EXTREME VALUE DISTRIBUTIONS FOR ONE-PARAMETER
ACTIONS ON HOMOGENEOUS SPACES

MAXIM SØLUND KIRSEBOM

Abstract. In this paper we study extreme value distributions for one-parameter actions on homogeneous spaces. We study both maximal distance excursions and closest distance returns of a one-parameter action. In the special case of the space of unimodular lattices we study extreme values for lengths of the shortest vector in a lattice. For certain sparse subsequences of the one-parameter action and by taking the maximum over a specific interval of indices we prove non-trivial estimates for the limiting distribution in all cases.

We also prove estimates for the limiting distribution of the $k$'th largest element in all of the above settings.

1. INTRODUCTION

Say we have a dynamical system consisting of a probability space $(X, \mu)$, a flow $\varphi_t : X \to X$, $t \in \mathbb{R}$ and a metric $d$. It is of interest to study the maximal distance that a typical orbit of the system gets away from a fixed point in the space in a certain period of time. That is,

$$ M_R(x) = \max_{t \in [0,R]} d(\varphi_t(x), x_0) $$

for $R \in \mathbb{R}$ and some fixed point $x_0 \in X$. In particular, we are keen to understand the behaviour of this maximal distance as $R \to \infty$. The growth rate of $M_R$ is well understood in many interesting cases. A well-known result of this type is Sullivan’s logarithm law for the maximal excursions of the geodesic flow on a cocompact, finite volume quotient of the real hyperbolic space, [14]. Kleinbock and Margulis [11] generalized the result of Sullivan by proving that the logarithm law holds for one-parameter actions on a certain class of cocompact, finite volume homogeneous spaces and for a more general class of functions than $d(\cdot, x_0)$. Later, Athreya, Ghosh and Prasad [1], [2] proved ultrametric analogues of Kleinbock and Margulis’ logarithm law.

In this paper we focus on the setting of Kleinbock and Margulis’ logarithm law, which we formally define in Subsection 1.1. Since the growth rate of $M_R$ is already well understood in this case it is natural to ask more precise questions about the behaviour of $M_R$. We do this by studying the distribution of $M_R$, that is the function $F_{M_R} : \mathbb{R} \to \mathbb{R}$ given by

$$ F_{M_R}(r) = \mu(x : M_R(x) \leq r). $$

Again we are keen to understand what happens when $R \to \infty$, that is, to determine the existence and form of the limit $\lim_{R \to \infty} F_{M_R}$. A result of this kind was proven by Pollicott
who was able to give an exact asymptotic formula for the distribution of the maximal
distance excursions of the geodesic flow on the modular surface. On the one hand Pollicott’s
statement is more precise than that of Sullivan, but on the other hand, it holds true only
in a special case of Sullivan’s setting.

The result of Pollicott is known as an extreme value distribution (EVD). One way to
approach the task of determining EVD’s in dynamical systems is to establish quantitative
mixing properties of the system and use these along with some classical results from the
field of extreme value theory. In the following subsection we elaborate on this idea.

**Extreme value theory and quantitative mixing.** For a real-valued stochastic process we may
study two types of extremes, namely the minimum or the maximum among a certain
collection of random variables in the process. The limiting distribution of either extreme
is known as an EVD. The conventional approach to determining an EVD is through the
techniques of extreme value theory. This is a branch of statistics which deals with the
distributional properties of extreme events in stochastic processes. When we consider
identically distributed stochastic processes, the theory roughly splits into two parts, the
case of independent and the case of dependent stochastic processes. The independent case
is simple and EVD’s of independent, identically distributed processes are well understood.
The dependent case is significantly more complicated and complex. If we make the stronger
assumption that our stochastic process is stationary, we obtain EVD’s when the dependence
is sufficiently weak. Here ”sufficiently weak” means that the stochastic process satisfies two
conditions commonly known as Condition $D$ and $D’$. For further details on these conditions
and extreme value theory, see [12]. For a short resume on extreme value theory, see [9].

Consider a dynamical system consisting of a probability space $(X, \mu)$, a flow $\varphi_t : X \to X$,
$t \in \mathbb{R}$ and an observable $D : X \to \mathbb{R}$. Assume that $\mu$ is $\varphi_t$-invariant. We are interested in
the stochastic process defined by the random variables $\xi_t = D(\varphi_t(x))$, $t \in \mathbb{R}$ which we refer
to as the stochastic process arising from the dynamical system. Notice that $\varphi_t$-invariance of
$\mu$ implies that $\xi_t$ is stationary. The prospect of proving Condition $D$ and $D’$ for $\xi_t$ depends
on the type and rate of mixing known for the system as well as the nature of the observable
$D$.

For many interesting dynamical systems, EVD’s have been established. The first example
came through the work of Collet [4] who proved an EVD for the closest distance returns
of typical orbits of a certain transformation of a closed interval. Collet’s result holds for
systems satisfying certain hyperbolicity assumptions. It is known that systems satisfying
these assumptions are exponentially mixing and Collet used this rate of mixing to prove
that the aforementioned conditions $D$ and $D’$ were satisfied. Collet’s work provided a
blueprint for how to apply extreme value theory to dynamical systems using quantitative
mixing properties. Since then, EVD’s have been determined for many other interesting
dynamical systems, see for example [3], [5], [6], [7], [8].

As mentioned previously, we are interested in determining EVD’s in the setting of Klein-
bock and Margulis’ logarithm law. It is known that the rate of mixing in this setting is
exponential. Despite this relatively fast rate, it proved beyond our capabilities to verify
the aforementioned condition $D$ in this setting. The problem we encounter when trying
to verify Condition $D$ is that the exponential mixing only holds true for observables that satisfy restrictive smoothness assumptions. This causes problems in our calculations and creates error terms which we are unable to control. Note that we are not suggesting that condition $D$ cannot be proven in this setting, we are merely saying that we were unable to do so.

Without Condition $D$ we do not have the results of extreme value theory available to us. Instead we take a more direct approach to estimating the EVD's of the system. Our idea is to apply the mixing property of the system directly to the distribution function of the extreme event. This results in an error term which we can control when we look at a special case. In the following we introduce our setup, including the special case mentioned.

1.1. Setup and main results. Let $G$ denote a connected semi-simple Lie group with finite center and let $\Gamma < G$ be an irreducible lattice such that $G/\Gamma$ is not compact. Set $X = G/\Gamma$ and let $a_t$ denote a one-parameter subgroup of $G$. Let $\mu$ denote the normalized Haar measure on $X$ and let $d$ denote the dimension of $G$. Let $d$ be the Riemannian metric on $X$ chosen by fixing a right invariant Riemannian metric on $G$ which is bi-invariant with respect to a maximal compact subgroup of $G$. Let $D : X \to \mathbb{R}$ denote a measurable function.

In order to control the aforementioned error terms we need to restrict ourselves to the following special case. The first restriction is that we only look at a sparse sequence of $a_t$. The second restriction is that the maximum which we study is taken over an interval of indices whose endpoints are increasing. More specifically, let $m_j \in \mathbb{R}$ be a sequence and let $\alpha_n < \beta_n$ be sequences of natural numbers both going to $\infty$ with $n$. Instead of studying the maximum $M_R$, we are looking at the maximum

$$\max_{m_{\alpha_n} \leq i \leq m_{\beta_n}} D(a_i x).$$

From here and onwards we fix the notation

$$I_n = \{m_{\alpha_n}, m_{\alpha_n + 1}, \ldots, m_{\beta_n}\}$$

for given sequences of natural numbers $\alpha_n < \beta_n$ and $m_j \in \mathbb{R}$. We then write the above maximum as

$$M_{I_n}(x) := \max_{i \in I_n} D(a_i x) = \max_{m_{\alpha_n} \leq i \leq m_{\beta_n}} D(a_i x).$$

We also fix the notation

$$N_n := \beta_n - \alpha_n + 1$$

throughout the paper. The necessary conditions on $m_j$, $\alpha_n$ and $\beta_n$ vary depending on the specific setting and will be made explicit in each of the forthcoming theorems.

We are also interested in the $k$'th largest element for some $k \in \mathbb{N}$. That is, instead of studying the maximum, i.e. the largest among $\{D(a_{m_{\alpha_n}} x), \ldots, D(a_{m_{\beta_n}} x)\}$, we study the $k$'th largest element in this collection. We will denote the $k$'th largest element by $\max^{(k)}$
and write
\[ M_{I_n}^{(k)}(x) = \max_{i \in I_n} (D(a_i, x)). \]

We are now ready to present the main results of the paper. We prove results for maximal distance excursions as well as closest distance returns in the full generality of the setup described above. Furthermore, in the special case of \( X \) being the space of unimodular lattices in \( \mathbb{R}^d \) we are able to obtain a more accurate result.

1.2. Maximal distance excursions. We study the maximal distance excursions by looking at the observable
\[ D(\cdot) = d(\cdot, x_0) \]
for a fixed point \( x_0 \in X \). We prove the following.

**Theorem 1.1.** Let \( m_j \in \mathbb{R} \) be a fixed sequence which satisfies that
\[ \lim_{j \to \infty} \frac{m_j - 1}{m_j} < C \]
where \( C \in (0, 1] \) is an explicit constant. Also, let \( \alpha_n < \beta_n \) be sequences in \( \mathbb{N} \) for which \( \alpha_n \to \infty \) and \( N_n \to \infty \). Assume that \( a_t \) is partially hyperbolic and that \( \text{Ad}(a_t) \) is diagonalizable. Then there exist positive constants \( w_1, w_2 \) and \( v \) such that for every \( x_0 \in X \)
\[ e^{-w_1v^{-r}} \leq \lim_{n \to \infty} \mu \left( M_{I_n} \leq u_n(r) \right) \leq \lim_{n \to \infty} \mu \left( M_{I_n} \leq u_n(r) \right) \leq e^{-w_2e^{-vr}}, \]
where \( u_n(r) = r + \frac{1}{n} \log N_n \).

For the \( k \)'th largest element we obtain the following result.

**Theorem 1.2.** Let \( m_j \in \mathbb{R} \) be a fixed sequence satisfying
\[ \sup_{j \in \mathbb{N}} \left( \frac{m_j - 1}{m_j} \right) < C \]
where \( C \in (0, 1] \) is an explicit constant. Set \( \rho = (\sup_{j \in \mathbb{N}} (m_j - 1/m_j))^{-1} \). Also let \( \alpha_n < \beta_n \) be sequences in \( \mathbb{N} \) for which \( \alpha_n \to \infty \) and \( N_n \to \infty \) in such a way that
\[ N_n = o \left( e^{\sigma \rho \alpha n} \right), \]
where \( \sigma > 0 \) is an explicit constant. Assume that \( a_t \) is partially hyperbolic and that \( \text{Ad}(a_t) \) is diagonalizable. Then there exist positive constants \( w_1, w_2 \) and \( v \) such that for every \( x_0 \in X \)
\[ e^{-w_1v^{-r}} \sum_{i=0}^{k-1} \frac{(w_2e^{-vr})^i}{i!} \leq \lim_{n \to \infty} \mu \left( M_{I_n}^{(k)} \leq u_n(r) \right) \]
\[ \leq \lim_{n \to \infty} \mu \left( M_{I_n}^{(k)} \leq u_n(r) \right) \leq e^{-w_2e^{-vr}} \sum_{i=0}^{k-1} \frac{(w_1e^{-vr})^i}{i!}, \]
where \( u_n(r) = r + \frac{1}{n} \log N_n \).
1.3. **Closest distance returns.** We study closest distance returns by looking at the observable

\[ D(\cdot) = -\log d(\cdot, x_0) \]

for a fixed point \( x_0 \in X \). We prove the following.

**Theorem 1.3.** Let \( m_j \in \mathbb{R} \) be a fixed sequence for which

\[ \sup_{j \in \mathbb{N}} \frac{m_{j-1}}{m_j} < C \]  \hspace{1cm} (1.2)

where \( C \in (0, 1] \) is an explicit constant. Set \( \rho = \left( \sup_{j \in \mathbb{N}} (m_{j-1}/m_j) \right)^{-1} \). Also let \( \alpha_n < \beta_n \) be sequences in \( \mathbb{N} \) for which \( \alpha_n \to \infty \) and \( N_n \to \infty \) in such a way that

\[ N_n = o \left( e^{\sigma \rho^n} \right), \]

where \( \sigma > 0 \) is an explicit constant. Assume that \( a_t \) is partially hyperbolic and that \( \text{Ad}(a_t) \) is diagonalizable. Let \( u_n(r) = r + \frac{1}{d} \log N_n \). Then there exists a positive constant \( w \) such that for every \( x_0 \in X \)

\[ \lim_{n \to \infty} \mu \left( M_{I_n} \leq u_n(r) \right) = e^{-we^{-dr}}. \]

A simple example for which the conditions of Theorem 1.3 are satisfied is \( \beta_n = 2n \), \( \alpha_n = n \) and \( m_j = q^j \) for some sufficiently large \( q \in \mathbb{R} \). For some \( w > 0 \) we then have for all \( x_0 \in X \)

\[ \lim_{n \to \infty} \mu \left( \max_{n \leq j \leq 2n} -\log d(q^j x, x_0) \leq u_n(r) \right) = e^{-we^{-dr}}. \]

For the \( k \)'th largest element we obtain the following result.

**Theorem 1.4.** Let \( m_j \in \mathbb{R} \) be a fixed sequence satisfying

\[ \sup_{s \in \mathbb{N}} \left( \frac{m_{j-1}}{m_j} \right) < C \]

where \( C \in (0, 1] \) is an explicit constant. Set \( \rho = \left( \sup_{j \in \mathbb{N}} (m_{j-1}/m_j) \right)^{-1} \). Also let \( \alpha_n < \beta_n \) be sequences in \( \mathbb{N} \) for which \( \alpha_n \to \infty \) and \( N_n \to \infty \) in such a way that

\[ N_n = o \left( e^{\sigma \rho^n} \right), \]

where \( \sigma > 0 \) is an explicit constant. Assume that \( a_t \) is partially hyperbolic and that \( \text{Ad}(a_t) \) is diagonalizable. Set \( u_n(r) = r + \frac{1}{d} \log N_n \). Then there exists a positive constant \( w \) such that for every \( x_0 \in X \)

\[ \lim_{n \to \infty} \mu \left( M_{I_n}^{(k)} \leq u_n(r) \right) = e^{we^{-dr}} \sum_{i=0}^{k-1} \frac{(we^{-dr})^i}{i!}. \]
1.4. Shortest vectors on the space of unimodular lattices. Let $X = \mathcal{L}_d$ denote the space of unimodular lattices in $\mathbb{R}^d$ and $\mu$ the normalized Haar measure on $\mathcal{L}_d$. Recall that $\mathcal{L}_d$ can be identified with $\text{SL}(d, \mathbb{R})/\text{SL}(d, \mathbb{Z})$ and hence can be thought of as a homogeneous space. Let $a_t$ denote a one-parameter subgroup of $\text{SL}(d, \mathbb{R})$. Let $\Delta : \mathcal{L}_d \to \mathbb{R}$ be given by

$$\Delta(\Lambda) = \max_{v \in \Lambda \setminus \{0\}} \log \left( \frac{1}{\|v\|} \right).$$

(1.3)

and set $\mathcal{D} = \Delta$. We choose to study the observable $\Delta$ because it is of great importance to the connections between flows on $\mathcal{L}_d$ and Diophantine approximation. Note that up to a change of variables $\Delta$ returns the length of the shortest non-zero vector in the lattice, hence the title of this subsection. We obtain the following result.

**Theorem 1.5.** Let $m_j \in \mathbb{R}$ be a fixed sequence such that

$$\lim_{j \to \infty} \frac{m_{j-1}}{m_j} < C$$

(1.4)

where $C \in (0, 1]$ is an explicit constant. Also, let $\alpha_n < \beta_n$ be sequences in $\mathbb{N}$ for which $\alpha_n \to \infty$ and $N_n \to \infty$. Assume that $a_t$ is partially hyperbolic and that $\text{Ad}(a_t)$ is diagonalizable. Set $u_n(r) = r + \frac{1}{d} \log N_n$. Then for $w = \frac{\sqrt{d}}{2d}$ we have

$$\lim_{n \to \infty} \mu \left( M_{I_n} \leq u_n(r) \right) = e^{-we^{-dr}}.$$

For the $k$'th largest element we obtain the following result.

**Theorem 1.6.** Let $m_j \in \mathbb{R}$ be a fixed sequence satisfying

$$\sup_{j \in \mathbb{N}} \left( \frac{m_{j-1}}{m_j} \right) < C$$

where $C \in (0, 1]$ is an explicit constant. Set $\rho = \left( \sup_{j \in \mathbb{N}} (m_{j-1}/m_j) \right)^{-1}$. Also let $\alpha_n < \beta_n$ be sequences in $\mathbb{N}$ for which $\alpha_n \to \infty$ and $N_n \to \infty$ in such a way that

$$N_n = o \left( e^{\sigma \rho^\alpha_n} \right),$$

where $\sigma > 0$ is an explicit constant. Assume that $a_t$ is partially hyperbolic and that $\text{Ad}(a_t)$ is diagonalizable. Set $u_n(r) = r + \frac{1}{d} \log N_n$. Then for $w = \frac{\sqrt{d}}{2d}$ we have

$$\lim_{n \to \infty} \mu \left( M_{I_n}^{(k)} \leq u_n(r) \right) = e^{-we^{-dr}} \sum_{i=0}^{k-1} \left( \frac{we^{-dr}}{i!} \right)^i.$$

2. Structure of the paper

We begin with Section 3 which contains all the necessary concepts, definitions and previous results. We also prove some basic estimates which will be used many times throughout the paper. In Section 4 we prove two general results from which Theorem 1.1, 1.3 and 1.5 will follow almost directly. In Section 5 we again prove two general results, which in this case imply Theorem 1.2, 1.4 and 1.6. Much of this section is repetition from Section
hence many repetitive details are left out and only the new ideas and differences are elaborated on.

3. Preliminaries

3.1. Asymptotic volume estimates. Let \((X, \mu)\) be a probability space and let \(D : X \to \mathbb{R}\) denote a measurable function. We define the tail distribution function of \(D\) as

\[
\Phi_D(z) = \mu(\{x : D(x) \geq z\}).
\]

In order to prove the main results it is necessary to have some estimate on the tail distribution function of the observable in question. We define two types of observables according to the accuracy with which the asymptotics of their tail distribution function is known.

**Definition 3.1.** For positive constants \(w_1, w_2\) and \(v\), we say that \(D\) is \((w_1, w_2, v)\)-DL ("Distance-Like") if it is uniformly continuous and satisfies

\[
w_1 e^{-vz} \leq \Phi_D(z) \leq w_2 e^{-vz}, \quad \forall z \in \mathbb{R}.
\]

For positive constants \(w\) and \(v\), we say that \(D\) is \((w, v)\)-SDL ("Strong-Distance-Like") if it is uniformly continuous and satisfies

\[
\Phi_D(z) = w e^{-vz} + o(e^{-vz}) \quad \text{as } z \to \infty.
\]

The notion of distance-like functions was introduced by Kleinbock and Margulis in \([11]\), and distance-like properties for two observables of particular interest are also proven therein. The first of these is the Riemannian distance to a fixed point which is a natural choice of observable when studying cuspidal excursions. As seen in the following theorem, the setup of Kleinbock and Margulis is slightly more general than that introduced in section 1.1.

**Theorem 3.2** (Kleinbock and Margulis, [11]). Let \(G\) be a connected semisimple Lie group and \(\Gamma\) a non-uniform irreducible lattice in \(G\). Let \(\mu\) be the normalized Haar measure on \(G/\Gamma\) and \(d\) a Riemannian metric on \(G/\Gamma\) chosen by fixing a right invariant Riemannian metric on \(G\) bi-invariant with respect to a maximal compact subgroup of \(G\). Let \(x_0\) be an arbitrary fixed point in \(G/\Gamma\). Then there exist positive constants \(w_1, w_2\) and \(v\) such that the observable \(d(\cdot, x_0)\) is \((w_1, w_2, v)\)-DL.

The second observable considered in [11] is \(\Delta : \mathcal{L}_d \to \mathbb{R}\), introduced in Section 1.4. Recall the well-known fact that \(\mathcal{L}_d\) is isomorphic to the homogeneous space \(\text{SL}(d, \mathbb{R})/\text{SL}(d, \mathbb{Z})\). We therefore write

\[
\mathcal{L}_d = \text{SL}(d, \mathbb{R})/\text{SL}(d, \mathbb{Z}),
\]

and in accordance with the focus of this paper, we think of \(\mathcal{L}_d\) as a homogeneous space. The following is known about the tail distribution function of \(\Delta\).
Theorem 3.3 (Kleinbock and Margulis, [11]). Let $\Delta : \mathcal{L}_d \to \mathbb{R}$ be defined as in (1.3) and let $\mu$ denote the normalized Haar measure on $\mathcal{L}_d$. Set $C_d = \frac{V_d}{2\pi(d)}$ where $V_d$ denotes the volume of the unit ball in $\mathbb{R}^d$. Then there exists a positive constant $C'_d$ such that

$$C_de^{-dz} \geq \Phi_\Delta(z) \geq Cde^{-dz} - C'_de^{-2dz}.$$

In other words, $\Delta$ is $(C_d, d)$-SDL.

Let $(X, \mu)$ denote a probability space which is also a metric space with a metric $d$. Another observable which we will study in this paper is $-\log d(\cdot, x_0)$ with $x_0 \in X$ an arbitrary fixed point. This is the observable we use to study closest distance returns. It is clear that small values of $d(x, x_0)$ correspond to large values of $-\log d(x, x_0)$. Hence, by looking at the successive maxima of $-\log d(\cdot, x_0)$ along some trajectory, we are actually looking at the closest distance returns of said trajectory. The choice of $-\log$ is not canonical and indeed we could have chosen any continuous functions $f$ with the property that $f(x) \to \sup_{x \in X} f(x)$ for $x \to 0$. However, $-\log$ turns out to be convenient from a technical point of view which is the reason why we make this choice.

The observable $-\log d(\cdot, x_0)$ is not uniformly continuous and hence it fails to be neither SDL nor DL. However, we can still determine the asymptotics of its tail distribution function. Indeed, the following lemma is easy to prove. By a smooth measure we will mean a $C^\infty$ function times the Lebesgue measure.

Lemma 3.4. Assume that $X$ is a Riemannian manifold of dimension $d$ with a smooth measure $\mu$ having positive density. Let $x_0 \in X$ be an arbitrary fixed point and set $D(\cdot) = -\log d(\cdot, x_0)$ where $d$ is the Riemannian metric on $X$. Then, for some $w > 0$,

$$\Phi_D(z) = we^{-dz} + o(e^{-dz}) \quad \text{as } z \to \infty.$$

Proof. Since $X$ is a Riemannian manifold we can find a coordinate chart $\sigma : X \to \mathbb{R}^d$ such that $\sigma(B_r(x_0)) = B_r(0)$. Let $\rho$ denote the density of $\mu$ with respect to the Lebesgue measure $m$ on $\mathbb{R}^d$. Then we can write

$$\mu(B_r(x_0)) = \int_{B_r(0)} \rho(x) \, dx.$$

Since $\rho$ is smooth, we know that we can write $\rho(x) = \rho(0) + O(r)$. Using this we get

$$\mu(B_r(x_0)) = \rho(0)m(B_r(0)) + O(r)m(B_r(0)) = \rho(0)r^d + O(r^{d+1}).$$

Notice that

$$\Phi_D(z) = \mu \left( \{ x : -\log d(x, x_0) \geq z \} \right) = \mu \left( \{ x : d(x, x_0) < e^{-z} \} \right) = \mu(B_{e^{-z}}(x_0)).$$

We conclude that

$$\Phi_D(z) = \rho(0)e^{-dz} + O(e^{-(d+1)z}).$$

In particular, with $w = \rho(0)$

$$\Phi_D(z) = we^{-dz} + o(e^{-dz}),$$

as $z \to \infty$. \hfill \Box
Remark 3.5. The assumption that the measure \( \mu \) is smooth is stronger than necessary. However, in the settings where we apply this lemma, we are equipped with a smooth measure, hence we made this assumption in the lemma to simplify the proof.

3.2. Exponential mixing for one-parameter flows. Decay of correlations often plays a central role in developing stochastic properties for a dynamical system. In this section we describe the rate of decay known for the one-parameter actions which we study.

We remind ourselves of the main setup introduced in Section 1.1. \( G \) denotes a connected semi-simple Lie group with finite center and \( \Gamma < G \) is an irreducible lattice such that \( G/\Gamma \) is not compact. \( a_t \) denotes a one-parameter subgroup of \( G \) and \( \mu \) is the normalized Haar measure on \( X := G/\Gamma \) induced by the Haar measure \( m \) on \( G \). The dimension of \( G \) is denoted \( d \) and \( g \) is the Lie algebra of \( G \).

The type of mixing which is known in this setting is exponential decay of correlations against the Sobolev norm of the observables. To precisely formulate this version of exponential mixing we need to define the Sobolev norm on the Sobolev space of \( X \) for which in turn we need to define what we mean by a derivative on \( X \). We therefore define as follows

**Definition 3.6 (Derivative on \( X \)).** The derivative of a function \( f : X \to \mathbb{R} \) in the direction of an element \( \zeta \in g \) will be denoted \( D_\zeta f \) and is defined by

\[
D_\zeta f(x) = \frac{d}{dt} f(\exp(\zeta t)x)|_{t=0}.
\]

**Definition 3.7 (#th Sobolev norm).** Let \( \{\zeta_1, \ldots, \zeta_d\} \) denote an arbitrary basis for \( g \). Then the \( k \)'th Sobolev norm of a function \( f : X \to \mathbb{R} \) with respect to \( \{\zeta_1, \ldots, \zeta_d\} \) is denoted \( S_k(f) \) and is given by

\[
S_k(f)^2 = \|f\|^2 + \sum_{l=1}^{k} \left( \sum_{n_1=1}^{d} \cdots \sum_{n_l=1}^{d} \|D_{\zeta_{n_1}} \cdots D_{\zeta_{n_l}} f\|^2 \right).
\]

Let \( \{\zeta_1, \ldots, \zeta_d\} \) be a fixed basis of \( g \). For \( \bar{l} = (n_1, \ldots, n_l) \in \{1, \ldots, d\}^* \) we define the differential operator \( D_{\bar{l}} = D_{\zeta_{n_1}} \cdots D_{\zeta_{n_l}} \). We may then define the set

\[
W^{(2,k)}(X) = \{ f \in L^2(X) : D_{\bar{l}} f \in L^2(X) \text{ for all } \bar{l} \in L_l \text{ and every } 0 \leq l \leq k \}.
\]

We then call the space \( W^{(2,k)}(X) \), equipped with the norm \( S_k \), the Sobolev space of \( X \).

**Definition 3.8 (Exponential mixing).** We say that \( a_t \) has exponential decay of correlations against \( W^{(2,k)} \) Sobolev observables if for all \( f, g \in W^{(2,k)}(X) \) there exist constants \( \delta > 0 \) and \( c > 0 \) such that

\[
\left| \int_X f(x) g(a_t x) d\mu(x) - \int_X f d\mu \int_X g d\mu \right| \leq ce^{-\delta t} S_k(f) S_k(g).
\]  

In the setup described above we have exponential mixing. This is a consequence of different works by Harish-Chandra, Howe, Cowling and Katok-Spatzier. In the context of our work it is most convenient to use the following formulation of the exponential mixing property for \( a_t \). First we need to define what we mean by \( a_t \) being partially hyperbolic.
Definition 3.9 (Partially hyperbolic). We say that the one-parameter subgroup $a_t \subset G$ is partially hyperbolic if the adjoint action $\text{Ad}(a_t)$ has at least one eigenvalue different from 1 in absolute value.

Theorem 3.10 (Kleinbock and Margulis [10]). Assume that $a_t$ is partially hyperbolic. Then there exist constants $c > 0$, $\delta > 0$ and $k \in \mathbb{N}$ such that for any two functions $\varphi, \psi \in W^{(2,k)}(X)$ and for any $t \geq 0$, equation (3.1) holds.

3.2.1. Smooth approximations of characteristic functions. Theorem 3.10 provides us with a fast rate of decay, but the upper bound being given by Sobolev observables causes a variety of problems when trying to estimate EVD’s for one-parameter actions. The precise nature of these problems will be clear later, however, in general terms, the problem is that the observables to which we want to apply exponential mixing are not Sobolev functions but characteristic functions. Hence, we can not use the exponential mixing property of $a_t$ directly. First we need to approximate the characteristic functions by smooth functions. The smooth functions that we use is the convolution of a particular smooth function and a characteristic function. In the following we discuss some of the technicalities necessary for constructing suitable smooth approximations.

Recall that for functions $\varphi : G \to \mathbb{R}$ and $\psi : X \to \mathbb{R}$, we define the convolution $\varphi * \psi : X \to \mathbb{R}$ as

$$(\varphi * \psi)(x) = \int_G \varphi(g) \psi(g^{-1}x) \, dm(g).$$

Lemma 3.11. Let $A \subset X$ be measurable and let $\varphi \in C^\infty(G)$ be such that $\int_G \varphi \, dm = 1$. Then

$$\int_X \varphi * 1_A \, d\mu = \mu(A).$$

Also, for any $\zeta \in \mathfrak{g}$ we have

$$D_\zeta(\varphi * 1_A) = (D_\zeta \varphi) * 1_A \quad \text{and} \quad S_k(\varphi * 1_A) \leq S_k(\varphi) \mu(A).$$

Proof. The first claim is an easy consequence of Fubini’s Theorem, the definition of convolution and the fact that the action of $G$ on $X$ is measure preserving. Namely, we see that

$$\int_X \varphi * 1_A \, d\mu = \int_X \int_G \varphi(g) 1_A(g^{-1}x) \, d\mu(x) \, dm(g)$$

$$= \int_G \varphi(g) \int_X 1_A(g^{-1}x) \, d\mu(x) \, dm(g)$$

$$= \int_G \varphi(g) \, dm(g) \int_X 1_A(x) \, d\mu(x)$$

$$= \mu(A).$$
To prove the second part let $\zeta \in \mathfrak{g}$ and look at the integral

$$\int_G \varphi(g) \mathbb{1}_A(g^{-1} \exp(t\zeta)x) \, dm(g).$$

We do a change of variables by setting $h = \exp(-t\zeta)g$. Then

$$\int_G \varphi(g) \mathbb{1}_A(g^{-1} \exp(t\zeta)x) \, dm(g) = \int_G \varphi(\exp(t\zeta)h) \mathbb{1}_A(h^{-1}x) \, dm(h).$$

Using this we can calculate the derivative of the convolution

$$D_\zeta (\varphi \ast \mathbb{1}_A) = \frac{d}{dt} \left[ \int_G \varphi(g) \mathbb{1}_A(g^{-1} \exp(t\zeta)x) \, dm(g) \right] \bigg|_{t=0}$$

$$= \frac{d}{dt} \left[ \int_G \varphi(\exp(t\zeta)h) \mathbb{1}_A(h^{-1}x) \, dm(h) \right] \bigg|_{t=0}$$

$$= \frac{d}{dt} \left[ \varphi(\exp(t\zeta)h) \right] \bigg|_{t=0} \mathbb{1}_A(h^{-1}x) \, dm(h)$$

$$= \left( D_\zeta \varphi \right)(h) \mathbb{1}_A(h^{-1}x) \, dm(h)$$

$$= (D_\zeta \varphi) \ast \mathbb{1}_A.$$

It clearly follows that

$$D_{\zeta_1} \ldots D_{\zeta_d} (\varphi \ast \mathbb{1}_A) = (D_{\zeta_1} \ldots D_{\zeta_d} \varphi) \ast \mathbb{1}_A. \quad (3.2)$$

Taking the $L^2$-norm and using the Young inequality for convolutions we get

$$\| (D_{\zeta_1} \ldots D_{\zeta_d} \varphi) \ast \mathbb{1}_A \|_2 \leq \| D_{\zeta_1} \ldots D_{\zeta_d} \varphi \|_2 \| \mathbb{1}_A \|_1$$

$$= \| D_{\zeta_1} \ldots D_{\zeta_d} \varphi \|_2 \mu(A).$$

It is then clear from the definition of the Sobolev norm that $S_k(\varphi \ast \mathbb{1}_A) \leq S_k(\varphi) \mu(A)$. □

Let $\varepsilon > 0$ and $A \subset X$ be measurable. We will approximate characteristic functions by smooth functions of the type $\varphi_\varepsilon \ast \mathbb{1}_A$ where $\varphi_\varepsilon \in C^\infty(G)$ is defined as follows. First pick a coordinate chart $\sigma$ such that $\sigma^{-1}(B(e, \varepsilon)) \subset B(0, \varepsilon) \subset \mathbb{R}^d$ where $e \in G$ denotes the identity. On $\mathbb{R}^d$ we pick a function $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\text{supp}(\varphi) \subset B(0, 1)$. Define then $\varphi_\varepsilon$ on $\mathbb{R}^d$ by

$$\varphi_\varepsilon(x) = \frac{\varphi(\varepsilon^{-1}x)}{\int_{B(0, \varepsilon)} \varphi(\varepsilon^{-1}x) \rho(x) \, dx},$$

where $\rho$ denotes the density of the measure $m$ with respect to the Lebesque measure on $\mathbb{R}^d$. Finally define

$$\varphi_\varepsilon := \varphi_\varepsilon \circ \sigma^{-1}. \quad (3.3)$$

This function is easily seen to have the desirable properties that $\int_G \varphi_\varepsilon \, dm = 1$ and $\text{supp}(\varphi_\varepsilon) \subset B(e, \varepsilon)$. Notice also that by doing a change of variables we get that for
\( \varepsilon > 0 \) sufficiently small,

\[
\int_{B(0,\varepsilon)} \varphi(\varepsilon^{-1}x)\rho(x)\,dx = \varepsilon^d \int_{B(0,\varepsilon)} \varphi(x)\rho(x)\,dx \sim \varepsilon^d,
\]

for some constant \( C > 0 \). Hence for \( \varepsilon > 0 \) sufficiently small, \( \varphi'_\varepsilon(x) = C\varepsilon^{-d}\varphi(\varepsilon^{-1}x) \) for some constant \( C > 0 \).

We will often need the following estimate.

**Lemma 3.12.** For sufficiently small \( \varepsilon > 0 \) we have

\[
S_k(\varphi_\varepsilon) = O\left( \varepsilon^{-\left(\frac{d}{2}+k\right)} \right).
\]

**Proof.** For arbitrary \( \zeta_r \in \mathfrak{g} \) we have

\[
D_{\zeta_r} \varphi_\varepsilon = D_{\zeta_r}(\varphi'_\varepsilon \circ \sigma^{-1}) = \frac{d}{dt} \left[ \varphi'_\varepsilon \circ \sigma^{-1}(\exp(t\zeta_r)x) \right] \bigg|_{t=0}.
\]

In the following let \( \bar{x} = (x_1, \ldots, x_d) = \sigma^{-1}(x) \) and \( \alpha(x, t) = (\alpha_1(x, t), \ldots, \alpha_d(x, t)) = \sigma^{-1}(\exp(t\zeta_r)x) \). Using the chain rule we compute the derivative,

\[
\frac{d}{dt} \left[ \varphi'_\varepsilon \circ \sigma^{-1}(\exp(t\zeta_r)x) \right] \bigg|_{t=0} = \sum_{i=1}^{d} \frac{d\alpha_i}{dt} \bigg|_{t=0} \cdot \frac{d\varphi'_\varepsilon}{dx_i}(\alpha(x, t)) \bigg|_{t=0}
\]

\[
= \sum_{i=1}^{d} \frac{d\alpha_i}{dt} \bigg|_{t=0} \cdot \frac{d\varphi'_\varepsilon}{dx_i}(\bar{x})
\]

\[
\ll \sum_{i=1}^{d} \frac{d\alpha_i}{dt} \bigg|_{t=0} \cdot \frac{d}{dx_i}(\varepsilon^{-d}\varphi(\varepsilon^{-1}\bar{x}))
\]

\[
= \varepsilon^{-(d+1)} \sum_{i=1}^{d} \frac{d\alpha_i}{dt} \bigg|_{t=0} \cdot \frac{d\varphi}{dx_i}(\varepsilon^{-1}\bar{x}).
\]

Now, set

\[
\psi = \sum_{i=1}^{d} \frac{d\alpha_i}{dt} \bigg|_{t=0} \cdot \frac{d\varphi}{dx_i}.
\]
and notice that $\psi$ is a smooth function. Let again $\rho$ denote the density of $\mu$ with respect to the Lebesgue measure on $\mathbb{R}^d$. From the above we get that

$$\|D_{\zeta} \varphi_\epsilon\|_2 = \|\epsilon^{-(d+1)} \psi(\epsilon^{-1}x)\|_2 = \epsilon^{-(d+1)} \left( \int_X |\psi(\epsilon^{-1}x)|^2 \rho(x) \, dx \right)^{\frac{1}{2}}$$

$$= \epsilon^{-(d+1)} \left( \int_X |\psi(x)|^2 \rho(\epsilon x) \epsilon^d \, dx \right)^{\frac{1}{2}}$$

$$= \epsilon^{-(d+1)} \left( \int_X |\psi(x)|^2 \rho(x) \, dx \right)^{\frac{1}{2}}$$

$$= O \left( \epsilon^{-\left(\frac{d}{2}+1\right)} \right).$$

It is easy to see that for higher order derivatives we get

$$\|D_{\zeta_1} \cdots D_{\zeta_l} \varphi_\epsilon\|_2 = O \left( \epsilon^{-\left(\frac{d}{2}+l\right)} \right), \quad (3.4)$$

and from the definition of the Sobolev norm,

$$S_k(\varphi_\epsilon) = O \left( \epsilon^{-\left(\frac{d}{2}+k\right)} \right).$$

□

4. General estimates for EVD’s of one-parameter actions

In this section we establish the general theory from which Theorem $1.1$, $1.3$ and $1.5$ will follow. First we introduce the necessary notation and definitions.

4.1. Notation and setup. Let $G$ denote a Lie group, $\Gamma < G$ a lattice and set $X := G/\Gamma$. Let $\mu$ denote the Haar measure on $X$ which is induced by the Haar measure $m$ on $G$. We assume that $\Gamma$ is chosen in such a way that $X$ has finite measure with respect to $\mu$. Assume also that $\mu$ has been renormalized to a probability measure. Let $a_t$ denote a one-parameter subgroup of $G$ and let $d$ denote the dimension of $G$. Let $\mathfrak{g}$ denote the Lie algebra of $G$. As usual $\text{Ad}(a_t)$ denotes the adjoint action of $a_t$. If $\text{Ad}(a_t)$ is diagonalizable, then there exists an eigenbasis $W = \{\zeta_1, \cdots, \zeta_d\}$ of $\mathfrak{g}$ such that for every $\zeta_r \in W$ there exists $\gamma_r \in \mathbb{R}$ such that

$$\text{Ad}(a_t) \zeta_r = e^{\gamma_r t} \zeta_r. \quad (4.1)$$

Let $m_j$ denote a sequence in $\mathbb{R}$. Also, let $\alpha_n$ and $\beta_n$ denote sequences of natural numbers for which $\alpha_n < \beta_n$, $\alpha_n \to \infty$ and $\beta_n - \alpha_n \to \infty$. Recall that $N_n := \beta_n - \alpha_n + 1$, $I_n = \{m_{\alpha_n}, m_{\alpha_n+1}, \ldots, m_{\beta_n}\}$ and for an observable $\mathcal{D} : X \to \mathbb{R}$ we write

$$M_{I_n}(x) = \max_{i \in I_n} \mathcal{D}(a_i x) = \max_{\alpha_n \leq i \leq \beta_n} \mathcal{D}(a_i x).$$

We make the following assumptions throughout the section.
(1) Assume that \( \text{Ad}(a_t) \) is diagonalizable and fix an eigenbasis \( W := \{ \zeta_1, \ldots, \zeta_d \} \) for \( \text{Ad}(a_t) \). Set also throughout

\[
\gamma := \max_{1 \leq i \leq d} \gamma_i,
\]

where \( \gamma_i \) satisfies (4.1).

(2) Assume throughout that the Sobolev norm \( S_k \) is defined with respect to the fixed eigenbasis \( W \).

(3) Assume that the one-parameter action \( a_t \) has exponential decay of correlations in the sense of Definition 3.1.

4.2. Maximal excursions. In this subsection we prove a general result from which [1.1] and Theorem [1.5] will follow. In those theorems we were concerned with the observables \( d(\cdot, x_0) \) and \( \Delta(\cdot) \). In this section we look at a general observable \( D : X \to \mathbb{R} \) which we characterize according to the asymptotic behavior of its tail distribution function. We state the first main theorem of this subsection.

**Theorem 4.1.** Assume that \( m_j \in \mathbb{R} \) satisfies

\[
\lim_{j \to \infty} \frac{m_j - 1}{m_j} < \min \left( 1, \frac{\delta}{k\gamma} \right),
\]

where \( k, \delta \) are as in (3.1) and \( \gamma \) as in (4.2).

A) Assume \( D \) is \((w, v)\)-SDL for some positive constants \( w \) and \( v \). Then for \( u_n(r) = r + \frac{1}{v} \log N_n \) we have

\[
\lim_{n \to \infty} \mu(M_{1n} \leq u_n(r)) = e^{-we^{-vr}}.
\]

B) Assume \( D \) is \((w_1, w_2, v)\)-DL for some positive constants \( w_1, w_2 \) and \( v \). Then for \( u_n(r) = r + \frac{1}{v} \log N_n \) we have

\[
e^{-w_1e^{-vr}} \leq \lim_{n \to \infty} \mu(M_{1n} \leq u_n(r)) \leq \lim_{n \to \infty} \mu(M_{1n} \leq u_n(r)) \leq e^{-w_2e^{-vr}}.
\]

**Remark 4.2.** For notational simplicity we will write \( u_n \) instead of \( u_n(r) \) whenever the dependence on \( r \) is not important.

**Preparations for the proof of Theorem 4.1.** In the following we prepare for the proof of Theorem 4.1. While the preparations are on the heavy side in terms of technicalities, they serve to lighten the proof of the theorem itself. The overall structure of the proof is described by the following three steps.

**Step 1**) Rewriting \( \mu(M_{1n} \leq u_n) \) as an integral of a product of characteristic functions and approximating the characteristic functions by smooth functions.

**Step 2**) Applying exponential decay of correlations to the integral of a product of smooth functions in order to obtain an expression for the error term between the integral of the product and the product of integrals.
**Step 3**) Deriving an estimate for the error term which shows that the error term vanishes in the limit and that the product of integrals of smooth approximations and the integral of the product of characteristic functions agree in the limit.

The technical and computationally tedious parts of these three steps are taken care of in this section. For \( k \in \mathbb{N} \) and \( C > 0 \) we say that \( \psi \in C^\infty(X) \) is \((C, k)\)-regular if

\[
S_k(\psi) \leq C \int_X \psi \, d\mu.
\]

For some function \( D : X \to \mathbb{R} \), set \( V(r) = \{ x : D(x) \leq r \} \). We are interested in approximating the characteristic functions \( \mathbb{1}_{V(u_n)} \) by \((C, k)\)-regular functions. This is done in the following lemma.

**Lemma 4.3.** Assume that \( D : X \to \mathbb{R} \) is uniformly continuous. Then for any \( k \in \mathbb{N} \) and any \( \delta > 0 \) there exist \( \varepsilon > 0 \), \( C_\varepsilon = O\left(\varepsilon^{-\left(\frac{d}{2} + k\right)}\right) > 0 \) and two \((C_\varepsilon, k)\)-regular functions \( g_{n,\varepsilon} \) and \( h_{n,\varepsilon} \) such that

\[
g_{n,\varepsilon} \leq \mathbb{1}_{V(u_n)} \leq h_{n,\varepsilon} \leq 1.
\]

and

\[
\mu(V(u_n - \delta)) \leq \int_X g_{n,\varepsilon} \, d\mu \leq \int_X h_{n,\varepsilon} \, d\mu \leq \mu(V(u_n + \delta)).
\]

**Proof.** For \( \varepsilon > 0 \), define the sets

\[
V'(u_n, \varepsilon) = \{ x \in V(u_n) : d(x, \partial V(u_n)) \geq \varepsilon \}
\]

\[
V''(u_n, \varepsilon) = \{ x \in X : d(x, V(u_n)) \leq \varepsilon \}.
\]

Clearly \( V'(u_n, \varepsilon) \subset V(u_n) \subset V''(u_n, \varepsilon) \). Let \( \delta > 0 \) be given. It follows from uniform continuity of \( D \) that we can find an \( \varepsilon = \varepsilon(\delta) > 0 \) such that

\[
d(x, y) < \varepsilon \text{ implies } |D(x) - D(y)| < \delta.
\]

This implies that \( V(u_n - \delta) \subset V'(u_n, \varepsilon) \). To see this, assume first that \( x \in V(u_n - \delta) \), i.e. \( D(x) \leq u_n - \delta \). Together this gives \( \delta \leq D(y) - D(x) \) which implies \( d(x, y) \geq \varepsilon \) by uniform continuity. This shows that \( x \in V'(u_n, \varepsilon) \). It follows by the same argument that \( V''(u_n, \varepsilon) \subset V(u_n + \delta) \). So we have that for any \( \delta > 0 \) there exists an \( \varepsilon > 0 \) such that

\[
V(u_n - \delta) \subset V'(u_n, \varepsilon) \subset V''(u_n, \varepsilon) \subset V(u_n + \delta).
\] (4.3)

For every \( \delta > 0 \) we define the functions \( g_{n,\varepsilon} \) and \( h_{n,\varepsilon} \) as

\[
g_{n,\varepsilon} = \varphi_\varepsilon \ast \mathbb{1}_{V'(u_n, \varepsilon)} \quad \text{and} \quad h_{n,\varepsilon} = \varphi_\varepsilon \ast \mathbb{1}_{V''(u_n, \varepsilon)},
\]

with \( \varphi_\varepsilon \) as defined in (3.3). For notational simplicity we will omit writing the dependency of \( \varepsilon \) on \( \delta \). By taking measures in (4.3) and using Lemma 3.11 we get

\[
\mu(V(u_n - \delta)) \leq \int_X g_{n,\varepsilon} \, d\mu \leq \int_X h_{n,\varepsilon} \, d\mu \leq \mu(V(u_n + \delta)).
\]
The definition of $g_{n,\varepsilon}$ and $h_{n,\varepsilon}$ also implies
\[ g_{n,\varepsilon} \leq 1_{V(u_n)} \leq h_{n,\varepsilon} \leq 1. \tag{4.4} \]
To see this, we first rewrite $g_{n,\varepsilon}$ as an integral,
\[ g_{n,\varepsilon}(x) = \int_G \varphi_\varepsilon(g) 1_{V'(u_n,\varepsilon)}(g^{-1}x) \, dm(g) \]
\[ = \int_{B(e,\varepsilon)} \varphi_\varepsilon(g) 1_{gV'(u_n,\varepsilon)}(x) \, dm(g). \]
If $x \notin gV'(u_n,\varepsilon)$ then the first inequality is trivial. Assume therefore that $x \in gV'(u_n,\varepsilon)$. This means that we can write $x = gy$ where $y \in V'(u_n,\varepsilon)$ and hence $d(y, \partial V(u_n)) \geq \varepsilon$. Since $g \in B(e,\varepsilon)$ we see that $gy \in V(u_n)$ which means that $gV'(u_n,\varepsilon) \subset V(u_n)$. This proves the first inequality in (4.4) and the second is proved the same way while the third is obvious.

The $(C_\varepsilon, k)$-regularity of $g_{n,\varepsilon}$ follows directly from Lemma 3.11 with $C_\varepsilon = S_k(\varphi_\varepsilon)$ since
\[ S_k(\varphi_\varepsilon * 1_{V'(u_n,\varepsilon)}) \leq S_k(\varphi_\varepsilon) \mu (V'(u_n,\varepsilon)) = S_k(\varphi_\varepsilon) \int_X g_{n,\varepsilon} \, d\mu. \]
By the same argument $h_{n,\varepsilon}$ is $(C_\varepsilon, k)$-regular. By Lemma 3.12 we have that for sufficiently small $\varepsilon > 0$, $C_\varepsilon = O\left(\varepsilon^{-\left(\frac{1}{2} + k\right)}\right)$. \hfill \Box

Exponential mixing, as we define it in Definition 3.1, is also known as exponential 2-mixing since it describes the exponential rate of decay of correlations between two observables. As will be clear later in the proof of Theorem 4.1 we will be given an integral of a product of multiple observables and we will be interested in estimating the decay of correlations of the entire product. In the following we show which estimate is obtained by repeatedly applying 2-mixing to said product.

For any integers $i_1 < i_2$, let
\[ G_{(i_1, i_2)}(x) = \prod_{i = i_1}^{i_2} g_{n,\varepsilon}(a_m, x) \quad \text{and} \quad H_{(i_1, i_2)}(x) = \prod_{i = i_1}^{i_2} h_{n,\varepsilon}(a_m, x). \]

Lemma 4.4. Let $m_j \in \mathbb{R}$ be a fixed sequence. Then for $\varepsilon > 0$ sufficiently small,
\[ \left| \int_X G_{(\alpha_n, \beta_n)} \, d\mu - \left( \int_X g_{n,\varepsilon} \, d\mu \right)^{N_n} \right| \leq \varepsilon^{-\left(\frac{\beta_n}{2} + k\right)} \sum_{s = \alpha_n + 1}^{\beta_n} e^{-\delta s} S_k \left( G_{(\alpha_n, s-1)} \right) \]
and
\[ \left| \int_X H_{(\alpha_n, \beta_n)} \, d\mu - \left( \int_X h_{n,\varepsilon} \, d\mu \right)^{N_n} \right| \leq \varepsilon^{-\left(\frac{\beta_n}{2} + k\right)} \sum_{s = \alpha_n + 1}^{\beta_n} e^{-\delta s} S_k \left( H_{(\alpha_n, s-1)} \right). \]
Proof. For any \( C > 0 \) and \( k \in \mathbb{N} \) let \( \psi \) denote a \((C, k)\)-regular function such that \( \int_X \psi \, d\mu \leq 1 \). Set \( \psi_i(x) = \psi(a_{m_i} x) \) and for any integers \( i_1 < i_2 \), set
\[
F_{(i_1, i_2)} = \psi_{i_1} \cdots \psi_{i_2}.
\]
The idea is to write
\[
F_{(\alpha_n, \beta_n)}(x) = F_{(\alpha_n, \beta_n-1)}(x)\psi(a_{m_n} x).
\]
and apply exponential decay of correlations to this product. Doing this we get
\[
\left| \int_X F_{(\alpha_n, \beta_n)} \, d\mu - \int_X F_{(\alpha_n, \beta_n-1)} \, d\mu \int_X \psi \, d\mu \right| \ll e^{-\delta m_n S_k(F_{(\alpha_n, \beta_n-1)})} S_k(\psi).
\]
(4.5)
We again apply decay of correlations, this time to \( F_{(\alpha_n, \beta_n-1)}(x) = F_{(\alpha_n, \beta_n-2)}(x)\psi(a_{m_n} x) \) and we obtain the estimate
\[
\left| \int_X F_{(\alpha_n, \beta_n-1)} \, d\mu - \int_X F_{(\alpha_n, \beta_n-2)} \, d\mu \int_X \psi \, d\mu \right| \ll e^{-\delta m_{\beta_n-1} S_k(F_{(\alpha_n, \beta_n-2)})} S_k(\psi).
\]
Inserting this in (4.5) gives
\[
\left| \int_X F_{(\alpha_n, \beta_n)} \, d\mu - \int_X F_{(\alpha_n, \beta_n-2)} \, d\mu \left( \int_X \psi \, d\mu \right)^2 \right| \ll e^{-\delta m_{\beta_n} S_k(F_{(\alpha_n, \beta_n-1)})} S_k(\psi)
\]
\[
+ \ e^{-\delta m_{\beta_n-1} S_k(F_{(\alpha_n, \beta_n-2)})} S_k(\psi) \left( \int_X \psi \, d\mu \right).
\]
We continue by estimating \( \int_X F_{(\alpha_n, \beta_n-s)} \, d\mu \) for all \( 1 \leq s \leq \beta_n - \alpha_n + 1 \) using the exponential 2-mixing. The calculation terminates for \( s = \beta_n - \alpha_n - 1 \), i.e. when we have the estimate
\[
\left| \int_X F_{(\alpha_n, \alpha_n+1)} \, d\mu - \int_X \psi_{\alpha_n} \, d\mu \int_X \psi \, d\mu \right| = \left| \int_X F_{(\alpha_n, \alpha_n+1)} \, d\mu - \left( \int_X \psi \, d\mu \right)^2 \right|
\]
\[
\ll e^{-\delta m_{\alpha_n+1} S_k(\psi_{\alpha_n})} S_k(\psi).
\]
Inserting all the estimates, one after the other, into (4.5) we get that
\[
\left| \int_X F_{(\alpha_n, \beta_n)} \, d\mu - \left( \int_X \psi \, d\mu \right)^N \right| \ll S_k(\psi) \sum_{s=\alpha_n+1}^{\beta_n} e^{-\delta m_s} S_k(F_{(\alpha_n, s-1)}) \left( \int_X \psi \, d\mu \right)^{\beta_n-s}.
\]
Since we have assumed \( \int_X \psi \, d\mu \leq 1 \), we can simplify the expression above to get
\[
\left| \int_X F_{(\alpha_n, \beta_n)} \, d\mu - \left( \int_X \psi \, d\mu \right)^N \right| \ll S_k(\psi) \sum_{s=\alpha_n+1}^{\beta_n} e^{-\delta m_s} S_k(F_{(\alpha_n, s-1)}).
\]
Finally, for \( \varepsilon > 0 \) sufficiently small \( g_{n, \varepsilon} \) and \( h_{n, \varepsilon} \) are both \((C_\varepsilon, k)\)-regular functions with integral bounded by 1 and Lemma 3.11 and Lemma 3.12 imply that \( S_k(g_{n, \varepsilon}) \) and \( S_k(h_{n, \varepsilon}) \) are both bounded by \( O\left(\varepsilon^{-\left(\frac{d}{2}+k\right)}\right) \). Hence, setting \( \psi = g_{n, \varepsilon} \) or \( \psi = h_{n, \varepsilon} \) completes the proof. \( \square \)
We need estimates on the error terms in Lemma 4.4 that are explicit in $n$ and $\varepsilon$. For this we need to understand the Sobolev norms of the functions $G(\alpha_n, c)$ and $H(\alpha_n, c)$ for $\alpha_n < c \leq \beta_n$. This is the content of the next lemma.

**Lemma 4.5.** Assume that $m_j$ is increasing. Then for $\varepsilon > 0$ sufficiently small

$$S_k (G(\alpha_n, c)) = O \left( (c - \alpha_n + 1)^k \varepsilon^{-\left(k(\frac{d}{2}+1)\right)} e^{k\gamma_m} \right)$$

and

$$S_k (H(\alpha_n, c)) = O \left( (c - \alpha_n + 1)^k \varepsilon^{-\left(k(\frac{d}{2}+1)\right)} e^{k\gamma_m} \right),$$

for any $\alpha_n < c \leq \beta_n$.

**Proof.** Throughout the proof, let $\psi = \varphi_* \mathbb{1}_A$ where $A \subset X$ is measurable and $\varphi_\varepsilon$ is defined as in (3.3). Notice that $\psi \leq 1$ since $\int \varphi_\varepsilon \, dm = 1$. Set again $\psi_t(x) = \psi(a_m x)$ and for any integer $\alpha_n < c \leq \beta_n$, set

$$F := F(\alpha_n, c) = \psi_{\alpha_n} \cdots \psi_c.$$

In order to obtain an upper bound on $S_k(F)$ we derive an upper bound on $\|D_{\zeta_1} \cdots D_{\zeta_k} F\|_2$ which is independent of $n_1, \ldots, n_l$. However, we begin by proving the lemma in the special case of $k = 1$ which we do by deriving an upper bound on $\|D_{\zeta} F\|_2$ which is independent of $r$. The $k = 1$ case is simpler in terms of notation but still demonstrates the main ideas of the general estimate.

Let $\zeta_r \in W$ be arbitrary. Using the product rule we get

$$D_{\zeta} F(x) = \frac{d}{dt} \left[ \psi_{\alpha_n}(\exp(t\zeta) x) \cdots \psi_c(\exp(t\zeta) x) \right]_{t=0} = \sum_{s=\alpha_n}^{c} \left[ \psi_{\alpha_n}(\exp(t\zeta) x) \cdots \frac{d}{dt} \psi_s(\exp(t\zeta) x) \cdots \psi_c(\exp(t\zeta) x) \right]_{t=0} = \sum_{s=\alpha_n}^{c} \psi_{\alpha_n}(x) \cdots \frac{d}{dt} \left[ \psi_s(\exp(t\zeta) x) \right]_{t=0} \cdots \psi_c(x).$$

We can rewrite $\psi_s(\exp(t\zeta) x)$ as follows.

$$\psi_s(\exp(t\zeta) x) = \psi(a_m \exp(t\zeta) x) = \psi(a_m \exp(t\zeta) a_{m_s} a_m x) = \psi(\exp(t\text{Ad}(a_m) \zeta) a_m x) = \psi(\exp(t e^{\gamma m_s} \zeta) a_m x).$$
Recall also the well-known fact that for any \( a \in \mathbb{R} \) we have \( D_{a \zeta} = aD_\zeta \). Using this we get

\[
D_{\zeta} \psi_s = \frac{d}{dt} \left[ \psi_s(\exp(t\zeta) x) \right] |_{t=0} = \frac{d}{dt} \left[ \psi(\exp(t e^{\gamma m_s \zeta} a_{m_s} x)) \right] |_{t=0} = D e^{\gamma m_s \zeta} \psi(a_{m_s} x) = e^{\gamma m_s} D_{\zeta} \psi(a_{m_s} x).
\]

Inserting this in (4.6) gives,

\[
D_{\zeta} F = \sum_{s=0}^{c} \psi_{\alpha_n}(x) \cdots e^{\gamma m_s} D_{\zeta} \psi(a_{m_s} x) \cdots \psi_{\epsilon}(x).
\]

Taking the \( L^2 \)-norm we get

\[
\|D_{\zeta} F\|_2 = \left\| \sum_{s=0}^{c} \psi_{\alpha_n}(x) \cdots e^{\gamma m_s} D_{\zeta} \psi(a_{m_s} x) \cdots \psi_{\epsilon}(x) \right\|_2
\leq \sum_{s=0}^{c} \|\psi_{\alpha_n}(x) \cdots e^{\gamma m_s} D_{\zeta} \psi(a_{m_s} x) \cdots \psi_{\epsilon}(x)\|_2
\leq \sum_{s=0}^{c} \|\psi\|_{\infty}^{(c-\alpha_n)} e^{\gamma m_s} \|D_{\zeta} \psi\|_2
\leq \sum_{s=0}^{c} e^{\gamma m_s} \|D_{\zeta} \psi\|_2,
\]

where the last inequality holds since \( \psi \leq 1 \). Recall that \( \gamma = \max_{1 \leq i \leq d} \gamma_i \) and recall also from the proofs of Lemma 3.11 and 3.12 that \( \|D_{\zeta} \psi\|_2 \leq \|D_{\zeta} \varphi\|_2 = O \left( \varepsilon^{-\left(\frac{d}{2}+1\right)} \right) \). Then

\[
S_1(F)^2 = \|F\|_2^2 + \sum_{n_1=1}^{d} \|D_{\zeta_{n_1}} F\|_2^2 \leq \|F\|_2^2 + \sum_{n_1=1}^{d} \sum_{s=0}^{c} \left( e^{\gamma m_{s_n}} \|D_{\zeta_{n_1}} \psi\|_2 \right)^2
\leq \|F\|_2^2 + \sum_{n_1=1}^{d} \left( (c - \alpha_n + 1)e^{\gamma m_{c_1}} \|D_{\zeta_{n_1}} \psi\|_2 \right)^2
\]

It follows from this that

\[
S_1(F) = O \left( (c - \alpha_n + 1)\varepsilon^{-\left(\frac{d}{2}+1\right)}e^{\gamma m_{c_1}} \right).
\]

Having understood how to estimate the first Sobolev norm of \( F \), we can proceed to estimate the \( k \)'th Sobolev norm of \( F \). In this case we are looking at uniformly estimating the \( L^2 \)-norm of all derivatives of the type \( D_{\zeta_{n_1}} \cdots D_{\zeta_{n_k}} F \). Writing down an exact expression for the result of applying the product rule to \( F \), \( l \) times, is notationally complicated and provides little enlightenment. Instead, consider that since \( F \) is a product of \( (c - \alpha_n + 1) \) functions,
$D_{\zeta_1} \cdots D_{\zeta_l} F$ becomes a sum containing a total of $(c - \alpha_n + 1)^l$ terms. Fix a number $1 \leq p \leq (c - \alpha_n + 1)^l$. Then the $p$'th term of this sum is of the form $D_p^{(\alpha_n)} \psi_{\alpha_n} \cdots D_p^{(c)} \psi_c$ where $D_p^{(s)}$ denotes the differential operator applied to the function $\psi_s$ in the $p$'th term of the sum. For every $p$, the total order of $D_p^{(\alpha_n)}, \ldots, D_p^{(c)}$ equals $l$. Say that the order of $D_p^{(s)}$ is $q(s) \leq l$. Then we can write $D_p^{(s)}$ as
\[
D_p^{(s)} = D_{\zeta_1} \cdots D_{\zeta_{q(s)}},
\]
where $n_1, \ldots, n_{q(s)} \in \{1, \ldots, d\}$. Repeated application of (4.7) gives
\[
D_p^{(s)} \psi_s(x) = e^{(\gamma_1 + \cdots + \gamma_{q(s)})} m_s D_p^{(s)} \psi(a_m, x) \leq e^{q(s) \gamma_m} D_p^{(s)} \psi(a_m, x).
\]
Since the total order of $D_p^{(\alpha_n)}, \ldots, D_p^{(c)}$ equals $l$, or in other words, $q(\alpha_n) + \cdots + q(c) = l$, we get
\[
\|D_p^{(s)} \psi_s(x)\|_2 \leq e^{l \gamma_m} \|\left(D_p^{(\alpha_n)} \psi\right)(a_m, x) \cdots \left(D_p^{(c)} \psi\right)(a_m, x)\|_2 \leq e^{l \gamma_m} \|D_p^{(\alpha_n)} \psi\|_\infty \cdots \|D_p^{(c)} \psi\|_\infty.
\]
It follows from equations (3.22), (3.24) and the Young inequality for convolutions that
\[
\|D_p^{(s)} \varphi\|_2 \leq \|D_p^{(s)} \varphi\|_2 \|1_A\|_2 \leq \|D_p^{(s)} \varphi\|_2 = O\left(e^{-\left(\frac{p}{2} + q(s)\right)}\right).
\]
Notice that for $c - \alpha_n + 1 \geq l$, at most $l$ of the operators $D_p^{(\alpha_n)}, \ldots, D_p^{(c)}$ are not the identity. This means that in the product $\left\|\left(D_p^{(\alpha_n)} \psi\right)\right\|_\infty \cdots \left\|\left(D_p^{(c)} \psi\right)\right\|_\infty$, we can bound $c - \alpha_n + 1 - l$ of the factors uniformly by 1 while we must use (4.8) to estimate the remaining $l$ factors. Using this observation we get
\[
\|D_p^{(\alpha_n)} \psi_{\alpha_n}(x) \cdots D_p^{(c)} \psi_c(x)\|_2 = O\left(e^{l \gamma_m} e^{-l\left(\frac{p}{2} + q(c)\right)}\right) = O\left(e^{l \gamma_m} e^{-l\left(\frac{p}{2} + 1\right)}\right).
\]
Since this bound holds for all $1 \leq p \leq (c - \alpha_n + 1)^l$ we see that
\[
\|D_{\zeta_1} \cdots D_{\zeta_l} F\|_2 = O\left((c - \alpha_n + 1)^l e^{l \gamma_m} e^{-l\left(\frac{p}{2} + 1\right)}\right),
\]
and consequently
\[
S_k(F) = S_k\left(F_{(\alpha_n, c)}\right) = O\left((c - \alpha_n + 1)^{k} e^{-k\left(\frac{p}{2} + 1\right)} e^{k \gamma_m}\right).
\]
Since $g_{n, \varepsilon}$ and $h_{n, \varepsilon}$ both satisfy the definition of $\psi$, the bound on the Sobolev norm holds for both $F = G_{(\alpha_n, c)}$ and $F = H_{(\alpha_n, c)}$.

Combining Lemma 4.4 and Lemma 4.5 we get the following estimates for the increasing sequence $m_j$ and all $\varepsilon > 0$ sufficiently small.
\[
\left|\int_X G_{(\alpha_n, \beta_n)} d\mu - \left(\int_X g_{n, \varepsilon} d\mu\right)^{N_n}\right| \ll e^{-\frac{p}{2}(k+1)-2k} \sum_{s=\alpha_n+1}^{\beta_n} e^{k \gamma_{m_s-1}} \sum_{s=\alpha_n+1}^{\beta_n} e^{k \gamma_{m_s-1}-s\gamma_s}(s - \alpha_n)^k \quad (4.9)
\]
and
\[ \left| \int_X H_{(\alpha_n, \beta_n)} \, d\mu - \left( \int_X h_{n, \varepsilon} \, d\mu \right)^N \right| \ll \varepsilon^{-\frac{\delta}{2}(k+1)-2k} \sum_{s=\alpha_n+1}^{\beta_n} e^{\gamma s_{\alpha_n-1} - \delta m_s (s - \alpha_n)^k}. \tag{4.10} \]

This completes the preparations for the proof of Theorem 4.1.

**Proof of Theorem 4.1.** In the following we prove part A) of the theorem. Part B) is almost identical and we make a comment on this at the end of the proof. So in the following assume \( D \) is \((w, v)\)-SDL for some constants \( w > 0 \) and \( v > 0 \). We begin by rewriting the quantity
\[ \mu(M_{I_n} \leq u_n) = \mu\left( \max_{\alpha_n \leq j \leq \beta_n} D(a_{m_j} x) \leq u_n \right) \]
as an integral. Recall the notation \( V(u_n) = \{ x : D(x) \leq u_n \} \). By definition of the maximum we have
\[ \mu\left( \max_{\alpha_n \leq j \leq \beta_n} D(a_{m_j} x) \leq u_n \right) = \mu\left( D(a_{m_{\alpha_n}} x) \leq u_n, \ldots, D(a_{m_{\beta_n}} x) \leq u_n \right) = \int_X \mathbb{1}_{D(a_{m_{\alpha_n}} x) \leq u_n, \ldots, D(a_{m_{\beta_n}} x) \leq u_n} \, d\mu \]
\[ = \int_X \mathbb{1}_{V(u_n)}(a_{m_{\alpha_n}} x) \cdots \mathbb{1}_{V(u_n)}(a_{m_{\beta_n}} x) \, d\mu. \tag{4.11} \]

Lemma 4.3 and equation (4.11) imply that
\[ \int_X G_{(\alpha_n, \beta_n)} \, d\mu \leq \mu(M_{I_n} \leq u_n) \leq \int_X H_{(\alpha_n, \beta_n)} \, d\mu. \]

Notice that the assumption that \( \lim_{j \to \infty} \frac{m_{j+1}}{m_j} < \min\left(1, \frac{\delta}{k \gamma} \right) \) means that \( m_j \) is increasing for \( j \geq j_0 \) for some sufficiently large \( j_0 \). It then follows from (4.9) and (4.10) that for sufficiently large \( n \) and sufficiently small \( \varepsilon > 0 \),
\[ \left( \int_X g_{n, \varepsilon} \, d\mu \right)^N - O\left( \varepsilon^{-\frac{\delta}{2}(k+1)-2k} \sum_{s=\alpha_n+1}^{\beta_n} e^{\gamma s_{\alpha_n-1} - \delta m_s (s - \alpha_n)^k} \right) \]
\[ \leq \mu(M_{I_n} \leq u_n) \leq \left( \int_X h_{n, \varepsilon} \, d\mu \right)^N + O\left( \varepsilon^{-\frac{\delta}{2}(k+1)-2k} \sum_{s=\alpha_n+1}^{\beta_n} e^{\gamma s_{\alpha_n-1} - \delta m_s (s - \alpha_n)^k} \right). \tag{4.12} \]

The task is now to determine the limit, if it exists, of the upper and lower bound. First we take care of the error term.
The assumption that \(\lim_{s \to \infty} \frac{m_{s-1}}{m_s} < \min\left(1, \frac{\delta}{k\gamma}\right)\) means that there exist some \(s_0 \in \mathbb{N}\) and \(\rho > 1\) such that for all \(s \geq s_0\) we have
\[
\frac{m_{s-1}}{m_s} \leq \frac{1}{\rho} < \min\left(1, \frac{\delta}{k\gamma}\right).
\]
From this we see that
\[
m_s \geq \rho m_{s-1} \geq \cdots \geq \rho^{s-s_0} m_{s_0}.
\]
Consequently, for all \(s \geq s_0\),
\[
k\gamma m_{s-1} - \delta m_s = m_s \left(k\gamma \frac{m_{s-1}}{m_s} - \delta\right) \leq m_{s_0} \rho^{-s_0} \left(k\gamma - \delta\right) \rho^s < 0.
\]
Set \(\sigma = -m_{s_0} \rho^{-s_0} \left(k\gamma - \delta\right) > 0\). Then, since \(\alpha_n \to \infty\) for \(n \to \infty\), we see that for sufficiently large \(n \in \mathbb{N}\) we have
\[
\sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_{s-1} - \delta m_s} (s - \alpha_n)^k \leq \sum_{s=\alpha_n+1}^{\beta_n} \left(e^{-\sigma}\rho^s \right) (s - \alpha_n)^k \leq \sum_{s=\alpha_n+1}^{\infty} \left(e^{-\sigma}\rho^s \right) (s - \alpha_n)^k.
\]
The ratio test shows that the series converges and hence the right hand side goes to 0 as \(n \to \infty\). This proves that
\[
O \left(\varepsilon^{-\frac{d}{2(k+1)-2k}} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_{s-1} - \delta m_s} (s - \alpha_n)^k\right) \to 0
\]
for \(n \to \infty\).

Returning to the main terms, we claim that
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \left(\int_X g_{n,\varepsilon} \, d\mu\right)^{N_n} = \lim_{\delta \to 0} \lim_{n \to \infty} \left(\int_X h_{n,\varepsilon} \, d\mu\right)^{N_n} = e^{-we^{-vr}},
\]
To see this we first look at the limits for \(n \to \infty\). Recall from Lemma 4.3 that for any \(\delta > 0\) we can find \(\varepsilon = \varepsilon(\delta) > 0\) such that
\[
\mu\left(V(u_n - \delta)\right) \leq \int_X g_{n,\varepsilon} \, d\mu \leq \int_X h_{n,\varepsilon} \, d\mu \leq \mu\left(V(u_n + \delta)\right);
\]
and hence
\[
\mu\left(V(u_n - \delta)\right)^{N_n} \leq \left(\int_X g_{n,\varepsilon} \, d\mu\right)^{N_n} \leq \left(\int_X h_{n,\varepsilon} \, d\mu\right)^{N_n} \leq \mu\left(V(u_n + \delta)\right)^{N_n}.
\]
Recall that \(u_n(r) = r + \frac{1}{v} \log N_n\). Using the \((w, v)\)-SDL property of \(D\) we get,
\[
\left(1 - \frac{we^{-v(r-\delta)}}{N_n} + o\left(\frac{e^{-v(r-\delta)}}{N_n}\right)\right)^{N_n} \leq \left(\int_X g_{n,\varepsilon} \right)^{N_n} \leq \left(\int_X h_{n,\varepsilon} \right)^{N_n} \leq \left(1 - \frac{we^{-v(r+\delta)}}{N_n} + o\left(\frac{e^{-v(r+\delta)}}{N_n}\right)\right)^{N_n}.
\]
To compute the limit of the upper and lower bound we first rewrite as follows,
\[
\left(1 - \frac{we^{-v(r+\delta)}}{N_n}\right) + o\left(\frac{e^{-v(r+\delta)}}{N_n}\right) = e^{N_n \log\left(1 - \frac{we^{-v(r+\delta)}}{N_n}\right)} + o\left(\frac{e^{-v(r+\delta)}}{N_n}\right).
\]
We then estimate the right hand side using the second order Taylor expansion of \(\log(1+x)\), that is, \(\log(1 + x) = x + O(x^2)\). Inserting the resulting estimate and taking the limit for \(n \to \infty\) gives
\[
e^{we^{-v(r-\delta)}} \leq \lim_{n \to \infty} \left(\int_X g_{n,\varepsilon} \, d\mu\right)^{N_n} \leq \lim_{n \to \infty} \left(\int_X h_{n,\varepsilon} \, d\mu\right)^{N_n} \leq e^{we^{-v(r+\delta)}}.
\]
Therefore the claim is proved by letting \(\delta \to 0\).

It then follows from (4.14) and (4.13), that when we take the limit for \(n \to \infty\) and then the limit for \(\delta \to 0\) in (4.12), we get
\[
\lim_{n \to \infty} \mu(M_{I_n} \leq u_n) = e^{-we^{-vr}}
\]
which proves part A) of the theorem.

The proof of part B) is essentially identical to the proof of part A). The only real difference occurs following equation (4.15) where we apply the DL property instead of the SDL property. For some positive constants \(w_1, w_2\) and \(v\) we then get
\[
1 - \frac{w_1 e^{-v(r-\delta)}}{N_n} \leq \int_X g_{n,\varepsilon} \leq \int_X h_{n,\varepsilon} \leq 1 - \frac{w_2 e^{-v(r+\delta)}}{N_n},
\]
which gives the desired inequalities when raising to the power \(N_n\) and taking the limit for \(n \to \infty\). In fact, the proof of part B) by itself would be easier as we can leave \(\varepsilon\) fixed since we do not have hopes of an exact limit. This would simplify the estimates of the Sobolev norms in Lemma 4.5 as we would not have to keep track of the dependency on the parameter \(\varepsilon\).

4.3. Closest distance returns. In this subsection we prove a general result from which Theorem 1.3 will follow. Recall that, for a fixed point \(x_0 \in X\),
\[
\mathcal{D}(\cdot) = -\log d(\cdot, x_0)
\]
does not satisfy the criteria for being a \((w, d)\)-SDL function since it is not uniformly continuous. However, recall from Lemma 3.1 that for some \(w > 0\)
\[
\mu(\{x : \mathcal{D}(x) > z\}) = we^{-dz} + o(e^{-dz})
\]
as \(z \to \infty\). The following result is a closest return analogue of Theorem 4.1, but the lack of uniform continuity means that we need to make stronger assumptions. The statement is as follows.

**Theorem 4.6.** Let \(\rho^{-1} = \left(\sup_{s \in \mathbb{N}} \frac{m_{s+1}}{m_s}\right)\). Assume that \(m_j \in \mathbb{R}\) satisfies
\[
\sup_{s \in \mathbb{N}} \frac{m_{k+1}}{m_k} < \min\left(1, \frac{\delta}{k\gamma}\right)
\]
where $k, \delta$ are as in (3.1) and $\gamma$ as defined in (4.2). Assume further that
\[
N_n = o \left( e^{\sigma \rho^{r_n}} \right),
\]
where $\sigma = -\frac{m_0}{k} \left( \frac{3}{2} + \frac{2}{4} \right) + \frac{k \gamma}{\rho} > 0$. Then for $u_n(r) = r + \frac{1}{2} \log N_n$ and all $x_0 \in X$ we have
\[
\lim_{n \to \infty} \mu \left( M_{I_n} \leq u_n(r) \right) = e^{-we^{-dr}}.
\]

Since the conditions of Theorem 4.6 are rather technical, we give a simple example for which the conditions are met.

**Example 4.7.** Let $q \in \mathbb{R}$ and set $m_j = q^j$. It is then clear that for sufficiently large $q \in \mathbb{R}$ we have the inequality
\[
\sup_{j \in \mathbb{N}} \frac{m_{j-1}}{m_j} = \frac{1}{q} < \min \left( 1, \frac{\delta}{k \gamma} \right)
\]
With this choice of $m_j$ we see from condition (4.16) that the assumptions of Theorem 4.6 are satisfied when we make the simple choice $\beta_n = 2n$ and $\alpha_n = n$. It then holds true that for some $w > 0$,
\[
\lim_{n \to \infty} \mu \left( \max_{n \leq j \leq 2n} -\log d(a_q, x, x_0) \leq u_n(r) \right) = e^{-we^{-dr}}.
\]

Notice that we can pick $\alpha_n$ and $\beta_n$ such that $N_n$ grows much faster. However, we made the given choice to obtain a simple and intuitive statement.

**Proof of Theorem 4.6.** The proof of Theorem 4.6 is very similar to the proof of Theorem 4.1. The strategy is the same and we can reuse Lemma 4.4 and 4.5 with only minimal alterations. However, we need a modified version of Lemma 4.3. This is a consequence of $-\log d(\cdot, x_0)$ not being uniformly continuous.

Set $V(r) = \{ x : D(x) \leq r \}$. We want to approximate the characteristic functions $1_{V(u_n)}$ by $(C, k)$-regular functions. Notice that
\[
1_{V(u_n)} = 1_{B(x_0, e^{-un})^c}.
\]

**Lemma 4.8.** For every $k \in \mathbb{N}$, any $\omega > 0$ and any $n \in \mathbb{N}$ we can find two $(C_{n, \omega}, k)$-regular functions $g_{n, \omega}$ and $h_{n, \omega}$ such that
\[
g_{n, \omega} \leq 1_{B(x_0, e^{-un})^c} \leq h_{n, \omega} \leq 1
\]
and
\[
\mu \left( B(x_0, e^{-un+\omega})^c \right) = \int_X g_{n, \omega} \, d\mu \leq \int_X h_{n, \omega} \, d\mu = \mu \left( B(x_0, e^{-un-\omega})^c \right).
\]

**Proof.** For any $\omega > 0$ we have
\[
B(x_0, e^{-un+\omega})^c \subset B(x_0, e^{-un})^c \subset B(x_0, e^{-un-\omega})^c.
\]
Set \( \varepsilon = \varepsilon_{n,\omega} = e^{-un}(1 - e^{-\omega}) \) and let \( \varphi_\varepsilon \) be defined as in (3.3). We then define the functions \( g_{n,\omega} \) and \( h_{n,\omega} \) as

\[
g_{n,\omega} = \varphi_\varepsilon \cdot 1_{B(x_0, e^{-un+\omega})^c} \quad \text{and} \quad h_{n,\omega} = \varphi_\varepsilon \cdot 1_{B(x_0, e^{-un-\omega})^c}.
\]

With this definition (4.17) follows directly from Lemma 3.11. We also get

\[
g_{n,\omega} \leq 1_{B(x_0, e^{-un})^c} \leq h_{n,\omega} \leq 1. \tag{4.18}
\]

To see this, we first rewrite \( g_{n,\omega} \) as an integral,

\[
g_{n,\omega}(x) = \int_G \varphi_\varepsilon(g) 1_{B(x_0, e^{-un+\omega})^c}(g^{-1}x) \, dm(g)
= \int_{B(\varepsilon,\varepsilon)} \varphi_\varepsilon(g) 1_{B(x_0, e^{-un+\omega})^c}(x) \, dm(g).
\]

It's clear that if \( gB(x_0, e^{-un+\omega})^c \subset B(x_0, e^{-un})^c \) then the first inequality of (4.18) is established. So assume \( x \in gB(x_0, e^{-un+\omega})^c \). Then we can write \( x = gy \) where \( y \in B(x_0, e^{-un+\omega})^c \). This means that \( d(y, \partial B(x_0, e^{-un})^c) \geq e^{-un+\omega} - e^{-un} = e^{-un}(e^\omega - 1) \geq \varepsilon \). Hence \( gy \in B(x_0, e^{-un})^c \) and \( gB(x_0, e^{-un+\omega})^c \subset B(x_0, e^{-un})^c \). The second inequality of (4.18) is proved similarly and the third inequality is trivial.

The \((C_{n,\omega}, k)\)-regularity of \( g_{n,\omega} \) follows directly from Lemma 3.11 with \( C_{n,\omega} = S_k(\varphi_\varepsilon) \) since

\[
S_k(\varphi_\varepsilon \cdot 1_{B(x_0, e^{-un+\omega})^c}) \leq S_k(\varphi_\varepsilon) \mu(B(x_0, e^{-un+\omega})^c) = S_k(\varphi_\varepsilon) \int X g_{n,\omega} \, d\mu.
\]

By the same argument \( h_{n,\omega} \) is \((C_{n,\omega}, k)\)-regular. \( \square \)

Again we begin the proof by rewriting the quantity

\[
\mu(M_{\alpha n} \leq u_n) = \mu \left( \max_{\alpha_n \leq \beta_n} \mathcal{D}(x) \leq u_n \right)
\]

as an integral. By definition of the maximum we have

\[
\mu \left( \max_{\alpha_n \leq \beta_n} \mathcal{D}(a_{mj}) \leq u_n \right) = \mu(\mathcal{D}(a_{m\alpha_n}) \leq u_n, \ldots, \mathcal{D}(a_{m\beta_n}) \leq u_n)
= \int_X 1_{V(u_n)}(a_{m\alpha_n}) \cdots 1_{V(u_n)}(a_{m\beta_n}) \, d\mu(x). \tag{4.19}
\]

For any integers \( i_1 < i_2 \), set

\[
G_{(i_1, i_2)}(x) = \prod_{i=i_1}^{i_2} g_{n,\omega}(a_{mi}) \quad \text{and} \quad H_{(i_1, i_2)}(x) = \prod_{i=i_1}^{i_2} h_{n,\omega}(a_{mi}).
\]

Lemma 4.8 and equation (4.19) then imply that

\[
\int_X G_{(\alpha_n, \beta_n)} \, d\mu \leq \mu(M_{\alpha n} \leq u_n) \leq \int_X H_{(\alpha_n, \beta_n)} \, d\mu.
\]
For $g_{n,\omega}$ and $h_{n,\omega}$ the analogues of (4.9) and (4.10) are, that for sufficiently large $n$,

$$\left| \int_X G(\alpha_n, \beta_n) \, d\mu - \left( \int_X g_{n,\omega} \, d\mu \right)^{N_n} \right| \ll \varepsilon_{n,\omega}^{(k+1)-2k} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_{s-1} - \delta m_s} (s - \alpha_n)^k$$

and

$$\left| \int_X H(\alpha_n, \beta_n) \, d\mu - \left( \int_X h_{n,\omega} \, d\mu \right)^{N_n} \right| \ll \varepsilon_{n,\omega}^{(k+1)-2k} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_{s-1} - \delta m_s} (s - \alpha_n)^k.$$

This implies

$$\left( \int_X g_{n,\omega} \, d\mu \right)^{N_n} - O\left( \varepsilon_{n,\omega}^{(k+1)-2k} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_{s-1} - \delta m_s} (s - \alpha_n)^k \right) \leq \mu(M \leq u_n)$$

$$\leq \left( \int_X h_{n,\omega} \, d\mu \right)^{N_n} + O\left( \varepsilon_{n,\omega}^{(k+1)-2k} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_{s-1} - \delta m_s} (s - \alpha_n)^k \right).$$

Again, we want to determine the limit, if it exists, of the upper and lower bound as $n \to \infty$ and $\omega \to 0$, and we begin by looking at the error term. First we make the trivial estimate

$$\varepsilon_{n,\omega}^{(k+1)-2k} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_{s-1} - \delta m_s} (s - \alpha_n)^k \leq \varepsilon_{n,\omega}^{(k+1)-2k} (\beta_n - \alpha_n)^k \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_{s-1} - \delta m_s}.$$

We now look at the sum in the error term. Since $\sup_{s \in \mathbb{N}} \frac{m_{s-1}}{m_s} = \frac{1}{\rho}$ we see that for all $s \in \mathbb{N}$

$$m_s \geq \rho m_{s-1} \geq \cdots \geq \rho^s m_0,$$

and consequently,

$$k\gamma m_{s-1} - \delta m_s = m_s \left( k\gamma \frac{m_{s-1}}{m_s} - \delta \right) \leq m_0 \left( \frac{k\gamma}{\rho} - \delta \right) \rho^s < 0.$$

Set $\sigma' = -m_0 \left( \frac{k\gamma}{\rho} - \delta \right) > 0$. Now, since $\rho > 1$, we know that for $s \in \mathbb{N}$ sufficiently large we have that $\rho^{s+i} \geq \rho^s + i$ and hence

$$\left( e^{-\sigma'} \right)^{\rho^{s+i}} \leq \left( e^{-\sigma'} \right)^{\rho^s + i}.$$
Consequently, for \( n \in \mathbb{N} \) sufficiently large we can write
\[
\sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma_{m,s-1} - \delta_m s} = \sum_{s=\alpha_n+1}^{\beta_n} \left( e^{-\sigma'} \right)^i \leq \sum_{i=\rho_n+1}^{\beta_n} \left( e^{-\sigma'} \right)^i \\
\leq \sum_{i=\rho_n}^{\infty} \left( e^{-\sigma'} \right)^i \\
= O\left( e^{-\sigma' \rho_n} \right).
\]

Finally we look at the rest of the error term. Recalling from Lemma 4.8 that \( \varepsilon_{n,\omega} = e^{-u_n (1 - e^{-\omega})} \), we get
\[
\varepsilon_{n, \omega}^{-\frac{d}{2}(k+1)-2k} (\beta_n - \alpha_n)^k = \left( e^{-r} e^{-\frac{1}{2} \log N_n (1 - e^{-\omega})} \right)^{-\frac{d}{2}(k+1)-2k} (\beta_n - \alpha_n)^k \\
= O\left( N_n^{\frac{k}{2} + \frac{d}{2} + \frac{1}{2}} \right) \tag{4.22}
\]

Here we considered \( r \) and \( \omega \) constants since we are currently focusing on the limit as \( n \to \infty \). Hence for \( n \in \mathbb{N} \) sufficiently large we have,
\[
\varepsilon_{n, \omega}^{-\frac{d}{2}(k+1)-2k} (\beta_n - \alpha_n)^k \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma_{m,s-1} - \delta_m s} = O\left( \frac{N_n^{\frac{k}{2} + \frac{d}{2} + \frac{1}{2}}}{e^{\sigma' \rho_n}} \right) \to 0 \tag{4.23}
\]
for \( n \to \infty \). The convergence follows from the assumption in (4.16).

For the main terms we claim that
\[
\lim_{\omega \to 0} \lim_{n \to \infty} \left( \int_X g_{n, \omega} d\mu \right)^{N_n} = \lim_{\omega \to 0} \lim_{n \to \infty} \left( \int_X h_{n, \omega} d\mu \right)^{N_n} = e^{-we^{-dr}} \tag{4.24}
\]
To see this we first look at the limits for \( n \to \infty \). It follows from Lemma 4.8 and Lemma 3.4 that
\[
\int_X g_{n, \omega} d\mu = \mu \left( B(x_0, e^{-(u_n - \omega)})^c \right) = 1 - w \left( e^{-d(u_n - \omega)} \right) + o \left( e^{-d(u_n - \omega)} \right)
\]
\[
\int_X h_{n, \omega} d\mu = \mu \left( B(x_0, e^{-(u_n + \omega)})^c \right) = 1 - w \left( e^{-d(u_n + \omega)} \right) + o \left( e^{-d(u_n + \omega)} \right).
\]
Recall that \( u_n(r) = r + \frac{1}{d} \log n \). Raising to the power \( N_n \) we get
\[
\left( \int_X g_{n, \omega} d\mu \right)^{N_n} = \left( 1 - \frac{we^{d(r + \omega)}}{N_n} + o \left( \frac{e^{d(r + \omega)}}{N_n} \right) \right)^{N_n} \\
\left( \int_X h_{n, \omega} d\mu \right)^{N_n} = \left( 1 - \frac{we^{d(r - \omega)}}{N_n} + o \left( \frac{e^{d(r - \omega)}}{N_n} \right) \right)^{N_n}.
\]
To compute the limit of the two right hand sides above, we first rewrite as follows
\[
1 - \frac{we^{d(-r \pm \omega)}}{N_n} + o \left( \frac{e^{d(-r \pm \omega)}}{N_n} \right) = e^{-N_n \log \left( 1 - \frac{we^{d(-r \pm \omega)}}{N_n} + o \left( \frac{e^{d(-r \pm \omega)}}{N_n} \right) \right)}.
\]
We then estimate the right hand side using the second order Taylor expansion of \( \log(1 + x) \), that is, \( \log(1 + x) = x + O(x^2) \). Inserting the resulting estimate and taking the limit for \( n \to \infty \) gives,
\[
\lim_{n \to \infty} \mu \left( M_I \leq u_n \right) = e^{-we^{-d}e^{-Dr}}.
\]
The claim is then proved by letting \( \omega \to 0 \). It then follows from (4.28) and (4.24) that when we take the limit for \( n \to \infty \) and then for \( \omega \to 0 \) in (4.20), we get
\[
\lim_{n \to \infty} \mu \left( M_I \leq u_n \right) = e^{-we^{-d}e^{-Dr}}.
\]
This proves the theorem.

4.4. Finalizing proofs of main results. The only thing left to do to prove Theorem 1.1, 1.3 and 1.5 is to argue that the assumptions of Theorem 4.1 and 4.6 are satisfied. It follows from Theorem 3.10 that in the setup of Section 1.2, 1.3 and 1.4, \( a_t \) has exponential decay of correlations. This shows that Theorem 1.3 follows from Theorem 4.6. Also, from Theorem 3.2 we know that \( d(\cdot, x_0) \) is \((w_1, w_2, v)\)-DL for some positive constants \( w_1, w_2 \) and \( v \). From Theorem 3.3 we know that \( \Delta \) is \((w, d)\)-SDL for \( w = \frac{V_d}{2 \zeta(d)} \). Theorem 1.1 and 1.5 then follow from Theorem 4.1.

5. General theory for exceedances by one-parameter actions

We now proceed to the general theory from which Theorem 1.2, 1.4 and 1.6 will follow. Many aspects of the proofs are similar to the proofs presented in Section 4, but some alterations and new ideas are necessary.

First we need to rewrite the distribution function of the \( k \)'th largest maximum of a general stationary process \( \xi_i \). The computation is basic and essentially only contains straightforward set and measure theoretic arguments. However, the resulting expression from this derivation will be used often throughout this section, hence we have dedicated a separate section to it.

5.1. The distribution of the \( k \)'th largest maximum for a stationary process. Let \((X, P)\) denote a probability space. Let \( \xi_i \) be a stationary sequence of random variables on \( X \). For any subset \( I \subset \mathbb{N} \) we will denote the \( k \)'th largest maximum by
\[
M_I^{(k)} = \max_{i \in I}^{(k)}(\xi_i),
\]
by which we mean the \( k \)'th largest \( \xi_i \) for \( i \in I \) where \( k \leq |I| \). We are interested in the limiting distribution of \( M_I^{(k)} \), i.e. the limit
\[
\lim_{n \to \infty} P \left( M_I^{(k)} \leq u_n \right)
\]
for certain choices of interval $I$. In order to estimate this limiting distribution we begin by rewriting the set $\{ M^{(k)}_i \leq u_n \}$. First we see that

\[
\{ M^{(k)}_i \leq u_n \} = \{ M^{(1)}_i \leq u_n \} \cup \{ M^{(1)}_i > u_n, M^{(2)}_i \leq u_n \} \\
\cup \cdots \cup \{ M^{(k-1)}_i > u_n, M^{(k)}_i \leq u_n \} \\
= \{ M^{(1)}_i \leq u_n \} \cup \bigcup_{i=2}^{k} \{ M^{(i-1)}_i > u_n, M^{(i)}_i \leq u_n \}.
\]

Clearly the sets in the union are disjoint, hence

\[
P(M^{(k)}_i \leq u_n) = P(M^{(1)}_i \leq u_n) + \sum_{i=2}^{k} P(M^{(i-1)}_i > u_n, M^{(i)}_i \leq u_n). \tag{5.1}
\]

In the previous sections we have already derived estimates on the limit of $P(M^{(1)}_i \leq u_n)$ so we focus on the limiting distribution of the sets $\{ M^{(i-1)}_i > u_n, M^{(i)}_i \leq u_n \}$ for $2 \leq i \leq k$. We can write this set as a union of disjoint sets as well, but first we need to introduce some additional notation. In the following let $m_j$ be a subsequence in $\mathbb{C}$, let $a < b$ be integers and set $I = \{ a, m_{a+1}, \ldots, b \}$. Set $N = |I| = b - a + 1$. Let $S^{(i)}_I$ denote the set of all distinct subsets of $I$ with cardinality $i - 1$. We can then write

\[
\{ M^{(i-1)}_i > u_n, M^{(i)}_i \leq u_n \} = \bigcup_{J \in S^{(i)}_I} \left( \bigcap_{j \in J} \{ \xi_j > u_n \} \cap \bigcap_{j \in I \setminus J} \{ \xi_j \leq u_n \} \right).
\]

As an example we see that for $i = 2$ this union simplifies to

\[
\{ M^{(1)}_i > u_n, M^{(2)}_i \leq u_n \} = \bigcup_{i=a}^{b} \left( \{ \xi_i > u_n \} \cap \bigcap_{j=a,j \neq i}^{b} \{ \xi_j \leq u_n \} \right) \\
= \{ \xi_a > u_n, \xi_{a+1} \leq u_n, \ldots, \xi_b \leq u_n \} \\
\cup \{ \xi_a \leq u_n, \xi_{a+1} > u_n, \xi_{a+2} \leq u_n, \ldots, \xi_b \leq u_n \} \\
\vdots \\
\cup \{ \xi_a \leq u_n, \xi_{a+1} \leq u_n, \ldots, \xi_b > u_n \}.
\]
As the sets in the union are clearly disjoint we get
\[
P \left( M_{j}^{(i-1)} > u_n, M_{j}^{(i)} \leq u_n \right) = P \left( \bigcup_{J \in S_{j}^{(i)}} \left( \cap_{j \in J} \{ \xi_j > u_n \} \cap \cap_{j \in I \setminus J} \{ \xi_j \leq u_n \} \right) \right)
\]
\[
= \sum_{J \in S_{j}^{(i)}} \left( \cap_{j \in J} \{ \xi_j > u_n \} \cap \cap_{j \in I \setminus J} \{ \xi_j \leq u_n \} \right).
\]

Re-inserting this in (5.1) gives
\[
P \left( M_{j}^{(k)} \leq u_n \right) = P(M_{j}^{(1)} \leq u_n)
\]
\[
+ \sum_{i=2}^{k} \sum_{J \in S_{j}^{(i)}} \left( \cap_{j \in J} \{ \xi_j > u_n \} \cap \cap_{j \in I \setminus J} \{ \xi_j \leq u_n \} \right).
\]

This way of writing the distribution of the \( k \)'th largest maximum will be used several times throughout the section. We notice that a way of deriving an estimate on the limiting distribution of the \( k \)'th largest maximum is to derive an estimate on the probability
\[
P \left( \cap_{j \in J} \{ \xi_j > u_n \} \cap \cap_{j \in I \setminus J} \{ \xi_j \leq u_n \} \right),
\]
which is independent of the choice of \( J \in S_{j}^{(i)} \).

5.2. Notation and setup. We return to the setup introduced in Section 4.1 which we briefly recall. \( m_j \) denotes a fixed sequence in \( \mathbb{R} \). \( \alpha_n \) and \( \beta_n \) denote sequences for which \( \alpha_n < \beta_n \), \( \alpha_n \to \infty \) and \( \beta_n - \alpha_n \to \infty \). Set \( N_n := \beta_n - \alpha_n + 1 \) and \( I_n = \{ m_{\alpha_n}, m_{\alpha_n+1}, \ldots, m_{\beta_n} \} \). We also recall the general assumptions of the section, namely

1. Assume that \( \text{Ad}(a_t) \) is diagonalizable and fix an eigenbasis \( W := \{ \zeta_1, \cdots, \zeta_d \} \) for \( \text{Ad}(a_t) \). That is for every \( \zeta_r \in W \) there exists \( \gamma_r \in \mathbb{R} \) such that
\[
\text{Ad}(a_t) \zeta_r = e^{\gamma_r t} \zeta_r.
\]

Set also throughout
\[
\gamma := \max_{1 \leq i \leq d} \gamma_i \quad \text{and} \quad \gamma_i \quad \text{satisfies} \quad (5.3)
\]

2. Assume that the Sobolev norm \( S_k \) is defined with respect to the fixed eigenbasis \( W \).

3. The one-parameter action \( a_t \) will be assumed to have exponential decay of correlations as defined in Definition 3.1.
5.3. **Maximal excursions.** In this section we prove a general result from which Theorem 1.2 and 1.6 will follow. Let $D : X \to \mathbb{R}$ denote a measurable function and set $\xi_i(x) = D(a_i x)$. In this section we will look at the $l$’th largest maximum to avoid notational confusion with the degree of the Sobolev norm for which we use the parameter $k$.

We are ready to state the first main theorem of the section.

**Theorem 5.1.** Let $\rho^{-1} = \left( \sup_{s \in \mathbb{N}} \left( \frac{m_{s-1}}{m_s} \right) \right)$. Assume that $m_j \in \mathbb{R}$ satisfies
\[ \sup_{s \in \mathbb{N}} \left( \frac{m_{s-1}}{m_s} \right) < \min \left( 1, \frac{\delta}{k \gamma} \right) \]
where $k, \delta$ are as in (3.1) and $\gamma$ as in (5.4). Assume further that $N_n = o \left( e^{\sigma \rho_n} \right)$, where $\sigma = -m_0 \left( \frac{k}{\rho} - \delta \right) > 0$.

A) Assume $D$ is $(w, v)$-SDL for some positive constants $w$ and $v$. Then for $u_n(r) = r + \frac{1}{v} \log N_n$ we have
\[ \lim_{n \to \infty} \mu \left( M_{I_n}^{(l)} \leq u_n(r) \right) = e^{-w e^{-vr}} \sum_{i=0}^{l-1} \frac{(we^{-vr})^i}{i!} \]

B) Assume $D$ is $(w_1, w_2, v)$-DL for some positive constants $w_1, w_2$ and $v$. Then for $u_n(r) = r + \frac{1}{v} \log N_n$ we have
\[ e^{-w_1 e^{-vr}} \sum_{i=0}^{l-1} \frac{(w_2 e^{-vr})^i}{i!} \leq \lim_{n \to \infty} \mu \left( M_{I_n}^{(l)} \leq u_n(r) \right) \leq \lim_{n \to \infty} \mu \left( M_{I_n}^{(l)} \leq u_n(r) \right) \leq e^{-w_2 e^{-vr}} \sum_{i=0}^{l-1} \frac{(w_1 e^{-vr})^i}{i!} \]

Recall equation (5.2) stating that
\[ \mu \left( M_{I_n}^{(l)} \leq u_n \right) = \mu(M_{I_n}^{(1)} \leq u_n) + \sum_{i=2}^{l} \sum_{J \in \mathcal{S}_n^{(i)}} \mu \left( \bigcap_{j \in J} \{ \xi_j > u_n \} \cap \bigcap_{j \in I_n \setminus J} \{ \xi_j \leq u_n \} \right) \] (5.5)

The idea is to derive upper and lower bounds on the probability
\[ \mu \left( \bigcap_{j \in J} \{ \xi_j > u_n \} \cap \bigcap_{j \in I_n \setminus J} \{ \xi_j \leq u_n \} \right) , \]
that are independent of the choice of $J \subset I_n$. We do this similarly to how we estimated
\[ \mu \left( \xi_{m_{\alpha n}} \leq u_n, \ldots, \xi_{m_{\beta n}} \leq u_n \right) \]
Lemma 4.3. The approximating functions are defined as

Proof. The proof is identical to the proof of Lemma 4.3. With

Assume that Lemma 5.2. Fortunately, this follows almost immediately from the proof of Lemma 4.3.

The following is an adaptation of Lemma 4.4.

Lemma 5.2. Assume that \(\mathcal{D} : X \to \mathbb{R}\) is uniformly continuous. Then for any \(k \in \mathbb{N}\) and any \(\delta > 0\) there exist \(\varepsilon > 0\) and \(C_\varepsilon = O\left(\varepsilon^{-\left(\frac{4}{3} + k\right)}\right) > 0\) such that for every \(r \in \mathbb{R}\) one can find two \((C_\varepsilon, k)\)-regular functions \(g'_{n,\varepsilon}\) and \(h'_{n,\varepsilon}\) on \(X\) such that

\[
g'_{n,\varepsilon} \leq 1_{U(u_n(r))} \leq h'_{n,\varepsilon} \leq 1
\]

and

\[
\mu (U(u_n(r) + \delta)) \leq \int_X g'_{n,\varepsilon} \, d\mu \leq \int_X h'_{n,\varepsilon} \, d\mu \leq \mu (U(u_n(r) - \delta)).
\]

Proof. The proof is identical to the proof of Lemma 4.3. With \(\varepsilon = \varepsilon(\delta) > 0\) being as in Lemma 4.3 the approximating functions are defined as

\[
g'_{n,\varepsilon} = \varphi_{\varepsilon} \ast 1_{U'(u_n,\varepsilon)} \quad \text{and} \quad h'_{n,\varepsilon} = \varphi_{\varepsilon} \ast 1_{U''(u_n,\varepsilon)}
\]

where

\[
U'(u_n,\varepsilon) = \{x \in U(u_n) : d(x, \partial U(u_n)) \geq \varepsilon\}
\]

\[
U''(u_n,\varepsilon) = \{x \in X : d(x, U(u_n)) \leq \varepsilon\}
\]

and \(\varphi_{\varepsilon}\) as defined in (3.3) \(\square\)

Recall the functions \(g_{n,\varepsilon} \leq 1_{V(u_n)} \leq h_{n,\varepsilon}\) defined in Lemma 4.3. For any subset \(J \subset I_n\) of cardinality \(i - 1\), define the functions

\[
g^{(j)}_J = \begin{cases} g'_{n,\varepsilon} & \text{if } j \in J \\ g_{n,\varepsilon} & \text{if } j \in I_n \setminus J. \end{cases} \quad \text{and} \quad h^{(j)}_J = \begin{cases} h'_{n,\varepsilon} & \text{if } j \in J \\ h_{n,\varepsilon} & \text{if } j \in I_n \setminus J. \end{cases}
\]

and also, for integers \(i_1 < i_2\)

\[
G^J_{(i_1,i_2)}(x) = \prod_{j=i_1}^{i_2} g^{(j)}_J(a_{m_j}x) \quad \text{and} \quad H^J_{(i_1,i_2)}(x) = \prod_{j=i_1}^{i_2} h^{(j)}_J(a_{m_j}x).
\]

The following is an adaptation of Lemma 4.4.

Lemma 5.3. Let \(J \subset I_n\) be any subset of cardinality \(i - 1\). Then

\[
\left| \int_X G^J_{(\alpha_n,\beta_n)} \, d\mu - \prod_{j=\alpha_n}^{\beta_n} \int_X g^{(j)}_J \, d\mu \right| \leq \sum_{s=\alpha_n+1}^{\beta_n} e^{-\delta_m} S_k \left( G^J_{(\alpha_n,s-1)} \right) S_k \left( g^{(s)}_J \right) \prod_{j=s+1}^{\beta_n} \int_X g^{(j)}_J \, d\mu \tag{5.6}
\]
and
\[
\left| \int_X H^J_{(\alpha_n, \beta_n)} \, d\mu - \prod_{j=\alpha_n}^{\beta_n} \int_X h_j^{(j)} \, d\mu \right| \lesssim \sum_{s=\alpha_n+1}^{\beta_n} e^{-\delta m_s} S_k \left( H^J_{(\alpha_n, s-1)} \right) S_k \left( h_j^{(s)} \right) \prod_{j=s+1}^{\beta_n} \int_X h_j^{(j)} \, d\mu. \tag{5.7}
\]

Remark 5.4. When \( s = \beta_n \) in the error term we use the convention that
\[
\prod_{j=\beta_{n+1}}^{\beta_n} := 1.
\]

Proof. Let \( J \subset I_n \) be an arbitrary subset of cardinality \( i - 1 \). In the following, let \( \varphi \) and \( \psi \) denote two \((C, k)\)-regular functions on \( X \). Define the function
\[
\eta_j^{(j)} = \begin{cases} 
\varphi & \text{if } j \in J \\
\psi & \text{if } j \in I_n \setminus J.
\end{cases}
\]

Define also the function
\[
F_{(\alpha_n, \beta_n)} = F^J_{(\alpha_n, \beta_n)}(x) = \prod_{j=\alpha_n}^{\beta_n} \eta_j^{(j)}(a_{m_j} x).
\]

As in Lemma 4.4, the idea is to write
\[
F_{(\alpha_n, \beta_n)}(x) = F_{(\alpha_n, \beta_n-1)}(x) \eta_j^{(\beta_n)}(a_{m_j} x),
\]
and use the exponential decay of correlations on this product. Indeed, the proof of the lemma is identical to that of Lemma 4.4 with the only exception being that we do not apply the estimate \( \int_X \eta_j^{(j)} \, d\mu \leq 1 \). We leave out the details to avoid unnecessary repetition. □

We need estimates on the error terms in Lemma 5.3 that are explicit in \( n \) and \( \varepsilon \). For this we need to understand the Sobolev norm of the functions \( G^J_{(\alpha_n, s-1)} \) and \( H^J_{(\alpha_n, s-1)} \). The following is an adaptation of Lemma 4.5.

Lemma 5.5. Let \( J \subset I_n \) be any subset of cardinality \( i - 1 \) and let \( i_1 = |J \cap \{m_{\alpha_n}, \ldots, m_c\}| \). Then for any \( \alpha_n < c \leq \beta_n \) we have that for \( \varepsilon > 0 \) sufficiently small,
\[
S_k(G^J_{(\alpha_n, c)}) = O \left( (c - \alpha_n + 1)^k \varepsilon^{-k(\frac{D}{2}+1)} \mu(U(u_n))^{\frac{1}{2}(i_1-1)+1} e^{k\gamma m_c} \right)
\]
and
\[
S_k(H^J_{(\alpha_n, c)}) = O \left( (c - \alpha_n + 1)^k \varepsilon^{-k(\frac{D}{2}+1)} \mu(U(u_n - \delta))^{\frac{1}{2}(i_1-1)+1} e^{k\gamma m_c} \right).
\]

Proof. This proof bears many similarities to the proof of Lemma 4.5; hence in the following we leave out several details which have already been discussed. First we prove the statement for \( S_k(G^J_{(\alpha_n, c)}) \).
For notational simplicity set \( G := G^J_{(a_n,c)} \), where \( \alpha_n < c \leq \beta_n \). We are looking to estimate

\[
\left\| D_{\zeta_{n_1}} \cdots D_{\zeta_{n_l}} G \right\|_2.
\]

By the product rule, \( D_{\zeta_{n_1}} \cdots D_{\zeta_{n_l}} G \) can be written as a sum containing a total of \((c-\alpha_n+1)^l\) terms. Let \( 1 \leq p \leq (c-\alpha_n+1)^l \). Then the \( p \)'th term is of the form \( D_p^{(\alpha_n)} g_j^{(\alpha_n)} \cdots D_p^{(c)} g_j^{(c)} \), where \( D_p^{(s)} \) is the differential operator associated to the function \( g_j^{(s)} \) in the \( p \)'th term of the sum. For every \( p \), the sum of the orders of \( D_p^{(\alpha_n)}, \ldots, D_p^{(c)} \) equals \( l \). We then get

\[
\left\| D_p^{(\alpha_n)} g_j^{(\alpha_n)}(a_{m_{\alpha_n}} x) \cdots D_p^{(c)} g_j^{(c)}(a_{m_c} x) \right\|_2 \\
\leq e^{\gamma_m \epsilon_c} \left\| D_p^{(\alpha_n)} g_j^{(\alpha_n)}(a_{m_{\alpha_n}} x) \right\|_2 \cdots \left\| D_p^{(c)} g_j^{(c)}(a_{m_c} x) \right\|_2 \\
\leq e^{\gamma_m \epsilon_c} \left\| D_p^{(\alpha_n)} g_j^{(\alpha_n)} \right\|_\infty \cdots \left\| D_p^{(c-1)} g_j^{(c-1)} \right\|_\infty \left\| D_p^{(c)} g_j^{(c)} \right\|_2. \\
\tag{5.8}
\]

Notice that we are free to choose which function should remain inside the \( L^2 \)-norm. It follows from equation \((3.2)\) and the Young inequality for convolutions that

\[
\left\| D_p^{(s)} g'_{n,\epsilon} \right\|_\infty \leq \left\| D_p^{(s)} \varphi_{\epsilon} \right\|_2 \left\| 1_{U'(u_n)} \right\|_2 \leq \left\| D_p^{(s)} \varphi_{\epsilon} \right\|_2 \mu(U(u_n))^{1/2} \\
\left\| D_p^{(s)} g_{n,\epsilon} \right\|_\infty \leq \left\| D_p^{(s)} \varphi_{\epsilon} \right\|_2 \left\| 1_{U'(u_n)} \right\|_2 \leq \left\| D_p^{(s)} \varphi_{\epsilon} \right\|_2 \mu(U(u_n))^{1/2} \leq \left\| D_p^{(s)} \varphi_{\epsilon} \right\|_2 \mu(U(u_n)). \\
\tag{5.9}
\]

Since \( u_n \to \infty \) for \( n \to \infty \) it is clear that \( \mu(V(u_n)) \leq 1 \) is increasing with \( n \) while \( \mu(U(u_n)) \) is decreasing with \( n \). So \( \mu(U(u_n)) \) has a role to play in the convergence of the error term in Lemma \(5.3\) while \( \mu(V(u_n)) \) will not affect the convergence. Consequently it makes sense in the following computations to apply the estimate \( \mu(V(u_n))^{1/2} \leq 1 \). Using the facts stated above we can rewrite \((5.8)\) further. Pick the \( L^2 \)-norm in \((5.8)\) to contain one of the \( g'_{n,\epsilon} \) as opposed to one of the \( g_{n,\epsilon} \). Using equation \((3.1)\) as well, we get

\[
\left\| D_p^{(\alpha_n)} g_j^{(\alpha_n)}(a_{m_{\alpha_n}} x) \cdots D_p^{(c)} g_j^{(c)}(a_{m_c} x) \right\|_2 \\
\leq e^{\gamma_m \epsilon_c} \mu(U(u_n))^{1/2} \left\| D_p^{(\alpha_n)} \varphi_{\epsilon} \right\|_2 \cdots \left\| D_p^{(c)} \varphi_{\epsilon} \right\|_2 \\
= O \left(e^{\gamma_m \epsilon_c} \mu(U(u_n))^{1/2} [1 + \epsilon^{1/2} - l(\epsilon + 1)^{1/2}] \right).
\]

Since this estimate is independent of \( p \) we get

\[
\left\| D_{\zeta_{n_1}} \cdots D_{\zeta_{n_l}} G \right\|_2 = O \left( (c - \alpha_n + 1)^l e^{\gamma_m \epsilon_c} \mu(U(u_n))^{1/2} [1 + \epsilon^{1/2} - l(\epsilon + 1)^{1/2}] \right),
\]

and finally

\[
S_k \left( G^J_{(\alpha_n,c)} \right) = O \left( (c - \alpha_n + 1)^k \epsilon^{-k(\epsilon + 1)^{1/2}} \mu(U(u_n))^{1/2} [1 + \epsilon^{1/2} + e^{k \gamma_m \epsilon_c}] \right).
\]
When proving the statement for $S_k\left(H^I_{(\alpha,n,c)}\right)$ we see that the only difference appears in (5.9) where we instead have
\[
\|D^{(s)}_p h_{n,c}\|_\infty \leq \|D^{(s)}_p \phi \|_2 \mu(U(u_n - \delta))^{\frac{1}{2}} \\
\|D^{(s)}_p h_{n,c}\|_\infty \leq \|D^{(s)}_p \phi \|_2 \mu(V(u_n + \delta))^{\frac{1}{2}} \\
\|D^{(s)}_p g_{n,c}\|_2 \leq \|D^{(s)}_p \phi \|_2 \mu(U(u_n - \delta)).
\]

For the same reasons as stated above we use the upper bound $\mu(V(u_n + \delta))^{\frac{1}{2}} \leq 1$. Clearly this leads to the estimate
\[
S_k\left(H^I_{(\alpha,n,c)}\right) = O\left((c - \alpha_n + 1)^k \varepsilon^{-k(\frac{d}{2}+1)} \mu(U(u_n - \delta))^{\frac{1}{2}i_1+\frac{1}{2}} e^{k\gamma m_c}\right).
\]

We now combine Lemma 5.3 and Lemma 5.5 and derive an upper bound on the error term which is independent of the choice of $J \subset I_n$.

**Lemma 5.6.** For any sufficiently small, but fixed $\varepsilon > 0$ and for any subset $J \subset I_n$ of cardinality $i_1 - 1$ we have
\[
\left|\int_X G_{(\alpha_n,\beta_n)}^J d\mu - \prod_{j=\alpha_n}^{\beta_n} \int_X g^{(j)}_J d\mu\right| \ll \mu(U(u_n))^{\frac{1}{2}i} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_s-1-\delta m_s}(s - \alpha_n)^k
\]
and
\[
\left|\int_X H^J_{(\alpha_n,\beta_n)} d\mu - \prod_{j=\alpha_n}^{\beta_n} \int_X h^{(j)}_J d\mu\right| \ll \mu(U(u_n - \delta))^{\frac{1}{2}i} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_s-1-\delta m_s}(s - \alpha_n)^k.
\]

**Proof.** We only prove the first inequality since the second one is proved identically.

Insert the estimate for $S_k\left(G^J_{(\alpha_n,s-1)}\right)$ in (5.6). This gives the error term
\[
\varepsilon^{-k(\frac{d}{2}+1)} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_s-1-\delta m_s}(s - \alpha_n)^k \mu(U(u_n))^{\frac{1}{2}i_1+\frac{1}{2}} S_k(g^{(s)}_J) \prod_{j=s+1}^{\beta_n} \int_X g^{(j)}_J d\mu.
\]

(5.10) where $i_1 = |J \cap \{m_{\alpha_n}, \ldots, m_{s-1}\}|$. Let $i_2 = 1$ if $s \in J$ and $i_2 = 0$ if $s \notin J$. It then follows from Lemma 3.11 and Lemma 3.12 that $S_k(g^{(s)}_J) \leq \varepsilon^{-(\frac{d}{2}+k)} \mu(U(u_n))^{i_2}$. Set $i_3 = |J \cap \{m_{s+1}, \ldots, m_{\beta_n}\}|$. It is then clear that
\[
\prod_{j=s+1}^{\beta_n} \int_X g^{(j)}_J d\mu \leq \mu(U(u_n))^{i_3}.
\]

This means that (5.10) is bounded by
\[
\varepsilon^{-\frac{d}{2}(k+1)-2k} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_s-1-\delta m_s}(s - \alpha_n)^k \mu(U(u_n))^{\frac{1}{2}i_1+i_2+i_3+\frac{1}{2}}.
\]
From this expression and the fact that \( \mu(U(u_n)) \) is decreasing with \( n \) it is clear that the error term is maximal when \( i_1 \) is maximal, which happens for choice \( J = \{ m_{\alpha_n}, \ldots, m_{\alpha_n+i-1} \} \). Picking \( i_1 = i - 1 \), \( i_2 = 0 \) and \( i_3 = 0 \) then gives us an error term which is independent of \( J \) and is given by

\[
\varepsilon^{-\frac{2}{3}(k+1)-2k} \mu(U(u_n))^{\frac{1}{2}} \sum_{s=\alpha_n+1}^{\beta_n} e^{k \gamma m_{s-1}-\delta m_s (s - \alpha_n)^k}.
\]

Since we consider \( \varepsilon > 0 \) fixed, the result follows. \( \square \)

This completes the necessary adaptations and we are ready to prove Theorem 5.1.

5.3.1. Proof of Theorem 5.1 In the following we prove part A) of the theorem. Part B) is almost identical and we make a comment on this at the end of the proof. So in the following assume \( \mathcal{D} \) is \((w,v)\)-SDL for some positive constants \( w \) and \( v \).

Let \( J \subset I_n \) be any subset of cardinality \( i - 1 \). Then

\[
\mu \left( \bigcap_{j \in J} \{ \xi_j > u_n \} \cap \bigcap_{j \in I_n \setminus J} \{ \xi_j \leq u_n \} \right) = \int_X \prod_{j \in J} 1_{U(u_n)}(a_{m_j}x) \prod_{j \in I_n \setminus J} 1_{V(u_n)}(a_{m_j}x) d\mu(x).
\]

(5.11)

Using Lemma 4.3 and Lemma 5.2 we get

\[
\int_X \prod_{j \in J} g'_{n,\varepsilon}(a_{m_j}x) \prod_{j \in I_n \setminus J} g_{n,\varepsilon}(a_{m_j}x) d\mu(x) \leq \int_X \prod_{j \in J} 1_{U(u_n)}(a_{m_j}x) \prod_{j \in I_n \setminus J} 1_{V(u_n)}(a_{m_j}x) d\mu(x)
\]

(5.12)

\[
\leq \int_X \prod_{j \in J} h'_{n,\varepsilon}(a_{m_j}x) \prod_{j \in I_n \setminus J} h_{n,\varepsilon}(a_{m_j}x) d\mu(x).
\]

Then (5.11) and (5.12) give

\[
\int_X G^J_{(\alpha_n,\beta_n)} d\mu \leq \mu \left( \bigcap_{j \in J} \{ \xi_j > u_n \} \cap \bigcap_{j \in I_n \setminus J} \{ \xi_j \leq u_n \} \right) \leq \int_X H^J_{(\alpha_n,\beta_n)} d\mu.
\]

(5.13)

While we have assumed \( \mathcal{D} \) to be \((w,v)\)-SDL we only need \((w_1, w_2, v)\)-DL for some positive constants \( w_1, w_2 \) and \( v \) in the following estimate. For any \( \delta > 0 \), we get

\[
\mu(U(u_n)) \leq w_2 e^{-v u_n} = \frac{w_2}{N_n} e^{-vr} = \mathcal{O}(N_n^{-1})
\]

\[
\mu(U(u_n - \delta)) \leq w_2 e^{-v (u_n - \delta)} = \frac{w_2 e^{v \delta}}{N_n} e^{-vr} = \mathcal{O}(N_n^{-1}).
\]
Since we are only interested in the dependence on $n$ for now we considered $r \in \mathbb{R}$ and $\delta > 0$ to be fixed in the estimate above. This implies that
\[
\int_X G_{(\alpha, \beta_n)}^J(x) d\mu(x) \geq \prod_{j=\alpha_n}^{\beta_n} \int_X g_j^{(j)} d\mu - O \left( N_n^{-\frac{1}{2i}} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma_{m_s-\delta m_s}(s-\alpha_n)^k} \right)
\]
and
\[
\int_X H_{(\alpha, \beta_n)}^J(x) d\mu(x) \leq \prod_{j=\alpha_n}^{\beta_n} \int_X h_j^{(j)} d\mu + O \left( N_n^{-\frac{1}{2i}} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma_{m_s-\delta m_s}(s-\alpha_n)^k} \right).
\]
Inserting this in (5.13) and summing over all $J \in S_{I_n}^{(i)}$ gives
\[
\sum_{J \in S_{I_n}^{(i)}} \left( \prod_{j=\alpha_n}^{\beta_n} \int_X g_j^{(j)} d\mu - O \left( N_n^{-\frac{1}{2i}} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma_{m_s-\delta m_s}(s-\alpha_n)^k} \right) \right)
\leq \sum_{J \in S_{I_n}^{(i)}} \mu \left( \bigcap_{j \in J} \{ \xi_j > u_n \} \cap \bigcap_{j \in I_n \setminus J} \{ \xi_j \leq u_n \} \right) \tag{5.14}
\leq \sum_{J \in S_{I_n}^{(i)}} \left( \prod_{j=\alpha_n}^{\beta_n} \int_X h_j^{(j)} d\mu + O \left( N_n^{-\frac{1}{2i}} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma_{m_s-\delta m_s}(s-\alpha_n)^k} \right) \right),
\]
Notice that
\[
\prod_{j=\alpha_n}^{\beta_n} \int_X g_j^{(j)} d\mu = \left( \int_X g_{n,c} d\mu \right)^{i-1} \left( \int_X g_{n,c} d\mu \right)^{N_n-(i-1)},
\]
\[
\prod_{j=\alpha_n}^{\beta_n} \int_X h_j^{(j)} d\mu = \left( \int_X h_{n,c} d\mu \right)^{i-1} \left( \int_X h_{n,c} d\mu \right)^{N_n-(i-1)},
\]
which means that both the upper and lower bound is independent of $J$. This turns (5.14) into
\[
\left| S_{I_n}^{(i)} \right| \prod_{j=\alpha_n}^{\beta_n} \int_X g_j^{(j)} d\mu - O \left( \left| S_{I_n}^{(i)} \right| N_n^{-\frac{1}{2i}} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma_{m_s-\delta m_s}(s-\alpha_n)^k} \right)
\leq \sum_{J \in S_{I_n}^{(i)}} \mu \left( \bigcap_{j \in J} \{ \xi_j > u_n \} \cap \bigcap_{j \in I_n \setminus J} \{ \xi_j \leq u_n \} \right) \tag{5.15}
\leq \left| S_{I_n}^{(i)} \right| \prod_{j=\alpha_n}^{\beta_n} \int_X h_j^{(j)} d\mu + O \left( \left| S_{I_n}^{(i)} \right| N_n^{-\frac{1}{2i}} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma_{m_s-\delta m_s}(s-\alpha_n)^k} \right),
\]
The task is now to determine the limit of the upper and lower bound. We first take care of the error term. Recall that
\[ |S_{I_n}^{(i)}| = \left( N_n^{i} \right) (i-1)!(N_n-(i-1))! \sim N_n^{i-1}. \]
In the following we use (4.21) from the proof of Theorem 4.1. For sufficiently large \( n \) we get
\[
\left| S_{I_n}^{(i)} \right| N_n^{-\frac{1}{2}i} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_{s-1} - \delta m_s} (s - \alpha_n)^k \sim N_n^{\frac{1}{2}i-1} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_{s-1} - \delta m_s} (s - \alpha_n)^k
\leq N_n^{\frac{1}{2}i+k-1} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_{s-1} - \delta m_s}
= O \left( N_n^{\frac{1}{2}i+k-1} e^{-\sigma' \rho^\alpha_n} \right),
\]
where \( \sigma' = -m_0 \left( \frac{k\gamma}{\rho} - \delta \right) \). Using the assumption that \( N_n = o \left( e^{\sigma' \rho^\alpha_n} \right) \) it follows that for any \( 2 \leq i \leq l \), we have
\[
O \left( N_n^{\frac{1}{2}i+k-1} e^{-\sigma' \rho^\alpha_n} \right) \to 0
\]
for \( n \to \infty \).

For the main term we get,
\[
\left| S_{I_n}^{(i)} \right| \prod_{j=\alpha_n}^{\beta_n} \int_X g_j^{(j)} d\mu = \left| S_{I_n}^{(i)} \right| \left( \int_X g_{n,x} d\mu \right)^{i-1} \left( \int_X g_{n,x} d\mu \right)^{N_n-(i-1)}
\geq \left| S_{I_n}^{(i)} \right| \mu(U(u_n + \delta))^{i-1} \mu(V(u_n - \delta))^{N_n-(i-1)}
\sim N_n^{i-1} \left( \frac{we^{-v(\delta+r)}}{N_n} \right)^{i-1} \left( e^{-we^{-vr} + o(1)} \right)
\to \frac{\left( \frac{we^{-v(\delta+r)}}{i-1} \right)^{i-1}}{e^{-we^{-vr}}} \text{ for } n \to \infty.
\]
Here we estimated the second integral similarly to the computation following equation (4.14). By equivalent calculations we get
\[
\left| S_{I_n}^{(i)} \right| \prod_{j=\alpha_n}^{\beta_n} \int_X h_j^{(j)} d\mu \to \frac{\left( \frac{we^{-v(\delta-r)}}{i-1} \right)^{i-1}}{e^{-we^{-vr}}} \text{ for } n \to \infty.
\]
Looking again at (5.5) we see that we need to sum over $i$ and take the limit for $n \to \infty$ in (5.15). This gives

$$e^{-we^{-vr}} \sum_{i=2}^{l} \frac{(we^{-v(\delta-r)})^{i-1}}{(i-1)!}$$

$$\leq \lim_{n \to \infty} \sum_{i=2}^{l} \sum_{j \in S_{l_n}^{(i)}} \mu \left( \bigcap_{j \in J} \{ \xi_j > u_n \} \cap \bigcap_{j \in I_n \setminus J} \{ \xi_j \leq u_n \} \right)$$

$$\leq e^{-we^{-vr}} \sum_{i=2}^{l} \frac{(we^{-v(\delta-r)})^{i-1}}{(i-1)!}.$$ 

Taking the limit for $\delta \to 0$ then gives

$$\lim_{n \to \infty} \sum_{i=2}^{l} \sum_{j \in S_{l_n}^{(i)}} \mu \left( \bigcap_{j \in J} \{ \xi_j > u_n \} \cap \bigcap_{j \in I_n \setminus J} \{ \xi_j \leq u_n \} \right) = e^{-we^{-vr}} \sum_{i=2}^{l} \frac{(we^{-v(\delta-r)})^{i-1}}{(i-1)!}.$$ 

And finally, by taking the limit for $n \to \infty$ in (5.5) and using Theorem 4.1 A) we get

$$\lim_{n \to \infty} \mu \left( M_{i_n}^{(l)} \leq u_n \right) = e^{-we^{-vr}} + e^{-\lambda e^{-vr}} \sum_{i=2}^{l} \frac{(we^{-v(\delta-r)})^{i-1}}{(i-1)!}$$

$$= e^{-we^{-vr}} \sum_{i=0}^{l-1} \frac{(we^{-v(\delta-r)})^{i}}{i!}.$$ 

This concludes the proof of part A) of the theorem.

The proof of part B) is identical with the exception that we use the DL assumption instead of the SDL assumption. This change of assumption automatically generates the upper and lower bounds in the statement of part B).

5.4. Closest return case. In this subsection we prove a general result from which Theorem 1.4 will follow. For a fixed point $x_0 \in X$ set

$$D(\cdot) = -\log d(\cdot, x_0)$$

Recall from Lemma 3.4 that for some $w > 0$ we have

$$\mu(x : D(x) > z) = we^{-dz} + o(e^{-dz}).$$

Set also,

$$\xi_i(x) = D(a_i x).$$

**Theorem 5.7.** Let $\rho^{-1} = \left( \sup_{s \in \mathbb{N}} \frac{m_{s-1}}{m_s} \right)$. Assume that $m_j \in \mathbb{R}$ satisfies

$$\sup_{s \in \mathbb{N}} \left( \frac{m_{s-1}}{m_s} \right) < \min \left( 1, \frac{\delta}{k\gamma} \right).$$
where \( k, \delta \) are as in (3.1) and \( \gamma \) as defined in (4.2). Assume further that
\[
N_n = o \left( e^{\sigma \rho \alpha n} \right),
\]
for \( \sigma = \frac{-m_0}{k(\frac{3}{2} + \frac{1}{2} l)} \left( \frac{k \gamma}{\rho} - \delta \right) \). Then, for \( u_n(r) = r + \frac{1}{2} \log N_n \) and all \( x_0 \in X \) we have
\[
\lim_{n \to \infty} \mu \left( M_{I_n}^{(l)} \leq u_n(r) \right) = e^{-w e^{-v r}} \sum_{i=0}^{l-1} \frac{(w e^{-v r})^i}{i!}.
\]

5.4.1. Proof of Theorem 5.7. In the same way as the proof of Theorem 5.1 was an adaptation of the proof of Theorem 4.1, the proof of Theorem 5.7 is an adaptation of the proof of Theorem 4.6. To avoid excessive repetition we provide less detail in this proof than in the proofs of the aforementioned theorems.

Lemma 4.8 provides smooth approximations of the characteristic function \( 1_{B(x_0, e^{-w u})} \). By the same argument we can get smooth approximations of \( 1_{B(x_0, e^{-w u})} \). We state the lemma.

Lemma 5.8. For every \( k \in \mathbb{N}, \) any \( \omega > 0 \) and any \( n \in \mathbb{N} \) we can find two \((C_n, \omega, k)\)-regular functions \( g'_{n, \omega} \) and \( h'_{n, \omega} \) such that
\[
g'_{n, \omega} \leq 1_{(B(x_0, e^{-u_n}))} \leq h'_{n, \omega} \leq 1
\]
and
\[
\mu \left( B(x_0, e^{-u_n - \omega}) \right) = \int_X g'_{n, \omega} d\mu \leq \int_X h'_{n, \omega} d\mu = \mu \left( B(x_0, e^{-u_n + \omega}) \right).
\]

Proof. The proof is identical to the proof of Lemma 4.8. For any \( \omega > 0 \), set \( \varepsilon = \varepsilon_{n, \omega} = e^{-u_n}(1 - e^{-\omega}) \). In this case the approximating functions are defined as
\[
g'_{n, \omega} = \varphi_{\varepsilon} \ast 1_{(B(x_0, e^{-u_n}))} \quad \text{and} \quad h'_{n, \omega} = \varphi_{\varepsilon} \ast 1_{(B(x_0, e^{-u_n + \omega}))},
\]
and \( \varphi_{\varepsilon} \) is as defined in (3.3). \( \square \)

Let again
\[
g^{(j)}_J = \begin{cases} g'_{n, \omega} & \text{if } j \in J \\ g_{n, \omega} & \text{if } j \in I_n \setminus J \end{cases} \quad \text{and} \quad h^{(j)}_J = \begin{cases} h'_{n, \omega} & \text{if } j \in J \\ h_{n, \omega} & \text{if } j \in I_n \setminus J \end{cases}
\]
and for integers \( i_1 < i_2 \)
\[
G_{i_1, i_2}^J(x) = \prod_{j=i_1}^{i_2} g^{(j)}_J(a_m x) \quad \text{and} \quad H_{i_1, i_2}^J(x) = \prod_{j=i_1}^{i_2} h^{(j)}_J(a_m x).
\]
The analogue of Lemma 5.9 holds in this setting too. The only difference is that since \( \varepsilon_{n, \omega} \) depends on \( n \) we cannot consider it a constant. The lemma therefore states
Lemma 5.9. For any subset \( J \subset I_n \) of cardinality \( i - 1 \) we have that for sufficiently large \( n \),

\[
\left| \int_X G_{(\alpha_n, \beta_n)}^J d\mu - \prod_{j=\alpha_n}^{\beta_n} \int_X g_j^{(j)} d\mu \right| \ll \varepsilon_n \omega \mu(B(x_0, e^{-u_n}))^{\frac{1}{2}i} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_s - \delta m_s} (s - \alpha_n)^k
\]

and

\[
\left| \int_X H_{(\alpha_n, \beta_n)}^J d\mu - \prod_{j=\alpha_n}^{\beta_n} \int_X h_j^{(j)} d\mu \right| \ll \varepsilon_n \omega \mu(B(x_0, e^{-u_n+\omega}))^{\frac{1}{2}i} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_s - \delta m_s} (s - \alpha_n)^k.
\]

The proof proceeds by making the same derivations as in equation (5.11) and through to equation (5.15). The only difference is in the estimate of the error term which in this case is

\[
\varepsilon_n \omega \left| S_{I_n}^{(i)} \right| N_n^{-\frac{1}{2}i} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_s - \delta m_s} (s - \alpha_n)^k
\]

\[
= O \left( N_n^{k \left( \frac{1}{2} + \frac{i}{2} \right) + \frac{i}{2}} \left| S_{I_n}^{(i)} \right| N_n^{-\frac{1}{2}i} \sum_{s=\alpha_n+1}^{\beta_n} e^{k\gamma m_s - \delta m_s} \right)
\]

\[
= O \left( N_n^{k \left( \frac{1}{2} + \frac{i}{2} \right) + \frac{i}{2} (i-1)} e^{-\sigma' \rho n} \right),
\]

where \( \sigma' = -m_0 \left( \frac{k\gamma}{\rho} - \delta \right) > 0 \). Here we used the computation from (4.21) and (4.22). By assumption \( N_n = o \left( e^{\sigma' \rho n} \right) \) from which it follows that for any \( 2 \leq i \leq l \), we have that

\[
O \left( N_n^{k \left( \frac{1}{2} + \frac{i}{2} \right) + \frac{i}{2} (i-1)} e^{-\sigma' \rho n} \right) \rightarrow 0 \text{ for } n \rightarrow \infty.
\]

The rest of the proof is identical to the to last part of the proof of Theorem 5.1.

5.5. Finalizing proofs of main results. We already discussed in Section 4.4 that in the setups of Theorem 1.2, 1.4 and 1.6 \( a_t \) has exponential decay of correlations. This follows from Theorem 3.10. Also, Theorem 3.3 gives that \( \Delta \) is \((w, d)\)-SDL for \( w = \frac{V_2}{2\zeta(d)} \) and Theorem 3.2 gives that \( d(\cdot, x_0) \) is \((w_1, w_2, v)\)-DL for some positive constants \( w_1, w_2 \) and \( v \). Hence Theorem 1.2 and 1.6 follow from Theorem 5.1 part B) and A) respectively while Theorem 1.4 follows from Theorem 5.7.
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Department of Mathematics, University of Bremen, Bremen, Germany
E-mail address: kirsebom@math.uni-bremen.de