ENDS OF SPACES VIA LINEAR ALGEBRA

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June 17, 2022

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Abstract. We develop a theory that may be considered as a prequel to the coarse theory. We are viewing ends of spaces as extra points at infinity. In order to discuss behaviour of spaces at infinity one needs a concept (a measure) of approaching infinity. The simplest way to do so is to list subsets of X that are bounded (i.e. far from infinity) and that list should satisfy certain basic properties. Such a list $S_X$ we call a scale on a set X (see Section 3). In order to use ideas from the Stone Duality Theorem we consider sub-Boolean algebras $BA_X$ of the power set $2^X$ of X that contain $S_X$ and that leads naturally to the concept of ends of a scaled Boolean algebra $(X, S_X, BA_X)$ which can be attached to $X$ and form a new scaled Boolean algebra $(\overline{X}, S_X, BA_X)$ that is compact at infinity.

Given a scaled space $(X, S_X)$ the most natural scaled Boolean algebra is $(X, S_X, 2^X)$ which can be too far removed from the geometry of $X$. Therefore we need to figure out how to trim $2^X$ to a smaller sub-Boolean algebra $BA_X$. More generally, how to trim a sub-Boolean algebra $BA_X$ to a smaller one. That is done using ideas from linear algebra. Namely, we consider a family $\mathcal{F}$ of naturally arising $S_X$-linear operators on $BA_X$ and the smaller sub-Boolean algebra $BA_{\mathcal{F}}$ consists of eigensets of $\mathcal{F}$, an analog of eigenvectors from linear algebra. We show that all ends defined in literature so far (Freudenthal ends, ends of finitely generated groups, Specker ends, Cornulier ends, ends of coarse spaces) are special cases of such a process.

Date: June 17, 2022.

2000 Mathematics Subject Classification. Primary 06E15; Secondary 15A42, 20F69, 54D35.

Key words and phrases. Boolean algebras, coarse geometry, eigenvectors, ends of groups, Freudenthal compactification, group actions, Svarc-Milnor Lemma.
1. Introduction

Historically, as noted in [6] on p.287, ends are the oldest coarse topological notion. Here is their internal description:

**Definition 1.1.** A Freudenthal space is a $\sigma$-compact, locally compact, connected and locally connected Hausdorff space $X$.

**Definition 1.2.** A Freudenthal end of a Freudenthal space is a decreasing sequence $(U_i)_{i \geq 1}$ of components of $X \setminus K_i$, where $(K_i)_{i \geq 1}$ is an exhausting sequence, i.e. $K_i$ is compact, $K_i \subset \text{int}(K_{i+1})$ for each $i \geq 1$, and $\bigcup_{i=1}^{\infty} K_i = X$. The set of ends of $X$ is denoted by $\text{Ends}(X)$.

Ends were used by Freudenthal in 1930 in his famous compactification (see [12] for information about theorems in this section and see [8] for results related to the theory of dimension):

**Theorem 1.3.** Suppose $X$ is a $\sigma$-compact, locally compact, connected and locally connected Hausdorff space. It has a compactification $\bar{X}$ such that $\bar{X} \setminus X$ is of dimension 0 and $\bar{X}$ dominates any compactification $\hat{X}$ of $X$ whose corona is of dimension 0.

**Notation 1.4.** Given a $\sigma$-compact, locally compact, connected and locally connected Hausdorff space $(X, T)$. If $U \in T$, we define the subset $U_{\text{end}} := \{(U_i) \in \text{Ends}(X) | U_i \subset U \text{ for some } i \geq 1\}$ and $\bar{U}_{\text{end}} := U_{\text{end}} \cup U$.

The family $T \cup \{\bar{U}_{\text{end}} | U \in T\}$ is a basis for a topology $T_{\text{end}}$ on $X \cup \text{Ends}(X)$. The topological space $\bar{X} := X \cup \text{Ends}(X)$ is a compactification of $X$ called the Freudenthal compactification. The space of ends $\text{Ends}(X) = \bar{X} \setminus X$ is of dimension 0, and $\bar{X}$ dominates any compactification $\hat{X}$ of $X$ whose corona is of dimension 0. Moreover, the number of ends of $X$ is the supremum of $n_i \geq 0$ where $n_i$ is the number of all mutually disjoint unbounded components of $X \setminus K_i$, for all $i \geq 1$.

Initially, ends were useful as properties of topological groups:

**Theorem 1.5.** (Freudenthal) A path connected topological group has at most two ends.

**Theorem 1.6.** (Leo Zippin [18]) If a locally compact, metrizable, connected topological group $G$ is two-ended, then $G$ contains a closed subgroup $T$ isomorphic to the group of reals such that the coset-space $G/T$ is compact; moreover, the space $G$ is the topological product of the axis of reals by a compact connected set homeomorphic to the space $G/T$.

**Theorem 1.7.** (H. Hopf) Let $G$ be a finitely generated discrete group acting on a space $X$ by covering transformations. Suppose the orbit space $B := X/G$ is compact. Then (i) and (ii), below, hold.

(i) The space of ends of $X$ has 0, 1 or 2 (discrete) elements or is a Cantor space.
(ii) If $G$ also acts on $Y$ satisfying the hypotheses above, then $X$ and $Y$ have homeomorphic end spaces.

Conclusion (ii) suggests to regard the space of ends of $X$ as an invariant of the group $G$ itself.
Definition 1.8. Let $p : X \to B$ be a covering map with compact base $B$ and the group of covering transformations $G$. The end space of $G$ is

$$\text{Ends}(G) := \text{Ends}(X).$$

When applied to a Cayley graph of $G$, it gives the standard definition of ends of finitely generated groups (see [6], p.295). See [?] for basic results in this theory and see [9] for more general facts in coarse geometry related to groups. [?] contains interesting results for ends of finitely generated groups.

Stone [15] assigns to each Boolean algebra $B$ a totally disconnected compact Hausdorff space $X_B$ called the Stone’s space associated to $B$; and conversely, every Stone’s space $X$ (totally disconnected compact Hausdorff space) is assigned a Boolean algebra $S_X$ that consists of all clopen subsets of $X$. Stone’s representation theorems assure us that two Boolean algebras are isomorphic if and only if their corresponding Stone’s spaces are homeomorphic; and that two Stone’s spaces are homeomorphic if and only if their Boolean algebras of their clopen subsets are isomorphic. See [16] for basic results on Stone Duality.

Realizing that the space of ends of a finitely generated group is a Stone’s space, and that its Boolean algebra of all clopen subsets is isomorphic to the quotient Boolean algebra of the Boolean algebra consists of all almost right invariant subsets by the ideal of finite subsets ($A \subset G$ is called almost right invariant if the symmetric difference $A \Delta A \cdot g$ is finite for all $g \in G$); E. Specker [13] defined the space of ends of an arbitrary group as the Stone’s space of the quotient Boolean algebra of the Boolean algebra of all almost right invariant subsets by the ideal of finite subsets.

Adopting Specker’s approach, W. Dicks and M. J. Dunwoody [5] consider ends of non-finitely generated groups. In particular, they proved the following result that is a generalization of the famous theorem of Stallings [14]:

Theorem 1.9. A group $G$ has infinitely many ends if and only if one of the following conditions holds:

(i) $G$ is countably infinite and locally finite,
(ii) $G$ can be expressed as an amalgamated free product $A \ast_C B$ or an HNN extension $A \ast_C$, where $C$ is a finite subgroup of $A$ and $B$ such that $[A : C] \geq 3$ and $[B : C] \geq 2$.

Yves Cornulier [4] has also studied the space of ends of infinitely generated groups.

When a group $G$ is equipped with a left-invariant proper metric, its almost right invariant subsets are exactly the coarsely clopen subsets of $G$ equipped with the large scale structure induced by its metric. The concept of coarsely clopen subsets is defined for any large scale space $X$. Furthermore, the family of all coarsely clopen subsets $\text{CC}(X)$ of a given large scale space $X$ is a Boolean algebra. Consequently, it is natural to define the space of ends $\text{Ends}(X)$ of a coarsely connected large scale space $X$ as the Stone’s space of the to the quotient Boolean algebra of $\text{CC}(X)$ by the ideal of all bounded subsets. This is done in [11]. In this paper we generalize all above approaches to ends using ideas from Linear Algebra, namely analogs of eigenvectors of linear operators.

The authors declare that all data supporting the findings of this study are available within the article.
2. Scaled spaces

In order to discuss behaviour of spaces at infinity one needs a concept (a measure) of approaching infinity. The simplest way to do so is to list subsets of $X$ that are bounded and that list should satisfy certain basic properties. This leads us to the next concept.

**Definition 2.1.** A scaled space is a pair $(X, S_X)$ of a set and a scale $S_X$ in $X$, i.e. a family $S_X$ of subsets of $X$ that is closed under finite unions and contains the empty set. We do not assume that $X = \bigcup S_X$, so $S_X$ may consist of the empty set only. A subset $A \subset X$ is called bounded if there exists $B \in S_X$ such that $A \subset B$; otherwise, $A \subset X$ is called unbounded. Given a scaled space $(X, S_X)$ and a subset $Y \subset X$, the family $B_Y := \{K \cap Y : K \in S_X\}$ is a scale in $Y$. The pair $(Y, B_Y)$ is a scaled subspace of $(X, S_X)$.

**Definition 2.2.** $C \equiv D$ mod $S_X$ means existence of $B_1, B_2 \in S_X$ such that $C \cup B_1 = D \cup B_2$. Equivalently, there is $B \in S_X$ such that $C \smallsetminus B = D \smallsetminus B$ ($B := B_1 \cup B_2$ works).

If a scale $S_X$ is a bornology, i.e. every subset of $B \in S_X$ belongs to $S_X$, then $C \equiv D$ mod $S_X$ is equivalent to the symmetric difference $C \Delta D$ belonging to $S_X$. For arbitrary scaled spaces, it is equivalent to the symmetric difference $C \Delta D$ being bounded.

**Observation 2.3.** The reason we do not assume each scale $S_X$ is a bornology is that when discussing topologies induced by certain families containing $S_X$ would quite often lead to discrete topologies and that is not what we are interested in.

**Lemma 2.4.** If $C \equiv D$ mod $S_X$, then $X \setminus C \equiv X \setminus D$ mod $S_X$.

**Proof.** Suppose $C \setminus B = D \setminus B$ for some $B \in S_X$. Then $(X \setminus C) \cup B = (X \setminus D) \cup B$. \qed

**Definition 2.5.** A connectivity structure $C_X$ on a set $X$ is a family of subsets of $X$ (called connected) containing all singletons with the property that any subset $Y$ of $X$ must belong to $C_X$ if every two points $x, y$ of $Y$ can be joined by a chain $C_i \subset C_X$, $1 \leq i \leq n$, i.e. $x \in C_1$, $y \in C_n$, and $C_i \cap C_{i+1} \neq \emptyset$ for each $i < n$.

**Proposition 2.6.** Suppose $(X, C_X)$ is a set equipped with a connectivity structure and $Y$ is a subset of $X$. Each point $y \in Y$ belongs to a maximal connected subset of $Y$ called the $C_X$-component of $Y$. The family of all $C_X$-components of $Y$ is a partition of $Y$.

**Proof.** Let $y \in Y$ and $A_y := \{A \subset Y : y \in A \text{ and } A \in C_y\}$. Clearly, $A_y \neq \emptyset$ as $\{y\} \in A_y$. Let $C_y$ be a chain in $A_y$, then $\bigcup_{A \in C_y} A$ is an upper bound of $C_y$ and $\bigcup_{A \in C_y} A \in A_y$. By Zorn’s Lemma, $A_y$ has a maximal. \qed

**Definition 2.7.** A scaled Boolean algebra is a triple $(X, S_X, BA_X)$ consisting of a set, a scale $S_X$, and a Boolean algebra $BA_X$ of subsets of $X$ such that $S_X \subset BA_X$.

Given a scaled Boolean algebra $(X, S_X, BA_X)$ and $Y \subset X$. The scaled Boolean algebra $(Y, B_Y, BA_Y)$ is a scaled sub-Boolean algebra of $(X, S_X, BA_X)$ if:

1. $(Y, B_Y)$ is a scaled subspace of $(X, S_X)$.
2. $\{A \cap Y : A \in BA_X\} = BA_Y$. 


Proposition 2.8. Given a scaled space \((X, S_X, C_X)\) equipped with a connectivity structure. If for each bounded set \(B\) the union of bounded \(C_X\)-components of \(X \setminus B\) is bounded, then the family \(BA_X\) of all subsets of \(X\) each equivalent to a union of components of \(X \setminus B\) for some bounded \(B\) is a \(S_X\)-scaled Boolean algebra.

Proof. If \(A \equiv C\) and \(C\) is a union of some \(C_X\)-components of \(X \setminus B\), then \(X \setminus C\) is equivalent to the union of the remaining \(C_X\)-components of \(X \setminus B\).

Suppose \(A_i \equiv C_i\), \(C_i\) being a union of \(C_X\)-components of \(X \setminus B_i\), \(i = 1, 2\). Put \(B := B_1 \cup B_2\) and consider all \(C_X\)-components of \(X \setminus B\) contained in either \(C_1\) or \(C_2\). Since the family of all such \(C_X\)-components partitions \(C_1 \cup C_2\), their union \(C\) is equivalent to \(A_1 \cup A_2\). Finally, consider all \(C_X\)-components of \(X \setminus B\) contained in both \(C_1\) and \(C_2\), then their union \(D\) is equivalent to \(A_1 \cap A_2\). This shows that if \(A_1, A_2 \in BA_X\), then \(A_1 \cup A_2, A_1 \cap A_2 \in BA_X\). \(\square\)

The scaled Boolean algebra as in 2.8 is called the Boolean algebra induced by the connectivity structure \(C_X\).

3. Compactifications of scaled Boolean algebras

In this section we define an analog of a compactification of a topological space.

Definition 3.1. Given a scaled Boolean algebra \((X, S_X, BA_X)\), \(x \in X\) is a point at infinity of \(X\) if \(\{x\}\) is unbounded. A scaled Boolean algebra is compact at infinity if every family \(\{A_i\}_{i \in I}\) of elements of the Boolean algebra \(BA_X\) covering all points at infinity has a finite subfamily \(\{A_i\}_{i \in F}\) covering \(X \setminus B\) for some \(B \in S_X\).

Observation 3.2. If a scaled Boolean algebra \((X, S_X, BA_X)\) is compact at infinity and has no points at infinity, then \(X\) is bounded.

Proof. Since each point \(x\) is bounded, there is \(B(x) \in S_X\) containing \(x\). There are finitely many points \(x_i, i \leq n\), so that \(B := X \setminus \bigcup_{i=1}^n B(x_i)\) is bounded. Hence, \(X\) is bounded. \(\square\)

Observation 3.3. Each scaled Boolean algebra \((X, S_X, BA_X)\) induces a topology on \(X\) in which elements of the Boolean algebra \(BA_X\) are clopen. If a scaled Boolean algebra is compact at infinity and every bounded union of elements of \(BA_X\) belongs to \(BA_X\), then each clopen set belongs to the \(BA_X\).

Proof. Notice that \(BA_X\) is a basis for a topology on \(X\) in which every element of \(BA_X\) is clopen.

Suppose \((X, S_X, BA_X)\) is compact at infinity and \(A\) is clopen. Express \(A\) as a union of elements \(A_i, i \in I\), of \(BA_X\) and do the same for \(X \setminus A\). The resulting cover has a finite subcover \(S\) of \(X \setminus B\) for some \(B \in S_X\). Notice that \(K := \bigcup_{i \in I} (B \cap A_i) \in BA_X\), and hence \(A\) is a finite union of elements of \(BA_X\) (the union of those from \(S\) contained in \(A\) and \(K\)). Therefore \(A\) belongs to \(BA_X\). \(\square\)

Proposition 3.4. Let \((X, S_X, BA_X)\) be a compact at infinity scaled Boolean algebra, and \((Y, B_Y, BA_Y)\) is a scaled sub-Boolean algebra of \((X, S_X, BA_X)\) with \(Y \subset X\) being a closed subset (i.e. its complement being a union of elements of \(BA_X\)), then \((Y, B_Y, BA_Y)\) is compact at infinity.
Proof. Let \( \{A_i\}_{i \in I} \) be a family of elements of the Boolean algebra \( BA_Y \) covering all points at infinity of \( Y \). There exists a family \( \{C_i\}_{i \in I} \) of elements of the Boolean algebra \( BA_X \) such that \( A_i = C_i \cap Y \) for all \( i \in I \). Express \( X \setminus Y \) as a union of elements \( \{D_j\}_{j \in J} \) of \( BA_X \), then the family \( \{C_i, D_j : i \in I \text{ and } j \in J\} \) covers all points at infinity of \( X \). A finite subfamily of it covers all of \( Y \) but a bounded subset of \( Y \). □

Definition 3.5. Let \( (X, S_X, BA_X) \) be a scaled Boolean algebra. An end of it is a family \( V = \{A_i\}_{i \in I} \) of unbounded elements of \( BA_X \) that is maximal with respect to the property that \( \bigcap_{i \in F} A_i \) is unbounded for each finite subset \( F \) of \( I \). An end \( V = \{A_i\}_{i \in I} \) is external if \( \bigcap_{i \in I} A_i \) is bounded, otherwise it is internal. The family of all ends of \( (X, S_X, BA_X) \) is denoted by \( \text{Ends}(X, S_X, BA_X) \) or by \( \text{Ends}(X) \) for simplicity if the Boolean algebra is clearly understood. Similarly, the family of all internal (respectively, external) ends of \( (X, S_X, BA_X) \) is denoted by \( \text{IntEnds}(X, S_X, BA_X) \) (respectively, \( \text{ExtEnds}(X, S_X, BA_X) \)) of \( \text{Ends}(X) \) for simplicity if the Boolean algebra is clearly understood.

Observation 3.6. If \( V = \{A_i\}_{i \in I} \) is an external end, then in fact \( \bigcap_{i \in I} A_i = \emptyset \).

Proof. Pick \( B \in S_X \) containing \( \bigcap_{i \in I} A_i \). For each \( i \in I \), \( A_i \setminus B \) must belong to \( V \) as it has unbounded intersections with any finite subfamily of \( V \). Therefore, \( \bigcap_{i \in I} A_i = \emptyset \). □

Observation 3.7. If \( V = \{A_i\}_{i \in I} \) is an internal end, then every point \( x \) of \( C := \bigcap_{i \in I} A_i \) is a point at infinity of \( X \). Moreover, if \( x \in C \) belongs to \( A \in BA_X \), then \( C \subset A \).

Proof. As above, \( B \cap C = \emptyset \) for each \( B \in S_X \), hence any \( x \in C \) is unbounded. If \( x \in A \in BA_X \), then for any finite \( F \subset I \) the set \( A \cap \bigcap_{i \in F} A_i \) is unbounded as it contains \( x \). Thus, \( A = A_i \) for some \( i \in I \) and \( C \subset A \). □

In view of Observation 3.7 the intersection of all elements of any internal end is a blob that should be treated as a singular point. That leads to the following:

Definition 3.8. A scaled Boolean algebra \( (X, S_X, BA_X) \) is Hausdorff if each point at infinity \( x \) of \( X \) is the intersection of all elements of \( BA_X \) containing it.

Proposition 3.9. Suppose \( (X, S_X, BA_X) \) is a scaled Boolean algebra.

a. If \( (X, S_X, BA_X) \) is compact at infinity, then it has no external ends.
b. If \( (X, S_X, BA_X) \) has no ends, then it is compact at infinity.

Proof. a. Assume that \( (X, S_X, BA_X) \) is compact at infinity. If \( V = \{A_i\}_{i \in I} \) is an external end of it, then \( \{X \setminus A_i\}_{i \in I} \) is a cover of \( X \) by elements of the Boolean algebra \( BA_X \). Choose a finite subset \( F \subset I \) such that \( \{X \setminus A_i\}_{i \in F} \) covers \( X \setminus B \) for some \( B \in S_X \). Then \( \bigcap_{i \in F} A_i \) is bounded, a contradiction.

b. Suppose that \( \text{Ends}(X) = \emptyset \) and that \( (X, S_X, BA_X) \) is not compact at infinity. Choose a cover \( \{A_i \in BA_X : i \in I\} \) of points at infinity of \( X \) that has no finite subcover of \( X \) mod \( S_X \). Notice that, for any finite subset \( F \subset I \), if \( \bigcap_{i \in F} X \setminus A_i \) is
bounded, then \( \bigcup_{i \in F} A_i \) is a finite subcover of \( X \mod S_X \), a contradiction. Therefore, \( \{ X \setminus A_i : i \in I \} \) is contained in some end, a contradiction. \( \square \)

Definition 3.10. A compactification at infinity of a scaled Hausdorff Boolean algebra \((X, S_X, BAX)\) is a compact at infinity scaled Hausdorff Boolean algebra \((\bar{X}, S_X, \bar{BAX})\) such that the following conditions are satisfied:
1. \((X, S_X, BAX)\) is a scaled sub-Boolean algebra of \((\bar{X}, S_X, \bar{BAX})\).
2. If \( A \in \bar{BAX} \) intersects \( \bar{X} \setminus X \), then it intersects \( X \) as well.

Observation 3.11. Conditions 1 and 2 in 3.10 can be summarized as follows: the function \( A \to A \cap X \) induces isomorphism of Boolean algebras \((\bar{X}, S_X, \bar{BAX})\) and \((X, S_X, BAX)\).

Proposition 3.12. Given a compactification at infinity \((\bar{X}, S_X, \bar{BAX})\) of a scaled Hausdorff Boolean algebra \((X, S_X, BAX)\) and given \( A \in BAX \), the unique element \( A' \) of \( \bar{BAX} \) such that \( A = A' \cap X \) is equal \( cl(A) \), where \( cl \) is the closure operator in \( \bar{X} \) induced by \( \bar{BAX} \). Moreover, for all \( A_1, A_2 \in BAX \), then \( cl(A_1 \cap A_2) = cl(A_1) \cap cl(A_2) \).

Proof. Since \( A' \) is a closed subset of \( \bar{X} \) containing \( A \), so \( cl(A) \subset A' \).

If \( x \notin cl(A) \), then there is \( C \in \bar{BAX} \) containing \( x \) and not intersecting \( A \). Look at \( C \cap A' \). It does not intersect \( A \), so \( X \cap C \cap A' = \emptyset \). Hence \( C \cap A' = \emptyset \) and \( x \notin A' \). Finally, assume that \( A_1, A_2 \in BAX \), then \( A_i = X \cap cl(A_i), i = 1, 2 \). Since \( cl(A_1) \cap cl(A_2) \in \bar{BAX} \), and \( A_1 \cap A_2 = X \cap (cl(A_1) \cap cl(A_2)) \), thus \( cl(A_1 \cap A_2) = cl(A_1) \cap cl(A_2) \). \( \square \)

Proposition 3.13. Given a compactification at infinity \((\bar{X}, S_X, \bar{BAX})\) of a scaled Hausdorff Boolean algebra \((X, S_X, BAX)\) and given \( x \in \bar{X} \setminus X \), the family \( V_x \) of all \( A \in BAX \) such that \( x \in cl(A) \) is an external end of \((X, S_X, BAX)\). Moreover, if \( x \neq y \in \bar{X} \setminus X \), then \( V_x \neq V_y \).

Proof. If \( x \in cl(A_i), i \leq n \), then \( C := \bigcap_{i=1}^{n} A_i \) being bounded implies \( \bigcap_{i=1}^{n} cl(A_i) = C \), a contradiction. If \( A \in BAX \) intersects each \( A_i \) along an unbounded subset and \( x \notin cl(A) \), then there is \( A' \in BAX \) containing \( x \) in its closure with \( cl(A') \cap A = \emptyset \).

Hence, \( A \cap A' = \emptyset \), a contradiction. \( \square \)

Our next result is the main point of this section and it is motivated by 3.13

Lemma 3.14. Let \((X, S_X, BAX)\) be a scaled Hausdorff Boolean algebra.
(a) The family \( \bar{BAX} := \{ A : A \in BAX \} \) is a sub-Boolean algebra of \( 2^{\bar{X}} \), and \((\bar{X}, S_X, \bar{BAX})\) is a scaled Hausdorff Boolean algebra, where \( \bar{A} := A \cup \{ V \in ExtEnds(X) : A \in V \} \) and \( \bar{X} := X \cup ExtEnds(X) \).
(b) Let \( A \in X \) and \( V \in ExtEnds(X) \). \( A \in V \) if and only \( V \in cl(\bar{X})(A) \).
(c) If \( A \in BAX \), then \( cl_{\bar{X}}(A) = \bar{A} \).
(d) If \( \{ cl_{\bar{X}}(A_i) : i \in I \} \) is a family of elements of \( \bar{BAX} \) covering \( ExtEnds(X) \), then there is a finite subset \( F \subset I \) such that \( X \setminus \bigcup_{i \in F} A_i \) is bounded.

Proof. (a) Notice that \( A_1 \cup A_2 = \bar{A}_1 \cup \bar{A}_2 \) and \( A_1 \cap A_2 = \bar{A}_1 \cap \bar{A}_2 \) for all \( A_1, A_2 \in BAX \). Hence, \( \bar{X} \setminus A = \bar{X} \setminus A \). Moreover, \( \bar{B} = B \), for all \( B \in S_X \).
Moreover, the space of ends $\text{ExtEnds}(X)$. Assume that $A \subseteq V$ and let $C \subseteq \overline{A}$ such that $\overline{C}$ is a neighborhood of $V$, then $A, C \subseteq V$ and so $C \cap A \neq \emptyset$. Thus, $V \subseteq \overline{C}(A)$. Conversely, assume that $V \subseteq \overline{C}(A)$ and $A \notin V$. Choose $C \subseteq V$ such that $C \cap A = \emptyset$. Hence, $\overline{C}$ is a neighborhood of $V$ that misses $A$, a contradiction.

(c) Let $A \subseteq \overline{A}$. Clearly $\overline{C}(A) \subseteq \overline{A}$. If $V \subseteq \overline{A} \cap \text{ExtEnds}(X)$, then $A \subseteq V$, and hence by part (b), $V \subseteq \overline{C}(A)$.

(d) Let $\mathcal{F}$ be the collection of all finite subsets of $I$. Seeking contradiction assume that for any $F \subseteq \mathcal{F}$, $A_F = X \setminus \bigcup_{i \in F} A_i$ is unbounded. The collection $\{A_F : F \subseteq \mathcal{F}\}$ is contained in some end $V \subseteq \text{Ends}(X)$. It cannot be internal, so it is external. Hence, $\{A_F : F \subseteq \mathcal{F}\} \subseteq V \subseteq \overline{cl(A_i)}$ for some $j \in I$ which implies that $A_j \subseteq V$ and $X \setminus A_j \subseteq V$, a contradiction. 

\begin{theorem}
Given a scaled Hausdorff Boolean algebra $(X, S_X, \overline{BAX})$, the scaled Boolean algebra $(\overline{X}, S_X, \overline{BAX})$ is the unique compactification at infinity of $(X, S_X, \overline{BAX})$.
Moreover, the space of ends $\text{Ends}(X)$ is compact and totally disconnected.
\end{theorem}

\begin{proof}
Given a family $\{\overline{cl(A_i)}\}_{i \in I}$ of all elements of $\overline{BAX}$ that covers all points at infinity of $(\overline{X}, S_X, \overline{BAX})$, then it must cover $\text{Ends}(X)$. By part (d) of 3.14 there is a finite $F \subseteq I$ such that $X \setminus (\bigcup_{i \in F} A_i)$ is bounded. Since $\text{Ends}(X) \subseteq \bigcup_{i \in F} \overline{cl(A_i)}$, $X \setminus (\bigcup_{i \in F} \overline{cl(A_i)})$ is bounded. This shows that $\text{Ends}(X)$ is compact and $(\overline{X}, S_X, \overline{BAX})$ is compact at infinity. 3.14 tells us that $(\overline{X}, S_X, \overline{BAX})$ is indeed a compactification at infinity and hence $\text{Ends}(X)$ is Hausdorff. The family $\{\overline{cl(A)} \cap \text{Ends}(X) : A \subseteq \overline{BAX}\}$ is a basis for $\text{Ends}(X)$ consisting of clopen subsets. 
\end{proof}

3.1. Freudenthal compactification at infinity. In this part we present a general theory of Freudenthal ends.

\begin{definition}
A Freudenthal scaled space is a triple $(X, S_X, C_X)$, where $S_X$ is a scale covering $X$ and $C_X$ is a connectivity structure, with the property that for any bounded set $B$ of $X$ there are only finitely many unbounded components of $X \setminus B$ and the union of remaining components of $X \setminus B$ is bounded.

If $\overline{BAX}$ is the Boolean algebra induced by the connectivity structure $C_X$, the compactification at infinity $(\overline{X}, S_X, \overline{BAX})$ is called the Freudenthal compactification at infinity of $(X, S_X, \overline{BAX})$.
\end{definition}

\begin{proposition}
If a triple $(X, S_X, C_X)$ is a Freudenthal scaled space and $\overline{BAX}$ is the induced Boolean algebra, then for each end $V$ of $\overline{BAX}$ and each bounded set $B$ of $X$ there is a unique unbounded component $C$ of $X \setminus B$ belonging to $V$.
Moreover, if $X = \bigcup_{n=1}^{\infty} B_n$, where $(B_n)_{n \in \mathbb{N}}$ is an increasing sequence, and $B_n$ is a basis of bounded subsets of $X$, then ends of $\overline{BAX}$ are in one-to-one correspondence with decreasing sequences $C_n$, where $C_n$ is an unbounded component of $X \setminus B_n$.
\end{proposition}

\begin{proof}
Let $V$ be an end of $\overline{BAX}$, $B$ be a bounded subset of $X$, and $C_1, \ldots, C_n$ be the unbounded components of $X \setminus B$. Clearly, each $C_i$ is contained in some end $V_i$ and that not two distinct components of $C_i$’s are in the same end.

Seeking contradiction, assume that $C_i \notin V$ for all $1 \leq i \leq n$, and for each $1 \leq i \leq n$, choose $A_i \subseteq V$ such that $C_i \cap A_i = \emptyset$. Notice that $X \setminus \bigcup_{i=1}^{n} C_i$ is bounded, $\bigcap_{i=1}^{n} A_i \subseteq V$.
and $\bigcap_{i=1}^n A_i \subset X \setminus \bigcup_{i=1}^n C_i$, a contradiction.

Suppose $X = \bigcup_{n=1}^\infty B_n$, where $B_n \subset B_{n+1}$ for all $n \in \mathbb{N}$ and $B_n$ is a basis of bounded subsets of $X$ (every bounded subset is contained in some $B_n$). From the above, every decreasing sequence $(C_n)_{n \in \mathbb{N}}$ of unbounded components of $X \setminus B_n$ is contained in exactly one end and every end contains exactly one decreasing sequence $(C_n)_{n \in \mathbb{N}}$ of unbounded components of $X \setminus B_n$. □

**Lemma 3.18.** Let $X$ be a Freudenthal space (see [1.1]). $X$ admits an exhausting sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets such that both $K_n$ and $\text{int}(K_n)$ are connected.

**Proof.** Since $X$ is $\sigma$-compact, it can be written as a countable union of compact subsets, say $X := \bigcup_{n \in \mathbb{N}} L_n$. As $X$ is locally compact and locally connected, each $L_n$ can be covered by finitely many open connected and relatively compact subsets. Therefore $X$ is a countable union of open connected relatively compact subsets, say $X := \bigcup_{n \in \mathbb{N}} O_n$. Now, we inductively construct an increasing sequence $(U_n)_{n \in \mathbb{N} \cup \{0\}}$ of open connected relatively compact subsets satisfying $\text{cl}_X(U_n) \subset U_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, and $X := \bigcup_{n \in \mathbb{N}} U_n$. Put $U_0 := \emptyset$, and assume that $U_1, \ldots, U_n$ are open connected relatively compact subsets such that $\text{cl}_X(U_i) \subset U_{i+1}$ for all $1 \leq i < n$. Since $\text{cl}_X(U_n)$ is compact, there exist $O_{n_1}, \ldots, O_{n_m}$ open connected relatively compact subsets such that $\text{cl}_X(U_n) \subset \bigcup_{i=1}^m O_{n_i}$ and $\text{cl}_X(U_n) \cap O_{n_i} \neq \emptyset$ for all $1 \leq i \leq m$. Put $U_{n+1} := \bigcup_{i=1}^m O_{n_i}$; since $X$ is connected, $\text{cl}_X(U_n)$ is properly contained in $U_{n+1}$. Moreover, $U_{n+1}$ is open connected relatively compact subset. Finally, since $\text{cl}_X(\bigcup_{n \in \mathbb{N}} U_n) = \bigcup_{n \in \mathbb{N}} \text{cl}_X(U_n) = \bigcup_{n \in \mathbb{N}} U_n$, and $X$ is connected, so $X = \bigcup_{n \in \mathbb{N}} U_n$.

Now, put $K_n := \text{cl}_X(U_n)$ for all $n \in \mathbb{N}$. Notice that for any open subset $U \subset X$, one has $U \subset \text{int}(\text{cl}_X(U))$. Notice that:

$U_n \subset \text{int}(\text{cl}_X(U_n)) = \text{int}(K_n) \subset K_n = \text{cl}_X(U_n) \subset U_{n+1} \subset \text{int}(\text{cl}_X(U_{n+1})) = \text{int}(K_{n+1})$,

and therefore, $\text{int}(K_n)$ is connected for all $n \in \mathbb{N}$, and $(K_n)_{n \in \mathbb{N}}$ is an exhausting sequence consists of compact connected subsets. □

**Theorem 3.19.** Let $X$ be a Freudenthal space. There exists a scale $S_X$ and a connectivity structure $C_X$ such that the ends of $(X, S_X, C_X)$ are in one-to-one correspondence with the Freudenthal ends of $X$.

**Proof.** Let $S_X$ be the family of all relatively compact subsets of $X$, and $K_0 := \emptyset$. By [1.1], we can find an exhausting sequence $(K_n)_{n \in \mathbb{N} \cup \{0\}}$ of compact subsets such that both $K_n$ and $\text{int}(K_n)$ are connected. Without loss of generality, we may assume that every component of $X \setminus K_1$ is unbounded component. Put $U_n := \text{int}(K_{n+2}) \setminus K_n$ and let $C_X$ be the connectivity structure generated by $U := \{U \subset X : U$ is a component of $U_n$, for some $n \in \mathbb{N}\}$, i.e. $C \in C_X$ if and only if $C$ is a singleton or every two points $x, y$ of $C$ can be joined by a chain $C_i \in U$, $1 \leq i \leq n$, i.e. $x \in C_1, y \in C_n$, and $C_i \cap C_{i+1} \neq \emptyset$ for each $i < n$. Since $X$ is locally connected, every $U \in \mathcal{U}$ is open connected and relatively compact subset. The ends of $(X, S_X, BA_X)$, where $BA_X$ is the Boolean algebra induced by the
connectivity structure $C_X$, are in one-to-one correspondence with the Freudenthal end of $X$. \qed

4. Scaled linear spaces

In this section we introduce analogs of eigenvectors in the context of scaled Boolean algebras.

**Definition 4.1.** Given a scaled Boolean algebra $(X, S_X, BA_X)$, a function $f : BA_X \to BA_X$ is a $S_X$-linear operator if:
1. $f(C \cup D) \equiv f(C) \cup f(D) \mod S_X$ for any $C, D \in BA_X$.
2. $f(B)$ is bounded if $B$ is bounded.
3. $f(X) \equiv X \mod S_X$.

**Definition 4.2.** A scaled linear space is a quadruple $(X, S_X, BA_X, \mathcal{F})$, where $(X, S_X, BA_X)$ is a scaled Boolean algebra with no internal ends and $\mathcal{F}$ is a non-empty family of $S_X$-linear operators on $BA_X$. When $BA_X := 2^X$ is the powerset Boolean algebra, we write $(X, S_X, \mathcal{F})$ for simplicity.

It is well-known that, for a symmetric 2x2 matrix $M$, if $v$ is an eigenvector of $M$, then so is any $w \neq \vec{0}$ perpendicular to $v$. The following is an analog of that phenomenon.

**Definition 4.3.** Given a $S_X$-linear operator $f$, $f$ is symmetric if, for any $A \in BA_X$, $f(A) \equiv A \mod S_X$ implies $f(X \setminus A) \equiv X \setminus A \mod S_X$.

Given a family $\mathcal{F}$ of $S_X$-linear operators, $\mathcal{F}$ is symmetric if for any $A \in BA_X$, $f(A) \equiv A \mod S_X$ for all $f \in \mathcal{F}$ implies $f(X \setminus A) \equiv X \setminus A \mod S_X$ for all $f \in \mathcal{F}$.

Given a family $\mathcal{F}$ of $S_X$-linear operators, an element $A$ of $BA_X$ is an eigenset of $\mathcal{F}$ if $f(A) \equiv A \mod S_X$ and $f(X \setminus A) \equiv X \setminus A \mod S_X$, for all $f \in \mathcal{F}$.

**Proposition 4.4.** If $C$ and $D$ are eigensets of $\mathcal{F}$, then so are $C \cup D, D \setminus C$ and $C \setminus D$.

**Proof.** $f(C \cup D) \equiv f(C) \cup f(D) \equiv C \cup D \mod S_X$ for any $f \in \mathcal{F}$.

According to our assumptions both $X \setminus C$ and $X \setminus D$ are eigensets of $\mathcal{F}$. Hence their union and its complement $C \setminus D$ are eigensets of $\mathcal{F}$. Finally, $D \setminus C = D \cap (X \setminus C)$. \qed

**Corollary 4.5.** Given a scaled linear space $(X, S_X, BA_X, \mathcal{F})$, the family of eigensets of $\mathcal{F}$ induces a scaled Boolean algebra $(X, S_X, BA_{\mathcal{F}})$.

**Definition 4.6.** Given a scaled linear space $(X, S_X, BA_X, \mathcal{F})$, an eigenset at infinity of $\mathcal{F}$ is an end of $(X, S_X, BA_{\mathcal{F}})$.

**Definition 4.7.** Given a scaled linear space $(X, S_X, BA_X, \mathcal{F})$, and $Y \subset X$. A scaled linear space $(Y, B_Y, BA_Y, \mathcal{F}_Y)$ is a scaled linear subspace of $(X, S_X, \mathcal{F})$ if:
1. $(Y, B_Y, BA_Y, \mathcal{F}_Y)$ is a scaled sub-Boolean algebra of $(X, S_X, BA_X)$.
2. $(Y, B_Y, BA_{\mathcal{F}_Y})$ is a scaled sub-Boolean algebra of $(X, S_X, BA_{\mathcal{F}})$.
3. Every element $f \in \mathcal{F}_Y$ admits an extensions $f_X \in \mathcal{F}$ such that $f(A) = f_X(A) \cap Y$ for each $A \subset Y$.

**Definition 4.8.** A scaled linear space $(X, S_X, BA_X, \mathcal{F})$ is compact at infinity if it has no eigensets at infinity. Equivalently, the induced scaled Boolean algebra $(X, S_X, BA_{\mathcal{F}})$ is compact at infinity.
**Observation 4.9.** A scaled Boolean algebra $(X,S_X,BA_X)$ with no internal ends is compact at infinity if and only if the scaled Boolean algebra $(X,S_X,BA_F)$ is compact at infinity for any family $F$ of $S_X$-linear operators.

**Proof.** Assume that $(X,S_X,BA_X)$ is compact at infinity and $F$ is any family of $S_X$-linear operators. Since $BA_F \subset BA_X$, $(X,S_X,BA_F)$ must be compact at infinity. Conversely, take $F := \{id_X\}$. □

**Example 4.10.** Suppose a group $G$ acts on a set $X$. Define $S_X$ as the family of all finite subsets of $X$, $BA_X$ as all subsets of $X$. We define $F$ as the family $f,g \in G$, where $f_g(A) := A \cdot g$.

$F$ is a family of $S_X$-linear operators that are symmetric and its eigensets consist of all almost invariant subsets of $X$.

**Proof.** Recalling $A$ is almost invariant if the symmetric difference $A\Delta(A \cdot g)$ is finite for each $g \in G$. Since $(A^c\Delta(A^c \cdot g^{-1})) \cdot g = (A \cdot g \setminus A) \cup (A \cdot g^{-1} \setminus A) \cdot g$ for all $A \subset X$ and all $g \in G$, $A$ is almost invariant if and only if $A^c$ is almost invariant. Consequently, any almost invariant subset $A$ of $X$ is an eigenset of $f_g$ for all $g \in G$.

Suppose $A$ is an eigenset of $F$ but $A^c\Delta(A^c \cdot g)$ is not finite for some $g \in G$. Without loss of generality, assume $D := A^c \cdot g \setminus A^c$ is not finite. Notice $D \subset A$ and $D \cdot g^{-1} \subset A^c$, so no element of $D$ can belong to $A \Delta(A \cdot g)$ resulting in $D \subset A \Delta(A \cdot g)$, a contradiction. Thus, each $f_g$ is symmetric. □

**Example 4.11.** Consider a scaled space $(X,S_X,C_X)$ equipped with a connectivity structure such that for each bounded set $B$ the union of bounded $C_X$-components of $X \setminus B$ is bounded. Let $BA_X$ be the Boolean algebra induced by $C_X$. Given $B \in S_X$ and given $A \subset X$, put:

$U_B^1 := \{C \subset X \setminus B : C \text{ is a component of } X \setminus B \text{ and } A \cap C \text{ is unbounded} \}$;

and define:

$$
\begin{align*}
f_B(A) = \begin{cases} A & \text{if } A \in BA_X \\
st(A,U_B^1) & \text{if } A \notin BA_X.
\end{cases}
\end{align*}
$$

$F := \{f_B\}_{B \in S_X}$ is a family of $S_X$-linear operators and its eigensets are exactly elements of $BA_X$.

**Proof.** Suppose $B, B' \in S_X$.

**Claim 1:** If $A$ is equivalent mod $S_X$ to the $C$ union of some unbounded components of $X \setminus B$ and $B \subset B'$, then $A$ is equivalent mod $S_X$ to a union of some unbounded components of $X \setminus B'$.

**Proof of Claim 1:** Each unbounded component of $X \setminus B'$ is contained in a unique unbounded component of $X \setminus B$. Let $C'$ be the union of all unbounded components of $X \setminus B'$ contained in $C$. Notice $A \equiv C'$ mod $S_X$.

**Claim 2:** $f_B(A_1 \cup A_2) \equiv f_B(A_1) \cup f_B(A_2)$ mod $S_X$.

**Proof of Claim 2:** The only interesting case is that of $f_B(A_1) = A_1$ and $A_2$ not being equivalent mod $S_X$ to a union of unbounded components of any $X \setminus B'$. Pick $B' \in S_X$ containing $B$ such that $A_1$ is equivalent to a union of some unbounded components of $X \setminus B'$. Notice that $f_B(A_1) \cup f_B(A_2)$ is equivalent to the union of unbounded components of $X \setminus B'$ that intersect $A_1 \cup A_2$ non-trivially. The same can be said about $f_B(A_1 \cup A_2)$.

Notice $f_B(D) = \emptyset$ if $D$ is bounded. Hence, each $f_B$ is $S_X$-linear. By the way the family $F$ is defined, $BA_X = BA_F$. □
Example 4.12. Suppose $X$ is a large scale space. Define $S_X$ as the family of all bounded subsets of $X$. Given a uniformly bounded cover $\mathcal{U}$ of $X$ and given $A \subset X$ define $f_{\mathcal{U}}(A)$ as the star of $A$ with respect to $\mathcal{U}$.

$\mathcal{F}$ is a symmetric family of $S_X$-linear operators, its eigensets consist of coarse clopen subsets of $X$ and its eigensets at infinity consist of coarse ends of $X$.

Proof. Recall that a large scale space is a set $X$ equipped with a family of covers (called uniformly bounded covers) $\mathcal{U}$ that is closed under refinement and under taking stars of uniformly bounded covers.

Coarse open subsets $A$ of $X$ (see [11]) are those for which $st(A, U) \cap st(X \setminus A, U)$ is bounded for each uniformly bounded cover $U$. That is the same as $f_{\mathcal{U}}(A) \equiv A$ for each $U$. \hfill \Box

Example 4.13. Given a locally compact topological group $G$, we define $\mathcal{F}$ as the family $f_K$, $K \subset G$ is compact, as follows: $f_K(A) = A \cdot K$.

$B_G$ consists of all pre-compact open subsets of $G$.

$\mathcal{F}$ is symmetric and its eigensets at infinity consist of ends of $X$.

Proof. Similar to Example 4.10. \hfill \Box

Definition 4.14. Given a scaled linear space $(X, S_X, \mathcal{F})$ and its ends $Ends(X)$, define $(\bar{X}, S_X, \mathcal{F})$ as follows:

1. $\bar{X} := X \cup Ends(X)$,
2. Given $f \in \mathcal{F}$ define an extension $\bar{f} : 2^\bar{X} \rightarrow 2^\bar{X}$ as follows:

$$
\bar{f}(A) = \begin{cases} 
  f(A) \cup \{V \in Ends(X) : A \cap D \text{ is unbounded, for all } D \in V\} & \text{if } A \subset X \\
  \{V \in Ends(X) : \forall D \in V, 3V_D \in A \text{ such that } D \in V_D\} & \text{if } A \subset Ends(X) \\
  \bar{f}(A \cap X) \cup \bar{f}(A \cap Ends(X)) & \text{if } A \subset \bar{X},
\end{cases}
$$

define $\mathcal{F} := \{\bar{f} : f \in \mathcal{F}\}$.

Proposition 4.15. $(\bar{X}, S_X, \mathcal{F})$ is a scaled linear space that is compact at infinity. The eigensets of $\mathcal{F}$ are unions of two eigensets of the following forms:

1. An eigenset $A$ of $(X, S_X, \mathcal{F})$ union all ends $V$ of $(X, S_X, \mathcal{F})$ containing $A$.
2. All ends $V$ of $(X, S_X, \mathcal{F})$ containing $C$ for some eigenset $C$ of $(X, S_X, \mathcal{F})$.

Proof. Given an eigenset $A$ of $(\bar{X}, S_X, \mathcal{F})$ it is clear that $A \cap X$ is an eigenset of $(X, S_X, \mathcal{F})$.

Given an eigenset at infinity $\{A_i\}_{i \in I}$ of $\mathcal{F}$, the family $\{A_i \cap X\}_{i \in I}$ has the property of every finite intersection being unbounded. Hence there is an eigenset at infinity $V$ of $(X, S_X, \mathcal{F})$ containing all of them and $V$ is in the intersection of $\{A_i\}_{i \in I}$.

Given an eigenset at infinity $\{A_i\}_{i \in I}$ of $\mathcal{F}$, assume the family $\{A_i \cap \bar{X}\}_{i \in I}$ does not have the property of every finite intersection being unbounded. Therefore assume $\{A_i \setminus \bar{X}\}_{i \in I}$ has the property of every finite intersection being non-empty.

Suppose $\bigcap_{i \in I} (A_i \setminus \bar{X}) = \emptyset$. For every $V \in \bar{X} \setminus X$ choose $i(V) \in I$ such that $V \not\in A_{i(V)}$, i.e. there is $C_V \in V$ with no end in $A_{i(V)}$ containing $C_V$.

There is no end containing all $\bar{X} \setminus C_V$, so $X \setminus B$ is contained in a finite union of $C_V$’s for some bounded $B$. Therefore each end $W$ of $(X, S_X, \mathcal{F})$ contains at least one of those $C_V$’s and there is no end in the intersection of corresponding $A_{i(V)}$’s, a contradiction. \hfill \Box
Definition 4.16. Let \((X, S_X, \mathcal{F})\) be a scaled linear subspace of a compact at infinity scaled linear space \((\tilde{X}, S_X, \tilde{\mathcal{F}})\). \((\tilde{X}, S_X, \tilde{\mathcal{F}})\) is a **compactification at infinity** of \((X, S_X, \mathcal{F})\) if the following conditions are satisfied:
1. \(\tilde{\mathcal{F}}\) consists exactly of extensions \(\tilde{f}\) of elements \(f\) of \(\mathcal{F}\).
2. The scaled Boolean algebra \((\tilde{X}, S_X, BA_{\tilde{\mathcal{F}}})\) is a compactification at infinity of \((X, S_X, BA_{\mathcal{F}})\) and \(BA_{\mathcal{F}}\) is a sub-Boolean algebra of \(BA_{\tilde{\mathcal{F}}}\).

Theorem 4.17. For each scaled linear space \((X, S_X, \mathcal{F}), (\tilde{X}, S_X, \tilde{\mathcal{F}})\) is its maximal compactification at infinity.

**Proof.** Given a compactification at infinity \((\tilde{X}, S_X, \tilde{\mathcal{F}})\) of \((X, S_X, \mathcal{F})\) define an extension \(f : \tilde{X} \to \tilde{X}\) of \(id_X\) as follows: for each end \(V\) of \((X, S_X, \mathcal{F})\) let \(f(V)\) be the intersection of all \(\tilde{A}, A \in V\). \(\square\)

4.1. **Topological scaled linear spaces.**

Definition 4.18. A topological scaled linear space \((X, S_X, \mathcal{F})\) is a scaled linear space \((X, S_X, \mathcal{F})\) where \(X\) has a topology and \(\mathcal{F}\) contains the corresponding closure operator \(cl\).

Observation 4.19. In Definition 4.18 we may talk about multiple topologies as long as the corresponding closure operator is either included in \(\mathcal{F}\) or adding it does not change the Boolean algebra \(BA_{\mathcal{F}}\). For example, it is so for the discrete topology. Also, if there is a symmetric \(f \in \mathcal{F}\) such that \(cl(A) \subseteq f(A)\) and \(cl(A) \subseteq BA_X\) for each \(A \in BA_{\mathcal{F}}\), then clearly \(f(A) \equiv A\) implies \(cl(A) \equiv A\), so adding \(cl\) to \(\mathcal{F}\) does not change the Boolean algebra \(BA_{\mathcal{F}}\). The reason it is important is that it implies the topology on the space of ends is independent on those topologies on \(X\).

Example 4.20. The structures in Section 4 easily can include the closure operator. More generally, any time there is \(f \in \mathcal{F}\) such that \(f(A)\) is a neighborhood of \(A\) containing star of the star of \(A\) with respect to an open cover of \(X\), adding the closure operator does not affect the family of eigensets. Such is the case of large scale spaces that have a uniformly bounded cover consisting of open subsets (see 4.12). It is so for metric spaces.

Definition 4.21. A topological scaled linear space \((X, S_X, \mathcal{F})\) is **compact at infinity** if for every cover of \(X\) mod \(S_X\) by unbounded open sets there is a finite subcover of \(X\) mod \(S_X\).

Observation 4.22. Notice in the Definition 4.21, in contrast to the case of scaled Boolean algebras (see 3.1), we do not mention points at infinity as our definition of scaled linear spaces assume they do not exist.

Proposition 4.23. Given a topological scaled linear space \((X, S_X, \mathcal{F})\), the space \((\tilde{X}, S_X, \tilde{\mathcal{F}})\) is also a topological scaled linear space in the topology induced by \(cl\) and it is compact at infinity.

**Proof.** Define a subset \(U\) of \(\tilde{X}\) to be open if \(U \cap X\) is open and, for each \(V \subseteq U \setminus X\), there is \(A \subseteq V, A \subseteq U\), such that \(W \subseteq U\) for every \(W\) containing \(A\). We need to show that \(U\) is open if and only if \(cl(\tilde{X} \setminus U) = \tilde{X} \setminus U\).

By the definition of \(cl\), \(cl(\tilde{X} \setminus U)\) is the union of \(X \setminus U\), of all ends \(V\) such that every \(A \subseteq V\) intersects \(X \setminus U\) along an unbounded subset, and of all ends \(W\) such that for every \(A \subseteq W\) there is an end in \(\tilde{X} \setminus U\) containing \(A\).
If $U$ is open and $V \in U \cap \cl(X \setminus U) \setminus X$, then there is $A \in V$, $A \subset U$, such that $W \in U$ for every end $W$ containing $A$, a contradiction.

Assume $\cl(X \setminus U) = X \setminus U$. If $U$ is not open, there is $V \in U \setminus X$, such that there is no $A \in V$, $A \subset U$, such that $W \in U$ for every end $W$ containing $A$.

Therefore each $A \in V$ either intersects $X \setminus U$ along an unbounded subset (not possible), so there is $A \subset U$ in which case there is $W$ containing $A$ but not in $U$. Hence $W \in \cl(X \setminus U)$ and $V \in \cl(X \setminus U)$ contradicting $V \in U$.

Suppose $\{U_i\}_{i \in I}$ is a family of open subsets of $X$ covering $X \setminus X$ such that no finite subfamily covers $X \setminus X$. For each end $V$ choose $i(V) \in I$ and $A_{i(V)} \in V$ contained in $U_{i(V)}$.

There is no end containing all $X \setminus A_{i(V)}$, so $X \setminus X$ is contained in a finite union of $A_{i(V)}$’s. Therefore each end $W$ of $(X, S_X, F)$ contains at least one of those $A_{i(V)}$’s and $X \setminus X$ is contained in the corresponding union of $U_{i(V)}$’s, a contradiction. \hfill \square

### 4.2. Coarse scaled linear spaces.

**Definition 4.24.** A **coarse scaled linear space** $(X, S_X, F)$ is a scaled linear space $(X, S_X, F)$ where $X$ has a coarse structure and $F$ contains the corresponding star operators $st$ with respect to uniformly bounded covers of $X$.

**Observation 4.25.** In Definition 4.24 we may talk about multiple coarse structures as long as the corresponding star operators are either included in $F$ or adding them does not change the Boolean algebra $B_F$. The reason it is important is that it implies the topology on the space of ends is independent on those coarse structures on $X$.

**Example 4.26.** The structures in Section 6 easily can include the star operators. More generally, any time, given a uniformly bounded cover $U$ of $X$, there is $f \in F$ such that $f(A)$ contains star of $A$ with respect to $U$, adding all the star operators does not affect the family of eigensets.

The following can be easily generalized to locally compact spaces.

**Proposition 4.27.** Suppose $X$ is a proper metric space. If $(X, S_X, F)$ is a topological scaled linear space and a coarse scaled linear space, then $X \cup \text{Ends}(X)$ is a compactification of $X$ that is dominated by the Higson compactification of $X$.

**Proof.** It is sufficient to show that, for any continuous function $f : X \cup \text{Ends}(X) \to [0, 1]$, its restriction $f|X$ to $X$ is a Higson function (i.e. both continuous and slowly oscillating).

Suppose $f$ is not slowly oscillating. That means there are two sequences $\{x_n\}_{n \geq 1}$ and $\{y_n\}_{n \geq 1}$ in $X$ both diverging to infinity and $|f(x_n) - f(y_n)| \to 0$ but there is $M > 0$ such that $d(x_n, y_n) < M$ for all $n \geq 1$. Reduce it to the case of $\lim f(x_n) = a$, $\lim f(y_n) = a + 4b$, $b > \epsilon/4$ and $f(x_n) \in (a - b, a + b)$, $f(y_n) \in (a + 2b, a + 5b)$ for all $n \geq 1$. We may find an end $V_1 \in \cl(\{x_n\}_{n \geq 1})$, such that $f(V_1) \subset (a - b, a + b)$. Hence there is an eigenset $A_1 \in V_1$ of $F$ such that $f(A_1) \subset [a - b, a + b]$. That eigenset contains infinitely many elements of $\{x_n\}_{n \geq 1}$. Look at the corresponding subsequence of $\{y_n\}_{n \geq 1}$ and find an end $V_2$ in its closure such that $f(V_2) \in [a + 2b, a + 5b]$, there is an eigenset $A_2 \in V_2$ of $F$ such that $f(A_2) \subset [a + 2b, a + 5b]$. That eigenset contains infinitely many elements of $\{y_n\}_{n \geq 1}$. Since $B(A_2, 2M) \setminus A_2$ is bounded, infinitely many elements of $\{y_n\}_{n \geq 1}$ and infinitely many elements of $\{x_n\}_{n \geq 1}$ are contained in $A_2$, a contradiction. \hfill \square
5. Actions of scaled groups

The following is a natural generalization of locally compact topological groups.

**Definition 5.1.** A scaled group is a pair \((G, S_G)\) consisting of a group \(G\) and a scale \(S_G\) on \(G\) covering it such that \(B_1 \cdot B_2^{-1} \in S_G\) whenever \(B_1, B_2 \in S_G\).

Notice that the family of compact (finite, countable) subsets in any topological (discrete) group forms a scale as in Definition 5.1.

**Proposition 5.2.** If \((G, S_G)\) is a scaled group, then it induces four scaled linear spaces:

1. \((G, S_G, \cdot)\), where \(\cdot\) is the family of functions \(rm_B : 2^G \to 2^G, B \in S_G\), defined by \(rm_B(A) = A \cdot B\).
2. \((G, S_G, S_G^-)\), where \(S_G^-\) is the family of functions \(lm_B : 2^G \to 2^G, B \in S_G\), defined by \(lm_B(A) = B \cdot A\).
3. \((G, S_G, G)\), where \(\cdot\) is the family of functions \(rm_g : 2^G \to 2^G, g \in G\), defined by \(rm_g(A) = A \cdot g\).
4. \((G, S_G, G^-)\), where \(\cdot\) is the family of functions \(lm_g : 2^G \to 2^G, g \in G\), defined by \(lm_g(A) = g \cdot A\).

Scaled linear spaces \((G, S_G, \cdot)\) and \((G, S_G, S_G^-)\) (\((G, S_G, G)\) and \((G, S_G, G^-)\)) have homeomorphic end spaces.

**Proof.** Notice that for all \(A, B, C \in 2^G\), then \((A \cup C) \cdot B = A \cdot B \cup C \cdot B\) and \(B \cdot (A \cup C) = B \cdot A \cup B \cdot C\).

\(A\) is an eigenset of \(S_G^-\) if and only if, for each \(B \in S_G\) there is \(K \in S_G\) such that \(A \cdot B \cup K = A \cup K\) which is equivalent to \(B^{-1} \cdot A^{-1} \cup K^{-1} = A^{-1} \cup K^{-1}\). That means \(A \to A^{-1}\) establishes homeomorphism of end spaces. 

**Definition 5.3.** An action of a scaled group \((G, S_G)\) on a scaled space \((X, S_X)\) is a group action \((g, x) \to g \cdot x\) such that \(B \cdot K\) is bounded in \(X\) whenever \(B \in S_G\) and \(K \in S_X\). It is **cobounded** if there is \(K_0 \in S_X\) such that \(X = G \cdot K_0\). The action is **proper** if for every \(K \in S_X\) the set \(\{g \in G | (g \cdot K) \cap K \neq \emptyset\}\) (called the stabilizer of \(K\)) is bounded in \(G\).

**Observation 5.4.** Notice that any action of a scaled group \((G, S_G)\) on a scaled space \((X, S_X)\) induces two scaled linear spaces:

1. One consisting of \(m_g : 2^X \to 2^X (g \in G), m_g(A) := g \cdot A\) is denoted by \((X, S_X, G)\).
2. The other consisting of \(m_B : 2^X \to 2^X (B \in S_G, B \neq \emptyset), m_B(A) := B \cdot A\) is denoted by \((X, S_X, G^-)\).

If \(S_G\) consists of all finite subsets of \(G\), then the two linear scaled spaces have the same eigensets, hence the same end spaces.

**Proof.** \(A\) being an eigenset of \((X, S_X, G^-)\) means that for each \(g \in G\) the set \((g \cdot A) \Delta A\) is bounded in \(X\). Therefore, if \(B \subset G\) is finite and non-empty, then \((B \cdot A) \Delta A\) is a finite union of bounded subsets of \(X\), hence it is bounded.

**Proposition 5.5.** Let \(H\) be a subgroup of a scaled group \((G, S_G)\) and \(S_H := S_G \cap H\).

1. \((H, S_H)\) is a scaled group.
2. The natural actions \((h, x) \to h \cdot x\) and \((h, x) \to x \cdot h^{-1}\) of \((H, S_H)\) on \((G, S_G)\) are proper. Moreover, if \(H\) is of bounded index, then the actions are cobounded.
Theorem 5.9. sends uniformly bounded covers of large scale groups structure. In [11] such groups are called large scale groups.

Definition 5.6. A large scale space is a set \( X \) equipped with a family \( \text{LSS} \) of covers (called uniformly bounded covers) satisfying the following two conditions:
1. \( \text{st}(\mathcal{U}, \mathcal{V}) \in \text{LSS} \) if \( \mathcal{U}, \mathcal{V} \in \text{LSS} \).
2. If \( \mathcal{U} \in \text{LSS} \) and every element of \( \mathcal{V} \) is contained in some element of \( \mathcal{U} \), then \( \mathcal{V} \in \text{LSS} \).

Sets which are contained in an element of \( \mathcal{U} \in \text{LSS} \) are called bounded. Thus, each large scale structure on \( X \) induces a natural scale on \( X \).

The following is in some sense dual to the famous Švarc-Milnor Lemma (see [2] and [11] for other generalizations of the classical version of that lemma). Proposition 5.7 establishes existence of certain large scale structures and our version of the Švarc-Milnor Lemmas (see 5.9) establishes uniqueness of such a structure.

Proposition 5.7. If a scaled group \((G, S_G)\) acts properly and coboundedly on a scaled space \((X, S_X)\), then it induces a large scale structure on \( X \) that is coarsely equivalent to that on \( G \).

Proof. Consider refinements of covers of \( X \) of the form \( C_B := \{g \cdot B\}_{g \in G} \) for some \( B \in S_X \) (hence \( G \cdot B = X \)). To see that \( \text{st}(C_B, C_B) \) is such a refinement, look at all \( g \in G \) such that for some fixed \( h \in G \), \((g \cdot B) \cap (h \cdot B) \neq \emptyset \). What that means is that \( h^{-1} \cdot g \in B' \), where \( B' \) is the stabilizer of \( B \). Hence, \( g \in h \cdot B' \) and the the cover \( \text{st}(C_B, C_B) \) refines \( C_{B' \cdot B} \). Indeed, given \( h \in G \), the union of all \( g \cdot B \) intersecting \( h \cdot B \) is contained in \( h \cdot B' \cdot B \).

As \( G \) acts on itself, the above recipe says that the large scale structure on \( G \) consists of refinements of covers of \( G \) of the form \( C_B := \{g \cdot B\}_{g \in G} \) for some \( B \in S_G \). Pick \( x_0 \in B_0 \), where \( G \cdot B_0 = X \), and consider \( f : G \to X \), \( f(g) := g \cdot x_0 \). \( f \) sends uniformly bounded covers of \( G \) to uniformly bounded covers of \( G \cdot x_0 \) and \( f^{-1} \) does the same. Indeed, \( f(g \cdot B) \), \( B \in S_G \), is equal to \( g \cdot B \cdot x_0 \). Conversely, if \( f(g) \in h \cdot B \), \( B \in S_X \) containing \( x_0 \), then \( g \cdot x_0 \in h \cdot B \) and \( h^{-1} \cdot g \) belongs to the stabilizer \( B' \) of \( B \). Thus, \( f^{-1}(C_B) \) refines \( C_{B'} \).

Moreover, \( \text{st}(G \cdot x_0, C_{B_0}) = X \), so the inclusion \( G \cdot x_0 \to X \) is a coarse equivalence.

Remark 5.8. In view of [12], every scaled group \((G, S_G)\) has a natural large scale structure. In [11] such groups are called large scale groups.

Theorem 5.9. Given a scaled group \((G, S_G)\) consider the large scale structure \( \text{LSS} \) on \( G \) induced by refinements of the covers \( C_B := \{g \cdot B\}_{g \in G}, B \in S_G \). Suppose \((G, S_G)\) acts by uniform coarse equivalences on a large scale space \( X \). If the action
Proof. Choose $B_0 \in S_X$, $S_X$ being the family of all bounded subsets of $X$, such that $G \cdot B_0 = X$ $(G, S_G)$ acts by uniform coarse equivalences on a large scale space $X$ means that, for each uniformly bounded cover $U$ of $X$, there is a uniformly bounded cover $V$ of $X$ so that each $g \cdot U$, $g \in G$, refines $V$. Each $U \in \mathcal{U}$ contains $x_U \in U$ belonging to $g_U \cdot B_0$ for some $g_U \in G$. Therefore $g_U^{-1} \cdot U$ is contained in an element $V(U)$ of $V$ and intersecting $B_0$. Hence, $g_U^{-1} \cdot U \subset W := st(B_0, V)$ and $U \subset g_U \cdot W \in S_X$. That means the large scale structure on $X$ is identical to the one described in 5.7. Apply 5.7.

The following result mixes right and left multiplication actions of a group $G$ on itself. It says that the two ways of defining end spaces on a scaled groups are identical.

**Proposition 5.10.** Given a scaled group $(G, S_G)$ consider the large scale structure $\mathcal{LSS}$ on $G$ induced by refinements of the covers $C_B := \{B \cdot g\}_{g \in G}$, $B \in S_G$. $A$ is coarsely clopen in $\mathcal{LSS}$ if and only if $A$ is an eigenset of $S_G \cdot \cdot \cdot$. Therefore, the end space of $(G, S_G, S_G \cdot \cdot \cdot)$ is identical with the end space of the coarse space $(G, \mathcal{LSS})$.

Proof. Let us show $st(A, C_B) \setminus A = B \cdot B^{-1} \cdot A \setminus A$. That suffices to prove 5.10 $g \in st(A, C_B) \setminus A$ if and only if $g \notin A$ and there is $h \in G$ such that $g \in B \cdot h$ and $(B \cdot h) \cap A \neq \emptyset$. Thus, $h \in B^{-1} \cdot A$ and $g \in B \cdot B^{-1} \cdot A \setminus A$.

In view of Propositions 5.10 and 5.7 the following question seems relevant:

**Question 5.11.** Suppose a scaled group $(G, S_G)$ acts properly, transitively, and coboundedly on a scaled space $(X, S_X)$. Are end spaces of $(G, S_G, S_G \cdot \cdot \cdot)$ and $(X, S_X, S_G \cdot \cdot \cdot)$ homeomorphic.

What if we replace transitive actions by cobounded actions?

In the remainder of this section we will rephrase results from [11] related to large scale groups in terms of scaled groups using Remark 5.8 and Proposition 5.10.

**Definition 5.12.** A scaled group $(G, S_G)$ is locally bounded if for every bounded subset $B$ of $G$ the subgroup $[B]$ of $G$ generated by $B$ is bounded.

**Proposition 5.13.** [11] Suppose the scale $S_G$ of the scale group $G$ has a countable basis. If $G$ is an unbounded and locally bounded group, then its number of ends of $(G, S_G)$ is infinite.

**Definition 5.14.** A scaled group $(G, S_G)$ is of bounded geometry if there is a bounded set $K$ such that for every bounded subset $B$ of $G$ there are elements $g_i \in G$, $i \leq k$, so that $B \subset \bigcup_{i=1}^{k} g_i \cdot K$.

**Theorem 5.15.** [11] Let $(G, S_G)$ be a boundedly generated scaled group of bounded geometry. If $(G, S_G, S_G \cdot \cdot \cdot)$ has two ends, then $G$ contains an unbounded cyclic subgroup of bounded index.

**Theorem 5.16.** Let $(G, S_G)$ be a $\sigma$-bounded scaled group of bounded geometry. If $(G, S_G, S_G \cdot \cdot \cdot)$ has two ends, then $G$ is boundedly generated.
6. Ends of geodesic spaces

Given a geodesic space $X$ and $x_0 \in X$ the most natural operation on a subset $A$ of $X$ is its cone $\text{Cone}(A, x_0)$ defined as the union of all geodesics $[x_0, x]$ joining $x_0$ and $x \in A$. There are two ways to mix it up: one is first to take a ball around $A$, then the cone. The other is to take the ball of the cone. We will show that the two processes yield the same end spaces.

**Definition 6.1.** One family of operators, $\mathcal{F}_{cb}$ consists of $cb_r : 2^X \to 2^X$, $r > 0$, defined as $cb_r(A) := \text{Cone}(B(A, r), x_0)$.

The other family of operators $\mathcal{F}_{bc}$ consists of $bc_r : 2^X \to 2^X$, $r > 0$, defined as $bc_r(A) := B(\text{Cone}(A, x_0), r)$.

To show that they have the same families of eigensets with respect to $S_X$ defined as all open bounded subsets of $X$, we introduce a combination of the above families: $g_r(A) := B(cb_r(A), r)$.

**Observation 6.2.** As $A \subset cb_r(A) \cup bc_r(A) \subset cb_r(A) \cup bc_r(A) \subset g_r(A)$ for any $A \subset X$, any eigenset of the family $\mathcal{F} = \{g_r\}_{r > 0}$ is an eigenset of the other two families. Our goal is to show all three families have the same eigensets and they do not depend on the basepoint $x_0$.

**Definition 6.3.** Given a pointed geodesic space $(X, x_0)$, its **geodesic eigenset** is a subset $A$ such that $A$ (and therefore $X \setminus A$ as well) is an eigenset of $(X, S_X, \mathcal{F})$, where $S_X$ consists of all open bounded subsets of $X$ and $\mathcal{F} = \{g_r\}_{r > 0}$ as in [6.1].

**Definition 6.4.** Given a pointed geodesic space $(X, x_0)$, its **geodesic ends** are defined as the ends of $(X, S_X, \mathcal{F})$, where $S_X$ consists of all open bounded subsets of $X$ and $\mathcal{F} = \{g_r\}_{r > 0}$.

**Proposition 6.5.** If $X$ is a geodesic space, then for any bounded subset $K$ of $X$ any union $C$ of components of $X \setminus K$ is a geodesic eigenset of $X$.

**Proof.** Without loss of generality, we may assume that $C$ is unbounded. Given $r > 0$ suppose $x_1 \in g_r(C) \setminus B(\text{Cone}(B(K, 2r), x_0), 2r)$. Choose $x_2 \in \text{Cone}(B(C, r), x_0)$ at the distance from $x_1$ less than $r$. There is a geodesic $[x_0, y]$ containing $x_1$, where $y \in B(C, r)$. Choose $z \in C$, $d(y, z) < r$. Notice that implies $y \in C$ as any geodesic $[y, z]$ is outside of $K$. Indeed, if $t \in [y, z] \cap K$, then $y \in B(K, r)$ and $x_2 \in \text{Cone}(B(K, r), x_0)$ resulting in $x_1 \in B(\text{Cone}(B(K, r), x_0), r)$, a contradiction.

Now, the sub-geodesic $[x_2, y]$ of $[x_0, y]$ is outside of $B(K, 2r)$ as well (otherwise $x_1 \in B(\text{Cone}(K, x_0), 2r)$), so it is contained in $C$. Hence, any geodesic $[x_1, x_2]$ misses $K$, so we can conclude $x_1 \in C$.

Finally, if $B := B(\text{Cone}(B(K, 2r), x_0), 2r)$, then $C \setminus B \subset g_r(C) \setminus B \subset C$ which implies $C \equiv g_r(C) \text{ mod } S_X$.

Since $(X \setminus K) \setminus C$ is also a union of some components of $X \setminus K$, $C$ is an eigenset of $(X, S_X, \mathcal{F})$. \hfill $\square$

**Proposition 6.6.** If $X$ is a geodesic space and $A \equiv cb_1(A)$ or $A \equiv bc_1(A) \text{ mod } S_X$, then there is a bounded closed subset $K$ of $X$ such that $A$ is equivalent mod $S_X$ to some union of components of $X \setminus K$.

**Proof.** Without loss of generality, we may assume that $A$ is unbounded. Pick $m > 0$ such that $cb_1(A) \setminus B(x_0, m) = A \setminus B(x_0, m)$ or $bc_1(A) \setminus B(x_0, m) = A \setminus B(x_0, m)$. Suppose $x \in A$ belongs to a component $C$ of $X \setminus \text{cl}(B(x_0, m+1))$ and assume $y \in C$. 


Choose a piece-wise geodesic path $P$ from $x$ to $y$ in $C$. If $z \in P$ and $d(x, z) < 1$, then $z \in cl_b(A)$ (or $z \in bc_1(A)$), so $z \in A$ as $z \notin cl(B(x_0, m + 1))$. By induction on finitely many elements of $P$, $y \in A$. That means $A$ contains all the components of $X \setminus cl(B(x_0, m + 1))$ that intersect $A$. In particular, $A$ is equivalent to the union of such components.

**Corollary 6.7.** If $X$ is a geodesic space, then all three families from 6.7 have the same eigensets, namely unions of some components of $X \setminus K$, $K$ being bounded. In particular, those eigensets do not depend on the basepoint in $X$.

**Corollary 6.8.** If $X$ is a proper geodesic space, then its geodesic ends correspond to Freudenthal ends of $X$.

Given a metric space $(X, d)$, the **Gromov product** of $x$ and $y$ with respect to $a \in X$ is defined by

$$\langle x, y \rangle_a = \frac{1}{2}(d(x, a) + d(y, a) - d(x, y)).$$

Recall that metric space $(X, d)$ is (Gromov) **$\delta$-hyperbolic** if it satisfies the $\delta/4$-inequality:

$$\langle x, y \rangle_a \geq \min\{\langle x, z \rangle_a, \langle z, y \rangle_a\} - \delta/4, \quad \forall x, y, z, a \in X.$$

$(X, d)$ is **Gromov hyperbolic** if it is $\delta$-hyperbolic for some $\delta > 0$.

The assumptions in the next Proposition are chosen specifically so that it applies to both visual hyperbolic spaces and visual CAT(0)-spaces.

**Proposition 6.9.** Suppose $X$ is a visual geodesic space (i.e. there is $x_0 \in X$ such that the union of all geodesic rays emanating from $x_0$ equals $X$), $\sim$ is an equivalence relation on the set of geodesic rays emanating from the basepoint $x_0 \in X$, and $T$ is a topology on the set $\partial X$ of equivalence classes induced by $\sim$. Given $A \subset \partial X$ define $\text{Cone}(A)$ as the union of all geodesic rays $r$ so that $[r] \in A$. Suppose the following conditions are satisfied:

1. If a sequence of rays $r_n$ converges pointwise to $r_0$, then $[r_n]$ converges to $[r_0]$ in $T$.
2. Given a sequence $[r_n] \in \partial X$, $n \geq 1$, there is a subsequence $n(k)$ of positive integers and rays $s_k$ converging pointwise to a ray $r$ so that $[s_k] = [r_{n(k)}]$ for each $k \geq 1$.
3. If $[r]$ belongs to the closure $cl(A)$ of $A \subset \partial X$, then there are exist rays $r_n$, $n \geq 0$, such that $[r_0] = [r]$, $[r_n] \in A$ for $n \geq 1$, and $r_n$ converges pointwise to $r_0$.
4. Given two geodesic rays $r$ and $s$ at $x_0$, $r \sim s$ if and only if there is a sequence $x_n \in r$ diverging to infinity and $M > 0$ such that $\text{dist}(x_n, s) < M$ for each $n \geq 1$.

If $A$ is a clopen subset of $\partial X$, then both $\text{Cone}(A)$ and $X \setminus \text{Cone}(A)$ are geodesic eigensets of $X$.

Conversely, given $D \subset X$ such that $D$ and $X \setminus D$ are eigensets of $(X, S_X, \mathcal{F})$, then there is a clopen subset $A$ of $\partial X$ of $X$ such that $D \equiv \text{Cone}(A)$ mod $S_X$.

**Proof.** Let $C := \text{Cone}(A)$. Suppose $g_r(C) \setminus C$ is not bounded for some $r > 0$. We can choose a sequence $x_n \in g_r(C) \setminus C$ diverging to infinity such that some geodesic rays $l_n$ containing $x_n$ converge pointwise to a ray $l$. Therefore $[l] \notin A$. Choose points $y_n \in C$ satisfying $d(x_n, y_n) < 2r$ for large $n$. We may choose, by passing to subsequences, rays $l'_n \in A$ containing $y_n$ and converging pointwise to a ray $l'$ equivalent to $l$, hence not in $C$, a contradiction.
Let us show that an unbounded geodesic eigenset is, up to a bounded set, equivalent to $\text{Cone}(A)$ for some clopen $A$. Suppose $D \subset X$ is a geodesic eigenset of $(X, S_X, F)$. By 6.10 we can choose $M > 0$ such that $D \setminus B(x_0, M)$ and $(X \setminus D) \setminus B(x_0, M)$ are union of components of $X \setminus B(x_0, M)$. Given $r > 0$ and given a geodesic ray $l$ at $x_0$, $l \setminus B(x_0, M)$ intersects either $D$ or $X \setminus D$ but not both. Put $A := \{[l] : l$ is a geodesic ray at $x_0$ and $l \setminus B(x_0, M) \cap D \neq \emptyset\}$. Notice that if two geodesic rays $t$ and $s$ at $x_0$, satisfy $t \sim s$ and $t \cap D$ contains a geodesic ray emanating from a point in $D$, then $s \cap D$ contains a geodesic ray emanating from a point in $D$. Indeed, by Condition 4), there is a sequence $x_n \in t$ diverging to infinity and $M > 0$ such that $\text{dist}(x_n, s) < M$ for each $n \geq 1$. Consider $r := M + 1$ and conclude that $s \cap D$ contains a geodesic ray emanating from a point in $D$. Therefore, $D = \text{Cone}(A)$, and similarly $X \setminus D = \text{Cone}(\partial X \setminus A)$. It remains to show that $A$ is closed in $\partial X$ as that implies it is clopen in $\partial X$. It is so by Condition 3). □

Corollary 6.10. Suppose $X$ is a proper geodesic space. If $X$ is Gromov hyperbolic and visual, then its geodesic ends correspond to the components of the Gromov boundary $\partial_0(X)$ of $X$.

Corollary 6.11. Suppose $X$ is a proper geodesic space. If $X$ is a visual CAT(0) space, then the geodesic ends of $X$ correspond to components of the visual boundary of $X$.

Question 6.12. Given a metric space $X$ and $p \in X$ one can introduce $S_X$-operators $(S_X$ being the family of open bounded subsets of $X)$ $f_r$, $r > 0$, as

$$f_r(A) = \{x \in X \mid \langle x, y \rangle_p > r \text{ for some } y \text{ in } A\}.$$ 

What can be said about the resulting ends of $X$?

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