Bound of Entanglement of Assistance and Monogamy Constraints

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We investigate the entanglement of assistance which quantifies capabilities of producing pure bipartite entangled states from a pure tripartite state. The lower bound and upper bound of entanglement of assistance are obtained. In the light of the upper bound, monogamy constraints are proved for arbitrary n-qubit states.

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I. INTRODUCTION

In quantum information theory, entanglement is a vital resource for some practical applications such as quantum cryptography, quantum teleportation and quantum computation [1, 2]. During the last decade, this inspired a great deal of effort for detecting and quantifying the entanglement [3, 4, 5, 6, 7, 8, 9, 10, 11]. On the other hand, the creation and distribution of entanglement is also of central interest in quantum information processing. More specially the distribution of bipartite entanglement is a key ingredient for performing certain quantum-information processing tasks such as teleportation.

One of the methods for generating bipartite entanglement is the entanglement of assistance that is defined in Refs. [12, 13]. It quantifies the entanglement which could be created by reducing a multipartite entangled state to an entangled state with fewer parties (e.g. bipartite) via measurements. Such producing of entanglement, also called “assisted entanglement”, is a special case of the localizable entanglement [14], which is especially important for quantum communication, where quantum repeaters are needed to establish bipartite entanglement over a long length scale [15]. For a pure \( d_2 \otimes d_2 \otimes n \) state, the analytical formula of entanglement of assistance has been derived by Laustsen et al. [16], whereas the calculation of entanglement of assistance is not easy for a general pure tripartite state [17].

In this paper, we explore the entanglement of assistance for a general pure tripartite state in terms of 1-concurrence [18]. We obtain a lower bound of entanglement of assistance, which is also the lower bound of a tripartite entanglement measure, the entanglement of collaboration. This may help to characterize the localizable entanglement. Furthermore, an upper bound is also obtained. Deducing from the upper bound of entanglement of assistance, we find a proper form of entanglement monogamy inequality for arbitrary N-qubit states, which is analogous to the monogamy constraints for concurrence proposed by Coffman et al. [19] and proven by Osborne et al. [20] for the general case.

The paper is organized as follows: In Sec. II, we derive a lower bound and upper bound of entanglement of assistance for pure tripartite states. In Sec. III, monogamy constraints are proved in terms of this upper bound. Finally in Sec. IV we conclude with a discussion of our results.

II. BOUND OF ENTANGLEMENT OF ASSISTANCE

We consider a pure \( (d_1 \times d_2 \times N) \) tripartite state shared by three parties referred to as Alice, Bob and Charlie, who performs a measurement on his party to yield a known bipartite entangled state shared by Alice and Bob. Charlie’s aim is to maximize the entanglement of the state between Alice and Bob. This maximum average entanglement that he can create is called entanglement of assistance, which was originally defined in terms of entropy of entanglement [12, 13]. In this paper, we define entanglement of assistance in terms of the entanglement measure 1-concurrence:

\[
E_a(\ket{\psi}_{ABC}) = E_a(\rho_{AB}) \equiv \max_i p_i C(\ket{\phi_i}_{AB}),
\]

which is maximized over all possible pure-state decompositions of \( \rho_{AB} = \text{Tr}_C(\ket{\psi}_{ABC}) = \sum_i p_i \ket{\phi_i}_{AB} \bra{\phi_i} \). By applying the method in Ref. [4], we can obtain the lower bound of entanglement of assistance for pure tripartite states.

For any given pure-state decomposition of \( \rho_{AB} = \sum_i p_i \ket{\phi_i}_{AB} \bra{\phi_i} \), we have

\[
E_a(\ket{\psi}_{ABC}) = \max \sum_i p_i C(\ket{\phi_i}_{AB})
= \max \sum_i p_i \sqrt{\sum_m |\bra{\phi_i} S_{mn} \phi_i^*\rangle|^2}
\geq \max \sqrt{\sum_m \sum_i p_i |\bra{\phi_i} S_{mn} \phi_i^*\rangle|^2},
\]

where \( S_{mn} = L_m \otimes L_n, L_m, m = 1, ..., d_1(d_1 - 1)/2, L_n, n = 1, ..., d_2(d_2 - 1)/2 \) are the generators of group \( SO(d_1) \) and \( SO(d_2) \) respectively. The inequality holds.
according to the Minkowski inequality \[ \left( \sum_{i=1}^{k} (x_i^p) \right)^{1/p} \leq \sum_{i=1}^{k} (x_i^p)^{1/p}, \quad p > 1. \]

Consider the eigenvalue decomposition of \( \rho_{AB} \), \( \rho_{AB} = \Psi M \Psi^\dagger \), where \( M \) is a diagonal matrix whose diagonal elements are the eigenvalues of \( \rho \), and \( \Psi \) is a unitary matrix whose columns are the eigenvectors of \( \rho \). Taking into account the relation \( \Phi W^{1/2} = \Psi M^{1/2} U \), where \( U \) is a right-unitary matrix, we can rewrite inequality (11) as

\[
E_a(\rho_{AB}) \geq \max \sum_{mn} \left| \sum_i \left| U^T W^{1/2} S_{mn} \Psi^\dagger \Phi_{ii} \right|^2 \right|
= \max \sum_{mn} \left| \sum_i \left| U^T M^{1/2} \Psi S_{mn} \Psi M^{1/2} U \right|^2 \right|
.
\]

In terms of the Cauchy-Schwarz inequality \( \sum_i (x_i^2) \geq \sum_i x_i y_i \), the inequality

\[
E_a(\rho_{AB}) \geq \max \sum_i \left| U^T \left( \sum_{mn} z_{mn} A_{mn} \right) U \right|_{ii} \tag{2}
\]

is implied for any \( z_{mn} = y_{mn} e^{i \theta_{mn}} \) with \( y_{mn} \geq 0 \) and \( \sum_{mn} y_{mn}^2 = 1 \), where \( A_{mn} = M^{1/2} \Psi S_{mn} \Psi M^{1/2} \). Since \( \sum_{mn} z_{mn} A_{mn} \) is a symmetric matrix, we can always find a unitary matrix \( U \) such that \( \sum_i \left| U^T \sum_{mn} z_{mn} A_{mn} U \right|_{ii} = \| \sum_{mn} z_{mn} A_{mn} \| \) as shown in Ref. [21], where \( \| \cdot \| \) stands for the trace norm defined by \( \| G \| = \text{Tr}(G G^\dagger)^{1/2} \). For an arbitrary unitary matrix \( V \), we have

\[
\sum_i \left| W^T \left( \sum_{mn} z_{mn} A_{mn} V \right) V \right|_{ii} = \sum_i \left| W^T (U^{-1})^T U^T \sum_{mn} z_{mn} A_{mn} U U^{-1} V \right|_{ii} = \sum_i \left| (U^{-1} V)_{ij} \right|^2 \lambda_i \leq \sum_i \lambda_i,
\]

where \( \lambda_i(z) \) is dependent on the choice of the \( y \) and \( \theta \), are the singular values of the matrix \( T = \sum_{mn} z_{mn} A_{mn} \), i.e., the square roots of the eigenvalues of the positive Hermitian matrix \( T T^\dagger \). Therefore the maximum of Eq. (2) is given by \( \max_{z \in \mathbb{C}} \left( \sum_i \lambda_i(z) \right) = \max_{z \in \mathbb{C}} \left( \sum_{mn} z_{mn} A_{mn} \right) \). Hence, we arrive at the lower bound of entanglement of assistance for a pure tripartite state as following:

\[
E_a(\rho_{AB}) \geq \max_{z \in \mathbb{C}} \left\| \sum_{mn} z_{mn} A_{mn} \right\| \tag{3}
.
\]

Furthermore the entanglement of collaboration \( \rho_{AB} \) quantifies the maximum amount of entanglement that can be generated between two parties from a tripartite state with collaborations composed of local operations and classical communication among the three parties. It has been shown by Gour et al. [22] that, for tripartite states, the entanglement of collaboration is greater than or equal to entanglement of assistance in terms of a given entanglement measure. Therefore our lower bound is also the one for entanglement of collaboration, which can be tightened by numerical optimization. Our bound may help to characterize localizable entanglement. For a pure 2 \( \times 2 \times N \) state, this lower bound is consistent with the result of Ref. [16].

We can also obtain the upper bound of entanglement of assistance. From the definition of entanglement of assistance, we have

\[
|E_a(\rho_{AB})|^2 = \left[ \max_i \sum p_i C(|\phi_i^a_{AB}) \right]^2 \leq \max_i \left[ \sum (\sqrt{p_i} C(|\phi_i^a_{AB}) \right]^2 \left\| C(|\phi_i^a_{AB}) \right\|^2 = \max_i 2 p_i [1 - \text{Tr}(\rho_A^2)] \leq 2 (1 - \text{Tr}(\rho_A^2)),
\]

where \( \rho_A^2 = \text{Tr}_B(|\phi_A \rangle \langle \phi_A|) \). The first inequality holds according to the Cauchy-Schwarz inequality [24]; the last one, which has also been proved in Ref. [25], holds due to the convex property of \( \text{Tr}(\rho_A^2) \).

Define the upper bound as the tangle of assistance \( \tau_a(\rho_{AB}) \equiv \max \sum_i p_i C(|\phi_i^a_{AB}) \right|^2 \) of assistance that satisfies the monogamy constraints for n-qubit pure state \( \rho_{AB} \). Similar to the entanglement of assistance that satisfies the monogamy constraints for arbitrary n-qubit states [26, 27], we show below that the tangle of assistance also exhibits monogamy constraints for arbitrary n-qubit states.

### III. MONOGAMY INEQUALITY

Consider a pure tripartite state \( |\Psi\rangle_{ABC} \). The tangle of assistance is defined by

\[
\tau_a(|\Psi\rangle_{ABC}) = \max_{\{p_x, |\psi_x\rangle\}} \sum_x p_x [C(|\phi_x\rangle)]^2 = \max_{\{p_x, |\psi_x\rangle\}} \sum_x p_x S_2[\text{Tr}_B(|\psi_x\rangle \langle \psi_x|)],
\]

where the linear entropy \( S_2[\rho] = 2 [1 - \text{Tr}(\rho^2)] \), and the maximum runs over all pure-state decompositions \( \{p_x, |\psi_x\rangle\} \) of \( \rho_{AB} = \text{Tr}_C(|\Psi\rangle_{ABC} \langle \Psi|) = \sum_x p_x |\psi_x\rangle \langle \psi_x| \). In the case of pure state \( \rho_{AB} \), the tangle of assistance is the square of concurrence of this state.

**Theorem 1** For an arbitrary n-qubit state, the tangle of assistance satisfies,

\[
\tau_a(\rho_{A_1A_2}) + \tau_a(\rho_{A_1A_3}) + \cdots + \tau_a(\rho_{A_1A_n}) \geq \tau_a(\rho_{A_1A_2A_3 \cdots A_n}),
\]

where \( \tau_a(\rho_{A_1A_2A_3 \cdots A_n}) \) denotes the tangle of assistance in the bipartite partition \( A_1 | A_2 A_3 \cdots A_n \).
Proof: First of all, we prove the following inequality
\[ \tau_a(\rho_{AB}) + \tau_a(\rho_{AC}) \geq \tau_a(\rho_{ABC}), \]  
for arbitrary tripartite states $\rho_{ABC}$ in $2 \times 2 \times 2^{n-2}$ system. We first prove Eq. (5) for pure states. In this case, due to the local-unitary invariance of $\tau_a(\rho_{AC})$, we can rotate the basis of subsystem $C$ into the local Schmidt basis $|V_k\rangle$, $k = 1, \ldots, 4$, given by the eigenvectors of $\rho_C = Tr_{\bar{A}B}(\rho_{ABC})$. In this way we can regard the $2^{n-2}$-dimensional qudit $C$ as an effective four-dimensional qudit. Therefore, we simply need to prove Eq. (5) for a $2 \times 2 \times 4$ pure state $ABC$.

For pure states of a tripartite system $ABC$ of two qubits $A$ and $B$ and a four-level system $C$, we have
\[ \tau_a(\rho_{ABC}) = \max_{\{M_x\}} \sum_x p_x S_2(\rho_x), \]  
where the maximum runs over all rank-1 POVMs on Bob’s system, $p_x = Tr(I_A \otimes M_x \rho_{AB})$ is the probability of outcome $x$, and $\rho_x = Tr_B(I_A \otimes M_x \rho_{AB})/p_x$ is the posterior state in Alice’s subsystem. For convenience, we take the definition
\[ I(\rho_{AB}) := S_2(\rho_A) - \max_{\{M_x\}} \sum_x p_x S_2(\rho_x). \]  
By comparing $I(\rho_{AB})$ with Eq. (5) for pure tripartite states, we see that it is sufficient to prove the inequality
\[ I(\rho_{AB}) \leq \tau_a(\rho_{AB}), \]  
for all two-qubit states $\rho_{AB}$.

We first derive a computable formula for $I(\rho_{AB})$. Any bipartite quantum state $\rho_{AB}$ may be written as
\[ \rho_{AB} = \Lambda \otimes I_B(|V_{B'}\rangle \langle V_{B'}|), \]  
where $V_{B'B}$ is the symmetric two-qubit purification of the reduced density operator $\rho_B$ on an auxiliary qubit system $B'$ and $\Lambda$ is a qubit channel from $B'$ to $A$. Deducing from Eq. (6) we have
\[ \rho_x = Tr_B(I_A \otimes M_x |V_{B'}\rangle \langle V_{B'}|)/p_x = Tr_B[(I_A \otimes M_x)(\Lambda \otimes I_B)|V_{B'}\rangle \langle V_{B'}|]/p_x = \Lambda[Tr_B(I_A \otimes M_x |V_{B'}\rangle \langle V_{B'}|)]/p_x. \]  
Since the rank of $M_x$ is 1, $Tr_B(I_A \otimes M_x |V_{B'}\rangle \langle V_{B'}|)$ is a pure state. Moreover, all pure-state decompositions of $\rho_B' = Tr_B(V_{B'}|V_{B'}\rangle \langle V_{B'}|)$ can be realized by the rank-1 POVM measurements $\{M_x\}$ operating on subsystem $B$ of $|V_{B'}\rangle \langle V_{B'}|$. Hence $I(\rho_{AB})$ satisfies
\[ I(\rho_{AB}) = S_2[\Lambda(\rho_B)] - \max_{\{p_x, |\psi_x\rangle\}} \sum_x p_x S_2[\Lambda(|\psi_x\rangle)], \]  
where the maximum runs over all pure-state decompositions $\{p_x, |\psi_x\rangle\}$ of $\rho_B$ such that $\sum_x p_x |\psi_x\rangle \langle \psi_x| = \rho_B$. The action of a qubit channel $\Lambda$ on a single-qubit state $\rho = (I + r \cdot \sigma)/2$, where $\sigma$ is the vector of Pauli operators, may be written as $\Lambda(\rho) = (I + Lr + 1 \cdot \sigma)/2$, where $L$ is a $3 \times 3$ real matrix and 1 is a three-dimensional vector. In this Pauli basis, the possible pure-state decompositions of $\rho_B$ are represented by all possible sets of probabilities $\{p_x\}$ and unit vectors $\{r_j\}$ such that $\sum_j p_j r_j = r_B$, where $(I + r_B \cdot \sigma)/2 = \rho_B$. In terms of the Bloch representation of one-qubit states, the linear entropy $S_2$ is given by $S_2[(I + r \cdot \sigma)/2] = 1 - |r|^2$. In this way we get the following equation $S_2[\Lambda(I + r \cdot \sigma)/2] = 1 - (Lr + 1)^T (Lr + 1)$. Substituting $r_j = r_B + x_j$, one can easily check that Eq. (8) reduces to the following one whose value is determined by $\{p_j, x_j\}$ subject to the conditions $\sum_j p_j x_j = 0$ and $|r_B + x_j| = 1$,
\[ I(\rho_{AB}) = S_2[\Lambda(\rho_B)] - \max_{\{p_x, |\psi_x\rangle\}} \sum_x p_x S_2[\Lambda(|\psi_x\rangle)] = \max_{\{p_x, |\psi_x\rangle\}} \sum_j p_j \left(1 - |L(r_B + x_j) + 1|^2 \right) \lambda_j. \]  
Without loss of generality, we assume that $L^T L$ is diagonal with diagonal elements $\lambda_j \leq \lambda_2 \leq \lambda_2$. The constraints $|r_B + x_j| = 1$ lead to the identities $(x_j^T)^2 = 1 - |r_j|^2 - 2r_j x_j - (x_j^T)^2$. Substituting this into Eq. (9), we get $I(\rho_{AB}) = \lambda_2 - (1 - |r_j|^2) \min_{\{p_x, |\psi_x\rangle\}} \sum_j p_j (|\lambda_j - \lambda_2| (x_j^T)^2 + (\lambda_j - \lambda_2)(x_j^T) + 2 \lambda_j - \lambda_2 \lambda_j)$. This expression is obviously minimized by choosing $x_j^T = 0$ for all $j$. Then from the condition $|r_B + x_j| = 1$, $x_j^T$ have two solutions. The ensemble of two states corresponding to such two solutions can reach the minimum $\lambda_2(1 - |r_B|^2)$. As $S_2(\rho_B) = (1 - |r_B|^2)$, we obtain the following computable expression: $I(\rho_{AB}) = \lambda_{\min} S_2(\rho_B)$. Note that a local filtering operation of the form $\rho_{AB}' = (I \otimes B) \rho_{AB} (I \otimes B')$ leaves $L$ invariant and transforms $S_2(\rho_B) = \frac{det(B)^2}{\det[I \otimes B'] \rho_{AB} (I \otimes B')}$.

If the local filtering operator $B$ is invertible, we can get the conclusion that there does not exist a pure-state decomposition $\{q_j, |\psi_j\rangle\}$ of $\rho_{AB}'$ such that $\tau_a(\rho_{AB}') > \frac{det(B)^2}{\det[I \otimes B'] \rho_{AB} (I \otimes B')} \tau_a(\rho_{AB})$ by the contradiction. For the case that the operator $B$ is not invertible, such pure-state decomposition also doesn’t
exist. Furthermore, there exists exactly an optimal pure-state decomposition \( \{p_i, |\phi_i\rangle\} \) of the state \( \rho_{AB} \) for \( \tau_a(\rho_{AB}) \) such that

\[
\sum_p p_i C\left[ \frac{\text{Tr}[(I \otimes B)(I \otimes \phi_i) | B\rangle \langle B| \rho_{AB}]}{\text{Tr}[(I \otimes B)(I \otimes \phi_i) \rho_{AB}]} \right]^2 = \frac{\text{det}(B)^2}{\text{Tr}[(I \otimes B)(I \otimes \phi_i) \rho_{AB}]^2} \tau_a(\rho_{AB}).
\]

Therefore, the tangle of assistance \( \tau_a(\rho_{AB}) = \frac{\text{det}(B)^2}{\text{Tr}[(I \otimes B)(I \otimes \phi_i) \rho_{AB}]^2} \tau_a(\rho_{AB}) \). Since

\[
I(\rho_{AB}) = \frac{\text{det}(B)^2}{\text{Tr}[(I \otimes B)(I \otimes \phi_i) \rho_{AB}]^2} \lambda_{\text{min}} \text{S}_2(\rho_{B}),
\]

it transforms exactly in the same way as the tangle of assistance \( \tau_a(\rho_{AB}) \) does. As there always exists a filtering operation for which \( \rho_B \propto I \), we can assume, without loss of generality, that \( \text{S}_2(\rho_B) = 1 \).

So let us consider \( \rho_{AB} \) with \( \rho_B = \text{Tr}_A(\rho_{AB}) = \frac{1}{4} I \). In terms of Pauli operators, we can rewrite the pure state as follows:

\[
\frac{(I \otimes B)[V_B'B][V_B'[I \otimes B]}{\text{Tr}[(I \otimes B)^2][V_B'[B]}(\frac{1}{4} + \sum_i m_i I \otimes \sigma_i + \sum_i n_i \sigma_i \otimes I + \sum_{ij} O_{ij} \sigma_i \otimes \sigma_j),
\]

where \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are \( \sigma_x, \sigma_y \) and \( \sigma_z \) respectively. Then we get the conclusion from its purity and unity reduced density, that \( m_i = n_i = 0 \) for all \( i \) and the \( 3 \times 3 \) real matrix \( O \) is orthogonal. Thus we have \( \rho_{AB} = \frac{1}{4} \Lambda \otimes I_B[I + \sum m_i I \otimes \sigma_i \otimes \sigma_j] = \frac{1}{4} [I + \sum_i l_i \sigma_i \otimes I + \sum_{ij} (LO) \sigma_i \otimes \sigma_j] \).

As unitary operator \( U_1 \) satisfies the equation \( U_1 \sigma_1 U_1^\dagger = \sum P_i \sigma_i' \), where \( P \) is a real orthogonal \( 3 \times 3 \) matrix, we can always find local unitary operators, in terms of the theorem of singular value decomposition, so that \( U_1 \otimes U_2 \rho_{AB} U_2^\dagger U_2^\dagger = \frac{1}{4} [I + \sum_i (IP) \sigma_i \otimes I + \sum_{ij} (QLO) \sigma_i \otimes \sigma_j] \).

where \( Q \) and \( P \) are real orthogonal matrices and \( L' \) is a diagonal matrix with its diagonal elements the singular values of \( L \). Because of the local-unitary invariance of \( \tau_a(\rho_{AB}) \) and \( I(\rho_{AB}) \), without loss of generality, we assume that \( \rho_{AB} = \frac{1}{4} [I + \sum_i l_i \sigma_i \otimes I + \sum_i (R_i \sigma_i \otimes \sigma_i)] \), where \( R \) is a diagonal matrix with its diagonal elements the singular values of \( L \). Due to the positivity of \( \rho_{AB} \)

\[
\rho_{AB} = \frac{1}{4} \left( \begin{array}{cccc}
1 + R_3 + t_3 & 0 & t_1 - it_2 & R_1 - R_2 \\
0 & 1 - R_3 + t_3 & R_1 + R_2 & t_1 - it_2 \\
t_1 + it_2 & R_1 + R_2 & 1 - R_3 - t_3 & 0 \\
R_1 - R_2 & t_1 + it_2 & 0 & 1 + R_3 - t_3
\end{array} \right),
\]

the inequality \( 1 - t_1^2 - t_2^2 - 3t_3^2 \geq 2R_3^2 \) must hold. Therefore we obtain

\[
\tau_a(\rho_{AB}) \geq \text{Tr}[(I \otimes B) \sigma_a \rho_{AB} \sigma_a \otimes I_B \rho_{AB}]
\]

\[
= \frac{1}{16} \left[ 4 + 4(R_1^2 + R_2^2 + R_3^2) - 4(t_1^2 + t_2^2 + t_3^2) \right]
\]

\[
\geq \frac{1}{4} \left( R_1^2 + R_2^2 + 2R_3^2 \right)
\]

\[
\geq \lambda_{\text{min}}(L^T L).
\]

This inequalities imply that \( I(\rho_{AB}) \leq \tau_a(\rho_{AB}) \) for all two-qubit states \( \rho_{AB} \), which then proves Eq. (5) for pure states.

Now we extend Eq. (5) to mixed state case. Consider the maximizing pure-state decomposition \( \{p_x, |\psi_x\rangle\} \) for \( \tau_a(\rho_{A(BC)}) \). By applying the inequality Eq. (5) and taking into account the concavity of \( \tau_a \), we have

\[
\tau_a(\rho_{A(BC)}) = \sum_x p_x \tau_a(\rho_{A}^{x}(BC)) \leq \sum_x p_x [\tau_a(\rho_{A}^{x}(BC)) + \tau_a(\rho_{AC})] \leq \tau_a(\rho_{AB}) + \tau_a(\rho_{AC}) \],

where \( \rho_{A}^{x}(BC) = |\psi_x\rangle \langle \psi_x| \).

Let \( C = C_1C_2 \) be a \( 2 \times 2^{n-3} \) system and apply Eq. (5), then we get

\[
\tau_a(\rho_{A(BC)}) \leq \tau_a(\rho_{AB}) + \tau_a(\rho_{AC}) \]

\[
\leq \tau_a(\rho_{AB}) + \tau_a(\rho_{AC}) + \tau_a(\rho_{AC}).
\]

Successively applying Eq. (5) to partitions of \( C \), we obtain the inequality Eq. (1) by induction.

In fact, Eq. (1) turns out to be an equality for product states under partition \( A(BC_1 \cdots C_n) \). For the generalized GHZ states, Eq. (2) is a strictly inequality.

\[ \text{IV. DISCUSSION} \]

In summary, as an important quantity in quantum computation, the entanglement of assistance has been investigated in terms of I-concurrency for pure tripartite states. We have obtained a lower bound of entanglement of assistance, which is also the lower bound of the tripartite entanglement measure, the entanglement of collaboration. In stead of great difficulty involved in computing the entanglement of collaboration, the lower bound Eq. (3) can be calculated in a numerical optimization to make a good estimation of entanglement of collaboration. Moreover, an upper bound is also obtained. In the light of the upper bound of entanglement of assistance, we find a proper form of entanglement monogamy inequality for arbitrary N-qubit states.

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