Markov Decision Processes with Continuous Side Information

Aditya Modi
Computer Science and Engineering, Univ. of Michigan Ann Arbor
admodi@umich.edu

Nan Jiang
Microsoft Research, New York
nanjiang@umich.edu

Satinder Singh
Computer Science and Engineering, Univ. of Michigan Ann Arbor
baveja@umich.edu

Ambuj Tewari
Department of Statistics, Univ. of Michigan Ann Arbor
tewaria@umich.edu

Abstract

We consider a reinforcement learning (RL) setting in which the agent interacts with a sequence of episodic MDPs. At the start of each episode the agent has access to some side-information or context that determines the dynamics of the MDP for that episode. Our setting is motivated by applications in healthcare where baseline measurements of a patient at the start of a treatment episode form the context that may provide information about how the patient might respond to treatment decisions.

We propose algorithms for learning in such Contextual Markov Decision Processes (CMDPs) under an assumption that the unobserved MDP parameters vary smoothly with the observed context. We also give lower and upper PAC bounds under the smoothness assumption. Because our lower bound has an exponential dependence on the dimension, we consider a tractable linear setting where the context is used to create linear combinations of a finite set of MDPs. For the linear setting, we give a PAC learning algorithm based on KWIK learning techniques.

Keywords: Reinforcement Learning, PAC bounds, KWIK Learning.

1. Introduction

Consider a basic sequential decision making problem in healthcare, namely that of learning a treatment policy for patients to optimize some health outcome of interest. One could model the interaction with every patient as a Markov Decision Process (MDP). In precision or personalized medicine, we want the treatment to be personalized to every patient. At the same time, the amount of data available on any given patient may not be enough to personalize well. This means that modeling each patient via a different MDP will result in severely suboptimal treatment policies. The other extreme of pooling all patients’ data results in more data but most of it will perhaps not be relevant to the patient we currently want to treat. We therefore face a trade-off between having a large amount of shared data to learn a single policy, and, finding the most relevant policy for each patient. A similar trade-off occurs in other applications involving humans as the agents environment such as online tutoring and web advertising.

©2017 A. Modi, N. Jiang, S. Singh & A. Tewari.
A key observation is that in many personalized decision making scenarios, we have some side information available about individuals which might help us in designing personalized policies and also help us pool the interaction data across the right subsets of individuals. Examples of such data include laboratory data or medical history of patients in healthcare, user profiles or history logs in web advertising, and student profiles or historical scores in online tutoring. Access to such side information should let us learn better policies even with a limited amount of interaction with individual users. We refer to this side-information as contexts and adopt an augmented model called Contextual Markov Decision Process (CMDP) proposed by Hallak et al. (2015). We assume that contexts are fully observed and available before the interaction starts for each new MDP.\footnote{Hallak et al. (2015) assumes latent contexts, which results in significant differences from our work in application scenarios, required assumptions, and results. See detailed discussion in Section 5}

In this paper we study the sample complexity of learning in CMDPs in the worst case. We consider two concrete settings of learning in a CMDP with continuous contexts. In the first setting, the individual MDPs vary in an arbitrary but smooth manner with the contexts, and we propose our Cover-Rmax algorithm in Section 3 with PAC bounds. The innate hardness of learning in this general case is captured by our lower bound construction in Section 3.1. To show that it is possible to achieve significantly better sample complexity in more structured CMDPs, we consider another setting where contexts are used to create linear combinations of a finite set of fixed but unknown MDPs. We use the KWIK framework to devise the KWIK\_LR-Rmax algorithm in Section 4.1 and also provide a PAC upper bound for the algorithm.

2. Contextual Markov Decision Process

2.1 Problem setup and notation

We start with basic definitions and notations for MDPs, and then introduce the contextual case.

**Definition 2.1 (Markov Decision Processes).** A Markov Decision Process (MDP) is defined as a tuple $(S, A, p(\cdot|\cdot, \cdot), r(\cdot, \cdot), \mu)$ where $S$ is the state space and $A$ is the action space; $p(s'|s, a)$ defines the transition probability function for a tuple $(s, a, s')$ where $s, s' \in S, a \in A$; and $\mu$ defines the initial state distribution for the MDP.

We consider the case of fixed horizon (denoted $H$) episodic MDPs. We use $\pi_h(s)$ to denote a policy’s action for state $s$ at timestep $h$. For each episode, an initial state is observed according to the distribution $\mu$ and afterwards, for $0 \leq h < H$, the agent chooses an action $a_h = \pi_h(s_h)$ according to a (non-stationary) policy. There is a reward $r_h$ and then a next state $s_{h+1}$ according to the reward and the transition functions. For policy $\pi$ we define its value as follows:

$$V_\pi^M = E_{s_0 \sim \mu, M, \pi} \left[ \frac{1}{H} \sum_{h=0}^{H-1} r(s_h, \pi_h(s_h)) \right].$$

An optimal policy $\pi^*$ is one that achieves the largest possible value (called optimal value and denoted $V_\pi^*$). Next we define the contextual model which is similar to the definition given by Hallak et al. (2015):

1. Hallak et al. (2015) assumes latent contexts, which results in significant differences from our work in application scenarios, required assumptions, and results. See detailed discussion in Section 5
Definition 2.2 (Contextual MDP). A contextual Markov Decision Process (CMDP) is defined as a tuple \((C, S, A, M)\) where \(C\) is the context space, \(S\) is the state space, and \(A\) is the action space. \(M\) is a function which maps a context \(c \in C\) to MDP parameters \(M(c) = \{p^c(\cdot | \cdot, \cdot), r^c(\cdot, \cdot), \mu^c(\cdot)\}\).

We denote the MDP for a context \(c\) by \(M^c\). We make the simplifying assumption that the initial state distribution is independent of the context and is same for all. We assume throughout the paper that the rewards are bounded between 0 and 1. We denote \(|S|, |A|\) by \(S, A\) respectively. We also assume that the context space is bounded, and for any \(c \in C\) the \(\ell_2\) norm of \(c\) is upper bounded by some constant. We consider the online learning scenario with the following protocol: For \(t = 1, 2, \ldots:\)

1. Observe context \(c_t \in C\).
2. Choose a policy \(\pi_t\) (based on \(c_t\) and previous episodes).
3. Experience an episode in \(M^{c_t}\) using \(\pi_t\).

We do not make any distributional assumptions over the context sequence. Instead, we allow the sequence to be chosen in an arbitrary and potentially adversarial manner. A natural criteria for judging the efficiency of the algorithm is to look at the number of episodes where it performs sub-optimally. The main aim of the PAC analysis is to bound the number of episodes where we have \(V^{\pi_t}_{M^{c_t}} < V^*_M - \epsilon\), i.e., the value of the algorithm’s policy is not \(\epsilon\)-optimal (Dann and Brunskill, 2015). Although, we do give PAC bounds for the Cover-Rmax algorithm given below, the reader should make note that, we have not made explicit attempts to achieve the tightest possible result. We use the Rmax (Brafman and Tennenholtz, 2002) algorithm as the base of our construction to handle exploration-exploitation because of its simplicity. Our approach can also be combined with the other PAC algorithms (Strehl and Littman, 2008; Dann and Brunskill, 2015) for improved dependence on \(S, A\) and \(H\).

3. Cover-Rmax

In this section, we present the Cover-Rmax algorithm and provide a PAC bound for it under a smoothness assumption. The key motivation for our contextual setting is that sharing information among different contexts is helpful. Therefore, it is natural to assume that the MDPs corresponding to similar contexts will themselves be similar. This can be formalized by the following smoothness assumption:

Definition 3.1 (Smoothness). Given a CMDP \((C, S, A, M)\), and a distance metric over the context space \(\phi(\cdot, \cdot)\), if for any two contexts \(c_1, c_2 \in C\), we have the following constraints:

\[
\|p^{c_1}(\cdot | s, a) - p^{c_2}(\cdot | s, a)\|_1 \leq L_p \phi(c_1, c_2) \\
|r^{c_1}(s, a) - r^{c_2}(s, a)| \leq L_r \phi(c_1, c_2)
\]

then, we call this a smooth CMDP with smoothness parameters \(L_p\) and \(L_r\).
We assume that the distance metric and the constants $L_p$ and $L_r$ are known. This smoothness assumption allows us to use a minimally tweaked version of Rmax (Brafman and Tennenholtz, 2002) and provide an analysis for smooth CMDPs similar to existing literature on MDPs (Kearns and Singh, 2002; Strehl et al., 2009; Strehl and Littman, 2008). If we know the transition dynamics and the expected reward functions for each state-action pair in a finite MDP, we can easily compute the optimal policy. The idea of Rmax is to distinguish the state-action pairs as *known* or *unknown*: a state-action pair is known if we have visited it for enough number of times, so that the empirical estimates of reward and transition probabilities are near-accurate due to sufficient data. A state $s$ becomes *known* when all for all actions $a$ the pairs $(s,a)$ become *known*. Rmax then constructs an auxiliary MDP which encourages optimistic behaviour by assigning maximum reward (hence the name Rmax) to the remaining *unknown* states. When we act according to the optimal policy in the auxiliary MDP, one of the following must happen: 1) we exploit the information available and achieve near-optimal value, or, 2) we visit unknown states and accumulate more information efficiently.

Formally, for a set of known states $K$, we define an (approximate) induced MDP $\hat{M}_K$ in the following manner. Let $n(s,a)$ and $n(s,a,s')$ denote the number of observations of state-action pair $(s,a)$ and transitions $(s,a,s')$ respectively. Also, let $R(s,a)$ denote the total reward obtained from state-action pair $(s,a)$. For each $s \in K$, define the values

$$p_{\hat{M}_K}(s'|s,a) = \frac{n(s,a,s')}{n(s,a)},$$

$$r_{\hat{M}_K}(s,a) = \frac{R(s,a)}{n(s,a)}.$$

For each $s \notin K$, define the values as $p_{\hat{M}_K}(s'|s,a) = \mathbb{I}\{s' = s\}$ and $r_{\hat{M}_K}(s,a) = 1$.

We use the certainty equivalent policy computed for this induced MDP and perform balanced wandering (Kearns and Singh, 2002) for unknown states. Balanced wandering ensures that all actions are tried equally and fairly for *unknown* states. Assigning maximum reward to the *unknown* states pushes the agent to visit these states and provides the necessary exploration impetus. The generic template of Rmax is given in Algorithm 1.

For the contextual case, we have an infinite number of such MDPs. The idea behind our algorithm is that, one can group close enough contexts and treat them as a single MDP. Utilizing the boundedness of the context space $\mathcal{C}$, we create a *cover* of $\mathcal{C}$ with finitely many balls $B_r(o_i)$ of radius $r$ centered at $o_i \in \mathbb{R}^d$. By tuning the radius $r$, we can control the bias introduced by ignoring the differences among the MDPs in the same ball. Doing so allows us to pool together the data from all MDPs in a ball, so that we avoid the difficulty of infinite MDPs and instead only deal with finitely many of them. The size of the cover, i.e., the number of balls can be measured by the notion of covering numbers (see e.g., Wu (2016)), defined as

$$N(\mathcal{C}, r) = \min\{|\mathcal{Y}| : \mathcal{C} \subseteq \cup_{y \in \mathcal{Y}} B_r(y)\}.$$

The resulting algorithm, Cover-Rmax, is obtained by using the subroutines in Algorithm 2, and we state its sample complexity guarantee in Theorem 3.2.
Algorithm 1: Rmax Template for CMDP

1. Initialize $(S, A, C, \epsilon, \delta)$;
2. for each episode $t = 1, 2, \cdots$ do
3. Receive context $c_t \in C$;
4. Set $K, M_K$ using $Predict(c_t, s, a)$ for all $(s, a)$.
5. $\pi \leftarrow \pi^*_M$;
6. for $h = 0, 1, \cdots H - 1$ do
7. if $s_h \in K$ then
8. Choose $a_h := \pi_h(s_h)$;
9. else
10. Choose $a_h : (s_h, a_h)$ is unknown;
11. Update $(c_t, s_h, a_h, (s_h+1, r_h))$;
12. end
13. end
14. end

Algorithm 2: Cover-Rmax

1. Function Initialize($S, A, C, \epsilon, \delta$)
2. $r_0 = \min(\frac{\epsilon}{8HLP}, \frac{\epsilon}{8Lr})$;
3. Create an $r_0$-cover of $C$;
4. Initialize counts for all balls $B(o_j)$;
5. Function Predict($c, s, a$)
6. Find $j$ such that $c \in B(o_j)$;
7. if $n_j(s, a) < m$ then
8. return $\hat{p}^c(\cdot|s, a)$ and $\hat{r}^c(s, a)$ using (2);
9. else
10. return unknown;
11. end
12. Function Update($c, s, a, (s', r)$)
13. Find $j$ such that $c \in B(o_j)$;
14. if $n_j(s, a) < m$ then
15. Increment counts and rewards in $B(o_j)$;
16. end

Theorem 3.2 (PAC bound for Cover-Rmax). For any input values $0 < \epsilon, \delta \leq 1$ and a CMDP with smoothness parameters $L_P$ and $L_r$, with probability at least $1 - \delta$, the Cover-Rmax algorithm produces a sequence of policies $\{\pi_t\}$ which yield at most

$$O\left(\frac{NH^2SA}{\epsilon^3} \left(S + \ln \frac{NSA}{\delta} \ln \frac{N}{\delta}\right)\right)$$

non-$\epsilon$-optimal episodes, where $N = N(C, r_0)$ and $r_0 = \min(\frac{\epsilon}{8HLP}, \frac{\epsilon}{8Lr})$.

Proof sketch. We first of all carefully adapt the analysis of Rmax by Kakade (2003) to get the PAC bound for an episodic MDP. Let $m$ be the number of visits to a state-action pair.
after which the model’s estimate \( \hat{p}(s, a) \) for \( p(s, a) \) has an \( \ell_1 \) error of at most \( \epsilon/4H \) and reward estimate \( \hat{r}(s, a) \) has an absolute error of at most \( \epsilon/4 \). We can show that:

**Lemma 3.3.** Let \( M \) be an MDP with the fixed horizon \( H \). If \( \hat{\pi} \) is the optimal policy for \( \hat{M}_K \) as computed by \( R_{\text{max}} \), then for any starting state \( s_0 \), with probability at least \( 1 - 2\delta \), we have \( V^\pi_M(s_0) \geq V^*_{M} - 2\epsilon \) for all but \( O \left( \frac{\ln S A}{\epsilon} \right) \) episodes.

Now instead of learning the model for each contextual MDP separately, we combine the data within each ball. Therefore, we have to take care of two things: choose the radius \( r_0 \) for a fine enough cover and a value of \( m \) which is the number of visits after which a state becomes known for a ball. For satisfying the conditions of Lemma 3.3 for all MDPs within a ball, we need the radius \( r \) of the cover to be \( r \leq r_0 = \min \left( \frac{\epsilon}{8HL}, \frac{\epsilon}{8Lr} \right) \) and the value of \( m = \frac{128(S \ln 2 + \ln \frac{SA}{\delta} H^2)}{\epsilon^2} \).

Using Lemma 3.3, we obtain an upper bound on number of non-\( \epsilon \) episodes in a single ball as \( O \left( \frac{H^2 SA}{\epsilon^2} \left( S + \ln \frac{SA}{\delta} \ln \frac{1}{\delta} \right) \right) \) with probability at least \( 1 - \delta \).

Setting the individual failure probability to be \( \delta/N(C, r_0) \) and using the union bound, we get the stated PAC bound.\(^2\)

We observe that the PAC bound has linear dependence on the covering number of the context space. In case of a \( d \)-dimensional Euclidean metric space, the covering number would be of the order \( O \left( \frac{1}{\epsilon^d} \right) \). However, we show in Section 3.1, that, the dependence would be at least linear, and hence, indicate the difficulty of optimally learning in such cases.

### 3.1 Lower Bound

We prove a lower bound on the number of sub-optimal episodes for any learning algorithm in a smooth CMDP which shows that a linear dependence on the covering number of the context space is unavoidable. As far as we know, there is no existing way of constructing PAC lower bounds for continuous state spaces with smoothness, so we cannot simply augment the state representation to include context information. Instead, we prove our own lower bound in Theorem 3.4 which builds upon the work of Dann and Brunskill (2015) on lower bounds for episodic finite MDPs and of Slivkins (2014) on lower bounds for contextual bandits.

**Theorem 3.4** (Lower bound for smooth CMDP). There exists constants \( \delta_0, \epsilon_0 \), such that for every \( \delta \in (0, \delta_0) \) and \( \epsilon \in (0, \epsilon_0) \), any algorithm that satisfies a PAC guarantee for \( (\epsilon, \delta) \) and computes a sequence of deterministic policies for each context, there is a hard CMDP \( (C, S, A, M) \) with smoothness constant \( L_p = 1 \), such that

\[
\mathbb{E}[B] = \Omega \left( \frac{N(C, \epsilon_1) SA}{\epsilon^2} \right)
\]

where \( B \) is the number of sub-optimal episodes and \( \epsilon_1 = \frac{1280H \epsilon^4}{(H-2)} \).

**Proof.** The overall idea is to embed multiple MDP learning problems in a CMDP, such that the agent has to learn the optimal policy in each MDP separately and cannot generalize across them. We show that the maximum number of problems that can be embedded scales

\(^2\) For detailed proofs, we refer the reader to the appendix.
Markov Decision Processes with Continuous Side Information

Figure 1: Hard instances for episodic MDP (Dann and Brunskill, 2015). The initial state 0 moves to a uniform distribution over states 1 to $n$ regardless of the action, and states $+/-$ are absorbing with 1 and 0 rewards respectively. States 0 to $n$ have 0 reward for all actions. Each state $i \in [n]$ essentially acts as a hard bandit instance, whose $A$ actions move to $+$ and $-$ randomly. Action $a_0$ satisfies $p(+|i, a_0) = \frac{1}{2} + \epsilon'$ and there is at most one other action $a_i$ with $p(+|i, a_i) = \frac{1}{2} + \epsilon'$. Any other action $a_j$ satisfies $p(+|i, a_j) = \frac{1}{2}$.

with the covering number, and the result follows by incorporating known PAC lower bound for episodic MDPs.

We start with the lower bound for learning in episodic MDPs. See Figure 1 and its caption for details. The construction is due to Dann and Brunskill (2015) and we adapt their lower bound statement to our setting in Theorem 3.5.

**Theorem 3.5 (Lower bound for episodic MDPs (Dann and Brunskill, 2015)).** There exists constants $\delta_0, \epsilon_0$, such that for every $\delta \in (0, \delta_0)$ and $\epsilon \in (0, \epsilon_0)$, any algorithm that satisfies a PAC guarantee for $(\epsilon, \delta)$ and computes a sequence of deterministic policies, there is a hard instance $M_{\text{hard}}$ so that $\mathbb{E}[B] = \Omega\left(\frac{SA}{\epsilon^2}\right)$, where $B$ is the number of sub-optimal episodes. The constants can be chosen as $\delta_0 = e^{-4} \frac{\epsilon}{8\epsilon'}$, $\epsilon_0 = \frac{H^2 - 2}{640He^{4}}$.\(^3\)

Now we discuss how to populate the context space with these hard MDPs. Note in Figure 1 that, the agent does not know which action is the most rewarding ($a_i$), and the adversary can choose $i$ to be any element of $[A]$ (which is essentially choosing an instance from a family of MDPs). In our scenario, we would like to allow the adversary to choose the MDP independently for each individual packing point to yield a lower bound linear in the packing number. However, this is not always possible due to the smoothness assumption, as committing to an MDP at one point may restrict the adversary’s choices in another point.

To deal with this difficulty, we note that any pair of hard MDPs differ from each other by $O(\epsilon')$ in transition distributions. Therefore, we construct a packing of $C$ with radius $r = 8\epsilon'$, defined as a set of points $Z$ such that any two points in $Z$ are at least $r$ away from

---

\(^3\) The lower bound here differs from that in the original paper by $H^2$, because our value is normalized (see Eq.(11)), whereas they allow the magnitude of value to grow with $H$. 

7
each other. The maximum size of such $Z$ is known as the packing number:

$$ \mathcal{D}(\mathcal{C}, r) = \max\{|Z| : Z \text{ is an } r\text{-packing of } \mathcal{C}\}, $$

which is related to the covering number as $N(\mathcal{C}, r) \leq \mathcal{D}(\mathcal{C}, r)$. The radius $r$ is chosen to be $O(\epsilon')$ so that arbitrary choices of hard MDP instances at different packing points always satisfy the smoothness assumption (recall that $L_p = 1$). Once we fix the MDPs for all $c \in \mathcal{Z}$, we specify the MDP for $c \in \mathcal{C} \setminus \mathcal{Z}$ as follows: for state $i$ and action $a$,

$$ p_c(+|i, a) = \max_{c' \in Z} \max(1/2, p_{c'}(+|i, a) - \phi(c, c')/2). $$

Essentially, as we move away from a packing point, the transition to $+/-$ become more uniform. We can show that:

**Claim 3.6.** The CMDP defined above is satisfies Definition 3.1 with constant $L_p = 1$.

We choose the context sequence given as input to be repetitions of an arbitrary permutation of $Z$. By construction, the learning at different points in $Z$ are independent, so the lower bound is simply the lower bound for learning a single MDP (Theorem 3.5) multiplied by the cardinality of $Z$ (the packing number). Using the well known relation that $N(\mathcal{C}, r) \leq \mathcal{D}(\mathcal{C}, r)$, we have the desired lower bound. We refer the reader to the appendix for proof of Claim 3.6 and a more detailed analysis.

### 4. Contextual Linear Combination of MDPs

From the previous section, it is clear that for a contextual MDP with just smoothness assumptions, exponential dependence on context dimension is unavoidable. Further, the computational requirements of our Cover-Rmax algorithm scales with the covering number of the context space. As such, in this section, we focus on a more structured assumption about the mapping from context space to MDPs and show that we can achieve substantially improved sample efficiency.

The specific assumption we make in this section is that the model parameters of an individual MDP $M_c$ is the linear combination of the parameters of $d$ base MDPs, i.e.,

$$

\begin{align*}
    p^c(s'|s, a) &= c^\top \begin{bmatrix} p_1(s'|s, a) \\ \vdots \\ p_d(s'|s, a) \end{bmatrix} := c^\top P(s, a, s'), \\
    r^c(s, a) &= c^\top \begin{bmatrix} r_1(s, a) \\ \vdots \\ r_d(s, a) \end{bmatrix} := c^\top R(s, a).
\end{align*}

\tag{4}

We use $P(s, a, s')$ and $R(s, a)$ as shorthand for the $d \times 1$ vectors that concatenate the parameters from different base MDPs for the same $s, a$ (and $s'$). The parameters of the base MDPs ($p_i$ and $r_i$) are unknown and need to be recovered from data by the learning process.

\[4\] The reward function does not vary with context hence reward smoothness is satisfied for all $L_r \geq 0$. The proof of the claim is deferred to the appendix.
agent, and the combination coefficients are directly available which is the context vector $c$ itself. This assumption can be motivated in an application scenario where the user/patient responds according to her characteristic distribution over $d$ possible behavioural patterns.

A mathematical difficulty here is that for an arbitrary context vector $c \in \mathbb{R}^d$, $p^c(\cdot | \cdot, \cdot)$ is not always a valid transition function and may violate non-negativity and normalization constraints. Therefore, we require that $c \in \Delta_{d-1}$, that is, $c$ stays in the probability simplex so that $p^c(\cdot | \cdot, \cdot)$ is always valid.\(^5\)

### 4.1 KWIK_LR-Rmax

We first explain how to estimate the model parameters in this linear setting, and then discuss how to perform exploration properly.

**Model estimation** Recall that in Section 3, we treat the MDPs whose contexts fall in a small ball as a single MDP, and estimate its parameters using data from the local context ball. In this section, however, we have a global structure due to our parametric assumption (base MDPs that are shared across all contexts). This implies that data obtained at a context may be useful for learning the MDP parameters at another context that is far away, and to avoid the exponential dependence on $d$ we need to leverage this structure and generalize globally across the entire context space.

Due to the linear combination setup, we use linear regression to replace the estimation procedure in Equation 2: in an episode with context $c$, when we observe the state-action pair $(s,a)$, a next-state $s_{\text{next}}$ will be drawn from $p^c(\cdot | s,a)$.\(^6\) Therefore, the indicator of whether $s_{\text{next}}$ is equal to $s'$ forms an unbiased estimate of $p^c(s' | s,a)$, i.e., $\mathbb{E}_{s_{\text{next}} \sim p^c(\cdot | s,a)} [\mathbb{I}[s_{\text{next}} = s']] = p^c(s'|s,a) = c^\top P(s,a,s')$. Based on this observation, we can construct a feature-label pair

$$\left( c, \mathbb{I}[s_{\text{next}} = s'] \right) \quad (5)$$

to recover $P(s,a,s')$ from data, we can simply collect the feature-label pairs that correspond to this particular $(s,a,s')$ tuple, and run linear regression to recover the coefficients. The case for reward function is similar, hence, not discussed.

If the data is abundant (i.e., $(s,a)$ is observed many times) and exploratory (i.e., the design matrix that consists of the $c$ vectors for $(s,a)$ is well-conditioned), we can expect to recover $P(s,a,s')$ accurately. But how to guarantee these conditions? Since the context is chosen adversarially, the design matrix can indeed be ill-conditioned.

Observe, however, when the matrix is ill-conditioned and new contexts lie in the subspace spanned by previously observed contexts, we can make accurate predictions despite the inability to recover the model parameters. An online linear regression (LR) procedure will take care of this issue, and we choose KWIK_LR (Walsh et al., 2009) as such a procedure.

The original KWIK_LR deals with scalar labels, which can be used to decide whether the estimate of $p^c(s'|s,a)$ is sufficiently accurate (known). A $(s,a)$ pair then becomes known if $(s,a,s')$ is known for all $s'$. This approach, however, generally leads to a loose analysis,

\(^5\) $\Delta_n$ is the $n$-simplex $\{ x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 1, x \geq 0 \}$.

\(^6\) Here we use $s_{\text{next}}$ to denote the random variable, and $s'$ to denote a possible realization.
because there is no need to predict $p^*(s'|s, a)$ for each individual $s'$ accurately: if the estimate of $p^*(s'|s, a)$ is close to the true distribution under $L_1$ error, the $(s, a)$ pair can already be considered as known. We extend the KWIK\_LR analysis to handle vector-valued outputs, and provide tighter error bounds by treating $p^*(s'|s, a)$ as a whole. Below we introduce our extended version of KWIK\_LR, and explain how to incorporate the knownness information in Rmax skeleton to perform efficient exploration.

**Identifying known $(s, a)$ with KWIK\_LR**

The KWIK\_LR-Rmax algorithm we propose for the linear setting still uses Rmax template (Algorithm 1) for exploration: in every episode we build the induced MDP $M_K$, and act greedily according to its optimal policy with balanced wandering. The major difference from Cover-Rmax lies in how the set of known states $K$ is identified and how $M_K$ is constructed, which we explain below (see pseudocode in Algorithm 3).

At a high level, the algorithm works in the following way: when constructing $M_K$, we query the KWIK procedure for estimates $\hat{p}^*(\cdot|s, a)$ and $\hat{c}^*(s, a)$ for every pair $(s, a)$ using $\text{Predict}(c, s, a)$. The KWIK procedure either returns $\bot$ (don’t know), or returns estimates that are guaranteed to be accurate. If $\bot$ is returned, then we consider $(s, a)$ as unknown and associate $s$ with $R_{\text{max}}$ reward for exploration. Such optimistic exploration ensures significant probability of observing $(s, a)$ pairs on which we have predicted $\bot$. If we do observe such pairs in an episode, we call $\text{Update}$ with feature-label pairs formed via Equation 5 to make progress on estimating parameters for unknown state-action pairs.

Next we walk through the pseudocode and explain how $\text{Predict}$ and $\text{Update}$ work in detail. Then we prove an upper bound on the number of updates that can happen (i.e., the if condition holds on Line 11), which forms the basis of our analysis of KWIK\_LR-Rmax.

In Algorithm 3, we initialize matrices $Q$ and $W$ for each $(s, a)$ using $\text{Initialize}()$ and update them over time. Let $C_t(s, a)$ be the design matrix at episode $t$, where each row is a context $c_t$ such that $(s, a)$ was observed in episode $\tau < t$. By matrix inverse rules, we can verify that the update rule on Line 10 essentially yields $Q_t(s, a) = (I + C_t^T C_t)^{-1}$, where $Q_t(s, a)$ is the value of $Q(s, a)$ in episode $t$. This is the inverse of the (unnormalized and regularized) empirical covariance matrix, which plays a central role in linear regression analysis. The matrix $W$ accumulates the outer product between the feature vector (context) $c$ and the one-hot vector label $y = (\{[s_{\text{next}} = s']\}_{s' \in S})^\top$. It is then obvious that $Q_t(s, a) W_t(s, a)$ is the linear regression estimate of $P(s, a)$ using the data up to episode $t$.

When a new input vector $c_t$ comes, we check whether $\|Q(s, a)c_t\|_2$ is below a predetermined threshold $\alpha_S$ (Line 5). Recall that $Q(s, a)$ is the inverse covariance matrix, so a small $\|Q(s, a)c_t\|_2$ implies that the estimate $Q_t(s, a) W_t(s, a)$ is close to $P(s, a)$ along the direction of $c_t$, so we predict $p^*(\cdot|s, a) = c_t^T P(s, a) \approx c_t^T Q(s, a) W(s, a)$; otherwise we return $\bot$. The KWIK subroutine for rewards is similar hence omitted. To ensure that the estimated transition probability is valid, we project the estimated vector onto $\Delta_{S-1}$, which can be done efficiently using existing techniques (Duchi et al., 2008).

Below we state the KWIK bound for learning the transition function; the KWIK bound for learning rewards is much smaller hence omitted here. We use the KWIK bound for scalar linear regression from Walsh et al. (2009) and the property of multinomial samples to get our KWIK bound.
Theorem 4.1 (KWIK_LR bound for learning multinomial vectors). For any \( \epsilon > 0 \) and \( \delta > 0 \), if the KWIK_LR algorithm is executed for probability vectors \( p_t(\cdot|s,a) \), with \( \alpha_S = \min\{b_1 \frac{\epsilon^2}{d^{3/2}}, b_2 \frac{\epsilon^2}{\sqrt{d \log(d^2/\delta)}} \} \), with suitable constants \( b_1 \) and \( b_2 \), then the number of \( \perp \)'s where updates take place (see Line 11) will be bounded by
\[
O\left( d^2 \epsilon^4 \max\{d^2, S^2 \log^2(d/\delta)\} \right),
\]
and, with probability at least \( 1 - \delta \), \( \forall c_t \) where a non-“\( \perp \)’’ prediction is returned, \( \|\hat{p}^c_t(\cdot|s,a) - p^c_t(\cdot|s,a)\|_1 \leq \epsilon \).

Proof sketch. (See full proof in the appendix.) We provide a direct reduction to KWIK bound for learning scalar values. The key idea is to notice that for any vector \( v \in \mathbb{R}^S \):
\[
\|v\|_1 = \sup_{f \in \{-1, 1\}^S} v^T f.
\]
So conceptually we can view Algorithm 3 as running \( 2^S \) scalar linear regression simultaneously, each of which projects the vector label to a scalar by a fixed linear transformation \( f \). We require every scalar regressor to have \( (\epsilon, \delta/2^S) \) KWIK guarantee, and the \( \ell_1 \) error guarantee for the vector label follows from union bound.

Algorithm 3: KWIK learning of \( p^c(\cdot|s,a) \)

1. Function Initialize\((S, d, \alpha_S)\)
   2. \( Q(s,a) \leftarrow I_d \) for all \((s,a)\);
   3. \( W(s,a) \leftarrow \{0\}^{d \times S} \) for all \((s,a)\);
2. Function Predict\((c, s, a)\)
   5. if \( \|Q(s,a)c\|_1 \leq \alpha_S \) then
   6. return \( \hat{p}^c(\cdot|s,a) = c^T Q(s,a) W(s,a) \);
   7. else
   8. return \( \hat{p}^c(\cdot|s,a) = \perp \);
3. Function Update\((c, s, a, s_{next})\)
   11. if \( \|Q(s,a)c\|_1 > \alpha_S \) (“\( \perp \)’’ prediction) then
   12. \( Q(s,a) \leftarrow Q(s,a) - \frac{Q(s,a)c}{1 + c^T Q(s,a)c} \); 
   13. \( y \leftarrow (\{I[s_{next} = s']\}_{s' \in S})^T \);
   14. \( W(s,a) \leftarrow W(s,a) + cy \);
4. end

With this result, we are ready to prove the formal PAC guarantee for KWIK_LR-Rmax.

Theorem 4.2 (PAC bound for KWIK_LR-Rmax). For any input values \( 0 < \epsilon, \delta \leq 1 \) and a linear CMDP model with \( d \) number of base MDPs, with probability \( 1 - \delta \), the KWIK_LR-Rmax algorithm, produces a sequence of policies \( \{\pi_t\} \) which yield at most
\[
O\left( \frac{d^2 H^4 S A}{\epsilon^5} \log \frac{1}{\delta} \max\{d^2, S^2 \log^2(d S A)\} \right)
\]
non-\( \epsilon \)-optimal episodes.
Proof. When the KWIK subroutine (Algorithm 3) makes non-"⊥" predictions \( \hat{p}^c(s,a,s') \), we require that
\[
\| \hat{p}^c(\cdot|s,a) - p^c(\cdot|s,a) \|_1 \leq \epsilon/8H.
\]
After projection onto \( \Delta_{S-1} \), we have:
\[
\| \Pi_{\Delta_{S-1}}(\hat{p}^c(s,a)) - p^c(\cdot|s,a) \|_1 \leq 2\| \hat{p}^c(\cdot|s,a) - p^c(\cdot|s,a) \|_1 \leq \epsilon/4H.
\]
Further, the update to the matrices \( Q \) and \( W \) happen only when an unknown state action pair \((s,a)\) is visited and the KWIK subroutine still predicts \( \perp \) (Line 10). The KWIK bound states that after a fixed number of updates to an unknown \((s,a)\) pair, the parameters will always be known with desired accuracy. The number of updates \( m \) can be obtained by setting the desired accuracy in transitions to \( \epsilon/8H \) and failure probability as \( \delta/SA \) in Theorem 4.1:
\[
m = O\left(\frac{d^2H^4}{\epsilon^4} \max\{d^2, S^2 \log^2(\frac{dSA}{\delta})\}\right)
\]
We now use Lemma 3.3 where instead of updating counts for number of visits, we look at the number of updates for unknown \((s,a)\) pairs. On applying a union bound over all state action pairs and using Lemma 3.3, it is easy to see that the sub-optimal episodes are bounded by \( O\left(\frac{mSA}{\epsilon} \ln \frac{1}{\delta}\right) \) with probability at least \( 1 - \delta \). The bound in Theorem 4.2 is obtained by substituting the value of \( m \).

We see that for this contextual MDP, the linear structure helps us in avoiding the exponential dependence in context dimension \( d \). The combined dependence on \( S \) and \( d \) is now \( O(\max\{d^4S, d^2S^3\}) \).

5. Related work

**Transfer in RL with latent contexts** The general definition of CMDPs captures the problem of transfer in RL and multi-task RL. See Taylor and Stone (2009) and Lazaric (2011) for surveys of empirical results. Recent papers have also advanced the theoretical understanding of transfer in RL. For instance, Brunskill and Li (2013) and Hallak et al. (2015) analyzed the sample complexity of CMDPs where each MDP is an element of a finite and small set of MDPs, and the MDP label is treated as the latent (i.e., unseen) context. Mahmud et al. (2013) consider the problem of transferring the optimal policies of a large set of known MDPs to a new MDP. The commonality of the above papers is that the MDP label (i.e., the context) is not observed. Hence, their methods have to initially explore in every new MDP to identify its label, which requires the episode length to be substantially longer than the planning horizon. This can be a problematic assumption in our motivating scenarios, where we interact with a patient / user / student for a limited period of time and the data in a single episode (whose length \( H \) is the planning horizon) is not enough for identifying the underlying MDP. In contrast to prior work, we propose to leverage observable context information to perform more direct transfer from previous MDPs, and our algorithm works with arbitrary episode length \( H \).
RL with side information  Our work leverages the available side-information for each MDP, which is inspired by the use of contexts in contextual bandits (Langford and Zhang, 2008; Li et al., 2010). The use of such side information can also be found in RL literature: Ammar et al. (2014) developed a multi-task policy gradient method where the context is used for transferring knowledge between tasks; Killian et al. (2016) used parametric forms of MDPs to develop models for personalized medicine policies for HIV treatment.

RL in metric space  For smooth CMDPs (Section 3), we pool observations across similar contexts and reduce the problem to learning policies for finitely many MDPs. An alternative approach is to consider an infinite MDP whose state representation is augmented by the context, and apply PAC-MDP methods for metric state spaces (e.g., C-PACE proposed by Pazis and Parr (2013)). However, doing so might increase the sample and computational complexity unnecessarily, because we no longer leverage the structure that a particular component of the (augmented) state, namely the context, remains the same in an episode. Concretely, the augmenting approach needs to perform planning in the augmented MDP over states and contexts, which makes its computational/storage requirement worse than our solution: we only perform planning in MDPs defined on $S$, whose computational characteristics have no dependence on the context space. In addition, we allow the context sequence to be chosen in an adversarial manner. This corresponds to adversarially chosen initial states in MDPs, which is usually not handled by PAC-MDP methods.

KWIK learning of linear hypothesis classes  Our linear combination setting (Section 4) provides an instance where parametric assumptions can lead to substantially improved PAC bounds. We build upon the KWIK-Rmax learning framework developed in previous work (Li et al., 2008; Szita and Szepesvári, 2011) and use KWIK linear regression as a sub-routine. For the resulting KWIK_LR-Rmax algorithm, its sample complexity bound inherently depends on the KWIK bound for linear regression. It is well known that even for linear hypothesis classes, the KWIK bound is exponential in input dimension in the agnostic case (Szita and Szepesvári, 2011). Therefore, the success of the algorithm relies on the validity of the modelling assumption.

Abbasi-Yadkori and Neu (2014) studied a problem similar to our linear combination setting, and proposed a no-regret algorithm by combining UCRL2 (Jaksch et al., 2010) with confidence set techniques from stochastic linear optimization literature (Dani et al., 2008; Filippi et al., 2010). Our work takes an independent and very different approach, and we provide a PAC guarantee which is not directly comparable to regret bound. Still, we observe that our dependence on $A$ is optimal for PAC whereas theirs is not ($\sqrt{A}$ is optimal for bandit regret analysis and they have $A$); on the other hand, their dependence on $T$ (the number of rounds) is optimal, and our dependence on $1/\epsilon$, its counterpart in PAC analysis, is suboptimal. It is an interesting future direction to combine the algorithmic ideas from both papers to improve the guarantees.

6. Conclusion

In this paper, we present a general setting of using side information for learning near-optimal policies in a large and potentially infinite number of MDPs. The proposed Cover-Rmax algorithm is a model-based PAC-exploration algorithm for the case where MDPs vary smoothly
with respect to the observed side information. Our lower bound construction indicates
the necessary exponential dependence of any PAC algorithm on the context dimension in
a smooth CMDP. We also consider another instance with a parametric assumption, and
using a KWIK linear regression procedure, present the KWIK_LR-Rmax algorithm for ef-
cient exploration in linear combination of MDPs. Our PAC analysis shows a significant
improvement with this structural assumption.

The use of context based modelling of multiple tasks has rich application possibilities
in personalized recommendations, healthcare treatment policies, and tutoring systems. We
believe that our setting can possibly be extended to cover the large space of multi-task RL
quite well with finite/infinite number of MDPs, observed/latent contexts, and determin-
istic/noisy mapping between context and environment. We hope our work spurs further
research along these directions.

Acknowledgments

This work was supported in part by a grant from the Open Philanthropy Project to the
Center for Human-Compatible AI, and in part by NSF Grant IIS 1319365. Ambuj Tewari
acknowledges the support from NSF grant CAREER IIS-1452099 and Sloan Research Fel-
lowship. Any opinions, findings, conclusions, or recommendations expressed here are those
of the authors and do not necessarily reflect the views of the sponsors.

References

Yasin Abbasi-Yadkori and Gergely Neu. Online learning in mdps with side information.
*arXiv preprint arXiv:1406.6812*, 2014.

Haitham B Ammar, Eric Eaton, Paul Ruvolo, and Matthew Taylor. Online multi-task
learning for policy gradient methods. In *Proceedings of the 31st International Conference
on Machine Learning (ICML-14)*, pages 1206–1214, 2014.

Ronen I Brafman and Moshe Tennenholtz. R-max-a general polynomial time algorithm
for near-optimal reinforcement learning. *Journal of Machine Learning Research*, 3(Oct):
213–231, 2002.

Emma Brunskill and Lihong Li. Sample complexity of multi-task reinforcement learning.
*arXiv preprint arXiv:1309.6821*, 2013.

Varsha Dani, Thomas P Hayes, and Sham M Kakade. Stochastic linear optimization under
bandit feedback. In *COLT*, pages 355–366, 2008.

Christoph Dann and Emma Brunskill. Sample complexity of episodic fixed-horizon rein-
fforcement learning. In *Advances in Neural Information Processing Systems*, pages 2818–
2826, 2015.

John Duchi, Shai Shalev-Shwartz, Yoram Singer, and Tushar Chandra. Efficient projections
onto the l 1-ball for learning in high dimensions. In *Proceedings of the 25th international
category on Machine learning*, pages 272–279. ACM, 2008.
Sarah Filippi, Olivier Cappe, Aurélien Garivier, and Csaba Szepesvári. Parametric bandits: The generalized linear case. In *Advances in Neural Information Processing Systems*, pages 586–594, 2010.

Assaf Hallak, Dotan Di Castro, and Shie Mannor. Contextual markov decision processes. *arXiv preprint arXiv:1502.02259*, 2015.

Thomas Jaksch, Ronald Ortner, and Peter Auer. Near-optimal regret bounds for reinforcement learning. *Journal of Machine Learning Research*, 11(Apr):1563–1600, 2010.

Sham Machandranath Kakade. *On the sample complexity of reinforcement learning*. PhD thesis, 2003.

Michael Kearns and Satinder Singh. Near-optimal reinforcement learning in polynomial time. *Machine Learning*, 49(2-3):209–232, 2002.

Taylor Killian, George Konidaris, and Finale Doshi-Velez. Transfer learning across patient variations with hidden parameter markov decision processes. *arXiv preprint arXiv:1612.00475*, 2016.

John Langford and Tong Zhang. The epoch-greedy algorithm for multi-armed bandits with side information. In *Advances in neural information processing systems*, pages 817–824, 2008.

A. Lazaric. Transfer in reinforcement learning: a framework and a survey. In M. Wiering and M. van Otterlo, editors, *Reinforcement Learning: State of the Art*. Springer, 2011.

Lihong Li, Michael L Littman, and Thomas J Walsh. Knows what it knows: a framework for self-aware learning. In *Proceedings of the 25th international conference on Machine learning*, pages 568–575. ACM, 2008.

Lihong Li, Wei Chu, John Langford, and Robert E Schapire. A contextual-bandit approach to personalized news article recommendation. In *Proceedings of the 19th international conference on World wide web*, pages 661–670. ACM, 2010.

MM Mahmud, Majd Hawasly, Benjamin Rosman, and Subramanian Ramamoorthy. Clustering markov decision processes for continual transfer. *arXiv preprint arXiv:1311.3959*, 2013.

Jason Pazis and Ronald Parr. Pac optimal exploration in continuous space markov decision processes. In *AAAI*, 2013.

Aleksandrs Slivkins. Contextual bandits with similarity information. *Journal of Machine Learning Research*, 15(1):2533–2568, 2014.

Alexander L Strehl and Michael L Littman. An analysis of model-based interval estimation for markov decision processes. *Journal of Computer and System Sciences*, 74(8):1309–1331, 2008.

Alexander L Strehl, Lihong Li, and Michael L Littman. Reinforcement learning in finite mdps: Pac analysis. *Journal of Machine Learning Research*, 10(Nov):2413–2444, 2009.
István Szita and Csaba Szepesvári. Agnostic kwik learning and efficient approximate reinforcement learning. In Proceedings of the 24th Annual Conference on Learning Theory, pages 739–772, 2011.

Matthew E Taylor and Peter Stone. Transfer learning for reinforcement learning domains: A survey. Journal of Machine Learning Research, 10(Jul):1633–1685, 2009.

Thomas J Walsh, István Szita, Carlos Diuk, and Michael L Littman. Exploring compact reinforcement-learning representations with linear regression. In Proceedings of the Twenty-Fifth Conference on Uncertainty in Artificial Intelligence, pages 591–598. AUAI Press, 2009. A corrected version is available as Technical Report DCS-tr-660, Department of Computer Science, Rutgers University, December, 2009.

Yihong Wu. Packing, covering, and consequences on minimax risk. EECS598: Information-theoretic methods in high-dimensional statistics, Statistics, Yale University, 2016.
Appendix A. Proofs from Section 3

A.1 Proof of Lemma 3.3

We adapt the analysis in Kakade (2003) for the episodic case which results in the removal of a factor of $H$, since complete episodes are counted as mistakes and we do not count every sub-optimal action in an episode. We reproduce the detailed analysis here for completion. For completing the proof of Lemma 3.3, firstly, we will look at a version of simulation lemma from Kearns and Singh (2002). Also, for the complete analysis we will assume that the rewards lie between 0 and 1.

**Definition A.1 (Induced MDP).** Let $M$ be an MDP with $K \subseteq S$ being a subset of states. Given, such a set $K$, we define an induced MDP $M_K$ in the following manner. For each $s \in K$, define the values

$$p_{MK}(s'|s,a) = p_M(s'|s,a)$$
$$r_{MK}(s,a) = r_M(s,a)$$

For all $s \notin K$, define $p_{MK}(s'|s,a) = \mathbb{1}\{s' = s\}$ and $r_{MK}(s,a) = 1$.

**Lemma A.2 (Simulation lemma for episodic MDPs).** Let $M$ and $M'$ be two MDPs with the same state-action space. If the transition dynamics and the reward functions of the two MDPs are such that

$$\|p_M(\cdot|s,a) - p_{M'}(\cdot|s,a)\|_1 \leq \epsilon_1 \forall s \in S, a \in A$$
$$|r_M(s,a) - r_{M'}(s,a)| \leq \epsilon_2 \forall s \in S, a \in A$$

then, for every (non-stationary) policy $\pi$ the two MDPs satisfy this property:

$$|V^\pi_M - V^\pi_{M'}| \leq \epsilon_2 + H\epsilon_1$$

**Proof.** Consider $T_h$ to be the set of all trajectories of length $h$ and let $P_M^\pi(\tau)$ denote the probability of observing trajectory $\tau$ in $M$ with the behaviour policy $\pi$. Further, let $U_M(\tau)$ the expected average reward obtained for trajectory $\tau$ in MDP $M$.

$$|V^\pi_M - V^\pi_{M'}| = \left| \sum_{\tau \in T_h} [P^\pi_M(\tau)U_M(\tau) - P^\pi_{M'}(\tau)U_{M'}(\tau)] \right|$$
$$\leq \left| \sum_{\tau \in T_h} [P^\pi_M(\tau)U_M(\tau) - P^\pi_M(\tau)U_{M'}(\tau) + P^\pi_M(\tau)U_{M'}(\tau) - P^\pi_{M'}(\tau)U_{M'}(\tau)] \right|$$
$$\leq \left| \sum_{\tau \in T_h} [P^\pi_M(\tau)(U_M(\tau) - U_{M'}(\tau))] \right| + \left| \sum_{\tau \in T_h} [U_{M'}(\tau)(P^\pi_M(\tau) - P^\pi_{M'}(\tau))] \right|$$
$$\leq \left| \sum_{\tau \in T_h} P^\pi_M(\tau)|\epsilon_2 + \left| \sum_{\tau \in T_h} [P^\pi_M(\tau) - P^\pi_{M'}(\tau)] \right|$$

The bound for the second term follows from the proof of lemma 8.5.4 in Kakade (2003). Combining the two expressions, we get the desired result. \qed
Lemma A.3 (Induced inequalities). Let $M$ be an MDP with $K$ being the set of known states. Let $M_K$ be the induced MDP as defined in A.1 with respect to $K$ and $M$. We will show that for any (non-stationary) policy $\pi$, all states $s \in S$, 

$$V^\pi_{M_K}(s) \geq V^\pi_M(s)$$

and 

$$V^\pi_M(s) \geq V^\pi_{M_K}(s) - P^\pi_M[\text{Escape to an unknown state}|s_0 = s]$$

where $V^\pi_M(s)$ denotes the value of policy $\pi$ in MDP $M$ when starting from state $s$.

Proof. See Lemma 8.4.4 from Kakade (2003).

Corollary A.4 (Implicit Explore and Exploit). Let $M$ be an MDP with $K$ as the set of known states and $M_K$ be the induced MDP. If $\pi^*_M$ and $\pi^*_K$ be the optimal policies for $M\_K$ and $M$ respectively, we have for all states $s$:

$$V^*_{M_K}(s) \geq V^*_{M}(s) - P^*_{M}[\text{Escape to an unknown state}|s_0 = s]$$

Proof. Follows from Lemma 8.4.5 from Kakade (2003).

Proof of Lemma 3.3. Let $\pi^*_M$ be the optimal policy for $M$. Also, using the assumption about $m$, we have an $\epsilon/2$-approximation of $M_K$ as the MDP $\hat{M}_K$. Rmax computes the optimal policy for $\hat{M}_K$ which is denoted by $\hat{\pi}_M$. Then, by Lemma A.2,

$$V^\hat{\pi}_{\hat{M}_K}(s) \geq V^\hat{\pi}_{\hat{M}}(s) - \epsilon/2$$

$$\geq V^{\pi_M}_{M_K}(s) - \epsilon/2$$

$$\geq V^{\pi^*_{M_K}}_{M_K}(s) - \epsilon$$

Combining this with Lemma A.3, we get

$$V^\hat{\pi}_{\hat{M}}(s) \geq V^\hat{\pi}_{\hat{M}}(s) - P^\pi_{\hat{M}}[\text{Escape to an unknown state}|s_0 = s]$$

$$\geq V^{\pi^*_{M_K}}_{\hat{M}_K}(s) - \epsilon - P^\pi_{\hat{M}}[\text{Escape to an unknown state}|s_0 = s]$$

$$\geq V^{\pi^*_{M_K}}_{\hat{M}_K}(s) - \epsilon$$

If this escape probability is less than $\epsilon$, then the desired relation is true. Therefore, we need to bound the number of episodes where this expected number is greater than $\epsilon$. Note that, due to balanced wandering, we can have at most $mSA$ visits to unknown states for the Rmax algorithm. In the execution, we may encounter an extra $H - 1$ visits as the estimates are updated only after the termination of an episode.

Whenever this quantity is more than $\epsilon$, the expected number of exploration steps in $mSA/\epsilon$ such episodes is at least $mSA$. By the Hoeffding’s inequality, for $N$ episodes, with probability, at least $1 - \delta$, the number of successful exploration steps is greater than

$$N\epsilon - \sqrt{\frac{N}{2} \ln \frac{1}{\delta}}$$

Therefore, if $N = O\left(\frac{mSA}{\epsilon} \ln \frac{1}{\delta}\right)$, with probability at least $1 - \delta$, the total number of visits to an unknown state is more than $mSA$. Using the upper bound on such visits, we conclude that these many episodes suffice. \qed
A.2 Proof of Theorem 3.2

We now need to compute the required resolution of the cover and the number of transitions m which will guarantee the approximation for the value functions as required in the previous lemma. The following result is the key result:

**Lemma A.5 (Cover approximation).** For a given CMDP and a finite cover, i.e., \( C = \bigcup_{i=1}^{N(C,r)} B_i \) such that \( \forall i, \forall c_1, c_2 \in B_i \) :

\[
\|p^{c_1}(\cdot|s,a) - p^{c_2}(\cdot|s,a)\|_1 \leq \epsilon/8H
\]

and

\[
|r^{c_1}(s,a) - r^{c_2}(s,a)| \leq \epsilon/8
\]

if we visit every state-action pair \( m = 128(S\ln 2+\ln 8A)H^2 \) times in a ball \( B_i \) summing observations over all \( c \in B_i \), then, for any policy \( \pi \) and with probability at least \( 1 - 2\delta \), the approximate MDP \( \hat{M}_i \) corresponding to \( B_i \) computed using empirical averages will satisfy

\[
|V_{\hat{M}_i} - V_{\hat{M}_i}^\pi| \leq \epsilon/2
\]

for all \( c \in B_i \).

**Proof.** For each visit to a state action pair \( (s,a) \), we observe a transition to some \( s' \in S' \) for context \( c_t \in B_i \) in \( t_{th} \) visit with probability \( p_{c_t}(s,a) \). Let us encode this by an \( S \)-dimensional vector \( I_t \) with 0 at all indices except \( s' \). After observing \( m \) such transitions, we create the estimate for any \( c \in B_i \) as \( p_{\hat{M}_i}(\cdot|s,a) = \frac{1}{m} \sum_{t=1}^{m} I_t \). Now for all \( c \in B_i \),

\[
\|p_{\hat{M}_i}(\cdot|s,a) - p^{\hat{\pi}}(\cdot|s,a)\|_1 \leq \|p_{\hat{M}_i}(\cdot|s,a) - \frac{1}{m} \sum_{t=1}^{m} p^{c_t}(\cdot|s,a)\|_1 + \epsilon/8H
\]

For bounding the first term, we use the Hoeffding’s bound:

\[
P\left[ \left\| p_{\hat{M}_i}(\cdot|s,a) - \frac{1}{m} \sum_{t=1}^{m} p^{c_t}(\cdot|s,a)\right\|_1 \geq \epsilon \right] = P\left[ \max_{s' \in A \subseteq S} (p_{\hat{M}_i}(s'|A,s,a) - \frac{1}{m} \sum_{t=1}^{m} p^{c_t}(s'|A,s,a)) \geq \epsilon/2 \right]
\]

\[
\leq \sum_{s' \in A \subseteq S} P\left[ (p_{\hat{M}_i}(s'|A,s,a) - \frac{1}{m} \sum_{t=1}^{m} p^{c_t}(s'|A,s,a)) \geq \epsilon/2 \right]
\]

\[
\leq (2^S - 2) \exp(-m\epsilon^2/2)
\]

Therefore, with probability at least \( 1 - \delta/2 \), for all \( s \in S, a \in A \), we have:

\[
\|p_{\hat{M}_i}(\cdot|s,a) - p^{\hat{\pi}}(\cdot|s,a)\|_1 \leq \sqrt{\frac{2(S\ln 2 + \ln 2SA/\delta)}{m}} + \epsilon/8H
\]

If \( m = \frac{128(S\ln 2+\ln 8A)H^2}{\epsilon^2} \), the error becomes \( \epsilon/4H \). One can easily verify using similar arguments that, the error in rewards for any context \( c \in B_i \) is less than \( \epsilon/4 \).

By using the simulation lemma A.2, we get the desired result. \( \square \)
Appendix B. Lower bound analysis

B.1 Proof of Claim 3.6

Once the instances at the packing points are assigned, the parameters for any other context $c \in C$, state $i$ and action $a$ are given by:

$$p_c(+|i, a) = \max_{c' \in Z} \max \{1/2, p_{c'}(+|i, a) - \phi(c, c')/2\}$$

We now prove that, with this definition, the smoothness requirements are satisfied:

**Claim B.1.** The contextual MDP defined above is a valid instance of a contextual MDP with smoothness constants $L_p = 1$. (The reward function does not vary with context hence reward smoothness is satisfied for all $L_r \geq 0$.)

**Proof.** We need to prove that the defined contextual MDP satisfies the constraints in Definition 3.1. Let us assume that the smoothness assumption is violated for a context pair $(c_1, c_2)$. The smoothness constraints for rewards are satisfied trivially for any value of $L_r$ as they are constant. This implies that there exists state $i \in [n]$ and action $a$ such that

$$||p_{c_1}(i, a) - p_{c_2}(i, a)||_1 > \phi(c_1, c_2)$$

$$\Rightarrow 2|p_{c_1}(+i, a) - p_{c_2}(+i, a)| > \phi(c_1, c_2)$$

We know that, $p_c(+|i, a) \in [1/2, 1/2 + \epsilon']$, which shows that $\phi(c_1, c_2) < 2\epsilon'$. Without loss of generality, assume $p_{c_1}(+|i, a) > p_{c_2}(+|i, a)$ which also leads to

$$p_{c_1}(+|i, a) > 1/2$$

$$\Rightarrow \exists c_0 \in Z \text{ such that } \phi(c_1, c_0) < 2\epsilon'$$

By triangle inequality, we have

$$\phi(c_2, c_0) < 4\epsilon'$$

Now, $\forall c_0 \in Z$, such that $\phi(c_0, c_0) \geq 8\epsilon'$, by triangle inequality, we have

$$\phi(c_0, c_1) > 6\epsilon'$$

$$\phi(c_0, c_2) > 4\epsilon'$$

This simplifies the definition of $p_c(+|i, a)$ for $c = c_1, c_2$ to

$$p_c(+|i, a) = \max(1/2, p_{c_0}(+|i, a) - \phi(c_0, c)/2)$$

Now,

$$|p_{c_1}(+|i, a) - p_{c_2}(+|i, a)| = p_{c_1}(+|i, a) - p_{c_2}(+|i, a)$$

$$= p_{c_0}(+|i, a) - \phi(c_0, c_1)/2 - \max(1/2, p_{c_0}(+|i, a) - \phi(c_0, c_2)/2)$$

$$\leq p_{c_0}(+|i, a) - \phi(c_0, c_1)/2 - (p_{c_0}(+|i, a) - \phi(c_0, c_2)/2)$$

$$= \frac{1}{2}(\phi(c_0, c_2) - \phi(c_0, c_1)) \leq \phi(c_1, c_2)/2$$

which leads to a contradiction. \qed
B.2 Lower bound for smooth CMDP

In the lower bound construction, we made a key argument that the defined contextual mapping with specific instances, requires any agent to learn the models of a significant number of MDPs separately to get decent generalization. The construction populates a set of packing points in the context space with hard MDPs and argues that these instances are independent of each other from the algorithm’s perspective. To formalize this statement, let \( Z \) be the \( 8\epsilon' \)-packing as before. The adversary makes the choices of the instances at each context \( c \in Z \), as follows: Select an MDP from the family of hard instances described in Figure 1 where the optimal action from each state in \( \{1, \ldots, n\} \) is chosen randomly and independently from the other assignments. The parameter \( \epsilon' \) deciding the difference in optimality of actions in Figure 1 is taken as \( \frac{160H\epsilon e^4}{(H-2)} \) which is required by the construction of Theorem 3.5.

We denote these instances by the set \( \mathcal{I} \) and an individual instance by \( I_z \) where \( z = \{z_1, z_2, \ldots, z_{|Z|}\} \) is the set of these random choices made by the adversary. By construction, we have a uniform distribution \( \Gamma \equiv \Gamma_1 \times \Gamma_2 \times \ldots \Gamma_{|Z|} \) over these possible set of instances \( \mathcal{I}_z \). From claim 5.3, any assignment of optimal actions to these packing points would define a valid smooth contextual MDP. Further, the independent choice of the optimal actions makes MDPs at each packing point at least as difficult as learning a single MDP. Formally, let the sequence of transitions and rewards observed by the learning agent for all packing points be \( T \equiv \{\tau_1, \tau_2, \ldots, \tau_{|Z|}\} \).

From claim 5.3, any assignment of optimal actions to these packing points would define a valid smooth contextual MDP. Further, the independent choice of the optimal actions makes MDPs at each packing point at least as difficult as learning a single MDP. Formally, let the sequence of transitions and rewards observed by the learning agent for all packing points be \( T \equiv \{\tau_1, \tau_2, \ldots, \tau_{|Z|}\} \). Due to the independence between individual instances, we can see that:

\[
P_{T}[\tau_1|\tau_2, \tau_3, \ldots, \tau_{|Z|}] = P_{T_1}[\tau_1]
\]

where \( P_{T}[\tau_i] \) denotes the distribution of trajectories \( \tau_i \). Thus, observing trajectories from MDP instances at other packing points does not let the algorithm deduce anything about the nature of the instance chosen for one point. With respect to this distribution, learning in the contextual MDP is equivalent to or worse than simulating a single MDP learning algorithm at each of these packing points. For any given contextual MDP algorithm \( \text{Alg} \), we have:

\[
E_{\Gamma}[B_i] = E_{\Gamma_{-i}}[E_{\Gamma_1, \text{Alg}}[B_i|T_{-i}]] \geq E_{\Gamma_{-i}}[E_{\Gamma_1, \text{Alg}}*[B_i]]
\]

where \( \text{Alg}^* \) is an optimal single MDP learning algorithm. The expectation is with respect to the distribution over the instances \( \mathcal{I}_z \) and the algorithm’s randomness. From Theorem 3.5, we can lower bound the expectation on the right hand side of the inequality by \( \Omega\left( \frac{SA}{\epsilon'^2} \right) \).

The total number of mistakes is lower bounded as:

\[
E_{\Gamma}\left| \sum_{i=1}^{\frac{|Z|}{S}} B_i \right| = \sum_{i=1}^{\frac{|Z|}{S}} E_{\Gamma}[B_i] \geq \Omega\left( \frac{|Z|SA}{\epsilon'^2} \right)
\]

Setting \( |Z| = D(C, \epsilon_1) \leq N(C, \epsilon_1) \) gives the stated lower bound with \( \epsilon_1 = 8\epsilon' \).

Appendix C. Proof of the Theorem 4.1

In this section, we present the proof of our KWIK bound for learning transition probabilities. Our proof uses a reduction technique that reduces the vector-valued label setting to the scalar setting, and combines the KWIK bound for scalar labels given by Walsh et al. (2009).
Proof. Fix a state action pair \((s, a)\). Consider a sequence of contexts \(c_1, c_2, \ldots\) for which the transitions were observed for pair \((s, a)\). Given a new context \(c\), we want to estimate:

\[
p^c(\cdot|s, a) = c^T P(s, a)
\]

In our KWIK\_LR algorithm, we estimate this as:

\[
\hat{p}^c(\cdot|s, a) = c^T Q(s, a)W(s, a)
\]

where \(Q(s, a)\) and \(W(s, a)\) are as described in Section 4.1.

We wish to bound the \(\ell_1\) error between \(\hat{p}^c(\cdot|s, a)\) and \(p^c(\cdot|s, a)\) for all \(c\) for which a prediction is made. We know that

\[
\|p^c(\cdot|s, a) - \hat{p}^c(\cdot|s, a)\|_1 = \sup_{f \in \{-1, 1\}^S} (p^c(\cdot|s, a) - \hat{p}^c(\cdot|s, a))f.
\]

We use this representation of \(\ell_1\)-norm to prove a tighter KWIK bound for learning transition probabilities. For every fixed \(f \in \{-1, 1\}^S\), we formulate a new linear regression problem with feature-label pair:

\((c, yf)\).

Recall that \(y = (\{I[s_{next} = s']\}_{s' \in S})^T\) is the vector label of real interest, and \(f\) projects \(y\) to a scalar value. Algorithm 3 can be viewed as implicitly running this regression thanks to linearity: since \(Q\) only depends on input contexts and \(W\) is linear in \(y\), \(\hat{p}^c(\cdot|s, a)f\) is simply equal to the linear regression prediction for the problem \((c, yf)\). As a result, the KWIK bound for the problem \((c, yf)\) (which we establish below) automatically applies as a property of \(\hat{p}^c(\cdot|s, a)\) thanks to Equation 6.

Now we establish the KWIK guarantee for the new regression problem. The groundtruth (expected) label is

\[
p^c(\cdot|s, a)f = c^T (P(s, a)f) := c^T \theta f.
\]

The noise in the label is then

\[
\eta f := (y - p^c(\cdot|s, a))f.
\]

This noise has zero-mean and constant magnitude: \(\|\eta f\| \leq \|y - p^c(\cdot|s, a)\|_1\|f\|_\infty \leq 2\).

With the above conditions, we can invoke the KWIK bound for scalar linear regression from Walsh et al. (2009):

**Theorem C.1** (KWIK bound for linear regression(Walsh et al., 2009)). Suppose the observation noise in a noisy linear regression problem has zero-mean and its absolute value is bounded by \(\beta\). Let \(M\) be an upper bound on the \(\ell_2\) norm of the true linear coefficients. For any \(\delta' > 0\) and \(\epsilon > 0\), if the KWIK linear regression algorithm is executed with \(\alpha_0 = \min\{b_1 \frac{\beta^2}{dM}, b_2 \frac{\beta^2 \log(d/M)}{dM}\}\), with suitable constants \(b_1\) and \(b_2\), then the number of \(\perp\)'s will be \(O(M^2 \max\{\frac{d^3}{\epsilon^2}, \frac{d \log^2(d/\delta')}{\epsilon^4}\})\), and with probability at least \(1 - \delta'\), for each sample \(x_t\) for which a prediction is made, the prediction is \(\epsilon\)-accurate.
For our purpose, $\beta = 2$ as $|\eta^f| \leq 2$ and $M = \sqrt{d}$ as $\|\theta^f\|_2 = \|P(s, a)f\|_2 \leq \sqrt{d}$.

Now set $\delta' = \frac{\delta}{2^S}$ in Theorem C.1. In the KWIK linear regression algorithm, the known status for a context $c$ is checked in the same manner as done in Line 5 in Algorithm 3. Therefore

$$
Pr\left[\|p^c(\cdot|s, a) - \hat{p}^c(\cdot|s, a)\|_1 \geq \epsilon\right] = Pr\left[\sup_{f \in \{-1, 1\}^S} (p^c(\cdot|s, a) - \hat{p}^c(\cdot|s, a))f \geq \epsilon\right] \quad \text{(Equation 6)}
$$

$$
\leq \sum_{f \in \{-1, 1\}^S} Pr\left[(p^c(\cdot|s, a) - \hat{p}^c(\cdot|s, a))f \geq \epsilon\right] \quad \text{(union bound)}
$$

$$
= \sum_{f \in \{-1, 1\}^S} Pr\left[c^\top \theta^f - c^\top Q(s, a)(W(s, a)f) \geq \epsilon\right] \quad \text{(regression w.r.t. } f \text{ implicitly run)}
$$

$$
\leq \sum_{f \in \{-1, 1\}^S} \delta/2^S = \delta.
$$

Substituting the values of $M$ and $\delta'$ in Theorem C.1, we get:

$$
\alpha_S = \min\{b_1 \frac{\epsilon^2}{d^{3/2}}, b_2 \frac{\epsilon^2}{\sqrt{d} \log(d/2^S \delta')} \frac{\epsilon}{2\sqrt{d}}\}
$$

and number of $\bot$’s is bounded as

$$
O(\max\{\frac{d^2 S^2 \log^2(d/\delta')}{\epsilon^4} \frac{\epsilon}{\epsilon^4}\}).
$$