The Effect of the Low Energy Constants on the Spectral Properties of the Wilson Dirac Operator

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Wilson term breaks $\chi -$ symmetry explicitly

Lattice spacing effects lead to new terms in $\chi - PT$

Sharpe and Singleton (1998), Rupak and Shoresh (2002), Baer,Rupak and Shoresh (2004)

$\epsilon -$ regime where in the thermodynamic, chiral and continuum limit $mV\Sigma, zV\Sigma$ and $a^2VW_i$ kept fixed.

At order $a^2$ it involves three Low Energy Constants (LECs)

$$Z_{N_f}(m, z; a) = \int_{\mathcal{M}} dU \ e^{-S[U]},$$

where the action is

$$S = -\frac{m}{2} \Sigma V \text{tr} (U + U^\dagger) - \frac{z}{2} \Sigma V \text{tr} (U - U^\dagger)$$

$$+ a^2VW_6[\text{tr} (U+U^\dagger)]^2 + a^2VW_7[\text{tr} (U-U^\dagger)]^2 + a^2VW_8\text{tr}(U^2+U^\dagger^2).$$
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The Goal

- Facilitate simulations in the deep chiral regime by an exact, analytical understanding of the average behavior of the smallest eigenvalues
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Introduction of the Model
Partition function of $D_W$ with $N_f$ flavors:

$$Z_{N_f}^{RMT,\nu} = \int dD_W \det^{N_f}(D_W + m)P(D_W)$$

$P(D_W)$ → is a Gaussian

$$D_W = \begin{pmatrix} aA & W \\ W^\dagger & aB \end{pmatrix} + am_6 + a\lambda_7\gamma_5$$

(Damgaard et al (2010), Akemann et al (2010), Kieburg et al (2011, 2012))

- $A : n \times n$ Hermitian
- $B : (n + \nu) \times (n + \nu)$ Hermitian
- $W : n \times (n + \nu)$ Complex
- $m_6$ and $\lambda_7$ scalar random variables
- At $a = 0 : D_W$ has $\nu$ generic zero modes
- At finite $a$ : definition of the index through spectral flow lines
  or equivalently \( \nu = \sum_{\lambda_k^W \in \mathbb{R}} \text{sign}(\langle k | \gamma_5 | k \rangle) \)

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$D_W = \frac{1}{2} \gamma_\mu (\nabla_\mu + \nabla^*_\mu) - \frac{1}{2} a \nabla^*_\mu \nabla_\mu$

- $a \neq 0$ is non-Hermitian but retains $\gamma_5$-Hermiticity
  
  $D_W^{\dagger} = \gamma_5 D_W \gamma_5$

- Eigenvalues of $D_W$ because of the $\gamma_5$-Hermiticity occur in complex conjugate pairs or are real

- ONLY eigenvectors corresponding to real eigenvalues have non-vanishing chirality $\langle k | \gamma_5 | k \rangle$
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The Eigenvalue Densities
\( \hat{a}_6 = \hat{a}_7 = 0.25, \hat{a}_8 = 0.7 \)
\( \hat{m} = 5.3 \)
\( \nu = 0 \) (top) and
\( \nu = 1 \) (bottom)

(Deuzeman, Wenger and Wuilloud (2011))

\( \hat{m} = 4.8, \nu = 2 \)

(Damgaard, Heller and Splittorff (2011))
The density of real eigenvalues of $D_W$

Damgaard, Heller and Splittorff (2012)

Cumulative eigenvalue distributions of $D_5$ with all $W_{6/7/8}$ included at $\nu = 0$

(Deuzeman, Wenger and Wuilloud (2011))
The effects of $W_6$ and $W_7$ when $W_8 = 0$

- $\hat{a}_6$ and $\hat{a}_7$ introduced through the addition of the Gaussian stochastic variable $\hat{m}_6 + \hat{\lambda}_7 \gamma_5$ to $D_W$

- $D = D_W + (m + \hat{m}_6)1 + \hat{\lambda}_7 \gamma_5$

- When $\hat{a}_8 = 0$ $D_W$ is anti-Hermitian,

- the eigenvalues of $D_W(\hat{\lambda}_7, \hat{m}_6) = D - m$ are given by

$$\hat{z}_\pm = \hat{m}_6 \pm i \sqrt{\hat{\lambda}^2_W - \hat{\lambda}^2_7}$$

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The effects of $W_6$ and $W_7$

Schematic plots of the effects of $W_6$ (left plot) and of $W_7$ (right plot). $W_6$ broadens the spectrum parallel to the real axis according to a Gaussian with width $4\hat{a}_6$, but does not change the continuum spectrum in a significant way. When $W_7 \neq 0$ and $W_6 = 0$ the purely imaginary eigenvalues invade the real axis through the origin and only the real (green crosses) are broadened by a Gaussian with width $4\hat{a}_7$.
Notice that the two curves for $\hat{a}_7 = \hat{a}_8 = 0.1$ (right plot) are two orders smaller than the other curves (left plot). Notice the soft repulsion of the additional real modes from the origin at large $\hat{a}_7$. The parameter $\hat{a}_6$ smooths the distribution.
Log-log plots of additional real modes vs $\hat{a}$ for $\nu = 0, 2$

Log-log plots of $N_{\text{add}}$ as a function of $\hat{a}_8$ for $\nu = 0$ (left plot) and $\nu = 2$ (right plot). $W_6$ has no effect on $N_{\text{add}}$. Saturation around zero due to a non-zero value of $\hat{a}_7$. For $\hat{a}_7 = 0$ (lowest curves) the average number of additional real modes behaves like $\hat{a}_8^{2\nu+2}$. Kieburg, Verbaarschot and SZ (2011)
Distribution of additional real modes for $\hat{a} \gg 1$

At $\hat{a} \gg 1$, $\rho_r$ develops square root singularities at the boundaries. Finite matrix size + finite lattice spacing $\rightarrow \rho_r$ has a tail dropping off much faster than the size of the support. The dependence on $W_6$ and $\nu$ is completely lost.
The distribution of the complex eigenvalues projected onto the imaginary axis for $\nu = 1$. Notice that $\hat{a}_6$ does not affect this distribution. The comparison of $\hat{a}_7 = \hat{a}_8 = 0.1$ with the continuum result (black curve) shows that $\rho_{\text{cp}}$ is still a good quantity to extract the chiral condensate $\Sigma$ at small lattice spacing.
The distribution is symmetric around the origin. At small $\hat{a}_8$ the distributions for $(\hat{a}_6, \hat{a}_7) = (1, 0.1), (0.1, 1)$ are almost the same Gaussian as the analytical result predicts. At large $\hat{a}_8$ the maximum reflects the predicted square root singularity which starts to build up. We have not included the case $\hat{a}_6/\hat{a}_7/\hat{a}_8 = 0.1$ since it exceeds the other curves by a factor of 10 to 100.
Extracting the LECs of Wilson chPT
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Extracting the LECs of Wilson chPT

\[ K^{(4)}_{n_1 n_2} \left( \frac{2 \pi}{2 \pi - 2 \pi} \right) \times \exp \left[ \sum_{n=1}^{\infty} \frac{\left( -\frac{1}{2} \right)^n}{n} \right] \times \left[ \text{sign}(x_1 - x_2) \exp \left[ \frac{n}{2a^2} \left( x_1 + x_2 - \frac{a^2}{n} \right)^2 \right] \right] \]

- the average number of the additional real modes for the lowest index:
  \[ N_{\text{add}} \lesssim 2V\sqrt{a^2(W_8 - 2W_7)}, \quad (75) \]
- the width of the Gaussian shaped strip of complex eigenvalues:
  \[ 2\sigma \lesssim \frac{4a\sqrt{W_8 - 2W_6}}{V\sqrt{2}}, \quad (76) \]
- the variance of the distribution of chirality over the real eigenvalues:
  \[ \langle (V\Sigma\Xi)^2 \rangle \lesssim 8V\sqrt{a^2(\nu W_8 - W_6 - W_7)}, \quad \nu > 0, \quad (77) \]

(Figure courtesy of M. Kieburg)
Conclusions

- Studied the effect of the three LECs on the spectrum of $D_W$.
- $W_6$ and $W_7$ can be interpreted as collective fluctuations of the spectrum while $W_8$ induces interactions among all modes.
- Analytical and numerical results of the eigenvalue densities of $D_W$
- At small lattice spacing we propose the following quantities for the extraction of LECs

\[
\tilde{a}^2 V \begin{bmatrix}
0 & -2 & 1 \\
-2 & 0 & 1 \\
-1 & -1 & 1 \\
-1 & -1 & 2
\end{bmatrix} \begin{bmatrix}
W_6 \\
W_7 \\
W_8
\end{bmatrix} = \frac{\pi^2}{8} \begin{bmatrix}
4 N_{\text{add}}^{\nu=0} / \pi^2 \\
2 \sigma^2 / \Delta^2 \\
\langle \tilde{x}^2 \rangle_{\rho_x}^{\nu=1} / \Delta^2 \\
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for upcoming results . . .
Thank you for your attention!

Collaborators:
Mario Kieburg 8E 17.30
A classification of 2− dim Lattice Theory
Jacobus Verbaarschot 7D 15.40
Discretization Effects in the $\epsilon$ Domain of QCD