Kaluza-Klein States versus Winding States: Can Both Be Above the String Scale?

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When closed strings propagate in extra compactified dimensions, a rich spectrum of Kaluza-Klein states and winding states emerges. Since the masses of Kaluza-Klein states and winding states play a reciprocal role, it is often believed that either the lightest Kaluza-Klein states or the lightest winding states must be at or below the string scale. In this paper, we demonstrate that this conclusion is no longer true for compactifications with non-trivial shape moduli. Specifically, we demonstrate that toroidal compactifications exist for which all Kaluza-Klein states as well as all winding states are heavier than the string scale. This observation could have important phenomenological implications for theories with reduced string scales, suggesting that it is possible to cross the string scale without detecting any states associated with spacetime compactification.

I. INTRODUCTION

Theories involving large extra spacetime dimensions and fundamental theories at reduced energy scales have attracted considerable attention in recent years [1,2]. As a result of such developments, it has become possible to contemplate experimental probes not only of compactification geometry, but also of quantum gravity, grand unification, and even string theory. Most recent phenomenological studies of such theories have focused on the effects of their low-energy Kaluza-Klein states. However, if the string scale itself is in the TeV range, then many additional string states may also play a role in the effective low-energy phenomenology.

In general, closed string theories (as well as certain closed-string sectors of Type I open string theories) give rise not only to Kaluza-Klein states, but also to string winding states as well as string oscillator states. These states may be differentiated by the manner in which their masses depend on the scale of compactification as well as on the string tension (equivalently the string scale). Kaluza-Klein states reflect the higher-dimensional momentum quantization induced by the spacetime compactification, and consequently have masses inversely proportional to the compactification radius but independent of the string scale. By contrast, winding states reflect the possibility that the string can stretch completely around the compactified dimension. These states therefore have masses that grow linearly with the compactification radius as well as with the string tension. Finally, string oscillator states correspond to the vibrational modes of the fundamental string. Their masses are thus set purely by the string tension (or equivalently the fundamental string scale). Note that while the masses of the Kaluza-Klein and winding states are sensitive to the compactification geometry, the spectrum of oscillator states is completely independent of this geometry. Likewise, the masses of the Kaluza-Klein excitations are independent of the string scale, whereas the masses of the winding and oscillator states depend critically on the string scale.

There are, of course, many instances in which it is legitimate to restrict our attention to various subsets of these states. For example, in theories whose fundamental constituents are point particles, there are no winding or oscillator states. The Kaluza-Klein spectrum therefore provides a complete low-energy description of the resonances associated with spacetime compactification. Likewise, even within the context of string theory, it is again legitimate to restrict our attention to the Kaluza-Klein states if the compactification radius is significantly larger than the fundamental string length. Indeed, in such cases, the winding states will be much heavier than the string scale, and will affect phenomenology only at correspondingly higher energies.

However, if the string scale is in the TeV range, some or all of the compactification radii may not differ significantly from the string scale [3,4]. In such cases, Kaluza-Klein states, winding states, and even oscillator states must be considered together. Their combined effects can therefore give rise to dramatic alternatives to traditional weak-scale supersymmetric or Kaluza-Klein approaches to the gauge hierarchy problem [5]. In such cases, however, an important question is to determine which of these states are truly the lightest in the string spectrum. In other words, which states can be expected to appear below, at, and above the string scale?

Ordinarily, the masses of Kaluza-Klein states and winding states play a reciprocal role: if the lightest Kaluza-Klein states are lighter than the string scale, then the corresponding winding states are necessarily heavier than the string scale. Similarly, the reverse situation in which the lightest Kaluza-Klein states are heavier than the string scale ordinarily results in winding states which are lighter than the string scale (and is equivalent to the
previous configuration as a result of $T$-duality). The expectation, then, is that either Kaluza-Klein states or the winding states must be lighter than the string scale, or must at least have masses equal to the string scale. Thus, it would seem that it is impossible to cross the string scale without seeing at least some states (either Kaluza-Klein or winding) associated with the compactification geometry.

In this paper, we shall demonstrate that this naïve expectation is incorrect in the case of compactifications with non-trivial shape moduli. Specifically, we shall show that it is possible for the string scale to be simultaneously lighter than all the Kaluza-Klein states as well as all the winding states. Thus, in such theories, it is possible to cross the string scale without seeing a single resonance associated with the spacetime compactification! Needless to say, this can therefore give rise to a low-energy phenomenology which is markedly different from that of the usual Kaluza-Klein effective field theories.

We shall begin by focusing purely on the spectrum of Kaluza-Klein and winding modes associated with toroidal compactification. After establishing our main result, we shall then proceed to discuss how this result emerges within the context of a full string theory when the string oscillator states are also taken into account.

II. SPECTRUM OF COMPACTIFICATION RESONANCES

We begin by considering the action for a string propagating in an $n$-dimensional spacetime governed by a metric $G_{i\beta}$ and a background antisymmetric tensor (torsion) field $B_{ij}$:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left( \sqrt{-g} g^{\alpha\beta} G_{ij} \partial_{\alpha} X^i \partial_{\beta} X^j + \epsilon^{\alpha\beta} B_{ij} \partial_{\alpha} X^i \partial_{\beta} X^j \right).$$

(1)

Here $\alpha' \equiv M_{\text{string}}^{-2}$ is the string Regge slope, and $\sigma$, $\partial_{\alpha}$, and $g_{\alpha\beta}$ are the coordinates, derivatives, and metric on the string worldsheet. The spacetime indices $i, j$ run from $1, ..., n$. Because our interest is in the spacetime compactification of this theory, we have restricted the above action to include only those bosonic worldsheet fields $X^i$ corresponding to the spacetime coordinates to be compactified. Later we will discuss how this may be embedded in a more complete string theory.

The next step is to compactify these $n$ spacetime dimensions. For simplicity, we shall take our compactification manifold to be a general, flat, $n$-dimensional torus. Such a torus can be specified by its periodicity radii $R_i$ ($i = 1, ..., n$) as well as by shape angles $\alpha_{ij}$ ($i, j = 1, ..., n$). These shape angles parametrize the relative orientations between the $i^{\text{th}}$ and $j^{\text{th}}$ toroidal periodicities.

Quantizing the string in the usual fashion, we then find that the resulting string states can be classified in terms of their Kaluza-Klein momentum numbers $n_i$ and their winding numbers $w_j$, where $i = 1, ..., n$. If we collect these momentum and winding numbers into a vector

$$\tilde{\mathcal{N}}^T = \left( \frac{n_1}{R_1}, ..., \frac{n_n}{R_n}, \frac{w_1 R_1}{\alpha'}, ..., \frac{w_n R_n}{\alpha'} \right),$$

(2)

we find that the mass of the corresponding state takes the form

$$\mathcal{M}^2_{n_i; w_j} = \tilde{\mathcal{N}}^T Q^{-1} \tilde{\mathcal{N}}$$

(3)

where

$$Q^{-1} = \begin{pmatrix} G^{-1} & G^{-1} B \\ -BG^{-1} & G - BG^{-1} B \end{pmatrix}.$$ 

(4)

In this expression, $G$ is the dimensionless $(n \times n)$-dimensional metric of the $n$-torus, now given by

$$G_{ij} \equiv \cos \alpha_{ij}$$

(5)

(where $\alpha_{ii} \equiv 0$), and $B$ is the $(n \times n)$-dimensional background antisymmetric tensor as in Eq. (3). The result in Eq. (3) is completely standard [5]; a full derivation will be given in Ref. [6].

Let us now examine specific examples of this mass formula. For a one-dimensional torus (i.e., a circle), there are no shape angles $\alpha_{ij}$. There is also no $B$-field background. We then obtain the standard spectrum associated with circle-compactification:

$$\mathcal{M}^2_{n_i; w_i} = \frac{n_i^2}{R^2} + \frac{w_i^2 R^2}{\alpha'^2}.$$ 

(6)

Note that this spectrum conforms to our usual expectations: depending on the value of the radius $R$, either the momentum (Kaluza-Klein) or the winding modes must be lighter than the string scale $M_{\text{string}} \equiv 1/\sqrt{\alpha'}$. Specifically, the mass of the lightest Kaluza-Klein and winding states are respectively given by

$$M_{\text{KK}}^2 = \frac{1}{R^2}, \quad M_{\text{winding}}^2 = \frac{R^2}{\alpha'}. $$

(7)

The self-dual radius $R_s$ (defined to be the radius at which these masses become equal) is therefore given by $R_s \equiv \sqrt{\alpha'}$, implying that

$$M_{\text{string}} = \frac{1}{R_s} = M_*$$

(8)

where $M_* \equiv M_{\text{KK}} = M_{\text{winding}}$ at the self-dual radius. These results are completely as expected, and it is possible to exploit $T$-duality in order to choose a convention such that $R^{-1} < M_{\text{string}}$ (or equivalently $M_{\text{KK}} < M_{\text{winding}}$) when we are not at the self-dual radius.
However, as we shall now demonstrate, these results no longer hold in the case of higher-dimensional toroidal compactifications with non-trivial shape moduli and background B-fields. In the case of a two-torus, the antisymmetric B-field has a single element \( b \equiv B_{12} \); there is also only one shape modulus \( \theta \equiv \alpha_{12} \). The corresponding mass formula then takes the form

\[
M^2_{R,w} = \frac{1}{\sin^2 \theta} \left( \frac{n_1^2}{R_1^2} + \frac{n_2^2}{R_2^2} - 2 \frac{n_1 n_2 \cos \theta}{R_1 R_2} \right) + \frac{b^2 + \sin^2 \theta}{\alpha'^2 \sin^2 \theta} \left( w_1^2 R_1^2 + w_2^2 R_2^2 + 2 w_1 w_2 R_1 R_2 \cos \theta \right) + \frac{2b}{\alpha' \sin^2 \theta} \left( n_1 w_1 \cos \theta - n_2 w_2 \cos \theta \right) + n_1 w_2 R_2 - n_2 w_1 R_1 \right) .
\]

(9)

Without loss of generality, let us take \( R_1 \geq R_2 \). The lightest Kaluza-Klein state then corresponds to the state with \((n_1, n_2; w_1, w_2) = (1, 0; 0, 0)\), and has mass

\[
M_{KK} = \frac{1}{R_1 \sin \theta} .
\]

(10)

By contrast, the lightest winding mode in the same direction associated with \( R_1 \) is the state \((n_1, n_2; w_1, w_2) = (0, 0; 1, 0)\), with mass

\[
M_{\text{winding}} = \frac{\sqrt{b^2 + \sin^2 \theta}}{\alpha' \sin \theta} \cdot R_1 .
\]

(11)

Solving for the self-dual radius, we find

\[
R^2_* = \frac{\alpha'}{\sqrt{b^2 + \sin^2 \theta}} .
\]

(12)

Thus, defining as before \( M_* \equiv M_{KK} = M_{\text{winding}} \) at the self-dual radius, we obtain the result

\[
M^2_{\text{string}} = \frac{1}{\sqrt{b^2 + \sin^2 \theta} \cdot R^2_*} = \frac{\sin^2 \theta}{\sqrt{b^2 + \sin^2 \theta}} M^2_* .
\]

(13)

This is to be compared with the case for a one-torus in Eq. (6).

Clearly, if \( \theta = \pi/2 \) and \( b = 0 \), we obtain the expected result that \( M_* = M_{\text{string}} \). In this case, it is impossible to make the Kaluza-Klein states heavier than the string scale without making at least one winding state lighter than the string scale. However, in all other cases, we find from Eq. (13) that

\[
M_* > M_{\text{string}} !
\]

(14)

This implies that it is possible (e.g., at the self-dual radius) for the string scale to be lighter than \( M_* \) the Kaluza-Klein modes as well as \( M_{\text{string}} \) the winding modes! Indeed, it is possible to cross the energy threshold associated with the string scale without having seen a single compactification resonance of either type! Choosing \( R_2 = R_* \) in this situation then guarantees that \( M_{KK} \) and \( M_{\text{winding}} \) all compactification resonances associated with this two-torus compactification will be heavier than the string scale.

This feature generalizes to higher-dimensional compactifications. In the case of a three-torus, for example, there are three radii \( R_i \), three shape angles \( \{\alpha_{12}, \alpha_{13}, \alpha_{23}\} \), and likewise three independent components \( \{b_{12}, b_{13}, b_{23}\} \) for the background antisymmetric B-field. Note that the shape angles must satisfy the constraint

\[
|\alpha_{12} - \alpha_{13}| < \alpha_{23} < \alpha_{12} + \alpha_{13}
\]

in order to guarantee that our three-torus is physical; these inequalities are saturated only in the degenerate limit when the direction associated with the third toroidal periodicity lies in the plane formed by the other two. Repeating the above procedure to calculate the Kaluza-Klein and winding masses associated with the first toroidal periodicity, we find that Eq. (13) becomes

\[
M^2_{\text{string}} = \frac{s_{23}}{\sqrt{K + K_1}} \frac{1}{R^2_*} = \frac{K}{s_{23} \sqrt{K + K_1}} M^2_*
\]

(16)

where \( s_{ij} \equiv \sin \alpha_{ij}, c_{ij} \equiv \cos \alpha_{ij} \), and where

\[
K \equiv \det G = 1 - c^2_{12} - c^2_{13} - c^2_{23} + 2 c_{12} c_{13} c_{23} ,
\]

\[
K_1 \equiv b^2_{12} s^2_{13} + b^2_{13} s^2_{12} - 2 b_{12} b_{13} (c_{23} - c_{12} c_{13}) .
\]

(17)

Similar results hold for the remaining toroidal periodicities. Note that this result reduces to Eq. (13) in the special case with \( c_{13} = c_{23} = b_{13} = b_{23} = 0 \).

Several features of this result are immediately apparent. First, it is straightforward to verify that

\[
\frac{K}{s_{23} \sqrt{K + K_1}} \leq 1
\]

(18)

whenever the shape angles satisfy the constraint in Eq. (15). Indeed, we find that \( K \) and \( K_1 \) are both necessarily positive when \( \alpha_{12} - \alpha_{13} \) is satisfied. Thus, once again, we verify that the self-dual Kaluza-Klein/winding mass scale is greater than \( M_{\text{string}} \). Indeed, this holds for all three periodicities of the torus.

More interestingly, however, we now observe a new feature: the self-dual radius, and indeed the mass of the lightest Kaluza-Klein/winding modes at the self-dual radius, are not universal for all toroidal directions! Instead, they depend on the specific configuration of shape angles and antisymmetric B-field components involved in the compactification. Thus, the whole notion of self-dual radius becomes a shape-dependent phenomenon, varying according to the specific direction of compactification.
III. GEOMETRIC BOUNDS

Thus far, we have demonstrated that in compactifications with non-trivial shape moduli, $M_*$ can generically be larger than $M_{\text{string}}$. As we have seen, this implies that compactifications exist for which the lightest Kaluza-Klein states as well as the lightest winding states are simultaneously heavier than the string scale.

Given this result, it is natural to wonder how large this separation between the lightest compactification states and the string scale can become. In other words, what is the maximum size for the ratio $M_*/M_{\text{string}}$?

At first glance, it may appear that this ratio is completely unbounded. For example, in the case of a two-torus in Eq. (13), it might initially appear that we can take $\theta \ll 1$, thereby making $M_*$ arbitrarily heavy. However, there is an important subtlety that must be taken into account. In our derivation of the two-torus result in Eq. (14), we assumed that the lightest Kaluza-Klein states as well as the lightest winding states are non-zero and large. Or, to phrase this restriction more mathematically in the case of a two-torus, we must ensure that $\hat{M}_{\alpha}^2/M_{\text{string}}^2 = 0; 0$ and $(0, 0; 1, 0)$ respectively. However, this assumption is true only if two conditions are satisfied. First, this is true only if $|\cos \theta|$ remains relatively small, so that no anomalous cancellations are induced in the mass formula in Eq. (14) when $n_1, n_2$ are both non-zero and large. Or, to phrase this restriction more mathematically, we must also restrict the components of the anti-symmetric compactifications for which each compactification has its own value of $M_*$. We have already seen this possibility, for example, in the three-torus case discussed above. These issues will be discussed further in Ref. [6].

IV. EMBEDDINGS INTO STRING THEORY

We now discuss the additional features that arise when these results are embedded into the full string framework. As we shall see, this is important because there are significant differences between the traditional Kaluza-Klein picture and its ultimate realization in string theory.

The full mass spectrum of a toroidally compactified string is generally governed by two equations of the form

$$\alpha' M_{\text{eff}}^2 = \alpha' M_{n_i w_i}^2 + 2(N + \overline{N}) + 2(a + \overline{a})$$

$$\overline{N} = N = \sum_i n_i w_i + a - \overline{a}.$$  \hspace{1cm} (21)

Here $M_{n_i w_i}^2$ represents the mass contribution from the Kaluza-Klein and winding excitations, as defined in Eq. (10), and $(N, \overline{N})$ are the total energies from the left-and right-moving string oscillator excitations. These energies are calculated as $N = \sum n N_n$ and $\overline{N} = \sum n \overline{N}_n$, where $N_n$ and $\overline{N}_n$ count the number of excitations of frequency $n$ of the underlying left- and right-moving worldsheet fields. Likewise, $(a, \overline{a})$ represent left- and right-moving vacuum-energy contributions which we shall keep arbitrary for now. The first of these equations indicates that the total mass of a given string state receives contributions not only from Kaluza-Klein and winding excitations, but also from the oscillators and from the total vacuum energy. By contrast, the second of these equations is a level-matching constraint which is required for the self-consistency of the string. This constraint ensures that the total mass of a given state receives equal contributions from the left- and right-moving worldsheet excitations, possibly offset by a vacuum energy difference $\overline{a} - a$. (The quantity $\sum_i n_i w_i$ represents the energy offset due to the Kaluza-Klein/winding modes.) While there may be additional GSO constraint equations governing the string spectrum in the case of realistic string models, the constraints listed in Eq. (21) are generic and appear as a minimal set for all toroidally compactified string theories.
Given the constraints in Eq. (21), we immediately see two important differences relative to the traditional Kaluza-Klein picture which focuses only on $\mathcal{M}_{n;w_i}$. First, we see that the total spacetime masses of our Kaluza-Klein and winding modes are generally offset by a non-zero vacuum energy $a + \pi$. The value of $a + \pi$ depends on the type of string in question (bosonic, superstring, etc.), as well as on the type of sector (Neveu-Schwarz, Ramond, higher-order twists, etc.) within a given string theory. The important point, however, is that $a + \pi$ can have either sign, and is often negative in many string sectors. This implies that the lightest Kaluza-Klein states according to the mass formula in Eq. (3) need not be the lightest Kaluza-Klein states in the actual string spectrum. We shall see several dramatic examples of this below.

The second important feature is the presence of a level-matching constraint which correlates the Kaluza-Klein/winding numbers $(n_i; w_i)$ with the string oscillator energies $(N, \overline{N})$. This implies that whether a given Kaluza-Klein or winding state is allowed in string theory depends not only on its total mass, but also on the particular combination $\sum_i n_i w_i$.

In order to see how these two features affect the mass spectrum, let us classify the various string states into sectors according to their values of $N + \overline{N} + a + \pi$.

In sectors with $N + \overline{N} + a + \pi > 0$, the string states are already massive even before any Kaluza-Klein or winding modes are excited. Such states are therefore not likely to be among the lightest states in the full string spectrum.

Next, we turn our attention to string sectors with $N + \overline{N} + a + \pi = 0$. Since this implies that $\mathcal{M}_{\text{tot}} = \mathcal{M}_{n;w_i}$, our previous results apply directly in such sectors. In other words, we have shown that there exist toroidal compactifications such that the Kaluza-Klein and winding states in such sectors are all heavier than the string scale. Note that the special case with $n_i = w_i = 0$ (corresponding to the absence of any Kaluza-Klein or winding excitations) corresponds to the massless string states (such as the graviton) which already exist in the spectrum prior to compactification. The remaining cases with non-zero $n_i$ and $w_i$ thus correspond to the usual Kaluza-Klein and winding excitations of these states, as well as the excitations of their Kaluza-Klein descendants [including the $U(1)$ graviphoton gauge fields which emerge from the metric and antisymmetric tensor via the original Kaluza-Klein mechanism]. It is important to note, however, that the spectrum of Kaluza-Klein and winding excitations in such sectors is restricted to those states with a common value of $\sum_i n_i w_i$ (typically zero since the lightest Kaluza-Klein states in such sectors have $\sum_i n_i w_i = 0$).

Finally, we must consider the sectors with $N + \overline{N} + a + \pi < 0$. In such sectors, the lightest Kaluza-Klein and winding excitations are tachyonic and must be GSO-projected out of the spectrum. Depending on the specific string model in question, there are two possible results of such GSO projections. First, it may happen that this entire sector of the string theory is GSO-projected out of the spectrum. In such cases, no further considerations are necessary. On the other hand, it may happen that these GSO projections affect only the lightest states, preserving states with $\alpha' \mathcal{M}_{\text{tot}}^2 > 0$. In such cases, then, we find that it is the multiply excited Kaluza-Klein/winding states in these sectors which are actually the lightest states which appear in the string spectrum!

As an explicit example of this phenomenon, let us consider the bosonic string (for which $a = \pi = -1$). If $(N, \overline{N}) = (1, 0)$ or $(0, 1)$, then $\alpha' \mathcal{M}_{\text{tot}}^2 = \alpha' \mathcal{M}_{n;w_i}^2 - 2$. Thus, Kaluza-Klein or winding states for which $\alpha' \mathcal{M}_{n;w_i}^2 = 2$ are actually massless. For example, if two dimensions in this theory are compactified on the so-called $SU(3)$ torus in Eq. (13), then the following twelve multiply excited Kaluza-Klein/winding states $(n_1, n_2; w_1, w_2)$ have $\alpha' \mathcal{M}_{n;w_i}^2 = 2$ and hence are massless:

$$\pm (1, 1; 1, 0), \pm (1, 0; 1, -1), \pm (0, 1; 0, 1)$$

$$\pm (1, 1; 0, -1), \pm (1, 0; -1, 0), \pm (0, 1; 1, -1). \quad (22)$$

Note that the first six states [top line of Eq. (22)] have $\sum_i n_i w_i = 1$, while the second six states [bottom line of Eq. (22)] have $\sum_i n_i w_i = -1$. Thus the first group of states can have $(N, \overline{N}) = (1, 0)$, while the second group can have $(N, \overline{N}) = (0, 1)$. Moreover, if these oscillator excitations correspond to excitations of the spacetime coordinates of uncompactified dimensions, then these states are spacetime vectors.

Clearly, massless vectors must correspond to gauge bosons. Indeed, it turns out that these twelve states combine with the four $U(1)$ graviphoton states which emerge from the Kaluza-Klein decomposition of the graviton and $B_{\mu\nu}$ field, thereby enhancing the total Kaluza-Klein gauge symmetry from $U(1)^4$ to $SU(3)_L \times SU(3)_R$. It is for this reason that the torus in Eq. (13) is called an $SU(3)$ torus. Likewise, a five-dimensional compactification on the five-dimensional torus of Eq. (23) produces eighteen multiply excited states with $\alpha' \mathcal{M}_{n;w_i}^2 = 2$ and $\sum_i n_i w_i = \pm 1$, thereby enhancing the total Kaluza-Klein gauge symmetry from $U(1)^{10}$ to $SO(10)_L \times SO(10)_R$.

Of course, this is nothing but the standard Narain mechanism by which one obtains non-abelian, simply laced, level-one affine gauge symmetries via toroidal compactifications in string theory: the Kaluza-Klein/winding quantum numbers become identified as the charges of a non-abelian gauge group $SU(1,1)$. (The analogous production of non-simply laced and higher-level affine gauge groups is discussed in Ref. (14).) Likewise, similar results hold for the superstring and the heterotic string.

These features indicate that the simple Kaluza-Klein effective field-theory picture becomes far richer, but also far more complex, when embedded within the full context of string theory. For example, if the $SU(3)$ gauge
symmetry of the strong interaction is realized in string theory as an enhanced gauge symmetry, as described above, then the massless gluons of the Standard Model are actually simultaneous combinations of Kaluza-Klein, winding, and oscillator modes! In other words, such gluons correspond to string states which simultaneously resonate in extra dimensions (Kaluza-Klein), wrap around those extra dimensions (winding), and also vibrate along their length with a certain frequency (oscillators). Likewise, the masses of the $W^\pm$ and $Z$ gauge bosons, ordinarily generated through the Higgs mechanism, can equivalently be generated by shifting various compactification radii away from the symmetric values which are needed to produce unbroken $SU(2) \times U(1)$ gauge symmetry.

Given these observations, we see that we must be very careful when interpreting our results within the context of string theory. As an example, let us again consider the case of the bosonic string with two dimensions compactified on the $SU(3)$ torus. We have already shown that such a toroidal compactification has $M^2/M^2_{\text{string}} = 4/3$. However, the presence of negative vacuum energies in the bosonic string requires that we must consider the higher Kaluza-Klein/winding excitations as well. It turns out that the complete spectrum of Kaluza-Klein/winding modes on this torus consists of

$$\alpha' M^2_{n_i;w_i} = 0, 4/3, 2, 10/3, 4, 16/3, \ldots$$

with values

$$\sum_i n_i w_i = 0, 0, \pm 1, \pm 1, 0, \{0, \pm 2\}, \ldots$$

respectively. [The states at $\alpha' M^2_{n_i;w_i} = 2$ are the gauge boson states listed in Eq. (22).] Likewise, there are only six sectors of the bosonic string which can possibly contain light states (defined as states with $\alpha' M^2_{\text{tot}} < 2$): these are the sectors with $(N, \overline{N}) = (0, 0), (1, 1), (0, 0), (2, 0)$, and $(0, 2)$. The allowed Kaluza-Klein/winding $(n_i; w_i)$ states in each of these sectors are those with $\sum_i n_i w_i = 0, 0, 1, -1, 2, -2$ respectively. Thus, proceeding sector by sector, we find that $(M_s/M_{\text{string}})^2 = 4/3$ in each sector. Thus, in each of these sectors, we conclude that all of the Kaluza-Klein and winding states are heavier than the string scale.

Note that the level-matching constraint is critical in reaching this conclusion. For example, the complete spectrum for a five-dimensional compactification on the $SO(10)$ torus in Eq. (23) is given by

$$\alpha' M^2_{n_i;w_i} = 0, 2, 5/2, 4, 9/2, 6, 13/2, \ldots$$

Clearly, our result would fail if a Kaluza-Klein/winding state with $\alpha' M^2_{n_i;w_i} = 2k + 1/2, k \in \mathbb{Z}$, were to emerge in a sector for which $\alpha' M^2_{\text{tot}} = \alpha' M^2_{n_i;w_i} - 2$, since this would yield a Kaluza-Klein/winding state with $M^2_{\text{tot}}/M^2_{\text{string}} = 1/2$. Such a state would then be lighter than the string scale. However, it is easy to show that the states with masses $\alpha' M^2_{n_i;w_i} = 2k + 1/2$ in Eq. (23) all have

$$\sum_i n_i w_i = \begin{cases} \text{even} & \text{if } k \text{ is odd} \\ \text{odd} & \text{if } k \text{ is even} \end{cases}$$

The level-matching constraint thus ensures that these states can only appear in sectors for which $\alpha' M^2_{\text{tot}} = \alpha' M^2_{n_i;w_i} + A$ where $A = -4, 0, 4, \ldots$. Thus, once again, we see that all of the Kaluza-Klein states and all of the winding states are heavier than the string scale.

Similar results also hold for the superstring. In this case, there are four sectors: the left- and right-moving worldsheet fermions can be either Neveu-Schwarz (for which $a = -1/2$) or Ramond (for which $a = 0$). Likewise, depending on this choice, the values of $N$ and $\overline{N}$ can be restricted either to integers (Ramond) or to integers and half-integers (Neveu-Schwarz). Working out the various combinations, we find that $M_{\text{tot}} = M_{n_i;w_i}$ in all relevant sectors which are capable of yielding light states. The only exception is the tachyonic NS/NS sector with $(N, \overline{N}) = (0, 0)$, but this sector is typically removed through the same GSO projection that introduces space-time supersymmetry. Thus, we again see that it is possible to choose compactification tori such that all Kaluza-Klein states and all winding states are heavier than the string scale. Such results also generalize to the heterotic string and (for Kaluza-Klein states) to the bulk sector of Type I strings as well.

We conclude this section with several important comments and caveats.

First, although our main result is that it is possible to cross the string scale without seeing a single Kaluza-Klein or winding-mode state, we stress that this statement refers only to those states which are beyond the massless level. Indeed, as we have seen, even at the massless level we generically have states which are composed of non-trivial combinations of Kaluza-Klein, winding, and oscillator excitations. Such states may be even be part of the Standard Model gauge group and matter content.

Second, we have not yet discussed the scale for the oscillator excitations. In general, the mass scale for the oscillator modes is set by $M_{\text{string}}$. However, we see from Eq. (21) that each oscillator excitation contributes two units to the net value of $\alpha' M^2_{\text{tot}}$. Thus, even the string oscillator modes are generically heavier than the string scale! Moreover, if we take the level-matching constraint into account, the oscillator mass gap becomes even greater. As an example, let us consider the oscillator excitations of the graviton. In the bosonic string, the graviton emerges as a state with $(N, \overline{N}) = (1, 1)$ and $n_i = w_i = 0$. However, we see from the level-matching constraint in Eq. (23) that it is impossible to excite a single additional oscillator mode, since this would require changing the value of $\sum_i n_i w_i$ by introducing simultaneous Kaluza-Klein/winding excitations. We therefore find...
that we need to excite oscillator modes in groups of two, keeping \( N = \mathcal{N} \). Thus, the first oscillator excitation of the graviton does not appear until \( \alpha' M^2_{\text{gal}} = 4 \).

Third, although we have given examples where our results hold in all string sectors, it is undoubtedly possible to construct string models where this fails to be the case. In such instances, the nature of the lightest string states may vary with the specific string sector in question. For example, it may happen that the lightest compactification states may exceed the string scale in one sector (e.g., for the gauge bosons), yet be below this scale in another sector (e.g., for quarks and leptons). This would then give rise to an interesting low-energy phenomenology in which different Standard Model particles have Kaluza-Klein/winding excitations of different masses, even though these excitations all correspond to the same extra dimensions with fixed radii!

Fourth, we emphasize that our discussion in this section has focused only on those generic features which are common to all string theories. As such, we have not focused on a particular class of string theories, nor have we focused on the compactification of all of the extra space-time dimensions predicted within a given class of string theories. In general, compactification of the full six (or seven) extra dimensions predicted by string theory will give rise to additional constraints beyond those which we have considered here. (For example, modular invariance requires that the total theory be compactified on tori for which the corresponding charge lattice is even and self-dual \( \mathbb{Z}^6 \), so that \( \alpha' M^2_{n_i \omega_i} \in 2\mathbb{Z} \) only.) In this paper, by contrast, we are only focusing on a subset of the full theory.

Fifth, we remark that our results hold for all values of \( M_{\text{string}} \). Thus, we have not focused on the various mechanisms \( \mathcal{E} \mathcal{E} \) by which the string scale might be reduced into the TeV range in a variety of different classes of string theories. Of course, one possibility is that this reduction might occur as a result of two or more extra dimensions taking values which are significantly larger than the string length \( \mathcal{E} \). Our results clearly do not apply for such extra dimensions. Instead, our focus here has been on those extra dimensions (so-called “TeV-sized extra dimensions”) whose sizes are relatively close to the fundamental string scale \( \mathcal{E} \mathcal{E} \). Indeed, it is only for these extra dimensions that our result applies, and for which it is possible to cross the string scale without seeing any corresponding Kaluza-Klein and/or winding modes.

Finally, we note that in this paper we have been discussing the properties of merely the tree-level string spectrum. We have therefore been identifying the Standard Model particles as massless string states, and focusing on light excitations thereof. Needless to say, field-theoretic effects such as electroweak symmetry breaking and supersymmetry breaking, and even string-theoretic effects such as vacuum restabilization, have the potential to shift the particle masses away from their tree-level values \( \mathcal{E} \). These effects can therefore be significant in cases where the string scale itself is in the TeV range.

V. CONCLUSIONS

In this paper, we have shown that there exist toroidal compactifications for which the lightest Kaluza-Klein states and the lightest winding states are both simultaneously heavier than the string scale. The key ingredient in these compactifications is the presence of non-trivial shape moduli and background antisymmetric tensor fields.

It is perhaps not surprising that non-trivial shape moduli have the potential to alter some of our naïve expectations concerning the masses of compactification states. In Ref. \( \mathcal{E} \), for example, it was shown that such moduli have the potential to render certain types of large extra dimensions invisible. Moreover, in Ref. \( \mathcal{E} \), it was shown that non-trivial shape moduli can trigger a so-called “shadowing” effect in which compactification geometry, much like other “constants” of nature, is effectively renormalized as a function of energy scale, with quantities such as compactification radii changing their apparent values as functions of the energy with which the compactification manifold is probed. Results such as these suggest that shape moduli have the potential to drastically change our naïve expectations based on studying simple compactifications in which shape moduli are ignored or held fixed.

It is also important to stress that such compactifications are not unusual in any way, and in fact are expected on rather general grounds. Even though we do not know the (presumably non-perturbative) dynamics which ultimately selects the preferred compactification, we expect that any effective potential which selects this preferred compactification should reflect the underlying symmetries of the torus and hence should be modular invariant. It then follows that the extrema of this potential should correspond to tori which sit at modular fixed points. However, tori such as the \( SU(3) \) and \( SO(10) \) tori discussed above correspond to highly symmetric modular fixed points. Indeed, as we have seen, these are exactly the sorts of tori which give rise to enhanced gauge symmetries in the full string theory. It is therefore reasonable to assume that tori such as these are preferred dynamically.

Needless to say, these results could have important phenomenological implications in theories involving TeV-sized extra dimensions \( \mathcal{E} \mathcal{E} \). Ordinarily, one might have assumed that the first experimental signals of such extra dimensions would be the appearance of Kaluza-Klein states, and that such Kaluza-Klein states (or their winding counterparts) must necessarily appear below the string scale. However, our results indicate that
this need not be the case: if certain extra dimensions are near the string scale, it is possible to cross the string scale without seeing a single corresponding state of either the Kaluza-Klein, winding, or oscillator variety! Thus, it is possible that the string scale may be lower than previously imagined.

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