A CLASS OF THE NON-DEGENERATE COMPLEX QUOTIENT EQUATIONS ON COMPACT KÄHLER MANIFOLDS

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Abstract. In this paper, we are concerned with the equations that are in the form of the linear combinations of the elementary symmetric functions of a Hermitian matrix on compact Kähler manifolds. Under the assumption of the cone condition, we obtain a priori estimates for the class of complex quotient equations. Then using the method of continuity, we prove an existence result.

1. Introduction. Let \((M, \omega)\) be an \(n\)-dimensional compact Kähler manifold and \(\chi\) be a smooth closed real \((1,1)\)-form on \(M\). In any local coordinate chart, we may write

\[
\omega = \sqrt{-1} g_{ij} dz^i d\bar{z}^j \quad \text{and} \quad \chi = \sqrt{-1} \chi_{ij} dz^i d\bar{z}^j.
\]

Define \(\Gamma^k_\omega\) as the set of all the real \((1,1)\)-forms whose eigenvalue sets with respect to \(\omega\) belong to a \(k\)-positive cone \(\Gamma^k\). For any \(u \in C^2(M)\) we obtain a new real \((1,1)\)-form \(\chi_u = \chi + \sqrt{-1} \partial \bar{\partial} u\). We are concerned with the following complex quotient equations on \(M\)

\[
\chi_u^k \land \omega^{n-k} = \sum_{l=0}^{k-1} \alpha_l \chi_u^l \land \omega^{n-l}, \quad \text{with} \quad \chi_u \in \Gamma^k_\omega.
\]  

Equations of the form (1.1) with \(\alpha_l(x) \geq 0\) for \(0 \leq l \leq k-1\) are first studied by Krylov [10], who solves the Dirichlet problem for domains in \(\mathbb{R}^n\). Recently, Guan-Zhang in [7] considered the \((k-1)\)-admissible solution without the sign of \(\alpha_{k-1}\) and obtained the global \(C^2\) estimates. Moreover, similar equations are also studied in [12].

Equations (1.1) include complex Monge-Ampère equation, the complex \(k\)-Hessian equations and \((k, l)\)-quotient equations as special cases. The importance of (1.1) comes from the fact that it is general enough to cover many natural geometric PDEs. The most well-known example is the complex Monge-Ampère equation, famously

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Let us now state our main result.

**Theorem 1.1.** With this background, we can now state our main result. There exists a unique smooth function \( u \) solving Equation (1.1) with coefficient functions \( \alpha_l \) for \( 0 \leq l \leq k-1 \).

A third example is the complex quotient equation which has the form

\[
\left( \frac{\sigma_k}{\sigma_l} \right)^{\frac{1}{k-l}} = \psi, \quad l < k,
\]

where \( \sigma_k \) denotes the \( k \)-th elementary symmetric function. When \( \psi \) is constant and assume the existence of a \( C \)-subsolution, Székelyhidi [18] proves a Liouville type theorem for \( \alpha \)-subharmonic functions for \( 0 \leq \alpha \leq 1 \). In this paper, we mainly concern Equation (1.1) with coefficient functions \( \alpha_l \) for \( 0 \leq l \leq k-1 \). For Equation (1.1), we need to introduce the cone condition which can be used to discover the \( C^0 \) and \( C^2 \) estimates. Following [5, 15, 17], we set

\[
\mathcal{C}(\alpha_{k-1}) := \{ \chi \in \mathcal{C}^0(M) \} \quad \text{for } 0 \leq l \leq k-1.
\]

In particular, we have

\[
\alpha_{k-1} \geq c := \frac{\int_M \left( \chi^k \wedge \omega^{n-k} - \sum_{l=0}^{k-2} \alpha_l \chi^l \wedge \omega^{n-l} \right)}{\int_M \chi^{k-1} \wedge \omega^{n-k+1}}.
\]

Then there exists a unique smooth function \( u \) and a unique real constant \( b \) solving Equation (1.1) up to rescaling, that is

\[
\frac{\chi^k}{\chi^{k-1}} \wedge \omega^{n-k+1} - \sum_{l=0}^{k-2} \alpha_l \chi^l \wedge \omega^{n-l} = e^b \alpha_{k-1}, \quad \text{with } \sup_M u = 0.
\]
Lemma 2.2. Equivalently, we can rewrite Equation (2.3) as
\[ F(x) = \sum_{l=0}^{k-2} \alpha_l F_l(x) = \alpha_{k-1}. \]
Lemma 2.3. If \( u \in C^2(M) \) is a solution of (1.1), \( \lambda(X) \in \Gamma^k_\omega \) and \( \alpha_l(x) > 0 \) for \( 0 \leq l \leq k-1 \), then

\[
\frac{\sigma_l(\lambda)}{\sigma_{k-1}(\lambda)} \leq C(n,k, \inf_{0 \leq i \leq k-2} \alpha_l); \quad (2.5)
\]

\[
\frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)} \leq C(n,k, \sum_{l=0}^{k-1} \sup \alpha_l). \quad (2.6)
\]

Proof. If \( \frac{\sigma_k}{\sigma_{k-1}} \leq 1 \), then we get from the equation (2.3)

\[
\alpha_l \frac{\sigma_l}{\sigma_{k-1}} \leq \frac{\sigma_k}{\sigma_{k-1}} \leq 1, \quad \text{for} \quad 0 \leq l \leq k-2.
\]

If \( \frac{\sigma_k}{\sigma_{k-1}} > 1 \), i.e. \( \frac{\sigma_{k-1}}{\sigma_k} < 1 \). We can get for \( 0 \leq l \leq k-2 \) by Lemma 2.1,

\[
\frac{\sigma_l}{\sigma_{k-1}} \leq \frac{(C^k_n)^{k-1-l}C^l_n(\sigma_{k-1})^{k-1-l}}{(C^k_n)^{k-l}C^l_n(\sigma_k)^{k-l}} \leq \frac{(C^k_n)^{k-1-l}C^l_n}{(C^k_n)^{k-l}} \leq C(n,k),
\]

and

\[
\frac{\sigma_k}{\sigma_{k-1}} = \sum_{l=0}^{k-1} \alpha_l \frac{\sigma_l}{\sigma_{k-1}} \leq C(n,k) \sum_{l=0}^{k-1} \sup \alpha_l. \quad \square
\]

Lemma 2.4. If \( u \in C^2(M) \) is a solution of (1.1), \( \lambda(X) \in \Gamma^k_\omega \) and \( \alpha_l(x) > 0 \) for \( 0 \leq l \leq k-1 \), then

\[
\frac{n-k+1}{k} \leq \sum \frac{\partial F}{\partial \lambda_i} = \sum F^{ii} \leq n-k+1. \quad (2.7)
\]

Proof. By direct computations, we can get

\[
\sum F^{ii} \geq \sum \frac{\partial \left( \frac{\sigma_k}{\sigma_{k-1}} \right)}{\partial \lambda_i} = \sum \frac{\sigma_{k-1}(\lambda|i)\sigma_{k-1} - \sigma_k \sigma_{k-2}(\lambda|i)}{\sigma_{k-1}^2}
\]

\[
= \frac{(n-k+1)\sigma_{k-1}^2 - (n-k+2)\sigma_k \sigma_{k-2}}{\sigma_{k-1}^2} \geq \frac{n-k+1}{k},
\]

and

\[
\sum F^{ii} = \sum \frac{\partial \left( \frac{\sigma_k}{\sigma_{k-1}} \right)}{\partial \lambda_i} - \sum_{l=0}^{k-2} \alpha_l \sum \frac{\partial \left( \frac{\sigma_l}{\sigma_{k-1}} \right)}{\partial \lambda_l}
\]

\[
= \sum \frac{\sigma_{k-1}(\lambda|i)\sigma_{k-1} - \sigma_k \sigma_{k-2}(\lambda|i)}{\sigma_{k-1}^2} - \sum_{l=0}^{k-2} \alpha_l \sum \frac{\sigma_{l-1}(\lambda|i)\sigma_{k-1} - \sigma_l \sigma_{k-2}(\lambda|i)}{\sigma_{k-1}^2}
\]

\[
= \frac{(n-k+1)\sigma_{k-1}^2 - (n-k+2)\sigma_k \sigma_{k-2}}{\sigma_{k-1}^2}
\]

\[
+ \sum_{l=0}^{k-2} \alpha_l \frac{(n-k+2)\sigma_k \sigma_{k-2} - (n-l+1)\sigma_{l-1} \sigma_{k-1}}{\sigma_{k-1}^2}
\]

\[
\leq (n-k+1) - \frac{(n-k+2)\sigma_{k-2}}{\sigma_{k-1}} \left( \frac{\sigma_k}{\sigma_{k-1}} - \sum_{l=0}^{k-2} \alpha_l \frac{\sigma_l}{\sigma_{k-1}} \right) \leq n-k+1. \quad \square
\]
When no confusion occurs, \( X \) and \( \chi \) also denote the corresponding Hermitian matrices at a given point \( p \). Under any local coordinate chart such that \( g_{ij} = \delta_{ij} \) at \( p \in M \), the cone condition (2.1) is equivalent to that
\[
\frac{\sigma_{k-1}(\chi|i)}{\sigma_{k-2}(\chi|i)} - \sum_{i=0}^{k-2} \alpha_i(p) \frac{\sigma_{i-1}(\chi|i)}{\sigma_{k-2}(\chi|i)} > \alpha_{k-1}(p), \quad 1 \leq i \leq n. \tag{2.8}
\]
Denote \( F^\omega = \frac{\partial F}{\partial x_i} \) and \( F = \sum F^\omega \). Using the cone condition, we obtain the following lemma which plays an important role in \( C^2 \) estimates.

**Lemma 2.5.** Let \( u \in C^2(M) \) is a solution of (1.1) on compact Kähler manifold \( (M, \omega) \), \( \alpha_l(x) > 0 \) for \( 0 \leq l \leq k-2 \) and \( \chi \) be a smooth closed real \((1,1)\)-form on \( M \). Suppose that \( \chi \in \mathcal{C}(\alpha_{k-1}) \), \( X \in \Gamma^k \) is diagonal and \( X_{1\overline{1}} = \max_{1 \leq i \leq n} X_{\overline{i}i} \) at a given point \( p \in M \). Then at \( p \) there are positive constants \( N, \theta \) such that when \( X_{1\overline{1}} \geq N \),
\[
F^{\overline{\omega}} u_{\overline{\omega}} \leq -\theta - \theta F \tag{2.9}
\]
or
\[
F^{1\overline{1}} X_{1\overline{1}} \geq \theta. \tag{2.10}
\]

**Proof.** Direct calculation yields that
\[
F^{\overline{\omega}} u_{\overline{\omega}} = F^{\overline{\omega}} (X_{\overline{i}i} - \chi_{\overline{i}i})
= \sum_{i=2}^{n} F^{\overline{\omega}} (X_{\overline{i}i} - (\chi_{\overline{i}i} - \epsilon)) + F^{1\overline{1}} (X_{1\overline{1}} - (\chi_{1\overline{1}} - \epsilon)) - \epsilon F. \tag{2.11}
\]
If \( \epsilon > 0 \) is small enough, then \( \chi - \epsilon I \in \mathcal{C}(\alpha_{k-1}) \). So there are \( N > 0 \) and \( \delta > 0 \) such that \( X_{1\overline{1}} > N \),
\[
F(B) = \frac{\sigma_{k-1}(B)}{\sigma_{k-2}(B)} - \sum_{i=0}^{k-2} \alpha_i(p) \frac{\sigma_{i-1}(B)}{\sigma_{k-2}(B)} > \alpha_{k-1}(p) + \delta, \tag{2.12}
\]
where
\[
B = \left( \begin{array}{cc} X_{1\overline{1}} & 0 \\ 0 & (\chi - \epsilon I) \end{array} \right)_{n \times n}.
\]
From (2.12) and \( F \) is concave in \( \Gamma^k \), we get
\[
\sum_{i=2}^{n} F^{\overline{\omega}} (X_{\overline{i}i} - (\chi_{\overline{i}i} - \epsilon)) = \sum_{i=1}^{n} F^{\overline{\omega}} (X_{\overline{i}i} - B_{\overline{i}i}) \leq F(X) - F(B) \leq -\delta. \tag{2.13}
\]
We may assume that \( X_{1\overline{1}} \gg -\chi_{1\overline{1}} + \epsilon \). Thus
\[
F^{1\overline{1}} (X_{1\overline{1}} - (\chi_{1\overline{1}} - \epsilon)) \leq 2F^{1\overline{1}} X_{1\overline{1}}. \tag{2.14}
\]
When \( X_{1\overline{1}} > N \), put (2.13) and (2.14) into (2.11),
\[
F^{\overline{\omega}} u_{\overline{\omega}} \leq -\delta + 2F^{1\overline{1}} X_{1\overline{1}} - \epsilon F.
\]
Set \( \theta = \min\{\frac{\delta}{2}, \epsilon\} \). If \( F^{1\overline{1}} X_{1\overline{1}} \leq \frac{\delta}{2} \), we have (2.9); otherwise, (2.10) must be true. \( \square \)

We may assume that there is a uniform constant \( \mu > 0 \) such that
\[
\chi - \mu \omega \in \Gamma^k \quad \text{and} \quad \omega - \mu \chi \in \Gamma^k. \tag{2.15}
\]
Lemma 2.6. [17] Let \((M, \omega)\) be a Hermitian manifold of complex dimension \(n \geq 2\). Suppose that smooth \(\chi \in \Gamma^k_\omega\) satisfies (2.15). If \(u \in C^2(M)\) satisfies \(\chi_u \in \Gamma^k_\omega\), then we have the following pointwise inequalities:

\[
\chi^{k-1}_{tu} \wedge \omega^{n-k} \geq (1-t)^{k-1} \mu^{k-1}_t \omega^{n-1}, \quad \text{for} \quad 0 \leq t \leq 1, \tag{2.16}
\]

\[
\chi^k_u \wedge \omega^{n-k} \leq \frac{1}{tk} \chi^k_t \omega^{n-k}, \quad \text{for} \quad 0 < t \leq 1. \tag{2.17}
\]

3. \(C^0\) estimates. In this section, we establish \(C^0\) estimate for Equation (1.1). According to Tosatti and Weinkove [19, 20], it suffices to prove

\[
\int_M |\partial e^{-\frac{u}{t}}|^2 \omega^n \leq C_p \int_M e^{-pu} \omega^n \tag{3.1}
\]

for \(p\) large enough. Following the work of Sun [17], we get the following proposition by the cone condition.

Proposition 3.1. Let \((M, \omega)\) be an \(n\)-dimensional compact Kähler manifold and \(\chi\) be a smooth closed real \((1,1)\)-form on \(M\). Suppose that \(\chi \in \mathcal{C}(\alpha_{k-1})\) and \(u\) is a \(C^2\) solution to Equation (1.1) with \(0 < \alpha_l \in C^2(M)\) for \(0 \leq l \leq k - 1\). Then there are uniform constants \(C\) and \(p_0\) such that for all \(p \geq p_0\), (3.1) holds true.

Proof. Direct calculation gives that

\[
C \int_M e^{-pu} \omega^n \geq C \int_M e^{-pu} \chi^{k-1} \wedge \omega^{n-k+1}
\]

\[
\geq \int_M e^{-pu} (F(\chi_u) - F(\chi)) \chi^{k-1} \wedge \omega^{n-k+1}. \tag{3.2}
\]

On the other hand, we have the pointwise equality

\[
(F(\chi_u) - F(\chi)) \chi^{k-1} \wedge \omega^{n-k+1}
\]

\[
= (\chi_u^k - \chi^k) \wedge \omega^{n-k} - \sum_{l=0}^{k-1} \alpha_l (\chi_u^l - \chi^l) \wedge \omega^{n-l} \tag{3.3}
\]

\[
= \int_0^1 \sqrt{-1} \partial \overline{\partial} u \wedge \left( k \chi^{k-1}_{tu} \wedge \omega^{n-k} - \sum_{l=0}^{k-1} \alpha_l \chi^{l-1}_{tu} \wedge \omega^{n-l} \right) dt.
\]

Using integration by parts, we have

\[
\int_M e^{-pu} (F(\chi_u) - F(\chi)) \chi^{k-1} \wedge \omega^{n-k+1}
\]

\[
= p \int_0^1 dt \int_M \sqrt{-1} e^{-pu} \partial u \wedge \overline{\partial} u \wedge \left( k \chi^{k-1}_{tu} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} \alpha_l \chi^{l-1}_{tu} \wedge \omega^{n-l} \right) - \frac{1}{p} \int_0^1 dt \int_M \sqrt{-1} e^{-pu} \sum_{l=0}^{k-1} \overline{\partial} \alpha_l \chi^{l-1}_{tu} \wedge \omega^{n-l} \tag{3.4}
\]

\[
\geq p \int_0^1 dt \int_M \sqrt{-1} e^{-pu} \partial u \wedge \overline{\partial} u \wedge \left( k \chi^{k-1}_{tu} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} \alpha_l \chi^{l-1}_{tu} \wedge \omega^{n-l} \right) - \frac{C_1}{p} \int_0^1 dt \int_M e^{-pu} \sum_{l=1}^{k-1} \chi^{l-1}_{tu} \wedge \omega^{n-l+1}.
\]
From (2.17) in Lemma 2.6, we obtain for $1 \leq l \leq k - 1$

$$2^l \int_0^1 dt \int_M e^{-pu} \chi_{tu}^{l-1} \wedge \omega^{n-l+1} \geq 2^{l-1} \int_0^1 dt \int_M e^{-pu} \chi_{2u}^{l-1} \wedge \omega^{n-l+1} \geq \int_0^1 dt \int_M e^{-pu} \chi_{lu}^{l-1} \wedge \omega^{n-l+1}. \quad (3.5)$$

Put (3.2) and (3.5) into (3.4)

$$C \int_M e^{-pu} \omega^n + \frac{C_2}{p} \int_0^\frac{1}{t} dt \int_M e^{-pu} \sum_{i=1}^{k-1} \chi_{tiu}^{l-1} \wedge \omega^{n-l+1} \geq p \int_0^1 dt \int_M \sqrt{1e^{-pu} \partial u} \wedge (k \chi_{iu}^{k-1} \wedge \omega^{n-k} - \sum_{i=1}^{k-1} l \alpha \chi_{iu}^{l-1} \wedge \omega^{n-l}) \quad (3.6)$$

Since operator $F$ is concave in $\Gamma^k_\omega$, we deduce that

$$F(\chi_{tu}|i) \geq (1-t)F(\chi|i) + tF(\chi_u|i). \quad (3.7)$$

Notice that for $1 \leq l \leq k - 1$

$$\frac{\sigma_{k-1}(\chi_{tu}|i)}{\sigma_{k-2}(\chi_{tu}|i)} > \frac{\sigma_{k-1}(\chi_u|i)}{\sigma_{k-2}(\chi_u|i)} \quad \text{and} \quad -\frac{\sigma_{k-1}(\chi_u|i)}{\sigma_{k-2}(\chi_u|i)} > -\frac{\sigma_{k-1}(\chi_u)}{\sigma_{k-2}(\chi_u)} \quad (3.8)$$

which implies that

$$F(\chi_{tu}|i) > \alpha_{k-1}(p). \quad (3.9)$$

Since $(M, \omega)$ is compact, (2.8) means that

$$F(\chi|i) \geq (1+\delta)\alpha_{k-1}(p), \quad (3.10)$$

where $\delta$ is a uniform constant, independent of the choice of $p$. Insert (3.8) and (3.9) into (3.7),

$$F(\chi_{tu}|i) \geq (1+(1-t)\delta)\alpha_{k-1}(p) \quad (3.11)$$

This is equivalent to that

$$k \chi_{tu}^{k-1} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} l \alpha \chi_{tu}^{l-1} \wedge \omega^{n-l} \geq (1-t)\delta(k-1)\alpha_{k-1} \chi_{tu}^{k-2} \wedge \omega^{n-k+1}. \quad (3.11)$$

From (3.11) and (2.16), we get

$$p \int_0^1 dt \int_M \sqrt{1e^{-pu}} \partial u \wedge \overline{\partial u} \wedge (k \chi_{tu}^{k-1} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} l \alpha \chi_{tu}^{l-1} \wedge \omega^{n-l}) \geq p \int_0^1 dt \int_M \sqrt{1e^{-pu}(1-t)\delta} \alpha_{k-1} \chi_{tu}^{k-1} \wedge \omega^{n-k+1} \quad (3.12)$$

and for $0 \leq t \leq \frac{1}{2}$,

$$\sqrt{1} \partial u \wedge \overline{\partial u} \wedge (k \chi_{tu}^{k-1} \wedge \omega^{n-k} - \sum_{l=1}^{k-1} l \alpha \chi_{tu}^{l-1} \wedge \omega^{n-l}) \geq c_2 \sqrt{1} \partial u \wedge \overline{\partial u} \wedge \chi_{tu}^{k-2} \wedge \omega^{n-k+1}. \quad (3.13)$$
Direct calculation deduces that

\[
\frac{1}{p} \int_{t_0}^{t_1} dt \int_M e^{-pu} \sum_{l=1}^{k-1} \chi_{tu}^{l-1} \wedge \omega^{n-l+1} \\
= \int_{t_0}^{t_1} dt \int_M \left( e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \sum_{l=1}^{k-1} (l-1) \chi_{tu}^{l-2} \wedge \omega^{n-l+1} \right) ds \\
+ \frac{1}{2p} \int_M e^{-pu} \sum_{l=1}^{k-1} \chi^{l-1} \wedge \omega^{n-l+1} \\
\leq \int_{t_0}^{t_1} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \sum_{l=2}^{k-1} (l-1) \chi_{tu}^{l-2} \wedge \omega^{n-l+1} + \frac{C_3}{p} \int_M e^{-pu} \omega^n. 
\]

(3.14)

We divide the right side term in (3.6) into two parts and insert (3.12), (3.13) and (3.14) into (3.6)

\[
\left( C + \frac{C_2 C_3}{p} \right) \int_M e^{-pu} \omega^n \\
+ C_2 \int_{t_0}^{t_1} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \sum_{l=2}^{k-1} \frac{l-1}{2} \chi_{tu}^{l-2} \wedge \omega^{n-l+1} \\
\geq \frac{pc_2}{2} \int_{t_0}^{t_1} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{k-2} \wedge \omega^{n-k+1} \\
+ \frac{pc_1}{2} \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega^{n-1}. 
\]

(3.15)

we will use the first term in the right side of (3.15) to deal with the second term in the left side of it. From integration by parts and Garding’s inequality, we get

\[
\int_{t_0}^{t_1} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{k-2} \wedge \omega^{n-k+1} \\
\geq \mu \int_{t_0}^{t_1} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{k-3} \wedge \omega^{n-k+2} \\
+ \frac{1}{k-2} \int_{t_0}^{t_1} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \frac{d}{dt} (\chi_{tu}^{k-2} \wedge \omega^{n-k+1}) \\
\geq \mu \int_{t_0}^{t_1} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{k-3} \wedge \omega^{n-k+2} \\
- \frac{1}{k-2} \int_{t_0}^{t_1} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{k-2} \wedge \omega^{n-k+1}, 
\]

(3.16)

which yields that

\[
\frac{k-1}{\mu (k-2)} \int_{t_0}^{t_1} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{k-2} \wedge \omega^{n-k+1} \\
\geq \int_{t_0}^{t_1} dt \int_M e^{-pu} \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{k-3} \wedge \omega^{n-k+2}. 
\]

(3.17)
Similar to (3.17), we can have for $2 \leq l \leq k - 2$

$$\frac{l}{\mu(l-1)} \int_0^2 dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{lu}^{l-1} \wedge \omega^{n-l} \geq \int_0^2 dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{lu}^{l-2} \wedge \omega^{n-l+1}.$$ 

By iterations, we obtain

$$C_4 \frac{2^l}{\mu(l-1)} \int_0^2 dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{lu}^{k-l+1} \wedge \omega^{n-k+1} \geq \int_0^2 dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{lu}^{l-2} \wedge \omega^{n-l+1}.$$ 

We may assume that $p_0 \geq \frac{(k-2)C_2C_4}{\epsilon_2}$, and thus for $p \geq p_0$,

$$(C + C_2C_3 \frac{p}{p}) \int_0^2 dt \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \chi_{lu}^k \wedge \omega^{n-k},$$

which means that (3.1) holds true.

4. $C^2$ estimates. In this section, we establish the $C^2$ estimate for Equation (1.1) by the cone condition. Our calculation mostly follows that of Hou-Ma-Wu [9]. One difference is that instead of using $g_\eta + u_\eta$ in suitable coordinates, we use $X_\eta = \chi_\eta + u_\eta$, where $\chi_\eta$ satisfies the cone condition.

**Proposition 4.1.** Let $u \in C^4(M)$ be a solution to Equation (1.1) on a compact Kähler manifold $(M, \omega)$ and $\chi$ be a smooth closed real $(1,1)$–form on $M$. Suppose that $0 < \alpha_l \in C^2(M)$ for $0 \leq l \leq k - 1$, $\chi \in C(\alpha_{k-1})$. Then there is an estimate

$$\sup_M |\partial \bar{\partial} u| \leq C \left( \sup_M |\nabla u|^2 + 1 \right),$$

where $C$ is a uniform constant.

**Proof.** Consider the function for $p \in M$ and a unit vector $\xi \in T_p^{(1,0)} M$

$$W(p, \xi) = \log(X_\xi \xi^\eta) + \varphi(|\nabla u|^2) + \psi(u).$$

Here

$$\varphi(t) = -\frac{1}{2} \log(1 - \frac{t}{2K}), \quad \psi(t) = -A \log(1 + \frac{t}{2L}),$$

where

$$K = \sup_M |\nabla u|^2 + 1, \quad L = \sup_M |u| + 1, \quad A = 2L(C_0 + 1),$$

and $C_0$ is to be determined later. Direct calculation gives that

$$0 < \frac{1}{4K} \leq \varphi' \leq \frac{1}{2K}, \quad \varphi'' = 2(\varphi')^2,$$

and

$$C_0 + 1 \leq -\psi' \leq 2(C_0 + 1), \quad \psi'' \geq \frac{2\epsilon}{1 - \epsilon} (\psi')^2, \quad \text{for all } \epsilon \leq \frac{1}{2A + 1}.$$ 

Since $M$ is compact, $W$ attains its maximum at some point $p \in M$ in some unit direction $\xi$. We choose a local normal coordinate system near $p$ such that $X$ is diagonal and $X_{1T} \geq \cdots \geq X_{nT}$. Without loss of generality, we can assume that $X_{1T} > 1$. From now on, all the calculations will be carried out at the point $p$ and
the Einstein summation convention will be used. Calculating covariant derivatives, we obtain

\[ 0 = W_i = \frac{X_{1\bar{\iota}}}{X_{1\iota}} + \varphi'(|\nabla u|^2)_i + \psi' u_i, \quad 1 \leq i \leq n, \quad (4.4) \]

\[ 0 \geq W_{\bar{\iota}i} = \frac{X_{1\bar{\iota}}}{X_{1\iota}} - \frac{X_{1\iota}X_{1\bar{\iota}}}{X_{1\bar{\iota}}} + \psi' u_i + \varphi'' |u_i|^2 + \varphi'(|\nabla u|^2)_i + \varphi'' (|\nabla u|^2)_i. \quad (4.5) \]

Multiplying (4.5) by \( F^{\iota\bar{\iota}} \) and summing it over index \( i \),

\[ 0 \geq F^{\iota i} \frac{X_{1\iota}}{X_{1\iota}} - \frac{X_{1\iota}X_{1\bar{\iota}}}{X_{1\bar{\iota}}} + \psi' F^{\bar{\iota}i} u_i + \varphi'' F^{\iota i} |u_i|^2 \]

\[ + \varphi' F^{\iota i} (|\nabla u|^2)_{\bar{\iota}i} + \varphi'' F^{\bar{\iota}i} (|\nabla u|^2)_{\bar{\iota}i}. \quad (4.6) \]

We will control some terms in (4.6). Covariant differentiating Equation (2.4) twice in the \( \frac{\partial}{\partial \bar{\iota}} \) direction and the \( \frac{\partial}{\partial \iota} \) direction, we have

\[ F^{\bar{\iota}i} X_{1\bar{\iota}} + \sum_{l=0}^{k-2} (\alpha_l)_{\bar{\iota}} F_l = (\alpha_{k-1})_{\bar{\iota}} \quad (4.7) \]

and

\[ F^{\bar{i}j\bar{\iota}} X_{1\iota} X_{1\bar{\iota}} + F^{\bar{i}j\bar{\iota}} X_{1\bar{\iota}} - 2 \text{Re} \left( \sum_{l=0}^{k-2} (\alpha_l)_{\iota} F_l X_{1\bar{\iota}} \right) + \sum_{l=0}^{k-2} (\alpha_l)_{\iota} F_l = (\alpha_{k-1})_{\iota}. \quad (4.8) \]

Direct calculation deduces that

\[ F^{\bar{i}j\bar{\iota}} X_{ij} = F^{\iota j} X_{ij} + \sum_{l=0}^{k-2} \alpha_l F^{\iota j} X_{ij} = \alpha_{k-1} - \sum_{l=0}^{k-2} (k-l) \alpha_l F_i. \quad (4.9) \]

By commuting the covariant derivatives, we obtain that

\[ X_{ij\bar{j}} - X_{j\bar{i}j} = \chi_{ij\bar{j}} - \chi_{j\bar{i}j}, \quad (4.10) \]

and

\[ X_{ij\bar{j}i} = X_{i\bar{j}j} - \chi_{i\bar{j}j} = R_{i\bar{j}i} X_{1\bar{i}} - R_{\bar{j}i \bar{j}} X_{1\iota} - G_{ij\bar{j}}, \quad (4.11) \]

where

\[ G_{ij\bar{j}} = \chi_{ij\bar{j}} - \chi_{ij\bar{j}} + \sum_m R_{ij\bar{m}} X_{m\bar{i}} - \sum_m R_{ij\bar{m}} X_{m\bar{i}}. \]

It follows from (4.8), (4.9) and (4.11) that

\[ F^{\iota i} \frac{X_{1\bar{\iota}}}{X_{1\iota}} \]

\[ = \frac{1}{X_{1\iota}} F^{\iota i} (X_{1\iota} + X_{1\bar{\iota}} R_{1\iota} - X_{1\iota} R_{1\bar{\iota}} - G_{1\iota}) \]

\[ \geq - \frac{F^{\bar{i}j\bar{\iota}} X_{1\iota} X_{1\bar{\iota}}}{X_{1\iota}} - \frac{2}{X_{1\iota}} \text{Re} \left( \sum_{l=0}^{k-2} (\alpha_l)_{\iota} F_l X_{1\bar{\iota}} \right) - \sum_{l=0}^{k-2} (\alpha_l)_{\iota} F_l + \frac{(\alpha_{k-1})_{\iota}}{X_{1\iota}} \]

\[ + \inf \frac{R_{1\iota}}{X_{1\iota}} (\alpha_{k-1} + \sum_{l=0}^{k-2} (k-l) \alpha_l F_i) - \frac{\sup G_{1\iota}}{X_{1\iota}}. \quad (4.12) \]

Direct calculation gives that

\[ (|\nabla u|^2)_{\bar{i}i} = \sum_{t} (|\chi_{t|\bar{i}|}^2 + |u_t|^2 - 2 \text{Re} (\chi_{t|\bar{i}|} u_t)) + \sum_{a,t} R_{t\bar{a}u} u_{a\bar{u}}. \]
where it was shown by Krylov in [10] that the which yields

\[ F_i \geq 0 \]

Direct computation gives that

\[ -\sum_{i,t} \text{Re}(\chi_{i,t} u_t) + \sum_{i,a,t} R_{i,a,t} u_{i,a,t} + \frac{1}{2} X_{i,t}^2 - 2 \sum_{i} \text{Re}(C_{i,t} u_t). \]  

Combining (4.13) and (4.7) yields that

\[ F_i (|\nabla u|^2) \geq -2 \sup_{i,t} (|\nabla u|) |\nabla u| F_i - \sup_{i,a,t} R_{i,a,t} |\nabla u|^2 F_i - 2 \sup_{i} (\chi_{i,t}^2) F_i \]

\[ + \frac{1}{2} \sum_{i} |\nabla \alpha_i| |\nabla u| F_i. \]

Take (4.12) and (4.14) into (4.6)

\[ 0 \geq \frac{\varphi'}{2} F_i \sum_{i} X_{i,t}^2 - \frac{F_i \sum_{i} X_{i,t}^2}{\chi_{i,t}^2} \]

\[ + \frac{1}{2} \sum_{i} |\nabla \alpha_i| |\nabla u| F_i \]

where \( C_1 \) and \( C_2 \) are uniform positive constants. We set

\[ \delta = \frac{1}{1 + 2A} \]

(4.16)

\[ C_0 = \frac{C_1 + C_2}{\theta}. \]

(4.17)

Then we have two cases to consider.

**Case 1** \( X_{i,t} < -\delta X_{i,t} \). Using the critical point condition (4.4), we obtain that

\[ -\frac{F_i \sum_{i} X_{i,t}^2}{\chi_{i,t}^2} = -\frac{F_i \sum_{i} X_{i,t}^2}{\chi_{i,t}^2} \varphi' (|\nabla u|^2) + \frac{\varphi'}{2} |\nabla u|^2 \]

\[ \geq -\frac{1}{2} \left( \varphi' \right)^2 F_i (|\nabla u|^2)^2 - 2 (\varphi')^2 |\nabla u|^2 F_i. \]

It was shown by Krylov in [10] that the \( \left( \frac{\sigma_{k-1}}{\sigma_{k}} \right)^{\frac{1}{k-1}} \) is concave in \( \Gamma_{k-1} \) for \( 0 \leq l \leq k-2 \), which means that

\[ \left( (-F_i)^{-\frac{l}{k-l}} \right)^{i,j} X_{i,t}^2 X_{j,t} \leq 0. \]

Direct computation gives that

\[ -\frac{F_i \sum_{i} X_{i,t}^2}{\chi_{i,t}^2} \geq \frac{k-l}{k-l-1} (-F_i)^{-1} |F_i \sum_{i} X_{i,t}^2|, \]

which yields

\[ -\frac{F_i \sum_{i} X_{i,t}^2}{\chi_{i,t}^2} \geq \frac{k-l}{k-l-1} \sum_{i} \text{Re}(\sum_{i,a,t} R_{i,a,t} u_{i,a,t} + \frac{1}{2} X_{i,t}^2 - 2 \sum_{i} \text{Re}(C_{i,t} u_t). \]

\[ \geq \sum_{l=0}^{k-2} \frac{k-l}{k-l-1} \text{Re}(\sum_{i} \text{Re}(\sum_{i,a,t} R_{i,a,t} u_{i,a,t} + \frac{1}{2} X_{i,t}^2 - 2 \sum_{i} \text{Re}(C_{i,t} u_t). \]

(4.19)
By Lemma 2.3 and Lemma 2.4, we see that 

\[
- \text{constant. Putting (4.2), (4.3), (4.18), (4.19) and (4.20) into (4.15), we obtain }
\]

\[
\text{Therefore, } \psi' F_{\tilde{\mu}l} u_{il} = \psi' F_{\tilde{\mu}} (X_{ii} - \chi_{ii}) \geq \psi' (\alpha_{k-1} - \sum_{l=0}^{k-2} (k-l)\alpha_l F_l + \sup_\chi \chi_{\tilde{\mu}}|F|). \quad (4.20)
\]

By Lemma 2.3 and Lemma 2.4, we see that \(-F_l\) and \(F\) can be controlled by uniform constant. Putting (4.2), (4.3), (4.18), (4.19) and (4.20) into (4.15), we obtain

\[
0 \geq \frac{1}{8K} F_{\tilde{\mu}}^2 - C_2(C_0 + 1)^2 - C_1(C_0 + 1)
\]

\[
\geq \frac{\delta^2}{8nK} F_{\tilde{\mu}}^2 - C_2(C_0 + 1)^2 - C_1(C_0 + 1). \quad (4.21)
\]

\(F\) has a uniform positive lower bound, so \(X_{1\tilde{\mu}} \leq C\).

**Case 2** \(X_{nn} \geq -\delta X_{1\tilde{\mu}}\). Let

\[
I = \{i \in \{1, \cdots, n\}| F_{\tilde{\mu}} > \delta^{-1} F_{1\tilde{\mu}}\}.
\]

For those indices which are not in \(I\), we have

\[
- \sum_{i \notin I} F_{\tilde{\mu}}^2 |X_{1i\tilde{\mu}}|^2 = - \sum_{i \notin I} F_{\tilde{\mu}}^2 (\varphi' (|\nabla u|^2)_i + \psi' u_i)^2 \geq -2(\varphi')^2 \sum_{i \notin I} F_{\tilde{\mu}}^2 (|\nabla u|^2)_i^2 - \frac{8n(C_0 + 1)^2}{\delta} K F_{1\tilde{\mu}}. \quad (4.22)
\]

Since

\[
-F_{1\tilde{\mu}, l} = \frac{F_{\tilde{\mu}} - F_{1\tilde{\mu}}}{X_{1\tilde{\mu}} - X_{l\tilde{\mu}}} \quad \text{and} \quad X_{l\tilde{\mu}} \geq -\delta X_{1\tilde{\mu}}.
\]

we have

\[
- \sum_{i \notin I} F_{1\tilde{\mu}, l} \geq \frac{1 - \delta}{1 + \delta X_{1\tilde{\mu}}} \sum_{i \notin I} F_{\tilde{\mu}}. \quad (4.23)
\]

Therefore,

\[
- \frac{F_{\tilde{\mu}, ij} X_{ij\tilde{\mu}}}{X_{1\tilde{\mu}}} \geq - \frac{F_{\tilde{\mu}}^2}{X_{1\tilde{\mu}}} - \frac{F_{\tilde{\mu}, ij} X_{ij\tilde{\mu}}}{X_{1\tilde{\mu}}} - \frac{F_{1\tilde{\mu}} X_{1i\tilde{\mu}} X_{ij\tilde{\mu}}}{X_{1\tilde{\mu}}} \geq - \frac{F_{\tilde{\mu}, ij} X_{ij\tilde{\mu}}}{X_{1\tilde{\mu}}} + \frac{1 - \delta}{1 + \delta X_{1\tilde{\mu}}} \sum_{i \notin I} F_{\tilde{\mu}} |X_{i\tilde{\mu}}|^2. \quad (4.24)
\]

Inserting (4.22) and (4.24) into (4.15), combining (4.19), we deduce that

\[
0 \geq \frac{\varphi'}{2} F_{\tilde{\mu}}^2 X_{1\tilde{\mu}}^2 + \frac{1 - \delta}{1 + \delta X_{1\tilde{\mu}}} \sum_{i \notin I} F_{\tilde{\mu}} X_{1i\tilde{\mu}}^2 - \sum_{i \notin I} F_{\tilde{\mu}} X_{1i\tilde{\mu}}^2 \geq \psi' F_{\tilde{\mu}} u_{ij} + \psi'' F_{\tilde{\mu}} |u_i|^2 - \frac{8n(C_0 + 1)^2}{\delta} F_{1\tilde{\mu}} + \sum_{i \notin I} \varphi'' F_{\tilde{\mu}} ((|\nabla u|^2)_i)^2 - C_1 - C_2. \quad (4.25)
\]
We need to control the terms in (4.25). It follows from (4.10) that
\[ |X_{iT}|^2 \geq |X_{iT}|^2 + 2\text{Re}(X_{iT} \bar{b}_i), \]
where \( b_i = X_{iT} - X_{iT}. \) Use (4.4),
\[ \sum_{i \in I} \varphi'' F_{i\bar{i}} \left( (|\nabla u|^2)_i \right)^2 \geq 2 \sum_{i \in I} F_{i\bar{i}} \left( \delta \frac{|X_{iT}|^2}{X_{iT}^2} - \frac{\delta}{1 - \delta} |\psi' u|^2 \right). \]
Noticing the fact that \( \psi'' \geq \frac{2\delta}{1 - \delta} (\psi')^2, \) we have from (4.26) and (4.27)
\[ \frac{1 - \delta}{1 + \delta} \frac{1}{X_{iT}^2} \sum_{i \in I} F_{i\bar{i}} |X_{iT}|^2 - \sum_{i \in I} F_{i\bar{i}} \frac{X_{iT} X_{iT}}{X_{iT}^2} + \psi' F_{i\bar{i}} |u|^2 + \sum_{i \in I} \varphi'' F_{i\bar{i}} \left( (|\nabla u|^2)_i \right)^2 \]
\[ \geq \frac{2\delta^2}{1 + \delta} \frac{1}{X_{iT}^2} \sum_{i \in I} F_{i\bar{i}} |X_{iT}|^2 - \frac{2(1 - \delta)}{1 + \delta} \frac{1}{X_{iT}^2} \sum_{i \in I} F_{i\bar{i}} \text{Re}(X_{iT} \bar{b}_i) \]
\[ \geq \frac{\delta^2}{1 + \delta} \frac{1}{X_{iT}^2} \sum_{i \in I} F_{i\bar{i}} |X_{iT}|^2 - \frac{1}{(1 + \delta) \delta^2} \sup |b_i|^2. \]
Recall \( F \) is controlled by uniform constant. Substituting (4.28) into (4.25) yields
\[ 0 \geq \frac{\varphi'}{2} F_{i\bar{i}} X_{iT}^2 + \psi' F_{i\bar{i}} u_{i\bar{i}} - \frac{8n(C_0 + 1)^2 K}{\delta} F_{i\bar{i}} - C_1 - C_2. \]
We may assume \( X_{iT} \geq N. \) If the first inequality of Lemma 2.5 is true, then
\[ 0 \geq \frac{1}{8K} F_{i\bar{i}} X_{iT}^2 + \theta(C_0 + 1)(F + 1) - \frac{8n(C_0 + 1)^2 K}{\delta} F_{i\bar{i}} - C_1 - C_2 \]
\[ \geq \frac{1}{8K} F_{i\bar{i}} X_{iT}^2 - \frac{8n(C_0 + 1)^2 K}{\delta} F_{i\bar{i}}, \]
which means that
\[ X_{iT}^2 \leq \frac{64n(C_0 + 1)^2 K^2}{\delta}. \]
If the second inequality of Lemma 2.5 is true, then, by (4.20),
\[ 0 \geq \frac{1}{8K} \theta X_{i\bar{i}} - (C_0 + 1)(C_1 + C_2) - \frac{8n(C_0 + 1)^2 K}{\delta} F - C_1 - C_2. \]
This implies that \( X_{i\bar{i}} \) is controlled by uniform constant. \( \square \)

5. \( C^1 \) estimates. In this section, we adapt the blowup method of Dinew and Kolodziej [4] to obtain the gradient estimate.

**Proposition 5.1.** Suppose that \( u \in C^4(M) \) be a solution to Equation (1.1) on a compact Kähler manifold \( (M, \omega) \), \( \chi_u \in \Gamma^{k} \omega \) and \( \sup_M u = 0. \) Let \( 0 < \alpha_l \in C^2(M) \) for \( 0 \leq l \leq k - 1 \) and \( \chi \in \mathcal{C}(\alpha_{k-1}). \) Then
\[ \sup_M |\nabla u|_\omega \leq C, \]
where \( C \) is a uniform constant.
Similar to the method of Dinew and Kolodziej [4], we can find a limit function \( \omega \) which we identify with an open ball \( B_p \). By Proposition 3.1, we know that the uniform norm of \( u \) and \( \alpha \) now the functions \( m \in \chi \in \Gamma^k_\omega \).

Fix \( \epsilon \) and consider any \( \phi \) and \( v \) that satisfy

\[
\frac{\partial^l \phi}{\partial x^l} < \epsilon, \quad \phi \in \Gamma^k_\omega,
\]

and

\[
\frac{k^l \chi^{k-1} \wedge \omega^{n-k}}{(k-1)^l \chi^{k-2} \wedge \omega^{n-k+1}} - \sum_{l=0}^{k-2} \alpha_l \frac{\chi^l \wedge \omega^{n-l}}{\chi^{k-1} \wedge \omega^{n-k+1}} = \phi, \quad \chi \in \Gamma^k_\omega,
\]

To argue by contradiction, suppose that the gradient estimate for \( v \) fails. For \( \phi_m \in C^2(M) \), we can find a sequence of \( C^4 \) smooth functions \( u_m \) satisfying

\[
C_m = \sup_M |\nabla u_m| \to \infty, \quad \text{as} \quad m \to \infty,
\]

and

\[
\frac{\partial^l \phi_m}{\partial x^l} < \epsilon, \quad \phi_m \in \Gamma^k_\omega, \quad \text{sup}_M u_m = 0,
\]

By Proposition 3.1, we know that the uniform norm of \( u_m \) is under control. Let \( p_m \in M \) be the point maximizing the functions \( |\nabla u_m| \) for each \( m \). After passing to a subsequence, we can assume that \( p_m \to p \in M \). Fix a coordinate chart around \( p \), which we identify with an open ball \( B_2(0) \) in \( \mathbb{C}^n \) with coordinates \( (z_1, \ldots, z^n) \) and \( \omega(0) = \beta := \delta_{ij}dz^i \wedge d\bar{z}^j \). We may assume that \( P_m \) are contained in \( B_1(0) \). Consider now the functions

\[
\tilde{u}(z) := u_m \left( \frac{z}{C_m} \right), \quad z \in B_{C_m}(0).
\]

Similar to the method of Dinew and Kolodziej [4], we can find a limit function \( v \in C^{1, \alpha}(\mathbb{C}^n) \) such that \( |\nabla v(0)| = 1 \).

On the other hand, we may show that \( v \) is a constant in \( \mathbb{C}^n \), which contradicts \( |\nabla v(0)| = 1 \). We assume \( \tilde{u}_m \) is convergent. In the new local coordinates, we get

\[
\left( \chi \left( \frac{z}{C_m} \right) + C_m^2 \sqrt{-1} \partial \bar{\partial} \tilde{u}_m(z) \right)^k \wedge \left( \omega \left( \frac{z}{C_m} \right) \right)^{n-k} - \sum_{l=0}^{k-2} \alpha_l \left( \chi \left( \frac{z}{C_m} \right) + C_m^2 \sqrt{-1} \partial \bar{\partial} \tilde{u}_m(z) \right)^l \wedge \left( \omega \left( \frac{z}{C_m} \right) \right)^{n-l} = \phi_m \left( \frac{z}{C_m} \right) \left( \chi \left( \frac{z}{C_m} \right) + C_m^2 \sqrt{-1} \partial \bar{\partial} \tilde{u}_m(z) \right)^{k-1} \wedge \left( \omega \left( \frac{z}{C_m} \right) \right)^{n-k+1}.
\]

Therefore,

\[
(C_m)^{2k} \left( \frac{1}{C_m^2} + \sqrt{-1} \partial \bar{\partial} \tilde{u}_m(z) \right)^k \wedge \left( 1 + O\left( \frac{|z|^2}{C_m^2} \right) \right)^{n-k} \]
We use the continuity method introduced in [16]. First, consider
\[ \chi \] where
\[ \text{generally does not work for all } \]
the cone condition
\[ \text{The obstacle for the uniform estimates of the solution flow is that the cone condition} \]
Obviously,
\[ \text{The proof of Theorem 1.1.} \]
\[ \text{In this section, we prove Theorem 1.1 by the} \]
\[ \text{method of continuity. From the standard regularity theory of uniformly elliptic} \]
\[ \text{partial differential equations, we can obtain the high order regularity. We refer} \]
\[ \text{readers to Tosatti, Wang, Weinkove and Yang [21]. Define } \phi \text{ by} \]
\[ \frac{\chi^k \wedge \omega^{n-k}}{\chi^{k-1} \wedge \omega^{n-k+1}} - \sum_{l=0}^{k-2} \alpha_l \frac{\chi^l \wedge \omega^{n-l}}{\chi^{k-1} \wedge \omega^{n-k+1}} = \phi, \quad \chi \in \Gamma_k. \]
\[ \text{Obviously,} \]
\[ \frac{\chi^{k-1} \wedge \omega^{n-k}}{(k-1)\chi^{k-2} \wedge \omega^{n-k+1}} - \sum_{l=0}^{k-2} \alpha_l \frac{\chi^l \wedge \omega^{n-l}}{(k-1)\chi^{k-2} \wedge \omega^{n-k+1}} > \phi. \]
\[ \text{The obstacle for the uniform estimates of the solution flow is that the cone condition} \]
\[ \text{generally does not work for all } \alpha \text{. We need to construct a new cone} \]
\[ \text{and impose some extra condition.} \]
\[ \text{We can find a smooth real function } h \text{ satisfying that for all } p \in M \]
\[ h(p) \geq \max \{ \phi(p), \alpha_{k-1}(p) \} \]
and
\[ \frac{\chi_{u_t}^k \wedge \omega^{n-k}}{\chi_{u_t}^{k-1} \wedge \omega^{n-k+1}} - \sum_{l=0}^{k-2} \alpha_l \frac{\chi_{u_t}^l \wedge \omega^{n-l}}{\chi_{u_t}^{k-1} \wedge \omega^{n-k+1}} = h(t)^{1-t}e^{a_t}, \quad t \in [0, 1], \]
where \( \chi_{u_t} \in \Gamma_k \) and \( a_t \) is a constant for each \( t \). Set
\[ T_1 := \{ t' \in [0, 1] | \exists u_t \in C^{2,\alpha}(M) \text{ and } a_t \text{ solving (6.2) for } t \in [0, t'] \}. \]
As shown in [16], the continuity method works if we can guarantee (1) \( 0 \in T_1 \) and
(2) uniform \( C^\infty \) estimates for all \( u_t \).
For \( T_1 \), the first requirement is naturally met. For the second requirement,
we only need to show a uniform cone condition for all the solution flow. At the
maximum point of \( u_t \), we get
\[ \frac{\chi_{u_t}^k \wedge \omega^{n-k}}{\chi_{u_t}^{k-1} \wedge \omega^{n-k+1}} - \sum_{l=0}^{k-2} \alpha_l \frac{\chi_{u_t}^l \wedge \omega^{n-l}}{\chi_{u_t}^{k-1} \wedge \omega^{n-k+1}} \]
This implies that condition (6.1) is uniform for all $u_t$. Therefore, there exists a solution $v$ to
\[
\frac{\chi^k_v \wedge \omega^{n-k}}{\chi^{k-1}_v \wedge \omega^{n-k+1}} - \sum_{l=0}^{k-2} \alpha_l \frac{\chi^l_v \wedge \omega^{n-l}}{\chi^{k-1}_v \wedge \omega^{n-k+1}} = h e^{a_l},
\]
for some $a_l \leq 0$.

Second, we start the continuity method from $\chi_v$ and consider the family of equations
\[
\frac{\chi^k_{u_t} \wedge \omega^{n-k}}{\chi^{k-1}_{u_t} \wedge \omega^{n-k+1}} - \sum_{l=0}^{k-2} \alpha_l \frac{\chi^l_{u_t} \wedge \omega^{n-l}}{\chi^{k-1}_{u_t} \wedge \omega^{n-k+1}} = (\alpha_{k-1})^t h^{1-t} e^{b_t}, \quad t \in [0, 1],
\]
where $\chi_{u_t} \in \Gamma^k_\omega$ and $b_t$ is a constant for each $t$. Set
\[
T_2 := \{ t' \in [0, 1] | \exists u_t \in C^{2,\alpha}(M) \text{ and } b_t \text{ solving } (6.3) \text{ for } t \in [0, t'] \}.
\]
We can see that $u_0 = v$ and $b_0 = a_1 \leq 0$, so $0 \in T_2$. From (6.3), we have
\[
\int_M \chi^k_{u_t} \wedge \omega^{n-k} - \int_M \sum_{l=0}^{k-2} \alpha_l \chi^l_{u_t} \wedge \omega^{n-l} = \int_M (\alpha_{k-1})^t h^{1-t} e^{b_t} \chi^{k-1}_{u_t} \wedge \omega^{n-k+1}
\geq e^{b_t} \int_M \alpha_{k-1} \chi^{k-1}_{u_t} \wedge \omega^{n-k+1}.
\]
From condition (1.3), we obtain $b_t \leq 0$. Hence, $(\alpha_{k-1})^t h^{1-t} e^{b_t} \leq h$. This means that the second requirement is also satisfied.

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