Maximal operators on variable Lebesgue spaces with weights related to oscillations of Carleson curves

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Abstract

We prove sufficient conditions for the boundedness of the maximal operator on variable Lebesgue spaces with weights \( \varphi_{t,\gamma}(\tau) = |(\tau - t)\gamma| \), where \( \gamma \) is a complex number, over arbitrary Carleson curves. If the curve has different spirality indices at the point \( t \) and \( \gamma \) is not real, then \( \varphi_{t,\gamma} \) is an oscillating weight lying beyond the class of radial oscillating weights considered recently by V. Kokilashvili, N. Samko, and S. Samko.

Key words: Maximal operator, weighted variable Lebesgue space, Dini-Lipschitz condition, oscillating weight, Carleson curve, indices of submultiplicative function, spirality indices.

1 Introduction and main result

Let \( \Gamma \) be a rectifiable curve in the complex plane equipped with arc-length measure \(|d\tau|\). We suppose that \( \Gamma \) is simple, that is, homeomorphic to a segment or to a circle. A measurable function \( w : \Gamma \to [0, \infty] \) is said to be a weight if it is positive and finite almost everywhere. Let \( p : \Gamma \to (1, \infty) \) be a continuous function. A weighted variable Lebesgue space \( L^{p(\cdot)}(\Gamma, w) \) is the set of all measurable complex-valued functions \( f \) on \( \Gamma \) such that

\[
\int_{\Gamma} |f(\tau)w(\tau)/\lambda|^{p(\tau)} |d\tau| < \infty
\]

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for some \( \lambda = \lambda(f) > 0 \). It is a Banach space when equipped with the Luxemburg-Nakano norm

\[
\|f\|_{L^p(\Gamma, w)} = \inf \left\{ \lambda > 0 : \int_{\Gamma} \frac{|f(\tau)w(\tau)|}{\lambda} |d\tau| \leq 1 \right\}.
\]

It is clear that \( L_p^w(\Gamma) \) coincides with the standard Lebesgue space whenever \( p \) is constant. It is a partial case of so-called Musielak-Orlicz spaces (see [16,17]).

Two weights \( w_1 \) and \( w_2 \) on \( \Gamma \) are said to be equivalent if there is a bounded and bounded away from zero function \( f \) on \( \Gamma \) such that \( w_1 = fw_2 \). It is easy to see that \( L_p^w(\Gamma, w_1) \) and \( L_p^w(\Gamma, w_2) \) are isomorphic whenever \( w_1 \) and \( w_2 \) are equivalent.

A curve \( \Gamma \) is said to be Carleson (or Ahlfors-David regular) if

\[
C_{\Gamma} := \sup_{t \in \Gamma} \sup_{\varepsilon > 0} \frac{\Gamma(t, \varepsilon)}{\varepsilon} < \infty
\]

where \( \Gamma(t, \varepsilon) := \{ \tau \in \Gamma : |\tau - t| < \varepsilon \} \) is the portion of the curve in the disk centered at \( t \) of radius \( \varepsilon \) and \( |\Omega| \) denotes the measure of a measurable set \( \Omega \subset \Gamma \). We are interested in the boundedness conditions for the maximal operator

\[
(Mf)(t) := \sup_{\varepsilon > 0} \frac{1}{\Gamma(t, \varepsilon)} \int_{\Gamma(t, \varepsilon)} |f(\tau)| \ |d\tau| \quad (t \in \Gamma)
\]

on weighted variable Lebesgue spaces. This operator is one of the main players in harmonic analysis. It is closely related to the Cauchy singular integral operator

\[
(Sf)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma(t, \varepsilon)} \frac{f(\tau)}{\tau - t} \ d\tau \quad (t \in \Gamma).
\]

The boundedness of both operators on standard weighted Lebesgue spaces is well understood (see e.g. [11,18]). If \( T \) is one of the operators \( M \) or \( S \) and \( 1 < p < \infty \), then \( T \) is bounded on \( L_p^w(\Gamma) \) if and only if \( w \) is a Muckenhoupt weight, \( w \in A_p(\Gamma) \), that is,

\[
\sup_{t \in \Gamma} \sup_{\varepsilon > 0} \left( \frac{1}{\varepsilon} \int_{\Gamma(t, \varepsilon)} w^p(\tau) \ |d\tau| \right)^{1/p} \left( \frac{1}{\varepsilon} \int_{\Gamma(t, \varepsilon)} w^{-q}(\tau) \ |d\tau| \right)^{1/q} < \infty
\]
where $1/p + 1/q = 1$. By Hölder’s inequality, if $w$ is a Muckenhoupt weight, then $\Gamma$ is a Carleson curve.

Let us define the weight we are interested in. Fix $t \in \Gamma$ and consider the function $\eta_t : \Gamma \setminus \{t\} \to (0, \infty)$ defined by

$$
\eta_t(\tau) := e^{-\text{arg}(\tau - t)},
$$

where $\text{arg}(\tau - t)$ denotes any continuous branch of the argument on $\Gamma \setminus \{t\}$. For every $\gamma \in \mathbb{C}$, put

$$
\varphi_{t,\gamma}(\tau) := |(\tau - t)^\gamma| = |\tau - t|^{|\text{Re}\eta_t(\tau)|^\gamma} \quad (\tau \in \Gamma \setminus \{t\}).
$$

In the Fredholm theory of singular integral operators with piecewise continuous coefficients on $L^p(\Gamma)$ (without weights!) it is important to know for which values of $\gamma$ the operator $S$ is bounded on $L^p(\cdot) (\Gamma, \varphi_{t,\gamma})$ (see e.g. [6,7] and also [1]). In fact, an attempt to answer this question is the our main motivation for this work.

The above question was completely studied for the case of standard Lebesgue spaces by A. Böttcher and Yu. Karlovich [1, Section 3.1]. To formulate their result explicitly, we need some definitions. A function $\varrho : (0, \infty) \to (0, \infty]$ is called regular if it is bounded from above in some open neighborhood of the point 1. A function $\varrho : (0, \infty) \to (0, \infty]$ is said to be submultiplicative if $\varrho(x_1 x_2) \leq \varrho(x_1) \varrho(x_2)$ for all $x_1, x_2 \in (0, \infty)$. A regular submultiplicative function is finite everywhere and one can define

$$
\alpha(\varrho) := \sup_{x \in (0, 1)} \frac{\log \varrho(x)}{\log x}, \quad \beta(\varrho) := \inf_{x \in (1, \infty)} \frac{\log \varrho(x)}{\log x}.
$$

In this case, by [1, Theorem 1.13], $-\infty < \alpha(\varrho) \leq \beta(\varrho) < \infty$. The numbers $\alpha(\varrho)$ and $\beta(\varrho)$ are called lower and upper indices of $\varrho$, respectively.

For $t \in \Gamma$, put $d_t := \max_{\tau \in \Gamma} |\tau - t|$. Following [1, Section 1.5], for a continuous function $\psi : \Gamma \setminus \{t\} \to (0, \infty)$, we define

$$
(W_t \psi)(x) := \begin{cases} 
\sup_{0 < R \leq d_t} \left( \max_{|\tau - t| = xR} \frac{\psi(\tau)}{\min_{|\tau - t| = xR} \psi(\tau)} \right) & \text{for } x \in (0, 1], \\
\sup_{0 < R \leq d_t} \left( \max_{|\tau - t| = x^{-1}R} \frac{\psi(\tau)}{\min_{|\tau - t| = x^{-1}R} \psi(\tau)} \right) & \text{for } x \in [1, \infty). 
\end{cases}
$$

This function is submultiplicative in view of [1, Lemma 1.15]. From [1, Theorem 1.18] it follows that $W_t \eta_t$ is regular for every $t \in \Gamma$ whenever $\Gamma$ is a
Carleson curve. Hence the lower and upper spirality indices $\delta_t^-$ and $\delta_t^+$ at $t \in \Gamma$ are correctly defined by
\[
\delta_t^- := \alpha(W_t \eta_t), \quad \delta_t^+ := \beta(W_t \eta_t).
\]

**Proposition 1**  (a) If $\Gamma$ is a piecewise smooth curve, then $\arg(\tau - t) = O(1)$ and $\delta_t^- = \delta_t^+ = 0$ for all $t \in \Gamma$.
(b) If $\Gamma$ is a Carleson curve satisfying
\[
\arg(\tau - t) = -\delta \log |\tau - t| + O(1) \quad \text{as} \quad \tau \to t
\]
at some $t \in \Gamma$ with some $\delta \in \mathbb{R}$, then $\delta_t^- = \delta_t^+ = \delta$.
(c) (R. Seifullayev) If $\Gamma$ is a Carleson curve, then
\[
\arg(\tau - t) = O(-\log |\tau - t|) \quad \text{as} \quad \tau \to t
\]
for every $t \in \Gamma$.
(d) (A. Böttcher, Yu. Karlovich) For any given real numbers $\alpha, \beta$ such that
\[-\infty < \alpha < \beta < +\infty,
\]
there exists a Carleson curve $\Gamma$ such that $\delta_t^- = \alpha$ and $\delta_t^+ = \beta$ at some point $t \in \Gamma$.

Parts (a) and (b) are trivial, a proof of part (c) is in [1, Theorem 1.10], and part (d) is proved in [1, Proposition 1.21]. From [1, Propistion 3.1] it follows that $W_t \varphi_{t, \gamma}$ is regular for every $\gamma \in \mathbb{C}$ and
\[
\alpha(W_t \varphi_{t, \gamma}) = \text{Re} \gamma + \min\{\delta_t^- \text{Im} \gamma, \delta_t^+ \text{Im} \gamma\},
\]
\[
\beta(W_t \varphi_{t, \gamma}) = \text{Re} \gamma + \max\{\delta_t^- \text{Im} \gamma, \delta_t^+ \text{Im} \gamma\}.
\]

These equalities in conjunction with [1, Theorem 2.33] yield the following.

**Theorem 2** (A. Böttcher, Yu. Karlovich) Let $\Gamma$ be a Carleson curve and $p \in (1, \infty)$ be constant. Suppose $t \in \Gamma$ and $\gamma \in \mathbb{C}$. Then $\varphi_{t, \gamma} \in A_p(\Gamma)$ if and only if
\[
0 < \frac{1}{p} + \text{Re} \gamma + \min\{\delta_t^- \text{Im} \gamma, \delta_t^+ \text{Im} \gamma\}
\]
\[
\leq \frac{1}{p} + \text{Re} \gamma + \min\{\delta_t^- \text{Im} \gamma, \delta_t^+ \text{Im} \gamma\} < 1.
\]

In the last decade many results from classical harmonic analysis for standard (weighted) Lebesgue spaces were extended to the setting of (weighted) variable
Lebesgue spaces (see e.g. [2,3,12,14,15] and the references therein). We recall only the most relevant result. Following [3,12], we will always suppose that \( p : \Gamma \to (1, \infty) \) is a continuous function satisfying the Dini-Lipschitz condition on \( \Gamma \), that is, there exists a constant \( C_p > 0 \) such that

\[
|p(\tau) - p(t)| \leq \frac{C_p}{-\log |\tau - t|}
\]

for all \( \tau, t \in \Gamma \) such that \( |\tau - t| \leq 1/2 \). For power weights one has the next criterion. For simplicity, we formulate it in the case of one singularity only. However, it is valid for power weights with a finite number of singularities (see [9,13]).

**Theorem 3 (V. Kokilashvili, V. Paatashvili, S. Samko)** Let \( \Gamma \) be a Carleson curve and \( p : \Gamma \to (1, \infty) \) be a continuous function satisfying the Dini-Lipschitz condition. For \( t \in \Gamma \) and \( \lambda \in \mathbb{R} \), define the power weight \( w(\tau) := |\tau - t|^\lambda \). If \( T \) is one of the operators \( M \) or \( S \), then \( T \) is bounded on \( L^p(\Gamma, w) \) if and only if

\[
0 < \frac{1}{p(t)} + \lambda < 1.
\]

Clearly, \( \phi_{t,\gamma} \) is equivalent to a power weight \( w(\tau) = |\tau - t|^\lambda \) if and only if \( \gamma \) is real or \( \Gamma \) satisfies (1) at \( t \). Hence, Theorem 3 is not applicable to the weight \( \eta_t \) for Carleson curves with \( \delta^- < \delta^+ \).

The sufficiency portion of Theorem 3 has been extended recently to the case of radial oscillating weights (see [10] for \( M \) and [11] for \( S \)). In the case of one singularity these weights have the form \( w(\tau) = f(|\tau - t|) \) where \( t \in \Gamma \) is fixed and \( f : (0, \text{diam}(\Gamma)] \to (0, \infty) \) is some continuous function with additional regularity properties. It is clear that the weight \( \eta_t \) is not of this form. Thus, in general, weights considered in this paper lie beyond the class of radial oscillating weights.

**Theorem 4 (Main result)** Let \( \Gamma \) be a Carleson curve and \( p : \Gamma \to (1, \infty) \) be a continuous function satisfying the Dini-Lipschitz condition. If \( t \in \Gamma \), \( \gamma \in \mathbb{C} \), and

\[
0 < \frac{1}{p(t)} + \text{Re} \gamma + \min\{\delta^- \text{Im} \gamma, \delta^+ \text{Im} \gamma\} \\
\leq \frac{1}{p(t)} + \text{Re} \gamma + \max\{\delta^- \text{Im} \gamma, \delta^+ \text{Im} \gamma\} < 1,
\]

then \( M \) is bounded on \( L^p(\Gamma, \phi_{t,\gamma}) \).
We conjecture that Theorem 4 is true with $M$ replaced by $S$ and that a check of the proof of [11, Theorem 4.3] will indicate the modifications needed to obtain the desired result. We also conjecture that inequalities (4) are necessary for the boundedness of $M$ and $S$ on $L^{p(\cdot)}(\Gamma, \varphi_{t,\gamma})$. To support the second conjecture, note that arguing as in [8], one can show that if $S$ is bounded on $L^{p(\cdot)}(\Gamma, \varphi_{t,\gamma})$, then

\[
0 \leq \frac{1}{p(t)} + \Re \gamma + \min \{ \delta^-_t \Im \gamma, \delta^+_t \Im \gamma \} \\
\leq \frac{1}{p(t)} + \Re \gamma + \max \{ \delta^-_t \Im \gamma, \delta^+_t \Im \gamma \} \leq 1.
\]

The paper is organized as follows. In Section 2 we formulate a sufficient condition for the boundedness of $M$ on $L^{p(\cdot)}(\Gamma, w)$ involving the classical Muckenhoupt condition. Further we apply it to the case of the weight $\varphi_{t,\gamma}$. In Section 3 we estimate a weight $w$ with the only singularity at $t$ by power weights with exponents $\alpha(W_t w) - \varepsilon$ and $\beta(W_t w) + \varepsilon$ where $\varepsilon$ is small enough. Section 4 contains the proof of Theorem 4. Here we follow an idea from [10] and represent the weighted maximal operator as the sum of four maximal operators. The first operator is the maximal operator over a small arc containing the singularity of the weight $\varphi_{t,\gamma}$. Its boundedness follows from the results of Section 2. The second and third maximal operators are estimated by maximal operators with power weights with exponents $\alpha(W_t \varphi_{t,\gamma}) - \varepsilon$ and $\beta(W_t \varphi_{t,\gamma}) + \varepsilon$ by using the results of Section 3. The boundedness of the latter operators follows from Theorem 3. The last maximal operator is over the complement of the small arc containing the singularity of the weight. Hence there is no influence of the weight on this operator and its boundedness follows trivially from Theorem 3.

2 Sufficient condition involving Muckenhoupt weights

Although a complete characterization of weights for which $M$ is bounded on weighted variable Lebesgue spaces is still unknown, one of the most significant recent results to achieve this aim is the following sufficient condition (see [10, Theorem A’]).

Theorem 5 (V. Kokilashvili, N. Samko, S. Samko) Let $\Gamma$ be a Carleson curve, $p : \Gamma \to (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition, and $w : \Gamma \to [0, \infty]$ be a weight such that $w^{p/p^*} \in A_{p^*}(\Gamma)$, where

\[
p_* := p_*(\Gamma) := \min_{\tau \in \Gamma} p(\tau).
\]
Then \( M \) is bounded on \( L^{p(*)}(\Gamma, w) \).

This theorem does not contain the sufficiency portion of Theorem \( 3 \) whenever \( p \) is variable because for the weight \( \rho(\tau) = |\tau - t|^h \) the condition \( \rho^{p/p_*} \in A_{p_*}(\Gamma) \)

is equivalent to \( -1/p(t) < \lambda < (p_* - 1)/p(t) \), while the “correct” interval for \( \lambda \) is wider: \( -1/p(t) < \lambda < (p(t) - 1)/p(t) \). This means that conditions of Theorem \( 5 \) cannot be necessary unless \( p \) is constant.

Now we apply Theorem \( 5 \) to the weight \( \varphi_{t,\gamma} \).

**Lemma 6** Let \( \Gamma \) be a Carleson curve and \( p : \Gamma \to (1, \infty) \) be a continuous function satisfying the Dini-Lipschitz condition. If \( t \in \Gamma, \gamma \in \mathbb{C} \), and

\[
0 < \frac{1}{p(t)} + \text{Re} \gamma + \min \{ \delta^-_t \text{Im} \gamma, \delta^+_t \text{Im} \gamma \} \\
\leq \frac{1}{p(t)} + \text{Re} \gamma + \max \{ \delta^-_t \text{Im} \gamma, \delta^+_t \text{Im} \gamma \} < \frac{p_*}{p(t)}.
\]

where \( p_* \) is defined by \( 5 \), then \( M \) is bounded on \( L^{p(*)}(\Gamma, \varphi_{t,\gamma}) \).

**Proof.** Inequalities \( 6 \) are equivalent to

\[
0 < \frac{1}{p_*} + \text{Re} \gamma \frac{p(t)}{p_*} + \min \left\{ \delta^-_t \text{Im} \gamma \frac{p(t)}{p_*}, \delta^+_t \text{Im} \gamma \frac{p(t)}{p_*} \right\} \\
\leq \frac{1}{p_*} + \text{Re} \gamma \frac{p(t)}{p_*} + \max \left\{ \delta^-_t \text{Im} \gamma \frac{p(t)}{p_*}, \delta^+_t \text{Im} \gamma \frac{p(t)}{p_*} \right\} < 1.
\]

By Theorem \( 2 \) the latter inequalities are equivalent to \( \varphi_{t,\gamma \rho(t)/p_*} \in A_{p_*}(\Gamma) \).

Observe that that the weights \( \varphi_{t,\gamma \rho(t)/p_*} \) and \( (\varphi_{t,\gamma})^{p/p_*} \) are equivalent and therefore belong to \( A_{p_*}(\Gamma) \) only simultaneously. Indeed, from Proposition \( 1 \) (c) and \( 3 \) it follows that

\[
\frac{[\varphi_{t,\gamma}(\tau)]^{p(\tau)/p_*}}{\varphi_{t,\gamma \rho(t)/p_*}(\tau)} = \frac{\exp \left\{ (\text{Re} \gamma \log |\tau - t| - \text{Im} \gamma \text{arg}(\tau - t)) \frac{p(\tau)}{p_*} \right\}}{\exp \left\{ \frac{\text{Re} \gamma \log |\tau - t| - \text{Im} \gamma \text{arg}(\tau - t)}{p_*} \arg(\tau - t) \right\}}
\]

\[
= \exp \left\{ \left( \frac{\text{Re} \gamma}{p_*} \log |\tau - t| - \frac{\text{Im} \gamma}{p_*} \text{arg}(\tau - t) \right) (p(\tau) - p(t)) \right\}
\]

\[
= \exp \left\{ \left( \frac{\text{Re} \gamma}{p_*} \log |\tau - t| + \frac{\text{Im} \gamma}{p_*} O(\log |\tau - t|) \right) O \left( \frac{1}{-\log |\tau - t|} \right) \right\}
\]

\[
= \exp\{O(1)\}
\]
as \( \tau \to t \). This immediately implies that the weights \( (\varphi_{t,\gamma})^{p/p_*} \) and \( \varphi_{t,\gamma p(t)/p_*} \) are equivalent because they are continuous on \( \Gamma \setminus \{ t \} \).

Finally, applying Theorem 5, we obtain that \( M \) is bounded on \( L^{p(\cdot)}(\Gamma, \varphi_{t,\gamma}) \). \( \Box \)

3 Estimates of weights with one singularity by power weights

Recall that there are more convenient formulas for calculation of indices of a regular submultiplicative function.

**Theorem 7** If \( \varrho : (0, \infty) \to (0, \infty) \) is regular and submultiplicative, then

\[
\alpha(\varrho) = \lim_{x \to 0} \frac{\log \varrho(x)}{\log x}, \quad \beta(\varrho) = \lim_{x \to \infty} \frac{\log \varrho(x)}{\log x},
\]

and

\[-\infty < \alpha(\varrho) \leq \beta(\varrho) < +\infty;\]

(b) \( \varrho(x) \geq x^{\alpha(\varrho)} \) for all \( x \in (0, 1) \) and \( \varrho(x) \geq x^{\beta(\varrho)} \) for all \( x \in (1, \infty) \);

(c) given any \( \varepsilon > 0 \), there exists an \( x_0 > 1 \) such that \( \varrho(x) \leq x^{\alpha(\varrho) - \varepsilon} \) for all \( x \in (0, x_0^{-1}) \) and \( \varrho(x) \leq x^{\beta(\varrho) + \varepsilon} \) for all \( x \in (x_0, \infty) \).

Part (a) is proved, for instance, in [1, Theorem 1.13]. Parts (b) and (c) follow from part (a), see e.g. [1, Corollary 1.14].

Fix \( t_0 \in \Gamma \). Let \( \omega(t_0, \delta) \) denote the open arc on \( \Gamma \) which contains \( t_0 \) and whose endpoints lie on the circle \( \{ \tau \in \mathbb{C} : |\tau - t_0| = \delta \} \). It is clear that \( \omega(t_0, \delta) \subset \Gamma(t_0, \delta) \), however, it may happen that \( \omega(t_0, \delta) \neq \Gamma(t_0, \delta) \).

**Lemma 8** Let \( \Gamma \) be a Carleson curve and \( t_0 \in \Gamma \). Suppose \( w : \Gamma \setminus \{ t_0 \} \to (0, \infty) \) is a continuous function and \( W_{t_0, w} \) is regular. Let \( \varepsilon > 0 \) and \( \delta \) be such that \( 0 < \delta < d_{t_0} \). Then there exist positive constants \( C_j = C_j(\varepsilon, \delta, w) \), where \( j = 1, 2 \), such that

\[
\frac{w(t)}{w(\tau)} \leq C_1 \frac{|t - t_0|^{\beta(W_{t_0, w})+\varepsilon}}{|\tau - t_0|^{\beta(W_{t_0, w})+\varepsilon}} \tag{7}
\]

for all \( t \in \Gamma \setminus \omega(t_0, \delta) \) and all \( \tau \in \omega(t_0, \delta) \); and

\[
\frac{w(t)}{w(\tau)} \leq C_2 \frac{|t - t_0|^{\alpha(W_{t_0, w}) - \varepsilon}}{|\tau - t_0|^{\alpha(W_{t_0, w}) - \varepsilon}} \tag{8}
\]

for all \( t \in \omega(t_0, \delta) \) and all \( \tau \in \Gamma \setminus \omega(t_0, \delta) \).
PROOF. Let us denote $\beta := \beta(W_{t_0}w)$. By Theorem 7(c), for every $\varepsilon > 0$ there exists an $x_0 \in (1, \infty)$ such that

$$(W_{t_0}w)(x) \leq x^{\beta + \varepsilon} \text{ for all } x \in (x_0, \infty).$$

From this inequality and the definition of $W_{t_0}w$ it follows that if $0 < R \leq d_{t_0}$ and $x \in (x_0, \infty)$, then

$$\max_{|\tau - t_0|=R} w(t) \leq x^{\beta + \varepsilon} \min_{|\tau - t_0|=x^{-1}R} w(\tau) = \left(\frac{R}{|\tau - t_0|}\right)^{\beta + \varepsilon} \min_{|\tau - t_0|=x^{-1}R} w(\tau).$$

Hence

$$w(t) \leq \left|\frac{t - t_0}{\tau - t_0}\right|^{\beta + \varepsilon} w(\tau)$$

(9)

for all $t \in \Gamma \setminus \{t_0\}$ and all $\tau \in \Gamma$ such that $|t - t_0|/|\tau - t_0| \in (x_0, \infty)$. Put

$$\Delta_{t_0} := \min_{t \in \Gamma \setminus \omega(t_0, \delta)} |t - t_0|.$$

It is clear that if $\tau \in \omega(t_0, \Delta_{t_0}/x_0)$ and $t \in \Gamma \setminus \omega(t_0, \delta)$, then (9) holds.

Since the function

$$f(\tau) := \frac{w(\tau)}{|\tau - t_0|^\beta}$$

is continuous on $\Gamma \setminus \{t_0\}$, we have

$$0 < M_1 := \inf_{\tau \in \omega(t_0, \delta) \setminus \omega(t_0, \Delta_{t_0}/x_0)} f(\tau), \quad M_2 := \sup_{\tau \in \Gamma \setminus \omega(t_0, \delta)} f(\tau) < \infty.$$

Hence

$$w(t) \leq M_2 |t - t_0|^\beta$$

for all $t \in \Gamma \setminus \omega(t_0, \delta)$ and

$$\frac{1}{w(\tau)} \leq \frac{1}{M_1 |\tau - t_0|^\beta}$$

9
for all \( \tau \in \omega(t_0, \delta) \setminus \omega(t_0, \Delta_{t_0}/x_0) \). Multiplying these inequalities, we obtain
\[
\frac{w(t)}{w(\tau)} \leq \frac{M_2}{M_1} \left| \frac{t - t_0}{\tau - t_0} \right|^{\beta + \varepsilon}
\]
for all \( t \in \Gamma \setminus \omega(t_0, \delta) \) and all \( \tau \in \omega(t_0, \delta) \setminus \omega(t_0, \Delta_{t_0}/x_0) \).

Thus (7) holds for \( t \in \Gamma \setminus \omega(t_0, \delta) \) and all \( \tau \in \omega(t_0, \Delta_{t_0}/x_0) \cup [\omega(t_0, \delta) \setminus \omega(t_0, \Delta_{t_0}/x_0)] = \omega(t_0, \delta) \)

with \( C_1 := \max\{1, M_2/M_1\} \). Estimate (8) is proved by analogy. \( \square \)

4 Proof of Theorem 4

The idea of the proof is borrowed from [10, Theorem B]. Fix \( t_0 \in \Gamma \) and \( \gamma \in \mathbb{C} \). Notice that we omitted the subscript for \( t_0 \) in the formulation of the theorem for brevity. It is easily seen that \( M \) is bounded on \( L^p(\Gamma; \varphi_{t_0,\gamma}) \) if and only if the operator
\[
(M_{t_0,\gamma}f)(t) := \sup_{\varepsilon > 0} \frac{\varphi_{t_0,\gamma}(t)}{|\Gamma(t, \varepsilon)|} \int_{[\Gamma(t, \varepsilon) / \varphi_{t_0,\gamma}(\tau)]} |f(\tau)| d\tau \quad (t \in \Gamma)
\]
is bounded on \( L^p(\cdot)(\Gamma) \).

As it was already mentioned in the introduction, the function \( W_{t_0,\varphi_{t_0,\gamma}} \) is regular and submultiplicative for every \( \gamma \in \mathbb{C} \) and

\[
\alpha := \alpha(W_{t_0,\varphi_{t_0,\gamma}}) = \text{Re} \gamma + \min\{\delta_{t_0}^{-}\text{Im} \gamma, \delta_{t_0}^{+}\text{Im} \gamma\}, \quad \beta := \beta(W_{t_0,\varphi_{t_0,\gamma}}) = \text{Re} \gamma + \max\{\delta_{t_0}^{-}\text{Im} \gamma, \delta_{t_0}^{+}\text{Im} \gamma\}.
\]

With these notations, conditions (4) have the form
\[
0 < \frac{1}{p(t_0)} + \alpha, \quad \frac{1}{p(t_0)} + \beta < 1.
\]

In this case there is a small \( \varepsilon > 0 \) such that
\[
0 < \frac{1}{p(t_0)} + \alpha - \varepsilon \leq \frac{1}{p(t_0)} + \beta + \varepsilon < 1. \quad (10)
\]
Since \( p : \Gamma \to (1, \infty) \) is continuous and \( 1/p(t_0) + \beta < 1 \), we can choose a number \( \delta \in (0, d_{t_0}) \) such that the arc \( \omega(t_0, \delta) \), which contains \( t_0 \) and has the endpoints on the circle \( \{ \tau \in \mathbb{C} : |\tau - t_0| = \delta \} \), is so small that \( 1 + \beta p(t_0) < p_* \), where

\[
p_* := p_*(\omega(t_0, \delta)) = \min_{\tau \in \omega(t_0, \delta)} p(\tau).
\]

Hence

\[
0 < \frac{1}{p(t_0)} + \alpha \leq \frac{1}{p(t_0)} + \beta < \frac{p_*}{p(t_0)}.
\]

(11)

Let us denote by \( \chi_\Omega \) the characteristic function of a set \( \Omega \subset \Gamma \). For \( f \in L^{p(\cdot)}(\Gamma) \), we have

\[
M_{t_0, \gamma} f \leq \chi_{\omega(t_0, \delta)} M_{t_0, \gamma} \chi_{\omega(t_0, \delta)} f + \chi_{\Gamma \setminus \omega(t_0, \delta)} M_{t_0, \gamma} \chi_{\omega(t_0, \delta)} f
\]

\[
+ \chi_{\omega(t_0, \delta)} M_{t_0, \gamma} \chi_{\Gamma \setminus \omega(t_0, \delta)} f + \chi_{\Gamma \setminus \omega(t_0, \delta)} M_{t_0, \gamma} \chi_{\Gamma \setminus \omega(t_0, \delta)} f.
\]

(12)

From (11) and Lemma 6 we conclude that \( M_{t_0, \gamma} \) is bounded on \( L^{p(\cdot)}(\omega(t_0, \delta)) \). Hence \( \chi_{\omega(t_0, \delta)} M_{t_0, \gamma} \chi_{\omega(t_0, \delta)} I \) is bounded on \( L^{p(\cdot)}(\Gamma) \).

From Lemma 8 it follows that

\[
\chi_{\Gamma \setminus \omega(t_0, \delta)} M_{t_0, \gamma} \chi_{\omega(t_0, \delta)} f \leq C_1 \chi_{\Gamma \setminus \omega(t_0, \delta)} M_{t_0, \alpha + \varepsilon} \chi_{\omega(t_0, \delta)} f \leq C_1 M_{t_0, \alpha + \varepsilon} f
\]

(13)

and

\[
\chi_{\omega(t_0, \delta)} M_{t_0, \gamma} \chi_{\Gamma \setminus \omega(t_0, \delta)} f \leq C_2 \chi_{\omega(t_0, \delta)} M_{t_0, \alpha - \varepsilon} \chi_{\Gamma \setminus \omega(t_0, \delta)} f \leq C_2 M_{t_0, \alpha - \varepsilon} f,
\]

(14)

where \( C_1 \) and \( C_2 \) are positive constants depending only on \( \varepsilon, \delta, \gamma, \) and \( t_0 \). Inequalities (10), Theorem 3 and inequalities (13)–(14) imply that the operators \( \chi_{\Gamma \setminus \omega(t_0, \delta)} M_{t_0, \gamma} \chi_{\omega(t_0, \delta)} I \) and \( \chi_{\omega(t_0, \delta)} M_{t_0, \gamma} \chi_{\Gamma \setminus \omega(t_0, \delta)} I \) are bounded on \( L^{p(\cdot)}(\Gamma) \).

Finally, since \( \Gamma \setminus \omega(t_0, \delta) \) does not contain the singularity of the weight \( \varphi_{t_0, \gamma} \) which is continuous on \( \Gamma \setminus \{t_0\} \), there exists a constant \( C_3 > 0 \) such that

\[
\chi_{\Gamma \setminus \omega(t_0, \delta)} M_{t_0, \gamma} \chi_{\Gamma \setminus \omega(t_0, \delta)} f \leq C_3 M f.
\]

Then Theorem 3 and the above estimate yield the boundedness of the operator \( \chi_{\Gamma \setminus \omega(t_0, \delta)} M_{t_0, \gamma} \chi_{\Gamma \setminus \omega(t_0, \delta)} I \) on \( L^{p(\cdot)}(\Gamma) \).
Thus all operators on the right-hand side of (12) are bounded on $L^{p(\cdot)}(\Gamma)$. Therefore the operator on the left-hand side of (12) is bounded, too. This completes the proof. □

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