The resurgence of the cusp anomalous dimension

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Abstract
This work addresses the resurgent properties of the cusp anomalous dimension’s strong coupling expansion, obtained from the integral Beisert–Eden–Staudacher (BES) equation. This expansion is factorially divergent, and its first non-perturbative corrections are related to the mass gap of the $O(6)$ $\sigma$-model. The factorial divergence can also be analyzed from a resurgence perspective. Building on the work of Basso and Korchemsky, a transseries ansatz for the cusp anomalous dimension is proposed and the corresponding expected large-order behaviour studied. One finds non-perturbative phenomena in both the positive and negative real coupling directions, which need to be included to address the analyticity conditions coming from the BES equation. After checking the resurgence structure of the proposed transseries, it is shown that it naturally leads to an unambiguous resummation procedure, furthermore allowing for a strong/weak coupling interpolation.

Keywords: cusp anomalous dimension, resurgence, strong coupling, transseries, resummation, Bethe ansatz, AdS/CFT

(Some figures may appear in colour only in the online journal)

1. Introduction and setup

The cusp anomalous dimension plays a central role in the study on many observables in four-dimensional gauge theories. In supersymmetric $\mathcal{N}=4$ Yang–Mills theory (SYM), it appears when studying the scaling behaviour of the anomalous dimension of a Wilson loop with a light-like cusp in the integration contour, in the $SL(2)$ sector of the theory [1, 2]. In the context of large spin operators, Wilson loops can be characterized by a Lorentz spin $\mathcal{S}$ and a twist $\mathcal{L}$. 
For the case of large spin and $L \sim \ln S$ the scaling behaviour of the minimal anomalous dimension is [3–7]

$$\gamma_{S,L}(g) = \left(2\Gamma_{\text{cusp}}(g) + \epsilon(g,j)\right)\ln S + O(L^{-1}),$$

where $(4\pi g)^2 = \lambda = g_{\text{SYM}}^2N$ is the 't Hooft coupling. Also, $j = L/\ln S$ is the only dependence on the twist from the leading contribution to the scaling, and taking $j = 0$ leaves us with the twist independent cusp anomalous dimension [2, 8]. It is a function solely of the coupling, and has been thoroughly studied in different regimes.

At weak coupling this function can be expanded in powers of $g^2$, with coefficients determined from perturbation theory [2, 9–13], and the corresponding series is convergent. At strong coupling, through the AdS/CFT correspondence [14], one can obtain an expansion in $g^{-1}$ from the semiclassical analysis of the energy of folded spinning strings in $\text{AdS}_5 \times S^5$, where the Lorentz spin and twist become angular momenta of the string solution [3, 15].

Studying the interpolating region between weak and strong coupling is difficult, and integrability played a crucial role. The all-loop Bethe ansatz for $\mathcal{N} = 4$ SYM [16–18] led to a set of integral equations, the BES equation [6, 19–21], describing the anomalous dimension, valid for any arbitrary coupling (with the FRS equations [6] valid for any scaling parameter $j$). In terms of an auxiliary function $\gamma(2gt)$, the BES equation can be written as

$$\frac{\gamma(2gt)}{2gt} = K(2gt, 0) - 2g \int_0^\infty \frac{dt'}{e^{2t'} - 1} K(2gt, 2gt') \gamma(2gt'),$$

where the BES kernel $K(t, t')$ can be found in [19, 20, 22]. This auxiliary function is related to the cusp anomalous dimension by

$$\Gamma_{\text{cusp}}(g) = 8 \lim_{t \to 0} \frac{\gamma(2gt)}{2gt}.$$

Solving these equations at weak coupling returned higher terms of the convergent perturbative expansion for the small $g \ll 1$ region [20, 23]. For intermediate coupling $g \sim 1$ a smooth solution to the BES equation was found numerically [24]. At strong coupling different attempts were made at solving the BES equations [23, 25–27], and in [28, 29] a solution was found leading to a strong coupling expansion. This approach consisted in noticing that a change of variables

$$\Omega(t) = \gamma(t) \left(1 + i \coth \left(\frac{t}{4g}\right)\right),$$

returns a simpler set of coupled integral equations for $\Omega(t)$, which are then solved using Fourier methods [30, 31]. One subsequently obtains a solution of the BES equation in the form [22, 32]

$$\Omega(t) = f_0(t) V_0(t) + f_1(t) V_1(t),$$

where\footnote{In [32] another unknown function $c(g)$ appeared in $f_1(t)$, but was seen to vanish for all coupling $g$.} (x = 8\pi g)
The functions $U_{0,1}^+$ and $V_{0,1}$ can be written in terms of Whittaker functions of first and second kinds, but for our purposes we only need their asymptotic expansions for large $x$, which can be found in appendix A. The coefficients $c_{±}(n, x)$ are determined from analyticity conditions on the solution $W((it)$ (given that $\gamma(it)$ is an entire function): from the expressions for $f_t(t)$ it already has the correct pole structure, but one still needs to impose the existence of zeros at $(x = 8\pi g)$

$$t_{\text{zeros}} = \frac{x}{2} \alpha_± \equiv \frac{x}{2}(\ell - \frac{1}{4}), \quad \ell \in \mathbb{Z}. \quad (2)$$

This condition can be rewritten as

$$1 = \sum_{n > 1} \frac{c_+(n, x)}{n - \alpha_±} U_{1,1}^- \left(\frac{nx}{2}\right) \alpha_± + U_{0,1}^+ \left(\frac{nx}{2}\right) n r(\alpha_±)$$

$$+ \sum_{n > 1} \frac{c_-(n, x)}{n + \alpha_±} U_{1,1}^- \left(\frac{nx}{2}\right) \alpha_± + U_{0,1}^+ \left(\frac{nx}{2}\right) n r(\alpha_±), \quad (3)$$

where $r(\alpha)$ is the ratio of functions $V_1(nx/2)$ and $V_0(nx/2)$, as defined in appendix A. This analyticity condition allows us to determine the coefficients $c_{±}(n, x)$ order by order as expansions in large coupling $x$. Once these coefficients are known, the cusp anomalous dimension is given by

$$\frac{\Gamma_{\text{cusp}}(g)}{2g} = 1 - 2 f_t(0)$$

$$= 1 - 2 \sum_{n > 1} \left[ c_+(n, x) U_{1,1}^- \left(\frac{nx}{2}\right) + c_-(n, x) U_{0,1}^+ \left(\frac{nx}{2}\right) \right]. \quad (4)$$

The strong coupling expansion found in this way is asymptotic. Moreover, the series is non-Borel summable for positive real coupling, due to the existence of singularities on the positive real axis of the Borel plane, which give rise to non-perturbative, exponentially suppressed corrections at strong coupling. In order to understand the analytic properties of the solution to the BES equation at strong coupling, one needs to account for all the non-perturbative phenomena in this limit. In [22, 32] the above procedure was taken a step further and the perturbative coefficients around the first non-perturbative correction were determined.

Both scaling function $\epsilon(g, j)$ and cusp anomalous dimension $\Gamma_{\text{cusp}}(g)$ have non-perturbative corrections. In [5] it was proposed that the scaling function $\epsilon(g, j)$ at strong coupling is directly related to the energy density of the ground state of the $O(6)$ nonlinear $\sigma$-model embedded in $\text{AdS}_5 \times S^5$ (taking $j/2$ to be the particle density). Moreover, the non-perturbative corrections appearing in $\epsilon(g, j)$ at strong coupling are given by the mass scale (mass gap) of the $O(6)$ model. Agreement between these two quantities was checked in [22, 33, 34],
at the level of the first non-perturbative correction to the scaling function. As for the cusp anomalous dimension, as it solves a different integral equation altogether, such a relation was less expected. Nevertheless, in [32], it was shown that the first non-perturbative correction to the anomalous dimension is exactly given by the square of the $O(6)$ mass gap.

Two important questions still remain at this point: are we aware of all of the non-perturbative phenomena defining the analytic properties of the cusp anomalous dimension? How can we systematically deal with a non-Borel summable asymptotic series? To answer both these questions we will now turn to the theory of resurgence.

Resurgent functions have been seen in a wide range of systems. In mathematics they appear for example as solutions of differential and finite difference equations (e.g. see the well studied cases of Painlevé I, II and Riccati nonlinear differential equations [35–38]). Analogously, often one can only determine physical observables in specific regimes of the coupling of the theory via a series expansion such as

$$\langle O(g) \rangle \simeq \sum_{k \geq 0} C_k g^{-k}. \quad (5)$$

However, these expansions are often asymptotic: the coefficients are factorially divergent, with large-order behaviour

$$C_k \sim \frac{\Gamma(k + \beta)}{A^{k+\beta}}, \quad k \gg 1. \quad (6)$$

$A, \beta$ are numbers related to the position and type of singularities of the related Borel transform.

It is well known that this divergence hints to the existence of non-perturbative phenomena unaccounted for in the perturbative series expansions. In physical settings, the existence of non-perturbative phenomena has been long noticed in the contexts of quantum mechanics [39, 40] and quantum field theories [41], associated to instantons [42] and renormalons [43]. In these examples, the existence of asymptotic multi-instanton sectors allowed for a complete unambiguous description of the energy eigenvalues via a transseries solution and resurgence [44–46]. Since then, the asymptotic behaviour of perturbation theory and the resurgence behind it was seen to exist in many different examples in physical systems, from quantum mechanics [47–49], to large $N$ gauge theories [37, 38, 50–57], quantum field theories [47, 58–64] and topological strings [65–67].

To account for all non-perturbative phenomena, one upgrades our perturbative expansions into a transseries [54]: a formal expansion in both perturbative variable $g$ and non-perturbative monomials $e^{-A g}$. Schematically

$$\langle O(g, \sigma) \rangle = \sum_{n \geq 0} \sigma^n e^{-n A g} \sum_{k \geq 0} O_k^{(n)} g^{-k}. \quad (7)$$

where $\sigma$ is a parameter to be fixed from some boundary conditions specific to each problem. The transseries is a formal object, since for each non-perturbative sector labeled by $n$ one has an associated asymptotic expansion $\Phi_n(g) \simeq g^{n} \sum_{k \geq 0} O_k^{(n)} g^{-k}$, with coefficients growing factorially at large orders. However, these sectors are not independent of each other: they are sectors of a resurgent transseries, whose large-order growth is intimately related [37]. A resurgent transseries is an expansion like (7), where the coefficients of one sector $O_k^{(m)}$ are related to, i.e. resurge in, the coefficients of neighbouring sectors $O_k^{(m')} (m$ close to $m')$. For example, for the perturbative sector $(n = 0)$, one expects a direct relation to the $n = 1$ sector.
The exact expressions for these large-order relations [37] can be determined via resurgent analysis (for an introduction to resurgence see [37, 68–70] and references therein). The associated Borel transforms of the series $B[\Phi_n](s)$, have a non-zero radius of convergence and singularities on the corresponding Borel $s$-plane at positions $s = nA, n \in \mathbb{N}$.

At this point we have a formal solution for our observable, and we now need to retrieve physical information from the asymptotic series $F_{\Phi_n}$. This is done via Borel resummation: the calculated Borel transform has a non-zero radius of convergence, and one can determine the analytic function associated with each series $B[\Phi_n](s)$, either exactly or by finding an approximate analytic result via the so-called Borel–Padé approximants [37, 54, 71]. Once the function or its approximant is known we then perform a resummation via a Laplace transform

$$S\Phi_n(g) = \int_0^{+\infty} ds \ e^{-s\cdot g} B[\Phi_n](s),$$

and the full answer for the observable is given by the transseries with each of its sectors resummed. This can only be performed if no poles exist in the direction of integration on the Borel plane, in this case on the positive real line. If instead $A$ is positive and real, the positive real line is called a Stokes line (singular direction on the Borel plane), and the series is said to be non-Borel summable: only lateral resummations can be defined

$$S_{L} \Phi_n(g) = \int_0^{+\infty} e^{i\sigma} ds \ e^{-s\cdot g} B[\Phi_n](s).$$

These lateral resummations differ by a non-perturbative ambiguity $(S_{L} - S_{R}) \Phi_n(g) \sim e^{-A g}$, which is purely imaginary when the coefficients $C_k^{(n)}$ are real and the Stokes line is along the real axis. Now the importance of having a resurgent transseries becomes apparent: due to the relations between different sectors, by taking into account the full transseries and a specific value for the transseries parameter $\sigma$, the ambiguities between different sectors cancel each other, and one is left with a non-ambiguous real-valued result. This is called median resummation (see [72] and references therein).

Recalling the transseries (7), let us assume that the positive real axis is a Stokes line and choose the lateral Borel resummation $S_{L}$ for every sector of the transseries. This resummation will have real and imaginary parts

$$S_{L} \Phi_n = \frac{1}{2} (S_{+} + S_{-}) \Phi_n + \frac{1}{2} (S_{+} - S_{-}) \Phi_n$$

$$\equiv S_{R} \Phi_n + i S_{L} \Phi_n.\quad (11)$$

The imaginary contribution $S_{L} \Phi_n$ is just the ambiguity coming from the sector $\Phi_n$. We can now determine the real and imaginary parts of the resummed transseries [72]

$$S_{R} \langle \mathcal{O}(g, \sigma) \rangle \equiv \sum_{n \geq 0} \sigma^n e^{-n A g} S_n \Phi_n(g) \equiv \sum_{n \geq 0} \sigma^n F^{(n)}(g)$$

$$\equiv S_{R} \langle \mathcal{O} \rangle + i S_{L} \langle \mathcal{O} \rangle.\quad (12)$$

$\text{2 Borel transforms are determined by inverse Laplace transforms to each term in the expansion, or equivalently } g^{-1} \rightarrow s^{1/\Gamma(k)}.$
The ambiguity in the resummation of the transseries is just its imaginary part \( \sigma = \sigma_R + i \sigma_I \)
\[
S_I(O) = \text{Im} \left( F^{(0)} \right) + \sigma_I \text{Re} \left( F^{(1)} \right) + \sigma_R \text{Im} \left( F^{(1)} \right) \\
+ 2\sigma_R \sigma_I \text{Re} \left( F^{(2)} \right) + \left( \sigma_R^2 - \sigma_I^2 \right) \text{Im} \left( F^{(2)} \right) + \cdots.
\] (13)

Median resummation is a specific prescription to cancel this imaginary contribution to the resummation of the transseries along the positive real axis to all orders, by some carefully chosen values of the transseries parameter \( \sigma = \sigma_0 \). This cancelation happens to all orders, and we are left with a real unambiguous answer
\[
S_R(O(g, \sigma_0)) = \text{Re} \left( F^{(0)} \right) + \sigma_{0,R} \text{Re} \left( F^{(1)} \right) \\
- \sigma_{0,I} \text{Im} \left( F^{(1)} \right) + \left( \sigma_{0,R}^2 - \sigma_{0,I}^2 \right) \text{Re} \left( F^{(2)} \right) + \cdots.
\] (14)

This resummed result can then be interpolated from the original regime where the asymptotic series were defined, into any complex value of the coupling \( g \), taking into consideration any crossing of singular Stokes lines. The systematic resummation and interpolation from asymptotic series using resurgence has recently been addressed for different problems \([57, 63, 67, 73, 74]\).

The aim of this paper is to perform a resurgent analysis of the expansions found in \([32]\) for the strong coupling regime of the cusp anomalous dimension. We start from the solution to the BES equation presented above, and enforce the analytic properties at the level of the expansion coefficients. We then determine the structure of singularities of the Borel transform associated to the perturbative sector by means of a Borel–Padé approximant. This allows us to finally propose a transseries ansatz for the cusp anomalous dimension which encompasses the expected non-perturbative phenomena existing at strong coupling up to the second non-perturbative sector. Using this ansatz in the analyticity conditions, we determine the coefficients of our transseries, solved order by order for the first three sectors.

Equipped with the series expansion for perturbative and first non-perturbative sectors of the cusp anomalous dimension, we then check its resurgent properties via the large-order relations. Along the way we determine the relevant Stokes constant associated with the Stokes transition across the positive real line.

We finish by using the methods of median resummation to systematically define a non-ambiguous resummed result valid at any value of the coupling, which encodes the analytic properties of the solution and can be used to interpolate between strong and weak coupling regimes.

2. Singularity structure of the cusp

In the interest of finding the correct transseries solution for the cusp anomalous dimension, we first analyze its perturbative asymptotic series. With that goal in mind we assume the coefficients \( c_\pm(n, x) \) have a simple (asymptotic) expansion in powers of \( x^{-1} \) where \( x = 8\pi g \):
\[
c_\pm(n, x) = x^{\beta_\pm(0)} n^{\pm 1/4} \sum_{k=0}^{+\infty} \sigma_k^{(0, \pm)}(n) x^{-k}.
\] (15)

\(^3\) In simple cases it was seen that \( i\sigma_I = \frac{S_1}{2} \), where \( S_1 \) is the so-called Stokes constant, was enough to cancel the ambiguity \([72]\), with residual freedom left in the real part \( \sigma_R \).
Substituting this ansatz into the analyticity conditions (3), and making use of the asymptotic expansions in appendix A, as well as properties of sums found in appendix B, one obtains for each power in $x^{-k}$ relations for the coefficients $\phi_k^{(0,\pm)}(n)$ depending on the ones for lower $k$. Solving these relations iteratively (in the same way as was done in [32] for the first few coefficients), we determine

$$\phi_k^{(0,\pm)}(n) = \sum_{m=0}^{k} Q_{k,m}^{(0,\pm)} n^{-m} \phi_0^{(0,\pm)}(n),$$  \hspace{1cm} (16)

where the expressions for $\phi_0^{(0,\pm)}(n)$ are very simple and can be found in appendix B. The analyticity condition imposes restrictions on the coefficients $\beta_k(0)$:

$$\beta_k(0) = \pm 1/4.$$  \hspace{1cm} (17)

The general solution for the numerical coefficients $Q_{k,m}^{(0,\pm)}$ is then simply given by

$$4Q_{k,k}^{(0,+)} = - \sum_{r=0}^{k-1} Q_{r,r}^{(0,+)} \sum_{\ell=0}^{k-r} M_{k-r,\ell}^{(0,+)} + \sum_{r=0}^{k-1} \sum_{\ell=0}^{k-r} M_{k-r,\ell}^{(0,+)},$$  \hspace{1cm} (18)

$$4Q_{k,k}^{(0,-)} = - \sum_{r=0}^{k-1} Q_{r,r}^{(0,-)} \sum_{\ell=0}^{k-r} M_{k-r,\ell}^{(0,-)} + \sum_{r=0}^{k-1} \sum_{\ell=0}^{k-r} M_{k-r,\ell}^{(0,-)},$$  \hspace{1cm} (18)

for $m = k$, and for $0 \leq a < k$

$$2Q_{k,a}^{(0,+)} = - 4 \sum_{m=a+1}^{k} Q_{k,m}^{(0,+)} K_{m-a}^{(0,+)} - \sum_{r=0}^{a} \sum_{\ell=0}^{k-r} M_{r,\ell}^{(0,+)} \sum_{m=a}^{k-r} Q_{k,r,m}^{(0,+)} K_{m-a}^{(0,+)}$$

$$- \sum_{r=a+1}^{k} \sum_{m=r}^{k} Q_{k-r,m}^{(0,-)} K_{m-a}^{(0,-)} \sum_{\ell=0}^{a} M_{r,\ell}^{(0,+)} + \sum_{r=a+1}^{k} \sum_{m=r}^{k} Q_{k-r,m}^{(0,-)} K_{m-a}^{(0,-)} M_{r,\ell}^{(0,+)} - 1,$$  \hspace{1cm} (19)

where similar solutions can be written for $Q_{k,a}^{(0,-)}$ by exchanging $K_m^{(0,+)} \leftrightarrow K_m^{(0,-)}$, and $M_r^{(0,+)} \rightarrow M_r^{(0,-)}$, $M_r^{(0,-)} \rightarrow M_r^{(0,+)}$. The definitions for the coefficients $M_r^{(0,\pm)}$ and $K_m^{(0,\pm)}$ can be found in the appendices. One can now determine several coefficients in the expansions (15), which was done numerically up to $k = 200$ and 200 decimal places accuracy. For the cusp anomalous dimension one then uses the expansion (4), and the expansions for the functions $U^{(m,\pm)}_k$, $k = 0, 1$ given in appendix A.

The strong coupling perturbative expansion for the cusp anomalous dimension becomes

$$\Gamma_{\text{cusp}}(g) = 2g \left( 1 + \Gamma_0^{(0)}(4\pi g) + O(e^{-x+\pi g}) \right),$$  \hspace{1cm} (20)

where the perturbative asymptotic expansion is

$$\Gamma_0^{(0)}(x) \equiv -2f_{\text{pert}}(0) \approx \sum_{k=1}^{\infty} \left( \frac{x}{2} \right)^{-k} \Gamma_k^{(0)}.$$  \hspace{1cm} (21)
The expansion coefficients are simply given by

\[
\Gamma_k^{(0)} \simeq 2^{1-k} \sum_{m=0}^{k-1} \sum_{s=0}^{m} S_+(s) Q_k^{(0,+)} F_{m+1}^{(0,+)} + 2^{1-k} \sum_{m=0}^{k-1} \sum_{s=0}^{m} S_-(s) Q_k^{(0,-)} F_{m+1}^{(0,-)},
\]

with

\[
S_+(s) = \frac{\Gamma(5/4 + s) \Gamma(1/4 + s)}{\Gamma(5/4) \Gamma(1/4 + s)} (-1)^s,
\]

\[
S_-(s) = \frac{\Gamma(3/4 + s) \Gamma(-1/4 + s)}{\Gamma(3/4) \Gamma(-1/4 + s)} (-1)^s.
\]

A numerical analysis of the coefficients \(\Gamma_k^{(0)}\) will easily show that these grow factorially for large-order \(k\). In fact they grow as \(\Gamma(k - 1/2)\), in agreement with the factorial growth found in [28]. Moreover, we find a subleading exponential growth:

\[
\Gamma_k^{(0)} \simeq \Gamma(k - 1/2) A^{-k},
\]

with \(A = \frac{1}{2}\). To show this, take the ratio of two consecutive coefficients and expand for large \(k\). The expected behaviour at large-order \(k\) is

\[
\frac{\Gamma_{k+1}^{(0)}}{\Gamma_k^{(0)}} A \simeq \frac{k}{k - 1/2} + O(k^{-2}) = 1 + \frac{1}{2k} + \cdots.
\]

In figure 1 we can see the confirmation of both the factorial growth and subleading exponential growth: plotting the ratio of two consecutive coefficients at large order converges to the constant \(c_0 = 1\) (thus confirming a \(k!\) and \(A^{-k}\) growth); the second plot presents the convergence of (25) to \(c_1 = 1/2\), showing that the factorial growth is in fact of the type \(\Gamma(k - 1/2)\).

Associated with this factorial growth there will be non-perturbative phenomena dictating the asymptotic nature of the series. These non-perturbative phenomena are most naturally represented as singularities on the complex Borel plane: we expect to find singularities at positions \(s = n A\), where \(A = 1/2\) - these will be associated to non-perturbative exponentially suppressed contributions of the type \(e^{-n A} \Gamma(\tau s)\).

To study the singularity structure associated with the perturbative expansion \(\Gamma^{(0)}(\tau g)\) we determine its Borel transform via the usual approach (by replacing \((4\pi g)^{-k} \rightarrow s^{-1} / \Gamma(k)\))

\[
B\left[ \Gamma^{(0)} \right](s) = \sum_{k=0}^{+\infty} \Gamma_k^{(0)} \frac{s^k}{\Gamma(k+1)}.
\]

This expansion will now have a non-zero radius of convergence and we approximate the corresponding function via the method of Padé approximants: using a diagonal approximant of order \(N = 100\) (half the order of coefficients calculated for the original series), we determine the best fit for a ratio of two polynomials \(B_p^{(0)}(\Gamma^{(0)})\) and analyze the structure of singularities for this function. This allows us to see the position and type of nearest singularities to the origin on the Borel plane: in particular, a condensation of poles hints to the existence of a branch cut. In figure 2 the structure of poles of the Borel–Padé approximant is given. We find the expected singularity at \(s = A = 1/2\), but we also find another type of singularity on the negative real axis at \(s = -4A = -2\).
Figure 1. Large-order ratio of perturbative coefficients and convergence to the expected factorial and exponential growths given in (25).

Figure 2. Poles of the diagonal Borel–Padé approximant of order 100 for the perturbative series of $\Gamma(2g)/(2g) - 1$. There is accumulation of poles in both positive and negative real directions, starting at $s = A \equiv 1/2$ and at $s = -4A = -2$. Note the existence of spurious poles away from the real line: non-stable numerical effects of the Padé method, which move away by choosing different non-diagonal approximants.
The first type of singularity had already been known, it is directly related to the square of the mass gap of the $O(6)$-$\sigma$-model embedded in $\text{AdS}_5 \times S^5$ [32]. The fact that it lies on the positive real axis prevents us from defining a resummation on this axis: we can only define lateral resummations (10) which differ by an imaginary ambiguous contribution. However, the second type of singularity lies on the negative real axis. Even though it will not give any ambiguous contribution to a resummation for real and positive coupling $g$, the analyticity conditions will not be blind to it. These types of singularities have been found before in the study of the Painlevé I and II equations [36–38].

If the perturbative expansion for some observable is asymptotic, one should upgrade the solution to include non-perturbative sectors, into what is called a transseries. The most important ingredient to writing a transseries solution fully describing our observable is to include all possible sectors associated with singularities on the Borel plane. In the present case, this means upgrading the expansions for $c_{\pm}(n, g)$ to a transseries including both types of singularities found.

3. Transseries and analyticity conditions

We now proceed to write a transseries ansatz for our coefficients $c_{\pm}(n, g)$ appearing in (4). A transseries ansatz will include at least as many sectors as the different singular directions found in the Borel–Padé analysis. Looking back at figure 2, we have two singular directions, the positive and negative real axis. Thus our transseries will need at least two types of sectors, each coming with a transseries parameter $s_i$. Note that even though we do not find other singular directions, we cannot exclude the existence of other independent sectors associated with singularities in the real axis further away from the origin. For the two types of singularities we did find we then write the following transseries

$$c_{\pm}(n, x; s_i) = \sum_{m_i, m_\pm = 0}^{+\infty} \sigma_{i\pm}^{m_i} \sigma_2^{m_\pm} e^{i\phi_{m_\pm, m_i}^2} \text{e}^{-\frac{1}{2} A_i} c_{\pm}^{(m_i)}(n, x),$$

where $\sigma_i$ are the transseries (or instanton counting) parameters, $A = \frac{1}{2}$, and $c_{\pm}^{(m_i)}(n, x)$ are perturbative expansions around each non-perturbative sector $(m_i, m_\pm)$, $c_{\pm}^{(0)}(n, x)$ are simply the perturbative expansions (15) found in the previous section, while the others will generically be

$$c_{\pm}^{(m_i)}(n, x) \approx x^{i\beta_i^{(m_i)}} \sum_{k=0}^{+\infty} x^{-k} \phi_{k}^{(m_i, m_\pm)}(n)n^{m_\pm + 1/4},$$

where $i\beta_i^{(m)}$ are numerical factors associated with the type of branch cuts on the Borel plane.

In general one is interested in the transseries solution at some particular value of the coupling $x$ (usually real positive), and the parameters $\sigma_i$ are fixed by some physical input. Moreover, having a non-zero $\sigma_2$ for positive real $x$ would lead to unstable, exponentially enhanced contributions, which physically we know should not be included. Nevertheless, such sectors need to be introduced in order to account for all the analytic properties of the problem at hand. On the real axis we will always have the parameter $\sigma_2$ fixed to zero and the parameter $\sigma_1$ fixed to some non-zero value. Still, the analytical properties of our function encompass the full complex plane, where both of these parameters can and will take different values: in fact they change values when one crosses the so-called Stokes lines, which are lines where there are singularities on the Borel plane.
Another important issue is that of resonance [36–38, 70]: for transseries with more than one type of exponential behaviour $e^{-m_1 A_1 - m_2 A_2}$, if there are values $m_i$ such that $m_1 A_1 + m_2 A_2 = 0$, a phenomena called resonance occurs. In the case of transseries solutions to nonlinear differential equations, it is seen that the associated recursion relations break down at these locations unless we enhance the perturbative expansions $e^{(m)}_\pm$ (28) to include other non-perturbative sectors, such as some finite number of powers of $\log(x)$. In the present case we will find resonance when $m_1 = 4m_2$, and one can expect a rich structure like the one found for the Painlevé solutions [37, 38]. One could be led to think that such structure could not be present, as when $m_1 = 4m_2$ the exponential cancels and these sectors would have been seen at the same order as the perturbative series. Nevertheless, these sectors are only present when $\sigma_2 = 0$, so even though they are part of the analytic and resurgent properties of the transseries, they will not be present when the analysis is restricted to the positive real axis.

As aforementioned, for the problem at hand we will be interested solely in positive real $x$, so the parameter $\sigma_1$ will just be fixed to zero, effectively turning (27) into a one-parameter transseries. Furthermore, the scope of this paper is to study the transseries contributions up to ‘two instantons’, $m = 2$ in (27), and in the following sections we will see that for this purpose the proposed ansatz will suffice.

Having written the transseries ansatz, we now need to substitute it in the analyticity conditions (3). After some algebra, we can rewrite these conditions as

$$2 = \sum_{m=0}^{\infty} \sigma^m e^{-A_2 m} \times \left[ \pi^{-1} \sum_{r=0}^{\left[ m/4 \right]} \sigma^{-4|\alpha|} r \left( F^{(m,r)}_+(n, x) + F^{(m,r)}_-(n, x) \right) \right],$$

which need to be obeyed for every zero $\alpha = \ell - \frac{1}{4}$, $\ell \in \mathbb{Z}$. As just mentioned, throughout this paper we focus on the contributions with $m \leq 2$. Therefore we will leave the issue of resonance and higher non-perturbative corrections for subsequent work. We briefly note that the last sum in (29) goes up to the integer part $\left[ \frac{m}{4 |\alpha|} \right]$, which will be non-zero if $m \geq 4 |\alpha|$.

This means that while for $m = 0$ we had two general equations: $\alpha_0 < 0$ (from which we obtained coefficients $Q_{(0,0)}^{(0,-)}$) and $\alpha_0 \geq \alpha_1 = \frac{3}{4} > 0$ (giving coefficients $Q_{(0,1)}^{(0,+)}$), for sectors $m = 1, 2$ we will have three types of equations $\alpha_0 < \alpha_0 = -\frac{1}{4}$, $\alpha_0 = -\frac{1}{4}$ and $\alpha_0 \geq \alpha_1 = \frac{3}{4} > 0$. For $m = 3$ one expects both zeros $\alpha_0 = -\frac{1}{4}$ and $\alpha_1 = \frac{3}{4}$ to have separate equations, and for $m = 4$ (taking $m_2 = 1$) one can expect to find the first instance of resonance: it is likely that the both the existence of extra zeros and resonance will mix in the analyticity conditions.

The functions $F^{(m,r)}_\pm$ are

---

4 For Painlevé [37, 38], these sectors played an essential role in obtaining the correct large-order behaviour of different sectors. In the present case their contribution will be suppressed by $4^{-\ell}$ for $\ell$ large, and will not be seen in the analysis which will follow.

5 For simplicity the transseries parameter was renamed $\sigma_1 \rightarrow \sigma$.

6 See appendices for the expansions used, and $R_k(x, \alpha) = R_k(x, |\alpha|)$ if $\alpha > 0$ while $\tilde{R}_k(x, \alpha) = \tilde{R}_k(x, |\alpha|)$ if $\alpha < 0$. 
\[ F_{+}^{(m,r)}(n, x) = x^{\beta_r(m-4|\alpha|r)-1/4} \times \sum_{k=0}^{+\infty} x^{-k} \sum_{n \geq 1} \phi_k^{(m-4|\alpha|(r,+)}(n) \times A_{0,1}(x, -n) \left( \delta_{r,0} \frac{A_{1,1}(x, -n)}{A_{0,1}(x, -n)} \alpha + x^{-1} \tilde{R}_r(x, \alpha) \right). \]  \] 

\[ F_{-}^{(m,r)}(n, x) = x^{\beta_r(m-4|\alpha|r)-3/4} \times \sum_{k=0}^{+\infty} x^{-k} \sum_{n \geq 1} \phi_k^{(m-4|\alpha|(r,-)}(n) \times A_{0,0}(x, -n) \left( \delta_{r,0} \frac{A_{1,0}(x, -n)}{A_{0,0}(x, -n)} \alpha + n \tilde{R}_r(x, \alpha) \right). \]  \]

We can now solve (29) for \( m = 1, 2 \), in the same manner as for the perturbative series in the previous section: one obtains similar (but lengthier) equations to (18) and (19). We found that

\[ \beta_+(1) = \frac{3}{4}, \beta_-(1) = \frac{1}{4}, \beta_+(2) = \pm \frac{1}{4}. \]  \]

and also

\[ \phi_k^{(m,+)}(n) = \sum_{\ell=0}^{k-1} Q_{k,\ell}^{(m,+)} \frac{1}{n^\ell} \phi_1^{(m,+)}(n), \]

\[ \phi_k^{(m,-)}(n) = \sum_{\ell=0}^k Q_{k,\ell}^{(m,-)} \frac{1}{n^\ell} \phi_0^{(m,-)}(n), \]  \]

where

\[ \phi_1^{(m,+)}(n) = - \mathcal{P}^{(m)} \phi_0^{(0,+)}(n), \]

\[ \phi_0^{(m,+)}(n) = 0, \]

\[ \phi_0^{(m,-)}(n) = \mathcal{P}^{(m)} \phi_0^{(0,-)}(n - 1). \]  \]

Looking back at (29) for \( m = 1, 2 \), we can now equate \( Q_{k,\ell}^{(m,-)} \) with \( \ell \geq 1 \) from the equations \( \alpha_\ell < \alpha_0 = -1/4 \), \( Q_{k,\ell}^{(m,+)} \) from \( \alpha_0 = -1/4 \) and \( Q_{k,\ell}^{(m,-)} \), \( r \geq 1 \) from \( \alpha_r = \alpha_1 = 3/4 > 0 \). How higher non-perturbative coefficients \( m \geq 3 \) will be equated from the analyticity conditions will depend on the how the transseries is generalized before introducing it into (3), but the procedure should be straightforward to generalize.

The numerical coefficients \( \mathcal{P}^{(m)} \) are also determined:

\[ \mathcal{P}^{(1)} = \sigma_0^{-1} e^{3\pi i/4} \frac{\Gamma(3/4)}{\Gamma(5/4)}, \]

\[ \mathcal{P}^{(2)} = - \left( \mathcal{P}^{(1)} \right)^2. \]  \]

It is worth noting that these coefficients seem to depend on the transseries parameter, which it is not yet fixed. The value of the transseries parameters \( \sigma \) can vary with the value of the coupling \( x \): even if we fix them to a particular value, when we move on the \( x \)-plane, these parameters will jump in value when crossing any Stokes line (lines where there are singularities on the Borel plane)—such a jump will be governed by the so-called Stokes automorphism. So how can we interpret the numerical values \( \mathcal{P}^{(m)} \) if they depend on a parameter which will change its value? In fact, the analyticity conditions are solved in a specific direction on the \( x \)-plane: the one which was chosen to perform the asymptotic
expansions. In this region we will have a specific value $\sigma \equiv \sigma_0$, and the $P^{(m)}$ will be fixed at that position on the $x$-plane (they depend on the numerical value $\sigma_0$, not on the general parameter $\sigma$). As we will see below, the value $\sigma_0$ will be chosen by the reality conditions of the cusp anomalous dimension for $x$ real and positive.

Once the $c^{(m)}_\pm(n, x)$ are determined for $m = 0, 1, 2$, we can write down the corresponding transseries for the cusp anomalous dimension $(4)$ with $m_2 = 0$

$$\Gamma_{\text{cusp}}(g, \sigma) = 2g \left( 1 + \sum_{m=0}^{+\infty} (\sigma P^{(1)})^m e^{-m\sigma} \Gamma^{(m)}(x) \right),$$

(36)

where $\Gamma^{(0)}$ was the perturbative contribution previously calculated (22). The two first non-perturbative corrections can be written as

$$\Gamma^{(m)}(x) \simeq \left( x^2 \right)^{-\beta(m)} \sum_{k=0}^{+\infty} \Gamma^{(m)}_k \left( \frac{x}{2} \right)^k,$$

(37)

where $\beta(m) = \frac{m}{2}$ and the coefficients are

$$\Gamma^{(m)}_0 = -4 \frac{P^{(m)}}{\left(P^{(1)}\right)^m},$$

(38)

$$\Gamma^{(m)}_k = -2^{k-1} \frac{P^{(m)}}{\left(P^{(1)}\right)^m} \times \left( \sum_{a=0}^{k} \sum_{s=0}^{d} S_-(s) Q^{(m, -)}_{\parallel a - r} \sum_{r=0}^{d} 4^{a+1-r} K^{(0, -)}_r \right.$$

$$\left. + \sum_{a=0}^{k} \sum_{s=0}^{d} S_+(s) Q^{(m, +)}_{\parallel a + r} K^{(0, +)}_r \right), \quad k \geq 1.$$  

(39)

We have now calculated the perturbative coefficients around the first two non-perturbative sectors (solved numerically up to $k = 200$ and 200 decimal places accuracy). If we analyze the growth of these coefficients, we find the same factorial growth $\Gamma(k - 1/2)$ for $m = 0, 1$, and the factorial growth $\Gamma(k + 1/2)$ for $m = 2$: not only the perturbative series is asymptotic, but so are the non-perturbative ones. Moreover the singularities which lead to these sectors lie on the positive real axis, and thus one cannot properly define a single integration contour on which to perform the resummation of the Borel transform.

Nevertheless, many lessons have been learned by now on the cases of resurgent transseries: if our transseries is resurgent, then there is a way of defining a single non-ambiguous result which properly cancels the imaginary ambiguity at all non-perturbative orders. But in order to use these results from resurgence, we first need to check that our transseries is indeed resurgent.

4. Resurgence and large order

Let us now check that the transseries (36) formed by the asymptotic series $\Gamma^{(m)}$ is indeed resurgent, i.e. that the coefficients $\Gamma^{(m)}_k$, $\Gamma^{(m)}_k$ of neighbouring sectors $m, m'$ are related. To perform this check we use the coefficients $\Gamma^{(m)}_k$ of the asymptotic series (37) and check if their large-order behaviour coincides with the large-order relations predicted by resurgence techniques. These are relations between the large-order $k \gg 1 \Gamma^{(m)}_k$ of one sector with the lower-order coefficients of a nearby sector $\Gamma^{(m)}_k$.

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7 The first coefficients around the $m = 1$ sector are in agreement with [32].
Take the transseries (36); assuming this transseries is resurgent, we can use the so-called alien calculus to determine the discontinuity of each asymptotic series \( \Gamma^{(m)} \) across singular directions (or Stokes lines). In our case (given we have taken \( m_2 = 0 \)) we only have one singular direction: the positive real axis \( \theta = 0 \). Resurgence then tells us [37] that the discontinuity of the perturbative series along this direction is (recall that \( A = 1/2 \))

\[
\text{Disc}_0 \Gamma^{(0)}(x) = -S_1 e^{-A_2} \Gamma^{(1)} - (S_1)^2 e^{-2A_2} \Gamma^{(2)} + \cdots.
\]  

(40)

There is one unknown constant in the above relation, the Stokes constant \( S_1 \). As we will see, the large-order relations will allow us to determine this constant with great accuracy. This step is extremely important as the Stokes constant plays a crucial role in the ambiguity cancelation and resummation.

From the discontinuity, we can use Cauchy’s theorem to determine large-order relations \([42]\). Schematically, one writes

\[
\Gamma^{(0)}(z) = \oint \frac{d\omega}{2\pi i} \frac{\Gamma^{(0)}(\omega)}{\omega - z} \approx \int_0^{+\infty} \frac{d\omega}{2\pi i} \text{Disc}_0 \Gamma^{(0)}(\omega) + \oint \frac{d\omega}{2\pi i} \frac{\Gamma^{(0)}(\omega)}{\omega - z} + \oint \frac{d\omega}{2\pi i} \frac{\Gamma^{(0)}(\omega)}{\omega - z} + \cdots.
\]

(41)

In certain conditions, it can be shown by scaling arguments that the integral at infinity does not contribute \([40, 75]\]. Expanding the rhs for large \( z \), using the resurgence relation for the discontinuity, and finally comparing equal powers of \( z \) for the expansions in both sides of the equation, we arrive at the relation

\[
\Gamma^{(0)}_{k} \approx -\frac{S_1 \mathcal{P}^{(1)}}{2\pi i} \sum_{h=0}^{+\infty} \Gamma^{(1)}_h \Gamma \left( k - \frac{1}{2} - h \right) + \frac{(S_1 \mathcal{P}^{(1)})^2}{2\pi i} \sum_{h=0}^{+\infty} \Gamma^{(2)}_h \Gamma \left( k - 1 - h \right) \frac{1}{(2A)^{k-1-h}}, \quad k \gg 1.
\]

(42)

This formula states that if resurgence is expected, then the large \( k \) behaviour of the perturbative series is dictated by the coefficients of the first non-perturbative sector, and then, more exponentially suppressed \((2^{-k})\), the coefficients of the second non-perturbative sector appear, and so on. The proportionality constant is once again the Stokes constant \( S_1 \). Taking the ratio of two consecutive coefficients (which removes the dependence on the yet unknown Stokes constant), and assuming \( k \gg 1 \), we have a series (asymptotic again)

\[
\frac{\Gamma^{(0)}_k}{\Gamma^{(0)}_{k+1}} \frac{1}{A} \approx \sum_{h=0}^{+\infty} c_h k^{-h},
\]

(43)

where the coefficients \( c_h \) can be predicted from the original large-order relation \((42)\). The first coefficients are

\[
c_0 = 1; \ c_1 = \frac{1}{2}; \ c_2 = \frac{1}{4} + A \frac{\Gamma^{(1)}_1}{\Gamma^{(0)}_1} \cdots.
\]

(44)

We can now check the convergence of \( \Gamma^{(0)}_k \) to the coefficients \( c_k \) by successively removing the previous coefficient from the ratio. For example to check the convergence to the coefficient \( c_2 \) we analyze
\[
\left( \frac{\Gamma^{(0)}_k}{\Gamma^{(0)}_{k+1}} k - c_0 \right) k - c_1 \rightarrow c_2 + O\left(k^{-1}\right).
\] (45)

In figure 3 we present the convergence to coefficient \(c_{10}\). In order for this convergence to be correct, all of the previous coefficients need to be correct to a very high accuracy, since factorial errors propagate rapidly. In this figure, the original ratio (in red) is shown, together with two related Richardson transforms which speed the convergence of this series in \(1/k\) (see [36, 53]). The error between the numerically calculated coefficient (via Richardson transforms) and the predicted result from the large-order formula is of order \(10^{-7}\).

If instead of dealing with the ratio of coefficients we analyze the following
\[
\frac{2\pi A k^{-\frac{1}{2}} \Gamma^{(0)}_k}{\Gamma\left( k - \frac{1}{2} \right) \Gamma^{(0)}_0} \simeq iS_{i1}\mathcal{D}^{(1)} \sum_{h=0}^{+\infty} \frac{\Gamma\left( k - \frac{1}{2} - h \right)}{\Gamma\left( k - \frac{1}{2} \right)} A^h
\]
\[
\sim iS_{i1}\mathcal{D}^{(1)} + O\left(k^{-1}\right),
\] (46)
we directly obtain a convergence to the unknown Stokes constant. In figure 4 this convergence is shown, with both the original series and a related Richardson transform. The increase in convergence speed from the Richardson interpolation method allows us to determine the Stokes constant to a very high accuracy. Up to an error of \(10^{-3}\) we find
\[
iS_{i1}\mathcal{D}^{(1)} = \frac{1}{2} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right).
\] (47)

The same ideas were repeated for the asymptotic series of \(\Gamma^{(1)}\), whose large order will be directly related to coefficients of \(\Gamma^{(2)}\), finding that resurgence predictions also worked in this case. In figure 5 we present the large-order behaviour of the ratio of coefficients of \(\Gamma^{(1)}\) and the convergence to the coefficient \(c_{12}^{(1)}\), where
\[
\frac{\Gamma^{(1)}_k}{\Gamma^{(1)}_{k+1}} A \simeq \sum_{h=0}^{+\infty} c_h^{(1)} k^{-h}.
\] (48)
The coefficients \(c_h^{(1)}\) are dictated by the coefficients \(\Gamma^{(2)}_k\) via the following large-order relations obtained from resurgence:
\[
\Gamma^{(1)}_k \simeq \frac{2\pi i}{2\pi i} \sum_{h=0}^{+\infty} \frac{\Gamma^{(2)}_h}{\Gamma\left( k - \frac{1}{2} - h \right)} A^{k-\frac{1}{2}-h}, k \gg 1.
\] (49)

Unlike the previous cases, the large-order behaviour of \(\Gamma^{(2)}\) is dictated not only by \(\Gamma^{(3)}\) but also by \(\Gamma^{(1)}\) (the two nearest singularities on the Borel plane will be equally distant from the origin, at \(s = \pm A\), as already expected from resurgence [37] for a one-parameter transseries). We conclude that indeed the transseries for the cusp anomalous dimension is resurgent (checked up to the second non-perturbative correction), and thus we can apply the methods of ambiguity cancelation known to exist for resurgent transseries. Note that another possible check of resurgence which can be done is to resum for each \(k\) the first line in (42), an asymptotic expansion itself, via a so-called Écalle–Borel–Padé resummation method, subtract it from the corresponding perturbative coefficients \(\Gamma^{(0)}_k\) and then compare the resulting large-order behaviour with the expected one from the second line of (42). Still, in the next section we will obtain an ambiguity cancelation by adding all sectors up to \(m = 2\), which would not happen if the second line of (42) was not accurate.
5. Ambiguity cancelation and interpolation

The next two questions which follow are: if our transseries is resurgent, can we use this knowledge to write a non-ambiguous result? And if this is possible, can we then interpolate from the strong coupling asymptotic expansion to small coupling?
The answer to the first question is simply yes. The fact that we have a resurgent transseries directly tells us how to obtain a non-ambiguous result even when resumming in directions which are non-Borel summable, i.e., along Stokes lines.

In order to verify the ambiguity cancelation it is sufficient to check that the imaginary part of the resummed lateral transseries \((12)\) cancels to higher and higher orders. The ambiguity coming from the perturbative series, i.e. the imaginary contribution from performing a lateral resummation, is of order \(e^{-4\pi g}\) for each value of the coupling \(g\). This is exactly the order at which the first non-perturbative sector starts contributing. In fact, if we add the two with the proper choice of parameter \(\sigma\) in \((7)\), we can see that the newfound imaginary part in \((12)\) will now be of order \(e^{-2A4\pi g}\)—the order of the second non-perturbative sector. In order to see this cancelation, and to determine the value of the combination \(\sigma \mathcal{P}(1)\) which brings about the cancelation, we first need to resum our asymptotic series.

The method for resumming our series is the so-called Écalle–Borel–Padé resummation method. It consists in calculating the Borel transforms for each sector \(\mathcal{B}[\Gamma^{(m)}](s)\), then determining a Padé approximant for each Borel transform, and subsequently perform a lateral Borel resummation (we have chosen \(S_+\) as in \((10)\)) for different values of the coupling \(0.1 \leq g \leq 4\) in order to obtain a resummed result for the full transseries for general values of \(g\). The imaginary part of the transseries for the cusp

\[
\frac{S_+ \Gamma}{2g} - 1 = S_+ \Gamma^{(0)} + \sigma e^{-4\pi g} S_+ \Gamma^{(1)} + \sigma^2 e^{-4\pi g} S_+ \Gamma^{(2)} + \ldots
\]

(50)

is given by \((13)\). In figure 6 the order \(10^{-6}\) of the imaginary part can be seen if we only include \(S_+ \Gamma^{(0)}\) (dark brown), or if we include both perturbative and first non-perturbative contributions (light brown), and finally if we include all calculated contributions (up to second order, in purple). The cancelation is shown for a range of values of the coupling.

Figure 5. Convergence of the large-order ratio of the coefficients of the sector \(m = 1\) to the predicted result related to the non-perturbative sector \(m = 2\): original ratio in red; in blue are Richardson transforms of order 2 (light blue) and 6 (dark blue). In light green the predicted value \(c^{(1)}_{15}\) is shown.

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ranging from weak to strong coupling. For example for \( g = 1.5 \) if we include only the perturbative part, the imaginary ambiguity is of order \( 10^{-5} \). Including the first non-perturbative sector cancels this imaginary part to \( 10^{-10} \). Including all three sectors cancels the imaginary part up to 16 decimal places. The value of the transseries parameter used was

\[
\sigma_0 = -(1 + i) \frac{i S_7 D^{(1)}}{2} = \frac{1}{2 \sqrt{2}} e^{-3 \pi i/4} \frac{\Gamma(3/4)}{\Gamma(5/4)},
\]

with the Stokes constant given by (47). This is in complete agreement with the proportionality constant appearing in front of the non-perturbative corrections in [32].

We are now ready to answer the second question posed in this section: once we have managed to write down a real unambiguous transseries result, given by (14) with the parameter \( \sigma_0 \) determined in (51), can we use this result to interpolate between strong and weak coupling? In figure 7 we show the truncated asymptotic series result in a dashed green line: this result is accurate at strong coupling, but diverges for weak coupling. In blue we plotted the weak coupling expansion for the cusp anomalous dimension as determined (up to seven loop order) by [20]. Naturally this result diverges for large values of the coupling. In red we show the resummed result including the perturbative series and the first two non-perturbative sectors. We see clearly that the red dots follow both the strong and weak regime closely, starting to diverge for \( g < 0.2 \). In order to obtain more accurate results after this point, we would need to include the next non-perturbative order.

In summary, once we have established resurgence of the transseries, we are able to write down an unambiguous result, and use that same result to reach values of the coupling as small as \( g = 0.2 \). Moreover, the information encoded in the transseries solution for the cusp anomalous dimension goes beyond the strong/weak interpolation for positive real values of the coupling. The resummation can be performed for \( g \in \mathbb{C} \): one can obtain a solution for any value of coupling, having in mind that to reach certain values one might have to cross a Stokes line and the transseries goes through a so-called Stokes transition (in a Stokes transition the transseries parameters will jump in value, and this jump is dictated by the Stokes transition).
constants and can be calculated from resurgence techniques. In other words, the transseries solution encodes the analytic properties of the cusp anomalous dimension as a function of the coupling. Another extremely insightful example of how the transseries encodes the analytic properties of the observable was recently achieved in the context of matrix models \[57\], where from a large \(N\) asymptotic expansion, resurgence and resummation techniques allowed the authors to reach not only finite values of the rank \(N\), but they were also able to analytically continue their results to any complex value of \(N\), once Stokes phenomena were taken into account.

6. Conclusions/outlook

In this work we presented a thorough analysis of the resurgent properties of the cusp anomalous dimension’s strong coupling expansion up to the second non-perturbative sector. When analyzing the perturbative series we found that there are at least two types of singularities on the Borel plane in both the positive and negative real axis. These need to be taken into account in order to fully solve the analyticity conditions coming from the BES equations. Nevertheless their physical interpretation has not yet been addressed. The non-perturbative behaviour associated with singularities at \(s = nA\) (on the positive real axis), were seen to be directly linked to the mass gap of the \(O(6)\sigma\)-model at least for \(n = 1\) \[32\]. The singularities at \(s = -4nA\) are much more suppressed and as such have not been addressed, but some questions can be immediately raised: does the same feature appear for the energy density of the \(O(6)\) model? Does the relation between scaling function and energy density hold for...
higher exponentially suppressed contributions? Since the two types of non-perturbative phenomena are collinear, will we witness resonance?

For the aims of the current paper we used a transseries ansatz for the cusp anomalous dimension which did not include the second type of non-perturbative phenomena, as we only studied the solution up to \( n = 2 \) in the non-perturbative order. Nevertheless this was enough to check the resurgent properties of the strong coupling expansion, with the large-order relations predicted by resurgence accurately solving the large-order behaviour of the perturbative expansion and first non-perturbative order.

In our case we could determine the asymptotic expansions around the non-perturbative sectors via the BES equation, and use these to check the resurgence of the transseries. In cases when only perturbation theory is known, one can go the opposite way and use the predictions of resurgence to determine the coefficients of the expansions around the non-perturbative sectors.

Knowing that the transeries proposed is indeed resurgent we then proceeded to determine the resummed transseries, using a lateral resummation procedure. This naturally introduces an imaginary ambiguity, which can then be seen to cancel given the proper choice of the transseries parameter: using the methods of median resummation [72] we determined the transseries parameter to be the one proposed in [32].

Finally the resummation procedure can be done for different values of the coupling, and we showed that including up to second order non-perturbative effects, we could systematically obtain accurate results for the cusp anomalous dimension all the way from strong coupling up to \( g \approx 0.2 \). Moreover we can perform the resummation for any values of the coupling, as long as we take into consideration the Stokes phenomena occurring when we analytically continue our results across Stokes lines.

This work is an example in the use and elegance of the resurgence techniques. Knowing the perturbative asymptotic expansion of an observable in some regime, we can determine the non-perturbative corrections via large-order relations, and upgrade the solution to a transseries. We can then use resummation methods such as Écalle–Borel–Padé to obtain a resummed solution for any value of the coupling (using resurgence techniques to analytically continue the solution across Stokes lines). This resummed transseries encodes the analytic properties of our observable as a function of the coupling.

Other open questions still to be addressed are whether one can use a transseries ansatz for the auxiliary function in the BES equation, and through this find an equivalent approach to determining the coefficients \( I_k^{(m)} \) to the one used in [28, 32]. More importantly, it remains to be understood how these results translate to the problem of the scaling function \( \epsilon(g, j) \) and the energy density of the \( O(6) \) \( \sigma \)-model. In these cases, one firstly needs know if it is possible to find a full transseries. If so, will we have the same type of singularities? Finally, in these two cases we have two parameters \( g, j \) and it would be important to understand how the different regimes dictated by these parameters appear in the resurgent context.

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Appendix A. Asymptotic expansions

In order to write down the asymptotic expressions appearing in the main text, we need first to define the following four asymptotic expansions

\[ A_{0,0}(x, \alpha) = \sum_{s=0}^{+\infty} \frac{\Gamma\left(\frac{3}{4} + s\right)\Gamma\left(-\frac{1}{4} + s\right)}{\Gamma\left(-\frac{1}{4}\right)s!(x \alpha)^s}, \]

(A1)

\[ A_{0,1}(x, \alpha) = \sum_{s=0}^{+\infty} \frac{\Gamma\left(\frac{1}{4} + s\right)\Gamma\left(\frac{5}{4} + s\right)}{\Gamma\left(\frac{3}{4}\right)s!(x \alpha)^s}, \]

(A2)

\[ A_{1,0}(x, \alpha) = \sum_{s=0}^{+\infty} \frac{\Gamma\left(\frac{3}{4} + s\right)^2}{\Gamma\left(\frac{1}{4}\right)s!(x \alpha)^s}, \]

(A3)

\[ A_{1,1}(x, \alpha) = \sum_{s=0}^{+\infty} \frac{\Gamma\left(\frac{1}{4} + s\right)^2}{\Gamma\left(\frac{3}{4}\right)s!(x \alpha)^s}. \]

(A4)

In all the expressions in this appendix we assume \( \alpha \neq 0 \). The functions \( U_n^\pm(x), n = 0, 1 \) have been defined in [32] in terms of Whittaker functions of the second kind. For the purposes of this paper however, we are only interested in their asymptotic expansions for large \( x \), which are:

\[ U_0^+\left(\frac{nx}{2}\right) \simeq (nx)^{-3/4}A_{0,1}(x, -n), \]

\[ U_0^-\left(\frac{nx}{2}\right) \simeq (nx)^{-1/4}A_{0,0}(x, -n), \]

\[ U_1^+\left(\frac{nx}{2}\right) \simeq \frac{1}{2}(nx)^{-1/4}A_{1,1}(x, -n), \]

\[ U_1^-\left(\frac{nx}{2}\right) \simeq \frac{1}{2}(nx)^{-3/4}A_{1,0}(x, -n). \]

(A5)

Note that even from these asymptotic expansions it is not difficult to see that these functions have a cut on the negative real axis. The other functions of interest appearing in the main text are \( V_n(x), n = 0, 1 \). Again, from [32] these are entire functions which were written in terms of Whittaker functions of the first kind, but for our purposes we are only interested in their asymptotic expansions. More specifically we will only need the asymptotic expansion of their ratio:

\[ 2r(\alpha) \equiv \frac{V_1\left(\frac{nx}{2}\right)}{V_0\left(\frac{nx}{2}\right)}. \]

(A6)

The asymptotic expansion of this ratio for large \( x \) depends on the sign of \( \alpha \). For \( \alpha = |\alpha| > 0 \)

\[ 2r(\alpha) \simeq \frac{n}{2} \left(1 + \frac{1}{4}(\alpha)^{-1} + \cdots\right). \]

(A7)
\[ 2r(|\alpha|) \approx \sum_{k=0}^{+\infty} e^{-k|\alpha|} R_k(x, |\alpha|), \]  

(A7)

where

\[ R_0(x, |\alpha|) = \frac{A_{1,0}(x, |\alpha|)}{A_{0,0}(x, |\alpha|)}, \]

\[ R_k(x, |\alpha|) = \left( R_0(x, |\alpha|) + |\alpha| x \frac{A_{1,1}(x, -|\alpha|)}{A_{0,1}(x, -|\alpha|)} \right) \]

\[ \times (-1)^k e^{\pm \pi i k/4} (|\alpha| x)^{k/2} \left( \frac{A_{0,1}(x, -|\alpha|)}{A_{0,0}(x, |\alpha|)} \right)^k. \]  

(A8)

For \( \alpha = -|\alpha| < 0 \) we then have

\[ 2r(-|\alpha|) \approx \sum_{k=0}^{+\infty} e^{-k|\alpha|} \bar{R}_k(x, |\alpha|), \]  

(A9)

where

\[ \bar{R}_0(x, |\alpha|) = x |\alpha| \frac{A_{1,1}(x, |\alpha|)}{A_{0,1}(x, |\alpha|)}, \]

\[ \bar{R}_k(x, |\alpha|) = \left( \bar{R}_0(x, |\alpha|) - |\alpha| x \frac{A_{1,0}(x, -|\alpha|)}{A_{0,0}(x, -|\alpha|)} \right) \]

\[ \times (-1)^k e^{\pm \pi i k/4} (|\alpha| x)^{k/2} \left( \frac{A_{0,0}(x, -|\alpha|)}{A_{1,1}(x, |\alpha|)} \right)^k. \]  

(A10)

In section 3 when solving the analyticity conditions, some particular combinations of the asymptotic expansions \( A_{i,j}(x, \alpha) \) repeatedly appeared. These were

\[ M_{0}^{(+)}(x) \equiv \frac{A_{0,0}(x, -n)}{x} \times \left( \frac{A_{1,1}(x, -n)}{A_{0,1}(x, -n)} |\alpha| x + A_{1,0}(x, |\alpha|) \right) \]

\[ \simeq |\alpha| \Gamma \left( \frac{5}{4} \right) \sum_{t=0}^{+\infty} \sum_{t=n}^{+\infty} \frac{M^{(0,+)\ell}_{r,t}}{n^t |\alpha|^{-t}}, \]  

(A11)

\[ M_{1}^{(+)}(x) \equiv \frac{A_{0,0}(x, -n)}{x(n + |\alpha|)} \times \left( \frac{A_{1,0}(x, -n)}{A_{0,1}(x, -n)} |\alpha| + n A_{1,0}(x, |\alpha|) \right) \]

\[ \simeq \frac{1}{n} \Gamma \left( \frac{3}{4} \right) \sum_{t=0}^{+\infty} x^{-r-1} \sum_{t=n}^{+\infty} \frac{M^{(1,+)\ell}_{r,t}}{n^t |\alpha|^{-t}}, \]  

(A12)

\[ M_{0}^{(-)}(x) \equiv \frac{A_{0,0}(x, -n)}{n + |\alpha|} |\alpha| \times \left( - \frac{A_{1,1}(x, -n)}{A_{0,1}(x, -n)} + A_{1,1}(x, |\alpha|) \right) \]

\[ \simeq \frac{1}{n} \Gamma \left( \frac{5}{4} \right) \sum_{t=0}^{+\infty} x^{-r-1} \sum_{t=n}^{+\infty} \frac{M^{(0,-)\ell}_{r,t}}{n^t |\alpha|^{-t}}, \]  

(A13)
Appendix B. Relations between sums

Given the two fundamental objects $\phi_0^{(0,+)}(n)$ independent of the coupling $x = 8\pi g$, appearing in the transseries of the coefficients $c_k(n, x)$ in (27), these can easily be determined from analyticity conditions to be

$$\phi_0^{(0,+)}(n) = 2 \frac{\Gamma(n - \frac{3}{4} + 1)}{n! \Gamma\left(\frac{1}{4}\right)^2},$$

$$\phi_0^{(0,-)}(n) = 2 \frac{\Gamma(n - \frac{1}{4} + 1)}{n! \Gamma\left(\frac{3}{4}\right)^2}.$$  \hspace{1cm} (B1)

With these definitions we calculate the following sums for $m \geq 1$

$$K_m^{(0,+)} \equiv -\frac{1}{8} m^{+}_{+} F_{m+1}(a^+, b; 1),$$

$$K_m^{(0,-)} \equiv -\frac{3}{8} m^{+}_{+} F_{m+1}(a^-, b; 1).$$  \hspace{1cm} (B4)

where $\left(p^{+}\right)_{+} F_{+}$ is a generalized hypergeometric function, $a^+$ are vectors with $m + 2$ entries: $a^+ = (1, \ldots, 1, 5/4)$ and $a^- = (1, \ldots, 1, 7/4)$, whereas $b = (2, \ldots, 2)$ is a vector with $m + 1$ entries. Defining also $K_0^{(0,\pm)} \equiv 1/2$, one can easily see that

$$\Gamma\left(\frac{5}{4}\right) \sum_{n \geq 0} \phi_0^{(0,+)}(n) \frac{\alpha}{n^m} = \sum_{\ell=0}^{m} K_\ell^{(0,+)} |\alpha|^{m-\ell},$$

$$\Gamma\left(\frac{3}{4}\right) \sum_{n \geq 0} \phi_0^{(0,-)}(n) \frac{\alpha}{n^m} = \sum_{\ell=0}^{m} K_\ell^{(0,-)} |\alpha|^{m-\ell}. \hspace{1cm} (B5)$$

In all the expressions in this appendix we assume $\alpha_\ell = \ell - 1/4$, for $\ell \in \mathbb{Z}$. Other useful identities of $\phi_0^{(0,-)}(n)$ are
\[
\sum_{n=1}^{\infty} \frac{\phi_0^{(0,-)}(n - 1)}{n - 0} \alpha_\ell = 0, \quad \ell \leq -1, \quad (B6)
\]

\[
\Gamma \left( \frac{3}{4} \right) \sum_{n=1}^{\infty} \frac{\phi_0^{(0,-)}(n - 1)}{n - 0} \alpha_\ell^m = - \sum_{s=0}^{m-1} \frac{1}{\alpha_\ell^s} \sum_{r=0}^{s} \frac{1}{\alpha_0^r} 1^{s-r+1}, \quad m - \ell \geq 1, \quad (B7)
\]

\[
\Gamma \left( \frac{3}{4} \right) \sum_{n=1}^{\infty} \frac{\phi_0^{(0,-)}(n - 1)}{n^m} = \sum_{\ell=0}^{\infty} \frac{K_\ell^{(0,-)}}{\alpha_0^{m-\ell}}, \quad m \geq 1, \quad (B8)
\]

\[
\Gamma \left( \frac{3}{4} \right) \alpha_\ell^{m} \sum_{n=1}^{\infty} \frac{\phi_0^{(0,-)}(n - 1)}{n - 0} \alpha_\ell^m = \sum_{\ell=0}^{\infty} \frac{K_\ell^{(1,-)}}{\alpha_0^{m-\ell}}, \quad m \geq 1, \quad (B9)
\]

where \( K_\ell^{(1,-)} = \pi / (4\sqrt{2}) \) and for \( \ell \geq 1 \), \( K_\ell^{(1,-)} = (-1)^{m-1} 4^{\ell-1} \alpha^{m-\ell} \). Useful identities of \( \phi_0^{(0,+)}(n) \) are

\[
\Gamma \left( \frac{5}{4} \right) \alpha_\ell^{m} \sum_{n=1}^{\infty} \frac{\phi_0^{(0,+)}(n)}{n + 0} \alpha_\ell^m = - \sum_{\ell=0}^{\infty} (-4)^{m-\ell} K_\ell^{(1,+)}, \quad (B10)
\]

where \( K_0^{(1,+)} = \frac{1}{2} - \frac{\pi}{4\sqrt{2}} \) and \( K_\ell^{(1,+)} = K_\ell^{(0,+)} \) for \( \ell \geq 1 \).

References

[1] Polyakov A M 1980 Nucl. Phys. B 164 171–88
[2] Korchemsky G and Radyushkin A 1987 Nucl. Phys. B 283 342–64
[3] Gubser S S, Klebanov I R and Polyakov A M 1998 Phys. Lett. B 428 105–14
[4] Belitsky A V, Gorsky A S and Korchemsky G P 2006 Nucl. Phys. B 748 24–59
[5] Alday L F and Maldacena J M 2007 J. High Energy Phys. JHEP11(2007)019
[6] Freyhult L, Rej A and Staudacher M 2008 J. Stat. Mech. P07015
[7] Frolov S, Tirziu A and Tseytlin A A 2007 Nucl. Phys. B 766 232–45
[8] Korchemsky G 1989 Mod. Phys. Lett. A 4 1257–76
[9] Belitsky A V, Gorsky A S and Korchemsky G P 2003 Nucl. Phys. B 667 3–54
[10] Kotikov A V, Lipatov L N, Onishchenko A I and Velizhanin V N 2004 Phys. Lett. B 595 521–9
Kotikov A V, Lipatov L N, Onishchenko A I and Velizhanin V N 2006 Phys. Lett. B 632 754 (erratum)
[11] Bern Z, Dixon L J and Smirnov V A 2005 Phys. Rev. D 72 085001
[12] Bern Z, Czakon M, Dixon L J, Kosower D A and Smirnov V A 2007 Phys. Rev. D 75 085010
[13] Cachazo F, Spradlin M and Volovich A 2007 Phys. Rev. D 75 105011
[14] Maldacena J M 1998 Adv. Theor. Math. Phys. 2 231–52
[15] Frolov S and Tseytlin A A 2002 J. High Energy Phys. JHEP06(2002)007
[16] Arutyunov G, Frolov S and Staudacher M 2004 J. High Energy Phys. JHEP10(2004)016
[17] Staudacher M 2005 J. High Energy Phys. JHEP05(2005)054
[18] Beisert N and Staudacher M 2005 Nucl. Phys. B 727 1–62
[19] Eden B and Staudacher M 2006 J. Stat. Mech. P11014
[20] Beisert N, Eden B and Staudacher M 2007 J. Stat. Mech. P01021
[21] Belitsky A 2006 Phys. Lett. B 643 354–61
[22] Basso B and Korchemsky G 2009 Nucl. Phys. B 807 397–423
[23] Kotikov A V and Lipatov L N 2007 Nucl. Phys. B 769 217–55
[24] Benni M, Benvenuti S, Klebanov I and Scardicchio A 2007 Phys. Rev. Lett. 98 131603
[25] Alday L F, Arutyunov G, Benni M, Eden B and Klebanov I 2007 J. High Energy Phys. JHEP04(2007)082
[26] Kostov I, Serban D and Volin D 2008 Nucl. Phys. B 789 413–51
[27] Beccaria M, Angelis F D and Forini V 2007 J. High Energy Phys. JHEP04(2007)066
[28] Basso B, Korchemsky G and Kotanski J 2008 Phys. Rev. Lett. 100 091601
[29] Kostov I, Serban D and Volin D 2008 J. High Energy Phys. JHEP08(2008)101
[30] Hasenfratz P, Maggiore M and Niedermayer F 1990 Phys. Lett. B 245 522–8
[31] Hasenfratz P and Niedermayer F 1990 Phys. Lett. B 245 529–32
[32] Basso B and Korchemsky G P 2009 J. Phys. A: Math. Theor. 42 254005
[33] Bajnok Z, Balog J, Basso B, Korchemsky G and Palla L 2009 Nucl. Phys. B 811 438–62
[34] Volin D 2010 Phys. Rev. D 81 105008
[35] Olde Daalhuis A 2005 Proc. R. Soc. A 461 3005–21
[36] Garoufalidis S, Its A, Kapaev A and Mariño M 2012 Int. Math. Res. Not. 2012 561
[37] Aniceto I, Schiappa R and Vonk M 2012 Commun. Numer. Theor. Phys. 6 339
[38] Schiappa R and Vaz R 2014 Commun. Math. Phys. 330 655–721
[39] Bender C M and Wu T T 1969 Phys. Rev. 184 1231
[40] Bender C M and Wu T 1973 Phys Rev. D 7 1620
[41] Dyson F 1952 Phys. Rev. 85 631–2
[42] Zinn-Justin J 1981 Phys. Rep. 70 109
[43] Beneke M 1999 Phys. Rep. 317 1
[44] Bogomolny E 1980 Phys. Lett. B 91 431
[45] Zinn-Justin J 1981 Nucl. Phys. B 192 125
[46] Zinn-Justin J 1983 Nucl. Phys. B 218 333–48
[47] Dunne G V and Ünsal M 2014 Phys. Rev. D 89 041701
[48] Başar G, Dunne G V and Ünsal M 2013 J. High Energy Phys. JHEP10(2013)041
[49] Dunne G V and Ünsal M 2014 Phys. Rev. D 89 105009
[50] David F 1991 Nucl. Phys. B 348 507–24
[51] David F 1993 Phys. Lett. B 302 403–10
[52] Mariño M 2008 J. High Energy Phys. JHEP03(2008)060
[53] Mariño M, Schiappa R and Weiss M 2008 Commun. Numer. Theor. Phys. 2 349
[54] Mariño M 2008 J. High Energy Phys. JHEP12(2008)114
[55] Mariño M, Schiappa R and Weiss M 2009 J. Math. Phys. 50 052301
[56] Pasquetti S and Schiappa R 2010 Ann. Henri Poincaré 11 351
[57] Couso-Santamaría R, Schiappa R and Vaz R 2015 Ann. Phys., NY 356 1–28
[58] Dunne G V and Ünsal M 2012 J. High Energy Phys. JHEP11(2012)170
[59] Dunne G V and Ünsal M 2013 Phys. Rev. D 87 025015
[60] Cherman A, Dorogoni D, Dunne G V and Ünsal M 2014 Phys. Rev. Lett. 112 021601
[61] Aniceto I, Russo J G and Schiappa R 2015 J. High Energy Phys. JHEP03(2015)172
[62] Shifman M 2015 J. Exp. Theor. Phys. 120 386–98
[63] Başar G and Dunne G V 2015 J. High Energy Phys. JHEP02(2015)160
[64] Dunne G V and Ünsal M 2015 J. High Energy Phys. JHEP09(2015)199
[65] Couso-Santamaría R, Edelstein J D, Schiappa R and Vonk M 2016 Ann. Henri Poincaré in press (arXiv:1308.1695)
[66] Couso-Santamaría R, Edelstein J D, Schiappa R and Vonk M 2015 Commun. Math. Phys. 338 285–346
[67] Grassi A, Mariño M and Zakany S 2015 J. High Energy Phys. JHEP05(2015)038
[68] Sauzin D 2014 arXiv:1405.0356
[69] Dorogoni D 2014 arXiv:1411.3585
[70] Aniceto I, Başar G and Schiappa R 2016 in preparation
[71] Bender C and Orszag S 1978 Advanced Mathematical Methods for Scientists and Engineers (New York; McGraw-Hill)
[72] Aniceto I and Schiappa R 2015 Commun. Math. Phys. 335 183–245
[73] Heller M P and Spalinski M 2015 Phys. Rev. Lett. 115 072501
[74] Hatsuda Y and Okuyama K 2015 J. High Energy Phys. JHEP09(2015)051
[75] Collins J C and Soper D E 1978 Ann. Phys., NY 112 209–34