No black hole bomb for $D$-dimensional extremal Reissner–Nordstrom black holes under charged massive scalar perturbation

Jia-Hui Huang$^{1,2,3,a}$

1 Guangdong Provincial Key Laboratory of Quantum Engineering and Quantum Materials, School of Physics and Telecommunication Engineering, Higher Education Mega Center, South China Normal University, West Waihuan Road No. 378, Guangzhou, China
2 Guangdong Provincial Key Laboratory of Nuclear Science, Institute of Quantum Matter, Higher Education Mega Center, South China Normal University, West Waihuan Road No. 378, Guangzhou, China
3 Guangdong-Hong Kong Joint Laboratory of Quantum Matter, Higher Education Mega Center, South China Normal University, West Waihuan Road No. 378, Guangzhou, China

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Abstract The superradiant stability of asymptotically flat $D$-dimensional extremal Reissner–Nordstrom black holes under charged massive scalar perturbation is analytically studied. Recently, an analytical method has been proposed by the author and used to prove that five and six-dimensional extremal Reissner–Nordstrom black holes are superradiantly stable under charged massive scalar perturbation. We apply this analytical method in the $D$-dimensional extremal Reissner–Nordstrom black hole case and prove that there is no black hole bomb for $D$-dimensional Reissner–Nordstrom black hole under charged massive scalar perturbation and the system is superradiantly stable.

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1 Introduction

Analysis of linear perturbation of black holes plays an important role in many topics, such as the (in)stability of black hole solutions, the black hole ringdown phase after binary merger and astrophysics [1–3]. Among various linear perturbation modes of black holes, superradiance mode is an interesting one, which can extract energy from the black holes [4–7]. When a charged bosonic wave is scattering off a charged rotating black hole, the wave is amplified by the black hole if the angular frequency $\omega$ of the wave satisfies

$$\omega < m\Omega_H + e\Phi_H. \quad (1.1)$$

where $e$ and $m$ are the charge and azimuthal number of the bosonic wave mode, $\Omega_H$ is the angular velocity of the black hole horizon and $\Phi_H$ is the electromagnetic potential of the black hole horizon. This superradiant scattering was studied long time ago [8–14], and has broad applications in various areas of physics (for a recent comprehensive review, see [4]).

For a superradiant black hole and perturbation system, if a mirror-like mechanism is introduced between the black hole
horizon and spatial infinity, the amplified perturbation will be scattered back and forth between the “mirror” and black hole horizon, and this will lead to the superradiant instability of the system. This is dubbed black hole bomb mechanism [15–18]. Superradiant (in)stability of various charged and rotating black holes has been studied extensively in the literature. The superradiant (in)stability of four-dimensional rotating Kerr black holes under massive scalar or vector perturbation has been studied in [19–33]. Rotating or charged black holes with certain asymptotically curved space are proved to be superradiantly unstable under massless or massive bosonic perturbation [34–45], where the asymptotically curved geometries provide natural mirror-like boundary conditions.

For asymptotically flat black holes, the four-dimensional extremal or non-extremal Reissner–Nordstrom (RN) black hole has been proved superradiantly stable against charged massive scalar perturbation in the full parameter space of the black hole and scalar perturbation system [46–51]. The argument in the proof is that the two conditions for the possible superradiant instability of the system, (i) existence of a trapping potential well outside the black hole horizon and (ii) superradiant amplification of the trapped modes, can’t be satisfied simultaneously in the RN black hole and scalar perturbation system [46,48].

For various higher dimensional black holes, the linear stability analysis has also been studied in the literature (for an incomplete list, see [52–62]). In Ref. [54], the asymptotically flat RN black holes in $D = 5, 6, \ldots, 11$ are shown to be stable by studying the time-domain evolution of the massless scalar perturbation with a numerical characteristic integration method. In Ref. [55], the authors have provided numerical evidence that asymptotically flat extremal RN black holes are stable for arbitrary $D$ under massless perturbation.

It is known that the mass term of a scalar perturbation may provide a natural mirror-like boundary condition for low frequency perturbation. Recently, an analytical method based on the Descartes’ rule of signs has been developed by the author to study the superradiant stability of higher dimensional RN black holes under charged massive scalar perturbation [63,64]. Explicitly, the superradiant stability of five and six dimensional extremal RN black holes and five dimensional non-extremal RN black holes has been studied and it is proved that there is no black hole bomb for each case.

In this work, we will go a step further and apply the above mentioned analytical method to study the superradiant stability of arbitrary $D$-dimensional ($D \geq 7$) extremal RN black hole under charged massive scalar perturbation. The effect on the dynamics of the scalar perturbation, which originates from the curved RN black hole, can be described by an effective potential. We will show that there is no potential well for the effective potential experienced by the scalar perturbation. The two conditions for the possible superradiant instability can not be satisfied simultaneously, so there is no black hole bomb for $D$-dimensional extremal RN black hole under charged massive scalar perturbation and the system is superradiantly stable.

The organization of this paper is as follows: In Sect. 2, we present a general description of the model and the asymptotic analysis of boundary conditions. In Sect. 3, the effective potential of the radial equation of motion is given and the asymptotic behaviors of the effective potential at the horizon and spatial infinity are discussed. In Sect. 4, we present a general description of the proof that there is no potential well outside the black hole horizon for the superradiant modes. Most of the details of the proof for $D = 7$ case and $D$-dimensional case are in the appendix. The final Section is devoted to the summary.

2 Scalar field in $D$-dimensional RN black holes

We first present our model with non-extremal $D$-dimensional RN black hole and then take the extremal limit for further discussion. The metric of the $D$-dimensional non-extremal RN black hole [63–65] is

$$ds^2 = - f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{D-2}. \quad (2.1)$$

The function $f(r)$ reads

$$f(r) = 1 - \frac{2m}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}}, \quad (2.2)$$

where the parameters $m$ and $q$ are related with the ADM mass $M$ and electric charge $Q$ of the RN black hole,

$$m = \frac{8\pi}{(D-2) Vol(S^{D-2})} M, \quad q = \frac{8\pi}{\sqrt{2(D-2)(D-3)}Vol(S^{D-2})} Q. \quad (2.3)$$

Here $Vol(S^{D-2}) = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}$ is the volume of unit $(D-2)$-sphere. $d\Omega^2_{D-2}$ is the common line element of a $(D-2)$-dimensional unit sphere $S^{D-2}$ and can be written as

$$d\Omega^2_{D-2} = d\theta_2^2 + \sum_{i=1}^{D-3} \prod_{j=i+1}^{D-2} \sin^2(\theta_i) d\theta_i^2, \quad (2.4)$$

where the ranges of the angular coordinates are taken as $\theta_i \in [0, \pi]$ ($i = 2, \ldots, D-2$), $\theta_i \in [0, 2\pi]$. The inner and outer horizons of this RN black hole are

$$r_{\pm} = (m \pm \sqrt{m^2 - q^2})^{1/(D-3)}. \quad (2.5)$$

It is obvious that we have the following two equalities

$$r_+^{D-3} + r_-^{D-3} = 2m, \quad r_+^{D-3} r_-^{D-3} = q^2. \quad (2.6)$$
The electromagnetic field outside the black hole horizon is described by the following 1-form vector potential

\( A = -\frac{D - 2}{2(D - 3)} \frac{q}{r^{D-3}} dt = -c_D \frac{q}{r^{D-3}} dt. \)  

The equation of motion for a charged massive scalar perturbation in this \( D \)-dimensional non-extremal black hole background is governed by the covariant Klein–Gordon equation

\[(D_v D^\nu - \mu^2)\phi = 0,\]  

where \( D_v = \nabla_v - i e A_v \) is the covariant derivative and \( \mu, e \) are the mass and charge of the scalar field respectively. The solution of this equation with definite angular frequency can be decomposed as

\[\phi(t, r, \theta_i) = e^{-i\omega t} R(r)\Theta(\theta_i).\]  

The angular eigenfunctions \( \Theta(\theta_i) \) are \((D - 2)\)-dimensional scalar spherical harmonics and the corresponding eigenvalues are given by \(-l(l + D - 3), (l = 0, 1, 2, \ldots)\) \([66–70]\).

The radial equation of motion is described by

\[\Delta \frac{d}{dr} \left( \Delta \frac{dR}{dr} \right) + UR = 0,\]  

where

\[\Delta = r^{D-2} f(r),\]  

\[U = (\omega + e A_r) r^{2(D-2)} - (l + D - 3) r^{D-4} \Delta - \mu^2 r^{D-2} \Delta.\]  

In order to analyze the physical boundary conditions needed here at the horizon and spatial infinity, we define the tortoise coordinate \( y \) by \( dy = r^{D-2} \Delta dr \) and a new radial function \( \tilde{R} = r^{\frac{D-2}{2}} R \), then the radial equation (2.10) can be rewritten as

\[\frac{d^2 \tilde{R}}{dy^2} + \tilde{U} \tilde{R} = 0,\]  

where

\[\tilde{U} = \frac{U}{r^{2(D-2)}} - \frac{(D - 2) f(r)(D - 4)f(r) + 2rf''(r)}{4r^2}.\]  

The asymptotic behaviors of \( \tilde{U} \) at the spatial infinity and the outer horizon are

\[\lim_{r \to +\infty} \tilde{U} = \omega^2 - \mu^2,\]  

\[\lim_{r \to r_+} \tilde{U} = \left( \omega - c_D \frac{eq}{r_+^{D-3}} \right)^2 = (\omega - e\Phi_h)^2,\]  

where \( \Phi_h \) is the electric potential of the outer horizon of the RN black hole. Here we need purely ingoing wave condition at the horizon and bound state condition at spatial infinity, which leads to the following two conditions

\[\omega < e\Phi_h = c_D \frac{eq}{r_+^{D-3}},\]  

\[\omega < \mu.\]  

The first inequality is the superradiance condition and the second inequality gives the bound state condition.

### 3 Effective potential and its asymptotic behaviors

In order to analyze the superradiant stability of the RN black hole and scalar perturbation system, we define a new radial function \( \psi = \Delta^{1/2} R \), then the radial equation of motion (2.10) can be written as a Schrodinger-like equation

\[\frac{d^2 \psi}{dr^2} + (\omega^2 - V) \psi = 0,\]

where \( V \) is the effective potential, which is the main object we will discuss. The explicit expression for the effective potential \( V \) is

\[V = \omega^2 + \frac{B_1}{A_1},\]

where

\[A_1 = 4r^2 (r^{2D-6} - 2mr^{D-3} + q^2)^2,\]

\[B_1 = 4(\mu^2 - \omega^2) r^{4D-10}\]

\[+ (2l + D - 2)(2l + D - 4) r^{4D-12}\]

\[- 8(\mu^2 - c_D e^{q\omega}) r^{3D-7}\]

\[+ 4q^2 (\mu^2 - c_D e^q \omega) r^{2D-4}\]

\[+ 2(2m^2 - q^2 (2\lambda + 3) (D - 4) (D - 2) + 2) r^{2D-6}\]

\[- 4m q^2 (D - 4) (D - 2) r^{D-3}\]

\[+ q^4 (D - 4) (D - 2),\]

where \( \lambda = l + D - 3 \).

#### 3.1 Extremal limit

Now we consider the extremal limit by taking \( m = q \). In this limit, the superradiance condition becomes into

\[\omega < c_D e.\]  

The expression of the effective potential \( V \) becomes into

\[V = \omega^2 + \frac{B}{A},\]  

where \( A \) and \( B \) read

\[A = 4r^2 (r^{2D-6} - 2mr^{D-3} + m^2)^2\]

\[= 4r^2 (r^{D-3} - m)^4,\]

\[B = 4(\mu^2 - \omega^2) r^{4D-10}\]

\[+ 2(2l + D - 2)(2l + D - 4) r^{4D-12}\]

\[+ 8(\mu^2 - c_D e^{q\omega}) r^{3D-7}\]

\[+ 4q^2 (\mu^2 - c_D e^q \omega) r^{2D-4}\]

\[+ 2(2m^2 - q^2 (2\lambda + 3) (D - 4) (D - 2) + 2) r^{2D-6}\]

\[- 4m q^2 (D - 4) (D - 2) r^{D-3}\]

\[+ q^4 (D - 4) (D - 2)\]
\[ +4m^2(\mu^2 - c_D e^2)r^{2D-4} + 2m^2(2\lambda_i + 3(D-4)(D-2))r^{2D-6} - 4m^3(D-4)(D-2)r^{D-3} + m^4(D-4)(D-2). \] 

(3.7)

In the extremal limit, the asymptotic behaviors of \( V \) at the horizon and spatial infinity are

\[ r \to r_h, \quad V \to -\infty; \] 

\[ r \to +\infty, \quad V \to \mu^2. \] 

(3.8)

(3.9)

At the spatial infinity, the asymptotic behavior of the derivative of the effective potential, \( V'(r) \), is

\[
V'(r) \rightarrow \begin{cases} 
\frac{-(D-2)(D-4)-4\lambda_i-8m(\mu^2 + c_D e^2 - 2\omega^2)}{2r^3}, & D = 5; \\
\frac{-(D-2)(D-4)-4\lambda_i}{2r^3}, & D \geq 6. 
\end{cases}
\]

(3.10)

Given the superradiance condition (3.4) and bound state condition (2.16), we can prove \( V'(r) < 0 \) at spatial infinity when \( D = 5 \). It is also obvious that \( V'(r) < 0 \) at spatial infinity when \( D \geq 6 \). This means that there is no potential well near the spatial infinity and one maximum exists for the effective potential \( V(r) \) outside the black hole horizon.

In the next section, we will prove that there is only one extreme (it is just the maximum mentioned above) outside the event horizon \( r_h \) for the effective potential in the \( D \)-dimensional extremal RN black hole case, no potential well exists outside the event horizon for the superradiance modes. So there is no black hole bomb and \( D \)-dimensional extremal RN black holes are superradiantly stable under charged massive scalar perturbation. In our proof, the mathematical theorem Descartes’ rule of signs plays an important role, which asserts that the number of positive roots of a polynomial equation with real coefficients is at most the number of sign changes in the sequence of the polynomial’s coefficients.

4 Analysis of the potential wells of \( V \)

In this section, we show that there is only one extreme for the effective potential outside the RN black hole horizon by analyzing the derivative of the effective potential \( V'(r) \). Explicitly, it is shown that only one real root exists for the following equation

\[ V'(r) = 0, \] 

when \( r > r_h \).

In the extremal RN black hole case, the derivative of effective potential (3.5) can be expressed as \( V'(r) = \frac{E'(r)}{F'(r)}, E(r) \) and \( F(r) \) are polynomials of \( r \), which read as follows

\[ F(r) = 2r^3(r^{D-3} - m)^5, \] 

\[ E(r) = -(D_1 + 4\lambda_i) r^{5(D-3)}. \] 

When \( a_5 = -(D_1 + 4\lambda_i), a_4 = m(5D_1 + (24 - 4D)\lambda_i), a_3 = -2m^2(5D_1 + (18 - 4D)\lambda_i), a_2 = 2m^3(5D_1 + (8 - 2D)\lambda_i), a_1 = -5m^4 D_1, a_0 = m^5 D_1, a'_3 = -4(D - 3)m(\mu^2 + \omega(c_D e^2 - 2\omega)), a'_2 = 4(D - 3)m^2(c_D e^2 + 2\mu^2 - 3c_D e\omega), \)

\[ + m(5D_1 + (24 - 4D)\lambda_i)r^{4(D-3)} - 2m^2(5D_1 + (18 - 4D)\lambda_i)r^{3(D-3)} + 2m^3(5D_1 + (8 - 2D)\lambda_i)r^{2(D-3)} - 4(D - 3)m^2(\mu^2 + \omega(c_D e^2 - 2\omega))r^{4D-10} + 4(D - 3)m^3(c_D e^2 + 2\mu^2 - 3c_D e\omega)r^{3D-7} + 4(D - 3)m^4(c_D e^2 - \mu^2)r^{2D-4} - 5m^4 D_1 r^{D-3} + m^5 D_1 \]

\[ = a_0 + a_1 r^{D-3} + a_2 r^{2D-6} + a_3 r^{5D-9} + a'_2 r^{3D-7} + a'_4 r^{4D-12} + a'_3 r^{5D-15} \]

\[ \text{where} \]

\[ a_5 = -(D_1 + 4\lambda_i), a_4 = m(5D_1 + (24 - 4D)\lambda_i), a_3 = -2m^2(5D_1 + (18 - 4D)\lambda_i), a_2 = 2m^3(5D_1 + (8 - 2D)\lambda_i), a_1 = -5m^4 D_1, a_0 = m^5 D_1, a'_3 = -4(D - 3)m(\mu^2 + \omega(c_D e^2 - 2\omega)), a'_2 = 4(D - 3)m^2(c_D e^2 + 2\mu^2 - 3c_D e\omega), a'_4 = 4(D - 3)m^3(c_D e^2 - \mu^2). \]

\[ \text{and } D_1 = D^2 - 6D + 8 = (D - 2)(D - 4). \]

Because we are interested in the real roots of the equation \( V'(r) = 0 \), only the numerator \( E(r) \) of \( V'(r) \) is important for our analysis. It is equivalent to consider the real roots of the equation \( E(r) = 0 \). After changing the variable \( z = r - r_h \), \( E(r) \) can be rewritten as a polynomial of \( z, E(z) \). A real root of \( E(r) = 0 \) when \( r > r_h \) is equivalent to a positive root of \( E(z) = 0 \). The polynomial \( E(z) \) can be expanded as

\[ E(z) = \sum_{i=0}^{5D-15} b_i z^i. \]

(4.5)

In the following of this section, we will prove that there is only one positive real root for the equation \( E(z) = 0 \), i.e., only one maximum outside the event horizon \( r_h \) for the effective potential in the \( D \)-dimensional extremal RN black hole case and no potential well exists outside the event horizon for the superradiance modes. This is achieved by showing

\[ \text{sign}(b_{p+1}) \leq \text{sign}(b_p), \quad 0 \leq p < 5D - 15. \]

(4.6)

Then, for the sequence of the real coefficients \( b_{5D-15}, b_{5D-16}, \ldots, b_0 \) in the polynomial \( E(z) \), the sign change is always 1 and according to Descartes’ rule of signs, the equation \( E(z) = 0 \) has at most one positive real root.

The constant term \( b_0 \) in \( E(z) \) is

\[ b_0 = a_0 + a_1 r_h^{D-3} + a_2 r_h^{2D-6} + a_3 r_h^{5D-9} + a'_2 r_h^{3D-7} + a'_3 r_h^{4D-12} + a'_4 r_h^{5D-15}. \]

(4.7)
Plugging (4.4) into the above equation and after a straightforward calculation, we can obtain

\[ b_0 = 8(D - 3)m^5 r_h^2 (\omega - c_D e)^2 > 0. \] (4.8)

where we use the equation \( r_h^D = m \).

It is easy to see that \( a_5 = -(D_1 + 4\lambda_i) < 0 \). After considering the superradiance condition (3.4) and bound state condition (2.16), it is also easy to verify that

\[ a'_4 = -4(D - 3)m(\mu^2 + \omega(cDe - 2\omega)) = -4(D - 3)m(\mu^2 - \omega^2 + \omega(cDe - \omega)) < 0. \] (4.9)

So we can immediately find that

\[ b_{5D - 1} = a_5 = -(D_1 + 4\lambda_i) < 0, \]
\[ b_{5D - 16} = a_5C_{5D - 1}^1 r_h < 0, \]
\[ \ldots \]
\[ b_{4D - 9} = a_5C_{5D - 15}^0 r_h^D - 6 < 0, \]
\[ b_{4D - 10} = a_5C_{5D - 15}^{10} r_h^D - 5 + a'_4 < 0, \]
\[ b_{4D - 11} = a_5C_{5D - 15}^{11} r_h^D - 4 + a'_4C_{4D - 10} r_h < 0. \] (4.10)

Then let us consider the coefficient \( b_{4D - 12} \) of \( z^{4D - 12} \), which is

\[ b_{4D - 12} = a_5C_{5D - 15}^{12} r_h^D - 3 + a'_4C_{4D - 10} r_h + a_4. \] (4.11)

The term involving \( a'_4 \) is negative. Now, we show that the sum of the left two terms, \( a_5C_{5D - 15}^{12} r_h^D + a_4 \), is also negative. It is easy to check that when \( D \geq 7, C_{5D - 15} > 5D - 15 > 15 \). Then

\[ a_5C_{5D - 15}^{12} r_h^D + a_4 = -mD_1(C_{5D - 15} - 5) - 4m\lambda_1(C_{5D - 15}^2 - 6 + D), \]

The first and second terms on the right of the above equation are both negative, so we have

\[ b_{4D - 12} < 0. \] (4.12)

4.1 Coefficients of \( z^p, 3D - 7 < p < 4D - 12 \)

For \( 3D - 7 < p < 4D - 12 \), the coefficient of \( z^p \) can be written as follows,

\[ b_p = a_5C_{5D - 15}^p r_h^{2D - 5} + a'_4C_{4D - 10} r_h^{4D - 10} + a_4C_{4D - 12} r_h. \]

On the right of the above equation, the term involving \( a'_4 \) is negative because \( a'_4 < 0 \). Now, let’s prove the sum of the left two terms is also negative in the following and we will neglect the positive factor \( r_h^{4D - 12 - p} \) for simplicity,

\[ a_5C_{5D - 15}^p + a'_4C_{4D - 12} \]
\[ = mC_{5D - 15}^p(-D_1 - 4\lambda_i) + mC_{4D - 12}^p(5D_1 + 4(6 - D)\lambda_i) \]
\[ = mD_1(-C_{5D - 15}^p + 5C_{4D - 12}^p) - 4m\lambda_1(C_{5D - 15}^p + (D - 6)C_{4D - 12}^p) \]
\[ = mD_1C_{4D - 12}^p(-5D - 15 - 5D - 16) \]
\[ \frac{5D - 15 - p + 1}{4D - 12 - p + 1} \cdot \frac{5D - 15 - p + 1}{4D - 12 - p + 1} < -\left( \frac{5}{4} \right)^p. \] (4.14)

The \( \lambda_1 \) term in the above equation is obviously negative when \( D \geq 7 \). Because

\[ - \frac{5D - 15}{4D - 12} \cdot \frac{5D - 15 - p + 1}{4D - 12 - p + 1} < -\left( \frac{5}{4} \right)^p. \] (4.15)

and \( \left( \frac{5}{4} \right)^p > 5 \) when \( p > 3D - 7 > 11 \), the \( D_1 \) term in (4.14) is also negative. So (4.14) is negative.

We finally obtain that

\[ b_p < 0, \quad 3D - 7 < p < 4D - 12. \] (4.16)

4.2 Coefficients of \( z^p, p = 3D - 7, 3D - 8, 3D - 9 \)

In this subsection, we prove the sign relations between pairs of adjacent coefficients of \( z^p, p = 3D - 7, 3D - 8, 3D - 9 \). The three coefficients are listed as following

\[ b_{3D - 7} = a_5C_{5D - 15}^{3D - 7} r_h^{2D - 8} + a'_4C_{4D - 10}^{3D - 7} r_h^{4D - 10} + a_4C_{4D - 12}^{3D - 7} r_h. \]

Plugging (4.4) into the above equations and after a straightforward calculation, we can obtain

\[ b_{3D - 7} = r_h^{p - 5} (a_{sm}C_{5D - 15}^{3D - 7} + a_4C_{4D - 12}^{3D - 7}) \]
\[ + a'_4mC_{4D - 10}^{3D - 7} + a'_4 \]
\[ = r_h^{p - 5} (-D_1 m(C_{5D - 15}^{3D - 7} - 5C_{4D - 12}^{3D - 7})) \]
\[ \frac{5D - 15 - p + 1}{4D - 12 - p + 1} < -\left( \frac{5}{4} \right)^p. \] (4.17)
that these coefficients satisfy

\[ \frac{C_{3D-7}^3}{2 C_{3D-7}^3 + C_{4D-10}^3} \omega = \omega \]

\[ + 4(D - 3)m^2(\mu^2 - \omega^2)(2C_{3D-7}^3 - C_{4D-10}^3). \]

(4.18)

\[ b_{3D-8} = r_h^{2D-4}(a_5 m C_{3D-8}^{3D-8} + a_4 C_{4D-12}^{3D-8} + r_h a_4' m C_{4D-10}^{3D-8} + a_3') \]

\[ + r_h^{2D-4}(-D_1 m C_{3D-8}^{3D-15} - 5 C_{4D-12}^{3D-8}) \]

\[ - 4m \lambda (C_{3D-8}^{3D-15} + 6(D - 6)C_{4D-12}^{3D-8}) \]

\[ + 4(D - 3)m^2 r_h (2C_{3D-7}^3 + C_{4D-10}^3)(c D e - \omega) \]

\[ \times \left( \frac{C_{3D-7}^3}{2 C_{3D-7}^3 + C_{4D-10}^3} - \omega \right) \]

\[ + 4(D - 3)m^2 r_h (\mu^2 - \omega^2)(2C_{3D-8}^3 - C_{4D-10}^3). \]

(4.19)

\[ b_{3D-9} = r_h^{2D-3}(a_5 m C_{3D-9}^{3D-9} + a_4 C_{4D-12}^{3D-9} + r_h a_4' m C_{4D-10}^{3D-9} + a_3') \]

\[ + r_h^{2D-3}(-D_1 m C_{3D-9}^{3D-15} - 5 C_{4D-12}^{3D-9} + 10) \]

\[ - 4m \lambda (C_{3D-9}^{3D-15} + 6(D - 6)C_{4D-12}^{3D-9} + 9 - 2D) \]

\[ + 4(D - 3)m^2 r_h^2 (2C_{3D-7}^{3D-9} + C_{4D-10}^{3D-9})(c D e - \omega) \]

\[ \times \left( \frac{C_{3D-7}^{3D-9}}{2 C_{3D-7}^{3D-9} + C_{4D-10}^{3D-9}} - \omega \right) \]

\[ + 4(D - 3)m^2 r_h (\mu^2 - \omega^2)(2C_{3D-9}^{3D-7} - C_{4D-10}^{3D-9}). \]

(4.20)

Here it is not easy to fix the signs of the three coefficients. Now we define the following three normalized coefficients

\[ \tilde{b}_{3D-7} = \frac{r_h^{3D-7}}{2 C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}} b_{3D-7}. \]

\[ \tilde{b}_{3D-8} = \frac{r_h^{3D-8}}{2 C_{3D-7}^{3D-8} + C_{4D-10}^{3D-8}} b_{3D-8}. \]

\[ \tilde{b}_{3D-9} = \frac{r_h^{3D-9}}{2 C_{3D-7}^{3D-9} + C_{4D-10}^{3D-9}} b_{3D-9}. \]

(4.21)

It is worth emphasizing that all the normalization factors are positive and the signs of \( \tilde{b}_a \) and \( \tilde{b}_b \) are the same. We show that these coefficients satisfy

\[ \tilde{b}_{3D-7} < \tilde{b}_{3D-8}, \quad \tilde{b}_{3D-8} < \tilde{b}_{3D-9}. \]

When \( \tilde{b}_{3D-7} < \tilde{b}_{3D-9} \), the possible signs of these two coefficients are \((-,-),(+,-),(++),\) which can be denoted as \( \text{sign}(b_{3D-7}) \leq \text{sign}(b_{3D-9}) \). Similarly, we have

\[ \text{sign}(b_{3D-7}) \leq \text{sign}(b_{3D-8}). \]

(4.22)

\[ \text{sign}(b_{3D-8}) \leq \text{sign}(b_{3D-9}). \]

(4.23)

4.3 Coefficients of \( z^p, 0 < p < 3D - 9 \)

Similarly, we can prove the sign relations between pairs of adjacent coefficients of \( z^p, 0 < p < 3D - 9 \). After complicated cases on a case by case basis, we show the following sign relations

\[ \text{sign}(b_{p+1}) \leq \text{sign}(b_p) \quad 0 < p < 3D - 9. \]

(4.24)

The details of the proofs for \( D = 7 \) case and \( D \)-dimensional case can be found in the Appendix.

5 Summary

In this work, superradiant stability of \( D \)-dimensional (\( D \geq 7 \)) extremal RN black hole under charged massive scalar perturbation is studied analytically. Based on the asymptotic analysis of the effective potential \( V(r) \) experienced by the scalar perturbation, we know there is one maximum for the effective potential outside the black hole horizon. Then we derive the numerator \( E(z) \) of the derivative of the effective potential, which is a polynomial of \( z = r - r_h \) with real coefficients. In Sect. 4, we show in \((4.8),(4.10),(4.12),(4.16)\) that

\[ b_0 > 0, \quad b_p < 0 \quad (3D - 7 < p < 5D - 15). \]

(5.1)

According the results in \((4.22),(4.23),(4.24)\), we obtain

\[ \text{sign}(b_p) \geq \text{sign}(b_{p+1}) \quad (0 < p < 3D - 7). \]

(5.2)

So the sign change in the following sequence of the real coefficients of \( E(z) \),

\[ (b_{5D-15}, b_{5D-16}, \ldots, b_{p+1}, b_p, \ldots, b_1, b_0), \]

(5.3)

is always 1. Then according to Descartes’ rule of signs, we know there is at most 1 positive root for the equation \( E(z) = 0 \) (i.e. \( V'(r) = 0 \) when \( r > r_h \)). Thus there is only one extreme for the effective potential outside the horizon, which is the maximum obtained by asymptotical analysis and there is no potential well outside the horizon for the superradiance modes. Given the superradiance condition and bound state condition, a typical shape of the effective potential outside the horizon is shown in Fig. 1. The two conditions for the possible superradiant instability can not be satisfied simultaneously, so there is no black hole bomb for \( D \)-dimensional extremal RN black hole under charged massive scalar perturbation.
The analytical method used in this paper seems to be efficient to analyze the superradiant stability of higher dimensional black holes. We have already applied it to study the superradiant stability of 5-dimensional non-extremal RN black hole under charged massive scalar perturbation in [63]. As a step further, it is interesting to apply it to study other higher dimensional non-extremal RN black hole cases and even the $D$-dimensional ($D \geq 6$) non-extremal RN black hole case. It is also interesting to apply it in studying higher dimensional rotating black hole cases.

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Appendix A: Proof of sign relations in $D=7$ case

In this section we will show that no black hole bomb exists for $D=7$ extremal RN black hole under charged massive scalar perturbation. In this case, the black hole horizon is located at $r_+ = r_− = r_h = m^{1/4}$. The bound state condition is $\omega < \mu$, and superradiance condition is $\omega < e^{\phi_h} = c_{De} = \sqrt{5/8}e$.

The denominator of the derivative of effective potential $V'(r)$ is $2r^3(r^4 - m)$. The numerator of $V'$ is

\[ n_\gamma(r) = a_5r^{20} + a_4'r^{18} + a_4r^{16} + a_3'r^{14} + a_3r^{12} + a_2'r^{10} + a_2r^8 + a_1r^4 + a_0, \]

where

\[ a_5 = -(4\lambda_i + 15), \quad a_4' = -16m(\mu^2 + c_{De}e\omega - 2\omega^2), \]
\[ a_4 = m(75 - 4\lambda_i), \]
\[ a_3' = 16m^2(c_{De}e^2 - 3c_{De}e\omega + 2\mu^2), \]
\[ a_3 = 10m^2(2\lambda_i - 15), \]
\[ a_2' = 16m^3(c_{De}e^2 - \mu^2), \]
\[ a_2 = 6m^3(25 - 2\lambda_i), \quad a_1 = -75m^4, \quad a_0 = 15m^3. \] (A.1)

We can see $a_5 < 0$. Given the bound state condition $\omega < \mu$ and superradiance condition $\omega < c_{De}e$, it is easy to check $a_4' < 0$. As explained in the main content of the paper, we are interested in the numerator of $V'$. Change the variable $r$ to $z = r - r_h$, then the numerator of $V'$ can be written as

\[ n_\gamma(z) = \sum_{i=0}^{20} b_i z^i. \] (A.2)

Now let’s consider the signs of the coefficients or the sign relations between adjacent coefficients in the sequence $(b_20, b_{19}, ..., b_1, b_0)$.

Because $a_5 < 0$, $a_4' < 0$, it is easy to see that

\[ b_{20} = a_5 = -(4\lambda_i + 15) < 0, \quad b_{19} = a_5c_{De}r_h < 0, \]
\[ b_{18} = a_5C_{20}r_h^2 + a_4' < 0, \quad b_{17} = a_5C_{20}r_h^3 + a_4C_{18}r_h < 0, \]
\[ b_{16} = a_5C_{20}r_h^4 + a_4C_{18}r_h^2 + a_4, \]
\[ = a_4C_{18}r_h^2 - 8m(9075 + 2423\lambda_i) < 0, \]
\[ b_{15} = a_4C_{20}r_h^5 + a_4C_{18}r_h^3 + a_4C_{16}r_h, \]
\[ = a_4C_{18}r_h^3 - 320m^{3/4}(723 + 194\lambda_i) < 0, \]
\[ b_0 = a_5r_h^{20} + a_4r_h^{18} + a_4r_h^{16} + a_4r_h^{14} + a_3r_h^{12} + a_3r_h^{10} + a_2r_h^8 + a_1r_h^4 + a_0, \]
\[ = 32m^{11}(\omega - c_{De}e)^2 > 0. \] (A.3)

The signs of the other coefficients cannot be judged as above. These coefficients are listed as follows

\[ b_{14} = a_5C_{20}r_h^6 + a_4C_{18}r_h^4 + a_4C_{16}r_h^2 + a_3, \]
\[ b_{13} = a_5C_{20}r_h^7 + a_4C_{18}r_h^5 + a_4C_{16}r_h^3 + a_4C_{14}r_h, \]
\[ b_{12} = a_5C_{20}r_h^8 + a_4C_{18}r_h^6 + a_4C_{16}r_h^4 + a_4C_{14}r_h^2 + a_3, \]
\[ b_{11} = a_5C_{20}r_h^9 + a_4C_{18}r_h^7 + a_4C_{16}r_h^5 + a_3C_{14}r_h^3 + a_2C_{12}r_h, \]
\[ b_{10} = a_5C_{20}r_h^{10} + a_4C_{18}r_h^8 + a_4C_{16}r_h^6 + a_3C_{14}r_h^4 + a_3C_{12}r_h^2 + a_2, \]
\[ b_9 = a_5 C_{20r}^{11}_{r_h} + a_4 C_{18r}^{9}_{r_h} + a_4 C_{16r}^{7}_{r_h} + a_3 C_{14r}^{5}_{r_h} + a_2 C_{12r}^{3}_{r_h} + a_2 C_{10r}^{9}_{r_h}, \]
\[ b_8 = a_5 C_{20r}^{12}_{r_h} + a_4 C_{18r}^{10}_{r_h} + a_4 C_{16r}^{8}_{r_h} + a_3 C_{14r}^{6}_{r_h} + a_2 C_{12r}^{4}_{r_h} + a_2 C_{10r}^{2}_{r_h} + a_2 + a_2, \]
\[ b_7 = a_5 C_{20r}^{13}_{r_h} + a_4 C_{18r}^{11}_{r_h} + a_4 C_{16r}^{9}_{r_h} + a_3 C_{14r}^{7}_{r_h} + a_2 C_{12r}^{5}_{r_h} + a_2 C_{10r}^{3}_{r_h} + a_2 C_{8r}^{1}_{r_h}, \]
\[ b_6 = a_5 C_{20r}^{14}_{r_h} + a_4 C_{18r}^{12}_{r_h} + a_4 C_{16r}^{10}_{r_h} + a_3 C_{14r}^{8}_{r_h} + a_2 C_{12r}^{6}_{r_h} + a_2 C_{10r}^{4}_{r_h} + a_2 C_{8r}^{2}_{r_h}, \]
\[ b_5 = a_5 C_{20r}^{15}_{r_h} + a_4 C_{18r}^{13}_{r_h} + a_4 C_{16r}^{11}_{r_h} + a_3 C_{14r}^{9}_{r_h} + a_2 C_{12r}^{7}_{r_h} + a_2 C_{10r}^{5}_{r_h} + a_2 C_{8r}^{3}_{r_h}, \]

Then we will normalized the above coefficients with positive factors and consider the sign relations between pairs of adjacent normalized coefficients. For adjacent coefficients \((b_{13}, b_{14})\)

\[ \tilde{b}_{13} - \tilde{b}_{14} = \frac{b_{13}}{2C_{14}^{13} + C_{18}^{13}} - \frac{b_{14}}{2C_{14}^{14} + C_{18}^{14}} = \frac{64m^2[2907150 + 7432155\lambda_l + 1071\sqrt{mcDE(c_{DE} - \omega) + 4284\sqrt{m(\mu^2 - \omega^2)}]} \quad \text{3290119} \quad (A.4) \]

\[ b_4 = a_5 C_{20r}^{16}_{r_h} + a_4 C_{18r}^{14}_{r_h} + a_4 C_{16r}^{12}_{r_h} + a_3 C_{14r}^{10}_{r_h} + a_3 C_{12r}^{8}_{r_h} + a_2 C_{10r}^{6}_{r_h} + a_2 C_{8r}^{4}_{r_h} + a_1, \]

Given the bound state condition \(\omega < \mu\) and superradiance condition \(\omega < c_{DE}\), we have

\[ \tilde{b}_{13} - \tilde{b}_{14} > 0, \quad \text{sign}(b_{13}) \equiv \text{sign}(b_{14}). \quad (A.5) \]

Similarly, we have the following differences between adjacent coefficients

\[ \tilde{b}_{12} - \tilde{b}_{13} = \frac{b_{12}}{2C_{14}^{12} + C_{18}^{12}} - \frac{b_{13}}{2C_{14}^{13} + C_{18}^{13}} = \frac{2m^2[53035800 + 13044565\lambda_l + 74256\sqrt{mcDE(c_{DE} - \omega) + 297024\sqrt{m(\mu^2 - \omega^2)}]} \quad \text{2877511} \quad (A.6) \]

\[ \tilde{b}_{11} - \tilde{b}_{12} = \frac{b_{11}}{2C_{14}^{11} + C_{18}^{11}} - \frac{b_{12}}{2C_{14}^{12} + C_{18}^{12}} = \frac{4m^2[19169400 + 4472699\lambda_l + 74256\sqrt{mcDE(c_{DE} - \omega) + 297024\sqrt{m(\mu^2 - \omega^2)}]} \quad \text{2933749} \quad (A.7) \]

\[ \tilde{b}_{10} - \tilde{b}_{11} = \frac{b_{10}}{2C_{14}^{10} + C_{18}^{10}} - \frac{b_{11}}{2C_{14}^{11} + C_{18}^{11}} = \frac{2m^2[41(4480678) + 9870121\lambda_l) + 15960828\sqrt{mcDE(c_{DE} - \omega) + 63648000\sqrt{m(\mu^2 - \omega^2)}]} \quad \text{186193371} \quad (A.8) \]

\[ \tilde{b}_{9} - \tilde{b}_{10} = \frac{b_{9}}{2C_{14}^{9} + C_{18}^{9}} - \frac{b_{10}}{2C_{14}^{10} + C_{18}^{10}} = \frac{16m^2[387480660 + 81912199\lambda_l + 6557980\sqrt{mcDE(c_{DE} - \omega) + 25826944\sqrt{m(\mu^2 - \omega^2)}]} \quad \text{401260671} \quad (A.9) \]
The bound state condition \( \omega < \mu \) and superradiance condition \( \omega < c_{DE} \), it is easy to see that all the differences above are positive. So we have

\[
\tilde{b}_i - \tilde{b}_{i+1} > 0, \quad \text{sign}(b_i) \geq \text{sign}(b_{i+1}), \quad (i = 1, 2, \ldots, 12).
\]  

(A.18)

According to the results (A.3), (A.5), (A.18), we conclude that the sign change in the sequence of coefficients \((b_20, b_19, \ldots, b_1, b_0)\) is always 1.

### Appendix B: Proof of sign relations in \( D \)-dimensional case

In this section, we give the details of the analytical proof of the following sign relations for \( b_p \) and \( b_{p+1} \) which are respectively the coefficients of \( z^p \) and \( z^{p+1} \) in the numerator of the derivative of the effective potential,

\[
\text{sign}(b_p) \geq \text{sign}(b_{p+1}), \quad (0 < p < 3D - 7).
\]  

(B.1)

B.1 Coefficients of \( z^p, \quad p = 3D - 7, 3D - 8, 3D - 9 \)

In this subsection, we prove the sign relations between pairs of adjacent coefficients in the sequence \((b_{3D-7}, b_{3D-8}, b_{3D-9})\).

#### B.1.1 \( \tilde{b}_{3D-7} < \tilde{b}_{3D-8} \)

Here we will prove the following inequality

\[
\tilde{b}_{3D-8} - \tilde{b}_{3D-7} > 0.
\]  

(B.2)
The difference $\bar{h}_{3D-8} - \tilde{h}_{3D-7}$ can be divided into four terms and we will consider term by term. First, let's see the $\mu^2 - \omega^2$ term in the difference, which can be written as

$$4(D - 3)m^2 \bar{h}_{3D-7} (\mu^2 - \omega^2)$$

$$\times \left[ \frac{2C_{3D-8}^{3D-7} - C_{3D-8}^{3D-7} - 2C_{3D-7}^{3D-7} - C_{4D-10}^{3D-7}}{2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7} - 2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}} \right]. \quad (B.3)$$

Given the bound state condition, the factor outside the square bracket in the above is positive. The factor in the square bracket is equivalent to

$$\frac{-2C_{3D-8}^{3D-7} + C_{3D-8}^{3D-7}}{2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7} + 2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}}. \quad (B.4)$$

With the following combinatorial identity

$$c_n^m = \frac{m + 1}{n - m} c_{n+1}^m,$$ \quad (B.5)

we have

$$\frac{C_{3D-8}^{3D-7}}{C_{3D-8}^{3D-7}} = (D - 2) \frac{C_{3D-7}^{3D-7}}{C_{3D-7}^{3D-7}}. \quad (B.6)$$

Then Eq. (B.4) can be rewritten as

$$\frac{-2}{2(D - 2)C_{3D-7}^{3D-7}/C_{4D-10}^{3D-7} + 2} + \frac{2}{2C_{3D-7}^{3D-7}/C_{4D-10}^{3D-7} + 1}. \quad (B.7)$$

which is obviously positive and then Eq. (B.3) is positive.

Second, let's see the $(\mu \mu - \omega \omega)$ term in the difference (B.2),

$$4(D - 3)m^2 \bar{h}_{3D-7} (\mu \mu - \omega \omega) \mu \mu$$

$$\times \left[ \frac{C_{3D-8}^{3D-7} - C_{3D-7}^{3D-7}}{2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7} - 2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}} \right]. \quad (B.8)$$

Using the Eq. (B.6), the factor in the square bracket of the above can be rewritten as

$$\frac{(D - 2)C_{3D-7}^{3D-7}}{2(D - 2)C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7} + 2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}}. \quad (B.9)$$

which is positive and then Eq. (B.8) is positive.

Thirdly, let's see the $D_1$ term in the difference (B.2),

$$D_1m^2 \left[ - \frac{C_{3D-8}^{3D-7} - 5C_{3D-8}^{3D-7}}{2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}} + \frac{C_{3D-7}^{3D-7} - 5C_{4D-12}^{3D-7}}{2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}} \right]. \quad (B.10)$$

Using a similar proof as (4.15), we can obtain $C_{3D-15}^{3D-7} - 5C_{3D-7}^{3D-7} - 4C_{4D-12}^{3D-7} > 0$, $C_{3D-15}^{3D-7} - 5C_{4D-12}^{3D-7} > 0$. In order to prove the factor in the square bracket in the above expression is positive, we can equivalently prove

$$\frac{C_{3D-8}^{3D-7} - 5C_{3D-8}^{3D-7}}{2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}} < \frac{2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}}{2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}} \quad \text{or} \quad \frac{C_{3D-15}^{3D-7} - 5C_{3D-7}^{3D-7}}{2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}}.$$ \quad (B.11)

Because

$$\frac{2C_{3D-7}^{3D-7} + 1}{2(D - 2)} \frac{C_{3D-7}^{3D-7}}{C_{3D-7}^{3D-7} + 4C_{4D-10}^{3D-7}} > \frac{1}{D - 2} \quad \frac{1}{2D - 7}.$$

we immediately obtain the expression (B.10) is positive.

Finally, let's see the $\lambda_1$ term in the difference (B.2),

$$-4m^5 \lambda_1 \left[ \frac{C_{3D-8}^{3D-7} + (D - 6)C_{3D-7}^{3D-7}}{2C_{3D-7}^{3D-7} + 4C_{4D-10}^{3D-7}} - \frac{C_{3D-7}^{3D-7} + (D - 6)C_{4D-12}^{3D-7}}{2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}} \right]. \quad (B.13)$$

The positivity of the above expression is equivalent to

$$\frac{C_{3D-8}^{3D-7} + (D - 6)C_{3D-7}^{3D-7}}{2C_{3D-7}^{3D-7} + 4C_{4D-10}^{3D-7}} < \frac{C_{3D-7}^{3D-7} + (D - 6)C_{4D-12}^{3D-7}}{2C_{3D-7}^{3D-7} + C_{4D-10}^{3D-7}} \quad \text{or} \quad \frac{C_{3D-8}^{3D-7} + (D - 6)C_{3D-7}^{3D-7}}{2C_{3D-7}^{3D-7} + 4C_{4D-10}^{3D-7}}.$$ \quad (B.14)

Finally, let's see the $\lambda_2$ term in the difference (B.2),

$$\frac{1}{2D - 7}C_{3D-15}^{3D-7} + (D - 6)\frac{1}{2D - 7}C_{3D-7}^{3D-7}$$

$$\frac{1}{2D - 7}C_{3D-15}^{3D-7} + (D - 6)\frac{1}{2D - 7}C_{3D-7}^{3D-7}.$$ \quad (B.15)

The left side of the above inequality is

$$\frac{1}{2D - 7} \frac{C_{3D-7}^{3D-7}}{C_{3D-15}^{3D-7}} + (D - 6)\frac{1}{2D - 7} \frac{C_{3D-7}^{3D-7}}{C_{3D-15}^{3D-7}}$$

$$= \frac{1}{k_D + 1} \left[ \frac{1}{k_D + 1} \right] = \frac{1}{2D - 7}.$$ \quad (B.17)

where

$$k_D = \frac{C_{3D-15}^{3D-7}}{(D - 6)C_{3D-12}^{3D-7}} = \frac{1}{D - 6} \cdot \frac{5D - 15}{4D - 12} \cdots \frac{2D - 7}{D - 4} > \frac{1}{D - 6} \cdot \left( \frac{5}{4} \right)^{3D-7} > 22.$$ \quad (B.18)
So we have
\[
\frac{1}{2D - 7} + \frac{1}{k_D + 1} \left( \frac{1}{D - 4} - \frac{1}{2D - 7} \right) < \frac{1}{2D - 7} + \frac{1}{23} \left( \frac{1}{D - 4} - \frac{1}{2D - 7} \right). \tag{B.19}
\]
Further, when \( D \geq 7 \) we have
\[
\frac{1}{D - 2} - \left( \frac{1}{2D - 7} + \frac{1}{23} \left( \frac{1}{D - 4} - \frac{1}{2D - 7} \right) \right) = \frac{1}{23(D - 2)(D - 4)(2D - 7)} \times (22D^2 - 202D + 454) > 0. \tag{B.20}
\]
The left side of the inequality (B.16) is smaller than \( \frac{1}{17.5} \). One can easily see that the right side of the inequality (B.16) satisfies
\[
\frac{2C_{3D-7}^3 - D - 2 C_{3D-9}^3 + 7 C_{4D-10}^3}{2C_{3D-7}^3 + C_{4D-10}^3} > \frac{1}{D - 2}. \tag{B.21}
\]
The inequality (B.16) holds when \( D \geq 7 \).

After showing that the four terms (B.3), (B.8), (B.10), (B.13) are all positive, the inequality (B.2) is proved when \( D \geq 7 \). The possible signs for \( b_{3D-7}, b_{3D-8} \) are \((-,-), (-,+), (+,+). The relation between signs of \( b_{3D-7}, b_{3D-8} \) is
\[
\text{sign}(b_{3D-7}) \leq \text{sign}(b_{3D-8}). \tag{B.22}
\]

B.1.2 \( \tilde{b}_{3D-8} < \tilde{b}_{3D-9} \)

Here let’s prove the following inequality
\[
\tilde{b}_{3D-9} - \tilde{b}_{3D-8} > 0. \tag{B.23}
\]
The difference \( \tilde{b}_{3D-9} - \tilde{b}_{3D-8} \) can be divided into four terms and we will consider term by term. First, let’s see the \( \mu^2 - \omega^2 \) term in the above difference, which is
\[
4(D - 3)m_1^2 r_h^{3D-7} (\mu^2 - \omega^2) \times \left[ \frac{2C_{3D-7}^3 - C_{3D-9}^3 - 10 C_{4D-10}^3}{2C_{3D-7}^3 + C_{4D-10}^3} - \frac{2C_{3D-8}^3 - C_{3D-9}^3 - 10 C_{4D-10}^3}{2C_{3D-8}^3 + C_{4D-10}^3} \right]. \tag{B.24}
\]
The factor in the square bracket is equivalent to
\[
\frac{-2C_{3D-9}^3}{2C_{3D-7}^3 + C_{4D-10}^3} + \frac{2C_{3D-8}^3}{2C_{3D-8}^3 + C_{4D-10}^3} = \frac{2C_{3D-8}^3}{(D-1)C_{3D-7}^3 + C_{4D-10}^3} + \frac{2C_{3D-8}^3}{2C_{3D-8}^3 + C_{4D-10}^3}. \tag{B.25}
\]
Because \( D \geq 7 \), the above expression is positive. Thus the \( \mu^2 - \omega^2 \) term in the difference \( \tilde{b}_{3D-9} - \tilde{b}_{3D-8} \) is positive given the bound state condition.

Second, let’s see the \( (c_D e - \omega) \) term in the difference
\[
\tilde{b}_{3D-9} - \tilde{b}_{3D-8} \), which is
\[
4(D - 3)m_1^2 r_h^{3D-7} (c_D e - \omega)e_D \times \left[ \frac{C_{3D-9}^3 - C_{3D-8}^3}{2C_{3D-7}^3 + C_{4D-10}^3} - \frac{C_{3D-9}^3 - C_{3D-8}^3}{2C_{3D-8}^3 + C_{4D-10}^3} \right]. \tag{B.26}
\]
Define the factors in the square brackets in (B.24) and (B.26) as following
\[
x_2 = \frac{2C_{3D-9}^3 - C_{3D-7}^3 - 10 C_{4D-10}^3}{2C_{3D-7}^3 + C_{4D-10}^3} - \frac{2C_{3D-8}^3 - C_{3D-9}^3 - 10 C_{4D-10}^3}{2C_{3D-8}^3 + C_{4D-10}^3}. \tag{B.27}
\]
\[
y_2 = \frac{C_{3D-8}^3}{2C_{3D-7}^3 + C_{4D-10}^3} - \frac{C_{3D-9}^3}{2C_{3D-8}^3 + C_{4D-10}^3}. \tag{B.28}
\]
One can check that \( x_2 - y_2 \approx 0 \). We have already proved \( x_2 > 0 \) in (B.25), so \( y_2 > 0 \), i.e. the \( (c_D e - \omega) \) term is positive.

Thirdly, let’s see the \( D_1 \) term in the difference \( \tilde{b}_{3D-9} - \tilde{b}_{3D-8} \), which is
\[
-D_1 m_1^4 r_h^{4D-12} \times \left[ \frac{C_{3D-9}^3 - 5C_{4D-12}^3 - C_{3D-9}^3 - 5C_{4D-12}^3}{2C_{3D-7}^3 + C_{4D-10}^3} - \frac{C_{3D-8}^3 - 5C_{4D-12}^3 - C_{3D-9}^3 - 5C_{4D-12}^3}{2C_{3D-8}^3 + C_{4D-10}^3} \right]. \tag{B.29}
\]
Because
\[
\frac{C_{3D-9}^3}{2C_{3D-12}^3} = \frac{C_{3D-8}^3}{5D - 15 5D - 16 4D - 12 4D - 13 \cdots 2D - 6}{D - 3}
\]
\[
> \left( \frac{5}{4} \right)^{3D-8} > \left( \frac{5}{4} \right)^{10} \approx 9.3 > 5, \tag{B.30}
\]
we have
\[
-2C_{3D-9}^3 - 5C_{4D-12}^3 > 0. \tag{B.31}
\]
Now let’s prove the factor in square bracket is negative. The factor is
\[
\frac{C_{3D-9}^3 - 5C_{4D-12}^3 + 10}{2C_{3D-7}^3 + C_{4D-10}^3} - \frac{C_{3D-8}^3 - 5C_{4D-12}^3}{2C_{3D-8}^3 + C_{4D-10}^3}. \tag{B.32}
\]
According to (B.5), the first term of the factor can be written as
\[
\frac{C_{3D-9}^3 - 5C_{4D-12}^3 + 10}{2C_{3D-7}^3 + C_{4D-10}^3} = \frac{3D-8}{2D-6} C_{3D-8}^3 - \frac{3D-8}{2D-6} C_{3D-9}^3 - 10 + \frac{2D-6}{2D-8} C_{3D-7}^3 + \frac{2D-6}{2D-8} C_{4D-10}^3
\]
\[
= \frac{C_{3D-8}^3}{2D-6} - 2 \times 5C_{3D-8}^3 - 2D-6 C_{4D-12}^3 + 2D-8 C_{4D-12}^3 \times (D - 3)C_{3D-7}^3 + \frac{2D-6}{2D-8} C_{4D-10}^3. \tag{B.33}
\]
Then the negativity of (B.32) is equivalent to
\[
\frac{C_{5D-8}^{3D-8} - 2 \times 5 C_{4D-12}^{3D-8} + 2D - 6}{C_{5D-15}^{3D-8} - 2 \times 5 C_{4D-12}^{3D-8} + 2D - 6} < 1.
\]
For the left side of the Eq. (B.34), we have
\[
\frac{C_{5D-15}^{3D-8} + 2 \times 5 C_{4D-12}^{3D-8} + 2D - 6}{C_{5D-15}^{3D-8} - 5 C_{4D-12}^{3D-8} + 2D - 6} < 1.
\]
For the right side of the Eq. (B.34), we have
\[
\frac{D - 3 \times 2 C_{5D-7}^{3D-8} + 2D - 6}{D - 1} < \frac{2D - 6}{D - 1}.
\]
According to (B.5), the left side of the above can be rewritten as
\[
\frac{C_{5D-8}^{3D-8} - 2 \times 5 C_{4D-12}^{3D-8} + 2D - 6}{C_{5D-15}^{3D-8} - 2 \times 5 C_{4D-12}^{3D-8} + 2D - 6} < 1.
\]
Thus, the inequality (B.39) is equivalent to
\[
\frac{C_{5D-15}^{3D-8} + (D - 6) C_{4D-12}^{3D-8} + 2D - 6}{C_{5D-15}^{3D-8} - 5 C_{4D-12}^{3D-8} + 2D - 6} < 1.
\]
For the left side of the Eq. (B.36), we have
\[
\frac{C_{5D-15}^{3D-8} + (D - 6) C_{4D-12}^{3D-8} + 9 - 2D}{C_{5D-15}^{3D-8} - C_{4D-12}^{3D-8} + 9 - 2D} < 1.
\]
Finally, let’s see the \( \lambda_1 \) term in the difference \( \delta_{3D-9} - \bar{b}_{3D-9} \), which is
\[
-4 m_1 T_1^{D-12} \times \left[ \frac{C_{5D-9}^{3D-9} + (D - 6) C_{4D-12}^{3D-9} + 9 - 2D}{C_{5D-15}^{3D-9} - C_{4D-12}^{3D-9} + 9 - 2D} \right].
\]
Now let’s prove the factor in square bracket is negative. This factor is
\[
\frac{C_{5D-15}^{3D-9} + (D - 6) C_{4D-12}^{3D-9} + 9 - 2D}{C_{5D-15}^{3D-9} - C_{4D-12}^{3D-9} + 9 - 2D} < 1.
\]
Since \( 9 - 2D < 0 \), a sufficient condition for the above to be negative
\[
\frac{C_{5D-15}^{3D-9} + (D - 6) C_{4D-12}^{3D-9}}{2C_{5D-7}^{3D-9} + C_{4D-10}^{3D-9}} < \frac{C_{5D-15}^{3D-8} + (D - 6) C_{4D-12}^{3D-8}}{2C_{5D-7}^{3D-8} + C_{4D-10}^{3D-8}}.
\]
The factor in the square bracket is equivalent to 
\[ \frac{r_h^{2D-9}}{a_3 C_{3D-9}} \]

Because \( \frac{4D-10-p}{3D-7-p} \) > 1, the above expression is positive. Then (B.47) is positive given the bound state condition.

Secondly, the \((c_D e - \omega)\) term in the difference (B.46) is

\[
4(D - 3)m^5 r_h^3 c_D (c_D e - \omega) c_D e
\]

The factor in the above square bracket can be denoted as

\[
y_3 \equiv \frac{C_{5D-9}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1}} - \frac{C_{5D-9}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1}}.
\]  
(B.50)

One can easily check that

\[ 2y_3 - x_3/2 = 0. \]  
(B.51)

And because \( x_3 > 0 \), we then have \( y_3 > 0 \). Together with the superradiance condition, we obtain that the \((c_D e - \omega)\) term is positive.

Thirdly, let’s see the \( \lambda_1 \) term in the difference (B.46), which is

\[
-4m^5 \lambda_1 \left( \frac{C_{5D-15}^p + (D - 6) C_{4D-12}^p + (9 - 2D) C_{3D-9}^p}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1}} - \frac{C_{5D-15}^{p+1} + (D - 6) C_{4D-12}^{p+1} + (9 - 2D) C_{3D-9}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1}} \right).
\]  
(B.52)

The factor in square bracket is

\[
\frac{C_{5D-15}^p + (D - 6) C_{4D-12}^p}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1}} - \frac{C_{5D-15}^{p+1} + (D - 6) C_{4D-12}^{p+1} + (9 - 2D) C_{3D-9}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1}} + \frac{(2D - 9) C_{3D-9}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1} - \frac{C_{5D-9}^p}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1}} - \frac{C_{5D-9}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1}}. \]  
(B.53)

For the second line of the above expression, we have

\[
\frac{C_{5D-9}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1}} - \frac{C_{5D-9}^p}{C_{4D-10}^p + 2C_{3D-7}^p}
\]

\[
= (2D - 9) \left[ \frac{(3D - 9 - p) C_{5D-9}^p}{(4D - 10 - p) C_{4D-10}^p + (3D - 7 - p) 2C_{3D-7}^p} - \frac{C_{5D-9}^p}{C_{4D-10}^p + 2C_{3D-7}^p} \right].
\]  
(B.54)

Since \( \frac{4D-10-p}{3D-7-p} > 1 \), the above equation is less than 0. So the second line of (B.53) is negative.
For the first line of (B.53), we will show that it is also negative, which is equivalent to
\[
\frac{C_{p}^D_{5D-15}}{C_{4D-10}^p + 2C_{3D-7}^p} < \frac{C_{5D-15}^{p+1} - (D-6)C_{4D-12}^{p+1}}{C_{4D-10}^p + 2C_{3D-7}^p} \iff \frac{C_{5D-15}^{p+1} + (D-6)C_{4D-12}^{p+1}}{C_{5D-15}} \Rightarrow \frac{(4D - 10 - p)C_{4D-10}^p + (3D - 7 - p)2C_{3D-7}^p}{C_{4D-10}^p + 2C_{3D-7}^p} < \frac{(5D - 15 - p)C_{5D-15}^p + (4D - 12 - p)(D - 6)C_{4D-12}^p}{C_{5D-15}}.
\]

For the left side of the above inequality, we have
\[
(4D - 10 - p)C_{4D-10}^p + (3D - 7 - p)2C_{3D-7}^p < 4D - 10 - p.
\]

For the right side of the inequality (B.55), we have
\[
\frac{(5D - 15 - p)C_{5D-15}^p + (4D - 12 - p)(D - 6)C_{4D-12}^p}{C_{5D-15}} = 4D - 12 - p + \frac{1}{1 + (D - 6)C_{4D-12}^p/C_{5D-15}^p}.
\]

Since
\[
\frac{C_{4D-12}^p}{C_{5D-15}^p} = \frac{4D - 12 - p}{5D - 15} \leq \frac{4D - 12 - p}{5D - 15 - p} = \left(\frac{4}{5}\right)^{2D-4} < \left(\frac{4}{5}\right)^{8} < 0.2,
\]
then for $D > 6$
\[
4D - 12 - p + \frac{D - 3}{1 + (D - 6)C_{4D-12}^p/C_{5D-15}^p} > 4D - 12 - p + \frac{D - 3}{1 + (D - 6)0.2} > 4D - 12 - p + 3 = 4D - 9 - p.
\]

Based on (B.56), (B.59), we obtain that the first line of (B.53) is also negative. So the $\lambda_1$, term (B.52), is positive.

Finally, let’s see the $D_1$ term in the difference (B.46), which is
\[
-m^2D_1 \left[ \frac{C_{5D-15}^p - 5C_{4D-12}^p + 10C_{3D-9}^p}{C_{4D-10}^p + 2C_{3D-7}^p} - \frac{C_{5D-15}^{p+1} - 5C_{4D-12}^{p+1} + 10C_{3D-9}^{p+1}}{C_{4D-10}^p + 2C_{3D-7}^p} \right].
\]

We will prove the difference in the above square bracket is negative and then $D_1$ term is positive.

For $D > 6$ and $p > 2D - 4 > 8$, we have the following inequalities
\[
\frac{C_{5D-15}^p}{C_{4D-10}^p} < \frac{5D - 15 5D - 16}{3D - 9 3D - 10} \frac{5D - 14 - p}{3D - 8 - p} > \left(\frac{5}{3}\right)^8 > 10,
\]

\[
\frac{C_{4D-12}^p}{C_{5D-15}^p} < \frac{(3/5)^8 < 0.02}.
\]

\[
\frac{C_{5D-15}^p}{C_{4D-12}^p} = \frac{5D - 15 5D - 16}{4D - 12 4D - 13} \frac{5D - 14 - p}{4D - 11 - p} > \left(\frac{5}{4}\right)^8 > 5,
\]

so $C_{5D-15}^p - 5C_{4D-12}^p + 10C_{3D-9}^p > 0$, i.e. the numerators in (B.60) are positive. Then
When $D > 6$, the left term in inequality (B.65) satisfies
\[
(4D - 10 - p)C_{4D-10}^p + (3D - 7 - p)2C_{3D-7}^p \\
< 4D - 10 - p.
\]
(B.66)

For $D > 6$, we also have $D - 3 > 3$. The right term in inequality (B.65) satisfies
\[
(5D - 15 - p)C_{5D-15}^p - (4D - 12 - p)5C_{4D-12}^p + (3D - 9 - p)10C_{3D-9}^p \\
= 4D - 12 - p + (D - 3)C_{5D-15}^p - (D - 3)10C_{3D-9}^p \\
= 4D - 12 - p + (D - 3)\frac{C_{5D-15}^p - 10C_{3D-9}^p}{C_{5D-15}^p - 5C_{4D-12}^p + 10C_{3D-9}^p} \\
> 4D - 12 - p + (D - 3)\frac{1 - 10C_{5D-15}^p/\bar{C}_{5D-15}^p}{1 + 10C_{5D-15}^p/\bar{C}_{5D-15}^p} \\
> 4D - 12 - p + 3 \frac{2}{3} = 4D - 10 - p.
\]
(B.67)

With the above inequality and (B.66), we obtain that the $D_1$ term (B.60) is positive.

So in this case, all four terms in the difference (B.46) are shown to be positive, we then obtain that
\[
\text{sign}(b_p) \geq \text{sign}(b_{p+1}).
\]
(B.68)

B.3 Coefficients of $z^p$, $p = 2D - 3, 2D - 4$

The two coefficients of $z^p$ for $p = 2D - 3, 2D - 4$ are listed as follows
\[
b_{2D-3} = a_5 C_{5D-15}^{2D-3} h^3 + a_4 C_{4D-10}^{2D-3} h^2 \\
+ a_5 C_{5D-15}^{2D-3} h^2 + a_3 C_{5D-15}^{2D-3} h \\
+ r_h^{-2}(m^3 \lambda_l) C_{5D-15}^{2D-3} h^2 + (D - 6) C_{4D-10}^{2D-3} h^2 \\
+ (9 - 2D) C_{3D-7}^{2D-3} h \\
+ r_h^{-2}(m^3 \lambda_l) C_{5D-15}^{2D-3} h^2 + (D - 6) C_{4D-10}^{2D-3} h^2 \\
+ (9 - 2D) C_{3D-7}^{2D-3} h \\
+ r_h^{-2}(m^3 \lambda_l) C_{5D-15}^{2D-3} h^2 + (D - 6) C_{4D-10}^{2D-3} h^2 \\
+ (9 - 2D) C_{3D-7}^{2D-3} h \\
+ r_h^{-2}(m^3 \lambda_l) C_{5D-15}^{2D-3} h^2 + (D - 6) C_{4D-10}^{2D-3} h^2 \\
+ (9 - 2D) C_{3D-7}^{2D-3} h
\]
(B.69)

Define two normalized new coefficients with positive factors,
\[
\tilde{b}_{2D-3} = \frac{b_{2D-3} h^3}{C_{2D-3}^{2D-3}}, \quad \tilde{b}_{2D-4} = \frac{b_{2D-4} h^2}{C_{2D-4}^{2D-4}}.
\]
(B.71)

Now, we consider the difference between $\tilde{b}_{2D-3}$ and $\tilde{b}_{2D-4}$
\[
\tilde{b}_{2D-4} - \tilde{b}_{2D-3} = \frac{b_{2D-4} h^2}{C_{2D-4}^{2D-4}} - \frac{b_{2D-3} h^3}{C_{2D-3}^{2D-3}} - \frac{b_{2D-3} h^3}{C_{2D-3}^{2D-3}}.
\]
(B.72)

The difference can be decomposed into four terms and we will analyze term by term.
First, let’s see the \((\mu^2 - \omega^2)\) term in (B.72), which is
\[
4(D - 3)m^3\gamma_h^2(\mu^2 - \omega^2) \left[ \frac{2C^{2D-4}}{C^{4D-10}} - \frac{2C^{2D-3}}{C^{4D-10} + 2C^{3D-7}} \right] \left( \frac{2C^{2D-4}}{C^{4D-10} - C^{2D-3}} \right)
\]
\[\frac{C^{2D-4} - C^{2D-4} - 1}{C^{4D-10} + 2C^{2D-3} - 1} \]
The positivity of the factor in the square bracket is equivalent to
\[
-2C^{2D-4}C^{4D-10} + 2C^{2D-3}C^{4D-10} + 2C^{2D-3}C^{3D-7} - 1
\]
\[\Rightarrow C^{2D-4}C^{4D-10} + 2C^{2D-3}C^{4D-10} + 2C^{2D-3}C^{3D-7} - 1
\]
\[
(2D - 6)C^{2D-4}C^{4D-10} + (D - 3)2C^{2D-4}C^{4D-10} + (D - 3)2C^{2D-4}C^{3D-7}
\]
\[\Rightarrow C^{2D-4}C^{4D-10} + 2C^{2D-3}C^{4D-10} + 2C^{2D-3}C^{3D-7} - 1
\]
\[
C^{2D-4}C^{4D-10} + 2C^{2D-3}C^{4D-10} + 2C^{2D-3}C^{3D-7} - 1
\]
\[\Rightarrow C^{2D-4}C^{4D-10} + 2C^{2D-3}C^{4D-10} + 2C^{2D-3}C^{3D-7} - 1
\]

The positivity of the factor in the square bracket is equivalent to
\[
-2C^{2D-4}C^{4D-10} + 2C^{2D-3}C^{4D-10} + 2C^{2D-3}C^{3D-7} - 1
\]
\[\Rightarrow C^{2D-4}C^{4D-10} + 2C^{2D-3}C^{4D-10} + 2C^{2D-3}C^{3D-7} - 1
\]
\[
(2D - 6)C^{2D-4}C^{4D-10} + (D - 3)2C^{2D-4}C^{4D-10} + (D - 3)2C^{2D-4}C^{3D-7}
\]
\[\Rightarrow C^{2D-4}C^{4D-10} + 2C^{2D-3}C^{4D-10} + 2C^{2D-3}C^{3D-7} - 1
\]
\[
C^{2D-4}C^{4D-10} + 2C^{2D-3}C^{4D-10} + 2C^{2D-3}C^{3D-7} - 1
\]
\[\Rightarrow C^{2D-4}C^{4D-10} + 2C^{2D-3}C^{4D-10} + 2C^{2D-3}C^{3D-7} - 1
\]

Then we find that
\[
y_1 - 2x_1 = \frac{3}{C^{2D-4} - 10 + C^{2D-4} - 1} < 0. \quad \text{(B.78)}
\]

So \(x_1 > 0\) and the \((cD\epsilon - \omega)\) term is positive given the superradiance condition.

Thirdly, let’s see the \(\lambda_i\) term in (B.72), which will be shown to be positive,
\[
-4m^3\lambda_i \left[ C^{2D-4}C^{4D-10} + (D - 6)C^{2D-4}C^{4D-12} + (9 - 2D)C^{2D-4}C^{3D-9} \right]
\]
\[\frac{C^{2D-4}C^{4D-10} + 2C^{2D-4}C^{3D-7} - 1}{C^{2D-4}C^{4D-12} + (9 - 2D)C^{2D-4}C^{3D-9}}
\]

The negativity of the factor in the square bracket is equivalent to
\[
C^{2D-4}C^{4D-10} + (D - 6)C^{2D-4}C^{4D-12} + (9 - 2D)C^{2D-4}C^{3D-9}
\]
\[\frac{C^{2D-4}C^{4D-10} + 2C^{2D-4}C^{3D-7} - 1}{C^{2D-4}C^{4D-12} + (9 - 2D)C^{2D-4}C^{3D-9}}
\]

\[
C^{2D-4}C^{4D-10} + 2C^{2D-4}C^{3D-7} - 1
\]

\[
C^{2D-4}C^{4D-10} + 2C^{2D-4}C^{3D-7} - 1
\]

\[
C^{2D-4}C^{4D-10} + 2C^{2D-4}C^{3D-7} - 1
\]

\[
C^{2D-4}C^{4D-10} + 2C^{2D-4}C^{3D-7} - 1
\]

Then we find that
\[
y_1 - 2x_1 = \frac{3}{C^{2D-4} - 10 + C^{2D-4} - 1} < 0. \quad \text{(B.78)}
\]

So \(x_1 > 0\) and the \((cD\epsilon - \omega)\) term is positive given the superradiance condition.

Thirdly, let’s see the \(\lambda_i\) term in (B.72), which will be shown to be positive,
\[
-4m^3\lambda_i \left[ C^{2D-4}C^{4D-10} + (D - 6)C^{2D-4}C^{4D-12} + (9 - 2D)C^{2D-4}C^{3D-9} \right]
\]
\[\frac{C^{2D-4}C^{4D-10} + 2C^{2D-4}C^{3D-7} - 1}{C^{2D-4}C^{4D-12} + (9 - 2D)C^{2D-4}C^{3D-9}}
\]

The negativity of the factor in the square bracket is equivalent to
\[
C^{2D-4}C^{4D-10} + (D - 6)C^{2D-4}C^{4D-12} + (9 - 2D)C^{2D-4}C^{3D-9}
\]
\[\frac{C^{2D-4}C^{4D-10} + 2C^{2D-4}C^{3D-7} - 1}{C^{2D-4}C^{4D-12} + (9 - 2D)C^{2D-4}C^{3D-9}}
\]

We then consider the left and right terms of the above inequality separately. For the left term of the above inequality, we have
\[
C^{2D-3}C^{4D-10} + 2C^{2D-4}C^{3D-7} - 1
\]
\[
\frac{(2D - 6)C^{2D-4}C^{4D-10} + (D - 3)2C^{2D-4}C^{3D-7}}{C^{2D-4}C^{4D-10} + 2C^{2D-4}C^{3D-7} - 1}
\]
\[
= 2D - 6 - (2D - 6)C^{2D-4}C^{3D-7} - 1 < 2D - 6. \quad \text{(B.81)}
\]
For the right term in (B.80), we have

\[
\begin{align*}
\frac{C_{5D-15}^{2D-3} + (D - 6)C_{4D-12}^{2D-3} + (9 - 2D)C_{3D-9}^{2D-3}}{C_{5D-15}^{2D-4} + (D - 6)C_{4D-12}^{2D-4} + (9 - 2D)C_{3D-9}^{2D-4}} &= (3D - 11)C_{5D-15}^{2D-4} + (2D - 8)(D - 6)C_{4D-12}^{2D-4} + (D - 5)(9 - 2D)C_{3D-9}^{2D-4} \\
&= 2D - 8 + \frac{(D - 3)C_{5D-15}^{2D-4} + (3 - D)(9 - 2D)C_{3D-9}^{2D-4}}{C_{5D-15}^{2D-4} + (D - 6)C_{4D-12}^{2D-4}} \\
&> 2D - 8 + \frac{(D - 3)C_{5D-15}^{2D-4}}{C_{5D-15}^{2D-4} + (D - 6)C_{4D-12}^{2D-4}} \\
&\geq \frac{(D - 3)}{1 + (D - 6)(4/5)^8} > 2D - 5.
\end{align*}
\]

For the left term of the above inequality, we have

\[
\begin{align*}
(2D - 6)C_{4D-10}^{2D-4} + (D - 3)2C_{3D-7}^{2D-4} \\
&\frac{C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4}}{D} - 1 \\
&= (2D - 6)C_{5D-15}^{2D-4} + 10C_{3D-9}^{2D-4} < 2D - 6. \quad \text{(B.85)}
\end{align*}
\]

So we obtain that the right term is greater than the left term in (B.80) and the \(\lambda_1\) term in difference (B.72) is positive.

Finally, let’s see the \(D_1\) term in the difference (B.72), which will be shown to be positive,

\[
-m^3D_1 \left[ \frac{C_{5D-15}^{2D-4} - 5C_{4D-12}^{2D-4} + 10C_{3D-9}^{2D-4}}{C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4} - 1} \right] < 0. \quad \text{(B.83)}
\]

The positivity of the above term is equivalent to the following inequality

\[
\begin{align*}
\frac{C_{5D-15}^{2D-4} - 5C_{4D-12}^{2D-4} + 10C_{3D-9}^{2D-4}}{C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4} - 1} &< 0 \\
\Rightarrow \frac{C_{5D-15}^{2D-3} - 5C_{4D-12}^{2D-3} + 10C_{3D-9}^{2D-3}}{C_{4D-10}^{2D-3} + 2C_{3D-7}^{2D-3} - 1} &< 0 \\
\Rightarrow \frac{C_{5D-15}^{2D-3} - 5C_{4D-12}^{2D-3} + 10C_{3D-9}^{2D-3}}{C_{4D-10}^{2D-3} + 2C_{3D-7}^{2D-3} - 1} &< 0
\end{align*}
\]

In the last line of the above equation, we use the fact that \(D > 6\) and

\[
y = \frac{C_{5D-15}^{2D-4}}{C_{3D-9}^{2D-4}} < (3/5)^{2D-4} < (3/5)^8 < 0.017. \quad \text{(B.87)}
\]

So the right term of (B.84) is greater than the left term of (B.84) and the \(D_1\) term in the difference (B.72) is positive.

Then according to the positivity of the four terms in the difference (B.72), we obtain

\[
\text{sign}(b_{2D-4}) \geq \text{sign}(b_{2D-3}). \quad \text{(B.88)}
\]
B.4 Coefficients of $z^p$, $p = 2D - 4, 2D - 5, 2D - 6$

In this subsection, we will consider the sign relations between the coefficients of $z^p$, $p = 2D - 4, 2D - 5, 2D - 6$. $b_{2D-4}$ is already given in the last subsection. The left two coefficients are

\[ b_{2D-5} = a_5 c_{2D-5}^{2D-5} r_{h}^{3D-10} + a_4 c_{2D-5}^{2D-5} r_{h}^{2D-5} + a_4 c_{2D-5}^{2D-5} r_{h}^{2D-7} + a_3 c_{2D-5}^{2D-5} r_{h}^{D-2} + a_3 c_{2D-5}^{2D-5} r_{h}^{D-4} + a_3 c_{2D-5}^{2D-5} r_{h}^{D-1} \]

\[ \begin{align*}
&= r_{h}^{(-m^3 D)}(c_{2D-5}^{2D-5} - 5c_{2D-5}^{2D-5} + 10c_{2D-5}^{2D-5}) \\
&\quad + r_{h}^{(-4m^3 D)(c_{2D-5}^{2D-5} + (D - 6)c_{2D-5}^{2D-5} + (9 - 2D)c_{2D-5}^{2D-5})} \\
&\quad + 4r_{h}(D - 3)m^3(c_{2D-5}^{2D-5} + 2c_{2D-5}^{2D-5} - C_{2D-5}^{2D-5} (c_{DE} - \omega)) \\
&\quad \times \left( \frac{(c_{2D-5}^{2D-5} + C_{2D-5}^{2D-5} c_{DE} - \omega)}{C_{4D-10}^{2D-5} + 2C_{2D-5}^{2D-5} - C_{2D-4}^{2D-5}} \right) \\
&\quad + 4r_{h}(D - 3)m^3(\mu^2 - \omega^2)(2c_{3D-7}^{2D-5} - C_{2D-5}^{2D-5} - C_{2D-10}^{2D-4}) \tag{B.89} \end{align*} \]

\[ b_{2D-6} = a_5 c_{2D-6}^{2D-6} r_{h}^{D-9} + a_4 c_{2D-6}^{2D-6} r_{h}^{D-4} + a_4 c_{2D-6}^{2D-6} r_{h}^{D-6} + a_3 c_{2D-6}^{2D-6} r_{h}^{D-1} + a_3 c_{2D-6}^{2D-6} r_{h}^{D-3} + a_3 c_{2D-6}^{2D-6} r_{h}^{D-2} \]

\[ \begin{align*}
&\quad + a_2 c_{2D-6}^{2D-6} \\
&\quad = (-m^3 D)(c_{2D-6}^{2D-6} - 5c_{2D-6}^{2D-6} + 10c_{2D-6}^{2D-6} - 10) \\
&\quad + (4m^3 D)(c_{2D-6}^{2D-6} + (D - 6)c_{2D-6}^{2D-6} + (9 - 2D)c_{2D-6}^{2D-6}) \\
&\quad + (9 - 2D)c_{2D-6}^{2D-6} + (D - 4) \\
&\quad + 4r_{h}^{(-m^3 D)(c_{2D-6}^{2D-6} + (D - 6)c_{2D-6}^{2D-6} + (9 - 2D)c_{2D-6}^{2D-6} + (D - 4) - 4)} \\
&\quad + 4r_{h}^{(-4m^3 D)(c_{2D-6}^{2D-6} + (D - 6)c_{2D-6}^{2D-6} + (9 - 2D)c_{2D-6}^{2D-6} + (D - 4) - 4)} \\
&\quad \times \left( \frac{(c_{2D-6}^{2D-6} + C_{2D-6}^{2D-6} c_{DE} - \omega)}{C_{4D-10}^{2D-6} + 2C_{2D-6}^{2D-6} - C_{2D-4}^{2D-6}} \right) \\
&\quad + 4r_{h}^{(-3m^3 D)(\mu^2 - \omega^2)(2c_{3D-7}^{2D-6} - C_{2D-6}^{2D-6} - C_{2D-10}^{2D-4})} \tag{B.90} \end{align*} \]

First, let’s consider the $(\mu^2 - \omega^2)$ term in the difference (B.91), which is

\[ 4(D - 3)m^3 r_{h}^{(-m^3 D)} \left[ \frac{2c_{3D-7}^{2D-5} - C_{2D-5}^{2D-5} - C_{2D-7}^{2D-5}}{C_{4D-10}^{2D-5} + 2C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} \right] \]

\[ \begin{align*}
&\quad - \frac{2c_{3D-7}^{2D-4} - C_{2D-4}^{2D-4} - 1}{2C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4} - C_{2D-4}^{2D-4}} \tag{B.92} \end{align*} \]

This term will prove to be positive. The positivity of the factor in the square bracket is equivalent to

\[ \begin{align*}
&2c_{3D-7}^{2D-5} - C_{2D-5}^{2D-5} - C_{2D-7}^{2D-5} \\
&\quad > \frac{2c_{3D-7}^{2D-4} - C_{2D-4}^{2D-4} - 1}{2C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4} - C_{2D-4}^{2D-4}} \\
&\quad \Leftrightarrow -2C_{3D-7}^{2D-5} + C_{2D-5}^{2D-5} + C_{2D-5}^{2D-5} \\
&\quad < \frac{2c_{3D-7}^{2D-4} - C_{2D-4}^{2D-4} - 1}{2C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4} - C_{2D-4}^{2D-4}} \\
&\quad \Leftrightarrow -2C_{3D-7}^{2D-5} + C_{2D-5}^{2D-5} + C_{2D-5}^{2D-5} \\
&\quad < \frac{2c_{3D-7}^{2D-4} - C_{2D-4}^{2D-4} - 1}{2C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4} - C_{2D-4}^{2D-4}} \\
&\quad \Leftrightarrow -2C_{3D-7}^{2D-5} + C_{2D-5}^{2D-5} + C_{2D-5}^{2D-5} \\
&\quad < \frac{2c_{3D-7}^{2D-4} - C_{2D-4}^{2D-4} - 1}{2C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4} - C_{2D-4}^{2D-4}} \\
&\quad \Leftrightarrow \frac{(2D - 5)c_{4D-10}^{2D-5} + (D - 2)c_{3D-7}^{2D-5} - C_{2D-5}^{2D-5}}{C_{4D-10}^{2D-5} + 2c_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} \tag{B.93} \end{align*} \]

The left and right terms of the above inequality can be reduced as follows

\[ \begin{align*}
&\frac{(2D - 5)c_{4D-10}^{2D-5} + (D - 2)c_{3D-7}^{2D-5} - C_{2D-5}^{2D-5}}{C_{4D-10}^{2D-5} + 2c_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} \\
&\quad = D - 2 + (D - 3) \frac{c_{2D-5}^{2D-5} + C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}}{C_{4D-10}^{2D-5} + 2c_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} \\
&\quad (B.94) \end{align*} \]

\[ \begin{align*}
&\frac{(2D - 5)c_{4D-10}^{2D-5} + (D - 2)c_{3D-7}^{2D-5} + C_{2D-5}^{2D-5}}{C_{4D-10}^{2D-5} - 2C_{3D-7}^{2D-5} + C_{2D-4}^{2D-4}} \tag{B.95} \end{align*} \]

B.4.1 $b_{2D-4} < b_{2D-5}$

Consider the difference between two normalized coefficients, $b_{2D-4} - b_{2D-5}$,

\[ b_{2D-5} - b_{2D-4} = \frac{r_{h}^{(-m^3 D)} - r_{h}^{(-4m^3 D)}}{C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4} - C_{2D-4}^{2D-4}} \tag{B.91} \]

This difference can be decomposed into four terms and we will analyze term by term.

\[ \odot \text{ Springer} \]
Now we can get the difference between the above two terms, which is obviously positive,
\[
(D - 3) \left( \frac{4C_{2D-5}}{C_{4D-10}} \right) \left( \frac{C_{2D-5} - C_{2D-4}}{C_{4D-10} + 2C_{3D-7} - C_{2D-4}} - C_{2D-5} + C_{2D-4} - C_{2D-7} - 2C_{2D-4} \right) > 0.
\]
(B.96)

So the \((\mu^2 - \omega^2)^2\) term in the difference (B.91) is positive.

Secondly, let’s see the \((c_D e - \omega)\) term in the difference (B.91), which is
\[
4(D - 3)m^3 r^2_n(c_D e - \omega)c_D e \left[ \frac{C_{2D-5} - C_{2D-4}}{C_{4D-10} + 2C_{3D-7} - C_{2D-4}} \right] - \frac{C_{2D-4}}{C_{4D-10} + 2C_{3D-7} - 1}.
\]
(B.97)

We will prove its positivity in the following. Given the superradiant condition, the factor \(4(D - 3)m^3 r^2_n(c_D e - \omega)c_D e\) is positive. We just need to prove

\[
\begin{align*}
\frac{C_{2D-5}}{C_{4D-10} + 2C_{3D-7} - C_{2D-4}} + (D - 6)\frac{C_{2D-5}}{C_{4D-12} + (9 - 2D)C_{3D-9}}
&< \frac{C_{2D-4}}{C_{4D-10} + 2C_{3D-7} - 1} \\
&\Leftrightarrow \frac{C_{2D-4}}{C_{4D-10} + 2C_{3D-7} - 1} - \frac{C_{2D-5}}{C_{4D-10} + 2C_{3D-7} - C_{2D-4}} \\
&< \frac{C_{2D-5}}{C_{4D-10} + 2C_{3D-7} - C_{2D-4}}
\end{align*}
\]
(B.100)

It is obvious that the right term of the above inequality is less than \(D - 2\). For the left term of the above inequality, we have
\[
(2D - 5)\frac{C_{2D-5}}{C_{4D-10}} + (D - 2)\frac{C_{2D-4}}{C_{3D-7} - C_{2D-4}}
= 2D - 5 + \left(\frac{6 - 2D)(C_{2D-5}}{C_{4D-10} + 2C_{3D-7} - C_{2D-4}} \right) < 2D - 5.
\]
(B.102)
For the right term of the inequality (B.101), when \( D > 6 \), we have

\[
(3D - 10)C_{5D-15}^{2D-5} + (2D - 7)(D - 6)C_{4D-12}^{2D-5} + (D - 4)(9 - 2D)C_{3D-9}^{2D-5} \\
C_{5D-15}^{2D-5} + (D - 6)C_{4D-12}^{2D-5} + (9 - 2D)C_{3D-9}^{2D-5} \\
= 2D - 7 + \frac{(D - 3)C_{5D-15}^{2D-5} + (3 - D)(9 - 2D)C_{3D-9}^{2D-5}}{C_{5D-15}^{2D-5} + (D - 6)C_{4D-12}^{2D-5} + (9 - 2D)C_{3D-9}^{2D-5}} \\
> 2D - 7 + \frac{D}{C_{5D-15}^{2D-5} + (D - 6)C_{4D-12}^{2D-5}} \\
> 2D - 7 + \frac{D - 3}{1 + (D - 6)C_{4D-12}^{2D-5}} \\
> 2D - 7 + 3 = 2D - 4. 
\]

(B.103)

So we prove that the right term is greater than the left term in the inequality (B.101) and the \( \lambda_l \) term in the difference (B.91) is positive.

Finally, let’s see the \( D_l \) term in the difference (B.91), which is

\[
-m^3 D_l \left[ \frac{C_{5D-15}^{2D-5} - 5C_{4D-12}^{2D-5} + 10C_{3D-9}^{2D-5}}{C_{4D-10}^{2D-5} + 2C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} \\
- \frac{C_{5D-15}^{2D-4} - 5C_{4D-12}^{2D-4} + 10C_{3D-9}^{2D-4}}{C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4} - C_{2D-4}^{2D-4}} \right]. 
\]

(B.104)

The positivity of the above term is equivalent to the negativity of the factor in the square bracket, i.e.

\[
\frac{C_{5D-15}^{2D-5} - 5C_{4D-12}^{2D-5} + 10C_{3D-9}^{2D-5}}{C_{4D-10}^{2D-5} + 2C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} \\
< \frac{C_{5D-15}^{2D-4} - 5C_{4D-12}^{2D-4} + 10C_{3D-9}^{2D-4}}{C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4} - C_{2D-4}^{2D-4}} \\
\iff \frac{C_{5D-15}^{2D-4} - 5C_{4D-12}^{2D-4} + 10C_{3D-9}^{2D-4}}{C_{4D-10}^{2D-4} + 2C_{3D-7}^{2D-4} - C_{2D-4}^{2D-4}} \\
< \frac{C_{5D-15}^{2D-5} - 5C_{4D-12}^{2D-5} + 10C_{3D-9}^{2D-5}}{C_{4D-10}^{2D-5} + 2C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} \\
\iff 2D - 5)\frac{C_{4D-10}^{2D-5} + (D - 2)C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}}{C_{4D-10}^{2D-5} + 2C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} \\
< \frac{(3D - 10)C_{5D-15}^{2D-5} - (2D - 7)5C_{4D-12}^{2D-5} + (D - 4)10C_{3D-9}^{2D-5}}{C_{5D-15}^{2D-5} - 5C_{4D-12}^{2D-5} + 10C_{3D-9}^{2D-5}}. 
\]

(B.105)

For the left term of the above inequality (B.105), we have

\[
\frac{2D - 5)C_{4D-10}^{2D-5} + (D - 2)2C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}}{C_{4D-10}^{2D-5} + 2C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} \\
< 2D - 5 + (6 - 2D)(C_{5D-15}^{2D-5} - C_{2D-4}^{2D-5}) \frac{C_{5D-15}^{2D-5} - 5C_{4D-12}^{2D-5} + 10C_{3D-9}^{2D-5}}{C_{4D-10}^{2D-5} + 2C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} < 2D - 5. 
\]

(B.106)

For the right term of the inequality (B.105), we have

\[
\frac{(3D - 10)C_{5D-15}^{2D-5} - (2D - 7)5C_{4D-12}^{2D-5} + (D - 4)10C_{3D-9}^{2D-5}}{C_{5D-15}^{2D-5} - 5C_{4D-12}^{2D-5} + 10C_{3D-9}^{2D-5}} \\
< 2D - 7 + \frac{(D - 3)C_{5D-15}^{2D-5} - 10C_{3D-9}^{2D-5}}{C_{5D-15}^{2D-5} - 5C_{4D-12}^{2D-5} + 10C_{3D-9}^{2D-5}} \\
< 2D - 7 + \frac{(D - 3)C_{5D-15}^{2D-5} - 10C_{3D-9}^{2D-5}}{C_{5D-15}^{2D-5} - 5C_{4D-12}^{2D-5} + 10C_{3D-9}^{2D-5}} \\
< 2D - 7 + \frac{(D - 3)10C_{3D-9}^{2D-5}}{1 + 10C_{3D-9}^{2D-5}C_{5D-15}^{2D-5}} \\
< 2D - 7 + 3 = 2D - 4. 
\]

(B.107)

So we prove that the right term is greater than the left term in the inequality (B.105).

According to the positivity of the four terms in the difference (B.91), we obtain

\[
\text{sign}(b_{2D-5}) \geq \text{sign}(b_{2D-4}). 
\]

(B.108)

\[ B.4.2 \]  \( \tilde{b}_{2D-5} < \tilde{b}_{2D-6} \)

\[
\tilde{b}_{2D-6} - \tilde{b}_{2D-5} = \frac{b_{2D-6}}{C_{4D-10}^{2D-6} + 2C_{3D-7}^{2D-6} - C_{2D-4}^{2D-6}} - \tilde{b}_{2D-5} 
\]

(B.109)

The difference can be decomposed into four terms and we will analyze term by term.

First, let’s see the \( (\mu^2 - \omega^2) \) term in the difference (B.109), which is

\[
4(D - 3)m^3 \frac{\mu^2 - \omega^2}{K} \left[ \frac{2C_{4D-10}^{2D-6} - C_{4D-10}^{2D-6} - C_{4D-10}^{2D-6}}{C_{4D-10}^{2D-6} + 2C_{3D-7}^{2D-6} - C_{2D-4}^{2D-6}} - 2C_{2D-5}^{2D-5} - C_{2D-5}^{2D-5} \right]. 
\]

(B.110)
The positivity of this term is equivalent to the positivity of the factor in the square bracket, i.e.

\[
2C^{2D-6}_{3D-10}C^{2D-6}_{3D-7}C^{2D-6}_{2D-4} > \frac{2C^{2D-5}_{3D-7}C^{2D-5}_{2D-4}}{C^{2D-5}_{3D-7} + C^{2D-5}_{2D-4}} \Rightarrow -2C^{2D-5}_{3D-7} + C^{2D-5}_{4D-10} + C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}
\]

\[
< -2C^{2D-5}_{3D-7} + C^{2D-5}_{4D-10} + C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4} \Rightarrow \frac{2C^{2D-5}_{3D-7} + C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}}{C^{2D-5}_{3D-7} + C^{2D-5}_{2D-4}}
\]

\[
< \frac{(2D - 4)C^{2D-6}_{4D-10} - (D - 1)2C^{2D-6}_{3D-7} + 2C^{2D-6}_{2D-4}}{C^{2D-6}_{4D-10} - 2C^{2D-6}_{3D-7} + C^{2D-6}_{2D-4}}. \tag{B.111}
\]

For the left term of the above inequality, we have

\[
(2D - 4)C^{2D-6}_{4D-10} + (D - 1)2C^{2D-6}_{3D-7} - 2C^{2D-6}_{2D-4} = D - 1 + \frac{(D - 3)C^{2D-6}_{4D-10} + (D - 3)C^{2D-6}_{3D-7} - C^{2D-6}_{2D-4}}{C^{2D-6}_{4D-10} + 2C^{2D-6}_{3D-7} - C^{2D-6}_{2D-4}}. \tag{B.112}
\]

For the right term of the inequality (B.111), we have

\[
(2D - 4)C^{2D-6}_{4D-10} - (D - 1)2C^{2D-6}_{3D-7} + 2C^{2D-6}_{2D-4} = D - 1 + \frac{(D - 3)C^{2D-6}_{4D-10} - (D - 3)C^{2D-6}_{2D-4}}{C^{2D-6}_{4D-10} - 2C^{2D-6}_{3D-7} + C^{2D-6}_{2D-4}}. \tag{B.113}
\]

Then the difference between the right term and the left term of the inequality (B.111) is

\[
\frac{4(D - 3)C^{2D-6}_{4D-10}(C^{2D-6}_{3D-7} - C^{2D-6}_{2D-4})}{(C^{2D-6}_{4D-10} + 2C^{2D-6}_{3D-7} - C^{2D-6}_{2D-4})(C^{2D-6}_{4D-10} - 2C^{2D-6}_{3D-7} + C^{2D-6}_{2D-4})}, \tag{B.114}
\]

which is obviously positive. So the \((\mu^2 - \omega^2)\) term in the difference (B.109) is positive.

Secondly, let’s see the \((c_D e - \omega)\) term in the difference (B.109), which is

\[
4(D - 3)m^3r^3_2(c_D e - \omega)c_D e[\frac{C^{2D-6}_{4D-10} + C^{2D-6}_{3D-7} - C^{2D-6}_{2D-4}}{C^{2D-5}_{4D-10} + 2C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}} - \frac{C^{2D-5}_{4D-10} + 2C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}}{C^{2D-5}_{4D-10} + 2C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}}]. \tag{B.115}
\]

The positivity of the above term is equivalent to the positivity of the factor in the square bracket, i.e.

\[
\frac{C^{2D-5}_{4D-10} + 2C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}}{C^{2D-5}_{4D-10} + 2C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}} > \frac{C^{2D-5}_{4D-10} + 2C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}}{C^{2D-5}_{4D-10} + 2C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}} \Rightarrow \frac{C^{2D-5}_{4D-10} + 2C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}}{C^{2D-5}_{4D-10} + 2C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}} \Rightarrow \frac{(2D - 4)C^{2D-6}_{4D-10} + (D - 1)2C^{2D-6}_{3D-7} - 2C^{2D-6}_{2D-4}}{C^{2D-6}_{4D-10} + 2C^{2D-6}_{3D-7} - C^{2D-6}_{2D-4}} > \frac{(D - 1)C^{2D-6}_{3D-7} + 2C^{2D-6}_{2D-4}}{C^{2D-6}_{3D-7} + C^{2D-6}_{2D-4}}. \tag{B.116}
\]

For the right term of the above inequality, when \(D > 6\), it is easy to see that

\[
\frac{(D - 1)C^{2D-6}_{3D-7} + 2C^{2D-6}_{2D-4}}{C^{2D-6}_{3D-7} + C^{2D-6}_{2D-4}} < D - 1. \tag{B.117}
\]

For the left term of the inequality (B.116), we have

\[
(2D - 4)C^{2D-6}_{4D-10} + (D - 1)2C^{2D-6}_{3D-7} - 2C^{2D-6}_{2D-4} = D - 1 + \frac{(D - 3)C^{2D-6}_{4D-10} + (D - 3)C^{2D-6}_{3D-7} - C^{2D-6}_{2D-4}}{C^{2D-6}_{4D-10} + 2C^{2D-6}_{3D-7} - C^{2D-6}_{2D-4}}. \tag{B.118}
\]

So the left term is greater than the right term of the inequality (B.116). The \((c_D e - \omega)\) term in the difference (B.109) is positive.

Thirdly, let’s see the \(\lambda_l\) term in the difference (B.109), which is

\[
-4m^3\lambda_l \left[ \frac{C^{2D-6}_{3D-15} + (D - 6)C^{2D-6}_{4D-12} + (9 - 2D)C^{2D-6}_{3D-9} + D - 4}{C^{2D-6}_{3D-10} + 2C^{2D-6}_{3D-7} - C^{2D-6}_{2D-4}} - \frac{C^{2D-5}_{3D-15} + (D - 6)C^{2D-5}_{4D-12} + (9 - 2D)C^{2D-5}_{3D-9} + D - 4}{C^{2D-5}_{4D-10} + 2C^{2D-5}_{3D-7} - C^{2D-5}_{2D-4}} \right]. \tag{B.119}
\]

The positivity of the above term is equivalent to the negativity of the factor in the square bracket, i.e.
\[
\begin{align*}
\frac{C_{5D-15}^{2D-6} + (D - 6)C_{4D-12}^{2D-6} + (9 - 2D)C_{3D-9}^{2D-6} + D - 4}{C_{4D-10}^{2D-6} + 2C_{3D-7}^{2D-6} - C_{2D-4}^{2D-6}} & \\
< \frac{C_{5D-15}^{2D-5} + (D - 6)C_{4D-12}^{2D-5} + (9 - 2D)C_{3D-9}^{2D-5}}{C_{4D-10}^{2D-5} + 2C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} \\
& \equiv \frac{D_{5D-15}^{2D-5} + (D - 6)C_{4D-12}^{2D-5} + (9 - 2D)C_{3D-9}^{2D-5} + D - 4}{C_{4D-10}^{2D-5} + 2C_{3D-7}^{2D-5} - C_{2D-4}^{2D-5}} \\
& \equiv \frac{(2D - 4)C_{4D-10}^{2D-6} + (D - 1)2C_{3D-7}^{2D-6} - 2C_{2D-4}^{2D-6}}{C_{4D-10}^{2D-6} + 2C_{3D-7}^{2D-6} - C_{2D-4}^{2D-6}}
\end{align*}
\]

(B.120)

For the left term of the above inequality, we have

\[
(2D - 4)C_{4D-10}^{2D-6} + (D - 1)2C_{3D-7}^{2D-6} - 2C_{2D-4}^{2D-6} = 2D - 4 + \frac{(6 - 2D)(C_{3D-7}^{2D-6} - C_{2D-4}^{2D-6})}{C_{4D-10}^{2D-6} + 2C_{3D-7}^{2D-6} - C_{2D-4}^{2D-6}} < 2D - 4.
\]

(B.121)

For the right term of the inequality (B.120), we have

\[
\begin{align*}
(3D - 9)C_{5D-15}^{2D-6} + (D - 6)(D - 6)C_{4D-12}^{2D-6} + (9 - 2D)C_{3D-9}^{2D-6} & \\
& \equiv \frac{D_{5D-15}^{2D-6} + (D - 6)(D - 6)C_{4D-12}^{2D-6} + (9 - 2D)C_{3D-9}^{2D-6} + D - 4}{C_{4D-10}^{2D-6} + 2C_{3D-7}^{2D-6} - C_{2D-4}^{2D-6}} \\
& \equiv \frac{(D - 3)C_{5D-15}^{2D-6} + (D - 6)(2D - 9)C_{3D-9}^{2D-6} - (2D - 6)(D - 4)}{C_{4D-10}^{2D-6} + 2C_{3D-7}^{2D-6} - C_{2D-4}^{2D-6}}
\end{align*}
\]

(B.122)

So the right term is greater than the left term of the inequality (B.120). The \(\lambda_1\) term in the difference (B.109) is positive.

Finally, let see the \(D_1\) term in the difference (B.109), which is

\[
\sum_{\text{Springer}}
\]
\[
\begin{align*}
& \frac{C_{5D}^{2D-6} - 5C_{5D}^{2D-6} + 10C_{3D}^{3D-6} - 10}{C_{5D}^{2D-6} + 2C_{5D}^{2D-6} - C_{2D}^{2D-4}} < \frac{C_{5D}^{2D-5} - 5C_{4D}^{2D-5} + 10C_{3D}^{3D-5}}{C_{5D}^{2D-5} + 2C_{5D}^{2D-5} - C_{2D}^{2D-4}} \\
& \quad \Leftrightarrow \frac{C_{5D}^{2D-5} - 5C_{4D}^{2D-5} + 10C_{3D}^{3D-5}}{C_{5D}^{2D-5} + 2C_{5D}^{2D-5} - C_{2D}^{2D-4}} < \frac{C_{5D}^{2D-4} - 5C_{4D}^{2D-4} + 10C_{3D}^{3D-4} - 10}{C_{5D}^{2D-4} + 2C_{5D}^{2D-4} - C_{2D}^{2D-4}} \\
& \quad \Leftrightarrow \frac{(2D - 4)C_{5D}^{2D-6} + (D - 1)C_{4D}^{2D-6} - 2C_{2D}^{2D-6}}{C_{5D}^{2D-6} + 2C_{5D}^{2D-6} - C_{2D}^{2D-6}} < 2D - 4.
\end{align*}
\]

(B.124)

For the left term of the above inequality, we have

\[
(2D - 4)C_{5D}^{2D-6} + (D - 1)C_{4D}^{2D-6} - 2C_{2D}^{2D-6} = 2D - 4 + \frac{(6 - 2D)(C_{5D}^{2D-6} - C_{4D}^{2D-6} - C_{2D}^{2D-6})}{C_{5D}^{2D-6} + 2C_{5D}^{2D-6} - C_{2D}^{2D-6}} < 2D - 4.
\]

(B.125)

For the right term of the inequality (B.124), we have

\[
(3D - 9)C_{5D}^{2D-6} - (2D - 6)C_{4D}^{2D-6} + (D - 3)10C_{3D}^{3D-6} - 10 = 2D - 6 + \frac{(D - 3)C_{5D}^{2D-6} + (3 - D)10C_{4D}^{2D-6} + 10(2D - 6)}{C_{5D}^{2D-6} + 5C_{4D}^{2D-6} + 10C_{3D}^{3D-6} - 10} \geq 2D - 4.
\]

(B.126)

So the right term is greater than the left term of the inequality (B.124). The \(D_1\) term in the difference (B.109) is positive.

Based on the positivity of the four terms in the difference (B.109), we obtain

\[
\text{sign}(b_{2D-6}) \geq \text{sign}(b_{2D-5}).
\]

(B.127)

B.5 Coefficients of \(z^p\), \(2D - 6 > p > D - 3\)

Since \(D > 6(sp 7)\), we have \(p \geq D - 2 \geq 5\) in this case. For \(C_{5D-15}^p\), \(C_{4D-12}^p\), \(C_{3D-9}^p\), we have the following inequalities

\[
\begin{align*}
C_{5D-15}^p & < (3/5)^p < (3/5)^5 \Rightarrow C_{5D-15}^p \\
& > (5/3)^5C_{3D-9}^p \geq 12C_{3D-9}^p, \\
C_{4D-12}^p & < (3/4)^p < (3/4)^5 \Rightarrow C_{4D-12}^p \\
& > (4/3)^5C_{3D-9}^p \geq 4C_{3D-9}^p. 
\end{align*}
\]

(B.128)

The coefficient of \(z^p\) when \(2D - 6 > p > D - 3\) is

\[
b_p = a_5C_{5D-15}^p - 5C_{5D-15}^p, \quad +a_4C_{4D-12}^p - 10C_{2D-6}^p, \quad +a_3C_{3D-9}^p - (D - 3)10C_{2D-6}^p, \quad +a_2C_{2D-4}^p - C_{2D-6}^p, \quad +a_1C_{2D-4}^p - C_{2D-6}^p, \quad +a_0C_{2D-4}^p - C_{2D-6}^p.
\]

Now we consider the difference between two normalized coefficients, \(\tilde{b}_p, \tilde{b}_{p+1}\).

\[
\tilde{b}_p - \tilde{b}_{p+1} = \frac{r_p^p}{C_{4D-10}^p + 2C_{5D-7}^p - C_{2D-4}^p} b_p - \frac{r_{p+1}^p}{C_{4D-10}^p + 2C_{5D-7}^p - C_{2D-4}^p} b_{p+1}
\]

(B.130)

The difference can be decomposed into four terms and we will analyze term by term.

First, let’s see the \((\mu^2 - \omega^2)\) term in the difference (B.130), which is

\[
4(D - 3)m^5r_h^p(\mu^2 - \omega^2) \left[ \frac{2C_{5D-7}^p - C_{4D-10}^p - C_{2D-4}^p}{C_{4D-10}^p + 2C_{5D-7}^p - C_{2D-4}^p} \right] - 2C_{5D-7}^p - C_{4D-10}^p - C_{2D-4}^p
\]

(B.131)
The positivity of the above term is equivalent to the positivity of the factor in the square bracket, i.e.

\[
\frac{2C_{3D-7}^p - C_{4D-10}^p - C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} > \frac{2C_{3D-7}^{p+1} - C_{4D-10}^{p+1} - C_{2D-4}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1} - C_{2D-4}^{p+1}}
\]

\[
\Leftrightarrow \frac{4D - p - 10)C_{4D-10}^p + (3D - p - 7)2C_{3D-7}^p - (2D - p - 4)C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} > 0.
\]

For the left term of the inequality (B.134), we have

After a straightforward calculation, the difference of the right and left terms of the above inequality is

\[
\frac{4D - 3)C_{4D-10}^p + (3D - p - 7)2C_{3D-7}^p - (2D - p - 4)C_{2D-4}^p}{(C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p)(C_{4D-10}^p - 2C_{3D-7}^p + C_{2D-4}^p)} > 0.
\]

So the \((\mu^2 - \omega^2)\) term in the difference (B.130) is positive.

Secondly, let’s see the \((cDe - \omega)\) term in the difference (B.130), which is

\[
4(D - 3)m^5r_{2(e-o)cDe}^2 [\frac{C_{3D-7}^p + C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} - \frac{C_{3D-7}^{p+1} + C_{2D-4}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1} - C_{2D-4}^{p+1}}].
\]

The positivity of the above term is equivalent to the positivity of the factor in the square bracket, i.e.

\[
\frac{C_{3D-7}^p + C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} > \frac{C_{3D-7}^{p+1} + C_{2D-4}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1} - C_{2D-4}^{p+1}}
\]

\[
\Leftrightarrow \frac{(4D - p - 10)C_{4D-10}^p + (3D - p - 7)2C_{3D-7}^p - (2D - p - 4)C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} > 3D - p - 7.
\]

So the left term is greater than the right term of the inequality (B.134). The \((cDe - \omega)\) term in the difference (B.130) is positive.

Thirdly, let’s see the \(\lambda_l\) term in the difference (B.130), which is

\[
\frac{4D - 3)m^5r_{2(e-o)cDe}^2 [\frac{C_{3D-7}^p + C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} - \frac{C_{3D-7}^{p+1} + C_{2D-4}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1} - C_{2D-4}^{p+1}}]}
\]

For the right term of the above inequality, it is easy to see that

\[
\frac{(3D - p - 7)C_{3D-7}^p + (2D - p - 4)C_{2D-4}^p}{C_{3D-7}^p + C_{2D-4}^p} < 3D - p - 7.
\]

(B.135)
\[-4m^{5} \lambda_{1} \left[ \frac{C_{p}^{D-15} + (D - 6)C_{4D-12}^{p} + (9 - 2D)C_{3D-9}^{p} + (D - 4)C_{2D-6}^{p}}{C_{4D-10}^{p} + 2C_{3D-7}^{p} - C_{2D-4}^{p}} \right] \]

\[-\frac{C_{5D-15}^{p+1} + (D - 6)C_{4D-12}^{p+1} + (9 - 2D)C_{3D-9}^{p+1} + (D - 4)C_{2D-6}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1} - C_{2D-4}^{p+1}} \]

(B.137)

The positivity of the above term is equivalent to the negativity of the factor in the square bracket, i.e.

\[
\frac{C_{5D-15}^{p} + (D - 6)C_{4D-12}^{p} + (9 - 2D)C_{3D-9}^{p} + (D - 4)C_{2D-6}^{p}}{C_{4D-10}^{p} + 2C_{3D-7}^{p} - C_{2D-4}^{p}} < \frac{C_{5D-15}^{p+1} + (D - 6)C_{4D-12}^{p+1} + (9 - 2D)C_{3D-9}^{p+1} + (D - 4)C_{2D-6}^{p+1}}{C_{4D-10}^{p+1} + 2C_{3D-7}^{p+1} - C_{2D-4}^{p+1}} \]

\[
\Rightarrow \frac{C_{5D-15}^{p+1} + (D - 6)C_{4D-12}^{p+1} + (9 - 2D)C_{3D-9}^{p+1} + (D - 4)C_{2D-6}^{p+1}}{C_{5D-15}^{p} + (D - 6)C_{4D-12}^{p} + (9 - 2D)C_{3D-9}^{p} + (D - 4)C_{2D-6}^{p}} < \frac{(4D - p - 10)C_{5D-15}^{p} + (3D - p - 7)2C_{3D-7}^{p} - (2D - p - 4)C_{2D-4}^{p}}{a_{5}C_{5D-15}^{p} + a_{4}(D - 6)C_{4D-12}^{p} + a_{3}(9 - 2D)C_{3D-9}^{p} + a_{2}(D - 4)C_{2D-6}^{p}} \]

(B.138)

In the above inequality, \(a_{i} = (D - 3)i - p\). For the left term of the above inequality, we have

\[
\frac{(4D - p - 10)C_{5D-15}^{p} + (3D - p - 7)2C_{3D-7}^{p} - (2D - p - 4)C_{2D-4}^{p}}{C_{4D-10}^{p} + 2C_{3D-7}^{p} - C_{2D-4}^{p}} = 4D - p - 10 + \frac{(6 - 2D)(C_{3D-7}^{p} - C_{2D-4}^{p})}{C_{4D-10}^{p} + 2C_{3D-7}^{p} - C_{2D-4}^{p}} < 4D - p - 10. \]

(B.139)

For the right term of the inequality (B.138), we have

\[
\frac{a_{5}C_{5D-15}^{p} + a_{4}(D - 6)C_{4D-12}^{p} + a_{3}(9 - 2D)C_{3D-9}^{p} + a_{2}(D - 4)C_{2D-6}^{p}}{C_{5D-15}^{p} + (D - 6)C_{4D-12}^{p} + (9 - 2D)C_{3D-9}^{p} + (D - 4)C_{2D-6}^{p}} > 4D - p - 12
\]

\[
+ \frac{(D - 3)C_{5D-15}^{p}}{C_{5D-15}^{p} + (D - 6)C_{4D-12}^{p}} > 4D - p - 12
\]

\[
+ \frac{D - 3}{1 + (D - 6)(4/3)^{4}} > 4D - p - 10. \]

(B.140)
So the right term is greater than the left term of the inequality (B.138). The $\lambda_1$ term in the difference (B.130) is positive.

Finally, let’s see the $D_1$ term in the difference (B.130), which is

$$-m^p D_1 \left[ C_p^p \frac{5C_p^p - 12 + 10C_p^{3D - 9} - 10C_p^{2D - 6}}{C_p^{4D - 10} + 2C_p^{3D - 7} - C_p^{2D - 4}} - \frac{C_p^{p+1}}{C_p^{4D - 10} + 2C_p^{3D - 7} - C_p^{2D - 4}} \right].$$

(B.141)

The positivity of the above term is equivalent to the negativity of the factor in the square bracket, i.e.

$$\frac{C_p^p}{C_p^{4D - 10} + 2C_p^{3D - 7} - C_p^{2D - 4}} - \frac{C_p^{p+1}}{C_p^{4D - 10} + 2C_p^{3D - 7} - C_p^{2D - 4}} < 0.$$

(B.142)

For the left term of the above inequality, we have

$$\frac{(4D - p - 10)C_p^{4D - 10} + (3D - p - 7)2C_p^{3D - 7} - (2D - p - 4)C_p^{2D - 4}}{C_p^{4D - 10} + 2C_p^{3D - 7} - C_p^{2D - 4}} = 4D - p - 10 + \frac{(6 - 2D)C_p^{3D - 7} - C_p^{2D - 4}}{C_p^{4D - 10} + 2C_p^{3D - 7} - C_p^{2D - 4}} < 4D - p - 10.$$

(B.143)

For the right term of the inequality (B.142), we have

$$\frac{asC_p^{p-1}}{C_p^{5D - 15} - 5C_p^{4D - 12} + 10C_p^{3D - 9} - 10C_p^{2D - 6}} = a_4 + \frac{(D - 3)C_p^{4D - 15} + (3 - D)10C_p^{3D - 9} - (6 - 2D)10C_p^{2D - 6}}{C_p^{5D - 15} - 5C_p^{4D - 12} + 10C_p^{3D - 9} - 10C_p^{2D - 6}}.$$

(B.144)

In the above proof, it is based on the following results, where the inequalities (B.128) are used. So the right term is greater than the left term of the inequality (B.142). The $D_1$ term in the difference (B.130) is positive.

Based on the positivity of the four terms in the difference (B.130), we obtain that when $2D - 6 > p > D - 3$, sign($b_p$) $\geq$ sign($b_{p+1}$).

(B.147)

B.6 Coefficients of $z^p$, $p = D - 2, D - 3$

In this subsection, we consider the coefficients of $z^p$, $p = D - 2, D - 3$, which are

$$b_{D-2} = a_5C_p^{5D - 15} - a_5C_p^{4D - 12} + a_310C_p^{3D - 9} - a_210C_p^{2D - 6},$$

$$b_{D-3} = a_4C_p^{4D - 10} + a_410C_p^{3D - 9} + 20C_p^{2D - 6}.$$
The difference can be decomposed into four terms and we will analyze term by term.

First, let’s see the $(\mu^2 - \omega^2)$ term in the difference (B.150), which is

\[ r_h^2 4(D - 3)m^4(\mu^2 - \omega^2) \left[ \frac{C_{3D}^{D-3} - C_{4D}^{D-10} - C_{2D}^{D-4}}{C_{3D}^{D-10} + 2C_{3D}^{D-7} - C_{2D}^{D-4}} \right. \]

\[ \left. - \frac{2C_{3D}^{D-2} - C_{4D}^{D-10} - C_{2D}^{D-4}}{C_{3D}^{D-10} + 2C_{3D}^{D-2} - C_{2D}^{D-4}} \right] \]

(B.151)

The second line in the above square bracket can be rewritten as

\[
\frac{2C_{3D}^{D-2} - C_{4D}^{D-10} - C_{2D}^{D-4}}{C_{3D}^{D-10} + 2C_{3D}^{D-2} - C_{2D}^{D-4}}
\]

\[
\left(2(\omega - \mu)C_{3D}^{D-3} - (D - 7)C_{4D}^{D-10} - (D - 1)C_{2D}^{D-4}\right) \]

\[
(3D - 7)C_{3D}^{D-10} + (2D - 4)2C_{4D}^{D-3} - (D - 1)C_{2D}^{D-4} \]

A straightforward calculation of the factor in the square bracket in (B.151) is

\[
\frac{4(D - 3)C_{4D}^{D-3}C_{3D}^{D-7} - C_{2D}^{D-4}}{C_{4D}^{D-10} + 2C_{3D}^{D-3} - C_{2D}^{D-4}}(3D - 7)C_{4D}^{D-10} + (2D - 4)2C_{3D}^{D-3} - (D - 1)C_{2D}^{D-4} \]

\[
> 0.
\]

So the $(\mu^2 - \omega^2)$ term in the difference (B.150) is positive given the bound state condition.

Secondly, let’s see the $(cD - \omega)$ term in the difference (B.150), which is

\[
r_h^2 4(D - 3)m^4(cD - \omega)cD \left[ \frac{C_{3D}^{D-3} + C_{2D}^{D-4}}{C_{3D}^{D-10} + 2C_{3D}^{D-7} - C_{2D}^{D-4}} \right. \]

\[ \left. - \frac{C_{3D}^{D-2} + C_{2D}^{D-4}}{C_{3D}^{D-10} + 2C_{3D}^{D-2} - C_{2D}^{D-4}} \right] \]

(B.152)

The second line in the above square bracket can be rewritten as

\[
\frac{C_{3D}^{D-2} + C_{2D}^{D-4}}{C_{3D}^{D-10} + 2C_{3D}^{D-2} - C_{2D}^{D-4}}
\]

\[
\left((2D - 4)C_{3D}^{D-3} + (D - 1)C_{2D}^{D-4}\right) \]

\[
(3D - 7)C_{4D}^{D-10} + (2D - 4)2C_{3D}^{D-3} - (D - 1)C_{2D}^{D-4} \]

A straightforward calculation of the factor in the square bracket in (B.152) is

\[
\frac{(D - 3)(C_{4D}^{D-3}C_{3D}^{D-7} + 2C_{4D}^{D-3}C_{2D}^{D-4} + 3C_{3D}^{D-7}C_{2D}^{D-4})}{(C_{4D}^{D-10} + 2C_{3D}^{D-7} - C_{2D}^{D-4})(3D - 7)C_{4D}^{D-10} + (2D - 4)2C_{3D}^{D-3} - (D - 1)C_{2D}^{D-4}} \]

\[
> 0.
\]
So the \((c_D e - \omega)\) term in the difference (B.150) is positive given the superradiance condition.

Thirdly, let’s see the \(\lambda_i\) term in the difference (B.150), which is

\[
-4m^4 \lambda_i \left[ C_D^{D-3}_{4D-10} + (D - 6)C_D^{D-3}_{4D-12} + (9 - 2D)C_D^{D-3}_{3D-9} + (D - 4)C_D^{D-3}_{2D-6} \\
+ C_D^{D-3}_{4D-10} + 2C_D^{D-3}_{3D-7} - C_D^{D-3}_{2D-4} \\
- C_D^{D-2}_{5D-15} + (D - 6)C_D^{D-2}_{4D-12} + (9 - 2D)C_D^{D-2}_{3D-9} + (D - 4)C_D^{D-2}_{2D-6} \\
+ C_D^{D-2}_{4D-10} + 2C_D^{D-2}_{3D-7} - C_D^{D-2}_{2D-4} \right]
\]

(B.153)

The positivity of the above inequality is equivalent to the negativity of the factor in the square bracket, i.e.

\[
-4m^4 \lambda_i \left[ \frac{C_D^{D-3}_{4D-10} + (D - 6)C_D^{D-3}_{4D-12} + (9 - 2D)C_D^{D-3}_{3D-9} + (D - 4)C_D^{D-3}_{2D-6}}{C_D^{D-3}_{4D-10} + 2C_D^{D-3}_{3D-7} - C_D^{D-3}_{2D-4}} \\
- \frac{C_D^{D-2}_{5D-15} + (D - 6)C_D^{D-2}_{4D-12} + (9 - 2D)C_D^{D-2}_{3D-9} + (D - 4)C_D^{D-2}_{2D-6}}{C_D^{D-2}_{4D-10} + 2C_D^{D-2}_{3D-7} - C_D^{D-2}_{2D-4}} \right] < \frac{C_D^{D-3}_{4D-10} + (D - 6)C_D^{D-3}_{4D-12} + (9 - 2D)C_D^{D-3}_{3D-9} + (D - 4)C_D^{D-3}_{2D-6}}{C_D^{D-3}_{4D-10} + 2C_D^{D-3}_{3D-7} - C_D^{D-3}_{2D-4}}
\]

\[
\iff \frac{C_D^{D-2}_{5D-15} + (D - 6)C_D^{D-2}_{4D-12} + (9 - 2D)C_D^{D-2}_{3D-9} + (D - 4)C_D^{D-2}_{2D-6}}{C_D^{D-2}_{4D-10} + 2C_D^{D-2}_{3D-7} - C_D^{D-2}_{2D-4}} < \frac{C_D^{D-3}_{4D-10} + (D - 6)C_D^{D-3}_{4D-12} + (9 - 2D)C_D^{D-3}_{3D-9} + (D - 4)C_D^{D-3}_{2D-6}}{C_D^{D-3}_{4D-10} + 2C_D^{D-3}_{3D-7} - C_D^{D-3}_{2D-4}}
\]

\[
\iff a_4 C_D^{D-3}_{5D-15} + a_3 (D - 6) C_D^{D-3}_{4D-12} + a_2 (9 - 2D) C_D^{D-3}_{3D-9} + a_1 (D - 4) C_D^{D-3}_{2D-6}
\]

where \(a_i = i (D - 3)\). For the left term of the above inequality, we have

\[
(3D - 7) C_D^{D-3}_{4D-10} + (2D - 4) C_D^{D-3}_{3D-7} - (D - 1) C_D^{D-3}_{2D-4}
\]

\[
= 3D - 7 + \frac{(6 - 2D)(C_D^{D-3}_{3D-7} - C_D^{D-3}_{2D-4})}{C_D^{D-3}_{4D-10}} < 3D - 7.
\]

(B.155)

For the right term of the inequality (B.154), we have

\[
(3D - 7) C_D^{D-3}_{5D-15} - (2D - 4) C_D^{D-3}_{4D-12} + (D - 7) C_D^{D-3}_{3D-9} - (D - 4) C_D^{D-3}_{2D-6} = 3D - 7
\]

\[
> (D - 7) C_D^{D-3}_{3D-9} + (2D - 4) (C_D^{D-3}_{3D-9} - C_D^{D-3}_{2D-6}) + (D - 4) C_D^{D-3}_{2D-6} > 0.
\]

(B.156)
So the right term is greater than the left term of the inequality (B.154). The $\lambda_I$ term in the difference (B.150) is positive.

Finally, let’s see the $D_1$ term in the difference (B.150), which is

$$-m^4 D_1 \left[ \frac{C_{5D-15}^{D-3} - 5C_{4D-12}^{D-3} + 10C_{3D-9}^{D-3} - 10C_{2D-6}^{D-3} + 5}{C_{4D-10}^{D-3} + 2C_{3D-7}^{D-3} - C_{2D-4}^{D-3}} \right]
$$

(B.157)

For $D = 7$, one can check directly that the above term is positive. In the next, we discuss $D \geq 8$ cases.

The positivity of the $D_1$ term is equivalent to the negativity of the factor in square bracket, i.e.

$$\frac{C_{5D-15}^{D-3} - 5C_{4D-12}^{D-3} + 10C_{3D-9}^{D-3} - 10C_{2D-6}^{D-3} + 5}{C_{4D-10}^{D-3} + 2C_{3D-7}^{D-3} - C_{2D-4}^{D-3}} < \frac{C_{5D-15}^{D-2} - 5C_{4D-12}^{D-2} + 10C_{3D-9}^{D-2} - 10C_{2D-6}^{D-2} + 5}{C_{4D-10}^{D-2} + 2C_{3D-7}^{D-2} - C_{2D-4}^{D-2}}$$

(B.158)

For the left term of the above inequality, we have

$$\frac{(3D - 7)C_{5D-15}^{D-3} + (2D - 4)2C_{4D-12}^{D-3} - (D - 1)C_{3D-7}^{D-3} - (D - 3)10C_{2D-6}^{D-3}}{C_{5D-15}^{D-3} - 5C_{4D-12}^{D-3} + 10C_{3D-9}^{D-3} - 10C_{2D-6}^{D-3} + 5} < 3D - 7 + \frac{(6 - 2D)(C_{5D-15}^{D-3} - C_{4D-12}^{D-3})}{C_{5D-15}^{D-3} - 5C_{4D-12}^{D-3} + 10C_{3D-9}^{D-3} - 10C_{2D-6}^{D-3} + 5}$$

(B.159)

For the right term of the inequality (B.158), we have

$$\frac{(4D - 12)C_{5D-15}^{D-3} - (3D - 9)5C_{4D-12}^{D-3} + (2D - 6)10C_{3D-9}^{D-3} - (D - 3)10C_{2D-6}^{D-3}}{C_{5D-15}^{D-3} - 5C_{4D-12}^{D-3} + 10C_{3D-9}^{D-3} - 10C_{2D-6}^{D-3} + 5} < 3D - 7 + \frac{(D - 5)C_{5D-15}^{D-3} + 10C_{4D-12}^{D-3} - (D - 1)10C_{3D-9}^{D-3} + (2D - 4)10C_{2D-6}^{D-3} - 5(3D - 7)}{C_{5D-15}^{D-3} - 5C_{4D-12}^{D-3} + 10C_{3D-9}^{D-3} - 10C_{2D-6}^{D-3} + 5}$$

(B.160)

In the last line of the above equation, we use the positivity of the following term

$$\frac{(D - 5)C_{5D-15}^{D-3} + 10C_{4D-12}^{D-3} - (D - 1)10C_{3D-9}^{D-3} + (2D - 4)10C_{2D-6}^{D-3} - 5(3D - 7)}{C_{5D-15}^{D-3} - 5C_{4D-12}^{D-3} + 10C_{3D-9}^{D-3} - 10C_{2D-6}^{D-3} + 5}.$$  

(B.161)
The positivity of the denominator of the above term can be checked directly for $D = 8, 9, 10$ and for $D > 10$, since $C_{5D-10}^p/C_{4D-12}^p > (5/4)^{D-3} > 5$, the denominator is positive. For the numerator of the above term, we have

$$
[(D - 5)C_{5D-15}^p + 10C_{4D-12}^p - (D - 1)10C_{3D-9}^p] \\
+[(2D - 4)40C_{2D-6}^p - 5(3D - 7)] \\
> [(D - 5)(5/3)C_{5D-3}^p + 10(4/3)^{D-3} - 10(D - 1)C_{3D-9}^p] \\
+[(2D - 4)10C_{2D-6}^p - 5(3D - 7)].
$$

The factor in the first square bracket is positive when $D > 7$ and the term in the second square bracket is obviously positive, then the numerator is also positive.

So the right term is greater than the left term in the inequality (B.158) and the $D_1$ term in the difference (B.150) is positive.

Based on the positivity of the four terms in the difference (B.150), we obtain that

$$\text{sign}(b_{D-3}) \geq \text{sign}(b_{D-2}). \quad (B.162)$$

### B.7 Coefficients of $z^p$, $D - 3 > p > 0$

When $D - 3 > p > 0$, the coefficient of $z^p$ is

$$
b_p = a_5C_{5D-15}^p + 40C_{4D-12}^p + a_4'C_{3D-9}^p + a_3C_{3D-9}^p + a_2C_{2D-6}^p + a_1C_{3D-10}^p + 10C_{3D-9}^p + 10C_{2D-6}^p + 5C_D^p + (9 - 2D)C_{3D-9}^p + (D - 4)C_{2D-6}^p + r_h^{-p}(4(D - 3)m^5(D - 3)C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p) + r_h^{-p}(4(D - 3)m^5(D - 3)C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p)

\times(C_{DE} - \omega)

+ r_h^{-p}(4(D - 3)m^5(D - 3)C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p)

\times(C_{DE} - \omega)

+ r_h^{-p}(4(D - 3)m^5(D - 3)C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p)

\times(C_{DE} - \omega)

+ r_h^{-p}(4(D - 3)m^5(D - 3)C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p)

\times(C_{DE} - \omega)

+ r_h^{-p}(4(D - 3)m^5(D - 3)C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p)

\times(C_{DE} - \omega).

\quad (B.163)

Now we consider the difference between two normalized coefficients, $b_p, b_{p+1}$.

\[\tilde{b}_p - \tilde{b}_{p+1} = \frac{r_p^p b_p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p}

- \frac{r_p^{p+1} b_{p+1}}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} \quad (B.164)
\]

The difference can be decomposed into four terms and we will analyze term by term.

First, let’s see the $(\mu^2 - \omega^2)$ term in the difference (B.164), which is

$$
r_h^24(D - 3)m^5(\mu^2 - \omega^2) \left[ \frac{2C_{3D-7}^p - C_{4D-10}^p - C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} \right]

- \frac{2C_{3D-7}^p - C_{4D-10}^p - C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} \quad (B.165)

\]

The positivity of the above term is equivalent to the positivity of the factor in the square bracket. After a straightforward calculation of this factor, we obtain

$$
\frac{2C_{3D-7}^p - C_{4D-10}^p - C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} > 0, \quad (B.166)
$$

where $a'_1 = 4D - p - 10, a'_2 = 3D - p - 7, a_2 = 2D - p - 4$. So the $(\mu^2 - \omega^2)$ term in the difference (B.164) is positive given the bound state condition.

Secondly, let’s see the $(c_{DE} - \omega)$ term in the difference (B.164), which is

$$
r_h^24(D - 3)m^5(c_{DE} - \omega) \left[ \frac{C_{3D-7}^p + C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} \right]

- \frac{C_{3D-7}^p + C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} \quad (B.167)
$$

The positivity of the above term is equivalent to the positivity of the factor in the square bracket. After a straightforward calculation of this factor, we obtain

$$
\frac{C_{3D-7}^p + C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} > 0, \quad (B.168)
$$

So the $(c_{DE} - \omega)$ term in the difference (B.164) is positive given the superradiance condition.

Thirdly, let’s see the $\lambda_l$ term in the difference (B.164), which is
The positivity of the above term is equivalent to the negativity of the factor in the square bracket, i.e.

\[
\begin{align*}
&-\sum_{i=1}^{4} a_i \left[ C_{5D-15}^p + (D - 6) C_{4D-12}^p + (9 - 2D) C_{3D-9}^p + (D - 4) C_{2D-6}^p \right. \\
&\quad \left. \frac{C_{5D-15}^p + (D - 6) C_{4D-12}^p + (9 - 2D) C_{3D-9}^p + (D - 4) C_{2D-6}^p}{C_{4D-10}^p + 2 C_{3D-7}^p - C_{2D-4}^p} \right]
\end{align*}
\]

(B.169)

In the last line of the above inequality, we need the following result

\[
\begin{align*}
& (D - 5) C_{5D-15}^p - 2(D - 6) C_{4D-12}^p + (D - 1) \\
& (2D - 9) C_{2D-9}^p \\
& -(2D - 4)(D - 4) C_{2D-6}^p > 0 \\
& \Leftrightarrow (D - 5) C_{5D-15}^p - 2(D - 6) C_{4D-12}^p + (D - 7) C_{3D-9}^p \\
& + (2D - 4)(D - 4)(C_{3D-9}^p - C_{2D-6}^p) > 0
\end{align*}
\]

In the above inequality, \( a_i = (D - 3)i - p \). For the left term of the above inequality, we have

\[
(4D - p - 10) C_{4D-10}^p + (3D - p - 7) 2 C_{3D-7}^p - (2D - p - 4) C_{2D-4}^p
\]

\[
= 4D - p - 10 + \frac{(6 - 2D)(C_{3D-7}^p - C_{2D-4}^p)}{C_{4D-10}^p + 2 C_{3D-7}^p - C_{2D-4}^p} < 4D - p - 10.
\]  

(B.171)

For the right term of the inequality (B.170), we have

\[
\begin{align*}
& a_5 C_{5D-15}^p + a_4(D - 6) C_{4D-12}^p + a_3(9 - 2D) C_{3D-9}^p + a_2(D - 4) C_{2D-6}^p \\
& \quad \frac{C_{5D-15}^p + (D - 6) C_{4D-12}^p + (9 - 2D) C_{3D-9}^p + (D - 4) C_{2D-6}^p}{C_{5D-15}^p + (D - 6) C_{4D-12}^p + (9 - 2D) C_{3D-9}^p + (2(3 - D)(D - 4) C_{2D-6}^p)} \\
& \quad = 4D - p - 10 + \frac{(D - 5) C_{5D-15}^p - 2(D - 6) C_{4D-12}^p + (D - 1)(2D - 9) C_{3D-9}^p - (2D - 4)(D - 4) C_{2D-6}^p}{C_{5D-15}^p + (D - 6) C_{4D-12}^p + (9 - 2D) C_{3D-9}^p + (D - 4) C_{2D-6}^p} > 4D - p - 10.
\end{align*}
\]  

(B.172)
It is easy to check that the above expression is positive when $D > 6$. Similar result holds for $p = 3$. Then the inequality (B.173) holds.

And the right term is greater than the left term of the inequality (B.170). The $\lambda_l$ term in the difference (B.164) is positive.

Finally, let’s see the $D_1$ term in the difference (B.164), which is

\[
-\frac{4}{m^5} \frac{1}{D_1} \left( \frac{C_{^{p+1}}_{^{5D-15}} - 5C_{^{p}}_{^{4D-12}} + 10C_{^{p-1}}_{^{3D-9}} - 10C_{^{p-2}}_{^{2D-6}} + 5C_{^{p-3}}_{^{1D-3}}}{C_{^{p+1}}_{^{4D-10}} + 2C_{^{p-1}}_{^{3D-7}} - C_{^{p-3}}_{^{2D-4}}} \right) \left( \frac{C_{^{p+1}}_{^{5D-15}} - 5C_{^{p}}_{^{4D-12}} + 10C_{^{p-1}}_{^{3D-9}} - 10C_{^{p-2}}_{^{2D-6}} + 5C_{^{p-3}}_{^{1D-3}}}{C_{^{p+1}}_{^{4D-10}} + 2C_{^{p-1}}_{^{3D-7}} - C_{^{p-3}}_{^{2D-4}}} \right).
\]

One can check that when $1 < p \leq 4$,

\[
C_{^{5D-15}}^{p} - 5C_{^{4D-12}}^{p} + 10C_{^{3D-9}}^{p} - 10C_{^{2D-6}}^{p} + 5C_{^{1D-3}}^{p} = 0.
\]

and when $p > 5$ and $D > 6$,

\[
C_{^{5D-15}}^{p} - 5C_{^{4D-12}}^{p} + 10C_{^{3D-9}}^{p} - 10C_{^{2D-6}}^{p} + 5C_{^{1D-3}}^{p} > 0.
\]

For the above inequality, one can check it directly when $p = 5, 6, 7$. When $p \geq 8$, $C_{^{5D-15}}^{p} / C_{^{4D-12}}^{p} > (5/4)^8 > 5$ and the above inequality holds.

Then the $D_1$ term is non-negative for $1 \leq p \leq 4$. In the next, we only discuss the $p > 4$ cases. The positivity of the $D_1$ term is equivalent to the negativity of the factor in the square bracket, i.e.

\[
\frac{C_{^{5D-15}}^{p} - 5C_{^{4D-12}}^{p} + 10C_{^{3D-9}}^{p} - 10C_{^{2D-6}}^{p} + 5C_{^{1D-3}}^{p}}{C_{^{4D-10}}^{p} + 2C_{^{3D-7}}^{p} - C_{^{2D-4}}^{p}} < (D - p - 10)C_{^{4D-10}}^{p} + (3D - p - 7)2C_{^{3D-7}}^{p} - (2D - p - 4)C_{^{2D-4}}^{p}
\]

or

\[
\frac{a_4C_{^{5D-15}}^{p} - a_15C_{^{4D-12}}^{p} + a_210C_{^{3D-9}}^{p} - a_310C_{^{2D-6}}^{p} + a_45C_{^{1D-3}}^{p}}{C_{^{5D-15}}^{p} - 5C_{^{4D-12}}^{p} + 10C_{^{3D-9}}^{p} - 10C_{^{2D-6}}^{p} + 5C_{^{1D-3}}^{p}},
\]

where $a_i = (D - 3)i - p$. 
For the left term of the above inequality, we have

\[
(4D - p - 10)C_{4D-10}^p + (3D - p - 7)2C_{3D-7}^p - (2D - p - 4)C_{2D-4}^p
\]

\[
= \frac{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p}{C_{4D-10}^p + 2C_{3D-7}^p - C_{2D-4}^p} < 4D - p - 10.
\]

(B.182)

For the right term of the inequality (B.181), we have

\[
\frac{a_5C_{5D-15}^p - a_45C_{4D-12}^p + a_310C_{3D-9}^p - a_210C_{2D-6}^p + a_15C_{D-3}^p}{C_{5D-15}^p - 5C_{4D-12}^p + 10C_{3D-9}^p - 10C_{2D-6}^p + 5C_{D-3}^p}
\]

\[
= 4D - p - 10 + \frac{(D - 5)C_{5D-15}^p + 10C_{4D-12}^p - 10(D - 1)C_{3D-9}^p + 20(D - 2)C_{2D-6}^p - 5(3D - 7)C_{D-3}^p}{C_{5D-15}^p - 5C_{4D-12}^p + 10C_{3D-9}^p - 10C_{2D-6}^p + 5C_{D-3}^p}
\]

\[
> 4D - p - 10.
\]

(B.183)

In the last line of the above equation, we use the result that when \( p > 4 \)

\[
(D - 5)C_{5D-15}^p + 10C_{4D-12}^p - 10(D - 1)C_{3D-9}^p + 20(D - 2)C_{2D-6}^p - 5(3D - 7)C_{D-3}^p > 0.
\]

(B.184)

The above inequality can be shown as follows

\[
(D - 5)C_{5D-15}^p + 10C_{4D-12}^p - 10(D - 1)C_{3D-9}^p + 20(D - 2)C_{2D-6}^p - 5(3D - 7)C_{D-3}^p
\]

\[
> C_{3D-9}^p(D - 5)(5/3)^p + 10(4/3)^p - 10(D - 1)
\]

\[
+ [20(D - 2)C_{2D-6}^p - 5(3D - 7)C_{D-3}^p].
\]

When \( p \geq 5 \) and \( D > 6 \), it is easy to check the term in the first square bracket is positive. The term in the second square bracket is obviously positive. Thus the inequality (B.184) holds. So the \( D_1 \) term in the difference (B.164) is positive.

Based on the positivity of the four terms in the difference (B.164), we obtain that

\[
\text{sign}(b_p) \geq \text{sign}(b_{p+1}), (D - 3 > p > 0).
\]

(B.185)

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