1. INTRODUCTION

A large body of recent work may be summarized as follows. Configurations carrying nonzero vorticity over sufficiently large scales occur in the vacuum with nonzero probability for all $\beta < \infty$. In fact this probability approaches unity. The statement implies the presence of arbitrarily long thick vortices in the vacuum. This in turn automatically implies area-law for large Wilson loops, i.e., nonvanishing string tension for all $\beta < \infty$.

A vortex carrying configuration may be characterized over a sufficiently large loop by a singular gauge transformation which is multivalued in $SU(N)$, but single-valued in $SU(N)/Z(N)$. The topological obstruction encountered in trying to extend the definition of the singular gauge transformation throughout the area enclosed by the loop forms then a surface of codimension 2. It is only the presence of the obstruction, rather than its exact location (which is gauge dependent) that is of course significant, and signals the presence of a ‘vortex core’ that cannot be deformed to the trivial vacuum by any continuous deformation. It must be stressed, however, that in general such a configuration can have ‘core’ vorticity distributed in a very nonlocal, patchwise manner. This in fact can be expected to be the case for generic individual configurations in the partition function sum. Individual vortex configuration with spacially well-defined localized core and long distance tail can of course be constructed and do occur in the path integral, but are not necessarily representative of most such configurations. One should not confuse individual vortex configurations in the partition sum with the canonical picture of the vortex realized as a soliton of an effective long distance action (resulting from some integration over configurations).

This makes identification and computation of the contribution of vortex-carrying configurations in the path measure somewhat tricky. Thus, e.g., attempts to gauge fix so that vortices in any one configuration can be associated with well-localized thin ‘projection vortices’ encounter ambiguities and problems that are discussed in other contributions in these proceedings. Here we report on recent studies which, for the first time, directly measured this contribution. The idea is to enforce the presence of a vortex (mod $N$) in every configuration by an appropriate singular transformation, and measure the corresponding expectation. This then measures directly the contribution of such configurations, and hence the excitation probability (equivalently, the free energy cost) for a vortex in the vacuum. It provides the most direct and physically transparent
method for assessing the presence of vortices in the vacuum at large $\beta$.

2. VORTEX FREE ENERGY

A coclosed set of plaquettes (2-cells) is a closed set of $(d - 2)$-cells on the dual lattice. Thus, in $d = 3$, it is a closed loop of dual bonds; in $d = 4$, a closed 2-dim surface of dual plaquettes. Let $\mathcal{V}$ denote a coclosed set of plaquettes that winds through every, say, 2-dim [12]-plane of the lattice $\Lambda$, i.e. a topologically nontrivial plaquette set wrapped around the periodic lattice ($d$-torus) in the $(d - 2)$ directions $\lambda \neq 1, 2$ perpendicular to $\mu = 1 \nu = 2$. We define the partition function

$$Z_\Lambda(\tau) = \int \prod_b dU_b \exp \left( -\sum_{p \notin \mathcal{V}} A_p(U_p) - \sum_{p \in \mathcal{V}} A_p(\tau U'_p) \right), \tag{1}$$

where the plaquette action $A_p(U_p)$ is replaced by the ‘twisted’ action $A_p(\tau U'_p)$ for each plaquette of $\mathcal{V}$. Here the ‘twist’ $\tau \in Z(\Lambda)$ is an element of the center. There are thus $(N - 1)$ different nontrivial choices for $\tau$. The trivial element $\tau = 1$ is the ordinary partition function $Z_\Lambda(1) \equiv Z_\Lambda$. As indicated by the notation on the l.h.s., the exact position or shape of $\mathcal{V}$ is irrelevant; the only dependence is on the presence of the $Z(\Lambda)$ flux winding through each [12]-plane. $\mathcal{V}$ can be moved around and distorted by a shift of integration variables, but not removed; it is rendered topologically stable by winding completely around the lattice. By the same token introducing two twists, $\tau$ on $\mathcal{V}$ and $\tau'$ on $\mathcal{V}'$, in (1) is equivalent to introducing one twist $\tau'' = \tau \tau'$ since $\mathcal{V}$ and $\mathcal{V}'$ can be brought together by a shift of integration variables. This expresses the mod $N$ conservation of the $Z(\Lambda)$ flux introduced by the twist. Thus, for $N = 2$, any odd number of such (nontrivial) twists is equivalent to one, and any even number to none.

The twist enforces the introduction of a discontinuous (singular) $SU(N)$ gauge transformation on every configuration in the path integral measure in (1) with multivaluedness in $Z(\Lambda)$; in other words, the introduction of a $\pi_1(SU(N)/Z(N))$ vortex. The set $\mathcal{V}$ represents the topological obstruction to having single-valuedness everywhere. Thus $Z_\Lambda(\tau)$, eq. (1), is the partition sum for configurations with a topologically stable vortex trapped inside and completely winding around the lattice. The vortex free energy (v.f.e.) order parameter, now defined as

$$\exp(-F_\mathcal{V}(\tau)) = \frac{Z_\Lambda(\tau)}{Z_\Lambda} = \left\langle \exp \left( -\sum_{p \in \mathcal{V}} (A_p(\tau U'_p) - A_p(U'_p)) \right) \right\rangle, \tag{2}$$

is then the normalized expectation for the excitation of such a vortex.

As it is evident from (2), measurement of this quantity presents a difficult sampling problem. It represents the expectation of an operator that can get significant contributions only from configurations having very small statistical weight, and thus becoming very difficult to measure with increasing lattice size. The problem can be addressed by use of a multihistogram method. Results of such a computation [2] for $SU(2)$ are presented in figure 1 for symmetric lattices and for three different values of $\beta$. The lattice spacings are $a = 0.165$ fm, $a = 0.119$ fm and $a = 0.085$ fm for $\beta = 2.3$, $\beta = 2.4$, and $\beta = 2.5$, resp. Notice that, with the lattice size expressed in physical units, the measurements for different $\beta$’s fall on the same curve, as they should. This indicates that the universal curve has been reached, and will not change at larger beta. Also, the onset of the sharp rise around 0.7 fm is in the region of the finite temperature deconfining phase transition providing another indirect consistency check. The amplitude goes to unity for sufficiently large lattice size. For comparison, an upper bound for Coulomb massless behavior (from maximizing action in spin wave approximation) gives $\sim \exp(-\beta(\pi/2)^2) \approx 0.085$ at $\beta = 2.3$.

Thus the excitation free energy cost for a sufficiently thick vortex vortex goes to zero at large $\beta$. The curve gives a direct estimate of ‘sufficiently thick’: slightly above 1 fm. This is the thickness at which vortices can become arbitrarily long as the free-energy cost goes to zero. In this sense the vacuum can be viewed as having a ‘condensate’
of long thick vortices.

Over distances well below 1 fm the thick vortices ("confiners") are not readily detectable among perturbative and other shorter range fluctuations such as instantons dominating the action. (The instanton distribution is generally estimated to be centered around \(0.3 - 0.4\) fm.) As a quantitative demonstration, let \(<\text{tr}U_p>\) denote the plaquette expectation in the presence of the twist, i.e. in the presence of a vortex winding around the lattice. For a \(10^4\) lattice at \(\beta = 2.4\) we find \(<\text{tr}U_p> = 1.2557(2)\), while for the usual plaquette expectation in the absence of the twist \(<\text{tr}U_p> = 1.2598(2)\). Thus asymptotically in the large volume limit a vortex becomes locally ‘invisible’.

3. RELATION TO WILSON AND ’T HOOFT LOOPS

The behavior of other order parameters such as the Wilson and ’t Hooft loops can be related to that of the v.f.e. In particular, the existence of non-vanishing string tension is implied by confining behavior for the v.f.e. We consider \(SU(2)\) for simplicity.

Consider a rectangular loop \(C\) in a [12]-plane. Then, for any reflection positive plaquette action, the Wilson \(W[C]\) loop obeys the bound [3]:

\[ W[C] \leq \left( \exp(-\hat{F}) \right)^{\frac{A_C}{A}}, \]  

(3)

where \(A_C\) is the minimal area bounded by \(C\), \(A\) the area of the lattice [12]-plane, and

\[
\exp(-\hat{F}) = \sum_{\tau=1}^{\tau=1-1} \tau \exp(-F_v(\tau)) = \left( 1 - \exp(-F_v(-1)) \right)
\]  

(4)

defines the \(Z(2)\) Fourier transform of the v.f.e.

Similarly, let \(B_{\text{min}}\) denote a ’t Hooft loop of minimal length, i.e. one consisting of the 4 cubes (in \(d=4\)) forming the coboundary of a plaquette (equivalently, the four dual bonds of the boundary of a dual plaquette), and having the flux attached to the loop winding completely around the lattice. This is nothing but [3] with one plaquette removed from the set \(V\) on which the \(Z(2)\) twist resides. Physically, this may be viewed as an operator creating a vortex ‘punctured’ by a \(Z(2)\) monopole current loop of minimal length (a ‘tagged vortex’). Then it is not hard to prove that

\[ B_{\text{min}} \geq \exp(-F_v(-1)) \exp[-\beta <\text{tr}U_p>((-1)) \]  

(5)

where, as above, \(<\text{tr}U_p>\) denotes the plaquette expectation in the presence of the (un-)punctured twist winding around the lattice. Now confining behavior for \(B_{\text{min}}\) can be shown [3] to provide a sufficient condition for the existence of a nonvanishing string tension through a lower bound on it. Hence, by (5), the v.f.e. does also.

4. AN ANALYTICAL APPROACH

It would clearly be very nice to get a (semi)analytic handle on the v.f.e. at large \(\beta\). This is of course a very difficult problem. Here we would like to briefly outline a possible approach that we have been pursuing. Consider the standard polymer expansion of the partition function.
It is based on the character expansion of the plaquette function Boltzmann factor:
\[ e^{\frac{4}{\beta}trU_p} = c_0(\beta) \left[ 1 + \sum_i c_i(\beta) \chi_i(U_p) \right]. \]
Inserting in \( Z_\Lambda \) and expanding one obtains the polymer expansion:
\[
Z_\Lambda = \int \prod p \mathcal{D} U \prod_p e^{\frac{4}{\beta}trU_p} = c_0(\beta)^{|\Lambda|} \left[ \sum_{\Gamma} \prod_{\gamma} z_\gamma \right] \]
\[
= c_0(\beta)^{|\Lambda|} \exp \left( \sum_C a(C) \prod_{\gamma \in C} z_\gamma \right). \]
The first sum (6) is over all sets \( \Gamma \) of disjoint polymers \( \gamma \) of activity \( z_\gamma \); it is a finite series on a finite lattice \( \Lambda \). The second sum (7) is over all clusters \( C \), i.e. sets of overlapping polymers allowing multiple copies of the same polymer. There are simple explicit formulas for the coefficients \( a(C) \). This is now an infinite series even on finite \( \Lambda \). As it is well-known, the free-energy cluster expansion (7) converges for strong coupling (small enough \( \beta \)), and also in the \( |\Lambda| \to \infty \) limit.

The v.f.e (8) now has a very nice expansion as the difference of two free energy cluster expansions
\[
\frac{Z_\Lambda(-)}{Z_\Lambda} = \exp \left( \sum_C a_C \left( \prod_{\gamma \in C} z_\gamma^{-1} - \prod_{\gamma \in C} z_\gamma \right) \right). \]
Here \( z_\gamma^{-1} \) are the activities in the presence of the twist. But (8) is independent of the location at which the twist intersects any \([12]\)-plane. Hence there is a huge cancelation of terms in the difference: only polymers that extend completely across \([12]\)-planes contribute. But (8) is, of course, still an infinite series even on a finite lattice.

Proceeding in this manner at large \( \beta \) appears hopeless unless one can work with a finite sum. Fortunately this can be accomplished by resuming all clusters with repeated multiplicities to convert the expansions (6), (8) into a finite sum on a finite lattice. The result is
\[
\frac{Z_\Lambda(-)}{Z_\Lambda} = \exp \left( \sum_C \sum_{S \in S} a_{SC} \left( \ln Z^{(-)}(S) - \ln Z(S) \right) \right) \]
where now the sum is over all skeleton clusters \( C \) and all sub-skeleton clusters \( S \) within each \( C \), with \( Z(S) \) the partition function for \( S \), and explicitly computable coefficients \( a_{SC} \). (9) is an exact expression that holds for all \( \beta \).

Having reached (9) as a suitable starting point, the next step is to define a decimation procedure for the sum over skeleton clusters on the lattice \( \Lambda \) in terms of skeleton clusters on a coarser lattice \( \Lambda' \). Various decimation schemes for sets of clusters may be devised. If a sufficiently accurate such scheme can be implemented, it should, by successive iterations, reach the strong coupling regime without encountering a fixed point.

5. CONCLUSIONS

The question of confinement at large \( \beta \) can be reduced to that of the excitation probability for an (arbitrarily long) vortex. Our recent numerical simulations have shown that this probability indeed tends to unity. Much work remains to be done. In particular, investigations on lattices of asymmetric length in the various directions are needed in order to obtain better understanding of the process of flux spreading leading to vortex thickening, as well as allow extraction of numerical values for the string tension directly from the v.f.e. Related work on ’t Hooft loops, free energy derivatives and strong coupling effective actions is being pursued by several groups [1].

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