Excited states of a static dilute spherical Bose condensate in a trap

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(December 31, 2021)

The Bogoliubov approximation is used to study the excited states of a dilute gas of $N$ atomic bosons trapped in an isotropic harmonic potential characterized by a frequency $\omega_0$ and an oscillator length $d_0 = \sqrt{\hbar/m\omega_0}$. The self-consistent static Bose condensate has macroscopic occupation number $N_0 \gg 1$, with nonuniform spherical condensate density $n_0(r)$; by assumption, the depletion of the condensate is small ($N' \equiv N - N_0 \ll N_0$). The linearized density fluctuation operator $\hat{\rho}'$ and velocity potential operator $\hat{\Phi}'$ satisfy coupled equations that embody particle conservation and Bernoulli’s theorem. For each angular momentum $l$, introduction of quasiparticle operators yields coupled eigenvalue equations for the excited states; they can be expressed either in terms of Bogoliubov coherence amplitudes $u_l(r)$ and $v_l(r)$ that determine the appropriate linear combinations of particle operators, or in terms of hydrodynamic amplitudes $\rho_l'(r)$ and $\Phi_l'(r)$. The hydrodynamic picture suggests a simple variational approximation for $l > 0$ that provides an upper bound for the lowest eigenvalue $\omega_l$ and an estimate for the corresponding zero-temperature occupation number $N'_l$; both expressions closely resemble those for a uniform bulk Bose condensate.

PACS numbers: 03.75.Fi, 05.30.Jp, 32.80.Pj, 67.90.+z

*Contributed paper for Low Temperature Conference LT21, Prague, August, 1996, to appear in Czechoslovak Journal of Physics
I. INTRODUCTION

Recent experimental verification of Bose condensation in dilute confined $^{87}\text{Rb}$ has stimulated extensive theoretical research. In the Bogoliubov approximation, $N_0$ atoms occupy the macroscopic condensate, with $N_0 \lesssim N$ and only small depletion ($N'/N \equiv 1 - N_0/N \ll 1$). For a repulsive scattering length $a > 0$ and mean atomic density $n$, this depletion is of order $\sqrt{na^3}$ and thus nonperturbative.

The balance between kinetic energy $\frac{\hbar^2}{2m}\xi^2$ and interaction energy $4\pi a \frac{\hbar^2 n}{m}$ defines the “healing length” $\xi = (8\pi an)^{-1/2}$, and the Bogoliubov approximation requires that $a \ll \xi$. In addition, a confined Bose gas with typical dimension $R$ differs qualitatively from an ideal Bose gas whenever $\xi \ll R$. The present work uses the Bogoliubov approximation to provide a hydrodynamic description of the excited states and a variational estimate of the eigenvalue $\omega_l$ and zero-temperature occupation number $N'_l$ for the lowest radial mode for each positive angular momentum $l \geq 1$.

II. BASIC FORMALISM

In [1], the harmonic trap $U = \frac{1}{2}m\omega_0^2 r^2$ (here taken as isotropic) has a size $d_0 = \sqrt{\frac{\hbar}{m\omega_0}} \approx 1 \mu\text{m}$, and the positive scattering length $a \approx 10 \text{ nm}$ acts to expand the atomic cloud. For $N \equiv Na/d_0 \gg 1$, the actual atomic density $\approx N/R^3$ is smaller than the naive estimate $\approx N/d_0^3$, with the radial expansion factor $R/d_0$ of order $\mathcal{N}^{1/5}$; thus the Bogoliubov approximation holds for $Na^3/R^3 \approx \mathcal{N}^{2/5}(a/d_0)^2 \ll 1$ [namely, $N \ll (d_0/a)^6 \sim 10^{12}$]. The additional condition $R \gg \xi$ for the failure of an ideal-Bose-gas model now requires $(8\pi \mathcal{N}^{4/5})^{1/2} \gg 1$, which here holds even for $N \approx 100$.

The condensate wave function $\Psi$ satisfies the nonlinear Gross-Pitaevskii equation [4,5]

$$\left(T + U - \mu + V\right)\Psi = 0,$$

where $T = -\hbar^2 \nabla^2 / 2m$ is the kinetic energy, $V \equiv 4\pi a \hbar^2 m^{-1}|\Psi|^2$ characterizes the repulsive interaction energy, and the chemical potential $\mu$ must be adjusted to ensure that $\int dV |\Psi|^2 = \ldots$
\( N_0 \) and that \( \Psi \to 0 \) at large distances. In the present case, the condensate is stationary with density \( n_0(r) = |\Psi(r)|^2 \). For large \( N \), the kinetic energy can be ignored \[8\], giving the “Thomas-Fermi” approximation

\[
V_{TF} = (\mu - U) \theta(\mu - U) \equiv \frac{1}{2} \hbar \omega_0 (R^2 - r^2) \theta(R - r),
\]

(2)

where lengths are expressed in units of \( d_0 \), \( \mu = \frac{1}{2} \hbar \omega_0 R^2 \) defines the dimensionless “radius” \( R \) of the condensate, and \( R^5 = 15N \) \[8\].

III. HYDRODYNAMIC DESCRIPTION

The (small) noncondensate field operator \( \hat{\phi} \) obeys the linear equation

\[
i \hbar \partial \hat{\phi}/\partial t = (T + U - \mu + 2V) \hat{\phi} + V \hat{\phi}^\dagger,
\]

(3)

along with the adjoint equation. In the Bogoliubov approximation \( (N' \ll N_0) \), the total particle density operator becomes \( \hat{\psi}^\dagger \hat{\psi} \approx n_0 + \hat{\rho}' \), where the operator \( \hat{\rho}' = \sqrt{n_0}(\hat{\phi} + \hat{\phi}^\dagger) \) characterizes the fluctuating noncondensate density; similarly, the particle current operator here becomes \( \hat{j}' \approx n_0 \nabla \hat{\Phi}' \), where \( \hat{\Phi}' = (\hbar/2mi \sqrt{n_0})(\hat{\phi} - \hat{\phi}^\dagger) \) is the velocity potential operator. Equation \( (3) \) readily yields

\[
\partial \hat{\rho}'/\partial t + \nabla \cdot (n_0 \nabla \hat{\Phi}') = 0,
\]

(4)

and the linearized Bernoulli’s equation \[8\]

\[
\frac{\partial \hat{\Phi}'}{\partial t} + \frac{4\pi a \hbar^2}{m^2} \hat{\rho}' + \frac{\hbar^2}{4m^2n_0} \left[ \nabla \cdot \left( \frac{\hat{\rho}' \nabla n_0}{n_0} \right) - \nabla^2 \hat{\rho}' \right] = 0.
\]

(5)

The latter can be recognized as a linearization of the exact classical Bernoulli’s theorem \[8\]

\[
U + \frac{1}{2}mv^2 + (e + p)/n + m\partial \Phi / \partial t = 0
\]

for a compressible irrotational isentropic fluid with density \( n_0 + \hat{\rho}' \) and velocity potential \( \Phi' \) [here, \( p(n) = 2\pi a \hbar^2 n^2 / m \) is the pressure functional and \( e(n) = \sqrt{n} T \sqrt{n} + p(n) \) is the (quantum) energy-density functional].
IV. BOGOLIUBOV EQUATIONS

The hydrodynamic description provides a valuable qualitative picture of the normal modes of the nonuniform compressible spherical condensate, but the resulting equations involve spatial derivatives of $n_0$; furthermore, in the Thomas-Fermi approximation (2), the appropriate boundary conditions at $R$ are not obvious. Thus it is preferable to return to Eq. (3), expressing the field operator $\hat{\phi}$ as a linear combination of quasiparticle operators $\alpha_j$ and $\alpha_j^\dagger$

$$\hat{\phi}(r, t) = \sum_j \left[ u_j(r) \alpha_j e^{-i\omega_j t} - v_j^*(r) \alpha_j^\dagger e^{i\omega_j t} \right];$$

(6)

here, the wave-function coefficients $u_j$ and $v_j$ constitute a two-component vector function $U_j$ that obeys the matrix “Bogoliubov” eigenvalue equation

$$(T + U - \mu + 2V) U_j - V \tau_3 U_j = \hbar \omega_j \tau_3 U_j.$$  

(7)

In contrast to Eqs. (4) and (5), only $V \propto n_0$ appears here, and the quantum interpretation provides the obvious boundary condition that $U_j$ vanish for $r \to \infty$ (note that the hydrodynamic amplitudes are essentially linear combinations of $u_j$ and $v_j$).

In the case of a static spherical condensate, the Bogoliubov Eqs. (7) have solutions of the form $u_l(r)Y_{lm}$ and $v_l(r)Y_{lm}$; in addition, they have a simple variational basis [5] that yields an upper bound for the lowest eigenvalue $\omega_l$ for each positive angular momentum $l > 0$. To ensure the proper normalization, take $u_l(r) = \cosh \chi_l f_l(r)$ and $v_l(r) = \sinh \chi_l f_l(r)$, where $\int_0^\infty r^2 dr |f_l(r)|^2 = 1$. With the Thomas-Fermi approximation (2), the analogy to acoustic waves in a sphere suggests the radial trial function $f_l(r) \propto x^l(1 - x^2) \theta(1 - x)$, where $x \equiv r/R$. A simple variational calculation yields the estimate $\omega_l/\omega_0 = \sqrt{T_l^2 + 2T_l V_l}$, with $T_l = \frac{1}{4} (2l + 3)(2l + 7)/R^2$ and $V_l = 3R^2/(2l + 9)$ the expectation values of the dimensionless kinetic energy $-\frac{1}{2} \nabla^2$ and Thomas-Fermi potential energy $\frac{1}{2} (R^2 - r^2)$. This frequency has a very different form for small and large $l$

$$\frac{\omega_l}{\omega_0} \approx \begin{cases} \sqrt{\frac{3}{2} (2l + 3)(2l + 7)/(2l + 9)}, & \text{for } T_l \ll V_l; \\ (2l + 3)(2l + 7)/4R^2, & \text{for } T_l \gg V_l. \end{cases}$$

(8)
Stringari [8] has obtained a related expression $\omega_l/\omega_0 = \sqrt{l}$ with a purely hydrodynamic model, adding that $\omega_l = \omega_0$ is an exact result. Note the close analogy to the Bogoliubov excitation energy $E_k = \sqrt{T_k^2 + 2T_kV_k}$ for a uniform bulk medium, where $T_k = \hbar^2 k^2 / 2m$ and $V_k = 4\pi a\hbar^2 n_0 / m$ [2].

Each of these lowest excited states for $l > 0$ has a zero-temperature occupation number $N'_l = \int_0^R r^2 dr |v_l(r)|^2$, with the variational estimate $N'_l = \sinh^2 \chi_l = \frac{1}{2}(T_l + V_l) / \sqrt{T_l^2 + 2T_lV_l - \frac{1}{2}}$, again very similar in structure to that for a uniform Bose gas [4]. In particular, $N'_l \approx \sqrt{V_l / 8T_l} \gg 1$ for $T_l \ll V_l$, and $N'_l \approx V_l^2 / 4T_l^2 \ll 1$ for $T_l \gg V_l$.

ACKNOWLEDGMENTS

Supported in part by the National Science Foundation under Grant No. DMR 94-21888; I am grateful for helpful correspondence with B. V. Svistunov.

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