A series representation of the discrete fractional Laplace operator of arbitrary order

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Abstract

Although fractional powers of non-negative operators have received much attention in recent years, there is still little known about their behavior if real-valued exponents are greater than one. In this article, we define and study the discrete fractional Laplace operator of arbitrary real-valued positive order. A series representation of the discrete fractional Laplace operator for positive non-integer powers is developed. Its convergence to a series representation of a known case of positive integer powers is proven as the power tends to the integer value. Furthermore, we show that the new representation for arbitrary real-valued positive powers of the discrete Laplace operator is consistent with existing theoretical results.

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1 Introduction

Due to its wide array of applications in multi-physical sciences, the construction and approx-
imation of fractional powers of the Laplace operator have been of great interest for nearly a
century (cf., e.g., [4, 28, 30, 36, 42] and references therein). Classically, only fractional powers
of the order $s \in (0, 1)$ are considered, and in this case, one can define the fractional Laplace
operator applied to a smooth enough function in a natural way. Specifically, for $d \in \mathbb{N},$
$s \in (0, 1)$ let $u: \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function, and for every $\varepsilon \in (0, \infty), x \in \mathbb{R}^d$ let $B_\varepsilon(x)$
be the $d$-dimensional ball of radius $\varepsilon$ centered at $x$ (with respect to the typical topology of
$\mathbb{R}^d$). Then for every $x \in \mathbb{R}^d$ we may define the $s$-order fractional Laplace operator applied
to $u$ at $x$ as

$$((-\Delta)^s u)(x) = c_{d,s} \lim_{\varepsilon \to 0^+} \left[ \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{d+2s}} \, dy \right],$$

where $c_{d,s} \in [0, \infty)$ is a known normalization constant.

It is worth noting that the recent rapid increase in interest in the fractional Laplace
operator is also due to the seminal work of Caffarelli and Silvestre [5]. In their work, it
was shown that one may study the non-local operator given by (1.1) via the Dirichlet-to-
Neumann operator associated with a particular extension problem posed in $\mathbb{R}^d \times [0, \infty)$
(albeit, one trades the non-locality for a problem which is either singular or degenerate
depending upon the value of $s \in (0, 1)$). The employed Dirichlet-to-Neumann operator is
a particular example of the Poincaré-Stecklov operator (cf., e.g., [26]). For a fixed domain,
the Poincaré-Stecklov operator is known to map the boundary values of a harmonic function
to the normal derivative values of the same harmonic function on the same boundary. We
can summarize the results of Caffarelli and Silvestre (cf., e.g., [5, Eq. (3.1)]) as follows. Let
$d \in \mathbb{N}, s \in (0, 1)$, let $u: \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function, and let $v: [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy
for all $x \in \mathbb{R}^d$ that $v(0, x) = u(x)$ and for all $t \in (0, \infty), x \in \mathbb{R}^d$ that

$$\left(\frac{\partial^2}{\partial t^2} v\right)(t, x) + \frac{1 - 2s}{t} \left(\frac{\partial}{\partial t} v\right)(t, x) + (\Delta_x v)(t, x) = 0.$$
Then there exists $c \in [0, \infty)$ such that for all $x \in \mathbb{R}^d$ it holds that
\[
((-\Delta)^s u)(x) = c \lim_{t \to 0^+} t^{1-2s}(\frac{\partial}{\partial t} v)(t, x).
\]
(1.3)

Interestingly, the constant $c \in [0, \infty)$ in (1.3) depends only upon the parameter $s \in (0, 1)$ and not upon $d \in \mathbb{N}$. More importantly, this demonstrates that one may trade out the highly non-local problem given by (1.1) for the local problem given by (1.2) and (1.3). This technique has also been recently further generalized to cases of arbitrary non-negative operators defined on Banach spaces [2, 16, 31, 32, 34].

While the above formulations (i.e., (1.1), (1.2), and (1.3)) may be used to provide insights into the continuous fractional Laplace operator with order $s \in (0, 1)$, they cannot be directly used to provide any insight into the discrete case or the case where $s \in (0, \infty)$. The discrete case is a natural consideration as it arises in the study of numerous physically relevant phenomena (cf., e.g., [22, 23, 35] and references therein) and also in an attempt to numerically approximate (1.1). The consideration of a truly discrete case—that is, the case which is the fractional power of the discrete Laplace operator rather than a direct approximation of (1.1)—was originally studied by Ciaurri et al. [9]. By employing the basic language of semigroups (e.g., a special case of Ciaurri et al. [8, Eq. (1)] combined with Padgett [34, Theorem 2.1]) Ciaurri et al. were able to develop the first series representation for the discrete fractional Laplace operator of order $s \in (0, 1)$ (cf. Definition 3.10, for clarity). Moreover, it was shown that this formulation did converge to the continuous case via adaptive mesh refinements (cf. Ciaurri et al. [8, Theorems 1.7 and 1.8]).

The consideration of higher-order fractional Laplace operators has recently received an increase in attention in continuous cases (cf., e.g., [7, 15, 17, 37, 43]), but to the authors' knowledge the only study in the discrete case has been carried out by Padgett et al. [35]. Rectifying this aforementioned gap in theory is the primary goal of this article. In particular, we will develop a series representation of the discrete fractional Laplace operator of order $s \in (0, \infty)$. This development is illustrated in Theorem 1.1, which is a partial description of the main result of this article focused on the case of positive non-integer powers of the discrete Laplace operator.

**Theorem 1.1.** Let $m \in \{1, 2, 3, \ldots\}$, $s \in (m - 1, m)$, let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, let $\mathbb{R}$ be the real number field, let $\ell^2(\mathbb{Z})$ be the set of all $u: \mathbb{Z} \to \mathbb{R}$ which satisfy that $\sum_{k \in \mathbb{Z}}|u(k)|^2 < \infty$, let $-\Delta: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ satisfy for all $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that $(-\Delta u)(n) = 2u(n) - u(n-1) - u(n+1)$, and let $v: [0, \infty) \times \mathbb{Z} \to \mathbb{R}$ satisfy for all $n \in \mathbb{Z}$ that $v(0,n) = ((-\Delta)^{m-1}u)(n)$ and for all $t \in (0, \infty)$, $n \in \mathbb{Z}$ that
\[
\left(\frac{\partial^2}{\partial t^2} v\right)(t, x) + \frac{1-2(s-m+1)}{t}(\frac{\partial}{\partial t} v)(t, x) + (\Delta v)(t, x) = 0.
\]
(1.4)

Then

\footnotesize
\[
1\text{Note that we define integer powers of } -\Delta: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \text{ inductively. That is, we have for all } k \in \mathbb{N}_0, u \in \ell^2(\mathbb{Z}), n \in \mathbb{Z} \text{ that if } k = 0 \text{ it holds that } ((-\Delta)^ku)(n) = -u(n) \text{ and if } k \in \mathbb{N} \text{ it holds that } ((-\Delta)^ku)(n) = (-\Delta(-\Delta)^{k-1}u)(n).
\]

3
there exists \( c \in [0, \infty) \) such that for all \( n \in \mathbb{Z} \) it holds that
\[
((-\Delta)^s-m+1)u) = c \left[ \lim_{t \to 0^+} t^{1-2(s-m+1)} \left( \frac{\partial}{\partial t} v \right)(t, n) \right] = \mathcal{L}_t^{1-2(s-m+1)}(\mathcal{L}_n^{s}u)(n) = c
\]
and
\[
((-\Delta)^s)u) = \sum_{k \in \mathbb{Z}} K(k)(u(n) - u(n-k)).
\] (1.6)

We now provide some clarifying remarks regarding the above result. Item (i) of Theorem 1.1 above is a direct consequence of combining Definition 3.11 and Padgett [34, Theorem 2.1] (applied for every $n \in \mathbb{Z}$ with $s \leq s - m + 1$, $\Delta \leq \Delta$, $u_0 \leq ((\Delta)^{m-1}u)(n)$, $(u(t))_{t \in [0,\infty)} \leq (v(t, n))_{t \in [0,\infty)}$ in the notation of Padgett [34, Theorem 2.1]). See the beginning of Section 2 for an explanation of this “applied with” notation (i.e., the symbol “\( \leq \)”). The right-hand-side of (1.5) is not considered in detail, herein, as it is an elementary consequence of Padgett [34, Theorem 2.1]. Item (ii) of Theorem 1.1 follows directly from Lemma 4.4 and Lemma 5.1.

The main result of this article is Theorem 5.4 in Section 5. This result provides a complete description of the series representation of the discrete fractional Laplace operator. The most surprising implication of Theorem 5.4 is that the formula for the function $K: \mathbb{Z} \to \mathbb{R}$ in Theorem 1.1 depends only on the parameter $s \in (0, \infty)$ (cf. Definition 4.1 below). In fact, this function is continuous with respect to the parameter $s$ for all $s \in (0, \infty) \setminus \{1, 2, 3, \ldots\}$ with the points $s \in \{1, 2, 3, \ldots\}$ all being removable singularities of the function $K: \mathbb{Z} \to \mathbb{R}$. Hence, we may extend the definition of $K: \mathbb{Z} \to \mathbb{R}$ to that of an analytic function (cf. (5.20) of Theorem 5.4).

The remainder of this article is organized as follows. In Section 2 we recall several basic definitions and properties of sequence spaces and introduce the so-called logarithmic norm. Afterwards, in Section 3 we define the discrete Laplace operator of arbitrary real-valued positive order. We do so by introducing the heat semigroup generated by the discrete Laplace operator and then defining higher-order powers via induction. In Section 4 we define a discrete fractional kernel function and provide a detailed investigation of its various quantitative and qualitative properties. Thereafter, in Section 5 we construct a series representation for real-valued positive powers of the discrete fractional Laplace operator by employing the results from Sections 3 and 4. Finally, in Section 6, a number of useful concluding remarks are provided. Continuing avenues of research based on the results developed in this article are also outlined.

## 2 Background

In this section we review several basic concepts regarding sequence spaces and the logarithmic norm. More specifically, in Subsection 2.1 we introduce the standard $\ell^2(\mathbb{Z})$ sequence space
and an associated function which we denote the semi-inner product. In particular, Lemma 2.5 demonstrates that the standard \( \ell^2(\mathbb{Z}) \) inner product coincides with our particular semi-inner product. Afterwards, in Subsection 2.2 we define the logarithmic norm and the so-called upper-right Dini derivative. We then demonstrate a very useful property in Lemma 2.8 regarding the upper-right Dini derivative of \( \ell^2(\mathbb{Z}) \) norms.

It is worth noting that the contents of this section have been studied in various parts of the scientific literature (although rarely together and in this particular setting). The concept of semi-inner products has been studied extensively in the literature; cf., e.g., [12, 19, 29]. They were originally introduced in an effort to extend standard Hilbert space-type arguments to the more general setting of normed vector spaces. Herein, we employ a slight abuse of notation as Definition 2.4 does not define a semi-inner product in the sense of Lumer (cf., e.g., [29]). However, the object defined in Definition 2.4 does possess many of the desired properties of a semi-inner product and Lemma 2.5 demonstrates that no generality is lost by employing this definition. It is also worth noting that we are not the only authors to employ such notation; cf. e.g., Söderlind [40, Definition 5.1]. Moreover, it is worth mentioning that Lemma 2.8 appears in Jones et al. [20, Lemma 2.4] in a more general setting but we include it below for clarity and completeness.

Throughout this article, \( \mathbb{R} \) and \( \mathbb{C} \) stand for the usual real and complex number fields, respectively. Further, let \( i = \sqrt{-1} \in \mathbb{C} \), let \( \mathbb{Z} \) denote the set of integers, let \( \mathbb{N} = \{1, 2, 3, \ldots \} \), and for every \( z \in \mathbb{C} \) let \( \Re(z) \in \mathbb{R} \) denote the real part of the complex number \( z \). In addition, we briefly mention a particular notation used throughout this article which emphasizes how various outside results are applied. If, for example, a result is referenced which names a particular mathematical object \( \mathcal{X} \), then in order to state results about a family of objects, herein, e.g., \( \mathcal{Y}_t \), \( t \in \mathbb{R} \), we will write “applied for every \( t \in \mathbb{R} \) with \( \mathcal{X} \cap \mathcal{Y}_t \) in the notation of . . .” in order to clarify its use. We generalize this approach in the natural way in the case if multiple mathematical objects are involved (cf., e.g., the proof of Lemma 3.7). Moreover, when carrying out mathematical induction on a variable, say \( n \in \mathbb{N}_0 \), we will use the notation “\( \mathbb{N}_0 \ni (n - 1) \rightarrow n \in \mathbb{N} \)” to emphasize and clarify both the inductive set and the inductive variable (cf., e.g., the proof of Lemma 3.8 below).

### 2.1 Sequence spaces

**Definition 2.1** (Set of all sequences). We denote by \( \mathbb{S} \) the set of all functions with domain \( \mathbb{Z} \) and codomain \( \mathbb{R} \).

**Definition 2.2** (The \( \ell^2(\mathbb{Z}) \) Hilbert space). We denote by \( \ell^2(\mathbb{Z}) \) the set of all \( u \in \mathbb{S} \) satisfying that \( \sum_{k \in \mathbb{Z}} |u(k)|^2 < \infty \) (cf. Definition 2.1). Furthermore, we denote by \( \|\cdot\|_2 : \mathbb{S} \rightarrow [0, \infty] \) the function which satisfies for all \( u \in \mathbb{S} \) that \( \|u\|_2^2 = \sum_{k \in \mathbb{Z}} |u(k)|^2 \).

**Definition 2.3** (The \( \ell^2(\mathbb{Z}) \) inner product). We denote by \( \langle \cdot, \cdot \rangle : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \rightarrow \mathbb{R} \) the function which satisfies for all \( u, v \in \ell^2(\mathbb{Z}) \) that \( \langle u, v \rangle = \sum_{k \in \mathbb{Z}} u(k)v(k) \) (cf. Definition 2.2).

**Definition 2.4** (Semi-inner product). We denote by \( [\cdot, \cdot] : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \rightarrow \mathbb{R} \) the function which satisfies for all \( u, v \in \ell^2(\mathbb{Z}) \) that

\[
[u, v] = \left[ \lim_{\varepsilon \to 0^+} \frac{\|v + \varepsilon u\|_2 - \|v\|_2}{\varepsilon} \right] \|v\|_2 \tag{2.1}
\]
Lemma 2.5. Let \( u, v \in \ell^2(\mathbb{Z}) \) (cf. Definition 2.2). Then

(i) it holds that \( \langle u, u \rangle = \| u \|_2^2 \) and

(ii) it holds that \( \langle u, v \rangle = [u, v] \)

(cf. Definitions 2.3 and 2.4).

Proof of Lemma 2.5. First, observe that item (i) follows immediately from Definitions 2.2 and 2.3. Next, note that item (i), Definition 2.4, and the fact that \( \langle \cdot, \cdot \rangle : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \mathbb{R} \) is a symmetric bilinear form\(^2\) assure that

\[
[u, v] = \lim_{\varepsilon \to 0^+} \left[ \frac{\| v + \varepsilon u \|_2 - \| v \|_2}{\varepsilon} \right] \| v \|_2 = \lim_{\varepsilon \to 0^+} \frac{\| v \|_2 - \| v + \varepsilon u \|_2 - \| v \|_2}{\varepsilon} \left( \| v + \varepsilon u \|_2 + \| v \|_2 \right) \| v \|_2 = \left[ \lim_{\varepsilon \to 0^+} \frac{\langle v, v \rangle + 2\varepsilon \langle u, v \rangle + \varepsilon^2 \langle u, u \rangle - \langle v, v \rangle}{\varepsilon(\| v + \varepsilon u \|_2 + \| v \|_2)} \right] \| v \|_2 = \left[ \lim_{\varepsilon \to 0^+} \frac{2\varepsilon \langle u, v \rangle + \varepsilon^2 \langle u, u \rangle}{2\| v \|_2} \right] \| v \|_2 = \langle u, v \rangle.
\]

This establishes item (ii). The proof of Lemma 2.5 is thus complete.\( \square \)

2.2 The logarithmic norm

Definition 2.6 (Logarithmic norm). We denote by \( \mu(A) \in \mathbb{R}, A : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \), the function which satisfies for all \( A : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) that

\[
\mu(A) = \sup_{v \in \ell^2(\mathbb{Z}), \| v \|_2 \neq 0} \frac{\langle Av, v \rangle}{\| v \|_2^2}
\]

(cf. Definitions 2.2 and 2.3).

Definition 2.7 (Upper-right Dini derivative). For every \( v : [0, \infty) \to \mathbb{R} \) we denote by \( D_t^+v(t) \in [-\infty, \infty], t \in [0, \infty) \), the function which satisfies for all \( t \in [0, \infty) \) that

\[
D_t^+v(t) = \lim_{\varepsilon \to 0^+} \frac{v(t + \varepsilon) - v(t)}{\varepsilon}
\]
Lemma 2.8. It holds for all differentiable \( v: [0, \infty) \to \ell^2(\mathbb{Z}) \), \( t \in [0, \infty) \) that

\[
D_t^+ \|v(t)\|_2 = \left[ \left( \frac{d}{dt} v(t), v(t) \right) \right]^{\frac{1}{2}} \|v(t)\|_2
\]

(cf. Definitions 2.2, 2.3, and 2.7).

Proof of Lemma 2.8. Throughout this proof let \( v: [0, \infty) \to \ell^2(\mathbb{Z}) \), let \( t \in [0, \infty) \), and assume without loss of generality that \( \|v(t)\|_2 \neq 0 \) (cf. Definition 2.2). Note that the hypothesis that \( v \) is differentiable and Taylor’s theorem (cf., e.g., Cartan et al. [6, Theorem 5.6.3]) yield that there exists \( \delta_t(\varepsilon) \in \ell^2(\mathbb{Z}) \), \( \varepsilon \in \mathbb{R} \), such that for all \( \varepsilon \in \mathbb{R} \) with \( |\varepsilon| \) sufficiently small

(A) it holds that \( v(t + \varepsilon) = v(t) + \varepsilon \frac{d}{dt} v(t) + |\varepsilon| \delta_t(\varepsilon) \) and

(B) it holds that \( \lim_{\varepsilon \to 0} \delta_t(\varepsilon) = 0 \).

Combining items (A) and (B) with Definition 2.7 and item (ii) of Lemma 2.5 hence shows that

\[
D_t^+ \|v(t)\|_2 = \lim_{\varepsilon \to 0^+} \sup_{\varepsilon} \frac{\|v(t) + \varepsilon \frac{d}{dt} v(t) + |\varepsilon| \delta_t(\varepsilon)\|_2 - \|v(t)\|_2}{\varepsilon}
\]

\[
= \lim_{\varepsilon \to 0^+} \frac{\|v(t) + \varepsilon \frac{d}{dt} v(t)\|_2 - \|v(t)\|_2}{\varepsilon}
= \lim_{\varepsilon \to 0^+} \frac{\|v(t)\|_2^2}{\varepsilon} = \left\| \frac{\frac{d}{dt} v(t)}{\|v(t)\|_2} \right\|_2 \|v(t)\|_2
\]

(cf. Definition 2.3). The proof of Lemma 2.8 is thus complete.

We close Subsection 2.2 with a brief discussion of Definition 2.6. For every \( A: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) let \( \|A\|_{\text{op}} = \inf\{ c \in [0, \infty] : \forall v \in \ell^2(\mathbb{Z}) \text{ it holds that } \|Av\|_2 \leq c\|v\|_2 \} \) (cf. Definition 2.2). Then Definitions 2.4 and 2.6 and the Rayleigh quotient theorem (cf., e.g., [13, Theorem A.26]) imply that for every \( A: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) with \( \|A\|_{\text{op}} \in [0, \infty) \), \( A = A^* \), and nonempty pure point spectrum it holds that

\[
\mu(A) = \sup_{v \in \ell^2(\mathbb{Z})} \frac{\langle Av, v \rangle}{\|v\|_2^2} = \max \{ \lambda \in \mathbb{R} : \exists v \in \ell^2(\mathbb{Z}) \text{ with } \|v\|_2 \neq 0 \text{ and } Av = \lambda v \}
\]

(e.g., \( \mu(A) \) is the maximal eigenvalue of \( A \)). This fact will prove quite useful in the proof of Lemma 3.7 in Subsection 3.1.

3 The discrete Laplace operator of arbitrary order

In this section we introduce the discrete fractional Laplace operator and define the notion of real-valued positive powers of this operator. First, in Subsection 3.1 we define the discrete Laplace operator as well as introduce and study its associated discrete heat semigroup. Proposition 3.2 is presented in order to clarify the fact that positive integer powers of the
discrete Laplace operator map elements of $\ell^2(\mathbb{Z})$ into $\ell^2(\mathbb{Z})$ (cf. Definition 2.2). The associated discrete heat semigroup is shown to be a strongly continuous contraction semigroup via the tools developed in Subsection 2.2. Note that Definition 3.1 is provided for clarity, as the evaluation of the Gamma function with arguments whose real parts are negative occurs frequently throughout the remainder of this article.

Afterwards, in Subsection 3.2 we provide a series representation of positive integer powers of the discrete Laplace operator. The result in Lemma 3.8 is well-known in the literature, but its proof is included for completeness (cf., e.g., Kelley and Peterson [21, Eq. (2.1)]). The series representation presented in Lemma 3.8 will be a crucial component in proving the main result of this article (cf. Theorem 5.4).

In Subsection 3.3 we define arbitrary real-valued positive powers of the discrete Laplace operator (cf. Definition 3.11). This is accomplished by first defining the case when the positive real-valued power is bounded above by one (cf. Definition 3.10). We then define higher-order positive real-valued powers via an inductive procedure. This definition is shown to be well-defined in $\ell^2(\mathbb{Z})$ in Lemma 3.9; i.e., it is shown that the discrete fractional Laplace operator maps $\ell^2(\mathbb{Z})$ into $\ell^2(\mathbb{Z})$. At this point, we wish to again emphasize that Definition 3.11 is not a direct “discretization” of the pointwise formula for the continuous case (cf., e.g., (1.1) for the case where $s \in (0,1)$), but rather the fractional power of the discrete Laplace operator.

### 3.1 The discrete Laplace operator and its associated semigroup

**Definition 3.1** (Gamma function). Let $X = \{z \in \mathbb{C}: \Re(z) \in (0, \infty)\}$ and let $\Gamma: X \to \mathbb{C}$ be the function which satisfies for all $z \in X$ that $\Gamma(z) = \int_{0}^{\infty} x^{z-1} \exp(-x) \, dx$. Then we denote$^3$ by $\tilde{\Gamma}: \mathbb{C} \to \mathbb{C}$ the function which satisfies for all $z \in X$ that $\Gamma(z) = \tilde{\Gamma}(z)$, for all $z \in \mathbb{C}$ with $\Re(z) \in (-\infty, 0] \setminus \{\ldots, -2, -1, 0\}$ that

$$\Gamma(z) = \frac{\tilde{\Gamma}(z + ||\Re(z)||)}{(z + ||\Re(z)|| - 1)(z + ||\Re(z)|| - 2) \cdots z},$$

and for all $z \in \mathbb{C}$ with $\Re(z) \in \{\ldots, -2, -1, 0\}$ that $\frac{1}{\Gamma(z)} = 0$.

**Proposition 3.2.** It holds$^4$ for all $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$ that

$$\sum_{n \in \mathbb{Z}} \left| \sum_{k=0}^{2s} (-1)^{k-s} \binom{2s}{k} u(n - s + k) \right|^2 < \infty$$

(cf. Definitions 2.2 and 3.1).

**Proof of Proposition 3.2.** Throughout this proof let $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$. Observe that the fact that $u \in \ell^2(\mathbb{Z})$ and Definition 2.2 ensure that $\sum_{k \in \mathbb{Z}} |u(k)|^2 < \infty$. This, the triangle inequality, the fact that $s \in \mathbb{N}$, and the fact Jensen’s inequality implies that for all $r, m \in \mathbb{N}$,

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$^3$Note that $|\cdot|: \mathbb{Z} \to \mathbb{R}$ satisfies for all $x \in \mathbb{R}$ that $|x| = \max\{n \in \mathbb{Z}: n \leq x\}$.

$^4$Note that for all $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, n\}$ it holds that $\binom{n}{k} = \frac{\Gamma(1+n)/\Gamma(1+k+\Gamma(1+n-k))}{\Gamma(n)}$ (cf. Definition 3.1).
\(v_1, v_2, \ldots, v_m \in [0, \infty)\) it holds that \(\sum_{k=1}^{m} v_k^r \leq m^{\max\{r-1,0\}} \sum_{k=1}^{m} v_k^r\) assure that

\[
\sum_{n \in \mathbb{Z}} \left| \sum_{k=0}^{2s} (-1)^{k-s} \binom{2s}{k} u(n - s + k) \right|^2 \leq \sum_{n \in \mathbb{Z}} \left| \sum_{k=0}^{2s} \binom{2s}{k} u(n - s + k) \right|^2 
\]

\[
\leq \sum_{n \in \mathbb{Z}} \left( 2s \sum_{k=0}^{2s} \binom{2s}{k} |u(n - s + k)|^2 \right) = 2s \sum_{k=0}^{2s} \binom{2s}{k} \left[ \sum_{n \in \mathbb{Z}} |u(n - s + k)|^2 \right] < \infty.
\] (3.3)

The proof of Proposition 3.2 is thus complete.

Proof of Proposition 3.2. Throughout this proof let \(I_k : [0, \infty) \to \mathbb{R}, k \in \mathbb{Z}\), satisfy for all \(z \in [0, \infty), k \in \mathbb{Z}\) that

\[
I_k(z) = \sum_{j \in \mathbb{N}_0} \frac{(z/2)^{2j+k}}{\Gamma(1+j)\Gamma(j+k+1)}
\] (3.6)

(cf. Definition 3.1). Observe that (3.6) and Olver et al. [33, Eq. 10.27.1] (applied for every \(k \in \mathbb{Z}\) with \(I_k \sim I_k, z \sim 2z\) in the notation of Olver et al. [33, Eq. 10.27.1]) assure that for all \(z \in [0, \infty)\) it holds that

\[
\sum_{k \in \mathbb{Z}} \left[ \sum_{j \in \mathbb{N}_0} \frac{z^{2j+k}}{\Gamma(1+j)\Gamma(j+k+1)} \right] = \sum_{k \in \mathbb{Z}} I_k(2z)
\] (3.7)

\[
= I_0(2z) + \sum_{k \in \mathbb{N}} (I_k(2z) + I_{-k}(2z)) = I_0(2z) + 2 \sum_{k \in \mathbb{N}} I_k(2z).
\]

Combining this, (3.6), the triangle inequality, Minkowski’s inequality, and Olver et al. [33, Eq. 10.35.5] (applied with \(I_k \sim I_k, z \sim 2z\) in the notation of Olver et al. [33, Eq. 10.35.5]) ensures that for all \(z \in [0, \infty)\) it holds that

\[
\sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left| \sum_{j \in \mathbb{N}_0} \frac{z^{2j+n-k}}{\Gamma(1+j)\Gamma(j+n-k+1)} u(k) \right|^2
\]
\[
\begin{align*}
&= \sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \exp(-2z) I_k(2z) u(n - k) \right|^2 \\
&\leq \exp(-4z) \left[ \sum_{k \in \mathbb{Z}} |I_k(2z)| \left( \sum_{n \in \mathbb{Z}} |u(n - k)|^2 \right)^{1/2} \right]^2 \\
&= \exp(-4z) \left[ \sum_{k \in \mathbb{Z}} I_k(2z) \|u\|_2 \right]^2 \\
&= \exp(-4z) \left[ I_0(2z) + 2 \sum_{k \in \mathbb{N}} I_k(2z) \right] \|u\|_2^2 = \exp(-4z) \|\exp(2z)\|_2^2 \|u\|_2^2 = \|u\|_2^2 < \infty.
\end{align*}
\]

The proof of Proposition 3.5 is thus complete. \(\square\)

**Definition 3.6** (Discrete heat semigroup). We denote by \(S_z(\Delta) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \ z \in [0, \infty), \) the function which satisfies for all \(z \in [0, \infty), \ u \in \ell^2(\mathbb{Z}), \ n \in \mathbb{Z} \) that

\[
(S_z(\Delta)u)(n) = \sum_{k \in \mathbb{Z}} \exp(-2z) \left[ \sum_{j \in \mathbb{N}_0} \Gamma(1 + j) \Gamma(j + n - k + 1) \right] u(k) \quad (3.9)
\]

(cf. Definitions 2.2, 3.1, 3.3, and 3.6).

**Lemma 3.7.** Let \(u \in \ell^2(\mathbb{Z}) \) (cf. Definition 2.2). Then

(i) it holds that \(S_z(\Delta) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \ z \in [0, \infty), \) is a strongly continuous semigroup,

(ii) it holds for all \(z \in [0, \infty), \ n \in \mathbb{Z} \) that \(\frac{d}{dz}(S_z(\Delta)u)(n) = (\Delta S_z(\Delta)u)(n), \)

(iii) it holds that \(\mu(\Delta) \in (-\infty, 0), \) and

(iv) it holds for all \(z \in [0, \infty) \) that \(\|S_z(\Delta)u\|_2 \leq \exp(z\mu(\Delta))\|u\|_2 \leq \|u\|_2\)

(cf. Definitions 2.6, 3.3, and 3.6).

**Proof of Lemma 3.7.** First, note that \(S_z(\Delta) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \ z \in [0, \infty), \) (cf. Definitions 3.3 and 3.6) is a strongly continuous semigroup if

(A) it holds that \(S_0(\Delta) = I \) (cf. Definition 3.4),

(B) it holds for all \(t, z \in [0, \infty) \) that \(S_{t+z}(\Delta) = S_t(\Delta) S_z(\Delta), \)

(C) it holds for all \(v \in \ell^2(\mathbb{Z}) \) that \(\lim_{z \to 0^+} \|S_z(\Delta)v - v\|_2 = 0\)

(cf., e.g., Jones et al. [20, Definition 2.6]). Observe that Definition 3.6, the fact that for all \(v \in \ell^2(\mathbb{Z}) \) it holds that \(\sup_{n \in \mathbb{Z}} |v(n)| < \infty, \) and, e.g., Ciaurri et al. [8, Proposition 1] (applied for every \(v \in \ell^2(\mathbb{Z}) \) with \(f \vee v, W_t \wedge S_z(\Delta) \) in the notation of Ciaurri et al. [8, Proposition 1]) show that \(S_z(\Delta) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \ z \in [0, \infty) \) satisfies items (A), (B), and (C). This establishes item (i). Next, note that Definition 3.6 and, e.g., Ciaurri et al. [8, Proposition 2] (applied for every \(n \in \mathbb{Z} \) with \(f \vee u, (u(n, t))_{t \in [0, \infty)} \wedge ((S_z(\Delta)u)(n))_{z \in [0, \infty)} \) in the notation of Ciaurri et al. [8, Proposition 2]) establish item (ii). In addition, observe that Definition 2.4 implies that if for all \(v \in \ell^2(\mathbb{Z}) \) with \(\|v\|_2 \neq 0 \) it holds that \(\langle \Delta v, v \rangle \in (-\infty, 0) \) then it holds...
that $\mu(\Delta) \in (-\infty, 0)$. To that end, note that Definition 2.3 guarantees that for all $v \in \ell^2(\mathbb{Z})$ it holds that
\[
\langle \Delta v, v \rangle = \sum_{k \in \mathbb{Z}} [(\Delta v)(k)] v(k) = \sum_{k \in \mathbb{Z}} (v(k - 1) - 2v(k) + v(k + 1))v(k)
\]
\[
= \sum_{k \in \mathbb{Z}} v(k - 1)v(k) - 2\sum_{k \in \mathbb{Z}} v(k)^2 + \sum_{k \in \mathbb{Z}} v(k + 1)v(k)
\]
\[
= \sum_{k \in \mathbb{Z}} v(k - 1)v(k) - \sum_{k \in \mathbb{Z}} v(k)^2 - \sum_{k \in \mathbb{Z}} v(k - 1)^2 + \sum_{k \in \mathbb{Z}} v(k)v(k - 1)
\]
\[
= -\sum_{k \in \mathbb{Z}} [v(k - 1)^2 - 2v(k - 1)v(k) + v(k)^2] = -\sum_{k \in \mathbb{Z}} [v(k - 1) - v(k)]^2.
\]
This demonstrates that for all $v \in \ell^2(\mathbb{Z})$ with $\|v\|_2 \neq 0$ it holds that $\langle \Delta v, v \rangle \in (-\infty, 0)$. This and Definition 2.6 establish item (iii). Moreover, observe that item (i), item (iii), and Jones et al. [20, Lemma 2.8] (applied with $x \cap u$, $A \cap \Delta$, $(\mathbb{T}_t(A))_{t \in [0, \infty)} \cap (S_z(\Delta))_{z \in (0, \infty)}$ in the notation of Jones et al. [20, Lemma 2.8]) hence prove item (iv). The proof of Lemma 3.7 is thus complete.

\[
3.2 \quad \text{Positive integer powers of the discrete Laplace operator}
\]

**Lemma 3.8.** It holds for all $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that
\[
((-\Delta)^s u)(n) = \sum_{k=0}^{2s} (-1)^{k-s} \binom{2s}{k} u(n - s + k) \quad (3.11)
\]
(cf. Definitions 2.2 and 3.11).

**Proof of Lemma 3.8.** We prove (3.11) by induction on $s \in \mathbb{N}$. For the base case $s = 1$ observe that Definition 3.3 establishes (3.11). This proves (3.11) in the case $s = 1$. For the induction step $\mathbb{N} \ni (s - 1) \rightarrow s \in \mathbb{N} \cap [2, \infty)$, let $s \in \mathbb{N} \cap [2, \infty)$ and assume for all $s \in \{1, 2, \ldots, s - 1\}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that (3.11) holds. Note that the fact that $s \in \mathbb{N} \cap [2, \infty)$ implies that for all $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ it holds that
\[
((-\Delta)^s u)(n) = (-\Delta((-\Delta)^{s-1} u))(n). \quad (3.12)
\]
This, the induction hypothesis, and Definition 3.3 ensure that for all $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ it holds that
\[
((-\Delta)^s u)(n) = \left(-\Delta \left(\sum_{k=0}^{2(s-1)} (-1)^{k-(s-1)} \binom{2(s-1)}{k} u(\cdot - (s-1) + k)\right)\right)(n)
\]
\[
= 2 \sum_{k=0}^{2s-2} (-1)^{k-s+1} \binom{2s-2}{k} u(n - s + 1 + k)
\]
\[
- \sum_{k=0}^{2s-2} (-1)^{k-s+1} \binom{2s-2}{k} u((n - 1) - s + 1 + k)
\]
\[
- \sum_{k=0}^{2s-2} (-1)^{k-s+1} \binom{2s-2}{k} u((n + 1) - s + 1 + k)
\]

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\[ \begin{align*}
&= 2 \sum_{k=0}^{2s-2} (-1)^{k-s+1} \binom{2s-2}{k} u(n - s + 1 + k) - \sum_{k=0}^{2s-2} (-1)^{k-s+1} \binom{2s-2}{k} u(n - s + k) \\
&\quad - \sum_{k=0}^{2s-2} (-1)^{k-s+1} \binom{2s-2}{k} u(n + 2 - s + k) \\
&= 2 \sum_{k=1}^{2s-1} (-1)^{k-s} \binom{2s-2}{k-1} u(n - s + k) + \sum_{k=0}^{2s-2} (-1)^{k-s} \binom{2s-2}{k} u(n - s + k) \\
&\quad + \sum_{k=2}^{2s} (-1)^{k-s} \binom{2s-2}{k-2} u(n - s + k) \\
&\quad = \left[ \sum_{k=1}^{2s-1} (-1)^{k-s} \binom{2s-2}{k-1} u(n - s + k) + \sum_{k=0}^{2s-2} (-1)^{k-s} \binom{2s-2}{k} u(n - s + k) \right] \\
&\quad + \left[ \sum_{k=1}^{2s-1} (-1)^{k-s} \binom{2s-2}{k-1} u(n - s + k) + \sum_{k=2}^{2s} (-1)^{k-s} \binom{2s-2}{k-2} u(n - s + k) \right].
\end{align*} \tag{3.13} \]

Combining this and the fact that for all \( n \in \mathbb{N}, k \in \{1, 2, \ldots, n-1\} \) it holds that \( \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \) assures that for all \( u \in \ell^2(\mathbb{Z}), n \in \mathbb{Z} \) it holds that

\[ \begin{align*}
\left( (-\Delta)^s u \right)(n) &= (-1)^{s-1} u(n + s - 1) + (-1)^{-s} u(n - s) + \sum_{k=1}^{2s-2} (-1)^{k-s} \binom{2s-1}{k} u(n - s + k) \\
&\quad + (-1)^{1-s} u(n + 1 - s) + (-1)^{s} u(n + s) \\
&\quad + \sum_{k=2}^{2s-1} (-1)^{k-s} \binom{2s-1}{k-1} u(n - s + k) \\
&\quad = (-1)^{s-1} u(n + s - 1) + (-1)^{-s} u(n - s) + (-1)^{1-s} u(n + 1 - s) \\
&\quad + (-1)^{s} u(n + s) + (-1)^{1-s} \binom{2s-1}{1} u(n - s + 1) \\
&\quad + (-1)^{s-1} \binom{2s-1}{2s-2} u(n - s + (2s-1)) \\
&\quad + \sum_{k=2}^{2s-2} (-1)^{k-s} \binom{2s-1}{k-1} u(n - s + k) \\
&\quad = (-1)^{s-1} u(n + s - 1) + (-1)^{-s} u(n - s) + (-1)^{1-s} u(n + 1 - s) \\
&\quad + (-1)^{s} u(n + s) + (-1)^{1-s} \binom{2s-1}{1} u(n - s + 1) \\
&\quad + (-1)^{s-1} \binom{2s-1}{2s-2} u(n + s - 1) + \sum_{k=2}^{2s-2} (-1)^{k-s} \binom{2s}{k} u(n - s + k). \tag{3.14}
\end{align*} \]

Next, observe that the fact that for all \( n \in \mathbb{N}, k \in \{1, 2, \ldots, n-1\} \) it holds that \( \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \) and the fact that for all \( n \in \mathbb{N}, k \in \{0, 1, \ldots, n\} \) it holds that \( \binom{n}{k} = \binom{n}{n-k} \) show that

\[ 1 + \binom{2s-1}{2s-2} = \binom{2s-1}{2s-1} + \binom{2s-1}{2s-2} = \binom{2s}{2s-1} \tag{3.15} \]

and

\[ 1 + \binom{2s-1}{1} = \binom{2s-1}{0} + \binom{2s-1}{1} = \binom{2s}{1}. \tag{3.16} \]

Combining (3.14), (3.15), and (3.16) hence implies that for all \( u \in \ell^2(\mathbb{Z}), n \in \mathbb{Z} \) it holds that

\[ \left( (-\Delta)^s u \right)(n) = (-1)^{-s} \binom{2s}{0} u(n - s) + (-1)^{s} \binom{2s}{2s} u(n + s) + (-1)^{1-s} \binom{2s}{1} u(n - s + 1) \]
\[ + (-1)^{s-1} \left( \frac{2s}{2s-1} \right) u(n + s - 1) + \sum_{k=2}^{2s-2} (-1)^{k-s} \left( \frac{2s}{k} \right) u(n - s + k) \]  
\[ = \sum_{k=0}^{2s} (-1)^{k-s} \left( \frac{2s}{k} \right) u(n - s + k). \]

Induction therefore establishes (3.11). The proof of Lemma 3.8 is thus complete.

\[\]  

3.3 The discrete fractional Laplace operator of arbitrary order

Lemma 3.9. Let \( m \in \mathbb{N}, s \in (m - 1, m) \). Then it holds for all \( u \in \ell^2(\mathbb{Z}) \) that

\[
\left\| \frac{1}{\Gamma(-(s-m+1))} \int_{0}^{\infty} z^{-s+m-2} [S_z(\Delta) - I] ((-\Delta)^{m-1}u) \, dz \right\|_2 < \infty
\]  

(cf. Definitions 2.2, 3.1, 3.3, 3.4, and 3.6).

Proof of Lemma 3.9. Throughout this proof let \( m \in \mathbb{N}, s \in (m - 1, m) \). We claim that for all \( z \in (0, \infty), u \in \ell^2(\mathbb{Z}) \) it holds that

\[
\left\| (S_z(\Delta) - I)u \right\|_2 \leq \int_{0}^{z} \exp((z-w)\mu(\Delta)) \left\| \Delta u \right\|_2 \, dw
\]  

(cf. Definitions 2.2, 2.6, 3.3, 3.4, and 3.6). Note that the fact that \( \langle \cdot, \cdot \rangle : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \rightarrow \mathbb{R} \) is a symmetric bilinear form and Definition 3.6 ensure that for all \( u \in \ell^2(\mathbb{Z}), z \in (0, \infty) \) it holds that

\[
\frac{d}{dz} \left\| (S_z(\Delta) - I)u \right\|_2 = \frac{\langle \Delta (S_z(\Delta) - I)u, (S_z(\Delta) - I)u \rangle}{\left\| (S_z(\Delta) - I)u \right\|_2}
\]  

\[
= \sup_{v \in \ell^2(\mathbb{Z})} \left\| \frac{\langle \Delta u, (S_z(\Delta) - I)u \rangle}{\left\| (S_z(\Delta) - I)u \right\|_2} \right\| + \frac{\langle \Delta u, (S_z(\Delta) - I)u \rangle}{\left\| (S_z(\Delta) - I)u \right\|_2}
\]  

(cf. Definition 2.3). This, the Cauchy-Schwarz inequality, and the fact that item \( (i) \) of Lemma 3.7 implies that for all \( u \in \ell^2(\mathbb{Z}) \) it holds that \( \lim_{z \to 0^+} \left\| (S_z(\Delta) - I)u \right\|_2 = 0 \) assure that for all \( z \in (0, \infty), u \in \ell^2(\mathbb{Z}) \) it holds that

\[
\left\| (S_z(\Delta) - I)u \right\|_2 \leq \int_{0}^{z} \exp((z-w)\mu(\Delta)) \left\| \Delta u \right\|_2 \, dw
\]  

Combining this and the fact that Proposition 3.2 (applied with \( s \prec 1 \) in the notation of Proposition 3.2) ensures that for all \( u \in \ell^2(\mathbb{Z}) \) it holds that \( \left\| \Delta u \right\|_2 < \infty \) proves (3.19). Next, observe that (3.19), Lemma 3.8, and Jensen’s inequality guarantee that for all \( u \in \ell^2(\mathbb{Z}) \) it holds that

\[
\left\| \frac{1}{\Gamma(-(s-m+1))} \int_{0}^{\infty} z^{-s+m-2} [S_z(\Delta) - I] ((-\Delta)^{m-1}u) \, dz \right\|_2
\]
\[
\leq \frac{1}{|\Gamma(-(s - m + 1))|} \int_0^\infty z^{-s+m-2} \| [S_z(\Delta) - I] ((-\Delta)^{m-1} u) \|_2 \, dz
\]  
(3.22)
\[
\leq \frac{\|(\cdot \Delta)^m u\|_2}{|\Gamma(-(s - m + 1))|} \int_0^\infty z^{-s+m-2} \int_0^z \exp((z - w)\mu(\Delta)) \, dw \, dz
\]

(cf. Definition 3.1). In addition, note that item (iii) of Lemma 3.7 and integration by parts show that
\[
0 \leq \int_0^1 z^{-s+m-2} \left[ \frac{1 - \exp(z\mu(\Delta))}{-\mu(\Delta)} \right] \, dz
\]
\[
= \int_0^1 z^{-s+m-2} \left[ \frac{1 - \exp(z\mu(\Delta))}{-\mu(\Delta)} \right] \, dz + \int_1^\infty z^{-s+m-2} \left[ \frac{1 - \exp(z\mu(\Delta))}{-\mu(\Delta)} \right] \, dz
\]
\[
\leq \int_0^1 z^{-s+m-2} \left[ \frac{1 - \exp(z\mu(\Delta))}{-\mu(\Delta)} \right] \, dz + \int_1^\infty z^{-s+m-2} \, dz
\]
\[
= \lim_{w \to 0^+} \left[ \frac{z^{-s+m-1}}{-s+m-1} \right] \left[ 1 - \exp(z\mu(\Delta)) \right] \bigg|_{z=w} - \int_0^1 \left[ \frac{z^{-s+m-1}}{-s+m-1} \right] \exp(z\mu(\Delta)) \, dz
\]
\[
+ \lim_{w \to \infty} \left[ \frac{z^{-s+m-1}}{-s+m-1} \right] \bigg|_{z=1} \,
\]
\[
= \left[ \frac{1}{-s+m-1} \right] \left[ \frac{1 + \mu(\Delta) - \exp(\mu(\Delta))}{-\mu(\Delta)} \right] - \int_0^1 \left[ \frac{z^{-s+m-1}}{-s+m-1} \right] \exp(z\mu(\Delta)) \, dz
\]
\[
\leq \left[ \frac{1}{-s+m-1} \right] \left[ \frac{1 + \mu(\Delta) - \exp(\mu(\Delta))}{-\mu(\Delta)} \right] + \int_0^1 \left[ \frac{z^{-s+m-1}}{-s+m-1} \right] \, dz
\]
\[
= \left[ \frac{1}{-s+m-1} \right] \left[ \frac{1 + \mu(\Delta) - \exp(\mu(\Delta))}{-\mu(\Delta)} \right] + \frac{1}{(-s+m)(s-m+1)} < \infty.
\]

Combining (3.22), (3.23), and the fact that Proposition 3.2 assures that for all \( u \in \ell^2(\mathbb{Z}) \) it holds that \( \|(\cdot \Delta)^m u\|_2 < \infty \) hence proves (3.18). The proof of Lemma 3.9 is thus complete. 

**Definition 3.10** (Discrete fractional Laplace operator for \( s \in (0, 1) \)). Let \( s \in (0, 1) \). Then we denote by \((-\Delta)^s: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})\) the function which satisfies for all \( u \in \ell^2(\mathbb{Z}), n \in \mathbb{Z}\) that
\[
((-\Delta)^s u)(n) = \frac{1}{\Gamma(-s)} \int_0^\infty z^{-s-1}[S_z(\Delta) - I]u(n) \, dz
\]
(3.24)

(cf. Definitions 2.2, 3.1, 3.3, 3.4, and 3.6).

**Definition 3.11** (Discrete fractional Laplace operator for \( s \in (0, \infty) \)). Let \( s \in (0, \infty) \). Then we denote by \((-\Delta)^s: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})\) the function which satisfies for all \( u \in \ell^2(\mathbb{Z}), n \in \mathbb{Z}\) that
\[
((-\Delta)^s u)(n) = ((-\Delta)^{s-[s]}(-\Delta)^{[s]} u)(n)
\]
(3.25)
\[
eq \begin{cases} 
((-\Delta)^s u)(n) & : s \in \mathbb{N} \\
\frac{1}{\Gamma(-s-[s])} \int_0^\infty z^{-(s-[s])}[-S_z(\Delta) - I]((-\Delta)^{[s]} u)(n) \, dz & : s \in (0, \infty) \setminus \mathbb{N}
\end{cases}
\]

(cf. Definitions 2.2, 3.1, 3.3, 3.4, 3.6, and 3.10).
4 The discrete fractional kernel

In this section we introduce a kernel function which will allow us to conveniently provide a series representation of \((3.25)\) in Definition 3.11. Proposition 4.2 and Lemma 4.3 are preliminary results which allow us to prove Lemma 4.4—a result which outlines useful properties exhibited by the kernel defined in Definition 4.1. It is worth noting that Proposition 4.2 is a well-known result and that Lemma 4.3 is a generalization of \([9, \text{Lemma 9.2 (a)}]\), which was only proven in the case where \(s \in (0,1)\). Proposition 4.5 and Lemma 4.7 are the main results of this section and allow us to prove Lemma 5.1 in Subsection 5.1.

**Definition 4.1 (Fractional kernel).** We denote\(^5\) by \(K_s : \mathbb{Z} \rightarrow \mathbb{R}, \ s \in \mathbb{R},\) the function which satisfies for all \(k \in \mathbb{Z}, \ m \in \mathbb{N}, \ s \in (m - 1, m)\) that

\[
K_s(k) = \frac{\mathbb{1}_{\mathbb{Z}\setminus\{0\}}(k) 4^s \Gamma(1/2 + s) \Gamma(|k| - s)}{\sqrt{\pi} \Gamma(-s) \Gamma(|k| + 1 + s)} \quad (4.1)
\]

(cf. Definition 3.1).

**Proposition 4.2.** It holds for all \(a, b \in \mathbb{R}, \ \lambda \in (0, \infty)\) with \(0 \leq a < b < \infty\) that

\[
\min\{\lambda, 1\} \leq \frac{b^\lambda - a^\lambda}{b^\lambda - 1 (b - a)} \leq \max\{\lambda, 1\}. \quad (4.2)
\]

**Proof of Proposition 4.2.** First, note that for all \(a, b \in \mathbb{R}\) with \(0 \leq a < b < \infty\) it holds that

\[
0 \leq a/b < 1 \quad (4.3)
\]

Observe that \((4.3)\) ensures that for all \(\lambda \in (0, 1), a, b \in \mathbb{R}\) with \(0 \leq a < b < \infty\) it holds that

\[
0 \leq a/b \leq (a/b)^\lambda < 1. \quad (4.4)
\]

Combining the mean value theorem and \((4.4)\) hence guarantee that for all \(\lambda \in (0, 1), a, b \in \mathbb{R}\) with \(0 \leq a < b < \infty\) it holds that there exists \(c \in (a/b, 1)\) such that

\[
\frac{b^\lambda - a^\lambda}{b^\lambda - 1 (b - a)} = \frac{1 - (a/b)^\lambda}{1 - a/b} \geq \lambda. \quad (4.5)
\]

Next, note that \((4.3)\) demonstrates that for all \(\lambda \in [1, \infty), a, b \in \mathbb{R}\) with \(0 \leq a < b < \infty\) it holds that \(0 \leq (a/b)^\lambda \leq a/b < 1.\) This shows that for all \(\lambda \in [1, \infty), a, b \in \mathbb{R}\) with \(0 \leq a < b < \infty\) it holds that

\[
\frac{b^\lambda - a^\lambda}{b^\lambda - 1 (b - a)} = \frac{1 - (a/b)^\lambda}{1 - a/b} \geq 1. \quad (4.6)
\]

\(^5\)Let \(A \subseteq \mathbb{R}.\) Then it holds for all \(x \in A\) that \(\mathbb{1}_A(x) = 1\) and for all \(x \in \mathbb{R}\setminus A\) that \(\mathbb{1}_A(x) = 0.\)
Combining (4.6) with the mean value theorem therefore proves that for all \( \lambda \in [1, \infty) \), \( a, b \in \mathbb{R} \) with \( 0 \leq a < b < \infty \) it holds that there exists \( d \in (a/b, 1) \) such that

\[
\frac{b^\lambda - a^\lambda}{b^{\lambda-1}(b-a)} = \frac{1 - (a/b)^\lambda}{1 - a/b} = \lambda d^{\lambda-1} \leq \lambda.
\] (4.7)

Combining (4.4), (4.5), (4.6), and (4.7) thus establishes (4.2). The proof of Proposition 4.2 is thus complete.

**Lemma 4.3.** It holds for all \( m \in \mathbb{N} \), \( s \in (m - 1, m) \) that there exists \( C \in \mathbb{R} \) such that for all \( k \in \mathbb{Z} \) with \( |k| \in [m, \infty) \) it holds that

\[
\left| \frac{\Gamma(|k| - s)}{\Gamma(|k| + 1 + s)} - \frac{1}{|k|^{1+2s}} \right| \leq \frac{C}{|k|^{2+2s}}
\] (4.8)

(cf. Definition 3.1).

**Proof of Lemma 4.3.** Throughout this proof let \( m \in \mathbb{N} \), \( s \in (m - 1, m) \) and without loss of generality let \( k \in \mathbb{Z} \) with \( k \in [m, \infty) \). Note that the triangle inequality assures that

\[
\left| \frac{\Gamma(|k| - s)}{\Gamma(|k| + 1 + s)} - \frac{1}{|k|^{1+2s}} \right| \leq \left| \frac{\Gamma(|k| - s)}{\Gamma(|k| + 1 + s)} - \frac{1}{|k - s|^{1+2s}} \right| + \left| \frac{1}{|k - s|^{1+2s}} - \frac{1}{|k|^{1+2s}} \right|
\] (4.9)

\[
= \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} - \frac{1}{(k - s)^{1+2s}} + \frac{1}{(k - s)^{1+2s}} - \frac{1}{k^{1+2s}}.
\]

Next, observe that Proposition 4.2 (applied with \( \lambda \propto 1 + 2s \), \( a \propto 1/k \), \( b \propto 1/(m-s) \) in the notation of Proposition 4.2) ensures that

\[
\left| \frac{1}{(k - s)^{1+2s}} - \frac{1}{k^{1+2s}} \right| = \left| \frac{(k - s)^{-1} + 2s}{(k - s)^{2+2s}} - \frac{1}{k^{1+2s}} \right| = \frac{1 + 2s}{k^{2+2s}} \left( \frac{1}{k} - \frac{k - s}{(k - s)^2} \right) = \frac{(1 + 2s)s}{k(k - s)^{1+2s}}.
\] (4.10)

This and the fact that \( k \in [m, \infty) \) show that

\[
\left| \frac{1}{(k - s)^{1+2s}} - \frac{1}{k^{1+2s}} \right| \leq \frac{(1 + 2s)s}{k^{2+2s}} \leq \left( \sup_{k \in [m, \infty)} \frac{k^{2+2s}}{(k - s)^{2+2s}} \right) \frac{1 + 2s}{k^{2+2s}}.
\] (4.11)

In addition, note that, e.g., Tricomi and Erdélyi [41, Eq. (15), page 140] (applied with \( z \propto k \), \( \alpha \propto -s \), \( \beta \propto 1 + s \) in the notation of Tricomi and Erdélyi [41, Eq. (15), page 140]) guarantees that

\[
\frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} = \frac{1}{\Gamma(1 + 2s)} \int_0^\infty \exp(-(k-s)v)(1 - \exp(-v))^{2s} dv.
\] (4.12)
This, the fact that for all \( r \in [0, \infty) \) it holds that \( \int_0^\infty \exp(-rv)v^{2s}\,dv = \Gamma(1 + 2s)r^{-(1 + 2s)} \), and Jensen’s inequality prove that

\[
\left| \frac{\Gamma(k-s)}{\Gamma(k+1+s)} - \frac{1}{(k-s)^{1+2s}} \right| = \left| \frac{1}{\Gamma(1+2s)} \int_0^\infty \exp(-(k-s)v)(1-\exp(-v))^{2s} \,dv - \frac{1}{(k-s)^{1+2s}} \right| \quad (4.13)
\]

Furthermore, observe that Proposition 4.2 (applied with \( \lambda \sim 2s, a \sim \frac{1-\exp(-v)}{v}, b \sim 1 \) in the notation of Proposition 4.2) ensures that for all \( v \in (0, \infty) \) it holds that

\[
1 - \left( \frac{1-\exp(-v)}{v} \right)^{2s} \leq \max\{2s, 1\} \left[ 1 - \frac{1-\exp(-v)}{v} \right]. \quad (4.14)
\]

Combining this, (4.13), the fact that for all \( v \in (0, \infty) \) it holds that \( v - (1-\exp(-v)) < v^2/2 \), the fact that for all \( r \in [0, \infty) \) it holds that \( \int_0^\infty \exp(-rv)v^{1+2s}\,dv = \Gamma(2 + 2s)r^{-(2 + 2s)} \), and the fact that Definition 3.1 implies that for all \( z \in (0, \infty) \) it holds that \( \Gamma(1 + z) = z\Gamma(z) \) assures that

\[
\left| \frac{\Gamma(k-s)}{\Gamma(k+1+s)} - \frac{1}{(k-s)^{1+2s}} \right| \leq \frac{\max\{2s, 1\}}{\Gamma(1+2s)} \int_0^\infty \exp(-(k-s)v)v^{2s} \left| 1 - \left( \frac{1-\exp(-v)}{v} \right) \right| \,dv \quad (4.15)
\]

\[
\leq \frac{\max\{s, 1/2\}}{\Gamma(1+2s)} \int_0^\infty \exp(-(k-s)v)v^{1+2s} \,dv = \frac{\max\{s, 1/2\}\Gamma(2 + 2s)}{\Gamma(1 + 2s)} \frac{1}{(k-s)^{2+2s}}
\]

\[
= \max\{s, 1/2\}(1 + 2s) \frac{1}{(k-s)^{2+2s}} \leq \sup_{k \in [m, \infty)} \frac{k^{2+2s}}{(k-s)^{2+2s}} \max\{s, 1/2\}(1 + 2s) \frac{1}{k^{2+2s}}.
\]

Combining this, (4.9), (4.11), and the fact that \( \sup_{k \in [m, \infty)} k^{2+2s}(k-s)^{-(2+2s)} \in \mathbb{R} \) hence proves (4.8). The proof of Lemma 4.3 is thus complete.

**Lemma 4.4.** Let \( m \in \mathbb{N} \), \( s \in (m-1, m) \). Then

(i) it holds for all \( k \in \mathbb{Z} \) that \( K_s(k) = K_s(-k) \)

(ii) it holds that there exists \( C \in \mathbb{R} \) such that for all \( k \in \mathbb{Z} \) with \( |k| \in \mathbb{Z} \setminus \{0\} \) it holds that

\[
|K_s(k)| \leq \frac{C}{|k|^{1+2s}} \quad (4.16)
\]
(cf. Definition 4.1).

Proof of Lemma 4.4. First, note that Definition 4.1 ensures that for all $k \in \mathbb{Z}$ it holds that

\[ K_s(-k) = \frac{-1_{\mathbb{Z}\setminus\{0\}}(-k) 4^s \Gamma(1/2 + s) \Gamma(-k - s)}{\sqrt{\pi} \Gamma(-s) \Gamma(-k - s + 1 + s)} = K_s(k) \] (4.17)

(cf. Definitions 3.1 and 4.1). This establishes item (i). Next, observe that for all $k \in \mathbb{Z} \cap (-m, m)$ with $k \neq 0$ it holds that

\[ |K_s(k)| \leq \frac{4^s \Gamma(1/2 + s) |\Gamma(k - s)|}{\sqrt{\pi} |\Gamma(-s)| \Gamma(k + 1 + s)} = \frac{4^s \Gamma(1/2 + s) |\Gamma(k - s)|}{\sqrt{\pi} |\Gamma(-s)| \Gamma(k + 1 + s)} \cdot \frac{|k|^{1+2s}}{|k|^{1+2s}} \leq \frac{4^s \Gamma(1/2 + s) m^{1+2s}}{\sqrt{\pi} |\Gamma(-s)|} \left( \sup_{j \in \mathbb{Z} \cap (-m, m)} \frac{|\Gamma(|j| - s)|}{\Gamma(|j| + 1 + s)} \right) \frac{1}{|k|^{1+2s}}. \] (4.18)

This, the fact that $s \in (0, \infty) \setminus \mathbb{N}$ implies that $\sup_{j \in \mathbb{Z} \cap (-m, m)} |\Gamma(|j| - s)|/\Gamma(|j| + 1 + s) \in \mathbb{R}$, and Lemma 4.3 establish item (ii). The proof of Lemma 4.4 is thus complete. \qed

Proposition 4.5. It holds for all $m \in \mathbb{N}$, $s \in (m - 1, m)$, $k \in \mathbb{Z}$ that

\[ K_s(k) = \frac{1_{\mathbb{Z}\setminus\{0\}}(k) (-1)^{k+1} \Gamma(2s + 1)}{\Gamma(1 + s + k) \Gamma(1 + s - k)} \] (4.19)

(cf. Definitions 3.1 and 4.1).

Proof of Proposition 4.5. Throughout this proof let $m \in \mathbb{N}$, $s \in (m - 1, m)$ and without loss of generality let $k \in \mathbb{N}$ (cf. item (i) of Lemma 4.4). Observe that Definition 4.1 and the Legendre duplication formula (cf., e.g., [1, Eq. (6.1.18), Page 256]) ensure that

\[ K_s(k) = \frac{-4^s \Gamma(1/2 + s) \Gamma(-k - s)}{\sqrt{\pi} \Gamma(-s) \Gamma(k + 1 + s)} = \frac{-4^s \Gamma(1/2 + s) \Gamma(k - s)}{\sqrt{\pi} \Gamma(-s) \Gamma(k + 1 + s)} \cdot \Gamma(s) \] (4.20)

This and the fact that Definition 3.1 implies that for all $z \in \mathbb{C}$ with $\Re(z) \in \mathbb{R}\setminus\{\ldots, -2, -1, 0\}$ it holds that $z \Gamma(z) = \Gamma(1 + z)$ assure that

\[ K_s(k) = \frac{-2 \Gamma(2s) \Gamma(k - s)}{\Gamma(-s) \Gamma(k + 1 + s)} \cdot \frac{s}{\Gamma(1 + s + k) \Gamma(k - s)} = \frac{\Gamma(2s + 1) \Gamma(k - s)}{\Gamma(1 + s + k) \Gamma(1 + s - k)}. \] (4.21)

Next, note that the Euler reflection formula (cf., e.g., [1, Eq. (6.1.17), Page 256]) guarantees that

\[ \Gamma(s) \Gamma(1 - s) = (-1)^{k+1} \Gamma(k - s) \Gamma(1 + s - k). \] (4.22)

Combining (4.21) and (4.22) hence yields (4.19). The proof of Proposition 4.5 is thus complete. \qed
Proposition 4.6. Let \( s \in (0, \infty) \setminus \mathbb{N} \). Then it holds for all \( m \in \mathbb{N} \) that

\[
\frac{\Gamma(m - s)}{2s\Gamma(m + s)} + \sum_{k=1}^{m-1} \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} = -\frac{\Gamma(-s)}{2\Gamma(1 + s)} \tag{4.23}
\]

(cf. Definition 3.1).

Proof of Proposition 4.6. We prove (4.23) by induction on \( m \in \mathbb{N} \). For the base case \( m = 1 \) note that the fact that Definition 3.1 ensures that for all \( z \in \mathbb{C} \) with \( \Re(z) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\} \) it holds that \( z\Gamma(z) = \Gamma(z + 1) \) guarantees that

\[
\frac{\Gamma(1 - s)}{2s\Gamma(1 + s)} + \sum_{k=1}^{0} \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} = \frac{\Gamma(1 - s)}{2s\Gamma(1 + s)} = -\frac{s\Gamma(-s)}{2s\Gamma(1 + s)} = -\frac{\Gamma(-s)}{2\Gamma(1 + s)} . \tag{4.24}
\]

This establishes (4.23) in the case \( m = 1 \). For the induction step \( \mathbb{N} \ni (m - 1) \rightarrow m \in \mathbb{N} \cap [2, \infty) \), let \( m \in \mathbb{N} \cap [2, \infty) \) and assume for all \( m' \in \{1, 2, \ldots, m - 1\} \) that (4.23) holds. Observe that the induction hypothesis shows that for all \( m \in \mathbb{N} \cap [2, \infty) \) it holds that

\[
\frac{\Gamma(m - s)}{2s\Gamma(m + s)} + \sum_{k=1}^{m-1} \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} = \left[ \frac{\Gamma(m - s)}{2s\Gamma(m + s)} + \frac{\Gamma((m - 1) - s)}{\Gamma((m - 1) + 1 + s)} \right] + \sum_{k=1}^{(m-1)-1} \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} = \frac{\Gamma(m - s)}{2s\Gamma(m + s)} + \frac{\Gamma(m - 1 - s)}{2s\Gamma(m - 1 + s)} + \frac{\Gamma(m - 1 - s)}{2s\Gamma(m - 1 + s)} + \frac{\Gamma(-s)}{2\Gamma(1 + s)} . \tag{4.25}
\]

Next, note that the fact that Definition 3.1 ensures that for all \( z \in \mathbb{C} \) with \( \Re(z) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\} \) it holds that \( z\Gamma(z) = \Gamma(z + 1) \) demonstrates that for all \( m \in \mathbb{N} \cap [2, \infty) \) it holds that

\[
\frac{\Gamma(m - s)}{2s\Gamma(m + s)} + \frac{-\Gamma(m - 1 - s)}{2s\Gamma(m - 1 + s)} = \frac{\Gamma(m - s)}{2s\Gamma(m + s)} + \frac{-(m - 1 + s)\Gamma(m - 1 - s)}{2s\Gamma(m - 1 + s)} = \frac{(m - 1 - s)\Gamma(m - s) - (m - 1 + s)\Gamma(m - s)}{2s\Gamma(m + s)} = \frac{-\Gamma(m - s)}{(m - 1 - s)\Gamma(m + s)} . \tag{4.26}
\]

Moreover, observe that Definition 3.1 assures that for all \( z \in \mathbb{C} \) with \( \Re(z) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\} \) it holds that \( z\Gamma(z) = \Gamma(z + 1) \) demonstrates that for all \( m \in \mathbb{N} \cap [2, \infty) \) it holds that

\[
\frac{-\Gamma(m - s)}{(m - 1 - s)\Gamma(m + s)} + \frac{\Gamma(m - 1 - s)}{\Gamma(m + s)} = \frac{-\Gamma(m - s) + (m - 1 - s)\Gamma(m - 1 - s)}{(m - 1 - s)\Gamma(m + s)} = \frac{-\Gamma(m - s) + \Gamma(m - s)}{(m - 1 - s)\Gamma(m + s)} = 0 . \tag{4.27}
\]

Combining (4.25), (4.26), and (4.27) therefore proves (4.23). The proof of Proposition 4.6 is thus complete. \( \square \)
Lemma 4.7. It holds for all \( m \in \mathbb{N} \), \( s \in (m-1,m) \) that

\[
\sum_{k \in \mathbb{Z}} K_s(k) = \frac{4^s \Gamma(1/2 + s)}{\sqrt{\pi} \Gamma(1 + s)}
\]  

(4.28)

(cf. Definitions 3.1 and 4.1).

Proof of Lemma 4.7. First, note that item (ii) of Lemma 4.4 ensures that for all \( m \in \mathbb{N} \), \( s \in (m-1,m) \) it holds that \( \sum_{k \in \mathbb{Z}} K_s(k) \in \mathbb{R} \). Next, observe that Definition 4.1 and item (i) of Lemma 4.4 assure that for all \( m \in \mathbb{N} \), \( s \in (m-1,m) \) it holds that

\[
\sum_{k \in \mathbb{Z}} K_s(k) = \frac{-4^s \Gamma(1/2 + s)}{\sqrt{\pi} \Gamma(-s)} \left[ \sum_{k \in \mathbb{Z}} 1_{\mathbb{Z} \setminus \{0\}}(k) \Gamma(|k| - s) \right] \Gamma(|k| + 1 + s)
\]  

(4.29)

\[
= \frac{-2 \cdot 4^s \Gamma(1/2 + s)}{\sqrt{\pi} \Gamma(-s)} \left[ \sum_{k \in \mathbb{N}} \Gamma(k - s) \right] \Gamma(k + 1 + s).
\]

In addition, note that, e.g., Artin [3, Eq. (2.13)] (applied for every \( m \in \mathbb{N} \), \( s \in (m-1,m) \), \( k \in \mathbb{N} \cap [m, \infty) \) with \( x \land k-s \), \( y \land 1 + 2s \) in the notation of Artin [3, Eq. (2.13)]) implies that for all \( m \in \mathbb{N} \), \( s \in (m-1,m) \), \( k \in \mathbb{N} \cap [m, \infty) \) it holds that

\[
\frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} = \frac{1}{\Gamma(1 + 2s)} \left[ \frac{\Gamma(k - s) \Gamma(1 + 2s)}{\Gamma(k + 1 + s)} \right] = \frac{1}{\Gamma(1 + 2s)} \int_0^1 (1 - z)^{(1+2s)-1} z^{k-s-1} dz
\]

\[
= \frac{1}{\Gamma(1 + 2s)} \int_0^1 (1 - z)^{2s} z^{k-s-1} dz.
\]

(4.30)

This and Fubini’s theorem guarantee that for all \( m \in \mathbb{N} \), \( s \in (m-1,m) \) it holds that

\[
\sum_{k \in \mathbb{N}} \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} = \sum_{k=1}^{m-1} \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} + \sum_{k=m}^{\infty} \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)}
\]

\[
= \sum_{k=1}^{m-1} \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} + \frac{1}{\Gamma(1 + 2s)} \int_0^1 (1 - z)^{2s} z^{k-s-1} dz
\]

\[
= \sum_{k=1}^{m-1} \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} + \frac{1}{\Gamma(1 + 2s)} \int_0^1 (1 - z)^{2s} z^{m-s-1} \left[ \sum_{k=m}^{\infty} z^k \right] dz
\]

(4.31)

\[
= \sum_{k=1}^{m-1} \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} + \frac{1}{\Gamma(1 + 2s)} \int_0^1 (1 - z)^{2s-1} z^{m-s-1} dz.
\]

Combining this, the fact that Definition 3.1 ensures that for all \( z \in \mathbb{C} \) with \( \Re(z) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\} \) it holds that \( z\Gamma(z) = \Gamma(z + 1) \), and, e.g., Artin [3, Eq. (2.13)] (applied
for every $m \in \mathbb{N}$, $s \in (m-1, m)$ with $x \bowtie m-s$, $y \bowtie 2s$ in the notation of Artin [3, Eq. (2.13)] demonstrates that for all $m \in \mathbb{N}$, $s \in (m-1, m)$ it holds that

$$
\sum_{k \in \mathbb{N}} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} = \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \frac{1}{\Gamma(1+2s)} \left[ \frac{\Gamma(m-s)\Gamma(2s)}{\Gamma(m+s)} \right]
$$

$$
= \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \frac{1}{2s\Gamma(2s)} \left[ \frac{\Gamma(m-s)\Gamma(2s)}{\Gamma(m+s)} \right] \quad (4.32)
$$

Combining this and Proposition 4.6 proves that for all $m \in \mathbb{N}$, $s \in (m-1, m)$ it holds that

$$
\sum_{k \in \mathbb{Z}} K_s(k) = \frac{-2 \cdot 4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(-s)} \left[ \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \frac{\Gamma(m-s)}{2s\Gamma(m+s)} \right] \quad (4.33)
$$

The proof of Lemma 4.7 is thus complete. \qed

5 Series representation for the discrete fractional Laplace operator

In this section we prove the main result of this article. First, in Subsection 5.1 we provide a series representation of the real-valued non-integer powers of the discrete Laplace operator (cf. Lemma 5.1). This representation employs the fractional kernel introduced in Definition 4.1 and its proof hinges upon the results developed in Section 4. It is particularly interesting to note that the representation obtained in Lemma 5.1 coincides with the representation presented in Ciaurri et al. [9, Theorem 1.1] (the case where $m = 1$) and Padgett et al. [35, Theorem 2] (the case where $m = 2$).

In Subsection 5.2 we demonstrate that the representation presented in Lemma 5.1 holds for $s \in \mathbb{N}$ if we consider the limiting values of the discrete kernel function. The main result of the article, Theorem 5.4, follows immediately from the combination of Lemmas 5.1 and 5.3. In particular, Theorem 5.4 demonstrates that all real-valued positive powers of the discrete Laplace operator may be represented with the same series (or, at least, as the limit of this series). Therefore, it is the case that the discrete fractional Laplace operator is, in some sense, a perturbation of the standard positive integer power case.

While it is not the purpose of this article to discuss such issues, we wish to emphasize the importance of the last sentence in the previous paragraph. The fact that the discrete fractional Laplace operator’s series representation coincides with the series representation for positive integer powers provides a framework to endow fractional calculus with potentially enlightening physical interpretations. A particularly lacking feature of the fractional calculus is the lack of meaningful physical interpretations in many situations, which has been one
of the primary limiting factor in its widespread application. However, Theorem 5.4 allows us to view the fractional powers as “transitional phases” between the positive integer cases, loosely speaking. Thus, we may use the known physical intuition for positive integer powers to provide a deeper understanding of the positive non-integer cases.

5.1 Series representation for positive non-integer order

Lemma 5.1. It holds for all $m \in \mathbb{N}$, $s \in (m - 1, m)$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that

$$((-\Delta)^s u)(n) = \sum_{k \in \mathbb{Z}} K_s(n - k)(u(n) - u(k))$$

(cf. Definitions 2.2, 3.11, and 4.1).

Proof of Lemma 5.1. Throughout this proof let $u \in \ell^2(\mathbb{Z})$, let $v : \mathbb{Z} \to \ell^2(\mathbb{Z})$ satisfy for all $n \in \mathbb{Z}$ that $v(n) = (-\Delta u)(n)$, and let $A_s \in \mathbb{R}$, $s \in (0, \infty)$, satisfy for all $s \in (0, \infty)$ that

$$A_s = \frac{4^s \Gamma(1/2 + s)}{\sqrt{\pi \Gamma(1 + s)}}$$

(cf. Definitions 3.1 and 3.3). Observe that Padgett et al. [35, Theorem 1] establishes (5.1) in the case that $m = 1$, $s \in (0, 1)$. Next, note that Definition 3.11 and the fact that for all $m \in \mathbb{N} \cap [2, \infty)$, $n \in \mathbb{Z}$ it holds that $((-\Delta)^m u)(n) = ((-\Delta)^{m-1}(-\Delta) u)(n)$ (i.e., we are invoking the fact that standard function composition is associative on its domain) ensure that for all $m \in \mathbb{N} \cap [2, \infty)$, $s \in (m - 1, m)$, $n \in \mathbb{Z}$ it holds that

$$((-\Delta)^s u)(n) = ((-\Delta)^{s-(m-1)}(-\Delta)^{m-1} u)(n) = ((-\Delta)^{s-(m-1)}((-\Delta)^{m-2}(-\Delta) u))(n)$$

$$= ((-\Delta)^{s-(m-1)}((-\Delta)^{m-2} v))(n) = ((-\Delta)^{s-1} v)(n). \quad (5.3)$$

We now claim that for all $m \in \mathbb{N} \cap [2, \infty)$, $s \in (m - 1, m)$, $n \in \mathbb{N}$ it holds that

$$((-\Delta)^s u)(n) = \sum_{k \in \mathbb{Z}} K_s(n - k)(u(n) - u(k)). \quad (5.4)$$

We prove (5.4) by induction on $m \in \mathbb{N} \cap [2, \infty)$. For the base case $m = 2$ observe that Padgett et al. [35, Theorem 2] establishes (5.4). For the induction step $\mathbb{N} \cap [2, \infty) \ni (m - 1) \rightarrow m \in \mathbb{N} \cap [3, \infty)$, let $m \in \mathbb{N} \cap [3, \infty)$ and assume for all $m \in \{2, 3, \ldots, m - 1\}$, $s \in (m - 1, m)$, $n \in \mathbb{Z}$ that (5.4) holds. Observe that the induction hypothesis, (5.2), (5.3), (5.4), and Lemma 4.7 demonstrate that for all $s \in (m - 1, m)$, $n \in \mathbb{Z}$ it holds that

$$((-\Delta)^s u)(n) = ((-\Delta)^{s-1} v)(n) = \sum_{k \in \mathbb{Z}} K_{s-1}(n - k)(v(n) - v(k)) \quad (5.5)$$

$$= v(n) \sum_{k \in \mathbb{Z}} K_{s-1}(n - k) - \sum_{k \in \mathbb{Z}} K_{s-1}(n - k)v(k) = A_{s-1}v(n) - \sum_{k \in \mathbb{Z}} K_{s-1}(n - k)v(k).$$

This, Definition 4.1, the fact that for all $n \in \mathbb{Z}$ it holds that $v(n) = (-\Delta u)(n)$, and item (i) of Lemma 4.4 show that for all $s \in (m - 1, m)$, $n \in \mathbb{Z}$ it holds that

$$((-\Delta)^s u)(n) = A_{s-1}v(n) - \sum_{k \in \mathbb{Z}} K_{s-1}(n - k)v(k)$$
implies that for all
\[ \implies \]
that for all
\[ \implies \]
Next, note that
\[ \implies \]
the fact that Definition 3.1 implies that for all \( z \in \mathbb{C} \) with \( \Re(z) \in \mathbb{R}\setminus \{-2, -1, 0\} \) it holds that \( \Gamma(z+1) = z\Gamma(z) \), and the fact that \( 1-s \in (-\infty, 0] \) guarantee that for all \( s \in (m-1, m) \) it holds that
\[
K_{s-1}(1) = \frac{-1_{\mathbb{Z}\setminus\{0\}}(1) 4^{s-1}\Gamma(\frac{1}{2} + s - 1)\Gamma(1 - s - 1)}{\sqrt{\pi}\Gamma(-s-1)\Gamma(1 - s)} = \frac{-4^{s-1}\Gamma(s - \frac{1}{2})\Gamma(2 - s)}{\sqrt{\pi}\Gamma(1 - s)\Gamma(1 + s)}
\]
and
\[
K_{s-1}(2) = \frac{-1_{\mathbb{Z}\setminus\{0\}}(2) 4^{s-1}\Gamma(\frac{1}{2} + s - 1)\Gamma(2 - s - 1)}{\sqrt{\pi}\Gamma(-s-2)\Gamma(1 - s)} = \frac{-4^{s-1}\Gamma(s - \frac{1}{2})\Gamma(3 - s)}{\sqrt{\pi}\Gamma(1 - s)\Gamma(2 + s)}
\]
Observe that (5.2), (5.7), and the fact that Definition 3.1 implies that for all \( z \in \mathbb{C} \) with \( \Re(z) \in \mathbb{R}\setminus \{-2, -1, 0\} \) it holds that \( \Gamma(z+1) = z\Gamma(z) \) hence ensure that for all \( s \in (m-1, m) \) it holds that
\[
2A_{s-1} + 2K_{s-1}(1) = 2 \left[ \frac{4^{s-1}\Gamma(s - \frac{1}{2})}{\sqrt{\pi}\Gamma(s)} \right] + \frac{4^{s-1}\Gamma(s - \frac{1}{2})(s - 1)}{\sqrt{\pi}s\Gamma(s)}
\]
\[
= \frac{2 \cdot 4^{s-1}\Gamma(s - \frac{1}{2})}{\sqrt{\pi}\Gamma(s)} \left[ 1 + \frac{s - 1}{s} \right] = \frac{2 \cdot 4^{s-1}\Gamma(s - \frac{1}{2})(s - 1)}{\sqrt{\pi}s\Gamma(s)} \left[ \frac{2s - 1}{s} \right]
\]
\[
= \frac{4^s(s - \frac{1}{2})\Gamma(s - \frac{1}{2})}{\sqrt{\pi}s\Gamma(s)} = \frac{4^s\Gamma(\frac{1}{2} + s)}{\sqrt{\pi}\Gamma(1 + s)} = A_s.
\]
In addition, observe that Proposition 4.5 and the fact that Definition 3.1 implies that for all $z \in \mathbb{C}$ with $\Re(z) \in \mathbb{R}\{\ldots,-2,-1,0\}$ it holds that $\Gamma(z+1) = z\Gamma(z)$ demonstrate that for all $k \in \mathbb{Z}\{\ldots,-1,0,1\}$, $s \in (m-1,m)$ it holds that

$$2K_{s-1}(k) - K_{s-1}(k-1) - K_{s-1}(k+1)$$

\begin{align*}
&= \frac{-2(-1)^k\Gamma(2s-1)}{\Gamma(s+k)\Gamma(s-k)} - \frac{(s+k-1)(-1)^k\Gamma(2s-1)}{(s-k)\Gamma(s+k)\Gamma(s-k)} - \frac{(s-k-1)(-1)^k\Gamma(2s-1)}{(s+k)\Gamma(s+k)\Gamma(s-k)} \\
&= \frac{(-1)^k\Gamma(2s-1)}{\Gamma(s+k)\Gamma(s-k)} \left[-2 \frac{s+k-1}{s-k} - \frac{s-k-1}{s+k} \right] \\
&= \frac{(-1)^k\Gamma(2s-1)}{\Gamma(s+k+1)\Gamma(s-k+1)}(2s-4s^2) = \frac{(-1)^{k+1}\Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)} = K_s(k). \tag{5.10}
\end{align*}

Moreover, note that Definition 4.1, (5.2), (5.7), (5.8), and the fact that Definition 3.1 implies that for all $z \in \mathbb{C}$ with $\Re(z) \in \mathbb{R}\{\ldots,-2,-1,0\}$ it holds that $\Gamma(z+1) = z\Gamma(z)$ assure that for all $s \in (m-1,m)$ it holds that

$$A_{s-1} + 2K_{s-1}(1) - K_{s-1}(2)$$

\begin{align*}
&= \frac{4s^{-1}\Gamma(s-1/2)}{\Gamma(s)} + 2 \left[ \frac{4s^{-1}\Gamma(s-1/2)(s-1)}{\sqrt{s}\Gamma(s)} \right] - \frac{4s^{-1}\Gamma(s-1/2)(2-s)(s-1)}{\sqrt{s(1+s)}\Gamma(s)} \\
&= \frac{4s^{-1}\Gamma(s-1/2)}{\Gamma(s)(s+1)} \left[ 1 + 2 \left( \frac{s-1}{s} \right) + \frac{(s-2)(s-1)}{s(1+s)} \right] = \frac{4s^{-1}\Gamma(s-1/2)}{\sqrt{s}\Gamma(s)} \left[ \frac{4s^{-1/2}}{s+1} \right] \\
&= \frac{4^{-1}\Gamma(s+1/2)}{\sqrt{s}\Gamma(s)} \cdot \frac{s\Gamma(-s)}{\Gamma(s+1)} = \frac{-4\Gamma(s+1/2)\Gamma(1-s)}{\sqrt{s}\Gamma(-s)\Gamma(2+s)} = K_s(1). \tag{5.11}
\end{align*}

Combining (5.6), (5.9), (5.10), (5.11), and item (i) of Lemma 4.4 therefore yields that for all $s \in (m-1,m)$, $n \in \mathbb{Z}$ it holds that

$$((-\Delta)^s u)(n) = \left[ 2A_{s-1} + 2K_{s-1}(1) \right] u(n)$$

\begin{align*}
&- \left[ A_{s-1} + 2K_{s-1}(1) - K_{s-1}(2) \right] (u(n-1) + u(n+1)) \\
&- \sum_{k \in \mathbb{Z}\{\ldots,-1,0,1\}} \left[ 2K_{s-1}(k) - K_{s-1}(k-1) - K_{s-1}(k+1) \right] u(n-k) \tag{5.12}
\end{align*}

\begin{align*}
&= A_s u(n) - K_s(1) (u(n-1) + u(n+1)) - \sum_{k \in \mathbb{Z}\{\ldots,-1,0,1\}} K_s(k) u(n-k) \\
&= A_s u(n) - \sum_{k \in \mathbb{Z}\{0\}} K_s(k) u(n-k) = \sum_{k \in \mathbb{Z}} K_s(n-k) (u(n) - u(k)).
\end{align*}

Induction hence establishes (5.4). The proof of Lemma 5.1 is thus complete. \hfill \square

### 5.2 Series representation for arbitrary positive order

**Lemma 5.2.** Let $s \in \mathbb{N}$. Then it holds for all $k \in \mathbb{Z}$ that

$$\lim_{z \to s^+} K_z(k) = \frac{\mathbb{1}_{\{1,2,\ldots,s\}}(k) (-1)^{k+1} \Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)} \tag{5.13}$$

(cf. Definitions 3.1 and 4.1).
Proof of Lemma 5.2. First, note that Definition 4.1 and Proposition 4.5 ensure that for all $k \in \mathbb{Z}$ it holds that

$$
\lim_{z \to s} K_z(k) = \lim_{z \to s} \frac{1_{\mathbb{Z}\setminus\{0\}}(k) (-1)^{k+1}\Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)}
$$

(cf. Definitions 3.1 and 4.1). Combining this and the fact that Definition 3.1 ensures that for all $z \in \{\ldots,-2,-1,0\}$ it holds that $\frac{1}{\Gamma(z)} = 0$ establishes (5.13). The proof of Lemma 5.2 is thus complete.

Lemma 5.3. It holds for all $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that

$$
((\Delta)^s u)(n) = \sum_{k=0}^{2s} (-1)^{k-s} \binom{2s}{k} u(n-s+k) = \lim_{z \to s} \left[ \sum_{k \in \mathbb{Z}} K_z(n-k)(u(n) - u(k)) \right]
$$

(cf. Definitions 2.2, 3.11, and 4.1).

Proof of Lemma 5.3. First, note that Definition 3.1 and the fact that for all $a \in \mathbb{N}$, $b \in \{0,1,\ldots,a\}$ it holds that $\binom{a}{b}$ assure that for all $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ it holds that

$$
\sum_{k=0}^{2s} (-1)^{k-s} \binom{2s}{k} u(n-s+k) = \sum_{k=-s}^{s} (-1)^{k} \binom{2s}{s+k} u(n-k)
$$

$$
= \sum_{k=-s}^{s} \left[ \frac{(-1)^{k}\Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)} \right] u(n-k)
$$

$$
= \left[ \frac{\Gamma(2s+1)}{\Gamma(1+s)\Gamma(1+s)} \right] u(n) - \sum_{k=1}^{s} \left[ \frac{(-1)^{k+1}\Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)} \right] (u(n-k) + u(n+k))
$$

(cf. Definitions 2.2 and 3.1). Next, observe that Lemma 4.7 ensures that for all $z \in (0,\infty)\setminus\mathbb{N}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ it holds that

$$
\sum_{k \in \mathbb{Z}} K_z(n-k)(u(n) - u(k)) = u(n) \sum_{k \in \mathbb{Z}} K_z(n-k) - \sum_{k \in \mathbb{Z}} K_z(n-k)u(k)
$$

$$
= \left[ \frac{4\pi\Gamma(1/2+z)}{\sqrt{\pi\Gamma(1+z)}} \right] u(n) - \sum_{k \in \mathbb{Z}} K_z(n-k)u(k)
$$

(cf. Definition 4.1). This, Definition 3.1, Definition 4.1, items (i) and (ii) of Lemma 4.4, Lemma 5.2, and, e.g., Rudin [38, Theorem 7.17] guarantee that for all $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ it holds that

$$
\lim_{z \to s} \left[ \sum_{k \in \mathbb{Z}} K_z(n-k)(u(n) - u(k)) \right] = \left[ \frac{4\pi\Gamma(1/2+z)}{\sqrt{\pi\Gamma(1+z)}} \right] u(n) - \lim_{z \to s} \left[ \sum_{k \in \mathbb{Z}} K_z(n-k)u(k) \right]
$$

$$
= \left[ \frac{4\pi\Gamma(1/2+s)}{\sqrt{\pi\Gamma(1+s)}} \right] u(n) - \lim_{z \to s} \left[ \sum_{k \in \mathbb{N}} K_z(k)(u(n-k) + u(n+k)) \right]
$$

25
\[
\frac{4^s \Gamma(1/2 + s)}{\sqrt{\pi} \Gamma(1 + s)} \sum_{k \in \mathbb{N}} \left[ \lim_{z \to s} \left( K_z(k) \left( u(n - k) + u(n + k) \right) \right) \right]
\]

\[
\frac{4^s \Gamma(1/2 + s)}{\sqrt{\pi} \Gamma(1 + s)} u(n) - \sum_{k \in \mathbb{N}} \left[ \lim_{z \to s} \frac{1}{\Gamma(1 + s + k) \Gamma(1 + s - k)} \left( u(n - k) + u(n + k) \right) \right]
\]

\[
= \frac{4^s \Gamma(1/2 + s)}{\sqrt{\pi} \Gamma(1 + s)} u(n) - \sum_{k = 1}^s \left[ \frac{(-1)^{k+1} \Gamma(2s + 1)}{\Gamma(1 + s + k) \Gamma(1 + s - k)} \left( u(n - k) + u(n + k) \right) \right].
\]

In addition, note that the fact that Definition 3.1 implies that for all \( z \in \mathbb{C} \) with \( \Re(z) \in \mathbb{R}\setminus\{-2, -1, 0\} \) it holds that \( z \Gamma(z) = \Gamma(1 + z) \) and the Legendre duplication formula (cf., e.g., [1, Eq. (6.1.18), Page 256]) demonstrate that for all \( s \in \mathbb{N} \) it holds that

\[
\frac{4^s \Gamma(1/2 + s)}{\sqrt{\pi} \Gamma(1 + s)} = \frac{4^s \Gamma(1/2 + s)}{\sqrt{\pi} \Gamma(1 + s)} \cdot s \Gamma(s) = 4^s \frac{2^{1-2s} \sqrt{\pi} s \Gamma(2s)}{\Gamma(1 + s) \Gamma(1 + s)} = \frac{\Gamma(2s + 1)}{\Gamma(1 + s) \Gamma(1 + s)}. \tag{5.19}
\]

Combining this, (5.16), (5.18), Lemma 3.8, and Definition 3.11 proves (5.15). The proof of Lemma 5.3 is thus complete. \( \square \)

**Theorem 5.4.** It holds for all \( s \in (0, \infty), u \in \ell^2(\mathbb{Z}), n \in \mathbb{Z} \) that

\[
((-\Delta)^s u)(n) = \lim_{z \to s} \left[ \sum_{k \in \mathbb{Z}} K_z(n - k) (u(n) - u(k)) \right] \tag{5.20}
\]

(cf. Definitions 2.2, 3.11, and 4.1).

**Proof of Theorem 5.4.** Note that combining Lemmas 5.1 and 5.3 establishes (5.20). The proof of Theorem 5.4 is thus complete. \( \square \)

## 6 Conclusions and future endeavors

### 6.1 Concluding remarks

In this article we developed novel results regarding real-valued positive fractional powers of the discrete Laplace operator. In particular, we defined a discrete fractional Laplace operator for arbitrary real-valued positive powers (cf. Definition 3.11) and then developed its series representation (cf. Theorem 5.4). This latter task was primarily accomplished through the development of two sets of results. First, we constructed the series representation for positive integer powers of the discrete Laplace operator (cf. Lemmas 3.8 and 5.3). Next, we developed series representations for positive non-integer powers of the discrete Laplace operator (cf. Lemma 5.1). The main result of the article (cf. Theorem 5.4) is obtained by showing that the series representations obtained in each of the previous steps in fact coincide.

The main results developed—i.e., the results of Section 5—required numerous preliminary results from various areas of mathematics. The results in Section 2 are of a functional analysis flavor and allow for a beautiful description of important properties of strongly continuous semigroups. These results were combined with results from discrete harmonic analysis...
in Section 3 in order to define and study the discrete fractional Laplace operator. Since the presented definition of this operator (cf. Definition 3.11) employs a so-called semigroup language, it was imperative that all novel mathematical objects are determined to be well defined in $\ell^2(\mathbb{Z})$ (cf. Definition 2.2). Finally, Section 4 provides a detailed study of the proposed fractional kernel function (cf. Definition 4.1) which is necessary for the development of the coefficients of the series representations presented in Section 5. Therein, it is shown that the proposed fractional kernel function is well-defined, symmetric, and continuous for all $s \in (0, \infty) \setminus \mathbb{N}$. It is later shown in Lemma 5.2 that the values $s \in \mathbb{N}$ are in fact removable singularities.

As a final remark, we wish to emphasize the importance of the presented results. Due to the rapidly growing interest in problems related to fractional calculus, there is a need to determine the validity of including fractional operators into existing models. The study of the discrete fractional Laplace operator, or its continuous counterpart, for the case when $s \in (0, 1)$ is well-understood and often used in physical sciences. In this setting, the operator may be used to model super-diffusive phenomena [35]. Moreover, there have been rigorous studies of the operator in this parameter regime which demonstrate that such considerations are well-defined and well-behaved. As such, it is natural to attempt to extend these studies to the case when $s \in (1, \infty)$, as well. The current article has demonstrated that such extensions are indeed well-defined in the discrete case. In addition, Theorem 5.4 shows that one may potentially use the existing understanding of the case of positive integer powers of the discrete Laplace operator to provide some much needed physical intuition to the discrete fractional Laplace operator. However, there are still numerous unanswered questions regarding important properties of these operators and we will outline a few such open problems and research directions in Subsection 6.2 below.

### 6.2 Related future endeavors

First and foremost, there is a need to continue the analytical work presented herein in order to obtain a better understanding of the discrete fractional Laplace operator. In this article, we have considered the setting where all objects are defined in $\ell^2(\mathbb{Z})$, however, this is not always the appropriate setting for physically relevant problems. As such, we intend to extend our study to the situation where the underlying function spaces have less regularity (e.g., Hölder spaces) and develop standard regularity estimates. We also intend to develop similar series representations for the case of real-valued negative exponents. Such representations are highly important for studying fractional Poisson-like problems, as they provide representations of the solution to these problems. Finally, we hope to develop an understanding of the spectral properties of the discrete fractional Laplace operator. While it is well-known that the discrete Laplace operator (cf. Definition 3.3) has purely absolutely continuous spectra (cf., e.g., Dutkay and Jorgensen [14]), to the authors’ knowledge this has not been rigorously proven in the case of the discrete fractional Laplace operator. Demonstrating this to be the case is of utmost importance and will have far-reaching implications in mathematics and physics (for clarification, see the techniques outlined in Liaw [27]).

The proposed discrete fractional Laplace operator is also of interest due to its importance in the physical sciences. An example of interest for future research is transport in turbulent plasmas. It has been experimentally observed that heat and particle transport in turbulent
plasmas is non-local (i.e., anomalous) in nature (cf., e.g., [11, 18, 39]). Comparison between transport models using the fractional Laplace operator and experimental results have demonstrated that electron transport in turbulent fusion plasmas is characterized by fractional exponents in the range \( s \in (0.6, 1) \), which indicates super-diffusive behavior [24, 25]. Moreover, a generalized approach to modeling anomalous diffusive transport in turbulent plasmas employs diffusion-type equations where fractional derivatives occur in both space and time (cf., e.g., [10]). Using the series representations presented herein, we intend to show that the fractional derivative in time can be incorporated into the spatial derivative, which can greatly simplify such equations.

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