Notes on diagonals of the product and symmetric variety of a surface

Luca Scala

Abstract

Let $X$ be a smooth quasi-projective algebraic surface and let $\Delta_n$ the big diagonal in the product variety $X^n$. We study cohomological properties of the ideal sheaves $I_{\Delta_n}$ and their invariants $(I_{\Delta_n})^{S_n}$ by the symmetric group, seen as ideal sheaves over the symmetric variety $S^n X$. In particular we obtain resolutions of the sheaves of invariants $(I_{\Delta_n})^{S_n}$ for $n = 3, 4$ in terms of invariants of sheaves over $X^n$ whose cohomology is easy to calculate. Moreover, we relate, via the Bridgeland-King-Reid equivalence, powers of determinant line bundles over the Hilbert scheme to powers of ideals of the big diagonal $\Delta_n$. We deduce applications to the cohomology of double powers of determinant line bundles over the Hilbert scheme with 3 and 4 points and we give universal formulas for their Euler-Poincaré characteristic. Finally, we obtain upper bounds for the regularity of the sheaves $I_{\Delta_n}$ over $X^n$ with respect to very ample line bundles of the form $L \boxtimes \cdots \boxtimes L$ and of their sheaves of invariants $(I_{\Delta_n})^{S_n}$ on the symmetric variety $S^n X$ with respect to very ample line bundles of the form $\mathcal{D}_L$.

Introduction

Let $X$ be a smooth quasi-projective algebraic surface and consider the product variety $X^n$, for $n \geq 2$. The big diagonal $\Delta_n$ is the closed subscheme of $X^n$ defined as the scheme-theoretic union of all pairwise diagonals $\Delta_I$, where $I$ is a cardinality 2 subset of $\{1, \ldots, n\}$. The aim of this article is the study of the ideal sheaf $I_{\Delta_n}$ of the big diagonal $\Delta_n$ of the product variety $X^n$ and its invariants $(I_{\Delta_n})^{S_n}$, seen as an ideal sheaf over the symmetric variety $S^n X$.

The main reason for studying diagonal ideals is that their geometry is tightly intertwined with the geometry of the Hilbert scheme of points $X^{[n]}$ and that of the isospectral Hilbert scheme $B^n$. As an example of this close interplay, we mention that in [Sca15b] we related the singularities of the isospectral Hilbert scheme in terms of the singularities of the pair $(X^n, I_{\Delta_n})$ and it is by studying the latter that we could prove that the singularities of $B^n$ are canonical if $n \leq 5$, log-canonical if $n \leq 7$ and not log-canonical if $n \geq 9$.

In this work we are concerned with cohomological properties of the ideal sheaf $I_{\Delta_n}$ and its invariants $(I_{\Delta_n})^{S_n}$. A first motivation comes from the study of symmetric powers $S^k L^{[n]}$ of tautological bundles over the Hilbert scheme of points. Let $\mu : X^{[n]} \longrightarrow S^n X$ be the Hilbert-Chow morphism. In [Sca15a] we built a natural filtration $W^k \mu_! S^k L^{[n]}$ of the push-forward $\mu_! S^k L^{[n]}$, indexed by partitions $\lambda$ of $k$ of length at most $n$, whose graded sheaves, at least for $k$ low — but we believe it is a general fact —, are given, up to tensorization by some line bundles, by invariants of diagonal ideals by certain subgroups of $S_n$: we indicate them with $L^\lambda(-2\lambda \Delta)$. The sheaves $L^\lambda(-2\lambda \Delta)$ are in general more complicated than the invariants $(I_{\Delta_n})^{S_n}$ of the big diagonal; however, for $\lambda = 1^k$ (in exponential notation), the sheaf $L^\lambda(-2\lambda \Delta)$ is directly related to the sheaf of invariants $(I_{\Delta_n})^{S_n}$, as we shall see. Therefore, understanding the invariants $(I_{\Delta_n})^{S_n}$ is instrumental for the investigation of symmetric powers of tautological bundles.

There is, moreover, another and more direct source of interest for the cohomological study of diagonal ideals. We recall that the Bridgeland-King-Reid transform

$$\Phi : \mathbb{R}p_* \circ q^* : D^b(X^{[n]}) \longrightarrow D^b_{S_n}(X^n),$$

where $p : B^n \longrightarrow X^n$ is the blow-up along the diagonal $\Delta_n$ and where $q : B^n \longrightarrow X^{[n]}$ is the (flat) quotient map by the symmetric group $S_n$, is an equivalence of derived categories between the derived category of coherent sheaves over the Hilbert scheme $X^{[n]}$ and the $S_n$-equivariant derived category of the
product variety $X^n$. Since the ideal $\mathcal{I}_{\Delta_n}$, or its powers $\mathcal{I}_{\Delta_n}^k$, are $\mathfrak{S}_n$-equivariant sheaves over $X^n$, they need to correspond, under the BKR-equivalence, to some remarkable object over the Hilbert scheme of points: indeed, it turns out that they are related, up to tensorization by the alternating representation $\varepsilon_n$ of the symmetric group, to powers of determinant line bundles on $X^n$. Considering slightly more general $\mathfrak{S}_n$-equivariant objects on $X^n$ we proved the following.

**Theorem 1.8** Let $F$ be a vector bundle of rank $r$ and $A$ be a line bundle over the smooth quasi-projective surface $X$. Consider, over the Hilbert scheme of points $X^{[n]}$, the rank $nr$ tautological bundle $F^{[n]}$ associated to $F$ and the natural line bundle $\mathcal{D}_A$ associated to $A$. Then, in $D_{\mathfrak{S}_n}(X^n)$ we have:

\[
\Phi((\det F^{[n]})^{\otimes k} \otimes \mathcal{D}_A) \simeq^{\text{g.s.}} \mathcal{I}_{\Delta_n}^k \otimes \left(\left(\left((\det F)^{\otimes k} \otimes A\right) \boxtimes \cdots \boxtimes \left((\det F)^{\otimes k} \otimes A\right)\right)\right) \otimes \varepsilon_n^r. \tag{*}
\]

Taking $\mathfrak{S}_n$-invariants, the previous theorem yields

\[
\mathbf{R}\mu_*((\det F^{[n]})^{\otimes k} \otimes \mathcal{D}_A) \simeq \pi_*(\mathcal{I}_{\Delta_n}^k \otimes \varepsilon_n^r) \otimes \mathcal{D}_{\det F} \otimes \mathcal{D}_A. \tag{**}
\]

Therefore, by means of the BKR-equivalence $\Phi$, or of the derived push forward $\mathbf{R}\mu_*$ of the Hilbert-Chow morphism, one might use facts about diagonal ideals and their invariants to deduce properties of determinants line bundles over Hilbert schemes; on the other hand, one can use known results about determinants on Hilbert schemes of points to enlighten properties of the ideal $\mathcal{I}_{\Delta_n}$, its powers and their invariants. This is precisely what happens, as we explain below.

In order to obtain resolutions of invariants $(\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n}$ in terms of simpler sheaves, at least from the point of view of cohomology computations, we consider, for cardinality 2 subsets $I \subseteq \{1, \ldots, n\}$, complexes

\[
K_I^0 : \mathcal{O}_{X^n} \longrightarrow \mathcal{O}_{\Delta_I} \longrightarrow 0
\]

which we take as right resolutions of the ideals $\mathcal{I}_{\Delta_I}$. Being the ideal $\mathcal{I}_{\Delta_n}$ the intersection of the ideals $\mathcal{I}_{\Delta_I}$, the former is isomorphic to the zero-cohomology sheaf $\mathcal{H}^0(\otimes_I K_I^0)$, where $I$ runs among cardinality 2 subsets of $\{1, \ldots, n\}$. However, the complex $\otimes_I K_I^0$ is far from being exact, because the partial diagonals are not transverse: we are then led to consider the derived tensor product $\otimes_I K_I^0$ of the complexes $K_I^0$ and its associated spectral sequence

\[
E_1^{p,q} := \bigoplus_{i_1 + \cdots + i_m = p} \text{Tor}_{-q}(K_{I_1}^{i_1}, \ldots, K_{I_m}^{i_m}) \tag{*}
\]

abutting to the sheaf cohomology $\mathcal{H}^{p+q}(\otimes_I K_I^0)$. Here $m = \binom{n}{2}$ and $\{I_1, \ldots, I_m\}$ are all cardinality 2 subsets of $\{1, \ldots, n\}$. It is now clear that the ideal $\mathcal{I}_{\Delta_n}$ of the big diagonal is given by the term $E_1^{0,0}$ of the spectral sequence above. In order to deal with the latter we now face two difficulties. The first is the understanding of the multitors appearing in $(\otimes_I K_I^0)$, which are of the form $\text{Tor}_{-q}(\mathcal{O}_{\Delta_{I_1}}, \ldots, \mathcal{O}_{\Delta_{I_m}})$, for some cardinality 2 multi-indexes $I_1, \ldots, I_m$. The second is the handling of combinatorial possibilities given by the multi-indexes $I_1, \ldots, I_m$. As for the first, we establish in section 2.2 the following general formula for multitors of structural sheaves of smooth subvarieties $Y_1, \ldots, Y_l$ of a smooth variety $M$ intersecting in a smooth variety $Z = Y_1 \cap \cdots \cap Y_l$ in terms of normal bundles $N_{Y_i}$ and $N_Z$ of $Y_i$ and $Z$ in $M$, respectively. For the second, we think of the multi-indexes $I_i$ as edges of a subgraph $\Gamma$ of the complete graph $\Gamma_n$ with $n$ vertices, such that no vertex of $\Gamma$ is isolated. Classifying all possible multitors appearing in $(\otimes_I K_I^0)$ up to isomorphisms is then reduced to classifying all possible graphs $\Gamma$ of this kind. The usefulness of the graph-theoretic approach extends further since several interesting properties of the multitor $\text{Tor}_q(\mathcal{O}_{\Delta_{I_1}}, \ldots, \mathcal{O}_{\Delta_{I_m}})$ can be understood in graph-theoretic terms. A fundamental fact is that if $I_1, \ldots, I_l$ identify a graph $\Gamma$ with $c$ independent cycles, then the associated multitor $\text{Tor}_q(\mathcal{O}_{\Delta_{\Gamma}}) := \text{Tor}_q(\mathcal{O}_{\Delta_{I_1}}, \ldots, \mathcal{O}_{\Delta_{I_l}})$ is isomorphic to the exterior power $\Lambda^q(Q_{\Gamma})$, where $Q_{\Gamma}$ is a rank $2c$ vector bundle over the intersection $\Delta_{\Gamma} = \Delta_{I_1} \cap \cdots \cap \Delta_{I_l}$. Resorting to the associated graphs is very helpful also when considering the $\mathfrak{S}_n$-action on the naturally equivariant spectral sequence $E_1^{p,q}$. The $\mathfrak{S}_n$-action on $E_1^{p,q}$ induces a $\text{Stab}_{\mathfrak{S}_n}(\Gamma)$-action on the multitor $\text{Tor}_q(\mathcal{O}_{\Delta_{\Gamma}}) \simeq \Lambda^q(Q_{\Gamma})$; the group $\text{Stab}_{\mathfrak{S}_n}(\Gamma)$ acts fiberwise on the vector bundle $Q_{\Gamma}$ via the representation $\mathbb{C}^2 \otimes q_{\Gamma}$, where $q_{\Gamma}$ is the vector space generated over $\mathbb{C}$ by independent cycles. These facts allow us to classify, for $n = 3, 4$ all
bounded above by

We proved a similar statement (Theorem 4.8) about vanishing and regularity for \( B \) of \( X \) has Picard number one. Indeed, suppose that \( \text{Pic}(\mathcal{O}) \) abelian. Then we have the vanishing of certain vanishing and regularity properties of \( \mathcal{O} \).

Theorem 3.14 [3.19] Let \( X \) be a smooth algebraic variety. We have the following natural resolutions of ideals of invariants \( \mathcal{T}_{\Delta}^{s} \) and \( \mathcal{I}_{\Delta}^{s} \) over the symmetric varieties \( S^{X} \) and \( S^{4}X \), respectively.

\[
\begin{align*}
0 \rightarrow (\mathcal{I}_{\Delta}^{s})^e_{s} & \rightarrow \mathcal{O}_{S^{X}} \rightarrow w_{2,n}(\mathcal{O}_{S^{X}}) \rightarrow w_{3,n}(\mathcal{O}_{S^{X}}) \rightarrow 0 \\
0 \rightarrow (\mathcal{I}_{\Delta}^{s})^e_{s} & \rightarrow \mathcal{O}_{S^{X}} \rightarrow w_{2,n}(\mathcal{O}_{S^{X}}) \rightarrow w_{3,n}(\mathcal{O}_{S^{X}}) \rightarrow 0 \\
\end{align*}
\]

The maps \( r, D, C \) are explicit. Here we indicated with \( w_{2,n}(\mathcal{O}_{S^{X}}) \) and \( w_{3,n}(\mathcal{O}_{S^{X}}) \) particular sub sheaves of \( w_{2,n}(\mathcal{O}_{S^{X}}) \) and \( w_{3,n}(\mathcal{O}_{S^{X}}) \) which will be made precise in subsection 3.3. Now, the terms appearing in the resolutions do not present any difficulty from the point of view of cohomology computations. Therefore, by theorems 3.14 [3.19] and formula (12) we deduce, for \( n = 3, 4 \), and for \( X \) a surface, a spectral sequence \( \mathcal{E}_{pq}^{l} \) abutting to the cohomology \( H^{p+q}(X[n], (\text{det} L[n])^{\otimes 2} \otimes \mathcal{D}_{A}) \) and universal formulas for the Euler-Poincaré characteristic \( \chi(X[n], (\text{det} L[n])^{\otimes 2} \otimes \mathcal{D}_{A}) \) of twisted dual powers of determinant line bundles over Hilbert schemes of points. 

These facts have a direct application to the sheaves \( \mathcal{E}_{\lambda}((-2\lambda), \lambda = (r, \ldots, r) \), \( |\lambda| = rl \), since the sheaf \( \mathcal{E}_{\lambda}((-2\lambda) \lambda \) is isomorphic to \( \mathcal{E}_{\lambda}((-2\lambda) \lambda \). where \( v_{l} \) is the finite map \( v_{l} : S^{X}X \rightarrow S^{X}X \) sending \( (x, y) \rightarrow x + y \). We also obtain a right resolution of the sheaf \( \mathcal{E}_{\lambda}^{2}((-2\lambda) \lambda \) over \( S^{X}X \), which is more difficult to treat, since it is not directly related to determinant line bundles over Hilbert schemes.

Finally, as anticipated, we can use properties of determinant line bundles over Hilbert scheme to deduce important facts about diagonal ideals and their invariants. We use formulas (12) and (13) and the positivity properties of \( \text{det} L[n] \) when \( L \) is \( n \)-very ample to study vanishing theorems and regularity for the ideal sheaves \( \mathcal{T}_{\Delta}^{s} \) over \( X \) and their invariants \( (\mathcal{T}_{\Delta}^{s})^{e_{s}} \) over \( S^{X}X \). In particular, we proved the following result.

Theorem 4.4 Let \( X \) be a smooth projective surface and \( L \) be a line bundle over \( X \). Suppose that, for a certain \( m \in N^{*} \), \( L^{m} \otimes K_{X}^{-1} = \otimes_{i=1}^{2} B_{i} \), with \( B_{i} \) \( n \)-very ample and the other \( B_{i} \), \( i \neq 1 \), \((n-1)\)-very ample. Then we have the vanishing

\[
H^{i}(S^{X}X, (\mathcal{T}_{\Delta}^{s})^{e_{s}} \otimes \mathcal{D}_{L}^{m} = 0 \quad \text{for} \quad i > 0.
\]

If, moreover, \( L \) is very ample over \( X \), then the ideal \( (\mathcal{T}_{\Delta}^{s})^{e_{s}} \) is \((m + 2n)\)-regular with respect to \( \mathcal{D}_{L} \). In particular the regularity \( \text{reg}((\mathcal{T}_{\Delta}^{s})^{e_{s}}) \) of the ideal \( (\mathcal{T}_{\Delta}^{s})^{e_{s}} \) with respect to the line bundle \( \mathcal{D}_{L} \) is bounded above by

\[
\text{reg}((\mathcal{T}_{\Delta}^{s})^{e_{s}}) \leq m_{0} + 2n
\]

where \( m_{0} \) is the minimum of all \( m \) satisfying the condition above.

We proved a similar statement (Theorem 4.3) about vanishing and regularity for \( \mathcal{T}_{\Delta}^{s} \) with respect to the line bundle \( L^{2} \otimes \cdots \otimes L^{2} \) on \( X \) for \( 2 \leq n \leq 7 \). These results can be written in a nicer way when the surface \( X \) has Picard number one. Indeed, suppose that \( \text{Pic}(X) = 2B \) and let \( r \) be the minimum positive power of \( B \) such that \( B^{r} \) is very ample and write \( K_{X} = B^{r} \) for some integer \( w \). Then the regularities \( \text{reg}((\mathcal{T}_{\Delta}^{s})^{e_{s}}) \) and \( \text{reg}((\mathcal{T}_{\Delta}^{s})^{e_{s}}) \), with respect to the line bundle \( B^{r} \otimes \cdots B^{r} \) on \( X \) and \( \mathcal{D}_{B^{r}} \) on \( S^{X}X \), respectively, are bounded above by

\[
\text{reg}((\mathcal{T}_{\Delta}^{s})^{e_{s}}) \leq (k + 3)n - k + [w/r] \quad \text{for} \quad 2 \leq n \leq 7
\]

\[
\text{reg}((\mathcal{T}_{\Delta}^{s})^{e_{s}}) \leq 2n((k + 1)/2) + 1 - 2[(k + 1)/2] + 1 + [w/r] \quad \text{for all} \quad n \in N, \; n \geq 2.
\]

Conventions. i). We work over the field of complex numbers. By point we will always mean closed point.
ii). Let $A$ a $\mathbb{C}$-algebra and $M$ an $A$-module. For $n \in \mathbb{N} \setminus \{0\}$, consider the symmetric power $S^n M$ of the module $M$. We consider $S^n M$ as the space of $\mathfrak{S}_n$-invariants of $M \otimes^n$ for the action of $\mathfrak{S}_n$ permuting the factors in the tensor product. Throughout this article, we will use the following convention for the symmetric product $u_1 \cdots u_n$ of elements $u_j \in M$:

$$u_1 \cdots u_n := \sum_{\sigma \in \mathfrak{S}_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)},$$

where the right hand side is seen in $M \otimes^n$. We use an analogous convention for the exterior product:

$$u_1 \wedge \cdots \wedge u_n := \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\sigma} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)},$$

where $(-1)^{\sigma}$ is the signature of the permutation $\sigma$ and where we see $\Lambda^n M$ as the space of anti-invariants for the action of $\mathfrak{S}_n$ over $M \otimes^n$.

Acknowledgements. This work is partially supported by CNPq, grant 307795/2012-8.

1 The Bridgeland-King-Reid transform of diagonal ideals

Consider a smooth quasi-projective algebraic surface $X$. Denote with $X^{[n]}$ the Hilbert scheme of $n$ points over $X$ and with $B^n$ the isospectral Hilbert scheme $[\text{Hai99} \text{ Hai01}]$, that is, the blow-up of $X^n$ along the big diagonal $\Delta_n$. We indicate with $p : B^n \to X^n$ the blow-up map, with $q : B^n \to X^{[n]}$ the quotient projection by the symmetric group $\mathfrak{S}_n$ and with $\mu : X^{[n]} \to S^n X$ the Hilbert-Chow morphism. The Bridgeland-King-Reid equivalence $[\text{BKR01} \text{ Hai01} \text{ Hai02}]$, in the case of the action of the symmetric group $\mathfrak{S}_n$ over the product variety $X^n$, provides an equivalence of derived categories

$$\Phi := \mathbb{R}p_* \circ q^* : \mathcal{D}^b(X^{[n]}) \xrightarrow{\sim} \mathcal{D}^b_{\mathfrak{S}_n}(X^n) \quad (1.1)$$

from the derived category of coherent sheaves over the Hilbert scheme of $n$ points over $X$ and the $\mathfrak{S}_n$-equivariant derived category of the product variety $X^n$. Any power $\mathcal{I}^m_{\Delta_n}$ of the ideal $\mathcal{I}_{\Delta_n}$ is naturally a $\mathfrak{S}_n$-equivariant coherent sheaf over $X^n$: it is then natural to ask what is the corresponding complex of coherent sheaves over the Hilbert schemes of points for the equivalence (1.1). In general we can twist the ideal $\mathcal{I}^m_{\Delta_n}$ with the line bundle $L \boxtimes \cdots \boxtimes L$ ($n$-factors) and ask the same question for $\mathcal{I}^m_{\Delta_n} \otimes L \boxtimes \cdots \boxtimes L$. To give a general statement, we need to introduce the line bundle $\mathcal{D}_L$.

Remark 1.1. If $L$ is a line bundle on $X$, the line bundle $L \boxtimes \cdots \boxtimes L$ ($n$-factors) on $X^n$ descends to a line bundle $\mathcal{D}_L$ on $S^n X$, in the sense that $\pi^* \mathcal{D}_L = L \boxtimes \cdots \boxtimes L$ [\text{DN89} Thm 2.3]. As a consequence, the line bundle $\mathcal{D}_L$ coincides with the sheaf of $\mathfrak{S}_n$-invariants, on $S^n X$, of the line bundle $L \boxtimes \cdots \boxtimes L$. Pulling-back the line bundle $\mathcal{D}_L$ via the Hilbert-Chow morphism $\mu : X^{[n]} \to S^n X$ we get a line bundle $\mu^* \mathcal{D}_L$ on the Hilbert scheme, called the natural line bundle on $X^{[n]}$ associated to the line bundle $L$ on $X$. For brevity’s sake, we will denote it again with $\mathcal{D}_L$.

We need as well a technical lemma about the local cohomology of ideal sheaves $\mathcal{I}^m_{\Delta_n}$, $s \in \mathbb{N}^*$ with respect to the closed subscheme $W$ given by the intersection of double diagonals in $X^n$. More precisely, we define $W$ as the scheme-theoretic intersection of pairwise diagonals

$$W := \bigcap_{|I|=|J|=2; I,J \subseteq \{1, \ldots, n\}; I \neq J} \Delta_I \cap \Delta_J.$$

It is a closed subscheme of $X^n$ of codimension 4.

Notation 1.2. We denote with $X^n_+$ the open subset $X^n \setminus W$ and with $B^n_+, S^n_+ X, X^{[n]}_+$ the open subsets $B^n_+ := p^{-1}(X^n_+)$, $S^n_+ X := \pi(X^n_+)$, $X^{[n]}_+ := \mu^{-1}(S^n_+ X)$. We denote moreover, with $j : X^n_+ \hookrightarrow X^n$ the open immersion of $X^n_+$ into $X^n$. We also denote with $j$ the open immersion of each of the open sets $B^n_+, S^n_+ X, X^{[n]}_+$ into their closure $B^n, S^n X, X^{[n]}$, respectively; it will be clear from the context over which variety we are working.

Remark 1.3. The following is an important result about powers of the ideal sheaf of the diagonal $\Delta_n$ in $X^n$, and it has been proven by Haiman in [\text{Hai01} Corollary 3.8.3]. Over $X^n$, for all $s \in \mathbb{N}$ one has:

$$\bigcap_{|I|=2; |I| \subseteq \{1, \ldots, n\}} \mathcal{I}^s_{\Delta_I} = \left( \bigcap_{|I|=2; |I| \subseteq \{1, \ldots, n\}} \mathcal{I}_{\Delta_I} \right)^s = \mathcal{I}^s_{\Delta_n} \quad (1.2)$$

4
The local cohomology property of the ideal sheaves we want to prove is the following.

**Lemma 1.4.** Let \( l : \{(i, j) \mid i, j \in \mathbb{N}, 1 \leq i < j \leq n\} \to \mathbb{N} \) be a function. Then

\[
j_*j^* \bigcap_{1 \leq i < j \leq n} \mathcal{I}^{(i,j)}_{\Delta_{ij}} = \bigcap_{1 \leq i < j \leq n} \mathcal{I}^{(i,j)}_{\Delta_{ij}}.
\]

**Proof.** We begin by proving recursively that, for any \( s \in \mathbb{N} \) and for any fixed natural numbers \( i, j, 1 \leq i < j \leq n \), we have

\[
\mathcal{H}^0_W(I^s_{\Delta_{ij}}) = 0 \quad \text{for all } 0 \leq l \leq 2.
\]

Indeed it is true for \( s = 0 \), since \( I^0_{\Delta_{ij}} \cong \mathcal{O}_{X^n} \), \( X^n \) is normal, and \( W \) is of codimension 4 in \( X^n \). Consider now \( s \in \mathbb{N} \). The local cohomology long exact sequence applied to the short exact sequence

\[
0 \to I^{s+1}_{\Delta_{ij}} \to I^s_{\Delta_{ij}} \to I^s_{\Delta_{ij}} / I^{s+1}_{\Delta_{ij}} \to 0
\]

yields:

\[
0 \to \mathcal{H}^0_W(I^{s+1}_{\Delta_{ij}}) \to \mathcal{H}^0_W(I^s_{\Delta_{ij}}) \to \mathcal{H}^0_W(I^s_{\Delta_{ij}} / I^{s+1}_{\Delta_{ij}}) \to \mathcal{H}^1_W(I^{s+1}_{\Delta_{ij}}) \to \mathcal{H}^1_W(I^s_{\Delta_{ij}}) \to \mathcal{H}^1_W(I^s_{\Delta_{ij}} / I^{s+1}_{\Delta_{ij}}) \to 0.
\]

Note that \( \mathcal{H}^l_W(I^s_{\Delta_{ij}} / I^{s+1}_{\Delta_{ij}}) = 0 \) for \( l = 0, 1 \) because the sheaf \( I^s_{\Delta_{ij}} / I^{s+1}_{\Delta_{ij}} \) is a vector bundle over the smooth subvariety \( \Delta_{ij} \), in which \( W \) is of codimension 2 and because of [Sca09, Lemma 3.1.9]. Now, if \( \mathcal{H}^l_W(I^s_{\Delta_{ij}}) = 0 \) for \( l = 0, 1, 2 \), the local cohomology long exact sequence above yields the vanishing for \( I^{s+1}_{\Delta_{ij}} \) and \( l = 0, 1, 2 \). Induction on \( s \) then yields (1.3) for any \( s \in \mathbb{N} \).

Since \( \mathcal{H}^l_W(I^s_{\Delta_{ij}}) = 0 \) for \( l = 0, 1, 2 \) and for any \( s \in \mathbb{N} \), an analogous argument via the long exact sequence in local cohomology applied to the short exact sequence

\[
0 \to I^s_{\Delta_{ij}} \to \mathcal{O}_{X^n} \to \mathcal{O}_{X^n} / I^s_{\Delta_{ij}} \to 0
\]

proves that \( \mathcal{H}^l_W(\mathcal{O}_{X^n}, I^s_{\Delta_{ij}}) = 0 \) for \( l = 0, 1 \) and for any \( s \); this last fact is equivalent to the isomorphism

\[
j_*j^* \mathcal{O}_{X^n} / I^s_{\Delta_{ij}} \cong \mathcal{O}_{X^n} / I^s_{\Delta_{ij}}.
\]

The above isomorphism, together with the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & I_{\Delta_{ij}}^{(i,j)} \\
& \uparrow & \uparrow \\
j_*j^* \bigcap_{1 \leq i < j \leq n} I_{\Delta_{ij}}^{(i,j)} & \to & j_*j^* \mathcal{O}_{X^n} \\
& \uparrow & \uparrow \\
0 & \to & \bigoplus_{1 \leq i < j \leq n} \mathcal{O}_{X^n} / I_{\Delta_{ij}}^{(i,j)}
\end{array}
\]

where the second and third vertical arrows are isomorphisms, because we proved so above, yields the statement of the lemma. \( \square \)

**Lemma 1.5.** Let \( k \in \mathbb{N}^* \). We have the isomorphism of sheaves of invariants over \( S^n X \):

\[
(T^{2k-1}_{\Delta_n})^{\Theta_n} \cong (T^k_{\Delta_n})^{\Theta_n}.
\]

**Proof.** The statement follows using (1.2), taking \( \Theta_n \)-invariant in the exact sequence

\[
0 \to T^{2k-1}_{\Delta_n} \to \mathcal{O}_{X^n} \to \oplus_{i < j} \mathcal{O}_{X^n} / T^{2k-1}_{\Delta_{ij}}
\]

and noting that \( \pi_* \left( \oplus_{i < j} \mathcal{O}_{X^n} / T^{2k-1}_{\Delta_{ij}} \right)^{\Theta_n} \cong \pi_* \left( \mathcal{O}_{X^n} / T^{2k-1}_{\Delta_{12}} \right)^{\Theta_n} \cong \pi_* \left( \mathcal{O}_{X^n} / T^{2k}_{\Delta_{12}} \right)^{\Theta_n} \cong \pi_* \left( T^{2k-1}_{\Delta_{12}} \right)^{\Theta_n} \cong \pi_* \left( T^k_{\Delta_{12}} \right)^{\Theta_n} \cong \pi_* \left( T^k_{\Delta_{12}} \right)^{\Theta_n} \). \( \square \)
Remark 1.6. Indicate now with $E$ the exceptional divisor (or the boundary) of $X^{[n]}$: it is the exceptional divisor for the Hilbert-Chow morphism and the branching divisor for the map $q : B^n \longrightarrow X^{[n]}$. It is well known [Leh99, Lemma 3.7] that

$$O_{X^{[n]}}(-E) \simeq (\det O_X^{[n]})^{\otimes 2}. $$

As a consequence there exists a divisor $e$ on the Hilbert scheme $X^{[n]}$ such that $E = 2e$, and such that $O_{X^{[n]}}(-e) = \det O_X^{[n]}$. It is also well known that $\det L_n \simeq D_L \otimes \det O_X^{[n]}$, which can be rewritten, with the notations just explained, as

$$\det L_n \simeq D_L(-e). \quad (1.5)$$

Denote now with $E_B$ the exceptional divisor over the isospectral Hilbert scheme, that is, the exceptional divisor for the blow-up map $p : B^n \longrightarrow X^n$. We have $O_{B^n}(-E_B) \simeq q^*O_{X^{[n]}}(-e).$

Denoting with $\varepsilon_n$ the alternating representation of $S_n$, we can now prove

**Theorem 1.7.** Let $X$ be a smooth quasi-projective algebraic surface. Then

$$\Phi(O_{X^{[n]}}(-le)) \simeq \Phi((\det O_X^{[n]})^{\otimes l}) \simeq T_{\Delta_n} \otimes \varepsilon_n^l$$

$$R_\mu_*O_{X^{[n]}}(-le) \simeq R_\mu_*(\det O_X^{[n]})^{\otimes l} \simeq \pi_*^n(T_{\Delta_n} \otimes \varepsilon_n^l).$$

The proof of theorem [1.7] is a consequence of the following more general result.

**Theorem 1.8.** Let $F$ be a vector bundle of rank $r$ and $A$ be a line bundle over the smooth quasi-projective surface $X$. Consider, over the Hilbert scheme of points $X^{[n]}$, the rank $nr$ tautological bundle $F^{[n]}$ associated to $F$ and the natural line bundle $D_A$ associated to $A$. Then, in $D(S_n(X^n))$ we have:

$$\Phi((\det F^{[n]})^{\otimes k} \otimes D_A) \simeq q_{\mu*} T_{\Delta_n} \otimes ((\det F)^{\otimes k} \otimes A) \boxtimes \cdots \boxtimes ((\det F)^{\otimes k} \otimes A) \otimes \varepsilon_n^{rk}.$$
and
\[ \det(\oplus_i p^* F_i|_{E_{ij}}) \simeq \bigotimes_i p^* \det(p^* F_i|_{E_{ij}}) \simeq \bigotimes_i \bigotimes_j \mathcal{O}_{B^n}(rE_{ij}) \simeq \mathcal{O}_{B^n}(rE_B). \] (1.6)

As for the first factor, we have, just as coherent sheaves, without considering the \( \mathfrak{S}_n \)-action:
\[ \det(\oplus_i p^* F_i) \simeq p^*(\bigotimes_i \det F_i) \simeq p^*(\det F \boxtimes \cdots \boxtimes \det F). \]

However, it is clear that a consecutive transposition \( \tau_{i,i+1} \) acts on the sheaf on the left hand side with the sign \((-1)^r\), while it acts trivially on the right hand side: hence, to have an isomorphism as \( \mathfrak{S}_n \)-equivariant sheaves we have to correct the previous formula by the representation \( \varepsilon_n^r \): that is, as \( \mathfrak{S}_n \)-sheaves:
\[ \det(\oplus_i p^* F_i) \simeq p^*(\det F \boxtimes \cdots \boxtimes \det F) \otimes \varepsilon_n^r. \] (1.7)

From (1.7) and (1.6) we get that, as \( \mathfrak{S}_n \)-equivariant sheaves, over \( B_n^* \):
\[ q^* \det F^{[n]} \simeq p^*(\det F \boxtimes \cdots \boxtimes \det F) \otimes p_* \mathcal{O}_{B^n}(-rE_B) \otimes \varepsilon_n^r. \]

Since this is an isomorphism of vector bundles, since \( B^n \) is normal and since the complementary of \( B_n^* \) in \( B^n \) is a closed subscheme of codimension 2, the previous isomorphism extends to the whole variety \( B^n \) as an isomorphism of \( \mathfrak{S}_n \)-equivariant vector bundles. Therefore, by projection formula:
\[ \mathcal{R}p_* q^* \det F^{[n]} \simeq (\det F \boxtimes \cdots \boxtimes \det F) \otimes p_* \mathcal{O}_{B^n}(-rE_B) \otimes \varepsilon_n^r. \]

To finish the proof we just have to show that \( p_* \mathcal{O}_{B^n}(-rE_B) \simeq T^r_{\Delta_n} \). Since \( \mathcal{O}_{B^n}(-rE_B) \) is a line bundle, since \( B^n \) is normal and \( B_n^* \) is the complementary of a closed of codimension 2, we have \( \mathcal{O}_{B^n}(-rE_B) \simeq j_* j^* \mathcal{O}_{B^n}(-rE_B) \); hence
\[ p_* \mathcal{O}_{B^n}(-rE_B) \simeq p_* j_* j^* \mathcal{O}_{B^n}(-rE_B) \simeq j_* (p|_{B_n^*})_* j^* \mathcal{O}_{B^n}(-rE_B) \]
\[ \simeq j_* (p|_{B_n^*})_* \mathcal{O}_{B^n}(-rE_B)|_{B_n^*} \simeq j_* j^* T^r_{\Delta_n} \simeq T^r_{\Delta_n} \]
since, over \( X_n^* \), \( p \) is a smooth blow-up and thanks to lemma 1.4.

**Step 2. Arbitrary A.** If \( A \) is non trivial we write:
\[ \Phi((\det F^{[n]} \otimes k \otimes \mathcal{D}_A) \simeq \mathcal{R}p_* q^* (\det F^{[n]} \otimes k \otimes \mathcal{D}_A) \simeq \mathcal{R}p_* (q^* (\det F^{[n]} \otimes k \otimes q^* \mathcal{D}_A)) \]
\[ \simeq \mathcal{R}p_* (q^* (\det F^{[n]} \otimes k \otimes p^*(A \boxtimes \cdots \boxtimes A))) \]
\[ \simeq (\mathcal{R}p_* q^* (\det F^{[n]} \otimes k) \otimes (A \boxtimes \cdots \boxtimes A)) \]

where in the third isomorphism we used that \( q^* \mathcal{D}_A \simeq p^*(A \boxtimes \cdots \boxtimes A) \), and in the third we used projection formula. Now the formula follows immediately from the previous case. \( \square \)

**Corollary 1.9.** Let \( F \) be a vector bundle of rank \( r \) and \( A \) a line bundle on the smooth quasi-projective surface \( X \). Then
\[ \mathcal{R}p_* ((\det F^{[n]} \otimes k \otimes \mathcal{D}_A) \simeq \pi_* \left( T^r_{\Delta_n} \otimes \varepsilon_n^r \right) \otimes \mathcal{D}_A. \]

## 2 Multitors of pairwise diagonals in \( X^n \)

Let \( X \) be a smooth algebraic variety and let \( n \in \mathbb{N}, n \geq 2 \). In this section we will study multitors of the form \( \text{Tor}^q_{X^n}(\mathcal{O}_{\Delta_1}, \ldots, \mathcal{O}_{\Delta_n}) \), where \( \Delta_{ij} \) are pairwise diagonals in \( X^n \). This study will be useful in order to prove the resolutions of \( \mathfrak{S}_n \)-invariants of diagonal ideals of subsections 3.2 and 3.3.

### 2.1 A general formula for multitors

In this section we will prove a general formula for multitors \( \text{Tor}^q_{X^n}(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_k}) \) of structural sheaves of smooth subvarieties \( Y_i \) of a smooth algebraic variety \( M \).
Theorem 2.1. Let $M$ be a smooth algebraic variety and $Y_1, \ldots, Y_l$ be smooth subvarieties of $M$ such that the intersection $Z := Y_1 \cap \cdots \cap Y_l$ is smooth. Then:

$$
\text{Tor}_q^{\mathcal{O}_M}(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_l}) = \Lambda^q(\oplus_{i=1}^l N_{Y_i}|_Z/N_Z)^*,
$$

(2.1)

where $N_{Y_i}$ and $N_Z$ denote the normal bundles of $Y_i$ and $Z$ in $M$, respectively.

Proof. Let $x$ be a point of $Z$, and $U$ an affine open neighbourhood of $x$. Restricting $U$ if necessary, we can find generators $f_{j_1}, \ldots, f_{j_l}$ of $\mathcal{T}_U(U)$, such that $c_i = \text{codim } Y_i$. It is possible to find them since $Y_j$ is complete intersection in $U$; we can moreover find $g_1, \ldots, g_n$ generators of $\mathcal{T}_U(U)$, with $c = \text{codim } Z \leq \sum c_i = d$.

Denote simply with $g$ the vector $(g_1, \ldots, g_n) \in \mathcal{O}_M(U)^\oplus_n$, with $f$ the vector $(f_1, \ldots, f_c, \ldots, f_{j_l}) \in \mathcal{O}_M(U)^\oplus_{j_l}$ and with $f_j$ the vector $(f_{j_1}, \ldots, f_{j_l}) \in \mathcal{O}_M(U)^{c_j}$. Denote with $E$ the vector bundle $\mathcal{O}_Z^E$, with $F_j$ the bundle $\mathcal{O}_Z^{c_j}$ and with $F$ the bundle $\mathcal{O}_Z^E$. It is clear that, over $U$, $g$ defines a section of $E$, $f_j$ defines a section of $F_j$ and $f$ a section of $F$. We can identify the conormal bundle $N_{Y_i}^c$ with the restriction $E^c|_Z$ and $N_{Y_i}$ with $F_j^c|_Y$.

Step 1. Since all the varieties $Y_i$ and $Z$ are smooth, by the exactness of the conormal sequence, we can identify conormal bundles $N_{Y_i}^c$ and $N_Z^c$ with differentials in $\Omega_M^1$ vanishing over $Y_i$ and $Z$, respectively. Since both $f_j$ and $g_i$ are generators of $\mathcal{I}_Z(U)$, there is a $d \times c$-matrix $B \in \mathcal{M}_{d \times c}(\mathcal{O}_M(U))$ and a $c \times d$-matrix $B \in \mathcal{M}_{c \times d}(\mathcal{O}_M(U))$ such that $g = Bf$, $f = Ag$. This means that, taking differentials: $dg = (dB)f + Bdf$ and $df = (dA)g + Adg$. On points $y \in Z \cap U$ we have just: $dg = Bdf$, $df = Adg$ and $dg = BAdg$. Now, since $Z$, over $U$, is smooth and complete intersection of $(g_1, \ldots, g_n) \in \mathcal{O}_M(U)^\oplus_n$, we have that $[g_i]$ are a basis of $\mathcal{I}_Z(U)/\mathcal{I}_Z(U)^2$, that is $dg_i$ are a local frame for $N_{Y_i}^c$ over $Z \cap U$ and $df_j$ are a local frame for $N_{Y_i}^c$. Hence $dg_i$ are linearly independent in $\Omega_M^1|_Z(y) \supseteq N_{Y_i}^c(y)$ for any $y \in Z \cap U$. Now on $\Omega_M^1|_Z(y)$ we have the relation $dg = B(y)A(y)dg$, meaning that the matrix $B(y)A(y)$ takes linearly independent into linearly independent, which implies that for any $y \in Z \cap U$, the matrix $B(y)$ is surjective and $A(y)$ is injective. Hence, $B(z)$ and $A(z)$ have to be surjective and injective, respectively, in a neighbourhood of $x$. Restricting $U$, we can suppose that $A$ and $B$ are injective and surjective, respectively, in any point of $U$.

Step 2. Let $E, F_i, F, g, f, U$ built as in the previous step. The matrix $A$ allows to define an injective morphism of vector bundles over $U$:

$$
\begin{array}{cccc}
0 & E & \rightarrow & F_1 \oplus \cdots \oplus F_l & \rightarrow & Q & \rightarrow & 0
\end{array}
$$

whose cokernel we call $Q$. It is a locally free sheaf on $U$ of rank $d - c$. Note that $A$ takes the section $g$ into the section $f$, and hence defines an injective morphism of pairs $A: (E, g) \rightarrow (F, f)$. Since we are on an affine open set, the sequence splits; hence we have a morphism $p_E: F \rightarrow E$ such that $p_E \circ A = \text{id}_E$; the splitting yields an isomorphism: $F \rightarrow E \oplus Q$, given by $(p_E, p_Q)$. Under this isomorphism the section $f$ of $F$ is carried onto the section $g \oplus 0$ of $Q$, since $(p_E, p_Q)f = p_E f \oplus p_Q f = p_E Ag \oplus p_Q Ag = g \oplus 0$. Hence we have an isomorphism of pairs $(F, f) \simeq (E \oplus Q, g \oplus 0)$.

Step 3. The previous step yields an isomorphism of Koszul complexes: $K^\bullet(F, f) \simeq K^\bullet(E \oplus Q, g \oplus 0) \simeq K^\bullet(E, g) \otimes K^\bullet(Q, 0)$. Note that

$$
Q|_Z \simeq (\oplus_j F_j/E)|_Z \simeq (\oplus_j F_j/E)|_Z \simeq \oplus_j N_{Y_j}|_Z/N_Z.
$$

Hence:

$$
\text{Tor}_q^{\mathcal{O}_M}(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_l}) \simeq H^{-q}(K^\bullet(F, f)) \simeq H^{-q}(K^\bullet(E \oplus Q, g \oplus 0))
\simeq H^{-q}(K^\bullet(E, g) \otimes K^\bullet(Q, 0))
\simeq \bigoplus_{r+s=q} H^r(K^\bullet(E, g)) \otimes K^s(Q, 0)
\simeq H^q(K^\bullet(E, g)) \otimes \Lambda^q Q^*
\simeq \Lambda^q Q^*|_Z
\simeq \Lambda^q(\oplus_{i=1}^l N_{Y_i}|_Z/N_Z)^*,
$$

where in the fourth isomorphism we used the fact that $K(E \oplus Q, g \oplus 0)$ is a tensor product of $K(E, g)$ with $K(Q, 0)$, which is a complex of locally free sheaves with zero differentials.
Step 4. We obtained the wanted isomorphism locally. That these local isomorphisms glue to a global one is an easy exercise and we leave it to the reader.

**Notation 2.2.** Let \( k_1, \ldots, k_l \) be positive integers. Let \( M \) be an algebraic variety and \( F_1, \ldots, F_l \) coherent sheaves over \( M \). We denote with \( \text{Tor}^{k_1, \ldots, k_l}_q(F_1, \ldots, F_l) \) the multitor \( \text{Tor}^{O_1, \ldots, O_l}_q(F_1, \ldots, F_l, 1, \ldots, 1) \), where, for all \( i \), the factor \( F_i \) is repeated \( k_i \) times.

The product of symmetric groups \( S_{k_1} \times \cdots \times S_{k_l} \) obviously acts on multitors of the form \( \text{Tor}^{k_1, \ldots, k_l}_q(F_1, \ldots, F_l) \), defined above. Details of this action are described in [Scal99 Appendix B]. Therefore one can study them as \( S_{k_1} \times \cdots \times S_{k_l} \)-representations. We have the following.

**Proposition 2.3.** Let \( M \) be a smooth algebraic variety and \( Y_1, \ldots, Y_l \) locally complete intersection subvarieties of \( M \) such that the intersection \( Z = Y_1 \cap \cdots \cap Y_l \) is locally complete intersection. Then for \( k_1, \ldots, k_l \) positive integers we have, as \( S_{k_1} \times \cdots \times S_{k_l} \)-representations:

\[
\text{Tor}^{k_1, \ldots, k_l}_q(O_{Y_1}, \ldots, O_{Y_l}) \simeq \bigoplus_{q_1 + q_2 = q} \text{Tor}^{O_1, \ldots, O_l}_{q_1}(O_{Y_1}, \ldots, O_{Y_l}) \otimes \Lambda^{q_2} (\bigoplus_{i=1}^m N_{Y_i/M} \otimes \rho_{k_i}) ,
\]

where \( \rho_{k_i} \) is the standard representation of the symmetric group \( S_{k_i} \).

**Proof.** We solve locally, on adequate open affine subsets \( U_i \), the structural sheaves \( O_{Y_i} \) with Koszul complex \( K^* (F_i, s_i) \), where \( F_i \) is a vector bundle of rank codim \( U_i \), \( Y_i \) and \( s_i \) is a section of \( F_i \) transverse to the zero section. Then, over \( U = U_1 \cap \cdots \cap U_l \), we have:

\[
\text{Tor}^{k_1, \ldots, k_l}_q(O_{Y_1}, \ldots, O_{Y_l}) = H^{-q}((\bigotimes_{i=1}^l \otimes_{j=1}^{k_i} K^* (F_i, s_i)) = H^{-q}((\bigotimes_{i=1}^l K^* (F_i \otimes R_{k_i}, \sigma_{k_i} \otimes s_i))
\]

where \( R_{k_i} \simeq \mathbb{C}^{k_i} \) is the natural representation of \( S_{k_i} \), with canonical basis \( e_i \), and \( \sigma_{k_i} \) is its invariant element, that is \( \sigma_{k_i} = \sum_{\epsilon_i = 1}^{k_i} \epsilon e_i \in R_{k_i} \). Then, since \( K^* (F_i \otimes R_{k_i}, \sigma_{k_i} \otimes s_i) = K^* (F_i \otimes \rho_{k_i}) \otimes K^* (F_i, s_i) \) by [Scal15a Remark B.5], we have:

\[
\text{Tor}^{k_1, \ldots, k_l}_q(O_{Y_1}, \ldots, O_{Y_l}) = H^{-q}((\bigotimes_{i=1}^l K^* (F_i, s_i) \otimes \bigotimes_{i=1}^l K^* (F_i \otimes \rho_{k_i}, 0))
\]

\[
= H^{-q}((\bigotimes_{i=1}^l K^* (F_i, s_i) \otimes K^* (\bigotimes_{i=1}^l F_i \otimes \rho_{k_i}))
\]

\[
= \bigoplus_{q_1 + q_2 = q} \text{Tor}^{O_1, \ldots, O_l}_{q_1}(O_{Y_1}, \ldots, O_{Y_l}) \otimes \Lambda^{q_2} (\bigoplus_{i=1}^m N_{Y_i/X} \otimes \rho_{k_i})
\]

as \( \times_{i=1}^l S_{k_i} \)-representations. Now the open sets of the form \( U \) cover the algebraic variety \( M \). It is an easy exercise to prove that the local isomorphism shown above glue to give a global isomorphism over \( M \).

**Corollary 2.4.** Let \( M \) be a smooth algebraic variety and \( Y_1, \ldots, Y_l \) smooth subvarieties of \( M \) such that the intersection \( Z := Y_1 \cap \cdots \cap Y_l \) is smooth. Let \( k_1, \ldots, k_l \) positive integers. Then we have, as \( S_{k_1} \times \cdots \times S_{k_l} \)-representations:

\[
\text{Tor}^{k_1, \ldots, k_l}_q(O_{Y_1}, \ldots, O_{Y_l}) \simeq \Lambda^q ([\bigotimes_{i=1}^l N_{Y_i} | Z] / N_Z] \otimes \bigoplus_{i=1}^m N_{Y_i} \otimes \rho_{k_i}).
\]

### 2.2 Multitors of pairwise diagonals

In this subsection we apply the previous general formula to the case of pairwise diagonals. We are concerned with multitors of the form \( \text{Tor}^{O_1, \ldots, O_l}_q(O_{\Delta_1}, \ldots, O_{\Delta_l}) \), where \( X \) is a smooth algebraic variety, and where \( I_j \) are subsets of \( \{1, \ldots, n\} \) of cardinality 2 such that \( I_i \neq I_j \) if \( i \neq j \). It is therefore convenient to think of the multi-indexes \( I_j \) as _edges of a simple graph_. We refer to [Die10] for basic concepts in graph theory. More precisely, given \( l \) distinct multi-indexes \( I_1, \ldots, I_l \) of cardinality 2, we can build the simple graph \( \Gamma \), whose set of vertices \( V_\Gamma \) is defined as the set \( I_1 \cup \cdots \cup I_l \) and whose edges are \( E_\Gamma = \{ I_1, \ldots, I_l \}. \) All the vertices of the graph \( \Gamma \) are non-isolated, that is, they have degree greater or equal than 1. On the other hand, given a simple graph \( \Gamma \), such that its vertices \( V_\Gamma \) are a subset of \( \{1, \ldots, n\} \) and are non isolated, its edges are a set of \( l \) distinct cardinality-2 multi-indexes \( \{I_1, \ldots, I_l\} \) such that \( V_\Gamma = I_1 \cup \cdots \cup I_l. \) For such a graph \( \Gamma \), we denote with \( \Delta_\Gamma \) the intersection of diagonals \( \Delta_i, i \in \Gamma \) and with \( \text{Tor}_q(\Delta, \Gamma) \) the multitor \( \text{Tor}^{O_1, \ldots, O_l}_q(\Delta_1, \ldots, \Delta_l) \). The isomorphism class of this multitor does not depend on the order in which the edges \( I_j \) are taken; however, the order of the diagonals is important when dealing with permutation of factors in a multitor: therefore, in the following section, we will always consider \( l \)-uples of cardinality 2-multi-indexes \( \{I_1, \ldots, I_l\} \), ordered via the lexicographic order. 

---

9
Remark 2.5. Let $S$ be a simple graph. Let $v$ the number of vertices, $l$ the number of edges and $k$ the number of connected components. The number of independent cycles $c$ of the graph $S$ is given by $c = l - v + k$.

Remark 2.6. Let $X$ be a smooth algebraic variety of dimension $d$. Let $S$ be a simple graph without isolated vertices. The subvariety $\Delta_{S}$ of $X^S$, intersection of the distinct pairwise diagonals $\Delta_{S}$, $I \in E_{T}$ is smooth of codimension $d(v - k)$, where $v = |V_{I}|$. This fact, together the possibility of using formula (2.1), since all varieties $\Delta_{I}$, $I \in E_{T}$ and $\Delta_{\Gamma}$ are smooth, allows us to translate properties of the graph $S$ into properties of the multitor $T_{(\Delta, \Gamma)}$. In particular, it is clear that

- the pairwise diagonals $\Delta_{I}$, $I \in E_{T}$, intersect transversely in the subvariety $\Delta_{\Gamma}$ if and only if $d(v - k) = \text{codim}X_{S} \cdot \Delta_{\Gamma} = \sum_{I \in E_{T}} \text{codim}X_{S} \cdot \Delta_{I} = dl$, that is, if and only if $c = l - v + k = 0$, that is, \textit{if and only if the graph $S$ is acyclic}; in this case $\text{Tor}_{q}(\Delta, \Gamma) = 0$ for all $q > 0$.

- the sheaf $Q_{T_{\gamma}} := \left( \oplus_{I \in E_{F}} N_{\Delta_{I}} \big|_{\Delta_{\Gamma}} \right) / N_{\Delta_{\Gamma}}$ is a vector bundle over $\Delta_{\Gamma}$ of rank $dc$; hence $\text{Tor}_{q}(\Delta, \Gamma) = 0$ for $q > dc$.

For $A \subseteq \{1, \ldots, n\}$ denote with $\mathcal{S}(A)$ the symmetric group of the set $A$ and with $\rho_{A}$ its the standard representation. Let $\tilde{\mathcal{S}}_{T} := \text{Stab}_{\rho_{A}}(\Gamma)$ be the subgroup of $\mathcal{S}_{n}$ transforming the graph $\Gamma$ into itself. It is a subgroup of $\mathcal{S}(V_{T}) \times \mathcal{S}(\overline{V}_{T})$. Indicate with $\tilde{\mathcal{S}}_{T}$ the subgroup $\mathcal{S}(V_{T}) \cap \tilde{\mathcal{S}}_{T} \cap \mathcal{S}(\overline{V}_{T})$.

Suppose now that $\Gamma_{1}, \ldots, \Gamma_{k}$ are the connected components of the graph $\Gamma$. Let $S \Gamma_{1}, \ldots, S \Gamma_{k}$ be the partition of $\{1, \ldots, k\}$ induced by the equivalence relation defined by $i \sim j$ if and only if $\Gamma_{i}$, is isomorphic to $\Gamma_{j}$. Denote with $\tilde{\mathcal{S}}_{T}^{k}$ the subgroup $\mathcal{S}(S_{1}) \times \cdots \times \mathcal{S}(S_{k})$ of $\mathcal{S}_{k}$ and with $\tilde{\mathcal{S}}_{T} = \text{Stab}_{\rho_{A}}(\Gamma) \cap \mathcal{S}(V_{T})$, where $\mathcal{S}(V_{T})$ is naturally seen as a subgroup of $\mathcal{S}_{n}$. Then there is a split exact sequence

$$1 \longrightarrow \tilde{\mathcal{S}}_{T}^{1} \times \cdots \times \tilde{\mathcal{S}}_{T}^{k} \longrightarrow \tilde{\mathcal{S}}_{T} \longrightarrow \tilde{\mathcal{S}}_{T}^{1} \longrightarrow 1. \quad (2.2)$$

In other words, the subgroup $\tilde{\mathcal{S}}_{T}$ of the stabilizer $\mathcal{S}_{T}$ is a semi-direct product $(\tilde{\mathcal{S}}_{T}^{1} \times \cdots \times \tilde{\mathcal{S}}_{T}^{k}) \times \tilde{\mathcal{S}}_{T}^{1}$; the proof of this fact is analogous to [Sca15a, Lemma 2.12]. The full stabilizer $\mathcal{S}_{T}$ is isomorphic to

$$\mathcal{S}_{T} \simeq \tilde{\mathcal{S}}_{T} \times \mathcal{S}(\overline{V}_{T}) \simeq \left( (\tilde{\mathcal{S}}_{T}^{1} \times \cdots \times \tilde{\mathcal{S}}_{T}^{k}) \times \tilde{\mathcal{S}}_{T}^{1} \right) \times \mathcal{S}(\overline{V}_{T}).$$

The multitor $\text{Tor}_{q}(\Delta, \Gamma)$ is naturally $\mathcal{S}_{T}$-linearized. Consider the standard representation $\rho_{V_{T}}$ of $\mathcal{S}(V_{T})$. It can naturally be seen as a $\mathcal{S}(V_{T}) \times \mathcal{S}(\overline{V}_{T})$-representation, since the second factor acts trivially on $V_{T}$. Denote as $\rho_{T} := \text{Res}_{\mathcal{S}_{T}} \rho_{V_{T}}$ the restriction of $\rho_{V_{T}}$ to $\mathcal{S}_{T}$.

Notation 2.7. If $\Gamma$ is a subgraph of $K_{n}$ without isolated points and with $k$ connected components $\Gamma_{1}, \ldots, \Gamma_{k}$, we denote with $\nu : X^{k} \longrightarrow X^{V_{T}}_{\Gamma_{1}} \times \cdots \times X^{V_{T}}_{\Gamma_{k}}$ the immersion defined by embedding each factor $X$ in the factor $X^{V_{T}}_{\Gamma_{i}}$ diagonally; for any connected component $\Gamma_{i}$ we indicate with $\nu_{\Gamma_{i}} : X^{n} \longrightarrow X^{V_{T}}_{\Gamma_{i}}$ the projection onto the factors in $V_{T}$; the morphism $\pi_{\Gamma} : X^{n} \longrightarrow X^{V_{T}}_{\Gamma_{1}} \times \cdots \times X^{V_{T}}_{\Gamma_{k}}$ is defined as $\pi_{\Gamma} := \nu_{\Gamma_{1}} \times \cdots \times \nu_{\Gamma_{k}}$. Finally, if $F$ is a sheaf over $X^{k}$, we indicate with $F_{\Gamma}$ the sheaf over $X^{n}$ defined as $\pi_{\Gamma}^{*} F$. If $\Gamma$ has a single edge, say $E_{T} = \{I\}$, we will denote, for brevity’s sake $F_{T}$ with $F_{I}$.

Notation 2.8. We will indicate with $W_{T}$ and $q_{T}$ the representations of $\mathcal{S}_{T}$ defined by $W_{T} := \oplus_{I \in E_{T}} \rho_{I}$ and by the exact sequence

$$0 \longrightarrow \rho_{T} \longrightarrow W_{T} \longrightarrow q_{T} \longrightarrow 0,$$

respectively. It is clear that, if $\Gamma_{i}$ are the connected components of the graph $\Gamma$, the vector space $\oplus_{i=1}^{k} q_{\Gamma_{i}}$ is naturally a $\mathcal{S}_{T}$-representation isomorphic to $q_{T}$.

Notation 2.9. Let $\Gamma$ be a simple graph without isolated vertices, and let $\gamma$ be an oriented cycle in $\Gamma$. If $\gamma$ is an edge of $\gamma$, $I = \{i, j\}$, $i < j$, then we define the sign $\eta_{I, \gamma}$ to be $+1$ if the vertex $j$ is the immediate successor of $i$ in $\gamma$, and $\eta_{I, \gamma} = -1$ otherwise.

Remark 2.10. The representation $q_{T}$ is generated over $\mathcal{C}$ by independent cycles. More precisely, we can consider $q_{T}$ as a subrepresentation of $\oplus_{I \in E_{T}} R_{I}$, where $R_{I} \simeq \mathcal{C}^{I}$, with basis $\epsilon_{i}$, $i \in I$, is the natural $\mathcal{S}(I)$-representation. If $I = \{i, j\}$, $i < j$, let’s indicate with $e_{I}$ the vector $e_{j} - e_{i}$. If $\gamma$ is an oriented cycle in $\Gamma$, then we can consider the vector $e_{\gamma} := \sum_{i \in \gamma} \eta_{i, \gamma} e_{I}$ in $W_{T} = \oplus_{I \in E_{T}} \rho_{I} \subseteq \oplus_{I \in E_{T}} R_{I}$. Now $e_{\gamma_{1}}, \ldots, e_{\gamma_{c}}$ are independent cycles in $\Gamma$ if and only if $e_{\gamma_{1}}, \ldots, e_{\gamma_{c}}$ are independent in $W_{T}$ and project to a basis in $q_{T}$. Hence we can identify $q_{T}$ with the subspace $\mathcal{C} e_{\gamma_{1}} \oplus \cdots \oplus \mathcal{C} e_{\gamma_{c}}$ of $W_{T}$.
Remark 2.11. Consider the vector bundles $\mathbb{H}_i TX \otimes q_r$, and $\mathbb{H}_i \Omega X \otimes q_r$, over $X^k$. They are naturally $G$-equivariant; indeed the $G$ acts on the variety $X^k$ via the surjective composition $G \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}^T$; this action lifts to the tangent and cotangent bundle. On the other hand the action of $G$ over the representations $q_r$, is induced by its action on $q_r$.

We have the following interpretation of a multitor of pairwise diagonals $\text{Tor}_q(\Delta, \Gamma)$ in terms of data attached to the graph $\Gamma$.

**Proposition 2.12.** As $G$-sheaves, we have that $\text{Tor}_q(\Delta, \Gamma) \simeq \Lambda^q(\mathbb{H}_i TX \otimes q_r)^{\Gamma}$.

**Proof.** Consider first the case in which $\Delta$ is connected. Then $\Delta^r \simeq X \times X^T$. It is then sufficient to prove the proposition when $n = v$, since the case $n > v$ can then be obtained by flat base change. The morphism $i_{\Gamma} : X \longrightarrow X^n$ is an isomorphism over the image $\Delta^r$. Hence pulling back the exact sequence over $\Delta^r$

\[
0 \longrightarrow N_{\Delta^r} \longrightarrow \bigoplus_{i \in E^r} N_i |_{\Delta^r} \longrightarrow Q_{\Gamma} \longrightarrow 0
\]

via $i_{\Gamma}$, we obtain over $X$, by definition of $\rho r$ and $q_r$, a $G$-equivariant exact sequence

\[
0 \longrightarrow TX \otimes \rho r \longrightarrow TX \otimes W_r \longrightarrow TX \otimes q_r \longrightarrow 0.
\]

Hence the vector bundle $Q_{\Gamma} = \bigoplus_{i \in E^r} N_i \big/ N_{\Delta^r}$ over $\Delta^r$ is isomorphic to $i_{r\ast}(TX \otimes q_r) = (TX \otimes q_r)^{\Gamma}$ and we conclude by formula (2.1).

Consider now a general $\Gamma$. We have that $\Delta^r \simeq X^k \times X^T$: it is then sufficient to consider the case $n = v$ for the same reason as above. In this case $i_{\Gamma} : X^k \longrightarrow X^n$ is an isomorphism over the image $\Delta^r$. Consider the connected components $\Gamma_i$, $i = 1, \ldots, k$, of the graph $\Gamma$. We have $N_{\Delta^r} \simeq \oplus_i N_{\Delta^r} |_{\Delta^r}$, since the partial diagonals $\Delta^r_i$ intersect transversely. For each $i = 1, \ldots, k$, we have sequences

\[
0 \longrightarrow N_{\Delta^r_i} \longrightarrow \bigoplus_{i \in E^r_i} N_i |_{\Delta^r_i} \longrightarrow Q_{\Gamma_i} \longrightarrow 0.
\]

Hence over $\Delta^r_i$ we have an exact sequence

\[
0 \longrightarrow N_{\Delta^r_i} \longrightarrow \bigoplus_{i \in E^r_i} N_i |_{\Delta^r_i} \longrightarrow \bigoplus_i Q_{\Gamma_i} \longrightarrow 0
\]

and hence pulling everything back to $X^k$ we get the exact sequence

\[
0 \longrightarrow \mathbb{H}_i TX \otimes \rho r_i \longrightarrow \mathbb{H}_i TX \otimes W_{\Gamma_i} \longrightarrow \mathbb{H}_i TX \otimes q_r \longrightarrow 0.
\]

Hence, as $G$-vector bundles over $\Delta^r$, $Q_{\Gamma} \simeq i_{r\ast}(\mathbb{H}_i TX \otimes q_r) \simeq (\mathbb{H}_i TX \otimes q_r)^{\Gamma}$, and we conclude by formula (2.1). \qed

**Notation 2.13.** Let $\Gamma$ a simple graph without isolated vertices, such that $V^r \subseteq \{1, \ldots, n\}$. If $J \subseteq \{1, \ldots, n\}$, $|J| = 2$, and $J \notin E^r$, we will indicate with $\Gamma \cup J$ the graph obtained by $\Gamma$ adding the edge $J$, that is, the graph defined by $V_{\Gamma \cup J} := V^r \cup J, E_{\Gamma \cup J} := E^r \cup \{J\}$.

**Proposition 2.14.** Let $X$ a smooth algebraic variety of dimension $d$. Let $\Gamma$ be a simple graph without isolated vertices such that $V^r \subseteq \{1, \ldots, n\}$ and with edges $E^r = \{I_1, \ldots, I_l\}$. Let $J \subseteq \{1, \ldots, n\}$, $|J| = 2$, and $J \notin E^r$. Identifying $\text{Tor}_q(\Delta, \Gamma)$ with $\text{Tor}_q(\Delta_{I_1}, \ldots, \Delta_{I_l}, \mathcal{O}_X^{\ast})$, the $G \cap \mathcal{G}_{\Gamma \cup J}$-equivariant map

\[
i_{r\ast}: \text{Tor}_q(\Delta, \Gamma) \longrightarrow \text{Tor}_q(\Delta, \Gamma \cup J),
\]

induced by the restriction $\mathcal{O}_X^{\ast} \longrightarrow \Delta^r$, can be identified with

- the restriction $\Lambda^q(Q^r_{\Gamma}) \longrightarrow \Lambda^q(Q^r_{\Gamma}) |_{\Delta_{\Gamma \cup J}}$ if $J \not\subseteq V^r$,

- the natural injection $\Lambda^q(Q^r_{\Gamma}) \longrightarrow \Lambda^q(Q^r_{\Gamma \cup J})$, induced by the injection of vector bundles $Q^r_{\Gamma} \longrightarrow Q^r_{\Gamma \cup J}$ over $\Delta^r$, if $J \subseteq V^r$.

**Proof.** Let $x$ be a point in $\Delta^r$. On a small affine open neighbourhood of $x$, we can find vector bundles $F_I$, $I \in E^r$, of rank $d$, such that $\Delta^r_I$ are the zero locus of sections $s_I$ of $F_I$ transverse to the zero section. The
It is sufficient to consider the case $\rho = \ast$. The structural sheaves $O_{\Delta}$ can then be resolved with Koszul complexes $K^{\bullet}(F_t, s_t)$. Denote with $F_t$ the vector bundle $\bigoplus_{t \in E_t} F_t$ and with $s_t$ the section $\bigoplus_{t \in s_t}$. Therefore

$$\text{Tor}_q(\Delta, \Gamma) \simeq H^{-q}(\bigoplus_{t \in E_t} K^{\bullet}(F_t, s_t)) \simeq H^{-q}(K^{\bullet}(F_t, s_t)) \simeq \Lambda^q(Q^*_{\Gamma})$$

The morphism $\text{Tor}_q(\Delta, \Gamma) \rightarrow \text{Tor}_q(\Delta, \Gamma \cup J)$ of the statement is induced by the injection of vector bundles $i_J : F_{\Gamma}^* = \bigoplus_{t \in E_{\Gamma}} F_t^* \rightarrow \bigoplus_{t \in E_{\Gamma \cup J}} F_t^* = F_{\Gamma \cup J}^*$; one then sees, as in the proof of theorem 2.1, that the injection $i_J$ induces the natural map:

$$Q^*_\Gamma = [\bigoplus_{t \in E_{\Gamma}} N_{\Delta_t}/N_{\Delta_t}]^* \rightarrow [(\bigoplus_{t \in E_{\Gamma}} N_{\Delta_t} \oplus N_{\Delta_t})]_{\Delta \cup J}/N_{\Delta \cup J}]^* = Q^*_{\Gamma \cup J}.$$

Now if $J \not\subseteq V_{\Gamma}$, we just have that $N_{\Delta \cup J} \simeq N_{\Delta} \oplus N_{J}$, and hence $Q^*_{\Gamma \cup J} \simeq Q^*_{\Gamma}$ and the previous map is the restriction; on the other hand, if $J \subseteq V_{\Gamma}$, $Q^*_{\Gamma \cup J}$ is a vector bundle over $\Delta \cup J = \Delta$ and the previous map is the natural injection. This proves the statement.

**Proposition 2.15.** Let $K_v$ the complete graph on $v$ vertices and suppose that $V_{K_v} \subseteq \{1, \ldots, n\}$. Then, as $\tilde{\mathcal{S}}_{K_v}$-representations,

$$\text{Tor}_q(\Delta, K_v) \simeq \Lambda^q(\Omega^1 \otimes \Lambda^2 \rho_v)_{K_v},$$

where $\rho_v$ denotes the standard representation of $\tilde{\mathcal{S}}_{K_v} \simeq \mathcal{S}_v$.

**Proof.** It is sufficient to consider the case $v = n$, since the case $v < n$ follows by flat base change. By proposition 2.12 it is sufficient to prove that the representation $q_{K_v}$ is isomorphic to $\Lambda^2 \rho_n$. Since $\rho_n \simeq \rho_n$, it is sufficient to prove that the representation $W_{K_v} = \bigoplus_{|I| = 2, I \subseteq \{1, \ldots, n\}} \rho_I \simeq \rho_n \oplus \Lambda^2 \rho_n$. In order to achieve this, it is sufficient to prove that the characters of the two representations are the same. Let $i$ the $n$-uple $(i_1, \ldots, i_n)$, with $\sum_j j i_j = n$. Denote with $C_i$ the conjugacy class in $\mathcal{S}_n$ of permutations having $i_j$ $j$-cycles.

By Frobenius formula [FH91] exercise 4.15, the character of $\rho_n \oplus \Lambda^2 \rho_n$ is valued, on $C_i$:

$$(\chi_{\rho_n} + \chi_{\Lambda^2 \rho_n})(C_i) = i_1 - 1 + \frac{1}{2}(i_1 - 1)(i_1 - 2) - i_2 = \binom{i_1}{2} - i_2.$$  

On the other hand, a basis of the representation $W_{K_v}$ is given by vectors $e_J$, $J \subseteq \{1, \ldots, n\}$, $|J| = 2$. If $I = \{i, j\}$, we have that $(i,j)e_I = -e_I$ and $(i,j)e_h = e_jh$ if $h \not\in I$. Hence, any cycle $\gamma_j$ of length $j \geq 3$ will act with trace zero. The 1-cycles $\gamma_1 \ldots \gamma_i$ act trivially on a $\binom{i_1}{2}$-dimensional space, and hence with trace $\binom{i_1}{2}$. Since the cycles are disjoint, the traces add up. Hence

$$\chi_{W_{K_v}}(C_i) = \binom{i_1}{2} - i_2 = (\chi_{\rho_n} + \chi_{\Lambda^2 \rho_n})(C_i).$$

**2.3 The cases $n = 3, 4$.**

We analyse here in detail the representations $q_{\Gamma}$ with $\Gamma$ a non-acyclic subgraph of $K_n$, $n = 3, 4$, with non isolated vertices.

**Non acyclic graphs.** There is a unique non-acyclic subgraph of $K_3$ without isolated vertices, the complete graph $K_3$ itself. On the other hand, up to isomorphism, the non-acyclic subgraphs of $K_4$ without isolated vertices are the following:

- for $l = 3$, the complete graph $K_3$ with 3 vertices; its stabilizer is the group $\mathcal{S}_3$;
- for $l = 4$, the 4-cycle $C_4$ and the graph $K_3 \cup J$, obtained by $K_3$ adding an edge $J \not\subseteq E_{K_3}$; since $C_4$ has dihedral symmetry, $\tilde{\mathcal{S}}_{C_4}$ is isomorphic to the dihedral group $D_4$; on the other hand, adding an edge to the graph $K_3$ reduces its symmetries to $\mathcal{S}_2$;
- for $l = 5$ the graph $C_4 \cup L$, obtained adding an edge $L \not\subseteq E_{C_4}$ to the 4-cycle $C_4$; its stabilizer is $\tilde{\mathcal{S}}_{C_4 \cup L} \simeq \mathcal{S}_2 \times \mathcal{S}_2$;
for $l = 6$, the complete graph $K_4$; its stabilizer is the full symmetric group $S_4$.

The non-acyclic subgraphs of $K_4$ without isolated vertices are represented in the figure below.

![Graphs](image)

**Classification of representations $q_1$.** In order to classify the representations $q_1$, we need the following lemma

**Lemma 2.16.** Let $n = r + s$, with $r, s \in \mathbb{N} \setminus \{0\}$. Consider the symmetric groups $S_{r+s}$, $S_r$, $S_s$. Then, as $S_r \times S_s$-representations, we have $r_{r+s} \simeq r_r \oplus r_s \oplus 1$. As a consequence, if $n = \sum_{i=1}^t k_i$, as $S_{k_1} \times \cdots \times S_{k_t}$-representations, we have $r_{k_1+\cdots+k_t} \simeq r_{k_1} \oplus \cdots \oplus r_{k_t} \oplus 1^{-1}$, where $1^{-1}$ is the $(r-1)$-dimensional trivial representation.

**Proof.** The natural $S_n$-representation $R_n$ splits as $R_n \simeq R_r \oplus R_s$ when seen as as $S_r \times S_s$-representation. The inclusion $r_{r+s} \hookrightarrow R_n = R_r \oplus R_s$ is $S_r \times S_s$-equivariant. Then, as $S_r \times S_s$-representations, $R_n \simeq R_r \oplus R_s = (r_r \oplus 1) \oplus (r_s \oplus 1) \simeq r_r \oplus r_s \oplus 1$. Hence the statement: $r_{r+s} \simeq r_r \oplus r_s \oplus 1$. \hfill $\Box$

**Remark 2.17.** The dihedral group $D_4$, generated by the reflection $\sigma$ and the rotation $\rho$, has 4 finite dimensional irreducible representations: the trivial, the standard representation $\theta$, the determinantal $\det := \det \theta$, the linear $\ell(1, -1, -1, -1)$, the linear $\ell(-1, 1, -1, -1)$. A description and a character table for these representation is given in.

We already know, by proposition 2.16, that $q_{K_3} \simeq \Lambda^2 \rho_3 \simeq \varepsilon$ and that $q_{K_4} \simeq \Lambda^2 \rho_4 \simeq \rho_4 \otimes \varepsilon$. As for the remaining representations $q_\Gamma$ for a subgraph of $K_4$ without isolated vertices we have the following.

- $\Gamma = K_3 \cup J$. Up to isomorphism we can think that $E_{K_3 \cup J} = \{\{1\}, \{1\}, \{2\}, \{3\}, \{3\}, \{4\}, \{4\}\}$. Hence $S_{E_{K_3 \cup J}} \simeq S(1, 2)$ and $W_{E_{K_3 \cup J}} = \rho_{12} \oplus \rho_{13} \oplus \rho_{23} \oplus \rho_{14}$. The character $\chi_{W_{E_{K_3 \cup J}}}$ is easily $(4, 0)$ according to the conjugacy classes of $1, 12$; but $(1, 4) = (1, -1) + (1, -1) + (1, 0) = 2\chi + 2\chi$. Hence $\chi_{q_{K_3 \cup J}} = \chi_{W_{E_{K_3 \cup J}}} - \chi_{v_{K_3 \cup J}} = 2\chi + 2\chi - 2\chi = \chi$ by lemma 2.16. Hence $q_{K_3 \cup J} \simeq \varepsilon$.

- $\Gamma = C_4$. Up to isomorphism, we can think that $E_{C_4} = \{\{1\}, \{1\}, \{2\}, \{3\}, \{3\}, \{4\}, \{4\}\}$. We easily have that the stabilizer $S_{C_4}$ is isomorphic to the dihedral group $D_4$, where the reflection $\sigma$ and the rotation $\rho$ are identified with $\sigma = (24)$ and $\rho = (1234)$, for example. Then $\chi_{W_{C_4}}$ according to conjugacy classes, $1, \sigma, \sigma \rho, \rho, \rho^2$, is given by $\chi_{W_{C_4}} = (4, 0, 2, 0, 0)$, and hence $W_{C_4}$ is isomorphic to $\det \oplus \ell(1, -1, -1) \oplus \theta$. Computing characters we get that $\chi_{p_{C_4}} = (3, 1, -1, -1, -1)$ and hence $\rho_{C_4}$ is isomorphic to $\ell(1, -1, -1, -1)$. Hence $q_{C_4} \simeq \det$ as $D_4$-representation.

- $\Gamma = C_4 \cup L$. Up to isomorphism, suppose that the graph $C_4 \cup L$ has edges $E_{C_4 \cup L} = \{\{1\}, \{1\}, \{2\}, \{2\}, \{3\}, \{3\}, \{4\}, \{4\}\}$, so that $S_{E_{C_4 \cup L}} \simeq S(1, 3) \times S(2, 4)$. Then, according to conjugacy classes $1, (13), (24), (13)(24)$, the character $\chi_{W_{C_4 \cup L}}$ is given by $(5, -1, -1, -1)$. By lemma 2.16 $\rho_{C_4 \cup L} \simeq \text{Res}_{S_{E_{C_4}} \times S_{E_2}}^{S_{E_{C_4 \cup L}}} \rho_5 \simeq (\varepsilon \oplus 1) \oplus (1 \otimes \varepsilon) \oplus 1$ and hence $\chi_{q_{C_4 \cup L}} = (3, 1, -1, -1, -1)$. Hence $\chi_{q_{C_4 \cup L}} = (2, -2, 0, 0)$, which yields $q_{C_4 \cup L} \simeq (\varepsilon \oplus 1) \oplus (\varepsilon \otimes \varepsilon)$.

**Multitors as $\mathfrak{g}$-representations.** For $n = 3, 4$, all non-acyclic graph are connected, hence the corresponding $\text{Tor}_n(\Delta, \Gamma)$ is isomorphic to $\Lambda^q(\Omega_X^1 \otimes q_\Gamma)$. We decompose the exterior power $\Lambda^q(\Omega_X^1 \otimes q_\Gamma)$ according to the Schur-functor decomposition

$$
\Lambda^q(\Omega_X^1 \otimes q_\Gamma) = \bigoplus_{\lambda} S^\lambda \Omega_X^1 \otimes S^{\lambda'} q_\Gamma
$$
where the direct sum is taken on partitions $\lambda$ of $q$ such that $\lambda$ has at most $\dim X$ rows and at most $\dim q\Gamma$ columns [FH91, Exercise 6.11]. In the next section it will be important to determine $\mathfrak{S}_\Gamma$-invariants of the previous exterior power. We have the following

**Lemma 2.18.** Let $n = 4$, let $\Gamma_1$ be a graph of the kind $C_4 \cup L$ and let $K_4$ the complete graph with 4-vertices. If $S^\lambda q\Gamma_1$ has nontrivial $\mathfrak{S}_\Gamma_1$-invariants, or if $S^\lambda qK_4$ has nontrivial $\mathfrak{S}_4$-invariants, then the partition $\lambda$ appears in the following table, which indicates as well the dimension of the space of invariants.

| $\lambda$ | (2) | (3) | (4) | (3,1) | (2,2) | (3,1,1) | (6) | (5,1) | (4,2) | (2,2,2) |
|-----------|-----|-----|-----|-------|-------|---------|-----|-------|-------|---------|
| $\dim(S^\lambda q\Gamma_1)^{\mathfrak{S}_\Gamma}$ | 2   | 0   | 3   | 1     | 1     | 0       | 4   | 2     | 2     | 0       |
| $K_4$     | 1   | 1   | 2   | 0     | 1     | 1       | 3   | 1     | 2     | 1       |

Table 1

**Proof.** The proof is straightforward in the case of the graph $\Gamma_1$. In the case of the complete graph $K_4$, we decompose $S^\lambda qK_4 \simeq S^\lambda (\rho_4 \otimes \varepsilon) \simeq S^\lambda V^{2,1,1}$ into irreducible $\mathfrak{S}_4$-representations via the script in **GAP** [GAP15] indicated in [mof].

3 Invariants of diagonal ideals for low $n$.

Let $X$ be a smooth algebraic variety. The ideal $I_{\Delta_n}$ of the big diagonal $\Delta_n$ is the intersection $I_{\Delta_n} = \cap_{I \subseteq \{1, \ldots, n\}, |I| = 2} I_{\Delta_I}$ of ideals of pairwise diagonals $\Delta_I$, $I \subseteq \{1, \ldots, n\}$, $|I| = 2$: it is then isomorphic to the kernel of the natural morphism

$$O_X^n \longrightarrow \bigoplus_{I \subseteq \{1, \ldots, n\}, |I| = 2} O_{\Delta_I}.$$  \hspace{1cm} (3.1)

Hence it is useful to consider, for each multi-index $I$ of cardinality 2, a right resolution $K_I^\bullet$ of the ideal sheaf $I_{\Delta_I}$:

$$K_I^\bullet : 0 \longrightarrow O_X^n \longrightarrow O_{\Delta_I} \longrightarrow 0,$$

concentrated in degree 0 and 1: indeed, the first nontrivial map of the $\mathfrak{S}_n$-equivariant complex $\bigotimes_I K_I^\bullet$ is indeed exactly (3.1); here the order in which the tensor product is taken is always the lexicographic order on the cardinality 2-multi-indexes. However, the complex $\bigotimes_I K_I^\bullet$ is not exact, and in order to deal with this problem, it is better to consider the derived tensor product of complexes

$$\bigotimes_I^L K_I^\bullet.$$

Indeed, let $r = n(n+1)/2$ and consider the spectral sequence

$$E_1^{p,q} := \bigoplus_{i_1 + \cdots + i_r = p} \text{Tor}_{-q}(K_{i_1}^{i_2}, \cdots, K_{i_r}^{i_1}).$$  \hspace{1cm} (3.2)

It abuts to the cohomology $H^{p+q}(\bigotimes_I^L K_I^\bullet)$ and the term $E_2^{0,0}$ is clearly isomorphic to the ideal $I_{\Delta_n}$.

**Remark 3.1.** Since $I_{\Delta_I}$ are sheaves and the complexes $K_I^\bullet$ are their resolutions, $H^{p+q}(\bigotimes_I^L K_I^\bullet) = \text{Tor}_{-p-q}(I_{\Delta_{i_2}}, \cdots, I_{\Delta_{i_r}}) = 0$ for $p + q > 0$. Consequently, the abutment of the spectral sequence is zero for $p + q > 0$.

We plan to get information on the sheaf of invariants $I_{\Delta_n}^{\mathfrak{S}_n} = (E_2^{0,0})^{\mathfrak{S}_n}$ from the vanishing of the abutment in positive degree and from studying the spectral sequence of invariants in detail.
3.1 The comprehensive $\mathfrak{S}_n$-action on the spectral sequence $E_1^{p,q}$.

We will here briefly explain the action of the symmetric group $\mathfrak{S}_n$ on the derived tensor product $\bigotimes_I^L K_i^\ast$ or, equivalently, on the spectral sequence $E_1^{p,q}$. This is analogous to what done in [Sca09] for the derived tensor power of a complex of sheaves $C^\ast$. The point here is that, when considering the multitors Tor$_{-q}(K_{i_1}^\ast, \ldots, K_{i_m}^\ast, n)$, the terms $K_i^\ast$ are not just sheaves, but terms of a complex $K_i^\ast$. In the following remark we will recall what we explained in detail in [Sca09] Section 4.1 and [Sca09] Appendix B.

Remark 3.2. Let $C_i^\ast, C_2^\ast$ complexes of sheaves over a variety $M$. If we have a tensor product of complexes of sheaves $C_i^\ast \otimes C_2^\ast$ the permutation of factors $\tau_{12}$: $C_i^\ast \otimes C_2^\ast \rightarrow C_2^\ast \otimes C_i^\ast$ is a morphism of complexes if and only if $\tau_{12}$ acts on the term of degree $h$, that is the term $(C_i^\ast \otimes C_2^\ast)_h := \oplus_{i+j=h} C_i^\ast \otimes C_j^\ast$, exchanging the terms $C_i^\ast \otimes C_j^\ast \rightarrow C_j^\ast \otimes C_i^\ast$ and twisting by the sign $(-1)^i$. The same argument can be applied to a tensor product of complexes $C_i^\ast \otimes C_j^\ast \otimes \ldots \otimes C_n^\ast$. Indeed, in order to understand how a general permutation of factors operate on a tensor product of complexes, it is sufficient to understand how a consecutive transposition acts, and this is completely analogous to the case $r = 2$.

If now we want to understand the effect of permuting factors in a derived tensor product $C_i^\ast \otimes \ldots \otimes C_n^\ast$, we have to resolve each of the complexes $C_i^\ast$ with a complex of locally free $R_i^\ast$ (at least locally), and apply the previous reasoning to $R_i^\ast \otimes \ldots \otimes R_i^\ast$. To be more explicit, at the level of spectral sequences, consider a consecutive transposition $\tau_{j,j+1}$ and consider the spectral sequences

\[
E_1^{p,q} = \oplus_{i_1+\ldots+i_m=p} \text{Tor}_{-q}(C_{i_1}^\ast, \ldots, C_{i_m}^\ast)
\]

\[
E_1^{p,q} = \oplus_{i_1+\ldots+i_m=p} \text{Tor}_{-q}(C_{i_1}^\ast, \ldots, C_{i_m}^\ast)
\]

abutting to $H^{p+q}(C_1^\ast \otimes \ldots \otimes L C_i^\ast \otimes \ldots \otimes L C_n^\ast)$ and $H^{p+q}(C_1^\ast \otimes \ldots \otimes L C_i^\ast \otimes \ldots \otimes L C_n^\ast)$. The consecutive transposition $\tau_{j,j+1}$ induces an isomorphism $E_1^{p,q} \rightarrow E_1^{p,q}$ and hence isomorphisms

\[
\tau_{j,j+1} : \text{Tor}_{-q}(C_{i_1}^\ast, \ldots, C_{i_j}^\ast, C_{i_{j+1}}^\ast, \ldots, C_{i_n}^\ast) \rightarrow \text{Tor}_{-q}(C_{i_1}^\ast, \ldots, C_{i_{j+1}}^\ast, C_{i_j}^\ast, \ldots, C_{i_n}^\ast).
\]

Now, considering the $C_j^\ast$ just sheaves, and not as terms of a complex $C_j^\ast$, one has the standard permutation of factors in a multitor

\[
\tau_{j,j+1} : \text{Tor}_{-q}(C_{i_1}^\ast, \ldots, C_{i_j}^\ast, C_{i_{j+1}}^\ast, \ldots, C_{i_n}^\ast) \rightarrow \text{Tor}_{-q}(C_{i_1}^\ast, \ldots, C_{i_{j+1}}^\ast, C_{i_j}^\ast, \ldots, C_{i_n}^\ast).
\]

We proved in [Sca09] section 4.1 that the two isomorphisms $\tau_{j,j+1}$ and $\tau_{j,j+1}$ are related by the sign

\[
\tau_{j,j+1} = (-1)^{i_j+i_{j+1}} \tau_{j,j+1}.
\]

Remark 3.3. When we say that the complex $K_1^\ast \otimes \ldots \otimes L K_{n-1,n}$ is $\mathfrak{S}_n$-equivariant, what we mean is that $\mathfrak{S}_n$ acts up to permutation of factors. More precisely, we can interpret the $\mathfrak{S}_n$-action in the following way. For brevity’s sake, denote with $m = n(n+1)/2$ and set $E_K := \{J_1, \ldots, J_m\}$ in lexicographic order. Consider the monomorphism $\theta : \mathfrak{S}_n \rightarrow \mathfrak{S}(E_K)$ induced by the natural action of $\mathfrak{S}_n$ on $E_K$: if $\sigma \in \mathfrak{S}_n$, we will briefly indicate with $\tilde{\sigma}$ its image in $\mathfrak{S}(E_K)$. Denote with $\Delta \mathfrak{S}_n$ the subgroup of $\mathfrak{S}_n \times \mathfrak{S}(E_K)$ given by the set of couples $(\sigma, \tilde{\sigma})$, such that $\sigma \in \mathfrak{S}_n$. The complex $K_{i_1}^\ast \otimes \ldots \otimes L K_{i_m}^\ast$ is now $\Delta \mathfrak{S}_n$-equivariant; any element $(\sigma, \tilde{\sigma})$ in $\Delta \mathfrak{S}_n$ acts via a composition

\[
K_{i_1}^\ast \otimes \ldots \otimes L K_{i_m}^\ast \xrightarrow{\lambda_\sigma} \sigma^*(K_{i_1}^\ast \otimes \ldots \otimes L K_{i_m}^\ast) \xrightarrow{\lambda_{\tilde{\sigma}}^{-1}} \sigma^*(K_{i_1}^\ast \otimes \ldots \otimes L K_{i_m}^\ast)
\]

where $\lambda_\sigma$ is the geometric action and is induced by isomorphisms

\[
K_{i_1}^\ast \otimes \ldots \otimes L K_{i_m}^\ast \rightarrow \sigma^*(K_{i_1}^\ast \otimes \ldots \otimes L K_{i_m}^\ast) \simeq \sigma^*(K_{i_1}^\ast \otimes \ldots \otimes L K_{i_m}^\ast),
\]

while $\lambda_{\tilde{\sigma}}^{-1}$ is the permutation of factors induced by $\tilde{\sigma}^{-1}$: it operates the same way as the permutations described in remark 3.2. At the level of the spectral sequence (3.2) and in the identification of $\text{Tor}_{-q}(K_{i_1}^\ast, \ldots, K_{i_m}^\ast)$ with $\text{Tor}_{-q}(\Delta, \sigma(\Gamma))$ for some subgraph $\Gamma$ of $K_n$ without isolated vertices, the action of an element $(\sigma, \tilde{\sigma})$ is expressed through compositions of isomorphisms

\[
\text{Tor}_{-q}(\Delta, \Gamma) \xrightarrow{\lambda_\sigma} \sigma^*\text{Tor}_{-q}(\Delta, \sigma(\Gamma)) \xrightarrow{\lambda_{\tilde{\sigma}}^{-1}} \sigma^*\text{Tor}_{-q}(\Delta, \sigma(\Gamma))
\]
where \( \lambda_\sigma \) is described by the geometric action seen throughout subsection 4.2 and where \( \lambda_{a-1} \) is the derived permutative action described in remark 3.2 in other words, after formula 3.3 \( \lambda_{a-1} \) operates with the sign \( \varepsilon_{E_{pq}}(\delta^{-1}) = \varepsilon_{E_{pq'}}(\delta) \), where \( \delta \) is naturally seen in \( \mathcal{S}(E_{\sigma'}(\Gamma)) \) and where \( \varepsilon_{E_{pq}}(\delta^{-1}) \) is the alternating representation of \( \mathcal{S}(E_{\sigma'}(\Gamma)) \). In particular, if \( \sigma \in \mathcal{S}_r \), for the comprehensive \( \mathcal{S}_r \)-action, we have the isomorphism of \( \mathcal{S}_r \)-representations:

\[
\text{Tor}_{-q}(\Delta, \Gamma) \simeq \Lambda^q(\mathbb{P}^k_{1,1} \otimes \Omega_1^1 \otimes \mathcal{F}_r)_{|\Gamma} \otimes \mathfrak{Res}_{\Gamma} \varepsilon_{\mathcal{F}_r}.
\]

From now on, we will omit talking about the group \( \mathcal{S}_n \) and, for brevity’s sake, when considering the \( \mathcal{S}_n \)-action on the derived tensor product \( K^*_{1,2} \otimes^L \cdots \otimes^L K^*_{n-1,n} \), we will always tacitly intend the \( \Delta_{\mathcal{S}_n} \)-action explained here above.

**Remark 3.4.** Denote with \( \mathcal{G}_n \) the set of subgraphs of the complete graph \( K_n \) without isolated vertices and with \( l \) edges. We can form \( \mathcal{S}_n \)-equivariant complexes of \( \mathcal{O}_X \)-modules \( (\mathcal{I}^*_q, \mathcal{O}^*_q) \) on \( X^n \) by setting \( \mathcal{I}^*_0 := \mathcal{O}_X \), \( \mathcal{I}^*_q := \oplus_{\Gamma \in \mathcal{G}_p,n} \mathcal{O}^*(\mathcal{I}^*_q) \) and where the differential \( \mathcal{O}^* : \mathcal{I}^*_q \rightarrow \mathcal{I}^*_{q+1} \) is defined, over the component \( \mathcal{O}^*(\mathcal{I}^*_q) \), \( \mathcal{I}^*_q \in \mathcal{G}_{p+1,n} \), as the alternating sum:

\[
\mathcal{O}^*(x) := \sum_{\Gamma' \subseteq \Gamma, p} \varepsilon_{\Gamma, \Gamma'} \mathcal{O}^*(x) - \sum_{\Gamma' \subseteq \Gamma, p} \varepsilon_{\Gamma, \Gamma'} \mathcal{O}^*(x)
\]

over the subgraphs of \( \Gamma' \) with \( p \)-edges of the inclusions \( i_{\Gamma, \Gamma'} \). The sign \( \varepsilon_{\Gamma, \Gamma'} \) is defined as \( (-1)^{a-1} \), where \( a \) is the position in \( \Gamma \) — according to the lexicographic order — of the only edge in \( \Gamma' \) which is not in \( \Gamma \). It is now immediate, using proposition 2.11 to show that the complexes \( \mathcal{I}^*_q \) are \( \mathcal{S}_n \)-equivariant and isomorphic to \( E^1_{\mathcal{I}^*_q} \). The complexes \( \mathcal{I}^*_q \) are not exact in general, as we will see in the sequel; however, they seem to arise in a pretty natural way as combinatorial objects, without the need to be linked to multitors; they might have an interest on their own. On the other hand it seems difficult to describe a general pattern for their cohomology.

**Notation 3.5.** In what follows, if \( H \subseteq \{1, \ldots, n\} \) is a cardinality 3 multi-index, we will indicate with \( K_3(H) \) the complete graph with vertices in \( H \), which is a 3-cycle. Sometimes, for brevity’s sake, and when there is no risk of confusion, we will indicate this 3-cycle directly with \( H \), instead of \( K_3(H) \).

**Notation 3.6.** For \( r, j \in \mathbb{N}, j \geq 1 \), we will write the sheaf \( (\Lambda^j \Omega^1_X \otimes \mathcal{O}_{X^{n-3}}) \otimes \mathcal{I}^*_{\Delta_{n-2}} \) over the variety \( X \times X^{n-3} \) just with \( \Lambda^j \Omega^1_X \otimes \mathcal{I}^*_{\Delta_{n-2}} \), or with \( \Lambda^j \Omega^1_X (-r \Delta_{n-2}) \). We will also indicate with \( \Lambda^j \Omega^1_X \otimes \mathcal{O}_{\Delta_{n-2}} \) the sheaf \( (\Lambda^j \Omega^1_X \otimes \mathcal{O}_{X^{n-3}}) \otimes \mathcal{O}_{\Delta_{n-2}} \).

**Lemma 3.7.** The kernel of the first differential \( d_1 : E_{1,0}^1 \rightarrow E_{1,0}^2 \) of the spectral sequence \( E_{pq}^{1,0} \) is given by

\[
\ker d_1 \simeq \left( \oplus_{|J|=2} \mathcal{O}_{\mathcal{I}_{\mathcal{I}_1}} \right) := \left\{ (f_J)_1 \in \oplus_{|J|=2} \mathcal{O}_{\mathcal{I}_{\mathcal{I}_1}} \mid (f_J - f_K) |_{\Delta_{J \cap \Delta_K}} = 0, \forall J, K, J \neq K \right\}.
\]

The term \( E_{2}^{3,1} \) is given by:

\[
E_{2}^{3,1} \simeq \bigoplus_{|H|=3} Q^1_{K_3(H)}(H) \otimes \bigcap_{|J|=2, J \not\subseteq H} \mathcal{I}_{\Delta_{J}} \simeq \bigoplus_{|H|=3} (\Omega^1_X \otimes \mathcal{I}_{\Delta_{n-2}})_{K_3(H)}.
\]

**Proof.** The first statement is a consequence of the fact that the map \( \partial^1 : \mathcal{I}_{0}^1 \rightarrow \mathcal{I}_{0}^0 \) in remark 3.3 is given by:

\[
(\partial^1(f_J)_1)_{\Gamma} = (\varepsilon_{J, \Gamma} f_J + \varepsilon_{K, \Gamma} f_K) |_{\Delta_{J \cap \Delta_K}} = \varepsilon_{J, \Gamma} (f_J - f_K) |_{\Delta_{J \cap \Delta_K}},
\]

where \( \Gamma \) is the graph with two edges \( J \) and \( K \).

The second statement follows in a similar way, considering that \( E_{3}^{3,1} \) is \( \mathcal{O}_{|H|=3} Q^2_{K_3(H)}(H) \) and that the differential \( \partial^1 : \mathcal{I}_{1}^1 \rightarrow \mathcal{I}_{1}^1 \) is induced by restrictions:

\[
\partial^3((x_H)_H)_{K_3(L) \cup J} = \varepsilon_{K_3(L), K_3(L) \cup J} L^* x_L |_{\Delta_{J \cap \Delta_J}},
\]

where \( H \) and \( L \) are cardinality 3 multi-indexes and \( J \) is a cardinality 2 multi-index. Hence \( (x_H)_H \in \bigoplus_{|H|=3} Q^2_{K_3(H)}(H) \) belongs to \( E_{3}^{3,1} \) if and only each restriction \( x_H |_{\Delta_{J \cap \Delta_J}} \) is zero. But this means exactly it belongs to \( \bigoplus_{|H|=3} Q^2_{K_3(H)}(H) \otimes \bigcap_{|J|=2, J \not\subseteq H} \mathcal{I}_{\Delta_{J}} \). Note that each sheaf \( Q^2_{K_3(H)}(H) \) is isomorphic to \( (\Omega^1_X \otimes \mathcal{I}_{\Delta_{n-2}})_{K_3(H)} \).

\[
\square
\]
Remark 3.8. By Danila’s lemma, we have the isomorphism of sheaves of invariants over $S^nX$

$$(E^3_i)^\Theta_n \cong \bigoplus_{[\Gamma] \in \mathcal{G}_n} \pi_*(\text{Tor}_{-q}(\Delta, \Gamma))^\Theta_i \cong \bigoplus_{[\Gamma] \in \mathcal{G}_n} \pi_*(\Lambda^q(Q_\Gamma))^\Theta_i$$

where on the right hand sides we consider the comprehensive $\mathcal{G}_n$-action. More in general, we consider a subgroup $G$ of $\mathcal{G}_n$. Hence the sheaves of $G$-invariants over the symmetric variety $S^nX$

$$\pi_*(E^3_i)^G \cong \bigoplus_{[\Gamma] \in \mathcal{G}_n/\mathcal{G}} \pi_*(\text{Tor}_{-q}(\Delta, \Gamma))^\text{Stab}_G(\Gamma) \cong \bigoplus_{[\Gamma] \in \mathcal{G}_n/\mathcal{G}} \pi_*(\Lambda^q(Q_\Gamma))^\text{Stab}_G(\Gamma)$$

The following table lists the groups $\mathcal{G}_n$ and the representation Res$\mathcal{G}_n \in E_n$ for all isomorphisms classes of non empty graphs $\Gamma \subseteq K_4$ without isolated vertices. Here we indicate with $A_1$ the graph with a single edge, with $A_2$ a graph with two intersecting edges, with $B_2$ a graph with two non-intersecting edges, with $A_3$ and $B_3$ the acyclic subgraphs of $K_4$ with three edges and with, respectively, no vertex of degree 3 and a single vertex of degree 3.

| $\mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ |
|-----------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ |
| $\mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ |
| $\mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ |
| $\mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ | $\mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n$ |

The case of the graph $\Gamma = B_2$ needs a line of explanation. We can suppose that $\Gamma$ is the graph consisting of the edges $\{1, 2\}, \{3, 4\};$ hence $\mathcal{G}_n = ((12)) \times ((34)) \times ((13)(24))$. The subgroup $((13)(24))$ is isomorphic to $\mathcal{G}_n^2$ and acts nontrivially in the representation $\text{Res}\mathcal{G}_n = \mathcal{G}_n \times \mathcal{G}_n \times \mathcal{G}_n = E_n$.

Lemma 3.9. The invariants $(E^1_0)^{\Theta_n}$ of the term $E^1_0$ of the spectral sequence $E^q_n$ are isomorphic to the sheaf $A_4(O_X) := \pi_*(O_{\Delta_{12} \otimes O_{\Delta_{23}}}) \otimes (\varepsilon \otimes 1)$ of $\mathcal{G}_n$-invariants. Over an affine open set $S^nU$ its module of sections is isomorphic to $\Lambda^2H^0(O_U) \otimes S^{n-4}H^0(O_U)$.

Proof. We first remark that, for any $n \geq 4$, the types $A_2$ and $B_2$ are the only isomorphism classes of subgraphs of $K_n$ with 2 edges and without isolated vertices. By remark 3.8 we have that $\pi_*(E^1_0)^{\Theta_n} \cong \pi_*(O_{\Delta_{12} \otimes \text{Res}\mathcal{G}_{12} \varepsilon_{12}}) \otimes \pi_*(O_{\Gamma_3 \otimes \text{Res}\mathcal{G}_{\Gamma_3} \varepsilon_{\Gamma_3}})$, where $\Gamma_1$ is a graph of type $A_2$, and $\Gamma_2$ is a graph of type $B_2$. Now the first summand is zero and the second identifies to the one in the statement, when taken $\Gamma_2$ be the graph with edges $\{1, 2\}$, $\{3, 4\}$.

Notation 3.10. Let $n, l \in \mathbb{N}^*$, $l < n$. We denote with $w_1$ the morphism $X \times S^{n-1}X \longrightarrow S^nX$ sending $(x, y)$ to the 0-cycle $lx + y$. It is a finite morphism if $l = 1$ and a closed immersion if $l \geq 2$.

Proposition 3.11. In the identifications $(E^0_1)^{\Theta_n} \cong w_{22}(O_X \otimes O_{S^n-X})$ and $(E^2_0)^{\Theta_n} \cong A_4(O_X)$, the invariant differential $d_1^{\Theta_n} : (E^1_1)^{\Theta_n} \longrightarrow (E^1_0)^{\Theta_n}$ of the spectral sequence is determined locally, over an affine open set of the form $S^nU = \text{Spec} S^nA$, by the formula

$$d_1^{\Theta_n}(a \otimes b_1 \ldots b_{n-2}) = \sum_{1 \leq i < j \leq n-2} (a \otimes b_ib_j - b_ib_j \otimes a) \otimes \tilde{b}_i,$$

where $a, b_i \in A$.

Proof. The expression is obtained — over an affine open set of the form $S^nU$ as in the statement — by identifying $\pi_*(E^1_0)^{\Theta_n}$ with $\pi_*(O_{\Delta_{12} \otimes \text{Res}\mathcal{G}_{\Delta_{12}} \varepsilon_{\Delta_{12}}}$, where $\Gamma_2$ is the graph with edges $\{1, 2\}$, $\{3, 4\}$ we considered in the proof of lemma 3.8. Hence the map of invariants $d_1^{\Theta_n}$ can be identified with the morphism

$$w_{22}(O_X \otimes O_{S^n-X}) \cong \pi_*(O_{\Delta_{12} \otimes \text{Res}\mathcal{G}_{\Delta_{12}} \varepsilon_{\Delta_{12}}}) \cong \pi_*(O_{\Delta_{12} \otimes O_{\Delta_{34}}}) \otimes \pi_*(O_{\Delta_{12}})^{\Theta_n} \longrightarrow \pi_*(O_{\Delta_{12} \otimes \text{Res}\mathcal{G}_{\Delta_{12}} \varepsilon_{\Delta_{12}}}) \cong A_4(O_X)$$

given by

$$d_1^{\Theta_n}(a \otimes b_1 \ldots b_{n-2}) = d_1(a \otimes b_1 \ldots b_{n-2} + (13)(24), a \otimes b_1 \ldots b_{n-2})) = d_1(a \otimes b_1 \ldots b_{n-2}) + (13)(24), d_1(a \otimes b_1 \ldots b_{n-2}) = \sum_{1 \leq i < j \leq n-2} (a \otimes b_ib_j - b_ib_j \otimes a) \otimes \tilde{b}_i.$$
where we saw the element \( b_1, \ldots, b_{n-2} \) as a \( \mathcal{S}(\{3, \ldots, n\}) \)-invariant element in \( H^0(U)^{\otimes n-2} \).

**Notation 3.12.** We denote the kernel \( \ker d_1^{3^n} \) with \( w_{2r}(O_X \boxtimes O_{X^{n-2}}) \).

The next lemma is immediate from lemma [3.7]

**Lemma 3.13.** The invariants \( (E_2^{3,-1})^{3^n} \) of the term \( E_2^{3,-1} \) of the spectral sequence \( E_1^{p,q} \) are isomorphic to the sheaf \( w_{3r}((\Omega_X \boxtimes I_{\Delta_{n-2}})^{3^n-3}) \), where \( \mathcal{S}_{n-3} \) acts on the factor \( X^{n-3} \) of the product \( X \times X^{n-3} \).

### 3.2 The case \( n = 3 \)

**Theorem 3.14.** Let \( X \) be a smooth algebraic variety. The complex of coherent sheaves over \( S^3X \)

\[
\begin{array}{cccc}
0 & \longrightarrow & O_{S^3X} & \longrightarrow \\
& r & & \\
& & w_{2r}(O_{X \times X}) & \longrightarrow \end{array}
\]

Here the sheaf \( O_{S^3X} \) is the restriction and \( d \) is given locally by \( D(a \otimes b) = 2a(b - b \otimes b) \) — is a resolution of the sheaf of invariants \( (\mathcal{I}_{\Delta_3})^{3^3} \).

**Proof.** All subgraphs \( \Gamma \subseteq K_3 \) are connected. Hence the multitors \( \text{Tor}_q(\Delta, \Gamma) \) are isomorphic to \( \Lambda^q(\Omega_X \otimes \mathcal{G}_T) \Gamma \otimes \text{Res}_{\mathcal{G}_T}(\mathcal{G}_T) \).

for the comprehensive \( \mathcal{S}_r \)-action. Remembering that \( q_{K_3} \simeq 3^3 \) and using table \([2] \) we see immediately that the term \( (E_1^{3,-2})^{3^3} \simeq \text{Tor}_2(\Delta, K_3)^{3^3} \simeq w_{3r}(\mathcal{A}^2(\Omega_X \otimes \mathcal{G}_T)^{3^3}) \) vanishes. Moreover, \( (E_1^{3,0})^{3^3} \simeq \pi_*(\mathcal{A}^0(\mathcal{D}_{123}) \otimes \mathcal{G}_T)^{3^3} \), \( (E_1^{3,1})^{3^3} \simeq \pi_*(\mathcal{A}^1(\mathcal{D}_{123}) \otimes \mathcal{G}_T)^{3^3} \): hence there are no \( \mathcal{S}_3 \)-invariants for \( q = 0, p = 2, 3 \). Therefore, for \( p + q \geq 0 \), the only nonzero terms in the spectral sequence of invariants are of the form \( (E_1^{3,0})^{3^3} \), \( (E_1^{3,1})^{3^3} \), \( (E_1^{3,-1})^{3^3} \) and \( (E_1^{3,-3})^{3^3} \). The first two are easily proven to be isomorphic to the sheaves \( O_{S^3X} \) and \( w_{2r}(O_{X \times X}) \), respectively. Moreover \( (E_1^{3,-1})^{3^3} \simeq \text{Tor}_3(\Delta, K_3)^{3^3} \simeq w_{3r}(\Omega_X) \) and, analogously, \( (E_1^{3,-3})^{3^3} \simeq w_{3r}(\Lambda^3(\Omega_X)) \).

Hence we have the resolution of the statement where the map \( D : w_{2r}(O_{X \times X}) \longrightarrow w_{3r}(\Omega_X) \) is induced by the second differential \( d_2^{3^3} \) of the spectral sequence of invariants; the precise local expression of the map \( D \) follows from proposition \([\ref{A.12}] \) in the appendix.

Let \( G = \mathcal{S}(\{23\}) \). In section \([4.3] \) we will need the following result about the invariants \( \pi_*(\mathcal{I}_{\Delta_3})^G \).

**Proposition 3.15.** Let \( X \) be a smooth algebraic variety. Over \( S^3X \), the sheaf of invariants \( \pi_*(\mathcal{I}_{\Delta_3})^G \) is resolved by the complex

\[
0 \longrightarrow w_{1r}(O_X \boxtimes O_{S^2X}) \longrightarrow [w_{2r}(O_X \boxtimes O_X)^{\otimes 2}]_0 \longrightarrow w_{3r}(\Omega_X) \longrightarrow 0 .
\]

Here the sheaf \( [w_{2r}(O_X \boxtimes O_X)^{\otimes 2}]_0 \) is the kernel of the map \( w_{2r}(O_X \boxtimes O_X)^{\otimes 2} \longrightarrow w_{2r}(O_{\Delta_3}) \) given locally by \( (a \otimes u, b \otimes v) \longmapsto au - bv \). The first map of the complex is locally defined as \( a \otimes u \cdot v \longmapsto (au \otimes v + av \otimes u, 2uv \otimes a) \), while the second is determined by \( (a \otimes u, b \otimes v) \longmapsto 2adu - vdb \).

**Proof.** We consider the invariants \( \pi_*(E_1^{p,q})^G \) of the spectral sequence \( E_1^{p,q} \) by the group \( G = \mathcal{S}(\{23\}) \). By proposition \([\ref{2.12}] \) and remark \([3.3] \) the terms \( \pi_*(E_1^{p,q})^G \), as \( G \)-representations, are

\[
\pi_*(E_1^{p,q})^G \simeq \bigoplus_{[\Gamma] \in \mathcal{G}(\{23\})} \Lambda^{-q}(\Omega_X^G \otimes \text{Res}_{\text{Stab}(\Gamma)}(\mathcal{G}_T)^G) \Gamma \otimes \text{Res}_{\text{Stab}(\Gamma)}(\mathcal{G}_T)^G \epsilon_{\mathcal{E}r} .
\]

It is then immediate to prove that \( \pi_*(E_1^{0,0})^G \simeq \pi_*(\Omega_X^G) \simeq w_{1r}(O_X \boxtimes O_{S^2X}) \), \( \pi_*(E_1^{1,0})^G \simeq \pi_*(\mathcal{D}_{123}) \oplus \pi_*(\mathcal{D}_{132}) \simeq w_{2r}(O_X \boxtimes O_X)^{\otimes 2} \), \( \pi_*(E_1^{2,0})^G \simeq \pi_*(\mathcal{D}_{123}) \simeq w_{2r}(O_{\Delta_3}) \). For \( q < 0 \) and \( p + q \geq 0 \), the only nontrivial terms are

\[
\pi_*(E_1^{3,-1})^G \simeq w_{3r}(\Omega_X)^G
\]

and \( \pi_*(E_1^{3,-3})^G \simeq \pi_*(\Lambda^3(\Omega_X)^G) \), since \( \pi_*(E_1^{3,-2})^G \simeq [w_{2r}(O_X \boxtimes O_X)^{\otimes 2}]_0^G \) hence, drawing the page \( E_2 \) of the spectral sequence, we get the complex in the statement. To prove that the maps are the ones mentioned above, one sees immediately that the first is induced by restrictions, while for the second one has just to track down the higher differential \( d_2^G \), but this is done easily taking \( G \)-invariants in the statement of corollary \([\ref{A.11}] \).
3.3 The case $n = 4$.

In order to understand the $\mathfrak{S}_4$-invariants of the sheaf $I_{\Delta_4}$, we have to work out the spectral sequence $\pi_*(E_{1}^{pq})^{\mathfrak{S}_4}$; the first step, by virtue of remark 3.8, is to compute, for each class $[\Gamma] \in \mathfrak{S}_4$, the invariants $\pi_*(\text{Tor}_q(D, \Gamma) \otimes \text{Res}_{\Gamma} \varepsilon E_r)^{\mathfrak{S}_r}$. For $q < 0$, we are just interested in graphs with at least one cycle, which are all connected: the above sheaves then have the form

$$\pi_*(\text{Tor}_q(D, \Gamma) \otimes \text{Res}_{\Gamma} \varepsilon E_r)^{\mathfrak{S}_r} \simeq \pi_*(\Lambda^{-q}(\Omega^1_X \otimes q_r) \Gamma \otimes \text{Res}_{\Gamma} \varepsilon E_r)^{\mathfrak{S}_r}$$

and hence can be computed easily by combining table 1 with table 2. For convenience of the reader we present the computation of the invariants $\pi_*(\text{Tor}_q(D, \Gamma) \otimes \text{Res}_{\Gamma} \varepsilon E_r)^{\mathfrak{S}_r}$ in the following table.

| $q$ | $\Gamma = K_3$ | $\Gamma = K_3 \cup J$ | $\Gamma = C_4$ | $\Gamma = C_4 \cup L$ | $\Gamma = K_4$ |
|-----|-----------------|----------------------|-----------------|----------------------|-----------------|
| -1  | $w_{34}(\Omega^1_X \boxtimes \mathcal{O}_X)$ | $w_{44}(\Omega^1_X)$ | 0               | 0                    | 0               |
| -2  | $w_{34}(\Lambda^2\Omega^1_X \boxtimes \mathcal{O}_X)$ | 0                   | 0               | $w_{44}(\Lambda^2\Omega^1_X)$ | $w_{44}(\Lambda^2\Omega^1_X)$ |
| -3  | $w_{34}(\Lambda^4\Omega^1_X \boxtimes \mathcal{O}_X)$ | $w_{44}(\Lambda^4\Omega^1_X)$ | 0               | 0                    | $w_{44}(\Lambda^4\Omega^1_X)$ |
| -4  | $w_{34}(\Lambda^6\Omega^1_X \boxtimes \mathcal{O}_X)$ | 0                   | 0               | $w_{44}(\Lambda^6\Omega^1_X)$ | $w_{44}(\Lambda^6\Omega^1_X)$ |
| -5  | $w_{34}(\Lambda^8\Omega^1_X \boxtimes \mathcal{O}_X)$ | $w_{44}(\Lambda^8\Omega^1_X)$ | 0               | $w_{44}(\Lambda^8\Omega^1_X)$ | $w_{44}(\Lambda^8\Omega^1_X)$ |
| -6  | $w_{34}(\Lambda^{10}\Omega^1_X \boxtimes \mathcal{O}_X)$ | 0                   | 0               | $w_{44}(\Lambda^{10}\Omega^1_X)$ | $w_{44}(\Lambda^{10}\Omega^1_X)$ |

Table 3

As for $q = 0$, it is clear that

$$\pi_*(E_{1}^{0})^{\mathfrak{S}_4} \simeq \pi_*(I_{0}^{0})^{\mathfrak{S}_4} \simeq \oplus_{[\Gamma] \in \mathfrak{S}_4} \pi_*(\mathcal{O}_{\Delta_4} \otimes \text{Res}_{\Gamma} \varepsilon E_r)^{\mathfrak{S}_r}.$$

We have the following lemmas

Lemma 3.16. The complex $\pi_*(E_{1}^{*})^{\mathfrak{S}_4}$ is quasi isomorphic to the complex

$$0 \to \mathcal{O}_{S^4X} \overset{r}{\to} w_{24}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X}) \overset{d_1^{\mathfrak{S}_4}}{\to} A_4(\mathcal{O}_X) \to 0,$$

which is exact in degree greater or equal than 2. The first map is the restriction, while the second is given locally by $a \otimes b \mapsto a \wedge b$.

Proof. With the help of table 2 we immediately have that the complex $\pi_*(E_{1}^{*})^{\mathfrak{S}_4}$ is quasi-isomorphic to the complex

$$0 \to \mathcal{O}_{S^4X} \overset{r}{\to} w_{24}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X}) \overset{d_1^{\mathfrak{S}_4}}{\to} A_4(\mathcal{O}_X) \to 0 \to 0 \to \pi_*(\mathcal{O}_{\Delta_{1234}}) \to \pi_*(\mathcal{O}_{\Delta_{1234}}) \to 0.$$

The map $\pi_*(\mathcal{O}_{\Delta_{1234}}) \to \pi_*(\mathcal{O}_{\Delta_{1234}})$ is immediately an isomorphism, being induced by the identity on the sheaf $\mathcal{O}_{\Delta_{1234}}$; the map $d_1^{\mathfrak{S}_4} : w_{24}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X}) \to A_4(\mathcal{O}_X)$ is surjective, by proposition 3.11 since it is given locally, on an affine open subset of the form $S^n U, U = \text{Spec}(A)$, by the map $A \otimes S^2 A \to \Lambda^2 A$, sending $a \otimes b$ to $a \wedge b = a \otimes b - b \otimes a$.

Lemma 3.17. Consider a graph $\Gamma$ of the kind $C_4 \cup L$. Then the vector space $\Lambda^4(\mathcal{C}^2 \otimes q_r)$ is completely $\mathfrak{S}_4$-invariant. Moreover, the composition

$$c : \Lambda^4(\mathcal{C}^2 \otimes q_r) \to \Lambda^4(\mathcal{C}^2 \otimes q_{K_4}) \to \Lambda^4(\mathcal{C}^2 \otimes q_{K_4})^{\mathfrak{S}_4}$$

where the first map is the injection $i_{\Gamma,K_4}$ and the second is the projection onto the invariants, is an isomorphism.
For brevity’s sake, we will do the proof in the case $\dim C$ is determined, up to a constant, by $C$. By the same table we can also deduce that the vector space $A^4(C^2 \otimes q_{K_4})^{\Theta_4}$ is one dimensional. Therefore, to prove the second statement, we just have to prove that the map $c$ is nonzero. It is not restrictive to suppose that $\Gamma$ is defined by edges $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{3, 4\}$. Indicating an oriented 3-cycle with a sequence of its vertices, the basis of $C^2 \otimes q_{r}$ is then given by vectors $e_i \otimes e_{123}$ and $e_i \otimes e_{134}$, $i = 1, 2$. Denote more briefly, for an oriented 3-cycle $H$, with $\gamma_H = e_1 \otimes e_{1H}$ and $\delta_H = e_2 \otimes e_{2H}$, where we write the elements in the set $H$ in an order according to the given orientation. Hence we can take as a basis of $C^2 \otimes q_{r}$ the vectors $\gamma_{123}$, $\delta_{123}$, $\gamma_{134}$, $\delta_{134}$: similarly, as a basis of $C^2 \otimes q_{K_4}$ we take the previous vectors, to which we add $\gamma_{124}$, $\delta_{124}$. For brevity’s sake, denote with $\alpha_H := \gamma_H \wedge \delta_H$. A basis of $A^4(C^2 \otimes q_{r})$, which is fully invariant, is then given by $\alpha = \alpha_{123} \wedge \alpha_{134}$. Since this element is invariant by $\Theta_4 \simeq ((13)) \times ((24))$, we have that the map $c$ is, up to a constant, given by

$$c(a) = \sum_{\sigma \in \Theta_4/\Theta_4} \sigma \cdot i_{\Gamma, K_4} a.$$ 

Now the cosets $\Theta_4/\Theta_4$ can be represented by the set $\{id, (12), (14), (23), (34), (12)(34)\}$, hence $c(a)$ is given, up to a constant by

$$c(a) = \alpha_{123} \wedge \alpha_{134} + \alpha_{213} \wedge \alpha_{234} + \alpha_{423} \wedge \alpha_{431} + \alpha_{112} \wedge \alpha_{124} + \alpha_{124} \wedge \alpha_{143} + \alpha_{214} \wedge \alpha_{243}$$

$$= \alpha_{123} \wedge \alpha_{134} + \alpha_{213} \wedge \alpha_{234} + \alpha_{423} \wedge \alpha_{434} + \alpha_{123} \wedge \alpha_{124} + \alpha_{124} \wedge \alpha_{134} + \alpha_{124} \wedge \alpha_{234}$$

Now, using that $\alpha_{234} = \alpha_{123} + \alpha_{134} - \alpha_{124}$, we get easily that, up to a constant,

$$c(a) = 3(\alpha_{123} \wedge \alpha_{124} + \alpha_{123} \wedge \alpha_{134} + \alpha_{124} \wedge \alpha_{134})$$

which is a nonzero element of $A^4(C^2 \otimes q_{K_4})^{\Theta_4} \subseteq A^4(C^2 \otimes q_{K_4})$.

**Notation 3.18.** Consider now the second invariant differential

$$A := a^2 \Theta_4^2 : w_{3*}(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_2}) \simeq \pi_*(E_2^{-1})^{\Theta_4} \simeq \pi_*(E_2^{-5})^{\Theta_4}.$$ 

Its precise expression is determined in the appendix, corollary \ref{cor:A} We denote with $w_{3*}(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_2})_0$ its kernel.

Recall notation \ref{eq:3.12} We have the following result.

**Theorem 3.19.** Let $X$ be a smooth algebraic variety. The sheaf of invariants $(\mathcal{I}_{\Delta_2})^{\Theta_4}$ over the symmetric variety $S^4X$ admits a right resolution given by a natural complex

$$\begin{array}{cccccccc}
\mathbb{I}_4^* := 0 & \to & \mathcal{O}_{S^4X} & \overset{r}{\to} & w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X})_0 & \overset{D}{\to} & w_{3*}(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_2})_0 & \overset{C}{\to} & w_{4*}(S^3\Omega^1_X) & \to 0 ,
\end{array}$$

where $r$ is the restriction, $D$ is defined locally as $D(a \otimes u, v) = (2adu - u da) \otimes v + (2adv - v da) \otimes u$ and $C$ is determined, up to a constant, by $C(\omega \otimes f) = \text{sym}((\omega \otimes f)_\omega)$, where $\omega \in \Omega^1_X$ and $f \in \mathcal{I}_{\Delta_2}$.

**Proof.** For brevity’s sake, we will do the proof in the case $\dim X \leq 2$: the proof in the general case is analogous, but slightly more technical: we indicate in the next remark how to adapt the present proof to the general case.

The abutment of the spectral sequence of invariants $\pi_*(E_1^{p,q})^{\Theta_4}$ is zero in positive degree, after remark \ref{rem:a}. The complex $\pi_*(E_1^{p,q})^{\Theta_4}$ has been studied in lemma \ref{lem:a}. At level $q = -1$, recalling table \ref{table:3} and notation \ref{not:3.6} we just have the complex $w_{3*}(\Omega^1_X \boxtimes \mathcal{O}_X) \to w_{3*}(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_2}) \simeq w_{4*}(\Omega^1_X)$, in degree 3 and 4, and at level $q = -2$ we have the complex $w_{4*}(K_X \boxtimes K_X) \to w_{4*}(K_X)$, in degree 5 and 6. Moreover, at level $q = -4$, we have the complex $w_{4*}(K_X^2) \to w_{4*}(K_X)$, in degree 5 and 6. The only other nonzero terms for $q < 0$ are $\pi_*(E_1^{6,-3})^{\Theta_4} \simeq w_{4*}(S^3\Omega^1_X)$ and $\pi_*(E_1^{6,-6})^{\Theta_4} \simeq w_{4*}(K_X^3)$.

We now prove that the map $w_{3*}(\Omega^1_X \boxtimes \mathcal{O}_X) \to w_{4*}(\Omega^1_X)$ is surjective. Indeed it can be seen as the map of $\Theta_4$-invariants

$$w_{3*}(\Omega^1_X \boxtimes \mathcal{O}_X) \to w_{4*}(\Omega^1_X) \simeq \pi_*(\Omega^1_X(K_{3}(123)))^{\Theta_4} \to \pi_*(\Omega^1_X(K_{3}(123)))^{\Theta_4}.$$
This map is naturally the composition

\[ \pi_*(\Omega_{X_1}^{1}(K_3(123)))^{\otimes^3} \subset \pi_*(\Omega_{X_1}^{1}(K_3(123)))^{\otimes^2} \to \pi_*(\Omega_{X_1}^{1}(K_3(123),\cup(134)))^{\otimes^2} \]

where the first map is the natural inclusion and the second is the map of \( \mathcal{E}_2 \)-invariants of the map \( \pi_*(\Omega_{X_1}^{1}(K_3(123))) \to \pi_*(\Omega_{X_1}^{1}(K_3(123),\cup(134))) \). But this last map is surjective, since the map \( (\Omega_{X_1}^{1})_{K_3(123)} \to (\Omega_{X_1}^{1})_{K_3(123),\cup(134)} \) is a restriction and hence surjective, and because \( \pi \) is finite. Moreover, the natural inclusion \( \pi_*(\Omega_{X_1}^{1}(K_3(123)))^{\otimes^3} \to \pi_*(\Omega_{X_1}^{1}(K_3(123)))^{\otimes^2} \) is an isomorphism, since both terms coincide with \( \pi_*(\Omega_{X_1}^{1}(K_3(123))) \). This means in particular that \( \pi_*(E_4^{-1})^{\otimes^3} = 0 \).

Let’s now look at the map \( w_4(K_X \oplus K_X) \to w_4(K_X) \). Its cokernel is isomorphic to \( \pi_*(E_4^{-2})^{\otimes^4} \).

Now, because of the form of the complexes \( \pi_*(E_4^{-q})^{\otimes^4} \) for \( q = 0, -1 \), and because we proved that \( \pi_*(E_4^{-1})^{\otimes^4} = 0 \), there are no nonzero higher invariant differentials \( d_i^{\otimes^4} \) targeting \( \pi_*(E_4^{-2})^{\otimes^4} \); hence this term has live till the \( \infty \)-page, as a graded sheaf of the abutment, but the abutment in degree 4 is zero. Hence \( \pi_*(E_4^{-2})^{\otimes^4} \) to vanish and the map \( w_4(K_X \oplus K_X) \to w_4(K_X) \) to be surjective. Hence \( \pi_*(E_4^{-2})^{\otimes^4} \simeq w_4(K_X) \).

Finally, we look at the map \( \pi_*(E_4^{-q})^{\otimes^4} \simeq w_4(K_X^2) \to w_4(K_X^2) \simeq \pi_*(E_1^{-4})^{\otimes^4} \) at level \( q = -4 \). It coincides with the map of invariants

\[ \pi_*(\Lambda^4(Q_1^p))^{\otimes^4} \to \pi_*(\Lambda^4(Q_{K_4}^p))^{\otimes^4}, \]

induced by the inclusion \( i_{R,K_4}: Q_1^p \to Q_{K_4}^p \), where \( \Gamma \) is a graph of the kind \( C_4 \cup L \). Now, since both \( \mathcal{E}_1 \) and \( \mathcal{E}_4 \) act trivially over \( \Delta_{1234} = \Delta_1 \), and \( \pi|_{\Delta_{1234}} \) is a closed immersion, the map \( (\mathcal{E}_4) \) is an isomorphism if and only if the map of line bundles \( i_{R,K_4}^*K_X \simeq \Lambda^4(Q_{K_4}^p)^{\otimes^4} \to \Lambda^4(Q_{K_4}^p)^{\otimes^4} \simeq i_{R,K_4}^*K_X \) over \( \Delta_{1234} \) is. Since \( \Lambda^4(Q_1^p) \) is fully \( \mathcal{E}_1 \)-invariant, the previous map coincide, up to constants, with the composition

\[ \Lambda^4(Q_1^p) \to \Lambda^4(Q_{K_4}^p) \to \Lambda^4(Q_{K_4}^p)^{\otimes^4}, \]

where the first is the inclusion and the second is the projection onto the invariants. Now, on the fibers, this map is precisely the map of lemma \( \ref{lem:inclusion} \). Hence it is an isomorphism.

Consequently, at level \( E_2 \), the only nonzero terms are

\[ \pi_*(E_2^{1,0})^{\otimes^4} \simeq (\mathcal{I}_{\Delta_1})^{\otimes^4}, \quad \pi_*(E_2^{-1})^{\otimes^4} \simeq \text{coker}(\mathcal{O}_{S^{1}\times X} \to w_4(\mathcal{O}_X \boxtimes \mathcal{O}_{S^{2}X})), \quad \pi_*(E_2^{-1})^{\otimes^4} \simeq w_{4}(\Omega_{X}^{1} \boxtimes \mathcal{I}_{\Delta_2}), \quad \pi_*(E_2^{-2})^{\otimes^4} \simeq w_4(K_X), \quad \pi_*(E_2^{-2})^{\otimes^4} \simeq w_4(S^{3} \Omega_{X}^{1}). \]

Therefore, drawing the page \( E_2 \) of the spectral sequence \( \pi_*(E_2^{p,q})^{\otimes^4} \), we deduce a complex

\[ 0 \to \pi_*(E_2^{1,0})^{\otimes^4} \to w_3(\Omega_{X}^{1} \boxtimes \mathcal{I}_{\Delta_2}) \to w_4(\mathcal{O}_X \boxtimes \mathcal{O}_{S^{2}X}) \to 0 \]

which is non exact only in the middle, with cohomology isomorphic to \( w_4(S^{3} \Omega_{X}^{1}) \); henceforth we have an exact sequence

\[ 0 \to \pi_*(E_2^{1,0})^{\otimes^4} \to w_3(\Omega_{X}^{1} \boxtimes \mathcal{I}_{\Delta_2}) \to w_4(S^{3} \Omega_{X}^{1}) \to 0. \]

The statement of the theorem follows, since one has as well an exact sequence

\[ 0 \to (\mathcal{I}_{\Delta_1})^{\otimes^4} \to \mathcal{O}_{S^{1}\times X} \to w_2(\mathcal{O}_X \times \mathcal{O}_{S^{2}X}) \to \pi_*(E_2^{1,0})^{\otimes^4} \to 0.\]

The precise expression of map \( D \) and \( C \) will be determined in the appendix, proposition \( \ref{prop:D} \) and proposition \( \ref{prop:C} \).

**Remark 3.20.** We sketch here how to adapt the proof in the case \( X \) is of higher dimension. Using table \( \ref{table:higherdim} \) we see that, for \( p \leq 4 \), we have to consider, for \( q \) negative odd, the surjective maps \( (E_1^{q})^{\otimes^4} \simeq w_4(\Lambda^{-q}\Omega_{X}^{1} \boxtimes \mathcal{O}_X) \to w_4(\Lambda^{-q}\Omega_{X}^{1} \boxtimes \mathcal{O}_X) \simeq (E_1^{q})^{\otimes^4} \). Hence, for \( q \) odd, \( (E_2^{q})^{\otimes^4} \simeq w_4(\Lambda^{-q}\Omega_{X}^{1} \boxtimes \mathcal{I}_{\Delta_2}) \) and \( (E_2^{q})^{\otimes^4} = 0 \). Consider now \( p \geq 5 \). Analogously as in the case of a surface, we have a surjective map \( (E_2^{-2})^{\otimes^4} \simeq w_4(\Lambda^{2}\Omega_{X}^{1} \boxtimes \Lambda^{2}\Omega_{X}^{1}) \to w_4(\Lambda^{2}\Omega_{X}^{1}) \simeq (E_2^{-2})^{\otimes^4} \). In higher dimension one can prove, in a way similar to lemma \( \ref{lem:inclusion} \), that the map \( (E_2^{-4})^{\otimes^4} \to (E_2^{-4})^{\otimes^4} \) is surjective with kernel \( w_4(\Lambda^{3}\Omega_{X}^{1} \boxtimes \Omega_{X}^{1}) \simeq (E_2^{-4})^{\otimes^4} \). Finally, we have that \( (E_2^{-5})^{\otimes^4} \simeq (E_2^{-5})^{\otimes^4} \simeq w_4(S^{3,1} \Omega_{X}^{1}) \). These are the only relevant differences in the page \( E_2 \). Analogously to corollary \( \ref{cor:higherdim} \) one
can prove that the second differential \((d_2^{2,-3})^{\otimes 4} : w_{3,4}(\Lambda^3 \Omega_X^1 \otimes T_{\Delta_4}) \to w_{4,4}(\Lambda^3 \Omega_X^1 \otimes \Omega_X^1)\) is surjective with kernel \(w_{3,4}(\Lambda^3 \Omega_X^1 \otimes T_{\Delta_4}^1) \simeq (E_3^{3,-3})^{\otimes 4}\); moreover, similarly to proposition \(\text{A}.19\) one can show that the third differential \((d_3^{3,-3})^{\otimes 4} : w_{3,4}(\Lambda^3 \Omega_X^1 \otimes \Omega_X^1) \to w_{4,4}(S^3 \Omega_X^1 \otimes \Omega_X^1)\) is surjective. Hence, at the page \(E_3\) all differences with the 2-dimensional case happen along the diagonal, so they are not relevant for the complex in the statement of theorem \(\text{3.19}\).

To finish this subsection, we present two immediate byproducts of the proof of theorems \(\text{3.14}\) and \(\text{3.19}\).

**Corollary 3.21.** For \(n\) equal to 3 or 4, the \(\mathcal{G}_n\)-invariants of the product ideal \(\prod_{I \in E_n} \mathcal{I}_{\Delta_{i}} = \mathcal{I}_{\Delta_{12}} \cdot \cdots \cdot \mathcal{I}_{\Delta_{n-1,n}}\) and of the ideal \(\mathcal{I}_{\Delta_n}\) coincide:

\[
\left( \mathcal{I}_{\Delta_{12}} \cdots \mathcal{I}_{\Delta_{n-1,n}} \right)^{\mathcal{G}_n} \simeq \left( \mathcal{I}_{\Delta_{n}} \right)^{\mathcal{G}_n} = \left( \mathcal{I}_{\Delta_{12}} \cdots \mathcal{I}_{\Delta_{n-1,n}} \right)^{\mathcal{G}_n}.
\]

**Proof.** The invariants \(\left( \mathcal{I}_{\Delta_{12}} \cdots \mathcal{I}_{\Delta_{n-1,n}} \right)^{\mathcal{G}_n}\) of the product of ideals coincide with the term \(\pi_*(E_1^{0,0})^{\mathcal{G}_n}\) of the spectral sequence \(\pi_*(E_1^{0,0})^{\mathcal{G}_n}\) above, since the abutment in degree 0 is the tensor product \(\mathcal{I}_{\Delta_{12}} \otimes \cdots \otimes \mathcal{I}_{\Delta_{n-1,n}}\) and since \(\pi_*(E_1^{0,0})\), being the first graded sheaf for the natural filtration on the abutment, is the image of the natural morphism \(\mathcal{I}_{\Delta_{12}} \otimes \cdots \otimes \mathcal{I}_{\Delta_{n-1,n}}^{\mathcal{G}_n} \to \mathcal{O}_{X \mathcal{G}_n}^{\mathcal{G}_n} \simeq \mathcal{O}_{S_n \mathcal{G}_n}\). But it is evident from the proof of theorems \(\text{3.14}\) and \(\text{3.19}\) that \(\pi_*(E_1^{0,0})^{\mathcal{G}_n} \simeq \pi_*(E_2^{0,0})^{\mathcal{G}_n} \simeq \left( \mathcal{I}_{\Delta_{n}} \right)^{\mathcal{G}_n}\).

**Remark 3.22.** It seems to us an interesting question whether the statement of corollary \(\text{3.21}\) is true for general \(n\); in some contexts (for example when taking inverse images) the product of ideals is better behaved than the intersection; therefore, knowing that, at least at level of invariants, the two coincide, might turn out useful in some applications.

**Remark 3.23.** When taking the tensor product of ideals, things are clearly different. Over \(S^3 X\), we have the isomorphism \(\left( \mathcal{I}_{\Delta_{12}} \otimes \mathcal{I}_{\Delta_{13}} \otimes \mathcal{I}_{\Delta_{23}} \right)^{\mathcal{G}_3} \simeq \left( \mathcal{I}_{\Delta_{12}} \cdot \mathcal{I}_{\Delta_{13}} \cdot \mathcal{I}_{\Delta_{23}} \right)^{\mathcal{G}_3}\) only if \(\dim X \leq 2\), while over \(S^4 X\) the two sheaves are definitely not isomorphic also for \(X\) of dimension 2; their difference, in this case, is measured by the sheaf \(w_{4,4}(K_X^3)\), as the next exact sequence proves:

\[
0 \to w_{4,4}(K_X^3) \to \left( \mathcal{I}_{\Delta_{12}} \otimes \cdots \otimes \mathcal{I}_{\Delta_{34}} \right)^{\mathcal{G}_4} \to \left( \mathcal{I}_{\Delta_{4}} \right)^{\mathcal{G}_4} \to 0.
\]

### 3.4 Twisting by the line bundle \(D_L\)

Let now \(F\) be a \(\mathcal{G}_n\)-equivariant coherent sheaf over \(X^n\). By definition of the line bundle \(D_L\) on the symmetric variety \(S^n X\) (which is valid for \(X\) of arbitrary dimension, see remark \(\text{1.1}\)), using projection formula and taking \(\mathcal{G}_n\)-invariants we have the following equation:

\[
(\pi_*(E_1^{0,n}) F) \otimes D_L \simeq \pi_*(E_1^{0,n} (F \otimes L^{\mathcal{G}_n})).
\]

Because of this fact, all results proved in this section continue to work when we tensorize the sheaf \(\mathcal{I}_{\Delta_n}\) with a line bundle of the form \(L \otimes \cdots \otimes L\), or its invariants by \(D_L\). In particular we have that for \(n\) equal to 3 or 4, the sheaf of invariants \(\pi_*(\mathcal{I}_{\Delta_n} \otimes L^{\mathcal{G}_n})^{\mathcal{G}_n} \simeq (\mathcal{I}_{\Delta_n})^{\mathcal{G}_n} \otimes D_L\) is resolved by the complex \(\mathbb{L}^* \otimes D_L\); in other words

**Corollary 3.24.** Let \(X\) be a smooth algebraic variety. Over \(S^3 X\) and \(S^4 X\), respectively, we have the following resolutions

\[
0 \to \pi_*(\mathcal{I}_{\Delta_3} \otimes L^{\mathcal{G}_3})^{\mathcal{G}_3} \to D_L \to w_{2,4}(L^2 \otimes L) \to w_{3,4}(\Omega_X^1 \otimes L^3) \to 0
\]

\[
0 \to \pi_*(\mathcal{I}_{\Delta_4} \otimes L^{\mathcal{G}_4})^{\mathcal{G}_3} \to D_L \to w_{2,4}(L^2 \otimes D_L) \to w_{3,4}(\Omega_X^1 \otimes L^3 \otimes \mathcal{I}_{\Delta_4}) \to w_{4,4}(S^3 \Omega_X^1 \otimes L^4) \to 0
\]

### 4 Applications

From now on \(X\) will always be a smooth algebraic surface.
4.1 Cohomology of \((\det L^n)^2\) for low n.

**Theorem 4.1.** Let \(X\) be a smooth quasi-projective surface and \(L\) and \(A\) two line bundles over \(X\). Then, for \(n = 3\) or \(n = 4\), the cohomology \(H^* (X^n, (\det L^n)^{\otimes 2} \otimes \mathcal{D}_A)\) is computed by the spectral sequence

\[
E_1^{pq} := H^q (X^n, \mathcal{P}_n^p \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A).
\]

**Proof.** By corollary [1.9] and by lemma [1.5] we have

\[
\mathcal{R} \mu_* ((\det L^n)^{\otimes 2} \otimes \mathcal{D}_A) \simeq (\mathcal{I}_{\Delta_n}^2)^{\otimes n} \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A \simeq (\mathcal{I}_{\Delta_n})^{\otimes n} \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A.
\]

Hence, applying the functor \(\mathcal{R} \Gamma\) on both sides, we get

\[
H^* (X^n, (\det L^n)^{\otimes 2} \otimes \mathcal{D}_A) \simeq \mathcal{R} \Gamma \mu_* ((\det L^n)^{\otimes 2} \otimes \mathcal{D}_A) \simeq H^* (X^n, (\mathcal{I}_{\Delta_n})^{\otimes n} \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A).
\]

The spectral sequence in the statement computes the hypercohomology of the complex \(\mathcal{I}_n^* \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A\), which is a resolution of the sheaf \((\mathcal{I}_{\Delta_n})^{\otimes n} \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A\), by theorems [3.14] and [3.19].

An immediate application yields the computation of the Euler-Poincaré characteristic of \((\det L^n)^{\otimes 2} \otimes \mathcal{D}_A\).

**Corollary 4.2.** Let \(X\) a smooth projective surface and \(L\) and \(A\) line bundles over \(X\). For \(n = 3\) and \(n = 4\) we have the following formulas for the Euler-Poincaré characteristic of \((\det L^n)^{\otimes 2} \otimes \mathcal{D}_A\) over the Hilbert scheme \(X^n\):

\[
\chi (X^n, (\det L^n)^{\otimes 2} \otimes \mathcal{D}_A) = \frac{\chi (L^2 \otimes A)^2 + 2}{3} - \frac{\chi (L^4 \otimes A^2) \chi (L^2 \otimes A) + \chi (A^3)}{3} + \frac{\chi (L^6 \otimes A^3)}{3} - \frac{\chi (L^8 \otimes A^4)}{3} - \frac{\chi (K_X \otimes A^4)}{3} - \frac{\chi (S^3 \Omega_X^1 \otimes L^8 \otimes A^4)}{3}.
\]

We now mention an effective vanishing result for the cohomology of \((\det L^n)^k \otimes \mathcal{D}_A\), for any \(n\) and \(k\).

**Remark 4.3.** We recall that a line bundle \(L\) on a smooth projective surface \(X\) is called \(m\)-very ample if, for any \(\xi \in X^{[m+1]}\), the restriction map \(H^0 (L) \to H^0 (L_\xi)\) is surjective. The property of being \(m\)-very ample generalizes the fact of being very ample, since “1-very ample” means exactly “very ample”. After results of [BS91] and [CG90], one can prove that \(\det L^n\) is globally generated if \(L\) is \(n\)-(1-very ample), and that it is actually very ample if \(L\) is \(n\)-very ample (see also [Sca15a, Cor. 5.10]).

**Corollary 4.4.** Let \(X\) be a smooth projective surface and \(L\) and \(A\) two line bundles over \(X\) such that \(L^k \otimes A \otimes K_X^{-1}\) is a product \(\otimes_{i=1}^k B_i\) of line bundles \(B_i\), with \(B_1\) \(n\)-very ample and \(B_j\) \((n-1)\)-very ample, for \(j = 2, \ldots, k\). Then we have the vanishing

\[
H^i (X^n, (\det L^n)^{\otimes k} \otimes \mathcal{D}_A) = 0 \quad \text{for } i > 0.
\]

In particular the statement is true if \(L^k \otimes A \otimes K_X^{-1}\) is a product of \((k-1)+1\) very ample line bundles over \(X\).

**Proof.** One has just to note that, in the hypothesis of the corollary, and since \(\mathcal{D}_K_X \simeq K_X^{[n]}\), \((\det L^n)^{\otimes k} \otimes \mathcal{D}_A \simeq (\otimes_{i=1}^k \det B_i^{[n]}) \otimes K_X^{[n]}\). Then one uses Kodaira vanishing, since all line bundles \(\det B_i^{[n]}\) are nef and \(\det B_1^{[n]}\) is very ample. The last statement follows from the fact that a product of \(l\) 1-very ample line bundles is \(l\)-very ample [BS91].

4.2 Regularity of \(\mathcal{I}_{\Delta_n}^k\) and \((\mathcal{I}_{\Delta_n})^{\otimes n}\)

**Notation 4.5.** If \(s \in \mathbb{Q}\) is a rational number, we denote with \([s]\) its integral part, or round-down, and with \([s]\) its round-up.

**Theorem 4.6.** Let \(X\) be a smooth projective surface and \(L\) be a line bundle over \(X\). Let \(n \in \mathbb{N}, n \geq 2\). Let \(m \in \mathbb{N}\) be an integer with the property
a) \( L^m \otimes K_X^{-1} = \bigotimes_{i=1}^{2\lfloor(k+1)/2\rfloor} B_i \), with \( B_i \) \( n \)-very ample and with \( B_j \) \((n-1)\)-very ample, for \( j > 1 \).

Then we have the vanishing
\[
H^i(S^n X, (\mathcal{I}_X^k)^{\otimes_n} \otimes \mathcal{D}_L^m) = 0 \quad \text{for } i > 0.
\]

If, moreover, \( L \) is very ample on \( X \), then \((\mathcal{I}_X^k)^{\otimes_n}\) is \((m+2n)\)-regular with respect to \( L \otimes \cdots \otimes L \). Therefore, if \( m_0 \) is the minimum of \( m \) such that condition a) is true, we have the upper bound
\[
\text{reg}((\mathcal{I}_X^k)^{\otimes_n}) \leq m_0 + 2n
\]
for the regularity of the ideal sheaf \((\mathcal{I}_X^k)^{\otimes_n}\) with respect to \( \mathcal{D}_L \).

Remark 4.7. We note that condition a) is true in particular if \( L^m \otimes K_X^{-1} \) is a tensor product \( 2n[(k+1)/2] - 2[(k+1)/2] + 1 \) very ample line bundles over \( X \). If this holds and, additionally, the line bundle \( L \) is very ample, then the ideal \( (\mathcal{I}_X^k)^{\otimes_n} \) is \((m+2n)\)-regular with respect to \( \mathcal{D}_L \).

Proof of theorem 4.6. For the first statement, by lemma 1.5 and by corollary 1.9, we have that
\[
(\mathcal{I}_X^k)^{\otimes_n} \otimes \mathcal{D}_L^m \simeq (2^{2\lfloor(k+1)/2\rfloor})^{\otimes_n} \otimes \mathcal{D}_L^m = R\mu_*((\det \mathcal{O}_X^{[n]})^{\otimes 2\lfloor(k+1)/2\rfloor} \otimes \mathcal{D}_L^m).
\]

Hence
\[
H^i(S^n X, (\mathcal{I}_X^k)^{\otimes_n} \otimes \mathcal{D}_L^m) \simeq H^i(X^{[\mu]}, \mathcal{D}_L = (-2[(k+1)/2] \cdot))
\]
and we conclude by Theorem 5.14.

For the second part, from the first statement and from the fact that, under the hypothesis, \( \mathcal{D}_L \) is very ample, we immediately have that \( H^i(S^n X, (\mathcal{I}_X^k)^{\otimes_n} \otimes \mathcal{D}_L^{m+i}) = 0 \) for \( i > 0 \) and for \( l \geq 0 \). Consequently, \((\mathcal{I}_X^k)^{\otimes_n}\) has to be \((m+2n)\)-regular with respect to \( \mathcal{D}_L \).

Theorem 4.8. Let \( X \) be a smooth projective surface and \( L \) be a line bundle over \( X \). Let \( n \in \mathbb{N}, 2 \leq n \leq 7 \). Let \( m \in \mathbb{N} \) be an integer with the property

b) \( L^m \otimes K_X^{-1} = \bigotimes_{i=1}^{k+1} B_i \), with \( B_i \) \( n \)-very ample and with \( B_j \) \((n-1)\)-very ample, for \( j > 1 \).

Then we have the vanishing
\[
H^i(X^n, \mathcal{I}_X^k \otimes (L^m \otimes \cdots \otimes L^m)) = 0 \quad \text{for } i > 0.
\]

If, moreover, \( L \) is very ample, then the ideal sheaf \( \mathcal{I}_X^k \) is \((m+2n)\)-regular with respect to \( L \otimes \cdots \otimes L \). Therefore, if \( m_0 \) is the minimum of \( m \) such that condition b) is true, we have the upper bound
\[
\text{reg}(\mathcal{I}_X^k) \leq m_0 + 2n
\]
for the regularity of the ideal sheaf \( \mathcal{I}_X^k \) with respect to \( L \otimes \cdots \otimes L \).

Remark 4.9. We note that condition b) is true if \( L^m \otimes K_X^{-1} \) is a tensor product of \((k+1)n - k\) very ample line bundles over \( X \). If this holds and, additionally, \( L \) is very ample, then, for \( 2 \leq n \leq 7 \), the ideal sheaf \( \mathcal{I}_X^k \) is \((m+2n)\)-regular with respect to \( L \otimes \cdots \otimes L \).

Proof of theorem 4.8. The first statement follows immediately by Theorem 5.15 and by the fact that, by Theorem 2.12, \( B^n \) has log-canonical singularities for \( n \leq 7 \). As for the second, its proof is analogous to the proof of the similar statement for the regularity of \((\mathcal{I}_X^k)^{\otimes_n}\) in theorem 4.6.

The previous regularity results are nicer to state when \( X \) has Picard number one.

Corollary 4.10. Let \( X \) be a smooth projective surface with Picard group \( \text{Pic}(X) \simeq \mathbb{Z} B \), where \( B \) is the ample generator. Let \( r \) be the minimum positive power of \( B \) such that \( B^r \) is very ample. Suppose, moreover, that \( K_X \simeq B^w \), for some integer \( w \). Then we have the following.

- The sheaf \((\mathcal{I}_X^k)^{\otimes_n}\) is \((m+2n)\)-regular with respect to \( \mathcal{D}_B^r \), if \( m \geq 2n[(k+1)/2] - 2[(k+1)/2] + 1 + w/r \).

Hence, with respect to \( \mathcal{D}_B^r \),
\[
\text{reg}((\mathcal{I}_X^k)^{\otimes_n}) \leq 2n([(k+1)/2] + 1) - 2[(k+1)/2] + 1 + [w/r].
\]
• If $2 \le n \le 7$, the sheaf $\mathcal{I}^{k}_{\Delta_n}$ is $(m+2n)$-regular with respect to $B^r \boxtimes \cdots \boxtimes B^r$, if $m \ge (k+1)n-k+w/r$. Hence, with respect to $B^r \boxtimes \cdots \boxtimes B^r$,

$$\text{reg}(\mathcal{I}^{k}_{\Delta_n}) \le (k+3)n - k + [w/r].$$

**Remark 4.11.** If $X = \mathbb{P}_2$, taking $B = \mathcal{O}_{\mathbb{P}_2}(1)$, we can say, more simply, that, $\text{reg}(\mathcal{I}^{k}_{\Delta_n}) \le 2n((k + 1)/2 + 1) - 2[(k + 1)/2] - 2$ and, for $2 \le n \le 7$, $\text{reg}(\mathcal{I}^{k}_{\Delta_n}) \le (k + 3)(n - 1)$.

**Remark 4.12.** The results proven in this subsection for $\mathcal{I}^{k}_{\Delta_n}$ are valid for $2 \le n \le 7$. We expect them to hold also for $n = 8$, since $B^n$ should have log-canonical singularities also in this case [Sca15b Conjecture 2]. However, we don’t know, and it seems to us an interesting question, if the previous bound is still a good upper bound for the regularity of $\mathcal{I}^{k}_{\Delta_n}$, for general $n$, or, if not, what would be a good one. The proof we gave here can’t go through in general since we proved that $B^n$ does not have log-canonical singularities for $n \ge 9$ [Sca15b Theorem 2.12].

### 4.3 The sheaves $\mathcal{L}^\mu(-2\mu\Delta)$

Let $n, k \in \mathbb{N}$, $n \ge 2$, and let $\mu$ be a partition of $k$ of length $l(\mu) \le n$. The symmetric group $\mathfrak{S}_n$ acts naturally on the set of compositions of $k$ supported in $\{1, \ldots, n\}$. Indicate with $L^\mu$ the line bundle on $X^n$ defined by $L^\mu := \bigotimes_{i=1}^n p_i^* L^{\otimes \mu_i}$, where $p_i : X^n \rightarrow X$ is the projector onto the $i$-th factor. In $\mathcal{Sca15a}$ we defined sheaves $\mathcal{L}^\mu(-2\mu\Delta)$ over the symmetric variety $S^n X$ as

$$\mathcal{L}^\mu(-2\mu\Delta) := \pi_* \left( L^\mu \otimes \bigcap_{1 \le i < j \le l(\mu)} \mathcal{I}^{2\mu_{ij}}_{\Delta_{\mu_{ij}}} \right)^{\mathfrak{S}_\mu},$$

where we see $\mu$ as a composition of $k$ supported in $\{1, \ldots, n\}$. If $\mu_2 = \cdots = \mu_{l(\mu)} = l$, we denote more simply this sheaf with $\mathcal{L}^\mu(-2l\Delta)$. We also use sheaves $\mathcal{L}^\mu(-m\Delta)$, for an integer $m \in \mathbb{N}$, whose definition is analogous. The interest in such sheaves comes from the fact that we believe they could be in all generality the graded sheaves for a natural filtration on the direct image $\pi_* (S^n L^{[n]})$ of symmetric powers of tautological bundles on $X^n$ via the Hilbert-Chow morphism.

**Remark 4.13.** The sheaf $\mathcal{L}^\mu(-2\mu\Delta)$ over the symmetric variety $S^n X$ is closely related to the same sheaf over $S(l\mu) X$; more precisely, if $v_1$ is the finite morphism $v_1 : S^l X \times S^{n-l} X \rightarrow S^n X$, sending the couple of $0$-cycles $(x, y)$ to $x + y$, we can write the isomorphism of sheaves over $S^n X$:

$$\mathcal{L}^\mu(-2\mu\Delta) \simeq v_1(\mu)_* (\mathcal{L}^\mu(-2\mu\Delta) \boxtimes \mathcal{O}_{S^{n-l}(\mu)_X})$$

and, in general, if $A$ is a line bundle over $X$, then $\mathcal{L}^\mu(-2\mu\Delta) \otimes D_A \simeq v_1(\mu)_* (\mathcal{L}^\mu(-2\mu\Delta) \otimes D_A \boxtimes D_A)$.

**Remark 4.14.** Let $\lambda = (r_1, \ldots, r_l)$ and set $l = l(\lambda)$. Then it is immediate to see that the sheaf $\mathcal{L}^\lambda(-2r\Delta)$ over $S^l X$ is isomorphic to

$$\mathcal{L}^\lambda(-2r\Delta) \simeq (\mathcal{I}^{2\mu_{ij}}_{\Delta_{\mu_{ij}}})^{\mathfrak{S}_l} \otimes D_{L^r} \simeq \mu_*(\det \mathcal{O}_X^{[l]} \otimes D_{L^r}).$$

If $n \ge l$, over $S^n X$, in general we have that $\mathcal{L}^\lambda(-2r\Delta) \otimes D_A \simeq v_1(\mu)_* ((\mathcal{I}^{2\mu_{ij}}_{\Delta_{\mu_{ij}}})^{\mathfrak{S}_l} \otimes D_{L^r \otimes A} \boxtimes D_A) \simeq v_1(\mu)_* (\det \mathcal{O}_X^{[l]} \otimes D_{L^r \otimes A} \boxtimes D_A)$.

We come to our results. Denote the partition $(1, \ldots, 1)$ of $l$ with $1^l$ (in exponential notation). As a consequence of the previous remark, as well as of theorems 4.14, 4.19 and of subsection 3.4 we obtain the following

**Corollary 4.15.** For $l = 3, 4$, and $n \ge l$, the sheaf $\mathcal{L}^{1^l}(-2\Delta) \otimes D_A$ over $S^n X$ is resolved by the complex

$$v_1(1^l \otimes D_{L^\otimes A} \boxtimes D_A): \mathcal{L}^{1^l}(-2\Delta) \otimes D_A \simeq \frac{\partial_0}{\partial_1} v_1(1^l \otimes D_{L^\otimes A} \boxtimes D_A).$$

Similarly, thanks to proposition 3.15 we obtain

**Corollary 4.16.** The sheaf $\mathcal{L}^{2,1^l}(-\Delta) \otimes D_A$ is resolved over $S^n X$ by the complex

$$0 \rightarrow w_3(v_1(L^2 \otimes A \boxtimes D_{L^\otimes A} \boxtimes D_A)) \rightarrow w_3(v_2(L^2 \otimes A^2 \boxtimes L \otimes A) \oplus v_2((L^2 \otimes A^2 \boxtimes L^2 \otimes A)[0] \boxtimes D_A)) \rightarrow w_3(v_4(\Omega_X^1 \otimes L^1 \otimes A^3) \boxtimes D_A)) \rightarrow 0.$$
It would be now immediate to give a formula for the Euler-Poincaré characteristic of $L^l(-2\Delta) \otimes \mathcal{D}_A$, using corollary \[3.2\] or corollary \[4.2\] and of $L^{2,1,1}(-\Delta) \otimes \mathcal{D}_A$. We leave this to the reader.

**The sheaf $L^{2,1,1}(-2\Delta)$** To finish this section we will describe the sheaf $L^{2,1,1}(-2\Delta)$, which is important for the work \[Sca15a\]. If $A \subseteq \{1, \ldots, n\}$ and if $\mu$ is a composition of some integer $l$ supported in $\{1, \ldots, n\}$, we define $\mu_A$ as the composition coinciding with $\mu$ over the set $A$, and with 0 over $\{1, \ldots, n\} \setminus A$. We define with $\vert \mu_A \vert = \sum_{i=1}^n \mu_A(i)$. In \[Sca15a\] Remark 4.6] we defined a natural $\text{Stab}_{\mathcal{E}_n}(\mu)$-equivariant differential

$$d^l_\Delta : L^\mu \otimes \mathcal{T}_{\Delta_{(\mu)}} \otimes I_{\mu}^{l(n)} \rightarrow \bigoplus_{I \subseteq \{1, \ldots, l(n)\}} (S^i \Omega_X^l \otimes L^{\mu_I})_I \otimes L^\mu$$

and an invariant version over $\mathbb{S}^n X$:

$$d^l_\Delta : L^\mu(-l\Delta) := \pi_* (L^\mu \otimes \mathcal{T}_{\Delta_{(\mu)}})^{\mathcal{E}_n(n)} \rightarrow \pi_* \left( \bigoplus_{I \subseteq \{1, \ldots, l(n)\}} (S^i \Omega_X^l \otimes L^{\mu_I})_I \otimes L^\mu \right)^{\mathcal{E}_n(n)}$$

whose kernel is $L^\mu(-(l+1)\Delta)$. Denote with $\mathcal{K}^l_{(1)(1)}(-l\Delta)$ the sheaf

$$\mathcal{K}^l_{(1)(1)}(-l\Delta) := \pi_* \left( (\Omega_X^1 \otimes L^3 \{12\} \otimes p_3^* L \otimes \mathcal{T}_{\Delta_{(2)}}) \otimes (4, \ldots, n) \right);$$

it will be denoted just with $\mathcal{K}^l_{(1)(1)}$ if $l = 0$. It is clear that, for $\mu = (2, 1, 1)$,

$$\pi_* \left( \bigoplus_{I \subseteq \{1, \ldots, l(n)\}} (\Omega_X^1 \otimes L^{\mu_I})_I \otimes L^\mu \right)^{\mathcal{E}_n(n)} \simeq \mathcal{K}^l_{(1)(1)} \oplus \pi_* \left( (\Omega_X^1 \otimes L^2 \{23\} \otimes p_3^* L^2) \otimes (2, 3) \otimes (4, \ldots, n) \right)$$

$$\simeq \mathcal{K}^l_{(1)(1)},$$

since $\mathcal{G}(2, 3)$ acts with a sign on the sheaf $(\Omega_X^1 \otimes L^2 \{23\})$. With these notations, we can prove the following fact.

**Proposition 4.17.** We have the exact sequence over $\mathbb{S}^n X$:

$$0 \rightarrow L^{2,1,1}(-2\Delta) \rightarrow L^{2,1,1}(-\Delta) \rightarrow \mathcal{K}^l_{(1)(1)}(-2\Delta) \rightarrow w_{3*}((S^3 \Omega_X^1 \otimes L^4) \otimes \mathcal{O}_{S^{n-3} X}) \rightarrow 0,$$  \[4.2\]

where the third map is the differential $d^l_\Delta$ and the fourth one is the composition:

$$\mathcal{K}^l_{(1)(1)}(-2\Delta) = \pi_* \left( (\Omega_X^1 \otimes L^3 \{12\} \otimes p_3^* L \otimes \mathcal{T}_{\Delta_{(2)}}) \otimes (4, \ldots, n) \right)$$

$$\rightarrow \pi_* \left( (\Omega_X^1 \otimes L^3 \{12\} \otimes p_3^* L \otimes \mathcal{T}_{\Delta_{(2)}} / \mathcal{T}_{\Delta_{(2)}}) \otimes (4, \ldots, n) \right) \simeq$$

$$w_{3*}((\Omega_X^1 \otimes S^2 \Omega_X^1 \otimes L^4) \otimes \mathcal{O}_{S^{n-3} X}) \rightarrow w_{3*}((S^3 \Omega_X^1 \otimes L^4) \otimes \mathcal{O}_{S^{n-3} X}).$$  \[4.3\]

**Proof.** It is clear that, by construction, the differential $d^l_\Delta$ takes values in $\mathcal{K}^l_{(1)(1)}(-2\Delta)$; it is also clear that $L^{2,1,1}(-2\Delta)$ is the kernel of the third map. Moreover, by construction, the map $\mathcal{K}^l_{(1)(1)}(-2\Delta) \rightarrow w_{3*}((S^3 \Omega_X^1 \otimes L^4) \otimes \mathcal{O}_{S^{n-3} X})$ is surjective. Hence it remains to prove that the sequence

$$L^{2,1,1}(-\Delta) \rightarrow \mathcal{K}^l_{(1)(1)}(-2\Delta) \rightarrow w_{3*}((S^3 \Omega_X^1 \otimes L^4) \otimes \mathcal{O}_{S^{n-3} X})$$

is exact. By GAGA principle it is actually sufficient to prove the exactness of the sequence \[4.4\] for $X = \mathbb{C}^2$ and $L$ trivial. More precisely, let $(\mathbb{S}^n X)_{an}$ be the complex analytic space associated to the complex algebraic variety $S^n X$. Then the natural morphism $(\mathbb{S}^n X)_{an}, \mathcal{O}_{(\mathbb{S}^n X)_{an}} \rightarrow (\mathbb{S}^n X, \mathcal{O}_{\mathbb{S}^n X})$ is faithfully flat, by GAGA principle. This implies, in particular, that the sequence \[4.4\] is exact over $\mathbb{S}^n X$ if and only if the induced sequence of complex analytic sheaves

$$L^{2,1,1}(-\Delta)_{an} \rightarrow \mathcal{K}^l_{(1)(1)}(-2\Delta)_{an} \rightarrow w_{3*}((S^3 \Omega_X^1 \otimes L^4) \otimes \mathcal{O}_{S^{n-3} X})_{an}$$

\[4.5\]
is exact over $(S^nX)_{an}$. But this is holds if and only if it holds for an arbitrary small open set of $(S^nX)_{an}$ in the complex topology. Now a sufficiently small open set of $(S^nX)_{an}$ in the complex topology is always biholomorphic to a sufficiently small open set of $(S^nC^2)_{an}$, in the complex topology. Hence it is sufficient to prove that the sequence (4.3) is exact analytically over $(S^nC^2)_{an}$ and $L$ trivial, but this is equivalent, invoking GAGA principle again, to proving the same fact algebraically over $S^nC^2$ and $L$ trivial.

It is also easy to see that it is sufficient to prove the statement for $n = 3$. In this case set $A = C[x, y]$, $A^{\otimes 3} = C[x_1, x_2, x_3, y_1, y_2, y_3]$. Identifying coherent sheaves with modules, it is sufficient to prove that the sequence of $S^3A$-modules

$$(\mathcal{I}_{\Delta_3})^{\Theta((2,3))} \xrightarrow{d_\Delta^3} (\Omega^1_A \otimes_C A)(-2\Delta) \xrightarrow{S^3\Omega^1_A}$$

is exact, where we wrote briefly $(\Omega^1_A \otimes_C A)(-2\Delta)$ for $(\Omega^1_A \otimes_C A) \otimes_{A \otimes A} \mathcal{I}_{\Delta_3}^2$, where $\mathcal{I}_{\Delta_3}^2$ is the ideal of the diagonal in $A \otimes A$ and where $A \otimes A$ acts componentwise on $\Omega^1_A \otimes C A$.

The ideal $\mathcal{I}_{\Delta_3}$ of the big diagonal in $X^3$ equals the ideal $(\mathcal{I}_{\Delta_3}, \mathcal{I}_{\Delta_3}, \mathcal{I}_{\Delta_3}, q)$, where $q$ is the quadric polynomial $q = (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)$.

The quadric $q$ is anti-invariant for $\Theta([2,3])$, hence $\mathcal{I}_{\Delta_3}^{\Theta((2,3))} = (\mathcal{I}_{\Delta_3}, \mathcal{I}_{\Delta_3}, \mathcal{I}_{\Delta_3}, \Theta((2,3)))$. Writing down all nine degree-3 generators of $\mathcal{I}_{\Delta_3}$, and taking $\Theta([2,3])$-invariants, we get that $\mathcal{I}_{\Delta_3}^{\Theta((2,3))}$ is generated, up to elements of degree 4, by $q(x_3 - x_2)$ and $q(y_3 - y_2)$. Now it is easy to see that the image of $d_{\Delta}^3$ contains $(\Omega^1_A \otimes_C A)(-3\Delta)$. Indeed, if $\alpha, \beta \in \mathbb{N}$, $\alpha + \beta > 1$, we have

$$d_{\Delta}^3 \left[ ((x_2 - x_1)(x_3 - x_1)^\alpha(y_3 - y_1)^\beta - (x_3 - x_1)(x_2 - x_1)^\alpha(y_2 - y_1)^\beta \right] = (x_3 - x_1)^{\alpha+1}(y_3 - y_1)^{\beta+1}dx$$

$$d_{\Delta}^3 \left[ ((x_2 - x_1)(x_3 - x_1)^\alpha(y_3 - y_1)^\beta - (x_3 - x_1)(x_2 - x_1)^\alpha(y_2 - y_1)^\beta \right] = (x_3 - x_1)^{\alpha+1}(y_3 - y_1)^{\beta+1}dx$$

$$d_{\Delta}^3 \left[ ((y_2 - y_1)(x_3 - x_1)^\alpha(y_3 - y_1)^\beta - (y_3 - y_1)(x_2 - x_1)^\alpha(y_2 - y_1)^\beta \right] = (x_3 - x_1)^{\alpha+1}(y_3 - y_1)^{\beta+1}dy$$

Consider now an element of the form

$$\tau = (adx + bdy)(x_3 - x_1)^2 + (edx + cdy)(x_3 - x_1)(y_3 - y_1) + (f dx + g dy)(y_3 - y_1)^2$$

in $(\Omega^1_A \otimes C A)(-2\Delta)$. The image of $\tau$ in $S^3\Omega^1_A$ is $adx^3 + (b + c)dx^2 dy + (e + f) dx dy^2 + g dy^3$. If $\tau$ is in the kernel of the map $(\Omega^1_A \otimes C A)(-2\Delta) \longrightarrow S^3\Omega^1_A$, then $a = g = 0$ and $b = -c$, $e = -f$ and $\tau$ is of the form $\tau = b(x_3 - x_1)^2 dy - b(x_3 - x_1)(y_3 - y_1) dy + e(x_3 - x_1)(y_3 - y_1) dy - e(y_3 - y_1)^2 dy$. But it is now easy to see that $\tau$ is a linear combination of $d_{\Delta}^3[q(x_3 - x_2)]$ and $d_{\Delta}^3[q(y_3 - y_2)]$ and hence in the image of $d_{\Delta}^3$. These facts show that the sequence (4.3) is exact.

A Appendix: Determination of higher differentials in the spectral sequence of invariants

We will determine here explicitly the higher differentials in the spectral sequence of invariants $E^{p,q}_{2n}$ for $n = 3, 4$, appearing in theorem 3.11 and 3.13 and 3.14. To fix ideas, we will always suppose that $\dim X = 2$, but everything can be straightforwardly generalized in the case $X$ is a smooth algebraic variety of arbitrary dimension.

[1] Here we mean: if it holds over any open set $V_j$ of an open cover $\{V_j\}$ of $(S^nX)_{an}$, where each $V_j$ is chosen to be sufficiently small.
Remark A.1. In order to prove that a certain higher differential $d_{\Delta}^{r,n} : (E^{p,q})^{\otimes n} \to (E^{p+r,q-r+1})^{\otimes n}$ has a certain expression, we define explicitly another morphism between the same coherent sheaves of invariants, say $D_{\gamma} : (E^{p,q})^{\otimes n} \to (E^{p+r,q-r+1})^{\otimes n}$, and then we prove that the two maps coincide. By GAGA principle (as done in proposition \[\ref{prop117}], or, alternatively, by localization an completion, in order to compare the two maps we can always reduce the problem to the case $X = \mathbb{C}^2$, where computations are much easier.

Remark A.2. Over $X^n = (\mathbb{C}^2)^n = \text{Spec} \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ we resolve the sheaves $K^*_n, I \subseteq \{1, \ldots, n\}$ with Koszul complexes $K^*_n(F_I, s_I)$, where $F_I$ is the trivial rank 2 bundle, with global frame $\gamma_I, \delta_I$, and with global section $s_I = x_I^{r_1} + y_I^{r_1}$, where, if $I = \{i, j\}, i < j$, we denoted briefly $x_I$ and $y_I$ the differences $x_j - x_i$ and $y_j - y_i$, respectively. We then build a term by term free resolution $R_I^{q,r}$ of the complex $K^*_n$. The spectral sequence $E_1^{q,r}$ can be seen as the spectral sequence associated to the bicomplex $L^{q,r} := \bigoplus_{|I| = 2} R_I^{q,r}$, where the tensor product is taken respecting the lexicographic order of the multi-indexes $I$: we see it as a bicomplex with respect to the sum of the first indexes and the sum of the second. We denote with $\partial$ and $\delta$ the (commuting) horizontal and vertical differentials, respectively. It is straightforward to see that, remembering the notation used in the proof of proposition \[\ref{prop114}],

$$L^{p,r} \simeq \bigoplus_{I_1, \ldots, I_p \subseteq \{1, \ldots, n\}} K^*(F_{I_1}, s_{I_1}) \otimes \cdots \otimes K^*(F_{I_p}, s_{I_p}) \quad \simeq \quad \bigoplus_{I_1, \ldots, I_p \subseteq \{1, \ldots, n\}} K^*(F_{I_1} \oplus \cdots \oplus F_{I_p}, s_{I_1} \oplus \cdots \oplus s_{I_p}) \quad \bigoplus_{\Gamma \in \mathcal{G}_{p,n}} K^*(F_{\Gamma}, s_{\Gamma})$$

where the direct sums are over distinct $I_1, \ldots, I_p \subseteq \{1, \ldots, n\}$, in lexicographic order.

Remark A.3. Let $X = \mathbb{C}^2$. After the description of $Q_{\Gamma}$ given in \[\ref{prop112}, it is practical to think of $F_I$ as $\mathbb{C}^2 \otimes \rho_I$ with $\gamma_I = e_1 \otimes e_I$ and $\delta_I = e_2 \otimes e_I$. The quotient bundle $Q_{\Gamma}$ can then be seen as

$$Q_{\Gamma} \simeq \Omega^{1}_{\Delta_{\Gamma}} \otimes (\Omega^{2}_{\Delta} \otimes q_{\Gamma}).$$

The isomorphism $Q_{\Gamma} \simeq (\Omega^{1}_{\Delta} \otimes q_{\Gamma})_{|V} \simeq \Omega_{\Delta_{\Gamma}} \otimes (\Omega^{2}_{\Delta} \otimes q_{\Gamma})$ is given by identifying, over $\Delta_{\Gamma}$, the vector $e_1 \otimes v$, for $v \in q_{\Gamma}$, with $dx \otimes v$, with and $e_2 \otimes v$ with $dy \otimes v$. Of course, since every bundle here is trivial and the representation $q_{\Gamma}$ is autodual, $Q_{\Gamma} \simeq Q^*_{\Gamma}$.

Notation A.4. Suppose that the graph $\Gamma \in \mathcal{H}_{\rho,n}$ contains an oriented 3-cycle $K_3(H)$. We will identify the 3-cycle $K_3(H)$ with the sequence of its vertices written in order according to the orientation. Hence we write $e_H$ for the corresponding vector in $q_{\Gamma}$. Moreover, we write $\gamma_H = e_1 \otimes e_H$ and $\delta_H = e_2 \otimes e_H$ the corresponding vectors in $\mathbb{C}^2 \otimes q_{\Gamma}$, which can be seen as elements of $Q_{\Gamma}$, by the previous remark.

Determination of the map $D$

Notation A.5. For $J \subseteq \{1, \ldots, n\}$ a cardinality 2 multi-index and for $i \in \mathbb{N}^*$, we denote with $d_{\Delta,J}$ the $i$-th order differential $d_{\Delta,J}^{r,n} : \mathcal{I}_{\Delta,J}^{r,n} \to \mathcal{I}_{\Delta,J}^{r+1,n}$ and with $r_J$ the restriction $r_J : \mathcal{O}_{X^n} \to \mathcal{O}_{\Delta,J}$. Sometimes we see the operator $d_{\Delta,J}$ as taking values in $(S^p \Omega^1_X)^J$, via the isomorphism $S^p \mathcal{I}_{\Delta,J} \simeq S^p \mathcal{O}_{\Delta,J} \simeq (S^p \Omega^1_X)^J$.

Remark A.6. Let $Y, Z$ be smooth subvarieties of a smooth variety $M$ intersecting transversely in a smooth subvariety $Y \cap Z$. It is easy to show that $N_{Y \cap Z/M} \simeq N_{Z/M}|_{Y \cap Z}$.

Remark A.7. Let $I$ and $J$ be two distinct cardinality 2 multi-indexes in $\{1, \ldots, n\}$, such that $I \neq J \neq \emptyset$. Let $I = \{i, j\}, J = \{j, k\}$, with $j < k$ and $i \neq k$. The diagonal $\Delta_{I,J}$ will be identified to $X^{(1, \ldots, n \setminus \{k\})} \simeq X^{n-1}$; therefore $\Delta_{I,J}$ defines a pairwise diagonal in $\Delta_{\text{II}} \simeq X^{n-1}$, that we still indicate with $\Delta_{I,J}$. By \[\ref{A.6}, we can define the composition

$$d_{\Delta_{I,J}} \circ r_J : \mathcal{I}_{\Delta_{I,J}} \longrightarrow \mathcal{I}_{\Delta_{I,J}} / \mathcal{I}_{\Delta,J} \simeq \mathcal{I}_{\Delta_{I,J}} / \mathcal{I}_{\Delta,J} \longrightarrow \mathcal{N}_{\Delta_{I,J}} \simeq \mathcal{N}_{\Delta_{I,J}} / \mathcal{N}_{\Delta_{I,J}}|_{\Delta_{I,J}}.$$

Remark A.8. Recall that $E_{\text{II}}^{3,n-1} \simeq \bigoplus_{H \subseteq \{1, \ldots, n\}, |H| = 3} Q_{\text{II}^{3}}^{*}(H) \otimes \mathcal{I}_{\Delta_{n-2}}$. Now each of the sheaves $Q_{\text{II}^{3}}^{*}(H) \otimes \mathcal{I}_{\Delta_{n-2}}$ can be identified with $(\Omega^2_X \otimes \mathcal{I}_{\Delta_{n-2}})(K_3(H))$ (see notations \[\ref{prop126}]). Hence elements in $Q_{\text{II}^{3}}^{*}(H) \otimes \mathcal{I}_{\Delta_{n-2}}$ can be identified with differential forms in $\Omega^2_X \otimes \mathcal{O}_{X^n-3}$ over the product $X \times X^{n-3}$ vanishing on the diagonal $\Delta_{n-2}$ in $\Delta_H \simeq X \times X^{n-3}$. For brevity’s sake, we will denote the sheaf $(\Omega^2_X \otimes \mathcal{I}_{\Delta_{n-2}})(H)$ more briefly with $(\Omega^2_X \otimes \mathcal{I}_{\Delta_{n-2}})(H)$. 28
Recalling notation $(\oplus_{|I|=2}\mathcal{O}_{\Delta I})_0$ for the kernel of $d_1 : \oplus_{|I|=2}\mathcal{O}_{\Delta I} \longrightarrow \oplus_{I \neq J,|I|=|J|=2} \mathcal{O}_{\Delta I \setminus J}$, we have the following

**Lemma A.9.** The morphism of coherent sheaves $\tilde{D} : (\oplus_{|I|=2}\mathcal{O}_{\Delta I})_0 \longrightarrow \oplus_{|I|=3} (\Omega_{\mathcal{X}}^1 \otimes \mathcal{I}_{\Delta_{n-2}})_H$, given by

$$\tilde{D}(f_I)_H = d_{\Delta I} r_J(\tilde{f}_I - \tilde{f}_J) - d_{\Delta I} r_K(\tilde{f}_K - \tilde{f}_I)$$

— where $E_{K_5(H)} = \{I, J, K\}$, $I < J < K$, and where $\tilde{f}_I$ are liftings to $\mathcal{O}_{\mathcal{X}^n}$ of functions $f_I$ in $\mathcal{O}_{\Delta L}$ — descends to a morphism of coherent sheaves $\tilde{D} : E^{1,0}_2 \longrightarrow E^{3,-1}_2$, which coincides with the differential $d_2$ of the spectral sequence $E^{p,q}_1$.

**Proof.** It is easy to prove that the formula for $\tilde{D}$ well defines a morphism of sheaves $(\oplus_{|I|=2}\mathcal{O}_{\Delta I})_0 \longrightarrow \oplus_{|I|=3} (\Omega_{\mathcal{X}}^1 \otimes \mathcal{I}_{\Delta_{n-2}})_H$ as in the statement, which is clearly zero on $d_1(E^{0,0}_1)$ and hence induces a morphism of sheaves $\tilde{D} : E^{1,0}_2 \longrightarrow E^{3,-1}_2$, by lemma A.3.

We now prove that $\tilde{D} = d_2$: we put ourselves in the situation explained in remarks A.1 A.2 A.3. Let $f_I$, $|I| = 2$, functions in $\mathcal{O}_{\Delta I}$, such that $(f_I)_H$ is in $E^{1,0}_1$. For all $I$, let $\tilde{f}_I$ be regular functions in $\mathcal{O}_{\mathcal{X}^n}$ restricting to $f_I$. The element $(\tilde{f}_I)_I$ is in $L^{1,0}$ and its image $\partial((\tilde{f}_I)_I) \in L^{2,0}$ is zero when projected to $E^{2,0}_1$. In other word we have that

$$\partial((\tilde{f}_I)_I)_{I \cup J} = \varepsilon_{I_1,I_2,J_1,J_2} \tilde{f}_{I_1} + \varepsilon_{I_2,I_3,J_1,J_2} \tilde{f}_{I_3} \in \mathcal{I}_{\Delta_{I_1 \cup I_2}}$$

for pairs of cardinality 2-multi-indexes $I_1, I_2$ with $I_1 \neq I_2$, where we indicated graphs with two edges just as a union of these. Therefore the element $\partial((\tilde{f}_I)_I)_{I \cup J}$ can be lifted to $L^{2,-1}$ as

$$\varepsilon_{I_1,I_2,J_1,J_2} \tilde{f}_{I_1} + \varepsilon_{I_2,I_3,J_1,J_2} \tilde{f}_{I_3} = a_{I_1} x_{I_1} + b_{I_1} y_{I_1} + c_{I_1} x_{I_2} + d_{I_1} y_{I_2} = \delta(w_{I_1})$$

for elements $w_{I_1, I_2} = a_{I_1} \gamma_{I_1}^* + b_{I_1} \delta_{I_1}^* + c_{I_1} \gamma_{I_2}^* + d_{I_1} \delta_{I_2}^*$, where, according to remark A.2, we indicated with $\gamma_{I_i}^*, \delta_{I_i}^*$ a frame of $F^*_I$. The image in $L^{3,1}$ via $\partial$ of liftings $w_{I_1, I_2}$ will represent the image of $d_2$. The complex $L^{3,1}$ is now a direct sum of Koszul complexes $K^*(F_I, s_I)$, where $\Gamma$ is a simple graph with 3 edges and without isolated vertices. But if $\Gamma$ is acyclic then the correspondant Koszul complex is acyclic in negative degree, and the $\Gamma$-component of the image via $\partial$ will be zero in vertical cohomology and hence in $E^{3,-1}_2$. Hence we are just interested in components of the second differential $d_2$ indexed by 3-cycles, determined by cardinality 3-multi-indexes. Suppose now $H$ is such a multi-index and that $\{I, J, K\}$ are the edges of the corresponding 3-cycles, with $I < J < K$. From $(\partial \circ \partial((\tilde{f}_I)_I))_{K_5(H)} = 0$ we deduce

$$\partial((\tilde{f}_I)_I)_{I \cup J} - \partial((\tilde{f}_I)_I)_{I \cup K} + \partial((\tilde{f}_I)_I)_{J \cup K} = 0,$$

which implies that

$$(a_{I,J} - a_{I,K})x_I + (c_{I,J} + a_{J,K})x_J + (-c_{I,K} + c_{J,K})x_K = 0$$

$$(b_{I,J} - b_{I,K})y_I + (d_{I,J} + b_{J,K})y_J + (-d_{I,K} + d_{J,K})y_K = 0$$

and hence, since $x_I - x_J + x_K = 0$ and $y_I - y_J + y_K = 0$, that

$$a_{I,J} - a_{I,K} = -(c_{I,K} + c_{J,K}) = -(c_{I,J} + a_{J,K}) \quad \text{(A.1a)}$$

$$b_{I,J} - b_{I,K} = -(d_{I,K} + d_{J,K}) = -(d_{I,J} + b_{J,K}) \quad \text{(A.1b)}$$

Finally the image of $d_2$ is represented in $L^{3,-1}$ by $w_{I \cup J} - w_{I \cup K} - w_{J \cup K}$, which is equal to

$$(a_{I,J} - a_{I,K})\gamma_{I,J}^* + (c_{I,J} + a_{J,K})\gamma_{J,K}^* + (-c_{I,K} + c_{J,K})\gamma_{I,K}^* +$$

$$+ (b_{I,J} - b_{I,K})\delta_{I,J}^* + (d_{I,J} + b_{J,K})\delta_{J,K}^* + (-d_{I,K} + d_{J,K})\delta_{I,K}^*$$

and which, using relations (A.1), can be rewritten as

$$(a_{I,J} - a_{I,K})\gamma_{I,J}^* + (b_{I,J} - b_{I,K})\delta_{I,J}^* ,$$

which, by notation A.4 and in the identifications explained in remark A.3 is precisely the formula in the statement.

□

29
Over an affine open set of the form $U^n$ or $S^nU$, with $U = \text{Spec}(A)$, we will identify sheaves with their modules of global sections. In particular, we will denote with $\Omega^1_A((-\Delta_{n-2}) = (\Omega^1_A \otimes A^{n-3}) \otimes I_{\Delta_{n-2}}$ the module of sections of the sheaf $\Omega^1_A \otimes A^{n-3}$ over $U^n$, where $I_{\Delta_{n-2}}$ is the ideal of the big diagonal in $U \otimes A^{n-3}$ and $\Omega^1_A \otimes A^{n-3}$ is seen as a $A^{\otimes n-2}$-module (see notation 3.10) and hence a $A^{\otimes n}$-module, via the contraction of the first three factors $A^{\otimes n} \rightarrow A \otimes A^{n-3}$. We denote with $\oplus_{|I|=2}(A \otimes A^{n-2})_0$ the module of global sections of the sheaf $(\oplus_{|I|=2} \Omega_{\Delta})_0$ over $U^n$ and with $(A \otimes A^{n-2})_0$ its $\mathfrak{S}_{n}$-invariants over $S^nU$.

**Notation A.10.** If $w_1 \otimes \cdots \otimes w_l$ is an element of $A^{|I|}$, and $1 \leq i \leq l$, we indicate with $\tilde{w}_i$ the element $w_1 \otimes \cdots \otimes w_{i-1} \otimes w_{i+1} \cdots \otimes w_l \in A^{n-1}$. We use an analogous notation for an element $w_1, \ldots, w_l \in S^4A$.

**Corollary A.11.** Over an affine open set $U^n = \text{Spec} A^{\otimes n}$, the differential $d_2: E^{1,0}_{2,2} \rightarrow E^{3,-1}_{2,2}$ is induced by the map $\tilde{D}: (\oplus_{|I|=2} A \otimes A^{\otimes n-2})_0 \rightarrow \oplus_{H \subseteq \{1, \ldots, n\}, |H|=3} \Omega^1_A((-\Delta_{n-2})$, determined by

$$\tilde{D}(f_J)(L)_H = adw_{k-2} \otimes \tilde{u}_{k-2} + bdv_{j-1} \otimes \tilde{v}_{j-1} - w_{j-1} \otimes \tilde{w}_j$$

where $H = \{i, j, k\}$, $i < j < k$, $I = \{i, j\}, J = \{i, k\}$, $K = \{j, k\}$, and where $f_I = a \otimes u_1 \otimes \cdots \otimes u_{n-2}, f_J = b \otimes v_1 \otimes \cdots \otimes v_{n-2}, f_K = c \otimes w_1 \otimes \cdots \otimes w_{n-2}$.

**Proposition A.12.** The invariant differential $d_2: (E^{1,0}_{2,2})^{\otimes n} \rightarrow (E^{3,-1}_{2,2})^{\otimes n}$ is induced locally, over an affine open set of the form $S^nU$, by the map $D: (A \otimes A^{n-2})_0 \rightarrow \Omega^1_A((-\Delta_{n-2}) \otimes (A^{\otimes n-3}$ determined by

$$D(a \otimes b_1, \ldots, b_{n-2}) = \sum_{i=1}^{n-2} (2adb_i - b_i da) \otimes \tilde{b}_i .$$

Here the group $\mathfrak{S}_{n-3}$ acts on the factor $A^{n-3}$ of the tensor product $A \otimes A^{\otimes n-3}$.

**Determination of the Map $A$ for $n = 4$.** In what follows we use the second convention of notation 3.5

Any graph of the kind $G \cup L$ has a distinguished edge $L$ (the only edge whose vertices are of degree 3) and therefore it can be decomposed uniquely as a union of two 3-cycles $H$ and $K$, determined by two cardinality 3 multi-indexes $H$ and $K$ such that $H \cap K = L$. In what follows we will write such a graph just as $H \cup K$, instead of $K_3(H) \cup K_3(K)$.

If $H$ is a cardinality 3 multi-index, say $H = \{i, j, k\}$, with $i = \min H$, we identify $\Delta_H$ with $X^{(1,2,4)} \setminus (j,k) \simeq X^2$. Moreover, suppose $K$ is a 3-cycle $K = \{i_1, i_2, i_3\}$ with $i_1 < i_2 < i_3$. We say that a simple path in $K$ is positively oriented if, in the orientation of the path, the vertex following $i_1$ is $i_2$, negatively oriented if it is not positively oriented. When writing the coefficient $\eta_{H,K}$ for $I$ and edge of $K$ (see notation 2.9), we will always assume that $K$ is positively oriented.

We introduce a general sign $\varepsilon_{\Gamma, \Gamma'}$ for a couple $(\Gamma, \Gamma')$ where $\Gamma$ is a subgraph of $\Gamma'$, if $\Gamma' \setminus \Gamma = \{I_1, \ldots, I_l\}$, in lexicographic order, then $\varepsilon_{\Gamma, \Gamma'} = \prod_{j=0}^{l-1} \varepsilon_{\Gamma \cup I_j \cup \cdots I_l, \Gamma \cup I_j \cup \cdots I_{j+1}}$.

According to these facts and notations, we have the first

**Lemma A.13.** Consider the map $\tilde{A}: \oplus_{|I|=3}(\Omega^1_X \otimes I_{\Delta_{2}})_H \rightarrow \oplus_{\Gamma \in \mathfrak{S}_{n-4}} A^2(\Omega^1_X \otimes q_{\Gamma})$, whose component $\tilde{A}_{\Gamma}$ is zero if $H$ is not a subgraph of $\Gamma$ and is defined by the formula

$$\tilde{A}_{\Gamma}(\omega \otimes f) = -\varepsilon_{\Gamma,H,K} \eta_{H,K}(\omega \otimes e_H) \wedge (d_{\Delta} f \otimes e_K)$$

if $\Gamma = H \cup K$, for some 3-cycle $K$, with $I = \min E_{H \cup K \setminus E_H}$, where $\omega \in \Omega^1_X$, $f \in I_{\Delta_2}$ and where $e_H$, $e_K$ are base elements in $q_{H \cup K}$. Then the image of $\tilde{A}$ is in $\ker d_1$; hence it induces a map $\tilde{A}: E^{3,-1}_{2,2} \rightarrow E^{5,-2}_{2,2}$, which coincides with the second differential $d_2$.

**Proof.** It is clear that the map $\tilde{A}$ is well defined and it is easy to see that its image lies in $\ker d_1$; this yields the good definition of the map $\tilde{A}: E^{3,-1}_{2,2} \rightarrow E^{5,-2}_{2,2}$. We just have to prove that it coincides with $d_2$. We put ourselves in the situation of remarks A.1, A.2, A.3 Consider a differential form $\omega \otimes f$ in $(\Omega^1_X \otimes I_{\Delta_{2}})_H$, where $\omega \in \Omega^1_X$ and $f \in I_{\Delta_2}$. In what follows we identify the element $\omega \otimes f \in (\Omega^1_X \otimes I_{\Delta_{2}})_H$ with the element in $\oplus_{|I|=3}(\Omega^1_X \otimes I_{\Delta_{2}})_L$ whose $H$-component is $\omega \otimes f$ and whose other $L$-component, with $L \neq H$ is zero.
To prove the statement, it is sufficient to compute the component $d_2(\omega \otimes f)_{H:\cup J}$ of the second differential, where $K = \{I, J, L\}$, where it will always be assumed that $I < J$ in the lexicographic order. The form $\omega \otimes f$ can be rewritten as

$$\omega \otimes f = h(dx \otimes 1) + g(dy \otimes 1)$$

over $X \times X \simeq \Delta_H$, where $h, g \in \mathcal{I}_{\Delta}$. Now $\Delta_H$ is generated over $X \times X$ by (classes in $\mathcal{O}_{\Delta_H}$ of) regular functions $x_I, y_I$ in $\mathcal{O}_X$: hence we can lift $h$ and $g$ to regular functions (which we will still call $h$ and $g$) $h = ax_I + by_I$, $g = cx_I + dy_I$ in $\mathcal{O}_X$. By remark [A.3] the differential form $\omega \otimes f$ can be represented, in $L^{3,-1}$ as

$$\omega \otimes f = h\gamma^*_H + g\delta^*_H = (ax_I + by_I)\gamma^*_H + (cx_I + dy_I)\delta^*_H .$$

Since the element $\omega \otimes f$ was chosen in $E_2^{3,-1} = \ker(\partial : E_1^{3,-1} \to E_1^{4,-1})$, this means that the vertical cohomology class of $\partial(\omega \otimes f)$ in $E_3^{3,-1} = H^g_1(L^4, \bullet)$ is zero. This is equivalent to saying that $\partial(\omega \otimes f)_{H:\cup I}$ and $\partial(\omega \otimes f)_{L:\cup J}$ come from elements in $L^{4,-2}$; for the first we can we can write

$$\partial(\omega \otimes f)_{H:\cup I} = \epsilon_{H, H; \cup I}(\omega \otimes f) = \delta \left( \epsilon_{H, H; I}(a(\gamma_I^* + b\delta^*_I) \cup \gamma_H^* + (c\gamma_J^* + d\delta^*_J) \cup \delta^*_H) \right) \quad (A.2)$$

For the second we have, taking into account that, by definitions of the signs $\eta_{H,K}$ and omitting for brevity’s sake the index $K$, we can write $\eta_{I}x_I + \eta_{J}x_J + \eta_{L}x_L = 0$:

$$\partial(\omega \otimes f)_{L:\cup J} = \epsilon_{H, H; \cup I}(\omega \otimes f) = \delta \left( \epsilon_{H, H; I}(a(\gamma_I^* + b\delta^*_I) \cup \gamma_H^* + (c\gamma_J^* + d\delta^*_J) \cup \delta^*_H) \right) . \quad (A.3)$$

Now the element between parenthesis in [A.2] has image

$$\epsilon_{H, H; \cup K} \epsilon_{H, H; \cup I}(a(\gamma_I^* + b\delta^*_I) \cup \gamma_H^* + (c\gamma_J^* + d\delta^*_J) \cup \delta^*_H) \quad \text{via } \partial \text{ in } L^{5,-2}, \text{ while the element between parenthesis in [A.3] has image}$$

$$\epsilon_{H, H; \cup K} \epsilon_{H, H; \cup I}(a(\gamma_I^* + b\delta^*_I) \cup \gamma_H^* + (c\gamma_J^* + d\delta^*_J) \cup \delta^*_H) \quad \text{via } \partial \text{ in } L^{5,-2}. \text{ The sum of the two terms is given by}$$

$$\epsilon_{H, H; \cup K} \epsilon_{H, H; \cup I}(a(\gamma_I^* - \eta\gamma^*_H - \eta\gamma^*_L) + b(-\eta\gamma^*_I - \eta\gamma^*_L) \cup \gamma_H^* + (c(-\eta\gamma^*_J - \eta\gamma^*_L) + d(-\eta\gamma^*_I - \eta\gamma^*_L) \cup \delta^*_H) \quad \text{since we can easily see that } \epsilon_{H, H; \cup K} \epsilon_{H, H; \cup I} = - \epsilon_{H, H; \cup K} \epsilon_{H, H; \cup I} \text{ because } \epsilon_{H, H; \cup I} = \epsilon_{H, H; \cup K} \text{ and } \epsilon_{H, H; \cup J} = - \epsilon_{H, H; \cup J}; \text{ Note now that } \gamma^*_K = \eta\gamma^*_I + \eta\gamma^*_J + \eta\gamma^*_L, \delta^*_K = \eta\delta^*_I + \eta\delta^*_J + \eta\delta^*_L; \text{ By lemma [Sc15a, Lemma A.3], we have that } d_2(\omega \otimes f)_{H:K} \text{ is represented by the vertical cohomology class of}$$

$$\epsilon_{H, H; \cup K} \epsilon_{H, H; \cup I}(a(\gamma_I^* + b\delta^*_I) \cup \gamma_H^* + (c\gamma_J^* + d\delta^*_J) \cup \delta^*_H) .$$

Since we identified the classes of $\gamma^*_H, \gamma^*_K, \delta^*_H, \delta^*_K$ with $dx \otimes e_H, dx \otimes e_K, dy \otimes e_H, dy \otimes e_K \in \Omega^1_X \otimes \eta_{H,K}, \epsilon_{H,K}$, respectively, we obtain the formula in the statement. \( \Box \)

Remark A.14. For future use the following computation wil turn out handy. Let $X = \mathbb{C}^2$. Consider the differential form $\omega \otimes f$, as above, but now lift it to the element $(ax_M + by_M)\gamma^*_H + (cx_M + dy_M)\delta^*_H \in L^{3,-1}$, for functions $a, b, c, d \in \mathcal{O}_X$, and let $K$ a 3-cycle as above with edges $\{I, J, L\}$, such that the edge $M$ satisfies $M \notin H, M \notin K$. The stairway process in order to compute the component $d_2(\omega \otimes f)_{H;K}$ provides the element

$$\epsilon_{H, H; \cup K} \epsilon_{H, H; \cup L}(\gamma_H^* \cup (a\gamma_K^* + b\delta_K^*) + \delta_H^* \cup (c\gamma_K^* + d\delta_K^*)) = \epsilon_{H, H; \cup K} \epsilon_{H, H; \cup L}(\omega \otimes e_H) \cup (d\Delta f \otimes e_K) \in L^{5,-2} ,$$

up to elements coming from $L^{4,-2}$. 31
As an immediate corollary, taking $\mathfrak{S}_4$-invariants, and using Danila’s lemma for morphisms, we have

**Corollary A.15.** The $\mathfrak{S}_4$-invariant higher differential $A = d_2^{\mathfrak{S}_4} : (E_2^{3,-1})^{\mathfrak{S}_4} \longrightarrow (E_2^{5,-2})^{\mathfrak{S}_4}$ takes an element $\omega \otimes f$ in $w_{3,+}(\Omega_X^1 \otimes T_{\Delta_2})$ to the element $\omega \wedge d_\Delta f$ in $w_{4,+}(\Lambda^2 \Omega_X^1)$. \hfill $\blacksquare$

**Proof.** Indeed it is sufficient to take $H = \{1, 2, 3\}$ and $K = \{1, 3, 4\}$. Then $L = \{1, 3\}$, $I = \{1, 4\}$, $J = \{3, 4\}$. In order to compute the invariant differential $d_2^{\mathfrak{S}_4}(\omega \otimes f)$ of an element $\omega \otimes f$, by [ScMa00] Lemma A.1, we just have to compute $d_3(\omega \otimes f + (24), \omega \otimes f) = (\omega \otimes e_K) \wedge (d_\Delta f \otimes e_K) + (\omega \otimes e_K) \wedge (d_\Delta f \otimes e_H)$ — where we omit writing the push-forward $\pi_*$ — and this can be identified with $\omega \otimes d_\Delta f - d_\Delta f \otimes \omega = \omega \wedge d_\Delta f$ in $\Omega_X^1 \otimes \Omega_X^1 \subseteq \Lambda^2(\Omega_X^1 \otimes \mathcal{O}_{H,R}(K))$. \hfill $\blacksquare$

**Determination of the map $C$.**

**Remark A.16.** Recall notation A.15. Let $X = \mathbb{C}^2$. The differential forms $x_{14}dx$, $y_{14}dy$ are in the image of $D : w_{2,+}(\mathcal{O}_X \otimes \mathcal{O}_{S_2X})_0 \longrightarrow w_{3,+}(\Omega_X^1 \otimes T_{\Delta_2})_0$. Indeed, by corollary A.15 it is clear that $A(x_{14}dx) = dx \wedge dx = 0 = dy \wedge dy = A(y_{14}dy)$. So both differential forms belong to $w_{3,+}(\Omega_X^1 \otimes T_{\Delta_2})_0$. On the other hand we have that $x \otimes 1, 1 \in w_{2,+}(\mathcal{O}_X \otimes \mathcal{O}_{S_2X})_0$, since $d_2^{\mathfrak{S}_4}(x \otimes 1.1) = 2x \otimes 1 - 2x \otimes 1 = 0$ and, by proposition A.12 $D(x \otimes 1.1) = (2x dx - x dx) \otimes 1 - dx \otimes x = (dx \otimes 1)(x \otimes 1 - 1 \otimes x)$, which can be identified with $x_{14}dx$. Similarly $y_{14}dy$ is $D(y \otimes 1.1)$, and $y \otimes 1$ is in $w_{2,+}(\mathcal{O}_X \otimes \mathcal{O}_{S_2X})_0$.

**Remark A.17.** Since we have surjective maps $E_3^{5,-3} \longrightarrow E_3^{6,-3}$ and $E_2^{5,-3} \longrightarrow E_3^{6,-3}$, and since $E_2^{6,-3} \cong \Lambda^3 \mathcal{Q}_{K_4}$, we can see $E_2^{6,-3}$ as a quotient of the bundle $\Lambda^3 \mathcal{Q}_{K_4}$ over the small diagonal $\Delta_{1234}$. If $a$ is an element of $\Lambda^3 \mathcal{Q}_{K_4}$, we denote with $[a]$ the class of its image in $E_3^{6,-3}$.

**Remark A.18.** The natural composition $w_{3,+}(\Omega_X^1 \otimes T_{\Delta_2}^3) \longrightarrow w_{3,+}(\Omega_X^1 \otimes T_{\Delta_2})_0 \longrightarrow (E_3^{3,-1})^{\mathfrak{S}_4}$ is surjective.

**Proof.** By GAGA principle, it is sufficient to prove the statement for $X = \mathbb{C}^2$. But in this case it follows by remark A.16 and by corollary A.15 indeed a differential form $\omega \otimes f \in w_{3,+}(\Omega_X^1 \otimes T_{\Delta_2})_0$ is in the kernel of $d_2^{\mathfrak{S}_4}$ if and only if it is of the form $\omega \otimes f = ax_{14}dx + by_{14}dy + \omega_1 \otimes g$, where $g$ is in $T_{\Delta_2}^3$. But now the term $ax_{14}dx + by_{14}dy$ is zero in $(E_3^{3,-1})^{\mathfrak{S}_4}$, because of remark A.16.

By the previous remark, we can represent any element in $(E_3^{3,-1})^{\mathfrak{S}_4}$ by an element in $w_{3,+}(\Omega_X^1 \otimes T_{\Delta_2}^3)$. In the proof of the next proposition we will use the following notation: if $I$ is a cardinality 2-multi-index in $\{1, \ldots, 4\}$, we will indicate with $\Gamma(I)$ the graph obtained by the complete graph $K_4$ removing the edge $I$, that is $V_{\Gamma(I)} = \{1, \ldots, 4\}$, $E_{\Gamma(I)} = E_{K_4} \setminus I$.

**Proposition A.19.** Consider the map $C : w_{3,+}(\Omega_X^1 \otimes T_{\Delta_2}^3) \longrightarrow w_{4,+}(S^3 \Omega_X^1)$ defined by the formula

$$C(\omega \otimes f) = \text{sym}(\omega \otimes d_3^\mathfrak{S}_4 f),$$

where $\omega \in \Omega_X^1$ and $f \in T_{\Delta_2}^3$. It descends to a map $C : E_3^{3,-1} \longrightarrow E_3^{6,-3}$, which coincides, up to a constant, with $d_3^{\mathfrak{S}_4}$.

**Proof.** It is clear that the formula induces a well defined map $C : E_3^{3,-1} \longrightarrow E_3^{6,-3}$. It is sufficient to prove that this map coincides up to constants with the invariant differential $d_3^{\mathfrak{S}_4}$. We put ourselves in the situation explained in remarks A.1, A.2, A.3. For brevity’s sake, we indicate with $H_0$ the cardinality 3-multi-index $\{1, 2, 3\}$ and the associated 3-cycle. We identify $(E_3^{3,-1})^{\mathfrak{S}_4}$ with $w_{3,+}(\Omega_X^1 \otimes T_{\Delta_2})_0 \cong \pi_*(\Omega_X^1)_{H_0} \otimes T_{\Delta_2}^3$; hence $w_{3,+}(\Omega_X^1 \otimes T_{\Delta_2}^3)$ can be identified with $\pi_*(\Omega_X^1)_{H_0} \otimes T_{\Delta_2}^3 \subseteq \ker d_2^{\mathfrak{S}_4}$. By Danila’s lemma for morphisms, if $\omega \otimes f \in \pi_*(\Omega_X^1)_{H_0} \otimes T_{\Delta_2}^3$, then

$$d_3^{\mathfrak{S}_4}(\omega \otimes f) = d_3\left(\sum_{[\tau] \in \mathfrak{S}_4/\mathfrak{S}_3} \tau_*(\omega \otimes f)\right) = \sum_{[\tau] \in \mathfrak{S}_4/\mathfrak{S}_3} \tau_* d_3(\omega \otimes f),$$

where, when writing $d_3(\omega \otimes f)$ we think of $\omega \otimes f$ as an element in ker $d_2 \subseteq \oplus_H(\Omega_X^1 \otimes T_{\Delta_2})_H$.

Hence we just need to compute $d_3(\omega \otimes f)$ in $E_3^{6,-3}$. The element $\omega \otimes f \in \pi_*(\Omega_X^1)_{H_0} \otimes T_{\Delta_2}^3$ can be written as $\omega \otimes f = h(dx \otimes 1) + g(dy \otimes 1)$, where $h, g \in T_{\Delta_2}^3$. Writing $h = ax_{14}^2 + bx_{14}y_{14} + cy_{14}^2$ and $g = a^2x_{14}^2 + b^2x_{14}y_{14} + c^2y_{14}^2$, we see that is sufficient to compute the image for $d_3$ of differential forms of the kind $a x_{14}^2 dx$, $a x_{14}y_{14} dx$, $a y_{14}^2 dx$, $a x_{14} dy$, $a x_{14}y_{14} dy$, $a y_{14}^2 dy$, for an arbitrary function $a \in \mathcal{O}_{X \times X}$, of...
course, thinking of these differential forms as elements in $\ker d_2$. Let’s begin with $\alpha x_{14}^2 dx$. We have that $d_2(\alpha x_{14}^2 dx) = 0$ in $E_2^{0,-2}$: this means that its components are zero:

$$d_2(\alpha x_{14}^2 dx)_{\Gamma(3\bar{1})} = 0, \quad d_2(\alpha x_{14}^2 dx)_{\Gamma(3\bar{1})} = 0, \quad d_2(\alpha x_{14}^2 dx)_{\Gamma(3\bar{1})} = 0.$$ 

By lemma [A.13] we have that, in terms of representants in $\Lambda^3(Q^*_{\Gamma(3\bar{1})})$ and $\Lambda^2(Q^*_{\Gamma(3\bar{1})})$

$$d_2(\alpha x_{14}^2 dx)_{\Gamma(3\bar{1})} = [\alpha x_{14} \gamma_{123} \wedge \gamma_{134}], \quad d_2(\alpha x_{14}^2 dx)_{\Gamma(3\bar{1})} = [\alpha x_{14} \gamma_{123} \wedge \gamma_{124}].$$

In order to compute the representant of $d_2(\alpha x_{14}^2 dx)_{\Gamma(3\bar{1})}$ in $L^{5,-2}$, we invoke remark [A.13] and we find, in terms of representants in $\Lambda^3(Q^*_{\Gamma(3\bar{1})})$:

$$d_2(\alpha x_{14}^2 dx)_{\Gamma(3\bar{1})} = [-\alpha x_{14} \gamma_{123} \wedge \gamma_{234}].$$

We now lift the elements we found to $L^{5,-3}$. We get

$$\alpha x_{14} \gamma_{123} \wedge \gamma_{134} = (\alpha \gamma_{14} \wedge \gamma_{123} \wedge \gamma_{134}) \in \Lambda^3(C^2 \otimes W_{\Gamma(3\bar{1})})$$

$$\alpha x_{14} \gamma_{123} \wedge \gamma_{124} = (\alpha \gamma_{14} \wedge \gamma_{123} \wedge \gamma_{124}) \in \Lambda^3(C^2 \otimes W_{\Gamma(3\bar{1})})$$

$$-\alpha x_{14} \gamma_{123} \wedge \gamma_{234} = (\alpha \gamma_{12} \wedge \gamma_{24} \wedge \gamma_{123} \wedge \gamma_{234}) \in \Lambda^3(C^2 \otimes W_{\Gamma(3\bar{1})}).$$

By [Sca15a] Lemma A.3], the term $d_3(\alpha x_{14}^2 dx)$ in $E_3^{0,-3}$ is represented by the sum of the images of the preceding liftings via the horizontal differential $\partial$: hence:

$$d_3(\alpha x_{14}^2 dx) = [\alpha \gamma_{14} \wedge \gamma_{123} \wedge (\gamma_{134} - \gamma_{124}) + \alpha (-\gamma_{12} - \gamma_{24}) \wedge \gamma_{123} \wedge \gamma_{234}]$$

Note now that $\gamma_{234} = \gamma_{123} + \gamma_{134} - \gamma_{124}$ and that the term $(-\gamma_{12} - \gamma_{24}) \wedge \gamma_{123} \wedge \gamma_{234}$ is zero in $E_3^{0,-3}$ since it comes from something in $L_3^{5,-3}$. Hence we get that

$$d_3(\alpha x_{14}^2 dx) = [-\alpha \gamma_{12} \wedge \gamma_{14} + \gamma_{24}] \wedge \gamma_{123} \wedge (\gamma_{134} - \gamma_{124}) = [\alpha \gamma_{123} \wedge \gamma_{124} \wedge \gamma_{134}]$$

where again we simplified terms coming from $L^{5,-3}$. The term $\alpha \gamma_{123} \wedge \gamma_{124} \wedge \gamma_{134}$ belongs to $\Lambda^3(Q^*_{\Gamma(3)}) \simeq \Lambda^3(C^2 \otimes q_{K_4})$ and can be identified with $\alpha (dx \otimes e_{123} \otimes (dx \otimes e_{124})) = \alpha (dx)^3 \otimes (e_{123} \otimes e_{124} \otimes e_{134}) \in S^3Q^*_{\Gamma X} \otimes \Lambda^3 q_{K_4} \subseteq \Lambda^3(Q^*_{\Gamma X} \otimes q_{K_4}).$

When computing $d_3(\alpha x_{14} y_{14} dx)$, we have, analogously to the previous case that $d_2(\alpha x_{14} y_{14} dx)_{\Gamma(3\bar{1})}$ is represented in $L^{5,-2}$ by $\alpha x_{14} \gamma_{123} \wedge \delta_{134}$ and hence we have the lifting $\alpha x_{14} \gamma_{123} \wedge \delta_{134} = (\alpha \gamma_{14} \wedge \gamma_{123} \wedge \delta_{134})$; moreover $d_2(\alpha x_{14} y_{14} dx)_{\Gamma(3\bar{1})}$ is represented by $\alpha x_{14} \gamma_{123} \wedge \delta_{134}$ and hence can be lifted to $\alpha \gamma_{14} \wedge \gamma_{123} \wedge \delta_{134}$ in $L^{5,-2}$; finally, by remark [A.14]

$$d_2(\alpha x_{14} y_{14} dx)_{\Gamma(3\bar{1})} = [-\alpha x_{14} \delta_{123} - \delta_{124}] \wedge \gamma_{123} = [\alpha x_{14} \gamma_{123} \delta_{123} - \delta_{124}]$$

and we have the lifting $\alpha x_{14} \gamma_{123} \delta_{123} - \delta_{124} = (\alpha \gamma_{12} \wedge \gamma_{24} \wedge \gamma_{123} \wedge (\delta_{123} - \delta_{124})$. Then, using that $\delta_{234} = \delta_{123} + \delta_{134} - \delta_{24}$, we get that $d_3(\alpha x_{14} y_{14} dx)$ is given by the class

$$d_3(\alpha x_{14} y_{14} dx) = [\alpha \gamma_{123} \delta_{134} \wedge \gamma_{14} \wedge \gamma_{12} \wedge \gamma_{24}] = [\alpha \gamma_{123} \wedge \gamma_{124} \wedge \delta_{134}]$$

were we simplified elements coming from $L^{5,-3}$. Analogously

$$d_3(\alpha y_{14}^2 dx) = [\alpha \gamma_{123} \wedge \gamma_{124} \wedge \delta_{134}].$$

The computation of all other elements is done by symmetry. We than finally have that, for a differential form $\omega \otimes f$ in $p_{a}(\Omega^1_{\bar{X}} \otimes \mathcal{O}_{\Delta_{a}})$ as in the beginning, the differential $d_3(\omega \otimes f)$ is represented by the class of $\omega \otimes d_3^a f$ in $(\Omega^1_{\bar{X}} \otimes S^2\Omega^*_{\bar{X}})_{K_4} \subseteq \Lambda^3(\Omega^1_{\bar{X}} \otimes q_{K_4})_{K_4}$.

To finish the computation we remark the following two facts. Firstly, the terms $\gamma_{123} \wedge \gamma_{124} \wedge \delta_{134}$, satisfies the equality $\gamma_{123} \wedge \gamma_{124} \wedge \delta_{134} = [\gamma_{123} \wedge \gamma_{124} \wedge \delta_{134}]$. Indeed, simplifying at each step elements coming from $L^{5,-3}$,

$$\gamma_{123} \wedge \gamma_{124} \wedge \delta_{134} = \gamma_{123} \wedge (\gamma_{123} \wedge \gamma_{134} - \gamma_{234}) \wedge \delta_{134} = -[\gamma_{123} \wedge \gamma_{234} \wedge \delta_{134}]
\begin{array}{l}
\quad = \quad \gamma_{123} \wedge \gamma_{234} \wedge (\delta_{123} - \delta_{124}) = -[\gamma_{123} \wedge \gamma_{234} \wedge \delta_{124}]
\quad = \quad -[\gamma_{123} \wedge \gamma_{234} \wedge \delta_{124}]
\quad = \quad [-\gamma_{123} \wedge \gamma_{234} \wedge \delta_{124}]
\end{array}$$
The same is true for any nontrivial triple wedge products of vectors associated to 3-cycles $\gamma^*_H$, $\delta^*_K$. Secondly, the class in $E^3_{\Delta}$ of any such nontrivial triple wedge product is $\mathbb{S}_4$-invariant. The proof of this fact is similar to that of the previous fact. Hence, up to positive constants

$$d_{\mathbb{S}_4}^3(\omega \otimes f) = \text{sym}(\omega \otimes d_{\Delta}^3 f)$$

in $w_4(\mathcal{S}^3 \Omega^1_X \otimes \Lambda^3 q_K)$.

References

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.*, 14(3):535–554 (electronic), 2001.

[BS91] M Beltrametti and Andrew J Sommese. Zero cycles and k-th order embeddings of smooth projective surfaces. In *Problems in the theory of surfaces and their classification*, Symposia Math, volume 32, pages 33–48, 1991.

[CG90] Fabrizio Catanese and Lothar Göttsche. d-very-ample line bundles and embeddings of Hilbert schemes of 0-cycles. *Manuscripta Mathematica*, 68(1):337–341, 1990.

[Dan01] Gentiana Danila. Sur la cohomologie d’un fibré tautologique sur le schéma de Hilbert d’une surface. *J. Algebraic Geom.*, 10(2):247–280, 2001.

[Die10] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer Heidelberg, fourth edition, 2010.

[DN89] J.-M. Drezet and M. S. Narasimhan. Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. *Invent. Math.*, 97(1):53–94, 1989.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[GAP15] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.8*, 2015.

[Hai99] Mark Haiman. Macdonald polynomials and geometry. In *New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97)*, volume 38 of *Math. Sci. Res. Inst. Publ.*, pages 207–254. Cambridge Univ. Press, Cambridge, 1999.

[Hai01] Mark Haiman. Hilbert schemes, polygraphs and the Macdonald positivity conjecture. *J. Amer. Math. Soc.*, 14(4):941–1006 (electronic), 2001.

[Hai02] Mark Haiman. Vanishing theorems and character formulas for the Hilbert scheme of points in the plane. *Invent. Math.*, 149(2):371–407, 2002.

[Leh99] Manfred Lehn. Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. *Invent. Math.*, 136(1):157–207, 1999.

[Sca09] Luca Scala. Cohomology of the Hilbert scheme of points on a surface with values in representations of tautological bundles. *Duke Math. J.*, 150(2):211–267, 2009.

[Sca15a] Luca Scala. Higher symmetric powers of tautological bundles on Hilbert schemes of points on a surface. arXiv: 1502.07595v1, 2015.

[Sca15b] Luca Scala. Singularities of the Isospectral Hilbert Scheme. arXiv: 1510.03071, 2015.

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.