Uni-variate and Bi-variate Inverted Exponential-Teissier Distribution in Bayesian and Non-Bayesian Framework to Model Stochastic Dynamic Variation of Climate data

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AIM

Literature Survey

Inverted Exponential-Teissier (IET) Distribution

Statistical Properties

Characterization of IET

Measure of Reliability

Bivariate Inverted Exponential-Teissier (IET) Distribution

Measure of Dependence

Survival Function and Hazard Function

Simulation Study

Application to Kerala Rainfall Data

Summary
AIM

- To construct univariate and Bivariate Inverted Exponential-Teissier (IET) Distribution.
- To discuss its statistical properties.
- To illustrate the model using a simulation study.
- To apply it to Kerala rainfall data and compare it with the existing models.
Uni-variate and Bi-Variate probability distributions are very much indispensable to modeling different types of extreme spatial and spatio-temporal events like environmental, climate, rainfall, drought, and pollution data sets. In the literature, we found the followings:

- (Van Montfort and Witter (1986)) first employ probability distribution to model rainfall depth by Generalized Pareto distribution.
- (Aksoy (2000)) modeled daily rainfall amount and ascension curve with the help of two-parameter Gamma distribution.
- (B. Moccia et al. (2021)) explained the behavior of daily rainfall data in two geographic locations, namely, Lazio and Sicily, located in central and south Italy, are explained using six probability distributions, Frechet, Gumbel, Pareto type-II, Weibull, and Log-normal.
- (G. Teissier (1934)) introduces Teissier Distribution (TD) to the model of the frequency of mortality because of aging.
(Hanum et al. (2015)) explained the extreme rainfall event by a new family of probability distribution with the help of T-X transformation.

(V.K. Sharma et al. (2022)) introduces Exponentiated Teissier Distribution (ETD).

(N. Poonia et al. (2022)) has introduced Alpha Power Exponentiated Teissier distribution (APETD) and estimated the parameter using MLE, and illustrated the utility of this probability distribution to model the rainfall and temperature data.
There are a few limitations:

1. They don’t pay enough attention to estimating the parameters in the Bayesian framework, and missing data are inevitable in real-life data sets.

2. They ignore extending those distributions in the Bi-variate or multivariate scenario. Acceptance of the absence of covariates causes significant inaccuracy in prediction.

3. They explain the extreme environmental events disregarding their dependency analysis.
Teissier Distribution

Let $X$ follows the Teissier distribution with parameters $\theta (> 0)$. Then, the cumulative distribution function (CDF) of $X$ is given by:

$$F_X(x) = 1 - \exp(\theta x - \exp(\theta x) + 1) \quad x > 0, \theta > 0. \quad (1)$$

In this case, we write $T \sim TD(\theta)$.

The probability density function (PDF) of random variable $X$ is given by:

$$f_X(x) = \theta (\exp(\theta x) - 1) \exp(\theta x - \exp \theta x + 1) \quad x > 0, \theta > 0. \quad (2)$$
A modified Exponential-Teissier distribution function (ET) using T-X transformation is given by:

\[
F_{ET}(x) = 1 - \exp\left[\alpha(\theta x - \exp(\theta x) + 1)\right] \quad x > 0, \theta > 0, \alpha > 0.
\]

(3)

where T is considered as Exponential distribution with rate parameter \(\frac{1}{\alpha}\), and the X is considered as Teissier Distribution with parameter \(\theta\).

Let \(X \sim ET(\alpha, \theta)\).

Then, \(Y = \frac{1}{X} \sim IET(\Theta)\), where \(\Theta = \{\alpha, \theta\}\).
The CDF and PDF of IET are given by:

\[ F_{IET}(y; \Theta) = P(Y \leq y) = P\left(\frac{1}{X} \leq y\right) = P\left(X \geq \frac{1}{y}\right) = e^{\alpha \left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)} \]

\[ f_{IET}(y; \Theta) = e^{\alpha \left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)} \cdot \frac{\alpha \theta \left(e^{\frac{\theta}{y}} - 1\right)}{y^2}; \]

where, \(y > 0, \ \Theta = \{(\alpha, \theta) : \alpha > 0; \theta > 0\}\)
Figure: The characteristics of CDF (left) and PDF (right) of $IET(\alpha, \theta)$ for different values of $(\alpha, \theta)$. 
Expection $E(Y)$

Let $Y \sim IET(\Theta)$.

\[
E[Y] = \int_0^\infty 1 - e^{\alpha \cdot (\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1)} dy
\]

\[
= \int_0^\infty 1 - \sum_{r=0}^{\infty} \frac{\left(\alpha \cdot (\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1)\right)^r}{r!} dy
\]

\[
= - \int_0^\infty \sum_{r=1}^{\infty} \frac{\left(\alpha \cdot (\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1)\right)^r}{r!} dy
\]

\[
= \sum_{r=1}^{\infty} \sum_{k=0}^{r} \frac{(-1)^{k+1} \alpha^r (r)}{r!} \cdot \int_0^\infty \left(e^{\frac{\theta}{y}} - \frac{\theta}{y}\right)^k dy
\]
Mode ($M_0$) and Quantile [$Q(p)$] of IET

- Mode($M_0$) of $Y$ is given by

\[ e^{\theta M_0} + e^{-\theta M_0} = \frac{1}{\alpha} + 2 \]

\[ \Rightarrow \cosh \left( \frac{\theta}{M_0} \right) = 1 + \frac{1}{2\alpha} \]  \hspace{1cm} (6)

\[ \Rightarrow M_0 = \frac{\theta}{\cosh^{-1} \left( \frac{2\alpha+1}{2\alpha} \right)} \]

- $p^{th}$ Quantile [$Q(p)$] of $Y$ can be calculated from the following equality

\[ \frac{\theta}{Q(p)} - e^{\frac{\theta}{Q(p)}} = \left( \frac{1}{\alpha} \cdot \log(p) - 1 \right) \]  \hspace{1cm} (7)
Theorem

\( Y \sim IET(\Theta) \) iff

(i) \( M_Y(t) = \alpha \theta \sum_{p=2}^{\infty} \sum_{k=1}^{\infty} \sum_{j=0}^{p} \frac{(t\alpha)^p \cdot (-1)^j \cdot (p) \cdot \theta^k}{p! k!} \int_{0}^{\infty} \left( e^{\frac{\theta}{y}} - \frac{\theta}{y} \right)^j \cdot y^{p-2-k} \, dy \)

(ii) \( \phi_Y(t) = \alpha \theta \sum_{p=2}^{\infty} \sum_{k=1}^{\infty} \sum_{j=0}^{p} \frac{(it\alpha)^p \cdot (-1)^j \cdot (p) \cdot \theta^k}{p! k!} \int_{0}^{\infty} \left( e^{\frac{\theta}{y}} - \frac{\theta}{y} \right)^j \cdot y^{p-2-k} \, dy \)

(iii) \( \phi_Y(t) = \frac{M_Y(t)}{(i-1)} \), where \( i = \sqrt{-1} \).
The probability of lifetime (Y) of a component surviving beyond time y is survival function $[S_Y(y)]$ and the corresponding conditional failure rate, i.e., hazard rate $[h_Y(y)]$ are given in the followings:

$$S_Y(y; \Theta) = P(Y \geq y) = 1 - e^{\left[\alpha \cdot \left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)\right]}$$

$$h_Y(y; \Theta) = \frac{e^{\left[\alpha \cdot \left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)\right]} \cdot \alpha \theta \left( e^{\frac{\theta}{y}} - 1 \right)}{y^2 \cdot \left[1 - \exp \left(\alpha \cdot \left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)\right)\right]} \quad (8)$$

where, $y > 0$; and $\Theta = \{(\alpha, \theta): \alpha > 0; \theta > 0\}$
Behaviour of Hazard Rate of IET

Theorem

If \( Y \sim IET(\Theta) \) then \( h_Y(y;\Theta) \) is

(i) Increasing if

\[
(2y + \theta) \cdot e^{\frac{\theta}{y} - 2y} \cdot e^{\alpha \cdot \left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)} + \alpha \theta e^{\frac{2\theta}{y}} \geq (2\alpha + 1)\theta + 2y \cdot e^{\frac{\theta}{y}} - 2y - \alpha \theta.
\]

(ii) Decreasing if

\[
(2y + \theta) \cdot e^{\frac{\theta}{y} - 2y} \cdot e^{\alpha \cdot \left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)} + \alpha \theta e^{\frac{2\theta}{y}} \leq (2\alpha + 1)\theta + 2y \cdot e^{\frac{\theta}{y}} - 2y - \alpha \theta.
\]
Figure: The characteristics of SF (left) and HR (right) of $IET(\Theta)$ for different values of $\Theta$. 
Multi-Dimensional Copulas

Definition (d-Dimensional Copulas)

A function $C : [0,1]^d \rightarrow [0,1]$ is a $d$-dimensional copula if the following conditions are satisfied:

1. $C(1,\ldots,y_j,\ldots,1) = y_j, \forall j = 1,2,\ldots,d$ with $y_j \in [0,1]$
2. $C(y_1,y_2,\ldots,y_d) = 0$ if at least one $y_j = 0$ for $j = 1,2,\ldots,d$
3. For any $y_{\{j,1\}}, y_{\{j,2\}} \in [0,1]$ with $y_{\{j,1\}} \leq y_{\{j,2\}}$, for $j = 1,2,\ldots,d$:

$$\sum_{a_1=1}^{2} \sum_{a_2=1}^{2} \ldots \sum_{a_d=1}^{2} (-1)^{a_1+a_2+\ldots+a_d} C(y_{\{1,a_1\}},y_{\{2,a_2\}},\ldots,y_{\{d,a_d\}}) \geq 0 \quad (9)$$
Multivariate Gaussian Copula

Example

The multivariate Gaussian copula [R. B. Nelsen (2007)] is given by the function:

\[ C(u_1, u_2, \ldots, u_d | R) = \Phi_R(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_d)), \quad (10) \]

where \( \Phi^{-1} \) is the inverse of the cumulative distribution function of the standard normal distribution and \( \Phi_R \) is the joint CDF of a standard multivariate normal distribution with correlation matrix \( R \).
FGM Copula

Example

The Farlie-Gumbel-Morgenstern (FGM) copula is given by

\[
C(x, y) = xy(1 + \rho(1 - x)(1 - y))
\]  

(11)

where \((x, y) \in [0, 1] \times [0, 1], -1 \leq \rho \leq 1\).
Theorem (Sklar (1959))

**Sklar’s Theorem:** Let $Y_1, Y_2, \ldots, Y_d$ be random variables with marginal distribution functions $F_1, F_2, \ldots, F_d$ and joint cumulative distribution function $F$, then the following are true:

1. There exists a d-dimensional copula $C$ such that for all $y_1, y_2, \ldots, y_d \in \mathbb{R}$

   $$F(y_1, y_2, \ldots, y_d) = C(F_1(y_1), F_2(y_2), \ldots, F_d(y_d)) \quad (12)$$

2. If $Y_1, Y_2, \ldots, Y_d$ are continuous then the copula $C$ is unique. Otherwise, $C$ can be uniquely determined on a d-dimensional rectangle $\text{Range}(F_1) \times \text{Range}(F_2) \times \ldots \times \text{Range}(F_d)$. 

Sklar’s Theorem

Outline

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Joint CDF and PDF of Bi-variate IET (BIET)

Let \( X \sim IET(\Theta_1) \) and \( Y \sim IET(\Theta_2) \). The joint CDF using copula \( C \) is given by

\[
H(x, y) = C(F_X(x), F_Y(y))
\]

In particular, if we consider FGM copula, the CDF, PDF and Conditional PDF are given, respectively

\[
H(x, y \mid \Theta_1, \Theta_2, k) = F(x) \cdot F(y) \cdot (1 + k \cdot (1 - F(x)) \cdot (1 - F(y)))
\]

\[
= e \left( \alpha_1 \left( \frac{\theta_1}{x} - e^{\frac{\theta_1}{x}} + 1 \right) \right) \cdot e \left( \alpha_2 \left( \frac{\theta_2}{y} - e^{\frac{\theta_2}{y}} + 1 \right) \right).
\]

\[
= e \left( \alpha_1 \left( \frac{\theta_1}{x} - e^{\frac{\theta_1}{x}} + 1 \right) \right) \cdot e \left( \alpha_2 \left( \frac{\theta_2}{y} - e^{\frac{\theta_2}{y}} + 1 \right) \right).
\]

\[
g(x, y \mid \Theta_1, \Theta_2, k) = f(x) \cdot f(y) \cdot (1 + k \cdot (1 - 2 \cdot F(x)) \cdot (1 - 2 \cdot F(y)))
\]

\[
g(x \mid y, \Theta_1, \Theta_2, k) = f(x) \cdot (1 + k \cdot (1 - 2 \cdot F(x)) \cdot (1 - 2 \cdot F(y))
\]

(13)
Behaviour of PDF and CDF of BIET

**Figure:** (left) The characteristic of PDF of $BIET(2,4,1,2,0.75)$ and in (right) that of CDF of $BIET(2,4,1,2,0.75)$. 
Behaviour of SF and HR of BIET

Figure: (left) The characteristic of HR of $BIET(2,4,1,2,0.75)$ and in (right) that of SF of $BIET(2,4,1,2,0.75)$. 
Dependency Measurements

- **Kendall’s τ:**
  \[
  \tau = P[(X_1 - X_2) \cdot (Y_1 - Y_2) > 0] - P[(X_1 - X_2) \cdot (Y_1 - Y_2) < 0]
  \]
  \[
  = 1 - 4 \cdot \int_0^1 \int_0^1 \frac{\partial H(x, y | \Theta_1, \Theta_2, k)}{\partial F(x)} \cdot \frac{\partial H(x, y | \Theta_1, \Theta_2, k)}{\partial F(y)} \, dF(x) \, dF(y)
  \]
  \[
  = 1 - \frac{(9 - 2k)}{9} = \frac{2 \cdot k}{9}
  \]

- **Speraman’s ρ:**
  \[
  \rho_s = 12k \cdot \int_0^1 \int_0^1 kF(x) \cdot F(y) (1 - F(x)) \cdot (1 - F(y)) \, dF(x) \, dF(y)
  \]
  \[
  = 12k \cdot \int_0^1 F(x) \cdot (1 - F(x)) \, dF(x) \int_0^1 F(y) \cdot (1 - F(y)) \, dF(y)
  \]
  \[
  = \frac{12k}{36} = \frac{k}{3}
  \]
Dependency Measurement

- Schweizer and Wolff’s $\sigma$:

$$\sigma_{X,Y} = 12k \cdot \int_0^1 F(x) \cdot (1 - F(x)) \, dF(x) \int_0^1 F(y) \cdot (1 - F(y)) \, dF(y)$$

$$= \frac{12k}{36} = \frac{k}{3}$$

(16)

- Local Dependence $\gamma(x, y)$:

$$\frac{\partial^2 \log(f(x, y \mid \Theta, k))}{\partial x \partial y} = 4 \cdot \left[ \frac{f(x)f(y)m_2(x,y) - 2kf(x)m_1(x,y)(1 - 2F(y))}{(1 + k(1 - 2F(x)) \cdot (1 - 2F(y)))^2} \right]$$

$$m_1(x,y) = f(y)(1 - 2F(x)); \ m_2(x,y) = 1 + k(1 - 2F(x)) \cdot (1 - 2F(y)).$$

(17)
Tail behaviour and Tail dependency

- Here $H(x,y) \geq F(x) \cdot F(y) \forall k > 0$ therefore, $X$ and $Y$ are Positive Quadrant Dependent (PQD). Moreover, $X$ and $Y$ are Negative Quadrant Dependent (NQD) $\forall k < 0$.

**Theorem**

Let $(X,Y) \sim BIET(\Theta_1,\Theta_2,k)$ if $k \geq 0$ then, $Y$ is Left-Tail Decreasing (LTD), Right-Tail Increasing (RTI), and Stochastically Increasing (SI) on $X$. 
Probabilistic Time-Series Forecasting

We assume that the joint Stochastic Process (SP), \( (Y(t), Y(t + \tau)) \sim BIET(\Theta_1, \Theta_2, k) \) where \( t \in T \), and \( \tau \) is a temporal lag. In this section, we introduce a novel method of probabilistic time series prediction outlined by \( BIET(\Theta_1, \Theta_2, k_\tau) \).

\[
P\left[ Y(t + \tau) \leq y_{t+\tau} \mid Y(t) = y_t \right] = \frac{1}{2}
\]

\[
\Rightarrow P\left[ F(Y(t + \tau)) \leq F(y_{t+\tau}) \mid F(Y(t)) = F(y_t) \right] = \frac{1}{2}
\]

\[
\Rightarrow \frac{\partial H(y_t, y_{t+\tau})}{\partial F(Y_t)} = \frac{1}{2}
\]

\[
\Rightarrow 2F(y_t) + 2k_\tau F(y_t) - 2k_\tau F^2(y_t) = F(y_{t+\tau}) \left( 2k_\tau F(y_t) - 2k_\tau F^2(y_t) \right)
\]

\[
\Rightarrow F(y_{t+\tau}) = \frac{F(y_t) \cdot \left( 1 + k_\tau (1 - F(y_t)) \right) - 1}{k_\tau \cdot F(y_t) \cdot (1 - F(y_t))}
\]

\[
\Rightarrow y_{t+\tau} = F^{-1}\left[ \frac{F(y_t) \cdot \left( 1 + k_\tau (1 - F(y_t)) \right) - 1}{k_\tau \cdot F(y_t) \cdot (1 - F(y_t))} \right]
\]
Simulate Data from IET Distribution

Algorithm 2 Simulated data set generated from Uni-variate $IET(\Theta)$

Input $n$
    $m$
Output $S \in \mathbb{R}^{m \times n}$

for each $i \leftarrow 1$ to $m$
do
    for each $j \leftarrow 1$ to $n$
do
        Generate $u_{ij} \sim Uniform(0, 1)$
        $S[i, j] \leftarrow F^{-1}(u_{ij})$ \hspace{1em} $\triangleright$ where $F^{-1}$ is the quasi-inverse of the IET CDF.
end for
end for
Simulate Data from BIET Distribution

Algorithm 3 Simulated data set generated from $BIET(\Theta_1, \Theta_2, k)$

Input $n$

$m$

Output $S_X, S_Y \in \mathbb{R}^{m \times n}$

for each $i \leftarrow 1$ to $m$
do

for each $j \leftarrow 1$ to $n$
do

Generate $(u_{ij}, t_{ij}) \sim Uniform[(0, 1) \times (0, 1)]$

$v_{ij} = C_u^{-1}(t_{ij})$

$S_X[i, j] \leftarrow F_1^{-1}(u_{ij})$

$S_Y[i, j] \leftarrow F_2^{-1}(v_{ij})$

end for

end for

$\triangleright n =$ Sample Size

$\triangleright m =$ Number of Iterations

$\triangleright C_u = \frac{\partial C(u,v)}{\partial u}$
**Table:** MLE of $\Theta$ in $IET(\Theta)$

| n   | Parameter | Sample Mean | MSE  | Bias  | BCI(0.95)         | BSE   |
|-----|-----------|-------------|------|-------|-------------------|-------|
| 100 | $\alpha = 2$ | 1.470       | 1.588| -0.529| (0.351, 4.723)   | 1.143 |
|     | $\theta = 3$  | 3.843       | 1.763| 0.843 | (2.057, 5.748)   | 1.025 |
| 200 | $\alpha = 2$ | 1.500       | 1.226| -0.499| (0.419, 4.337)   | 0.988 |
|     | $\theta = 3$  | 3.670       | 1.181| 0.676 | (2.166, 5.448)   | 0.851 |
| 300 | $\alpha = 2$ | 1.498       | 1.335| -0.501| (0.471, 4.358)   | 1.041 |
|     | $\theta = 3$  | 3.646       | 1.021| 0.648 | (2.122, 5.191)   | 0.777 |
### Bayesian Estimation

**Table:** BE of $\Theta$ in $IET(\Theta)$

| n   | Parameter      | Sample Mean | MSE  | Bias  | BCI(0.95)       | BSE  |
|-----|----------------|-------------|------|-------|-----------------|------|
| 100 | $\alpha = 1$   | 0.755       | 0.151| -0.244| (0.202, 1.294)  | 0.304|
|     | $\theta = 3.5$ | 4.132       | 0.909| 0.632  | (3.220, 5.785)  | 0.713|
| 200 | $\alpha = 1$   | 0.729       | 0.167| -0.270 | (0.223, 1.323)  | 0.167|
|     | $\theta = 3.5$ | 4.166       | 0.974| 0.666  | (3.198, 5.801)  | 0.974|
| 300 | $\alpha = 1$   | 0.707       | 0.181| -0.292 | (0.224, 1.291)  | 0.309|
|     | $\theta = 3.5$ | 4.221       | 1.107| 0.721  | (3.206, 5.868)  | 0.766|
### Weighted Least Square Estimation

**Table:** WLS of $\Theta$ in $IET(\Theta)$

| n   | Parameter | Sample Mean | MSE   | Bias | BCI(0.95)       | BSE  |
|-----|-----------|-------------|-------|------|-----------------|------|
| 100 | $\alpha = 1$ | 1.219       | 1.425 | 0.219| (0.171, 4.595)  | 1.173|
|     | $\theta = 3$ | 3.357       | 1.335 | 0.357| (1.567, 5.552)  | 1.099|
| 200 | $\alpha = 1$ | 1.091       | 0.896 | 0.091| (0.225, 3.863)  | 0.942|
|     | $\theta = 3$ | 3.308       | 0.892 | 0.308| (1.698, 5.204)  | 0.893|
| 300 | $\alpha = 1$ | 1.123       | 0.923 | 0.123| (0.241, 3.937)  | 0.953|
|     | $\theta = 3$ | 3.267       | 0.838 | 0.266| (1.690, 5.058)  | 0.876|
### Cramer Von-Mises Estimation

**Table: CVM estimate of $\Theta$ in $IET(\Theta)$**

| n  | Parameter | Sample Mean | MSE | Bias | BCI(0.95)     | BSE  |
|----|-----------|-------------|-----|------|--------------|------|
| 100| $\alpha = 1$ | 1.067       | 1.128 | 0.066 | (0.170, 4.800) | 1.060 |
|    | $\theta = 3$ | 3.516       | 1.450 | 0.516 | (1.608, 5.619) | 1.088 |
| 200| $\alpha = 1$ | 1.064       | 0.886 | 0.064 | (0.211, 3.919) | 0.939 |
|    | $\theta = 3$ | 3.364       | 0.966 | 0.364 | (1.691, 5.287) | 0.913 |
| 300| $\alpha = 1$ | 1.004       | 0.588 | 0.004 | (0.255, 3.149) | 0.767 |
|    | $\theta = 3$ | 3.326       | 0.742 | 0.326 | (1.870, 4.924) | 0.797 |
**EM Estimation**

**Table:** EM estimate of $\Theta$ in $IET(\Theta)$

| n  | Parameter | Sample Mean | MSE  | Bias  | BCI(0.95)      | BSE |
|----|-----------|-------------|------|-------|----------------|-----|
| 100| $\alpha = 2$ | 1.461       | 2.013| -0.538| (0.280, 5.146) | 1.313 |
|    | $\theta = 3$ | 3.991       | 2.315| 0.991 | (1.981, 5.909) | 1.154 |
| 200| $\alpha = 2$ | 1.408       | 1.956| -0.591| (0.308, 5.100) | 1.267 |
|    | $\theta = 3$ | 4.028       | 2.357| 1.028 | (1.988, 5.877) | 1.140 |
| 300| $\alpha = 2$ | 1.472       | 1.762| -0.527| (0.342, 5.021) | 1.218 |
|    | $\theta = 3$ | 3.896       | 1.954| 0.896 | (2.024, 5.786) | 1.073 |
Table 9: MLE estimate of $\Theta_1$, and $\Theta_2$ in $BIETD(\Theta_1, \Theta_2, k)$

| n  | Parameter | True Value | MLE     | MSE       | Bias     | BSE       | CI(0.95)              |
|----|-----------|------------|---------|-----------|----------|-----------|-----------------------|
|    | $\alpha_1$ | 2          | 1.91894 | 1.318571  | -0.08105857 | 1.17219 | (0.713956, 5.048073) |
| 100| $\theta_1$ | 4          | 4.3918  | 1.156687  | 0.3918002 | 0.9293735 | (2.6501084, 5.905896) |
|    | $\alpha_2$ | 1          | 1.18307 | 0.999449  | 0.1830769 | 1.01724  | (0.18919609, 4.393151) |
|    | $\theta_2$ | 2          | 2.19754 | 0.4182949 | 0.1975495 | 0.6473151 | (1.09674574, 3.567457) |
|    | $k$       | 0.5        | 0.542354| 0.0710688 | 0.04235385| 0.25683  | (0.06647816, 0.99987)  |
| 200| $\alpha_1$ | 2          | 1.810679| 0.9559668 | -0.1893206 | 1.04497  | (0.7186286, 4.7243299) |
|    | $\theta_1$ | 4          | 4.442419| 0.932662  | 0.4424185 | 0.8734426 | (2.779292, 5.89826)   |
|    | $\alpha_2$ | 1          | 1.111495| 0.3376843 | 0.1114951 | 0.7936206 | (0.2978534, 3.4017358) |
|    | $\theta_2$ | 2          | 2.110467| 0.2221527 | 0.1104672 | 0.4794214 | (1.2047094, 3.1016042) |
|    | $k$       | 0.5        | 0.5030928| 0.04734568| 0.00309286| 0.204922 | (0.1140134, 0.9121645) |
| 300| $\alpha_1$ | 2          | 1.69689 | 0.715872  | -0.3031097 | 0.9249394| (0.7393568, 4.3671022) |
|    | $\theta_1$ | 4          | 4.511125| 0.7528599 | -0.5111251 | 0.8093461| (2.8509058, 5.8702773) |
|    | $\alpha_2$ | 1          | 1.060169| 0.3384617 | 0.0601688 | 0.628002 | (0.3535171, 2.7387518) |
|    | $\theta_2$ | 2          | 2.087978| 0.150146 | 0.087978 | 0.3983438 | (1.3251008, 2.8927158) |
|    | $k$       | 0.5        | 0.5004218| 0.02269951| 0.00042178| 0.1655991| (0.178608, 0.8276942)  |
Kerala Rainfall Data

Table: Fitted PDF on Summer Rainfall Data

| PDF     | MLE                        | -logL | AIC        | BIC        |
|---------|----------------------------|-------|------------|------------|
| IET     | \( \hat{\alpha} = 14.54, \hat{\theta} = 45.23 \) | 664.5 | 1333.006   | 1338.53    |
| ETD     | \( \hat{\alpha} = 3.66, \hat{\theta} = 0.00343 \) | 666.844 | 1337.689   | 1343.213   |
| APETD   | \( \hat{\alpha} = 6.95, \hat{\theta} = 0.00544, \hat{\gamma} = 0.367 \) | 677.322 | 1358.644   | 1368.93    |
| Weibull | \( \hat{\alpha} = 2.621, \hat{\theta} = 152.744 \) | 973.4278 | 1950.856   | 1951.618   |
| Gamma   | \( \hat{\alpha} = 86.47, \hat{\theta} = 0.76 \) | 2047.882 | 4099.764   | 4100.526   |
Kerala Rainfall Data

Table: Fitted PDF on Monsoon Rainfall Data

| PDF   | MLE                      | -logL  | AIC      | BIC      |
|-------|--------------------------|--------|----------|----------|
| IETD  | $\hat{\alpha} = 0.01137, \hat{\theta} = 2360$ | 736.852 | 1477.705 | 1483.229 |
| ETD   | $\hat{\alpha} = 0.5084, \hat{\theta} = 0.0024773$ | 765.3679 | 1534.736 | 1540.26  |
| APETD | $\hat{\alpha} = 2.459, \hat{\theta} = 0.0026, \hat{\gamma} = 4.216$ | 724.7513 | 1453.503 | 1463.789 |
| Weibull | $\hat{\alpha} = 6.326, \hat{\theta} = 590.17$ | 1153.67  | 2311.34  | 2312.102 |
| Gamma | $\hat{\alpha} = 579.3027, \hat{\theta} = 1.2$ | 1783.327 | 3570.654 | 3571.416 |
Kerala Rainfall Data

Figure: Behaviour of Kerala rainfall in summer and winter
Fitting suitable PDFs

Comparison between PDFs of Summer rainfall

Comparison between PDFs of Monsoon rainfall

Figure: Comparison between the Five PDFs in Rainfall Data in Kerala.
Copula-Based Median Regression in Kerala Rainfall Data

Figure: BIET-based Median regression between annual Summer and Annual Monsoon rainfall data in Kerala.
Figure: Temporal behavior of CDF of summer and monsoon rainfall.
We introduced a new IET distribution, derived its different statistical properties and characterizations, and measured various reliability properties and also estimate model parameters using techniques such as MLE, EM, WLSE, CVM, BE, etc.

The proposed IET distribution shows better modeling compatibility than existing distributions in the literature.

We extended IET to BIET and explored different measures of dependency and also introduced a new temporal probabilistic median regression model to explain the probability that a variable is less than some particular value.

We illustrate our entire derivations using Kerala rainfall data of summer and monsoon from 1901 to 2017 as a case study.

The proposed model is applied to real data and compared with existing models in the literature.
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Thank You