Entanglement of resonantly coupled field modes in cavities with vibrating boundaries

M. A. Andreata, A. V. Dodonov and V. V. Dodonov

Departamento de Física, Universidade Federal de São Carlos,
Via Washington Luiz, km 235, 13565-905 São Carlos, SP, Brasil

Abstract

We study time dependence of various measures of entanglement (covariance entanglement coefficient, purity entanglement coefficient, normalized distance coefficient, entropic coefficients) between resonantly coupled modes of the electromagnetic field in ideal cavities with oscillating boundaries. Two types of cavities are considered: a three-dimensional cavity possessing eigenfrequencies $\omega_3 = 3\omega_1$, whose wall oscillates at the frequency $\omega_w = 2\omega_1$, and a one-dimensional (Fabry–Perot) cavity with an equidistant spectrum $\omega_n = n\omega_1$, when the distance between perfect mirrors oscillates at the frequencies $\omega_1$ and $2\omega_1$. The behaviour of entanglement measures in these cases turns out to be completely different, although all three coefficients demonstrate qualitatively similar time dependences in each case (except for some specific situations, where the covariance entanglement coefficient, based on traces of covariance submatrices, seems to be essentially more sensitive to entanglement than other measures, which are based on determinants of covariance submatrices). Different initial states of the field are considered: vacuum, squeezed vacuum, thermal, Fock, and even/odd coherent states.

PACS: 42.50.Lc; 42.50.Dv; 03.65.-w

Key words: Dynamical Casimir effect; Vibrating boundary; Parametric resonance; Coupled modes; Entanglement; Quantum purity; Entropy; Distance; Covariances; Fock states; Gaussian states; Even/odd coherent states; Squeezed states; Thermal states

1 Introduction

During the past decade it was recognized that the concept of entanglement, introduced by Schrödinger in 1935 [1,2], is not only one of the most profound in quantum mechanics (as was shown in the same year by Einstein, Podolsky and Rosen in their famous paper [3], albeit without using explicitly this word), but it is crucial for many promising new applications, such as quantum cryptography, quantum communication and teleportation, quantum computing, etc. This explain a burst of interest to various problems connected to this concept observed for the past few years. One of such problems is a search for quantitative measures of entanglement.

In the most cases, the measures based on different kinds of entropies have been considered [4,5,6,7,8]. For example, if the total system, consisting of parts 1 and 2, is described by means of the statistical operator $\hat{\rho}_0$, then the entanglement measure is frequently expressed in terms of the total and “partial” entropies as the “index of correlation” [4]

$$I_c = S_1 + S_2 - S_0, \quad S_k = -\text{Tr}_k \hat{\rho}_k \ln \hat{\rho}_k,$$

where the reduced statistical operator is defined as, e.g., $\hat{\rho}_1 = \text{Tr}_2 \hat{\rho}_0$. For the total pure states formula [4] is reduced to $I_c = 2S_1 = 2S_2$.  

*on leave from Lebedev Physical Institute and Moscow Institute of Physics and Technology, Russia
However, despite of many advantages, the measures such as (1) are not very convenient from the practical point of view: in order to calculate them, one has to diagonalize the reduced statistical operators, and this is rather difficult problem in the generic case, especially for infinite-dimensional Hilbert spaces (corresponding to the so called “continuous variable systems”), except for a few simple special cases. Therefore, many authors looked for other measures, which could be calculated more easily.

In our paper, we consider several families of the simplified measures. The first one is based on the notions of quantum purity $\mu = \text{Tr} \bar{\rho}^2$ or “linear entropy” $S_L = 1 - \mu$. Different measures containing these quantities were proposed in [1, 2, 3, 12, 13]. Measures based on the Hilbert–Schmidt distance between the given state and its “disentangled” counterpart were proposed in [12, 13]. On the other hand, even simpler (although non-universal) measures of entanglement of continuous variable quantum systems, expressed in terms of the cross-covariances of the quadrature components or the annihilation/creation operators, have been introduced recently in [14, 15]. These measures are discussed in Section 2.

One of numerous possible applications of the entanglement measures is a compact quantitative characterization of the evolution of coupled quantum mechanical systems. We began these studies in [14], where two harmonic oscillators with constant frequencies but with the most general time-dependent resonance couplings were considered. The aim of the present paper is to compare different measures in the case when the field modes in a cavity are entangled due to the motion of its boundary (the physical reason of entanglement in this case is the Doppler effect). This case is reduced to the models of two or many oscillators with time-dependent frequencies and a specific time-dependent coupling (of the “coordinate–momentum” type).

We consider two types of cavities, beginning with a three-dimensional cavity with accidental degeneracy of the spectrum (which happens, e.g., in cubical cavities), when only two modes can occur in resonance with an oscillating wall (Section 3). A one-dimensional (Fabry–Perot) cavity is considered in Section 4. In this case all modes are coupled due to the equidistance of the (unperturbed) spectrum of the field eigenfrequencies. It was discovered as far back as in [10] that the field evolution in three- and one-dimensional cavities is qualitatively different. For example, in the 3D case the number of photons in the resonance modes grows with time exponentially, whereas in the 1D case this growth is only linear. It was pointed out in [10] that the growth of the number of photons in the 1D cavity is slowed down due to a strong intermode interaction, which is equivalent in this case to entanglement. Now we are able to give a quantitative characterization of such an entanglement. The results of our study are discussed in Section 5.

2 Purity, distance and covariance measures of entanglement

2.1 Purity entanglement measure

By analogy with definition (1), the “linear entropy of entanglement” can be defined as

$$\mathcal{L} = S_L^1 + S_L^2 - S_L^0 = 1 + \text{Tr} \bar{\rho}^2 - \text{Tr} \bar{\rho}_1^2 - \text{Tr} \bar{\rho}_2^2. \quad (2)$$

Such a definition seems reasonable if the total system is in a pure quantum state. Then $\text{Tr} \bar{\rho}^2 = 1$ and $\text{Tr} \bar{\rho}_1^2 = \text{Tr} \bar{\rho}_2^2$, so that $\mathcal{L} = 2S_L^1 = 2S_L^2$. As a matter of fact, only this case was considered in the earlier studies [11, 12], where measures of entanglement were identified with the linear entropy of the state of a subsystem or with some equivalent quantities, such as the purity itself, the “participation ratio” $1/\text{Tr} \bar{\rho}_k^2$, or the “Renyi entropy” $S_R = -\ln (\text{Tr} \bar{\rho}_k^2)$.

However, if the state of the total system is mixed, then definition (2) leads to some unexpected consequences. Consider, for example, a generic Gaussian two-mode state described by means of the Wigner function (we assume $\hbar = 1$ throughout the paper)

$$W(q) = |\det (Q)|^{-1/2} \exp \left[-\frac{1}{2} (q - \langle q \rangle) Q^{-1} (q - \langle q \rangle) \right], \quad \int W(q)dq/(2\pi)^2 = 1, \quad (3)$$

where $q = (x_1, p_1, x_2, p_2)$, and the symmetrical $4 \times 4$ covariance matrix $Q$ consists of $2 \times 2$ blocks

$$Q = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{pmatrix}, \quad Q_{11} = \bar{Q}_{11}, \quad Q_{22} = \bar{Q}_{22}, \quad Q_{12} = \bar{Q}_{21} \quad (4)$$
(a tilde over matrices means matrix transposition). The symmetrical real covariances are defined as
\[ q_{\alpha \beta} \equiv \frac{1}{2} (q_{\alpha}q_{\beta} + q_{\beta}q_{\alpha}) \equiv q_{\alpha}q_{\beta}, \quad \overline{ab} \equiv \langle \hat{a}\hat{b} \rangle - \langle \hat{a} \rangle \langle \hat{b} \rangle \] (5)

(in other words, a straight line over the product of two observables means the ordered centralized average value, whereas a wide tilde means the symmetrized centralized average value). Then

\[ \mu \equiv \text{Tr} \hat{\rho}^2 = \int \text{d}q [W(q)]^2 \frac{1}{(2\pi)^2} = \text{det}(2Q) \] (6)

For factorized (disentangled) states, \( Q_{12} = 0 \), therefore \( \text{det} Q = \text{det} Q_{11} \text{det} Q_{22} \) and \( \mu = \mu_1 \mu_2 \), which results in the relations

\[ \mathcal{L}_{\text{fact}} = 1 + \mu_1 \mu_2 - \mu_1 - \mu_2 = (1 - \mu_1)(1 - \mu_2) \]

Consequently, for mixed states (\( \mu < 1 \)), one can meet the situation when \( \mathcal{L} > 0 \) in the absence of any entanglement, if \( \mu_1 \neq 1 \) and \( \mu_2 \neq 1 \).

It seems better to use the difference \( \mathcal{L} = \mathcal{L}_{\text{fact}} - \mu = \mu_1 \mu_2 \). But it tends to zero when \( \mu \to 0 \). For this reason, we introduce the normalized purity entanglement coefficient

\[ \tilde{\mathcal{L}} = 1 - \frac{\mu_1 \mu_2}{\mu} \] (7)

For the Gaussian states (3) it can be expressed as

\[ \tilde{\mathcal{L}} = 1 - \sqrt{\frac{\text{det} Q}{\text{det} Q_{11} \text{det} Q_{22}}} = 1 - \sqrt{\text{det} (E - Q_{12}^{-1} Q_{21} Q_{12}^{-1})} \] (8)

The second equality (where \( E \) stands for the unit matrix) is obtained with the aid of the known formula for the determinant of a block matrix

\[ \text{det} Q = \text{det} (Q_{11} - Q_{12}^{-1} Q_{21}) \text{det} Q_{22}. \]

In particular, for pure composite states (\( \mu = 1 \)) we have

\[ \tilde{\mathcal{L}} = 1 - \mu_1^2 = 1 - \mu_2^2 = \frac{1}{4} \mathcal{L}(4 - \mathcal{L}). \] (9)

A measure of entanglement between two coupled modes resembling (8) was introduced in [18] (where it was named “group correlation coefficient”):

\[ K^2 = 1 - \frac{\text{det} Q}{\text{det} Q_{11} \text{det} Q_{22}} = \tilde{\mathcal{L}} \left( 2 - \tilde{\mathcal{L}} \right). \] (10)

In principle, the measures (8) and (10) can be used for arbitrary (not only Gaussian) states, although sometimes they can give zero value even for truly entangled states (if the matrix of the second-order variances is factorized, but intermode correlations exist for higher-order moments). Also, instead of (7) one could use the following extension of formula (10) to arbitrary states:

\[ \tilde{K}^2 = 1 - \left( \frac{\mu_1 \mu_2}{\mu} \right)^2. \] (11)

### 2.2 Distance entanglement measure

Another possibility to characterize entanglement is to use the Hilbert-Schmidt distance between the given state and different “disentangled” states. It was considered, e.g., in [1] [2] [13] (analogous approach was developed in [10] to quantify the “degree of nonclassicality” of quantum states). In [13], the entanglement measure was defined as \( \text{Tr} (\hat{\rho} - \hat{\rho}_1 \otimes \hat{\rho}_2)^2 \). However, we prefer to normalize it by \( \text{Tr} \hat{\rho}^2 \), in order that the
entanglement measure would not go to zero for highly mixed states. Thus we shall consider the following quantity:

\[ Z = \frac{\left( \text{Tr}(\rho - \hat{\rho}_1 \otimes \hat{\rho}_2)^2 \right)}{\text{Tr}\rho^2} \equiv 1 + \frac{\mu_1 \mu_2}{\mu} - \frac{2}{\mu} \text{Tr}(\hat{\rho} \cdot [\hat{\rho}_1 \otimes \hat{\rho}_2]). \]  

(12)

For any states \( \hat{\rho} \) and \( \hat{R} \) one has (the normalization factor corresponds here to the two-mode case)

\[ \text{Tr}(\hat{\rho} \hat{R}) = \int W_{\rho}(q)W_{R}(q)dq/(2\pi)^2. \]  

(13)

For the Gaussian states the integrals can be calculated with the aid of the known formula

\[ \int \exp(-qAq + bq) dq = [\det(A/\pi)]^{-1/2} \exp\left(\frac{1}{4}bA^{-1}b\right). \]  

(14)

The inverse matrix \( Q^{-1} \) can be represented in the block form with the aid of the Frobenius formula

\[ \begin{vmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{vmatrix}^{-1} = \begin{vmatrix} Q_{11}^{-1} + Q_{12}^{-1}Q_{21}Q_{22}^{-1}Q_{11}^{-1} & -Q_{11}^{-1}Q_{12}Q_{22}^{-1} \\ -Q_{21}Q_{11}^{-1} & Q_{22}^{-1} \end{vmatrix}, \]  

(15)

where matrices \( Q_{11} \) and \( Q_{22} \) are the same as in \( \square \) (this is obvious from the physical point of view). Thus we arrive at the following expression for the \( Z \)-measure (it is equivalent, except for the normalizing factor \( \mu^{-1} \), to that given in \( \square \), but it is written in more simple explicit form):

\[ Z = 1 + \sqrt{\frac{\det Q}{\det Q_{11} \det Q_{22}}} - 2 \sqrt{\frac{\det(2Q)}{\det Q_z}} = Q + Q_d = \begin{vmatrix} 2Q_{11} & Q_{12} \\ Q_{21} & 2Q_{22} \end{vmatrix}. \]  

(16)

### 2.3 Covariance entanglement measures

Other measures of entanglement have been introduced recently in \[ \square, \square, \square, \square \]. They are expressed directly in terms of the cross-covariances of the quadrature components or the equivalent annihilation/creation operators as follows:

\[ \hat{Y} = \left( \frac{\text{Tr}(Q_{12}Q_{21})}{\text{Tr}Q_{11}\text{Tr}Q_{22}} \right)^{1/2} \]  

(17)

\[ = \left[ \frac{|a_1a_2|^2 + |a_1a_2|^2}{2 \left( a_1^*a_1^2 + 1/2 \right) \left( a_2^*a_2^2 + 1/2 \right)} \right]^{1/2} = \left[ \frac{(x_1x_2)^2 + (p_1p_2)^2 + (x_1p_2)^2 + (p_1x_2)^2}{4E_1E_2} \right]^{1/2}, \]  

(18)

\[ \hat{\gamma} = 2\sqrt{\frac{\text{Tr}(Q_{12}Q_{21})}{\text{Tr}Q}} = \frac{\sqrt{2 \left( |a_1a_2|^2 + |a_1a_2|^2 \right)}}{a_1^*a_1 + a_2^*a_2 + 1} = \sqrt{\frac{(x_1x_2)^2 + (p_1p_2)^2 + (x_1p_2)^2 + (p_1x_2)^2}{E_1 + E_2}}, \]  

(19)

where (we use properly normalized dimensionless quadrature variables)

\[ \hat{a}_k = (\hat{x}_k + i\hat{p}_k)/\sqrt{2}, \quad \hat{E}_k = a_k^*a_k + \frac{1}{2} = \frac{1}{2} (x_kx_k + p_kp_k), \quad k = 1, 2. \]  

(20)
Since the coefficients (18) and (19) are expressed in terms of traces of products of the off-diagonal blocks of the total covariance matrix $Q$, they are obviously invariant with respect to the rotations in the phase plane of each subsystem. (Another invariant quantity, namely the determinant of the off-diagonal blocks, det $Q_{12}$, plays an important role for the problem of separability of continuous variable systems [20]). It can be shown that $0 \leq \tilde{Y} \leq Y < 1$.

We would like to emphasize that the coefficients $Y$ and $\tilde{Y}$ are defined for any (not only Gaussian) quantum state. They are significantly simpler than other entanglement measures from the point of view of calculations (to calculate traces of matrices is much more easy than to calculate determinants, not speaking on calculating eigenvalues of density operators or matrices, which are necessary to obtain the entropic measures). A disadvantage of the coefficients $Y$ and $\tilde{Y}$ is that in some cases they are equal to zero even when the state is entangled, but the second-order moments of quadrature components are equal to zero. However, this does not happen for Gaussian and many other important quantum states.

2.4 Entropic measures for Gaussian states

In order to demonstrate how simple are expressions given in the preceding subsections, compared with the “standard” entropic measure (1), we give here the formula for the entropy of a generic Gaussian state. It is also determined by the covariance matrix, but in a more complicated way than the coefficients considered above. For an arbitrary $N$-mode Gaussian state the entropy was found in different but equivalent forms in [21, 24] and recently in [23]. The most simple expression is [24]

$$S_N = -\sum_{j=1}^{N} \left[ (\kappa_j + 1/2) \ln (\kappa_j + 1/2) - (\kappa_j - 1/2) \ln (\kappa_j - 1/2) \right], \quad (21)$$

where $\kappa_j \geq 1/2$ ($j = 1, \ldots, N$) are $N$ positive eigenvalues of matrix $X$, which is a “ratio” of the symmetric covariance matrix $Q$ and antisymmetric commutator matrix:

$$X = Q \Omega^{-1}, \quad Q_{jk} = \frac{1}{2} \langle \hat{q}_j \hat{q}_k + \hat{q}_k \hat{q}_j \rangle, \quad \Omega_{jk} = \langle \hat{q}_j \hat{q}_k - \hat{q}_k \hat{q}_j \rangle. \quad (22)$$

One can easily verify that if $\kappa$ is an eigenvalue of $X$, then $-\kappa$ is another eigenvalue. Also, it can be shown that all eigenvalues of $X$ are real. It is worth emphasizing that formula (21) is valid for arbitrary sets of operators with $c$-number commutators (canonical coordinates and momenta, “annihilation” and “creation” operators, kinetic momenta and relative coordinates for particles moving in homogeneous magnetic fields, etc.).

In the one-mode case, the eigenvalues of matrix $X$ are equal to $\pm \kappa$ (and $\kappa^2 = \kappa^2 E_2$), where

$$\kappa = h^{-1} \sqrt{\Delta}, \quad \Delta = \langle \hat{x}^2 \tilde{p}^2 - (\tilde{x} \hat{p})^2 \rangle \geq h^2/4 = \det Q. \quad (23)$$

(The last inequality is the Schrödinger–Robertson uncertainty relation [25]). In this case, different expressions equivalent to formula (21) were found in [21, 28, 29].

Calculating the characteristic polynomial of the $4 \times 4$ matrix $X$ in the two-mode case, one arrives at the biquadratic equation (for $h = 1$) [22]

$$\kappa^4 - D_2 \kappa^2 + D_0 = 0, \quad (24)$$

where coefficients $D_2$ and $D_0$ are nothing but quantum universal invariants, i.e., functions which are invariant with respect to arbitrary linear canonical (preserving commutation relations) transformations [30].

$$D_2 = \Delta_1 + \Delta_2 + 2 \langle \hat{x}_1 \hat{x}_2 \hat{p}_1 \hat{p}_2 - \hat{p}_1 \hat{x}_2 \hat{p}_2 \hat{x}_1 \rangle, \quad (25)$$

$$D_0^{(2)} = \det Q = \left( p_1^2 p_2^2 - \hat{p}_1 \hat{p}_2 \right)^2 \left[ \hat{x}_1^2 \hat{x}_2^2 - \hat{x}_1 \hat{x}_2 \hat{p}_1 \hat{p}_2 \right]^2 + \hat{x}_1^2 \hat{x}_2^2 \hat{p}_1^2 \hat{p}_2^2 \left( \hat{x}_1 \hat{x}_2 \hat{p}_1 \hat{p}_2 \right)^2 - \hat{x}_1^2 \hat{x}_2^2 \hat{p}_1^2 \hat{p}_2^2 \left( \hat{x}_1 \hat{x}_2 \hat{p}_1 \hat{p}_2 \right)^2 + 2 \hat{x}_1 \hat{x}_2 \hat{p}_1^2 \hat{p}_2^2 \left[ \hat{x}_1 \hat{x}_2 \hat{p}_1 \hat{p}_2 \right] + 2 \hat{p}_1 \hat{p}_2 \left[ \hat{x}_1 \hat{x}_2 \hat{p}_1 \hat{p}_2 \right] - 2 \hat{x}_1 \hat{x}_2 \hat{p}_1 \hat{p}_2 \left[ \hat{x}_1 \hat{x}_2 \hat{p}_1 \hat{p}_2 \right]. \quad (26)$$
The symbol $\Delta_k$, obviously, means the combination defined by (23) and related to the $k$th mode.

Positive solutions of Eq. (24) read

$$\kappa_{1,2} = \frac{1}{2} \left[ \sqrt{D_2 + 2\sqrt{D_0}} \pm \sqrt{D_2 - 2\sqrt{D_0}} \right].$$

(27)

The reality of $\kappa_{1,2}$ is ensured by the inequalities

$$D_2 \geq 2\sqrt{D_0} \geq \frac{\hbar^2}{2},$$

(28)

which can be considered as generalized uncertainty relations for two-mode systems (for systematic studies of such generalizations see, e.g., [27, 31]).

Formulae (21), (23) and (27) permit us to express the entropic index of correlation (1) analytically in terms of the covariances of quadrature components for arbitrary Gaussian states. However, the corresponding expression is very cumbersome, and it is much more complicated than any other entanglement measure discussed in the preceding subsections. In the next sections we compare the behaviour of different entanglement measures for various concrete physical models.

3 A three-dimensional cavity with a vibrating wall and two resonantly coupled modes

Classical and quantum phenomena in cavities with moving boundaries attracted attention of many researchers for a long time (see review [32]). Especially popular this topic became in the last decade, being known now under the names nonstationary Casimir effect [33], dynamical Casimir effect [34], or mirror (motion) induced radiation [35, 36]. One of several theoretical results obtained in the last years was the prediction of the exponential growth of the energy of the field under the resonance conditions, when the wall performs vibrations at the frequency which is a multiple of the unperturbed field eigenfrequency [16, 36, 37].

A unified description of the field inside an ideal cavity with moving boundaries can be achieved in the frameworks of the Hamiltonian approach proposed by Law [37] and developed in [38] (for other references see [32], and for the most recent publications see [39, 40, 41]). Consider a scalar massless field $\Phi(r, t)$, satisfying the wave equation $\Phi_{tt} = \nabla^2 \Phi$ inside the cavity and the Dirichlet boundary condition $\Phi = 0$ on the boundary (we assume $c = \hbar = 1$). We assume that we know the complete orthonormalized set of eigenfunctions (and eigenfrequencies) of the Laplace equation $\nabla^2 f_\alpha(r) + \omega_\alpha^2 f_\alpha(r) = 0$ in the case of stationary cavity. Now suppose that a part of the boundary is a plane surface moving according to a prescribed law of motion $L(t)$ (for the most recent study of the case when $L(t)$ is a dynamical variable due to the back reaction of the field see [42]). Expanding the field $\Phi(r, t)$ over “instantaneous” eigenfunctions $f_\alpha(r, L(t))$,

$$\Phi(r, t) = \sum_\alpha q_\alpha(t)f_\alpha(r, L(t)),$$

(29)

we satisfy automatically the boundary conditions. Then the dynamics of the field is described completely by the dynamics of the generalized coordinates $q_\alpha(t)$, which, in turn, can be derived from the time-dependent Hamiltonian [38]

$$H(t) = \frac{1}{2} \sum_\alpha \left[ p_\alpha^2 + \omega_\alpha^2(L(t))q_\alpha^2 \right] + \frac{\dot{L}(t)}{L(t)} \sum_{\alpha \neq \beta} p_\alpha m_{\alpha\beta} q_\beta$$

(30)

with antisymmetrical time-independent coefficients

$$m_{\alpha\beta} = -m_{\beta\alpha} = L \int dV \frac{\partial f_\alpha(r, L)}{\partial L} f_\beta(r, L).$$

(31)

For example, in the case of a rectangular three-dimensional cavity with dimensions $L_x, L_y, L_z$, the eigenmodes are well known products of sine functions like $\sin(\pi k_x x/L_x)$ (or sine and cosine functions in the case of


electromagnetic field), labeled by three natural numbers \(k_x, k_y, k_z\), whereas unperturbed eigenfrequencies are given by the formula

\[
\omega_{k_x, k_y, k_z} = \pi \sqrt{\left(\frac{k_x}{L_x}\right)^2 + \left(\frac{k_y}{L_y}\right)^2 + \left(\frac{k_z}{L_z}\right)^2}.
\]  

If one surface of the parallelepiped, perpendicular to the \(x\)-axis, moves in the \(x\)-direction (so that the \(L_x\)-dimension of the cavity is a function of time), then \([13]\)

\[
m_{kj} = (-1)^{k_x+j_x} \frac{2k_x j_x}{j_x^2 - k_x^2} \delta_{k_x, j_x} \delta_{k_z, j_z}.
\]

(In the case of electromagnetic field, one should take into account polarizations of the modes, i.e., that \(f_\alpha\) and \(f_\beta\) in Eq. \([11]\) are vector functions, whose directions are perpendicular, respectively, to the vectors \((k_x, k_y, k_z)\) and \((j_x, j_y, j_z)\). But one can always choose two modes with coinciding polarizations, directed along the perpendicular to the plane formed by these two vectors. Then all formulae are the same as in the scalar case.)

We are interested in the case when one of the cavity’s walls performs small oscillations with the frequency \(\Omega\) close to the double frequency of some unperturbed mode \(\omega_1^{(0)} = 1\) (i.e., we normalize all frequencies by \(\omega_1^{(0)}\)), so that the time-dependent frequency \(\omega_1(t)\) reads

\[
\omega_1(t) = 1 + 2\epsilon \cos(2\pi t\tau), \quad \tau = 1 + \delta,
\]

where we assume that \(|\delta| \ll 1\) and \(|\epsilon| \ll 1\). Also we suppose that the unperturbed field frequency spectrum includes the frequency \(\omega_3^{(0)} = 3 + \Delta\) with \(|\Delta| \ll 1\), but it does not contain frequencies close to \(5\omega_1^{(0)}\).

A possibility of such a situation was pointed out in \([13]\). An example is a cubic cavity with the pair of \(k_x, k_y, k_z\) labeled by three natural numbers \(k\), \(k\), \(k\), \(m\) and \(n\) (for the quadrature components of the field, whereas the letter \(x\) without indices will mean the usual space coordinate inside the cavity):

\[
H_{13} = \frac{1}{2} (\dot{x}^2 + \dot{p}^2) + \frac{1}{2} [1 + 4\epsilon \cos(2\pi t)] \dot{x}^2 + \frac{1}{2} [9 + 6\Delta + \tilde{\epsilon} \cos(2\pi t)] \dot{x}^3 + 3\mu \epsilon \sin(2\pi t) (p_1 x_3 - p_3 x_1).
\]

The constant parameter \(\mu\) is proportional to the coefficient \(m_{12}\) in \([30]\). For the rectangular cavity, \(\mu = j_x/(12 k_x)\) if the modes \(\{k_x, m, n\}\) and \(\{j_x, m, n\}\) are in resonance. Writing \([33]\) we have neglected the second order terms with respect to \(\epsilon\) and \(\Delta\). Parameter \(\tilde{\epsilon}\) has the same order of magnitude as \(\epsilon\), but it does not affect the solution in the zeroth order approximation \([14]\).

Hamiltonian \([33]\) results in the following differential equations for the generalized coordinates \(x_1\) and \(x_3\) (we neglect corrections of the second order):

\[
\dot{x}_1 = -\left[1 + 4\epsilon \cos(2\pi t)\right] x_1 + 24\mu \epsilon [\cos(2\pi t) x_3 + \sin(2\pi t) \dot{x}_3],
\]

\[
\dot{x}_3 = -\left[9 + 6\Delta + \tilde{\epsilon} \cos(2\pi t)\right] x_3 - 24\mu \epsilon [\cos(2\pi t) x_1 + \sin(2\pi t) \dot{x}_1].
\]

These equations have been solved, using the method of slowly varying amplitudes, in \([14]\). We consider here two special cases.

### 3.1 Exact (symmetric) resonance

In the case of exact resonance, \(\delta = \Delta = 0\), the solutions of Eqs. \([36]\) and \([37]\) read

\[
x_1(t) = x_1(0) \left[ C_1^+ \cos(\rho t) + C_1^- \sin(\rho t) \right] - p_1(0) \left[ S_1^+ \cos(\rho t) + C_1^- \sin(\rho t) \right]
\]

\[
+ 8\mu \left. \frac{\sin(\rho t)}{\rho} \right| \left[ 3S_1^- x_3(0) + C_1^- p_3(0) \right],
\]

\[
(38)
\]
Following relations hold: 

\[ x_3(t) = x_3(0) \left[ C_3^+ \cos(\rho t) - S_3^+ \frac{\sin(\rho t)}{\rho} \right] + \frac{1}{3} p_3(0) \left[ S_3^+ \cos(\rho t) - C_3^+ \frac{\sin(\rho t)}{\rho} \right] - 8\mu \frac{\sin(\rho t)}{\rho} \left[ S_3^+ x_1(0) - C_3^+ p_1(0) \right], \]  

\[ p_1(t) = -x_1(0) \left[ S_1^+ \cos(\rho t) + C_1^+ \frac{\sin(\rho t)}{\rho} \right] + p_1(0) \left[ C_1^+ \cos(\rho t) + S_1^+ \frac{\sin(\rho t)}{\rho} \right] - 8\mu \frac{\sin(\rho t)}{\rho} \left[ 3C_1^+ x_3(0) + S_1^+ p_3(0) \right], \]

\[ p_3(t) = 3x_3(0) \left[ S_3^- \cos(\rho t) - C_3^- \frac{\sin(\rho t)}{\rho} \right] + p_3(0) \left[ C_3^- \cos(\rho t) - S_3^- \frac{\sin(\rho t)}{\rho} \right] - 24\mu \frac{\sin(\rho t)}{\rho} \left[ C_3^- x_1(0) - S_3^- p_1(0) \right], \]  

where 

\[ C_k^\pm(\tau; t) = \cosh \tau \cos(k\omega t) \pm \sinh \tau \sin(k\omega t), \quad S_k^\pm(\tau; t) = \sinh \tau \cos(k\omega t) \pm \cosh \tau \sin(k\omega t), \]  

\[ \tau = \frac{1}{2} \varepsilon t, \quad \rho = \sqrt{2\nu - 1}, \quad \nu \equiv 96\mu^2. \]  

The arguments \( t \) ("fast time") and \( \tau \) ("slow time") of the functions \( C_k^\pm(\tau; t) \) and \( S_k^\pm(\tau; t) \) can be considered as independent variables. Then the following relations hold: 

\[ \frac{\partial C_k^\pm}{\partial t} = \pm kS_k^\mp, \quad \frac{\partial S_k^\pm}{\partial t} = \pm kC_k^\mp. \]  

For the modes \( \{111\} \) and \( \{511\} \) of the cubical cavity or \( \{110\} \) and \( \{510\} \) of the rectangular cavity with \( L_x = \sqrt{2} L_y \) we have \( \nu = 50/3 \). Due to this explicit example, we assume that parameter \( \nu \) is large: \( \nu \gg 1 \).

Symbols \( x_k \) and \( p_k \) in equations (38)-(41) can be considered both as classical variables and quantum operators in the Heisenberg picture, due to the linearity of the problem (or due to the quadratic nature of Hamiltonian \( (30) \)). Using equations (38)-(41), one can calculate mean values of squares and products of canonical variables (operators) at any moment of time, provided such mean values were known at the initial moment \( t = 0 \). We confine ourselves to the case when initially the field modes were in thermal states with the mean photon numbers \( (\theta_1 - 1)/2 \) and \( (\theta_3 - 1)/2 \), where \( \theta_k = \coth(k\beta_k/2), \beta_k \) being inverse absolute temperature in dimensionless units. In the natural case of equal initial temperatures of the modes, the following relations hold: 

\[ \theta_{31} = \frac{\theta_3}{\theta_1} = \frac{\theta_1^{-1} + 3}{3\theta_1^{2} + 1}, \quad 1 \geq \theta_{31} \geq \frac{1}{3}. \]  

The normalized mean energies in each mode, \( \varepsilon_k = \langle p_k^2 + \omega_k^2 x_k^2 \rangle / (2\omega_k) \) (namely these quantities are used in the definitions of the covariance entanglement coefficients \( \{13\} \) and \( \{14\} \)), depend on time as follows \( \{44\} \), 

\[ \varepsilon_1 = \frac{\theta_1}{2} \cosh(2\tau) \left[ \frac{\sin^2(\rho t)}{\rho^2} (1 + 2\nu \theta_{31}) + \cos^2(\rho t) \right] + \sinh(2\tau) \frac{\sin(2\rho t)}{\rho}, \]

\[ \varepsilon_3 = \frac{\theta_1}{2} \cosh(2\tau) \left[ \frac{\sin^2(\rho t)}{\rho^2} (1 + 2\nu \theta_{13}) + \cos^2(\rho t) \right] - \sinh(2\tau) \frac{\sin(2\rho t)}{\rho}. \]  

Calculating the covariance entanglement coefficient, one should use, instead of variables \( x_k \) and \( p_k \), the normalized variables \( \hat{x}_k = \sqrt{\omega_k} x_k \) and \( \hat{p}_k = p_k / \sqrt{\omega_k} \) (in our case \( \omega_k \equiv k \)): see equation \( \{20\} \). After some algebra we have obtained the following expressions: 

\[ \tilde{Y} = \sqrt{\frac{F}{4\varepsilon_1\varepsilon_3}}, \quad \tilde{Y} = \frac{\sqrt{F}}{\varepsilon_1 + \varepsilon_3}, \]  

8
\[
F = \frac{\nu}{2\nu - 1} \sin^2(\rho \tau) \left\{ \cosh(4\tau) \left[ \cos^2(\rho \tau) (\theta_1 - \theta_3)^2 + \frac{\sin^2(\rho \tau)}{\rho^2} (\theta_1 + \theta_3)^2 \right] + \frac{\sin(2\rho \tau)}{\rho} \sinh(4\tau) (\theta_1^2 - \theta_3^2) \right\}.
\] (49)

The determinants of the covariance matrices for each mode have been calculated in \[44\]. For the first mode,
\[
\det Q_{11} = \frac{1}{4} g_1^2 g_1^2, \quad g_1^2 = \cos^2(\rho \tau) + \sin^2(2\rho \tau \frac{2\nu \theta_3 - 1}{2(2\nu - 1)} + \sin^4(\rho \tau) \left( \frac{2\nu \theta_3 + 1}{2\nu - 1} \right)^2.
\] (50)

For another excited mode one should interchange indices 1 and 3 in (50). Since the evolution of the total system is unitary in the case discussed, the total determinant does not depend on time: \[\det \mathcal{Q} = \theta_1^2 \theta_3^2 / 16\]. Therefore the purity entanglement coefficient \[\tilde{\mathcal{L}}\] has the form
\[
\tilde{\mathcal{L}} = 1 - (g_1 g_3)^{-1}.
\] (51)

Eqs. (21) and (23) lead to the following explicit formula for the entropic entanglement measure \[I_c\]:
\[
I_c = \frac{1}{2} \sum_{i=1,3} \left[ (\theta_i g_i + 1) \ln (\theta_i g_i + 1) - (\theta_i g_i - 1) \ln (\theta_i g_i - 1) - (\theta_i + 1) \ln (\theta_i + 1) - (\theta_i - 1) \ln (\theta_i - 1) \right].
\] (52)

We see that despite the exponential (although non-monotonous in the high-temperature case \[\theta_k \gg 1\]) growth of energy of each mode, all entanglement coefficients exhibit strong (quasi)periodic oscillations as functions of the “slow time” \[\tau\], going to zero when \[\rho \tau = n\pi\].

In the simplest case of the initial vacuum states of each mode (\[\theta_1 = \theta_3 = 1\]) we have
\[
F = \frac{4\nu}{(2\nu - 1)^2} \sin^4(\rho \tau) \cosh(4\tau),
\] (53)
\[
g_1^2 = g_3^2 = g_3^2 = 1 + \frac{8\nu}{(2\nu - 1)^2} \sin^4(\rho \tau),
\] (54)
\[
\tilde{\mathcal{L}} = \frac{8\nu \sin^4(\rho \tau)}{(2\nu - 1)^2 + 8\nu \sin^4(\rho \tau)} \approx \frac{2}{\nu} \sin^4(\rho \tau),
\] (55)
\[
I_c = (g_0 + 1) \ln (g_0 + 1) - (g_0 - 1) \ln (g_0 - 1) - 2 \ln 2 \approx \frac{\sin^4(\rho \tau)}{\nu} \ln \left( \frac{2e\nu}{\sin^4(\rho \tau)} \right).
\] (56)

The approximate equalities in (53) and (54) hold for \[\nu \gg 1\]. Under this condition, \[\mathcal{E}_1 \approx \mathcal{E}_3 \approx \frac{1}{2} \cosh(2\tau)\], so that for \[\tau > 1\] we obtain
\[
\mathcal{Y} \approx \sqrt{\frac{2}{\nu}} \sin^2(\rho \tau) \approx \sqrt{\tilde{\mathcal{L}}}. \quad \text{(The last approximate equality in (57) holds for } \tau > 1\).
\]

The evolution of functions \[\tilde{\mathcal{L}}(\tau)\] and \[\mathcal{Y}^2(\tau)\] for the initial vacuum case is shown in Figure [8].

For high-temperature initial states (\[\theta_{1,3} \gg 1\]), the entanglement coefficients do not depend on the parameter \[\nu\] (if \[\nu \gg 1\]) for almost all instants of time, more precisely, under the condition \[|\cos(\rho \tau)| \gg \rho^{-1} \sim \nu^{-1/2}\]:
\[
\tilde{\mathcal{L}} \approx \frac{\sin^2(2\rho \tau) (\theta_{31} + \theta_{13} - 2)}{4 + \sin^2(2\rho \tau) (\theta_{31} + \theta_{13} - 2)} \approx \mathcal{Y}^2,
\] (57)
\[
I_c \approx \ln (g_1 g_3) \approx \ln \left[ 1 + \frac{1}{4} \sin^2(2\rho \tau) (\theta_{31} + \theta_{13} - 2) \right].
\] (58)

(The last approximate equality in (57) holds for \[\tau > 1\]. The simple formula for \[I_c\] is obtained in the limit case \[\theta_{1,3} \rightarrow \infty\]; there are some corrections of the order of \[\theta_{1,3}^{-1}\] for finite initial mean numbers of photons.)

For the maximal possible value of the coefficient \[\theta_{13} = 3\] (in the case of true initial thermal equilibrium), the maximum values of the expressions (57) and (58) (which are achieved when \[\sin^2(2\rho \tau) = 1\]) are equal
to \( \hat{L}_{\text{max}} = 1/4 \) and \( I_c^{\text{(max)}} = \ln(4/3) \approx 1/3 \). Note that in the limit high-temperature case, the purity entanglement coefficient coincides identically with one of possible forms of the “compact entropy” (another compact parameter, \( \tanh(I_c) \), was introduced in \[14\])

\[
J_c = 1 - \exp(-I_c) \, .
\]  

(59)

Figure 2 shows the evolution of entropic entanglement measure \( L_c(\tau) \) for vacuum and high-temperature initial states. Note that one has \( \theta_1 \approx 140 \), if \( L_0 = 1 \, \text{cm} \) and \( T = 300 \, \text{K} \). For this value of \( \theta_1 \), the plot of the compact entropy \( J_c(\tau) \) becomes indistinguishable from the plot of the purity entanglement coefficient \( \tilde{L}(\tau) \).

The functions \( \tilde{L}(\tau) \) and \( Y^2(\tau) \) are compared in Figure 3. We see that two functions are very close in some intervals, although their maxima are different (because the value \( \rho \approx 5.7 \) is not very large for the chosen parameter \( \nu = 50/3 \)).

In the high-temperature case, intermediate nonzero minima of the entanglement coefficients (besides exact zero minima at the instants \( \tau_\nu = n\pi/\rho \)) are observed at the moments of "slow time" when the modes approximately exchange their purities \[14\]. The positions of these additional minima for \( \tilde{L} \) and \( I_c \) are determined by the condition \( \cos(\rho \tau) = 0 \), so that

\[
\tilde{L}_{\text{min}} = \frac{2\nu (\theta_{31} + \theta_{13} + 2)}{4\nu^2 + 1 + 2\nu (\theta_{31} + \theta_{13})}, \quad I_c^{(\text{min})} = \ln \left[ 1 + \frac{2\nu (\theta_{31} + \theta_{13} + 2)}{(2\nu - 1)^2} \right].
\]  

(60)

For \( \nu \gg 1 \) we have

\[
\tilde{L}_{\text{min}} \approx I_c^{(\text{min})} \approx \frac{\theta_{31} + \theta_{13} + 2}{2\nu}.
\]  

(61)

On the other hand, the intermediate minima of \( Y \) are much smaller. Indeed, the minimum of the expression inside figure brackets in Eq. \[13\] is achieved for (neglecting corrections of the order of \( \rho^{-3} \))

\[
\tan(2\rho \tau) = \frac{2}{\rho} \tanh(4\tau) \frac{\theta_1 + \theta_3}{\theta_1 - \theta_3}.
\]

At this moment of time we obtain

\[
F \approx \frac{(\theta_1 + \theta_3)^2}{4\nu \cosh(4\tau)}, \quad 4\mathcal{E}_1 \mathcal{E}_3 \approx \theta_1 \theta_3 \cosh^2(2\tau),
\]

so that for \( \tau > 1 \),

\[
Y \approx e^{-4\tau} \sqrt{\frac{2}{\nu} (\theta_{31} + \theta_{13} + 2)} \equiv Y_* \approx 2e^{-4\tau} \sqrt{\tilde{L}_{\text{min}}},
\]

and it is clear that the intermediate minimum of \( Y \) does not exceed the value \( Y_* \).

Therefore, we arrive at rather paradoxical situation, especially for realistic values of parameters \( \nu \) and \( \theta_{1,3} \). According to Figure 3, the intermediate minimum value of \( L \)-coefficient in the high-temperature case is only twice less than the maximal value. Moreover, this high-temperature intermediate minimum value is bigger than the maximum value in the vacuum case (see Figure 1). Thus, the \( L \)-coefficient tells us that for \( \cos(\rho \tau) = 0 \), two modes are “more entangled” in the high-temperature case than in the case of initial vacuum state (or at least have the same order of entanglement, according to the \( I_c \)-coefficient in Figure 3), whereas the covariance entanglement coefficient \( Y \) shows that two modes become practically disentangled at this instant of time.

The resolution of this “paradox” is as follows. According to Eqs. \[18\], \[18\] and \[19\], the function \( F \) gives the upper limit for squares of any elements of the “off-diagonal” block \( Q_{12} \) of the covariance matrix \( Q \) \( \[6\] \), whereas functions \( \mathcal{E}_k \) give the bounds for the elements of “diagonal” blocks \( Q_{kk} \). This happens because \( F \) and \( \mathcal{E}_k \) are based on traces of the covariance submatrices. Therefore, if \( Y \to 0 \), this means that all elements of matrix \( Q_{12} \) responsible for the intermode correlations (at least for the Gaussian states considered in this section) become negligible in comparison with the variances \( \mathcal{E}_k \) and \( \mathcal{E}_k \) of the quadrature components. From the physical point of view, it is equivalent to disappearance of correlations between the two subsystems, i.e., their disentanglement.
On the other hand, the coefficients \( \hat{L} \) and \( I_x \) are based on determinants of the covariance submatrices. But it is well known that the determinant of a matrix can be quite small even if all elements of the matrix are big, and this is the reason of the qualitative difference in the behaviour of the “covariance” and “entropic” entanglement coefficients. This is clearly seen from the last expression in Eq. (30), which shows that the value of the purity entanglement coefficient \( \hat{L} \) depends on the matrix \( R = Q_{12} Q_{22}^{-1} Q_{21} Q_{11}^{-1} \). Using easily verified formula \( \det(E + \alpha) \approx \text{Tr} \alpha \), which holds provided all elements of matrix \( \alpha \) are small with respect to unity, we can simplify formula (30) in the case of small entanglement as follows:

\[
\hat{L} \approx \frac{1}{2} \text{Tr} (Q_{12} Q_{22}^{-1} Q_{21} Q_{11}^{-1}).
\]  

But each matrix \( Q_{kk}^{-1} \) \( (k = 1, 3) \) contains the denominator \( \det Q_{kk} \), which can be much less than any element of matrix \( Q_{kk} \). If this happens, then the inequality \( \text{Tr} (Q_{12} Q_{22}^{-1} Q_{21} Q_{11}^{-1}) \gg \text{Tr} (Q_{12} Q_{21}) / (\text{Tr} Q_{11} \text{Tr} Q_{22}) \) becomes quite possible. Just such a situation takes place in the example considered. Although diagonal elements \( x_k x_k \) and \( p_k p_k \) of matrices \( Q_{kk} \) grow exponentially with time, these matrices have also exponentially growing off-diagonal covariance elements \( x_k p_k \) (this means that each mode occurs in \( \text{highly-correlated quantum state} \) \( |2\rangle \) with quadrature correlation coefficient \( r = \frac{x_k p_k}{(x_k x_k p_k p_k)^{1/2}} \) approaching the unit value), so that \( \det Q_{kk} \) does not grow unlimitedly with time, exhibiting only relatively small oscillations. For this reason, elements of matrices \( Q_{kk}^{-1} \) have the same order of magnitude \( (\sim \exp(2\tau)) \) as elements of matrices \( Q_{kk} \) themselves. On the other hand, elements of matrix \( Q_{12} \) have an order of \( \exp(-2\tau) \) at the moments of intermediate minima. Therefore, the exponential time dependences are canceled in the measures based on determinants, resulting in the inequalities \( \mathcal{J}_e, \hat{L} \gg \mathcal{Y} \) for \( \cos(\rho \tau) \approx 0 \).

This example permits us to make a conjecture that the covariance entanglement coefficient \( \mathcal{Y} \) is not only simpler from the point of view of calculations, but it could be preferable from the physical point of view, because it is more sensitive to entanglement than entropic and purity measures. Other arguments in favour of \( \mathcal{Y} \) can be found in [3].

### 3.2 Asymmetric resonance

An interesting feature of the Hamiltonian (33) discovered in [10] is a possibility to compensate one detuning (e.g., \( \delta \)) at the expense of another. In particular, an exponential growth of the energies of both modes can be obtained under the conditions of “asymmetric resonance”

\[
\delta = \epsilon, \quad 3\delta - \Delta = \epsilon \nu / 2.
\]  

In this case the quadrature components depend on time as follows:

\[
x_1(t) = x_1(0) \left[ \left( 1 - \frac{2}{\nu} \right) C_1^-(2R\tau; t) + \frac{2}{\nu} \cos \phi_1 \right] - p_1(0) \left[ \left( 1 - \frac{2}{\nu} \right) S_1^-(2R\tau; t) - \frac{2}{\nu} \sin \phi_1 \right] \\
+ \frac{x_3(0)}{4\mu} \left[ C_1^- (2R\tau; t) - \cos \phi_1 \right] - \frac{p_3(0)}{12\mu} \left[ S_1^- (2R\tau; t) + \sin \phi_1 \right],
\]

\[
x_3(t) = x_3(0) \left[ \left( 1 - \frac{2}{\nu} \right) \cos \phi_3 + \frac{2}{\nu} C_3^- (2R\tau; t) \right] + \frac{1}{3} p_3(0) \left[ \left( 1 - \frac{2}{\nu} \right) \sin \phi_3 - \frac{2}{\nu} S_3^- (2R\tau; t) \right] \\
+ \frac{x_1(0)}{12\mu} \left[ C_3^- (2R\tau; t) - \cos \phi_3 \right] - \frac{p_1(0)}{12\mu} \left[ S_3^- (2R\tau; t) + \sin \phi_3 \right],
\]

\[
p_1(t) = -x_1(0) \left[ \left( 1 - \frac{2}{\nu} \right) S_1^+(2R\tau; t) + \frac{2}{\nu} \sin \phi_1 \right] + p_1(0) \left[ \left( 1 - \frac{2}{\nu} \right) C_1^+(2R\tau; t) + \frac{2}{\nu} \cos \phi_1 \right] \\
- \frac{x_3(0)}{4\mu} \left[ S_1^+ (2R\tau; t) - \sin \phi_1 \right] + \frac{p_3(0)}{12\mu} \left[ C_1^+ (2R\tau; t) - \cos \phi_1 \right],
\]
subsystems are essentially different. In the case of the strict resonance discussed in the preceding subsection, the rates of increase of the energies of each mode are almost twice bigger than they were in the case of the main terms (neglecting corrections of the order of $\delta \sim \epsilon$ in the amplitude coefficients).

The (normalized) mean energies of each mode depend on time as follows:

$$E_1 = \frac{\theta_1}{2} \left[ \left( 1 - \frac{4}{\nu} \right) \cosh(4R\tau) + \frac{4}{\nu} \psi(\tau) \right] + \frac{\theta_3}{\nu} \left[ \cosh(4R\tau) + 1 - 2\psi(\tau) \right],$$

$$E_3 = \frac{\theta_3}{2} \left[ 1 - \frac{4}{\nu} + \frac{4}{\nu} \psi(\tau) \right] + \frac{\theta_1}{\nu} \left[ \cosh(4R\tau) + 1 - 2\psi(\tau) \right],$$

where

$$\psi(\tau) \equiv \cosh(2R\tau) \cos(2J\tau).$$

The energy of the third mode is significantly less than the energy of the first mode, if $\nu \gg 1$. For this reason this regime of excitation was named “asymmetrical”. For $\tau > 1$, $E_3/E_1 \approx 6/\nu$. Note, however, that for the cubical cavity with $\nu = 50/3$, the energy of the third mode is only three times less than that of the first one. It is important, nonetheless, that the rates of increase of the energies of each mode are almost twice bigger than they were in the case of the strict resonance discussed in the preceding subsection.

The covariance entanglement coefficients can be written again in the form (48), but with $E_{1,3}$ given by (68) and (69). The function $F$ in the asymmetric case reads (neglecting corrections of the order of $\nu^{-2}$ with respect to the main terms)

$$F = 2\nu^{-1} \left\{ \theta_1^2 \left[ \cosh^2(4R\tau) + \sinh^2(2R\tau) \right] \cosh(6R\tau) \cos \phi_0 \right. \\
+ 2\nu^{-1} \left[ \cos \phi_0 \left\{ \cosh(2R\tau) + 3 \cosh(6R\tau) - 2 \cosh^2(2R\tau) \cos \phi_0 \right. \right. \\
\left. - 2 \cosh(4R\tau) - 2 \sinh^2(4R\tau) \right] \right\} \\
+ \theta_3^2 \left[ \cosh^2(2R\tau) \cos \phi_0 \right. \\
+ 2\nu^{-1} \left[ \cos \phi_0 \left\{ \cosh(6R\tau) + 3 \cosh(2R\tau) \right. \right. \\
\left. - 2 \cosh^2(2R\tau) \cos \phi_0 \right. \right. \\
\left. - 2 \cosh(4R\tau) \right] \right\} \\
+ \left. 2\theta_1 \theta_3 \left[ \cosh(4R\tau) \left\{ \cosh(2R\tau) \cos \phi_0 \right. \right. \\
\left. - 1 \right] \right\} \\
\left. + 2\nu^{-1} \left[ -2 \cos \phi_0 \left\{ \cosh(2R\tau) + \cosh(6R\tau) \right. \right. \\
\left. - 2 \cosh^2(2R\tau) \cos \phi_0 + 4 \cosh^4(2R\tau) - 2 \right] \right),$$

where $\phi_0 = -2J\tau$.

If $\tau \to \infty$, then (for $\nu \gg 1$)

$$F \approx \frac{\theta_1^2}{2\nu} \phi^{8R\tau}, \quad E_1 \approx \frac{\theta_1}{4} \phi^{4R\tau}, \quad E_3 \approx \frac{\theta_1}{2\nu} \phi^{4R\tau},$$

so that $Y \to 1$, whereas $\tilde{Y} \to \sqrt{8/\nu}$. Consequently, the coefficient $Y$ is preferable when the energies of subsystems are essentially different.

The purity and entropic entanglement coefficients are given by Eqs. (51) and (52), with

$$g_1^2(\tau) = 1 + \frac{8}{\nu} \left[ (1 - \theta_{31}) \psi(\tau) - 1 + \theta_{31} \cosh^2(2R\tau) \right]$$

(71)
and $g_3$ obtained from (71) by means of the replacement $1 \leftrightarrow 3$. We see a significant difference from the strict resonance case: now functions $g_{1,3}(\tau)$ increase exponentially with time for $\tau \gg 1$. Asymptotically, each mode appears in a highly mixed quantum state, with $\det Q_{11} = \det Q_{33} = \theta_j \theta_3 \exp(4R\tau)/(2\nu)$. The purity entanglement coefficient (53) tends asymptotically to the unit value independently of the initial temperature (or coefficients $\theta_k$):

$$\widetilde{\mathcal{E}} \approx 1 - \frac{\nu}{2} \exp(-4R\tau), \quad \tau \gg 1.$$ 

For $\tau \gg 1$ the entropic entanglement coefficient grows unlimitedly: $I_c \sim \ln(g_1g_3) \sim 4R\tau$. Therefore in Fig. 4 we compare the compact parameter $J_c(\tau)$ (53) with the functions $\Upsilon(\tau)$ (51), $\widetilde{\mathcal{E}}$ (51), and $|\widetilde{\mathcal{E}}(\tau)|^{1/2}$ for the initial vacuum state. Since all formulae in the asymmetric case are obtained neglecting terms of the order of $\nu^{-2}$, we use the value $\nu = 100$ in the illustrations. The difference between the asymptotical values of the functions for $\tau \gg 1$ and the correct value 1 shows the accuracy of approximation (about 2%). The dependences of the entanglement covariance and purity coefficients $\Upsilon$ and $\widetilde{\mathcal{E}}$ on the “slow time” $\tau$ for the initial vacuum and high-temperature state are shown in Fig. 4. Remember that in the high-temperature case the coefficient $\widetilde{\mathcal{E}}$ tends to the compact entropic coefficient $J_c$.

## 4 Fabry-Perot cavity with an oscillating boundary

The problem of the scalar massless field in a 1D cavity formed by two infinite ideal plates whose positions are given by $x_{left} \equiv 0$ and

$$x_{right} \equiv L(t) = L_0 (1 + \varepsilon \sin[p\omega_1 t]), \quad |\varepsilon| \ll 1, \quad \omega_1 = \pi c/L_0, \quad p = 1, 2, \ldots$$

was solved in [13]. The only component of the operator vector potential of the electromagnetic field $\hat{A}(x,t)$ in the Heisenberg representation can be written as

$$\hat{A}(x,t) = \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} \left[ \hat{b}_n \psi^{(n)}(x,t) + \text{h.c.} \right], \quad \left[ \hat{b}_n, \hat{b}_k^{\dagger} \right] = \delta_{nk}, \quad (73)$$

where

$$\psi^{(n)}(x,t) = \sqrt{L_0/L(t)} \sum_{k=1}^{\infty} \sin \left[ \frac{\pi k x}{L(t)} \right] \left\{ \rho_{k}^{(n)}(\tau)e^{-i\omega_k t} - \rho_{-k}^{(n)}(\tau)e^{i\omega_k t} \right\},$$

$$\tau = \frac{1}{2} \varepsilon \omega_1 t, \quad \omega_n = n \omega_1. \quad (74)$$

The normalization factors $2/\sqrt{n}$ in (73) are chosen in such a way that the energy of the field in the stationary case can be represented as a sum of energies of independent mode oscillators. The coefficients $\rho_{k}^{(n)}(\tau)$ satisfy an infinite system of coupled equations ($k = \pm 1, \pm 2, \ldots; n = 1, 2, \ldots$)

$$\frac{d}{d\tau} \rho_{k}^{(n)} = \sigma \left[ (k+p)\rho_{k+p}^{(n)} - (k-p)\rho_{k-p}^{(n)} \right], \quad \sigma \equiv (-1)^p, \quad (76)$$

which was solved in [13] (here we confine ourselves to the simplest special case of solutions found in [13], corresponding to the strict resonance).

Due to the initial conditions $\rho_{k}^{(n)}(0) = \delta_{kn}$ the solutions to (76) satisfy the relation $\rho_{j+mp}^{(k+np)} \equiv 0$ if $j \neq k$. The non-zero coefficients $\rho_{m}^{(n)}$ read

$$\rho_{j+mp}^{(j+np)}(\tau) = \frac{\Gamma(1+n+j/p)(\sigma k)^{n-m}}{\Gamma(1+m+j/p)\Gamma(1+n-m)} F\left(n+j/p, -m-j/p; 1+n-m; \kappa^2 \right), \quad (77)$$

where

$$\kappa = \tanh(p\tau) \quad (78)$$

13
and $F(a, b; c; z)$ is the Gauss hypergeometric function. The functions (74) are exact solutions to the set of equations (76) relating the coefficients with different lower indices. Besides, these functions satisfy another set of equations, which can be treated as recurrence relations with respect to the upper indices [45]

$$\frac{d}{dt} \rho^{(n)}_m = n \left\{ \sigma \left[ \rho^{(n-p)}_m - \rho^{(n+p)}_m \right] \right\}, \quad n \geq p, \quad \rho^{(0)}_m \equiv 0 \quad (79)$$

$$\frac{d}{dt} \rho^{(n)}_m = n \left\{ \sigma \left[ \rho^{(p-n)*}_m - \rho^{(p+n)}_m \right] \right\}, \quad n = 1, 2, \ldots, p - 1 \quad (80)$$

The consequences of equations (79), (79) and (80) are the identities

$$\sum_{m=-\infty}^{\infty} m \rho^{(n)*}_{m} \rho^{(k)}_{m} = n \delta_{nk}, \quad n, k = 1, 2, \ldots \quad (81)$$

$$\sum_{n=1}^{\infty} \frac{m}{n} \left[ \rho^{(n)*}_{m} \rho^{(n)}_{j} - \rho^{(n)*}_{-j} \rho^{(n)}_{-m} \right] = \delta_{mj}, \quad m, j = 1, 2, \ldots \quad (82)$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \left[ \rho^{(n)*}_{m} \rho^{(n)}_{-j} - \rho^{(n)*}_{j} \rho^{(n)}_{-m} \right] = 0, \quad m, j = 1, 2, \ldots \quad (83)$$

We suppose that after some interval of time $T$ the wall comes back to its initial position $L_0$. For $t \geq T$, the field operator assumes the form

$$\hat{A}(x, t) = \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} \sin \left( \pi nx/L_0 \right) \left[ \hat{a}_n e^{-i\omega_n t} + \text{h.c.} \right] \quad (84)$$

where operators $\hat{a}_m$ are related to the initial operators $\hat{b}_n$ and $\hat{b}_n^\dagger$ by means of the Bogoliubov transformation ($\tau_T \equiv \frac{1}{2} \hbar \omega_1 T$)

$$\hat{a}_m = \sum_{n=1}^{\infty} \sqrt{\frac{m}{n}} \left[ \hat{b}_n \rho^{(n)*}_{m} (\tau_T) - \hat{b}_n^\dagger \rho^{(n)*}_{-m} (\tau_T) \right], \quad m = 1, 2, \ldots \quad (85)$$

The commutation relations $[\hat{a}_n, \hat{a}_k^\dagger] = \delta_{nk}$ hold due to the identities (81)-(83) which are nothing but the unitarity conditions of the transformation (83). These commutation relations together with the expression for the energy of the field

$$\hat{H} \equiv \frac{1}{8\pi} \int_{0}^{L_0} \! dx \left[ \left( \frac{\partial \hat{A}}{\partial t} \right)^2 + \left( \frac{\partial \hat{A}}{\partial x} \right)^2 \right] = \sum_{n=1}^{\infty} \omega_n \left( \hat{a}_n^\dagger \hat{a}_n + \frac{1}{2} \right) \quad (86)$$

convince us that $\hat{a}_n$ and $\hat{a}_n^\dagger$ are true photon annihilation and creation operators at $t \geq T$ (like the operators $\hat{b}_n$ and $\hat{b}_n^\dagger$ were ‘physical’ ones at $t < 0$).

### 4.1 Intermode entanglement in the parametric resonance case ($p = 2$)

Our first goal is to calculate the entanglement coefficients between different modes in the case of the parametric resonance, $p = 2$. If the initial state of the field was vacuum with respect to the initial operators $\hat{b}_n$: $\hat{b}_n|0\rangle = 0$ (we use here the Heisenberg picture), then the covariance entanglement coefficient between the $r$th and $s$th modes is

$$\gamma_{r,s} = \left[ \frac{|\langle \hat{a}_r \hat{a}_s \rangle|^2 + |\langle \hat{a}_r^\dagger \hat{a}_s \rangle|^2}{2 \left( \langle \hat{a}_r^\dagger \hat{a}_r \rangle + 1/2 \right) \left( \langle \hat{a}_s^\dagger \hat{a}_s \rangle + 1/2 \right)} \right]^{1/2} \quad (87)$$
Using (83), one can express the average values contained in formula (87) as (assuming hereafter ω₁ = 1)

\[
\langle \hat{a}_r \hat{a}_s \rangle = -\sqrt{rs} \sum_{n=1}^{\infty} \frac{1}{n} \rho_{r}^{(n)} \rho_{-s}^{(n)*} = -\sqrt{rs} \sum_{n=1}^{\infty} \frac{1}{n} \rho_{s}^{(n)} \rho_{-r}^{(n)*},
\]

(88)

\[
\langle \hat{a}_r^2 \rangle = \sqrt{rs} \sum_{n=1}^{\infty} \frac{1}{n} \rho_{r}^{(n)} \rho_{-r}^{(n)*}, \quad \langle \hat{a}_r \hat{a}_r \rangle = r \sum_{n=1}^{\infty} \frac{1}{n} |\rho_{r}^{(n)}|^{2},
\]

(89)

where the coefficients \(\rho_{\pm n}^{(n)}\) should be taken at the moment \(T\), thus their argument is \(\tau_T\). Strictly speaking, the expressions in (88) and (89) have physical meanings at those moments of time \(T\) when the wall returns to its initial position, i.e., for \(T = N\pi/p\) with an integer \(N\). Consequently, the argument \(\tau_T\) of the coefficients \(\rho_{\pm n}^{(n)}\) in (88) and (89) assumes discrete values \(\tau^{(N)} = N\pi/(2p)\). One should remember, however, that something interesting in our problem happens for the values \(\tau \sim 1\) (or larger). Then \(N \sim \varepsilon^{-1} \gg 1\), and the minimal increment \(\Delta\tau \sim \varepsilon\) is so small that \(\tau_T\) can be considered as a continuous variable (under the realistic conditions, \(\varepsilon \leq 10^{-8}\)). For this reason, we omit hereafter the subscript \(T\), writing simply \(\tau\) instead of \(\tau_T\) or \(\tau^{(N)}\).

Differentiating the right-hand sides of equations (88) and (89) with respect to the ‘slow time’ \(\tau\), one can remove the fraction 1/\(n\) with the aid of the recurrence relations (79) and (80). After that, changing if necessary the summation index \(n\) to \(n \pm p\), one can verify that almost all terms in the right-hand sides are cancelled, and the infinite series are reduced to the finite sums. For \(p = 2\) we obtain the equations (taking into account that all functions \(\rho_{m}^{(n)}\) are real in the strict resonance case, according to Eq. (77))

\[
d\langle \hat{a}_r \hat{a}_s \rangle/d\tau = -\sqrt{rs} \left[ \rho_{r}^{(1)} \rho_{s}^{(1)} + \rho_{-r}^{(1)} \rho_{-s}^{(1)} \right],
\]

(90)

\[
d\langle \hat{a}_r^2 \rangle/d\tau = \sqrt{rs} \left[ \rho_{r}^{(1)} \rho_{-r}^{(1)} + \rho_{s}^{(1)} \rho_{-s}^{(1)} \right], \quad d\langle \hat{a}_r \hat{a}_r \rangle/d\tau = 2r \rho_{r}^{(1)} \rho_{-r}^{(1)}.
\]

(91)

For \(p = 2\), only odd modes can be excited from the initial vacuum state. In this case, the hypergeometric functions in the formula (77) for coefficients \(\rho_{r}^{(n)}\) with \(j = 1\) are reduced to some combinations of the complete elliptic integrals of the first and the second kinds \[43\]

\[
K(\kappa) = \int_{0}^{\pi/2} \frac{d\alpha}{\sqrt{1 - \kappa^2 \sin^2 \alpha}} = \frac{\pi}{2} F \left( \frac{1}{2}, \frac{1}{2}; 1; \kappa^2 \right),
\]

\[
E(\kappa) = \int_{0}^{\pi/2} \frac{\sin \alpha}{\sqrt{1 - \kappa^2 \sin^2 \alpha}} = \frac{\pi}{2} F \left( -\frac{1}{2}, \frac{1}{2}; 1; \kappa^2 \right),
\]

so that equations (90) and (91) can be integrated for any values of \(r\) and \(s\): see \[45, 46\] or Appendix \[3\] for technical details. In particular, for the first few modes we find

\[
\langle \hat{a}_1^2 \rangle = \frac{2}{\pi^2 \kappa} \left[ \kappa^2 K^2 - 2EK + E^2 \right],
\]

(92)

\[
\langle \hat{a}_3^2 \rangle = \frac{2}{9\pi^2 \kappa^3} \left[ \kappa^2 (4 - \kappa^2) K^2 - 2(2\kappa^4 - 3\kappa^2 + 4)EK + (4\kappa^4 - \kappa^2 + 4)E^2 \right],
\]

(93)

\[
\langle \hat{a}_1 \hat{a}_3 \rangle = -\frac{2\sqrt{3}}{3\pi^2 \kappa} \left[ \kappa^2 K^2 - 2EK + (1 + \kappa^2) E^2 \right],
\]

(94)

\[
\langle \hat{a}_1^2 \rangle = \frac{2\sqrt{3}}{\pi^2 \kappa} \left[ \frac{\kappa^2}{3} K^2 + \frac{2}{3}(\kappa^2 - 2)EK + E^2 \right],
\]

(95)

\[
\langle \hat{a}_1 \hat{a}_5 \rangle = \frac{2\sqrt{5}}{45\pi^3 \kappa^3} \left[ \kappa^2 (\kappa^2 + 8) K^2 - 2(\kappa^4 + 8)EK + (8\kappa^4 + 7\kappa^2 + 8)E^2 \right],
\]

(96)

\[
\langle \hat{a}_1^2 \rangle = -\frac{2\sqrt{5}}{3\pi^2 \kappa^2} \left[ \frac{\kappa^2}{5} (2\kappa^2 + 1) K^2 + \frac{2}{5}(2\kappa^4 - 2\kappa^2 - 3)EK + (\kappa^2 + 1)E^2 \right],
\]

(97)
\[ \langle \hat{a}_3 \hat{a}_5 \rangle = \frac{2\sqrt{15}}{45\pi^2\kappa^4} \left[ \kappa^2(\kappa^2 - 2)(\kappa^2 - 2)K^2 + 2(2\kappa^6 - \kappa^4 - 2\kappa^2 + 4)EK \right. \\
-4(\kappa^2 + 1)(\kappa^4 - \kappa^2 + 1)E^2 \right], \tag{98} \]
\[ \langle \hat{a}_3^\dagger \hat{a}_5 \rangle = \frac{2\sqrt{15}}{45\pi^2\kappa^4} \left[ \kappa^2(7\kappa^2 - 4)K^2 + 2(8\kappa^4 - 15\kappa^2 + 4)EK - (4\kappa^4 - 19\kappa^2 + 4)E^2 \right], \tag{99} \]
\[ \mathcal{E}_1 = \frac{2}{\pi^2} K (2E - \kappa^2K), \tag{100} \]
\[ \mathcal{E}_3 = \frac{2}{3\pi^2\kappa^2} \left[ (3\kappa^2 - 2) K (2E - \kappa^2K) + 2 (1 + \kappa^2) E^2 \right] \tag{101} \]
\[ \mathcal{E}_5 = -\frac{2}{45\pi^2\kappa^4} \left[ \kappa^2(47\kappa^4 - 30\kappa^2 - 8)K^2 + 2(4\kappa^6 - 47\kappa^4 + 26\kappa^2 + 8)EK \\
-2(\kappa^2 + 1)(4\kappa^4 + 11\kappa^2 + 4)E^2 \right], \tag{102} \]

where \( \kappa = \sqrt{1 - \kappa^2} \) and we used \( \mathcal{E}_r = \langle \hat{a}_r^\dagger \hat{a}_r \rangle + 1/2. \)

In Figure 3 we show \( Y_{1,3} \) and \( Y_{3,5} \). We see that the entanglement is strongest for the lowest modes. However, for any pair \( r, s \) the coefficient \( \gamma_{r,s} \) tends asymptotically to the unit value when \( \kappa \to 1 \). To prove this property, one should use the asymptotical forms of the coefficients \( \rho_m^{(z)} \) for \( \tau \to \infty \), i.e., for \( \kappa \to 1 \).

Namely, replacing the hypergeometric functions in (77) by their values for the unit argument \( \kappa = 1 \), and the purity entanglement coefficient can be calculated by means of formula (8). In a Gaussian case, the determinant of the symmetrical 4 \( \times \) 4 matrix \( Q \) (24) contains 17 different terms. However, in the specific case involved all covariances between the \( \text{“coordinate”} \) and \( \text{“momentum”} \) operators turn out to be equal to zero identically: \( \langle \hat{x}_i \hat{p}_j \rangle = 0 \), and for this reason the determinant of the covariance matrix for the \( i \)th and \( j \)th modes can be factorized in the following simple form:

\[ \det Q = \left( \sigma_{p,p_{i,p_{j,p_{j}}} - \sigma_{p_{j,p_{j}}}^2} \right) \left( \sigma_{x,x_{i}} \sigma_{x_{j},x_{j}} - \sigma_{x_{i},x_{j}}^2 \right). \tag{105} \]

Nonzero covariances are given by the following expressions:

\[ \sigma_{x_{i},x_{j}} \equiv \langle x_{i}x_{j} \rangle = \frac{1}{2} \langle \hat{a}_{i}^\dagger \hat{a}_{j} + \hat{a}_{j}^\dagger \hat{a}_{i} \rangle + \text{Re} \langle \hat{a}_{i} \hat{a}_{j} \rangle, \quad \sigma_{p_{i},p_{j}} \equiv \langle p_{i}p_{j} \rangle = \frac{1}{2} \langle \hat{a}_{i}^\dagger \hat{a}_{j} + \hat{a}_{j}^\dagger \hat{a}_{i} \rangle - \text{Re} \langle \hat{a}_{i} \hat{a}_{j} \rangle. \tag{106} \]
Introducing the correlation coefficients,
\[ r_{x,x_j} = \frac{\sigma_{x,x_j}}{\sqrt{\sigma_{x,x} \sigma_{x_j,x_j}}} , \quad r_{p,p_j} = \frac{\sigma_{p,p_j}}{\sqrt{\sigma_{p,p} \sigma_{p_j,p_j}}} , \]  
we can represent the \( \tilde{L} \) and \( Z \) entanglement coefficients between the \( i \)th and \( j \)th modes as
\[ \tilde{L}_{ij} = 1 - \sqrt{\left( 1 - r_{x,x_j} \right) \left( 1 - r_{p,p_j}^2 \right)} , \]  
\[ Z_{ij} = 1 + \sqrt{\left( 1 - r_{x,x_j}^2 \right) \left( 1 - r_{p,p_j}^2 \right) - 2 \frac{\left( 1 - r_{x,x_j}^2 \right) \left( 1 - r_{p,p_j}^2 \right)}{1 - \frac{1}{4} r_{x,x_j}^2 \left( 1 - \frac{1}{4} r_{p,p_j}^2 \right)}} . \]  
If all correlation coefficients are small (in particular, if \( \tau \ll 1 \)), then
\[ \tilde{L}_{ij} \approx 2Z_{ij} \approx \frac{1}{2} \left( r_{x,x_j}^2 + r_{p,p_j}^2 \right) . \]  
When \( \tau \to \infty \), then, due to equations \( 104, 107 \) and \( 107 \), the coefficients \( \sigma_{p,p_j} \) linearly grow with time in such a way that the momentum correlation coefficient \( r_{p,p_j} \) tends to the unit value. At the same time, the coefficients \( \sigma_{x,x_j} \) and \( r_{x,x_j} \) tend to some finite limit values. Therefore, the purity entanglement coefficient \( \tilde{L} \) and the distance entanglement coefficient \( Z \) approach the unit value. Using the asymptotic formulae for the complete elliptic integrals \( 17 \),
\[ K(\kappa) \approx \frac{4}{\kappa} + \frac{1}{4} \left( \ln \frac{4}{\kappa} - 1 \right) \kappa^2 + \cdots , \quad E(\kappa) \approx \frac{1}{2} \left( \ln \frac{4}{\kappa} - \frac{1}{2} \right) \kappa^2 + \cdots , \quad \kappa \to 1 , \]
one can see that for \( \tau \gg 1 \), \( 1 - \tilde{L} \sim 1 - Z \sim \tau^{-1/2} \). In particular,
\[ 1 - \tilde{L}_{13} \approx \sqrt{\frac{44}{57\tau}} \approx 0.88 \frac{1}{\sqrt{\tau}} , \quad 1 - Z_{13} \approx \sqrt{\frac{44}{3\tau} \left( \frac{8}{\sqrt{219}} - \frac{1}{\sqrt{19}} \right)} \approx \frac{1.19}{\sqrt{\tau}} . \]
Calculating the entropic entanglement measure \( \Omega \) one should take into account that the reduced entropy of any two-mode subsystem depends on time in the case involved (in contradistinction to the case considered in the preceding section), because the evolution of each finite-dimensional subsystem is not unitary. This entropy is determined by two eigenvalues of the corresponding \( 4 \times 4 \) matrix \( Q \Omega^{-1} \), which are given by formula \( 27 \). The reduced entropy of the \( k \)th mode is determined by the single number
\[ f_k = \sqrt{\sigma_{p_k,p_k} \sigma_{x_k,x_k}} , \]  
as soon as the coordinate-momentum covariances are equal to zero in the strict resonance case considered. The explicit formula for the entropic entanglement measure between the \( k \)th and \( n \)th modes becomes (for the initial vacuum state of the field)
\[ \Omega_{kn} = \sum_{j=k,n} \left[ (f_j + 1/2) \ln (f_j + 1/2) - (f_j - 1/2) \ln (f_j - 1/2) \right] \]
\[ - \sum_{j=\delta,\delta+1} \left[ (f_{kn}^\delta + 1/2) \ln (f_{kn}^\delta + 1/2) - (f_{kn}^\delta - 1/2) \ln (f_{kn}^\delta - 1/2) \right] , \]  
where
\[ 2f_{kn}^\delta = \left[ p_k p_k x_k x_k + p_n p_n x_n x_n + 2p_k p_n x_k x_n + 2 \sqrt{p_k p_k p_n p_n - p_k p_n^2} \left( x_k x_k x_n x_n - x_k x_n^2 \right) \right]^{1/2} \]
\[ + \delta \left[ p_k p_k x_k x_k + p_n p_n x_n x_n + 2p_k p_n x_k x_n - 2 \sqrt{p_k p_k p_n p_n - p_k p_n^2} \left( x_k x_k x_n x_n - x_k x_n^2 \right) \right]^{1/2} . \]  
The behaviour of different entanglement coefficients is shown in Fig. \( \text{F} \). All of them monotonously tend to unity with the course of time, but much more slowly than in the case of asymmetric resonance in the 3D cavity (due to interaction with other resonant modes).
4.2 Entanglement in the “semi-resonance case” \((p = 1)\)

A qualitatively different behaviour of all characteristics of the field is observed in the “semi-resonance case”, when the frequency of the oscillations of the boundary coincides with the fundamental field eigenfrequency \((p = 1)\) \([18, 18]\). In this case one should put \(j = 0\) in formula \((77)\), and all coefficients \(\rho_m(\nu)\) with negative lower indices \(m\) are equal to zero identically. As a consequence, no photons can be created from the initial vacuum state, which is clearly seen from equation \((83)\). If initially the field was in non-vacuum state (at least for some mode), then the total number of photons in all modes is conserved, although the total energy grows exponentially due to “heating” the high-frequency modes (at the expense of “cooling” the low-frequency modes).

We suppose that initially only the first mode was excited, while all the others were in the vacuum state. Then the dynamics of all modes is described by means of the unique coefficient

\[
\rho_m^{(1)} = (\tanh \tau)^{m-1}/\cosh^2 \tau.
\]

If initially the excited mode was in a coherent state, then all second-order central moments connecting different modes are equal to zero, resulting in zero covariance entanglement coefficient: \(\gamma_{r,s}^{coh} = 0\). However, for other initial states we obtain nonzero values of \(\gamma_{r,s}\).

If initially the first mode was in the Fock state \(|n\rangle\), then

\[
\gamma_{r,s}^{Fock} = \frac{n \zeta_r \zeta_s}{\sqrt{2(n \zeta_r^2 + 1/2)(n \zeta_r^2 + 1/2)}},
\]

where

\[
\zeta_m = \sqrt{m} \rho_m^{(1)} = \sqrt{m} (\tanh \tau)^{m-1}/\cosh^2 \tau \leq 1.
\]

If initially the first mode was in a squeezed vacuum state \(|\psi\rangle = \exp \left[ R(\hat{b}_1^2 - \hat{b}_1^2)/2 \right] |0\rangle\) with the average number of photons \(\nu_1 = \sinh^2(R)\), then

\[
\gamma_{r,s}^{sqz} = \frac{\zeta_r \zeta_s \sqrt{\nu_1(2\nu_1 + 1)}}{\sqrt{2(\nu_1 \zeta_r^2 + 1/2)(\nu_1 \zeta_s^2 + 1/2)}},
\]

If initially the first mode was in an even/odd coherent state \([19]\)

\[
|\alpha_\pm\rangle = \frac{|\alpha_1\rangle \pm | - \alpha_1\rangle}{\sqrt{2[1 \pm \exp(-2|\alpha_1|^2)]}},
\]

then the mean numbers of photons are given by the formulae

\[
\nu_1^{(+)} = |\alpha_1|^2 \tanh(|\alpha_1|^2), \quad \nu_1^{(-)} = |\alpha_1|^2 \coth(|\alpha_1|^2).
\]

In both cases, the entanglement coefficient can be written as

\[
\gamma_{r,s}^{ev/od} = \frac{\zeta_r \zeta_s \sqrt{\nu_1(\nu_1 + |\alpha_1|^2)}}{\sqrt{2(\nu_1 \zeta_r^2 + 1/2)(\nu_1 \zeta_s^2 + 1/2)}}.
\]

For big enough number of photons in the initial squeezed and even/odd states, \(\nu_1 \gg 1\), the entanglement coefficient becomes very close to the maximal possible unit value, if \(\nu_1 \zeta_r^2 \gg 1\), but with increase of time \(\gamma\) eventually goes to zero, because \(\zeta_{r,s}(\tau) \to 0\) for \(\tau \to \infty\). In the case of the initial Fock state, the maximal value of \(\gamma\) does not exceed \(1/\sqrt{2}\). A typical behaviour of the covariance entanglement coefficient between the first and second modes for the initial Fock and squeezed states is shown in Fig. 3. The behaviour of \(\gamma_{m,n}\) for the initial thermal and even/odd states is very similar, especially for large mean numbers of photons. The evolution of the mean number of photons in the first and second modes is shown in Fig. 3.

The momentum-coordinate covariances turn out to be equal to zero again (as in the case of \(p = 2\)), therefore we need only two correlation coefficients defined in \([107]\), in order to calculate the purity and
distance entanglement coefficients (in the case of the initial squeezed state of the first mode) with the aid of Eqs. (108) and (109). These correlation coefficients are as follows,

\[ r_{x,x_j} = \frac{\chi \zeta_i(\tau) \zeta_j(\tau)}{\sqrt{[1 + \chi \zeta^2_i(\tau)][1 + \chi \zeta^2_j(\tau)]}} \]
\[ r_{p,p_j} = -\frac{\lambda \zeta_i(\tau) \zeta_j(\tau)}{\sqrt{[1 - \lambda \zeta^2_i(\tau)][1 - \lambda \zeta^2_j(\tau)]}} \]

(117)

where \( \chi = e^{2R} - 1 \), \( \lambda = 1 - e^{-2R} \).

The time dependences of the \( \bar{L} \) and \( \bar{Z} \) entanglement coefficients are compared in Fig. 10. We see that the full and dashed curves are very close, especially for large mean numbers of photons.

5 Conclusion

We have compared time dependences of several functions characterizing the degree of entanglement between field modes of ideal cavities with resonantly vibrating walls for different models of such cavities. All these functions (the “standard” entropic entanglement measure for Gaussian states, covariance entanglement coefficient introduced in [13, 14], distance entanglement coefficient introduced in [13], and purity entanglement coefficient introduced here) are based on the second-order covariance matrix of the field quadrature components. In spite of having different analytical forms, the coefficients concerned show similar qualitative (and in certain cases even quantitative) behaviour for each fixed model. Therefore, the covariance entanglement coefficient, being the simplest from the point of view of calculations, seems to be the most convenient, especially compared with the entropic entanglement measure, whose calculation requires tremendous efforts, giving practically the same information on the degree of entanglement. Moreover, an example at the end of section 3.1 shows that the covariance entanglement coefficient (based on traces of covariance submatrices) can be more sensitive to entanglement than other measures (which are based on determinants of covariance submatrices).

On the other hand, the behaviour of each selected entanglement coefficient turns out to be completely different for different kinds of cavities. For the three-dimensional cavities with accidental degeneracy of the spectrum of eigenfrequencies, the entanglement coefficients exhibit oscillations in the case of “symmetric” resonance, remaining relatively small for all instants of time. Moreover, they go to zero periodically, despite that the energy of each mode increases unlimitedly. In the case of “asymmetric” resonance, fast (in the “slow time” scale) oscillations of the entanglement coefficients are also observed, but all these coefficients tend to the maximal possible unit value with increase of time. For the model of one-dimensional (Fabry–Perot) cavity with equidistant spectrum, all entanglement coefficients monotonously go to the unit value in the parametric resonance case. In the “semiresonance” case, they rapidly reach the values close to unity and remain at this level for some interval of time (which increases with increase of the initial mean number of quanta), but eventually they decay to zero. Therefore, this study adds some new features to our understanding of the behaviour of fields in cavities with vibrating boundaries, in addition to results obtained earlier in [14, 15, 16, 20, 21].

Acknowledgement

The authors acknowledge a full support of the Brazilian agency CNPq.

A The Bogoliubov coefficients in the 1D parametric resonance case

The nonzero coefficients \( \rho_{2m+1}^{(1)} \) in the parametric resonance case (\( p = 2 \)) read [45, 46]

\[ \rho_{2m+1}^{(1)} = \frac{(-1)^m \Gamma (m + 1/2) \kappa^m}{\Gamma(1/2) \Gamma(1 + m)} F \left( m + 1/2, -1/2; 1 + m ; \kappa^2 \right) \]  

(A.1)
\[ \rho_{-2m-1}^{(1)} = \frac{(-1)^m \Gamma(m + 1/2) \Gamma(3/2) \kappa^{m+1}}{\pi \Gamma(2 + m)} F(m + 1/2, 1/2; 2 + m; \kappa^2). \] (A.2)

In particular (\(\tilde{\kappa} \equiv \sqrt{1 - \kappa^2}\)),

\[ \rho_{1}^{(1)} = \frac{2}{\pi} E(\kappa), \quad \rho_{-1}^{(1)} = \frac{2}{\pi \kappa} [E(\kappa) - \tilde{\kappa}^2 K(\kappa)], \] (A.3)

\[ \rho_{3}^{(1)} = \frac{2}{3\pi \kappa} [(1 - 2\kappa^2) E(\kappa) - \tilde{\kappa}^2 K(\kappa)] \quad \rho_{-3}^{(1)} = -\frac{2}{3\pi \kappa^2} [(2 - \kappa^2) E(\kappa) - 2\tilde{\kappa}^2 K(\kappa)], \] (A.4)

The general structure of the coefficients \(\rho_{2m+1}^{(1)}\) in terms of the complete elliptic integrals is

\[ \rho_{2m+1}^{(1)} = \frac{2}{\pi \kappa m} [f_m(\kappa^2) E(\kappa) + \tilde{\kappa}^2 g_m(\kappa^2) K(\kappa)] \] (A.5)

\[ \rho_{-2m-1}^{(1)} = \frac{2}{\pi \kappa m + 1} [r_m(\kappa^2) E(\kappa) + \tilde{\kappa}^2 s_m(\kappa^2) K(\kappa)] \] (A.6)

where \(f_m(x), g_m(x), r_m(x), s_m(x)\) are polynomials of the degree \(m\) which can be found from the recurrence relations [16].

**B Calculation of integrals**

To calculate, for instance, the average value \(\langle \hat{a}_3^3 \hat{a}_3 \rangle\), we use equations (A.3), (A.4) and (A.5), replacing the derivative over \(\tau\) by the derivative with respect to \(\kappa\) in accordance with the relation \(d\kappa = 2\tilde{\kappa}^2 d\tau\). In this way we arrive at the equation

\[ \frac{d\langle \hat{a}_3^3 \hat{a}_3 \rangle}{d\kappa} = \frac{2\sqrt{3}}{3\pi^2 \tilde{\kappa}^2} [\langle 1 + \kappa^2 \rangle E^2(\kappa) - \tilde{\kappa}^4 K^2(\kappa) - 2\kappa^2 \tilde{\kappa}^2 E(\kappa) K(\kappa)]. \] (A.7)

Taking into account the differentiation rules [17]

\[ \frac{dK(\kappa)}{d\kappa} = \frac{E(\kappa)}{\kappa \tilde{\kappa}^2}, \quad \frac{dE(\kappa)}{d\kappa} = \frac{E(\kappa) - K(\kappa)}{\kappa}, \] (A.8)

we may suppose that the factor \(\tilde{\kappa}^2\) in the denominator of the right-hand side of equation (A.7) comes from the derivative \(dK/d\kappa\). Thus it is natural to look for the solution in the form

\[ \langle \hat{a}_3^3 \hat{a}_3 \rangle = \frac{2\sqrt{3}}{3\pi^2 \kappa} [A(\kappa) K^2(\kappa) + B(\kappa) K(\kappa) E(\kappa) + C(\kappa) E^2(\kappa)], \] (A.9)

where \(A(\kappa), B(\kappa)\) and \(C(\kappa)\) are some polynomials of \(\kappa\). Putting the expression (A.9) into equation (A.7) we obtain a set of coupled equations for the coefficients of these polynomials, which can be resolved recursively. The equations for other second-order moments can be integrated in the same manner.

**References**

[1] E. Schrödinger, *Proc. Camb. Phil. Soc.*, 31, 555 (1935).

[2] E. Schrödinger, *Naturwissenschaften*, 23, 807, 823, 844 (1935) [English translation in: *Quantum Theory and Measurement* (J. A. Wheeler and W. H. Zurek, eds.), p. 152, Princeton Univ. Press, Princeton (1983)].
[3] A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.*, **47**, 777 (1935).

[4] S. M. Barnett and S. J. D. Phoenix, *Phys. Rev. A*, **40**, 2404 (1989); **44**, 535 (1991).

[5] A. Mann, B. C. Sanders, and W. J. Munro, *Phys. Rev. A*, **51**, 989 (1995).

[6] C. H. Bennett, H. J. Herbert, S. Popescu, and B. Schumacher, *Phys. Rev. A*, **53**, 2046 (1996); S. Popescu and D. Rohrlich, *Phys. Rev. A*, **56**, R3319 (1997); S. L. Braunstein, *Phys. Lett. A*, **219**, 169 (1996); V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, *Phys. Rev. Lett.*, **78**, 2275 (1997).

[7] V. Vedral and M. B. Plenio, *Phys. Rev. A*, **57**, 1619 (1998).

[8] W. K. Wootters, *Phys. Rev. Lett.*, **80**, 2245 (1998).

[9] M. G. A. Paris, *J. Opt. B*, **1**, 299 (1999); M. J. Donald and M. Horodecki, *Phys. Lett. A*, **264**, 257 (1999); M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. Lett.*, **84**, 2014 (2000), *Phys. Rev. Lett.*, **84**, 2263 (2000); S. Parker, S. Bose, and M.B. Plenio, *Phys. Rev. A*, **61**, 032305 (2000); T. Hiroshima, *Phys. Rev. A*, **63**, 022305 (2001).

[10] K. Furuya, M. C. Nemes, and G. Q. Pellegrino, *Phys. Rev. Lett.*, **80**, 5524 (1998); R. M. Angelo, K. Furuya, M. C. Nemes, and G. Q. Pellegrino, *Phys. Rev. A*, **64**, 043801 (2001); J. Gemmer and G. Mahler, *Eur. Phys. J. D*, **17**, 385 (2001).

[11] C. Witte and M. Trucks, *Phys. Lett. A*, **257**, 14 (1999); M. Ozawa, *Phys. Lett. A*, **268**, 158 (2000).

[12] V. I. Man’ko, G. Marmo, E. C. G. Sudarshan, and F. Zaccaria, *J. Phys. A*, **35**, 7137 (2002).

[13] A. S. M. de Castro and V. V. Dodonov, *J. Russ. Laser Research*, **23**, 93 (2002).

[14] V. V. Dodonov, A. S. M. de Castro, and S. S. Mizrahi, *Phys. Lett. A*, **296**, 73 (2002).

[15] V. V. Dodonov and A. B. Klimov, *Phys. Rev. A*, **53**, 2664 (1996).

[16] F. R. Gantmakher, *The Theory of Matrices*, Nauka, Moscow (1966).

[17] A. S. Holevo, M. Sohma, and O. Hirota, *Phys. Rev. A*, **59**, 1820 (1999).

[18] V. V. Dodonov, in: A. A. Komar, (Ed.), *Group Theory, Gravitation and Elementary Particle Physics*, Proc. Lebedev Phys. Inst., vol. 167, Nauka, Moscow (1986), p. 7 [translated by Nova Science, Commack (1987), p. 7].

[19] A. S. Holevo, M. Sohma, and O. Hirota, *Phys. Rev. A*, **59**, 1820 (1999).

[20] V. V. Dodonov and V. I. Man’ko, in: A. A. Komar, (Ed.), *Group Theory, Gravitation and Elementary Particle Physics*, Proc. Lebedev Phys. Inst., vol. 167, Nauka, Moscow (1986), p. 7 [translated by Nova Science, Commack (1987), p. 7].
[25] E. Schrödinger, *Ber. Kgl. Akad. Wiss. Berlin*, 24, 296 (1930); H. P. Robertson, *Phys.Rev.*, 35, 667 (1930).

[26] V. V. Dodonov, E. V. Kurmyshev, and V. I. Man’ko, *Phys. Lett.* A, 79, 150 (1980).

[27] V. V. Dodonov and V. I. Man’ko, *Invariants and the Evolution of Nonstationary Quantum Systems* (Proc. Lebedev Phys. Inst. 183), ed. M. A. Markov, Nova Science, Commack (1989).

[28] G. S. Agarwal, *Phys. Rev.* A, 3, 828 (1971).

[29] V. Perinová, J. Křepelka, J. Peřina, A. Lukš, and P. Szlachetka, *Opt. Acta*, 33, 15 (1986).

[30] V. V. Dodonov, *J. Phys.* A, 33, 7721 (2000); V. V. Dodonov and O. V. Man’ko, *J. Russ. Laser Res.*, 21, 438 (2000); *J. Opt. Soc. Am.* A, 17, 2403 (2000).

[31] D. A. Trifonov, *J. Opt. Soc. Am.* A, 17, 2486 (2000).

[32] V. V. Dodonov, in: M. W. Evans (Ed.), *Modern Nonlinear Optics*, Advances in Chem. Phys. Series, vol. 119, Wiley, New York (2001), Part 1, p. 309.

[33] V. V. Dodonov, A. B. Klimov, and V. I. Man’ko, *Phys. Lett.* A, 142, 511 (1989).

[34] J. Schwinger, *Proc. Nat. Acad. Sci. USA*, 90, 958 (1993).

[35] G. Barton and C. Eberlein, *Ann. Phys.* (NY), 227, 222 (1993).

[36] A. Lambrecht, M.-T. Jaekel, and S. Reynaud, *Phys. Rev. Lett.*, 77, 615 (1996).

[37] C. K. Law, *Phys. Rev.* A, 49, 433 (1994); 51, 2537 (1995).

[38] R. Schützhold, G. Plunien, and G. Soff, *Phys. Rev.* A, 57, 2311 (1998).

[39] R. Schützhold, G. Plunien, and G. Soff, *Phys. Rev.* A, 65, 043820 (2002); G. Schaller, R. Schützhold, G. Plunien, and G. Soff, *Phys. Rev.* A, 66, 023812 (2002).

[40] H. Saito and H. Hyuga, *Phys. Rev.* A, 65, 053804 (2002).

[41] L. A. S. Machado and P. A. Maia Neto, *Phys. Rev.* D, 65, 125005 (2002).

[42] C. K. Cole and W. C. Schieve, *Phys. Rev.* A, 64, 023813 (2001).

[43] M. Crocce, D. A. R. Dalvit, and F. D. Mazzitelli, *Phys. Rev.* A, 64, 013808 (2001).

[44] A. V. Dodonov and V. V. Dodonov, *Phys. Lett.* A, 289, 291 (2001).

[45] V. V. Dodonov, *J. Phys.* A, 31, 9835 (1998).

[46] V. V. Dodonov and M. A. Andreata, *J. Phys.* A, 32, 6711 (1999).

[47] I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Academic, New York (1994).

[48] V. V. Dodonov, *Phys. Lett.* A, 213, 219 (1996).

[49] V. V. Dodonov, I. A. Malkin, and V. I. Man’ko, *Physica*, 72, 597 (1974).

[50] M. A. Andreata and V. V. Dodonov, *J. Phys.* A, 33, 3209 (2000).
Figure 1: The covariance entanglement coefficient squared $\mathcal{Y}^2$ (thick line) and the purity entanglement coefficient $\tilde{\mathcal{L}}$ (thin line) versus “slow time” $\tau$ for two interacting modes $\{1,1,1\}$ and $\{5,1,1\}$ in a 3D cubical cavity ($\nu = 50/3$) under the condition of strict (“symmetric”) resonance and for the initial vacuum state.

Figure 2: The entropic entanglement measure $I_c$ versus “slow time” $\tau$ for two interacting modes $\{1,1,1\}$ and $\{5,1,1\}$ in a 3D cubical cavity ($\nu = 50/3$) under the condition of strict (“symmetric”) resonance, for the initial vacuum state (thick line; $\theta_1 = \theta_3 = 1$) and high-temperature state (thin line; $\theta_1 = 3\theta_3$).

Figure 3: The functions $\tilde{\mathcal{L}}(\tau)$ (thin line) and $\mathcal{Y}^2(\tau)$ (thick line) for two interacting modes $\{1,1,1\}$ and $\{5,1,1\}$ in a 3D cubical cavity ($\nu = 50/3$) under the condition of strict (“symmetric”) resonance and for the high-temperature initial state with $\theta_1 = 140$.

Figure 4: Time dependences of different entanglement measures under the condition of “asymmetric resonance” (63), for $\nu = 100$ and the initial vacuum state. Thick line: the covariance entanglement coefficient $\mathcal{Y}(\tau)$ (18). Thin lines from bottom to top: the purity entanglement coefficient $\tilde{\mathcal{L}}$ (51), the compact entropic entanglement measure $J_c(\tau)$ (59), the function $[\tilde{\mathcal{L}}(\tau)]^{1/2}$.

Figure 5: The covariance entanglement coefficient $\mathcal{Y}(\tau)$ (18) and the purity entanglement coefficient $\tilde{\mathcal{L}}(\tau)$ (51) under the condition of “asymmetric resonance” (63) with $\nu = 100$, for the initial vacuum state with $\theta_1 = \theta_3 = 1$ (monotonous dependences) and high-temperature state with $\theta_1 = 3\theta_3$ (oscillating functions). In both cases, upper curves correspond to $\mathcal{Y}(\tau)$ and lower curves correspond to $\tilde{\mathcal{L}}(\tau)$.

Figure 6: The covariance entanglement coefficient $\mathcal{Y}_{n,m}$ (87) in the 1D resonance ($p = 2$) cavity versus the compact parameter $\kappa = \tanh(2\tau)$ for the vacuum initial state. Full curve: $\mathcal{Y}_{1,3}$; dashed curve: $\mathcal{Y}_{3,5}$.

Figure 7: Different coefficients characterizing entanglement between the first and third modes in the 1D resonance ($p = 2$) cavity versus “slow time” $\tau$ (in the insertion) and the compact parameter $\kappa = \tanh(2\tau)$, for the vacuum initial state. The order of the curves in the main figure, from top to bottom: covariance entanglement coefficient $\mathcal{Y}$ (18); compact entropic coefficient $J_c$ (59); purity entanglement coefficient $\tilde{\mathcal{L}}$ (51); the square of the covariance entanglement coefficient $\mathcal{Y}^2$ (dashed curve in the insertion); distance entanglement coefficient $Z$ (16).

Figure 8: The covariance entanglement coefficient $\mathcal{Y}_{1,2}$ (87) in the 1D “semiresonance” ($p = 1$) cavity versus the “slow time” $\tau$, for the Fock (dashed curves) and squeezed vacuum (full curves) initial states of the first mode with mean photon numbers $\nu_1 = 1, 50, 1000$.

Figure 9: The mean number of photons in the first and second modes of the 1D “semiresonance” ($p = 1$) cavity versus the “slow time” $\tau$, for the initial Fock state $|1\rangle$.

Figure 10: The purity entanglement coefficient $\tilde{\mathcal{L}}_{1,2}$ (108) (full curves) and distance entanglement coefficient $Z_{1,2}$ (109) (dashed curves) versus the “slow time” $\tau$, for the 1D “semiresonance” ($p = 1$) cavity and the initial squeezed vacuum state of the first mode with different mean numbers of photons $\nu_1 = 1, 50, 1000$. 

23
\[ \gamma_{1,2} \]

- \( \nu_1 = 1000 \)
- \( \nu_1 = 50 \)
- \( \nu_1 = 1 \)

\[ \tau \]
