N-sided Radial Schramm-Loewner Evolution

Vivian Olsiewski Healey* and Gregory F. Lawler†

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Abstract

We use the interpretation of the Schramm-Loewner evolution as a limit of path measures tilted by a loop term in order to motivate the definition of $n$-radial SLE going to a particular point. In order to justify the definition we prove that the measure obtained by an appropriately normalized loop term on $n$-tuples of paths has a limit. The limit measure can be described as $n$ paths moving by the Loewner equation with a driving term of Dyson Brownian motion. While the limit process has been considered before, this paper shows why it naturally arises as a limit of configurational measures obtained from loop measures.

1 Introduction

Multiple Schramm-Loewner evolution has been studied by a number of authors including [KL07], [Dub07], [JL18a], [PW19], [BPW] (chordal) and [Zhaa], [Zhab] (2-sided radial). For $\kappa \leq 4$, domain $D$, and $n$-tuples $x$ and $y$ of boundary points, multiple chordal SLE from $x$ to $y$ in $D$ is defined as the measure absolutely continuous with the $n$-fold product measure of chordal SLE in $D$ with Radon-Nikodym derivative

$$ Y(\gamma) = I(\gamma) \exp \left\{ \frac{c}{2} \sum_{j=2}^{n} m[K_j(\gamma)] \right\}, \quad (1) $$

where $I(\gamma)$ is the indicator function of

$$ \{ \gamma^j \cap \gamma^k = \emptyset, \ 1 \leq j < k \leq n \}, $$

and $m[K_j(\gamma)]$ is the Brownian loop measure of loops that intersect at least $j$ paths (see, e.g., [JL18a] for this result; see [LSW04] for the construction of Brownian loop measure). We would like to define multiple radial SLE by direct analogy with the chordal case, but this is not possible for two reasons. First, in the radial case the event

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\[ I(\gamma) \text{ would have measure } 0, \text{ and second, the Brownian loop measure } m[K_j(\gamma)] \text{ would be infinite, since all paths approach } 0. \] Instead, the method will be to construct a measure on \( n \) paths that is absolutely continuous with respect to the product measure on \( n \) independent radial SLE curves with Radon-Nikodym derivative analogous to \([1]\) but for both \( I(\gamma) \) and \( m[K_j(\gamma)] \) depending only on the truncations of the curves at a large time \( T \). Taking \( T \) to infinity then gives the definition of multiple radial SLE. The precise details of this construction, the effect on the driving functions, and the rate of convergence of the partition function are the main concern of this work.

Schramm-Loewner evolution, originally introduced in [Sch00], is a distribution on a curve in a domain \( D \subset \mathbb{C} \) from a boundary point to either another boundary point (chordal SLE) or an interior point (radial SLE). In both the chordal and radial cases, there are various ways to define SLE measure. Schramm’s original observation was that any probability measure on curves satisfying conformal invariance and the domain Markov property can be described in the upper half plane or the disc using the Loewner differential equation. More precisely, after a suitable time change, it is the measure on parameterized curves \( \gamma \) such that for each \( t \in [0, \infty) \), \( D = g_t(D \setminus \gamma[0,t]) \), where \( g_t \) solves the Loewner equation:

\[
\begin{align*}
\text{Chordal: } & \quad \dot{g}_t(z) = \frac{a}{g_t(z) - B_t}, \quad g_0(z) = z, \\
\text{Radial: } & \quad \dot{g}_t(w) = 2ag_t(w)\frac{z_t + g_t(w)}{z_t - g_t(z)}, \quad g_0(z) = z,
\end{align*}
\]

where \( a = 2/\kappa \), \( B_t \) is a standard Brownian motion, and \( z_t = e^{2iB_t} \). However, this dynamical interpretation is somewhat artificial in the sense that the curves typically arise from limits of models in equilibrium physics and are not “created” dynamically using this equation. Indeed, the dynamic interpretation is just a way of describing conditional distributions given certain amounts of information. When studying SLE, one goes back and forth between such dynamical interpretations and configurational or “global” descriptions of the curve.

One aspect of the global perspective is that radial SLE measure in different domains may be compared by also considering the partition function \( \Psi_D(z, w) \), which assigns a total mass to the set of SLE curves from \( z \) to \( w \) in the domain \( D \). It is defined as the function with normalization \( \Psi_D(1, 0) = 1 \) satisfying conformal covariance:

\[
\Psi_D(z, w) = |f'(z)|^b |f'(w)|^{\tilde{b}} \Psi_{D'}(z', w'),
\] (2)

where \( f(D) = D', \ f(z) = z', \ f(w) = w' \), and

\[
\begin{align*}
b = \frac{6 - \kappa}{2\kappa} = \frac{3a - 1}{2}, \\
\tilde{b} = b \frac{\kappa - 2}{4} = b \frac{1 - a}{2a}
\end{align*}
\]

are the boundary and interior scaling exponents. (This definition requires sufficient smoothness of the boundary near \( z \).) Another convention defines the partition function with an additional term for the determinant of the Laplacian, however, the benefit of our convention is that value of the partition function is equal to the total mass.
Considering SLE as a measure with total mass allows for direct comparison between SLE measure in $D$ with SLE measure in a smaller domain $D' \subset D$. This comparison is called either boundary perturbation or the restriction property, and is stated precisely in Proposition 4 [JL18a].

Multiple chordal SLE was first considered in [BBK05, Dub07, KL07]. Dubédat [Dub07] shows that two (or more) SLEs commute only if a system of differential equations is satisfied, and the construction holds until the curves intersect. Using this framework, the uniqueness of global multiple SLE is shown in [KPI6, PW19] and [BPW]. In these works, the term local SLE is used to refer to solutions to the Loewner equation up to a stopping time, while global SLE refers to the measure on entire paths.

This work builds on the approach of [KL07], which relies on the loop interpretation to give a global definition for $0 < \kappa \leq 4$. However, because we have to take limits, we will need to use both global and dynamical expressions. The dynamical description relies on computations concerning the radial Bessel process (Dyson Brownian motion on the circle) and go back to [Car03], and hold in the more general setting of $\kappa < 8$.

Our main result is the following. Let $n$ be a positive integer and $\gamma = (\gamma^1, \ldots, \gamma^n)$ an $n$-tuple of curves from $z^j_0 \in \partial \mathbb{D}$ to 0 with driving functions $z^j_t = e^{2i\theta^j_t}$. We will assume that the curves are parameterized using the $a$-common parameterization, which is defined in [3.2]. Let $\mathbb{P}$ denote the $n$-fold product measure on independent radial SLE curves from $\gamma^j(0)$ to 0 in $\mathbb{D}$ with this parameterization. (See Figure 1.) Let $L^j_t = L^j_t(\gamma_t)$ be the set of loops $\ell$ that hit the curve $\gamma^j$ and at least one initial segment $\gamma^k_t$ for $k = 1, \ldots, n$, $k \neq j$ but do not hit $\gamma^j$ first. (See the lefthand side of Figure 2.) Here we are measuring the “time” on the curves $\gamma$ and not on the loops. Define

$$L_t = I_t \exp \left\{ \frac{c}{2} \sum_{j=1}^n m_\mathbb{D}(L^j_t) \right\}$$

where $I_t$ is the indicator function that $\gamma^j_t \cap \gamma^k_t = \emptyset$ for $j \leq k$, and $m_\mathbb{D}$ is the Brownian loop measure.
Theorem 3.12. Suppose $0 < \kappa \leq 4$ and $t > 0$. For each $T > t$, let $\mu_T = \mu_{T,t}$ denote the measure whose Radon-Nikodym derivative with respect to $\mathbb{P}$ is

$$\frac{\mathcal{L}_T}{\mathbb{E}^{\theta_0}[\mathcal{L}_T]}.$$}

Then as $T \to \infty$, the measure $\mu_{T,t}$, viewed as a measure on curves stopped at time $t$, approaches a probability measure with respect to the variation distance.

Moreover, the measures are consistent and give a probability measure on curves $\{\gamma(t) : t \geq 0\}$. This measure can be described as the solution to the $n$-point Loewner equation with driving functions $z^j_t = e^{2i\theta^j_t}$ satisfying

$$d\theta^j_t = 2a \sum_{k \neq j} \cot(\theta^j_t - \theta^k_t) dt + dW^j_t,$$

where $W^j_t$ are independent standard Brownian motions.

A key step in the proof is Theorem 3.7 which gives exponential convergence of a particular partition function for $n$-radial Brownian motion. This theorem is valid for $0 < \kappa < 8$, but only in the $\kappa \leq 4$ case can we apply this to our model and give a corollary that we now describe. Let $X = X_n$ denote the set of ordered pairs $\theta = (\theta^1, \ldots, \theta^n)$ in the torus $[0, \pi)^n$ for which there are representatives with $0 \leq \theta^1 < \theta^2 < \ldots < \theta^n < \theta^1 + \pi$. Denote

$$F_a(\theta) = \prod_{1 \leq j < k \leq n} |\sin(\theta^k - \theta^j)|^a,$$

$$I_a = \int_X F_a(\theta) d\theta,$$

$$\beta = \beta(a, n) = \frac{a(n^2 - 1)}{4}.$$
Corollary 3.10. If $a \geq 1/2$, there exists $u = u(2a, n) > 0$ such that
\[
\mathbb{E}^{\theta_0}[L_t] = e^{-2an\beta t} \frac{I_{2a}}{I_{4a}} F_a(\theta)[1 + O(e^{-ut})].
\]

The paper is organized as follows. Section 2.1 describes the multiple $\lambda$-SAW model, a discrete model which provides motivation and intuition for the perspective we take in the construction of $n$-radial SLE. Section 2 gives an overview of the necessary background for the radial Loewner equation. Section 3 contains the construction of $n$-radial SLE (Theorem 3.12) as well as locally independent SLE. The necessary results about the $n$-radial Bessel process are stated here in the context of $\kappa \leq 4$ without proof. Finally, section 5 contains our results about the $n$-radial Bessel process, including Theorem 3.7. These results hold for all $\kappa < 8$ and include proofs of the statements that were needed in section 3.

2 Preliminaries

2.1 Discrete Model

Although we will not prove any results about convergence of a discrete model to the continuous, much of the motivation for our work comes from a belief that SLE is a scaling limit of the “$\lambda$-SAW” described first in [KL07]. In particular, the key insight needed to prove Theorem 3.12, the use of the intermediate process locally independent SLE$\kappa$ as a step between independent SLE$\kappa$ and $n$-radial SLE$\kappa$, was originally formulated by considering the partition function of multiple $\lambda$-SAW paths approaching the same point. For this reason, we describe the discrete model in detail here.

The model weights self-avoiding paths using the random walk loop measure, so we begin by defining this. A (rooted) random walk loop in $\mathbb{Z}^2$ is a nearest neighbor path $\ell = [\ell_0, \ell_1, \ldots, \ell_{2k}]$ with $\ell_0 = \ell_{2k}$. The loop measure gives measure $\hat{m}(\ell) = (2k)^{-1} 4^{-2k}$ to each nontrivial loop of length $2k > 0$. If $V \subset A \subset \mathbb{Z}^2$, we let
\[
F_V(A) = \exp \left\{ \sum_{\ell \subset A, \ell \cap V \neq \emptyset} m(\ell) \right\},
\]
that is, $\log F_V(A)$ is the measure of loops in $A$ that intersect $V$.

We fix $n$ and some $r_n > 0$ such that there exists $n$ infinite self-avoiding paths starting at the origin that have no intersection after they first leave the ball of radius $r_n$. (For $n \leq 4$, we can choose $r_n = 0$ but for larger $n$ we need to choose $r_n$ bigger because one cannot have five nonintersecting paths starting at the origin. This is a minor discrete detail that we will not worry about.) If $A \subset \mathbb{Z}^2$ is a finite, simply connected set containing the disk of radius $r_n$ about the origin, we let $\mathcal{W}_A$ denote the set of self-avoiding walks $\eta$ starting at $\partial A$, ending at 0, and otherwise staying in $A$. As a slight abuse of notation, we will write $\eta^1 \cap \eta^2 = \emptyset$ if the paths have no intersections other then the beginning of the reversed paths up to the first exit from the ball of radius $r_n$. (If $n \leq 4$ and $r_n = 0$, this means that the paths do not intersect anywhere except their terminal point which is the origin.)
Conjecture 2.1. Suppose that $\nu$ is an $n$-tuple of such paths, we let $I(\eta)$ be the indicator function of the event that $\eta^j \cap \eta^k = \emptyset$ for all $j \neq k$. We write $|\eta|$ for the number of edges in $\eta$ and $|\eta| = |\eta^1| + \cdots + |\eta^j|$. Let $W_A = W_{A,n}$ denote the set of $n$-tuples $\eta$ in $W_A$ with $I(\eta) = 1$. We then consider the measure on configurations given by

$$\nu_{A,c}(\eta) = \exp\{-\beta|\eta|\} I(\eta) F_\eta(A)^{c/2}.$$ 

Here $\beta = \beta_c$ is a critical value under which the measure becomes critical. If $z \in (\partial A)^n$, we write $\bar{\nu}$ for the $\nu$ restricted to $A$. Let $\mathcal{W}_A(z)$ be the set of $\eta \in \mathcal{W}_A$ such that $\eta^j$ starts at $z^j$. Suppose $D$ is a bounded, simply connected domain in $\mathbb{C}$ containing the origin and let $z = (z^1, \ldots, z^n)$ be an $n$-tuple of distinct points in $\partial D$ oriented counterclockwise. For ease, we assume that for each $j$, $\partial D$ in a neighborhood of $z^j$ is a straight line segment parallel to the coordinate axes (e.g., $D$ could be a rectangle and none of the $z^j$ are corner points). For each lattice spacing $N^{-1}$, let $A_N$ be an approximation of $ND$ in $\mathbb{Z}^2$ and let $z_N = (z_{N,1}, \ldots, z_{N,n})$ be lattice points corresponding to $Na$. We can consider the limit as $N \to \infty$ of the measure on scaled configurations $N^{-1} \eta$ given by $\nu_{A,N,c}$ restricted to $W_{A,N}(z_N)$.

**Conjecture 2.1.** Suppose $c \leq 1$. Then there exist $b, \tilde{b}_n$ and, critical $\beta = \beta_c$ and a partition function $\Psi_*(D; z, 0)$ such that as $N \to \infty$,

$$\nu_{A,N,c}(W_{A,N}(z_N)) \sim \Psi_*(D; z, 0) N^{nb} N^{\tilde{b}_n}.$$ 

Moreover, the scaling limit $N^{-nb} N^{-\tilde{b}_n} \nu_{A,N,c}$ is $n$-radial $SLE_{\kappa}$, $\mu_D(z, 0)$ with partition function $\Psi(D; z, 0)$. If $f : D \to f(D)$ is a conformal transformation with $f(0) = 0$, then

$$f \circ \mu_D(z, 0) = |f'(z)|^{b} |f''(0)|^{\tilde{b}_n} \mu_{f(D)}(f(z), 0).$$ 

Here $f(z) = (f(z^1), \ldots, f(z^n))$ and $f'(z) = f'(z^1) \cdots f'(z^n)$.

This conjecture is not precise, and since we are not planning on proving it, we will not make it more precise. The main goal of this paper is to show that assuming the conjecture informs us as to what $n$-radial $SLE_{\kappa}$ should be and what the exponents $b, \tilde{b}_n$ are.

The case $n = 1$ is usual radial $SLE_\kappa$ for $\kappa \leq 4$ and the relation is

$$b = \frac{6 - \kappa}{2\kappa}, \quad \tilde{b}_1 = \tilde{b} = b \frac{\kappa - 2}{4}, \quad c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}.$$ 

This is understood rigorously in the case of $c = -2, \kappa = 2$ since the model is equivalent to the loop-erased random walk. For other cases it is an open problem. For $c = 0$, it is essentially equivalent to most of the very hard open problems about self-avoiding walk. However, assuming the conjecture and using the fact that the limit should satisfy the restriction property, one can determine $\kappa = 8/3, c = 0$, $b = 5/8, \tilde{b} = 5/48$. The critical exponents for SAW can be determined (exactly but nonrigorously) from these values.

The case $n = 2$ is related to two-sided radial $SLE_{\kappa}$ which can also be viewed as chordal $SLE_{\kappa}$ from $z^1$ to $z^2$, restricted to paths that go through the origin. In this case, $b_2 = d - 2$ where $d = 1 + \frac{\kappa}{8}$ is the fractal dimension of the paths.
2.2 Radial SLE and the restriction property

The radial Schramm-Loewner evolution with parameter \( \kappa = 2/a \) (SLE\(_{\kappa}\)) from \( z = e^{2i\theta} \) to the origin in the unit disk is defined as the random curve \( \gamma(t) \) with the following properties. Let \( D_t \) be the component of \( \mathbb{D} \setminus \gamma[0,t] \) containing the origin. If \( g_t : D_t \rightarrow \mathbb{D} \) is the conformal transformation with \( g_t(0) = 0, g_t'(0) > 0 \), then \( g_t \) satisfies

\[
\dot{g}_t(w) = 2a g_t(z) \frac{e^{2iB_t} + g_t(w)}{e^{2iB_t} - g_t(w)}, \quad g_0(w) = w,
\]

where \( B_t \) is a standard Brownian motion. More precisely, this is the definition of radial SLE\(_{\kappa}\) when the curve has been parameterized so that \( g'_t(0) = e^{2at} \).

We will view SLE\(_{\kappa}\) as a measure on curves modulo reparameterization (there is a natural parameterization that can be given to the curves, but we will not need this in this paper). We extend SLE\(_{\kappa}\) to be a probability measure \( \mu_D^\#(z,w) \) where \( D \) is a simply connected domain, \( z \in \partial D \) and \( w \in D \) by conformal transformation. It is also useful for us to consider the non-probability measure \( \mu_D(z,w) = \Psi_D(z,w) \mu_D^\#(z,w) \). Here \( \Psi_D(z,w) \) is the radial partition function that can be defined by \( \Psi_D(1,0) = 1 \) and the scaling rule

\[
\Psi_D(z,w) = |f'(z)|^b |f'(w)|^{b} \Psi_{f(D)}(f(z),f(w)),
\]

where

\[
b = \frac{6 - \kappa}{2\kappa} = \frac{3a - 1}{2}, \quad \tilde{b} = b - \frac{\kappa - 2}{4} = b + \frac{1 - a}{2a},
\]

are the boundary and interior scaling exponents. This definition requires sufficient smoothness of the boundary near \( z \). However, if \( D' \subset D \) agree in neighborhoods of \( z \), then the ratio

\[
\Psi(D, D'; z, w) := \frac{\Psi_{D'}(z, w)}{\Psi_D(z, w)}
\]

is a conformal invariant and hence is well defined even for rough boundary points.

We will need the restriction property for radial SLE\(_{\kappa}\), \( \kappa \leq 4 \). We state here it in a way that does not depend on the parameterization, which is the form that we will use.

**Definition** If \( D \) is a domain and \( K_1, K_2 \) are disjoint subsets of \( D \), then \( m_D(K_1, K_2) \) is the Brownian loop measure of loops that intersect both \( K_1 \) and \( K_2 \).

**Proposition 2.2** (Restriction property). Suppose \( \kappa \leq 4 \) and \( D = \mathbb{D} \setminus K \) is a simply connected domain containing the origin. Let \( z \in \partial D \) with \( \text{dist}(z,K) > 0 \), and let \( \gamma \) be a radial SLE\(_{\kappa}\) path from \( z \) to \( 0 \) in \( \mathbb{D} \). Let

\[
M_t = 1\{\gamma_t \cap K = \emptyset\} \exp \left\{ \frac{c}{2} m_D(\gamma_t, K) \right\} \Psi_t,
\]

where \( \Psi_t = \Psi(\mathbb{D} \setminus \gamma_t, D \setminus \gamma_t; \gamma(t), 0) \). Then \( M_t \) is a uniformly integrable martingale, and the probability measure obtained from Girsanov’s theorem by tilting by \( M_t \) is SLE\(_{\kappa}\) from \( z \) to \( 0 \) in \( D \). In particular,

\[
\mathbb{E} \left[ 1\{\gamma \cap K = \emptyset\} \exp \left\{ \frac{c}{2} m_D(\gamma, K) \right\} \right] = \mathbb{E}[M_0] = \mathbb{E}[M_\infty] = M_0 = \Psi(D, D; z, 0). \quad (4)
\]
See [JL18a] for a proof. It will be useful for us to discuss the ideas in the proof. We parameterize the curve as above and we consider $\Psi_t$, the ratio of partition functions at time $t$ of $SLE$ in $D \setminus \gamma_t$ with $SLE$ in $\mathbb{D} \setminus \gamma_t$. Using the scaling rule and Itô’s formula, one computes the SDE for $\Psi_t$,

$$d\Psi_t = \Psi_t \left[ A_t \, dt + R_t \, dB_t \right].$$

This is not a local martingale, so we find the compensator and let

$$M_t = \Psi_t \exp \left\{ - \int_0^t A_s \, ds \right\},$$

which satisfies

$$dM_t = R_t \, M_t \, dB_t.$$

This is clearly a local martingale. The following observations are made in the calculations:

- The compensator term is the same as $\exp \left\{ \frac{c}{2} m_D(\gamma, K) \right\}$.
- If we use Girsanov theorem and tilt by this martingale, we get the same distribution on paths as $SLE_\kappa$ in $D$. The latter distribution was defined by conformal invariance.

All of this is valid for all $\kappa$ up to the first time $t$ that $\gamma(t) \in K$. For $\kappa \leq 4$, we now use the fact that radial $SLE$ in $D$ never hits $K$ and is continuous at the origin. This allows us to conclude that it is a uniformly integrable martingale. With probability one in the new measure we have $\gamma \cap K = \emptyset$ and hence we can conclude the proposition.

We sketched the proof in order to see what happens when we allow the set $D$ to shrink with time. In particular, let $D_t = \mathbb{D} \setminus K_t$, where $K_t$ grows with time, and let

$$\Psi_t = \Psi(\mathbb{D} \setminus \gamma_t, D_t \setminus \gamma_t; \gamma(t), 0),$$

$$T = \inf \{ t : \gamma_t \cap K_t \neq \emptyset \}.$$

For $t < T$, we can again consider $SLE$ tilted by $\Psi_t$. However, since $K_t$ is growing, the loop term is more subtle in this case. Roughly speaking, the relevant loops are those that intersect $K_s$ for some $s$ smaller than their first intersection time with $\gamma_t$. More precisely, the local martingale has the form

$$L_{t}^{c/2} \exp \left\{ - \int_0^t A_s \, ds \right\} \Psi_t,$$

where

- $\log L_t$ is the Brownian measure of loops $\ell$ that hit $\gamma_t$ and satisfy the following: if $s_\ell$ is the smallest time with $\gamma(s_\ell) \in \ell$, then $l \cap K_{s_\ell} \neq \emptyset$.
- $A_t = \rho'(0)$, for $\rho(\epsilon) = \rho_t(\epsilon) = \Psi(\mathbb{D} \setminus \gamma_t, D_{t+\epsilon} \setminus \gamma_t; \gamma(t), 0)$.

We assume that $A_t$ is well defined, that is, that $\rho$ is differentiable.

When we tilt by $\Psi_t$, the process at time $t$ moves like $SLE$ in $D_t$. We will only consider this up to time $T$. 
3 Measures on $n$-tuples of paths

We will use a similar method to define two measures on $n$-tuples of paths which can be viewed as process taking values in $D^n$. We start with $n$ independent radial SLE paths. First, we will tilt independent SLE by a loop term to define a process with the property that each of the $n$ paths locally acts like SLE in the disk minus all $n$ initial segments. We will tilt this process again by another loop term and take a limit to give the definition of global multiple radial SLE. Splitting up the construction into two distinct tiltings will allow us to analyze the contribution of $t$-measurable loops separately from that of “future loops.” Furthermore, each of these processes is interesting in its own right, and we show that in each case the driving function satisfies the radial Bessel equation. (See equations (17) and (28).)

This clarifies which terms cause the multiple paths to avoid each other’s past versus the terms that ensure that the paths continue to avoid each other in the future until all curves reach the origin.

3.1 Notation

We will set up some basic notation; some of the notation that was used in the single SLE setting above will be repurposed here in the setting of $n$ curves. (See Figure 3)
• We fix positive integer $n$ and let $\mathbf{\theta} = (\theta^1, \ldots, \theta^n)$ with

$$\theta^1 < \cdots < \theta^n < \theta^1 + \pi.$$ 

Let $z^j = \exp\{2i\theta^j\}$ and $z = (z^1, \ldots, z^n)$. Note that $z^1, \ldots, z^n$ are $n$ distinct points on the unit circle ordered counterclockwise.

• Let $\gamma = (\gamma^1, \ldots, \gamma^n)$ be an $n$-tuple of curves $\gamma^j : (0, \infty) \to \mathbb{D} \setminus \{0\}$ with $\gamma^j(0+) = z^j$ and $\gamma^j(\infty) = 0$. We write $\gamma_t^j$ for $\gamma^j[0, t]$ and $\gamma_t = (\gamma_t^1, \ldots, \gamma_t^n)$. In a slight abuse of notation, we will use $\gamma_t^j$ to refer to both the set $\gamma_t^j[0, t]$ and the function $\gamma^j$ restricted to times in $[0, t]$.

• Let $D_t^1, D_t^2$ be the connected components of $\mathbb{D} \setminus \gamma_t^1, \mathbb{D} \setminus \gamma_t^2$, respectively, containing the origin. Let $g_t^j : D_t^1 \to \mathbb{D}, g_t : D_t \to \mathbb{D}$ be the unique conformal transformations with

$$g_t^j(0) = g_t(0) = 0, \quad (g_t^j)'(0), g_t'(0) > 0.$$ 

• Let $T$ be the first time $t$ such that $\gamma_t^j \cap \gamma_k^l \neq \emptyset$ for some $1 \leq j < k \leq n$.

• Define $z_t^j = \exp\{2i\theta_t^j\}$ by $g_t(\gamma^j(t)) = z_t^j$. Let $z_t = (z_t^1, \ldots, z_t^n), \mathbf{\theta}_t = (\theta_t^1, \ldots, \theta_t^n)$. For $\zeta \in \mathbb{H}$ define $h_t(\zeta)$ to be the continuous function of $t$ with $h_0(\zeta) = \zeta$ and

$$g_t(e^{2i\zeta}) = e^{2ih_t(\zeta)}.$$ 

Note that if $\zeta \in \mathbb{R}$ so that $e^{2i\zeta} \in \partial\mathbb{D}$, we can differentiate with respect to $\zeta$ to get

$$|g_t'(e^{2i\zeta})| = h_t'(\zeta).$$ 

(5)

• More generally, if $t = (t_1, \ldots, t_n)$ is an $n$-tuple of times, we define $\gamma_t, D_t, g_t$. We let $\alpha(t) = \log g_t'(0)$.

• We will say that the curves have the common (capacity $a$-)parameterization if for each $t$,

$$\partial_j \alpha(t, t, \ldots, t) = 2a, \quad j = 1, \ldots, n.$$ 

(6)

In particular,

$$g_t^j(0) = e^{2ant}.$$ 

(7)

Note that (6) is a stronger condition than (7).

The following form of the Loewner differential equation is proved in the same way as the $n = 1$ case,

**Proposition 3.1.** [Radial Loewner equation] If $\gamma_t$ has the common parameterization, then for $t < T$, the functions $g_t, h_t$ satisfy

$$\dot{g}_t(w) = 2a g_t(w) \sum_{j=1}^n \frac{z_t^j}{z_t^j - g_t(w)}, \quad \dot{h}_t(\zeta) = a \sum_{j=1}^n \cot(h_t(\zeta) - \theta^j_t).$$

If $\partial D_t$ contains an open arc of $\partial\mathbb{D}$ including $w = e^{2i\zeta}$, then

$$|g_t'(w)| = \exp \left\{-a \int_0^t \sum_{j=1}^n \csc^2(h_s(\zeta) - \theta^j_t) \, ds \right\}.$$ 

(8)
3.2 Common parameterization and local independence

Suppose that $\gamma_1, \ldots, \gamma_n$ are independent radial $SLE_\kappa$ paths in $\mathbb{D}$ starting at $z_1, \ldots, z_n$, respectively, going to the origin. Then we can parameterize the paths so that they have the common parameterization. (This parameterization is only possible until the first time that two of the paths intersect, but this will not present a problem since we will usually restrict to nonintersecting paths.) Indeed, suppose $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ are independent $SLE_\kappa$ paths with the usual parameterization as in Section 2.2. It is not true that $\tilde{\gamma}_t = (\tilde{\gamma}_1(t), \ldots, \tilde{\gamma}_n(t))$ has the common parameterization. We will write $\gamma_j(t) = \tilde{\gamma}_{\sigma_j(t)}(t)$ where $\sigma_j(t)$ is the necessary time change. Define $g_t$ by $g_t = g_{t,j} \circ g_t$. The driving functions for $\tilde{\gamma}_t$ are independent standard Brownian motions; denote these by $\tilde{\theta}_j(t)$. Define $\xi_j(t)$ by $\xi_j(t) = \tilde{\theta}_j(t)$ so that $e^{2i\xi_j(t)} = g_t(\gamma_j(t))$. Furthermore, define define $h_t$ and $h_{t,j}$ so that

$$h_t(w) = h_{t,j} \circ h_t(w),$$

and

$$g_t(e^{2i\omega}) = e^{2i\varnothing_t}.$$ (9)

(See Figure 3.)

**Lemma 3.2.** The derivative $\dot{\sigma}_j(t)$ depends only on $\gamma_t$ and is given by

$$\dot{\sigma}_j(t) = h_{t,j}^\prime(\xi_j(t))^{-2}. $$ (10)

**Proof.** Differentiating both sides of equation (9), we obtain

$$\dot{h}_t(w) = \dot{h}_{t,j}(h_t(w)) + h_{t,j}^\prime(h_t(w)) \times \dot{h}_t(w). $$ (11)

Since $g_t$ satisfies the (single-slit) radial Loewner equation with an extra term for the time change, $h_t$ satisfies

$$\dot{h}_t(w) = a \cot(h_t(w) - \xi_t) \times \dot{\sigma}_j(t). $$

On the other hand, $h_{t,j}$ satisfies

$$\dot{h}_{t,j}(w) = a \sum_{k \neq j} \cot(h_{t,j}(w) - \theta_t^k). $$

Substituting these expressions for $\dot{h}_t(w)$ and $\dot{h}_j(w)$ into (11) and using the equation for $h_t(w)$ given in Proposition 3.1 shows that

$$a \sum_{k=1}^n \cot(h_t(w) - \theta_t^k) = a \sum_{k \neq j} \cot(h_t(w) - \theta_t^k) + h_{t,j}(h_t(w)) \times a \dot{\sigma}_j(t) \cot(h_t(w) - \xi_t). $$ (12)

Solving for $\dot{\sigma}_j(t)$ and taking the limit as $w \to \gamma_j(t)$ verifies (10).
The components of $\gamma$ are not quite independent because the rate of “exploration” of the path $\gamma^j$ depends on the other paths. However, the paths are still independent in the sense that the conditional distribution of the remainder of the paths given $\gamma_t$ are independent $SLE$ paths; in the case of $\gamma^j$ it is $SLE$ in $D^j_t$ from $\gamma^j(t)$ to 0.

We will define another process, which we will call locally independent $SLE_\kappa$ that has the property that locally each curve grows like $SLE_\kappa$ from $\gamma^j(t)$ to 0 in $D^j_t$ (rather than in $D^j_t$). This will be done similarly as for a single path. Intuitively, at time $t$ each curve $\gamma^j_t$ can “see” $\gamma_t$, but not the future evolution of the other curves.

Recall that $SLE_\kappa$ in $D \subset \mathbb{D}$ is obtained from $SLE_\kappa$ in $D$ by weighting by the appropriate partition function. Since the partition function is not a martingale, this is done by finding an appropriate differentiable compensator so that the product is a martingale, and then applying Girsanov’s theorem.

Let 
\[
\Psi^j_t = \Psi(D^j_t, D_t; \gamma^j(t), 0), \quad \Psi_t = \prod_{j=1}^n \Psi^j_t, \tag{13}
\]

\[
\psi(\theta_t) = \sum_{j=1}^n \sum_{k \neq j} \csc^2(\theta^j_t - \theta^k_t), \tag{14}
\]

\[
\tau = \inf \{ t : \exists j \neq k \text{ such that } \gamma^j_t \cap \gamma^k_t = \emptyset \}. \tag{15}
\]

For any loop, let 
\[
s^j(\ell) = \inf \{ t : \gamma^j(t) \in \ell \}, \quad s(\ell) = \min_j s^j(\ell).
\]

We make a simple observation that will make the ensuing definitions valid.

**Lemma 3.3.** Let $\gamma^1, \ldots, \gamma^n$ be nonintersecting curves. Then except for a set of loops of Brownian loop measure zero, either $s(\ell) = \infty$ or there exists a unique $j$ with $s^j(\ell) = s(\ell)$.

**Sketch of proof.** We consider excursions between the curves $\gamma^1, \ldots, \gamma^n$, that is, times $r$ such that $\ell(r) \in \gamma^k$ for some $k$ and the most recent visit before time $r$ was to a different curve $\gamma^j$. There are only a finite number of such excursions. For each one, the probability of hitting a point with the current smallest index is zero. \(\square\)

Let $\mathcal{L}^j_t = \mathcal{L}^j_t(\gamma_t)$ be the set of loops $\ell$ with $s(\ell) < s^j(\ell) \leq t$, and let
\[
\mathcal{L}_t = \mathcal{L}_t \exp \left\{ \frac{c}{2} \sum_{j=1}^n m_D(\mathcal{L}^j_t) \right\}. \tag{15}
\]

(See Figure 2) Here $\hat{I}_t$ is the indicator function that $\gamma^j_t \cap \gamma^k_t = \emptyset$ for $j \neq k$.

We note that while the definitions of $s^j$ and $s$ (and hence $\mathcal{L}^j_t$) depend on the parameterization of the curve, $\mathcal{L}_t$ depends only on the traces of the curves $\gamma^1_t, \ldots, \gamma^n_t$. For this reason, we could also define $\mathcal{L}_t$ for an $n$-tuple $\mathbf{t} = (t_1, \ldots, t_n)$. 12
Proposition 3.4. Let $0 < \kappa \leq 4$. If $\gamma_t$ is independent SLE$_\kappa$ with the common parameterization, and

$$M_t = \hat{\mathcal{L}}_t \Psi_t \exp \left\{ ab \int_0^t \psi(\theta_s) \, ds \right\}, \quad (16)$$

then $M_t$ is a local martingale for $0 \leq t < \tau$. If $P_*$ denotes the measure obtained by tilting $P$ by $M_t$, then

$$d\theta^j_t = a \sum_{k \neq j} \cot(\theta^j_t - \theta^k_t) \, dt + dW^j_t, \quad (17)$$

where $W^1_t, \ldots, W^n_t$ are independent standard Brownian motions with respect to $P_*$. Furthermore,

$$P_* \{ \tau < \infty \} = 0.$$

**Definition** We call the $n$-tuple of curves $\gamma_t$ under the measure $P_*$ locally independent SLE$_\kappa$.

The idea of the proof will be to express $M_t$ as a product of martingales

$$M_t = \prod_{j=1}^n M_t^j$$

with the following property: after tilting by the martingale $M_t^j$ the curve $\gamma^j$ locally at time $t$ evolves as SLE$_\kappa$ in the domain $D_t = D \setminus \gamma_t$. The martingales $M_t^j$ are found by following the method of proof in [Proposition 5, [JL18a]]. The construction shows that under $P_*$, at each time $t$ the curves $\gamma^1, \ldots, \gamma^n$ are locally growing as $n$ independent SLE$_\kappa$ curves in $D_t$, which is the reason for the name locally independent SLE. Locally independent SLE is revisited in §4.

**Proof of Proposition 3.4.** Since the $\xi^j_t$ are independent standard Brownian motions under the time changes $\sigma^1, \ldots, \sigma^n$, there exist independent standard Brownian motions $B^1_t, \ldots, B^n_t$ such that

$$d\xi^j_t = \sqrt{\dot{\sigma}^j(t)} \, dB^j_t, \quad j = 1, \ldots, n.$$

By Lemma 3.2,

$$dB^j_t = h_{t,j}(\xi^j_t) \, d\xi^j_t, \quad j = 1, \ldots, n,$$

and Itô’s formula shows that each $\theta^j_t$ satisfies

$$d\theta^j_t = h_{t,j}(\xi^j_t) \, dt + \frac{h''_{t,j}(\xi^j_t)}{2 \left( h'_{t,j}(\xi^j_t) \right)^2} dt + dB^j_t.$$

Define

$$M_t^j = I_t^j \Psi_t \exp \left\{ \frac{c}{2} m_B(\hat{L}^j_t) \right\} \exp \left\{ ab \int_0^t \sum_{k \neq j} \csc^2(\theta^j_s - \theta^k_s) \, ds \right\}, \quad t < T, \quad (18)$$
so that
\[ M_t = \prod_{j=1}^{n} M^j_t. \] (19)

Applying the method of proof of the boundary perturbation property for single slit radial SLE [Proposition 5, [JL18a]], we see that \( \Psi^j_t \) satisfies
\[ d\Psi^j_t = \Psi^j_t \left[ \left( \frac{-c}{2} m^j_{\Gamma}(L^j_t) - ab \int_0^t \sum_{k:k \neq j} \csc^2(\theta^j_s - \theta^k_s)ds \right) dt + \frac{b}{2} h''_{t,j}(\xi^j_t) d\xi^j_t \right], \]
and \( M^j_t \) is a local martingale satisfying
\[ dM^j_t = M^j_t \frac{b}{2} h''_{t,j}(\xi^j_t) d\xi^j_t, \quad M^j_0 = 1. \]

Since the \( \xi^j_t \) are independent, \( M_t \) satisfies
\[ dM_t = M_t \left[ \sum_{j=1}^{n} \frac{b}{2} h''_{t,j}(\xi^j_t) d\xi^j_t \right]. \]

Therefore, \( M_t \) is a local martingale, and equation (17) follows by the Girsanov theorem. \( \square \)

### 3.3 Dyson Brownian Motion on the Circle

The construction of \( n \)-radial SLE\( \kappa \) in Section 3.4 will require some results about the \( n \)-radial Bessel process (Dyson Brownian motion on the circle), which we state here. However, the proofs of these results are postponed until Section 5 since they hold in the more general setting of \( 0 < \kappa < 8 \) and do not rely on Brownian loop measure.

A note about parameters: we state the results here using parameters \( \alpha \) and \( b_\alpha \) since the results hold outside of the SLE setting. When we apply these results to SLE\( \kappa \) in the next section, we will set \( \alpha = a = 2/\kappa \) or \( \alpha = 2a = 4/\kappa \). In particular, when \( \alpha = a, b_\alpha = b = (3a - 1)/2. \)

Define
\[ F_\alpha(\theta) = \prod_{1 \leq j < k \leq n} |\sin(\theta^j - \theta^k)|^\alpha, \quad \tau = \inf\{t : F_\alpha(\theta) = 0\}, \]
and recall the definition of \( \psi(\theta) \) from (14). The next result will be verified in the discussion following the proof of Lemma 5.1.

**Proposition 3.5.** Let \( \theta^1, \ldots, \theta^n \) be independent standard Brownian motions, and let \( \alpha > 0 \). If
\[ M_{t,\alpha} = F_\alpha(\theta_t) \exp \left\{ \frac{\alpha^2 n (n^2 - 1)}{6} t \right\} \exp \left\{ \frac{\alpha - \alpha^2}{2} \int_0^t \psi(\theta_s)ds \right\}, \quad 0 \leq t < \tau, \] (20)
then $M_{t,\alpha}$ is a local martingale for $0 \leq t < \tau$ satisfying
\[
dM_{t,\alpha} = M_{t,\alpha} \sum_{j=1}^{n} \left( \sum_{k \neq j} \alpha \cot(\theta_j^t - \theta_k^t) \right) d\theta_j^t.
\]

If $P_\alpha$ denotes the probability measure obtained after tilting by $M_{t,\alpha}$, then
\[
d\theta_j^t = \alpha \sum_{k \neq j} \cot(\theta_j^t - \theta_k^t) dt + dW_j^t, \quad 0 \leq t < \tau,
\]
where $W_1^t, \ldots, W_n^t$ are independent standard Brownian motions with respect to $P_\alpha$.

Furthermore, if $\alpha \geq 1/2$,
\[
P_\alpha(\tau = \infty) = 1.
\]

**Proposition 3.6.** Suppose that $\alpha \geq 1/4$ and
\[
N_t = N_{t,\alpha,2\alpha} = F_{\alpha}(\theta_t) \exp \left\{ \frac{\alpha^2 n(n^2 - 1)}{2} t \right\} \exp \left\{ -\alpha b_\alpha \int_0^t \psi(\theta_s) ds \right\},
\]
where $b_\alpha = (3\alpha - 1)/2$. Then $N_t$ is a $P_\alpha$-martingale, and the measure obtained by tilting $P_\alpha$ by $N_t$ is $P_{2\alpha}$.

**Proof.** See Proposition 5.6 and its proof.

We will also require the following theorem, which is proven immediately after Proposition 5.6.

**Theorem 3.7.** If $\alpha \geq 1/2$, there exists $u = u(2\alpha, n) > 0$ such that
\[
E_\theta^\alpha \left[ \exp \left\{ -\alpha b_\alpha \int_0^t \psi(\theta_s) ds \right\} \right] = e^{-2\alpha \beta t} F_{\alpha}(\theta_t) \frac{I_{3\alpha}}{I_{4\alpha}} \left[ 1 + O(e^{-ut}) \right],
\]
where
\[
\beta = \beta(\alpha, n) = \frac{\alpha(n^2 - 1)}{4},
\]
and $E_\alpha$ denotes expectation with respect to $P_\alpha$.

### 3.4 $n$-Radial $SLE_\kappa$

The remainder of the section is devoted to the construction of $n$-radial $SLE_\kappa$, which may also be called *global multiple radial SLE*$_\kappa$. As we have stated before, we will consider three measures on $n$-tuples of curves with the common parameterization:

- $P, E$ will denote independent $SLE_\kappa$ with the common parameterization;
- $P_*, E_*$ will denote locally independent $SLE_\kappa$;
- $P, E$ will denote $n$-radial $SLE_\kappa$. 

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In Section 3.2, we obtained $\mathbb{P}_*$ from $\mathbb{P}$ by tilting by a $\mathbb{P}$-local martingale $M_t$. We will obtain $\mathcal{P}$ from $\mathbb{P}_*$ by tilting by a $\mathbb{P}_*$-local martingale $N_{t,T}$ and then letting $T \to \infty$. Equivalently, we obtain $\mathcal{P}$ from $\mathbb{P}$ by tilting by $\tilde{N}_{t,T} := M_t N_{t,T}$ and letting $T \to \infty$.

Let $L_t^j = L_t^j(\gamma^j)$ be the set of loops $\ell$ with $s(\ell) < s^j(\ell)$ and $s(\ell) \leq t$, as in Figure 2.

Define
\[
L_t = I_t \exp \left\{ \frac{c}{2} \sum_{j=1}^{n} m_D(L_t^j) \right\}.
\]

Here $I_t$ is the indicator function that $\gamma_t^j \cap \gamma^k = \emptyset$ for $j \neq k$.

Let
\[
\tilde{N}_{t,T} = E_{\theta_0}^t [L_T | \gamma_t], \quad 0 \leq t \leq T,
\]

where the conditional expectation is with respect to $\mathbb{P}$. By construction, $\tilde{N}_{t,T}$ is a martingale for $0 \leq t \leq T$ with $\tilde{N}_{0,T} = E_{\theta_0}^t [L_T]$.

For the next proposition, recall that $\hat{L}_t$ weights by loops that hit at least two curves before time $t$; the precise definition is given in (15).

**Proposition 3.8.** Let $T \geq 0$. If $\gamma_t$ is independent SLE$_\kappa, 0 < \kappa \leq 4$, with the common parameterization, then
\[
\tilde{N}_{t,T} = \hat{L}_t \Psi_t E_{\theta_t}^t [L_{T-t}], \quad 0 \leq t \leq T. \tag{24}
\]

In particular, if
\[
N_{t,T} = \exp \left\{ -ab \int_0^t \psi(\theta_s) \, ds \right\} E_{\theta_t}^t [L_{T-t}], \quad 0 \leq t \leq T,
\]

then $N_{t,T}$ is a $\mathbb{P}_*$-martingale for $0 \leq t \leq T$, and
\[
E_{\theta_t}^t [L_{T-t}] = E_{\theta_*}^t \left[ \exp \left\{ -ab \int_0^{T-t} \psi(\theta_s) \, ds \right\} \right]. \tag{25}
\]

Note that the expectation on the righthand side of (25) is with respect to $\mathbb{P}_*$.

**Proof.** We may write
\[
L_T = \hat{L}_t \frac{L_T}{\hat{L}_t} L_{T,t},
\]

where
\[
L_{T,t} = \exp \left\{ \frac{c}{2} \sum_{j=1}^{n} m_D(\ell : t < s(\ell) \leq T, s(\ell) < s^j(\ell)) \right\},
\]

The term $L_{T,t}$ should be thought of as the “future loop” term, since it accounts for loops that hit at least two curves with the first hit occurring during $(t,T]$.

The restriction property shows that
\[
E_{\theta_0}^t \left[ \hat{L}_t \left( \frac{L_T}{\hat{L}_t} \right) | \gamma_t \right] = \hat{L}_t \Psi_t.
\]
Moreover, the conditional distribution on $\gamma_T \setminus \gamma_t$, after tilting by $\hat{L}_t \left( L_t / \hat{L}_t \right)$ is that of independent SLE in $D_t$. Since $L_{T,t}$ depends only on $\gamma_T \setminus \gamma_t$, this gives (24).

For the second part of the proposition, notice that $N_{t,T} = \tilde{N}_{t,T}$, which is a $P$-martingale by construction, so $N_{t,T}$ is a $P^*$-martingale. Since $E^{\theta_0} [L_0] = 1$, this implies that

$$N_{t,T} = E^{\theta_0} [N_{T,T} | \gamma_t] = E^{\theta_0} \left[ \exp \left\{ -ab \int_0^T \psi(\theta_s) \, ds \right\} \bigg| \gamma_t \right],$$

which verifies (25).

**Proposition 3.9.** Let $P^T_*$ denote the probability measure obtained by tilting $P$ by $\tilde{N}_{t,T}$. Under $P^*_T$, conditionally on $\hat{\gamma}_j T$, the distribution of $\gamma_j$ is SLE$_\kappa$ in $D \setminus \hat{\gamma}_j T$.

**Proof.** The result follows by an application of the restriction property.

The next result, which gives the exponential rate of convergence of $E^{\theta_0}[L_T]$, is a direct application of Theorem 3.7.

**Corollary 3.10.** There exists $u = u(2a,n) > 0$ such that as $T \to \infty$,

$$E^{\theta_0} [L_T] = \frac{T^{3a}}{I_{4a}} F_a(\theta_0) e^{-2a\beta nT} \left[ 1 + O(e^{-uT}) \right],$$

where $\beta$ is given by (23).

**Proof.** Notice that (25) implies that

$$\tilde{N}_{0,T} = E^{\theta_0} [L_T] = E^{\theta_0} \left[ \exp \left\{ -ab \int_0^T \psi(\theta_s) \, ds \right\} \bigg| \gamma_t \right].$$

Substituting this into Theorem 3.7 gives the result.

We define the $n$-interior scaling exponent:

$$\beta_n = \beta - \tilde{b}(n-1) = \frac{4(n^2 - 1) + (6 - \kappa)(\kappa - 2)}{8\kappa},$$

where $\beta$ is defined by (23).

**Proposition 3.11.** With respect to $P$,

$$\tilde{M}_t := e^{2a\tilde{b}_n t} \hat{L}_t F_a(\theta_t)$$

is a local martingale. If $P$ denotes the measure obtained by tilting by $\tilde{M}_t$, then

$$d\theta_t^j = 2a \sum_{k \neq j} \cot(\theta_t^j - \theta_t^k) \, dt + dW_t^j,$$

where $W_t^1, \ldots, W_t^n$ are independent standard Brownian motions with respect to $P$.  

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Proof. Comparing (17) and (21), we see that tilting $n$ independent Brownian motions by $M_{t,a}$ gives the SDE satisfied by the driving functions of locally independent $SLE_\kappa$. By Proposition 3.6, tilting further by $N_{t,a,2a}$ (defined in (22) above) gives driving functions that satisfy (28), which is the $n$-radial Bessel equation (21) for $\alpha = 2a$. This implies that $P_{2a}$ is obtained by tilting $P$ by $M_t N_{t,a,2a}$.

To verify that $M_t N_{t,a,2a} = \tilde{M}_t$, we use the fact that

$$\Psi_t = \exp \left\{ -2a \tilde{b} n (n-1) t \right\},$$

which follows from conformal covariance of the partition function. \qed

As above, let $P$ denote the measure on $n$ independent radial $SLE_\kappa$ curves from $\theta_0$ to 0 with the $a$-common parameterization.

Theorem 3.12. Let $0 < \kappa \leq 4$. Let $t > 0$ be fixed. For each $T > t$, let $\mu_T = \mu_{T,t}$ denote the measure whose Radon-Nikodym derivative with respect to $P$ is

$$\frac{L_T}{E^{\theta_0} [L_T]}.$$

Then as $T \to \infty$, the measure $\mu_{T,t}$ approaches $P$ with respect to the variation distance. Furthermore, the driving functions $z^j_t = e^{2i \theta^j_t}$ satisfy

$$d\theta^j_t = 2a \sum_{k \neq j} \cot(\theta^j_t - \theta^k_t) \, dt + dW^j_t,$$

(29)

where $W^j_t$ are independent standard Brownian motions in $P$.

Proof. We see that

$$\frac{d\mu_{T,t}}{dP_t} = \frac{E^{\theta_T} [L_T | \gamma_t]}{E^{\theta_0} [L_T]} = \frac{\tilde{N}_{t,T}}{\tilde{N}_{0,T}}.$$

(30)

By Proposition 3.11, $P$ is obtained by tilting $P$ by $\tilde{M}_t$, so we compare $\tilde{N}_{t,T}$ to $\tilde{M}_t$ and apply Corollary 3.10:

$$\frac{d\mu_{T,t}}{dP_t} = \frac{\tilde{N}_{t,T}}{\tilde{M}_t} = \frac{E^{\theta_{T-t}} [L_{T-t} | F_a(\theta_0)]}{E^{\theta_0} [L_T] F_a(\theta_t) \exp \left\{ \frac{a^2 n (n^2-1)}{2} t \right\}}.$$

$$= \frac{E^{\theta_{T-t}} [L_{T-t} | \gamma_t]}{E^{\theta_0} [L_T | \gamma_t]} \exp \left\{ \frac{a^2 n (n^2-1)}{2} t \right\}.$$

$$= 1 + O(e^{-u(T-t)}).$$

Therefore,

$$\lim_{T \to \infty} \left[ \frac{dP_t}{dP_t} \left( \frac{d\mu_{T,t}}{dP_t} - \frac{dP_t}{dP_t} \right) \right] = 0.$$

But $\frac{dP_t}{dP_t}$ is constant (since $t$ is fixed), so this implies convergence of $\mu_{T,t}$ to $P_t$ in the variation distance. \qed
Definition Let $0 < \kappa \leq 4$. If the curves $\gamma^1, \ldots, \gamma^n$ are distributed according to $\mathcal{P}$, we call $\gamma$ (global) $n$-radial $SLE_\kappa$.

Corollary 3.13. Let $\gamma$ be $n$-radial $SLE_\kappa$ for $0 < \kappa \leq 4$. With probability one, $\gamma$ is an $n$-tuple of simple curves.

Proof. By construction, $n$-radial $SLE_\kappa$ is a measure on $n$-tuples of curves that is absolutely continuous with respect to $n$-independent $SLE_\kappa$. But since $0 < \kappa \leq 4$, each independent $SLE_\kappa$ curve is almost surely simple. \qed

To conclude this section, we remark that the results above do not address the question of continuity at $t = \infty$. Additionally, it would be natural to extend the definition of $n$-radial $SLE$ to apply to $\kappa \in (0, 8)$ by using the measure $\mathbb{P}_{2a}$ instead of $\mathcal{P}$, but we will not consider this here.

4 Locally independent $SLE$

Here we discuss locally independent $SLE$ and explain how it arises as a limit of processes that act like “independent $SLE$ paths in the current domain.” For ease we will do the chordal case and 2 paths, but the same idea works for any number of paths and for radial $SLE$. Locally independent $SLE$ is defined here for all $\kappa < 8$, but when $\kappa \leq 4$ the radial version is the same as the process defined in Proposition 3.4.

This construction clarifies the connection between locally independent $SLE$ and commuting $SLE$ defined in [Dub07]. Intuitively, given a sequence of commuting $SLE$ increments, as the time duration of the increments goes to 0, the curves converge to locally independent $SLE$.

Throughout this section we write $\mathbf{B}_t = (\mathbf{B}^1_t, \mathbf{B}^2_t)$ for a standard two-dimensional Brownian motion, that is, two independent one-dimensional Brownian motions. We will use the fact that $\mathbf{B}_t$ is H"older continuous. We give a quantitative version here which is stronger than we need.

- Let $E_h$ denote the event that for all $0 \leq t \leq 1/h$ and all $0 \leq s \leq h$,
  $$|\mathbf{B}_{t+s} - \mathbf{B}_t| \leq s^{1/2} \log(1/s).$$

  Then as $h \to 0$, $\mathbb{P}(E_h^c)$ decays faster than any power of $h$.

We will define the discrete approximation using the same Brownian motions as for the continuum and then the convergence follows from deterministic estimates coming from the Loewner equation. Since these are standard we will not give full details. We first define the process. Let $a = 2/\kappa$.

Definition Let $\mathbf{X}_t = (X^1_t, X^2_t)$ be the solution to the SDEs,

$$dX^1_t = \frac{a}{X^1_t - X^2_t} dt + dB^1_t, \quad dX^2_t = \frac{a}{X^2_t - X^1_t} dt + dB^2_t,$$

with $X^1_0 = x_1, X^2_0 = x_2$. Let $\tau_u = \inf\{t : |X^2_t - X^1_t| \leq u\}$, $\tau = \tau_{0+}$.
Note that \( Z_t := X_t^2 - X_t^1 \) satisfies
\[
dZ_t = \frac{2a}{Z_t} \, dt + \sqrt{2} \, dW_t,
\]
where \( W_t := (B_t^2 - B_t^1) / \sqrt{2} \) is a standard Brownian motion. This is a (time change of a) Bessel process from which we see that \( \mathbb{P}\{\tau < \infty\} = 0 \) if and only if \( \kappa \leq 4 \). If \( 4 < \kappa < 8 \) we can continue the process for all \( \tau < \infty \) by using reflection. We will consider only \( \kappa < 8 \).

**Definition** If \( \kappa < 8 \), locally independent \( SLE_\kappa \) is defined to be the collection of conformal maps \( g_t \) satisfying the Loewner equation
\[
\partial_t g_t(z) = \frac{a}{g_t(z) - X_t^1} + \frac{a}{g_t(z) - X_t^2}, \quad g_0(z) = z.
\]
This is defined up to time
\[
T_z = \sup\{t : \text{Im}[g_t(z)] > 0\}.
\]
Locally independent \( SLE_\kappa \) produces a pair of curves \( \gamma(t) = (\gamma^1(t), \gamma^2(t)) \). Note that \( \text{hcap}[\gamma_t] = 2at \). If \( \kappa \leq 4 \), then \( \gamma^1_t \cap \gamma^2_t \neq \emptyset \); this is not true for all \( t < \tau \) if \( 4 < \kappa < 8 \).

Let us fix a small number \( h = 1/n \) and consider the process viewed at time increments \( \{kh : k = 0, 1, \ldots\} \). The following estimates hold uniformly for \( k \leq 1/h \) on the event \( E_h \). The first comes just by the definition of the SDE and the second uses the Loewner equation. Let \( \Delta^j_k = \Delta_{k,h}^j = B_{kh}^j - B_{(k-1)h}^j \).

- If \( |X_{kh}^{2} - X_{kh}^{1}| \geq h^{1/8} \), then
  \[
  X_{(k+1)h}^j = X_{kh}^j + \frac{ah}{X_{kh}^j - X_{kh}^{3-j}} + \Delta_{k+1}^j + o(h^{4/3}). \tag{32}
  \]
- If \( \text{Im}[g_{kh}(z)] \geq u/2 \), and \( 0 \leq s \leq h \),
  \[
  g_{kh+s}(z) = g_{kh}(z) + \frac{as}{g_{kh}(z) - X_{kh}^1} + \frac{as}{g_{kh}(z) - X_{kh}^2} + o_u(h^{4/3}). \tag{33}
  \]

We will compare this process to the process which at each time \( kh \) grows independent \( SLE_\kappa \) paths in the current domain, increasing the capacity of each path by \( h \). Let us start with the first time period in which we have independent \( SLE \) paths. Again, we restrict to the event \( E_h \).

- Let \( \tilde{\gamma}^1, \tilde{\gamma}^2 \) be independent \( SLE_\kappa \) paths starting at \( x_1, x_2 \) respectively with driving function \( \dot{X}_t^j = B_t^j \), each run until time \( h \). To be more precise if \( \tilde{g}_t^j : \mathbb{H} \setminus \gamma_t^j \to \mathbb{H} \) is the standard conformal transformation, then
  \[
  \partial_t \tilde{g}_t^j(z) = \frac{a}{\tilde{g}_t^j(z) - X_t^j}, \quad \tilde{g}_0^j(z) = z, \quad 0 \leq t \leq h
  \]
  Note that \( \text{hcap}[\gamma_t^1] = \text{hcap}[\gamma_t^2] = ah \). Although \( \text{hcap}[\gamma_t] < 2ah \), if \( |x_2 - x_1| \geq h^{1/8} \),
  \[
  \text{hcap}[\gamma_t] = 2ah - o(h^{4/3}).
  \]
This defines $\gamma_t$ for $0 \leq t \leq h$ and we get corresponding conformal maps

$$\tilde{g}_t : \mathbb{H} \setminus \gamma_t \to \mathbb{H}, \quad 0 \leq t \leq h.$$ 

If $\text{Im}[z] \geq 1/2$, then

$$\tilde{g}_h(z) = z + \frac{ah}{z-x^1} + \frac{ah}{z-x^2} + o(h^{4/3}).$$

Also, by writing $\tilde{g}_h = \phi \circ \tilde{g}_j^h$, we can show that

$$\hat{X}_j^h := \tilde{g}_h(\tilde{\gamma}^j(h)) = \hat{X}_{j}^h + \frac{ah}{x_j - x_{3-j}} + o(h^{4/3}).$$

- Recursively, given $\hat{X}_{kh}$ and $\hat{\gamma}^1_{kh}, \hat{\gamma}^2_{kh}$ and $\tilde{g}_t$ for $0 \leq t \leq kh$ (the definition of these quantities depends on $h$ but we have suppressed that from the notation), let

$$\hat{X}_{kh+t} = \hat{X}_{kh} + [B_{kh+t}^j - B_k^j], \quad 0 \leq t \leq h,$$ 

and let $\hat{\gamma}^1_{kh+t}, \hat{\gamma}^2_{kh+t}, 0 \leq t \leq h$ be independent $SLE_\kappa$ paths with driving functions $\hat{X}_{kh+t}$. For $j = 1, 2$, define

$$\hat{\gamma}^j_{kh+t} = \tilde{g}^{-1}_k(\hat{\gamma}^j_{t,k}), \quad 0 \leq t \leq h.$$ 

This defines $\hat{\gamma}_{kh+t}, 0 \leq t \leq h$ and $\tilde{g}_{kh+t} : \mathbb{H} \setminus \hat{\gamma}_{kh+t} \to \mathbb{H}$ is defined as before. Set

$$\hat{X}_{(k+1)h} = \tilde{g}_{(k+1)h}(\hat{\gamma}^j((k+1)h)).$$

Note that if $|\hat{X}_{kh}^1 - \hat{X}_{kh}^1| \geq h^{1/8}$, then

$$\hat{X}_{(k+1)h}^j = \hat{X}_{kh}^j + \Delta_{(k+1)h}^j + \frac{ah}{\hat{X}_{kh}^j - \hat{X}_{kh}^{3-j}} + o(h^{4/3}). \quad (34)$$

Also, if $\text{Im}[\tilde{g}_{kh}(z)] \geq u/2$,

$$\tilde{g}_{(k+1)h}(z) = \tilde{g}_{kh}(z) + \frac{ah}{\tilde{g}_{kh}(z) - \hat{X}_{kh}^1} + \frac{ah}{\tilde{g}_{kh}(z) - \hat{X}_{kh}^2} + o_u(h^{4/3}). \quad (35)$$

- If at any time $\hat{\gamma}^1_{h,k} \cap \hat{\gamma}^2_{h,k} \neq \emptyset$ this procedure is stopped.

Note that we are using the same Brownian motions as we used before.

**Proposition 4.1.** With probability one, for all $t < \tau$ and all $z \in \mathbb{H} \setminus \gamma_t$,

$$\lim_{h \downarrow 0} \tilde{g}_t(z) = g_t(z).$$
Proof. We actually prove more. Let
\[ K(u, h) = \sup \{|\tilde{g}_t(z) - g_t(z)| : \text{Im}[g_t(z)] \geq u, t \leq \tau_u \wedge u^{-1}\} \, . \]
Then for each \( u > 0 \), with probability one,
\[ \lim_{h \downarrow 0} K(u, h) = 0. \]
We fix \( u \) and allow constants to depend on \( u \) and assume that \( \text{Im}[g_t(z)] \geq u \). Then, if
\[ \Theta_k = \max \{|X^i_{kh} - \tilde{X}^i_{kh}|\} \]
Then (32) and (34) imply that
\[ \Theta_{k+1} \leq \Theta_k[1 + O(h)] + O(h^{4/3}), \]
or if \( \tilde{\Theta}_k = \Theta_k + kh^{4/3} \), then \( \tilde{\Theta}_k \leq \Theta_k[1 + O(h)] \). This shows that \( \Theta_k \) is bounded for \( k \leq (hu)^{-1} \) and hence
\[ \Theta_k \leq ch^{1/3}, \quad k \leq (hu)^{-1}. \quad (36) \]
We now let
\[ D_k = \max_{r \leq k} |g_{kh}(z) - \tilde{g}_{kh}(z)| \]
and see that (33), (35), and (36) imply
\[ D_{k+1} \leq D_k[1 + O(h)] + O(h^{4/3}), \]
which then gives
\[ D_k \leq ch^{1/3}, \quad k \leq (hu)^{-1}. \]
Note that for \( kh \leq t \leq (k+1)h \).
\[ g_t(z) = g_{kh}(z) + O(h), \quad \tilde{g}_t(z) = \tilde{g}_{kh}(z) + O(h), \]
and hence for all \( t \leq \tau_u \wedge u^{-1} \)
\[ \tilde{g}_t(z) = g_t(z) + O(h^{1/3}). \]
\[ \square \]

5 \( n \)-Radial Bessel process

In this section we study the process that we call the \( n \)-particle radial Bessel process. The image of this process under the map \( z \mapsto e^{2iz} \) will be called Dyson Brownian motion on the circle. We fix integer \( n \geq 2 \) and allow constants to depend on \( n \). Let \( \mathcal{X}' = \mathcal{X}'_n \) be the torus \([0, \pi)^n\) with periodic boundary conditions and \( \mathcal{X} = \mathcal{X}_n \) the set of \( \theta = (\theta^1, \ldots, \theta^n) \in \mathcal{X}' \) such that we can find representatives with
\[ \theta^1 < \theta^2 < \cdots < \theta^n < \theta^{n+1} := \theta^1 + \pi. \quad (37) \]
Let \( \mathcal{X}_n^* \) be the set of \( z = (z^1, \ldots, z^n) \) with \( z^j = \exp\{2i\theta^j\} \) and \( \theta \in \mathcal{X} \). In other words, \( \mathcal{X}_n^* \) is the set of \( n \)-tuples of distinct points on the unit circle ordered counterclockwise (with a choice of a first point). Note that \( |z^j - z^k| = 2|\sin(\theta^k - \theta^j)| \). We let

\[
\psi(\theta) = \sum_{j=1}^n \sum_{k \neq j} \csc^2(\theta^j - \theta^k) = 2 \sum_{1 \leq j < k \leq n} \csc^2(\theta^j - \theta^k),
\]

\[
F(\theta) = \prod_{1 \leq j < k \leq n} |\sin(\theta^k - \theta^j)| = 2^{-n(n-1)/2} \prod_{1 \leq j < k \leq n} |z_k - z_j|,
\]

\[
F_\alpha(\theta) = F(\theta)\alpha,
\]

\[
d(\theta) = \min_{1 \leq j < k \leq n} |\sin(\theta^{j+1} - \theta^j)|
\]

\[
f_\alpha(\theta) = T_\alpha^{-1} F_\alpha(\theta), \quad T_\alpha = \int_\mathcal{X} F_\alpha(\theta) \, d\theta.
\]

Here \( d\theta \) denotes integration with respect to Lebesgue measure restricted to \( \mathcal{X} \).

**Remark** We choose to represent points \( z^j \) on the unit circle as \( \exp\{2i\theta^j\} \) (rather than \( \exp\{i\theta^j\} \)) because the relation

\[
F_\alpha(\theta) = 2^{-\alpha n(n-1)/2} \prod_{1 \leq j < k \leq n} |z_k - z_j|^\alpha,
\]

makes it easy to relate measures on \( \mathcal{X}_n^* \) with measures that arise in random matrices. (See, for example, Chapter 2 of [For10] for the distribution of the eigenvalues of the circular \( \beta \)-ensemble.) Note that if \( \theta^1, \ldots, \theta^n \) are independent standard Brownian motions, then \( z^1, \ldots, z^n \) are independent driftless Brownian motions on the circle with variance parameter 4.

We will use the following trigonometric identity.

**Lemma 5.1.** If \( \theta \in \mathcal{X}_n^* \),

\[
\sum_{j=1}^n \left( \sum_{k \neq j} \cot(\theta^j - \theta^k) \right)^2 = \psi(\theta) - \frac{n(n^2 - 1)}{3}.
\]

**Proof.** We first note that if \( x, y, z \) are distinct points in \([0, \pi)\), then

\[
\cot(x - y) \cot(x - z) + \cot(y - x) \cot(y - z) + \cot(z - x) \cot(z - y) = -1
\]

Indeed, without loss of generality, we may assume that \( 0 = x < y < z \) in which case the lefthand side is

\[
\cot(y - z) \left[ \cot y - \cot z \right] + \cot y \cot z
\]

which equals \(-1\) using the sum formula

\[
\cot(y - z) = \frac{\cot y \cot z + 1}{\cot z - \cot y}
\]
When we expand the square on the lefthand side of (38) we get the sum of two terms,

\[ \sum_{j=1}^{n} \sum_{k \neq j} \cot^2(\theta^j - \theta^k) \]  \hspace{1cm} (40)

\[ \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \{ j \neq k, j \neq l, k \neq l \} \cot(\theta^j - \theta^k) \cot(\theta^j - \theta^l). \]  \hspace{1cm} (41)

Using \( \cot^2 y + 1 = \csc^2 y \), we see that (40) equals \( \psi(\theta) - n(n-1) \). We write (41) as 2 times

\[ \sum \left[ \cot(\theta^j - \theta^k) \cot(\theta^j - \theta^l) + \cot(\theta^k - \theta^l) \cot(\theta^l - \theta^j) + \cot(\theta^k - \theta^j) \cot(\theta^j - \theta^l) \right], \]

where the sum is over all 3 elements subsets \( \{ j, k, l \} \) of \( \{ 1, \ldots, n \} \). Using (39), we see that (41) equals

\[ -2 \left( \frac{n}{3} \right) = -\frac{n(n-1)(n-2)}{3}. \]

Therefore, the lefthand side of (38) equals

\[ \psi(\theta) - n(n-1) - \frac{n(n-1)(n-2)}{3} = \psi(\theta) - \frac{n(n^2 - 1)}{3}. \]

We will let \( \theta_t = (\theta_1^t, \ldots, \theta_n^t) \) be a standard \( n \)-dimensional Brownian motion in \( \mathcal{X}^* \) starting at \( \theta_0 \in \mathcal{X} \) and stopped at

\[ T = \inf \{ t : \theta_t \notin \mathcal{X} \} = \inf \{ t : d(\theta_t) = 0 \}, \]

defined on the filtered probability space \( (\Omega, \mathcal{F}_t, P) \).

Differentiation using (38) shows that

\[ \partial_j F_\alpha(\theta) = F_\alpha(\theta) \sum_{k \neq j} \alpha \cot(\theta^j - \theta^k), \]

\[ \partial_{jj} F_\alpha(\theta) = F_\alpha(\theta) \left[ \left( \sum_{k \neq j} \alpha \cot(\theta^j - \theta^k) \right)^2 - \alpha \sum_{k \neq j} \csc^2(\theta^j - \theta^k) \right], \]

\[ \Delta F_\alpha(\theta) = F_\alpha(\theta) \left[ -\frac{\alpha^2 n(n^2 - 1)}{3} + (\alpha^2 - \alpha) \psi(\theta) \right]. \]

Hence, if we define

\[ M_{t,\alpha} := F_\alpha(\theta_t) \exp \left\{ -\frac{1}{2} \int_0^t \frac{\Delta F_\alpha(\theta_s)}{F_\alpha(\theta_s)} \, ds \right\} \]

\[ = F_\alpha(\theta_t) \exp \left\{ \frac{\alpha^2 n(n^2 - 1)}{6} t \right\} \exp \left\{ \frac{\alpha - \alpha^2}{2} \int_0^t \psi(\theta_s) \, ds \right\}, \quad t < T, \]  \hspace{1cm} (42)

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then $M_{t,\alpha}$ is a local martingale for $0 \leq t < T$ satisfying

$$dM_{t,\alpha} = M_{t,\alpha} \sum_{j=1}^{n} \left( \sum_{k \neq j} \alpha \cot(\theta_{1}^{j} - \theta_{1}^{k}) \right) d\theta_{1}^{j}.$$ 

We will write $\mathbb{P}_{\alpha}, \mathbb{E}_{\alpha}$ for the probability measure obtained after tilting $\mathbb{P}$ by $M_{t,\alpha}$ using the Girsanov theorem. Then

$$d\theta_{1}^{j} = \alpha \sum_{k \neq j} \cot(\theta_{1}^{j} - \theta_{1}^{k}) dt + dW_{1}^{j}, \quad t < T,$$  

for independent standard Brownian motions $W_{1}^{1}, \ldots, W_{1}^{n}$ with respect to $\mathbb{P}_{\alpha}$. If $\alpha \geq 1/2$, comparison with the usual Bessel process shows that $\mathbb{P}_{\alpha}(T = \infty) = 1$. In particular, $M_{t,\alpha}$ is a martingale and $\mathbb{P}_{\alpha} \ll \mathbb{P}$ on $\mathcal{F}_{t}$ for each $t$. (It is not true that $\mathbb{P} \ll \mathbb{P}_{\alpha}$ since $\mathbb{P} \{ T < t \} > 0$.)

This leads to the following definitions.

**Definition.** The $n$-radial Bessel process with parameter $\alpha$ is the process satisfying (43) where $W_{1}^{1}, \ldots, W_{1}^{n}$ are independent Brownian motions.

**Proposition 5.2.** If $\theta_{t}$ satisfies (43) and $\tilde{\theta}_{t} = \theta_{t/n}$, then $\tilde{\theta}_{t}$ satisfies

$$d\tilde{\theta}_{t} = \frac{\alpha}{n} \sum_{k \neq j} \cot(\tilde{\theta}_{1}^{j} - \tilde{\theta}_{1}^{k}) dt + \frac{1}{\sqrt{n}} d\tilde{W}_{1}^{j}, \quad t < \tilde{T},$$

where $\tilde{W}_{1}^{1}, \ldots, \tilde{W}_{1}^{n}$ are independent Brownian motions and $\tilde{T} = nT$.

We also refer to a process satisfying (44) as the $n$-radial Bessel process. If $n = 2$, $\tilde{\theta}_{1}^{1}, \tilde{\theta}_{1}^{2}$ satisfy (44) and

$$X_{t} = \tilde{\theta}_{1}^{2} - \tilde{\theta}_{1}^{1}, \quad B_{t} = \frac{1}{\sqrt{2}} [\tilde{W}_{1}^{2} - \tilde{W}_{1}^{1}],$$

then $B_{t}$ is a standard Brownian motion and $X_{t}$ satisfies

$$dX_{t} = \alpha \cot X_{t} dt + dB_{t}.$$ 

This equation is called the radial Bessel equation.

**Proposition 5.3.** Let $p_{t,\alpha} (\theta, \theta')$ denote the transition density for the system (43). Then for all $t$ and all $\theta, \theta'$,

$$p_{t,\alpha} (\theta, \theta') = \frac{F_{2\alpha} (\theta')}{F_{2\alpha} (\theta)} p_{t,\alpha} (\theta', \theta).$$  

**Proof.** Let $p_{t} = p_{t,0}$ be the transition density for independent Brownian motions killed at time $T$. Fix $t, \theta, \theta'$. Let $\gamma : [0, t] \to \mathcal{X}$ be any curve with $\gamma(0) = \theta, \gamma(t) = \theta'$ and note that the Radon-Nikodym derivative of $\mathbb{P}_{\alpha}$ with respect to $P$ evaluated at $\gamma$ is

$$Y(\gamma) := \frac{F_{\alpha} (\theta')}{F_{\alpha} (\theta)} A_{t}(\gamma), \quad A_{t}(\gamma) = e^{\alpha^{2}(n-1)t/2} \exp \left\{ \frac{-\alpha^{2}}{2} \int_{0}^{t} \psi(\gamma(s)) ds \right\}.$$  

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If \( \gamma^R \) is the reversed path, \( \gamma^R(s) = \gamma(t-s) \), then \( A_t(\gamma^R) = A_t(\gamma) \) and hence
\[
Y(\gamma^R) = \frac{F_\alpha(\theta)}{F_\alpha(\theta')} A_t.
\]
Therefore,
\[
\frac{Y(\gamma)}{Y(\gamma^R)} = \frac{F_\alpha(\theta')} {\frac{F_\alpha(\theta)}{F_\alpha(\theta')}} = \frac{F_{2\alpha}(\theta')}{F_{2\alpha}(\theta)}.
\]
Since the reference measure \( P \) is time reversible and the above holds for every path, (45) holds.

**Proposition 5.4.** If \( \alpha \geq 1/2 \) and \( \theta_t \) satisfies (43), then with probability one \( T = \infty \).

**Proof.** This follows by comparison with a usual Bessel process; we will only sketch the proof. Suppose \( P\{T < \infty\} > 0 \). Then there would exist \( j < k \) such that with positive probability \( \gamma^j_T = \gamma^k_T \) but \( \gamma^{j-1}_T < \gamma^j_T \) and \( \gamma^{k+1}_T > \gamma^k_T \) (here we are using “modular arithmetic” for the indices \( j, k \) in our torus). If \( k = j + 1 \), then by comparison to the process
\[
dX_t = \left( \frac{a}{X_t} - r \right) dt + dB_t,
\]
one can see that this cannot happen. If \( k > j + 1 \), we can compare to the Bessel process obtained by removing the points with indices \( j + 1 \) to \( k - 1 \). \( \square \)

**Proposition 5.5.** If \( \alpha \geq 1/2 \), then the invariant density for (43) is \( f_{2\alpha} \). Moreover, there exists \( u > 0 \) such that for all \( \theta, \theta' \),
\[
p_t(\theta', \theta) = f_{2\alpha}(\theta) \left[ 1 + O(e^{-ut}) \right]. \tag{46}
\]

**Proof.** The fact that \( f_{2\alpha} \) is invariant follows from
\[
\int p_{t,\alpha}(\theta, \theta') f_{2\alpha}(\theta) d\theta = \int p_{t,\alpha}(\theta, \theta') \frac{F_{2\alpha}(\theta')}{F_{2\alpha}(\theta)} f_{2\alpha}(\theta) d\theta
\]
\[
= f_{2\alpha}(\theta') \int p_{t,\alpha}(\theta', \theta) d\theta = f_{2\alpha}(\theta').
\]

Proposition 5.7 below shows that there exist \( 0 < c_1 < c_2 < \infty \) such that for all \( x, y \in \mathcal{X} \),
\[
c_1 F_{2\alpha}(y) \leq p_1(x, y) \leq c_2 F_{2\alpha}(y). \tag{47}
\]
The proof of this fact is the subject of §5.1. The exponential rate of convergence (46) then follows by a standard coupling argument (see, for example, §4 of [Law15]). \( \square \)

**Proposition 5.6.** Suppose \( \alpha \geq 1/4 \) and
\[
N_{t,\alpha,2\alpha} = F_\alpha(\theta_t) \exp \left\{ \frac{\alpha^2 n(n^2 - 1)}{2} t \right\} \exp \left\{ -\alpha b_\alpha \int_0^t \psi(\theta_s) ds \right\},
\]
where \( b_\alpha = (3\alpha - 1)/2 \). Then \( N_{t,\alpha,2\alpha} \) is a \( \mathbb{P}_\alpha \)-martingale, and the measure obtained by tilting \( \mathbb{P}_\alpha \) by \( N_{t,\alpha,2\alpha} \) is \( \mathbb{P}_{2\alpha} \).
Proof. Note that

\[ M_{t,2\alpha} = F_{2\alpha}(\theta_t) \exp \left\{ \frac{(2\alpha)^2 n(n^2 - 1)}{6} \right\} \exp \left\{ \frac{2\alpha - (2\alpha)^2}{2} \int_0^t \psi(\theta_s) ds \right\}, \]

\[ = M_{t,\alpha} N_{t,\alpha,2\alpha}. \]

Since \( M_{t,\alpha}, M_{t,2\alpha} \) are both local martingales, we see that \( N_{t,\alpha,2\alpha} \) is a local martingale with respect to \( \mathbb{P}_\alpha \). Also, the induced measure by “tilting first by \( M_{t,\alpha} \) and then tilting by \( N_{t,\alpha,2\alpha} \)” is the same as tilting by \( M_{t,2\alpha} \). Since \( 2\alpha \geq 1/2 \), we see that with probability one, \( T = \infty \) in the new measure, from which we conclude that it is a martingale.  

We now prove Theorem 3.7.

Proof of Theorem 3.7. Using the last two propositions, we see that

\[ \mathbb{E}^\theta_\alpha \left[ \exp \left\{ -\alpha b \int_0^t \psi(\theta_s) ds \right\} \right] = \exp \left\{ \frac{-\alpha^2 n(n^2 - 1)}{2} \right\} \mathbb{E}^\theta_\alpha [N_t F_{-\alpha}(\theta_t)] \]

\[ = \exp \left\{ \frac{-\alpha^2 n(n^2 - 1)}{2} \right\} F_\alpha(\theta) \mathbb{E}^\theta_2 [F_{-\alpha}(\theta_t)] \]

\[ = e^{-2\alpha n \beta t} F_\alpha(\theta) \frac{T_{3\alpha}}{T_{4\alpha}} [1 + O(e^{-ut})]. \]

In particular, the last equality follows by applying Proposition 5.5 for \( \alpha = 2a \).  

Setting \( \alpha = a \), we can write the result as

\[ e^{2n(n-1)\beta t} \mathbb{E}^\theta_\alpha \left[ \exp \left\{ -ab \int_0^t \psi(\theta_s) ds \right\} \right] = e^{-2an\beta t} F_\alpha(\theta) \frac{T_{3\alpha}}{T_{4\alpha}} [1 + O(e^{-ut})] \]

where

\[ \beta_n = \beta - \tilde{b}(n - 1), \]

is the \( n \)-interior scaling exponent, as in (27).  

5.1 Rate of convergence to invariant density

It remains to verify the bounds (47) used in the proof of Proposition 5.5. While related results have appeared elsewhere, including \[\text{EY17}\], we have not found Proposition 5.7 in the literature, and so we provide a full proof here. This is a sharp pointwise result; however, unlike results coming from random matrices, the constants depend on \( n \) and we prove no uniform result as \( n \to \infty \). Our argument uses the general idea of a “separation lemma” (originally \[\text{Law96}\]).

We consider the \( n \)-radial Bessel process given by the system (43) with \( \alpha \geq 1/2 \). By Proposition 5.5 its invariant density is \( f_{2\alpha} \), so that for \( x, y \in \mathcal{X} \),

\[ \frac{p_t(x,y)}{F_{2\alpha}(y)} = \frac{p_t(y,x)}{F_{2\alpha}(x)}. \]

We will prove the following.
Proposition 5.7. For every positive integer \( n \) and \( \alpha \geq 1/2 \), there exist \( 0 < c_1 < c_2 < \infty \), such that for all \( x, y \in \mathcal{X} \),
\[
c_1 F_{2\alpha}(y) \leq p_t(x, y) \leq c_2 F_{2\alpha}(y).
\]

For the remainder of this section, we fix \( n \) and \( \alpha \geq 1/2 \) and allow constants to depend on \( n, \alpha \). We will let \( \tilde{p}_t(x, y) \) denote the transition density for independent Brownian motions killed upon leaving \( \mathcal{X} \). If \( U \subset \mathcal{X} \) we will write \( p_t(x, y; U) \) or \( p_t(x, y; \bar{U}) \) for the density of the \( n \)-radial Bessel process killed upon leaving \( U \); we write \( \tilde{p}_t(x, y; U) \) or \( \tilde{p}_t(x, y; \bar{U}) \) for the analogous densities for independent Brownian motions. Then we have
\[
p_t(x, y; U) = \tilde{p}_t(x, y; U) \frac{M_{t,\alpha}}{M_{0,\alpha}}.
\]  

(48)

We can use properties of the density of Brownian motion to conclude analogous properties for \( p_t(x, y) \). For example we have the following:

- For every open \( U \) with \( \bar{U} \subset \mathcal{X} \) and every \( t_0 \), there exists \( C = C(U, t_0) \) such that
\[
C^{-1} \tilde{p}_t(x, y, U) \leq p_t(x, y; U) \leq C \tilde{p}_t(x, y; U), \quad 0 \leq t \leq t_0.
\]  

(49)

Indeed, \( M_{t,\alpha}/M_{0,\alpha} \) is uniformly bounded away from 0 and \( \infty \) for \( t \leq t_0 \) and paths staying in \( \bar{U} \). Another example is the following easy lemma which we will use in the succeeding lemma.

Lemma 5.8. Suppose that \( 0 \leq \theta_0^1 < \theta_0^2 < \theta_0^n < \theta_0^{n+1} \) where \( \theta_0^{j+1} = \theta_0^j + \pi \). For every \( r > 0 \), there exists \( q > 0 \) such that the following holds. If \( \epsilon \leq 1/(8n) \) and \( \theta_0^{j+1} - \theta_0^j \geq r \epsilon \) for \( j = 1, \ldots, n \), then with probability at least \( q \) the following holds:
\[
\theta_t^{j+1} - \theta_t^j \geq 2 \epsilon, \quad \epsilon^2/2 \leq t \leq \epsilon^2, \quad j = 1, \ldots, n,
\]
\[
\theta_t^{j+1} - \theta_t^j \geq \epsilon r/2, \quad 0 \leq t \leq \epsilon, \quad j = 1, \ldots, n,
\]
\[
|\theta_t^j - \theta_0^j| \leq 4 \epsilon, \quad 0 \leq t \leq \epsilon^2, \quad j = 1, \ldots, n.
\]

Proof. If \( \theta_0^j \) were independent Brownian motions, then scaling shows that the probability of the event is independent of \( \epsilon \) and it is easy to see that it is positive. Also, on this event \( M_{t,\alpha}/M_{0,\alpha} \) is uniformly bounded away from 0 uniformly in \( \epsilon \). \( \square \)

The next lemma shows that there is a constant \( \delta > 0 \) such that from any initial configuration, with probability at least \( \delta \) all the points are separated by \( 2 \epsilon \) by time \( \epsilon^2 \).

Lemma 5.9. There exists \( \delta > 0 \) such that if \( 0 < d(\theta_0) \leq \epsilon < \delta \), then
\[
P\{d(\theta_{t^2}) \geq 2 \epsilon \} \geq \delta.
\]

Moreover, if \( \tau = \tau_\epsilon = \inf\{t : d(\theta_t) = 2 \epsilon\} \), then for all positive integers \( k \),
\[
P\{\tau > k \epsilon^2\} \leq (1 - \delta)^k.
\]
Proof. The second inequality follows immediately from the first and the Markov property. We will prove a slightly stronger version of the first inequality result. Let

\[ Y_t = \max_{j=1, \ldots, n} \max_{0 \leq s \leq t} |\theta_s^j - \theta_0^j|. \]

Then we will show that there exists \( \delta_n \) such that

\[ \mathbb{P}\{d(\theta, \varepsilon^2) \geq 2\varepsilon; Y_{\varepsilon^2} \leq \delta_n^{-1} \varepsilon\} \geq \delta_n. \]  

(50)

We have put the explicit \( n \) dependence on \( \delta_n \) because our argument will use induction on \( n \). Without loss of generality we assume that

\[ 0 = \theta_1 < \theta_2 < \cdots < \theta_n < \pi \]

and \( \pi - \theta_n \geq \theta_j - \theta_{j-1} \) for \( j = 2 \ldots n \).

For \( n = 2 \), \( \theta_2^t - \theta_1^t \) is a radial Bessel process which for small \( \epsilon \) is very closely approximated by a regular Bessel process. Either by using the explicit transition density or by scaling, we see that there exists \( c_1 > 0 \) such that for all \( \epsilon \leq 1 \), if \( \theta_2^t - \theta_1^t \leq \epsilon \),

\[ \mathbb{P}\{\theta_2^t - \theta_1^t \geq 2\epsilon; \theta_2^t - \theta_1^t \leq 4\epsilon \text{ for } 0 \leq t \leq \epsilon^2\} \geq c_1. \]

Let \( A_\epsilon \) denote the event

\[ A_\epsilon = \{\theta_2^t - \theta_1^t \geq 2\epsilon; \theta_2^t - \theta_1^t \leq 4\epsilon \text{ for } 0 \leq t \leq \epsilon^2; |W_1^t|, |W_2^t| \leq u\epsilon, 0 \leq t \leq \epsilon^2\}, \]

where \( u \) is chosen so that

\[ \mathbb{P}\left\{ \max_{0 \leq t \leq 1} |W_j^t| \geq u \right\} = c_1/4. \]

Then \( \mathbb{P}(A_\epsilon) \geq c_1/2. \) Since

\[ \theta_1^t = \alpha \int_0^t \cot(\theta_1^s - \theta_3^s) \, ds + W_1^t, \]

we see that

\[ 2\alpha \int_0^{\epsilon^2} \cot(\theta_2^s - \theta_1^s) \, ds = \theta_2^\epsilon - \theta_1^\epsilon - W_2^\epsilon + W_1^\epsilon \leq (4 + 2u) \epsilon. \]

and for \( 0 \leq t \leq \epsilon^2 \),

\[ |\theta_1^t - \theta_0^t| \leq \alpha \int_0^t \cot(\theta_1^s - \theta_3^s) \, ds + |W_1^t| \leq (2 + 2u) \epsilon. \]

This establishes (50) for \( n = 2 \).

We now assume that (50) holds for all \( j < n \). We claim that it suffices to prove that there exists \( \delta_n > 0 \) such that if \( d(\theta_0) \leq \epsilon < \delta_n \), then

\[ \mathbb{P}\{\tau_\epsilon \leq \delta_n^{-2} \epsilon^2, Y_{\tau_\epsilon} \leq \delta_n^{-1} \epsilon\} \geq \delta_n. \]

(51)
If we apply (51) to $\tau_{\delta_n/2}$ we can use Lemma 5.8 to conclude (50) for $n$.

Let us first assume that $\epsilon \leq \delta_{n-1}$ and that there exists $j \in \{1, \ldots, n - 1\}$ with $\theta^{j+1} - \theta^j \geq \epsilon$. Consider independent $j$-radial and $(n-j)$-radial Bessel processes $(\theta^1_t, \ldots, \theta^j_t)$ and $(\theta^{j+1}_t, \ldots, \theta^n_t)$. In other words, remove the terms of the form

$$\pm \cot(\theta^m_t - \theta^k_t), \quad k \leq j < j + 1 \leq m$$

from the drift in the $n$-radial Bessel process, so that now particles $\{1, \ldots, j\}$ do not interact with particles $\{j + 1, \ldots, n\}$. Using the inductive hypothesis, we can find a $\lambda$ such that with probability at least $\lambda$ we have

$$d(\theta^{j+1}_t, \theta^j_t) \geq 2\lambda \epsilon, \quad Y^{j+1}_t \leq \epsilon^2.$$

This calculation is done with respect to the two independent processes but we note that on this event,

$$\theta^{j+1}_t - \theta^j_t \geq \frac{\epsilon}{2}, \quad 0 \leq t \leq \lambda^2 \epsilon^2.$$

Hence we get a lower bound on the Radon-Nikodym derivative between the $n$-radial Bessel process and the two independent processes.

Now suppose there is no such separation. Let $\sigma_{\epsilon} = \inf \{t : |\theta^o_t - \theta^1_t| \geq n \epsilon\}$; it is possible that $\sigma_{\epsilon} = 0$. If there were no other particles, $\theta^o_t - \theta^1_t$ would be a radial Bessel process. The addition of other particles pushes the first particle more to the left and the $n$th particle more to the right. Hence by comparison, we see that

$$\mathbb{P}\{\sigma_{\epsilon} \leq n^2 \epsilon^2\} \geq c_1$$

and as above we can find $u$ such that

$$\mathbb{P}\left\{\sigma_{\epsilon} \leq n^2 \epsilon^2 ; \max_{0 \leq t \leq n^2 \epsilon^2} |W^j_t| \leq u \epsilon\right\} \geq \frac{c_1}{2},$$

and hence (with a different value of $u$)

$$\mathbb{P}\{\sigma_{\epsilon} \leq n^2 \epsilon^2 ; Y_{\sigma_{\epsilon}} \leq u \epsilon\} \geq \frac{c_1}{2}.$$

Note that on this good event there exists at least one $j = 1, \ldots, n-1$ with $\theta^{j+1}_{\sigma_{\epsilon}} - \theta^j_{\sigma_{\epsilon}} \geq \epsilon$.

For $\zeta > 0$, let $V_{\zeta} = \{\theta \in X : d(\theta) \geq 2^{-\zeta}\}$ and let $\sigma_{\zeta}$ denote the first time that the process enters $V_{\zeta}$:

$$\sigma_{\zeta} = \inf \left\{t : d(\theta_t) \geq 2^{-\zeta}\right\}.$$

Define

$$r = r(\delta) = \min\{k \in \mathbb{Z} : 2^{-k} < \delta\},$$

where $\delta$ is as in Lemma 5.9. Note that $r$ is a fixed constant for the remainder of this proof.
Lemma 5.10. There exists \( q > 0 \) such that for any \( x \in \mathcal{X} \),
\[
\mathbb{P}^x\{\sigma_r \leq 1/4\} \geq q. \tag{53}
\]

Proof. We will construct a sequence of times \( \frac{1}{8} \leq t_1 \leq t_2 \leq \cdots \leq \frac{1}{4} \) such that if
\[
g_k = \inf_{x \in V_k} \mathbb{P}^x\{\sigma_r \leq t_k\}, \quad k \in \mathbb{N},
\]
then \( q = \inf_k g_k > 0 \). Using (49), we can see that \( g_k > 0 \) for each \( k \). To show that \( q > 0 \), it suffices to show that there is a summable sequence such that for all \( k \) sufficiently large, \( g_{k+1} \geq g_k (1 - u_k) \). We will do this with \( u_k = (1 - \delta)^k \) where \( \delta \) is as in Lemma 5.9.

For this purpose, denote
\[
s_k = k^2 2^{-2(k+1)}, \tag{54}
\]
and let \( l \geq r \) be sufficiently large so that
\[
\sum_{k=l}^{\infty} s_k \leq \frac{1}{8}.
\]
Define the sequence \( \{t_k\} \) by
\[
t_k = \begin{cases} \frac{1}{4}, & k \leq l \\ t_{k-1} + s_{k-1}, & k > l. \end{cases} \tag{55}
\]
This sequence satisfies \( \frac{1}{8} \leq t_1 \leq t_2 \cdots \leq \frac{1}{4} \). Applying Lemma 5.9, we see that if \( d(x) \leq 2^{-(k+1)} \), then
\[
\mathbb{P}^x \{\sigma_{k+1} \leq s_{k+1}\} \geq 1 - u_k.
\]
Therefore,
\[
\mathbb{P}^x \{\sigma_r \leq t_{k+1}\} \geq \mathbb{P}^x \{\sigma_{k+1} \leq s_{k+1}\} \mathbb{P}^x \{\sigma_r \leq t_{k+1}|\sigma_{k+1} \leq s_{k+1}\} \geq (1 - u_k) \inf_{z \in \partial V_k} \mathbb{P}^z \{\sigma_r \leq t_k\},
\]
so that for all \( k > r \),
\[
g_{k+1} \geq g_k (1 - u_k).
\]

We are now prepared to prove Proposition 5.7; we prove the upper and lower bounds separately.

Proof of Proposition 5.7, lower bound. We let
\[
\hat{\epsilon} = \inf \left\{ p_t(z, y) : z \in \partial V_r, y \in V_r, \frac{1}{4} \leq t \leq 1 \right\}. \tag{56}
\]
To see that this is positive, we use (49) to see that
\[
\hat{\epsilon} \geq c \inf \left\{ \tilde{p}_t(z, y; V_{r+1}) : z \in \partial V_r, y \in V_r, \frac{1}{4} \leq t \leq 1 \right\},
\]

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and straightforward arguments show that the righthand side is positive. Lemma 5.10 implies that for \( y \in V_r, x \in \mathcal{X} \), and \( \frac{1}{2} \leq t \leq 1 \),

\[
p_t(x, y) \geq \mathbb{P}^{x} \left\{ \sigma_{r} \leq \frac{1}{4} \right\} \hat{\epsilon} = q \hat{\epsilon}.
\]

Also, since \( F_{2\alpha} \) is bounded uniformly away from 0 in \( V_r \), we have

\[
p_t(y, x) = \frac{F_{2\alpha}(x)}{F_{2\alpha}(y)} p_t(x, y) \geq c F_{2\alpha}(x).
\]

This assumes \( x \in \mathcal{X}, y \in V_r \). More generally, if \( x, y \in \mathcal{X} \),

\[
p_1(x, y) \geq \int_{V_r} p_{1/2}(x, z) p_{1/2}(z, y) \, dz = F_{2\alpha}(y) \int_{V_r} p_{1/2}(x, z) p_{1/2}(y, z) F_{2\alpha}(z)^{-1} \, dz \geq c F_{2\alpha}(y).
\]

In order to prove the upper bound, we will need two more lemmas.

**Lemma 5.11.** If \( r \) is as defined in (52), then

\[
p^{\ast} := \sup \{ p_t(z, y) : z \in \partial V_{r+1}, y \in V_r, 0 \leq t \leq 1 \} < \infty.
\]

**Proof.** Let \( p^{\#} = \sup \{ p_t(z, y; V_{r+2}) : z \in \partial V_{r+1}, y \in V_r, 0 \leq t \leq 1 \} \).

By comparison with Brownian motion using (49) we can see that \( p^{\#} < \infty \). Also by comparison with Brownian motion, we can see that there exists \( \rho > 0 \) such that for all \( x \in \partial V_{r+2} \),

\[
\mathbb{P}^{x}\{ \theta[0, 1] \cap V_{r+1} = \emptyset \} \geq \rho.
\]

From this and the strong Markov property, we see that

\[
p^{\ast} \leq p^{\#} + (1 - \rho) p^{\ast}.
\]

(The term \( p^{\#} \) on the righthand side corresponds to paths from \( x \) to \( y \) that stay within \( V_{r+2} \). The term \( (1 - \rho) p^{\ast} \) corresponds to paths from \( x \) to \( y \) that hit \( \partial V_{r+2} \).) Therefore,

\[
p^{\ast} \leq \frac{p^{\#}}{\rho} < \infty.
\]

\[\square\]

**Lemma 5.12.** There exist \( c, \beta < \infty \) such that if \( t \geq 1, k > r, x \in \mathcal{X}, y \in V_{k+1} \setminus V_k \), then

\[
p_{2-2\delta t}(x, y; V_{k}^{c}) \leq c 2^{\beta k} (1 - \delta)^{t} F_{2\alpha}(y),
\]

where \( \delta > 0 \) is the constant in Lemma 5.9.
Proof. We first consider $t = 1$. For independent Brownian motions, we have

$$
\tilde{p}_2(\mathbf{x}, \mathbf{y}) \leq c \ 2^{nk}, \quad \mathbf{x} \in V_{k+1};
$$

$$
\tilde{p}_t(\mathbf{x}, \mathbf{y}) \leq c \ 2^{nk}, \quad 0 \leq t \leq 2^{-k}, \quad \mathbf{x} \in \partial V_{k+2}.
$$

There exist $\lambda, c'$ such that $F_{2\alpha}(z) \geq c' \ 2^{-\lambda k}$ for $\mathbf{z} \in V_{k+2}$. Using the first inequality and Lemma 5.11 implies that for each $k$

$$
p_{2^{-2k}}(\mathbf{x}, \mathbf{y}; V_{k+3}) \leq c \ 2^{(n+\lambda)k} \leq c \ 2^{2\beta n} F_{2\alpha}(y)
$$

$$
p_t(\mathbf{z}, \mathbf{y}; V_{k+3}) \leq c \ 2^{\beta k} F_{2\alpha}(y) \quad t \leq 2^{-2k}, \quad \mathbf{z} \in \partial V_{k+2},
$$

where $\beta = n + 2\lambda$. Arguing as in the previous lemma, we can conclude that

$$
p_{2^{-2k}}(\mathbf{x}, \mathbf{y}) \leq c \ 2^{\beta n} F_{2\alpha}(y).
$$

This gives the first inequality and the second follows from Lemma 5.9 using

$$
p_{2^{-2k}}(\mathbf{x}, \mathbf{y}; V_k^c) \leq \mathbb{P}^x \{ \mathbf{t}[0, 2^{-2k}(t-1)] \subset V_k^c \sup_{\mathbf{z} \in X} p_{2^{-2k}}(\mathbf{z}, \mathbf{y}) \}
$$

Proof of Proposition 5.7 upper bound. For each $k \in \mathbb{N}$, define

$$
J_k = \sup_{y \in V_k} \frac{p_t(\mathbf{x}, \mathbf{y})}{F_{2\alpha}(\mathbf{y})},
$$

where $\tilde{t}_k = t_k + 1/2$, so that

$$
\tilde{t}_k = \begin{cases} 
\frac{3}{4}, & k \leq l \\
\tilde{t}_{k-1} + s_{k-1}, & k > l,
\end{cases}
$$

where $s_k$ and $t_k$ are defined in (54) and (55) above. Since $F_{2\alpha}(y) \geq d(y)^{n(n-1)/2}$, Lemma 5.11 implies that for each $k$, $J_k < \infty$. We will show that $J_{k+1} \leq J_k + c_k$ for a summable sequence $\{c_k\}$, which implies that $\lim_{k \to \infty} J_k < \infty$.

To bound $J_{k+1}$, notice that if $y \in V_{k+1}$ and $t_{k+1} \leq t \leq 1$, we have the decomposition for arbitrary $\mathbf{x}$:

$$
p_t(\mathbf{x}, \mathbf{y}) = \frac{p_t(\mathbf{x}, \mathbf{y})}{F_{2\alpha}(\mathbf{x})} = \frac{p_t(\mathbf{y}, \mathbf{x}; \sigma_k \leq s_k)}{F_{2\alpha}(\mathbf{x})} + \frac{p_t(\mathbf{y}, \mathbf{x}; \sigma_k > s_k)}{F_{2\alpha}(\mathbf{x})}.
$$

The strong Markov property implies that the first term on the righthand side is equal to

$$
\int_{\partial V_k} p_{\sigma_k}(\mathbf{y}, \mathbf{z}; \sigma_k \leq s_k) \frac{p_{t-\sigma_k}(\mathbf{z}, \mathbf{x})}{F_{2\alpha}(\mathbf{x})} d\mathbf{z} \leq \mathbb{P}^y \{ \sigma_k \leq s_k \} \sup_{\mathbf{z} \in \partial V_k} \frac{p_s(\mathbf{x}', \mathbf{z})}{F_{2\alpha}(\mathbf{z})} \leq J_k.
$$
Using (58), we can see that the second term on the righthand side of (59) may be bounded by
\[
\frac{p_t(y, x; \sigma_k > s_k)}{F_{2\alpha}(x)} = \int_{Y_n} p_{s_k}(y, z; \sigma_k > s_k) \frac{p_{t-s_k}(z, x)}{F_{2\alpha}(x)} \, dz
\]
\[
= \int_{V_k^c} p_{s_k}(y, z; V_k^c) \frac{p_{t-s_k}(x, z)}{F_{2\alpha}(z)} \, dz
\]
\[
= \frac{1}{F_{2\alpha}(y)} \int_{V_k^c} p_{s_k}(z, y; V_k^c) p_{t-s_k}(x, z) \, dz
\]
\[
\leq e^{-O(k^2)} \int_{V_k^c} p_{t-s_k}(x, z) \, dz \leq e^{-O(k^2)}.
\]

Therefore,
\[
J_{k+1} \leq J_k + e^{-O(k^2)},
\]
completing the proof.

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