Quantum corrections to gravity

Yukio Tomozawa

(Dated: January 30, 2013)

Abstract

This paper revisits quantum corrections to gravity. It was shown previously by other authors that quantum field theories in curved space time provide quadratic curvature forms as quantum corrections to gravity in a conformally flat metric. Application to a spherically symmetric and static (SSS) metric shows that only the Gauss Bonnet combination (GB) yields the correct expression. Using a variational method, the author shows that the metric he obtained in 1985 as an example in a simplified case was indeed the exact solution for a SSS metric. This proves that gravity becomes repulsive at short distances by quantum corrections.
I. INTRODUCTION

Quantum field theories on a curved space time were extensively studied in 1970-80 in the case of a conformally flat (CF) metric, and it was concluded that quantum corrections to gravity are expressed as quadratic curvature forms in that metric[1]. The application of that formalism to the Friedman Walker (FW) metric gave the interesting observation that a collapsed universe will bounce back to an expansion in some cases[2]. This provides a hint that quantum corrections to gravity might yield a repulsive component at short distances. The author studied quantum corrections for a SSS metric and obtained a repulsive force at short distance in the special case of a Gauss Bonnet (GB) combination, and also by a numerical computation. Based on this analysis, the author proposed a model for high energy cosmic rays produced from AGN in 1985[3], [4], [5]. This turns out to predict recent data from the Pierre Auger Project that suggests a possible correlation between high energy cosmic rays and locations of AGN[6]. Further discussion of the model yielded a new mass scale at the knee energy of the cosmic ray spectrum and a prediction for the mass of a dark matter particle, which is supported by recent data from HESS[7], [8], [9]. Because of successful predictions that agree with observational reality, it is important to solidify a theoretical basis for the model. From this point of view, the author has revisited the subject of quantum corrections on gravity. Using an explicit form for a SSS metric, it is shown that among proposed expressions in a conformally flat metric, only the Gauss Bonnet combination satisfies the finiteness of renormalization. This implies that the metric obtained by the author in 1985 is in fact the rigorous solution for quantum corrections to gravity and the repulsive nature of gravity at short distances has been established. A derivation of the final result by a variational method is presented.

II. QUANTUM CORRECTIONS IN A SSS METRIC

Extensive studies of quantum corrections to gravity in a CF metric can be summarized[1], [5] as

\[ H^{\mu\nu} = \frac{1}{(n-4)} \frac{1}{\sqrt{|g|}} \delta g_{\mu\nu} \int F \sqrt{|g|} d^n x, \]  

(1)
where

\[ F = AR^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + BR^{\alpha\beta} R_{\alpha\beta} + CR^2 \]  

\[ = f_0 + (n - 4)f_1 + O((n - 4)^2), \]  

and A, B and C are constants. It is well known that the GB combination

\[ A = 1, B = -4, C = 1 \]  

is finite for any metric. For later reference, the GB combination is shown explicitly as

\[ GB = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta} R_{\alpha\beta} + R^2. \]  

For a CF metric, the combinations

\[ (A, B, C) = (0, 1, 0) \]  

and

\[ (A, B, C) = (0, 0, 1/3) \]

give finite values and contribute to a quantum correction term, although the two yield an identical correction in a CF metric. (It will be shown later that the latter is an erroneous statement.) Using an explicit expression for curvature in n dimensions, I will examine this property for a SSS metric in n dimensions

\[ ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\Omega^2, \]  

where \( \nu \) and \( \lambda \) are functions of the radial coordinate, \( r \), and the angular part can be expressed symbolically as

\[ d\Omega^2 = d\omega_1^2 + d\omega_2^2 + \cdots + d\omega_{n-2}^2. \]

**Theorem 1** For any metric, the F-term for quantum corrections is a quadratic curvature form.

**Proof.** For any metric other than a CF metric, the F-term can be written as

\[ F = F_1 + \delta F, \]
where \( F_1 \) is the \( F \)-term for quantum corrections to a CF metric, which are quadratic curvature forms, as described above, and

\[
\delta F = 0 \quad (11)
\]

for a CF metric. A CF metric is characterized by the vanishing of

\[
H = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2. \quad (12)
\]

Then, \( \delta F \) is nothing but

\[
\delta F = \lambda H, \quad (13)
\]

where \( \lambda \) is a non-dimensional constant, since this is the only dimensionally accessible quantity. In other words, Eq. (10) is a sum of quadratic curvature forms for any metric. 

In the rest of this section, it will be shown that the GB combination is the only possible solution if Eq. (1) is to be finite in the limit as \( n \) approaches 4 for a SSS metric. For that purpose, I will start with Eq. (1) and Eq. (3) for a SSS metric.

**Theorem 2** The \( F \)-term for quantum correction of a SSS metric is a GB quadratic combination.

**Proof.** From the computation of curvatures in the appendix, one can express the integrand in Eq. (3), where \( f_0 \) and \( f_1 \) are coefficients in the expansion in \( (n-4) \) and

\[
f_0 = (A + B/2 + C)g_1 + (B + 4C)g_2
- 4Cg_3, \quad (14)
\]

where

\[
g_1 = (\nu'' + \nu'^2/2 - \nu'\lambda'/2)^2 e^{-2\lambda} + 2(\nu'^2 + \lambda'^2)(e^{-\lambda}/r)^2
+ 4((e^{-\lambda} - 1)e^{-\lambda}/r^2)^2, \quad (15)
\]

\[
g_2 = ((\nu'' + \nu'^2/2 - \nu'\lambda'/2)(\nu' - \lambda' - \nu'\lambda'/r)e^{-2\lambda}/r
+ \frac{1}{2}(\nu'^2 + \lambda'^2)(e^{-\lambda}/r)^2 - 2(\nu' - \lambda')((e^{-\lambda} - 1)/r^2)e^{-2\lambda}/r., \quad (16)
\]

and

\[
g_3 = ((\nu'' + \nu'^2/2 - \nu'\lambda'/2)(e^{-\lambda} - 1) + \nu'\lambda')e^{-2\lambda}/r^2. \quad (17)
\]
First manipulate the last term. Multiplying by the \( r \) dependent term,
\[
\sqrt{|g|} = e^{(\nu+\lambda)/2} r^{n-2}
\]  
(18)
and noticing that a total derivative does not contribute to a variation, one can reformulate
\[
G = (\nu'' + \nu'^2/2)(e^\lambda - 1)e^{-2\lambda} e^{(\nu+\lambda)/2} r^{n-4}
\]
\[
= 2(e^{\nu/2})''(e^{-\lambda/2} - e^{-3\lambda/2}) r^{n-4}
\]
\[
= -(2(e^{\nu/2})'((e^{-\lambda/2} - e^{-3\lambda/2}) r^{n-4})'
\]
\[
= \nu'\lambda'(e^\lambda - 3)/2e^{-2\lambda} e^{(\nu+\lambda)/2} r^{n-4}
\]
\[
= (n - 4)\nu'(e^\lambda - 1)e^{-2\lambda} e^{(\nu+\lambda)/2} r^{n-5}
\]  
(19)
then
\[
-4Cg_1 e^{(\nu+\lambda)/2} r^{n-2}
\]
\[
= 4C(n - 4)\nu'(e^\lambda - 1)e^{-2\lambda} e^{(\nu+\lambda)/2} r^{n-5},
\]  
(20)
where all total derivative terms are dropped.

Let us show that the last term of \( g_1 \), which does not contain a derivative, cannot be eliminated by manipulation of a partial derivative of \( g_1 e^{(\nu+\lambda)/2} r^{n-2}. \) This would be possible if differentiation of the \( r^m \) term were to be performed and the derivative order were to be reduced. However, by doing that a factor of \( n - 4 \) would be multiplied in the transition from the second derivative to the first derivative (or from a product of two first order derivatives to a first order derivative). Therefore, the last term of \( g_1 \) cannot be eliminated by partial differentiation, since all the terms without derivatives which arise in such a manner contain \( n-4 \) as a factor, and hence vanish for \( n \) approaching 4. By the same argument, \( g_1 \) does not contain terms with a first order derivative. Likewise, the last term of \( g_2 \) cannot be eliminated by the technique of partial differentiation.

The last term of \( g_2 e^{(\nu+\lambda)/2} r^{n-2} \) can be split into two terms. While one of them is reduced to
\[
(\nu' - \lambda') e^{(\nu-\lambda)/2} r^{n-5} = -2(n - 5)e^{(\nu-\lambda)/2} r^{n-6}
\]  
(21)
by partial differentiation, the other term is expressed as
\[
(\nu' - \lambda') e^{(\nu-3\lambda)/2} r^{n-5} = 2\lambda' e^{(\nu-3\lambda)/2} r^{n-5} - 2(n - 5)e^{(\nu-3\lambda)/2} r^{n-6}
\]  
(22)
\[
\frac{2}{3}\nu' e^{(\nu-3\lambda)/2} r^{n-5} - \frac{2}{3}(n - 5)e^{(\nu-3\lambda)/2} r^{n-6}.
\]  
(23)
From a linear combination of the two terms on the right hand side, one gets

\[
(\nu' - \lambda')e^{(\nu-3\lambda)/2}r^{n-5} = \frac{2}{3(a + b)}((a\nu' + 3b\lambda')e^{(\nu-3\lambda)/2}r^{n-5} - (a + 3b)(n - 5)e^{(\nu-3\lambda)/2}r^{n-6})
\]

\[
= (\nu' - \lambda')e^{(\nu-3\lambda)/2}r^{n-5} - \frac{a + 3b}{3(a + b)}((\nu' - 3\lambda')e^{(\nu-3\lambda)/2}r^{n-5} + 2(n - 5)e^{(\nu-3\lambda)/2}r^{n-6})
\]

\[
= (\nu' - \lambda')e^{(\nu-3\lambda)/2}r^{n-5}.
\] (24)

This means that the first derivative term in Eqs. (22) or (23) cannot be eliminated by partial differentiation. Since the last term of \(g_2\) is the only term which contains a first derivative, and cannot be eliminated, the coefficient of \(g_2\) must vanish.

\[B + 4C = 0\] (27)

Then the last term of \(g_1\) is the only term which contains a non-derivative term and cannot be eliminated by partial differentiation, so the coefficient of \(g_1\) should vanish

\[A + B/2 + C = 0\] (28)

These results yield

\[B = -4C, \quad \text{and} \quad A = C\] (29)

which is a multiple of the GB condition, Eq. (4). The last term of \(f_0\) is proportional to \((n - 4)\) as is seen in Eq. (20). This term should be added to the quantum correction term \(f_1\) in the variational calculation. This proves that quantum corrections for a SSS metric should consist of a GB combination exclusively, where

\[f_0 = 0.\] (30)

This completes the proof of the Theorem. ■

Before proceeding to the computation of \(f_1\) for the purpose of the variational calculation, a comment on the relationship between the two quadratic curvature terms is in order. From the relationship

\[R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{3}R^2 = \frac{1}{2}(H - GB),\] (31)
the difference between $R_{\alpha\beta}R_{\alpha\beta}$ and $\frac{1}{3}R^2$ in a CF metric is $-\frac{1}{2}GB$, instead of zero, as is claimed in literature\[1\].

Using expressions from the appendix, the expansion of $F$ in $(n - 4)$ yields

$$f_1 = A((\nu'^2 + \lambda'^2)(e^{-\lambda}/r)^2 + 6(e^\lambda - 1)^2(e^{-\lambda}/r)^2)$$

$$+ B(\frac{1}{2}(\nu'' + \nu'^2/2 - \nu'\lambda'/2)(\nu' - \lambda')e^{-2\lambda}/r)$$

$$+ \frac{5}{4}(\nu'^2 + \lambda'^2)(e^{-\lambda}/r)^2 - \nu'\lambda'(e^{-\lambda}/r)^2/2$$

$$- 3(\nu' - \lambda')(e^\lambda - 1)/r)(e^{-\lambda}/r)^2$$

$$+ 5((e^\lambda - 1)/r)^2(e^{-\lambda}/r)^2)$$

$$+ C(2(\nu'' + \nu'^2/2 - \nu'\lambda'/2)(\nu' - \lambda')e^{-2\lambda}/r$$

$$+ 4(\nu'^2 + \lambda'^2)(e^{-\lambda}/r)^2 - 8\nu'\lambda'(e^{-\lambda}/r)^2$$

$$- 6(\nu'' + \nu'^2/2 - \nu'\lambda'/2)(e^\lambda - 1)(e^{-\lambda}/r)^2$$

$$- 16(\nu' - \lambda')(e^\lambda - 1)/r)(e^{-\lambda}/r)^2$$

$$+ 12(e^\lambda - 1)^2(e^{-\lambda}/r)^2)$$

$$+ 4C\nu'((e^\lambda - 1)/r)(e^{-\lambda}/r)^2,$$

where the last term is added from Eq. (20). Rearranging appropriate terms, one gets

$$f_1 = (\nu'^2 + \lambda'^2)(e^{-\lambda}/r)^2(A + B/4 + B + 4C)$$

$$+ (B + 4C)g_4$$

$$- 6Cg_3$$

$$+ ((e^\lambda - 1)(e^{-\lambda}/r)^2)(6A + 5B + 12C)$$

$$- (\nu' - \lambda')(e^\lambda - 1)/r)(e^{-\lambda}/r)^2(3B + 16C)$$

$$+ 4C\nu'((e^\lambda - 1)/r)(e^{-\lambda}/r)^2,$$

where

$$g_4 = \frac{1}{2}((\nu'' + \nu'^2/2 - \nu'\lambda'/2)(\nu' - \lambda')r - \nu'\lambda')(e^{-\lambda}/r)^2$$

(34)

In this expression, the first two terms vanish by the BG condition, Eq. (4), while the third term becomes higher order in $(n - 4)$ by Eq. (20). In the last two terms, the $\nu'$ terms cancel and finally one gets

$$f_1 = 4\lambda'((e^\lambda - 1)/r)(e^{-\lambda}/r)^2 - 2(e^\lambda - 1)^2(e^{-\lambda}/r)^2).$$

(35)
Multiplying by $e^{(\nu+\lambda)/2} r^2$, variation of

$$g_{00} = e^\nu$$

(36)

yields

$$H^{00} = \frac{1}{e^{(\nu+\lambda)/2} r^2} \frac{\delta}{\delta(e^\nu)} (f_1 e^{(\nu+\lambda)/2} r^2)$$

(37)

$$= \frac{1}{2} e^{-\nu} f_1$$

(38)

or

$$H^0_0 = \frac{1}{2} f_1$$

(39)

$$= 2\lambda'((e^\lambda - 1)/r)(e^{-\lambda}/r)^2 - (e^\lambda - 1)^2(e^{-\lambda}/r^2)^2.$$  

(40)

For the variation of

$$g_{11} = -e^\lambda,$$

(41)

$$H^{11} = \frac{1}{e^{(\nu+\lambda)/2} r^2} \frac{\delta}{\delta(-e^\lambda)} (f_1 e^{(\nu+\lambda)/2} r^2)$$

$$- \left(\frac{\delta}{\delta(-e^\lambda)}(f_1 e^{(\nu+\lambda)/2} r^2)\right)'$$

$$= \frac{1}{e^{(\nu+\lambda)/2} r^2} (4(e^\lambda)^{3/2} e^{-5\lambda/2} - \frac{5}{2} e^{-7\lambda/2} e^{\nu/2}$$

$$+ (e^{-\lambda/2} + 2e^{-3\lambda/2} - 3e^{-5\lambda/2})e^{\nu/2}$$

$$+ 4(e^{-3\lambda/2} - e^{-5\lambda/2})e^{\nu/2})'$$

$$= \frac{1}{e^{(\nu+\lambda)/2} r^2} (2\nu' e^\lambda - \frac{1}{r} e^{-3\lambda} e^{(\nu+\lambda)/2}$$

$$+ (e^\lambda - 1)^2 e^{-3\lambda} e^{(\nu+\lambda)/2}),$$

(42)

and then

$$H^1_1 = -e^\lambda H^{11}$$

(43)

$$= -2\nu'((e^\lambda - 1)/r)(e^{-\lambda}/r)^2 - (e^\lambda - 1)^2(e^{-\lambda}/r^2)^2.$$  

(44)

These expressions, Eq. (39) and Eq. (43), have been derived by a different method in 1985[3,5], and were used for solving the Einstein equation. This process will be recapitulated in the next section. The difference from 1985 is that this time we have shown that this is the exact solution for a SSS metric.
Finally one can estimate the coefficient of the GB term for a SSS metric. In a CF metric, the $F$-term is expressed as
\[
\frac{\alpha}{6} R^2 - \beta GB,
\] (45)
where
\[
\alpha = \frac{1}{2880\pi^2} (N_S + 6N_\nu - 18N_\nu),
\] (46)
and
\[
\beta = \frac{1}{2880\pi^2} (N_S + 11N_\nu + 62N_\nu),
\] (47)
$N_s$, $N_\nu$ and $N_V$ being the numbers of scalar fields, four-component neutrino fields and vector fields respectively. Then the $F$-term in a SSS metric becomes
\[
\frac{\alpha}{6} R^2 - \beta GB \to \frac{\alpha}{6} (R^2 - 3R^{\alpha\beta}R_{\alpha\beta} + \frac{3}{2}H) - \beta GB
\] (48)
\[
= -\kappa GB,
\] (49)
where
\[
\kappa = \beta - \frac{1}{4} \alpha
\] (50)
\[
= \frac{1}{2880\pi^2} (\frac{3}{4}N_s + \frac{19}{2}N_\nu + \frac{133}{2}N_V).
\] (51)
Eq. (49) indicates that the quantum corrections for any metric are the GB combination of quadratic curvature forms, unless it is a CF metric.

III. EXPLICIT FORM OF QUANTUM CORRECTIONS FOR A SSS METRIC

The Einstein equation for a SSS metric, Eq. (8), is expressed
\[
G^\mu_\rho = R^\mu_\rho - \delta^\mu_\rho R = -l^2 \kappa H^\mu_\rho,
\] (52)
where
\[
l^2 = 16\pi G
\] (53)
and $\kappa$ is given in Eq. (51). An explicit form of the Einstein equation reads
\[
e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}
\] \[
= \frac{\xi}{2} \left( e^{-2\lambda} \left( \frac{2\lambda'}{r^3} + \frac{1}{r^4} \right) + 2e^{-\lambda} \left( -\frac{\lambda'}{r^3} - \frac{1}{r^4} \right) + \frac{1}{r^4} \right)
\] (54)
and
\[ e^{-\lambda}\left(-\frac{\nu'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = \frac{\xi}{2}(e^{-2\lambda}\left(-\frac{2\nu'}{r^3} + \frac{1}{r^4}\right) + 2e^{-\lambda}\left(\frac{\nu'}{r^3} - \frac{1}{r^4}\right) + \frac{1}{r^4}), \tag{55} \]

where
\[ \xi = l^2\kappa. \tag{56} \]

Subtraction of the two Einstein equations yields
\[ \nu' + \lambda' = 0 \tag{57} \]
or
\[ \nu + \lambda = 0 \tag{58} \]
by the boundary condition at
\[ r = \infty. \tag{59} \]

Multiplying Eq. 54 by \( r^2 \),
\[ e^{-\lambda}(r\lambda' - 1) + 1 = \frac{\xi}{2}(e^{-2\lambda}\left(\frac{2\lambda'}{r} + \frac{1}{r^2}\right) \nonumber + 2e^{-\lambda}\left(-\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^4}), \tag{60} \]

which can be expressed as
\[ -(re^{-\lambda})' + r' = \frac{\xi}{2}(-(\frac{e^{-2\lambda}}{r})') \nonumber + 2(\frac{e^{-\lambda}}{r})' - (\frac{1}{r})', \tag{61} \]

which can be integrated
\[ -re^{-\lambda} + r \nonumber = \frac{\xi}{2}(\frac{e^{-2\lambda}}{r} + 2\frac{e^{-\lambda}}{r} - \frac{1}{r}) + K, \tag{62} \]
or
\[ e^{-2\lambda} - 2\left(1 + \frac{r^2}{\xi}\right)e^{-\lambda} + 1 + \frac{2r^2}{\xi} - \frac{2Kr}{\xi} = 0, \tag{63} \]
where $K$ stands for an integration constant. The solution of this quadratic equation is

$$e^{-\lambda} = e^{\nu} = 1 + \frac{r^2}{\xi} - \sqrt{\frac{r^4}{\xi^2} + \frac{2Kr}{\xi}}. \quad (64)$$

This is the solution\cite{3,5} that was obtained by the author in 1985. In the limit, $r \to \infty$ or $\xi \to 0$, one obtains

$$e^{\nu} \to 1 - \frac{K}{r} + \frac{K^2 \xi}{2r^4} + \cdots, \quad (65)$$

so that the integration constant $K$ is determined to be

$$K = 2GM. \quad (66)$$

The important point is that this is the exact solution for quantum corrections to a SSS metric. This solution was obtained by the author in August, 1985\cite{3}. The same solution was obtained independently by Boulware and Deser\cite{10} for a string-generated gravity model.

IV. REPULSIVE NATURE OF GRAVITY AT SHORT DISTANCES

As $r \to 0$, the solution of Eq. (64) becomes

$$e^{\nu} \to 1 - \sqrt{\frac{2Kr}{\xi}} \quad (67)$$

which shows the repulsive nature of quantum corrections to gravity at short distances. While the outer horizon is at $K=2GM$, the inner horizon is

$$\frac{\xi}{2K}. \quad (68)$$

If the sign of $\xi$ is reversed, the solution becomes

$$e^{-\lambda} = e^{\nu} = 1 + \frac{r^2}{|\xi|} + \sqrt{\frac{r^4}{|\xi|^2} + \frac{2Kr}{|\xi|}} \quad (69)$$

The solution is terminated at

$$r = (2K\xi i)^{1/3}, \quad (70)$$

at which point

$$e^{-\lambda} = e^{\nu} = -(\frac{2K}{\sqrt{1/\xi i}})^{2/3}. \quad (71)$$
In other words, the gravitational potential is attractive up to the radius \( r \) in Eq. (70), inside of which there is no solution. This is equivalent to a repulsive core at that radius. Hence, irrespective of the sign of \( \xi \), the quantum corrections make gravity repulsive at short distances. This is a useful information, since the sign convention in general relativity is reversed from author to author, as to that of the curvature and/or the right hand side of the Einstein equation.

As is well known by now, this result is the basis for the author’s model of high energy cosmic ray emission from AGN[3]-[5].

V. APPENDIX: CURVATURE IN A SSS METRIC IN N DIMENSIONS

The extension of the curvature in a SSS metric to \( n \) dimensions reads

\[
R_{0101} = \left( \frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu' \lambda'}{4} \right) e^{-\lambda} \tag{72}
\]

\[
R_{0202} = R_{0303} = \cdots = R_{0,n-1,n-1} = \frac{\nu'}{2} e^{-\lambda} \frac{e^{-\lambda}}{r} \tag{73}
\]

\[
R_{1212} = R_{1313} = \cdots = R_{1,n-1,n-1} = -\frac{\lambda'}{2} e^{-\lambda} \frac{e^{-\lambda}}{r} \tag{74}
\]

\[
R_{2323} = R_{2424} = \cdots = R_{2,n-1,n-1} = R_{3434} = \cdots = R_{n-2,n-1,n-1} = -(\nu^2 - 1) \frac{e^{-\lambda}}{r^2} \tag{75}
\]

\[
P_{c,0} = P_{c,01} + P_{c,02} + \cdots + P_{c,0,n-1} = \left( \frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu' \lambda'}{4} \right) e^{-\lambda} + (n - 2) \frac{\nu'}{2} e^{-\lambda} \frac{e^{-\lambda}}{r} \tag{76}
\]
\[ R_{1,1} = R_{10}^{10} + R_{12}^{12} + R_{13}^{13} + \cdots + R_{1,n-1}^{1,n-1} \]
\[ = \left( \frac{\nu'' + \nu'^2 - \nu' \lambda'}{4} - \frac{\nu' \lambda'}{2} \right) e^{-\lambda} - (n - 2) \frac{\nu' \lambda' e^{-\lambda}}{2} r \]  
\[ (77) \]

\[ R_{2,2} = R_{20}^{20} + R_{21}^{21} + R_{23}^{23} + \cdots + R_{2,n-1}^{2,n-1} \]
\[ = \frac{\nu' - \lambda' e^{-\lambda}}{2} r - (n - 3) (e^\lambda - 1) \frac{e^{-\lambda}}{r^2} \]
\[ = R_{3,3} = R_{4,4} = \cdots = R_{n,n-1}^{n-1} \]  
\[ (78) \]

\[ R = R_{0,0}^{0} + R_{1,1}^{1} + \cdots + R_{n,n-1}^{n-1} \]
\[ = \left( \frac{\nu'' + \nu'^2}{2} - \frac{\nu' \lambda'}{2} \right) e^{-\lambda} + (n - 2) (\nu' - \lambda') \frac{e^{-\lambda}}{r} \]
\[ - (n - 2) (n - 3) (e^\lambda - 1) \frac{e^{-\lambda}}{r^2} \]  
\[ (79) \]

For the quadratic curvature form,

\[ R^\alpha_{\cdot \gamma} R^\gamma_{\cdot \beta} R^\delta_{\cdot \alpha \beta} \]
\[ = 4 \left( (R^0_{10})^2 + (R^0_{02})^2 + \cdots + (R^0_{0,n-1})^2 \right) + (R^1_{12})^2 + (R^1_{13})^2 + \cdots + (R^1_{1,n-1})^2 \]
\[ + (R^3_{23})^2 + (R^3_{24})^2 + \cdots + (R^3_{2,n-1})^2 \]
\[ + (R^4_{34})^2 + \cdots \]
\[ + (R^{n-2}_{n-2,n-1})^2 \]
\[ = \left( (\nu'' + \frac{\nu'^2}{2} - \frac{\nu' \lambda'}{2}) e^{-\lambda} \right)^2 + (n - 2) (\nu'^2 + \lambda'^2) (\frac{e^{-\lambda}}{r})^2 \]
\[ + 2(n - 2)(n - 3) ((e^\lambda - 1) \frac{e^{-\lambda}}{r^2})^2 \]  
\[ (80) \]
\[ R^\alpha_{\cdot\beta} R^\beta_{\cdot\alpha} = (R^0_{\cdot\cdot})^2 + (R^1_{\cdot\cdot})^2 + \cdots (R^{n-1}_{\cdot\cdot})^2 \]

\[ = \frac{1}{2} ((\nu'' + \frac{\nu'^2}{2} - \frac{\nu' \lambda'}{2}) e^{-\lambda})^2 \]

\[ + \frac{n-2}{2} (\nu'' + \frac{\nu'^2}{2} - \frac{\nu' \lambda'}{2}) (\nu' - \lambda') \frac{e^{-2\lambda}}{r} \]

\[ + \frac{(n-2)^2}{4} (\nu'^2 + \lambda'^2) \frac{e^{-\lambda}}{r}^2 \]

\[ + \frac{(n-2)}{4} (\nu' - \lambda')^2 \frac{e^{-\lambda}}{r}^2 \]

\[ - (n-2)(n-3) (\nu' - \lambda') \frac{e^\lambda - 1}{r} (\frac{e^{-\lambda}}{r})^2 \]

\[ + (n-2)(n-3)^2 \frac{e^\lambda - 1}{r}^2 (\frac{e^{-\lambda}}{r})^2 \]

\[ \text{(81)} \]

and

\[ R^2 = ((\nu'' + \frac{\nu'^2}{2} - \frac{\nu' \lambda'}{2}) e^{-\lambda} + (n-2)(\nu' - \lambda') \frac{e^{-\lambda}}{r} \]

\[ - (n-2)(n-3)(e^\lambda - 1) \frac{e^{-\lambda}}{r^2})^2. \]

\[ \text{(82)} \]

**Acknowledgments**

It is a great pleasure to thank David N. Williams for reading the manuscript.

[1] Birrell, N. D. and Davies, P. C. W., Quantum Fields in Curved Space (Cambridge University Press 1982).

[2] Anderson, P., Phys. Rev. D 28, 271 (1983); ibid., D 29, 615 (1984).

[3] Tomozawa, Y., Magnetic Monopoles, Cosmic Rays and Quantum Gravity, in the Proc. of 1985 INS International Symposium on Composite Models of Quarks and Leptons (Tokyo, edit. Terazawa, H. and Yasue, M., 1985), pp. 386. See also UM-Th-17 and C85-08-13.

[4] Tomozawa, Y., The Origins of Cosmic Rays and Quantum Effects of Gravity, in Quantum Field Theory (ed. Mancini, F., Ersever Science Publishers B. V., 1986) pp. 241. This book is the Proceedings of the International Symposium in honor of Hiroomi Umezawa held in Positano, Salerno, Italy, June 5-7, 1985.
[5] Tomozawa, Y., Cosmic Rays, Quantum Effects on Gravity and Gravitational Collapse, Lectures given at the Second Workshop on Fundamental Physics, University of Puerto Rico, Humacao, March 24-28, 1986. The lecture notes can be retrieved from KEK Kiss NO 200035789 at [http://www-lib.kek.jp/KISS/kiss_prepri.html](http://www-lib.kek.jp/KISS/kiss_prepri.html)

[6] The Pierre Auger Collaboration, Science 318, 938 (2007); Correlation of the Highest-energy Cosmic Rays with the Positions of Nearby Active Galactic Nuclei, arXiv: 0712.2843 (2007).

[7] Tomozawa, Y., Evidence for a Dark Matter Particle, arXiv, astro-ph. 1002.1938 (2010).

[8] Tomozawa, Y., Cosmic Rays from AGN, the Knee Energy Mass Scale and Dark Matter Particles, arXiv, astro-ph. 1002.1327 (2010).

[9] Aharonian, F. et.al., Astron. & Astroph. 477, 353 (2008).

[10] Boulware, D. G. and Deser, S., Phys. Rev. Letters 55, 2656 (1985); Frolov, V. P. and Shapiro, I. L., Phys. Rev, D80, 044034 (2009).