SINGULAR HARDY-TRUDINGER-MOSER INEQUALITY AND
THE EXISTENCE OF EXTREMALS ON THE UNIT DISC

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Abstract. We present the singular Hardy-Trudinger-Moser inequality and the existence of their extremal functions on the unit disc $B$ in $\mathbb{R}^2$. As our first main result, we show that for any $0 < t < 2$ and $u \in C_\infty^0(B)$ satisfying
$$\int_B |\nabla u|^2 dx - \int_B \frac{u^2}{(1-|x|^2)^t} dx \leq 1,$$
there exists a constant $C_0 > 0$ such that the following inequality holds
$$\int_B e^{4\pi(1-t/2)|x|} dx \leq C_0.$$
Furthermore, by the method of blow-up analysis, we establish the existence of extremal functions in a suitable function space. Our results extend those in Wang and Ye [36] from the non-singular case $t = 0$ to the singular case for $0 < t < 2$.

1. Introduction. Let us present a brief history of some classical Trudinger-Moser inequalities to motivate our work. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Classical Sobolev embedding asserts that $W_0^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous when $kp < n$ and $1 \leq q \leq \frac{np}{n-kp}$, where $k \in \mathbb{N}$ and $1 \leq p < \infty$. In the case $k = 1$ and $p = n$, it is well known that $W_0^{1,n}(\Omega) \nsubseteq L^q(\Omega)$ for $1 \leq q < +\infty$, however, $W_0^{1,n}(\Omega) \nsubseteq L^\infty(\Omega)$. In this borderline case, the optimal imbedding is an Orlicz space imbedding. In [35], Trudinger showed that $W_0^{1,n}(\Omega)$ can be embeded into the Orlicz space $L_{\phi_\alpha}(\Omega)$, which is determined by the Young function $\phi_\alpha(t) = \exp(\alpha|t|^\frac{n}{n-1}) - 1$ for some $\alpha > 0$. (The readers can also see Yudovich [38] and Pohozaev [34]). In 1971, Moser sharpened the exponent $\alpha$. More precisely, he showed the following Trudinger-Moser inequality in [33].

**Theorem A.** Let $u \in W_0^{1,n}(\Omega)$, for $0 \leq \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where $\omega_{n-1}$ denotes the surface area of the unit sphere in $\mathbb{R}^n$, then there exists a positive constant $C_n$ such that the following inequality holds
$$\sup_{\|\nabla u\|_{L^n(\Omega)} \leq 1} \int_\Omega \exp(\alpha_n |u|^\frac{n}{n-1}) dx \leq C_n |\Omega|.$$

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The constant $\alpha_n$ is sharp in the sense that if we replace $\alpha > \alpha_n$, then the inequality (1.1) can no longer hold with some $C_n$ independent of $u$.

There has been a lot of research about the Trudinger-Moser inequalities. Some results have played fundamental roles in solving partial differential equations. Here, we only recall some related results.

In 2012, Wang and Ye [36] established the following Hardy-Trudinger-Moser inequality on the unit disc $B$ in $\mathbb{R}^2$ by using the Schwartz rearrangement argument. They also proved the supremum can be achieved in a suitable function space by blow-up analysis.

**Theorem B.** For $u \in C_c^\infty (B)$, where $B$ denotes the unit disk in $\mathbb{R}^2$, let $H(u) := \int_B |\nabla u|^2 dx dy - \int_B \frac{u^2}{(1-|z|^2)^2} dx dy$, then there exists a constant $C > 0$ such that

$$\int_B e^{4\pi u^2} dx dy \leq C.$$

Furthermore, the supremum can be achieved in a suitable function space.

We can observe the second term of the $H(u)$ is the square of $L^2$ norm on the hyperbolic disk. Thus, it is closely related to the Trudinger-Moser inequality on the hyperbolic spaces (see Mancini and Sandeep [31], and Lu and Tang [23, 24], etc.).

Wang and Ye conjectured the Hardy-Trudinger-Moser inequality will be true for any bounded and convex domain with smooth boundary. Lu and Yang gave an affirmative answer to this conjecture. They confirmed this conjecture indeed holds for any bounded and convex domain in $\mathbb{R}^2$ in [25]. They proved Theorem C via an argument from local inequalities to global ones using the level sets of functions under consideration developed by Lam and Lu in [13, 14] (see also [5, 16, 39]), together with the Riemann mapping theorem.

**Theorem C.** Let $\Omega$ be a bounded and convex domain in $\mathbb{R}^2$. Then there exists a constant $C(\Omega) > 0$ such that for any $u \in C_c^\infty (\Omega)$, the following inequality holds

$$\int_\Omega e^{4\pi u^2} dx dy \leq C(\Omega),$$

where $H_d(u) = \int_\Omega |\nabla u|^2 dx dy - \frac{1}{4} \int_\Omega \frac{u^2}{d(z, \partial \Omega)^2} dx dy$ and $d(z, \partial \Omega) = \min_{z_1 \in \partial \Omega} |z - z_1|$.

Later, Mancini, Sandeep and Tintarev proved another modified Trudinger-Moser inequality on $B$ in [32] by rearrangement. They proved for all $u \in C_c^\infty (B)$ satisfying

$$\int_B |\nabla u|^2 dx dy - \int_B \frac{u^2}{(1-|z|^2)^2} dx dy \leq 1,$$

then there exists a constant $C > 0$ such that the following inequality holds

$$\int_B e^{4\pi u^2} - 1 - 4\pi u^2 \frac{dx}{(1-|z|^2)^2} \leq C.$$
These Hardy-Adams inequalities are the borderline case of the high order Hardy-Sobolev-Maz’Ya inequality proved in [27]. Employing a rearrangement argument and a change of variables, Adimurthi and Sandeep generalized the Trudinger-Moser inequality (1.1) to a singular version in [2]. They proved the following theorem.

**Theorem D.** Let \( \Omega \subset \mathbb{R}^n \), \( \alpha \geq 0 \), \( 0 \leq \beta < n \) satisfy \( \frac{\alpha}{\alpha + \frac{\beta}{n}} \leq 1 \), then the following inequality holds
\[
\sup_{u \in W^{1,n}_0(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} \exp\left(\frac{\alpha}{\alpha + \frac{\beta}{n}} |x|^{\beta}\right) |x|^{\beta} dx \leq C_n. 
\] **(1.2)**

This inequality was extended to \( \Omega = \mathbb{R}^n \) in Adimurthi and Yang [3] which gave a singular version of the inequality in Li and Ruf [21]. Then, Dong and Lu [7], and Dong, Lam and Lu [6] established the two weighted Trudinger-Moser inequality in \( \mathbb{R}^n \). (See also Lam and Lu [11, 12] for more singular Trudinger-Moser inequalities on \( \mathbb{R}^n \).)

Another interesting question is whether there exist extremal functions for the Trudinger-Moser inequality (1.1). The first breakthrough was due to the celebrated work of Carleson and Chang [4], who proved the existence of extremal functions when \( \Omega \) is a ball in \( \mathbb{R}^n \) by the symmetrization argument. Later, Carleson and Chang’s results were extended by Flucher [9] to arbitrary bounded domain in \( \mathbb{R}^2 \) and by Lin [22] to general bounded domain in \( \mathbb{R}^n \) (\( n \geq 2 \)). Li [17, 19] and [18] developed a blow-up method to establish the existence of extremal functions for the Trudinger-Moser inequality on Riemannian manifolds (see also [10, 19, 20, 28, 29, 30]).

As we mentioned before, Wang and Ye [36] established Theorem B. In the spirit of Wang and Ye [36], in this paper, our main result is to establish an improved Hardy-Trudinger-Moser inequality and the existence of the extremals, which are stated as Theorem 1 and Theorem 2 respectively. The main tools we use are the rearrangement argument and blow-up analysis. Because of the presence of weights, we will encounter additional difficulties than the non-weighted case. Our main results are as follows.

**Theorem 1.1.** Let \( 0 < t < 2 \), \( u \in C_0^\infty(B) \) and \( H(u) \) be defined as follows
\[
H(u) := \int_B |\nabla u|^2 dx - \int_B \frac{u^2}{(1 - |x|^2)^{2}} dx, 
\] then there exists a constant \( C_0 > 0 \) such that the following inequality holds
\[
S := \sup_{H(u) \leq 1} \int_B e^{4\pi(1-t/2)u^2}/|x|^t dx \leq C_0. \] **(1.4)**

**Theorem 1.2.** Let \( 0 < t < 2 \), then there exists \( u_0 \in \mathcal{H}^1(B) \) satisfying
\[
H(u_0) := \int_B |\nabla u_0|^2 dx - \int_B \frac{u_0^2}{(1 - |x|^2)^{2}} dx = 1, 
\] such that
\[
S = \int_B \frac{e^{4\pi(1-t/2)u_0^2}}{|x|^t} dx. 
\] Here, we mention that the definition of \( \mathcal{H}^1(B) \) will be given in Section 2.
It’s worthy to mention that when $t = 0$, Theorem 1 and Theorem 2 are the results of Wang and Ye [36].

Here, we describe the organization of this article. In Section 2, We reduce the problem to radially symmetric and nonincreasing function by the rearrangement argument. We prove the subcritical singular Hardy-Trudiger-Moser inequality and the existence of their extremals. In Section 3, we investigate the asymptotic behavior of the maximizing sequence $\{u_k\}$ near and far away from the origin by carrying out blow-up analysis procedure. Hence we complete the proof of Theorem 1.1. In Section 4, we assume the sequence $\{u_k\}$ blows up at the origin, then we give the explicit upper bound for critical function. In Section 5, by constructing proper test function sequence, we can verify that $S$ exceeds the upper bound, which implies no blowing up occurs. Thus we complete the proof of Theorem 1.2.

2. Maximizers of subcritical version. To prove Theorem 1, we first reduce the problem to radially symmetric decreasing functions by the rearrangement argument. For any $u \in H_0^1(B)$, where $B$ is the unit disk in $\mathbb{R}^2$ and $H_0^1(B)$ denotes the classical Sobolev space. Let $u^*$ denote the radially nonincreasing rearrangement with respect to $dV = \frac{4}{(1-|x|^2)^2} dx$. Then we know $\|\nabla u^*\|_2 \leq \|\nabla u\|_2$ and

$$\int_B \frac{u^*}{(1-|x|^2)^2} dx = \int_B \frac{u}{(1-|x|^2)^2} dx.$$  

We note $H(u) := \int_B |\nabla u|^2 dx - \int_B \frac{u}{(1-|x|^2)^2} dx$, hence $\|u\|_H := \sqrt{H(u)}$ defines a norm over $H_0^1(B)$. The completion of $C_\infty^0(B)$ with respect to the norm $\| \cdot \|_H$ is a Hilbert space, which is denoted by $\mathcal{H}(B)$. For simplicity, we denote $\mathcal{H}(B)$ by $\mathcal{H}$ and $\| \cdot \|_H$ by $\| \cdot \|$. We know $H(u) \leq 1$ implies $H(u^*) \leq 1$.

Notice the function $\frac{(1-|x|^2)^2}{|x|^t}$ is radially nonincreasing for $x \in B$ and $0 < t < 2$, by calculation we can get the rearrangement of $\frac{(1-|x|^2)^2}{|x|^t}$ is also itself (with respect to $dV = \frac{4}{(1-|x|^2)^2} dx$). Using the Hardy-Littlewood inequality

$$\int_B f^* g^* dV \geq \int_B f g dV,$$

we can get

$$\int_B \frac{e^{4\pi(1-t/2)u^*^2}}{|x|^t} dx \geq \frac{1}{4} \int_B \frac{e^{4\pi(1-t/2)u^*^2} (1-|x|^2)^2}{|x|^t} dV$$

$$\geq \frac{1}{4} \int_B e^{4\pi(1-t/2)u^*^2} \frac{(1-|x|^2)^2}{|x|^t} dV$$

$$= \int_B e^{4\pi(1-t/2)u^*^2} \frac{1}{|x|^t} dx.$$  

Therefore, we only need to consider nonincreasing radially symmetric functions. We denote by

$$\Sigma := \{u : u \in C_0^\infty(B), u(x) = u(r) \text{ with } r = |x|, \ u' \leq 0 \text{ and } \|u\| \leq 1\},$$

and let $\mathcal{H}^1(B)$ be the closure of $\Sigma$ in $\mathcal{H}$.

We present the following subcritical version first.
Lemma 2.1. Let \(0 < t < 2\), \(\alpha_k\) be an increasing sequence which converges to \(4\pi(1 - \frac{t}{2})\) as \(k \to +\infty\), then there exists a constant \(C > 0\) such that

\[
\sup_{u \in \mathcal{H}^1(B)} \int_B \frac{e^{\alpha_k u^2}}{|x|^t} \, dx \leq C,
\]  
(2.1)

and the supremum can be attained by some \(u_k \in \mathcal{H}^1(B)\).

Proof. We only need to show the inequality (2.1) holds for the subspace \(\Sigma\). Moreover, it is enough to show the function \(u \in \Sigma\) with \(u(0) > 1\), since if \(u(0) \leq 1\), we have

\[
\left\| \frac{e^{\alpha_k u^2}}{|x|^t} \right\|_{L^1(B)} \leq \frac{2\pi e^{\alpha_k}}{2 - t}.
\]

Then as mentioned in the proof of Theorem 3 in [36], we claim that there exists two constants \(r_2 \in (0, 1)\) and \(C > 0\) independent of \(u \in \Sigma \cap \{u(0) > 1\}\) such that

\[
\|\nabla u\|_{L^2(B_{r_2})} \leq 1 \text{ and } u(r_2) \leq C.
\]

We omit the detailed proof of this claim but refer the reader to [36]. For any \(r < r_2\), we have

\[
\alpha_k u^2 \leq 4\pi(1 - t/2)(u(r) - u(r_2))^2 + C_k.
\]

Here \(C_k\) depends only on \(k\). We use the singular Trudinger-Moser inequality on the bounded domain and get the following inequality

\[
\int_B \frac{e^{\alpha_k u^2}}{|x|^t} \, dx = \int_{B_{r_2}} \frac{e^{\alpha_k u^2}}{|x|^t} \, dx + \int_{B_{r_2}^c} \frac{e^{\alpha_k u^2}}{|x|^t} \, dx
\]

\[
\leq \int_{B_{r_2}} \frac{e^{4\pi(1 - \frac{t}{2})(u - u(r_2))^2} + C_k}{|x|^t} \, dx + \frac{1}{2 - t} 2\pi(1 - r_2^2) e^{\alpha_k u(r_2)^2}
\]

\[
\leq C_k \int_B \frac{e^{4\pi(1 - \frac{t}{2})(u - u(r_2))^2}}{|x|^t} \, dx + \frac{2\pi e^{4\pi C^2(1 - \frac{t}{2})}}{2 - t}
\]

\[
< +\infty.
\]

Thus we complete the proof of inequality (2.1). Next, we will show the existence of extremals. Fix \(k\), consider a maximizing sequence \(v_j \in \mathcal{H}^1(B)\) for inequality (2.1) with \(\|v_j\| \leq 1\). Take a subsequence (we still denote as \(v_j\)) satisfying

\[
v_j \rightharpoonup u_k, \text{ weakly in } \mathcal{H}^1(B),
\]

\[
v_j \to u_k, \text{ for almost everywhere } x \in B,
\]

\[
v_j \to u_k, \text{ strongly in } L^q(B) \text{ for } 1 < q < +\infty.
\]

It implies that \(\frac{e^{\alpha_k v_j^2}}{|x|^t}\) converges to \(\frac{e^{\alpha_k u^2}}{|x|^t}\) a.e. and is bounded in \(L^q(B)\) for some \(q > 1\). Naturally, we have

\[
\int_B \frac{e^{\alpha_k u^2}}{|x|^t} \, dx = \lim_{j \to +\infty} \int_B \frac{e^{\alpha_k v_j^2}}{|x|^t} \, dx.
\]

Thus, \(u_k\) is the extremal of

\[
\sup_{u \in \mathcal{H}^1(B)} \int_B \frac{e^{\alpha_k u^2}}{|x|^t} \, dx.
\]
Lemma 2.2. \( u_k \) is a maximal sequence for \( S \), that is to say

\[
\lim_{k \to +\infty} \int_B e^{\alpha_k u_k^2} \cdot dx = \sup_{\|u\| \leq 1} \int_B e^{e^{\pi(1-t/2)u^2}} \cdot dx.
\]

Proof. On the one hand, it is obvious that \( \lim_{k \to +\infty} \int_B e^{\alpha_k u_k^2} \cdot dx \leq S \). On the other hand, for any \( \|u\| \leq 1 \), we have

\[
\int_B e^{e^{\pi(1-t/2)u^2}} \cdot dx \leq \liminf_{k \to +\infty} \int_B e^{\alpha_k u_k^2} \cdot dx \leq \liminf_{k \to +\infty} \int_B e^{\alpha_k u_k^2} \cdot dx,
\]

which implies

\[
S \leq \liminf_{k \to +\infty} \int_B e^{\alpha_k u_k^2} \cdot dx.
\]

Therefore, we obtain

\[
\lim_{k \to +\infty} \int_B e^{\alpha_k u_k^2} \cdot dx = \sup_{\|u\| \leq 1} \int_B e^{e^{\pi(1-t/2)u^2}} \cdot dx.
\]

\( \square \)

3. Proof of Theorem 1.1. We next consider the convergence of the sequence \( \{u_k\} \) as \( k \to +\infty \). If \( \|u_k\|_\infty = u_k(0) \) does not go to infinite as \( k \to +\infty \), then there exists subsequence, which we still denote as \( \{u_k\} \), and \( \|u_k\|_\infty \leq C \). We can assume \( u_k \to u_0 \) in \( H \) and \( u_k \to u_0 \) a.e. in \( B \). Hence \( \|u_0\| \leq 1 \) and \( u_0 \in L^\infty(B) \). For any \( \omega \in H \) and \( \|\omega\| \leq 1 \), the following inequality holds

\[
\int_B e^{\alpha_k \omega^2} \cdot dx \leq \int_B e^{\alpha_k u_k^2} \cdot dx.
\]

Exploiting monotone and dominated convergence theorem, we have

\[
\int_B e^{e^{\pi(1-t/2)\omega^2}} \cdot dx = \lim_{k \to +\infty} \int_B e^{\alpha_k \omega^2} \cdot dx \leq \lim_{k \to +\infty} \int_B e^{\alpha_k u_k^2} \cdot dx = \int_B e^{e^{\pi(1-t/2)u_0^2}} \cdot dx.
\]

That is to say

\[
\int_B e^{e^{\pi(1-t/2)u_0^2}} \cdot dx = \sup_{\|u\| \leq 1} \int_B e^{e^{\pi(1-t/2)u^2}} \cdot dx.
\]

In the following, we will suppose Theorem 1 does not hold true, then

\[
\lim_{k \to +\infty} \int_B e^{e^{\pi(1-t/2)u_0^2}} \cdot dx = +\infty = \lim_{k \to +\infty} \|u_k\|_\infty.
\]

Next, we will use blow-up analysis to give a proof of Theorem 1 as in \[17, 18, 19, 28, 29, 30\] and \[36\]. Since \( u_k \) be the extremal function of the subcritical inequality, it is easy to check that \( \|u_k\| = 1 \). By Euler-Lagrange multiplier theorem, one can easily calculate that \( \{u_k\} \) satisfies the following Laplace equation

\[
L_H(u_k) := -\Delta u_k - \frac{u_k}{1 - |x|^2} = \frac{u_k e^{\alpha_k u_k^2}}{\lambda_k |x|}, \quad \text{where} \quad \lambda_k = \int_B e^{\alpha_k u_k^2} \cdot dx. \tag{3.2}
\]

Lemma 3.1. Let \( \lambda_k \) be defined in (3.2), then \( \inf_k \lambda_k > 0 \).
Proof. Using the element inequality $e^t \leq 1 + te^t$ for $t \geq 0$, one has
\[
\int_B \frac{e^{\alpha_k u_k^2} u_k^2}{|x|^t} \, dx \leq \int_B \frac{e^{\alpha_k u_k^2} \alpha_k u_k^2}{|x|^t} \, dx + \int_B \frac{1}{|x|^t} \, dx = \alpha_k \lambda_k + \int_B \frac{1}{|x|^t} \, dx.
\]
By the proof of Lemma 2.2, we know that
\[
\int_B \frac{1}{|x|^t} \, dx < \lim_{k \to +\infty} \int_B \frac{e^{\alpha_k u_k^2}}{|x|^t} \, dx = S.
\]
Consequently, we have \( \inf \lambda_k > 0 \). \( \square \)

**Definition 3.2.** A sequence \( \{u_k\} \in \mathcal{H}^1(B) \) is a Sobolev-normalized concentrating sequence at \( x_0 \), if
i) \( \lim_{k \to +\infty} \|u_k\| = 1 \);
ii) \( u_k \to 0 \) weakly in \( \mathcal{H}^1(B) \);
iii) For any \( \delta > 0 \), \( \lim_{k \to +\infty} \left( \int_{B(B(x_0))} |\nabla u_k|^2 \, dx - \int_{B(B(x_0))} \frac{u_k^2}{(1-|x|^2)^{\frac{1}{2}}} \, dx \right) = 0 \).

**Lemma 3.3.** Assume \( \{u_k\} \) satisfies (3.1) and \( u_k \to u_0 \in \mathcal{H}^1(B) \), then \( u_0 \equiv 0 \).

Proof. Suppose \( u_0 \neq 0 \), then there exists \( r_0 \in (0, \frac{1}{2}) \) such that \( u_0(r_0) > 0 \) and exists \( r_1 \in (0, r_0) \) and \( \eta > 0 \) such that when \( k \) is big enough, we have \( \|\nabla u_k\|_{L^2(B_{r_1})} \leq 1 - \eta < 1 \). We omit the detailed statement here but refer the readers to Lemma 5 of [36]. By singular Trudinger-Moser inequality (1.2), we have
\[
\int_{B_{r_1}} e^{4\pi(1-\eta)|x|^{-t}u_k^2} \, dx \leq C, \quad \text{where} \quad \lim_{k \to +\infty} b_k = u_k(r_1).
\]
Similarly to the proof of Lemma 5 in Wang and Ye [36], we can conclude that \( ||e^{4\pi(1-\eta)|x|^{-t}u_k^2}||_1 \leq C < +\infty \), which contradicts to (3.1). Thus \( u_0 \equiv 0 \). \( \square \)

Next, we define \( c_k = u_k(0) = \max_{x \in B} u_k(x) \) and \( r_k = \frac{\lambda_k}{c_k \exp(\alpha_k c_k^2)} \).

By Lemma 2.1, we can obtain
\[
\lambda_k = \int_B e^{\alpha_k u_k^2} u_k^2 \, dx \leq \int_B \frac{e^{\alpha_k u_k^2} u_k^2}{|x|^t} \, dx + C
\]
\[
\leq c_k^2 e^{(\alpha_k - 2\pi(1-\eta)c_k^2) \frac{r_k^2}{c_k^2}} \int_B \frac{e^{2\pi(1-\frac{1}{2})u_k^2}}{|x|^t} \, dx + 1
\]
\[
\leq c_k^2 e^{(\alpha_k - 2\pi + \pi\eta)c_k^2} + 1.
\]

By the definition of \( r_k \) and notice \( t < 2 \), we have
\[
r_k c_k = \frac{\lambda_k}{\exp(\alpha_k c_k^2)} \leq c_k^2 e^{(-2\pi + \pi\eta)c_k^2} + e^{-\alpha_k c_k^2} \to 0, \quad \text{as} \quad k \to +\infty.
\]

Hence \( \lim_{k \to +\infty} r_k c_k = 0 \) and \( \lim_{k \to +\infty} r_k = 0 \). Define \( v_k(x) = u_k(r_k^{-1} x) \) and \( \xi_k(x) = c_k(v_k(x) - c_k) \). A straightforward calculation yields
\[
- \Delta \xi_k = \frac{v_k \exp(\alpha_k (v_k^2 - c_k^2))}{c_k |x|^t} + \frac{\frac{r_k}{r_k^2} c_k^2}{(1 - \frac{r_k}{|x|^2})^2} \cdot \frac{v_k}{c_k} \text{ in } D'(B_{r_k^{1/(t-2)}}).
\] (3.3)
Now we investigate the asymptotic behavior of \( \{u_k\} \). In view of \( 0 \leq v_k \leq c_k \), for any \( R > 0 \), \(-\triangle \xi_k = O(1)\) in \( B_R \) for big \( k \). Since \( \xi_k(0) = 0 \), standard elliptic estimate implies that \( \xi_k \) converges to \( \xi \) in \( C^1_{\text{loc}}(\mathbb{R}^2) \). Therefore

\[
v_k - c_k = \frac{\xi_k}{c_k} \to 0, \quad \frac{v_k}{c_k} \to 1 \quad \text{and} \quad v_k^2 - c_k^2 = 2\xi_k + \frac{\xi_k^2}{c_k^2} \to 2\xi \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}^2).
\]

Taking \( k \to +\infty \) in equation (3.3), the equation verified by \( \xi \in C^1(\mathbb{R}^2) \) is

\[
-\triangle \xi = \frac{e^{8\pi(1-t)/2}\xi}{|x|^t}.
\]

Recall \( \xi(0) = 0 \) and \( \xi \) is radially symmetric and nonincreasing function, then the solution of equation (3.4) is uniquely determined as

\[
\xi(x) = -\frac{1}{2\pi(2-t)}\ln\left(1 + \frac{2\pi}{2-t}|x|^{2-t}\right).
\]

By the change of variables and careful calculation, we derive

\[
\int_{\mathbb{R}^2} \frac{e^{8\pi(1-t)/2}\xi}{|x|^t} \, dx = \int_{\mathbb{R}^2} \frac{(1 + \frac{2\pi}{2-t}|x|^{2-t})^{-2}}{|x|^t} \, dx = \int_0^{+\infty} \frac{1}{(1 + s)^2} \, ds = 1.
\]

Thus we know the asymptotic behavior of \( \{u_k\} \) near the original point.

Divide \( B \) into three parts: the interior part \( B_{R \sqrt{\frac{c_k}{L}}} \), the outer part

\[
\{Lu_k \leq c_k\} := \left\{ x \in B \, | \, u_k(x) \leq \frac{c_k}{L} \right\}
\]

and the rest region

\[
\{Lu_k \geq c_k\} \backslash B_{R \sqrt{\frac{c_k}{L}}} := \left\{ x \in B \backslash B_{R \sqrt{\frac{c_k}{L}}} \, | \, u_k(x) \geq \frac{c_k}{L} \right\}.
\]

To investigate \( \{u_k\} \) in the outer part and the rest region, we denote \( u_{k,L} = \min\{u_k, \frac{c_k}{L}\} \). We have the following lemma.

**Lemma 3.4.** For any \( L > 1 \), \( \limsup_{k \to +\infty} H(u_{k,L}) \leq L^{-1} \).

**Proof.** One only need to notice the fact that \( \int_{\mathbb{R}^2} \frac{e^{8\pi(1-t)/2}\xi}{|x|^t} \, dx = 1 \). The proof is the same as Lemma 6 of [36], we omit the details here. \( \square \)

**Lemma 3.5.** The following two equalities hold

\[
\lim_{k \to +\infty} \frac{c_k^2}{\lambda_k} = 0 \quad (3.7)
\]

and

\[
\lim_{k \to +\infty} \frac{c_k}{\lambda_k} \int_B \frac{u_k e^{\alpha_k u_k^2}}{|x|^t} \, dx = 1. \quad (3.8)
\]

**Proof.** Firstly, we estimate \( \left\| \frac{e^{\alpha_k u_k^2}}{|x|^t} \right\|_1 \). Fix \( L > 2 \). On the one hand, note the fact that \( u_{k,L} \to 0 \) a.e. in \( B \), we have

\[
I_k := \int_{\{Lu_k \leq c_k\}} \frac{e^{\alpha_k u_k^2}}{|x|^t} \, dx \leq \int_B \frac{e^{\alpha_k u_k^2}}{|x|^t} \, dx \to \frac{2\pi}{2-t}, \quad \text{as} \quad k \to +\infty.
\]
On the other hand, by the uniform convergence of \( u_k \) to zero in \( B_r^c \) for any \( r \in (0, 1) \), we obtain
\[
I_k := \int_{\{Lu_k \leq c_k\}} \frac{e^{\alpha_k u_k^2}}{|x|^t} \, dx \geq \int_{B_r^c} \frac{e^{\alpha_k u_k^2}}{|x|^t} \, dx \to \frac{2\pi}{2 - t} (1 - r^2 - t), \text{ as } k \to +\infty.
\]
Consequently, \( \lim_{k \to +\infty} I_k = \frac{2\pi}{2 - t} \) when we take \( r \to 0 \). Furthermore,
\[
J_k := \int_{\{Lu_k \geq c_k\}} \frac{e^{\alpha_k u_k^2}}{|x|^t} \, dx = \frac{L^2 \lambda_k}{c_k^2} \int_{\{Lu_k \geq c_k\}} \frac{1}{\lambda_k} u_k^2 e^{\alpha_k u_k^2} \, dx \leq \frac{L^2 \lambda_k}{c_k^2} \int_{B} \frac{1}{\lambda_k} u_k^2 e^{\alpha_k u_k^2} \, dx = \frac{L^2 \lambda_k}{c_k^2}.
\]
In this way, we get the following
\[
+\infty = \lim_{k \to +\infty} \int_{B} \frac{e^{\alpha_k u_k^2}}{|x|^t} \, dx = \lim_{k \to +\infty} (I_k + J_k) \leq \frac{2\pi}{2 - t} + \limsup_{k \to +\infty} \frac{L^2 \lambda_k}{c_k^2}, \tag{3.9}
\]
which means \( \liminf_{k \to +\infty} \lambda_k^{-1} c_k^2 = 0 \). Hence we acquire equation (3.7).

In order to prove the equation (3.8), we estimate the integral on three parts respectively. First, it is obvious that
\[
\lambda_k^{-1} c_k \int_{\{Lu_k \leq c_k\}} \frac{u_k e^{\alpha_k u_k^2}}{|x|^t} \, dx \leq \lambda_k^{-1} c_k^2 I_k \to 0, \text{ as } k \to +\infty.
\]
Then, for any \( R > 0 \), by the change of variables, we can get
\[
\lambda_k^{-1} c_k \int_{B_r} \frac{u_k e^{\alpha_k u_k^2}}{|x|^t} \, dx = \int_{B_{\frac{R}{c_k}}} \frac{u_k e^{\alpha_k u_k^2}}{|x|^t} \, dx \to \int_{B_R} \frac{e^{8\pi(1-t/2)\xi}}{|x|^t} \, dx.
\]
For the neck region, we have
\[
\lambda_k^{-1} c_k \int_{\{Lu_k \geq c_k\} \setminus B} \frac{u_k e^{\alpha_k u_k^2}}{|x|^t} \, dx \leq L \int_{\{Lu_k \geq c_k\} \setminus B} \frac{\lambda_k^{-1} u_k^2 e^{\alpha_k u_k^2}}{|x|^t} \, dx \leq L \int_{B \setminus B_{\frac{R}{c_k}}} \frac{\lambda_k^{-1} u_k^2 e^{\alpha_k u_k^2}}{|x|^t} \, dx = L \int_{B \setminus B_{\frac{R}{c_k}}} \frac{\lambda_k^{-1} u_k^2 e^{\alpha_k u_k^2}}{|x|^t} \, dx \to L \int_{B_R} \frac{e^{8\pi(1-t/2)\xi}}{|x|^t} \, dx.
\]
With the help of (3.6), we can complete the proof of equation (3.8) by tending \( R \) to \( +\infty \). \( \square \)
The equation (3.8) and its proof illustrates that $g_\delta$ is a Green's function. The family $g_k$ converges to $G_0$ in $W^{1,p}_\text{loc}(B)$ weakly if $p \in (1, 2)$, strongly in $L^q(B)$ for all $q \geq 1$ and also in $C(B^c_r)$, $\forall r \in (0,1)$. Here $G_0$ is defined as follows

$$G_0(r) = -\frac{\ln r}{2\pi} + C_1 + O(r^{1+\alpha}), \quad r \to 0. \quad (3.11)$$

Now, we give the proof of Theorem 1.

**Proof.** Proof of Theorem 1. Suppose Theorem 1 is false and let $\rho \in (0,1)$ be a small constant. By virtue of Proposition 1, we have

$$\lim_{k \to \infty} c_k u_k(\rho) = G_0(\rho). \quad (3.12)$$

As mentioned in Section 5 of [36], for a fixed $\rho$, it is easy to see that

$$\int_{B_\rho} |\nabla u_k|^2 \, dx < 1, \quad \text{for } k \text{ big enough.}$$

With the help of singular Trudinger-Moser inequality to $(u_k - u_k(\rho))^+ \in H_0^1(B)$, we get

$$\int_{B_\rho} e^{4\pi (1-t/2)(u_k - u_k(\rho))^2/dx} = \int_B e^{4\pi (1-t/2)(u_k - u_k(\rho))^2} \frac{dx}{|x|^t} \leq C.$$

On the other hand, there holds

$$u_k^2(r) = (u_k(r) - u_k(\rho))^2 + 2u_k(r)u_k(\rho) - u_k^2(\rho) \leq (u_k(r) - u_k(\rho))^2 + 2c_k u_k(\rho).$$

Letting $k$ tend to $+\infty$, we have

$$\int_B e^{4\pi (1-t/2)u_k^2} \frac{dx}{|x|^t} \leq \int_{B_\rho} \frac{e^{4\pi (1-t/2)(u_k(r) - u_k(\rho))^2 + 8\pi (1-t/2)c_k u_k(\rho)}}{|x|^t} \, dx$$

$$\leq \frac{2\pi}{2 - t} e^{4\pi (1-t/2)u_k^2(\rho)} + \frac{2\pi}{2 - t} e^{4\pi (1-t/2)u_k^2(r)}$$

$$\to e^{8\pi (1-t/2)G_0(\rho)}C + \frac{2\pi}{2 - t}.$$

Obviously, this contradicts to the hypothesis (3.1). Hence we finish the proof of Theorem 1.
4. The upper bound of \( S \). Let \( \{u_k\} \) be the maximum sequence of \( S \). We will prove the existence of the extremal function by contradiction. Assume Theorem 1.2 is not valid, then \( \lim_{k \to +\infty} \|u_k\|_\infty = +\infty \). Define

\[
S_k := \int_B \frac{e^{\alpha_k u_k^2}}{|x|^t} \, dx = \max_{u \in \mathcal{H}, \|u\| \leq 1} \int_B \frac{e^{\alpha_k u^2}}{|x|^t} \, dx.
\]

Recall Lemma 2.2

\[
\lim_{k \to +\infty} S_k = \sup_{u \in \mathcal{H}, \|u\| \leq 1} \int_B \frac{e^{4\pi(1-t/2)u^2}}{|x|^t} \, dx := S.
\]

We already know \( S < +\infty \) by Theorem 1.1. And it is obvious \( S > \frac{2\pi}{2-t} \). All arguments and the properties acquired for \( u_k \) in the previous context are true, except two properties. One is the proof of Lemma 3.3 and the other is Property (3.7).

We give a new approach of Lemma 3.3. Fix \( \rho \in (0, 1) \), in view of \( u_k \) is uniformly bounded in \( \overline{B_\rho} \) and \( \lim c_k = +\infty \), then for any \( L > 1 \), \( u_{k,L} = u_k \) in \( \overline{B_\rho} \) for \( k \) big enough. It yields

\[
\limsup_{k \to +\infty} \|u_k\|_{L^\infty(B_\rho)} = \limsup_{k \to +\infty} \|u_{k,L}\|_{L^\infty(B_\rho)} \leq C_\rho \limsup_{k \to +\infty} H(u_{k,L}) \leq \frac{C_\rho}{L}.
\]

When \( L \to +\infty \), we have \( u_0 = 0 \) in \( B_\rho \). Notice \( \rho > 0 \) is arbitrary, consequently \( u_0 = 0 \).

We will show that Property (3.7) is not true. Actually, we have the next lemma.

**Lemma 4.1.**

\[
\lim_{k \to +\infty} \frac{c_k^2}{\lambda_k} = \frac{2 - t}{2(S - \pi)}.
\]

**Proof.** Similarly to the proof of (3.9), we get

\[
S = \lim_{k \to +\infty} S_k \leq \frac{2\pi}{2-t} + \limsup_{k \to +\infty} \frac{L^2 \lambda_k}{c_k^2},
\]

this means \( \limsup_{k \to +\infty} \lambda_k c_k^2 < +\infty \) since \( S > \frac{2\pi}{2-t} \). Therefore, \( \limsup_{k \to +\infty} \lambda_k c_k^2 = 0 \). Let \( p_1 = \frac{L^2}{4} > 1 \)(since \( L > 2 \)). Since \( \|u_k\|_q \to 0 \) for any \( q \geq 1 \), we get the following by Hölder inequality

\[
\int_{\{u_k \leq c_k\}} \frac{u_k e^{\alpha_k u_k^2}}{|x|^t} \, dx \leq \left\| \frac{e^{\alpha_k u_k^2}}{|x|^t} \right\|_{p_1} \|u_k\|_{q_1} \to 0, \quad \text{where} \quad \frac{1}{p_1} + \frac{1}{q_1} = 1.
\]

The same approach derives that

\[
\frac{c_k}{\lambda_k} \int_{\{u_k \leq c_k\}} \frac{u_k e^{\alpha_k u_k^2}}{|x|^t} \, dx \to 0, \quad \text{as} \quad k \to +\infty.
\]

Moreover, (3.8) and Proposition 1 still hold true. Let \( \Omega_{\rho,r} = B_\rho \setminus \overline{B_r} \) with \( 0 < r < \rho < 1 \), since \( L_H(G_0) = 0 \) in \( \Omega_{\rho,r} \subset \mathbb{R}^2 \), the Pohozaev identity shows that

\[
\int_{\Omega_{\rho,r}} \frac{\text{div}(a(x)u)}{2} G_0^2(x) \, dx = -\pi \left( s^2 G_0^2(s) + a(s) s^2 G_0^2(s) \right)_{s^2} = 0.
\]
Lemma 4.2. Carleson-Chang [4] type result.

1. Combining (4.1) with (4.2), we obtain for any $\rho \in (0, 1)$,

$$\int_{B_{\rho}} \frac{\text{div}(a(x)x)}{2} G^2_{0}(x)dx - \pi \rho^2 G^2_{0}(\rho) - \pi a(\rho) \rho^2 G^2_{0}(\rho) = -\frac{1}{4\pi}, \quad \forall \rho \in (0, 1). \quad \text{(4.1)}$$

Similarly, employing the Pohozaev identity to $L_H(u_k) = \frac{u_k e^{\alpha_k u^2}}{\lambda_k |x|^t}$ in $B_{\rho}$ and multiplying by $c_k^2$, we get for any $\rho \in (0, 1)$,

$$\int_{B_{\rho}} \frac{\text{div}(a(x)x)}{2} g^2_{k}(x)dx - \pi \rho^2 g^2_{k}(ho) - \pi a(\rho) \rho^2 g^2_{k}(\rho) = c_k^2 \left( \frac{\pi \rho^{2-t} e^{\alpha_k u^2}}{\lambda_k} - \int_{B_{\rho}} \frac{e^{\alpha_k u^2}}{\lambda_k |x|^t} \right). \quad \text{(4.2)}$$

Finally, $g_k \to G_0$ in $C^1_{\text{loc}}(B \setminus \{0\})$ and $L^2(B)$ by standard elliptic theory and Proposition 1. Combining (4.1) with (4.2), we obtain for any $\rho \in (0, 1)$,

$$c_k^2 \left( \int_{B_{\rho}} \frac{e^{\alpha_k u^2}}{\lambda_k |x|^t} dx - \pi \rho^{2-t} \right) \to 1 - \frac{t}{2}, \quad \text{as} \quad k \to +\infty.$$

Letting $\rho \to 1$ and applying the uniform convergence of $u_k$ to 0 in $\overline{B_R}$ for $r > 0$, we complete the proof. \hfill $\square$

We proceed as in [36] to reach a contradiction. It is essential to give a claim of Carleson-Chang [4] type result.

Lemma 4.2. If $\lim_{k \to +\infty} \|u_k\|_{\infty} = +\infty$, then the upper bound

$$S \leq \frac{2\pi}{2 - t} \left( 1 + e^{1+4\pi(1-\frac{1}{2})C_G} \right),$$

where $C_G$ is given in (3.11).

Proof. Fix $L > 2$, we divide the integral areas into three regions. Firstly, it is easy to check

$$\int_{\{L_{u_k} \leq c_k\} \setminus B} \frac{e^{\alpha_k u^2}}{|x|^t} dx \leq \int_{B} \frac{e^{\alpha_k u^2}}{|x|^t} dx \to \frac{2\pi}{2 - t}, \quad \text{as} \quad k \to +\infty.$$  

Secondly, we have the following

$$\int_{\{L_{u_k} \geq c_k\} \setminus B} \frac{e^{\alpha_k u^2}}{|x|^t} dx \leq \frac{L^2 \lambda_k}{c_k} \int_{\{L_{u_k} \geq c_k\} \setminus B} \lambda_k^{-1} \frac{u_k^2 e^{\alpha_k u^2}}{|x|^t} dx$$

$$\leq \frac{L^2 \lambda_k}{c_k} \left( 1 - \int_{B} \lambda_k^{-1} \frac{u_k^2 e^{\alpha_k u^2}}{|x|^t} dx \right)$$

$$\to \frac{2L^2(S - \pi)}{2 - t} \left( 1 - \int_{B_R} \frac{e^{8\pi(1-\frac{1}{2})\xi}}{|x|^t} dx \right) \to 0 \quad \text{(as} \quad R \to +\infty).$$

Thirdly, for the integral over the interior region $B \setminus \overline{B_R}$, we borrow some ideas from [36]. Fix a small constant $\rho \in (0, 1)$, we have

$$\lim_{k \to +\infty} \int_{B_{\rho}} a(x)c_k^2 u_k^2 dx = \lim_{k \to +\infty} \int_{B_{\rho}} a(x)G_0^2 dx =: J_1(\rho).$$
\[ H_{B_{\rho}}(g_k) = H(g_k) - H_{B_{\rho}}(g_k) = \int_{B_{\rho}} \frac{g_k^2 e^{\alpha_k u_k^2}}{\lambda_k |x|^t} \, dx - \int_{\partial B_{\rho}} \frac{\partial g_k}{\partial \nu} g_k \, d\sigma. \]

The first term goes to 0 when \( k \to +\infty \), we get
\[
- \int_{\partial B_{\rho}} \frac{\partial g_k}{\partial \nu} g_k \, d\sigma = - g_k(\rho) \int_{B_{\rho}} \Delta g_k \, dx
\]
\[
= g_k(\rho) \left( \int_{B_{\rho}} a(x) g_k \, dx + \int_{B_{\rho}} \frac{g_k e^{\alpha_k u_k^2}}{\lambda_k |x|^t} \, dx \right)
\]
\[
\to G_o(\rho) \left( \int_{B_{\rho}} a(x) G_0 \, dx + 1 \right) =: J_2(\rho).
\]

Then we know
\[
\int_{B_{\rho}} |\nabla u_k|^2 \, dx = 1 - \frac{E_\rho + o_k(1)}{c_k^2},
\]
where \( \lim_{k \to +\infty} o_k(1) = 0 \) and
\[ E_\rho := J_2(\rho) - J_1(\rho) = G_o(\rho) + G_o(\rho) \int_{B_{\rho}} a(x) G_0 \, dx - \int_{B_{\rho}} a(x) G_o^2 \, dx. \quad (4.3) \]

Let \( l_k(x) = \| \nabla u_k \|_{L^2(B_{\rho})}^{-1} (u_k(x) - u_\rho(x)) \). Clearly, \( l_k \in H^1_0(B_{\rho}) \| \nabla l_k \|_2 = 1 \) and \( l_k \) converges weakly to 0 in \( \mathcal{D}'(B_{\rho}) \). Using the upper bound estimate of singular Trudinger-Moser inequality, there holds
\[
\limsup_{k \to +\infty} \int_{B_{\rho}} \frac{e^{4 \pi (1-t/2) l_k^2}}{|x|^t} - 1 \, dx \leq \frac{2\pi}{2-t} e^{2-t}. \quad (4.4)
\]

Moreover, \( c_k^{-1} l_k \to 1 \) uniformly in \( B \_{R_k} \frac{2}{\pi} \) since \( c_k^{-1} u_k \to 1 \) uniformly in \( B_{\rho} \). As analyzed in [36], we have
\[
u_k^2(x) = l_k^2(x) + 2G_0(\rho) - E_\rho + o_k(1),
\]
where \( o_k(1) \) tends to 0 uniformly in \( B \_{R_k} \frac{2}{\pi} \). In view of equation (4.4), we have
\[
\limsup_{k \to +\infty} \int_{B \_{R_k} \frac{2}{\pi}} e^{4 \pi (1-t/2) \nu_k^2} \, dx \leq \limsup_{k \to +\infty} \int_{B \_{R_k} \frac{2}{\pi}} e^{4 \pi (1-t/2) \nu_k^2} \, dx
\]
\[
\leq e^{(1-t/2)(8\pi G_0(\rho)-4\pi E_\rho)} \limsup_{k \to +\infty} \int_{B \_{R_k} \frac{2}{\pi}} e^{4 \pi (1-t/2) \nu_k^2} \, dx
\]
\[
\leq e^{(1-t/2)(8\pi G_0(\rho)-4\pi E_\rho)} \limsup_{k \to +\infty} \int_{B_{\rho}} e^{4 \pi (1-t/2) \nu_k^2} \, dx
\]
\[
\leq 2\pi e^{2-t} e^{1+(1-t/2)(8\pi G_0(\rho)-4\pi E_\rho)}.
\]

For any small \( \rho > 0 \), combing the three parts of estimation and let \( R \) go to \( +\infty \), we can get
\[
S = \lim_{k \to +\infty} \int_{B} e^{\alpha_k u_k^2} \, dx \leq \frac{2\pi}{2-t} + \frac{2\pi}{2-t} e^{1+(1-t/2)(8\pi G_0(\rho)-4\pi E_\rho)}.
\]
Employing the expansion of $G_0$, we can get
\[
2(1 - \frac{t}{2})\ln r + (1 - \frac{t}{2})(8\pi G_0(r) - 4\pi E_\rho) \to (1 - \frac{t}{2})4\pi C_G, \text{ as } r \to 0.
\]
Consequently, it follows $S \leq \frac{2\pi}{2-t}(1 + e^{1-t(1-\frac{t}{2})4\pi C_G})$.

5. **The proof of Theorem 1.2.** By constructing proper test function sequence, we can verify that $S$ exceeds the upper bound, which implies no blowing up occurs. Then we complete the proof of Theorem 2 by contradiction.

**Lemma 5.1.** There holds $S > \frac{2\pi}{2-t}(1 + e^{1+t(1-\frac{t}{2})4\pi C_G})$.

**Proof.** We will construct a test function as follows
\[
f_\varepsilon(r) = \begin{cases} 
\beta_\varepsilon + \frac{1}{\beta_\varepsilon} \left( -\frac{1}{2\pi(2-t)} \ln(1 + \frac{2\pi}{2-t}|r^{2-t}) + \gamma_\varepsilon \right) & \text{if } r \leq \varepsilon R, \\
\text{otherwise} & \text{if } \varepsilon R \leq r \leq 1,
\end{cases}
\]
where $R = (\ln \varepsilon)^{\frac{1}{2-t}}$, $\beta_\varepsilon$ and $\gamma_\varepsilon$ are constants to be chosen later. First, choose $\gamma_\varepsilon$ such that
\[
\frac{G_0(\varepsilon R)}{\beta_\varepsilon} = \beta_\varepsilon + \frac{1}{\beta_\varepsilon} \left( -\frac{1}{2\pi(2-t)} \ln(1 + \frac{2\pi}{2-t}|R^{2-t}) + \gamma_\varepsilon \right),
\]
which satisfies the function continuous. We can get the following by expanding $G_0$
\[
2\pi(2-t)(\beta_\varepsilon^2 + \gamma_\varepsilon) \\
= -(2-t)\ln(\varepsilon R) + 2\pi(2-t)C_G + \ln(1 + \frac{2\pi}{2-t}|R^{2-t}) + o(\varepsilon R) \\
= -(2-t)\ln \varepsilon + 2\pi(2-t)C_G + \ln \frac{2\pi}{2-t} + O(R^{t-2}).
\]
(5.1)
It is easy to see that $f_\varepsilon \in \mathcal{H}$. Now we estimate $\|f_\varepsilon\|$. Let $0 < r < \rho < 1$, as mentioned in [36], taking $\rho \to 1$, we get
\[
H_{B_{\varepsilon R}}(f_\varepsilon) \leq \frac{1}{4\pi \beta_\varepsilon^2} \left( -2\ln(\varepsilon R) + 4\pi C_G + o(\varepsilon R) \right).
\]
On the other hand, by change of variable and integration by parts, we have
\[
\int_{B_{\varepsilon R}} |\nabla f_\varepsilon|^2 \, dx = \frac{1}{\beta_\varepsilon^2} \int_{B_{\varepsilon R}} |\nabla \xi|^2 \, dx \\
= \frac{2\pi}{\beta_\varepsilon^2(2-t)^2} \int_0^1 \frac{r^{3-2t}}{(1 + \frac{2\pi}{2-t}r^{2-t})^2} \, dr \\
= \frac{1}{2\pi(2-t)\beta_\varepsilon^2} \left( \ln(1 + \frac{2\pi}{2-t}R^{2-t}) - 1 + \frac{2-t}{2-t + 2\pi R^{2-t}} \right).
\]
Hence, we have
\[
H(f_\varepsilon) \leq H_{B_{\varepsilon R}}(f_\varepsilon) + \int_{B_{\varepsilon R}} |\nabla f_\varepsilon|^2 \, dx \\
\leq \frac{1}{4\pi \beta_\varepsilon^2} \left( -2\ln(\varepsilon R) + 4\pi C_G \right) \\
+ \frac{1}{2\pi(2-t)\beta_\varepsilon^2} \left( \ln(1 + \frac{2\pi}{2-t}R^{2-t}) - 1 + O(R^{t-2}) \right) \\
\leq -\frac{\ln \varepsilon}{2\pi \beta_\varepsilon^2} + \frac{C_G}{\beta_\varepsilon^2} + \frac{1}{2\pi(2-t)\beta_\varepsilon^2} \left( \ln \frac{2\pi}{2-t} - 1 + O(R^{t-2}) \right), \\
\]
(5.2)
We choose $\beta_\varepsilon > 0$ such that $H(f_\varepsilon) = 1$, it leads to

$$4\pi \left(1 - \frac{t}{2}\right) \beta_\varepsilon^2 \leq \left(1 - \frac{t}{2}\right) \left(-2 \ln \varepsilon + 4\pi C_G\right) + \ln \frac{2\pi}{2-t} - 1 + O(R^{t-2}) = O(|\ln \varepsilon|).$$

Hence, $\beta_\varepsilon = O(|\ln \varepsilon| \varepsilon^{\frac{1}{2}})$. From (5.1), it follows

$$4\pi \left(1 - \frac{t}{2}\right) \gamma_\varepsilon \geq 1 + O(R^{t-2}).$$

Next we will estimate $\left\| \frac{4\pi(1-\frac{t}{2})f_\varepsilon^2}{|x|^t} \right\|_1$. Applying $e^t \geq 1 + t$, we have

$$\int_{B_{\varepsilon R}} \frac{e^{4\pi(1-t/2)f_\varepsilon^2}}{|x|^t} \, dx \geq \frac{2\pi}{2-t} - \frac{2\pi}{2-t} (\varepsilon R)^2 + \frac{2\pi(2-t)}{\beta_\varepsilon^2} \int_{B_{\varepsilon R}} \frac{G_0^2}{|x|^t} \, dx$$

$$\geq \frac{2\pi}{2-t} + \frac{2\pi(2-t)}{\beta_\varepsilon^2} \left( \int_B G_0^2 \right) + o_\varepsilon(1).$$

Furthermore, in $B_{\varepsilon R}$, notice $f_\varepsilon = \beta_\varepsilon + \frac{\xi(\varepsilon^{-1} r) + \gamma_\varepsilon}{\beta_\varepsilon}$ in $B_{\varepsilon R}$, combine equation (5.1) with (5.3), we have

$$4\pi \left(1 - \frac{t}{2}\right) f_\varepsilon^2(r) \geq 4\pi \left(1 - \frac{t}{2}\right) \beta_\varepsilon^2 + 8\pi \left(1 - \frac{t}{2}\right) \gamma_\varepsilon + 8\pi \left(1 - \frac{t}{2}\right) \xi(\varepsilon^{-1} r)$$

$$= 4\pi \left(1 - \frac{t}{2}\right) \left(\beta_\varepsilon^2 + \gamma_\varepsilon\right) + 4\pi \left(1 - \frac{t}{2}\right) \gamma_\varepsilon + 8\pi \left(1 - \frac{t}{2}\right) \xi(\varepsilon^{-1} r)$$

$$\geq -(2-t) \ln \varepsilon + 2\pi(2-t) C_G + \ln \frac{2\pi}{2-t} + 1$$

$$+ 8\pi \left(1 - \frac{t}{2}\right) \xi(\varepsilon^{-1} r) + O(R^{t-2}).$$

Then in $B_{\varepsilon R}$, recall the equation (3.5), we can estimate as follows

$$\int_{B_{\varepsilon R}} \frac{e^{4\pi(1-t/2)f_\varepsilon^2}}{|x|^t} \, dx \geq e^{-\varepsilon(2-t)} + 2\pi(2-t) C_G + \ln \frac{2\pi}{2-t} + O(R^{t-2}) \int_{B_{\varepsilon R}} \frac{e^{8\pi(1-t/2)f_\varepsilon^2}}{|x|^t} \, dx$$

$$= \frac{2\pi}{2-t} e^{2\pi(2-t) C_G + 1 + O(R^{t-2})} \int_{B_{\varepsilon R}} \frac{e^{8\pi(1-t/2)f_\varepsilon^2}}{|x|^t} \, dx$$

$$= \frac{2\pi}{2-t} e^{2\pi(2-t) C_G + 1 + O(R^{t-2})} \int_{B} G_0^2 \, dx + o_\varepsilon(1).$$

Finally, since $R^{t-2} \beta_\varepsilon^2 = o_\varepsilon(1)$, there holds

$$\int_{B} \frac{e^{4\pi(1-t/2)f_\varepsilon^2}}{|x|^t} \, dx \geq \frac{2\pi}{2-t} + \frac{2\pi(2-t)}{\beta_\varepsilon^2} \left( \int_B G_0^2 \right) + o_\varepsilon(1)$$

$$+ \frac{2\pi}{2-t} e^{2\pi(2-t) C_G + 1 + O(R^{t-2})} \int_{B} G_0^2 \, dx + o_\varepsilon(1)$$

$$\geq \frac{2\pi}{2-t} + \frac{2\pi}{2-t} e^{2\pi(2-t) C_G + 1 + O(R^{t-2})} \int_{B} G_0^2 \, dx + o_\varepsilon(1).$$

By choosing a small $\varepsilon > 0$, we conclude

$$S \geq \left\| \frac{e^{4\pi(1-t/2)f_\varepsilon^2}}{|x|^t} \right\|_1 > \frac{2\pi}{2-t} + \frac{2\pi}{2-t} e^{2\pi(2-t) C_G + 1}. $$

It contradicts to Lemma 4.2, hence we complete the proof.

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