Interaction of Vortices in Complex Vector Field and Stability of a “Vortex Molecule”

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We consider interaction of vortices in the vector complex Ginzburg–Landau equation (CVGLE). In the limit of small field coupling, it is found analytically that the interaction between well-separated defects in two different fields is long-range, in contrast to interaction between defects in the same field which falls off exponentially. In a certain region of parameters of CVGLE, we find stable rotating bound states of two defects—a “vortex molecule”.

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The complex Ginzburg–Landau equation (CGLE) is a paradigm model for qualitative description of weakly nonlinear oscillatory media (see for review [1]). This equation is a generic form which is obtained as the amplification of nonlinear oscillatory media (see for review [1]). This is a vast literature on vortex solutions of CGLE and on equation (VGLE) is a possibility of transition between two “phases”, which can be characterized by either “mixing” or “separation” of two “superfluids”. Defects (vortices) can exist in both “superfluids”, and transitions between alternative core structures are possible [2,4]. In this sense, VGLE could be viewed as a toy model of \(^{3}\)He.

A particularly intriguing possibility, suggested in Ref. [4], is the formation of a bound pair of defects in the two fields, i.e. a vortex “molecule” with dipole structure. We shall show in this Letter that such a “molecule” cannot in fact exist in the model with real coefficients, but is readily formed in the vector model with complex coefficients (CVGLE). The latter form appears, in particular, as an amplitude equation near the lasing transition [3,4]. Recent simulations of CVGLE [8] have shown spiral wave patterns with an exceptionally rich structure where both separated (but closely packed) defects in the two fields and “vector” defects with a common core (called argument and director vortices in Ref. [3]) could be seen. There has been so far, however, no theoretical studies of isolated defects in CVGLE and their interactions.

In this Letter we study interaction of defects in CVGLE in the limit of small coupling between two complex fields. We have found that the interaction between a well-separated pair of defects in two different fields is always long-range (power-like), in contrast to the interaction between defects in the same field which falls off exponentially as in a single CGLE [1]. In a certain region of parameters of CVGLE we found stable rotating bound states of two defects—a “vortex molecule”. Analytical results are in excellent agreement with simulations.

Under appropriate scaling of the physical variables, the vector equation for two symmetric interacting complex fields \(A_{\pm}\) acquires the universal form

\[
\partial_t A_{\pm} = A_{\pm} - (1 + i c) \left( |A_{\pm}|^2 + g |A_{\mp}|^2 \right) A_{\pm} + (1 + i b) \nabla^2 A_{\pm},
\]

where real parameters \(b\) and \(c\) are, respectively, the ratio of dispersion to diffusion and the ratio of conservative to dissipative nonlinearity, and the complex parameter \(g = g_+ + ig_\mp\) characterizes the magnitude of the coupling.

A scalar defect of Eq. (0) with unit topological charge is a one-armed spiral in \(A_+\) (or \(A_-\) field), while a vector defect is formed by two scalar defects with a common core. Simple energy considerations, applicable in the case of VGLE with real coefficients, point out that scalar defects in different fields tend to separate or unite, respectively, when \(g\) is positive or negative. For the CVGLE, no energy integral can be defined, but the limit \(g \to 0\) is, of course, distinguished, since the interaction ceases, and defects in both fields, described by the usual Hagan’s [10] rotating spiral solution, are mutually independent.

Defects in two different fields. At \(|g| \ll 1\) the interaction can be treated perturbatively. For \(g = 0\) the scalar defect of Eq. (0) in either field centered at the origin is

\[
A(r, \theta) = F(r) \exp i (\omega t + \theta + \psi(r)),
\]

where \((r, \theta)\) are polar coordinates, \(\omega = c + (b - c) k_0^2\), are the rotation frequency, and \(k_0\) is an asymptotic wavenumber emitted by the spiral. The real functions \(F\) and \(\psi\) have the following asymptotic behavior for \(r \to \infty\): \(F \to \sqrt{1 - k_0^2}\) and \(\chi \to k_0\), where \(\chi = \psi'(r)\) is a local wavenumber. The wavenumber \(k_0\) is determined uniquely for given \(b, c\) [10].

A form more convenient for the analysis, obtained after setting \(A_{\pm} = u_{\pm} \sqrt{(1 + i \omega b)/(1 + i \omega b/e^{-i \omega t})}\) and rescaling \(\nabla \to \nabla \sqrt{(1 + \omega b)/(1 + \omega b)}\partial_t\), is

\[
\nabla \to \nabla \sqrt{(1 + \omega b)/(1 + \omega b)}\partial_t,
\]
(1 - ib)∂tu_± = (1 + iΩ)u_± \\
- (1 + iq) (|u_±|^2 + g|u_±|^2) u_± + ∇^2 u_±, \tag{3}

where \( q = (c - b)/(1 + bc) \), \( Ω = (ω - c)/(1 + ωc) \).

Due to the interaction, the positions of defects \( r_±(t) \) become slow functions of time, so that the instantaneous drift velocity \( \dot{r}_± \equiv \mathbf{v}_± = O(|q|) \). Rewriting Eq. \((3)\) in the comoving frame, we obtain in the first order

\[
(1 - ib)\mathbf{v}_± \cdot \nabla u_± + (1 + iq) (|u_±|^2 + g|u_±|^2) u_± + ∇^2 u_± = 0, \tag{4}
\]

The imaginary part of the advective term can be absorbed by transforming \( u_± \to u_± \exp \left( \frac{1}{2}ib\mathbf{v}_± \cdot \mathbf{x} \right) \), which accounts for the Doppler shift in the emitted wave \([11]\].

We concentrate upon one defect, say, that marked by the index + (which we will further omit), take its position as the origin and view interaction with its counterpart as a perturbation. Expanding in \( g \), we write

\[
u_± = (F(|r - r_±|) + w_±) \exp \left( \frac{1}{2}ib\mathbf{v}_± \cdot \mathbf{x} \right) \exp[\frac{1}{2}iq|\mathbf{r} - \mathbf{r}_±|], \tag{5}\]

where \( \mathbf{r}_± \) are positions of the zeroes of the respective fields and \( θ_± \) are polar angles about these points; \( w_± \) is an \( O(|q|) \) correction. Substituting the ansatz \((5)\) into Eq. \((3)\), we obtain the first-order equation \( \mathcal{H} + \Psi = 0 \), containing the linear operator (cf. \([1]\))

\[
\mathcal{H} = -(1 + iq)F^2(w + w^*) + ∆w + \frac{2i}{r^2} \partial_θ w + 2i\chi F\partial_r \left( \frac{w}{F} \right),
\]

and the inhomogeneity

\[
Ψ = -(1 + iq)gF^2(\bar{r}) + (F^* + iF\chi)\mathbf{v} \cdot \mathbf{n} - \frac{F}{r}\mathbf{v} \times \mathbf{n}.
\]

Here \( \bar{r} = |r - \mathbf{R}| \), \( \mathbf{R} = r_- - r_+ \) is the separation between the defects, \( \mathbf{n} = (\cos θ, \sin θ) \) is the unit vector along \( \mathbf{R} \), and \( ∆ = ∇^2 - F^{-1}∇^2 F \).

The operator \( \mathcal{H} \) has two Goldstone modes corresponding to the translational symmetry in the plane. Therefore the inhomogeneous equation \( \mathcal{H} + Ψ = 0 \) has bounded solution if \( Ψ \) is orthogonal to the adjoint zero modes \( w^* \) of the homogeneous problem. Thus, the drift velocity can be derived from the solvability condition (see Ref. \([3]\))

\[
\text{Re} \int w^*(\mathbf{r})Ψ(\mathbf{r})d^2\mathbf{r} = 0, \tag{6}\]

The operator \( \mathcal{H} \) is not self-adjoint for any \( q \neq 0 \). As a result, the adjoint zero modes cannot be expressed through the translational modes \( ∇u_0 \), and have to be computed directly by solving the equation \( \mathcal{H}^+(w^*, w^*) = 0 \). The adjoint operator \( \mathcal{H}^+ \) is

\[
\mathcal{H}^+ = -F^2(w + w^* - iq(w - w^*)) + ∆w^* + \frac{2i}{r^2} \partial_θ w^* + 2i\chi F\partial_r (r\chi Fw^*). \tag{7}\]

We take note that the Goldstone modes of Eq. \((3)\) are first harmonics \( w = e^{-i(θ + ψ)}∇u_0 \). In view of the orthogonality of the eigenfunctions of the operator and its adjoint with distinct eigenvalues, the zero modes of the adjoint operator must contain the first harmonics as well. To solve the equation \( \mathcal{H}^+ = 0 \), we first separate the real and imaginary parts of \( w^* = u^* + is^* \) and then take the latter’s first harmonics \( (t^+, s^+) = (T^+(r), S^+(r)) e^{it} + c.c. \). Thus, we derive from Eq. \((7)\)

\[
\Delta T^* + \frac{2i\chi S^*}{r^2} - \frac{2}{rF}(r\chi FST^*)' - 2F^2(T^* + qS^*) = 0, \tag{8}\]

\[
\Delta S^* - \frac{2}{rF}(r\chi FST^*') = 0,
\]

where \( ∆_r \equiv ∂_r^2 + r^{-1}∂_r - r^{-2} - F^{-1}∇^2 F \). A typical structure of the adjoint mode in shown in Fig. 1. One can find that the adjoint mode decays exponentially for \( r \to ∞ \) (see also Ref. \([12]\)).

![FIG. 1. The real and imaginary parts (solid/dashed lines) of \( T^+ \) and \( S^+ \) components of the adjoint mode for \( q = 1.3 \).](image)

The solvability condition \((3)\) can be rewritten in a compact form by defining the complex velocity \( v = v_x - iv_y \), so that \( \mathbf{v} \cdot \mathbf{n} = \text{Re}(ve^{iθ}) \), \( \mathbf{v} \times \mathbf{n} = \text{Im}(ve^{iθ}) \). Assuming that \( \mathbf{R} \) is directed along the \( x \) axis, \( v_x \) and \( v_y \) coincide with the radial and tangential velocity components, respectively, \( v_r \) and \( v_θ \). Expressing also the first harmonics of the cross-coupling term through the angular integral \( F(r) = π^{-1} \int_0^∞ \cos θ(F(\bar{r}))^2dθ \), we derive from Eq. \((3)\) the equation for the complex velocity

\[
v = \mathcal{I}^{-1}γ_r \int_0^∞ \left[ T^* + \frac{2i\chi}{γ_r}S^* \right] F(r)dr, \tag{9}\]

where \( γ_r \equiv γ_r + iγ_i = (1 + iq)g \), and the friction factor is

\[
\mathcal{I} = \frac{1}{2} \int_0^∞ \left[ F'(r)T^* + \left( \chi F + \frac{iF}{r} \right) S^* \right] dr.
\]

This expression is the principal analytical result of our work. One can see immediately that, with a generic complex \( T^+(r) \) and \( S^+(r) \), the velocity can be modified simply by rotating the argument of the complex interaction parameter \( g \). In this way, one can ensure that the radial
component \( v_r = \text{Re}(v) \) vanishes at some finite \( R \) at least in a certain interval of \( \arg(g) \). If this equilibrium position is stable, it signals the formation of a bound state, which we call a vortex molecule. Since at the equilibrium distance the tangential component of the velocity given by \( v_r = \text{Im}(v) \) is generally non-zero, this bound state must rotate with a certain angular velocity.

The bound states cannot form in the real VGLE. In this case the operator \( \mathcal{H} \) is self-adjoint, and the adjoint mode is just the translation mode, \( (T^+, S^+) = (F', -iF/r) \). From Eq. (1) one derives then \( v_r \sim g \int r F' F \, dr \) and \( v_r = 0 \). Since \( F(r) \) is a monotonic function, \( v_r \) does not change sign. One finds that the vector defect is unstable for \( g > 0 \) and stable otherwise. The eigenvalue \( \lambda \) of the mode responsible for splitting the vector defect as function of \( g \) is shown in Fig. 4 (inset).

For \( R \gg 1 \) and \( R \ll 1 \) Eq. (3) can be calculated analytically. Using the asymptotic expansions \( F^2(\bar{r}) \approx 1 - \frac{k_0}{2qR^2} \) valid for \( R \gg 1 \) we obtain

\[
F = \frac{k_0}{2q} \int_0^\pi \frac{\cos \theta d\theta}{\sqrt{r^2 + R^2 - 2rR \cos \theta}} \approx \frac{k_0 r}{2qR^2}.
\] (10)

For \( R \ll 1 \) one uses \( F(\bar{r}) = F(r) + F'(r)(R/r - R \cos \theta) \), leading to \( \bar{F} = -RFF'(r) \). The solvability condition (1) yields then \( v = -R\gamma_0 \alpha_1 \) for \( R \ll 1 \), \( v = R^{-2}\gamma_0 \alpha_2 \) for \( R \gg 1 \) with the constants \( \alpha_{1,2} \) given by

\[
\alpha_1 = \bar{I}^{-1} \int_0^\infty F^2 F' \left( T^+ + \frac{\gamma_1}{\gamma_r} S^+ \right) r \, dr,
\]
\[
\alpha_2 = \frac{k_0}{2q} \bar{I}^{-1} \int_0^\infty F F' \left( T^+ + \frac{\gamma_i}{\gamma_r} S^+ \right) r^2 \, dr.
\] (11)

FIG. 2. The real (solid line) and imaginary (dashed line) parts of \( \alpha_1 \) and \( \alpha_2 \) as functions of \( g \) (for \( g \) real).

Provided \( F(r), \chi \) and the adjoint mode \( T^+, S^+ \) are known, the dependence of velocity on distance \( R \) can be found explicitly. We used a matching-shooting algorithm to find a stationary spiral solution and a corresponding adjoint eigenmode. Then, we evaluated numerically the integrals in Eq. (1) and calculated both the constants \( \alpha_{1,2} \) (Fig. 2) and radial and tangential velocities as functions of the distance \( R \) (Fig. 3). We find that for \( g_i = 0, g_r > 0 \) and \( q < q_c \approx 0.52 \) the defects repel one another (\( v_r < 0 \)) at small distances, so that a vector defect is unstable, as in the case of real VGLE. Surprisingly, for larger \( q > q_c \) the real part of \( \alpha_1 \) changes sign, and the defects bind at \( R = 0 \) forming a stable vector defect. Correspondingly, for \( g_i < 0 \) the vector defect is stable at \( q < q_c \) as for real VGLE, but becomes unstable at \( q > q_c \).

FIG. 3. The radial and tangential velocities \( v_r, v_\tau \) as functions of the separation distance \( R \) for \( q = 1.3 \) and \( g = -0.01 \). The solid line has been computed using Eq. (3), circles present results of simulations of Eq. (1).

FIG. 4. Equilibrium separation \( R \) vs. \( q \) obtained from Eq. (4). For \( g_r < 0 \) and \( g_i = 0 \) solid lines indicates stable radii and dashed line unstable ones. Inset: The eigenvalue \( \lambda \) vs. \( g \) for real VGLE.

The bifurcation at \( q = q_c \) is subcritical for \( g_r > 0 \) but supercritical for \( g_r < 0 \). In the latter case, it generates a stable solution with finite \( R \). Under these conditions, the dependence of the radial velocity \( v_r \) on \( R \) has
a zero at some equilibrium distance, as shown in Figs. 3, 4. For \( g_r > 0 \) this bound state is unstable. As \( q \) further increases, a new pair of stable/unstable bound states emerges as a result of a saddle-node bifurcation. This is shown in Fig. 4. The number of equilibrium radii is finite since the asymptotic behavior of the radial velocity is always monotonic. Certainly, only the bound state with the smallest radius is important, since at large \( R \) the “binding strength” decreases.

The analytical results were compared with numerical simulations of Eq. (3). We used the Crank-Nicholson method in the domain of \( 250 \times 250 \) units with \( 500 \times 500 \) grid points and with non-reflecting boundary conditions (see for details Ref. [13]). The analytical and numerical results appears to be in excellent agreement, as seen in Fig. 3.

**Defect pair in the same field.** The interaction of oppositely-charged defects in the same field can be reduced to interaction of a single defect with a plane boundary. The problem of interaction in the scalar CGLE is determined by the growing exponential solution of the stationary linearized problem (see for details Ref. [9]). As a result, the velocity due to interaction with the boundary and \( q > 0 \) is the root of the corresponding characteristic equation obtained at \( r \gg 1 \). As shown in Ref. [3], \( p \) is real for \( q \leq 0.85 \) and complex for \( q > 0.85 \). Thus, interaction of the defect with a plane boundary (or two defects) is oscillatory, and a variety of (unstable) bound states is possible.

![Figure 5](image_url)  
**FIG. 5.** The radial velocity \( v_r \) vs. distance to the boundary \( R \) for \( q = 1.5 \) and \( g = 0.04 \). The dashed line with circles shows the results of simulations with Eq. (1), the solid line is a fit by \( v \sim e^{-pR} + c.c. \), where \( p = 0.746 + 0.94i \) is the root of the characteristic equation of the stationary problem.

The interaction problem in the framework of CVGLE is similar to that of the scalar CGLE. Looking at stationary perturbations of CVGLE, \( w_{\pm} \), we find that generic solutions grow exponentially away from the defect core \( w_{\pm} \sim A_0 e^{pr} \), where \( p \) is the corresponding root of the characteristic equation and \( A_0 \) is the eigenvector. Thus, we expect exponentially decaying interaction of the defects in CVGLE, \( v \sim e^{-pR} \). For \( g \to 0 \) the corresponding exponents should be close to those of the scalar CGLE. This exponential interaction is not surprising, since the waves emitted by the defects collide and form shocks which screen the cores of the defects.

This exponential interaction is verified by numerical simulation of CVGLE. As one sees in Fig. 5, the dependence of the radial velocity (i.e. the component along the line connecting the cores) vs. \( R \) is very well approximated by an exponential dependence with the exponent derived from the solution of the stationary problem.

We have shown that the interaction between two well-separated defects in different complex field is always long-range. In a certain parametric domain, these defects may form stable rotating bound states. In contrast, the interaction between defects in the same field falls off exponentially. Our analytical results are limited to a small coupling constant \( g \). Additional core instabilities may be encountered as \( g \) grows.

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[1] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. 65, 851 (1993).
[2] L.M. Pismen, Vortices in Nonlinear Fields, Oxford University Press (1999).
[3] L. Gil, Phys. Rev. Lett. 70, 162 (1993).
[4] L.M. Pismen, Phys. Rev. Lett. 72, 228 (1995); Physica D 73, 244 (1994).
[5] M. Haelterman and A.P. Sheppard, Phys. Rev. E 49, 4512 (1994).
[6] T.J. Bridges, Physica D 57, 375 (1992).
[7] M. San Miguel, Phys. Rev. Lett. 75, 425 (1995).
[8] E. Hernández-García, A. Amengual, R. Montagne, M. San Miguel, P. Colet, and M. Hoyuelos, Europhys. News 29, 184 (1998).
[9] I.S. Aranson, L. Kramer and A. Weber, Phys. Rev. E 47, 3221 (1993); ibid 47, 4337 (1993).
[10] P.S. Hagan, SIAM J. Appl. Math. 42, 762 (1982).
[11] The value of \( b \) should not be close to the limit of spiral core instability, see I.S. Aranson, L. Kramer and A. Weber, Phys. Rev. Lett. 72, 2316 (1994).
[12] I.S. Aranson, A.R. Bishop, and L. Kramer, Phys. Rev. E 57, 5726 (1998).
[13] H. Châ tê and P. Manneville, Physica A 224 348 (1996).