New Results on the “a-theorem” in Four Dimensional Supersymmetric Field Theory

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In four dimensional $N = 1$ supersymmetric field theory it is often the case that the $U(1)_R$ current that becomes part of the superconformal algebra at the infrared fixed point is conserved throughout the renormalization group (RG) flow. We show that when that happens, the central charge $a$ decreases under RG flow. The main tool we employ is an extension of recent ideas on “$a$-maximization” away from fixed points of the RG. This extension is useful more generally in studying RG flows in supersymmetric theories.
1. Introduction

A Quantum Field Theory (QFT) is traditionally defined as an ultraviolet (UV) fixed point connected to an infrared (IR) fixed point by an RG flow. At the UV fixed point, which describes the very short distance physics, the theory is scale invariant, and in all known examples conformal. A non-trivial RG flow is typically induced by perturbing the UV fixed point by adding to the Lagrangian a relevant\(^1\) operator. In the presence of the perturbation, the short distance behavior of correlation functions is still described by the UV conformal field theory (CFT), while at long distances the theory flows to an IR fixed point, where it becomes conformal again.

RG flow is in general irreversible. For example, if CFT \( B \) is obtained by perturbing CFT \( A \) by a relevant operator and going to long distances, one cannot (in all known examples) get back from \( B \) to \( A \) by perturbing \( B \) by a relevant operator. Therefore, it is natural to ask whether one can define an intrinsic characteristic of fixed points which keeps track of this irreversibility – a real number \( \mathcal{M} \) associated with each CFT which has the property that if CFT’s \( A \) and \( B \) are connected by an RG flow and \( \mathcal{M}(A) > \mathcal{M}(B) \), it must be that \( A \) is the UV fixed point of the flow and \( B \) the IR one, and vice versa. It would be nice if such an \( \mathcal{M} \) also counted the “number of degrees of freedom” of the fixed point, due to the intuition that RG flow proceeds by decoupling of high energy degrees of freedom as the distance scale is increased. A necessary condition for such an interpretation seems to be that \( \mathcal{M}(A) \) should be positive for all\(^2\) CFT’s \( A \).

In two dimensional QFT it was shown in \[1\] that one can choose \( \mathcal{M} \) to be the Virasoro central charge \( c \). The Zamolodchikov \( c \)-theorem states that one can define a quantity with the following properties:

1. It is positive and monotonically decreasing throughout the RG flow.
2. At the UV and IR fixed points it coincides with the central charge of the relevant CFT, \( c \).

Moreover, at fixed points of the RG one can think of the central charge as counting the number of degrees of freedom of the theory \[2\].

Despite much work (see e.g. \[3\]-\[12\]), an analog of the \( c \)-theorem in four dimensional QFT has not been proven so far. At the same time, increasing evidence has been accumulating for the conjecture \[3\], often referred to as the “\( a \)-theorem,” that the analog of \( c \) in

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1. Here and below “relevant” includes “marginally relevant,” as in asymptotically free gauge theories.
2. We restrict attention to unitary theories.
four dimensions is the central charge $a$, the coefficient of the Euler density in the conformal anomaly on a curved spacetime manifold. By studying examples of RG flows for which the UV and IR fixed points are sufficiently well understood to compute the value of $a$, it was found that in all cases $a_{UV} > a_{IR}$.

Most of the examples alluded to above correspond to supersymmetric field theories. One reason for that is that our understanding of four dimensional supersymmetric field theories is significantly better than that of non-supersymmetric ones, and in many cases we now understand the infrared behavior of these theories well enough to compute $a$. Another reason is that in supersymmetric theories the conformal anomaly that gives rise to $a$ is related by supersymmetry to an anomaly associated with the $U(1)_R$ current that belongs to the $N = 1$ superconformal multiplet \[8\]. Therefore, if we can identify the superconformal $U(1)_R$ current at the UV and IR fixed points of a supersymmetric RG flow, we can check whether the $a$-theorem is satisfied.

There are two basic tools that have been used to determine the superconformal $U(1)_R$ symmetry at strongly coupled fixed points. One is Seiberg duality \[13\], which relates the infrared behavior of different gauge theories. This duality often relates a theory that is strongly coupled in some variables to a theory which is weakly coupled, or free, in other variables.

The second tool, which is more important for the purpose of the present discussion, is the following. It has been known for a long time that in many interacting four dimensional QFT’s, the $U(1)_R$ that becomes part of the superconformal group at the IR fixed point is conserved throughout the RG flow, i.e. it is part of the symmetry group of the full theory. The symmetry of the full theory can be analyzed by studying the vicinity of the UV fixed point, where the dynamics is often simpler due to asymptotic freedom. If the symmetry group contains a unique $U(1)_R$ which satisfies the physical consistency conditions, one can identify it with the IR superconformal R-symmetry, and use 't Hooft anomaly matching to compute the central charge $a$.

A major stumbling block in implementing the second method of determining the superconformal R symmetry has been that in many cases there is more than one $U(1)_R$ that is preserved throughout the RG flow and satisfies all the necessary conditions. Progress on this problem was recently reported by Intriligator and Wecht (IW), who showed that there are in fact additional consistency conditions, that were not taken into account previously, which uniquely determine the superconformal R-charge in these cases \[14\]. Let $R$ be any...
of the global R-charges which are conserved throughout the RG flow. Compute the central charge $a$, which is given by a particular combination of 't Hooft anomalies

$$a = 3\text{tr}R^3 - \text{tr}R \ .$$  \ (1.1)

The fact that there is more than one possible choice of $U(1)_R$ implies that the anomaly $a$ depends on some continuous parameters, which parametrize the particular $U(1)_R$ whose anomaly is being computed. IW proved that the superconformal $U(1)_R$ of the infrared theory is the unique current that corresponds to a local maximum of $a$ (1.1) as a function of all the continuous parameters mentioned above.

The results of IW allow one to obtain a more detailed understanding of the infrared behavior and phase structure of many gauge theories that were previously mysterious. Recent investigations using these results \[15,16\] revealed a rich structure of fixed points and flows between them, consistent with previous work on Seiberg duality \[13,17-20\] and with the predictions of the $a$-theorem.

A natural question that arises from the results of \[14-16\] is whether they imply the $a$-theorem, at least within their domain of validity. In this note we will show that they do. More precisely, we will prove that if the $U(1)_R$ that becomes part of the superconformal algebra in the IR is preserved throughout the RG flow, then

$$a_{UV} > a_{IR} \ .$$  \ (1.2)

The main observation we will use is that the idea of $a$-maximization introduced in \[14\] can be extended away from fixed points of the RG, and used to construct a function on the space of field theories that monotonically decreases throughout RG flows and coincides with the usual central charge $a$ at fixed points. This extension has other uses as well, as we will demonstrate with a few examples.

To illustrate the method, in the next section we discuss the case of supersymmetric non-abelian gauge theory with vanishing superpotential. In section 3 we include the effect of superpotentials. Section 4 contains a discussion of some generalizations of the results of sections 2 and 3 to situations where some of the assumptions do not apply.

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3 After rescaling by a factor $32/3$. 

2. Supersymmetric gauge theories with vanishing superpotential

Consider a supersymmetric gauge theory with gauge group $G$ and chiral superfields $\Phi_i$ in the representations $r_i$ of the gauge group. One can choose a basis of generators of the gauge group in the representation $r$, $T^a$, such that

$$\text{Tr}_r(T^a T^b) = T(r) \delta^{ab} .$$ (2.1)

The invariant (2.1) corresponding to the adjoint representation will be denoted by $T(G)$. For example, for $G = SU(N_c)$, $T(\text{fundamental}) = 1/2$, $T(\text{adjoint}) = T(G) = N_c$.

The $\beta$-function for the gauge coupling $g$, or more conveniently for $\alpha = g^2 / 4\pi$ is [21,22]

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi} \left( 3T(G) - \sum_i T(r_i)(1 - \gamma_i(\alpha)) \right) .$$ (2.2)

Here $\gamma_i$ is the anomalous dimension of $\Phi_i$; at fixed points of the RG, the scaling dimension of $\Phi_i$ is given by

$$\Delta(\Phi_i) = 1 + \frac{1}{2} \gamma_i .$$ (2.3)

At weak coupling one has

$$\gamma_i(\alpha) = -\frac{\alpha}{\pi} C_2(r_i) + O(\alpha^2)$$ (2.4)

where

$$C_2(r) = \frac{|G|}{|r|} T(r) .$$ (2.5)

$|G|$ and $|r|$ are the dimensions of the group $G$ and the representation $r$, respectively. Eq. (2.2) implies that $\alpha$ is (marginally) relevant at weak coupling if the theory is asymptotically free,

$$3T(G) - \sum_i T(r_i) > 0 .$$ (2.6)

In this case $\alpha$ grows as the distance scale increases, and the theory approaches a non-trivial fixed point at long distances. At that fixed point, where $\alpha = \alpha^*$, one must have

$$3T(G) - \sum_i T(r_i)(1 - \gamma_i(\alpha^*)) = 0 ,$$ (2.7)

assuming, as we will do below, that $\alpha^*$ is sufficiently small so that the denominator of (2.2) never vanishes along the RG flow.
The condition (2.7) has a well known interpretation in terms of the R-charges at the IR fixed point of the flow. Using (2.3) and the relation between the scaling dimension $\Delta$ and superconformal $U(1)_R$ charge, $R$,

$$\Delta = \frac{3}{2} R \quad (2.8)$$

one can rewrite (2.7) as

$$T(G) + \sum_i T(r_i)(R_i(\alpha^*) - 1) = 0. \quad (2.9)$$

This equation is also the condition that the R-symmetry with $R(\Phi_i) = R_i(\alpha^*)$ be anomaly free and thus conserved throughout the RG flow labeled by $\alpha$. We conclude that supersymmetric gauge theory with any matter that satisfies the asymptotic freedom condition (2.6) has the property that the IR $U(1)_R$ is conserved throughout the RG flow, as long as the NSVZ $\beta$-function (2.2) is reliable.

In general, eq. (2.9) does not determine the R-charges at the IR fixed point uniquely, but since the IR $U(1)_R$ is a symmetry of the full theory, we can use the results of [14] to determine the $R_i$. The anomaly (1.1) takes in this case the form

$$a(R_i) = 2|G| + \sum_i |r_i| \left[ 3(R_i - 1)^3 - (R_i - 1) \right] \quad (2.10)$$

where the first contribution is due to the gauginos and the second to the quarks in the chiral multiplets. The IR superconformal $U(1)_R$ charges can be found by (locally) maximizing $a$, (2.10), subject to the constraint (2.9). The value of the central charge $a_{IR}$ is obtained by substituting the resulting R-charges into (2.10). To compute $a_{UV}$ one notes that at short distances the theory is free, and all the R-charges are given by their free field value, $R_i = 2/3$.

It is natural to ask whether the prediction of the $a$-theorem (1.2) is satisfied in this case. We will next show that (1.2) indeed holds under the assumptions stated above. In order to prove this, it is convenient to start with a generalization of (2.10) that takes into account the constraint (2.9):

$$a(R_i, \lambda) = 2|G| + \sum_i |r_i| \left[ 3(R_i - 1)^3 - (R_i - 1) \right] - \lambda \left[ T(G) + \sum_i T(r_i)(R_i - 1) \right]. \quad (2.11)$$

4 In a slight abuse of notation, we will denote this generalization by $a$ as well.
One can think of the new variable $\lambda$ as a Lagrange multiplier imposing the constraint (2.9). The procedure for finding the IR $U(1)_R$ described in the previous paragraph is equivalent to the following.

Find the local maximum of $a(R_i, \lambda)$ with respect to the $R_i$, keeping $\lambda$ fixed and arbitrary. Note, in particular, that all the $R_i$ are taken to be independent - we do not impose the constraint (2.9). At the local maximum, the $R_i$ are functions of $\lambda$. Substituting their values into (2.11), one finds $a = a(R_i(\lambda), \lambda)$. In order to impose the constraint (2.9) one now extremizes $a$ with respect to $\lambda$, by imposing

$$\frac{da(R_i(\lambda), \lambda)}{d\lambda} = 0.$$  

(2.12)

This fixes $\lambda$ and thus $R_i(\lambda)$. We will denote the value of $\lambda$ that solves (2.12) by $\lambda^*$. We will see below that $R_i(\lambda^*)$ satisfy the constraint (2.9); in general, the procedure described here is equivalent to maximizing (2.10) subject to the constraint (2.9).

So far we simply restated the original determination of the R-charges at the IR fixed point of a super-Yang-Mills (SYM) theory in a slightly different way. However, one might wonder whether there is some further information in the $\lambda$ dependence of the generalized central charge $a(R_i(\lambda), \lambda)$. An interesting fact is that for $\lambda = 0$, the generalized central charge (2.11) is simply that of free field theory without any gauge interaction, and maximizing it with respect to the $R_i$ gives the free field values $R_i = 2/3$. Thus, as we vary $\lambda$ between 0 and $\lambda^*$, the central charge $a(R_i(\lambda), \lambda)$ varies continuously between the UV, free field theory, value, and the IR value corresponding to the interacting fixed point.

It is now easy to prove that (1.2) holds in this case. One has

$$\frac{da(R_i(\lambda), \lambda)}{d\lambda} = \frac{\partial a}{\partial \lambda} + \sum_i \frac{\partial a}{\partial R_i} \frac{\partial R_i}{\partial \lambda}.$$  

(2.13)

The second term on the r.h.s. vanishes by construction, since $R_i(\lambda)$ is found by solving the equation $\partial_{R_i} a = 0$. The first term can be read off (2.11):

$$\frac{da}{d\lambda} = - \left[ T(G) + \sum_i T(r_i)(R_i(\lambda) - 1) \right].$$  

(2.14)

At $\lambda = 0$, $R_i(0) = 2/3$ and the r.h.s. is nothing but the coefficient in the one loop $\beta$-function which is negative by assumption (see (2.6)). It remains negative for all positive $\lambda$ up to $\lambda = \lambda^*$ (as we will see momentarily, $\lambda^*$ is positive), where it vanishes. We see that $a(R_i(\lambda), \lambda)$ is a decreasing function of $\lambda$ between $\lambda = 0$, which corresponds to the UV
fixed point, and $\lambda = \lambda^*$, which corresponds to the IR fixed point. Hence, $a_{UV} > a_{IR}$, in accordance with (1.2). Note also that $\lambda^*$, which is by definition an extremum of $a(R_i(\lambda), \lambda)$ (2.12), is in fact a (local) minimum of that function.

It is useful to write explicitly the solution of the equations described above. Maximizing (2.11) with respect to $R_i$ leads to

$$R_i(\lambda) = 1 - \frac{1}{3} \left[ 1 + \frac{\lambda T(r_i)}{|r_i|} \right]^\frac{1}{2}. \quad (2.15)$$

$\lambda^*$ is the solution of the constraint (2.9),

$$T(G) + \sum_i T(r_i)(R_i(\lambda^*) - 1) = 0. \quad (2.16)$$

Using (2.15) it is easy to see that $\lambda^*$ is positive when the theory is asymptotically free (2.6).

Although $\lambda$ was originally introduced in eq. (2.11) as a Lagrange multiplier, its properties are reminiscent of those of the gauge coupling $\alpha$. We next argue that $\lambda$ and $\alpha$ are in fact related.

To see that, it is useful to recall the following well known story. If the matter content of a gauge theory is such that the theory is just barely asymptotically free, then the IR fixed point can be studied perturbatively in $\alpha$ [23,24]. For supersymmetric theories, this is the case when the coefficient in the one loop $\beta$-function, $3T(G) - \sum_i T(r_i)$ is positive (2.6), but very small. To find the perturbative fixed point, one then looks for solutions of the fixed point condition (2.9), where the R-charges $R_i(\alpha)$ are related to the anomalous dimensions $\gamma_i(\alpha)$ via the relation (see (2.3), (2.8))

$$3R_i(\alpha) = 2 + \gamma_i(\alpha). \quad (2.17)$$

Both the anomalous dimensions, $\gamma_i(\alpha)$, and the R-charges, $R_i(\alpha)$, have an expansion in $\alpha$. For the R-charges one has

$$R_i(\alpha) = \frac{2}{3} + R^{(1)}_i(\alpha) + R^{(2)}_i(\alpha^2) + \cdots \quad (2.18)$$

where the $R^{(n)}_i$ can be computed (in a particular scheme, or parametrization of coupling space) by performing loop calculations in the gauge theory. The value of $\alpha$ at the IR fixed point is obtained by plugging (2.18) into (2.3) and solving the resulting equation for $\alpha^*$. 

7
If $\epsilon \equiv 3T(G) - \sum_i T(r_i)$ is very small and positive, one is led to an expansion of $\alpha^*$ (and thus of $R_i(\alpha^*)$ (2.18)), in a power series in $\epsilon$.

The results of IW reviewed above allow one to compute the IR R-charges $R_i(\alpha^*)$ directly, by using $a$-maximization. In the description of the method of IW given above, the $R_i$ are functions of $\lambda$ (2.15). They have a Taylor expansion analogous to (2.18), and the value of $\lambda$ at the IR fixed point is obtained by solving (2.16), which has the same form as the fixed point condition (2.9).

The similarity of the way $\alpha$ and $\lambda$ enter the problem leads us to propose that they should be identified, in the sense that both parametrize the space of gauge couplings. Thus, they should be related as follows:

$$\lambda = A_1 \alpha + A_2 \alpha^2 + A_3 \alpha^3 + \cdots$$

(2.19)

where $A_1, A_2, A_3 \cdots$ are numerical coefficients to be determined. Analytic redefinitions of the form $\alpha = \tilde{\alpha} + a\tilde{\alpha}^2 + b\tilde{\alpha}^3 + \cdots$, which relate different renormalization schemes, change the coefficients $A_2, A_3, \cdots$ but the leading coefficient $A_1$ is invariant under such redefinitions. It can be determined by studying the leading weak coupling behavior of the theory. Using (2.4), one finds

$$R_i(\alpha) = \frac{2}{3} - \frac{\alpha}{3\pi} \frac{|G|}{|r_i|} T(r_i) + O(\alpha^2) .$$

(2.20)

On the other hand, expanding (2.15) one has

$$R_i(\lambda) = \frac{2}{3} - \frac{\lambda}{6} \frac{T(r_i)}{|r_i|} + O(\lambda^2) .$$

(2.21)

Comparing the two, one concludes that

$$\lambda = \frac{2\alpha}{\pi} |G| + O(\alpha^2) .$$

(2.22)

Thus, the relation between $\alpha$ and $\lambda$ is independent of the particular representation $r_i$. This had to be the case, given the fact that we are solving essentially the same equations (2.9), (2.16), and expanding in a power series in $\alpha$ and $\lambda$, respectively. Nevertheless, the agreement supports the relation between $\alpha$ and $\lambda$ proposed above.

We see that the $a$-maximization procedure of [14] together with the postulate that $\lambda$ is proportional to $\alpha$ at weak coupling (2.19) predict the dependence of the anomalous

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5 One expects that $\lambda$ should be a monotonic function of $\alpha$, at least for $0 \leq \alpha \leq \alpha^*$. 
dimensions on the representation $r_i$ given in eq. (2.4) \( i.e. \) the fact that $\gamma_i(\alpha) \propto \alpha T(r_i)/|r_i|$ to leading order in $\alpha$). This is a non-trivial result which is normally derived by evaluating one loop diagrams in the SYM theory.

The above discussion can be extended to higher loops \( i.e. \) higher orders in $\alpha, \lambda \). One can expand $R_i$ (2.15) in a power series in $\lambda$, and use (2.17) and (2.19) to find constraints on the anomalous dimensions $\gamma_i(\alpha)$ at higher loop order. For example, to order $\alpha^2$ one finds

$$\gamma_i(\alpha) = -\frac{\alpha}{\pi}C_2(r_i) + \frac{\alpha^2}{2\pi^2}(C_2(r_i))^2 - \frac{A_2}{2|G|}\alpha^2C_2(r_i) + O(\alpha^3). \tag{2.23}$$

The two loop \( O(\alpha^2) \) term in the anomalous dimension is scheme-dependent. We see that the term that goes like $(C_2(r_i))^2$ is in fact scheme-independent, and the scheme dependence appears in the term proportional to $C_2(r_i)$ via the coefficient $A_2$ (2.19). The two loop anomalous dimensions in $N = 1$ SYM theories can be found for example in [25], which also contains references to the original literature on the subject. One can readily check that the structure of the two loop anomalous dimension is as in (2.23), and the coefficient of $(C_2(r_i))^2$ agrees. The coefficient of $C_2(r_i)$ in [25] can be used to compute $A_2$ for the particular renormalization scheme used there.

Some comments are appropriate at this point:

1. The result (2.15) shows that the R-charges, and thus the scaling dimensions (2.8), monotonically decrease along the RG flow. This is not a general feature of four dimensional QFT. We will see in the next section that superpotential terms often have the opposite effect.

2. In analogy to two dimensional QFT, one might want to require that the central charge $a(R_i(\lambda), \lambda)$ satisfy the relation

$$\frac{da}{d\alpha} = \beta(\alpha)G(\alpha) \tag{2.24}$$

where $G(\alpha)$ is the metric on coupling space. Using (2.2), (2.14) and (2.19) one finds that this metric is

$$G(\alpha) = \frac{d\lambda}{d\alpha} \frac{2\pi}{3\alpha^2} \left[ 1 - \frac{\alpha}{2\pi}T(G) \right] \approx \frac{4|G|}{3\alpha^2} \tag{2.25}$$

where in the last approximate equality we used (2.22) to evaluate $G$ for small $\alpha$. In this metric, the distance to the free field fixed point $\alpha = 0$ from any finite $\alpha$ is infinite – it goes like $\int_0^\alpha d\alpha \sqrt{G(\alpha)} \simeq \int_0^\alpha \frac{d\alpha}{\alpha}$. 

9
The relation between $\lambda$ and $\alpha$ in the vicinity of the UV fixed point, (2.22), shows that $\lambda$ must be positive. Similar restrictions on Lagrange multipliers associated with superpotentials will play a role in our discussion later.

The authors of [9] proposed a different interpolating central charge, which in our notation is given by

$$\tilde{a}(\lambda) = 2|G| + \sum_i |r_i| \left[ 3(R_i(\lambda) - 1)^3 - (R_i(\lambda) - 1) \right].$$

(2.26)

This charge decreases as well under RG flow:

$$\frac{d\tilde{a}}{d\lambda} = \lambda \sum_i T(r_i) \frac{dR_i}{d\lambda} = -\frac{\lambda}{6} \sum_i \frac{T^2(r_i)}{|r_i|} \left( 1 + \frac{\lambda T(r_i)}{|r_i|} \right)^{-\frac{1}{2}},$$

(2.27)

where we have used (2.15).

We end this section with two examples. The first is $N = 1$ supersymmetric QCD (SQCD). The gauge group is $G = SU(N_c)$; the matter consists of $N_f$ chiral superfields $Q_i$, $\tilde{Q}^i (i = 1, \ldots, N_f)$ transforming in the $N_c, \bar{N}_c$ of $SU(N_c)$, respectively.

This theory is asymptotically free for $N_f < 3N_c$. The results of [14] are not needed to determine the superconformal $U(1)_R$ of the IR theory in this case. The R-charges $R_Q, R_{\tilde{Q}}$ must be taken to be equal, due to the $Z_2$ symmetry of exchanging $Q$ and $\tilde{Q}$. The anomaly constraint (2.29) takes the form $N_c + N_f(R_Q - 1) = 0$, with the solution

$$R_Q = R_{\tilde{Q}} = 1 - \frac{N_c}{N_f}.$$

(2.28)

It is known [13] that this result is correct for $3N_c/2 \leq N_f \leq 3N_c$. For $N_f < 3N_c/2$, the NSVZ $\beta$-function breaks down in a way that is difficult to understand from the present perspective. The physics is then believed to be described by the Seiberg-dual theory, which has gauge group $G_m = SU(N_f - N_c)$. We will have nothing new to say about this here; thus, we will restrict our attention to the region $N_f \geq 3N_c/2$.

The UV and IR central charges $a_{UV}$ and $a_{IR}$ are:

$$a_{UV} = 2(N_c^2 - 1) + 2N_fN_c\frac{2}{9},$$

$$a_{IR} = 2(N_c^2 - 1) + 2N_fN_c \left[ 3(R_Q - 1)^3 - (R_Q - 1) \right],$$

(2.29)

with $R_Q$ given by (2.28). It can be checked directly that $a_{UV} > a_{IR}$ in the region under discussion [8,14] but one can also apply our reasoning above. Eq. (2.15) takes in this case the form

$$R_Q(\lambda) = 1 - \frac{1}{3} \left( 1 + \frac{\lambda}{2N_c} \right)^{\frac{4}{3}}.$$

(2.30)
Substituting this into (2.11) one finds
\[ a(R_Q(\lambda), \lambda) = 2(N_c^2 - 1) + \frac{4}{9} N_f N_c \left( 1 + \frac{\lambda}{2N_c} \right)^{\frac{3}{2}} - \lambda N_c. \] (2.31)

At \( \lambda = 0 \) this agrees with the UV value (2.29). Differentiating with respect to \( \lambda \) one finds, as in (2.14):
\[ \frac{da}{d\lambda} = -N_c + \frac{N_f}{3} \left( 1 + \frac{\lambda}{2N_c} \right)^{\frac{1}{2}} = -N_c + N_f (1 - R_Q(\lambda)) . \] (2.32)

We see that \( a(R_Q(\lambda), \lambda) \) decreases as \( \lambda \) grows. \( \lambda^* \), the solution of (2.16), is given here by
\[ \frac{\lambda^*}{2N_c} = \left( \frac{3N_c}{N_f} \right)^2 - 1 \] (2.33)
which is indeed positive for \( N_f < 3N_c \).

A point that is nicely illustrated by the SQCD example is that \( \alpha \)-maximization is useful beyond the determination of the IR \( U(1)_R \) charge. As discussed above, the fact that the IR R-charges are given by (2.28) does not require it. Instead, \( \alpha \)-maximization is used here to continuously interpolate between the UV and IR fixed points and define a monotonically decreasing function throughout the RG flow, (2.31).

As mentioned above, the discussion must break down for \( N_f < 3N_c/2 \). Presumably, this is due to the fact that in this range, during the RG flow \( \alpha \) reaches the value at which the denominator in (2.4) has a pole, after which the NSVZ \( \beta \)-function (2.2) might become unreliable. If we knew the relation between \( \lambda \) and \( \alpha \) (2.19), we would be able to check this quantitatively, but unfortunately the precise relation between the two is not known at present.

Our second example is adjoint SQCD, a model with gauge group \( G = SU(N_c) \), \( N_f \) flavors of fundamentals \( Q_i, \tilde{Q}^{i} \) as before, but now with an extra chiral superfield \( X \) transforming in the adjoint representation of the gauge group. This model is asymptotically free for \( N_f < 2N_c \), and is believed to flow in the IR to a non-trivial fixed point for all \( N_f \) in this range.

As before, \( Q_i \) and \( \tilde{Q}^{i} \) have the same R-charge \( R_Q \), while \( X \) has R-charge \( R_X \). The anomaly constraint (2.9) is
\[ N_f R_Q + N_c R_X = N_f. \] (2.34)

It does not fix \( R_Q, R_X \) uniquely and one has to use the results of [14,15] to do that.
Our general discussion gives rise in this case to the following $\lambda$-dependent R-charges:

$$R_Q(\lambda) = 1 - \frac{1}{3} \left( 1 + \frac{\lambda}{2N_c} \right)^{\frac{1}{2}}$$

$$R_X(\lambda) = 1 - \frac{1}{3} \left( 1 + \frac{\lambda N_c}{N_c^2 - 1} \right)^{\frac{1}{2}} .$$

(2.35)

The IR fixed point is at $\lambda = \lambda^* > 0$, for which the R-charges (2.35) satisfy (2.34). One can check that the resulting IR R-charges are consistent with those given in [14,15].

Adjoint SQCD is an example where our construction interpolates smoothly between two fixed points in each of which one needs to maximize $a$ with respect to the $R_i$. As in the general case, (2.14), the central charge monotonically decreases throughout the flow.

One other comment that needs to be made here is that, as pointed out in [15], when the R-charges of $Q$, $\tilde{Q}$ or $X$ drop below $1/3$, one needs to modify the procedure of IW to take into account unitarity constraints. We will not attempt to incorporate these corrections here; it would be interesting to do so.

3. Models with non-vanishing superpotentials

So far we only discussed supersymmetric non-abelian gauge interactions. In this section we will include the effects of superpotentials.

3.1. Chiral superfields with non-zero superpotentials

In this subsection we consider theories with no gauge fields. We expect no interesting dynamics in this case, since such theories are always IR free in four dimensions (at least at weak coupling), but we can still apply our methods to them and see what we get. This discussion will also serve as preparation for the case with gauge interactions.

Let $\Phi$ be a chiral superfield with standard kinetic term, and consider the effect of various polynomial superpotentials. For example, take

$$W = m\Phi^2 .$$

(3.1)

This is a mass term, and we expect it to lead to an RG flow with a trivial IR fixed point. From the point of view of our analysis one can include the effect of (3.1) as follows.

Let $R_\phi$ be the R-charge of $\Phi$. Consider the generalized central charge

$$a(R_\phi, \lambda) = 3(R_\phi - 1)^3 - (R_\phi - 1) + \lambda(R_\phi - 1) .$$

(3.2)
At $\lambda = 0$, this is just the central charge of a free massless chiral superfield; the local maximum of $a(R_\phi, 0)$ is at $R_\phi = 2/3$, the free value [14]. $\lambda$ is a Lagrange multiplier enforcing the condition $R_\phi = 1$, which is necessary for the R-symmetry to be preserved by the superpotential (3.1). We can again find the local maximum of (3.2) at fixed $\lambda$. This occurs at

$$R_\phi = 1 - \frac{1}{3} \sqrt{1 - \lambda}.$$  \hspace{1cm} (3.3)

Substituting back into (3.2) we find

$$a(R_\phi(\lambda), \lambda) = \frac{2}{9} (1 - \lambda)^{\frac{2}{3}}.$$  \hspace{1cm} (3.4)

We see that $a$ interpolates between the value $a = 2/9$ (corresponding to a free superfield) at $\lambda = 0$, and $a = 0$ at $\lambda = 1$ where $\frac{da}{d\lambda} = 0$ (see (2.12)). Between the two values, $a$ monotonically decreases, since

$$\frac{da(R_\phi(\lambda), \lambda)}{d\lambda} = R_\phi(\lambda) - 1$$  \hspace{1cm} (3.5)

which is negative for all $\lambda < 1$ (compare to (2.13), (2.14)). One can again argue that $\lambda$ in (3.2), which starts its life as a Lagrange multiplier, actually parametrizes the coupling space (3.1). In the field theory this space is labeled by $m/E$, where $E$ is the energy scale at which we are working. The “coupling” $m/E$ varies between 0 and $\infty$. In the description (3.4) this is translated into the dependence on $\lambda$, which varies between 0 and 1.

Note also that the discussion above provides a simple example in which the R-charge increases under RG flow. This should be contrasted with gauge interactions, which lower the R-charges, see (2.15).

Next we discuss the case of a cubic superpotential

$$W = \mu \Phi^3.$$  \hspace{1cm} (3.6)

This interaction is marginally irrelevant, so one does not expect to find any interacting fixed points at $\mu \neq 0$. In this case, the central charge takes the form

$$a(R_\phi, \lambda) = 3(R_\phi - 1)^3 - (R_\phi - 1) + \lambda(R_\phi - \frac{2}{3}).$$  \hspace{1cm} (3.7)

The local maximum in $R_\phi$ is again at (3.3), and

$$a(R_\phi(\lambda), \lambda) = \frac{2}{9} (1 - \lambda)^{\frac{2}{3}} + \frac{1}{3} \lambda.$$  \hspace{1cm} (3.8)
Imposing (2.12) we find
\[\frac{da(R_\phi(\lambda), \lambda)}{d\lambda} = -\frac{1}{3} \sqrt{1 - \lambda} + \frac{1}{3} = 0 .\] (3.9)

The only solution is \(\lambda = 0\), the original trivial fixed point with \(\mu = 0\) (3.6). Note in particular that \(a\) (3.8) “knows” about the fact that the cubic superpotential (3.6) is marginally irrelevant. In (3.8) this is the statement that while the term linear in \(\lambda\) vanishes, the quadratic term is positive, so \(\lambda = 0\) is a local minimum of \(a\).

More generally, one can consider a superpotential of the form \(W = \mu \Phi^n\). Repeating the analysis for this case one finds that \(R_\phi(\lambda)\) is given by (3.3) and the central charge is
\[a(R_\phi(\lambda), \lambda) = \frac{2}{9} (1 - \lambda)^{\frac{3}{2}} + \lambda \frac{n - 2}{n} .\] (3.10)

The stationarity condition is
\[\frac{da}{d\lambda} = -\frac{1}{3} \sqrt{1 - \lambda} + \frac{n - 2}{n} = 0 .\] (3.11)

For \(n > 3\), this equation has a solution, \(\lambda^*\), but it is negative. We have seen before that the Lagrange multiplier \(\lambda\) must have a particular sign, both in the gauge theory example of section 2, where this sign is correlated with that of \(\alpha\) (see (2.22)), and in the case of the mass deformation described earlier in this subsection. The general rule is that the sign of \(\lambda\) is constrained to be such that when we perturb a fixed point by a relevant operator, \(a(\lambda)\) should decrease when we turn on \(\lambda\), and vice-versa. In other words, if we choose \(\lambda\) to be positive by convention, it must be that \(\partial_\lambda a < 0\) at \(\lambda = 0\) for a relevant perturbation, and \(\partial_\lambda a > 0\) for an irrelevant one. This is why in section 2, \(\lambda\) was constrained to be positive (see (2.14)) and it also implies that in (3.11) \(\lambda\) should be taken to be positive.

Thus, the solutions of (3.11) with \(\lambda < 0\) should be discarded, and we do not find any fixed points with non-zero \(\lambda\) (or \(\mu\)) when we turn on an irrelevant superpotential, as one would expect.

The only remaining case is a linear superpotential, \(W = \mu \Phi\). The stationarity condition is in this case
\[-\frac{1}{3} \sqrt{1 - \lambda} - 1 = 0 .\] (3.12)

It has no solutions, in agreement with the fact that for a linear superpotential, the condition for a supersymmetric ground state, \(W' = 0\) has no solutions.
We can also study models in which the superpotential is a more general polynomial

\[ W = \sum_{n=2}^{k} a_n \Phi^n. \]  

(3.13)

The most important new effect here is that the R-symmetry is in general completely broken throughout the RG flow. For example, if \( a_2 \neq 0 \) and we focus on the vacuum with \( \langle \Phi \rangle = 0 \), the R-charge of \( \Phi \) still approaches 1 in the IR, as in (3.3) in the limit \( \lambda \to 1 \), but now this symmetry is present only at the IR fixed point, where one can neglect the higher powers of \( \Phi \) in (3.13).

One can generalize our analysis of the different cases above and write the central charge corresponding to (3.13) as

\[ a(R_\phi, \lambda_2, \cdots, \lambda_k) = 3(R_\phi - 1)^3 - (R_\phi - 1) + \sum_{n=2}^{k} \lambda_n (R_\phi - \frac{2}{n}), \]  

(3.14)

where \( (\lambda_2, \cdots, \lambda_k) \) parametrize the space of couplings \( (a_2, \cdots, a_k) \) in (3.13). After solving for \( R_\phi \) as a function of \( (\lambda_2, \cdots, \lambda_k) \) one gets a central charge defined on the space of \( \lambda \)'s, \( a(R_\phi(\lambda_2, \cdots, \lambda_k), \lambda_2, \cdots, \lambda_k) \). We will not pursue this here since most of the couplings \( \lambda_n \) are irrelevant. Instead we will move on to the case where both gauge interactions and superpotentials are present.

### 3.2. Models with gauge interactions

In the last subsection we have seen that in the absence of non-abelian gauge fields there is little interesting dynamics, essentially because the only relevant perturbation of a massless free chiral superfield is a mass term.

The situation is different in non-abelian gauge theories. As we have seen in section 2, gauge interactions decrease the scaling dimensions and R-charges of chiral superfields (see (2.15)); thus, it often happens that the IR fixed point of a gauge theory has relevant perturbations that did not exist at the (free) UV fixed point. In this subsection we will study the effect of these perturbations on the discussion of section 2.

The basic setup will be the same as in section 2. We have an \( N = 1 \) SYM theory with gauge group \( G \) and chiral superfields \( \Phi_i \) in representations \( r_i \). We assume that the theory is asymptotically free and flows in the IR to a fixed point which is described by the NSVZ
\(\beta\)-function \((2.2)\) \(\text{i.e. satisfies (2.4), (2.9)}\). We further assume that at the IR fixed point of this gauge theory, the gauge invariant chiral operator

\[ O = \prod_i \Phi_i^{n_i} \] (3.15)

is relevant, \textit{i.e.}

\[ \sum_i n_i R_i (\alpha^*) < 2 . \] (3.16)

We thus can add to the Lagrangian the superpotential

\[ W = \mu O \] (3.17)

which drives the theory to a new infrared fixed point, which we will refer to as \(\text{IR}'\). At that new fixed point, the R-charges \(R_i\) must satisfy the constraint

\[ \sum_i n_i R_i = 2 . \] (3.18)

We will furthermore assume that the anomaly condition \((2.9)\) is compatible with \((3.18)\), so that the R-symmetry at the new fixed point \(\text{IR}'\), which satisfies both \((2.9)\) and \((3.18)\), is preserved throughout the RG flow \(\text{UV} \rightarrow \text{IR} \rightarrow \text{IR}'\).

We would like to show that under these conditions

\[ a_{\text{IR}} > a_{\text{IR}'} . \] (3.19)

To do that, we consider the generalized central charge

\[ a(R_i, \lambda, \lambda') = 2|G| + \sum_i |r_i| [3(R_i - 1)^3 - (R_i - 1)] \]

\[ -\lambda \left[ T(G) + \sum_i T(r_i)(R_i - 1) \right] - \lambda' \left( 2 - \sum_i n_i R_i \right) . \] (3.20)

\(\lambda, \lambda'\) are again Lagrange multipliers enforcing the constraints \((2.16)\) and \((3.18)\). As discussed in the last subsection, they are both non-negative.

Following the same logic as in our previous discussions, we find the local maximum of \((3.20)\) with respect to the \(R_i\), at fixed \((\lambda, \lambda')\). This occurs at

\[ R_i(\lambda, \lambda') = 1 - \frac{1}{3} \left( 1 + \frac{\lambda T(r_i) - \lambda' n_i}{|r_i|} \right)^{\frac{1}{2}} . \] (3.21)
Substituting this back into (3.20) we arrive at a central charge \( a(R_i(\lambda, \lambda'), \lambda, \lambda') \), (which we will denote by \( a(\lambda, \lambda') \) for brevity) defined on the space of theories labeled by the gauge coupling (which is related to \( \lambda \), as discussed in section 2), and the superpotential coupling \( \mu \) (3.17), which is related to \( \lambda' \).

The derivation of eq. (2.14) can be repeated for this case. One finds:

\[
\frac{\partial a}{\partial \lambda} = -\left[ T(G) + \sum_i T(r_i)(R_i(\lambda, \lambda') - 1) \right],
\]

\[
\frac{\partial a}{\partial \lambda'} = -\left[ 2 - \sum_i n_i R_i(\lambda, \lambda') \right].
\]

(3.22)

The three fixed points discussed above are described as follows in terms of \( a(\lambda, \lambda') \). The free UV fixed point corresponds to \( \lambda = \lambda' = 0 \). The IR fixed point of the RG flow with vanishing superpotential is at \( \lambda' = 0, \lambda = \lambda_1^* \), with \( \lambda_1^* \) determined by solving

\[
\frac{\partial a(\lambda, 0)}{\partial \lambda} = 0.
\]

(3.23)

The fixed point \( IR' \) to which the system is driven by (3.17) (i.e. \( \lambda = 0 \)) is at \( \lambda = \lambda_2^*, \lambda' = \lambda_2'^* \) which are obtained by minimizing \( a(\lambda, \lambda') \) with respect to both \( \lambda \) and \( \lambda' \):

\[
\frac{\partial a}{\partial \lambda} = \frac{\partial a}{\partial \lambda'} = 0 \quad \text{at} \quad (\lambda, \lambda') = (\lambda_2^*, \lambda_2'^*).
\]

(3.24)

In order to derive (3.19), it is useful to recall some aspects of the analysis of section 2. Before turning on the superpotential (3.17) (i.e. at \( \lambda' = 0 \)), we saw that \( a(\lambda, 0) \) has a local minimum at \( \lambda = \lambda_1^* \). Turning on \( \lambda' \) should not change the picture qualitatively. As we see from the first line of (3.22), \( \partial_\lambda a(\lambda, \lambda') \) is negative for \( 0 \leq \lambda < \lambda^*(\lambda') \), where \( \lambda^*(\lambda') \) is obtained by solving the equation

\[
\frac{\partial a(\lambda, \lambda')}{\partial \lambda} = 0.
\]

(3.25)

For \( \lambda' = 0 \) we have \( \lambda^*(0) = \lambda_1^* \). Now consider following a trajectory from the fixed point \( IR \) to \( IR', \) i.e. from \( (\lambda, \lambda') = (\lambda_1^*, 0) \) to \( (\lambda, \lambda') = (\lambda_2^*, \lambda_2'^*) \), along the curve \( \lambda^*(\lambda') \). Along this curve one has

\[
\frac{da(\lambda^*(\lambda'), \lambda')}{d\lambda'} = \frac{\partial a}{\partial \lambda} + \frac{\partial a}{\partial \lambda} \frac{\partial \lambda^*}{\partial \lambda'}.
\]

(3.26)
As in (2.13), the second term vanishes since along the trajectory in question \( \partial_\lambda a = 0 \) (3.25). The first term is given by (3.22). Thus, we conclude that

\[
\frac{da(\lambda^*(\lambda'), \lambda')}{d\lambda'} = -\left[ 2 - \sum_i n_i R_i(\lambda^*(\lambda'), \lambda') \right].
\]  

The r.h.s. is negative at \( \lambda' = 0 \) (3.16) and it remains negative all the way to \( \lambda' = \lambda'^* \) where it vanishes.

Eq. (3.27) implies the inequality (3.19). Along the trajectory \( \lambda^*(\lambda') \) connecting the fixed points \( IR \) and \( IR' \), the central charge \( a \) is monotonically decreasing, so \( a_{IR} > a_{IR'} \). This concludes the proof of the inequality (3.19).

Note that we assumed above that \( \lambda'^* \) is positive. The reason for that is that, as explained in the previous subsection, \( \lambda' \) is by definition non-negative. Thus, any fixed point obtained by adding the relevant perturbation (3.17) must give rise to a critical point of \( a(\lambda, \lambda') \) (3.24) at positive \( \lambda \) and \( \lambda' \). If there are no solutions of eq. (3.24) with \( \lambda'^* > 0 \), then we conclude that this perturbation breaks supersymmetry, like the linear superpotential discussed in the previous subsection.

There are many additional aspects of the construction that are worth exploring. Since our primary interest was in establishing the hierarchy \( a_{UV} > a_{IR} > a_{IR'} \), we focused on a very specific trajectory in the space of couplings. Starting at the UV fixed point, we turned on only the gauge coupling, went to the IR fixed point, and only then turned on the relevant superpotential (3.17), which drove the system to the fixed point \( IR' \).

One could study more general RG trajectories in coupling space, that lead directly from \( UV \rightarrow IR' \), without passing through \( IR \). Such trajectories involve turning on both the gauge coupling and \( \mu \) (3.17), or both \( \lambda \) and \( \lambda' \), as one leaves the UV fixed point.

As mentioned in the beginning of this subsection, the operator (3.15) is in general irrelevant near the UV fixed point, and only becomes relevant at some point along the flow of the gauge coupling. Thus, if we turn on both the gauge coupling \( \alpha \), and \( \mu \) (3.17), at first \( \alpha \) will grow and \( \mu \) decrease along the RG flow, but at some point, when \( \alpha \) passes a critical value such that (3.15) becomes relevant, \( \mu \) will start growing as well.

The generalized central charge \( a(\lambda, \lambda') \) provides a good quantitative tool for studying such flows. The second line of (3.22) shows that for small \( \lambda \), \( \partial_{\lambda'} a \) is positive, while beyond a critical value \( \lambda_{c_r}(\lambda') \) it becomes negative. Qualitatively, RG flows correspond to rolling in a landscape with height function \( a(\lambda, \lambda') \). It would be nice to understand quantitatively the shape of the resulting trajectories in the \((\lambda, \lambda')\) plane.

18
An interesting logical possibility related to these trajectories is that one might think that it is possible that even if the operator $\mathcal{O}$ is irrelevant at the IR fixed point of the gauge theory, the fixed point $IR'$ could still exist, and one could reach it without going through the $W = 0$ IR fixed point. We will see later that this possibility is not realized in an example that we study in detail. A more general analysis of this issue will be left to future work.

Another natural generalization is to multi-step cascades, where one flows from $UV \rightarrow IR^{(1)} \rightarrow IR^{(2)} \rightarrow IR^{(3)} \rightarrow \cdots$. To go from $IR^{(n)} \rightarrow IR^{(n+1)}$ one turns on a superpotential $W = \mu_n \mathcal{O}_n$ that is relevant at the fixed point $IR^{(n)}$. As long as the assumptions of our analysis are valid, i.e. if the R-symmetry under which $R(\mathcal{O}_n) = 2$ is a symmetry of the full RG flow, one can repeat the analysis above, and it leads to the hierarchy

$$a_{UV} > a_{IR}^{(1)} > a_{IR}^{(2)} > a_{IR}^{(3)} > \cdots . \tag{3.28}$$

This can be shown by studying the generalized central charge

$$a(R_i, \lambda, \lambda_n) = 2|G| + \sum_i |r_i| \left[ 3(R_i - 1)^3 - (R_i - 1) \right] - \lambda \left[ T(G) + \sum_i T(r_i)(R_i - 1) \right] - \sum_n \lambda_n \left[ 2 - R(\mathcal{O}_n) \right]. \tag{3.29}$$

Maximizing in $R_i$ leads to a central charge $a(\lambda, \lambda_n)$ which can be used as in equations (3.22) – (3.27) to prove (3.28).

Another application of our formalism is to the study of moduli spaces of CFT’s. A generic way to obtain a non-trivial moduli space of fixed points is to study a gauge theory of the sort discussed above, with a superpotential

$$W = \sum_n \mu_n \mathcal{O}_n \tag{3.30}$$

in a situation where the conditions $T(G) + \sum_i T(r_i)(R_i - 1) = 0$ and $R(\mathcal{O}_n) = 2$ are not linearly independent (viewed as functions of the $R_i$). Thus, there exist constants $A$, $A_n$, not all of which are zero, such that

$$A \left[ T(G) + \sum_i T(r_i)(R_i - 1) \right] + \sum_n A_n \left[ 2 - R(\mathcal{O}_n) \right] = 0. \tag{3.31}$$
In this case, one generically expects (see e.g. [20]) the IR fixed point of the flow to be a manifold labeled by the couplings of exactly marginal operators. The number of moduli is the same as the number of independent relations (3.31).

Relations of the form (3.31) have the following effect on our discussion. One can use each such relation to express one of the terms in square brackets on the second line of (3.29) in terms of the others, and thus eliminate it from the equation. This decreases the number of independent terms, but the coefficients of the remaining ones still depend on the original couplings \((\lambda, \lambda_n)\). Thus, \(a(\lambda, \lambda_n)\) is in this case independent of some linear combinations of \(\lambda\) and the \(\{\lambda_n\}\). These combinations are the moduli of the IR CFT. This establishes in general that both the central charge \(a\), and the R-charges \(R_i\) are independent of the moduli.

Although we have presented the arguments of this section in terms of deformations of a free field fixed point by a combination of gauge interactions and superpotentials, it should be clear that one can in fact phrase them abstractly as follows. Let \(A\) be an \(N = 1\) superconformal field theory, and \(O\) a chiral operator, such that the superpotential (3.17) corresponds to a relevant perturbation of the SCFT \(A\), leading at long distances to SCFT \(B\). Assume further that the \(U(1)_R\) symmetry of the IR SCFT \(B\) is a subgroup of the symmetry group of the full theory. Then \(a_A > a_B\).

To prove this, start with the central charge \(a\) corresponding to a generic \(U(1)_R\) symmetry at the fixed point \(A\). Under the assumptions stated above, this central charge depends on some continuous parameters \((s_1, \cdots, s_n)\), which parametrize the particular \(U(1)_R\) symmetry whose anomaly is being computed. The special \(U(1)_R\) which belongs to the superconformal multiplet is obtained by locally maximizing \(a(s_1, \cdots, s_n)\) with respect to the \(s_i\) [14]. The effect of the perturbation (3.17) is captured by studying the generalized central charge

\[
a(\lambda, s_1, \cdots, s_n) = a(s_1, \cdots, s_n) - \lambda [2 - R_O(s_1, \cdots, s_n)] ,
\]

(3.32)

where \(R_O\) is the R-charge of the chiral operator \(O\) under the symmetry labeled by \((s_1, \cdots, s_n)\). Locally maximizing \(a(\lambda, s_1, \cdots, s_n)\) with respect to \((s_1, \cdots, s_n)\) and solving for the \(s_i\) as a function of \(\lambda\) leads to an effective central charge \(a(\lambda)\), which interpolates between the fixed point \(A\) at \(\lambda = 0\), and the fixed point \(B\) at \(\lambda = \lambda^*\), corresponding to a local minimum of \(a(\lambda)\) at a positive value of \(\lambda\). Between the two fixed points, \(a\) is monotonically decreasing, since

\[
\frac{da}{d\lambda} = - [2 - R_O(s_1, \cdots, s_n)] .
\]

(3.33)
To conclude this section we briefly discuss an example of a model with a non-vanishing superpotential, a deformation of adjoint SQCD, which was introduced at the end of section 2.

It is not difficult to check, using equations (2.34), (2.35), that the R-charge of $X$ at the IR fixed point of adjoint SQCD with vanishing superpotential goes like $N_f/N_c$ for small $N_f/N_c$, and in particular it can become arbitrarily small, at least when $N_f, N_c$ are large (see \[14\]-\[15\] for a more detailed discussion). Thus, if $N_f/N_c$ is small enough, the operator $\text{tr}X^{k+1}$ is relevant at the IR fixed point, and we can add to the Lagrangian the superpotential

$$W = g_k \text{tr}X^{k+1}, \quad (3.34)$$

which is of interest in the study of Seiberg duality \[17\]-\[19\]. The generalized central charge (3.20) is now:

$$a(R_X, R_Q, \lambda, \lambda') = 2(N_c^2 - 1) + (N_c^2 - 1) \left[3(R_X - 1)^3 - (R_X - 1)\right] + 2N_f N_c \left[3(R_Q - 1)^3 - (R_Q - 1)\right] - \lambda [N_c R_X + N_f (R_Q - 1)] - \lambda' [2 - (k + 1)R_X] - \lambda [N_c R_X + N_f (R_Q - 1)] - \lambda' [2 - (k + 1)R_X] \quad (3.35)$$

Maximizing with respect to $R_X$ and $R_Q$ one finds (3.21)

$$R_X(\lambda, \lambda') = 1 - \frac{1}{3} \left(1 + \frac{\lambda N_c - \lambda'(k + 1)}{N_c^2 - 1}\right)^{\frac{1}{2}} \quad (3.36)$$

$$R_Q(\lambda, \lambda') = 1 - \frac{1}{3} \left(1 + \frac{\lambda}{2N_c}\right)^{\frac{1}{2}}$$

The superpotential (3.34) is relevant when $R_X(\lambda, \lambda') < 2/(k + 1)$. The non-trivial fixed point $IR'$ is obtained by setting (see (2.34), (3.22), (3.24), (3.36))

$$R_X(\lambda_2^*, \lambda_2'^*) = \frac{2}{k + 1}, \quad R_Q(\lambda_2^*, \lambda_2'^*) = 1 - \frac{2N_c}{(k + 1)N_f}. \quad (3.37)$$

For example, for the case of a cubic superpotential, $k = 2$, one finds

$$\frac{\lambda_2^*}{2N_c} = \left(\frac{2N_c}{N_f}\right)^2 - 1, \quad (3.38)$$

$$\lambda_2'^* = \frac{\lambda_2^* N_c}{3}. \quad (3.39)$$
Note that both $\lambda_2^*$ and $\lambda'_2^*$ are positive, as implied by our general discussion, when the theory is asymptotically free, $N_f < 2N_c$.

For $k > 2$, one finds

\[
\frac{\lambda_2^*}{2N_c} = \left[ \frac{6N_c}{(k + 1)N_f} \right]^2 - 1,
\]

\[
\lambda'_2^*(k + 1) = 2N_c^2 \left[ \left( \frac{6N_c}{(k + 1)N_f} \right)^2 - 1 \right] - (N_c^2 - 1) \left[ 9 \left( \frac{k - 1}{k + 1} \right)^2 - 1 \right].
\] (3.39)

Using the results of [14,15] one can show that (3.39) leads to a sensible picture. In particular, the condition $\lambda'_2^* > 0$ (which also implies $\lambda_2^* > 0$) is the same as the condition that the $\text{tr}X^{k+1}$ perturbation (3.34) is relevant in adjoint SQCD with vanishing superpotential. We see that the fixed point (3.37) exists if and only if the operator (3.34) corresponds to a relevant perturbation of the IR limit of adjoint SQCD with $W = 0$, something that, as mentioned above, is not apriori obvious.

4. Generalizations

Many of the examples of RG flows in which the central charge $a$ was observed to decrease in [14-16] fall into the class of models satisfying the assumptions of sections 2, 3 of this paper, and thus the behavior of $a$ for them is explained by our analysis. However, there are other cases studied in [14-16] which do not satisfy some of the assumptions on which our proof was based, and it is natural to ask what happens in these cases. We will not attempt to provide a general understanding of these issues here, but just comment on some of them.

The assumption that the IR $U(1)_R$ of an RG flow is a symmetry of the full theory is certainly one that we would like to relax. Our techniques are actually useful for treating cases in which it is violated. We next describe an example which, as discussed in [15], provides a sensitive test of the $a$-theorem.

In the last section we outlined the structure of adjoint SQCD in the presence of a superpotential (3.34) that goes like $\text{tr}X^{k+1}$. In particular, we saw that despite appearances, this deformation can be relevant and drive the system to a non-trivial fixed point.

\[\text{The latter condition is given by eq. (3.15) in [15] for large } N_f, N_c.\]
Suppose we now further deform this model by adding to the superpotential a term that goes like $\text{tr}X^k$, and study the superpotential

$$W = g_k\text{tr}X^{k+1} + g_{k-1}\text{tr}X^k.$$  \hspace{1cm} (4.1)

The UV fixed point of the RG flow we are interested in is the IR fixed point to which the system is driven by the $\text{tr}X^{k+1}$ superpotential (3.34). At this fixed point, the operator $g_{k-1}\text{tr}X^k$ is relevant, and adding it to the superpotential (4.1) leads to a new fixed point, in which the superpotential is effectively

$$W_{IR} \simeq g_{k-1}\text{tr}X^k,$$  \hspace{1cm} (4.2)

and the R-charge of $X$ is $R_{X}^{(IR)} = 2/k$, in contrast to the UV fixed point (3.34), where $W = g_k\text{tr}X^{k+1}$ and $R_X = 2/(k+1)$.

A natural question is whether $a$ decreases along this RG flow. As pointed out in [9], this is far from obvious. It was checked to be true in [15] by using the results of [14]. Here we would like to see whether the fact that $a$ decreases in this flow can be understood more conceptually, from the general perspective of sections 2, 3.

The technical complication here is that the IR R-symmetry, for which $R_X = 2/k$, is not a symmetry of the full theory – it is broken by the $X^{k+1}$ term in (4.1). It is rather an accidental symmetry of the IR theory, associated with the fact that $g_k \to 0$ in the IR. Nevertheless, the construction of sections 2, 3 is useful for showing that $a$ decreases along this RG flow. To see that one can proceed as follows.

As in eq. (3.14), the generalized central charge for the system with superpotential (4.1) has the form

$$a(R_X, R_Q, \lambda, \lambda_1, \lambda_2) = 2(N_c^2 - 1) + (N_c^2 - 1) [3(R_X - 1)^3 - (R_x - 1)] + 2N_f N_c \left[3(R_Q - 1)^3 - (R_Q - 1) \right] - \lambda [N_c R_X + N_f (R_Q - 1)] - \lambda_1 \left[2 - (k + 1)R_X \right] - \lambda_2 \left[2 - kR_X \right]$$  \hspace{1cm} (4.3)

Maximizing with respect to $R_X, R_Q$ gives

$$R_X(\lambda, \lambda_1, \lambda_2) = 1 - \frac{1}{3} \left(1 + \frac{\lambda N_c - \lambda_1(k + 1) - \lambda_2 k}{N_c^2 - 1}\right)^{\frac{1}{2}}$$  \hspace{1cm} (4.4)

$$R_Q(\lambda, \lambda_1, \lambda_2) = 1 - \frac{1}{3} \left(1 + \frac{\lambda}{2N_c}\right)^{\frac{1}{2}}$$
Substituting in (4.3), one finds the central charge \( a \) as a function of \((\lambda, \lambda_1, \lambda_2)\) which, as discussed above, provide a particular parametrization of the space labeled by the couplings \((\alpha, g_k, g_{k-1})\).

The UV fixed point of the flow we are interested in, corresponding to the superpotential (3.34) is at
\[
\frac{\partial a}{\partial \lambda} = \frac{\partial a}{\partial \lambda_1} = 0; \quad \lambda_2 = 0 .
\] (4.5)

The IR fixed point corresponding to \( W = g_{k-1} \text{tr} X^k \) is at
\[
\frac{\partial a}{\partial \lambda} = \frac{\partial a}{\partial \lambda_2} = 0; \quad \lambda_1 = 0 .
\] (4.6)

We would like to show that the value of \( a \) at the solution of (4.5) is larger than at the solution of (4.6).

As a first step, one can reduce \( a \) to a function of two variables, \( \lambda_1 \) and \( \lambda_2 \), by imposing the relation
\[
\frac{\partial a(\lambda, \lambda_1, \lambda_2)}{\partial \lambda} = 0 ,
\] (4.7)
which is common to (4.5) and (4.6), and using it to solve for \( \lambda \) as a function of \( \lambda_1, \lambda_2 \). Substituting \( \lambda(\lambda_1, \lambda_2) \) into \( a \) we find a central charge \( a(\lambda_1, \lambda_2) \), which satisfies
\[
\frac{\partial a}{\partial \lambda_1} = - [2 - (k + 1) R_X(\lambda_1, \lambda_2)]
\]
\[
\frac{\partial a}{\partial \lambda_2} = - [2 - k R_X(\lambda_1, \lambda_2)]
\] (4.8)

The UV fixed point of our flow corresponds to \( \partial_{\lambda_1} a = 0, \lambda_2 = 0 \); the IR fixed point to \( \partial_{\lambda_2} a = 0, \lambda_1 = 0 \).

Now, consider the behavior of \( R_X(\lambda_1, \lambda_2) \) as a function of \((\lambda_1, \lambda_2)\). The point \((\lambda_1, \lambda_2) = (0, 0)\) corresponds to the IR fixed point of adjoint SQCD with vanishing superpotential studied in section 2. We have assumed above that the perturbation (3.34) is relevant there:
\[
R_X(0, 0) < \frac{2}{k + 1} .
\] (4.9)

Now, the UV fixed point of our RG flow has \( R_X = 2/(k + 1) \) (see (4.3), (4.8)), while the IR fixed point has \( R_X = 2/k \).

Consider the curve
\[
R_X(\lambda_1, \lambda_2) = \frac{2}{k + 1}
\] (4.10)
in the \((\lambda_1, \lambda_2)\) plane. It must intersect the \(\lambda_1\) and \(\lambda_2\) axes at two points, \((\lambda_1^*, 0)\) and \((0, \lambda_2^*)\), respectively. As before, \((\lambda_1^*, \lambda_2^*)\) are positive. Similarly, the curve

\[ R_X(\lambda_1, \lambda_2) = \frac{2}{k} \]  

intersects the axes at the points \((\lambda_1^*, 0)\) and \((0, \lambda_2^*)\). Since at \((\lambda_1, \lambda_2) = (0, 0)\), \(R_X < 2/(k + 1)\), by continuity one has

\[ \lambda_1^* > \lambda_1^*, \quad \lambda_2^* > \lambda_2^*. \]  

We would like to show that

\[ a(\lambda_1^*, 0) > a(0, \lambda_2^*). \]  

To show that, consider the following trajectory connecting the two points. First go from \((\lambda_1^*, 0) \rightarrow (0, \lambda_2^*)\) following the curve \((4.10)\). Along this trajectory, \(\partial_{\lambda_1} a = 0\), and therefore \(a\) decreases:

\[ \frac{da(\lambda_1(\lambda_2), \lambda_2)}{d\lambda_2} = \frac{\partial a(\lambda_1(\lambda_2), \lambda_2)}{\partial \lambda_2} = -\frac{2}{k + 1} < 0, \]  

where in the last step we used the second line of \((4.8)\).

Now proceed on the \(\lambda_2\) axis, from \((0, \lambda_2^*) \rightarrow (0, \lambda_2^*)\). Here, too, \(a\) decreases, using the second line of \((4.8)\) and the fact that \(R_X < 2/k\) for all \(\lambda_2^* < \lambda_2 < \lambda_2^*\). Thus, we conclude that the central charge \(a\) decreases along the flow from \(W \sim \text{tr}X^{k+1}\) to \(W \sim \text{tr}X^k\), as found in \([15]\).

We see that the generalized central charge \(a\) away from fixed points is useful in cases that do not satisfy the assumptions used in sections 2, 3. What replaced those assumptions in the example discussed here is the dynamical assumption that the RG flow \(k \rightarrow k - 1\) associated with the superpotential \((4.1)\) has the property that \(g_{k-1} \rightarrow 0\) in the UV and \(g_k \rightarrow 0\) in the IR. It is in fact likely that by thinking of RG flow as rolling down in the landscape \(a(\lambda_1, \lambda_2)\) one can derive this assumption. It is also possible that one can generalize the basic idea to a wide range of circumstances. We will not attempt to do that here.

It would be nice to generalize our construction to other situations which fall outside the range of validity of the discussion of sections 2, 3. These include the following:

1. Often, the IR fixed point of an RG flow has additional accidental symmetries of a different kind than those encountered in the example above. One class of such accidental symmetries is associated with fields becoming free in the IR. In fact, such
symmetries are relevant in the adjoint SQCD example discussed at the end of section 3, in a certain range of parameters (see [13]). The effect of such accidental symmetries on $a$-maximization is well understood [15,16] and it should not be difficult to include them in our construction as well. We will leave this interesting problem for future work.

(2) We did not discuss the effect of Higgsing, which is another way of generating RG flows. In [15,16] it was found that the $a$-theorem seems to hold for such flows as well. It would be interesting to generalize our discussion to this case.

(3) As we mentioned above, our approach is based on the NSVZ $\beta$-function, which is known to break down rather abruptly at strong coupling. In all known situations of this sort, one has a Seiberg dual description of the physics, and the $a$-theorem is known to be satisfied. In order to incorporate these situations in our framework one needs a better understanding of Seiberg duality than is presently available. This is an interesting direction for further research.

Acknowledgments: I thank D. Sahakyan for discussions. This work was supported in part by DOE grant DE-FG02-90ER-40560.
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