Double robust inference for continuous updating GMM

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May 19, 2021

Abstract

We propose the double robust Lagrange multiplier (DRLM) statistic for testing hypotheses specified on the pseudo-true value of the structural parameters in the generalized method of moments. The pseudo-true value is defined as the minimizer of the population continuous updating objective function and equals the true value of the structural parameter in the absence of misspecification. The (bounding) $\chi^2$ limiting distribution of the DRLM statistic is robust to both misspecification and weak identification of the structural parameters, hence its name. To emphasize its importance for applied work, we use the DRLM test to analyze the return on education, which is often perceived to be weakly identified, using data from Card (1995) where misspecification occurs in case of treatment heterogeneity; and to analyze the risk premia associated with risk factors proposed in Adrian et al. (2014) and He et al. (2017), where both misspecification and weak identification need to be addressed.

Keywords: weak identification, misspecification, robust inference, Lagrange multiplier.

1 Introduction

Little more than twenty years ago, inference procedures for analyzing possibly weakly identified structural parameters using the generalized method of moments (GMM) of Hansen (1982) were mostly lacking. Since then huge progress has been made to develop such procedures, see e.g. Staiger and Stock (1997), Dufour (1997), Stock and Wright (2000), Kleibergen (2002, 2005, 2009), Moreira (2003), Andrews and Cheng (2012), Andrews and Mikusheva (2016a, b), and Han and McCloskey (2019). At present, we therefore have a variety of so-called weak identification robust inference methods. Given the prevalence of weak identification in applied work, a lot of emphasis has also been put in raising awareness amongst practitioners, see e.g. Kleibergen and Mavroeidis (2009), Beaulieu et al. (2013), Mavroeidis et al. (2014), Andrews et al. (2019), and Kleibergen and Zhan (2020).

The weak identification robust inference procedures in GMM lead to inference that is centered around the continuous updating estimator (CUE) of Hansen et al. (1996). GMM requests the moment condition

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to hold at a (unknown) true value of the parameter which is then also the minimizer of the population continuous updating objective function. The inference resulting from weak identification robust inference procedures concerning hypotheses specified on the true value of the structural parameters remains reliable under varying degrees of identification. When there is no value of the structural parameters where the GMM moment conditions exactly hold, the structural model is rendered misspecified and we refer to the minimizer of the (population continuous updating) GMM objective function as the pseudo-true value. The pseudo-true value depends on the (population) objective function at hand and different objective functions lead to distinct pseudo-true values. We use the minimizer of the population continuous updating objective function as the pseudo-true value because of its invariance properties and since weak identification robust tests lead to inference that is centered around it. In case of misspecification, these inference procedures for testing hypotheses specified on the pseudo-true value become size distorted for just small amounts of misspecification. This would not sound as much of a problem if it was possible to efficiently detect such misspecification. This is, however, not so since misspecification tests, like the Sargan-Hansen test (Sargan (1958) and Hansen (1982)), are virtually powerless in settings of joint misspecification and weak identification; see Gospodinov et al. (2017). Weak identification robust inference procedures thus came about to overcome the general critique of non-robustness of traditional inference procedures to varying identification strengths, see e.g. Staiger and Stock (1997) and Dufour (1997), but are similarly non-robust to misspecification.

Arguably, the first to emphasize the importance of misspecification in the presence of weak (or no) identification were Kan and Zhang (1999). With the surge in applied work on structural estimation, awareness of misspecification has grown further, see Hall and Inoue (2003). In asset pricing models, for example, it is now generally accepted that misspecification, alongside weak identification, is an important empirical issue, see e.g. Kan et al. (2013) and Kleibergen and Zhan (2020). Kan et al. (2013) therefore developed misspecification robust \( t \)-statistics for the Fama-MacBeth (FM) (1973) two-pass estimator, i.e. the typical estimator employed to estimate risk premia in linear asset pricing models. Similarly, Hansen and Lee (2021) construct the limiting distribution of an iterated GMM estimator in misspecified GMM, which can be used to conduct Wald tests on the pseudo-true value of the structural parameters; furthermore, Evdokimov and Kolesár (2018) and Lee (2018) analyze testing the treatment effect resulting from multiple instruments whose local average treatment effects might differ, leading to misspecification of the moment equation of the underlying linear instrumental variables (IV) regression model. These misspecification robust tests on the pseudo-true value are, however, not robust to weak identification, so identical to the weak identification robust inference procedures, they cannot deal with the empirically relevant setting of both misspecification and weak identification for which Hansen and Lee (2021) state “... this extension would be desirable but considerably more challenging.”
In this paper, we therefore extend the weak identification robust score or Lagrange multiplier (KLM) test from Kleibergen (2002, 2005, 2009) to a double robust Lagrange multiplier (DRLM) test. This DRLM test is size correct and robust to both misspecification and weak identification, hence its name. The DRLM statistic is a quadratic form of the score function, which equals zero at all stationary points of the CUE sample objective function. This is also the case for the KLM statistic and explains the power problems of the KLM test, see e.g. Andrews et al. (2006). To overcome the power problems of the KLM test, the KLM statistic can be combined in a conditional or unconditional manner with the Anderson-Rubin (AR) (1949) statistic, see e.g. Andrews (2016). Andrews et al. (2006) show that the conditional likelihood ratio test of Moreira (2003) provides the optimal manner of combining these statistics for the homoskedastic linear IV regression model with one included endogenous variable. We use the maximal invariant to show that in case of misspecification, it is not obvious how to improve the power of the DRLM test by such combination arguments, since the statistics with which the DRLM statistic is to be combined to improve power have non-central limiting distributions with parameters that cannot be consistently estimated under misspecification. We therefore improve the power of the DRLM test by exploiting the specification of the derivative of the DRLM statistic with respect to the structural parameters.

The rest of the paper is organized as follows. In the second section, we present continuous updating GMM with misspecification, and discuss how and when a structural interpretation can be obtained from the pseudo-true value. We introduce a measure of the identification strength which has to (considerably) exceed the minimal value of the population continuous updating objective function for the pseudo-true value to be structurally interpretable. In the third section, we introduce the DRLM test and prove that it is size correct. For ease of exposition, we also illustrate the latter using a simulation experiment. The fourth section conducts a power study of the DRLM test and other weak identification robust tests. It shows that weak identification robust tests on the pseudo-true value of the structural parameters are size distorted for just small amounts of misspecification while the DRLM test is not. It also proposes the power improvement rule and shows that the resulting test procedure has generally good power. The fifth section conducts a simulation experiment using nonlinear GMM with an asset pricing Euler moment equation that results from a constant relative rate of risk aversion (CRRA) utility function. The sixth section applies the DRLM test to risk premia using asset pricing data from Adrian et al. (2014) and He et al. (2017), and to analyze the return on education using data from Card (1995) for which local average treatment effects that differ over the instruments can lead to misspecification, see Imbens and Angrist (1994). Especially for the risk premium parameters, we show that usage of other inference procedures understates the uncertainty of the risk measures because of the misspecification and weak identification present. The seventh section concludes. Technical details and additional material are relegated to the Online Appendix.
2 GMM with potential misspecification

We analyze the $m \times 1$ parameter vector $\theta = (\theta_1 \ldots \theta_m)'$ whose parameter region is the $\mathbb{R}^m$. The $k_f \times 1$ dimensional function $f(\ldots)$ is a continuously differentiable function of the parameter vector $\theta$ and a Borel measurable function of a data vector $X_t$ which is observed for time/individual $t$. Since we focus on misspecification, the model is overidentified, i.e. there are more moment equations than structural parameters so $k_f > m$. The population moment function of $f(\theta, X_t)$ equals $\mu_f(\theta)$:

$$E_X(f(\theta, X_t)) = \mu_f(\theta),$$  \hspace{1cm} (1)

with $\mu_f(\theta)$ a $k_f$-dimensional continuously differentiable function. Unlike regular GMM, see Hansen (1982), we do not request that there is a specific value of $\theta$, say $\theta_0$, at which $\mu_f(\theta_0) = 0$. Our analysis thus differs from a recent one proposed by Cheng et al. (2021), who construct a model selection procedure for evaluating potentially misspecified models with possibly weakly identified structural parameters, which explicitly uses a set of base moments contained in all considered models that are guaranteed to hold. We analyze $\theta$ using the continuous updating setting of Hansen et al. (1996). We use it because of its invariance properties and since it leads to inference using identification robust statistics in standard GMM, see e.g. Stock and Wright (2000) and Kleibergen (2005). The accompanying population continuous updating objective function is:

$$Q_p(\theta) = \mu_f(\theta)'V_{ff}(\theta)^{-1}\mu_f(\theta),$$  \hspace{1cm} (2)

with $V_{ff}(\theta)$ the covariance matrix of the sample moment $f_T(\theta, X) = \frac{1}{T} \sum_{t=1}^{T} f_t(\theta)$, $f_t(\theta) = f(\theta, X_t)$.

$$V_{ff}(\theta) = \lim_{T \to \infty} E \left[ f_T(\theta, X) - \mu_f(\theta) \right] \left[ f_T(\theta, X) - \mu_f(\theta) \right]' ,$$  \hspace{1cm} (3)

so $f_T(\theta, X)$ is the sample analog of $\mu_f(\theta)$ for a data set of $T$ observations: $X_t$, $t = 1, \ldots, T$.

We define the pseudo-true value of $\theta$, $\theta^*$, as the minimizer of the population objective function:

$$\theta^* = \arg \min_{\theta \in \mathbb{R}^m} Q_p(\theta).$$  \hspace{1cm} (4)

Lateron we discuss if this $\theta^*$ is our object of interest, which depends amongst others on whether a measure of the amount of misspecification is less than a measure of the strength of identification. The minimizer of the population objective function satisfies the first order condition (FOC) stated in Theorem 1.  

\footnote{Throughout the paper, we use recentered covariance matrices while the continuous updating estimator is identical under a recentered or uncentered version of the covariance matrix estimator; see Theorem 1 of Hansen and Lee (2021).
Theorem 1: The FOC (divided by two) for a stationary point $\theta^*$ of the population objective function reads:

$$\frac{1}{2} \frac{\partial}{\partial \theta} Q_p(\theta^*) = 0 \iff \mu_f(\theta^*)' V_{ff}(\theta^*)^{-1} D(\theta^*) = 0,$$

(5)

with

$$\begin{align*}
D(\theta) &= J(\theta) - [V_{\theta f}(\theta) V_{ff}(\theta)^{-1} \mu_f(\theta)] \\
J(\theta) &= \frac{\partial}{\partial \theta} \mu_f(\theta),
\end{align*}$$

(6)

and $V_{\theta f}(\theta) = \lim_{T \to \infty} E \left[ T \left( \frac{\partial}{\partial \theta} f_T(\theta, X) - \mu_f(\theta) \right) \left( f_T(\theta, X) - \mu_f(\theta) \right) \right]$, $i = 1, \ldots, m$.

(7)

Proof. See the Online Appendix and Kleibergen (2005). □

Theorem 1 shows that if there is a unique value of $\theta$, $\theta_0$, for which $\mu_f(\theta_0) = 0$, then also $\theta^* = \theta_0$ and $D(\theta_0) = J(\theta_0)$. The misspecification thus implies that the recentered Jacobian $D(\theta^*)$ differs from the population Jacobian that results from the moment equations, $J(\theta^*)$, in other instances.

2.1 Running example 1: Linear asset pricing model

The linear asset pricing model shows the extent to which the mean of an $(N+1)$-dimensional vector of asset returns $R_t$ is spanned by the betas of $m$ risk factors contained in the $m$-dimensional vector $F_t$. It is reflected by the moment function:

$$\mu_f(\lambda_0, \lambda_F) = E(R_t) - \iota_{N+1}\lambda_0 - B\lambda_F,$$

(8)

with $\iota_{N+1}$ an $(N+1)$-dimensional vector of ones, $B$ an $(N + 1) \times m$ dimensional matrix:

$$B = \text{cov}(R_t, F_t) \text{var}(F_t)^{-1},$$

(9)

and $\lambda_0$ is the zero-beta return, $\lambda_F$ is the $m$-dimensional vector of risk premia.

The asset pricing moment equation in (8) can be more compactly written by removing the zero-beta return which we accomplish by taking the asset returns in deviation of the $(N+1)$-th asset return:

$$R_t = \begin{pmatrix} R_{1t} \\ \vdots \\ R_{Nt} \end{pmatrix} - \iota_N R_{(N+1)t}, \quad \beta = \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix} - \iota_N B_{N+1},$$

(10)

\footnote{Our results are invariant with respect to the asset return which is subtracted; see Kleibergen and Zhan (2020).}
for \( R_t = (R_{1t} \ldots R_{(N+1)t})' \), \( B = (B_1' \ldots B_{N+1}')' \). The removal of the zero-beta return leads to the moment function:

\[ \mu_f(\lambda_F) = \mu_R - \beta \lambda_F, \quad (11) \]

with \( \mu_R = E(R_t) \) and \( \beta = \text{cov}(R_t, F_t) \text{var}(F_t)^{-1} \).

The mean asset returns are not necessarily fully spanned by the \( \beta \)'s. We therefore analyze the pseudo-true value of the risk premia \( \lambda_F \), which is the minimizer of the population continuous updating objective function:

\[ Q_p(\lambda_F) = (\mu_R - \beta \lambda_F)' \left[ \text{Var} \left( \sqrt{T} \left( \tilde{R} - \tilde{\beta} \lambda_F \right) \right) \right]^{-1} (\mu_R - \beta \lambda_F), \quad (12) \]

since \( f_T(\lambda_F, X) = \tilde{R} - \tilde{\beta} \lambda_F \), with \( \tilde{R} = \frac{1}{T} \sum_{t=1}^{T} R_t \) and \( \tilde{\beta} = \frac{1}{T} \sum_{t=1}^{T} \tilde{R}_t \tilde{F}_t' \left( \frac{1}{T} \sum_{t=1}^{T} \tilde{F}_t \tilde{F}_t' \right)^{-1} \), \( \tilde{R}_t = R_t - \tilde{R} \), \( \tilde{F}_t = F_t - \tilde{F} \). The population continuous updating objective function results from a generalized reduced rank problem, see also Kleibergen (2007):

\[ Q_p(\lambda_F) = \min_{D \in \mathbb{R}^{N \times m}} Q_p(\lambda_F, D) \quad (13) \]

with \( D(\lambda_F) = \arg \min_{D \in \mathbb{R}^{N \times m}} Q_p(\lambda_F, D) \) and

\[ Q_p(\lambda_F, D) = \left[ \text{vec} \left( \left( \mu_R \hat{\beta} \right) + D \left( \lambda_F \hat{I}_m \right) \right) \right]' \left[ \text{Var} \left( \sqrt{T} \left( \tilde{R}' \text{vec}(\tilde{\beta})' \right) \right) \right]^{-1} \left[ \text{vec} \left( \left( \mu_R \hat{\beta} \right) + D \left( \lambda_F \hat{I}_m \right) \right) \right]. \quad (14) \]

The minimal value of (14) over \((\lambda_F, D)\) is invariant to the reduced rank specification implied by \( D(\lambda_F \hat{I}_m) \). When using another reduced rank specification, say, \( A(I_m \hat{\phi}) \), with \( A \) an \( N \times m \) matrix and \( \phi \) an \( m \)-dimensional vector, it leads to an identical value of the optimized objective function over \((\phi, A)\). Hence, restrictions imposed on this specification, like, for example, \( \phi_1 = 0 \), with \( \phi_1 \) the top element of \( \phi \), which imposes a reduced rank value on just \( \beta \), lead to a larger (or equal) value of the minimized objective function. This restricted specification is thus such that the objective function reflects the identification strength of \( \lambda_F \) as reflected by the distance of \( \beta \) from a reduced rank value. If the minimal value of the objective function in (13) coincides with the one resulting from this restricted specification, some or even all elements of the resulting pseudo-true value \( \lambda_F^* \) will be very large or even infinite since they now result from a reduced rank value of \( \beta \), and do not reflect risk premia. For the pseudo-true value \( \lambda_F^* \) to reflect risk premia, so to have a structural interpretation and be of interest, the strength of identification has to exceed a measure of the amount of misspecification. We can therefore use the minimal value of the population continuous updating objective function resulting from (13) as a measure of the amount of misspecification, and compare it with
a measure of the identification strength \((IS)\), whose sample analog corresponds with a statistic testing the rank of \(\beta\), see e.g. Cragg and Donald (1997), Kleibergen and Paap (2006), and Robin and Smith (2000)\).

\[
IS = \min_{\xi \in \mathbb{R}^{m-1}} Q_r(\xi)
\]

\[
Q_r(\xi) = (\frac{1}{\xi})' \beta' \left( \left( \frac{1}{\xi} \otimes I_N \right)' \text{Var} \left( \sqrt{T} \text{vec}(\hat{\beta}) \right) \left( \frac{1}{\xi} \otimes I_N \right) \right)^{-1} \beta \left( \frac{1}{\xi} \right)
\]

\[
= \min_{G \in \mathbb{R}^{N \times (m-1)}} Q_r(\xi, G)
\]

\[
Q_r(\xi, G) = \left[ \text{vec} \left( \beta + G \left( \frac{\xi}{I_{m-1}} \right) \right) \right]' \left[ \text{Var} \left( \sqrt{T} \text{vec}(\hat{\beta}) \right) \right]^{-1} \left[ \text{vec} \left( \beta + G \left( \frac{\xi}{I_{m-1}} \right) \right) \right].
\]

The Online Appendix provides a proof that the \(IS\) identification strength measure in (15) equals the minimal value of the restricted objective function alluded to previously, where we used the reduced rank specification \(A(I_m: \phi)\), with \(A\) an \(N \times m\) matrix and \(\phi\) an \(m\)-dimensional vector with its top element restricted to zero. The identification strength measure is thus always larger than or equal to the minimal value of the population continuous updating objective function. When the minimal value of the population continuous updating objective function is then just slightly smaller than \(IS\) in (15), we have to be cautious with interpreting the pseudo-true value as risk premia which is then also reflected by their very large values. We next further illustrate this for a simplified setting of the linear asset pricing model.

When \(\mu_F = E(F_t) = 0\) and \(\hat{\beta}\) results from the regression of \(\tilde{R}_t\) on \(\tilde{F}_t\) in which the error term is assumed to be i.i.d. with \(N \times N\) dimensional covariance matrix \(\Omega\), Lemma 1 in the Online Appendix shows that \(\tilde{R}\) and \(\hat{\beta}\) are independently normally distributed in large samples, see also Shanken (1992) and Kleibergen (2009). The population continuous updating objective function (12) then simplifies to:

\[
Q_p(\lambda_F) = \frac{1}{1 + \lambda_F Q_{F_F} \lambda_F} (\mu_R - \beta \lambda_F)' \Omega^{-1} (\mu_R - \beta \lambda_F),
\]

with \(Q_{F_F} = \text{var}(F_t)\), so its minimal value equals the smallest root of the characteristic polynomial:

\[
\tau \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{F_F}^{-1} \end{array} \right) - \left( \mu_R : \beta \right)' \Omega^{-1} \left( \mu_R : \beta \right) = 0.
\]

**Proposition 1.** Using a value of \(\lambda_F, \lambda_F^*\), that satisfies the FOC in Theorem 1, the smallest root of the characteristic polynomial in (17) equals either

\[
\frac{1}{1 + \lambda_F^* Q_{F_F} \lambda_F^*} (\mu_R - \beta \lambda_F^*)' \Omega^{-1} (\mu_R - \beta \lambda_F^*)
\]

In the homoskedastic linear IV regression model with one included endogenous variable, the counterpart for the identification measure “\(IS\)” in (15) equals the number of instruments times the population analog of the first stage \(F\)-statistic.
or the smallest root of the characteristic polynomial:

$$|\tau(Q_{F\bar{F}} + \lambda_F^*\lambda_F^{*\prime})^{-1} - D(\lambda_F^*)\Omega^{-1}D(\lambda_F^*)| = 0,$$

(19)

with

$$D(\lambda_F) = -\beta - (\mu_R - \beta\lambda_F)\lambda'_F Q_{F\bar{F}}^{-1}(1 + \lambda'_F Q_{F\bar{F}}^{-1}\lambda_F)^{-1} = -(\beta Q_{F\bar{F}} + \mu_R \lambda_F')(Q_{F\bar{F}} + \lambda_F \lambda_F^{*\prime})^{-1}.$$

**Proof.** The rewriting of (17) to obtain the above is conducted in the Online Appendix.

Without misspecification, there is a value of $\lambda_F^*$ for which (18) is equal to zero, so it is the smallest root of the characteristic polynomial. Proposition 1 therefore shows that in models with misspecification, the minimizer of the population objective function is not necessarily our object of interest or put differently, has a structural interpretation. For example, when $m = 1$, $\beta = 0$ and $\mu_R \neq 0$, the roots of the characteristic polynomial in (17) equal zero, attained when $\lambda_F \to \pm \infty$; and $\mu'_R \Omega^{-1} \mu_R$, attained at $\lambda_F = 0$. The smallest root then corresponds with the IS identification strength measure in (15), so the resulting pseudo-true value $\lambda_F \to \pm \infty$ cannot be interpreted as a risk premium. The pseudo-true value is only of interest when it has a structural interpretation, so it represents risk premia, which occurs when the IS identification strength measure (15) strictly exceeds the minimal value of the population objective function. This condition clearly fails when $\beta = 0$ so $IS = 0$, but $\mu_R \neq 0$. This setting is used in Kan and Zhang (1999) to point at the misbehavior of traditional inference methods; see also Gospodinov et al. (2017).

### 2.2 Running example 2: Linear IV regression model

For the linear IV regression model:

$$y = X\theta + \varepsilon,$$

$$X = Z\Pi + V,$$

(20)

with $\theta$ and $\Pi$ $m \times 1$ and $k \times m$ matrices containing unknown parameters, $y = (y_1 \ldots y_T)'$ and $X = (X_1 \ldots X_T)'$ $T \times 1$ and $T \times m$ dimensional matrices containing the endogenous variables, $Z = (Z_1 \ldots Z_T)'$ a $T \times k$ matrix containing the instrumental variables, $\varepsilon = (\varepsilon_1 \ldots \varepsilon_T)'$ and $V = (V_1 \ldots V_T)'$ are $T \times 1$ and $T \times m$ matrices of errors. The population moment function is:

$$\mu_f(\theta) = \sigma_{Zy} - \Sigma_{ZX}\theta,$$

(21)

with $\sigma_{Zy} = E((Z_t - \mu_Z)(y_t - \mu_y))$, $\Sigma_{ZX} = E((Z_t - \mu_Z)(X_t - \mu_X)') = Q_{Z\Pi}$, $Q_{Z\Pi} = E((Z_t - \mu_Z)(Z_t - \mu_Z)'),$ $\mu_y = E(y_t), \mu_X = E(X_t), \mu_Z = E(Z_t).$ When $u_t = \varepsilon_t + V_t^'\theta$ and $V_t$ are i.i.d. distributed with mean zero and covariance matrix $\Omega = \begin{pmatrix} \omega_{uu} & \omega_{uv} \\ \omega_{vu} & \omega_{vv} \end{pmatrix}$, the population continuous updating objective function of the linear IV
regression model is:

\[
Q_p(\theta) = \frac{1}{\omega_u - 2\omega_{uv} + 2\omega_{vV}} (\sigma_{Zy} - \Sigma_{Zx}\theta)'Q_{ZZ}^{-1}(\sigma_{Zy} - \Sigma_{Zx}\theta).
\] (22)

Along the same lines as for the linear asset pricing model, the minimal value of this population continuous updating objective function equals the smallest root of a characteristic polynomial:

\[
\left| \tau \Omega - \left( \sigma_{Zy} : \Sigma_{Zx} \right)'Q_{ZZ}^{-1} \left( \sigma_{Zy} : \Sigma_{Zx} \right) \right| = 0.
\] (23)

If there is no value of \( \theta \) for which \( \mu_f(\theta) = 0 \), identical to the characteristic polynomial of the linear asset pricing model, the smallest root of the characteristic polynomial is only associated with misspecification when the amount of misspecification is less than the identification strength, so the IS identification strength measure adapted to the linear IV regression model exceeds the minimal value of the population objective function.

Misspecified linear IV regression models are of interest in several settings, for example, when analyzing treatment effects. In case of multiple discrete instruments and heterogeneous treatment effects, the local average treatment effects of Imbens and Angrist (1994) differ over the instruments, so the linear IV regression model using all these instruments is misspecified. The pseudo-true value is then a function of these local average treatment effects. We lateron provide an empirical illustration of this using data from Card (1995) in Section 6. Kolesár et al. (2015) provide another example of how a misspecified linear IV regression model can render a structural interpretation. Similarly, Kan et al. (2013) give a structural interpretation to the misspecified linear factor model as minimizing the pricing errors. In the Online Appendix, we provide further discussions on how a structural interpretation can be given to these models in case of misspecification.

It is also important to realize that the identification of the structural parameters is often rather weak in applied settings in which case misspecification tests have very little power, see Gospodinov et al. (2017). The identification robust tests needed because of weak identification then become size distorted for testing the pseudo-true value in the presence of misspecification, so it is important to have tests which remain size correct for these empirically relevant settings.

### 3 Double robust score test

The sample analog of the population continuous updating objective function is the sample objective function for the continuous updating estimator (CUE) of Hansen et al. (1996):

\[^4\]This adaptation is just the population analog of a rank statistic testing for a reduced rank value of \( \Pi \).
\[ \hat{Q}_s(\theta) = f_T(\theta, X)' \hat{V}_{ff}(\theta)^{-1} f_T(\theta, X), \] (24)

with \( \hat{V}_{ff}(\theta) \) a consistent estimator of \( V_{ff}(\theta) \), \( \hat{V}_{ff}(\theta) \to V_{ff}(\theta) \), so the CUE, \( \hat{\theta} \), is:

\[ \hat{\theta} = \arg \min_{\theta \in \mathbb{R}^m} \hat{Q}_s(\theta). \] (25)

To construct the large sample behavior of test statistics centered around the CUE, we make Assumption 1 as in Kleibergen (2005) except that it concerns the large sample behavior of the sample moments and their derivative at the pseudo-true value \( \theta^* \) instead of the true value.

**Assumption 1.** For a value of \( \theta \) equal to the minimizer of the continuous updating population objective function, \( \theta^* \), the \( k_f \times 1 \) dimensional derivative of \( f_i(\theta) \) with respect to \( \theta_i \),

\[ q_{it}(\theta) = \frac{\partial f_i(\theta)}{\partial \theta_i} : k_f \times 1, \quad i = 1, \ldots, m, \] (26)

is such that the joint limiting behavior of the sums of the series \( \bar{f}_i(\theta)' = f_i(\theta) - E(f_i(\theta)) \) and \( \bar{q}_i(\theta) = (\bar{q}_{i1}(\theta)' \ldots \bar{q}_{im}(\theta)' \ldots)' \), with \( \bar{q}_{it}(\theta) = q_{it}(\theta) - E(q_{it}(\theta)) \), accords with the central limit theorem:

\[ \frac{1}{\sqrt{T}} \sum_{t=1}^T \begin{pmatrix} \bar{f}_i(\theta) \\ \bar{q}_i(\theta) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \psi_f(\theta) \\ \psi_\theta(\theta) \end{pmatrix} \sim N(0, V(\theta)), \] (27)

where \( \psi_f : k_f \times 1, \psi_\theta : k_\theta \times 1, k_\theta = mk_f \), and \( V(\theta) \) is a positive semi-definite symmetric \( (k_f + k_\theta) \times (k_f + k_\theta) \) matrix,

\[ V(\theta) = \begin{pmatrix} V_{ff}(\theta) & V_{f\theta}(\theta) \\ V_{\theta f}(\theta) & V_{\theta \theta}(\theta) \end{pmatrix}, \] (28)

with \( V_{\theta f}(\theta) = V_{f\theta}(\theta)' = (V_{\theta,1f}(\theta)' \ldots V_{\theta,mf}(\theta)' \ldots)' \), \( V_{\theta \theta}(\theta) = (V_{\theta,ij}(\theta)) : i, j = 1, \ldots, m \); and \( V_{ff}(\theta), V_{\theta f}(\theta), V_{\theta \theta}(\theta) \) are \( k_f \times k_f \) dimensional matrices for \( i, j = 1, \ldots, m \), and

\[ V(\theta) = \lim_{T \to \infty} \text{var} \left( \sqrt{T} \begin{pmatrix} f_T(\theta, X) \\ \text{vec}(q_T(\theta, X)) \end{pmatrix} \right), \] (29)

with \( q_T(\theta, X) = \frac{\partial f_T(\theta, X)}{\partial \theta} \bigg|_\theta = \frac{1}{T} \sum_{t=1}^T (q_{1t}(\theta) \ldots q_{mt}(\theta)) \).

Assumption 1 requests a joint central limit theorem to hold for the sample moments and their derivative with respect to \( \theta \). It is satisfied under mild conditions which are listed in Kleibergen (2005), like, for
example, finite \( r \)-th moments for \( r > 2 \), mixing conditions for the sample moments in case of time-series data. Allowing for a positive semi-definite covariance matrix \( V(\theta) \) is important for applications, like, for example, dynamic linear panel data models. We next also use Assumption 2 from Kleibergen (2005) which concerns the convergence of the covariance matrix estimator \( \hat{V}(\theta) \).

**Assumption 2.** The convergence behavior of the covariance matrix estimator \( \hat{V}(\theta) \) towards \( V(\theta) \) is such that

\[
\hat{V}(\theta) \xrightarrow{p} V(\theta) \text{ and } \frac{\partial \text{vec}(\hat{V}_f(\theta))}{\partial \theta} \xrightarrow{p} \frac{\partial \text{vec}(V_f(\theta))}{\partial \theta}.
\]

(30)

The CUE satisfies the FOC for a minimum of the CUE sample objective function.

**Theorem 2:** The FOC (divided by two) for a stationary point \( \hat{\theta}^* \) of the CUE sample objective function reads:

\[
\frac{1}{2} \frac{\partial}{\partial \theta^T} \hat{Q}_s(\hat{\theta}^*) = 0 \quad \iff \quad f_T(\hat{\theta}^*, X) \hat{V}_f(\hat{\theta}^*)^{-1} \hat{D}(\hat{\theta}^*) = 0,
\]

(31)

with

\[
\hat{D}(\theta) = q_T(\theta, X) - \begin{bmatrix} \hat{V}_f(\theta)\hat{V}_f(\theta)^{-1}f_T(\theta, X) & \cdots & \hat{V}_0(\theta)\hat{V}_f(\theta)^{-1}f_T(\theta, X) \end{bmatrix}
\]

(32)

and

\[
\hat{V}(\theta) = \begin{pmatrix} \hat{V}_f(\theta) & \hat{V}_0(\theta) \\ \hat{V}_0(\theta) & \hat{V}_0(\theta) \end{pmatrix},
\]

(33)

with \( \hat{V}_f(\theta) = \hat{V}_f(\theta)' = (\hat{V}_{0,1}(\theta) \ldots \hat{V}_{m,1}(\theta))' \), \( \hat{V}_0(\theta) = (\hat{V}_{0,1}^{ij}(\theta)) : i, j = 1, \ldots, m \); and \( \hat{V}_f(\theta), \hat{V}_0(\theta), \hat{V}_{0,1}(\theta) \) are \( k_f \times k_f \) dimensional matrices for \( i, j = 1, \ldots, m \).

**Proof.** It follows along the lines of the proof of Theorem 1; see also Kleibergen (2005).

Theorem 2 shows that the FOC of the sample CUE objective function can in an identical manner be factorized as the FOC of the population continuous updating objective function provided in Theorem 1. Theorem 3 further shows that the two components in which the FOC of the sample objective function factorizes are independently distributed in large samples.

**Theorem 3:** When Assumptions 1 and 2 hold and for \( \theta^* \) the pseudo-true value minimizing the population continuous updating objective function:

\[
\sqrt{T} (f_T(\theta^*, X) - \mu_f(\theta^*)) \xrightarrow{d} \psi_f(\theta^*),
\]

\[
\sqrt{T} \text{vec} \left( \hat{D}(\theta^*) - D(\theta^*) \right) \xrightarrow{d} \psi_{0,f}(\theta^*),
\]

(34)
where $\psi_{0,f}(\theta^{*}) = \psi_{0}(\theta^{*}) - V_{0f}(\theta^{*})V_{ff}(\theta^{*})^{-1}\psi_f(\theta^{*})$ and

$$
\psi_f(\theta^*) \sim N(0, V_{ff}(\theta^*)), \\
\psi_{0,f}(\theta^*) \sim N(0, V_{00,f}(\theta^*)�)
$$

with $V_{00,f}(\theta) = V_{00}(\theta) - V_{0f}(\theta)V_{ff}(\theta)^{-1}V_{f0}(\theta)$, and $\psi_{0,f}(\theta^*)$ is independent of $\psi_f(\theta^*)$.

**Proof.** See the Online Appendix and Lemma 1 in Kleibergen (2005).

In standard GMM using the CUE objective function, the sample moment $f_T(\theta, X)$ is centered at zero at the true value, so we can use different identification robust statistics, like the score, GMM-Anderson-Rubin and extensions of the conditional likelihood ratio statistic of Moreira (2003); see Stock and Wright (2000), Kleibergen (2005), Andrews (2016) and Andrews and Mikusheva (2016a, b). In our misspecified GMM setting the sample moment is not centered at zero, so we can not use any of these statistics. We therefore propose a misspecification robust score statistic, which uses that the expected value of the limit of the derivative of the sample objective function:

$$
s(\theta) = \frac{1}{2n} \hat{Q}_s(\theta) = f_T(\theta, X)'\hat{V}_{ff}(\theta)^{-1}\hat{D}(\theta), \tag{36}
$$

is equal to zero at the pseudo-true value $\theta^*$, as shown in Theorem 4 below.

**Theorem 4:** When Assumptions 1 and 2 hold, $\theta^*$ is the minimizer of the population continuous updating objective function, and

$$
\bar{\mu}_f(\theta^*) = \lim_{T \to \infty} E \left[ \sqrt{T} f_T(\theta^*, X) \right] \\
\bar{D}(\theta^*) = \lim_{T \to \infty} E \left[ \sqrt{T} q_T(\theta^*, X) \right] - \left[ V_{00,f}(\theta^*)V_{ff}(\theta^*)^{-1}\bar{\mu}_f(\theta^*)\ldots V_{0m,f}(\theta^*)V_{ff}(\theta^*)^{-1}\bar{\mu}_f(\theta^*) \right] \tag{37}
$$

with $\bar{\mu}_f(\theta^*)$ and $\bar{D}(\theta^*)$ finite valued $k_f$ and $k_f \times m$ dimensional continuously differentiable functions of $\theta^*$, so $\bar{\mu}_f(\theta^*)'V_{ff}(\theta^*)^{-1}\bar{D}(\theta^*) = 0$, the limit behavior of $s(\theta^*)$ is characterized by:

$$
T s(\theta^*) \xrightarrow{d} \bar{\mu}_f(\theta^*)'V_{ff}(\theta^*)^{-1}\Psi_{0,f}(\theta^*) + \psi_f(\theta^*)'V_{ff}(\theta^*)^{-1}\bar{D}(\theta^*) + \psi_f(\theta^*)'V_{ff}(\theta^*)^{-1}\Psi_{0,f}(\theta^*), \tag{38}
$$

with $\text{vec}(\Psi_{0,f}(\theta^*)) = \psi_{0,f}(\theta^*)$, so the expected value of the limit of the derivative of the sample CUE objective function is equal to zero at the pseudo-true value $\theta^*$:

$$
\lim_{T \to \infty} E [ T \times s(\theta^*) ] = 0. \tag{39}
$$
Proof. See the Online Appendix. ■

The limit behavior of the score in (38) equals the sum of three distinct elements. Since all normal random variables involved in the limit expression are independently distributed according to Theorem 3, the mean of the limit behavior of the score is equal to zero. Theorem 4 uses local to zero sequences for \( \mu_f(\theta) \) and \( D(\theta) \) which are orthogonal at the pseudo-true value \( \theta^* \). This is without loss of generality. We just use them to save on notation, since it avoids that certain bounded random variables get multiplied by diverging objects which would imply that the expectation becomes ill defined. This treatment is analogous to the weak instrument asymptotics (see, e.g. Staiger and Stock (1997)) that lead to weak identification robust tests.

3.1 DRLM statistic

If the limit expression of the score in (38) would just consist of the first two elements, it would be straightforward to construct the weight matrix for a score statistic, since these two components are independently distributed. The weight matrix would then consist of the sum of the covariance matrices of each of these two components, and the limiting distribution of the score statistic would be \( \chi^2(m) \). Since \( \bar{\mu}_f(\theta^*) \) and \( \bar{D}(\theta^*) \) are not consistently estimable and the third component present in the limit expression (38) is not independent of both the first and second component, we cannot use this weight matrix for the score statistic. We provide this argument since it provides the insight into how we do obtain an appropriate weight matrix for constructing our test statistic, as we show next.

We note that the sum of the second and third component of the limit expression in (38) equals the limit of the score used in the KLM statistic from Kleibergen (2005). The limit behavior of the KLM statistic can be expressed as:

\[
\psi_f(\theta^*) V_{ff}(\theta^*)^{-1} \left[ \bar{D}(\theta^*) + \Psi_{\theta,f}(\theta^*) \right] \\
\left( \left[ \bar{D}(\theta^*) + \Psi_{\theta,f}(\theta^*) \right]' V_{ff}(\theta^*)^{-1} \left[ \bar{D}(\theta^*) + \Psi_{\theta,f}(\theta^*) \right] \right)^{-1} \\
\left[ \bar{D}(\theta^*) + \Psi_{\theta,f}(\theta^*) \right]' V_{ff}(\theta^*)^{-1} \psi_f(\theta^*) \\
\sim \chi^2(m). \tag{40}
\]

In an identical manner, we can add the first and third component of the limit expression in (38) to obtain:

\[
\left[ \bar{\mu}_f(\theta^*) + \psi_f(\theta^*) \right]' V_{ff}(\theta^*)^{-1} \Psi_{\theta,f}(\theta^*) \\
\left( \left( I_m \otimes V_{ff}(\theta^*)^{-1} \left[ \bar{\mu}_f(\theta^*) + \psi_f(\theta^*) \right] \right)' \left( I_m \otimes V_{ff}(\theta^*)^{-1} \left[ \bar{\mu}_f(\theta^*) + \psi_f(\theta^*) \right] \right) \right)^{-1} \\
\Psi_{\theta,f}(\theta^*) V_{ff}(\theta^*)^{-1} \left[ \bar{\mu}_f(\theta^*) + \psi_f(\theta^*) \right] \sim \chi^2(m). \tag{41}
\]

The weight function
involved in the limit behavior of the KLM statistic in (40) takes account of the dependence between the second and third component of the limit expression of the score in (38). Similarly, the weight function

\[ \left( I_m \otimes V_{ff}(\theta^*)^{-1} [\bar{\mu}_f(\theta^*) + \psi_f(\theta^*)] \right) \left( I_m \otimes V_{ff}(\theta^*)^{-1} [\bar{\mu}_f(\theta^*) + \psi_f(\theta^*)] \right) \] (43)

in (41) takes account of the dependence between the first and third component of the limit expression of the score in (38). Identical to the case where we just have the first two components present in the limit expression of the score, we sum the weight functions in (42) and (43) for our score statistic. Hence, the limit expression of our score statistic presented below in Definition 1 is:

\[ \left\{ [\bar{\mu}_f(\theta^*) + \psi_f(\theta^*)]' V_{ff}(\theta^*)^{-1} \Psi_{\theta,f}(\theta^*) + \psi_f(\theta^*) V_{ff}(\theta^*)^{-1} \bar{D}(\theta^*) \right\} \]

\[ \left\{ [\bar{D}(\theta^*) + \Psi_{\theta,f}(\theta^*)]' V_{f \theta,f}(\theta^*)^{-1} \bar{D}(\theta^*) + \Psi_{\theta,f}(\theta^*) \right\} + \left\{ \bar{D}(\theta^*)' V_{f \theta,f}(\theta^*)^{-1} [\bar{\mu}_f(\theta^*) + \psi_f(\theta^*)] \right\}^{-1} \]

The double robust score or Lagrange multiplier statistic then results by plugging in estimators that lead to the appropriate limit behavior of the score statistic in (44): \( \sqrt{T} \bar{\mu}_f(\theta^*) = \sqrt{T} f_T(\theta^*, X) \rightarrow_d \bar{\mu}_f(\theta^*) + \psi_f(\theta^*), \sqrt{T} \bar{D}(\theta^*) \rightarrow_d \bar{D}(\theta^*) + \Psi_{\theta,f}(\theta^*) \).

**Definition 1.** The double robust score or Lagrange multiplier (DRLM) statistic for testing \( H_0 : \theta = \theta^* \), with \( \theta^* \) the pseudo-true value, is:

\[ DRLM(\theta^*) = T^2 \times f_T(\theta^*, X)' \bar{V}_{ff}(\theta^*)^{-1} \bar{D}(\theta^*) \]

\[ = \left[ T \times \left( I_m \otimes \bar{V}_{ff}(\theta^*)^{-1} f_T(\theta^*, X) \right)' \bar{V}_{f \theta,f}(\theta^*)' \left( I_m \otimes \bar{V}_{ff}(\theta^*)^{-1} f_T(\theta^*, X) \right) \right]^{-1} \bar{D}(\theta^*)' \bar{V}_{f \theta,f}(\theta^*)^{-1} f_T(\theta^*, X). \]

The component resulting from the weight matrix (42) in the overall weight matrix makes the quadratic form of the second and third component of the limit expression of the score in (38) with it \( \chi^2(m) \) distributed. Similarly, the weight matrix (43) does so for the quadratic form of the first and third component of the limit expression of the score in (38). Since the first and second component are independently distributed, the third component is therefore “double” counted in the overall weight matrix. This makes the limit behavior
in (44) bounded by a $\chi^2(m)$ distributed random variable. This bound is sharp when the third component of the limit behavior of the score is negligible, which occurs for large values of $\hat{\mu}_f(\theta^*)$ and/or $\hat{D}(\theta^*)$. When $\hat{\mu}_f(\theta^*)$ and $\hat{D}(\theta^*)$ are both equal to zero, the limit behavior reduces to:

$$
\psi_f(\theta^*)' V_{ff}(\theta^*)^{-1} \Psi_{\theta.f}(\theta^*) \left[ \Psi_{\theta.f}(\theta^*)' V_{ff}(\theta^*)^{-1} \Psi_{\theta.f}(\theta^*) \right]^{-1} \Psi_{\theta.f}(\theta^*)' V_{ff}(\theta^*)^{-1} \psi_f(\theta^*)
$$

(46)

which is obviously bounded by a $\chi^2(m)$ distributed random variable. For intermediate values of $\hat{\mu}_f(\theta^*)$ and $\hat{D}(\theta^*)$, the $\chi^2(m)$ bound remains, which is further articulated in the proof of Theorem 5.

**Theorem 5:** When Assumptions 1 and 2 hold and given the specifications in (37), the limit behavior of $DRLM(\theta^*)$ under $H_0: \theta = \theta^*$, with $\theta^*$ the minimizer of the population continuous updating objective function, is bounded according to:

$$
\lim_{T \to \infty} \Pr \left[ DRLM(\theta^*) > cv_{\chi^2(m)}(\alpha) \right] \leq \alpha,
$$

(47)

with $cv_{\chi^2(m)}(\alpha)$ the $(1 - \alpha) \times 100\%$ critical value for the $\chi^2(m)$ distribution.

**Proof.** See the Online Appendix, which also provides an extension to Assumptions 1 and 2 by stating the parameter space of the distributions which render the DRLM test size correct; see also Andrews and Guggenberger (2017). $lacksquare$

### 3.2 DRLM test for the linear asset pricing model

For further exposition, we use the DRLM statistic to test the risk premia in the linear asset pricing model with i.i.d. errors.

**Running example 1: Linear asset pricing model** For a DRLM test of the risk premia, we need the specification of the different components of the DRLM statistic for the linear asset pricing model with i.i.d. errors:

$$
\begin{align*}
\hat{f}_T(\lambda_F, X) &= \bar{R} - \hat{\beta} \lambda_F \\
\hat{D}(\lambda_F) &= -\hat{\beta} - (\bar{R} - \hat{\beta} \lambda_F)(1 + \lambda_F \hat{Q}_{\bar{F}_F}^{-1} \lambda_F)^{-1} \lambda_F \hat{Q}_{\bar{F}_F}^{-1} \\
&= -\frac{1}{T} \sum_{t=1}^{T} \bar{R}_t (\bar{F}_t + \lambda_F)' \left[ \frac{1}{T} \sum_{t=1}^{T} (\bar{F}_t + \lambda_F)(\bar{F}_t + \lambda_F)' \right]^{-1} \\
\hat{V}_{ff}(\lambda_F) &= (1 + \lambda_F \hat{Q}_{\bar{F}_F}^{-1} \lambda_F) \hat{\Omega} \\
\hat{V}_{\theta \theta.f}(\lambda_F) &= (\hat{Q}_{\bar{F}_F} + \lambda_F \hat{\Lambda}_{\bar{F}_F})^{-1} \otimes \hat{\Omega},
\end{align*}
$$

(48)
so the specification of the DRLM statistic reads:

\[
DRLM(\lambda_F) = T(1 + \lambda_F^T \hat{Q}_{FF}^{-1} \lambda_F)^{-\frac{1}{2}} (\hat{R} - \hat{\beta} \lambda_F)^T \hat{\Omega}^{-\frac{1}{2}} \hat{D}(\lambda_F)
\]

\[
= \hat{\mu}(\lambda_F)^T \hat{D}(\lambda_F)^* \left[ \hat{\mu}(\lambda_F)^T \hat{\mu}(\lambda_F) I_m + \hat{D}(\lambda_F)^T \hat{D}(\lambda_F) \right]^{-\frac{1}{2}} \hat{D}(\lambda_F)^* \hat{\mu}(\lambda_F),
\]

with \( \hat{\mu}(\lambda_F)^* = \sqrt{T} \hat{\Omega}^{-\frac{1}{2}} (\hat{R} - \hat{\beta} \lambda_F)(1 + \lambda_F^T \hat{Q}_{FF}^{-1} \lambda_F)^{-\frac{1}{2}} = \sqrt{T} \hat{V}_{fT}(\lambda_F)^{-\frac{1}{2}} f_T(\lambda_F, X), \) and \( \hat{D}(\lambda_F)^* = \sqrt{T} \hat{\Omega}^{-\frac{1}{2}} \hat{D}(\lambda_F)(\hat{Q}_{FF} + \lambda_F \lambda_F^T)^\frac{1}{2}. \)

**Corollary 1.** When Assumptions 1 and 2 hold and under i.i.d. errors, the limit behavior of the DRLM statistic under \( H_0 : \lambda_F = \lambda_F^* \) is characterized by:

\[
DRLM(\lambda_F^*) \xrightarrow{d} \left[ \psi_f^T (\hat{D} + \Psi_{\theta,f}) I_m + (\hat{D} + \Psi_{\theta,f})^T \hat{\psi}_f + \psi_f^T \right]^{-\frac{1}{2}} \hat{D}(\lambda_F)^* \hat{\mu}(\lambda_F) \leq \chi^2(m),
\]

with \( \hat{\mu} = \Omega^{-\frac{1}{2}} \hat{\mu}(\lambda_F^*)(1 + \lambda_F^T Q_{FF}^{-1} \lambda_F)^{-\frac{1}{2}}, \hat{D} = \Omega^{-\frac{1}{2}} \hat{D}(\lambda_F^*)(Q_{FF} + \lambda_F^T \lambda_F)^\frac{1}{2}, \hat{\mu}^T \hat{D} \equiv 0, \psi_f \) and \( \Psi_{\theta,f} N \times 1 \) and \( N \times m \) dimensional random matrices that consist of independent standard normal random variables, and “\( \leq \)” indicates stochastically dominated.

The limit behavior of the DRLM statistic in Corollary 1 shows that it under \( H_0 \) only depends on two parameters, the “lengths” of \( \hat{\mu} \) and \( \hat{D} \), which reflect the amount of misspecification and the strength of identification respectively, and is dominated by a \( \chi^2(m) \) distribution.

### 3.3 Size of the DRLM test

Next, we illustrate the size of the DRLM test for the linear asset pricing model discussed above. In particular, Figure 1 shows the rejection frequencies of 5% significance DRLM tests with a 95% \( \chi^2(1) \) critical value as a function of the lengths of \( \hat{\mu} \) and \( \hat{D} \) for a single factor setting, so \( m = 1 \), and \( N = 25 \). The latter number corresponds with the twenty-five Fama-French size and book-to-market sorted portfolios, which are the default in the asset pricing literature; see Fama and French (1993). Also when \( m = 1 \), \( \hat{D} \) reduces to a vector with the same dimension as \( \hat{\mu} \), so their lengths result from the inner products of the elements in each vector.

---

\(^5\)For a continuous non-negative scalar random variable \( u \): \( u \leq \chi^2(m) \) implies that \( \Pr \left[ u > cv_{\chi^2(m)}(\alpha) \right] \leq \alpha \) for \( \alpha \in (0,1] \).
Figure 1: Rejection frequency of 5% significance DRLM tests of $H_0: \lambda_F = \lambda_F^*$ using a 95% $\chi^2(1)$ critical value as a function of the lengths of $\bar{\mu}$ and $\bar{D}$, $m = 1$, $N = 25$.

Figure 2: Rejection frequency of 5% significance KLM tests of $H_0: \lambda_F = \lambda_F^*$ using a 95% $\chi^2(1)$ critical value as a function of the lengths of $\bar{\mu}$ and $\bar{D}$, $m = 1$, $N = 25$.

Figure 1 shows that the DRLM test is size correct, since its rejection frequency does not exceed 5% for any length of $\bar{\mu}$ and $\bar{D}$. For comparison, Figure 2 presents the rejection frequencies of the KLM test, see Kleibergen (2005), as a function of the lengths of $\bar{\mu}$ and $\bar{D}$. It shows that the KLM test is only size correct when there is no misspecification so $\bar{\mu} = 0$, and can be severely size distorted for small values of the length of $\bar{\mu}$, especially when paired with small values of the length of $\bar{D}$.

Figure 1 also shows that the DRLM test is conservative when the lengths of both $\bar{\mu}$ and $\bar{D}$ are small. This is comparable to the subset Anderson-Rubin test for the homoskedastic linear IV regression model which
Guggenberger et al. (2012) show to be conservative in case of weak identification when using standard $\chi^2$ critical values. In Guggenberger et al. (2019) a data-dependent conditional critical value function is therefore proposed, which makes the subset Anderson-Rubin test near optimal. To reduce the conservativeness of the DRLM test, we follow Guggenberger et al. (2019) and calibrate a feasible conditional critical value function based on the maximum of $\hat{\mu}(\lambda_F)^*\hat{\mu}(\lambda_F)^*$ and $\hat{D}(\lambda_F)^*\hat{D}(\lambda_F)^*$. Specifically, when the maximum of these is less than two-hundred and fifty, we computed a 95% conditional critical value function based on $\max(\hat{\mu}(\lambda_F)^*\hat{\mu}(\lambda_F)^*, \hat{D}(\lambda_F)^*\hat{D}(\lambda_F)^*)$. Using the conditional critical value, the contour lines in Figure 3 show that the conservativeness of a 5% significance DRLM test has been reduced substantially from an area where the maximal length of $\bar{\mu}$ and $\bar{D}$ is less than twenty to an area where their sum is less than ten.

Figure 3: Rejection frequency of 5% significance DRLM tests of $H_0: \lambda_F = \lambda_F^*$ using a conditional 95% critical value as a function of the lengths of $\bar{\mu}$ and $\bar{D}$, $m = 1$, $N = 25$.

4 Power

The score is equal to zero at all stationary points of the CUE sample objective function, so the same holds for tests based on a quadratic form of it, like, for example, the DRLM and KLM tests, as well. This leads to the somewhat oddly behaved power of the KLM test in regular GMM. Tests with better power properties therefore exist in GMM that, implicitly or explicitly, combine the KLM test with an asymptotically independent $J$-test in either a conditional or unconditional manner, see Moreira (2003), Kleibergen (2005), Andrews et al. (2006), Andrews (2016), and Andrews and Mikusheva (2016a, b). In our misspecified GMM setting, this is, however, not possible since the limiting distribution of the $J$-statistic is a non-central $\chi^2$ distribution with an unknown non-centrality parameter. Hence, we can not combine this limiting distribution

---

6 The conditional critical value function we calibrated for Figure 3 is $f(r) = 2.4 + (\lfloor r \rfloor^{0.35}) \times (3.84 - 2.4)/(250^{0.35})$ for $r \leq 250$ and $f(r) = 3.84$ for $r > 250$, with $r$ the conditioning variable and $\lfloor \cdot \rfloor$ the entier function.
with that of the DRLM statistic to obtain the (conditional) critical values for a combination test.

4.1 Power improvement

To improve the power of the DRLM test, we can further reject hypothesized values of $\theta$ which are close to a stationary point of the CUE sample objective function other than the CUE. This would be similar to the, conditional or unconditional, identification robust combination tests in regular GMM, which use that while the KLM test does not reject at such values of $\theta$, $J$ and/or GMM Anderson-Rubin (AR) tests, see Anderson and Rubin (1949) and Stock and Wright (2000), likely do. For hypothesized values of $\theta$ close to the CUE, these combination tests put most weight on the KLM test but shift the weight towards the $J$ and GMM-AR tests when $\theta$ is close to other stationary points, see Andrews (2016) and Kleibergen (2007). Since the limiting distributions of the $J$ and GMM-AR statistics depend on unknown nuisance parameters in our misspecified GMM setting, it is not clear how we can use these statistics to improve power. To improve the power of the DRLM test, we can further reject values of $\theta$ when in between the hypothesized value and the CUE there are significant values of the DRLM statistic.

We next lay out the steps needed to turn the above idea into a size correct test for stylized linear GMM settings.

Theorem 6: a. For a given data set of realized values and a linear moment equation, the sum of $f_T(\theta, X)'\hat{V}_{ff}(\theta)^{-1}f_T(\theta, X)$ and vec($\hat{D}(\theta)'\hat{V}_{\theta f}(\theta)^{-1}$vec($\hat{D}(\theta)$)) does not vary over $\theta$.

b. When $m = 1$ and $f_T(\theta, X)$ is linear in $\theta$, the derivative of DRLM($\theta$) with respect to $\theta$ reads:

$$
\frac{1}{2} \frac{\partial}{\partial \theta} \text{DRLM}(\theta) = T \left( f_T(\theta, X)'\hat{V}_{ff}(\theta)^{-1}f_T(\theta, X) + D(\theta)'\hat{V}_{\theta f}(\theta)^{-1}D(\theta) \right) \times 
\left\{ \hat{D}(\theta)'\hat{V}_{ff}(\theta)^{-1}\hat{D}(\theta) - 2f_T(\theta, X)'\hat{V}_{ff}(\theta)^{-1}\hat{D}(\theta) \right\}
$$

$$
= f_T(\theta, X)'\hat{V}_{ff}(\theta)^{-1}f_T(\theta, X) - 2f_T(\theta, X)'\hat{V}_{ff}(\theta)^{-1}\hat{D}(\theta)
$$

The above equation simplifies to:

$$
\frac{1}{2} \frac{\partial}{\partial \theta} \text{DRLM}(\theta) = \left( f_T(\theta, X)'\hat{V}_{ff}(\theta)^{-1}f_T(\theta, X) + D(\theta)'\hat{V}_{\theta f}(\theta)^{-1}D(\theta) \right) \times 
\left( T \times \hat{D}(\theta)'\hat{V}_{\theta f}(\theta)^{-1}\hat{D}(\theta) - T \times f_T(\theta, X)'\hat{V}_{ff}(\theta)^{-1}f_T(\theta, X) \right) \left( \hat{V}_{\theta f}(\theta)^{-1} \right)^{\frac{1}{2}}
$$

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Proof. See the Online Appendix.

Running example 1: Linear asset pricing model  Theorem 6c shows that for the one factor linear asset pricing model with i.i.d. errors, the derivative of the DRLM statistic is proportional to the difference between the GMM-AR statistic, \( T \times f_T(\theta, X)\hat{V}_{ff}(\theta)^{-1}f_T(\theta, X) \), and an independently distributed statistic reflecting the strength of identification, \( T \times \hat{D}(\theta)^{\prime}\hat{V}_{\theta\theta.f}(\theta)^{-1}\hat{D}(\theta) \). Theorem 6a further shows that, for a given data set of realized values, the sum of these two statistics does not depend on \( \theta \). Given a realized data set, the DRLM statistic considered as a function of \( \theta \) thus attains its maximum when both statistics are identical so they equal half their sum.

Corollary 2. For a given data set of realized values for the one factor linear asset pricing model with i.i.d. errors, the maximal value of the DRLM statistic as a function of \( \lambda_F \) is attained at the value of \( \lambda_F \) where the GMM-AR statistic, \( T \times f_T(\lambda_F, X)^{\prime}\hat{V}_{ff}(\lambda_F)^{-1}f_T(\lambda_F, X) \), equals half the sum of \( T \times f_T(\lambda_F, X)^{\prime}\hat{V}_{ff}(\lambda_F)^{-1}f_T(\lambda_F, X) \) and \( T \times \hat{D}(\lambda_F)^{\prime}\hat{V}_{\theta\theta.f}(\lambda_F)^{-1}\hat{D}(\lambda_F) \), which, as stated in Theorem 6a, is constant over \( \lambda_F \).

Using Corollary 2 and the sample equivalent of the characteristic polynomial in (17), we can solve for the value of \( \lambda_F \) that maximizes the DRLM statistic for a given data set of realized values. We do so by not equating the characteristic polynomial to zero but to half the sum of \( T \times f_T(\lambda_F, X)^{\prime}\hat{V}_{ff}(\lambda_F)^{-1}f_T(\lambda_F, X) \) and \( T \times \hat{D}(\lambda_F)^{\prime}\hat{V}_{\theta\theta.f}(\lambda_F)^{-1}\hat{D}(\lambda_F) \), which, as stated in Theorem 6a, is constant over \( \lambda_F \). We can then straightforwardly solve for the value of \( \lambda_F \) that maximizes the DRLM statistic in a data set of realized values. We use this maximizer to improve the power of the DRLM test, as follows.

The power of a 100 \( \times \alpha \% \) significance DRLM test of \( H_0 : \lambda_F = \lambda_F^1 \) can be improved by rejecting \( H_0 \) alongside for significant values of DRLM(\( \lambda_F^1 \)) also when both:

1. The maximal value of the DRLM statistic for the analyzed data set is significant at the 100 \( \times \alpha \% \) level.
2. The DRLM statistic evaluated at \( \lambda_F^1 \) is insignificant at the 100 \( \times \alpha \% \) level but \( \lambda_F^1 \) lies inside the closed interval indicated by the significant maximizers of the DRLM statistic that does contain the CUE.

The above algorithm rejects \( H_0 \) alongside for significant values of DRLM(\( \lambda_F^1 \)) also when there is a significant value of the DRLM statistic on the line between \( \lambda_F^1 \) and the CUE. To show that the above algorithm leads to a size correct test, we compute its rejection frequency when testing \( H_0 : \lambda_F = 0 \) using the setup from Figures 1-3. While the generic specification of the DRLM test is for a stationary point of the population continuous updating objective function, the above algorithm explicitly tests for the minimizer. When computing the size of the test at the hypothesized value, of, say, zero, we therefore have to ascertain that it
is the minimizer of the population objective function. For the setup in Figures 1-3, which uses the limit expression of the DRLM statistic in (50), the population minimizer is at zero if the amount of misspecification is less than the strength of identification so the length of $\bar{\mu}$ is less than that of $\bar{D}$, since the IS identification measure (15) equals the quadratic form of $\bar{D}$. When the length of $\bar{\mu}$ exceeds that of $\bar{D}$, the minimizer of the population objective function is at $\pm\infty$ as discussed in Section 2. In standard GMM, there is no misspecification so the amount of misspecification is then always less than or equal to the identification strength, i.e. the hypothesized value automatically corresponds with the minimizer of the population objective function.

Figure 4: Rejection frequency of 5% significance tests of $H_0 : \lambda_F = 0$ using power improved DRLM and a calibrated conditional 95% critical value as a function of the lengths of $\bar{\mu}$ and $\bar{D}$, $m = 1$, $N = 25$.

Figure 4 shows the rejection frequency of the power improved DRLM test when the minimizer of the population continuous updating objective function equals the hypothesized value which is zero. Figure 4 does therefore not show the rejection frequency for values where the length of $\bar{\mu}$ exceeds that of $\bar{D}$, since the hypothesized value does then not correspond with the minimizer of the population objective function which is at $\pm\infty$. The rejection frequencies in Figure 4 are computed using the calibrated conditional critical values explained previously. Figure 4 shows that the power improvement does not affect the size of the DRLM test when the hypothesized value equals the minimizer of the population continuous updating objective function.

4.2 Power analysis

We use the one factor linear asset pricing model to compare the power and size of different identification robust test procedures with that of the DRLM test. For the power analysis, the minimizer of the population continuous updating objective function is the pseudo-true value $\lambda_F^*$ while we test for a zero value under the null hypothesis. We then map out the power curve by changing the pseudo-true value and keeping the
hypothesized value, zero, fixed. Theorem 7 states the limiting distributions of the different components of the DRLM statistic for testing the hypothesis of interest used for the power analysis.

\textbf{Theorem 7:} For testing \( H_0 : \lambda_F = \lambda^1_F = 0 \), the limit behaviors of the components of the DRLM statistic in the one factor linear asset pricing model with i.i.d. errors, \( m = 1 \) and \( Q_{FP} = 1 \), while the pseudo-true value equals \( \lambda^*_F \), are characterized by:

\[
\sqrt{T} \Omega^{-\frac{1}{2}} \hat{R} \rightarrow_d \bar{\mu} (1 + (\lambda^*_F)^2)^{-\frac{1}{2}} - \bar{D} (1 + (\lambda^*_F)^2)^{-\frac{1}{2}} \lambda^*_F + \psi^*_f (\lambda^*_F = 0),
\]

\[
\sqrt{T} \Omega^{-\frac{1}{2}} \bar{D} (\lambda^*_F = 0) \rightarrow_d \bar{D} (1 + (\lambda^*_F)^2)^{-\frac{1}{2}} + \bar{\mu} (1 + (\lambda^*_F)^2)^{-\frac{1}{2}} \lambda^*_F + \psi^*_{\theta, f} (\lambda^*_F = 0),
\]

with \( \psi^*_f (\lambda^*_F = 0), \psi^*_{\theta, f} (\lambda^*_F = 0) \) independent standard normal \( N \) dimensional random vectors, \( \mu^* = \lim_{T \rightarrow \infty} \sqrt{T} \mu_f (\lambda^*_F), \mu_f (\lambda^*_F) = \mu_R - \beta \lambda^*_F, D^* = \lim_{T \rightarrow \infty} \sqrt{T} D (\lambda^*_F), D (\lambda^*_F) = -\beta - \mu_f (\lambda^*_F) \lambda^*_F (Q_{FP} + \lambda^*_F \lambda^*_F)^{-1} \), \( \bar{\mu} = \Omega^{-\frac{1}{2}} \mu^* (1 + \lambda^*_F Q_{FP}^{-1} \lambda^*_F)^{-\frac{1}{2}}, \bar{D} = \Omega^{-\frac{1}{2}} D^* (Q_{FP} + \lambda^*_F \lambda^*_F)^{-\frac{1}{2}}, \) so \( \bar{\mu}' \bar{D} = 0 \).

\textbf{Proof.} See the Online Appendix. \( \blacksquare \)

The specification in Theorem 7 is such that, since \( \bar{\mu}' \bar{D} = 0, \lambda^*_F \) is the minimizer of the population continuous updating objective function when the length of \( \bar{D} \), whose quadratic form equals the IS identification strength measure \([15]\), is larger than or equal to the length of \( \bar{\mu} \), which reflects misspecification. The product of the limit behavior of both components in \([52]\):

\[
T \bar{\mu}' \bar{D} (\lambda^*_F = 0) \rightarrow_d (1 + (\lambda^*_F)^2)^{-\frac{1}{2}} \lambda^*_F (\bar{\mu}' \bar{D} - \bar{D}' \bar{D}) + (1 + (\lambda^*_F)^2)^{-\frac{1}{2}} \left[ \psi^*_f (\lambda^*_F = 0)' (\bar{D} + \bar{\mu} \lambda^*_F) + \psi^*_f_{\theta, f} (\lambda^*_F = 0)' (\bar{\mu} - \bar{D} \lambda^*_F) \right],
\]

further shows that identification is problematic when the lengths of \( \bar{\mu} \) and \( \bar{D} \) are equal so the amount of misspecification equals the identification strength.

We next analyze the power of identification robust tests and the DRLM test for two settings of misspecification: no misspecification, and weak misspecification. The power analysis for a mildly misspecified setting is discussed in the Online Appendix.

\subsection{No misspecification}

We first compare the power of the DRLM test with existing identification robust tests when no misspecification is present, so all of these tests are size correct. Figures 5–7 show the different power curves. Figure 5 shows the power curves of the KLM test of Kleibergen (2002, 2005, 2009) and the DRLM test for various identification strengths and no misspecification. The power of the KLM test is known to be non-monotonic which is in line with Panel 5.1. Panel 5.2 shows that power curves of the DRLM test are non-monotonic as
Figure 5: Power of 5% significance KLM and DRLM tests of

\[ H_0 : \lambda_F = 0 \text{ with no misspecification, } N = 25, \ Q_{FF} = 1 \]

Panel 5.1: KLM

Panel 5.2: DRLM

Figure 6: Power of 5% significance LR and size and power improved

DRLM tests of \( H_0 : \lambda_F = 0 \) with no misspecification, \( N = 25, \ Q_{FF} = 1 \)

Panel 6.1: LR

Panel 6.2: DRLM with size and power improvements

Panel 6.2 in Figure 6 shows that the size and power improved DRLM test, which uses the size and power improvement procedures discussed previously, has a nearly monotonic power curve. Panel 6.1 in Figure 6 shows power curves of the conditional likelihood ratio (LR) test of Moreira (2003) which is known to be optimal for this setting, see Andrews et al. (2006). Figure 7 shows power curves of the factor Anderson-Rubin (AR) test, see Anderson and Rubin (1949) and Kleibergen (2009). Overall, Figures 5-7 show that
without misspecification, DRLM is comparable to several existing identification robust tests.

Figure 7: Power of 5% significance AR tests of $H_0: \lambda_F = 0$ with no misspecification, $N = 25$, $Q_{FF} = 1$.

4.2.2 Weak misspecification

We next compare the power of the different test procedures in a setting of weak misspecification where $\bar{\mu}'\bar{\mu} = 4.4$. Figure 8 therefore shows power curves of the KLM and DRLM tests for various identification strengths, while Figure 9 shows power curves of the LR and size and power corrected DRLM test. Figure 10 shows power curves of the factor AR test. The power curves of the different test procedures are comparable to the ones in the previous Figures 5-7 except that we observe size distortion of the identification robust factor AR, KLM and LR tests in Figures 8-10. Except for the factor AR test, these size distortions become less when the identification strength increases. For the conditional LR test, the rejection frequency at zero decreases from 15% to 9% when the identification strength increases. It equals 13% when the amount of misspecification equals the identification strength. For the KLM test, it decreases from 7% to 5%. For the factor AR test, the rejection frequency at zero equals 15% for all settings of the identification strength, since no estimator of the identification strength is involved in the factor AR test. For the DRLM and size and power improved DRLM tests, we observe no size distortion.

What is striking is that, for small values of the identification strength, the power of the identification robust factor AR and LR tests decreases when $\lambda_F$ moves away from zero. This results since when the amount of misspecification exceeds the identification strength, the population continuous updating objective function is maximized at zero instead of minimized. The population continuous updating objective function is then minimized when $\lambda_F$ equals $\pm\infty$. When the strength of identification equals zero, so the length of $\bar{D} = 0$, the moment equation (11) is, however, still not satisfied at these values of $\lambda_F$ so the LR, KLM and factor AR tests remain size distorted even at these values. Moving away from zero at these settings of the identification strength, however, in general reduces the sample continuous updating objective function, which then leads
to a lower rejection frequency of these tests. For values of the identification strength exceeding the amount of misspecification, the population continuous updating objective function is minimized at zero, so we then no longer observe a reduction of the rejection frequency when $\lambda^*_F$ moves away from zero.

Figure 8: Power of 5% significance KLM and DRLM tests of $H_0: \lambda_F = 0$ with misspecification, $\bar{\mu}'\bar{\mu} = 4.4$, $N = 25$, $Q_{FF} = 1$

Panel 8.1: KLM

Panel 8.2: DRLM

Figure 9: Power of 5% significance LR and size and power improved DRLM tests of $H_0: \lambda_F = 0$ with misspecification, $\bar{\mu}'\bar{\mu} = 4.4$, $N = 25$, $Q_{FF} = 1$

Panel 9.1: LR

Panel 9.2: DRLM with size and power improvements
Figure 10: Power of 5% significance AR tests of $H_0 : \lambda_F = 0$ with misspecification, $\bar{\mu}' \bar{\mu} = 4.4$, $N = 25$, $Q_{\bar{F}\bar{F}} = 1$

Figure 11: Distribution function of $J$-statistic for misspecification when $H_0 : \lambda_F = 0$ holds, solid line: $\bar{D}' \bar{D} = 0$, dash-dot: $\bar{D}' \bar{D} = 4.4 =$ strength of misspecification, dashed: $\bar{D}' \bar{D} = 100$.

To show the difficulty of detecting the weak misspecification used in Figures 8-10, Figure 11 presents the simulated distribution function of the misspecification $J$-statistic, which equals the minimal value of the factor AR statistic for the simulated data, when the null hypothesis holds, so for values of $\lambda_F^*$ equal to zero. In particular, Figure 11 shows the distribution function of the $J$-statistic for three different values of the identification strength $\bar{D}' \bar{D}$: 0, 4.4 and 100. In Guggenberger et al. (2012), it is shown that the distribution function of the $J$-statistic is a non-increasing function of the identification strength. Recognizing that the 95% critical value of the $\chi^2(24)$ distribution, since $N - 1 = 24$, equals 36.42, Figure 11 shows that we never reject no misspecification at the 5% significance level when $\bar{D}' \bar{D}$ equals 0 or 4.4, and we only do so in 15% of the cases when $\bar{D}' \bar{D}$ equals 100. Thus, Figure 11 illustrates the difficulty of detecting weak misspecification. In the Online Appendix, we also discuss a setting of mild misspecification with $\bar{\mu}' \bar{\mu} = 10$ where it is also very hard for the $J$-test to detect misspecification, and the size distortions of the weak identification robust tests become even more pronounced while the DRLM test remains size correct.
4.3 More power improvements?

We further analyze the power of invariant tests for which we use that they are a function of the maximal invariant. We therefore construct the maximal invariant for a stylized setting of the linear asset pricing model with independent normal errors, a known value of the covariance matrix and a fixed number of observations, see also Andrews et al. (2006) which uses an identical setting for the linear IV regression model. In order to do so, we first conduct a singular value decomposition of $\Omega^{-\frac{1}{2}} \left( \begin{pmatrix} \tilde{\mu}_R \\ \tilde{\beta} \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & Q_{FF}^{\frac{1}{2}} \end{pmatrix}$, with $\tilde{\mu}_R = \sqrt{T} \mu_R$, $\tilde{\beta} = \sqrt{T} \beta$, which is invariant to transformations and whose least squares estimator has an identity covariance matrix.

**Theorem 8:** A singular value decomposition of $\Omega^{-\frac{1}{2}} \left( \begin{pmatrix} \tilde{\mu}_R \\ \tilde{\beta} \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & Q_{FF}^{\frac{1}{2}} \end{pmatrix}$ results in:

$$
\Omega^{-\frac{1}{2}} \left( \begin{pmatrix} \tilde{\mu}_R \\ \tilde{\beta} \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & Q_{FF}^{\frac{1}{2}} \end{pmatrix} = USV' = -\Omega^{-\frac{1}{2}} D(\lambda_F') \left( \begin{pmatrix} \lambda_F^* \\ I_m \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & Q_{FF}^{\frac{1}{2}} \end{pmatrix} + \Omega^{\frac{1}{2}} D(\lambda_F') \delta \left( \begin{pmatrix} \lambda_F^* \\ I_m \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & Q_{FF}^{\frac{1}{2}} \end{pmatrix},
$$

(54)

with $U$ an $N \times N$ dimensional orthonormal matrix, $V$ an $(m+1) \times (m+1)$ dimensional orthonormal matrix, and $S$ an $N \times (m+1)$ dimensional diagonal matrix with the singular values in decreasing order on the main diagonal:

$$
U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},
$$

(55)

where $U_{11}, S_1, V_{21}$ are $m \times m$ dimensional matrices; $S_2$ is an $(N - m) \times 1$ dimensional matrix, $V_{11}, V_{22}$ are $m \times 1$ dimensional vectors, $U_{12}, U_{21},$ and $U_{22}$ are $m \times (N - m)$, $(N - m) \times m$ and $(N - m) \times (N - m)$ dimensional matrices and $V_{12}$ is a scalar. The $N \times (N - m)$ dimensional matrix $D(\lambda_F')_\perp$ is the orthogonal complement of $D(\lambda_F')$, $D(\lambda_F')_\perp D(\lambda_F') \equiv 0$, $D(\lambda_F')_\perp \Omega D(\lambda_F')_\perp \equiv I_{N-m}$; and $\left( \begin{pmatrix} \lambda_F^* \\ I_m \end{pmatrix} \right)_\perp \equiv 0$ and $\left( \begin{pmatrix} \lambda_F^* \\ I_m \end{pmatrix} \right)_\perp \equiv 1$, so $\left( \begin{pmatrix} \lambda_F^* \\ I_m \end{pmatrix} \right)_\perp = \left( 1 - \lambda_F'^* \right) \left( 1 + \lambda_F'^* Q_{FF}^{-1} Q_{FF}^* \lambda_F'^* \right)^{-\frac{1}{2}}$:

$$
D(\lambda_F') = -\Omega^{\frac{1}{2}} U_1 S_1 V_{21} Q_{FF}^{\frac{1}{2}}, \quad \lambda_F^* = Q_{FF}^{\frac{1}{2}} V_{21}^{-1} V_1, \quad \delta = (U_{22} U_{22}')^{-\frac{1}{2}} U_{22} S_2 V_{12} (V_{12} V_{12}')^{-\frac{1}{2}}.
$$

(56)

**Proof.** See the Online Appendix and also Kleibergen and Paap (2006).
The squared singular values are the roots of the characteristic polynomial in (17), so \( \lambda^*_F \) in Theorem 8 is the pseudo-true value of the risk premia. The population moment \( \mu_f(\lambda_F) \) results from post-multiplying \( \Omega^{-\frac{1}{2}} \begin{pmatrix} \mu_R & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q_{\bar{FF}}^\frac{1}{2} \end{pmatrix} \), which is spanned by \( \begin{pmatrix} 1 & 0 \\ 0 & Q_{\bar{FF}}^\frac{1}{2} \lambda_F \end{pmatrix}' \), and pre-multiplying by \( \Omega^{-\frac{1}{2}} \). The derivative of the population continuous updating objective function at \( \lambda_F \) then results as:

\[
\sqrt{T} \mu_f(\lambda_F)' \Omega^{-1} D(\lambda_F) = -\left( \begin{array}{c} 1 \\ -\lambda_F \end{array} \right)' \begin{pmatrix} \lambda^*_F & I_m \end{pmatrix}' D(\lambda^*_F)' \Omega^{-1} D(\lambda_F) + \left( \begin{array}{cc} 1 & 0 \\ -\lambda_F & 0 \end{array} \right)' \begin{pmatrix} \lambda^*_F & I_m \end{pmatrix}' \delta' D(\lambda^*_F)'_{\perp} D(\lambda_F),
\]

which equals zero when \( \lambda_F \) is the pseudo-true value but also at the other stationary points. When there is no misspecification, \( \delta = 0 \) and \( D(\lambda^*_F) = -\ddot{\beta} \) so

\[
\sqrt{T} \mu_f(\lambda_F)' \Omega^{-1} D(\lambda_F) = \begin{pmatrix} 1 \\ -\lambda_F \end{pmatrix}' \begin{pmatrix} \lambda^*_F & I_m \end{pmatrix}' \ddot{\beta}' \Omega^{-1} D(\lambda_F) = (\lambda^*_F - \lambda_F)' \ddot{\beta}' \Omega^{-1} D(\lambda_F),
\]

and \( \ddot{\beta} \) is the only nuisance parameter.

Andrews et al. (2006) construct the two-sided power envelope for testing the single structural parameter in a linear IV regression model with independent normal errors and a known value of the reduced form covariance matrix. This power envelope directly extends to the linear one factor asset pricing model with independent normal errors and no misspecification. It is then of interest to determine if such a power envelope can be constructed in case of misspecification. Andrews et al. (2006) construct the power envelope using the maximal invariant, which is stated in Theorem 9 alongside its distribution for the one factor linear asset pricing model with independent normal errors and known covariance matrices of the errors and factors.

**Theorem 9:** The maximal invariant, \( S = \begin{pmatrix} S_{\perp \perp} & S_{\lambda^*_F \perp}' \\ S_{\lambda^*_F \perp} & S_{\lambda^*_F \lambda^*_F} \end{pmatrix} \), for testing \( H_0 : \lambda_F = \lambda^1_F \) in the one factor linear asset pricing model with independent normal errors and known values of the covariance matrices of the errors, \( \Omega \), and factors, \( Q_{\bar{FF}} \), is the quadratic form of:
\[
\sqrt{T} \Omega^{-\frac{1}{2}} \left( \begin{bmatrix} R \\ \hat{\beta} \end{bmatrix} \right) \\
\left( \begin{array}{c}
1 \\
-\lambda_F^1
\end{array} \right) (1 + \lambda_F^1 Q_{FF}^{-1} \lambda_F^1)^{-\frac{1}{2}} : \left( \begin{array}{rr}
1 & 0 \\
0 & Q_{FF}
\end{array} \right) \left( \begin{array}{c}
\lambda_F^1 \\
I_m
\end{array} \right) (Q_{FF} + \lambda_F^1 \lambda_F^1)^{-\frac{1}{2}}
\right). 
\] (59)

When \( m = 1 \), it has a non-central Wishart distribution with \( T \) degrees of freedom, identity scale matrix and non-centrality parameter:

Correct specification:
\[
\left( \begin{array}{c}
(\lambda_F^* - \lambda_F^1)(1 + (\lambda_F^1)^2 Q_{FF}^{-1})^{-\frac{1}{2}} \\
(Q_{FF} + (\lambda_F^1)^2)^{-\frac{1}{2}} (Q_{FF} + \lambda_F^1 \lambda_F^1)
\end{array} \right) \\
\vec{\beta}’ \Omega^{-\frac{1}{2}} \vec{\beta} \\
\left( \begin{array}{c}
(\lambda_F^* - \lambda_F^1)(1 + (\lambda_F^1)^2 Q_{FF}^{-1})^{-\frac{1}{2}} \\
(Q_{FF} + (\lambda_F^1)^2)^{-\frac{1}{2}} (Q_{FF} + \lambda_F^1 \lambda_F^1)
\end{array} \right)’,
\]

Misspecification:
\[
\left( \begin{array}{c}
(\lambda_F^* - \lambda_F^1)(1 + (\lambda_F^1)^2 Q_{FF}^{-1})^{-\frac{1}{2}} \\
(Q_{FF} + (\lambda_F^1)^2)^{-\frac{1}{2}} (Q_{FF} + \lambda_F^1 \lambda_F^1)
\end{array} \right) D(\lambda_F^*)’ \Omega^{-1} D(\lambda_F^*) \\
\left( \begin{array}{c}
(1 + \lambda_F^1)^2 Q_{FF}^{-1} - \frac{1}{2} (1 + \lambda_F^1 Q_{FF}^{-1} \lambda_F^1) \\
-(Q_{FF} + (\lambda_F^1)^2)^{-\frac{1}{2}} (\lambda_F^* - \lambda_F^1)^{-\frac{1}{2}} (1 + (\lambda_F^1)^2 Q_{FF}^{-1})^{-\frac{1}{2}} (1 + \lambda_F^1 Q_{FF}^{-1} \lambda_F^1)
\end{array} \right)’,
\]

where the specifications of \( D(\lambda_F^*) \) and \( \delta \) are stated in Theorem 8.

**Proof.** See the Online Appendix. 

The elements of the maximal invariant in Theorem 9 are such that:
\[
S_{\lambda_F^1 \lambda_F^1} = T \hat{D}(\lambda_F^1)^’ \tilde{V}_{\theta \theta, f}(\lambda_F^1)^{-1} \hat{D}(\lambda_F^1) \\
S_{\lambda_F^1} = T f_T(\lambda_F^1, X)’ \tilde{V}_{f f}(\lambda_F^1)^{-1} f_T(\lambda_F, X) \\
S_{\lambda_F^1} = T \left( \tilde{V}_{f f}(\lambda_F^1)^{-\frac{1}{2}} f_T(\lambda_F, X) \right)’ \left( \tilde{V}_{\theta \theta, f}(\lambda_F^1)^{-\frac{1}{2}} \hat{D}(\lambda_F^1) \right).
\] (61)

Since \( 1 + (\lambda_F^1)^2 Q_{FF}^{-1} \) is known, the distribution of the maximal invariant in Theorem 9 is a function of three unknown parameters: \( D(\lambda_F^*)’ \Omega^{-1} D(\lambda_F^*) \), \( \delta’ \delta \), and \( (\lambda_F^* - \lambda_F^1) \). Under \( H_0 : \lambda_F = \lambda_F^1 = \lambda_F^* \), \( \lambda_F^1 - \lambda_F^1 = 0 \), so one of these three parameters is pinned down.

**Corollary 3.** Under \( H_0 : \lambda_F = \lambda_F^1 \), the non-centrality parameter of the non-central Wishart distribution of the maximal invariant equals
the score test in case of misspecification. It is thus not obvious how to improve the power of invariant tests of $H^0$ 

4.4 Testing multiple and subsets of the structural parameter vector

The expressions of the DRLM statistic apply as well to settings where the structural parameter vector has multiple elements. The power enhancement procedure directly extends as well. Hence, we can improve the power of testing a hypothesis on the structural parameter vector by also rejecting it when there are significant values of the statistic on every line going from the hypothesized parameter value to the CUE.
Many times, we are interested in constructing confidence sets on the individual elements of the structural parameter vector. Subset DRLM tests of hypotheses specified on a selection of the elements of the structural parameter vector which result from substituting the CUE for the parameters left unspecified under the hypothesis of interest, are not necessarily size correct, see Guggenberger et al. (2012). Confidence sets with the correct coverage therefore result by projecting the joint confidence set that applies to all structural parameters on the different axes, see also Dufour and Taamouti (2005).

5 Nonlinear GMM

The DRLM test is applicable to general non-linear GMM settings with unrestricted covariance matrices. In this section we present a small simulation study using the non-linear moment equation resulting from a CRRA utility function, see e.g. Hansen and Singleton (1982), to illustrate the size and power properties of the DRLM test in a non-linear GMM setting.

Running example 3: Constant relative risk aversion (CRRA) The moment function resulting from the CRRA utility function (see e.g. Hansen and Singleton (1982)) is:

$$E\left[ \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} (\iota_N + R_{t+1}) - \iota_N \right] = \mu_f(\delta, \gamma),$$

with $\delta$ the discount factor, which is kept fixed at the value used in the simulation experiment, $\delta_0 = 0.95$, $\gamma$ the relative rate of risk aversion, $C_t$ consumption at time $t$, $R_{t+1}$ an $N$-dimensional vector of asset returns, and $\iota_N$ an $N$-dimensional vector of ones. The sample moment function and its derivative therefore only depend on $\gamma$:

$$f_T(\gamma, X) = \frac{1}{T} \sum_{t=1}^T f_t(\gamma), \quad f_t(\gamma) = \delta_0 \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} (\iota_N + R_{t+1}) - \iota_N,$$

$$q_T(\gamma, X) = \frac{1}{T} \sum_{t=1}^T q_t(\gamma), \quad q_t(\gamma) = -\delta_0 \ln \left( \frac{C_{t+1}}{C_t} \right) \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} (\iota_N + R_{t+1}).$$

The covariance matrix estimators are the Eicker-White ones, see White (1980):

$$\hat{V}_{ff}(\gamma) = \frac{1}{T} \sum_{t=1}^T (f_t(\gamma) - f_T(\gamma, X)) (f_t(\gamma) - f_T(\gamma, X))',$$

$$\hat{V}_{\theta f}(\gamma) = \frac{1}{T} \sum_{t=1}^T (q_t(\gamma) - q_T(\gamma, X)) (f_t(\gamma) - f_T(\gamma, X))',$$

$$\hat{V}_{\theta \theta}(\gamma) = \frac{1}{T} \sum_{t=1}^T (q_t(\gamma) - q_T(\gamma, X)) (q_t(\gamma) - q_T(\gamma, X))',$$

$$\hat{V}_{\theta \theta f}(\gamma) = \hat{V}_{\theta \theta}(\gamma) - \hat{V}_{\theta f}(\gamma) \hat{V}_{ff}(\gamma)^{-1} \hat{V}_{\theta f}(\gamma)'.$$

We use a log-normal data generating process to jointly simulate consumption growth and asset returns
in accordance with the moment equation. Since the discount factor is fixed at its true value, \( \gamma \) is the single structural parameter of interest; see, for example, Savov (2011) and Kroencke (2017). The population moment function then reads:

\[
\mu_f(\gamma) = \begin{pmatrix}
\exp (\ln(\delta_0) + \mu_{2,1,0} + \frac{1}{2} (V_{rr,11,0} + \gamma^2 V_{cc,0} - 2\gamma V_{rc,1,0})) \\
\vdots \\
\exp (\ln(\delta_0) + \mu_{2,N,0} + \frac{1}{2} (V_{rr,NN,0} + \gamma^2 V_{cc,0} - 2\gamma V_{rc,N,0}))
\end{pmatrix} - \iota_N,
\] (67)

with \( \mu_{2,0} = (\mu_{2,1,0} \ldots \mu_{2,N,0})' \) the mean of \( r_{t+1} = \ln(1 + R_{t+1}) \), \( V_{cc,0} \) the (scalar) variance of \( \Delta c_{t+1} = \ln \left( \frac{C_{t+1}}{C_t} \right) \), \( V_{rr,0} = V_{rr,0}' = (V_{rr,1,0} \ldots V_{rr,N,0})' \) the \( N \times 1 \) dimensional covariance between \( r_{t+1} \) and \( \Delta c_{t+1} \) and \( V_{rr,0} = V_{rr,ij,0} : i, j = 1, \ldots, N \), the \( N \times N \) dimensional covariance matrix of \( r_{t+1} \). The Online Appendix provides the expression of the population covariance matrix \( V_{ff}(\gamma) \) needed to compute the pseudo-true value \( \gamma^* \):

\[
\gamma^* = \arg \min_{\gamma} \mu_f(\gamma)' V_{ff}(\gamma)^{-1} \mu_f(\gamma).
\] (68)

Unlike for the linear asset pricing model, we need to compute the pseudo-true value \( \gamma^* \) numerically since no closed-form expression is available when there is misspecification. This also explains why we use the log-normal setting so we have an analytical expression of the population moment function, and only use one structural parameter since numerical optimizing in higher dimensions is both computationally demanding and can be imprecise.

We analyze GMM-AR and DRLM tests for correctly specified and misspecified settings.

**Correct Specification and \( N = 5 \)** Standard GMM operates under correct specification so (67) equals zero, which implies that:

\[
\mu_{2,0} = -\iota_N \ln(\delta_0) - \frac{1}{2} \begin{pmatrix}
V_{rr,11,0} \\
\vdots \\
V_{rr,NN,0}
\end{pmatrix} + \iota_N \gamma^2 V_{cc,0} - 2\gamma V_{rc,0}.
\] (69)

We revisit the simulation study in Kleibergen and Zhan (2020), who examine the GMM-AR test on \( \gamma \). We augment their simulation study by the DRLM test. Figure 12 shows the resulting power curves of GMM-AR and DRLM tests. It indicates that GMM-AR and DRLM are both size-correct with good power in the correctly specified setting.

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\(^7\)See the Online Appendix for its construction and for further details on the simulation setup.
Figure 12: Simulated power curves of GMM-AR (solid blue) and DRLM (dashed red) tests with 5% significance under correct specification. The CRRA moment condition is imposed in the data generation process with $\delta = 0.95$ and $N = 5$. The null hypothesis is $H_0: \gamma = 15$.

In addition, since we consider $N = 5$ in the data generation process (DGP), there is over-identification, which helps explain the difference in power between the GMM-AR and DRLM tests.

**Misspecification and $N = 5$** For misspecification, we no longer impose (69) in the DGP. Instead, we just test for the pseudo-true value of $\gamma$, denoted by $\gamma^*$. Specifically, we start with an auxiliary $\tilde{\mu}_2$ that satisfies (69), and then subtract a vector of constants ($c$) to introduce misspecification in the DGP:

$$
\tilde{\mu}_2 = -\iota_N \ln(\delta_0) - \frac{1}{2} \begin{pmatrix} V_{rr,11,0} \\ \vdots \\ V_{rr,NN,0} \end{pmatrix} + \iota_N \gamma^2 V_{cc,0} - 2\gamma V_{rc,0},
$$

$$
\mu_{2,0} = \tilde{\mu}_2 - \iota_N \cdot c.
$$

(70)

Figure 13 illustrates the simulation design. When $c = 0$, $\gamma^* = 15$, and $\min \mu_f' V_f^{-1} \mu_f = 0$, as in the previous correct specification case. When $c$ deviates from zero, the pseudo-true value $\gamma^*$ starts to differ from 15 in Panel 13.1, and the objective function $\mu_f' V_f^{-1} \mu_f$ in Panel 13.2 is no longer equal to zero at the pseudo-true value $\gamma^*$. 

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Figure 13: Pseudo-true value and population objective function as functions of the misspecification

Panel 13.1: Pseudo-true value function  Panel 13.2: Population objective function at $\gamma^*$

Figure 14 shows the rejection frequencies of GMM-AR and DRLM tests of $H_0: \gamma^* = 24$ which corresponds, according to Panel 13.1, with a degree of misspecification of 0.1. We consider a range of values of $c$ from 0 to 0.2 in the DGP while we test for $H_0: \gamma^* = 24$, or put differently, $H_0: c = 0.1$. Figure 14 shows that the GMM-AR test rejects the null more often than the nominal significance level of 5% to reflect that the moment condition is misspecified. In contrast, since the DRLM test allows for misspecification, it has the correct rejection frequency at the hypothesized value.

Figure 14: Simulated power curves of GMM-AR (solid blue) and DRLM (dashed red) tests at the 5% significance level under misspecification. The null hypothesis $H_0: \gamma = \gamma^* = 24$ corresponds with misspecification equal to $c = 0.1$ where $c$ reflects the deviation for misspecification.
Figure 15: Rejection frequencies of GMM-AR and DRLM tests of $H_0: \gamma = \gamma^*$ at the 5% significance level with $N = 5$ as a function of the strengths of identification, $\tilde{c}$, and misspecification $c$.

**Size of AR and DRLM tests with $N = 5$** Furthermore, Figure 15 shows the trade-off between the identification strength and the amount of misspecification for the rejection frequencies of GMM-AR and DRLM tests. The DGP is such that the correlation coefficient between the log-consumption growth and the log asset returns, $\rho_i = \frac{V_{\omega i,0}}{\sqrt{V_{cc,0}V_{rr,ii,0}}}$, is scaled by a constant $\tilde{c}$ to vary identification. Figure 15 shows the rejection frequencies of tests of $H_0: \gamma = \gamma^*$ as a function of the misspecification $c$ and strength of identification which is (partly) reflected by $\tilde{c}$. We note that the pseudo-true value $\gamma^*$ is a function of $(c, \tilde{c})$, so the reported rejection frequencies in Figure 15 are for different hypothesized values of $\gamma^*$. Figure 15 shows that the GMM-AR test gets size distorted when the misspecification increases. This is unlike the DRLM test, which remains size correct for all values of the identification and misspecification strengths.

6 Applications

We apply the DRLM test and the identification robust AR, KLM and LR tests to data for two different models discussed previously: the linear asset pricing model and the linear IV regression model.

6.1 Running example 1: Linear asset pricing model

We briefly revisit the linear factor models considered in Adrian et al. (2014) and He et al. (2017) using our DRLM test and the identification robust factor AR, KLM and LR tests; see also Kleibergen (2009) and Kleibergen and Zhan (2020).

Adrian et al. (2014) propose a leverage risk factor (“LevFac”) for asset pricing. The leverage level is
the ratio of total assets over the difference between total assets and liabilities, and the leverage risk factor equals its log change. The empirical study of Adrian et al. (2014) uses quarterly data between 1968Q1 and 2009Q4. Following Lettau et al. (2019), we extend the time period to 1963Q3 - 2013Q4 and use $N = 25$ size and book-to-market sorted portfolios as test assets. Adrian et al. (2014) show that the leverage factor prices the cross-section of many test portfolios, as reflected by the significant Fama-MacBeth (FM) (1973) and Kan-Robotti-Shanken (KRS) $t$-statistics on the risk premium reported in Table I. The KRS $t$-statistic is robust to misspecification but not to weak identification, see Kan et al. (2013).

He et al. (2017) propose the banking equity-capital ratio factor (“$EqFac$”) for asset pricing. We consider one of their specifications with “$EqFac$” and the market return “$R_m$” as the two factors. Table I shows significant FM and KRS $t$-statistics for the risk premium on “$EqFac$”.

For both Adrian et al. (2014) and He et al. (2017), the risk premia are, however, weakly identified, as indicated by the large $p$-values of both the $\chi^2$ and $F$ rank tests reported in Table I.

**DRLM: Adrian, Etula, and Muir (2014)** Using the same data as for Table I Figure 16 shows the $p$-values for testing the risk premium on the leverage factor (horizontal line) using the DRLM, AR, KLM, and LR tests. Most of the $p$-values in Figure 16 are above the 5% level, which implies that none of the DRLM, AR, KLM, and LR tests leads to tight 95% confidence intervals for the risk premium on the leverage factor as shown in Table I. Given the smallish $p$-value of the $J$-test, 0.20, and the weak identification of the risk premium on the leverage factor reflected by the unbounded 95% confidence sets, it is likely that there is misspecification so it would be appropriate to use the DRLM test.

Figure 16: Adrian, Etula and Muir (2014). $p$-value from the DRLM (dashed red), AR (dashed blue), KLM (solid black), LR (dash-dotted green) and the 5% level (dotted black). $J$-statistic (=minimum AR) equals 28.42, with $p$-value of 0.20 resulting from $\chi^2(N - 2)$. 
Table 1: Inference on Risk Premia $\lambda_F$ in Adrian, Etula, and Muir (2014) and He, Kelly, and Manela (2017)

The test assets are the $N = 25$ size and book-to-market portfolios from 1963Q3 to 2013Q4 taken from Lettau, Ludvigson, and Ma (2019). “LevFac” is the leverage factor of Adrian, Etula, and Muir (2014). “EqFac” is the banking equity-capital ratio factor of He, Kelly, and Manela (2017). “$R_m$” is the market return. The estimate of $\lambda_F$ and the FM $t$-statistic result from the Fama-MacBeth (1973) two-pass procedure. The KRS $t$-statistic is based on the KRS $t$-test of Kan, Robotti, and Shanken (2013). The point estimates of $\lambda_F$ are identical to those reported in Lettau, Ludvigson, and Ma (2019). The rank test is adopted from Kleibergen and Zhan (2020) for testing $H_0: \text{rank}(\beta) = m - 1$, where $m$ is the number of risk factors.

|                  | Adrian, Etula, and Muir (2014) | He, Kelly, and Manela (2017) |
|------------------|---------------------------------|-----------------------------|
|                  | $LevFac$                        | $R_m$                       | $EqFac$                     |
| Estimate of $\lambda_F$ | 13.91                          | 1.19                        | 6.88                        |
| FM $t$           | 3.58                            | 0.81                        | 2.14                        |
| KRS $t$          | 2.55                            | 0.77                        | 2.10                        |
| CUE of $\lambda_F$ | 51.77                          | 23.22                       | 94.02                       |
| 95% confidence set |                               |                             |                             |
| FM $t$           | (6.29, 21.54)                   | (-1.67, 4.05)               | (0.57, 13.19)               |
| KRS $t$          | (3.22, 24.60)                   | (-1.84, 4.22)               | (0.46, 13.30)               |
| DRLM             | $(−\infty, -91.4) \cup (−9.2, 1.7) \cup (17.8, +\infty)$ | $(-\infty, +\infty)$       | $(−\infty, +\infty)$       |
| DRLM (power enh.) | $(−\infty, -91.4) \cup (17.8, +\infty)$ | $(−\infty, +\infty)$       | $(−\infty, +\infty)$       |
| AR               | $(−\infty, -101.4) \cup (18.3, +\infty)$ | $(−\infty, -64.6) \cup (8.1, +\infty)$ | $(−\infty, -244.1) \cup (37.7, +\infty)$ |
| KLM              | $(−\infty, -185.7) \cup (−5.7, -0.9) \cup (20.8, +\infty)$ | $(−\infty, -7.2) \cup (−4.7, -0.3) \cup (1.0, +\infty)$ | $(−\infty, -23.8) \cup (−8.1, 1.7) \cup (11.8, +\infty)$ |
| LR               | $(−\infty, -274.2) \cup (21.8, +\infty)$ | $(−\infty, -9.6) \cup (2.2, +\infty)$ | $(−\infty, -33.8) \cup (16.2, +\infty)$ |
| Rank test        | $\chi^2$-statistic (p-value)   | 31.97 (0.13)                | 35.88 (0.04)                |
|                  | $F$-statistic (p-value)         | 1.17 (0.28)                 | 1.33 (0.16)                 |
The $p$-values of the DRLM test in Figure 16 are equal to one at two different points. The $p$-values of the AR test show that one of these two points relates to the minimal value of the AR test and the other one to the maximal value of the AR test. Using the power enhancement rule for the DRLM test, we can reject non-significant values that lie within the closed interval indicated by the significant maximizers of the DRLM statistic that does not contain the CUE, so the non-significant $p$-values of the DRLM test which occur around the maximizer of the AR test can all be categorized as significant ones according to the power enhancement rule. The resulting 95% confidence set for the DRLM test rejects a zero value of the risk premium of the leverage factor and is reported in Table 1 alongside the one which results from just applying the DRLM test. The FM and KRS $t$-statistics reported in Table 1 also reject a zero value of the risk premium, but these tests are not reliable because of the weak identification of the risk premium of the leverage factor and the likely misspecification reflected by the smallish $p$-value of the $J$-test.

The $\chi^2$ rank statistic reported in Table 1, 31.97, corresponds with the sample analog of the IS identification strength measure (15) and is always larger than or equal to the minimal value of the CUE objective function whose value corresponds with the $J$-statistic reported in Figure 16, 28.42. The just slightly larger value of the rank statistic implies that the CUE can very well result from a reduced rank value of the $\beta$’s which then further explains its huge value in Table 1, 51.77, and the unbounded 95% confidence sets. We thus have to be cautious with interpreting the CUE as reflecting the risk premium on the leverage factor.

**DRLM: He, Kelly, and Manela (2017)** Figure 17 shows the joint 95% confidence sets (shaded areas) of the risk premia on the banking equity-capital ratio factor “EqFac” and the market return “$R_m$”, from using the DRLM, AR, KLM, and LR tests. The $p$-value of the $J$-test shows that misspecification is present, so it is appropriate to use the DRLM test for the confidence set of the minimizer of the population continuous updating objective function. The 95% confidence sets of the DRLM and KLM tests have two rather disjoint areas. The power enhancement rule for the DRLM test shows that the smaller disjoint area can be discarded for the joint 95% confidence set that results from the DRLM test. The resulting 95% confidence set from the DRLM test includes a zero value for the risk premium on “EqFac”, which indicates that the pricing ability of “EqFac” is under doubt.

The minimal value of the CUE objective function reported in Figure 17 of 35.32, which equals the $J$-statistic, is just slightly below the $\chi^2$ rank statistic reported in Table 1, 35.88. The sample analog of the IS identification strength measure (15) is therefore just above the minimal value of the CUE objective function. This makes it likely that the CUE estimates result from a lower rank value of the $\beta$-matrix of the risk factors which then further explains the very large values of the CUE estimates, (23.22, 94.02), and their unbounded 95% confidence sets shown in Figure 17. It is thus difficult to interpret the CUE as reflecting
the risk premium on the two risk factors.

Figure 17: He, Kelly and Manela (2017). 95% confidence sets from DRLM, AR, KLM and LR. 
$J$-statistic (minimum of AR) equals 35.32, with $p$-value of 0.036 resulting from $\chi^2(N - 3)$.

![Panel 17.1: DRLM](image1)

![Panel 17.2: AR](image2)

![Panel 17.3: KLM](image3)

![Panel 17.4: LR](image4)

Figure 18: $R_m$ and SMB. 95% confidence sets from DRLM, AR, KLM and LR. 
$J$-statistic (minimum of AR) equals 59.34, with $p$-value of 0.00 resulting from $\chi^2(N - 3)$.

![Panel 18.1: DRLM](image5)

![Panel 18.2: AR](image6)

![Panel 18.3: KLM](image7)

![Panel 18.4: LR](image8)
To compare with Figure 17, we further replace the “EqFac” risk factor with the “SMB” (small minus big) factor from Fama and French (1993) and similarly construct Figure 18. The risk premia on “$R_m$” and “SMB” are well-known to be strongly identified, and the rank test of Kleibergen and Zhan (2020) yields a $p$-value close to zero for testing their $\beta$’s (with the rank test $\chi^2$-statistic equal to 128.35, $F$-statistic equal to 4.89). The AR test now signals model misspecification, since it rejects every hypothesized risk premia as shown in Panel 18.2, so the 95% confidence set that results from the AR test is empty. Our DRLM test, which allows for misspecification, yields a tight confidence set in Panel 18.1. This tight confidence set, in contrast with the wide one in Panel 17.1, indicates that the pricing ability of “EqFac” differs substantially from “SMB”. Because of the misspecification, the 95% confidence sets resulting from the KLM and LR tests are not representative for the minimizer of the population objective function.

The minimal value of the sample objective function of 59.34 reported in Figure 18, which equals the $J$-statistic, is now well below the sample measure of the identification strength of 128.35. The sample measure of the identification strength corresponds with the rank test $\chi^2$ statistic. Despite the misspecification, the CUE estimates can therefore be straightforwardly interpreted as representing the risk premia on “$R_m$” and “SMB”. This is further reflected by their convex 95% confidence sets shown in Figure 18.

### 6.2 Running example 2: Linear IV regression for the return on education using Card (1995) data

To further show the ease of implementing the DRLM test for applied work, we use the return on education data from Card (1995). Card (1995) uses proximity to college as the instrument in an IV regression of (the log) wage on (length of) education. For more details on the data, we refer to Card (1995). The instruments used in our specification are three binary indicator variables which show the proximity to a two-year college, a four-year college and a four-year public college, respectively. The included exogenous variables are a constant term, age, age$^2$, and racial, metropolitan, family and regional indicator variables. All three binary instruments have their own local average treatment effects, which in case of heterogeneous treatment effects leads to misspecification of the linear IV regression model since it considers them to be identical, see Imbens and Angrist (1994).

Figure 19 presents the values of the AR, LR, KLM and DRLM statistics around the CUE. It also shows their critical value functions at the 5% level. The other area of small values of the DRLM statistic is left out since it would be discarded by the power enhancement rule. The $J$-statistic, which equals the minimal value of the AR statistic, is 2.99 with a $p$-value of 0.22. The first stage $F$-statistic is 7.01 so the return on education is weakly identified, see Stock and Yogo (2005), which then also implies that the $J$-test does not have much
power. Its quite low p-value can thus as well indicate misspecification, which would result from distinct local average treatment effects for the different instruments. Lee (2018) constructs misspecification-robust standard errors for the two stage least squares estimator when the local average treatment effects differ, but the resulting t-test is not valid here because of the weak identification of the return on education indicated by the small first stage F-statistic. This makes the DRLM test more appealing, since it is robust to both misspecification and weak identification. Kitagawa (2015) further shows that the validity of the instruments for the Card (1995) data depends on the specification of the model. Figure 19 then shows that allowing for misspecification further enlarges the identification-robust confidence set for the return on education.

Figure 19: Tests of the return on education using Card (1995) data with the DRLM (solid black), KLM (dashed black), LR (solid red) and AR (solid blue) statistics and their 95% (conditional) critical value lines (dotted in the color of the test they refer to).

Since the number of instruments is equal to three, the sample analog of the IS identification strength measure \( IS \) of 21.03 (=three times the first stage F-statistic) is well above the minimal value of the CUE objective function, 2.99. A structural interpretation could therefore be rendered to the CUE, which is also reflected by the bounded 95% confidence sets shown in Figure 19.

7 Conclusions

We show that it is generally feasible to conduct reliable inference on the pseudo-true value of the structural parameters resulting from the population continuous updating GMM objective function using the DRLM test. For linear moment equations, we also propose a measure of the identification strength that can be
compared with the minimal value of the CUE objective function to gauge whether the pseudo-true value can be interpreted in a structural manner. While settings of weak identification paired with misspecification are empirically relevant, it was so far not possible to conduct reliable inference in these settings. This holds since weak identification robust tests are size distorted when the model is misspecified, while the misspecification tests which are typically used to detect misspecification, are virtually powerless under weak identification. Hence, the DRLM test removes an important obstacle for conducting reliable inference in these empirically relevant settings. We propose some straightforward power improvements for the DRLM test which make it work well, and hope to conduct further power improvements in future work. We also use the DRLM test to analyze data from three studies which are plagued by both weak identification and misspecification issues: Card (1995), Adrian et al. (2014), and He et al. (2017). It shows that other inference procedures can seriously underestimate the uncertainty concerning the structural parameters when both misspecification and weak identification matter.
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Online Appendix for
“Double robust inference for continuous updating GMM”

1 Structural interpretation for misspecified settings

In case of misspecification, the structural specification resulting from the pseudo-true value depends on the involved population objective function. For the linear asset pricing and instrumental variables regression models, it is thus instructive to see how the population continuous updating objective function comes to a structural specification at the pseudo-true value. We therefore first lay out the unrestricted specification of the population moments used by the population continuous updating objective function to obtain its structural specification at the pseudo-true value for the linear factor and instrumental variables regression models with i.i.d. errors:

Factor model:

\[
\begin{pmatrix}
\mu_R \\
\beta
\end{pmatrix}
= -D^*\begin{pmatrix}
\lambda_F^* & I_m
\end{pmatrix}
+ \Omega D^*_{\perp} \delta^* \begin{pmatrix}
\lambda_F^* & I_m
\end{pmatrix}_{\perp}
\begin{pmatrix}
1 & 0 \\
0 & Q_{\bar{F}\bar{F}}^{-1}
\end{pmatrix}
\]

Linear instrumental variables regression model:

\[
\begin{pmatrix}
\sigma_{Zy} \\
\Sigma_{ZX}
\end{pmatrix}
= -D^*\begin{pmatrix}
\theta^* & I_m
\end{pmatrix}
+ Q_{ZZ} D^*_{\perp} \delta^* \begin{pmatrix}
\theta^* & I_m
\end{pmatrix}_{\perp}
\begin{pmatrix}
1 & 0 \\
0 & Q_{\bar{Z}\bar{Z}}^{-1}
\end{pmatrix}
\]

where \(D^*\) is an \(N \times m\) dimensional matrix for the factor model and a \(k \times m\) dimensional matrix for the linear instrumental variables regression model, \(D^*_{\perp}\) is the orthogonal complement of \(D^*\), so an \(N \times (N - m)\) dimensional matrix for the factor model: \(D^*_{\perp} = 0\), \(D^*_{\perp} \Omega D^*_{\perp} = I_{N-m}\); and a \(k \times (k - m)\) dimensional matrix for the linear instrumental variables regression model: \(D^*_{\perp} = 0\), \(D^*_{\perp} Q_{ZZ} D^*_{\perp} = I_{k-m}\); in an identical manner: \(\lambda_F^* \begin{pmatrix} I_m \end{pmatrix}_{\perp} = (1 - \lambda_F^* \Omega (\lambda_F^*)^{-\frac{1}{2}} \begin{pmatrix} I_m \end{pmatrix}_{\perp} \) and \(\theta^* \begin{pmatrix} I_m \end{pmatrix}_{\perp} = (1 - \theta^* \left(\frac{1}{-\theta^*} \Omega \left(-\theta^*\right)\right)^{-\frac{1}{2}} \begin{pmatrix} I_m \end{pmatrix}_{\perp} \) and \(\delta^*\) is an \((N-m)\) dimensional vector for the factor model and a \((k-m)\) dimensional vector for the linear
instrumental variables regression model reflecting the misspecification so in case of correct specification, $\delta^* = 0$. The matrix $\left( \begin{array}{c} \gamma_1 \\ \Gamma_2 \end{array} \right)$ is $N \times (m + 1)$ dimensional for the linear factor model and $k \times (m + 1)$ dimensional for the instrumental variables regression model. The above specification results from a singular value decomposition of the normalized population moments, see e.g. Theorem 8 and Kleibergen and Paap (2006).

The unrestricted specifications show that the population continuous updating objective function at the pseudo-true value equals:

$$Q_p(\lambda^*_\gamma) = \frac{1}{1 + \lambda^*_\gamma Q^{-1}_{PF} \lambda^*_F} (\mu_R - \beta \lambda^*_F)' \Omega^{-1} (\mu_R - \beta \lambda_F) = \delta^* \delta^*,$$

$$Q_p(\theta^*) = \frac{1}{\omega_{\theta^*} - 2 \omega_{\theta^*} \theta^* \Omega^{-1} \theta^*} (\sigma_{Zy} - \Sigma_{ZX} \theta^*)' Q^{-1}_{ZZ}(\sigma_{Zy} - \Sigma_{ZX} \theta^*) = \delta^* \delta^*,$$

which further illustrates that $\delta^* \delta^*$ equals the squared smallest singular value of either $\Omega^{-\frac{1}{2}} \left( \begin{array}{c} \mu_R \\ \beta \end{array} \right)$, factor model, or $Q^{-\frac{1}{2}}_{ZZ} \left( \sigma_{Zy} \Sigma_{ZX} \right) \Omega^{-\frac{1}{2}}$, linear instrumental variables regression model.

For the unrestricted specification to have a structural interpretation, we need that:

Factor model:

$$\gamma_1' \Omega^{-1} D^* = 0, \quad \Gamma_2' \Omega^{-1} D^* = 0, \quad \left( \begin{array}{c} \gamma_1 \\ \Gamma_2 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} \lambda^*_F \\ I_m \end{array} \right) = 0.$$

Linear instrumental variables regression model:

$$\gamma_1' Q^{-1}_{ZZ} D^* = 0, \quad \Gamma_2' Q^{-1}_{ZZ} D^* = 0, \quad \left( \begin{array}{c} \gamma_1 \\ \Gamma_2 \end{array} \right) \Omega \left( \begin{array}{c} \theta^* \\ I_m \end{array} \right) = 0.$$

The restrictions for the linear instrumental variables regression model are identical to those in Kolesár et al. (2015), except that they also assume that $\Gamma_2 = 0$ who show that they allow for a causal interpretation. Kolesár et al. (2015) motivate them by means of a random coefficients assumption with potentially many instruments where direct, channeled through $\gamma_1$, and indirect effects, channeled through $D^*$, are independently distributed. In asset pricing, the factors are often considered as proxies for true underlying risk factors. The measurement error between the observed proxy risk factors and the true underlying risk factors can then similarly be represented by a random coefficient specification where the measurement error reflected by $\left( \begin{array}{c} \gamma_1 \\ \Gamma_2 \end{array} \right)$ is uncorrelated with the true risk factor $D^*$ after correcting for the covariance matrix of the errors.

The unrestricted specifications crucially hinge on that the largest singular values, identifying the struc-

---

1 We note that Assumption 2 in Kolesár et al. (2015) is imposed on $Q^{-1}_{ZZ}(\sigma_{Zy} \Sigma_{ZX})$ so $\gamma_1' Q^{-1}_{ZZ} D^* = 0$ in their specification.
tural specification, and the smallest one, which represents the misspecification, differ considerably. The largest singular values reflect the identification strength of the structural parameters so when these are close to the singular value reflecting the misspecification, the pseudo-true value is weakly identified. Furthermore, when the singular value representing the misspecification exceeds (some of) the singular values reflecting the identification strength, we can no longer attribute a structural interpretation to the pseudo-true value.

2 Additional numerical results

2.1 Mild misspecification

We increase the amount of misspecification to $\bar{\mu}'\bar{\mu} = 10$, which is still quite small since there are twenty-five moment equations. Figures A1, A2, and A3 show that the increased misspecification exacerbates the size distortion of the AR, KLM and LR tests compared to the setting of weak misspecification in the paper.

For the conditional LR test, the rejection frequency at zero decreases from 30% to 8% when the identification strength increases. When the amount of misspecification and the identification strength coincide, the rejection frequency of the LR test is 27% when $\lambda_{F} = 0$. For the KLM test, the rejection frequency decreases from 10% to 5%. For the DRLM and size and power improved DRLM test, we observe either no size distortion and a rejection frequency of 8% which decreases to 5% when the identification strength increases. The minor size distortion of the size and power improved DRLM test only occurs when the amount of misspecification exceeds the strength of identification, so the hypothesized value is not the minimizer of
the population objective function, and is not present when the identification strength is larger than or equal to the amount of misspecification. The rejection frequency of the AR test is equal to 36% for all identification strengths. When the amount of misspecification exceeds the identification strength, the maximum of the population continuous updating objective function is situated at $\lambda^*_F = 0$, which explains why the rejection frequency of the AR and LR tests decreases away from $\lambda^*_F = 0$ for low values of the identification strength. For values of the identification strength which exceed the amount of misspecification, we see no decrease of the rejection frequency when $\lambda^*_F$ moves away from zero.

Figure A2: Power of 5% significance LR and size and power improved DRLM tests of $H_0 : \lambda_F = 0$ with misspecification, $\bar{\mu}' \bar{\mu} = 10$, $N = 25$, $Q_{\bar{F}\bar{F}} = 1$

Panel A2.1: LR

Panel A2.2: DRLM with size and power improvements

Figure A3: Power of 5% significance AR tests of $H_0 : \lambda_F = 0$

with misspecification, $\bar{\mu}' \bar{\mu} = 10$, $N = 25$, $Q_{\bar{F}\bar{F}} = 1$
Figure A4 shows the distribution function of the misspecification $J$-statistic, which equals the minimal value of the AR statistic, when the null hypothesis holds, so for values of $\lambda_F$ equal to zero. It shows the distribution function for three different values of the identification strength $\bar{D}'\bar{D}$: 0, 10 and 100. Recognizing that the 95% critical value of the $\chi^2(24)$ distribution, since $N - 1 = 24$, equals 36.42, Figure A4 shows that we never reject no misspecification at the 5% significance level when $\bar{D}'\bar{D}$ equals 0, 7% of the times when $\bar{D}'\bar{D}$ equals 10 and 33% when $\bar{D}'\bar{D}$ equals 100. This indicates the difficulty of detecting the mild misspecification present in the simulated data.

To show that the power issues discussed previously for both the identification robust tests and the misspecification $J$-test do not result from the somewhat large number of moment equations, 25, we next discuss a somewhat smaller simulation experiment with fewer moment conditions.

2.2 Power of $J$-test and identification-robust tests with fewer moment conditions

To show that the low power of the $J$-test for misspecification is not just resulting from the large number of moment equations, we repeat the simulation exercise with fewer moment equations, $N = 5$, and weak misspecification: $\bar{\mu}'\bar{\mu} = 2.5$. Panels A5.1 and A5.2 in Figure 5 show the power curves for the conditional LR and size and power improved DRLM tests. Panel A5.1 shows that the conditional LR test is size distorted and its rejection frequency equals 17% when the misspecification and strength of identification are identical. The size and power improved DRLM test shows no size distortion. Figure A6 shows the simulated distribution function of the $J$-statistic. Since $N = 5$, the limiting distribution of the $J$-statistic is a $\chi^2(4)$ distribution whose 95% critical value equals 9.48. The simulated distribution function shows
that we never reject no misspecification when $\bar{D}'\bar{D} = 0$, 2.5% of the times when $\bar{D}'\bar{D} = 2.5$ which equals the strength of misspecification, and 20% of the times when $\bar{D}'\bar{D} = 100$. This reiterates the difficulty of detecting misspecification, which leads to size distorted identification-robust tests, using the $J$-test when the identification is weak; see also Gospodinov et al. (2017).

Figure A5: Power of 5% significance LR and DRLM tests of $H_0: \lambda_F = 0$ with misspecification, $\bar{\mu}'\bar{\mu} = 2.5$, $N = 5$, $Q_{FF} = 1$

Panel A5.1: LR Panel A5.2: DRLM

Figure A6. Distribution function of $J$-statistic for misspecification when $H_0: \lambda_F = 0$ holds, solid line $\bar{D}'\bar{D} = 0$, dash-dot: $\bar{D}'\bar{D} = 2.5 = \text{strength of misspecification}$, dashed: $\bar{D}'\bar{D} = 100$. 
3 Lemmas and Proofs

3.1 Lemma

Lemma 1. The estimators \( \hat{R} \) and \( \hat{\beta} \) in the linear regression model:

\[
R_t = c + \beta F_t + u_t,
\]

with \( c \) an \( N \)-dimensional vector of constants, \( F_t = G_t - \bar{G} \), with \( G_t \) an \( m \)-dimensional vector of factors and \( \bar{G} = \frac{1}{T} \sum_{t=1}^{T} G_t \), so \( \bar{F} = 0 \), and \( u_t \) an \( N \)-dimensional vector which contains the errors which are i.i.d. distributed with mean zero and covariance matrix \( \Omega \), are independently distributed in large samples.

Proof: Since \( \hat{\bar{R}} = \hat{\beta} \bar{F} = \hat{\beta} \), and the joint limit behavior of \( \hat{c} \) and \( \hat{\beta} \) accords with

\[
\sqrt{T} \left[ \begin{pmatrix} \hat{c} \\ \text{vec}(\hat{\beta}) \end{pmatrix} - \begin{pmatrix} c \\ \text{vec}(\beta) \end{pmatrix} \right] \xrightarrow{d} \begin{pmatrix} \psi_c \\ \psi_\beta \end{pmatrix},
\]

with

\[
\begin{pmatrix} \psi_c \\ \psi_\beta \end{pmatrix} \sim N \left( 0, (Q^{-1} \otimes I_N) \Sigma(Q^{-1} \otimes I_N) \right),
\]

since \( \frac{1}{T} \sum_{t=1}^{T} \begin{pmatrix} 1 \\ F_t \end{pmatrix} \begin{pmatrix} 1 \\ F_t \end{pmatrix}' \xrightarrow{p} Q = \begin{pmatrix} 1 & \mu_F' \\ \mu_F & Q_{FF} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & Q_{FF} \end{pmatrix}, \mu_F = 0, Q_{FF} = E(F_t F_t') = Q_{F\bar{F}} + \mu_F \mu_F', \) and \( \frac{1}{T} \sum_{t=1}^{T} \begin{pmatrix} 1 \\ F_t \end{pmatrix} \begin{pmatrix} 1 \\ F_t \end{pmatrix}' \otimes u_t u_t' \xrightarrow{p} \Sigma. \) When \( u_t \) is i.i.d., \( \Sigma = (Q \otimes \Omega) \), with \( \Omega = \text{var}(u_t) \), so

\[
\begin{pmatrix} \psi_c \\ \psi_\beta \end{pmatrix} \sim N \left( 0, \begin{pmatrix} 1 & 0 \\ 0 & Q_{FF}^{-1} \end{pmatrix} \otimes \Omega \right),
\]

so the limit behaviors of \( \hat{\bar{R}} = \hat{\beta} \) and \( \hat{\beta} \) are independent.

Lemma 2. a. When \( \hat{V}_{ff}(\theta)^{-\frac{1}{2}} = \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{V}_{ff}(\theta)^{-\frac{1}{2}}, \theta : 1 \times 1, \) it holds that

\[
\frac{\partial}{\partial \theta} \hat{V}_{ff}(\theta)^{-\frac{1}{2}} = -\hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{V}_{ff}(\theta)^{-1}. 
\]

b. \( \frac{\partial}{\partial \theta} \hat{V}_{ff}(\theta)^{-\frac{1}{2}} f_T(\theta, X) = \hat{V}_{ff}(\theta)^{-\frac{1}{2}} D(\theta). \)

7
Proof: a. Because $\hat{V}_f(\theta)^{-1} = \hat{V}_f(\theta)^{-\frac{1}{2}}\hat{V}_f(\theta)^{-\frac{1}{2}}$, $\hat{V}_f(\theta)^{-\frac{1}{2}}\hat{V}_f(\theta)^{-\frac{1}{2}} = I_k$, and

$$\left(\frac{\partial V_f{\theta}}{\partial \theta}\right)^{-\frac{1}{2}}\hat{V}_f(\theta)^{-\frac{1}{2}} = \hat{V}_f(\theta)^{-\frac{1}{2}}\hat{V}_f(\theta)^{-\frac{1}{2}}\left(\frac{\partial V_f{\theta}}{\partial \theta}\right)^{-\frac{1}{2}}$$

such that $\frac{\partial V_f{\theta}}{\partial \theta}^{-\frac{1}{2}} = -\hat{V}_f(\theta)^{-\frac{1}{2}}\hat{V}_f(\theta)^{-\frac{1}{2}}$, since $\frac{\partial V_f{\theta}}{\partial \theta} = \hat{V}_f(\theta)^{-\frac{1}{2}}\hat{V}_f(\theta)^{-\frac{1}{2}}$ which results from the definition of $q_T(\theta, X) = \frac{\partial}{\partial \theta} f_T(\theta, X)$.

b. Using the product rule of differentiation:

$$\frac{\partial}{\partial \theta} \hat{V}_f(\theta)^{\frac{1}{2}} = \left(\frac{\partial}{\partial \theta} \hat{V}_f(\theta)^{\frac{1}{2}}\right) f_T(\theta, X) + \hat{V}_f(\theta)^{\frac{1}{2}} \left(\frac{\partial}{\partial \theta} f_T(\theta, X)\right)$$

$$= -\hat{V}_f(\theta)^{\frac{1}{2}}\hat{V}_f(\theta)^{-1} f_T(\theta, X) + \hat{V}_f(\theta)^{\frac{1}{2}} q_T(\theta, X)$$

$$= \hat{V}_f(\theta)^{-\frac{1}{2}} \hat{D}(\theta).$$
c. The specification of \( \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{D}(\theta) \) is \( \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{D}(\theta) = \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \left[ q_{T}(\theta, X) - \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} f_{T}(\theta, X) \right] \), so:

\[
\frac{\partial}{\partial \theta} \left( \hat{V}_{ff}(\theta)^{-\frac{1}{2}} D_{T}(\theta, X) \right) = \left( \frac{\partial}{\partial \theta} \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \right) D_{T}(\theta, X) + \frac{\partial}{\partial \theta} \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \left[ q_{T}(\theta, X) - \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} f_{T}(\theta, X) \right] - \hat{V}_{\theta f}(\theta) \left( \frac{\partial}{\partial \theta} \hat{V}_{ff}(\theta)^{-1} \right) f_{T}(\theta, X) - \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} \left( \frac{\partial}{\partial \theta} f_{T}(\theta, X) \right) = - \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} D_{T}(\theta, X) - \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} f_{T}(\theta, X) + \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} f_{T}(\theta, X). 
\]

\[
\hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} D_{T}(\theta, X) = -2 \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} D_{T}(\theta, X) - \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} f_{T}(\theta, X)
\]

d. The specification of \( f_{T}(\theta, X)^{\prime} \hat{V}_{ff}(\theta)^{-1} \hat{D}(\theta) \) is:

\[
\frac{\partial}{\partial \theta} f_{T}(\theta, X)^{\prime} \hat{V}_{ff}(\theta)^{-1} \hat{D}(\theta) = \left( \frac{\partial}{\partial \theta} \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \right) \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{D}(\theta) + f_{T}(\theta, X)^{\prime} \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \left( \frac{\partial}{\partial \theta} \hat{V}_{ff}(\theta)^{-\frac{1}{2}} \hat{D}(\theta, X) \right) = \hat{D}(\theta)^{\prime} \hat{V}_{ff}(\theta)^{-1} \hat{D}(\theta) - 2 f_{T}(\theta, X)^{\prime} \hat{V}_{ff}(\theta)^{-1} \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} D_{T}(\theta, X) - f_{T}(\theta, X)^{\prime} \hat{V}_{ff}(\theta)^{-1} \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} f_{T}(\theta, X).
\]

e. The specification of \( V_{00, f}(\theta) = V_{00}(\theta) - \hat{V}_{\theta f}(\theta) \hat{V}_{ff}(\theta)^{-1} \hat{V}_{\theta f}(\theta)^{\prime} \) is such that:

\[
\frac{\partial}{\partial \theta} V_{00, f}(\theta)
\]

f. The specification of \( f_{T}(\theta, X)^{\prime} \hat{V}_{ff}(\theta)^{-1} \hat{V}_{00, f}(\theta) \hat{V}_{ff}(\theta)^{-1} \hat{V}_{\theta f}(\theta)^{\prime} \) is such that:

\[
\frac{\partial}{\partial \theta} V_{00, f}(\theta)
\]
\[
\frac{\partial}{\partial \theta} \left( f_T(\theta, X) V_{ff}(\theta)^{-1} \right) 
= 2 \left( \frac{\partial}{\partial \theta} V_{ff}(\theta)^{-1} f_T(\theta, X) \right) + f_T(\theta, X) V_{ff}(\theta)^{-1} \left( \frac{\partial}{\partial \theta} f_T(\theta, X) \right) 
\]

h. It follows from e and g above.

3.2 Proof of Theorem 1

The derivative of \( Q_p(\theta) \) with respect to \( \theta \) consists of two parts. The derivative of \( \mu_f(\theta) \) with respect to \( \theta \) : \( J(\theta) = \frac{\partial}{\partial \theta} \mu_f(\theta) \), and the derivative of \( V_{ff}(\theta)^{-1} \) with respect to \( \theta \). To obtain the derivative of \( V_{ff}(\theta)^{-1} \) with respect to \( \theta \), we start out with the derivative of \( V_{ff}(\theta) \) with respect to \( \theta \):

\[
\frac{\partial}{\partial \theta} \text{vec}(V_{ff}(\theta)) = \lim_{T \to \infty} \text{vec}(\text{var}(\sqrt{T} f_T(\theta, X))) 
= \lim_{T \to \infty} \text{vec} \left( E \left( \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{T} (f_t(\theta) - \mu_f(\theta))(f_j(\theta) - \mu_f(\theta))' \right) \right) 
= \lim_{T \to \infty} E \left( \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{T} \left( (f_t(\theta) - \mu_f(\theta)) \odot (f_j(\theta) - \mu_f(\theta)) \right) \right) 
\]

\[
\frac{\partial}{\partial \theta} \text{vec}(V_{ff}(\theta)) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{T} \left( (f_t(\theta) - \mu_f(\theta)) \odot (f_j(\theta) - \mu_f(\theta)) \right) 
= \lim_{T \to \infty} E \left( \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{T} \left( (f_t(\theta) - \mu_f(\theta)) \odot (f_j(\theta) - \mu_f(\theta)) \right) \right) 
\]

\[
= (\text{vec}(V_{0,f}(\theta)) \ldots \text{vec}(V_{0,m_f}(\theta))) + (\text{vec}(V_{0,f}(\theta))' \ldots \text{vec}(V_{0,m_f}(\theta)')) 
\]
with \( q_j(\theta) = \frac{\partial}{\partial \theta} f_j(\theta) = (q_{1,j}(\theta) \ldots q_{m,j}(\theta)) \) and

\[
V_{0,f}(\theta) = \lim_{T \to \infty} E \left( T \left( \frac{\partial}{\partial \theta} f_T(\theta, X) - \mu_f(\theta) \right) (f_T(\theta, X) - \mu_f(\theta))^\prime \right), \quad i = 1, \ldots, m.
\]

We can now specify the derivative of the objective function with respect to \( \theta \):

\[
\frac{1}{2} \frac{\partial}{\partial \theta} Q_p(\theta) = \mu_f(\theta)' V_{ff}(\theta)^{-1} \frac{\partial \mu_f(\theta)}{\partial \theta} - \frac{1}{2} \left( (\mu_f(\theta) \otimes \mu_f(\theta))' (V_{ff}(\theta)^{-1} \otimes V_{ff}(\theta)^{-1}) \frac{\partial}{\partial \theta} \text{vec}(V_{ff}(\theta)) \right)
\]

\[
= \mu_f(\theta)' V_{ff}(\theta)^{-1} J(\theta) - \frac{1}{2} ((\mu_f(\theta) \otimes \mu_f(\theta))' (V_{ff}(\theta)^{-1} \otimes V_{ff}(\theta)^{-1})
\]

\[
\quad \times \text{vec}(V_{0_f}(\theta)) \ldots \text{vec}(V_{0_m,f}(\theta)) + \text{vec}(V_{0_f}(\theta)^\prime) \ldots \text{vec}(V_{0_m,f}(\theta)^\prime)) \]

\[
= \mu_f(\theta)' V_{ff}(\theta)^{-1} [J(\theta) - (V_{01,f}(\theta) V_{ff}(\theta)^{-1} \mu_f(\theta) \ldots V_{0m,f}(\theta) V_{ff}(\theta)^{-1} \mu_f(\theta))]
\]

\[
= \mu_f(\theta)' V_{ff}(\theta)^{-1} D(\theta),
\]

with \( D(\theta) = J(\theta) - [V_{01,f}(\theta) V_{ff}(\theta)^{-1} \mu_f(\theta) \ldots V_{0m,f}(\theta) V_{ff}(\theta)^{-1} \mu_f(\theta)] \).

### 3.3 Proof that the IS identification measure is larger than or equal to the minimal value of the population continuous updating objective function

The minimal value over \((\lambda_F, D)\) of

\[
Q_p(\lambda_F, D) = \left[ \text{vec} \left( \left( \mu_F : \beta \right) + D \left( \lambda_F : I_m \right) \right)^\prime \right]' \left[ \text{Var} \left( \sqrt{T} \left( \hat{R}' : \text{vec}(\hat{\beta})^\prime \right) \right) \right]^{-1}
\]

equals the minimal value of

\[
Q_p(\phi, A) = \left[ \text{vec} \left( \left( \mu_R : \beta \right) + A(I_m : \phi) \right)^\prime \right]' \left[ \text{Var} \left( \sqrt{T} \left( \hat{R}' : \text{vec}(\hat{\beta})^\prime \right) \right) \right]^{-1}
\]

over \((A, \phi)\), with \( A \) an \( N \times m \) matrix and \( \phi \) an \( m \)-dimensional vector. This results since \( D(\lambda_F : I_m) \) and \( A(I_m : \phi) \) are equivalent representations of an \( N \times (m + 1) \) dimensional matrix of rank \( m \) (except for a measure zero space). Restricting the top element of \( \phi = (\phi_1 : \phi_2)' \), with \( \phi_1 \) a scalar and \( \phi_2 \) an \( (m - 1) \)-dimensional vector, to zero, so \( \phi_1 = 0 \), does not decrease the minimal value of the above function. The resulting restricted
specification reads

\[
Q_p(\phi_2, A) = \left[ \text{vec} \left( \left( \mu_R : \beta \right) + A \left( I_m : \phi_2 \right) \right) \right]' \\
\left( \begin{array}{cc}
-\Sigma^{-1}_{\beta\beta} & -\Sigma^{-1}_{\beta\beta} \\
-\Sigma^{-1}_{\beta\beta} & -\Sigma^{-1}_{\beta\beta}
\end{array} \right)
\left( \begin{array}{c}
-\Sigma^{-1}_{R,R,\beta} \\
-\Sigma^{-1}_{R,R,\beta}
\end{array} \right)
\]

where we used that \( A = (a_1 : A_2), a_1 \) an \( N \)-dimensional vector, \( A_2 \) an \( N \times (m - 1) \) dimensional matrix and the partitioned inverse of \( \text{Var} \left( \sqrt{T} \left( \bar{R}' : \text{vec}(\bar{\beta})' \right) \right) \):

\[
\text{Var} \left( \sqrt{T} \left( \bar{R}' : \text{vec}(\bar{\beta})' \right) \right) = \left( \begin{array}{cc}
\Sigma_{\bar{R}\bar{R}} & \Sigma_{\bar{R}\beta} \\
\Sigma_{\bar{R}\beta} & \Sigma_{\beta\beta}
\end{array} \right),
\]

\[
\left[ \text{Var} \left( \sqrt{T} \left( \bar{R}' : \text{vec}(\bar{\beta})' \right) \right) \right]^{-1} = \left( \begin{array}{cc}
\Sigma^{-1}_{R,R,\beta} & -\Sigma^{-1}_{R,R,\beta} \\
-\Sigma^{-1}_{\beta\beta} & \Sigma^{-1}_{\beta\beta}
\end{array} \right)
\left( \begin{array}{cc}
-\Sigma^{-1}_{R,R,\beta} & \Sigma^{-1}_{R,R,\beta} \\
-\Sigma^{-1}_{\beta\beta} & \Sigma^{-1}_{\beta\beta}
\end{array} \right)
\]

with \( \Sigma_{\bar{R}\bar{R}} = \Sigma'_{\bar{R}}, \Sigma_{\bar{R}\beta} = \Sigma'_{\bar{R} \beta} \) and \( \Sigma_{\beta\beta} N \times N, N \times Nm \) and \( Nm \times Nm \) dimensional matrices respectively, and \( \Sigma_{R,R,\beta} = \Sigma^{-1}_{R,R,\beta} - \Sigma^{-1}_{R,R,\beta} \Sigma^{-1}_{\beta\beta} \).

Stepwise minimization of \( Q_p(\phi_2, A) = Q_p(\phi_2, a_1, A_2) \) now results in

\[
\hat{a}_1(\phi_2, A_2) = \arg\min_{a_1 \in \mathbb{R}} Q_p(\phi_2, a_1, A_2)
\]

so

\[
Q_p(\phi_2, A_2) = \min_{a_1 \in \mathbb{R}} Q_p(\phi_2, a_1, A_2)
\]

\[
= \left[ \text{vec} \left( \beta + A_2 \left( I_{m-1} : \phi_2 \right) \right) \right]' \Sigma^{-1}_{\beta\beta} \left[ \text{vec} \left( \beta + A_2 \left( I_{m-1} : \phi_2 \right) \right) \right],
\]

whose minimal value over \( (\phi_2, A_2) \) corresponds with the IS identification measure so it is always larger than or equal to the minimal value of the population continuous updating objective function.
3.4 Proof of Proposition 1

We pre and post-multiply the matrices in the characteristic polynomial:

\[
\left| \tau \begin{pmatrix} 1 & 0 \\ 0 & Q_{FF}^{-1} \end{pmatrix} - \left( \mu_R : \beta \right) \right| \Omega^{-1} \left( \mu_R : \beta \right) = 0,
\]

by

\[
\begin{pmatrix} 1 & 0 \\ -\lambda_F & I_k \end{pmatrix},
\]

which since the determinant of this matrix equals one does not alter the roots:

\[
\left| \tau \begin{pmatrix} 1 + \lambda_F Q_{FF}^{-1} & -\lambda_F Q_{FF}^{-1} \\ -Q_{FF}^{-1} & Q_{FF}^{-1} \end{pmatrix} - \left( \mu_R - \beta \lambda_F : \beta \right) \right| \Omega^{-1} \left( \mu_R - \beta \lambda_F : \beta \right) = 0.
\]

We next do so again using:

\[
\begin{pmatrix} 1 & \lambda_F Q_{FF}^{-1}(1 + \lambda_F Q_{FF}^{-1} \lambda_F)^{-1} \\ 0 & I_k \end{pmatrix},
\]

to obtain:

\[
\left| \tau \begin{pmatrix} 1 + \lambda_F Q_{FF}^{-1} \lambda_F & 0 \\ 0 & Q_{FF}^{-1} - Q_{FF}^{-1} \lambda_F(1 + \lambda_F Q_{FF}^{-1} \lambda_F)^{-1} \lambda_F Q_{FF}^{-1} \end{pmatrix} - \left( \mu_R - \beta \lambda_F : -D(\lambda_F) \right) \right| \Omega^{-1} \left( \mu_R - \beta \lambda_F : -D(\lambda_F) \right) = 0.
\]

with \(D(\lambda_F) = -\beta - (\mu_R - \beta \lambda_F) \lambda_F Q_{FF}^{-1}(1 + \lambda_F Q_{FF}^{-1} \lambda_F)^{-1}\). For a value of \(\lambda_F, \lambda_F^*\), which satisfies the FOC, so \((\mu_R - \beta \lambda_F^*) \Omega^{-1} D(\lambda_F^*) = 0\), the characteristic polynomial then becomes:

\[
\left| \begin{pmatrix} \tau(1 + \lambda_F^* Q_{FF}^{-1} \lambda_F^*) - (\mu_R - \beta \lambda_F^*) \right| \Omega^{-1} (\mu_R - \beta \lambda_F^*) \right) \\
0 \\
\tau(Q_{FF}^{-1} - Q_{FF}^{-1} \lambda_F(1 + \lambda_F^* Q_{FF}^{-1} \lambda_F^*)^{-1} \lambda_F^* Q_{FF}^{-1} - D(\lambda_F^*) \Omega^{-1} D(\lambda_F^*) \right) \right| = 0.
\]

We can further use that \(Q_{FF}^{-1} - Q_{FF}^{-1} \lambda_F(1 + \lambda_F^* Q_{FF}^{-1} \lambda_F^*)^{-1} \lambda_F^* Q_{FF}^{-1} = (Q_{FF} + \lambda_F^* \lambda_F)^{-1}\).
3.5 Proof of Theorem 3

The joint limit behavior of \(f_T(\theta, X)\) and \(q_T(\theta, X)\) at the pseudo-true value \(\theta^*\) reads:

\[
\sqrt{T} \left( \begin{array}{c} f_T(\theta^*, X) - \mu_f(\theta^*) \\ \text{vec}(q_T(\theta^*, X) - J(\theta^*)) \end{array} \right) \xrightarrow{d} \left( \begin{array}{c} \psi_f(\theta) \\ \psi_0(\theta) \end{array} \right).
\]

We pre-multiply it by

\[
\hat{R}(\theta^*) = \left( \begin{array}{cc} I_{k_f} & 0 \\ -\hat{V}_f(\theta^*)\hat{V}_{ff}(\theta^*)^{-1} & I_{k_{jm}} \end{array} \right) \xrightarrow{p} \left( \begin{array}{cc} I_{k_f} & 0 \\ -V_{\theta f}(\theta^*)V_{ff}(\theta^*)^{-1} & I_{k_{jm}} \end{array} \right) = R(\theta^*),
\]

to obtain

\[
\sqrt{T} \left[ \hat{R}(\theta^*) \left( \begin{array}{c} f_T(\theta^*, X) \\ \text{vec}(q_T(\theta^*, X)) \end{array} \right) - R(\theta^*) \left( \begin{array}{c} \mu_f(\theta^*) \\ \text{vec}(J(\theta^*)) \end{array} \right) \right] \xrightarrow{d} R(\theta^*) \left( \begin{array}{c} \psi_f(\theta^*) \\ \psi_0(\theta^*) \end{array} \right) \Leftrightarrow \sqrt{T} \left( \begin{array}{cc} f_T(\theta^*, X) - \mu_f(\theta^*) \\ \text{vec}(\hat{D}(\theta^*) - D(\theta^*)) \end{array} \right) \xrightarrow{d} \left( \begin{array}{c} \psi_f(\theta^*) \\ \psi_\theta(\theta^*) \end{array} \right),
\]

with \(\psi_{\theta, f}(\theta^*) = \psi_0(\theta^*) - V_{\theta f}(\theta^*)V_{ff}(\theta^*)^{-1}\psi_f(\theta^*)\) which is independent of \(\psi_f(\theta^*)\) since

\[
R(\theta^*)V(\theta^*)R(\theta^*)' = \left( \begin{array}{cc} V_{ff}(\theta^*) & 0 \\ 0 & V_{\theta \theta, f}(\theta^*) \end{array} \right),
\]

where \(V_{\theta \theta, f}(\theta^*) = V_{\theta \theta}(\theta^*) - V_{\theta f}(\theta^*)V_{ff}(\theta^*)^{-1}V_{\theta f}(\theta^*)'\), so \(\psi_f(\theta^*)\) and \(\psi_{\theta, f}(\theta^*)\) are uncorrelated and independent since they are normal distributed random variables.

3.6 Proof of Theorem 4

The joint limit behaviors of \(f_T(\theta^*, X), \hat{D}(\theta^*)\) and \(\hat{V}_{ff}(\theta^*)\) are such that:

\[
T_s(\theta^*) = \left( \sqrt{T}f_T(\theta^*, X) \right)' \hat{V}_{ff}(\theta^*)^{-1} \left( \sqrt{T}\hat{D}(\theta^*) \right) \xrightarrow{d} [\bar{\mu}_f(\theta^*) + \psi_f(\theta^*)]' V_{ff}(\theta^*)^{-1} [\hat{D}(\theta^*) + \Psi_{\theta, f}(\theta^*)]
\]

\[
= \bar{\mu}_f(\theta^*)' V_{ff}(\theta^*)^{-1} \Psi_{\theta, f}(\theta^*) + \psi_f(\theta^*)' V_{ff}(\theta^*)^{-1} [\hat{D}(\theta^*) + \Psi_{\theta, f}(\theta^*)]
\]

\[
= (\bar{\mu}_f(\theta^*) + \psi_f(\theta^*))' V_{ff}(\theta^*)^{-1} \Psi_{\theta, f}(\theta^*) + \psi_f(\theta^*)' V_{ff}(\theta^*)^{-1} \hat{D}(\theta^*),
\]

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where $\text{vec}(\Psi_{\theta,f}(\theta^*)) = \psi_{\theta,f}$, since $\bar{\mu}_f(\theta^*)'V_{ff}(\theta^*)^{-1}\bar{D}(\theta^*) = 0$. Since $\psi_f(\theta^*)$ and $\psi_{\theta,f}(\theta^*)$ are independently distributed, this shows that the expected value of the limit of the score of the CUE sample objective function equals zero at the pseudo-true value $\theta^*$.

### 3.7 Proof of Theorem 5

We can specify the limit behavior of $Ts(\theta^*)$ as:

$$Ts(\theta^*)' \xrightarrow{d} a + b + c,$$

with $a = \Psi_{\theta,f}(\theta^*)'V_{ff}(\theta^*)^{-1}\bar{\mu}_f(\theta^*)$, $b = \bar{D}(\theta^*)'V_{ff}(\theta^*)^{-1}\psi_f(\theta^*)$ and $c = \Psi_{\theta,f}(\theta^*)'V_{ff}(\theta^*)^{-1}\psi_{\theta,f}(\theta^*)$. To obtain the bound on the limiting distribution of the DRLM statistic, we next further characterize the limit behavior of the above components. We first do so for $m = 1$.

**m=1:** We specify $a$, $b$ and $c$ as:

- $a = \Psi_{\theta,f}'G'\mu^*$,
- $b = D^*G'\psi_f^*$,
- $c = \Psi_{\theta,f}'G'\psi_{f}^*$,

which results from a singular value decomposition of $V_{ff}(\theta^*)^{-\frac{1}{2}}V_{\theta\theta,f}(\theta^*)^{\frac{1}{2}}$:

$$V_{ff}(\theta^*)^{-\frac{1}{2}}V_{\theta\theta,f}(\theta^*)^{\frac{1}{2}} = LGK',$$

with $L$ and $K$ $k \times k$ dimensional orthonormal matrices and $G$ a diagonal $k \times k$ dimensional matrix with the non-negative singular values in decreasing order on the main diagonal and we used that $\mu^* = L'V_{ff}(\theta^*)^{-\frac{1}{2}}\bar{\mu}_f(\theta^*)$, $D^* = K'V_{\theta\theta,f}(\theta^*)^{-\frac{1}{2}}\bar{D}(\theta^*)$, $\psi_f^* = L'V_{ff}(\theta^*)^{-\frac{1}{2}}\psi_f(\theta^*) \sim N(0,I_k)$, $\Psi_{\theta,f}^* = K'V_{\theta\theta,f}(\theta^*)^{-\frac{1}{2}}\Psi_{\theta,f}(\theta^*) \sim N(0,I_k)$ and independent of $\psi_f^*$.

Using the above, the limit behavior of the DRLM statistic can be specified as:

$$DRLM(\theta^*) \xrightarrow{d} \frac{\Psi_{\theta,f}'G'\mu^* + D^*G'\psi_f^* + \Psi_{\theta,f}'G'\psi_{f}^*}{\left[\left(\mu^* + \psi_f^*\right)'GG'\left(\mu^* + \psi_f^*\right) + \left(D^* + \Psi_{\theta,f}^*\right)'G'G\left(D^* + \Psi_{\theta,f}^*\right)\right]^{-1}},$$

$$= \frac{\Psi_{\theta,f}'G'\mu^* + D^*G'\psi_f^* + \Psi_{\theta,f}'G'\psi_{f}^*}{(\mu^* + \psi_f^*)G^2(\mu^* + \psi_f^*) + (D^* + \Psi_{\theta,f}^*)G^2(D^* + \Psi_{\theta,f}^*)}.$$
The largest element of $G, G_{11}$, can be set to one. This implies that $G$ contains $k - 1$ non-negative elements which are not preset to 0 or 1. The number of elements in both $\mu^*$ and $D^*$ equals $k$.

When $\mu^*$ and $D^*$ equal zero, the limit behavior of $DRLM(\theta^*)$ becomes:

\[
DRLM(\theta^*)|_{\mu^*=D^*=0} \xrightarrow{d} \frac{[\psi_{*,f}^* G' \psi_{*,f}^*]^2}{[\psi_{*,f}^* G' \psi_{*,f}^* + \Psi_{*,f}^* G' \Psi_{*,f}^*]} \leq \chi^2(1),
\]

since both $[\psi_{*,f}^* G' \psi_{*,f}^*]^2 \sim \chi^2(1)$ and $[\psi_{*,f}^* G' \psi_{*,f}^*]^2 \sim \chi^2(1)$ and where “$\leq$” indicates stochastically dominated so for a continuous non-negative scalar random variable $u \leq \chi^2(m)$: $\Pr[u > cv \chi^2(m)(\alpha)] \leq \alpha$, for $\alpha \in (0, 1]$ and with $cv \chi^2(m)(\alpha)$ the $(1 - \alpha) \times 100\%$ critical value for the $\chi^2(m)$ distribution.

Similarly, when the length of $\mu^*$ and/or $D^*$ goes to infinity:

\[
\left\{ \begin{array}{l}
\lim_{\mu^* \rightarrow \infty, \mu^* = \ldots} DRLM(\theta^*) \\
\lim_{D^* \rightarrow \infty, D^* = \ldots} DRLM(\theta^*) \\
\lim_{\mu^* \rightarrow \infty, D^* = \ldots} DRLM(\theta^*)
\end{array} \right\} \xrightarrow{d} \chi^2(1).
\]

The limit behavior is identical with respect to the different elements of $\mu^*$ and $D^*$. Figure A7 shows for a pre-specified fixed value of $G$ that the distribution function associated with the limit behavior of $DRLM(\theta^*)$ is a non-increasing function of either the length of $\mu^*$ or $D^*$. Figure A7 also shows the difference with the $\chi^2(1)$ distribution function which makes it clear that the $\chi^2(1)$ distribution dominates the limiting distribution of the DRLM statistic for this specific value of $G$. Since $G$ is a diagonal matrix with only non-negative elements, this behavior holds also for all other values of $G$ so the limit behavior of $DRLM(\theta^*)$ is bounded by the $\chi^2(1)$ distribution:

\[
\lim_{T \rightarrow \infty} \Pr[DRLM(\theta^*) > cv \chi^2(1)(\alpha)] \leq \alpha.
\]

$m > 1$: We specify $a, b$ and $c$ as:

\[
a = \Psi_{\theta,f}^* \mu^*, \quad b = D^* \psi_f^*, \quad c = \Psi_{\theta,f}^* \psi_f^*,
\]

with $G = (I_m \otimes V_{f \theta}(\theta)^{-\frac{1}{2}}) V_{\theta \theta,f}(\theta)^{\frac{1}{2}}, \Psi_{\theta,f} = V_{f \theta}(\theta)^{-\frac{1}{2}} \Psi_{\theta,f}(\theta^*), \text{vec}(\Psi_{\theta,f}) = G \psi_{\theta,f}, \psi_{\theta,f} \sim N(0, I_m), \psi_f^* = V_{f \theta}(\theta^*)^{-\frac{1}{2}} \psi_f(\theta^*) \sim N(0, I_k)$ and independent of $\psi_{\theta,f}, \mu^* = V_{f \theta}(\theta)^{-\frac{1}{2}} \tilde{\mu_f}(\theta^*), D^* = V_{f \theta}(\theta^*)^{-\frac{1}{2}} \tilde{D}(\theta^*), \text{vec}(D^*) = G \text{vec}(\tilde{D}^*)$ and $\tilde{D}^*$ is a $k \times m$ dimensional matrix.
Figure A7. Distribution function of the DRLM statistic for a fixed value of $G$ as a function of the length of either $\mu^*$ or $D^*$. 
Using the above, the limit behavior of the DRLM statistic can be specified as:

\[
DRLM(\theta^*) \xrightarrow{d} \left[ \Psi_{\theta,f}^* \mu^* + D^* \psi_f^* + \Psi_{\theta,f}^* \Psi_f^* \right]' \\
\left[ (I_m \otimes (\mu^* + \psi_f^*))' GG' \left( I_m \otimes (\mu^* + \psi_f^*) \right) + \left( D^* + \Psi_{\theta,f}^* \right)' \left( D^* + \Psi_{\theta,f}^* \right) \right]^{-1} \\
\Psi_{\theta,f}^* \mu^* + D^* \psi_f^* + \Psi_{\theta,f}^* \Psi_f^* .
\]

The limiting distribution of the DRLM statistic depends on the \(k^2m^2 + km + k + 1\) parameters present in: \(G, D^*, \mu^*, k\) and \(m\). Since the limiting distribution is invariant to multiplying \(G\) by a positive scalar, we normalize \(G\) such that one diagonal element of \(G\), say \(G_{11}\), is equal to one. This explains the number of parameters affecting the limiting distribution of the DRLM statistic.

When \(\mu^*\) and \(D^*\) equal zero, the limit behavior of \(DRLM(\theta^*)\) becomes:

\[
DRLM(\theta^*)|_{\mu^*=D^*=0} \xrightarrow{d} \psi_f^* \Psi_{\theta,f}^* \left[ (I_m \otimes \psi_f^*)' GG' \left( I_m \otimes \psi_f^* \right) + \Psi_{\theta,f}^* \Psi_{\theta,f}^* \right]^{-1} \Psi_{\theta,f}^* \psi_f^* \sim \chi^2(m),
\]
since \(\psi_f^* \Psi_{\theta,f}^* \left[ (I_m \otimes \psi_f^*)' GG' \left( I_m \otimes \psi_f^* \right) + \Psi_{\theta,f}^* \Psi_{\theta,f}^* \right]^{-1} \Psi_{\theta,f}^* \psi_f^* \sim \chi^2(m).

Similarly, when using a singular value decomposition of \(D^*\):

\[
D^* = L_D G_D K_D',
\]

with \(L_D\) and \(K_D\) \(k \times k\) and \(m \times m\) dimensional orthonormal matrices and \(G_D\) a diagonal \(k \times m\) dimensional matrix with the non-negative singular values in decreasing order on the main diagonal, we can specify the limiting behavior of the DRLM statistic:

\[
DRLM(\theta^*) \xrightarrow{d} \left[ K_D \tilde{\Psi}_{\theta,f} L_D' \mu^* + K_D G_D' L_D' \psi_f^* + K_D \tilde{\Psi}_{\theta,f} L_D' \psi_f^* \right]' \\
\left[ \left( I_m \otimes (\mu^* + \psi_f^*) \right)' GG' \left( I_m \otimes (\mu^* + \psi_f^*) \right) \right]^{-1} \\
\left[ K_D \tilde{\Psi}_{\theta,f} L_D' \mu^* + K_D G_D' L_D' \psi_f^* + K_D \tilde{\Psi}_{\theta,f} L_D' \psi_f^* \right] .
\]

where \(\tilde{\Psi}_{\theta,f} = L_D \tilde{\Psi}_{\theta,f} K_D', \tilde{\psi}_f = L_D' \psi_f^*, \tilde{\mu} = L_D' \mu^*, \tilde{G} = (K_m \otimes L_D V_{ff}(\theta^*)^{-\frac{1}{2}}) V_{\theta,f}(\theta^*)^{\frac{1}{2}}, vec(\tilde{\Psi}_{\theta,f}) = \tilde{G}_\theta^* f, \psi_f^* \sim N(0,1).\) The resulting limiting behavior is such that when the length of \(\mu^*\) or the \(m\) singular values
in $G_D$ go to infinity:

$$\lim_{\mu^*, \mu \to \infty} DRLM(\theta^*) \quad \begin{cases} 
\lim_{G_{D,ii} \to \infty, \ i=1,\ldots,m} DRLM(\theta^*) \to \chi^2(m). 
\end{cases}$$

Since $G$ is positive semi-definite, it can be verified numerically that for any fixed $G$, the distribution function associated with the limit behavior of $DRLM(\theta^*)$ is non-increasing when any element of $\mu^*$ or $G_{D,ii}^* \to \infty$, $i = 1, \ldots, m$ increases. The limit behavior of $DRLM(\theta^*)$ is therefore bounded by the $\chi^2(m)$ distribution:

$$\lim_{T \to \infty} \Pr [DRLM(\theta^*) > cv\chi^2(m)(\alpha)] \leq \alpha.$$
behavior as:

$$DRLM(\theta^*) \rightarrow_d \left[ \Psi_{0,f}' \tilde{\mu} + G_D' \tilde{\psi}_f + \Psi_{0,f}' \tilde{\psi}_f \right]'$$

$$\left( (\mu + \tilde{\psi}_f)' (\mu + \tilde{\psi}_f) I_m + (G_D' + \Psi_{0,f})' (G_D' + \Psi_{0,f}) \right)^{-1} \left[ \Psi_{0,f}' \tilde{\mu} + G_D' \tilde{\psi}_f + \Psi_{0,f}' \tilde{\psi}_f \right],$$

which only depends on the $m$ singular values in $G_D$ and the length of $\mu$. The distribution function of the limit behavior is again a non-decreasing function of the length of $\mu$ and the $m$ singular values in $G_D$ so its limit behavior is bounded by the $\chi^2(m)$ distribution.

**Definition of the parameter space** In Andrews and Guggenberger (2017), the asymptotic size of the KLM test is proven to equal the nominal size and the accompanying parameter space on the distributions of the observations is stated for both i.i.d. and dependent data settings.

To start out with the i.i.d. setting, define for some $\kappa, \tau > 0$ and $M < \infty$, the parameter space:

$$\mathcal{F} = \{ F : \{ X_t : t \geq 1 \} \text{ are i.i.d. under } F, \ E(f_t(\theta^*)) = \mu_f(\theta^*) \},$$

$$\theta^* = \arg \min_{\theta \in \mathcal{F}} \mu_f(\theta)' V_{ff}(\theta)^{-1} \mu_f(\theta), \quad V_{ff}(\theta) = E \left( (f_t(\theta) - \mu_f(\theta)) (f_t(\theta) - \mu_f(\theta))' \right),$$

$$E \left( \left\| \begin{pmatrix} f_t(\theta^*)' \end{pmatrix} \left( \text{vec} \left( \frac{\partial}{\partial \theta} f_t(\theta^*) \right) \right)^{2+\kappa} \right\| \right) \leq M \quad \text{and} \quad \lambda_{\min}(V_{ff}(\theta^*)) \geq \tau,$$

where $\lambda_{\min}(A)$ is the smallest characteristic root of the matrix $A$. The parameter space above is identical to the one in Andrews and Guggenberger (2017) Equation (3.3) except that it is defined for the pseudo-true value $\theta^*$ defined as the minimizer of the population continuous updating objective function for which $\mu_f(\theta^*)$ is not necessarily equal to zero.

Since we are after proving the size correctness of the DRLM test which tests hypotheses specified on the pseudo-true value $\theta^*$, we define the recentered Jacobian:

$$D(\theta) = J(\theta) - \left[ V_{0,f}(\theta) V_{ff}(\theta)^{-1} \mu_f(\theta) \ldots V_{0,n,f}(\theta) V_{ff}(\theta)^{-1} \mu_f(\theta) \right], \quad J(\theta) = \frac{\partial}{\partial \theta} \mu_f(\theta),$$

$$V_{0,i,f}(\theta) = E \left[ \left( \frac{\partial}{\partial \theta} (f_i(\theta) - \mu_f(\theta)) \right) (f_i(\theta) - \mu_f(\theta))' \right], \quad i = 1, \ldots, m,$$

$$V_{ff}(\theta) = E \left( (f_t(\theta) - \mu_f(\theta)) (f_t(\theta) - \mu_f(\theta))' \right).$$

The pseudo-true value is then such that

$$\mu_f(\theta^*)' V_{ff}(\theta^*)^{-1} D(\theta^*) = 0.$$

To guarantee with probability one, a non-singular value of the limit value of the sample analog of $V_{ff}(\theta^*)^{-1} D(\theta^*)$, 


\[ \hat{V}_{ff}(\theta^* \theta^*)^{-1} \hat{D}(\theta^*) \], Andrews and Guggenberger (2017) provide a number of additional conditions on the parameter space \( \mathcal{F} \). Since we allow for misspecification, these conditions have to hold when using the recentered Jacobian \( D(\theta) \) instead of the Jacobian \( J(\theta) \) as in Andrews and Guggenberger (2017). Taken together these conditions imply that the singular values of \( V_{ff}(\theta^*)^{-1} D(\theta^*) \) should be bounded away from zero and the same applies for the quadratic form of the orthonormal vectors resulting from the singular value decomposition of \( V_{ff}(\theta^*)^{-1} D(\theta^*) \) with respect to the covariance matrix of \( \text{vec}(\hat{D}(\theta^*)) \). We refer to Andrews and Guggenberger (2017) for the definition of this reduced parameter space.

The parameter spaces in Andrews and Guggenberger (2017) imply Lemma 10.2 in their Supplementary Appendix which coincides with our Theorem 3 except that Theorem 3 allows for a population mean function \( \mu_f(\theta^*) \) different from zero. Jointly with some weak laws of large numbers, the limiting distributions resulting from Lemma 10.2 in the Supplementary Appendix of Andrews and Guggenberger (2017) provide the building blocks for their Theorem 11.1 which states that the asymptotic size of the KLM test equals the nominal size. Since the parameter spaces also imply our Theorem 3 whose resulting limiting distributions alongside some weak laws of large numbers imply Theorems 4 and 5, which states that the limiting distribution of the DRLM statistics is bounded by a \( \chi^2(m) \) distribution, the parameter spaces thus also imply that the asymptotic size of the DRLM test equals the nominal size.

For the dependent times-series setting, \( \kappa, \tau > 0, \ d > (2 + \kappa)/\kappa \) and \( M < \infty \), the space of distributions is defined by:

\[
\mathcal{F}_{ts} = \{ F : \{ X_t : t = 0, 1, \ldots \} \text{ are stationary and strong mixing under } F \text{ with strong mixing numbers } \{ \alpha_F(p) : p \geq 1 \} \text{ that satisfy } \alpha_F(p) \leq C p^{-d}, \ E(f_t(\theta^*)) = \mu_f(\theta^*), \ \theta^* = \arg \min_{\theta \in \mathbb{R}^p} \mu_f(\theta) V_{ff}(\theta)^{-1} \mu_f(\theta), \ V_{ff}(\theta) = E \left[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{T} (f_t(\theta) - \mu_f(\theta)) (f_j(\theta) - \mu_f(\theta))' \right], \ E \left( \left\| \left( f_t(\theta) : \text{vec} \left( \frac{\partial}{\partial \theta} f_t(\theta) \right) \right)' \right\|_2^{2+\kappa} \right) \leq M \text{ and } \lambda_{\min}(V_{ff}(\theta^*)) \geq \tau \}
\]

which again, except for the usage of the pseudo-true value \( \theta^* \) and a possibly non-zero mean of \( f_t(\theta^*) \), is identical to Equation (7.2) in Andrews and Guggenberger (2017). Identical to the i.i.d. setting, Andrews and Guggenberger (2017) provide a number of additional conditions on the parameter space \( \mathcal{F}_{ts} \), to guarantee with probability one, a non-singular value of the limit value of the sample analog of \( V_{ff}(\theta^*)^{-1} D(\theta^*) \), \( \hat{V}_{ff}(\theta^*)^{-1} \hat{D}(\theta^*) \). Replacing the value of the Jacobian, \( J(\theta) \), by the recentered Jacobian, \( D(\theta) \), in the conditions from Andrews and Guggenberger (2017) then implies that also for our setting the limit value of the \( \hat{V}_{ff}(\theta^*)^{-1} \hat{D}(\theta^*) \) is non-singular with probability one. The resulting parameter space then again implies our Theorem 3 from which Theorem 5 follows so the asymptotic size of the DRLM test coincides with the
3.8 Proof of Theorem 6

a. Starting out from a linear moment equation, like, for example, the one for the linear asset pricing model, $f_T(\lambda_F, X) = \bar{R} - \hat{\beta}\lambda_F$, which is WLOG:

$$d = \left( \begin{array}{c} \bar{R} \\ \text{vec}(\hat{\beta}) \end{array} \right)^T \sqrt{T} \left( \begin{array}{c} \bar{R} \\ \text{vec}(\hat{\beta}) \end{array} \right)^{-1} \left( \begin{array}{c} \bar{R} \\ \text{vec}(\hat{\beta}) \end{array} \right) = \left( R - \hat{\beta}\lambda_F \right)^T \sqrt{T} \left( R - \hat{\beta}\lambda_F \right)^{-1} \left( R - \hat{\beta}\lambda_F \right)$$

which shows that, given a realized data set and since $d$ does not depend on $\lambda_F$, the sum of $f_T(\lambda_F, X)\hat{V}_{ff}(\lambda_F)^{-1}f_T(\lambda_F, X)$ and $\left( \text{vec}(\hat{D}(\lambda_F)) \right)^T \hat{V}_{\theta_f}(\lambda_F)^{-1} \left( \text{vec}(\hat{D}(\lambda_F)) \right)$ does not depend on $\lambda_F$.

b. Given the specifications of the derivatives in Lemma 2, the derivative of DRLM($\theta$) when $m = 1$ and $f_T(\theta, X)$ is linear in $\theta$ reads:

$$\frac{1}{2} \frac{\partial}{\partial \theta} \text{DRLM}(\theta)$$

$$= -\frac{1}{2} T \frac{\partial}{\partial \theta} \left\{ f_T(\theta, X)^T \hat{V}_{ff}(\theta)^{-1} \hat{D}(\theta) \left[ f_T(\theta, X)^T \hat{V}_{ff}(\theta)^{-1} \hat{V}_{\theta_f}(\theta) \hat{V}_{ff}(\theta)^{-1} f_T(\theta, X) \right] + \hat{D}(\theta)^T \hat{V}_{ff}(\theta)^{-1} \hat{D}(\theta) \right\}$$

$$= T \left[ f_T(\theta, X)^T \hat{V}_{ff}(\theta)^{-1} \hat{V}_{\theta_f}(\theta) \hat{V}_{ff}(\theta)^{-1} f_T(\theta, X) + \hat{D}(\theta)^T \hat{V}_{ff}(\theta)^{-1} \hat{D}(\theta) \right]^{-1} f_T(\theta, X)^T \hat{V}_{ff}(\theta)^{-1} \hat{D}(\theta)$$

which is WLOG:  

$$\left( \frac{\partial}{\partial \theta} \text{DRLM}(\theta) \right)^T \hat{V}_{\theta_f}(\lambda_F)^{-1} \left( \frac{\partial}{\partial \theta} \text{DRLM}(\theta) \right)$$

By the delta method, $DRLM(\theta)$ is normally distributed as $N(0, \hat{V}_{\theta_f}(\lambda_F)^{-1})$. 

By case of i.i.d. data, $m = 1$, and $f_T(\theta, X)$ linear in $\theta$, $\hat{V}(\theta)$ has a Kronecker product structure so $\hat{V}_{ff}(\theta) = \hat{v}_{ff}(\theta)\hat{V}, \hat{V}_{\theta_f}(\theta) = \hat{v}_{\theta_f}(\theta)\hat{V}$ and $\hat{V}_{\theta_0}\lambda_f(\theta) = \hat{V}_{\theta_0}\lambda_f(\theta)\hat{V}$ with $\hat{v}_{ff}(\theta), \hat{v}_{\theta_f}(\theta), \hat{v}_{\theta_0}\lambda_f(\theta)$ scalar and $\hat{V}$ a
We first construct the limit behavior of $\hat{\lambda}_F$ and $\mu_f(\lambda_F)$ when the (pseudo-) true value of $\lambda_F$ equals $\lambda_F^*$ so we use that

$$R_t = \mu_R - \beta \lambda_F^* + \beta (\bar{F}_t + \lambda_F^*) + u_t,$$

with $\frac{1}{T} \sum_{t=1}^{T} \bar{F}_t = 0$:

$$-\hat{D}(\lambda_F^*) = \frac{1}{T} \sum_{t=1}^{T} R_t (\bar{F}_t + \lambda_F^*)' \left[ \frac{1}{T} \sum_{t=1}^{T} (\bar{F}_t + \lambda_F^*)(\bar{F}_t + \lambda_F^*)' \right]^{-1}$$

$$= \frac{1}{T} \sum_{t=1}^{T} \left( \mu_R - \beta \lambda_F^* + \beta (\bar{F}_t + \lambda_F^*) + u_t \right) (\bar{F}_t + \lambda_F^*)' \left[ \frac{1}{T} \sum_{t=1}^{T} (\bar{F}_t + \lambda_F^*)(\bar{F}_t + \lambda_F^*)' \right]^{-1}$$

$$= \frac{1}{T} \sum_{t=1}^{T} \left( \mu_R - \beta \lambda_F^* + \beta (\bar{F}_t + \lambda_F^* - \lambda_F^*) + u_t \right) (\bar{F}_t + \lambda_F^*)' \left[ \frac{1}{T} \sum_{t=1}^{T} (\bar{F}_t + \lambda_F^*)(\bar{F}_t + \lambda_F^*)' \right]^{-1}$$

$$= \beta + \frac{1}{T} \sum_{t=1}^{T} \left( \mu_R - \beta \lambda_F^* + u_t \right) (\bar{F}_t + \lambda_F^*)' \left[ \frac{1}{T} \sum_{t=1}^{T} (\bar{F}_t + \lambda_F^*)(\bar{F}_t + \lambda_F^*)' \right]^{-1}$$

so

$$\sqrt{T} \left( \hat{D}(\lambda_F^*) - D(\lambda_F^*) \right) \rightarrow_d \psi_{\theta,f}(\lambda_F^*)$$

$$\sqrt{T} \left( \hat{D}(\lambda_F^*) - D(\lambda_F^*) - (\mu_R - \beta \lambda_F^*) \lambda_F^*(Q_{F\overline{F}} + \lambda_F^* \lambda_F^*)^{-1} + (\mu_R - \beta \lambda_F^*) \lambda_F^*(Q_{F\overline{F}} + \lambda_F^* \lambda_F^*)^{-1} \right) \rightarrow_d \psi_{\theta,f}(\lambda_F^*)$$

3.9 Proof of Theorem 7

We first construct the limit behavior of $\hat{D}(\lambda_F^*)$ and $\mu_f(\lambda_F^*)$ when the (pseudo-) true value of $\lambda_F$ equals $\lambda_F^*$.
with $-D(\lambda_F) = \beta + (\mu_R - \beta \lambda_F) \lambda_F(\lambda_F Q_{F \bar{F}} + \lambda_F \lambda_F')^{-1}$, $\psi_{0,f}(\lambda_F') \sim N(0, (\lambda_F Q_{F \bar{F}} + \lambda_F \lambda_F')^{-1} \otimes \Omega)$, and

$$R - \beta \lambda_F' = \mu_R + \frac{1}{T} \sum_{t=1}^{T} u_t - \beta \lambda_F' + \frac{1}{T} \sum_{t=1}^{T} u_t \bar{T}_t \left( \frac{1}{T} \sum_{t=1}^{T} \bar{T}_t \right) \lambda_F'$$

$$= \mu_R - \beta \lambda_F' + \frac{1}{T} \sum_{t=1}^{T} u_t - \beta \lambda_F' + \frac{1}{T} \sum_{t=1}^{T} u_t \bar{T}_t \left( \frac{1}{T} \sum_{t=1}^{T} \bar{T}_t \right) \lambda_F'$$

$$= (\mu_R - \beta \lambda_F') \left( [1 - \lambda_F'(\lambda_F Q_{F \bar{F}} + \lambda_F \lambda_F')^{-1} \lambda_F'] - D(\lambda_F')(\lambda_F' - \lambda_F') + \frac{1}{T} \sum_{t=1}^{T} u_t - \beta \lambda_F' + \frac{1}{T} \sum_{t=1}^{T} u_t \bar{T}_t \left( \frac{1}{T} \sum_{t=1}^{T} \bar{T}_t \right) \lambda_F' \right)$$

$$= (\mu_R - \beta \lambda_F') \left( [1 - \lambda_F'(\lambda_F Q_{F \bar{F}} + \lambda_F \lambda_F')^{-1} \lambda_F'] - D(\lambda_F')(\lambda_F' - \lambda_F') + \frac{1}{T} \sum_{t=1}^{T} u_t - \beta \lambda_F' + \frac{1}{T} \sum_{t=1}^{T} u_t \bar{T}_t \left( \frac{1}{T} \sum_{t=1}^{T} \bar{T}_t \right) \lambda_F' \right)$$

$$\sqrt{T} \left( (R - \beta \lambda_F') - \mu_f(\lambda_F') (1 + \lambda_F' Q_{F \bar{F}} \lambda_F')^{-1} - D(\lambda_F') (\lambda_F' - \lambda_F') + \mu_f(\lambda_F') \lambda_F'(\lambda_F Q_{F \bar{F}} + \lambda_F \lambda_F')^{-1} \lambda_F' \right) \overset{d}{\rightarrow} \psi_f(\lambda_F')$$

with $\mu_f(\lambda_F') = \mu_R - \beta \lambda_F'$ and $\psi_f(\lambda_F') \sim N(0, (1 + \lambda_F' Q_{F \bar{F}} \lambda_F') \Omega)$ and independent of $\psi_{0,f}(\lambda_F')$.

For testing $H_0 : \lambda_F = 0$, so $\lambda_F' = 0$, the above expressions simplify to:

$$\sqrt{T} \left( \hat{D}(\lambda_F') - \mu_f(\lambda_F') (1 + \lambda_F' Q_{F \bar{F}} \lambda_F')^{-1} - D(\lambda_F') (\lambda_F' - \lambda_F') \right) \overset{d}{\rightarrow} \psi_f(\lambda_F')$$

with $\psi_f(\lambda_F') \sim N(0, (1 + \lambda_F' Q_{F \bar{F}} \lambda_F') \Omega)$ and $\psi_f(\lambda_F') \sim N(0, \Omega)$. We next use that $\mu^* = \lim_{T \rightarrow \infty} \sqrt{T} \mu_f(\lambda_F')$, $D^* = \lim_{T \rightarrow \infty} \sqrt{T} D(\lambda_F')$, $\bar{\mu} = \Omega^{-\frac{1}{2}} \mu^* (1 + \lambda_F^* Q_{F \bar{F}} \lambda_F^*)^{-\frac{1}{2}}$, $\bar{D} = \Omega^{-\frac{1}{2}} D^* (Q_{F \bar{F}} + \lambda_F^* \lambda_F^*)^\frac{1}{2}$ so for $m = 1$, $Q_{F \bar{F}} = 1$:

$$\sqrt{T} \Omega^{-\frac{1}{2}} \bar{R} \overset{d}{\rightarrow} \bar{\mu}(1 + (\lambda_F)^2)^{-\frac{1}{2}} - \bar{D} (1 + (\lambda_F)^2)^{-\frac{1}{2}} \lambda_F + \psi_f(\lambda_F') = 0$$

$$\sqrt{T} \Omega^{-\frac{1}{2}} \bar{D}(\lambda_F') = 0 \overset{d}{\rightarrow} \bar{D} (1 + (\lambda_F')^2)^{-\frac{1}{2}} + \bar{\mu}(1 + (\lambda_F')^2)^{-\frac{1}{2}} \lambda_F^* + \psi^*_{0,f}(\lambda_F') = 0$$

with $\psi^*_f(\lambda_F')$ and $\psi^*_{0,f}(\lambda_F')$ independent standard normal $N$ dimensional random vectors.

### 3.10 Proof of Theorem 8

We first specify: $U_1 S_1 V_1' = -\Omega^{-\frac{1}{2}} D(\lambda_F') \left( \begin{array}{c} \lambda_F' \\ I_m \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{F \bar{F}}^{-\frac{1}{2}} \end{array} \right)$, so $D(\lambda_F') = -\Omega^{-\frac{1}{2}} U_1 S_1 V_1' Q_{F \bar{F}}^{-\frac{1}{2}} = Q_{F \bar{F}}^{-\frac{1}{2}} V_{11}^{-1} V_{11}'$. We next specify: $U_2 S_2 V_2' = \Omega^\frac{1}{2} D(\lambda_F') \left( \begin{array}{c} \lambda_F' \\ I_m \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{F \bar{F}}^{-\frac{1}{2}} \end{array} \right)$, so for $D(\lambda_F') =$
\[ \Omega^{\frac{1}{2}}(D(\lambda_F')_1 D(\lambda_F')_2)' , \text{ with } D(\lambda_F')_1 = -Q_{FP}^{-\frac{1}{4}} \nu_{21} S_1 U'_1 : m \times m, \quad D(\lambda_F')_2 = -Q_{FP}^{-\frac{1}{4}} \nu_{21} S_1 U'_2 : m \times (N - m) : \]

\[ D(\lambda_F')_\perp = \Omega^{-\frac{1}{2}} \left( \begin{array}{c}
-\nu_{11}^{-1} S_1^{-1} \nu_{21}^{-1} Q_{FP}^{-\frac{1}{4}} Q_{FP}^{-\frac{1}{4}} \nu_{21} S_1 U'_1 \\
I_{N-m}
\end{array} \right) \]

\[ (I_{N-m} + \nu_{21} S_1^{-1} \nu_{21}^{-1} S_1^{-1} \nu_{21}^{-1} S_1^{-1} \nu_{21} S_1 U'_1)^{-\frac{1}{2}} \]

\[ = \Omega^{-\frac{1}{2}} \left( \begin{array}{c}
-\nu_{11}^{-1} U'_1 \\
I_{N-m}
\end{array} \right) \left( \begin{array}{c}
I_{N-m} + \nu_{21}^{-1} \nu_{11}^{-1} U'_1 U'_1 \\
I_{N-m}
\end{array} \right)^{-\frac{1}{2}} \]

\[ = \Omega^{-\frac{1}{2}} \left( \begin{array}{c}
U_{12} \\
U_{22}
\end{array} \right) \left( \begin{array}{c}
U_{22}^{-1} (U_{22}^{-1} U_{12} + U_{22} U_{22})^{-\frac{1}{2}}
\end{array} \right) \]

\[ = \Omega^{-\frac{1}{2}} \left( \begin{array}{c}
U_{12} \\
U_{22}
\end{array} \right) \left( \begin{array}{c}
U_{22}^{-1} U_{22} U_{22}^{-1}
\end{array} \right)^{-\frac{1}{2}} \]

\[ = \Omega^{-\frac{1}{2}} U_{12}^{-1} (U_{22} U_{22})^{-\frac{1}{2}} \]

\[ = \Omega^{-\frac{1}{2}} U_{12}^{-1} (U_{22} U_{22})^{-\frac{1}{2}} \]

since \( U'_{11} U_{12} + U'_{21} U_{22} = 0 \) (because of the orthogonality of \( U \)), \( U_{12} U_{22}^{-1} = -U_{11}^{-1} U'_{21} \), and \( U'_{12} U_{12} + U'_{22} U_{22} = I_{N-m} \), and

\[ \left( \begin{array}{c}
\lambda_F \\
I_m
\end{array} \right)_\perp = (1 + \nu_{11}^{-1} \nu_{12}^{-1} \nu_{21}^{-1} \nu_{11})^{-\frac{1}{2}} \left( \begin{array}{c}
1 \\
-\nu_{11}^{-1} \nu_{21}^{-1}
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right) \]

\[ = (1 + \nu_{12}^{-1} \nu_{22}^{-1} \nu_{12}^{-1})^{-\frac{1}{2}} \left( \begin{array}{c}
1 \\
\nu_{12}^{-1} \nu_{22}^{-1}
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right) \]

\[ = (\nu_{12}^{-1} (\nu_{12} \nu_{12} + \nu_{22} \nu_{22}) \nu_{12}^{-1} \nu_{12}^{-1})^{-\frac{1}{2}} \left( \begin{array}{c}
\nu_{12} \nu_{22}^{-1} \\
\nu_{12}^{-1} \nu_{22}^{-1}
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right) \]

\[ = (\nu_{12}^{-1} \nu_{12}^{-1} \nu_{12}^{-1} \nu_{22}^{-1})^{-\frac{1}{2}} \left( \begin{array}{c}
\nu_{12} \nu_{22}^{-1} \\
\nu_{12}^{-1} \nu_{22}^{-1}
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right) \]

\[ = (\nu_{12} \nu_{12}^{-1} \nu_{12}^{-1} \nu_{22}^{-1})^{-\frac{1}{2}} \left( \begin{array}{c}
\nu_{12} \nu_{22}^{-1} \\
\nu_{12}^{-1} \nu_{22}^{-1}
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right) \]

\[ = \left( \begin{array}{c}
1 \\
0
\end{array} \right) \]

\[ \text{as desired.} \]
since $\gamma_{11} \gamma_{21} + \gamma_{21} \gamma_{22} = 0$, so $-\gamma_{21}^{-1} \gamma_{11} = \gamma_{22} \gamma_{12}^{-1}$, and $\gamma_{12} \gamma_{12} + \gamma_{22} \gamma_{22} = 1$, from which it then results that

$$\delta = (U_{22}U_{22}')^{-\frac{1}{2}} U_{22} S_2 \gamma_{12}' (\gamma_{12} V_{12})^{-\frac{1}{2}}.$$  

### 3.11 Proof of Theorem 9

The proof that the quadratic form of

$$\Omega^{-\frac{1}{2}} \left( \begin{array}{c} \bar{R} \\ \bar{\beta} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ -\lambda_F^1 & Q_{FP}^{-1} \lambda_F^1 \end{array} \right)^{-\frac{1}{2}} \left( \begin{array}{cc} 0 & Q_{FP}^{-1} \\ 0 & -\lambda_F^1 \end{array} \right) \left( \begin{array}{c} \bar{\mu}_R \\ \bar{\beta} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{FP}^{-1} \end{array} \right) \left( \begin{array}{c} \lambda_F^* \\ I_m \end{array} \right) (Q_{FP} + \lambda_F^1 \lambda_F^1)^{-\frac{1}{2}}$$

is a maximal invariant follows along the lines of Andrews et al. (2006). It uses that

$$\sqrt{T} \Omega^{-\frac{1}{2}} \left( \begin{array}{c} \bar{R} \\ \bar{\beta} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{FP}^{-1} \end{array} \right) = \Omega^{-\frac{1}{2}} \left( \begin{array}{c} \bar{\mu}_R \\ \bar{\beta} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{FP}^{-1} \end{array} \right) + \psi_{R\beta},$$

with $\text{vec}(\psi_{R\beta}) \sim N(0, I_{N(m+1)})$, is post-multiplied by the orthonormal matrices

$$\left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{FP}^{-1} \end{array} \right) \left( \begin{array}{cc} \lambda_F^1 & I_m \end{array} \right) (Q_{FP} + \lambda_F^1 \lambda_F^1)^{-\frac{1}{2}}$$

We next construct the distributions of the two elements in the above expression for the cases of correct specification and misspecification. For the latter we use the specification from Theorem 8.

**Correct specification.** Without misspecification, $\bar{\mu}_R = \bar{\beta} \lambda_F^*$ so $\Omega^{-\frac{1}{2}} \left( \begin{array}{c} \bar{\mu}_R \\ \bar{\beta} \end{array} \right) = \Omega^{-\frac{1}{2}} \bar{\beta} \left( \begin{array}{cc} \lambda_F^* & I_m \end{array} \right)$ and

$$\Omega^{-\frac{1}{2}} \bar{\mu}(\lambda_F)^* = \Omega^{-\frac{1}{2}} \left( \begin{array}{c} \bar{R} \\ \bar{\beta} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{FP}^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{FP}^{-1} \end{array} \right) \left( \begin{array}{c} \lambda_F^* \\ I_m \end{array} \right) (1 + \lambda_F^1 Q_{FP}^{-1} \lambda_F^1)^{-\frac{1}{2}}$$

$$= \Omega^{-\frac{1}{2}} \left( \bar{R} - \bar{\beta} \lambda_F^1 \right) (1 + \lambda_F^1 Q_{FP}^{-1} \lambda_F^1)^{-\frac{1}{2}}$$

$$= \Omega^{-\frac{1}{2}} \bar{\beta} (\lambda_F^* - \lambda_F^1) (1 + \lambda_F^1 Q_{FP}^{-1} \lambda_F^1)^{-\frac{1}{2}} + \psi_\perp,$$

with $\psi_\perp = \psi_{R\beta} \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{FP}^{-1} \end{array} \right) \left( \begin{array}{c} 1 \\ -\lambda_F^1 \end{array} \right) (1 + \lambda_F^1 Q_{FP}^{-1} \lambda_F^1)^{-\frac{1}{2}} \sim N(0, I_N)$, and
\[ \Omega^{-\frac{1}{2}D(\lambda_F^\dagger)^*} = \Omega^{-\frac{1}{2}} \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{\hat{F}\hat{F}} \end{array} \right) \left( \begin{array}{cc} \lambda_F^\dagger & I_m \end{array} \right) \left( Q_{\hat{F}\hat{F}} + \lambda_F^\dagger \lambda_F^\perp \right)^{-\frac{1}{2}} \]

\[ = \Omega^{-\frac{1}{2}} \left( \begin{array}{cc} \lambda_F^\dagger & I_m \end{array} \right) \left( Q_{\hat{F}\hat{F}} + \lambda_F^\dagger \lambda_F^\perp \right)^{-\frac{1}{2}} + \psi_{\lambda_F^\perp}, \]

with \( \psi_{\lambda_F^\perp} = \psi_{R \beta} \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{\hat{F}\hat{F}}^\perp \end{array} \right) \left( \begin{array}{c} \lambda_F^1 \\ I_m \end{array} \right)^r (Q_{\hat{F}\hat{F}} + \lambda_F^1 \lambda_F^\perp)^{-\frac{1}{2}} + \psi_{\lambda_F^\perp} \psi_{\lambda_F^\perp}. \]

The maximal invariant is the quadratic form of the above two components so it consists of the three elements:

\[ S_{\lambda_F^\perp, \lambda_F^\dagger} = (Q_{\hat{F}\hat{F}} + \lambda_F^\dagger \lambda_F^\perp)^{-\frac{1}{2}} (Q_{\hat{F}\hat{F}} + \lambda_F^\dagger \lambda_F^\perp)^{\frac{1}{2}} (Q_{\hat{F}\hat{F}} + \lambda_F^\dagger \lambda_F^\perp)^{-\frac{1}{2}} (Q_{\hat{F}\hat{F}} + \lambda_F^\perp \lambda_F^\perp)^{-\frac{1}{2}} + \]

\[ S_{\lambda_F^\perp \perp} = (1 + \lambda_F^\dagger Q_{\hat{F}\hat{F}}^-1 \lambda_F^\perp)^{-1}(\lambda_F^\dagger - \lambda_F^\perp)^{-\frac{1}{2}} \lambda_F^\perp (Q_{\hat{F}\hat{F}} + \lambda_F^\perp \lambda_F^\perp) \]

\[ S_{\lambda_F^\perp \perp} = (Q_{\hat{F}\hat{F}} + \lambda_F^\perp \lambda_F^\perp)^{-\frac{1}{2}} (Q_{\hat{F}\hat{F}} + \lambda_F^\dagger \lambda_F^\perp)^{-\frac{1}{2}} (Q_{\hat{F}\hat{F}} + \lambda_F^\dagger \lambda_F^\perp)^{-\frac{1}{2}} + \psi_{\lambda_F^\perp} \psi_{\lambda_F^\perp}. \]

**Misspecification.** To specify the maximal invariant under misspecification, we use the singular value decomposition from Theorem 8:

\[ \Omega^{-\frac{1}{2}D(\lambda_F^\dagger)^*} = \Omega^{-\frac{1}{2}} \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{\hat{F}\hat{F}} \end{array} \right) \left( \begin{array}{cc} \lambda_F^\dagger & I_m \end{array} \right) \left( Q_{\hat{F}\hat{F}} + \lambda_F^\dagger \lambda_F^\perp \right)^{-\frac{1}{2}} \]

\[ = \Omega^{-\frac{1}{2}} D(\lambda_F^\dagger) \left( \begin{array}{cc} \lambda_F^\dagger & I_m \end{array} \right) \left( Q_{\hat{F}\hat{F}} + \lambda_F^\perp \lambda_F^\perp \right)^{-\frac{1}{2}} + \psi_{\lambda_F^\perp}, \]

\[ = \Omega^{-\frac{1}{2}} D(\lambda_F^\dagger) (Q_{\hat{F}\hat{F}} + \lambda_F^\perp \lambda_F^\perp)^{-\frac{1}{2}} + \psi_{\lambda_F^\perp}, \]
with $\text{vec}(\psi_{\lambda^*_F}) \sim N(0, I_{N_m})$, and

$$\Omega^{-\frac{1}{2}} \bar{\mu}(\lambda_F)^* = \Omega^{-\frac{1}{2}} \left( R - \beta \lambda^*_F \right) \left( 1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F \right)^{-\frac{1}{2}}$$

$$= \Omega^{-\frac{1}{2}} \left( \begin{array}{cc} \bar{R} & \beta \\ 0 & Q_{FF} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & Q_{FF} \end{array} \right) \left( 1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F \right)^{-\frac{1}{2}}$$

$$= \Omega^{-\frac{1}{2}} D(\lambda_F) \left( \lambda_F - \lambda^*_F \right) \left( 1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F \right)^{-\frac{1}{2}} + \Omega \hat{\psi}_{\lambda_F} (1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F)^{-\frac{1}{2}} + \psi \perp,$$

with $\psi \perp \sim N(0, I_N)$ and independent of $\psi_{\lambda^*_F}$. The maximal invariant is the quadratic form of the above two components so it consists of the three elements:

$$S_{\lambda_F \lambda^*_F} = (Q_{FF} + \lambda^*_F \lambda_F^{-1})^{-\frac{1}{2}} (Q_{FF} + \lambda^*_F \lambda_F^{-1})^T D(\lambda_F)^\ast (Q_{FF} + \lambda^*_F \lambda_F^{-1}) (Q_{FF} + \lambda^*_F \lambda_F^{-1})^{-\frac{1}{2}} + \delta (\lambda^*_F - \lambda_F) \left( (Q_{FF} + \lambda^*_F \lambda_F^{-1})^T - \frac{1}{2} \right) \left( \Omega^{-\frac{1}{2}} D(\lambda_F) (Q_{FF} + \lambda^*_F \lambda_F^{-1}) (Q_{FF} + \lambda^*_F \lambda_F^{-1})^{-\frac{1}{2}} + \Omega \hat{\psi}_{\lambda_F} (1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F)^{-\frac{1}{2}} + \psi \perp = 1 \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F \right)^{-\frac{1}{2}} \right) \psi_{\lambda_F} + \psi \perp \psi_{\lambda_F}.$$

$$S_{\lambda_F \perp} = (1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F)^{-1} \left( (Q_{FF} + \lambda^*_F \lambda_F^{-1})^T D(\lambda_F)^\ast (Q_{FF} + \lambda^*_F \lambda_F^{-1}) (1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F)^{-1} + \delta' \left( (Q_{FF} + \lambda^*_F \lambda_F^{-1})^T - \frac{1}{2} \right) \left( \Omega^{-\frac{1}{2}} D(\lambda_F) (Q_{FF} + \lambda^*_F \lambda_F^{-1}) (Q_{FF} + \lambda^*_F \lambda_F^{-1})^{-\frac{1}{2}} + \Omega \hat{\psi}_{\lambda_F} (1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F)^{-\frac{1}{2}} + \psi \perp = 1 \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F \right)^{-\frac{1}{2}} \right) \psi_{\lambda_F} \psi \perp.$$

$$S_{\lambda_F \perp} = (Q_{FF} + \lambda^*_F \lambda_F^{-1})^{-\frac{1}{2}} \left( (Q_{FF} + \lambda^*_F \lambda_F^{-1})^T D(\lambda_F)^\ast (Q_{FF} + \lambda^*_F \lambda_F^{-1}) (1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F)^{-\frac{1}{2}} + \delta' \left( (Q_{FF} + \lambda^*_F \lambda_F^{-1})^T - \frac{1}{2} \right) \left( \Omega^{-\frac{1}{2}} D(\lambda_F) (Q_{FF} + \lambda^*_F \lambda_F^{-1}) (Q_{FF} + \lambda^*_F \lambda_F^{-1})^{-\frac{1}{2}} + \Omega \hat{\psi}_{\lambda_F} (1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F)^{-\frac{1}{2}} + \psi \perp = 1 \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F \right)^{-\frac{1}{2}} \right) \psi_{\lambda_F} \psi \perp.$$

Using further that $D(\lambda_F)^\ast \Omega D(\lambda_F)^\ast = I_{N-m}$, $m = 1$ so $(1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F)^{-1} = (1 + \lambda^*_F \bar{Q}^{-1}_{FF} \lambda_F)^{-1} = Q_{FF}^{-1}$.
\( \lambda_1 \lambda_1' \), the above can be specified as:

\[
S_{\lambda_1 \lambda_1'} = (Q_{\bar{F} \bar{F}} + (\lambda_1')^2)^{-\frac{1}{2}} (Q_{\bar{F} \bar{F}} + \lambda_1 \lambda_1')^2 D(\lambda_1')' \Omega^{-1} D(\lambda_1') + (1 + \lambda_1^2 Q_{\bar{F} \bar{F}}^{-1} \lambda_1^2)^{-1} (Q_{\bar{F} \bar{F}} + (\lambda_1')^2)^{-1} (\lambda_1^2 - \lambda_1')^2 \delta \delta + 2 \left( Q_{\bar{F} \bar{F}} + (\lambda_1')^2 \right)^{-\frac{1}{2}} \psi_{\lambda_1'}' \left[ \Omega^{-\frac{1}{2}} D(\lambda_1') (Q_{\bar{F} \bar{F}} + \lambda_1 \lambda_1') \right] + \Omega^\frac{1}{2} D(\lambda_1') \delta (\lambda_1^2 - \lambda_1') (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} \psi_{\lambda_1'} \psi_{\lambda_1'}
\]

\[
S_{\perp \perp} = (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-1} (\lambda_1^2 - \lambda_1')' D(\lambda_1')' \Omega^{-1} D(\lambda_1') (\lambda_1^2 - \lambda_1') + \delta' \delta (1 + \lambda_1^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} + 2 \psi_{\perp} \left[ \Omega^{-\frac{1}{2}} D(\lambda_1') (\lambda_1^2 - \lambda_1') (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} \right] + \psi_{\perp} \psi_{\perp}
\]

\[
S_{\lambda_1 \perp} = (\lambda_1^2 - \lambda_1') (Q_{\bar{F} \bar{F}} + (\lambda_1')^2)^{-\frac{1}{2}} (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} \left[ (Q_{\bar{F} \bar{F}} + \lambda_1 \lambda_1') D(\lambda_1')' \Omega^{-1} D(\lambda_1') - (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-1} \delta' \delta (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} \right] + \left[ (Q_{\bar{F} \bar{F}} + (\lambda_1')^2)^{-\frac{1}{2}} (Q_{\bar{F} \bar{F}} + \lambda_1 \lambda_1')' D(\lambda_1')' \Omega^{-\frac{1}{2}} \right] \psi_{\perp} + \psi_{\perp} \left[ \Omega^{-\frac{1}{2}} D(\lambda_1') (\lambda_1^2 - \lambda_1') (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} \right] + \psi_{\perp} \psi_{\perp}.
\]

Since \( \Omega^{-\frac{1}{2}} \hat{D}(\lambda_1')^* \) and \( \Omega^{-\frac{1}{2}} \hat{\mu}(\lambda_1')^* \) are independently normal distributed with identity covariance matrices, the quadratic form of \( (\Omega^{-\frac{1}{2}} \hat{D}(\lambda_1')^* : \Omega^{-\frac{1}{2}} \hat{\mu}(\lambda_1')^* ) \) with \( T \) degrees of freedom, identity scale matrices and a non-centrality parameter which is the quadratic form of the mean of the distribution of \( (\Omega^{-\frac{1}{2}} \hat{D}(\lambda_1')^* : \Omega^{-\frac{1}{2}} \hat{\mu}(\lambda_1')^* ) \), which read:

Correct specification:

\[
\begin{pmatrix}
(\lambda_1^2 - \lambda_1') (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} \\
(Q_{\bar{F} \bar{F}} + (\lambda_1')^2)^{-\frac{1}{2}} (Q_{\bar{F} \bar{F}} + \lambda_1 \lambda_1')
\end{pmatrix}
\begin{pmatrix}
(\lambda_1^2 - \lambda_1') (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} \\
(Q_{\bar{F} \bar{F}} + (\lambda_1')^2)^{-\frac{1}{2}} (Q_{\bar{F} \bar{F}} + \lambda_1 \lambda_1')
\end{pmatrix}
\]

Misspecification:

\[
\begin{pmatrix}
(\lambda_1^2 - \lambda_1') (1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} \\
(Q_{\bar{F} \bar{F}} + (\lambda_1')^2)^{-\frac{1}{2}} (Q_{\bar{F} \bar{F}} + \lambda_1 \lambda_1')
\end{pmatrix}
\begin{pmatrix}
D(\lambda_1')' \Omega^{-1} D(\lambda_1') \\
(Q_{\bar{F} \bar{F}} + (\lambda_1')^2)^{-\frac{1}{2}} (Q_{\bar{F} \bar{F}} + \lambda_1 \lambda_1')
\end{pmatrix}
\]

\[
\begin{pmatrix}
(1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} (1 + \lambda_1^2 Q_{\bar{F} \bar{F}}^{-1}) \\
-(Q_{\bar{F} \bar{F}} + (\lambda_1')^2)^{-\frac{1}{2}} (\lambda_1^2 - \lambda_1')
\end{pmatrix}
\begin{pmatrix}
(1 + (\lambda_1')^2 Q_{\bar{F} \bar{F}}^{-1})^{-\frac{1}{2}} (1 + \lambda_1^2 Q_{\bar{F} \bar{F}}^{-1}) \\
-(Q_{\bar{F} \bar{F}} + (\lambda_1')^2)^{-\frac{1}{2}} (\lambda_1^2 - \lambda_1')
\end{pmatrix}
\]

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4 Simulation setup for the CRRA moment function

We use a log-normal data generating process to simulate consumption growth and asset returns in accordance with the CRRA moment condition.

Let \( \Delta c_{t+1} = \ln \left( \frac{C_{t+1}}{C_t} \right) \) and \( r_{t+1} = \ln(\iota_N + R_{t+1}) \), which are i.i.d. normally distributed:

\[
\begin{bmatrix}
\Delta c_{t+1} \\
r_{t+1}
\end{bmatrix} \sim \text{NID}(\mu, V)
\]

\[
\equiv \text{NID}\left(\begin{bmatrix} 0 \\ \mu_{2,0} \end{bmatrix}, \begin{bmatrix} V_{cc,0} & V_{cr,0} \\ V_{rc,0} & V_{rr,0} \end{bmatrix}\right),
\]

with \( \mu_{2,0} = (\mu_{2,1,0} \ldots \mu_{2,N,0})' \) the mean of \( r_{t+1} \), \( V_{cc,0} \) the (scalar) variance of \( \Delta c_{t+1} \), \( V_{cr,0} = V'_{cr,0} = (V_{r,1,0} \ldots V_{r,N,0})' \) the \( N \times 1 \) dimensional covariance between \( r_{t+1} \) and \( \Delta c_{t+1} \) and \( V_{rr,0} = (V_{rr,ij,0}) : i, j = 1, \ldots, N \), the \( N \times N \) dimensional covariance matrix of \( r_{t+1} \).

This DGP has also been used in Kleibergen and Zhan (2020), where the covariance matrix \( V = [V_{cc,0}, V_{cr,0}; V_{rc,0}, V_{rr,0}] \) is calibrated to data. We will change the value of \( \mu_{2,0} \) to vary the magnitude of the misspecification. We will also alter the correlation coefficient of \( \Delta c_{t+1} \) and \( r_{t+1} \) to vary identification.

Given pre-set values of \( \delta_0, \mu_{2,0}, V_{cc,0}, V_{rc,0} \) and \( V_{rr,0} \), the CRRA moment equation is such that:

\[
\mu_f(\gamma) = E \left[ \delta_0 \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} (\iota_N + R_{t+1}) - \iota_N \right]
\]

\[
= E \left[ \begin{bmatrix}
\exp (\ln(\delta_0) - \gamma \Delta c_{t+1} + r_{t+1,1}) \\
\vdots \\
\exp (\ln(\delta_0) - \gamma \Delta c_{t+1} + r_{t+1,N})
\end{bmatrix} \right] - \iota_N
\]

\[
= \begin{bmatrix}
\exp (\ln(\delta_0) + \mu_{2,1,0} + \frac{1}{2} (V_{rr,11,0} + \gamma^2 V_{cc,0} - 2\gamma V_{rc,1,0})) \\
\vdots \\
\exp (\ln(\delta_0) + \mu_{2,N,0} + \frac{1}{2} (V_{rr,NN,0} + \gamma^2 V_{cc,0} - 2\gamma V_{rc,N,0}))
\end{bmatrix} - \iota_N.
\]

We also need the explicit expression of \( V_{ff}(\gamma) \):
\[ V_{ff}(\gamma) = E[(f_t(\gamma) - \mu_f(\gamma))(f_t(\gamma) - \mu_f(\gamma))'] = Var\left(e^{\ln(\delta) - \gamma \Delta c_{t+1} + r_{t+1}}\right) \]

\[ = \begin{pmatrix}
\exp(\ln(\delta) + \mu_{2,1.0} + \frac{1}{2} (V_{rr,11.0} + \gamma^2 V_{cc,0} - 2\gamma V_{rc,1.0})) \\
\vdots \\
\exp(\ln(\delta) + \mu_{2,N.0} + \frac{1}{2} (V_{rr,N.N,0} + \gamma^2 V_{cc,0} - 2\gamma V_{rc,N,0})) \\
\end{pmatrix} \times \begin{pmatrix}
\exp(\ln(\delta) + \mu_{2,1.0} + \frac{1}{2} (V_{rr,11.0} + \gamma^2 V_{cc,0} - 2\gamma V_{rc,1.0})) \\
\vdots \\
\exp(\ln(\delta) + \mu_{2,N.0} + \frac{1}{2} (V_{rr,N.N,0} + \gamma^2 V_{cc,0} - 2\gamma V_{rc,N,0})) \\
\end{pmatrix}
\]

\[ \exp\left(-\gamma I_N : I_N \right) \begin{bmatrix}
V_{cc,0} & V_{cr,0} \\
V_{rc,0} & V_{rr,0}
\end{bmatrix} \left(-\gamma I_N : I_N \right)' - \gamma I_N \right), \]

where \( \odot \) stands for element-by-element multiplication.