Fusion Rules in Navier-Stokes Turbulence: First Experimental Tests

*Adrienne L. Fairhall, ‡Brindesh Dhruva, *Victor S. L’vov, *Itamar Procaccia and ‡Katepalli R. Sreenivasan

*Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel
‡Mason Laboratory, Yale University, New Haven, CT 06520-8186

We present the first experimental tests of the recently derived fusion rules for Navier-Stokes (N-S) turbulence. The fusion rules address the asymptotic properties of many-point correlation functions as some of the coordinates coalesce, and form an important ingredient of the nonperturbative statistical theory of turbulence. Here we test the fusion rules when the spatial separations lie within the inertial range, and find good agreement between experiment and theory. An unexpected result is a simple linear law for the Laplacian of the velocity fluctuation conditioned on velocity increments across large separations.

In this Letter the predictions of the recently proposed fusion rules [1] are tested by analyzing the turbulent velocity signal at high Reynolds number. We start with a short theoretical summary.

The theory focuses on two-point velocity differences

\[ w(r, r', t) \equiv u(r', t) - u(r, t), \]

where \( u(r', t) \) is the Eulerian velocity field accessible to experiment. One attempts to extract predictable and computable results by considering the statistical properties of \( w \). The most informative statistical quantities are the equal-time rank-\( n \) tensor correlation functions of velocity differences

\[ F_n(r_1, r'_1; r_2, r'_2; \ldots; r_n, r'_n) \equiv \langle w(r_1, r'_1)w(r_2, r'_2)\ldots w(r_n, r'_n) \rangle, \]

where \( \langle \cdot \rangle \) denotes averaging, and all coordinates are distinct. We consider N-S turbulence for which the scaling exponents are presumed to be universal (i.e., they do not depend on the detailed form of forcing), and where the correlations \( F_n \) are homogeneous functions \( \Box \), namely

\[ F_n(\lambda r_1, \lambda r'_1, \ldots, \lambda r'_n) = \lambda^{\zeta_n} F_n(r_1, r'_1, \ldots, r'_n), \]

\( \zeta_n \) being the homogeneity (or scaling) exponent. This form applies whenever all the distances \( |r_i - r'_i| \) are in the “inertial range”, i.e. between the outer scale \( L \) and the dissipative scale \( \eta \) of the system. Our aim here is to describe the behavior of such functions as pairs of coordinates approach one another, or “fuse”. The fusion rules, derived in [1][3][4], govern the analytical structure of the correlation functions under this coalescence.

The statistical function that has been most commonly studied [1][3][4][5] is the structure function \( S_n(R) \)

\[ S_n(R) = \langle |w(r, r')|^n \rangle, \quad R \equiv r' - r. \]

Clearly the structure function is obtained from \( \Box \) by the fusing of all coordinates \( r_i \) into one point \( r \), and all coordinates \( r'_i \) into \( r' = r + R \). In doing so, one crosses the viscous dissipation length-scale. One then expects a change of behavior, reflecting the role of the viscosity in the theory for \( S_n(R) \). In developing a N-S based theory in terms of \( S_n(R) \), one encounters the notorious closure problem: one must balance terms arising from the convective term \( u \cdot \nabla u \) and the dissipative \( \nu \nabla^2 u \) term, neither of which can be neglected. Hence determining \( S_n(R) \) requires information about \( S_{n+1}(R) \). All known closures of this hierarchy of equations are arbitrary. However, according to the theory of Refs. [1][3][4], the fully unfused \( F_n \) does not suffer from this problem. When all separations are in the inertial range, the viscous term may be neglected, and one obtains \( \Box \) homogeneous equations for \( F_n \) in terms of \( F_n \) only, with no hierarchic connections to higher or lower order correlations. Such homogeneous equations may exhibit new, anomalous scaling solutions for the correlation functions \( F_n \).

There are various possible configurations of coalescence. We will test only those fusions in which the coalescing points are those of velocity differences. One can also consider the coalescence of points from different velocity differences, but they are experimentally more difficult to measure; we will “precoalesce” all such points here and comment on its effects [1]. The first set of fusion rules that we examine concerns \( F_n \) when \( p \) pairs of coordinates \( r_1, r'_1 \ldots r_p, r'_p \) \((p < n)\) of \( p \) velocity differences coalesce, with typical separations between the coordinates \( |r_i - r'_i| \sim r \) for \( i \leq p \), and all other separations of the order of \( R, r \ll R \ll L \). The fusion rules predict

\[ F_n(r_1, r'_1; \ldots; r_n, r'_n) \]

\[ = \mathcal{F}_p(r_1, r'_1; \ldots; r_p, r'_p)\Psi_{n,p}(r_{p+1}, r'_{p+1}; \ldots; r_n, r'_n), \]

where \( \mathcal{F}_p \) is a tensor of rank \( p \) associated with the first \( p \) tensor indices of \( F_n \), and it has a homogeneity exponent \( \zeta_p \). The \((n-p)\)-rank tensor \( \Psi_{n,p}(r_{p+1}, r'_{p+1}; \ldots; r_n, r'_n) \) is a homogeneous function with a scaling exponent
FIG. 1. Log-log plot of the structure functions $S_n(R)$ as a function of $R$ for $n = 2, 4, 6, 8$ denoted by $\times$, $\ast$, $\circ$ respectively.

The mean wind speed was 7.6 ms\(^{-1}\) and was later converted to velocity through an in-situ calibration. The wind direction, measured independently by a vane anemometer, was approximated by the RMS velocity. This figure shows that we have around three decades of “inertial range” (between, say 10 and 10\(^4\) sampling units) but that the dissipative range is not well-resolved. Structure functions of orders higher than 8 are less reliable and will not be considered.

While the fusion rules are formulated for differences in $d$-dimensional space, the surrogated data represent a 1-dimensional cut. This has implications for the choice of positioning of the coordinates. In $d$-dimensional space we can choose separations to fall within balls of size $R$ and $r$ respectively. In our case this ball collapses onto a line, and best results are obtained when the pairs of coordinates in the two groups coincide. As a simple demonstration consider the second order quantity $F_2(r_1, r_1'; r_2, r_2')$ with the three different choices: (i) $|r_1 - r_1'| = x$, $r_2 = r_2'$ = $y$, $r = |y-x|$, (ii) $|r_1 - r_1'| = |r_2 - r_2'| = r$, $r_1' = r_2$, (iii) $|r_1 - r_1'| = |r_2 - r_2'| = r$ where $r_1$ and $r_2$ are also separated by $r$. In one dimension one can simply compute the ratio of the correlation functions in cases (ii) and (iii) with respect to case (i). One finds a reduction factor $2^{5/2} - 1 \approx 0.2$ and $(-2^{5/2} + 1 + 3^2)/2 \approx -0.05$. We thus see that there is a rapid decrease in amplitude when the distances are not enmeshed, and so all averaging is done using maximally enmeshed configurations (i.e., case (i)). As remarked above, in doing so, all the “unprimed” points not across velocity differences are already fused. This procedure does not affect predictions and, see [1].

Explicitly, therefore, we examine the behavior of the correlation function

$$F_{p+q}(r, R) = \langle [u(x + r) - u(x)]^p [u(x + R) - u(x)]^q \rangle$$

as a function of both $r$ and $R$ for several values of the powers $p$ and $q$. From Eq. (5) one expects

$$F_{p+q}(r, R) \sim S_p(r) S_{q+p}(R) / S_p(R).$$

FIG. 2. Log-log plot of $F_{p+q}(r, R)$ as a function of $r$ at fixed $R$ for $q = 2$ and $p = 2, 4, 6$ denoted by $\times$, $\ast$ and $\circ$ respectively with dashed lines. Shown with dotted lines are the same quantities divided by $S_p(r)$.

FIG. 3. As in Fig. 2 as a function of $R$ at fixed $r$, with the dotted lines representing the quantities divided by $S_{p+q}(R)/S_p(R)$.

FIG. 4. Log-log plot of $F_{1+q}(r, R)$ as a function of $R$ for $q = 1, 3, 5$ denoted by $\times$, $\ast$ and $\circ$ respectively. Shown with dotted lines are the same quantities divided by $S_{1+q}(R)/R$.

FIG. 5. log\(_{10}\)[J\(_n\)(R)] as a function of the fusion rule predictions $\log_{10}[J_2 S_n(R)/S_2(R)]$ for $n = 2, 4, 6, 8$ denoted by $\times$, $\ast$, $\circ$ respectively. Inset: the coefficient $C_n$ with the same notation.
In Fig. 6 we display the results for $q = 2$ with even values of $p$ as a function of $r$ for $r$ in the inertial range. Only even values are displayed as the odd correlations fluctuate in sign. The large scale $R$ was fixed at the upper end of the inertial range. The data show clean scaling in the inertial range. Overlaid are the averages corrected by the prediction of the fusion rule (12). Here and in all other figures the averages themselves are connected with dashed lines, whereas compensated results are shown dotted. One observes a change of behavior as $r$ approaches $R$ at the upper limits and the average increases in size towards the “fully-fused” quantity, $S_{p+q}(R)$. Similarly convincing results were obtained for other values of $p$ and $q$.

In Fig. 6 we show $F_{p+q}(r, R)$ as a function of the large scale $R$ with the small separation $r$ fixed at the low end of the inertial range, together with the values corrected by (12). There is a clear trend towards zero slope in the corrected quantity in the upper inertial range.

We consider now the special case that a single pair of points in a velocity difference approach one another. The prediction of Eq. (1) is tested for fixed $r$ and the expected dependence on $R$ is well-verified in Fig. 6. The results found by varying $r$ are not shown here: the linear configuration of all the measurement points leads to a competition between the leading and next order of scaling.

The function $J_2$ was computed and confirmed to be constant throughout the inertial range. In Fig. 3 we present $J_n(R)$ as a function of $nJ_2S_n(R)/2S_2(R)$ for $n, n = 2, 4, 6, 8$ and inertial range $R$. The finite difference surrogate of the Laplacian (cf. Eq. (8)) was computed with $\rho = 10$. The straight line $y = x$ passing through the data is not a fit. It appears from these results that (8) is obeyed well with $C_n = 1$. The $R$ independence of $C_n$ is a direct confirmation of the fusion rules for the fusion of two points. On the other hand, the fact that $C_n$ is independent of $n$ is a surprise that does not follow from fusion rules, and has interesting implications for the statistical theory of turbulence. A more sensitive check of the value of $C_n$ is obtained by dividing $nJ_2S_n(R)/S_2(R)$ by $J_n(R)$ for individual values of $n$ and $R$. This is displayed in the inset in Fig. 5. Clearly, there are statistical fluctuations that increase with increasing $n$ but the data show that $C_n$ is approximately constant in $R$ and $n$, with a value of about unity.

If indeed $C_n$ is $n$-independent, there are surprising consequences for the conditional statistics of our field. To see this, $J_n(R)$ may be rewritten

$$J_n(R) = \int d\delta u_R P(\delta u_R) \langle \nabla^2 \mathbf{u}(r) | \delta u_R \rangle \delta u_R^{n-1}. \quad (13)$$

Here $\langle \nabla^2 \mathbf{u}(r) | \delta u_R \rangle$ is the average of the finite difference Laplacian conditioned on a value of $\delta u_R$. The only way of satisfying both (13) and (8) with $C_n$ that is independent of $n$ and $R$ is to assert that the conditional average, which is in general a function of $\delta u_R$ and $R$, can be factored into a function of $R$ and a linear function of $\delta u_R$:

$$\langle \nabla^2 \mathbf{u}(r) | \delta u_R \rangle = \frac{J_2}{2S_2(R)} \delta u_R. \quad (14)$$

Such linear laws have been discussed in the context of conditional statistics of passive scalar advection (14), and were thought to be reasonable because of the linear nature of the advection-diffusion equation for the scalar. Thus, linear laws for N-S turbulence as well were unexpected. In Fig. 6 we display a direct calculation of the conditional average of the surrogate Laplacian with $\rho = 10$, multiplied by $2S_2(R)/J_2$ as a function of $\delta u_R$ for nine values of $R$ spanning the inertial range. All the data collapse on a single straight line whose slope is unity. For ease of presentation we have displaced the different values of $R$ from each other. It appears that the prediction (14) is amply supported by the data.

Unfortunately we cannot test (14) with the present data because sub-dissipation scales are not resolved. The prediction implies that the nature of the conditional average changes qualitatively when $\rho$ decreases below the dissipative scale. Such changes have important consequences for the ultraviolet properties of the statistical theory of turbulence, and a rich variety of predictions are already available (12). It is thus worthwhile generating high-Reynolds-number data that resolve sub-dissipative scales. Efforts to acquire such data are under way.

This work was supported in part by the German Israeli Foundation, the US-Israel Bi-National Science Foundation, the Minerva Center for Nonlinear Physics, and the Naftali and Anna Backenroth-Bronicki Fund for Research in Chaos and Complexity. We thank the Brookhaven National Laboratory for permission to use their facilities and Mr. Victor Cassella for his help in setting up the experiment.

[1] V.S. L’vov and I. Procaccia, Phys. Rev. Lett. 76, 2896 (1996).
[2] A. S. Monin and A. M. Yaglom. 
Statistical Fluid Mechanics: Mechanics of Turbulence, volume II. (MIT Press, Cambridge, Mass., 1973).
[3] U. Frisch. Turbulence: The Legacy of A.N. Kolmogorov (Cambridge University Press, Cambridge, 1995).
[4] V.S. L’vov and I. Procaccia, Phys. Rev. E, 54, 6268 (1996).
[5] F. Anselmet, Y. Gagne, E.J. Hopfinger, and R.A. Antonia, J. Fluid Mech. 140, 63 (1984).
[6] K.R. Sreenivasan and R.A. Antonia, Annu. Rev. Fluid Mech., 29, 435 (1997)
[7] V. Yakhot and Y. Sinai, Phys. Rev. Lett., 63, 1963 (1991)
[8] P. Kailasnath, K.R. Sreenivasan and J.R. Saylor, Phys. Fluids A, 5, 3207 (1993)
[9] R.H. Kraichnan, Phys. Rev. Lett., 72, 1016 (1994)
[10] A. Fairhall, O. Gat, V.S. L’vov and I. Procaccia, Phys. Rev. E, 53, 3518 (1996)
[11] E.S.C. Ching, V.S. L’vov, E. Podivilov and I. Procaccia, Phys. Rev. E, 54, 6364 (1996).
[12] V.S. L’vov and I. Procaccia, Phys. Rev. Lett., 7, 3541 (1996).