On the Classification of Fifth Order Quasi-linear Non-constant Separant Scalar Evolution Equations of the KdV-type

Gülcan ÖZKUM\textsuperscript{1} and Ayşe Hümayra BİLGE\textsuperscript{2}

\textsuperscript{1}Science and Letter Faculty, Kocaeli University, Umuttepe Campus, Kocaeli, Turkey
\textsuperscript{2}Faculty of Sciences and Letters, Kadir Has University Istanbul, Turkey
e-mail: ozkumg@itu.edu.tr and ayse.bilge@khas.edu.tr

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Abstract

Fifth order, quasi-linear, non-constant separant evolution equations are of the form \( u_t = A \frac{\partial^5 u}{\partial x^5} + \tilde{B} \), where \( A \) and \( \tilde{B} \) are functions of \( x \), \( t \), \( u \) and of the derivatives of \( u \) with respect to \( x \) up to order 4. We use the existence of a “formal symmetry”, hence the existence of “canonical conservation laws” \( \rho_{(i)} \), \( i = -1, \ldots, 5 \) as an integrability test. We define an evolution equation to be of the KdV-Type, if all odd numbered canonical conserved densities are nontrivial. We prove that fifth order, quasi-linear, non-constant separant evolution equations of KdV type are polynomial in the function \( a = A^{1/5} \), \( a = (\alpha u_3^2 + \beta u_3 + \gamma)^{-1/2} \), where \( \alpha \), \( \beta \) and \( \gamma \) are functions of \( x \), \( t \), \( u \) and of the derivatives of \( u \) with respect to \( x \) up to order 2. We determine the \( u_2 \) dependency of \( a \) in terms of \( P = 4\alpha \gamma - \beta^2 > 0 \) and we give an explicit solution, showing that there are integrable fifth order non-polynomial evolution equations.

Keywords: Evolution equations, Integrability, Classification, Recursion operator, Formal symmetry

1 Introduction

In the literature, the term “integrable equations” refers to nonlinear equations that can either be transformed to a linear equation or solved by an inverse
spectral transformation. The methods to determine whether a given equation is likely to be integrable or not are called “integrability tests”. An extensive review of the integrability tests and a comprehensive list of integrable equation have been given in [1].

The Korteweg-deVries (KdV), Sawada-Kotera and Kaup hierarchies are well known hierarchies of integrable equations; the KdV hierarchy appears at all odd orders, while the Sawada-Kotera and Kaup hierarchies exist at odd orders that are not multiples of 3. Most of the constant separant third order integrable equations listed in [1] are transformable to an equation belonging to one of these hierarchies; it is in fact conjectured that the only exception is the Krivchev-Novikov equation [2]. The search for new hierarchies of integrable equations starting at order seven didn’t also give any positive result [3]; this situation was clarified for the polynomial and scale invariant evolution equations by the remarkable result of Wang and Sanders [4], stating that in the class of polynomial scale invariant equations, any integrable equation of order 7 or higher is a symmetry of a lower order equation. Thus, the open problems were reduced to the classifications at the third and fifth orders and to the classification of higher order non-polynomial equations.

In a series of papers we have applied the “formal symmetry” method of [1] to generic, non-polynomial equations; we have first proved that integrable evolution equations of order 7 and greater are quasi-linear [5]; then we have shown that they are polynomial in top three derivatives [6]. Motivated by these polynomiality results, we decided to investigate the structure of fifth order quasi-linear integrable equations with non-constant separant to see whether they would be polynomial and transformable to the constant separant case classified in [1].

In the present work we consider quasi-linear fifth order evolution equations

$$u_t = A \frac{\partial^5 u}{\partial x^5} + \tilde{B}$$

and we focus on the case where $\rho(3)$ is nontrivial, a case naturally distinguished by the classification algorithm using canonical conserved densities. Evolution equations with an unbroken sequence of canonical densities will be called of “Korteweg-deVries (KdV)-Type” as opposed to the Sawada-Kotera and Kaup equations [7] for which $\rho(3)$ is trivial.

We introduce our notation and give basic definitions in Section 2. The dependency on $u_4$ is easily determined as presented in Section 3.1 and it is shown that $u_t$ is quadratic in $u_4$. The form of the dependency on $u_3$ depends on whether the canonical conserved density $\rho(3)$ is nontrivial or not. In Section 3.2.1 we show that for the KdV-Type equations, the non-triviality of $\rho(3)$ implies that $A^{1/5} = a = (\alpha u_3^2 + \beta u_3 + \gamma)^{-1/2}$, where $\alpha$, $\beta$ and $\gamma$ are certain functions of $x$, $t$, $u$, $u_1$ and $u_2$. We use the conserved density conditions to determine the form of the evolution equation; in particular, in Section 3.2.3, we show that they are
polynomial in $a$ and in $u_3$, by proving that the coefficients of logarithmic and other transcendental functions of $a$ vanish by virtue of the partial differential equations satisfied by $\alpha$, $\beta$ and $\gamma$. The discriminant of the quadratic expression $\alpha u_3^2 + \beta u_3 + \gamma$ is denoted by $-P$ and assumed to be negative, to avoid real roots. The $u_2$ dependency of the functions $\alpha$, $\beta$ and $\gamma$ are determined in terms of $P$ but the $u_1$, $u$, $x$ and $t$ dependencies could not be obtained. In Section 3.2.4, we present a fifth order integrable equation, assuming that $u_t$ has no $x$, $t$, $u$, $u_1$ dependency, hence we present a new integrable equation in this class.

As discussed in Section 4, these results show that at the fifth order there are non-constant separant integrable equations which are possibly higher order analogues of the essentially nonlinear class of equations at the third order [8]. Furthermore, as part of ongoing work, we have shown that there are candidates of integrable equations of the form $u_t = a^{2k+1}u_{2k+1} + \ldots$ at orders $k = 3, 4, 5, 6$, which supports the existence of a new hierarchy, which nevertheless is expected to be transformable to the KdV equation by a Miura type transformation involving third order derivatives.

2 Preliminaries

2.1 Notation

We shall work with evolution equations in one space dimension $x$ and use the following notation for the partial derivatives of the dependent variable $u$.

$$u_t = \frac{\partial u}{\partial t}, \quad u_1 = \frac{\partial u}{\partial x} = u_x, \quad u_k = u_{x\ldots x}^{k\text{ times}} = \frac{\partial^k u}{\partial x^k}.$$

In the following we shall assume that all functions are sufficiently differentiable. Infinitely differentiable functions of $x$, $t$, $u$ and of the derivatives of $u$ up to an arbitrary but finite order are called "differential functions" [9]. We shall use subscripts for partial derivatives with respect to the derivatives of $u$, as below

$$\varphi_k = \frac{\partial \varphi}{\partial u_k}, \quad \varphi_{kj} = \frac{\partial^2 \varphi}{\partial u_k \partial u_j}.$$

The labels of functions will be denoted by indices in parenthesis, such as $\sigma_{(i)}$ or $\rho_{(i)}$.

A scalar evolution equation of order $m$ is of the form

$$u_t = F(x, t, u, u_1, \ldots, u_m). \quad (1)$$

The total derivative operators with respect to $x$ and $t$ are respectively denoted by $D$ and $D_t$. That is, if $\varphi$ is a differentiable function of $x$, $t$, $u$ and the partial
derivatives of $u$ with respect to $x$ up to order $k$, then

$$D\varphi = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u} u_1 + \frac{\partial \varphi}{\partial u_1} u_2 + \ldots + \frac{\partial \varphi}{\partial u_k} u_{k+1} = \varphi_x + \varphi_0 u_1 + \cdots + \varphi_k u_{k+1}. \quad (2)$$

If $u(x,t)$ satisfies (1) and $\varphi$ is as above, then

$$D_t\varphi = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial u} F + \frac{\partial \varphi}{\partial u_1} DF + \ldots + \frac{\partial \varphi}{\partial u_k} D^k F = \varphi_t + \varphi_0 F + \cdots + \varphi_k D^k F. \quad (3)$$

### 2.2 The “Level Grading”

The notions of “level grading” and level homogeneity introduced in [6] are analogues of scaling and scale homogeneity for polynomials; in the level grading context, the “level above a base level $k$” reflects the number of differentiations applied to a function that depends on the derivatives of order at most $k$. The crucial property of the level grading is its invariance under integrations by parts. This property implies that “top level” terms of a conserved density $\rho$ will give top level terms in its time derivative $D_t\rho$, hence we may omit lower level terms and still track correctly higher level integrability conditions. The “level of $u_n$ above the base level $k$” is defined as $n - k$ and clearly it is equal to the number of differentiations above $k$. The level of a monomial $u_{n_1}^{b_1} \ldots u_{n_m}^{b_m}$ is then defined as $b_1(n_1 - k) + \cdots + b_m(n_m - k)$. For example, if the base level is $k = 4$, $u_5$ has level 1, while if the base level is $k = 3$, $u_5$ and $u_2^2$ are both monomials of level 2, hence, for example, the right hand side of (14) is a sum of level homogeneous expressions of levels 2, 1 and 0 above the base level $k = 3$. In the present work, our results are based on explicit computations including derivatives of all orders, however we still indicate level related results whenever necessary.

### 2.3 Formal Symmetries

The Frechet derivative of a differential function $F$ is defined by

$$F_s = \sum_{i=0}^{m} \frac{\partial F}{\partial u_i} D^i = \sum_{i=0}^{m} F_i D^i. \quad (4)$$

Let $\sigma = \sigma(x,t,u,\ldots,u_n)$ and $\rho = \rho(x,t,u,\ldots,u_n)$ be differential functions. If

$$\sigma_t = F_s \sigma, \quad (5)$$

then $\sigma$ is called a “symmetry”. If one can find a differential function $\eta$ such that

$$\rho_t = D_\eta \quad (6)$$

then $\rho$ is called a “conserved density”. Let $R$ be a linear operator. If for any symmetry $\sigma$, $R\sigma$ is also a symmetry then $R$ is called a “recursion operator”. If $R\sigma$ is a symmetry, then the equation

$$ (R\sigma)_t = F_*(R\sigma) \tag{7} $$

gives $(R\sigma)_t = R_t \sigma + RF_\sigma = F_* R\sigma$, hence

$$ (R_t + [R,F_\sigma])\sigma = 0. \tag{8} $$

Therefore, if $R$ satisfies the operator equation (8), it certainly satisfies (7). Note however that in this case there is no guarantee that the quantity $R\sigma$ which satisfies the symmetry equation (7) is a local function. This suggests that if we would solve the operator equation (8), we could as well work with a pseudo-differential operator $R$. In addition, we may only be interested in solving the operator equation (8) up to a finite order. A pseudo-differential operator that solves (8) up to a finite order is called a “formal symmetry”. The existence of a formal symmetry has been proposed as an integrability test in [1]. This approach has the advantage that the existence of the recursion operator, hence the classification of integrable equations is insensitive to the order and the functional form of the operator $R$. We have used the symbolic programming language REDUCE in order to compute the partial differential equations implied by these conserved density conditions.

### 3 The form of integrable equations

The criterion of integrability by the existence of a formal symmetry [1] implies the existence of certain conserved densities $\rho_{(i)}$, $i = -1, 0, 1, \ldots$. It is well known that for any order $m$, if $F_m = \frac{\partial F}{\partial u_m}$, $F_{m-1} = \frac{\partial F}{\partial u_{m-1}}$ then $\rho_{(-1)} = F_m^{-1/m}$ and $\rho_{(0)} = F_{m-1}/F_m$ are always conserved densities. The expression of the conserved densities $\rho_{(i)}$, $i = 1, \ldots, 5$ are given in Appendix A[10].

#### 3.1 Dependency on $u_4$

We assume that the evolution equation is of the form

$$ u_t = a^5 u_5 + \tilde{B}, \tag{9} $$

where $a$ and $\tilde{B}$ are functions of $x$, $t$, $u$, $u_i$, $i = 1, \ldots, 4$. Substituting (9) in the conserved densities given in Appendix A, we can obtain explicitly the form of the canonical conserved densities $\rho_{(1)}$ and $\rho_{(2)}$. However the conserved densities $\rho_{(3)}$, $\rho_{(4)}$ and $\rho_{(5)}$ involve the expressions $\sigma_{(-1)}$, $\sigma_{(0)}$ and $\sigma_{(1)}$ defined
Differentiating this expression with respect to $u$ by $D_{t} \rho_{(i)} = D_{t} \sigma_{(i)}$, hence they can be explicitly determined only after the lower order conserved density conditions are solved. Note that, since $\sigma_{(3)}$, $\sigma_{(4)}$ and $\sigma_{(5)}$ are not used anywhere else, we may omit the total derivatives in $\rho_{(i)}$, for $i = 3, 4, 5$. In practice we solve the lower order conserved density conditions up to a certain order, replace the results in the other conserved densities and proceed iteratively. After substitutions and integrations by parts, the conserved densities are of the form given below.

$$
\begin{align*}
\rho_{(-1)} &= a^{-1}, \\
\rho_{(0)} &= 5a^{4}a_{4}u_{5} + \tilde{B}_{4}, \\
\rho_{(1)} &= \tilde{P}_{(1)}u_{5}^{2} + \tilde{Q}_{(1)}u_{5} + \tilde{R}_{(1)}, \\
\rho_{(2)} &= \tilde{M}(2)u_{5}^{3} + \tilde{P}(2)u_{5}^{2} + \tilde{Q}(2)u_{5} + \tilde{R}(2), \\
\rho_{(3)} &= \tilde{P}(3)u_{6}^{2} + \tilde{Q}(3)u_{6}^{2} + \tilde{R}(3)u_{6}^{2} + \tilde{S}(3)u_{5}^{2} + \tilde{T}(3)u_{5} + \tilde{U}(3), \\
\rho_{(4)} &= \tilde{M}(4)u_{6}^{5} + \tilde{P}(4)u_{6}^{4} + \tilde{Q}(4)u_{6}^{3} + \tilde{R}(4)u_{6}^{3} + \tilde{S}(4)u_{5}^{3} + \tilde{T}(4)u_{5}^{3} + \tilde{U}(4)u_{5} \\
&+ \tilde{V}(4), \\
\rho_{(5)} &= \tilde{P}(5)u_{7}^{2} + \tilde{Q}(5)u_{6}^{3} + \tilde{R}(5)u_{6}^{2}u_{5} + \tilde{S}(5)u_{5}^{2} + \tilde{T}(5)u_{5}^{2} + \tilde{U}(5)u_{5}^{2} + \tilde{V}(5)u_{5}^{2} \\
&+ \tilde{W}(5)u_{5}^{2} + \tilde{X}(5)u_{5} + \tilde{Y}(5). \tag{10}
\end{align*}
$$

In the equations above $\tilde{B}_{4} = \frac{\partial \tilde{B}_{4}}{\partial a_{4}}$ and the coefficients $\tilde{P}_{(i)}$, $\tilde{Q}_{(i)}$, $\tilde{R}_{(i)}$ ($i = 1, \ldots, 5$), $\tilde{S}_{(i)}$, $\tilde{T}_{(i)}$, $\tilde{U}_{(i)}$ ($i = 3, 4, 5$), $\tilde{M}(2)$, $\tilde{M}(4)$, $\tilde{V}(4)$, $\tilde{V}(5)$, $\tilde{W}(5)$, $\tilde{X}(5)$ and $\tilde{Y}(5)$ are functions of $x$, $t$, $u$, $u_{1}$, $u_{2}$, $u_{3}$, $u_{4}$ that can be computed from the expressions of the canonical densities given in the Appendix A. We omit their explicit expressions except for

$$
\tilde{P}_{(1)} = -\frac{a_{4}^{2}}{a}, \quad \tilde{P}_{(3)} = \frac{3}{2}a[a_{44}a - 4a_{4}^{2}]. \tag{11}
$$

In [5], Corollary 4.3, it has been shown that conserved densities of order $n$ larger than the order of the evolution equation should be at most quadratic in their highest derivatives; thus in particular, we should have $\tilde{M}(2) = 0$.

The coefficient of $u_{7}^{2}u_{5}$ in $D_{t} \rho_{1}$ gives

$$
10a_{4}a^{3}(a_{44}a - 4a_{4}^{2}) = 0. \tag{12}
$$

Thus we should either have $a_{4} = 0$, or if $a_{4} \neq 0$, then $a_{44}a - 4a_{4}^{2} = 0$. If $a_{4} = 0$, then the coefficient of $u_{5}^{3}$ in $D_{t} \rho_{(-1)}$ is an expression that involves $a$ and $\tilde{B}_{4}$. Differentiating this expression with respect to $u_{4}$ twice we obtain $\tilde{B}_{444} = 0$.

On the other hand if $a_{44}a - 4a_{4}^{2} = 0$, then, $\tilde{P}_{(5)} = -20a_{4}^{2}u_{5}^{2}$ and $\tilde{Q}_{(5)} = \frac{880}{3}a_{4}^{3}u_{5}^{2}$ and the coefficient of $u_{5}^{2}u_{6}u_{5}$ gives $a_{4} = 0$. When $a_{4} = 0$, in $\rho_{(4)}$, $M(4) = \tilde{P}(4) = \tilde{Q}(4) = \tilde{R}(4) = 0$. By the Corollary 4.3 of [5], $\rho_{(4)}$ should be
quadratic in $u_5$, hence $\tilde{S}(4)$ should be zero. This condition gives again $\tilde{B}_{444} = 0$. Thus in either case, the $u_4$ dependency is determined as below.

$$a_4 = 0, \quad \tilde{B}_{444} = 0. \quad (13)$$

### 3.2 Dependency on $u_3$

As a result of (13) the evolution equation reduces to the form

$$u_t = a^5 u_5 + B u_4^2 + C u_4 + G \quad (14)$$

where $a$, $B$, $C$ and $G$ are now functions of $x$, $t$, $u$, $u_1$, $u_2$ and $u_3$.

With this form of $F$ the canonical conserved densities reduce to the form below. Although the explicit expressions of the conserved densities have been obtained, we use the generic expressions given below for computational efficiency.

$$\rho_{(-1)} = a^{-1},$$
$$\rho_{(1)} = P_{(1)} u_4^2 + Q_{(1)} u_4 + R_{(1)},$$
$$\rho_{(3)} = (P_{(2)} u_5^2 + Q_{(2)} u_4^4) + R_{(2)} u_4^3 + S_{(2)} u_4^2 + T_{(2)} u_4 + U_2,$$
$$\rho_{(5)} = (P_{(3)} u_6^2 + Q_{(3)} u_5^4 + R_{(3)} u_5^3 u_4^2 + S_{(3)} u_4^6) + (T_{(3)} u_5^2 u_4 + U_{(3)} u_4^5)$$
$$+ (V_{(3)} u_4^4 + W_{(3)} u_2^2) + X_{(3)} u_4^3 + Y_{(3)} u_4^2 + Z_{(3)} u_4 + M_{(3)} \quad (15)$$

The coefficients $P_{i(j)}$, $Q_{i(j)}$, $R_{i(j)}$ ($i = 1, 2, 3$), $S_{i(j)}$, $T_{i(j)}$, $U_{i(j)}$ ($i = 2, 3$), $V_{(3)}$, $W_{(3)}$, $X_{(3)}$, $Y_{(3)}$, $Z_{(3)}$ and $M_{(3)}$ are functions of $x$, $t$, $u$, $u_1$, $u_2$, $u_3$.

**Definition 3.1.** An evolution equation $u_t = F(x, t, u, \ldots, u_m)$ of odd order $m$ is called of “KdV-Type”, if it admits an infinite sequence of non-trivial conserved densities at all orders.

In terms of canonical densities the condition above means that all the odd numbered canonical densities $\rho_{(2k+1)}$, in particular, the canonical conserved density $\rho_{(3)}$ is non-trivial. Note that these conserved densities are respectively of levels 0, 2, 4 and 6 above the base level 3, hence their total derivatives with respect to time will be respectively of levels 5, 7, 9 and 11.

We shall use the conditions obtained by equating to zero the coefficients of top 3 levels terms in the conserved density conditions as listed in Table 1. The coefficients of these monomials denoted as $K_{ijk}$, where the first index $i$ denotes the label of the conserved density, the second index $j$ denotes the level of that term (as a hexadecimal number) and $k$ counts the number of such conditions. For example, $K_{3b4}$ is the 4th condition obtained as a coefficient of a term of level 11 ($b$ as hexadecimal) in $D_t \rho_{(3)}$. 

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Table 1: The conserved density conditions for $\rho^{(i)}$ getting from the coefficients of terms of top 3 levels.

| $\rho^{(i)}$ | Top level | Top-1 level | Top-2 level |
|---------------|------------|-------------|-------------|
| $\rho(-1)$    | $u_5^2 u_4 - K_{051}$, | $u_5^3 - K_{041}$, | $u_4^3 - K_{031}$ |
|               | $u_5^4 - K_{052}$         | $u_4^4 - K_{042}$         |             |
| $\rho(1)$     | $u_6^2 u_4 - K_{171}$,   | $u_6^2 - K_{161}$,       |             |
|               | $u_6^3 u_4 - K_{172}$,   | $u_6^3 - K_{162}$,       |             |
|               | $u_6^3 u_5^3 - K_{173}$, | $u_6^3 u_4^3 - K_{163}$, | $u_6^3 u_4 - K_{151}$, |
|               | $u_6^4 - K_{174}$        | $u_6^4 - K_{164}$        | $u_6^4 - K_{152}$ |
| $\rho(2)$     | $u_7^2 u_4 - K_{291}$,   | $u_7^2 - K_{281}$,       |             |
|               | $u_7^3 - K_{292}$        |             |             |
|               | $u_7^3 u_5 u_4 - K_{293}$| $u_7^3 u_5 u_4 - K_{282}$ | $u_7^3 u_4 - K_{271}$ |
|               | $u_7^4 u_3^2 - K_{294}$, | $u_7^4 u_3^2 - K_{283}$, |             |
|               | $u_7^4 u_4 - K_{295}$,   | $u_7^4 - K_{284}$,       |             |
|               | $u_7^4 u_5^3 - K_{296}$, | $u_7^4 u_5^3 - K_{285}$, | $u_7^4 u_4 - K_{272}$ |
|               | $u_7^5 u_4^2 - K_{297}$, | $u_7^5 u_4^2 - K_{286}$, | $u_7^5 u_4^2 - K_{273}$ |
|               | $u_7^6 - K_{298}$        | $u_7^6 - K_{287}$        | $u_7^6 - K_{274}$ |
| $\rho(3)$     | $u_8^2 u_4 - K_{381}$,   | $u_8^2 - K_{371}$,       |             |
|               | $u_8^3 u_5 - K_{392}$,   |             |             |
|               | $u_8^3 u_5 u_4 - K_{393}$|             |             |
|               | $u_8^4 u_3^2 - K_{394}$, | $u_8^4 u_3^2 - K_{393}$, |             |
|               | $u_8^4 u_4 - K_{395}$,   | $u_8^4 - K_{392}$        |             |
|               | $u_8^5 u_3^2 u_4 - K_{396}$| $u_8^5 u_3^2 u_4 - K_{395}$, | $u_8^5 u_5 u_4 - K_{393}$ |
|               | $u_8^5 u_4^2 - K_{397}$, | $u_8^5 u_4^2 - K_{396}$, |             |
|               | $u_8^6 u_4^3 - K_{398}$, |             |             |

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3.2.1 Top level computations

We start by collecting those top level equations that will determine $B$ and give a third order ordinary differential equation for the $u_3$ derivative of $a$. Although the explicit forms of the $P_{(i)}$’s are known, it will be more convenient to consider them as new unknown functions.

From the condition $K_{051}$ we get

$$K_{051} : a_{333}a^6 - 7a_{33}a_3a_5 - \frac{4}{5} a_{33}aB + 8a_3^3a^4 - \frac{8}{5} a_3^2B = 0. \quad (16)$$

Recall that as we are studying KdV-Type equations we are assuming that $\rho^{(3)}$ is nontrivial, hence $P^{(2)} \neq 0$. We then use the equations

$$K_{361} : 35a_3a^4P_{(3)} - 5(P_{(3)})_3 a^5 + 4BP_{(3)} = 0,$$
$$K_{171} : 15a_3a^4P_{(1)} - 5(P_{(1)})_3 a^5 + 4BP_{(1)} = 0,$$
$$K_{291} : 25a_3a^4P_{(2)} - 5(P_{(2)})_3 a^5 + 4BP_{(2)} = 0,$$
$$K_{292} : 20a_3a^4P_{(2)} - 5(P_{(2)})_3 a^5 + 6BP_{(2)} = 0. \quad (17)$$

Taking the difference of the last two equations, we solve $B$ as

$$B = \frac{5}{2} a_3a^4. \quad (18)$$

From the remaining equations in (17) we find that the derivatives of $P_{(i)}$ and $a$ with respect to $u_3$ are proportional, and we solve them as below

$$P_{(1)} = P_{(1o)} a^5, \quad P_{(2)} = P_{(2o)} a^7, \quad P_{(3)} = P_{(3o)} a^9. \quad (19)$$

where $P_{(io)}$ are functions of $x, t, u, u_1$ and $u_2$. Finally, $K_{051}$ gives

$$a_{333}a^2 - 9a_{33}a_3a + 12a_3^3 = 0. \quad (20)$$

Assuming $a = Z^k$ and substituting in (20), we see that for $k = 1/2$, $Z_{333} = 0$, hence $a$ is of the form

$$a = (\alpha u_3^2 + \beta u_3 + \gamma)^{-\frac{1}{2}} \quad (21)$$

where $\alpha, \beta$ and $\gamma$ are functions of $x, t, u, u_1$ and $u_2$. From (21) we can see that

$$a_3 = -\frac{1}{2} a^3(2\alpha u_3 + \beta), \quad a_i = -\frac{1}{2} a^3(\alpha_i u_3^2 + \beta_i u_3 + \gamma_i), \quad i = x, 0, 1, 2. \quad (22)$$

If we require $a$ to be real for every $u_3$, we see that

$$\beta^2 - 4\alpha\gamma < 0, \quad \text{and} \quad \alpha > 0$$

and we define

$$P = 4\alpha\gamma - \beta^2. \quad (23)$$

From the expressions, $K_{293}, K_{362}, K_{363}$ and $K_{368}$, we solve respectively $Q_{(2)}, Q_{(3)}, R_{(3)}$ and $S_{(3)}$ but we omit the explicit expressions here. The rest of the top level conditions are automatically satisfied.
3.2.2 Top-1 level computations

By computing the coefficients of \( u_4^2 \), \( u_5^2 \) and \( u_6^2 \) respectively in \( \rho(1) \), \( \rho(3) \) and \( \rho(5) \), we find that
\[
\begin{align*}
P_{(1o)} &= \frac{1}{8} P, \\
P_{(2o)} &= -\frac{3}{8} P, \\
P_{(3o)} &= \frac{5}{2} P.
\end{align*}
\] (24)

From the (Top-1) level condition \( K_{281} \), \( C \) is obtained as,
\[
C = \frac{5DP}{2}a^5 + 5(Da - a_3u_4)a^4.
\] (25)

We can again solve \( R_{(2)} \), \( T_{(3)} \) and \( V_{(3)} \) respectively from \( K_{282} \), \( K_{3a2} \) and \( K_{3a6} \) but we omit these expressions.

Then from \( K_{162} \), the coefficients of \( u_3^2 \) we obtain
\[
\alpha = \alpha_{(0)}P^2,
\] (26)

where \( \alpha_{(0)} = \alpha_{(0)}(u,u_1) \), and the coefficient of \( u_3 \) gives
\[
\frac{\partial \beta}{\partial u_2} - \frac{1}{2} \beta \frac{\partial P}{\partial u_2} P = 3\alpha_{(0)}P \left( \frac{\partial P}{\partial u_1}u_2 + \frac{\partial P}{\partial u}u_1 + \frac{\partial P}{\partial x} \right).
\] (27)

The remain conditions of the (Top-1) level conditions are satisfied automatically.

3.2.3 Top-2 level computations

All of the (Top-2) level conditions except for \( K_{292} \) are identically satisfied. From the condition \( K_{292} \) we get a second order ordinary differential equation for the \( u_3 \) derivative of \( G \). The form of this equation suggests that we have to multiply it by \( a^{-1} \) in order to integrate it. After substituting
\[
(2\alpha u_3 + \beta) = -2a_3 a^{-3},
\] (28)

the resulting expression is
\[
a^{-3}G_{33} - 3a^{-4}a_3G_3 + \sum_{i=1}^{4} \left( \lambda_{(i)} u_3 + \mu_{(i)} \right) a^{2i} = 0,
\] (29)

where the \( \lambda_{(i)} \)'s and \( \mu_{(i)} \)'s depend on \( x, t, u, u_1 \) and \( u_2 \). In order to integrate this equation, we need to know the integrals of various powers of \( a \). Note that \( a^{-2} \) is an irreducible quadratic polynomial in \( u_3 \), hence the integrals of \( a^2 u_3 \) and \( a^2 \) will involve logarithms and arctangent functions. However, the coefficients of these terms will be zero, hence, instead of writing these integrals explicitly,
we just indicate these as new functions. These integration formulas are given below.

\[
\int a \, du_3 = \alpha^{-1/2} \ln \left[ P^{-1/2} \left( 2\alpha^{1/2} a^{-1} + 2\alpha u_3 + \beta \right) \right],
\]

\[
\int a u_3 \, du_3 = \frac{1}{\alpha a} - \frac{\beta}{2\alpha} \int a \, du_3,
\]

\[
\int a^2 \, du_3 = \psi = 2P^{-1/2} \tan^{-1} \left( P^{-1/2} (2\alpha u_3 + \beta) \right),
\]

\[
\int a^2 u_3 \, du_3 = \varphi = -\frac{1}{\alpha} \ln(a) - \frac{\beta}{2\alpha} \int a^2 \, du_3,
\]

\[
\int a^3 \, du_3 = \frac{2}{P} (2\alpha u_3 + \beta) a. \tag{30}
\]

The integrals of \(a^k\) and \(a^k u_3\) for \(k \geq 3\) are evaluated using the iterative formulas given below.

\[
\int a^k \, du_3 = \frac{2}{k-2} P^{1/2} (2\alpha u_3 + \beta) a^{k-2} + \frac{4(k-3)}{k-2} \frac{1}{P} \alpha \int a^{k-2} \, du_3, \tag{31}
\]

\[
\int a^k u_3 \, du_3 = \frac{1}{2\alpha} \left( -\frac{k}{2} + 1 \right)^{-1} a^{k-2} - \frac{\beta}{2\alpha} \int a^k \, du_3. \tag{32}
\]

The resulting integral is of the form

\[
Ga^3 + \kappa(1) \psi + \kappa(2) \varphi + \sum_{i=1}^{3} \left( \nu(i) u_3 + \eta(i) \right) a^{2i} = \chi, \tag{33}
\]

where the \(\kappa(i)\)'s, \(\nu(i)\)'s, \(\eta(i)\)'s and \(\chi\) depend on \(x, t, u, u_1\) and \(u_2\). The coefficients of the functions \(\psi\) and \(\varphi\) defined in Eqn.(33) are zero, after we substitute the equations for \(\alpha\) and \(\beta\), hence actually

\[
\kappa(1) = \kappa(2) = 0.
\]

Multiplying (33) by \(a^3\), we have

\[
G_3 a^3 = \chi a^3 - \sum_{i=1}^{3} \left( \nu(i) u_3 + \eta(i) \right) a^{2i+3}. \tag{34}
\]

Integrating one more time, \(G\) is obtained as

\[
G = \left( G_{(1)} u_3 + G_{(2)} \right) a^7 + \left( G_{(3)} u_3 + G_{(4)} \right) a^5 + \left( G_{(5)} u_3 + G_{(6)} \right) a^3 + \left( G_{(7)} u_3 + G_{(8)} \right) a + G_{(9)}. \tag{35}
\]

The coefficients are found explicitly from \(K_{292}\), but we omit these expressions. It follows that for \(P \neq 0\) integrable evolution equations with nontrivial \(\rho(3)\) are of the form

\[
u_t = a^5 u_5 + \frac{5}{2} a_3 a^4 u_4^2 + \left( \frac{5}{2} \frac{DP}{\rho} a^5 + 5 (Da - a_3 u_4) a^4 \right) u_4 + \left( G_{(1)} u_3 + G_{(2)} \right) a^7.
\]
\[ + \left( G(3)u_3 + G(4) \right) a^5 + \left( G(5)u_3 + G(6) \right) a^3 + \left( G(7)u_3 + G(8) \right) a + G(9), \] (36)

where \( a \) and \( P \) are given respectively by (21) and (23), the \( G(i), i = 1, \ldots, 6 \) are certain functions of \( \alpha, \beta \) and \( \gamma, G(7) = G(7o)P, \) where \( G(70) \) is a function of \( x, t, u \) and \( u_1 \) and \( G(8), G(9) \) are functions of \( x, t, u, u_1 \) and \( u_2. \)

**Proposition 3.1.** Let \( u_t = a^5u_5 + \tilde{B} \) be an evolution equation where \( a \) and \( \tilde{B} \) depend on \( x, t, u, u_i, i = 1, \ldots, 4. \) If the canonical densities \( \rho(i), i = -1, \ldots, 5 \) are conserved, and \( \frac{\partial^2 \rho(-1)}{\partial u_3^2} \) and \( \frac{\partial^2 \rho(3)}{\partial u_3^2} \) are nonzero, then integrable evolution equations are of the form in (36), where \( a \) and \( P \) are given respectively by (21) and (23), the \( G(i), i = 1, \ldots, 6 \) are certain functions of \( \alpha, \beta \) and \( \gamma, G(7) = G(7o)P, \) where \( G(70) \) is a function of \( x, t, u \) and \( u_1 \) and \( G(8), G(9) \) are functions of \( x, t, u, u_1 \) and \( u_2. \)

3.2.4 A new solution

The equation (36) obtained above gives the form of the candidates for integrable fifth order quasi-linear equations. In this section we assume that \( a \) is independent of \( x, u \) and \( u_1 \) and obtain an equation for which all of the canonical densities \( \rho(i), i = -1, \ldots, 5 \) are conserved.

**Proposition 3.2.** Let \( u_t = a^5u_5 + \tilde{B} \) be an evolution equation where \( a \) and \( \tilde{B} \) depend on \( u_i, i = 2, \ldots, 4. \) If the canonical densities \( \rho(i), i = -1, \ldots, 5 \) are conserved, and \( \frac{\partial^2 \rho(-1)}{\partial u_3^2} \) and \( \frac{\partial^2 \rho(3)}{\partial u_3^2} \) are nonzero, then

\[
 u_t = a^5u_5 + Bu_4^2 + Cu_4 + (G(1)a^7 + G(3)a^5 + G(5)a^3 + G(7)a)u_3 + G(9), \] (37)

where

\[
 a = (\alpha u_3^2 + \beta u_3 + \gamma)^{-1/2}, \quad \alpha = \alpha_0 P^2, \quad \beta = 0, \quad \gamma = \frac{P + \beta^2}{4\alpha},
\]

\[
 P = (\mu u_3 + \nu u_2 + \kappa)^{-1}, \quad q = 4\mu \kappa - \nu^2,
\]

\[
 B = -\frac{5}{2} \alpha_0 P^2 a^7 u_3, \quad C = \left(-\frac{15}{8} \alpha_0^{-1} a^2 + \frac{5}{2} P \right) (2\mu u_2 + \nu) a^5 u_3,
\]

\[
 G(1) = \frac{45}{128} \alpha_0^{-3} P^{-3} (qP - 4\mu), \quad G(3) = \frac{27}{16} \alpha_0^{-2} P^{-2} \left(-qP + \frac{40}{9} \mu \right),
\]

\[
 G(5) = \alpha_0^{-1} P^{-1} \left(qP - \frac{15}{2} \mu \right), \quad G(7) = G(70)P, \quad G(9) = G(90),
\]

and \( \mu, \nu, \kappa, \alpha_0, G(70) \) and \( G(90) \) are constants.

It has been checked that all even canonical densities are trivial and all odd canonical densities up to \( \rho(5) \) are conserved. Thus the equation above is integrable in the sense of admitting a formal symmetry [1]. We have observed that all the even canonical densities up to \( \rho(4) \) are trivial while the odd canonical densities are nontrivial.
4 Discussion of the Results

In the course of this study, the existence of non-polynomial integrable equations came as surprise, because we were expecting that the polynomiality in top three derivatives would be generalized to lower orders. We then remarked that this class seems to be related to the essentially nonlinear class of third order equations [1],

\[ u_t = H = (A_1 u_3^2 + A_2 u_3 + A_3)^{-1/2}(2A_1 u_3 + A_2) + A_4, \]

where \( A_i = A_i(x, u, u_1, u_2) \). This similarity is suggested by the fact that the conserved density \( \rho(-1) \) is the same for both equations. Note that a scaling of the \( A_i \)'s result in a scaling of \( H \); replacing \( A_i \) by \( (2/P)^{1/2} A_i, i = 1, 2, 3 \) one can obtain

\[ \frac{\partial H}{\partial u_3} = a^3. \]

Hence \( \rho(-1) \) is the same for the essentially nonlinear third order class and the fifth order quasi-linear non-constant separant class.

Then, in order to see whether these equations were part of a hierarchy, we studied the top level terms of level homogeneous odd order equations up to order 13; surprisingly we have seen that at each order, these flows were polynomial in \( u_k, k \geq 3 \) of the form \( u_t = a^m u_m + \ldots \), i.e, with separant \( a^m \).

Finally we note that the classification of essentially nonlinear class of equations is discussed in [8] where it is suggested to use a formulation where the dependent variable is replaced by the canonical density \( \rho(-1) \). With this method, the classification of the essentially nonlinear equations is partly completed in [11]; in particular the classification of the equations with \( A_1 = 0 \) corresponding to \( \alpha(0) = 0 \) in our case is obtained, while the case \( A_1 \neq 0 \) is still open.

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Appendix A. Canonical Densities

\[ \rho_{(-1)} = a^{-1}, \]  
\[ \rho_{(0)} = \frac{F_4}{F_5}, \]  
\[ \rho_{(1)} = 2(D^2a)^{-1} - (Da)^2a^{-1} + 2(Da)a^{-5}F_4 - \frac{2}{5}(DF_4)a^{-4} + \frac{1}{5}a^{-4}F_3 \]  
\[ -\frac{2}{25}a^{-9}F_4^2, \]  
\[ \rho_{(2)} = -2(D^2a)a^{-4}F_4 + 12(Da)^2a^{-5}F_4 - 4(Da)(DF_4)a^{-4} \]  
\[ + 3(Da)a^{-4}F_3 - \frac{12}{5}(Da)a^{-9}F_4^2 - \frac{3}{5}(DF_3)a^{-3} + \frac{2}{5}(D^2F_4)a^{-3} \]  
\[ + \frac{12}{25}(DF_4)a^{-8}F_4 + \frac{2}{5}a^{-3}F_2 - \frac{6}{25}a^{-8}F_3F_4 + \frac{8}{125}a^{-13}F_4^3, \]  
\[ \rho_{(3)} = -\frac{8}{5}(D^4a)a^2 - \frac{16}{5}(D^3a)(Da)a - (D^3a)a^{-3}F_4 \]  
\[ -\frac{12}{5}(D^2a)^2a + \frac{12}{5}(D^2a)(Da)^2 + 2(D^2a)(Da)a^{-4}F_4 \]  
\[ -\frac{7}{5}(D^2a)(DF_4)a^{-3} - \frac{9}{5}(D^2a)a^{-3}F_3 + \frac{28}{25}(D^2a)a^{-8}F_4^2 \]  
\[ -\frac{3}{5}(Da)^4a^{-1} + 2(Da)^3a^{-5}F_4 + \frac{14}{5}(Da)^2(DF_4)a^{-4} \]  
\[ + \frac{48}{5}(Da)^2a^{-4}F_3 - \frac{371}{25}(Da)^2a^{-9}F_4^2 - \frac{14}{5}(Da)(DF_3)a^{-3} \]  
\[ -\frac{7}{5}(Da)(D^2F_4)a^{-3} + \frac{126}{25}(Da)(DF_4)a^{-8}F_4 + 3(Da)a^{-3}F_2 \]  
\[ -\frac{21}{5}(Da)a^{-8}F_3F_4 + \frac{42}{25}(Da)a^{-13}F_4^3 - \frac{3}{5}(DF_2)a^{-2} \]  
\[ + \frac{1}{5}(D^2F_3)a^{-2} + \frac{9}{25}(DF_3)a^{-7}F_4 + \frac{1}{5}(D^3F_4)a^{-2} \]  
\[ -\frac{4}{25}(D^2F_4)a^{-7}F_4 - \frac{7}{25}(DF_4)^2a^{-7} + \frac{12}{25}(DF_4)a^{-7}F_3 \]  
\[ -\frac{42}{125}(DF_4)a^{-12}F_4^2 - \frac{3}{5}a^{-1}\sigma(-1) + \frac{3}{5}a^{-2}F_1 - \frac{6}{25}a^{-7}F_2F_4 \]  
\[ -\frac{3}{25}a^{-7}F_3^2 + \frac{21}{125}a^{-12}F_3F_4^2 - \frac{21}{625}a^{-17}F_4^3. \]
\[
\rho^{(i)} = \frac{4}{5} (D^4a) a^{-2} F_4 - \frac{66}{5} (D^3a) (Da) a^{-3} F_4 + \frac{16}{5} (D^3a) (DF_4) a^{-2} \\
-(D^3a) a^{-2} F_3 + \frac{4}{5} (D^3a) a^{-7} F_4^2 - \frac{42}{5} (D^2a)^2 a^{-3} F_4 \\
+ \frac{318}{5} (D^2a) (Da)^2 a^{-4} F_4 - \frac{138}{5} (D^2a) (Da) (DF_4) a^{-3} \\
-8 (D^2a) (Da) a^{-8} F_4^2 - \frac{6}{5} (D^2a) (DF_3) a^{-2} + \frac{18}{5} (D^2a) (D^2F_4) a^{-2} \\
+ \frac{64}{25} (D^2a) (DF_4) a^{-7} F_4 - \frac{6}{5} (D^2a) a^{-2} F_2 + \frac{28}{25} (D^2a) a^{-7} F_3 F_4 \\
- \frac{44}{125} (D^2a) a^{-12} F_4^3 - \frac{264}{5} (Da)^4 a^{-5} F_4 + \frac{192}{5} (Da)^3 (DF_4) a^{-4} \\
+ 3 (Da)^3 a^{-4} F_3 + \frac{48}{5} (Da)^3 a^{-9} F_4^2 + \frac{9}{5} (Da)^2 (D^3F_3) a^{-3} \\
-12 (Da)^2 (D^2F_4) a^{-3} - \frac{316}{25} (Da)^2 (DF_4) a^{-8} F_4 + \frac{24}{5} (Da)^2 a^{-3} F_2 \\
- \frac{372}{25} (Da)^2 a^{-8} F_3 F_4 + \frac{1056}{125} (Da)^2 a^{-13} F_4^3 - \frac{6}{5} (Da) (DF_2) a^{-2} \\
- \frac{6}{5} (Da) (D^2F_3) a^{-2} + \frac{48}{25} (Da) (DF_3) a^{-7} F_4 + 2 (Da) (D^3F_4) a^{-2} \\
+ \frac{64}{25} (Da) (D^2F_4) a^{-7} F_4 + \frac{44}{25} (Da) (DF_4)^2 a^{-7} \\
+ \frac{68}{25} (Da) (DF_4) a^{-7} F_3 - \frac{352}{125} (Da) (DF_4) a^{-12} F_4^2 + 2 (Da) a^{-2} F_1 \\
- \frac{12}{5} (Da) a^{-7} F_2 F_4 - \frac{6}{5} (Da) a^{-7} F_3^2 + \frac{66}{25} (Da) a^{-12} F_3 F_4 \\
- \frac{88}{125} (Da) a^{-17} F_4^4 - \frac{2}{5} (DF_1) a^{-1} + \frac{4}{25} (DF_2) a^{-6} F_4 \\
+ \frac{1}{5} (D^3F_3) a^{-1} - \frac{6}{25} (DF_3) (DF_4) a^{-6} + \frac{6}{25} (DF_3) a^{-6} F_3 \\
- \frac{18}{125} (DF_3) a^{-11} F_4^2 - \frac{4}{25} (D^4F_4) a^{-1} - \frac{4}{25} (D^3F_4) a^{-6} F_4 \\
- \frac{8}{25} (D^2F_4) (DF_4) a^{-6} - \frac{2}{25} (D^2F_4) a^{-6} F_3 + \frac{4}{125} (D^2F_4) a^{-11} F_4^2 \\
+ \frac{24}{125} (DF_4)^2 a^{-11} F_4 + \frac{8}{25} (DF_4) a^{-6} F_2 - \frac{48}{125} (DF_4) a^{-11} F_3 F_4 \\
+ \frac{88}{625} (DF_4) a^{-16} F_4^3 + 2 (D\sigma_{(-1)}) + \frac{4}{5} a^{-1} F_0 + \frac{4}{25} a^{-1} \sigma_{(0)} \\
- \frac{4}{25} a^{-6} F_1 F_4 - \frac{4}{25} a^{-6} F_2 F_3 + \frac{12}{125} a^{-11} F_2 F_4^2 + \frac{12}{125} a^{-11} F_3^2 F_4 \\
- \frac{44}{625} a^{-16} F_3 F_4^3 + \frac{176}{15625} a^{-21} F_4^5 , \tag{A.6}
\]

\[
\rho^{(5)} = 4 (a^{-1} \sigma_{(1)} - \rho_{(1)} \sigma_{(-1)}) , \tag{A.7}
\]

where \( F_m = \partial F/\partial u_m , m = 1, \ldots, 5 \) and \( \sigma_{(i)} , i = -1, 0, 1 \) are differential polynomials such that \( (\rho_{(i)})_t = D\sigma_{(i)} \).
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