Three-loop soft anomalous dimensions for top-quark processes

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Abstract

I present results for soft anomalous dimensions through three loops for several processes involving the production of top quarks. In particular, I discuss single-top and top-pair production. I also present some numerical results for double-differential distributions in $t\bar{t}$ production through approximate N$^3$LO.

1 Introduction

The inclusion of soft-gluon corrections in theoretical predictions for top-quark processes is required for better accuracy, and it involves calculations of soft anomalous dimensions. The first calculations at one loop were done in the mid 90’s [1], but two-loop calculations appeared much later. The current state-of-the-art has been extended to three loops for some processes.

For partonic processes $f_1(p_1) + f_2(p_2) \rightarrow t(p_t) + X$, we define a kinematical threshold variable $s_4 = s + t + u - \sum_i m_i^2$ where $s = (p_1 + p_2)^2$, $t = (p_1 - p_t)^2$, and $u = (p_2 - p_t)^2$. At partonic threshold $s_4 \rightarrow 0$, and the soft-gluon corrections involve logarithms of the form $[\ln^k(s_4/m_t^2)/s_4]$, with $k \leq 2n - 1$ at perturbative order $\alpha^n$.

We define transforms of the partonic cross section as
$$\tilde{\sigma}(N) = \int (ds_4/s) e^{-Ns_4/s} \sigma(s_4),$$
with transform variable $N$. The factorized expression for the cross section is [1]

$$\sigma^{f_1f_2\rightarrow tX}(N) = H^{f_1f_2\rightarrow tX} S^{f_1f_2\rightarrow tX} \left( \frac{m_t}{N\mu_F} \right) \psi_1(N_1, \mu_F) \psi_2(N_2, \mu_F) \prod_i J_i(N, \mu_F)$$

(1)

where $H^{f_1f_2\rightarrow tX}$ is an $N$-independent hard function, $S^{f_1f_2\rightarrow tX}$ is a soft function [1], while the $\psi$ and $J$ describe collinear emission from initial- and final-state particles [2].

The soft function $S^{f_1f_2\rightarrow tX}$ satisfies the renormalization group equation

$$\left( \frac{\mu}{\mu_R} \frac{\partial}{\partial \mu} + \beta(g_s) \frac{\partial}{\partial g_s} \right) S^{f_1f_2\rightarrow tX} = -\Gamma_S^{f_1f_2\rightarrow tX} S^{f_1f_2\rightarrow tX} - S^{f_1f_2\rightarrow tX} \Gamma_S^{f_1f_2\rightarrow tX}$$

(2)

where the soft anomalous dimension $\Gamma_S^{f_1f_2\rightarrow tX}$ controls the evolution of $S^{f_1f_2\rightarrow tX}$, which gives the exponentiation of logarithms of $N$ in the resummed cross section. The resummation of these soft
corrections at NNLL accuracy requires knowledge of two-loop soft anomalous dimensions while at N^3LL accuracy it requires three-loop soft anomalous dimensions. The resummed cross sections may be expanded at finite order and produce, upon inversion to momentum space, physical predictions.

A recent review of calculations of cusp and soft anomalous dimensions for many processes can be found in Ref. [3] (see also [4–6] for the cusp, [7–11] for single top, and [1, 12] for t\bar{t}).

2 Cusp anomalous dimension

The cusp anomalous dimension, \( \Gamma_{\text{cusp}} \), involves two eikonal lines, and it is a basic ingredient in calculations of soft anomalous dimensions for partonic processes. For two lines with momenta \( p_i \) and \( p_j \), the cusp angle is \( \theta = \cos^{-1}(p_i \cdot p_j / \sqrt{p_i^2 p_j^2}) \), and we write the perturbative series

\[
\Gamma_{\text{cusp}} = \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n \Gamma_{\text{cusp}}^{(n)}.
\]

At one loop, \( \Gamma_{\text{cusp}}^{(1)} = C_F (\theta \coth \theta - 1) \). In the case of two heavy quarks, this can be written in terms of the speed \( \beta = \tanh(\theta/2) \) as

\[
\Gamma_{\text{cusp}}^{(1)\beta} = -C_F \left[ \frac{(1 + \beta^2)}{2\beta} \ln \left( \frac{1 - \beta}{1 + \beta} \right) + 1 \right].
\]  

(3)

At two loops, we have [4]

\[
\Gamma_{\text{cusp}}^{(2)} = K_2 \Gamma_{\text{cusp}}^{(1)} + \frac{1}{2} C_F C_A \left[ 1 + \zeta_2 + \theta^2 - \coth \theta \left[ \zeta_2 \theta + \theta^2 + \frac{3}{3} \sum \ln (1 - e^{-2\theta}) \right] \right]
+ \coth^2 \theta \left[ -\zeta_3 + \zeta_2 \theta + \frac{3}{3} \sum \ln (e^{-2\theta}) + \frac{3}{3} \sum \ln (e^{-2\theta}) \right].
\]

(4)

where \( K_2 = C_A \left( \frac{67 - \zeta_2}{36} \right) = \frac{5}{18} n_f \). This can be written in terms of \( \beta \) and denoted as \( \Gamma_{\text{cusp}}^{(2)\beta} \).

The three-loop result [5, 6] can be written as [3, 6]

\[
\Gamma_{\text{cusp}}^{(3)} = K_3 \Gamma_{\text{cusp}}^{(1)} + 2 K_2 \left( \Gamma_{\text{cusp}}^{(2)} - K_2 \Gamma_{\text{cusp}}^{(1)} \right) + C^{(3)}
\]

(5)

where \( K^{(3)} \) and \( C^{(3)} \) have long expressions. Again, the result can be expressed in terms of \( \beta \).

If eikonal line \( i \) represents a massive quark, with mass \( m_i \), and eikonal line \( j \) a massless quark, then we find simpler expressions. At one loop, \( \Gamma_{\text{cusp}}^{(1) m_i} = C_F \ln(2p_i \cdot p_j / (m_i \sqrt{s})) - 1/2 \). At two loops [9, 11], \( \Gamma_{\text{cusp}}^{(2) m_i} = K_2 \Gamma_{\text{cusp}}^{(1) m_i} + C_F C_A (1 - \zeta_3) / 4 \). At three loops [11],

\[
\Gamma_{\text{cusp}}^{(3) m_i} = K_3 \Gamma_{\text{cusp}}^{(1) m_i} + \frac{1}{2} K_2 C_F C_A (1 - \zeta_3) + C_F C_A^2 \left( -\frac{1}{4} + \frac{3}{8} \zeta_2 - \frac{3}{8} \zeta_3 - \frac{3}{8} \zeta_2 \zeta_3 + \frac{9}{16} \zeta_5 \right).
\]

(6)

If both eikonal lines are massless, then \( \Gamma_{\text{cusp}}^{\text{massless}} = C_F \ln(2p_i \cdot p_j / s) \sum_{n=1}^{\infty} (\alpha_s / \pi)^n K_n \).

3 Single-top production

Next, we discuss various single-top production processes [7–11].
3.1 Single-top t-channel production

The soft anomalous dimension for t-channel single-top production, $\Gamma_S^{bq\to tq'}$, is a $2 \times 2$ matrix in color space. We use a t-channel singlet-octet color basis. The one-loop [7, 10, 11] and two-loop [10, 11] results are well known.

At three loops, we have

\[
\begin{align*}
\Gamma_S^{(3)bq\to tq'} &= K_3 \Gamma_S^{(1)bq\to tq'} + \frac{1}{2} K_2 C_F C_A (1 - \zeta_3) + C_F C_A^2 \left(-\frac{1}{4} + \frac{3}{8} \zeta_2 - \frac{3}{8} \zeta_2 \zeta_3 + \frac{9}{16} \zeta_5\right), \\
\Gamma_S^{(3)bq\to tq'} &= K_3 \Gamma_S^{(1)bq\to tq'} + \frac{1}{2} K_2 C_F C_A (1 - \zeta_3) + C_F C_A^2 \left(-\frac{1}{4} + \frac{3}{8} \zeta_2 - \frac{3}{8} \zeta_2 \zeta_3 + \frac{9}{16} \zeta_5\right), \\
\Gamma_S^{(3)bq\to tq'} &= K_3 \Gamma_S^{(1)bq\to tq'} + \frac{1}{2} K_2 C_F C_A (1 - \zeta_3) + C_F C_A^2 \left(-\frac{1}{4} + \frac{3}{8} \zeta_2 - \frac{3}{8} \zeta_2 \zeta_3 + \frac{9}{16} \zeta_5\right) + X_S^{(3)bq\to tq'}. \\
\end{align*}
\]

(7)

The first element, i.e. the "11" element, of the matrix at three loops was calculated in [11]. Due to the relatively simple color structure of the hard matrix for this process, it is the only three-loop element that contributes to the N^3LO soft-gluon corrections. Here we have also provided three-loop results for the other three matrix elements up to unknown terms from four-parton correlations, which are denoted as $X_S^{(3)bq\to tq'}$ in the above equation.

3.2 Single-top s-channel production

We continue with results for the s-channel, for which $\Gamma_S^{sq\to t\bar{b}}$ is a $2 \times 2$ matrix, and we use an s-channel singlet-octet color basis. The one-loop [7, 8, 11] and two-loop [8, 11] results are, again, well known.

At three loops, we have

\[
\begin{align*}
\Gamma_S^{(3)aq\to t\bar{b}} &= K_3 \Gamma_S^{(1)aq\to t\bar{b}} + \frac{1}{2} K_2 C_F C_A (1 - \zeta_3) + C_F C_A^2 \left(-\frac{1}{4} + \frac{3}{8} \zeta_2 - \frac{3}{8} \zeta_2 \zeta_3 + \frac{9}{16} \zeta_5\right), \\
\Gamma_S^{(3)aq\to t\bar{b}} &= K_3 \Gamma_S^{(1)aq\to t\bar{b}} + \frac{1}{2} K_2 C_F C_A (1 - \zeta_3) + C_F C_A^2 \left(-\frac{1}{4} + \frac{3}{8} \zeta_2 - \frac{3}{8} \zeta_2 \zeta_3 + \frac{9}{16} \zeta_5\right), \\
\Gamma_S^{(3)aq\to t\bar{b}} &= K_3 \Gamma_S^{(1)aq\to t\bar{b}} + \frac{1}{2} K_2 C_F C_A (1 - \zeta_3) + C_F C_A^2 \left(-\frac{1}{4} + \frac{3}{8} \zeta_2 - \frac{3}{8} \zeta_2 \zeta_3 + \frac{9}{16} \zeta_5\right) + X_S^{(3)aq\to t\bar{b}}. \\
\end{align*}
\]

(8)

Again, the "11" element of the matrix at three loops was calculated in [11] and is the only three-loop element to contribute to the N^3LO soft-gluon corrections. We have also provided in the above equation three-loop results for the other three matrix elements up to unknown terms from four-parton correlations, which are denoted as $X_S^{(3)aq\to t\bar{b}}$.

3.3 Associated tW production

The soft anomalous dimension for tW production has only one element (not a matrix). It is known at one loop [7, 9], two-loops [9], and three loops [11]. The three-loop result is [11]

\[
\Gamma_S^{(3)tg\to tW} = K_3 \Gamma_S^{(1)tg\to tW} + \frac{1}{2} K_2 C_F C_A (1 - \zeta_3) + C_F C_A^2 \left(-\frac{1}{4} + \frac{3}{8} \zeta_2 - \frac{3}{8} \zeta_2 \zeta_3 + \frac{9}{16} \zeta_5\right). \\
\]

(9)
4 Top-antitop pair production

We continue with top-antitop pair production which can proceed via the $q\bar{q} \to t\bar{t}$ and the $gg \to t\bar{t}$ channels.

In the $q\bar{q} \to t\bar{t}$ channel, $\Gamma^{q\bar{q} \to t\bar{t}}_S$ is a $2 \times 2$ matrix, and we use an $s$-channel singlet-octet color basis. Here we will concentrate on the "22" matrix element which at one-loop contributes already to the soft-gluon corrections at NLO. At one loop, this element is [1, 12]

$$\Gamma^{(1)q\bar{q} \to t\bar{t}}_{22} = \left(1 - \frac{C_A}{2C_F}\right) \Gamma^{(1)\beta}_{\text{cusp}} + 4C_F \ln \left(\frac{t - m_t^2}{u - m_t^2}\right) - \frac{C_A}{2} \left[1 + \ln \left(\frac{\sin^2(t - m_t^2)}{(u - m_t^2)^4}\right)\right]$$

(10)

while at two loops it is [3, 12]

$$\Gamma^{(2)q\bar{q} \to t\bar{t}}_{22} = K_2 \Gamma^{(1)q\bar{q} \to t\bar{t}}_{22} + \left(1 - \frac{C_A}{2C_F}\right) \left(\Gamma^{(2)\beta}_{\text{cusp}} - K_2 \Gamma^{(1)\beta}_{\text{cusp}}\right) + \frac{C_A^2}{2} (1 - \zeta_3).$$

(11)

At three loops, we find the following expression:

$$\Gamma^{(3)q\bar{q} \to t\bar{t}}_{S_{22}} = K_3 \Gamma^{(3)q\bar{q} \to t\bar{t}}_{S_{22}} + \left(1 - \frac{C_A}{2C_F}\right) \left(\Gamma^{(3)\beta}_{\text{cusp}} - K_3 \Gamma^{(1)\beta}_{\text{cusp}}\right) + \frac{K_2 C_A^2}{2} (1 - \zeta_3)$$

$$+ C_A \left[\frac{1}{4} + \frac{2}{8} \zeta_2 + \frac{3}{8} \zeta_3 + \frac{3}{16} \zeta_2 \zeta_3 + \frac{9}{16} \zeta_5\right] + X^{(3)q\bar{q} \to t\bar{t}}_{S_{22}}$$

(12)

where $X^{(3)q\bar{q} \to t\bar{t}}_{S_{22}}$ denotes unknown three-loop contributions from four-parton correlations. The other three-loop matrix elements are not fully known either but have analogous structure to that at two loops (see also [3]).

In the $gg \to t\bar{t}$ channel, $\Gamma^{gg \to t\bar{t}}_S$ is a $3 \times 3$ matrix, and we use the color basis $c_1 = \delta^a_{12}$, $c_2 = d^{abc} T^c_{12}$, $c_3 = i F^{abc} T^c_{12}$. At one loop for $gg \to t\bar{t}$, the "22" matrix element is [1, 12]

$$\Gamma^{(1)gg \to t\bar{t}}_{S_{22}} = \left(1 - \frac{C_A}{2C_F}\right) \Gamma^{(1)\beta}_{\text{cusp}} + \frac{C_A}{2} \left[\ln \left(\frac{(t - m_t^2)(u - m_t^2)}{s m_t^2}\right) - 1\right]$$

(13)

while at two loops it is [3, 12]

$$\Gamma^{(2)gg \to t\bar{t}}_{S_{22}} = K_2 \Gamma^{(1)gg \to t\bar{t}}_{S_{22}} + \left(1 - \frac{C_A}{2C_F}\right) \left(\Gamma^{(2)\beta}_{\text{cusp}} - K_2 \Gamma^{(1)\beta}_{\text{cusp}}\right) + \frac{C_A^2}{4} (1 - \zeta_3).$$

(14)

At three loops, we find the expression

$$\Gamma^{(3)gg \to t\bar{t}}_{S_{22}} = K_3 \Gamma^{(1)gg \to t\bar{t}}_{S_{22}} + \left(1 - \frac{C_A}{2C_F}\right) \left(\Gamma^{(3)\beta}_{\text{cusp}} - K_3 \Gamma^{(1)\beta}_{\text{cusp}}\right) + \frac{K_2 C_A^2}{2} (1 - \zeta_3)$$

$$+ C_A \left[\frac{1}{4} + \frac{2}{8} \zeta_2 + \frac{3}{8} \zeta_3 + \frac{3}{16} \zeta_2 \zeta_3 + \frac{9}{16} \zeta_5\right] + X^{(3)gg \to t\bar{t}}_{S_{22}}$$

(15)

where $X^{(3)gg \to t\bar{t}}_{S_{22}}$ denotes unknown three-loop contributions from four-parton correlations.

As an application of the soft-gluon formalism at NNLL accuracy, in Figure 1 we show top-quark double-differential distributions in $p_T$ and rapidity with soft-gluon corrections through approximate $N^3$LO [13]. The theoretical predictions describe very well the CMS data at 13 TeV [14].
5 Conclusion

I have presented results for soft anomalous dimensions at one, two, and three loops. The cusp anomalous dimension was discussed first followed by results for the soft anomalous dimensions in single-top production and in top-antitop pair production.

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