MAGNETIC BRAIDING AND PARALLEL ELECTRIC FIELDS

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ABSTRACT

The braiding of the solar coronal magnetic field via photospheric motions—with subsequent relaxation and magnetic reconnection—is one of the most widely debated ideas of solar physics. We readdress the theory in light of developments in three-dimensional magnetic reconnection theory. It is known that the integrated parallel electric field along field lines is the key quantity determining the rate of reconnection, in contrast with the two-dimensional case where the electric field itself is the important quantity. We demonstrate that this difference becomes crucial for sufficiently complex magnetic field structures. A numerical method is used to relax a braided magnetic field toward an ideal force-free equilibrium; the field is found to remain smooth throughout the relaxation, with only large-scale current structures. However, a highly filamentary integrated parallel current structure with extremely short length-scales is found in the field, with the associated gradients intensifying during the relaxation process. An analytical model is developed to show that, in a coronal situation, the length scales associated with the integrated parallel current structures will rapidly decrease with increasing complexity, or degree of braiding, of the magnetic field. Analysis shows the decrease in these length scales will, for any finite resistivity, eventually become inconsistent with the stability of the coronal field. Thus the inevitable consequence of the magnetic braiding process is a loss of equilibrium of the magnetic field, probably via magnetic reconnection events.

Key words: magnetic fields – MHD – Sun: corona

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1. INTRODUCTION

The heating of the solar corona as a result of the braiding of its constituent magnetic loops is one of the fundamental theories proposed to explain coronal heating. Despite its long history relative to the subject area—originating with the early notion of “topological dissipation” (Parker 1972)—the braiding concept is highly controversial, with no general consensus reached as to the exact mechanism or even its general viability.

Coronal loops are typically modeled as consisting of ideal plasma threaded by a force-free magnetic field. With the ends of each loop anchored in the turbulent photosphere and Alfvén travel times along a loop much faster than the timescales associated with convection, the field evolution should be via a sequence of force-free states. Parker (1972) argues that relaxation to a smooth equilibrium following an arbitrary perturbation is impossible (excepting artificial cases with certain symmetries) and instead tangential discontinuities, corresponding to current sheets, must develop in the field. The claim was first refuted in an analytical treatment by van Ballegooijen (1985) who demonstrated the existence of smooth equilibria in non-symmetric three-dimensional (3D) configurations and under arbitrary continuous boundary perturbations. He found singular current sheets appearing only with the application of discontinuous velocity fields at the boundary or in the case of a discontinuous magnetic field, corresponding to isolated magnetic flux patches in the photosphere (an idea followed up in Priest et al. 2002).

Several other authors have also demonstrated classes of smooth equilibria (Zweibel & Li 1987; Longcope & Strauss 1994; Craig & Sneyd 2005) and numerous authors have tackled the problem via numerical simulations (e.g., Mikić et al. 1989; Longcope & Sudan 1994; Galsgaard & Nordlund 1996) finding equilibria that are nonsingular—although they do often contain current structures on a smaller scale than that of the footpoint displacements. However, it can be argued that numerical simulations may have an inherent resolution problem in allowing current sheet formation to be recognized. In their analytical treatment Craig & Sneyd (2005) demonstrated the existence of smooth equilibria for any footpoint disturbance and for arbitrary compressibility. Significantly, their argument was further advanced through the use of a Lagrangian relaxation scheme that enabled the stability of smooth solutions to be demonstrated, even for significant footpoint displacements.

While no firm conclusion has been reached, either analytically or numerically, on the capacity of magnetic braiding to produce current sheets in the corona, it is clear that the present thinking regarding magnetic braiding is fundamentally based on ideas developed from classical two-dimensional (2D) reconnection theory (Sweet 1958; Parker 1957; Petschek 1964), namely that, in order to allow for rapid reconnection, extremely small scales in the electric current (compared with the large-scale magnetic field structure) are required. In three dimensions, a necessary condition for the occurrence of reconnection is a nonzero integrated electric field parallel to the magnetic field (Hesse & Schindler 1988; Schindler et al. 1988). In particular, reconnection may take place in the absence of any topological features such as magnetic null points. As such it has quite different characteristics, both in terms of onset and evolution, to the 2D case. For some examples of 3D reconnection in the absence of a null point, the situation relevant to magnetic braiding see, for example, Hornig & Priest (2003) and Pontin et al. (2005). Hence the results of magnetic braiding should be reassessed with the proper 3D reconnection criteria in mind. In particular, the development of parallel electric fields as a result of braiding processes should be examined, together with their consequences. We begin to address these issues in this paper, within the framework of resistive MHD.

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The outline of the paper is as follows. We build an analytical model for a braided magnetic field between two parallel plates (Section 2.1) and describe a numerical relaxation scheme in which we relax this field to a force-free state, as expected for the solar corona (Section 2.2). We then analyze the result of the relaxation process in terms of both the current structure and integrated parallel currents (Section 3) and discuss how these structures depend on the degree of braiding of the field (Section 4). Results are discussed in Section 5 and conclusions drawn in Section 6.

2. METHODS

In this paper, we are interested in how braided magnetic fields relax to a force-free equilibrium and specifically in the nature of such equilibria. We begin by describing a general method for constructing braided fields and then concentrate on one particular field structure, a configuration based on the Borromean rings. This situation has been chosen since it represents a truly braided field with a number of advantageous solarlike properties (as described in Section 2.1), lends itself to analytical modeling and, in addition, can be modified in a way to allow for easy comparison with more and less complex fields. We do not concentrate on the details of how this particular field would be built up in the corona, rather taking it as a representative example of a complex field. The construction method for this and similar fields is described in the following section.

2.1. Braid Construction

The braid considered here is modeled on a standard "pigtail" braid. The pigtail braid consists of three strands, braided as shown in Figure 1 (left). By way of motivation for this choice we suppose that the braiding of field lines anchored in the photosphere occurs in a random manner, that is, left-handed and right-handed twists occur with the same probability. We therefore wish to take a braid which has no net twist. More precisely, if we close the braid to form a link, by identifying the upper and lower planes in Figure 1, then the pairwise linking of strands should vanish. The simplest link with this property is the Borromean rings (see the right-hand image of Figure 1), which have the property that any two of the three rings are unlinked while the whole link itself is nontrivial, i.e., cannot be taken apart. Our pigtail braid is the braid equivalent of the Borromean rings. The vanishing net twist of our configuration is also the most conservative case. It is easy to drive currents in a model by highly twisting the field, a situation that is not observed in the corona, while our choice represents an essentially twist-free situation. A further advantage of the braid is that it can be considered as the concatenation of three identical parts, each consisting of a pair of twists (intersection of strands), one positive and one negative. As such it is easy to study the effect of increasing braid complexity through a concatenation of increasing numbers of these elementary parts.

The magnetic fields representing our braids are constructed as follows. Each field consists initially of a uniform background field \( \mathbf{b} \) superimposed on which are a number of isolated magnetic flux rings, corresponding to the crossings of strands in our braid. The basic form of these flux rings is given in cylindrical polar coordinates by

\[
\mathbf{B}_c = \frac{2b_0 k}{a} \exp \left( -\frac{r^2}{a^2} - \frac{z^2}{l^2} \right) \mathbf{e}_\phi,
\]

with the parameters included allowing for variation in the field strength \((k, b_0)\), as well as the radius \((a)\) and vertical extent \((l)\) of the region of toroidal field. The effect of adding the field given by Equation (1) to the background field \( b_0 \mathbf{z} \) is to create a region of nonuniform twist within the otherwise straight vertical field, the maximum degree of twist controlled by the parameter \( k \). An expression corresponding to Equation (1) may be found in Cartesian coordinates with the field center transposed from \( r = 0, z = 0 \) to \( x = x_c, y = y_c, z = z_c \). We label this field \( \mathbf{B}_c \):

\[
\mathbf{B}_c = 2 \frac{b_0 k}{a} \exp \left( \frac{-(x-x_c)^2-(y-y_c)^2}{a^2} - \frac{(z-z_c)^2}{l^2} \right) \times \left( - (y - y_c) \mathbf{e}_x + (x - x_c) \mathbf{e}_y \right),
\]

and let \( \mathbf{B}_{ci} \) denote the field \( \mathbf{B}_c \) taken with the parameters \( \mathbf{c}_i = (x_{ci}, y_{ci}, z_{ci}, k, a, l) \). Braided fields of various complexities may then be constructed as

\[
\mathbf{B} = b_0 \mathbf{z} + \sum_{i=1}^{n} \mathbf{B}_{ci},
\]

with appropriately chosen parameter sets \( \mathbf{c}_i \) \((i = 1, \ldots, n)\). In this paper, we set \( b_0 = 1 \) throughout.

The braided fields constructed in this manner are taken as an initial condition in numerical relaxation experiments. An alternative would have been to begin with a uniform magnetic field between lower and upper boundaries and use a numerical technique to simulate photospheric motions and obtain a braided field and then proceed to relaxation techniques. However, here our primary interest lies in the relaxation process and its consequences and so we take the analytical models as a proxy for a field constructed through simulated photospheric motions. The advantages of this approach lie in a significant saving in computational time, the ability to easily adopt fields with the aforementioned braid properties and a model that lends itself to analytical analysis.

To construct the pigtail braid on which we base the majority of our analysis (Sections 2 and 3) we take the parameter sets \( \mathbf{c}_1 = (1, 0, -20, 1, \sqrt{2}, 2) \), \( \mathbf{c}_2 = (-1, 0, -12, -1, \sqrt{2}, 2) \), \( \mathbf{c}_3 = (1, 0, -4, 1, \sqrt{2}, 2) \), \( \mathbf{c}_4 = (-1, 0, 4, -1, \sqrt{2}, 2) \), and \( \mathbf{c}_5 = (1, 0, -3, 1, \sqrt{2}, 2) \).

Figure 1. Left: illustration of the pigtail braid, on which the braided magnetic field modeled here is based. The pigtail braid is the braid equivalent of the Borromean rings (right).

(A color version of this figure is available in the online journal.)
For the analytical work that follows, the lower boundary can be conveniently considered by taking the limit $z \to -\infty$ and, similarly, the upper boundary as $z \to \infty$. Practically however, due to the exponential decay of the toroidal field components, a short distance above and below the regions of toroidal field centered at $z = \pm 20$, the field can be approximated as vertical. Accordingly, for the numerical investigations we define the domain as $x \in [-4, 4], y \in [-4, 4], z \in [-24, 24]$. Taking this domain it is worthwhile to note that the difference in magnetic energy between $E^3$, as defined by Equation (4), and the potential field satisfying the same boundary conditions (i.e., the uniform field $b_0\hat{z}$) is just 3.08%; the field is close to potential in this respect. This is in accordance with observations of coronal fields (Solanki et al. 2006).

We now proceed to describe the numerical method we will use to relax $E^3$ to a force-free state while keeping the field on the boundaries fixed.

### 2.2. Numerical Methods

While the field in the solar corona is expected to be largely force-free, i.e., in an equilibrium state satisfying $(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$, the initial state for $E^3$ (see Equation (4)) is not in such an equilibrium. Therefore, we seek to relax the field toward a force-free equilibrium while preserving the topology of the field, i.e., the relaxation must correspond to an ideal plasma evolution within the volume and have vanishing velocity on the boundaries. For this we use the 3D Lagrangian magneto-frictional ideal relaxation scheme described in Craig & Sneyd (1986). The advantage of a Lagrangian scheme is that since the numerical grid and, due to the ideal evolution, the magnetic field lines, moves with the plasma elements, grid points accumulate in regions where length scales become small. This enables a better resolution of these small-scale features than would be achieved with a similar Eulerian code and the same number of grid points.

Details of the code can be found in Craig & Sneyd (1986, 1990) and are only briefly summarized here for convenience. Since the relaxation is ideal, both the line element $\delta \mathbf{x}$ that joins two fluid particles and the vector $\mathbf{B}/\rho$ must follow the same time evolution, that is

$$
\frac{D}{Dt} \left( \frac{\mathbf{B}}{\rho} \right) = \left( \frac{\mathbf{B}}{\rho} \cdot \nabla \right) \mathbf{v},
$$

and similarly for $\delta \mathbf{x}$; the code makes use of this by invoking the same Lagrangian description for both. In order to realize the equilibria, a fictitious equation of motion that guarantees a monotonic decrease in the magnetic energy of the system is adopted. The momentum equation takes the form

$$
\mathbf{v} \cdot \mathbf{v} = \mathbf{j} \times \mathbf{B},
$$

i.e., a frictional term proportional to the velocity of the fluid, $\mathbf{v}$, is taken ($\nu \sim 1$) and inertial effects ignored. The choice is also advantageous in that it is a parabolic system, so the possibility of shock waves, and is justified since our primary interest is in the final stable equilibrium rather than the path to it. Additionally, this choice of equation has been shown not to alter the stability or state of the final equilibrium. Magnetic flux and $\nabla \cdot \mathbf{B}$ are both conserved by the relaxation. The numerical technique itself is an implicit (ADI) unconditionally stable scheme.
3. FORCE-FREE APPROXIMATION

3.1. Numerical Results

We now take the initial state for $E^3$ (as given by Equation (4)) and use the Lagrangian numerical scheme of Craig & Sneyd (1986), described in Section 2.2, to relax it toward a force-free equilibrium, with the field on the upper and lower boundaries ($z = \pm 24$) held fixed by line-tying at those boundaries. The experiment has been run at various resolutions and the results presented here are from a 1613 computational grid and a state in which $(j \times B)_{\text{max}} \approx 2 \times 10^{-2}$, a figure which may be compared with the maximum Lorentz force in the initial state of 1.38. Henceforth, we refer to this state as the “relaxed state.” There are several numerical issues with the accuracy of the Lagrangian scheme that prevent further relaxation toward $j \times B = 0$. For a discussion of possible effects of a further approach to the limiting state $j \times B = 0$, see Section 3.2.

Figure 3(a) shows an isosurface of current, $|j|$, in the relaxed state, while Figure 3(b) shows $j_\parallel$ in the central plane $z = 0$ at the same stage. Two spiraling current structures are seen to extend vertically throughout the domain; the current structure is smooth and large scale with no (singular or nonsingular) current sheets present and the maximum value of $|j|$ has decreased from 2.83 to 1.47. Similarly, no small scales in $B$ are found. Some field lines illustrative of $E^3$ in both the initial and final states are shown in Figure 4, where the same three field lines are plotted in both cases (the boundary conditions allow us to identify these exactly). As illustrated in the figure, the twist in the initial configuration appears to have become distributed evenly along field lines (which are, of course, still braided since the relaxation was ideal). The magnetic energy in the relaxed state is just 0.96% in excess of potential, a figure which may be compared with the 3.08% excess of the initial state.

As discussed in Section 1, in 3D it is the integrated electric field component parallel to the magnetic field that is the crucial quantity for reconnection. Since within the framework of resistive MHD, as considered here, Ohm’s law is given by $E + v \times B = \eta j$, the parallel electric fields are directly related to parallel currents by the relation $\int E_\parallel dl = \int \eta j_\parallel dl = \int \eta j_\parallel dl$, where the last equality holds in the case of uniform resistivity $\eta$ (in practice, $\eta$ may be spatially varying, dependent, for example, on the current density itself). These considerations motivate us to consider the nature of the integrated parallel currents in both the initial and relaxed states for $E^3$. For ease of notation we label $J_\parallel = \int j_\parallel dl$. Results are detailed in this section and their implications for the corona discussed in Section 5.

We calculate the integrated parallel currents, $J_\parallel$, along field lines in $E^3$, both for the initial and relaxed states. Specifically, for
Figure 5. Contours of $|\int j_\parallel dl|$ for $E^3$ in the initial (left), and equilibrium (right), states. The integral is taken along field-lines starting at the given location on the lower boundary through to the upper boundary.

(A color version of this figure is available in the online journal.)

a large number of points on the lower boundary of the domain ($z = -24$) we integrate the parallel current along the field line from that location to upper boundary ($z = +24$). Figure 5 shows a contour plot of the result for a subsection of the domain with the contour value indicating the absolute value of $J_l$, $|\int j_\parallel dl|$, for the field line starting at that given location. Two important, related points about the structure of $J_l$ are illustrated which are made more precise in the following text.

(1) The width of the layers of enhanced $J_l$ is very small compared with both the size of the domain and the typical scales of $j$ and $B$.

(2) The structure, or distribution, of $J_l$ is approximately preserved in the relaxation process, while its peak value increases by just over 25% (from 27.04 to 33.86).

In order to aid examination of the first of these points, the width of the $J_l$ layers, we show in Figure 6 contours of that quantity in the initial state of $E^3$ (comparable to the relaxed state by point 2 above) for both the full domain and for a small subsection of it. The integrated parallel currents, $J_l$, are typically concentrated in thin layerlike structures; the right-hand image of this figure shows one such typical layer in some detail. This layer has a half-width at half-maximum (HWHM) of 0.0167, around 3% of the typical horizontal scales of $j$ in the relaxed state.

The nature of $J_l$ in $E^3$ throughout the relaxation may be explained, at least in part, by consideration of the mapping $F(x, y)$ of the locations of field lines from the lower to the upper boundary of the braid and recalling this mapping remains unchanged by the relaxation. The mapping is complex in the sense that there is a strong dependence on initial conditions—neighboring field lines on the lower boundary can thread the upper boundary some distance apart. Since $J_l$ is, by definition, constant for each field line, its distribution on the upper boundary may be deduced from that on the lower boundary through an application of the field line mapping $F(x, y)$. A result of the complexity of the mapping is that even if $J_l$ were to have large scales on the lower boundary it would show small scales on the upper boundary (excepting the case where $J_l$ is constant throughout the domain). This effect is illustrated via a pedagogical example in Figure 7. By design the field $E^3$ has no preferred boundary—a certain symmetry exists between the lower and upper boundaries by the construction of the braid. Accordingly a map of the integrated parallel currents will show small scales on both the upper and lower boundaries.

Thus, the properties of $J_l$ arise through the complexity of the field line mapping for the braided field. The apparent structure of the quantity will depend on the plane from which it is viewed, with the largest scales being present in the central plane. A particular point to note is that the fine scales in $J_l$ are independent of the scales of $j$ itself—which has remained large scale in the relaxation thus far.

Since the field line mapping from lower to upper boundary is unchanged by the relaxation the preservation in the structure of $J_l$ in those locations is partially explained. Additionally, an increase in the maximum value of $J_l$ between the initial and relaxed states has been observed, resulting in an increase in the spatial gradients $\nabla J_l$. We address the importance and possible consequences of the existence of large gradients in $J_l$ in Section 5.

3.2. Theoretical Considerations

As mentioned in the previous section, the relaxed state for $E^3$, in which $(j \times B)_{\text{max}} \approx 2 \times 10^{-2}$, is only an approximation to a true force-free state $j \times B = 0$. The question of how good an approximation we are making then arises. In general, a measure of the proximity of a given field (which contains no nulls) to a force-free state can be found using the quantities

$$
\epsilon := \max_d \frac{\mathbf{J} \times \mathbf{B}}{B^2} = \max_d \frac{\mathbf{J} \times \mathbf{B}}{|B|} \quad \text{and} \quad l = \max_d \frac{\nabla B}{|B|},
$$

where $D$ is the domain under consideration. Now $\epsilon l$ is a dimensionless measure of the proximity to a force-free state, being at most one and vanishing for a force-free field ($0 \leq \epsilon l \leq 1$). In our relaxed state for $E^3$, and taking $D$ as the full domain,
we find $\epsilon l = 0.4965$ for the initial state and $\epsilon l = 0.0087$ for the final state, again suggesting a reasonable approximation to a force-free equilibrium.

There is, however, a possibility that additional characteristics, such as current sheets may form in the limit $j \times B \to 0$. While this is not suggested by the relaxation results, it may be deduced by the consideration of the quantity $\alpha$, the proportionality constant between $B$ and $j$. We have $j = \alpha B$, and this, together with the solenoidal constraint on $B$, implies $B \cdot \nabla \alpha = 0$, that is, the parameter $\alpha$ is constant along field lines. In the previous section, we found that the result of $\mathcal{J}_F$ being constant along field lines, together with the complexity of the field line mapping, is that small scales in the integrated parallel current occur. By the same argument then, in a force-free state $\alpha$ and hence $\mathcal{J}_F$ is likely to show small scales, at least toward the upper and lower boundaries, due to the topology of the braid. Note that the presence of small scales in the current do not necessarily high currents—while the scales are small the variation in magnitude across them need not be large.

How should this consideration be reconciled with our observations that the current $j$ has not developed small scales during a relaxation in which the perpendicular currents have been reduced by a factor of 70? It is possible, as with any numerical relaxation experiment, that the force-free state for $E^3$ has additional properties not seen in the numerical approximation. The braided field $E^3$ has been chosen as a special (physically motivated) case with no net twist. Accordingly, the parameter $\alpha$ must be close to zero for the braided proportion of the magnetic flux in a force-free field. Letting $\alpha^* = j_\parallel / B$ then in the relaxed state considered here $d\alpha^*/ds$, the change in $\alpha^*$ along field lines, will be small but nonzero. The small variations in $\alpha^*$ along field lines will, due to the approximate nature of the equilibrium, become important over the total length of the field line and can result in changes of sign in $\alpha^*$ along these lines.

We now continue our analysis of integrated parallel currents in braided fields. Our motivation for doing so arises from the consideration that, just as in the numerical experiments, in any physical situation the state $j \times B = 0$ will only be obtained in a limit. While additional physical characteristics may appear in this limit, it is the neighborhood of the limit and not only the limit itself that is the physically relevant case since in nature the force-free condition will also be obtained only approximately.

One of the consequences of the preservation of the structure of $\mathcal{J}_F$ during the relaxation is that the key properties of the relaxed state may be deduced from the initial state. To examine the distribution of $\mathcal{J}_F$ in various braided field configurations we may examine the initial state which has been imposed as a physically motivated closed form from which we may quickly and accurately determine $\mathcal{J}_F$. Accordingly we now proceed to consider how changing the complexity of the braided field affects the scales found in $\mathcal{J}_F$.

### 4. EFFECT OF BRAID COMPLEXITY

The initial state in $E^3$, as given by Equation (4), consists of six isolated magnetic flux rings superimposed onto a uniform background field, the effect being to create six regions of isolated twist within that otherwise straight field. As discussed
in Section 2.1 fields similar to $E^3$ may also be defined, using Equation (3) and the appropriate choice of parameters (guided by Equation (4)) to maintain the same essential pattern in the braid, including the alternating sign of twist with height, but with varying numbers of twisted field regions. More precisely, various fields $E^i (i = 1, 2, 3, \ldots)$ may be constructed by reference to the “elementary” unit, $E$, with magnetic field given by

$$B = h_0 \hat{z} + \sum_{i=1}^{2} B_{\hat{c}_i},$$

where $\hat{c}_1 = (1, 0, -4, 1, \sqrt{2}, 2)$ and $\hat{c}_2 = (-1, 0, 4, -1, \sqrt{2}, 2)$, and choosing a suitable composition. We name these fields according to the number of concatenations taken, so $E^2$ consists of four twisted regions, $E^3$ of eight, etc. Hence, the field construction method allows the complexity of braided field $E^n$ to be denoted by $n$; we take an increase in $n$ to be generally representative of an increase in the degree of braiding of the coronal field due to additional photospheric motions.

We now analyze the pattern of $J_1$ along field lines for a number of fields $E^n$, in a similar manner to that carried out for $E^3$. The aim is to deduce how the width of the $J_1$ structures varies with braid complexity. Note that the previously described relaxation experiment allows us to deduce that gradients in the quantity, $\nabla J_1$, may increase during the relaxation process, and so here we are deducing only a lower limit on this quantity.

Figure 8 shows contours of $|J_1|$ for portions of the domain for both $E$ and $E^3$. The scale of the domain shown in the pictures has been chosen to allow the length scales of $J_1$ to be clearly seen. For $E$ the maximum value of $|J_1|$ is lower (at 11.38) and the characteristic widths of the distribution are much broader. For $E^3$ the peaks in $|J_1|$ are much narrower (as evidenced by the scale of the domain shown) and have a higher value of 33.62 in the section of the domain shown which has been arbitrarily chosen and, as such, is unlikely to encompass the global maximum in $|J_1|$.

From these simple considerations we expect that as the complexity of a braided field increases then the maximum in $|J_1|$ increases while the width of the $|J_1|$ structures, $d$, decreases. In order to determine the functional dependency $d(n)$ and the maximum value of $|J_1|$ with $n$ (number of elementary braids, $E$) we would have to make 2D contour plots such as those in Figure 8 for many higher values of $n$. Due to the increasing resolution required for these plots such a method is numerically costly. Instead we choose to consider $J_1$ along field lines originating along the line $y = x, 0 \leq x \leq 2$ on the lower boundary. Taking just a 1D line cut has the disadvantages that the line may not be (indeed is unlikely to be) perpendicular to $|J_1|$ structures and may not cross regions of maximal $|J_1|$. However, the method might still be sufficient to allow a rough functional dependence of $d(n)$ to be determined. For a number $E^n$ we consider the HWHM, $d$, of the maximum peak in $J_1$ value along this line. We note that the value obtained may not be representative of the thinnest structures within the full domain but the method should suffice in giving us an indication of the nature of the variation of $d(n)$.

The result is shown as a scatter plot in Figure 9; we deduce that the width of the $J_1$ structures decreases exponentially with increasing braid complexity—the line superimposed is a fit to the data with equation $d = \kappa 10^{-n/2}$, where $\kappa = 0.77$. Although this equation should not be taken to given any exact dependence of $d$ with $n$, it gives a rough guide as to what we may expect. In addition, we note that the distance between adjacent peaks in $|J_1|$ also becomes smaller with increasing $n$.

The above described dependence of $d$ with $n$ might be expected from the following theoretical consideration on the nature of the field line mapping for $E$ (and hence for the composition $E^n$). If we assume that the maximum value of $|J_1|$ in $E$ coincides with the location where the field lines are stretched the most then we would expect $d \sim d_0/\lambda$, where $\lambda$ is the eigenvalue of the field line mapping and $d_0$ the length scale of the current distribution. Subsequently, an iteration of this map, as appropriate for a concatenation of $E$ to form various braided fields $E^n$, would result in the width $d$ of $|J_1|$ layers in $E^n$ of $d \sim d_0/\lambda^n$. This would occur provided there are sufficiently many regions of maximal field line stretching that it becomes likely that a region with this property is mapped onto a similar region in the appended braid. By the same argument we would expect the maximum value of $|J_1|$ to increase linearly with $n$ and this value may further increase in a relaxation toward force-free equilibrium.

We now proceed to discuss the implications of the presence of narrow $J_1$ layers within a magnetic field structure.
5. IMPLICATIONS FOR CORONAL EVOLUTION

In Section 3, we took an initially imposed magnetic field, $E^0$, and relaxed it toward a force-free equilibrium using a numerical scheme. Small-scale integrated parallel current structures, $J_{||}$, have been found throughout the relaxation and, in addition, gradients in $J_{||}$ increased during the process. These small scales in $J_{||}$ are present while the structure of $j$ itself is of large scale. Since the relaxed state presented is only an approximation to a force-free state it could be that the structure of the exact force-free state differs from that presented here, perhaps with small scales in $j$ additionally present. However, since any physical relaxation must also approach a force-free state in a limiting process, the consequences of small scales in $J_{||}$ in that approach should be considered. While it is well known that in two dimensions thin current sheets can lead to rapid magnetic reconnection, there may also be weaker conditions for such an event in three dimensions. In this section, we show that small scales in the integrated parallel currents are a likely candidate.

Consider first the equation of motion. The motivation for assuming the solar corona is largely force-free is that magnetic forces are expected to dominate other forces and so, assuming $Dv/Dt = j \times B$, we may approximate our relaxed state as being stationary. In an ideal situation then the ideal Ohm’s law, $E + v \times B = 0$, may be satisfied with $v = 0$ and $\phi = 0$, where $E = -\nabla \phi$ due to the stationarity of the state. In a real resistive plasma, such as the solar corona, a very small but finite resistivity, $\eta$, is present and Ohm’s law may be given by

$$E + v \times B = \eta j.$$

Then, since we are approximating the equilibrium as being time independent, we have that $E = -\nabla \phi$, and the electric potential $\phi$ may be deduced by integrating along field lines. For a spatially uniform $\eta, \phi$ is given by

$$\phi = -\int \eta j_\parallel \, dl = -\eta \int j_{||} \, dl.$$

The plasma velocity $v$ then follows as

$$v = \frac{(-\nabla \phi - \eta j) \times B}{B^2}.$$

We may make an order of magnitude estimate of the relative sizes of the terms in Equation (7), with the aim of comparing them with the Alfvén velocity of the plasma. Considering the first term in this expression, $(-\nabla \phi \times B) / B^2$, we have

$$O\left(\frac{-\nabla \phi \times B}{B^2}\right) \sim \frac{\phi}{d} \frac{1}{B^2},$$

since the narrow width $d$ of the parallel current structures will induce the largest gradients in the electric potential. Estimating the size of the electric potential we have

$$O(\phi) \sim \eta \int j_{||} \, dl \sim \frac{B}{\mu_0 l} L,$$

where $l$ denotes the width of the electric current structures and $L$ the length over which the integration should be carried out. Now considering the second term in our estimate of $v$, Equation (7), we have

$$O\left(\frac{\eta j \times B}{B^2}\right) \sim \frac{\eta \epsilon}{\mu_0 l},$$

where $\epsilon$ is given by Equation (5). Combining these estimates, and comparing to the Alfvén velocity, we find

$$O\left(\frac{v}{v_A}\right) \sim \frac{\eta}{\mu_0 l} \frac{L}{d} \frac{1}{v_A} + \frac{\eta \epsilon}{\mu_0 \mu_0 v_A} = \frac{1}{S} \left(\frac{L}{d} + \epsilon l\right),$$

where $S$ is the Lundquist number based on the length scale, $l$, of the current structures. The term $\epsilon l$ in the above expression is just the previously described measure of the proximity of a field to a force-free state and vanishes for a force-free field (in the relaxed state presented for $E^0$ it is found to be 0.0087). We therefore neglect this term and finally obtain and estimate for the plasma velocity as

$$O\left(\frac{v}{v_A}\right) \sim \frac{1}{S} \frac{L}{d}.$$

Taking $E^0$ as an example, and taking $L$ fixed (since in a realistic situation the length of a coronal loop should remain roughly constant with braiding), an approximate relation for $d$ is $d \sim 0.77 \times 10^{-n/2}$ (as discussed in Section 4) and, accordingly (taking $L \sim 48$, as for $E^0$)

$$O\left(\frac{v}{v_A}\right) \sim \frac{1}{S} \frac{70^{n/2}}.$$

Therefore, for any finite value of $S$ we can find a value $n$ such that $v \gg v_A$ and the $E^0$ cannot be in equilibrium. In other words, the ideal approximation ($S \to \infty$) in which $E^0$ can relax toward a force-free equilibrium for arbitrary $n$ must eventually break down as a result of the thin integrated parallel current layers.

Within the more general theory of the magnetic braiding of the solar corona (where $L$ may be, say, roughly the length of a coronal loop and so remains approximately fixed with increasing braiding), we expect the width of integrated parallel current layers, $d$, to decrease exponentially as the coronal field becomes more and more braided by photospheric motions. After a finite amount of braiding $d$ will become so small that, no matter how small the resistivity, ability of the state to relax toward a
force-free equilibrium will break down and a loss of equilibrium will result. One possibility is that the loss of equilibrium will be via magnetic reconnection. In three dimensions the rate of reconnection is given by $\int E_\parallel d\ell$ where the integral is taken along the field line where the quantity has its maximum value. In order to determine where and how magnetic reconnection is triggered we would need to take a full 3D MHD simulation of the domain. However, the small-scale structures found in $|\int j_\parallel d\ell|$ will necessitate a very high spatial resolution for any such accurate MHD simulation. For sufficiently high $n$ and low $\eta$ this is beyond the reach of currently available computing power. Whether for moderate $n$ and comparatively high $\eta$ effects due to the existence of small length scales in $|\int j_\parallel d\ell|$ are already evident is the subject of a current investigation.

6. DISCUSSION AND CONCLUSIONS

In this paper, we have readdressed the magnetic braiding process (Parker 1972) in view of the consideration that in three dimensions it is the integrated parallel electric fields along field lines that are important for reconnection. In resistive MHD these are related to the integrated parallel currents not just the current structure itself. The key outcome of the investigation is a theory that as the coronal field is braided, integrated parallel current along field lines will develop smaller and smaller structures. As a result we conclude that relaxation to a smooth equilibrium is not possible for a magnetic field with an arbitrary degree of braiding; magnetic braiding by photospheric motions will, at some point, lead to a lack of equilibrium in the coronal loop.

In Section 2.1, a particular magnetic field configuration is constructed, modeled on the Borromean rings. We refer to this field as $E^3$. The initial configuration for $E^3$ is taken as a magnetic field between two parallel plates containing six isolated twisted regions in an otherwise vertical field. Each twisted region has only a small amount of twist and the overall twist of the configuration is zero. Thus the braid can be taken to model a (straightened-out) coronal loop. Its aspect ratio is such that, together with the amount of twist, it would likely appear straight in coronal conditions.

A Lagrangian numerical scheme was taken to relax the field $E^3$ toward a force-free equilibrium with the field on the boundaries held fixed. In the relaxed state the maximum Lorentz force inside the volume is 2 orders of magnitude lower, at $\sim 10^{-2}$, than that initially present. With the technique presently employed, numerical difficulties prevent further asymptotic relaxation toward $j \times B = 0$. This is believed to be a result of the implementation of second-order differencing on a highly deformed grid, rather than a real physical effect, and is the subject of a current investigation.

The relaxed state is described in Section 3. The twist in the configuration has become evenly distributed along field lines and the current, initially in the form of six isolated closed current regions, is in two smooth, twisted layers extending throughout the volume. While noting that, as with any numerical experiment, there may be additional features present in the state $j \times B = 0$ that have not been found thus far, we went on to examine the properties of the relaxed state, motivated by the consideration that any physical force-free equilibrium will also be achieved only approximately. In three dimensions it is the presence of an integrated electric field component parallel to the magnetic field that is crucial for magnetic reconnection. In resistive MHD this is related to the integrated parallel current structure along field lines (here abbreviated to $J_\parallel$), rather than the current itself. This consideration motivated us to consider the nature of $J_\parallel$ in the braided field $E^3$.

The variation in $J_\parallel$ between field lines originating on the lower boundary of the domain was examined in Section 3. Its structure is found to be nearly the same in both cases, i.e., approximately conserved in the relaxation process, and there is seen to be significant variation in the value of this quantity between field lines that intersect the lower boundary of the domain only a small distance apart. We refer to this characteristic as “thin integrated parallel current layers.” During the relaxation process the width of these layers is preserved, while the peak values in $J_\parallel$ are enhanced, so creating larger gradients in $J_\parallel$ between the layers. The small scales of $J_\parallel$ are found to arise from the complexity of the field, specifically the mapping of the locations of field lines on the lower to the upper boundary. This mapping shows strong sensitivity to initial conditions and as a result quantities that are constant along field lines (such as $J_\parallel$) and smooth in one plane will show small scales in other planes as they are mapped along field lines.

In Section 4, the variation in the width, $d$, of these integrated parallel current layers with the degree of braiding of the magnetic field is considered through an examination of the fields $E^3$ for various $n$. Here, each increase in $n$ can be thought of as being the result of additional braiding of the coronal field via photospheric motions. With increasing $n$, $d$ is found to decrease exponentially. More generally then, the width of the integrated parallel current layers within a configuration is expected to decrease with increasing braiding.

In Section 5, arguments are given relating to the consequences of narrow $J_\parallel$ layers on the possibility of finding a force-free equilibrium. It is concluded that once $L/d \sim S$ (where $L$ represents the length of the braided field lines and $S$ the Lundquist number)—a state that is inevitable as the magnetic braiding by photospheric motions continues—a smooth equilibrium state can no longer be attained. The precise manifestation of this lack of equilibrium is yet to be determined but we consider it is likely to be via magnetic reconnection events.

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