Introduction - Recently, there has been a surge of interest in the correlated electrons in flat bands [1–4]. In particular, twisted bilayer graphene [5–10], in which interlayer twist angle tunes the bandwidth to zero, has spurred a great deal of community interest, exhibiting strong correlations of electrons and nontrivial band topology [11–14]. The advent of magic angle twisted bilayer graphene has renewed the interest in the realization of flat bands in diverse systems [15–17]. Notable examples include the recent study by Zhu, Prodan, and Ahn (ZPA) [18]. Utilizing the two-dimensional (2D) array of the Su-Schrieffer-Heeger (SSH) chains [19–21], ZPA have shown that one-dimensional (1D) flat bands can arise, being stabilized via the formation of topological domain walls (DWs). Encouragingly, the proposed 1D flat bands have been experimentally confirmed in a mechanical metamaterial [22], whereas their condensed-matter realization is yet to be discovered.

Regarding efforts to search for topological materials, these days have witnessed remarkable developments, based on topological quantum chemistry and symmetry indicators [23–32]. Due to these high-throughput approaches, mass topological materials have been catalogued [33–36], started with the seminal work by Fu and Kane [37]. The Fu-Kane formula expresses the $Z_2$ topological indices using parity eigenvalues, enabling the discovery of archetypal topological materials, such as Zn$_2$(PS$_3$)$_3$. The new insight from our work could help efforts to realize topological flat bands in solid-state systems.

Two-dimensional weak-type insulators in inversion-symmetric crystals

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The Su-Schrieffer-Heeger (SSH) chain is an one-dimensional lattice that comprises two dimerized sublattices. Recently, Zhu, Prodan, and Ahn (ZPA) proposed in [L. Zhu, E. Prodan, and K. H. Ahn, Phys. Rev. B 99, 041117 (2019)] that one-dimensional flat bands can occur at topological domain walls of a two-dimensional array of the SSH chains. Here, we newly suggest a two-dimensional topological insulator that is protected by inversion and time-reversal symmetries without spin-orbit coupling. It is shown that the two-dimensional SSH chains realize the proposed topological insulator. Utilizing the first Stiefel-Whitney numbers, a weak type of $Z_2$ topological indices are developed, which identify the proposed topological insulator, dubbed a two-dimensional Stiefel-Whitney insulator (2DSWI). The ZPA model is employed to study the topological phase diagrams and topological phase transitions. It is found that the phase transition occurs via the formation of the massless Dirac points that wind the entire Brillouin zone. We argue that this unconventional topological phase transition is a characteristic feature of the 2DSWI, manifested as the one-dimensional domain wall states. Using first-principles calculations, we find the suggested 2DSWI should be realized in eleven known materials, such as Zn$_2$(PS$_3$)$_3$. The new insight from our work could help efforts to realize topological flat bands in solid-state systems.

$Z_2$ topological indices - We consider a 2D Bloch Hamiltonian $\mathcal{H}(k)$ that is invariant under $PT$ and $T^2 = 1$. Since $[PT, \mathcal{H}(k)] = 0$, 1D families of real Hamiltonians on a closed loop $\mathcal{C}$ in the BZ, $\mathcal{H}(k)|_{\mathcal{C}}$, can be characterized by the $Z_2$-quantized first SW number $\nu_C = \frac{\pi}{2} \oint_{\mathcal{C}} A(k) \cdot dk$ [53], where $P$ is the path-ordering operator, $A(k) = i(u(k)\nabla_k u(k))$ is the Berry connection, and $u(k)$ is the cell-periodic part of the occupied Bloch function at $k \in \text{BZ}$. The number refers to a topological invariant that characterizes the twist of real Bloch states in momentum space [53]. The Zak phase [54] that is $Z_2$-quantized by the reality condition corresponds to the first SW number. The SW number is defined for any
given $k_1$ and $k_2$ lines as $v_1(k_1) = \frac{1}{\pi} P \int_{-\pi}^{\pi} A_1(k_1, k_2) dk_2$
and $v_2(k_2) = \frac{1}{\pi} P \int_{-\pi}^{\pi} A_2(k_1, k_2) dk_1$, where $A_i$ ($i = 1, 2$) is $k_i$-component of $A(k)$. In the presence of a direct band gap, one-parameter families of real Hamiltonians
on a $k_i$ line ($i = 1, 2$) are adiabatically connected to those on a $k_i'$ line for any $k_i' \neq k_i'$, resulting in the equivalent SY numbers $v_i(k_i) \neq v_i(k_i')$. Therefore, the $Z$ indices $v_1$ and $v_2$ can unambiguously characterize a $\mathcal{PT}$-symmetric insulating phase.

**Inversion-symmetry indicators** - The parity eigenvalues $\xi_n(\Gamma_i) = \pm 1$ of the occupied Bloch states at the four time-reversal invariant momenta (TRIMs) $\Gamma_{i=1,2} = \frac{1}{2}(n_1 b_1 + n_2 b_2)$, where $n_i = 0, 1$ and $b_i$ are primitive lattice vectors [See Fig. 1(a)]. They provide a symmetry indicator that diagnoses the 2DSWIs. The SY numbers $v_1$ and $v_2$ on the $k_1 = 0$ ($\pi$) and $k_2 = 0$ ($\pi$) loops satisfy

$$(-1)^{v_1} = \xi_a \xi_d \xi_b \xi_c; \quad (-1)^{v_2} = \xi_a \xi_b \xi_d \xi_c,$$

where

$$\xi_a = \prod_n \xi_n(\Gamma_a).$$

Equations (1) - (3) are well-defined if $\xi_a \xi_d = \xi_b \xi_c$ and $\xi_a \xi_b = \xi_d \xi_c$, guaranteed by the strong $\mathbb{Z}_2$ index $v_0 = 0$ because

$$(-1)^{v_0} = \xi_a \xi_b \xi_c \xi_d = 1.$$

Note that the strong index $v_0 = 0$ is also a necessary condition for an insulating phase since the strong SY $v_0 = 1$ dictates the presence of the massless DPs in momentum space [55]. Figure 1(b) shows possible configurations of parity eigenvalues compatible with $v_0 = 0$. We identify the 2DSWI as a $\mathcal{PT}$-symmetric topological insulator in vanishing SOC, characterized by $v_0 = 0$ and $(v_1 v_2) \neq (00)$.

**2DSWI as a stack of SSH chains** - Similar to the weak topological insulators in three dimensions [37, 56], which can be viewed as a stack of 2D quantum spin Hall insulators, the 2DSWI with $(v_1 v_2) \neq (00)$ can be viewed as a stack of 1D SSH chains along $G_n = v_1 b_1 + v_2 b_2$. In stark contrast to the 3D weak topological insulators, however, by stacking the SSH chains in two dimensions, every 1D line in the BZ has the first SY number. This results in the boundary mode, for example, at any edge momentum $\hat{k}$ when projected along the $k_j$-direction with $v_1 = 1$, where $i,j = 1, 2$ and $i \neq j$. In general, the band topology of a 2DSWI manifests as 1D topological states at the $(v_1 v_2)$ edge, as illustrated in Fig. 1(b).

**Zhu-Prodan-Ahn model** - Using the ZPA model, we suggest a material realization of the 2DSWI in 2D coupled SSH chains. As shown in Fig. 2, a family of 2D coupled SSH chains are constructed by an alternating array of two SSH chains. The ZPA model considers inter- and intra-chain dimerizations, parameterized by $d$ and $e$. Here, $d = (d_x, d_y)$ and $e$ describe an in-plane staggered distortion and a strain of the unit cell, respectively [See Figs. 2(a) and 2(b)]. This family of 2D coupled SSH chains includes various 2D lattices, such as a square lattice with $d = 0$ and $e = 0$ [Figs. 2(c)] and a graphene-like honeycomb lattice with $e = 1/3$, $d = (1/3, 0)$ [Fig. 2(d)].

A Hamiltonian, describing hopping of electrons in the 2D coupled SSH chains, is given by

$$H(k) = h^*(k) \sigma^+ + h(k) \sigma^-,$$

where $\sigma^\pm = \sigma_x \pm i \sigma_y$ are the Pauli matrices associated with the sublattices and $h(k) = -t_1 - t_2 e^{ik_2} - t_3 e^{-ik_1} - t_4 e^{-(k_1+k_2)}$. The hopping parameters $t_j$ ($j = 1, 2, 3, 4$) are given by $t_1 = 1 - e + 2d_x$, $t_2 = 1 + e + 2d_y$, $t_3 = 1 - e - 2d_x$, and $t_4 = 1 + e - 2d_y$, so that they describe the deforma-
Topological phase diagrams - The DPs that occur when $\nu_0 = 1$ are stable, existing in the finite regions of $d$-space $e \sin \left(\frac{\theta_d - \pi}{4}\right) \sin \left(\frac{\theta_d + \pi}{4}\right) < 0$. The trajectories of the DPs as a function of the staggered dimerization angle $\theta_d$ capture the topological phase transition of the 2DSWIs, driven by the dimerization. Figures 3(c) and 3(d) show the trajectories of the DPs from $\theta_d = -\pi/4$ to $\pi/4$ for $e = 0.25$ and from $\theta_d = \pi/4$ to $3\pi/4$ for $e = -0.25$, respectively. When $e > 0$, the DPs traverse the BZ from $\Gamma_1$ to $\Gamma$, winding the 2D BZ twice (once) along the $k_1$-($k_2$)-direction [Fig. 3(c)]. On the other hand, when $e < 0$, the DPs travel from $\Gamma_1$ to $\Gamma$, winding the BZ once only along the $k_2$-direction [Fig. 3(d)]. The BZ winding by DPs characterizes the 2DSWIs. The one-parameter families of real Hamiltonians on any $k_i$ line ($i = 1, 2$) defines a 1D topological insulator indexed by $\nu_i$. The weak $Z_2$ indices $(\nu_1, \nu_2)$ can change by changing all the topological insulators at any $k_i \in [-\pi, \pi]$. This necessitates the existence of the Dirac nodal line that winds the BZ during the phase transition.

1D flat-band DW states - We argue that the ZPA 1D boundary modes are a physical manifestation of the DPs that wind the BZ. ZPA have shown that the DWs of the 2D SSH chain host the one-dimensional flat bands [18], which we reproduce in Fig. 4. In view of the topological phase transition, the DW geometry [Fig. 4(a)], in which the dimerization parameter $d$ smoothly varies from 0 to $2\pi$ along $y$, seemingly visits the four electronic phases of the phase diagram [Fig. 3(a)]. Therefore, the topological phase transition should be captured in between the insulating domains via the occurrence of two DPs that wind the edge BZ $\Gamma_1 \in [-\pi, \pi]$ twice. Figure 4(b) shows the electronic energy spectrum calculated from the DW structure. In good agreement with the previous results [18, 22], our calculations reproduce the flat bands at $E = 0$ in otherwise all gapped bulk states. We find that there exist two mid-gap states per DW [See Fig. 4(b)], which cover the whole $E_d$ edge BZ twice. Associated with the topological phase transition of $\nu_1$, these DW states can be considered as the projection of the DP trajectories from $\theta_d = -\pi/4$ to $\pi/4$ along $k_2$. Since the the DW states...
wind the $X_1$ twice, the resulting change of $\nu_1$ should be zero, which is in line with the adjacent 2DSWI phases $(\nu_0; \nu_1 \nu_2) = (0; 11)$ and $0; 10)$. This supports that the 1D DW states are a physical manifestation of the DPs that wind the BZ. We note that the degeneracy of the zero modes are stabilized by the crystalline symmetry of the DW geometry as discussed by ZPA [18].

It is interesting to note that the trajectories of the DPs and, thus, the 1D DW states can be viewed as a 1D projection of a Dirac nodal line that lives in higher three dimensions. A 3D BZ can be constructed by adding additional dimension from the polar coordinate $\theta_d \in [-\pi, \pi]$ to the 2D BZ. Viewed from the three dimensions ($k_1, k_2, \theta_d$), one can consider one-parameter families of the Hamiltonian $\mathcal{H}(k_1, k_2, \theta_d)$ in class AI [57], since satisfy $[\mathcal{H}(k_1, k_2, \theta_d), \mathcal{P}\mathcal{T}] = 0$ and $(\mathcal{P}\mathcal{T})^{-1}\theta_d\mathcal{P}\mathcal{T} = \theta_d$. Therefore, the strong index $(\nu_0 = 1)$ is well-defined, dictating the presence of the Dirac nodal line that threads the 2D BZ at $\theta_d = 0$ ($\theta_d = \pi/2$) for $e > 0$ ($e < 0$). This idea may help access higher-dimensional band topology realized in a globally smooth spatially varying geometry of lower-dimensional materials.

**Material realizations** - Finally, we use first-principles calculations based on density functional theory (DFT) to predict the materials that realize the first 2DSWI phase [58]. Using the suggested indices indicated by parity eigenvalues, eleven known materials are identified as the first 2DSWI [59]. As a representative example, here we demonstrate the 2DSWI in Zn$_2$(PS$_3$)$_3$. Figure 5(a) shows the monoclinic atomic structure of Zn$_2$(PS$_3$)$_3$ in space group C2/m ($\#12$), which includes most importantly inversion symmetry. Zn$_2$(PS$_3$)$_3$ has a well-defined band gap throughout the entire BZ as shown in Fig. 5. Evaluating the parity eigenvalues, we find the weak indices $(\nu_1 \nu_2) = (11)$ for the delineated atomic structure. The nontrivial indices dictate the presence of the edge states along the $k_1 + k_2$ direction, confirmed by our DFT calculations in Fig. 5(c). Our DFT first 2DSWI belongs to an important class of materials, realized in realistic materials with intriguing physical manifestation as nearly-flat edge states. Furthermore, our first-principles calculations show that the parity eigenvalues that we suggest in the present study serve as a symmetry indicator to discern the 2DSWI, potentially leading to further discovery of material realizations and the experimental observations.

**Conclusion** - In summary, we have demonstrated that the combination of inversion and time-reversal symmetries allows for the $\mathbb{Z}_2$ classification of topological insulators under vanishing spin-orbit interactions. The proposed topological insulators are characterized by the strong $\mathbb{Z}_2$ index $\nu_0 = 0$ and the weak $\mathbb{Z}_2$ indices $(\nu_1 \nu_2) \neq (00)$, indicated by parity eigenvalues at four TRIMs. Diagnosing with the topological indices, we have shed light on the centrosymmetric 2D coupled SSH chains based...
on the ZPA model as a material realization of the proposed topological insulators, dubbed the two-dimensional Stiefel-Whitney insulators (2DSWIs). The 2DSWI features 1D topological boundary modes as a physical manifestation of their characteristic topological phase transitions. The proposed inversion-symmetry indicators can help identify the 2DSWIs in solid-state materials. Hopefully, our results simulate further experimental and theoretical studies and lead to the discover a solid-state platform that realizes topological flat bands.

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