Evolution of inspiral orbits around a Schwarzschild black hole

Niels Warburton
Collaborators: S. Akcay, L. Barack, J. Gair, N. Sago
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Overview

- Key advance: Lorenz gauge GSF via the frequency domain
- Orbit evolution
- Sample inspiral
- Future prospects
Lorenz gauge GSF via the frequency domain (FD)

Goal: solve the linearized Einstein equation in the Lorenz gauge

\[ \Box \bar{h}_{\mu\nu} + 2 \bar{R}^\alpha_{\mu \beta} \bar{h}^{\alpha\beta} = -16\pi T_{\mu\nu}, \quad \nabla_{\mu} \bar{h}^{\mu\nu} = 0 \]

Method: decompose metric perturbation and source into spherical harmonic and frequency modes

\[ \bar{h}_{\mu\nu}(t, r, \theta, \varphi) = \frac{\mu}{r} \sum_{lmn}^{10} \sum_{i=1} R^{(i)}_{lmn}(r) e^{-i\omega t} Y_{\mu\nu}^{(i)lm}(\theta, \varphi) \]

where the mode frequency is \( \omega = m\Omega_\varphi + n\Omega_r \). Result is a set of 10 coupled ODEs governed by

\[
\frac{d^2 R^{(i)}_{lmn}(r)}{dr_*^2} - \left[ V_l(r) - \omega^2 \right] R^{(i)}_{lmn}(r) - 4\tilde{M}^{(i)l}_j R^{(j)}_{lmn}(r) = J^{(i)}_{lmn}
\]
Key task: provide suitable boundary conditions for the homogeneous radial equation [Akçay 2010]

For a fixed radius, $r_0$, with $r_{\text{min}} < r_0 < r_{\text{max}}$ the $\bar{h}_{\mu\nu}^{(i)}(t, r)$ is not a smooth function of time. To ensure rapid convergence of the Fourier sum we employ the ‘method of extended homogeneous solutions’ [Barack, Ori, Sago 2008]

Solve for the full retarded field mode-by-mode and regularize using the mode-sum procedure
Lorenz gauge GSF via the FD: efficiency

For low orbital eccentricities our code is very efficient. Harmonic decomposition makes for straightforward parallelization.
Orbit evolution

Aim

To track the phase evolution of gravitational waveforms from EMRIs to within one radian over the entire inspiral

From Hinderer and Flanagan’s two-time scale analysis we know the contributions to the orbital phase evolution scale as:

\[ \mathcal{O}(M/\mu) : \text{Orbit averaged dissipative piece of the SF} \]

\[ \mathcal{O}(1) : \begin{cases} 
\text{Oscillatory component of the dissipative SF} \\
\text{Conservative component of the SF} \\
\text{Orbit averaged dissipative piece of the second-order SF}
\end{cases} \]
Up to orientation, bound geodesic orbits in Schw. spacetime are uniquely specified by $E$ and $L$.

We use an alternative parametrization: the semi-latus rectum, $p$, and orbital eccentricity, $e$.

$$p \equiv \frac{2r_{\text{max}} r_{\text{min}}}{M(r_{\text{max}} + r_{\text{min}})} \quad e \equiv \frac{r_{\text{max}} - r_{\text{min}}}{r_{\text{max}} + r_{\text{min}}}$$

Also introduce a relativistic anomaly parameter, $\chi$, such that

$$r(t) = \frac{pM}{1 + e \cos[\chi(t) - \chi_0]}$$

where $\chi_0$ is the periastron phase.
Orbit evolution: osculating orbits

**Basic idea**

At $t_1$ the position and velocity of the inspiralling particle corresponds to an osculating (‘kissing’) geodesic. In general, at later times $x_p(t)$ and the osculating geodesic will diverge. If instead \(\{p, e, \chi_0\} \to \{p(t), e(t), \chi_0(t)\}\) the trajectory can be described by a sequence of osculating geodesics.

**Evolution equations**

Pound and Poisson derived the evolution equations in Schw. spacetime.

\[
\begin{align*}
\dot{p} &= \mathcal{F}_p[p, e, \chi - \chi_0, F_{\text{diss}}(t)] \\
\dot{e} &= \mathcal{F}_e[\cdots] \\
\dot{\chi}_0 &= \mathcal{F}_{\chi_0}[p, e, \chi - \chi_0, F_{\text{cons}}(t)]
\end{align*}
\]
Orbit evolution: key assumption

- The true self-force felt by an inspiralling particle at a given instance is approximated by the self-force of a particle that has spent its entire history on the corresponding osculating geodesic.

\[
\begin{align*}
\dot{p} &= \mathcal{F}_p[p, e, \chi - \chi_0, F_{\text{diss}}^\text{self}(p, e, \chi - \chi_0)] \\
\dot{e} &= \mathcal{F}_e[\cdots] \\
\dot{\chi}_0 &= \mathcal{F}_{\chi_0}[p, e, \chi - \chi_0, F_{\text{cons}}^\text{self}(p, e, \chi - \chi_0)]
\end{align*}
\]

- It is not clear what the order of magnitude of the error is from making this assumption. We believe the error scales as $\mu/M$ but the coefficient is unknown.

- Quantifying this error is very important. Will discuss further at the end.
Self-force Interpolation Model

The need for a model

- Though we now have fast GSF codes it is still impractical to calculate $F^r$ and $F^\varphi$ for given $(p, e, \nu \equiv \chi - \chi_0)$ at each step of the orbital evolution.
- Instead we fit as much data as we can produce (see later) to a model.

Fourier representation

4-components: \( \{ F^r_{\text{cons}}, F^\varphi_{\text{diss}}, F^r_{\text{diss}}, F^\varphi_{\text{cons}} \} \)

\[ F^r_{\text{cons}} = (\mu/M)^2 \sum_{n=0}^{\bar{n}} A_n(p, e) \cos(n\nu) \]

\[ A_n(p, e) = p^{-2} \sum_{j=\bar{j}}^{\bar{J}} \sum_{k=\bar{k}}^{\bar{k}} a^r_{jk} p^{-k} e^j \]

Fit the \( a^r_{jk} \) using standard $\chi^2$ minimization and seek a global accuracy of

\[ \delta F \equiv \frac{F(\text{model}) - F(\text{data})}{F(\text{data})} < 10^{-3} \]
Fit the model with data from over 1000 geodesics
Verified the fit by checking the results of the model against Barack and Sago’s TD code.
Sample inspiral: snapshots

\[ \mu = 10M_\odot, \quad M = 10^6 M_\odot \]

Initial conditions:
\[ (p_0, e_0, \chi_{00}) = (12, 0.2, 0) \]

Inspiral completes 75,550 periastron passages

One hour snapshots of orbital motion at (reading from top left) 1443 days, 500 days, 75 days and 1 hour to plunge.
Sample inspiral: evolution of osculating parameters

Black dots mark (from the right) 500 days, 100 days, 10 days and 1 hour to plunge. [note the left and right hand axes]
Sample inspiral: evolution of azimuthal phase

Construct a radiative approximation:

\[
\dot{p} = \langle F_p[p, e, v, F_{\text{diss}}] \rangle \\
\dot{e} = \langle F_e \cdots \rangle \\
\dot{\chi}_0 = \langle F_{\chi_0}[p, e, v, F_{\text{cons}}] \rangle = 0
\]

where \( \langle \cdot \rangle \) is a \( t \)-average over instantaneous osculating geodesic.

Then consider difference in accumulated phase:

\[
\Delta \varphi_{RA} \equiv \varphi^{\text{full}} - \varphi^{RA}
\]

\[
t_{RR} = \left( \frac{M}{\mu} \right) T_c, \quad T_c = \frac{2\pi}{6^{3/2}M}
\]

(Play waveform for \( \mu/M = 10^{-4} \))
Future prospects

- Greater coverage of the \((p, e)\) parameter space with high accuracy data. Use FD for low eccentricity orbits and TD otherwise. Recent progress with higher order regularization params assists with improving accuracy [Heffernan et al.]
- Include 2nd order orbit averaged dissipative effects, see Pound’s talk.
- Key: assess the error from the osculating orbit method. Recent project: compare scalar self-force (SSF) osculating orbit inspirals with self consistent SSF evolutions [Diener et al.]