Multiple list colouring of planar graphs

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Abstract

This paper proves that for each positive integer \(m\), there is a planar graph \(G\) which is not \((4m + \lfloor \frac{2m-1}{9} \rfloor, m)\)-choosable. Then we pose some conjectures concerning multiple list colouring of planar graphs.

Keywords: fractional choice number, multiple list colouring, planar graph.

1 Introduction

A \(b\)-fold colouring of a graph \(G\) is a mapping \(\phi\) which assigns to each vertex \(v\) of \(G\) a set \(\phi(v)\) of \(b\) colours, so that adjacent vertices receive disjoint colour sets. An \((a, b)\)-colouring of \(G\) is a \(b\)-fold colouring \(\phi\) of \(G\) such that \(\phi(v) \subseteq \{1, 2, \ldots, a\}\) for each vertex \(v\). The fractional chromatic number of \(G\) is

\[\chi_f(G) = \inf \left\{ \frac{a}{b} : G \text{ is } (a, b)\text{-colourable} \right\}.\]

An \(a\)-list assignment of \(G\) is a mapping \(L\) which assigns to each vertex \(v\) a set \(L(v)\) of \(a\) permissible colours. A \(b\)-fold \(L\)-colouring of \(G\) is a \(b\)-fold colouring \(\phi\) of \(G\) such that \(\phi(v) \subseteq L(v)\) for each vertex \(v\). We say \(G\) is \((a, b)\)-choosable if for any \(a\)-list assignment \(L\) of \(G\), there is a \(b\)-fold \(L\)-colouring of \(G\). The fractional choice number of \(G\) is

\[ch_f(G) = \inf \left\{ \frac{a}{b} : G \text{ is } (a, b)\text{-choosable} \right\}.\]

It was proved by Alon, Tuza and Voigt [1] that for any finite graph \(G\), \(\chi_f(G) = ch_f(G)\) and moreover the infimum in the definition of \(ch_f(G)\) is attained and hence can be replaced by minimum. This implies that if \(G\) is \((a, b)\)-colourable, then for some integer \(m\), \(G\) is \((am, bm)\)-choosable. As every planar graph is 4-colourable (which is equivalent to \((4, 1)\)-colourable), we know for each planar graph \(G\), there is an integer \(m\) such that \(G\) is \((4m, m)\)-choosable. However, the integer \(m\) depends on \(G\). We prove in this paper that there is no integer \(m\) so that every planar graph is \((4m, m)\)-choosable.

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**Theorem 1** For each positive integer $m$, there is a planar graph $G$ which is not $(4m + \lfloor \frac{2m-1}{9} \rfloor, m)$-choosable.

The $m = 1$ case of Theorem 1 is equivalent to say that there are non-4-choosable planar graphs, which was proved by Voigt [8]. A smaller non-4-choosable planar graph was constructed [6], a 3-colourable non-4-choosable planar graph was constructed by Gutner [3, 9] and it was prove in [3] that it is NP-complete to decide if a given planar graph is 4-choosable.

## 2 The proof of Theorem 1

In this section, $m$ is a fixed positive integer. Let $k = \lfloor \frac{2m-1}{9} \rfloor$. We shall construct a planar graph $G$ which is not $(4m + k, m)$-choosable.

![Figure 1: The graph $G$](image)

**Lemma 2** Let $G$ be the graph as shown in Figure 1. There is a list assignment $L$ of $G$ for which the following hold:

1. $|L(s)| = 4m + k$ for each vertex $s$, except that $|L(u)| = |L(u')| = m$.

2. There is no $m$-fold $L$-colouring of $G$. 

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Proof. Let $A, B, C, D$ be pairwise disjoint sets of colours such that $|A| = |B| = m$ and $|C| = |D| = 2m + k$. Let $X, X'$ be disjoint subsets of $C$ with $|X| = |X'| = m$. Define list assignment $L$ as follows:

- $L(u) = A$ and $L(u') = B$.
- $L(v) = L(w) = L(t) = L(t') = A \cup B \cup C$.
- $L(x) = L(a) = X \cup A \cup D$, and $L(x') = L(a') = X' \cup A \cup D$.
- $L(y) = L(b) = X \cup B \cup D$, and $L(y') = L(b') = X' \cup B \cup D$.
- $L(z) = L(c) = L(z') = L(c') = A \cup B \cup D$.

We shall show that there is no $m$-fold $L$-colouring of $G$. Assume to the contrary that $\phi$ is an $m$-fold $L$-colouring of $G$. Then $\phi(u) = A$ and $\phi(u') = B$ and $\phi(v), \phi(w)$ are two disjoint $m$-subsets of $C$.

Let $S = G - (\phi(v) \cup \phi(w))$. We have $|S| = k$. As $X, X'$ are disjoint subsets of $C$, we have

$$|X \cap S| + |X' \cap S| \leq |S| = k.$$  

By symmetry between the two triangles $(u, v, w)$ and $(u', v, w)$, we may assume that

$$|X \cap S| \leq |X' \cap S|,$$

and hence $|X \cap S| \leq k/2$.

(In case $|X' \cap S| \leq \frac{k}{2}$, then we consider the subgraph contained in the triangle $(u', v, w)$). Thus

$$|X \cap \phi(v)| + |X \cap \phi(w)| = |X \cap (\phi(v) \cup \phi(w))| \geq m - \frac{k}{2}.$$

By symmetry between triangles $(u, v, t)$ and $(u, w, t)$, we may assume that

$$|X \cap \phi(v)| \geq |X \cap \phi(w)|,$$

and hence $|X \cap \phi(v)| \geq \frac{m - k}{2} - \frac{k}{4}$.

(In case $|X \cap \phi(w)| \leq \frac{m}{2} - \frac{k}{4}$, then we consider the subgraph contained in the triangle $(u, w, t)$).

Let $T = X - \phi(v)$. We have

$$|T| = |X| - |X \cap \phi(v)| \leq \frac{m}{2} + \frac{k}{4}.$$

Let $R = B - \phi(t)$. As $\phi(t)$ is disjoint from $\phi(u) \cup \phi(v) \phi(w)$, we know that $\phi(t) \subseteq B \cup S$. Hence

$$|R| \leq |S| = k.$$  

By deleting the colours used by the neighbours of $a, b, c$, respectively, we have

- $\phi(a) \subseteq D \cup T$.  

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• \( \phi(b) \subseteq D \cup R \),
• \( \phi(c) \subseteq D \cup T \cup R \).

As \( \phi(a), \phi(b), \phi(c) \) are pairwise disjoint, we have

\[
3m = |\phi(a) \cup \phi(b) \cup \phi(c)| \leq |D| + |T| + |R| \leq (2m + k) + \left( \frac{m}{2} + \frac{k}{4} \right) + k = \frac{5m}{2} + \frac{9k}{4} < 3m,
\]
a contradiction. ■

Let \( p = \binom{4m+k}{m, m, 2m+k} \), and let \( H \) be obtained from the disjoint union of \( p \) copies of \( G \) by identifying all the copies of \( u \) into a single vertex (also named as \( u \)) and all the copies of \( u' \) into a single vertex (also named as \( u' \)), and then add an edge connecting \( u \) and \( u' \). It is obvious that \( H \) is a planar graph.

Now we show that \( H \) is not \((4m + k, m)\)-choosable. Let \( Z \) be a set of \( 4m + k \) colours. Let \( L(u) = L(u') = Z \). There are \( p \) possible \( m \)-fold \( L \)-colourings of \( u \) and \( u' \). Each such a colouring \( \phi \) corresponds to one copy of \( G \). In that copy of \( G \), define the list assignment as in the proof of Lemma 2 by replacing \( A \) with \( \phi(u) \) and \( B \) with \( \phi(u') \). Now Lemma 2 implies that no \( m \)-fold colouring of \( u \) and \( u' \) can be extended to an \( m \)-fold \( L \)-colouring of \( H \). This completes the proof of Theorem 1.

### 3 Some open problems

Thomassen proved that every planar graph is 5-choosable [7]. The proof can be easily adopted to show that for any positive integer \( m \), every planar graph is \((5m, m)\)-choosable. Given a positive integer \( m \), let \( a(m) \) be the minimum integer such that every planar graph is \((a(m), m)\)-choosable. Combining Thomassen’s result and Theorem 1, we have

\[
4m + \left\lfloor \frac{2m - 1}{9} \right\rfloor + 1 \leq a(m) \leq 5m.
\]

For \( m = 1 \), the upper bound and the lower bound coincide. So \( a(1) = 5 \). As \( m \) becomes bigger, the gap between the upper and lower bounds increases. A natural question is what is the exact value of \( a(m) \). We conjecture that the upper bound is not always tight.

**Conjecture 3** There is a constant integer \( m \) such that every planar graph is \((5m-1, m)\)-choosable.

It is proved recently in [4] that every planar graph is 1-defective \((9, 2)\)-paintable, which implies that every planar graph is 1-defective \((9, 2)\)-choosable (i.e., if each vertex has 9 permissible colours, then there is a 2-fold colouring of the vertices of \( G \) with permissible colours so that each colour class induces a graph of maximum degree at most 1.) The following conjecture, which is stronger than Conjecture 3 asserts that the 1-defective can be replaced by 0-defective.
Conjecture 4 Every planar graph is (9, 2)-choosable.

It follows from the Four Colour Theorem that for any integer $m$, every planar graph is $(4m, m)$-colourable. Without using the Four Colour Theorem, it is proved very recently by Cranston and Rabern [2] that every planar graph is (9, 2)-colourable (an earlier result in [5] shows that every planar graph $G$ is $(5m - 1, m)$-colourable with $m = |V(G)| + 1$).

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