Pulsation frequency distribution in δ Scuti stars

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Accepted 2015 July 6. Received 2015 July 6; in original form 2015 May 25

ABSTRACT

We study the frequency distributions of Delta Scuti (δ Scuti) stars observed by the Kepler satellite in short-cadence mode. To minimize errors in the estimated stellar parameters, we divided the instability strip into 10 regions and determined the mean frequency distribution in each region. We confirm that the presence of low frequencies is a property of all δ Scuti stars, rendering meaningless the concept of δ Scut/γ Dor hybrids. We obtained the true distribution of equatorial rotational velocities in each region and calculated the frequency distributions predicted by pulsation models, taking into account rotational splitting of the frequencies. We confirm that rotation cannot account for the presence of low frequencies. We calculated a large variety of standard pulsation models with different metal and helium abundances, but were unable to obtain unstable low-frequency modes driven by the k mechanism in any model. We also constructed models with modified opacities in the envelope. Increasing the opacity at a temperature log T = 5.06 by a factor of 2 does lead to instability of low-degree modes at low frequencies, but also decreases the frequency range of δ Sct-type pulsations to some extent. We also re-affirm the fact that less than half of the stars in the δ Scut instability strip have pulsations detectable by Kepler. We also point out the huge variety of frequency patterns in stars with roughly similar parameters, suggesting that non-linearity is an important factor in δ Sct pulsations.

Key words: stars: oscillations – stars: variables: δ Scuti.

1 INTRODUCTION

Delta Scuti (δ Scut) stars are main-sequence dwarfs and giants with spectral types A0–F5 which pulsate in multiple p, g and mixed modes. They lie on the extension of the Cepheid instability strip towards low luminosities. From ground-based observations, most δ Scut stars are found to pulsate with frequencies in the range 5–50 d⁻¹. The pulsations are driven by the k mechanism due to partial ionization of He ii (Pamyatnykh 2000).

The γ Doradus (γ Dor) stars lie in a fairly small region on, or just above, the main sequence that partly overlaps the cool edge of the δ Sct instability strip. They pulsate in multiple g modes with low frequencies (typically less than 5 d⁻¹) and are thought to be driven by a mechanism known as ‘convective blocking’ (Guzik et al. 2000; Dupret et al. 2004). A subset of the δ Scut stars pulsate in both the high- and low-frequency ranges and are known as δ Scut/γ Dor hybrids.

Space photometry from the Kepler observatory has completely changed this view. It is found that all δ Scut stars pulsate in low-frequency modes, i.e. they are all hybrids (Balona 2014a). The misconception of hybrid δ Scut/γ Dor stars is due to the fact that low frequencies in δ Scut stars have sufficiently high amplitudes to be observed from the ground only in a narrow band close to the red edge of the instability strip. The low frequencies in δ Scut stars present a serious problem because standard models predict that low frequencies are stable for δ Scut stars hotter than the granulation boundary. In these hot stars, the subsurface convection zone is too thin for convective blocking to drive low-frequency pulsations.

Apart from the problem of low frequencies in hot δ Scut stars, there is the additional problem that the majority of stars in the δ Scut instability strip do not pulsate (Balona & Dziembowski 2011). It is difficult to understand why in two stars with the same physical parameters one star should pulsate as a δ Scut star and the other not pulsate at all, or at least have pulsational amplitudes not detectable in Kepler observations. It is possible that non-linear mode coupling may stabilize the pulsations (Dziembowski 1982; Dziembowski & Krolikowska 1985; Dziembowski, Krolikowska & Kosovichev 1988) or that the opacity-driving mechanism may be saturated (Nowakowski 2005). Since non-linear models of non-adiabatic non-radial pulsations do not exist, it is not possible, at present, to test these ideas.
Another issue is the fact that A stars appear to be not as simple as previously thought. It has always been supposed that stellar activity ceases for stars hotter than the granulation boundary because the subsurface convection zone is too thin to generate a magnetic field by the dynamo mechanism. Analysis of Kepler photometry shows that about 40 per cent of A stars have light variations whose periods closely match the expected rotation periods of these stars, suggesting the presence of star-spots (Balona 2011, 2013). Furthermore, about 2 per cent of A stars have flares (Balona 2012, 2013). In fact, the relative number of A-type flare stars is about the same as F and G flare stars and not much smaller than K and M flare stars (Balona 2015). The star-spot and flare activity in A stars indicates that we do not fully understand the physics of stellar envelopes which may be related to our lack of understanding of low frequencies in hot δ Sct stars.

Using Kepler data, Balona (2014a) showed that low frequencies in δ Sct stars are not stochastically driven, nor can they be explained as non-linear combinations of high-frequency modes. By comparing the observed distribution of frequencies in Kepler δ Sct stars with those from models, Balona (2014a) also showed that the low frequencies cannot be explained as a result of rotational splitting of high-frequency modes.

In order to understand the origin of the low frequencies in δ Sct stars, it is necessary to explore what effects contribute most to the stability or instability of these frequencies in the models. A simple examination of the instability parameter, $\eta$, calculated by a non-adiabatic pulsation model will tell us if a particular mode is stable or unstable. However, in order to compare the predicted frequency distribution with observations, it is necessary to simulate rotational splitting of the frequencies as well as other effects. In the absence of mode identification and sufficiently accurate stellar parameters, this is the only way to test if model predictions agree with observations. In order to simulate the effect of rotational splitting, the distribution of true equatorial rotational velocities for the group of stars under study is required. In Balona (2014a), all δ Sct stars were treated as a single group using rotation periods inferred from the light curve.

In this paper, we compare the observed and simulated frequency distributions in several regions across the instability strip. Different physical conditions are used for each set of models in an attempt to determine which effects most closely reproduce the observed frequency distributions. In this way, we hope to determine whether it is possible to understand the low frequencies under the assumption that they are driven by the $\kappa$ mechanism. We study mode stability using standard models, i.e. using different abundances and standard opacities, as well as non-standard models, i.e. using opacities which are artificially increased in certain regions in the envelope.

## 2 THE DATA

The Kepler light curves are available as uncorrected simple aperture photometry (SAP) and with pre-search data conditioning (PDC) in which instrumental effects are removed. The vast majority of the stars are observed in long-cadence (LC) mode with exposure times of about 30 min. Several thousand stars were also observed in short-cadence (SC) mode with exposure times of about 1 min. These data are publicly available on the Barbara A. Mikulski Archive for Space Telescopes (MAST, archive.stsci.edu).

In order to identify δ Sct stars in the Kepler field, we calculated the periodograms of over 20000 stars, including all stars observed in SC mode. By visually examining the light curves and periodograms and using the effective temperatures in the Kepler Input Catalogue (KIC; Brown et al. 2011) as a guide, we were able to identify over 1600 δ Sct stars in LC mode and 403 δ Sct stars in SC mode with known stellar parameters. These do not include δ Sct stars in eclipsing binary systems.

The highest pulsation frequency that can be detected in LC mode is about 24 d$^{-1}$. Since the pulsation frequencies in many δ Sct stars exceed this value, only SC observations allow the unambiguous identification of all pulsation frequencies. Due to variations in heliocentric time correction, the Kepler data are not sampled at exactly equal time intervals. In principle, this allows frequencies higher than 24 d$^{-1}$ to be identified in LC data (Murphy, Shibahashi & Kurtz 2013). In practice, however, the chance of frequency misidentification is very high for low-amplitude peaks because the difference in peak amplitude between the high- and low-frequency aliases is slight. For this reason, we restrict our analysis to the 403 δ Sct stars observed in SC mode.

To extract all significant frequencies, we first calculated the standard Lomb periodogram for unequally spaced data (Press & Rybicki 1989). The usual technique to extract peaks is that of successive pre-whitening. Great caution needs to be exercised in using this method as indiscriminate use leads to a large number of spurious frequencies (Balona 2014b). In effect, only those peaks which are visible in the periodogram of the raw, un-prewhitened data can be considered as significant. Indiscriminate successive pre-whitening will extract frequencies which cannot be resolved in the periodogram, even though the extracted amplitudes of such spurious frequencies may be deemed significant. In determining the frequency content of a star, we terminated pre-whitening when the peak can no longer be resolved in the original periodogram.

## 3 SYSTEMATIC AND RANDOM ERRORS

In order to compare the observed frequency distribution with the calculated frequency distribution, we need to determine the accuracy of the stellar parameters for individual stars. The stellar parameters derived from Sloan multicolour photometry (without the $u$ band) are listed in the KIC. A full explanation of how these parameters were determined is given in Brown et al. (2011). It is important to determine the systematic and random errors that might be present in the values of effective temperature, $T_{\text{eff}}$, surface gravity, $\log g$, and relative luminosity, $\log (L/L_\odot)$, derived from the KIC parameters.

We do not know the true values of $T_{\text{eff}}$ and $\log (L/L_\odot)$ for any δ Sct star, but we can compare the values in the KIC derived from multicolour photometry with the more precise values that can be derived from spectroscopy. Recently, Lehmann et al. (2011), Catanzaro et al. (2010), Niemczura et al. (2015) and Tkachenko et al. (2013) have obtained spectroscopy of several Kepler stars from which stellar parameters are derived. The most numerous observations are those of Niemczura et al. (2015) for which we find a systematic difference in effective temperature $\Delta T_{\text{eff}} = T_{\text{eff}}(\text{spec}) - T_{\text{eff}}(\text{KIC}) = 144 \pm 24$ K from 107 stars with a standard deviation of 253 K per star. There is no significant dependence of $\Delta T_{\text{eff}}$ on $T_{\text{eff}}$. Since the standard deviation of $T_{\text{eff}}(\text{spec}) \approx 150$ K (Niemczura et al. 2015), we estimate that the standard deviation of $T_{\text{eff}}(\text{KIC}) \approx 200$ K on the assumption that the variances add in quadrature. The top panel of Fig. 1 shows $T_{\text{eff}}(\text{spec})$ from Niemczura et al. (2015) as a function of $T_{\text{eff}}(\text{corr}) = T_{\text{eff}}(\text{KIC}) + 144$.

The surface gravities in the KIC are compared to the spectroscopic surface gravities from Niemczura et al. (2015) in the middle panel of Fig. 1. The mean difference is $\log g(\text{spec}) - \log g(\text{KIC}) = 0.03 \pm 0.02$ dex. The standard deviation is 0.24 dex while Niemczura et al. (2015) quote the typical standard deviation in $\log g$ at about 0.15 dex. This suggests that
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Figure 1. Top panel: the corrected KIC effective temperature as a function of the effective temperature determined by spectroscopic observations of Niemczura et al. (2015). The straight line has unit slope and zero intercept. Middle panel: the surface gravity, $\log g$, listed in the KIC as a function of $\log g$ from Niemczura et al. (2015). The straight line has unit slope and zero intercept. Bottom panel: the luminosity, $\log (L/L_\odot)$, derived from the KIC parameters as a function of $\log (L/L_\odot)$ derived from the spectroscopic parameters. The straight line has unit slope and zero intercept.

the standard deviation in the KIC value of $\log g$ is about 0.2 dex. According to Brown et al. (2011) the standard deviation of $\log g$ in the KIC for dwarfs is about 0.4 dex.

Given $T_{\text{eff}}$ and the surface gravity, $\log g$, the stellar radius may be estimated using the relationship in Torres, Andersen & Giménez (2010). This relationship requires the metal abundance which is usually listed among the spectroscopically derived parameters. When the metal abundance is not available, the solar value was used. From the effective temperature and the radius, the relative stellar luminosity, $\log (L/L_\odot)(\text{spec})$, can be determined. For stars where only KIC values are available, the luminosity, $\log (L/L_\odot)(\text{KIC})$ can likewise be estimated from the tabulated effective temperature and radius.

The bottom panel of Fig. 1 shows $\log (L/L_\odot)(\text{spec})$ as a function of $\log (L/L_\odot)(\text{KIC})$. The correlation is rather poor, no doubt due to the poor precision of $\log g$. As the figure shows, a line of unit slope and zero intercept gives a satisfactory fit. The standard deviation in $\log (L/L_\odot)$ is 0.26 if the regression is performed with $\log (L/L_\odot)(\text{KIC})$ as the independent variable or 0.29 as the dependent variable. The standard deviation of $\log (L/L_\odot)$ derived from the KIC parameters must therefore be about 0.20 dex, since the total variance is the quadratic sum of the two variances.

We used spectroscopically derived parameters whenever possible for each star. When only KIC values are available, we corrected $T_{\text{eff}}$ in the KIC by 144 K and estimated the luminosity from the KIC radius and uncorrected effective temperature. The luminosities, $\log (L/L_\odot)$, using the Torres et al. (2010) relationship with $T_{\text{eff}}$ and $\log g$ from the KIC and adopting solar abundances for all stars differ from the KIC luminosities by about 0.1 dex. The standard deviation in $\log (L/L_\odot)$ is about twice this value.

Given the large errors in $T_{\text{eff}}$ and $\log (L/L_\odot)$, the lack of mode identification and the profound effect of rotation on the frequency spectrum, it is not possible to model the pulsation frequencies of individual stars. In fact, we are not interested in detailed modelling of a specific star, but only in a comparison of the overall observed and calculated frequency distributions. For this reason we have taken a different approach. We can minimize the uncertainties in effective temperature and luminosity by using the mean values of $\log T_{\text{eff}}$ and $\log (L/L_\odot)$ of many stars in approximately the same location in the instability strip. A good estimate of the mean frequency distribution at this position in the instability strip is obtained by averaging the frequency distributions of these stars. For this purpose we divided the theoretical H–R diagram into 10 regions as shown in Fig. 2.

These regions were chosen to cover the instability strip as evenly as possible with reasonable resolution while, at the same time, allowing a fairly large number of stars in each region. The mean effective temperature, luminosity, surface gravity and radius of all $\delta$ Scct stars in each region are shown in Table 1.

Figure 2. The theoretical H–R diagram showing the SC $\delta$ Sct stars (dots) and the designated regions (labelled). The line is the theoretical zero-age main sequence.
Table 1. For each region number, \( N_{\text{reg}} \), the number of \( \delta \) Sct stars, \( N \), in that region and the average number of modes, \( N_{\text{modes}} \), having \( v > 5 \) d\(^{-1} \) are shown. Also shown is the corresponding range in spectral type, the mean effective temperature, \( T_{\text{eff}} \), luminosity, \( \log(L/L_\odot) \), surface gravity, \( \log g \) and radius, \( R/R_\odot \). The last column, \( \langle v_e \rangle \), is the mean equatorial rotational velocity (in km s\(^{-1} \)) derived from observations of many field stars of the particular spectral type and luminosity class range.

| \( N_{\text{reg}} \) | \( N \) | \( N_{\text{modes}} \) | Sp. ty. | \( \log T_{\text{eff}} \) | \( \log (L/L_\odot) \) | \( \log g \) | \( R/R_\odot \) | \( \langle v_e \rangle \) |
|---|---|---|---|---|---|---|---|---|
| 1 | 29 | 90 | F8V–F3V | 3.8253 ± 0.0015 | 0.6889 ± 0.0152 | 4.100 ± 0.011 | 1.654 ± 0.024 | 40 |
| 2 | 60 | 252 | F4V–A9V | 3.8546 ± 0.0012 | 0.8896 ± 0.0119 | 4.043 ± 0.009 | 1.822 ± 0.020 | 90 |
| 3 | 32 | 183 | F0V–A7V | 3.8858 ± 0.0019 | 1.0922 ± 0.0178 | 4.003 ± 0.011 | 1.993 ± 0.030 | 150 |
| 4 | 25 | 138 | A8V–A4V | 3.9146 ± 0.0012 | 1.2671 ± 0.0190 | 3.985 ± 0.013 | 2.136 ± 0.039 | 150 |
| 5 | 35 | 236 | F5IV–F1IV | 3.8537 ± 0.0014 | 1.0808 ± 0.0136 | 3.889 ± 0.010 | 2.280 ± 0.030 | 70 |
| 6 | 61 | 245 | F1IV–A8IV | 3.8837 ± 0.0012 | 1.3335 ± 0.0130 | 3.781 ± 0.010 | 2.663 ± 0.035 | 110 |
| 7 | 34 | 220 | A8IV–A5IV | 3.9121 ± 0.0014 | 1.4897 ± 0.0148 | 3.777 ± 0.013 | 2.793 ± 0.047 | 140 |
| 8 | 18 | 103 | F5III–F1III | 3.8471 ± 0.0022 | 1.3958 ± 0.0210 | 3.566 ± 0.016 | 3.381 ± 0.070 | 80 |
| 9 | 58 | 365 | F1III–A8III | 3.8746 ± 0.0013 | 1.5508 ± 0.0118 | 3.551 ± 0.009 | 3.560 ± 0.041 | 100 |
| 10 | 15 | 83 | A9III–A6III | 3.9043 ± 0.0025 | 1.7030 ± 0.0181 | 3.553 ± 0.012 | 3.692 ± 0.059 | 110 |

4 MODE VISIBILITY

In determining the frequency distribution of a particular star, we are faced with the problem that individual frequencies are of widely different amplitudes. It would, of course, be preferable to calculate the amplitudes from the models as well. Unfortunately non-linear calculations of non-radial modes are beyond existing technical capabilities. One approach is to take the linear growth rate as a proxy for the amplitude. However, the final pulsational amplitude attained by a star depends on many factors and not only the initial growth rate. We decided that the most satisfactory method is to ignore amplitude altogether. In other words, a mode of very high amplitude is given the same weight as a mode of very low amplitude. In determining the observed frequency distribution, we simply count the number of significant frequency peaks in the periodogram within a given frequency interval, ignoring their amplitudes.

Using this approach requires that each unstable mode in the models be weighted according to its visibility. It is evident that modes of very high spherical harmonic degree, \( l \), will not be observed because of cancellation effects and cannot therefore be given the same weight as modes of low \( l \). In other words, each unstable mode derived from a model should be assigned a weight equal to its visibility before it can be compared with observations. To calculate the visibility of a mode, we need to discuss the expected relative light amplitude caused by a mode of particular spherical harmonic degree, \( l \), and azimuthal order, \( m \).

The monochromatic light amplitude, \( A_i(i) \), for a pulsating star whose axis of rotation is inclined at angle \( i \) with respect to the observer can be expressed as (Daszyńska-Daszkiewicz et al. 2002)

\[
A_i(i) = \epsilon Y^m_i(i,0)b_j^i \left\{ D_{1,i}^j + D_{2,i}^j + D_{3,i}^j \right\}.
\]

Here \( \epsilon \) is the intrinsic amplitude, \( Y^m_i(i,\phi) = N_{lm} P^m_i(\cos \phi) e^{i\phi} \) is the spherical harmonic with \( N^m_i \) the normalizing factor and \( P^m_i(\cos \phi) \) the associated Legendre polynomial (\( i = \sqrt{1 - \lambda} \)). The disc averaging factor, \( b_j^i \), is given by

\[
b_j^i = \int_0^\pi h_j(\mu)\mu P_i(\mu)d\mu,
\]

where \( \mu = \cos \theta \) and \( \theta \) is the angle in the spherical coordinate system centred on the star. We used the simple limb-darkening law \( h(\mu) = 1 + \frac{3}{4} \mu \) independent of wavelength. Values of \( b_j \) drop sharply with \( \mu = 1 \), \( b_0 = 0.708 \), . . . \( b_0 = 0.008 \).

In calculating \( D_{1,i} \) and \( D_{1,i} \), we used the partial derivatives for the Johnson V band computed by Kowalczuk & Daszyńska-Daszkiewicz (2007). The value of \( D_{1,i} \) is typically around 40 and \( D_{3,i} \) is in the range 2–3. The geometric term, \( D_{2,i} \), increases in absolute value with \( l \) and is comparable with \( D_{1,i} \) for high \( l \). The temperature variation together with \( b_j \) dominates the visibility.

5 THE EFFECT OF ROTATION

In a non-rotating star, pulsation modes with the same spherical harmonic degree, \( l \), but with different azimuthal numbers, \( m \), have
the same frequency. To first order, the effect of rotation is to remove this degeneracy so that a mode with frequency \( v_0 \) in the non-rotating star appears as \( 2l + 1 \) equally-spaced multiplets with frequencies, \( v \), given by

\[
v = v_0 + m v_{\text{rot}} (1 - C_{\text{nl}}),
\]

where \( v_{\text{rot}} \) is the rotation frequency and \( C_{\text{nl}} \) is a constant which depends on the structure of the star, \( l \) and the radial order, \( n \) (Ledoux 1951). The value of \( C_{\text{nl}} \) is quite small for \( p \) modes and hence the frequency splitting may be several cycles \( d^{-1} \) in the extreme case of high \( m \) and large \( v_{\text{rot}} \). The effect of rotation to first order is to introduce a symmetric spread in frequencies.

The above formula breaks down as the rotation rate increases and the splitting is no longer symmetric. The frequency of each multiplet decreases with increasing rotation rate, but the decrease in frequency is greater for larger values of \( |m| \). Thus the spread in frequencies is skewed towards low frequencies. It is therefore important to take this effect into account by including higher order rotational perturbations.

The following formula (Goupil 2011) is accurate to third order:

\[
\omega = \omega_0 - m(1 - C_{\text{nl}})\Omega + (D_1 + m^2 D_2)\Omega^2 + m(T_1 + m^2 T_2)\Omega^3,
\]

where \( \omega, \omega_0 \) are the perturbed and unperturbed angular pulsation frequencies and \( \Omega \) is the angular rotation frequency. The values of \( D_1, D_2, T_1 \) and \( T_2 \) depend on the mode. In this expression, the pulsation and rotation frequencies are in units of the dynamical frequency \( \omega_{\text{dyn}} = \sqrt{GM/R^3} \), where \( G \) is the gravitational constant, \( M \) the stellar mass and \( R \) the stellar polar radius. The coefficients have not been calculated for stars in the \( \delta \) Sct instability strip. However, Reese, Li`ni`eres & Rieutord (2006) lists values for a polytrope of index 3 for \( l \leq 3 \) and \( n \leq 10 \) for solid body rotation.

Since most A stars are moderate or rapid rotators, it is very important to include rotation effects as accurately as possible. Since calculations are only available for polytropes and for limited values of \( l \) and \( n \), we are faced with a problem. We decided that the best approach is to use the third-order formula and polytropic values for all \( p \) modes. For \( p \) modes with \( l > 3 \), we use the coefficients for \( l = 3 \). For radial orders \( n > 10 \), we use the values for \( n = 10 \). This might overestimate the rotational splitting for modes with \( l > 3 \). Rotational splitting for \( g \) modes is smaller than for \( p \) modes. We decided to use the first-order formula for all \( g \) modes. The distinction between \( p \) and \( g \) modes is made on the basis of the ratio of the mode kinetic energy in the gravity-wave propagation zone, \( E_{\text{kg}} \), to the total kinetic energy of a mode, \( E_k \), as calculated by the models. We treated modes with \( E_{\text{kg}}/E_k < 0.5 \) as \( p \) modes.

To calculate the frequencies of the rotational multiplets in the way just described requires knowledge of the rotation frequency, \( v_{\text{rot}} \). In Balona (2014a), the value of \( v_{\text{rot}} \) was obtained from a peak in the periodogram which could be attributed to rotational modulation. We cannot do this because the number of stars with known photometric rotation period is too small (and sometimes zero) in each region. To overcome this problem, we have to make the assumption that the distribution of equatorial rotational velocities in a particular region closely corresponds to the distribution of equatorial rotational velocities for field stars with a spectral type and luminosity class appropriate to that region. We used the catalogue of projected rotational velocities compiled by Glebocki & Stawikowski (2000) to determine the distribution of \( v \sin i \) for stars in each region. To obtain the distribution of true equatorial rotational velocities, we used the procedure described in Balona (1975). Fig. 3 shows the distribution of \( v \sin i \) and the polynomial approximation to the true distribution of equatorial rotational velocities.
6 ASSIGNING WEIGHTS

For a stellar model with a given mass and radius, we may calculate the rotational frequency, \( \nu_{\text{rot}} \), for any given equatorial rotational velocity, \( v_\text{e} \). Given \( \nu_{\text{rot}} \) and the calculated frequency of an unstable mode of degree \( l \) and radial order \( n \), we may calculate the \( 2l + 1 \) rotationally-perturbed frequencies using the first-order Ledoux formula for \( g \) modes and the third-order formula for the \( p \) modes. To each of these frequency multiplets, we assign the same weight, \( w_{\nu_{\text{rot}}} \), which is the probability of finding a star with the particular value of \( v_\text{e} \). This weight is calculated using the polynomial fit to the distribution of true equatorial velocities for stars in the appropriate region. We calculated the frequencies and weights of all rotationally-split multiplets using values of \( v_\text{e} \) from zero to the maximum value in steps of 10 km s\(^{-1}\).

We also assign a weight \( w_i = (Y_0^m(i, 0)) \) which accounts for the visibility of the mode due to random orientation of the axis of rotation by numerically calculating the integral. This term is relatively unimportant because the \( w_i \) does not change much. The visibility due to cancellation effects, \( w_i = b_i \), is pre-calculated for any given value of \( l \). Finally we calculate the weight \( w_i = D_{1,1} + D_{2,1} + D_{3,1} \) using the values of \( f, \omega \) and \( \tilde{\rho} \) from the model as well as the partial derivatives applicable to the given model. The total weight for the mode with frequency \( v = W_i = w_{\nu_{\text{rot}}}w_\nu w_i. \) In calculating the predicted frequency distribution we chose a bin size of 1 d\(^{-1}\). The probability at any given frequency, \( P(\nu) \), is \( P(\nu) = \sum W_i \) for all frequencies within this interval.

In principle, the frequency distribution, \( P(\nu) \), needs to be calculated for all possible values of \( l \). However, the mode visibility drops sharply with \( l \), so that modes with \( l > l_{\text{max}} \) have amplitudes which are so low that they can be ignored. The simplest way to determine \( l_{\text{max}} \) is to construct frequency distributions using increasing values of \( l_{\text{max}} \). We find that there is scarcely any difference in the distributions with \( l_{\text{max}} > 4 \). We adopted \( l_{\text{max}} = 6 \) which is certainly adequate for our purposes.

In the above procedure, we have made assumptions that may not perhaps be entirely justifiable. For example, we assume that all rotationally-split multiplets have the same intrinsic amplitude. We cannot calculate these amplitudes so we do not really know, but this is probably a fair assumption when averaged over many stars. It seems reasonable to assume that stars with approximately the same stellar parameters and rotational velocities have similar frequency distributions. This is certainly the case given our current understanding and that is what the models predict. We shall see below that even this seemingly secure assumption may not be correct.

Combination frequencies are known to occur in high-amplitude \( \delta \) Sct (HADS) stars. There are only four HADS in the Kepler field, though it is possible that stars with lower amplitudes might have some combination frequencies. Removing these four stars has a negligible effect on the frequency distribution for the corresponding regions. In any case, since combination frequencies are roughly distributed evenly over the frequency range, the effect will be a slight uniform increase in the distribution, but no overall change in shape.

7 COMPARISON WITH THEORETICAL PREDICTIONS

Equilibrium models were computed using the Warsaw–New Jersey evolution code (Paczyński 1970), assuming two initial hydrogen fractions, \( X_0 = 0.70, 0.65 \) and four metal abundances, \( Z = 0.005, 0.015, 0.030 \) and 0.050. We adopted the chemical element mixture of Asplund et al. (2009) and two sources of the opacity data: OPAL (Rogers & Iglesias 1992) and OP (Seaton 2005). We did not include overshooting from the convective core and used the mixing length parameter \( \alpha_{\text{MLT}} = 0.5 \) for the convective scaleheight. We chose this low value of \( \alpha_{\text{MLT}} \) to eliminate, as far as possible, the fictitious instability of low-frequency modes in the cooler and lower mass models. This instability is a consequence of the frozen convective flux approximation that we adopted.

We calculated a grid of equilibrium models covering the \( \delta \) Sct instability strip. The models take rotation into account to first order by applying the centrifugal force correction to local gravity while keeping spherical symmetry. We used a typical initial equatorial rotational velocity of \( v_\text{e} = 150 \) km s\(^{-1}\) except for the models with the lowest mass where we used \( v_\text{e} = 50 \) km s\(^{-1}\). The effect of rotation is a slight shift of the evolutionary tracks to lower effective temperatures and higher luminosities. The main-sequence band is also somewhat extended (see for example Breger & Pamyatnykh 1998). Given the fact that we need to determine the rotational frequency splitting over a wide range of rotational velocities corresponding to the known equatorial rotational velocity distribution for a particular group of stars, the value of \( v_\text{e} \) in the equilibrium model is not an important factor. A non-rotating model would serve our purposes equally well. As explained in the previous section, we used the third-order rotational perturbation only for \( p \) modes, while for \( g \) modes the first-order Ledoux splitting formula was used.

The linear non-adiabatic code of Dziembowski (1977) was used to determine the frequencies and the instability parameter, \( \eta \), as well as the ratio of luminous flux to displacement, \( f \), for each mode. Only modes with \( l \leq 6 \) were considered.

In calculating the predicted frequency distribution, we use individual values of log \( T_{\text{eff}} \) and log \( (L/L_\odot) \) for each star in a particular region. The model with temperature and luminosity closest to these values is used to determine the predicted frequency distribution. The average of all frequency distributions for stars in a given region is taken as the best approximation of the predicted frequency distribution for that region.

The observed and predicted frequency distributions are shown in Fig. 4 for \( Z = 0.015 \) and \( Z = 0.030 \) using OPAL opacities and an initial hydrogen abundance \( X_0 = 0.70 \). The unstable low frequencies predicted for the models of region 1 are a result of the frozen convection approximation and are fictitious. It can be seen that low frequencies are observed in stars across the whole instability strip. In other words, all \( \delta \) Sct stars are hybrids. Also, it can be seen that the predicted frequency distributions fail to show frequencies less than about 5 d\(^{-1}\). The same result was obtained with models calculated using OP opacities.

Apart from the disagreement in the low-frequency range, we also note that for the more luminous stars of regions 8, 9 and 10, the observed distributions extend to much higher frequencies than predicted. In general, we expect the observed distribution to be broader than the predicted distribution because each region includes a fraction of stars with true values of \( T_{\text{eff}} \) and log \( (L/L_\odot) \) which are outside the region. Even so, it does not seem possible to explain the high-frequency tails of the luminous stars in this way because there is no such tail among the less-luminous stars.

What we can definitely conclude from Fig. 4 is that a change in metal abundance cannot explain the mismatch between observations and the models. We also calculated models with \( Z = 0.005 \) and \( Z = 0.050 \) which we do not show in Fig. 4 to avoid confusion. The distributions with lower and higher metal abundance are quite similar to those shown in the figure. We also calculated models where the helium abundance is substantially increased so that the initial
Frequency distribution in δ Scuti stars

Figure 4. The observed frequency distributions (filled) and the corresponding theoretical frequency distributions derived from pulsation models with metal abundance $Z = 0.015$ (solid blue curves) and $Z = 0.030$ (dotted green curves). Only unstable modes with the degrees, $l \leq 6$ are considered. Each panel shows the frequency distribution corresponding to the labelled region in the H–R diagram of Fig. 2.

Hydrogen abundance is decreased from $X_0 = 0.70$ to $X_0 = 0.65$, but this does not affect mode stability at all at low frequencies. However, there is a slight increase of the instability parameter, $\eta$, at high frequencies. We also studied the effect of modes of higher degree and found a negligible difference in the distributions for modes with $l_{\text{max}} = 6$ and $l_{\text{max}} = 10$.

It could be argued that we have not taken the effect of rotation fully into account and that agreement could be achieved by a more realistic treatment of rotational splitting. Unfortunately, we do not have access to non-adiabatic models which take into account stellar distortion, gravity darkening and rapid rotation in a more realistic way. We can, however, take the effect of rotation to extremes by using the third-order perturbation for all modes. We can also enhance the visibility of modes of high degree (which have maximum rotational splitting) by artificially suppressing the disc-averaging factor, $b_l$ (i.e. setting $b_l = 1$ for all modes). The distributions calculated in this way are shown in Fig. 5 for $Z = 0.015$.

Although there is a slight increase in the numbers of low-frequency modes, it is still far too little to match observations. Artificially increasing the effect of rotation in this way also does not assist in explaining the high-frequency tail in luminous stars. We may confidently conclude that there is no possibility that rotational splitting of high-frequency modes can explain the low frequencies.

Since the low frequencies cannot be explained by modifications of standard models, we need to consider other possibilities. Low frequencies not predicted by models also occur in the β Cep stars. These are early B stars with multiple p, g and mixed modes driven by the opacity bump due to iron group elements (the Z bump). Some low frequencies in these stars can be explained by standard models and some demand increasing the opacity in the Z-bump region (Pamyatnykh, Handler & Dziembowski 2004). Whether this is the correct solution on how to create the increased opacity is not known. As in the B stars, it is possible that artificially increasing the opacity of δ Sct models in the Z-bump region at $\log T \approx 5.35$ might destabilize the low frequencies.

Recently, Cugier (2012, 2014) found that the OPAL and OP opacities are markedly underestimated in comparison with the Rosseland mean opacities taken from the Castelli & Kurucz (2003) model atmospheres. As a result, a new opacity bump appears at $\log T \approx 5.06$. This bump is due to an attempt to include all spectral lines of atoms in the opacities (Kurucz 2011). This additional opacity bump affects the stability of stellar pulsations and also needs to be considered.
Figure 5. The same as Fig. 4, but we have artificially increased the effect of rotational splitting by using the third-order formula for all modes and setting the disc-averaging factor, $b_l$, to unity so as to maximize the visibility of modes of high degree where rotational splitting is the greatest. Only predicted distributions for $Z = 0.015$ are shown.

In testing the effect of this bump (which we will call the ‘Kurucz’ bump), we did not use the actual Kurucz opacities where the bump occurs, but simulate this additional opacity bump by artificially increasing the standard OPAL opacities in the appropriate temperature range.

In Fig. 6, we show the instability parameter, $\eta$, as a function of frequency for $l \leq 6$ in five models with the same effective temperature and luminosity but different metal abundances and enhanced opacities. As one can see, an increase in $\eta$ in the low-frequency range can be obtained either by increasing the value of $Z$ or by an increased opacity in the Z bump or by increasing the standard OPAL opacities to simulate the Kurucz bump. However, low-frequency modes remain stable except when the Kurucz bump opacity is included. Instability of $l = 1$ modes at low frequencies begins when the standard OPAL opacity in the Kurucz-bump region is increased by a factor of 2. Maximum instability occurs when this opacity is increased by a factor of 3. Any further opacity increase results in saturation and no further increase in $\eta$ can occur. Note that an increase in opacity in the Z-bump or Kurucz-bump regions leads to somewhat reduced $\eta$ for modes with high frequencies. For example, the frequency range of unstable high-frequency modes is decreased from the 15–35 d$^{-1}$ range for a standard model to 16–32 d$^{-1}$ for a model where the OPAL opacity in the Kurucz-bump region is increased by a factor of 2.

Fig. 7 illustrates the opacity-driving mechanism for low- and high-frequency modes in a $\delta$ Scuti model with a simulated Kurucz opacity bump. The model parameters are the same as panel (e) in Fig. 6. In the top panel, we plot the mean Rosseland opacity, $\kappa$ (the left y-axis), and its corresponding temperature derivative, $\kappa_T$ (the right y-axis), as a function of temperature. The dashed lines correspond to the model with artificially enhanced opacities at log $T = 5.06$. The solid lines correspond to the standard model. The layer that is crucial in driving the oscillations is where the opacity derivative, $\kappa_T$, is increasing outwards in the stellar envelope. This region coincides approximately with the local opacity maximum. Hence we use the term ‘driving due to an opacity bump’. In the top panel, we see four such regions which, potentially, can contribute to driving. The region at log $T \approx 4.1$ is due to H I and He I ionization. The region at log $T \approx 4.7$ is due to He II ionization. The region at log $T \approx 5.06$ corresponds to the simulated Kurucz bump. The region at log $T \approx 5.3$ corresponds to the Z bump. We can see that the Kurucz bump has clearly the potential for driving pulsations,
The instability parameter, $\eta$, as a function of frequency for modes with $l \leq 6$ for representative models of region 3. All models have $\log T_{\text{eff}} = 3.8855$ and $\log (L/L_\odot) = 1.093$. Panel (a) is a standard model with mass $M = 1.76 M_\odot$ and $Z = 0.015$. Panel (b) is a standard model with $M = 2.25 M_\odot$ and $Z = 0.050$. Panel (c) is a standard model with $M = 1.48 M_\odot$ and $Z = 0.005$. In panel (d), the opacity at $\log T = 5.35$ (the $Z$-bump region) has been increased by a factor of 2. In panel (e), the opacity at $\log T = 5.06$ has been increased by a factor of 2 to simulate a new opacity bump in the Kurucz opacity data (see main text). All models have initial hydrogen abundance $X_0 = 0.70$.

though actual pulsational instability also depends on other factors, as described below.

The middle and bottom panels show the differential work integral, $dW/d\log T$, in models with standard opacities and with enhanced opacities in the Kurucz bump for two $l = 1$ modes with frequencies 0.86 $d^{-1}$ (the gravity mode $g_{30}$) and 23.51 $d^{-1}$ (the acoustic mode $p_{3}$). Driving occurs in the layer where this differential work integral is positive.

As can be seen, the high-frequency mode (bottom panel) is driven mainly by the He II ionization opacity bump with some contribution from the Kurucz bump. The hydrogen bump does not have any effect because the thermal time-scale in this superficial region is too short. Even for such high frequencies, the region remains in thermal equilibrium during pulsation (neutral stability). The much deeper $Z$ bump also does not contribute to driving of this acoustic mode. This is due to the small amplitude of the mode in the deep layers (see for example, Pamyatnykh 1999 for a discussion of the main features of the $\kappa$-mechanism).

For the low-frequency mode (middle panel of Fig. 7), the gravity mode is stable in a model with standard opacities. However, in the model with enhanced opacities in the Kurucz bump, the conditions for pulsational driving are fulfilled in the deeper layers. This happens to be just where the simulated Kurucz bump is located. There is also an additional contribution from the $Z$ bump. At such a low
frequency, both the He II and H I opacity bumps do not contribute to driving because these regions remain in thermal equilibrium.

In summary, the Kurucz opacity bump is crucial for the excitation of low-frequency modes in δ Sct stellar models. However, we do not suggest this as the solution to the low frequencies in δ Sct stars. Any effect which increases the opacity in this region will be important and it is possible that the calculated opacities need upward revision near log $T \approx 5.06$.

8 FURTHER UNSOLVED PROBLEMS

Even a cursory inspection of the periodograms of δ Sct stars is sufficient to show the wide variety of frequency patterns in these stars. In fact, each star is unique and can be identified by its periodogram. This is strange because one expects stars with similar effective temperatures, luminosities and rotational velocities to have similar frequency spectra, but this does not seem to be the case. Examples of this effect are shown in Fig. 8 for stars belonging to region 3. The stars shown all have well-determined spectroscopic effective temperatures and surface gravities.

While it is true that the stars may individually have considerably different parameters due to substantial errors in $T_{\text{eff}}$ and log ($L/L_{\odot}$), the differences in frequency spectra are very pronounced. This disparity cannot be proved until more accurate parameters are obtained, of course. Nevertheless, it seems that small differences in the parameters may lead to large differences in frequency patterns, implying that non-linearities in δ Sct envelopes may be very important.

The question of the number of non-pulsating stars in the δ Sct instability strip was addressed by Balona & Dziembowski (2011). They found that most δ Sct stars have effective temperatures in the range 7000 $< T_{\text{eff}} < 8500$ K and that even in this range no more than 40–50 per cent of stars pulsate as δ Sct variables. Guzik et al. (2014, 2015) confirm the presence of several apparently constant stars within the δ Sct instability strip. However, Murphy et al. (2015) concluded from a study of 54 stars that all stars within the δ Sct instability strip pulsate. We have found 1165 δ Sct stars in the above temperature range in the Kepler field observed in LC mode. There are 2839 stars (including the δ Sct stars) in the same temperature range, meaning that only about 41 per cent of stars in the instability strip are detected as δ Sct stars in the Kepler photometry.

It could be argued that most of the non-δ Sct stars are outside the instability strip due to errors in the effective temperature. Errors in luminosity alone play no role because the stars will still be within the instability strip as they can only lie between the zero-age main sequence and TAMS. There are no known high-luminosity stars in this temperature range and in any case, none are expected due to the very short lifetimes such stars will have in this evolutionary stage. We have seen that a typical error of about 200 K in $T_{\text{eff}}$ can be expected, which is considerably less than the range of 1500 K that we are discussing. The probability that a star in the middle of the instability strip lies, in actual fact, outside the strip is less than 0.01. Of course, the probability will be higher if the star is closer to the edge of the instability strip, but it means that the probability that all 1674 non-pulsating stars are outside the instability strip is the product of the individual probabilities which is essentially zero.

There can be no question that non-pulsating stars are present in the δ Sct instability strip. Of course, it can be argued that the pulsations are less than the detectable limit for Kepler (typically less than about 50 ppm). This might be the case, though one can argue against this idea on the basis of the amplitude distribution (Balona & Dziembowski 2011). Such an amplitude disparity, if it exists, still needs an explanation.

9 CONCLUSIONS

We have compared the frequency distributions of groups of δ Sct stars with similar effective temperatures and luminosities with the frequency distributions calculated from pulsation models taking into account, in an approximate but fairly realistic way, the effect of rotational splitting. We have come to the same conclusion as Balona (2014a): that it is impossible to account for frequencies with $\nu < 5 \text{d}^{-1}$ on the basis that these are high-frequency modes shifted to low frequencies by rotation. Other factors examined in detail in Balona (2014a) are also excluded. The presence of low frequencies is a general feature of all δ Sct stars, no matter where they lie in the instability strip. This is clearly seen in the observed distributions.
shown in Fig. 4. We also found other inconsistencies between the observed and predicted frequency distributions. For example, the observed distribution for the more luminous stars is much wider than expected. However, this is a minor problem compared to absence of predicted low frequencies.

We calculated a large variety of standard models with differing helium and metal abundances, but in no case could we find unstable modes at low frequencies. The instability parameter, \( \eta \), does however tend to increase as the metal abundance, \( Z \), increases. We also studied the effect of increasing the opacity at temperatures \( \log T = 5.35 \) (the Z bump) and the effect of simulating a new opacity bump at \( \log T = 5.06 \) (the Kurucz bump). The Kurucz bump does not occur in the OPAL and OP opacities, but appears in model atmospheres, as discussed by Cugier (2012, 2014). We found that increasing the opacity in these two regions does increase the value of \( \eta \). However, low-frequency modes remain stable unless the OPAL opacities at \( \log T = 5.06 \) are increased by at least a factor of 2 to simulate the Kurucz bump or if the opacity in the Z bump is increased by a factor of at least 3. At the same time, the range of high-frequency modes is decreased to some extent.

We do not claim that this opacity increase is the solution to the problem of low frequencies in \( \delta Sc \) stars, but only that there is a likely problem with current opacity data and that further investigation into the sources of opacity in this region is required (Bailey et al. 2009, 2015; Le Pennec & Turk-Chièze 2014; Le Pennec et al. 2014).

A problem of equal importance is the question of why there are so many non-pulsating stars in the \( \delta Sc \) instability strip. This could simply be that the Kepler photometry is not sufficiently precise to detect such pulsations. This does not resolve the problem because it would not explain the very large disparity in amplitudes among the \( \delta Sc \) stars. We are clearly faced with substantial problems in the physics of A–F stars.

Recently, further problems in our current understanding of stellar pulsations have come to light. It seems that there are a group of pulsating stars with high frequencies lying between the cool end of the \( \beta Cep \) instability strip and the hot end of the \( \delta Sc \) instability strip. These were recently investigated by Balona et al. (2015) using Kepler and K2 data. Standard models cannot reproduce these pulsations, though it is possible that these so-called Maia variables may be rapidly-rotating SPB stars. The unexplained presence of low frequencies in a well-studied group such as the \( \delta Sc \) stars shows that we still do not fully understand the physics and/or envelopes of hot stars. Until we have a better understanding of the low frequencies in A stars we are not likely to make much progress with the more complex problem of explaining the origin of high frequencies in Maia variables.

ACKNOWLEDGEMENTS

The authors wish to thank the Kepler team for their generosity in allowing the data to be released to the Kepler Asteroseismic Science Consortium (KASC) ahead of public release and for their outstanding efforts which have made these results possible. Funding for the Kepler mission is provided by NASA’s Science Mission Directorate.

LAB wishes to thank the National Research Foundation of South Africa for financial support. JDD and AAP acknowledge partial financial support from the Polish NCN grants 2011/01/B/ST9/05448 and 2011/01/M/ST9/05914.

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