Braiding properties of Majorana Kramers Pairs

Konrad Wölms, Ady Stern, and Karsten Flensberg

1 Center for Quantum Devices, Niels Bohr Institute, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark
2 Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel

(Dated: July 13, 2015)

We consider the braiding of Kramers pairs of Majorana bound states. We derive the most general transformation on the many-body ground state that is applied as the result of such a braiding process. The result is derived in the context of a simple toy model, but we will show that it has the most general form that is compatible with local and global conservation of electron parity. In accordance with earlier work the resulting transformation turns out to be path dependent, which shows that Kramers pairs of Majorana bound states cannot be used for topological quantum computation. We also discuss under which conditions the result is path independent and corresponds to two independent exchanges of pairs of Majorana bound states.

For certain classes of topological superconductors (D, BDI), Majorana bound states (MBS) appear at topological phase boundaries or in vortices. Those MBS are a signature of the non-trivial topological phases and they can be used for topological quantum computation [1]. As such MBS have received a lot of attention, both from the theoretical and the experimental side [2-4].

Another closely related topological class is DIII. This class has time reversal symmetry that squares to minus one and exhibits Kramers pairs of MBS at topological phase boundaries or in vortices. Even though there have not been experimental attempts yet to realize this topological phase, there have been many theoretical proposals on how to obtain this phase[5-13]. A natural question to ask is whether those Kramers pairs of MBS can be used for topological quantum computation similar to MBS appearing in classes D and BDI. This question is equivalent to the questions whether Kramers pairs of MBS are non-Abelian anyons and whether their adiabatic exchange generates a transformation that is independent of the details of the exchange path.

In earlier work we argued that this cannot be the case because Kramers pairs of MBS have a local degree of freedom that can be manipulated adiabatically [14]. This makes any exchange of two such Kramers pairs path dependent, which means that they are not anyons or equivalent that they cannot be used for topological quantum computation.

In this article, we study the general transformation that occurs when braiding two Kramers pairs of MBS. We study it in the context of a simple toy model, but this will not limit the generality of the results because even in this simple model we find the most general transformation that is allowed by local parity conservation.

The toy model we study is the time-reversal-symmetric analog of braiding by tuning couplings [15]. It consists out of four Kramers pairs of Majorana fermions \( \chi_i, \tilde{\chi}_i = T \chi_i T^{-1} \), which we write as a vector \( \chi_i = (\chi_i, \tilde{\chi}_i)^T \). The most general time-reversal-symmetric coupling between two Majorana Kramers pairs is

\[
H_{jk} = i t \chi_j^T \sigma_x e^{i \theta_{jk} \sigma_y} \chi_k. \tag{1}
\]

In the toy model there are only three coupling terms, \( H_{41} \), \( H_{42} \) and \( H_{43} \). In order to have the appropriate ground state degeneracy at all times, there is at least one non-zero coupling and at most two at each point in parameter space. Furthermore, we take the Hamiltonian with two couplings to be of the form

\[
H = \cos \theta H_{ij} + \sin \theta H_{ik}, \tag{2}
\]

such that \( \theta \) controls the relative strength of the couplings. The effect of switching from one coupling to another, by means of tuning \( \theta \) from 0 to \( \pi/2 \), is to move a zero energy Kramers pair from one site to another. The situation is depicted in Fig. 1.

In order to study the adiabatic braiding process we have to specify a basis at each point in parameter space. At each such point we have two zero energy Kramers pairs of Majorana bound states, which we denote with \( X_1 \) and \( X_2 \). We assume that those are local and initially start out as \( X_1 \sim X_1, X_2 \sim X_2 \), where \( \sim \) means up to (generally different) local rotations \( e^{i \alpha \sigma_i} \). We construct a local basis by forming local fermions \( D_\eta = \frac{1}{2}(X_\eta + i \tilde{X}_\eta) \), where we used a Greek index which is either 1 or 2 to distinguish it from the Latin indices which run from 1 to 4. The basis we use at each point in parameter space is
now given as
\[
\begin{align*}
    |00\rangle &= D^\dagger_2|00\rangle, \\
    |01\rangle &= D^\dagger_1|00\rangle, \\
    |10\rangle &= D^\dagger_1|00\rangle, \\
    |11\rangle &= D^\dagger_2 D^\dagger_1|00\rangle.
\end{align*}
\]

Note that there is a problem with our local basis choice \([3]\) when completing one braid. By definition our \(X_2\) are local and after the braiding is done they’ll end up as \(X_1 \sim X_2\) and \(X_2 \sim X_1\). This means that our local basis \([3]\) does not go back to itself after completing a loop in parameter space. We can correct for that by an explicit basis transformation at the end. This way the unitary transformation, \(U\), due to braiding, factors into local Berry phases, \(U_{\text{local}}\), followed by a global basis transformation, \(B\). We have
\[
U = BU_{\text{local}}.
\]

In principle one could perform a basis transformation that exchanges \(X_1\) and \(X_2\) at an earlier point during the exchange. In that case one still has to make sure that the phases of the states at the end match up exactly with the ones in the beginning, otherwise one needs to do another basis transformation. Therefore it is most convenient to perform everything as a single bases transformation, which includes potential phases, at the end of the process.

In our earlier work \([14]\) we studied \(U_{\text{local}}\). It generally takes the form
\[
U_{\text{local}} = e^{\varphi_+ X_1 \hat{x}_1 + \varphi_\times X_2 \hat{x}_2},
\]
where the \(\varphi_i\)s are calculated as
\[
\varphi_\eta = \frac{1}{2} \int_C \{X_\eta, \nabla_\lambda \hat{x}_\eta\} \cdot d\lambda,
\]
with \(C\) being the path in parameter space during the exchange. We refer to this as mixing by an angle \(\varphi_1\) or \(\varphi_2\) of the Kramers pair 1 and 2 respectively. Note that the integration path is not closed. Therefore a gauge transformation will change (6) according to the initial and final gauge choice. We will discuss this issue in detail later. In this work, we will focus on finding the basis transformation \(B\) in order to calculate (1).

To find the basis transformation we have to determine the final form of \(X_\eta\). The transformation \(B\) is then determined as the transformation that fulfills
\[
BX_\eta\text{final}B^\dagger = X_\eta\text{initial}.
\]

This way \(B\) is, of course, only defined up to an abelian phase, but this phase will be strongly system dependent in any case and we do not consider it. From the definition it is also obvious that \(B\) depends on the gauge choice at the initial and final position. We will discuss this dependence in detail below.

\[
\begin{align*}
\text{initial basis} & \quad \text{exchange} \\
\text{basis transformation} & \quad \text{final basis}
\end{align*}
\]

FIG. 2. Graphical representation of (1). A loop in parameter space is achieved by a continuous exchange followed by a basis transformation. The circles represent Kramers pairs of Majorana fermions. The ambiguity of dividing a circle into two represents the gauge freedom in choosing Kramers partners. Two different lines and colors are used to divide the circles to distinguish the two Kramers pairs.

A pictorial representation of the decomposition (4) is given in figure 2 where the circles represent Kramers pairs and the gauge freedom in the choice of Kramers partners is represented by choice of angle at which the circles are split into two halves.

One might be worried about the transformation properties of equation (4) under gauge transformations. First of all it is clear that the decomposition can only hold as long as we restrict ourselves to local gauge transformations. Under a gauge transformation \(W(\lambda)\) the operator \(U\) should transform as
\[
U \rightarrow W(\lambda_{\text{initial}})UU^\dagger(\lambda_{\text{initial}}),
\]
which means that the transformation \(U\) is basis dependent as one would expect. We explicitly show that this relation holds for our decomposition (4) under local gauge transformations, which have the form
\[
W_{\text{local}} = e^{{f_1(\lambda)x_1}\hat{x}_1 + {f_2(\lambda)x_2}\hat{x}_2}.
\]

This is important, because we will show below that \(B\) and \(U_{\text{local}}\) do not satisfy the same transformation property individually.

We denote \(W_{\text{local}}(\lambda_{\text{initial/final}})\) by \(W_{i/f}\) respectively. It follows directly from \(BX_{\eta\text{final}}B^\dagger = X_{\eta\text{initial}}\) that
\[
W_i B W_i^\dagger W_i X_{\eta\text{final}} W_i^\dagger W_i B W_i^\dagger = W_i X_{\eta\text{initial}} W_i^\dagger.
\]
Hence \(B\) transforms as \(B \rightarrow W_i B W_i^\dagger\). If one replaces \(X_\eta, \hat{x}_\eta\) by \(W_{\text{local}} X_\eta W_{\text{local}}^\dagger, W_{\text{local}} \hat{x}_\eta W_{\text{local}}^\dagger\) in (6) one easily finds that the local phases transform as \(\varphi_\eta \rightarrow \varphi_\eta + (2f_\eta(\lambda_{\text{final}}) - 2f_\eta(\lambda_{\text{initial}}))\), from which it follows that \(U_{\text{local}}\) transforms as \(U_{\text{local}} \rightarrow W_i U_{\text{local}} W_i^\dagger\). Hence neither \(B\) nor \(U_{\text{local}}\) transform according to \(\delta\), in particular
because they depend on the gauge choice at $\lambda_{\text{final}}$. This dependence however drops out of their product, such that $U$ indeed fulfills (8), proving that our decomposition (11) is valid for any local gauge choice.

Before we continue and calculate $B$ let us comment on a particular interesting local gauge choice. This gauge choice is defined by $U_{\text{local}} = 1$, which is always possible. This way one simply has $U = B$, which is useful for numerical calculations, because in this way the calculation of the transformation does not require taking derivatives. A discretization of a path in parameter space only needs to be fine grained enough in order to fix the gauge choice.

To calculate $B$ we have to find $X_{\eta}^{\text{final}}$ and therefore we need to understand how $X_{\eta}$ changes during one switching process (2) when taking $\theta = 0 \rightarrow \frac{\pi}{2}$. We require locality of the zero energy Kramers pairs. Therefore one of the $X_{\eta}$ will stay constant through the switching process (up to parameter dependent local gauge choices). The other of the $X_{\eta}$ will move between sites and by solving $[H, X_{\eta}] = 0$ we find

$$X_{\eta} = e^{i\sigma_{\eta}} \left( \cos \theta e^{i\beta_{ij}\sigma_{\eta}} X_j - \sin \theta e^{i\beta_{ik}\sigma_{\eta}} X_k \right),$$

where $\alpha$ is an arbitrary gauge choice that may depend on the $\beta$s and $\theta$.

We want to patch three switching processes together such that $X_{\eta}$ is continuous. In order to do that conveniently we pick a particular parameter dependence for $\alpha$, such that $X_{\eta}(\theta = 0) = \pm X_j$ and $X_{\eta}(\theta = \frac{\pi}{2}) = \mp X_k$. The sign change is motivated by analogy to switching processes of single Majorana fermions in class D systems. A particular gauge choice of $\alpha$ that gives the desired $X_{\eta}$ at $\theta = 0, \frac{\pi}{2}$ is $\alpha_{\pm} = -\beta_{ij} \cos \theta - \beta_{ik} \sin \theta + (1 \mp 1) \frac{\pi}{2}$.

Table I shows $X_{\eta}$ initially and after each switching process. It has exactly the same structure as for single, class D, MBS. As the transformation from initial to final Kramers pairs of MBS we get

$$X_{\text{initial}} = \chi_1 \rightarrow X_{\text{final}}^{\text{initial}} = \chi_2,$$

$$X_{\text{initial}} = \chi_2 \rightarrow X_{\text{final}}^{\text{final}} = -\chi_1.$$ 

Therefore the basis transformation that generates this transformation is simply

$$B = e^{\frac{\pi}{4}(\chi_1 \chi_2 + \chi_1 \chi_2)},$$

which is structurally the same as two independent braiding transformations of pairs of MBS. According to (11) the total braiding transformation is

$$U = e^{\frac{\pi}{4}(\chi_1 \chi_2 + \chi_1 \chi_2)} e^{\frac{\pi}{4}\chi_1 \chi_1 + \frac{\pi}{4} \chi_2 \chi_2}.$$ 

The question arises how general this transformation is, because we obtained it in the context of our simple toy model. From the construction it is clear that we can always choose a parameter dependent basis such that Eqs. 11 and 12 are true. Therefore the question reduces to whether $U_{\text{local}}$ can be more complicated than (9). Since we assume that both Kramers pairs of Majorana fermions are always decoupled, the local parity operators $i X_{\eta} \tilde{X}_{\eta}$ are conserved quantities. The only local transformations which we can construct out of those are given by $U_{\text{local}}$. Hence our toy model already describes the most general possible braiding transformation for Kramers pairs of MBS.

Because the phases in Eq. (5) are path dependent, the exchange of Kramers pairs of MBS cannot be used for topological quantum computation. The natural question arises whether there are additional conditions under which the phases become path independent. We will show that this is the case in the absence of local mixing. By local mixing we mean the transformations that can arise when changing the parameters along any closed trajectory in parameter space that does not braid $X_1, X_2$. The transformation describing this process will simply be $U = U_{\text{local}}$, with the difference that the integral in Eq. 9 is over a closed trajectory in this case. The absence of local mixing means that $\varphi_\eta$ is zero for any such closed loop. This result is useful because the presence or absence of local mixing is a property of individual Kramers pairs of MBS, but we can use it to make a statement about braiding of several Kramers pairs.

Before we discuss the implications of the presence or absence of local mixing, note that if the phases in Eq. (5) are path independent, then there cannot be local mixing, because otherwise we could always add a closed path with non-vanishing mixing phase to the open path in Eq. (9), violating the path independence of the phase. The absence of local mixing is therefore necessary in order to have constant phases in Eq. (5). We now proceed to show that it is also sufficient and moreover that in the absence of local mixing the braiding transformation can be reduced to two independent braidings of $\chi_1, \chi_2$ and $\chi_1, \chi_2$ respectively.

We first show that $\varphi_\eta$ are path independent. Assume we have two braiding paths 1 and 2 such that the mixing angles along those paths are $\varphi_\eta$ and $\bar{\varphi}_\eta$ respectively. The braiding transformations are given by $U$ and $\bar{U}$. We can now form a local mixing operation $\bar{U} U$ with mixing angles $\varphi_\eta - \bar{\varphi}_\eta$, but since there is no local mixing by assumption we have $\varphi_\eta - \bar{\varphi}_\eta = 0$. Therefore the $\varphi_\eta$ are path independent. The situation is illustrated in figure 3.

We now argue that $\varphi_1 = -\varphi_2$. In our simple toy model this can be checked by means of a straightforward calcu-
a) $$U : \quad \begin{array}{c} 1 \\ \rightarrow \\ 2 \end{array} \quad \varphi_2 - \varphi_2 = 0$$

$$\bar{U} \bar{U} : \quad \begin{array}{c} 1 \\ \rightarrow \\ 2 \end{array} \quad \varphi_1 - \bar{\varphi}_1 = 0$$

b) $$\begin{array}{c} 1 \\ \rightarrow \\ 2 \\ \rightarrow \\ 1 \begin{array}{c} 1 \\ \rightarrow \\ 2 \end{array} \quad \varphi_1 \end{array} \quad \varphi_2$$

FIG. 3. (a) Two exchanges along different paths can be combined into two local operation (solid and dashed line). Consequently the mixing angles are path independent if local mixing angles are zero. (b) One Kramers pair gets moved out of the way for the other to make a loop. During the first half pair 1 rotates by $$\varphi_1$$ and during the second half it rotates by $$\varphi_2$$. If there is no local mixing this implies $$\varphi_1 + \varphi_2 = 0$$.

It should be noted that a symmetry condition (or some other kind of condition) that guarantees the absence of local mixing does not need to be global. It only needs to be true locally around the Kramers pair. That means that one could try to engineer a system such that the symmetry condition is approximately satisfied close to the topological phase boundaries, where the Kramers pairs of MBS are localized. If this remains true as the phase boundary and hence the Kramers pair moves, than one can get path independent braiding statistics.

Even with the ideal transformation of Eq. (14) the question remains whether the braiding can yield something more interesting then moving localized fermions around. The answer is yes, but it would require schemes that initialize and read out non-local states. One can for instance think of bringing two Kramers pairs of MBSs close together such that they split in energy and one can initialize them in the lowest energy state of the split states, after which one brings them apart again. Operations on those states can then be performed similarly to the operations performed on class D Majorana qubits. The non-local states in class DIII states are actually entangled with respect to the local bases. However, local mixing will decohere the superpositions and therefore absence of local mixing is crucial for any operations involving entanglement in the local basis.

In conclusion, we derived the most general braiding transformation for Kramers pairs of Majorana bound states. This transformation naturally decomposes into a transformation that describes the independent exchange of two pairs of MBS and additional local rotations of the Kramers pairs that are being exchanged. The angles of those local rotations are generally path dependent and independent for the two Kramers pairs. The fact that the angles are path dependent shows again that Kramers pairs of MBS cannot be used for topological quantum computation. The derived form of the transformation is the most general one allowed by global and local parity conservation and therefore the result is not limited to the simple toy model, which we used to study it. We also defined the concept of local mixing, and showed that whether or not the phases are path dependent is equivalent to the presence or absence of local mixing. We also argued that in the absence of local mixing the exchange transformation reduces to two independent exchange of pairs of MBS without any path dependence. This is helpful because it might be easier to check/engineer the presence or absence of local mixing, because it is a local property.

We thank P. Brouwer for useful discussions. The Center for Quantum Devices is funded by the Danish National Research Foundation. The research was supported by The Danish Council for Independent Research | Natural Sciences. A. S. acknowledges support from the European Research Council (ERC), Minerva foundation, the U.S.-Israel BSF, and Microsofts Station Q.
[1] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Reviews of Modern Physics 80, 1083 (2008).
[2] C. Beenakker, Annual Review of Condensed Matter Physics 4, 113 (2013).
[3] M. Leijnse and K. Flensberg, Semiconductor Science and Technology 27, 124003 (2012).
[4] J. Alicea, Physical Review B 81, 125318 (2010).
[5] C. L. M. Wong and K. T. Law, Physical Review B 86, 184516 (2012).
[6] F. Zhang, C. L. Kane, and E. J. Mele, Physical Review Letters 111, 056402 (2013).
[7] S. Nakosai, J. C. Budich, Y. Tanaka, B. Trauzettel, and N. Nagaosa, Physical Review Letters 110, 117002 (2013).
[8] S. Nakosai, Y. Tanaka, and N. Nagaosa, Physical Review Letters 108, 147003 (2012).
[9] S. Deng, L. Viola, and G. Ortiz, Physical Review Letters 108, 036803 (2012).
[10] A. Keselman, L. Fu, A. Stern, and E. Berg, Physical Review Letters 111, 116402 (2013).
[11] E. Gaidamauskas, J. Paaske, and K. Flensberg, Physical Review Letters 112, 126402 (2014).
[12] A. Haim, A. Keselman, E. Berg, and Y. Oreg, Physical Review B 89, 220504 (2014).
[13] J. Danon and K. Flensberg, Physical Review B 91, 165245 (2015).
[14] K. Wöllms, A. Stern, and K. Flensberg, Physical Review Letters 113, 246401 (2014).
[15] J. D. Sau, D. J. Clarke, and S. Tewari, Physical Review B 84, 094505 (2011).