MAPPING SPACES AND $R$-COMPLETION

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Abstract. We study the questions of how to recognize when a simplicial set $X$ is of the form $X = \text{map}^\ast(Y, A)$, for a given space $A$, and how to recover $Y$ from $X$, if so. A full answer is provided when $A = K(R, n)$, for $R = \mathbb{F}_p$ or $\mathbb{Q}$, in terms of a mapping algebra structure on $X$ (defined in terms of product-preserving simplicial functors out of a certain simplicially-enriched sketch $\Theta$). In addition, when $A = \Omega^\infty A$ for a suitable connective ring spectrum $A$, we can recover $Y$ from $\text{map}^\ast(Y, A)$, given such a mapping algebra structure. Most importantly, our methods provide a new way of looking at the classical Bousfield-Kan $R$-completion.

Introduction

Given pointed topological spaces $A$ and $Y$, one can form the simplicial mapping space $\text{map}^\ast(Y, A)$, which models the topological space $\text{Hom}_{\mathbb{T}op}^\ast(Y, A)$ of all pointed continuous maps (with the compact-open topology). Such mapping spaces play a central role in modern homotopy theory, so it is natural to ask when a simplicial set $X$ is of the form $\text{map}^\ast(Y, A)$, up to weak equivalence. We assume that one of the two spaces $A$ and $Y$ is given, but not both.

This problem has been extensively studied in the case where $Y = S^n$ is a sphere, so $X$ is an $n$-fold loop space (see, e.g., [Su, Sta, M2, Ba]). A general answer for any pointed $Y$ was provided in [BBD].

Here we consider the dual problem: given a space $A$ and a simplicial set $X$,

(a) when is $X$ of the form $\text{map}^\ast(Y, A)$ for some space $Y$?

(b) If $X$ satisfies the conditions prescribed in the answer to (a), how can we recover $Y$ from it?

Observe that the best we can hope for is to recover $Y$ up $A$-equivalence – where a map $f : Y \to Y'$ is called an $A$-equivalence if it induces a weak equivalence $\text{map}^\ast(Y', A) \simeq \text{map}^\ast(Y, A)$.

Unfortunately, the methods of [BBD] do not carry over in full to the dual problem: we can answer both questions only when $A = K(R, n)$ for $R = \mathbb{Q}$ or $\mathbb{F}_p$. However, if we know that $X$ is a mapping space, and only wish to recover $Y$, we can do a little better: in this case we can allow $A$ to be $\Omega^\infty A$ for a suitable ring spectrum $A$ (in particular, $A$ can be $K(R, n)$ for any commutative ring $R$). Nevertheless, this is probably the most useful example of the problem under consideration, since the most important representable functors studied in homotopy theory are the generalized cohomology theories corresponding to such $\Omega$-spectra.

0.1. Remark. Note that when $A = K(V, n)$, $X := \text{map}^\ast(Y, A)$ is itself a GEM – i.e., it is homotopy equivalent to a product of Eilenberg-Mac Lane spaces – so

Date: May 11, 2014.

2010 Mathematics Subject Classification. Primary: 55P48; secondary: 55P20, 55P60, 55U35.

Key words and phrases. Mapping space, mapping algebra, $p$-completion, rationalization, cosimplicial resolution.
it appears to carry very little homotopy-invariant information. Thus at first sight it
seems unlikely that one could recover anything but the $R$-cohomology of $Y$ from $X$.

In particular, any GEM $X \simeq \prod_{i=1}^{n} K(M_i, i)$ for finitely-generated abelian groups
$M_0, \ldots, M_n$ is homotopy equivalent to $\map_*(Y, K(\mathbb{Z}, n))$ for some space $Y$ –
e.g., for $Y \simeq \bigvee_{i=0}^{m} L(M_i, i)$, where $L(M_i, i)$ is a co-Moore space (cf. [Ka]). Thus
the answer to (a) is always positive, for such an $X$.

There are two difficulties with this point of view:

(i) This will not necessarily work for other rings $R$, or if the abelian groups $M_i$
are not finitely generated (see [GC] and [GR below).

(ii) There are usually many other possible choices of $Y$, and the homotopy type
of $X$ alone does not enable us to distinguish between them.

With reference to the second point, we must therefore impose sufficient addi-
tional structure on $X = \map_*(Y, A)$ to allow us to recover $Y$, uniquely up to
$A$-equivalence. This structure is defined as follows:

0.2. Mapping algebras. In the original example of loop space recognition men-
tioned above, the additional data on $\Omega Y = \map_*(S^1, A)$ consisted of the group
structure, induced by the cogroup structure map $\nabla : S^1 \rightarrow S^1 \vee S^1$, its itera-
tes, and various (higher) homotopies between them. These are all encoded in the actions, by
composition, of pointed mapping spaces between wedges of circles on $\Omega Y$ (and on
$\Omega A \times \Omega A = \map_*(S^1 \vee S^1, A)$, and so on). When $Y = S^n$, this is described in
terms of a suitable operad (see [M2]), but a codification of the additional structure
needed for an arbitrary pointed space $Y$ was provided in [BB1, BBD].

The latter suggests that in the dual case one should look the action of the mapping
spaces $\map_*(\prod_{i=1}^{n} A, A)$ on $\map_*(Y, \prod_{i=1}^{n} A) = \prod_{i=1}^{n} \map_*(Y, A)$. Moreover,
since $\Omega \map_*(Y, A) = \map_*(Y, \Omega A)$, we should consider also maps products of
various loop spaces of $A$.

Although some of our results are valid for any $H$-group $A$, their full force requires
the ability to deloop $A$ arbitrarily, so we start with an $\Omega$-spectrum $A := (A_n)_{n \in \mathbb{Z}}$,
where we might have $A = A_0$ in mind for our original questions (a) and (b) above.
We can then formalize the additional structure mentioned above as follows:

Let $\Theta_A$ be the sub-category of $\mathcal{Top}_*$ (or any other simplicial category $\mathcal{C}$) whose
objects are generated by the spaces $A_n$ ($n \in \mathbb{Z}$) under products of cardinality
$< \lambda$, for a suitably chosen cardinal $\lambda$. Thus the objects of $\Theta_A$ have the form
$\prod_{i \in I} A_{n_i}$, for some $(n_i)_{i \in I}$ in $\mathbb{Z}$. This will be called an enriched sketch, since it
inherits a simplicial enrichment from $\mathcal{C}$.

For example, if $\mathcal{A} = \mathcal{HR} := (K(R, n))_{n=0}^{\infty}$ is the $R$-Eilenberg-Mac Lane spectrum
for some ring $R$, and $\lambda = \aleph_0$, then the objects of $\Theta_A$ are all finite type free
$R$-module GEMs.

A simplicial functor $\mathcal{X} : \Theta_A \rightarrow \mathcal{S}_*$ which preserved all loops and products will
be called a $\Theta_A$-mapping algebra. The main example we have in mind is a realizable
$\Theta_A$-mapping algebra, of the form $B \mapsto \map_*(Y, B)$ for some fixed object $Y$ (and
all $B \in \Theta_A$, of course).

Note that if $X = \map_*(Y, A_\lambda)$, the realizable $\Theta_A$-mapping algebra $\mathcal{X}$ corresponding to $Y$
has $\mathcal{X}(\prod_{i \in I} A_{n_i}) = \prod_{i \in I} \Omega^{k-n_i} X$, so we can think of $\mathcal{X}$ as additional
structure on the simplicial set $X$, including choices of deloopings of $X$.

It turns out that this structure is precisely what is needed to recover $Y$ from $X$,
under suitable assumptions. In fact, we show:
**Theorem A.** Let \( A = \left( \Lambda_n \right)_{n=0}^\infty \) be an \( \Omega \)-spectrum model of a connective ring spectrum, with \( R = \pi_0 A \) a commutative ring, and let \( \mathcal{X} \) be a \( \Theta_A \)-mapping algebra structure on \( X = \text{map}_*(Y, A_0) \) for a simply-connected space \( Y \). We then can construct functorially from \( \mathcal{X} \) a cosimplicial space \( W^\bullet \) with total space \( \text{Tot} W^\bullet \) weakly equivalent to the \( R \)-completion of \( Y \), and \( X \simeq \text{map}_*(\text{Tot} W^\bullet, A_0) \).

See Theorem 3.33 Corollary 4.28 and Theorem 4.29 below.

**0.3. \( \Theta \)-algebras.** If \( \Theta_A \) is an enriched sketch as above, applying \( \pi_0 \) to each of its mapping spaces yields a category \( \Theta_A := \pi_0 \Theta_A \), which is an ordinary (algebraic) sketch in the sense of Ehresmann (see §2.1). If \( \mathcal{X} : \Theta_A \rightarrow S_* \) is a \( \Theta_A \)-mapping algebra, composing with \( \pi_0 \) yields an \( \Theta_A \)-algebra \( \Lambda := \pi_0 \mathcal{X} \) — that is, a product-preserving functor \( \Lambda : \Theta_A \rightarrow \text{Set}_* \). We can think of such an \( \Theta_A \)-algebra as an algebraic version of a \( \Theta_A \)-mapping algebra. Note that the category of simplicial \( \Theta_A \)-algebras has a model category structure as in [Q II, §4], which allows us to define free simplicial resolutions of \( \Lambda \) (§5.1).

For example, if \( A = \mathcal{H} \mathbb{F}_p \) (and \( \lambda = \mathbb{N}_0 \)), \( \Theta_A \) is the homotopy category of finite type \( \mathbb{F}_p \)-GEMs, and a \( \Theta_A \)-algebra is just an algebra over the mod \( p \) Steenrod algebra, as in [Sc] §1.4.

Our main technical tool in this paper, which we hope will be of independent use, is the following:

**Theorem B.** If \( \mathcal{X} \) is a \( \Theta_A \)-mapping algebra and \( \Lambda = \pi_0 \mathcal{X} \) is the corresponding \( \Theta_A \)-algebra, then any algebraic CW-simplicial resolution \( V_* \rightarrow \Lambda \) can be realized by a cosimplicial space \( W^\bullet \).

See Theorem 5.6 below.

**0.4. Warning.** We do not claim that the space \( Y := \text{Tot} W^\bullet \) obtained by means of Theorem B in fact realizes the \( \Theta_A \)-mapping algebra \( \mathcal{X} \). In particular, if \( \mathcal{X} \) is an \( \Theta_A \)-mapping algebra structure on a simplicial set \( X \), it need not be true that \( X \simeq \text{map}_*(Y, A) \), even for \( A = \mathbb{K}(R, n) \) (see §6.5 below).

However, it turns out that, for suitable fields \( R \), the only obstruction to realizability is a purely algebraic condition on \( \Lambda = \pi_0 \mathcal{X} \). In particular, we show:

**Theorem C.** When \( A = \mathcal{H} \mathbb{R} \) for \( R = \mathbb{Q} \) or \( \mathbb{F}_p \), any simply-connected finite type \( \Theta_A \)-mapping algebra \( \mathcal{X} \) is realizable by an \( R \)-complete space \( Y \), unique up to \( R \)-equivalence.

See Theorem 6.12 below (which is somewhat more general).

Theorem C implies that for \( R = \mathbb{Q} \) or \( \mathbb{F}_p \), a finite type \( R \)-GEM \( X \) which can be endowed with a simply-connected \( \Theta_A \)-mapping algebra structure (also involving deloopings of \( X \)) is realizable as a mapping space \( X \simeq \text{map}_*(Y, \mathbb{K}(R, n)) \) for some \( R \)-complete \( Y \). Moreover, \( Y \) is uniquely determined up to \( R \)-equivalence by the choice of \( \Theta_A \)-mapping algebra structure.

**0.5. A new look at \( R \)-completion.** The three theorems above, taken together, provide us with a new way of looking at the concept of \( R \)-completion: more precisely, we have three notions associated to any commutative ring \( R \):

(a) An \( \Theta_A \)-mapping algebra \( \mathcal{X} \), for \( A = \mathcal{H} \mathbb{R} \), with the accompanying algebraic notion of the \( \Theta_A \)-algebra \( \Lambda = \pi_0 \mathcal{X} \);
(b) A cosimplicial $R$-resolution $W^\bullet$ of the $\Theta_A$-mapping algebra $\mathfrak{x}$, with the accompanying algebraic notion of a free simplicial $\Theta_A$-algebra resolution $V_\bullet$ of $\pi_0 \mathfrak{x}$ (see §1.18);
(c) The realization $\text{Tot} W^\bullet$ of the $R$-resolution $W^\bullet$, which is $R$-complete in the simply-connected case.

When $R = \mathbb{F}_p$ or $\mathbb{Q}$, under mild assumptions on $\mathfrak{x}$ (e.g., when it is simply connected and of finite type) we show that these notions are equivalent, inasmuch as the original $\mathfrak{x}$ is in fact the $\Theta_A$-mapping algebra of $\text{Tot} W^\bullet$, up to weak equivalence.

Moreover, each of these three notions has certain advantages:

(a) The $\Theta_A$-mapping algebra $\mathfrak{x}$ exhibits in an explicit form all the information about a space $Y$ which is retained by its $R$-completion.
(b) The cosimplicial space $W^\bullet$, realizing the algebraic resolution $V_\bullet$ of $H^\ast(Y; R)$, encodes the higher order cohomology operations for $Y$ in a visible manner (see §5.32 below)
(c) The $R$-complete space $\text{Tot} W^\bullet = R^\infty Y$ allows us to work with a single space which retains all the above information.

0.6. **Notation.** The category of topological spaces will be denoted by $\mathcal{Top}$, and that of pointed topological spaces by $\mathcal{Top}_\ast$. The category of simplicial sets will be denoted by $\mathcal{S} = \mathcal{sSet}$, that of pointed simplicial sets by $\mathcal{S}_\ast = \mathcal{sSet}_\ast$, and that of pointed Kan complexes by $\mathcal{S}^\ast_{\mathcal{Kan}}$.

Unless otherwise stated, $\mathcal{C}$ will be a pointed simplicial model category in which every object is cofibrant (cf. [Q, II, §2]) – so in particular it is left proper. We assume moreover that both $\otimes$ and $\times$ preserve cofibrations. The objects of $\mathcal{C}$ will be denoted by boldface letters: $X, Y, \ldots$. The main example we shall be concerned with is $\mathcal{S}_\ast$, and by space we always mean a (pointed) simplicial set.

0.7. **Organization.** In Section 1 we recall some facts about (co)simplicial objects, and in particular Bousfield’s resolution model category structure. In Section 2 we define the notions of enriched sketches and mapping algebras, and in Section 3 we explain how this structure can be used to recover $Y$ from $X = \text{map}_\ast(Y, A)$ by means of a suitable cosimplicial resolution $Y \to W^\bullet$, when $A = K(R, n)$ for a field $R$. In Section 4 we modify the construction of this cosimplicial resolution to obtain Theorem A. In Section 5 we prove our main technical result, Theorem B, showing how any algebraic resolution of $\pi_0 \mathfrak{x}$ can be realized in $\mathcal{C}$. This is used in Section 6 to prove Theorem C, which allows us to recognize mapping spaces of the form $\text{map}_\ast(Y, K(R, n))$ and recover $Y$ up to $R$-completion.

0.8. **Acknowledgements.** We wish to thank Pete Bousfield for many useful comments and elucidations of his work, and Paul Goerss for a helpful pointer.

1. **The Resolution Model Category of Cosimplicial Objects**

The main technical tool for reconstructing the source $Y$ of a mapping space $X := \text{map}_\mathcal{C}(Y, A)$ in a pointed simplicial model category $\mathcal{C}$, such as $\mathcal{S}_\ast$, is the construction of a suitable cosimplicial resolution $W^\bullet$ of the putative $Y \in \mathcal{C}$.

The proper framework for obtaining such a $W^\bullet$ is Bousfield’s resolution model category of cosimplicial objects over $\mathcal{C}$, which generalizes and dualizes the Dwyer-Kan-Stover theory of the $E^2$-model category of simplicial spaces (cf. [DKSt]).
1.1. Simplicial and cosimplicial objects. We first collect some standard facts and constructions related to (co)simplicial objects in any category $C$.

Let $\Delta$ denote the category of finite ordered sets and order-preserving maps (cf. [ML §2]), and $\Delta^+$ the subcategory with the same objects, but only monic maps. A cosimplicial object $G^\bullet$ in a category $C$ is a functor $\Delta \to C$, and a restricted cosimplicial object is a functor $\Delta^+ \to C$. More concretely, we write $G^n$ for the value of $G^\bullet$ at the ordered set $[n] = (0 < 1 < \ldots < n)$. The maps in the diagram $G^\bullet$ are generated by the coface maps $d^i = d^i_n : G^n \to G^{n+1}$ $(0 \leq i \leq n + 1)$, as well as codegeneracy maps $s^j = s^j_n : G^n \to G^{n-1}$ $(0 \leq j < n)$ in the non-restricted case, satisfying the usual cosimplicial identities.

Dually, a simplicial object $G_\bullet$ in $C$ is a functor $\Delta^{\text{op}} \to C$. The category $\text{C}^\Delta$ of cosimplicial objects over $C$ will be denoted by $cC$, and the category $\text{C}^{\Delta^{\text{op}}}$ of simplicial objects over $C$ will be denoted by $sC$.

There are natural embeddings $c(-)^\bullet : C \to cC$ and $c(-)_\bullet : C \to sC$, defined by letting $c(A)^\bullet$ denote the constant cosimplicial object which is $A$ in every cosimplicial dimension, and similarly for $c(A)_\bullet$.

1.2. Latching and matching objects. For a cosimplicial object $G^\bullet \in cC$ in a complete category $C$, the $n$-th matching object for $G^\bullet$ is defined to be

\begin{equation}
M^nG^\bullet := \lim_{\phi : [n] \to [k]} G^k,
\end{equation}

where $\phi$ ranges over the surjective maps $[n] \to [k]$ in $\Delta$. There is a natural map $\zeta^n : G^n \to M^nG^\bullet$ induced by the structure maps of the limit, and any iterated codegeneracy map $s^l = \phi_* : G^k \to G^n$ factors as

\begin{equation}
s^l = \text{proj}_\phi \circ \zeta^n,
\end{equation}

where $\text{proj}_\phi : M^nG^\bullet \to G^k$ is the structure map for the copy of $G^k$ indexed by $\phi$ (cf. [BKL X, §4.5]).

Similarly, the $n$-th latching object for $G^\bullet \in cC$ is the colimit

\begin{equation}
L^nG^\bullet := \text{colim}_{\theta : [k] \to [n]} G^k,
\end{equation}

where $\theta$ ranges over the injective maps $[k] \to [n]$ in $\Delta$ (for $k < n$), with $\sigma^n : L^nG^\bullet \to G^n$ defined by the structure maps of the colimit.

These two constructions have analogues for a simplicial object $G_\bullet$ over a cocomplete category $C$: the latching object $L_nG_\bullet := \text{colim}_{\theta : [k] \to [n]} G_k$, and the matching object $M_nG_\bullet := \text{lim}_{\theta : [n] \to [k]} G_k$, equipped with the obvious canonical maps.

1.6. Definition. If $C$ is pointed and complete, the $n$-th Moore chains object of $G_\bullet \in sC$ is defined to be:

\begin{equation}
C_nG_\bullet := \bigcap_{i=1}^n \text{Ker}(d_i : G_n \to G_{n-1})
\end{equation}

with differential $\partial^nG_\bullet = \partial_n := (d_0)(C_nG_\bullet : C_nG_\bullet \to C_{n-1}G_\bullet$. The $n$-th Moore cycles object is $Z_nG_\bullet := \text{Ker}(\partial^nG_\bullet)$.
Dually, if $\mathcal{C}$ is pointed and cocomplete, the $n$-th Moore cochains object of $G^\bullet \in s\mathcal{C}$, written $C^n G^\bullet$, is defined to be the colimit of:

$$
\prod_{i=1}^{n} G^{n-1} \xrightarrow{\perp d^i} G^n
$$

with differential $\delta^n : C^{n-1} G^\bullet \to C^n G^\bullet$ induced by $d^0$.

1.9. Definition. A simplicial object $G^\bullet \in s\mathcal{C}$ over a pointed category $\mathcal{C}$ is called a CW object if it is equipped with a CW basis $(G_n)_{n=0}^\infty$ in $\mathcal{C}$ such that $G_n = G_n \amalg L_n G^\bullet$, and $d_i|_{G_n} = 0$ for $1 \leq i \leq n$. In this case $d_0^n := d_0|_{G_n} : G_n \to G_{n-1}$ is called the attaching map for $G_n$. By the simplicial identities $d_0^n$ factors as

$$
\partial d_0^n : G_n \to Z_{n-1}G^\bullet \subset G_{n-1}.
$$

In this case we have an explicit description

$$
L_n G^\bullet := \bigcup_{0 \leq k \leq n} \bigcap_{0 \leq i_1 < \ldots < i_{n-k-1} \leq n-1} G_k
$$

for its $n$-th latching object, in which the iterated degeneracy map $s_{i_{n-k-1}} \ldots s_{i_2}s_{i_1}$, restricted to the basis $G_k$, is the inclusion into the copy of $G_k$ indexed by $(i_1, \ldots, i_{n-k-1})$.

A cosimplicial CW object may be defined analogously, but we shall only need the following variant:

1.12. Definition. A cosimplicial pointed space $W^\bullet \in cS_*$ equipped with a CW basis $\overline{W}^n$ ($n \geq 0$) in $S_*$ is called a weak CW object if

(a) For each $n \geq 0$, we have a weak equivalence $\varphi^n : W^n \xrightarrow{\sim} \overline{W}^n \times M^n W^\bullet$, and we set

$$
\overline{\varphi}^n := \text{proj}_{\overline{W}^n} \circ \varphi^n : W^n \to \overline{W}^n.
$$

where $\text{proj}_{\overline{W}^n} : W^n \to \overline{W}^n$ is the projection.

(b) $\overline{\varphi}^n \circ d_{n-1}^i \sim 0$ for $1 \leq i \leq n$.

(c) If we define the attaching map for $\overline{W}^n$ to be $\overline{d}_{n-1}^0 := \overline{\varphi}^{n-1} \circ d_{n-1}^0 : W^{n-1} \to \overline{W}^n$, we require that it be a "Moore cochain" in the sense that

$$
\overline{d}_{n-1}^0 \circ d_{n-2}^i = 0
$$

for all $1 \leq i \leq n - 2$.

1.15. Remark. Recall that a simplicial model category $\mathcal{C}$ is one in which, for each (finite) $K \in S_*$ and $X \in \mathcal{C}$, we have objects $X \otimes K$ and $X^K$ in $\mathcal{C}$ equipped with appropriate adjunction-like isomorphisms and axiom SM7 (see [Q, II, §1-2]). In particular, such model categories are simplicially enriched.

1.16. Reedy model structure. If $\mathcal{C}$ is a model category, the Reedy model structure on $s\mathcal{C}$ (cf. [Bo2, §2.2]) is defined by letting a cosimplicial map $f : X^\bullet \to Y^\bullet$ in $s\mathcal{C}$ be:
(i) a Reedy weak equivalence when \( f : X^n \to Y^n \) is a weak equivalence in \( C \) for \( n \geq 0 \);
(ii) a Reedy cofibration when \( X^n \prod_{L^n X} L^n Y^\bullet \to Y^n \) is a cofibration in \( C \) for \( n \geq 0 \);
(iii) a Reedy fibration when \( X^n \to Y^n \prod_{M^n X} M^n X^\bullet \) is a fibration in \( C \) for \( n \geq 0 \).

The Reedy model category structure on \( \mathcal{C} \) is defined dually (see [Hi, §15.3]).

1.17. Definition. The total space \( \text{Tot} W^\bullet \) of a cosimplicial space \( W^\bullet \in \mathcal{C} \) is defined to be the simplicial mapping space \( \text{map}_{\mathcal{C}}(\Delta^n, W^\bullet) \), where \( \Delta^n \in \mathcal{C} \) has \( \Delta^n := \Delta[n] \in \mathcal{S} \) (the standard \( n \)-simplex) – see [BK1, X, §3].

We write \( \text{Tot} W^\bullet := \text{Tot} V^\bullet \), where \( W^\bullet \to V^\bullet \) is a functorial Reedy fibrant replacement.

1.18. \( \mathcal{G} \)-resolution model structure. Let \( \mathcal{G} \) be a class of homotopy group objects in a pointed model category \( \mathcal{C} \), closed under loops. We shall be mainly interested in the case where \( \mathcal{G} \) consists of the \( \Omega^\infty \)-spaces in a class of \( \Omega \)-spectra in \( \mathcal{C} = S_* \).

A map \( i : A \to B \) in \( \text{ho} \mathcal{C} \) is called \( \mathcal{G} \)-monic if \( i^* : [B,G] \to [A,G] \) is onto for each \( G \in \mathcal{G} \). An object \( Y \) in \( \mathcal{C} \) is called \( \mathcal{G} \)-injective if \( i^* : [B,Y] \to [A,Y] \) is onto for each \( \mathcal{G} \)-monic map \( i : A \to B \) in \( \text{ho} \mathcal{C} \). A fibration in \( \mathcal{C} \) is called \( \mathcal{G} \)-injective if it has the right lifting property for the \( \mathcal{G} \)-monic cofibrations in \( \mathcal{C} \).

The homotopy category \( \text{ho} \mathcal{C} \) is said to have enough \( \mathcal{G} \)-injectives if each object is the source of a \( \mathcal{G} \)-monic map to a \( \mathcal{G} \)-injective target. In this case \( \mathcal{G} \) is called a class of injective models in \( \text{ho} \mathcal{C} \).

Recall that a homomorphism in the category \( s\mathcal{G} \) of simplicial groups is a weak equivalence or fibration when its underlying map in \( \mathcal{S} \) is such. A map \( f : X^\bullet \to Y^\bullet \) in \( \mathcal{C} \) is called

(i) a \( \mathcal{G} \)-equivalence if \( f^* : [Y^\bullet, G] \to [X^\bullet, G] \) is a weak equivalence in \( s\mathcal{G} \) for each \( G \in \mathcal{G} \);
(ii) a \( \mathcal{G} \)-cofibration if \( f \) is a Reedy cofibration and \( f^* : [Y^\bullet, G] \to [X^\bullet, G] \) is a fibration in \( s\mathcal{G} \) for each \( G \in \mathcal{G} \);
(iii) a \( \mathcal{G} \)-fibration if \( f : X^n \to Y^n \prod_{M^n Y} M^n X^\bullet \) is a \( \mathcal{G} \)-injective fibration in \( \mathcal{C} \) for each \( n \geq 0 \).

In [Bo2, Theorem 3.3], Bousfield showed that if \( \mathcal{C} \) is a left proper pointed model category and \( \mathcal{G} \) is a class of injective models in \( \text{ho}(\mathcal{C}) \), the above defines a left proper pointed simplicial model category structure on \( \mathcal{C} \).

1.19. Definition. Given a class \( \mathcal{G} \) of homotopy group objects in a model category \( \mathcal{C} \) as above, a cosimplicial object \( W^\bullet \in \mathcal{C} \) is called weakly \( \mathcal{G} \)-fibrant if it is Reedy fibrant, and every \( W^n \) is in \( \mathcal{G} \) \( (n \geq 0) \). A weak \( \mathcal{G} \)-resolution of an object \( Y \in \mathcal{C} \) is a weakly \( \mathcal{G} \)-fibrant \( W^\bullet \) which is \( \mathcal{G} \)-equivalent to \( c(Y)^\bullet \) \( (\text{cf. §11}) \). See [Bo2, §6].

2. ENRICHED SKETCHES AND MAPPING ALGEBRAS

We now set up the categorical framework needed to describe the relevant extra structure on a mapping space.

2.1. Definition. Let \( \Theta \) be an sketch, in the sense of Ehresmann (cf. [E], [Bo, §5.6]): that is, a small pointed category with a distinguished set \( \mathcal{P} \) of (small) products (including the empty product \( * \)). A \( \Theta \)-algebra is a functor \( A : \Theta \to \text{Set}_* \) which preserves the products in \( \mathcal{P} \). We think of a map \( \phi : \prod_{i < \kappa} a_i \to \prod_{j < \lambda} b_j \) in \( \Theta \) as
representing an $\lambda$-valued $\kappa$-ary operation on $\Theta$-algebras, with gradings indexed by $(a_i)_{i<\kappa}$ and $(b_j)_{j<\lambda}$, respectively.

The category of $\Theta$-algebras is denoted by $\Theta$-$\text{Alg}$. If each object of $\Theta$ is uniquely representable (up to order) as a product of elements in a set $O \subseteq \text{Obj } \Theta$, there is a forgetful functor $U : \Theta$-$\text{Alg} \to \text{Set}^O$ into the category of $O$-graded sets, with left adjoint the free $\Theta$-algebra functor $F : \text{Set}^O \to \Theta$-$\text{Alg}$.

2.2. Example. The simplest kind of a sketch is a theory in the sense of Lawvere (cf. [L]), in which Obj $\Theta = \mathbb{N}$ is generated under products by a single object, so that $\Theta$-algebras are simply sets with additional structure. For example, the theory $\mathcal{G}$ whose algebras are groups is just the opposite category of the homotopy category of finite wedges of circles.

2.3. Definition. We define a $\mathcal{G}$-sketch to be a sketch $\Theta$ equipped with an embedding of sketches $\mathcal{G}^O \hookrightarrow \Theta$, for $O$ as above. In this case, any $\Theta$-algebra $\Lambda$ has a natural underlying $O$-graded group structure. We do not require the operations of a $\mathcal{G}$-sketch to be homomorphisms (that is, commute with the $\mathcal{G}$-structure).

2.4. Example. Almost all varieties of (graded) universal algebras, in the sense of [Mc, V, §6] – such as groups, associative, or Lie algebras, and so on – have an underlying (graded) group structure, so they are categories of $\Theta$-algebras for a suitable $\mathcal{G}$-sketch $\Theta$.

2.5. Proposition. If $\Theta$ is a $\mathcal{G}$-sketch, the category $s\Theta$-$\text{Alg}$ of simplicial $\Theta$-algebras has a model category structure, in which the weak equivalences and fibrations are defined objectwise.

Proof. See [BP, §6], which is a slight generalization of [Q, II, §4].

2.6. Enriched sketches and algebras. There is also an enriched version of the notions defined above, introduced in [BB1], in which we assume that the theory, or sketch, is simplicially enriched, and the algebras over it are simplicial. This takes place in the context of a simplicially enriched model category. Note that in fact any model category $C$ can be enriched over $S$ – that is, for each $X,Y \in C$, there is a simplicial mapping space $\text{map}_C(X,Y)$, with continuous compositions, such that $[X,Y]_{hoC}$ is equal to its set of components $\pi_0\text{map}_C(X,Y)$ (cf. [DK1]).

2.7. Definition. Let $C$ be a simplicial model category as in §0.6, and $\lambda$ some limit cardinal (to be determined by the context – see Remark 5.7 below). An enriched sketch $\Theta$ is a small full sub-simplicial category of $C$, closed under loops (cf. [Q, I, §2]), with a distinguished set of products $P$. We assume all objects in $\Theta$ are fibrant (and cofibrant) homotopy group objects in $C$.

2.8. Example. Let $F = \{A^i\}_{i \in I}$ be a set of $\Omega$-spectra, so each $A^i_n \cong \Omega A^i_{n+1}$ is an $\Omega^\infty$-space, and let $\Theta_F$ denote the full sub-simplicial category of $C$ whose objects are products of the spaces $A^i_n$ of cardinality $< \lambda$. We write

\begin{equation}
\hat{F} := \{A^i_n : A^i \in F, n \in \mathbb{Z}\} \subseteq \text{Obj } \Theta_F .
\end{equation}

When $F = \{A\}$ consists of a single spectrum, we write $\Theta_A$ for $\Theta_F$. In particular, for any ring (or abelian group) $R$ we let $A = HR$ be the corresponding Eilenberg-Mac Lane spectrum, and denote the resulting enriched sketch $\Theta_A$ (for
\[ \lambda = \aleph_0 \] by \( \Theta_R \). Thus the objects of \( \Theta_R \) are finite type \( R \)-GEMs (generalized Eilenberg-Mac Lane spaces) — that is, spaces of the form \( \prod_{i=1}^{N} K(R, m_i) \) \( (m_i \geq 1) \).

Note that in this case we may assume that each object in \( \Theta_R \) is a strict abelian group object.

2.10. \textit{Remark.} We can always assume that the cardinal \( \lambda \) for \( \Theta_F \) is 1, by including all products of \( \Omega \)-spectra of the requisite cardinality in the original list \( F \) itself. The reason for specifying \( \lambda \) in the definition of an enriched sketch is because we want to think of elements in \( \text{map}_{\Theta}([\prod_{i=0}^{n} B_i, B]) \) as continuous \( n \)-ary operations (as in the discrete case — cf. \cite{Ek}).

2.11. \textbf{Definition.} In a pointed simplicial model category \( \mathcal{C} \) (cf. \cite{BB1}), the inclusions \( i_0, i_1 : * \to \Delta[1] \) induce natural “evaluation maps” \( ev_0, ev_1 : X^{\Delta[1]} \to X \), which are trivial fibrations, for any \( X \in \mathcal{C} \). This allows one to define the \textit{path} and \textit{loop} objects in \( \mathcal{C} \) by the pullback diagrams:

\begin{equation}
\begin{array}{ccc}
PX & \xrightarrow{\sim} & X_{\Delta[1]} \\
\downarrow_{PB} \downarrow_{X} & & \downarrow_{ev_0} \downarrow_{X} \\
* \sim \downarrow_{X} & \xleftarrow{\sim} & PX
\end{array}
\end{equation}

These will also be our models for simplicial path and loop spaces \( PK \) and \( \Omega K \) for any Kan complex \( K \in S^*_{Kan} \).

2.13. \textbf{Definition.} For \( \mathcal{C} \) a model category as in \cite{BB1} and \( \Theta \subseteq \mathcal{C} \) an enriched sketch as above, a \( \Theta \)-mapping algebra \( \text{ is } \) a pointed simplicial functor \( X : \Theta \to S_* \), written \( X : B \mapsto X\{B\} \) for any \( B \in \Theta \), taking values in Kan complexes, and satisfying the following three conditions:

(a) The natural map \( X\{\prod_{i<\lambda} B_i\} \to \prod_{i<\lambda} X\{B_i\} \) is an isomorphism for products \( \prod_{i<\lambda} B_i \) in our distinguished set \( \mathcal{P} \).

(b) Using the convention that \( X\{B^K\} := (X\{B\})^K \) for any finite simplicial set \( K \), and that \( X\{PB\} := PX\{B\} \), we require that \( X \) preserve the right hand pullback squares of (2.12) for all \( X \in \Theta \), so that we also have a natural identification

\[ X\{\Omega B\} = \Omega X\{B\} \].

(c) Any cofibration \( i : B \hookrightarrow B' \) in \( \Theta \) induces an inclusion \( i_\# : X\{B\} \to X\{B'\} \) for all \( B \in \Theta \).

The category of \( \Theta \)-mapping algebras will be denoted by \( \text{Map}_{\Theta} \).

2.14. \textbf{Definition.} For a given object \( Y \in \mathcal{C} \), we have a \textit{realizable} \( \Theta \)-mapping algebra \( \mathcal{M}_\Theta \text{Y} \) defined for any \( B \in \Theta \) by \( \mathcal{M}_\Theta \text{Y}\{B\} := \text{map}_\mathcal{C}(Y, B) \). When \( C = S_* \) and \( \Theta = \Theta_R \), we shall denote this by \( \mathcal{M}_R \text{Y} \). The \textit{realizable} \( \Theta \)-mapping algebra \( \mathcal{M}_\Theta \text{B} \) for \( B \in \Theta \) will be called \textit{free}.

2.15. \textbf{Lemma (cf. \cite{BB1} 8.17).} If \( \mathcal{M} \) is an \( \Theta \)-mapping algebra and \( \mathcal{M}_\Theta \text{B} \) is a free \( \Theta \)-mapping algebra (for \( B \in \Theta \)), there is a natural isomorphism

\[ \Phi : \text{map}_{\text{Map}_\Theta}(\mathcal{M}_\Theta \text{B}, \mathcal{M}) \to \mathcal{M}\{B\} \],

with

\[ \Phi(f) = f(Id_B) \in \mathcal{M}\{B\} \text{ for any } f \in \text{Hom}_{\text{Map}_\Theta}(\mathcal{M}_\Theta \text{B}, \mathcal{M}) = \text{map}_{\text{Map}_\Theta}(\mathcal{M}_\Theta \text{B}, \mathcal{M})_0. \]
2.16. Definition. Given an Ω-spectrum $\mathcal{A} = (A_n)_{n \in \mathbb{Z}}$ in a model category $C$ as in [0G] we have an associated enriched sketch $\mathcal{C} = \Theta$ whose objects are of the form $B = \prod_{i \in I} A_n$ with $|I| < \lambda$. In this case an $\Theta$-mapping algebra structure on a simplicial set $X$ is a $\Theta$-mapping algebra $X$ with

\[(2.17)\]

$$X(\prod_{i \in I} A_n) = \prod_{i \in I} \Omega^{-n_i} X,$$

which implicitly involves choices of deloopings of $X = X(\Lambda_0)$. This defect can be remedied by defining a suitable $\Theta$-mapping algebra structure has no implications for any $X' \simeq X$. This defect can be remedied by defining a suitable $\Theta$-mapping algebra structure.

2.18. Remark. The $\Theta$-mapping algebra structures we define here are rigid, in the sense that the action of the mapping spaces between objects of $\Theta$ is strict. In particular, the fact that $X = map_C(Y, A_n)$ has a $\Theta_A$-mapping algebra structure has no implications for any $X' \simeq X$. This defect can be remedied by defining a suitable notion of a lax $\Theta$-mapping algebra, as was done in the dual case in [BBD] §6.

2.19. The associated algebraic sketch. To any enriched sketch $\Theta$ in a simplicial model category $C$ we can associate an algebraic $\Theta$-sketch $\Theta := \pi_0 \Theta$, with the same objects as $\Theta$, where $\text{Hom}_\Theta(A, B) := \pi_0 \text{map}_\Theta(A, B)$.

A $\Theta$-algebra $\Lambda : \pi_0 \Theta \rightarrow \text{Set}_*$ is called enrichable if it is of the form $\Lambda_X := \pi_0 X$ for some $X \in Map_\Theta$ (not necessarily unique).

We define a map of $\Theta$-mapping algebras $f : X \rightarrow Y$ to be a weak equivalence if it induces an isomorphism $f_\# : \pi_0 X \rightarrow \pi_0 Y$ of the corresponding $\Theta$-algebras.

2.20. Definition. A $\Theta$-algebra $\Lambda : \pi_0 \Theta \rightarrow \text{Set}_*$ is realizable (in $C$) if it is enrichable by a realizable $\Theta$-mapping algebra $M_\Theta Y$ — that is, $\Lambda \simeq \Lambda_{M_\Theta Y}$ for some $Y \in C$ (again, not necessarily unique). In this case we say that $Y$ realizes $\Lambda$. Any $\Theta$-algebra of the form $\pi_0 X$, where $X$ is a free $\Theta$-mapping algebra, will be called a free $\Theta$-algebra.

2.21. Remark. In principle, any coproduct of free $\Theta$-mapping algebras or $\Theta$-algebras is also free (in the sense of being in the image of the left adjoint of an appropriate forgetful functor). However, in order to avoid the question of realizability for arbitrary free $\Theta$-mapping algebras (or $\Theta$-algebras), we restrict attention to coproducts of monogenic objects of cardinality $< \lambda$.

We also have an algebraic version of Lemma 2.15 which follows from the usual Yoneda Lemma:

2.22. Lemma. If $\Lambda$ is an $\Theta$-algebra and $\pi_0 M_\Theta B$ is a free $\Theta$-algebra (for $B \in \Theta$), there is a natural isomorphism $\text{Hom}_{\Theta\text{-Alg}}(\pi_0 M_\Theta B, \Lambda) \cong \Lambda\{B\}$.

2.23. Example. For any ring $R$, with $\Theta_R$ as in [2.8] we obtain an $\Theta$-sketch $\Theta_R := \pi_0 \Theta_R$, namely, the full subcategory of $\text{ho}S_*$ whose objects are finite type $R$-GEMs (which are abelian group objects in $\text{ho}S$). Note that the cohomology functor $H^*(-; R)$ in fact lands in $\Theta_R\text{-Alg}$, and realizable $\Theta_R$-algebras are those which correspond to actual spaces.

2.24. Simplicial $\Theta$-mapping algebras. Unfortunately, there seems to be no useful model category structure on the category $Map_\Theta$ of $\Theta$-mapping algebras.
However, we do have a model category structure on the functor category \( S^\otimes \), in which weak equivalences and fibrations are defined objectwise (cf. [DK2, §7], and compare [BB2 §8]), and any free \( \Theta \)-mapping algebra is in fact a homotopy cogroup object for \( S^\otimes \). Thus we obtain a resolution model category structure (cf. [Bo2]) on the category \( sS^\otimes_s \) of simplicial simplicially-enriched functors \( \Theta \to S^\otimes_s \), in which a map \( f : \mathcal{Y}_* \to \mathcal{Y}_* \) is a weak equivalence if for each \( B \in \Theta \), the map \( \pi_0 \mathcal{Y}_*\{B\} \to \pi_0 \mathcal{Y}_*\{B\} \) is a weak equivalence of simplicial groups (see [BB2 §2.2]).

3. Mapping algebras and cosimplicial resolutions

We now explain how the \( \Theta \)-mapping algebra structure on \( X = \text{map}_S(Y, A) \) described in §3.3. suffices to construct a cosimplicial resolution of \( Y \) from \( X \), generalizing the results of [BB2, §3] for \( \Theta = \Theta_R \) when \( R \) is a field.

Let \( F \) be a set of \( \Omega \)-spectra. By construction we have a functor \( \pi_0 : \text{Map}_{\Theta_F} \to \Theta_F\text{-Alg} \) associating to any \( \Theta_F\)-mapping algebra \( \mathcal{X} \) its \( \Theta_F \)-algebra \( \pi_0 \mathcal{X} \) (cf. §2.19). Since \( \Theta_F \)-mapping algebras are rather complicated objects, it is natural to ask whether this functor factors through some simpler category. Evidently, for any fibrant simplicial set \( K \), \( \pi_0 K \) depends only on the 0-simplices and their homotopies, i.e., on the 1-truncation \( \tau_1 K \) of \( K \). However, we need even less information if \( K \) is a group object in \( \text{ho} S_\ast \):

3.1. Definition. In a model category \( C \) as in [1,2] an \( H \)-group is a fibrant (and cofibrant) homotopy group object – that is, an object \( X \in C \) equipped with structure maps \( \mu : X \times X \to X \) and \( (-)^{-1}X \to X \), (with \( * \to X \) as the identity element), as well as chosen homotopies for each of the identities in \( \Theta \) (cf. §2.2) such as: \( H : \mu \circ (\mu \times \text{Id}) \sim \mu \circ (\text{Id} \times \mu) \) for the associativity, \( G : \mu \circ ((-)^{-1} \times \text{Id}) \circ \text{diag} \sim c_\ast \) for the left inverse, and so on.

For any cofibrant \( Y \in C \), the Kan complex \( K := \text{map}_C(Y, X) \) inherits an \( H \)-group structure in \( S_\ast \). We define an equivalence relation on its 0-simplices by setting:

\[
(3.2) \quad f \sim g \iff \exists \alpha \in K_1 \text{ such that } d_0 \alpha = \mu_\ast(f^{-1}) \text{ and } d_1 \alpha = * .
\]

for \( f, g, \in K_0 \). Note that \( \alpha \in (PK)_0 \) (see §2.11).

3.3. Lemma. For any \( H \)-group \( X \in C \) as above,

\[
\pi_0 \text{map}_C(Y, X) \cong (\text{map}_C(Y, X))_0/ \sim .
\]

3.4. Definition. If \( F = \{A^i\}_{i \in I} \) is some set of \( \Omega \)-spectra in a pointed model category \( C \), and \( \hat{F} := \{\hat{A}^i\}_{i \in F, n \in \mathbb{Z}} \) as above, a discrete \( F \)-mapping algebra is a function \( \mathcal{Y} : \hat{F} \to \text{Set}^J_s \), written \( \mathcal{A}^i \mapsto (P\mathcal{Y}\{\hat{A}^i\} \xrightarrow{\rho_n} \mathcal{Y}\{\hat{A}^i\}) \) \( (A \in F, n \in \mathbb{Z}) \), where \( J \) denotes the single-arrow category \( 0 \to 1 \). The category of discrete \( F \)-mapping algebras will be denoted by \( \text{Map}_{F,d} \).

3.5. Definition. Let \( \rho : S^\text{Kan} \to \text{Set}^J_s \) be the functor assigning to a Kan complex \( K \) the 0-simplices \( p_0 : (PK)_0 \to K_0 \) of its path fibration \( p : PK \to K \) (cf. §2.11).

If \( \mathcal{X} \) is an \( \Theta_F \)-mapping algebra for \( F \) as above, the associated discrete \( F \)-mapping algebra \( \rho \mathcal{X} \) is defined by setting

\[
\rho \mathcal{X}(\mathcal{A}^i) := p_0 : \mathcal{X}(P\mathcal{A}^i)_0 \to \mathcal{X}(\mathcal{A}^i)_0 .
\]

This defines a functor \( \rho : \text{Map}_{\Theta_F} \to \text{Map}_{F,d} \), since \( \mathcal{X} \) takes values in Kan complexes. Moreover, we may define a covariant functor \( \mathcal{L}_F : C \to \text{Map}_{F,d}^{op} \) by \( (\mathcal{L}_F \mathcal{Y})(\mathcal{A}^i) := \rho \mathcal{M}_F \mathcal{Y}(\mathcal{A}^i) \) for each \( A \in F \).
3.6. Remark. Note that \( \text{Map}_{\mathcal{F}, d} \) is just the diagram category \( \text{Set}_{\mathcal{F}}^T \), indexed by a linear category \( \Gamma \) consisting of a single non-identity arrow \( q_n : \mathbb{A}_n \to \mathbb{A}_n \) for each \( n \in \mathbb{Z} \) and \( \mathcal{A} \in \mathcal{F} \), and thus no non-trivial compositions.

Therefore, we have a pullback diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\text{Set}_{\mathcal{F}}^T}(\mathcal{X}, \mathcal{Y}) & \xrightarrow{\text{PB}} & \prod_{\mathcal{A} \in \mathcal{F}} \prod_{n \in \mathbb{Z}} \text{Hom}_{\text{Set}}(\mathcal{X}(\mathbb{A}_n), \mathcal{Y}(\mathbb{A}_n)) \\
\downarrow & & \downarrow \mathcal{Y}(q_n)_* \\
\prod_{\mathcal{A} \in \mathcal{F}} \prod_{n \in \mathbb{Z}} \text{Hom}_{\text{Set}}(\mathcal{X}(\mathbb{A}_n), \mathcal{Y}(\mathbb{A}_n)) & \xrightarrow{\text{x}(q_n)_*} & \prod_{\mathcal{A} \in \mathcal{F}} \prod_{n \in \mathbb{Z}} \text{Hom}_{\text{Set}}(\mathcal{X}(\mathbb{A}_n), \mathcal{Y}(\mathbb{A}_n))
\end{array}
\]

for any \( \mathcal{X}, \mathcal{Y} \in \text{Set}_{\mathcal{F}}^T \). Thus \( \text{Hom}_{\text{Set}_{\mathcal{F}}^T}(\mathcal{X}, \mathcal{Y}) \) is a product over \( n \in \mathbb{Z} \) and \( \mathcal{A} \in \mathcal{F} \) (of pullback squares).

We deduce from Lemma 3.3 and the fact that each \( \mathbb{A}_n \) in \( \mathcal{A} \) is an (infinite) loop space:

3.8. Lemma. For any \( \Theta_\mathcal{F} \)-mapping algebra \( \mathcal{X} \), the \( \Theta_\mathcal{F} \)-algebra \( \pi_0 \mathcal{X} \) is determined by \( \rho \mathcal{X} \) (together with the maps \( \mu_n : \mathcal{X}(\mathbb{A}_n \times \mathbb{A}_n)_0 \to \rho \mathcal{X}(\mathbb{A}_n)_0 \)).

3.9. The dual Stover construction. In [Sto], Stover described a certain comonad on topological spaces which eventually was shown to produce simplicial resolutions in the \( E^2 \)-model category of [DKS1]. We now give a more conceptual description of the dual construction.

Since the functor \( L_\mathcal{F} : S_* \to \text{Map}_{\mathcal{F}, d}^\text{op} \) of (3.5) has sufficient information to calculate the homotopy groups \( \pi_0(M_{\mathcal{A} Y}(\mathbb{A}_n)) \cong [Y, \mathbb{A}_n] \) — that is, the \( \mathcal{A} \)-cohomology of \( Y \) for each \( \mathcal{A} \in \mathcal{F} \) — if we can construct a right adjoint \( R_\mathcal{F} : \text{Map}_{\mathcal{F}, d}^\text{op} \to L_\mathcal{F} \), taking value in \( \mathcal{G} \)-injective for some class \( \mathcal{G} \supseteq \mathcal{F} \), we might be able to use it to produce a cosimplicial weak \( \mathcal{G} \)-resolution of any space \( Y \), and thus its \( \mathcal{G} \)-completion \( L_\mathcal{G} Y \).

Since \( L_\mathcal{F} \) is contravariant, we need a natural isomorphism

\[
\text{Hom}_\mathcal{C}(Y, R_\mathcal{F} \mathcal{X}) \cong \text{Hom}_{\text{Map}_{\mathcal{F}, d}^\text{op}}(L_\mathcal{F} Y, \mathcal{X}) \cong \text{Hom}_{\text{Set}_{\mathcal{F}}^T}(\mathcal{X}, L_\mathcal{F} Y)
\]

for any \( Y \in \mathcal{C} \) and discrete \( \mathcal{F} \)-mapping algebra \( \mathcal{X} \), where \( \text{Set}_{\mathcal{F}}^T = \text{Map}_{\mathcal{F}, d}^\text{op} \).

By (3.7), the right hand side naturally splits as a product over \( \mathcal{A} \in \mathcal{F} \) and \( n \in \mathbb{Z} \) of pullback squares in \( \text{Set}_{\mathcal{F}}^T \):

\[
\begin{array}{ccc}
M_n & \xrightarrow{\text{PB}} & \text{Hom}_{\text{Set}_{\mathcal{F}}^T}(\mathcal{X}(\mathbb{A}_n), L_\mathcal{F} Y(\mathbb{A}_n)) \\
\downarrow & & \downarrow x(q_n)_* \\
\text{Hom}_{\text{Set}_{\mathcal{F}}^T}(\mathcal{X}(\mathbb{A}_n), L_\mathcal{F} Y(\mathbb{A}_n)) & \xrightarrow{x(q_n)_*} & \text{Hom}_{\text{Set}_{\mathcal{F}}^T}(\mathcal{X}(\mathbb{A}_n), L_\mathcal{F} Y(\mathbb{A}_n))
\end{array}
\]
Since each pointed set \( X\{P\underline{A}_n\} \) is a coproduct in \( \text{Set}_* \) of its non-zero element singletons, we can re-write this as

\[
\begin{array}{ccc}
M_n & \longrightarrow & \prod_{x\{P\underline{A}_n\}} \text{Hom}_C(Y, P\underline{A}_n) \\
\downarrow & & \downarrow \text{(p\underline{A}_n)\#} \\
\prod_{x(\underline{A}_n)} \text{Hom}_C(Y, \underline{A}_n) & \longrightarrow & \prod_{x(P\underline{A}_n)} \text{Hom}_C(Y, \underline{A}_n) \\
\end{array}
\]

using the convention that the factor indexed by the basepoint of \( X\{P\underline{A}_n\} \) or \( X\{\underline{A}_n\} \) is identified to zero.

Therefore, the right-hand side of (3.10) splits naturally as a product over \( \mathcal{A} \in \mathcal{F} \) and \( n \in \mathbb{Z} \) of certain pointed sets, each of which factors in turn as a product of two types of pointed sets: namely, the products sets

\[
\prod_{x(\underline{A}_n) \not\subseteq \text{Im} X\{q_n\}} \text{Hom}_C(Y, \underline{A}_n) \times \prod_{* \neq \Phi \in \text{Im} X\{q_n\}^{-1}(*)} \text{Hom}_C(Y, \Omega\underline{A}_n)
\]

and pullback squares of the form:

\[
\begin{array}{ccc}
M_\phi & \longrightarrow & \prod_{x(q_n)^{-1}(\phi)} \text{Hom}_C(Y, P\underline{A}_n) \\
\downarrow & & \downarrow \text{(p\underline{A}_n)\#} \\
\text{Hom}_C(Y, \underline{A}_n) & \longrightarrow & \prod_{x(q_n)^{-1}(\phi)} \text{Hom}_C(Y, \underline{A}_n) \\
\end{array}
\]

for each \( * \neq \phi \in \text{Im} X\{q_n\} \subseteq X\{\underline{A}_n\} \).

Thus we have a natural identification of the left-hand term \( \text{Hom}_C(Y, R\mathcal{F}X) \) in (3.10) with a limit of sets of the form \( \text{Hom}_C(Y, -) \). Since \( \text{Hom}_C(Y, -) \), commutes with limits, we set

\[
R\mathcal{F}X := \prod_{\mathcal{A} \in \mathcal{F}} \prod_{n \in \mathbb{Z}} \prod_{\phi \in X\{\underline{A}_n\}} Q_\phi ,
\]

where we define \( Q_\phi \) for \( \phi \in X\{\underline{A}_n\} \) as follows:

(a) If \( * \neq \phi \in \text{Im} X\{q_n\} \), then \( Q_\phi \) is defined by the pullback square

\[
\begin{array}{ccc}
Q_\phi & \longrightarrow & P\underline{A}_n \\
\downarrow & & \downarrow \text{p\underline{A}_n} \\
\underline{A}_n & \longrightarrow & \prod_{x(q_n)^{-1}(\phi)} \underline{A}_n \\
\end{array}
\]

in \( \mathcal{C} \).

(b) If \( \phi \not\in \text{Im} X\{q_n\} \), we set \( Q_\phi := \underline{A}_n \).

(c) If \( \phi = * \), we set

\[
Q_\phi := \prod_{X\{q_n\}^{-1}(*) \setminus \{*\}} \Omega\underline{A}_n .
\]
3.17. **Definition.** We call $\mathcal{R}_F\mathcal{Y}$ of (3.15) the dual Stover construction for $\mathcal{F}$, applied to the discrete $\mathcal{F}$-mapping algebra $\mathcal{Y}$. This defines a functor $\mathcal{R}_F : \text{Map}^{\text{op}}_{\mathcal{F},d} \to \mathcal{C}$, right adjoint to $L_F$, and thus a monad $\mathcal{T}_F := \mathcal{R}_F \circ L_F$ on $\mathcal{C}$, with unit $\eta = \text{Id}_{L_F} : \text{Id} \to \mathcal{T}_F$ and multiplication $\mu = \mathcal{R}_F \text{Id}_{\mathcal{T}_F} : \mathcal{T}_F \circ \mathcal{T}_F \to \mathcal{T}_F$, as well as a comonad $\mathcal{S}_F := L_F \circ \mathcal{R}_F$ on $\text{Map}^{\text{op}}_{\mathcal{F},d}$, with counit $\varepsilon = \text{Id}_{L_F} : \mathcal{S}_F \to \text{Id}$ and comultiplication $\delta = \mathcal{L}_F \text{Id}_{\mathcal{S}_F} : \mathcal{S}_F \to \mathcal{S}_F \circ \mathcal{S}_F$ (cf. [Mc, VI, §1]).

Recall that a coalgebra over $\mathcal{S}_F$ is an object $\mathcal{Y} \in \text{Map}^{\text{op}}_{\mathcal{F},d}$, equipped with a section $\zeta : \mathcal{Y} \to \mathcal{S}_F \mathcal{Y}$ for the counit $\varepsilon_{\mathcal{Y}} : \mathcal{S}_F \mathcal{Y} \to \mathcal{Y}$, such that

\[
\mathcal{S}_F \zeta \circ \zeta = \delta_{\mathcal{Y}} \circ \zeta
\]

(see [Mc, VI, §2]).

3.19. **Example.** Any realizable discrete $\mathcal{F}$-mapping algebra $L_F Y = \rho \mathcal{M}_F Y$ has a canonical structure of a coalgebra over $\mathcal{S}_F$, with $\zeta : L_F Y \to L_F \mathcal{R}_F L_F Y$ (in $\text{Map}^{\text{op}}_{\mathcal{F},d}$) equal to $L_F(\text{Id}_{L_F Y})$, where $\text{Id}_{L_F Y} : Y \to \mathcal{R}_F L_F Y$ is the adjoint of $\text{Id} : L_F Y \to L_F Y$.

3.20. **The Stover category.** If $\mathcal{F}$ is a set of $\Omega$-spectra in $\mathcal{C}$, and $\kappa$ is some (infinite) cardinal, we define an $\mathcal{F}$-Stover object (for $\kappa$) to be a product of at most $\kappa$ objects which are either spaces $A_n$ from an $\Omega$-spectrum $A \in \mathcal{F}$, or are pullbacks $Q$ of the form:

\[
\begin{array}{ccc}
Q & \longrightarrow & \coprod_T P A_n \\
\downarrow & & \downarrow \rho \\
\coprod_T A_n & \xrightarrow{\text{diag}} & \prod_T A_n
\end{array}
\]

(3.21)

for various $A \in \mathcal{F}$, where $T$ is some set of cardinality $\leq \kappa$.

Thus if $\kappa$ bounds the cardinality of all sets $\mathfrak{X}(\{A_n\})$ ($A \in \mathcal{F}, n \in \mathbb{Z}$), for some discrete $\mathcal{F}$-mapping algebra $\mathfrak{X}$, then $\mathcal{R}_F \mathfrak{X}$ is an $\mathcal{F}$-Stover object for $\kappa$. Moreover, since each pullback $Q$ in (3.21) is weakly equivalent to a product of copies of $\Omega A_n$, we see that any $\mathcal{F}$-Stover object is $\mathcal{G}$-injective for any class $\mathcal{G} \supseteq \mathcal{F}$.

We denote by $\Theta^{\mathcal{S}, \kappa}_F$ the full simplicial subcategory of $\mathcal{C}$ consisting of all $\mathcal{F}$-Stover objects for $\kappa$. We shall always assume that $\Theta_{\mathcal{F}} \subseteq \Theta^{\mathcal{S}, \kappa}_F$, which holds as long as $\kappa \geq \lambda$ (see Remark 2.10). Note that $\Theta^{\mathcal{S}, \kappa}_F$ is itself an enriched sketch, whose mapping algebras will be called $\mathcal{F}$-Stover mapping algebras (for $\kappa$). The category of $\mathcal{F}$-Stover mapping algebras will be denoted by $\text{Map}^{\mathcal{S}}_{\mathcal{F}}$, and the corresponding free $\mathcal{F}$-Stover mapping algebra functor will be written $\mathcal{M}^{\mathcal{S}}_{\mathcal{F}} : \mathcal{C} \to \text{Map}^{\mathcal{S}}_{\mathcal{F}}$.

We thus have a forgetful functor $U^{\mathcal{S}} : \text{Map}^{\mathcal{S}}_{\mathcal{F}} \to \text{Map}_{\Theta_{\mathcal{F}}}$, and when there is no danger of confusion we shall denote the composite $\rho \circ U^{\mathcal{S}}$ simply by $\rho$, so we have $\rho \mathcal{M}_F = \rho \mathcal{M}^{\mathcal{S}}_{\mathcal{F}}$.

3.22. **Remark.** It is difficult to keep track of the (co)monads obtained from adjoint functors when they are not covariant, which is why Definition 3.17 was given in terms of $\text{Map}^{\text{op}}_{\mathcal{F},d}$. However, for the purposes of the following Lemma and Proposition, we prefer to work in $\text{Map}^{\text{op}}_{\mathcal{F},d}$ itself, so that we will refer to $\varepsilon_{\mathcal{Y}}^{\text{op}} : \mathcal{Y} \to \mathcal{S}_F \mathcal{Y}$ rather than the original $\varepsilon_{\mathcal{Y}} : \mathcal{S}_F \mathcal{Y} \to \mathcal{Y}$, and so on.
3.23. Lemma. The coalgebra structure map $\zeta : \mathcal{L}_F Y \to \mathcal{L}_F \mathcal{R}_F \mathcal{L}_F Y$ (in $\text{Map}_{\mathcal{F}, \mathcal{A}}^{\text{op}}$), of a realizable discrete $\mathcal{F}$-mapping algebra $\mathcal{L}_F Y$ is induced by a map of $\mathcal{F}$-Stover mapping algebras $\zeta' : \mathcal{M}_{\text{St}}^F(\mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F Y)) \to \mathcal{M}_{\text{St}}^F Y$ (in $\text{Map}_{\mathcal{F}}^{\text{St}}$), so $\zeta^{\text{op}} = \rho \zeta'$.

Proof. Since $\mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F Y)$ is an $\mathcal{F}$-Stover object, we see that $\mathcal{M}_{\text{St}}^F(\mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F Y))$ is a free $\mathcal{F}$-Stover mapping algebra, so in order to define the map $\zeta'$, by Lemma 2.15 it suffices to produce a “tautological element” $\langle \zeta' \rangle$ in

$$\mathcal{M}_{\text{St}}^F \{\mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F Y)\}_0 = \text{map}_{\mathcal{F}}(Y, \mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F Y))_0 = \text{Hom}_{\mathcal{F}}(Y, \mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F Y))$$

$$\cong \text{Hom}_{\text{Map}_{\mathcal{F}, \mathcal{A}}^{\text{op}}}(\mathcal{L}_F Y, \mathcal{L}_F Y),$$

where the last isomorphism is just (3.10). We may thus choose $\langle \zeta' \rangle$ to be the adjoint of $\text{Id} : \mathcal{L}_F Y \to \mathcal{L}_F Y$.

More explicitly, since $\mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F Y)$ is defined by a limit, $\langle \zeta' \rangle$ will be determined by choosing compatible elements in each component of (3.13). However, each component $A_n$ is indexed by an element $\phi$ in $\mathcal{M}_{\text{St}}^F \{A_n\}_0$, and each component $P A_n$ is indexed by an element $\Phi$ in $\mathcal{M}_{\text{St}}^F \{P A_n\}_0$, and all these choices are compatible, yielding the required $\langle \zeta' \rangle$.

$\square$

3.24. Proposition. For every $\mathcal{F}$-Stover mapping algebra $\mathcal{X}$, the corresponding discrete $\mathcal{F}$-mapping algebra $\mathcal{Y} : = \rho \mathcal{X}$ has a natural structure of a coalgebra over $S_\mathcal{F}$.

Compare [BBD] Proposition 3.7 and [BB2] Proposition 3.16.

Proof. Using Remark 3.22, we must produce a section $\zeta^{\text{op}} : S_\mathcal{F} \mathcal{Y} \to \mathcal{Y}$ for $\zeta^{\text{op}} : \mathcal{Y} \to S_\mathcal{F} \mathcal{Y}$ satisfying (3.18). In fact, we will construct a map of $\mathcal{F}$-Stover mapping algebras $\zeta' : \mathcal{M}_{\text{St}}^F \mathcal{R}_F \mathcal{X} \to \mathcal{X}$ fitting into a commuting square

$$\begin{array}{ccc}
\mathcal{M}_{\text{St}}^F(\mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F \mathcal{R}_F \mathcal{Y})) & \xrightarrow{\mathcal{M}_{\text{St}}^F \zeta'} & \mathcal{M}_{\text{St}}^F \mathcal{R}_F \mathcal{Y} \\
\mathcal{M}_{\text{St}}^F \mathcal{R}_F \mathcal{Y} & \xrightarrow{\zeta'} & \mathcal{Y} \\
\rho \mathcal{Y} & \xrightarrow{\rho \delta} & \mathcal{Y}
\end{array}$$

(3.25)

in $\text{Map}_{\mathcal{F}}$, with $\zeta = (\rho \zeta')^{\text{op}}$ and $\delta_\mathcal{Y} = \rho \delta'$.

As in the proof of Lemma 3.23, in order to define the map $\delta' : \mathcal{M}_{\text{St}}^F(\mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F \mathcal{R}_F \mathcal{Y})) \to \mathcal{M}_{\text{St}}^F \mathcal{R}_F \mathcal{Y}$, by Lemma 2.15 it suffices to produce a “tautological element” $\langle \delta' \rangle$ in

$$\mathcal{M}_{\text{St}}^F \mathcal{R}_F \mathcal{Y} \{\mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F \mathcal{R}_F \mathcal{Y})\}_0 = \text{map}_{\mathcal{F}}(\mathcal{R}_F \mathcal{Y}, \mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F \mathcal{R}_F \mathcal{Y}))_0,$$

that is, a map $\theta : \mathcal{R}_F \mathcal{Y} \to \mathcal{R}_F(\rho \mathcal{M}_{\text{St}}^F (\mathcal{R}_F \mathcal{Y}))$, which we choose to be the adjoint of $\text{Id} : \rho \mathcal{M}_{\text{St}}^F (\mathcal{R}_F \mathcal{Y}) \to \rho \mathcal{M}_{\text{St}}^F (\mathcal{R}_F \mathcal{Y})$.

The verification that (3.25) commutes is as in [BB2] Proposition 3.16. $\square$

3.26. Notation. Given a set $\mathcal{F}$ of $\Omega$-spectra and an $\Theta_\mathcal{F}$-mapping algebra $\mathcal{X}$, we write

$$\kappa_\mathcal{X} : = \sup \{|\mathcal{X}\{A_n\}| : A \in \mathcal{F}, n \in \mathbb{Z}\}.$$
Proof. The fact that the discrete $\mathcal{F}$-mapping algebra $\rho \mathcal{M}_F W^\bullet$ is homotopy equivalent to $c(\rho \mathcal{X})_\bullet$ follows from the usual properties of the “standard construction” for a (co)monad (cf. [God, App., §3] and compare [We, §8.6]). However, since we need the precise details of the contravariant case in our setting, we first summarize these as follows:

We may define an augmented simplicial object $\tilde{\mathfrak{U}}_\bullet \to \mathfrak{Y}$ in $\text{Map}_{F,d}^{op}$ by iterating the corresponding comonad $\mathcal{S}_F : \text{Map}_{F,d}^{op} \to \text{Map}_{F,d}^{op}$ on $\mathfrak{Y} := \rho \mathcal{X}$, so $\tilde{\mathfrak{U}}_k = \mathcal{S}_F^{k+1} \mathfrak{Y}$ (cf. [We, §8.6.4]). By Proposition 3.24, $\mathfrak{Y}$ is a coalgebra over $\mathcal{S}_F$, so it is $\mathcal{S}_F$-projective (cf. [We, §8.6.6]), and the structure map $\zeta_\mathfrak{Y} : \mathfrak{Y} \to \tilde{\mathfrak{U}}_0 = \mathcal{S}_F \mathfrak{Y}$ provides an extra degeneracy map $s_{k+1} : \mathcal{S}_F^{k+1} \mathfrak{Y} : \tilde{\mathfrak{U}}_k \to \tilde{\mathfrak{U}}_{k+1}$ (see [We, Proposition 8.6.8]).

An explicit description of $\tilde{\mathfrak{U}}_\bullet \to \mathfrak{Y}$ in low dimensions is given by the following diagram in $\text{Map}_{F,d}^{op}$:

$$
\begin{align*}
\mathfrak{Y}_{-1} := \mathfrak{Y} & \quad \mathfrak{Y}_0 := \mathcal{S}_F \mathfrak{Y} & \quad \mathfrak{Y}_1 := \mathcal{S}_F^2 \mathfrak{Y} \\
\tilde{\mathfrak{U}}_{-1} := \mathfrak{Y} & \quad \tilde{\mathfrak{U}}_0 := \mathcal{S}_F \mathfrak{Y} & \quad \tilde{\mathfrak{U}}_1 := \mathcal{S}_F^2 \mathfrak{Y}
\end{align*}
$$

Applying the functor $\mathcal{R}_F$ dimensionwise to $\tilde{\mathfrak{U}}_\bullet \to \mathfrak{Y}$ yields an augmented simplicial object $\mathcal{R}_F \tilde{\mathfrak{U}}_\bullet \to \mathcal{R}_F \mathfrak{Y}$. We set

$$
\mathcal{W}^n := \mathcal{R}_F \tilde{\mathfrak{U}}_{n-1} = \mathcal{R}_F \mathcal{S}_F^n \mathfrak{Y} \quad \text{with} \quad \mathcal{W}^0 := \mathcal{R}_F \mathfrak{Y}.
$$

Moreover, we have an extra map $d^{n+1} : \mathcal{W}^n = \mathcal{R}_F \mathcal{S}_F^n \mathfrak{Y} \to \mathcal{R}_F \mathcal{S}_F^{n+1} \mathfrak{Y} = \mathcal{W}^{n+1}$, adjoint to $\text{Id}_{\mathcal{S}_F^{n+1} \mathfrak{Y}}$, so we actually obtain a cosimplicial object $\mathcal{W}^\bullet$.

Once again, we have an explicit description of $\mathcal{W}^\bullet$ in low dimensions as a diagram in $\mathcal{C}$:

$$
\begin{align*}
\mathcal{W}^0 = \mathcal{R}_F \mathfrak{Y} & \quad \mathcal{W}^1 = \mathcal{R}_F \mathcal{S}_F \mathfrak{Y} & \quad \mathcal{W}^2 = \mathcal{R}_F \mathcal{S}_F^2 \mathfrak{Y} \\
\tilde{\mathcal{W}}^0 = \mathcal{R}_F \tilde{\mathfrak{U}}_{-1} & \quad \tilde{\mathcal{W}}^1 = \mathcal{R}_F \tilde{\mathfrak{U}}_0 & \quad \tilde{\mathcal{W}}^2 = \mathcal{R}_F \tilde{\mathfrak{U}}_1
\end{align*}
$$

As we saw in the proof of Proposition 3.24, the opposite of $\mathcal{S}_F$-coalgebra structure map of $\mathfrak{Y} = \rho \mathcal{X}$ lifts to a map of $\mathcal{F}$-Stover mapping algebras $\zeta' : \mathcal{M}_F^\text{St} \mathcal{W}^\bullet \to \mathcal{X}$, making $\mathcal{M}_F^\text{St} \mathcal{W}^\bullet \to \mathcal{X}$ into an augmented simplicial $\mathcal{F}$-Stover mapping algebra, which is described in low dimensions by the diagram (in $\text{Map}_{F,d}^{\text{St}}$):

$$
\begin{align*}
\mathcal{X} & \xleftarrow{\zeta'} \mathcal{M}_F^\text{St} \mathcal{W}^0 & \mathcal{M}_F^\text{St} \mathcal{W}^1 & \cdots
\end{align*}
$$
By construction, applying $\rho$ to this yields $\tilde{N}_* \to \mathbb{J}$ (viewed as a coaugmented cosimplicial object $Map^{op}_{\mathcal{F}} d$). Since each $A_n$ of $\mathcal{A}$ is an (infinite) loop space, for each $n, k \geq 0$, each of the pointed sets $\tilde{N}_k\{A_n\}$ (in the notation of [3.6]) is equipped with a binary operation $\mu_\#: \tilde{N}_k\{A_n\} \times \tilde{N}_k\{A_n\} \to \tilde{N}_k\{A_n\}$ (induced by the $H$-space multiplication $\mu: A_n \times A_n \to A_n$), and we can therefore use Lemma 3.3 to calculate the homotopy groups of the augmented simplicial (abelian) group

$$G_* := \pi_0(\Omega^{St}_{\mathcal{F}} W^*\{A_n\}) \xrightarrow{\zeta(A_n)^\#} \pi_0 X\{A_n\} =: G_1$$

from the object $\tilde{N}_*\{P A_n \to A_n\}$ in $sSet^I$.

Note that the indexing of (3.30) does not match with that of (3.28), even after taking into account the fact that these diagrams are in opposite categories. It will be convenient to choose our indexing convention for the augmented simplicial object $\rho\Omega^{St}_{\mathcal{F}} W^* \to \mathbb{J}$ (in $sMap^{St}_{\mathcal{F}}$) so that the extra degeneracy map is indexed as:

$$s_{k-1} = (\varepsilon S^{+1} \mathbb{I})^{op}: \rho\Omega^{St}_{\mathcal{F}} W^k \to \rho\Omega^{St}_{\mathcal{F}} W^{k+1}$$

with

$$d_is_{-1} = \begin{cases} 
\text{Id} & \text{if } i = 0 \\
 s_{-1}d_{i-1} & \text{if } i \geq 1 
\end{cases}$$

(3.31)

(which follows from the fact that $\varepsilon \mathbb{J} \circ \zeta \mathbb{I} = \text{Id}$ in $Map^{op}_{\mathcal{F} d}$, and $\varepsilon_*$ is a natural transformation). This completes the survey of the known facts about the “standard construction”.

If we knew that $s_{k-1}$ induced a group homomorphism $G_k \to G_{k+1}$ for each $k \geq 0$, we could deduce directly that the augmented simplicial group $G_* \to G_{-1}$ is acyclic. However, unlike the rest of the simplicial structure on $\tilde{N}_*$, the maps $s_{-1}$ are not induced from maps of full $\mathcal{F}$-Stover mapping algebras, so they need not respect the $H$-space structure induced by that of $A_n$.

Nevertheless, we can modify the usual proof of the acyclicity of $G_* \to G_{-1}$ as follows: any Moore cycle $\alpha \in C_k G_* \subseteq G_k$ (with $d_0 \alpha = 0$ for $0 \leq i \leq k$) may be represented by an element $a \in \tilde{N}_k\{A_n\}$, equipped with elements $b_i \in \tilde{N}_{k-1}\{P A_n\}$ such that $d_i a = (\tilde{N}_{k-1}\{q_n\})(b_i)$ ($0 \leq i \leq k$).

If we let $c := s_{k-1}a \in \tilde{N}_{k+1}\{A_n\}$, by (3.31) we have $d_0(c) = a$ and $d_i c = d_is_{k-1}a = s_{k-1}d_i a = s_{k-1}(\tilde{N}_{k-1}\{q_n\})(b_i) = (\tilde{N}_{k-1}\{q_n\})(s_{k-1}b_i)$ for $1 \leq i$, so that $[c] \in \pi_0\tilde{N}_{k+1}\{A_n\} = G_{k+1}$ is a Moore chain, with $\partial_{k+1}([c]) = \alpha$.

Thus we have shown that $\zeta': \Omega^{St}_{\mathcal{F}} W^* \to c(\mathcal{X})_*$ is a weak equivalence of simplicial $\mathcal{F}$-Stover mapping algebras. □

3.32. Corollary. Given a set $\mathcal{F}$ of $\Omega$-spectra in a model category $\mathcal{C}$, and an object $Y \in \mathcal{C}$, let $W^* \in \mathcal{C}$ be obtained from the $\mathcal{F}$-Stover mapping algebra $X = \Omega^{St}_{\mathcal{F}} Y$ as in Proposition 3.27. Then for every $\mathcal{A} = (A_k)_{k \in \mathbb{Z}}$ in $\mathcal{F}$ and $n \in \mathbb{Z}$, the augmented simplicial abelian group $[W^*, A_n] \to [Y, A_n]$ is acyclic.

Although the construction above works for any family of $\Omega$-spectra $\mathcal{F}$, its relevance to our original question on recovering $Y$ from $\text{map}_C(Y, A)$ is restricted to the following special case:

3.33. Theorem. Assume that $R$ is a field, $Y \in S_*$ is $R$-good (cf. [BK1] 1, 5.1), and $X := \text{map}_R(Y, A)$ for $A = K(R, n)$. Let $X$ be an $\mathcal{F}$-Stover mapping
algebra-structure on $X$ for $\mathcal{F} := \mathcal{H}(R)$ and $\kappa = \kappa_{\mathcal{H}(R)}$ (cf. §3.20), and let $W^*$ be constructed from $\mathcal{X}$ as in Proposition 3.27. Then $\hat{X} \simeq \text{map}_s(\text{Tot}W^*, A)$, which means that we can recover $Y$ from $\mathcal{X}$ up to $A$-equivalence.

Proof. Any simplicial $R$-module is of the form $M \simeq \prod_{n=0}^{\infty} K(V_n, n)$ for some $\mathbb{N}$-graded $R$-vector space $(V_n)_{n=0}^{\infty}$, up to weak equivalence. Moreover, each $V_n = \bigoplus_{\lambda_n} R$ embeds as a split summand in $V'_n = \prod_{\lambda_n} R$. Thus $M$ is a homotopy retract of $M' = \prod_{n=0}^{\infty} K(V'_n, n)$ so that for any (augmented) cosimplicial space $Y \to W^*$ the (augmented) simplicial abelian group $[W^*, M] \to [Y, M]$ is a retract of $[W^*, M'] \to [Y, M']$. However, for any space $X$ we have $[X, M'] \cong \prod_{n=0}^{\infty} \prod_{\lambda_n} [X, K(R, n)]$, so if $[W^*, K(R, n)] \to [Y, K(R, n)]$ is acyclic for each $n \geq 0$, so is $[W^*, M] \to [Y, M]$.

Now let $\mathcal{G}$ denote the class of $\Omega^\infty$-spaces of $\mathcal{H}(R)$-module spectra – or equivalently, all simplicial $R$-modules (cf. [EKMM, IV, §2.7]). If $\mathcal{X} := \mathcal{M}_R Y$, then the cosimplicial resolution $W^*$ is a weak $\mathcal{G}$-resolution of $Y$ (cf. [Bo2, Definition 6.1]), by the above and Corollary 3.32 so $\text{Tot}W^* \simeq \text{Tot} R \simeq Y$ by [Bo2, §7.7 & Theorem 9.5]. Therefore, $Y \to \text{Tot}W^*$ induces a weak equivalence $\text{map}_s(Y, K(R, n)) \to \text{map}_s(\text{Tot}W^*, K(R, n))$ by [Bo2, Proposition 8.5].

4. Mapping algebras and completions

By Theorem 3.33, the construction $W^*$ of Section 3 can be used to recover $Y$ from a realizable $\mathcal{F}$-Stover mapping algebra $\mathcal{M}_R Y$ when working over a field. However, for more general rings or ring spectra, we need a more elaborate choice of $\mathcal{G}$ consisting of all $\mathcal{H}(R)$-module spectra (or equivalently, all simplicial $R$-modules) – a proper class. For this purpose, one could restrict attention to the collection of simplicial $R$-modules of cardinality $< \nu$, for some inaccessible cardinal $\nu$, but in order to avoid unnecessary set-theoretic assumptions we describe a modified version of the dual Stover construction, adapted to the particular $Y \in S_*$ at hand.

4.1. Definition. For any commutative ring $R$, consider the category $sR$-Mod of all simplicial $R$-modules. Let $sM_R$ denote (a skeleton of) the subcategory of $s$-$R$-Mod with the same objects, but only proper monomorphisms as maps. For every cardinal $\lambda$ we denote by $(sM_R)_\lambda$ the full subcategory of $sM_R$ consisting of objects of cardinality $\leq \lambda$.

4.2. The modified dual Stover construction.

When $R$ is not a field, the model category structure of Bousfield (cf. §1.18) defines weak equivalences of cosimplicial spaces using the class $\mathcal{G}$ of all $\Omega^\infty$-spaces of $\mathcal{H}(R)$-module spectra. However, since $\mathcal{G}$ is a proper class, the dual Stover construction of §3.9 makes no sense for $\hat{F} := \mathcal{G}$ (cf. (2.2)). Thus, we need to replace $\mathcal{G}$ by a set of $\Omega^\infty$-spaces of $\mathcal{H}(R)$-module spectra, which still suffice to recover $Y$ from $\mathcal{M}_R Y$. We shall use the set $\hat{F} := (sM_R)_\lambda$, for a suitable cardinal $\lambda = \hat{\lambda}_Y$ depending on $Y$ – see §4.2.6 below.

4.3. Definition. If $R$ is a commutative ring, $\lambda$ is a cardinal, and $\hat{F} := (sM_R)_\lambda$, a semi-discrete $R$-mapping algebra (for $\lambda$) is a functor $\mathcal{G} : \hat{F} \to \text{Set}_*$, written $M \mapsto (P\mathcal{G} \{M\} \xrightarrow{\text{proj}} \mathcal{G} \{M\})$ where $J$ denotes the single-arrow category $0 \to 1$ (cf. §3.4). The category of semi-discrete $R$-mapping algebras will be denoted by $\text{Map}_{R, d}$.

4.4. Definition. Note that $\hat{F} = (sM_R)_\lambda$ is indeed the set of $\Omega^\infty$-spaces in a certain collection $\mathcal{F}$ of $\Omega$-spectra which are $\mathcal{H}(R)$-module spectra, as in (2.9).
If $\mathcal{X}$ is an $\Theta_F$-mapping algebra for this $\mathcal{F}$, the associated semi-discrete $R$-mapping algebra $\rho \mathcal{X}$ is defined by setting

$$\rho \mathcal{X}\{M\} := (\mathcal{X}\{p_M\}_0 : \mathcal{X}\{PM\}_0 \to \mathcal{X}\{M\}_0)$$

(see §3.5).

This defines a functor $\rho : \text{Map}_{\Theta_F} \to \text{Map}_{\mathcal{F}^d}$, since $\mathcal{X}$ takes values in Kan complexes. Moreover, we may define a covariant functor $\hat{\mathcal{L}}^\lambda_R : S_* \to \text{Map}_{\mathcal{F}^d}$ by $(\hat{\mathcal{L}}^\lambda_R Y)\{M\} := \rho \mathcal{F} Y\{M\}$ for each $M \in \hat{\mathcal{F}}$.

4.5. Remark. Note that $\text{Map}_{\mathcal{F}^d}$ is just the diagram category $\text{Set}_\Gamma^*$, indexed by a category $\Gamma$ whose maps consist of the arrows in squares of the form:

$$PM_0 \xrightarrow{p_j} PM_1$$

for any map $j : M_0 \to M_1$ in $(\mathcal{M}_R)_\lambda$, where the horizontal arrows may be composed for $M_0 \xhookrightarrow{\lambda} M_1 \xhookrightarrow{\lambda} M_2$.

Therefore, we can give an explicit description for the right adjoint $\hat{\mathcal{R}}^\lambda_R : \text{Map}_{\mathcal{F}^d} \to \mathcal{S}_*$ as in §3.6 with $\hat{\mathcal{R}}^\lambda R \mathcal{X}$ described by the limit of a diagram described as follows:

(a) For each $M_0 \in \hat{\mathcal{F}}$, consider the map of sets $\mathcal{X}\{q_{M_0}\} : \mathcal{X}\{PM_0\} \to \mathcal{X}\{M_0\}$.

$$\prod_{\phi \in \mathcal{X}\{q_{M_0}\}^{-1}(\phi)} PM_0 \xrightarrow{j} \prod_{\phi \in \mathcal{X}\{q_{M_1}\}^{-1}(\phi)} PM_1$$

for each square (4.6).

(b) On the other hand, if $\phi \in \mathcal{X}\{M_0\}$ is not contained in $\text{Im}(\mathcal{X}\{q_{M_0}\}) \subseteq \mathcal{X}\{M_0\}$, then for any $j : M_0 \to M_1$ also $\mathcal{X}\{j\}(\phi) \in \mathcal{X}\{M_0\}$ is contained in $\text{Im}(\mathcal{X}\{q_{M_1}\}) \subseteq \mathcal{X}\{M_1\}$, and we have a “corner of corners” of the form:

$$\Pi_{\phi \in \mathcal{X}\{q_{M_0}\}^{-1}(\phi)} PM_0 \xrightarrow{j} \Pi_{\phi \in \mathcal{X}\{q_{M_1}\}^{-1}(\phi)} PM_1$$

for each square (4.6).

4.8. Definition. The right adjoint $\hat{\mathcal{R}}^\lambda_R : \text{Map}_{\mathcal{F}^d} \to \mathcal{S}_*$ to $\hat{\mathcal{L}}^\lambda_R$ defines a monad $\hat{\mathcal{F}}^\lambda_R := \hat{\mathcal{R}}^\lambda_R \circ \hat{\mathcal{L}}^\lambda_R$ on $\mathcal{S}_*$, with unit $\hat{\eta} = \text{Id}_{\hat{\mathcal{F}}^\lambda_R} : \text{Id} \to \hat{\mathcal{F}}^\lambda_R$ and multiplication
\[ \hat{\mu} = \hat{R}_R^\Lambda \text{Id}_{\hat{T}_R^\Lambda} : \hat{T}_R^\Lambda \circ \hat{T}_R^\Lambda \rightarrow \hat{T}_R^\Lambda, \] as well as a comonad \( \hat{\mathcal{S}}_R^\Lambda := \hat{\mathcal{L}}_R^\Lambda \circ \hat{R}_R^\Lambda \) on \( \text{Map}_{R,\text{op}}^\text{op} \),

with counit \( \hat{\varepsilon} = \text{Id}_{\hat{R}_R^\Lambda} : \hat{\mathcal{S}}_R^\Lambda \rightarrow \text{Id} \) and comultiplication \( \hat{\delta} = \hat{\mathcal{L}}_R^\Lambda \text{Id}_{\hat{\mathcal{S}}_R^\Lambda} : \hat{\mathcal{S}}_R^\Lambda \rightarrow \hat{\mathcal{S}}_R^\Lambda \circ \hat{\mathcal{S}}_R^\Lambda. \)

4.9. Remark. Since all the limits in the construction of \( \hat{\mathcal{R}}_R^\Lambda \mathcal{X} \) take place in \( s\mathcal{M}_R \), the analogue of Proposition 3.24 stating that \( \rho \mathcal{X} \) has natural structure of a coalgebra over \( \hat{\mathcal{S}}_R^\Lambda \) will hold for any \( \Theta_F \)-mapping algebra \( \mathcal{X} \) which can be extended to a \( \Theta_F \)-mapping algebra for \( \hat{\mathcal{F}}' = (s\mathcal{M}_R)_\mu \) where \( \mu \) bounds the cardinalities of all objects in \( \hat{\mathcal{R}}_R^\Lambda \mathcal{X} \).

As in Section 3 we then have:

4.10. **Proposition.** Assume given a commutative ring \( R \) and a \( \Theta_F \)-mapping algebra \( \mathcal{X} \) for \( \hat{\mathcal{F}} := (s\mathcal{M}_R)_\lambda \), which is a coalgebra over \( \hat{\mathcal{S}}_R^\Lambda \), and let \( \hat{\mathcal{W}}^*_\lambda \) be obtained from \( \mathcal{X} \) as in Proposition 3.27 by iterating \( \hat{T}_R^\Lambda \) on \( \hat{\mathcal{S}}_R^\Lambda \rho \mathcal{X} \), with an augmentation \( \mathfrak{M}_F \hat{\mathcal{W}}^*_\lambda \rightarrow \mathcal{X} \). Then each \( \hat{\mathcal{W}}^*_\lambda \) is in \( s\mathcal{M}_R \), and for every \( M \in \hat{\mathcal{F}} \), the augmented simplicial \( R \)-module \( [\hat{\mathcal{W}}^*_\lambda, M] \rightarrow \pi_0 \mathcal{X} \{M\} \) is acyclic.

**Proof.** We obtain a cosimplicial object \( \hat{\mathcal{W}}^*_\lambda \) over \( S_* \) using the fact that \( \rho \mathcal{X} \) is a coalgebra over the comonad \( \hat{\mathcal{S}}_R^\Lambda \). By the description in §4.5 we see that for any semi-discrete \( R \)-mapping algebra \( \mathcal{Y} \), \( \hat{\mathcal{R}}_R^\Lambda \mathcal{Y} \) is a limit of simplicial \( R \)-modules (and all the maps in question are maps in \( s\mathcal{M}_R \text{-Mod} \)), so in particular, \( \hat{T}_R^\Lambda Y := \hat{\mathcal{R}}_R^\Lambda (\hat{\mathcal{L}}_R^\Lambda (\hat{T}_R^\Lambda)^n Y) \) is a simplicial \( R \)-module.

Note that for any \( M \in \hat{\mathcal{F}} \), the functor \( [-, M] : S_* \rightarrow \text{R-Mod} \) factors through \( \hat{\mathcal{L}}_R^\Lambda (\hat{\mathcal{S}}_R^\Lambda) \{M\} \), by Lemma 3.3 so to show that \( [\hat{\mathcal{W}}^*_\lambda, M] \rightarrow \pi_0 \mathcal{X} \{M\} \) is acyclic, it suffices to show that the augmented simplicial semi-discrete \( R \)-mapping algebra \( \rho \mathcal{X} \leftarrow \hat{\mathcal{L}}_R^\Lambda \hat{\mathcal{W}}^*_\lambda \) is contractible, which follows from [We, Proposition 8.6.10] as in the proof of Proposition 3.27.

4.11. **Corollary.** Given a commutative ring \( R \), a cardinal \( \lambda \), and \( \mathcal{Y} \in S_* \), we obtain a coaugmented cosimplicial space \( \mathcal{Y} \rightarrow \hat{\mathcal{W}}^*_\lambda \mathcal{Y} \) (by taking \( \mathcal{X} := \mathfrak{M}_F \mathcal{Y} \)), such that for every \( M \in \hat{\mathcal{F}} \) the augmented simplicial \( R \)-module \( [\hat{\mathcal{W}}^*_\lambda, M] \rightarrow \{\mathcal{Y}, M\} \) is acyclic.

Note that if \( \mathcal{X} \) is realizable, it automatically has an \( \hat{\mathcal{S}}_R^\Lambda \)-coalgebra structure, as in §3.19.

4.12. **Definition.** For any simplicial \( R \)-module \( M \), \( \mathcal{X} \in S_* \), and \( \phi : \mathcal{X} \rightarrow M \) in \( S_* \), we denote by \( \hat{\text{Im}}(\phi) \) the smallest simplicial submodule of \( M \) containing \( \text{Im}(\phi) \). If \( \hat{\text{Im}}(\phi) = M \), we call \( \phi \) an effective epimorphism, denoted by \( \phi : \mathcal{X} \rightarrow \mathcal{M} M \).

Moreover, the path space construction \( PM \) of (2.11) is also a simplicial \( R \)-module,

and \( \Phi : \mathcal{X} \rightarrow PM \) is a nullhomotopy of \( \phi : \mathcal{X} \rightarrow M \) if \( p_M \Phi = \phi \). However, \( \hat{\text{Im}}(\Phi) \subseteq PM \) need not be a path space, so we need the following:

Let \( \Phi : \mathcal{X} \rightarrow M \) denote the adjoint of \( \Phi \) (see (2.11) and [Q, Ch. II, 1.3]), with \( \Phi \circ \text{inc} = \phi \) for the inclusion \( \text{inc} : \mathcal{X} \leftarrow C\mathcal{X} \). Then \( M' := \hat{\text{Im}}(\phi) \) is contained in
and the original nullhomotopies fit into an adjoint commutative diagram:

\[ (4.13) \]

We write

\[ (4.14) \]

and call a nullhomotopy \( \Phi \) **effectively surjective** if \( M = M'' \) in (4.13) (that is, if \( PM = \Im(\Phi) \)). We denote the set of all effectively surjective nullhomotopies of \( \phi \) by \( \hat{\Hom}(X, PM) \).

**4.15. Definition.** For any \( M \in sM_R \) and effective epimorphism \( \phi : X \rightarrow M \) we define \( \hat{Q}_\phi \) by the pullback square in \( S_* \):

\[ (4.16) \]

where \( \prod' \) indicates that empty factors are to be omitted from the product (so the limit is in fact taken over a small diagram).

Note that if \( j : M \hookrightarrow CM \cong M' \) is the inclusion into the cone, \( j \circ \phi \) is nullhomotopic, so the upper right hand corner of (4.16), and thus \( \hat{Q}_\phi \), are never empty.

We now define \( \mathcal{T}_R : S_* \rightarrow S_* \) by

\[ (4.17) \]

\[ \mathcal{T}_R X := \prod'_{M \in sM_R} \prod_{\phi : X \rightarrow M} \hat{Q}_\phi . \]
Since there is only a set of choices of \( M \) and \( \phi \) for which \( \widehat{Q}_\phi \neq \emptyset \), the products in (4.17) are in fact taken over a set, rather than a proper class, of indices.

4.18. Lemma. The construction \( T_R \) is functorial in \( f : X \to Y \).

Proof. Let \( \psi : Y \to M \) be an effective epimorphism, \( j : M \hookrightarrow M' \) an inclusion in \( sM_R \), and \( \Psi : Y \to PM' \) an effectively surjective nullhomotopy of \( j \circ \psi : Y \to M' \).

Set \( M'' := \widetilde{\text{Im}}(\psi \circ f) \) and \( M''' := \widetilde{\text{Im}}(\Psi \circ f) \), as in (4.14), with \( j'' : M'' \hookrightarrow M \) and \( j' : M''' \to M' \) the inclusions. As in (4.13) we also have an inclusion \( j''' : M'' \hookrightarrow M''' \), fitting into the diagram:

![Diagram](image)

The map \( T_Rf : T_RX \to T_RY \) is defined into the factor \( \widehat{Q}_\psi \) of \( T_RY \) by projecting from \( T_RX \) onto \( \widehat{Q}_\phi \) and then onto the copy of \( PM'' \) indexed by \( \Phi \), which then maps by \( Pj' \) to the copy of \( PM' \) in \( \widehat{Q}_\psi \) indexed by \( \Psi \). The copy of \( M'' \) in the lower left corner of (4.16) for \( \widehat{Q}_\phi \) maps by \( j''' \) to the corresponding copy of \( M \) for \( \widehat{Q}_\psi \). The commutativity of the lower right hand parallelogram in (4.19) ensures that this induces a well-defined map on the limits (4.16) and (4.17).

4.20. Definition. For any \( X \in S_\ast \) we define the cardinal \( \lambda_X \) to be

\[
\lambda_X := \sup_{M \in sM_R} \{ |\text{Im}(\phi)| \mid \phi : X \to M \} \cup \{ |\text{Im}(\Phi)| \mid \Phi : X \to PM \} .
\]

4.21. Proposition. For any \( X \in S_\ast \) and \( \lambda \geq \lambda_X \) we have a canonical isomorphism \( T_R \lambda X \cong \widehat{T}_R \lambda X \).

Proof. By construction, \( \widehat{T}_R \lambda X \) is obtained from the semi-discrete \( R \)-mapping algebra \( X := \rho MFX \) by taking the limit over all maps \( \phi : X \to M_0 \) for \( M_0 \in (sM_R)_\lambda \) of the diagram described in (4.5) (for various inclusions \( j_0 : M_0 \hookrightarrow M_1 \)), where if the copy of \( M_0 \) in the diagram is indexed by \( \phi_0 \), then the two copies of \( M_1 \) are indexed by \( \phi_1 := j_0 \circ \phi_0 \).

This limit therefore splits up into connected components, one for each effective epimorphism \( \phi : X \to M_0 \), with all other simplicial \( R \)-modules \( M_1 \) in that component indexed by \( j \circ \phi \) for various inclusions \( j : M_0 \hookrightarrow M_1 \). Thus \( M_0 \) is initial among all the simplicial modules \( M_1 \) in its component of the diagram.
Moreover, the following diagram is cofinal in (4.7):

\[
\begin{array}{c}
\prod_{(X^q_{M_0})^{-1}(\phi)} PM_0 \\
\downarrow^\Pi j_{PM_0} \\
M_0 \\
\end{array}
\begin{array}{c}
\prod_{(X^q_{M_1})^{-1}(\phi)} PM_1 \\
\downarrow^\Pi_{PM_1} \\
M_1 \\
\end{array}
\begin{array}{c}
\prod_{(X^q_{M_1})^{-1}(j\phi)} PM_1 \\
\end{array}
\]

(4.22)

Note that since \( \phi : X \to M_0 \) is an effective epimorphism, any nullhomotopy \( \Phi : X \to PM_0 \) must be effectively surjective (see (4.13)), so we may replace the index set \( (X^q_{M_0})^{-1}(\phi) \) by \( \widehat{\text{Hom}}(X, PM_0)_{\phi} \).

Moreover, the map \( \prod (X^q_{j})^* \) into the lower middle product (of copies of \( M_1 \), indexed again by \( \widehat{\text{Hom}}(X, PM_0)_{\phi} \)) is simply the projection onto those factors of \( \prod_{\Phi \in (X^q_{M_1})^{-1}(j\phi)} M_1 \) indexed by nullhomotopies \( \Psi \) for \( j \circ \phi \) which are \textit{not} in the image of \( j_* : \text{Hom}(X, PM_0) \to \text{Hom}(X, PM_1) \).

Thus we may decompose this index set as a disjoint union

\[
(X^q_{M_1})^{-1}(j\phi) = j_* ((X^q_{M_0})^{-1}(\phi)) \amalg \text{New}^0_1,
\]

where \( \text{New}^0_1 \) consists of those nullhomotopies \( \Psi : X \to PM_1 \) of \( j \circ \phi \) which do not come from nullhomotopies of \( \phi \) itself.

The factors in the right hand products in (4.22) indexed by \( \text{New}^0_1 \) therefore fit into a cospan of the form

\[
\begin{array}{c}
M_0 \\
\downarrow^{(j)} \\
\prod_{\text{New}^0_1} M_1 \\
\end{array}
\begin{array}{c}
\prod_{\text{New}^0_1} PM_1 \\
\end{array}
\]

(4.23)

while the remainder of (4.22) becomes

\[
\begin{array}{c}
M_0 \\
\downarrow^{\text{diag}} \\
\prod_{\widehat{\text{Hom}}(X, PM_0)_{\phi}} M_0 \\
\end{array}
\begin{array}{c}
\prod_{\widehat{\text{Hom}}(X, PM_0)_{\phi}} PM_0 \\
\end{array}
\]

(4.24)

where if \( \widehat{\text{Hom}}(X, PM_0)_{\phi} \) is empty (that is, \( \phi \) itself is not nullhomotopic), we replace (4.24) by a single copy of \( M_0 \) mapping diagonally to \( \prod_{\text{New}_1^0} M_1 \), by (4.15)(b).

Note that any nullhomotopy \( \Phi : X \to PM_1 \) of \( j \circ \phi \) which is not effectively surjective in the sense of Definition 4.12 necessarily factors through \( X \xrightarrow{\Phi'} PM_2 \xrightarrow{Pj'} PM_1 \) for some \( M_0 \subseteq M_2 \subseteq M_1 \) in \((s\mathcal{M}_R)_\lambda\), as in (4.13).

Therefore, if we replicate Diagram (4.22) for the inclusion \( j' : M_2 \to M_1 \), again all such nullhomotopies \( \Phi \) will be omitted from the new set \( \text{New}_2^1 \), so in the full diagram for all of \((s\mathcal{M}_R)_\lambda\) the corner (4.23) for \( M_1 \) is replaced by

\[
\begin{array}{c}
M_0 \\
\downarrow^{(j)} \\
\prod_{\widehat{\text{Hom}}(X, PM_1)_{\phi}} M_1 \\
\end{array}
\begin{array}{c}
\prod_{\widehat{\text{Hom}}(X, PM_1)_{\phi}} PM_1 \\
\end{array}
\]

(4.25)

as in (4.16).
Thus whenever \( \lambda > \lambda_X \), we have a cofinal diagram defining \( \hat{\mathcal{T}}^R X \) in which only those \( M \in (sM_R)_\lambda \) appear for which there is either an effective epimorphism \( X \to M \), or there is an effectively surjective nullhomotopy \( X \to PM \) for some map \( X \to M \): that is, in fact we take only \( M \in (sM_R)_{\lambda_X} \), by Definition 4.20. This shows that in fact the natural map \( \mathcal{T}_R X \to \hat{\mathcal{T}}^R X \) is an isomorphism. \( \square \)

4.26. **Notation.** For any commutative ring \( R \) and \( Y \in \mathcal{S}_* \), let

\[
\hat{\lambda}_Y := \sup \{ \lambda_{\gamma Y} : n \in \mathbb{N} \},
\]

let \( \mathcal{F}_Y \) be the collection of \( \Omega \)-spectra whose \( \Omega^\infty \)-spaces are the objects of \( (sM_R)_{\lambda_Y} \), and let \( Y \to \hat{W}^* \) denote the coaugmented cosimplicial space \( Y \to \hat{W}^* \) of Corollary 4.11.

4.27. **Proposition.** Given a commutative ring \( R \), let \( \mathcal{G} \) denote the class of all simplicial \( R \)-modules in \( \text{ho} \mathcal{S}_* \). For any \( Y \in \mathcal{S}_* \), \( Y \to \hat{W}^* \) is a weak \( \mathcal{G} \)-resolution of \( Y \) (cf. [Bo2, Definition 6.1]).

**Proof.** Given any simplicial \( R \)-module \( M \in \mathcal{G} \), then as long as \( \kappa > |M|_1 \), \( [Y, M] \leftarrow [\hat{W}_n, M] \) is an acyclic generated simplicial abelian group, and each \( \hat{W}_n \) is in \( \mathcal{G} \), by Proposition 4.10. However, if \( \kappa \geq \hat{\lambda}_Y \), then \( \hat{W}_n \) is naturally isomorphic to \( \mathcal{T}^n+1 Y \) for every \( n \geq 0 \), by Proposition 4.21, so in particular, \( \hat{W}^* \cong \hat{W}_n \). \( \square \)

4.28. **Corollary.** For \( \mathcal{G} \) and \( Y \to \hat{W}^* \) as above, \( \text{Tot} \hat{W}^* \) (cf. (1.17) is weakly equivalent to the \( R \)-completion \( R_\infty Y \) of \( Y \).

**Proof.** The cosimplicial space \( \hat{W}^* \) is a weak \( \mathcal{G} \)-resolution of \( Y \), so in particular it is a \( \mathcal{G} \)-complete expansion (cf. [Bo2, Definition 9.4]), and thus \( \text{Tot} \hat{W}^* \cong \hat{L} \mathcal{G} Y \) by [Bo2, Theorem 9.5], while \( \hat{L} \mathcal{G} Y \cong R_\infty Y \) by [Bo2, §7.7]. \( \square \)

We thus obtain the following generalization of Theorem 3.33.

4.29. **Theorem.** Let \( \mathcal{A} = (A_n)_{n=0}^\infty \) be an \( \Omega \)-spectrum model of a connective ring spectrum, with \( R := \pi_0 \mathcal{A} \). Assume that \( Y \in \mathcal{S}_* \) is \( R \)-good, and let \( \mathcal{X} \) be a \( \Theta_F \)-mapping algebra structure on \( X = \text{map}(Y, A_0) \), for \( F = \mathcal{F}_Y \). If \( \hat{W}^* \) is the cosimplicial space obtained from \( \mathcal{X} \) as in (4.20), then \( X \cong \text{map}(\text{Tot} \hat{W}^*, A_0) \).

**Proof.** Let the class \( \mathcal{G}' \subseteq \text{Obj ho} \mathcal{S}_* \) consist of all spaces \( \Omega^\infty N \) where \( N \) is an \( A \)-module spectrum, and let \( \mathcal{G} \subseteq \text{Obj ho} \mathcal{S}_* \) consist of all \( \Omega^\infty M \), where \( M \) is an \( \mathcal{H}R \)-module spectrum. By [EKMM, IV, §2.7], in fact all such spectra \( M \) are \( \mathcal{G}' \)-GEMs, so \( \mathcal{G} = \text{Obj ho} sM_R \). Moreover, as the proof of [Bo2, Theorem 9.7], \( \mathcal{G} \subseteq \mathcal{G}' \) and each \( \mathcal{G}' \)-injective object is \( \mathcal{G} \)-complete, so by [Bo2, Theorem 9.6] we have a natural weak equivalence \( \hat{L} \mathcal{G}' Y \cong \hat{L} \mathcal{G} Y \) for every \( Y \in \mathcal{S}_* \), and \( \hat{L} \mathcal{G} Y \cong R_\infty Y \) by [Bo2, §7.7]. Therefore, \( \text{Tot} \hat{W}^* \cong \hat{L} \mathcal{G} Y \) by Corollary 4.28 and \( Y \) is \( R \)-good (that is, \( \mathcal{G} \)-good) if and only if it is \( \mathcal{G}' \)-good, by [Bo2, Definition 8.3].

Therefore, by [Bo2, Proposition 8.5] \( Y \to \text{Tot} \hat{W}^* \) is a \( \mathcal{G}' \)-equivalence, so in particular \( \text{map}(Y, A) \to \text{map}(\text{Tot} \hat{W}^*, A) \) is a weak equivalence. \( \square \)

4.30. **Example.** Ring spectra as in Theorem 4.29 include \( MU \) or \( BP \), connective \( ko \) or \( ku \), and of course the Eilenberg-Mac Lane spectrum \( \mathcal{H}R \) for any commutative ring \( R \).
Any simply-connected \( Y \in S_* \) is \( R \)-good. If \( R \) is a solid ring (cf. \[BK2\]) – e.g., \( R \subseteq \mathbb{Q} \) or \( R = \mathbb{R}_p \) – and \( \pi_1 Y \) is \( R \)-perfect (that is, \( (\pi_1 Y)_{ab} \otimes R = 0 \)), then \( Y \) is \( R \)-good by \[BK1\] VII, 3.2.

Note that for any commutative ring \( R \) and \( Y \in S_* \), \( (\text{core } R)_\infty Y \simeq R_\infty Y \) by \[BK1\] I, Lemma 9.1, where \( \text{core } R \subseteq R \) is the maximal solid subring of \( R \) (cf. \[BK2\]). Therefore, we can replace \( R := \pi_0 \mathcal{A} \) by \( \text{core } R \) in the Theorem (and in the construction \( \hat{\mathcal{W}}^* \) in \[1.26\]).

4.31. Remark. For such an \( \mathcal{A} \), Theorem \[1.29\] provides us with a recovery procedure for retrieving \( Y \) from the \( \Theta_F \)-mapping algebra structure \( X = \mathcal{M}_F Y \) on \( X = \text{map}(Y, \mathcal{A}) \), for \( F \) as in \[1.4\] Of course, we cannot expect to recover \( Y \) up to weak equivalence, but only to the extent that \( \tau \) is determined by \( X \) (namely, up to \( \mathcal{G}' \)-completion, for \( \mathcal{G}' \) as above). However, it does not allow us to recognize when an abstract \( \Theta_F \)-mapping algebra is in fact realizable.

Moreover, it implies that we can actually define a \( \mathcal{G} \)-completion for an abstract \( \Theta_F \)-mapping algebra \( X \): namely, \( \hat{\mathcal{L}}_c X := \text{Tot} \hat{\mathcal{W}}^* \) for \( \hat{\mathcal{W}}^* \) as in \[1.26\]. However, without additional assumptions, it need not be true that \( X \simeq \mathcal{M}_F (\hat{\mathcal{L}}_c X) \).

5. Realizing simplicial \( \Theta \)-algebra resolutions

Our goal here is to show how a free simplicial (algebraic) resolution \( V_* \) of an enrichible \( \Theta \)-algebra \( \Lambda \) can be realized in \( C \). For this purpose, we require the following:

5.1. Definition. For any \( \Theta \)-sketch \( \Theta \) (\[2.3\]), a \( CW \)-resolution of a \( \Theta \)-algebra \( \Lambda \) is a cofibrant replacement \( \varepsilon : G_* \xrightarrow{\sim} c\Lambda \) (in the model category of simplicial \( \Theta \)-algebras given by Proposition \[2.5\]), equipped with a CW basis \( (\overline{G}_n)_{n=0}^\infty \) (as in \[2.9\]), with each \( \overline{G}_n \) a free \( \Theta \)-algebra.

5.2. Remark. In fact, any CW object \( G_* \) for which each \( \overline{G}_n \) is a free \( \Theta \)-algebra, and each attaching map \( \partial_0 \overline{G}_n \) \((n \geq 0)\) surjects onto \( Z_{n-1}G_* \), is a CW-resolution. Here we set \( Z_{-1}G_* := \Lambda \) and \( \partial_0 \overline{G}_n := \varepsilon \), so that
\[
(5.3) \quad \varepsilon \circ \partial_0 \overline{G}_n = 0.
\]

5.4. Lemma. Let \( W^* \in cC \) be a Reedy cofibrant cosimplicial object over a model category \( C \) (as in \[0.4\]), and \( B \) a homotopy group object in \( C \). Then for any Moore chain \( \beta \in C_n[W^*, B] \) for the simplicial group \( [W^*, B] \):

(a) \( \beta \) can be realized by a map \( b : W^n \to B \) with \( b \circ d_{n-1} = 0 \) for all \( 1 \leq i \leq n \).

(b) If \( \beta \) is an algebraic Moore cycle, we can choose a nullhomotopy \( H : W^{n-1} \to PB \subseteq B^{[0,1]} \) for \( b \circ d_{n-1} \) such that \( H \circ d_{j-1} = 0 \) for \( 1 \leq j \leq n-1 \).

Proof. Since \( W^* \) is Reedy cofibrant, the simplicial space \( U_* = \text{map}_*(W^*, B) \in sS_* \) is Reedy fibrant, so we have an isomorphism
\[
(5.5) \quad t_* : \pi_1 C_n U_* \to C_n \pi_* U_*
\]
(cf. \[BK1\] X, 6.3]). Thus we can represent \( \alpha \in C_n \pi_0 U_* \) by a map \( a \in C_n U_* \), which implies (i).

If \( \alpha \) is a cycle, then \( \partial_n(\alpha) = [a \circ d_{n-1}] \) vanishes in \( \pi_0 C_{n-1} U_* \), so we have a nullhomotopy \( H \) for \( a \circ d_{n-1} \) in
\[
PC_{n-1} \text{map}_*(W^*, B) = C_{n-1} \text{map}_*(W^*, PB) \subseteq \text{map}_*(W^{n-1}, PB),
\]
which implies (ii). \qed

The following result essentially dualizes (and extends) [Bl, Theorem 3.16]:

5.6. **Theorem.** Assume given an enriched sketch \( \Theta \) in a model category \( \mathcal{C} \) as in \( \text{(0.6)} \) with \( \Theta := \pi_0 \Theta \) the associated algebraic sketch, and let \( \Lambda \) be a \( \Theta \)-algebra equipped with a CW-resolution \( V_* \). If \( \Lambda = \Lambda_\Theta \) is enriched by a \( \Theta \)-mapping algebra \( \bar{\chi} \), then there is a CW cosimplicial object \( W^* \in \mathcal{C} \) realizing \( V_* \) — that is, there is an augmentation \( \varepsilon_{| \Theta |} : \mathcal{M}_{\Theta} W^0 \to \bar{\chi} \) such that \( \pi_0(\mathcal{M}_{\Theta} W^*) \to \pi_0 \bar{\chi} \) is isomorphic to \( V_* \to \Lambda \). If \( \bar{\chi} = \mathcal{M}_{\Theta} Y \) for some \( Y \in \mathcal{C} \), then \( V_* \to \Lambda \) can be realized by a coaugmented cosimplicial object \( Y \to W^* \).

5.7. **Remark.** The cardinal \( \lambda \) which bounds the size of the products in \( \Theta \) must be chosen so that all the free \( \Theta \)-algebras \( \overline{V}_n \) in the CW basis for \( V_* \) are represented by objects in \( \Theta \) (see (2.20)).

**Proof.** We first choose once and for all objects \( W^k \) in \( \Theta \) realizing \( \overline{V}_n \), in the sense of (2.20) — so \( \pi_0 \mathcal{M}_{\Theta} W^0 \cong \overline{V}_n \) as \( \Theta \)-algebras. This is possible by assumption 5.7.

We will construct the cosimplicial object \( W^* \in \mathcal{C} \) by a double induction: in the outer induction, we construct a sequence of cosimplicial objects and maps:

\[
\ldots \to W^*_{[n]} \xrightarrow{\pi_{[n]}} W^*_{[n-1]} \xrightarrow{\pi_{[n-1]}} W^*_{[n-2]} \to \ldots \to W^*_{[0]},
\]

such that:

(a) \( W^* \) is the dimensionwise limit of \( (5.8) \).
(b) Each cosimplicial object \( W^*_{[n]} \) is weakly \( G \)-fibrant for \( G := \text{Obj} \Theta \) (cf. \( (1.19) \)), as well as being cofibrant in the Reedy model category \( (4.16) \).
(c) The simplicial \( \Theta \)-mapping algebra \( \mathcal{M}_{\Theta} W^*_{[n]} \) has an augmentation \( \varepsilon_{[n]} : \mathcal{M}_{\Theta} W^0_{[n]} \to \bar{\chi} \), which we can identify with a 0-simplex \( \varepsilon_{[n]} \) in \( \bar{\chi}(\mathcal{M}_{\Theta} W^0_{[n]}) \) by Lemma 2.15.
(d) \( W^*_{[n]} \) is an \( n \)-coskeletal weak CW cosimplicial object (cf. \( (1.12) \), with CW basis \( (\mathcal{M}_{\Theta} W^0)_{k=0} \)) (and zero CW basis object in dimensions \( > n \)).
(e) The augmented simplicial \( \Theta \)-mapping algebra \( \mathcal{M}_{\Theta} W^*_{[n]} \) realizes \( V_* \to \Lambda \) through simplicial dimension \( n \).
(f) The maps \( \pi_{[n]} \) restrict to a fibration weak equivalence \( \pi_{[n]}^k : W^k_{[n]} \to W^k_{[n-1]} \) for each \( 0 \leq k < n \), so \( W^k \) is the homotopy limit of the objects \( W^k_{[n]} \) \( (n \geq 0) \).
(g) The augmentation \( \varepsilon_{[n-1]} : \mathcal{M}_{\Theta} W^*_{[n-1]} \to \bar{\chi} \) extends along the \( \Theta \)-mapping algebra map \( \pi_{[n]}^* : \mathcal{M}_{\Theta} W^*_{[n-1]} \to \mathcal{M}_{\Theta} W^*_{[n]} \) to \( \varepsilon_{[n]} : \mathcal{M}_{\Theta} W^*_{[n]} \to \bar{\chi} \).

**Step 0 of the outer induction.**

We start the induction with \( W^*_{[0]} : c(\mathcal{M}_{\Theta} W^0)^* \) (the constant cosimplicial object), which is both Reedy cofibrant and weakly \( G \)-fibrant. Note that because \( \overline{V}_0 \) is a free \( \Theta \)-algebra, the \( \Theta \)-algebra augmentation \( \varepsilon : \overline{V}_0 \to \Lambda \) corresponds to a unique element in \( \mathcal{M}_{\Theta} \overline{V}_0 = \pi_0 \bar{\chi}(\mathcal{M}_{\Theta} W^0) \), for which we may choose a representative \( \varepsilon_{[0]} \in \bar{\chi}(\mathcal{M}_{\Theta} W^0_{[0]}) \), which by Lemma 2.15 corresponds to a map of \( \Theta \)-mapping algebras \( \varepsilon_{[0]} : \mathcal{M}_{\Theta} W^0_{[0]} \to \bar{\chi} \).
Step 1 of the outer induction.

We choose a map \( \overline{d}_0^0 : \overline{W}^0 \to \overline{W}^1 \) realizing the first attaching map \( \overline{d}_1^0 : \overline{V}_1 \to \overline{V}_0 = \overline{\nabla}_0 \), and define (the 1-truncation of) \( \overline{W}^*_{[1]} \) by the diagram:

\[
\begin{align*}
\overline{W}^0_{[1]} & = \overline{W}^0 \\
d_0^0 \downarrow \quad d_0^1 & = d_0^1 = \text{Id} \\
\overline{W}^1 & = \overline{W}^0 \times \overline{P} \overline{W}^1 \\
d_0^1 & = d_0^1 = \text{Id} \\
\overline{P} \overline{W}^1 & \quad \text{with } d_1 
\end{align*}
\]

Here \( p : PX \to X \) is the path fibration in \( C \), defined as in (2.12).

To define the augmentation \( \varepsilon_{[1]} \) as a 0-simplex in \( (\mathfrak{X}(\overline{W}^0_{[1]}))_0 \) extending \( \varepsilon_{[0]} \in \mathfrak{X}(\overline{W}^0_{[0]}) = \mathfrak{X}(\overline{W}^0) \) (see (b) above), we use the fact that

\[
\mathfrak{X}(\overline{W}^0_{[1]}) = \mathfrak{X}(\overline{W}^0 \times \overline{P} \overline{W}^1) = \mathfrak{X}(\overline{W}^0) \times \mathfrak{X}(\overline{P} \overline{W}^1) = \mathfrak{X}(\overline{W}^0) \times P \mathfrak{X}(\overline{W}^1),
\]

by (2.13a)-(b), so we need only to find a 0-simplex \( H \) in \( P \mathfrak{X}(\overline{W}^1) \) – which, by (2.12), is a 1-simplex in \( \mathfrak{X}(\overline{W}^1) \) with \( d_1 H = 0 \).

In order to qualify as an augmentation \( \mathfrak{Y}^*_{[1]} \to \mathfrak{X} \) of simplicial \( \Theta \)-mapping algebras, \( \varepsilon_{[1]} \) must satisfy the simplicial identity

\[
\varepsilon_{[1]} \circ d_0 = \varepsilon_{[1]} \circ d_1 : \mathfrak{Y}^*_{[1]} \to \mathfrak{X}
\]

as maps of \( \Theta \)-mapping algebras – or equivalently, these must correspond to the same 0-simplex in

\[
\mathfrak{X}(\overline{W}^1_{[1]}) = \mathfrak{X}(\overline{W}^0 \times \overline{W}^1 \times \overline{W}^i) = \mathfrak{X}(\overline{W}^0) \times \mathfrak{X}(\overline{W}^1) \times \mathfrak{X}(\overline{W}^i).\]

In the first factor and third factor this obviously holds, so we need only consider the two 0-simplices \( \mathfrak{X}(\overline{W}^1) \): in other words, since the path fibration \( p \) in (5.9) becomes \( d_0 \) in the simplicial set \( \mathfrak{X}(\overline{W}^1) \) (since it is induced by an inclusion \( \Delta[0] \hookrightarrow \Delta[1] \)), we must choose the nullhomotopy \( H \) so that \( d_0 H \) is the 0-simplex \( (d_0)^\# \varepsilon_{[0]} \).

But by (5.8) we know that \( \varepsilon \circ \overline{d}_0^0 = 0 \) in \( \Theta \)-Alg, which implies (by our choices of \( \overline{d}_0^0 \) and \( \varepsilon_{[0]} \) representing \( d_0^0 \) and \( \varepsilon \), respectively) that \( (d_0)^\# \varepsilon_{[0]} \) is nullhomotopic, so we can choose an \( H \) as required.

Step \( n \) of the outer induction \((n \geq 2)\):

Assume given \( \overline{W}^*_{[n-1]} \) satisfying (a)-(g) above, we construct an intermediate \( n \)-coskeletal restricted cosimplicial object \( \overline{W}^*_{[n]} \) (cf. (1.1)) by a descending induction on the cosimplicial dimension \( 0 \leq k \leq n \).

We require that, for each \( 0 \leq k < n \), the object \( \overline{W}^k_{[n]} \in C \) is defined by the (homotopy) pullback diagram:

\[
\begin{align*}
\overline{W}^k_{[n]} & \quad \xrightarrow{q^k} (\Omega^{n-k-1}\overline{W}^k)^{\Delta[1]} \\
\psi_{[n]} & \quad \xrightarrow{\text{PB}} \\
\overline{W}^k_{[n-1]} & \quad \xrightarrow{\eta^k} \Omega^{n-k-1}\overline{W}^n
\end{align*}
\]
for some map $\eta^k$ such that
\begin{equation}
\eta^k \circ d_{k-1}^i = 0 \quad \text{for all } 1 \leq i \leq k.
\end{equation}

The idea is that the projection of the coface map $d_0^k : W_{k-1}^{i} \to W_{[n]}^i$ onto $\Omega^{n-k-1}W^n$ in (5.11) describes the value $a^{k-1}$ of a certain “universal $(n-k-1)$–th order cohomology operation” (see §5.32 below), while the projection onto $(\Omega^{n-k-1}W^n)^{\Delta[1]}$ describes a homotopy $F^{k-1}$ between this value and the corresponding element $\eta^k \circ d_0^{k-1}$ in the cohomology of $W_{k-1}^{i}$. This higher order operation is defined by composing the homotopy of a lower order operation with some map (which we think of as a primary cohomology operation) — in this case, $F^{k-1} \circ d_0^{k-1}$.

At the $k$-th stage, we assume that we have defined $\tilde{W}_{[n]}^i$ for $n \geq i \geq k + 1$, with all coface maps $d_j^i : \tilde{W}_{[n]}^i \to \tilde{W}_{[n]}^{i+1}$ for all $j$ and $n \geq i \geq k + 1$, as well as $d_0^k : W_{[n-1]}^k \to \tilde{W}_{[n]}^k$ (if $k < n$), with the 0-th coface map of $\tilde{W}_{[n]}^* \to W_{[n-1]}^*$ given by:
\begin{equation}
\tilde{d}_0^k := \eta^k \circ \psi_n^k : \tilde{W}_{[n]}^k \to \tilde{W}_{[n]}^{k+1}.
\end{equation}
and $d_0^k : W_{[n-1]}^k \to W_{[n]}^k$ equal to $\psi_n^k \circ d_0^k$.

We write:
\begin{equation}
F^k := \eta^{k+1} \circ \tilde{d}_0^k : W_{[n-1]}^k \to (\Omega^{n-k-2}W^n)^{\Delta[1]}
\end{equation}
for the homotopy given in the previous stage, so the map $\tilde{d}_0^k$ into the pullback $\tilde{W}_{[n]}^{k+1}$ in (5.11) is determined by the two compatible maps $F^k$ and $d_0^k$. We also set:
\begin{equation}
a^k := ev_1 \circ F^k : W_{[n-1]}^k \to \Omega^{n-k-2}W^n
\end{equation}
for the previous value of the corresponding higher order operation, so
\begin{equation}
F^k : \eta^{k+1} \circ d_0^k \sim a^k.
\end{equation}

We also assume by induction that:
\begin{equation}
F^k \circ d_i^k = 0 \quad \text{for all } 1 \leq i \leq k.
\end{equation}

This implies that $a^k \circ d_i^k = 0$ for $i \geq 1$, but in fact we require that:
\begin{equation}
(a^k \circ d_i^k = 0 \quad \text{for all } 0 \leq i \leq k.
\end{equation}

**Step $k = n$ of the descending induction:**

We start the induction by setting
\begin{equation}
\tilde{W}_{[n]}^n := W_{[n-1]}^n \times W^n.
\end{equation}

By assumption $W_{[n-1]}^*$ is Reedy cofibrant, so the bisimplicial set $U_* := map_C(W_{[n-1]}^*, W^n)$
is Reedy fibrant. Moreover, since \( \pi_0 \mathcal{M}_\Theta W_{[n-1]}^k \cong V_k \) for all \( 0 \leq k < n \) by (c) above, the algebraic attaching map \( \overline{d}_0^n : \overline{W}_n \to V_{n-1} \) is a homotopy class
\[
(5.20) \quad \alpha \in \pi_0 U_{n-1} = [W_{[n-1]}^{n-1}, \overline{W}] = \pi_0 (\mathcal{M}_\Theta W_{[n-1]}^{n-1}(\overline{W})) = V_{n-1}(\overline{W}) \,
\]
where the last equality follows from Lemma 2.22. This \( \alpha \) is a Moore chain in \( \pi_0 U_* \) by Definition 1.9, so by Lemma 5.4(a), \( \overline{d}_0^i \) can be represented by a continuous map \( \overline{d}_{n-1}^0 : W_{[n-1]}^{n-1} \to \overline{W} \)
satisfying:
\[
(5.21) \quad \overline{d}_{n-1}^0 \circ \overline{d}_{n-2}^j = 0 \text{ for all } 1 \leq j \leq n-1.
\]
This defines \( \overline{d}_{n-1}^0 : W_{[n-1]}^{n-1} \to \overline{W} \) into the product \( \overline{W}^n \), extending the given face map \( \overline{d}_{n-1}^0 : W_{[n-1]}^{n-1} \to W_{[n-1]}^n \).

**Step k = n − 1 of the descending induction:**

We define \( W_{[n]}^{n-1} \) by the pullback diagram \( (5.11) \), with \( \eta^{-1} = 0 \). Thus:
\[
(5.22) \quad W_{[n]}^{n-1} := W_{[n-1]}^{n-1} \times P\overline{W}^n.
\]
By \( (1.10) \), the class \( \alpha \) of \( (5.20) \) is in fact a Moore cycle, so by Lemma 5.4(b) we have a nullhomotopy
\[
(5.23) \quad F^{n-2} : \overline{d}_{n-1}^0 \circ \overline{d}_{n-2}^0 \sim 0
\]
satisfying \( (5.17) \).

We define \( \overline{d}_{n-2}^0 : W_{[n-1]}^{n-2} \to W_{[n]}^{n-1} \) into the new factor \( P\overline{W}^n \) (and extending the given face map \( \overline{d}_{n-2}^0 : W_{[n-1]}^{n-2} \to W_{[n-1]}^{n-1} \)) to be
\[
(5.24) \quad F^{n-2} : W_{[n-1]}^{n-2} \to P\overline{W}^n.
\]
We define the coface map \( \overline{d}_{n-1}^1 : W_{[n]}^{n-1} \to W_{[n]}^{n} \) by the given \( \overline{d}_{n-1}^1 : W_{[n-1]}^{n-1} \to W_{[n]}^{n} \)
to be the first factor of \( (5.19) \), and the composite
\[
(5.25) \quad W_{[n]}^{n-1} \overset{\text{proj}_{\overline{W}^n}}{\longrightarrow} P\overline{W}^n \overset{p}{\longrightarrow} \overline{W}^n,
\]
on the second factor of \( (5.19) \) (where \( p : P\overline{W}^n \to \overline{W}^n \) is the path fibration).

The remaining face maps \( \overline{d}_{n-1}^i : W_{[n]}^{n-1} \to W_{[n]}^{n} \) \( (2 \leq i \leq n) \) extend the given \( d_{n-1}^i : W_{[n-1]}^{n-1} \to W_{[n]}^{n} \) by the zero map into the CW basis \( \overline{W}^n \).

The only cosimplicial identity that can be verified at this stage is
\[
(5.25) \quad \overline{d}_{n-1}^1 \circ \overline{d}_{n-2}^0 = \overline{d}_{n-1}^0 \circ \overline{d}_{n-2}^0,
\]
which follows from the fact that \( p \circ F^{n-2} = d_{n-1}^0 \circ d_{n-2}^0 \), by \( (5.23) \).

**Step k of the descending induction** \( (0 < k \leq n - 2) \):

We define the map \( \eta^k : W_{[n]}^k \to \Omega^{n-k-1}\overline{W}^n \) as follows:
By assumption we are given a homotopy \( F^k : \eta^{k+1} \circ d_{k}^0 \sim a^k \), so we get a homotopy:
\[
(5.26) \quad F^k \circ d_{k-1}^0 : \eta^{k+1} \circ a^k \circ d_{k-1}^0 \sim a^k \circ d_{k-1}^0
\]
where \( \eta^{k+1} \circ d^k \circ d^0 : W_{n-1}^{k-1} \to \Omega^{n-k-2}W^n \) is the zero map by (5.12), Definition 1.12(c), and the identity \( d^1d^0 = d^0d^0 \). Since also \( a^k \circ d^0_{k-1} = 0 \) by (5.18), \( F^k \circ d^0_{k-1} : W_{n-1}^{k-1} \to (\Omega^{n-k-2}W^n)_{\Delta[1]} \) is a self-nullhomotopy, so it factors through the inclusion \( i_k : \Omega^{n-k-1}W^n \hookrightarrow (\Omega^{n-k-2}W^n)_{\Delta[1]} \) and thus defines a map \( a^{k-1} : W_{n-1}^{k-1} \to \Omega^{n-k-1}W^n \) with

\[
i_k \circ a^{k-1} = F^k \circ d^0_{k-1}.
\]

Moreover,

\[
i_k \circ a^{k-1} \circ d^0_{k-2} = F^k \circ d^0_{k-1} \circ d^0_{k-2} = F^k \circ d^0_{k-1} \circ d^0_{k-2} = 0
\]

for all \( 0 \leq i \leq k-1 \) by (5.17), so (5.18) holds for \( a^{k-1} \) since \( i_k \) is monic.

Therefore, \( a^{k-1} \) is a \((k-1)\)-cycle for the Reedy fibrant bisimplicial set \( V_* := \text{mapc}(W_{n-1}^*, \Omega^{n-k-1}W^n) \), and thus in particular represents a \((k-1)\)-cycle \([a^{k-1}]\) for \( V_* \{ \Omega^{n-k-1}W^n \} \), as in (5.20).

Because \( V_* \to \Lambda \) is a resolution, and thus acyclic, there is a class \( \gamma_k \in V_k \{ \Omega^{n-k-1}W^n \} \) with \( \partial_0(\gamma_k) = [a^{k-1}] \). Moreover, the map \( \overline{\gamma}_n \in W_k W_{n-1} \to W_k \) of (1.13) induces the inclusion \( (\overline{\gamma}_n)^* : V_k \to V_k \), so we have a class \([\gamma_k] := (\overline{\gamma}_n)^*(\gamma_k) \in V_k \{ \Omega^{n-k-1}W^n \} \) which is a Moore chain by (1.12(b)).

By Lemma 5.4(a) we can represent this class by a map \( \eta^k : W_{n-1}^{k-1} \to \Omega^{n-k-1}W^n \) satisfying (5.12), while by Lemma 5.4(b) we have a homotopy

\[
F^k-1 : \eta^k \circ d^0_{k-1} \sim a^{k-1} : W_{n-1}^{k-1} \to \Omega^{n-k-1}W^n
\]

satisfying (5.17) for \( k-1 \).

We now define \( W_n^k \) by the pullback diagram (5.11), in which both vertical arrows are weak equivalences. To define the coface map \( d^k_{k-1} : W_{n-1}^{k-1} \to W_n^k \) extending \( d^0_{k-1} : W_{n-1}^{k-1} \to W_n^k \), we set \( d^k \circ \psi^k : W_{n-1}^k \to W_{n-1}^{k+1} \), it suffices to specify the composite

\[
q^{k+1} \circ d^1_k : W_n^k \to (\Omega^{n-k-2}W^n)_{\Delta[1]},
\]

which we set equal to the composite:

\[
W_n^k \xrightarrow{q^k} (\Omega^{n-k-1}W^n)_{\Delta[1]} \xrightarrow{ev} \Omega^{n-k-1}W^n \xrightarrow{i_k} (\Omega^{n-k-2}W^n)_{\Delta[1]}.
\]

This indeed defines a map into the pullback (5.11) for \( k+1 \), since the composite

\[
\Omega^{n-k-1}W^n \xrightarrow{i_k} (\Omega^{n-k-2}W^n)_{\Delta[1]} \xrightarrow{ev} \Omega^{n-k-2}W^n
\]

is zero, which matches (5.12).

The remaining face maps \( d^i_k : W_n^k \to W_n^{k+1} \) \((2 \leq i \leq k + 1)\) are defined by extending the given \( d^i_k : W_{n-1}^{k-1} \to W_{n-1}^{k+1} \) by the zero map into \((\Omega^{n-k-2}W^n)_{\Delta[1]}\) (which again matches (5.12)).

To verify the cosimplicial identity

\[
d^1_k \circ d^0_{k-1} = d^0_k \circ d^0_{k-1} : W_{n-1}^{k-1} \to W_{n-1}^{k+1},
\]
it suffices to check the post-composition with $q^{k+1}$, where:

\[
q^{k+1} \circ d_k^1 \circ d_{k-1}^0 = i_k \circ \text{ev}_1 \circ q^k \circ d_{k-1}^0 = i_k \circ \text{ev}_1 \circ F^{k-1}
\]

by \((5.24)\), \((5.28)\), \((5.27)\), and \((5.13)\). The identities $d^1 \circ d_{k-1}^0 = d^1 \circ d_{k-1}^0$ hold trivially, with both sides vanishing, for all $k + 2 \geq j > i \geq 0$ except for $(j, i) \in \{(1, 0), (2, 0), (2, 1)\}$. We check these three cases:

(a) The case $d^1 \circ d_{k-1}^0 = d^1 \circ d_{k-1}^0$ is \((5.30)\), which was already verified in step $k + 1$.

(b) To show $d^1 \circ d_{k-1}^0 = d^1 \circ d_{k-1}^0$, it suffices to check the post-composition with $q^{k+2}$, where $q^{k+2} \circ d^2 \circ d_{k-1}^0 = 0$ by definition, while

\[
q^{k+2} \circ d^2 \circ d_{k-1}^0 = i_{k-1} \circ \text{ev}_1 \circ q^{k+1} \circ d^1 \circ d_{k-1}^0 = i_{k-1} \circ \text{ev}_1 \circ i_k \circ \text{ev}_1 \circ q^k = 0
\]

since $\text{ev}_1 \circ i_k : \Omega^{n-k-1}\W^n \rightarrow \Omega^{n-k-2}\W^n$ is the zero map.

(c) To show $d^2 \circ d_{k-1}^0 = d^2 \circ d_{k-1}^0$, it suffices to check the post-composition with $q^{k+2}$, where again $q^{k+2} \circ d^2 \circ d_{k-1}^0 = 0$ by definition, while

\[
q^{k+2} \circ d^2 \circ d_{k-1}^0 = F^{k+1} \circ \psi_{[n]} \circ d^1 \circ d_{k-1}^0 = 0
\]

(in the notation of \((5.11)\)), by \((5.17)\).

**Step $k = 0$ of the descending induction:**

If $X = \mathcal{M}_\Theta Y$, the last step of the induction is no different from the general $k$, with $W_{[n-1]} = Y$ However, in the general case we no longer have an object $W_{[n-1]}$ in $C$ for $k = 0$, so we must modify our construction somewhat:

By induction the homotopy $F^0$ is a 1-simplex in $(\mathcal{M}_\Theta W_{[n-1]}(\Omega^{n-2}W^n)\}_{1}$, (for which \((5.17)\) is vacuous). Its simplicial face maps are $d_0 F^0 = \eta^1 \circ d_0$ and $d_1 F^0 = \eta^0$, respectively.

Applying $\varepsilon_{[n-1]} : \mathcal{M}_\Theta W_{[n-1]} \rightarrow X$ to $F^0$ yields a 1-simplex $\varepsilon_{[n-1]}(F^0) \in X(\Omega^{n-2}W^n)$, with

\[
d_0 \varepsilon_{[n-1]}(F^0) = \varepsilon_{[n-1]}(\eta^1 \circ d_0) = (\varepsilon_{[n-1]} \circ d_0^\#)(\eta^1) = (\varepsilon_{[n-1]} \circ d_0^\#)(\eta^1 \circ \psi_{[n-1]})
\]

\[
= \varepsilon_{[n-1]}(\eta^1 \circ d_0) = 0
\]

by \((5.10)\) and \((5.12)\).

Similarly, since $i_1 \circ \eta^1 = F^1 \circ d_0 : W_{[n-1]} \rightarrow (\Omega^{n-3}W^n)^{\Delta[1]}$ for $i_1 : \Omega^{n-2}W^n \rightarrow (\Omega^{n-3}W^n)^{\Delta[1]}$, we see that:

\[
(i_1)_{\#}d_1 \varepsilon_{[n-1]}(F^0) = d_1 \varepsilon_{[n-1]}(i_1 \circ F^0) = \varepsilon_{[n-1]}(i_1 \circ \eta^0) = \varepsilon_{[n-1]}(F^1 \circ d_0^\#)
\]

\[
= (\varepsilon_{[n-1]} \circ d_0^\#)(F^1) = (\varepsilon_{[n-1]} \circ d_0^\#)(F^1) = \varepsilon_{[n-1]}(F^1 \circ d_0) = 0
\]

by \((5.17)\).

Since $i_1$ is a cofibration in $C$ by \((2.12)\) (and the fact that any $W^n \in \Theta$ is cofibrant), by \((2.13)\) \((5.14)\) is monic, so $d_1 \varepsilon_{[n-1]}(F^0) = 0$, too.
Thus the 1-simplex \( \varepsilon_{[n-1]}(F^0) \in \mathcal{X}\{\Omega^{n-2}W^0\} \) actually defines a 0-simplex \( a^{-1} \) in \( \mathcal{X}\{\Omega^{n-1}W^n\} \), representing some class \([a^{-1}] \in \pi_0\mathcal{X}\{\Omega^{n-1}W^n\} = \Lambda\{\Omega^{n-1}W^n\} \).

Because \( V_* \) is a \( \Theta \)-algebra resolution of \( \Lambda \), \( \varepsilon : V_0 \to \Lambda \) is surjective, so there is a class \([\eta^0] \in V_0\{\Omega^{n-1}W^0\}\) with \( \varepsilon([\eta^0]) = [a^{-1}] \), which we can represent by a map \( \eta^0 : W^0_{[n-1]} \to \Omega^{n-1}W^n \), together with a 1-simplex \( F^{-1} \in \mathcal{X}\{\Omega^{n-1}W^n\} \) with \( d_0F^{-1} = \varepsilon_{[n-1]}(\eta^0) \) and \( d_0F^{-1} = a^{-1} \).

We can thus use \( \eta^0 \) to define \( W^0_{[n]} \) by the pullback diagram (5.11), and use \( F^{-1} \) to extend \( \varepsilon_{[n-1]} \) to \( \varepsilon_{[n]} : \mathcal{X}\{\Omega^{n-1}W^0\} \to \mathcal{X} \).

**Completing the \( n \)-th step of the outer induction:**

Once the inner (descending) induction is complete, we define a full cosimplicial object \( \tilde{W}^*_{[n]} \) by ascending induction on the cosimplicial dimension \( k \). This will be equipped with a map of restricted cosimplicial objects \( g : \tilde{W}^*_{[n]} \to W^*_{[n]} \), such that the composite map \( f := \psi_{[n]} \circ g : \tilde{W}^*_{[n]} \to W^*_{[n-1]} \) is a map of cosimplicial objects which is a dimensionwise weak equivalence.

We start with \( \tilde{W}^0_{[n]} := W^0_{[n]} \) and \( g^0 := \text{Id} \), and define \( \tilde{W}^k_{[n]} \) by the pullback diagram:

\[
\begin{array}{ccc}
\tilde{W}^k_{[n]} & \xrightarrow{\zeta_{[n]}^k} & M^k\tilde{W}^*_{[n]} \\
\downarrow{g^k} & & \downarrow{M^k f} \\
W^k_{[n]} & \xrightarrow{\psi_{[n]}^k} & W^k_{[n-1]} \\
\end{array}
\]

(5.31)

(used by the notation of \( \{1, 2\} \).

The codegeneracy maps of \( \tilde{W}^*_{[n]} \) are defined by (1.14), while the coface map \( d_k^{-1} : \tilde{W}^k_{[n-1]} \to \tilde{W}^k_{[n]} \) is defined by the maps \( \tau_{k-1} J_k^{-j} : \tilde{W}^k_{[n-1]} \to \tilde{W}^k_{[n]} \) constructed in the descending induction and the induced map \( M^{k-1}\tilde{W}^*_{[n]} \to M^k\tilde{W}^*_{[n]} \) determined by the current induction, (1.14), and the cosimplicial identities.

Finally, we let \( h : W^*_{[n]} \xrightarrow{\sim} \tilde{W}^*_{[n]} \) be any Reedy cofibrant replacement (cf. \$1.16\$). Note that \( W^*_{[n]} \) is still a weak CW object, with CW basis \( (\tilde{W}^k)_{k=0}^n \), and the map \( \varphi_{[n]} : W^k_{[n]} \to \tilde{W}^k_{[n]} \) defined to be the composite:

\[
W^k_{[n]} \xrightarrow{h^k} \tilde{W}^k_{[n]} \xrightarrow{g^k} W^k_{[n]} \xrightarrow{\psi_{[n]}^k} W^k_{[n-1]} \xrightarrow{\varphi_{[n]}^k} \tilde{W}^k_{[n-1]} \to \tilde{W}^k_{[n-1]}.
\]

Since \( h, g, \) and \( \psi_{[n]} \) are maps of (restricted) cosimplicial objects, by construction (and (5.17)) we see that (1.14) is indeed satisfied.

This completes the \( n \)-th step of the outer induction, and thus the proof of the Theorem. □

5.32. **Cosimplicial resolutions and higher operations.** The particular construction used to produce the realization \( \tilde{W}^* \) of the algebraic resolution \( V_* \to \Lambda \) actually encodes, in an explicit form, the additional higher order information that is needed to distinguish between different objects \( Y \) realizing the given \( \Theta \)-algebra \( \Lambda \).
For example, if $\mathcal{C} = \mathcal{S}$, and $\mathcal{A} = \mathcal{HR}$, this additional data takes the form of the higher order cohomology operations (see, e.g., \textcolor{red}{[Ad]}, \textcolor{blue}{[BM]}, \textcolor{violet}{[Ma]}, \textcolor{green}{[Wa]}), as follows:

Note that the $n$-th object $\overline{V}_n$ of the algebraic CW basis for $V_\bullet$ is a coproduct of free monogenic $\Theta_R$-algebras of the form $H^*(K(R, n_1); R)$, each of which is indexed by a map $\phi_i : H^*(K(R, n_1); R) \to V_{n-1}$ (the restriction of the attaching map $\partial_0^n : \overline{V}_n \to V_{n-1}$). Moreover, $\phi_i$ factors through some finite coproduct $\coprod_{j=1}^{k_i} H^*(K(R, n_j); R)$ of free monogenic $\Theta_R$-algebras in $V_{n-1}$, each of which is in turn indexed by a map $\psi_{i,j} : H^*(K(R, n_j); R) \to V_{n-2}$, and so on. Thus the original summand $H^*(K(R, n_1); R)$ of $\overline{V}_n$ is ultimately indexed by a composable sequence of $n + 1$ maps between finitely generated free $\Theta_R$-algebras, except for the very last, which lands in $H^*(Y; R) = \pi_0 G$.

Moreover, the composite $(\coprod_{j=1}^{k_i} \psi_{i,j}) \circ \phi_i$ vanishes, since $d_0 \circ \overline{\partial}_0^n$ in a CW object. This means that when we realize $V_\bullet$ by the cosimplicial space $W^\bullet$, the corresponding composite will be null-homotopic. This is the source of the map $\eta^{n-2} : W^{n-2}_{[n-1]} \to \Omega \overline{W}^n$ (in step $k = n - 2$ of the descending induction in the proof), which is in fact just the Toda bracket, or secondary cohomology operation, associated to this nullhomotopy (see \textcolor{red}{[Ha]} \S 4.1). The 0-th face map into the higher loops $\Omega \overline{W}^n$ are analogously associated to higher order cohomology operations, corresponding to the initial segments of the above composable sequence of homotopy classes of maps indexing each summand $H^*(K(R, n_1); R)$.

See \textcolor{red}{[BJT]} for a detailed discussion of higher homotopy operations and simplicial spaces in the dual setting.

6. Recognizing and realizing mapping algebras

We are now in a position to address the fundamental question of recognizing mapping spaces of the form $X = \text{map}_\pi(Y, A)$, given $A$. Unfortunately, we are able to give a satisfactory answer only when $A = K(R, n)$ for $R = \mathbb{Q}$ or $\mathbb{F}_p$.

First, we note the following elementary fact:

6.1. **Lemma.** If $R$ is a field, and $G \to B^\bullet$ is a coaugmented cosimplicial $R$-module such that the dual augmented simplicial $R$-module $(B^\bullet)^\dagger \to G^\dagger$ is acyclic, then $G \to B^\bullet$ is acyclic, too.

Here $W^\dagger := \text{Hom}_R(W, R)$ is the $R$-dual of an $R$-module $W$.

**Proof.** Write $A^\bullet$ for $(B^\bullet)^\dagger$. By Definition \textcolor{red}{[1.6]} the $n$-th Moore chains object $C_n A^\bullet$ of $A^\bullet \to G^\dagger$ is obtained by applying $\text{Hom}_R(-, R)$ to the $n$-th Moore cochains object $C^n B^\bullet$ of $G \to B^\bullet$, since the latter is defined by the colimit (\textcolor{red}{[1.8]}), and the former by the corresponding limit (\textcolor{red}{[1.7]}).

If we think of the cocoin complex $C^n B^\bullet$ as a negatively-indexed chain complex $E_*$, and thus of $C_n A^\bullet$ as the cocoin complex $\text{Hom}_R(E_*, R)$, we see that

$0 = \pi_i A^\bullet = H_i(C_n A^\bullet) = H^{-i}(E_*, R) \xrightarrow{\cong} \text{Hom}_R(H_{-i}(E_*), R)$,

by the Universal Coefficient Theorem (cf. \textcolor{red}{[We]} Theorem 3.6.5]). Since the homology groups $H_{-i}(E_*) \cong \pi^i B^\bullet$ are free $R$-modules, they must all vanish. \hfill $\square$

6.2. **Definition.** For any ring $R$, and $\Theta_R$ as in \textcolor{red}{[2.8]} a $\Theta_R$-algebra $\Lambda$ is

(a) $k$-connected if $\Lambda(K(R, i)) = 0$ for $0 \leq i \leq k$.

(b) **Finite type** if $\Lambda(K(R, n))$ is a finitely generated $R$-module for each $n \geq 0$. 

(c) *Finite* if it is finite type, and there is an \( N \) such that \( \Lambda\{K(R, n)\} \) vanishes for \( n \geq N \).

(d) *Allowable* if there is a partially ordered set \( J \), with the under category \( J/J \) finite for each \( j \in J \), and a diagram of finite \( \Theta_R \)-algebras \( \Lambda : J \to \Theta_R\text{-Alg} \) such that

\[
\Lambda = \lim_{j \in J} \Lambda_j,
\]

in \( \Theta_R\text{-Alg} \).

We say that an \( \Theta_R \)-mapping algebra \( \mathfrak{X} \) is *simply-connected*, *finite type*, *finite*, or *allowable* if the \( \Theta_R \)-algebra \( \pi_0 \mathfrak{X} \) is such.

### 6.4 Example.

Any finite-type \( \Theta_R \)-algebra \( \Lambda \) is allowable, with \( (6.3) \) given by the directed system of finite truncations of \( \Lambda \).

### 6.5 Remark.

If \( R \) is a field with \( |R| \leq \aleph_0 \), such as \( \mathbb{F}_q \) or \( \mathbb{Q} \), and \( \dim_R V = \aleph_0 \), then \( V \cong \bigoplus_{j=1}^{\infty} R \), so \( V^\dagger \cong \prod_{j=1}^{\infty} R \) and thus \( |V^\dagger| = |R|^\aleph_0 > |R| \cdot \aleph_0 \). Therefore, no \( R \)-module \( W \) of dimension \( \aleph_0 \) can be isomorphic to \( V^\dagger \) for any \( R \)-module \( V \). Thus a \( \Theta_R \)-algebra \( \Lambda \) for which some \( \Lambda\{K(R, n)\} \) is \( \aleph_0 \)-dimensional is not allowable. Similar phenomena occur for other fields and other cardinals.

### 6.6 Lemma.

Let \( R \) be a field.

(a) If \( \Lambda \) is an allowable \( \Theta_R \)-algebra, then there is a graded \( R \)-module \( M_\ast \) with \( \Lambda\{K(R, i)\} \cong M_i^\dagger \) \((as \ R\text{-modules}) for each \( i \geq 0 \).

(b) Any realizable \( \Theta_R \)-mapping algebra \( \mathfrak{X} = \mathfrak{M}_A Y \) is allowable, and \( M_\ast = H_\ast(Y; R) \).

**Proof.** (a) Since (co)limits in functor categories are defined object-wise, this follows from the fact that for a finite dimensional \( R \)-module \( W \), \( W^\dagger \) has a natural isomorphism \( (W^\dagger)^\dagger \cong W \), and thus

\[
\Lambda\{K(R, i)\} = (\lim_{j \in J} \Lambda_j)\{K(R, i)\} \cong \lim_{j \in J} (U\Lambda_j\{K(R, i)\})^\dagger \cong (\text{colim}_{j \in J} U\Lambda_j\{K(R, i)\})^\dagger,
\]

where \( U : \Theta_R\text{-Alg} \to \text{gr} R\text{-Mod} \) is the forgetful functor to graded \( R \)-modules, which creates all limits in \( \Theta_R\text{-Alg} \) since it is a right adjoint.

(b) For any \( Y \in S_\ast \) there is a natural weak equivalence

\[
H^i(Y; R) \cong \lim_{j \in J} H^i(Y_j; R) \quad \text{for all } i \geq 0,
\]

where \( (Y_j)_{j \in J} \) is the directed system of finite subcomplexes of \( Y \) (see \( \text{[LS VIII (F)]} \) and \( \text{[HM \S 2]} \)), and \( \pi_0 \mathfrak{X}\{K(R, i)\} = H^i(Y, R) \cong \text{Hom}_R(H_i Y, R) \) by the Universal Coefficient Theorem.

In particular, any free \( \Theta_R \)-algebra is realizable, and thus allowable.

### 6.9 Proposition.

Let \( R = \mathbb{F}_p \) or a field of characteristic 0, let \( \Lambda \) be an allowable \( \Theta_R \)-algebra with \( \varepsilon : V_\ast \to \Lambda \) a free simplicial \( \Theta_R \)-algebra resolution, and let \( M_\ast \) and \( N_\ast \) be the graded \( R \)-modules of Lemma 6.6 for \( \Lambda \) and \( V_0 \), respectively. Then there is a graded \( R \)-linear map \( \psi_\ast : M_\ast \to N_\ast \) with \( \varepsilon\{K(R, i)\} = \psi_i^\dagger \) for each \( i \geq 0 \).
Proof. Since $V_0$ is a free $\Theta_R$-algebra, it is of the form $V_0 = H^*(W^0; R)$ (for $W^0 \simeq \prod_{a \in A} K(R, n_a)$), so it is allowable. By (6.3) we know that $\varepsilon$ is determined by a compatible system of $\Theta_R$-algebra-maps $\varepsilon_j : V_0 \rightarrow \Lambda_j$. Moreover, $V_0 \cong \prod_{a \in A} H^*(K(R, n_a); R)$ is a coproduct (over $A$) of monogenic free $\Theta_R$-algebras. Thus $\varepsilon_j : V_0 \rightarrow \Lambda_j$ is completely determined by choices of maps of $\Theta_R$-algebras

\begin{equation}
\varepsilon_{a,j} : H^*(K(R, n_a); R) \rightarrow \Lambda_j,
\end{equation}

with compatibility requirements only with respect to the various $j \in J$.

Since both source and target in (6.10) are finite $\Theta_R$-algebras, such maps are completely determined by their duals, which are maps of $R$-coalgebras.

When $\operatorname{char}(R) = 0$, by the Milnor-Moore Theorem (cf. [MM, App.]) we have:

\begin{equation}
H_\ast(W^0; R) = H_\ast(\prod_{a \in A} K(R, n_a); R) \cong \bigotimes_{a \in A} H_\ast(K(R, n_a); R),
\end{equation}

where the right-hand side is the product in the category of cocommutative coalgebras (cf. [Goe 1.1b]), since $H_\ast(W^0; R)$ is then the cofree cocommutative coalgebra $V(G_\ast)$ on the graded $R$-module $G_\ast := \pi_\ast W^0$, and the functor $V$ preserves products since it is right adjoint to the forgetful functor.

When $R = \mathbb{F}_p$ the duals $\varepsilon^\dagger_{a,j} : \Lambda_j^\dagger \rightarrow H_\ast(K(R, n_a); R)$ of (6.10) are maps of $\mathbb{F}_p$-coalgebras over the mod $p$ Steenrod algebra. In this case, (6.11) still holds by [Bo1 §4.4], where the right hand side is now the product in the category of $\mathbb{F}_p$-coalgebras over the Steenrod algebra. Therefore, all the maps $\varepsilon^\dagger_{a,j}$ define a unique map $\varepsilon^\dagger_j : \Lambda_j^\dagger \rightarrow H_\ast(W^0; R)$ as required. \hfill \Box

As noted in §6.1, the homotopy type of an $R$-GEM $X$ alone does not allow us to determine whether it is of the form $X \simeq \mathcal{M}_R(Y)$ for some space $Y \in S_\ast$, and recover $Y$ uniquely – for this we need an $\Theta_R$-mapping algebra structure $\mathcal{X}$ on $X$ (cf. §2.16). Thus we are naturally led to the following:

**Question:** Given an $\Theta_R$-mapping algebra $\mathcal{X}$, is there a space $Y$ with $\mathcal{X} \simeq \mathcal{M}_R(Y)$, and if so, is $Y$ uniquely determined?

The answer is given by the following:

**6.12. Theorem.** When $R$ is $\mathbb{Q}$ or $\mathbb{F}_p$, any simply-connected allowable $\Theta_R$-mapping algebra $\mathcal{X}$ is weakly equivalent to $\mathcal{M}_R(Y)$ (cf. §2.19) for a simply-connected $R$-complete space $Y \in S_\ast$, unique up to weak equivalence.

If we only assume that $Y \in S_\ast$ realizes $\mathcal{X}$, it is unique up to $R$-equivalence.

**Proof.** Let $V_\ast \rightarrow \Lambda$ be any a free simplicial resolution of the allowable $\Theta_R$-algebra $\Lambda$. It is readily verified that if $\Lambda$ is simply-connected, this resolution may be chosen so that each $V_k$ is simply-connected, since $H^i(K(R, n); R) = 0$ for $0 \leq i < n$.

By Theorem 5.6, $V_\ast$ may be realized by a cosimplicial space $W^\bullet$, with each $W^n$ a simply-connected $R$-GEM, and the corresponding simplicial $\Theta_R$-mapping algebra $\mathcal{W}_n := \mathcal{M}_R(W^\bullet)$ is augmented to $\mathcal{X}$. We may assume that $W^\bullet$ is Reedy fibrant. Since $V_\ast \rightarrow \Lambda$ is acyclic, for each $K(R, i) \in \Theta_R$ we have:

\begin{equation}
\pi_n V_\ast(K(R, i)) \cong \pi_n H^{i-n}(W^\bullet; R) \cong \begin{cases} 
\Lambda\{K(R, i)\} & \text{if } n = 0 \\
0 & \text{otherwise}.
\end{cases}
\end{equation}
Therefore, by Lemma 6.1 the cosimplicial graded $R$-module $H_*(W^*; R)$ satisfies:

$$\pi^n H_{i-n}(W^*; R) \cong \begin{cases} M_i & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

for $M_*$ as in Lemma 6.6.

Therefore, because $R = \mathbb{Q}$ or $\mathbb{F}_p$, the homology spectral sequence for the simplicial space $W^*$ (cf. [A1, R, Bo1]), with:

$$E^2_{m,t} = \pi^m H_t(W^*; R) \implies H_t(Tot W^*; R),$$

satisfies the hypotheses of [Bo1, Theorem 3.4], so it converges strongly, and $Y := \text{Tot } W^*$ is simply-connected. Moreover, $H_*(Y; R) \cong \Lambda$ since (6.15) collapses at the $E^2$-term, so as in Theorem 3.3 we have a weak equivalence for the class $G$ of all simplicial $R$-modules. This implies that $\hat{L}_G Y \simeq Y$ by [Bo2, Theorem 6.5] – that is, $Y$ is $R$-complete (cf. [BK1, I, §5] and [Bo2, §7.7]).

Note that the natural map $c(Y)^* \to W^*$ induces a weak equivalence of simplicial $\Theta_R$-mapping algebras $\mathfrak{W}_* \to c(M_R Y)_*$ (cf. [2.24]), where $\mathfrak{W}_* := M_R W^*$. On the other hand, by Theorem 5.6 we also have a weak equivalence $\mathfrak{W}_* \to c(X)_*$.

For any simplicial $\Theta_R$-mapping algebra $\mathfrak{W}_*$, we can define its realization $\mathfrak{Z} := \|\mathfrak{W}_*\| : \Theta \to S_*$ by letting $\mathfrak{Z}(B)$ denote the diagonal of the bisimplicial group $\mathfrak{W}_* \{B\}$, for any $B \in \Theta$. Since $P\mathfrak{Z}(B) = \mathfrak{Z}(PB) \to \mathfrak{Z}(B)$ is a fibration by [A2, Theorem 6.2], and the diagonal preserves products and cofibrations, we see that $\mathfrak{Z}$ satisfies the three conditions of Definition 2.13, so it is an $\Theta_R$-mapping algebra.

Moreover, by (6.16) the Quillen spectral sequence for the bisimplicial group $\mathfrak{W}_* \{B\}$ collapses for any $B \in \Theta_R$, and thus the natural maps of $\Theta_R$-mapping algebras

$$M_R Y \leftarrow \mathfrak{Z} = \|\mathfrak{W}_*\| \rightarrow X$$

induced by the simplicial augmentations $M_R Y \leftarrow \mathfrak{W}_* \to X$ are both weak equivalences. Thus we see that $Y$ indeed realizes $X$, up to weak equivalence of $\Theta_R$-mapping algebras. The uniqueness up to weak equivalence (for $R$-complete $Y$) follows from [BK1 I, Lemma 5.5].

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