EIGENVALUE ESTIMATES FOR BELTRAMI-LAPLACIAN UNDER BAKRY-ÉMERY RICCI CURVATURE CONDITION

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Abstract. On closed Riemannian manifolds with Bakry-Émery Ricci curvature bounded from below and bounded gradient of the potential function, we obtain lower bounds for all positive eigenvalues of the Beltrami-Laplacian instead of the drifted Laplacian. The lower bound of the $k$th eigenvalue depends on $k$, Bakry-Émery Ricci curvature lower bound, the gradient bound of the potential function, and the dimension and diameter upper bound of the manifold, but the volume of the manifold is not involved. Especially, these results apply to closed manifolds with Ricci curvature bounded from below.

1. Introduction

Let $(M, g)$ be a Riemannian manifold, $f$ a smooth function on $M$. The Bakry-Émery Ricci curvature tensor $\text{Ric} + \text{Hess } f$, first introduced in [2], is a natural generalization of the classical Ricci curvature tensor (the case where $f$ is a constant). Here, $\text{Ric}$ and $\text{Hess } f$ represent the Ricci curvature tensor and the hessian of $f$, respectively.

Bakry-Émery Ricci curvature being bounded below is the concept of “Ricci curvature bounded below” for smooth metric space $(M, g, e^{-f}dV)$, namely, $M$ equipped with the distance induced by $g$ and measure $e^{-f}dV$, where $dV$ is the volume element. It can also be extended to general metric measure spaces and used to study Ricci limit spaces (see e.g. [19], [20], [14]). Moreover, manifolds with constant Bakry-Émery Ricci curvature are so called Ricci solitons, which play a crucial role in the singularity analysis of the Ricci flow (see e.g. [18], [22], [7], [4]). Therefore, the question that whether the results for manifolds with Ricci curvature bounded below can also be established when Bakry-Émery Ricci curvature is bounded below has drawn a lot of attention.

In this paper, we study the eigenvalue estimates of Beltrami Laplacian $\Delta$ on closed manifolds. The basic assumptions are that $(M^m, g)$ is an $m$-dimensional closed Riemannian manifold with

\begin{equation}
\text{Ric} + \text{Hess } f \geq -Kg,
\end{equation}

and

\begin{equation}
|\nabla f| \leq L,
\end{equation}

where $\nabla f$ is the gradient of $f$, and $K$ and $L$ are nonnegative constants.

On manifolds with Ricci curvature bounded below, there have been numerous results on eigenvalue estimates (see e.g. [13], [17], [3], [8], [12], [26], [11]). For manifolds with Bakry-Émery Ricci curvature bounded from below, normally the weighted measure $e^{-f}dV$ is considered, and the corresponding self-adjoint Laplace operator is the drifted Laplacian $\Delta_f = \Delta - \nabla f \cdot \nabla$. Under the assumptions (1.1) and (1.2), Munteanu-Wang [16], Su-Zhang [21], and Wu [24] independently obtained a Cheng type upper bound for the first positive eigenvalue of $\Delta_f$. On the other hand, Charalambous-Lu-Rowlett [6] proved lower bound
estimates for all positive eigenvalues of $\Delta f$. An eigenvalue comparison for the first positive eigenvalue of $\Delta f$ is also given in [3] and [1].

Different from the above setting, we consider here the standard measure $dV$ and Beltrami-Laplacian $\Delta$ under conditions (1.1) and (1.2). A main difficulty rising in this case is that the hessian of $f$ does not appear in the Bochner formula for $\Delta$, as opposed to the Bochner formula for $\Delta f$. Thus, to utilize the lower boundedness of the Bakry-Émery Ricci curvature, we need to manually add $\text{Hess } f$, which causes an extra bad term $-\text{Hess } f(\nabla \cdot, \nabla \cdot)$.

By using integration by parts and Moser iteration, we are able to overcome this difficulty.

Denote the eigenvalues of $\Delta$ by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$, we derive lower bounds for all $\lambda_k$’s. More precisely, we show that

**Theorem 1.1.** Let $(M^m, g)$ be an $m$-dimensional closed Riemannian manifold. Assume that conditions (1.1) and (1.2) are satisfied. Then

1. we have
   \[ \lambda_1 \geq c_0; \tag{1.3} \]
2. for $m \geq 3$,\n   \[ \lambda_k \geq c_1 k^{\frac{2}{m}}, \quad \forall k \geq 2, \tag{1.4} \]
3. and for $m = 2$,
   \[ \lambda_k \geq c_2 k^{\frac{1}{2}}, \quad \forall k \geq 2. \tag{1.5} \]

Here $c_0$, $c_1$ and $c_2$ are constants depending on $m$, $K$, $L$, and the upper bound $D$ of the diameter of $M$.

We prove (1) and (2) of Theorem 1.1 separately in sections 2 and 3 (see Theorem 2.9 and Theorem 3.7), where explicit expressions of $c_0$, $c_1$ and $c_2$ can also be found. In section 2, we establish the estimate (1.3) by finding a lower bound of Cheeger’s isoperimetric constant $IN_1(M)$. Actually, we obtain lower bound for the general isoperimetric constant $IN_\alpha(M)$, $\alpha > 0$, defined in [11]. The proof follows a method of Dai-Wei-Zhang [9] and uses the volume comparison result of Q. Zhang and the third author [25]. In section 3, following the method in [23] (see also [15]), estimates (1.4) and (1.5) are proved by using (1.3) and gradient estimates for eigenfunctions. The gradient estimates are done by Moser iteration, in which the Sobolev inequality required comes from the isoperimetric constant estimate in section 2.

2. Isoperimetric constant estimate and lower bound of $\lambda_1$

In this section, we prove part (1) of Theorem 1.1. According to [5], it suffices to bound Cheeger’s isoperimetric constant from below. Firstly, let us recall the definitions of isoperimetric constants. We adapt the notations and definitions in [11].

**Definition 2.1.** Let $(M, g)$ be a compact Riemannian manifold (with or without boundary). For $\alpha > 0$, The Neumann $\alpha$-isoperimetric constant of $M$ is defined by

\[ IN_\alpha(M) = \inf_{\substack{\partial \Omega_1 = \partial \Omega_2 \\ M = \Omega_1 \cup H \cup \Omega_2}} \frac{\text{Vol}(H)}{\min\{\text{Vol}(\Omega_1), \text{Vol}(\Omega_2)\}^{\alpha}}, \]

where the infimum is taken over all hypersurfaces $H$ dividing $M$ into two parts, denoted by $\Omega_1$ and $\Omega_2$, and $\text{Vol}(\cdot)$ denotes the volume of a region.
In [5], Cheeger showed that

**Lemma 2.2.** Let \((M, g)\) be a closed Riemannian manifold. Then

\[
\lambda_1 \geq \frac{IN_1(M)^2}{4}.
\]

Thus, one can get a lower bound of \(\lambda_1\) by bounding \(IN_1(M)\) from below. As indicated in [9], this can be done by using the method therein. For completeness, we state the result and also include the proof in the following.

**Theorem 2.3.** Let \((M^m, g)\) be an \(m\)-dimensional complete Riemannian manifold, \(m \geq 2\). Assume that (1.1) and (1.2) are satisfied. Let \(\Omega\) be a bounded convex domain in \(M\). Then for \(1 \leq \alpha \leq m - 1\), we have

\[
IN_\alpha(\Omega) \geq d^{-1}2^{2m-15-m}e^{-((24-\frac{2}{\alpha})Ld-(104-\frac{1}{5})Kd^2)}Vol(\Omega)^{1-\frac{1}{\alpha}},
\]

and for \(0 < \alpha < 1\), we have

\[
IN_\alpha(\Omega) \geq d^{-1}2^{2m-15-m}e^{-22Ld-103Kd^2}Vol(\Omega)^{1-\frac{1}{\alpha}},
\]

where \(d\) is the diameter of the domain \(\Omega\).

In particular, if \(M\) is closed, then

\[
IN_1(M) \geq D^{-1}2^{2m-15-m}e^{-103K}\]

and

\[
IN_{m-1}(M) \geq D^{-1}2^{2m-15-m}e^{-22Ld-103Kd^2}Vol(M)^{\frac{1}{m}},
\]

where \(D\) is an upper bound of the diameter of \(M\).

Before starting the proof of Theorem 2.3, let us present some results needed. First of all, Q. Zhang and the third author [25] proved a volume comparison theorem for manifolds satisfying (1.1) and (1.2).

**Theorem 2.4.** Let \((M^m, g)\) be an \(m\)-dimensional complete Riemannian manifold. Suppose that \(\text{Ric} + \frac{1}{2}L_V g \geq -Kg\) for some constant \(K \geq 0\) and smooth vector field \(V\) with \(|V| \leq L\), where \(L_V\) means the Lie derivative in the direction of \(V\). Then the following conclusions are true.

(a) Let \(A(s, \theta)\) denote the volume element of the metric \(g\) on \(M\) in geodesic polar coordinates. Then for any \(0 < s_1 < s_2\), we have

\[
\frac{A(s_2, \theta)}{s_2^{m-1}} \leq e^{2Ls_2 + Ks_2^2}A(s_1, \theta) \frac{s_1^{m-1}}{s_1^{m-1}},
\]

(b) For any \(0 < r_1 < r_2\), we have

\[
\frac{\text{Vol}(B_{r_2}(x))}{r_2^m} \leq e^{K(r_2^2 - r_1^2) + 2L(r_2 - r_1)}\frac{\text{Vol}(B_{r_1}(x))}{r_1^m},
\]

where \(B_r(x)\) is the geodesic ball centered at \(x \in M\) with radius \(r\).

**Remark 2.5.** When \(V = \nabla f\), the assumptions in the above Theorem become (1.1) and (1.2).

Next, we need the following lemma by Gromov.
Lemma 2.6 ([10]). Let \((M^m, g)\) be a complete Riemannian manifold. Let \(\Omega\) be a convex domain in \(M\), and \(H\) a hypersurface dividing \(\Omega\) into two parts \(\Omega_1, \Omega_2\). For any Borel subsets \(W_i \subset \Omega_i, i = 1, 2\), there exists an \(x_1\) in one of \(W_i\), say \(W_1\), and a subset \(W\) in the other part \(W_2\), such that

\[
\text{Vol}(W) \geq \frac{1}{2} \text{Vol}(W_2),
\]

and for any \(x_2 \in W\), there is a unique minimal geodesic \(\gamma_{x_1, x_2}\) between \(x_1\) and \(x_2\) which intersects \(H\) at some \(z\) with

\[
\text{dist}(x_1, z) \geq \text{dist}(x_2, z),
\]

where \(\text{dist}(x_1, z)\) denotes the distance between \(x_1\) and \(z\).

Combining Theorem 2.4 and Lemma 2.6 we get

Lemma 2.7. Let \(H, W\) and \(x_1\) be as in Lemma 2.6. Then

\[
\text{Vol}(W) \leq D_1 2^{m-1} e^{4LD_1 + 4KD_1^2} \text{Vol}(H'),
\]

where \(D_1 = \sup_{x \in W} \text{dist}(x_1, x)\), and \(H'\) is the set of intersection points with \(H\) of geodesics \(\gamma_{x_1, x}\) for all \(x \in W\).

Proof. Let \(S_{x_1}\) be the set of unit tangent vectors of \(M\) at \(x_1\), and \(\Gamma \subset S_{x_1}\) the subset of vectors \(\theta\) such that \(\gamma_{\theta} = \gamma_{x_1, x_2}\) for some \(x_2 \in W\). The volume element of the metric \(g\) is written as \(dV = A(\theta, t)d\theta \wedge dt\) in polar coordinates \((\theta, t) \in S_{x_1} \times \mathbb{R}^+\). For any \(\theta \in \Gamma\), let \(r(\theta)\) be the radius such that \(\exp_{x_1}(r(\theta)) \in H\). Then it follows from Lemma 2.6 that \(W \subset \{ \exp_{x_1}(r(\theta)) | r(\theta) \leq r \leq 2r(\theta), \theta \in \Gamma \}\), and hence

\[
\text{Vol}(W) \leq \int_{r(\theta)}^{2r(\theta)} A(\theta, t)dtd\theta.
\]

For \(r(\theta) \leq t \leq 2r(\theta) \leq 2D_1\), by (2.5), we have

\[
\frac{A(\theta, t)}{t^{m-1}} \leq e^{2Lt + Kt^2} \frac{A(\theta, r(\theta))}{r(\theta)^{m-1}},
\]

which implies that

\[
A(\theta, t) \leq e^{4LD_1 + 4KD_1^2} 2^{m-1} A(\theta, r(\theta)).
\]

Plugging the above inequality into (2.10) gives

\[
\text{Vol}(W) \leq e^{4LD_1 + 4KD_1^2} 2^{m-1} \int_{\Gamma} r(\theta) A(\theta, r(\theta))d\theta \leq D_1 2^{m-1} e^{4LD_1 + 4KD_1^2} \text{Vol}(H').
\]

When \(W\) is the intersection of \(\Omega\) and a ball in \(M\), the above lemma implies that

Corollary 2.8. Let \(H\) be any hypersurface dividing a convex domain \(\Omega\) into two parts \(\Omega_1, \Omega_2\). For any ball \(B_r(x)\) in \(M\), we have

\[
\min(\text{Vol}(B_r(x) \cap \Omega_1), \text{Vol}(B_r(x) \cap \Omega_2)) \leq 2^{m+1} r e^{4Ld + 4Kd^2} \text{Vol}(H \cap (B_{2r}(x))),
\]

where \(d = \text{diam}(\Omega)\), the diameter of \(\Omega\). In particular, if \(B_r(x) \cap \Omega\) is divided equally by \(H\), then

\[
\text{Vol}(B_r(x) \cap \Omega) \leq 2^{m+2} r e^{4Ld + 4Kd^2} \text{Vol}(H \cap B_{2r}(x)).
\]
Proof. Put $W_i = B_r(x) \cap \Omega_i$ in the above lemma and use $D_1 \leq 2r$ and $H' \subset H \cap B_{2r}(x)$. □

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Let $H$ be any hypersurface dividing $M$ into two parts, $\Omega_1$ and $\Omega_2$. We may assume that $\text{Vol}(\Omega_1) \leq \text{Vol}(\Omega_2)$. For any $x \in \Omega_1$, let $r_x$ be the smallest radius such that

$$\text{Vol}(B_{r_x}(x) \cap \Omega_1) = \text{Vol}(B_{r_x}(x) \cap \Omega_2) = \frac{1}{2} \text{Vol}(B_{r_x}(x) \cap \Omega).$$

By (2.12), we have,

$$\text{Vol}(B_{r_x}(x) \cap \Omega) \leq 2^{m+2} r_x e^{4Ld+4Kd^2} \text{Vol}(H \cap B_{2r_x}(x)).$$

(2.13)

The domain $\Omega_1$ has a covering

$$\Omega_1 \subset \bigcup_{x \in \Omega_1} B_{2r_x}(x).$$

By Vitali Covering Lemma, we can choose a countable family of disjoint balls $B_i = B_{2r_{x_i}}(x_i)$ such that $\bigcup_i B_{10r_{x_i}}(x_i) \supset \Omega_1$. So

$$\text{Vol}(\Omega_1) \leq \sum_i \text{Vol}(B_{10r_{x_i}}(x_i) \cap \Omega_1).$$

Applying the volume comparison Theorem 2.4 in $\Omega_1$ gives

$$\frac{\text{Vol}(B_{10r_{x_i}}(x_i) \cap \Omega_1)}{(10r_{x_i})^m} \leq e^{99Kr_{x_i}^2 + 18Lr_{x_i}} \frac{\text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega_1)}{(r_{x_i})^m}.$$ 

On the other hand, since $\text{Vol}(\Omega_1) \leq \text{Vol}(\Omega_2)$, we have $r_x \leq d$ for any $x \in \Omega_1$. Thus,

$$\text{Vol}(B_{10r_{x_i}}(x_i) \cap \Omega_1) \leq 10^m e^{99Kd^2 + 18Ld} \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega_1)$$

$$= 2^{-1} 10^m e^{99Kd^2 + 18Ld} \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega_1).$$

Therefore,

$$\text{Vol}(\Omega_1) \leq 2^{-1} 10^m e^{99Kd^2 + 18Ld} \sum_i \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega).$$

(2.14)

Moreover, since the balls $B_i$ are disjoint, (2.13) gives

$$\text{Vol}(H) \geq \sum_i \text{Vol}(B_i \cap H) \geq 2^{-m-2} e^{-4Ld-4Kd^2} \sum_i r_{x_i}^{-1} \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega).$$

(2.15)

When $1 \leq \alpha \leq \frac{m}{m-1}$, it follows from (2.14) and (2.15) that

$$\frac{\text{Vol}(H)}{\text{Vol}(\Omega_1)^{\frac{1}{\alpha}}} \geq \frac{2^{-m-2} e^{-4Ld-4Kd^2} \sum_i r_{x_i}^{-1} \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega)}{(2^{-1} 10^m e^{99Kd^2 + 18Ld})^{\frac{1}{\alpha}} \left(\sum_i \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega)\right)^{\frac{1}{\alpha}}}$$

$$\geq \frac{2^{-m-2} e^{-4Ld-4Kd^2} \sum_i r_{x_i}^{-1} \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega)}{2^{-1} 10^m e^{99Kd^2 + 18Ld} \sum_i \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega)^{\frac{1}{\alpha}}}$$

$$\geq \frac{2^{-m-5} e^{-22Ld-103Kd^2} \inf_i r_{x_i}^{-1} \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega)}{\text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega)^{\frac{1}{\alpha}}}$$

$$= 2^{-m-5} e^{-22Ld-103Kd^2} \inf_i r_{x_i}^{-1} \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega)^{1-\frac{1}{\alpha}}.$$
Applying the volume comparison Theorem 2.1 in $\Omega$ gives
\[
\frac{\text{Vol}(B_d(x_i) \cap \Omega)}{d^m} \leq e^{K_d^2 + 2L_d} \frac{\text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega)}{r_{x_i}^m}.
\]

Since $1 - \frac{1}{\alpha} \geq 0$, and $m(1 - \frac{1}{\alpha}) - 1 \leq 0$, we derive
\[
\inf r_{x_i}^{-1} \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega)^{1 - \frac{1}{\alpha}} \geq d^{m(1 - \frac{1}{\alpha}) - 1} \inf r_{x_i}^{-m(1 - \frac{1}{\alpha})} \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega)^{1 - \frac{1}{\alpha}} \\
\geq d^{-1} e^{-(K_d^2 + 2L_d)(1 - \frac{1}{\alpha})} \text{Vol}(\Omega)^{1 - \frac{1}{\alpha}}.
\] (2.17)

From (2.16) and (2.17), we conclude that
\[
IN_{\alpha}(\Omega) \geq d^{-1} 2^{-m} \frac{1}{15} m e^{-(24 - \frac{2}{\alpha})L_d - (104 - \frac{4}{\alpha})K_d^2} \text{Vol}(\Omega)^{1 - \frac{1}{\alpha}}.
\]

On the other hand, when $0 < \alpha < 1$, similarly to (2.16), we have
\[
\frac{\text{Vol}(H)}{\text{Vol}(\Omega)} = \frac{\text{Vol}(H)}{\text{Vol}(\Omega) \text{Vol}(\Omega)^{\frac{1}{\pi} - 1}} \geq \frac{\text{Vol}(H)}{\text{Vol}(\Omega) \text{Vol}(\Omega)^{\frac{1}{\pi} - 1}} \\
\geq \frac{2^{-m} e^{-4L_d - 4K_d^2}}{2^{-1} 10^9 e^{90K_d^2 + 18L_d}} \sum_i \text{Vol}(B_{r_{x_i}}(x_i) \cap \Omega) \text{Vol}(\Omega)^{1 - \frac{1}{\alpha}} \\
\geq d^{-1} 2^{-m} \frac{1}{15} m e^{-22L_d - 103K_d^2} \text{Vol}(\Omega)^{1 - \frac{1}{\alpha}}.
\] (2.18)

Taking infimum over $H$ finishes the proof. \qed

From Lemma 2.2 and Theorem 2.3, we immediately have the estimate of the first eigenvalue.

**Theorem 2.9.** Let $(M^m, g)$ be an $m$-dimensional closed Riemannian manifold with diameter bounded from above by $D$, and $m \geq 2$. Suppose that (1.1) and (1.2) are satisfied. Then
\[
\lambda_1 \geq \frac{1}{16} D^{-2} 400^{-m} e^{-14L_d - 206K_d^2} := c_0.
\] (2.19)

To derive the lower bound of higher order eigenvalues, we need to use gradient estimates for eigenfunctions, which in term require a Sobolev inequality. According to section 9 in [11], the desired Sobolev inequality follows from the lower bound estimate of $IN_{\frac{m}{m-1}}(M)$.

**Definition 2.10 ([11]).** Let $(M^m, g)$ be an $m$-dimensional compact Riemannian manifold (with or without boundary). For any $\alpha > 0$, the Neumann $\alpha$-Sobolev constant of $M$ is defined by
\[
SN_{\alpha}(M) = \inf_{f \in H^{1,1}(M)} \frac{\int_M |\nabla f|^2}{\{\inf_{k \in \mathbb{R}} \int_M |f - k|^\alpha \}^{\frac{1}{\alpha}}},
\]
where $H^{1,1}(M)$ is the Sobolev space.

As pointed out in [11], when $\alpha > \frac{m}{m-1}$, it holds that $IN_{\alpha}(M) = SN_{\alpha}(M) = 0$. In general, the relation between $IN_{\alpha}(M)$ and $SN_{\alpha}(M)$ is as follows.

**Lemma 2.11** (section 9 in [11]). For any $\alpha > 0$, we have
\[
\min\{1, 2^{1 - \frac{1}{\alpha}}\} IN_{\alpha}(M) \leq SN_{\alpha}(M) \leq \max\{1, 2^{1 - \frac{1}{\alpha}}\} IN_{\alpha}(M).
\]
Moreover, a lower bound of the Sobolev constant $SN_\alpha(M)$ provides a Sobolev inequality. In fact, we have

**Lemma 2.12** (Corollary 9.9 in [11]). Let $(M^m, g)$ be a compact Riemannian manifold (with or without boundary). There exist constants $C_1(\alpha), C_2(\alpha) > 0$ depending only on $\alpha$, such that

$$\int_M |\nabla f|^2 \geq C_1(\alpha) S N_\alpha(M)^2 \left( \int_M |f|^{\frac{2m}{m-2}} - C_2(\alpha) \text{Vol}(M)^{\frac{2-2\alpha}{\alpha}} \int_M |f|^2 \right)$$

for all $f \in H^{1,2}(M)$.

Then by choosing $\alpha = \frac{m}{m-1}$ for $m \geq 3$ and $\alpha = \frac{4}{3}$ for $m = 2$, and combining Lemma 2.11, Lemma 2.12, and Theorem 2.3, one can get the following Sobolev inequalities.

**Corollary 2.13.** Let $(M^m, g)$ be an $m$-dimensional compact Riemannian manifold (with or without boundary). Assume that (1.1) and (1.2) are satisfied. Then for any $f \in H^{1,2}(M)$,

1. when $m \geq 3$, we have

$$\int_M |\nabla f|^2 \geq C_1(m) \tilde{C}^2 \text{Vol}(M)^{\frac{2}{m}} \left( \int_M |f|^{\frac{2m}{m-2}} - C_2(m) \text{Vol}(M)^{-\frac{2}{m}} \int_M |f|^2 \right),$$

   where $\tilde{C} = D^{-1}2^{-m-1}e^{-(22+\frac{2}{m})LD-(103+\frac{1}{m})KD^2}$, and $C_1(m)$ and $C_2(m)$ are dimensional constants;

2. when $m = 2$, one has

$$\int_M |\nabla f|^2 \geq \tilde{S}_1 \tilde{S}^2 \text{Vol}(M)^{\frac{1}{2}} \left( \int_M |f|^4 \right)^{\frac{1}{2}} - \tilde{S}_2 \text{Vol}(M)^{-\frac{1}{2}} \int_M |f|^2,$$

   where $\tilde{S}_1$ and $\tilde{S}_2$ are pure constants, and $\tilde{S} = D^{-1}2^{-5}e^{-(22+\frac{1}{2})LD-(103+\frac{1}{4})KD^2}$.

**Remark 2.14.** By carefully following the proof of Corollary 9.9 in [11], one can check that we may take $C_1(m) = \frac{(m-2)^2}{4(m-1)^2}2^{\frac{2m}{m-1}}$, $C_2(m) = 2^{\frac{2m^3-7m^2+2m+4}{m(m-1)(m-2)^2}}$, $\tilde{S}_1 = 3^{-2}2^{-\frac{1}{5}}$, and $\tilde{S}_2 = 2^{\tilde{S}}$.

3. GRADIENT AND HIGHER ORDER EIGENVALUE ESTIMATES

   In this section, we use a method in [23] (see also [15]) to show the lower bound estimates of high order eigenvalues. Firstly, we prove a gradient estimate of eigenfunctions by Moser iteration.

**Proposition 3.1.** Let $(M^m, g)$, $m \geq 3$, be an $m$-dimensional closed Riemannian manifold. Suppose that (1.1) and (1.2) are satisfied. Let $\lambda$ be an eigenvalue of the Laplace operator, and $u$ an eigenfunction satisfying $\Delta u = -\lambda u$. Then we have the following gradient estimate.

$$|\nabla u|^2 \leq 2^m \left( \frac{m}{m-2} \right)^{\frac{m(m-2)}{2}} \left( \frac{3\lambda + 2K + 2L^2 + C_2}{C_1} \right) \left( \lambda + L^2 \right) \text{Vol}(M)^{-1} \int_M u^2, \quad (3.1)$$
where \( C_1 = C_1(m)\hat{C}^2 \), and \( C_2 = C_1(m)\hat{C}^2 C_2(m) \) with \( C_1(m), C_2(m), \hat{C} \) the constants in (2.20).

In particular, when \( ||u||_{L^2} = 1 \), we have

\[
|\nabla u|^2 \leq 2^m \left( \frac{m}{m-2} \right)^{\frac{m-2}{2}} \left( \frac{3\lambda + 2K + 2L^2 + C_2}{C_1} \right) \frac{m}{2} (\lambda + L^2) \text{Vol}(M)^{-1}. \tag{3.2}
\]

**Proof.** Let \( v = |\nabla u|^2 + L^2 u^2 \). The Bochner formula and assumptions (1.1) and (1.2) induce that

\[
\Delta v = 2|\text{Hess } u|^2 + 2 \nabla \Delta u, \nabla u > + 2\text{Ric}(\nabla u, \nabla u) + 2L^2 u\Delta u + 2L^2 |\nabla u|^2 \\
\geq 2|\text{Hess } u|^2 - 2\lambda |\nabla u|^2 - 2K |\nabla u|^2 - 2f_{ij}u_iu_j - 2L^2 \lambda u^2 + 2L^2 |\nabla u|^2 \\
= 2|\text{Hess } u|^2 - 2\lambda v + (2L^2 - 2K) |\nabla u|^2 - 2f_{ij}u_iu_j \\
\geq 2u_{ij}^2 - 2(\lambda + K)v - 2f_{ij}u_iu_j.
\]

Multiple both sides above by \( v^{p-1}, p \geq 2 \), and take integrals over \( M \). Notice that

\[
\int_M v^{p-1} \Delta v = -\int_M <\nabla v^{p-1}, \nabla v >= -\int_M (p-1)v^{p-2} <\nabla v, \nabla v > \\
= -(p-1) \int_M v^{p-2} |\nabla v|^2 = -\frac{4(p-1)}{p^2} \int_M |\nabla v|^2 v^{p-1}.
\]

Hence, we have

\[
\frac{4(p-1)}{p^2} \int_M |\nabla v|^2 v^{p-1} \leq -2 \int_M u_{ij}^2 v^{p-1} + 2(\lambda + K) \int_M v^p + 2 \int_M f_{ij}u_iu_j v^{p-1}. \tag{3.3}
\]

For the third term on the right hand side above, integrating by part yields

\[
2 \int_M f_{ij}u_iu_j v^{p-1} = -2 \int_M f_i(u_iu_j v^{p-1})_j \\
\quad = -2 \int_M f_i u_iu_j v^{p-1} - 2 \int_M f_i u_iu_j v^{p-1} - 2 \int_M f_i u_iu_j (p-1)v^{p-2} v_j. \tag{3.4}
\]

For \( I \) above, using Cauchy-Schwarz inequality and the bound of \( |\nabla f| \) gives

\[
I = -2 \int_M f_i u_iu_j v^{p-1} \leq 2 \int_M \left( u_{ij}^2 v^{p-1} + \frac{1}{4} f_i^2 u_j^2 v^{p-1} \right) \\
= 2 \int_M u_{ij}^2 v^{p-1} + \frac{1}{2} \int_M f_i^2 u_j^2 v^{p-1} \\
\leq 2 \int_M u_{ij}^2 v^{p-1} + \frac{L^2}{2} \int_M v^p.
\]

Next, noticing that \( v \geq 2L|\nabla u||u| \), we have

\[
II = -2 \int_M f_i u_iu_j v^{p-1} = 2\lambda \int_M f_i u_iu_j v^{p-1} \leq 2\lambda L \int_M |\nabla u||u|v^{p-1} \leq \lambda \int_M v^p.
\]
Finally, by applying Cauchy-Schwarz inequality inequality again to $III$, we deduce

$$III = -2 \int_M f_j u_j (p - 1) v_j v_j \leq 2(p - 1) L \int_M \left| \nabla u_j^2 \right| \left| \nabla v \right| v_j v_j \leq 2(p - 1) L \int_M \left| \nabla v \right| v_j v_j$$

$$\leq 2(p - 1) L (\frac{1}{4\epsilon_1} \int_M v^p + \epsilon_1 \int_M \left| \nabla v \right|^2 v_j v_j)$$

$$= \frac{(p - 1) L}{2\epsilon_1} \int_M v^p + \frac{8(p - 1) L \epsilon_1}{p^2} \int_M \left| \nabla v \right|^2,$$

where $\epsilon_1 > 0$ is any constant. Thus, by combining the above estimates in (3.3), we arrive at

$$\left( \frac{4(p - 1)}{p^2} - \frac{8(p - 1) L \epsilon_1}{p^2} \right) \int_M \left| \nabla v \right|^2 \leq (3\lambda + \frac{L^2}{2} + 2K + \frac{(p - 1) L}{2\epsilon_1}) \int_M v^p.$$ 

Assume for now that $L > 0$. Then, by choosing $\epsilon_1 = \frac{1}{4\lambda}$ and noticing that $\frac{2(p - 1)}{p^2} \geq \frac{1}{p}$ for $p \geq 2$, one gets

$$\int_M \left| \nabla \tilde{v} \right|^2 \leq p^2 (3\lambda + 2L^2 + 2K) \int_M v^p. \tag{3.5}$$

If $L = 0$, then $f$ is a constant, and from (3.3) we conclude that

$$\int_M \left| \nabla \tilde{v} \right|^2 \leq \frac{1}{2} p^2 (\lambda + K) \int_M v^p,$$

which is better than (3.5). Therefore, we always have

$$\int_M \left| \nabla \tilde{v} \right|^2 \leq p^2 (3\lambda + 2L^2 + 2K) \int_M v^p. \tag{3.6}$$

Recall the Sobolev inequality (2.20),

$$\int_M \left| \nabla f \right|^2 \geq C_1 \text{Vol}(M) \frac{2}{m} \left( \int_M |f|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} - C_2 \int_M |f|^2 \quad \tag{3.7}$$

for all $f \in H^{1,2}(M)$, where $C_1 = C_1(m)\bar{C}^2$, and $C_2 = C_1(m)\bar{C}^2\bar{C}_2(m)$. Putting $f = \tilde{v}$ and using (3.6) yield

$$\left( \int_M \left| \tilde{v} \right|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq p^2 \left( \frac{3\lambda + 2K + 2L^2 + C_2}{C_1 \text{Vol}(M)^{\frac{2}{m}}} \right) \int_M v^p.$$ 

Denote $Q = \frac{3\lambda + 2K + 2L^2 + C_2}{C_1 \text{Vol}(M)^{\frac{2}{m}}}$ for convenience. The inequality above means that

$$||v||^{\frac{m}{m-2}} \leq \left( p^2 Q \right)^{\frac{1}{m}} ||v||_p$$

for all $p \geq 2$.

Setting $\beta = \frac{m}{m-2}$, $p = 2\beta^j$ for $j = 0, 1, 2, \ldots$, it implies that

$$||v||_{2\beta^j+1} \leq 2^{\frac{1}{\beta^j}} \beta^\frac{j}{\beta} Q^{\frac{1}{2\beta^j}} \frac{1}{\beta} ||v||_{2\beta^j}.$$ 

Iterating this estimate, we conclude that

$$||v||_{2\beta^j+1} \leq 2^{\sum_{l=0}^{\infty} \frac{1}{\beta^l}} \beta^\frac{1}{\beta} Q^{\frac{1}{2\beta^l}} \frac{1}{\beta} ||v||_2.$$
Letting $j \to \infty$, we obtain
\[ ||v||_\infty \leq 2^m \left( \frac{m}{m-2} \right)^{m-2} Q^m ||v||_2. \]
Notice that $\int_M v^2 \leq ||v||_\infty \int_M v$. Therefore, the above estimate reduces to
\[ \max_M v \leq 2^m \left( \frac{m}{m-2} \right)^{m-2} Q^m \int_M v. \]
This finishes the proof, since
\[ \int_M v = \int_M (|\nabla u|^2 + L^2 u^2) = (\lambda + L^2) \int_M u^2. \]
\[ \Box \]

When $m = 2$, by using the Sobolev inequality (2.21) instead of (2.20), one can similarly obtain the following gradient estimate for $u$.

**Proposition 3.2.** If $(M, g)$ is a Riemann surface, $u$ is an eigenfunction associated to eigenvalue $\lambda$, and (1.1) and (1.2) are satisfied, then
\[ |\nabla u|^2 \leq 2^8 \left( \frac{3\lambda + 2K + 2L^2 + S_2}{S_1} \right)^2 (\lambda + L^2) Vol(M)^{-1} \int_M u^2, \]
where $S_1 = \tilde{S}_1 \tilde{S}_2$, and $S_2 = \tilde{S}_1 \tilde{S}_2 \tilde{S}_2$ with $\tilde{S}_1, \tilde{S}_2, \tilde{S}$ the constants in (2.21).

Next, we prove a similar gradient estimate for linear combinations of eigenfunctions.

**Proposition 3.3.** Let $(M^m, g_{ij})$ be an $m$-dimensional closed Riemannian manifold satisfying (1.1) and (1.2). Let $\phi_j$ be a normalized eigenfunction associated to $\lambda_j$, $j = 1, 2, ..., k$ i.e., $\Delta \phi_j = -\lambda_j \phi_j$ and $\int_M |\phi_j|^2 dV = 1$. Then for any sequence of real numbers $b_j$, $j = 1, 2, ..., k$, with $\sum_{j=1}^k b_j^2 \leq 1$, the linear combination $w = \sum_{j=1}^k b_j \phi_j$ satisfies that, for $m \geq 3$,
\[ |\nabla w|^2 + L^2 w^2 \leq 2^m \left( \frac{m}{m-2} \right)^{m-2} \left( \frac{6\lambda_k + 2K + 2L^2 + C_2}{C_1} \right)^2 (\lambda_k + L^2) Vol(M)^{-1}, \]
and for $m = 2$,
\[ |\nabla w|^2 + L^2 w^2 \leq 2^8 \left( \frac{6\lambda_k + 2K + 2L^2 + S_2}{S_1} \right)^2 (\lambda_k + L^2) Vol(M)^{-1}, \]
where $C_1, C_2, S_1, S_2$ are constants in Propositions 3.1 and 3.2.

**Proof.** Here, we only present the proof of (3.8). The proof of (3.9) is similar by using (2.21) instead of (2.20). First of all, since $\lambda_k > 0$, we can write
\[ \Delta w = -\sum_{j=1}^k \lambda_j b_j \phi_j = -\lambda_k \eta, \]
where \( \eta = \sum_{j=1}^{k} \frac{\lambda_j}{\lambda_k} b_j \phi_j \).

Let \( v = |\nabla w|^2 + L^2 w^2 \). Then
\[
\Delta v = 2|Hess w|^2 + 2 < \nabla \Delta w, \nabla w > + 2Ric(\nabla w, \nabla w) + 2L^2 \Delta w + 2L^2 |\nabla w|^2
\]
\[
\geq 2w_{ij}^2 - 2\lambda_k \eta_i w_i - 2K|\nabla w|^2 - 2f_{ij} w_i w_j - 2L^2 \lambda_k \eta w.
\]
\[
\geq 2w_{ij}^2 - 2\lambda_k \eta_i w_i - 2K v - 2f_{ij} w_i w_j - 2L^2 \lambda_k \eta w.
\]

Multiplying both sides by \( v^{p-1}, p \geq 2 \), and integrating over \( M \) give
\[
\frac{4(p-1)}{p^2} \int_M |\nabla v|^2 \leq -2 \int_M w_{ij}^2 v^{p-1} + 2\lambda_k \int_M \eta_i w_i v^{p-1}
\]
\[
+ 2K \int_M v^p + 2 \int_M f_{ij} w_i w_j v^{p-1} + 2\lambda_k L^2 \int_M \eta v v^{p-1}.
\]
(3.10)

Using Hölder inequality yields
\[
2\lambda_k \int_M \eta_i w_i v^{p-1} \leq 2\lambda_k \int_M |\nabla \eta| v^{p-\frac{1}{2}} \leq 2\lambda_k \left( \int_M |v|^p \right)^{\frac{p-\frac{1}{2}}{p}} \left( \int_M |\nabla \eta|^2p \right)^{\frac{1}{2p}}.
\]
(3.11)

Notice that the coefficients in \( \nabla \eta \) satisfy \( \sum_{j=1}^{k} (\frac{\lambda_j}{\lambda_k} b_j)^2 \leq \sum_{j=1}^{k} \frac{b_j}{\lambda_k} \leq 1 \) and \( \int_M v^p \geq \int_M |\nabla w|^{2p} \). Thus,
\[
\int_M |\nabla \eta|^{2p} \leq \max_{b_1, \ldots, b_k} \int_M v^p.
\]
(3.12)

By combining (3.11) and (3.12), we obtain
\[
2\lambda_k \int_M \eta_i w_i v^{p-1} \leq 2\lambda_k \max_{b_1, \ldots, b_k} \int_M v^p.
\]
(3.13)

Here and in the rest of the proof, the maximum is taken for all real numbers \( b_1, \ldots, b_k \) such that \( \sum_{j=1}^{k} b_j^2 \leq 1 \).

Similarly, for the last term of (3.10), we have
\[
2\lambda_k L^2 \int_M \eta v v^{p-1} \leq 2\lambda_k L \int_M |\nabla \eta| v^{p-\frac{1}{2}} \leq 2\lambda_k L \left( \int_M |v|^p \right)^{\frac{p-\frac{1}{2}}{p}} \left( \int_M |\nabla \eta|^2p \right)^{\frac{1}{2p}}
\]
\[
\leq 2\lambda_k \max_{b_1, \ldots, b_k} \int_M v^p.
\]
(3.14)

Finally, we need to deal with the fourth term on the right hand side of (3.10). Using integration by parts gives
\[
2 \int_M f_{ij} w_i w_j v^{p-1} = -2 \int_M f_i w_{ij} w_j v^{p-1} - 2 \int_M f_i w_i w_j v^{p-1} + 2 \int_M f_i w_i w_j (p-1) v^{p-2} v_j.
\]
\[
\text{(I)} \quad \text{(II)} \quad \text{(III)}
\]
(3.15)

Using Cauchy-Schwarz inequality and the bound of \( |\nabla f| \), we have
\[
I = -2 \int_M f_i w_{ij} w_j v^{p-1} \leq \frac{L^2}{2} \int_M w_{ij}^2 v^{p-1} \leq 2 \int_M w_{ij}^2 v^{p-1} + \frac{L^2}{2} \max_{b_1, \ldots, b_k} \int_M v^p,
\]
\[
\text{II} = \frac{L^2}{2} \int_M v^p.
\]
\[
\text{III} \leq \frac{L^2}{2} \max_{b_1, \ldots, b_k} \int_M v^p.
\]
Thus, we get

$$II = -2 \int_M f_i w_i w_j v^{p-1} \leq 2 \lambda_k \int_M |\nabla f||\nabla w||\eta|v^{p-1} \leq 2 \lambda_k L \int_M |\eta|v^{p-\frac{1}{2}} \leq 2 \lambda_k \max_{b_1, \ldots, b_k} \int_M v^p,$$

and

$$III = -2 \int_M f_i w_i w_j (p-1)v^{p-2} v_j \leq (p-1) L \int_M |\nabla w|^2 v^{p-2} |\nabla v| \leq 2(p-1) L \int_M v^{p-1} |\nabla v|$$

$$\leq 2(p-1) L \left( \frac{1}{4 \varepsilon_2} \int_M v^p + \varepsilon_2 \int_M v^{p-2} |\nabla v|^2 \right)$$

$$= \left( \frac{p-1}{2 \varepsilon_2} \right) \max_{b_1, \ldots, b_k} \int_M v^p + \frac{8(p-1) L \varepsilon_2}{p^2} \int_M |\nabla v|^2,$$

where $\varepsilon_2 > 0$ is arbitrary constant. Hence, plugging the estimates above in (3.10) asserts that

$$\left( \frac{4(p-1)}{p^2} - \frac{8(p-1) L \varepsilon_2}{p^2} \right) \int_M |\nabla v|^2 \leq \left( 6 \lambda_k + \frac{L^2}{2} + 2 K + \frac{(p-1) L}{2 \varepsilon_2} \right) \max_{b_1, \ldots, b_k} \int_M v^p.$$

Choosing $\varepsilon_2 = \frac{1}{12}$, it follows that

$$\max_{b_1, \ldots, b_k} \int_M |\nabla v|^2 \leq 2^p \left( 6 \lambda_k + 2 K + 2 L^2 \right) \max_{b_1, \ldots, b_k} \int_M v^p.$$

Again, by (3.16) and the Sobolev inequality (2.20), we have

$$\max_{b_1, \ldots, b_k} \left( \int_M v^{m-2} \right) \frac{m-2}{m} \leq 2^p \left( \frac{6 \lambda_k + 2 K + 2 L^2 + C_2}{C_1 \operatorname{Vol}(M)^{\frac{2}{m}}} \right) \max_{b_1, \ldots, b_k} \left( \int_M v^p \right).$$

Denoting $Q = \frac{6 \lambda_k + 2 K + 2 L^2 + C_2}{C_1 \operatorname{Vol}(M)^{\frac{2}{m}}}$ and using Moser iteration as in Proposition 3.1, it follows that

$$\max_{b_1, \ldots, b_k} ||v||_\infty \leq 2^m \left( \frac{m}{m-2} \right)^{\frac{m(m-2)}{4}} Q^\frac{m}{2} \max_{b_1, \ldots, b_k} ||v||_2.$$

Square both sides above and notice that

$$\max_{b_1, \ldots, b_k} \int_M v^2 \leq \max_{b_1, \ldots, b_k} ||v||_\infty \max_{b_1, \ldots, b_k} \int_M v.$$

Thus, we get

$$\max_{b_1, \ldots, b_k} ||v||_\infty \leq 2^m \left( \frac{m}{m-2} \right)^{\frac{m(m-2)}{4}} Q^\frac{m}{2} \max_{b_1, \ldots, b_k} \int_M v. \quad (3.18)$$

On the other hand, since $\phi_1$, $\phi_2$, $\ldots$, $\phi_k$ are orthonormal, we have

$$\int_M v = \int_M (|\nabla w|^2 + L^2 w^2) = - \int_M w \Delta w + L^2 \int_M w^2$$

$$= \int_M \left( \sum_{j=1}^k b_j \phi_j \right) \left( \sum_{i=1}^k \lambda_i b_i \phi_i \right) + \sum_{j=1}^k \lambda_j b_j^2 \geq \left( \lambda_k + L^2 \right) \sum_{j=1}^k b_j^2 \leq \lambda_k + L^2.$$
This, together with (3.18), completes the proof.

The above gradient estimate for linear combinations of eigenfunctions allows us to derive the arithmetic inequality of the eigenvalues below.

**Lemma 3.4.** Under the same assumptions and notations as in Proposition 3.3, we have for \( m \geq 3 \),

\[
\lambda_1 + \lambda_2 + \ldots + \lambda_k \leq m^2 \left( \frac{m}{m-2} \right)^{\frac{m(m-2)}{2}} \left( \frac{6\lambda_k + 2K + 2L^2 + C_2}{C_1} \right)^{\frac{m}{2}} (\lambda_k + L^2), \tag{3.19}
\]

and for \( m = 2 \),

\[
\lambda_1 + \lambda_2 + \ldots + \lambda_k \leq 2^9 \left( \frac{6\lambda_k + 2K + 2L^2 + S_2}{S_1} \right)^2 (\lambda_k + L^2). \tag{3.20}
\]

**Proof.** We only prove (3.19) by using (3.8). The proof of (3.20) follows similarly from (3.9).

If \( k \leq m \), the conclusion follows immediately from Proposition 3.1 by integrating both sides of (3.2) for each \( \phi_j \), \( j = 1, 2, \ldots, k \).

When \( k > m \), for each \( x \in M \), we can find an orthogonal matrix \((a_{ij})_{k \times k}\) such that \( \varphi_i = \sum_{j=1}^k a_{ij} \phi_j, i = 1, 2, \ldots, k \) satisfy that

\[
\nabla_l \varphi_i(x) = 0, \quad l = 1, 2, \ldots, m, \quad m + 1 \leq i \leq k.
\]

Indeed, since the rank of the matrix

\[
J = \begin{pmatrix}
\nabla_1 \phi_1 & \ldots & \nabla_1 \phi_k \\
\vdots & & \vdots \\
\nabla_m \phi_1 & \ldots & \nabla_m \phi_k
\end{pmatrix}
\tag{3.21}
\]

is no more than \( m \), there are \( k - m \) linearly independent solutions of \( J \vec{e} = \vec{0} \), and then Schmidt orthogonalization gives \((a_{ij})\).

Thus, we derive from Proposition 3.3 that

\[
|\nabla \phi_1|^2 + \ldots + |\nabla \phi_k|^2 = |\nabla \varphi_1|^2 + \ldots + |\nabla \varphi_k|^2 = |\nabla \varphi_1|^2 + \ldots + |\nabla \varphi_m|^2 \leq m^2 \left( \frac{m}{m-2} \right)^{\frac{m(m-2)}{2}} \left( \frac{6\lambda_k + 2K + 2L^2 + C_2}{C_1} \right)^{\frac{m}{2}} (\lambda_k + L^2) \text{Vol}(M)^{-1}.
\]

Thus, integrating both sides gives Lemma 3.4.

**Remark 3.5.** Notice that the above Lemma cannot be deduced directly from Propositions 3.1 and 3.2, which will enlarge the coefficient \( m \) on the right hand side of (3.19) and (3.20) to be \( k \).

From (3.19) and (3.20), in order to get a lower bound of \( \lambda_k \), we only need the following lemma.
Lemma 3.6 ([23]). For $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$, if the inequality
\[ \lambda_1 + \lambda_2 + \ldots + \lambda_k \leq C_3 \lambda_k^{\frac{m}{m+1}} \]  
holds for any $k \geq 1$, then one has
\[ \lambda_k \geq C_4 k^{\frac{2}{m}}, \]  
where
\[ C_4 = \min \left\{ \lambda_1, \left( \frac{m}{C_3(m+2)} \right)^{\frac{2}{m}} \right\}, \]
and $m \geq 1$ is an integer.

Now we can see that a lower bound of $\lambda_k$ follows immediately from Theorem 2.9, Lemma 3.4 and Lemma 3.6.

Theorem 3.7. Assume that $(M^m, g)$ is an $m$-dimensional closed Riemannian manifold such that (1.1) and (1.2) are satisfied. Let $c_0$ be the lower bound of $\lambda_1$ in Theorem 2.9. Then
(1) for $m \geq 3$,
\[ \lambda_k \geq c_1 k^{\frac{2}{m}}, \quad \forall k \geq 2, \]  
where $c_1 = \min \left\{ c_0, \left( \frac{m}{C_5(m+2)} \right)^{\frac{2}{m}} \right\}$, and
\[ C_5 = m^{2m} \left( \frac{m}{m-2} \right)^{\frac{m(m-2)}{2}} c_0^{m(m-2)} \left( \frac{6c_0 + 2K + 2L^2 + C_1}{C_1} \right)^m (c_0 + L^2); \]
(2) for $m = 2$,
\[ \lambda_k \geq c_2 k^{\frac{1}{2}}, \quad \forall k \geq 2, \]  
where $c_2 = \min \left\{ c_0, \left( \frac{2}{3c_0} \right)^{\frac{1}{2}} \right\}$, and $C_6 = 2^9 c_0^{-3} \left( \frac{6c_0 + 2K + 2L^2 + S_2}{S_1} \right)^2 (c_0 + L^2)$. 

Proof. To prove (3.24), from Lemma 3.4, we have
\[ \lambda_1 + \lambda_2 + \ldots + \lambda_k \leq \lambda_k^{\frac{m}{m+1}} m^{2m} \left( \frac{m}{m-2} \right)^{\frac{m(m-2)}{2}} \left( \frac{6 + 2K + 2L^2 + C_1}{C_1} \right)^m (1 + \frac{L^2}{\lambda_k}). \]
Since $\lambda_k \geq \lambda_1 \geq c_0$, it follows that
\[ \lambda_1 + \lambda_2 + \ldots + \lambda_k \leq C_5 \lambda_k^{\frac{m}{m+1}}. \]  
From Lemma 3.6 we can easily get the conclusion. The proof of (3.25) is similar. \(\square\)

Remark 3.8. Recall that the constants $C_1$, $C_2$, $S_1$, and $S_2$ have explicit expressions according to Corollary 2.13 and Remark 2.14. Thus, the lower bound of $\lambda_k$ in the above theorem can also be expressed explicitly.
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