ON THE EXISTENCE OF LEFT AND RIGHT EIGENVALUES

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Abstract. In this note, we consider arbitrary finite-dimensional real algebras containing a copy of complex numbers. It is proved that matrices with entries from an arbitrary finite-dimensional real algebra containing a square root of negative one in its left (resp. right) associate set have left (resp. right) eigenvalues. A quick consequence of our main result is the existence of left and right eigenvalues for matrices with entries from finite-dimensional alternatives algebras containing a copy of complex numbers, e.g., octonions, and more generally matrices with entries from the real Cayley-Dickson algebras.

1. Introduction

In this short note, we consider finite-dimensional real algebras, not necessarily commutative or associative, that contain a copy of complex numbers. We prove that matrices of size greater than one with entries in such algebras have left (resp. right) eigenvalues in their copies of complex numbers provided that their left (resp. right) associate sets contain a square root of negative one, thereby proving a special case of a conjecture suggested in [3]. A quick consequence of this result is that every matrix of size greater than one with entries in a real Cayley-Dickson algebra has left and right eigenvalues.

Let us begin by setting the stage. As is usual, we use the symbols \( \mathbb{R}, \mathbb{C}, \) and \( \mathbb{H} \) to, respectively, denote the field of real numbers, the complex field, and the division ring of quaternions. More generally, we use \( \mathbb{R}_n \) to denote the real Cayley-Dickson algebra of dimension \( 2^n \), where \( n \in \mathbb{N} \). A vector space \( A \) over reals together with a multiplication coming from a bilinear form on \( A \) is said to be a real algebra. The algebra \( A \) is said to be unital or to have an identity element if its
multiplication has an identity element, denoted by 1. Customarily, the identity element of the addition operation of the algebra is denoted by 0. The algebra $\mathbb{A}$ is called associative (resp. commutative) if its multiplication is associative (resp. commutative). Throughout, by an algebra we mean an arbitrary real algebra which is not necessarily associative or commutative. A nonzero element $a \in \mathbb{A}$ is said to be invertible if there exists a unique element of the algebra, denoted by $a^{-1}$, satisfying the relations $aa^{-1} = a^{-1}a = 1$. The symbol $\mathbb{A}^{-1}$ is used to denote the set of all invertible elements of the algebra $\mathbb{A}$. Note that if an algebra $\mathbb{A}$ is associative, then the uniqueness of the inverse element is a redundant hypothesis in the definition of the invertible elements of $\mathbb{A}$.

Let $R$ be a ring, not necessarily associative or commutative, and $A \in M_n(R)$, the set of all $n \times n$ matrices with entries from $R$. An element $\lambda \in R$ is said to be a left (resp. right) eigenvalue of the matrix $A$ if there is a nonzero $n \times 1$ column matrix $X \in R^n := M_{n \times 1}(R)$ such that $AX = \lambda X$ (resp. $AX = X\lambda$). An element $\lambda \in R$ is said to be an eigenvalue of the matrix $A$ if there is a nonzero $n \times 1$ column matrix $X \in R^n := M_{n \times 1}(R)$ such that $AX = \lambda X = X\lambda$.

2. Main result

Here is our main result.

**Theorem 2.1.** Let $n \in \mathbb{N}$ with $n > 1$ and $\mathbb{A}$ be a finite-dimensional real algebra containing a copy $\mathbb{C}_I := \{a + bI : a, b \in \mathbb{R}\}$ of the complex numbers, where $I \in \text{las}(\mathbb{A})$ (resp. $I \in \text{ras}(\mathbb{A})$) and $I^2 = -1$. Then every $A \in M_n(\mathbb{A})$ has left (resp. right) eigenvalues in $\mathbb{C}_I$.

**Proof.** First, view the elements of $M_n(\mathbb{A})$ as linear transformations acting on the left of the (finite-dimensional) real vector space $\mathbb{A}^n$ via the matrix multiplication. The crucial trick in the proof is the following observation. For a given $p \in \mathbb{C}_I$ with $I^2 = -1$ and $I \in \text{las}(\mathbb{A})$ (resp. $I \in \text{ras}(\mathbb{A})$), it is plain that the transformations $L_p, R_p : \mathbb{A}^n \to \mathbb{A}^n$, where
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defined by $L_p(x) := px$ and $R_p(x) = xp$, respectively, are real linear transformations on the (finite-dimensional) real vector space $A^n$. Moreover, $L_pL_q = L_{pq}$ (resp. $R_pR_q = R_{qp}$) for all $p, q \in \mathbb{C}_I$.

Let $A \in M_n(A)$ be given and $I \in \text{las}(A)$ (resp. $I \in \text{ras}(A)$) be such that $I^2 = -1$. We prove the assertion for left eigenvalues. The assertion for right eigenvalues is proved in a similar fashion except that one must make use of the linear transformations $R_p$'s $(p \in \mathbb{C}_I)$. Since $A^n$ is finite-dimensional, the real linear transformations $A - L_p$ with $p \in \mathbb{C}_I$ is invertible if and only if the kernel of $A - L_p$ is zero. Thus it suffices to show that there exist $x, y \in \mathbb{R}$ such that $A - L_x + Iy$ is not invertible, equivalently its kernel is nonzero. Let $J = I_I$ and note that $J^2 = -I_{A^n}$, where $I_{A^n}$ denotes the identity linear transformation on $A^n$. We can write

$$A - L_x + Iy = A - (xI_{A^n} + yJ).$$

Let $N = \text{dim} A^n$. Fixing a basis for the real vector space $A^n$, the assertion reduces to the following: Let $B, J \in M_N(\mathbb{R})$ with $J^2 = -I_N$ and $I_N$ being the identity matrix of size $N$. Then there exist $x, y \in \mathbb{R}$ such that $B - (xI_N + yJ)$ is not invertible, or equivalently its kernel is nonzero. To prove this, view the matrices $B, J, I_N$ as matrices with quaternion entries and note that $B - (xI_N + yJ)$ is invertible as a matrix with real entries if and only if it is invertible as a matrix with quaternion entries. Next, note that, over quaternions, the matrix $J$ is similar to the diagonal matrix $J_N := \text{diag}(i, \ldots, i) \in M_N(\mathbb{H})$ of size $N$. To see this, just note that from $J^2 = -I_N$ we see that, over complex numbers, $J$ is similar to a diagonal matrix whose diagonal entries are $\pm i$. But then again, since $-i = j^{-1}ij$, we get that $J$ is similar to the diagonal matrix $J_N = \text{diag}(i, \ldots, i) \in M_N(\mathbb{H})$. So the assertion boils down to finding $x, y \in \mathbb{R}$ such that $B - (xI_N + yJ_N) \not\in M_N(\mathbb{H})^{-1}$, where $B \in M_N(\mathbb{H})$ is arbitrarily given. But this, in view of the fact that every quaternion is similar to a complex number, is a quick consequence of the main result of [2]. This completes the proof. □

The following corollaries are quick consequences of the preceding theorem.

**Corollary 2.2.** Let $n \in \mathbb{N}$ with $n > 1$ and $A$ be a finite-dimensional real alternative algebra containing a copy of the complex numbers, namely, $\mathbb{C}_I := \{a + bI : a, b \in \mathbb{R}\}$ with $I \in A$ and $I^2 = -1$. Then every $A \in M_n(A)$ has left and right eigenvalues in $\mathbb{C}_I$. 
Proof. Let \( I \in A \) with \( I^2 = -1 \). Just note that \( I \in \text{las}(A) \cap \text{ras}(A) = A \). The assertion now follows from the preceding theorem. 

Corollary 2.3. Every matrix of size greater than 1 with entries in a Cayley-Dickson algebra has left and right eigenvalues in the complex numbers.

Proof. For \( n \in \mathbb{N} \), let \( \mathbb{R}_n \) denote the real Cayley-Dickson algebra of dimension \( 2^n \). A straightforward induction on \( n \in \mathbb{N} \) reveals that \( i \in \text{las}(\mathbb{R}_n) \cap \text{ras}(\mathbb{R}_n) \) for all \( n \in \mathbb{N} \). The assertion is now immediate from Theorem 2.1.

References

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