] |\psi_\varepsilon| + \alpha_1 M |\psi_\varepsilon| = C(\alpha_1, M, F)|\psi_\varepsilon|. \tag{20}$$

We now provide a formal argument, which can be made rigorous by regularizing arguments. We multiply the equation at the first line of (34) times $\psi_\varepsilon$ and integrate by parts to get

$$\frac{d}{dt} \int_0^R \frac{\psi_\varepsilon^2}{2} dx + \int_0^R (\partial_x \psi_\varepsilon)^2 dx + \int_0^R \psi_\varepsilon^2 d\mu^*_\varepsilon(x) = - \int_0^R \psi_\varepsilon \varphi_\varepsilon d\nu_\varepsilon(x) + \int_0^R G_\varepsilon \psi_\varepsilon dx$$
Step 2. We establish compactness of the family \{\psi_\varepsilon(t, \cdot)\} and eventually we get that

\[ \|\varphi_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|\varphi_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} + C(\alpha_1, M, F) \int_0^R \psi_\varepsilon^2 dx \]

\[ \leq M \|\varphi_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} + C(\alpha_1, M, F) \int_0^R \psi_\varepsilon^2 dx \]  

\[ \leq C(M, R) \|\varphi_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} + C(\alpha_1, M, F) \int_0^R \psi_\varepsilon^2 dx. \]  

Next, we fix a constant \( k > 0 \) (to be determined in the following), we use the Young Inequality and we infer that

\[ \|\psi_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})} \leq \frac{k}{2} \|\psi_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{2k} \|\varphi_\varepsilon\|_{L^2(\mathbb{R})}^2 + \frac{k}{2} \|\partial_x \psi_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{2k} \|\varphi_\varepsilon\|_{L^2(\mathbb{R})}^2. \]  

Next, we choose the constant \( k \) in such a way that \( C(M, R)k = 1 \) and by combining (38) and (39) we arrive at

\[ \frac{d}{dt} \int_0^R \frac{\psi_\varepsilon^2}{2} dx + \frac{1}{2} \int_0^R (\partial_x \psi_\varepsilon)^2 dx + \int_0^R \psi_\varepsilon^2 d\mu_\varepsilon(x) \leq C(M, R) \|\varphi_\varepsilon\|_{L^2(\mathbb{R})}^2 + C(\alpha_1, M, F, R) \int_0^R \psi_\varepsilon^2 dx. \]  

Owing to the Gronwall Lemma and to the fact that \( \psi_\varepsilon(t = 0) = 0 \), the above inequality implies that

\[ \int_0^R \psi_\varepsilon^2(t, x) dx \leq C(\alpha_1, M, F, T, R) \|\varphi_\varepsilon\|_{L^2(\mathbb{R})}^2 \]  

for every \( t \in [0, T] \).

Step 2. We establish compactness of the family \{\psi_\varepsilon\}. We rely on the Aubin-Lions Lemma [14]. We recall that

\[ H^1([0, R]) \hookrightarrow C^0([0, R]) \hookrightarrow H^*([0, R]), \]

and the first inclusion is compact and the second is continuous. Owing to the Aubin-Lions Lemma [14], to establish the compactness of the family \{\psi_\varepsilon\} in the space \( L^2([0, T]; C^0([0, R])) \) it suffices to show that

\[ \{\psi_\varepsilon\} \text{ is uniformly bounded in } L^2([0, T]; H^1([0, R])), \]

\[ \{\partial_t \psi_\varepsilon\} \text{ is uniformly bounded in } L^2([0, T]; H^*([0, R])). \]

To establish (43) we use (42). To establish (44), we use the equation at the first line of (34), and eventually we get that

\[ \frac{d}{dt} \int_0^R \frac{\psi_\varepsilon^2}{2} dx + \frac{1}{2} \int_0^R (\partial_x \psi_\varepsilon)^2 dx + \int_0^R \psi_\varepsilon^2 d\mu_\varepsilon(x) \leq C(M, R) \|\varphi_\varepsilon\|_{L^2(\mathbb{R})}^2 + C(\alpha_1, M, F, R) \int_0^R \psi_\varepsilon^2 dx. \]
\[
\|\partial_t \psi_\varepsilon\|_{L^2([0,T];H^1([0,R]))} \leq \|\psi_\varepsilon\|_{L^2([0,T];H^1([0,R]))} \left[ K + C(R) \text{ess sup}_{t \in [0,T]} \|\mu_\varepsilon\|_{L^1([0,R])} \
+ C(\alpha_1, M, F) \right] + C(M, R, T) \text{ess sup}_{t \in [0,T]} \|\nu_\varepsilon\|_{L^1([0,R])}.
\]

Owing to the Aubin-Lions Lemma [14], we infer that the family \(\{\psi_\varepsilon\}\) is compact in \(L^2([0,T];C^0([0,R]))\) and hence that there is a sequence \(\psi_{\varepsilon_k}\) and a function \(\varphi_1 \in L^2([0,T];C^0([0,R]))\) such that
\[
\psi_{\varepsilon_k} \to \varphi_1 \quad \text{in } L^2([0,T];C^0([0,R])),
\]
\[
\psi_{\varepsilon_k}(t,\cdot) \to \varphi_1(t,\cdot) \quad \text{uniformly in } C^0([0,R]), \quad \text{for a.e. } t \in [0,T].
\]

By using the bound (40) we infer that the family \(\{\partial_x \psi_\varepsilon\}\) is weakly compact in \(L^2([0,T]\times[0,R])\) and hence that
\[
\partial_x \psi_{\varepsilon_k} \rightharpoonup \partial_x \varphi_1 \quad \text{weakly in } L^2([0,T]\times[0,R]).
\]

**Step 3.** We show that the accumulation point \(\varphi_1\) is a weak solution of (30). We argue by passing to the limit in each of the terms in (36). First, we point out that by using (45) and (47) we get
\[
\int_0^T \int_0^R (\partial_t v \psi_{\varepsilon_k} - \partial_x v \partial_x \psi_{\varepsilon_k}) dt dx \to \int_0^T \int_0^R (\partial_t v \varphi_1 - \partial_x v \partial_x \varphi_1) dt dx.
\]

Next, we point out that
\[
\left| \int_0^T \int_0^R v \psi_{\varepsilon_k} d\mu_\varepsilon(x) dt - \int_0^T \int_0^R v \varphi_1 d\mu_\varepsilon(x) dt \right|
\leq \int_0^T \|v\|_{C^0([0,R]\times[0,T])} \|\psi_{\varepsilon_k}(t,\cdot) - \varphi_1(t,\cdot)\|_{C^0([0,R])} \|\mu_\varepsilon\|_{L^1([0,R])} dt
\leq \text{ess sup}_{0 \leq t \leq T} \|\mu_\varepsilon\|_{L^1([0,R])} \|v\|_{C^0([0,R]\times[0,T])} C(T) \|\psi_{\varepsilon_k} - \varphi_1\|_{L^2([0,T]\times C^0([0,R]))}
\xrightarrow{(45)} 0.
\]

We now want to show that
\[
\left| \int_0^T \int_0^R v \varphi_{\varepsilon_k} d\nu_\varepsilon(x) dt - \int_0^T \int_0^R v \varphi_\ast d\nu_\varepsilon(x) dt \right| \to 0.
\]

To this end, we recall that \(\varphi_\varepsilon\) is the weak solution of the initial-boundary problem (10) in the case when \(\mu = \mu^* + \varepsilon \nu\) and we apply the stability estimate (23) with \(\varphi = \varphi_\ast, \hat{\varphi} = \varphi_\varepsilon, \mu = \mu^*, \hat{\mu} = \mu^* + \varepsilon \nu\). By recalling the continuous embedding (9), we infer that
\[
\varphi_{\varepsilon_k} \to \varphi_\ast \quad \text{in } L^2([0,T];C^0([0,R]))
\]
and by arguing as in (49) we arrive at (50). We are left with the last term in (36). To handle it, we recall that \(0 \leq \varphi_\ast, \varphi_\varepsilon \leq M\) owing to (20), we use the Taylor formula with Lagrange remainder and we infer that
\[
g(t,x,\varphi_\varepsilon) = g(t,x,\varphi_\ast) + \partial_\varphi g(t,x,\varphi_\ast)[\varphi_\varepsilon - \varphi_\ast] + \frac{1}{2} \partial^2_{\varphi^2} g(t,x,y)[\varphi_\varepsilon - \varphi_\ast]^2, \quad y \in [0,M].
\]

Owing to (32), (33) and (35), this implies that
\[
|G_\varepsilon(t,x) - \partial_\varphi g(t,x,\varphi_\ast)\psi_\varepsilon| \leq K \max_{(t,x,y) \in [0,T]\times[0,M]\times[0,M]} |\partial^2_{\varphi^2} g(t,x,y)||\psi_\varepsilon||\varphi_\varepsilon - \varphi_\ast|
\]
(52)
and hence that
\[
\int_0^T \int_0^R |G_{\varepsilon_k}(t,x) - \partial_x g(t,x,\varphi_*)\varphi_1| dt dx \leq \int_0^T \int_0^R |G_{\varepsilon_k}(t,x) - \partial_x g(t,x,\varphi_*)\psi_{\varepsilon_k}| dt dx
\]
\[
+ \int_0^T \int_0^R |\partial_x g(t,x,\varphi_*)\psi_{\varepsilon_k} - \partial_x g(t,x,\varphi_*)\varphi_1| dt dx
\]
\[
(52) \quad \leq K \max_{(t,x,y)\in[0,T] \times [0,R] \times [0,M]} |\partial_x^2 g(t,x,y)| \int_0^T \int_0^R |\varphi_1| |\varphi_{\varepsilon_k} - \varphi_*| dt dx
\]
\[
+ \int_0^T \int_0^R |\partial_x g(t,x,y)| \int_0^T \int_0^R |\psi_{\varepsilon_k} - \varphi_1| dt dx
\]
\[
(53) \quad \Rightarrow \quad \max_{(t,x,y)\in[0,T] \times [0,R] \times [0,M]} |\partial_x^2 g(t,x,y)||\psi_{\varepsilon_k}||L^2| ||\varphi_{\varepsilon_k} - \varphi_*||L^2
\]

This implies that
\[
\int_0^T \int_0^R v G_{\varepsilon_k} dx dt - \int_0^T \int_0^R v \partial_x g(t,x,\varphi_*)\varphi_1 dx dt \rightarrow 0
\]
and show that \( \varphi_1 \) is a weak solution of (30).

**Step 4.** We establish (31). We recall that \( \varepsilon_k > 0 \), that \( \mu^* \) is an optimal strategy and \( \mu^* + \varepsilon_k \nu \) is a competitor provided that \( \varepsilon_k \geq 0 \) is sufficiently small. We conclude that
\[
J(\mu^* + \varepsilon_k \nu) - J(\mu^*) \leq 0.
\]

By using the explicit expression of \( J(\mu^* + \varepsilon_k \nu) \) we infer
\[
\frac{J(\mu^* + \varepsilon_k \nu) - J(\mu^*)}{\varepsilon_k} = \int_0^T \int_0^R \psi_{\varepsilon_k} d\mu'_k(x) dt + \int_0^T \int_0^R \varphi_{\varepsilon_k} d\nu_k(x) dt
\]
\[
- \Psi \left( \int_0^T \int_0^R c(t,x) d(\mu'_k + \varepsilon_k \nu_k)(x) dt \right) - \Psi \left( \int_0^T \int_0^R c(t,x) d\mu'_k(x) dt \right)
\]

By combining the above two expressions we get
\[
0 \geq \int_0^T \int_0^R \psi_{\varepsilon_k} d\mu'_k(x) dt + \int_0^T \int_0^R \varphi_{\varepsilon_k} d\nu_k(x) dt
\]
\[
- \Psi \left( \int_0^T \int_0^R c(t,x) d(\mu'_k + \varepsilon_k \nu_k)(x) dt \right) - \Psi \left( \int_0^T \int_0^R c(t,x) d\mu'_k(x) dt \right)
\]
\[
(49),(50) \quad \Rightarrow \quad \int_0^T \int_0^R \varphi_1 d\mu'_k(x) dt + \int_0^T \int_0^R \varphi_* d\nu_k(x) dt
\]
\[
- \Psi' \left( \int_0^T \int_0^R c(t,x) d\mu'_k(x) dt \right) \int_0^T \int_0^R c(t,x) d\nu_k(x) dt
\]
that is (31). This concludes the proof of Proposition 2. \( \square \)
5. **Uniqueness of optimal solutions.** In this section we discuss the uniqueness of optimal solutions of the optimization problem (29). We refine hypothesis (H.1) by introducing the following condition.

(H.6) The function $h$ in (12) is a constant. In other words,

$$f(t, x, \varphi) > 0 \quad \text{if and only if} \quad \varphi < h.$$  \hspace{1cm} (54)

We recall that the function $\partial_\varphi f$ is continuous and by recalling (11) we define the constant

$$-h_* := \max_{(t,x) \in [0,T] \times [0,R]} \partial_\varphi f(t, x, h) < 0. \hspace{1cm} (55)$$

Also, in the following we term $\alpha_2$ a Lipschitz constant for $\partial_\varphi f$, namely

$$|\partial_\varphi f(t, x, \varphi_1) - \partial_\varphi f(t, x, \varphi_2)| \leq \alpha_2 |\varphi_1 - \varphi_2| \hspace{1cm} (56)$$

for every $(t, x) \in [0, T] \times [0, R]$ and $\varphi_1, \varphi_2 \in [0, M]$. We explicitly point out that the constants $h$ and $h_*$ are data of the problem and are therefore fixed.

The main result of the present section states that, if the initial datum $\varphi_0$ is sufficiently close to the constant $h$, then the solution of the optimization problem (29) is unique within a class of measures with sufficiently small total variation.

**Theorem 5.1.** Assume (H.1)-(H.6). There is a constant $0 < \delta < 1$, which only depends on the constants $\alpha_1, \alpha_2, M, h, T, R$, and $h_*$, such that, if

$$\operatorname{ess sup}_{t \in [0,T]} \|\mu_t\|_{\mathcal{M}([0,R])} + \|\varphi_0 - h\|_{H^1([0,R])} \leq \delta, \hspace{1cm} (57)$$

then the solution $\mu$ of the constrained optimization problem (29) is unique. More precisely, assume that $\tilde{\mu}$ and $\bar{\mu}$ are two points of maximum of $J$ such that

$$\max \left\{ \operatorname{ess sup}_{t \in [0,T]} \|\tilde{\mu}_t\|_{\mathcal{M}([0,R])}, \operatorname{ess sup}_{t \in [0,T]} \|\bar{\mu}_t\|_{\mathcal{M}([0,R])} \right\} \leq \delta. \hspace{1cm} (58)$$

Then $\tilde{\mu} = \bar{\mu}$.

Note that, owing to (28), to achieve the bound

$$\operatorname{ess sup}_{t \in [0,T]} \|\tilde{\mu}_t\|_{\mathcal{M}([0,R])} \leq \delta$$

it suffices to have $1/b_0 \leq \delta$, where $b_0$ is the bound from below on the function $b$, see (27), and is therefore a datum of the problem.

**Proof of Theorem 5.1.** We fix two points of maximum $\tilde{\mu}$ and $\bar{\mu}$ satisfying (58) and we argue by contradiction assuming that $\tilde{\mu} \neq \bar{\mu}$. We set

$$\varepsilon := \operatorname{ess sup}_{t \in [0,T]} \|\tilde{\mu}_t\|_{\mathcal{M}([0,R])} \leq 2\delta. \hspace{1cm} (59)$$

We define the (signed) measure $\nu \in L^\infty([0,T]; \mathcal{M}([0, R]))$, by setting

$$\nu := \frac{\tilde{\mu} - \bar{\mu}}{\varepsilon}, \hspace{1cm} (60)$$

we define the map $j : [0, \varepsilon] \to \mathbb{R}$, by setting

$$j(\zeta) := J(\tilde{\mu} + \zeta \nu), \hspace{1cm} (61)$$

and we point out that by construction $j$ attains its maximum at both $\zeta = 0$ and $\zeta = \varepsilon$. 

By using Lemma 5.2, we conclude that the map $j$ is continuous and concave on $[0,\varepsilon]$. This contradicts the fact that $j$ attains its maximum at $\zeta = \varepsilon$ and hence concludes the proof.

Lemma 5.2. Let $j$ be the same function as in (61). Then $j$ is continuous on $[0,\varepsilon]$ and twice differentiable in $[0,\varepsilon]$. Also, if the constant $\delta$ in (58) is sufficiently small, then

$$j''(\zeta) < 0 \quad \text{for every } \zeta \in [0,\varepsilon].$$

Proof. First, we point out that the map $j$ is continuous: this can be seen by using the stability estimate (23). Also, by arguing as in the proof of Proposition 2 we infer as in the proof of Lemma 5.2 we infer that $j$ is twice differentiable and that, for every $\zeta \in [0,\varepsilon]$, we have

$$j''(\zeta) = \int_0^T \int_0^R \varphi_2(t,x)d\mu^*_1(x)dt + 2 \int_0^T \varphi_1(t,x)d\nu_1(x)dt$$

$$- \Psi^\prime \left( \int_0^T \int_0^R c(t,x)d\mu^*_1(x)dt \right) \left( \int_0^T \int_0^R c(t,x)d\nu_1(x)dt \right)^2$$

provided that the measure $\mu^*$ is given by

$$\mu^* := \tilde{\mu} + \zeta \nu$$

and the functions $\varphi_1$ and $\varphi_2$ are defined as follows. The functions $\varphi_1$, $\varphi_2$, and $\varphi_*$ are the weak solutions of

$$\begin{cases}
\partial_t \varphi_1 = \partial^2_{xx} \varphi_1 - \varphi_1 \mu^* - \varphi_* \nu + \partial_x g(\cdot, \cdot, \varphi_*) \varphi_1, \\
\partial_x \varphi_1(t,0) = \partial_x \varphi_1(t,R) = 0, \\
\varphi_1(0,x) = 0,
\end{cases}$$

$$\begin{cases}
\partial_t \varphi_2 = \partial^2_{xx} \varphi_2 - \varphi_2 \mu^* - 2\nu \varphi_1 + \partial_x g(\cdot, \cdot, \varphi_*) \varphi_2 + \partial^2_{\varphi} g(\cdot, \cdot, \varphi_*) \varphi_2^2, \\
\partial_x \varphi_2(t,0) = \partial_x \varphi_2(t,R) = 0, \\
\varphi_2(0,x) = 0,
\end{cases}$$

and

$$\begin{cases}
\partial_t \varphi_* = \partial^2_{xx} \varphi_* - \varphi_* \mu^* + g(\cdot, \cdot, \varphi_*), \\
\partial_x \varphi_*(t,0) = \partial_x \varphi_*(t,R) = 0, \\
\varphi_*(0,x) = \varphi_0(x),
\end{cases}$$

respectively. By using the convexity of the function $\Psi$, we infer from (63) that

$$j''(\zeta) \leq \int_0^T \int_0^R \varphi_2(t,x)d\mu_1^*(x)dt + 2 \int_0^T \varphi_1(t,x)d\nu_1(x)dt$$

and hence the proof of Lemma 5.2 is an easy consequence of Lemma 5.3 below.

Lemma 5.3. Assume (H.1)-(H.6). Let $\mu^*$, $\varphi_1$ and $\varphi_2$ be the same as in (64), (65) and (66), respectively. There is a sufficiently small constant $\delta$ such that, if (57) holds, then

$$2 \int_0^T \int_0^R \varphi_1(t,x)d\nu_1(x)dt + \int_0^T \int_0^R \varphi_2(t,x)d\mu_1^*(x)dt < 0.$$

The proof of Lemma 5.3 is rather long and technical and it is established in § 7.
6. Solutions of the differential game and Nash equilibria. This section aims at discussing the differential game modeling the case when there are several competing fish companies and at establishing the existence of Nash equilibria. More precisely, we define our differential game as follows: we assume that there are \( m \geq 1 \) players (i.e., fish companies) and we denote by \( \mu_i \) the fishing intensity of the \( i \)-th company. We term \( \phi \) the fish population density and we consider the initial-boundary value problem

\[
\begin{aligned}
&\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} - \phi \sum_{i=1}^{m} \mu_i + f(\cdot, \cdot, \phi), \\
&\frac{\partial \phi}{\partial x}(t, 0) = \frac{\partial \phi}{\partial x}(t, R) = 0, \\
&\phi(0, x) = \phi_0(x).
\end{aligned}
\]  

(69)

The goal of the \( i \)-th player (i.e., fish company) is to maximize his payoff \( J_i \), which is defined by setting

\[
J_i(\mu) := \int_0^T \int_0^R \phi(t, x) d\mu_{i,t}(x) dt - \Psi_i\left(\int_0^T \int_0^R c_i(t, x) d\mu_{i,t}(x) dt\right).
\]  

(70)

The admissible controls satisfy \( \mu_i \in L^\infty(0, T; \mathcal{M}_+(0, R)) \) and the constraint

\[
\int_0^R b_i(t, x) d\mu_{i,t}(x) \leq 1, \quad \text{for a.e. } t \in [0, T].
\]  

(71)

The functions \( \Psi_i, c_i \) and \( b_i \) in (70) and (71) satisfy the following assumptions.

(H.7) The functions \( \Psi_1, \ldots, \Psi_m : \mathbb{R} \to \mathbb{R} \) are twice continuously differentiable, nondecreasing, and convex.

(H.8) The functions \( c_1, \ldots, c_m : [0, T] \times [0, R] \to \mathbb{R}^+ \cup \{+\infty\} \) are all lower semi-continuous. The functions \( b_1, \ldots, b_m : [0, T] \times [0, R] \to \mathbb{R} \cup \{+\infty\} \) are lower semi-continuous and satisfy

\[
b_i(t, x) \geq b_0 > 0, \quad \text{for all } (t, x) \in [0, T] \times [0, R] \text{ and } i = 1, \ldots, m,
\]  

(72)

for some positive constant \( b_0 > 0 \).

We now provide the definition of Nash equilibrium.

**Definition 6.1.** A Nash equilibrium solution for the differential game (69) is an \( m \)-tuple \( (\mu_1, \ldots, \mu_m) \) such that, for every \( i \in \{1, \ldots, m\} \), \( \mu_i \in L^\infty([0, T]; \mathcal{M}_+(0, R)) \) is a solution of the problem:

\[
\text{maximize } J_i(\mu_i)
\]  

defined as in (70) among \( \mu_i \in L^\infty([0, T]; \mathcal{M}_+(0, R)) \) satisfying (71).

The main result of the present section establishes the existence of Nash equilibria.

**Theorem 6.2.** Assume (H.1)-(H.2) and (H.6)-(H.8). There is a constant \( \delta > 0 \), which only depends on the constants \( \alpha_1, \alpha_2, M, h, T, R \) and \( h_\ast \) such that, if

\[
||\phi_0 - h||_{H^1(0, R)} \leq \delta,
\]  

(73)

then the differential game (69) has a Nash equilibrium \( (\mu_1, \ldots, \mu_m) \) such that

\[
\text{ess sup}_{t \in [0, T]} \|\mu_{i,t}\|_{\mathcal{M}(0, R)} \leq \delta \quad \text{for every } i = 1, \ldots, m.
\]  

(74)
Proof. We follow the same argument as in [5, §6] and, to simplify the exposition, we only discuss the case when \( m = 2 \) and we assume \( c_1 = c_2, \Psi_1 = \Psi_2 \). The proof straightforwardly extends to the general case. We proceed according to the following steps.

**Step 1.** we introduce some notation and make some preliminary considerations. First, we fix \( \eta \in L^\infty([0,T]; \mathcal{M}_+(\Omega)) \) and we consider the function

\[
J_\eta : L^\infty([0,T]; \mathcal{M}_+(\Omega)) \to \mathbb{R},
\]

\[
J_\eta(\mu) := \int_0^T \int_0^R \varphi(t,x) d\mu(x) dt - \Psi \left( \int_0^T \int_0^R c(t,x) d\mu(x) dt \right),
\]

where \( \varphi \) is the weak solution of the initial-boundary value problem

\[
\begin{aligned}
\partial_t \varphi &= \partial_x^2 \varphi - [\eta + \mu] \varphi + g(\cdot, \cdot, \varphi), \\
\partial_x \varphi(t,0) &= \partial_x \varphi(t,R) = 0, \\
\varphi(0,x) &= \varphi_0(x).
\end{aligned}
\]

Next, we fix a small constant \( \delta > 0 \). The precise value of \( \delta \) will be determined in the following. We define the set \( \mathcal{C}_\delta \) by setting

\[
\mathcal{C}_\delta := \left\{ \mu \in L^\infty([0,T]; \mathcal{M}_+(\Omega)) : \text{ess sup}_{t\in[0,T]} ||\mu||_{\mathcal{M}(\Omega)} \leq \delta \right\}.
\]

By using the same argument as in the proof of Theorem 1 (existence) and Theorem 5.1 (uniqueness) we arrive at the following result.

**Lemma 6.3.** Under the same assumptions as in the statement of Theorem 6.2, there is a sufficiently small constant \( \delta \) such that, if (73) holds, then for every \( \eta \in C_\delta \) there is a unique \( \mu^{\text{opt}}(\eta) \in C_\delta \) such that

\[
J_\eta(\mu^{\text{opt}}(\eta)) \geq J_\eta(\mu) \quad \text{for every } \mu \in C_\delta.
\]

By relying on Lemma 6.3 we can define the map \( T \) by setting

\[
T : C_\delta \times C_\delta \to C_\delta \times C_\delta
\]

\[
T(\mu_1, \mu_2) := (\mu^{\text{opt}}(\mu_2), \mu^{\text{opt}}(\mu_1)).
\]

We now show that \( C_\delta \) is compact with respect to the weak-* convergence. First, we fix a sequence \( \{\mu_n\} \) in \( C_\delta \) and we recall that the Borel measure \( \mu_n \) on \([0,T] \times [0,R] \) is defined by setting

\[
\mu_n(E) := \int_0^T \int_0^R \mathbf{1}_E(t,x) d\mu_n(t,x) dt.
\]

Note that, if \( \mu_n \in C_\delta \), then the total variation \( |\mu_n| \leq \delta T \). Hence, there is a Borel measure \( \mu \) such that, up to subsequences, \( \mu_n \rightharpoonup^* \mu \) in \( \mathcal{M}([0,T] \times [0,R]) \), namely

\[
\int_0^T \int_0^R v(t,x) d\mu(t,x) dt \to \int_0^T \int_0^R v(t,x) d\mu(t,x)
\]

for every \( v \in C^0([0,T] \times [0,R]) \). We now have to show that the limit measure \( \mu \in C_\delta \), namely it admits a representation like (80). To this end, we term \( \pi \) the projection

\[
\pi : \mathbb{R}^2 \to \mathbb{R} \quad (t,x) \mapsto t
\]

and we point out that

\[
\pi_1 \mu_n = f_n \mathcal{L}^1_{|[0,T]} \quad \text{for every } n.
\]
In the above expression, \( \pi_\sharp \mu_n \) denotes the push-forward of the measure \( \mu_n \) and \( L^1_{|[0,T]} \) the restriction of the Lebesgue measure. Also, the density \( f_n \) is given by

\[
f_n(t) := \| \mu_n,I \|_{M([0,T])} \leq \delta \quad \text{for a.e. } t \in [0,T].
\]

Next, we point out that, by possibly extracting a further subsequence, we can assume that the sequence

\[
f_n \overset{\Delta}{\rightharpoonup} f \quad \text{weakly}^* \text{ in } L^\infty([0,T])
\]

for some accumulation point \( f \) satisfying \( \| f \|_{L^\infty([0,T])} \leq \delta \). By passing to the weak* limit on both sides of the equality (81) we obtain

\[
\pi_\sharp \mu = f L^1_{|[0,T]}.
\]

By using the Disintegration Theorem [1, Theorem 2.28] and recalling the inequality \( \| f \|_{L^\infty([0,T])} \leq \delta \) we eventually conclude that \( \mu \in C_\delta \). This implies that the set \( C_\delta \) is compact. Also, it is obviously convex. Owing to the Schauder-Tychonoff Fixed Point Theorem, if the map \( T \) defined as in (79) is continuous, then it admits a fixed point, which is by construction a Nash equilibrium in the sense of Definition 6.1. Hence, the proof of Theorem 6.2 boils down to the proof of the continuity of \( T \).

**Step 2.** we prove that the map \( T \) defined as in (79) is continuous with respect to the weak-* convergence. To prove the continuity of \( T \) it suffices to show that the map \( \eta \mapsto \mu^{\text{opt}}(\eta) \) defined as in the statement of Lemma 6.3 is continuous. Hence, we fix

\[
\sigma_n \overset{\Delta}{\rightharpoonup} \sigma.
\]

We want to show that

\[
\tau_n := \mu^{\text{opt}}(\sigma_n) \overset{\Delta}{\rightharpoonup} \mu^{\text{opt}}(\sigma) := \tau \quad \text{as } n \to +\infty.
\]

Owing to the weak-*compactness of \( C_\delta \) we have that, up to subsequences,

\[
\tau_n \overset{\Delta}{\rightharpoonup} \tau_\infty \quad \text{as } n \to +\infty
\]

for some \( \tau_\infty \in C_\delta \). Owing to the uniqueness part in Lemma 6.3, to establish (83) it suffices to show that

\[
J_\sigma(\tau_\infty) \geq J_\sigma(\tau).
\]

To establish (85) we argue as follows. First, we term \( \varphi_n \) the weak solution of the initial-boundary value problem (76) in the case when \( \eta = \sigma_n \) and \( \mu = \tau_n \), namely

\[
\begin{aligned}
\partial_t \varphi_n &= \partial^2_{xx} \varphi_n - [\sigma_n + \tau_n] \varphi_n + g(\cdot,\cdot, \varphi_n), \\
\partial_x \varphi_n(t,0) &= \partial_x \varphi_n(t,R) = 0, \\
\varphi_n(0,x) &= \varphi_0(x).
\end{aligned}
\]

Note that, owing to (75),

\[
J_{\sigma_n}(\tau_n) = \int_0^T \int_0^R \varphi_n(t,x) d\tau_n(t,x) dt - \Psi \left( \int_0^T \int_0^R c(t,x) d\tau_n(t,x) dt \right).
\]

Also, we term \( \tilde{\varphi}_n \) the solution of the initial-boundary value problem (76) in the case when \( \eta = \sigma_n \) and \( \mu = \tau \), namely

\[
\begin{aligned}
\partial_t \tilde{\varphi}_n &= \partial^2_{xx} \tilde{\varphi}_n - [\sigma_n + \tau] \tilde{\varphi}_n + g(\cdot,\cdot, \tilde{\varphi}_n), \\
\partial_x \tilde{\varphi}_n(t,0) &= \partial_x \tilde{\varphi}_n(t,R) = 0, \\
\tilde{\varphi}_n(0,x) &= \varphi_0(x).
\end{aligned}
\]
We recall that \( \tau_n = \mu^{opt}(\sigma_n) \) and we infer that

\[
J_{\sigma_n}(\tau_n) \geq J_{\sigma_n}(\tau) = \int_0^T \int_0^R \varphi_n(t,x) d\tau_t(x) dt - \Psi \left( \int_0^T \int_0^R c(t,x) d\tau_t(x) dt \right). \tag{89}
\]

Next, we recall the estimate (23), we use the Aubin-Lions Lemma [14] as in the proof of Theorem 3.2 and we conclude that there are functions \( \varphi \) and \( \tilde{\varphi} \) such that

\[
\varphi_n \to \varphi, \quad \tilde{\varphi}_n \to \tilde{\varphi} \quad \text{strongly in} \quad L^2([0,T];C^0([0,\bar{R}])) \quad \text{as} \quad n \to +\infty. \tag{90}
\]

By standard arguments, we conclude that \( \varphi \) and \( \tilde{\varphi} \) are solutions of the initial-boundary value problem (76) in the case when \( \eta = \sigma, \mu = \tau_\infty \) and \( \eta = \sigma, \mu = \tau, \) respectively, namely

\[
\begin{align*}
\partial_t \varphi &= \partial_{xx} \varphi - [\sigma + \tau_\infty] \varphi + g(\cdot, \cdot, \varphi), \\
\partial_x \varphi(t,0) &= \partial_x \varphi(t,R) = 0, \\
\varphi(0,x) &= \varphi_0(x),
\end{align*}
\]

and

\[
\begin{align*}
\partial_t \tilde{\varphi} &= \partial_{xx} \tilde{\varphi} - [\sigma + \tau] \varphi + g(\cdot, \cdot, \tilde{\varphi}), \\
\partial_x \tilde{\varphi}(t,0) &= \partial_x \tilde{\varphi}(t,R) = 0, \\
\tilde{\varphi}(0,x) &= \varphi_0(x).
\end{align*}
\]

Next, we use the convergence \( \varphi_n \to \varphi \) and the lower semicontinuity of \( c \) to pass to the limit in (87). By using the fact that \( \Psi \) is nondecreasing, we get

\[
\begin{align*}
\limsup_{n \to +\infty} J_{\sigma_n}(\tau_n) &\leq \limsup_{n \to +\infty} \int_0^T \int_0^R \varphi_n(t,x) d\tau_{n,t}(x) dt \\
&\quad + \limsup_{n \to +\infty} -\Psi \left( \int_0^T \int_0^R c(t,x) d\tau_{n,t}(x) dt \right) \\
&= \lim_{n \to +\infty} \int_0^T \int_0^R \varphi_n(t,x) d\tau_{n,t}(x) dt \\
&\quad - \liminf_{n \to +\infty} \Psi \left( \int_0^T \int_0^R c(t,x) d\tau_{n,t}(x) dt \right) \\
&\quad - \Psi \left( \liminf_{n \to +\infty} \int_0^T \int_0^R c(t,x) d\tau_{n,t}(x) dt \right) \\
&\quad + \Psi \left( \int_0^T \int_0^R c(t,x) d\tau_{\infty,t}(x) dt \right) \\
&\quad - \Psi \left( \int_0^T \int_0^R c(t,x) d\tau_{\infty,t}(x) dt \right) \\
&\quad \leq \int_0^T \int_0^R \varphi(t,x) d\tau_{\infty,t}(x) dt \\
&\quad - \Psi \left( \int_0^T \int_0^R c(t,x) d\tau_{\infty,t}(x) dt \right) \\
&\quad \leq J_\sigma(\tau_\infty). \tag{91}
\end{align*}
\]

By using the convergence \( \tilde{\varphi}_n \to \tilde{\varphi} \), we can then pass to the limit in the expression at the right hand side of (89) and conclude that

\[
J_\sigma(\tau_\infty) \geq \limsup_{n \to +\infty} J_{\sigma_n}(\tau_n) \geq \lim_{n \to +\infty} J_{\sigma_n}(\tau) \geq \int_0^T \int_0^R \tilde{\varphi}(t,x) d\tau_t(x) dt - \Psi \left( \int_0^T \int_0^R c(t,x) d\tau_t(x) dt \right) \tag{91}
\]

and

\[
J_\sigma(\tau). \tag{91}
\]
The above chain of inequalities implies (85) and hence concludes the proof of Theorem 6.2. □

7. Proof of Lemma 5.3. To establish the proof of Lemma 5.3 we proceed as follows: in §§ 7.1, 7.2 and § 7.3 we provide a formal argument which is completely justified only in the case when all the functions are sufficiently regular. In § 7.4 we first conclude this formal argument by (formally) establishing (68) and then we explain how the formal argument can be made rigorous by relying on an approximation argument. To simplify notation, in the formal argument given in §§ 7.1, 7.2 and § 7.3 we write sup_{t \in [0,T]} \| \mu_t \|_{\mathcal{M}(\mathbb{R})} to denote ess sup_{t \in [0,T]} \| \mu_t \|_{\mathcal{M}(\mathbb{R})}.

7.1. Proof of Lemma 5.3: estimates on \( \varphi_\ast \). We first control the distance of the function \( \varphi_\ast \) from the constant \( h \).

Lemma 7.1. Let \( \varphi_\ast \) be the weak solution of the initial-boundary value problem (10), then
\[
\| \varphi_\ast (t, \cdot) - h \|_{L^\infty([0,R])} \leq C(\alpha_1, M, T, R) \left( \| \varphi_0 - h \|_{H^1([0,R])} + \sup_{t \in [0,T]} \| \mu_t^\ast \|_{\mathcal{M}(\mathbb{R})} \right)
\]
for every \( t \in [0,T] \). In particular, there is a threshold \( \delta \), which only depends on \( \alpha_1, M, T, R \) and \( h \) such that, if (57) holds, then
\[
\varphi_\ast (t, x) \geq \frac{h}{2} > 0 \quad \text{for a.e. } (t, x) \in [0,T] \times [0,R].
\]

Proof. Owing to the Duhamel Representation Formula (see the Appendix) we have
\[
\varphi_\ast (t, x) = \int_0^t \int_0^R D(t,x,y)\varphi_0(y)dy - \int_0^t \int_0^R D(t-s,x,y)\varphi_\ast (s,y)d\mu_s^\ast (y)ds
\]
\[
+ \int_0^t \int_0^R D(t-s,x,y)f(s,y,\varphi_\ast (s,y))\varphi_\ast (s,y)dyds.
\]
(95)

We recall that \( f(t,x,h) \equiv 0 \), which implies that \( \varphi \equiv h \) is a weak solution of the initial-boundary value problem (10) in the case when \( \mu^\ast \equiv 0 \). We deduce the following representation formula:
\[
\varphi_\ast (t, x) - h = \int_0^t \int_0^R D(t,x,y)[\varphi_0(y) - h]dy - \int_0^t \int_0^R D(t-s,x,y)\varphi_\ast (s,y)d\mu_s^\ast (y)ds
\]
\[
+ \int_0^t \int_0^R D(t-s,x,y)f(s,y,\varphi_\ast (s,y))\varphi_\ast (s,y)dyds.
\]
(96)

In the previous expressions, \( D \) is the same kernel as in (159). Since \( f(\cdot,\cdot,h) \equiv 0 \), then
\[
|f(t,x,\varphi_\ast)\varphi_\ast| = |f(t,x,\varphi_\ast)\varphi_\ast - f(t,x,h)\varphi_\ast| \leq \alpha_1|\varphi_\ast||\varphi_\ast - h|
\]
\[
\leq C(\alpha_1, M)|\varphi_\ast (t, x) - h| \quad \text{for every } (t, x) \in [0,T] \times [0,R].
\]
(97)

By plugging the above inequality into (96) we infer
\[
\| \varphi_\ast (t, \cdot) - h \|_{L^\infty([0,R])} \leq \|D(t,x,\cdot)||_{L^1([0,R])} \| \varphi_0 - h \|_{L^\infty([0,R])}
\]
\[
+ M \sup_t \| \mu_t^\ast \|_{\mathcal{M}(\mathbb{R})} \int_0^t \|D(t-s,x,\cdot)||_{L^\infty([0,R])}ds
\]
(20)
We control the first term by arguing as follows:

\[
+ \int_0^t \|D(t - s, x, \cdot)\|_{L^1([0,R])} \|f(s, \cdot, \varphi_*)(s, \cdot)\|_{L^\infty([0,R])} ds
\]

\[
\leq K\|\varphi_0 - h\|_{L^\infty([0,R])} + C(M, R) \sup_t \|\mu_t^*\|_{\mathcal{M}([0,R])} \int_0^t \frac{1}{\sqrt{t-s}} ds
\]

\[
+ K \int_0^t \|f(s, \cdot, \varphi_*)(s, \cdot)\|_{L^\infty([0,R])} ds
\]

\[
\leq C(R)\|\varphi_0 - h\|_{H^1([0,R])} + C(M, T, R) \sup_t \|\mu_t^*\|_{\mathcal{M}([0,R])}
\]

\[
+ C(\alpha_1, M) \int_0^t \|\varphi_*(s, \cdot) - h\|_{L^\infty([0,R])} ds.
\]

Owing to the Gronwall Lemma, the above inequality implies (93). \(\square\)

Next, we control the derivative \(\partial_x \varphi_*\) in \(L^\infty([0,T]; L^2([0,R]))\).

**Lemma 7.2.** Under the same assumptions as in the statement of Lemma 7.1, we have

\[
\|\partial_x \varphi_*(t, \cdot)\|_{L^2([0,R])} \leq C(\alpha_1, M, T, R) \left[\|\varphi_0 - h\|_{H^1([0,R])} + \sup_{t \in [0,T]} \|\mu_t^*\|_{\mathcal{M}([0,R])}\right]
\]

for every \(t \in [0,T]\).

**Proof.** By differentiating the representation formula (95) with respect to \(x\) and using (166) we get

\[
\partial_x \varphi_*(t, x) = \int_0^R \hat{D}(t, x, y)\varphi'_0(y)dy - \int_0^t \int_0^R \partial_x D(t - s, x, y)\varphi_*(s, y)dyds
\]

\[
+ \int_0^t \int_0^R \partial_x D(t - s, x, y)f(s, y, \varphi_*)\varphi_*(s, y)dyds.
\]

We control the first term by arguing as follows:

\[
\|I_1(t, \cdot)\|_{L^2([0,R])} = \left[\int_0^R \left(\int_0^R \hat{D}(t, x, y)\varphi'_0(y)dy\right)^2 dx\right]^{1/2}
\]

Hölder\n
\[
\leq \left[\int_0^R \left(\int_0^R \hat{D}(t, x, y)(\varphi'_0)^2(y)dy\right)\left(\int_0^R \hat{D}(t, x, y)dy\right) dx\right]^{1/2}
\]

\[
\leq K \left[\int_0^R \int_0^R \hat{D}(t, x, y)(\varphi'_0)^2(y)dydx\right]^{1/2}
\]

\[
\leq K \left(\int_0^R (\varphi'_0)^2(y)\int_0^R \hat{D}(t, x, y)dy dx\right)^{1/2}
\]

\[
\leq K \left[\int_0^R (\varphi'_0)^2(y)\right]^{1/2} \leq K\|\varphi_0 - h\|_{H^1([0,R])}.
\]
To establish the last inequality, we have used the fact that $h$ is a constant. Owing to the Bochner Theorem [13, p.473], we have

$$
\|I_2(t, \cdot)\|_{L^2([0,R])} \leq \int_0^t \left\| \int_0^R \partial_x D(t - s, y) \varphi^*_s(s, y) dy \right\|_{L^2([0,R])} ds
$$

(20)

$$
\leq M \int_0^t \left[ \int_0^R \left( \int_0^R |\partial_x D(t - s, x, y)|^2 dy \right) dx \right]^{1/2} ds
$$

Hölder

$$
\leq M \int_0^t \left[ \int_0^R \left( \int_0^R (\partial_x D)^2(t - s, x, y) d\mu^*_s(y) \right) \right]^{1/2} ds
$$

\leq M \left( \sup_{s \in [0,T]} \|\mu^*_s\|_{\mathcal{M}([0,R])} \right)^{1/2} \int_0^t \left( \int_0^R \left( \int_0^R (\partial_x D)^2(t - s, x, y) dx d\mu^*_s(y) \right) \right)^{1/2} ds
$$

(164)

$$
\leq C(M) \left( \sup_{s \in [0,T]} \|\mu^*_s\|_{\mathcal{M}([0,R])} \right)^{1/2} \int_0^t \left[ \int_0^R \frac{1}{(t-s)^{3/2}} d\mu^*_s(y) \right]^{1/2} ds
$$

$$
\leq C(M, T) \sup_{s \in [0,T]} \|\mu^*_s\|_{\mathcal{M}([0,R])}.
$$

(101)

By using again the Bochner Theorem [13, p.473] and arguing as before we get

$$
\|I_3(t, \cdot)\|_{L^2([0,R])} \leq \int_0^t \left\| \int_0^R \partial_x D(t - s, y) f(s, y, \varphi)_* \varphi^*_s(s, y) dy \right\|_{L^2([0,R])} ds
$$

(20),(97)

$$
\leq C(M, \alpha_1) \int_0^t \left\| \int_0^R |\partial_x D(t - s, y)| |\varphi^*_s - h|(s, y) dy \right\|_{L^2([0,R])} ds
$$

$$
\leq C(M, \alpha_1) \int_0^t \int_0^R \|\partial_x D(t - s, y)\|_{L^2([0,R])} |\varphi^*_s - h|(s, y) dy ds
$$

(164)

$$
\leq C(\alpha_1, M) \int_0^t \|\varphi^*_s(s, \cdot) - h\|_{L^\infty([0,R])} \frac{1}{(t-s)^{3/4}} ds
$$

(93)

$$
\leq C(\alpha_1, M, T, R) \left[ \sup_{t \in [0,T]} \|\mu^*_t\|_{\mathcal{M}([0,R])} + \|\varphi_0 - h\|_{H^1([0,R])} \right] \int_0^t \frac{1}{(t-s)^{3/4}} ds
$$

$$
\leq C(\alpha_1, M, T, R) \left[ \sup_{t \in [0,T]} \|\mu^*_t\|_{\mathcal{M}([0,R])} + \|\varphi_0 - h\|_{H^1([0,R])} \right].
$$

(102)

By plugging (100), (101) and (102) into (99) we establish (98).

7.2. **Proof of Lemma 5.3: estimates on $\varphi_1$.** We can now control the first term in (68).
Lemma 7.3. Under the same assumptions as in the statement of Lemma 7.1, we have

\[
\int_0^T \int_0^R \varphi_1(t,x) d\nu_t(x) dt \leq \|\varphi_1\|_{L^2([0,T];H^1([0,R]))}^2 \left[-C(M,h_\ast)+C(\alpha_1,\alpha_2, M, h, T, R)\delta\right].
\]  

(103)

Proof. We argue according to the following steps.

Step 1. We find a more convenient expression for the left hand side of (103). Owing to (94), the function \( \varphi_\ast \) is bounded away from 0. We recall that by definition

\[
\partial_\varphi g(\cdot, \cdot, \varphi) = f(\cdot, \cdot, \varphi) + \partial_\varphi f(\cdot, \cdot, \varphi_\ast) \varphi_\ast.
\]

We divide the equation at the first line of (30) times \( \varphi_\ast \) to (94), the function \( \varphi_\ast \) is bounded away from 0. We recall that by definition

\[
g(\cdot, \cdot, \varphi) = f(\cdot, \cdot, \varphi) + \partial_\varphi f(\cdot, \cdot, \varphi_\ast) \varphi_\ast.
\]

By using the fact that

\[
\varphi_\ast = \frac{\partial_\varphi \varphi_1}{\varphi_\ast} + f(\cdot, \cdot, \varphi_\ast) + \partial_\varphi f(\cdot, \cdot, \varphi_\ast) \varphi_1.
\]

We divide the equation at the first line of (30) times \( \varphi_\ast \) and we use the above expression for \( \partial_\varphi g \) we eventually get

\[
\nu = -\frac{\partial_\varphi \varphi_1}{\varphi_\ast} + \frac{\partial^2 \varphi_1}{\varphi_\ast} - \varphi_1 \mu_\ast + f(\cdot, \varphi_\ast) \varphi_1 + \partial_\varphi f(\cdot, \cdot, \varphi_\ast) \varphi_1.
\]

By using the above expression for \( \nu \), we can formally rewrite the left hand side of (103) as

\[
\int_0^T \int_0^R \varphi_1(t,x) d\nu_t(x) dt = -\int_0^T \int_0^R \frac{\varphi_1}{\varphi_\ast} \partial_\varphi \varphi_1 dx dt + \int_0^T \int_0^R \varphi_1 \frac{\partial^2 \varphi_1}{\varphi_\ast} dx dt
\]

and

\[
-\int_0^T \int_0^R \varphi_1 \partial_\varphi \varphi_1 dx dt - \frac{1}{2} \int_0^T \int_0^R \frac{\partial_\varphi \varphi_1}{\varphi_\ast} dx dt
\]

\[
= -\frac{1}{2} \int_0^T \int_0^R \frac{\varphi_1}{\varphi_\ast} (T,x) dx - \frac{1}{2} \int_0^T \int_0^R \frac{\varphi_1}{\varphi_\ast} \partial_\varphi \varphi_\ast dx dt
\]

\[
\leq -\frac{1}{2} \int_0^T \int_0^R \frac{\varphi_1}{\varphi_\ast} (T,x) dx + \frac{1}{2} \int_0^T \int_0^R \frac{\varphi_1}{\varphi_\ast} \partial_\varphi \varphi_\ast dx dt
\]

\[
= -\frac{1}{2} \int_0^T \int_0^R \frac{\varphi_1}{\varphi_\ast} \partial_{xx} \varphi_\ast dx dt + \frac{1}{2} \int_0^T \int_0^R \frac{\varphi_1}{\varphi_\ast} d\mu_\ast (x) dt
\]

\[
- \frac{1}{2} \int_0^T \int_0^R \frac{\varphi_1}{\varphi_\ast} f(t,x,\varphi_\ast) dx dt.
\]

(104)

Step 2. We separately control each of the terms in (104). Owing to Cauchy condition \( \varphi_1(0, \cdot) \equiv 0 \) in (30) and to the inequality (94) we have

\[
-\frac{1}{2} \int_0^T \int_0^R \frac{\varphi_1}{\varphi_\ast} \partial_{xx} \varphi_\ast dx dt = \int_0^T \int_0^R \partial_\varphi \varphi_\ast \partial_\varphi \varphi_1 \frac{\varphi_1}{\varphi_\ast} dx dt - \int_0^T \int_0^R (\partial_\varphi \varphi_\ast)^2 \frac{\varphi_1}{\varphi_\ast} dx dt.
\]

(106)
By plugging (105), (107) and (106) into (104) we arrive at

\[
\int_0^T \int_0^R \frac{\phi_1}{\phi_*} \partial_{xx}^2 \phi_1 \, dx \, dt = - \int_0^T \int_0^R \frac{\partial_x \phi_1}{\phi_*} \, dx \, dt + \int_0^T \int_0^R \frac{\phi_1}{\phi_*} \partial_x \phi_1 \partial_x \phi_* \, dx \, dt.
\]

(107)

By plugging (105), (107) and (106) into (104) we arrive at

\[
\int_0^T \int_0^R \phi_1(t, x) \, dx \, dt \overset{(104), (105)}{\leq} - \frac{1}{2} \int_0^T \int_0^R \frac{\phi_1^2}{\phi_*^2} \partial_x \phi_* \, dx \, dt
\]

\[
+ \frac{1}{2} \int_0^T \int_0^R \frac{\phi_1^2}{\phi_*^2} \, dx \, dt
\]

\[
- \frac{1}{2} \int_0^T \int_0^R \frac{\phi_1}{\phi_*} f(t, x, \phi_*) \, dx \, dt + \int_0^T \int_0^R \frac{\phi_1}{\phi_*} \partial_{xx} \phi_1 \, dx \, dt
\]

\[
- \frac{1}{2} \int_0^T \int_0^R \frac{\phi_1}{\phi_*} d\mu_1^*(x) \, dt
\]

\[
+ \int_0^T \int_0^R \partial_x f(t, x, \phi_*) \phi_*^2 \, dx \, dt
\]

\[
\leq 2 \int_0^T \int_0^R \partial_x \phi_* \partial_x \phi_1 \frac{\phi_1}{\phi_*^2} \, dx \, dt - \int_0^T \int_0^R (\partial_x \phi_*)^2 \frac{\phi_1^2}{\phi_*^3} \, dx \, dt
\]

\[
- \frac{1}{2} \int_0^T \int_0^R \frac{\phi_1^2}{\phi_*^2} \, dx \, dt
\]

\[
+ \frac{1}{2} \int_0^T \int_0^R \frac{\phi_1}{\phi_*} f(t, x, \phi_*) \, dx \, dt - \int_0^T \int_0^R (\partial_x \phi_1)^2 \frac{\phi_*^2}{\phi_*^3} \, dx \, dt
\]

\[
+ \int_0^T \int_0^R \partial_x f(t, x, \phi_*) \phi_*^2 \, dx \, dt.
\]

(108)

Next, we use the Young Inequality and we get

\[
2 \int_0^T \int_0^R \frac{\phi_1}{\phi_*} \partial_x \phi_1 \partial_x \phi_* \, dx \, dt \leq \frac{1}{a} \int_0^T \int_0^R \frac{(\partial_x \phi_1)^2}{\phi_*} \, dx \, dt
\]

\[
+ \frac{1}{a} \int_0^T \int_0^R (\partial_x \phi_*)^2 \frac{\phi_*^2}{\phi_*^3} \, dx \, dt
\]
for some $a > 0$ to be determined in the following. We plug the above inequality into (108) and we get

$$
\int_0^T \int_0^R \varphi_1(t,x) d\nu_t(x) dt \leq \left[ \frac{1}{a} - 1 \right] \int_0^T \int_0^R \frac{(\partial_x \varphi_1)^2}{\varphi_*^2} dx dt
$$

$$
+ \left[ a - 1 \right] \int_0^T \int_0^R \frac{(\partial_x \varphi_*)^2}{\varphi_*^2} dx dt
$$

$$
- \frac{1}{2} \int_0^T \int_0^R \varphi_*^2 \mu_t^*(x) dt + \frac{1}{2} \int_0^T \int_0^R \varphi_*^2 f(t,x,\varphi_*) dx dt
$$

$$
\leq \int_0^T \int_0^R \frac{(\partial_x \varphi_1)^2}{\varphi_*^2} dx dt + \int_0^T \int_0^R \partial_x f(t,x,\varphi_*) \varphi_*^2 dx dt
$$

$$
= \frac{1}{a} \int_0^T \int_0^R \frac{(\partial_x \varphi_1)^2}{\varphi_*^2} dx dt + \int_0^T \int_0^R \partial_x f(t,x,\varphi_*) \varphi_*^2 dx dt.
$$

(109)

Owing to (55), we get

$$
\int_0^T \int_0^R \partial_x f(t,x,\varphi_*) \varphi_*^2 dx dt
$$

$$
\leq \int_0^T \int_0^R \left[ -h_* + [\partial_x f(t,x,\varphi_*) - \partial_x f(t,x,h)]\right] \varphi_*^2(t,x) dx dt
$$

(110)

$$
\leq \int_0^T \int_0^R \left[ -h_* + \alpha_2|\varphi_* - h|\right] \varphi_*^2 dx dt.
$$

Next, we choose $a = 2$ and by recalling (20) we infer

$$
\left[ \frac{1}{a} - 1 \right] \int_0^T \int_0^R \frac{(\partial_x \varphi_1)^2}{\varphi_*^2} dx dt = -\frac{1}{2} \int_0^T \int_0^R \frac{(\partial_x \varphi_1)^2}{\varphi_*^2} dx dt
$$

$$
\leq -C(M) \int_0^T \int_0^R (\partial_x \varphi_1)^2 dx dt.
$$

(20)

By combining the above equation with (110) and plugging the result in (109) we arrive at

$$
\int_0^T \int_0^R \varphi_1(t,x) d\nu_t(x) dt \leq -C(M) \int_0^T \int_0^R (\partial_x \varphi_1)^2 dx dt - h_* \int_0^T \int_0^R \varphi_*^2 dx dt
$$

$$
+ \int_0^T \int_0^R (\partial_x \varphi_1)^2 \varphi_*^2 dx dt
$$

$$
+ \frac{1}{2} \int_0^T \int_0^R \varphi_*^2 f(t,x,\varphi_*) dx dt + \alpha_2 \int_0^T \int_0^R |\varphi_* - h| \varphi_*^2 dx dt
$$

$$
\leq -C(M, h_*) \Vert \varphi_1 \Vert_{L^2([0,T];H^1([0,h]))}^2 + \int_0^T \int_0^R (\partial_x \varphi_1)^2 \varphi_*^2 dx dt
$$

$$
+ C(\alpha_1) \int_0^T \int_0^R \varphi_*^2 |\varphi_* - h| dx dt + \alpha_2 \int_0^T \int_0^R |\varphi_* - h| \varphi_*^2 dx dt
$$

(111)
Lemma 7.4. Under the same assumptions as in the statement of Lemma 7.1, we have
\[
\int_0^T \|\varphi_2(t, \cdot)\|_{L^\infty([0,R])} dt \leq C(\alpha_1, \alpha_2, M, h, T, R, F) \|\varphi_1\|_{L^2([0,T]; H^1([0,R]))}^2. \tag{114}
\]

Proof. First, we recall that, since \(g(t, x, \varphi) = f(t, x, \varphi)\varphi\), then
\[
\partial_x g = \partial_x f \cdot \varphi + f, \quad \partial_x^2 g = \partial_x^2 f \cdot \varphi + 2\partial_x f.
\]
This implies
\[
\|\partial_\varphi \varphi(t,\cdot,\varphi_\ast)\|_{L^\infty([0,T]\times[0,R])} \leq \alpha_1 M + \|f(\cdot,\cdot,\varphi_\ast) - f(\cdot,\cdot,h)\|_{L^\infty([0,T]\times[0,R])}
\]
(11), (20)

\[
\leq C(\alpha_1, M) + C(\alpha_1, M, T, R)\delta
\]
(57), (93)
\[
\leq C(\alpha_1, M, T, R)
\]
(115)
and, by combining (11), (20) and (56),
\[
\|\partial_\varphi^2 \varphi(t,\cdot,\varphi_\ast)\|_{L^\infty([0,T]\times[0,R])} \leq C(\alpha_1, \alpha_2, M).
\]
(116)

By applying the Duhamel Representation Formula (see the Appendix) to the linear equation (66) we arrive at
\[
\varphi_2(t, x) = -\int_0^t \int_0^R D(t - s, x, y)\varphi_2(s, y)d\mu^*_s(y)ds
- 2 \int_0^t \int_0^R D(t - s, x, y)\varphi_1(s, y)d\nu_s(y)ds
+ \int_0^t \int_0^R D(t - s, x, y)\partial_\varphi g(s, y, \varphi_s, \varphi_2(s, y)dyds
+ \int_0^t \int_0^R D(t - s, x, y)\partial_\varphi^2 g(s, y, \varphi_\ast)\varphi_2^2(s, y)dyds.
\]

The above representation formula implies
\[
|\varphi_2(t, x)| \leq \int_0^t \|D(t - s, x, \cdot)\|_{L^\infty([0,R])} \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])} \|\mu^*_s\|_{M([0,R])}ds
+ I(t) + C(\alpha_1, M, T, R) \int_0^t \|D(t - s, x, \cdot)\|_{L^1([0,R])} \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])}ds
+ C(\alpha_1, \alpha_2, M) \int_0^t \|D(t - s, x, \cdot)\|_{L^1([0,R])} \|\varphi_1(s, \cdot)\|_{L^\infty([0,R])}^2 ds,
\]
(117)
where we have defined the function \(I(t)\) by setting
\[
I(t) := 2 \sup_{x \in [0,R]} \left|\int_0^t \int_0^R D(t - s, x, y)\varphi_1(s, y)d\nu_s(y)ds\right|.
\]
(118)

From (117) we get
\[
\|\varphi_2(t, \cdot)\|_{L^\infty([0,R])} \leq \sup_t \|\mu^*_t\|_{M([0,T])} \int_0^t \frac{C(R)}{\sqrt{t - s}} \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])}ds
+ I(t) + C(\alpha_1, M, T, R) \int_0^t \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])}ds
+ C(\alpha_1, \alpha_2, M) \int_0^t \|\varphi_1(s, \cdot)\|_{L^\infty([0,R])}^2 ds
\]
\[
\leq \delta \int_0^t \frac{C(R)}{\sqrt{t - s}} \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])}ds + I(t)
\]
(57), (9)
Let Lemma 7.5.

This implies that, by time integrating (7.3), we get that, if $\tau$ implies that

$$\int_0^\tau \frac{1}{\sqrt{T-s}} \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])} ds \leq C(T) \int_0^\tau \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])} ds.$$ 

Next, we point out that, if $\tau \leq T$, then

$$\int_0^\tau \frac{1}{\sqrt{T-s}} \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])} ds \leq C(T, R) \delta \int_0^\tau \|\varphi_2(t, \cdot)\|_{L^\infty([0,R])} dt + \int_0^T I(t) dt$$

$$+ C(\alpha_1, M, T, R) \int_0^\tau \int_0^t \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])} ds dt$$

$$+ C(\alpha_1, \alpha_2, M, R, T) \|\varphi_1\|_{L^2([0,T]; H^1([0,R]))}^2.$$ 

If the constant $\delta$ is sufficiently small to have $C(T, R)\delta \leq 1/2$, the above inequality implies that

$$\int_0^\tau \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])} ds \leq C(\alpha_1, M, T, R) \int_0^\tau \int_0^t \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])} ds dt$$

$$+ 2 \int_0^T I(t) dt + C(\alpha_1, \alpha_2, M, R, T) \|\varphi_1\|_{L^2([0,T]; H^1([0,R]))}^2.$$ 

Next, we apply the Gronwall Lemma to the function

$$\tau \mapsto \int_0^\tau \|\varphi_2(s, \cdot)\|_{L^\infty([0,R])} ds$$

and we infer that (114) holds true provided that Lemma 7.5 below holds true.

To conclude the proof of Lemma 7.4 we are left to establish the following result.

\textbf{Lemma 7.5.} Let $I$ be the same function as in (118), then we have

$$\int_0^T I(t) dt \leq C(\alpha_1, M, h, T, R, h, F) \|\varphi_1\|_{L^2([0,T]; H^1([0,R]))}^2.$$ 

\textit{Proof.} First, we use the formula in the proof of Lemma 7.3 for $\nu$, we recall that $I$ is defined as in (118) and we get

$$\int_0^T I(t) dt \leq 2 \int_0^T \left( \sup_{x \in [0,R]} \left| \int_0^t \int_0^R D(t-s, x, y) \frac{\partial \varphi_1}{\varphi_1} \varphi_1(s, y) dy ds \right| dt \right)$$

$$+ 2 \int_0^T \left( \sup_{x \in [0,R]} \left| \int_0^t \int_0^R D(t-s, x, y) \frac{\partial^2 \varphi_1}{\varphi_1} \varphi_1(s, y) dy ds \right| dt \right).$$
By recalling (94) and (161), we conclude that
\[
\begin{align*}
\partial_t \phi_1(s,x,y) &= \partial_t \phi_2(s,x,y) + \frac{\partial_2 \phi_1}{\phi_2} \partial_1 \phi_2.
\end{align*}
\] (122)

We now separately control the terms \(J_1, \ldots, J_5\). First, we consider the term \(J_1\): we point out that
\[
\begin{align*}
\frac{\partial_t \phi_1}{\phi_2} &= \frac{\partial_1 \phi_1}{\phi_2} = \partial_t \left[ \frac{\phi_1^2}{\phi_2} \right] + \frac{\phi_1^2}{\phi_2} \partial_t \phi_2.
\end{align*}
\] (122)

and we get
\[
\begin{align*}
J_1^{(122)} &\leq \int_0^T \sup_{x \in [0,R]} \left| \int_0^t \int_0^R D(t-s,x,y) \partial_s \left[ \frac{\phi_1^2}{\phi_2} \right] (s,y)\,dy \,ds \right| \,dt \\
&\quad + \int_0^T \sup_{x \in [0,R]} \left| \int_0^t \int_0^R D(t-s,x,y) \frac{\phi_1^2}{\phi_2} \partial_s \phi_2(s,y) \,dy \,ds \right| \,dt.
\end{align*}
\] (123)

By using the Integration by Parts Formula and the initial condition \(\phi_1(t=0) \equiv 0\) we get
\[
\begin{align*}
\int_0^T \sup_{x \in [0,R]} \left| \int_0^t \int_0^R D(t-s,x,y) \partial_s \left[ \frac{\phi_1^2}{\phi_2} \right] (s,y)\,dy \,ds \right| \,dt \\
&\leq \int_0^T \sup_{x \in [0,R]} \left| \int_0^t \int_0^R \partial_s D(t-s,x,y) \frac{\phi_1^2}{\phi_2} (s,y)\,dy \,ds \right| \,dt \\
&\quad + \int_0^T \lim_{s \to t^-} \sup_{x \in [0,R]} \int_0^R D(t-s,x,y) \frac{\phi_1^2}{\phi_2} (s,y) \,dy \,dt
\end{align*}
\]

Next, we recall that the kernel \(D\) is defined as in (159) and hence satisfies the heat equation, which implies that in the above formula we can replace \(\partial_s D\) with \(-\partial_{xx} D\). By recalling (94) and (161), we conclude that
\[
\begin{align*}
\int_0^T \sup_{x \in [0,R]} \left| \int_0^t \int_0^R D(t-s,x,y) \partial_s \left[ \frac{\phi_1^2}{\phi_2} \right] (s,y)\,dy \,ds \right| \,dt \\
&\leq \int_0^T \sup_{x \in [0,R]} \int_0^t \int_0^R \partial_{xx} D(t-s,x,y) \frac{\phi_1^2}{\phi_2} (s,y) \,dy \,ds \,dt \\
&\quad + C(h) \int_0^T \|\phi_1(t,\cdot)\|_{L^\infty([0,R])}^2 \,dt
\end{align*}
\] (124)
We infer that

\[ J_{11} \leq J_{11} + C(h, R) \int_0^T \| \varphi_1(t, \cdot) \|_{H^1([0, R])}^2 dt \]

\[ \leq J_{11} + C(h, R) \| \varphi_1 \|_{L^2([0, T]; H^1([0, R]))}^2. \]

By plugging the above formula into (123) we get

\[ J_1 \leq J_{11} + J_{12} + C(h, R) \| \varphi_1 \|_{L^2([0, T]; H^1([0, R]))}^2. \quad (125) \]

We now focus on the term \( J_{11} \) defined in (124). First, we point out that

\[ \partial_y \left[ \frac{\varphi_1^2}{\varphi_*} \right] = 2 \frac{\varphi_1}{\varphi_*} \partial_y \varphi_1 - \frac{\varphi_1^2}{\varphi_*^2} \partial_y \varphi_. \quad (126) \]

We infer that

\[
J_{11} \leq \int_0^T \int_0^t \sup_{x \in [0, R]} \left| \int_0^R \partial_x \tilde{D}(t - s, x, y) \partial_y \left[ \frac{\varphi_1}{\varphi_*} \right] (s, y) dy \right| ds dt
\]

\[
\leq K \int_0^T \int_0^t \sup_{x \in [0, R]} \left| \int_0^R \partial_x \tilde{D}(t - s, x, y) \frac{\varphi_1}{\varphi_*} \partial_y \varphi_1 (s, y) dy \right| ds dt
\]

\[
+ K \int_0^T \int_0^t \sup_{x \in [0, R]} \left| \int_0^R \partial_x \tilde{D}(t - s, x, y) \frac{\varphi_1^2}{\varphi_*^2} \partial_y \varphi_1 (s, y) dy \right| ds dt.
\]

We now control \( J_{11} \):

\[
J_{11} \leq \int_0^T \int_0^t \sup_{x \in [0, R]} \left| \partial_x \tilde{D}(t - s, x, y) \frac{\varphi_1}{\varphi_*} \partial_y \varphi_1 (s, y) \right| dy ds dt
\]

\[
\leq \int_0^T \int_0^t \sup_{x \in [0, R]} \left\| \partial_x \tilde{D}(t - s, x, y) \right\|_{L^2([0, R])} \left\| \frac{\varphi_1}{\varphi_*} \right\|_{L^2([0, R])} \left\| \partial_y \varphi_1 (s, y) \right\|_{L^2([0, R])} ds dt
\]

\[
\leq C(h) \int_0^T \int_0^t \frac{1}{(t - s)^{3/4}} \left( \| \varphi_1(s, \cdot) \|_{L^\infty([0, R])} + \| \partial_y \varphi_1 \|_{L^2([0, R])} \right) ds dt
\]

\[
\leq C(h, R) \int_0^T \int_0^t \frac{1}{(t - s)^{3/4}} \| \varphi_1(s, \cdot) \|_{H^1([0, R])} ds dt
\]

\[
\leq C(h, T, R) \| \varphi_1 \|_{L^2([0, T]; H^1([0, R]))}^2.
\]

To control \( J_{12} \) we use (98) and by arguing as before we get

\[ J_{12} \leq C(\alpha_1, M, h, T, R) \| \varphi_1 \|_{L^2([0, T]; H^1([0, R]))}^2. \quad (129) \]

By combining (128) and (129) and recalling that \( \delta \leq 1 \) we get

\[ J_{11} \leq K (J_{111} + J_{112}) \leq C(\alpha_1, M, h, T, R) \| \varphi_1 \|_{L^2([0, T]; H^1([0, R]))}^2. \quad (130) \]
We now control $J_{12}$: we recall that $J_{12}$ is defined as in (123) and that $\varphi_*$ satisfies (67). We get

$$J_{12} \leq 2 \int_0^T \int_0^t \sup_{x \in [0,R]} \left| \int_0^R D(t-s,x,y) \frac{\varphi_*^2}{\varphi_*^2} \partial_y \varphi_*(s,y) dy \right| ds dt$$

$$+ 2 \int_0^T \int_0^t \sup_{x \in [0,R]} \left| \int_0^R D(t-s,x,y) \frac{\varphi_*^2}{\varphi_*^2} d\mu_*(y) \right| ds dt$$

$$+ 2 \int_0^T \int_0^t \sup_{x \in [0,R]} \left| \int_0^R D(t-s,x,y) \frac{\varphi_*^2}{\varphi_*^2} f(s,y,\varphi_*) dy \right| ds dt. \quad (131)$$

To control $J_{121}$, we first point out that

$$\partial_y \left[ \frac{\varphi_*^2}{\varphi_*^2} \partial_y \varphi_* \right] = \frac{\varphi_*^2}{\varphi_*^2} \partial_y^2 \varphi_* + 2 \frac{\varphi_*}{\varphi_*} \partial_y \varphi_1 \partial_y \varphi_* - 2 \frac{\varphi_*}{\varphi_*^2} (\partial_y \varphi_*)^2. \quad (132)$$

This implies

$$J_{121} \leq K \int_0^T \int_0^t \sup_{x \in [0,R]} \left| \int_0^R \partial_y D(t-s,x,y) \frac{\varphi_*^2}{\varphi_*^2} \partial_y \varphi_*(s,y) dy \right| ds dt$$

$$+ K \int_0^T \int_0^t \sup_{x \in [0,R]} \left| \int_0^R D(t-s,x,y) \frac{\varphi_*}{\varphi_*} \partial_y \varphi_1 \partial_y \varphi_*(s,y) dy \right| ds dt$$

$$+ K \int_0^T \int_0^t \sup_{x \in [0,R]} \left| \int_0^R D(t-s,x,y) \frac{\varphi_*^2}{\varphi_*^2} (\partial_y \varphi_*)^2(s,y) dy \right| ds dt. \quad (133)$$

We now separately control each of the above terms: for simplicity, here and in the following we will skip some computations that are similar to the ones we performed before. However, the detailed computations can be found in the online version of the present paper, available as arXiv:1602.05737.

By arguing as before we get

$$J_{1211} \leq C(\alpha_1, M, h, T, R) \delta \| \varphi_1 \|_{L^2([0,T];H^1([0,R]))}^2 \quad (134)$$

and

$$J_{1212} \leq C(\alpha_1, M, h, T, R) \delta \| \varphi_1 \|_{L^2([0,T];H^1([0,R]))}^2. \quad (135)$$

Finally, we have

$$J_{1213} \leq \int_0^T \int_0^t \| \frac{\varphi_*^2}{\varphi_*^2} (s, \cdot) \|_{L^2([0,R])} \sup_{x \in [0,R]} \| D(t-s, x, \cdot) \|_{L^\infty([0,R])} ds dt$$

and after some computations one can show that

$$J_{1213} \leq C(\alpha_1, M, h, T, R) \delta^2 \| \varphi_1 \|_{L^2([0,T];H^1([0,R]))}^2. \quad (136)$$
By combining (134), (135) and (136) and recalling that \( \delta \leq 1 \) we obtain
\[
J_{121} = J_{1211} + J_{1212} + J_{1213} \leq C(\alpha_1, M, h, T, R) \delta \|\varphi_1\|_{L^2([0,T];H^1([0,R]))}^2.
\] (137)

Next, we recall that \( J_{122} \) is defined as in (131) and after some computations we arrive at
\[
J_{122} \leq C(h, T, R) \delta \|\varphi_1\|_{L^2([0,T];H^1([0,R]))}^2.
\] (138)

To control \( J_{123} \), we first point out that
\[
\|f((\cdot, \cdot), \varphi_*)\|_{L^\infty([0,T]\times[0,R])} \leq \alpha_1 \|\varphi_* - h\|_{L^\infty([0,T]\times[0,R])} \leq C(\alpha_1, M, T, R) \delta.
\] (139)

By combining (137), (138), (139) and recalling that \( \delta \leq 1 \) we arrive at
\[
J_{12} \leq J_{121} + J_{122} + J_{123} \leq C(\alpha_1, h, M, R, T) \|\varphi_1\|_{L^2([0,T];H^1([0,R]))}^2.
\] (140)

Finally, we recall (125) and we conclude that
\[
J_1 \leq C(h, R) \|\varphi_1\|_{L^2([0,T];H^1([0,R]))}^2 + J_{11} + J_{12} \leq C(\alpha_1, h, M, R, T) \|\varphi_1\|_{L^2([0,T];H^1([0,R]))}^2.
\] (142)

We now focus on the term \( J_2 \). We recall that \( J_2 \) is defined as in (121) and we preliminary point out that
\[
\frac{\varphi_1}{\varphi_*} \frac{\partial^2_x \varphi_1}{\varphi_*} = \partial_x \left[ \frac{\varphi_1}{\varphi_*} \frac{\partial_x \varphi_1}{\varphi_*} \right] - \left( \frac{\partial_x \varphi_1}{\varphi_*} \right)^2 + \frac{\varphi_1 \partial_x^2 \varphi_1}{\varphi_*^2} - \partial_x \varphi_*.
\]

Owing to the Integration by Parts Formula, this implies
\[
\int_0^T \int_0^t \sup_{x \in [0,R]} \left| \int_0^R D(t-s, x, y) \frac{\varphi_1}{\varphi_*} \frac{\partial^2_x \varphi_1(s, y)}{\varphi_*} ds \right| dt \leq \int_0^T \int_0^t \sup_{x \in [0,R]} \left| \int_0^R \partial_y D(t-s, x, y) \frac{\varphi_1}{\varphi_*} \partial_y \frac{\varphi_1}{\varphi_*} ds \right| dt.
\]

\[
\int_0^T \int_0^t \sup_{x \in [0,R]} \left| \int_0^R D(t-s, x, y) \frac{\partial^2_x \varphi_1}{\varphi_*^2} ds \right| dt \leq \int_0^T \int_0^t \sup_{x \in [0,R]} \left| \int_0^R D(t-s, x, y) \frac{\varphi_1}{\varphi_*} \partial_x \varphi_1 \partial_x \varphi_* ds \right| dt.
\]

We can separately control each of the above terms and (after some computations) we arrive at
\[
J_2 \leq J_{21} + J_{22} + J_{23} \leq C(\alpha_1, M, h, T, R) \|\varphi_1\|_{L^2([0,T];H^1([0,R]))}^2.
\] (144)
Finally, we recall that the terms $J_3$, $J_4$ and $J_5$ are defined as in (121). After some computations, we conclude that
\begin{equation}
J_3 + J_4 + J_5 \leq C(\alpha_1, h, M, F, T, R)\|\varphi_1\|_{L^2([0,T];H^1([0,R]))}^2.
\end{equation}
By combining (125), (144) and (145) we eventually arrive at
\begin{equation}
\int_0^T I(t)dt \leq J_1 + J_2 + J_3 + J_4 + J_5 \leq C(\alpha_1, M, h, T, R, h, F)\|\varphi_1\|_{L^2([0,T];H^1([0,R]))}^2
\end{equation}
and this concludes the proof of Lemma 7.5.

### 7.4. Conclusion of the proof of Lemma 5.3.

We use (114) and we get
\begin{align*}
\left| \int_0^T \int_0^R \varphi_2(t,x)d\mu^*_t(x)dt \right| &\leq \int_0^T \left| \int_0^R \mu^*_t \varphi_2(t,x)dt \right|d\mu^*_t(x)dt \\
&\leq \sup_{t \in [0,R]} \|\mu^*_t\|_{\mathcal{M}([0,R])} \int_0^T \|\varphi_2(t,\cdot)\|_{L^\infty([0,R])}dt \\
&\leq \sup_{t \in [0,R]} \|\mu^*_t\|_{\mathcal{M}([0,R])} \int_0^T \|\varphi_2(t,\cdot)\|_{L^\infty([0,R])}dt \\
&\leq C(\alpha_1, \alpha_2, M, h, T, R, F)\|\varphi_1\|_{L^2([0,T];H^1([0,R]))}^2.
\end{align*}

Next, we combine (103) with the above inequality and we conclude that
\begin{align*}
2 \int_0^T \int_0^R \varphi_1(t,x)d\nu_t(x)dt + \int_0^T \int_0^R \varphi_2(t,x)d\mu^*_t(x)dt \\
&\leq \|\varphi_1\|_{L^2([0,T];H^1([0,R]))}^2 \left[ -C(M, h, T, R, F) \right].
\end{align*}
In the previous expression, the quantity at the right hand side is negative provided that the constant $\delta$ is sufficiently small. This establishes (68) and concludes the formal proof of Lemma 5.3.

To complete the proof of Lemma 5.3 we are left to make rigorous the formal argument given so far. To this end, we rely on an approximation argument. First, we recall the equality
\begin{equation}
\esssup_{t \in [0,T]} \|\nu_t\|_{\mathcal{M}([0,R])} \overset{(59),(60)}{=} 1
\end{equation}
and we point out that, by passing to the limit in the inequality (42), we get that $\varphi_1$ satisfies
\begin{align*}
\int_0^R \varphi_1^2(t,x)dx + \int_0^T \int_0^R (\partial_x \varphi_1)^2 dxdt + \int_0^T \int_0^R \varphi_1^2 d\mu^*_t(x)dt \\
&\leq C(\alpha_1, M, F, T, R) \esssup_{t \in [0,T]} \|\nu_t\|_{\mathcal{M}([0,R])}^2 \\
&\overset{(149)}{\leq} C(\alpha_1, M, F, T, R), \quad \text{for every } t \in [0,T].
\end{align*}
By relying on analogous computations and by using (150) we infer that
\begin{align*}
\int_0^R \varphi_2^2(t,x)dx + \int_0^T \int_0^R (\partial_x \varphi_2)^2 dxdt + \int_0^T \int_0^R \varphi_2^2 d\mu^*_t(x)dt \\
&\leq C(\alpha_1, \alpha_2, M, F, T, R), \quad \text{for every } t \in [0,T].
\end{align*}
Next, we fix three sequences of smooth functions \( \{ \nu_k \}, \{ \mu^*_k \} \subseteq C^\infty_c(\mathbb{R}^2), \{ \varphi_{0k} \} \subseteq C^\infty_c(\mathbb{R}) \) such that
\[
\nu_{kt} \xrightarrow{\ast} \nu_t \quad \text{in} \quad M([0,R]), \quad \| \nu_{kt} \|_{M([0,R])} \leq \sup_{t \in [0,T]} \| \nu_t \|_{M([0,R])} \quad \text{a.e.} \quad t \in [0,T], \quad (152)
\]
\[
\mu^*_{kt} \xrightarrow{\ast} \mu^*_t \quad \text{in} \quad M([0,R]), \quad \| \mu^*_{kt} \|_{M([0,R])} \leq \sup_{t \in [0,T]} \| \mu^*_t \|_{M([0,R])} \quad \text{a.e.} \quad t \in [0,T], \quad (153)
\]
\[
\varphi_{0k} \to \varphi_0 \quad \text{strongly in} \quad H^1([0,R]). \quad (154)
\]
We term \( \varphi_{*k}, \varphi_{1k}, \) and \( \varphi_{2k} \) the corresponding solutions of the initial-boundary value problems (67), (65) and (66), respectively. Since the coefficient \( \mu^*_{kt} \) and \( \nu_{kt} \) and the initial datum \( \varphi_{0k} \) are all smooth, then one can show that the solutions \( \varphi_{*k}, \varphi_{1k} \) and \( \varphi_{2k} \) are also smooth. This implies that the formal argument given at the previous paragraphs is completely justified and one gets
\[
2 \int_0^T \int_0^R \varphi_{1k}(t,x) d\nu_{tk}(x) dt + \int_0^T \int_0^R \varphi_{2k}(t,x) d\mu^*_{tk}(x) dt \\
\leq \| \varphi_{1k} \|^2_{L^2([0,T];H^1([0,R]))} - C(M,h) + C(\alpha_1, \alpha_2, M, h, T, R) \delta \\
\leq -\frac{C(M,h_*)}{2} \| \varphi_{1k} \|^2_{L^2([0,T];H^1([0,R]))} \quad \text{for every} \quad k
\]
provided that the constant \( \delta \) is sufficiently small. By standard arguments, one can show that
\[
\varphi_{*k} \to \varphi_*, \quad \varphi_{1k} \to \varphi_1, \quad \varphi_{2k} \to \varphi_2 \quad \text{strongly in} \quad L^2([0,T], C^0([0,R])).
\]
Also,
\[
\partial_x \varphi_{*k} \to \partial_x \varphi_*, \quad \partial_x \varphi_{1k} \to \partial_x \varphi_1, \quad \partial_x \varphi_{2k} \to \partial_x \varphi_2 \quad \text{weakly in} \quad L^2([0,T]\times[0,R]).
\]
In particular, the above convergence results imply that
\[
\| \varphi_{1k} \|^2_{L^2([0,T];H^1([0,R]))} \leq \liminf_{k \to +\infty} \| \varphi_{1k} \|^2_{L^2([0,T];H^1([0,R]))} \quad (156)
\]
and so
\[
\int_0^T \int_0^R \varphi_{1k} d\nu_{tk}(x) dt \rightarrow \int_0^T \int_0^R \varphi_1 d\nu_t(x) dt, \quad (157)
\]
\[
\int_0^T \int_0^R \varphi_{2k} d\mu^*_{tk}(x) dt \rightarrow \int_0^T \int_0^R \varphi_2 d\mu^*_t(x) dt.
\]
By passing to the limit in (155) we get
\[
2 \int_0^T \int_0^R \varphi_1(t,x) d\nu_t(x) dt + \int_0^T \int_0^R \varphi_2(t,x) d\mu^*_t(x) dt \\
\leq -\frac{C(M,h_*)}{2} \| \varphi_{1k} \|^2_{L^2([0,T];H^1([0,R]))} \quad (158)
\]
This establishes (68) and hence concludes the proof of Lemma 5.3.
Appendix A. Fundamental solutions of the heat equation. For the readers’ convenience, we collect in this appendix some basic facts about the fundamental solutions of the heat equation in one-dimensional, bounded domains. We refer to [7] for an extended discussion.

First, we fix an interval $]0, R]$ and we define the function $D : [0, +\infty[ \times ]0, R[ \times ]0, R[ \rightarrow \mathbb{R}$ by setting
\[
D(t, x, y) := \sum_{m = -\infty}^{m = +\infty} G(t, x + 2mR - y) + G(t, x + 2mR + y),
\]
where $G$ is the standard Green kernel
\[
G(t, x) := \frac{1}{2\sqrt{\pi t}} \exp \left( -\frac{x^2}{4t} \right).
\]

Note that, for every $u_0 \in L^2(]0, R[)$, the function
\[
u(t, x) := \int_0^R D(t, x, y) u_0(y) dy
\]
is a weak solution of the initial-boundary value problem
\[
\begin{cases}
\partial_t u = \partial_{xx}^2 u, \\
\partial_x u(t, 0) = \partial_x u(t, R) = 0, \\
u(0, x) = u_0(x).
\end{cases}
\]

Note furthermore that, owing to Duhamel’s principle, for every measurable, bounded function $\ell : [0, +\infty[ \times ]0, R[ \rightarrow \mathbb{R}$ the function
\[
u(t, x) := \int_0^R D(t, x, y) u_0(y) dy + \int_0^t \int_0^R D(t - s, x, y) \ell(s, y) dy ds
\]
is a weak solution of the initial-boundary value problem
\[
\begin{cases}
\partial_t u = \partial_{xx}^2 u + \ell(t, x), \\
\partial_x u(t, 0) = \partial_x u(t, R) = 0, \\
u(0, x) = u_0(x).
\end{cases}
\]

By direct computations, one can show that the kernel $D$ satisfies the following estimates:
\[
\|D(t, x, \cdot)\|_{L^\infty(]0, R[)} \leq \frac{C(R)}{\sqrt{t}} \quad \forall \ t > 0, x \in ]0, R[, \quad \tag{160}
\]
\[
\|D(t, x, \cdot)\|_{L^1(]0, R[)} \leq K \quad \forall \ t > 0, x \in ]0, R[, \quad \tag{161}
\]
\[
\|\partial_y D(t, x, \cdot)\|_{L^2(]0, R[)} \leq \frac{K}{t^{3/4}} \quad \forall \ t > 0, x \in ]0, R[, \quad \tag{162}
\]
\[
\|D(t, \cdot, y)\|_{L^1(]0, R[)} \leq K \quad \forall \ t > 0, y \in ]0, R[, \quad \tag{163}
\]
\[
\|\partial_y D(t, \cdot, y)\|_{L^2(]0, R[)} \leq \frac{K}{t^{3/4}} \quad \forall \ t > 0, y \in ]0, R[. \quad \tag{164}
\]

Finally, we define the kernel \( \hat{D} :[0, +\infty[ \times ]0, R[\times ]0, R[\rightarrow \mathbb{R} \) associated with the Dirichlet boundary conditions by setting

\[
\hat{D}(t, x, y) := \sum_{m=\pm \infty} G(t, x + 2mR - y) - G(t, x + 2mR + y),
\]

(165)

and we point out that

\[
\int_0^R \partial_x D(t, x, y)u_0(y)dy = \int_0^R \hat{D}(t, x, y)u'_0(y)dy,
\]

(166)

\[
\int_0^R \partial_{xx} D(t, x, y)u_0(y)dy = \int_0^R \partial_x \hat{D}(t, x, y)u'_0(y)dy
\]

for every continuously differential function \( u_0 \). By direct computations, we get the estimates

\[
\| \hat{D}(t, x, \cdot)\|_{L^\infty(0, R]} \leq \frac{C(R)}{\sqrt{t}} \quad \text{for every } t > 0, x \in ]0, R[, \tag{167}
\]

\[
\| \hat{D}(t, x, \cdot)\|_{L^1(0, R]} \leq K \quad \text{for every } t > 0, x \in ]0, R[, \tag{168}
\]

\[
\| \partial_x \hat{D}(t, x, \cdot)\|_{L^2(0, R]} \leq \frac{K}{t^{3/4}} \quad \text{for every } t > 0, x \in ]0, R[, \tag{169}
\]

\[
\| \hat{D}(t, \cdot, y)\|_{L^1(0, R]} \leq K \quad \text{for every } t > 0, y \in ]0, R[. \tag{170}
\]

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