Asynchronous Consensus Dynamics for Group of High-Order Agents Under Switching Topologies and Time-Varying Delays

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ABSTRACT This paper investigates the consensus problem of high-order multi-agent systems (MASs) under multiple communication constraints, including asynchronous communication, arbitrary switching topologies and time-varying delays. In the scenario of asynchronous communication, each agent only receives its neighbors’ states information at the time instants when it needs to adjust the state. For each agent, an distributed control protocol involving its own all-order state information and its neighbors’ first-order delayed state information is designed. With the help of state vector transformation technology, the high-order consensus problem under multiple communication constraints can be equivalently transformed into a convergence problem of products of nonnegative matrices. The related knowledge of nonnegative matrix theory is used to solve this convergence problem, and further establish a sufficient algebraic condition of achieving high-order consensus. Finally, the achievement of high-order consensus is demonstrated through a numerical example.

INDEX TERMS High-order dynamics, consensus multi-agent systems, asynchronous scenario.

I. INTRODUCTION

Since the 21st century, multi-agent distributed coordination has become an active research field because it has comprehensive applications, such as opinion dynamics of social networks [1], flocking control [2], tracking control [3]–[6] and formation control [7]. It is known that consensus is an important and fundamental topic of distributed coordination. In general, the consensus on control usually means designing a distributed update rule so that networked agents can reach consensus on a certain amount of interest.

Starting with the distributed algorithm in [8], the consensus problem for first-order MASs has been paid close attention in the existing literature [9]–[16]. In the past ten years, plenty of progress toward second-order consensus problem have been found in [17]–[23], [25]–[27], just to name a few. A distributed consensus strategy was proposed in [17] to implement second-order consensus for a class of agents with discrete-time dynamics. The consensus problem of second-order MASs with time-varying reference signal was considered in [18]. Tang et al. investigated the second-order consensus problem of MASs with nonuniform time-varying delays in [19]. The reference [20] established an algebraic criteria for second-order consensus of MASs with intrinsic nonlinear dynamics. The event-based second-order consensus problems were studied in [21]–[24]. In addition, the second-order consensus problems for sampled systems, stochastic systems, heterogeneous systems and adaptive systems were explored in the references [25]–[27], and [28], respectively.

Remarkably, although a great quantity of results with regard to first/second-order consensus of MASs have been reported, accelerations will inevitably have an impact on the dynamics of the agents in addition to positions and velocities in many practical applications. For instance, in the unmanned aerial vehicles (UAV) group, some UAVs that are not equipped with sensors suddenly change direction for avoiding danger when danger comes. Obviously, in this
case, the UAVs also need to change the accelerations in addition to changing the positions and speeds. In addition, as mentioned in [29], when the food or source of danger is perceived, the bird flock sometimes suddenly changes the direction of movement. There is no doubt that it is of practical significance to study the consensus for the high-order MASs. Jiang & Wang [30] explored the problem of continuous-time high-order consensus in the presence of fixed network topology and switching network topologies, respectively, and further analyzed the setting with multiple communication delays in [31]. The reference [32] discussed the consensus dynamics of agent group with high-order dynamics in the discrete-time setting. Under the situation that the communication delays are unknown, the authors of [33] examined the consensus of networked agents with heterogeneous high-order dynamics. In [34], the authors analyzed the consensus convergence speed for high-order MASs in a random network with arbitrary weights. Rezaee et al. discussed the high-order static average consensus under the undirected topology in [35] and analyzed the high-order consensus for nonlinear models with input-affine in [36]. The reference [37] investigated the high-order consensus via an adaptive control method.

All the above documents are concerned with a communication environment in which all agents take a common clock, namely, all agents receive their neighbors’ data information synchronously. When it comes to actual engineering scenarios, the clocks on different microprocessors are not synchronized in general. Further, this clock difference can seriously affect system performance and may even cause instability in the system if the microprocessors are time-driven. For all we know, the asynchronous consensus problem of groups composed of first/second-order dynamic agents has been deeply and comprehensively studied so far. In [38], Xiao et al. explored the asynchronous continuous-time consensus control for first-order dynamic agents, where all agents are required to be in a closed convex constraint set while reaching consensus. The stationary asynchronous second-order consensus under switching topologies was investigated in [39], where the velocities of all agents gradually converges to zeros. The asynchronous second-order consensus problem was studied in [40] via a sampled-control method. In addition, Shi et al. [41] examined the group consensus of heterogeneous MASs in which the agents take first-order or second-order dynamics. In recent years, some literatures have also studied the asynchronous consensus problem of second-order and heterogeneous multi-agent systems in the presence of one or multiple leader agents (e.g., see [42],[43]).

It’s remarkable that the existing works on asynchronous consensus so far mainly considered first-order or second-order dynamic agents, and few asynchronous consensus results for agents with high-order dynamics has been found. Inspired by this fact, our purpose is to investigate the asynchronous high-order consensus problem in the discrete-time setting in this paper. In addition, communication delays and switching topologies are usually considered in consensus research due to factors such as limited agent communication capabilities and external environmental interference. Therefore, we further consider the effect of time-varying delays and arbitrary switching topologies on consensus dynamics in this paper. The proposed model is closely related to the works concerning the asynchronous second-order dynamic model [39] and the synchronous high-order dynamic model [32]. The differences between our work and that in [32] and [39] can be summarized as follows. Compared with [32], we study the asynchronous case, which largely generalizes the synchronous case in [32]. And compared with [39], our focus is on the more extensive asynchronous high-order model under switching topologies. In a word, the asynchronous high-order dynamic model proposed in this paper includes the asynchronous second-order dynamic model [39] and the synchronous high-order dynamic model [32] as two special cases.

From the technical level, it is not easy to directly analyze the stability of the high-order system because of the time-varying nature of topology and the complexity of asynchronous environment. For this reason, we first convert the asynchronous high-order system into an augmented system with the time-varying coefficient matrices through the construction of mixed state vectors. Then, the effective parameters selection strategy is proposed to guarantee that the time-varying coefficient matrices in the augmented system are stochastic indecomposable and aperiodic (SIA). At last, the system stability is analyzed by the properties of SIA matrix.

The rest of this contribution is specifically organized as follows. Section II introduces basic concepts for digraph, and Section III shows the problem formulation. A sufficient criterion for asynchronous high-order consensus is presented in Section IV. Section V provides a simulation to show the achievement of asynchronous high-order consensus. The conclusions about this article are shown in Section VI.

II. PRELIMINARIES

A digraph is represented by \( \mathcal{G} = (\mathcal{E}, \mathcal{V}) \), which is consisting of a vertex set \( \mathcal{V} = \{v_1, v_2, \ldots, v_n\} \) and an edge set \( \mathcal{E} \). A directed edge from \( v_j \) to \( v_i \) is represented by \( (v_i, v_j) \). The neighbor set of vertex \( v_i \) is defined as

\[
\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}.
\]

Let \( \mathcal{A} \) represent the adjacency matrix of \( \mathcal{G} \). The relationship between adjacency matrix and edge set satisfies: \( a_{ij} > 0 \) if \( (v_j, v_i) \in \mathcal{E} \), and \( a_{ij} = 0 \) if \( (v_j, v_i) \notin \mathcal{E} \). The digraph \( \mathcal{G} \)'s in-degree matrix is defined as

\[
\mathcal{D} = \text{diag}\left[\sum_{j=1}^{n} a_{1j}, \sum_{j=1}^{n} a_{2j}, \ldots, \sum_{j=1}^{n} a_{nj}\right].
\]

The Laplacian matrix can be expressed as \( \mathcal{L} = \mathcal{D} - \mathcal{A} \). A path is an ordered sequence of different edges so that consecutive edges are joined together. A digraph contains a directed spanning tree if there exists a root vertex such that for any non-root vertex, a directed path from the root to the non-root vertex can be found.
For a real matrix $H = [h_{ij}] \in \mathbb{R}^{n \times n}$, $\text{diag}(H) = \text{diag}[h_{ii}, h_{22}, \ldots, h_{nn}]$ denotes a diagonal matrix in which the diagonal entries are $h_{11}, h_{22}, \ldots, h_{nn}$; $H(i, :) \in \mathbb{R}^{1 \times n}$ denotes the $i$th row of $H$; $|H| = |[h_{ij}]| \in \mathbb{R}^{n \times n}$ denotes a nonnegative matrix with each element $[h_{ij}]$ being the absolute value of the element $h_{ij}$ in $H$. A matrix $H \in \mathbb{R}^{n \times n}$ is called row-stochastic if it is nonnegative and $H1_n = 1_n$, where $1_n \in \mathbb{R}^{n \times 1}$ is an all 1’s column vector. Furthermore, the row-stochastic matrix $H$ is stochastic indecomposable and aperiodic (SIA) if $\lim_{k \to \infty} H^k = 1_n \gamma^T$, where $\gamma \in \mathbb{R}^{n \times 1}$. $\mathbb{N}$ stands for the set of natural numbers. Let $r! = r(r - 1)(r - 2) \cdots 1$ for any $r \in \mathbb{N}$, in particular, $0! = 1$. Let $\mathcal{V}[H] = \{\mathcal{V}[H], \mathcal{E}[H], H\}$ denote a digraph, where $H$ is the adjacency matrix and $\mathcal{V}[H] = \{1, 2, \ldots, n\}$ is the vertex set. For nonnegative matrices $H = [h_{ij}]_{n \times n}$ and $R = [r_{ij}]_{n \times n}$, we say that the digraphs $\mathcal{V}[H]$ and $\mathcal{V}[R]$ are the same type, denoted by $\mathcal{V}[H] \sim \mathcal{V}[R]$, if $h_{ij} > 0 \iff r_{ij} > 0$ and $h_{jj} = 0 \iff r_{jj} = 0$. $\otimes$ represents the Kronecker product. $I_n \in \mathbb{R}^{n \times n}$ denotes an identity matrix.

Lemma 1 [38]: Define the following two matrices:

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & A_h \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$N_0 = \begin{pmatrix} I_n & 0 & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & I_n & \cdots & 0 \end{pmatrix},$$

where $A_1, A_2, \ldots, A_h$ are $n \times n$ nonnegative matrices. The digraph $\mathcal{G}[N_k]$ has a spanning tree if the digraph $\mathcal{G}[\sum_{j=1}^{k} \Psi_j]$ has a spanning tree, where $N(k) = \Psi + N_0^k$ for any positive integer $k$.

Lemma 2 [38]: A row stochastic matrix $H \in \mathbb{R}^{n \times n}$ is SIA if digraph $\mathcal{G}[H]$ has a spanning tree with a self-loop on the root vertex.

Lemma 3 [44]: For a compact set $Q$ consisting of $r \times r$ SIA matrices, if $\prod_{i=1}^{k} Q_i$ is SIA for any nonnegative integer $k$ and any $Q_1, Q_2, \ldots, Q_k \in Q$ (repetitions permitted), then there exists a column vector $y \in \mathbb{R}^{r \times 1}$ such that $\lim_{k \to \infty} \prod_{i=1}^{k} Q_i = 1_y y^T$ for any infinite sequence $Q_1, Q_2, \ldots, Q_k, \ldots$ (repetitions permitted) of matrices from $Q$.

III. MODEL FORMULATION

A. SYNCHRONOUS CONTINUOUS-TIME MODEL

Consider a system with $n$ agents, labelled as $v_1, v_2, \ldots, v_n$. The information flows among all agents is depicted by a digraph $\mathcal{G}$. In the adjacency matrix, $a_{ij} > 0$ and $a_{ij} = 0$ indicate that agent $v_i$ can and cannot receive the information of agent $v_j$. The continuous-time dynamics of high-order dynamic agents are given by

$$\dot{x}_i^{(1)}(t) = \frac{x_i^{(2)}(t)}{u_i(t)},$$

$$\dot{x}_i^{(2)}(t) = \frac{x_i^{(3)}(t)}{u_i(t)},$$

$$\vdots$$

$$\dot{x}_i^{(p)}(t) = u_i(t),$$

where $x_i^{(s)}(t_k) \in \mathbb{R}$, $s = 1, 2, \ldots, p$ and $u_i(t_k)$ represent agent $v_i$’s states and control protocol. A widely-used control input designed in [30] is described as

$$u_i(t) = \beta_1 \sum_{q \in N(i)} a_{ij}(t) \left( x_i^{(1)}(t) - x_j^{(1)}(t) \right) - \sum_{s=2}^{p} \beta_x x_i^{(s)}(t),$$

where the dependence of $a_{ij}(t)$ on $t$ means that the topology is dynamically changing; $N_i(t)$ is the neighbor set of agent $v_i$; $\beta_x > 0$, $s = 1, 2, \ldots, p$ are gain parameters.

B. SYNCHRONOUS DISCRETE-TIME MODEL

Due to the unreliability of communication channels in most practical applications, information flows may not be transmitted continuously between agents due to the unreliability of communication channels. This makes us consider a discrete communication scenario where each agent only detects its neighbors’ information at discrete time instants $t_k$, $k \in \mathbb{N}$. The information interactions among agents is described by switching digraphs $\mathcal{G}(t_k) = (\mathcal{V}, \mathcal{E}(t_k), A(t_k))$. In the adjacency matrix $A(t_k) = [a_{ij}(t_k)]_{n \times n}$, we assume that all non-zero weight factors are uniformly lower and upper bounded, i.e. $a_{ij}(t_k) \in [\underline{a}, \overline{a}]$ whenever $a_{ij}(t_k) > 0$, where $0 < \underline{a} < \overline{a}$. The neighbor set of agent $v_i$ is denoted by $N_i(t_k) = \{v_j : (v_i, v_j) \in \mathcal{E}(t_k)\}$, and the Laplacian matrix is $L(t_k)$. Each agent $v_i$ is driven by the following dynamics:

$$\dot{x}_i^{(1)}(t_{k+1}) = x_i^{(1)}(t_k) + T x_i^{(2)}(t_k),$$

$$\dot{x}_i^{(2)}(t_{k+1}) = x_i^{(2)}(t_k) + T x_i^{(3)}(t_k),$$

$$\vdots$$

$$\dot{x}_i^{(p)}(t_{k+1}) = x_i^{(p)}(t_k) + Tu_i(t_k),$$

where $x_i^{(s)}(t_k) \in \mathbb{R}$, $s = 1, 2, \ldots, p$ are the states of agent $v_i$ at $t_k$; $T > 0$ is the constant step-size; and the consensus protocol $u_i(t_k)$ given in [32] is expressed as

$$u_i(t_k) = \beta_1 \sum_{q \in N_i(t_k)} a_{ij}(t_k) \left( x_i^{(1)}(t_k) - x_j^{(1)}(t_k) \right) - \sum_{s=2}^{p} \beta_x x_i^{(s)}(t_k),$$

in which we can see that all agents obtain information from their neighbors synchronously, and the step size of update process is time-invariant.
Unlike the synchronous environment, we consider in this paper an asynchronous environment, in which each agent only receives its neighbors’ states information at the instants when it needs to adjust the state. We assume that the time instants, at which each agent $v_i$ receives information from its neighbors’, are $\{t^i_k\} = \{t^i_0, t^i_1, \ldots, t^i_k, \ldots\}$, which is not necessarily evenly spaced. Let time sequence $\{t^i_k\}$ satisfy the following conditions.

1) There exist real numbers $\tilde{k}$ and $\hat{k}$ such that $\tilde{k} \leq t^i_{k+1} - t^i_k \leq \hat{k}$ for any $k \in \mathbb{N}$.

2) $0 \leq \tau_j(t^i_k) \leq S\tilde{k}$ holds for all $i, j, k$, where $S \in \mathbb{N}$ and $\tau_j(t^i_k)$ is the communication delay when agent $v_i$ receives the state information from $v_j$ at $t^i_k$.

Similar to [38], all the time sequences $\{t^i_k\}$, $i = 1, 2, \ldots, n$ are merged into a single ordered sequence $T = \{t_0, t_1, t_2, \ldots, t_k, \ldots\}$, where $t_k < t_{k+1}$, $k \in \mathbb{N}$. It implies that there exists at least one agent that receives its neighbors’ information at each $t_k$. And thus, there always exists $\omega \in \mathbb{N}$ such that $t^i_{\omega} \leq t_k < t_{k+1} \leq t^i_{\omega+1}$ for sequential times $t_k, t_{k+1}$ in $T$ and agent $v_i$. Let $T_k = t_{k+1} - t_k$, and thus, $T_k \in (0, \tilde{k}]$.

According to the above analysis about asynchronous setting, the asynchronous high-order dynamics is expressed as

$$
\begin{align*}
\dot{x}_i^1(t_{k+1}) & = x_i^1(t_k) + T_k x_i^2(t_k), \\
\dot{x}_i^2(t_{k+1}) & = x_i^2(t_k) + T_k x_i^3(t_k), \\
& \vdots \\
\dot{x}_i^p(t_{k+1}) & = x_i^p(t_k) + T_k u_i(t_k),
\end{align*}
$$

where

$$
u_i(t_k) = \beta_1 \sum_{v_j \in N(v_i)} a_{ij}(t_k) \left( x_i^1(t_{\omega}) - \tau_j(t_{\omega}) - x_j^1(t_k) \right) - \sum_{s=2}^p \beta_3 x_i^{(s)}(t_k).
$$

In addition, we let the initial values $x_i^1(t_0) = x_i^1(t_0)$ for all $t_0 \leq t_0$, where $i = 1, 2, \ldots, n$ and $s = 1, 2, \ldots, p$.

**Definition 1:** The asynchronous high-order consensus for system (5) with protocol (6) is said to be achieved if

$$
\begin{align*}
\lim_{t_k \to \infty} \left\| x_i^1(t_k) - x_j^1(t_k) \right\| & = 0, \\
\lim_{t_k \to \infty} \left\| x_i^{(s)}(t_k) \right\| & = 0
\end{align*}
$$

for any $s \in \{2, 3, \ldots, p\}$ and $i, j \in \{1, 2, \ldots, n\}$.

For transforming the asynchronous high-order system into an augmented system, the following lemma is presented.

**Lemma 4 [38]:** For any $i \in \{1, 2, \ldots, n\}$ and $k \in \mathbb{N}$, the element number of the set $\{t_k : t_k \in [t^i_k, \tau_i(t^i_k)]\}$ is no more than $\tilde{h} = (\lceil \frac{2}{\tilde{k}} \rceil)(n - 1) + 1$, where $\tilde{h}$ is the maximum integer not greater than $\frac{2}{\tilde{k}}$. Denote $\tilde{h} \leq (S + 1)\tilde{h}$, then the element number of $\{t_k \in [t^i_k, \tau_i(t^i_k), t^i_{k+1}]\}$ is no more than $\tilde{h}$.

**IV. MAIN RESULT**

The tool about the product of nonnegative matrices is utilized to solve the asynchronous high-order consensus problem in this section. Firstly, we let

$$
\begin{align*}
x_i(t_k) & = \left[ x_i^1(t_k), x_i^2(t_k), \ldots, x_i^p(t_k) \right]^T, \\
x_k(t_k) & = \left[ x_k^1(t_k), x_k^2(t_k), \ldots, x_k^p(t_k) \right]^T,
\end{align*}
$$

$$
\begin{align*}
E(t_k) & = \begin{pmatrix} 1 & T_k & 0 & \cdots & 0 & 0 \\
0 & 1 & T_k & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & T_k \\
0 & \beta_2 T_k & \beta_3 T_k & \cdots & -\beta_p T_k & T_k \end{pmatrix},
\end{align*}
$$

$$
F(t_k) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\beta_1 T_k & 0 & \cdots & 0 \end{pmatrix}.
$$

Then using protocol (6), the dynamics of agents is written as the following form

$$
x(t_{k+1}) = \left[ I_n \otimes E(t_k) - D(t_k) \otimes F(t_k) \right] x(t_k) - \sum_{y=1}^h \left[ A_y(t_k) \otimes F(t_k) \right] x(t_{k-y+1}),
$$

where the matrices $A_y(t_k), y = 1, 2, \ldots, h$ are nonnegative and satisfy $[A_y(t_k)]_{ij} = [A(t_k)]_{ij}$, $1 \leq k' \leq h$ and $[A_y(t_k)]_{ij} = 0, y \neq k'$ if $t_{\omega} - \tau_i(t_{\omega}) = t_k - k' + 1$. It can be easily observed that $\sum_{y=1}^h A_y(t_k) = A(t_k)$.

The asynchronous second-order consensus problem was studied in [39], where the dynamic model is written as

$$
x_i(t_{k+1}) = x_i(t_k) + T_k \dot{\theta}_i(t_k), \\
\dot{\theta}_i(t_{k+1}) = \theta_i(t_k) + T_k u_i(t_k),
$$

in which $x_i(t_k) \in \mathbb{R}, \theta_i(t_k) \in \mathbb{R}$ and

$$
u_i(t_k) = -\gamma \dot{\theta}_i(t_k) + \sum_{v_j \in N(v_i)} a_{ij}(t_k) \left( x_j(t_k) - x_i(t_k) \right)
$$

represent $v_i$’s position, velocity and protocol, respectively. In order to transform the asynchronous system into an augmented system with coefficient matrices being nonnegative, the authors used some technical model transformation methods. For example, let

$$
y_i(t_k) = \left[ x_i(t_k), x_i(t_k) + (2/\gamma) \nu_i(t_k) \right]^T, \\
y(t_k) = \left[ y_1^T(t_k), y_2^T(t_k), \ldots, y_n^T(t_k) \right]^T, \\
\varphi(t_k) = \left[ y^T(t_k), y^T(t_{k-1}), \ldots, y^T(t_{k-b+1}) \right]^T,
$$

then the asynchronous system is converted to the following form:

$$
\varphi(t_{k+1}) = \Phi(t_k) \varphi(t_k),
$$

where matrices $\Phi(t_k), k \in \mathbb{N}$ can be guaranteed to be nonnegative under appropriate gain parameter $\gamma$. 

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Similar to [39], the purpose of this paper is also to transform the asynchronous high-order multi-agent system into an equivalent augmented system with coefficient matrices being nonnegative. To achieve this goal, we use new model transformation methods. It is noteworthy that the model transformations in this paper are more complex than the literature [39] due to the existence of a large number of gain parameters in protocol (6), furthermore, the asynchronous second-order consensus in [39] is regarded as a special case of our paper. Let
\[
\begin{align*}
    z_i^{(1)}(t_k) &= \gamma x_i^{(1)}(t_k) \\
    z_i^{(2)}(t_k) &= \gamma^2 C_1 x_i^{(1)}(t_k) + \gamma^2 C_1 x_i^{(2)}(t_k) \\
    z_i^{(3)}(t_k) &= \gamma^3 C_2 x_i^{(1)}(t_k) + \gamma^2 C_2 x_i^{(2)}(t_k) + \gamma^3 C_2 x_i^{(3)}(t_k) \\
    &\vdots \\
    z_i^{(p)}(t_k) &= \gamma^p C_{p-1} x_i^{(1)}(t_k) + \gamma^2 C_{p-1} x_i^{(2)}(t_k) \\
    &\quad + \cdots + \gamma^p C_{p-1} x_i^{(p)}(t_k),
\end{align*}
\]
then \( z_i(t_k) = \left[z_i^{(1)}(t_k), z_i^{(2)}(t_k), \ldots, z_i^{(p)}(t_k)\right]^T \) can be written as \( z_i(t_k) = \Theta x_i(t_k) \), where
\[
\Theta = \begin{pmatrix}
    \gamma & 0 & 0 & \cdots & 0 \\
    \gamma C_0 & \gamma^2 C_1 & 0 & \cdots & 0 \\
    \gamma C_2 & \gamma^2 C_2 & \gamma^3 C_2 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \gamma^p C_{p-1} & \gamma^2 C_{p-1} & \gamma^3 C_{p-1} & \cdots & \gamma^p C_{p-1}
\end{pmatrix},
\]
where \( C_n^m = \frac{n!}{m!(n-m)!}, \ n \geq m, \) and \( \gamma > 0 \) is a constant to be determined. Furthermore, let \( z(t_k) = \left[z_1^T(t_k), z_2^T(t_k), \ldots, z_n^T(t_k)\right]^T \), then system (7) is transformed as
\[
z(t_{k+1}) = \left[I_n \otimes G(t_k) - D(t_k) \otimes H(t_k)\right]z(t_k) - \sum_{y=1}^{h} [A_y(t_k) \otimes H(t_k)]z(t_{k-y+1}), \tag{8}
\]
where \( H(t_k) = \Theta F(t_k) \Theta^{-1} = \gamma^{p-1} F(t_k) \) and
\[
G(t_k) = \begin{pmatrix}
    1 - T_k & T_k & 0 & \cdots & 0 & 0 \\
    0 & 1 - T_k & T_k & \cdots & 0 & 0 \\
    0 & 0 & 1 - T_k & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & 1 - T_k & T_k \\
    g_1(t_k) & g_2(t_k) & g_3(t_k) & \cdots & g_{p-1}(t_k) 1 + g_p(t_k)
\end{pmatrix},
\]
in which \( g_i(t_k), \ i = 1, 2, \ldots, p, \) are constants which can be used to indicate parameters \( \beta_1, \beta_2, \ldots, \beta_p, \) such as
\[
\begin{align*}
    \beta_1 &= \gamma^p C_{p-1} \\
    &\quad - \frac{\gamma^p}{T_k} \left[g_2(t_k) C_1^1 + g_3(t_k) C_2^1 + \cdots + g_p(t_k) C_{p-1}^1\right], \\
    \beta_2 &= \gamma^3 C_{p-1} \\
    &\quad - \frac{\gamma^3}{T_k} \left[g_3(t_k) C_2^2 + g_4(t_k) C_3^2 + \cdots + g_p(t_k) C_{p-1}^2\right], \\
    \beta_p &= \gamma^p C_{p-1} \\
    &\quad - \frac{\gamma^p}{T_k} \left[g_p(t_k) C_{p-1}^p\right],
\end{align*}
\]
Let \( \xi(t_k) = \left[z^T(t_k), z^T(t_{k-1}), \ldots, z^T(t_{k-h+1})\right]^T \), then system (8) can be rewritten as an augmented system with the following matrix-vector form:
\[
\Xi(t_{k+1}) = \Xi(t_k) \xi(t_k), \tag{10}
\]
where
\[
\Xi(t_k) = \begin{pmatrix}
    I_n \otimes G(t_k) - D(t_k) \otimes H(t_k) & A_1(t_k) \otimes H(t_k) \\
    I_{np} & 0 \\
    \vdots & \vdots \\
    \vdots & 0 \\
    \vdots & 0 \\
    \vdots & \vdots \\
    0 & I_{np} \\
\end{pmatrix}
\]
\[
\Xi(t_k) = \begin{pmatrix}
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
    \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
    \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
    \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
    \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
    \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
Lemma 5: \( \Xi(t_k) \) is a row-stochastic matrix in which the block \( I_n \otimes G(t_k) - D(t_k) \otimes H(t_k) \) contains positive diagonal elements if
\[
\gamma > \hat{k}, \quad g_{1,\min} > \gamma^{p-1} \hat{k} \Delta, \quad g_{1,\min} > 0, \ i = 2, 3, \ldots, p-1, \quad 1 + g_{p,\min} > 0, \quad g_{1,\min} + g_{2,\min} + \cdots + g_{p,\min} = 0, \tag{11}
\]
where \( g_{i,\min} = \min_{k \in \mathbb{N}} |g_i(t_k)|, \ i = 1, 2, \ldots, p, \) and \( \Delta = (n-1)\hat{\alpha}. \)

Proof: When the conditions in (11) hold, we get that the matrix \( I_n \otimes G(t_k) - D(t_k) \otimes H(t_k) \) is nonnegative and takes positive diagonal elements. Furthermore, since the matrix \( A_1(t_k) \otimes H(t_k) \) is nonnegative, we have that \( I_{n-m} \otimes G(t_k) - \left[D_J(k) - A_{g_{\{0\}}(k)}\right] \otimes H(t_k) \) is a nonnegative matrix with positive diagonal elements.

The following part is the proof for the conclusion that \( \Xi(t_k) \) is a row-stochastic matrix.

It is known that \( G(t_k) = \Theta E(t_k) \Theta^{-1} \), thus
\[
\begin{align*}
    \left[g_1(t_k), g_2(t_k), \ldots, g_{p-1}(t_k), 1 + g_p(t_k)\right] \\
    &\quad = \left[\gamma^p C_{p-1}^1, \gamma^2 C_{p-1}^1, \cdots, \gamma^p C_{p-1}^1\right] \left[(E(t_k) - I) + I\right] \Theta^{-1} \\
    &\quad = \left[\gamma^p C_{p-1}^1, \gamma^2 C_{p-1}^1, \cdots, \gamma^p C_{p-1}^1\right] (E(t_k) - I) \Theta^{-1} \\
    &\quad = [0, \gamma^p C_{p-1}^1 - \beta_1 T_k, \gamma^p C_{p-1}^1 - \beta_2 T_k, \cdots, \gamma^p C_{p-1}^1 - \beta_p T_k, (1 + g_p(t_k))] \Theta^{-1} \\
    &\quad = [0, \gamma^p C_{p-1}^1 - \beta_1 T_k, \gamma^p C_{p-1}^1 - \beta_2 T_k, \cdots, \gamma^p C_{p-1}^1 - \beta_p T_k, (1 + g_p(t_k))] \Theta^{-1} \\
\end{align*}
\]
\[
\begin{align*}
    \beta_1 &= \gamma^p C_{p-1}^1 \\
    &\quad - \frac{\gamma^p}{T_k} \left[g_2(t_k) C_1^1 + g_3(t_k) C_2^1 + \cdots + g_p(t_k) C_{p-1}^1\right], \\
    \beta_2 &= \gamma^3 C_{p-1}^1 \\
    &\quad - \frac{\gamma^3}{T_k} \left[g_3(t_k) C_2^2 + g_4(t_k) C_3^2 + \cdots + g_p(t_k) C_{p-1}^2\right], \\
    \beta_p &= \gamma^p C_{p-1}^1 \\
    &\quad - \frac{\gamma^p}{T_k} \left[g_p(t_k) C_{p-1}^p\right],
\end{align*}
\]
Let

$$\Theta_1 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
C_0^1 & C_1^1 & 0 & \cdots & 0 \\
C_0^2 & C_1^2 & C_2^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_0^{p-1} & C_1^{p-1} & C_2^{p-1} & \cdots & C_1^{p-1}
\end{pmatrix},$$

$$\Gamma = \begin{pmatrix}
\gamma_1 & 0 & 0 & \cdots & 0 \\
0 & \gamma_1 & 0 & \cdots & 0 \\
0 & 0 & \gamma_1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & \gamma_1
\end{pmatrix},$$

then we have $\Theta = \Theta_1 \Gamma$. Through observation, we find that the first column of $\Theta_1$ is $1_p$. As a result, the row sums of $\Theta_1^{-1}$ except the first row are all $0$. The fact means

$$[g_1(t_k), g_2(t_k), \ldots, g_{p-1}(t_k), g_p(t_k)]1_p
= [0, (\gamma C_0^0 - \beta_1)T_k, \ldots, (\gamma^p C_0^{p-2} - \beta_{p-1})T_k] \Theta_1^{-1}
= [0, (\gamma C_0^0 - \beta_1)T_k, \ldots, (\gamma^p C_0^{p-2} - \beta_{p-1})T_k] \Gamma^{-1} \Theta_1^{-1}
= 0.$$

Furthermore, $[g_1(t_k), g_2(t_k), \ldots, g_{p-1}(t_k), 1 + g_p(t_k)]1_p = 1$. It thus follows that $I_n \otimes G(t_k)1_{pn} = 1_{pn}$. Besides, we can also obtain that $[-D(t_k) \otimes H(t_k) + \sum_{j=1}^{h} A_{j\cup i}(t_k) \otimes H(t_k) + \sum_{j=1}^{h} A_{j}(t_k) \otimes H(t_k)]1_{pn} = 0$. Therefore $\Xi(t_k)$ is a row-stochastic matrix.

In Lemma 5, the inequalities have been given to ensure that the coefficients matrices of system (10) are nonnegativity by selecting appropriate constants $g_i, i = 1, 2, \ldots, p$ satisfying (11). Then according to (9) we can find a parameters selection strategy:

$$\beta_1 = \gamma C_0^0 - \frac{\gamma^2}{T_{\min}} \sum_{i=1}^{p} g_i C_1^{p-1},$$

$$\beta_2 = \gamma^2 C_1^0 - \frac{\gamma^3}{T_{\min}} \sum_{i=1}^{p} g_i C_2^{p-1},$$

$$\beta_3 = \gamma^3 C_2^0 - \frac{\gamma^4}{T_{\min}} \sum_{i=1}^{p} g_i C_3^{p-1},$$

$$\cdots$$

$$\beta_{p-1} = \gamma^p C_{p-2}^0 - \frac{\gamma^p}{T_{\min}} \sum_{i=1}^{p} g_i C_{p-1}^{p-1}. \quad (13)$$

Clearly, appropriate gain parameters $\beta_1, \beta_2, \ldots, \beta_{p-1}$ can be always found to ensure the nonnegativity of $\Xi(t_k)$ for any given $g_i, i = 1, 2, \ldots, p$ satisfying the conditions in (11). Before proceeding, the following conditions related to interaction topologies are required to introduced.

**Lemma 6:** Suppose that the conditions in (11) are satisfied, and all time instants in $T$ are divided into infinite number of uniformly bounded time intervals $[t_j, t_{j+1}), j = 1, 2, \ldots, \text{starting at } k_1 = 0$. Then one can derive the following results: (1) If the union digraph $G(t_k) \cup G(t_{k+1}) \cup \cdots \cup G(t_{k+i-1})$ associated with each interval $[t_j, t_{j+1})$ contains a spanning tree, then the digraph $\prod_{t_{j+1} \in I_{t_j}} G_{\Xi(t_k)}$ contains a spanning tree. (2) The matrix $\prod_{t_{j+1} \in I_{t_j}} G_{\Xi(t_k)}$ is SIA.

**Proof:** (1) Since the union digraph $G(t_k) \cup G(t_{k+1}) \cup \cdots \cup G(t_{k+i-1})$ has a spanning tree, then the union digraph $G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_{i-1}$ contains a spanning tree due to the fact that $\sum_{j=1}^{h} A_{j}(t_k) = A(t_k)$ for any $k \in \mathbb{N}$. It follows that the digraph $\prod_{t_{j+1} \in I_{t_j}} G_{\Xi(t_k)}$ contains a spanning tree. Define a new matrix $\Xi'(t_k)$ (see (14) at the bottom of the next page), then $\Xi(t_k) \geq \Xi'(t_k)$. Let $N_0 = \begin{pmatrix} I_{np} & 0 & \cdots & 0 \\
I_{np} & 0 & \cdots & 0 \\
0 & I_{np} & \cdots & 0 \\
0 & 0 & \cdots & I_{np}
\end{pmatrix}$ and

$$D(t_k) = \begin{pmatrix}
A_1(t_k) \otimes H(t_k) & \cdots & A_h(t_k) \otimes H(t_k)
\end{pmatrix}.$$
spanning tree with \( ip \) being the root vertex. Furthermore, \( G[N(t_0)] \) has a spanning tree with root vertex \( ip \) because of \( \prod_{i=k}^{k+1} N(t_i) \). Based on Lemma 7, we also know that the SIA matrix if \( \prod_{i=k}^{k+1} N(t_i) \) has a spanning tree with root vertex \( ip \) as the root agent. Moreover, we get from Lemma 5 that there exist positive diagonal elements in \( I_p \otimes G(t_k) - D(t_k) \otimes F(t_k) \) under condition (10), which means that the root vertex \( ip \) has self-loop in \( G[N(t_k)] \). From Lemma 2, we know that \( \prod_{i=k}^{k+1} G[N(t_i)] \) is SIA.

**Remark 1:** Since \( \{T_k : T_k \in (0, k] \} \) is not a compact set, the set \( \Gamma \) consisting of all possible matrices \( G(t_k) \) is not compact. In the following, it is required to prove the compactness of \( \widetilde{\Gamma} \) since all \( \widehat{\mathcal{X}}(t_k) \), \( k \in \mathbb{N} \) must belong to a compact set for guaranteeing the establishment of \( \lim_{k \to \infty} \prod_{i=0}^{k} G(t_i) = I_{np}y^T \), where \( y \) is a column vector.

**Lemma 7:** \( \widetilde{\Gamma} \) is a compact set under the conditions in (11). Proof: According to Lemma 6, one knows that \( \mathcal{X}(t_k) \) is SIA if the conditions in (11) hold. Denote

\[
\mathcal{S}(t_k) = I_p \otimes G(t_k) - D(t_k) \otimes H(t_k) + \sum_{j=1}^{h} \mathcal{A}_j(t_k) \otimes H(t_k) = I_p \otimes G(t_k) - \mathcal{L}(t_k) \otimes H(t_k).
\]

Let \( \mathcal{H} = \{ \mathcal{S}(t_k) : T_k \in [0, k], \forall k \in \mathbb{N} \} \). It is clear that \( T_k \in [0, k], k \in \mathbb{N} \) is compact. And matrix \( \mathcal{L}(t_k) \) is compact because \( a_{ij}(t_k) \in [\alpha, \bar{\alpha}] \) whenever \( a_{ij}(t_k) > 0 \), where \( 0 < \alpha < \bar{\alpha} \). Therefore, the set \( \mathcal{H}(t_k) \) is a compact set. For each matrix \( \mathcal{X}(t_k) \), we have \( I_p \otimes G(t_k) - D(t_k) \otimes H(t_k) + \mathcal{A}_1(t_k) \otimes H(t_k), \ldots, \mathcal{A}_h(t_k) \otimes H(t_k) \in \Lambda(\mathcal{H}) \), where \( \Lambda(\mathcal{H}) = \{ \mathcal{S}(t_k) \in \mathcal{H} : [\mathcal{S}(t_k)]_{ij} = [\mathcal{S}(t_k)]_{ij}, \mathcal{S}(t_k)]_{ij}, \text{ or } [\mathcal{S}(t_k)]_{ij}, \mathcal{S}(t_k)]_{ij} = 0, i, j = 1, 2, \ldots, np \} \) by observing that all the possible choices of \( I_p \otimes G(t_k) - D(t_k) \otimes H(t_k) + \mathcal{A}_1(t_k) \otimes H(t_k), \ldots, \mathcal{A}_h(t_k) \otimes H(t_k) \) are finite for any \( \mathcal{S}(t_k) \in \mathcal{H} \), we know that \( \mathcal{H} \) is a compact set. As a result, \( \widetilde{\Gamma} \) is a compact set.

**Theorem 1:** There is a series of uniformly bounded time intervals \( [t_j, t_{j+1}], j = 1, 2, \ldots, \) starting at \( k = 0 \). The asynchronous high-order consensus problem for system (5) under protocol (6) is solvable if the conditions in (11) hold and the union digraph \( \mathcal{G}(t_k) \cup \mathcal{G}(t_{k+1}) \cup \cdots \cup \mathcal{G}(t_{k+1}) \) has a spanning tree associated with each time interval \( [t_j, t_{j+1}] \).

**Proof:** Known from Lemma 6, \( G[N(t_k)] \) is an SIA matrix if \( \mathcal{G}(t_k) \cup \mathcal{G}(t_{k+1}) \cup \cdots \cup \mathcal{G}(t_{k+1}) \) has a spanning tree. Based on Lemma 7, we also know that the set \( \Gamma \) is compact. Therefore, we can obtain by Lemma 3 that \( \prod_{i=0}^{k} G(t_i) = I_{np}y^T \), where \( y \in \mathbb{R}^{np \times 1} \).

The rest can be analyzed by a proof similar to Theorem 2 in [45]. For each \( t_i > t_0 \), set \( k_i \) represent the largest integer that satisfies \( k_i \leq y \). Note that matrix \( \prod_{i=k_i}^{k} G(t_i) \) is row-stochastic, and then we have \( \prod_{i=0}^{k} G(t_i) = \lim_{k \to \infty} \prod_{i=k_i}^{k} G(t_i) \prod_{i=0}^{k_i} G(t_i) \). Since all \( \mathcal{X}(t_k) \in \Gamma \) for any \( t_k \), the row sums of matrix products \( \prod_{i=k}^{k+1} G(t_i) \) are bounded. Therefore, we have \( \prod_{i=k}^{k+1} G(t_i) = I_{np}y^T \). Combining this with the fact that \( \xi (t_{k+1}) = \mathcal{X}(t_k) \xi (t_k) = \prod_{i=k_i}^{k} G(t_i) \xi (t_i) \) yields \( \lim_{k \to \infty} \xi (t_k) = y^T \xi (t_0) \), which implies that \( \lim_{k \to \infty} x_{kp}^k (t_k) = c, \) where \( c \in \mathbb{R} \) is a constant, and \( \lim_{k \to \infty} x_{kp}^k (t_k) = 0, s = 2, 3, \ldots, p \). Known from Definition 1, we know that the asynchronous high-order consensus is solvable.

**Remark 2:** In the existing some literatures on the high-order consensus problem [30]–[37], there is no state vector conversion similar to the one used in this paper, so the coefficient matrices of the system contain both positive and negative elements. This means that it is necessary to analyze the convergence of products of matrices with both positive and negative elements for realizing consensus. In fact, it is not easy to study such a convergence problem because of the existence of negative elements in matrices, and there is no effective way to solve it so far. Unlike these literatures, the asynchronous high-order consistency problem is equivalently transformed into a convergence problem of products of row-stochastic matrices by designing novel state transformation vectors in this paper. As a result, the existing

\[
\mathcal{X}'(t_k) = \begin{pmatrix}
\text{diag}(I_p \otimes G(t_k)) - [D(t_k) - A_1(t_k)] \otimes H(t_k) & A_2(t_k) \otimes H(t_k) & \cdots & A_{h-1}(t_k) \otimes H(t_k) & A_h(t_k) \otimes H(t_k)
\end{pmatrix} = \begin{pmatrix}
I_{np} & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & I_{np} \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & I_{np} & 0 & 0
\end{pmatrix}, \quad (14)
\]
results about row-stochastic matrix can be used to solve the asynchronous high-order consensus problem.

V. SIMULATION RESULTS

Consider a third-order system with 5 agents. All asynchronous time series \( t^i_k \), \( i = 1, 2, \ldots, n \) are randomly selected under the constraint that \( \hat{\gamma} = 0.15 \) and \( \hat{\delta} = 0.3 \), which guarantees \( T_{\min} = \min_k \{ T_k \} = 0.1 \). We merge all the time sequences \( t^i_k \), \( i = 1, 2, \ldots, n \) into a single ordered sequence \( t_0, t_1, t_2, \ldots, t_k, \ldots \). Choose the time delays as \( \tau_{21}(t^i_k) = \tau_{42}(t^i_k) = 0.1, \tau_{31}(t^i_k) = \tau_{53}(t^i_k) = \tau_{14}(t^i_k) = 0.2 \) and \( \tau_{32}(t^i_k) = \tau_{54}(t^i_k) = \tau_{55}(t^i_k) = 0.3 \). All time delays satisfy \( \tau_{ij}(t^i_k) \leq \hat{S} \hat{\delta} \), where \( \hat{S} = 2 \). All possible interaction topologies are illustrated in Figure 1. Suppose the interaction topologies in each time interval \( [t_{4k}, t_{4(k+1)}] \) obey the following rule: from \( G_a \) to \( G_b \), \( G_b \) to \( G_c \), \( G_c \) to \( G_d \), and then, \( G_a \) to \( G_d \). Thus, the union of digraphs in \( [t_{4k}, t_{4(k+1)}] \) has a spanning tree. To satisfy the inequalities in (11), we take \( \gamma = 1, \beta_{1, \min} = 0.6, \beta_{2, \min} = 0.2 \) and \( \beta_{3, \min} = -0.8 \). Then we get from (13) that \( \beta_1 = 15 \) and \( \beta_2 = 10 \). The agents’ state trajectories of different orders are plotted in Figure 2, Figure 3 and Figure 4, respectively, from which we observe that all agents gradually reach the consensus state. That is to say, the asynchronous high-order consensus is achieved asymptotically.

VI. CONCLUSION

This dissertation has investigated the consensus for group of agents with high-order dynamics under multiple communication constraints, such as asynchronous communication, time-varying delays and arbitrary switching topologies. By combining all asynchronous time sequences into a single ordered sequence, the asynchronous high-order system has been transformed into a synchronous augmented system. Then the convergence of the augmented system has been discussed by employing the tools of infinite products of nonnegative matrices. Accordingly a sufficient condition has been obtained. Finally a numerical example has been provided to confirm the correctness of the theoretical result.

It is worth noting that the asynchronous high-order consensus problem in an ideal environment is considered in this paper, in which the information transmission between agents is not disturbed by external noise. In reality, external disturbance is common in MASs due to various uncertainties such as model mismatch, channel noise, measurement error, etc. Therefore, our future work is mainly to solve the asynchronous high-level consensus problem under the external disturbance environment.

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