COMPLETING LIE ALGEBRA ACTIONS TO LIE GROUP ACTIONS

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Abstract. For a finite dimensional Lie algebra \( g \) of vector fields on a manifold \( M \) we show that \( M \) can be completed to a \( G \)-space in a universal way, which however is neither Hausdorff nor \( T_1 \) in general. Here \( G \) is a connected Lie group with Lie-algebra \( g \). For a transitive \( g \)-action the completion is of the form \( G/H \) for a Lie subgroup \( H \) which need not be closed. In general the completion can be constructed by completing each \( g \)-orbit.

1. Introduction. In [7], Palais investigated when one could extend a local Lie group action to a global one. He did this in the realm of non-Hausdorff manifolds, since he showed, that completing a vector field \( X \) on a Hausdorff manifold \( M \) may already lead to a non-Hausdorff manifold on which the additive group \( \mathbb{R} \) acts. We reproved this result in [3], being unaware of Palais' result. In [4] this result was extended to infinite dimensions and applied to partial differential equations like Burgers' equation: Solutions of the PDE were continued beyond the shocks and the universal completion was identified.

Here we give a detailed description of the universal completion of a Hausdorff \( g \)-manifold to a \( G \)-manifold. For a homogeneous \( g \)-manifold (where the finite dimensional Lie algebra \( g \) acts infinitesimally transitive) we show that the \( G \)-completion (for a Lie group \( G \) with Lie algebra \( g \)) is a homogeneous space \( G/H \) for a possibly non-closed Lie subgroup \( H \) (theorem 7). In example 8 we show that each such situation can indeed be realized. For general \( g \)-manifolds we show that one can complete each \( g \)-orbit separately and replace the \( g \)-orbits in \( M \) by the resulting \( G \)-orbits to obtain the universal completion \( G/M \) (theorem 9). All \( g \)-invariant structures on \( M \) ‘extend’ to \( G \)-invariant structures on \( G/M \). The relation between our results and those of Palais are described in 10.

2. \( g \)-manifolds. Let \( g \) be a Lie algebra. A \( g \)-manifold is a (finite dimensional Hausdorff) connected manifold \( M \) together with a homomorphism of Lie algebras \( \zeta = \zeta^M : g \to \mathfrak{X}(M) \) into the Lie algebra of vector fields on \( M \). We may assume without loss that it is injective; if not replace \( g \) by \( g/\ker(\zeta) \). We shall also say that \( g \) acts on \( M \).

The image of \( \zeta \) spans an integrable distribution on \( M \), which need not be of constant rank. So through each point of \( M \) there is a unique maximal leaf of that distribution; we also call it the \( g \)-orbit through that point. It is an initial submanifold of \( M \) in the sense that a mapping from a manifold into the orbit is smooth if and only if it is smooth into \( M \), see [3], 2.14ff.

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Let $\ell : G \times M \to M$ be a left action of a Lie group with Lie algebra $\mathfrak{g}$. Let $t_a : M \to M$ and $t^X : G \to M$ be given by $t_a(x) = \ell^a(a) = \ell(a, x) = a \cdot x$ for $a \in G$ and $x \in M$. For $X \in \mathfrak{g}$ the fundamental vector field $\zeta_X = \zeta_X^M \in \mathfrak{X}(M)$ is given by $\zeta_X(x) = -T_{t_a}(\ell^a)X = -T_{t_a}(a\cdot x)\ell_a(X, 0) = -\partial_{\log} \exp(tX) \cdot x$. The minus sign is necessary so that $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ becomes a Lie algebra homomorphism. For a right action the fundamental vector field mapping without minus would be a Lie algebra homomorphism. Since left actions are more common, we stick to them.

3. **The graph of the pseudogroup.** Let $M$ be a $\mathfrak{g}$-manifold, effective and connected, so that the action $\zeta = \zeta^M : \mathfrak{g} \to \mathfrak{X}(M)$ is injective. Recall from [1], 2.3 that the pseudogroup $\Gamma(\mathfrak{g})$ consists of all diffeomorphisms of the form

$$\mathcal{F}_{t_{n}} \circ \ldots \circ \mathcal{F}_{t_{2}} \circ \mathcal{F}_{t_{1}} | U$$

where $X_{i} \in \mathfrak{g}$, $t_{i} \in \mathbb{R}$, and $U \subset M$ are such that $\mathcal{F}_{t_{1}}^{X_{1}}$ is defined on $U$, $\mathcal{F}_{t_{2}}^{X_{2}}$ is defined on $\mathcal{F}_{t_{1}}^{X_{1}}(U)$, and so on.

Now we choose a connected Lie group $G$ with Lie algebra $\mathfrak{g}$, and we consider the integrable distribution of constant rank $d = \text{dim}(\mathfrak{g})$ on $G \times M$ which is given by

$$\{(L_X(g), \zeta^M_X(x)) : (g, x) \in G \times M, X \in \mathfrak{g}\} \subset TG \times TM,$$

where $L_X$ is the left invariant vector field on $G$ generated by $X \in \mathfrak{g}$. This gives rise to the foliation $\mathcal{F}_{\zeta}$ on $G \times M$, which we call the graph foliation of the $\mathfrak{g}$-manifold $M$.

Consider the following diagram, where $L(e, x)$ is the leaf through $(e, x)$ in $G \times M$, $O_g(x)$ is the $g$-orbit through $x$ in $M$, and $W_x \subset G$ is the image of the leaf $L(e, x)$ in $G$. Note that $\text{pr}_1 : L(e, x) \to W_x$ is a local diffeomorphism for the smooth structure of $L(e, x)$.

$$\begin{array}{ccc}
L(e, x) & \xrightarrow{\text{pr}_2} & O_g(x) \\
& \searrow & \downarrow \\
& 0, 1 & \searrow \text{open} \\
\downarrow & & \downarrow \\
\mathcal{F} & \searrow & \mathcal{F} \\
G \times M & \xrightarrow{\text{pr}_2} & M \\
& \searrow & \downarrow \\
& \text{pr}_1 & \downarrow \\
\mathcal{F} & \searrow & \mathcal{F} \\
G & \xrightarrow{\text{pr}_1} & G
\end{array}$$

Moreover we consider a piecewise smooth curve $c : [0, 1] \to W_x$ with $c(0) = e$ and we assume that it is liftable to a smooth curve $\tilde{c} : [0, 1] \to L(e, x)$ with $\tilde{c}(0) = (e, x)$. Its endpoint $\tilde{c}(1) \in L(e, x)$ does not depend on small (i.e. liftable to $L(e, x)$) homotopies of $c$ which respect the ends. This lifting depends smoothly on the choice of the initial point $x$ and gives rise to a local diffeomorphism $\gamma_x(c) : U \to \{c(1)\} \times U' \to U'$, a typical element of the pseudogroup $\Gamma(\mathfrak{g})$ which is defined near $x$. See [1], 2.3 for more information and example 4 below. Note, that the leaf $L(g, x)$ through $(g, x)$ is given by

$$\{(gh, y) : (h, y) \in L(e, x)\} = (\mu_g \times \text{Id})(L(e, x))$$

where $\mu : G \times G \to G$ is the multiplication and $\mu_g(h) = gh = \mu^h(g)$. 

4. Examples. It is helpful to keep the following examples in mind, which elaborate upon [11], 5.3. Let $G = \mathfrak{g} = \mathbb{R}^2$, let $W$ be an annulus in $\mathbb{R}^2$ containing 0, and let $M_1$ be a simply connected piece of finite or infinite length of the universal cover of $W$. Then the Lie algebra $\mathfrak{g} = \mathbb{R}^2$ acts on $M$ but not the group. Let $p : M_1 \to W$ be the restriction of the covering map, a local diffeomorphism.

Here $G \times_\mathfrak{g} M_1 \cong G = \mathbb{R}^2$. Namely, the graph distribution is then also transversal to the fiber of $pr_2 : G \times M_1 \to M_1$ (since the action is transitive and free on $M_1$), thus describes a principal $G$-connection on the bundle $pr_2 : G \times M_1 \to M_1$.

Each leaf is a covering of $M_1$ and hence diffeomorphic to $M_1$ since $M_1$ is simply connected. For $g \in \mathbb{R}^2$ consider $j_g : M_1 \xrightarrow{\text{ins}} \{g\} \times M_1 \subset G \times M_1 \xrightarrow{\pi} G \times_\mathfrak{g} M_1$ and two points $x \neq y \in M_1$. We may choose a smooth curve $\gamma$ in $M_1$ from $x$ to $y$, lift it into the leaf $L(g, x)$ and project it to a curve $c$ in $g + W$ from $g$ to $c(1) = g + p(y) - p(x) \in g + W$. Then $(g, x)$ and $(c(1), y)$ are on the same leaf. So $j_g(x) = j_g(y)$ if and only if $p(x) = p(y)$. So we see that $j_g(x) = g + p(x)$, and thus $G \times_\mathfrak{g} M_1 = \mathbb{R}^2$. This will also follow from 7.

Let us further complicate the situation by now omitting a small disk in $M_1$ so that it becomes non simply connected but still projects onto $W$, and let $M_2$ be a simply connected component of the universal cover of $M_1$ with the disk omitted. What happens now is that homotopic curves which act equally on $M_1$ act differently on $M_2$.

It is easy to see with the methods described below that the completion $G M_i = \mathbb{R}^2$ in both cases.

5. Enlarging to group actions. In the situation of 3 let us denote by $G M = G \times_\mathfrak{g} M = G \times M / \mathcal{F}_C$ the space of leaves of the foliation $\mathcal{F}_C$ on $G \times M$, with the quotient topology. For each $g \in G$ we consider the mapping

\[
j_g : M \xrightarrow{\text{ins}} \{g\} \times M \subset G \times M \xrightarrow{\pi} G M = G \times_\mathfrak{g} M.
\]

Note that the submanifolds $\{g\} \times M \subset G \times M$ are transversal to the graph foliation $\mathcal{F}_C$. The leaf space $G M$ of $G \times M$ admits a unique smooth structure, possibly singular and non-Hausdorff, such that a mapping $f : G M \to N$ into a smooth manifold $N$ is smooth if and only if the compositions $f \circ j_g : M \to N$ are smooth. For example we may use the structure of a Frölicher space or smooth space induced by the mappings $j_g$ in the sense of [5], section 23 on $G M = G \times_\mathfrak{g} M$. The canonical open maps $j_g : M \to G M$ for $g \in G$ are called the charts of $G M$.
each \( x \in M \) and for \( g'g^{-1} \) near enough to \( e \) in \( G \) there exists a curve \( c : [0, 1] \to W_x \) with \( c(0) = e \) and \( c(1) = g'g^{-1} \) and an open neighborhood \( U \) of \( x \) in \( M \) such that for the smooth transformation \( \gamma_x(c) \) in the pseudogroup \( \Gamma(g) \) we have

\[
(5.2) \quad j_{g'}|U = j_g \circ \gamma_x(c).
\]

Thus the mappings \( j_g \) may serve as a replacement for charts in the description of the smooth structure on \( gM \). Note that the mappings \( j_g \) are not injective in general. Even if \( g = g' \) there might be liftable smooth loops \( c \) in \( W_x \) such that (5.2) holds. Note also some similarity of the system of ‘charts’ \( j_g \) with the notion of an orbifold where one uses finite groups instead of pseudogroup transformations.

The leaf space \( G = G \times gM \) is a smooth \( G \)-space where the \( G \)-action is induced by \( (g', x) \to (gg', x) \) in \( G \times M \).

**Theorem.** The \( G \)-completion \( gM \) has the following universal properties:

\[
(5.3) \quad \text{Given any Hausdorff \( G \)-manifold \( N \) and \( g \)-equivariant mapping \( f : M \to N \) there exists a unique \( G \)-equivariant continuous mapping \( \tilde{f} : gM \to N \) with } \quad \tilde{f} \circ j_x = f. \quad \text{Namely, the mapping } \tilde{f} : M \times M \to N \text{ given by } \tilde{f}(g, x) = g.f(x) \text{ is smooth and factors to } \tilde{f} : gM \to N.
\]

\[
(5.4) \quad \text{In the setting of (5.3), the universal property holds also for the } T_1 \text{-quotient of } gM, \text{ which is given as the quotient } gM/\mathcal{F}_\zeta \text{ of } gM \text{ by the equivalence relation generated by the closure of leaves.}
\]

\[
(5.5) \quad \text{If } M \text{ carries a symplectic or Poisson structure or a Riemannian metric such that the } g \text{-action preserves this structure or is even a Hamiltonian action then the structure ‘can be extended to } gM \text{ such that the enlarged } G \text{-action preserves these structures or is even Hamiltonian’}.
\]

**Proof.** (5.3) Consider the mapping \( \bar{f} = \ell^N \circ (\text{Id}_G \times f) : M \times M \to N \) which is given by \( \bar{f}(g, x) = g.f(x) \). Then by (3.1) and (3.2) we have for \( X \in G \)

\[
T\bar{f}.(L_X(g), \zeta_X^N(x)) = T\ell.(L_X(g), T_xf.\zeta_X^N(x))
\]

\[
= T\ell.(R_{Ad(g)}X(g), 0_{f(x)}) + T\ell(0_g, \zeta_X^N(f(x)))
\]

\[
= -\zeta_{Ad(g)}X(g.f(x)) + T\ell_g.\zeta_X^N(f(x)) = 0.
\]

Thus \( \bar{f} \) is constant on the leaves of the graph foliation on \( M \times M \) and thus factors to \( \tilde{f} : gM \to N \). Since \( \bar{f}(g.g_1, x) = g.g_1.f(x) = g.f(g, x) \), the mapping \( \tilde{f} \) is \( G \)-equivariant. Since \( N \) is Hausdorff, \( \tilde{f} \) is even constant on the closure of each leaf, thus (5.4) holds also.

(5.5) Let us treat Poisson structure \( P \) on \( M \). For symplectic structures or Riemannian metrics the argument is similar and simpler. Since the Lie derivative along fundamental vector fields of \( P \) vanishes, the pseudogroup transformation \( \gamma_x(c) \) in (5.2) preserves \( P \). Since \( gM \) is the quotient of the disjoint union of all spaces \( \{g\} \times M \) for \( g \in G \) under the equivalence relation described by (5.2), \( P \) ‘passes down to this quotient’. Note that we refrain from putting too much meaning on this statement.

The universal property (5.3) holds also for smooth \( G \)-spaces \( N \) which need not be Hausdorff, nor \( T_1 \), but should have tangent spaces and foliations so that it is meaningful to talk about \( g \)-equivariant mappings. We will not go into this, but see \cite{G}, section 23 for some concepts which point in this direction.
As an application of the universal property of the $G$-completion $G_M$, we see that $G_M$ depends on the choice of $G$ in the following way. We write $G = \Gamma \setminus \tilde{G}$, where $\tilde{G}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and $\Gamma \subset \tilde{G}$ is the discrete central subgroup such that $\Gamma \cong \pi_1(G)$. Then we have $G_M \cong \Gamma \setminus \tilde{G}M$ as $G$-spaces, so that $\tilde{G}M$ is potentially less singular than $G_M$.

6. Example. Let $\mathfrak{g} = \mathbb{R}^2$ with basis $X, Y$, let $M = \mathbb{R}^3 \setminus \{(0,0,z) : z \in \mathbb{R}\}$, and let $\zeta^\alpha : \mathfrak{g} \to X(M)$ be given by

$$
(6.1) \quad \zeta^\alpha_X = \partial_x + \alpha \frac{yz}{x^2 + y^2} \partial_z, \quad \zeta^\alpha_Y = \partial_y - \alpha \frac{xz}{x^2 + y^2} \partial_z, \quad \alpha > 0
$$

which satisfy $[\zeta^\alpha_X, \zeta^\alpha_Y] = 0$. By construction of the graph foliation $\mathcal{F}_{\zeta^\alpha}$ in (3.1) and the procedure summarized in diagram (3.2), the leaves of $\mathcal{F}_{\zeta^\alpha}$ are determined explicitly as follows. For any smooth curve $c(t) = (\xi(t), \eta(t)) \in G$ starting at $(\xi_0, \eta_0)$ we have $\dot{c}(t) = \xi(t) X + \eta(t) Y \in \mathfrak{g}$ and the lifted curve $(c(t), y(t))$ is in the leaf $L((\xi_0, \eta_0), y_0)$ if and only if it satisfies the first order ODE

$$
(6.2) \quad (y(t), \dot{y}(t)) = \xi(t) \zeta^\alpha_X (y(t)) + \eta(t) \zeta^\alpha_Y (y(t))
$$

with initial value $y(0) = y_0 = (x_0, y_0, u = x_0) \in M$. Substituting (6.1) into (6.2), we see that this ODE is linear, that is $\dot{x} = \xi, \dot{y} = \eta$ and $\dot{z} = -\alpha \frac{xy^2}{x^2 + y^2}$, where $r^2 = x^2 + y^2$.

Thus the projection $x(t)$ of $y(t)$ to the $(x, y)$-plane is given by $x(t) = c(t) - ((\xi_0, \eta_0) - x_0) = c(t) - (\xi_0 - x_0, \eta_0 - y_0)$, whereas the third equation leads to

$$
(6.3) \quad z(t) = u \ e^{-\alpha (\theta(t) - \theta_0)} = u \ e^{\alpha \theta_0} \ e^{-\alpha \theta(t)},
$$

where $\theta$ is the angle function in the $(x, y)$-plane. This depends only on the endpoints $x_0$, $x(t)$ and the winding number of the curve $x$ and is otherwise independent of $x$. Incompleteness occurs whenever the curve $x$ goes to $(0, 0) \in \mathbb{R}^2$ in finite time $t < \infty$, that is $x(t) \to (0,0)$, $t \uparrow t$ or equivalently $c(t) \to (\xi_0, \eta_0) - x_0$, $t \uparrow t$. It follows that the leaf $L((\xi_0, \eta_0), y_0)$ is parametrized by $(r, \theta) \in \mathbb{R}_+ \times \mathbb{R}$ with $z = z(\theta)$ being independent of $r > 0$ and that

$$
(6.4) \quad \text{pr}_1 : L((\xi_0, \eta_0), y_0) \to W_{(\xi_0, \eta_0), y_0} = \mathbb{R}^2 \setminus \{(\xi_0, \eta_0) - x_0\}
$$

in (3.2) is a universal covering. This is visibly consistent with (3.3). In order to parametrize the space of leaves $G_M$, we observe that the parameter $x_0$ can be eliminated. In fact, from the previous formulas we see that

$$
(6.5) \quad L((\xi_0', \eta_0'), (x_0', u')) = L((\xi_0, \eta_0), (x_0, u)),
$$

if and only if $(\xi_0', \eta_0') - x_0' = (\xi_0, \eta_0) - x_0$ and $u' = u e^{\alpha (\theta_0 - \theta_0')}$, so that we have $z'(\theta) = u' e^{\alpha \theta_0} e^{-\alpha \theta(t)} = u e^{\alpha \theta_0} e^{-\alpha \theta(t)} = z(\theta)$. In particular, it follows that

$$
(6.6) \quad L((\xi_0, \eta_0), y_0) = L((\xi_0 + 1, \eta_0), (1,0, u')),
$$

where $(\xi_0, \eta_0) - x_0$, $u' = u e^{\alpha \theta_0}$, $\theta_0' = 0$, projecting to $\mathbb{R}^2 \setminus \{(\xi_0', \eta_0')\}$. Therefore the leaves of the form $L((\xi_0 + 1, \eta_0), (1,0,u))$ are distinct for different values of $(\xi_0, \eta_0)$ and fixed value of $u$ and from the relation (3.3) we conclude that

$$
(6.7) \quad L((\xi_0 + 1, \eta_0), (1,0,u)) = (\xi_0, \eta_0) + L((1,0), (1,0,u)),
$$

that is $G = \mathbb{R}^2$ acts without isotropy on $G_M$. We also need to determine the range for the parameter $u$. Obviously, we have $L((1,0), (1,0,u')) = L((1,0), (1,0,u))$ if and only if $u' = e^{2\pi \alpha n} u$ for $n \in \mathbb{Z}$. Thus these leaves are parametrized by $[u]$,
taking values in the quotient of the additive group $\mathbb{R}$ under the multiplicative group 
\[ \{e^{2\pi an} : n \in \mathbb{Z}\}, \]
that is
\[ (6.8) \quad \{0\} \cup \mathbb{S}^1_+ \cup \mathbb{S}^1_- \cong \{0\} \cup \mathbb{R}_+^\times / \{e^{2\pi an} : n \in \mathbb{Z}\} \cup \mathbb{R}_-^\times / \{e^{2\pi an} : n \in \mathbb{Z}\}. \]

The topology on the above space is determined by the leaf closures, respectively the orbit closures. First we have $L((\xi_0 + 1, \eta_0), (1, 0, u)) = (\xi_0, \eta_0) + L((1, 0), (1, 0, u))$ in $G \times M$ and it is sufficient to determine the closures of $L((1, 0), (1, 0, u))$. For $(1, 0, u) \in M$ with $u \neq 0$ we consider the curve $c(\theta) = e^{i\theta} \in G = \mathbb{R}^2$. It is liftable to $G \times M$ and determines on $M$ the curve $y(t) = (\cos \theta, \sin \theta, u e^{-\alpha \theta})$. Thus the curve $(c(\theta), y(\theta))$ in the leaf through $(1, 0; 1, 0, u) \in G \times M \subset \mathbb{R}^3$ has a limit cycle for $\theta \to \infty$ which lies in the different leaf through $(1, 0; 1, 0, 0)$ which is closed, given by the $(x, y)$–plane $(\mathbb{R}^2 \times 0) \setminus 0$ at level $(1, 0) \in G$. Thus we have
\[ (6.9) \quad L((1, 0), (1, 0, u)) = L((1, 0), (1, 0, u)) \cup L((1, 0), (1, 0, 0)). \]

Hence the leaf $L((1, 0), (1, 0, u))$ is not closed and the topological space $G M$ is not $T_1$ and not a manifold. The orbits of the $g$-action are determined by the leaf structure via $\pi_g$ in diagram (3.2) and they look here as follows: The $(x, y)$–plane $(\mathbb{R}^2 \times 0) \setminus 0$ is a closed orbit. Orbits above this plane are helicoidal staircases leading down and accumulating exponentially at the $(x, y)$–plane. Orbits below this plane are helicoidal staircases leading up and again accumulating exponentially. Thus the orbit space $M/g$ of the $g$-action is given by (6.8), with the point $0$ being closed. By (6.9), the closure of any orbit represented by a point $[u]$ on one of the circles is given by $\{[u], [0]\}$. From (6.6) and (6.7), we see that the $G$-completion $G M$ has a section over the orbit space $G M/G \cong M/g$ given by $[u] \mapsto L((1, 0), (1, 0, u))$. Therefore $G M \cong G \times M/g = \mathbb{R}^2 \times \{\{0\} \cup \mathbb{S}^1_+ \cup \mathbb{S}^1_-\}$.

The structure of the completion and the orbit spaces are independent of the deformation parameter $\alpha > 0$ in (6.1). However for $\alpha \downarrow 0$, the completion just means adding in the $z$–axis, that is we get $G M \cong \mathbb{R}^3$ with $G = \mathbb{R}^2$ acting by parallel translation on the affine planes $z = c$, and $M/g \cong G M/G \cong \mathbb{R}$ as it should be.

It was pointed out to us [2] that one can make this example still more pathological: Consider the above example only in a cylinder over the annulus $0 < x^2 + y^2 < 1$. Add an open handle to the disk and continue the $\mathbb{R}^2$-action on the cylinder over the disk with an open handle added in such a way that there is a shift in the $z$-direction when one traverses the handle. Then one of the helicoidal staircases is connected to the disk itself, so it accumulates onto itself. This is called a ‘resilient leaf’ in foliation theory.

**7. Theorem.** Let $M$ be a connected transitive effective $g$-manifold. Let $G$ be a connected Lie group with Lie algebra $g$. Then we have:

(7.1) Then there exists a subgroup $H \subset G$ such that the $G$-completion $G M$ is diffeomorphic to $G/H$.

(7.2) The Hausdorff quotient of $G M$ is the homogeneous manifold $G\tilde{H}$. It has the following universal property: For each smooth $g$-equivariant mapping $f : M \to N$ into a Hausdorff $G$-manifold $N$ there exists a unique smooth $G$-equivariant mapping $\tilde{f} : G\tilde{H} \to N$ with $f = \tilde{f} \circ \pi \circ j_c : M \to G/H \xrightarrow{\pi} G/\tilde{H} \to N$.

(7.3) For each leaf $L(g, x_0) \subset G \times M$ the projection $\text{pr}_2 : L(g, x_0) \to M$ is a smooth fiber bundle with typical fiber $H$. 


Proof. Since the action is transitive we have the exact sequence of vector bundles over $M$
\[ 0 \to \text{iso} \to M \times \mathfrak{g} \xrightarrow{\xi} TM \to 0. \]

(7.1) We choose a base point $x_0 \in M$. The $G$-completion is given by $G \cdot M = G \times \mathfrak{g}$
and the orbit space of the $\mathfrak{g}$-action on $G \times M$ which is given by $\mathfrak{g} \ni \xi \to LX \times c_\mathfrak{g}^X$,
and the $G$-action on the completion is given by multiplication from the left. The submanifold $G \times \{x_0\}$ meets each $\mathfrak{g}$-orbit in $G \times M$ transversely, since
\[ T_{(g,x_0)}(G \times \{x_0\}) + T_{(g,x_0)}L(g, x_0) = \{LX(g) \times 0_{x_0} + LY(g) \times \xi_Y(x_0) : X, Y \in \mathfrak{g}\} \]
\[ = T_{(g,x_0)}(G \times M). \]

By (3.3) we have $L(g, x) = g,L(e, x)$ so that we can define the linear subspace $\mathfrak{g}_{x_0} = \mathfrak{h} \subset \mathfrak{g}$ by
\[ X \in \mathfrak{h} \iff X \times 0_{x_0} \in T_{(e, x_0)}(G \times \{x_0\}) \cap T_{(e, x_0)}L(e, x_0) \]
\[ \iff LX(g) \times 0_{x_0} \in T_{(g,x_0)}G \times \{x_0\} \cap T_{(g,x_0)}L(g, x_0) \]
Since $G \times \{x_0\}$ is a leaf of a foliation and the $L(e, x)$ also form a foliation, $\mathfrak{h}$ is a
Lie subalgebra of $\mathfrak{g}$. Let $H_0$ be the connected Lie subgroup of $G$ which corresponds to $\mathfrak{h}$. Then clearly $H_0 \times \{x_0\} \subset G \times \{x_0\} \cap L(e, x_0)$. Let the subgroup $H \subset G$ be given by
\[ H = \{g \in G : (g, x_0) \in L(e, x_0)\} = \{g \in G : L(g, x_0) = L(e, x_0)\}, \]
then the $C^\infty$-curve component of $H$ containing $e$ is just $H_0$. So $H$ consists of at
most countably many $H_0$-cosets. Thus $H$ is a Lie subgroup of $G$ (with a finer
topology, perhaps). By construction the orbit space $G \times \mathfrak{g} \cdot M$ equals the quotient of
the transversal $G \times \{x_0\}$ by the relation induced by intersecting with leaves $L(g, x_0)$,
i.e., $G \times \mathfrak{g} \cdot M = G/H$.

(7.2) Obviously the $T_1$-quotient of $G/H$ equals the Hausdorff quotient $G/\overline{H}$
which is a smooth manifold. The universal property is easily seen.

(7.3) Let $x \in M$ and $(g, x) \in L(e, x_0) = L(g, x) = g,L(e, x)$. So it suffices to treat
the leaf $L(e, x)$. We choose $X_1, \ldots, X_n \in \mathfrak{g}$ such that $\xi_X(x_1), \ldots, \xi_X(x_n)$ form a basis
of the tangent space $T_x M$. Let $u : U \to \mathbb{R}^n$ be a chart on $M$ centered at $x$ such
that $u(U)$ is an open ball in $\mathbb{R}^n$ and such that $\xi_X(x_1), \ldots, \xi_X(x_n)$ are still linearly
independent for all $y \in U$. For $y \in U$ consider the smooth curve $c_y : [0, 1] \to U$
given by $c_y(t) = u^{-1}(t, u(y))$. We consider
\[ \partial_t c_y(t) = c_y'(t) = \sum_{i=1}^n f_y^i(t) \xi_{X_i}(c_y(t)), \quad f_y^i \in C^\infty([0, 1], \mathbb{R}) \]
\[ X_y(t) = \sum_{i=1}^n f_y^i(t) X_i \in \mathfrak{g}, \quad X \in C^\infty([0, 1], \mathfrak{g}) \]
\[ g_y \in C^\infty([0, 1], G), \quad T(\mu_{g_y(t)}) \partial_t g_y(t) = X_y(t), \quad g_y(0) = e, \]
and everything is also smooth in $y \in U$. Then for $h \in H$ we have $(h,g_y(t), c_y(t)) \in
L(e, x)$ since
\[ \partial_t(h,g_y(t), c_y(t)) = (L_{X_y(t)}(h,g_y(t)), \xi_{X_y(t)}(c_y(t))). \]
Thus $U \times H \ni (y, h) \mapsto \text{pr}_2^{-1}(U) \cap L(e, x)$ is the required fiber bundle parameteri-

8. Example. Let $G$ be simply connected Lie group and let $H$ be a connected Lie group of $G$ which is not closed. For example, let $G = \text{Spin}(5)$ which is compact of rank 2 and let $H$ be a dense 1-parameter subgroup in its 2-dimensional maximal torus. Let $\text{Lie}(G) = \mathfrak{g}$ and $\text{Lie}(H) = \mathfrak{h}$. We consider the foliation of $G$ into right $H$-cosets $gH$ which is generated by $\{ L_X : X \in \mathfrak{h} \}$ and is left invariant under $G$. Let $U$ be a chart centered at $e$ on $G$ which is adapted to this foliation, i.e. $u : U \to u(U) = V_1 \times V_2 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that the sets $u^{-1}(V_1 \times \{x\})$ are the leaves intersected with $U$. We assume that $V_1$ and $V_2$ are open balls, and that $U$ is so small that $\exp : W \to U$ is a diffeomorphism for a suitable convex open set $W \subset \mathfrak{g}$. Of course $\mathfrak{g}$ acts on $U$ and respects the foliation, so this $\mathfrak{g}$-action descends to the leaf space $M$ of the foliation on $U$ which is diffeomorphic to $V_2$.

Lemma. In this situation, for the $G$-completion we have $G \times_{\mathfrak{g}} M = G/H$

Proof. We use the method described in the end of the proof of theorem 7: $G/M = G \times_{\mathfrak{g}} M$ is the quotient of the transversal $G \times \{x_0\}$ by the relation induced by intersecting with leaves $L(g,x_0)$. Thus we have to determine the subgroup $H_1 = \{ g \in G : (g,x_0) \in L(e,g) \}$.

Obviously any smooth curve $c_1 : [0,1] \to H$ starting at $e$ is liftable to $L(e,x_0)$ since it does not move $x_0 \in M$. So $H \subseteq H_1$, and moreover $H$ is the $C^\infty$-path component of the identity in $H_1$.

Conversely, if $c = (c_1,c_2) : [0,1] \to L(e,x_0) \subset G \times M$ is a smooth curve from $(e,x_0)$ to $(g,x_0)$ then $c_2$ is a smooth loop through $x_0$ in $M$ and there exists a smooth homotopy $h$ in $M$ which contracts $c_2$ to $x_0$, fixing the ends. Since $\text{pr}_2 : L(e,x_0) \to M$ is a fiber bundle by (7.3) we can lift the homotopy $h$ from $M$ to $L(e,x_0)$ with starting curve $c$, fixing the ends, and deforming $c$ to a curve $c'$ in $L(e,x_0) \cap \text{pr}_2^{-1}(x_0)$. Then $\text{pr}_1 \circ c'$ is a smooth curve in $H_1$ connecting $e$ and $g$.

Thus $H_1 = H$, and consequently $G/M = G/H$.

9. Theorem. Let $M$ be a connected $\mathfrak{g}$-manifold. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Then the $G$-completion $G/M$ can be described in the following way:

(9.1) Form the leaf space $M/\mathfrak{g}$, a quotient of $M$ which may be non-Hausdorff and not $T_1$ etc.

(9.2) For each point $z \in M/\mathfrak{g}$, replace the orbit $\pi^{-1}(z) \subset M$ by the homogeneous space $G/H_z$ described in theorem 7, where $x$ is some point in the orbit $\pi^{-1}(z) \subset M$. One can use transversals to the $\mathfrak{g}$-orbits in $M$ to describe this in more detail.

(9.3) For each point $z \in M/\mathfrak{g}$, one can also replace the orbit $\pi^{-1}(z) \subset M$ by the homogeneous space $G/\overline{\mathfrak{m}}_z$ described in theorem 7, where $x$ is some point in the orbit $\pi^{-1}(z) \subset M$. The resulting $G$-space has then Hausdorff orbits which are smooth manifolds, but the same orbit space as $M/\mathfrak{g}$.

See example 6 above.

Proof. Let $O(x) \subset M$ be the $\mathfrak{g}$-orbit through $x$, i.e., the leaf through $x$ of the singular foliation (with non-constant leaf dimension) on $M$ which is induced by the $\mathfrak{g}$-action. Then the $G$-completion of the orbit $O(x)$ is $G O(x) = G H_z$ for the Lie subgroup $H_z \subset G$ described in theorem (7.1). By the universal property of the $G$-completion we get a $G$-equivariant mapping $G O(x) \to G M$ which is injective and a homeomorphism onto its image, since we can repeat the construction of theorem
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(7.1) on \( M \). Clearly the mapping \( j_e : M \to gM \) induces a homeomorphism between the orbit spaces \( M/g \to gM/G \).

Now let \( s : V \to M \) be an embedding of a submanifold which is a transversal to the \( g \)-foliation at \( s(v_0) \): We have \( Ts \cdot T_{v_0}V \oplus \zeta_{s(v_0)}(g) = T_{s(v_0)}M \). Then \( s \) induces a mapping \( V \to G \times M \) and \( V \to gM \) and we may use the point \( s(v) \) in replacing \( O(s(v)) \) by \( G/H_{s(v)} \) for \( v \) near \( v_0 \).

The following diagram summarizes the relation between the preceding constructions.

\[
\begin{array}{ccc}
M & \xrightarrow{\bigcup_{[x] \in M/g} G/H_x} & G/M/G \\
\downarrow & & \downarrow \\
M & \xrightarrow{\bigcup_{[x] \in M/g} G/H_x} & G/M/\mathcal{F}_\zeta \\
\downarrow & & \downarrow \\
M/g & \xrightarrow{\cong} & gM/G \\
\downarrow & & \downarrow \\
G/M/G & \xrightarrow{\pi_G} & (G \times M/\mathcal{F}_\zeta)/G
\end{array}
\]

Note that taking the \( T_1 \)-quotient \( G \times M/\mathcal{F}_\zeta \) of the leaf space \( gM \) may be a very severe reduction. In example 6 the isotropy groups \( H_x \) are trivial and we have \( G \times M/\mathcal{F}_\zeta = \mathbb{R}^2 \times \{0\} \) and \( (G \times M/\mathcal{F}_\zeta)/G = \{0\} \)

10. Palais’ treatment of \( g \)-manifolds. In [7], Palais considered \( g \)-actions on finite dimensional manifolds \( M \) in the following way. He assumed from the beginning, that \( M \) may be a non-Hausdorff manifold, since the completion may be non-Hausdorff. Then he introduces notions which we can express as follows in the terms introduced here:

(10.1) \((M, \zeta)\) is called \textit{generating} if it generates a local \( G \)-transformation group. See [7], II,2, Def. V and II,7, Thm. XI. This holds if and only if the leaves of the graph foliation on \( G \times M \) described in section 3 are Hausdorff. For Hausdorff \( g \)-manifolds this is always the case.

(10.2) \((M, \zeta)\) is called \textit{uniform} if \( \text{pr}_1 : L(e, x) \to G \) in (3.2) is a covering map for each \( x \in M \). See [7], III,6, Def. VIII and III,6, Thm. XVII, Cor., Cor.2. In the Hausdorff case the \( g \)-action is then complete and it may be integrated to a \( \tilde{G} \)-action, where \( \tilde{G} \) is a simply connected Lie group with Lie algebra \( g \), so that \( \tilde{G}M \cong M \).

(10.3) \((M, \zeta)\) is called \textit{univalent} if \( \text{pr}_1 : L(e, x) \to G \) in (3.2) is injective for \( \forall x \). See [7], III,2, Def. VI and III,4, Thm. X.

(10.4) \((M, \zeta)\) is called \textit{globalizable} if there exists a (non-Hausdorff) \( G \)-manifold \( N \) which contains \( M \) equivariantly as an open submanifold. See [7], III,1, Def. II and III,4, Thm. X. This is a severe condition which is not satisfied in examples 4 and 6 above.

Palais’ main result on (non-Hausdorff) manifolds with a vector field says that (10.1), (10.3), and (10.4) are equivalent. See [7], III,7, Thm. XX.

On (non-Hausdorff) \( g \)-manifolds his main result is that (10.3) and (10.4) are equivalent. See [7], III,1, Def. II and III,4, Thm. X, and also III,2, Def. VI and III,4, Thm. X.
11. Concluding remarks. (11.1) A suitable setting for further development might be the class of discrete $g$-manifolds, that is $g$-manifolds for which the $\tilde{G}$-space $\tilde{G}M$ is $T_1$, or equivalently the leaves of the graph foliation $F_\zeta$ on $\tilde{G} \times M$ are closed. In this case, the charts $j_\phi : M \to \tilde{G}M$ in (5.1) are local diffeomorphisms with respect to the unique smooth structure on $\tilde{G}M$ and $\tilde{G}M$ is a smooth manifold, albeit not necessarily Hausdorff.

(11.2) In the context of (11.1), there are several definitions of proper $g$-actions, all of which are equivalent to saying that the $\tilde{G}$-action on $\tilde{G}M$ is proper. Many properties of proper actions will carry over to this case.

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