MINIMIZING EXPECTED TIME TO REACH A GIVEN CAPITAL LEVEL BEFORE RUIN

XIAOQING LIANG
School of Sciences, Hebei University of Technology
Tianjin 300401, China

LIHUA BAI
School of Mathematical Sciences, Nankai University
Tianjin 300071, China

(Communicated by Hailiang Yang)

ABSTRACT. In this paper, we consider the optimal investment and reinsurance problem for an insurance company where the claim process follows a Brownian motion with drift. The insurer can purchase proportional reinsurance and invest its surplus in one risky asset and one risk-free asset. The goal of the insurance company is to minimize the expected time to reach a given capital level before ruin. By using the Hamilton-Jacobi-Bellman equation approach, we obtain explicit expressions for the value function and the optimal strategy. We also provide some numerical examples to illustrate the results obtained in this paper, and analyze the sensitivity of the parameters.

1. Introduction. In the past decades, optimal investment or reinsurance problems for risk models have gained a great deal of attention in actuarial literature. This is due to the fact that the insurance company can reduce its risk exposure by purchasing reinsurance and increase its profit by investing its surplus into the risky and risk-free assets. In fact, in a diffusion model blending with either of these two controls (investment and reinsurance) or both, much research has been done concerning various optimal criteria.

The most common criterion of optimization is to minimize the probability of ruin or maximize the utility of terminal wealth. By controlling investment or reinsurance strategies, quite a number of results have been done related to this criterion. See, for example, [4], [5], [6], [13], [15], [20], [21], [22], [23], [25]. As dividend is another important element in insurance business, many researchers incorporate it with reinsurance and investment in one risk model to maximize the expected present value of total dividends. See, for instance, [1], [2], [3], [7], [10], [11], [12], [14], [17], [18], [24], [26].

In this paper, we will consider another important risk-measure criterion, minimizing the expected time to reach a given capital level before ruin. This criterion is...
first suggested in [19]. In their work, the optimal investment problem without any constraint for an ordinary investor (no external risk process) was solved by maximizing the square of the drift divided by the square of the volatility (see (4.3) in [19]). Another related work is [16], where the authors explored the minimal expected time to reach a goal before ruin within the framework of proportional reinsurance. The optimal proportional reinsurance strategy was obtained by solving the corresponding HJB equation.

In this work, we model the claim process as a Brownian motion with drift. Furthermore, the company is allowed to purchase cheap proportional reinsurance and invest its surplus in a financial market consisting of a risky asset and a risk-free asset. The aim of the insurance company is to minimize the expected time to reach a goal before ruin by choosing control variables. In the end, we provide complete solution. In particular, the results indicate that the optimal investment strategy is not always proportional, which is different from the results in [19].

Compared to [19] and [16], the innovations of our work are as follows. (1) In our setting, short-selling is prohibited, the proportion retained for the insurer is constrained to the interval [0, 1] and the claim process is correlated to the price process of risky asset. These lead to the failure of the approach in [19]. (2) To tackle the difficulties mentioned in (1), we apply the HJB equation approach which is also used in [16]. However, the optimization problem becomes much more complicated by involving investment, which can be seen in later sections. Most importantly, taking risk-free asset into consideration brings great difficulties in solving HJB equation. In order to solve the HJB equation, we employ tools such as the Legendre transform. Hence, our work is not a simple extension of [19] and [16]. There are many future research problems to work on.

The remainder of the paper is organized as follows. In Section 2 we give a rigorous mathematical formulation of the model. The detailed discussion of the optimization problem and the strategy are presented in Section 3. Moreover, it turns out that the minimal expected time to reach a goal before ruin and the corresponding optimal strategy have explicit expressions. In Section 4, we give some numerical examples to illustrate the results we obtained.

2. Model formulation. To motivate this model, we first introduce the classical Cramér-Lundberg model

\[ R_t = x + pt - \sum_{i=1}^{N_t} X_i, \]

where \( x \geq 0 \). \( \{N_t\}_{t \geq 0} \) is a Poisson process with intensity \( \beta \) and the individual claim sizes \( X_1, X_2, \cdots \), independent of \( \{N_t\} \), are i.i.d. positive random variables with common continuous distribution \( F \) and finite first and second moments \( \mu_\infty \) and \( \sigma_\infty^2 \), respectively. The premium rate \( p \) is calculated via the expected value principle, that is,

\[ p = (1 + \eta) \beta \mu_\infty, \]

where \( \eta > 0 \) is the relative safety loading of the insurance company.

In order to reduce risk due to the insurance claims, insurer is allowed to transfer part of its risk to a reinsurer. Then the reinsurer is required to pay a fraction of the claims. In return, the insurer diverts a certain amount of premiums to the reinsurer. Let \( q \) be a (fixed) retention level and let \( X_i^{(q)} \) denote the part of the claims held by the insurer. Assume that the reinsurer uses safety loading \( k(q)\eta \),
where the proportional factor $k(q)$ depends only on $q$, then the insurer’s reserve at time $t$ is described by

$$R_t^{(q,n)} = x + p^{(q,n)} t - \sum_{i=1}^{N_t} X_i^{(q)},$$

where the premium rate $p^{(q,n)}$ is

$$p^{(q,n)} = (1 + \eta)\beta\mu_{\infty} - (1 + k(q)\eta)\beta \left( \mu_{\infty} - \mathbb{E}X_i^{(q)} \right)$$

$$= \beta\mathbb{E}X_i^{(q)} + \eta\beta \left( \mu_{\infty} - k(q) \left( \mu_{\infty} - \mathbb{E}X_i^{(q)} \right) \right).$$

According to [9], one can show that

$$\{\eta R_t^{(q,n)}\}_{t \geq 0} \rightarrow \mu_{\infty}(q) t + \sqrt{\sigma_{\infty}^2(q)} \omega_t$$

in $D[0, \infty)$ (the space of right continuous functions with left limits endowed with the Skorohod topology) as $\eta \downarrow 0$, where

$$\mu_{\infty}(q) = \beta \left( \mu_{\infty} - k(q) \left( \mu_{\infty} - \mathbb{E}X_i^{(q)} \right) \right), \quad \sigma_{\infty}^2(q) = \beta \mathbb{E} \left( X_i^{(q)} \right)^2,$$

and $\omega_t$ is a standard Brownian motion.

Throughout this paper, we mainly focus on the case of cheap proportional reinsurance in which $k(q) = 1$ and $X_i^{(q)} = qX_i$, such that

$$\mu_{\infty}(q) = aq, \quad \sigma_{\infty}^2(q) = b^2q^2,$$

where $a = \beta\mu_{\infty}$ and $b = \sqrt{\beta\sigma_{\infty}^2}$. Thus, for a fixed $q$, the diffusion process $\{R_t\}_{t \geq 0}$ approximating the Cramér-Lundberg model is governed by the SDE

$$dR_t^{(q)} = aqdt + bqd\omega_t.$$

To give a rigorous mathematical formulation, we start with a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with filtration $\{\mathcal{F}_t\}_{t \geq 0}$. The filtration $\mathcal{F}_t$ represents the information available at time $t$ and all decisions are made based on this information.

The uncontrolled reserve process is given by

$$dR_t = adt + Bd_t,$$

where $B_t$ is a standard Brownian motion adapted to $\mathcal{F}_t$.

Apart from the reserve process $R_t$, we suppose that the insurer is allowed to invest its surplus in a risky asset (stock or mutual fund) and a risk-free asset with interest rate $r > 0$. Specifically, the price process of the risky asset follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t d_1 B_t,$$

where $\mu > r$ and $\sigma > 0$ are constants. $B_t^S$ is a standard Brownian motion with respect to $\mathcal{F}_t$. We use $\rho_S(0 < \rho_S < 1)$ to denote the correlation coefficient between $B_t$ and $B_t^S$, that is, $\mathbb{E}(B_t B_t^S) = \rho_S t$.

Let $\pi_t$ be a strategy described by a two-dimensional stochastic process $(\pi_t, q_t)$, where $\pi_t$ represents the amount invested in the risky asset at time $t$ and $q_t$ represents the proportion retained at time $t$. Then the controlled reserve process $R_t^\pi$ evolves according to the dynamics

$$\begin{align*}
\left\{ \begin{array}{ll}
    dR_t^\pi &= r R_t^\pi dt + aq_t dt + bq_t dB_t + (\mu - r)\pi_t dt + \sigma \pi_t dB_t^S, \\
    R_0^\pi &= x.
\end{array} \right.
\end{align*}$$

(2.1)
Definition 2.1. A strategy $\alpha$ is said to be admissible if
(i) $(\pi_t, q_t)$ is $\mathcal{F}_t$-progressively measurable;
(ii) $0 \leq q_t \leq 1$;
(iii) $\pi_t \geq 0$ (short-selling is prohibited) and $\int_0^t \pi_s^2 ds < \infty$ almost surely for all $t \geq 0$.
Denote the set of all admissible strategies by $\alpha_S$.

3. Minimizing expected time to reach a given capital level. Suppose that
the insurer is interested in minimizing the expected time to reach a goal $\bar{x}$ before
ruin. For any $\alpha \in \alpha_S$, let $\tau_\alpha := \inf\{t : R_t^\alpha \leq 0\}$ be the ruin time, and $T_\alpha^\bar{x} := \inf\{t : R_t^\alpha \geq \bar{x}\}$ be the first hitting time to $\bar{x}$. For the Brownian motion risk model, it is
well-known that $\tau_\alpha = \inf\{t : R_t^\alpha = 0\}$ and $T_\alpha = \inf\{t : R_t^\alpha = \bar{x}\}$. Define
$$T_\alpha := \begin{cases} T_\alpha^{\bar{x}}, & \text{if } \tau_\alpha > T_\alpha^{\bar{x}}, \\ \infty, & \text{if } \tau_\alpha < T_\alpha^{\bar{x}}, \end{cases}$$
the first time to reach $\bar{x}$ before ruin. The insurer aims at minimizing the expected
value of $T_\alpha$, that is, the insurer concentrates on the following optimization problem
$$\min_{(\pi, q) \in \alpha_S} E[T_\alpha],$$
subject to $$(\pi, q) \in \alpha_S, (R_t, \pi_t, q_t) \text{ satisfies (2.1)}.$$ (3.1)

To solve problem (3.1), we use dynamic programming HJB equation approach
described in [8]. Before this, it is necessary to show that the value function in (3.1)
is well-defined.

Lemma 3.1. Let $\alpha'$ be defined by
$$\alpha' = \{\alpha \in \alpha_S : E[T_\alpha] < \infty\}.$$ 
Then, $\alpha'$ is not empty, that is, there exists at least one admissible strategy $\alpha \in \alpha'$.

Proof. Suppose the initial surplus $0 < x < \bar{x}$, we consider a strategy $\alpha$ such that
$q_0 = 0$ and $\pi_t = \lambda R_t$ for some constant $\lambda > 0$. Based on (2.1), the corresponding
surplus process $R_t^\alpha$ is then given by
$$dR_t^\alpha = rR_t^\alpha dt + \lambda(\mu - r)R_t^\alpha dt + \lambda \sigma R_t^\alpha dB_t^S.$$ (3.2)
The solution of (3.2) is $R_t^\alpha = x \exp\{Y_t\}$, where $Y_t = (r+\lambda(\mu-r)-\lambda^2 \sigma^2/2)t + \lambda \sigma B_t^S$,
which is a Brownian motion with drift. This drift coefficient is positive if $\lambda^2 \sigma^2/2 - \lambda(\mu-r) - r < 0$. Thus, for $\lambda$ satisfying this condition, $\{Y_t\}_{t \geq 0}$ reaches $\ln(\bar{x}/x)$ with
probability 1. Hence, $\{R_t^\alpha\}_{t \geq 0}$ reaches $\bar{x}$ with probability 1 with $R_t^\alpha > 0$. That is $P(T_\alpha < \infty) = P(T_\alpha = T_\alpha^\bar{x} < \infty) = 1$. On the other hand, from the optional stopping
theorem, we have
$$EY_{T_\alpha \wedge t} = \left(r + \lambda(\mu-r) - \frac{1}{2} \lambda^2 \sigma^2\right) E(T_\alpha \wedge t), \forall t > 0.$$ 
Hence, by applying the dominated convergence theorem and the Fatou lemma, we
obtain $E[T_\alpha] < \infty$. 

In view of Lemma 3.1, we restrict the optimization problem (3.1) to $\alpha'$. In
addition, for control problems with infinite time horizon, it is natural to focus on
stationary control strategies as the state dynamics are time-independent. Thus, we can write the objective function as

\[ V_\alpha(x) = E[T_\alpha | R_\alpha^0 = x] := E_x[T_\alpha]. \]

Then, our aim is to compute the value function

\[ V(x) = \inf_{\alpha \in \alpha'} V_\alpha(x) \]

and the associated optimal investment-reinsurance strategy \( \alpha^* \in \alpha' \) such that \( V(x) = V_{\alpha^*}(x) \).

According to the standard arguments, for example, [8], we know that if the value function \( V(x) \in C^2 \), then \( V(x) \) satisfies the following HJB equation

\[
\begin{align*}
&\inf_{\pi \geq 0, 0 \leq q \leq 1} \left\{ (rx + (\mu - r)\pi +aq)V'(x) + \frac{1}{2}(\sigma^2\pi^2 + 2\sigma\pi b\rho S + q^2 b^2)V''(x) + 1 \right\} = 0, \\
&\text{for } x \in (0, \bar{x}), \text{ with boundary conditions} \\
&V(0) = \infty, \\
&V(\bar{x}) = 0. 
\end{align*}
\]

(3.3)

In the following, we first find a solution of (3.3)-(3.5) on the class of strictly decreasing convex functions on \((0, \bar{x})\), then verify that it is indeed the value function.

The proofs of theorems and lemmas are deferred to the Appendix.

Define

\[ M = \frac{(\mu - r)bps - a\sigma}{\sigma b^2(1 - \rho_S^2)}, \quad (3.6) \]

\[ N = \frac{a\sigma bps - (\mu - r)b}{\sigma^2 b(1 - \rho_S^2)}, \quad (3.7) \]

\[ A = \frac{(\mu - r)^2b^2 + a^2\sigma^2 - 2a\sigma(\mu - r)bps}{2\sigma^2b^2(1 - \rho_S^2)} > 0. \quad (3.8) \]

These constants will appear in the subsequent theorems and lemmas.

**Theorem 3.2.** There exists a strictly decreasing convex solution \( W(x) \) to HJB equation (3.3) with boundary conditions (3.4) and (3.5). The solution \( W(x) \) and the corresponding strategy \((\pi^*(x), q^*(x))\) are as follows:

(i) If \( a\sigma bps \geq (\mu - r)b \) and \( b^2 \geq a\bar{x} \), then

\[
W(x) = \frac{2b^2}{a^2 + 2b^2 r} \ln \frac{\bar{x}}{x},
\]

and

\[
(\pi^*(x), q^*(x)) = \left( 0, \frac{a}{b^2 x} \right).
\]

(ii) If \( a\sigma bps \geq (\mu - r)b \) and \( b^2 < a\bar{x} \), then

\[
W(x) = \begin{cases} 
\frac{2b^2}{a^2 + 2b^2 \rho S} \ln \frac{b^2}{a^2} - \int_{b^2/a}^\bar{x} h(v)dv, & 0 \leq x < b^2/a, \\
- \int_x^{\bar{x}} h(v)dv, & b^2/a \leq x \leq \bar{x}, 
\end{cases}
\]

and

\[
(\pi^*(x), q^*(x)) = \begin{cases} 
(0, \frac{a}{b^2 x}), & 0 < x < b^2/a, \\
(0, 1), & b^2/a \leq x < \bar{x},
\end{cases}
\]

(3.9)
where
\[
h(x) = e^{-\frac{a}{
\left(\frac{2a}{b^2} - \frac{2b}{a^2}\right) x^2 - \frac{2a}{b^2} x^2 - \frac{2b}{a^2} x + \frac{2b}{a^2}} u^2 + \frac{2b}{a^2}} du\]
(iii) If \( \alpha \leq (\mu - r)\rho_S \), then
\[
W(x) = \frac{2\sigma^2}{(\mu - r)^2 + 2\sigma^2 r \ln \frac{x}{\bar{x}}} \quad \text{and} \quad (\pi^*(x), q^*(x)) = \left(\frac{\mu - r}{\sigma^2} x, 0\right).
\]
(iv) If \((\mu - r)\rho_S^2 < \alpha\rho_S < (\mu - r)b \) and \( M\bar{\epsilon} \geq -1 \), then
\[
W(x) = \frac{1}{r + A} \ln \frac{x}{\bar{x}},
\]
and
\[
(\pi^*(x), q^*(x)) = (-N x, -M x).
\]
(v) If \((\mu - r)\rho_S^2 < \alpha\rho_S < (\mu - r)b \) and \( M\bar{\epsilon} < -1 \), then
\[
W(x) = -\frac{1}{r + A} \ln x + k_1, \quad 0 \leq x < -1/M,
\]
and \( W(x) \) satisfies the following differential equation:
\[
\left(rx + a - \frac{(\mu - r)\rho_S}{\sigma}\right) W'(x) - \frac{(\mu - r)^2 W'^2(x)}{2\sigma^2 W''(x)} + \frac{1}{2} b^2 (1 - \rho_S^2) W''(x) + 1 = 0,
\]
on \([-1/M, \bar{x}]\) with boundary conditions
\[
W(\bar{x}) = 0, \quad W'(\frac{-1}{M}) = \frac{M}{r + A},
\]
and
\[
(\pi^*(x), q^*(x)) = \begin{cases} (-N x, -M x), & 0 < x < -1/M, \\
\left(-\frac{\mu - r}{\sigma^2} W'(x) - \frac{b\rho_S}{\sigma}, 1\right), & -1/M \leq x < \bar{x}, \end{cases}
\]
where the constants \( M, N, A \) and \( k_1 \) are given in (3.6), (3.7), (3.8), and (A.25), respectively.

Now, we have found a candidate solution \( W(x) \) and a candidate optimal strategy
\[
\alpha^* = (\pi^*(R^*_t), q^*(R^*_t)),
\]
for the optimization problem (3.1), where \( R^*_t \) denotes the surplus process under the strategy \( \alpha^* \). We only need to verify that \( W(x) \) is indeed the value function and \( \alpha^* \) given in (3.15) is the associated optimal strategy. Before this, we present the following lemma which shows that \( \alpha^* \in \alpha' \).

Lemma 3.3. Let \( \alpha^* \) be defined in (3.15) with \((\pi^*(x), q^*(x)) \) given in Theorem 3.2. Then \( \alpha^* \in \alpha' \), i.e., \( E_x[T_{\alpha^*}] < \infty \).

Theorem 3.4. Let \( W(x) \) and \((\pi^*(x), q^*(x)) \) be given in Theorem 3.2. Then \( W(x) = V(x) \). Furthermore \( \alpha^* = (\pi^*(R^*_t), q^*(R^*_t)) \) is the associated optimal investment-reinsurance strategy.
Assume that there is no risk-free asset, that is, \( r = 0 \), then the minimal expected time for the insurer to reach the goal before ruin and the associated optimal strategy can be derived similarly as that in Theorem 3.2. Here we provide the results for this case in the following corollary without detailed proof.

**Corollary 3.5.** For the case \( r = 0 \), the value function \( V(x) \) and the corresponding optimal strategy \((\pi^*(x), q^*(x))\) are presented as follows:

(i) If \( a\sigma \rho S \geq \mu b \) and \( b^2 \geq \bar{x} \alpha \), then

\[
V(x) = \frac{2b^2}{a^2} \ln \frac{\bar{x}}{x},
\]

and

\[(\pi^*(x), q^*(x)) = \left(0, \frac{a}{b^2} x\right)\].

(ii) If \( a\sigma \rho S \geq \mu b \) and \( b^2 < \bar{x} \alpha \), then

\[
V(x) = \begin{cases} 
\frac{2b^2}{a^2} \ln \frac{d}{x}, & 0 \leq x < \frac{b^2}{a}, \\
-\frac{\bar{x}}{a} + c_1 e^{-\frac{2a}{b^2} x} + c_2, & \frac{b^2}{a} \leq x \leq \bar{x},
\end{cases}
\]

and

\[(\pi^*(x), q^*(x)) = \begin{cases} 
(0, \frac{a}{b^2} x), & 0 < x < \frac{b^2}{a}, \\
(0, 1), & \frac{b^2}{a} \leq x < \bar{x},
\end{cases}
\]

where

\[
d = \frac{b^2}{a} \exp \left\{ \frac{a \bar{x}}{2b^2} - \frac{1}{4} \left[ 1 + \exp \left( 2 - \frac{2a}{b^2} \bar{x} \right) \right] \right\},
\]

\[
c_1 = \frac{b^2 e^2}{2a^2}, \quad c_2 = -\frac{\bar{x}}{a} - \frac{b^2}{2a^2} \exp \left( 2 - \frac{2a}{b^2} \bar{x} \right).
\]

(iii) If \( a \sigma \leq \mu b \rho S \), then

\[
V(x) = \frac{2\sigma^2}{\mu^2} \ln \frac{\bar{x}}{x} \quad \text{and} \quad (\pi^*(x), q^*(x)) = \left(0, \frac{\mu}{\sigma^2} x\right).
\]

(iv) If \( \mu b^2 \sigma^2 < a\sigma \rho S < \mu b \) and \( M_0 \bar{x} \geq -1 \), then

\[
V(x) = \frac{1}{A_0} \ln \frac{\bar{x}}{x},
\]

and

\[ (\pi^*(x), q^*(x)) = (-N_0 x, -M_0 x). \]

(v) If \( \mu b^2 \sigma^2 < a\sigma \rho S < \mu b \) and \( M_0 \bar{x} < -1 \), then

\[
V(x) = \begin{cases} 
-\frac{1}{A_0} \ln x + k_1, & 0 \leq x < -1/M_0, \\
\int_{-1/M_0}^{x} G^{-1}(y + c_3) dy + c_4, & -1/M_0 \leq x \leq \bar{x},
\end{cases}
\]

(3.16)

and

\[ (\pi^*(x), q^*(x)) = \begin{cases} 
(-N_0 x, -M_0 x), & 0 < x < -1/M_0, \\
\left(\frac{-\mu}{\sigma^2} G^{-1}(x + c_3) - \frac{b \rho S}{\sigma^2}, 1\right), & -1/M_0 \leq x < \bar{x},
\end{cases} \]
where the constants \( M_0, \ N_0 \) and \( A_0 \) are given in (3.6), (3.7) and (3.8) by setting \( r = 0 \), respectively. The function \( G(\cdot) \) is given by

\[
G(y) = \int_{-\infty}^{y} \frac{1}{g(z)} \, dz, \quad y < 0,
\]

with

\[
g(z) = \sqrt{(1 + (a - \mu b \rho_S / \sigma)z)^2 + \frac{b^2 \mu^2(1 - \rho_S^2)z^2}{\sigma^2} - 1 - (a - \mu b \rho_S / \sigma)z},
\]

and

\[
c_3 = \int_{-\infty}^{M_0/A_0} \frac{1}{g(y)} \, dy + \frac{1}{M_0}, \quad c_4 = -\int_{-1/M_0}^{\hat{x}} G^{-1}(y + c_3) \, dy,
\]

\[
k_1 = c_4 - \frac{1}{A_0} \ln(-M_0).
\]

4. Numerical examples. In this section, we provide some numerical examples to illustrate the analytical results obtained in Section 3. The parameters are given as follows: \( a = 0.01, r = 0.02, \mu = 0.06, \rho_S = 0.6 \), and the goal \( \hat{x} = 1 \).

In Figs. 1 and 2, we fix \( b = 0.04 \) and focus on the effects of the stock volatility \( \sigma \) on the minimal expected time and the corresponding optimal strategies. In the left panel of each figure, the insurer is allowed to invest in both the risky asset and the risk-free asset; while in the right panel, the insurer only invests in the risky asset. Specifically, we define \( \zeta(x) = x - \pi^*(x) \), the amount invested in the risk-free asset. Note that, \( \zeta(x) < 0 \) means that the insurer borrows money \( |\zeta(x)| \) at rate \( r \). In Fig. 1(a), we set \( \sigma = 0.1 \), the results are in coincidence with case (iv) of Theorem 3.2. The optimal proportion retained is 0.3906 and the optimal amount borrowed at rate \( r \) is 2.9062. Comparing with Fig. 1(a), we plot the value function and the associated optimal strategies when the interest rate equals zero in Fig. 1(b). Based on Corollary 3.5, the optimal choice for the insurer is to divert its entire risk incurred by claims to the reinsurer, and totally invest money in the risky asset.

In Fig. 2, we set \( \sigma = 0.01 \), and we find \( a\sigma - (\mu - r)b \rho_S < 0 \). In accordance with Theorem 3.2 and Corollary 3.5, the results in this example coincide with case (iii). The proportion of claims retained is zero. Moreover, in light of Figs. 1 and 2, we observe that the insurer will spend much more time to reach the goal by investing in two assets than only in one risky asset. This seems intuitively reasonable when the return of the risky stock with small variation is higher than that of a risk-free bond.

In Figs. 3 and 4, we fix \( \sigma = 0.3 \), and change the values of \( b \), the volatility induced by claims. In Fig. 3, we set \( b = 0.03 \), for both of the cases \( r \neq 0 \) and \( r = 0 \), the optimal choice for the insurer is investing no money in the risky stock, and taking part of the claims when the surplus of the insurer is less than 0.09, and then taking total claims without buying any reinsurance when the surplus of the insurer becomes larger than 0.09.

Fig. 4 shows the value function and the corresponding optimal strategies for \( b = 0.3 \). Similarly to the examples considered in Figs. 1 and 2, the insurer in this situation will not buy any reinsurance, but invests a positive amount in the risk-free asset. Meanwhile, contrast to the graphs plotted in Figs. 1 and 2, we find the minimal expected time spent by the insurer for the case \( r \neq 0 \) is less than that for \( r = 0 \). Because the value of volatility of the risky stock we use in these examples is...
bigger than that used in Figs. 1 and 2, which implies that more risks will be taken by the insurer.

In Fig. 5, we set \( b = 0.04, \sigma = 0.1 \) and fix the initial surplus \( x = 0.5 \). We plot the graphs of the minimal expected time for various values of goal. In this case, we find the expected time for \( r = 0 \) is smaller than that for \( r \neq 0 \).

![Figure 1](image1)

**Figure 1.** The minimal expected time and the associated optimal strategies for \( \sigma = 0.1 \).

![Figure 2](image2)

**Figure 2.** The minimal expected time and the associated optimal strategies for \( \sigma = 0.01 \).

**Appendix.** Proof of Theorem 3.2. Of course, one can directly verify that \( W(x) \) solves (3.3)-(3.5). But in order to illustrate the method, we will solve (3.3)-(3.5) analytically. Suppose that it has a solution \( W(x) \in C^2 \) satisfying \( W''(x) < 0 \) and \( W'''(x) > 0 \) on \( (0, \bar{x}) \). Then, for \( \pi \) and \( q \) without restriction, the left-hand side of (3.3) attains its minimum at

\[
(\pi(x), q(x)) = \left( N \frac{W'(x)}{W''(x)}, M \frac{W''(x)}{W''(x)} \right),
\]  
(A.1)
Figure 3. The minimal expected time and the associated optimal strategies for $b = 0.03$.

Figure 4. The minimal expected time and the associated optimal strategies for $b = 0.3$.

Figure 5. Expected time vs goal for $x = 0.5$. 
and the minima with \( q = 0 \), \( q = 1 \) and \( \pi = 0 \) are respectively attained at

\[
\pi^0(x) = -\frac{\mu - r}{\sigma^2} \frac{W'(x)}{W''(x)} > 0,
\]

\[
\pi^1(x) = -\frac{\mu - r}{\sigma^2} \frac{W'(x)}{W''(x)} - \frac{b\rho_s}{\sigma},
\]

and

\[
q^0(x) = -\frac{a}{b^2} \frac{W'(x)}{W''(x)} > 0,
\]

where \( M \) and \( N \) are given in (3.6) and (3.7). Due to the different values of \( \pi(x) \) and \( q(x) \), we define the following regions:

\[
O_1 = \{ 0 < x < \bar{x} : \pi(x) \leq 0, q^0(x) < 1 \}, \quad (A.3)
\]

\[
O_2 = \{ 0 < x < \bar{x} : \pi(x) \leq 0, q^0(x) \geq 1 \}, \quad (A.4)
\]

\[
O_3 = \{ 0 < x < \bar{x} : \pi^0(x) > 0, q(x) \leq 0 \}, \quad (A.5)
\]

\[
O_4 = \{ 0 < x < \bar{x} : \pi(x) \geq 0, 0 \leq q(x) \leq 1 \}, \quad (A.6)
\]

\[
O_5 = \{ 0 < x < \bar{x} : \pi^1(x) \geq 0, q(x) \geq 1 \}. \quad (A.7)
\]

Based on the above analysis, we have the following three cases to deal with.

1. **The case of** \( a\sigma\rho_s \geq (\mu - r)b \)

   In this case, from (3.6) and (3.7), we see \( M \leq 0 \) and \( N \geq 0 \). Therefore, we only need to consider \( O_1 \) and \( O_2 \). First, for \( x \in O_1 \), the minimum of the left-hand side of (3.3) is attained at

\[ (\pi^*_i(x), q^*_i(x)) = \left( 0, -\frac{a}{b^2} \frac{W'(x)}{W''(x)} \right). \]

Substituting it into (3.3) and rearranging, we get

\[ rxW'(x) - \frac{a^2}{2b^2} \frac{W'^2(x)}{W''(x)} + 1 = 0. \]

We consider the Legendre transform \( W_0(y) \) of \( W(x) \) defined by

\[ W_0(y) = \min_x \{ W(x) + xy \}. \]

Then we can recover \( W(x) \) from \( W_0(y) \) by

\[ W(x) = \max_y \{ W_0(y) - xy \}. \]

The minimizing value of \( x \) in (A.9) equals \( I(-y) = W'_0(y) \), in which \( I \) is the inverse function of \( W' \). Therefore, the maximizing value of \( y \) in (A.10) equals \(-W'(x)\). Substituting \( x = I(-y) \) in Eq. (A.8), we get

\[ -ryW'_0(y) + \frac{a^2}{2b^2} y^2 W''_0(y) + 1 = 0. \]

The general solution of (A.11) is

\[ W_0(y) = C_1 y^{\frac{a^2 + 2b^2}{2b^2}r} + \frac{2b^2}{a^2 + 2b^2r} \ln y + C_2, \]
in which $C_1$ and $C_2$ are constants to be determined.

Since $W(0) = \infty$, so there exists a point $y_0$, such that $W_0(y_0) = \infty$ and $W'(y_0) = 0$. Moreover, we see

$$W_0'(y) = C_1 \frac{a^2 + 2b^2r}{a^2} y \frac{2a^2}{a^2 + 2b^2r} + \frac{2b^2}{a^2 + 2b^2r}.$$

It follows that $y_0 = \infty$ and $C_1 = 0$. Hence,

$$W_0(y) = \frac{2b^2}{a^2 + 2b^2r} \ln y + C_2. \quad (A.12)$$

Substituting (A.12) into (A.10), we obtain

$$W(x) = \frac{2b^2}{a^2 + 2b^2r} \ln \left(\frac{2b^2}{a^2 + 2b^2r}x\right) - \frac{2b^2}{a^2 + 2b^2r} + C_2. \quad (A.13)$$

This together with (A.2) and (A.3) gives

$$\mathcal{O}_1 = \{x : 0 < x < b^2/a\},$$

which implies that if $b^2 \geq a\bar{x}$, then $W(x)$ given in (A.13) satisfies (3.3) with boundary conditions (3.4)-(3.5) on $(0, \bar{x})$. In this case, it is easy to show that

$$C_2 = -\frac{2b^2}{a^2 + 2b^2r} \ln \left(\frac{2b^2}{a^2 + 2b^2r}\bar{x}\right) + \frac{2b^2}{a^2 + 2b^2r}$$

and the corresponding strategy is

$$\left(\pi^*(x), q^*(x)\right) = (0, ax/b^2), \quad (A.14)$$

for $0 < x < \bar{x}$. Hence, (i) is proved.

If $b^2 < a\bar{x}$, then $W(x)$ given in (A.13) only holds on $0 < x < b^2/a$ and $(\pi^*(x), q^*(x))$ given in (A.14) is the associated strategy on $0 < x < b^2/a$. When $b^2/a \leq x < \bar{x}$, we turn to $\mathcal{O}_2$ and guess that

$$\mathcal{O}_2 = \{x : b^2/a \leq x < \bar{x}\}.$$

Then, the left-hand side of (3.3) attains its minimum at

$$(\pi^*(x), q^*(x)) = (0, 1), \quad b^2/a \leq x < \bar{x}. \quad (A.15)$$

Substituting (A.15) into (3.3) yields

$$(rx + a)W'(x) + \frac{b^2}{2} W''(x) + 1 = 0. \quad (A.16)$$

Denote $h(x) := W'(x)$, then (A.16) can be rewritten as

$$h'(x) + \frac{2}{b^2} (rx + a)h(x) + \frac{2b^2}{b^2} = 0. \quad (A.17)$$

Thus,

$$h(x) = e^{-\frac{rx}{b^2} - \frac{2a^2}{2b^2}} \left(C_1 - \int_{b^2/a}^{x} \frac{2}{b^2} e^{\frac{r}{b^2} u^2 + \frac{2a^2}{2b^2}} du\right),$$

where $C_1$ is a constant to be determined later. Hence, we get the expression for $W$ on $b^2/a \leq x \leq \bar{x}$ as

$$W(x) = \int_{b^2/a}^{x} h(v)dv + C_2.$$
From the boundary condition (3.5), we obtain that \( \bar{C}_2 = -\int_{b^2/a}^{\bar{x}} h(v)dv \). Moreover, the values of constants \( \bar{C}_1 \) and \( C_2 \) can be determined by the fact that \( W(x) \) is continuous and continuously differentiable at \( x = b^2/a \), that is,

\[
\begin{align*}
\bar{C}_1 &= -\frac{2a}{a^2 + 2b^2r} e^{\frac{2a}{a^2 + 2b^2r} x^2}, \\
C_2 &= -\int_{b^2/a}^{\bar{x}} h(v)dv - \frac{2b^2}{a^2 + 2b^2r} \ln \frac{2a}{a^2 + 2b^2r} + \frac{2b^2}{a^2 + 2b^2r}.
\end{align*}
\]

Therefore, we obtain the function \( W \) on \((0, \bar{x})\) as follows

\[
W(x) = \begin{cases} 
\frac{2b^2}{a^2 + 2b^2r} \ln \frac{b^2}{a} - \int_{b^2/a}^{\bar{x}} h(v)dv, & 0 < x < b^2/a, \\
-\int_{b^2/a}^{\bar{x}} h(v)dv, & b^2/a \leq x \leq \bar{x}, 
\end{cases} \tag{A.18}
\]

where

\[
h(x) = e^{-\frac{x}{2b^2} - \frac{2a}{b^2} x} \left(-\frac{2a}{a^2 + 2b^2r} e^{\frac{2a}{a^2 + 2b^2r} x^2} - \int_{b^2/a}^{\bar{x}} 2 \frac{b^2}{a^2 + 2b^2r} e^{\frac{2a}{b^2} x^2 + \frac{2a}{b^2} u} du \right).
\]

Finally, to confirm that \( W(x) \) in (A.18) indeed solves (3.3) with boundary conditions (3.4) and (3.5), we need to check that (A.3) and (A.4) hold. For \( 0 < x < b^2/a \), it can be directly verified by substituting \( W(x) \) in (A.18) into (A.3). For \( b^2/a \leq x \leq \bar{x} \), clearly, \( h(x) < 0 \), and we also claim that \( h'(x) > 0 \) on \((b^2/a, \bar{x})\), otherwise one could find a point \( x_1 \in (b^2/a, \bar{x}) \), such that \( h'(x_1) < 0 \). By the continuously differentiable property of \( W(x) \) at \( x = b^2/a \), it follows easily that 

\[
h'(b^2/a) = \frac{2\sigma^2}{b(0 + 2\sigma^2)} > 0.
\]

Therefore, we can find one point \( \bar{x} \in (b^2/a, \bar{x}] \) such that \( h'(\bar{x}) = 0 \). Hence, we get

\[
h''(\bar{x}) = \lim_{x \to \bar{x}} \frac{h'(x) - h'(\bar{x})}{x - \bar{x}} < 0.
\]

On the other hand, taking derivatives on both sides of equation (A.17), and replacing \( x \) by \( \bar{x} \), we have

\[
h''(\bar{x}) + \frac{2}{b^2} r h(\bar{x}) = 0,
\]

which contradicts with the fact that \( h''(\bar{x}) < 0 \) and \( h(\bar{x}) < 0 \). Hence, we conclude that \( h'(x) > 0 \) on \((b^2/a, \bar{x}]\). Moreover, (A.4) can be verified similarly as in Proposition A.1, which ends the proof of (ii).

(2) The case of \( a\sigma \leq (\mu - r)b\sigma_2 \)

In this case, we find that \( M \geq 0 \) and \( N < 0 \) given in (3.6) and (3.7), respectively. Thus, only \( \mathcal{O}_3 \) needs to be considered. In this region, the left-hand side of (3.3) attains its minimum at

\[
(\pi^*(x), q^*(x)) = \left(-\frac{\mu - r}{\sigma^2} W'(x), 0\right).
\]

Substituting it into (3.3), we get the differential equation satisfied by \( W \):

\[
x W(x) - \frac{(\mu - r)^2}{2\sigma^2} W''(x) + 1 = 0.
\]

Analysis similar to (A.8) shows that

\[
W(x) = \frac{2\sigma^2}{(\mu - r)^2 + 2\sigma^2 r} \ln \frac{2\sigma^2}{((\mu - r)^2 + 2\sigma^2 r)x} - \frac{2\sigma^2}{(\mu - r)^2 + 2\sigma^2 r} + \bar{C}_3,
\]

where

\[
\bar{C}_3 = -\int_{b^2/a}^{\bar{x}} h(v)dv.
\]
with

\[ C_3 = \frac{2\sigma^2}{(\mu - r)^2 + 2\sigma^2 r} \ln \frac{2\sigma^2}{(\mu - r)^2 + 2\sigma^2 r} \bar{x} + \frac{2\sigma^2}{(\mu - r)^2 + 2\sigma^2 r} \]

by the boundary condition (3.5). Therefore, we have (3.10) and \( O_3 = \{ x : 0 < x < \bar{x} \} \).

(3) The case of \((\mu - r)b\rho_S^2 < a\sigma\rho_S < (\mu - r)b\)

Recalling the constants \( M \) and \( N \) defined in (3.6) and (3.7), we see that \( M < 0 \) and \( N < 0 \). Therefore, we will only consider \( O_4 \) and \( O_5 \). If \( x \in O_4 \), then

\[ (\pi^*(x), q^*(x)) = (\pi(x), q(x)) \]

and HJB equation (3.3) reduces to

\[ rx W'(x) - A \frac{W'^2(x)}{W''(x)} + 1 = 0 \]

with \( A \) defined in (3.8). As in the arguments of (A.8), the solution of (A.19) is

\[ W(x) = -\frac{1}{r + A} \ln x + k_1, \]

where \( k_1 \) is a constant and will be determined later. Now the region \( O_4 \) becomes

\[ O_4 = \{ x : 0 < x < -\frac{1}{M} \} \]

which implies that if \( M \bar{x} \geq -1 \), then \( W(x) \) given in (A.20) satisfies (3.3) with boundary conditions (3.4)-(3.5) on \((0, \bar{x})\). In this case, \( k_1 = \frac{1}{r + A} \ln \bar{x} \) and the minimizing function is

\[ (\pi^*(x), q^*(x)) = (-Nx, -Mx) \]

for \( 0 < x < \bar{x} \). Hence, (iv) is proved.

If \( M \bar{x} < -1 \), then \( W(x) \) given in (A.20) solves (3.3) only on \((-1/M, 0)\) and (A.21) holds only on \((-1/M, 0)\). For \(-1/M < x < \bar{x}\), we guess the region \( O_5 \) has the following form

\[ O_5 = \{ x : -\frac{1}{M} \leq x < \bar{x} \} \]

Then, the left-hand side of (3.3) attains its minimum at

\[ (\pi^*(x), q^*(x)) = \left( -\frac{\mu - r}{\sigma^2} \frac{W'(x)}{W''(x)} - \frac{b\rho_S}{\sigma}, 1 \right), \quad -\frac{1}{M} \leq x < \bar{x}. \]

Applying this to (3.3) results in

\[ \left( rx + a - \frac{(\mu - r)b\rho_S}{\sigma} \right) W'(x) - \frac{(\mu - r)^2}{2\sigma^2} \frac{W'^2(x)}{W''(x)} + \frac{1}{2} b^2 (1 - \rho_S^2) W''(x) + 1 = 0. \]

Since \( W(x) \) is continuous and continuously differentiable at \( x = -1/M \), we have

\[ W \left( -\frac{1}{M} \right) = -\frac{1}{r + A} \ln \left( -\frac{1}{M} \right) + k_1, \]

\[ W' \left( -\frac{1}{M} \right) = \frac{M}{r + A}. \]

The boundary conditions (3.5) and (A.26) allow us to solve the second-order equation (A.24) numerically. Once we have a solution on \([-1/M, \bar{x})\), we can use (A.25) to
Proof. First, we prove that \(W(x)\) is strictly decreasing and convex on \([-1/M, \bar{x})\). Otherwise, we can find a point \(c \in [-1/M, \bar{x})\) such that \(W'(c) = 0\). Since \(W'(-1/M) = M/(1 + A) < 0\), there is a point \(\bar{c} \in [-1/M, \bar{x})\) such that \(W'(\bar{c}) = 0\). Hence, \(W''(\bar{c}) = \lim_{x \to \bar{c}} W'(x) - W'(\bar{c})/x - \bar{c} > 0\).

Substituting \(\bar{c}\) into equation (A.24) yields
\[
\frac{1}{2} b^2(1 - \rho_S^2)W''(\bar{c}) + 1 = 0,
\]
which is a contradiction. Therefore, \(W'(x) < 0\) on \([-1/M, \bar{x})\). We also claim that \(W''(x) > 0\) on \([-1/M, \bar{x})\). Otherwise, since \(W''(-1/M) = M^2/(1 + A) > 0\), we can find a point \(\bar{x} \in [-1/M, \bar{x})\) such that \(W''(\bar{x}) = 0\). Then, (A.24) cannot hold when \(x \to \bar{x}\). Hence \(W(x)\) is strictly convex on \([-1/M, \bar{x})\).

For the remaining part, we verify (A.27) and (A.28). Denote function \(f\) by
\[
f(q) = aq W'(x) + \left(\frac{1}{2} q^2 b^2 + \sigma q b \rho_S\right) W''(x),
\]
for \(x \in [-1/M, \bar{x})\), then
\[
f'(q) = aW'(x) + (qb^2 + \sigma qb \rho_S) W''(x)
= \left( a - \frac{(\mu - r)b \rho_S}{\sigma} \right) W'(x) + qb^2 W''(x)(1 - \rho_S^2),
\]
where the last equality is obtained by the fact that \(\pi(x) = \frac{\mu - r}{\sigma} W'(x) - \frac{b \rho_S}{\alpha}\). Therefore, proving (A.27) is equivalent to showing that
\[
f'(1) = \left( a - \frac{(\mu - r)b \rho_S}{\sigma} \right) W'(x) + b^2 W''(x)(1 - \rho_S^2) \leq 0,
\]
for \(x \in [-1/M, \bar{x})\), that is to prove \(W'(x)/W''(x) \leq \frac{1}{\rho_S}\) on \([-1/M, \bar{x})\).

To demonstrate the claim, we first prove that for arbitrary \(x_0 \in (-1/M, \bar{x})\), the inequality \(W'(x)/W''(x) > -x\) cannot hold on \((-1/M, x_0)\). Otherwise, we can find a point \(x_0 \in (-1/M, \bar{x})\), such that \(W'(x)/W''(x) > -x\) holds on \((-1/M, x_0)\). Suppose \(\psi(x) = W''(x)/W'(x)\), then
\[
\psi(x) < -\frac{1}{x}, \tag{A.29}
\]
Therefore, \((A.24)\) can be rewritten as

\[
W'(x) = \frac{M}{r + A} e^{\int_{x_1/M}^x \psi(u)du} > -\frac{1}{(r + A)x}. \tag{A.30}
\]

Substituting \((A.29)\) and \((A.30)\) into \((A.31)\) yields

\[
r x + a - \frac{(\mu - r)b \sigma}{\sigma} + \frac{(\mu - r)^2}{2\sigma^2} \psi(x) + \frac{1}{2}b^2(1 - \rho_S^2) \frac{1}{x} - (r + A)x > 0,
\]

which is equivalent to

\[
\left(\frac{(\mu - r)^2}{2\sigma^2} - A\right)x^2 + \left( a - \frac{(\mu - r)b \sigma}{\sigma}\right)x - \frac{1}{2}b^2(1 - \rho_S^2) > 0. \tag{A.32}
\]

Moreover, since

\[
\frac{(\mu - r)^2}{2\sigma^2} - A = -\frac{(a \sigma - (\mu - r)b \sigma)}{2\sigma^2 b^2 (1 - \rho_S^2)} < 0,
\]

\((A.32)\) can be rewritten as

\[
-\frac{1}{2b^2(1 - \rho_S^2)} \left(\frac{a \sigma - (\mu - r)b \sigma}{\sigma}\right)x^2 + \frac{1}{b^2(1 - \rho_S^2)} > 0,
\]

for \(x \in (-1/M, x]\), which is impossible. Therefore, \(W'(x) > -x\) cannot hold on \((-1/M, x_0)\).

Now, we claim that there is no constant \(x \in (-1/M, x]\), such that \(W'(x) = -x\), i.e., \(\psi(x) = -1/x\). Otherwise there exists \(x^* = \inf\{-1/M < x \leq x : \psi(x) = -1/x\}\), then combining the above conclusion we obtain \(\psi(x) \geq -1/x\) for \(x \in (-1/M, x^*]\).

Thus,

\[
\psi'(x^*-) = \lim_{x \uparrow x^*} \frac{\psi(x^*) - \psi(x)}{x^* - x} \leq \lim_{x \uparrow x^*} \frac{-1}{x^* + \frac{1}{x^*}} = \frac{1}{(x^*)^2}. \tag{A.33}
\]

Replacing \(x\) by \(x^*\) in \((A.24)\), we get

\[
r x^* + a - \frac{(\mu - r)b \sigma}{\sigma} + \frac{(\mu - r)^2}{2\sigma^2} x^* - \frac{1}{2}b^2(1 - \rho_S^2) \frac{1}{x^*} + \frac{1}{W'(x^*)} = 0,
\]

and

\[
W'(x^*) = \frac{-2x^*}{(\frac{(\mu - r)^2}{\sigma^2} + 2r)(x^*)^2 + 2(a - \frac{b \sigma (\mu - r)}{\sigma})x^* - b^2(1 - \rho_S^2)}. \tag{A.34}
\]

Taking derivatives on both sides of \((A.24)\) at \(x = x^*\) implies

\[
r + \frac{(\mu - r)^2}{2\sigma^2} (x^*)^2 \psi'(x^*) - \frac{1}{2}b^2(1 - \rho_S^2) \psi'(x^*) + \frac{1}{x^* W'(x^*)} = 0,
\]
and substituting (A.34) into the above equation, we obtain
\[(x^*)^2 \psi'(x^*) = \frac{(\mu-r)^2}{\sigma^2}(x^*)^2 + \frac{2(a - \frac{b \rho_S (\mu-r)}{\sigma})}{\sigma} x^* - b^2(1 - \rho_S^2). \tag{A.35}\]

From (A.33), we see the right-hand side of (A.35) should be less than 1, which can be simplified to \(x^* \leq -1/M\). This contradicts with the fact that \(x^* \in (-1/M, \bar{x}]\). Hence, we conclude that \(\frac{W'(x)}{W''(x)} \leq -x\) on \((-1/M, \bar{x}]\). Consequently, we see \(\frac{W'(x)}{W''(x)} \leq \frac{1}{\bar{x}}\). Therefore, (A.27) is proved.

For (A.28), we have
\[
\pi^1(x) = -\frac{\mu - r}{\sigma^2} \frac{W'(x)}{W''(x)} - \frac{b \rho_S}{\sigma} \geq -\frac{\mu - r}{\sigma^2} \frac{1}{M} - \frac{b \rho_S}{\sigma} \geq 0. \tag{A.36}\]

The first inequality is derived by the fact that \(\frac{W'(x)}{W''(x)} \leq \frac{1}{M}\) and the second inequality holds by (3.6) and the assumption \((\mu - r)b \rho_S^2 < a \sigma \rho_S < (\mu - r)b\). \(\square\)

**Proof of Lemma 3.3.** In view of Theorem 3.2, we need to verify that \(E_x[T_{\alpha^*}] < \infty\) in the following three cases.

1. **The case of** \(a \sigma \rho_S \geq (\mu - r)b\)
   It is obvious that the problem under (i) is a special case of that under (ii). Therefore, we only provide its proof under (ii). In view of (3.9) and (3.15), we have
   \[
   \alpha^* = (\pi^*(R^*_s), q^*(R^*_s)) = \begin{cases} 
   (0, \frac{a}{b} R^*_s), & 0 < R^*_s < b^2/a, \\
   (0, 1), & b^2/a \leq R^*_s < \bar{x}. \end{cases} \tag{A.37}\]
   For the initial surplus 0 < \(x < b^2/a\), the surplus process under strategy \(\alpha^*\) evolves as
   \[
   dR^*_t = r R^*_t dt + \frac{a^2}{b^2} R^*_t dt + \frac{a}{b} R^*_t dB_t, \tag{A.38}\]
   for \(t < T_{b^2/a} \wedge \tau_{\alpha^*}\), where \(T_{b^2/a} = \inf\{t \geq 0 : R^*_t = b^2/a\}\). The solution of (A.38) is
   \[
   R^*_t = x \exp \left\{ \left( \frac{a^2}{2b^2} + r \right) t + \frac{a}{b} B_t \right\}. \]
   Since \(a^2/(2b^2) + r\) is positive, so \(R^*_t\) reaches \(b^2/a\) before ruin with probability 1. Moreover, once \(R^*_t\) is again less than \(b^2/a\), it will still reach \(b^2/a\) before ruin with probability 1. On the other hand, when \(R^*_t\) is larger than \(b^2/a\), by (A.37), the corresponding surplus process becomes
   \[
   dR^*_t = (r R^*_t + a) dt + dB_t. \]
   The positive drift implies that \(R^*_t\) tends to infinity when \(t\) goes to infinity with probability 1. Hence, all the above statements yield \(P(T_{\alpha^*} < \infty) = 1\). Choosing \(f(x) = \ln x\) and a large constant \(n\), applying Itô’s formula to \(f(R^*_t)\) with \(R^*_0 = x\), we get
   \[
   f(R^*_{T_{\alpha^*} \wedge n}) = f(x) + \int_0^{T_{\alpha^*} \wedge n} \mathcal{L} f(R^*_s) ds + \int_0^{T_{\alpha^*} \wedge n} f'(R^*_s) b q^*(R^*_s) dB_s, \tag{A.39}\]
where
\[ \mathcal{L} f(x) = \begin{cases} \quad f'(x)(r + \frac{a^2}{b^2})x + \frac{a^2}{2b^2}x^2f''(x), & 0 < x < b^2/a, \\ \quad f'(x)(rx + a) + \frac{1}{2}b^2f''(x), & b^2/a < x < \bar{x}. \end{cases} \]

Since \( f(x) = \ln x \), we see \( \mathcal{L} f(x) \geq r + \frac{2a\bar{x} - b^2}{2\bar{x}} > 0 \). Taking expectations of (A.39) and letting \( n \to \infty \) yield
\[ \ln \bar{x} \geq \ln x + \left( r + \frac{2a\bar{x} - b^2}{2\bar{x}} \right) E_x[T_{a^*}]. \]

Therefore, \( E_x[T_{a^*}] \) is finite.

In parallel to the above proof, one can also show that \( E_x[T_{a^*}] < \infty \), for \( b^2/a \leq x < \bar{x} \).

(2) **The case of** \( a\sigma \leq (\mu - r)b\rho_S \)

In this case, according to (3.10) and (3.15), the candidate optimal strategy is given as follows
\[ (\pi^*(R^*_t), q^*(R^*_t)) = \left( \frac{\mu - r}{\sigma^2} R^*_t, 0 \right). \]

Then the corresponding surplus process is described by
\[ dR^*_t = rR^*_t dt + \frac{(\mu - r)^2}{\sigma^2} R^*_t dt + \frac{(\mu - r)}{\sigma} R^*_t dB^S_t. \]

Similar analysis as in the proof of Lemma 3.1, we have \( \alpha^* \in \alpha' \).

(3) **The case of** \( (\mu - r)b\rho_S^2 < a\sigma \rho_S < (\mu - r)b \)

By the similar arguments in Case (1), we only prove \( E_x[T_{a^*}] < \infty \) under (v). By \( (\pi^*(x), q^*(x)) \) given in (3.14) and \( \alpha^* \) given in (3.15), we get
\[ (\pi^*(R^*_t), q^*(R^*_t)) = \begin{cases} (-NR^*_t, -MR^*_t), & 0 < R^*_t < -1/M, \\ \left( \frac{\mu - r}{\sigma^2} W^c(R^*_t) - \frac{b\rho_S}{\sigma}, 1 \right), & -1/M \leq R^*_t < \bar{x}. \end{cases} \]

We first prove that \( P(T_{a^*} < \infty) = 1 \) for \( 0 < x < -1/M \). Consider the surplus process
\[ dR^*_t = rR^*_t dt - aMR^*_t dt - bMR^*_t dB_t - (\mu - r)NR^*_t dt - \sigma N R^*_t dB^S_t, \]  \quad (A.40)

for \( t < T_{a^*}^{-1/M} \wedge \tau_{a^*} \), where \( T_{a^*}^{-1/M} = \inf \{ t \geq 0 : R^*_t = -1/M \} \). The solution for (A.40) is \( R^*_t = x \exp \{ Y^*_t \} \), where
\[ Y^*_t = \beta t - bMB_t - \sigma N B^S_t, \]

with the drift coefficient
\[ \beta = r - \left( aM + (\mu - r)N + b\rho_S M N + \frac{1}{2}b^2 M^2 + \frac{1}{2}\sigma^2 N^2 \right) = r + A > 0. \]

Therefore, \( R^*_t \) reaches \(-1/M\) before ruin with probability 1. Moreover, once \( R^*_t \) is again less than \(-1/M\), it will still reach \(-1/M\) before ruin with probability 1. On
the other hand, when $R^*_t$ is larger than $-1/M$, then
\[ dR^*_t = \left( rR^*_t + a - \frac{(\mu - r)^2}{\sigma^2} \frac{W'(R^*_t)}{W''(R^*_t)} - \frac{b\rho S(\mu - r)}{\sigma} \right) dt + bdB_t \]
\[ + \left( -\frac{\mu - r}{\sigma} \frac{W'(R^*_t)}{W''(R^*_t)} - b\rho S \right) dB_t^S. \]

By (A.36), we see that the drift of $R^*_t$ is positive, thus $R^*_t$ tends to infinity with probability 1. Moreover, using analogous discussion of Case (1) and applying Itô's formula to $W(R^*_t)$, where $W(x)$ is the solution given in Theorem 3.2, we find that $E_x[T_{\alpha^*}^r] = W(x) < \infty$.

Similarly, we can prove $E_x[T_{\alpha^*}] < \infty$ for $-1/M < x \leq \bar{x}$. \hfill \Box

**Proof of Corollary 3.5.** Here we only give the proof of Case(v), other cases can be directly obtained by setting $r = 0$ in the results obtained in Theorem 3.2. When $r = 0$, (3.11) becomes
\[ \left( a - \frac{\mu b \rho S}{\sigma} \right) W'(x) - \frac{\mu^2}{2\sigma^2} W''(x) + \frac{1}{2} \left( 1 - \rho_S^2 \right) W'''(x) + 1 = 0. \]

(A.42)

Since $W'(x) < 0$ and $W''(x) > 0$, we can rewrite (A.42) as

\[ W''(x) = g(W'(x)), \quad -1/M_0 \leq x < \bar{x}, \]

where $g(z)$ is given in (3.18). Then
\[ W'(x) = G^{-1}(x + c_3), \quad -1/M_0 \leq x < \bar{x}, \]

with function $G(\cdot)$ given in (3.17). Integrating this equation, we get
\[ W(x) = \int_{-1/M_0}^x G^{-1}(y + c_3) dy + c_4, \quad -1/M_0 \leq x \leq \bar{x}. \]

(A.43)
Therefore, we obtain the closed-form expression for \( W(x) \):

\[
W(x) = \begin{cases} 
-\frac{1}{\sigma^2} \ln x + k_1, & 0 \leq x < -1/M_0, \\
\int_{-1/M_0}^{x} G^{-1}(y + c_3)dy + c_4, & -1/M_0 \leq x \leq \bar{x},
\end{cases}
\]  
(A.44)

where \( c_4 \) can be obtained by the boundary condition (3.5), and \( k_1, c_3 \) can be determined by the continuous and continuously differentiable property of \( W(x) \) at \( x = -1/M_0 \). As a result, (3.16) is obtained. Inserting \( W(x) \) of (A.43) into (A.23), we get

\[
(\pi^*(x), q^*(x)) = \left( \frac{\mu}{\sigma^2} G^{-1}(x + c_3) - \frac{b\rho}{\sigma}, 1 \right), -1/M_0 \leq x < \bar{x}.
\]  
(A.45)

It remains for us to verify that \( q(x) \) given in (A.1) is bigger than 1, i.e., \( q(x) \geq 1 \). In accordance with (A.43), we get

\[
q(x) = M_0 \frac{G^{-1}(x + c_3)}{g(G^{-1}(x + c_3))}.
\]

By straightforward calculation, we obtain that \( q(-1/M_0) = 1 \) and

\[
q'(x) = M_0 \frac{(G^{-1}(x + c_3))' [g(G^{-1}(x + c_3)) - G^{-1}(x + c_3)g'(G^{-1}(x + c_3))]}{[g(G^{-1}(x + c_3))]^2}.
\]

On the other hand, simple computation yields

\[
g(G^{-1}(x + c_3)) - G^{-1}(x + c_3)g'(G^{-1}(x + c_3)) < 0,
\]

Hence, \( q'(x) > 0 \), i.e., \( q(x) \) is increasing. Thus, \( q(x) \geq 1 \) on \([-1/M_0, \bar{x}]\). Similarly, we can also verify that \( \pi^1(x) \geq 0 \) on \([-1/M_0, \bar{x}]\). It is easy to check that \( \mathcal{O}_3 \) in (A.7) coincides with \( \{x : -1/M_0 \leq x < \bar{x}\} \). Therefore, \( W(x) \) in (A.44) solves (3.3) with boundary conditions (3.4)-(3.5) and (A.45) holds on \([-1/M_0, \bar{x}]\). According to Theorem 3.4, \( W(x) \) is indeed the value function and \( (\pi^*(x), q^*(x)) \) is the corresponding optimal strategy. Hence, the proof is complete. \( \square \)

REFERENCES

[1] S. Asmussen, B. Højgaard and M. Takasaki, Optimal risk control and dividend distribution policies: Example of excess-of-loss reinsurance for an insurance corporation, Finance and Stochastics, 4 (2000), 299–324.
[2] S. Asmussen and M. Takasaki, Controlled diffusion models for optimal dividend pay-out, Insurance: Mathematics and Economics, 20 (1997), 1–15.
[3] P. Azcue and N. Muler, Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model, Mathematical Finance, 15 (2005), 261–308.
[4] L. Bai and J. Guo, Optimal proportional reinsurance and investment with multiple risky assets and no-shorting constraint, Insurance: Mathematics and Economics, 42 (2008), 968–975.
[5] S. Browne, Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin, Mathematics of Operations Research, 20 (1995), 937–958.
[6] L. Chen and H. Yang, Optimal reinsurance and investment strategy with two piece utility function, Journal of Industrial and Management Optimization, 12 (2016).
[7] T. Choulli, M. Takasaki and X. Zhou, A diffusion model for optimal dividend distribution for a company with constraints on risk control, SIAM Journal on Control and Optimization, 41 (2003), 1946–1979.
[8] W. Fleming and H. Soner, Controlled Markov Processes and Viscosity Solutions, Springer Science & Business Media, 2006.
[9] J. Grandell, Aspects of Risk Theory, Springer, 1991.
[10] B. Højgaard and M. Takasaki, Controlling risk exposure and dividends pay-out schemes: Insurance company example, Mathematical Finance, 9 (1999), 153–182.
[11] B. Højgaard and M. Taksar, Optimal risk control for a large corporation in the presence of returns on investments, *Finance and Stochastics*, 5 (2001), 527–547.
[12] B. Højgaard and M. Taksar, Optimal dynamic portfolio selection for a corporation with controllable risk and dividend distribution policy, *Quantitative Finance*, 4 (2004), 315–327.
[13] C. Irgens and J. Paulsen, Optimal control of risk exposure, reinsurance and investments for insurance portfolios, *Insurance: Mathematics and Economics*, 35 (2004), 21–51.
[14] Z. Liang and V. Young, Dividends and reinsurance under a penalty for ruin, *Insurance: Mathematics and Economics*, 50 (2012), 437–445.
[15] Z. Liang and K. Yuen, Optimal dynamic reinsurance with dependent risks: Variance premium principle, *Scandinavian Actuarial Journal*, 2016 (2016), 18–36.
[16] S. Luo, M. Wang and X. Zeng, Optimal reinsurance: Minimize the expected time to reach a goal, *Scandinavian Actuarial Journal*, 2016 (2015), 741–762.
[17] J. Paulsen, Optimal dividend payouts for diffusions with solvency constraints, *Finance and Stochastics*, 7 (2003), 457–473.
[18] J. Paulsen and H. Gjessing, Optimal choice of dividend barriers for a risk process with stochastic return on investments, *Insurance: Mathematics and Economics*, 20 (1997), 215–223.
[19] V. Pestien and W. Sudderth, Continuous-time red and black: how to control a diffusion to a goal, *Mathematics of Operations Research*, 10 (1985), 599–611.
[20] H. Schmidli, Optimal proportional reinsurance policies in a dynamic setting, *Scandinavian Actuarial Journal*, 2001 (2001), 55–68.
[21] M. Taksar and C. Markussen, Optimal dynamic reinsurance policies for large insurance portfolios, *Finance and Stochastics*, 7 (2003), 97–121.
[22] N. Wang, Optimal investment for an insurer with exponential utility preference, *Insurance: Mathematics and Economics*, 40 (2007), 77–84.
[23] H. Yang and L. Zhang, Optimal investment for insurer with jump-diffusion risk process, *Insurance: Mathematics and Economics*, 37 (2005), 615–634.
[24] C. Yin and K. C. Yuen, Optimal dividend problems for a jump-diffusion model with capital injections and proportional transaction costs, *Journal of Industrial and Management Optimization*, 11 (2015), 1247–1262.
[25] X. Zhang, M. Zhou and J. Guo, Optimal combinational quota-share and excess-of-loss reinsurance policies in a dynamic setting, *Applied Stochastic Models in Business and Industry*, 23 (2007), 63–71.
[26] M. Zhou and K. Yuen, Optimal reinsurance and dividend for a diffusion model with capital injection: Variance premium principle, *Economic Modelling*, 29 (2012), 198–207.

Received November 2015; 1st revision February 2016; final revision October 2016.

E-mail address: liangxiaoqing15@hotmail.com
E-mail address: lhbai@nankai.edu.cn