Homological face-width condition forcing $K_6$-minors in graphs on surfaces

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Abstract

It is proved that every graph embedded on a (non-spherical) surface with non-separating face-width at least 7 contains a minor isomorphic to $K_6$. It is also shown that face-width four yields the same conclusion for graphs on the projective plane.

1 Introduction

A surface is a connected compact 2-manifold. Unless explicitly stated otherwise, surfaces will be assumed to be non-simply connected and have no boundary. If there is a nonempty boundary, then we speak of a bordered surface and every component of the boundary is called a cuff. A simple closed curve $\gamma$ on a surface $\Sigma$ is said to be surface separating or zero-homologous if cutting $\Sigma$ along $\gamma$ results in a disconnected (bordered) surface. Two disjoint simple closed curves are said to be homologous if they are either both zero-homologous, or none of them is zero-homologous, but cutting the surface along both of these curves disconnects the surface.

Let $G$ be a graph embedded on a surface $\Sigma$. We regard $G$ as a subset of $\Sigma$ (that is, we identify $G$ with its embedding on $\Sigma$). The face-width of $G$, denoted by $\text{fw}(G)$, is the maximum number $k$ so that every non-contractible simple closed curve in $\Sigma$ intersects $G$ in at least $k$ points. The homology version, the non-separating face-width of $G$, denoted by $\text{nsfw}(G)$, is the maximum number $k$ so that every surface non-separating simple closed curve

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in \( \Sigma \) intersects \( G \) in at least \( k \) points. We refer to [10] for additional terminology involving graphs embedded in surfaces.

A graph \( H \) is a minor of a graph \( G \), in symbols \( H \leq_m G \), if \( H \) can be obtained from a subgraph of \( G \) by a series of contractions of edges.

The theory of graph minors (Robertson and Seymour [15]) shows that for every surface \( \Sigma \) there exists a constant \( c_{\Sigma} \) (depending only on \( \Sigma \)) such that if \( G \) embeds in \( \Sigma \) with face-width at least \( c_{\Sigma} \), then \( G \) contains \( K_6 \) as a minor. We are interested in finding the best possible value for \( c_{\Sigma} \). If \( G \) is an apex graph, then \( G \) does not contain \( K_6 \) as a minor. It is known that there are apex graphs that can be embedded on non-spherical surfaces with face-width at least three, see [9]. Hence, there are surfaces \( \Sigma \) with \( c_{\Sigma} \geq 4 \). In fact, there are examples showing that \( c_{\Sigma} \geq 4 \) for every surface \( \Sigma \). We first show that \( c_{\Sigma} = 4 \) in the special case when \( \Sigma \) is the projective plane.

**Theorem 1.1.** Let \( G \) be a graph embedded on the projective plane. If \( \text{fw}(G) \geq 4 \), then \( K_6 \leq_m G \).

We suspect that \( c_{\Sigma} = 4 \) for every \( \Sigma \); however a proof (or disproof) seems to be out of reach. Our main result given below extends Theorem 1.1 to arbitrary surfaces and strengthens the afore-mentioned result of Robertson and Seymour from [15] in two ways. First, we obtain an upper bound on \( c_{\Sigma} \) that is independent of the surface. In addition to this, we are able to loosen the face-width requirement by involving a condition on the non-separating face-width instead. Note that for graphs on the projective plane, we have \( \text{nsfw}(G) = \text{fw}(G) \) and that \( \text{nsfw}(G) \geq \text{fw}(G) \) holds in general.

**Theorem 1.2.** Every graph \( G \) embedded on a non-spherical surface with \( \text{nsfw}(G) \geq 7 \) contains the complete graph \( K_6 \) as a minor.

There is a continuing interest in the structure of graphs that do not contain \( K_6 \) as a minor. An outstanding open problem in this area is a conjecture of Jørgensen [3] that every 6-connected graph has no \( K_6 \)-minor if and only if it can be made planar by removing one vertex. An asymptotic version of Jørgensen’s Conjecture has been recently proved by Kawarabayashi et al. [7]. The known structure of such graphs is used in [5] in the design of an efficient algorithm for constructing linkless embeddings of graphs in 3-space. As for graphs embedded in surfaces, several papers [6,11,13,12] concern \( K_6 \)-minors in triangulations of surfaces of small genus, while [2] obtained a general result about \( K_6 \)-minors in graphs in the projective plane.

All graphs in this paper are finite and simple. Paths and cycles have no “repeated vertices”. A path \( P = x_0x_1\ldots x_n \) is given by the sequence of its consecutive vertices \( x_0, x_1, \ldots, x_n \), but it is considered as a subgraph. If a path \( P \) has endvertices \( u \) and \( v \), then \( P \) is called a \((u,v)\)-path (also \((v,u)\)-path). The order of a path \( P \), denoted as \(|P|\), is its number of vertices. For vertices \( a \) and \( b \) on a path \( P \), \( P[a,b] \) denotes the \((a,b)\)-path contained in \( P \), and \( P[a,b] = P[a,b] - b \) denotes the path from \( a \) to the predecessor of \( b \). The paths \( P[a,b] \) and \( P(a,b) \) are defined analogously. The same notation is used for cycles with given clockwise orientation, where \( C[a,b] \) denotes the path from \( a \) to \( b \) in the clockwise direction.
For $A_i \subseteq V(G)$ or $A_i \subseteq G$ ($i = 1, 2$), an $(A_1, A_2)$-path is an $(a_1, a_2)$-path $P$ with $V(P) \cap V(A_i) = \{a_i\}$ for $i = 1, 2$, an $(A_1)$-path is an $(a_1, a_2)$-path with $V(P) \cap V(A_1) = \{a_1, a_2\}$, where $a_1 \neq a_2$ and $P$ contains an edge that is not in $A_1$.

## 2 Face-chains

Let $G$ be a graph embedded in a surface $\Sigma$. We denote by $F(G)$ the set of all facial walks of $G$. Each facial walk is also considered as being a subgraph of $G$ consisting of all vertices and edges on the boundary of a face of the embedding. The open face corresponding to the face-chain $\Lambda$ is an ($\Lambda$)-path condition.

We shall abuse terminology and call a face-chain $\Lambda$ surface separating or contractible when $\Gamma(\Lambda)$ has that property.

The following result is well-known (cf. [10]) and is referred to as the 3-path condition.

**Theorem 2.1.** Let $G$ be a graph embedded on $\Sigma$, and let $x, y \in V(G)$. Suppose $G$ contains three $(x, y)$-paths, $P_1, P_2, P_3$, pairwise disjoint except for their ends. Let $C_{ij}$ ($1 \leq i < j \leq 3$) be the cycle $P_i \cup P_j$. Then the following holds:

(a) If two of the three cycles $C_{ij}$ are contractible, then so is the third.
(b) If two of the three cycles $C_{i,j}$ are surface separating, then so is the third.

Let $\Lambda$ be a closed face-chain. Let $\Lambda' = w_0, F'_1, w_1, \ldots, w_{k-1}, F'_k, w_k$ be a face-chain (not closed) of length $k$ such that $w_0$ is incident with a face $F_i$ and $w_k$ is incident with a face $F_j$ ($0 \leq i < j \leq n - 1$) in the face-chain $\Lambda$. There are two face-chains in $\Lambda$ whose first and last faces are $F_i$ and $F_j$. We can combine each of these with $\Lambda'$ to get a closed face-chain containing $\Lambda'$. By using the 3-path property, we deduce the following.

**Theorem 2.2.** Let $G$ be a graph embedded in $\Sigma$, and let $\Lambda$ be as in (H1). Let $\Lambda' = w_0, F'_1, w_1, \ldots, w_{k-1}, F'_k, w_k$ be a face chain of length $k \geq 0$, where $w_0$ and $w_k$ vertices or edges that are incident with faces $F_i$ and $F_j$ in $\Lambda$ ($0 \leq i < j \leq n - 1$). Then the closed face-chain formed by $\Lambda'$ and the shorter one of the two face-chains from $F_i$ to $F_j$ in $\Lambda$ is a face-chain of length $\leq 2k + 2$.

**Proof.** Let $\Lambda_1 = F_i, x_{i+1}, F_{i+1}, \ldots, x_j, F_j$ and $\Lambda_2 = F_j, x_{j+1}, F_{j+1}, \ldots, x_0, F_0, \ldots, x_i, F_i$ be the two face-subchains from $F_i$ to $F_j$ contained in $\Lambda$. They together use $n + 2$ faces. Let us now consider the two closed face-chains $\Lambda_1 \cup \Lambda'$ and $\Lambda_2 \cup \Lambda'$. Clearly,

$$|\Lambda_1 \cup \Lambda'| + |\Lambda_2 \cup \Lambda'| = n + 2 + 2k.$$

By the 3-path condition, at least one of them is surface non-separating, thus it is of length at least $n$ by (H1)(ii). So, it follows that the length of the other one is at most $2k + 2$. □

The following theorem is well-known for the face-width (cf. [10]); the proof for non-separating face-width is essentially the same.

**Theorem 2.3.** Let $G$ be a 3-connected graph embedded on a surface $\Sigma$ with $\text{nsfw}(G) \geq 3$. Then all facial walks of $G$ are cycles, and any two of them are either disjoint or intersect in a single vertex or a single edge.

The following is an easy corollary of Theorem 2.3, Theorem 2.2 (with $k = 0$) and the 3-path condition.

**Theorem 2.4.** Let $G$ be a 3-connected graph embedded on a surface $\Sigma$ with $\text{nsfw}(G) \geq 3$, and let $\Lambda$ be as in (H1). Then $\Lambda$ is clean.

The following result is an easy corollary of the 3-path condition. Its proof for the edge-width can be found in [16]; for the proof of the face-width version, see [10]; the proof for the non-separating face-width is essentially the same as in [10].

**Theorem 2.5.** Let $G$ be embedded in a surface $\Sigma$, and let $\Lambda$ be as in (H1). Let $G'$ be obtained from $G$ by cutting $\Sigma$ along $\Gamma(\Lambda)$ and capping off the resulting cuffs. Then $\text{fw}(G') \geq \left\lceil \frac{1}{2} \text{fw}(G) \right\rceil$ and $\text{nsfw}(G') \geq \left\lceil \frac{1}{2} \text{nsfw}(G) \right\rceil$.

Let $G$ be a graph embedded on $\Sigma$ and let $p \in \Sigma \setminus G$ be a preselected point on the surface. If $C$ is a surface-separating cycle of $G$, we denote by $\text{Int}(C)$ the subgraph of $G$ contained in the part of the surface separated by $C$ that contains $p$; in particular, $C \subseteq \text{Int}(C)$. Let
Let $f \in F(G)$ be a face of $G$. We define subgraphs $B_0(f), B_1(f), B_2(f), \ldots$ of $G$ recursively as follows: $B_0(f) = f$, and for $k \geq 1$, $B_k(f)$ is the union of $B_{k-1}(f)$ and all facial walks that have a vertex in $B_{k-1}(f)$. Let $\partial B_k(f)$ be the set of edges of $B_k(f)$ (together with their ends) that are not incident with a vertex of $B_{k-1}(f)$. With this notation we have the following result (see [8]).

**Theorem 2.6.** Let $G$ be a graph embedded on $\Sigma$ with $\text{nsfw}(G) \geq 2$. Let $f \in F(G)$ and let $k = \lfloor \frac{1}{2} \text{nsfw}(G) \rfloor - 1$. Then there exist pairwise disjoint surface-separating cycles $C_0(f), \ldots, C_k(f)$ such that for $i = 0, \ldots, k$, $C_i(f) \subseteq \partial B_i(f)$ and $B_i(f) \subseteq \text{Int}(C_i(f))$ (where $\text{Int}$ is defined with respect to a point $p$ in the face $f$). Moreover, if $l = \lfloor \frac{1}{2} \text{fw}(G) \rfloor - 1$, then the cycles $C_0, C_1, \ldots, C_k$ are contractible in $\Sigma$.

In addition to having large $\text{nsfw}(G)$, we will also need $\text{fw}(G)$ to be large. This will be made possible by the following theorem.

**Theorem 2.7.** Let $G$ be a graph embedded in a surface $\Sigma$ with $k = \text{nsfw}(G) \geq 6$. Then $G$ contains a minor $G'$ such that $G'$ is 3-connected and has an embedding in a surface $\Sigma'$ with $\text{nsfw}(G') = k$ and $\text{fw}(G') \geq 6$.

**Proof.** Let $G'$ be a minor of $G$ with the minimum number of vertices and edges such that $G'$ has an embedding in a surface $\Sigma'$ with $\text{nsfw}(G') = k$. Clearly, $G'$ exists. We claim that $\text{fw}(G') \geq 6$. If not, let $1 \leq l \leq 5$ be the smallest integer such that there exists a closed face-chain $\Lambda = x_0, F_0, x_1, \ldots, F_{l-1}, x_0$ with $\Gamma(\Lambda)$ non-contractible. Since $k \geq 6$, $\Gamma(\Lambda)$ is surface-separating. Let $\Sigma'_1$ and $\Sigma'_2$ be the surfaces obtained by cutting $\Sigma'$ along $\Gamma(\Lambda)$ and capping off the resulting cuff. For $i = 1, 2$, let $G'_i$ be the subgraph of $G'$ in $\Sigma'_i$, and let $G''_i$ be obtained from $G'_i$ by adding a vertex of degree $l$ and joining it to all vertices in $X = \{x_0, \ldots, x_{l-1}\}$ (and embedding the vertex and these edges into the capped disk). Each face $F'_j$ ($0 \leq j < l$) determines a face $F''_j$ in $\Sigma'_i$. This correspondence makes it possible to convert every face-chain in $G''_i$ to a face-chain in $G'$. Note that $G''_i$ is a proper minor of $G'$ (since $l$ is smallest, $G''_i \setminus X$ contains a connected component adjacent to all vertices in $X$ and can thus be contracted into the added vertex of $G''_i$). By the minimality of $G'$, we conclude that the embedding of $G''_i$ in $\Sigma'_i$ has $\text{nsfw}(G''_i) < k$. Let $\Lambda'$ be a non-separating closed face-chain of length $k' < k$ confirming this fact. It is easy to see that $\Lambda'$ determines a non-separating face-chain in $G'$ of the same length (since $l \leq 5$). This contradiction proves that $\text{fw}(G') \geq 6$.

Finally, since $\text{fw}(G') \geq 3$, $G'$ contains a 3-connected minor whose face-width and non-separating face-width are the same ([10]). By the minimality of $G'$, this minor is equal to $G'$. This completes the proof. \qed

## 3 Disjoint paths on a surface

Let $G$ be a graph embedded on $\Sigma$. Let $C_1, C_2 \subseteq G$ be disjoint, homologous, surface non-separating cycles in $G$. Note that $C_1$ and $C_2$ are 2-sided since pairs of 1-sided homologous cycles always intersect each other. Let $\Sigma_0$ and $\Sigma_1$ be bordered surfaces, whose cuffs coincide
Lemma 3.1. Each of $G_0$ and $G_1$ contains nsw(G) pairwise disjoint $(C_1, C_2)$-paths.

For the proof of Lemma 3.1, we need the following result whose weaker form for contractible curves has appeared in [1].

Lemma 3.2. Let $G$ be a graph embedded on a surface $\Sigma$, and let $A$ (possibly $A = \emptyset$) be a set of vertices such that $G' = G - A$ is disconnected. Let $\hat{C}_1$ and $\hat{C}_2$ be distinct connected components of $G'$. Then $\Sigma$ contains a simple closed curve $\Gamma$ such that $\Gamma \cap G \subseteq A$ and if $\Gamma$ is surface separating, then $\hat{C}_1$ and $\hat{C}_2$ are contained in different connected components of $\Sigma \setminus \Gamma$.

Proof. Consider the disconnected graph $G'$ with its induced embedding on $\Sigma$. We claim that

(1) $\Sigma \setminus G'$ contains a 2-sided simple closed curve $\Gamma$ that intersects $G$ only in edges joining $\hat{C}_1$ with $A$, and $\Gamma$ is either surface non-separating in $\Sigma$, or separates $\Sigma$ into two components, one containing $\hat{C}_1$ and the other one containing $\hat{C}_2$.

To see this, let us first delete all components of $G'$ distinct from $\hat{C}_1$ and $\hat{C}_2$. Next, let us add an edge $e$ joining a vertex in $\hat{C}_1$ with a vertex in $\hat{C}_2$ so that the resulting graph $G'' = \hat{C}_1 \cup \hat{C}_2 + e$ is embedded in $\Sigma$. Since $e$ is a cut-edge of $G''$, the unique facial walk $F$ containing $e$ in the induced embedding of $G''$ contains $e$ twice and $e$ is traversed in opposite directions. Following the part of this facial walk in $\hat{C}_1$, we see that $\Sigma$ contains a simple closed curve $\Gamma$ that follows the boundary of $F$ close to $\hat{C}_1$ so that $\Gamma$ crosses $e$ exactly once, and $\Gamma$ intersects only $e$ and the edges of $G$ joining $\hat{C}_1$ with $A$. In particular, $\Gamma$ does not intersect any of the removed components of $G'$. If $\Gamma$ separates $\Sigma$, then each component of $\Sigma \setminus \Gamma$ contains exactly one of the components $\hat{C}_1$ or $\hat{C}_2$ since the edge $e$ crosses $\Gamma$. This proves (1).

Let us consider all simple closed curves satisfying the conclusion of (1), except that we allow them to intersect $G$ not only at interior points of the edges joining $\hat{C}_1$ with $A$, but also allow that $\Gamma$ passes through vertices in $A$. Among all such curves, choose $\Gamma \subseteq \Sigma$ having minimum number of crossings with interior points on the edges joining $\hat{C}_1$ with $A$. Note that $\Gamma$ intersects $G$ only in $A$ or in edges joining $A$ to vertices in $\hat{C}_1$. By possibly altering $\Gamma$, we may assume that each intersection of $\Gamma$ with an edge of $G$ is a crossing.

If $\Gamma \cap E(G) = \emptyset$, then $\Gamma$ is of the desired form and the claim follows. Hence $\Gamma$ intersects an edge $a = uv \in E(G)$, where $u \in V(\hat{C}_1)$ and $v \in A$. Replace a short segment of $\Gamma$ around this intersection with a simple curve which follows $a$ to its endvertex $v$ in $A$, crosses through $v$ and returns back on the other side of $a$ (if $\Gamma$ intersects $a$ in more than one point, choose the intersection point which is closest to $v$). The resulting curve $\Gamma'$ is homotopic to $\Gamma$. By the minimality property of $\Gamma$, $\Gamma'$ is not simple, and is hence composed of two simple closed curves $\Gamma_1$ and $\Gamma_2$ that intersect at $v$. 
We may assume that both $\Gamma_1$ and $\Gamma_2$ separate $\Sigma$, for if $\Gamma_i$ ($1 \leq i \leq 2$) does not separate $\Sigma$, then $\Gamma_i$ can be chosen instead of $\Gamma$, contradicting the minimality of $\Gamma$. By our choice of $\Gamma$ and $a$, it is easy to see that there must exist $i \in \{1, 2\}$ such that cutting $\Sigma$ along $\Gamma_i$ disconnects $\Sigma$ into two components each containing exactly one component $C_1$ or $C_2$. But then $\Gamma_i$ can be chosen instead of $\Gamma$, contradicting the minimality of $\Gamma$. This completes the proof. 

\begin{proof}[Proof of Lemma 3.1] By symmetry, it suffices to prove the lemma for $G_1$. Let $r$ be the maximum number of disjoint $(C_1, C_2)$-paths contained in $G_1$. To prove the claim, we have to show that $r \geq \text{nsfw}(G)$. By Menger’s theorem there exists $A \subseteq V(G_1)$ with $|A| = r$ that separates $C_1$ and $C_2$. Let $G_2 \supseteq G_1$ be the graph embedded in $\Sigma_1'$ that is obtained from $G_1$ by adding two vertices $v_1, v_2$, where $v_i$ is adjacent to all vertices in $C_i$ ($i = 1, 2$). For $i = 1, 2$, let $\hat{C}_i$ be the connected component of $G_2 - A$ containing $v_i$. Then $A$ satisfies assumptions of Lemma 3.2. Let $\Gamma$ be a simple closed curve on $\Sigma_1'$ as promised to exist by Lemma 3.2. If $\Gamma$ is surface-separating in $\Sigma_1'$, then it separates $v_1$ from $v_2$. Moreover, $\Gamma \cap G_2 \subseteq A$ and we may assume that $\Gamma$ is disjoint from the interior of $\Sigma_0$. However, in the surface $\Sigma$, $\Gamma$ is surface non-separating since the two components of $\Sigma_1' \setminus \Gamma$ are connected together in $\Sigma \setminus \Gamma$ through $\Sigma_0$. Thus, we conclude that $\Gamma$ is always surface non-separating. Therefore, $r \geq |\Gamma \cap G| = |\Gamma \cap G_2| \geq \text{nsfw}(G)$. 

For a path $P$, we denote by $\text{int}(P)$ the path obtained from $P$ by removing its end-vertices (and incident edges).

\begin{theorem} 3.3 \end{theorem}

Let $G, C_1, C_2$ and $G_1, \Sigma_0, \Sigma_1$ be as introduced at the beginning of the section. Let $P$ be a set of pairwise disjoint $(C_1, C_2)$-paths in $G_1$ of maximum cardinality. Let $i \in \{1, 2\}$, and let $w, w' \in V(C_i)$ be two vertices of $C_i$. Let $X, \overline{X} \subseteq C_i$ be the two $(w, w')$-paths on $C_i$, i.e., $C_i = X \cup \overline{X}$ and $X \cap \overline{X} = \{w, w'\}$. Then one of the following holds:

(a) There exists an $(\text{int}(\overline{X}), P)$-path in $G_1$ disjoint from $X$. (Here we consider $P$ as a subgraph of $G$.)

(b) There exist $v, u \in V(X)$ such that $v$ and $u$ are incident with a common face $f \in F(G_1)$ in $\Sigma_1$, and the closed curve in $\Sigma$ formed by a simple arc in $f^0$ from $u$ to $v$ together with the segment $X[v, u]$ on $X$ is surface non-separating in $\Sigma$.

\begin{proof} By symmetry we may assume that $i = 1$. We may also assume that $\text{int}(\overline{X}) \neq \emptyset$ since otherwise (b) holds with $\{v, u\} = \{w, w'\}$. Moreover, no path in $P$ has an end in $\text{int}(\overline{X})$ since otherwise (a) holds. We will assume that (a) fails, and show that (b) holds. In particular, we will show that there exists a simple arc $\gamma$ in $\Sigma_1$, so that $\gamma \cap G_1 = \{v, u\}$, where $v, u \in V(X)$, and $\gamma \cup X[v, u]$ is a surface non-separating closed curve. This will imply (b).

So, suppose (a) does not hold. The maximality of $|P|$ and the assumption that (a) does not hold, imply that in $G_1 - V(X)$, there is no $(\text{int}(\overline{X}), P \cup C_2)$-path. Hence, $G_1 - V(X)$ is disconnected, with $C_2$ and $\text{int}(\overline{X})$ belonging to distinct connected components.

Let $G_2$ (embedded on $\Sigma_1'$) be obtained from $G_1 - V(X)$ by adding an edge $e$ (embedded along the deleted path $X$, but drawn inside the capped face in $\Sigma_1'$ so that it does not intersect
$G_1$) connecting the two end vertices of $\text{int}(X)$. Note that $G_2$ has the same connected components as $G_1 - V(X)$, since $e$ connects two vertices of $\text{int}(X)$ that are in the same component of $G_1 - V(X)$. Let $F_1$ be the face of $G_2$ in $\Sigma_1$ bounded by the cycle $C'_1 = \text{int}(X) + e$.

Clearly, $C'_1$ is a cycle in $G_2$ which is homotopic to $C_1$ in $\Sigma$. Let $\hat{C}_1'$ and $\hat{C}_2$ be the connected components of $G_2$ containing $C_1'$ and $C_2$, respectively. Let $\Gamma \subseteq \Sigma_1'$ be the closed curve obtained by applying Lemma 3.2 to the embedded graph $G_1 + e \subseteq \Sigma_1'$, the separating vertex set $V(X)$ playing the role of $A$, and considering the connected components $\hat{C}_1'$ and $\hat{C}_2$ of $G_2 = (G_1 + e) - V(X)$. Then $\Gamma \cap \text{int}(F_1) = \emptyset$ since $\Gamma \cap \hat{C}_1' = \emptyset$. In particular, $\Gamma \subseteq \Sigma_1$ and $\Gamma \cap G_1 \subseteq V(X)$. We claim that

(1) $\Gamma$ is surface non-separating in $\Sigma$.

This is clear if $\Gamma$ is surface non-separating in $\Sigma_1'$. Otherwise, $\Gamma$ is surface separating in $\Sigma_1'$. As guaranteed by the use of Lemma 3.2, $\Gamma$ separates $\hat{C}_1'$ from $\hat{C}_2$ in $\Sigma_1'$. However, in $\Sigma$, these two parts are connected together via the surface part $\Sigma_0$, so $\Gamma$ is not surface separating in $\Sigma$. This proves (1).

In the sequel we will consider curves in $\Sigma_1' \setminus \text{int}(F_1)$. We can view $\Sigma_1 \subset \Sigma_1' \setminus \text{int}(F_1) \subset \Sigma$ and therefore talk about homology properties of such curves in $\Sigma$.

Let $\Gamma_1$ be a closed curve in $\Sigma_1' \setminus \text{int}(F_1)$ so that the following conditions hold:

(i) $\Gamma_1 \cap (G_1 + e) \subseteq V(X)$, $\Gamma_1$ is surface non-separating in $\Sigma$, and every arc $\gamma \subseteq \Gamma$ with ends $x, y \in V(X)$ and which is otherwise disjoint from $\Sigma_1$ is homotopic to $X[x, y]$;

(ii) subject to (i), the number of connected components of $\Gamma_1 \cap \Sigma_1$ is minimum.

Note that such a choice of $\Gamma_1$ is possible since $\Gamma$ satisfies (i).

The curve, $\Gamma_1$ is surface non-separating in $\Sigma$. By (i) we deduce that there exists an arc $\gamma \subseteq \Gamma_1$ contained in $\Sigma_1$ such that $\gamma \cap G_1 = \{x, y\}$ where $x, y \in V(X)$. Let $\gamma'$ be a curve in $\Sigma_1' \setminus \Sigma_1$ along $X$ with ends $x$ and $y$ that is homotopic to $X[x, y]$.

If $\gamma \cup \gamma'$ is surface non-separating in $\Sigma$, then (b) holds. Otherwise, by replacing $\gamma$ in $\Gamma_1$ with $\gamma'$, we obtain a new curve $\Gamma_2$ that satisfies (i), by the 3-path condition. But then the existence of $\Gamma_2$ contradicts (ii) in the choice of $\Gamma_1$. This contradiction concludes the proof.

The following result is a well-known corollary of Menger’s theorem.

**Lemma 3.4.** Let $\Sigma$ be a cylinder and let $F_1$ and $F_2$ be the two cuffs. Let $G$ be a graph embedded on $\Sigma$ and suppose that for $i = 1, 2$, $S_i := F_i \cap G \subseteq V(G)$. Let $r \geq 0$ be an integer. Suppose that every simple closed curve $\Gamma$ with $\Gamma \cap G \subseteq V(G)$ and $|\Gamma \cap V(G)| < r$ is contractible in $\Sigma$. Then there are $r$ pairwise disjoint $(S_1, S_2)$-paths in $G$. 

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4 A grid on a cylinder

Let $C$ be a cycle and let $S \subseteq V(C)$ be a subset of its vertices. For $x,y \in V(C)$, let $A$ and $B$ be the two components (possibly empty) of $C - \{x,y\}$. We define the distance between $x$ and $y$ on $C$ with respect to $S$, denoted by $\text{dist}_{(C,S)}(x,y)$ to be $\min\{|V(A) \cap S|, |V(B) \cap S|\}.$

Theorem 4.1. Let $G$ be a graph embedded in a cylinder, and suppose that $G$ has three pairwise disjoint homotopic cycles $C_1,C_2,C_3$, such that $C_1$ and $C_3$ coincide with the cuffs of the cylinder. Let $k \geq 7$ be an integer and let $P_0,\ldots,P_{k-1}$ be pairwise disjoint ($C_1,C_3$)-paths in $G$ such that for every $0 \leq i \leq k-1$ and $1 \leq j \leq 3$, the intersection of $P_i$ and $C_j$ is a single vertex. For $i = 0,\ldots,k-1$, let $s_i$ and $t_i$ be the ends of $P_i$ on $C_1$ and $C_3$, respectively. Set $S := \{s_0,\ldots,s_{k-1}\}$ and $T := \{t_0,\ldots,t_{k-1}\}$. Let $a_1,a_2 \in V(C_1)$ and $b_1,b_2 \in V(C_3)$ such that $b_1 \neq b_2$ and $\text{dist}_{(C_1,S)}(a_1,a_2) \geq 2$. Then the following holds:

(i) If $\text{dist}_{(C_3,T)}(b_1,b_2) \geq 1$, then $G' = G + a_1b_1 + a_2b_2$ contains a $K_6$-minor.

(ii) If $\text{dist}_{(C_3,T)}(b_1,b_2) = 0$, and there exist vertices $a_3 \in V(C_1)$ and $b_3 \in V(C_3)$, such that $\text{dist}_{(C_3,T)}(b_1,b_3) \geq 1$ or $\text{dist}_{(C_3,T)}(b_2,b_3) \geq 1$, then $G' = G + a_1b_1 + a_2b_2 + a_3b_3$ contains a $K_6$-minor.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graphs.png}
\caption{The graphs $\Delta_1,\ldots,\Delta_5$}
\end{figure}

Proof. Throughout the proof all indices are taken modulo $k$. We introduce the graphs $\Delta_1,\ldots,\Delta_5$ which are depicted in Figure 1. Each $\Delta_i$ contains a $K_6$-minor as is evident by the labelling of the vertices in the figure. Thus, it suffices to prove that one of these graphs is a minor of $G'$. 

9
By relabelling the paths \( P_i \) \((0 \leq i < k)\), we may assume that the vertices \( s_0, s_1, \ldots, s_{k-1} \)
(resp., \( t_0, t_1, \ldots, t_{k-1} \)) appear on \( C_1 \) (resp., on \( C_3 \)) in this cyclic order, and we consider
the corresponding orientation of \( C_1 \) and \( C_3 \) as clockwise orientation. For \( i, j \in \{0, \ldots, k-1\} \),
let \( C_1[s_i, s_j] \) (resp., \( C_3[t_i, t_j] \)) be the \((s_i, s_j)\)-path (resp., \((t_i, t_j)\)-path) on \( C_1 \) (resp., \( C_3 \)) in
the clockwise direction on the cycle. Set \( C := C_1 \cup C_2 \cup C_3 \subseteq G \).

By replacing \( G \) with a minor of \( G \), we may assume that \( G \) is the union of the cycles
\( C_1, C_2, C_3 \) and the paths \( P_0, \ldots, P_{k-1} \). Moreover, any contraction of an edge on \( C_1 \) or \( C_3 \)
either identifies two vertices in \( S \cup T \), violates one of distance assumptions on \( a_1, a_2, b_1, b_2, b_3 \),
or identifies \( b_1 \) and \( b_2 \).

As a consequence, by (possibly) relabelling the paths, we may assume that the following
conditions are satisfied:

1. \( a_1 = s_0, \ a_2 = s_j \) \((3 \leq j \leq k - 3)\).
2. There are indices \( 0 \leq \ell < r \leq k - 1 \) such that \( b_1 = t_\ell \) and \( b_2 = t_r \)
or \( b_1 = t_r \) and \( b_2 = t_\ell \).
3. If (ii) holds, then \( a_3 \in S \) and \( b_3 \in T \).

Proof of (i). We proceed according to three cases.

Case 1. Suppose that \( r, \ell \in \{1, \ldots, j - 1\} \) or \( \ell, r \in \{j + 1, \ldots, k - 1\} \). By symmetry
(i.e., after possibly changing the labelling of the paths to the reverse cyclic labelling), we
may assume the former. Since \( r - \ell > 1, s_{\ell+1} \neq s_r \).

If \( b_1 = t_\ell \) and \( b_2 = t_r \), then \( \Delta_1 \leq_m C \cup P_{\ell+1} \cup P_{\ell+1} \cup P_r \cup P_j \cup \{a_1b_1, a_2b_2\} \). To see
this, consider the outer cycle of \( \Delta_1 \) to correspond to \( C_1 \) and the inner-most cycle to corre-
spond to \( C_3 \). The four paths shown correspond in clockwise order, starting on the left \( \left[ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right] \) to
\( P_{\ell+1}, P_{\ell+1}, P_r, \) and \( P_j \), and the two crossed edge are obtained by contracting \( C_1(a_1, s_{\ell+1}) \) and
\( C_3(t_{j+1}, b_1) \).

If \( b_1 = t_r \) and \( b_2 = t_\ell \) then \( \Delta_2 \leq_m C \cup P_0 \cup P_\ell \cup P_r \cup P_j \cup P_{j+1} \cup \{a_1b_1, a_2b_2\} \).

Case 2. Suppose \( \{r, \ell\} \cap \{0, j\} \neq \emptyset \). By symmetry we may assume that \( \ell = 0 \) and
\( 2 \leq r \leq j \).

Suppose first that \( r < j \). If \( b_1 = t_0 \) and \( b_2 = t_r \), then \( \Delta_1 \leq_m C \cup P_{j+1} \cup P_1 \cup P_r \cup P_j \cup \{a_1b_1, a_2b_2\} \) (this is obtained after contracting \( C_1(a_1, s_1) \) and \( C_3(t_{j+1}, b_1) \)). If \( b_1 = t_r \) and
\( b_2 = t_0 \), then \( \Delta_2 \leq_m C \cup P_{k-1} \cup P_1 \cup P_r \cup P_j \cup P_{j+1} \cup \{a_1b_1, a_2b_2\} \) (after contracting \( C_1(s_{k-1}, a_1) \) and
\( C_3(b_2, t_1) \)).

Suppose now that \( r = j \). Since \( k \geq 7 \), we may assume by symmetry that \( j \leq k - 4 \). If
\( b_1 = t_0 \) and \( b_2 = t_j \) then \( \Delta_1 \leq_m C \cup P_{k-1} \cup P_1 \cup P_{j+1} \cup P_{j+1} \cup \{a_1b_1, a_2b_2\} \). If \( b_1 = t_j \) and
\( b_2 = t_0 \), then \( \Delta_2 \leq_m C \cup P_{k-1} \cup P_1 \cup P_{j+1} \cup P_{j+2} \cup \{a_1b_1, a_2b_2\} \) (we contract \( C_1(s_{k-1}, a_1), \)
\( C_1(a_2, s_j+1), C_3(b_2, s_1) \) and \( C_3(t_{j-1}, b_1) \)).

Case 3. Suppose that \( 1 \leq \ell \leq j - 1 \) and \( j + 1 \leq r \leq k - 1 \). Then \( \Delta_1 \leq_m C \cup P_r \cup P_0 \cup
P_\ell \cup P_j \cup \{a_1b_1, a_2b_2\} \).

\(^1\)We will stick with similar assumptions in the remaining cases: \( C_1 \) and \( C_3 \) will correspond to the outer
and inner cycle, respectively, and the order of the paths in \( \Delta_i \) will correspond to the listed order, starting
on the left and continuing clockwise.
Proof of (ii). We have \( r = \ell + 1 \) (since by assumption \( \text{dist}_{(C_3,T)}(b_1, b_2) = 0 \) and \( b_1 \neq b_2 \)). By symmetry, we may assume that \( 0 \leq \ell \leq j - 1 \).

Case 1. Suppose that \( \ell = 0 \) or \( \ell = j - 1 \). By symmetry we may assume that \( \ell = 0 \) and then \( r = 1 \). If \( b_1 = t_0 \) and \( b_2 = t_1 \) then \( \Delta_3 \leq m \ C \cup P_0 \cup P_1 \cup P_{j-1} \cup P_j \cup P_{j+1} \cup P_{k-1} \cup \{a_1b_1, a_2b_2\} \).

If \( b_1 = t_1 \) and \( b_2 = t_0 \) then \( \Delta_4 \leq m \ C \cup P_0 \cup P_1 \cup P_{j-1} \cup P_j \cup P_{j+1} \cup P_{k-1} \cup \{a_1b_1, a_2b_2\} \).

Case 2. Suppose that \( 1 \leq \ell \leq j - 2 \). If \( b_1 = t_{\ell+1} \) and \( b_2 = t_\ell \), then \( \Delta_2 \leq m \ C \cup P_0 \cup P_\ell \cup P_{\ell+1} \cup P_j \cup P_{j+1} \cup \{a_1b_1, a_2b_2\} \).

Suppose now that \( b_1 = t_\ell \) and \( b_2 = t_{\ell+1} \). We may assume that \( \ell = 1 \) and \( \ell + 2 = j \). For if \( \ell \neq 1 \), then \( \Delta_5 \leq m \ C \cup P_1 \cup P_\ell \cup P_{\ell+1} \cup P_j \cup P_{j+1} \cup P_0 \cup \{a_1b_1, a_2b_2\} \), and the case when \( \ell \neq j - 2 \) is symmetric to the case when \( \ell \neq 1 \).

![Figure 2: The graphs obtained in Case 2 of proof of (ii) contain K6-minors](image)

Hence, we are left with the case where \( \ell = 1 \) and \( j = 3 \). By assumption, \( b_3 \in T \setminus \{t_1, t_2\} \). Suppose that \( b_3 \in \{t_5, \ldots, t_{k-2}\} \). Then \( \text{dist}_{(C_3,T)}(b_3, b_1) \geq 2 \) and \( \text{dist}_{(C_3,T)}(b_3, b_2) \geq 2 \). In addition, since \( \text{dist}_{(C_1,S)}(a_1, a_2) \geq 2 \), there is \( z \in \{1, 2\} \) such that \( \text{dist}_{(C_1,S)}(a_3, a_z) \geq 1 \). Then the proof follows by the proof of (i) by interchanging the roles of \( C_1 \) and \( C_3 \) and \( S \) and \( T \), with \( b_3 \) and \( b_z \) playing the role of \( a_1 \) and \( a_2 \), and \( a_3 \) and \( a_z \) playing the role of \( b_1 \) and \( b_2 \), respectively. Thus, we may assume that \( b_3 \in \{t_0, t_3, t_4, t_{k-1}\} \).

Suppose that \( b_3 \in \{t_0, t_3\} \). By symmetry, we may assume that \( b_3 = t_3 \). Let \( H := C \cup \{P_0, P_1, P_2, P_3, P_4, P_5\} \cup \{a_1b_1, a_2b_2, a_3b_3\} \). By contracting edges on \( C_1 \), we obtain a minor \( H' \) of \( H \) such that \( a_3 \in \{s_0, s_1, \ldots, s_5\} \). For each of these six possibilities for \( a_3 \), we see that \( H' \) contains a \( K_6 \)-minor (see Figure 2).

Finally, suppose that \( b_3 \in \{t_4, t_{k-1}\} \). By symmetry, we may assume that \( b_3 = t_4 \). Let \( H := C \cup \{P_0, P_1, P_2, P_3, P_{k-2}, P_{k-1}\} \cup \{a_1b_1, a_2b_2, a_3b_3\} \). Let \( H' \) be obtained from \( H \) by contracting \( C_3(t_3, b_3) \). The proof now follows as in the previous paragraph.
5 The projective plane

In this section we present the proof of Theorem 1.1. The projective plane contains graphs of face-width 3 that do not contain $K_6$ as a minor. In fact the graphs obtained from $K_6$ by performing one or more $\Delta Y$-transformations\(^2\) on facial triangles of $K_6$ provide such examples. On the other hand, face-width four forces $K_6$ minor as claimed by Theorem 1.1. In this section we give a proof of this theorem.

It suffices to prove Theorem 1.1 for minor-minimal graphs embedded in the projective plane with face-width 4. It was proved by Randby [14] that every such graph can be obtained from the projective $4 \times 4$ grid (the first graph depicted in Figure 3) by a series of $Y\Delta$ and $\Delta Y$-transformations. Let $G_4$ be the family of such graphs. It is known [14] that $G_4$ contains precisely 270 graphs.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{The triangle-free graphs in $G_4$ contain $K_6$ minors}
\end{figure}

It is easy to see that if $G$ is obtained from $H$ by a $\Delta Y$-transformation and $G$ has a $K_6$ minor, then so does $H$. Therefore it suffices to prove that all triangle-free graphs in $G_4$

\(^2\)We say that a graph $H$ is obtained from a graph $G$ by a $\Delta Y$-transformation if the edges of a triangle $T = uwv$ are removed from $G$ and replaced by a new vertex $y$ and three edges joining $y$ with each of $u, v, w$. The inverse operation is said to be a $Y\Delta$-transformation.
contain \( K_6 \) as a minor. To justify this conclusion, note that graphs in \( G_4 \) have face-width 4; thus every triangle in such a graph \( G \) is facial in any embedding of \( G \) on the projective plane. Also observe that every \( \Delta Y \)-transformation increases the number of vertices, thus performing these transformations as long as possible, we end up with a triangle-free graph.

Examining the 270 graphs in \( G_4 \), we found that precisely eight of them are triangle-free. They are shown in Figure 3 (drawn in the projective plane), where also a \( K_6 \) minor is exhibited for each of them (the thick edges should be contracted in order to obtain a \( K_6 \) minor). This observation completes the proof of Theorem 1.1.

6 Proof of the main result

In this section we prove Theorem 1.2. Let \( G \) and \( \Sigma \) be as in the theorem. By Theorem 2.7 we may assume that \( G \) is 3-connected and that \( \text{fw}(G) \geq 6 \). Let \( \Lambda \) and \( n \) be as in (H1). By Theorem 2.4 \( \Lambda \) is clean. We shall also assume that \( x_i \in V(G) \) \((i = 0, \ldots, n-1)\), for if \( x_i \) is an edge then we contract \( x_i \) and work with the resulting minor of \( G \). The only danger is that such contractions lower \( \text{nsfw}(G) \). However, this will not be a problem, since any further arguments involving large \( \text{nsfw}(G) \) will refer to the original graph.

Let \( \Sigma \) be a cylinder with cuffs \( F_1 \) and \( F_2 \). Let \( G \) be a graph embedded on \( \Sigma \), and let \( n \geq 2 \) be an integer. Let \( C_1, \ldots, C_n \) be pairwise disjoint homotopic cycles in \( G \) such that \( F_1 = C_1, \ F_2 = C_n \) and \( C_1, \ldots, C_n \) appear along \( \Sigma \) in order. For \( i = 1, \ldots, n-1 \), we say that \( C_i+1 \) is tight in \( G \) with respect to \( C_i \), if there does not exist a \((C_i+1)\)-path \( P \) that is disjoint from \( C_i+1 \) except for its two ends, \( P \) is disjoint from \( C_i \), and \( P \) is embedded in the sub-cylinder of \( \Sigma \) bounded by \( C_i \) and \( C_i+1 \).

The proof proceeds according two two cases, depending whether \( \Lambda \) is 2-sided or 1-sided.

6.1 Proof of Theorem 1.2 when \( \Lambda \) is 2-sided

As \( \Gamma(\Lambda) \) is 2-sided, then as we traverse along \( \Gamma(\Lambda) \) on \( \Sigma \), one side is naturally the “left-hand side” and the other is the “right-hand side”. The curve \( \Gamma(\Lambda) \) splits each face \( F_i \) into two closed disks. Each of these closed disks is bounded by the portion of \( \Gamma(\Lambda) \) in \( F_i \) and a part of the boundary of \( F_i \). For \( i = 0, \ldots, n-1 \), let \( \partial_L(F_i) \) \((\partial_R(F_i)) \) be the portion of the boundary of \( F_i \) to the left (right) of \( \Gamma(\Lambda) \). Then each of \( \partial_L(F_i) \) and \( \partial_R(F_i) \) is a path in \( G \) from \( x_i \) to \( x_{i+1} \) (indices modulo \( n \)). All these paths are pairwise disjoint except for their ends. Set \( C_L(\Lambda) := \bigcup_{i=0}^{n-1} \partial_L(F_i) \) and \( C_R(\Lambda) := \bigcup_{i=0}^{n-1} \partial_R(F_i) \). Since \( \Lambda \) is clean, each of \( C_L(\Lambda) \) and \( C_R(\Lambda) \) is a cycle in \( G \).

Cutting \( \Sigma \) along \( \Gamma(\Lambda) \), results in a new graph \( G' \) embedded on \( \Sigma' \), where \( \Sigma' \) is the surface obtained from \( \Sigma \) by cutting along \( \Gamma(\Lambda) \) and capping off the resulting two cuffs. Let \( F_L \) and \( F_R \) be the two added faces of \( G' \) whose boundaries coincide with \( C_L(\Lambda) \) and \( C_R(\Lambda) \), respectively. By Theorem 2.5 we have \( \text{nsfw}(G') \geq 4 \) and \( \text{fw}(G') \geq 3 \), since \( \text{nsfw}(G) \geq 7 \) and \( \text{fw}(G) \geq 6 \). Let us now apply Theorem 2.6 to \( G' \) and its faces \( F_L \) and \( F_R \), respectively. Let \( B_1(F_L), B_1(F_R), B_1(F_L) \) and \( C_1(F_R) \) be the disks (cycles) as obtained by the application of Theorem 2.6 and the facts that \( \text{nsfw}(G') \geq 4 \) and \( \text{fw}(G') \geq 3 \). Set \( \Omega_R := C_1(F_R) \) and
Lemma 6.1. In $G'$, the cycle $\Omega_L$ (resp., $\Omega_R$) is tight with respect to $F_L$ (resp., $F_R$), and
$\Omega_L \subseteq B_1(F_L)$ (resp., $\Omega_R \subseteq B_1(F_R)$).

Next we observe that

Lemma 6.2. $\Omega_R$ and $\Omega_L$ are disjoint.

Proof. For suppose not, then we let $v \in V(\Omega_R \cap \Omega_L)$. By the definition of $B_1(F_R)$ and
$B_1(F_L)$, $v$ is co-facial with some vertex of $F_R$, say $w_R$, and some vertex of $F_L$, say $w_L$. In $G$,
the vertices $v, w_R, w_L$, define a face-chain $\Lambda'$ of length two (not closed), starting and ending
in $\Lambda$, whose faces are distinct from the faces of $\Lambda$. Let $\Lambda_1$ and $\Lambda_2$ be the two face-subchains in $\Lambda$ with ends $w_R$ and $w_L$. As $\Gamma(\Lambda')$ connects the left side of $\Gamma(\Lambda)$ with its right side, we see that both $\Gamma(\Lambda' \cup \Lambda_1)$ and $\Gamma(\Lambda' \cup \Lambda_2)$ are surface non-separating in $G$.

To obtain a contradiction, note that by Theorem 2.2 (with $k = 2$), one of $\Lambda' \cup \Lambda_1$ and
$\Lambda' \cup \Lambda_2$ is of length at most 6. Since both $\Gamma(\Lambda' \cup \Lambda_1)$ and $\Gamma(\Lambda' \cup \Lambda_2)$ are surface non-separating
in $G$, we have a contradiction to the assumption that $\text{nsfw}(G) \geq 7$. \qed

In $G$, the cycles $\Omega_R$ and $\Omega_L$ are homotopic to $\Gamma(\Lambda)$ (and homotopic to each other).
Therefore, there exists $\Sigma' \subseteq \Sigma$ such that $\Sigma'$ is homeomorphic to a cylinder, the cuffs
of which coincide with $\Omega_R$ and $\Omega_L$, and $\Gamma(\Lambda) \subseteq \Sigma'$. Let $G(\Omega_L, \Omega_R) \subseteq G$, be the subgraph of $G$
embedded in $\Sigma'$ (including $\Omega_L$ and $\Omega_R$).

Let $Q = \{Q_1, Q_2, \ldots\}$ be a set of pairwise disjoint paths, such that each $Q_i$ is an $(\Omega_L, \Omega_R)$-
path, disjoint from $G(\Omega_L, \Omega_R)$ except for its ends. If $Q$ is of maximum cardinality, then we
say that $Q$ is an exterior $(\Omega_L, \Omega_R)$-linkage. By Lemma 3.1, $|Q| \geq \text{nsfw}(G) \geq 7$.

By two applications of Lemma 3.4 and using Lemma 6.1, we see that $G(\Omega_L, \Omega_R)$ contains a set
$P_R^0, \ldots, P_R^{n-1}$ and $P_L^0, \ldots, P_L^{n-1}$ of pairwise disjoint paths, satisfying the following
properties:

(1) For $i = 0, \ldots, n - 1$, $P_i^L$ (resp., $P_i^R$) has ends $x_i$ and $l_i \in \Omega_L$ (resp., $r_i \in \Omega_R$) and is
otherwise disjoint from $\Omega_L$ (resp., $\Omega_R$) and $X(\Lambda)$. For $i = 0, \ldots, n - 1$, set $P_i := P_i^L \cup P_i^R$.
Note that $P_i$ is an $(l_i, r_i)$-path contained in $G(\Omega_L, \Omega_R)$. Also note that the vertices
$r_0, r_1, \ldots, r_{n-1}$ (indices modulo $n$) appear on $\Omega_R$ ($\Omega_L$) in order.

(2) Let $X \in \{L, R\}$. For $i = 0, \ldots, n - 1$, $P_i$ is disjoint from $\partial_X(F_i)$, if $j \neq \{i - 1, i\}$
(indices modulo $n$). In addition, we may assume that $P_i - x_i$ intersects at most one
of $\partial_X(F_i)$ and $\partial_X(F_{i-1})$. If $P_i - x_i$ intersects $F_i$ we say that $F_i$ is the $X$-support of $P_i$,
otherwise $F_{i-1}$ is the $X$-support of $P_i$.

A set $\mathcal{P} = \{P_0, \ldots, P_{n-1}\}$ of paths satisfying properties (1) and (2) above, is called an
internal $(\Omega_L, \Omega_R)$-linkage. Figure 4 shows part of an internal linkage and the corresponding
notation as used in the sequel.
For $i = 0, \ldots, n - 1$, let $\Omega_R(i)$ (resp., $\Omega_L(i)$) be the path on $\Omega_R$ (resp., $\Omega_L$) from $r_i$ to $r_{i+1}$ (resp., $l_i$ to $l_{i+1}$) not passing thorough $r_{i+2}$ (resp., $l_{i+2}$).

Let $X \in \{R, L\}$. For a subset of indices $I = \{i_0, \ldots, i_{|I|-1}\} \subseteq \{0, \ldots, n - 1\}$, let $F_X(P, I) \subseteq F(\Lambda)$ be a set of consecutive faces of $\Lambda$ satisfying the following:

1. For every vertex $v \in \bigcup_{i \in I} \Omega_X(i)$ there exists $f \in F_X(P, I)$ such that $v$ is co-facial with some vertex of $f$.

2. Subject to (1), $|F_X(P, I)|$ is minimum.

Observe that $F_X(P, I)$ exists by Lemma 6.1. Of special interest is the case when $I = \{i\}$ or $I = \{i, i+1\}$ for some $0 \leq i \leq n - 1$. Note that $1 \leq |F_X(P, \{i\})| \leq 3$ and $1 \leq |F_X(P, \{i, i+1\})| \leq 4$.

For $i = 0, \ldots, n - 1$ and $X \in \{R, L\}$, let $D_X(i)$ be the closed disk bounded by $\Omega_X(i)$, $P^X_i$, $P^X_{i+1}$ and a path in $C_X(\Lambda)$ on the boundary of the faces in $F_X(P, \{i\})$.

The following is a direct consequence of the definition of $F_X(P, \{i\})$.

Lemma 6.3. For $i = 0, \ldots, n - 1$, each of $r_i$ and $r_{i+1}$ (resp., $l_i$ and $l_{i+1}$) is co-facial in $G(\Omega_L, \Omega_R)$ with some vertex in $V(F_R(P, \{i\}))$ (resp., $V(F_L(P, \{i\}))$).

A system is a pair $(Q, P)$, where $Q$ is an exterior $(\Omega_L, \Omega_R)$-linkage and $P$ is an interior $(\Omega_L, \Omega_R)$-linkage. For $X \in \{L, R\}$ and a subset $A \subseteq Q \cup P$ of paths, we denote by

\begin{itemize}
\item $\Lambda_1$ and $\Lambda_2$ need not be subchains since $w_L$ and $w_R$ need not be constituents of $\Lambda$. But all other faces and vertices in $\Lambda_1$ and $\Lambda_2$ are taken from $\Lambda$.
\end{itemize}
Ends\((A, \Omega_X)\) the set of endvertices of the paths in \(A\) contained in \(\Omega_X\). If \(A\) is a single path, we set \(End(A, \Omega_X) = Ends(\{A\}, \Omega_X)\). The following is the key ingredient in the proof.

**Lemma 6.4.** Let \(k \geq n \geq 7\), and let \(\mathcal{Z} = (\mathcal{Q}, \mathcal{P})\) be a system, where \(\mathcal{Q} = \{Q_0, \ldots, Q_{k-1}\}\) and \(\mathcal{P} = \{P_0, \ldots, P_{n-1}\}\). Let \(Z \in \{L, R\}\) and \(Y \in \{L, R\}\setminus Z\). Then there exists a system \((\mathcal{Q}', \mathcal{P}')\), where \(\mathcal{Q}' = \{Q_0', \ldots, Q'_{k-1}\}\) and \(\mathcal{P}' = \{P_0', \ldots, P'_{n-1}\}\) satisfying the following:

\[
\begin{align*}
(P1) & \text{ For } i = 0, \ldots, n-1, \ P_i^Z = P_i^Z. \\
(P2) & \text{ For } i = 0, \ldots, k-1, \ End(Q_i, \Omega_Z) = End(Q'_i, \Omega_Z). \\
(P3) & \text{ There exist paths } A, B \in \mathcal{Q}', \text{ such that } End(A, \Omega_Y) \text{ and } End(B, \Omega_Y) \text{ are at distance at least two on } \Omega_Y \text{ with respect to } Ends(\mathcal{P}', \Omega_Y).
\end{align*}
\]

**Proof.** Assume for a contradiction that the claim is false. By symmetry we may assume that \(Z = R\) and \(Y = L\).

For every system \((\mathcal{Q}', \mathcal{P}')\) satisfying (P1) and (P2), let \(S \subseteq \Omega_L\) be a minimal segment on the cycle \(\Omega_L\) such that \(Ends(\mathcal{Q}', \Omega_L) \subseteq V(S)\) and \(S\) contains in its interior at most one vertex from the set \(Ends(\mathcal{P}', \Omega_L)\). Note that \(S\) exists since otherwise (P3) would hold. Let \(T \subseteq \Omega_L\) be the minimal segment on \(\Omega_L\) containing \(S\) such that the endpoints of \(T\) are in \(Ends(\mathcal{P}', \Omega_L)\). Among all systems \((\mathcal{Q}', \mathcal{P}')\), we choose one with the following properties:

\[
\begin{align*}
(J0) & \text{ } (\mathcal{Q}', \mathcal{P}') \text{ satisfies (P1) and (P2).} \\
(J1) & \text{ The number of vertices of } Ends(\mathcal{P}', \Omega_L) \text{ contained in } T \text{ is as large as possible and } |V(T) \setminus V(S)| \text{ is minimum.}
\end{align*}
\]

Such a choice of \((\mathcal{Q}', \mathcal{P}')\) is clearly possible as \((\mathcal{Q}, \mathcal{P})\) satisfies (P1) and (P2). By symmetry, we may assume that \(T = \Omega_L(0)\) or \(T = \Omega_L(0) \cup \Omega_L(1)\) and that \(|Ends(\mathcal{Q}, \Omega_L) \cap V(\Omega_L(0))| \geq 4\). To simplify notation, we may also assume that \((\mathcal{Q}, \mathcal{P}) = (\mathcal{Q}', \mathcal{P}')\).

Set \(X := \Omega_L(n-1) \cup \Omega_L(0) \cup \Omega_L(1)\). Then \(X \subseteq \Omega_L\) is an \((l_{n-1}, l_2)\)-path on \(\Omega_L\) containing \(l_0\). Let \(\bar{X}\) be the other \((l_{n-1}, l_2)\)-path on \(\Omega_L\) such that \(X \cup \bar{X} = \Omega_L\) and \(X\) and \(\bar{X}\) are disjoint except for their ends. We may further assume that \(\mathcal{Z}'\) is chosen so that

\[
(J2) \text{ Subject to (J1), } |V(X)| \text{ is minimum.}
\]

Observe that by the maximality of the number of disjoint paths in \(\mathcal{Q}\), there does not exist an \((int(\bar{X}), \Omega_R)\)-path which is internally disjoint from \(G(\Omega_L, \Omega_R)\). Let us apply Theorem 3.3 to \(G\) with \(\mathcal{Q}\), \(X\) and its ends (that is, the endpoints of \(X\) playing the role of \(w\) and \(w'\) in Theorem 3.3). Note that outcome (b) of Theorem 3.3 is obtained. Let \(u, v \in V(X)\) and \(f\) be as promised to exist by Theorem 3.3(b). Without loss of generality, assume that \(v\) is closer to \(l_{n-1}\) than \(u\) on \(X\).

The property of \(f\) stated in Theorem 3.3 implies the following.

\[
(1) \text{ Every face-chain in } G(\Omega_L, \Omega_R) \text{ with ends } v \text{ and } u \text{ has length at least 6.}
\]
To see this, suppose that $\Lambda'$ is a face-chain in $G(\Omega_L, \Omega_R)$ of length at most five with ends $u$ and $v$. Then $\Lambda' \cup f$ is a closed face-chain in $G$ of length at most six, and by the property of $f$, $\Gamma(\Lambda' \cup f)$ is surface non-separating; contradicting the assumption that $\text{nsfw}(G) \geq 7$. This proves (1).

Property (1) immediately implies that

(2) $v \in V(\Omega_L(n-1))$ and $u \in V(\Omega_L(1))$.

Next we claim the following:

(3) In $G(\Omega_L, \Omega_R)$ there exist face chains $g_v, g_u$ of length at most two such that $g_v$ has ends $v, l_0$, and $g_u$ has ends $l_1, u$. Moreover, if $g_v$ (resp., $g_u$) is of length two, then $v$ (resp., $u$) is co-facial with some vertex in the $L$-support of $P_0$ (resp., $P_1$).

We will prove existence of $g_u$ (the proof for $g_v$ is exactly the same; in fact it is even easier since no $Q \in Q$ has an end in the interior of $\Omega_L(n-1)$).

Let $j \in \{0, 1\}$ so that $F_j$ is the $L$-support of $P_j$. Let $w$ be the end of the path $P_1 \cap \partial_L(F_j)$ of $P_1$ so that $w \neq x_1$, unless the path $P_1 \cap \partial_L(F_j)$ is the single vertex $x_1$. It suffices to show that every vertex $x \in \Omega_L(l_1, u)$ is of degree two in $G(\Omega_R, \Omega_L)$ except for $w$ and the two neighbors of $x$ on the cycle $\Omega_L$. (Note that $x$ is adjacent in $G(\Omega_L, \Omega_R)$ with only two vertices in $\Omega_L$, since $\Omega_L$ is tight.)

Suppose to the contrary that there exists $x \in \Omega_L(l_1, u)$ such that $x$ has a neighbor in $G(\Omega_L, \Omega_R)$ that is distinct from $w$ and from the two neighbors of $x$ on the cycle $\Omega_L$. Since $G$ is 3-connected (and hence $G - w$ is 2-connected), by Menger’s theorem there exists a $(\partial D_L(1))$-path $P$ with ends $x$ and $y$, where $y \in V(\partial D_L(1)) \setminus \{w\}$. Since $\Omega_L$ is tight, we have $y \in V(F_L)$.

**Case 1.** Suppose $j = 1$. By Lemma 6.1, $y \in V(F_1 \cup F_2) \setminus \{w\}$. Let $P'_2$ be the path obtained from $P_2$ by rerouting $P'_2$ so that it passes via $P$. Let $P'$ be the new collection of paths. We claim that $P'$ contradicts our choice of $\{Q, P\}$.

If $\text{Ends}(Q, \Omega_L(l_1, l_2)) = \emptyset$, then $\{Q, P'\}$ contradicts (J2). Suppose that $\text{Ends}(Q, \Omega_L(x, u)) \neq \emptyset$ for some $Q \in Q$, and let $Q'$ be a path in $Q$ with an end in $\Omega_L(l_0, l_1)$. Such a path exists since $|\text{Ends}(Q, \Omega_L(0))| \geq 3$ and hence $|\text{Ends}(Q, \Omega_L(l_0, l_1))| \geq 1$. Then $Q$ and $Q'$ are at distance at least two with respect to $\text{Ends}(P', \Omega_L)$ and hence $\{Q, P'\}$ satisfies (P3). It follows that $\text{Ends}(Q, \Omega_L) \subseteq \Omega_L(0) \cup \Omega_L[l_1, x]$, but then $\{Q, P'\}$ contradicts (J1).

**Case 2.** $j = 0$. By Lemma 6.1, $y \in V(F_0 \cup F_1 \cup F_2) \setminus \{w\}$. We first observe that $u$ is not co-facial in $G(\Omega_L, \Omega_R)$ with any vertex in $V(F_0)$. For suppose $u$ is co-facial in $G(\Omega_L, \Omega_R)$ with $F_0$, say via a face $g_0$. By Lemma 6.3, $v$ is co-facial in $G(\Omega_L, \Omega_R)$, say via a face $g_1$, with some vertex of $V(\partial D_L(F_i))$, for some $i \in \{n - 2, n - 1, 0\}$. Then using $g_0, g_1, F_{n-2}, F_{n-1}$ and $F_0$ we can construct a face-chain of length at most five in $G(\Omega_L, \Omega_R)$ with ends $u$ and $v$, contradicting (1).

If $y \in V(F_1 \cup F_2) \setminus \{x_1\}$, the proof proceeds exactly as in Case 1 (by replacing $P'_2$ by another path using $P'$ and thus obtaining a contradiction). Hence, we may assume that $y \in V(F_0) \setminus \{w\}$. Since $u$ is not co-facial with any vertex in $V(F_0)$, Lemma 6.1 implies that
there exists a path $P'$ in $D_L(1)$ with one end in $V(F_1 \cup F_2) \setminus \{x_1\}$ and the other end in $V((P - F_0) \cup \Omega_L[x,u])$. We then see that there exists a path $P''$ with one end in $V(\Omega_L[x,u])$ and another end in $V(F_1 \cup F_2) \setminus \{x_1\}$, and so $P''$ playing the role of $P$. This proves (3).

Observe that at least one of $P_0$ and $P_1$ is not $L$-supported by $F_0$. For if both $P_0$ and $P_1$ are $L$-supported by $F_0$, then by Lemma 6.3, each of $l_0$ and $l_1$ is co-facial with some vertex in $V(F_0)$. By (3), there is a face-chain of length at most two from $v$ (resp., $u$) to a vertex in $V(F_0)$, since by (3), $v$ is either co-facial with $l_0$ (resp., $u$ is co-facial with $l_1$) in $G(\Omega_L, \Omega_R)$ or co-facial with a vertex in $V(F_0)$. Combining these faces together with $F_0$, we obtain a face-chain from $v$ to $u$ in $G(\Omega_L, \Omega_R)$ that is of length at most 5, contradicting (1).

We will assume henceforth that $P_1$ is $L$-supported by $F_1$ (if $P_1$ is $L$-supported by $F_0$, the proof follows the same arguments). Now we distinguish two cases: either $P_0$ is $L$-supported by $F_0$ or by $F_{n-1}$. We consider the latter case (the former case is proved by the same arguments).

By (1) and (3), $l_0$ and $l_1$ are not co-facial in $G(\Omega_R, \Omega_L)$. Hence there exists $x \in V(\Omega_L[l_0, l_1])$ of degree at least three in $G(\Omega_R, \Omega_L)$. By Menger’s theorem and since $\Omega_L$ is tight, there is a $\partial D_L(0)$-path $P$ in $D_L(0)$ with ends $x$ and $y$, where $y \in \partial D_L(0) \cap (F_{n-1} \cup F_0 \cup F_1)$.

**Case 1.** Suppose that $y \in V(F_0) \setminus \{x_0, x_1\}$. Let $P'_0$ (resp., $P'_1$) be obtained from $P_0$ (resp., $P_1$) by re-routing it so that it passes via $P$ rather via $P^L_0$ (resp., $P^L_1$). Let $P' = (P \setminus \{P_0\}) \cup \{P'_0\}$ and let $P'' = (P \setminus \{P_0\}) \cup \{P'_1\}$. We claim that one of $(Q, P')$ or $(Q, P'')$ contradicts our choice of $(Q, P)$. We argue as follows.

We may assume that $\text{Ends}(Q, \Omega_L[l_0, x]) \neq \emptyset$, for otherwise $(Q, P')$ contradicts (J1). Further, we may assume that $\text{Ends}(Q, \Omega_L[l_1, x]) = \emptyset$, for otherwise $(Q, P')$ satisfies (P3) (since $\text{Ends}(Q, \Omega_L[l_0, x]) \neq \emptyset$). Now it is easy to see that $(Q, P'')$ contradicts one of the two conditions stated in (J1).

**Case 2.** Suppose that $y \in V(F_1 \cup F_{n-1})$. We may assume that $y \in V(F_1)$ (if $y \in V(F_{n-1})$ the proof follows by the same arguments). Let $w$ be the end of the path $P_0 \cap \partial_L(F_{n-1})$ so that $w \neq x_0$ unless $V(P_0 \cap \partial_L(F_{n-1})) = \{x_0\}$. We proceed according to two cases, depending on whether $l_0$ is co-facial with a vertex in $V(F_1)$ or not.

**Case 2.1.** If $l_0$ is not co-facial with a vertex in $V(F_1)$, there exists a path $R$ in $D_L(0)$ with one end in $V(\Omega_L[l_0, x])$ and the other in $V(F_{n-1} \cup (F_0 \setminus x_1))$ (since $\Omega_L$ is tight). Let $P'_0$ (resp., $P'_1$) be obtained from $P_0$ (resp., $P_1$) by re-routing it so that it passes via $R$ (resp., $P$) rather than via $P^L_0$ (resp., $P^L_1$). The proof now proceed exactly as in Case (1).

**Case 2.2.** Suppose that $l_0$ is co-facial with some vertex in $V(F_1)$ via a face $f_0$. By (1) and (3), $g_0$ must be a face-chain of length two. By (the proof of) (3), $v$ and $w$ are co-facial in $G(\Omega_L, \Omega_R)$ and there exists a vertex $z \in \Omega_L(v, l_0)$ such that $zw \in E(G(\Omega_L, \Omega_R))$. Hence by (1), $w$ is not co-facial with a vertex in $V(F_1)$. Thus, there exists a path $R$ in $D_L(0)$ with one end in $V(F_{n-1} \cup (F_0 \setminus x_1))$ and the other end, say $a$, in $V(P^L_0 - V(F_{n-1}))$. Note that the path $R$ cannot end up in $\Omega_L(0) \setminus \{l_0\}$ because of the face $f_0$.

Let $P'_0$ (resp., $P'_{n-1}$) be obtained from $P_0$ (resp., $P_{n-1}$) by re-routing it so that it passes via $R$ (resp., the edge $wz$) rather than via $P^L_0$ (resp., $P^L_{n-1}$). Let $P'$ be the new collection of
paths obtained by replacing \( P_6 \) and \( P_{n-1} \) with \( P'_6 \) and \( P'_{n-1} \). Then \((Q, P')\) contradicts \((J2)\). This completes the proof of Lemma \[6.4\].

Now we can complete the proof of Theorem \[1.2\] when \( \Lambda \) is 2-sided. Let \((Q, P)\) be a system. By two applications of Lemma \[6.4\] we may assume that \((Q, P)\) satisfies \((P3)\) for \( X \in \{L, R\} \).

Let \( A, B \in Q \), such that \( \text{End}(A, \Omega_L) \) and \( \text{End}(B, \Omega_L) \) are at distance at least two with respect to \( \text{Ends}(P, \Omega_L) \). Set \( q_A = \text{End}(A, \Omega_R) \) and \( q_B = \text{End}(B, \Omega_R) \). If \( q_A \) and \( q_B \) are at distance at least one on \( \Omega_R \) with respect to \( \text{Ends}(P, \Omega_R) \), set \( H = G(\Omega_L, \Omega_R) \cup A \cup B \). Otherwise, \( q_A, q_B \in V(\Omega_R(i)) \) for some \( 0 \leq i \leq n - 1 \). By \((P3)\), there exists a path \( C \in Q \) such that \( \text{End}(C, \Omega_R) \not\in V(\Omega_R(i)) \). Set \( H = G(\Omega_L, \Omega_R) \cup A \cup B \cup C \).

For each \( i \in \{0, \ldots, n-1\} \), \( P_i \) intersect each of \( \Omega_L \) and \( \Omega_R \) in a single vertex. In addition, by definition, \( P'_i = P_i \cap G(\Lambda) \) is a sub-path of \( P_i \) with \( x_i \in V(P'_i) \). Let \( H_1 \) be obtained from \( H \) by contracting the path \( P'_i \) into the vertex \( x_i \), for \( i = 0, \ldots, n - 1 \). By Theorem \[4.1\] \( H_1 \) contains a \( K_6 \) minor, and hence also \( G \). This completes the proof.

### 6.2 Proof of Theorem \[1.2\] when \( \Lambda \) is 1-sided

Since the deletion of any vertex decreases the non-separating face-width at most by 1, we may assume that \( \text{nsfw}(G) = 7 \). By Theorem \[2.7\] we may also assume that \( G \) is 3-connected and that \( \text{fw}(G) \geq 6 \). Moreover, we shall assume throughout this subsection that \( \Lambda \) is 1-sided. Let \( G' \) be the embedded graph obtained from \( G \) by cutting the surface along \( \Gamma(\Lambda) \) and capping off the resulting cuff with a disk \( F \). In \( G' \), every vertex \( x_i \in X(\Lambda) \) \( (i = 0, 1, \ldots, 6) \) is split into two copies, \( x'_i \) and \( x''_i \), and the vertices \( x'_0, x'_1, \ldots, x'_6, x''_0, x''_1, \ldots, x''_6 \) appear on the boundary of \( F \) in the listed cyclic order. By Theorem \[1.1\] we may assume that \( \Sigma \) is not the projective plane, thus the resulting surface \( \Sigma' \) is not the sphere. By Theorem \[2.5\] \( \text{nsfw}(G') \geq \left[ \frac{1}{2} \text{nsfw}(G) \right] \geq 4 \) and \( \text{fw}(G') \geq 3 \).

Let us first consider the possibility that \( \text{fw}(G') = 3 \). Let \( \Gamma' \) be the corresponding non-contractible curve. Clearly, \( \Gamma' \) involves the face \( F \) and two other faces \( A, B \) that are also faces of \( G \) in \( \Sigma \). Let \( x, F, y, B, z, A, x \) be the corresponding closed face-chain in \( \Sigma' \). The curve \( \Gamma' \) is surface-separating on \( \Sigma' \) since \( \text{nsfw}(G') \geq 4 \). We can view \( \Gamma' \) as a simple closed curve \( \Gamma_0 \) in \( \Sigma \) by replacing the part of \( \Gamma' \) in \( F \) by a segment of \( \Gamma(\Lambda) \). We can do this in two ways, so we may take a segment of \( \Gamma(\Lambda) \) such that \( \Gamma_0 \) intersects at most 3 vertices in \( X(\Lambda) \) that are different from \( x \) and \( y \). Since \( \text{nsfw}(G) \geq 7 \), \( \Gamma_0 \) is surface-separating in \( \Sigma \) (possibly contractible) and it separates the surface into two non-spherical surfaces \( \Sigma_1 \) and \( \Sigma_2 \). One of them, say \( \Sigma_1 \), contains a 1-sided curve corresponding to \( \Lambda \). The part of \( \Gamma' \) disjoint from the interior of the face \( F \) can be combined in \( \Sigma \) with two segments contained in \( \Lambda \) to give two closed curves. One of them is \( \Gamma_0 \), and we call the other one \( \Gamma_1 \). Observe that \( \Gamma_1 \) is homologous to \( \Lambda \) (thus 1-sided) and \( \Gamma_0 \) is surface-separating in \( \Sigma \). Since \( \text{nsfw}(G) \geq 7 \) and \( \text{fw}(G) \geq 6 \), \( \Gamma_1 \) necessarily passes through four consecutive vertices in \( X(\Lambda) \), and \( \Gamma_0 \) passes through the remaining three vertices in \( X(\Lambda) \). We may assume that \( \Gamma_0 \) passes through \( x_0, x_1, x_2 \) and through the vertices \( x, y, z \). One particular observation is that \( x, y, z \notin X(\Lambda) \) and that the face-chain of \( \Gamma_0 \) is \( x, A, z, B, y, F_2, x_2, F_1, x_1, F_0, x_0, F_0, x \) (see Figure 5).
Let us consider the set $\mathcal{F}$ of faces of $G$ that lie inside $\Sigma_0$ and are incident with vertices in $X(\Gamma_0)$. Each such face is incident with precisely one vertex in $X(\Gamma_0)$. If not, we would either contradict 3-connectivity of $G$ or the fact that $\text{fw}(G) \geq 6$. If a face $Q \in \mathcal{F}$ is incident with $t \in X(\Gamma_0)$, we say that $Q$ is a $t$-face, and we let $\mathcal{F}_t$ denote the set of all $t$-faces in $\mathcal{F}$.

We say that two distinct vertices $s, t \in X(\Gamma_0)$ are at distance $d$ if their minimum face-distance in the closed face-chain of $\Gamma_0$ is equal to $d$. Note that $d \in \{1, 2, 3\}$.

Suppose that $s, t \in X(\Gamma_0)$ are at distance 3 and that $A \in \mathcal{F}_s$, $B \in \mathcal{F}_t$. If $A$ and $B$ have a vertex $v$ in common, then the face-chain $s, A, v, B, t$ and the two face-subchains of $\Gamma_0$ give rise to two closed face-chains in $\Sigma$ of length 5, so they determine contractible closed walks. The 3-path-property implies that $\Gamma_0$ is also contractible. This contradiction shows that $A \cap B = \emptyset$.

If $s, t \in X(\Gamma_0)$ are at distance 2 and $A \in \mathcal{F}_s$, $B \in \mathcal{F}_t$ have a vertex $v$ in common, then we similarly see that one of the face-chains in $\Sigma$ obtained in the same way as above is of length 4, the other one of length 6. The first one determines a contractible curve in $\Sigma$. We can re-route $\Gamma_0$ through $v$, thus making $\Sigma_2$ smaller. By repeating this process as long as necessary, we may assume that faces in $\mathcal{F}_s$ and $\mathcal{F}_t$ are disjoint whenever $s$ and $t$ are at distance 2.

If $s$ and $t$ are at distance 1 and two faces, $C \in \mathcal{F}_s$ and $D \in \mathcal{F}_t$, have a vertex $v$ in common (e.g. the faces $C, D$ depicted in Figure 5), then there is a face-chain of length 3 through $s, t, v$ and the two faces. The corresponding closed curve $\Gamma$ in $\Sigma$ is contractible, and we add all faces in the interior of $\Gamma$ into $\mathcal{F}$. After doing this for all possible choices of $s, t, C, D$, we define $\partial \mathcal{F}$ as the set of edges that belong to precisely one face in $\mathcal{F}$ and do not belong to any of the faces of $\Gamma_0$. The properties stated in the preceding paragraphs imply that $\partial \mathcal{F}$ is a simple cycle in $G$ that is homotopic to $\Gamma_0$. (In Figure 5, this cycle is represented as the boundary of the darker shaded area. All faces in the lighter shaded area belong to $\mathcal{F}$ and form a disk in $\Sigma$.) Now we delete all edges and vertices in $\Sigma_2$ that do not belong to any of the faces in $\mathcal{F}$ and cap off the cycle $\partial \mathcal{F}$ by pasting a disk onto it. This gives rise to a subgraph $G_1$ of $G$ embedded into the capped surface $\Sigma_1$. It is easy to see by using the 3-path-property that $\text{nsfw}(G_1) \geq 7$ since every surface non-separating face-chain
through the disk of $\partial F$ can be rerouted to use the face-chain of $\Gamma_0$ without increasing its length. Since the genus decreases by the reduction from $\Sigma$ to $\Sigma_1$, such a reduction can be made only a finite number of times, eventually yielding a case where $\text{fw}(G') \geq 4$.

From now on, we shall assume that $\text{fw}(G') \geq 4$. Let us apply Theorem 2.6 to the embedding of $G'$ in $\Sigma'$ and the face $F$. For $i = 0, 1$, let $C_i = C_i(F)$ be the cycle as in Theorem 2.6. Since $\text{fw}(G') \geq 4$, these two cycles are contractible in $\Sigma'$.

The boundary of $F$ is a cycle in $G'$. In $G$, it corresponds to a closed walk which intersects itself transversally when passing through the vertices in $\Lambda$, but it does not cross itself on the surface. In this sense we view $C_0$ as a closed walk in $G$. Theorem 2.6 assures that $C_0$ is homotopic to $C_1$.

Consider the cycle $C_1$ in $G$. Cutting $G$ along $C_1$ separates $\Sigma$ into two components, one of which contains $\Lambda$ and $C_0$. This surface is homeomorphic to the Möbius strip. By capping off the cuff (pasting a disk onto $C_1$), we obtain a graph embedded into the projective plane $\Sigma_1$. We denote by $F_1$ the face in $\Sigma_1$ bounded by the cycle $C_1$. We also denote by $\Sigma_2$ the other bordered surface obtained after cutting $\Sigma$ along $C_1$.

Let $v \in V(C_1)$. Since $G$ is 3-connected and the embedding of $G$ in $\Sigma$ has face-width more than 3, the facial neighborhood of $v$ forms a disk on the surface that is bounded by a cycle $N_v$. This cycle contains a path $P_v$ whose ends $x, y$ are on $C_1$ but all edges and other vertices on this path lie in $\Sigma_2 \setminus C_1$. Moreover, $P_v$ can be selected so that the cycle $Q_v$ consisting of $P_v$ and the $(x, y)$-segment of $C_1$ containing $v$ is contractible in $\Sigma_2$, and the interior of $Q_v$ contains all faces that are incident with $v$ and are contained in $\Sigma_2$. (The proof of this fact is essentially the same as the main argument in the proof of Theorem 2.6; cf. [8] or [10].) If $u, v \in V(C_1)$ and the ends of the paths $P_v$ and $P_u$ interlace on $C_1$, contractibility of the cycles $Q_v$ and $Q_u$ implies that $P_v$ and $P_u$ intersect. This property has the following consequence. Let $H'$ be the minor of $G \cap \Sigma_2$ obtained from $P = \bigcup_{v \in V(C_1)} P_v \cup C_1$ by contracting all edges in $P$ whose both ends are outside $C_1$. Then $H'$ consists of $C_1$ together with some chords of $C_1$ and some vertices whose all neighbors lie on $C_1$. The interlacing property stated earlier implies that $H'$ can be drawn in the disk so that $C_1$ is on the boundary of the disk. By inserting this disk into the face $F_1$ in $\Sigma_1$ we obtain a minor $G''$ of $G$ that is embedded into the projective plane. It is easy to see that the face-width of $G''$ is at least 4, and by Theorem 1.1 $G''$ contains $K_6$ as a minor. Since $G''$ is a minor of $G$, we conclude that $G$ has $K_6$ minor.

This completes the proof of Theorem 1.2.

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