Massive “spin-2” theories in arbitrary $D \geq 3$ dimensions

D. Dalmazi, A.L.R. dos Santos, E.L. Mendonça

UNESP - Campus de Guaratinguetá - DFQ
Avenida Dr. Ariberto Pereira da Cunha, 333
CEP 12516-410 - Guaratinguetá - SP - Brazil.

August 29, 2014

Abstract

Here we show that in arbitrary dimensions $D \geq 3$ there are two families of second order Lagrangians describing massive “spin-2” particles via a nonsymmetric rank-2 tensor. They differ from the usual Fierz-Pauli theory in general. At zero mass one of the families is Weyl invariant. Such massless theory has no particle content in $D = 3$ and gives rise, via master action, to a dual higher order (in derivatives) description of massive spin-2 particles in $D = 3$ where both the second and the fourth order terms are Weyl invariant, contrary to the linearized New Massive Gravity. However, only the fourth order term is invariant under arbitrary antisymmetric shifts. Consequently, the antisymmetric part of the tensor $e_{[\mu \nu]}$ propagates at large momentum as $1/p^2$ instead of $1/p^4$. So, the same kind of obstacle for the renormalizability of the New Massive Gravity reappears in this nonsymmetric higher order description of massive spin-2 particles.
1 Introduction

In the last years there has been a growing interest in massive gravity theories, see the review works [1, 2] and references therein. From the theoretical point of view, much of the recent interest in the subject has been triggered by the ghost free model of [3, 4], see also [5], both in $D = 4$. Recent developments in $D = 3$ on the other hand, were partially motivated by the New Massive Gravity (NMG) of [6]. All those works have the Fierz-Pauli (FP) model [7] of massive spin-2 particles as underlying free theory. The model of [3, 4] reduces to [7] at quadratic (free) level. Although this is not true for the NMG, the linearized version of NMG is closely connected with the FP theory, see [6] and also [8].

It is desirable to look for alternative non Fierz-Pauli (nFP) descriptions of massive spin-2 particles and their possible nonlinear extensions, in order to verify which physical features of recent massive gravities are model independent. The basic field in the FP theory is a symmetric rank-2 tensor $e_{\mu\nu} = e_{(\mu\nu)}$. Recently, other second order models for massive spin-2 particles in $D = 4$ have been suggested [9, 10, 11, 12]. They make use of a general rank-2 tensor with symmetric and antisymmetric parts $e_{\mu\nu} = e_{(\mu\nu)} + e_{[\mu\nu]}$. In [11, 12] we have found two families of Lagrangians $\mathcal{L}(a_1)$ and $\mathcal{L}_{n\text{FP}}(c)$ in $D = 4$ which differ from the usual FP theory nontrivially, in the sense that, they can not be brought to the FP form by any local field redefinition. They depend on the arbitrary real parameters $a_1$ and $c$ respectively.

In the next section we generalize the models $\mathcal{L}(a_1)$ and $\mathcal{L}_{n\text{FP}}(c)$ as obtained in [12] in $D = 4$ to arbitrary dimensions $D \geq 3$. The generalization of $\mathcal{L}_{n\text{FP}}(c)$ is less trivial if compared to $\mathcal{L}(a_1)$. The resulting Lagrangian explicitly depends on the space-time dimension, see (12). At zero mass we show that $\mathcal{L}_{n\text{FP}}(c)$ contains the same particle content of the massless FP theory. In the special case of $D = 3$ this means that the massless theory has no particle content. We use this fact to deduce via master action a fourth order dual theory similar to the NMG. We show that there is still a mismatch of local symmetries between the second and fourth order terms just like in the NMG case although both are now Weyl invariant.

Regarding the family $\mathcal{L}(a_1)$ we study its massless limit which contains an additional scalar field. We derive a range, see (21), for the constant $a_1$ via unitarity of the massless theory.

In section 4 we draw our conclusions.

2 “Spin-2” particles in $D \geq 3$ dimensions

There are three families [11] of second order models in $D = 4$ which describe a massive spin-2 particle via a rank-2 tensor $e_{\mu\nu}$, one of them is the well known massive FP theory. Those families are independent models in the sense that they can not be interconnected by any local field redefinition. Moreover they have different local symmetries in their massless cases. However, it was demonstrated in [12] that they can be related with the help of a decoupled and non dynamic extra field, for this reason we call this kind of field a spectator. We start with the generalization of $\mathcal{L}_{n\text{FP}}[c]$ to arbitrary dimensions. An important ingredient in our spectator approach is the usual FP model whose coefficients are $D$ independent:

\footnote{Throughout this work we use $\eta_{\mu\nu} = \text{diag}(-,+,\cdots,+)$ and $e_{(\mu\nu)} = (e_{\mu\nu} + e_{\nu\mu})/2$, $e_{[\mu\nu]} = (e_{\mu\nu} - e_{\nu\mu})/2$.}
\[ \mathcal{L}_{FP}[e_{\alpha\beta}] = -\frac{1}{2} \partial^\mu e^{(\alpha\beta)} \partial_\mu e_{(\alpha\beta)} + \frac{1}{4} \partial^\mu e \partial_\mu e + \left[ \partial^\alpha e_{(\alpha\beta)} - \frac{1}{2} \partial_\beta e \right]^2 - \frac{m^2}{2} (e_{\mu\nu} \epsilon^{\mu\nu} - e^2). \] (1)

The \( m = 0 \) limit of (1) is the usual linearized Einstein-Hilbert Lagrangian and describes massless spin-2 particles, it is invariant under reparametrizations plus antisymmetric shifts \( \delta e_{\mu\nu} = \partial_\nu \xi_\mu + \Lambda_{\mu\nu} \), where \( \Lambda_{\mu\nu} = -\Lambda_{\nu\mu} \).

The non Fierz-Pauli (nFP) model suggested in [10] in \( D = 4 \) deserves such name due to a real parameter \( c \) in the mass term which is given now by the combination \( (e_{\mu\nu} \epsilon^{\mu\nu} + c e^2) \). The absence of ghosts in the free theory does not depend on the FP fine tuning, see [10], we have in \( D = 4 \):

\[ \mathcal{L}_{nFP}^{D=4}(c) = -\frac{1}{2} \partial^\mu e^{(\alpha\beta)} \partial_\mu e_{(\alpha\beta)} + \frac{1}{6} \partial^\mu e \partial_\mu e + \left[ \partial^\alpha e_{(\alpha\beta)} \right]^2 - \frac{1}{3} (\partial^\alpha e_{\alpha\beta})^2 - \frac{1}{3} \partial^\alpha e_{\alpha\beta} \partial^\beta e - \frac{m^2}{2} (e_{\mu\nu} \epsilon^{\mu\nu} + c e^2). \] (2)

Contrary to the Lagrangian (1), we are going to verify that the coefficients in the derivative terms of \( \mathcal{L}_{nFP}^{D}(c) \) depend on the spacetime dimension. In order to generalize to \( D \) dimensions we add a scalar spectator \( \varphi \) to the FP model following the same steps of [12],

\[ \mathcal{L}_b = \mathcal{L}_{FP}[h_{\alpha\beta}] - b m^2 \varphi^2 + h_{\alpha\beta} T^{\alpha\beta}, \] (3)

where \( \mathcal{L}_{FP}[h_{\alpha\beta}] \) is the usual (symmetric) massive FP theory given in (1) with the replacement by a symmetric tensor \( e_{\alpha\beta} \rightarrow h_{\alpha\beta} = h_{\beta\alpha} \). We have also added a symmetric external source \( T^{\alpha\beta} \). The additional decoupled mass term with arbitrary real constant \( b \) clearly does not change the particle content of the massive FP theory. After a generalized shift with arbitrary real constants \( s \) and \( t \):

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + s \eta_{\mu\nu} \varphi + t \partial_\mu \partial_\nu \varphi, \] (4)

the Lagrangian \( \mathcal{L}_b \) becomes

\[ \mathcal{L}_b = \mathcal{L}_{FP}[h_{\alpha\beta}] + \left[ s(D-2) - t m^2 \right] (\partial_\mu h \partial^\mu \varphi - \partial^\mu h_{\mu\nu} \partial_\nu \varphi) + \frac{m^2}{2} \left[ s^2 D(D-1) - 2b \right] \varphi^2 + \frac{s^2}{2} \left( (D-2)(D-1) - s t m^2 (D-1) \right) \partial^\mu \varphi \partial_\mu \varphi + s m^2 (D-1) \varphi h + h_{\alpha\beta} T^{\alpha\beta} + s \varphi T + t \varphi \partial_\mu \partial_\nu T^{\mu\nu}. \] (5)

By requiring that derivative couplings between \( \varphi \) and \( h \) vanish, we fix \( t = s(D-2)/m^2 \). Introducing an auxiliary vector field and integrating by parts we can rewrite the \( \partial_\mu \varphi \partial^\mu \varphi \) term in a first order form

\[ \mathcal{L}_b = \mathcal{L}_{FP}[h_{\alpha\beta}] + \frac{m^2 s^2}{2} [(D-1)(D-2)] A^\mu A_\mu + \frac{m^2}{2} \left[ s^2 D(D-1) - 2b \right] \varphi^2 + h_{\mu\nu} T^{\mu\nu} - s \varphi \left\{ [(D-1)(D-2)] s m \partial \cdot A - (D-1) m^2 h - T - \frac{(D-2)}{m^2} \partial_\mu \partial_\nu T^{\mu\nu} \right\}. \] (6)
Due to the specific form of the usual Fierz-Pauli mass term, it is possible to generate a Maxwell term by making another shift in $\mathcal{L}_b$ and using the identity

$$
\mathcal{L}_{FP}[h_{\mu\nu} + r(\partial_{\mu}A_{\nu} + \partial_{\nu}A_{\mu})] = \mathcal{L}_{FP}[h_{\mu\nu}] - \frac{m r^2}{2} F_{\mu\nu}^2(A) + 2 m^2 r A^\mu(\partial^\alpha h_{\alpha\mu} - \partial_\mu h).
$$

After the shift $h_{\mu\nu} \rightarrow h_{\mu\nu} + r(\partial_{\nu}A_{\mu} + \partial_{\mu}A_{\nu})$ in (3) we decouple $A_\mu$ and $\varphi$ by choosing $r = -s/(D - 2)/2m$. Next, we introduce an antisymmetric field $B_{\mu\nu}$ by rewriting the Maxwell term in a first order form, we end up with a master Lagrangian which now involves three extra fields ($\varphi, A_\mu, B_{\mu\nu}$) besides $h_{\mu\nu}$:

$$
\mathcal{L}_{M1} = \mathcal{L}_{FP}[h_{\mu\nu}] + \frac{m^2 s^2}{2} (D - 1)(D - 2)A^\mu A_\mu + \frac{m^2}{2} [s^2 D(D - 1) - 2b] \varphi^2 + (D - 2) m s A^\mu \left( \partial^\alpha B_{\alpha\mu} + \partial^\alpha h_{\alpha\mu} - \partial_\mu h - \frac{\partial^\alpha T_{\alpha\mu}}{m^2} \right) + \frac{m^2}{2} B_{\mu\nu}^2 + s \varphi \left[(D - 1) m^2 h + T + \frac{2}{m^2} \partial_\mu \partial_\nu T_{\mu\nu}\right] + h_{\mu\nu} T_{\mu\nu}.
$$

We can define the generating function

$$
Z_{M1}[T] = \int D h_{\mu\nu} D \varphi D A_\mu D B_{\mu\nu} e^{i \int d^D x \mathcal{L}_{M1}}.
$$

If we functionally integrate over the extra field $B_{\mu\nu}$ in (9) and reverse the shift in (7), integrate over $A_\mu$ and reverse the shift (4) we come back to (3). On the other hand, if we integrate over $\varphi$ and $A_\mu$ in first place we obtain the Lagrangian

$$
\mathcal{L}(s, b) = \mathcal{L}_{FP}[h_{\mu\nu}] + \frac{m^2}{2} B_{\mu\nu}^2 - \frac{(D - 2)}{2(D - 1)} \left( \partial^\alpha B_{\alpha\mu} + \partial^\alpha h_{\alpha\mu} - \partial_\mu h - \frac{\partial^\alpha T_{\alpha\mu}}{m^2} \right)^2 - \frac{s^2}{2 m^2 [s^2 D(D - 1) - 2b]} \left[(D - 1) m^2 h + T + \frac{(D - 2)}{m^2} \partial_\mu \partial_\nu T_{\mu\nu}\right]^2.
$$

Defining $e_{\mu\nu} = h_{\mu\nu} + B_{\mu\nu}$, the Lagrangian $\mathcal{L}(s, b)$ can be rewritten as

$$
\mathcal{L}(s, b) = \mathcal{L}_{nFP}^D(c) + h_{\mu\nu}^* T_{\mu\nu} + \mathcal{O}(T^2),
$$

where $\mathcal{O}(T^2)$ stands for quadratic terms in the source and:

$$
\mathcal{L}_{nFP}^D(c) = -\frac{1}{2} \partial_\mu e^{(\alpha \beta)} \partial^\mu e^{(\alpha \beta)} + \frac{1}{2(D - 1)} \partial_\mu e^{(\mu)} e - 2 \partial_\mu e^{(\mu)} + [\partial_\mu e^{(\mu)}]^2 - \frac{(D - 2)}{2(D - 1)} [\partial_\mu e^{(\mu)}]^2 - \frac{m^2}{2} (e_{\mu\nu} e^{\mu\nu} + c e^2).
$$

The linear term in the source in (10) defines the dual field:

$$
h_{\mu\nu}^* = e_{(\mu\nu)} - \frac{1 + c}{(D - 1)} \eta_{\mu\nu} e - \frac{(1 + c)(D - 2)}{m^2 (D - 1)} \partial_{\mu} \partial_{\nu} e - \frac{(D - 2)}{2 m^2 (D - 1)} (\partial_\mu \partial^\alpha e_{\alpha\nu} + \partial_\nu \partial^\alpha e_{\alpha\mu}),
$$

\footnote{We assume $s^2 D(D - 1) - 2b \neq 0$ in order to integrate over $\varphi$ in (8).}
The arbitrary parameter $c$ in the mass term of (12) is defined through

$$c = \frac{2b - s^2(D - 1)}{s^2 D(D - 1) - 2b}. \quad (14)$$

Since (3) and (11) stem from the same generating function (9) we conclude that $\mathcal{L}_{nFP}(c)$ and $\mathcal{L}_{FP}$ are dual to each other in the sense that there is an equivalence of correlation functions up to contact terms via dual map $h^*_{\mu\nu} \leftrightarrow h_{\mu\nu}$ i.e,

$$\langle h^*_{\mu_1\nu_1}(x_1)...h^*_{\mu_N\nu_N}(x_N) \rangle_{nFP}(c) = \langle h_{\mu_1\nu_1}(x_1)...h_{\mu_N\nu_N}(x_N) \rangle_{FP} + \text{contact terms}. \quad (15)$$

It can be shown that the correlation functions involving the antisymmetric tensor $B_{\mu\nu}$ in $\mathcal{L}_{nFP}(c)$ vanish identically up to contact terms. Notice also that (12) reduces to (2) in $D = 4$.

The equations of motion from (12) are given by:

$$\Box e^{(\mu\nu)} + \frac{1}{(D - 1)} \partial^\mu \partial^\nu e - \partial^\mu \partial^\lambda e^{(\mu\lambda)} - \partial^\nu \partial^\lambda e^{(\lambda\mu)} = \frac{\eta^{\mu\nu}}{(D - 1)} \left[ \Box e - \partial_\alpha \partial_\beta e^{\alpha\beta} \right] = m^2 (e^{\mu\nu} + c \eta^{\mu\nu} e) - \frac{(D - 2)}{(D - 1)} \partial^\mu \partial^\nu e^{\lambda\nu}. \quad (16)$$

Applying $\eta_{\mu\nu}$ and $\partial_\nu$ on the equation (16), we find $(cD + 1)e = 0$ and $\partial_\mu e^{\mu\nu} = -c \partial^\nu e$, respectively. First, let us assume that $c \neq -1/D$. In this case the traceless condition $e = 0$ arises naturally as well as $\partial_\mu e^{\mu\nu} = 0$. Then, if we apply $\partial_\mu$ on the equation (16), using the fact that $e_{\mu\nu}$ is traceless and transverse with respect to the first indice, we obtain $\partial_\mu e^{\nu\mu} = 0$.

Coming back with these results in (16) we can easily verify that $e_{[\mu\nu]} = 0$ and rewrite the equations of motion as a Klein-Gordon equation:

$$((\Box - m^2)e^{(\mu\nu)} = 0. \quad (17)$$

Therefore, we have a massive particle with $D(D - 1)/2 - 1$ degrees of freedom for arbitrary values of $c$. On the other hand if $c = -1/D$, then (12) becomes Weyl invariant, and we could fix the gauge $e = 0$ and check that the Fierz-Pauli conditions still remain satisfied. Anyway, the equations of motion describe a massive “spin-2” particle in $D$ dimensions.

It is straightforward to check that for an arbitrary value of $D$, the massless version of (12) is invariant under linearized reparametrizations plus Weyl transformations, i.e, $\delta e_{\mu\nu} = \partial_\nu \xi_\mu + \eta_{\mu\nu} \phi$. It is possible to rewrite the derivative terms in (12) in terms of the traceless tensor $e_{\mu\nu} - \eta_{\mu\nu} e/D$ but the explicit dependence on $D$ does not disappear.

We remark that if $b = 0$, the arbitrary parameter $s$ disappears from (10) and we end up with an off-shell traceless description of massive spin-2 particles $\mathcal{L}_{nFP}(c = -1/D)$, see (14), confirming that the arbitrariness of the $\mathcal{L}_{nFP}(c)$ family stems indeed from the arbitrary mass term in (3) and not from the arbitrariness in the shift (4).

Now, it is a good moment to introduce the third family of Lagrangians describing a massive spin-2 particle. This family depends on a real constant $a_1$, see [12] for a detailed discussion. It is given by:
\[
\mathcal{L}_{a_1} = -\frac{1}{2} \partial^\mu e^{(\alpha\beta)} \partial_\mu e_{(\alpha\beta)} + \left( a_1 + \frac{1}{4} \right) \partial^\mu e \left[ \partial_\mu e - 2 \partial^\alpha e_{(\alpha\mu)} \right] + \left[ \partial^\alpha e_{(\alpha\beta)} \right]^2 \\
+ \left( a_1 - \frac{1}{4} \right) \left( \partial^\mu e_{\alpha\beta} \right)^2 - \frac{m^2}{2} (e_\mu e^{\mu\nu} - e^2) .
\] (18)

It can be shown that the equations of motion of \( \mathcal{L}_{a_1} \) lead to the Fierz-Pauli conditions \( e_{[\mu\nu]} = 0, \partial^\mu e_{\mu\nu} = 0, e = 0 \) and the Klein-Gordon equation \( (\Box - m^2) e^{(\mu\nu)} = 0 \) for arbitrary \( D \). There are three special values of \( a_1 \). First, if \( a_1 = -1/4 \) the model (18) reduces to the one obtained in [9] via a different dualization procedure. Secondly, if \( a_1 = 1/4, \mathcal{L}_{a_1} \) reduces to the FP theory (1). In the third case \( a_1 = (3 - D)/[4(D - 1)] \), the Lagrangian (18) becomes \( \mathcal{L}_{nFP}(c = -1) \). So \( \mathcal{L}(a_1) \) intersects both previous families.

In its massless case \( \mathcal{L}_{m=0}^{a_1} \) is invariant only under linearized reparametrizations in general, \( \delta e_{\mu\nu} = \partial_\mu \xi_\nu \). In the special cases \( a_1 = 1/4 \) and \( a_1 = (3 - D)/[4(D - 1)] \) it is also invariant under \( \delta e_{\mu\nu} = \Lambda_{\mu\nu} \), where \( \Lambda_{\mu\nu} = -\Lambda_{\nu\mu} \), and \( \delta e_{\mu\nu} = \phi \eta_{\mu\nu} \) respectively.

In order to check the particle content of \( \mathcal{L}_{a_1}^{m=0} \) it is convenient to rewrite it with the help of a non-dynamical vector field \( v_\mu \) as follows

\[
\mathcal{L}_{a_1}^{m=0} = -\frac{1}{2} \partial^\mu e^{(\alpha\beta)} \partial_\mu e_{(\alpha\beta)} + \left( a_1 + \frac{1}{4} \right) \partial^\mu e \left[ \partial_\mu e - 2 \partial^\alpha e_{(\alpha\mu)} \right] + \left[ \partial^\alpha e_{(\alpha\beta)} \right]^2 \\
- \left( a_1 - \frac{1}{4} \right) \left[ v_\mu v^\mu - 2v_\mu (\partial_\lambda e^{(\lambda\mu)} + \partial_\lambda B^{\lambda\mu}) \right] ,
\] (19)

where we have used \( e_{\mu\nu} = e_{(\mu\nu)} + B_{\mu\nu} \) with \( B_{\mu\nu} = -B_{\nu\mu} \). In first place, if we integrate over \( v_\mu \) we go back to \( \mathcal{L}_{a_1} \). However if we functionally integrate over \( B_{\mu\nu} \) we have a constraint whose general solution is \( v_\mu = \partial_\mu \psi \), where \( \psi \) is an arbitrary scalar field. Substituting this result back in (19) and changing variables \( \psi = \phi - e \) where \( \phi \) is an arbitrary scalar field, we will find after a field redefinition \( e_{(\mu\nu)} \rightarrow \tilde{e}_{(\mu\nu)} = \tilde{e}_{(\mu\nu)} - 2 \frac{(a_1 - 1/4)}{(D - 2)} \eta_{\mu\nu} \phi \), the decoupled theory

\[
\mathcal{L}_{a_1}^{m=0} = \mathcal{L}_{FP}^{m=0} [\tilde{e}_{\alpha\beta}] - 2 \frac{(D - 1)}{(D - 2)} \left( a_1 - \frac{1}{4} \right) \left[ a_1 + \frac{(D - 3)}{4(D - 1)} \right] \partial_\mu \phi \partial^\mu \phi .
\] (20)

where \( \mathcal{L}_{FP}^{m=0} \) is given in (1) and corresponds to the linearized version of the Einstein-Hilbert theory. Therefore, one can see that the massless version of \( \mathcal{L}_{a_1} \) describes a massless spin 2 particle plus a massless scalar field which disappears at \( a_1 = 1/4 \) and \( a_1 = -(D - 3)/4(D - 1) \). The scalar-tensor theory \( \mathcal{L}_{a_1}^{m=0} \) is unitary if:

\[
a_1 \leq -\frac{(D - 3)}{4(D - 1)} ; \quad a_1 \geq \frac{1}{4}.
\] (21)

### 3 Fourth order massive spin-2 model in \( D = 3 \)

In this section we check the particle content of the massless version of (12) in arbitrary \( D \geq 3 \) and derive in \( D = 3 \) a fourth order spin-2 model similar to the linearized NMG of [6]. In order to verify the particle content of the massless version of (12), we introduce a non-dynamical vector field \( C_\mu \) and rewrite that Lagrangian as:
\[
\mathcal{L}_{nFP}^{D}(m = 0) = -\frac{1}{2} \partial_{\mu}e^{(\alpha\beta)}\partial^{\mu}e^{(\alpha\beta)} + \frac{1}{2(D-1)} \partial_{\mu}e\partial^{\mu}e + \partial_{\mu}e^{(\mu\nu)}\partial^{\lambda}e_{(\lambda\nu)} - \frac{1}{(D-1)} \partial_{\mu}e\partial_{\nu}e^{(\nu\mu)} + \frac{(D-2)}{(2D-2)} C_{\mu} \left[ C^{\mu} + 2\partial_{\nu}(e^{(\nu\mu)} + B^{\nu\mu}) \right].
\]

Integrating over \(C_{\mu}\) one can recover the original kinetic term of \(\mathcal{L}_{nFP}^{D}\). Otherwise, the functional integration over \(B_{\mu\nu}\) in the path integral gives us a constraint whose general solution is \(C_{\mu} = \partial_{\mu}\phi\), where \(\phi\) is an arbitrary scalar field. Putting this back in \(\mathcal{L}_{m=0}^{D}nFP\) and making the change of variables \(e^{(\mu\nu)} \rightarrow \tilde{e}^{(\mu\nu)} - (\tilde{e} + \phi)\eta_{\mu\nu}\) we get rid of \(\phi\) and obtain the linearized Einstein-Hilbert theory given in (11) with \(m = 0\). Namely,

\[
\mathcal{L}_{nFP}^{m=0} \leftrightarrow \mathcal{L}_{FP}^{m=0}.
\]

Thus \(\mathcal{L}_{nFP}^{m=0}\) describes a massless particle with \(D(D-3)/2\) degrees of freedom which corresponds to a massless spin-2 particle in \(D = 4\).

Since \(\mathcal{L}_{nFP}^{m=0}\) has no \(3\) degrees of freedom in \(D = 3\), it can be used as a “mixing term” to build up a master action \([16]\) and deduce a higher order dual description of massive spin-2 particles in \(D = 3\).

We suggest the following master action in \(D = 3\):

\[
S_{M} = S_{nFP}^{D=3}[e_{\mu\nu}] - S_{nFP}^{m=0}[e_{\mu\nu} - f_{\mu\nu}],
\]

where we have introduced another rank-2 tensor \(f_{\mu\nu}\). Let us introduce sources \(j_{\mu\nu}\) which will allow us to derive a dual map between correlation functions in the dual theories via the generating function:

\[
W[j] = \int \mathcal{D}e_{\mu\nu} \mathcal{D}f_{\mu\nu} \exp i \left( S_{M} + \int d^{3}x j_{\mu\nu}e^{\nu\mu} \right).
\]

First of all, it is straightforward to verify that if we make the shift \(f_{\mu\nu} \rightarrow f_{\mu\nu} + e_{\mu\nu}\) in (24) we decouple \(f_{\mu\nu}\) and we end up with the particle content of the massive action \(S_{nFP}^{D=3}[e]\) since \(S_{nFP}^{m=0}\) has no content at all. So (24) certainly describes a massive spin-2 particle in \(D = 3\).

On the other hand, if we do not realize any shift and integrate over the field \(e_{\mu\nu}\) we obtain the following higher order dual theory\(^3\) written in terms of a traceless nonsymmetric tensor \(\tilde{f}_{\mu\nu}\)

\[
\mathcal{L}_{\text{Weyl}} = -\mathcal{L}_{nFP}^{m=0}[\tilde{f}_{\mu\nu}] + \mathcal{L}_{K}[\tilde{f}_{\mu\nu}] + j_{\mu\nu}F_{\mu\nu}(\tilde{f}) + \mathcal{O}(j^{2}),
\]

where \(\tilde{f}_{\mu\nu} = f_{\mu\nu} - \eta_{\mu\nu} f/3\). The Lagrangian \(\mathcal{L}_{nFP}^{m=0}\) is given in (12) with \(D = 3\) and \(m = 0\), while \(\mathcal{L}_{K} = (R_{\mu\nu}^{2} - \frac{3}{8}R^{2})ff/(2m^{2})\) is the linearized version of the so called \(K\)-term of the New Massive Gravity of \([6]\) with \(g_{\mu\nu} = \eta_{\mu\nu} + f_{(\mu\nu)}\). The nonsymmetric tensor \(F_{\mu\nu}(\tilde{f})\) plays the role of a dual field as we explain below. It is given by the traceless combination

\(^3\)This is consistent with (23) since the Einstein-Hilbert theory propagates no degrees of freedom in \(D = 3\).

\(^4\)The theory \(\mathcal{L}_{W\text{eyl}}\) has been obtained before in \([18]\) via a dimensional reduction of the massless FP theory in \(D = 4\).
\[ F^{\mu\nu}(\tilde{f}) = \frac{1}{m^2} \left[ -\Box \tilde{f}^{(\mu\nu)} + \partial^\rho \partial_\lambda \tilde{f}^{(\lambda\mu)} + \frac{1}{2} \partial^\mu \partial_\lambda \tilde{f}^{\nu\lambda} - \frac{\eta^{\mu\nu}}{2} \partial_\alpha \partial_\beta \tilde{f}^{\alpha\beta} \right]. \]  

(27)

Dropping the sources, \( \mathcal{L}_{Weyl} \) can be written as

\[ \mathcal{L}_{Weyl} = \frac{\tilde{f}_{\mu\nu}(\Box - m^2)\tilde{f}^{(\mu\nu)}}{2 m^2} + \frac{\partial^\rho \tilde{f}_{\mu\nu}(\Box - m^2)\partial_\lambda \partial_\rho \tilde{f}^{(\mu\nu)}}{m^2} + \frac{\left( \partial_\mu \partial_\nu \tilde{f}^{(\mu\nu)} \right)^2}{4 m^2} + \frac{\left( \partial_\mu \tilde{f}^{(\mu\nu)} \right)^2}{4}. \]  

(28)

The Weyl symmetry is hidden in the definition of \( \tilde{f}_{\mu\nu} \). The Weyl theory is also invariant under transverse linearized reparametrizations \( \delta \tilde{f}_{\mu\nu} = \partial_\rho \zeta^\rho_{\mu\nu} \), with \( \partial_\rho \zeta^\rho_{\mu\nu} = 0 \). The antisymmetric components \( \tilde{f}_{[\mu\nu]} \) only appear in the last term of (28). The equations of motion \( K^{\mu\nu} = \delta S_{Weyl}/\delta \tilde{f}_{\mu\nu} = 0 \) at vanishing sources can be written as

\[ K^{\mu\nu} = \left[ (\Box - m^2) \tilde{f}_{\mu\nu} - \partial^\rho \partial_\alpha \tilde{f}^{(\rho\alpha\nu)} - \partial^\rho \partial_\alpha \tilde{f}^{(\rho\alpha\mu)} + \eta^{\mu\nu} \partial_\alpha \partial_\beta \tilde{f}^{\alpha\beta} \right] - \frac{1}{2} \partial^\rho \partial_\alpha \tilde{f}^{\alpha\nu} = 0. \]  

(29)

From the antisymmetric components \( K^{[\mu\nu]} = 0 \) we have

\[ \partial_\alpha \tilde{f}^{\alpha\nu} = \partial^\rho \Phi, \]  

(30)

where \( \Phi \) is some scalar field. Using such information, the equations of motion (29) can be written as a Klein-Gordon equation for the dual field (27)

\[ K^{\mu\nu} = (\Box - m^2) F^{\mu\nu} = 0. \]  

(31)

Due to (30) we have \( F^{[\mu\nu]} = 0 \). In summary, besides the Klein-Gordon equation, all Fierz-Pauli conditions are satisfied by \( F^{\mu\nu} \),

\[ \eta_{\mu\nu} F^{\mu\nu} = 0, \]  

(32)

\[ \partial_\mu F^{\mu\nu} = 0, \]  

(33)

\[ F^{[\mu\nu]} = 0. \]  

(34)

Thus, \( \mathcal{L}_{Weyl} \) correctly describes a massive “spin-2” particle in \( D = 3 \).

It is typical of dual theories that equations of motion on one side may turn into identities on the dual side. In the usual Fierz-Pauli theory written in terms of a symmetric tensor \( h_{\mu\nu} \), the traceless and transverse conditions are dynamic while \( h^{[\mu\nu]} = 0 \) is an identity. In the dual Weyl theory \( \eta_{\mu\nu} F^{\mu\nu} = 0 \) and \( \partial_\mu F^{\mu\nu} = 0 \) are identities which do not depend on (30) as one can check directly from (27). The other FP conditions \( F^{[\mu\nu]} = 0 \) follow from the equations of motion: \( \delta S_{Weyl}/\delta \tilde{f}_{\mu\nu} = 0 \).

One can go beyond the duality at classical level and obtain the quantum equivalence between correlation functions by deriving with respect to the source in (25) and (26) obtaining:
\langle e_{\mu_1\nu_1}(x_1)...e_{\mu_N\nu_N}(x_N)\rangle_{nFP} = \langle F_{\mu_1\nu_1}[\tilde{f}(x_1)]...F_{\mu_N\nu_N}[\tilde{f}(x_N)]\rangle_{Weyl} + \text{contact terms}, \quad (35)

where the contact terms appear due to the quadratic terms in the sources in (26). In conclusion we have the dual map below between $S_{nFP}$ and $S_{Weyl}$:

\[ e_{\mu\nu} \leftrightarrow F_{\mu\nu}(\tilde{f}) \] \quad (36)

Since the equations of motion in general are enforced at quantum level up to contact terms we can use (13), (15), (35) and the remark below (15) to establish the direct dual map between the massive FP theory and $S_{Weyl}$:

\[ \langle e_{\mu_1\nu_1}(x_1)...e_{\mu_N\nu_N}(x_N)\rangle_{FP} = \langle F_{\mu_1\nu_1}[\tilde{f}(x_1)]...F_{\mu_N\nu_N}[\tilde{f}(x_N)]\rangle_{Weyl} + \text{contact terms}, \quad (37) \]

Therefore, the equivalence between $S_{FP}$ and $S_{Weyl}$ holds true beyond the on shell demonstration of \[18\].

Notice that the identification (37) links gauge invariant quantities since $F_{\mu\nu}[\tilde{f}]$ is invariant under Weyl transformations and transverse linearized reparametrizations while there is no local symmetry in the FP theory.

In summary, $S_{Weyl}$ describes a massive spin-2 particle in $D = 3$ (helicities $+2$ and $-2$). Although of fourth-order in derivatives the theory is unitary, just like $S_{FP}$. Differently from NMG, both fourth and second order terms can be written in terms of a traceless tensor (Weyl symmetry). However, only the fourth order term is invariant under antisymmetric shifts $\delta \tilde{f}_{\mu\nu} = \Lambda_{\mu\nu} = -\Lambda_{\nu\mu}$. Consequently $\tilde{f}_{[\mu\nu]}$ is only present in the second order term.

## 4 Conclusion

Here we have shown that besides the paradigmatic Fierz-Pauli (FP) theory, there are other two families of second order (in derivatives) Lagrangians describing massive “spin-2” particles in arbitrary $D \geq 3$ dimensions. The new families require the use of a nonsymmetric second rank tensor ($e_{\mu\nu} \neq e_{\nu\mu}$). In particular, one of the families is called non Fierz-Pauli ($\mathcal{L}_{nFP}$) since the mass term does not need to fit in the usual FP form, see (12). We have shown that at zero mass $\mathcal{L}_{nFP}(m = 0)$ is equivalent to the massless FP theory. Therefore, $\mathcal{L}_{nFP}^{D=3}(m = 0)$ has no particle content. In a master action approach [16], Lagrangian terms with empty spectrum may be used to generate dual theories of higher order in derivatives. In particular, this is how one can generate the $D = 3$ new massive gravity (NMG) of [6] as a dual theory to the usual (second order) massive FP theory, see comments in [8].

We have shown here by means of a master action that we can start with $\mathcal{L}_{nFP}$ in $D = 3$ and arrive at a dual theory describing a massive spin-2 particle in $D = 3$ which contains a second order and a fourth order (in derivatives) term just like the linearized new massive gravity, see (26). The fourth order term is the same one of the NMG, the so called K-term.

\[5\] See the third footnote of [12].
The K-term is invariant under linearized reparametrizations, linearized Weyl transformations and antisymmetric shifts: $\delta e_{\mu \nu} = \partial_{\mu} \zeta_{\nu} + \partial_{\nu} \zeta_{\mu} + \eta_{\mu \nu} \phi + \Lambda_{\mu \nu}$, with $\Lambda_{\mu \nu} = -\Lambda_{\nu \mu}$. However, the second order term in (28) is not the linearized Einstein-Hilbert theory as in the NMG case, the new second order term is invariant only under $\delta e_{\mu \nu} = \partial_{\nu} \zeta_{\mu} + \eta_{\mu \nu} \phi$. The master action (24) has allowed us to prove that (26) is off-shell equivalent to the massive FP theory. The correlation functions of $e_{\mu \nu}(x)$ in the massive nFP theory are mapped into correlation functions of $F_{\mu \nu}(f)$, given in (36), in the dual theory (26) up to contact terms.

The lack of Weyl symmetry in the second order term (linearized Einstein-Hilbert) is a key obstacle for the renormalizability of NMG [15, 19] since there will be a scalar degree of freedom [15] which will be present only in the second order term. Contrary to the linearized NMG, both terms of (26) are invariant under Weyl transformations. Unfortunately, it turns out that also in the case of (26), the second order term contains more degrees of freedom than the fourth order one. The antisymmetric components $e_{[\mu \nu]}$ are not present in the K-term which is invariant under antisymmetric shifts. Consequently the propagator $\langle e_{[\mu \nu]}(p) e_{[\alpha \beta]}(-p) \rangle$ behaves like $1/p^2$ for large momentum. So even if we were able to find a nonlinear completion of the model (26), the renormalizability of such model would be jeopardized. It seems impossible, using a rank-2 tensor, to formulate a ghost free massive spin-2 model of higher order in derivatives where all degrees of freedom are present in the highest order term. This might be a signal that there is no renormalizable massive gravity even in $D = 3$.

In section 2 we have shown that the second family of models $\mathcal{L}(a_1)$ found in [11], just like the usual Fierz-Pauli theory, has the same form in arbitrary dimensions $D \geq 3$. At zero mass $\mathcal{L}(a_1)$ contains an additional massless scalar particle besides the expected massless spin-2 particle unless $a_1 = 1/4$ or $a_1 = -(D-3)/(4(D-1))$ where $\mathcal{L}(a_1)$ reduces to the $\mathcal{L}_{FP}$ and $\mathcal{L}_{nFP}(c = -1)$ respectively. The unitarity bounds on the propagation of the massless scalar particle depends upon the space-time dimension, see (21).

5 Acknowledgements

The work of D.D. is supported by CNPq (307278/2013-1) and FAPESP (2013/00653-4) while A.L.R.S. is supported by Capes. We thank Gabriel B. de Gracia for a discussion.

References

[1] K. Hinterbichler, Rev.Mod.Phys. 84 (2012) 671-710, see also arXiv:1105.3735.
[2] C. de Rham, “Massive Gravity”, arXiv:1401.4173 (2014).
[3] C.de Rham, G. Gabadadze, Phys.Rev. D82 (2010) 044020
[4] C.de Rham, G. Gabadadze and A. J. Tolley, Phys.Rev.Lett. 106 (2011) 231101.
[5] S. Hassan and R.A. Rosen, Phys. Rev. Lett. 108 (2012) 041101.

6The case $a_1 = -1/4$ has been found before in [9]
[6] E. Bergshoeff, O. Hohm and P.K. Townsend, Phys.Rev.Lett. 102:201301, 2009.

[7] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173 (1939) 211.

[8] D. Dalmazi and E.L. Mendonça, JHEP 0909 (2009) 011.

[9] K. Morand and S. N. Solodukhin, Phys.Lett. B715 (2012) 260-266.

[10] D. Dalmazi, Phys.Rev. D86 (2012) 125036.

[11] D. Dalmazi, Phys. Rev. D87 (2013) 125027.

[12] D. Dalmazi, Phys. Rev. D88 (2013) 045003.

[13] H. Casini, R. Montemayor, and L.F. Urrutia, Phys. Rev. D68, 065011 (2003).

[14] D. Dalmazi, Phys.Rev. D80 (2009) 085008.

[15] S. Deser, Phys.Rev.Lett. 103 (2009) 101302.

[16] S. Deser, R. Jackiw, Phys. Lett. B139 (1984) 371.

[17] D. Dalmazi, A.L.R. dos Santos, E.L. Mendonça, “Massive spin-2 particles via embedment of the Fierz-Pauli equations of motions”, arXiv:1405.5546 (2014).

[18] E. Joung, K. Mkrtchyan, JHEP 1302 (2013) 134.

[19] K. Muneyuki and N. Ohta, Phys.Rev. D85 (2012) 101501.