SMALL ALGEBRAIC CENTRAL VALUES OF TWISTS OF ELLIPTIC $L$-FUNCTIONS

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ABSTRACT. We consider heuristic predictions for small non-zero algebraic central values of twists of the $L$-function of an elliptic curve $E/\mathbb{Q}$ by Dirichlet characters. We provide computational evidence for these predictions and consequences of them for instances of an analogue of the Brauer-Siegel theorem.

1. Introduction

Let $E$ be an elliptic curve defined over the rational field $\mathbb{Q}$ with $L$-function $L(E/\mathbb{Q}, s)$, conductor $N_E$, and denote by $w_E$ the sign of its functional equation. Then the Birch & Swinnerton-Dyer Conjecture relates the leading term of the Taylor expansion of $L(E/\mathbb{Q}, s)$ at $s = 1$ to the arithmetic invariants of $E/\mathbb{Q}$. In particular, it predicts that the order of vanishing of $L(E/\mathbb{Q}, s)$ at $s = 1$ is equal to the $\mathbb{Z}$-rank $r$ of the Mordell-Weil group $E(\mathbb{Q})$ and that

$$
\frac{L^{(r)}(E/\mathbb{Q}, 1)}{r!} = \frac{|\Omega(E(\mathbb{Q}))| \Omega_{E, R}(E(\mathbb{Q})) \prod_{p|N_E} c_p}{(|E(\mathbb{Q})_{\text{tors}}|^2}
$$

where the right hand side involves the usual invariants of $E/\mathbb{Q}$.

There are corresponding conjectures for $E$ over finite extensions $K/\mathbb{Q}$.

In the case of abelian extensions $K/\mathbb{Q}$, these questions can be investigated via the twists $L(E, s, \chi)$ of $L(E/\mathbb{Q}, s)$ by primitive Dirichlet characters $\chi$ associated to the extension $K/\mathbb{Q}$. In this context, the vanishing of $L(E, 1, \chi)$ have been studied (see [MR], [DFK1], [DFK2]) and some predictions were made regarding the frequency of such vanishings. Specifically, Conjecture 10.1 of [MR] predicts that there are only finitely many even primitive Dirichlet characters $\chi$ of order $k$ with $\phi(k) \geq 6$ such that $L(E, 1, \chi) = 0$. Rephrasing this in terms of the Birch & Swinnerton-Dyer conjecture, they predict (Conjecture 10.2 [MR]) for any (infinite)
real abelian extension $F/\mathbb{Q}$ which has only a finite number of subfields of degree 2, 3 or 5, that the Mordell-Weil group $E(F)$ is finitely generated.

In this article we will be concerned with statistics regarding the non-vanishing central values $L(E, 1, \chi)$ as $\chi$ varies over families of primitive Dirichlet characters of order at least 3. The heuristics of [MR], [DFK1], [DFK2] would indicate that for almost all such characters (100%), we would have $L(E, 1, \chi) \neq 0$.

In order to develop a heuristic from which we derive our speculations, we must make several assumptions. These hypotheses are both of an arithmetic nature and of a statistical nature.

1.1. HA – Arithmetic Hypotheses. In the following we assume both the Birch & Swinnerton-Dyer conjectures and the generalized Lindelöf hypothesis (see [GHP]) that $L(E, 1, \chi) = O(f^\epsilon_{\chi})$, where $f_{\chi}$ is the conductor of $\chi$, and the implied constants depend only on $E$.

1.2. HS – Statistical Hypotheses. In §3 we consider a totally real field $F/\mathbb{Q}$, with $[F : \mathbb{Q}] = n$, and the usual embedding $\psi : F \rightarrow \mathbb{R}^n$, where for $\alpha \in F$, $\psi(\alpha) = (\gamma_1(\alpha), \gamma_2(\alpha), \ldots, \gamma_n(\alpha)) \in \mathbb{R}^n$ where $1 = \gamma_1, \gamma_2, \ldots, \gamma_n$ are the distinct embeddings of $F$ into $\mathbb{R}$. Then the image of the ring of integers $\psi(O_F) \subset \mathbb{R}^n$ is a sublattice of $\mathbb{R}^n$. We assume that the the probability that the image of an integer lies in a region $\mathcal{T} \subset \mathbb{R}^n$ is given by the relative volume of $\mathcal{T}$, and further that the coordinates $\gamma(\alpha)$ are (generically) independent identically distributed random variables. (We say “generically” since this is certainly not the case for $\alpha$ lying in a subfield of $F$).

In §2.2 we follow [MTT] to write $L(E, 1, \chi) = (\text{known factors}) \times L^\text{alg}_E(\chi)$ where $L^\text{alg}_E(\chi) \in \mathbb{Q}(\chi)$ is an algebraic integer. Then (under the Birch & Swinnerton-Dyer conjectures) we can think of the integer $|\text{Nm}_{\mathbb{Q}(\chi)/\mathbb{Q}}(L^\text{alg}_E(\chi))|$ as essentially the order of the “$\chi$–part” of the Shafarevich-Tate group $\text{III}(E(K))$.

Let $B = \sigma_0(k) + \phi(k)/2 - 3$ where $\sigma_0(k)$ is the number of positive divisors of $k$ (sometimes denoted $d(k)$ or $\tau(k)$). Based on the model we propose, we make the following predictions:
Prediction 1.1. Let $L \gg 0$ be a fixed integer. Fix $k \geq 3$ and let $X > 0$. Then under the hypotheses above, the number $n_{k,E}(X,L)$ of primitive characters $\chi$ of order $k$ and conductor $f_{\chi} \leq X$ such that $0 \neq |\text{Nm}_{\mathbb{Q}(\chi)/\mathbb{Q}}(L_{E}^{\text{alg}}(\chi))| \leq L$ grows as

$$n_{k,E}(X,L) \sim b_{E}X^{1/2}\log^{B}(X) \quad \text{if} \quad \phi(k) = 2$$

$$\sim b_{E}'\log^{B+1}(X) \quad \text{if} \quad \phi(k) = 4$$

$$\text{is bounded} \quad \text{if} \quad \phi(k) \geq 6$$

as $X \to \infty$, for some non-zero constants $b_{E}$, and $b_{E}'$. For $k = 3$, we have $B = \sigma_{0}(3) + \phi(3)/2 - 3 = 0$, so the predicted growth rate for “small” non-zero algebraic values of cubic twists $0 \neq |\text{Nm}_{\mathbb{Q}(\chi)/\mathbb{Q}}(L_{E}^{\text{alg}}(\chi))| \leq L$ is of the order $b_{E}X^{1/2}$. This is in contrast with the situation for the distribution of class numbers of quadratic fields, where for imaginary quadratic fields for any $L > 0$ the class number is at most $L$ finitely often, and the unit group is finite, and for real quadratic fields we expect infinitely many to have class number 1, and large fundamental units. For the elliptic curve $E$, we expect both infinitely many occurrences of $0 \neq |\text{Nm}_{\mathbb{Q}(\chi)/\mathbb{Q}}(L_{E}^{\text{alg}}(\chi))| \leq L$ and also $E(K)/E(\mathbb{Q})$ finite.

For $k = 5$, we have $B = \sigma_{0}(5) + \phi(5)/2 - 3 = 1$, so the predicted growth rate for “small” non-zero algebraic values of quintic twists $0 \neq |\text{Nm}_{\mathbb{Q}(\chi)/\mathbb{Q}}(L_{E}^{\text{alg}}(\chi))| \leq L$ is of the order $b_{E}'\log^{2}(X)$.

These predictions seem to agree with the experimental computations in the cases below.

Prediction 1.2. Fix an integer $L \gg 0$. Let $\mathcal{C}$ be any set of primitive Dirichlet characters $\chi$ whose orders $k_{\chi}$ satisfy $\phi(k_{\chi}) \geq 6$. Then there are only finitely many $\chi \in \mathcal{C}$ such that $|\text{Nm}_{\mathbb{Q}(\chi)/\mathbb{Q}}(L_{E}^{\text{alg}}(\chi))| \leq L$.

Prediction 1.3. Under the hypotheses above, for almost all (100%) Dirichlet characters of order $k \geq 3$ and any $\epsilon > 0$, we have

$$f_{\chi}^{1/2-\epsilon} \ll |L_{E}^{\text{alg}}(\chi)| \ll f_{\chi}^{1/2+\epsilon}$$

Similar questions are discussed in [DW].
The Brauer-Siegel theorem (see [4]) states that if $K$ is an infinite sequence of Galois extensions of $\mathbb{Q}$ such that

$$\frac{[K : \mathbb{Q}]}{\log |D(K)|} \to 0$$

then

$$\frac{\log(h(K)R_K)}{\log \sqrt{|D(K)|}} \to 1$$

where $h_K$, $R_K$ and $D(K)$ are the class number, the regulator, and the discriminant of $K$ respectively.

Since families of cyclic cubic extensions $K/\mathbb{Q}$ with $|\text{Nm}_{\mathbb{Q}/\mathbb{Q}}(L_{E,\chi}(\mathbb{Q}))| \leq L$ are predicted by [4] to be infinite, we can take the upper limit (for $K$ in such a family and $\chi < X$) as $X \to \infty$ to get an analogue of the Brauer-Siegel limits.

**Prediction 1.4.** For such a family of cyclic cubic extensions $K/\mathbb{Q}$,

$$\lim_{K} \frac{\log(\text{III}(E(K))R(E(K)))}{\log(\sqrt{|D(K)|})} = 0$$

where the Birch & Swinnerton-Dyer conjecture predicts that the elliptic regulator $R(E(K)) = R(E(\mathbb{Q}))$ up to a bounded factor, since $L(E, 1, \chi) \neq 0$.

In this article we will generally be concerned with “orders of growth” (rather than asymptotics), and will use the notation $f(x) \sim g(x)$ to indicate that $f = O(g)$ and $g = O(f)$ as $x \to \infty$ (or that $0 < c < |f(x)/g(x)| < C$ as $x \to \infty$ without necessarily specifying $c$ or $C$).

In an appendix by Jungbae Nam we provide the results of numerical computations which (to us) seem to support these predictions. We would like to thank Evan Dummit and Andrew Granville for the proof of Lemma 3.1.

2. Preliminaries and Notation

2.1. Number of primitive characters.

Let $f$ be a positive integer, either odd or divisible by 4. Then the multiplicative group $(\mathbb{Z}/f\mathbb{Z})^*$ is naturally isomorphic with the Galois group $G = \text{Gal}(\mathbb{Q}(\zeta_f)/\mathbb{Q})$ of the cyclotomic field of $f^{th}$ roots of unity. The group of characters $\hat{G}$ of $G$ can then be identified with the group of Dirichlet characters $(\mathbb{Z}/f\mathbb{Z})^*$. Since each Dirichlet character modulo $f$ is induced from a unique primitive character of conductor $f$. 

dividing \( f \), we identify \( G = (\mathbb{Z}/f\mathbb{Z})^* \) with the set of primitive characters \( \chi \) of conductor \( f \chi \), dividing \( f \). The trivial character with \( f \chi = 1 \) will be denoted 1, and for \( \chi \in (\mathbb{Z}/f\mathbb{Z})^* \) let \( \text{ord}(\chi) \) denote the order of \( \chi \).

In the following we restrict to using only primitive characters as this allows for simpler functional equations and factorisations of \( L \)-functions (without extra Euler factors).

For \( k \geq 2 \) define the following sets of primitive Dirichlet characters:

\[ A_k := \{ \chi \mid \chi^k = 1 \} \quad \text{and} \quad B_k := \{ \chi \in A_k \mid \text{ord}(\chi) = k \} \]

Fix \( N \geq 1 \) a positive integer and let \( X > 0 \). Define

\[ A_{k,N}(X) = \{ \chi \in A_k \mid \gcd(f \chi, N) = 1, f \chi \leq X \} \quad \text{and} \quad B_{k,N}(X) = \{ \chi \in B_k \mid \gcd(f \chi, N) = 1, f \chi \leq X \} \]

For positive integers \( 0 < f \in \mathbb{Z} \), let \( a_k(f) = \#(\{ \chi \in A_k \mid f \chi = f \}) \) and \( b_k(f) = \#(B_k(f)) := \#(\{ \chi \in B_k \mid f \chi = f \}) \). Then by considering the series

\[
F_k(s, A_k) = \sum_{f \leq X, \gcd(f,N) = 1} a_k(f) \frac{f^s}{f^s} \quad \text{and} \quad F_k(s, B_k) = \sum_{f \leq X, \gcd(f,N) = 1} b_k(f) \frac{f^s}{f^s}.
\]

we obtain for some positive constant \( c_k = c_k(N) > 0 \) the asymptotic estimate (see \( \mathcal{C} \) Calculating \( \#(B_{k,N}(X)) \))

\[
#(B_{k,N}(X)) = \sum_{f \leq X, \gcd(f,N) = 1} b_k(f) \sim c_k X \log^{\sigma_0(k)-2}(X).
\]

where \( \sigma_0(k) \) is the number of divisors of \( k \). Hence for characters of prime order \( p \) we have

\[
#(B_{p,N}(X)) = \sum_{f \leq X, \gcd(f,N) = 1} b_p(f) \sim c_p X.
\]

where \( c_p = c_{p,N} > 0 \) (see \( e.g. \), [Na], Chapter 8, Cor.1).

2.2. Algebraic central values.

For this we follow [MTT], [MR].
Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N_E$, and for a primitive Dirichlet character $\chi$, let $L(E, s, \chi)$ denote the $L$-function $L(E/\mathbb{Q}, s)$ twisted by $\chi$. Then there are real numbers $\Omega^{\pm}$ such that

$$L(E, 1, \chi) = \frac{\Omega_{\chi}}{2\tau(\overline{\chi})} \sum_{a \mod f_{\chi}} \overline{\chi}(a) c(a, f_{\chi}; E)$$

where $\Omega_{\chi}$ equals $\Omega^{\pm}$ according as $\chi(-1) = \pm 1$ and the $c(a, f_{\chi}; E)$ are integers that do not depend on $\chi$ (but only on $a, f_{\chi}$ and $E$) and $\tau(\chi)$ is the Gauss sum corresponding to the character $\chi$. (One can ensure that the integers $c(a, f_{\chi}; E)$ have no common divisor by choosing the numbers $\Omega^{\pm}$ appropriately (see [W-W])).

From [MTT] the algebraic part of $L(E, 1, \chi)$ is defined by

$$L_{E}^{\text{alg}}(\chi) := \frac{2\tau(\overline{\chi})L(E, 1, \chi)}{\Omega_{\chi}} = \sum_{a \mod f_{\chi}} \overline{\chi}(a) c(a, f_{\chi}; E).$$

Then $L_{E}^{\text{alg}}(\chi)$ is an algebraic integer in the cyclotomic field $\mathbb{Q}(\chi)$ generated over $\mathbb{Q}$ by the values of $\chi$ and satisfies $\sigma(L_{E}^{\text{alg}}(\chi)) = L_{E}^{\text{alg}}(\chi^\sigma)$ for all $\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$.

Noting that $\chi(-1) = \overline{\chi}(-1)$ we see that $\Omega_{\chi} = \Omega_{\overline{\chi}}$, so from the functional equation, we have

$$L_{E}^{\text{alg}}(\chi) = \frac{2\tau(\overline{\chi})L(E, 1, \chi)}{\Omega_{\chi}} = \frac{2\tau(\overline{\chi})w_{E\chi}(N_E)\tau(\chi^2)}{f_{\chi}} \frac{L(E, 1, \chi)}{\Omega_{\overline{\chi}}}$$

$$= \frac{w_{E\chi}(N_E)\tau(\chi)\tau(\overline{\chi})}{f_{\chi}} \frac{2\tau(\chi)L(E, 1, \chi)}{\Omega_{\overline{\chi}}} = \frac{w_{E\chi}(N_E)\chi(-1)f_{\chi}}{f_{\chi}} \cdot L_{E}^{\text{alg}}(\chi)$$

$$= w_{E\chi}(-N_E)L_{E}^{\text{alg}}(\chi).$$

If $z \in \mathbb{C}^*$ is any non-zero complex number satisfying $z = w_{E\chi}(-N_E)\overline{\tau}$, then it follows that $L_{E}^{\text{alg}}(\chi)/z = x \in \mathbb{R}$ is real. Let $\zeta_{\chi} = w_{E\chi}(-N_E)$. Then $\zeta_{\chi}$ is a primitive $n^{\text{th}}$ root of unity for some $n \geq 1$ dividing $2k$, where $k$ is the order of $\chi$.

Suppose now that $\chi$ is a complex Dirichlet character of order $k \geq 3$. 
If $\zeta \neq \pm 1$, choose
\[ \lambda = \frac{1}{1 + \zeta} \] so that
\[ \lambda = \zeta \frac{1}{1 + \zeta} = \zeta \bar{\lambda}, \]
and
\[ L_{E}^{\text{alg}}(\chi) = \lambda \alpha \]
with $\alpha \in \mathcal{O}_{\chi}^+$ where $\mathcal{O}_{\chi}^+$ is the ring of integers in $\mathbb{Q}(\chi)^+$, the maximal real subfield of $\mathbb{Q}(\chi)$.

If $\zeta = -1$, let $c$ be the least positive integer such that the order of $\chi(c)$ is equal to $k$, the order of $\chi$. Choose
\[ \lambda = \frac{1}{\chi(c) - \chi(c)} \] so that
\[ \lambda = -\bar{\lambda} = \zeta \bar{\lambda} \]
and
\[ L_{E}^{\text{alg}}(\chi) = \lambda \alpha \]
with $\alpha \in \mathcal{O}_{\chi}^+$.

If $\zeta = 1$, then choose $\lambda = 1$ and so
\[ L_{E}^{\text{alg}}(\chi) = \lambda \alpha \]
with $\alpha \in \mathcal{O}_{\chi}^+$.

We have proved the following:

**Proposition 2.1.** Let $E/\mathbb{Q}$ be an elliptic curve defined over $\mathbb{Q}$, and let $\chi$ be a primitive Dirichlet character of order $k \geq 3$ and conductor $f_{\chi}$. Let $\zeta_{\chi} = w_{E} \chi(-N_{E})$. Then
\[ L_{E}^{\text{alg}}(\chi) = \lambda \alpha \]
where
\[ \lambda = \begin{cases} 
\frac{1}{1 + \zeta} & \text{if } \zeta \neq \pm 1 \\
\frac{1}{\chi(c) - \chi(c)} & \text{if } \zeta = -1 \\
1 & \text{if } \zeta = 1,
\end{cases} \]
and $\alpha \in \mathcal{O}_{\chi}^+$ are real cyclotomic integers. Also we have
\[ \sigma(\alpha) = \alpha \sigma \quad \text{and} \quad \sigma(\lambda) = \lambda \sigma \]
for all $\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. 
Remark 1. The choice of $\lambda_\chi$ is not unique. For example, in the case that $E$ is an elliptic curve with $w_E = 1$ and $\chi$ is a character of odd order $k$, then we could take
\[ \lambda_\chi' = \zeta_{k+1} = \chi(N_E)^{\frac{k+1}{2}}. \]
Then $\lambda_\chi'/\lambda_\chi^2 = \zeta_{k+1} = \zeta_\chi$ and
\[ L_E^{\text{alg}}(\chi) = \lambda_\chi' / \lambda_\chi' \]
where $\beta_\chi \in \mathcal{O}_\chi^+$ are real cyclotomic integers and we again have
\[ \sigma(\beta_\chi) = \beta_\chi^\sigma \quad \text{and} \quad \sigma(\lambda_\chi') = \lambda_\chi'^\sigma \]
for all $\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. Note that in this case that
\[ \beta_\chi = \lambda_\chi' / \lambda_\chi \]
and that $\lambda_\chi / \lambda_\chi'$ is a circular unit.

3. Probabilities for non-zero values

In this section we shall be consider a naïve probabilistic model for non-zero values of $L_E^{\text{alg}}(\chi)$. From Proposition 2.1 we see that
\[ L_E^{\text{alg}}(\chi) = \lambda_\chi \alpha_\chi \]
where the $\lambda_\chi$ are taken form a finite set and the $\alpha_\chi \in \mathcal{O}_\chi^+$ are real cyclotomic integers satisfying
\[ \sigma(\alpha_\chi) = \alpha_\chi^\sigma \]
for all $\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$.

We are interested in the distribution of the norms $A_\chi = \text{Nm}_{\mathbb{Q}(\chi)/\mathbb{Q}}(\alpha_\chi) \in \mathbb{Z}$ as $\chi$ varies over various families of primitive Dirichlet characters. Recall that the generalized Lindelöf hypothesis implies that $|\gamma(\alpha_\chi)| = O(\frac{1}{\gamma^4})$ for all $\gamma \in \text{Gal}(\mathbb{Q}(\chi)^+/\mathbb{Q})$ and therefore that
\[ 0 \leq A_\chi = \text{Nm}_{\mathbb{Q}(\chi)^+/\mathbb{Q}}(\alpha_\chi) = O\left(\int_{\chi}^{\frac{1}{4}+\epsilon}\right). \]

For a totally real field $F/\mathbb{Q}$, with $[F : \mathbb{Q}] = n$, let $\psi : F \rightarrow \mathbb{R}^n$ be the usual map
where for $\alpha \in F$, $\psi(\alpha) = (\sigma_1(\alpha), \sigma_2(\alpha), \ldots, \sigma_n(\alpha)) \in \mathbb{R}^n$ where $1 = \sigma_1, \sigma_2, \ldots, \sigma_n$ are the distinct embeddings of $F$ into $\mathbb{R}$. Then the image of the ring of integers $\psi(O_F) \subset \mathbb{R}^n$ is a sublattice of $\mathbb{R}^n$. We will be interested in the case that $F = \mathbb{Q}(\chi)^+$, and $O_F = \mathcal{O}_\chi^+$. 

Lemma 3.1. Let \( n \geq 1 \) be a positive integer and let \( L \) and \( M \) be real numbers with \( 0 < L \leq M^n \).

Define subsets \( \mathcal{T} \subseteq \mathcal{R} \subseteq \mathbb{R}^n \) by

\[
\mathcal{R} = \mathcal{R}(M) = \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq M, 1 \leq i \leq n \}
\]

and

\[
\mathcal{T} = \mathcal{T}(L, M) = \{ x = (x_1, x_2, \ldots, x_n) \in \mathcal{R} \mid x_1x_2 \cdots x_n \leq L \}.
\]

Then

\[
\mu(\mathcal{T}) / \mu(\mathcal{R}) = \frac{L}{M^n} P_{n-1}\left( \frac{L}{M^n} \right)
\]

where \( \mu \) is Lebesgue measure and \( P_m(x) = \sum_{j=0}^{m} x^j/j! \) is the \( m \)th Taylor polynomial of \( e^x \) at \( x = 0 \). Then the order of growth of the ratio of their measures as \( M \to \infty \) is

\[
\frac{\mu(\mathcal{T})}{\mu(\mathcal{R})} \sim \frac{L^n - \log^n(M)}{(n-1)!M^n}.
\]

Proof. (due to A. Granville)

Clearly \( \mu(\mathcal{R}) = M^n \), so we compute \( \mu(\mathcal{T}) \). Re-scaling and letting \( x_i' = x_i/M \) we see that

\[
\mu(\mathcal{T}) = M^n \int_{0 \leq x_1', x_2', \ldots, x_n' \leq 1} dx_1' dx_2' \cdots dx_n' = M^n I(n).
\]

and so \( \mu(\mathcal{T}) / \mu(\mathcal{R}) = I(n) \).

Set \( C = L/M^n \) and \( x_j' = e^{-y_j} \), so that \( dx_j' = -e^{-y_j} dy_j \), and then \( I(n) \) becomes

\[
I(n) = \int_{y_1, y_2, \ldots, y_n \geq 0} e^{-(y_1 + y_2 + \cdots + y_n)} dy_1dy_2 \cdots dy_n
\]

\[
= \int_{x \geq \log(1/C)} e^{-x} \cdot \int_{y_1 + y_2 + \cdots + y_{n-1} \leq x} dy_1dy_2 dy_{n-1}dx.
\]

Let

\[
J_n(x) := \int_{y_1 + y_2 + \cdots + y_n \leq x} dy_1dy_2 \cdots dy_n
\]
Then \( J_1(x) = x \) and
\[
J_n(x) = \int_{y=0}^{x} J_{n-1}(x-y) \, dy = \int_{t=0}^{x} J_{n-1}(t) \, dt = x^n / n! \text{ by induction.}
\]

Hence we have
\[
I(n) = \int_{x \geq \log (1/C)} e^{-x} x^{n-1} \, dx.
\]
Integrating by parts we see
\[
\int_{x \geq A} e^{-x} x^n \, dx = \left[ -e^{-x} x^n \right]_A^{\infty} + \int_{x \geq A} e^{-x} x^{n-1} \, dx,
\]
so by induction we have
\[
I(n) := \int_{0 \leq x_1, x_2, \ldots, x_n \leq 1} \prod_{x_1, x_2 \cdot x_n \leq C} dx_1 dx_2 \cdots dx_n = C \sum_{m=0}^{n-1} \frac{(-\log (C))^m}{m!}.
\]

Recalling that \( C = L/M^n \) yields the statement. \( \square \)

In order to estimate the frequency with which \( A_\chi \) assumes “small” non-zero values \( (0 < |A_\chi| \leq L_\chi \text{ for some choices of } L_\chi) \) for primitive Dirichlet characters \( \chi \) of fixed order \( k \), we use Lemma 3.1 to estimate probabilities for the set \( \{ x = (\gamma(\alpha_\chi)) | \gamma \in \text{Gal}(Q(\chi)^+/Q) \} \) to lie in a given region \( T \subset \mathbb{R}^{\phi(k)/2} \).

Our main assumption is that the probabilities are proportional to relative volume of the region \( T \) and that the coordinates \( \gamma(\alpha_\chi) \) are independent identically distributed random variables.

We note that
\[
A_\chi = 0 \iff \gamma(\alpha_\chi) = 0 \text{ for all } \gamma \in \text{Gal}(Q(\chi)^+/Q)
\]
\[
\iff \alpha_\chi \gamma = 0 \text{ for all } \gamma \in \text{Gal}(Q(\chi)^+/Q).
\]
and that such characters \( \chi \) are treated (via different probability models) in \([\text{MR}], [\text{DFK1}], \) and \([\text{DFK2}]\), and that they contribute 0 to the probability calculations below.

We consider the sums
\[
\frac{1}{\phi(k)} \sum_{\chi \in \mathcal{B}_{k,N}(X)} \text{Prob}(0 < |A_\chi| \leq L) = \sum_{\ell \leq X} \sum_{\gcd(f,N) = 1} \sum_{l_\chi = \ell} \text{Prob}(0 < |A_\chi| \leq L)
\]
for \( N \geq 1, X \gg 0 \), where \( L = L_\chi \), and determine the convergence of
\[
\frac{1}{\phi(k)} \sum_{\chi \in B_k, \gcd(f_\chi, N) = 1} \text{Prob}(0 < |A_\chi| \leq L).
\]

For characters \( \chi \) of order \( k \) and conductor \( f_\chi \), the generalized Lindelöf hypothesis implies that the image \( \psi(\alpha_\chi) \) lies in \( \mathcal{R}'(M) \) where
\[
\mathcal{R}'(M) = \{ \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid |x_i| \leq M, 1 \leq i \leq n \}
\]
with \( M \sim f_\chi^{1/2 + \epsilon} \) and \( n = \phi(k)/2 = [\mathbb{Q}(\chi)^+ : \mathbb{Q}] \). Taking into account the possible signs of the \( x_i \) we have \( \mu(\mathcal{R}'(M)) = 2^n \mu(\mathcal{R}(M)) \). Similarly, \( \mu(\mathcal{T}'(L, M)) = 2^n \mu(\mathcal{T}'(L, M)) \) where
\[
\mathcal{T}'(L, M) = \{ \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{R}'(M) \mid \prod_{i} x_i \leq L \}.
\]

Under our assumptions (that the number of lattice points in a region is proportional to the relative volume of the region and that the coordinates are independent identically distributed random variables), we find that for fixed \( k \),\( n = \phi(k)/2 \), and as \( M \sim f_\chi^{1/2 + \epsilon} \to \infty \) by Lemma 3.1 we have
\[
\text{Prob}(0 < |A_\chi| \leq L) = \frac{\mu(\mathcal{T}'(L, M))}{\mu(\mathcal{R}'(M))} \sim \frac{L \log^{n-1}(f_\chi^{1/2})}{f_\chi^{(\phi(k))/2 + \epsilon}}.
\]

Then for the first sum (3.1) we have
\[
\frac{1}{\phi(k)} \sum_{\chi \in B_k, N(X)} \text{Prob}(0 < |A_\chi| \leq L) = \sum_{f < X} \sum_{\gcd(f, N) = 1} b_k(f) \text{Prob}(0 < |A_\chi| \leq L)
\]
\[
= \sum_{f < X} \sum_{\gcd(f, N) = 1} b_k(f) \frac{L \log^{n-1}(f^{1/2})}{f^{(\phi(k))/2 + \epsilon}}.
\]

We will consider these sums taking \( L = L_\chi = O(f_\chi^c) \) with \( 0 \leq c < \phi(k)/4 \). In the following, since the values of \( \phi(k)/4 \) are discrete (half integers for \( k \geq 3 \)) and \( \epsilon > 0 \) can be taken arbitrarily small, the convergence of (3.4) below is determined by the value of \( \phi(k)/4 - c \).
Then
\[
\frac{1}{\phi(k)} \sum_{\chi \in \mathcal{B}_{k,N}(X)} \text{Prob}(0 < |A| \leq L) = \sum_{i \leq X} \sum_{f_i = 1} \sum_{\chi \in \mathcal{B}_{k,N}(X)} \text{Prob}(0 < |A| \leq L) = \sum_{i \leq X} b_k(f) \text{Prob}(0 < |A| \leq L)
\]
\[
\sim \sum_{i \leq X} b_k(f) \log^{n-1}(f^{1/2}) \frac{u \log B u}{u^{1+(\phi(k)/4)-c}}.
\]

(3.4)

Then by partial summation, we have, the growth as \(X \to \infty\)
\[
\frac{1}{\phi(k)} \sum_{\chi \in \mathcal{B}_{k,N}(X)} \text{Prob}(0 < |A| \leq L) \sim C \frac{X \log B(X)}{X^{(\phi(k)/4)-c}} + D \int_{1}^{X} \frac{u \log B u}{u^{1+(\phi(k)/4)-c}} du
\]

where \(C, D\) are non-zero constants and \(B = \sigma_0(k) + \phi(k)/2 - 3\).

4. Some Consequences

Fix an elliptic curve \(E/\mathbb{Q}\) defined over the rational field \(\mathbb{Q}\) with conductor \(N_E\). In the computations in the appendix, in order to accelerate the computations, we consider only those characters \(\chi\) of order \(k\) with conductors \(\gcd(f_\chi, N_E) = 1\). However, for our predictions we allow \(\gcd(f_\chi, N_E) > 1\).

4.1. \(c = 0\).

If we take \(c = 0\), then \(L = L_\chi\) is assumed to be a fixed bounded constant. As \(X \to \infty\), the sum (3.3) converges for \(\phi(k) \geq 6\) and diverges for \(\phi(k) = 2\) or \(4\). The Borel-Cantelli lemma then implies that
\[
\#\{\chi \in \mathcal{B}_k \mid \gcd(f_\chi, N) = 1, 0 < |A_\chi| \leq L\}
\]
is finite for \(\phi(k) \geq 6\), and infinite for \(\phi(k) = 2\) or \(4\).

Asymptotically as \(X \to \infty\), we have,
\[
\#\{\chi \in \mathcal{B}_{k,N}(X) \mid 0 < |\text{Nm}_{\mathbb{Q}(\chi)+/\mathbb{Q}(\alpha_\chi)}| \leq L\} \sim b_E X^{1/2} \log B(X) \quad \text{if} \quad \phi(k) = 2
\]
\[
\sim b'_E \log B^1(X) \quad \text{if} \quad \phi(k) = 4
\]
is bounded \(\text{if} \quad \phi(k) \geq 6\).
for some non-zero constants $b_E$, and $b'_E$. This is prediction 1.1 of the introduction.

For $k = 3$, we have $B = \sigma_0(3) + \phi(3)/2 - 3 = 0$, so the predicted growth rate for “small” non-zero algebraic values of cubic twists $L^{alg}(E, \chi, 1)$ is of the order $b_E X^{1/2}$.

For $k = 5$, we have $B = \sigma_0(5) + \phi(5)/2 - 3 = 1$, so the predicted growth rate for “small” non-zero algebraic values of quintic twists $A_\chi$ is of the order $b'_E \log^2(X)$.

These predicted growth rates seem to agree with the computations in the cases below.

For prediction 1.2 of the introduction, we note that

$$\# \{\chi \in B_k(X) \mid 0 < |\text{Nm}_{Q(\chi)}/Q(\alpha_\chi)| \leq L\}$$

is predicted to be finite for $\phi(k) \geq 6$. Note that since $B_k(f)$ consists of all primitive characters mod $f$ of order $k$, then $\cup_k B_k(f)$ is the set of all primitive characters mod $f$ and therefore is a subset of the set of all characters mod $f$. Hence

$$\sum_k b_k(f) = |\cup_k B_k(f)| \leq |(\mathbb{Z}/f\mathbb{Z})^\times| = \phi(f) \leq |f|.$$

If the series

$$\sum_{k \geq k_0} \sum_{\chi \in B_k(X)} \text{Prob}(0 < |A_\chi| \leq L_\chi)$$

converges for some $k_0$, it converges absolutely and so its value would equal the value of the re-arranged series

$$\sum_{f \leq X} \sum_{k \geq k_0} b_k(f) \text{Prob}(0 < |A_\chi| \leq L_\chi) \leq \sum_{f \leq X} \frac{\phi(f)}{\phi(k_0)/4} \leq \sum_{f \leq X} \frac{1}{\phi(k_0)/4 - 1}$$

which converges absolutely as $X \to \infty$ for $\phi(k_0)/4 > 2$, i.e. for $\phi(k_0) \geq 10$. Since we predict that there are only finitely many characters $\chi$ of order $k$ with $\phi(k) = 6$ or 8 for which $0 < |\text{Nm}_{Q(\chi)}/Q(\alpha_\chi)| \leq L$, then the Borel-Cantelli lemma predicts that

$$\# \{\chi \in \cup_{\phi(k)>4} B_k(X) \mid 0 < |\text{Nm}_{Q(\chi)}/Q(\alpha_\chi)| \leq L\}$$

is finite. Finally it is a consequence of of [MR], that

$$\# \{\chi \in \cup_{\phi(k)>4} B_k(X) \mid \text{Nm}_{Q(\chi)}/Q(\alpha_\chi) = 0\}$$

is finite if $\phi(k) \geq 6$ so prediction 1.2 would follow.
4.2. $0 < c < \phi(k)/4$.

As $X \to \infty$, the sum \[ \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} \] converges for $\phi(k) > 4(1 + c)$ and diverges for $\phi(k) \leq 4(1 + c)$. The Borel-Cantelli lemma then implies that

\[
\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \int_{1}^{\infty} \frac{\phi(t)}{t^s} \, dt
\]

is finite for $\phi(k) > 4(1 + c)$, and asymptotically as $X \to \infty$, we have

\[
\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \int_{1}^{\infty} \frac{\phi(t)}{t^s} \, dt, \quad \text{if } c = \phi(k)/4 - 1
\]

for some non-zero constants $d_E$ and $d'_E$. Note that if for some character $\chi$ we have $|\gamma(\alpha_\chi)| < f^{\frac{1}{c} + \delta}$ for some $\gamma \in \text{Gal}(\mathbb{Q}(\chi)^+/\mathbb{Q})$, then $|\text{Nm}_{\mathbb{Q}(\chi)^+/\mathbb{Q}}(\alpha_\chi)| \leq f^{k - \phi(k) - 1}$ which by the above can only happen for at most $O(X^{1 - \delta} \log(B(X)))$ characters, i.e. 0% of all characters. This is the content of prediction 1.3.

5. Brauer-Siegel Limits

Taking $L = L_{f_x} = O(f^c_x)$, and considering $\chi \in \mathcal{B}_{k,N}(X)$, then since $L \leq M^n = X^{\phi(k)/4}$, we have $0 \leq c \leq \phi(k)/4$. Then from 4.3 above we find that the model predicts that

\[
\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \int_{1}^{\infty} \frac{\phi(t)}{t^s} \, dt, \quad \text{if } c = \phi(k)/4 - 1
\]

is finite for $c < \phi(k)/4 - 1$ and is infinite for $\phi(k)/4 - 1 \leq c \leq \phi(k)/4$. The model then implies that

\[
\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \int_{1}^{\infty} \frac{\phi(t)}{t^s} \, dt, \quad \text{if } c = \phi(k)/4 - 1
\]

for any character $\chi \in \mathcal{B}_{k,N}(X)$, let $K = K_\chi/\mathbb{Q}$ be the associated cyclic extension of degree $k$. Then $E/K$ may be viewed as an elliptic curve over $K$ whose $L$-function satisfies

\[
L(E/K, s) = \prod_{i=0}^{k-1} L(E, s, \chi^i).
\]
Recall, we are always taking the primitive character giving \( \chi^i \) so for example \( L(E, s, \chi^0) = L(E/\mathbb{Q}, s) \). This allows us to express the leading term in the Taylor expansion at \( s = 1 \) of \( L(E/K, s) \) in terms of the corresponding leading terms of the twists \( L(E, s, \chi^i) \).

By restriction of scalars we may view \( E/K \) is an abelian variety over \( \mathbb{Q} \) whose "\( \chi \)-component" is also an abelian variety \( \mathcal{A}(\chi) \) over \( \mathbb{Q} \) with \( L \)-function equal to

\[
L(\mathcal{A}(\chi), s) = \prod_{1 \leq i < k \atop (i, k) = 1} L(E, s, \chi^i).
\]

The model suggests that for \( \phi(k)/4 - 1 \leq c \leq \phi(k)/4 \), there is an infinite family

\[
\mathcal{F}_{c, N} = \mathcal{F}_c = \{ \chi \in \mathcal{B}_{k, N}(X) \mid 0 < |\text{Nm}_{\mathbb{Q}(\chi)/\mathbb{Q}}(\alpha_{\chi})| \leq f^c \}
\]

of characters \( \chi \) (and therefore fields \( K_{\chi} \)) such that \( 0 < A_{\chi} \leq f^c \) and hence that \( L(\mathcal{A}(\chi), 1) \neq 0 \).

Then the Birch & Swinnerton-Dyer Conjecture implies that the group of rational points \( \mathcal{A}(\chi)(K_{\chi}) \) is finite, and so the corresponding regulator \( R_{\chi} = 1 \). In this case we have

\[
L^\text{alg}(\mathcal{A}(\chi)) = \prod_{1 \leq i < k \atop (i, k) = 1} L^\text{alg}_{E}(\chi^i) = d A_{\chi}^2.
\]

for some \( d \) bounded only in terms of \( k \) and that \( A_{\chi}^2 \) is essentially (up to constants) the order of the Shafarevich-Tate group \( \Sha(\mathcal{A}(\chi)) \). Hence for \( \chi \in \mathcal{F}_c \) we have

\[
\log(\Sha(\mathcal{A}(\chi)) \cdot R_{\chi}) \sim \log(\prod_{1 \leq i < k \atop (i, k) = 1} L^\text{alg}_{E}(\chi^i))
\]

\[
\sim \log(A_{\chi}^2)
\]

\[
\leq \log(f_{\chi}^{2c})
\]

\[
= 2c \log(f_{\chi}).
\]

(5.1)

Fix a character \( \chi \in \mathcal{B}_{k, N}(X) \) and let \( K = K_{\chi}/\mathbb{Q} \) be the corresponding cyclic extension of degree \( k \). Viewing \( E \) as an elliptic curve over \( K \) with \( L \)-function \( L(E/K, s) \) we have

\[
L(E/K, s) = \prod_{0 \leq i < k} L(E, s, \chi^i).
\]
Suppose for simplicity that \( k \) is an odd prime, and that \( L(E, 1, \chi) \neq 0 \). Then \( \chi(-1) = 1 \) and comparing leading terms at \( s = 1 \) the Birch & Swinnerton-Dyer conjecture predicts that

\[
\frac{\Omega_{E_K} R(E(K))|\Sha(E(K))| \prod p c_p}{|E(K)_{\text{tors}}|^2 \sqrt{|D(K/Q)|}} = \frac{\Omega^+_E R(E(Q))|\Sha(E(Q))| \prod p c_p}{|E(Q)_{\text{tors}}|^2} \prod_{1 \leq i < k} \frac{\Omega^+_E \mathcal{A}_E(\chi^i)}{2\tau(\chi^i)}
\]

where \( \Omega_{E_K} = (\Omega^+)^k \) and the \( c_p, c_p \) are the Tamagawa numbers for \( E/K, E/Q \). Since for \( \chi(-1) = 1 \) we have

\[
\sqrt{|D(K/Q)|} = \prod_{0 \leq i < k} \tau(\chi^i) = \psi^{(k-1)/2}.
\]

For more general \( k \), for the character \( \chi \) and \( \mathcal{A}(\chi) \) as above, we take

\[
\sqrt{|D(\chi)|} = \prod_{1 \leq i < k \atop (i,k)=1} \tau(\chi^i) = (f_\chi)^{\phi(k)/2}.
\]

up to possible constants, so that \( \log(\sqrt{|D(\chi)|}) = \frac{\phi(k)}{2} \log(f_\chi) \) up to an additive constant. Since the families \( \mathcal{F}_c \) are predicted to be infinite we can take the upper limit (as \( X \to \infty \)) of the Brauer-Siegel quotients.

\[
\lim_{\chi \in \mathcal{F}_c} \frac{\log(\Sha(\mathcal{A}(\chi)) \cdot R_\chi)}{\log(\sqrt{|D(\chi)|})} \leq \frac{4c}{\phi(k)}
\]

with \( \phi(k)/4 - 1 \leq c \leq \phi(k)/4 \). If we choose \( \phi(k)/4 - 1 \leq c' < c \leq \phi(k)/4 \), then the model would predict that \( \mathcal{F}_c - \mathcal{F}_{c'} \) would be an infinite set and for \( \chi \in \mathcal{F}_c - \mathcal{F}_{c'} \),

\[
\frac{4c'}{\phi(k)} \leq \lim_{\chi \in \mathcal{F}_{c'}} \frac{\log(\Sha(\mathcal{A}(\chi)) \cdot R_\chi)}{\log(\sqrt{|D(\chi)|})} \leq \lim_{\chi \in \mathcal{F}_c} \frac{\log(\Sha(\mathcal{A}(\chi)) \cdot R_\chi)}{\log(\sqrt{|D(\chi)|})} \leq \frac{4c}{\phi(k)}
\]

This contains the statements in predicion 1.4.

For \( c = 0 \) and \( \phi(k) = 2 \) or 4, the families of \( \S 3 \) are infinite and the same limits hold.

Note that for characters \( \chi \) of prime order \( k \), the fields \( K_\chi/Q \) are cyclic extensions of prime degree \( k \), and \( \Sha(\mathcal{A}(\chi)) \) is essentially just the relative Shafarevich-Tate group \( \Sha(E(K_\chi))/\Sha(E(Q)) \).
6. Calculating \( \#(B_{k,N}(X)) \)

**Remark 2.** This calculation has been done by many authors numerous times in differing cases, and an account can be found in Narkiewicz ([Na], 8.4.2, Notes to Chapter 8). A more general version of this result appears in [MR] and they essentially attribute this result to Kubota.

Fix \( k \geq 2 \) and let \( f = \prod p^{b_p} \) be the prime factorization of \( f \in \mathbb{Z} \) with \( f \) odd or \( 4 \mid f \). The Chinese Remainder Theorem states that

\[
\left( \mathbb{Z}/f\mathbb{Z} \right)^\times \simeq \prod_{p \mid f} \left( \mathbb{Z}/p^{b_p}\mathbb{Z} \right)^\times
\]

so that

\[
\left( \mathbb{Z}/f\mathbb{Z} \right)^\times \simeq \prod_{p \mid f} \left( \mathbb{Z}/p^{b_p}\mathbb{Z} \right)^\times,
\]

and hence every \( \chi \in \left( \mathbb{Z}/f\mathbb{Z} \right)^\times \) factors *uniquely* as \( \chi = \prod_{p \mid f} \chi_p \) with \( \chi_p \in \left( \mathbb{Z}/p^{b_p}\mathbb{Z} \right)^\times \). Also the conductor \( f \chi \) of \( \chi \) is equal to \( f \) if and only if the conductor \( f \chi_p \) of \( \chi_p \) is \( p^{b_p} \) for each prime \( p \) dividing \( f \). Finally,

\[
\chi \in A_k(f) \iff \chi^k = 1
\]

\[
\iff \chi_p^k = 1 \quad \text{for every } p \mid f
\]

\[
\iff \chi_p \in A_k(p^{b_p}) \quad \text{for every } p \mid f
\]

and therefore \( a_k(f) = \prod_{p \mid f} a_k(p^{b_p}) \).

Note that

\[
\left( \mathbb{Z}/p^{b_p}\mathbb{Z} \right)^\times \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^{b-1}\mathbb{Z} \quad \text{for } p > 2
\]

\[
\simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{b-2}\mathbb{Z} \quad \text{for } p = 2.
\]

Let \( k = p^a k' \) with \( (p, k') = 1 \), and let \( d_p = \gcd(p-1, k') = \gcd(p-1, k) \).

(Case a).

Suppose that \( \gcd(p, k) = 1 \). Then for \( 1 \neq \chi \in A_k(p^b) \) and \( p > 2 \), we must have \( b = 1 \), and the order \( \text{ord}(\chi) \) divides \( d_p \). Hence in this case \( a_k(p) = d_p - 1 \) and \( a_k(p^b) = 0 \) for \( b \geq 2 \). For \( p = 2 \), \( \gcd(2, k) = 1 \) implies that \( a_k(2^b) = 0 \) for all \( b \geq 1 \).

(Case b).

Suppose that \( p \mid k \) so \( k = p^a p' \) with \( a_p = \nu_p(k) \geq 1 \).
Let $p > 2$. For $\chi \in A_k(p^h)$ we have $\chi = \chi_1 \times \chi_2$ where the order $\text{ord}(\chi_2)$ divides $d_p$ and is prime to $p$ and $\text{ord}(\chi_1) = p^{b-1}$ and $f_{\chi_1} = p^b$ if $b > 1$. Therefore $a_k(p^h) = d_p - 1$ if $b = 1$ and $a_k(p^h) = (p^{b-1} - p^{b-2})d_p - 1$ if $2 \leq b \leq a + 1$ and $a_k(p^b) = 0$ for $b \geq a + 2$.

Now let $p = 2$, then any primitive Dirichlet character $\chi \neq 1$ with $f_{\chi} = 2^b$ must have $b \geq 2$. Then $a_k(p^3) = a_k(4) = 1$, $a_k(p^h) = 2 \cdot 2^{b-2} - 2 \cdot 2^{b-3} = 2^{b-2}$ for $3 \leq b \leq a + 2$ and $a_k(p^b) = 0$ for $b \geq a + 3$.

Let $N \geq 1$ be a positive integer. We consider the series

$$F_k(s, A_k) = \sum_{f} \frac{a_k(f)}{f^s} \quad \text{and} \quad F_{k,N}(s, A_k) = \sum_{\gcd(f,N) = 1} \frac{a_k(f)}{f^s}.$$ 

By the multiplicativity of the coefficients $a_k(f)$ we have the Euler product factorization

$$F_k(s, A_k) = \prod_{\gcd(p,k) = 1} \left( 1 + \frac{d_p - 1}{p^s} \right) \times \left( 1 + \frac{1}{2^s} + \frac{2}{2^s} + \cdots + \frac{2^{a_2}}{2(2a_2 + 2s)} \right)$$

$$\times \prod_{2 \leq p | k} \left( 1 + \frac{d_p - 1}{p^s} + \frac{(p-1)d_p - 1}{p^{2s}} + \cdots + \frac{p^{a_p-1}(p-1)d_p - 1}{p^{(a_p+1)s}} \right)$$

where the 2-Euler factor is 1 if $k$ is odd.

For functions $f(s), g(s)$, write $f(s) \approx g(s)$ if $f(s) = g(s)h(s)$ for $h(s)$ analytic near $s = 1$, and $h(1) \neq 0$. Then since $F_{k,N}(s, A_k)$ differs from $F_k(s, A_k)$ by a finite number of Euler factors, we see that

$$F_{k,N}(s, A_k) \approx F_k(s, A_k)$$

and we have

$$F_k(s, A_k) \approx \prod_{\gcd(p,k) = 1} \left( 1 + \frac{d_p - 1}{p^s} \right)$$

$$\approx \prod_{p | k} \left( 1 - \frac{1}{p^s} \right)^{-(d_p-1)}.$$ 

Then since $d_p = \sum_{d|d_p} \phi(d)$ we can write

$$\left( 1 - \frac{1}{p^s} \right)^{-(d_p-1)} = \prod_{1 \neq d|d_p} \left( 1 - \frac{1}{p^s} \right)^{-\phi(d)}$$
and therefore
\[
F_k(s, A_k) \approx \prod_{p \mid k} \left( 1 - \frac{1}{p^s} \right)^{(d_p-1)}
\approx \prod_{p \mid k, 1 \neq d \mid d_p} \prod_{d \neq 1} \left( 1 - \frac{1}{p^s} \right)^{-\phi(d)}.
\]
The factor \((1 - 1/p^s)^{-\phi(d)}\) appears if and only if \(p \nmid k, 1 \neq d \mid k, \) and \(p \equiv 1 \pmod{d}\) so that
\[
F_k(s, A_k) \approx \prod_{d \mid k, d \neq 1} \prod_{d \equiv 1 \pmod{d}} \left( 1 - \frac{1}{p^s} \right)^{-\phi(d)}
\]
where \(\zeta_d(s)\) is the Dedekind \(\zeta\)-function of the \(d\)th cyclotomic field \(\mathbb{Q}(e^{2\pi i/d})\).

Since \(F_{k,N}(s, A_k) \approx F_k(s, A_k)\) it follows from the Delange–IkeharaTauberian (see [Na], Appendix II, Theorem 1) theorem that
\[
\#(A_{k,N}(X)) = \sum_{\mathcal{I} \leq X, \gcd(\mathcal{I}, N) = 1} a_k(\mathcal{I}) \sim c_k X \log^{\sigma_0(k) - 2}(X)
\]
where \(c_k\) is a non-zero constant and \(\sigma_0(k)\) is the number of divisors of \(k\). Also by inclusion-exclusion we see that
\[
\#(B_{k,N}(X)) = \#(A_{k,N}(X)) + \sum_{1 \neq d \mid k} \mu(d) \#(A_{k/d, N}(X)).
\]
Since for \(1 \neq d \mid k\) the asymptotic power of \(\log X\) in \(\#(A_{k/d, N}(X))\) is strictly smaller than that in \(\#(A_{k,N}(X))\), we see that
\[
\#(B_{k,N}(X)) = \sum_{\mathcal{I} \leq X, \gcd(\mathcal{I}, N) = 1} b_k(\mathcal{I}) \sim c_k X \log^{\sigma_0(k) - 2}(X).
\]

**APPENDIX: COMPUTATIONAL RESULTS**

Jungbae Nam
In this section, we present computational results to support the statistical predictions above for the statistics of some small integer values $A_\chi$ associated with the central critical $L$-values for $L(E, s, \chi)$ for $E : 11a1, 14a1$ and $B_{k,N}(X)$ for $k = 3$ with $X = 3 \times 10^6$ and $k = 5$ with $X = 10^6$. Here we let $N = N_E$ for a given $E$.

Then, we compute the $L$-values at $s = 1$ by the following well-known formula:

$$L(E, 1, \chi) = \sum_{n \geq 1} \frac{(a_n \chi(n) + w_E c_\chi a_n \bar{\chi}(n))}{n} \exp \left( -\frac{2\pi n}{f_\chi \sqrt{N_E}} \right),$$

where $a_n$ is the coefficients of $L(E, s)$ and $c_\chi = \chi(N_E)\tau(\chi)/f_\chi$. Knowing that $\chi(-1) = 1$ for $k$ odd, we use non-zero complex values $\Omega_\chi = \Omega^+ \chi$ computed by SageMath [S+09] for 11a1 and 14a1. Using those values we compute $L_E^{\text{alg}}(\chi)$ and then $A_\chi \in \mathbb{Z}$ using Proposition 2.1.

In our numerical computations, the numerical values of $L(E, 1, \chi)$ for 11a1 and 14a1 with $f_\chi \leq 2 \times 10^6$ were already computed by Jack Fearnley for the articles [DFK1] and [DFK2], and we used his $L$-values. Those central $L$-values for $k = 3$ with $2 \times 10^6 \leq f_\chi \leq 3 \times 10^6$ and $k = 5$ with $f_\chi \leq 10^6$ were obtained by using around 40 threads in the cluster of the Centre de Recherches Mathématiques (CRM) for a couple of months. The codes to evaluate $L(E, 1, \chi)$ were created by using Cython and SageMath [S+09]. Moreover, we maintained at least 4 decimal place accuracy in computing the values of $A_\chi$. Table 1 shows $\#(B_{k,N}(X))$ for $E, k$ and $X$ used in our experimental computations.

|  | 11a1  | 14a1  | $X$         |
|---|-------|-------|-------------|
| 3 | 951116| 739810| $3 \times 10^6$|
| 5 | 192516| 262536| $10^6$      |

Table 1. $\#(B_{k,N}(X))$ for each $E$ and $k = 3$ with $X = 3 \times 10^6$ and $k = 5$ with $X = 10^6$.

For the support of our predictions on the frequencies of integer values, we use the values of $A_\chi$ divided the greatest common divisor (gcd), depending on $k$ and $E$, of all non-zero $A_\chi$ for $\chi \in B_{k,N}(X)$. For $E : 11a1$ and 14a1, those gcds are not trivial. By abuse of notation we still denote those values of $A_\chi$ by $A_\chi$. 

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(Case \( c = 0 \))

Given \( k, E \) with \( N = N_E \) and an integer \( l \), denote

\[
    n_{k,E}(x; l) = \#\{\chi \in B_{k,N}(x) \mid A_{\chi} = l\}.
\]

Then, in particular, \( n_{k,E}(x; 0) = \#\{\chi \in B_{k,N}(x) \mid L(E, 1, \chi) = 0\} \). Let’s consider the frequencies of vanishings we predicted. Define the function \( f_{k,E}(x) \) on the real numbers \( x > 0 \) by

\[
    f_{k,E}(x) = \begin{cases} 
        x^{1/2} \log^{1/4}(x) & \text{if } k = 3, E : 11a1 \\
        x^{1/2} \log^{9/4}(x) & \text{if } k = 3, E : 14a1 \\
        \log^{17/4}(x) & \text{if } k = 5, E : 11a1 \\
        \log^{3/2}(x) & \text{if } k = 5, E : 14a1
    \end{cases}
\]

Note that the larger powers of \( \log(x) \) come from the existences of \( k \)-torsion points of \( E \); 11a1 has 5-torsion and 14a1 has 3-torsion points and we numerically find more vanishings for these elliptic curves than those without \( k \)-torsion points. Refer to [DFK1] and [DFK2] for these choices of powers of \( \log(x) \). Then, our predictions assert

\[
    n_{k,E}(x; 0) \sim c_{k,E} f_{k,E}(x) \text{ as } x \to \infty
\]

with a constant \( c_{k,E} \) depending on \( k \) and \( E \). In order to see this, we compute the ratios \( n_{k,E}(x; 0)/f_{k,E}(x) \). Figure 1 and 5 depict those ratios for 11a1 and 14a1, \( k = 3 \) and \( x \leq 3 \times 10^6 \). Furthermore, Figure 9 and 13 do the same for \( k = 5 \) and \( x \leq 10^6 \).

Now, let’s consider the frequencies of some small non-zero integer values we predicted for twists for \( k = 3, 5 \). Define a function \( g_k(x) \) on the real numbers \( x > 0 \) by

\[
    g_k(x) = \begin{cases} 
        x^{1/2} & \text{if } k = 3 \\
        \log^2(x) & \text{if } k = 5
    \end{cases}
\]

We choose \(|l| = 1, 2, \ldots, 9\) for \( k = 3 \) and \( 1, 4, 5, 9, 11, 16, 19, 20, 25 \) for \( k = 5 \). Note that for \( k = 5 \), those values of \(|l|\) are the first 9 possible absolute values of non-zero \( A_{\chi} \). Then, we compute the ratios \( n_{k,E}(x; l)/g_k(x) \). For \( k = 3 \) with \( x \leq 3 \times 10^6 \), Figure 2, 3 and 4 depict those ratios for 11a1. Figure 6, 7 and 8 do the same for 14a1. For \( k = 5 \) with \( x \leq 10^6 \), Figure 10, 11 and 12 depict those ratios for 11a1. Figure 14, 15 and 16 do the same for 14a1.
Similarly, for an integer \( L > 0 \), denote 

\[ s_{k,E}(x; L) = \#\{\chi \in B_{k,N}(x) \mid 0 < |A_{x}| \leq L\} \]

Then, the above figures for \( n_{k,E}(x; L) \) seem to support our predictions asserting

\[ s_{k,E}(x; L) \sim d_{k,E}g_{k}(x) \text{ as } x \to \infty \]

with a constant \( d_{k,E} \) depending on \( k \) and \( E \). Note that these \( s_{k,E}(x; L) \) are the sum of \( n_{k,E}(x; l) \) over \( 0 < |l| \leq L \). We compute the ratios \( s_{k,E}(x; L)/g_{k}(x) \) for \( L = 1, 2, \ldots, 9 \) for \( k = 3 \) and \( 1, 4, \ldots, 25 \) for \( k = 5 \). Figure 17 and 18 depict those ratios for 11a1 and 14a1, \( k = 3 \) and \( x \leq 3 \times 10^6 \). Furthermore, Figure 19 and 20 do the same but for \( k = 5 \) and \( x \leq 10^6 \). Note that as expected, the lines in Figure 19 and 20 lie higher as \( L \) increases.

(Case \( 0 < c < \phi(k)/4 \))

Given \( k, E \) with \( N \) and a real number \( c \) with \( 0 \leq c \leq \phi(k)/4 \), denote

\[ m_{k,E}(x; c) := \#\{\chi \in B_{k,N}(x) \mid |A_{x}| \leq t_{x}^{c}\} \]

For each \( E : 11a1, 14a1 \), we compute \( m_{k,E}(x; c) \) for \( k = 3 \) with \( x \leq 3 \times 10^6 \) and \( k = 5 \) with \( x \leq 10^6 \) and \( c = 0.1, 0.2, \ldots, \phi(k)/4 \) and depict the following ratios

\[ m_{k,E}(x; c)/x^{1/2+c} \text{ for } k = 3, \]

\[ m_{k,E}(x; c)/(x^{c} \log(x)) \text{ for } k = 5 \]

in Figure 21 and 22 respectively.
Figure 1. Ratio $n_{k,E}(x;l)/f_{k,E}(x)$ for $k = 3$ and $x \leq 3 \times 10^6$ where $l = 0$ and $f_{k,E}(x) = x^{1/2} \log^{1/4}(x)$.

Figure 2. Ratio $n_{k,E}(x;l)/g_{k}(x)$ for $k = 3$ and $x \leq 3 \times 10^6$ where $l = \pm 1$ and $g_{k}(x) = x^{1/2}$. The upper line is for $l = 1$ and the lower one is for $l = -1$. 
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Figure 3. 11a1: Ratio $n_{k,E}(x;l)/g_k(x)$ for $k = 3$ and $x \leq 3 \times 10^6$
where $l = \pm 2, \pm 3, \pm 4, \pm 5$ and $g_k(x) = x^{1/2}$.
Small Values

(a) \( l = \pm 6 \): upper for \( l = -6 \) and lower for \( l = 6 \).

(b) \( l = \pm 7 \): upper for \( l = 7 \) and lower for \( l = -7 \).

(c) \( l = \pm 8 \): upper for \( l = -8 \) and lower for \( l = 8 \).

(d) \( l = \pm 9 \): upper for \( l = 9 \) and lower for \( l = -9 \).

Figure 4. 11a1: Ratio \( n_{k,E}(x; l)/g_k(x) \) for \( k = 3 \) and \( x \leq 3 \times 10^6 \) where \( l = \pm 6, \pm 7, \pm 8, \pm 9 \) and \( g_k(x) = x^{1/2} \).
Figure 5. 14a1: Ratio $n_{k,E}(x;l)/f_{k,E}(x)$ for $k = 3$ and $x \leq 3 \times 10^6$ where $l = 0$ and $f_{k,E}(x) = x^{1/2} \log^{9/4}(x)$.

Figure 6. 14a1: Ratio $n_{k,E}(x;l)/g_k(x)$ for $k = 3$ and $x \leq 3 \times 10^6$ where $l = \pm 1$ and $g_k(x) = x^{1/2}$. The upper line is for $l = 1$ and the lower one is for $l = -1$.
SMALL VALUES

(a) $l = \pm 2$: upper for $l = 2$ and lower for $l = -2$.

(b) $l = \pm 3$: upper for $l = 3$ and lower for $l = -3$.

(c) $l = \pm 4$: upper for $l = 4$ and lower for $l = -4$.

(d) $l = \pm 5$: upper for $l = -5$ and lower for $l = 5$.

**Figure 7.** 14a1: Ratio $n_{k,E}(x; l)/g_k(x)$ for $k = 3$ and $x \leq 3 \times 10^6$ where $l = \pm 2, \pm 3, \pm 4, \pm 5$ and $g_k(x) = x^{1/2}$. 
(a) $l = \pm 6$: upper for $l = 6$ and lower for $l = -6$.

(b) $l = \pm 7$: upper for $l = 7$ and lower for $l = -7$.

c) $l = \pm 8$: upper for $l = 8$ and lower for $l = -8$.

d) $l = \pm 9$: upper for $l = 9$ and lower for $l = -9$.

Figure 8. 14a1: Ratio $n_{k,E}(x; l)/g_k(x)$ for $k = 3$ and $x \leq 3 \times 10^6$
where $l = \pm 6, \pm 7, \pm 8, \pm 9$ and $g_k(x) = x^{1/2}$. 
Figure 9. 11a1: Ratio $n_{k,E}(x;1)/f_{k,E}(x)$ for $k = 5$ and $x \leq 10^6$ where $l = 0$ and $f_{k,E}(x) = \log^{17/4}(x)$.

Figure 10. 11a1: Ratio $n_{k,E}(x;l)/g_k(x)$ for $k = 5$ and $x \leq 10^6$ where $l = \pm 1$ and $g_k(x) = \log^2(x)$. The upper line is for $l = 1$ and the lower one is for $l = -1$. 
(a) $l = \pm 4$: upper for $l = -4$ and lower for $l = 4$. (b) $l = \pm 5$: upper for $l = 5$ and lower for $l = -5$.

(c) $l = \pm 9$: upper for $l = 9$ and lower for $l = -9$. (d) $l = \pm 11$: upper for $l = 11$ and lower for $l = -11$.

**Figure 11.** 11a1: Ratio $n_{k,E}(x; l)/g_k(x)$ for $k = 5$ and $x \leq 10^6$ where $l = \pm 4, \pm 5, \pm 9, \pm 11$ and $g_k(x) = \log^2(x)$.
\( l = \pm 16 \): upper for \( l = -16 \) and lower for \( l = 16 \). (b) \( l = \pm 19 \): upper for \( l = -19 \) and lower for \( l = 19 \).

(c) \( l = \pm 20 \): upper for \( l = -20 \) and lower for \( l = 20 \). (d) \( l = \pm 25 \): upper for \( l = 25 \) and lower for \( l = -25 \).

**Figure 12.** 11a1: Ratio \( n_{k,E}(x; l)/g_k(x) \) for \( k = 5 \) and \( x \leq 10^6 \) where \( l = \pm 16, \pm 19, \pm 20, \pm 25 \) and \( g_k(x) = \log^{2}(x) \).
Figure 13. 14a1: Ratio $n_{k,E}(x;l)/f_{k,E}(x)$ for $k = 5$ and $x \leq 10^6$ where $l = 0$ and $f_{k,E}(x) = \log^{3/2}(x)$.

Figure 14. 14a1: Ratio $n_{k,E}(x;l)/g_{k}(x)$ for $k = 5$ and $x \leq 10^6$ where $l = \pm 1$ and $g_{k}(x) = \log^2(x)$. The upper plot is for $l = -1$ and the lower one is for $l = 1$. 
(a) $l = \pm 4$: upper for $l = 4$ and lower for $l = -4$.

(b) $l = \pm 5$: upper for $l = -5$ and lower for $l = 5$.

(c) $l = \pm 9$: upper for $l = -9$ and lower for $l = 9$.

(d) $l = \pm 11$: upper for $l = -11$ and lower for $l = 11$.

**Figure 15.** 14a1: Ratio $n_{k,E}(x; l)/g_k(x)$ for $k = 5$ and $x \leq 10^6$

where $l = \pm 4, \pm 5, \pm 9, \pm 11$ and $g_k(x) = \log^2(x)$. 
(a) $l = \pm 16$: upper for $l = -16$ and lower for $l = 16$. (b) $l = \pm 19$: upper for $l = 19$ and lower for $l = -19$. 

(c) $l = \pm 20$: upper for $l = 20$ and lower for $l = -20$. (d) $l = \pm 25$: upper for $l = 25$ and lower for $l = -25$.

Figure 16. 14a1: Ratio $n_{k,E}(x;l)/g_k(x)$ for $k = 5$ and $x \leq 10^6$ where $l = \pm 16, \pm 19, \pm 20, \pm 25$ and $g_k(x) = \log^2(x)$. 
**Figure 17.** 11a1: Ratio $s_{k,E}(x;L)/g_k(x)$ for $k = 3$ and $x \leq 3 \times 10^6$ where $L = 1, 2, \ldots, 9$ and $g_k(x) = x^{1/2}$. The lines for $L$ lie higher as $L$ increases.

**Figure 18.** 14a1: Ratio $s_{k,E}(x;L)/g_k(x)$ for $k = 3$ and $x \leq 3 \times 10^6$ where $L = 1, 2, \ldots, 9$ and $g_k(x) = x^{1/2}$. The lines for $L$ lie higher as $L$ increases.
Figure 19. 11a1: Ratio \( s_{k,E}(x; L)/g_k(x) \) for \( k = 5 \) and \( x \leq 10^6 \) where \( L = 1, 4, 5, 9, 11, 16, 19, 20, 25 \) and \( g_k(x) = \log^2(x) \). The lines for \( L \) lie higher as \( L \) increases.

Figure 20. 14a1: Ratio \( s_{k,E}(x; L)/g_k(x) \) for \( k = 5 \) and \( x \leq 10^6 \) where \( L = 1, 4, 5, 9, 11, 16, 19, 20, 25 \) and \( g_k(x) = \log^2(x) \). The lines for \( L \) lie higher as \( L \) increases.
(a) 11a1: $c = 0.1, 0.2 \ldots , 0.5$ from the top  
(b) 14a1: $c = 0.1, 0.2 \ldots , 0.5$ from the top

**Figure 21.** $m_{k,E}(x; c)/x^{1/2+c}$ for $k = 3$

(a) 11a1: $c = 0.2, 0.1, 0.3, \ldots , 1$ from the top  
(b) 14a1: $c = 0.1, 0.2, 0.3, \ldots , 1$ from the top

**Figure 22.** $m_{k,E}(x; c)/(x^c \log(x))$ for $k = 5$
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