On quantization of Semenov-Tian-Shansky Poisson bracket on simple algebraic groups

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Dedicated to the memory of Joseph Donin

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Abstract

Let $G$ be a simple complex factorizable Poisson Lie algebraic group. Let $U_{\hbar}(g)$ be
the corresponding quantum group. We study $U_{\hbar}(g)$-equivariant quantization $\mathbb{C}_{\hbar}[G]$ of
the affine coordinate ring $\mathbb{C}[G]$ along the Semenov-Tian-Shansky bracket. For a simply
connected group $G$ we prove an analog of the Kostant-Richardson theorem stating that
$\mathbb{C}_{\hbar}[G]$ is a free module over its center.

Key words: Poisson Lie manifolds, quantum groups, equivariant quantization

1 Introduction

Let $G$ be a simple complex algebraic group. Suppose $G$ is a Poisson Lie group relative to
a quasitriangular Lie bialgebra structure on $g = \text{Lie } G$. Consider $G$ as a $G$-manifold with
respect to the conjugation action. In the present paper we study quantization of a special
Poisson structure on $G$ making it a Poisson Lie $G$-manifold. This (STS) Poisson structure
is due to Semenov-Tian-Shansky. In fact, the STS bracket makes $G$ a Poisson Lie manifold

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over $DG$, where $DG = G \times G$ is the Poisson Lie group corresponding to the double Lie bialgebra $Dg \simeq g \oplus g$.

The affine coordinate ring $\mathbb{C}[G]$ can be quantized along the STS Poisson bracket to a $U_\hbar(Dg)$-algebra $\mathbb{C}_\hbar[G]$. For $G$ connected, this quantization can be realized as a subalgebra in $U_\hbar(g)$. The algebra $\mathbb{C}_\hbar[G]$ is also realized as a quotient of the so called reflection equation (RE) algebra associated with $U_\hbar(g)$. For $G$ being a classical matrix group with the standard (zero weight) DS bracket, the corresponding ideal in the RE algebra is explicitly described.

Our main result is a quantum analog of the Kostant-Richardson theorem. In [K] Kostant proved that the algebras $\mathbb{C}[g]$ and $\mathcal{U}(g)$ are free modules over their subalgebras of $g$-invariants. Richardson generalized the case of $\mathbb{C}[g]$ to the affine coordinate ring of a semisimple complex algebraic group, [R]. Namely, if the subalgebra of invariants $I(G)$ (class functions) is polynomial, then $\mathbb{C}[G]$ is a free $I(G)$-module generated by a $G$-submodule in $\mathbb{C}[G]$ with finite dimensional isotypical components. We prove the analogous statement for $\mathbb{C}_\hbar[G]$.

The main result of the present paper can be formulated as follows.

Theorem. Let $G$ be a simple complex algebraic group and let $\mathbb{C}_\hbar[G]$ be the $U_\hbar(Dg)$-equivariant quantization of $\mathbb{C}[G]$ along the STS bracket. Then

i) the subalgebra $I_\hbar(G)$ of $U_\hbar(g)$-invariants coincides with the center of $\mathbb{C}_\hbar[G]$,

ii) $I_\hbar(G) \simeq I(G) \otimes \mathbb{C}[[\hbar]]$ as a $\mathbb{C}$-algebra.

Suppose that $I(G)$ is a polynomial algebra. Then

iii) $\mathbb{C}_\hbar[G]$ is a free $I_\hbar(G)$-module generated by a $U_\hbar(g)$-submodule $\mathcal{E} \subset \mathbb{C}_\hbar[G]$. Each isotypic component in $\mathcal{E}$ is $\mathbb{C}[[\hbar]]$-finite.

Remark that for connected simply connected $G$ the algebra of invariants is a polynomial algebra generated by the characters of fundamental representations, [St]. That also is true for some non-simply connected groups, for example, for $SO(2n + 1)$.

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2 Quantized universal enveloping algebras

Throughout the paper $g$ is a simple complex Lie algebra equipped with a quasitriangular Lie bialgebra structure. That is, we fix a classical solution $r \in g \otimes g$ to the Yang-Baxter
equation

\[ [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \] (1)

and normalize it so that the symmetric part \( \Omega := \frac{1}{2} (r_{12} + r_{21}) \) of \( r \) is the inverse (canonical element) of the Killing form on \( \mathfrak{g} \). Recall that quasitriangular solutions to the equation (1) are parameterized by combinatorial objects called Belavin-Drinfeld triples, \([\text{BD}]\).

By \( \mathbf{U}_h(\mathfrak{g}) \) we denote the quantization of the Lie bialgebra \((\mathfrak{g}, r)\), \([\text{Dr1, EK}]\). It is a quasitriangular topological Hopf \( \mathbb{C}[\hbar] \)-algebra isomorphic (as an algebra) to the space of \( \mathbf{U}(\mathfrak{g})[[\hbar]] \) of formal power series in \( \hbar \) with coefficients in \( \mathbf{U}(\mathfrak{g}) \) completed in the \( \hbar \)-adic topology.

Let \( \mathbf{R} \in \mathbf{U}_h(\mathfrak{g}) \) be the quasitriangular structure (universal R-matrix) on \( \mathbf{U}_h(\mathfrak{g}) \), the quantization of \( r \in \mathfrak{g} \otimes \mathfrak{g} \). Consider the twisted tensor square \( \mathbf{U}_h(\mathfrak{g}) \hat{\otimes} \mathbf{U}_h(\mathfrak{g}) \) of \( \mathbf{U}_h(\mathfrak{g}) \) constructed as follows, \([\text{RS}]\). The Hopf algebra \( \mathbf{U}_h(\mathfrak{g}) \hat{\otimes} \mathbf{U}_h(\mathfrak{g}) \) is obtained by the twist of the ordinary tensor square \( \mathbf{U}_h(\mathfrak{g}) \hat{\otimes} \mathbf{U}_h(\mathfrak{g}) \) by the cocycle \( \mathbf{R}_{23} \in \mathbf{U}_h(\mathfrak{g})^{\otimes 4} \). The symbol \( \hat{\otimes} \) means completed tensor product (in the \( \hbar \)-adic topology). The diagonal embedding \( \Delta: \mathbf{U}_h(\mathfrak{g}) \to \mathbf{U}_h(\mathfrak{g}) \hat{\otimes} \mathbf{U}_h(\mathfrak{g}) \) via comultiplication is a homomorphism of Hopf algebras. The algebra \( \mathbf{U}_h(\mathfrak{g}) \hat{\otimes} \mathbf{U}_h(\mathfrak{g}) \) is a quantization of the double \( \mathcal{D}\mathfrak{g} \), which in the simple quasitriangular (factorizable) case is isomorphic to \( \mathfrak{g} \oplus \mathfrak{g} \) as a Lie algebra. We will use notation \( \mathbf{U}_h(\mathcal{D}\mathfrak{g}) \) for \( \mathbf{U}_h(\mathfrak{g}) \hat{\otimes} \mathbf{U}_h(\mathfrak{g}) \).

### 3 Simple groups as Poisson Lie manifolds

Given an element \( \xi \in \mathfrak{g} \) let \( \xi^l \) and \( \xi^r \) denote, respectively, the left- and right invariant vector fields on \( G \). Namely, we put

\[
(\xi^l f)(g) = \frac{d}{dt} f(e^{t\xi} g)|_{t=0}, \quad (\xi^r f)(g) = \frac{d}{dt} f(e^{t\xi} g)|_{t=0}
\] (2)

for every smooth function \( f \) on \( G \).

There are two important Poisson structures on \( G \). First of them, the Drinfeld-Sklyanin (DS) Poisson bracket \([\text{Dr1}]\), is defined by the bivector field

\[
\varpi_{DS} = r^{l,l} - r^{r,r}.
\] (3)

This bracket makes \( G \) a Poisson Lie group, \([\text{STS}]\).

The Semenov-Tian-Shansky (STS) Poisson structure on the group \( G \) is defined by the bivector field

\[
\varpi_{STS} = r_{-}^{l,l} + r_{-}^{r,r} - r_{+}^{r,l} + r_{+}^{l,r} + \Omega^{l,l} - \Omega^{r,r} + \Omega^{r,l} - \Omega^{l,r} - r_{-}^{ad,ad} + (\Omega^{r,l} - \Omega^{l,r}).
\] (4)
Here $r_-$ is the skew symmetric part $\frac{1}{2}(r_{12} - r_{21})$ of $r$.

Consider the group $G$ as a $G$-space with respect to conjugation. Then the STS bracket makes $G$ a Poisson-Lie manifold over $G$ endowed with the Drinfeld-Sklyanin bracket, \[ \text{STS}. \]

We assume $G$ to be a linear algebraic group, e.g. a subgroup of $GL(V)$, where $V$ be a finite dimensional $G$-module. Then $G$ is an affine variety. Its irreducible (connected) component is an affine variety as well, \[ \text{VO}. \] Unless otherwise explicitly stated, $G$ is assumed to be connected.

The Lie algebra $\mathfrak{g}$ generates the left and right invariant vector fields on $\text{End}(V)$ defined similarly to \[2\]. Introduce a bivector field on $\text{End}(V)$ by the formula \[4\], where the superscripts $l,r$ mark the left-and right invariant vector field on $\text{End}(V)$. This bivector field is Poisson on the $G \times G$-invariant variety $\text{End}(V)^\Omega$ of matrices $A \in \text{End}(V)$ satisfying the quadratic equation $[A \otimes A, \Omega] = 0$. Restriction of this Poisson structure to $G \subset \text{End}(V)$ coincides with \[4\].

In the basic representation of $SL(n)$ the variety $\text{End}(V)^\Omega$ is the entire matrix space. Let $G$ be an orthogonal or symplectic group and $V$ its basic representation with the invariant form $B \in V \otimes V$. The variety $\text{End}(V)^\Omega$ coincides with the set of matrices fulfilling

$$BX^tB^{-1}X = f^2, \quad XBX^tB^{-1} = f^2,$$

(5)

Here $f$ is a numeric parameter. The condition $f \neq 0$ specifies a principal open set in $\text{End}(V)$, which is a group and a trivial central extension of $G$ ($f = 1$). Such an extension can be defined for an arbitrary matrix algebraic group and it will play a role in our consideration.

4 Quantization of the STS bracket on the group

By quantization of a Poisson affine variety $\mathbb{C}[M]$ we understand a $\mathbb{C}[[\hbar]]$-free $\mathbb{C}[[\hbar]]$-algebra $\mathbb{C}_\hbar[M]$ such that $\mathbb{C}_\hbar[M]/\hbar\mathbb{C}_\hbar[M] \simeq \mathbb{C}[M]$. The quantization is called equivariant if equipped with an action of a quantum group $U_\hbar(\mathfrak{g})$ that is compatible with the multiplication, namely

$$x \rhd (ab) = (x^{(1)} \rhd a)(x^{(2)} \rhd a) \quad \text{for all} \quad x \in U_\hbar(\mathfrak{g}) \quad \text{for all} \quad a, b \in \mathbb{C}_\hbar[M]$$

For an equivariant quantization to exist, $M$ must be a Poisson Lie manifold over the Poisson Lie group $G$ corresponding to the Lie bialgebra $\mathfrak{g}$.  

4
4.1 Some commutative algebra

In the present subsection we collect, for reader’s convenience, some standard facts about \( \mathbb{C}[[h]] \)-modules that we use in what follows.

**Lemma 4.1.** Let \( E \) be a free finite \( \mathbb{C}[[h]] \)-module. Then every \( \mathbb{C}[[h]] \)-submodule of \( E \) is finite and free.

This assertion holds true for modules over principal ideal domains, see e.g. [Jac].

Given an \( \mathbb{C}[[h]] \)-module \( E \) we denote by \( E_0 \) its ”classical limit”, the complex vector space \( E/hE \). A \( \mathbb{C}[[h]] \)-linear map \( \Psi: E \to F \) induces a \( \mathbb{C} \)-linear map \( E_0 \to F_0 \), which will be denoted by \( \Psi_0 \).

**Lemma 4.2.** Let \( E \) be a finite and \( W \) an arbitrary \( \mathbb{C}[[h]] \)-modules. A \( \mathbb{C}[[h]] \)-linear map \( W \to E \) is an epimorphism if the induced map \( W_0 \to E_0 \) is an epimorphism of vector spaces.

This is a particular case of the Nakayama lemma for modules over local rings, see e.g. [GH].

We say that a \( \mathbb{C}[[h]] \)-module \( E \) has no torsion (is torsion free) if \( hx = 0 \Rightarrow x = 0 \) for \( x \in E \).

**Lemma 4.3.** A finitely generated \( \mathbb{C}[[h]] \)-module is free if it is torsion free.

The latter assertion easily follows from the Nakayama lemma.

**Lemma 4.4.** Every submodule and quotient module of a finite \( \mathbb{C}[[h]] \)-module is finite.

This statement is obvious for quotient modules. For submodules, it follows from Lemma 4.1.

**Lemma 4.5.** Let \( \Psi: E \to F \) be a morphism of free finite \( \mathbb{C}[[h]] \)-modules such that the induced map \( \Psi_0: E_0 \to F_0 \) is an isomorphism of \( \mathbb{C} \)-vector spaces. Then \( \Psi \) is an isomorphism.

Using Lemma 4.1, the latter assertion can be reduced to the case \( E = F \) and \( \Psi \) being an endomorphism of \( E \). An endomorphism of a free module is invertible if and only if its residue mod \( h \) is invertible.

**Lemma 4.6.** Let \( \Psi: E \to F \) be a morphism of a \( \mathbb{C}[[h]] \)-modules. Suppose that \( E \) is finite, \( F \) is torsion free, and \( \Psi_0: E_0 \to F_0 \) is injective. Then \( E \) is free, and \( \Psi \) is injective.

**Proof.** First let us prove that \( \Psi \) is embedding assuming \( E \) to be free. In this case the image \( \text{im} \Psi \) is finite and has no torsion. Therefore it is free, by Lemma 4.3. The map \( \Psi_0 \) factors through the composition \( E_0 \to (\text{im} \Psi)_0 \to F_0 \), and the left arrow is surjective by construction.
Since $\Psi_0$ is injective, the map $E_0 \to (\im \Psi)_0$ is also injective and hence an isomorphism, by Lemma 4.5. Therefore $E \simeq \im \Psi$.

Now let $E$ be arbitrary and let $\{e_i\}$ be a set of generators such that their projections mod $\hbar$ form a base in $E_0$. Such generators do exist in view of the Nakayama lemma. Let $\hat{E}$ be the $\mathbb{C}[[\hbar]]$-free covering of $E$ generated by $\{e_i\}$. The composite map $\hat{\Psi}: \hat{E} \to E \to F$ satisfies the hypothesis of the lemma with free $\hat{E}$. We conclude that $\hat{\Psi}$ is injective. This implies that $E = \hat{E}$, i.e. $E$ is free, and that $\Psi$ is injective.

4.2 Quantization of the DS and STS brackets

Let $G$ be a simple complex algebraic group and $(V, \rho)$ its faithful representation. The affine ring $\mathbb{C}[G]$ is realized as a quotient of $\mathbb{C}[\End(V)]$ by an ideal generated by a finite system of polynomials $\{p_i\}$. We do not require $G$ to be connected but assume $G \subset \End(V)^\Omega$. Recall that $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ stands for the split Casimir.

Let $G^\sharp$ denote the smooth affine variety $G \times \mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of the field $\mathbb{C}$. The variety $G^\sharp$ is an algebraic group, however we will not use this fact until Section 3.

The affine coordinate ring $\mathbb{C}[G^\sharp]$ is isomorphic to the tensor product $\mathbb{C}[G] \otimes \mathbb{C}[f, f^{-1}]$. It can be realized as the quotient of $\mathbb{C}[\End(V)] \otimes \mathbb{C}[f, f^{-1}]$ by the ideal $(p^\sharp_i)$, where $p^\sharp_i(f, X) = f^{k_i}p_i(f^{-1}X)$ and $k_i$ is the degree of the polynomial $p_i$.

The algebra $\mathbb{C}[\End(V)] \otimes \mathbb{C}[f, f^{-1}]$ is equipped with a $\mathbb{Z}$-grading by setting $\deg \End^*(V) = 1$, $\deg f = 1$, and $\deg f^{-1} = -1$. The polynomials $\{p^\sharp_i\}$ are homogeneous, hence $\mathbb{C}[G^\sharp]$ is a $\mathbb{Z}$-graded algebra. Let us select in $\mathbb{C}[G^\sharp]$ the subalgebra which is the quotient of $\mathbb{C}[\End(V)]\{f\}$ by the ideal $(p^\sharp_i)$. This subalgebra is identified with the affine ring of the Zariski closure $\bar{G}^\sharp$ in $\End(V) \times \mathbb{C}$. It is graded, with finite dimensional homogeneous components. Clearly $\mathbb{C}[G^\sharp]$ is generated by $\mathbb{C}[\bar{G}^\sharp]$ over $\mathbb{C}[f^{-1}]$.

Define a two-sided $G$-action on $\mathbb{C}[G^\sharp]$ by setting it trivial on $\mathbb{C}[f, f^{-1}]$. This makes $\mathbb{C}[G^\sharp]$ a $U(\mathfrak{g})$-bimodule algebra. The action preserves grading and preserves the subalgebra $\mathbb{C}[\bar{G}^\sharp]$. The DS and STS brackets (3) and (4) are naturally defined on $\mathbb{C}[G^\sharp]$ and $\mathbb{C}[\bar{G}^\sharp]$ via the right and left $\mathfrak{g}$-actions on $\mathbb{C}[G^\sharp]$ and $\mathbb{C}[\bar{G}^\sharp]$ . They make both $\mathbb{C}[G^\sharp]$ and $\mathbb{C}[\bar{G}^\sharp]$ Poisson Lie algebras over the Lie bialgebras $\mathfrak{g}_{op} \oplus \mathfrak{g}$ and $\mathfrak{D} \mathfrak{g}$, correspondingly. The Poisson Lie manifolds $G_{DS}$ and $G_{STS}$ are sub-manifolds in $\bar{G}_{DS}^\sharp$ and $\bar{G}_{STS}^\sharp$ (as well as in $\bar{G}_{DS}^{\sharp}$ and $\bar{G}_{STS}^{\sharp}$) defined by the equation $f = 1$.

Recall the Takhtajan quantization of the DS Poisson structure on $G$, [14]. Consider the
quasitriangular quasi-Hopf algebra \((U(\mathfrak{g})[[h]], \Phi, R_0)\), where \(U(\mathfrak{g})[[h]]\) is equipped with the standard comultiplication, \(\Phi\) is a \(\mathfrak{g}\)-invariant associator, and \(R_0 = e^{\frac{\pi i}{h}}\) is the universal R-matrix. Since \(\Phi\) and \(R_0\) are \(G\)-invariant, \(\mathbb{C}[G] \otimes \mathbb{C}[[h]]\) is a commutative algebra in the quasi-tensor category of \(U(\mathfrak{g})_{op}[[h]] \hat{\otimes} U(\mathfrak{g})[[h]]\)-modules. The latter is a quasi-Hopf algebra with the associator \((\Phi^{-1})\Phi^\prime\) and the universal R-matrix \((R_0^{-1})R''_0\), [Dr3]. Here the prime is relative to the \(U(\mathfrak{g})_{op}[[h]]\)-factor while the double prime to the \(U(\mathfrak{g})[[h]]\)-factor.

Let \(\mathcal{J} \in U(\mathfrak{g}) \hat{\otimes}^2 [[h]]\) be a twist making \(U(\mathfrak{g})[[h]]\) the quasitriangular Hopf algebra \(U_h(\mathfrak{g})\). Then \((\mathcal{J}^{-1}) \mathcal{J}''\) converts \(U(\mathfrak{g})_{op}[[h]] \hat{\otimes} U(\mathfrak{g})[[h]]\) into the Hopf algebra \(U_h(\mathfrak{g})_{op} \hat{\otimes} U_h(\mathfrak{g})\). Applied to \(\mathbb{C}[G] \otimes \mathbb{C}[[h]]\), this twist makes it a \(U_h(\mathfrak{g})_{op} \hat{\otimes} U_h(\mathfrak{g})\)-module algebra, \(\mathbb{C}_h[G_{DS}]\). This algebra is commutative in the category of \(U_h(\mathfrak{g})\)-bimodules. It is a quantization of the DS-Poisson Lie bracket on \(G\).

The above quantization extends to the algebras \(\mathbb{C}_h[G^g_{DS}]\) and \(\mathbb{C}_h[G'^g_{DS}]\); the construction is literally the same. Since the two-sided action of \(\mathfrak{g}\) preserves the grading, the algebras \(\mathbb{C}_h[G^g_{DS}]\) and \(\mathbb{C}_h[G'^g_{DS}]\) are \(\mathbb{Z}\)-graded. The algebra \(\mathbb{C}_h[G_{DS}]\) is obtained from \(\mathbb{C}_h[G^g_{DS}]\) or from \(\mathbb{C}_h[G'^g_{DS}]\) as the quotient by the ideal \((f - 1)\).

Now consider \(\mathbb{C}_h[G_{DS}], \mathbb{C}_h[G^g_{DS}],\) and \(\mathbb{C}_h[G'^g_{DS}]\) as \(U_h(\mathfrak{g})_{op} \hat{\otimes} U_h(\mathfrak{g})\)-algebras, using identification between \(U_h(\mathfrak{g})_{op}\) and \(U_h(\mathfrak{g})_{op}\) via the antipode. Perform the twist from \(U_h(\mathfrak{g})_{op} \hat{\otimes} U_h(\mathfrak{g})\) to \(U_h(D \mathfrak{g})\) and the corresponding transformation of the algebras \(\mathbb{C}_h[G_{DS}], \mathbb{C}_h[G^g_{DS}],\) and \(\mathbb{C}_h[G'^g_{DS}]\). The resulting algebras \(\mathbb{C}_h[G_{STS}], \mathbb{C}_h[G^g_{STS}],\) and \(\mathbb{C}_h[G'^g_{STS}]\) are \(U_h(D \mathfrak{g})\)-equivariant quantizations along the STS bracket, [DM]. They are commutative in the braided category of \(U_h(D \mathfrak{g})\)-modules.

The algebras \(\mathbb{C}_h[G^g_{STS}]\) and \(\mathbb{C}_h[G'^g_{STS}]\) are \(\mathbb{Z}\)-graded and \(\mathbb{C}_h[G^g_{STS}] = \mathbb{C}_h[G_{STS}][f^{-1}]\). The homogeneous components in \(\mathbb{C}_h[G^g_{STS}]\) are \(\mathbb{C}[[h]]\)-finite and vanish for negative degrees. The algebra \(\mathbb{C}_h[G_{STS}]\) is obtained from \(\mathbb{C}_h[G^g_{STS}]\) (or from \(\mathbb{C}_h[G'^g_{STS}]\)) by factoring out the ideal \((f - 1)\).

5 The algebra \(\mathbb{C}_h[G_{STS}]\) as a module over its center

In the present section \(G\) is connected and \(\mathbb{C}_h[G]\) stands for \(\mathbb{C}_h[G_{STS}]\), that is, for the \(U_h(D \mathfrak{g})\)-equivariant quantization of \(\mathbb{C}[G]\) along the STS bracket. The action of \(U_h(\mathfrak{g})\) is induced by the diagonal embedding \(\Delta: U_h(\mathfrak{g}) \to U_h(D \mathfrak{g})\) and can be expressed through the left and right coregular actions of \(U_h(\mathfrak{g})\) on \(\mathbb{C}_h[G_{DS}]\) as

\[x(a) = x^{(2)} \triangleright a \triangleleft \gamma(x^{(1)}).\]
Here $\gamma$ stands for the antipode in $U_h(\mathfrak{g})$ and the actions are defined by $\xi \cdot a = \xi^l(a)$, and $a \triangleleft \xi = \xi^r(a)$ for $\xi \in \mathfrak{g}$, cf. \[2\]. We use that fact that $\mathbb{C}[G_{DS}]$ and $\mathbb{C}[G_{STS}]$ coincide as $U_h(\mathfrak{g})$-bimodules (but not algebras) and the $U_h(\mathfrak{g})$-actions is the actions of $U(\mathfrak{g})[[\hbar]]$.

**Proposition 5.1.** Let $G$ be a simple complex algebraic group equipped with the STS bracket. Let $\mathfrak{g}$ be its Lie bialgebra, $\mathcal{D}\mathfrak{g}$ the double of $\mathfrak{g}$, and let $\mathbb{C}_h[G]$ be the $U_h(\mathcal{D}\mathfrak{g})$-equivariant quantization of the affine ring $\mathbb{C}[G]$ along the STS bracket. Then the subalgebra $I_h(G)$ of $U_h(\mathfrak{g})$-invariants in $\mathbb{C}_h[G]$ coincides with the center.

**Proof.** The statement holds true for $\hat{G}^2$ too. Let us prove it for $\hat{G}^2$ first. The case of $G$ will be obtained by factoring out the ideal $(f - 1)$.

The subalgebra $I_h(\hat{G}^2)$ lies in the center of $\mathbb{C}_h[\hat{G}^2]$. Indeed, let $\hat{R}$ be the universal $R$-matrix of $U_h(\mathcal{D}\mathfrak{g})$. It is expressed through the universal $R$-matrix $R \in U^h_0(\mathfrak{g})$ by $\hat{R} = R_{14}R_{23}R_{12}R_{43}$, therefore $\hat{R} \in U_h(\mathcal{D}\mathfrak{g}) \hat{\otimes} U_h(\mathfrak{g})$. The algebra $\mathbb{C}_h[\hat{G}^2]$ is commutative in the category of $U_h(\mathcal{D}\mathfrak{g})$-modules, hence $(\hat{R} \triangleright a)(\hat{R} \triangleleft b) = ba$ for any $a, b \in \mathbb{C}_h[\hat{G}^2]$. Hence $ab = ba$ for $a \in I_h(\hat{G}^2)$.

Conversely, suppose that $ab = ba$ for some $a$ and all $b \in \mathbb{C}_h[\hat{G}^2]$. Present $a$ as $a = a_0 + O(\hbar)$, where $a_0 \in \mathbb{C}[\hat{G}^2]$. We have $0 = \hbar \varpi_{STS}(a_0, b) + O(\hbar^2)$ and therefore $\varpi(a_0, b) = 0$. The Poisson bivector field $\varpi_{STS}$ is induced by the classical $r$-matrix of the double $\xi^l \otimes \xi_i \in (\mathcal{D}\mathfrak{g})^{\otimes 2}$. The element $\xi \in \mathfrak{g}^*$ acts on $\hat{G}^2$ by vector field $r_-(\xi)^l - r_-(\xi)^r + \frac{1}{2}(\Omega(\xi)^l + \Omega(\xi)^r)$ (here we consider the elements of $\mathfrak{g} \otimes \mathfrak{g}$ as operators $\mathfrak{g}^* \to \mathfrak{g}$ by paring with the first tensor component). Let $e$ be the identity of the group $G$. At every point $(e \otimes c) \in G \times \mathbb{C}^* = G^z \subset G^2$ this vector field equals $\Omega(c)$. Since the Killing form is non-degenerate, $\zeta \triangleright a_0 = 0$ for all $\zeta \in \mathfrak{g}$ in an open set in $\hat{G}^2$ (in Euclidean topology). Therefore $\zeta \triangleright a_0 = 0$ for all $\zeta \in \mathfrak{g}$ and $a_0$ is $\mathfrak{g}$-invariant.

We can assume that $a$ is homogeneous with respect to the grading in $I_h(\hat{G}^2)$. Let $a'_0$ be $U_h(\mathfrak{g})$-invariant element such that $a'_0 = a_0$ mod $\hbar$. We can choose $a'_0$ of the same degree as $a$ (in fact, we can take $a'_0 = a_0 \triangleleft \theta^{-\frac{1}{2}}$, see the proof of Proposition 3.2). Then $a - a'_0$ is central and divided by $\hbar$. Acting by induction, we present $a$ as a sum $a = \sum_{\ell=0}^\infty \hbar^\ell a'_\ell$, where each summand is $U_h(\mathfrak{g})$-invariant. Since all $a'_\ell$ have the same degree, they lie in a finite $\mathbb{C}[[\hbar]]$-module. Hence the above sum converges in the $\hbar$-adic topology.

An immediate corollary of Proposition 5.1 is the analogous statement for $G^2$ and $\hat{G}^2$.

The following proposition asserts that the subalgebra of invariants in $\mathbb{C}_h[G]$ is not quantized.

**Proposition 5.2.** Let $\mathbb{C}_h[G]$ be the $U_h(\mathcal{D}\mathfrak{g})$-equivariant quantization of the STS bracket on $G$. Then $I_h(G)$ is isomorphic to $I(G) \otimes \mathbb{C}[[\hbar]]$ as a $\mathbb{C}$-algebra.
Proof. Consider two subspaces $\mathcal{I}_1$ and $\mathcal{I}_2$ in $\mathcal{A} = \mathbb{C}_h[G_{DS}]$ defined by the following conditions:

$$\mathcal{I}_1 = \{a \in \mathcal{A} : x \triangleright a = a \triangleleft x, \forall x \in U_h(\mathfrak{g})\}, \quad \mathcal{I}_2 = \{a \in \mathcal{A} : x \triangleright a = a \triangleleft \gamma^2(x), \forall x \in U_h(\mathfrak{g})\}. \quad (6)$$

Since $\mathcal{A}$ is an $U_h(\mathfrak{g})$-bimodule algebra, and the square antipode $\gamma^2$ is a Hopf algebra automorphism of $U_h(\mathfrak{g})$, both $\mathcal{I}_1$ and $\mathcal{I}_2$ are subalgebras in $\mathcal{A}$. The algebra $\mathcal{I}_1$ is isomorphic to $I(G) \otimes \mathbb{C}[[h]]$ as a $\mathbb{C}$-algebra. This readily follows from the Takhtajan construction of $\mathbb{C}_h[G_{DS}]$ rendered in Subsection 4.2.

Let us show that the algebra $\mathcal{I}_2$ is isomorphic to $\mathcal{I}_1$. Indeed, the fourth power of the antipode in $U_h(\mathfrak{g})$ is implemented by the similarity transformation with a group-like element $\theta \in U_h(\mathfrak{g})$, $[\text{Dr}2]$. This element has a group-like square root $\theta^{\frac{1}{2}} = e^{\frac{1}{2} \ln \theta} \in U_h(\mathfrak{g})$. The logarithm is well defined, because $\theta = 1 + O(h)$. In the case of the Drinfeld-Jimbo or standard quantization of $U(\mathfrak{g})$ the element $\theta^{\frac{1}{2}}$ belongs to $U_h(\mathfrak{h})$, where $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra. The map $a \mapsto a \triangleleft \theta^{\frac{1}{2}}$ is an automorphism of $\mathcal{A}$, and this automorphism sends $\mathcal{I}_1$ to $\mathcal{I}_2$.

Thus we have proven that $\mathcal{I}_2$ is isomorphic to $I(G) \otimes \mathbb{C}[[h]]$ as a $\mathbb{C}$-algebra. Consider the RE twist converting $\mathbb{C}_h[G_{DS}]$ into $\mathbb{C}_h[G_{STS}]$. This twist relates multiplications by the formula $[17]$, where $\mathcal{T}$ should be replaced by $\mathbb{C}_h[G_{DS}]$ and $\mathcal{K}$ by $\mathbb{C}_h[G_{STS}]$. It is straightforward to see that these multiplications coincide on $\mathcal{I}_2$. $\square$

Remark 5.3. In the proof of Proposition $[5,2]$ we used the observation that the multiplications in $\mathbb{C}_h[G_{DS}]$ and $\mathbb{C}_h[G_{STS}]$ coincide on $\mathcal{I}_2$. In fact, formula $[15]$ implies that $\mathbb{C}_h[G_{DS}]$ and $\mathbb{C}_h[G_{STS}]$ are the same as left $\mathcal{I}_2$-modules. Therefore the structure of left $\mathcal{I}_1$-module on $\mathbb{C}_h[G_{DS}]$ is the same as the structure of $\mathcal{I}_h(G)$-module on $\mathbb{C}_h[G_{STS}]$. This assertion also holds for $G^2$ and $G^2$.

Let $T$ be the maximal torus in $G$. Then $I(G) \simeq \mathbb{C}[T]^W$, where $W$ is the Weyl group, $[\text{St}]$. Suppose that the subalgebra of invariants in $\mathbb{C}[G]$ is polynomial. For example, that is the case when $G$ is simply connected; then $\mathbb{C}[T]^W$ is generated by characters of the fundamental representations $[\text{St}]$. Under the above assumption, the algebra $\mathbb{C}[G]$ is a free module over $I(G)$, $[R]$. There exists a $G$-submodule $\mathcal{E}_0 \subset \mathbb{C}[G]$ such that the multiplication map $I(G) \otimes \mathcal{E}_0 \to \mathbb{C}[G]$ gives an isomorphism of vector spaces. Each isotypic component in $\mathcal{E}_0$ has finite multiplicity. We will establish the quantum analog of this fact.

Theorem 5.4. Let $\mathbb{C}_h[G]$ be the $U_h(\mathfrak{g})$-equivariant quantization of the STS bracket on $G$. Suppose that the subalgebra $I(G)$ of $\mathfrak{g}$-invariants is a polynomial algebra. Then

i) $\mathbb{C}_h[G]$ is a free $I_h(G)$-module generated by a $U_h(\mathfrak{g})$-submodule $\mathcal{E} \subset \mathbb{C}_h[G]$.

ii) each isotypic component in $\mathcal{E}$ is $\mathbb{C}[[h]]$-finite.
Proof. Let \( \mathcal{E}_0 \) be the \( \mathcal{U}(\mathfrak{g}) \)-module generating \( \mathbb{C}[G] \) over \( I(G) \). Naturally considered as a subspace in \( \mathbb{C}[G^\sharp] \), it obviously generates \( \mathbb{C}[G^\sharp] \) over \( I(G^\sharp) \). Using invertibility of \( f \), we can make every isotypic component of \( \mathcal{E}_0 \) homogeneous and regard \( \mathcal{E}_0 \) as a graded submodule in \( \mathbb{C}[G^\sharp] \).

Put \( \mathcal{E} = \mathcal{E}_0 \otimes \mathbb{C}[\hbar] \). Let \( V_0 \) be a simple finite dimensional \( \mathfrak{g} \)-module and \( V = V_0 \otimes \mathbb{C}[\hbar] \) the corresponding \( \mathcal{U}_h(\mathfrak{g}) \)-module. Let \( (\mathcal{E}_0)_V \) denote the isotypic component of \( \mathcal{E}_0 \). The isotypic component \( \mathbb{C}[G^\sharp]_V \) is isomorphic to \( I(G^\sharp) \otimes (\mathcal{E}_0)_V \otimes \mathbb{C}[\hbar] \), as a \( \mathcal{U}_h(\mathfrak{g}) \)-module.

Let \( \tilde{m} \) denote the multiplication in \( \mathbb{C}_h[G^\sharp] \). The map \( \tilde{m} : I_h(G^\sharp) \otimes_{\mathbb{C}[\hbar]} \mathcal{E}_V \rightarrow \mathbb{C}_h[G^\sharp]_V \) is \( \mathcal{U}_h(\mathfrak{g}) \)-equivariant and respects grading. Let the superscript \((k)\) denote the homogeneous component of degree \( k \). The map \( \tilde{m} \) induces \( \mathcal{U}_h(\mathfrak{g}) \)-equivariant maps

\[
\bigoplus_{i+j=k} I^i_h(G^\sharp) \otimes_{\mathbb{C}[\hbar]} \mathcal{E}^{(i)}_V \rightarrow \mathbb{C}_h[G^\sharp]_V, \quad I^k_h(G^\sharp) \otimes_{\mathbb{C}[\hbar]} \mathcal{E}_V \rightarrow \mathbb{C}_h[G^\sharp]_V \subset \mathbb{C}_h[G^\sharp]_V. \tag{8}
\]

The left map has a \( \mathbb{C}[\hbar] \)-finite target, while the right one has a \( \mathbb{C}[\hbar] \)-finite source. All the \( \mathbb{C}[\hbar] \)-modules in \( \mathcal{S} \) are free. Modulo \( \hbar \), the left map is surjective, and the right one injective. Therefore they are surjective and injective, respectively, by Lemmas 4.2 and 4.6. Since \( \mathbb{C}_h[G^\sharp] = \mathbb{C}_h[G^\sharp][f^{-1}] \) and \( I_h(G^\sharp) = I_h[\bar{G}^\sharp][f^{-1}] \), this immediately implies that the map \( \tilde{m} \) is surjective and injective and hence an isomorphism.

Now recall that \( I_h(G^\sharp) \) is isomorphic to \( I_h(G)[f, f^{-1}] \). Taking quotient by the ideal \((f - 1)\) proves the theorem for \( G \). \qed

6 Quantization in terms of generators and relations

In this section we describe the quantization of \( \mathbb{C}[G] \) along the DS and STS brackets in terms of generators and relations for \( G \) being a classical matrix group. We give a detailed consideration to the DS-case. The case of STS is treated similarly, upon obvious modifications. Alternatively, the defining ideal \( \mathbb{C}_h[G_{STS}] \) can be derived from the ideal of \( \mathbb{C}_h[G_{DS}] \) using Proposition A.1 and the twist-equivalence between \( \mathbb{C}_h[G_{DS}] \) and \( \mathbb{C}_h[G_{STS}] \).

Function algebras on quantum classical matrix groups from the classical series were defined in terms of generators and relations in [FRT]. Here we prove that the algebras of [FRT] are included in flat \( \mathbb{C}[\hbar] \)-algebras, \( \mathbb{C}_h[G_{DS}] \).
6.1 FRT and RE algebras

In this subsection we recall the definition of the FRT and RE algebras, [FRT, KSkl].

Let $V_0$ be the basic representation of $G$ and let $V$ be the corresponding $U_\hbar(g)$-module. Let $R$ denote the image of the universal R-matrix of $U_\hbar(g)$ in $\text{End}(V^{\otimes 2})$. Put $N := \dim V_0$.

The FRT algebra $T$ is generated by the matrix coefficients $\{T^i_j\} \subset \text{End}^\ast(V)$ subject to the relations

$$RT_1T_2 = T_2T_1R,$$

where $T = ||T^i_j||$. So $T$ is the quotient of the free algebra $\mathbb{C}[[\hbar]]\langle T^i_j \rangle$. The latter is a $U_\hbar(g)$-bimodule algebra, the two-sided action being extended from the two-sided action on $\text{End}^\ast(V)$. The ideal $[9]$ is invariant, so $T$ is also a $U_\hbar(g)$-bimodule algebra. It is $\mathbb{Z}$-graded with $\text{deg} \: \text{End}^\ast(V) = 1$, and the grading is equivariant with respect to the two-sided $U_\hbar(g)$-action.

The RE algebra $K$ is also generated by the matrix coefficients of the basic representation, this time denoted by $K^i_j$. Let $K = ||K^i_j||$ be the matrix of the generators. The RE algebra $K$ is the quotient of the free algebra $\mathbb{C}[[\hbar]]\langle K^i_j \rangle$ by the ideal generated by the relations

$$R_{21}K_1R_{12}K_2 = K_2R_{21}K_1R_{12}.$$

The algebra $K$ is a $U_\hbar(Dg)$-module algebra, [DM]. It is $\mathbb{Z}$-graded, and the grading is invariant with respect to the $U_\hbar(Dg)$-action.

Recall from [DM] that the RE twist of the Hopf algebra $U_\hbar^{op}(g) \hat{\otimes} U_\hbar(g)$ to the twisted tensor square $U_\hbar(Dg)$, converts the algebra $T$ to $K$ (cf. also Subsection 4.2).

6.2 Algebra $\mathbb{C}_h[G_{DS}]$ in generators and relations.

From now one we assume the standard (zero weight) Lie bialgebra structure on $g$. In this section we describe the algebra $\mathbb{C}_h[G] = \mathbb{C}_h[G_{DS}]$ in terms of generators and relations.

We will use the group structure on $G^\sharp$, which is the trivial central extension of $G$. For $G$ orthogonal and symplectic, $G^\sharp$ is defined by equation [11] with $f \neq 0$. The basic representation of $G$ on $V_0$ naturally extends to a representation of of $G^\sharp$ on $V_0 \oplus \mathbb{C}$, since the subgroup $\mathbb{C}^* \subset G$ acts on the module $V_0$ by the delations. The indeterminant $f$ is the matrix coefficient of the one dimensional representation of $\mathbb{C}^*$.

Suppose $f \neq 0$. The group $G^\sharp$ can be identified with the $G^\sharp \times G^\sharp$-orbit in $\text{End}(V_0 \oplus \mathbb{C})$,
which for $G$ orthogonal and symplectic is defined by the equation

\[ B_0 T^i B_0^{-1} T = f^2, \quad T B_0 T^i B_0^{-1} = f^2, \quad \det(T) = f^N \quad (11) \]

and by (12) for $G = SL(n)$. The element $B_0 \in V_0 \otimes V_0$ in equation (11) is the classical invariant of the (orthogonal or symplectic) group $G$.

Clearly the ideals in $\mathbb{C}[\text{End}(V_0)][f]$ generated by (11) and by (12) are radical. This is obvious for the $G = SL(n)$ and follows from We for $G$ orthogonal and symplectic. The corresponding quotients of $\mathbb{C}[\text{End}(V_0)][f]$ are the affine coordinate rings of $G^2$.

Recall from [FRT] and [F] that there exists a central group-like two-sided $U_\hbar(sl(n))$-invariant $\det_q(T) \in T$ of degree $n$ such that $\det_q(T) = \det(T)$ modulo $\hbar$. For $G$ orthogonal and symplectic let $B$ denote the $U_\hbar(\mathfrak{g})$-invariant element from $V \otimes V$, see [FRT].

**Proposition 6.1.** Let $G$ be a classical unimodular matrix group. The $U_\hbar(\mathfrak{g})_{op} \otimes U_\hbar(\mathfrak{g})$-equivariant quantization $\mathbb{C}_\hbar[\hat{G}^2]$ can be realized as the quotient of $T[f]$ by the ideal of relations

\[ \det_q(T) = f^N \quad (13) \]

and, for $\mathfrak{g}$ orthogonal or symplectic,

\[ BT^i B^{-1} T = f^2, \quad T B T^i B^{-1} = f^2. \quad (14) \]

The quantization $\mathbb{C}_\hbar[G]$ is obtained from $\mathbb{C}_\hbar[\hat{G}^2]$ by factoring out the ideal $(f - 1)$.

**Proof.** Denote by $\mathfrak{S}$ the algebra $T[f]$, by $\mathfrak{T}$ the algebra $\mathbb{C}_\hbar[G^2]$, and by $\mathfrak{J}$ the ideal in $T[f]$ generated by the relations (11) and (12), depending on the type of $G$. The algebras $\mathfrak{S}$, $\mathfrak{T}$, and $\mathfrak{J}$ are graded, and the grading is $U_\hbar(\mathfrak{g}^i)$-compatible. Note that homogeneous components in $\mathfrak{S}$ and hence in $\mathfrak{J}$ are $U_\hbar(\mathfrak{g}^i)$-finite. There is an obvious bialgebra structure on $\mathfrak{S}$, with $f$ being group-like.

The Takhtajan construction of the quantization, see Subsection 4.2 implies that the evaluation at the identity $\varepsilon: a \mapsto a(e)$ is a character of the algebra $\mathfrak{T}$. Define a pairing between $\mathfrak{S}$ and $U_\hbar(\mathfrak{g}^i)$ setting $\langle a, x \rangle := \varepsilon(x \triangleright a) = \varepsilon(a \triangleleft x)$. This pairing is non-degenerate, because $G$ is connected.

The matrix coefficients of the basic representation are naturally considered as the elements of $\mathfrak{T}$. They satisfy the RTT relation, because $\mathfrak{T}$ is commutative in the category of $U_\hbar(\mathfrak{g}^i)$-bimodules. This defines an equivariant algebra homomorphism $\Psi: \mathfrak{S} \to \mathfrak{T}$. Clearly
the composition map \( \varepsilon \circ \Psi \) coincides with the counit of the bialgebra \( \mathfrak{S} \). From this we conclude that the invariant ideal \( \mathfrak{J} \) is annihilated by \( \varepsilon \) (the counit gives 1 on group-like elements, including \( \det_q(T) \) and \( f \)). Therefore \( \mathfrak{J} \) annihilates \( \mathcal{U}_h(\mathfrak{g}^\sharp) \) through the pairing \( \langle ., . \rangle \). Since this pairing is non-degenerate, the ideal \( \mathfrak{J} \) lies in the kernel of \( \Psi \).

The homomorphism \( \Psi \) preserves grading and it is identical on \( \text{End}^*(V) \oplus \mathbb{C}[[\hbar]]f \). As the image of \( \Psi \) is \( \mathbb{C}[[\hbar]]\)-free, we have the direct sum decomposition \( \mathfrak{S} = \ker \Psi \oplus \text{im} \Psi \) of \( \mathbb{C}[[\hbar]]\)-modules. Therefore \( \text{im} \Psi \) is embedded in \( \mathfrak{S}_0 \). Let us show that \( \mathfrak{J} = \ker \Psi \). Since both ideals are graded and the homogeneous components are finite, it suffices to show that the map \( \mathfrak{J}_0 \to (\ker \Psi)_0 \) induced by the embedding \( \mathfrak{J} \hookrightarrow \ker \Psi \) is surjective. Then we can apply the Nakayama lemma to each homogeneous component.

Denote by \( \mathfrak{J}_0^0 \) the image of \( \mathfrak{J}_0 \) in \( (\ker \Psi)_0 \subset \mathfrak{S}_0 \). This is a \( G^2 \times G^2 \)-invariant ideal, and it is easy to show that \( \mathfrak{J}_0^0 \) contains no positive integer powers of \( f \). On the other hand, the defining ideal \( \mathcal{N}(\hat{G}^2) \subset \mathfrak{S}_0 \) is maximal among such ideals, and it lies in \( \mathfrak{J}_0^0 \). Therefore \( \mathcal{N}(\hat{G}^2) = \mathfrak{J}_0^0 \cap \ker \Psi \), since otherwise the map \( \Psi \) would be zero. This proves \( \mathfrak{J} = \ker \Psi \). Another consequence is that \( \text{im} \Psi \) is a quantization of \( \mathbb{C}[G^2] \) that lies in \( \mathbb{C}_h[\hat{G}^2] \). Hence it coincides with \( \mathbb{C}_h[\hat{G}^2] \), because that is so in the classical limit.

The quantization \( \mathbb{C}_h[G^2] \) is isomorphic to \( \mathbb{C}_h[G^2][f^{-1}] \), as easily follows from the Takhtajan construction. Therefore \( \mathbb{C}_h[G^2] \) is realized as the quotient of the algebra \( \mathcal{T}[f, f^{-1}] \) by the ideal of the relations \( \{13\}, \{14\} \). On the other hand, \( \mathbb{C}_h[G^2] \) is a free module over \( \mathbb{C}[[\hbar]][f, f^{-1}] \). The quotient of \( \mathbb{C}_h[G^2] \) by the ideal \( (f - 1) \) is \( \mathbb{C}[[\hbar]]\)-free and thus is a quantization of \( \mathbb{C}[G] \). \( \square \)

For orthogonal and symplectic \( \mathfrak{g} \) a stronger assertion can be proven. Now let \( G \) be \( O(N) \) or \( Sp(n) \).

**Proposition 6.2.** The \( \mathcal{U}_h(\mathfrak{g})_{op} \otimes \mathcal{U}_h(\mathfrak{g}) \)-equivariant quantization \( \mathbb{C}_h[\hat{G}^2] \) can be realized as the quotient of \( \mathcal{T}[f] \) by the ideal of relations \( \{14\} \). The quantization \( \mathbb{C}_h[G] \) is obtained from \( \mathbb{C}_h[\hat{G}^2] \) by factoring out the ideal \( (f - 1) \).

**Proof.** For the group \( Sp(n) \) the statement is, in fact, already proven, because \( Sp(n) \) defined by \( \{11\} \) with \( f = 1 \) is unimodular and the condition \( \det_q = 1 \) is excessive. Thus let us focus on the orthogonal case.

The group \( \mathbb{Z}_2 \) acts on \( \mathcal{U}_h(\mathfrak{g}) \) by Hopf algebra automorphisms. This action is trivial for \( \mathfrak{g} = so(2n+1) \) and induced by the flip \( \sigma \) of the simple roots \( \alpha_{n-1} \) and \( \alpha_n \) (the automorphism of the Dynkin diagram) in the quantum Chevalley basis for \( \mathfrak{g} = so(2n) \). Consider the smash product \( \mathbb{Z}_2 \ltimes \mathcal{U}_h(\mathfrak{g}) \) with the natural structure of Hopf algebra, a deformation of the Hopf algebra \( \mathbb{Z}_2 \ltimes \mathcal{U}(\mathfrak{g}) \). Let \( \{e_j^+\} \) be the standard matrix base of \( \text{End}(V) \). The representation of
\( U_h(\mathfrak{g}) \) on \( V \) extends to a representation of \( \mathbb{Z}_2 \rtimes U_h(\mathfrak{g}) \) by assigning \( \sigma \mapsto -1 \) for \( \mathfrak{g} = \mathfrak{so}(2n+1) \) and \( \sigma \mapsto 1 - e_n^n - e_{n+1}^{n+1} + e_{n+1}^n + e_{n+1}^n \) for \( \mathfrak{g} = \mathfrak{so}(2n) \) (in the realization of [FRT]). Note that these matrices are characters of the algebra \( \mathcal{T} \) with \( \det = -1 \) in the classical limit.

Now repeat the proof of Proposition 6.1 replacing \( U_h(\mathfrak{g}^\ast) \) by \( \mathbb{Z}_2 \rtimes U_h(\mathfrak{g}^\ast) \).

It is crucial in the proofs of Propositions 6.2 and 6.2 that the defining ideal of the group \( G^\ast \) should be maximal invariant. Without the condition \( \det = f^N \), the group \( G^\ast \) has two connected components, which are orbits of the two-sided action \( U(\mathfrak{g}^\ast) \). Hence the defining ideal of \( G^\ast \) is not maximal among the invariant proper ideals in \( \mathbb{C}[G^\ast] \). It becomes so if we extend the symmetries and consider the algebra \( \mathbb{Z}_2 \rtimes U_h(\mathfrak{g}) \), instead of \( U_h(\mathfrak{g}) \).

### 6.3 Algebra \( \mathbb{C}_h[G_{STS}] \) in generators and relations

Under the twist from \( U_h^{op}(\mathfrak{g}^\hat{\ast}) \odot U_h(\mathfrak{g}) \) to \( U_h(\mathfrak{Dg}) \), the defining relations of \( \mathbb{C}_h[G_{STS}] \) in \( \mathcal{T} \) transform to certain relations in the RE algebra \( \mathcal{K} \) and generate a \( U_h(\mathfrak{Dg}) \)-invariant ideal in \( \mathcal{K} \), see Appendix A. Let us compute this ideal.

The multiplications in \( \mathcal{T} \) and \( \mathcal{K} \) are related by the formula (see [DM])

\[
m_{\mathcal{T}}(a \otimes b) = m_{\mathcal{K}}(R_1 \cdot a \cdot R_1^{-1} \otimes b \cdot R_2^{-1} \cdot R_2).
\]

Let \( G \) be the orthogonal or symplectic groups \( G \). Formula (15) applied to the equation

\[
T^d B^{-1} T = B^{-1}, \quad TBT^d = B
\]

(16) gives

\[
R_1^d K^d ((R_1^d)^{-1} B^{-1} (R_2^d)^{-1}) R_2 K = B^{-1}, \quad KR_1 BK^d R_2^d = R_1^d BR_2^d.
\]

The ideal generated by (17) lies in the kernel of the \( U_h(\mathfrak{Dg}) \)-equivariant projection \( \mathcal{K} \to \mathbb{C}_h[G_{STS}] \).

Similarly one can express the element \( \det_q(T) \) through the generators \( K_j^d \). We denote by \( \det_q(K) \) the resulting form of degree \( n \). The ideal \( (\det_q(K) - 1) \) is annihilated by the \( U_h(\mathfrak{Dg}) \)-equivariant projection \( \mathcal{K} \to \mathbb{C}_h[G_{STS}] \).

**Proposition 6.3.** Let \( G \) be a classical complex matrix group. Then the algebra \( \mathbb{C}_h[G_{STS}] \) is isomorphic to the quotient of \( \mathcal{K} \) by the ideal \( \mathfrak{J} \), where

i) \( \mathfrak{J} \) is generated by relations (17) and \( \det_q(K) = 1 \) for the \( G = SO(N) \).

ii) \( \mathfrak{J} = (\det_q(K) - 1) \) for \( G = SL(n) \).

iii) \( \mathfrak{J} \) is generated by relations (14) for \( G = O(N) \) or \( G = Sp(n) \).
Proof. This proposition can be proven by a straightforward modification of the proof of Proposition 6.1. Another way is to start from Proposition 6.1 and use the RE twist applied to the DS quantization, cf. Appendix A. 

A On twist of module algebras

In this subsection we study how twist of Hopf algebras affects defining relations of their module algebras. Let $\mathcal{H}$ be a Hopf algebra, $V$ a finite dimensional left $\mathcal{H}$-module, and $T(V)$ the tensor algebra of $V$. Let $W$ be an $\mathcal{H}$-submodule in $T(V)$ generating an ideal $J(W)$ in $T(V)$. Denote by $\mathcal{A}$ the quotient algebra $T(V)/J(W)$.

Let $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ be a twisting cocycle and $\tilde{\mathcal{H}}$ the corresponding twist of $\mathcal{H}$. Denote by $\tilde{\mathcal{A}}$ the twist of the module algebra $\mathcal{A}$. The multiplication in $\tilde{\mathcal{A}}$ is expressed through the multiplication in $\mathcal{A}$ by $m_{\tilde{\mathcal{A}}} = m_{\mathcal{A}} \circ \mathcal{F}$ and similarly for $\tilde{T}(V)$ and $T(V)$.

For each $n = 0, 1, \ldots$, introduce an automorphism of $V^{\otimes n}$ by induction:

$$\Omega_n = \text{id}, \quad n = 0, 1, \quad \Omega_n = (\Delta^n \otimes \Delta^k)(\mathcal{F})(\Omega_m \otimes \Omega_k), \quad k + m = n.$$ 

This definition does not depend on the partition $k + m = n$. The elements $\Omega_n$ amounts to a linear automorphism $\Omega$ of $T(V)$.

Proposition A.1. The algebra $\tilde{\mathcal{A}}$ is isomorphic to the quotient algebra $T(V)/J(\Omega^{-1} W)$.

Proof. Since the ideal $J(W) \subset T(V)$ is invariant, it is also an ideal in $\tilde{T}(V)$. It is easy to see that the quotient $\tilde{T}(V)/J(W)$ is isomorphic to $\tilde{\mathcal{A}}$. On the other hand, the algebra $\tilde{T}(V)$ is isomorphic to $T(V)$. The isomorphism is given by the maps $T(V) \ni m(v_1 \otimes \ldots \otimes v_n) \mapsto (\hat{m} \circ \Omega_n)(v_1 \otimes \ldots \otimes v_n)$, $n \in \mathbb{N}$, where $m$ and $\hat{m}$ are multiplications in $T(V)$ and $\tilde{T}(V)$. This implies the proposition. 

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