Nonlinear Internal Models for Output Regulation *

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Abstract

In this paper we show how nonlinear internal models can be effectively used in the design of output regulators for nonlinear systems. This result provides a significant enhancement of the non-equilibrium theory for output regulation, which we have presented in the recent paper “Limit Sets, Zero Dynamics, and Internal Models in the Problem of Nonlinear Output Regulation”.

Keywords: Limit Sets, Zero Dynamics, Internal Model, Regulation, Tracking, Nonlinear Control.

1 Introduction

In the recent paper [1], we have laid the foundations for a non-equilibrium theory of nonlinear output regulation, giving a more general (non-equilibrium) definition of the problem, deriving necessary conditions for the existence of solutions, and describing how the necessary conditions thus found, complemented with an additional set of appropriate hypotheses, can be used in the design of output regulators.

We recall that, generally speaking, the problem of output regulation is to have the regulated variables of a given controlled plant to asymptotically track (or reject) all desired trajectories (or disturbances) generated by some fixed autonomous system, known as exosystem. The hypotheses assumed in [1] for the design of output regulators no longer include the assumption, common to all earlier literature, that the zero-dynamics of the controlled plant have a globally asymptotically stable equilibrium. Rather, this assumption is replaced with

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*This work was partially supported by NSF under grant ECS-0314004, by the AFOSR under grant F49620-01-10039, and by the Boeing-McDonnell Douglas Foundation.
the (substantially weaker) hypothesis that the zero dynamics of the plant “augmented by exosystem” have a compact attractor. In [1], though, we have retained the (rather strong) assumption, itself also common to all earlier literature, that the set of all “feedforward inputs capable to secure perfect tracking” is a subset of the set of solutions of a suitable linear differential equation. In this note we show that, within the new framework, the assumption of linearity can also be dropped.

Since this technical note is to be viewed as a continuation of the work [1], we retain the same notation and – to avoid duplication – we refer the reader to a number key concepts introduced and/or described in that paper, among which the notion of omega limit set $\omega(B)$ of a set $B$, plays a major role. This concept is a deep generalization of the classical concept, due to Birkhoff, of omega limit set of a point and provides a rigorous definition of steady-state response in a nonlinear system (see [1] for details).

2 Problem Statement

In [1], as an illustration of how the new non-equilibrium concepts can be applied to the design of regulators, we have shown how the problem of output regulation can be solved, under appropriate assumptions, for a system which can be put in the form

$$
\begin{align*}
\dot{z} &= f_0(z, w) + f_1(z, \zeta, w) \zeta \\
\dot{\zeta} &= q(z, \zeta, w) + u \\
e &= \zeta \\
y &= \zeta,
\end{align*}
$$

(1)

with state $z, \zeta \in \mathbb{R}^n \times \mathbb{R}$, control input $u \in \mathbb{R}$, regulated output $e \in \mathbb{R}$, measured output $w \in \mathbb{R}$ and exogenous (disturbance) input $w \in \mathbb{R}^r$ generated by an exosystem

$$
\dot{w} = s(w).
$$

(2)

The functions $f_0(z, w), f_1(z, \zeta, w), q(z, \zeta, w)$ are $C^k$ functions (for some large $k$) of their arguments.

Remark. System (1) may look very particular, as it has relative degree 1 between control input $u$ and regulated output $e$. However, the design methodology described in [1], and pursued in what follows under much weaker hypotheses, lends itself to a straightforward extension to systems with higher relative degree, namely systems having the form of equation (33) in [1]. Details are somewhat lengthy and for this reason they are not included here.  

The analysis in [1] was based on three standing hypotheses. The first of these hypotheses is that the exosystem is Poisson stable, namely that:

Assumption 0. The set $W \subset \mathbb{R}^r$ of admissible initial conditions for the exosystem (2) is compact and $W = \bigcup_{w \in W} \omega(w)$.  

Letting \( Z \times E \subset \mathbb{R}^n \times \mathbb{R} \) be the compact set of initial states of (1) for which the problem of output regulation is to be solved, the second hypothesis is that the trajectories of the zero dynamics of (1), augmented with (2), are bounded, namely that:

**Assumption 1.** The positive orbit of \( Z \times W \) under the flow of

\[
\begin{align*}
\dot{z} &= f_0(z, w) \\
\dot{w} &= s(w)
\end{align*}
\]

has a compact closure, and \( \omega(Z \times W) \subset \text{int}(Z) \times W \). \( \triangleright \)

If follows from this assumption that the set \( A_0 := \omega(Z \times W) \), i.e. the \( \omega \)-limit set – under the flow of (3) – of the set \( Z \times W \), is a nonempty, compact, invariant set which is stable in the sense of Lyapunov and uniformly attracts \( Z \times W \).

The third assumption was that:

**Assumption 2.** There exist an integer \( d \) and real numbers \( a_0, a_1, \ldots, a_{d-1} \) such that, for any \( (z_0, w_0) \in A_0 \), the solution \((z(t), w(t))\) of (3) passing through \((z_0, w_0)\) at \( t = 0 \) is such that the function \( \varphi(t) := -q(z(t), 0, w(t)) \) satisfies

\[
\varphi^{(d)} + a_{d-1}\varphi^{(d-1)} + \cdots + a_1\varphi^{(1)} + a_0\varphi = 0 \ . \ \triangleright
\]

It is well-known that the function \( \varphi(t) \) considered above is the input needed to keep the regulated output \( e(t) \) identically zero (so long as \( e(0) = 0 \)). The assumption above calls for the existence of a linear differential equation of which \( \varphi(t) \) be a solution. In this paper, we drastically weaken this assumption, by simply calling for the existence of a nonlinear differential equation of which \( \varphi(t) \) be a solution, namely:

**Assumption 2-nl.** There exists an integer \( d \) and a locally Lipschitz function \( f : \mathbb{R}^d \to \mathbb{R} \) such that, for any \( (z_0, w_0) \in A_0 \), the solution \((z(t), w(t))\) of (3) passing through \((z_0, w_0)\) at \( t = 0 \) is such that the function \( \varphi(t) := -q(z(t), 0, w(t)) \) satisfies

\[
\varphi^{(d)} + f(\varphi, \varphi^{(1)}, \ldots, \varphi^{(d-1)}) = 0 \ . \ \triangleright
\]

### 3 Output Regulation via Nonlinear Internal Models

We proceed now with the construction of a controller which solves the problem of output regulation for system (1). To this end, consider the sequence of functions recursively defined as

\[
\tau_1(z, w) = -q(z, 0, w) \ , \ldots, \tau_{i+1}(z, w) = \frac{\partial \tau_i}{\partial z} f_0(z, w) + \frac{\partial \tau_i}{\partial w} s(w) ,
\]

for \( i = 1, \ldots, d - 1 \), and consider the map

\[
\tau : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^d \ ; \quad (z, w) \mapsto \text{col}(\tau_1(z, w), \tau_2(z, w), \ldots, \tau_d(z, w)) .
\]
If \( k \), the degree of continuous differentiability of the functions in \([1]\), is large enough, the map \( \tau \) is well defined and \( C^1 \). In particular \( \tau(\mathcal{A}_0) \), the image of \( \mathcal{A}_0 \) under \( \tau \) is a compact subset of \( \mathbb{R}^d \), because \( \mathcal{A}_0 \) is a compact subset of \( \mathbb{R}^n \times \mathbb{R}^r \).

Let \( f_c : \mathbb{R}^d \to \mathbb{R} \) be any locally Lipschitz function of compact support which agrees on \( \tau(\mathcal{A}_0) \) with the function \( f \) defined in \([4]\), i.e. a function such that, for some compact superset \( \mathcal{S} \) of \( \tau(\mathcal{A}_0) \) satisfies
\[
\begin{align*}
\forall \eta \in \mathcal{S} : & \quad f_c(\eta) = 0 \\
\forall \eta \in \tau(\mathcal{A}_0) : & \quad f_c(\eta) = f(\eta)
\end{align*}
\]

Note that, by definition, there is a number \( C > 0 \) such that
\[
|f_c(\eta)| \leq C, \quad \text{for all } \eta \in \mathbb{R}^d. \tag{5}
\]

Let now \( \Phi_c(\cdot) \) be the smooth vector field of \( \mathbb{R}^d \) defined as
\[
\Phi_c(\eta) = \begin{pmatrix}
\eta_2 \\
\eta_3 \\
\vdots \\
-\frac{d}{\tau(\eta, w(t))}
\end{pmatrix}
\]
and let \( \Gamma \) be the \( \mathbb{R}^{1 \times d} \) matrix
\[
\Gamma = \begin{pmatrix}
1 & 0 & \cdots & 0
\end{pmatrix}.
\]

Trivially, for any \((z_0, w_0) \in \mathcal{A}_0\) the function \( \tau(z(t), w(t)) \) is such that
\[
\frac{d}{dt} \tau_i(z(t), w(t)) = \tau_{i+1}(z(t), w(t))
\]
for \( i = 1, \ldots, d - 1 \). If, in addition, Assumption 2-nl holds, then
\[
\frac{d}{dt} \tau_d(z(t), w(t)) = -f(\tau(z(t), w(t))) = -f_c(\tau(z(t), w(t))).
\]

Thus, if Assumption 2-nl holds, for any \((z_0, w_0) \in \mathcal{A}_0\) the function \( \tau(z(t), w(t)) \) satisfies
\[
\frac{d}{dt} \tau(z(t), w(t)) = \Phi_c(\tau(z(t), w(t))). \tag{6}
\]

Remark. In other words, the restriction of \([3]\) to \( \mathcal{A}_0 \), which is invariant under the flow of \([3]\), and the restriction of
\[
\dot{\eta} = \Phi_c(\eta) \tag{7}
\]
to \( \tau(\mathcal{A}_0) \), which is invariant under the flow of \([7]\), are \( \tau \)-related systems. \( \triangleleft \)

Moreover,
\[
q(z, 0, w) = -\Gamma \tau(z, w), \quad \text{for all } (z, w) \in \mathcal{A}_0. \tag{8}
\]
Consider now a control law of the form
\[
\dot{\xi} = \Phi_c(\xi) + Gv \quad u = \Gamma \xi + v, 
\] (9)
with
\[
v = -ky. 
\] (10)
A simple check shows that, if Assumptions 0, 1 and 2-nl hold, this controller does have the internal-model property (see [1]), relative to the set $A_0$.

The control of (1) by means of (9) results in a system
\[
\begin{align*}
\dot{z} &= f_0(z, w) + f_1(z, \zeta, w)\zeta \\
\dot{\zeta} &= q(z, \zeta, w) + \Gamma \xi + v \\
\dot{\xi} &= \Phi_c(\xi) + Gv \\
\dot{w} &= s(w),
\end{align*} 
\] (11)
which, viewing $v$ as input and $\zeta$ as output, has relative degree 1 and a zero-dynamics
\[
\begin{align*}
\dot{z} &= f_0(z, w) \\
\dot{\zeta} &= \Phi_c(\xi) + G(-q(z, 0, w) - \Gamma \xi) \\
\dot{w} &= s(w).
\end{align*} 
\] (12)

The asymptotic properties of the latter are summarized in the following results.

**Lemma 1** Suppose Assumptions 0, 1 and 2-nl hold. Consider the triangular system
\[
\begin{align*}
\dot{z} &= f_0(z, w) \\
\dot{w} &= s(w) \\
\dot{\xi} &= \Phi_c(\xi) + G(-q(z, 0, w) - \Gamma \xi).
\end{align*} 
\] (12)
Let $\Xi \subset \mathbb{R}^d$ be a bounded subset such that $\tau(A_0) \subset \text{int}(\Xi)$. Choose numbers $c_0, c_1, \ldots, c_{d-1}$ such that the polynomial
\[
p(\lambda) = \lambda^d + c_{d-1}\lambda^{d-1} + \cdots + c_1\lambda + c_0
\] is Hurwitz, set
\[
G_0 = \text{col}(c_0, c_1, \ldots, c_{d-1}) \quad D_\kappa = \text{diag}(\kappa, \kappa^2, \ldots, \kappa^d)
\] and
\[
G = D_\kappa G_0.
\] Then, there is a number $\kappa^*$ such that, if $\kappa \geq \kappa^*$,
\[
\text{graph}(\tau|_{A_0}) = \omega(Z \times W \times \Xi)
\] under the flow of (12). In particular, $\text{graph}(\tau|_{A_0})$ is a compact invariant set which uniformly attracts $Z \times W \times \Xi$. 

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Proof. Set, for convenience, \( B = Z \times W \times \Xi \).

First of all, we show that the positive orbit of \( B \) is bounded, so that \( \omega(B) \) is nonempty, compact, invariant and uniformly attracts \( B \) (see Lemma 2.1 of [1]). To this end, set

\[
\tilde{\xi} = D^{-1}_\kappa \xi,
\]

and observe that the variable thus defined obeys

\[
\dot{\tilde{\xi}} = \kappa(A - G_0 \Gamma) \tilde{\xi} + D^{-1}_\kappa B f_c(D \kappa \tilde{\xi}) - G_0 q(z, 0, w)
\]

in which

\[
A = \begin{pmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix}.
\]

It is seen from (5) that, for any \( \kappa > 1 \) (which we can assume without loss of generality) and for any \( \tilde{\xi} \in \mathbb{R}^d \),

\[
|D^{-1}_\kappa B f_c(D \kappa \tilde{\xi})| \leq C.
\]

Thus, the integral curves of (13) are maximally defined on the entire real axis. Knowing that \( \xi(t) \) exists for any \( t > 0 \), and setting

\[
u(t) = D^{-1}_\kappa B f_c(D \kappa \xi(t)) - G_0 q(z(t), 0, w(t)),
\]

system (13) can be rewritten in the form

\[
\dot{\tilde{\xi}} = \kappa(A - G_0 \Gamma) \tilde{\xi} + \nu.
\]

Since the matrix \((A - G_0 \Gamma)\) is Hurwitz by construction, and \( \nu(t) \) is bounded (by some fixed number which only depends on the choice of \( W \) and \( Z \)), it is concluded that also \(|\xi(t)|\) is bounded. As a consequence, \( \xi(t) \) itself is bounded, by number which depends on the choice of \( W, Z \) and \( \Xi \) and of the parameter \( \kappa \). This concludes the proof that the positive orbit of \( B \) is bounded.

To find \( \omega(B) \), consider now the (closed) solid cylinder \( C = A_0 \times \mathbb{R}^d \). A simple argument shows that \( \omega(B) \subset C \). In fact, for any point \( p = (z, w, \xi) \in \omega(B) \), there must exist sequences \( p_k = (z_k, w_k, \xi_k) \) and \( t_k \), with \( t_k \to \infty \) as \( k \to \infty \), such that \( \lim_{k \to \infty} \phi(t_k, p_k) = p \). Because of the triangular structure of (12), the point \((z, w)\) is by definition in \( A_0 \) and hence \((z, w, \xi) \in C \).

It is immediate to see that graph(\( \tau|_{A_0} \)) is invariant for (12). In fact, observe that, for any \((z_0, w_0) \in A_0\), the solution \((z(t), w(t))\) of the top two equations of (12) passing through \((z_0, w_0)\) at time \( t = 0 \) remains in \( A_0 \) for all \( t \in \mathbb{R} \), because the latter is invariant for (3). Thus, bearing in mind (3), for all such trajectories the bottom equation of (12) can be rewritten as

\[
\dot{\xi} = \Phi_c(\xi) + G(\Gamma \tau(z, w) - \Gamma \xi).
\]
Set $\chi(t) = \xi(t) - \tau(z(t), w(t))$. Then, using (6), we have
\[
\dot{\chi} = \Phi_c(\chi + \tau(z, w)) - \Phi_c(\tau(z, w)) - GT\chi.
\] (14)

The point $\chi = 0$ is an equilibrium of this equation and, therefore, graph($\tau|_{A_0}$) is invariant for (12). In particular, for any point $p \in$ graph($\tau|_{A_0}$) it is possible to find sequences $p_k \in$ graph($\tau|_{A_0}$) and $t_k$, with $t_k \to \infty$ as $k \to \infty$, such that $\lim_{k \to \infty} \phi(t_k, p_k) = p$. This shows that graph($\tau|_{A_0}$) $\subset$ $\omega(B)$.

To complete the proof, we need to show that no point in $C \setminus$ graph($\tau|_{A_0}$) can be a point of $\omega(B)$. For, pick any point $p_0 \in C \setminus$ graph($\tau|_{A_0}$), i.e. a point $p_0 = (z_0, w_0, \xi_0)$ in which $(z_0, w_0) \in A_0$ and $\xi_0 \neq \tau(z_0, w_0)$. The complete orbit of (12) through $p_0$ is in $C$, because $A_0$ is invariant under (3). Thus, the $\xi$-component of the corresponding trajectory is such that $\chi(t) = \xi(t) - \tau(z(t), w(t))$ obeys (14), with $\chi(0) \neq 0$.

Set
\[
\bar{\chi} = D_{\kappa}^{-1}\chi,
\]
and observe that the variable thus defined obeys
\[
\dot{\bar{\chi}} = \kappa(A - G_0\Gamma)\bar{\chi} + B\Delta(\bar{\chi}, \tau(z, w), \kappa)
\] (15)

with
\[
\Delta(\bar{\chi}, \tau(z, w), \kappa)) = \frac{1}{\kappa^d}\left[f_c(D_{\kappa}\bar{\chi} + \tau(z, w)) - f_c(\tau(z, w))\right].
\]

Since the function $f_c(\cdot)$ is locally Lipschitz, and bounded, there exists a number $L$ such that, for any $\xi_1, \xi_2 \in \mathbb{R}^d$,
\[
|f_c(\xi_1) - f_c(\xi_2)| \leq L|\xi_1|.
\]

Thus, for any $\kappa > 1$,
\[
|\Delta(\bar{\chi}, \tau(z, w), \kappa))| \leq \frac{1}{\kappa^d} L|D_{\kappa}\bar{\chi}| \leq L|\bar{\chi}|.
\]

On the other hand, the matrix $(A - G_0\Gamma)$ is Hurwitz by construction and hence there exists a positive definite matrix $P$ such that
\[
P(A - G_0\Gamma) + (A - G_0\Gamma)^TP = -I.
\]

From this, it is immediately seen that the function $V(\bar{\chi}) = \bar{\chi}^TP\bar{\chi}$ satisfies, along the trajectories of (15),
\[
\dot{V}(\bar{\chi}) \leq -(\kappa - 2L|P|)|\bar{\chi}|^2.
\]

If $\kappa > 2L|P|$, there exist numbers $M > 0$ and $\alpha > 0$, both depending on $\kappa$, such that
\[
|\chi(t)| \leq Me^{-\alpha t}|\chi(0)|.
\] (16)

This property can be used to show that no point of $C \setminus$ graph($\tau|_{A_0}$) can be a point of $\omega(B)$. For, suppose by contradiction that a point $p_0 = (z_0, w_0, \xi_0)$, with $(z_0, w_0) \in A_0$ and $\xi_0 \neq \tau(z_0, w_0)$, be a point of $\omega(B)$. Set
\[
d_0 = |\xi_0 - \tau(z_0, w_0)|.
\] (17)
As the trajectory of Equation (12) through this point is defined for all \( t \in \mathbb{R} \) and bounded, there exists a number \( K \) such that \( |\chi(t)| \leq K \) for all \( t < 0 \). Pick now a number \( T > 0 \) such that 
\[
M e^{-\alpha T} K \leq 0.5 d_0.
\]
Then, using (16), we obtain 
\[
|\xi_0 - \tau(z_0, w_0)| \leq M e^{-\alpha T} |\xi(-T) - \tau(z(-T), w(-T))| \leq M e^{-\alpha T} K \leq 0.5 d_0,
\]
which contradicts (17). This concludes the proof of the Lemma. \( \triangleright \).

**Lemma 2** Suppose Assumptions 0, 1 and 2-nl hold. Choose \( G \) as indicated in Lemma 1, with \( \kappa \geq \kappa^* \). If \( \mathcal{A}_0 \) is locally exponentially attractive for (3), then \( \text{graph}(\tau|_{\mathcal{A}_0}) \) is locally exponentially attractive for (12).

**Proof.** To say that \( \mathcal{A}_0 \) is locally exponentially attractive for (3) is to say that there exists positive numbers \( \rho, A, \lambda \) such that, for any \((z_0, w_0)\) in the (closed) set 
\[
\mathcal{A}_\rho = \{(z, w) \in \mathbb{R}^n \times W : \text{dist}((z, w), \mathcal{A}_0) \leq \rho\}
\]
the solution \((z(t), w(t))\) of (3) passing through \((z_0, w_0)\) at \( t = 0 \) is such that 
\[
\text{dist}((z(t), w(t)), \mathcal{A}_0) \leq A e^{-\lambda t} \text{dist}((z_0, w_0), \mathcal{A}_0).
\]
Set, as in the proof of the previous Lemma, \( \chi(t) = \xi(t) - \tau(z(t), w(t)) \). Then, 
\[
\dot{\chi} = \Phi_c(\chi + \tau(z, w)) - \Phi_c(\tau(z, w)) - G\Gamma \chi + Gp(z, w), \tag{18}
\]
in which 
\[
p(z, w) = \Gamma \tau(z, w) + q(z, 0, w).
\]
From (8), it is seen that the function \( p(z, w) \) vanishes on \( \mathcal{A}_0 \). Since the function in question is \( C^1 \) and \( \mathcal{A}_\rho \) is compact, there exist a number \( N > 0 \) such that 
\[
|p(z, w)| \leq N \text{dist}((z, w), \mathcal{A}_0), \quad \text{for all } (z, w) \in \mathcal{A}_\rho,
\]
and therefore, if \( \text{dist}((z_0, w_0), \mathcal{A}_0) \leq \rho', \) with \( \rho' = \rho/A, \)
\[
|p(z(t), w(t))| \leq N A e^{-\lambda t} \text{dist}((z_0, w_0), \mathcal{A}_0). \tag{19}
\]
Changing \( \chi \) into \( \bar{\chi} = D_{\kappa}^{-1} \chi \) in (18) yields (see proof of the previous Lemma) 
\[
\dot{\bar{\chi}} = \kappa(A - G_0 \Gamma)\bar{\chi} + B\Delta(\bar{\chi}, \tau(z, w), \kappa) + G_0 p(z, w)
\]
from which, using again the same Lyapunov function \( V(\bar{\chi}) \) introduced above and bearing in mind (19), standard arguments show that, if \( \kappa > 2L|\kappa| \), an estimate of the form 
\[
|\chi(t)| \leq \bar{M} e^{-\alpha t} (|\chi(0)| + \text{dist}((z_0, w_0), \mathcal{A}_0))
\]
would contradict (17). This concludes the proof of the Lemma. \( \triangleright \).
holds, for every $\chi(0)$ and for every $(z_0, w_0)$ satisfying $\text{dist}((z_0, w_0), A_0) \leq \rho'$. From this, the result follows. $\triangle$

As shown in [11], the properties indicated in these two Lemma (which in [11] were proven to hold under the much stronger Assumption 2) guarantee that, if the input $v$ of (11) is chosen as in (10), the resulting closed-loop system has the desired asymptotic properties. As a matter of fact, the following conclusion holds.

**Proposition 1** Consider system (20). Let $Z, W$ be fixed compact sets of initial conditions and suppose Assumptions 0, 1 and 2-nl hold. Pick any compact set $\Xi$ as indicated in Lemma 1 and any closed interval $E$ of $\mathbb{R}$. Choose $G$ as indicated in Lemma 1, with $\kappa$ large enough so that the conclusions of Lemma 1 hold. Then, for every $\varepsilon > 0$, there exists a number $k^*$ such that, if $k \geq k^*$, the positive orbit of $Z \times W \times \Xi \times E$ is bounded and there exists $\bar{t}$ such that $|e(t)| \leq \varepsilon$ for all $t \geq \bar{t}$. If, in addition, $A_0$ is locally exponentially attractive for (3), then $e(t) \to 0$ as $t \to \infty$.

The proof of this Proposition follows a standard paradigm. Changing the variable $\xi$ into $\eta = \xi - Ge$ and setting $\bar{k} = k - \Gamma G$ yields a system of the form

$$
\begin{align*}
\dot{z} &= f_0(z, w) + f_1(z, e, w)e \\
\dot{w} &= s(w) \\
\dot{\eta} &= \Phi_c(\eta + Ge) - G\Gamma(\eta + Ge) - Gq(z, e, w) \\
\dot{e} &= q(z, e, w) + \Gamma \eta - \bar{k} e.
\end{align*}
$$

This can be viewed as interconnection of two subsystems, one with state $(z, w, \eta)$ and input $e$, the other with state $e$ and input $(z, w, \eta)$. Then, small-gain arguments can be invoked to show that, if $\bar{k}$ is large enough, the results of the Proposition hold. Details on the proof that the subsystems in question have the appropriate input-to-state stability properties can be found in [2].

**References**

[1] C.I. Byrnes, A. Isidori, Limit sets, zero dynamics and internal models in the problem of nonlinear output regulation, *IEEE Trans. on Automatic Control*, AC-48, pp. 1712-1723, (2003).

[2] C.I. Byrnes, A. Isidori, L. Praly, On the Asymptotic Properties of a System Arising in Non-equilibrium Theory of Output Regulation, Preprint of the Mittag-Leffler Institute, Stockholm, June 2003.