THE GROUP OF SELF HOMOTOPY EQUIVALENCES OF SOME LOCALIZED ASPHERICAL COMPLEXES

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ABSTRACT. By studying the group of self homotopy equivalences of the localization (at a prime $p$ and/or zero) of some aspherical complexes, we show that, contrary to the case when the considered space is a nilpotent complex, $E^m_\#(X_p)$ is in general different from $E^m_\#(X)$. That is the case even when $X = K(G, 1)$ is a finite complex and/or $G$ satisfies extra finiteness or nilpotency conditions, for instance, when $G$ is finite or virtually nilpotent.

1. INTRODUCTION

Given a pointed topological space, denote by $E(X)$, as usual, the group of homotopy classes of pointed self-homotopy equivalences of $X$ with the group structure given by composition. At the present, there is no standard procedure to the computation of $E(X)$ and, even for very “simple” classes of spaces, this group is still unknown. Ideas and techniques arising from completion and localization theory have been applied in different contexts to approximate $E(X)$ and some particularly interesting subgroups.

We shall mainly be concern in one of them, namely the subgroup $E^m_\#(X)$ of $E(X)$ formed by those classes inducing the identity on the homotopy groups up to $m$, i.e.,

$$E^m_\#(X) = \ker (E(X) \to \prod_{i=1}^m \text{aut} \pi_i(X)).$$

Whenever $X$ is either a finite complex, or a space with a finite number of non trivial homotopy groups, and $m$ is greater than or equal to the “homotopical” or “homological” dimension (we denote it $\dim X$ henceforth), this group is known to be nilpotent [6, 10] as it acts trivially on the homotopy groups up to $\dim X$. Therefore, it can be localized in the classical sense on any set (possibly empty) of primes $P$ [13]. What is then the relation between $E^m_\#(X_p)$ and $E^m_\#(X_p)$? A complete answer is given whenever the space is nilpotent [15, 17]. In fact, for such spaces, the morphism $E^m_\#(X) \to E^m_\#(X_p)$, induced by localization, is the localization morphism and thus $E^m_\#(X_p) \cong E^m_\#(X_p)$ for any $m \geq \dim X$. In [16] the same is proved for completion, that is to say, $E^m_\#(X_p) \cong E^m_\#(X_p)$.

A natural question arises: can this theorem be extended to a larger class of spaces? the first step in all the available proofs is proving the result for aspherical complexes. Therefore, we proceed by analyzing the group $E^m_\#(X_p)$ as $X$ runs through different classes of aspherical spaces, i.e., Eilenberg MacLane spaces of the type $K(G, 1)$. It is worth to mention that, out of the class of nilpotent spaces, $p$-localization functors do not, in general, preserve asphericity, even for virtually nilpotent $K(G, 1)$'s, i.e., when $G$ contains...
a nilpotent normal subgroup of finite index \([2, 9]\). Moreover, for many of these aspherical complexes, \(K(G, 1)_p\) has an infinite number of non trivial homotopy groups.

Consider such a space and observe that for any \(m \geq 1\), \(E^m_\#(K(G, 1)) = \{1\}\). Hence, realizing a non trivial group as \(E^m_\#(K(G, 1)_p)\) exhibits a counterexample of the classical results above for nilpotent spaces. In this note, we shall do that for aspherical complexes with any extra nilpotency or finiteness assumptions other than nilpotent of course (finite aspherical complexes and \(K(G, 1)\)'s with \(G\) finite or virtually nilpotent), any prime \(p\) and/or zero, and any \(m\).

It is known that there are different choices of \(p\)-localization which extend the classical one for nilpotent spaces. Except in theorem 2.1 below, in which we explicitly indicate that Bousfield homology localization \([4]\) is being used, we shall be working with the standard \(p\)-localization with respect to self maps of \(S^1\) \([5]\). In other words, a connected space \(X\) is \(p\)-local if \(\pi_1(X)\) is a \(p\)-local group and each \(\pi_k(X)\) is a \(p\)-local \(\pi_1(X)\)-module, \(k \geq 2\).

The last part of this note heavily rely on basic facts from rational homotopy theory for which we refer to the standard reference \([11]\).

2. Realizing \(E^m_\#(K(G, 1)_p)\)

We shall start by showing that, given a finite nilpotent space \(X\), using Bousfield homology localization \([4]\) and without any extra assumption on the group \(G\), we can easily realize the classical \(p\)-localization of the nilpotent group \(E^m_\#(X)\) as a group of the form \(E^m_\#(K(G, 1)_p)\).

**Theorem 2.1.** Let \(X\) be a nilpotent finite complex. Then, there exists a group \(G\) such that, for any \(p\) (prime number or zero) and any \(m \geq \text{dim} X\),

\[
E^m_\#(K(G, 1)_p) \cong E^m_\#(X)_p.
\]

**Proof.** By the Kan-Thurston theorem \([14]\) applied to the complex \(X\), we find a group \(G\) and a map \(f : K(G, 1) \to X\) such that, for any coefficient system \(A\), the map

\[
H^*(f; A) : H^*(K(G, 1); A) \to H^*(X; A)
\]

is an isomorphism. In particular, this occurs when \(A = \mathbb{Z}_p, \mathbb{Q}\), and therefore, the Bousfield homology \(p\)-localization (\(p\) a prime number or zero) of \(K(G, 1)\) coincides with that of \(X\), \(K(G, 1)_p \cong X_p\).

However, the Bousfield localization is an extension of the classical localization for nilpotent spaces. Therefore, the theorem of Maruyama stated above \([15, 17]\) implies that, for any \(m \geq \text{dim} X\), \(E^m_\#(X)_p \cong E^m_\#(X)_p\) and therefore \(E^m_\#(K(G, 1)_p) \cong E^m_\#(X)_p\). \(\square\)

**Corollary 2.2.** Let \(H\) be any finite abelian \(p\)-local group without 2-torsion. Then, for any \(m > 2\), there exists a group \(G\) such that

\[
E^m_\#(K(G, 1)_p) \cong H.
\]

**Proof.** Indeed, by \([1]\) Remark 4.7 (4)), any finite abelian group without 2-torsion \(H\) is isomorphic to \(E^m_\#(X)\) in which \(X\) is a co-Moore space of dimension \(m > 2\) and of type \((\mathbb{Z} \oplus H, m - 1)\). To finish apply theorem above to \(X\) taking into account that \(H_p = H\). \(\square\)

This is our first class of spaces of the form \(K(G, 1)\) for which \(E^m_\#(K(G, 1)_p)\) is not the trivial group for some \(p\) and some \(m \geq 1\). However, the group \(G\) obtained from Kan-Thurston theorem is far from satisfying finiteness or nilpotency restrictions. In the next class of examples we impose this kind of restricted behavior. We begin by the following observation:
Lemma 2.3. Let $G$ be a perfect, virtually nilpotent group. Then for any prime $p$
\[ \mathcal{E}_p^1(K(G,1)_p) = \mathcal{E}(K(G,1)_p). \]

Proof. Recall \cite{3} that a group $G$ is called generically trivial if, for any prime $p$, $G_p = \{1\}$. It has been proved, first for finite groups \cite{3} and then for virtually nilpotent groups \cite{9}, that generically trivial groups coincide with perfect groups. Hence, taking into account that $\pi_1(K(G,1)_p) = G_p = \{1\}$ \cite{5}, any self-equivalence of $\mathcal{E}(K(G,1)_p)$ is trivially in $\mathcal{E}_p^1(K(G,1)_p)$.

\[ \square \]

Theorem 2.4. For any group $H$ and any prime $p$, there exists a group $G$ such that $H$ is a subgroup of $\mathcal{E}_p^1(K(G,1)_p)$. Moreover, if $H$ is a finite group, then $G$ can also be chosen finite.

Proof. Consider the “regular” representation of $H$ as a subgroup of the group $\Sigma(H)$ of bijections of $H$. Choose any group $F$ for which the space $K(F,1)_p$ is non contractible and simply connected (i.e., $F_p = \{1\}$). For instance, any group as in lemma above satisfies this for all $p$. In \cite{2}[9], the reader may find many examples of these groups which satisfy additional finiteness or nilpotency properties. Under these assumptions consider the free product $G = \ast_{h \in H} F$ and observe that $BG = K(G,1) = \vee_{h \in H} K(F,1) = \vee_{h \in H} BF$. Hence $K(G,1)_p = (\vee_{h \in H} K(F,1))_p$ and therefore, via Van Kampen theorem, this space is also 1-connected and non contractible as, for instance, it has non trivial mod $p$ homology. Next, consider the composition

\[ \gamma : \Sigma(H) \to \mathcal{E}(\vee_{h \in H} K(F,1)_p) \to \mathcal{E}(K(G,1)_p) \]

where $\varphi$ is the injective morphism defined by $\varphi(\alpha)(x_h)_{h \in H} = (x_{\alpha(h)})_{h \in H}$ and $\psi$ is the map $\mathcal{E}(X) \to \mathcal{E}(X_p)$ induced by localization. We now show that $\gamma$ is an injective morphism: indeed, for each $h \in H$ consider the canonical inclusion $i_h : F \to G$ and “projection” $q_h : G \to F$ so that $q_h \circ i_h = 1_F$ and $q_h \circ i_{h'}$ is the trivial map for $h \neq h'$. Now, assume there is a bijection $\alpha \in \Sigma(H)$, $\alpha \neq 1_H$, such that $\gamma(\alpha) = 1_{K(G,1)_p}$. Hence, there exists $h \in H$ for which $\alpha(h) = h' \neq h$. For this element we have:

\[ 1_{K(F,1)_p} = (B1_F)_p = (Bq_{h'})_p \circ (Bi_{h'})_p = (Bq_{h})_p \circ (\gamma(\alpha)) \circ (Bi_{h})_p = (Bq_{h})_p \circ (Bi_{h'})_p = * \]

However, $K(F,1)_p$ is non contractible and therefore $\gamma$ is an injective morphism. To finish, consider the restriction of $\gamma$ to $H$,

\[ H \to K(G,1)_p, \]

and observe that, since $K(G,1)_p$ is simply connected, $\mathcal{E}_p^1(K(G,1)_p) = \mathcal{E}(K(G,1)_p)$.

On the other hand, note that the group $G$ is not finite even when $H$ is. Hence, whenever this is the case, to prove the second part of the theorem, we follow the same argument choosing the finite group $G = \prod_{h \in H} F$. Again, $K(G,1)_p = \prod_{h \in H} K(F,1)_p$ is also 1-connected and non contractible. As before, consider the restriction to $H$ of the injective morphism

\[ \Sigma(H) \to \mathcal{E}(K(G,1)_p) \cong \mathcal{E}(\prod_{h \in H} K(F,1)_p) \]

\[ \square \]

The aspherical spaces studied up to this point are not, in general, finite complexes. To cover this case, we now present a $K(G,1)$ which is a virtually nilpotent (i.e., $G$ is virtually nilpotent) finite complex whose rationalization has non trivial self homotopy equivalences which fix the homotopy groups up to any dimension. Recall that, from the homotopical
point of view, an *infranilmanifold* is a manifold of the homotopy type of an aspherical complex $K(G, 1)$ in which $G$ is a torsion free, finitely generated, virtually nilpotent group containing no non trivial finite normal subgroups [8, 7]. In fact, many of such spaces are flat Riemannian manifolds.

**Theorem 2.5.** There exist a compact infranilmanifold of the form $K(G, 1)$ for which $\mathcal{E}_m^m(K(G, 1)_Q) \neq \{1\}$ for all $m \geq \dim K(G, 1)$.

**Proof.** Let $K(F, 1)$ be the non orientable $4$--manifold [7, 2] in which $F$ is the virtually nilpotent group generated by $\{x_1, x_2, x_3, x_4, \alpha, \beta\}$ with relations:

- $[x_i, x_j] = 1$, $i \neq j$,
- $\alpha^2 = x_3$, $\beta^2 = x_4$, $\alpha \beta = x_2^{-1}x_3x_4^{-1}\beta \alpha$,
- $\alpha x_1 = x_1^{-1}\alpha$, $\alpha x_2 = x_2^{-1}\alpha$, $\alpha x_3 = x_3\alpha$, $\alpha x_4 = x_4^{-1}\alpha$,
- $\beta x_1 = x_1^{-1}\beta$, $\beta x_2 = x_2^{-1}\beta$, $\beta x_3 = x_3^{-1}\beta$, $\beta x_4 = x_4\beta$.

It is indeed an infranilmanifold of dimension $4$ whose rationalization turns out to be [2, Example 5.2]:

$$K(F, 1)_Q \cong (S^2 \vee S^3 \vee S^3)_Q.$$ 

Define $G = F \times \mathbb{Z}$ and observe that $K(G, 1) = K(F, 1) \times S^1$ is in fact a $5$--dimensional manifold whose rationalization is

$$K(G, 1)_Q = (S^2 \vee S^3 \vee S^3)_Q \times S^1_Q.$$ 

We shall prove that $\mathcal{E}_m^m(K(G, 1)) \neq \{1\}$ for all $m$. For that observe in the first place that $(S^2 \vee S^3 \vee S^3) \times S^1$ is a nilpotent space so that, in view of [15],

$$\mathcal{E}_m^m((S^2 \vee S^3 \vee S^3)_Q \times S^1_Q) \cong \mathcal{E}_m^m((S^2 \vee S^3 \vee S^3) \times S^1)_Q.$$ 

Moreover, this group coincides with the group $\mathcal{E}_m^m(\Lambda V, d)$ defined as follows:

1. $(\Lambda V, d)$ is the minimal model of the space $(S^2 \vee S^3 \vee S^3) \times S^1$. In other words, $\Lambda V$ is the commutative free algebra generated by the graded vector space $V \cong \pi_*(S^2 \vee S^3 \vee S^3) \times S^1 \otimes \mathbb{Q}$, and $d$ is a certain differential satisfying $dV \subset \Lambda^{\geq 2}V$, i.e., for each generator $v \in V$, $dv$ is a polynomial in $\Lambda^V$ with no linear terms.

2. $\mathcal{E}_m^m(\Lambda V, d)$ is the subgroup of homotopy classes of differential graded algebra automorphisms $f : (\Lambda V, d) \to (\Lambda V, d)$ which satisfy $f(v) - v \in \Lambda^{\geq 2}V$ for any $v \in V$ of degree less than or equal to $m$.

To compute $\mathcal{E}_m^m(\Lambda V, d)$ observe in the first place that $(\Lambda V, d) = (\Lambda W, d) \otimes (\Lambda x, 0)$ where $(\Lambda W, d)$ is the minimal model of $(S^2 \vee S^3 \vee S^3)$ and $(\Lambda x, 0)$, where $x$ is a single element of degree $1$, is the minimal model of $S^1$. On the other hand, it is well known [11] that the space $(S^2 \vee S^3 \vee S^3)$ is both formal and coformal, that is to say, its rational homotopy type depends only on either, its rational cohomology algebra, or its rational homotopy Lie algebra. To fix notation let $H = H^*(S^2 \vee S^3 \vee S^3; \mathbb{Q})$ be the commutative algebra generated by an element $e_2$ of degree $2$ and two elements $e_3, e_3$ of degree $3$, with trivial multiplication. Then, the model $(\Lambda W, d)$ of $(S^2 \vee S^3 \vee S^3)$ is classically constructed as follows [11]: the vector space $W = W^*_e$ is built inductively bigraded and this bigrading is inherited by $\Lambda W$, according to the usual rule, so that it satisfies:

- $W_0 \cong H$, $dW_0 = 0$,
- $W_{m+1} \cong (\Lambda^2 W_{\leq m})_m \cap \ker d$; $d : W_{m+1} \longrightarrow (\Lambda^2 W_{\leq m})_m$,
- $H^*(\Lambda W, d) \cong H$,
- Any cohomology class in $H^*(\Lambda W, d)$ represented by a decomposable cycle vanishes.
Our theorem will then be established once we prove the following:

**Lemma 2.6.** \( E^m_\#(\Lambda V, d) \neq \{1\} \) for all \( m \).

We shall define an automorphism \( \phi \) of \((\Lambda V, d) = (\Lambda W, d) \otimes (\Lambda x, 0)\), inductively on \( W_m \), satisfying:

- \( \phi(a_2) = a_2, \phi(b_3) = b_3, \phi(c_3) = c_3 + xa_2, \phi(x) = x \),
- For each \( w \) generator of \( W_m, m \geq 0 \), \( \phi(w) = w + w'x, w' \in W \).

Obviously \( \phi|_{W_0} \) satisfy our hypothesis. Assume it has been extended to \( \Lambda W_{m-1} \otimes \Lambda x \) and let \( w \in W_m \). Then, \( dw \in (\Lambda^2 W_{m-1})_{m-1} \), i.e., \( dw = \sum_i u_i v_i, u_i, v_i \in W_{m-1} \).

Hence, \( \phi(dw) = \sum_i \phi(u_i)\phi(v_i) = \sum_i (u_i + w'_i x)(v_i + w''_i x) = \sum_i u_i v_i + \sum_i (u_i w''_i \pm w'_i v_i)x = dw + \alpha x \), being \( \alpha = \sum_i (u_i w''_i \pm w'_i v_i) \). On the other hand \( \phi(dw) \) is obviously a cycle so is \( \alpha \). But, since \( \alpha \) is decomposable, it must be a boundary: \( \alpha = d w', w' \in W \).

Thus \( \phi(dw) = d(w + w'x) \).

Finally define \( \phi(w) = w + w'x \) and observe that the automorphism \( \phi \) of \((\Lambda V, d)\) just defined is not homotopic to the identity as \( H^*(\phi) \neq 1_{H^*(\Lambda V, d)} \). Indeed \( H^*(\phi)[c_3] = [c_3] + [x][a_2] \). This finishes the proof of the lemma and thus, that of theorem \( \frac{1}{2} \). \( \square \)

**References**

1. M. Arkowitz and K. Maruyama, Self homotopy equivalences which induce the identity on homology, cohomology or homotopy groups, Topology and its Appl. 87(2), 133–154 (1998).
2. G. Bastardas and A. Descheemaeker, On the homotopy type of \( \mu \)-completions of infranilmanifolds, Math. Zeitschr. 241, 685–696 (2002).
3. A. Berrick and C. Casacuberta, Groups and spaces with all localization trivial, Lecture Notes in Math. 1509, 20–29 (1992).
4. A.K. Bousfield, The localization of spaces with respect to homology, Topology 14, 133–150 (1975).
5. C. Casacuberta and G. Peschke, Localizing with respect to self-maps of the circle, Trans. Amer. Math. Soc. 339, 117–140 (1993).
6. M. Cuvilliez, A. Murillo and A. Viruel, Nilpotency of self homotopy equivalences with coefficients, Submitted, 2004.
7. L.S. Charlap, Bieberbach Groups and Flat Manifolds, Universitext, Springer (1986).
8. K. Dekimpe, Almost Bieberbach Groups: Affine and Polynomial Structures, Lecture Notes in Math. 1639, Springer (1996).
9. A. Descheemaeker and W. Malfait, Virtually nilpotent groups with (almost) all localizations trivial, Math. Nachr. 217, 43–51 (2000).
10. E. Dror and A. Zabrodsky, Unipotency and nilpotency in homotopy equivalences, Topology 18, 187–197 (1979).
11. Y. Félix, J.C. Thomas and S. Halperin, Rational Homotopy Theory, Graduate Texts in Math. 205, Springer (2001).
12. A. Garvín, A. Murillo, P. Pavesic and A. Viruel, Nilpotency and localization of groups of fibre homotopy equivalences, Contemporary Math. 274, 145–157 (2001).
13. P. Hilton, G. Mislin and J. Roitberg, Localization of Nilpotent Groups and Spaces, Mathematics Studies 15, North-Holland (1975).
14. D.M. Kan and W.P. Thurston, Every connected space has the homology of a \( K(\pi, 1) \), Topology 15, 253–258 (1976).
15. K. Maruyama, Localization of a certain subgroup of self-homotopy equivalences, Pacific Journal of Math. 136, 293–301 (1989).
16. J. Möller, Self-homotopy equivalences of \( H_\ast(-; \mathbb{Z}/p) \)-local spaces, Kodai Math. Jour. 12, 270–281 (1989).
[17] P. Pavesic, On the group $\text{Aut}_3(X)$, Proceedings of the Edinburgh Mathematical Society 45, 673–680 (2002).

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