An Indefinite Convection-Diffusion Operator
With Real Spectrum

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1 Introduction

For $0 < \varepsilon < 2$, we consider the operator

$$(H f)(\theta) := \varepsilon \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial f}{\partial \theta} \right) + \frac{\partial f}{\partial \theta}$$

initially defined on all $C^2$ periodic functions $f \in L^2(-\pi, \pi)$; the exact domain is given by taking the closure of the operator defined on the above functions. That such a closure exists will be shown later. In a recent paper [1] Benilov, O’Brien and Sazonov showed that the equation

$$\frac{\partial f}{\partial t} = H f$$

approximates the evolution of a liquid film inside a rotating horizontal cylinder.

We shall show that the eigenvalue problem

$$-i H f = \lambda f$$

has only real eigenvalues, which were shown to exist by Davies in [3]. In the same paper, he showed that the spectrum of $-i H$ is equal to the set of its eigenvalues, so this implies that the spectrum is real. This was conjectured in [1], and Chugunova and Pelinovsky proved in [2] that all but finitely many eigenvalues are real, and gave numerical evidence that all eigenvalues are real. Our approach is to show that it is sufficient to consider $H$ acting on the Hardy space $H^2(-\pi, \pi)$ and analytically continue any solution of (3) to the unit disc, where the corresponding ODE (8) is now self-adjoint on $[0, 1]$.
with regular singularities at the end points. In order to do this we make use of a bound on the Fourier coefficients proved by Davies in [3].

As in [3], by expanding \( f \in L^2(-\pi, \pi) \) in the form

\[
f(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} v_n e^{in\theta},
\]

one may rewrite the eigenvalue problem in the form \( Av = \lambda v \), where \( A = -iH \) is given by

\[
(Av)_n = \frac{\varepsilon}{2} n(n-1)v_{n-1} - \frac{\varepsilon}{2} n(n+1)v_{n+1} + nv_n.
\]

Here we have identified \( l^2(\mathbb{Z}) \) and \( L^2(-\pi, \pi) \) using the Fourier transform \( \mathcal{F} : l^2(\mathbb{Z}) \to L^2(-\pi, \pi) \). We have

\[
A = -i\mathcal{F}^{-1}H\mathcal{F}
\]

and define \( \text{Dom}(H) = \mathcal{F}(\text{Dom}(A)) \).

The (unbounded) tridiagonal matrix \( A \) is of the form

\[
A = \begin{pmatrix} A_- & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_+ \end{pmatrix}
\]

where \( A_- \) acts in \( l^2(\mathbb{Z}_-) \), the central 0 acts in \( \mathbb{C} \) and \( A_+ \) acts in \( l^2(\mathbb{Z}_+) \). We assume that \( A_+ \) has its natural maximal domain

\[
\mathcal{D} = \{ v \in l^2(\mathbb{Z}_+) : A_+v \in l^2(\mathbb{Z}_+) \}.
\]

Davies has shown in [3] that \( A_+ \) is closed and that \( \mathcal{D} \) is the closure under the graph norm of \( A_+ \) of the subspace consisting of those \( v \in l^2(\mathbb{Z}_+) \) that have finite support. Let \( \tau \) be the natural identification between \( l^2(\mathbb{Z}_+) \) and \( l^2(\mathbb{Z}_-) \); then \( \text{Dom}(A) = \tau(\mathcal{D}) \oplus \mathbb{C} \oplus \mathcal{D} \) and \( \tau \) induces a unitary equivalence between \( A_+ \) and \( A_- \). Therefore, in order to prove that all eigenvalues of \( A \) are real, we only need to prove that all eigenvalues of \( A_+ \) are real. The Fourier transform identifies \( l^2(\mathbb{Z}_+) \) with \( \{ f \in H^2(-\pi, \pi) : \int_{-\pi}^{\pi} f(\theta) d\theta = 0 \} \).

Let \( H_0 \) be the restriction of \( H \) to \( \mathcal{C}^2_{\text{per}}([-\pi, \pi]) \), which is clearly a subspace of \( \text{Dom}(H) \). We now show that \( H \) is the closure of \( H_0 \).

**Proposition 1.1** Where \( H \) and \( H_0 \) are as above, \( H \) is the closure of \( H_0 \).

**Proof** It follows from Davies’ result on the domain of \( A_+ \) that the trigonometric polynomials are dense in \( \text{Dom}(H) \) with respect to the graph norm. Since the trigonometric polynomials are contained in \( \mathcal{C}^2_{\text{per}}([-\pi, \pi]) \), this space is also dense in \( \text{Dom}(H) \), which is closed in graph norm since \( \text{Dom}(A) \) is.
2 Reality Of The Eigenvalues

If $A_+v = \lambda v$, then $v$ is a solution of the recurrence relation

$$
\frac{\varepsilon}{2}(n+1)(n+2)v_{n+2} + (\lambda - n - 1)v_{n+1} - \frac{\varepsilon}{2}n(n+1)v_n = 0
$$

(4)
satisfying the initial condition $\varepsilon v_2 = (1 - \lambda)v_1$. We shall study the generating function, $\sum_{k=1}^{\infty} v_k z^k$ of $(v_k)$.

Lemma 2.1 Let $v \in l^2(\mathbb{Z}_+)$ be such that $A_+v = \lambda v$. Then the function $u(z) := \sum_{k=1}^{\infty} v_k z^k$, defined for $|z| < 1$, satisfies the differential equation

$$
u'' - 2\frac{z + 1/\varepsilon}{(1 - z)(1 + z)}u' + \frac{2\lambda/\varepsilon}{z(1 - z)(1 + z)}u = 0.
$$

(5)

Proof The constant term in

$$z(1 - z)(1 + z)u'' - 2(z + 1/\varepsilon)zu' + (2\lambda/\varepsilon)u$$

is clearly 0, and the coefficient of $z$ is

$$2v_2 - 2v_1/\varepsilon + 2\lambda v_1/\varepsilon = 2 \left( v_2 - \frac{(1 - \lambda)}{\varepsilon}v_1 \right) = 0.$$

The coefficient of $z^n$ is

$$n(n+1)v_{n+1} - (n-1)(n-2)v_{n-1} - \frac{2}{\varepsilon}nv_n - 2(n-1)v_{n-1} + \frac{2\lambda}{\varepsilon}v_n$$

$$= \frac{2}{\varepsilon} \left[ \varepsilon n(n+1)v_{n+1} + (\lambda - n)v_n - \frac{\varepsilon}{2}n(n-1)v_{n-1} \right] = 0$$

for $n \geq 2$.

From here on we assume that $\lambda \in \mathbb{C}$ is an eigenvalue of the operator $A_+$ and $v$ is a corresponding non-zero eigenvector. In \[3\], Davies proved that there exist constants $b, m$ such that

$$||v||_{\infty,c} \leq b|\lambda|^m ||v||_2,$$

(6)

where $c = 1 + 1/\varepsilon$ and $||w||_{\infty,c} := \sup\{ |w_n|n^c : 1 \leq n < \infty \}$. Since $\varepsilon > 0$, this implies that $v \in l^1(\mathbb{Z}_+)$, and hence that $\sum_{k=1}^{\infty} v_k z^k$ is absolutely convergent for $|z| \leq 1$. The equation (5) can be written in the form of Heun’s equation

$$u'' + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) u' + \frac{\alpha \beta z - \mu}{z(z-1)(z-a)} u = 0$$

(7)
with $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1 + 1/\varepsilon$, $\epsilon = 1 - 1/\varepsilon$, $a = -1$ and $\mu = 2\lambda/\varepsilon$.

This is a Fuchsian equation with four regular singular points, at $0, 1, -1$ and $\infty$, with $\{0, 1 - \gamma\}, \{0, 1 - \delta\}, \{0, 1 - \epsilon\}$ and $\{\alpha, \beta\}$ as the roots of the corresponding indicial equations (for the Frobenius series at each regular singular point). For background information on Heun’s equation, see [1].

**Lemma 2.2** Suppose that $0 < \varepsilon < 2$, $1/\varepsilon \notin \mathbb{Z}$, $\lambda \in \mathbb{C}$ is an eigenvalue of $A_+$ and $v \in l^2(\mathbb{Z}_+)$ is a corresponding non-zero eigenvector. Then there exists a solution $u$ of (7) which is analytic in an open set containing $[0, 1]$ and such that $u(z) = \sum_{k=1}^{\infty} u_k z^k$ for all $z$ such that $|z| \leq 1$.

**Proof** Define $u(z) = \sum_{k=1}^{\infty} v_k z^k$ on $\{z \in \mathbb{C} : |z| < 1\}$. Let $u_1$ be the solution of (7) with exponent 0 about 1 and $u_2$ be the solution with exponent $-1/\varepsilon$ about 1. Let $U$ be the intersection of the open discs of unit radius about 0 and 1. The space of solutions of (7) in $U$ is two-dimensional, and $u$, $u_1$ and $u_2$ lie in this space. Hence there exist constants $a, b$ such that $u = au_1 + bu_2$ in $U$. Since $v \in l^2(\mathbb{Z}_+)$, $u(z)$ converges to a finite limit as $z \to 1$ in $U$. Also $u_1(1)$ is finite, but $u_2(z) \to \infty$ as $z \to 1$ in $U$. Therefore we must have $b = 0$ and $u = au_1$ in $U$. Let $W$ be the union of the open discs of unit radius about 0 and 1. We now extend $u$ to all of $W$ by $u(z) = au_1(z)$ on the open disc of unit radius about 1. Now $u$ is an analytic solution of (7) on $W$, which is an open set containing $[0, 1]$, such that $u(z) = \sum_{k=1}^{\infty} v_k z^k$ for all $z$ such that $|z| \leq 1$.

**Theorem 2.3** Suppose that $0 < \varepsilon < 2$, $1/\varepsilon \notin \mathbb{Z}$ and $\lambda \in \mathbb{C}$ is an eigenvalue of $A_+$. Then $\lambda \in \mathbb{R}$.

**Proof** Let $v \in l^2(\mathbb{Z}_+)$ be a non-zero eigenvector corresponding to $\lambda$. Let $u$ be as in Lemma 2.2 and put $\mu = 2\lambda/\varepsilon$. Following [1], the equation (7) can be written as

$$-(pu')' + qu = \mu w$$

on the complex plane cut along $[1, \infty)$ and $(-\infty, 0]$, where

$$
\begin{align*}
p(z) &= z^{\gamma}(1-z)^{\delta}(z-a)^{\epsilon} = (1-z)^{1+1/\varepsilon}(z+1)^{1-1/\varepsilon} \\
q(z) &= \alpha \beta z^{\gamma}(1-z)^{\delta-1}(z-a)^{-1} = 0 \\
w(z) &= z^{\gamma-1}(1-z)^{\delta-1}(z-a)^{-1} = z^{-1}(1-z)^{1/\varepsilon}(z+1)^{-1/\varepsilon}.
\end{align*}
$$

We now restrict $u$ to $[0, 1]$. It is clear that $u \in \mathcal{C}^{\infty}([0, 1])$ and

$$-(pu')' = \mu w$$
on $(0, 1)$. Note that $p > 0$ on $[0, 1)$ and $w > 0$ on $(0, 1)$ with $p(x), w(x) \to 0$ as $x \to 1$ from below. Since $u$ has a zero of order 1 at 0 and $w$ has a pole of order 1 at 0, $wu \in C([0, 1])$. Therefore $|u|^2w \in L^1(0, 1)$. Now

$$
\mu \int_0^1 |u(x)|^2 w(x) \, dx = \lim_{n \to \infty} \left\{ - \int_{1/n}^{1-1/n} (pu')(x)u(x) \, dx \right\}
= \lim_{n \to \infty} \left\{ - \left[ p(x)u'(x)u(x) \right]_{1/n}^{1-1/n} + \int_{1/n}^{1-1/n} p(x)u'(x)u'(x) \, dx \right\}
= \lim_{n \to \infty} \left\{ \left[ p(x) \left\{ u(x)u'(x) - u'(x)u(x) \right\} \right]_{1/n}^{1-1/n} - \int_{1/n}^{1-1/n} (pu')'(x)u(x) \, dx \right\}
= - \int_0^1 (pu')'(x)u(x) \, dx
= \frac{\mu}{\pi} \int_0^1 |u(x)|^2 w(x) \, dx.
$$

Since $u$ is a non-zero solution of (8) and $w > 0$ a.e. we have $\mu = \pi\bar{\mu}$ and hence $\mu \in \mathbb{R}$. Since $\lambda = \frac{\pi}{\bar{\mu}}$, we also have $\lambda \in \mathbb{R}$. 

\section*{References}

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