Billiard Representation for Multidimensional Quantum Cosmology near the Singularity

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ABSTRACT

The degenerate Lagrangian system describing a lot of cosmological models is considered. When certain restrictions on the parameters of the model are imposed, the dynamics of the model near the "singularity" is reduced to a billiard on the Lobachevsky space. The Wheeler-DeWitt equation in the asymptotical regime is solved and a third-quantized model is suggested.

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1 Introduction

In the present paper we deal with a stochastic behavior in multidimensional cosmological models near the singularity (see [1-8] and references therein). A large variety of these models may be described by the following Lagrangian [9]

\[ L = L(z^a, \dot{z}^a, N) = \frac{1}{2} N^{-1} \eta_{ab} \dot{z}^a \dot{z}^b - NV(z), \]  

where \( N > 0 \) is the Lagrange multiplier (modified lapse function), \( (\eta_{ab}) = \text{diag}(-1, +1, \ldots, +1) \) is matrix of minisuperspace metric, \( a, b = 0, \ldots, n-1 \), and

\[ V(z) = A_0 + \sum_{\alpha=1}^{m} A_{\alpha} \exp(u_\alpha^a z^a) \]  

is the potential. \( A_0 > 0 \) corresponds to Zeldovich matter and \( A_\alpha \neq 0 \). We impose the following restrictions on the vectors \( u^a = (u_0^a, \vec{w}^a) \) in the potential (1.2)

1) \( A_\alpha > 0 \), if \( (u^0)^2 = -(u_0^a)^2 + (\vec{w}^a)^2 > 0 \);  
2) \( u^0 > 0 \) for all \( \alpha = 1, \ldots, m \). (1.3) (1.4)

Here we consider the classical and quantum behavior of the dynamical system (1.1) for \( n \geq 3 \) in the limit

\[ z^2 = -(z^0)^2 + (\vec{z})^2 \to -\infty, \quad z = (z^0, \vec{z}) \in \mathcal{V}_-, \]  

where \( \mathcal{V}_- \equiv \{(z^0, \vec{z}) \in \mathbb{R}^n | z^0 < -|\vec{z}|\} \) is the lower light cone. The limit (1.5) implies \( z^0 \to -\infty \) and under certain additional assumptions describes the approaching to the singularity.

2 Billiard representation

We describe briefly our recent results on billiard representation near the "singularity" for the dynamical system (1.1) [10,11]. We restrict the Lagrange system (1.1) on the lower light cone and introduce the analogues of the Misner-Chitre coordinates in \( \mathcal{V}_- \) [12,13]

\[ z^0 = -\exp(-y^0) \frac{1 + \vec{y}^2}{1 - \vec{y}^2}, \]  
\[ \vec{z} = -2 \exp(-y^0) \frac{\vec{y}}{1 - \vec{y}^2}, \]

|\( \vec{y} \)| < 1. We also fix the gauge (or time parametrization)

\[ N = \exp(-2y^0) = -z^2. \]  

We get for the restriction of the Lagrangian on \( \mathcal{V}_- \)

\[ L_- = \frac{1}{2} [-(\dot{y}^0)^2 + h_{ij}(\vec{y})\dot{y}^i \dot{y}^j] - \exp(-2y^0)V. \]  

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Here
\[ h_{ij}(\vec{y}) = 4\delta_{ij}(1 - \vec{y}^2)^{-2}, \] (2.5)
i, j = 1, \ldots, n - 1, are the components of the Riemannian metric on the \((n - 1)\)-dimensional open unit disk (ball)
\[ D^{n-1} \equiv \{ \vec{y} = (y^1, \ldots, y^{n-1})|\|\vec{y}\| < 1 \} \subset \mathbb{R}^{n-1}. \] (2.6)

The pair \((D^{n-1}, h = h_{ij}(\vec{y})dy^i \otimes dy^j)\) is one of the realizations of the \((n - 1)\)-dimensional Lobachevsky space \(H^{n-1}\). We also get the energy constraint
\[ E_- = -\frac{1}{2}(\dot{y}^0)^2 + \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j + \exp(-2y^0)V = 0. \] (2.7)

Now we are interested in the behavior of the dynamical system in the limit \(y^0 \to -\infty\) (or, equivalently, in the limit (1.5)). Using the restrictions (1.3), (1.4) we obtain [11]
\lim_{y^0 \to -\infty} \exp(-2y^0)(V - A_0) = V(\vec{y}, B) \equiv 0, \quad \vec{y} \in B, \quad +\infty, \quad \vec{y} \in D^{n-1} \setminus B, \] (2.8)
where
\[ B = \bigcap_{\alpha \in \Delta_+} B(u^\alpha) \subset D^{n-1}, \] (2.9)
\[ \Delta_+ \equiv \{ \alpha | (u^\alpha)^2 > 0 \}, \] (2.10)
and
\[ B(u^\alpha) \equiv \{ \vec{y} \in D^{n-1} | \|\vec{y} + \vec{v}^\alpha|u_0^\alpha| > \sqrt{(\vec{v}^\alpha)^2 - 1} \}, \] (2.11)
\(\alpha \in \Delta_+\). \(B\) is an open domain. Its boundary \(\partial B = \overline{B} \setminus B\) is formed by certain parts of \(m_+ = |\Delta_+|\) (\(m_+\) is the number of elements in \(\Delta_+\)) of \((n - 2)\)-dimensional spheres with the centers in the points
\[ \vec{v}^\alpha = -\vec{v}^\alpha/u_0^\alpha, \quad \alpha \in \Delta_+, \] (2.12)
(|\(\vec{v}^\alpha| > 1\) and radii
\[ r_\alpha = \sqrt{(\vec{v}^\alpha)^2 - 1} \] (2.13)
respectively. So, in the limit \(y^0 \to -\infty\) we are led to the dynamical system
\[ L_\infty = -\frac{1}{2}(\dot{y}^0)^2 + \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j - V_\infty, \] (2.14)
\[ E_\infty = -\frac{1}{2}(\dot{y}^0)^2 + \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j + V_\infty = 0, \] (2.15)
where
\[ V_\infty = A_0 \exp(-2y^0) + V(\vec{y}, B). \] (2.16)
After the separating of \(y^0\) variable
\[ y^0 = \omega(t - t_0), \quad A_0 = 0, \]
\[ \frac{1}{2} \ln \left| \frac{2A_0}{\omega^2} \sinh^2(\omega(t - t_0)) \right|, \quad A_0 \neq 0. \] (2.17)
(\omega > 0 , t_0 are constants) the dynamical system is reduced to the Lagrange system with the Lagrangian
\[ L_B = \frac{1}{2} h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j - V(\vec{y}, B). \] (2.18)
Due to (2.17)
\[ E_B = \frac{1}{2} h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j + V(\vec{y}, B) = \frac{\omega^2}{2}. \] (2.19)
The limits \( t \to -\infty \) for \( A_0 = 0 \) and \( t \to t_0 + 0 \) for \( A_0 \neq 0 \) describe the approach to the singularity. When the set (2.10) is empty (\( \Delta_+ = \emptyset \)) we have \( B = D^{n-1} \) and the Lagrangian (2.18) describes the geodesic flow on the Lobachevsky space \( H^{n-1} = (D^{n-1}, h_{ij}dy^i \otimes dy^j) \).

When \( \Delta_+ \neq \emptyset \) the Lagrangian (2.18) describes the motion of the particle of unit mass, moving in the \((n-1)\)-dimensional billiard \( B \subset D^{n-1} \) (see (2.9)). The geodesic motion in \( B \) corresponds to a "Kasner epoch" and the reflection from the boundary corresponds to the change of Kasner epochs. Let \( A_0 = 0 \). When the volume of \( B \) is finite: \( volB < +\infty \), we have a stochastic behaviour near the singularity. Such situation takes place in Bianchi-IX cosmology [2]. When the billiard \( B \) has an infinite volume: \( volB = +\infty \) there are open zones of non-zero measure at the infinite sphere \( S^{n-2} \). After a finite number of reflections from the boundary the particle moves toward one of these open zones. For corresponding cosmological model we get the "Kasner-like" behavior in the limit \( t \to -\infty \). For \( A_0 \neq 0 \) (when Zeldovich matter is present) in the limit \( t \to t_0 + 0 \) we have \( y^0 \to -\infty \) and \( \vec{y}(t) \to \vec{y}_0 \in B \). So, the stochastic behavior near the singularity is absent in this case.

Proposition [11]. The billiard \( B \) (2.9) has a finite volume if and only if the point-like sources of light located at the points \( \vec{v}^\alpha \) (2.12) illuminate the unit sphere \( S^{n-2} \). The problem of illumination of convex body in multidimensional vector space by point-like sources for the first time was considered in [14, 15]. For the case of \( S^{n-2} \) this problem is equivalent to the problem of covering the spheres with spheres. There exist a topological bound on the number of point-like sources \( m_+ \) illuminating the sphere \( S^{n-2} \) [15]:
\[ m_+ \geq n. \] (2.20)
So, the stochastic behaviour for the solutions of Lagrange equations for the Lagrangian (1.1) with the gauge fixing (2.3) the limit (1.5) (near the "singularity") may take place only if \( A_0 = 0 \) and the number of terms in the potential with \( (u^\alpha)^2 > 0 \) is no less than the minisuperspace dimension.

3 Quantum case

The quantization of zero-energy constraint (2.7) leads to the Wheeler-DeWitt (WDW) equation in the gauge (2.3) [16,17]
\[ (-\frac{1}{2} \Delta[\bar{G}] + a_n R[\bar{G}] + \exp(-2y^0) V) \Psi = 0. \] (3.1)
Here \( \Psi = \Psi(y) \) is "the wave function of the Universe", \( V = V(y) \) is the potential (1.2), \( a_n = (n - 2)/8(n - 1) \), \( \Delta[\bar{G}] \) and \( R[\bar{G}] \) are the Laplace-Beltrami operator and the scalar
curvature of the minisuperspace metric

\[ \bar{G} = -dy^0 \otimes dy^0 + h, \quad h = h_{ij}(\bar{y})dy^i \otimes dy^j. \]  (3.2)

(We remind that, \( h \) is the metric on Lobachevsky space \( \mathbb{D}^{n-1} \)). The form of WDW eq. (3.1) follows from the demands of minisuperspace invariance and conformal covariance. Using

\[ \Delta[\bar{G}] = -\left(\frac{\partial}{\partial y^0}\right)^2 + \Delta[h], \quad R[\bar{G}] = R[h] = -(n-1)(n-2), \]  (3.3)

we rewrite (3.1) in the form

\[ \left(\frac{1}{2}\left(\frac{\partial}{\partial y^0}\right)^2 - \frac{1}{2}\Delta[h] - \frac{(n-2)^2}{8} + \exp(-2y^0)V\right)\Psi = 0. \]  (3.4)

In the limit \( y^0 \to -\infty \) the WDW eq. reduces to the relations

\[ \left(\left(\frac{\partial}{\partial y^0}\right)^2 + 2A_0 \exp(-2y^0) - \Delta_*(h)\right)\Psi_\infty = 0, \quad \Psi_\infty|_{\partial B} = 0, \]  (3.5)

where \( \partial B = \bar{B} \setminus B \) is the boundary of the billiard \( B \) (2.9) (in \( \mathbb{D}^{n-1} \)) and

\[ \Delta_*(h) = \Delta[h] + \frac{(n-2)^2}{4}. \]  (3.6)

Now, we suppose that \( \bar{B} \) is compact and the operator (3.6) with the boundary condition (3.5) has a negative spectrum, i.e.

\[ \Delta_*(h) u_n = -\omega_n^2 u_n, \]  (3.7)

\( \omega_n > 0, \ n = 0, 1, \ldots \) (this is valid at least for “small enough” \( B \)). Using (3.7) we get the general solution of the asymptotic WDW eq. (3.5)

\[ \Psi_\infty(y^0, \bar{y}) = \sum_{n=0}^{\infty} [a_n \Psi_n(y^0, \bar{y}) + a_n^* \Psi_n^*(y^0, \bar{y})]. \]  (3.8)

Here

\[ \Psi_n(y^0, \bar{y}) = f_n(y^0) u_n(\bar{y}), \]  (3.9)

where the set \( u_n \) in (3.7) is orthonormal basis in \( L_2(B) \):

\[ (u_n, u_m) = \delta_{mn}, \]  (3.10)

\( m, n = 0, 1, \ldots \). For \( A_0 = 0 \)

\[ f_n(t) = (2\omega_n)^{-1/2} \exp(-i\omega_n y^0). \]  (3.11)

The second quantized model (in gravity also called as ”third quantized”) in this case is defined by eq. (3.8) with the familiar relations imposed

\[ [a_n, a_m^*] = \delta_{mn}, \quad a_m^* = (a_m)^+, \quad a_n|0 >= 0, \]  (3.12)
\( m, n = 0, 1, \ldots \) For \( A_0 \neq 0 \) we consider two basis

\[
  f^{\text{in}}_n(y^0) = \left[ \frac{\pi}{2 \sinh(\pi \omega_n)} \right]^{1/2} J_{-i\omega_n} \left( \sqrt{2A_0} e^{-y^0} \right),
\]

(3.13)

\[
  f^{\text{out}}_n(y^0) = \frac{1}{2} (\pi)^{1/2} \exp(-\pi \omega_n/2) H^{(1)}_{i\omega_n} \left( \sqrt{2A_0} e^{-y^0} \right),
\]

(3.14)

where \( J_\nu, H^{(1)}_{\nu} \) are Bessel and Hankel functions respectively. The functions (3.13), (3.14) are related by Bogoljubov transformation [18]

\[
  f^{\text{out}}_n = \alpha_n f^{\text{in}}_n + \beta_n (f^{\text{in}}_n)^*,
\]

(3.15)

where

\[
  \alpha_n = \left( e^{\pi \omega_n} \right) \left( \frac{2}{2 \sinh \pi \omega_n} \right)^{1/2}, \quad \beta_n = -\left( e^{-\pi \omega_n} \right) \left( \frac{2}{2 \sinh \pi \omega_n} \right)^{1/2}.
\]

(3.16)

A standard calculation [18] gives the mean number of "out-Universes" containing in "in-vacuum"

\[
  N_n = \langle 0, \text{in} | a^\dagger_{n, \text{out}} a_{n, \text{out}} | 0, \text{in} \rangle = |\beta_n|^2 = (\exp(2\pi \omega_n) - 1)^{-1}.
\]

(3.17)

So, we obtained the Planck distribution with the temperature \( T = 1/2\pi \). We note that for Bianchi-IX case the considered scheme was suggested in [19].
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