I. INTRODUCTION

Bayesian image restoration, in which restore images degraded by noise are restored using the Bayesian framework, is an important generic technique for various kinds of signal processing. Suppose that there is an original image expressed by $I$ and that the original image is degraded to $h$ through a specific noise process expressed as $P(h \mid I)$. We observe only the degraded image. The purpose of the image restoration is to estimate the original image from the degraded image. In the Bayesian image restoration system, we compute the posterior distribution of the original image, $P(I \mid h) \propto P(h \mid I)P(I)$, where $P(I)$ is known as the prior distribution, which is designed by a specific model such as Markov random field (MRF), and use it to estimate the original image. The scheme of Bayesian image restoration is shown in figure 1.

Bayesian image restoration was developed in statistical mechanics as in engineering. In the statistical mechanics point of view, the standard framework of Bayesian image restoration is almost equivalent to the framework of a two-dimensional ferromagnetic spin model in random fields. Since, unfortunately, true statistical quantities cannot be obtained in the model used in the Bayesian image restoration system, an approximate technique must be employed to implement it. Loopy belief propagation (LBP), which is known as the Bethe approximation in statistical mechanics, is often used for this purpose.

For practical uses of the Bayesian image restoration system, it is important to evaluate the statistical performance of the implemented system, to achieve which we have to compute the quenched average of LBP over random fields. For this purpose, for Ising systems, the authors proposed an analytical evaluation method. In the previous method, the evaluation of the quenched average is reduced to solving simultaneous integral equations with respect to the distributions of messages. However, the method is not very practical, because the computational cost of solving the integral equations is considerable and its approximation accuracy is not very high. Furthermore, the method cannot evaluate the quenched average of the Bethe free energy and is formulated only in Ising systems.

In this paper, we propose a new analytic method of evaluating the quenched average of LBP over random fields based on the idea of the replica cluster variation method (RCVM). The presented method allows the quenched average of the Bethe free energies for general pair-wise MRFs to be evaluated, unlike the previous method. The remaining part of this paper is organized as follows. A brief explanation of LBP is given in section 1. The subsequent sections constitute the main part of the paper.
the proposed method is shown in section IV and some numerical results for checking its validity are shown in section V. In section VI we show a case that is exactly solvable by the present method. In section VII we compute the statistical performance of the Bayesian image restoration system using the proposed method. Finally, section VIII closes the paper with concluding remarks.

II. LOOPY BELIEF PROPAGATION IN RANDOM FIELDS

Let us consider undirected graph \( G(V, E) \) consisting of \( n \) vertices and some edges, where \( V = \{1, 2, \ldots, n\} \) is the set of vertices and \( E = \{(i, j)\} \) is the set of edges between a pair of vertices, where \( \{i, j\} \) denotes the undirected edges between vertices \( i \) and \( j \). On the graph, with discrete random variables \( S \in \{S_i \mid i \in V\} \), let us define the pair-wise Markov random field (MRF) expressed by

\[
P(S | h, \beta) := \frac{1}{Z(h, \beta)} \exp \left( \sum_{i \in V} \phi_i(S_i, h_i) + \sum_{\{i, j\} \in E} \psi_{i,j}(S_i, S_j) \right),
\]

where \( \phi_i(S_i, h_i) \) is a specific function of \( S_i \) and external field \( h_i \) on vertex \( i \) and \( \phi_{i,j}(S_i, S_j) \) is a specific function on edge \( \{i, j\} \). The notations \( Z \) and \( \beta \) are the partition function and the inverse temperature, which takes a positive value, respectively. The external fields directly correspond to the degraded images in the Bayesian image restoration described in section VII.

It is known that LBP is derived from the minimum condition of the Bethe free energy \( F_{\text{bethe}} \). According to the cluster variation method (CVM) \cite{11,12}, the Bethe free energy for the MRF in equation (1) is expressed by \( F_{\text{bethe}} \)

\[
F_{\text{bethe}}(h, \beta) := -\sum_{i \in V} \sum_{S_i} \phi_i(S_i, h_i) b_i(S_i) - \sum_{\{i, j\} \in E} \psi_{i,j}(S_i, S_j) b_{i,j}(S_i, S_j) + \frac{1}{\beta} \sum_{i \in V} H_1[b_i] + \frac{1}{\beta} \sum_{\{i, j\} \in E} (H_2[b_{i,j}] - H_1[b_i] - H_1[b_j]),
\]

where

\[
H_1[b_i] := \sum_{S_i} b_i(S_i) \ln b_i(S_i)
\]

and

\[
H_2[b_{i,j}] := \sum_{S_i, S_j} b_{i,j}(S_i, S_j) \ln b_{i,j}(S_i, S_j)
\]

are the one-vertex and the two-vertices negative entropies, respectively. Beliefs \( b_i(S_i) \) and \( b_{i,j}(S_i, S_j) \) are the one-vertex and the two-vertices marginal distributions, respectively, which are computed by

\[
b_i(S_i) \propto \exp \left( \beta \phi_i(S_i, h_i) \right) \prod_{j \in \partial i} M_{j \rightarrow i}(S_i),
\]

\[
b_{i,j}(S_i, S_j) \propto \exp \left( \beta \phi_i(S_i, h_i) + \phi_j(S_j, h_j) + \psi_{i,j}(S_i, S_j) \right) \prod_{k \in \partial\{i,j\}} M_{k \rightarrow i}(S_i) \prod_{l \in \partial\{i,j\}} M_{l \rightarrow j}(S_j) \prod_{M_{i \rightarrow j}} M_{i \rightarrow j}(S_i)^{-1} M_{i \rightarrow j}(S_j)^{-1}
\]

where \( \partial i \) is the set of vertices connected to vertex \( i \): \( \partial i = \{j \mid j \in V, \{i, j\} \in E\} \). Quantity \( M_{i \rightarrow j}(S_i) \) is the message (or the effective field) from vertex \( i \) to vertex \( j \), which is the solution of the message-passing equation:

\[
M_{i \rightarrow j}(S_j) \propto \sum_{S_i} \exp \left( \beta \phi_i(S_i, h_i) + \psi_{i,j}(S_i, S_j) \right) \prod_{k \in \partial\{i,j\}} M_{k \rightarrow i}(S_i) \prod_{l \in \partial\{i,j\}} M_{l \rightarrow j}(S_j)
\]

The main proposal presented in the paper is a method for evaluating the quenched average of the Bethe free energy over the random fields:

\[
[F_{\text{bethe}}]_h = \int dh \prod_{i \in V} p_i(h_i) F_{\text{bethe}}(h, \beta),
\]

where notation \([\cdots]_h\) represents the average value over the random fields and \( p_i(h_i) \) is the distribution of the external field on vertex \( i \) and these distributions vary by vertex in general.

III. PROPOSED METHOD

According to equation (6), in principle, we have to perform the averaging operation after constructing the Bethe free energy to obtain \([F_{\text{bethe}}]_h\) (see figure 2(a)). However, it is not straightforward to directly integrate the Bethe free energy. Thus, we adopt another strategy. In this paper, we propose an approximate method based on the idea of RCVM \cite{B2}. Figure 2(b) shows the procedure of our method. In the method, we first take the average of the free energy using the replica method and the CVM, namely, RCVM (section III A), and then, we apply the Bethe approximation to the resulting form of the RCVM (section III B).

A. Replica Cluster Variation Method

First, we obtain the quenched average of the true free energy of the MRF in equation (1), that is

\[
[F]_h = -\frac{1}{\beta} \int dh \prod_{i \in V} p_i(h_i) \ln Z(h, \beta).
\]
In the context of the replica method \[2, 13\], we have

\[
[F]_h = -\frac{1}{\beta} \lim_{x \to \infty} \frac{Z_x - 1}{x},
\]

where

\[
Z_x := \int dh \prod_{i \in V} p_i(h_i) Z(h, \beta)^x.
\]

By assuming \( x \) is a natural number, we obtain

\[
Z_x = \sum_{S_x} \exp \beta \left( \sum_{i \in V} e_i(S_i) + \sum_{\{i,j\} \in E} x \sum_{\alpha=1}^x \psi_{i,j}(S_i^\alpha, S_j^\alpha) \right),
\]

where \( S_x = \{ S_i^\alpha | i \in V, \alpha = 1, 2, \ldots, x \} \), \( S_i = \{ S_i^\alpha | \alpha = 1, 2, \ldots, x \} \), and function \( e_i(S_i) \) is defined by

\[
e_i(S_i) := \frac{1}{\beta} \ln \int dh_p(h) \exp \left( \beta \sum_{\alpha=1}^x \phi_i(S_i^\alpha, h) \right).
\]

We regard \( Z_x \) as the partition of \( x \)-replicated system and define the \( x \)-replicated free energy as

\[
F_x := -\frac{1}{\beta} \ln Z_x = -\sum_{i \in V} \sum_{S_i} e_i(S_i) Q_i(S_i) + \sum_{\alpha=1}^x \sum_{S^\alpha} H_{\text{int}}(S^\alpha) Q^\alpha(S^\alpha) + \frac{1}{\beta} \sum_{S_x} P_x(S_x) \ln P_x(S_x),
\]

where

\[
P_x(S_x) := \frac{1}{Z_x} \exp \beta \left( \sum_{i \in V} e_i(S_i) - \sum_{\alpha=1}^x H_{\text{int}}(S^\alpha) \right)
\]

is the Gibbs distribution of the \( x \)-replicated system, and \( S^\alpha = \{ S_i^\alpha | i \in V \} \). Energy function \( H_{\text{int}}(S) \) is defined by

\[
H_{\text{int}}(S) := -\sum_{\{i,j\} \in E} \psi_{i,j}(S_i, S_j),
\]

and is the interaction term of the original system. The distributions, \( Q_i(S_i) \) and \( Q^\alpha(S^\alpha) \), are the marginal distributions of distribution \( P_x(S_x) \). The factor graph representation of the \( x \)-replicated system is shown in figure 3(a).

In accordance with the cluster decomposition based on the CVM shown in figure 3(b), by using the marginal distributions, \( \{ Q_i(S_i), Q^\alpha(S^\alpha) \} \), together with one-variable marginal distributions, \( \{ Q^\alpha_i(S^\alpha_i) \} \), we approximate the Gibbs distribution of the \( x \)-replicated system as

\[
P_x(S_x) \approx \frac{\prod_{i \in V} Q_i(S_i) \prod_{\alpha=1}^x Q^\alpha_i(S^\alpha_i)}{\prod_{i \in V} \prod_{\alpha=1}^x Q^\alpha_i(S^\alpha_i)}. \tag{9}
\]

By applying this approximation to \( P_x(S_x) \) in the logarithmic function in the last term of equation (8), we obtain the expression of the variational free energy as

\[
F_x^{RCVM}[\{ Q_i, Q^\alpha \}] := \sum_{i \in V} \mathcal{V}_i[Q_i] + \sum_{\alpha=1}^x \mathcal{V}_{\text{int}}[Q^\alpha] - \frac{1}{\beta} \sum_{i \in V} \sum_{\alpha=1}^x H_i[Q^\alpha_i], \tag{10}
\]

where the functionals, \( \mathcal{V}_i[Q_i] \) and \( \mathcal{V}_{\text{int}}[Q^\alpha] \), are defined as

\[
\mathcal{V}_i[Q_i] := -\sum_{S_i} e_i(S_i) Q_i(S_i) + \frac{1}{\beta} \sum_{S^\alpha} H_{\text{int}}(S^\alpha) Q^\alpha_i(S^\alpha_i) \ln Q_i(S_i), \tag{11}
\]

\[
\mathcal{V}_{\text{int}}[Q^\alpha] := \sum_{S^\alpha} H_{\text{int}}(S^\alpha) Q^\alpha(S^\alpha) \ln Q^\alpha_i(S^\alpha_i). \tag{12}
\]

In the context of the CVM, the \( x \)-replicated free energy in equation (8) is approximated by the minimum point of the variational free energy in equation (10) with respect to marginal distributions \( \{ Q_i, Q^\alpha_i \} \), i.e.,

\[
F_x \approx \min_{\{ Q_i, Q^\alpha_i \}} F_x^{RCVM}[\{ Q_i, Q^\alpha_i, Q^\alpha_i \}]. \tag{14}
\]

It should be noted that, at the minimum point, the normalization constraints for \( \{ Q_i, Q^\alpha_i, Q^\alpha_i \} \) and the marginal constraints,

\[
Q^\alpha_i(S^\alpha_i) = \sum_{S_i \setminus \{ S^\alpha_i \}} Q_i(S_i), \tag{13}
\]
and
\[ Q^\alpha_i(S^\alpha_i) = \sum_{S^\alpha \setminus S^\alpha_i} Q^\alpha(S^\alpha), \]  
(14)

should hold.

In order to minimize the variational free energy with respect to \( \{Q_i(S_i)\} \), by using the Lagrange multipliers, we variate

\[ \mathcal{L}(\{Q_i\}) := \sum_{i \in V} V_i[Q_i] - \sum_{i \in V} \lambda_i \left( \sum_{S_i} Q_i(S_i) - 1 \right) \]

with respect to \( \{Q_i(S_i)\} \). From the result of this variation, we obtain

\[ Q_i(S_i) \propto \exp \left( c_i(S_i) + \sum_{\alpha=1}^x \Lambda_{i}^\alpha(S_i^\alpha) \right). \]  
(15)

The Lagrange multipliers, \( \Lambda_i = \{\Lambda_i^\alpha(S_i) \mid \alpha = 1, 2, \ldots, x\} \), are determined such that they satisfy equation (13). By substituting equation (15) into equation (11), while noting the marginal constraints in equation (13), we obtain the partially minimized variational free energy as

\[ \mathcal{F}_{\text{RCVM}}^{\text{Bethe}}[[Q^\alpha, Q^\alpha]] := \min_{\{Q_i\}} \mathcal{F}_{\text{RCVM}}[[Q_i, Q^\alpha, Q^\alpha]] \]

\[ = \sum_{i \in V} \text{extr} \left\{ \sum_{\alpha=1}^x \sum_{S^\alpha_i} \Lambda_i^\alpha(S_i^\alpha) Q^\alpha_i(S_i^\alpha) \right. \]

\[ - \frac{1}{\beta} \ln \int dh p_i(h) \prod_{\alpha=1}^x \exp \left( \phi_i(S_i^\alpha), h \right) \]

\[ + \Lambda_i^\alpha(S_i^\alpha) \left\} + \sum_{\alpha=1}^x \mathcal{V}_{\text{intr}}[Q^\alpha] - \frac{1}{\beta} \sum_{i \in V} \sum_{\alpha=1}^x H_i[Q^\alpha], \]  
(16)

where the notation “extr” denotes the extremum with respect to the assigned parameters.

**B. Bethe Approximation**

Functional \( \mathcal{V}_{\text{intr}}[Q^\alpha] \) can be interpreted as the variational free energy for the interaction term of the original system (see equation (12)). Since this variational free energy is the intractable functional in general, we approximate the variational free energy by the Bethe free energy. To do this, we approximate distribution \( Q^\alpha(S^\alpha) \) by 6

\[ Q^\alpha(S^\alpha) \approx \prod_{i \in V} Q^\alpha_i(S^\alpha_i) \prod_{\{i,j\} \in E} \frac{Q^\alpha_{ij}(S^\alpha_i, S^\alpha_j)}{Q^\alpha_{ij}(S^\alpha_i, S^\alpha_j)}, \]  
(17)

where the distributions, \( Q^\alpha_i(S^\alpha_i) \) and \( Q^\alpha_{ij}(S^\alpha_i, S^\alpha_j) \), are the marginal distributions of distribution \( Q^\alpha(S^\alpha) \), so that the marginal constraints,

\[ \sum_{S_i} Q^\alpha_{ij}(S^\alpha_i, S^\alpha_j) = Q^\alpha_j(S^\alpha_j) \]

and

\[ \sum_{S_j} Q^\alpha_{ij}(S^\alpha_i, S^\alpha_j) = Q^\alpha_i(S^\alpha_i), \]

are satisfied. By applying equation (17) to \( Q^\alpha(S^\alpha) \) in the logarithmic function in the last term of equation (12), we obtain the approximation, that is

\[ \mathcal{V}_{\text{intr}}[Q^\alpha] \approx \mathcal{V}_{\text{bethe}}[Q^\alpha] \]

\[ := - \sum_{\{i,j\} \in E} \mathcal{V}_{\text{bethe}}[Q^\alpha_{ij}] \]

\[ = - \frac{1}{\beta} \ln \left[ \mathcal{V}_{\text{intr}}[Q^\alpha] - \frac{1}{\beta} \sum_{i \in V} \sum_{\alpha=1}^x H_i[Q^\alpha] \right]. \]  
(18)

**C. Replica Symmetric Ansatz**

At the minimum point of the variational free energy, \( \mathcal{F}_{\text{Bethe}}[Q^\alpha, Q^\alpha] \), we make the replica symmetric (RS) assumption 2-13, that is, the relations \( Q^\alpha_i(S_i) = Q_i(S_i) \) and \( Q^\alpha_{ij}(S_i, S_j) = Q_{ij}(S_i, S_j) \) hold for any \( \alpha \) at the minimum point. Under this assumption, the minimum point of \( \mathcal{F}_{\text{Bethe}}[Q^\alpha, Q^\alpha] \) is equivalent to the minimum point of the RS variational free energy expressed as

\[ \mathcal{F}_{\text{Bethe}}^{\text{RS}}[[Q_i, Q_{ij}]] := \sum_{i \in V} \text{extr} \left\{ \sum_{\alpha=1}^x \sum_{S^\alpha_i} \Lambda_i^\alpha(S_i^\alpha) Q_i(S_i^\alpha) \right. \]

\[ - \frac{1}{\beta} \ln \int dh p_i(h) \prod_{\alpha=1}^x \exp \left( \phi_i(S_i^\alpha), h \right) \]

\[ + \mathcal{V}_{\text{bethe}}[Q^\alpha] \]  
(19)

From the convexity of the first term of the above equation with respect to \( \Lambda_i \) and the extremal conditions of \( \Lambda_i \), it
can be ensured that relation $\Lambda^a(S_i) = \Lambda_i(S_i)$ holds for any $a$ at the extremal points of $\Lambda_i$. Therefore, equation (23) can be reduced to

$$F^LBP(RS)[\{Q_i, Q_{i,j}\}] = \sum_{i \in V} \text{extr} \left\{ x \sum_{S_i} \Lambda_i(S_i) Q_i(S_i) \right\} - \frac{1}{\beta} \int dh p_i(h) \ln \sum_{S_i} \exp \left( \beta \phi_i(S_i, h) + \Lambda_i(S_i) \right)$$

$$+ x \psi_{\text{int}}[\{Q_i, Q_{i,j}\}] - \frac{x}{\beta} \sum_{i \in V} H_1[Q_i],$$  \hspace{1cm} (20)

and we regard the minimum point of this variational free energy as the approximation of the true $x$-replicated free energy in equation (8).

From equations (7), (8), and (20), we obtain the variational free energy at the limit of $x \to 0$ as

$$F^LBP(RS)[\{Q_i, Q_{i,j}\}] := \sum_{i \in V} \text{extr} \left\{ x \sum_{S_i} \Lambda_i(S_i) Q_i(S_i) \right\}$$

$$- \frac{1}{\beta} \int dh p_i(h) \ln \sum_{S_i} \exp \left( \beta \phi_i(S_i, h) + \Lambda_i(S_i) \right)$$

$$- \sum_{\{i,j\} \in E} \sum_{S_i, S_j} \psi_{i,j}(S_i, S_j) Q_{i,j}(S_i, S_j)$$

$$+ \frac{1}{\beta} \sum_{\{i,j\} \in E} (H_2[Q_{i,j}] - H_1[Q_i] - H_1[Q_j]).$$  \hspace{1cm} (21)

We expect that the minimum point of this variational free energy is the approximation of the quenched average of the Bethe free energy in equation (6):

$$[F_{\text{bethe}}]_h \approx F^LBP(RS) := \min_{\{Q_i, Q_{i,j}\}} F^LBP(RS)[\{Q_i, Q_{i,j}\}].$$

It should be noted that at the minimum of the variational free energy, $F^LBP(RS)[\{Q_i, Q_{i,j}\}]$, the normalization constraints,

$$\sum_{S_i} Q_i(S_i) = \sum_{S_i, S_j} Q_{i,j}(S_i, S_j) = 1,$$  \hspace{1cm} (22)

and the marginal constraints,

$$\sum_{S_i} Q_{i,j}(S_i, S_j) = Q_j(S_j)$$  \hspace{1cm} (23)

and

$$\sum_{S_j} Q_{i,j}(S_i, S_j) = Q_i(S_i),$$  \hspace{1cm} (24)

should hold.

### D. Message-passing Equation

In this section, we show the message-passing equation for minimizing the variational free energy in equation (21) obtained in the previous section.

In order to perform the conditional minimization of the variational free energy, we use the Lagrange multipliers as

$$\mathcal{L}^LBP(RS)[\{Q_i, Q_{i,j}\}] := F^LBP(RS)[\{Q_i, Q_{i,j}\}] - \sum_{i \in V} a_i \left( \sum_{S_i} Q_i(S_i) - 1 \right)$$

$$- \sum_{\{i,j\} \in E} \sum_{S_i, S_j} \gamma_{i,j}(S_i) \left( \sum_{S_j} Q_{i,j}(S_i, S_j) - Q_i(S_i) \right) + \sum_{S_j} \gamma_{i,j}(S_j) \left( \sum_{S_i} Q_{i,j}(S_i, S_j) - Q_j(S_j) \right),$$

where the Lagrange multipliers, $\{a_i, b_{i,j}\}$ and $\{\gamma_{i,j}(S_j)\}$, correspond to the normalization constraints in equation (22) and the marginal constraints in equations (23) and (24), respectively. From the minimum conditions of $\mathcal{L}^LBP(RS)[\{Q_i, Q_{i,j}\}]$ with respect to $Q_i(S_i)$ and $Q_{i,j}(S_i, S_j)$, we obtain

$$Q_i(S_i) \propto \exp \left[ \frac{\beta}{|\partial_i|} \left( \Lambda_i(S_i) + \sum_{k \in \partial_i} \gamma_{k,i}(S_i) \right) \right]$$  \hspace{1cm} (25)

and

$$Q_{i,j}(S_i, S_j) \propto \exp \left[ \beta \left( \psi_{i,j}(S_i, S_j) + \gamma_{i,j}(S_j) + \gamma_{j,i}(S_i) \right) \right].$$  \hspace{1cm} (26)

respectively, where the notation $|\cdots|$ denotes the number of entries of the assigned set. By introducing the messages as the form,

$$M_{j \rightarrow i}(S_i) := \exp \left\{ \frac{\beta}{|\partial_i| - 1} \left( \Lambda_i(S_i) - \beta^{-1} \ln Q_i(S_i) \right) + \sum_{j \in \partial i} \gamma_{k,i}(S_i) - \beta \gamma_{j,i}(S_i) \right\},$$

we obtain

$$\exp \left( \beta \gamma_{j,i}(S_i) \right) = Q_i(S_i) \exp \left( - \beta \Lambda_i(S_i) \right)$$

and
From equations (26) and (27), we obtain the relation
\[ \beta \Lambda_i(S_i) = \sum_{k \in \partial_i \setminus \{i\}} \ln \mathcal{M}_{k \rightarrow i}(S_i) + c, \tag{28} \]
where \( c \) is a constant unrelated to \( S_i \).

By substituting equations (25) and (26) into the marginal constraints in equations (24) and (24), together with equation (27), we obtain the message-passing equation as
\[ \mathcal{M}_{j \rightarrow i}(S_i) \propto \sum_{S_j} Q_j(S_j) \exp \left( -\Lambda_j(S_j) + \psi_{i,j}(S_i, S_j) \right) \times \prod_{k \in \partial_j \setminus \{j\}} \mathcal{M}_{k \rightarrow j}(S_j) \times \prod_{k \in \partial_j \setminus \{j\}} \mathcal{M}_{k \rightarrow j}(S_j) \]
\[ \propto \sum_{S_j} Q_j(S_j) \exp \left( \beta \psi_{i,j}(S_i, S_j) \right) \mathcal{M}_{i \rightarrow j}(S_j)^{-1}. \tag{29} \]

Here, from the first line to the second line in this equation, we use the relation in equation (28). From the extremal conditions for \( \{ \Lambda_i(S_i) \} \) in the first term in equation (21), we obtain
\[ Q_i(S_i) = \int dh p_i(h) \frac{\exp \left( \beta \phi_i(S_i, h) + \Lambda_i(S_i) \right)}{\sum_{S_i} \exp \left( \beta \phi_i(S_i, h) + \Lambda_i(S_i) \right)}. \tag{30} \]

From equations (26)–(28), two-vertices marginal distributions \( \{ Q_{i,j}(S_i, S_j) \} \) can be expressed as
\[ Q_{i,j}(S_i, S_j) \propto Q_i(S_i) Q_j(S_j) \exp \left( \beta \psi_{i,j}(S_i, S_j) \right) \times \mathcal{M}_{j \rightarrow i}(S_i)^{-1} \mathcal{M}_{i \rightarrow j}(S_j)^{-1}. \tag{31} \]

After numerically solving simultaneous equation (28)–(31), by substituting the values of \( Q_i(S_i), Q_{i,j}(S_i, S_j), \) and \( \Lambda_i(S_i) \) into equation (21), we obtain the minimum values of the variational free energy, \( \mathcal{F}_{\text{LBP}}^{\text{RS}} \), and regard it as the approximation of \( [F_{\text{bethe}}]_h \).

For the moment, we suppose that function \( \phi_i(S_i, h_i) \) can be divided as \( \phi_i(S_i, h_i) = \phi_i^{(0)}(S_i, h_i) + \phi_i^{(1)}(S_i) \). The variations in the quenched average of the Bethe free energy in equation (6) with respect to \( \phi_i^{(1)}(S_i) \) and \( \psi_{i,j}(S_i, S_j) \) are
\[ \delta \frac{\mathcal{F}_{\text{LBP}}^{\text{RS}}(S_i)}{\phi_i^{(1)}(S_i)} |_{[F_{\text{bethe}}]_h} = -[b_i(S_i)]_h \tag{32} \]
and
\[ \delta \frac{\mathcal{F}_{\text{LBP}}^{\text{RS}}(S_i, S_j)}{[F_{\text{bethe}}]_h} = -[b_{i,j}(S_i, S_j)]_h, \tag{33} \]
respectively, which are the quenched average of beliefs obtained from the LBP. On the other hand, the variations of \( \mathcal{F}_{\text{LBP}}^{\text{RS}} \) with respect to \( \phi_i^{(1)}(S_i) \) and \( \psi_{i,j}(S_i, S_j) \) are obtained as
\[ \frac{\delta}{\delta \phi_i^{(1)}(S_i)} \mathcal{F}_{\text{LBP}}^{\text{RS}}(S_i) = -Q_i(S_i) \tag{34} \]
and
\[ \frac{\delta}{\delta \psi_{i,j}(S_i, S_j)} \mathcal{F}_{\text{LBP}}^{\text{RS}}(S_i, S_j) = -Q_{i,j}(S_i, S_j), \tag{35} \]
respectively. By comparing equations (32) and (33) with equations (34) and (35), it can be expected that, if \( \mathcal{F}_{\text{min}}^{\text{LBP}}(S_i, S_j) \) is a good approximation of \( [F_{\text{bethe}}]_h \), the marginal distributions, \( Q_i(S_i) \) and \( Q_{i,j}(S_i, S_j) \), are also good approximations of the quenched averages of the beliefs, \( [b_i(S_i)]_h \) and \( [b_{i,j}(S_i, S_j)]_h \), respectively.

### E. Numerical Experiment

In this section, we describe the evaluation of the validity of our method by using the numerical experiments. In the experiment, we used the model expressed as
\[ P(S \mid h) = \frac{1}{Z(h)} \exp \left( \sum_{i \in V} h_i S_i + J \sum_{(i,j) \in E} S_i S_j \right), \tag{36} \]
which is defined on a graph of \( 8 \times 8 \) square lattice with the free boundary condition, where \( S_i \) takes \( q \) values as \( S_i \in \{ 2S/(q-1) - 1 \mid S = 0, 1, 2, \ldots, q-1 \} \). Fields \( h \) are i.i.d. random fields drawn from the Gaussian distribution with the mean of zero and variance of \( \sigma^2 \), \( \mathcal{N}(h \mid 0, \sigma^2) \). We compared the free energy per variable obtained by our method, \( \mathcal{F}_{\text{LBP}}^{\text{RS}}/n \), with the quenched average of the Bethe free energy (per variable) shown in equation (6), which was obtained by numerically averaging the Bethe free energy in equation (2) over 10000 realizations of the random fields.

Figures 4–6 shows the plots for \( q = 2, 3 \), and \( q = 4 \). “LBP” represents the results obtained by numerically averaged Bethe free energy and “RLBP” represents the results obtained by our method. In almost all cases the results of our method are consistent with the numerically averaged free energies, as expected. However, in the cases of large \( J \), mismatches between the two methods are observed.

Figure 7 shows the plot of the quenched magnetizations of the model shown in equation (36), which is defined on a graph of \( 14 \times 14 \) square lattice with the periodic boundary condition. We observe that the two methods show the different nature of the magnetizations. The magnetization obtained by the standard LBP slowly rises with the increase in \( J \), whereas, that obtained by the proposed method drastically rises, like the first-order transition, around \( J \approx 0.88 \). This different physical picture probably causes the mismatches between the two methods in the cases of large \( J \) in figures 4–6.
F. Exactly Solvable Case – Ferromagnetic Mean-field Model in Random Fields

In this section, we consider the ferromagnetic mean-field model in random fields expressed as

$$ P(S | h, \beta) \propto \exp \beta \left( \sum_{i \in V} \phi(S_i, h_i) + \frac{1}{n} \sum_{i < j} g(S_i)g(S_j) \right) $$

where the second summation represents the summation taken over all the distinct pairs of vertices and \{h_i\} are i.i.d. random fields drawn from a distribution \( p_h(h_i) \). By using the Hubbard-Stratonovich transformation, the free energy (per variable) of this model can be expressed as

$$ f = \frac{1}{n\beta} \ln \int dm \exp \left\{ -\frac{\beta}{2} m^2 \right\} $$

over \( i \in V \) in the exponent by using the law of large numbers, so that the free energy can be expressed as

$$ f = \frac{\beta}{2} m^2 - \frac{1}{\beta} \int dh p_h(h) \ln \sum_S \exp \beta \left( \phi(S, h) + mg(S) \right) $$

at the thermodynamical limit, where \( m \) is the solution of the saddle point equation and satisfies

$$ m = \int dh p_h(h) \frac{\sum_S g(S) \exp \beta \left( \phi(S, h) + mg(S) \right)}{\sum_S \exp \beta \left( \phi(S, h) + mg(S) \right)}. $$

Since the free energy in equation (38) does not depend on \{h_i\}, the quenched average of the free energy over the random fields, \( \langle F \rangle_h/n \), coincides with equation (38).

Since \( \psi_{i,j}(S_i, S_j) = g(S_i)g(S_j)/n \), the message-passing equation in equation (29) can be expanded as

$$ \ln \mathcal{M}_{j \rightarrow i}(S_i) = \ln \sum_{S_j} Q_{j}(S_j) \exp \left( \beta \psi_{i,j}(S_i, S_j) \right) \mathcal{M}_{i \rightarrow j}(S_j)^{-1} + c_0 $$

$$ = \frac{\beta}{n} \frac{g(S_i) \sum_{S_j} g(S_j)Q_{j}(S_j) \mathcal{M}_{i \rightarrow j}(S_j)^{-1}}{\sum_{S_j} Q_{j}(S_j) \mathcal{M}_{i \rightarrow j}(S_j)^{-1}} + c_1 + O(n^{-2}) $$

(40)
we can conclude that our message-passing method can exactly compute the quenched average of the free energy of the model in equation (37) at the thermodynamical limit.

FIG. 7. Quenched magnetizations versus $J$, where $q = 2$ and $\sigma = 1$.

where $c_0$ and $c_1$ are constants unrelated to $S_i$. For a sufficiently large $n$, from the above equation, since $M_{j\to i}(S_i) = \exp(c_1 + O(n^{-1})) \approx \exp(c_1)$, we ensure that all the messages are constants unrelated to $S$. Therefore, from equations (28) and (40), we obtain

$$\Lambda_i(S_i) = \frac{g(S_i)}{n} \sum_{j \in \partial_i} \sum_{S_j} g(S_j)Q_j(S_j) + c_2 = mg(S_i) + c_2$$

(41)

for a sufficiently large $n$, where $c_2$ is a constant unrelated to $S_i$ and we redefine $m := n^{-1} \sum_{i \in V} \sum_{S} g(S_i)Q_j(S_i)$. By substituting this equation into equation (40), we obtain the same expression for $m$ as equation (39). Since all the messages are constants, from equation (31), we have

$$Q_{i,j}(S_i, S_j) = Q_i(S_i)Q_j(S_j).$$

By substituting this equation and equation (41) into equation (21), we find that the equality $f = J_{\text{min}}^{\text{LBP(RS)}}/n$ holds at the thermodynamical limit. From this result, we can conclude that our message-passing method can

FIG. 6. Quenched Bethe free energies per variable for $q = 4$. The left panel shows the free energies versus $\sigma$ with $J = 0.2$ and the right panel shows the free energies versus $J$ with $\sigma = 1$. The error bars are the standard deviation of 10000 realizations.

IV. APPLICATION TO BAYESIAN IMAGE RESTORATION

In this section, we describe the estimation of the statistical performance of the Bayesian image restoration system using LBP by using the proposed method. In images, each pixel is allocated in a two-dimensional grid and has the intensity value corresponding to the color at its position. For an image $I \in \{I_i | i \in V\}$, the entry $I_i \in \{0, 1, \ldots, q - 1\}$ represents the intensity of the $i$th pixel. The scheme of the Bayesian image restoration system is shown in figure 1.

A. Framework of Bayesian Image Restoration using LBP

For original image $I$, let degraded image $h$ be generated by adding additive white Gaussian noise, i.e. $h_i = I_i + \eta_i$ where $\eta_i$ is the random noise drawn from Gaussian distribution $N(\eta | 0, \sigma^2)$. Given a specific degraded image, the posterior distribution of restored image $S$ is designed according to the method in [3]:

$$P(S \mid h, \alpha, \sigma^2) \propto \exp \left(- \sum_{i \in V} \frac{(S_i - h_i)^2}{2\sigma^2} - \frac{\alpha}{2} \sum_{\{i,j\} \in E} (S_i - S_j)^2 \right).$$

(42)

As mentioned in section III the degraded image is regarded as the random fields in this model. The posterior distribution is defined on a square lattice corresponding to the configuration of pixels. The parameter $\alpha$ controls the smoothness of the restored image. In the maximum posterior marginal (MPM) estimation, the $i$th pixel of the restored image is obtained by

$$S_i^{\text{MPM}} := \arg \max_{S_i} P(S_i \mid h, \alpha, \sigma^2),$$

(43)
where \( P(S_i \mid h, \alpha, \sigma^2) \) is the marginal distribution of the posterior distribution in equation (42). In practice, we approximate the marginal distributions by the beliefs obtained by LBP for the posterior distribution,

\[
S_i^{\text{MPM}} \approx S_i^{\text{MPM}(\text{LBP})} := \arg \max_{S_i} b_i(S_i),
\]

(44)

where the belief \( b_i(S_i) \) is obtained by equation (33) with \( \phi_i(S_i, h_i) = -(S_i - h_i)^2 / 2\sigma^2 \), \( \psi_i(S_i, S_j) = -\alpha(S_i - S_j)^2 / 2 \), and \( \beta = 1 \). The performance of the restoration is often measured by the mean square error (MSE) between the original image and the restored image,

\[
D(I, h, \alpha, \sigma^2) := \frac{1}{n} \sum_{i \in V} (I_i - S_i^{\text{MPM}(\text{LBP})})^2.
\]

(45)

Here, for the original image, we attempt to estimate the average value of the MSE over all possible degraded images, which is defined as

\[
D_{av}(I, \alpha, \sigma^2) := [D(I, h, \alpha, \sigma^2)]_h,
\]

(46)

where notation \([\cdots]_h \) expresses the average value over the distribution of additive noise process,

\[
P(h \mid I, \sigma^2) := \prod_{i \in V} \mathcal{N}(h_i \mid I_i, \sigma^2).
\]

(47)

The averaged MSE in equation (10) expresses the statistical performance of our Bayesian restoring system for the original image and for specific values of the parameters. We want to obtain it analytically using our message-passing method presented in section III D.

B. Estimation of Statistical Performance using Proposed Message-passing Method

By restoring \( \beta \) in \( b_i(S_i) \) in equation (13) as the pseudo temperature (recall that we vanish \( \beta \) in \( b_i(S_i) \) by setting it to one in the previous section), we obtain the relation

\[
\left( \arg \max_{S_i} b_i(S_i) \right)_h^k = \lim_{\beta \to \infty} \sum_{S_i} S_i^k b_i(S_i)_h.
\]

(48)

As mentioned in section III D, function \( Q_i(S_i) \) in equation (30) is regarded as the approximation of \([b(S_i)]_h \). We approximate equation (48) by

\[
\left( \arg \max_{S_i} b_i(S_i) \right)_h^k \approx \lim_{\beta \to \infty} \sum_{S_i} S_i^k Q_i(S_i) = \int dh \mathcal{N}(h \mid I_i, \sigma^2) X_i(h)^k,
\]

(49)

where

\[
X_i(h) := \arg \max_S \left( \phi_i(S, h) + \Lambda_i(S) \right)
\]

and \( \Lambda_i(S) \) is the solution of our message-passing method for the posterior distribution. In the approximation, it is implied that Lagrange multipliers \( \Lambda(S_i) \) is \( O(1) \) with respect to \( \beta \). From the approximation in equation (49), equation (46) is approximated as

\[
D_{av}(I, \alpha, \sigma^2) \approx \frac{1}{n} \sum_{i \in V} \int dh \mathcal{N}(h \mid I_i, \sigma^2)(I_i - X_i(h))^2.
\]

(50)

C. Numerical Experiment

In this section, we describe the estimation of the performance of Bayesian image restoration for the \( 64 \times 64 \) original colored images shown in figure 8. Colored images consist of three different channels: red, green and blue (RGB) channels, \( I = \{I_{\text{red}}, I_{\text{green}}, I_{\text{blue}}\} \). The pixels in each channel in the original images have eight intensities, i.e., \( q = 8 \). In the experiment, we assume that the different three channels are independently degraded by the same noise process shown in equation (47), and we restore the generated degraded images, \( h = \{h_{\text{red}}, h_{\text{green}}, h_{\text{blue}}\} \), by separately applying the Bayesian image restoration based on the posterior distribution shown in equation (42) to RGB channels. The total MSE of the restoration is obtained by the average of the MSEs of RGB channels, i.e.,

\[
D(I, h, \alpha, \sigma^2) = \{D(I_{\text{red}}, h_{\text{red}}, \alpha, \sigma^2) + D(I_{\text{green}}, h_{\text{green}}, \alpha, \sigma^2) + D(I_{\text{blue}}, h_{\text{blue}}, \alpha, \sigma^2)\} / 3.
\]

![FIG. 8. Original colored images I: (a) lenna, (b) parrots and (c) sailboat.](image)

For the original colored images in figure 8 we estimated the averaged MSE, \( D_{av}(I, \alpha, \sigma^2) \), by using two kinds of methods: the LBP and the proposed analytic method shown in equation (46). In the LBP, we approximated \( D_{av}(I, \alpha, \sigma^2) \) by the sample average of \( D(I, h, \alpha, \sigma^2) \) over 10000 different degraded images. In figures 9 and 10 we show the plots of \( D_{av}(I, \alpha, \sigma^2) \) versus \( \alpha \) and \( \sigma \), respectively, obtained by the two different methods: the sample average of the LBP restorations, “LBP,” and the proposed analytic method, “RLBP.” We can observe that the results obtained by our method are in good agreement with those obtained by the LBP restoration.
FIG. 9. Plots of $D_{av}(I, \alpha, \sigma^2)$ versus $\alpha$, when $\sigma = 0.5$, for the original images shown in figure 8: the left, the middle and the right plots show the results for figure 8(a), 8(b) and figure 8(c), respectively. The error bars are the standard deviation of 10000 LBP restorations.

FIG. 10. Plots of $D_{av}(I, \alpha, \sigma^2)$ versus $\sigma$, when $\alpha = 0.4$, for the original images shown in figure 8: the left, the middle and the right plots show the results for figure 8(a), 8(b) and figure 8(c), respectively. The error bars are the standard deviation of 10000 LBP restorations.

V. CONCLUDING REMARKS

In this paper, we proposed an analytic method based on the idea of the RCVM to approximately evaluate the quenched average of the Bethe free energies of general pair-wise MRF over the random fields drawn from a specific distribution. The results obtained by our method are consistent with those obtained by the numerical method in the numerical experiments in the artificial model presented in section III.E and in the Bayesian image restoration presented in section IV.C.

Although we focused only on the classical Bayesian image restoration as the application of our method, the method is applicable also to other various problems based on the MRF and the Bayesian framework, because our formulation allows any form of the interactions, any structure of graph, and any form of distribution of the random fields (except for the cases where correlations among fields exist). Moreover, by employing other mean-field methods, e.g., the naive mean-field approximation or higher-order approximations, instead of the Bethe approximation described in section III.B, we can develop the presented method to evaluate the statistical behaviors of the employed mean-field methods without large modifications.

Acknowledgment

This work was supported by CREST, JST and Grant-In-Aid (Nos. 24700220 and 25280089) for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

[1] S. Geman and D. Geman, IEEE Trans. on Pattern Analysis and Machine Intelligence 6, 721 (1984).
[2] H. Nishimori, Statistical Physics of Spin Glass and Information Processing – Introduction – (Oxford University Press, 2001).
[3] K. Tanaka, J. Phys. A: Math. and Gen. 35, R81 (2002).
[4] J. Pearl, Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference (San Francisco, CA: Morgan Kaufmann, 1988).
[5] Y. Kabashima and D. Saad, Europhys. Lett. 44, 668 (1998).
[6] J. S. Yedidia, W. T. Freeman, and Y. Weiss, IEEE Transactions on Information Theory 51, 2282 (2005).
[7] S. Kataoka, M. Yasuda, and K. Tanaka, J. Phys. Soc. Jpn. 79, 025001 (2010).
[8] K. Tanaka, S. Kataoka, and M. Yasuda, J. Phys.: Conf. Ser. 233, 012013 (2010).
[9] T. Rizzo, A. Lage-Castellanos, R. Mulet, and F. Ricci-Tersenghi, J. Stat. Phys. 139, 375 (2010).
[10] A. Lage-Castellanos, R. Mulet, F. Ricci-Tersenghi, and T. Rizzo, J. Phys. A: Math. and Theor. 46, 135001 (2013).
[11] R. Kikuchi, Phys. Rev. 81, 988 (1951).
[12] A. Pelizzola, J. Phys. A: Math. and Gen. 38, R309 (2005).
[13] M. Mezard, G. Parisi, and M. Virasoro, Spin Glass Theory and Beyond: An Introduction to the Replica Method and Its Applications (Singapore: World Scientific, 1987).
[14] T. Schneider and E. Pytte, Phys. Rev. B 15, 1519 (1977).