D-Branes on Noncompact Calabi-Yau Manifolds: K-Theory and Monodromy

Xenia de la Ossa, Bogdan Florea and Harald Skarke

Mathematical Institute, University of Oxford,
24-29 St. Giles’, Oxford OX1 3LB, England

We study D-branes on smooth noncompact toric Calabi-Yau manifolds that are resolutions of abelian orbifold singularities. Such a space has a distinguished basis \( \{ S_i \} \) for the compactly supported K-theory. Using local mirror symmetry we demonstrate that the \( S_i \) have simple transformation properties under monodromy; in particular, they are the objects that generate monodromy around the principal component of the discriminant locus. One of our examples, the toric resolution of \( \mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \), is a three parameter model for which we are able to give an explicit solution of the GKZ system.
1. Introduction

The simplest manifestation of mirror symmetry is an exchange of the Hodge numbers $h_{ii}$ and $h_{i,n-i}$ of a Calabi-Yau $n$-fold $X$ and its mirror $\tilde{X}$. Interpreted naively, this would seem to imply identifications between $n$-cycles on $\tilde{X}$ and holomorphic cycles on $X$. This leads to the following puzzle. Monodromy in the complex structure moduli space of $\tilde{X}$ can take $n$-cycles to arbitrary other $n$-cycles, so this would lead to the counterintuitive picture of mixing cycles of arbitrary even dimension on $X$.

Mathematically this puzzle is resolved by Kontsevich’s conjecture [1] that the relevant objects on $X$ are the elements of the bounded derived category $D^b$ of coherent sheaves. In terms of physics, we now have the following intuitive picture. We should not think of a cycle as a geometric object per se, but as something that a D-brane can wrap. A D-brane corresponds to a cycle with a vector bundle on it only in a semiclassical limit. In a more general construction a D-brane can be obtained from higher dimensional branes and anti-branes, leading to an interpretation in terms of K-theory [2] that is consistent with Kontsevich’s approach.

Monodromy in the complexified Kähler moduli space of Calabi-Yau manifolds has been the object of recent studies both by mathematicians [3]-[8] and by physicists [9]-[20]. One particular approach [3,13,16,17] uses well known results on McKay correspondence [21]-[28] to obtain a special basis for the K-theory on $X$. These authors study noncompact toric Calabi-Yau manifolds that are resolutions of singularities of the type $\mathbb{C}^d/\mathbb{Z}_n$ (or more general Calabi-Yau singularities in [19]) with a single exceptional divisor, mainly in order to describe compact Calabi-Yau manifolds as hypersurfaces in the exceptional divisor.

In this work we study D-branes on non-compact toric Calabi-Yau manifolds in their own right, with the aim of getting a better understanding of what the fundamental D-brane degrees of freedom are and how they behave under monodromy. We show how to construct a distinguished basis for the compactly supported K-theory with a number of remarkable properties, the most striking being the fact that the elements of this basis seem to generate the monodromy around the principal component of the discriminant locus in the same way as the structure sheaf $\mathcal{O}_X$ does in the compact case. We consider cases with more than one exceptional divisor, and we test the applicability of the above statements beyond the realm of McKay correspondence. We do not have general proofs for our statements, but we demonstrate their validity in various examples with the help of local mirror symmetry.

The outline of this paper is as follows. In the next section we present necessary material on toric varieties, their Mori and Kähler cones, and the secondary fan. In section 3
we introduce local mirror symmetry, toric moduli spaces and the GKZ system. While the material in these sections is known, its presentation relying on the holomorphic quotient approach to toric varieties may be useful; besides, it serves to establish notations and to introduce some of our examples. Section 4 is the core of this paper. There we discuss K-theory and known results related to McKay correspondence and proceed to define the distinguished generators $S_i$ of the compactly supported K-theory. We find that these generators are the ones that are responsible for monodromy around the principal component of the discriminant locus. In section 5 we demonstrate that our methods work in cases that are more complicated than examples of the type $\mathbb{C}^d/\mathbb{Z}_n$. We consider the case of $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and find that it is possible to solve the corresponding GKZ system, with results that agree precisely with our assertions.

2. Toric Calabi-Yau manifolds

We start with presenting some general considerations on non-compact toric Calabi-Yau manifolds and their Kähler and Mori cones that will be useful later. The results obtained here are standard [29]-[32], but our derivations from basic facts in toric geometry are possibly simpler than what can be found in the literature.

The data of a $d$-dimensional toric variety $X$ can be specified in terms of a fan $\Sigma$ in a lattice $N$ isomorphic to $\mathbb{Z}^d$. $X$ is smooth whenever each of the $d$-dimensional cones in $\Sigma$ is generated over $\mathbb{R}_+$ by exactly $d$ lattice vectors that generate $N$ over $\mathbb{Z}$. We will only consider this case.

Perhaps the simplest way of describing $X$ is as follows: Assume that there are $k$ one dimensional cones in $\Sigma$ generated by lattice vectors $v_1, \ldots, v_k$. Assign a homogeneous variable $z_i$ to each of the $v_i$ and a multiplicative equivalence relation among the $z_i$,

$$(z_1, \ldots, z_k) \sim (\lambda^{q_1} z_1, \ldots, \lambda^{q_k} z_k)$$

with $\lambda \in \mathbb{C}^*$ for any linear relation $q_1 v_1 + \cdots + q_k v_k = 0$ among the generators $v_i$. The $q_i$ can be normalized to be integers without common divisor; in the context of a gauged linear sigma model they are the charges with respect to the $U(1)$ fields. The number of independent relations of the type (2.1) is $k - d$.

Define a subset $\mathbb{C}^k \setminus F_\Sigma$ of $\mathbb{C}^k = \{(z_1, \ldots, z_k)\}$ as the set of all $k$-tuples of $z_i$ with the following property: If $z_i$ vanishes for all $i \in I \subset \{1, \ldots, k\}$, then all $v_i$ with $i \in I$ belong to the same cone. Then $X$ is $(\mathbb{C}^k \setminus F_\Sigma)/(\mathbb{C}^*)^{k-d}$, where the division by $(\mathbb{C}^*)^{k-d}$
is implemented by taking equivalence classes with respect to the multiplicative relations (2.1).

Every one dimensional cone generated by \( v_i \) corresponds in a natural way to the divisor \( D_i \) determined by \( z_i = 0 \). Similarly, an \( l \) dimensional cone spanned by \( v_{i_1}, \ldots, v_{i_l} \) determines the codimension \( l \) subspace \( z_{i_1} = \ldots = z_{i_l} = 0 \) of \( X \).

Monomials of the type \( z_1^{a_1} \ldots z_k^{a_k} \) are sections of line bundles \( O(a_1D_1 + \cdots + a_kD_k) \). If we denote by \( M \) the lattice dual to \( N \) and by \( \langle \ , \ \rangle \) the pairing between \( N \) and \( M \), it is easily checked that monomials of the form \( z_1^{(v_1,m)} \ldots z_k^{(v_k,m)} \) with \( m \in M \) are meromorphic functions (i.e., invariant under (2.1)) on \( X \). This implies the linear equivalence relations

\[
\langle v_1, m \rangle D_1 + \ldots + \langle v_k, m \rangle D_k \sim 0 \quad \text{for any } m \in M. \tag{2.2}
\]

Conversely, if a divisor of the form \( a_1D_1 + \cdots + a_kD_k \) belongs to the trivial class, then there exists an \( m \in M \) such that \( a_i = \langle v_i, m \rangle \) for all \( i \).

A calculation similar to the way the canonical divisor of \( \mathbb{P}^d \) is determined shows that the canonical divisor of \( X \) is given by \( -D_1 - \cdots - D_k \). Thus \( X \) is Calabi-Yau if and only if \( D_1 + \cdots + D_k \) is trivial, i.e. if and only if there exists an \( m \in M \) such that \( \langle v_i, m \rangle = 1 \) for every \( i \). Therefore the \( v_i \) must all lie in the same affine hyperplane. We will make use of this fact by drawing toric diagrams in dimension \( d - 1 \) that display only the endpoints of the \( v_i \).

We will be interested in the Kähler moduli space of \( X \). The dual of the Kähler cone is the Mori cone spanned by effective curves. Toric curves are determined by \( (d-1) \)-dimensional cones \( \sigma_{d-1} \) in \( \Sigma \). If a curve is compact, the corresponding cone is the boundary between two \( d \)-dimensional cones \( \sigma_d^{(1)}, \sigma_d^{(2)} \). If we denote the integer generators of \( \sigma_{d-1}, \sigma_d^{(1)}, \sigma_d^{(2)} \) by \( \{v_1, \ldots, v_{d-1}\}, \{v_1, \ldots, v_{d-1}, v_d\}, \{v_1, \ldots, v_{d-1}, v_{d+1}\} \), respectively (remember that we are assuming that our cones are simplicial and their generators generate \( N \)), we find that \( v_d + v_{d+1} \) must lie in the intersection of the hyperplane of \( \sigma_{d-1} \) with \( N \) and so there exists a unique linear relation of the form \( l_1v_1 + \ldots + l_{d+1}v_{d+1} = 0 \) with \( l_d = l_{d+1} = 1 \) and all \( l_i \) integer.

We will now argue that the \( l_i \) are actually the intersection numbers between the curve \( C = D_1 \cdot \ldots \cdot D_{d-1} \) determined by \( \sigma_{d-1} \) and the toric divisors \( D_i \). Our general rules imply that intersection numbers between \( d \) different toric divisors are 1 or 0 depending on whether these divisors form a cone in \( \Sigma \). This implies \( C \cdot D_d = l_d = 1, C \cdot D_{d+1} = l_{d+1} = 1 \) and \( C \cdot D_i = 0 \) for \( i > d + 1 \). For calculating \( C \cdot D_i \) with \( i < d \) we have to use linear
equivalence relations of the type (2.2). To calculate $C \cdot D_1$ we may choose $m$ to fulfill
\[ \langle v_1, m \rangle = 1 \text{ and } \langle v_2, m \rangle = \cdots = \langle v_d, m \rangle = 0. \]
Then
\[ 0 \sim \sum \langle v_i, m \rangle D_i = \langle v_1, m \rangle D_1 + \langle v_{d+1}, m \rangle D_{d+1} + \cdots = D_1 + \langle -l_1 v_1 - \cdots - l_d v_d, m \rangle D_{d+1} + \cdots = D_1 - l_1 D_{d+1} + \cdots, \]
i.e. $D_1 \sim l_1 D_{d+1} + \cdots$ where ‘\ldots’ stands for $D_i$ with $i > d + 1$ which do not intersect $C$. Thus we find that $C \cdot D_1 = l_1 C \cdot D_{d+1} = l_1$. As our choice of $D_1$ among the $D_i$ with $i < d$ was arbitrary, we have indeed shown that $C \cdot D_i = l_i$ for any $i$.

A set of generators for the Mori cone is then given by all those curves $C^{(i)}$ whose $l^{(i)}$ cannot be written as nonnegative linear combinations of the other $l^{(j)}$. The matrix $L$ whose lines are the $l^{(i)}$ of the Mori cone generators has the following remarkable properties: Any Matrix $Q$ consisting of $d - k$ independent (linear combinations of) lines of $L$ serves as a ‘charge matrix’ for the relations (2.1). If the Mori cone is simplicial, we just have $L = Q$. This will be the case in most of our examples, so we will not distinguish between $L$ and $Q$ in these cases. Any column of $L$ is associated with a toric divisor $D_i$. If a linear combination $\sum_j L_{ij} a_j$ of column vectors of $Q$ vanishes, then the corresponding divisor $\sum_j a_j D_j$ has vanishing intersection with any effective curve, i.e. it is trivial. Therefore a diagram displaying the column vectors of $L$ or $Q$ encodes the linear equivalence relations among the toric divisors $D_i$. We may interpret these vectors as one dimensional cones of a fan, the so called ‘secondary fan’ of $X$. Note, however, that two distinct but linearly equivalent toric divisors correspond to the same vector in the secondary fan. As the entries of $L$ are the intersections between the generators of the Mori cone and the divisors, the Kähler cone of $X$ is determined by those $\sum_j a_j D_j$ such that the corresponding linear combinations of the columns of $L$ only have nonnegative entries.

We should stress that our analysis was in terms of a single fixed triangulation. If we allow several distinct triangulations, the Mori cone vectors of any of them will lead to correct charge matrices $Q$ but the Kähler condition will depend on which combinations of the charge vectors correspond to the Mori cone, i.e. on the choice of triangulation. In this way several regions of a secondary fan constructed from some charge matrix $Q$ can correspond to different ‘geometric phases’ in the sense of [33][34].

We will now present some of the examples that we are going to use in this paper.

**Example 1:**

The toric resolution of $\mathbb{P}^2 / \mathbb{Z}_n$: We have toric divisors $D_0, \ldots, D_n$ corresponding to vectors
\[ v_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \cdots, \quad v_n = \begin{pmatrix} n \\ 1 \end{pmatrix}. \]
$D_0$ and $D_n$ are non-compact and correspond to the coordinates of the original $\mathbb{C}^2$ on which $\mathbb{Z}_n$ acts by $(z_0, z_n) \to (\epsilon z_0, \epsilon^{n-1} z_n)$ with $\epsilon = e^{2\pi i/n}$. All other $D_i$ are compact and are nothing but the effective curves. The Mori cone vectors are determined by $v_{i-1} - 2v_i + v_{i+1} = 0$, leading to

$$Q = \begin{pmatrix} 1 & -2 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & \ldots & -2 & 1 \end{pmatrix}.$$  \hspace{1cm} (2.5)

Upon dropping the first and the last column, this becomes $-M_{SU(n)}$, where $M_{SU(n)}$ is the Cartan matrix of $SU(n)$. Thus the generators of the Kähler cone, corresponding to linear combinations of the $D_i$ that turn the columns of $L$ into unit vectors, are given by $-\sum_{j=1}^{n-1} (M_{SU(n)})_{ij}^{-1} D_j$ or, alternatively, by

$$D_0, \ D_1 + 2D_0, \ D_2 + 2D_1 + 3D_0, \ \ldots, \ D_{n-2} + 2D_{n-3} + \cdots + (n-1)D_0.$$  \hspace{1cm} (2.6)

**Example 2:**

*The toric resolution of $\mathbb{C}^n/\mathbb{Z}_n$:*) The resolution of a singular space of the type $\mathbb{C}^n/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts on the coordinates of $\mathbb{C}^n$ by

$$(z_1, \ldots, z_n) \to (\epsilon z_1, \ldots, \epsilon z_n) \text{ with } \epsilon = e^{2\pi i/n}$$  \hspace{1cm} (2.7)

can be represented torically by vectors $v_1, \ldots, v_{n+1}$ subject to the single relation $v_1 + v_2 + \cdots + v_n = nv_{n+1}$; the $N$ lattice is just the lattice generated by the $v_i$. The first $n$ vectors $v_1, \ldots, v_n$ correspond to the original coordinates $z_i$ whereas $v_{n+1}$ corresponds to the single exceptional divisor $D_{n+1} = \{ z_{n+1} = 0 \}$ isomorphic to $\mathbb{P}^{n-1}$. The Mori cone is determined by the single relation, leading to

$$Q = (1,1,\ldots,1,-n).$$  \hspace{1cm} (2.8)

**Example 3:**

*The toric resolution of $\mathbb{C}^3/\mathbb{Z}_5$:* We first consider a singular space of the type $\mathbb{C}^3/\mathbb{Z}_5$, where $\mathbb{Z}_5$ acts on the coordinates of $\mathbb{C}^3$ by

$$(z_1, z_2, z_3) \to (\epsilon z_1, \epsilon^2 z_2, \epsilon z_3) \text{ with } \epsilon = e^{2\pi i/5}.$$  \hspace{1cm} (2.9)

As a toric variety $\mathbb{C}^3/\mathbb{Z}_5$ is determined by three vectors

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$  \hspace{1cm} (2.10)
in a lattice $N$ isomorphic to $\mathbb{Z}^3$, the singularity resulting from the fact that $v_1$, $v_2$ and $v_3$ generate only a sublattice of $N$. A complete crepant (i.e., canonical class preserving) toric resolution $X \to \mathbb{C}^3/\mathbb{Z}_5$ is obtained by adding two further rays

$$v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(2.11)

and triangulating the resulting diagram (this triangulation is unique in the present case).

![Diagram](image_url)

**Fig. 1:** The resolution of $\mathbb{C}^3/\mathbb{Z}_5$.

The resulting fan, with the redundant third coordinate suppressed, is shown in fig. 1. The structure of the resolution is easily read off from this diagram: We have two exceptional divisors $D_4$ and $D_5$ corresponding to $v_4$ and $v_5$, respectively. The star fans of $v_4$ and $v_5$ tell us that $D_4$ is a $\mathbb{P}^2$ and $D_5$ is a Hirzebruch surface $\mathbb{F}_3$. $D_4$ and $D_5$ intersect along a curve $h$ which is a hyperplane of the $\mathbb{P}^2$ and at the same time the negative section of $\mathbb{F}_3$. We denote by $D_1$, $D_2$ and $D_3$ the noncompact toric divisors corresponding to the vertices $v_1$, $v_2$ and $v_3$ respectively (i.e., the zero loci of the coordinates of our original $\mathbb{C}^3$). Intersection numbers can be calculated by using the linear equivalences

$$D_1 \sim D_3 \sim D_5 + 2D_2 \quad \text{and} \quad D_1 + D_2 + D_3 + D_4 + D_5 \sim 0$$

(2.12)

and the fact that three distinct toric divisors have an intersection number of 1 if they belong to the same cone and 0 otherwise. As $D_1$ is linearly equivalent to $D_3$ we omit expressions
involving $D_3$ in the following. Intersections of divisors are well defined whenever they involve at least one of $D_4$ and $D_5$. Triple intersections are given by
\begin{align*}
D_3^3 &= 9, \quad D_4^2 \cdot D_5 = -3, \quad D_4 \cdot D_5^2 = 1, \quad D_5^3 = 8, \\
D_4^2 \cdot D_1 &= -3, \quad D_4 \cdot D_5 \cdot D_1 = 1, \quad D_5^2 \cdot D_1 = -2, \quad D_5^2 \cdot D_2 = -5,
\end{align*}
(2.13)
and the vanishing of $D_4 \cdot D_2 = 0$. Intersections of two distinct divisors are determined by
\begin{align*}
D_4 \cdot D_5 &= D_4 \cdot D_1 = h, \quad D_4 \cdot D_2 = 0, \quad D_5 \cdot D_1 = f, \quad D_5 \cdot D_2 = h + 3f,
\end{align*}
(2.14)
where $f$ is the fibre of the $\mathbb{F}_3$. The self-intersections of $D_4$ and $D_5$ are
\begin{align*}
D_4^2 &= -3h, \quad D_5^2 = -2h - 5f.
\end{align*}
(2.15)
We also have:
\begin{align*}
D_4 \cdot h &= -3, \quad D_4 \cdot f = 1, \quad D_5 \cdot h = 1, \quad D_5 \cdot f = -2, \\
D_1 \cdot h &= 1, \quad D_1 \cdot f = 0, \quad D_2 \cdot h = 0, \quad D_2 \cdot f = 1.
\end{align*}
(2.16)
This implies that $(C_1, C_2) = (h, f)$ and $(D_1, D_2)$ form mutually dual bases of the Mori cone and the Kähler cone of $X$. In terms of codimension one (here, two dimensional) cones $\sigma$ and the linear relations between the rays in the two cones of maximal dimension that contain $\sigma$, we obtain the following linear relations among the vectors $v_1, \ldots, v_5$ of the fan:
\begin{align*}
l^{(1)} &= (1, 0, 1, -3, 1), \\
l^{(2)} &= (0, 1, 0, 1, -2).
\end{align*}
(2.17)
As a check on our intersection numbers, we observe that indeed $D_i \cdot C_j = l^{(j)}_i$.

Fig. 2: The secondary fan of the resolution of $\mathbb{F}_3/\mathbb{Z}_5$. 

7
If we consider the matrix whose lines are the generators (2.17) of the Mori cone and draw the rays corresponding to the columns of this matrix, we obtain the secondary fan for $X$ as shown in fig. 2. The linear relations among the vectors in this fan encode the linear equivalences (2.12) among the divisors.

3. Local mirror symmetry

In our study of D-brane states we will have to address issues that involve quantum geometry. A standard tool for this problem is the use of mirror symmetry. In particular, classical periods in the mirror geometry get mapped to quantum corrected expressions related to the middle cohomology of the original space. In the non-compact case one has to use local mirror symmetry. For our applications of this subject we have relied mainly on [35] and we refer to this paper for further references. The authors of [35] consider decompactifications of Calabi-Yau hypersurfaces in toric varieties such that the volumes of certain cycles remain compact. They show that in the decompactification limit these cycles lead to differential equations that are identical with the GKZ differential systems of a lower dimensional geometry. We will assume that this remains true even for cases where the non-compact Calabi-Yau geometry cannot be identified with a limiting case of a compact Calabi-Yau hypersurface.

The local mirror of a $d$-dimensional noncompact Calabi-Yau geometry is determined by interpreting the diagram of the hyperplane containing the end points of the $v_i$ now as a polytope $P$ in a $(d-1)$-dimensional lattice $\tilde{M}$. A polytope corresponds to a line bundle $\mathcal{L}$ over a toric variety $\mathcal{V}$ by the following construction: Fix any point in $\tilde{M}$ to be the origin. Describe the facets of $P$ by equations $E_j(\tilde{m}) := \langle \tilde{v}_j, \tilde{m} \rangle + c_j = 0$, where $\tilde{v}_j \in \tilde{N}$, the lattice dual to $\tilde{M}$ and fix the sign ambiguity about $\tilde{v}_j$ in such a way that $E_j(\tilde{m})$ is nonnegative for points $\tilde{m}$ of $P$. Choose $\mathcal{V}$ to be a toric variety whose one dimensional rays are the $\tilde{v}_j \in \tilde{N}$ corresponding to a variable $x_j$ as in the previous section. To every point $\tilde{m} \in \tilde{M}$ assign the monomial $\prod_j x_j^{E_j(\tilde{m})}$. Then $\mathcal{L}$ is the bundle whose sections are determined by polynomials of the type

$$P(a; x) = \sum_{i=1}^{k} a_i \prod_j x_j^{E_j(\tilde{m}_i)}.$$  \hspace{1cm} (3.1)

The ‘local mirror’ $\tilde{X}$ of $X$ is defined to be the vanishing locus of a section (3.1) of $\mathcal{L}$.

In the present context we can give an alternative description of the $E_j$: We have $N \simeq \tilde{M} \oplus \mathbb{Z}$ and may choose coordinates such that $v_i = (\tilde{m}_i, 1)$. Then we can write the
affine function $E_j(\tilde{m}_i)$ as a linear function of the form $\langle v_i, \tilde{v}_j' \rangle$ with $\tilde{v}_j' \in \text{Hom}(N, \mathbb{Z}) = M$ (it is easy to check that the $\tilde{v}_j'$ are the elements of $M$ dual to the $(d-1)$-dimensional cones at the boundary of the support of $\Sigma$).

Obviously the complex structure moduli space of $\tilde{X}$ is parametrized by the $a_i$. It is important to note, however, that different sets of $a_i$ need not correspond to different complex structures. In particular, a scaling $x_j \to \lambda_j x_j$ does not amount to a change in the complex structure but leads to a redefinition of the $a_i$, implying the equivalences

$$(a_1, a_2, \ldots, a_k) \sim (\lambda_j E_j(\tilde{m}_1) a_1, \lambda_j E_j(\tilde{m}_2) a_2, \ldots, \lambda_j E_j(\tilde{m}_k) a_k) = (\lambda_j^{\langle v_1, \tilde{v}_j' \rangle} a_1, \lambda_j^{\langle v_2, \tilde{v}_j' \rangle} a_2, \ldots, \lambda_j^{\langle v_k, \tilde{v}_j' \rangle} a_k)$$

for any $j$. Given identifications of this type it is natural to seek a description in terms of toric geometry. If we interpret the exponents of the $\lambda$’s as linear relations among vectors $u_i$ in a toric diagram and notice that the $\tilde{v}_j'$ generate $M$ (at least over the rational numbers), we find that the $u_i$ fulfill

$$\langle m, v_1 \rangle u_1 + \langle m, v_2 \rangle u_2 + \cdots + \langle m, v_k \rangle u_k = 0$$

for any $m \in M$. These are just the relations among the vectors of the secondary fan which encodes, as we saw, the linear equivalence relations (2.2) of the divisors $D_i$ corresponding to the $v_i$. There are some subtleties, however: As we saw in the previous section, it is possible that two distinct (but linearly equivalent) toric divisors lead to the same vector in the secondary fan. We will show how to interpret this in the context of the examples. Besides, it is possible that there are identifications in the moduli space that do not come from rescalings of the type $x_j \to \lambda_j x_j$ and hence have a structure different from (3.2). If this occurs, the toric variety associated with the secondary fan is called the ‘simplified moduli space’ $M_{\text{simp}}$. Depending on whether we have extra identifications or not, the toric variety corresponding to the secondary fan is a compactification of $M_{\text{smooth}}$ (the moduli space of all smooth local mirror hypersurfaces) or a covering space of a compactification of $M_{\text{smooth}}$.

$\tilde{X}$ will degenerate over various loci in $M_{\text{simp}}$ where $\partial P(a; x)/\partial x_j = 0$ can be solved for all $j$ without violating the conditions on which $x_i$ are allowed to vanish simultaneously. Some of these loci may just be toric divisors, but usually there is also at least one connected piece given by a polynomial equation in the $a_i$ to which we will refer as the primary or principal component of the discriminant locus.
If we want to relate the mirror geometry to the original one, we have to find a region in
the moduli space where quantum corrections are strongly suppressed. This is the case for
the deep interior of the Kähler cone, the so called large volume limit, which is dual to the
large complex structure limit. As we saw in section 2, the Kähler cone can be determined
by writing any divisor as a linear combination of toric divisors and demanding that the
Corresponding linear combination of columns of the matrix $L$ contain only nonnegative
entries. If the resulting generators do not belong to the secondary fan, we have to blow up
the moduli space in order to be able to change to the large complex structure variables.

In those cases where the Mori cone is simplicial we can draw the secondary fan by
displaying the columns of $L$ and the generators of the Kähler cone will be nothing but
the unit vectors. If we then write the linear relations among the vectors in the secondary
fan in such a way that we express every vector in terms of the unit vectors and use the
corresponding rules (2.1) to set all variables except the large complex structure variables
$z_i$ to 1, we find that the $z_i$ can be expressed as

$$z_i = \prod_{j=1}^{k} a_i^{l^{(i)}_j}.$$  \hspace{1cm} (3.4)

Note that we do not include a sign here (compare with e.g. [36]).

If $X$ is the resolution of an orbifold singularity of the type $\mathbb{C}^d/\mathbb{Z}_n$ there is another
distinguished coordinate patch in the moduli space containing the orbifold locus where all
$a_i$ except the ones corresponding to the coordinates of the $\mathbb{C}^d$ are set to zero. At this point
the conformal field theory is expected to acquire a quantum symmetry. We find that the
moduli space in this case always has a singularity that looks locally like $\mathbb{C}^{\dim M}/\mathbb{Z}_n$.

The GKZ differential operators are calculated by using the following recipe: For every
linear relation $\sum l_j v_j = 0$, where $l$ corresponds to any curve in the Mori cone (see [35])
we define a differential operator in terms of the $a_i$,

$$\mathcal{D} = \prod_{j: \ l_j > 0} \partial^{l_j}_{a_j} - \prod_{j: \ l_j < 0} \partial^{-l_j}_{a_j}.$$ \hspace{1cm} (3.5)

Assume that we work in a specific coordinate patch given by some $\phi_i = \prod a^{\mu_{ij}}_j$. In order
to transform (3.5) to a system involving the $\phi_i$ we can rewrite it in terms of operators

\footnote{We hope that no confusion arises from the fact that we use the same symbol $z_i$ for the
coordinates of $X$ and the large complex structure variables.}
\[ \Theta_{a_j} := a_j \partial_{a_j}, \text{ commute all } a_j \text{ to the left using } \Theta_{a_j} a_j^{-1} = a_j^{-1}(\Theta_{a_j} - 1) \text{ and then express } \Theta_{a_i} \text{ as } \sum_i \mu_{ij} \Theta_{\phi_i} \text{ with } \Theta_{\phi_i} := \phi_i \partial_{\phi_i}. \]

We stress that the solutions of the GKZ system are not the periods on \( \tilde{X} \) but rather the logarithmic integrals of the periods. While the periods are finite and non-vanishing on the moduli space wherever \( \tilde{X} \) is non-degenerate, the GKZ solutions have extra singularities at the zero loci of moduli space coordinates coming from the logarithmic integration. The GKZ solutions are multivalued and undergo monodromy transformations around codimension one loci where they are not holomorphic. We will be interested mainly in monodromies around the large complex structure divisors \( z_i = 0 \) and around the principal component of the discriminant locus. In addition, there is the possibility of a non-trivial transformation (‘orbifold monodromy’, which, strictly speaking, is not a monodromy) if the moduli space looks locally like \( \mathbb{C}^{\dim M}/\mathbb{Z}_n \).

We will now show how these concepts can be applied to our examples.

**Example 1:**
The mirror geometry of \( \mathbb{P}^2/\mathbb{Z}_n \): Here \( \mathcal{V} \) is \( \mathbb{P}^1 \) and the polynomial is given by

\[ a_0 x_1^n + a_1 x_1^{n-1} x_2 + \cdots + a_n x_2^n, \quad (3.6) \]

so the hypersurface \( \tilde{X} \) is just a collection of \( n \) points in \( \mathbb{P}^1 \). A ‘singularity’ of \( \tilde{X} \) occurs whenever two or more of these points coincide. The secondary fan is determined by the columns of \( (2.5) \). For \( n \geq 3 \) we have to blow up the moduli space in order to have a coordinate patch described by the large complex structure variables \( z_i = a_{i-1} a_{i+1}/(a_i)^2 \) (with \( 1 \leq i \leq n - 1 \)). The GKZ operators corresponding to the Mori cone generators,

\[ \partial_{a_0} \partial_{a_2} - \partial_{a_1}^2, \quad \partial_{a_1} \partial_{a_3} - \partial_{a_2}^2, \quad \ldots, \quad \partial_{a_{n-2}} \partial_{a_n} - \partial_{a_{n-1}}^2 \quad (3.7) \]

become

\[ \Theta_{a_0} \Theta_{a_2} - z_1 (\Theta_{a_1} - 1) \Theta_{a_1}, \quad \ldots, \quad \Theta_{a_{n-2}} \Theta_{a_n} - z_{n-1} (\Theta_{a_{n-1}} - 1) \Theta_{a_{n-1}} \quad (3.8) \]

with

\[ \Theta_{a_0} = \Theta_{z_1}, \quad \Theta_{a_1} = -2\Theta_{z_1} + \Theta_{z_2}, \]
\[ \Theta_{a_i} = \Theta_{z_{i-1}} - 2\Theta_{z_i} + \Theta_{z_{i+1}} \quad \text{for } 2 \leq i \leq n - 3, \]
\[ \Theta_{a_{n-1}} = \Theta_{z_{n-2}} - 2\Theta_{z_{n-1}}, \quad \Theta_{a_n} = \Theta_{z_{n-1}}. \quad (3.9) \]

We note that the space of solutions of \( (3.8) \) is too large unless we introduce further operators corresponding to linear combinations of the Mori cone generators.
The case of \( n = 2 \) allows for an explicit solution \[36\]: Here we have
\[
D = (\Theta_z - 2z(2\Theta_z + 1))\Theta_z
\]  
(3.10)
and \( D = 0 \) has a basis of solutions of the form
\[
\varpi_0 = 1, \quad \varpi_1 = \frac{1}{2\pi i} \ln \frac{1 - \sqrt{1 - 4z}}{1 + \sqrt{1 - 4z}}.
\]  
(3.11)
Special points in the moduli space are the large complex structure limit \( z = 0 \), the analog of the primary component of the discriminant locus at \( z = 1/4 \), and the orbifold point at \( z = \infty \) where we introduce a new coordinate \( \varphi \) by \( z\varphi^2 = 1 \). We find the following transformation properties upon taking loops around these points:
\[
z = 0: \quad \varpi_1 \to \varpi_1 + 1, \quad z = 1/4: \quad \varpi_1 \to -\varpi_1, \quad \varphi \to e^{\pi i} \varphi: \quad \varpi_1 \to -1 - \varpi_1. \]  
(3.12)
Example 2:
The local mirror geometry \( \tilde{X} \) of the resolution \( X \) of \( \mathbb{C}^n/\mathbb{Z}_n \) is just the mirror geometry of a compact Calabi-Yau manifold realised as a degree \( n \) hypersurface in \( \mathbb{P}^{n-1} \), i.e. \( \tilde{X} \) is a degree \( n \) hypersurface
\[
a_1x_1^n + \cdots + a_nx_n^n + a_{n+1}x_1\cdots x_n = 0
\]  
(3.13)
in \( \mathbb{P}^{n-1}/(\mathbb{Z}_n)^{n-2} \). The GKZ operator \( \partial_{a_1} \cdots \partial_{a_n} - \partial_{a_{n+1}}^n \) becomes
\[
\Theta_z^n - z(-n\Theta_z - n + 1)(-n\Theta_z - n + 2)\cdots(-n\Theta_z)
\]  
(3.14)
in terms of the large complex structure variable \( z = a_1\cdots a_n/(a_{n+1})^n \).
Example 3:
The mirror geometry of \( \mathbb{P}^3/\mathbb{Z}_5 \), a genus two Riemann surface:

Fig. 3: The fan for \( \mathbb{P}^2/\mathbb{Z}_5 \).
Here $\mathcal{V}$ is $\mathbb{P}^2/\mathbb{Z}_5$ with the $\mathbb{Z}_5$ acting on the homogeneous coordinates of $\mathbb{P}^2$ as $(x_1, x_2, x_3) \rightarrow (\epsilon x_1, x_2, \epsilon^{-1} x_3)$. The polynomial corresponding to fig. 1 is given by

$$a_1 x_1^5 + a_2 x_2^5 + a_3 x_3^5 + a_4 x_1^2 x_2 x_3^2 + a_5 x_1 x_2^3 x_3,$$ \hspace{1cm} (3.15)

where we have chosen the subscripts of the $a_i$ to correspond to those of the $v_i$ in fig. 1. The local mirror of $\mathbb{C}^3/\mathbb{Z}_5$ is given by the vanishing locus of (3.15) in $\mathbb{P}^2/\mathbb{Z}_5$. The action of $\mathbb{Z}_5$ on $\mathbb{P}^2$ has fixed points whenever two of the three $x_i$ vanish. The vanishing locus of (3.15) passes through one of these fixed points if and only if one of $a_1, a_2, a_3$ vanishes. Thus the generic hypersurface misses the fixed points. A quintic polynomial in $\mathbb{P}^2$ defines, by a standard calculation, a Riemann surface of Euler number $\chi = -10$. As the $\mathbb{Z}_5$ acts without fixed points on this surface, the Euler number is divided by 5, showing that the local mirror geometry is that of a Riemann surface $\mathcal{R}$ with $2 - 2g = \chi = -2$, i.e. genus $g = 2$.

Scalings $x_i \rightarrow \lambda_i x_i$ imply the equivalences

$$(a_1, a_2, a_3, a_4, a_5) \sim (\lambda_1^5 a_1, \lambda_2^5 a_2, \lambda_3^5 a_3, \lambda_1^2 \lambda_2 \lambda_3^3 a_4, \lambda_1 \lambda_2^3 \lambda_3^3 a_5).$$ \hspace{1cm} (3.16)

If we naively interpret the exponents of the $\lambda_i$ as a charge matrix

$$\begin{pmatrix} 5 & 0 & 0 & 2 & 1 \\ 0 & 5 & 0 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \end{pmatrix}$$ \hspace{1cm} (3.17)

with entries $q_i^j$ and try to find a fan with rays $v_i$ fulfilling $\sum_i q_i^j v_i = 0$ for $j = 1, 2, 3$, we find that we have to take $v_1 = v_3$, meaning that we should not distinguish between $a_1$ and $a_3$. This can be explained by the fact that taking $\lambda_3 = \lambda_1^{-1}$ implies that we can multiply $a_1$ with any nonzero number provided we divide $a_3$ by the same number without affecting the other $a_i$, i.e. as long as $a_1$ and $a_3$ are nonzero the complex structure of $\mathcal{R}$ depends only on $a_{13} := a_1 a_3$. This is even true if one of $a_1, a_3$ becomes zero, since an exchange of $a_1$ and $a_3$ can be compensated by exchanging $x_1$ with $x_3$ which does not affect the complex structure. Thus we can consistently drop the third line and the third column of (3.17) to obtain a matrix

$$\begin{pmatrix} 5 & 0 & 2 & 1 \\ 0 & 5 & 1 & 3 \end{pmatrix}.$$ \hspace{1cm} (3.18)

This is just the matrix of linear relations for the secondary fan of fig. 2. The corresponding compact toric variety

$$\mathcal{M}_{\text{toric}} = \{(a_{13}, a_2, a_4, a_5) \setminus \{(a_{13} = a_4 = 0) \lor (a_2 = a_5 = 0)\} \} / \sim \hspace{1cm} (3.19)$$
with $\sim$ as in (3.16) is closely related to the moduli space $\mathcal{M}_{\text{smooth}}$ of smooth hypersurfaces of the type (3.15): Smoothness implies $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$, so

$$\mathcal{M}_{\text{smooth}} \subset \{(a_{13}, a_2, a_4)\} \setminus \{(a_{13} = 0) \lor (a_2 = 0)\} / \sim \subset \mathcal{M}_{\text{toric}},$$

i.e. $\mathcal{M}_{\text{toric}}$ is a compactification of $\mathcal{M}_{\text{smooth}}$ (other sensible compactifications correspond to omitting $v_{13}$ or $v_2$ from fig. 2). $\mathcal{M}_{\text{smooth}}$ is contained in the single coordinate patch of $\mathcal{M}_{\text{toric}}$ defined by the cone spanned by $v_4$ and $v_5$. In this patch we can parametrize the hypersurface as the vanishing locus of

$$P_{\psi,\phi}(x_1, x_2, x_3) = x_1^5 + x_2^5 + x_3^5 - 5\psi x_1^2 x_2 x_3^2 - 5\phi x_1 x_2^3 x_3. \quad (3.21)$$

Having set $a_1$, $a_2$ and $a_3$ to one has used up most of the freedom coming from (3.16), the remaining relation being

$$(\psi, \phi) \sim (\epsilon^2 \psi, \epsilon \phi).$$

(3.22)

As we just noticed, $\mathcal{R}$ becomes singular along the divisors $a_{13} = 0$ and $a_2 = 0$ of $\mathcal{M}_{\text{toric}}$. The remaining singularities can be found by looking for values of $\psi, \phi$ where $\partial P_{\psi,\phi}/\partial x_i = 0$ for $i = 1, 2, 3$ can be solved by some $(x_1, x_2, x_3) \neq (0, 0, 0)$. This results in the equation

$$16\psi^5 + 40\psi^4 \phi^2 + 25\psi^3 \phi^4 + 20\psi^2 \phi + 45\psi \phi^3 + 27\phi^5 = 1$$

for the primary component of the discriminant locus.

We note that while $\mathcal{M}_{\text{toric}}$ contains some of the singular loci, it misses others such as $a_1 = a_4 = 0$, $a_2 = a_5 = 0$ and any points with three or four of the $a_i$ vanishing. The divisor $a_{13} = 0$ in $\mathcal{M}_{\text{toric}}$ corresponds to two one dimensional loci $a_1 = a_3 = 0$ and $a_1 a_3 = 0, a_1 + a_3 \neq 0$. Our main concern with the moduli space has to do with the study of monodromies. Thus we want to know what happens when we move around singularities at codimension one rather than what happens when we hit them. For example, the monodromy around $a_{13} = 0$ depends only on nonvanishing values of $a_{13}$ and not on how we interpret the locus $a_{13} = 0$. Therefore $\mathcal{M}_{\text{toric}}$ is sufficient for our purposes.
A schematic representation of $\mathcal{M}_{\text{toric}}$ is given in fig. 4, with the the toric divisors shown as straight lines and the primary component of the discriminant locus indicated by curved lines. The locus $a_{13} = 0$ is tangent to the discriminant locus at their point of intersection $4a_2a_4 = a_5^2$ whereas $27a_{13}a_5 + a_4^3 = 0$ corresponds to a transverse intersection with $a_2 = 0$. At $a_4 = a_5 = 0$ (i.e., $\phi = \psi = 0$) the moduli space has a singularity where the Riemann surface remains smooth; in addition there are $\mathbb{Z}_2$ and $\mathbb{Z}_3$ singularities at $a_{13} = a_5 = 0$ and $a_2 = a_4 = 0$, respectively.

As we saw above, there is a distinguished coordinate patch in $\mathcal{M}_{\text{toric}}$ which contains all loci where $\mathcal{R}$ is smooth. Now we want to study another distinguished set of coordinates corresponding to the ‘large complex structure limit’. We remember that the Kähler cone of $X$ (the resolution of $\mathbb{C}^3/\mathbb{Z}_5$) was spanned by $D_1 \sim D_3$ and $D_2$ corresponding to the vectors $v_{13}$ and $v_2$ in the secondary fan (fig. 2) respectively. The large radius limit of $X$ corresponds to the deep interior of the Kähler cone, so by local mirror symmetry the large complex structure limit is determined by the $v_{13} - v_2$–coordinate patch in $\mathcal{M}_{\text{toric}}$ given by

$$z_1 = a_{13} \frac{a_5}{a_4} = \frac{\phi}{52\psi^3}, \quad z_2 = a_2 \frac{a_2}{a_5} = -\frac{\psi}{5\phi^2}. \quad (3.24)$$

In terms of $z_1$, $z_2$ the principal component of the discriminant locus is determined by

$$1 + 27z_1 - 8z_2 - 225z_1z_2 + 16z_2^2 + 500z_1z_2^2 + 3125z_1^2z_2^3 = 0. \quad (3.25)$$

The GKZ system can be determined and solved with the methods described above. There are five independent solutions, as expected, which are described in appendix A.
4. D-branes and tautological bundles

We want to find out about the D-brane vacuum states in type II string theory on $X$. The mathematical structure that captures the largest number of properties of brane states is, at present knowledge, the bounded derived category $D^b$ of coherent sheaves on $X$ \footnote{\cite{37,38} (but see the remarks in \cite{39,40}). While we will make several remarks concerning $D^b$, we will work mainly with the somewhat coarser (but easier to handle) concepts of K-theory. Let $K(X)$ be the Grothendieck group of coherent sheaves on $X$. We expect compact brane states on a non-compact space $X$ to correspond to classes of the compactly supported K-theory group $K_c(X)$. Using the duality between $K(X)$ and $K_c(X)$ we can determine a basis for $K_c(X)$ by first finding a basis for $K(X)$.

Let us consider the situation where $X$ is a smooth crepant resolution of a singularity of the type $\mathbb{C}^d/G$, where $G$ is a finite subgroup of $SU(d)$. Since $X$ is smooth, $K(X)$ is generated by vector bundles (see e.g. \cite{41}). Moreover, if $\pi : X \rightarrow \mathbb{C}^d/G$ is a crepant resolution of an abelian singularity, $K(X)$ is in fact generated by $n$ line bundles, where $n$ is the order of $G$ (at least for $d \leq 3$) \footnote{We thank A. Craw for emphasizing the importance of choosing the G-Hilb resolution to us.}. Thus, for finding a basis for the group $K_c(X)$ related to fractional branes it is convenient to first determine a set \{\( R_i \)\} ($0 \leq i < n$) of line bundles whose K-theory classes generate $K(X)$. Clearly there is no choice for the $R_i$ that should be preferred a priori. Rather, there are two distinct constructions, each of which is related to McKay correspondence:

1. **Mathematicians’ construction** \footnote{\cite{22–25}: There is a vector bundle $R$ (the ‘tautological vector bundle’) transforming in the regular representation of $G$ whose decomposition into irreducibles gives the line bundles $R_i^{M}$. In particular, the $R_i^{M}$ are generated by their sections and the action of $G$ on the sections determines a one-to-one correspondence between the $R_i^{M}$ and the characters of the irreducible representations of $G$. In the case of a resolution of $\mathbb{C}^2/G$ with some finite group $G$ the first Chern classes $c_1(R_i^{M})$, $i \geq 1$ form a basis of $H^2(X, \mathbb{Z})$ dual to the basis of $H_2(X, \mathbb{Z})$ given by the homology classes of a basis of effective curves $C_i$ in the resolution. In the case of a singularity of the type $\mathbb{C}^3/G$ with $G$ an abelian subgroup of $SL(3, \mathbb{C})$ in general there exist several crepant resolutions and not for every resolution it is possible to define line bundles as above. However, it was shown in \cite{24} that there exists a distinguished crepant resolution, named G-Hilb, on which it is still possible to define the tautological line bundles (see also \cite{23,27,28}). The advantage of this approach is that it is rigorously proven for $d = 2$ and $d = 3$.} \footnote{\footnote{We thank A. Craw for emphasizing the importance of choosing the G-Hilb resolution to us.} 2}:

- There is a vector bundle $R$ (the ‘tautological vector bundle’) transforming in the regular representation of $G$ whose decomposition into irreducibles gives the line bundles $R_i^{M}$. In particular, the $R_i^{M}$ are generated by their sections and the action of $G$ on the sections determines a one-to-one correspondence between the $R_i^{M}$ and the characters of the irreducible representations of $G$. In the case of a resolution of $\mathbb{C}^2/G$ with some finite group $G$ the first Chern classes $c_1(R_i^{M})$, $i \geq 1$ form a basis of $H^2(X, \mathbb{Z})$ dual to the basis of $H_2(X, \mathbb{Z})$ given by the homology classes of a basis of effective curves $C_i$ in the resolution. In the case of a singularity of the type $\mathbb{C}^3/G$ with $G$ an abelian subgroup of $SL(3, \mathbb{C})$ in general there exist several crepant resolutions and not for every resolution it is possible to define line bundles as above. However, it was shown in \cite{24} that there exists a distinguished crepant resolution, named G-Hilb, on which it is still possible to define the tautological line bundles (see also \cite{23,27,28}). The advantage of this approach is that it is rigorously proven for $d = 2$ and $d = 3$.}
2. Physicists’ constructions: The authors of [13] suggest to consider, in the style of [42],
the world-volume theory of D0-branes, which is a theory of $n - 1$ $U(1)$ gauge fields and
d $n \times n$ matrices. It is conjectured (and shown in several examples) that the vacua of
such a theory in the different phases corresponding to different choices of Fayet-Iliopoulos
parameters all lead to moduli spaces that are nothing but the geometric phases of the
resolutions $X$ of $\Phi^d / G$. Now repeat this construction with an extra field transformi ng in a
specific one dimensional representation $\rho_i$ of $G$. It is conjectured that, independently of the
phase, this should lead to a space that is the total space of a line bundle $R^P_i$ over $X$, and
that repeating this for all characters $\rho_i$ should give a basis $\{R^P_i\}$ of $K(X)$. However, this
construction is extremely tedious to work with. A different method for determining $\{R^P_i\}$
based on the boundary chiral ring associated to a certain two dimensional gauge theory
has been proposed in [17]. The implications of this approach have been worked out for the
case of a single exceptional divisor that is a weighted projective space $W = \mathbb{P}^{d-1}_{n_1,\ldots,n_d}$ with
Fermat weights [17] or a Grassmannian $[19]$. In all examples we are aware of, the $R^P_i$ have
no sections. The advantage of this approach is that it appears to lead to dual classes $S^P_i$
whose interpretations in terms of D-branes are very well behaved.

Roughly, the resulting $R_i$ can be summarized in the following way. There is a set of
divisor classes $\{[F_i]\}$ containing all Kähler cone generators $[T_i]$ and the trivial class $[0]$ such
that all $F_i$ are nef, i.e. have nonnegative intersection with any curve in the Mori cone.
If we denote by $R^{\pm}_i$ the line bundles $\mathcal{O}(\pm F_i)$, then $\{R^M_i\} = \{R^+_i\}$ and $\{R^P_i\} = \{R^-_i\}$.
In two dimensions the $[F_i]$ are just the trivial class and the Kähler cone generators. In
higher dimensions we have to add extra divisor classes which are nonnegative integer linear
combinations of the $[T_i]$. For the $R^M_i$ with G-Hilb and $d = 3$ the authors of [23],[27] have
given an explicit construction. In terms of the language used in this paper this can be
summarised in the following way.

Through the sections we can assign a character to any $T_i$. It is also possible to assign
characters to toric curves. Such a curve $C$ corresponds up to a sign to some $m \in M$ leading
to a linear equivalence as in (2.2). By collecting expressions with the same sign this can be
written as $D \sim D'$ where $D$, $D'$ are effective divisors corresponding to the same character.
We then assign this character to $C$ and the corresponding line segment in the diagram,
and find that all the characters obtainable in this way also occur in the list of characters
corresponding to the $T_i$. Then every interior point $I$ of the toric diagram is of one of the

\footnote{P. Mayr informs us that this approach works in more general situations as well.}
following types:
1. There are three pairs of line segments with the same character meeting in $I$. In this case we add nothing to the list of $[F_i]$ (the classes assigned by $[23], [27]$ in this case are already among the Kähler cone generators).

2. There are two pairs of line segments with characters $\chi_m, \chi_n$ meeting in $I$ (and possibly an extra line segment). Then add $[T_m + T_n]$ to the list of $[F_i]$.

3. There are three line segments with the same character $\chi_m$. In this case add $[2T_m]$ to the list of $[F_i]$.

It turns out that this procedure always leads to a one to one correspondence between the $R_i^+$ and the character table of $G$ through the action of $G$ on the sections.

In many cases the $[F_i]$ are the same in the mathematicians' and physicists' constructions, i.e. $R_i^P = (R_i^M)^*$. However, $[17]$ seems to suggest partial resolutions in the case with a single interior point where the exceptional divisor is a weighted projective space. We note that the G-Hilb resolution may be incompatible with such a resolution or any refinement of it, as the following example shows.

In fig. 5 we have displayed the G-Hilb resolution of $\mathbb{C}^3/\mathbb{Z}_6$ constructed according to the rules of $[23], [27]$ and the partial resolution by an exceptional divisor $\mathbb{P}^{2}_{(1,2,3)}$. Clearly the former cannot be obtained as a refinement of the latter.

In the following we always follow the mathematicians’ approach.

The next step in our construction of D-brane states is to find a basis for $K^c(X)$ that is dual to the basis of $K(X)$ defined in terms of line bundles $R_i$. According to $[25]$, there is a pairing $(R, S)$ between representatives $R$ of $K(X)$ and $S$ of $K^c(X)$ that can be evaluated in terms of Chern characters

$$ (R, S) = \int_X \text{ch}(R) \cup \text{ch}^c(S) \ Td(X), $$

(4.1)
with $\text{ch}^c(S)$ the localized Chern character\footnote{Let $i : Y \hookrightarrow X$ be the embedding of a compact submanifold $Y$ in a noncompact manifold $X$. Elements of the compactly supported K-theory can be represented by either coherent sheaves $S_Y$ on $Y$ or by their finite resolution by vector bundles on $X$, that is by complexes $S$ of vector bundles on $X$ which are exact off $Y$ and whose homology is precisely the push-forward of $S_Y$ to $X$ \cite{43}. Then, the local Chern character is defined such that $\text{ch}^c(S) = \text{ch}(i_*S_Y)$ \cite{44}.} of the complex $S$ and $\text{Td}(X)$ the Todd class of $X$. There is also a closely related pairing which will become important when we study monodromies. It is defined as

$$
\langle R, S \rangle = (R^*, S) \quad (4.2)
$$

with $R^*$ the line bundle (or, more generally, the complex) dual to $R$. If we restrict $R$ to $K^c(X)$, these pairings become well defined under the exchange of $R$ and $S$ and we find that $\langle R, S \rangle$ is always symmetric whereas $\langle R, S \rangle$ is symmetric in even dimensions and skew in odd dimensions, as a consequence of the fact that $\text{Td}(X)$ is even when $c_1(X)$ is trivial.

The generally accepted way of obtaining a basis for $K^c(X)$ is to choose classes dual to those given by the line bundles $R_i$ with respect to $(\ , \ )$. Following this convention, we define classes of $K^c(X)$ by demanding that their representatives $S_j$ fulfill $(R_i, S_j) = \delta_{ij}$. Thus we obtain $S_j^+$ dual to $R_i^+$ and $S_j^-$ dual to $R_i^-$ with respect to $(\ , \ )$ and note that the $S_j^+$ are dual to the $R_i^-$ and $S_j^-$ are dual to the $R_i^+$ with respect to $(\ , \ )$.

So far we have not been specific about the representatives $S_i$ of the compactly supported K-theory. In the spirit of \cite{2} we may interpret them as bound states of $X$–filling branes. In mathematical terms this amounts to specifying a complex of vector bundles on $X$ that is exact outside a compact locus $Y$. It is not hard to check in every example that we may indeed represent every $S_i$ as a formal linear combination of line bundles of the form $O_X(\sum a_i D_i)$ and that the $\text{ch}^c(S_i)$ obtained from the line bundles $R_i$ form a basis for all Chern characters with support on the compact toric cycles.

Alternatively, one may wish to consider ‘pure’ branes defined in terms of the structure sheaves of the independent lower dimensional compact holomorphic cycles. Given the structure sheaves $\mathcal{O}_{C_i}$ where the $C_i$ form a basis for all compact holomorphic cycles on $X$, applying $k$ push-forwards for every cycle of codimension $k$ leads to sheaves $\tilde{S}_{C_i}$ on $X$. In order to relate these objects to Chern characters on $X$ we have to use the Grothendieck-Riemann-Roch theorem,

$$
i_*(\text{ch}(S_D)\text{Td}(D)) = \text{ch}(i_*S_D)\text{Td}(X) \quad (4.3)$$


for embeddings of the type \( i: D \hookrightarrow X \). Writing \( \text{ch}^c(S) = \sum n^{(C_i)} \text{ch}(\tilde{S}_{C_i}) \) allows us to define the charge vectors \( \vec{n}(S) \). Alternatively we may calculate the charge vectors by first calculating

\[
(R_j, \tilde{S}_{C_i}) = \int_X \text{ch}(R_j) \text{ch}(\tilde{S}_{C_i}) \text{Td}(X) = \int_{C_i} \text{ch}(R_j|_{C_i}) \text{Td}(C_i) = \chi(R_j|_{C_i}, C_i) =: \chi_{ji} \quad (4.4)
\]

and noticing that \((R_j, S_k) = \delta_{jk}\) implies \(\sum_i \chi_{ji} n^{(C_i)} = \delta_{jk}\), i.e. \(n^{(C_i)} = (\chi^{-1})_{ik}\). We note that the compact holomorphic cycles generate the compact homology of \(X\), so the number of \(C_i\) is equal to \(\chi^c(X) = \sum_{i=0}^{2d-1} (-1)^i H^c_i\) which is just the number of \(d\)-dimensional cones in \(\Sigma\) [29].

At the large volume limit the Mukai vector \(\text{ch}^c(S) \sqrt{\text{Td}(X)}\) determines the central charge \(Z_{\text{lv}}(t_i; S)\) of the brane configuration, where the \(T_i\) are generators of the Kähler cone. In particular we obtain

\[
Z_{\text{lv}}(t; \tilde{S}_p) = -1 \quad \text{and} \quad Z_{\text{lv}}(t; \tilde{S}_{C_i}) = t_i - 1 \quad (4.6)
\]

for the central charges of D0-branes and D2-branes wrapping (with trivial bundle) the generators \(C_i\) of the Mori cone dual to the \(T_i\). These are the objects related by local mirror symmetry to the solutions of the GKZ system at the large complex structure point. More precisely we expect the exact central charge \(Z(z; S)\) to be a linear combination of the GKZ solutions such that

\[
\lim_{z \to 0} (Z(z; S) - Z_{\text{lv}}(t_i; S)) = 0. \quad (4.7)
\]

If we demand that \(Z_{\text{lv}}(t; \tilde{S}_{C_i})\) measure the complexified Kähler class at the large Kähler limit we have to make the identification

\[
t_i - 1 = \frac{\ln z_i}{2\pi i} + O(z). \quad (4.8)
\]

Note that this is different from the conventions usually adopted in the literature, but we find that this is precisely the identification that works.

---

5 This formula occurs implicitly in [45] and explicitly in [13]; see also the remarks in [17].
Linearity implies that the central charge corresponding to any $S$ is given in terms of the charge vector by

$$Z(S) = \sum n^{(C_i)}Z(\tilde{S}_{C_i}).$$

(4.9)

Finally, we return to the subject of monodromy. In [1] it was conjectured (and pushed further in the work of [3][4]) that the monodromies around loci in the moduli space where the mirror $\tilde{X}$ of a Calabi-Yau threefold $X$ becomes singular induce autoequivalences of $D^b(X)$, the bounded derived category of coherent sheaves on $X$. Moreover, in the case of a Fano surface embedded in a Calabi-Yau threefold, a relationship of these autoequivalences of $D^b(X)$ with mutations of exceptional collections supported on the Fano surface was pointed out in [4]. For our purposes we will view the various monodromies mainly as automorphisms of $K^c(X)$. However, in some examples we will identify the monodromy actions on the exceptional collections of coherent sheaves supported on the compact divisors. As in the case of the local mirror geometry, we will be interested in the following three types of transformations:

— Monodromy around large Kähler structure divisors in the moduli space,
— Monodromy around the primary component of the discriminant locus,
— ‘Orbifold monodromy’ in the case $\mathbb{C}^d/G$.

Only the monodromy around a divisor $z_i = 0$ in the moduli space where the Kähler parameter $t_i$ (associated with the divisor class $[T_i]$ in $X$) becomes infinite allows for a classical analysis. In this case we just take $t_i \to t_i + 1$ in (4.5). Because of the multiplicativity of Chern characters, the fact that the Chern character of a line bundle is the exponential of its first Chern class and the form of (4.5), this transforms the $S_j$ by tensoring them with $O_X(-T_i)$. By (4.1), the $R_i$ transform by tensoring with $O_X(T_i)$.

According to the observations in [21][24], ‘orbifold monodromy’ should cyclically permute the $S_i$ if $X$ is a resolution of $\mathbb{C}^d/\mathbb{Z}_n$.

For the primary component of the discriminant locus we have the following picture: In the case of a compact Calabi-Yau variety $X$ it is conjectured (see [1][3][4][5][20]) that a sheaf $\mathcal{F}$ is subjected to a Fourier-Mukai transform whose kernel is the structure sheaf $O_X$, implying that the Chern character of $\mathcal{F}$ transforms as

$$\text{ch}(\mathcal{F}) \to \text{ch}(\mathcal{F}) - \langle O_X, \mathcal{F} \rangle \text{ch}(O_X),$$

(4.10)

where $\langle \ , \ \rangle$ is the pairing (4.2). In our case of non-compact $X$ this cannot work because it would violate compact support conditions, but we make the following observation:
In all of our examples we obtain expressions for $\text{ch}^c(S_0^-)$ that allow us to choose $S_0^-$ in such a way that its restriction $S_0^-|_{C_i}$ to any compact toric cycle $C_i$ is equal to $O_{C_i}$. For the case of a resolution $\pi$ of an orbifold singularity this means that our expressions for $\text{ch}^c(S_0^-)$ are consistent with taking $S_0^-$ to be the push-forward of the restriction of $O_X$ to $\pi^{-1}(0)$.

Wherever we have the possibility of comparison with the mirror geometry, we find that the monodromy around the primary component of the discriminant locus is given by

$$\text{ch}(\mathcal{F}) \rightarrow \text{ch}(\mathcal{F}) - \langle S_0^-, \mathcal{F} \rangle \text{ch}^c(S_0^-).$$  \hfill (4.11)

More precisely, the following happens: For one parameter models the principal component is pointlike. If we decompose the GKZ solutions into logarithms and holomorphic pieces at $z = 0$, the principal component is at the boundary of the radius of convergence of the holomorphic pieces. In this case we find that the monodromy is given precisely by (4.11) provided we choose the simplest anti-clockwise path, $\ln(-1) = \pi i$ and the identification (4.8). With more than one parameter the discriminant locus consists of several disjoint pieces in the $z$ coordinate patch (these pieces join in the other coordinate patches), and there is no unambiguous choice of component or path. We find, however, that at every branch one of the $S_i^-$ (possibly transformed by large complex structure monodromy) becomes massless. This is consistent with the picture that when we take $S_0^-$ along some non-trivial paths like the ones corresponding to ‘orbifold monodromy’ we turn it into one of the other $S_i^-$. 

If $d$ is even there is a simple consistency check: If we require (4.11) to respect the pairing $\langle \cdot, \cdot \rangle$ then it is easily checked that this is equivalent to $\langle S_i^-, S_i^- \rangle = 2$ (in odd dimensions the analogous condition $\langle S_i^-, S_i^- \rangle = 0$ is fulfilled automatically because of the skew symmetry of $\langle \cdot, \cdot \rangle$). This is indeed true in all of our examples.

**Example 1:**
Resolution of $\mathbb{C}^2/\mathbb{Z}_n$: The case of $\mathbb{C}^2/G$ with $G$ any discrete subgroup of $SU(2)$ is well understood by mathematicians in the context of McKay correspondence. If $G$ is abelian and resolved by the introduction of a set $\{C_i\}$ of exceptional curves, and if $\{(C_i^\vee)\}$ is a basis of divisor classes dual to $\{C_i\}$ then the $R_i^M$ are given by $R_0^M = O_X$ and $R_i^M = O_X(C_i^\vee)$ for $i \geq 1$. By (2.6) sections of the $R_i^+$ are given, for example, by $1$, $z_0$, $z_0^2 z_1$, $z_0^3 z_1^2 z_2$, ..., so the action of $\mathbb{Z}^n$ on these sections through $z_0 \rightarrow \epsilon z_0$, $z_n \rightarrow \epsilon^{n-1} z_n$ indeed reproduces the characters $1$, $\epsilon$, $\epsilon^2$, ... of $\mathbb{Z}^n$. 


Again the restriction of that cycle.

\[ \text{and therefore } \text{ch}^c(S_0^-) = \sum_{i=1}^{n-1} \text{ch}(\tilde{S}_{C_i}) - (n-2)\text{ch}(\tilde{S}_p). \]

The restriction of \( \mathcal{O}_X \) to the union of the \( C_i \) is the same as \( \sum \tilde{S}_{C_i} \) except for the \( n-2 \) points of the form \( C_i \cdot C_{i+1} \) where \( \sum \tilde{S}_{C_i} \) has rank two. Upon subtracting the \( n-2 \) sheaves with support on these points we arrive at a class that matches \( \text{ch}^c(S_0^-) \). It is easily checked that \( \langle S_i^-, S_i^- \rangle = 2 \) for all \( i \). The large volume central charges are given by \( Z^{kv}(t; S_i^-) = -1 + \sum_i t_i \) and \( Z^{kv}(t; S_i^-) = -t_i. \)

In the case of \( n = 2 \) this implies \( Z(S_0^-) = \varpi_1 \) and \( Z(S_1^-) = -1 - \varpi_1 \) and we see that the principal component and orbifold monodromies found in the mirror geometry are precisely the ones generated by \( (4.11) \) and permutations \( S_0^- \leftrightarrow S_1^- \), respectively.

**Example 2:**

For \( \mathbb{C}^n / \mathbb{Z}_n \) with \( \mathbb{Z}_n : (z_1, \ldots, z_n) \to (e^{2\pi i/n}z_1, \ldots, e^{2\pi i/n}z_n) \) the restrictions of the \( R_i^M \) to the exceptional divisor \( D \simeq \mathbb{P}^{n-1} \) are nothing but \( \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n-1) \). The independent holomorphic cycles are of the form \( \mathbb{P}^j \) with \( 0 \leq j \leq n-1 \) and the \( R_i^\pm \) restrict to \( \mathcal{O}_{\mathbb{P}^j}(\pm i) \).

This example has been previously considered in [17][18][17]. We include it as further evidence that the \( S_i \) have the properties stated above. Defining

\[ \chi_{kj} := \chi(\mathcal{O}_{\mathbb{P}^j}(k), \mathbb{P}^j) = \int_{\mathbb{P}^j} \text{ch}(\mathcal{O}_{\mathbb{P}^j}(k)) \text{Td}(\mathbb{P}^j) = \int_{\mathbb{P}^j} e^{kH} \left( \frac{H}{1-e^{-H}} \right)^{j+1} \] (4.13)

with \( H \) the hyperplane divisor, we find that

\[ \chi_{kj} - \chi_{k-1,j} = \int_{\mathbb{P}^j} e^{kH} (1-e^{-H}) \left( \frac{H}{1-e^{-H}} \right)^{j+1} = \]

\[ = \int_{\mathbb{P}^j} e^{kH} H \left( \frac{H}{1-e^{-H}} \right)^j = \int_{\mathbb{P}^j} e^{kH} \left( \frac{H}{1-e^{-H}} \right)^j = \chi_{k,j-1}. \] (4.14)

With \( \chi_{k0} = \chi_{00} = 1 \) this simple recursion is solved by \( \chi_{kj} = \binom{k+j}{j} \) and we obtain \( (R_i^\pm, \tilde{S}_{\mathbb{P}^j}) = \binom{j+i}{j} \), implying \( (R_0^-, \tilde{S}_{\mathbb{P}^j}) = 1 \) for any \( j \), \( (R_i^-, \tilde{S}_{\mathbb{P}^j}) = 0 \) for \( 1 \leq i \leq j \) and \( (R_i^-, \tilde{S}_{\mathbb{P}^j}) = (-1)^j \binom{i-1}{j-1} \) for \( i > j \). This leads to the following expressions for the \( S^-: \)

\[ S_0^- = \tilde{S}_{\mathbb{P}^{n-1}}, \quad (-1)^k S_k^- = \binom{n-1}{k} \tilde{S}_{\mathbb{P}^{n-1}} - \sum_{j=k+1}^{n-2} \binom{j}{k-1} \tilde{S}_{\mathbb{P}^j} \quad \text{for } k \geq 1. \] (4.15)

Again the restriction of \( S_0^- \) to any compact toric cycle is the same as the structure sheaf of that cycle.
Alternatively we may determine the $S_i^-$ by the ansatz $S_i^- = \sum a_{ik} i_* \mathcal{O}_{\mathbb{P}^n-1}(k)$. With 
\[(R_i^-, i_* \mathcal{O}_{\mathbb{P}^n-1}(k)) = \chi_{k-i,n-1} = \binom{n-1+k-i}{n-1} \] (4.16)
we get $(a^{-1})_{ki} = \chi_{k-i,n-1}$ which leads to 
\[S_i^- = \sum_{k=0}^n a_{ik} i_* \mathcal{O}_{\mathbb{P}^n-1}(k) = \sum_{k=0}^i (-1)^{i-k} \binom{n}{i-k} i_* \mathcal{O}_{\mathbb{P}^n-1}(k); \] (4.17)

\[\langle S_i^-, S_i^- \rangle = \sum_{k,l} a_{ik} a_{il} \int_X \text{ch}(i_* \mathcal{O}_{\mathbb{P}^n-1}(k))^* \text{ch}(i_* \mathcal{O}_{\mathbb{P}^n-1}(l)) \text{Td}(X) = \sum_{k,l} a_{ik} a_{il} \int_{\mathbb{P}^n-1} (1 - e^{-nH}) e^{-kH} e^{lH} Td(\mathbb{P}^{n-1}) = \sum_{k,l} a_{ik} a_{il} (\chi_{l-k,n-1} - \chi_{l-k-n,n-1}). \] (4.18)

Using $\chi_{l-k-n,n-1} = (-1)^{n-1} \chi_{k-l,n-1}$ and the reciprocity of $a$ and $\chi$ we get
\[\langle S_i^-, S_i^- \rangle = a_{ii} (1 - (-1)^{n-1}) = 0/2 \text{ for } n \text{ odd/even}, \] (4.19)
as it should be.

For $n = 3$ we find $Z^\text{lv}(t; S_0^-) = -t^2/2 + 3t/2 - 5/4$. The corresponding GKZ system has been studied at various places in the literature, e.g. in [30] and [9]. In terms of solutions
\[\varpi_0 = 1, \quad \varpi_1 = \frac{\ln z}{2\pi i} + O(z), \quad \varpi_2 = \left(\frac{\ln z}{2\pi i}\right)^2 + O(z \ln z), \] (4.20)
the rule $t \sim 1 + \ln z/(2\pi i)$ leads to $Z(S_0^-) = -\varpi_2/2 + \varpi_1/2 - 1/4$. Comparing with [3], we find that this is precisely the expression denoted there by $t_d$ which vanishes at the discriminant point $z = -1/27$.

**Example 3:**

The Kähler cone is generated by $[D_1]$ and $[D_2]$ corresponding to the characters $\epsilon = e^{2\pi i/5}$ and $\epsilon^3$, respectively. By applying the rules outlined above, we assign the character $\epsilon$ to each of the three line segments meeting at $v_4$ in fig. 1 and to the line segment between $v_5$ and $v_2$, whereas the remaining two line segments (from $v_5$ to $v_1$ and $v_3$) correspond to $\epsilon^3$. Thus we get $F_2 = 2D_1$ because of the three line segments with equal characters meeting.
at $v_4$ and $D_1 + D_2$ because of the two pairs of line segments at $v_5$. Altogether we get representatives $R^\pm_i = \mathcal{O}(\pm F_i)$ with
\[
F_0 = 0, \quad F_1 = D_1, \quad F_2 = 2D_1, \quad F_3 = D_2, \quad F_4 = D_1 + D_2
\]
for the bases of $K(X)$, where we have chosen the labels such that sections of $R^+_i$ transform as $\epsilon^i$ under $(z_1, z_2, z_3) \rightarrow (\epsilon z_1, \epsilon^3 z_2, \epsilon z_3)$. Using (4.1) we find that the localized Chern characters of the basis of $K^c(X)$ are given by
\[
\begin{align*}
\text{ch}^c(S^+_0) &= D_4 + D_5 \mp \left(\frac{3}{2}h + \frac{5}{2}f\right) + \frac{11}{6}p, \\
\text{ch}^c(S^+_1) &= -2D_4 - D_5 \mp \left(2h + \frac{3}{2}f\right) - \frac{4}{3}p, \\
\text{ch}^c(S^+_2) &= D_4 \mp \frac{1}{2}h + \frac{1}{2}p, \\
\text{ch}^c(S^+_3) &= -D_5 \pm \frac{5}{2}f - \frac{1}{3}p, \\
\text{ch}^c(S^+_4) &= D_5 \mp \frac{3}{2}f + \frac{1}{3}p, \\
\end{align*}
\]
with $p$ the class of a point.

Let us now consider the branes defined in terms of the structure sheaves $\mathcal{O}_p, \mathcal{O}_h, \mathcal{O}_f, \mathcal{O}_{\mathbb{F}_2}, \mathcal{O}_{\mathbb{F}_3}$ of the independent lower dimensional cycles. Denoting by $\tilde{S}_p$ the result of three successive inclusion maps acting on $\mathcal{O}_p$, etc., we arrive with the help of the Grothendieck-Riemann-Roch theorem (4.3) at the following result:
\[
\begin{align*}
\text{ch}(\tilde{S}_{D_4}) &= D_4 + \frac{3}{2}h + \frac{3}{2}p, \\
\text{ch}(\tilde{S}_{D_5}) &= D_5 + h + \frac{5}{2}f + \frac{4}{3}p, \\
\text{ch}(\tilde{S}_p) &= p, \\
\text{ch}(\tilde{S}_h) &= h + p, \\
\text{ch}(\tilde{S}_f) &= f + p.
\end{align*}
\]
This allows us to determine the D-brane charges $n_i = (n_{D_4}, n_{D_5}, n_p, n_h, n_f)$ with $n_p$ the D0-brane charge, $n_h, n_f$ D2-brane charges and $n_{D_4}, n_{D_5}$ D4-brane charges of the $S^-_i$ as
\[
\begin{align*}
n_0 &= (1, 1, 0, -1, 0) \\
n_1 &= (-2, -1, 0, 2, 1) \\
n_2 &= (1, 0, 0, -1, 0) \\
n_3 &= (0, -1, 0, 1, 0) \\
n_4 &= (0, 1, 1, -1, -1).
\end{align*}
\]
In particular, this means that $S_0^- = \tilde{S}_{D_4} + \tilde{S}_{D_5} - \tilde{S}_h$. Note how $\tilde{S}_{D_4} + \tilde{S}_{D_5}$ has rank 1 on $D_4$ and on $D_5$ except on their intersection $h$, where it has rank 2 which is compensated by subtracting $\tilde{S}_h$.

At this point we would like to mention that we have performed a similar analysis for $\mathbb{C}^3/\mathbb{Z}_n$ with arbitrary odd $n$ and an action of the type $(z_1, z_2, z_3) \rightarrow (e z_1, e^{n-2} z_2, e z_3)$. In that case the resolution requires a $\mathbb{P}^2$ and $(n - 3)/2$ Hirzebruch surfaces and we again obtain $R_i^+$ whose sections transform by the characters of $\mathbb{Z}_n$ and an expression for $S_0^-$ that reduces to the structure sheaf on every compact toric cycle.

Returning to $\mathbb{C}^3/\mathbb{Z}_5$ we now give an alternative description of the compactly supported K-theory classes in terms of non-trivial sheaves on the exceptional divisors. Again with the help of the Grothendieck-Riemann-Roch theorem, we find that we may choose representatives $S_i$ in terms of the following combinations of push-forwards of sheaves:

$$
S_0^- = j_*(O_{D_5}(-h)) + g_* O_{D_4},
S_1^- = -j_*(O_{D_5}(-h - f)) - g_* V,
S_2^- = g_* O_{D_4}(-1),
S_3^- = -j_*(O_{D_5}(-h)),
S_4^- = j_*(O_{D_5}(-h - f)),
$$

with $g : D_4 \hookrightarrow X$ and $j : D_5 \hookrightarrow X$ inclusion maps. By $V$ we denote a stable bundle on $\mathbb{P}^2$ with the Chern character given by

$$
\text{ch}(V) = 2 - h - \frac{1}{2} p
$$

where $p$ is the class of a point on $\mathbb{P}^2$. Note that $\{O_{\mathbb{P}^2}(-1), V, O_{\mathbb{P}^2}\}$ is a foundation of the helix of exceptional bundles on $\mathbb{P}^2$ and that $\{O_{\mathbb{F}_3}(-h - f), O_{\mathbb{F}_3}(-h)\}$ is a regular exceptional pair on $\mathbb{F}_3$ (see [19]).

In terms of the pure brane basis $\tilde{S}$ the large volume central charge is

$$
Z^{lv}(t_1, t_2; \tilde{S}) = - \int_X e^{-(t_1 D_1 + t_2 D_2)} \text{ch}(\tilde{S}) \sqrt{Td(X)} = \begin{bmatrix}
-\frac{1}{2} t_1^2 + \frac{3}{2} t_1 - \frac{5}{4} & -\frac{1}{2} t_2^2 + 2 t_1 t_2 + t_1 + \frac{5}{2} t_2 - \frac{7}{6} \\
-1 & t_1 - 1 \\
t_1 - 1 & t_2 - 1
\end{bmatrix}.
$$

We will now discuss monodromy by assuming that the assertions made in this section are correct. The comparison with the mirror geometry is rather technical and can be
found in appendix A. We want to find monodromy matrices acting on the charge vectors, \( n \rightarrow n \cdot M \cdot Z(\tilde{\mathcal{S}}) \) such that \( n \cdot Z(\tilde{\mathcal{S}}) = n \cdot M \cdot Z_{\text{mt}}(\tilde{\mathcal{S}}) \), where \( Z_{\text{mt}}(\tilde{\mathcal{S}}) = M^{-1} Z(\tilde{\mathcal{S}}) \) is the monodromy transformed version of \( Z \). The monodromy around the orbifold locus cyclically permutes the charge vectors (4.24). Therefore, we obtain:

\[
M_{\text{orb}} = \begin{bmatrix}
-2 & 0 & 1 & 1 & 0 \\
-2 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
-2 & 1 & 1 & 0 & 0 \\
-1 & -2 & 0 & 2 & 1 \\
\end{bmatrix}
\] (4.28)

Also, we can easily compute the large radius limit monodromies, \( M_{t_1} \) and \( M_{t_2} \). On the sheaves defined on the exceptional divisors the actions of the monodromies come from tensoring with the restrictions of \( \mathcal{O}_X(D_1) \) and \( \mathcal{O}_X(D_2) \). Therefore, the large radius limit monodromy \( M_{t_1} \) acts as following: the exceptional collection \( \{ \mathcal{O}_{\mathbb{P}^2}(-n), \mathcal{V}, \mathcal{O}_{\mathbb{P}^2}(-n + 1) \} \) on \( \mathbb{P}^2 \) is mutated to another exceptional collection, \( \{ \mathcal{O}_{\mathbb{P}^2}(-n + 1), \tilde{\mathcal{V}}, \mathcal{O}_{\mathbb{P}^2}(-n + 2) \} \), while on \( \mathbb{F}_3 \) is given by the tensoring with \( \mathcal{O}_{\mathbb{F}_3}(f) \), therefore taking regular exceptional pairs into regular exceptional pairs.

The action of the monodromy \( M_{t_2} \) is represented by tensoring any sheaf supported on \( \mathbb{F}_3 \) with \( \mathcal{O}_{\mathbb{F}_3}(h + 3f) \), hence again transforming regular pairs into regular pairs, while leaving any sheaf supported on \( \mathbb{P}^2 \) invariant.

Using (4.11) it is possible to compute the action of the monodromy around the principal component of the discriminant on the generators of \( K(X) \): \( R_i^- \) with \( i = 1, \ldots, 4 \) are invariant under this transformation, but \( R_0^- \mapsto \mathcal{O}_X(-2D_1 - D_2) \). With the help of (4.4), we readily obtain the monodromy around the conifold locus:

\[
M_{\text{con}} = \begin{bmatrix}
2 & 1 & 0 & -1 & 0 \\
1 & 2 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
2 & 2 & 0 & -1 & 0 \\
1 & 1 & 0 & -1 & 1 \\
\end{bmatrix}
\] (4.29)

The conifold monodromy, although preserving exceptional collections, acts in a very different way on \( K^c(X) \). For example, we have \(-[\mathcal{O}_{\mathbb{F}_3}(-h)] \mapsto [\mathcal{O}_{\mathbb{P}^2}] \) and \([\mathcal{O}_{\mathbb{P}^2}(-1)] \mapsto -[\mathcal{O}_{\mathbb{F}_3}] \), that is the D4-branes can ‘jump’ from one exceptional divisor to another. However, as remarked in [20], this is not very surprising since autoequivalences of \( D^b(X) \) need not preserve the D-branes.
5. Beyond $\mathbb{C}^d/\mathbb{Z}_n$

Up to now we have only considered cases of the type $\mathbb{C}^d/\mathbb{Z}_n$ with a single triangulation. We now want to examine the range of validity of our statements regarding the $S_i^-$ and monodromy. We first present another example, the resolution of $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, which is still an orbifold but has several interesting features: It is not of the simple $\mathbb{Z}_n$ type, it allows for more than one triangulation, its resolution involves three new non-compact toric divisors but no compact toric divisor, and finally it is a three parameter model whose GKZ system can be solved explicitly. We will be able to show explicitly that the $S_i^-$ vanish at (branches of) the principal component of the discriminant locus and nowhere else. Aspects of D-brane states on this model have been studied previously in e.g. [50][51]. Finally we examine the possibility of extending our results to cases not of the McKay type. We find that they still hold in many examples but not in general.

Example 4:

A toric resolution of $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$: A singular space of the type $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ where every non-trivial element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts by flipping the sign of two of the three coordinates of $\mathbb{C}^3$ can be resolved by introducing three additional non-compact divisors and three compact curves. There are several distinct possibilities for choosing the curves.

![Fig. 6: G-Hilb resolution of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.](image)

We use the G-Hilb resolution depicted in fig. 6. The Mori cone is generated by the following vectors:

\begin{align}
 l^{(1)} &= (1, 0, 0, 1, -1, -1), \\
 l^{(2)} &= (0, 1, 0, -1, 1, -1), \\
 l^{(3)} &= (0, 0, 1, -1, -1, 1). \tag{5.1}
\end{align}
The generators of the Kähler cone are the divisors $D_1$, $D_2$ and $D_3$ corresponding to the vanishing of the coordinates of $C_3$. The mirror geometry is determined by
\[ a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_1' x_2 x_3 + a_2' x_1 x_3 + a_3' x_1 x_2 = 0. \] (5.2)

The large complex structure coordinates $z_i$ are
\[ z_1 = \frac{a_1 a_1'}{a_2 a_3'}, \quad z_2 = \frac{a_2 a_2'}{a_1 a_3'}, \quad z_3 = \frac{a_3 a_3'}{a_1 a_2'} \] (5.3)

and the orbifold coordinates are
\[ \phi_1 = \frac{a_1'}{\sqrt{a_2 a_3}}, \quad \phi_2 = \frac{a_2'}{\sqrt{a_1 a_3}}, \quad \phi_3 = \frac{a_3'}{\sqrt{a_1 a_2}} \] (5.4)

The principal component of the discriminant locus is determined by
\[ 4z_1 z_2 z_3 - z_1 - z_2 - z_3 + 1 = 0. \] (5.5)

The simplest formulation of the GKZ system can be obtained by mixing large complex structure and orbifold coordinates. We find the operator
\[ D_1 = (\Theta_{\phi_2} + \Theta_{\phi_3})\Theta_{\phi_1} - 2z_1 \Theta_{\phi_2} \Theta_{\phi_3} \] (5.6)

from the first Mori cone vector and the same operator with cyclically permuted indices for the other two Mori cone vectors. This simply implies
\[ \Theta_{\phi_2} \Theta_{\phi_3} \Pi = \Theta_{\phi_1} \Theta_{\phi_3} \Pi = \Theta_{\phi_1} \Theta_{\phi_2} \Pi = 0 \] (5.7)

for any solution $\Pi$, i.e. there must be a basis of solutions depending only on at most one of the $\phi_i$. The sums of two Mori cone vectors lead to operators of the type
\[ D_1' = (\Theta_{\phi_1} + \Theta_{\phi_3})(\Theta_{\phi_1} + \Theta_{\phi_2}) - 4z_1 z_2 \Theta_{\phi_1}(\Theta_{\phi_1} - 1), \] (5.8)

which upon using (5.4) and (5.7) implies
\[ ((1 - 4\phi_1^{-2})\Theta_{\phi_1} + 4\phi_1^{-2}) \Theta_{\phi_1} \Pi = 0. \] (5.9)

The whole GKZ system has three solutions of the type $\ln((\phi_i + \sqrt{\phi_i^2 - 4})/2)$ and, as always, a constant solution. Upon returning to large complex structure variables, we obtain
\[ \Pi_0 = 1, \quad \Pi_1 = \ln \left( \frac{1 + \sqrt{1 - 4z_2 z_3}}{2} \right) - \frac{1}{2}(\ln z_2 + \ln z_3) \] (5.10)
and the corresponding index-permuted expressions for \( \Pi_2 \) and \( \Pi_3 \).

The divisors \( F_i \) determining the line bundles \( R_i \) are just \( D_1, D_2 \) and \( D_3 \) and we find

\[
\text{ch}^c(S^-_0) = p + C_1 + C_2 + C_3, \quad \text{ch}^c(S^-_1) = -C_1, \quad \text{ch}^c(S^-_2) = -C_2, \quad \text{ch}^c(S^-_3) = -C_3 \tag{5.11}
\]

where \( C_1 \) is the compact curve at the intersection of \( D_2' \) and \( D_3' \) etc. In terms of structure sheaves we can represent \( \tilde{S}_0^- \) as \( \tilde{S}_C + \tilde{S}_C + \tilde{S}_C - 2\tilde{S}_p \). Noticing that all three curves intersect in the same point, we find that we can again view \( \tilde{S}_0^- \) as the object whose restriction to any compact toric cycle is the structure sheaf of that cycle.

The central charges are determined by \( Z^{lv}(t; S^-_0) = -1 + t_1 + t_2 + t_3 \) and \( Z^{lv}(t; S^-_1) = -t_i \) for \( i \in \{1, 2, 3\} \), leading to

\[
Z(S^-_0) = 2 - \frac{\Pi_1 + \Pi_2 + \Pi_3}{2\pi i}, \quad Z(S^-_1) = -1 + \frac{-\Pi_1 + \Pi_2 + \Pi_3}{2\pi i}, \quad \text{etc.} \tag{5.12}
\]

At the orbifold point \( \phi_1 = \phi_2 = \phi_3 = 0 \) we have the following situation: The moduli space develops a \( \mathbf{Z}_2 \times \mathbf{Z}_2 \) singularity. Provided we make the right choice of sheets for the square roots and logarithms, we find \( \Pi_1 = \Pi_2 = \Pi_3 = 3\pi i/2 \) and thus \( Z(S^-_i) = -1/4 \) for any \( i \). The ‘orbifold monodromy’ \( \phi_2 \rightarrow -\phi_2, \phi_3 \rightarrow -\phi_3 \) acts as

\[
\Pi_2 \leftrightarrow 3\pi i - \Pi_2, \quad \Pi_3 \leftrightarrow 3\pi i - \Pi_3, \quad S^-_0 \leftrightarrow S^-_1, \quad S^-_2 \leftrightarrow S^-_3, \tag{5.13}
\]

and the other elements of the orbifold monodromy act in similar ways.

\( S^-_0 \) can become massless only if

\[
(1 + \sqrt{1 - 4z_2 z_3})(1 + \sqrt{1 - 4z_1 z_3})(1 + \sqrt{1 - 4z_1 z_2}) = 8z_1 z_2 z_3. \tag{5.14}
\]

We can rewrite this in the form \( \sqrt{1 - 4z_2 z_3} (E_1) = (E_2) \) such that \( E_1 \) and \( E_2 \) are expressions that do not contain \( \sqrt{1 - 4z_2 z_3} \). Then a necessary condition for (5.14) to hold is \( (1 - 4z_2 z_3)(E_1)^2 = (E_2)^2 \) and we can proceed to eliminate the other square roots in the same way. The result is an equation proportional to the square of the expression determining the principal component of the discriminant locus (5.5). Conversely, if we solve (5.5), e.g. by setting \( z_1 = (1 - z_2 - z_3)/(1 - 4z_2 z_3) \), plug this into (5.14) and choose the right sheets, we find that \( Z(S^-_0) \) indeed vanishes. The same type of analysis works for the other \( S^-_i \).

At this point it is natural to ask whether the analog of \( S^-_0 \), i.e. the sheaf that is equal to the structure sheaf upon restriction to any compact toric cycle but of rank zero away from these cycles, might lead to monodromies in cases that are not related to McKay correspondence. It turns out that this is very often the case (at least for sufficiently simple examples), but not true in general.
Three examples where this works are shown in fig. 7. The first of these is the resolution of a conifold singularity and exactly solvable. The other two (anticanonical line bundles over $\mathbb{F}_0 \simeq \mathbb{P}_1 \times \mathbb{P}_1$ and $\mathbb{F}_1$, respectively) are two parameter models that we treated with analyses similar to the ones used for example 3 ($\mathbb{C}^3/\mathbb{Z}_5$).

As a counterexample, consider the Calabi-Yau manifold depicted in fig. 8, whose GKZ system is again solvable. Here we find the following: If we choose as the line bundles $R_i$ the ones determined by the generators of the Kähler cone, we still find that the corresponding $S_0^-$ has the same restriction to compact toric cycles as $\mathcal{O}_X$. However, only two of the three generators of $K^c(X)$ become massless at the conifold locus (these statements are true for any triangulation). In particular, for the symmetric triangulation the vanishing locus of the central charge of $S_0^-$ does not coincide with the conifold locus.

Acknowledgements

We would like to thank Philip Candelas and Duiliu Diaconescu for very useful conversations.
Appendix A. Comparison of GKZ solutions and K-theory results for $\Phi^3/\mathbb{Z}_5$

The GKZ operators\(^6\) corresponding to the Mori cone generators (2.17) are given by

$$D^{(1)} = \partial_{a_1} \partial_{a_3} \partial_{a_5} - \partial_{a_4}^3, \quad D^{(2)} = \partial_{a_2} \partial_{a_4} - \partial_{a_5}^2.$$  \hspace{1cm} (A.1)

This can be turned into a system involving $z_1, z_2$ by standard manipulations described above. In this way we arrive at the following expressions in terms of $\Theta_{z_i} := z_i \frac{\partial}{\partial z_i}$:

$$D_1 = \Theta_{z_1}^2(\Theta_{z_1} - 2\Theta_{z_2}) - (\Theta_{z_2} - 3\Theta_{z_1} + 1)(\Theta_{z_2} - 3\Theta_{z_1} + 2)(\Theta_{z_2} - 3\Theta_{z_1} + 3)z_1$$
$$D_2 = \Theta_{z_2}(\Theta_{z_2} - 3\Theta_{z_1}) - (\Theta_{z_1} - 2\Theta_{z_2} + 1)(\Theta_{z_1} - 2\Theta_{z_2} + 2)z_2,$$  \hspace{1cm} (A.2)

Solutions to this system can be obtained by considering

$$\Pi(z_1, z_2; \rho_1, \rho_2) := \sum_{n_1, n_2=0}^{\infty} z_1^{n_1+\rho_1} z_2^{n_2+\rho_2} \frac{\Gamma(1+\rho_1)^2 \Gamma(1+\rho_2)}{\Gamma(1+n_1+\rho_1)^2 \Gamma(1+n_2+\rho_2)}$$
$$\times \frac{\Gamma(1+\rho_1-2\rho_2)}{\Gamma(1+n_1-2n_2+\rho_1-2\rho_2)} \frac{\Gamma(1+\rho_2-3\rho_1)}{\Gamma(1+n_2-3n_1+\rho_2-3\rho_1)}$$  \hspace{1cm} (A.3)

(the coefficients of $n_i$, $\rho_i$ in the $\Gamma$–functions are the entries of the Mori cone vectors (2.17)) and its partial derivatives w.r.t the $\rho_i$ at $\rho_1 = \rho_2 = 0$. We use:

$$\frac{\Gamma(1+\rho)}{\Gamma(1+n+\rho)} = \frac{1}{[(1+\rho)(2+\rho)\cdots(n+\rho)]^{-1}} \begin{cases} 1 & n = 0 \\ \rho(\rho-1)\cdots(\rho+n+1) & n > 0 \\ \rho(\rho-1)\cdots(\rho+n+1) & n < 0 \end{cases}$$  \hspace{1cm} (A.4)

$$\frac{\partial}{\partial \rho} \left( \frac{\Gamma(1+\rho)}{\Gamma(1+n+\rho)} \right)_{\rho=0} = \begin{cases} 0 & n = 0 \\ -S_n/n! & n > 0 \\ (-1)^{n-1}(-n-1)! & n < 0 \end{cases}$$  \hspace{1cm} (A.5)

$$\frac{\partial^2}{\partial \rho^2} \left( \frac{\Gamma(1+\rho)}{\Gamma(1+n+\rho)} \right)_{\rho=0} = \begin{cases} 0 & n = 0 \\ \text{finite} & n > 0 \\ 2(-1)^{n-2}S_{-n-1}(-n-1)! & n < 0 \end{cases}$$  \hspace{1cm} (A.6)

where $S_n = 1 + 1/2 + \ldots + 1/n$. This yields the constant solution $\Pi(z_1, z_2; 0, 0) = 1$ and, with $\Pi_{i_1\ldots i_k}$ for $(\partial^k \Pi/\partial \rho_{i_1}\ldots\partial \rho_{i_k})|_{\rho=0}$,

$$\Pi_1 = \ln z_1 + \sum z_1^{n_1} z_2^{n_2} A_{n_1 n_2} - 3 \sum z_1^{n_1} z_2^{n_2} B_{n_1 n_2},$$
$$\Pi_2 = \ln z_2 - 2 \sum z_1^{n_1} z_2^{n_2} A_{n_1 n_2} + \sum z_1^{n_1} z_2^{n_2} B_{n_1 n_2},$$

\(^6\) This GKZ system has also been studied in [53].
\[
\Pi_{11} = (\ln z_1)^2 + 2 \ln z_1 \left( \sum z_1^{n_1} z_2^{n_2} A_{n_1 n_2} - 3 \sum z_1^{n_1} z_2^{n_2} B_{n_1 n_2} \right) - 6 \sum z_1^{n_1} z_2^{n_2} C_{n_1 n_2} \\
+ \sum z_1^{n_1} z_2^{n_2} A_{n_1 n_2} (-2 S_{2n_2-n_1-1} + 6 S_{n_2-3n_1} - 4 S_{n_1}) \\
+ \sum z_1^{n_1} z_2^{n_2} B_{n_1 n_2} (-18 S_{3n_1-n_2-1} + 6 S_{n_1-2n_2} + 12 S_{n_1}), \\
\Pi_{12} = \ln z_1 \ln z_2 + \ln z_1 \left( -2 \sum z_1^{n_1} z_2^{n_2} A_{n_1 n_2} + \sum z_1^{n_1} z_2^{n_2} B_{n_1 n_2} \right) \\
+ \ln z_2 \left( \sum z_1^{n_1} z_2^{n_2} A_{n_1 n_2} - 3 \sum z_1^{n_1} z_2^{n_2} B_{n_1 n_2} \right) + 7 \sum z_1^{n_1} z_2^{n_2} C_{n_1 n_2} \\
+ \sum z_1^{n_1} z_2^{n_2} A_{n_1 n_2} \left( 4 S_{2n_2-n_1-1} - 7 S_{n_2-3n_1} + 4 S_{n_1} - S_{n_2} \right) \\
+ \sum z_1^{n_1} z_2^{n_2} B_{n_1 n_2} \left( 6 S_{3n_1-n_2-1} - 7 S_{n_1-2n_2} - 2 S_{n_1} + 3 S_{n_2} \right), \\
\Pi_{22} = (\ln z_2)^2 + 2 \ln z_2 \left( -2 \sum z_1^{n_1} z_2^{n_2} A_{n_1 n_2} + \sum z_1^{n_1} z_2^{n_2} B_{n_1 n_2} \right) - 4 \sum z_1^{n_1} z_2^{n_2} C_{n_1 n_2} \\
+ \sum z_1^{n_1} z_2^{n_2} A_{n_1 n_2} (-8 S_{2n_2-n_1-1} + 4 S_{n_2-3n_1} + 4 S_{n_2}) \\
+ \sum z_1^{n_1} z_2^{n_2} B_{n_1 n_2} (-2 S_{3n_1-n_2-1} + 4 S_{n_1-2n_2} - 2 S_{n_2}),
\]

where

\[
A_{n_1 n_2} = \frac{(2n_2-n_1-1)!}{(n_2-3n_1)! (n_1)! z_1^{n_2}} (-1)^{2n_2-n_1-1}, \\
B_{n_1 n_2} = \frac{(3n_1-n_2-1)!}{(n_2-3n_1)! (n_1)! z_2^{n_2}} (-1)^{3n_1-n_2-1}, \\
C_{n_1 n_2} = \frac{(2n_2-n_1-1)! (3n_1-n_2-1)!}{(n_1)! z_1^{n_2} z_2^{n_2}} (-1)^{2n_2-n_1-1+3n_1-n_2-1}
\]

and the summations are taken over those values of \(n_1, n_2\) where the arguments of all factorials are non-negative. Of the three expressions obtained by taking second derivatives only the first one and the linear combination \(3\Pi_{22} + 2\Pi_{12}\) of the other two actually solve the GKZ system (A.2). We note that there is also a linear combination of third derivatives (involving third powers of logarithms) that is annihilated by both operators occurring in (A.2). The reason is that this system is not yet complete as we have written it: In principle we should write down a GKZ operator for every curve in the Mori cone. Taking as an additional charge vector the sum \(l^{(1)} + l^{(2)}\) of our Mori cone generators, we see that the triple-log solution is excluded.

We now want to study monodromies of the GKZ solutions around the loci where \(R\) degenerates. This is an easy exercise for the divisors \(z_1 = 0, z_2 = 0\) where the monodromy is determined by \(\ln z_i \rightarrow \ln z_i + 2\pi i\). We find that the following set of solutions is well behaved (i.e., transforms by an \(SL(5, \mathbb{Z})\) matrix) under the monodromies \(z_i \rightarrow e^{2\pi i z_i}:
\]

\[
\begin{pmatrix}
1 \\
\frac{1}{2\pi i} \Pi_1 + \text{const.} \\
\frac{1}{2\pi i} \Pi_2 + \text{const.} \\
\frac{1}{2(2\pi i)^2} \Pi_{11} - \frac{1}{2(2\pi i)} \Pi_1 + \text{const.} \\
\frac{1}{2(2\pi i)^2} (3\Pi_{22} + 2\Pi_{12}) - \frac{1}{2(2\pi i)} \Pi_2 + \text{const.}
\end{pmatrix}
\]

(A.7)

33
In the large complex structure coordinate patch the discriminant locus consists of several different branches. The slice through real $z_1$, $z_2$ is shown in fig. 9. There are two branches with $z_1 \leq 0$. The one with $z_2 > 0$ is tangent to the axis $z_1 = 0$ in $z_2 = 1/4$. Parts of this branch are at the boundary of the domain of convergence of the $\Pi$'s in such a way that there is convergence like $1/n^2$. Through a numerical analysis we found that at this locus

$$-\frac{1}{2}(2\Pi_{12} + 3\Pi_{22}) + \frac{1}{2}\Pi_2 - \frac{1}{6} = 0,$$

which corresponds to the vanishing of $Z(S_{\frac{3}{4}}^{-})$ if we make the identifications $t_1 \sim 1 + \ln z_1$, $t_2 \sim \ln z_2$. This is equivalent to the vanishing of a $z_2$-monodromy transformed version of $S_{\frac{3}{4}}^{-}$ with (4.8). Similarly we find at the other branch with $z_1 < 0$ which intersects $z_2 = 0$ at $z_1 = -1/27$ that

$$-\frac{1}{2}\Pi_{11} + \frac{1}{2}\Pi_1 - \frac{1}{4} = 0.$$

This corresponds to a $z_1$-monodromy transformed version of $S_{\frac{5}{2}}^{-}$ vanishing. The third branch, with $z_1 > 0$ and $z_2 < 0$ is beyond the region of convergence. We have preliminary evidence that at this branch a $z_2$-monodromy transformed version of $S_{0}^{-}$ becomes massless.
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