BF models, Duality and Bosonization on higher genus surfaces

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Abstract

The generating functional of two dimensional $BF$ field theories coupled to fermionic fields and conserved currents is computed in the general case when the base manifold is a genus $g$ compact Riemann surface. The lagrangian density $L = dB \wedge A$ is written in terms of a globally defined 1-form $A$ and a multi-valued scalar field $B$. Consistency conditions on the periods of $dB$ have to be imposed. It is shown that there exist a non-trivial dependence of the generating functional on the topological restrictions imposed to $B$. In particular if the periods of the $B$ field are constrained to take values $4\pi n$, with $n$ any integer, then the partition function is independent of the chosen spin structure and may be written as a sum over all the spin structures associated to the fermions even when one started with a fixed spin structure. These results are then applied to the functional bosonization of fermionic fields on higher genus surfaces. A bosonized form of the partition function which takes care of the chosen spin structure is obtained.

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1 Introduction

In this paper we compute the generating functional of $BF$ topological systems coupled to fermions on a two dimensional compact manifold of arbitrary genus and apply the result to discuss the bosonization of the fermions. This work has two different motivations. Firstly, it has been observed, that the allowed world hyper-surfaces described by classical sources (p-branes) coupled to a $BF$ theory are subject to restrictions of topological nature. One is then led to ask the question of how this effect is translated to the quantum theory. The second aspect which motivates this work concerns the relation between topological models and duality transformations. For a large class of systems, duality transformations have been devised along the lines of the T-duality transformation in the sigma models. The method consists essentially in a two step elimination of one field in terms of it dual variable. First one introduces an auxiliary gauge field constrained at the beginning to have zero curvature. This allows to decouple the original variable from the currents and then one may perform the remaining integral in the quadratic approximation. At this point the connection to the $BF$ theory appears, since to impose the zero curvature condition into the functional integral one may introduce the partition function of a $BF$ topological model. When the duality transformation is applied to the generating functional of free fermionic fields on genus zero manifolds, it leads to the bosonized representation of the theory. In the operatorial approach bosonization in two and three dimensions has been related to the construction of dual soliton operators in bosonic theories and this give an additional meaning to the duality transformation discussed in and . At the intermediate step after introducing the gauge field one is dealing with a $BF$ theory coupled to fermions which is the subject of this paper. This point of view has been also extended to higher dimensions.

Over topologically trivial manifolds, the procedure described above allows in some cases to determine exact equivalences between fields theories. When the base manifold has genus $g$ one has to take care of the global definition of the geometrical objects appearing in the formulation. The global aspects introduced by the auxiliary fields in the path integral has been explicitly tested for example in the purely bosonic self-dual vectorial model in $3-D$. This model is known to be locally equivalent to the topologically massive model and in fact can be viewed as a gauge fixed version.
Nevertheless it has been shown that in topologically non trivial manifolds this equivalence has to be reinterpreted \cite{17}, \cite{18}, \cite{19} since the partition function of the topologically massive model has an additional factor of topological origin. When matter fields are included, the coupling with the topological field theory may be related to self interaction terms for the fermions \cite{20}.

In the specific case of the fermionic models there are other reasons to explore the consequences of defining the system on higher genus surfaces. Even in genus zero surfaces, when coupled to gauge fields with non-trivial topological properties, the fermions show dynamical effects. The most notorious of these is the non vanishing of fermion condensates \cite{21}, \cite{22}, \cite{23} due to the contributions of the instantons associated in four dimensions to the resolution of the $U(1)$ problem \cite{24}. The vanishing of such condensates in the topological trivial case is enforced by gauge symmetry. The non vanishing result in the most general situation may be traced in the functional approach to an explicit contribution of the zero modes of the fermionic fields\cite{21}. In higher genus surfaces one expects a more rich structure in the gauge field sector, but also further complications are introduced in the duality transformation when gauge fields of non-vanishing topological index have to be considered. For this reason in this article we exclude this possibility.

This paper is organized as follows. In section 2 we review some useful concepts and notation and discuss the result of the computation of the fermionic determinant in genus $g$ manifolds. In section 3 we compute the generating functional of a particular $BF$ topological theory coupled to fermions. This model which as we will see later appears naturally in the bosonization of fermions in higher genus surfaces is described in terms of a 1-form $A$ globally defined and a multivalued $B$ field. Since $dB$ should remain univalued one has to impose restrictions over the periods of $dB$. When these periods are chosen to be integral multiples of $4\pi$ the partition function is shown to be a sum over all spin structures even if one starts with a fixed spin structure. This is an interesting result which in particular implies that the partition function is independent of the spin structure originally chosen. In section 4 we discuss the bosonization \cite{3} of fermionic fields on higher genus Riemann surfaces. Here again, bosonization may be understood as a duality transformation \cite{8}, \cite{7} between the fermionic current and the Hodge dual of the field intensity tensor of a vector field. A careful treatment of the global aspects in the formulation leads naturally to a bosonized effective action in terms of
a multi-valued 0-form.

2 The fermionic determinant on higher genus compact Riemann surfaces

We will consider $BF$ models coupled to fermionic fields over higher genus Riemann surfaces. In order to compute its partition function one needs the explicit formula for the fermionic determinant in the case of zero curvature gauge potentials $A$ on trivial $U(1)$ line bundles. This determinant was computed in references [25], [26]. To express the result, let us introduce some notation concerning the properties of the manifold and the fields. We take $a_i$ and $b_j$ to be a basis of homology of closed curves over $\Sigma$, a compact Riemann surface of genus $g$. The set of curves $a_i$ and $b_j$ will be denoted by $C'$. If one deforms continuously the fermionic field along the curves of the basis, after returning to the original point the fermionic field may change sign or not. A spin structure over $\Sigma$ is determined by a combination of one of this two possibilities for each of the curves of the basis. The gauge potentials $A$ on a trivial $U(1)$ bundle are characterized by the vanishing of the Chern class

$$\int_{\Sigma} F(A) = \int_{\Sigma} dA = 0. \quad (1)$$

The index of the corresponding Dirac operator is then zero and consequently there are no zero modes in the fermionic sector.

The potential may be decomposed into its exact, co-exact and harmonic parts:

$$A = ds + dp + A_h. \quad (2)$$

The harmonic part of the field is expressed in terms of a base of real harmonic forms $\alpha_i$ and $\beta_i$, $i, j = 1, \cdots, g$ as follows,

$$A_h = 2\pi \sum_{i}^{g} (u_i \alpha_i - v_i \beta_i). \quad (3)$$

The real harmonic basis $\alpha_i$ and $\beta_j$ is constructed from two normalized holomorphic basis $\omega_j$ and $\hat{\omega}_j$, $j = 1, \cdots, g$

$$\int_{a_i} \omega_j = \delta_{ij}, \quad \int_{a_i} \hat{\omega}_j = \hat{\Omega}_{ij}$$
\[ \int_{b_i} \omega_j = \Omega_{ij}, \quad \int_{b_i} \hat{\omega}_j = \delta_{ij}, \tag{4} \]

where \( \Omega \) is the period matrix. In terms of \( \omega_j \) and \( \hat{\omega}_j \), \( \alpha_i \) and \( \beta_j \) are given by

\[
\beta_j = \frac{1}{2i} (\omega_k - \bar{\omega}_k)[Im \Omega]_{kj}^{-1},
\]

\[
\alpha_i = \frac{1}{2i} (\hat{\omega}_k - \bar{\hat{\omega}}_k)[Im \hat{\Omega}]_{ki}^{-1}.
\tag{5}
\]

The imaginary part of the period matrix \( Im \Omega \) is always an invertible matrix.

Let us now consider a fermionic field defined over a genus \( g \) compact Riemann surface with a definite but arbitrary spin structure. The spin structure is fixed by specifying two \( g \)-dimensional vectors \( \epsilon_i \) and \( \kappa_j \) with components 0 or \( \frac{1}{2} \) so that the periodicities of the fermions about the cycles \( a_i \) and \( b_j \) are respectively \( \exp(2\pi i \epsilon_i) \) and \( \exp(-2\pi i \kappa_j) \). The partition function which defines the fermionic determinant is

\[
\hat{Z}_f[A, \epsilon, \kappa] = \int D\psi D\bar{\psi} e^{\int d^2x \sqrt{g} \bar{\psi} (-i \partial + A) \psi} = \det[-i \mathcal{D} + A],
\tag{6}
\]

where we take \( \mathcal{D} \) to be the covariant derivative for the fermions.

The fermionic determinant in this situation may be obtained from the results in \cite{25},\cite{26} and is given by

\[
\hat{Z}_f[A, \epsilon, \kappa] = e^{-\frac{1}{2\pi} \int \Sigma F(A) \mathcal{D}^{-1} + \frac{1}{2} \det \left[ \frac{\det Im \Omega Vol(\Sigma)}{\det \Delta_0} \right] \frac{1}{2} \left| \theta \left[ \begin{array}{c} u + \epsilon \\ v + \kappa \end{array} \right] (0|\Omega) \right|^2}.
\tag{7}
\]

Here \( \Delta_0 \) is the Laplacian operator acting on 0-forms and \( Vol(\Sigma) \) is the area of the Riemann surface. The third factor is a \( \theta \)-function given by

\[
\theta \left[ \begin{array}{c} u \\ v \end{array} \right] (0|\Omega) = \sum_{n \in \mathbb{Z}^g} exp[i \pi (n + u) \Omega (n + u) + i2\pi (n + u)v].
\tag{8}
\]

We note that in \cite{7} the first factor depends only on the co-exact component of the gauge field. This contribution corresponds to the result for genus zero surfaces \cite{27} which is usually written in the form,

\[
\frac{\det(-i \mathcal{D} + A)}{\det(-i \mathcal{D})} = e^{-\frac{1}{2\pi} \int d^2x \sqrt{g} \mathcal{A}_\nu (\delta^{\nu\mu} - \mathcal{A}_\mu) \mathcal{A}_\nu}.
\tag{9}
\]
The other two factors in (7) give the Dirac determinant for a purely harmonic potential $A_h$ of the form (3). The result (7) has been used to investigate the Schwinger model in higher genus surfaces [28].

Finally by summing over all spin structures, we may also define,

$$\hat{Z}_f[A] = \sum_{\epsilon, \kappa} \hat{Z}_f[A, \epsilon, \kappa]$$ (10)

which will play a role in what follows.

### 3 Two dimensional BF theories coupled to fermions

In this section we will compute the generating functional for a particular BF system coupled to fermions over a genus $g$ Riemann surface. The action functional of a BF theory is written in terms of a connection $A$ and a field $B$ which may be interpreted as a Lagrange multiplier which enforces the $A$ field to have zero curvature [1]. In its usual form it is given by,

$$S_{BF} = \int_{\Sigma} dA \wedge B$$ (11)

Here the 0-form $B$ and $dA$ are defined globally on the manifold $\Sigma$. The connection $A$ may be allowed to have transitions over $\Sigma$. The computation of the partition function of this system was discussed in [2]. The off-shell BRST charge was computed in [29]. We will consider a modification of this system which appears naturally in the context of bosonization. We consider the action,

$$S_{BF}^{mod} = \int_{\Sigma} dB \wedge A$$ (12)

which may be different from the action above, for trivial bundles, only over higher genus surfaces. The one forms $A$ and $dB$ have to be globally defined but $B$ may be multi-valued. Due to the non trivial topological structure of the manifold, one may distinguish three cases in the definition of the generating functional. One may consider the following conditions on the periods of $dB$:

$$\oint_{C_i} dB = 0$$ (13)
\[ \oint_{C_i} dB = 2\pi m^I \]  
\[ \oint_{C_i} dB = 4\pi m^I. \] (14)  
(15)

The first case is the usual BF model. In the second and third cases we consider the summation on all the values of \( m^I \) in the functional integral \( (16) \). Each of the choices defines a different model.

The generating functional of these systems coupled to conserved currents \( j \) and \( J \) is given in all the cases by

\[ Z[j, J, \epsilon, \kappa] = \sum_{m^I} \int \mathcal{D}C \mathcal{D}A \mathcal{D}B \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_{\text{eff}}}, \] (16)

\[ S_{\text{eff}} = \int d^2x \sqrt{g} \bar{\psi}(iD - A - J)\psi + L_g] + \int_{\Sigma} \left( \frac{i}{2\pi} dB \wedge A - *J \wedge A \right) \] (17)

where \( L_g \) includes the gauge fixing term and the contributions of the auxiliary fields (ghosts fields and Lagrange multipliers) and \( \mathcal{D}C \) stands for the integration measure in those fields. The sum in \( m^I \) is included to stress the fact that we are summing over the \( B \) field configurations which satisfy either \( (13) \), \( (14) \) or \( (15) \). The spin connection is fixed and identified by the \( g \)-dimensional vectors \( \epsilon_i \) and \( \kappa_j \).

The functional integration on the Lagrange multiplier \( B \) of course provides the factor \( \delta(F(A)) \) in the measure of the generating functional but as we will see presently, the additional summation over the periods gives rise to a factor which constraints also the periods of \( A \). Let us see how this works. Suppose for example that we compute the generating functional \( (16) \) summing over the \( B \) field configurations which fulfills \( (15) \). Given two different configurations of \( B \), say \( B_1 \) and \( B_2 \), satisfying this condition we have

\[ B_2 - B_1 = b, \] (18)

where \( b \) is univalued over \( \Sigma \). In general we may then write,

\[ B = B_{m^I} + b, \] (19)

with \( B_{m^I} \) a specific configuration satisfying \( (15) \) with a set of values \( m^I \). The functional integration on the multi-valued \( B \) field has been expressed as an integration on the univalued function \( b \) and a sum over all possible
choices \( m^I \). Consider now the \( BF \) action in the sector defined by one of such choices. We have
\[
\int_{\Sigma} dB \wedge A = \int_{\Sigma} d(B_{m^I} A) + \int_{\Sigma} (-B_{m^I}) \wedge dA + \int_{\Sigma} (-b) \wedge dA.
\] (20)

The generating functional becomes,
\[
Z[j,J,\epsilon,\kappa] = \sum_{m^I} \int DADbDCD\psi\bar{\psi}e^{-S_{\text{eff}}}
\] (21)

\[
S_{\text{eff}} = \int_{\Sigma} d^2x \sqrt{g} [ \bar{\psi}(iD^\dagger - \bar{J} - A^\dagger) \psi + L_g ]
\]
\[
+ \frac{i}{2\pi} \int_{\Sigma} [d(B_{m^I} A) - (B_{m^I} + b)\wedge dA] - \int_{\Sigma} \ast J \wedge A
\]

At this point, one recovers the factor \( \delta(dA) \) in the measure of the generating functional by performing the functional integral in \( b \) and in the ghost fields introduced to guarantee the \( BRST \) invariance of the effective action. In particular this makes the second term in (20) to vanish and to disappear also from (21).

To evaluate the remaining functional integral, consider a triangulation of \( \Sigma \) in terms of elementary domains \( U_i, \ i \in [1,\ldots,N] \). Since \( \Sigma \) is compact the triangulation exists and the covering is provided by a finite number of elementary domains. Let \( A^i \) and \( B_{m^I}^i \) be the restrictions of the fields to the domain \( U_i \). Then, in the functional space projected by \( \delta(dA) \), we have
\[
\left[ \int_{\Sigma} dB \wedge A \right]_{dA=0} = \int_{\Sigma} d(B_{m^I} A) = \sum_{i=1}^N \int_{U_i} d(B_{m^I}^i A^i)
\]
\[
= \sum_{U_i \cap U_j \neq \emptyset} \int_{U_i \cap U_j} (B_{m^I}^i - B_{m^I}^j) A = \sum_{U_i \cap U_j \neq \emptyset} 4\pi m^{(ij)} \int_{U_i \cap U_j} A
\] (22)

where \( m^{(ij)} \) are integers. Using again that the connection is flat we finally get,
\[
e^{-\frac{i}{2\pi} \int_{\Sigma} dB \wedge A} = e^{i \sum_I 2m^I \bar{f}_{C^I} A}.
\] (23)

Here one recognizes the coefficients of the Fourier expansion of a delta function with period \( \pi \). Upon summing over all \( m^I \) the total contribution is
\[
\delta(F(A)) \sum_I \delta(\int_{C^I} A - \pi n^I).
\] (24)
When the $B$ field in (16) is taken to satisfy (13) only the factor $\delta (F(A))$ appears. We will not discuss this case furthermore. When the $B$ field is taken to satisfy (14), the second delta function has period $2\pi$ and the factor turns out to be

$$\delta (F(A)) \sum_t \delta (\oint_{C_t} A - 2\pi n^t).$$

Let us see now how the conditions (24) or (25) enter in the complete evaluation of (16). Using the decomposition (2) for the $A$ field, the factor $\delta (dA)$ in the measure of (16) allows the integration of the co-exact part of $A$ and we are left with the task of determining which are the configurations of $A_h$ that contribute. It is now straightforward to show that the delta functions in (24) (or respectively (25)) constrain the values of the coefficients in the expansion (3) of $A_h$ to be half-integers (or integers). To continue we use this fact and perform the functional integration in the fermions. Defining $u^0$ and $v^0$ to be the coefficients in the expansion of the harmonic part of $j$,

$$j_h = 2\pi \sum_i (u^0_i \alpha_i - v^0_i \beta_i).$$

we obtain:

$$Z[j, J, \epsilon, \kappa] = \sum_{u, v} \left[ \frac{det I m \Omega \text{Vol} (\Sigma) \Omega'}{det \Delta_0} \right]^{1/2} \left| \theta \left[ \begin{array}{c} u + u^0 + \epsilon \\ v + v^0 + \kappa \end{array} \right] (0|\Omega) \right|^2 e^{\int_{\Sigma} (\ast J \wedge A_h - \frac{1}{2\pi} \delta^\ast \frac{1}{2\pi} \delta J)}.$$  

The sum in (27) is over the allowed values of $u$ and $v$ which as we already said are all the integers or all the half-integers depending which case we are considering. From here on we have to distinguish between the two cases.

Let us take first the case when the $B$ field satisfies (14). Then in (27) we have a sum over the $g$-tuples with integral entries which we label by $m$ and $l$. The factor with the theta function in (24) takes the form,

$$\left| \theta \left[ \begin{array}{c} u^0 + m + \epsilon \\ v^0 + l + \kappa \end{array} \right] (0|\Omega) \right|^2$$

It is straightforward to see from (8) that this becomes independent of $l$ due to the square norm that we are taking. Moreover (28) also is independent of
m, since in (8) one may redefine \( n + m = n' \) and one still will have summation in all \( n' \). We can then factorize the contribution of the harmonic part of the field to the partition function in the form,

\[
Z_{2n}[j, J, \epsilon, \kappa] = \hat{Z}_f[j, \epsilon, \kappa] \sum_{m,l} e^{\int_{\Sigma} *J \wedge A_h}
\]  

(29)

with \( \hat{Z}_f[j, \epsilon, \kappa] \) given by (7)

\[
\hat{Z}_f[j, \epsilon, \kappa] = \left[ \frac{det m\Omega Vol(\Sigma)}{det' \Delta_0} \right]^{\frac{1}{4}} \left| \theta \left[ \begin{array}{c} u^0 + \epsilon \\ v^0 + \kappa \end{array} \right] (0|\Omega) \right|^2 e^{-\frac{1}{2} \int_{\Sigma} \frac{1}{\Omega} \theta^{-1} \theta^{-1} \ast dj}
\]  

(30)

Note that the external current \( J \) only couples to the harmonic part of the vector field. When \( J \) is zero we obtain,

\[
Z_{2n}[j, 0, \epsilon, \kappa] = \mathcal{N} \hat{Z}_f[j, \epsilon, \kappa]
\]  

(31)

with \( \mathcal{N} \) a constant which measures the volume of the harmonic space. This factor is expected from the original expression (16) since in that case the volume of the zero modes factorizes from the functional integral.

Consider now the situation when (13) holds. We have instead of (28) the expression,

\[
\left| \theta \left[ \begin{array}{c} u^0 + \frac{m}{2} + \epsilon \\ v^0 + \frac{l}{2} + \kappa \end{array} \right] (0|\Omega) \right|^2
\]  

(32)

where \( \frac{m}{2} \) and \( \frac{l}{2} \) are the half integer periods of \( A \). We consider the following decomposition,

\[
\begin{align*}
\frac{m}{2} &= m' + \eta, \\
\frac{l}{2} &= l' + \mu
\end{align*}
\]  

(33)

where \( m' \) and \( l' \) are integer numbers while \( \eta \) and \( \mu \) are \( g \)-tuples with components 0 or \( \frac{1}{2} \). Summation in all \( m \) and \( l \) is equivalent to summation in all \( (\eta, \mu) \) and all \( (m', l') \). The summation in the integers may be handled as before. Then the summation in the half integers \( (\eta, \mu) \) may be reinterpreted as a sum over all spin structures (weighted by a factor which depends on \( J \)).

When \( J \) is zero we have,

\[
Z_{4n}[j, 0, \epsilon, \kappa] = \mathcal{N} \sum_{\epsilon', \kappa'} \hat{Z}[j, \epsilon', \kappa'] = \mathcal{N} \hat{Z}_f[j, \cdot]
\]  

(34)
The factor \( \mathcal{N} \) here gives the same measure of the space of harmonic 1-forms with integral periods as in (31). We started with a fixed spin structure, however the final result corresponds to the partition function of spinor fields with summation \( n \) in all spin structures. In particular it shows that \( Z_{4n}[j,0,\epsilon,\kappa] \) is independent of the spin structure (i.e of \( \epsilon \) and \( \kappa \)).

4 Bosonization in higher genus surfaces

As an application we use the results of the previous section to discuss the bosonization of fermions over higher genus compact Riemann surfaces. The equations (29) and (34) already establish the relation between the partition function of the fermions and the partition function of the BF model. In this section we obtain this result using the the constructive approach of [7] and [8].

Let us begin with a quick review of the situation in the topologically trivial case. Consider the generating functional of a fermion field coupled to a conserved current \( j \),

\[
Z_f[j] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{\int d^2x \sqrt{g} \bar{\psi} (-i\partial + j^a)\psi}.
\]

(35)

We suppose here that the current \( j \) has a topological index zero. In two dimensions, on a genus zero surface, this fermion determinant is explicitly known [27] and given by (9). The duality-bosonization transformation allows to express this result in terms of a bosonic field. To construct this transformation one begins observing that the system has a global \( U(1) \) gauge invariance. Then [8],[7],[6] one makes a change of variables with the functional form of a local gauge transformation and identify the spurious contributions which appear in the action as coupling terms with a gauge field of zero curvature. The adequate change of variables in this case is,

\[
\psi(x) \rightarrow e^{i\Lambda(x)} \psi(x)
\]

(36)

where \( \Lambda(x) \) is an arbitrary parameter with local dependence on \( x \). The fermionic generating functional turns out to be,

\[
Z_f[j] = \mathcal{K} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{\int d^2x \sqrt{g} \bar{\psi} (-i\partial + j^{\mu} + \partial\Lambda)\psi}.
\]

(37)
where $K$ is the Jacobian of the transformation (which in this case is a non-relevant constant). This can be re-interpreted as the partition function of a model consisting of a flat connection $A_\mu$ coupled to the fermions in the particular gauge where,

$$A_\mu = \partial_\mu \Lambda.$$  \hfill (38)

The zero curvature condition on $A_\mu$ implies, of course, that the connection is locally a pure gauge. Since the vanishing of $^*F(A) = \epsilon^{\mu\nu} F_{\mu\nu}(A)$ implies that of $F_{\mu\nu}(A)$, one introduces the 1-form connection restricted by the condition

$$^*F(A) = \epsilon^{\mu\nu} F_{\mu\nu}(A) = 0.$$  \hfill (39)

After imposing this constraint in the functional integral one gets,

$$Z_j[j] = \int DAD\psi D\bar{\psi} \frac{\delta (^*F(A))}{Vol(G_A)} e^{\int d^2x \sqrt{g} \bar{\psi}(-i\partial + j + \Lambda)\psi},$$  \hfill (40)

where $G_A$ is the gauge group of $A$. Now one introduces a Lagrange multiplier $B$ to raise the $\delta(F)$ to the exponential but has to take into account that since there are infinitely many solutions of the equation $^*F(A) = 0$, the functional $\delta(^*F)$ has to be defined with some care. It is properly defined, \[15\] in terms of the generating functional of a BF topological field theory \[29\]. Using the BRST invariance as a guide to guarantee that the functional integral remains well defined we get,

$$Z_j[j] = \int DADBD\psi D\bar{\psi} DC e^{\int d^2x \sqrt{g} \bar{\psi}(-i\partial + j + \Lambda)\psi - \frac{i}{4} \epsilon^{\mu\nu} \partial_\mu BA_\nu + L_g},$$  \hfill (41)

where again $DC$ stands for the measure of the ghosts and auxiliary fields and $L_g$ for the contributions of those fields plus the gauge fixing term to the Lagrangian. The appearance of the BF effective action should be expected since the factor which comes from the exterior derivative in $\delta(^*F(A))$ may be expressed as a function of the Ray-Singer torsion an hence related to the BF effective action \[2\]. In two dimensions the Ray-Singer torsion turns out to be equal to one.

To complete the bosonization of the generating functional one makes a shift $A + j \rightarrow A$. The fermionic field remains coupled only to the new $A$ field. Then one uses the result (3) for the fermionic determinant, chooses an adequate gauge fixing condition which allow to make the quadratic functional integral in $A$ and ends up with,
\[ Z_f[j] = Z[0] \mathcal{N} \int \mathcal{D} B e^{-\int d^2 x \sqrt{g} \left( \frac{i}{4} \partial_\mu B \partial^\mu B - \frac{1}{4 \pi} \epsilon_{\mu \nu} \partial_\mu B j^\nu \right)} , \]  

(42)

where \( \mathcal{N} \) is the factor which appears after the quadratic integral on \( A \) has been performed. This is the bosonized effective action. The external current \( j \) appears in this expression coupled to the topological current of the Lagrange multiplier \( B \).

Let us now turn to the general case on an arbitrary genus \( g \), compact Riemann surface. On the light of (34) we start with,

\[
\hat{Z}_f[j] = \sum_{\epsilon_i, \kappa_j} \int \mathcal{D} \psi \mathcal{D} \bar{\psi} e^{\int d^2 x \sqrt{\bar{g}} \bar{\psi} (\partial^\mu + j^\mu + \partial^\mu \Lambda) \psi}.
\]  

(43)

Instead of using directly (34), let us argue how one can adapt the discussion presented for the genus zero surfaces and recover the \( BF \) partition function in a constructive way. Let us introduce the change of variables (36). In order to have a uniform change of variables in the functional integral, \( \Lambda(x) \) must satisfy

\[ \oint_{\mathcal{C}_I} d\Lambda = \pi n^I \]  

(44)

where \( n^I \) are integers. If all the \( n^I \) are even the change of variables does not change the spin structure that we have defined over \( \Sigma \). Otherwise we change from one to another spin structure but since we are summing over all of them this is not a problem here. We get again,

\[
\hat{Z}_f[j] = \sum_{\epsilon_i, \kappa_j} \int \mathcal{D} \psi \mathcal{D} \bar{\psi} e^{\int d^2 x \sqrt{\bar{g}} \bar{\psi} (\partial^\mu + j^\mu + \partial^\mu \Lambda) \psi}.
\]  

(45)

In this case we also wish to rewrite this in terms of a globally defined flat connection \( A \). For two dimensional surfaces this means that \( A \) should be a flat connection over a trivial \( U(1) \) line bundle. To achieve consistency with (44) we have to impose that

\[ G(A) = \oint_{\mathcal{C}_I} A = \pi n^I. \]  

(46)

This is exactly the condition forced by (24) and in fact its appearance at this point provided the original motivation to the discussion presented in the previous section. Things now follow smoothly. First, in order to introduce \( A \)
satisfying (46) in the functional integral one extends the functional integral to the space of connections and introduces factors \( \delta(F(A)) \) and \( \delta(G(A)) \) in the measure. We get,

\[
\hat{Z}_f[j] = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \frac{\delta(F(A))\delta(G(A))}{Vol(G_A)} e^{\int_\Sigma d^2x \sqrt{g} (-i\bar{\psi}j + j + A)\psi},
\]

(47)

where \( G_A \) is the group of allowed gauge transformations of \( A \), that is of those gauge transformations with an uniform gauge function.

Now we want to raise the \( \delta \) functions to the exponential. From our results of the previous section, the right way to do that is to take a multi-valued Lagrange multiplier \( B \) over \( \Sigma \) satisfying

\[
\oint_{c_1} dB = 4\pi m^I
\]

(48)

and to integrate over the functional space of \( B \) with all possible \( m^I \). In order to have a well defined functional integral, the measure has to be defined in terms of precisely the \( BF \) topological field theory we considered previously. We then recover (44)

\[
\hat{Z}_f[j] = Z_{4n}[j, 0, \epsilon, \kappa] = \sum_{m^I} \int \mathcal{D}C^I \mathcal{D}A \mathcal{D}B \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_{\text{eff}}},
\]

(49)

\[
S_{\text{eff}} = \int d^2x \sqrt{g} [\bar{\psi}(iD^j - A^j - j)\psi + L_g] + \frac{i}{2\pi} \int_\Sigma (dB \wedge A)
\]

(50)

Here as we discussed earlier the result does not depend on the spin structure \( (\epsilon, \kappa) \). To obtain the bosonized representation of (43) we now choose the gauge fixing and ghost terms in (49) and perform the fermionic integral. We can work more generally with \( J \neq 0 \) and use (14). Making first a shift

\[
\tilde{A} = A + j
\]

(51)

in (44), taking the gauge condition

\[
* d \ast\tilde{A} = 0
\]

(52)

and performing the fermionic integral we have,

\[
Z_{4n}[j, J, \epsilon, \kappa] = [det\Delta_0]^\frac{1}{2} [det Im \Omega Vol(\Sigma)]^\frac{1}{2} \theta \left[ \frac{u + \epsilon}{v + \kappa} \right] (0|\Omega)|^2
\]
\[
S[\tilde{A}, B] = \frac{1}{2\pi} \int_{\Sigma} [d\tilde{A} \frac{1}{\Delta_0} \star d\tilde{A} + i dB \wedge (\tilde{A} - j) + \frac{1}{2} d \star \tilde{A} \wedge \star d \star \tilde{A} - 2\pi \star J \wedge (\tilde{A} - j)]
\]

where a factor \([\text{det}' \Delta_0]\) arises from the integration on the ghost and anti ghost fields. The arguments \(u\) and \(v\) in the theta function are the coefficients in the expansion of \(\tilde{A}_h\) and are not restricted until now. To write out our final expression we introduce the decomposition (2) for \(\tilde{A}\),

\[
\tilde{A} = d\tilde{s} + \star d\tilde{p} + \tilde{A}_h
\]

and observe that, \(i)\) Integration in \(\tilde{s}\) contributes with a factor \((\text{det}' \Delta_0)^{-1}\) \(ii)\) Integration in \(\tilde{p}\) and the Jacobian of the transformation contribute a factor \((\text{det}' \Delta_0)^{\frac{1}{2}}\) and a term in the action of the form,

\[
S(B, J) = -\frac{1}{2\pi} \int_{\Sigma} (dB + 2\pi i \star J)_{\text{exact}} \wedge \star (dB + 2\pi i \star J)_{\text{exact}}
\]

since only the exact part of \((dB + 2\pi i \star j)\) couples with \(\tilde{p}\) \(iii)\) One is left with the integration in \(\tilde{A}_h\). Using the decomposition (19), for the field \(B\) one may show again that the summation over the periods of \(B\) leads to the half integral periodicity conditions in \(\tilde{A}_h\). The integral in \(\tilde{A}_h\) is then a summation over the half-integral periods. We finally obtain,

\[
Z_{4n}[j, J, \epsilon, \kappa] = \sum_{l,m} \int Dbe^{-S[\tilde{A}_h, b]} [\text{det} Im \Omega \text{Vol}(\Sigma)]^{\frac{1}{2}} \left| \theta \left[ \frac{l}{2\pi} \right] (0|\Omega) \right|^2
\]

where the contribution of the spin structure is included in the argument of the \(\theta\) function and we define,

\[
S[\tilde{A}_h, b] = \frac{1}{2\pi} \int_{\Sigma} (db + 2\pi i \star J_{\text{exact}}) \wedge \star (db + 2\pi i \star J_{\text{exact}}) - i(db + 2\pi \star J_{\text{exact}}) \wedge j + \int_{\Sigma} \star J \wedge \tilde{A}_h
\]

with \(A_h\) given by (3) restricted to half-integer periods. When \(J\) is zero this gives the bosonized expression for the fermionic partition function in higher
genus surfaces. A similar expression for the partition function over a single spin structure may be obtained straightforwardly, following the same lines, starting from (29). The results presented in this paper allow to encode the information concerning the spin structure of the manifold in terms of the topological properties of the fields of a $BF$ model. They show also the non-trivial way in which the bosonization rules are generalized to higher genus surfaces.

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