A LOCAL DISCONTINUOUS GALERKIN APPROXIMATION FOR THE $p$-NAVIER–STOKES SYSTEM, PART III: CONVERGENCE RATES FOR THE PRESSURE

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Abstract. In the present paper, we prove convergence rates for the pressure of the Local Discontinuous Galerkin (LDG) approximation, proposed in Part I of the paper (cf. [14]), of systems of $p$-Navier–Stokes type and $p$-Stokes type with $p \in (2, \infty)$. The results are supported by numerical experiments.

Key words. discontinuous Galerkin, $p$-Navier–Stokes system, error bounds, pressure

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1. Introduction. In this paper, we continue our study of the Local Discontinuous Galerkin (LDG) scheme, proposed in Part I of the paper (cf. [14]), of steady systems of $p$-Navier–Stokes type. In this paper, as we already did in Part II of the paper (cf. [15]), we restrict ourselves to the homogeneous problem, i.e.,

$$
-\text{div} \mathcal{S}(Dv) + [\nabla v]v + \nabla q = g \quad \text{in } \Omega,
$$

$$
\text{div} v = 0 \quad \text{in } \Omega,
$$

$$
v = 0 \quad \text{on } \partial \Omega.
$$

(1.1)

This system describes the steady motion of a homogeneous, incompressible fluid with shear-dependent viscosity. More precisely, for a given vector field $g : \Omega \to \mathbb{R}^d$ describing external body forces and a homogeneous Dirichlet boundary condition (1.1)$_3$, we seek for a velocity vector field $v = (v_1, \ldots, v_d)^\top : \Omega \to \mathbb{R}^d$ and a scalar kinematic pressure $q : \Omega \to \mathbb{R}$ solving (1.1). Here, $\Omega \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded polyhedral domain having a Lipschitz continuous boundary $\partial \Omega$. The extra stress tensor $\mathcal{S}(Dv) : \Omega \to \mathbb{R}^{d \times d}$ depends on the strain rate tensor $Dv := \frac{1}{2}(\nabla v + \nabla v^\top) : \Omega \to \mathbb{R}^{d \times d}$, i.e., the symmetric part of the velocity tensor $L := \nabla v : \Omega \to \mathbb{R}^{d \times d}$. The convective term $[\nabla v]v : \Omega \to \mathbb{R}^d$ is defined via $([\nabla v]v)_i := \sum_{j=1}^d v_j \partial_j v_i$ for all $i = 1, \ldots, d$.

Throughout the paper, we assume that the extra stress tensor $\mathcal{S}$ has $(p, \delta)$-structure (cf. Assumption 2.1). The relevant example falling into this class is

$$
\mathcal{S}(Dv) = \mu (\delta + |Dv|)^{p-2}Dv,
$$

where $p \in (1, \infty)$, $\delta \geq 0$, and $\mu > 0$.

For a discussion of the model and the state of the art, we refer to Part I of the paper (cf. [14]). As already pointed out, to the best of the authors’ knowledge, there are no investigations using DG methods for the $p$-Navier–Stokes problem (1.1). In this paper, we continue the investigations of Part I and Part II of the paper (cf. [14, 15]), and prove convergence rates for the pressure of the homogeneous $p$-Navier–Stokes problem (1.1) under the assumption that the velocity and $g$ satisfy natural regularity conditions and a smallness condition for the velocity in the energy norm. In doing so, we

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restrict ourselves to the case \( p \in (2, \infty) \). Our approach is inspired by the results in [11], [13], and [3]. The same results are obtained for the \( p \)-Stokes problem without the smallness condition.

This paper is organized as follows: In Section 2, we introduce the employed notation, define relevant function spaces, basic assumptions on the extra stress tensor \( \mathbf{S} \) and its consequences, the weak formulations Problem (Q) and Problem (P) of the system (1.1), and the discrete operators. In Section 3, we define our numerical fluxes and derive the flux and the primal formulation, i.e, Problem (Q_0) and Problem (P_0), of the system (1.1). In Section 4, we derive error estimates for our problem (cf. Theorem 4.1, Corollary 4.2). These are the first convergence rates for a DG-method for systems of \( p \)-Navier–Stokes type. In Section 5, we present numerical experiments.

2. Preliminaries.

2.1. Function spaces. We use the same notation as in Part I of the paper (cf. [14]). For the convenience of the reader, we repeat some of it.

We employ \( c, C > 0 \) to denote generic constants, that may change from line to line, but are not depending on the crucial quantities. For \( k \in \mathbb{N} \) and \( p \in [1, \infty] \), we employ the customary Lebesgue spaces \( (L^p(\Omega), \|\cdot\|_p) \) and Sobolev spaces \( (W^{k,p}(\Omega), \|\cdot\|_{k,p}) \), where \( \Omega \subseteq \mathbb{R}^d \), \( d \in \{2, 3\} \), is a bounded, polyhedral Lipschitz domain. The space \( W^{1,p}_0(\Omega) \) is defined as the space of functions from \( W^{1,p}(\Omega) \) whose trace vanishes on \( \partial \Omega \). We equip \( W^{1,p}_0(\Omega) \) with the norm \( \|\nabla \cdot \|_p \).

We do not distinguish between spaces for scalar, vector- or tensor-valued functions. However, we always denote vector-valued functions by boldface letters and tensor-valued functions by capital boldface letters. The mean value of a locally integrable function \( f \) is denoted by \( \langle f \rangle_M := \frac{1}{|M|} \int_M f \, dx \). Moreover, we employ the notation \( (f, g) := \int_{\Omega} fg \, dx \), whenever the right-hand side is well-defined.

From the theory of Orlicz spaces (cf. [18]) and generalized Orlicz spaces (cf. [12]), we use \( \psi \colon \mathbb{R}^+ \to \mathbb{R}^+ \) and generalized \( \psi \colon \Omega \times \mathbb{R}^+ \to \mathbb{R}^+ \), i.e., \( \psi \) is a Carathéodory function such that \( \psi(x, \cdot) \) is an \( \psi \)-function for a.e. \( x \in \Omega \), respectively. For \( f \in L^\psi(\Omega) \), the modular is defined via \( \rho_\psi(f) := \rho_\psi(|f|) := \int_\Omega \psi(|f|) \, dx \) if \( \psi \) is an \( \psi \)-function and \( \rho_\psi(f) := \rho_\psi(|f|) := \int_\Omega \psi(|f|) \, dx \), if \( \psi \) is a generalized \( \psi \)-function. Then, for a (generalized) \( \psi \)-function \( \psi \), we denote by \( L^\psi(\Omega) := \{ f \in L^\psi(\Omega) \mid \rho_\psi(f) < \infty \} \), the (generalized) Orlicz space. Equipped with the induced Luxembourg norm, i.e., \( \|f\|_\psi := \inf \{ \lambda > 0 \mid \rho_\psi(f/\lambda) \leq 1 \} \), the space (generalized) Orlicz space \( L^\psi(\Omega) \) is a Banach space. If \( \psi \) is a generalized \( \psi \)-function, then, for every \( f \in L^\psi(\Omega) \) and \( g \in L^{\psi^*}(\Omega) \), there holds the generalized Hölder inequality

\[
(f, g) \leq 2 \|f\|_\psi \|g\|_{\psi^*}. \tag{2.1}
\]

An \( \psi \)-function satisfies the \( \Delta_2 \)-condition, if there exists \( K > 2 \) such that for all \( t \geq 0 \), it holds \( \psi(2t) \leq K \psi(t) \). We denote the smallest such constant by \( \Delta_2(\psi) > 0 \). We need the following version of the \( \varepsilon \)-Young inequality: for every \( \varepsilon > 0 \), there exists a constant \( c_\varepsilon > 0 \), depending only on \( \Delta_2(\psi), \Delta_2(\psi^*) < \infty \), such that for every \( s, t \geq 0 \), it holds

\[
t s \leq \varepsilon \psi(t) + c_\varepsilon \psi^*(s). \tag{2.2}
\]

2.2. Basic properties of the extra stress tensor. Throughout the entire paper, we always assume that the extra stress tensor \( \mathbf{S} \) has \((p, \delta)\)-structure, which is defined here in a more stringent way compared to Part I of the paper (cf. [14]). A detailed discussion and full proofs can be found, e.g., in [9, 19]. For a given tensor \( \mathbf{A} \in \mathbb{R}^{d \times d} \), we denote its symmetric part by \( \mathbf{A}^{\text{sym}} := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) \in \mathbb{R}^{d \times d} \), \( \mathbf{A} : \mathbb{R}^{d \times d} := \{ \mathbf{A} \in \mathbb{R}^{d \times d} \mid \mathbf{A} = \mathbf{A}^T \} \).

\[1\]Here, \( L^0(\Omega) \) denotes the set of Lebesgue measurable scalar function defined on \( \Omega \).
For $p \in (1, \infty)$ and $\delta \geq 0$, we define a special N-function $\varphi := \varphi_{p,\delta} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ by

$$\varphi(t) := \int_0^t \varphi'(s) \, ds, \quad \text{where} \quad \varphi'(t) := (\delta + t)^{p-2} t, \quad \text{for all } t \geq 0. \quad (2.3)$$

The properties of $\varphi$ are discussed in detail in [9, 19, 14].

An important tool in our analysis is shifting N-functions $\psi_a : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, for a given N-function $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, we define the family of shifted N-functions $\psi_a : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, $a \geq 0$, via

$$\psi_a(t) := \int_0^t \psi'_a(s) \, ds, \quad \text{where} \quad \psi'_a(t) := \frac{\psi'(a + t)}{a + t}, \quad \text{for all } t \geq 0. \quad (2.4)$$

**Assumption 2.1** (Extra stress tensor). We assume that the extra stress tensor $S : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ belongs to $C^0(\mathbb{R}^{d \times d},\mathbb{R}^{d \times d}) \cap C^1(\mathbb{R}^{d \times d} \setminus \{0\};\mathbb{R}^{d \times d})$ and satisfies $S(A) = S(A^{\text{sym}})$ for all $A \in \mathbb{R}^{d \times d}$ and $S(0) = 0$. Moreover, we assume that the tensor $S = (S_{ij})_{i,j=1,\ldots,d}$ has $(p,\delta)$-structure, i.e., for some $p \in (1, \infty)$, $\delta \in [0, \infty)$, and the N-function $\varphi = \varphi_{p,\delta}$ (cf. (2.3)), there exist constants $C_0, C_1 > 0$ such that

$$\sum_{i,j,k,l=1}^d \partial_{kl} S_{ij}(A) R_{ij} R_{kl} \geq C_0 \varphi'(\frac{|A^{\text{sym}}|}{|A^{\text{sym}}|}) |B^{\text{sym}}|^2, \quad (2.5)$$

$$|\partial_{kl} S_{ij}(A)| \leq C_1 \varphi'(\frac{|A^{\text{sym}}|}{|A^{\text{sym}}|}), \quad (2.6)$$

are satisfied for all $A, B \in \mathbb{R}^{d \times d}$ with $A^{\text{sym}} \neq 0$ and all $i, j, k, l = 1, \ldots, d$. The constants $C_0, C_1 > 0$ and $p \in (1, \infty)$ are called the characteristics of $S$.

**Remark 2.2.** (i) It is well-known (cf. [19]) that the conditions (2.5), (2.6) imply the conditions in the definition of the $(p,\delta)$-structure in Part I of the paper (cf. [14]).

(ii) Assume that $S$ satisfies Assumption 2.1 for some $\delta \in [0, \delta_0]$. Then, if not otherwise stated, the constants in the estimates depend only on the characteristics of $S$ and on $\delta_0 \geq 0$, but are independent of $\delta \geq 0$.

(iii) Let $\varphi$ and $\{\varphi_a\}_{a \geq 0}$ be defined in (2.3) and (2.4), respectively. Then, the shifted operators $S_a : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$, $a \geq 0$, defined, for every $a \geq 0$ and $A \in \mathbb{R}^{d \times d}$, via

$$S_a(A) := \varphi'_a(|A^{\text{sym}}|) \frac{|A^{\text{sym}}|}{|A^{\text{sym}}|} A^{\text{sym}}, \quad (2.7)$$

have $(p,\delta+a)$-structure. In this case, the characteristics of $S_a$ depend only on $p \in (1, \infty)$ and are independent of $\delta \geq 0$ and $a \geq 0$.

Closely related to the extra stress tensor $S$ with $(p,\delta)$-structure is the non-linear function $F : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$, for every $A \in \mathbb{R}^{d \times d}$, defined via

$$F(A) := (\delta + |A^{\text{sym}}|)^{\frac{p-2}{2}} A^{\text{sym}}. \quad (2.8)$$

The connections between $S, F : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ and $\varphi_a, (\varphi_a)^* : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, $a \geq 0$, are best explained by the following result (cf. [9, 19, 11]).

**Proposition 2.3.** Let $S$ satisfy Assumption 2.1, let $\varphi$ be defined in (2.3), and let $F, F^*$ be defined in (2.8). Then, uniformly with respect to $A, B \in \mathbb{R}^{d \times d}$, we have that
The constants in (2.9) depend only on the characteristics of $\mathcal{S}$.

Remark 2.4. For the operators $\mathcal{S}_a : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$, $a \geq 0$, defined in (2.7), the assertions of Proposition 2.3 hold with $\varphi : \mathbb{R} \geq 0 \to \mathbb{R} \geq 0$ replaced by $\varphi_a : \mathbb{R} \geq 0 \to \mathbb{R} \geq 0$, $a \geq 0$.

The following results can be found in [10, 19].

Lemma 2.5 (Change of Shift). Let $\varphi$ be defined in (2.3) and let $F$ be defined in (2.8). Then, for each $\varepsilon > 0$, there exists $c_\varepsilon \geq 1$ (depending only on $\varepsilon > 0$ and the characteristics of $\varphi$) such that for every $A, B \in \mathbb{R}^{d \times d}$ and $t \geq 0$, it holds

$$
\varphi|_{B}(t) \leq c_\varepsilon \varphi|_{A}(t) + \varepsilon |F(A) - F(B)|^2,
$$

$$
\varphi|_{B}(t) \leq c_\varepsilon \varphi|_{A}(t) + \varepsilon |\varphi|_{A}(|B| - |A|)|,
$$

$$
(\varphi|_{B})^*(t) \leq c_\varepsilon (\varphi|_{A})^*(t) + \varepsilon |F(B) - F(A)|^2,
$$

$$
(\varphi|_{B})^*(t) \leq c_\varepsilon (\varphi|_{A})^*(t) + \varepsilon |\varphi|_{A}(||B| - |A||).
$$

2.3. The $p$-Navier–Stokes system. Let us briefly recall some well-known facts about the $p$-Navier–Stokes system (1.1). For $p \in (1, \infty)$, we define the function spaces

$$
\tilde{V} := (W_0^{1,p}(\Omega))^d, \quad \tilde{Q} := L_0^p(\Omega) := \{ f \in L^p(\Omega) \mid \langle f, \Omega \rangle = 0 \}.
$$

With this particular notation, the weak formulation of problem (1.1) is the following:

**Problem (Q).** For given $g \in L^p(\Omega)^d$, find $(v, q) \in \tilde{V} \times \tilde{Q}$ such that for all $(z, z)^T \in \tilde{V} \times \tilde{Q}$, it holds

$$
(\mathcal{S}(Dv), Dz) + (|\nabla v|v, z) - (q, \text{div} z) = (g, z),
$$

$$
(\text{div} v, z) = 0.
$$

Alternatively, we can reformulate Problem (Q) “hidding” the pressure.

**Problem (P).** For given $g \in L^p(\Omega)^d$, find $v \in \tilde{V}(0)$ such that for all $z \in \tilde{V}(0)$, it holds

$$
(\mathcal{S}(Dv), Dz) + (|\nabla v|v, z) = (g, z),
$$

where $\tilde{V}(0) := \{ z \in \tilde{V} \mid \text{div} z = 0 \}$.

The theory of pseudo-monotone operators yields the existence of a weak solution of Problem (P) for $p > \frac{2d}{d+2}$ (cf. [16]). DeRham’s lemma, the solvability of the divergence equation, and the negative norm theorem, then, ensure the solvability of Problem (Q).

2.4. DG spaces, jumps and averages.

2.4.1. Triangulations. We always denote by $T_h$, $h > 0$, a family of uniformly shape regular and conforming triangulations of $\Omega \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$, cf. [5], each consisting of $d$-dimensional simplices $K$. The parameter $h > 0$, refers to the maximal mesh-size of $T_h$, for which we assume for simplicity that $h \leq 1$. Moreover, we assume that the chunkiness is bounded by some constant $\omega > 0$, independent on $h$. By $\Gamma^i_h$ we denote the interior faces, and put $\Gamma_h := \Gamma^i_h \cup \partial \Omega$. We assume that each simplex $K \in T_h$ has at most one face from $\partial \Omega$. We introduce the following scalar product on $\Gamma_h$

$$
\langle f, g \rangle_{\Gamma_h} := \sum_{\gamma \in \Gamma_h} \langle f, g \rangle_{\gamma}, \quad \text{where} \quad \langle f, g \rangle_{\gamma} := \int_{\gamma} fg \, ds \quad \text{for all} \quad \gamma \in \Gamma_h,
$$

if all the integrals are well-defined. Similarly, we define the products $\langle \cdot, \cdot \rangle_{\partial \Gamma}$ and $\langle \cdot, \cdot \rangle_{\Gamma_h^i}$. We extend the notation of modulii to the sets $\Gamma^i_h \setminus \partial \Omega$, and $\Gamma_h$, i.e., we define the modulii $\rho_{\psi, B}(f) := \int_B \psi(|f|) \, ds$ for every $f \in L^p(B)$, where $B = \Gamma^i_h$ or $B = \partial \Omega$ or $B = \Gamma_h$. 


2.4.2. Broken function spaces and projectors. For every \( m \in \mathbb{N}_0 \) and \( K \in T_h \), we denote by \( P^m(K) \), the space of polynomials of degree at most \( m \) on \( K \). Then, for given \( k \in \mathbb{N}_0 \) and \( p \in (1, \infty) \), we define the spaces
\[
Q_h^k := \{ q_h \in L^1(\Omega) \mid q_h|_K \in P_k(K) \text{ for all } K \in T_h \}, \\
V_h^k := \{ v_h \in L^1(\Omega)^d \mid v_h|_K \in P_k(K)^d \text{ for all } K \in T_h \}, \\
X_h^k := \{ X_h \in L^1(\Omega)^{d \times d} \mid X_h|_K \in P_k(K)^{d \times d} \text{ for all } K \in T_h \}, \\
W^{1,p}(T_h) := \{ w_h \in L^1(\Omega) \mid w_h|_K \in W^{1,p}(K) \text{ for all } K \in T_h \}.
\]
In addition, for given \( k \in \mathbb{N}_0 \), we set \( Q_h^k := Q_h^k \cap C^0(\Omega) \). Note that \( W^{1,p}(\Omega) \subseteq W^{1,p}(T_h) \) and \( V_h^k \subseteq W^{1,p}(T_h) \). We denote by \( \Pi_h^k : L^1(\Omega)^d \rightarrow V_h^k \) the (local) \( L^2 \)-projection into \( V_h^k \), which for every \( v \in L^1(\Omega) \) and \( z_h \in V_h^k \) is defined via \( (\Pi_h^k v, z_h) = (v, z_h) \). Analogously, we define the (local) \( L^2 \)-projection into \( X_h^k \), i.e., \( \Pi_h^k : L^1(\Omega)^{d \times d} \rightarrow X_h^k \).

For every \( w_h \in W^{1,p}(T_h) \), we denote by \( \nabla_h w_h \in L^p(\Omega) \), the local gradient, defined via \( (\nabla_h w_h)_K := \nabla (w_h|_K) \) for all \( K \in T_h \). For every \( w_h \in W^{1,p}(T_h) \) and interior faces \( \gamma \in \Gamma_h^+ \) shared by adjacent elements \( K^-_\gamma, K^+_\gamma \in T_h \), we denote by
\[
\{ w_h \}_\gamma := \frac{1}{2} (\text{tr}_\gamma^{+}(w_h) + \text{tr}_\gamma^{-}(w_h)) \in L^p(\gamma), \\
[w_h \otimes n]_\gamma := \text{tr}_\gamma^{+}(w_h) \otimes n^+ + \text{tr}_\gamma^{-}(w_h) \otimes n^- \in L^p(\gamma),
\]
the average and normal jump, resp., of \( w_h \) on \( \gamma \). Moreover, for boundary faces \( \gamma \in \partial \Omega \), we define boundary averages and boundary jumps, resp., via
\[
\{ w_h \}_\gamma := \text{tr}_\gamma^0 (w_h) \in L^p(\gamma), \\
[w_h \otimes n]_\gamma := \text{tr}_\gamma^0 (w_h) \otimes n \in L^p(\gamma),
\]
where \( n : \partial \Omega \rightarrow \mathbb{S}^{d-1} \) denotes the unit normal vector field to \( \Omega \) pointing outward. Analogously, we define \( \{ X_h \}_\gamma \) and \( [X_h n]_\gamma \) for all \( X_h \in X_h^k \) and \( \gamma \in \Gamma_h \). Furthermore, if there is no danger of confusion, then we will omit the index \( \gamma \in \Gamma_h \), in particular, when we interpret jumps and averages as global functions defined on whole \( \Gamma_h \).

2.4.3. DG gradient and jump operators. For every \( k \in \mathbb{N}_0 \) and face \( \gamma \in \Gamma_h \), we define the (local) jump operator \( \mathcal{R}^k_{h,\gamma} : W^{1,p}(T_h) \rightarrow X_h^k \) for every \( w_h \in W^{1,p}(T_h) \) (using Riesz representation) via \( (\mathcal{R}^k_{h,\gamma} w_h, X_h)_\gamma = \langle [w_h \otimes n]_\gamma, \{ X_h \}_\gamma \rangle_\gamma \) for all \( X_h \in X_h^k \).

For every \( k \in \mathbb{N}_0 \), the (global) jump operator \( \mathcal{R}^k_h := \sum_{\gamma \in \Gamma_h^+} \mathcal{R}^k_{h,\gamma} : W^{1,p}(T_h) \rightarrow X_h^k \), by definition, for every \( w_h \in W^{1,p}(T_h) \) and \( X_h \in X_h^k \) satisfies
\[
(\mathcal{R}^k_h w_h, X_h) = \langle [w_h \otimes n], \{ X_h \} \rangle_{\Gamma_h}.
\]
Then, for every \( k \in \mathbb{N}_0 \), the DG gradient operator \( \mathcal{G}^k_h := \nabla_h - \mathcal{R}^k_h : W^{1,p}(T_h) \rightarrow L^p(\Omega) \), for every \( w_h \in W^{1,p}(T_h) \) and \( X_h \in X_h^k \) satisfies
\[
(\mathcal{G}^k_h w_h, X_h) = (\nabla_h w_h, X_h) - \langle [w_h \otimes n], \{ X_h \} \rangle_{\Gamma_h}.
\]
Apart from that, for every \( w_h \in W^{1,p}(T_h) \), we introduce the DG norm as
\[
\| w_h \|_{\nabla, p, h} := \| \nabla_h w_h \|_p + h^\frac{3}{2} \| h^{-1} [w_h \otimes n] \|_{p, \Gamma_h}.
\]
There exists a constant \( c > 0 \) (cf. \([11, (A.26)-(A.28)]) such that for every \( w_h \in W^{1,p}(T_h) \), it holds
\[
c^{-1} \| w_h \|_{\nabla, p, h} \leq \| \mathcal{G}^k_h w_h \|_p + h^\frac{3}{2} \| h^{-1} [w_h \otimes n] \|_{p, \Gamma_h} \leq c \| w_h \|_{\nabla, p, h}.
\]
The following result extends the embedding results for classical Sobolev spaces \( W^{1,p}(\Omega) \) and broken polynomial spaces \( V_h^k \) to DG Sobolev spaces \( W^{1,p}(T_h) \).
Proposition 2.6. Let $p, q \in [1, \infty)$ be such that $W^{1, p}(\Omega) \hookrightarrow L^q(\Omega)$. If $p > q$, then we additionally assume that $h \sim h_K$ uniformly with respect to $K \in \mathcal{T}_h$. Then, there exists a constant $c = c(p, q, \omega_0) > 0$ such that for every $w_h \in W^{1, p}(\mathcal{T}_h)$, it holds
\[ \|w_h\|_q \leq c \|w_h\|_{\nabla, p, h}, \] (2.22)
i.e., $W^{1, p}(\mathcal{T}_h) \hookrightarrow L^q(\Omega)$.

Proof. Note that $\Pi_0^p : L^1(\Omega) \rightarrow V_h^1$ satisfies [4, Assumption A.1] (with $S_K$ replaced by $K$ and $r_0 = 0$). Therefore, proceeding (with some simplifications) as in the proof of [4, Proposition A.2], we deduce that there exists a constant $c = c(p, q, \omega_0) > 0$ such that for every $w_h \in W^{1, p}(\mathcal{T}_h)$, it holds
\[ \|w_h - \Pi_0^p w_h\|_q \leq c h^{1 + d\min\{0, \frac{1}{p} - \frac{1}{q}\}} \|
abla w_h\|_p. \] (2.23)
Using in (2.23) that $1 + d\min\{0, \frac{1}{p} - \frac{1}{q}\} \geq 0$, the discrete embedding [7, Theorem 5.3] for functions from $V_h^1$, and the approximation properties of $\Pi_0^p$ (cf. [11, Appendix A.1], [13, Corollary A.8, Corollary A.19]), we obtain
\[ \|w_h\|_q \leq \|w_h - \Pi_0^p w_h\|_q + \|\Pi_0^p w_h\|_q \leq c \|
abla w_h\|_p + c \|\Pi_0^p w_h\|_{p, \nabla, h} \leq c \|
abla w_h\|_p, \] (2.24)
for every $w_h \in W^{1, \psi}(\mathcal{T}_h)$ via
\[ m_{\psi, h}(w_h) := h \rho_{\psi, \Gamma_h}(h^{-1}\|w_h \otimes n\|). \] (2.25)

For $\psi = \varphi_{\rho, 0}$, we have that $m_{\psi, h}(w_h) = h \|h^{-1}[w_h \otimes n]\|_{p, \Gamma_h}$ for all $w_h \in W^{1, \psi}(\mathcal{T}_h)$.

2.4.4. Symmetric DG gradient and symmetric jump operators. For every $w_h \in W^{1, p}(\mathcal{T}_h)$, we denote by $D_h w_h := [\nabla_h w_h]_{\text{sym}} \in L^p(\Omega; \mathbb{R}^{d \times d})$ the local symmetric gradient. In addition, for every $k \in \mathbb{N}_0$ and $X_h^{k, \text{sym}} := X_h^k \cap L^p(\Omega; \mathbb{R}^{d \times d})$, we define the symmetric DG gradient operator $\mathcal{D}_h^k : W^{1, p}(\mathcal{T}_h) \rightarrow L^p(\Omega; X_h^{k, \text{sym}})$, for every $w_h \in W^{1, p}(\mathcal{T}_h)$, via $D_h w_h := [\mathcal{G}_h w_h]_{\text{sym}} \in L^p(\Omega; X_h^{k, \text{sym}})$, i.e., for every $X_h \in X_h^{k, \text{sym}}$, we have that
\[ (\mathcal{D}_h^k w_h, X_h) = (D_h w_h, X_h) - \langle [w_h \otimes n], \{X_h\}\rangle_{\Gamma_h}. \] (2.26)
Apart from that, for every $w_h \in W^{1, p}(\mathcal{T}_h)$, we introduce the symmetric DG norm as
\[ \|w_h\|_{\mathcal{D}, p, h} := \|D_h w_h\|_p + h^{\frac{3}{2}} \|h^{-1}[w_h \otimes n]\|_{p, \Gamma_h}. \] (2.27)

The following discrete Korn type inequalities play an important role in the numerical analysis of the p-Navier-Stokes system (1.1).

Proposition 2.7 (Discrete Korn inequality). For every $p \in (1, \infty)$ and $k \in \mathbb{N}$, there exists a constant $c_{\text{Korn}} > 0$ such that for every $v_h \in V_h^k$, it holds
\[ \|v_h\|_{\nabla, p, h} \leq c_{\text{Korn}} \|v_h\|_{\mathcal{D}, p, h}. \] (2.28)

Proof. See [14, Proposition 2.4].
PROPOSITION 2.8 (Korn type inequality). For every \( p \in (1, \infty) \) and \( k \in \mathbb{N} \), there exists a constant \( c > 0 \) such that for every \( v_h \in V_h^k \) and every \( w_h \in W^{2,p}(T_h) \), it holds
\[
\|w_h - v_h\|_{V^p_h} \leq c \|w_h - v_h\|_{D_{p,h}} + c h^p \|\nabla_h w_h\|_p.
\]

Proof. See [15, Proposition 2.8].

For the symmetric DG norm, there holds a similar relation like (2.21).

PROPOSITION 2.9. For every \( p \in (1, \infty) \) and \( k \in \mathbb{N} \), there exists a constant \( c > 0 \) such that for every \( w_h \in W^{1,p}(T_h) \), it holds
\[
c^{-1} \|w_h\|_{D_{p,h}} \leq \|\nabla_h w_h\|_p + h^\frac{p}{2} \|h^{-1}\|w_h\|_p \leq c \|w_h\|_{D_{p,h}}. \quad (2.28)
\]

Proof. See [14, Proposition 2.5].

2.4.5. DG divergence operator. For every \( w_h \in W^{1,p}(T_h) \), we denote by \( \text{div}_h w_h := \text{tr}(\nabla_h w_h) \in L^p(\Omega) \), the local divergence. In addition, for every \( k \in \mathbb{N} \), the DG divergence operator \( \text{Div}_h^k : W^{1,p}(T_h) \rightarrow L^p(\Omega) \), for every \( w_h \in W^{1,p}(T_h) \), is defined via \( \text{Div}_h^k w_h := \text{tr}(\nabla_h w_h) \in Q_h^k \), i.e., for every \( z_h \in Q_h^k \), we have that
\[
(\text{Div}_h^k w_h, z_h) = (\text{div}_h w_h, z_h) - (\|w_h \cdot n\|, z_h)_{\Gamma_h}.
\]

Therefore, for every \( v \in W^1_0(\Omega) \) and \( z_h \in Q_h^k \), we have that
\[
(\text{Div}_h^k \Pi_h^k v, z_h) = -(v, \nabla z_h) = (\text{div} v, z_h). \quad (2.29)
\]

3. Fluxes and LDG formulations. To obtain the LDG formulation of (1.1) for \( k \in \mathbb{N} \), we proceed as in Part I [14, Sec. 3] to get the discrete counterpart of Problem (Q). Recall that the numerical fluxes are, for any stabilization parameter \( \alpha > 0 \), defined via
\[
\tilde{v}_{h,c}(v_h) := \begin{cases} \{v_h\} & \text{on } \Gamma_h^i, \\ \{0\} & \text{on } \partial \Omega, \end{cases} \quad \tilde{v}_{h,q}(v_h) := \begin{cases} \{v_h\} + h[hq_n] & \text{on } \Gamma_h^i, \\ \{0\} & \text{on } \partial \Omega, \end{cases} \quad (3.1)
\]
\[
\tilde{q}(q_h) := \{q_h\} \quad \text{on } \Gamma_h, \quad (3.2)
\]
\[
\hat{S}(v_h, s_h, L_h) := \{s_h\} - \alpha S([n_h^\text{sym}](h^{-1}[v_h \otimes n])) \quad \text{on } \Gamma_h, \quad (3.3)
\]
\[
\hat{K}(v_h) := \{K_h\} \quad \text{on } \Gamma_h, \quad (3.4)
\]
\[
\hat{G}(\Pi_h^k G) := \{\Pi_h^k G\} \quad \text{on } \Gamma_h, \quad (3.5)
\]

where \( S([n_h^\text{sym}]) \) is defined as in (2.7). Analogously to Part I of the paper (cf. [14]), to arrive at an inf-sup stable system without using pressure stabilization, we assume that \( q_h \in Q_h^k \cap \hat{Q} \). In particular, we then have that \( \tilde{v}_{h,c}(v_h) = \tilde{v}_{h,q}(v_h) \) on \( \Gamma_h \).

As in Part I of the paper (cf. [14]), we arrive at the flux formulation of (1.1), which reads: For given \( g \in L^p(\Omega) \), find \( (L_h, S_h, K_h, v_h, q_h)^T \in X_h^k \times X_h^k \times X_h^k \times V_h^k \times Q_h^k \) such that for all \( (X_h, Y_h, Z_h, z_h, \eta_h)^T \in X_h^k \times X_h^k \times X_h^k \times V_h^k \times Q_h^k \), it holds
\[
(L_h, X_h) = (G_h^k v_h, X_h) ,
\]
\[
(S_h, Y_h) = (S(H_h^\text{sym}), Y_h) ,
\]
\[
(K_h, Z_h) = (v_h \otimes v_h, Z_h) ,
\]
\[
(S_h - \frac{1}{2} K_h - g_h, L_h, D_h^k z_h) = (g - \frac{1}{2} L_h v_h, z_h) - \alpha \langle S([n_h^\text{sym}](h^{-1}[v_h \otimes n])), [z_h \otimes n] \rangle_{\Gamma_h} ,
\]
\[
(\text{Div}_h^k v_h, z_h) = 0 .
\]
Now, we eliminate in the system (3.6) the variables \( \mathbf{L}_h \in X_h^k \), \( \mathbf{S}_h \in X_h^k \) and \( \mathbf{K}_h \in X_h^k \) to derive a system only expressed in terms of the two variables \( \mathbf{v}_h \in V_h^k \) and \( q_h \in Q_h^k \). Using, in particular, that \( \mathbf{L}_h \) sym = \( D_h^k \mathbf{v}_h \), we get the discrete counterpart of Problem (Q):

**Problem (Q\( _h \))**. For given \( \mathbf{g} \in L^p(\Omega)^d \), find \( (\mathbf{v}_h, q_h)^T \in V_h^k \times Q_h^k \) such that for all \( (z_h, \omega_h)^T \in V_h^k \times Q_h^k \), it holds

\[
(\mathcal{S}(D_h^k \mathbf{v}_h) - \frac{1}{2} \mathbf{v}_h \otimes \mathbf{v}_h - q_h \mathbf{I}_d, D_h^k \omega_h) = (\mathbf{g} - \frac{1}{2} \mathcal{G}_h^k \mathbf{v}_h | \mathbf{v}_h, z_h) \quad (3.7)
\]

\[
- \alpha \left( \mathcal{S}_{(\mathcal{G}_h^k \mathbf{v}_h)} | (h^{-1}[\mathbf{v}_h \otimes \mathbf{n}]_+), [z_h \otimes \mathbf{n}], \right)_{\Gamma_h}.
\]

Next, we eliminate in the system (3.7), the variable \( q_h \in Q_h^k \) to derive a system only expressed in terms of the single variable \( \mathbf{v}_h \in V_h^k \). To this end, we introduce the space

\[
V_h^k(0) := \{ \mathbf{v}_h \in V_h^k | (Div_h^k \mathbf{v}_h, z_h) = 0 \} \text{ for all } z_h \in Q_h^k.
\]

Consequently, since \( (z_h, \omega_h)^T = (z_h, Div_h^k \omega_h) = 0 \) for all \( z_h \in Q_h^k \) and \( \omega_h \in V_h^k(0) \), we get the discrete counterpart of Problem (P):

**Problem (P\( _h \))**. For given \( \mathbf{g} \in L^p(\Omega)^d \), find \( \mathbf{v}_h \in V_h^k(0) \) such that for all \( z_h \in V_h^k(0) \), it holds

\[
(\mathcal{S}(D_h^k \mathbf{v}_h) - \frac{1}{2} \mathbf{v}_h \otimes \mathbf{v}_h, D_h^k \omega_h) = (\mathbf{g} - \frac{1}{2} \mathcal{G}_h^k \mathbf{v}_h | \mathbf{v}_h, z_h) \quad (3.8)
\]

\[
- \alpha \left( \mathcal{S}_{(\mathcal{G}_h^k \mathbf{v}_h)} | (h^{-1}[\mathbf{v}_h \otimes \mathbf{n}]_+), [z_h \otimes \mathbf{n}], \right)_{\Gamma_h}.
\]

Problem (Q\( _h \)) and Problem (P\( _h \)) are called primal formulations of the system (1.1).

Well-posedness (i.e., solvability), stability (i.e., a priori estimates), and (weak) convergence of Problem (Q\( _h \)) and Problem (P\( _h \)) are proved in Part I of the paper (cf. [14]).

4. Convergence rates for the pressure. Let us start with the main result of this paper:

**Theorem 4.1**. Let \( \mathcal{S} \) satisfy Assumption 2.1 with \( p \in (2, \infty) \) and \( \delta > 0 \), let \( k \in \mathbb{N} \), and let \( \mathbf{g} \in L^p(\Omega) \). Moreover, let \( (\mathbf{v}, q)^T \in V(0) \times Q \) be a solution of Problem (Q) (cf. (2.10), (2.11)) with \( \mathbf{F}(\mathbf{Dv}) \in W^{1,2}(\Omega) \) and let \( (\mathbf{v}_h, q_h)^T \in V_h^k(0) \times Q_h^k \) be a solution of Problem (Q\( _h \)) (cf. (3.7)) for \( \alpha > 0 \). Then, there exists a constant \( c_0 > 0 \), depending only on the characteristics of \( \mathcal{S} \), \( \delta^{-1}, \omega_0, \alpha^{-1} \), and \( k \), such that if \( \|\nabla \mathbf{v}\|_2 \leq c_0 \), then, it holds

\[
\|q_h - q\|_{\nu} \leq ch + c(\rho(\varphi|\mathbf{Dv})), (h \nabla q))^\frac{3}{2}
\]

with a constant \( c > 0 \) depending only on the characteristics of \( \mathcal{S} \), \( \|\mathbf{F}(\mathbf{Dv})\|_{1,2} \), \( \|\nabla q\|_{\nu} \), \( \|\mathbf{g}\|_{\nu} \), \( \delta|\Omega| \), \( \delta^{-1} > 0 \), \( \omega_0 \), \( \alpha^{-1} \), \( k \), and \( c_0 \).

**Corollary 4.2**. Let the assumptions of Theorem 4.1 be satisfied. Then, it holds

\[
\|q_h - q\|_{\nu} \leq c h^\frac{3}{2}
\]

with a constant \( c > 0 \) depending only on the characteristics of \( \mathcal{S} \), \( \|\mathbf{F}(\mathbf{Dv})\|_{1,2} \), \( \|\nabla q\|_{\nu} \), \( \|\mathbf{g}\|_{\nu} \), \( \delta|\Omega| \), \( \delta^{-1} > 0 \), \( \omega_0 \), \( \alpha^{-1} \), \( k \), and \( c_0 \). If, in addition, \( \mathbf{g} \in L^2(\Omega) \), then

\[
\|q_h - q\|_{\nu} \leq c h
\]

with a constant \( c > 0 \) depending only on the characteristics of \( \mathcal{S} \), \( \|\mathbf{F}(\mathbf{Dv})\|_{1,2} \), \( \|\mathbf{g}\|_{\nu} \), \( \|\delta + |\mathbf{Dv}|\|^{\frac{3}{2}} \), \( \delta|\Omega| \), \( \delta^{-1} \), \( \omega_0 \), \( \alpha^{-1} \), \( k \), and \( c_0 \).
Due to the modified numerical flux (3.3), the error analysis can no longer be performed in terms of modulars as in [3], but in terms of Luxembourg norms only. However, since all error estimates proved in [15] are formulated in terms of modulars, we need to translate them in terms of Luxembourg norms. The following lemma helps us to do this.

**Lemma 4.3.** Let \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) be defined by (2.3) for \( p \in [2, \infty) \) and \( \delta \geq 0 \) and let \( a \in L^p(\Omega) \) with \( a \geq 0 \) a.e. in \( \Omega \). Then, the following statements apply:

(i) For every \( c_0 \geq 1, \gamma \leq 1, \) and \( f \in L^p(\Omega) \) from \( \rho_{\varphi_a, \Omega}(f) \leq c_0 \gamma \), it follows that \( \|f\|_{\varphi_a} \leq (2c_0)^{\frac{1}{2}} \Delta_2(\varphi) \gamma \frac{\lambda}{p} \).

(ii) For every \( c_0 \geq 1, \gamma \leq 1, \) and \( g \in L^p(\Omega) \) from \( \rho_{\varphi_a, \Omega}(g) \leq c_0 \gamma \), it follows that \( \|g\|_{(\varphi_a)} \leq (c_0 c(p))^{\frac{1}{2}} \gamma \frac{\lambda}{p} \).

**Proof.** (i) Observing that, owing to [19, Lemma 5.1, (5.11)], (2.4), (2.3), \( p \geq 2 \), the \( \Delta_2 \)-condition of \( \varphi_a(x) : \mathbb{R}^n \to \mathbb{R}^n \) for a.e. \( x \in \Omega \) and [19, Lemma 5.3], for every \( \lambda \leq 1, c \geq 1, t \geq 0 \) and for a.e. \( x \in \Omega \), it holds

\[
\varphi_a(x) \frac{t}{\lambda} \leq (\varphi_a(x))' \frac{t}{\lambda} = \left( \delta + a(x) \frac{t}{\lambda} \right)^{p-2} \left( \frac{t}{\lambda} \right)^2
\]

\[
\leq \frac{1}{\lambda c^p} \left( \delta + a(x) + t \right)^{p-2} t^2 = \frac{1}{\lambda c^p} (\varphi_a(x))'(t) t
\]

\[
\leq \frac{1}{\lambda c^p} \varphi_a(x)(2t) \leq \frac{\Delta_2(\varphi_a(x))}{\lambda c^2} \varphi_a(x)(t) \leq \frac{2 \Delta_2(\varphi)}{\lambda c^2} \varphi_a(x)(t),
\]

choosing \( \lambda = \gamma \frac{\lambda}{p} \) and \( c = (2c_0)^{\frac{1}{2}} \Delta_2(\varphi) \), we find that

\[
\rho_{\varphi_a, \Omega} \left( \frac{f}{(c_0 \Delta_2(\varphi))^{\frac{1}{2}} \gamma \frac{\lambda}{p}} \right) \leq \frac{1}{c_0 \gamma} \rho_{\varphi_a, \Omega}(f) \leq 1,
\]

so that, from the definition of the Luxembourg norm, we conclude the assertion.

(ii) Observing that, due to \( p \geq 2 \), for every \( \lambda \leq 1, c \geq 1, t \geq 0 \), it holds

\[
(\varphi_a(x))' \frac{t}{\lambda} \leq \frac{c(p)}{\lambda^2} \left( \delta^{p-1} + a(x)^{p-1} \frac{t}{\lambda} \right)^{p-2} \left( \frac{t}{\lambda} \right)^2
\]

\[
\leq \frac{c(p)}{\lambda^2} \left( \delta^{p-1} + a(x)^{p-1} + t \right)^{p-2} \left( \frac{t}{\lambda} \right)^2
\]

\[
\leq \frac{c(p)}{\lambda^2 c^p} \left( \delta^{p-1} + a(x)^{p-1} + t \right)^{p-2} = \frac{c(p)}{\lambda^2 c^p} (\varphi_a(x))'(t),
\]

where \( c(p) > 0 \) depends only on \( p \geq 2 \), choosing \( \lambda = \gamma \frac{\lambda}{p} \) and \( c = (c_0 c(p))^{\frac{1}{2}} \), we find that

\[
\rho_{\varphi_a, \Omega} \left( \frac{g}{(c_0 c(p))^{\frac{1}{2}} \gamma \frac{\lambda}{p}} \right) \leq \frac{1}{c_0 \gamma} \rho_{\varphi_a, \Omega}(g) \leq 1,
\]

so that, from the definition of the Luxembourg norm, we conclude the assertion. ☐

In order to prove the results in Theorem 4.1 and Corollary 4.2, we need to derive a system similar to (3.7), which is satisfied by a solution of our original problem (1.1). Using the notation \( L = \nabla \nu, S = S(L_{\text{sym}}), K = \nu \otimes \nu, \) we find that \( (v, L, S, K) \in W^{1,p}(\Omega) \times L^p(\Omega) \times L^p(\Omega) \times L^p(\Omega) \). If, in addition, \( S, K, q \in W^{1,1}(\Omega), \) we observe as in [11], i.e., using integration-by-parts, the projection property of \( \Pi_{h}^s, \)
the definition of the discrete gradient and jump functional, that
\[
\begin{align*}
(L, X_h) &= \langle \nabla v, X_h \rangle, \\
(S, Y_h) &= \langle S(L)_{\text{sym}}, Y_h \rangle, \\
(K, Z_h) &= \langle v \otimes v, Z_h \rangle,
\end{align*}
\]
\[
(S - \frac{1}{2}K - qI_d, D_h^k z_h) = (g - \frac{1}{2}Lv, z_h) + \langle \{S\} - \{I_h^k S\}, [z_h \otimes n] \rangle_{\Gamma_h} + \frac{1}{2} \langle \{I_h^k K\} - \{K\}, [z_h \otimes n] \rangle_{\Gamma_h} + \langle \{I_h^k(qI_d)\} - \{qI_d\}, [z_h \otimes n] \rangle_{\Gamma_h},
\]
(4.1)
is satisfied for all \((X_h, Y_h, Z_h, z_h)^T \in X_h^k \times X_h^k \times X_h^k \times V_h^k\). As a result, using (4.1), (3.7) and (2.29), we arrive at
\[
\begin{align*}
(S(D_h^k v_h) - S(Dv), D_h^k z_h) + \alpha \langle S(\{I_h^k D_h^k v_h\}) (h^{-1} [v_h \otimes n]), [z_h \otimes n] \rangle_{\Gamma_h} \\
= (g_h - g, Div_h^k z_h) + b_h(v, v, z_h) - b_h(v_h, v_h, z_h) + \langle \{S\} - \{I_h^k S\}, [z_h \otimes n] \rangle_{\Gamma_h} + \frac{1}{2} \langle \{I_h^k K\} - \{K\}, [z_h \otimes n] \rangle_{\Gamma_h} + \langle \{I_h^k(qI_d)\} - \{qI_d\}, [z_h \otimes n] \rangle_{\Gamma_h},
\end{align*}
\]
(4.2)
which is satisfied for all \((z_h, z_h)^T \in V_h^k \times Q_{h,c}^k\). Here, we denoted the discrete convective term by \(b_h : W^{1,p}(T_h) \times W^{1,p}(T_h) \times W^{1,p}(T_h) \rightarrow \mathbb{R}\), which is defined via
\[
b_h(x_h, y_h, z_h) := \frac{1}{2}(z_h \otimes x_h, \mathcal{G}_h^k y_h) - \frac{1}{2}(y_h \otimes x_h, \mathcal{G}_h^k z_h)
\]
for all \((x_h, y_h, z_h)^T \in W^{1,p}(T_h) \times W^{1,p}(T_h) \times W^{1,p}(T_h)\).

Now we have prepared everything to prove our main result Theorem 4.1.

**Proof of Theorem 4.1.** From our assumptions, resorting to [15, Lemma 2.6], follows that \(\nabla q \in L^p(\Omega)\), which together with \((\varphi_{Dv})^t(t) \leq ct^p\), valid for every \(t \geq 0\), yields \(\rho(\varphi_{Dv})^t, \Omega(\nabla q) \leq \|\nabla q\|_{\rho}^p < \infty\). Moreover, appealing to [7, Lemma 6.10], we deduce the existence of a constant \(\beta > 0\) such that for every \(z_h \in Q_{h,c}^k\), it holds the LBB condition
\[
\beta \|z_h\|_p \leq \sup_{z_h \in V_h^k, \|\cdot\|_{L^p(\Omega)}} (z_h, D_h^k z_h).
\]
(4.3)
On the other hand, due to (4.2), for every \(z_h \in V_h^k\), we have that
\[
(g_h - g, Div_h^k z_h) = (S(D_h^k v_h) - S(Dv), D_h^k z_h) + \alpha \langle S(\{I_h^k D_h^k v_h\}) (h^{-1} [v_h \otimes n]), [z_h \otimes n] \rangle_{\Gamma_h} - b_h(v, v, z_h) + b_h(v_h, v_h, z_h) + \langle \{S\} - \{I_h^k S\}, [z_h \otimes n] \rangle_{\Gamma_h} + \frac{1}{2} \langle \{K\} - \{I_h^k K\}, [z_h \otimes n] \rangle_{\Gamma_h} + \langle \{qI_d\} - \{I_h^k(qI_d)\}, [z_h \otimes n] \rangle_{\Gamma_h}
\]
\[
= I_1 + \alpha I_2 + I_3 + \cdots + I_6.
\]
(4.4)
So, let us next estimate \(I_1, \ldots, I_6\) for some arbitrary \(z_h \in V_h^k\) with \(\|z_h\|_{L^p(\Omega)} \leq 1\): (ad \(I_1\)). Using the generalized Hölder inequality (2.1), we find that
\[
|I_1| \leq 2 \|S(D_h^k v_h) - S(Dv)\|_p (\varphi_{Dv})^t \|D_h^k z_h\|_{\rho(\varphi_{Dv})^t}.
\]
(4.5)
Appealing to (2.9) and [15, Theorem 4.1], we have that
\[
\rho_{(\varphi;Dv^r)*,\Omega}(S(\mathcal{D}_h^k v_h) - S(Dv)) \leq c \left\| F(\mathcal{D}_h^k v_h) - F(Dv) \right\|_2^2 \\
\leq c h^2 \left\| F(Dv) \right\|_{1,2}^2 + c \rho_{(\varphi;Dv^r)*,\Omega}(h \nabla q). \quad (4.6)
\]

Since, by assumption, we have that \( h \leq 1 \), for
\[
c_0 := \max \left\{ 1, c \left\| F(Dv) \right\|_{1,2}^2 + c \rho_{(\varphi;Dv^r)*,\Omega}(\nabla q) \right\} \geq 1,
\gamma := c_0^{-1} \left( c h^2 \left\| F(Dv) \right\|_{1,2}^2 + c \rho_{(\varphi;Dv^r)*,\Omega}(h \nabla q) \right) \leq 1,
\]
Lemma 4.3 yields a constant \( c(p) > 0 \), depending only on \( p \in (2, \infty) \), such that
\[
I_{1,1} \leq c_0^{\frac{1}{2}} c(p)^{\frac{1}{2}} \left( c h^2 \left\| F(Dv) \right\|_{1,2}^2 + c \rho_{(\varphi;Dv^r)*,\Omega}(h \nabla q) \right)^{\frac{1}{2}}. \quad (4.7)
\]
Using the shift change in Lemma 2.5, that \( \varphi(t) \leq c(p) (\delta^p + t^p) \) for all \( t \geq 0 \), and (2.28) we find that
\[
\rho_{(\varphi;Dv^r),\Omega}(\mathcal{D}_h^k z_h) \leq c \rho_{\varphi,\Omega}(\mathcal{D}_h^k z_h) + c \rho_{\varphi,\Omega}(Dv)
\leq c \left\| \mathcal{D}_h^k z_h \right\|_p^p + c \delta^p \left\| \Omega \right\| + c \rho_{\varphi,\Omega}(Dv)
\leq c + c \delta^p \left\| \Omega \right\| + c \rho_{\varphi,\Omega}(Dv)
\leq \max \left\{ 1, c + c \delta^p \left\| \Omega \right\| + c \rho_{\varphi,\Omega}(Dv) \right\},
\]
so that Lemma 4.3, where \( \lambda = 1 \) and \( c_0 = \max \left\{ 1, c + c \delta^p \left\| \Omega \right\| + c \rho_{\varphi,\Omega}(Dv) \right\} \), yields that
\[
I_{1,2} \leq (2 \max \left\{ 1, c + c \delta^p \left\| \Omega \right\| + c \rho_{\varphi,\Omega}(Dv) \right\})^{\frac{1}{2}} \Delta_2(\varphi) \cdot (4.8)
\]
Then, combining (4.7) and (4.8) in (4.6), we deduce that
\[
|I_1| \leq c h + c \left( \rho_{(\varphi;Dv^r)*,\Omega}(h \nabla q) \right)^{\frac{1}{2}}. \quad (4.9)
\]
(\textit{ad} \( I_2 \)). Using the generalized Hölder inequality (2.1), we find that
\[
|I_2| \leq 2 h^{1 - \frac{1}{2}} \left\| S_{(\mathcal{D}_h^k \mathcal{D}_h^k v_h)}(h^{-1} [v_h \otimes n]) \right\|_{(\varphi(\mathcal{D}_h^k \mathcal{D}_h^k v_h), v_h)}
\times \left\| h^{\frac{1}{2}} [z_h \otimes n] \right\|_{\varphi(\mathcal{D}_h^k \mathcal{D}_h^k v_h), v_h} \leq 2 h^{1 - \frac{1}{2}} I_{2,1} \cdot I_{2,2}. \quad (4.10)
\]
Appealing to (2.9) and [15, Theorem 4.1], we have that
\[
\rho_{(\varphi(\mathcal{D}_h^k \mathcal{D}_h^k v_h), v_h)}(S_{(\mathcal{D}_h^k \mathcal{D}_h^k v_h)}(h^{-1} [v_h \otimes n]))
\leq c h^{-1} \rho_{\varphi(\mathcal{D}_h^k \mathcal{D}_h^k v_h), h}(v_h)
\leq c h \left\| F(Dv) \right\|_{1,2}^2 + c h^{-1} \rho_{(\varphi;Dv^r)*,\Omega}(h \nabla q). \quad (4.11)
\]
Since, by assumption, we have that \( h \leq 1 \), for
\[
c_0 := \max \left\{ 1, c \left\| F(Dv) \right\|_{1,2}^2 + c \rho_{(\varphi;Dv^r)*,\Omega}(\nabla q) \right\} \geq 1,
\gamma := c_0^{-1} \left( c h \left\| F(Dv) \right\|_{1,2}^2 + c h^{-1} \rho_{(\varphi;Dv^r)*,\Omega}(h \nabla q) \right) \leq 1,
\]
Lemma 4.3 yields a constant \( c(p) > 0 \), depending only on \( p \in (2, \infty) \), such that

\[
I_{2,1} \leq c_0^{-\frac{1}{2}} c(p)^{\frac{1}{2}} (ch \| F(Dv) \|_{1,2}^2 + ch^{-1} \rho_m(h \nabla q))^\frac{1}{2}.
\]

(4.12)

Using that \( \varphi(t) \leq c \varphi(t) (\delta^p + t^p) \) for all \( t \geq 0 \) and \( h \in [0,1] \), the shift change in Lemma 2.5, that \( \varphi(t) \leq c \varphi(t) (\delta^p + t^p) \) for all \( t \geq 0 \), \( h \mathcal{H}^d(\Gamma_h) \leq c |\Omega| \), the discrete trace inequality \([13, (A.23)]\), the Orlicz-stability properties of \( \Pi_h \) \([13, (A.12)]\), and the a priori estimate \([14, Proposition~5.7] \), we find that

\[
\rho_{\varphi(t)}(||D_k v_h||_r, r, t) \leq c \rho_{\varphi(t)}(h^{-1} ||z_h \otimes n||_r) + c \rho_{\varphi(t)}(||\Pi_h D_k v_h||_r)
\]

\[
\leq c h ||h^{-1} ||z_h \otimes n||_r||^p + c \delta^p h \mathcal{H}^d(\Gamma_h)
\]

(4.13)

so that Lemma 4.3, where \( \lambda = 1 \) and \( c_0 = \max\{1, c + c \delta^p |\Omega| + c \|g\|_{p'}^p\} \), yields that

\[
I_{2,2} \leq 2 \max\{1, c + c \delta^p |\Omega| + c \|g\|_{p'}^p\} \frac{1}{2} \Delta_2(\varphi).
\]

(4.14)

Then, combining (4.12) and (4.14) in (4.10), we deduce that

\[
|I_2| \leq c h + c \left( \rho_{\varphi(t)}(r, t) (h \nabla q) \right)^\frac{1}{2}.
\]

(4.15)

(Ad I3). Introducing the notation \( e_h := v_h - v \in W^{1,p}(T_h) \), \( I_3 \) can be re-written as

\[
I_3 = b_h(v, v - \Pi_h^k v, z_h) - b_h(e_h, \Pi_h^k v, z_h) + b_h(v_h, \Pi_h^k e_h, z_h)
\]

(4.16)

So, we have to estimate \( I_{3,i} \), \( i = 1, 2, 3 \):

(Ad I3,1). The definition of \( b_h : W^{1,p}(T_h) \times W^{1,p}(T_h) \times W^{1,p}(T_h) \rightarrow \mathbb{R} \) yields:

\[
2 I_{3,1} = (z_h \otimes v, \mathcal{G}^k_h(v - \Pi_h^k v)) - (v - \Pi_h^k v) \otimes v, \mathcal{G}^k_h(z_h) =: I_{3,1}^1 + I_{3,1}^2.
\]

(4.17)

As already observed in \([15, Proof~of~Lemma~2.6] \), we have that \( v \in W^{2,2}(\Omega) \), where

\[
\|v\|_{2,2} \leq c \delta^{2-p} \|F(Dv)\|_{1,2}.
\]

(4.18)

Thus, exploiting that, by the Sobolev embedding theorem, \( v \in W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega) \), the discrete Sobolev embedding theorem (cf. \([8, Theorem~5.3] \)), the identities \( \mathcal{G}_h^k = \nabla_h - \mathcal{R}_h^k \) and \( v - \Pi_h^k v = v - \Pi_h^k v - \Pi_h^k (v - \Pi_h^k v) \), the approximation properties of \( \Pi_h^k \) (cf. \([13, Corollary~A.8, Lemma~A.1, Corollary~A.19] \)), and (4.18), we find that

\[
|I_{3,1}^1| \leq \|v\|_{\infty} \|z_h\|_2 \|\mathcal{G}^k_h(v - \Pi_h^k v)\|_2
\]

\[
\leq c h \|z_h\|_{\nabla h} \|\nabla^2 v\|_2
\]

\[
\leq c h \|\nabla F(Dv)\|_2.
\]

(4.19)

Similarly, we get, also using (2.21) and \( p > 2 \), that

\[
|I_{3,1}^2| \leq \|v\|_{\infty} \|v - \Pi_h^k v\|_2 \|\mathcal{G}^k_h z_h\|_2
\]

\[
\leq c h^2 \|\nabla^2 v\|_2 \|z_h\|_{\nabla h}
\]

\[
\leq c h^2 \|\nabla F(Dv)\|_2.
\]

(4.20)
Then, exploiting (2.21), the discrete Sobolev embedding theorem (cf. Proposition 2.6), the DG-stability property of $\Pi_h^k$ (cf. [11, (A.19)]), the Korn type inequality in Proposition 2.8, the estimates (4.18), [15, (4.50)], and [15, Theorem 4.1], we find that

$$|I_{3,2}^1| \leq \|G_h^k \Pi_h^k v\|_2 \|e_h\|_4 \|z_h\|_4$$

$$\leq c \|\Pi_h^k v\|_{\nabla,2,h} \|e_h\|_{\nabla,p,h} \|z_h\|_{\nabla,p,h} \leq c \|\nabla v\|_2 \|e_h\|_{\nabla,2,h} \|z_h\|_{\nabla,p,h} \leq c (h \|F(Dv)\|_{1,2} + (\rho(\varphi_{\Delta \psi})^* \Omega(h \nabla q))^{\frac{1}{2}}) \tag{4.22}$$

Similarly, we find that

$$|I_{3,3}^2| \leq \|G_h^k z_h\|_2 \|\Pi_h^k v\|_4 \|e_h\|_4$$

$$\leq c \|z_h\|_{\nabla,p,h} \|\Pi_h^k v\|_{\nabla,2,h} \|e_h\|_{\nabla,2,h} \leq c (h \|F(Dv)\|_{1,2} + (\rho(\varphi_{\Delta \psi})^* \Omega(h \nabla q))^{\frac{1}{2}}) \tag{4.23}$$

Then, exploiting (2.21), the discrete Sobolev embedding theorem (cf. [8, Theorem 5.3]), the DG-stability properties of $\Pi_h^k$ (cf. [11, (A.18)]), the apriori estimate in [14, Proposition 5.7], the Korn type inequality in Proposition 2.8, the estimates (4.18), [15, (4.50)], and [15, Theorem 4.1], we find that

$$|I_{3,3}^1| \leq \|G_h^k \Pi_h^k e_h\|_2 \|z_h\|_4 \|v_h\|_4$$

$$\leq c \|\Pi_h^k e_h\|_{\nabla,2,h} \|z_h\|_{\nabla,p,h} \|v_h\|_{\nabla,p,h} \leq c \|e_h\|_{\nabla,2,h} \|z_h\|_{\nabla,p,h} \leq c (h \|F(Dv)\|_{1,2} + (\rho(\varphi_{\Delta \psi})^* \Omega(h \nabla q))^{\frac{1}{2}}) \tag{4.25}$$

Similarly, we find that

$$|I_{3,3}^2| \leq \|G_h^k z_h\|_2 \|\Pi_h^k e_h\|_4 \|v_h\|_4$$

$$\leq c \|z_h\|_{\nabla,p,h} \|\Pi_h^k e_h\|_{\nabla,2,h} \|v_h\|_{\nabla,p,h} \leq c \|\Pi_h^k e_h\|_{\nabla,2,h} \|v_h\|_{\nabla,p,h} \leq c (h \|F(Dv)\|_{1,2} + (\rho(\varphi_{\Delta \psi})^* \Omega(h \nabla q))^{\frac{1}{2}}) \tag{4.26}$$

Eventually, combining (4.16)–(4.26), we conclude that

$$|I_3| \leq c h + c (\rho(\varphi_{\Delta \psi})^* \Omega(h \nabla q))^{\frac{1}{2}} \tag{4.27}$$

Then, using the generalized Hölder inequality (2.1), we find that

$$|I_4| \leq 2 h^{-\frac{1}{2}} \|\Pi_h^k S(Dv)\| - \{S(Dv)\} \|\varphi_{\Delta \psi} \|_{1,2} \|\varphi_{\Delta \psi} \|_{1,2} \|\varphi_{\Delta \psi} \|_{1,2} \|\varphi_{\Delta \psi} \|_{1,2} \|\varphi_{\Delta \psi} \|_{1,2} \|\varphi_{\Delta \psi} \|_{1,2} \leq c h^{-\frac{1}{2}} I_{4,1} + I_{4,2} \tag{4.28}$$
Appealing to [15, (4.43), (4.45)], we have that
\[\rho_{(\mathcal{Dv})^*,\Gamma_h} \left(\{\mathcal{I}_h^k \mathcal{S}(\mathcal{Dv})\} - \{\mathcal{S}(\mathcal{Dv})\}\right) \leq c h^2 \|\nabla \mathcal{F}(\mathcal{Dv})\|_2^2.\] (4.29)

Since, by assumption, we have that \(h \leq 1\), for
\[c_0 := \max \left\{1, c \|\nabla \mathcal{F}(\mathcal{Dv})\|_2^2 \right\},\quad \gamma := c_0^{-1} c h^2 \|\nabla \mathcal{F}(\mathcal{Dv})\|_2^2,
\]
Lemma 4.3 yields a constant \(c(p) > 0\), depending only on \(p \in (2, \infty)\), such that
\[I_{4,1} \leq c p^{-\frac{1}{p}} c(p) h \|\nabla \mathcal{F}(\mathcal{Dv})\|_2.\] (4.30)

Using a shift change in Lemma 2.5, [15, Lemma 4.11], (4.13), [15, Theorem 4.1], the convexity of \((\mathcal{Dv}(x))^*: \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}\) for a.e. \(x \in \Omega\) to together with \(\sup_{u \geq 0} \Delta_2((\mathcal{Dv})^*) < \infty\) and \(h \leq 1\), we find that
\[\rho_{(\mathcal{Dv}),\Gamma_h} \left(h^{\frac{1}{2}-1} [\mathcal{z}_h \otimes \mathcal{n}]\right) \leq \rho_{\mathcal{Dv},\Gamma_h} \left(h^{\frac{1}{2}-1} [\mathcal{z}_h \otimes \mathcal{n}]\right) + \rho_{\mathcal{Dv},\Gamma_h} \left(\mathcal{Dv} - \{\mathcal{I}_h^k \mathcal{Dv}\}\right) \leq \rho_{\mathcal{Dv},\Gamma_h} \left(h^{\frac{1}{2}-1} [\mathcal{z}_h \otimes \mathcal{n}]\right) + c h \|\nabla \mathcal{F}(\mathcal{Dv})\|_2^2 + c h \|\mathcal{F}(\mathcal{Dv})\|_2^2 \leq \max \left\{1, c + c \delta^p |\Omega| + c \|\mathcal{g}\|_{p,\gamma}^p + c h \|\mathcal{F}(\mathcal{Dv})\|_2^2 + c \rho_{\mathcal{Dv},\Gamma_h} \|\nabla \mathcal{F}(\mathcal{Dv})\|_2\right\}.\] (4.31)

Thus, Lemma 4.3, where \(\gamma = 1\) and \(c_0 = \max \{1, c + c \delta^p |\Omega| + c \|\mathcal{g}\|_{p,\gamma}^p + c \|\mathcal{F}(\mathcal{Dv})\|_2^2 + c \rho_{\mathcal{Dv},\Gamma_h} \|\nabla \mathcal{F}(\mathcal{Dv})\|_2\},\) yields that
\[I_{4,2} \leq (2 \max \{1, c + c \delta^p |\Omega| + c \|\mathcal{g}\|_{p,\gamma}^p + c \|\mathcal{F}(\mathcal{Dv})\|_2^2 + c \rho_{\mathcal{Dv},\Gamma_h} \|\nabla \mathcal{F}(\mathcal{Dv})\|_2\})^\frac{1}{2} \Delta_2(\mathcal{Dv}).\] (4.32)

Combining (4.30) and (4.32), we deduce that
\[I_4 \leq c h.\] (4.33)

(ad \(I_5\)). Using Hölder’s inequality, we find that
\[|I_5| \leq h^{\frac{1}{2}-\frac{1}{p}} \|\{\mathcal{K}\} - \{\mathcal{I}_h^k \mathcal{K}\}\|_{2,\Gamma_h} \|h^{\frac{1}{2}-1} [\mathcal{z}_h \otimes \mathcal{n}]\|_{2,\Gamma_h}.\] (4.34)

Taking into account \(\mathcal{v} \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)\), we have that \(\mathcal{K} = \mathcal{v} \otimes \mathcal{v} \in W_0^{1,2}(\Omega)\), where \(\|\nabla \mathcal{K}\|_2 \leq c \|\mathcal{v}\|_{\infty} \delta^{2-p} \|\mathcal{F}(\mathcal{Dv})\|_2\). Thus, [13, Corollary A.19] yields
\[\|\{\mathcal{K}\} - \{\mathcal{I}_h^k \mathcal{K}\}\|_{2,\Gamma_h} \leq c h^2 \|\nabla \mathcal{K}\|_2 \leq c h^{\frac{1}{2}} \|\mathcal{F}(\mathcal{Dv})\|_2,\] (4.35)

which together with \(\|h^{\frac{1}{2}-1} [\mathcal{z}_h \otimes \mathcal{n}]\|_{2,\Gamma_h} \leq c |\Omega| \delta^{-\frac{1}{2}} \|h^{\frac{1}{2}-1} [\mathcal{z}_h \otimes \mathcal{n}]\|_{p,\Gamma_h} \leq c\) yields
\[|I_5| \leq c h.\] (4.36)

(ad \(I_6\)). Using the generalized Hölder inequality (2.1), we find that
\[|I_6| \leq 2 h^{\frac{1}{2}-\frac{1}{p}} \|\{\mathcal{q}_I\} - \{\mathcal{I}_h^k \mathcal{q}_I\}\|_{(\mathcal{Dv})^*,\Gamma_h} \|h^{\frac{1}{2}-1} [\mathcal{z}_h \otimes \mathcal{n}]\|_{(\mathcal{Dv})^*,\Gamma_h} \] (4.37)
\[\leq 2 h^{\frac{1}{2}} I_{6,1} \cdot I_{4,2}.\] (4.38)
Appealing to [15, Proposition 4.9], we have that
\[\rho(\phi(Dv)^{-}, \gamma_h \{\{qL_d - \{\Pi^k_h \{qL_d\}\}\} \le c_h \|\nabla F(Dv)\|_2^2 + c_h^{-1} \rho(\phi(Dv)^{-}, \Omega(h \nabla q)).\]

Since, by assumption, we have that \(c \le 1\), for
\[c_0 := \max\{1, c \|F(Dv)\|_2^2 + c \rho(\phi(Dv)^{-}, \Omega(\nabla q)\}, \]
\[\gamma := c_0^{-1}(c \|F(Dv)\|_2^2 + c_h^{-1} \rho(\phi(Dv)^{-}, \Omega(h \nabla q)).\]

Lemma 4.3 yields a constant \(c(p) > 0\), depending only on \(p \in (1, \infty)\), such that
\[\|\{qL_d - \{\Pi^k_h \{qL_d\}\}\|_{(\phi(Dv)^{-}, \gamma_h} \le c_0 c^{-\frac{1}{2}} c \|\nabla F(Dv)\|_2^2 + c_h^{-1} \rho(\phi(Dv)^{-}, \Omega(h \nabla q))^{\frac{1}{2}}.\]

As a result, also using (4.32), we deduce that
\[|I_h| \le c_h + c \rho(\phi(Dv)^{-}, \Omega(h \nabla q))^{\frac{1}{2}}.\] (4.39)

Putting it all together, for every \(z_h \in V_h^k\) with \(\|z_h\|_{\nabla, p, h} \le 1\), we conclude that
\[(q_h - g, Div_h^k z_h) \le c_h + c \rho(\phi(Dv)^{-}, \Omega(h \nabla q))^{\frac{1}{2}}.\] (4.40)

Therefore, for every \(z_h \in Q_h^{k, \ell}\), we find that
\[\|q_h - g\|_{p'} \le \|q_h - z_h\|_{p'} + \|z_h - q\|_{p'}\]
\[\le \sup_{z_h \in V_h^k, \|z_h\|_{\nabla, p, h} \le 1} (q_h - z_h, Div_h^k z_h) + \|z_h - q\|_{p'}\]
\[\le \sup_{z_h \in V_h^k, \|z_h\|_{\nabla, p, h} \le 1} (q_h - q, Div_h^k z_h) + c \|z_h - q\|_{p'}\]
\[\le c_h + c \rho(\phi(Dv)^{-}, \Omega(h \nabla q))^{\frac{1}{2}} + c \|z_h - q\|_{p'}.\] (4.41)

Next, denote by \(\Pi_h^{Q, k} : L^{p'}(\Omega) \rightarrow Q_h^{k, \ell}\), the Clemént quasi-interpolation operator (cf. [6]), for which we have that
\[\|q - \Pi_h^{Q, k} q\|_{p'} \le c_h \|\nabla q\|_{p'}.\] (4.42)

Noting that the infimum of \(\|z_h - q\|_{p'}\) over \(Q_h^{k, \ell}\) and \(Q_h^{k, \ell}\) are comparable for \(q \in Q_h^{k, \ell}\), the assertion of Theorem 4.1 follows from (4.41) and (4.42), if we choose \(z_h = \Pi_h^{Q, k} q\).

**Proof of Corollary 4.2.** Using that \(\phi^*(h t) \le c h^p \phi^*(t)\) for all \(t \ge 0\), valid for \(p > 2\) (cf. [3]), we deduce from Theorem 4.1 that
\[\|q_h - q\|_{p'} \le c_h + c h^{\ell} \rho(\phi^*(\nabla q))^{\frac{1}{2}}.\]

If, in addition, \(g \in L^2(\Omega)\), then [15, Lemma 2.6] implies \((\delta + |Dv|)^{2-p} |\nabla q|^2 \in L^1(\Omega)\). Moreover, it holds \((\phi_x)^*(h t) \sim ((\delta + a)^{\ell-1} + h t)^{\ell-2} h^2 t^2 \le \delta + a)^{2-\ell} h^2 t^2\) for all \(t, a \ge 0\), since \(p > 2\). Therefore, from Theorem 4.1, we deduce that
\[\|q_h - q\|_{p'} \le c_h + c h \|\phi(\phi^*)^{2-p} \nabla q\|_2,\]
which is the assertion. \(\square\)
The same method of proof of course also works for the $p$-Stokes problem, i.e., we neglect the convective term in (1.1), Problem (P), Problem (P$_h$), Problem (Q), and Problem (Q$_h$). Note that the dependence on $\delta^{-1} > 0$ comes solely from the convective term. Thus, we obtain for the $p$-Stokes problem a better dependence on the constants.

**Theorem 4.4.** Let $\mathbf{S}$ satisfy Assumption 2.1 with $p \in (2, \infty)$ and $\delta \geq 0$, let $k \in \mathbb{N}$, and let $\mathbf{g} \in L^{p'}(\Omega)$. Moreover, let $(v, q) \in \tilde{V}(0) \times \tilde{Q}$ be a solution of Problem (Q) without the convective term (cf. (2.10), (2.11)) with $\mathbf{F}(\mathbf{D}v) \in W^{1,2}(\Omega)$ and let $(\mathbf{v}_h, q_h) \in V^k_h(0) \times \tilde{Q}_{h,c}$ be a solution of Problem (Q$_h$) without the terms coming from the convective term (cf. (3.7)) for $\alpha > 0$. Then, there exists a constant $c > 0$, depending only on the characteristics of $\mathbf{S}$, $\|\mathbf{F}(\mathbf{D}v)\|_{1,2}, \|\nabla q\|_{p'}, \|\mathbf{g}\|_{p'}, \delta^p|\Omega|, \omega_0, \alpha^{-1}, \text{ and } k$, such that

$$\|q_h - q\|_{p'} \leq c h + c \left( \rho_{\mathbf{v}(\mathbf{D}v)} \right)^*_\alpha (h \nabla q) \frac{\delta}{2}.$$

**Corollary 4.5.** Let the assumptions of Theorem 4.4 be satisfied. Then, it holds

$$\|q_h - q\|_{p'} \leq c h \frac{\delta}{2}$$

with a constant $c > 0$ depending only on the characteristics of $\mathbf{S}$, $\|\mathbf{F}(\mathbf{D}v)\|_{1,2}, \|\nabla q\|_{p'}, \|\mathbf{g}\|_{p'}, \delta^p|\Omega|, \omega_0, \alpha^{-1}, \text{ and } k$. If, in addition, $\mathbf{g} \in L^2(\Omega)$, then

$$\|q_h - q\|_{p'} \leq c h$$

with a constant $c > 0$ depending only on the characteristics of $\mathbf{S}$, $\|\mathbf{F}(\mathbf{D}v)\|_{1,2}, \|\mathbf{g}\|_{p'}, \|\delta + |\mathbf{D}v|\|_{2} \nabla q\|_{2}, \delta^p|\Omega|, \omega_0, \alpha^{-1}, \text{ and } k$.

**Proof of Theorem 4.4.** We proceed analogously to the proof of Theorem 4.1. In view of the absence of the convective term, the equality (4.4) now reads

$$(q_h - q, \nabla h) = I_1 + \alpha I_2 + I_4 + I_6,$$  \hspace{1cm} (4.43)

where $I_i, i = 1, 2, 4, 6$ are defined in (4.4). Then, resorting in (4.43) to (4.9), (4.15), (4.33), (4.39), we conclude that for every $\mathbf{z}_h \in V^k_h$ with $\|\mathbf{z}_h\|_{\nabla, p, h} \leq 1$, we have that

$$(q_h - q, \nabla h) \leq c h + c \left( \rho_{\mathbf{v}(\mathbf{D}v)} \right)^*_\alpha (h \nabla q) \frac{\delta}{2}$$  \hspace{1cm} (4.44)

with a constant $c > 0$ depending only on the characteristics of $\mathbf{S}$, $\|\mathbf{F}(\mathbf{D}v)\|_{1,2}, \|\nabla q\|_{p'}, \|\mathbf{g}\|_{p'}, \delta^p|\Omega|, \omega_0, \alpha^{-1}, \text{ and } k$. Having at our disposal (4.44), we conclude the proof as in the proof of Theorem 4.1. \hfill $\Box$

**Proof of Theorem 4.4.** We follow the arguments in the proof of Corollary 4.2, but now resort to Theorem 4.4. \hfill $\Box$

5. Numerical experiments. In this section, we apply the LDG scheme (3.6) (or (3.7) and (3.8)) to solve numerically the system (1.1) with $\mathbf{S} : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$, for every $\mathbf{A} \in \mathbb{R}^{d \times d}$ defined via $\mathbf{S}(\mathbf{A}) := (\delta + |\mathbf{A}_{\text{sym}}|)^{p-2} \mathbf{A}_{\text{sym}}$, where $\delta := 1e^{-4}$ and $p > 2$. We approximate the discrete solution $\mathbf{v}_h \in V^k_h$ of the non-linear problem (3.6) deploying the Newton solver from PETSc (version 3.17.3), cf. [17], with an absolute tolerance of $\tau_{abs} = 1e-8$ and a relative tolerance of $\tau_{rel} = 1e-10$. The linear system emerging in each Newton step is solved using a sparse direct solver from MUMPS (version 5.5.0), cf. [1]. For the numerical flux (3.3), we choose the fixed parameter $\alpha = 2.5$. This choice is in accordance with the choice in [11, Table 1]. In the implementation, the uniqueness of the pressure is enforced via a zero mean condition.
All experiments were carried out using the finite element software package FEniCS (version 2019.1.0), cf. [17].

For our numerical experiments, we choose \( \Omega = (-1, 1)^2 \) and linear elements, i.e., \( k = 1 \). We choose \( g \in L^2(\Omega) \) and boundary data \( v_0 \in W^{1,1-\frac{2}{p}}(\partial \Omega) \) such that \( v \in W^{1,2}(\Omega) \) and \( q \in Q \), for every \( x := (x_1, x_2) \in \Omega \) defined by

\[
v(x) := |x|^{2/3}(x_2, -x_1)^T, \quad q(x) := \eta (|x|^{\gamma} - (|x|^{\gamma})_0) \tag{5.1}
\]

are a solutions of (1.1). Here, we choose \( \beta = 1e-2 \), which implies \( F(Dv) \in W^{1,2}(\Omega) \).

Concerning the pressure regularity, we consider two cases: Namely, we choose either \( \gamma = 1 - \frac{2}{p} + 1e-4 \) and \( \eta = 25 \), which just yields \( q \in W^{1,2}(\Omega) \) (case 1), or we choose \( \gamma = \alpha + 1e-4 \) and \( \eta = 1e+3 \), which just yields \( (\delta + |Dv|)^{\frac{2}{2} - 2} \nabla q \in L^2(\Omega) \) (case 2). Thus, for \( \gamma = 1 - \frac{2}{p} + 1e-4 \) and \( \eta = 25 \) (case 1), we can expect the convergence rate \( h^{-2} \), while for \( \gamma = \alpha + 1e-4 \) and \( \eta = 1e+3 \) (case 2), we can expect the convergence rate 1. (cf. Corollary 4.2).

We construct a initial triangulation \( T_{h_0} \), where \( h_0 = \frac{1}{4} \), by subdividing a rectangular cartesian grid into regular triangles with different orientations. Finer triangulations \( T_{h_i} \), \( i = 1, \ldots, 5 \), where \( h_{i+1} = \frac{h_i}{2} \) for all \( i = 1, \ldots, 5 \), are obtained by regular subdivision of the previous grid: Each triangle is subdivided into four equal triangles by connecting the midpoints of the edges, i.e., the red-refinement rule, cf. [2, Definition 4.8 (i)].

Then, for the resulting series of triangulations \( T_{h_i} \), \( i = 1, \ldots, 5 \), we apply the above Newton scheme to compute the corresponding numerical solutions \( (v_i, L_i, S_i)^T := (v_{h_i}, L_{h_i}, S_{h_i})^T \in V_{h_i}^k \times X_{h_i}^k \times X_{h_i}^k \), \( i = 1, \ldots, 5 \), and the error quantities

\[ e_{q,i} := \| q_i - q \|_{p'}, \quad i = 1, \ldots, 5. \]

As estimation of the convergence rates, the experimental order of convergence (EOC)

\[
\text{EOC}_i(e_{q,i}) := \frac{\log(e_{q,i}/e_{q,i-1})}{\log(h_i/h_{i-1})}, \quad i = 1, \ldots, 5.
\]

For different values of \( p \in \{2.25, 2.5, 2.75, 3, 3.25, 3.5\} \) and a series of triangulations \( T_{h_i} \), \( i = 1, \ldots, 5 \), obtained by regular, global refinement as described above, the EOC is computed and presented in Table 1. In it, we observe a convergence ratio of about \( \text{EOC}_i(e_{q,i}) \approx p, \quad i = 1, \ldots, 5 \), which is higher than the expected convergence rate of 1 (cf. Corollary 4.2). This indicates that the error estimates in Corollary 4.2 are yet sub-optimal and it might, therefore, be possible to prove the estimate \( \| q_i - q \|_{p'} \leq c h^2 \).

| \( \gamma \) | \( p \) | \( \text{case 1} \) | \( \text{case 2} \) |
|---|---|---|---|
| \| \| \| |
| 1 | 0.988 | 0.986 | 0.984 | 0.983 | 0.982 | 1.096 | 1.175 | 1.237 | 1.285 | 1.324 | 1.357 |
| 2 | 0.997 | 0.995 | 0.994 | 0.993 | 0.992 | 0.991 | 1.107 | 1.191 | 1.258 | 1.312 | 1.356 | 1.392 |
| 3 | 0.999 | 0.999 | 0.998 | 0.997 | 0.996 | 1.111 | 1.198 | 1.267 | 1.323 | 1.370 | 1.410 |
| 4 | 1.000 | 1.000 | 0.999 | 0.999 | 0.999 | 0.998 | 1.112 | 1.201 | 1.272 | 1.322 | 1.364 | 1.403 |
| 5 | 1.000 | 1.000 | 1.000 | 0.999 | 0.999 | 0.998 | 1.112 | 1.202 | 1.277 | 1.324 | 1.323 | 1.334 |

Table 1: Experimental order of convergence: \( \text{EOC}_i(e_{q,i}), \quad i = 1, \ldots, 5 \).
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