A NOTE ON FREE PARATOPOLOGICAL GROUPS

FUCAI LIN

ABSTRACT. In this paper, we mainly discuss some generalized metric properties and the character of the free paratopological groups, and extend several results valid for free topological groups to free paratopological groups.

1. Introduction

In 1941, free topological groups were introduced by A.A. Markov in [13] with the clear idea of extending the well-known construction of a free group from group theory to topological groups. Now, free topological groups have become a powerful tool of study in the theory of topological groups and serve as a source of various examples and as an instrument for proving new theorems, see [2, 6, 10, 14].

It is well known that paratopological groups are good generalizations of topological groups, see e.g. [2]. The Sorgenfrey line ([3, Example 1.2.2]) with the usual addition is a first-countable paratopological group but not a topological group. The absence of continuity of inversion, the typical situation in paratopological groups, makes the study in this area very different from that in topological groups. Paratopological groups attract a growing attention of many mathematicians and articles in recent years, see [1, 3, 8, 9, 10, 11, 12]. As in free topological groups, S. Romaguera, M. Sanchis and M.G. Tkackenko in [19] define free paratopological groups. Recently, N.M. Pyrch has investigated some properties of free paratopological groups, see [16, 17, 18]. In this paper, we will discuss some generalized metric properties and the character of the free paratopological groups, and extend several results valid for free topological groups to free paratopological groups.

In section 3, we show, for example, that the groups $FP(X)$ and $AP(X)$ are $P$-spaces if and only if $X$ is a $P$-space, that if $X$ is a $P$-space then the groups $FP(X)$ and $AP(X)$ have the direct limit property, that if $Y$ is a closed subspace of a Tychonoff space $X$ then the subgroup $PG(Y, X)$ of $PG(X)$ generated by $Y$ is closed in $PG(X)$. In section 4, we mainly discuss the character of free abelian paratopological groups, and extend several results valid for free abelian topological groups to free abelian paratopological groups.

2. Preliminaries

All spaces are $T_0$ unless stated otherwise. We denote by $\mathbb{N}$ the set of all natural numbers. The letter $e$ denotes the neutral element of a group. Readers may consult [2, 4, 7] for notations and terminology not explicitly given here.
Firstly, we introduce some notions and terminology.

Recall that a topological group $G$ is a group $G$ with a (Hausdorff) topology such that the product mapping of $G 	imes G$ into $G$ is jointly continuous and the inverse mapping of $G$ onto itself associating $x^{-1}$ with an arbitrary $x \in G$ is continuous. A paratopological group $G$ is a group $G$ with a topology such that the product mapping of $G 	imes G$ into $G$ is jointly continuous.

**Definition 2.1.** [13] Let $X$ be a subspace of a topological group $G$. Assume that

1. The set $X$ generates $G$ algebraically, that is $<X>=G$;
2. Each continuous mapping $f : X \to H$ to a topological group $H$ extends to a continuous homomorphism $\hat{f} : G \to H$.

Then $G$ is called the Markov free topological group on $X$ and is denoted by $F(X)$.

**Definition 2.2.** [19] Let $X$ be a subspace of a paratopological group $G$. Assume that

1. The set $X$ generates $G$ algebraically, that is $<X>=G$;
2. Each continuous mapping $f : X \to H$ to a paratopological group $H$ extends to a continuous homomorphism $\hat{f} : G \to H$.

Then $G$ is called the Markov free paratopological group on $X$ and is denoted by $FP(X)$.

Again, if all the groups in the above definitions are Abelian, then we get the definitions of the Markov free Abelian topological group and the Markov free Abelian paratopological group on $X$ which will be denoted by $A(X)$ and $AP(X)$ respectively.

By a quasi-uniform space $(X, \mathcal{U})$ we mean the natural analog of a uniform space obtained by dropping the symmetry axiom. For each quasi-uniformity $\mathcal{U}$ the filter $\mathcal{U}^{-1}$ consisting of the inverse relations $U^{-1} = \{(y, x) : (x, y) \in U\}$ where $U \in \mathcal{U}$ is called the conjugate quasi-uniformity of $\mathcal{U}$.

We also recall that the universal quasi-uniformity $\mathcal{U}_X$ of a space $X$ is the finest quasi-uniformity on $X$ that induces on $X$ its original topology. Throughout this paper, if $\mathcal{U}$ is a quasi-uniformity of a space $X$ then $\mathcal{U}^*$ denotes the smallest uniformity on $X$ that contains $\mathcal{U}$, and $\tau(\mathcal{U})$ denotes the topology of $X$ generated by $\mathcal{U}$. A quasi-uniform space $(X, \mathcal{U})$ is called bicomplete if $(X, \mathcal{U}^*)$ is complete.

**Definition 2.3.** A quasi-pseudometric $d$ on a set $X$ is a function from $X \times X$ into the set of non-negative real numbers such that for $x, y, z \in X$: (a) $d(x, x) = 0$ and (b) $d(x, y) \leq d(x, z) + d(z, y)$. If $d$ satisfies the additional condition (c) $d(x, y) = 0 \Leftrightarrow x = y$, then $d$ is called a quasi-metric on $X$.

Every quasi-pseudometric $d$ on $X$ generates a topology $\mathcal{T}(d)$ on $X$ which has as a base the family of $d$-balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

A topological space $(X, \mathcal{T})$ is called quasi-(pseudo)metrizable if there is a quasi-(pseudo)metric $d$ on $X$ compatible with $\mathcal{T}$, where $d$ is compatible with $\mathcal{T}$ provided $\mathcal{T} = \mathcal{T}(d)$.

Denote by $\mathcal{U}^*$ the upper quasi-uniformity on $\mathbb{R}$ the standard base of which consists of the sets $U_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y < x + r\}$, where $r$ is an arbitrary positive real number.
Definition 2.4. Given a group $G$ with the neutral element $e$, a function $N : G \to [0, \infty)$ is called a quasi-prenorm on $G$ if the following conditions are satisfied:

1. $N(e) = 0$; and
2. $N(gh) \leq N(g) + N(h)$ for all $g, h \in G$.

Let $X$ be a space and $p \in X$. The character for $p$, character for $X$ and weight for $X$ are defined as follows respectively:

$$\chi(p, X) = \min\{|\mathcal{V}| : \mathcal{V} \text{ is a local base for } p\};$$
$$\chi(X) = \sup\{\chi(p, X) : p \in X\} + \omega;$$
$$\omega(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\}.$$

Throughout this paper, we use $G(X)$ to denote the topological groups $F(X)$ or $A(X)$, and $PG(X)$ to denote the paratopological groups $FP(X)$ or $AP(X)$. For a subset $Y$ of a space $X$, we use $G(Y, X)$ and $PG(Y, X)$ to denote the subgroups of $G(X)$ and $PG(X)$ generated by $Y$ respectively. Moreover, we denote the abstract groups of $F(X)$, $FP(X)$ by $F_a(X)$ and of $A(X)$ and $AP(X)$ by $A_a(X)$, respectively.

Since $X$ generates the free group $F_a(X)$, each element $g \in F_a(X)$ has the form $g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $x_1, \ldots, x_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$. This word for $g$ is called reduced if it contains no pair of consecutive symbols of the form $xx^{-1}$ or $x^{-1}x$. It follows that if the word $g$ is reduced and non-empty, then it is different from the neutral element of $F_a(X)$. In particular, each element $g \in F_a(X)$ distinct from the neutral element can be uniquely written in the form $g = x_1^{r_1}x_2^{r_2} \cdots x_n^{r_n}$, where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$, and $x_i \neq x_{i+1}$ for each $i = 1, \ldots, n - 1$. Such a word is called the normal form of $g$. Similar assertions are valid for $A_a(X)$.

We now outline some of the ideas of [19] in a form suitable for our applications.

Suppose that $e$ is the neutral element of the abstract free group $F_a(X)$ on $X$, and suppose that $\rho$ is a fixed quasi-pseudometric on $X$ which is bounded by 1. Extend $\rho$ from $X$ to a quasi-pseudometric $\rho_e$ on $X \cup \{e\}$ by putting

$$\rho_e(x, y) = \begin{cases} 
0, & \text{if } x = y, \\
\rho(x, y), & \text{if } x, y \in X, \\
1, & \text{otherwise}
\end{cases}$$

for arbitrary $x, y \in X \cup \{e\}$. By [19], we extend $\rho_e$ to a quasi-pseudometric $\rho^*$ on $X = X \cup \{e\} \cup X^{-1}$ defined by

$$\rho^*(x, y) = \begin{cases} 
0, & \text{if } x = y, \\
\rho_e(x, y), & \text{if } x, y \in X \cup \{e\}, \\
\rho_e(y^{-1}, x^{-1}), & \text{if } x, y \in X^{-1} \cup \{e\}, \\
2, & \text{otherwise}
\end{cases}$$

for arbitrary $x, y \in \hat{X}$.

Let $A$ be a subset of $\mathbb{N}$ such that $|A| = 2n$ for some $n \geq 1$. A scheme on $A$ is a partition of $A$ to pairs $\{a_i, b_i\}$ with $a_i < b_i$ such that each two intervals $[a_i, b_i]$ and $[a_j, b_j]$ in $\mathbb{N}$ are either disjoint or one contains the other.

If $\mathcal{S}$ is a word in the alphabet $\hat{X}$, then we denote the reduced form and the length of $\mathcal{S}$ by $[\mathcal{S}]$ and $\ell(\mathcal{S})$ respectively.

For each $n \in \mathbb{N}$, let $\mathcal{S}_n$ be the family of all schemes $\varphi$ on $\{1, 2, \cdots, 2n\}$. As in [19], define

$$\Gamma_\rho(\mathcal{S}_n, \varphi) = \frac{1}{2} \sum_{i=1}^{2n} \rho^*(x_i^{-1}, x_{\varphi(i)}).$$
Then we define a quasi-prenorm \(N_\rho : F_\rho(X) \to [0, +\infty)\) by setting \(N_\rho(g) = 0\) if \(g = e\) and
\[
N_\rho(g) = \inf\{\Gamma_\rho(\mathcal{F}, \varphi) : [\mathcal{F}] = g, \ell(\tilde{X}) = 2n, \varphi \in \mathcal{F}_n, n \in \mathbb{N}\}
\]
if \(g \in F_\rho(X) \setminus \{e\}\). It follows from Claim 3 in [19] that \(N_\rho\) is an invariant quasi-prenorm on \(F_\rho(X)\). Put \(\hat{\rho}(g, h) = N_\rho(g^{-1}h)\) for all \(g, h \in F_\rho(X)\). We refer to \(\hat{\rho}\) as the Graev extension of \(\rho\) to \(F_\rho(X)\).

Given a word \(\mathcal{F}\) in the alphabet \(\tilde{X}\), we say that \(\mathcal{F}\) is almost irreducible if \(\tilde{X}\) does not contain two consecutive symbols of the form \(u, u^{-1}\) or \(u^{-1}, u\) (but \(\mathcal{F}\) may contain several letters equal to \(e\)), see [19].

3. Some generalized metric properties on free paratopological groups

Let \(X\) be a metrizable space and the topology be generated by a metric \(d\). It follows from Theorem 3.2 in [19] that \(d\) can be extended to an invariant metric \(\hat{d}\) on \(PG(X)\). Therefore, \(PG(X)\) admits a weaker metrizable paratopological group topology.

For every non-negative integer \(n\), denote by \(B_n(X)\) the subspace of the free (Abelian) paratopological group \(PG(X)\) that consists of all words of reduced length \(\leq n\) with respect to the free basis \(X\).

**Lemma 3.1.** [2, Proposition 7.6.1] Let \((X, d)\) be a metric space, and \(\mathcal{F}_d\) be the topology on \(F_\rho(X)\) generated by the Graev extension \(d\) of \(d\) to \(F(X)\). Then \(B_n(X)\) is closed in \((F_\rho(X), \mathcal{F}_d)\) for each \(n \in \mathbb{N}\).

**Theorem 3.2.** Let \((X, d)\) be a metric space, and \(\mathcal{F}_d\) be the topology on \(FP(X)\) generated by the Graev extension \(d\) of \(d\) to \(FP(X)\). Then \(B_n(X)\) is closed in \((F_\rho(X), \mathcal{F}_d)\) for each \(n \in \mathbb{N}\).

**Proof.** Since \(X\) is Tychonoff, \(FP(X)\) coincides algebraically with \(F(X)\), and hence \(d\) on \(FP(X)\) coincides with Graev’s extension of \(d\) over the topological group \(F(X)\). By Lemma 3.1, we complete the proof. \(\square\)

By the similar proof of Theorem 3.2 we have the following theorem.

**Theorem 3.3.** Let \((X, d)\) be a metric space, and \(\mathcal{F}_d\) be the topology on \(A_\rho(X)\) generated by the Graev extension \(d\) of \(d\) to \(AP(X)\). Then \(B_n(X)\) is closed in \((A_\rho(X), \mathcal{F}_d)\) for each \(n \in \mathbb{N}\).

**Proposition 3.4.** The sets \(B_n(X)\) is closed in \(FP(X)\).

**Proof.** Let \(i : FP(X) \to F(X)\) be the identity mapping. Then \(i\) is a continuous isomorphism, and then \(B_n(X)\) is closed in \(FP(X)\) since each \(B_n(X)\) is closed in \(FP(X)\). \(\square\)

By the similar proof of Proposition 3.4 we have the following proposition.

**Proposition 3.5.** The sets \(B_n(X)\) is closed in \(AP(X)\).

By the definition of free paratopological groups, it is easy to obtain the following lemma.
Lemma 3.6. [19] The topology of the group $F_n(X)$ is the finest paratopological group topology on $F_nX$ that generates on $X$ its original topology. The same is valid for $AP(X)$.

Recalled that $X$ is a $P$-space if each $G_δ$-set in $X$ is open.

Theorem 3.7. The groups $FP(X)$ and $AP(X)$ are $P$-spaces if and only if $X$ is a $P$-space.

Proof. Necessity is a consequence of the fact that subspaces of $P$-spaces are $P$-spaces.

Sufficiency. Let $X$ be a $P$-space. Suppose that $\mathcal{F}$ is the topology of $PG(X)$. Put $\mathcal{F}_1$ be the sets consisting of all $G_δ$-sets in $PG(X)$. It is easy to see that $\mathcal{F}_1$ is a topology on $PG(X)$. Moreover, $(PG(X), \mathcal{F}_1)$ is a paratopological group topology. Since $X$ is a $P$-space, the restrictions of both $\mathcal{F}$ and $\mathcal{F}_1$ to $X$ coincide the original topology of $X$. Therefore, ones have $\mathcal{F} = \mathcal{F}_1$ by Lemma 3.6. Therefore, $PG(X)$ is a $P$-space.

We say that the topology of a space $X$ is determined by a family $\mathcal{C}$ of its subsets if a set $F \subset X$ is closed in $X$ iff $F \cap C$ is closed in $C$ for each $C \in \mathcal{C}$; We say that $G(X)$ ($PG(X)$) has the direct limit property if the topology of $G(X)$ ($PG(X)$) is determined by the family $\{B_n(X) : n \in \mathbb{N}\}$.

Theorem 3.8. If $X$ is a $P$-space, then the groups $FP(X)$ and $AP(X)$ have the direct limit property.

Proof. Assume that $K$ is a subsets of $PG(X)$ such that $K \cap B_n(X)$ is closed in $B_n(X)$ for each $n \in \mathbb{N}$. It follows from Propositions 3.4 and 3.5 that the sets $B_n(X)$ are closed in $PG(X)$. Hence the sets $B_n(X) \cap K$ are closed in $PG(X)$. It follows that $K$ is an $F_σ$-set in $PG(X)$. By Theorem 3.7 we can see that $PG(X)$ is a $P$-space, which implies that $K$ is closed in $PG(X)$.

By the group reflexion $G^p = (G, \tau^p)$ of a paratopological group $(G, \tau)$ we understand the group $G$ endowed with the strongest topology $\tau^p \subset \tau$ such that $(G, \tau^p)$ is a topological group.

A space is a functionally Hausdorff space if two distinct points $x$ and $y$ there is a continuous real-valued mapping $f$ on $X$ such that $f(x) \neq f(y)$.

In [17], N.M. Pyrch and A.V. Ravsky proved that if $X$ is a Functionally Hausdorff space then the topological group $FP(X)^p$ and $F(X)$ are topological isomorphic. In fact, we can prove that if $X$ is a Functionally Hausdorff space then the topological group $AP(X)^p$ and $AP(X)$ are topological isomorphic by an argument similar to the one in [17]. Therefore, we have the following lemma.

Lemma 3.9. If $X$ is a Functionally Hausdorff space then the topological group $PG(X)^p$ and $G(X)$ are topological isomorphic.

By Lemma 3.9 we can easily obtain the following lemma.

Lemma 3.10. If $X$ is a Functionally Hausdorff space and $f : X \to Y$ is the continuous embedding mapping, then $f$ admits an extension to a continuous monomorphism $\hat{f} : PG(X) \to G(Y)$.

Theorem 3.11. Let $X$ be a Functionally Hausdorff space, and $A$ be an arbitrary subset of $PG(X)$. If $A \cap B_n(X)$ is finite for each $n \in \mathbb{N}$, then $A$ is closed and discrete in $PG(X)$. 

Lemma 4.1.

Lemma 4.2.

Lemma 4.3.

Proof. Let \( f : X \to Y \) be a topological embedding of \( X \) to a compact space \( Y \). It follows from Lemma 3.10 that we can extend \( f \) to a continuous monomorphism \( \hat{f} : PG(X) \to G(Y) \). Let \( B \subset A \), and put \( C = \hat{f}(B) \). Then \( C \cap B_n(Y) \) is finite for each \( n \in \mathbb{N} \). Since \( Y \) is compact, \( G(Y) \) has the direct limit property. Therefore, \( C \) is closed in \( G(Y) \). Since \( \hat{f} \) is a continuous monomorphism, we have that \( B \) is closed in \( PG(X) \). Hence, all subsets of \( A \) are closed in \( PG(X) \), which implies that \( A \) is discrete.

\[ \square \]

Theorem 3.12. If \( X \) is a Functionally Hausdorff space, and \( K \) is a countably compact subspace of \( PG(X) \), then \( K \subset B_n(X) \) for some \( n \in \mathbb{N} \).

Proof. Suppose that \( K \setminus B_n(X) \neq \emptyset \) for each \( n \in \mathbb{N} \). It is easy to see that we can find an infinite subset \( A \subset K \) such that \( A \cap B_n(X) \) is finite for each \( n \in \mathbb{N} \). It follows from Theorem 3.11, Propositions 3.4 and 3.5 that \( A \) is closed and discrete in \( PG(X) \) and in \( K \), which is a contradiction.

\[ \square \]

Theorem 3.13. If \( Y \) is a closed subspace of a Tychonoff space \( X \), then the subgroup \( PG(Y, X) \) of \( PG(X) \) generated by \( Y \) is closed in \( PG(X) \).

Proof. Let \( bX \) be a Hausdorff compactification of \( X \), and \( f : X \to bX \) be a topological embedding of \( X \) to a compact space \( bX \). It follows from Lemma 3.10 that we can extend \( f \) to a continuous monomorphism \( \hat{f} : PG(X) \to G(bX) \). Denote by \( Z \) the closure of \( Y \) in \( bX \). It follows from the compactness of \( Z \) that \( G(Z, bX) \cap B_n(bX) \) is compact for each \( n \in \mathbb{N} \). Since \( bX \) is compact, \( G(bX) \) has the direct limit property, and hence \( G(Z, bX) \) is closed in \( G(bX) \). Thus \( G(Y, bX) = G(Z, bX) \cap G(X, bX) \) is a closed subgroup of \( G(X, bX) \). Since \( \hat{f} \) is a continuous monomorphism, ones have \( \hat{f}(PG(Y, X)) = G(Y, bX) \). Therefore, \( PG(Y, X) \) is closed in \( PG(X) \).

\[ \square \]

4. The character of free Abelian paratopological groups

Firstly, we give some technical lemmas.

The following two lemmas are essentially claims in the proof of Theorem 3.2 in [19].

Lemma 4.1. [19] Let \( g \) be a quasi-pseudometric on \( X \) bounded by 1. If \( g \) is a reduced word in \( F_n(X) \) distinct from \( e \), then there exists an almost irreducible word \( \mathcal{X}_g = x_1x_2 \cdots x_{2n} \) of length \( 2n \geq 2 \) in the alphabet \( \tilde{X} \) and a scheme \( \varphi_g \in \mathcal{S}_n \) that satisfy the following conditions:

1. For \( i = 1, 2, \cdots, 2n \), either \( x_i \) is \( e \) or \( x_i \) is a letter in \( g \);
2. \( [\mathcal{X}_g] = g \) and \( n \leq \ell(g) \); and
3. \( N_p(g) = \Gamma(\mathcal{X}_g, \varphi_g) \).

Lemma 4.2. [19] The family \( \mathcal{N} = \{ U_\rho(\varepsilon) : \varepsilon > 0 \} \) is a base at the neutral element \( e \) for a paratopological group topology \( \mathcal{F}_\rho \) on \( F_0(X) \), where \( U_\rho(\varepsilon) = \{ g \in F_0(X) : N_\rho(g) < \varepsilon \} \). The restriction of \( \mathcal{F}_\rho \) to \( X \) coincides with the topology of the space \( X \) generated by \( \rho \).

Lemma 4.3. [3] For every sequence \( V_0, V_1, \cdots \), of elements of a quasi-uniformity \( \mathcal{U} \) on a set \( X \), if

\[
V_0 = X \times X \quad \text{and} \quad V_{i+1} \circ V_{i+1} \circ V_{i+1} \subset V_i, \quad \text{for} \ i \in \mathbb{N},
\]

then
where ‘◦’ denotes the composition of entourages in the quasi-uniform space $(X, V)$, then there exists a quasi-pseudometric $\rho$ on the set $X$ such that, for each $i \in \mathbb{N}$,

$$V_i \subset \{(x, y) : \rho(x, y) \leq \frac{1}{2^i}\} \subset V_{i-1}.$$

**Lemma 4.4.** [10] For every quasi-uniformity $\mathcal{V}$ on a set $X$ and each $V \in \mathcal{V}$ there exists a quasi-pseudometric $\rho$ bounded by 1 on $X$ which is quasi-uniform with respect to $\mathcal{V}$ and satisfies the condition

$$\{(x, y) : \rho(x, y) < 1\} \subset V.$$

**Lemma 4.5.** [10] Suppose that $\rho$ is a quasi-pseudometric on a set $X$, and suppose that $m_1x_1 + \cdots + m_nx_n$ is the normal form of an element $h \in F_n(X) \setminus \{e\}$ of the length $l = \sum_{i=1}^n |m_i|$. Then there is a representation

$$h = (-u_1 + v_1) + \cdots + (-u_k + v_k),$$

where $2k = l$ if $l$ is even and $2k = l+1$ if $l$ is odd, $u_1, v_1, \ldots, u_k, v_k \in \{\pm x_1, \ldots, \pm x_n\}$ (but $v_k = e$ if $l$ is odd), and such that

$$\hat{\rho}_A(e, h) = \sum_{i=1}^k \rho^*(u_i, v_i).$$

In addition, if $\hat{\rho}_A(e, h) < 1$, then $l = 2k$, and one can choose $y_1, z_1, \ldots, y_k, z_k \in \{x_1, \ldots, x_n\}$ such that $h = (-y_1 + z_1) + \cdots + (-y_k + z_k)$ and $\hat{\rho}_A(e, h) = \sum_{i=1}^k \rho^*(y_i, z_i)$.

**Theorem 4.6.** [10] Let $X$ be a Tychonoff space, and let $\mathcal{P}_X$ be the family of all continuous quasi-pseudometrics from $(X \times X, \mathcal{V}^{-1} \times \mathcal{V}_X)$ to $(\mathbb{R}, \mathcal{V}^*)$ which are bounded by 1. Then the sets

$$V_\rho = \{g \in AP(X) : \hat{\rho}_A(e, g) < 1\}$$

with $\rho \in \mathcal{P}_X$ form a local base at the neutral element $e$ of $AP(X)$.

**Lemma 4.7.** [2] Let $k \in \omega, p, k_1, \ldots, k_p \in \mathbb{N}$ such that $\sum_{i=1}^p 2^{-k_i} < 2^{-k}$. Then we have

1. If $(X, V)$ is a quasi-uniform space and $\{U_n : n \in \omega\}$ a countable subcollection of $V$ such that $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n$ for each $n \in \omega$, then $U_k \circ \cdots \circ U_k \subset U_k$.
2. If $\{V_i : i \in \omega\}$ is a sequence of subsets of a group $G$ with the identity $e$ such that $e \in V_i$ and $V_{i+1}^G \subset V_i$ for each $i \in \omega$, then we have $V_k \cdots V_{n} \subset V_r$.

**Lemma 4.8.** Let $m_1x_1 + \cdots + m_nx_n$ be the normal form of an element $g \in A_0(X) \setminus \{e\}$ and let $d$ be a quasi-pseudometric on $X$. If $\sum_{i=1}^n m_i = 0$, then there is an reduced representation of $g$ in the form

$$g = (-z_1 + t_1) + \cdots + (-z_k + t_k)$$

such that $2k = \sum_{i=1}^n |m_i|$, $z_j, t_j \in \{x_1, \ldots, x_n\}$ for each $j \leq k$ and $\hat{d}_A(e, g) = \sum_{j=1}^k d(z_j, t_j)$.

**Proof.** It follows from $\sum_{i=1}^n m_i = 0$ that the number $m = \sum_{i=1}^n |m_i|$ has to be even. Let $m = 2k$ for some $k \in \mathbb{N}$. By Lemma 4.5, the element $g$ has a reduced representation $\varphi$ of $g$ of the form

$$g = (-u_1 + v_1) + \cdots + (-u_k + v_k)$$

and such that $\varphi(e, g) < 1$. Therefore, $\varphi(e, g) \leq \frac{1}{2^k}$, which implies $g \in AP(X)$ and hence $\hat{\rho}_A(e, g) < 1$.
such that
\[ \hat{d}_A(e, g) = \Gamma(\varphi) = \sum_{j=1}^{k} d^*(u_j, v_j), \]
where \( u_j, v_j \in \{\pm x_1, \ldots, \pm x_n\} \) for each \( j \leq k \). Obviously, each \( -u_j + v_j \) has one of the following four forms: \( a - b, -a + b, a + b, -a - b \) for some \( a, b \in X \).

Suppose that \( -u_1 + v_1 \) has the third form. Then we have \( -u_1 = a \in X \) and \( v_1 \in X \), and hence \( -u_1 + v_1 = a + v_1 \). Since \( \sum_{i=1}^{n} m_i = 0 \), there exists a \( 2 \leq j \leq k \) such that \( -u_j + v_j \) has the fourth form. Without loss of generality, we may assume that \( j = 2 \). Then \( u_2 \in X \) and \( v_2 = -b \) for some \( b \in X \). Therefore, from our definition of the quasi-pseudometric \( d^* \) on \( X \cup \{e\} \cup \{-X\} \), it follows that the sum \( \Gamma(\varphi) \) contains the part corresponding to \( -u_1 + v_1 \) and \( -u_2 + v_2 \)
\[ d^*(u_1, v_1) + d^*(u_2, v_2) = d^*(-a, v_1) + d^*(u_2, -b) \]
\[ = d(e, a) + d(e, v_1) + d(u_2, e) + d(b, e) \]
\[ \geq d^*(b, a) + d^*(u_2, v_1) \]
Replace the sum \( -u_1 + v_1 + (-u_2 + v_2) \) in the reduced representation \( \varphi \) by \( -u_2 + v_1 + (-b + a) \). Therefore, we get another reduced representation \( \varphi' \) of \( g \) of the form
\[ g = (-u_2 + v_1) + (-b + a) + (-a_3 + v_3) + \cdots + (-a_k + v_k). \]
It follows from the above inequality that \( \Gamma(\varphi') \leq \Gamma(\varphi) \). Since \( k \) is finite, by induction, we can give another reduced representation \( \varphi'' \) of \( g \) of the form
\[ g = (-z_1 + t_1) + \cdots (-z_k + t_k), \]
where \( z_j, t_j \in \{x_1, \ldots, x_n\} \) for each \( j \leq k \). Moreover, it is easy to see that we have \( \Gamma(\varphi'') \leq \Gamma(\varphi) \). However, the definition of \( \hat{d}_A(e, g) = \Gamma(\varphi) \leq \Gamma(\varphi'') \) whence it follows that \( \hat{d}_A(e, g) = \sum_{j=1}^{k} d(z_j, t_j) \).

Suppose that \( \mathcal{U}_X \) is the finest quasi-uniformity of a space \( X \). Put \( \mathcal{U}_X = \{P : P \) is a sequence of \( \mathcal{U}_X \} \). For each \( P \in \mathcal{U}_X \), denote by \( P = \{U_n : n \in \omega\} \) or \( P = \{U_n : n \in \omega\} \).

For each \( P = \{U_1, U_2, \cdots \} \in \mathcal{U}_X \), let
\[ W(P) = \{x_1 + y_1 + \cdots + x_k + y_k : (x_i, y_i) \in U_i \text{ for } i = 1, 2, \cdots, k, k \in \mathbb{N}\}, \]
and
\[ \mathcal{W} = \{W(P) : P \in \mathcal{U}_X\}. \]
Moreover, fixed any \( n \in \mathbb{N} \). Let
\[ \mathcal{D}_n(P) = \{Q \subset P : |Q| = n\}; \]
\[ W_n(P) = \{x_1 + y_1 + \cdots + x_n + y_n : (x_j, y_j) \in U_i \text{ for } i = 1, 2, \cdots, n, \}
{U_{i_1}, U_{i_2}, \cdots, U_{i_n} \in \mathcal{D}_n(P)\}, \]
and
\[ \mathcal{W}_n = \{W_n(P) : P \in \mathcal{U}_X\}. \]

**Remark** In the above definition, for \( P \in \mathcal{U}_X \), there may be the same elements in \( P \). In particular, for every \( U \in \mathcal{U}_X \), we have \( \{U, U, \cdots\} \) is also in \( \mathcal{U}_X \). Moreover, the reader should note that the representation of elements of \( W(P) \) and \( W_n(P) \) need not be a reduced representation.

Put \( \mathcal{R}_n(P) = \{Q \subset P : |Q| \leq n\} \). It is easy to see that
\[ W_n(P) = \{x_1 + y_1 + \cdots + x_k + y_k : (x_j, y_j) \in U_i \text{ for } i = 1, 2, \cdots, k, \}
{U_{i_1}, U_{i_2}, \cdots, U_{i_k} \in \mathcal{R}_n(P)\}. \]
The following theorem follows from an adaptation of the characterization of the neighborhoods of the identity for free topological groups obtained by V.G. Pestov in [15].
Theorem 4.9. The family $\mathcal{W}$ is a neighborhood base of $e$ in $AP(X)$.

Proof. It is easy to prove that the family $\mathcal{W}$ satisfies the following conditions (i)-(iv):

(i) for each $W \in \mathcal{W}$, there exists a $V \in \mathcal{W}$ such that $V + V \subset W$;
(ii) for each $W \in \mathcal{W}$ and each $g \in V$, there exists a $V \in \mathcal{W}$ such that $g + V \subset W$;
(iii) for every $U, V \in \mathcal{W}$, there exists a $W \in \mathcal{W}$ such that $W \subset U \cap V$;
(iv) $\{0\} = \cap \mathcal{W}$.

Therefore, the topology $\mathcal{F}_1$ generated by $\mathcal{W}$ on $A_0(X)$ is a paratopological group topology. Pick $P = \{U_1, U_2, \ldots \} \in \omega \mathcal{W}_X$ and $x \in X$, and put $W(x) = \{y \in X : (x, y) \in U_1\}$. Then $W(x)$ is open in $X$ since $\mathcal{W}_X$ is compactible with the original topology for $X$. Furthermore, we can prove that $x \in W(x) \subset (x + W(P)) \cap X$, which implies that $\mathcal{F}_1\lvert_X$ is weaker than the original topology for $X$.

Claim: The topology $\mathcal{F}_1$ is stronger than the topology of $AP(X)$.

Indeed, let $V$ be an open neighborhood of $e$ in $AP(X)$. Put $V_0 = V$ and pick a sequence $\{V_n : n \in \omega\}$ of neighborhoods of $e$ in $AP(X)$ such that $V_n + V_n + V_n \subset V_{n-1}$. For each $n \in \mathbb{N}$, put

$$U_n = \{(x, y) \in X \times X : -x + y \in V_n\},$$

then $U_n \in \mathcal{W}_X$. Hence $P = \{U_1, U_2, \ldots \} \in \omega \mathcal{W}_X$. For each point $g \in W(P)$, there exists an $n \in \mathbb{N}$ such that $g = -x_1 + y_1 - \cdots - x_n + y_n$ for some $(x_i, y_i) \in U_i$ for $i = 1, 2, \ldots, n$. Therefore, it follows from Lemma 4.7 that $g \in V_1 + V_2 + \cdots + V_n \subset V_0 = V$. Then we have $W(P) \subset V$.

It follows from Claim that $\mathcal{F}_1\lvert_X$ coincides with the original topology for $X$. Therefore, $\mathcal{F}_1$ is weaker than the topology for $AP(X)$. Hence $\mathcal{F}_1$ coincides with the topology for $AP(X)$. Thus the family $\mathcal{W}$ is a neighborhood base of $e$ in $AP(X)$. □

Using Lemma 4.7 and Theorem 4.9, we can show the following theorem by a similar proof of [20, Theorem 2.4].

Theorem 4.10. For each $n \in \mathbb{N}$, the family $\mathcal{W}_n$ is a neighborhood base of $e$ in $A_{2n}(X)$.

Theorem 4.11. Let $d$ be a quasi-pseudometric on $X$. Then $\hat{d}_A(kx, ky) = kd(x, y)$ for all $x, y \in X$ and $k \in \mathbb{N} \cup \{0\}$.

Proof. If $x = y$ or $k = 0$ then it is trivially true. Therefore, we can suppose that $x \neq y$ and $k \in \mathbb{N}$. Let $g = -kx + ky$. It follows from Lemma 4.8 that $g$ has an reduced representation of the form

$$g = (-z_1 + t_1) + \cdots (-z_k + t_k),$$

where $z_j, t_j \in \{x, y\}$ for each $j \leq k$ and $\hat{d}_A(e, g) = \sum_{j=1}^{k} d(z_j, t_j)$. Since the above representation of $g$ is reduced and $k > 0$, each $-z_j + t_j$ is equal to $-x + y$. Therefore, we have

$$\hat{d}_A(kx, ky) = \hat{d}_A(e, g) = \sum_{j=1}^{k} d(z_j, t_j) = \sum_{j=1}^{k} d(x, y) = kd(x, y).$$

□

A pair $(P, \leq)$ is call a quasi-ordered set if $\leq$ is a reflexive transitive relation on the set $P$. If $(P, \leq)$ has the additional property of antisymmetry, then it is a partially ordered set. A set $D \subset P$ is called dominating in the quasi-ordered set
(P, ≤) if for each p ∈ P there exists q ∈ D such that p ≤ q. Similarly, a subset E of P is said to be dense in (P, ≤) if for every p ∈ P there exists q ∈ E with q ≤ p. The minimal cardinality of a dominating family in (P, ≤) is denoted by D(P, ≤), while we use d(P, ≤) for the minimal cardinality of a dense set in (P, ≤). The notions of dominating and dense sets are dual: if a set S is dense in (P, ≤), then it is dominating in (P, ≥) and vice versa. Therefore, d(P, ≤) = D(P, ≥) and d(P, ≥) = D(P, ≤). Note that in any homogeneous space G we have d(N(e), <) = χ(G).

If (P, ≤) and (Q, ≪) are quasi-ordered sets, then a mapping f : P → Q is called order-preserving if x ≤ y implies f(x) ≪ f(y), where x, y ∈ P. Similarly, f is order-reversing if x ≤ y implies f(x) ≫ f(y).

Lemma 4.12. Let (P, ≤) and (Q, ≪) be quasi-ordered sets, and let f : P → Q be an order-preserving map. If f(P) is a dominating set in Q, then D(Q) ≤ D(P).

Let X be a Tychonoff space, and let PX be the family of all continuous quasi-pseudometrics from (X × X, W X−1 × W X) to (R, W*).

Theorem 4.13. Let X be a Tychonoff space. Then χ(AP(X)) = D(PX, ≤).

Proof. It follows from Theorem 4.9 that there exists a natural correspondence between the family PX and a base at the neutral element e of AP(X). Indeed, the map d → V d from (PX, ≤) to the partially ordered set (NM(e), ≥) of open neighborhoods of e in AP(X) is order-preserving and maps (PX, ≤) to a base at e in AP(X), that is, a dominating set in (NM(e), ≥), which implies that χ(AP(X)) ≤ D(PX, ≤) by Lemma 4.12.

Assume that a subset Q ⊂ PX is such that {V d : d ∈ Q} is a base at the neutral element in AP(X).

Claim: For every ρ ∈ PX, there exists d ∈ Q such that ρ ≤ 2d.

Indeed, fix a ρ ∈ PX, since {V d : d ∈ Q} is a base at the neutral element in AP(X), there exists d ∈ Q such that V d ⊂ V ρ. Now, we shall show that ρ ≤ 2d. Let x, y ∈ X. If d(x, y) < 1 then we have −x + y ∈ V d ⊂ V ρ, and hence ρ(x, y) < 1. Given an n ∈ N. Similarly, if n ∈ N and d(x, y) < 2−n then it follows from Theorem 4.11 that d A(e, 2n(x + y)) = d A(2n x, 2n y) = 2n d(x, y) < 1. Therefore, we have 2n(x + y) ∈ V d ⊂ V ρ, and hence ρ A(2n x, 2n y) = 2n ρ(x, y) < 1, that is, ρ(x, y) < 2−n. Thus we have showed that d(x, y) < 2−n implies ρ(x, y) < 2−n for n ∈ N ∪ {0}. Hence, it is easy to see that ρ(x, y) ≤ 2d(x, y) if d(x, y) = 0 or d(x, y) = 1. If 0 < d(x, y) < 1, then we can choose an n ∈ N such that 2−n−1 ≤ d(x, y) < 2−n. Therefore, we have ρ(x, y) < 2−n, which implies that ρ(x, y) ≤ 2d(x, y). Hence we have ρ ≤ 2d.

For each d ∈ Q, let d* = min{2d, 1}, and put Q* = {d* : d ∈ Q}. Obviously, we have Q* ⊂ PX. It follows from our claim that Q* is a dominating family in PX. Thus D(PX, ≤) ≤ |Q*| ≤ |Q|.

Suppose that B is a base at e in AP(X). For each B ∈ B, it follows from Theorem 4.6 that there exists dB ∈ PX such that V dB ⊂ B. Put A = {dB : B ∈ B}. Therefore, the set {V dB : B ∈ B} is a base at e and satisfies |A| ≤ |B|, and hence D(PX, ≤) ≤ |A| ≤ |B|. Thus D(PX, ≤) ≤ χ(AP(X)).

Therefore, we have D(PX, ≤) = χ(AP(X)).

From the proof of Theorem 4.13 it is easy to show the following theorem.
Theorem 4.14. Let $X$ be a Tychonoff space and $\mathcal{D} \subset \mathcal{P}_X$. Then the collection of open sets
\[ \{ g \in AP(X) : d_A(e,g) < \varepsilon \}, \text{ for } d \in \mathcal{D} \text{ and } \varepsilon > 0, \]
is a base at $e$ for the topology of the free Abelian paratopological group $AP(X)$ if and only if for each $\rho \in \mathcal{P}_X$ there is $d \in \mathcal{D}$ such that $\rho \leq 2d$.

Given two sequences $s = \{ U_n : n \in \omega \}$ and $t = \{ V_n : n \in \omega \}$ in $\omega \mathcal{U}_X$, we write $s \leq t$ provided that $U_n \subset V_n$ for each $n \in \omega$.

Theorem 4.15. Let $X$ be a Tychonoff space. Then $\chi(AP(X)) = d(\omega \mathcal{U}_X, \leq)$.

Proof. It follows from Theorem 4.9 that $\chi(AP(X)) \leq d(\omega \mathcal{U}_X, \leq)$. By Theorem 4.13, it is suffice to show that $d(\omega \mathcal{U}_X, \leq) \leq D(\mathcal{P}_X, \leq)$.

Indeed, for each $d \in \mathcal{P}_X$ and $n \in \omega$, let
\[ U_n(d) = \{(x, y) \in X \times X : d(x, y) \leq 2^{-n}\}. \]

Obvious, the correspondence $d \mapsto \{ U_n(d) : n \in \omega \}$ is an order-reversing mapping of $D(\mathcal{P}_X, \leq)$ to $(\omega \mathcal{U}_X, \leq)$. Put $\mathcal{A} = \{ \{ U_n(d) : n \in \omega \} : d \in \mathcal{P}_X \}$. Then $\mathcal{A}$ is a dense set in $\omega \mathcal{U}_X$. In fact, take an arbitrary sequence $\{ U_n : n \in \omega \}$ such that $3V_{n+1} \subset V_n \subset U_n$ for each $n \in \omega$. It follows from Theorem 4.10 that there exists $d \in \mathcal{P}_X$ such that $U_n(d) \subset V_n$ for each $n \in \omega$. Therefore, we have $\{ V_n : n \in \omega \} \leq \{ U_n : n \in \omega \}$.

Corollary 4.16. Let $X$ be a Tychonoff space. Then $\omega(X, \mathcal{U}_X) \leq \chi(AP(X)) \leq \omega(X, \mathcal{U}_X)^{\omega_0}$.

Proof. It is clear that
\[ \omega(X, \mathcal{U}_X) = d(\mathcal{U}_X, \leq) \leq d(\omega \mathcal{U}_X, \leq) \leq d(\mathcal{U}_X, \leq)^{\omega_0} = \omega(X, \mathcal{U}_X)^{\omega_0}. \]

Now, the result easily follows from Theorem 4.14.

We say that $X$ is a quasi-uniform $P$-space if the intersection of countably many elements of $\mathcal{U}_X$ is again an element of $\mathcal{U}_X$.

Theorem 4.17. If $X$ is a Tychonoff quasi-uniform $P$-space, then $\chi(AP(X)) = \omega(X, \mathcal{U}_X)$.

Proof. The correspondence $U \mapsto \{ U, U, \ldots \}$ from $(\mathcal{U}_X, \leq)$ to $(\omega \mathcal{U}_X, \leq)$ is an order-preserving embedding. Put $\mathcal{A} = \{ \{ U, U, \ldots \} : U \in \mathcal{U}_X \}$. Then $\mathcal{A}$ is a dense set in $\omega \mathcal{U}_X$. Indeed, since $X$ is a quasi-uniform $P$-space, it follows that, for an arbitrary $\{ U_0, U_1, \ldots \} \in \omega \mathcal{U}_X$, we have $U = \bigcap_{n \in \omega} U_n \in \mathcal{U}_X$, and hence $\{ U, U, \ldots \} \leq \{ U_0, U_1, \ldots \}$. Therefore, we have $\chi(AP(X)) = \omega(X, \mathcal{U}_X)$ by Theorem 4.14.

Let $\omega^\omega$ denote the family of all functions from $\mathbb{N}$ into $\mathbb{N}$. For $f, g \in \omega^\omega$ we write $f <^* g$ if $f(n) < g(n)$ for all but finitely many $n \in \mathbb{N}$. A family $\mathcal{F}$ is bounded if there is a $g \in \omega^\omega$ such that $f <^* g$ for all $f \in \mathcal{F}$, and is unbounded otherwise. We denote by $b$ the smallest cardinality of an unbounded family in $\omega^\omega$. It is easy to see that $\omega < b \leq c$, where $c$ denotes the cardinality of the continuum.

By means of an argument similar to the one used in [14, Lemma 2.14] we obtain the following theorem.

Theorem 4.18. If a Tychonoff space $X$ is not a quasi-uniform $P$-space, then $b \leq \chi(AP(X))$. 
Theorem 4.19. If a Tychonoff space $X$ contains an infinite compact set, then $\flat \leq \chi(\text{AP}(X))$.

Proof. Let $K$ be an infinite compact set of $X$. Since an infinite compact set cannot be a $P$-space, $X$ is not a quasi-uniform $P$-space. Then it follows from Theorem 4.18 that we have $\flat \leq \chi(\text{AP}(X))$. □

Acknowledgements. I wish to thank the reviewers for the detailed list of corrections, suggestions to the paper, and all her/his efforts in order to improve the paper.

References
[1] A.V. Arhangel’skii, E.A. Rezichenko, Paratopological and semitopological groups versus topological groups, Topology Appl. 151(2005) 107–119.
[2] A.V. Arhangel’skii, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press and World Sci., 2008.
[3] J. Cao, R. Drozdowski, Z. Piotrowski, Weak continuity properties of topological groups, Czech. Math. J. 60(135)(2010) 133–148.
[4] R. Engelking, General Topology (revised and completed edition), Heldermann Verlag, Berlin, 1989.
[5] P. Fletcher, W.F. Lindgren, Quasi-uniform spaces, Marcel Dekker, New York, 1982.
[6] M.I. Graev, Free topological groups, Izvestiya Akad. Nauk SSSR Ser. Mat. 12(1948) 279–381.
[7] G. Gruenhage, Generalized metric spaces, K. Kunen, J.E. Vaughan eds., Handbook of Set-Theoretic Topology, North-Holland, (1984), 423–501.
[8] F. Lin, R. Shen, On rectifiable spaces and paratopological groups, Topology Appl. 158(2011) 597–610.
[9] F. Lin, S. Lin, Pseudobounded or $\omega$-pseudobounded paratopological groups, Filomat 25(3)(2011) 93–103.
[10] F. Lin, Topological monomorphism between free paratopological groups, Bulletin of the Belgian Mathematical Society-Simon Stevin, to appear.
[11] C. Liu, A note on paratopological groups, Comment. Math. Univ. Carolin. 47(2006) 633–640.
[12] C. Liu, S. Lin, Generalized metric spaces with algebraic structures, Topology Appl. 157(2010) 1966–1974.
[13] A.A. Markov, On free topological groups, Dokl. Akad. Nauk SSSR 31(1941) 299–301.
[14] P. Nicholas, M.G. Tkackenko, The character of free topological groups I, Applied General Topology 6(2005) 15–41.
[15] V.G. Pestov, Neighborhoods of identity in free topological groups (Russian), Vestnik Moskov. Univ. Ser I. Mat. Mekh 3(1985) 8–10.
[16] N.M. Pyrch, On isomorphisms of the free paratopological groups and free homogeneous spaces I, Ser. Mech-Math. 63(2005) 224–232.
[17] N.M. Pyrch, A.V. Ravsky, On free paratopological groups, Matematychni Studii 25(2006) 115–125.
[18] N.M. Pyrch, Free paratopological groups and free products of paratopological groups, Journal of Mathematical Sciences 174(2)(2011) 190–195.
[19] S. Romaguera, M. Sanchis, M.G. Tkackenko, Free paratopological groups, Topology Proceedings 27(2002) 1–28.
[20] K. Yamada, Characterizations of a metrizable space $X$ such that every $A_n(X)$ is a $k$-space, Topol. Appl. 49(1993) 75–94.

Fucai Lin: Department of Mathematics and Information Science, Zhangzhou Normal University, Zhangzhou 363000, P. R. China
E-mail address: linfucai2008@yahoo.com.cn