ON THE PRIMARY DECOMPOSITION OF SOME DETERMINANTAL HYPEREDGE IDEAL

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Abstract. In this paper we describe the method which we applied to successfully compute the primary decomposition of a certain ideal coming from applications in combinatorial algebra and algebraic statistics regarding conditional independence statements with hidden variables. While our method is based on the algorithm for primary decomposition by Gianni, Trager and Zacharias, we were not able to decompose the ideal using the standard form of that algorithm, nor by any other method known to us.

1. Introduction

The aim of this paper is to describe the method which we applied to compute the primary decomposition of a certain ideal, and to give the result. The importance of this ideal comes from applications in combinatorial algebra and algebraic statistics, in particular regarding conditional independence statements with hidden variables, see [M], [MR], [EH] or [EHHM].

It turns out that the ideal is equidimensional and the intersection of two prime ideals. We tried several algorithms for primary decomposition as well as different implementations thereof, but none of them succeeded to decompose the ideal.

The outline of this paper is quite simple: The input ideal and some of its combinatorial properties are given in Section 2. This ideal can be decomposed using the computational method presented in Section 3. This method combines four improvements to the well-known algorithm for primary decomposition by Gianni, Trager and Zacharias [GTZ] which are described in detail in separate subsections. Finally, in Section 4 we specify the result of the primary decomposition and reveal some of the combinatorial properties of the computed prime components. However, the combinatorial structure of the second prime component is not yet fully understood.

2. Input ideal

The ideal $I$ of which we would like to compute a primary decomposition is a determinantal hyperedge ideal which is defined as follows: Over the polynomial ring $R := \mathbb{Q}[x_1, \ldots, x_{12}, y_1, \ldots, y_{12}, z_1, \ldots, z_{12}]$, consider the $3 \times 12$-matrix

$$M := \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 & z_9 & z_{10} & z_{11} & z_{12} \end{pmatrix}.$$
Following the notation for determinantial hyperedge ideals in [CMR], set
\[
R_1 := \{1, 2, 3\} =: N,
\]
\[
R_2 := \{4, 5, 6\},
\]
\[
R_3 := \{7, 8, 9\},
\]
\[
R_4 := \{10, 11, 12\},
\]
and denote the minor of \( M \) with row indices \( A \) and column indices \( B \) by \([A|B]_M\). Furthermore, for any set \( X \) and any non-negative integer \( s \), we denote by \( \binom{X}{s} \) the set of all subsets of \( X \) which contain exactly \( s \) elements.

With this notation, we define the ideal \( I \subseteq R \) as
\[
I := \left< [N|B]_M \mid B \in \left( \frac{R_i}{3} \right), i \in \{1, 2, 3, 4\} \text{ or } B \in \left( \frac{C_j}{3} \right), j \in \{1, 2, 3\} \right>,
\]
that is,
\[
I = \left< x_{1y4}z_7 - x_{1y7}z_4 - x_{4y1}z_7 + x_{4y7}z_1 + x_{7y1}z_4 - x_{7y4}z_1, \right.
\]
\[
\left. x_{1y4}z_{10} - x_{1y10}z_4 - x_{4y1}z_{10} + x_{4y10}z_1 + x_{10y1}z_4 - x_{10y4}z_1, \right.
\]
\[
\left. x_{1y7}z_{10} - x_{1y10}z_7 - x_{7y1}z_{10} + x_{7y10}z_1 + x_{10y1}z_7 - x_{10y7}z_1, \right.
\]
\[
\left. x_{4y7}z_{10} - x_{4y10}z_7 - x_{7y4}z_{10} + x_{7y10}z_7 + x_{10y4}z_7 - x_{10y7}z_7, \right.
\]
\[
\left. x_{2y5}z_{11} - x_{2y11}z_5 - x_{5y2}z_{11} + x_{5y11}z_2 + x_{11y2}z_5 - x_{11y5}z_2, \right.
\]
\[
\left. x_{2y8}z_{11} - x_{2y11}z_8 - x_{8y2}z_{11} + x_{8y11}z_2 + x_{11y2}z_8 - x_{11y8}z_2, \right.
\]
\[
\left. x_{5y8}z_{11} - x_{5y11}z_8 - x_{8y5}z_{11} + x_{8y11}z_5 + x_{11y5}z_8 - x_{11y8}z_5, \right.
\]
\[
\left. x_{3y6}z_9 - x_{3y9}z_6 - x_{6y3}z_9 + x_{6y9}z_3 + x_{9y3}z_6 - x_{9y6}z_3, \right.
\]
\[
\left. x_{3y6}z_{12} - x_{3y12}z_6 - x_{6y3}z_{12} + x_{6y12}z_3 + x_{12y3}z_6 - x_{12y6}z_3, \right.
\]
\[
\left. x_{3y9}z_{12} - x_{3y12}z_9 - x_{9y3}z_{12} + x_{9y12}z_3 + x_{12y3}z_9 - x_{12y9}z_3, \right.
\]
\[
\left. x_{3y6}z_{12} - x_{3y12}z_6 - x_{6y3}z_{12} + x_{6y12}z_3 + x_{12y3}z_6 - x_{12y6}z_3, \right.
\]
\[
\left. x_{3y9}z_{12} - x_{3y12}z_9 - x_{9y3}z_{12} + x_{9y12}z_3 + x_{12y3}z_9 - x_{12y9}z_3, \right.
\]
\[
\left. x_{6y3}z_{12} - x_{6y12}z_3 - x_{3y6}z_{12} + x_{3y12}z_6 + x_{12y3}z_6 + x_{12y6}z_3 - x_{12y9}z_6, \right.
\]
\[
\left. x_{1y2}z_3 - x_{1y3}z_2 - x_{2y1}z_3 + x_{2y3}z_1 + x_{3y1}z_2 - x_{3y2}z_1, \right.
\]
\[
\left. x_{4y5}z_6 - x_{4y6}z_5 - x_{5y4}z_6 + x_{5y6}z_4 + x_{6y4}z_5 - x_{6y5}z_4, \right.
\]
\[
\left. x_{7y8}z_9 - x_{7y9}z_8 - x_{8y7}z_9 + x_{8y9}z_7 + x_{9y7}z_8 - x_{9y8}z_7, \right.
\]
\[
\left. x_{10y12}z_{12} - x_{10y12}z_{11} - x_{11y10}z_{12} + x_{11y12}z_{10} + x_{12y10}z_{11} - x_{12y11}z_{10} \right> \subseteq \mathbb{Q}[x_1, \ldots, x_{12}, y_1, \ldots, y_{12}, z_1, \ldots, z_{12}].
\]

The ideal \( I \) exhibits a certain kind of symmetry in the sense that it stays invariant under certain permutations of the variables in the polynomial ring \( R \). In fact, the following operations on the matrix \( M \) in the above construction lead to the same ideal \( I \):

- permuting the rows
- permuting the sets of columns which correspond to the index sets \( R_1, R_2, R_3 \) and \( R_4 \)
- permuting the sets of columns which correspond to the index sets \( C_1, C_2 \) and \( C_3 \)

Let \( S_n \) be the symmetric group of order \( n \) as usual. Then the above observations imply that the symmetry group \( S \leq S_{36} \) of \( I \) has a subgroup
$S'$ is isomorphic to $S_3 \times S_4 \times S_3$. We suppose that indeed $S \cong S'$. However, this is more difficult to prove.

3. Computational method

Our computational method is based on the GTZ algorithm for primary decomposition by Gianni, Trager and Zacharias [GTZ] as presented in [GP, Chapter 4]. Recall that the GTZ algorithm computes some maximal dimensional associated primary ideals $Q_1, \ldots, Q_s$ of $I$ using maximal independent sets to reduce the problem to the zero-dimensional case, as well as a polynomial $h^m \notin I$ such that $I : \langle h^m \rangle = I : \langle h^\infty \rangle = Q_1 \cap \ldots Q_s$ and $I = (I : \langle h^m \rangle) \cap \langle I, h^m \rangle$.

An implementation of this algorithm can be found in the SINGULAR library primdec.lib [DLPS]. However, neither this implementation nor the one of the algorithm of Shimoyama and Yokoyama [SY] in the same library succeeded to compute a primary decomposition of the input ideal $I$ defined in the previous section.

We propose four improvements for applying this algorithm to our particular example:

(1) carefully choosing the maximal independent sets, based on data from intermediate steps
(2) if an ideal $P$ is expected to be a prime ideal, applying a special algorithm to check the primality of $P$
(3) choosing a monomial order for saturation which is compatible with the monomial orders used in the previous steps
(4) making use of the symmetry of the input ideal for the saturation step

These four improvements will be explained in detail in the following subsections. The last improvement cannot easily be applied in general if the input ideal does not admit any useful symmetry. On the other hand, we expect that especially the first and the third improvement are also beneficial for other classes of ideals.

3.1. Choosing maximal independent sets. In the GTZ algorithm, the reduction to the zero-dimensional case requires a maximal independent set, see [GP] Algorithm 4.3.2]. However, the input ideal might admit several maximal independent sets, any one of which can be used in the subsequent steps.

In our case, the SINGULAR command `indepSet(., 0)`, when applied to a Gröbner basis of $I$ w.r.t. the degree reverse lexicographic order with $x_1 > \ldots > x_{12} > y_1 > \ldots > y_{12} > z_1 > \ldots > z_{12}$, yields 17,223 different maximal independent sets. The ideal $I$ may even have more maximal independent sets, but it is computationally hard to list all of them.

Depending on the choice made in the algorithm, the running time needed for the computation of the next primary component, especially for the saturation step, may vary enormously. Thus we propose to proceed as follows: Let $K[X]$ with $X = \{X_1, \ldots, X_n\}$ be the polynomial ring of the input ideal. Randomly choose a maximal independent set $u \subseteq X$ of $I$ and compute a minimal Gröbner basis $G_u \subseteq K[X]$ of the zero-dimensional ideal
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IK(u)[X \ u] ⊆ K(u)[X \ u], see [GP] Proposition 4.3.1 (1). Further compute

d_u := \dim_{K(u)[X \ u]} (K(u)[X \ u] / IK(u)[X \ u])

as well as the degrees and the number of terms of the leading coefficients of the elements in G_u, considered as polynomials in K[u]. Repeat this for several maximal independent sets u and sort them, in this order,

(1) by increasing vector space dimension d_u,
(2) by increasing (maximal) degree and finally
(3) by the (maximal) number of terms of the leading coefficients of the elements in G_u.

Then choose the first set u in this sequence and proceed with the GTZ algorithm as usual.

It is reasonable to expect, and in fact our experiments confirm, that on average the above choice leads to improved running times for the saturation step compared to other, for example random, choices.

In our example, we chose subsequently

u_1 := \{x_1, \ldots, x_{12}, y_1, \ldots, y_{12}, z_1, z_2\} and then
u_2 := \{x_1, x_2, x_4, x_5, x_8, y_9, x_{11}, x_{12}, y_1, \ldots, y_{12}, z_1, z_4, z_8, z_{11}, z_{12}\}

as maximal independent sets to find the two primary components described in Section 4. Especially the computation of the second primary component was far out of reach for many other choices of maximal independent sets.

3.2. Checking ideals for primality. While trying to compute a primary decomposition of some given ideal I by whatever means, one may face the situation that one of the computed ideals P is expected to be one of the prime components of I, but it is not known a priori whether or not P is indeed a prime ideal. For example, this may happen if I is defined in a polynomial ring over a field of characteristic zero, but P has been found via computations in positive characteristic. In that situation, the following algorithm can be used to check the primality of P. This algorithm may be known to some experts in the field, but we did not find any reference.

Algorithm 1 Primality check

Input: P ⊆ K[X] = K[X_1, \ldots, X_n]
Output: true if P is a prime ideal, false otherwise
1: choose a maximal independent set u ⊆ X of P
2: compute a minimal Gröbner basis G ⊆ K[X] of PK(u)[X \ u]
3: if PK(u)[X \ u] is not a maximal ideal in K(u)[X \ u] then
4: return false
5: let c_1, \ldots, c_s ∈ K[u] be the leading coefficients of the elements in G
6: for i = 1, \ldots, s do
7: if P \neq P : (c_i^\infty) then
8: return false
9: return true

The correctness of this algorithm is based on the following statement:
Proposition 3.1. Let $P \subseteq K[X] = K[X_1, \ldots, X_n]$ be an ideal and let $u \subseteq X$ be a maximal independent set of $P$. Then $P$ is a prime ideal if and only if $PK(u)[X \setminus u]$ is a maximal ideal in $K(u)[X \setminus u]$ and $P = PK(u)[X \setminus u] \cap K[X]$.

Proof. ($\Leftrightarrow$): If $PK(u)[X \setminus u]$ is a maximal ideal in $K(u)[X \setminus u]$ and $P = PK(u)[X \setminus u] \cap K[X]$, then $P \subseteq K[X]$ is prime by the definition of primality.

($\Rightarrow$): Let $P = \langle g_1, \ldots, g_r \rangle$ be prime. Obviously $P \subseteq PK(u)[X \setminus u] \cap K[X]$, so let us consider $a \in PK(u)[X \setminus u] \cap K[X]$. Then $a$ can be written as $a = \sum_{i=1}^r h_ig_i$ for some $h_i \in K(u)[X \setminus u]$. By clearing denominators, there exists an element $d \in K[u] \setminus \{0\} \subseteq K[X]$ such that $dh_i \in K[X]$ for all $i = 1, \ldots, r$. Thus $da = \sum_{i=1}^r (dh_i)g_i \in P$, and the primality of $P$ implies $d \in P$ or $a \in P$. The former would yield $d \in P \cap K[u] = \{0\}$ by the assumption on $u$, which contradicts the choice of $d$. Therefore $a \in P$ as desired.

It remains to show that $PK(u)[X \setminus u]$ is prime. Consider elements $a, b \in K(u)[X \setminus u]$ with $ab \in PK(u)[X \setminus u]$. By clearing denominators as above, choose $d_a, d_b \in K[u] \setminus \{0\}$ with $d_a a \in K[X]$ and $d_b b \in K[X]$. Then $(d_a a)(d_b b) \in PK(u)[X \setminus u] \cap K[X] = P$, and the primality of $P$ implies that $(d_a a) \in P$ or $(d_b b) \in P$, which in turn yields $a \in PK(u)[X \setminus u]$ or $b \in PK(u)[X \setminus u]$. Therefore $PK(u)[X \setminus u]$ is a prime ideal. It is maximal since $PK(u)[X \setminus u]$ has dimension zero by [GP Proposition 4.3.1 (1)]. □

Corollary 3.2. The output of Algorithm 2 is correct.

Proof. Let $G \subseteq K[X]$ be a minimal Gröbner basis of $PK(u)[X \setminus u]$, let $c_1, \ldots, c_s \in K[u]$ be the leading coefficients of the elements in $G$ as in the algorithm, and define $h := \text{LCM}(c_1, \ldots, c_s) \in K[u]$. Then

$$PK(u)[X \setminus u] \cap K[X] = P : \langle h^\infty \rangle = \langle \ldots (P : \langle c_1^\infty \rangle) \ldots : \langle c_s^\infty \rangle \rangle$$

by [GP Proposition 4.3.1 (2)] and the basic properties of ideal quotients. Thus the statement follows from Proposition 3.1. □

In our case, we made use of the above algorithm to confirm the primality of the component $P_2$, see Section 3. Computationally, this was by far the hardest part of finding a primary decomposition of the input ideal $I$ defined in Section 2.

A variant of Algorithm 1 is the following improvement to the GTZ algorithm: Let $I = \langle f_1, \ldots, f_m \rangle \subseteq K[X]$ be the remaining ideal to decompose, of dimension greater than zero, and let $u \subseteq X$ be a maximal independent set of $I$. Then as the next step, the algorithm computes a primary decomposition $IK(u)[X \setminus u] = Q_1 \cap \ldots \cap Q_s$ in dimension zero. As the algorithm proceeds, the intersections $Q_i \cap K[X]$ are computed via saturation of $Q_i$ w.r.t. some $d_i \in K[u]$ to obtain primary components of $I$, see [GP paragraph below Algorithm 4.3.4].

Now suppose $s = 1$, that is, $IK(u)[X \setminus u]$ is a primary ideal in $K(u)[X \setminus u]$. Let $\{g_1, \ldots, g_r\} \subseteq I \subseteq K[X]$ be a Gröbner basis of $IK(u)[X \setminus u] = Q_1$. Such a Gröbner basis always exists: For example, take a lexicographical Gröbner basis of $I$ with $X \setminus u > u$ and discard the elements not needed for a minimal Gröbner basis of $Q_1$. Then $I_0 := \langle g_1, \ldots, g_r \rangle_K[X] \subseteq I$ and $I_0 : \langle d_1^\infty \rangle = I : \langle d_1^\infty \rangle$. Thus in most cases, it is computationally easier to
saturate $I$ than $I_0$ because loosely speaking, $I$ is already closer to the final result $I : \langle d_1^\infty \rangle$ than $I_0$.

In the case where $Q_1$ is even a maximal ideal, this trick is computationally equivalent to Algorithm 1.

3.3. **Choosing a monomial order for saturation.** For ideals of dimension greater than zero, the GTZ algorithm requires a saturation step, see [GP, Chapter 4.3]. As for many applications of Gröbner bases, the running time of this step heavily depends on the choice of the monomial order. Experiments have shown that it is beneficial to choose a monomial order which is compatible with the maximal independent set that has been used in the preceding steps of the algorithm.

More precisely, let $K[\mathbf{X}]$ with $\mathbf{X} = \{X_1, \ldots, X_n\}$ be the polynomial ring of the input ideal as in the previous subsection and let $u \subseteq \mathbf{X}$ be the chosen maximal independent set. We propose to use an elimination order for $\mathbf{X} \setminus u$ on $K[\mathbf{X}]$ for the saturation step. For implementational reasons, the lexicographical order with $X \setminus u > u$ is a good choice, see [GP, Algorithm 4.3.2, step 2].

3.4. **Making use of symmetry for the saturation step.** With notation as in Algorithm 1, if the input ideal $P$ has some kind of symmetry which also occurs among the coefficients $c_i$, then this can be used to speed the computation up. More precisely, let $\varphi$ be a $K$-algebra automorphism of $K[\mathbf{X}]$ which leaves $P$ invariant, that is, $\varphi(P) = P$. Furthermore suppose that $\varphi(c_j) = c_i$ for some indices $i, j \in \{1, \ldots, s\}$ with $i \neq j$, and that $P = P : \langle c_j^\infty \rangle$ has already been checked in line 7 of Algorithm 1. Because applying $\varphi$ commutes with saturation, we then have

$$P : \langle c_j^\infty \rangle = \varphi(P) : \langle \varphi(c_j)^\infty \rangle = \varphi(P : \langle c_j^\infty \rangle) = \varphi(P) = P.$$  

Thus the check can be left out for $c_i$.

For computing a primary decomposition of the input ideal $I$ defined in Section 2, we chose $\varphi$ to be the $\mathbb{Q}$-algebra automorphism of

$$\mathbb{Q}[x_1, \ldots, x_{12}, y_1, \ldots, y_{12}, z_1, \ldots, z_{12}]$$

given by

$$\varphi(x_i) := x_{\sigma(i)},$$  
$$\varphi(y_i) := y_{\sigma(i)},$$  
$$\varphi(z_i) := z_{\sigma(i)},$$

where $\sigma \in S_{12}$ is the permutation $\sigma = (14)(25)(36)$. The automorphism $\varphi$ leaves $I$ and the two prime components $P_1$ and $P_2$ presented in Section 4 invariant, which is easy to see for $I$ and $P_1$ and can be checked computationally for $P_2$. We used Algorithm 1 to check the primality of $P_2$, so in our case $P = P_2$. It turned out that indeed $\varphi(c_j) = c_i$ for some indices $i, j \in \{1, \ldots, s\}$ with $i \neq j$. Thus computing the saturation $P : \langle c_j^\infty \rangle$ is superfluous. This step would have taken almost 500 hours which is more than four times as long as the saturation of $P$ w.r.t. $c_j$, see Section 4. Since in this case, $\varphi$ is just a permutation of variables, this examples underlines once more the importance of choosing a monomial order for the saturation step as discussed in Subsection 3.3.
4. Result

Using the computational method from the previous section, we were able to show that the ideal \( I \subseteq R = \mathbb{Q}[x_1, \ldots, x_{12}, y_1, \ldots, y_{12}, z_1, \ldots, z_{12}] \) defined in Section 2 is the intersection

\[ I = P_1 \cap P_2 \]

of two different prime ideals \( P_1, P_2 \) where all three ideals \( I, P_1 \) and \( P_2 \) have Krull dimension 26. In particular, this implies that the above intersection is the unique irredundant primary decomposition of \( I \).

In our case, we first found the prime component \( P_1 \), with \( I \subseteq P_1 \), which was relatively easy. We then found the ideal \( P_2 \) for which we were able to show that \( I = P_1 \cap P_2 \) and \( I \not\subseteq P_2 \). But the hardest part of the computation were the saturations for the primality check of \( P_2 \), see line 7 in Algorithm 1.

While checking that \( P_2 \) is already saturated w.r.t. all the coefficients \( c_i \) was easy in some cases, the longest of these computations took 117 hours on an Intel® CoreTM i7-6700 CPU with up to 4 GHz. For one coefficient, it would have even taken almost 500 hours, but this computation can be avoided by the method described in Subsection 3.4.

Furthermore, \( P_1 \) is the ideal generated by all \( 3 \times 3 \)-minors of the matrix \( M \) defined in Section 2. However, the structure of the prime component \( P_2 \) is more involved. Using the SINGULAR command \texttt{mstd()}1, one can find a generating set \( G \) of \( P_2 \) which has the following properties:

- \( G \) consists of 44 polynomials \( p_1, \ldots, p_{44} \). The elements \( p_1, \ldots, p_{16} \) are just the generators of \( I \) listed in Section 2.
- Each polynomial in \( G' := \{p_{17}, \ldots, p_{44}\} \) is homogeneous of degree 12.
- The coefficients in \( p_1, \ldots, p_{44} \) are all 1, \(-1\), 2 or \(-2\) where the vast majority is just 1 or \(-1\).
- In each monomial in \( G' \), each index \( i \) from 1 to 12 appears exactly once, as \( x_i, y_i \) or \( z_i \).
- For each single element in \( G' \), all monomials have the same number of \( x \)'s, \( y \)'s and \( z \)'s. For example, all monomials of \( g_{17} \) have exactly six \( y \)'s and six \( z \)'s, but no \( x \)'s. We denote this partition by \( (0,6,6) \).
- The 28 elements of \( G' \) are in one-to-one correspondence to the 28 possible partitions of twelve items into three categories with no more then six items belonging to the same category.
- For \( g_{17} \) which corresponds to the partition \( (0,6,6) \) as mentioned above, we get all monomials as follows: From the \( 3 \times 4 \)-matrix

\[
\begin{pmatrix}
1 & 4 & 7 & 10 \\
2 & 5 & 8 & 11 \\
3 & 6 & 9 & 12
\end{pmatrix}
\]

choose two different columns A, B. Choose two indices from A. Choose two indices from B such that they are not both in the same rows as the two indices chosen from A. Let C, D be the remaining columns. Choose one index from C. Choose one index from D which is in a different row then the one chosen in C. The chosen indices.

\[1 \text{The set } G \text{ can be found in SINGULAR-readable form at the very end of the TeX file of this article which can be downloaded from } \text{https://arxiv.org/} \]
are the $y_i$’s, all the others are the $z_i$’s (or vice versa, of course). Repeating this process for all possible choices, we get exactly the 216 monomials appearing in $g_{17}$.

- The generators corresponding to the partitions $(6, 6, 0)$ and $(6, 0, 6)$ are just the same as $g_{17}$ for some permutation of $(x, y, z)$.
- The generator $g_{19}$ corresponds to the partition $(1, 5, 6)$. It can be mapped to $g_{17}$ via the map $x_i \mapsto y_i$. Some of the 252 terms of $g_{19}$ cancel under this map.

The above properties indicate that the set $G$ has a combinatorial structure which would be interesting to understand completely, including the coefficients. So far, however, we were not able to reveal every detail of this structure.
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