Abstract

We derive a closed expression for the $SU(2)$ Born-Infeld action with the symmetrized trace for static spherically symmetric purely magnetic configurations. The lagrangian is obtained in terms of elementary functions. Using it, we investigate glueball solutions to the flat space NBI theory and their self-gravitating counterparts. Such solutions, found previously in the NBI model with the 'square root – ordinary trace' lagrangian, are shown to persist in the theory with the symmetrized trace lagrangian as well. Although the symmetrized trace NBI equations differ substantially from those of the theory with the ordinary trace, a qualitative picture of glueballs remains essentially the same. Gravity further reduces the difference between solutions in these two models, and, for sufficiently large values of the effective gravitational coupling, solutions tends to the same limiting form. The black holes in the NBI theory with the symmetrized trace are also discussed.

PACS numbers: 04.20.Jb, 04.50.+h, 46.70.Hg
1 Introduction

Recent development in the superstring theory suggests that the low-energy dynamics of $N$ coincident $D$-branes is described by the $SU(N)$ Yang-Mills theory governed by the Born-Infeld type action \[ S = \frac{1}{4\pi} \int \left\{ 1 - \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} \right\} d^4x \] (1)

A precise definition of such non-Abelian Born-Infeld (NBI) action was the subject of a vivid discussion during past few years [2, 3, 4, 5, 6, 7, 8], for an early discussion see [9]. An ambiguity is encoded in specifying the trace operation over the gauge group generators. Formally a number of possibilities can be envisaged. Starting with the determinant form of the $U(1)$ Dirac-Born-Infeld action

\[
S = \frac{1}{4\pi} \int \left\{ 1 - \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} \right\} d^4x
\]

one can use the usual trace, the symmetrized or antisymmetrized \[2\] trace, a calculation of the determinant both with respect to Lorentz and the gauge matrix indices \[6\]. Alternatively one can start with the 'square root' form, which is most easily derived from (1) using the identities

\[
det(g_{\mu\nu} + F_{\mu\nu}) = det(g_{\mu\nu} - F_{\mu\nu}) = det(g_{\mu\nu} + i\tilde{F}_{\mu\nu}) =
\]

where $F_{\mu\nu} = F_{\mu}^{\alpha}F_{\nu}^{\alpha}$ (similarly for $\tilde{F}_{\mu\nu}$), and

\[
F_{\mu\nu}F_{\rho\sigma} - \tilde{F}_{\mu\nu}\tilde{F}_{\rho\sigma} = \frac{1}{2} g_{\mu\nu}F_{\alpha\beta}F_{\gamma\delta},
\]

\[
F_{\mu\nu}\tilde{F}_{\rho\sigma} = \frac{1}{4} g_{\mu\nu}F_{\alpha\beta}\tilde{F}_{\gamma\delta},
\]

leading to the equality

\[
\sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} = \sqrt{-\det(g)} \sqrt{1 + \frac{1}{2} F^2 - \frac{1}{16}(F\tilde{F})^2},
\]

with $F^2 = F_{\mu\nu}F^{\mu\nu}$, $F\tilde{F} = F_{\mu\nu}\tilde{F}^{\mu\nu}$. For a non-Abelian gauge group the relations \[3\] are no longer valid, so there is no direct relationship between the 'determinant' and the 'square root' form of the lagrangian. Therefore the latter can be chosen as an independent starting point for a non-Abelian generalization. There is, however, a particular trace operation – symmetrized trace – under which generators commute with each other and therefore both forms of the lagrangian remain equivalent. This definition is favored by the adiabaticity argument, as was clarified by Tseytlin \[2\]. Restricting the validity of the effective action by the no-derivative approximation, in the non-Abelian case one has to drop the commutators of the matrix-valued $F_{\mu\nu}$ since they can be reexpressed through the derivatives of $F_{\mu\nu}$. This corresponds to the following definition of the action

\[
S = \frac{1}{4\pi} \text{Str} \int \left\{ 1 - \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} \right\} d^4x
\]
where symmetrization applies to the field strength (not to potentials \[2\]). This action reproduces an exact string theory result for non-Abelian gauge fields up to \(\alpha'^2\) order. Although there is no reason to believe that this will be true in higher orders in \(\alpha'\), the Str action is an interesting model providing a minimal generalization of the Abelian action \[2\].

Some general technique was developed \[7\] to deal with the symmetrized products of gauge generators in a symbolic form, but for many purposes it is more desirable to have the lagrangian explicitly. In the general case an evaluation of the action can be performed through an expansion of the square root in powers of \(F_{\mu\nu}\), then the symmetrized trace over generators can be computed explicitly. The next step, resummation of the series into a closed expression, is very problematic. However this can be done if we restrict the field by some particular configurations (the ansatz has to be consistent with the full equations of motion). One example of this kind is the computation of an explicit action for \(D0\)-branes in three dimensions \[9\]. Here we present such a calculation in the four-dimensional \(SU(2)\) theory restricting the field configurations by requirements of the spherical symmetry and staticity. Such configurations are encountered in the study of magnetic monopoles and other solitons in the NBI theory (we use a term ‘soliton’ in a wide sense including unstable sphaleron solutions).

Soliton solutions to non-Abelian Born-Infeld theory were discussed recently in a number of papers using both the square root action with the ordinary and the symmetrized trace \[10, 11, 12, 13\], in the latter case, however, only perturbatively. It was shown that in the Born-Infeld theory, apart from monopoles and instantons, which also exist in the theory with the usual Yang-Mills quadratic action, new particle-like solutions of the glueball type are brought to existence \[14\]. For these solutions the full non-linear structure of the NBI lagrangian is essential, so earlier they could be explored only within the model with the ordinary trace. Here we show that these solutions persist in the theory with the action \(6\). Although the explicit Str lagrangian looks very differently from that with the ordinary tr, the glueball solutions remain qualitatively the same. We also study the static spherically symmetric NBI system coupled to gravity and demonstrate that gravity further reduces the difference between solutions obtained in these two models.

### 2 Symmetrized trace NBI action for static \(SO(3)\)-symmetric fields

We define the NBI Lagrangian in four spacetime dimensions as

\[
L_{NBI} = \frac{\beta^2}{4\pi} \text{Str} \left( 1 - \sqrt{-\det(g_{\mu\nu} + \frac{1}{\beta} F_{\mu\nu})} \right) = \frac{\beta^2}{4\pi} \text{Str}(1 - \mathcal{R}),
\]

where

\[
\mathcal{R} = \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16\beta^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2},
\]
with the parameter $\beta$ of the dimension of length$^{-2}$ (the 'critical field'). The normalization of the gauge group generators is chosen as follows

$$F_{\mu\nu} = F^a_{\mu\nu} t_a, \quad \text{tr} t_a t_b = \delta_{ab}. \quad (9)$$

The symmetrized trace of the product of $p$ matrices is defined as

$$\text{Str}(t_{a_1} \ldots t_{a_p}) \equiv \frac{1}{p!} \text{Str} \left( t_{a_1} \ldots t_{a_p} \text{ + all permutations} \right), \quad (10)$$

and it is understood that the general matrix function like (9) has to be series expanded. It has to be noted that while under the Str operation the generators obviously can be treated as commuting objects, the gauge algebra cannot be applied, i.e. $\tau^2 \neq 1$ until the symmetrization of the expansion is completed.

A general $SO(3)$ symmetric $SU(2)$ gauge field is described by the Witten’s ansatz

$$\sqrt{2} A = a_0 t_1 \, dt + a_1 t_1 \, dr + \{\tilde{w} \, t_2 - (1 - w) \, t_3\} \, d\theta + \{(1 - w) \, t_2 + \tilde{w} \, t_3\} \sin \theta \, d\phi, \quad (11)$$

where the functions $a_0, a_1, w, \tilde{w}$ depend on $r, t$ and $\sqrt{2}$ is introduced to maintain the standard normalization. Here we use a rotated basis $t_i, i = 1, 2, 3$ for the $SU(2)$ generators defined as

$$t_1 = n^a \tau^a / \sqrt{2}, \quad t_2 = \partial_\theta t_1, \quad \sin \theta t_3 = \partial_\phi t_1, \quad (12)$$

where $n^a = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, and $\tau^a$ are the Pauli matrices. These generators obey the commutation relations $[t_i, t_j] = \frac{1}{\sqrt{2}} \epsilon_{ijk} t_k$.

From four functions entering this ansatz one can be gauged away. In the static case we can further reduce the number of independent functions to two, while static purely magnetic configurations can be described by a single variable $w(r)$. Here we deal with this simplest case

$$\sqrt{2} A_\theta = -(1 - w) t_3, \quad \sqrt{2} A_\varphi = \sin \theta (1 - w) t_2, \quad A_r = A_t = 0. \quad (13)$$

The field strength tensor has the following non-zero components

$$\sqrt{2} F_{\theta \varphi} = w' t_3, \quad \sqrt{2} F_{r \varphi} = -\sin \theta w' t_2, \quad \sqrt{2} F_{\theta r} = \sin \theta (w^2 - 1) t_1, \quad (14)$$

where prime denotes derivatives with respect to $r$. For purely magnetic configurations the second term under the square root is zero, while the substitution of (14) gives

$$R^2 = 1 + \frac{(1 - w^2)^2}{\beta^2 r^4} t_1^2 + \frac{w^2}{\beta^2 r^2} (t_2^2 + t_3^2). \quad (15)$$

Now we have to expand the square root in a triple series in terms of the even powers of generators $t_1, t_2, t_3$. This can be achieved applying the formula

$$\sqrt{1 + x} = 1 - 2 \sum_{m=1}^{\infty} \frac{(2m-2)!}{m!(m-1)!} \left( -\frac{x}{4} \right)^m, \quad (16)$$
and further repeatedly using binomial expansions. Finally we obtain
\[ L_{NBI} = \frac{\beta^2}{4\pi} \sum_{i+j+k \geq 1} \frac{(-1)^{i+j+k}(2i + 2j + 2k - 2)!}{(i + j + k - 1)!i!j!k!} \left( \frac{V}{2} \right)^{2i} \left( \frac{K}{2} \right)^{2j+2k} f(i, j, k), \]  
(17)

where the sum over all positive \( i, j, k \) subject to a condition \( i + j + k \geq 1 \) is understood and
\[ V^2 = \left( 1 - \frac{w^2(r)}{2}\beta^2 r^4 \right), \quad K^2 = \frac{w^2(r)}{2\beta^2 r^2}. \]  
(18)

and the final factor is the symmetrized trace of the product of even powers of Pauli matrices:
\[ f(i, j, k) = \text{Str} \left( \tau_1^{2i} \tau_2^{2j} \tau_3^{2k} \right). \]  
(19)

To compute \( f(i, j, k) \) explicitly one can derive the following recurrent relation:
\[ (2i + 2j + 2k)(2i + 2j + 2k - 1)f(i, j, k) = 2i(2i - 1)f(i - 1, j, k) + 2j(2j - 1)f(i, j - 1, k) + 2k(2k - 1)f(i, j, k - 1). \]  
(20)

When only one index is non-zero one easily finds
\[ f(i, 0, 0) = f(0, i, 0) = f(0, 0, i) = 2. \]  
(21)

The full solution of the Eq.(2), satisfying the border conditions (21) reads
\[ f(i, j, k) = \frac{2(2i)!(2j)!(2k)!}{(2i + 2j + 2k)!i!j!k!}. \]  
(22)

Substituting this explicit expression into the expansion (17) of the lagrangian one obtains the following representation as a triple series
\[ L_{NBI} = \frac{\beta^2}{4\pi} \sum_{i+j+k \geq 1} \frac{(-1)^{i+j+k}(2i)!(2j)!(2k)!}{i!^2j!^2k!^2(2i + 2j + 2k - 1)} \left( \frac{V}{2} \right)^{2i} \left( \frac{K}{2} \right)^{2j+2k}. \]  
(23)

Remarkably, one can perform this summation explicitly. First we observe that, once the factor \( 1/(2i + 2j + 2k - 1) \) is omitted, and the summation is extended to all values of power indices including zero, we obtain a triple series expansion for the function
\[ Z(V, K) = \frac{1}{\sqrt{1 + V^2(1 + K^2)}}. \]  
(24)

Indeed, treating this function as a product of three square roots, two of which are equal, we find the following representation:
\[ Z(V, K) = \sum_{i,j,k=0}^{\infty} \frac{(2i)!(2j)!(2k)!}{i!^2j!^2k!^2} (-1)^{i+j+k} \left( \frac{V}{2} \right)^{2i} \left( \frac{K}{2} \right)^{2j+2k}. \]  
(25)

Now it is easy to see that the desired sum (23) is related to \( Z(V, K) \) through the following differential equation
\[ K \frac{\partial}{\partial K} L(K, V) + V \frac{\partial}{\partial V} L(K, V) - L(K, V) = \frac{\beta^2}{4\pi} (Z(V, K) - 1), \]  
(26)
where \( L(K, V) \) stands for \( L_{\text{NBI}} \). Its solution satisfying the initial conditions
\[
L(0, 0) = 0, \quad \frac{\partial}{\partial V} L(0, 0) = \frac{\partial}{\partial K} L(0) = 0,
\]
(27)

following from an initial definition of the lagrangian (7), reads
\[
L_{\text{NBI}} = \frac{\beta^2}{4\pi} \left( 1 - \frac{1 + V^2 + K^2 A}{\sqrt{1 + V^2}} \right),
\]
(28)

where
\[
A = \sqrt{\frac{1 + V^2}{V^2 - K^2}} \arctanh \sqrt{\frac{V^2 - K^2}{1 + V^2}}.
\]
(29)

Here we assumed that \( V^2 > K^2 \), otherwise an arctan form is more appropriate. Note that when the difference \( V^2 - K^2 \) changes sign, the function \( A \) remains real valued. It can be checked that when \( \beta \to \infty \), the standard Yang-Mills lagrangian (restricted to monopole ansatz) is recovered. In the strong field region our expression differs essentially from the square root/ordinary trace lagrangian.

3 Glueballs

The standard Yang-Mills theory does not admit classical particle-like solutions \[15, 16, 17\]. More precisely, this famous no-go result asserts that there exist no finite-energy nonsingular solutions to the four-dimensional Yang-Mills equations which would be either static, or non-radiating time-dependent \[17\]. This result follows from the conformal invariance of the Yang-Mills theory with the quadratic action, which implies that the stress–energy tensor is traceless: \( T_{\mu}^{\mu} = 0 = -T_{00} + T_{ii} \), where \( \mu = 0, ..., 3, \ i = 1, 2, 3 \). Since \( T_{00} > 0 \), the sum of the principal pressures \( T_{ii} \) is everywhere positive, \( i.e. \) the Yang-Mills matter is repulsive. This makes the mechanical equilibrium impossible \[18\]. In the spontaneously broken gauge theories conformal invariance is also broken by scalar fields. Thus the above obstruction is removed what opens the possibility of particle-like solutions: magnetic monopoles and sphalerons. Monopoles are essentially related to the presence of the Higgs field with the \( SO(3) \) component, while for sphalerons Higgs can be replaced by other attractive agent. Recall that the sphaleron was first obtained in the gauge theory with doublet Higgs \[19\] and its existence was explained by Manton \[20\] as a consequence of non–triviality of the third homotopy group of the broken phase manifold. Later it was found that similar solutions arise in the theories without Higgs like Einstein-Yang-Mills and Yang-Mills with dilaton (for a review see \[21\]), in all such cases conformal invariance is broken. Recently it was observed that the Born-Infeld modification of the Yang-Mills action also breaks conformal invariance \[14\], and gives rise to particle-like solutions. They form a discrete sequence labeled by the number of nodes of the function \( w(r) \), and the lower one-node solution is similar to the sphaleron of the Weinberg-Salam theory.

In \[14\] the NBI lagrangian was adopted in the ‘square-root’–ordinary trace form. Now we are able to perform similar calculations in the Str version of the NBI theory. It is worth noting that to study particle-like solutions in the pure NBI theory without Higgs
an exact in $\alpha'$ form of the lagrangian is needed since these solutions are formed in the strong field region. Therefore we assume the following one-dimensional action

$$S_1 = \int r^2 \left( 1 - \frac{1 + V^2 + K^2 A}{\sqrt{1 + V^2}} \right) dr.$$  \hspace{1cm} (30)

Note that the rescaling $\sqrt{\beta} r \to r$ does not change the action (an overall factor appearing in the lagrangian can be removed by the corresponding rescaling of time). Therefore without loss of generality $\beta$ can be fixed, it is convenient to choose $\beta = 1/\sqrt{2}$. Then the equation of motion for $w$ will read

$$\frac{d}{dr} \left\{ \frac{w'}{2(V^2 - K^2)} \left( \frac{K^2 \sqrt{1 + V^2}}{1 + K^2} - \frac{(2V^2 - K^2)A}{\sqrt{V^2 - K^2}} \right) \right\} = \frac{Vw(K^2 A - V^2)}{(V^2 - K^2)\sqrt{1 + V^2}}.$$  \hspace{1cm} (31)

where now

$$V^2 = \frac{(1 - w^2(r))^2}{r^4}, \quad K^2 = \frac{w'^2(r)}{r^2}.$$

(32)

Analyzing the extremal points of $w$ as discussed in [21] one finds that $w$ can not have local minima for $0 < w < 1$, $w < -1$ and can not have local maxima for $-1 < w < 0$, $w > 1$. Thus any finite energy solution which starts at the origin on the interval $-1 < w < 1$ lies entirely within the strip $-1 < w < 1$. Once $w$ leaves the strip, it diverges within a finite distance. We are interested in particle-like solution with finite total energy (mass) given for the present model by

$$\mathcal{E} = \int_0^\infty r^2 \left( \frac{1 + V^2 + K^2 A}{\sqrt{1 + V^2}} - 1 \right) dr.$$  \hspace{1cm} (33)

Boundary conditions at the origin can be derived combining the equation of motion with the requirement of convergence of this integral. Two classes of solutions are possible: one with $w(0) = 0$ (leading to embedded $U(1)$ solutions) and $|w(0)| = 1$, relevant for glueballs. Choosing without loss of generality $w(0) = 1$, we find the following series solution near the origin

$$w = 1 - br^2 + 3b^2 r^4 \frac{32b^4 + 20b^2 + 3}{2(32b^4 + 20b^2 + 15)} + O(r^6).$$  \hspace{1cm} (34)

To ensure convergence of the total energy (33) at infinity, $w$ must tend to $\pm 1$, 0. In the case $w(\infty) = 0$ the global solution is $w \equiv 0$, i.e. an embedded Abelian. For non-Abelian solutions one has

$$w = \pm \left( 1 - \frac{c}{r} \right) + O\left( \frac{1}{r^2} \right),$$  \hspace{1cm} (35)

where $c$ is another free parameter. The proof of existence of global solutions starting at the origin as (34) and approaching (33) at infinity may be given along the lines of [14].

The integer $n$ is equal to the number of zeroes of $w$. The $n = 1$ solution is an analog of the sphaleron known in the Weinberg-Salam theory [13, 20], it is expected to have one decay mode. Higher odd-$n$ solutions may be interpreted as excited sphalerons, they
are expected to have $n$ decay directions in the configuration space. Even-$n$ solutions are topologically trivial, they can be regarded as sphaleronic excitation of the vacuum. Qualitatively picture is the same as for the 'square root – ordinary trace form' \cite{14} but the quantized values of $b$ are rather different in particular, in that case $b_1 = 12.7463$.

Numerical solutions are shown on Fig. \ref{fig:1}, where for comparison first four solutions with ordinary trace are also shown. The difference for the main $n = 1$ sphaleron in both theories is rather small, it increases for higher-$n$ solutions which move closer to the origin (where two models differ substantially).

### Table 1: Values of $b$ and $M$ for first six solutions

| $n$ | $b$         | $M$       |
|-----|-------------|-----------|
| 1   | $1.23736 \times 10^2$ | 1.20240   |
| 2   | $5.05665 \times 10^4$ | 1.234583  |
| 3   | $1.67739 \times 10^5$ | 1.235979  |
| 4   | $7.11885 \times 10^6$ | 1.236046  |
| 5   | $4.94499 \times 10^8$ | 1.2360497 |
| 6   | $4.52769 \times 10^{10}$ | 1.2360497 |

4 Gravitating glueballs and black holes

Now let us consider the Str NBI theory coupled to gravity. The corresponding action reads:

$$S_{ENBI} = -\frac{1}{16\pi G} \int \left\{ R \sqrt{-g} + 4G\beta^2 \left( \text{Str} \sqrt{-\det(g_{\mu\nu} + \beta^{-1}F_{\mu\nu})} - \sqrt{-g} \right) \right\} d^4x, \quad (36)$$

where $R$ is the scalar curvature and $G$ is the Newton constant. For static spherically symmetric solutions the metric can be parameterized as follows:

$$ds^2 = N\sigma^2 dt^2 - \frac{dr^2}{N} - r^2 (d\theta^2 + \sin^2 \theta d\phi). \quad (37)$$

A computation of the symmetrized trace for the action \ref{eq:36} is a straightforward generalization of the above procedure for the flat spacetime, the main difference being in using curved metric \ref{eq:37} instead of flat. After suitable rescaling, two dimensional parameters of the theory $G, \beta$ combine in one dimensionless coupling constant

$$g = G\beta, \quad (38)$$

which is the only substantial parameter of the theory. The reduced one dimensional action reads

$$S_1 = \int \left\{ \frac{\sigma}{2} \left( 1 + N \left( 1 + 2r \left( \frac{\sigma'}{\sigma} + \frac{N'}{2N} \right) \right) \right) + gr^2 \sigma \left( 1 - \frac{1 + K^2 + V^2A}{\sqrt{1 + K^2}} \right) \right\} dr, \quad (39)$$
where now
\[ K^2 = \frac{(1 - w^2)^2}{r^4}, \quad V^2 = \frac{Nw'^2}{r^2}, \]
and \( A \) is still defined by (29).

The equations of motion derived from this action consist of an equation for the metric function \( \sigma \):
\[ \frac{\sigma'}{\sigma} = \frac{gK^2r((2V^2 + 2K^2V^2 - K^2 - K^4)A - K^2 - K^2V^2)}{N\sqrt{1 + V^2}(V^2 + K^2V^2 - K^2 - K^4)}, \]
an equation for the local mass function \( m(r) \) defined via \( N = 1 - 2m/r \):
\[ m' = gr^2\left(\frac{1 + V^2 + AK^2}{\sqrt{1 + V^2}} - 1\right), \]
and the following equation for \( w \):
\[ \frac{d}{dr}\left\{ \frac{N\sigma w'}{2(V^2 - K^2)} \left( K^2\sqrt{1 + V^2} + (2V^2 - K^2)A \right) \right\} = \frac{\sigma V w(K^2A - V^2)}{(V^2 - K^2)\sqrt{1 + V^2}}. \]

The equation for \( \sigma \) decouples from the rest of the system, and this function can be found by a simple integration once \( N \) and \( w \) are known. Therefore we concentrate on a coupled system for \( N, w \) which is obtained after using the \( \sigma \)-equation in the \( w \)-equation. Like in the ordinary trace version, these equations admit an embedded \( U(1) \) solution \( \sigma \equiv 1, w \equiv 0 \) corresponding to the unit magnetic charge. The corresponding metric was given in [22, 23] (see also [24]):
\[ N = 1 - \frac{2}{r} \left( m_0 + g \int_0^r \left( \sqrt{1 + x^4} - x^2 \right) dx \right), \]
where \( m_0 \) is a free parameter which can be positive, zero or negative. For \( m_0 = 0 \) the black hole solutions (with the event horizon) exist for \( g > g_{cr} = 1/2 \), otherwise there is no horizon. The role of this critical value in the non-Abelian case was discussed in [24] and [25] (for the ordinary trace \( SU(2) \) Born-Infeld action). The metric for regular solutions approaches that of an Abelian solution (without horizon and \( m_0 = 0 \)) for \( g < g_{cr} \) and large \( n \), the limiting mass being the corresponding Abelian mass.

The present model also have regular gravitating solutions for all values of \( g \) which can be thought as interpolating between the flat case solutions discussed above and Bartnik-McKinnon [26] solutions of Einstein-quadratic Yang-Mills model (for a detailed discussion see [21]). The situation is very much alike to the case of the ordinary trace ENBI model [25].

Regular gravitating solutions start at the origin with the following series expansion
\[ w = 1 - br^2 + O(r^4) \]
\[ N = 1 - \frac{2g(1 + 8b^2 - \sqrt{1 + 4b^2})}{3\sqrt{1 + 4b^2}}r^2 + O(r^4) \]
where \( b \) is again a free parameter. At infinity one should have

\[
w \to \pm 1, \quad N \sim 1 - \frac{2M}{r},
\]

where \( M \) is the Schwarzschild (ADM) mass. Numerical solutions interpolating between these asymptotics are shown in Figs. 2, 3. Note that the difference between Str and tr solutions is decreased with respect to the flat space case, especially for the first \( n = 1 \) sphaleron. The metric bending is slightly more pronounced in the Str case. In Fig. 4 we show the dependence of the parameter \( b_1 \) for the \( n = 1 \) solution on the gravitational coupling constant for both Str and tr versions of the NBI theory. Qualitatively their behavior is the same. For \( g \to \infty \) both curves converge to the rescaled Bartnik-McKinnon’s value.

In weak gravity limit (as \( g \to 0 \)), \( w(r) \)-functions for regular solutions do not differ considerably from the flat case, especially the sequence \( b_n \) is unbounded while the number \( n \) of nodes of \( w \) tends to infinity. The metric function \( N(t) \) with increasing \( n \) approaches the metric of the abelian solution at some interval which moves more and more close to the origin. The ADM masses of these solutions behave like

\[
\lim_{g \to 0} \frac{M_n(g)}{g} = M^\text{flat}_n
\]

where \( M^\text{flat}_n \) are the flat case masses defined by (33) and shown in table 1.

But if \( g > g_{cr} = 1/2 \) the situation changes. With increasing \( n \) the parameters \( b_n \) tend to a limiting value and the metric functions tend to the metric of the limiting abelian solution with a degenerate horizon. This situation resembles that of the Einstein-Yang-Mills model and indeed this model could be recovered in the limit \( g \to \infty \) after a rescaling \( \sqrt[2]{gr} \to r \). In this limit the glueball masses behave like

\[
\lim_{g \to \infty} \frac{M_n(g)}{\sqrt{g}} = M^\text{BK}_n
\]

where \( M^\text{BK}_n \) is the mass of corresponding Bartnik-McKinnon solution for the Einstein-Yang-Mills model. This behavior could be observed in Fig 5.

Let us discuss the black holes. Instead of (45) now we specify boundary conditions at the horizon \( r = r_h \) using the following series expansions:

\[
N = N'_h(r - r_h) + O((r - r_h)^2),
\]

\[
w = w_h + \frac{w_h(w_h^2 - 1)}{A r_h^2 N'_h}(r - r_h) + O((r - r_h)^2),
\]

where

\[
N'_h = \frac{1}{r_h} 2gr_h^2(\sqrt{1 + K^2} - 1) + 1.
\]

Asymptotically flat black hole solutions exist for all horizon radii \( r_h \) and some discrete sequence of \( w_h \), labeled by the number of nodes of \( w \) like in the regular case. Qualitative behavior of \( w \) outside the horizon is also the same. Numerical results for the metric function \( N(r) \) of the \( n = 1, 3 \) black holes are shown in Fig. 5 for a critical \( g = 1/2 \) and different values of the horizon radius. For non-small \( r_h \) the function \( N \) is monotonous outside the horizon, while for smaller \( r_h \) one observes a local minimum near the black hole.
5 Discussion

Two main results of this paper should be noted. First, we obtained a closed analytical expression for the symmetrized trace version of the $SU(2)$ NBI action restricted to monopole ansatz. This expression is particularly simple and can be further used in various problems. Moreover, similar technique can be applied to other related field configurations, the results will be given elsewhere.

Second, we have extended results of [14] about the existence of sphaleronic glueballs in the $SU(2)$ NBI theory without Higgs field to the symmetrized trace version of the NBI model. In both cases the classical scale invariance of the ordinary Yang-Mills lagrangian is broken by the Born-Infeld non-linearity, what removes an obstruction for classical glueballs. We have found that qualitatively the Str NBI glueball solutions remain the same, although a certain difference is observed in the core. In particular, quantized values of the parameter determining the derivative of the Yang-Mills field at the origin are much larger in the Str case. Nevertheless, the masses are not very different and with growing node number converge rapidly to the same limit. This is due to the fact that most of the energy is localized in the outer core region where the difference between two models is less pronounced.

When gravity is taken into account, we have demonstrated that there exist a continuous transition between the flat space glueballs and Bartnik-McKinnon’s particle-like solutions of the Einstein-Yang-Mills theory with the usual quadratic lagrangian. The latter can therefore be viewed as the strong gravity limit of purely gluonic sphalerons on $D3$ branes. Gravity was shown to further reduce the difference between the results of the Str and $tr$ models. This is well understood, since for the strong gravity (in units of the Born-Infeld critical field) both models have the same limit.

We thank A. Koshelev for a useful discussion. This work was supported in part by the RFBR grant 00-02-16306.

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Figure 1: First six solutions $w_n(r)$ for flat space glueballs in the NBI theory with symmetrized trace (solid line) and their four ordinary trace counterparts (dotted line).
Figure 2: Functions $w_n(r)$, $n = 1, 2, 3$ for gravitating glueballs in the Str (solid line) and Tr (dotted line) NBI models with $g = g_{cr} = \frac{1}{2}$. Solutions are practically indistinguishable for $n = 1$, becoming slightly different for $n = 2, 3$. 
Figure 3: Metric functions $N_n(r)$, $n = 1, 2, 3$ for gravitating glueballs in the Str (solid line) and Tr (dotted line) NBI models with $g = 1/2$. Gravitational binding in the Str case is slightly greater than in the Tr case.
Figure 4: Parameter $b$ versus an effective coupling $g$ (in log variables) for the gravitating NBI model $n = 1$ solutions in the Str (solid line) and Tr (dotted line) models.
Figure 5: Dependence of mass on $g$ for the $n = 1$ gravitating glueball (in units $\sqrt{g}$) in the ENBI theory with symmetrized trace (solid line) and ordinary trace (dotted line).
Figure 6: Metric functions $N_n(r)$ for black holes in the Str model for $n = 1$ (solid line) and $n = 3$ (dotted line) with $r_h = 1, 0.1, 0.01$ and $g = \frac{1}{2}$. 