SYMMETRIC PAIRS AND BRANCHING LAWS

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Abstract. Let $G$ be a compact connected Lie group and let $H$ be a subgroup fixed by an involution. A classical result assures that the $H$-action on the flag variety $F$ of $G$ admits a finite number of orbits. In this article we propose a formula for the branching coefficients of the symmetric pair $(G, H)$ that is parametrized by $H_C \backslash F$.

1. Introduction

Let $G$ be a compact connected Lie group equipped with an involution $\theta$. Let $G^\theta := \{g \in G, \theta(g) = g\}$ be the subgroup fixed by the involution. We consider a subgroup $H \subset G$ such that $(G^\theta)_0 \subset H \subset G^\theta$. The purpose of this paper is the study of the branching laws between $G$ and $H$.

Let $T$ be a maximal torus of $G$ that we choose $\theta$-invariant. Let $\mathfrak{t}$ be the Lie algebra of $T$. Let $\Lambda \subset \mathfrak{t}^*$ be the lattice of weights, and let $\mathfrak{t}_+^*$ be a Weyl chamber. The irreducible representations of $G$ are parametrized by the semi-group $\Lambda_+ := \Lambda \cap \mathfrak{t}_+^*$ of dominant weights.

Let $\lambda \in \Lambda_+$. In order to study the restriction $V^G_\lambda|_H$ of the irreducible $G$-representation $V^G_\lambda$, we consider the $H$-action on the flag variety $F = G/T$ of $G$. An important object is the $H$-invariant subset

$$Z_\theta \subset F$$

formed of the elements $x \in F$ for which the stabilizer subgroup $G_x := \{g \in G, gx = x\}$ is stable under $\theta$. In other words, $gT \in Z_\theta$ if and only if $g^{-1}\theta(g)$ belongs to the normalizer subgroup $N(T)$. A well-known result tells us that the group $H$ has finitely many orbits in $Z_\theta$, and that the map $O \in H_C \backslash F \longleftrightarrow O \cap Z_\theta \in H \backslash H_\theta$ is bijective [8, 14, 12, 9].

Let $x \in Z_\theta$. The stabilizer subgroup $G_x$ is a maximal torus in $G$, stable under $\theta$, with Lie algebra $\mathfrak{g}_x$. We will also consider the abelian subgroup $H_x := G_x \cap H$ (that is

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not necessarily connected). Any weight $\mu \in \Lambda$ determines a character $C_{\mu_x}$ of the torus $G_x$ by taking $\mu_x = g \cdot \mu$ if $x = gT \in F$.

We denote by $R_x \subset g_x^*$ the set of roots relative to the action of the Cartan subalgebra $g_x$ on $g \otimes \mathbb{C}$. The map $\mu \in R \mapsto \mu_x \in R_x$ is an isomorphism, and we take $R_x^+ \subset R_x$ as the image of $R^+ \subset R$ through this isomorphism.

The involution $\theta$ leaves the set $R_x$ invariant, and $\alpha \in R_x$ is an imaginary root if $\theta(\alpha) = \alpha$. If $\alpha$ is imaginary, the subspace $(g \otimes \mathbb{C})_\alpha$ is $\theta$-stable. There are two cases. If the action of $\theta$ on $(g \otimes \mathbb{C})_\alpha$ is trivial then $\alpha$ is compact imaginary. If the action of $-\theta$ on $(g \otimes \mathbb{C})_\alpha$ is trivial, then $\alpha$ is non-compact imaginary. We denote respectively by $R_x^c$ and by $R_x^{nci}$ the subsets of compact imaginary and non-compact imaginary roots, and we introduce the following $G_x$-modules

$$E^c_x := \sum_{\alpha \in R_x^c \cap R_x^+} (g \otimes \mathbb{C})_\alpha; \quad E^{nci}_x := \sum_{\alpha \in R_x^{nci} \cap R_x^+} (g \otimes \mathbb{C})_\alpha.$$  

The weight

$$\delta(x) := \frac{1}{2} \sum_{\alpha \in R_x^+ \cap \theta(\mathbb{R}_x^+)} \alpha \quad \text{subject to } \theta(\alpha) \neq \alpha$$

defines a character $C_{\delta(x)}$ of the abelian group $H_x$.

We denote by $R(H)$ and by $R(H_x)$ the representations rings of the compact Lie groups $H$ and $H_x$. An element $E \in R(H)$ can be represented as a finite sum $E = \sum_{V \in \hat{R}} m_v V$, with $m_v \in \mathbb{Z}$. We denote by $\hat{R}(H)$ (resp. $\hat{R}(H_x)$) the space of $\mathbb{Z}$-valued functions on $\hat{R}$ (resp. $\hat{R}_x$). An element $E \in \hat{R}(H)$ can be represented as an infinite sum $\sum_{V \in \hat{R}} m_v V$, with $m_v \in \mathbb{Z}$. The induction map $\text{Ind}_{H_x}^H : \hat{R}(H_x) \rightarrow \hat{R}(H)$ is the dual of the restriction morphism $R(H) \rightarrow R(H_x)$.

Let $m_x = \frac{1}{2} |R_x^+ \cap \theta(R_x^+) \cap \{\theta(\alpha) \neq \alpha\}| + \dim E^{nci}_x \in \mathbb{N}$.

The main result of this paper is the following theorem.

**Theorem 1.1.** Let $\lambda \in \Lambda_+$. We have the decomposition

$$V_\lambda^G|_H = \sum_{H_x \in H/\theta} Q_{H_x}(\lambda)$$

(1.1)

where the terms $Q_{H_x}(\lambda) \in \hat{R}(H)$ are defined by the following relation:

$$Q_{H_x}(\lambda) = (-1)^{m_x} \text{Ind}_{H_x}^H \left( C_{\lambda_x + \delta(x)} \otimes \text{Sym}(E^{nci}_x) \otimes \bigwedge E^c_x \right).$$

Here $\text{Sym}(E^{nci}_x)$, which is the symmetric algebra of $E^{nci}_x$, is an admissible representation of $H_x$ and $\bigwedge E^c_x = \bigwedge^+ E^c_x \otimes \bigwedge^- E^c_x$ is a virtual representation of $H_x$.

We give now another formulation for decomposition (1.1) using the (right) action of the Weyl group $W = N(T)/T$ on the flag variety $F$. If $x = gT \in F$ and $w \in W$ we take $xw := gwT$. We notice that $Z_\theta$ is stable under the action of $W$ and that the quotient $Z_\theta/W$ parametrizes the set of maximal tori of $G$ stable under $\theta$.

We associate to an element $x = gT \in Z_\theta$ the subgroup $W_x \subset W$ defined by the relation $w \in W_x \iff H_xw = H_x$. We denote by $H/\bigwedge Z_\theta/W$ the quotient of $Z_\theta$ by the
action of \( H \times W \), and by \( \bar{x} \in H \setminus Z_\theta / W \) the image of \( x \in Z_\theta \) through the quotient map. We associate to \( \bar{x} \in H \setminus Z_\theta / W \) the element \( Q_x(\lambda) \in \hat{R}(H) \) defined as follows

\[
Q_x(\lambda) = \sum_{w \in W^{H} \setminus W} Q_{Hxw}(\lambda).
\]

The previous theorem says then that \( V_\lambda^G|_H = \sum_{\bar{x} \in H \setminus Z_\theta / W} Q_x(\lambda) \). Here is a new formulation of Theorem 1.1.

**Theorem 1.2.** We have \( V_\lambda^G|_H = \sum_{\bar{x} \in H \setminus Z_\theta / W} Q_x(\lambda) \) where \( Q_x(\lambda) \in \hat{R}(H) \) has the following description

\[
Q_x(\lambda) = \text{Ind}_{H_n}^H \left( M_\lambda(\lambda) \otimes C_{\delta(x)} \otimes \bigwedge E^c_x \right),
\]

for some \( M_\lambda(\lambda) \in \hat{R}(H_n) \).

We finish this section by giving two basic examples associated to the group \( SU(2) \). Here the flag variety of \( SU(2) \) is the 2-dimensional sphere \( S^2 \). For \( n \geq 0 \), we denote by \( V_n \) the irreducible representation of \( SU(2) \) of dimension \( n + 1 \).

**Example 1.** \( G = SU(2) \) and the involution \( \theta \) is the conjugation by the matrix \[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]. The subgroup fixed by \( \theta \) is the torus \( T \simeq U(1) \) and the critical set \( Z_\theta \subset S^2 \) is composed of the poles \( S, N \) and the equator \( E \), so that \( T \setminus Z_\theta \) has three terms. We take \( \lambda = n \) in \( SU(2) \simeq \mathbb{N} \).

For \( Hx = E \), we have \( E^c_x = E^c_x = \{0\} \), \( H_x \simeq \mathbb{Z}_2 \), and \( C_{\lambda_x + \delta(x)} = C_n|\mathbb{Z}_2 \). The contribution of \( E \) is then \( \text{Ind}_{x_2}^{U(1)}(C_n|\mathbb{Z}_2) = C_n \otimes \sum_{k \in \mathbb{Z}} C_{2k} \).

For \( Hx = N \), we have \( H_x = T \), \( E^c_x = \mathbb{C}_2 \), \( E^c_x = \{0\} \), and \( C_{\lambda_x + \delta(x)} = C_n \). The contribution of \( N \) is then \( -C_{n+2} \otimes \text{Sym}(\mathbb{C}_2) \).

For \( Hx = S \), we have \( H_x = T \), \( E^c_x = \mathbb{C}_{-2} \), \( E^c_x = \{0\} \), and \( C_{\lambda_x + \delta(x)} = C_{-n} \). The contribution of \( S \) is then \( -C_{-n-2} \otimes \text{Sym}(\mathbb{C}_{-2}) \).

Finally, Relations (1.1) become

\[ V_n|_T = C_n \otimes \sum_{k \in \mathbb{Z}} C_{2k} = C_{-n-2} \otimes \text{Sym}(\mathbb{C}_{-2}) = C_{n+2} \otimes \text{Sym}(\mathbb{C}_2) \]

\[ = \sum_{k = -n}^{0} C_{2k+n} \].

**Example 2.** \( G = SU(2) \times SU(2) \) and the involution \( \theta \) is the map \((a, b) \mapsto (b, a)\). The subgroup fixed by \( \theta \) is \( SU(2) \) embedded diagonally and the critical set \( Z_\theta \subset S^2 \times S^2 \) is equal to the union of the orbits \( SU(2) \cdot (N, N) \) and \( SU(2) \cdot (S, N) \). Let \( \lambda = (n, m) \in \hat{G} \).

For \( x = (N, N) \) or \( x = (S, N) \) we have \( E^c_x = E^c_x = \{0\} \) and \( H_x \simeq T \). For \( x = (N, N) \) we have \( \lambda_x + \delta(x) = m+n+2 \), and for \( x = (S, N) \) we have \( \lambda_x + \delta(x) = m-n \). Relations (1.1) give then

\[ V_n \otimes V_m = \text{Ind}_T^{SU(2)}(C_{m-n}) - \text{Ind}_T^{SU(2)}(C_{m+n+2}) \].

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1The precise expression of \( M_x(\lambda) \) is given in Proposition 3.8.
It is not difficult to see that the previous identities correspond to the classical Clebsch-Gordan relations (see Example 4.2).

Here is a brief overview of the article. Sections 2 and 3 are devoted to the proof of our main result. In Section 4, we detail the case of $U(p) \times U(q) \subset U(n)$: in particular, we explain the branching formula we obtain for the restriction of $U(n)$ to $U(n - 1)$. In the last section, we recall Kostant’s branching formula and explain the formula it gives in the case of the restriction of $U(n)$ to $U(n - 1)$, in order to compare it with our own formula.

Notations

Throughout the paper:

- $G$ denotes a compact connected Lie group with Lie algebra $\mathfrak{g}$.
- $T$ is a maximal torus in $G$ with Lie algebra $\mathfrak{t}$.
- $\Lambda \subset \mathfrak{t}^*$ is the weight lattice of $T$: every $\mu \in \Lambda$ defines a 1-dimensional $T$-representation, denoted by $C_\mu$, where $\mathfrak{t}^\mu := \exp(i(\mu, X))$ acts by $t^\mu := e^{i(\mu, X)}$.
- The coadjoint action of $g \in G$ on $\xi \in \mathfrak{g}^*$ is denoted by $g \cdot \xi$.
- When a Lie group $K$ acts on a manifold $M$, we denote by $X \cdot m := \frac{d}{dt}e^{tX} \cdot m|_{t=0}$, $m \in M$, the infinitesimal action of $X \in \mathfrak{k}$ on $M$.

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2. NON ABELIAN LOCALIZATION

Our main result is obtained by means of a non-abelian localization of the Riemann-Roch character on the flag variety $\mathcal{F}$ of $G$. For that purpose we will use the family $(\Omega_r)_r$ of symplectic structure parametrized by the interior of the Weyl chamber $\mathfrak{t}^*_+$. The symplectic structure $\Omega_r$ comes from the identification $gT \rightarrow g \cdot r$ of $\mathcal{F}$ with the coadjoint orbit $Gr$. The moment map $\Phi_r : \mathcal{F} \rightarrow \mathfrak{g}^*$ associated to the action of $G$ on $(\mathcal{F}, \Omega_r)$ is the map $gT \mapsto g \cdot r$.

At the level of Lie algebras we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ where $\mathfrak{h} = \mathfrak{g}^\theta$ and $\mathfrak{q} = \mathfrak{g}^{-\theta}$. For any $\xi \in \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, we denote by $\xi^+$ his $\mathfrak{h}$-part and by $\xi^-$ his $\mathfrak{q}$-part. We use a $G$-invariant scalar product $(-, -)$ on $\mathfrak{g}$ such that the involution $\theta$ is an orthogonal map. It induces identifications $\mathfrak{g}^+ \simeq \mathfrak{g}$, $\mathfrak{h}^* \simeq \mathfrak{h}$ and $\mathfrak{q}^* \simeq \mathfrak{q}$.

The moment map $\Phi_r^H : \mathcal{F} \rightarrow \mathfrak{h}^*$ associated to the action of $H$ on $(\mathcal{F}, \Omega_r)$ is the map $gT \mapsto (g \cdot r)^+$.

2.1. Matsuki duality. Consider the complex reductive groups $G_C$ and $H_C$ associated to the compact Lie groups $G$ and $H$. Let $L \subset G_C$ be the real form such that $H \subset L$ is a maximal compact subgroup of $L$.

Matsuki duality is the statement that a one-to-one correspondence exists between the $H_C$-orbits and the $L$-orbits in $\mathcal{F}$; two orbits are in duality when their intersection is a single orbit of $H$. 
Uzawa, and Mirkovic-Uzawa-Vilonen [16, 9] proved the Matsuki correspondence by showing that both $H_C$-orbits and $L$-orbits in $\mathcal{F}$ are parametrized by the $H$-orbits in the set of critical points of the function $\|\Phi^H_r\|^2 : \mathcal{F} \to \mathbb{R}$.

First we recall the elementary but fundamental fact that the subset $Z_\theta$ is equal to the set of critical points of the function $\|\Phi^H_r\|^2$ [9, 3].

**Lemma 2.1.** Let $x = gT \in \mathcal{F}$ and $r \in \text{Interior}(t^*_+)$. The following statements are equivalent:

i) the subalgebra $\mathfrak{g}_x$ is invariant under $\theta$ (i.e. $x \in Z_\theta$),

ii) $g^{-1} \theta(g) \in N(T),$

iii) $x$ is a critical point of the function $\|\Phi^H_r\|^2,$

iv) $(g \cdot r)^+$ and $(g \cdot r)^-$ commute.

**Proof.** Let $n_g = g^{-1} \theta(g)$ and let $r$ be a regular element of $t^* \simeq t$. Since $\mathfrak{g}_x = \text{Ad}(g)t$ we see that

$$\theta(\mathfrak{g}_x) = \mathfrak{g}_x \iff n_g \in N_G(T)$$

$$\iff [n_g, \theta(r), r] = 0$$

$$\iff [\theta(g \cdot r), g \cdot r] = 0$$

$$\iff [(g \cdot r)^+, (g \cdot r)^-] = 0.$$ 

A small computation shows that for any $X \in \mathfrak{g}$ the derivative of the function $t \mapsto \|\Phi^H_r(e^{tX})\|^2$ at $t = 0$ is equal to $(X, [g \cdot r, \theta(g \cdot r)])$. Hence $x = gT$ is a critical point of the function $\|\Phi^H_r\|^2$ if and only if $[g \cdot r, \theta(g \cdot r)] = 0$. Finally we have proved that the statements i), ii), iii) and iv) are equivalent. □

Let us check the other easy fact.

**Lemma 2.2.** The set $H \setminus Z_\theta$ is finite.

**Proof.** Let $x = gT \in Z_\theta$. A neighborhood of $x$ is defined by elements of the form $e^{X}e^{Y}x$ where $X \in \mathfrak{h}$ and $Y \in \mathfrak{q}$. Now we see that $e^{X}e^{Y}gT \in Z_\theta$ if and only if $e^{-2g^{-1}Y} \in N(T)$. If $Y$ is sufficiently small the former relation is equivalent to $g^{-1}Y \in t$, and in this case $e^{X}e^{Y}x = e^{X}x$. We have proved that any element in $H \setminus Z_\theta$ is isolated. As $H \setminus Z_\theta$ is compact, we can conclude that $H \setminus Z_\theta$ is finite. □

2.2. Borel-Weil-Bott theorem. We first recall the Borel-Weil-Bott theorem. The flag manifold $\mathcal{F}$ is equipped with the $G$-invariant complex structure such that

$$T_{eT}\mathcal{F} \simeq \sum_{\alpha \in \mathbb{N}^+} (\mathfrak{g} \otimes \mathbb{C})_{\alpha}$$

is an identity of $T$-modules. Let us consider the tangent bundle $T\mathcal{F}$ as a complex vector bundle on $\mathcal{F}$ with the invariant Hermitian structure $h_F$ induced by the invariant scalar product on $\mathfrak{g}$.

Any weight $\lambda \in \Lambda$ defines a line bundle $\mathcal{L}_\lambda \simeq \mathbb{C} \times_T C_\lambda$ on $\mathcal{F}$.

**Definition 2.3.** We associate to a weight $\lambda \in \Lambda$

- the spin-c bundle on $\mathcal{F}$

$$S_\lambda := \bigwedge_C T\mathcal{F} \otimes \mathcal{L}_\lambda,$$
• The Riemann-Roch character \( RR_G(F, \mathcal{L}_\lambda) \in R(G) \) which is the equivariant index of the Dirac operator \( D_\lambda \) associated to the spin-c structure \( S_\lambda \).

The Borel-Weil-Bott theorem asserts that \( V_\lambda^G = RR_G(F, \mathcal{L}_\lambda) \) when \( \lambda \) is dominant. Now we consider the restriction \( V_\lambda^G|_H = RR_H(F, \mathcal{L}_\lambda) \). In the next section, we will explain how we can localize the \( H \)-equivariant Riemann-Roch character \( RR_H(F, \mathcal{L}_\lambda) \) on the critical set of the function \( ||\Phi^g_{r,H}||^2 \).

2.3. Localization of the Riemann-Roch character. In this section, we recall how we perform the “Witten non-abelian localization” of the Riemann-Roch character with the help of the moment map \( \Phi \).

The computation of the characters \( RR_H(F, \mathcal{L}_\lambda, \Phi^H R, Hx) \) will be handle in Section 3.1. To undertake these calculations we need to describe geometrically a neighborhood of \( Hx \) in \( F \). This is the goal of the next section.
2.4. Local model near \( Hx \subset Z_\theta \). Let \( x = gT \in Z_\theta \). We need to compute a symplectic model of a neighborhood of \( Hx \) in \((\mathcal{F}, \Omega_r)\). Here we use the identification \( \mathfrak{g} \simeq \mathfrak{g}^* \) given by the choice of an invariant scalar product. Let \( \mu = g \cdot r \) that we write \( \mu = \mu^+ + \mu^- \) where \( \mu^+ \in \mathfrak{h} \) and \( \mu^- \in \mathfrak{q} \).

The tangent space \( T_x\mathcal{F} \) is equipped with the symplectic two form \( \Omega_r|_x \):
\[
\Omega_r|_x(X \cdot x, Y \cdot x) = (\mu, [X, Y]), \quad X, Y \in \mathfrak{g}.
\]

We need to understand the structure of the symplectic vector space \((T_x\mathcal{F}, \Omega_r|_x)\). If \( a \subset \mathfrak{g} \) is a vector subspace we denote by \( a \cdot x := \{X \cdot x, X \in a\} \) the corresponding subspace of \( T_x\mathcal{F} \). The symplectic orthogonal of \( a \cdot x \) is denoted by \((a \cdot x)^\bot\).

If \( a, b \) are two subspaces, a small computation gives that
\[
(a \cdot x)^\bot \cap b \cdot x \simeq a^\bot \cap [b, \mu],
\]
where \( a^\bot \subset \mathfrak{g} \) is the orthogonal of \( a \) relatively to the scalar product.

We denote by \( \mathfrak{g}_{\mu^+} = \mathfrak{h}_{\mu^+} \oplus \mathfrak{q}_{\mu^+} \) the subspaces fixed by \( ad(\mu^+) \). Notice that \( \mathfrak{g}_\mu = \mathfrak{g}_x \) is an abelian subalgebra containing \( \mu^+ \) since \([\mu^+, \mu^-] = 0\). It follows that \( \mathfrak{g}_x \subset \mathfrak{g}_{\mu^+} \).

**Lemma 2.6.** \([\mathfrak{g}, \mu^+] \cdot x, \mathfrak{g}_{\mu^+} \cdot x \) and \([\mathfrak{h}, \mu^+] \cdot x \) are symplectic subspaces of \( T_x\mathcal{F} \).

**Proof.** It is a direct consequence of (2.2). \( \square \)

We consider now the symplectic subspace \( V_x \subset T_x\mathcal{F} \) defined by the relation
\[
(2.3) \quad V_x = ([\mathfrak{h}, \mu^+] \cdot x)^\bot \cap [\mathfrak{g}, \mu^+] \cdot x.
\]
A small computation shows that \( X \cdot x \in V_x \) if and only if \([X, \mu] \in [\mathfrak{g}, \mu^+]\).

We have the following important Lemma.

**Lemma 2.7.**

- We have the following decomposition
\[
(2.4) \quad T_x\mathcal{F} = \mathfrak{g}_{\mu^+} \cdot x \oplus [\mathfrak{h}, \mu^+] \cdot x \oplus V_x
\]
where \( \bot \) stands for the orthogonal relative to \( \Omega_r|_x \).
- \( \mathfrak{g}_{\mu^+} \cdot x \) is symplectomorphic to \( \mathfrak{h}_{\mu^+}/\mathfrak{h}_x \oplus (\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \).
- \([\mathfrak{h}, \mu^+] \cdot x \) is symplectomorphic to \( \mathfrak{h}/\mathfrak{h}_{\mu^+} \) equipped with the symplectic structure \( \Omega_{\mu^+}(\bar{u}, \bar{v}) = ([\mu^+, [u, v]]) \).
- \( V_x \) is symplectomorphic to \( (\mathfrak{h} \cdot x)^\bot \Omega / [(\mathfrak{h} \cdot x)^\bot \Omega \cap \mathfrak{h} \cdot x] \).

**Proof.** If we use the decomposition \( \mathfrak{g} = \mathfrak{g}_{\mu^+} \oplus [\mathfrak{g}, \mu^+] \) and the fact that the abelian subalgebra \( \mathfrak{g}_x \) is contained in \( \mathfrak{g}_{\mu^+} \) we obtain
\[
T_x\mathcal{F} = \mathfrak{g}_{\mu^+} \cdot x \oplus [\mathfrak{g}, \mu^+] \cdot x.
\]
It is obvious to check that the subspaces \([\mathfrak{g}, \mu^+] \cdot x \) and \( \mathfrak{g}_{\mu^+} \cdot x \) are orthogonal relatively to the symplectic form \( \Omega_r|_x \). Since \([\mathfrak{h}, \mu^+] \cdot x \) is a symplectic subspace we have \([\mathfrak{g}, \mu^+] \cdot x = [\mathfrak{h}, \mu^+] \cdot x \oplus V_x \) where \( V_x \) is defined by (2.3). The first point is proved.

The identities \( \theta(\mathfrak{g}_x) = \theta(\mathfrak{h}_x) \oplus \mathfrak{q}_x \) imply the decompositions \( \mathfrak{g}_x = \mathfrak{h}_x \oplus \mathfrak{q}_x \) and \( \mathfrak{g}_{\mu^+} \cdot x = \mathfrak{q}_{\mu^+} \cdot x \oplus \mathfrak{h}_{\mu^+} \cdot x \). The vector subspace \( \mathfrak{h}_{\mu^+} \cdot x \) is isomorphic to \( \mathfrak{h}_{\mu^+}/\mathfrak{h}_x \), and the map \( v \mapsto \Omega_r|_x(v, -) \) defines an isomorphism between \( \mathfrak{q}_{\mu^+} \cdot x \) and the dual of \( \mathfrak{h}_{\mu^+} \cdot x \). The second point is proved.
For the third point we use the isomorphism \( j : [\mathfrak{h}, \mu^+] \to \mathfrak{h}/\mathfrak{h}_\mu^+ \) induced by the projection \( \mathfrak{h} \to \mathfrak{h}/\mathfrak{h}_\mu^+ \). Then the map \( \tilde{u} \mapsto j(\tilde{u}) \cdot x \) defines a symplectomorphism between \((\mathfrak{h}/\mathfrak{h}_\mu^+, \Omega_{\mu^+})\) and \([\mathfrak{h}, \mu^+] \cdot x\).

Now we see that (2.4) together with the decomposition \( \mathfrak{h} \cdot x = \mathfrak{h}_\mu^+ \cdot x \oplus [\mathfrak{h}, \mu^+] \cdot x \) leads to
\[
(\mathfrak{h} \cdot x)^\perp_{\mathfrak{h}} = ([\mathfrak{h}, \mu^+] \cdot x)^\perp_{\mathfrak{h}} \cap (\mathfrak{h}_\mu^+ \cdot x)^\perp_{\mathfrak{h}}
= \mathfrak{h}_\mu^+ \cdot x \oplus V_x
= [(\mathfrak{h} \cdot x)^\perp_{\mathfrak{h}} \cap \mathfrak{h} \cdot x] \oplus V_x.
\]

The last point follows. \( \square \)

We denote by \( \Omega_{V_x} \) the restriction of \( \Omega_x \rvert_{x} \) on the symplectic vector subspace \( V_x \). The action of \( H_x \) on \((V_x, \Omega_{V_x})\) is Hamiltonian, with moment map \( \Phi_{V_x} : V_x \to \mathfrak{h}_x^* \) defined by the relation
\[
\langle \Phi_{V_x}(v), A \rangle = \frac{1}{2} \Omega_{V_x}(v, Av), \quad v \in V_x, \ A \in \mathfrak{h}_x.
\]

Thanks to Lemma [2.7], we know that the \( H_x \)-symplectic vector space \((T_x \mathcal{F}, \Omega_{V_x})\) admits the following decomposition
\[
T_x \mathcal{F} \simeq \mathfrak{h}_\mu^+ / \mathfrak{h}_x \oplus ([\mathfrak{h}_\mu^+ / \mathfrak{h}_x] \ast \oplus \mathfrak{h} / \mathfrak{h}_\mu^+ \oplus V_x.
\]

Thanks to the normal form Theorem of Marle [7] and Guillemin-Sternberg [5], we get the following result.

**Corollary 2.8.** An \( H \)-equivariant symplectic model of a neighborhood of \( Hx \) in \( \mathcal{F} \) is \( \mathcal{F}_x := H \times_{\mathfrak{h}_\mu^+} Y_x \) where
\[
Y_x = H_{\mu^+} \times_{H_x} ([\mathfrak{h}_\mu^+ / \mathfrak{h}_x] \ast \times V_x).
\]

The corresponding moment map on \( \mathcal{F}_x \) is
\[
\Phi_{\mathcal{F}_x}([h; \eta, v]) = h(\eta + \mu^+ + \Phi_{V_x}(v))
\]
for \([h; \eta, v] \in H \times_{H_x} ([\mathfrak{h}_\mu^+ / \mathfrak{h}_x] \ast \times V_x).

We finish this section by computing a compatible complex structure on \( V_x \).

By definition, the map that sends \( X \cdot x \) to \([X, \mu]\) defines an isomorphism \( i : V_x \to [\mathfrak{q}, \mu^+] \). The adjoint map \( \text{ad}(\mu) \) defines also an automorphism of \([\mathfrak{g}, \mu^+]\): for any \( X \in [\mathfrak{g}, \mu^+] \) we denote by \( \bar{X} \in [\mathfrak{g}, \mu^+] \) the unique element such that \( \text{ad}(\mu)\bar{X} = X \).

The symplectic structure \( \Omega_\mu := (i^{-1})^* \Omega_{V_x} \) satisfies the relations
\[
\Omega_\mu(X, Y) = (\mu,[\bar{X}, \bar{Y}]) = (X, \bar{Y}) = -(\bar{X}, Y), \quad \forall X, Y \in [\mathfrak{g}, \mu^+].
\]

We work with the following \( H_x \)-equivariant maps
- the one to one map \( T_\mu := -\text{ad}(\mu)\text{ad}(\theta(\mu)) : [\mathfrak{g}, \mu^+] \to [\mathfrak{g}, \mu^+] \).
- the complex structure \( J_{\mu^+} = \text{ad}(\mu^+)\text{ad}(\mu^+)^{1/2} \) on \([\mathfrak{g}, \mu^+]\).
The map $T_\mu$ restricts to a one to one map $T_x : [q, \mu^+] \to [q, \mu^+]$ and $J_{\mu^+}$ defines a complex structure on $[q, \mu^+]$ (still denoted by $J_{\mu^+}$).

Let $S_x := (T_x^2)^{-1/2}T_x$. The map $J_{V_x} := J_{\mu^+} \circ S_x$ defines a $H_x$-invariant complex structure on $[q, \mu^+]$.

**Lemma 2.9.** The $H_x$-symplectic space $(V_x, \Omega_{V_x})$ is isomorphic to $[q, \mu^+]$ equipped with the symplectic form $\Omega_{V_x}^{1/2}(v, w) = (J_{V_x}v, w)$.

**Proof.** We know already that $(V_x, \Omega_{V_x}) \simeq ([q, \mu^+], \Omega_\mu)$. If one takes $L = T_x \circ (-\text{ad}(\mu^+)^2)^{-1/4} \circ (T_x^2)^{-1/4}$, we check easily that $\Omega_{\mu}(L(v), L(w)) = (J_{V_x}v, w)$. \hfill \Box

### 3. Proof of the main theorem

We start with the following lemma.

**Lemma 3.1.** The quantity $\text{RR}_H(\mathcal{F}, L_\lambda, \Phi_r^H, Hx)$ does not depend on the choice of the regular element $r$ in the Weyl chamber. In the following we will denote it by $Q_{Hx}(\lambda) \in \tilde{R}(H)$.

**Proof.** Let $r_0, r_1$ be two regular elements of the Weyl chamber. For $t \in [0, 1]$, we consider the regular element $r(t) = tr_1 + (1-t)r_0$: the Kirwan vector field $\kappa_{r(t)}$ vanishes exactly on $Z_0$ for any $t \in [0, 1]$. If $U$ is an invariant neighboorhood of $Hx$ so that $U \cap Z_0 = Hx$, then $t \in [0, 1] \mapsto \sigma_{r(t)}(\mathcal{F}) \otimes L_\lambda|_U$ defines an homotopy of transversally elliptic symbols. Accordingly, the equivariant index of $\sigma_{r_0}(\mathcal{F}) \otimes L_\lambda|_U$ and $\sigma_{r_1}(\mathcal{F}) \otimes L_\lambda|_U$ are equal. \hfill \Box

#### 3.1. Computation of $Q_{Hx}(\lambda)$.

The computation of $Q_{Hx}(\lambda)$ is done in three steps.

##### 3.1.1. Step 1: holomorphic induction.

Let $H_{\mu^+} \subset H$ be the stabilizer subgroup of $\mu^+ := \Phi_r^H(x)$. By Corollary 2.8, a symplectic $H$-equivariant model of a neighborhood of $Hx$ in $\mathcal{F}$ is the manifold $H \times_{H_{\mu^+}} Y_x$ where

$$Y_x = H_{\mu^+} \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x).$$

The symplectic two from on $Y_x$ is built from the canonical symplectic structure on $H_{\mu^+} \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x)$ and the symplectic structure on $V_x$. The moment map relative to the action of $H_{\mu^+}$ on $Y_x$ is

$$\Phi_{Y_x}([h; \eta, v]) = h(\eta + \mu^+ + \Phi_{V_x}(v)) \in \mathfrak{h}_{\mu^+},$$

for $[h; \eta, v] \in H_{\mu^+} \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x)$.

Let $\kappa_{Y_x}$ the Kirwan vector field on $Y_x$. It is immediate to check that $[h; \eta, v] \in \{\kappa_{Y_x} = 0\}$ if and only if $\eta = 0$ and $(\mu^+ + \Phi_{V_x}(v)) \cdot v = 0$. The map $v \in V_x \mapsto (\mu^+ + v) \cdot v \in V_x$ is bijective and $v \mapsto (\mu^+ + v) \cdot v$ is homogeneous of degree equal to 3. Then there exists $\epsilon > 0$ such that

$$(\mu^+ + \Phi_{V_x}(v)) \cdot v = 0 \quad \text{and} \quad \|v\| \leq \epsilon \implies v = 0.$$  

In $Y_x$, we still denote by $x$ the point $[e, 0, 0]$. We equip $Y_x$ with an invariant almost complex structure that is compatible with the symplectic structure, and we denote by $\text{RR}_{H_{\mu^+}}(Y_x, L_\lambda|_{Y_x}, \Phi_{Y_x}, H_{\mu^+}x)$ the Riemann-Roch character on $Y_x$ localized on the component $H_{\mu^+}x \subset \{\kappa_{Y_x} = 0\}$. 

The quotient $\mathfrak{h}/\mathfrak{h}_{\mu^+}$, which is equipped with the invariant complex structure $J_{\mu^+} := ad(\mu^+)(-ad(\mu^+))^2)^{-1/2}$, is a complex $H_{\mu^+}$-module.

In [10][Theorem 7.5], we proved that $Q_{Hx}(x) = \text{RR}_{H}(F, \mathcal{L}_{\lambda}, \Phi_{r}^{H}, Hx)$ is equal to

\[(3.5) \quad \text{Ind}_{H_{\mu^+}}^{H} (\text{RR}_{H_{\mu^+}}(Y_{x}, \mathcal{L}_{\lambda}|_{Y_{x}}, \Phi_{Y_{x}}, H_{\mu^+}x)) \cap \bigwedge \mathfrak{h}/\mathfrak{h}_{\mu^+})].
\]

3.1.2. Step 2: cotangent induction. The map $\Phi_{x}(v) := \mu^+ + \Phi_{V_{x}}(v)$ is a moment map for the Hamiltonian action of $H_{\mu^+}$ on $V_{x}$. The moment map on the $H_{\mu^+}$-manifold

\[Y_{x} = H_{\mu^+} \times_{H_{\mu^+}} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_{\mu^+})^* \times V_{x})\]

is $\Phi_{Y_{x}}([h, \eta, v]) = h(\eta + \Phi_{x}(v)) \in \mathfrak{h}_{\mu^+}^*$. Let $\kappa_{V_{x}}(v) = -\Phi_{x}(v) \cdot v$ be the Kirwan vector field on $V_{x}$. We are interested in the connected component $\{0\}$ of $\{\kappa_{V_{x}} = 0\}$. We choose a compatible almost complex structure on the symplectic vector space $V_{x}$ and we denote by $\text{RR}_{H_{x}}(V_{x}, \Phi_{x}, \{0\}) \in \widehat{R}(H_{x})$ the Riemann-Roch character localized on $\{0\} \subset \{\kappa_{V_{x}} = 0\}$.

In Section 3.3 of [11] we have proved that

\[(3.6) \quad \text{RR}_{H_{\mu^+}}(Y_{x}, \mathcal{L}_{\lambda}|_{Y_{x}}, \Phi_{Y_{x}}, H_{\mu^+}x) = \text{Ind}_{H_{\mu^+}}^{H_{\mu^+}} (\text{RR}_{Hx}(V_{x}, \Phi_{x}, \{0\}) \otimes \mathcal{L}_{|_{Y_{x}}}).\]

3.1.3. Step 3: linear case. We write $q/\mathfrak{q}_{\mu^+}$ for the vector space $[q, \mu^+]$ equipped with the complex structure $J_{\mu^+}$. So $q/\mathfrak{q}_{\mu^+}$ is a $H_{\mu^+}$-module and we denote by $\text{Sym}(q/\mathfrak{q}_{\mu^+})$ the corresponding symmetric algebra.

We need to compare the virtual $H_{x}$-modules $\bigwedge_{J_{V_{x}}} V_{x}$ and $\bigwedge_{-J_{\mu^+}} V_{x}$. The weight

\[\delta(x) := \frac{1}{2} \sum_{\alpha \in \mathfrak{h}_{\mu^+}^* \cap \theta(\mathfrak{h}_{\mu^+})^* \cap \{\theta(\alpha) \neq \alpha\}} \alpha\]

defines a character $\mathbb{C}_{\delta(x)}$ of the abelian group $H_{x}$. Recall that $m_{x} \in \mathbb{N}$ corresponds to the quantity $\frac{1}{2} |\mathfrak{h}_{\mu^+}^* \cap \theta(\mathfrak{h}_{\mu^+})^* \cap \{\theta(\alpha) \neq \alpha\}| + \dim \mathbb{E}_{x}^{\text{nci}}$.

The following lemma will be proved in Section 3.2.

**Lemma 3.2.** The following identity holds:

\[\bigwedge_{J_{V_{x}}} V_{x} \simeq (-1)^{m_{x}} \mathbb{C}_{\delta(x)} \otimes \text{det}(\mathbb{E}_{x}^{\text{nci}}) \otimes \bigwedge_{-J_{\mu^+}} V_{x}.\]

On the vector space $V_{x}$, we can work with two localized Riemann-Roch characters:

- $\text{RR}_{H_{x}}(V_{x}, \Phi_{x}, \{0\})$ is defined with the complex structure $J_{V_{x}}$;
- $\tilde{\text{RR}}_{H_{x}}(V_{x}, \Phi_{x}, \{0\})$ is defined with the complex structure $-J_{\mu^+}$.

Thanks to the previous Lemma we know that $\text{RR}_{H_{x}}(V_{x}, \Phi_{x}, \{0\})$ is equal to $(-1)^{m_{x}} \mathbb{C}_{\delta(x)} \otimes \text{det}(\mathbb{E}_{x}^{\text{nci}}) \otimes \tilde{\text{RR}}_{H_{x}}(V_{x}, \Phi_{x}, \{0\})$.

**Proposition 3.3.** We have

\[(3.7) \quad \text{RR}_{Hx}(V_{x}, \Phi_{x}, \{0\}) = (-1)^{m_{x}} \mathbb{C}_{\delta(x)} \otimes \text{det}(\mathbb{E}_{x}^{\text{nci}}) \otimes \text{Sym}(q/\mathfrak{q}_{\mu^+}).\]

**Proof.** For $s \in [0, 1]$, we consider the $H_{x}$-equivariant map $\Phi^{s} : V_{x} \to \mathfrak{h}_{\mu^+}^*$ defined by the relation $\Phi^{s}(v) = \mu^+ + s\Phi_{V_{x}}(v)$. The corresponding Kirwan vector field on $V_{x}$ is $\kappa^{s}(v) = -\Phi^{s}(v) \cdot v$. It is not difficult to see that there exists $\epsilon > 0$ such that
\( \{ \kappa = 0 \} \cap \{ \| v \| \leq \epsilon \} = \{ 0 \} \) for any \( s \in [0,1] \). Then a simple deformation argument gives that \( \overline{RR}_{H_s}(V_x, \Phi^s, \{ 0 \}) \) does not depend on \( s \in [0,1] \). We have proved that
\[
\overline{RR}_{H_s}(V_x, \Phi^s, \{ 0 \}) = \overline{RR}_{H_s}(V_x, \mu^+, \{ 0 \})
\]
where \( \mu^+ \) denotes the constant map \( \Phi^0 \). Standard computations give \( \overline{RR}_{H_s}(V_x, \mu^+, \{ 0 \}) = \text{Sym}(q/q_{\mu^+}) \) (see [10] [Proposition 5.4]). Our proof is completed. \( \square \)

3.1.4. Conclusion. If we use the formulas (3.5), (3.6) and (3.7) we obtain the following expression
\[
Q_{Hx}(\lambda) = (-1)^{m_x} \text{Ind}^H_x \left( \mathbb{C}_{\lambda_x+\delta(x)} \otimes \det(E_x^{\text{nci}}) \otimes \text{Sym}(q/q_{\mu^+}) \otimes \bigwedge C h/h_{\mu^+} \right)
\]
in \( \tilde{R}(H) \). Here \( \mathbb{C}_{\lambda_x} \) is the character of \( G_x \) associated to the weight \( \lambda_x = g\lambda \).

The previous formula depends on a choice of a regular element \( r \) in the Weyl chamber. In the next section we will propose another expression for \( Q_{Hx}(\lambda) \) that does not depend on this choice.

3.2. Another expression for \( Q_{Hx}(\lambda) \). Let \( \mathfrak{A}_x \subset \mathfrak{g}_x^* \) be the roots for the action of the torus \( G_x \) on \( \mathfrak{g} \otimes \mathbb{C} \). The involution \( \theta : \mathfrak{t}^* \rightarrow \mathfrak{t}^* \) leaves the set \( \mathfrak{A}_x \) invariant and a root \( \alpha \in \mathfrak{A}_x \) is called \textit{imaginary} if \( \theta(\alpha) = \alpha \). We denote respectively by \( \mathfrak{A}_x^{\text{ci}} \) and by \( \mathfrak{A}_x^{\text{nci}} \) the subsets of \textit{compact imaginary} and \textit{non-compact imaginary} roots.

We choose a generic element \( r \in \mathfrak{t}_+^* \) such that \( \mu^+ = (g \cdot r)^* \) satisfies the following relation: for any \( \alpha \in \mathfrak{A}_x \), we have
\[
(\alpha, \mu^+) = 0 \iff \theta(\alpha) = -\alpha.
\]

Notice that an imaginary roots \( \alpha \) is positive if and only if \( (\alpha, \mu^+) > 0 \).

\textbf{Definition 3.4.} We consider the subset \( \mathfrak{A}_x \subset \mathfrak{A}_x \) defined by the following relations:
\[
\alpha \in \mathfrak{A}_x \iff \alpha(\mu^+) > 0 \quad \text{and} \quad \theta(\alpha) \neq \alpha.
\]

The involution \( \theta \) defines a free action of \( \mathbb{Z}_2 \) on the set \( \mathfrak{A}_x \). We denote by \( \mathfrak{A}_x/\mathbb{Z}_2 \) its quotient. For any \( \alpha \in \mathfrak{A}_x \), we denote by \( \mathbb{C}_\alpha \) the corresponding 1-dimensional representation of \( G_x \), and \( \mathbb{C}_\alpha|_{H_x} \) its restriction to the subgroup \( H_x \). We have a natural map \( [\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2 \mapsto \mathbb{C}_\alpha|_{H_x} \in \tilde{R}(H_x) \).

For any \( \alpha \in \mathfrak{A}_x \) we define
\[
\tilde{\alpha} = \pm \alpha
\]
where \( \pm \) is the sign of \( \alpha(\mu)\alpha(\theta(\mu)) \).

We consider the \( H_x \)-modules \( \mathfrak{h}/h_{\mu^+} := (\mathfrak{h}, \mu^+), q/q_{\mu^+} := (V_x, J_{\mu^+}) \) and \( (V_x, J_{V_x}) \).

\textbf{Lemma 3.5.} We have the following isomorphisms of \( H_x \)-modules
\[
\mathfrak{h}/h_{\mu^+} \simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} \mathbb{C}_\alpha|_{H_x} \oplus \bigoplus_{\alpha \in \mathfrak{A}_x^{\text{ci}} \cap \mathfrak{m}_x^+} \mathbb{C}_\alpha|_{H_x} \quad [A],
\]
\[
q/q_{\mu^+} \simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} \mathbb{C}_\alpha|_{H_x} \oplus \bigoplus_{\alpha \in \mathfrak{A}_x^{\text{nci}} \cap \mathfrak{m}_x^+} \mathbb{C}_\alpha|_{H_x} \quad [B],
\]
\[
(V_x, J_{V_x}) \simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} \mathbb{C}_{\tilde{\alpha}}|_{H_x} \oplus \bigoplus_{\alpha \in \mathfrak{A}_x^{\text{nci}} \cap \mathfrak{m}_x^+} \mathbb{C}_{\tilde{\alpha}}|_{H_x} \quad [C].
\]
\textbf{Proof.} Thanks to Lemma 2.9, we know that the $H_x$-module $(V_x, J_{V_x})$ is isomorphic to the vector space $[q, \mu^+]$ equipped with the complex structure $J_{V_x} := J_{\mu^+} \circ S_x$. We consider the vector spaces $[q, \mu^+]$ and $[g, \mu^+]$ equipped with the complex structure $J_{\mu^+}$. The projection (taking the real part) $r : g \otimes \mathbb{C} \to g$ induces an isomorphism of $G_x$-modules

$$r : \bigoplus_{\alpha(\mu^+) > 0} (g \otimes \mathbb{C})_\alpha \longrightarrow [g, \mu^+] .$$

The orthogonal projections $p_1 : [g, \mu^+] \to [q, \mu^+]$ and $p_2 : [g, \mu^+] \to [h, \mu^+]$ commute with the $H_x$-action, so the maps

$$p_1 \circ r : \bigoplus_{\alpha(\mu^+) > 0} (g \otimes \mathbb{C})_\alpha \longrightarrow [q, \mu^+] ,$$

$$p_2 \circ r : \bigoplus_{\alpha(\mu^+) > 0} (g \otimes \mathbb{C})_\alpha \longrightarrow [h, \mu^+]$$

are surjective morphisms of $H_x$-modules.

Let $V^1_x(\alpha) = p_1 \circ r((g \otimes \mathbb{C})_\alpha)$. We notice that $\dim_{\mathbb{C}} V^1_x(\alpha) \in \{0, 1\}$: $V^1_x(\alpha) = \{0\}$ only if $\alpha$ is a non-compact imaginary root and $V^1_x(\alpha) \simeq \mathbb{C}_\alpha|_{H_x}$ when $V^1_x(\alpha) \neq \{0\}$. We notice also that $V^1_x(\alpha) = V^1_x(\theta(\alpha))$, hence

$$q/q_{\mu^+} = ([q, \mu^+], J_{\mu^+}) \simeq \bigoplus_{[\alpha] \in \mathfrak{a}_x/\mathbb{Z}_2} V^1_x(\alpha) \oplus \bigoplus_{\alpha \in \mathfrak{h}_{\mu^+} \cap \mathfrak{h}_{\mu^+}^+} V^1_x(\alpha) .$$

The identity $[B]$ is proved.

Similarly we consider $V^2_x(\alpha) = p_2 \circ r((g \otimes \mathbb{C})_\alpha)$. We notice that $\dim_{\mathbb{C}} V^2_x(\alpha) \in \{0, 1\}$: $V^2_x(\alpha) = \{0\}$ only if $\alpha$ is a compact imaginary root and $V^2_x(\alpha) \simeq \mathbb{C}_\alpha|_{H_x}$ when $V^2_x(\alpha) \neq \{0\}$. We notice also that $V^2_x(\alpha) = V^2_x(\theta(\alpha))$, hence

$$\mathfrak{h}/\mathfrak{h}_{\mu^+} = ([\mathfrak{h}, \mu^+], J_{\mu^+}) \simeq \bigoplus_{[\alpha] \in \mathfrak{a}_x/\mathbb{Z}_2} V^2_x(\alpha) \oplus \bigoplus_{\alpha \in \mathfrak{h}_{\mu^+} \cap \mathfrak{h}_{\mu^+}^+} V^2_x(\alpha) .$$

The identity $[A]$ is proved.

Finally we check that the complex structures $J_{\mu^+}$ and $J_{V_x}$ preserve each $V^1_x(\alpha)$ and that $(V_x(\alpha), J_{V_x}) \simeq \mathbb{C}_{\alpha}|_{H_x}$ when $(\alpha, \mu^+) > 0$. The identity $[C]$ follows. \hfill $\square$

We consider the $H_x$-module $V_x := \sum_{[\alpha] \in \mathfrak{a}_x/\mathbb{Z}_2} \mathbb{C}_\alpha|_{H_x}$, and the $G_x$-modules $E^\text{nci}_x := \sum_{\alpha \in \mathfrak{h}_{\mu^+} \cap \mathfrak{h}_{\mu^+}^+} \mathbb{C}_\alpha$ and $E^\text{ci}_x := \sum_{\alpha \in \mathfrak{h}_{\mu^+} \cap \mathfrak{h}_{\mu^+}^+} \mathbb{C}_\alpha$. In the previous lemma we have proved that $H_x$-modules $\mathfrak{h}/\mathfrak{h}_{\mu^+}$ and $q/q_{\mu^+}$ are respectively isomorphic to $V_x \oplus E^\text{ci}_x$ and $V_x \oplus E^\text{nci}_x$. If we use the fact that $\text{Sym}(V_x) \otimes \bigwedge V_x = 1$, we get the following corollary.

\textbf{Corollary 3.6.} \textit{We have the following identity of virtual $H_x$-modules:}

$$\text{Sym}(q/q_{\mu^+}) \otimes \bigwedge \mathfrak{h}/\mathfrak{h}_{\mu^+} \simeq \text{Sym}(E^\text{nci}_x) \otimes \bigwedge E^\text{ci}_x .$$

\textbf{Proof of Lemma 3.2.} Let $B := \mathfrak{a}_x/\mathbb{Z}_2 \cup (\mathfrak{h}_{\mu^+} \cap \mathfrak{h}_{\mu^+}^+)$, and $\sum_{\alpha \in B} V_x = \Pi_{\alpha \in B}(1 - t^\alpha)$ whereas $\bigwedge_{J_{\mu^+}} V_x = \Pi_{\alpha \in B}(1 - t^{-\alpha})$.

Accordingly we get $\bigwedge_{J_{\mu^+}} V_x \simeq (-1)^{|B'|} \mathbb{C}_\eta \otimes \bigwedge_{J_{\mu^+}} V_x$ where $B' = \{\alpha \in B, \tilde{\alpha} = \alpha\}$ and $\eta = \sum_{\alpha \in B' \tilde{\alpha} = \alpha}$. Now it is easy
to check that an element $\alpha \in \mathcal{B}$ belongs to $\mathcal{B}'$ if and only if $\alpha$ and $\theta(\alpha)$ both belong to $\mathfrak{A}_x^+$. In other words

$$\mathcal{B}' = \{ \alpha \in \mathfrak{A}_x^+ \cap \theta(\mathfrak{A}_x^+), \ \theta(\alpha) \neq \alpha \} / \mathbb{Z}_2 \bigcup \mathfrak{A}_x^{nci} \cap \mathfrak{A}_x^+.$$  

We have proved that

$$\bigwedge_{J_{V_x}} V_x \simeq (-1)^{m_x} C_{\delta(x)} \otimes \det(\mathbb{E}_x^{nci}) \otimes \bigwedge_{-J_{\mu^+}} V_x.$$  

$\square$

Finally, thanks to Lemma 3.2 and Corollary 3.6 we obtain the final formula for $Q_{H^x}(\lambda)$ (that does not depend on the choice of $r$):

$$Q_{H^x}(\lambda) = (-1)^{m_x} \text{Ind}^H_{H_x} \left( C_{\lambda_x + \delta(x)} \otimes \det(\mathbb{E}_x^{nci}) \otimes \text{Sym}(\mathbb{E}_x^{nci}) \otimes \bigwedge \mathbb{E}_x^\lambda \right).$$

3.3. Computation of the virtual module $\mathbb{M}_x(\lambda)$. According to Theorem 1.1, we have the decomposition $V_x^{\lambda} |_{H} = \sum_{\lambda} Q_{x}(\lambda)$ where $Q_{x}(\lambda) = \text{Ind}^H_{H_x}(\mathbb{A}_x(\lambda))$, and $\mathbb{A}_x(\lambda) \in \hat{R}(H_x)$ has the following description

$$\mathbb{A}_x(\lambda) = \frac{1}{|W_x^H|} \sum_{w \in W} (-1)^{m_x w} C_{\lambda_x + \delta(xw)} \otimes \det(\mathbb{E}_x^{nci}) \otimes \text{Sym}(\mathbb{E}_x^{nci}) \otimes \bigwedge \mathbb{E}_x^\lambda_w.$$  

The aim of this section is to simplify the expression of the virtual $H^x$-module $\mathbb{A}_x(\lambda)$. We start by comparing the $G^x$-modules $\mathbb{E}_x^{nci}$ and $\mathbb{E}_x^{nci}$. We use the decomposition $\mathbb{E}_x^{nci} = (\mathbb{E}_x^{nci})^+ + (\mathbb{E}_x^{nci})^-$ where

$$(\mathbb{E}_x^{nci})^+_w := \sum_{\alpha \in \mathfrak{B}_x^{nci} \cap \mathfrak{S}_x \cap \mathfrak{A}_x^{ci}_w} C_{\alpha}, \quad \text{and} \quad (\mathbb{E}_x^{nci})^-_w = \sum_{\alpha \in \mathfrak{B}_x^{nci} \cap \mathfrak{S}_x \cap \mathfrak{A}_x^{ci}_w} C_{\alpha}.$$  

We have the following basic lemma (see Lemma 3.10).

**Lemma 3.7.** The $G^x$-module $|\mathbb{E}_x^{nci}|_w := (\mathbb{E}_x^{nci})^+_w + (\mathbb{E}_x^{nci})^-_w$ is isomorphic to $\mathbb{E}_x^{nci}_w$.

Let $\rho = \frac{1}{2} \sum_{\alpha \in \mathfrak{S}_x^+} \alpha$. We denote by $w \cdot \lambda = w(\lambda + \rho) - \rho$ the affine action of the Weyl group on the lattice $\Lambda$.

The main result of this section is the following proposition.

**Proposition 3.8.** Let $x \in \mathbb{Z}_\theta$. We have

$$\mathbb{A}_x(\lambda) = \mathbb{M}_x(\lambda) \otimes C_{\delta(x)} \otimes \bigwedge \mathbb{E}_x^\lambda$$  

where $\mathbb{M}_x(\lambda) \in \hat{R}(H_x)$ is defined by the following expression

$$\mathbb{M}_x(\lambda) = \frac{(-1)^{n_x}}{|W_x^H|} \sum_{w \in W} (-1)^{k_{x,w}} C_{w \cdot \lambda} \otimes \det((\mathbb{E}_x^{nci}_w)^+) \otimes \text{Sym}(|\mathbb{E}_x^{nci}|_w),$$  

and

- $k_{x,w} = |\mathfrak{A}_x^+ \cap \mathfrak{B}_x^{nci} | \{ \theta(\alpha) \neq \pm \alpha \} | + |\mathfrak{A}_x^+ \cap \mathfrak{B}_x^{nci} | \mathfrak{A}_x^{ci} |,$
- $n_x := |\theta(\mathfrak{A}_x^+) \cap \mathfrak{A}_x^+ - \frac{1}{2} |\theta(\mathfrak{A}_x^+) \cap \mathfrak{A}_x^+ | \{ \theta(\alpha) \neq \alpha \}|.$
Remark 3.9. We can describe $Q_x(\lambda)$ differently by taking $\{w_1, \ldots, w_p\} \subset W$ such that $W_x \setminus W \simeq \{\bar{w}_1, \ldots, \bar{w}_p\}$. We have $Q_x(\lambda) = \text{Ind}_{H_x}^H \left( \tilde{A}_x(\lambda) \right)$ with

$$\tilde{A}_x(\lambda) = \tilde{M}_x(\lambda) \otimes \mathbb{C}_{\delta(x)} \otimes \bigwedge E_x$$

and where $\tilde{M}_x(\lambda) \in \tilde{R}(H_x)$ is defined by the following expression

$$\tilde{M}_x(\lambda) = (-1)^n \sum_{k=1}^p (-1)^{k_x, w_k} \mathbb{C}_{(w_k \bullet \lambda)_k} \otimes \det((E_x^{nci})_w^+) \otimes \text{Sym}(|E_x^{nci}|_w).$$

We need to introduce some notations. To $x \in Z_\theta$, we associate :

- The polarized roots : to $\alpha \in \mathcal{R}_x$ and $w \in W$, we associate $|\alpha|_w \in \mathcal{R}_x$ defined as follows

$$|\alpha|_w = \begin{cases} \alpha & \text{if } \alpha \in \mathcal{R}_x^+, \\ -\alpha & \text{if } \alpha \notin \mathcal{R}_x^+. \end{cases}$$

- The following $G_x$-weights :

$$\gamma_{x,w}^{ci} := \sum_{\alpha \in \mathcal{R}_x^+, \alpha \not| \alpha|_w} \alpha, \quad \gamma_{x,w}^{nci} := \sum_{\alpha \in \mathcal{R}_x^{nci}, \alpha \not| \alpha|_w} \alpha, \quad \gamma_{x,w} := \sum_{\alpha \in \mathcal{R}_x^+} \alpha.$$

The proof of Proposition 3.8 is based on the following Lemma.

Lemma 3.10. Let $x \in Z_\theta$ and $w \in W$. Let $d_{ci}^{x,w}$ be the cardinal of the set $\{\alpha \in \mathcal{R}_x^{ci} \cap \mathcal{R}_x^+, |\alpha|_w \neq \alpha\}$. We have the following relations

1. $E_{x,w}^{nci} \simeq \sum_{\alpha \in \mathcal{R}_x^{nci} \cap \mathcal{R}_x^+} \mathbb{C}_{|\alpha|_w}$ and $E_{x,w}^{ci} \simeq \sum_{\alpha \in \mathcal{R}_x^{ci} \cap \mathcal{R}_x^+} \mathbb{C}_{|\alpha|_w}$.
2. $\det(E_{x,w}^{nci}) = C_{-\gamma_{x,w}^{nci}} \otimes \det((E_{x,w}^{ci})^+)$,
3. $\bigwedge E_{x,w}^{ci} = (-1)^{d_{ci}^{x,w}} C_{-\gamma_{x,w}^{ci}} \otimes \bigwedge E_{x,w}^{ci}$.
4. The $H_x$-weight $\delta(xw) - \delta(x)$ is equal to the restriction of $G_x$-weight $\gamma_{x,w}^{nci} + \gamma_{x,w}^{ci} - \gamma_{x,w}$ to $H_x$.

Proof. We remark that $\mathcal{R}_x = \mathcal{R}_x^{nci}$, $\mathcal{R}_x^+ = g(\mathcal{R}_x^+)$ and $\mathcal{R}_x^{nci} = g(w \mathcal{R}_x^+)$. The first point follows and points (ii) and (iii) derive from the first.

Let us check the last point. The term $\rho_x := \frac{1}{2} \sum_{\alpha \in \mathcal{R}_x^+} \alpha$ is the image of $\rho := \frac{1}{2} \sum_{\alpha \in \mathcal{R}_x^+} \alpha'$ through the map $\mu \mapsto \mu_x$. We see that

$$\rho_x + \theta(\rho_x) = \sum_{\alpha \in \mathcal{R}_x^+} \alpha = 2\delta(x) + 2\rho_x^{nci} + 2\rho_x^{ci}$$

where $\rho_x^{nci} = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_x^{nci} \cap \mathcal{R}_x^+} \alpha$ and $\rho_x^{ci} = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_x^{ci} \cap \mathcal{R}_x^+} \alpha$. Similarly we have

$$\rho_{xw} + \theta(\rho_{xw}) = 2\delta(xw) + 2\rho_{xw}^{nci} + 2\rho_{xw}^{ci}.$$

Thus the $H_x$-weight $\delta(xw) - \delta(x)$ is equal to the restriction to $H_x$ of the $G_x$-weight

$$\beta(x, w) := \rho_{xw} - \rho_x + (\rho_x^{nci} - \rho_{xw}^{nci}) + (\rho_x^{ci} - \rho_{xw}^{ci}).$$
We notice that \( \rho_{xw} - \rho_x = (w \rho - \rho)_x = -\gamma_{x,w} \). Furthermore, small computations give that \( \rho_{x}^{\text{nci}} - \rho_{xw}^{\text{nci}} = \gamma_{x,w}^{\text{nci}} \) and \( \rho_x^{\text{ci}} - \rho_{xw}^{\text{ci}} = \gamma_{x,w}^{\text{ci}} \). We have proved that \( \beta(x, w) = \gamma_{x,w}^{\text{nci}} + \gamma_{x,w}^{\text{ci}} = -\gamma_{x,w} \). The last point follows. \( \square \)

Now, we can finish the proof of the Proposition 3.8. We must check that the virtual \( H_x \)-module

\[
A := (-1)^{m_{xw}} C_{\lambda_x + \delta(x)} \otimes \det(E_{xw}^{\text{nci}}) \otimes \bigwedge E_{xw}^{\text{ci}}
\]

is equal to the virtual \( H_x \)-module

\[
B := (-1)^{n_x + k_{x,w}} C_{(w \rho \lambda_x) + \delta(x)} \otimes \det((E_{x}^{\text{nci}})^+) \otimes \bigwedge E_x^{\text{ci}}
\]

If we use Lemma 3.10, we get

\[
A = (-1)^{m_{xw} + c_{x,w}} C_{w(\lambda + \rho) - \rho_x + \delta(x)} \otimes \det((E_{x}^{\text{nci}})^+) \otimes \bigwedge E_x^{\text{ci}}
\]

Thus the equality \( A = B \) follows from the following lemma.

**Lemma 3.11.** For any \( x \in Z_\theta \) and \( w \in W \), we have \( n_x + k_{x,w} = m_{xw} + d_{x,w}^{\text{ci}} \) mod 2.

**Proof.** In order to simplify our notations, we write \( a \equiv b \) for \( a = b \) mod 2.

We have \( \dim E^{\text{nci}}_{xw} = \dim E_{xw}^{\text{nci}} \) and \( \dim E_x^{\text{ci}} = \dim E_{xw}^{\text{ci}} \), then

\[
m_{xw} - m_x = \frac{1}{2} \left( |\mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) \cap \{ \theta(\alpha) \neq \alpha \} | - | \mathfrak{R}_{x}^+ \cap \theta(\mathfrak{R}_{x}^+) \cap \{ \theta(\alpha) \neq \alpha \} | \right)
\]

\[
= \frac{1}{2} \left( |\mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) | - | \mathfrak{R}_{x}^+ \cap \theta(\mathfrak{R}_{x}^+) | \right).
\]

We remark now that

\[
\mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) = A_{++} \cup A_{+-} \cup A_{-+} \cup A_{--}
\]

with \( A_{++} = \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) \), \( A_{--} = -\mathfrak{R}_{x}^+ \cap -\theta(\mathfrak{R}_{x}^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) \), \( A_{+-} = \mathfrak{R}_{x}^+ \cap -\theta(\mathfrak{R}_{x}^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) \) and \( A_{-+} = -\mathfrak{R}_{x}^+ \cap \theta(\mathfrak{R}_{x}^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) \).

Similarly we have

\[
\mathfrak{R}_{x}^+ \cap \theta(\mathfrak{R}_{x}^+) = B_{++} \cup B_{+-} \cup B_{-+} \cup B_{--}
\]

with \( B_{++} = \mathfrak{R}_{x}^+ \cap \theta(\mathfrak{R}_{x}^+) \cap \mathfrak{R}_{x}^+ \cap \theta(\mathfrak{R}_{x}^+) \), \( B_{--} = -\mathfrak{R}_{x}^+ \cap -\theta(\mathfrak{R}_{x}^+) \cap -\mathfrak{R}_{xw}^+ \cap \theta(-\mathfrak{R}_{xw}^+) \), \( B_{+-} = \mathfrak{R}_{x}^+ \cap -\theta(\mathfrak{R}_{x}^+) \cap -\mathfrak{R}_{xw}^+ \cap \theta(-\mathfrak{R}_{xw}^+) \) and \( B_{-+} = \mathfrak{R}_{x}^+ \cap \theta(\mathfrak{R}_{x}^+) \cap -\mathfrak{R}_{xw}^+ \cap \theta(-\mathfrak{R}_{xw}^+) \).

We have the obvious relations : \( A_{++} = B_{++}, \ A_{--} = -B_{--}, \ A_{+-} = B_{-+}, \ A_{-+} = B_{+-} \). So we get \( m_{xw} - m_x \equiv |A_{+-}| + |B_{-+}| \).

Let consider \( A := \mathfrak{R}_{x}^+ \cap \mathfrak{R}_{xw}^+ \) and \( B := \mathfrak{R}_{x}^+ \cap -\mathfrak{R}_{xw}^+ \). We have

\[
m_{xw} - m_x \equiv |A \cap \theta(B)| + |A \cap -\theta(B)|
\]

\[
\equiv |A| + |A \cap \theta(A)| + |A \cap -\theta(A)|.
\]

Now we remark that

\[
|A \cap \theta(A)| \equiv |A \cap \theta(A) \cap \{ \theta(\alpha) = \alpha \}|
\]

\[
\equiv |\mathfrak{R}_{x}^+ \cap \mathfrak{R}_{xw}^+ \cap \{ \theta(\alpha) = \alpha \}|.
\]

Similarly

\[
|A \cap -\theta(A)| \equiv |A \cap -\theta(A) \cap \{ \theta(\alpha) = -\alpha \}|
\]

\[
\equiv |\mathfrak{R}_{x}^+ \cap \mathfrak{R}_{xw}^+ \cap \{ \theta(\alpha) = -\alpha \}|.
\]
At this stage we have proved that
\[ m_{xw} - m_x \equiv |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^- \cap \{\theta(\alpha) = \alpha\}| + |\mathfrak{R}_x^- \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = -\alpha\}|.
\]
\[ \equiv |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^- \cap \{\theta(\alpha) \neq -\alpha\}| + |\mathfrak{R}_x^- \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = \alpha\}|.
\]
As \( \delta_{x,w} = |\mathfrak{R}_x^+ \cap -\mathfrak{R}_x^- \cap \mathfrak{R}_{xw}^\mathfrak{ci}| \), we have \( |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^- \cap \{\theta(\alpha) = \alpha\}| + \delta_{x,w} \) is equal to \( \dim E_x^\mathfrak{ci} + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^- \cap \mathfrak{R}_{xw}^\mathfrak{ci}| \). This implies that \( m_{xw} + \delta_{x,w} \) is equal, modulo 2, to
\[ m_x + \dim E_x^\mathfrak{ci} + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^- \cap \mathfrak{R}_{xw}^\mathfrak{ci}| + |\mathfrak{R}_x^- \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq -\alpha\}|.
\]
\[ \equiv m_x + \dim E_x^\mathfrak{ci} + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^- \cap \mathfrak{R}_{xw}^\mathfrak{ci}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq \pm \alpha\}|.
\]
By definition \( m_x = \frac{1}{2}|\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^\mathfrak{ci})| + \dim E_x^\mathfrak{ci} \) and then
\[ m_x + \dim E_x^\mathfrak{ci} \equiv \frac{1}{2}|\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^\mathfrak{ci})| + \{\theta(\alpha) \neq \alpha\}| + |\mathfrak{R}_x^+ \cap \{\theta(\alpha) = \alpha\}| \equiv n_x.
\]
Finally we have proved that \( m_{xw} + \delta_{x,w} \) is equal, modulo 2, to \( n_x + k_{x,w} \).

\[ \square \]

4. EXAMPLES

In this section we will study in details some examples of our formula
\[ V^G_\lambda|_H = \sum_{k \in H \setminus Z_\theta/W} Q_k(\lambda)
\]
where \( Q_k(\lambda) = \text{Ind}_H^G(M_k(\lambda) \otimes \mathbb{C}_{\delta(z)} \otimes \wedge E_x^{\mathfrak{ci}}) \) and
\[ M_k(\lambda) = \frac{(-1)^{n_x}}{|W_x|} \sum_{w \in W} (-1)^{k_{x,w}} \mathbb{C}_{w \lambda} \otimes \det((E_x^{\mathfrak{ci}})_w) \otimes \text{Sym}(|E_x^{\mathfrak{ci}}|_w).
\]
Here the integers \( k_{x,w} \) and \( n_x \) are defined as follows:

- \( k_{x,w} = |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq \pm \alpha\}| + |\mathfrak{R}_x^- \cap \mathfrak{R}_{xw}^- \cap \mathfrak{R}_{xw}^{\mathfrak{ci}}|,
\)
- \( n_x = |\theta(\mathfrak{R}_x^+)| + \frac{1}{2}|\theta(\mathfrak{R}_x^+)| - \frac{1}{2} |\theta(\mathfrak{R}_x^-) \cap \mathfrak{R}_x^+ \cap \{\theta(\alpha) \neq \alpha\}|.
\)

4.1. \( K \subset K \times K \). Let \( K \) be a connected compact Lie group. Here we work with the Lie group \( G = K \times K \) and the involution \( \theta(k_1, k_2) = (k_2, k_1) \). The subgroup \( H = G^\theta \) is the group \( K \) embedded diagonally in \( G \).

Let \( T \) be a maximal torus of \( K \) and let \( W_K = N_K(T)/T \) be the Weyl group of \( K \). We denote by \( \mathfrak{R}_K \) the set of roots for \((K, T)\), and we make the choice of a set \( \mathfrak{R}_K^+ \) of positive roots.

In the next lemma we describe the critical set \( Z_\theta \) in the flag manifold \( F = K/T \times K/T \) of \( G \).

**Lemma 4.1.** We have \( Z_\theta = \bigcup_{w \in W_K} Z_w \) with \( Z_w = K \cdot (wT, T) \). In other words, the set \( H \setminus Z_\theta/W \) is a singleton.

**Proof.** The element \( x = (aT, bT) \in F \) belongs to \( Z_\theta \) if and only if \( (a^{-1}b, b^{-1}a) \in W \times W \). If \( b^{-1}a = w \in W \) then \( (aT, bT) \in Z_w \). \( \square \)
We take \(x = (T,T) \in Z_\theta\). For each \(w \in W_K\), we write \(xw = (wT,T)\). We take \(\lambda = (a,b) \in \Lambda_+^+ \times \Lambda_+^+ = \hat{G}\).

Our data are as follows:
- the group \(G_x\) is the maximal torus \(T \times T \subset K\),
- the group \(H_x\) is the maximal torus \(T \subset K\),
- \(\mathbb{C}_{(w\bullet \lambda)x + \delta(x)} = \mathbb{C}_{w(a+\rho)+b+\rho}\) as a character of \(T\),
- \(n_x = |\mathcal{R}_K^+|\),
- \(k_{x,w}\) is equal to \(|w \mathcal{R}_K^+ \cap \mathcal{R}_K^+| + |\mathcal{R}_K^+|\), so \((-1)^{k_{x,w}} = (-1)^w\),
- the vector spaces \(E^c_x, E^{n^c}_x\) are reduced to \(\{0\}\).

In this context we obtain the following relation
\[
V^K_a \otimes V^K_b = (-1)^{\dim(K/T)/2} \sum_{w \in W_K} (-1)^w \text{Ind}^K_T (\mathbb{C}_{w(a+\rho)+b+\rho}).
\]
This type of generalized Clebsch-Gordan formula what first noticed by Steinberg [15] (see Section 5).

**Example 4.2.** The irreducible representation \(SU(2)\) are parametrized by \(\mathbb{N}\). If \(n \geq 0\), the irreducible representation \(V_n\) of \(SU(2)\) satisfies
\[
V_n = \text{Ind}_{U(1)}^{SU(2)} ((\mathbb{C}_0 - \mathbb{C}_2) \otimes \mathbb{C}_n).
\]
If we take \(m \geq n \geq 0\), then (4.8) gives
\[
V_n \otimes V_m = \text{Ind}_{U(1)}^{SU(2)} (\mathbb{C}_{m-n}) - \text{Ind}_{U(1)}^{SU(2)} (\mathbb{C}_{m+n+2})
\]
\[
= \sum_{k=0}^{n} \text{Ind}_{U(1)}^{SU(2)} ((\mathbb{C}_0 - \mathbb{C}_2) \otimes \mathbb{C}_{m+n-2k})
\]
\[
= \sum_{k=0}^{n} V_{m+n-2k}.
\]
We recognize here the classical Clebsch-Gordan relations.

4.2. \(U(p) \times U(q) \subset U(p+q)\). Let \(p \geq q \geq 1\) and \(n = p + q\). We take \(G = U(n)\) with maximal torus \(T \simeq U(1)^n\) the subgroup formed by the diagonal matrices. We use the canonical map \(\tau\) from the symmetric group \(\mathfrak{S}_n\) into \(G\). It induces an isomorphism between \(\mathfrak{S}_n\) and the Weyl group \(W\) of \(G\).

We work with the involution \(\theta(g) = \Delta g \Delta^{-1}\) where \(\Delta := \text{diag}(I_p,-I_q)\): the subgroup fixed by \(\theta\) is \(H = U(p) \times U(q)\).

In the next section we describe the critical set \(Z_\theta \subset \mathcal{F}\). For another type of parametrization of \(H_C \setminus \mathcal{F}\), see Section 5 of [13].

4.2.1. The critical set. We consider the following elements of \(O(2)\):
\[
R = \begin{pmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

The element \(R\) is of order 8, \(R^2 = -J\) and \(R^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R = S\).

To any \(j \in \{0, \ldots, q\}\) we associate:
4.2.2. Localized indices.

\[ g_j := \text{diag}(1, \ldots, 1, R, \ldots, R, 1, \ldots, 1) \in G, \]

- the permutation \( w_j \in \mathfrak{S}_n \) that fixes the elements of \([1, \ldots, p - j] \cup [p + j + 1, \ldots, n]\) and such that
  \[ w_j(p - j + 2k - 1) = p - j + k, \quad w_j(p - j + 2k) = p + k, \quad \text{for} \quad 1 \leq k \leq j, \]

- \( k_j = \tau_jg_j \in G \), where \( \tau_j = \tau(w_j) \in N(T) \),
- \( x_j = k_jT \in \mathcal{F} \).

The adjoint map \( Ad(\tau_j) : G \to G \) sends the matrix \( \text{diag}(a_1, \ldots, a_{p-j}, b_1, \ldots, b_{2j}, c_1, \ldots, c_{q-j}) \) to the matrix \( \text{diag}(a_1, \ldots, a_{p-j}, b_1, b_3, \ldots, b_{2j-1}, b_2, b_4, \ldots, b_{2j}, c_1, \ldots, c_{q-j}) \).

We see then that
\[
\sigma_j := k_j^{-1}\Delta k_j = \text{diag}(1, \ldots, 1, S, \ldots, S, -1, \ldots, -1) \leq \text{n times} \quad \text{j times} \quad \text{q-j times}
\]
and \( k_j^{-1}\theta(k_j) = \sigma_j\Delta \) belong to \( N(T) \). Thus the elements \( x_0, \ldots, x_q \) belongs to \( Z_\theta \).

**Lemma 4.3.** In the flag manifold \( \mathcal{F} \) the set \( Z_\theta \) has the following description:
\[
Z_\theta = \bigcup_{0 \leq j \leq q} \bigcup_{\omega \in W_{x_j} \setminus W} Hx_jw
\]

So we have \( H \setminus Z_\theta / W = \{ \bar{x}_0, \ldots, \bar{x}_q \} \).

**Proof.** If \( 1 \leq a < b \leq n \), we denote by \( \tau_{a,b} \in N(T) \) the permutation matrix associated to the transposition \((a, b)\).

Let \( gT \in Z_\theta \). Then \( k := g^{-1}\theta(g)\Delta = g^{-1}\Delta g \) is an element of order two in \( N(T) \). The Weyl group element \( \bar{k} \in W \) is of order two, then there exists \( 0 \leq l \leq n/2 \), and a family \((a_1 < b_1), \ldots, (a_l < b_l)\) of disjoint couples in \( \{1, \ldots, n\} \) such that \( kT = \tau_{a_1,b_1} \ldots \tau_{a_l,b_l}T \).

Now, if we use the fact that the characteristic polynomial of \( k \in G \) is equal to \((X - 1)^p(X + 1)^q\) with \( p \geq q \geq 1 \), we see that

- \( l \leq q \),
- there exists \( n \in N(T) \) such that \( nkn^{-1} = \sigma_1 = k_l^{-1}\Delta k_l \).

If we take \( w = \bar{n} \in W \), the previous identity says that \( g \in Hk_lwT \). \( \square \)

4.2.2. Localized indices. We work with the groups \( T \subset H = U(p) \times U(q) \subset G = U(n) \) and the corresponding Lie algebras \( \mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{g} \). Let \( \mathfrak{R} = \{ \varepsilon_r - \varepsilon_s \} \) be the set of non-zero roots for the action of \( T \) on \( \mathfrak{g} \otimes \mathbb{C} \). We choose the Weyl chamber so that \( \mathfrak{R}^+ := \{ \varepsilon_r - \varepsilon_s, \ 1 \leq r < s \leq n \} \).

Let \( j \in \{0, \ldots, q\} \). The aim of this section is to compute the localized index \( Q_{x_j}(\lambda) \in \hat{R}(H) \). In order to have a fairly simple expression we will rewrite the terms of the form \( \text{Ind}^H_{\mathfrak{h}_{x_j}}(\mathbb{C}_\beta \otimes \Lambda L_{x_j}^j) \).

Let \( \{1, \ldots, n\} = I_1^j \cup I_2^j \cup I_3^j \cup I_4^j \) where \( I_1^j = \{ 1 \leq k \leq p - j \} \), \( I_2^j = \{ p - j + 1 \leq k \leq p \} \), \( I_3^j = \{ p + 1 \leq k \leq p + j \} \), and \( I_4^j = \{ p + j + 1 \leq k \leq n \} \).

For the maximal torus \( T \subset G \) we have a decomposition
\[ T \simeq T_1^j \times T_2^j \times T_3^j \times T_4^j \]
where $T^0_j = \{(t_k)_{k=1}^n, t_k \in U(1), t_k = 1 \text{ unless } k \in I_j^0\}$. Let $T_j \subset T^2_j \times T^3_j$ be the subtorus defined by the relations: an element $((t_k)_{k=1}^n, (s_k)_{k=1}^n) \in T^2_j \times T^3_j$ belongs to $T_j$ if and only if $t_{p-j+k} = s_{p+k}$ for all $1 \leq k \leq j$.

The elements of order two $\sigma_j \in G$ induce involutions on $G$ (by conjugation) that we still denote by $\sigma_j$. We start with a basic lemma whose proof is left to the reader.

**Lemma 4.4.** Let $x_j = k_j T \in F$.

- The adjoint map $Ad(k_j) : g \rightarrow g$ realizes an isomorphism between the vector space $t$ equipped with the involution induced by $\sigma_j$ and the vector space $g_{x_j}$ equipped with the involution $\theta$.
- The group $N(T)^{\sigma_j}/T^{\sigma_j}$ is isomorphic with $\mathfrak{g}_{p-j} \times \mathfrak{g}_{q-j} \times \mathfrak{g}_j \times \{\pm\}^j$.
- The adjoint map $Ad(k_j) : G \rightarrow G$ induces an isomorphism $\hat{N}(T)^{\sigma_j}/T^{\sigma_j} \simeq W_{x_j}$.
- The stabilizer subgroup $H_{x_j}$ is equal to $T^1_j \times T^3_j \subset T$.
- If $C_n$ is a character of $T$, then $C_{k_j,\alpha}$ is a character of $G_{x_j}$ and $C_{x_j,\alpha}$ is a character of $T$. We have the relation

$$C_{k_j,\alpha}|_{H_{x_j}} = C_{x_j,\alpha}|_{H_{x_j}}.$$

- The set of roots $\mathfrak{m}^{\mathfrak{ci}}_{x_j}$ is equal to

$$k_j \cdot \{\varepsilon_r - \varepsilon_s, 1 \leq r < s \leq p-j\} \bigcup k_j \cdot \{\varepsilon_r - \varepsilon_s, p+j+1 \leq r < s \leq n\}$$

and $\mathfrak{m}^{\mathfrak{ci}}_{x_j} = k_j \cdot \{\varepsilon_r - \varepsilon_s, 1 \leq r < p-j \land p+j+1 \leq s \leq n\}$.

We denote by $M_j$ the $T$-module $\mathbb{C}^{p-j} \otimes (\mathbb{C}^{q-j})^*$ where the subgroup $T^2_j \times T^3_j$ acts trivially and the $T^1_j \times T^4_j$-action is the canonical one. Thanks to Lemma 4.4, we have the following isomorphisms of $H_{x_j}$-modules: $E^{\mathfrak{ci}}_{x_j} \simeq M_j$. Following Lemma 3.7, one can associate the modules $(M_j)^{\mathfrak{ci}}_w$ and $|M_j|_w$ to each $w \in W$.

We consider the Lie group

$$K_j := U(p-j) \times U(q-j)$$

that we view as a subgroup of $H$ in such a way that $T^1_j \times T^4_j$ is a maximal torus of $K_j$. A set of positive roots for $(K_j, T^1_j \times T^4_j)$ is $\varepsilon_r - \varepsilon_s$ for $1 \leq r < s \leq p-j$ and $p+j+1 \leq r < s \leq n$. We equip $\mathfrak{k}_j/\left[\mathfrak{t}^1_j \times \mathfrak{t}^4_j\right]$ with a complex structure such that

$$\mathfrak{m}^{\mathfrak{ci}}_{x_j} \simeq \mathfrak{k}_j/\left[\mathfrak{t}^1_j \times \mathfrak{t}^4_j\right]$$

is an isomorphism of $T^1_j \times T^4_j$-modules.

The holomorphic induction map $\text{Hol}_{T^1_j \times T^4_j}^{K_j} : \hat{R}(T^1_j \times T^4_j) \rightarrow \hat{R}(K_j)$ is defined as follows:

$$\text{Hol}_{T^1_j \times T^4_j}^{K_j}(V) := \text{Ind}_{T^1_j \times T^4_j}^{K_j}(V \otimes \wedge \mathfrak{k}_j/\left[\mathfrak{t}^1_j \times \mathfrak{t}^4_j\right]).$$

If $a = (a_1 \geq \cdots \geq a_{p-j}) \in \mathbb{Z}^{p-j}$ and $b = (b_1 \geq \cdots \geq b_{q-j}) \in \mathbb{Z}^{q-j}$, then $C_{(a,b)}$ defines a character of $T^1_j \times T^4_j$ and

$$\text{Hol}_{T^1_j \times T^4_j}^{K_j}(C_{(a,b)}) = V_{\text{a}}^{U(p-j)} \otimes V_{\text{b}}^{U(q-j)}$$

is the irreducible representation of $K_j$ with highest weight $(a, b)$. 
A character $\mathbb{C}_\beta$ of the torus $T$ can be written $\mathbb{C}_\beta = \mathbb{C}_{\beta^{14}} \otimes \mathbb{C}_{\beta^{23}}$ where $\mathbb{C}_{\beta^{14}}$ is a character of $T^4_j \times T^4_j$ and $\mathbb{C}_{\beta^{23}}$ is a character of $T^3_j \times T^3_j$. Note that $\mathbb{C}_{\tau_j,\beta^{14}} |_{H_{x_j}} = \mathbb{C}_{\beta^{14}} \otimes \mathbb{C}_{\beta'}$ where $\beta' = \tau_j \beta^{23}$ defines a character of $T_j \subset T^2_j \times T^3_j$.

**Lemma 4.5.** Let $\mathbb{C}_\beta$ be a character of $T$. Then $\text{Ind}^H_{H_{x_j}}(\mathbb{C}_\beta |_{H_{x_j}} \otimes \wedge E^{ci}_{x_j})$ is equal to

$$\text{Ind}^H_{K_j \times T^2_j \times T^3_j}(\text{Hol}^{K_j}_{T^2_j \times T^3_j}(\mathbb{C}_{\beta^{14}} \otimes \mathbb{C}_{\beta^{23}}) \otimes L^2([T^2_j \times T^3_j] / T_j)),$$

where $L^2([T^2_j \times T^3_j] / T_j) = \text{Ind}^{T^2_j \times T^3_j}_{T^2_j}(1) \in \widehat{R}(T^2_j \times T^3_j)$.

**Remark 4.6.** To gain some space in our formulas, we will write $\text{Hol}^{K_j}_{T^2_j \times T^3_j}(\mathbb{C}_\beta)$ instead of $\text{Hol}^{K_j}_{T^2_j \times T^3_j}(\mathbb{C}_{\beta^{14}} \otimes \mathbb{C}_{\beta^{23}})$

We need to fix some notations.

**Definition 4.7.** • Let $\chi : H \to \mathbb{C}$ be the character $(A, B) \mapsto \det(A)^{-1}\det(B)^{-1}$.
• Let $\psi_j$ be the character
defined at the weight

$$\sum_{1 \leq k \leq j} (q - p + 2 + 2j - 4k)\varepsilon_{p-j+k}.$$

• For any $(j, w) \in [0, q] \times W$, we define the integer $d_{j,w}$ by the relation

$$d_{j,w} = \dim(M^+_j)_w + |\{1 \leq k \leq j, w^{-1}(p - j + 2k - 1) < w^{-1}(p - j + 2k)\}|.$$

A small computation gives the following lemma.

**Lemma 4.8.** • The $H_{x_j}$-character $\mathbb{C}_{\delta(x_j)}$ is equal to $\chi \otimes \psi_j |_{H_{x_j}}$.
• For any $(j, w) \in [0, q] \times W$, we have $(-1)^{n_{x_j} + k_{x_j,w}} = (-1)^j (-1)^{-j} (-1)^d_{j,w}$.

The main result of this section is the following proposition.

**Proposition 4.9.**

$$V^U_{\lambda}(\text{Ind}^{U(p) \times U(q)}_{U(p) \times U(q)}) = \sum_{j=0}^q Q_{x_j}(\lambda)$$

where $Q_{x_j}(\lambda) \in \widehat{R}(U(p) \times U(q))$ is determined by the relation

$$Q_{x_j}(\lambda) = \frac{(-1)^j (-1)^{d_{j,w}}}{|W_{x_j}|} \chi \otimes \sum_{w \in W} (-1)^w (-1)^d_{j,w} \text{Ind}^{U(p) \times U(q)}_{K_j \times T^2_j \times T^3_j}(A^w_j(\lambda) \otimes \psi_j).$$

Here the elements $A^w_j(\lambda) \in \widehat{R}(K_j \times T^2_j \times T^3_j)$ are defined as follows:

$$A^w_j(\lambda) = \text{Hol}^{K_j}_{T^2_j \times T^3_j}(\mathbb{C}_{\tau_j(\lambda)} \otimes \det((M^+_j)_w) \otimes \text{Sym}((M^+_j)_w)) \otimes L^2([T^2_j \times T^3_j] / T_j).$$

We finish this section by considering particular situations.

\[2\text{Remark that } \psi_j \text{ is trivial } T^1_j \times T^3_j \times T^4_j.\]
4.2.3. The extreme cases: $j = 0$ or $j = q$. When $j = 0$, the torus $T_0^2$ and $T_0^3$ are trivial and $K_0 = U(p) \times U(q) = H$. Moreover $\mathbb{M}_0 = \mathbb{C}^p \otimes (\mathbb{C}^q)^*$ and $d_{0,w} = \dim(\mathbb{M}_0)_w^+$. Thanks to Lemma 4.4, we know also that $W_{x_0} \simeq \mathcal{G}_p \times \mathcal{G}_q$. So we get the formula

$$Q_{x_0}(\lambda) = \frac{1}{p!q!} \sum_{w \in W} (\pm)_{w} \text{Hol}^H_T (\mathcal{C}_{w,\lambda} \otimes \det((\mathbb{M}_0)_w^+) \otimes \text{Sym}(|\mathbb{M}_0|_w))$$

where $(\pm)_w = (-1)^w(-1)^{\dim(\mathbb{M}_0)_w^+}$.

Remark 4.10. An useful exercise is to consider the term

$$A_w := (\pm)_{w} \text{Hol}^H_T (\mathcal{C}_{w,\lambda} \otimes \det((\mathbb{M}_0)_w^+) \otimes \text{Sym}(|\mathbb{M}_0|_w))$$

and verify that $A_{w'}w = A_w$ when $w' \in W_{x_0}$.

When $j = q$, the torus $T_q^4$ is trivial, $K_q = U(p - q)$ and $\mathbb{M}_q = \{0\}$. Moreover $W_{x_q} \simeq \mathcal{G}_{p-q} \times \mathcal{G}_q \times \{\pm\}^q$. In this case we obtain

$$Q_{x_q}(\lambda) = \frac{(-1)^{q(n+1)}}{(p-q)!q!2^q} \lambda_1^q \otimes \sum_{w \in W} (-1)^w(-1)^{d_{q,w}} Q_q^w(\lambda)$$

with

$$Q_q^w(\lambda) = \text{Ind}_{U(p-q) \times T_q^2 \times T_q^3}^{U(p) \times U(q)} \left( \text{Hol}_{T_q^3}^{U(p-q)} (\mathcal{C}_{\tau_q(w,\lambda)} \otimes \psi_q \otimes L^2([T_q^2 \times T_q^3] / T_q)) \right).$$

4.2.4. $U(n-1) \times U(1) \subset U(n)$. Here we are in the case where $q = 1$, and so

$$V_{\lambda}^{U(n)}|_{U(n-1) \times U(1)} = Q_{x_0}(\lambda) + Q_{x_1}(\lambda).$$

To simplify the expression of $Q_{x_0}(\lambda)$ we use the fact that the quotient $W_{x_0} \setminus W$ is represented by the class of the elements $\tau_{k,n} \in G$ associated to the transposition $(k,n)$ for $1 \leq k \leq n$. We write $T = T' \times U(1)$ where $T'$ is a maximal torus of $U(n-1)$. The $T'$-module $\mathbb{C}^{n-1}$ can be decomposed as $\mathbb{V}_k \otimes \mathbb{V}'_k$ where $\mathbb{V}_k = \sum_{j=1}^{k-1} \mathbb{C}_{\varepsilon_j}$ and $\mathbb{V}'_k = \sum_{j=k}^{n-1} \mathbb{C}_{\varepsilon_j}$.

The T-module $\mathbb{M}_0$ is equal to $\mathbb{C}^{n-1} \otimes \mathbb{C}^* = \mathbb{V}_k \otimes \mathbb{C}_{-\varepsilon_n} \oplus \mathbb{V}_k' \otimes \mathbb{C}_{-\varepsilon_n}$ and the polarized $T$-module $|\mathbb{M}_0|_{\tau_{k,n}}$ is equal to $\mathbb{V}_k \otimes \mathbb{C}_{-\varepsilon_n} \oplus \mathbb{V}_k' \otimes \mathbb{C}_{-\varepsilon_n}$. We have $\dim(\mathbb{M}_0)_{\tau_{k,n}}^+ = k-1$ and $\det(\mathbb{M}_0)_{\tau_{k,n}} = \mathbb{C}_{\mu_k} \otimes \mathbb{C}_{-\varepsilon_{n-k}}$ with $\mu_k = \sum_{j=1}^{k-1} \varepsilon_j$.

So we obtain

$$Q_{x_0}(\lambda) = \sum_{a,k \geq 0, 1 \leq k \leq n} (\pm)_k \text{Hol}_{T}^{U(n-1)} (\mathcal{C}_{\tau_{k,n},\lambda+\mu_k} \otimes \text{Sym}^b(\mathbb{V}_k) \otimes \text{Sym}^a(\overline{\mathbb{V}_k})) \otimes \mathbb{C}_{\varepsilon_n}^{1+a-b-k},$$

where $(\pm)_k = (-1)^k$ if $k < n$ and $(\pm)_n = (-1)^{n-1}$.

We consider now the term $Q_{x_1}(\lambda)$. When $j = q = 1$, the torus $T_1^4$ is trivial, $K_1 = U(n-2)$ and $\mathbb{M}_1 = \{0\}$. Moreover $W_{x_1} \simeq \mathcal{G}_{n-2} \times \{\pm\}$, $\tau_1 = Id$ and $\psi_1 = (2-n)\varepsilon_{n-1}$. Hence the quotient $W_{x_1} \setminus W$ is represented by the class of the elements $\tau_{l,n} \tau_{k,n-1}$ for $1 \leq k < l \leq n$. We denote by $\lambda_{kl}$ the term $\tau_{l,n} \tau_{k,n-1} \bullet \lambda$. 
In this case we obtain
\[ Q_{\bar{z}_1}(\lambda) = (-1)^n \chi \otimes \left( Q_{\bar{z}_1}^{n-1,n}(\lambda) - \sum_{1 \leq k < n-1} Q_{\bar{z}_1}^{k,n}(\lambda) + \sum_{1 \leq k < \ell \leq n-1} Q_{\bar{z}_1}^{k,l}(\lambda) \right) \]

with
\[ Q_{\bar{z}_1}^{k,l}(\lambda) = \text{Ind}_{U(n-1) \times T^3}^{U(n-2) \times T^3} \left( \text{Hol}_{T^1}^{U(n-2)} (\mathbb{C}_{\lambda_{kl}}) \otimes \psi_1 \otimes L^2([T^2 \times T^3] / T^1) \right) \]
\[ = \sum_{a \in \mathbb{Z}} \text{Ind}_{U(n-2) \times T^2}^{U(n-1) \times T^2} \left( \text{Hol}_{T^1}^{U(n-2)} (\mathbb{C}_{\lambda_{kl}}) \otimes \mathbb{C}_{\epsilon_{n-1}}^{\otimes a} \right) \otimes \mathbb{C}_{\epsilon_n}^{\otimes 2-n-a}. \]

Let us finish this section by considering the simplest example: \( U(1) \times U(1) \subset U(2) \). Take \( \lambda = (\lambda_1, \lambda_2) \in \hat{U}(2) \). We have \( V_\lambda^{U(2)}|_{U(1) \times U(1)} = Q_{\bar{z}_0}(\lambda) + Q_{\bar{z}_1}(\lambda) \) where
\[ Q_{\bar{z}_0}(\lambda) = - \mathbb{C}_\lambda \otimes \sum_{-\infty}^{\lambda_2 - \lambda_1 - 1} \mathbb{C}^{\otimes k}_{\epsilon_1 - \epsilon_2} - \mathbb{C}_\lambda \otimes \sum_{k \geq 1} \mathbb{C}^{\otimes k}_{\epsilon_1 - \epsilon_2} \]
and \( Q_{\bar{z}_1}(\lambda) = \mathbb{C}_\lambda \otimes \sum_{k \in \mathbb{Z}} \mathbb{C}^{\otimes k}_{\epsilon_1 - \epsilon_2} \). We recover the basic relation
\[ V_\lambda^{U(2)}|_{U(1) \times U(1)} = \mathbb{C}_\lambda \otimes \sum_{k = \lambda_2 - \lambda_1}^{0} \mathbb{C}^{\otimes k}_{\epsilon_1 - \epsilon_2}. \]

4.2.5. \( U(n-1) \subset U(n) \). If we restrict the representation \( V_\lambda^{U(n)} \) to the subgroup \( U(n-1) \), we get
\[ V_\lambda^{U(n)}|_{U(n-1)} = Q_0(\lambda) + Q_1(\lambda), \]
where the characters \( Q_0(\lambda), Q_1(\lambda) \in \hat{R}(U(n-1)) \) are given by the relations
\[ Q_0(\lambda) = \sum_{k=1}^{n} (\pm)_k \text{Hol}_{T^1}^{U(n-1)} (\mathbb{C}_{\tau_{\lambda,n} \cdot \lambda + \mu_k} \otimes \text{Sym}(V_k) \otimes \text{Sym}(\overline{V}_k)), \]
and
\[ Q_1(\lambda) = (-1)^{n \det} \otimes \left( Q_{1,n}^{n-1,n}(\lambda) - \sum_{1 \leq k < n-1} Q_{1,n}^{k,n}(\lambda) + \sum_{1 \leq k < \ell \leq n-1} Q_{1,n}^{k,l}(\lambda) \right), \]
with \( Q_{1,n}^{k,l}(\lambda) = \text{Ind}_{U(n-2)}^{U(n-1)} \left( \text{Hol}_{T^1}^{U(n-2)} (\mathbb{C}_{\lambda_{kl}}) \right) \).

Let’s detail expression 4.9 when \( n = 3 \).
Small calculations give \( Q_0(\lambda) = B_1(\lambda) + B_2(\lambda) + B_3(\lambda) \) with
\[ B_1(\lambda) = \text{Hol}_{T^1}^{U(2)} (\mathbb{C}_{\tau_{\lambda} \cdot \lambda + \mu_1} \otimes \text{Sym}(\mathbb{C}^2)) = \sum_{\lambda_2 - 1 \geq a \geq \lambda_3 - 1 \geq b} V_{(a,b)}^{U(2)}, \]
\[ B_2(\lambda) = \text{Hol}_{T^1}^{U(2)} (\mathbb{C}_{\tau_{\lambda} \cdot \lambda + \mu_2} \otimes \text{Sym}(\mathbb{C}_{\epsilon_1}) \otimes \text{Sym}(\overline{\mathbb{C}_{\epsilon_2}})) = \sum_{a \geq \lambda_1 + 1, \lambda_3 - 1 \geq b} V_{(a,b)}^{U(2)}, \]
\[ B_3(\lambda) = \text{Hol}_{T^1}^{U(2)} (\mathbb{C}_{\tau_{\lambda} \cdot \lambda + \mu_3} \otimes \text{Sym}(\mathbb{C}^2)) = \sum_{a \geq \lambda_1 + 1, b \geq \lambda_2 + 1} V_{(a,b)}^{U(2)}. \]
For the other term, we obtain $Q_1(\lambda) = A_1(\lambda) - A_2(\lambda) - A_3(\lambda)$ with

$$A_1(\lambda) = \det \otimes Q_1^{1,3}(\lambda) = \det \otimes \text{Ind}_{U(1)}^{U(2)}(C_{\lambda_2-1}) = \sum_{a \geq \lambda_2 \geq b} V_{U(2)}^{U(2)}(\lambda),$$

$$A_2(\lambda) = \det \otimes Q_1^{2,3}(\lambda) = \det \otimes \text{Ind}_{U(1)}^{U(2)}(C_{\lambda_1}) = \sum_{a \geq \lambda_1+1 \geq b} V_{U(2)}^{U(2)}(\lambda),$$

$$A_3(\lambda) = \det \otimes Q_1^{1,2}(\lambda) = \det \otimes \text{Ind}_{U(1)}^{U(2)}(C_{\lambda_3-2}) = \sum_{a \geq \lambda_3-1 \geq b} V_{U(2)}^{U(2)}(\lambda).$$

Finally one checks that the decomposition

$$V_{\lambda}^{U(3)}|_{U(2)} = A_1(\lambda) - A_2(\lambda) - A_3(\lambda) + B_1(\lambda) + B_2(\lambda) + B_3(\lambda)$$

permits to recover the classical relation $V_{\lambda}^{U(3)}|_{U(2)} = \sum_{\lambda_1 \geq a \geq \lambda_2 \geq b \geq \lambda_3} V_{U(2)}^{U(2)}(\lambda)$ (see [4]).

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In Figure 1 we can visualise the supports of the different characters: we have

$$A_1(\lambda) = \sum_{\mu \in A_1} V_{\mu}^{U(2)}, \quad A_2(\lambda) = \sum_{\mu \in A_2 \cup B_1} V_{\mu}^{U(2)}, \quad A_3(\lambda) = \sum_{\mu \in A_3 \cup B_1} V_{\mu}^{U(2)},$$

$$B_1(\lambda) = \sum_{\mu \in B_1} V_{\mu}^{U(2)}, \quad B_2(\lambda) = \sum_{\mu \in B_2} V_{\mu}^{U(2)}, \quad B_3(\lambda) = \sum_{\mu \in B_3} V_{\mu}^{U(2)},$$

so that $V_{\lambda}^{U(3)}|_{U(2)} = \sum_{\mu \in C} V_{\mu}^{U(2)}$.

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5. Kostant multiplicity formula

The aim of this section is first to recall the Kostant multiplicity formula: we follow the line of [4], Section 8.2. Then, we rewrite it in a form similar to the one we use in this article (see Proposition 5.4). Finally, we detail Kostant’s multiplicity formula for
the restriction of $U(n)$ to $U(n - 1)$, in order to compare it with the calculations done in Section 4.2.3.

Let $G' \subset G$ be two connected compact Lie groups with maximal tori $T' \subset T$. The corresponding Lie algebras are $\mathfrak{t} \subset \mathfrak{g}$ and $\mathfrak{t}' \subset \mathfrak{g}'$. In this section, we make the following regularity assumption:

(R) The centralizer $Z_{\mathfrak{g}}(\mathfrak{t}')$ of $\mathfrak{t}'$ in $\mathfrak{g}$ is abelian.

We recall the following well-known fact.

Lemma 5.1. The assumption (R) is valid when $G'$ is the connected component of a fixed-point subgroup of an involution.

Proof: Suppose that $G' = (G^*)_0$ for some involution $\tau$. Then, we have a decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{q}$ where $\mathfrak{q} = \{X \in \mathfrak{g}, \tau(X) = -X\}$. The centralizer $Z_{\mathfrak{g}}(\mathfrak{t}')$ is stable under the involution $\tau$ and under the adjoint action of $T$. Thus $\mathfrak{t} \subset Z_{\mathfrak{g}}(\mathfrak{t}') = \mathfrak{t}' \oplus Z_{\mathfrak{q}}(\mathfrak{t}')$: in particular the torus $T$ is invariant under $\tau$.

If $Z_{\mathfrak{g}}(\mathfrak{t}')$ is not abelian, there exists a roots $\alpha \in \mathfrak{R}$ such that $(\mathfrak{g}_C)_\alpha \subset Z_{\mathfrak{g}}(\mathfrak{t}')_C = \mathfrak{t}'_C \oplus Z_{\mathfrak{q}}(\mathfrak{t}')_C$. Then we obtain a contradiction: on one hand $(\mathfrak{g}_C)_\alpha \subset Z_{\mathfrak{g}}(\mathfrak{t}')_C$ implies that $\alpha|_{\mathfrak{t}'} = 0$ and on the other hand since $(\mathfrak{g}_C)_\alpha \subset \mathfrak{q}_C$, we must have $\sigma(\alpha) = \alpha$. The two conditions $\alpha|_{\mathfrak{t}'} = 0$ and $\sigma(\alpha) = \alpha$ implies that $\alpha = 0$. □.

Let $\mathfrak{R}$ and $\mathfrak{R}'$ and be the set of roots for the pairs $T \subset G$ and $T' \subset G'$. Note that assumption (R) is equivalent to:

(R') There exists $X_o \in \mathfrak{t}'$ such that $\langle \alpha, X_o \rangle \neq 0$ for all $\alpha \in \mathfrak{R}$.

If $\xi \in \mathfrak{t}^*$ we write $\overline{\xi}$ for the restriction of $\xi$ to $(\mathfrak{t}')^*$. Because of our assumption, $\overline{\pi} = 0$ for all $\alpha \in \mathfrak{R}$.

The positive roots are $\mathfrak{R}_+ := \{\alpha \in \mathfrak{R}, \langle \alpha, X_o \rangle > 0\}$ and $\mathfrak{R}_+ := \{\beta \in \mathfrak{R}', \langle \beta, X_o \rangle > 0\}$. We write $\overline{\mathfrak{R}}_+ := \{\overline{\pi}, \alpha \in \mathfrak{R}_+\}$ for the set of positive restricted roots: we keep track of the multiplicity $n_a = \#\{\alpha \in \mathfrak{R}_+, \overline{\pi} = a\}$ of each element $a \in \overline{\mathfrak{R}}_+$. Since $\mathfrak{R}_+'$ is contained in $\overline{\mathfrak{R}}_+$, we may consider the set of roots $\Sigma := \overline{\mathfrak{R}}_+ - \mathfrak{R}_+$: the multiplicity of $\beta \in \mathfrak{R}_+$ in $\Sigma$ is equal to

$$m_{\beta} := \begin{cases} n_{\beta} & \text{if } \beta \notin \mathfrak{R}_+ ', \\ n_{\beta} - 1 & \text{if } \beta \in \mathfrak{R}_+ '. \end{cases}$$

Let $\Lambda' \subset (\mathfrak{t}')^*$ be the lattice of weights for the torus $T'$. Let $(\mathfrak{t}')^*_+ \subset \mathfrak{t}'^*$ be the Weyl chamber associated to the system $\mathfrak{R}_+$. The irreducible representations of $G'$ are parameterized by $\Lambda'_+ = \Lambda' \cap (\mathfrak{t}')^*_+$.

Definition 5.2. We denote by $P_\Sigma : \Lambda' \to \mathbb{N}$ the partition function associated to the set $\Sigma$. For all $\xi' \in \Lambda'$, $P_\Sigma(\xi')$ is the number of way of writing $\xi' = \sum_{\beta \in \Sigma} x_\beta \beta$, where $x_\beta \in \mathbb{N}$ and each $\beta$ that occurs is counted with multiplicity $m_\beta$.

For dominant weights $\lambda \in \Lambda_+$ and $\mu \in \Lambda'_+$, we denote by $m_\lambda(\mu)$ the multiplicity of the irreducible $G'$-representation $V^G_\mu$ with highest weight $\mu$ in the irreducible $G$-representation $V^G_\lambda$ with highest weight $\lambda$.

If $w \in W$, we note $w \cdot \lambda := w(\lambda + \rho) - \rho$ where $\rho$ is the half sum of the positive roots.
Theorem 5.3. The branching multiplicities are

\begin{equation}
(5.10) \quad m_\lambda(\mu) = \sum_{w \in W} (-1)^w \mathcal{P}_\Sigma(w \bullet \lambda - \mu).
\end{equation}

We briefly recall how to obtain (5.10). Let $\chi^G_\lambda$ be the character of $V^G_\lambda$. The Weyl relation gives

\[ \chi^G_\lambda |_{T} \prod_{\alpha \in R_+} (1 - e^{-\alpha}) = \sum_{w \in W} (-1)^w e^{w \bullet \lambda}. \]

When we restrict this identity to $T' \subset T$, the relations

\[ \prod_{\alpha \in R_+} (1 - e^{-\alpha}) |_{T'} = \prod_{\beta \in \Pi'_+} (1 - e^{-\beta}) \prod_{\gamma \in \Sigma} (1 - e^{-\gamma}) \]

and

\[ \prod_{\gamma \in \Sigma} (1 - e^{-\gamma}) \left( \sum_{\xi' \in \Lambda'} \mathcal{P}_\Sigma(\xi') e^{-\xi'} \right) = 1 \]

permit to obtain

\[ \chi^G_\lambda |_{T'} \prod_{\beta \in \Pi'_+} (1 - e^{-\beta}) = \left( \sum_{w \in W} (-1)^w e^{w \bullet \lambda} \right) \left( \sum_{\xi' \in \Lambda'} \mathcal{P}_\Sigma(\xi') e^{-\xi'} \right) \]

\[ = \sum_{\xi' \in \Lambda'} N_\lambda(\xi') e^{\xi'}. \]

with $N_\lambda(\xi') := \sum_{w \in W} (-1)^w \mathcal{P}_\Sigma(w \bullet \lambda - \xi')$.

On the other hand, we have the decomposition $\chi^G_\lambda |_H = \sum_{\mu \in \Lambda'_+} m_\lambda(\mu) \chi^G_\mu$ and then

\[ \chi^G_\lambda |_{T'} \prod_{\beta \in \Pi'_+} (1 - e^{-\beta}) = \sum_{\mu \in \Lambda'_+} m_\lambda(\mu) \chi^G_\mu \prod_{\beta \in \Pi'_+} (1 - e^{-\beta}) \]

\[ = \sum_{\mu \in \Lambda'_+} \sum_{w' \in W'} (-1)^w' m_\lambda(\mu) e^{w' \bullet \mu}. \]

Finally, we obtain the identity

\[ \sum_{\xi' \in \Lambda'} N_\lambda(\xi') e^{\xi'} = \sum_{\mu \in \Lambda'_+} \sum_{w' \in W'} (-1)^w' m_\lambda(\mu) e^{w' \bullet \mu}, \]

that shows two things:

- $N_\lambda(\mu) = m_\lambda(\mu)$ if $\mu$ is dominant,
- $N_\lambda(w' \bullet \xi') = (-1)^w N_\lambda(\xi')$, for every $(w', \xi') \in W' \times \Lambda'$.

At this stage, we have proved Kostant’s multiplicity formula. In the following we rewrite this formula in another form. Let’s consider the following $T'$-module

\[ n_\Sigma := \bigoplus_{\beta \in \Sigma} \mathbb{C}_{-\beta}. \]

\footnote{Here $w' \bullet \xi' := w'(\lambda + \rho') - \rho'$ where $\rho' = \frac{1}{2} \sum_{\beta \in \Pi'_+} \beta$.}
Proposition 5.4. For any $\lambda \in \Lambda_+$, we have the following restriction formula

\[(5.11) \quad V^G_\lambda|_{G'} = \frac{1}{\# W'} \sum_{w \in W} (-1)^w \text{Hol}^G_{T'}(\mathbb{C}_{w\cdot\lambda} \otimes \text{Sym}(n_\Sigma)).\]

Proof: Since $\text{Sym}(n_\Sigma) = \sum_{\xi' \in \Lambda'} P_\Sigma(\xi') \mathbb{C}_{-\xi'}$ we have

$$\sum_{w \in W} (-1)^w \mathbb{C}_{w\cdot\lambda} \otimes \text{Sym}(n_\Sigma) = \sum_{w \in W} \sum_{\xi' \in \Lambda'} (-1)^w P_\Sigma(\xi') \mathbb{C}_{w\cdot\lambda - \xi'} = \sum_{\xi' \in \Lambda'} N_\lambda(\xi') \mathbb{C}_{\xi'}.$$ 

Hence the right hand side of (5.11) is equal to $\frac{1}{\# W'} \sum_{\xi' \in \Lambda'} N_\lambda(\xi') \text{Hol}^G_{T'}(\mathbb{C}_{\xi'})$. We use now the following facts:

- $N_\lambda(w' \cdot \xi') \text{Hol}^G_{T'}(\mathbb{C}_{w'\cdot\xi'}) = N_\Sigma(\xi') \text{Hol}^G_{T'}(\mathbb{C}_{\xi'})$ for every $(w', \xi') \in W' \times \Lambda'$.
- $N_\lambda(\xi') \text{Hol}^G_{T'}(\mathbb{C}_{\xi'}) = 0$ if $\xi' \notin W' \cdot \Lambda'_+$.
- $N_\lambda(\mu) \text{Hol}^G_{T'}(\mathbb{C}_\mu) = m_\lambda(\mu) V^G_\mu$ if $\mu \in \Lambda'_+$.

We have completed the proof of (5.11). $\blacksquare$

We conclude this section with a few examples.

5.1. $K \subseteq K \times K$. Let $K$ be a connected compact Lie group. Here we work with the Lie group $G = K \times K$ containing $K$ diagonally. Here $\Sigma \subseteq \mathfrak{t}^*$ is equal to the set $\mathfrak{a}_+$ of positive roots for $K$. We denote by $P : \Lambda \to \mathbb{N}$ the partition function associated to the set $\mathfrak{a}_+$.

If $\lambda, \mu, \nu$ are three dominant weights, we denote by $c'_{\lambda, \mu}$ the multiplicity of $V^K_\nu$ in $V^K_\lambda \otimes V^K_\mu$. The branching formula (5.10) becomes

$$c'_{\lambda, \mu} = \sum_{w_1, w_2 \in W} (-1)^{w_1 w_2} P(w_1 \cdot \lambda + w_2 \cdot \mu - \nu).$$

This formula was first observed by Steinberg [15].

Let's take a closer look at the branching formula (5.11). The $T$-module $n_\Sigma$ is equal to $n := \sum_{\alpha > 0} C_{-\alpha}$. By definition of the holomorphic induction map $\text{Hol}^K_T$, we have

$$\text{Hol}^K_T(\Theta \otimes \text{Sym}(n)) = (-1)^d \text{Ind}^K_T(\Theta \otimes \mathbb{C}_{\Sigma_2})$$

for any $\Theta \in R(T)$, with $d = \frac{1}{2} \dim K/T$. Finally, (5.11) becomes

$$V^K_\lambda \otimes V^K_\mu|_K = \frac{(-1)^d}{\# W} \sum_{w_1, w_2 \in W} (-1)^{w_1 w_2} \text{Ind}^K_T(\mathbb{C}_{w_1(\lambda + \rho) + w_2(\mu + \rho)})$$

$$= (-1)^d \sum_{w \in W} (-1)^w \text{Ind}^K_T(\mathbb{C}_{w(\lambda + \rho) + \mu + \rho}).$$

The latter formula is also obtained in (4.8).
5.2. $U(p) \times U(q) \subset U(n)$. In this example the torus $T$ of diagonal matrices is the maximal torus for both $U(n)$ and the subgroup $U(p) \times U(q)$. Here the $T$-module $n_{\Sigma}$ is the $T$-restriction of the $U(p) \times U(q)$-module $(C^p)^* \otimes C^q$, and the quotient $W' \setminus W$ is isomorphic to the subset $\text{Shuffle}(p, q)$ formed by the elements $w \in S_n$ satisfying $w(1) < \cdots < w(p)$ and $w(p + 1) < \cdots < w(p + q)$. Here, the branching formula (5.11) gives

$$V_{\lambda}^{U(n)}|_{U(p) \times U(q)} = \left( \sum_{w \in \text{Shuffle}(p,q)} (-1)^w \text{Hol}_{T}^{U(p) \times U(q)}(C^w \lambda) \right) \otimes \text{Sym}((C^p)^* \otimes C^q).$$

Let’s consider the case $q = 1$. From (5.12), we derive the following branching formula for the restriction to the subgroup $U(n-1)$:

$$V_{\lambda}^{U(n)}|_{U(n-1)} = \left( \sum_{1 \leq k \leq n} (-1)^{n-k} V_{\lambda[k]}^{U(n-1)} \right) \otimes \text{Sym}((C^{n-1})^*),$$

with $\lambda[n] = (\lambda_1, \ldots, \lambda_{n-1})$ and $\lambda[k] = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1} - 1, \ldots, \lambda_{n-1})$ for $1 \leq k \leq n-1$.

Let’s consider the case $n = 3$. For any $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3)$ we obtain the following formula

$$V_{\lambda}^{U(3)}|_{U(2)} = \left( V_{\lambda[3]}^{U(2)} - V_{\lambda[2]}^{U(2)} + V_{\lambda[1]}^{U(2)} \right) \otimes \text{Sym}((C^2)^*).$$
Hence $V^{U(3)}_\lambda|_{U(2)} = A_1(\lambda) - A_2(\lambda) + A_3(\lambda)$ with
\[
A_3(\lambda) = V^{U(2)}_{\lambda[3]} \otimes \Sym((\mathbb{C}^2)^*) = \sum_{\lambda_1 \geq a \geq \lambda_2 \geq b} V^{U(2)}_{(a,b)} ,
\]
\[
A_2(\lambda) = V^{U(2)}_{\lambda[2]} \otimes \Sym((\mathbb{C}^2)^*) = \sum_{\lambda_1 \geq a \geq \lambda_3 - 1 \geq b} V^{U(2)}_{(a,b)} ,
\]
\[
A_1(\lambda) = V^{U(2)}_{\lambda[1]} \otimes \Sym((\mathbb{C}^2)^*) = \sum_{\lambda_2 - 1 \geq a \geq \lambda_3 - 1 \geq b} V^{U(2)}_{(a,b)} .
\]

We recover the classical branching formula $V^{U(3)}_\lambda|_{U(2)} := \sum_{\lambda_1 \geq a \geq \lambda_2 \geq b \geq \lambda_3} V^{U(2)}_{(a,b)}$ (see [4], section 8.1). In Figure 2, one can visualize the support of each characters $A_k(\lambda)$.

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