GROUND STATE HOMOCLINIC SOLUTIONS FOR A SECOND-ORDER HAMILTONIAN SYSTEM

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Abstract. Consider the second-order Hamiltonian system

\[ \ddot{u} - L(t)u + \nabla W(t,u) = 0, \]

where \( t \in \mathbb{R}, u \in \mathbb{R}^N, L : \mathbb{R} \to \mathbb{R}^{N \times N} \) and \( W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \). We mainly study the case when both \( L \) and \( W \) are periodic in \( t \) and 0 belongs to a spectral gap of \( \sigma \left( -\frac{d^2}{dt^2} + L \right) \). We prove that the above system possesses a ground state homoclinic solution under assumptions which are weaker than the ones known in the literature.

1. Introduction. Consider the second-order Hamiltonian system

\[ \ddot{u} - L(t)u + \nabla W(t,u) = 0, \tag{1.1} \]

where \( t \in \mathbb{R}, u \in \mathbb{R}^N, L : \mathbb{R} \to \mathbb{R}^{N \times N} \) and \( W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) satisfy the following basic assumptions:

(L0) \( L \in C(\mathbb{R}, \mathbb{R}^{N \times N}) \) is \( T \)-periodic \( (T > 0) \) and \( L(t) \) is an \( N \times N \) symmetric matrix;

(W1) \( W \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}), W(t,x) \) is continuously differentiable with respect to \( x \in \mathbb{R}^N \) for each \( t \in \mathbb{R}, W(t,0) \equiv 0, W(t,x) \) is \( T \)-periodic in \( t \), and \( W(t,x) \geq 0 \) for all \( (t,x) \in \mathbb{R} \times \mathbb{R}^N \);

(W2) \( \nabla W(t,x) = o(|x|) \) as \( x \to 0 \) uniformly for \( t \in \mathbb{R} \).

We say that a solution \( u(t) \) of (1.1) is homoclinic (to 0) if \( u(t) \to 0 \) as \( t \to \pm \infty \). In addition, if \( u(t) \not\equiv 0 \) then \( u(t) \) is called a nontrivial homoclinic solution. A ground state homoclinic solution is a nontrivial homoclinic solution that minimizes the energy among all nontrivial homoclinic solutions.

In recent decades, the existence and multiplicity of homoclinic orbits for second-order Hamiltonian systems and their discrete analogues have been widely investigated by many authors, see for example [1–3, 5–9, 12, 14–17, 19–21, 26–30, 32–40] and the references therein. For the case when the condition

(L1) \( L(t) \) is positive definite uniformly in \( t \in \mathbb{R} \)

is satisfied and 0 is a local minimum of the energy functional associated with (1.1), the Mountain-pass theorem proved itself as an effective tool to establish the existence and multiplicity of homoclinic solutions for (1.1), see for example

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such that

\[ 0 < \mu W(t, x) \leq \nabla W(t, x) \cdot x, \quad \forall (t, x) \in \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}). \]

The condition (AR) allows to control the growth of \( W(t, x) \) as \( |x| \to 0 \) and \( |x| \to \infty \).

Later on, many authors tried to weaken (AR). For example, Ding and Lee [9] introduced the following weaker conditions than (AR):

\( (\text{SQ}) \lim_{|x| \to \infty} \frac{|W(t, x)|}{|x|^2} = \infty, \) uniformly in \( t \in \mathbb{R}; \)

\( (\text{DL}) \ \tilde{W}(t, x) := \frac{1}{2} \nabla W(t, x) \cdot x - W(t, x) > 0 \) for all \((t, x) \in \mathbb{R} \times \mathbb{R}^N \) and \( x \neq 0 \), and there exist positive constants \( c_0, R_1 \) and \( \sigma \in (0, 1) \) such that

\[ \nabla W(t, x) \cdot x \leq c_0 |x|^{2-\sigma} \tilde{W}(t, x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \ |x| \geq R_1. \]

However, for the case when (L1) is not satisfied, i.e., \( L(t) \) is not global positive definite, the problem is much more difficult since 0 is a saddle point rather than a local minimum of the energy functional, and it is not an easy task to establish the boundedness of Palais-Smale sequences. In this paper, we address this case assuming that 0 lies in a gap of \( \sigma \left(-\frac{d^2}{dt^2} + L\right) \), i.e.,

\[ (\text{L2}) \quad \sup \left[ \sigma \left(-\frac{d^2}{dt^2} + L\right) \cap (-\infty, 0) \right] < 0 < \inf \left[ \sigma \left(-\frac{d^2}{dt^2} + L\right) \cap (0, +\infty) \right]. \quad (1.2) \]

So far, there are few papers dealing with Hamiltonian system (1.1) under (L2) (see [12] in which much stronger assumptions than (DL) were required). Also, [12] is the first result in this field.

Motivated by the above works [4, 6, 9, 12, 23, 25], our first object is to generalize and improve the results obtained in [9] by relaxing (L1) to (L2), and (SQ) and (DL) to the following conditions:

\( (\text{W3}) \lim_{|x| \to \infty} \frac{W(t, x)}{|x|^2} = \infty, \) a.e. \( t \in \mathbb{R}; \)

\( (\text{W4}) \ \tilde{W}(t, x) = \frac{1}{2} \nabla W(t, x) \cdot x - W(t, x) \geq 0, \ \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \) and there exists \( g \in C((1, \infty), (0, \infty)) \) with \( \lim_{s \to \infty} g(s) = \infty \) such that

\[ \frac{|\nabla W(t, x)|}{|x|} \geq \frac{1}{4\gamma_2} \Rightarrow |\nabla W(t, x)| \leq \frac{|x|}{g(|x|)} \tilde{W}(t, x), \]

where \( \gamma_2 \) is the Sobolev imbedding constant, see (2.6).

We are now in a position to state the first result of this paper.

**Theorem 1.1** Assume that \( L \) and \( W \) satisfy (L0), (L2) and (W1)-(W4). Then system (1.1) possesses a ground state homoclinic solution.
where $E = E^+ \oplus E^−$ corresponds to the spectral decomposition of $-\frac{d^2}{dx^2} + L$ with respect to the positive and negative part of the spectrum, and $u = u^+ + u^- \in E^+ \oplus E^-$. If $\sigma\left(-\frac{d^2}{dx^2} + L\right) = (0, +\infty)$, then $\dim E^- = 0$. Otherwise, $E^-$ is infinite-dimensional, for more detail, see Section 2. Pankov [18] introduced the following set,

$$M = \{u \in E \setminus E^- : \langle \Phi'(u), u \rangle = \langle \Phi'(u), v \rangle = 0, \forall v \in E^- \}. \quad (1.4)$$

By definition, $M$ contains all nontrivial critical points of $\Phi$. If $u_0 \in E \setminus \{0\}$ satisfies $\Phi'(u_0) = 0$ and $\Phi(u_0) = \inf_M \Phi$, then it is a ground state solution of (1.1). There are many works on the existence of a ground state solution which minimizes the energy among all functions of the Nehari-Pankov mainfold for Schrödinger equations, see for example [12, 18, 22–24]. However, to the best of our knowledge, there are no similar results on the existence of a ground state homoclinic solution for Hamiltonian system (1.1). The main reason is that for system (1.1), there is no analog of the Nehari-type monotone condition used for Schrödinger equation.

Being motivated by [22–24], in the present paper we generalize the Nehari-type monotone condition to high-dimensional case in such a way that (1.1) possesses a ground state homoclinic solution $u_0$ with $\Phi(u_0) = \inf_M \Phi$.

Before presenting our second theorem, we define a set $\mathcal{ND}$ as follows:

$$\mathcal{ND} := \{h \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+) : h(t, 0) \equiv 0, h(t, s) \text{ is } T\text{-periodic in } t \text{ and nondecreasing in } s \in [0, \infty) \text{ for every } t \in \mathbb{R}\}.$$

We make the following assumptions:

(W5) for all $\theta \geq 0$, $x, y \in \mathbb{R}^N$, one has

$$\frac{1 - \theta^2}{2} \nabla W(t, x) \cdot x - \theta \nabla W(t, x) \cdot y + W(t, \theta x + y) - W(t, x) \geq 0,$$

(W6) there exist $\zeta_i \in \mathbb{R}^N \setminus \{0\}$ and symmetric positive definite matrix $A_j \in \mathbb{R}^{N \times N}$ such that

$$W(t, x) = \sum_{i=1}^k \int_0^{|\zeta_i|} g_i(t, s)ds + \sum_{j=1}^l \int_0^{(A_jx)^1} h_j(t, s)ds,$$

where $g_i, h_j \in \mathcal{ND}$.

Now, we are ready to state the second result of this paper.

**Theorem 1.2** Assume that $L$ and $W$ satisfy (L0), (L2) and (W1)-(W3) and (W5). Then system (1.1) possesses a homoclinic solution $\bar{u} \in E \setminus \{0\}$ such that $\Phi(\bar{u}) = \inf_M \Phi$.

**Corollary 1.3** Assume that $L$ and $W$ satisfy (L0), (L2) and (W1)-(W3) and (W6). Then system (1.1) possesses a homoclinic solution $\bar{u} \in E \setminus \{0\}$ such that $\Phi(\bar{u}) = \inf_M \Phi$.

Next, we give several nonlinear examples to illustrate assumptions (W3), (W4) and (W6).

**Example 1.4.** Let $W(t, x) = (\sin^2 t)|x|^2 \ln(1 + |x|^2)$. It is easy to see that $W$ satisfies (W1), (W2), (W3) and (W4) with $g(s) = s^{1/2}$, but neither of (AR), (SQ) and (DL).
Example 1.5. There are many functions which satisfy (W6), for example

\[ W(t,x) = \int_0^\zeta x h(t,s)ds \]

and

\[ W(t,x) = \int_0^{\|x\|} h(t,s)ds \]

where \( \zeta \in \mathbb{R}^N \setminus \{0\} \) and \( h \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+) \), \( h(t,0) \equiv 0 \), \( h(t,s) \) is \( T \)-periodic in \( t \) and nondecreasing in \( s \in [0, \infty) \) for every \( t \in \mathbb{R} \).

Remark 1.6. (W4) and (W5) (or (W6)) are complementary, but (W4) is satisfied by more functions. The ground state homoclinic solutions \( u_0 \) and \( \bar{u} \) provided by Theorems 1.1 and 1.2, respectively, have the same characterization – they minimize the energy among all nontrivial homoclinic solutions. However, \( \bar{u} \) has another (minimax) characterization given by

\[ \Phi(\bar{u}) = \inf_{\mathcal{M}} \Phi = \inf_{w \in E^+ \setminus \{0\}} \max_{u \in \hat{E}} \Phi(u), \]

where \( \hat{E} := E^- \oplus \mathbb{R}u \).

Remark 1.7. (L0) can be weakened: it is enough to assume that \( L \) is a \( T \)-periodic \( L^\infty \) function. Similarly, in (W1) one can assume that \( W \) and its derivative with respect to the \( \mathbb{R}^N \)-variable are Caratheodory functions with the obvious \( L^\infty \)-type boundedness with respect to the \( \mathbb{R} \)-variable.

2. Variational setting and preliminaries. Let \( X \) be a real Hilbert space with \( X = X^- \oplus X^+ \) and \( X^- \perp X^+ \). A functional \( \varphi \in \mathcal{C}^1(X, \mathbb{R}) \) is said to be weakly sequentially lower semi-continuous if for any \( u_n \rightharpoonup u \) in \( X \), one has \( \varphi(u) \leq \liminf_{n \to \infty} \varphi(u_n) \), and \( \varphi' \) is said to be weakly sequentially continuous if \( \lim_{n \to \infty} \langle \varphi'(u_n), v \rangle = \langle \varphi'(u), v \rangle \) for each \( v \in X \).

Lemma 2.1. ([12, 13]) Let \( (X, \| \cdot \|) \) be a real Hilbert space with \( X = X^- \oplus X^+ \) and \( X^- \perp X^+ \), and let \( \varphi \in \mathcal{C}^1(X, \mathbb{R}) \) be of the form

\[ \varphi(u) = \frac{1}{2} (\|u^+\| - \|u^-\|) - \psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+. \]

Assume that the following conditions are satisfied:

(KS1) \( \psi \in \mathcal{C}^1(X, \mathbb{R}) \) is bounded from below and weakly sequentially lower semi-continuous;

(KS2) \( \psi' \) is weakly sequentially continuous;

(KS3) there exist \( r > \rho > 0 \) and \( e \in X^+ \) with \( \|e\| = 1 \) such that

\[ \kappa := \inf \varphi(S^+_{\rho}) > \sup \varphi(\partial Q), \]

where

\[ S^+_{\rho} = \{ u \in X^+: \|u\| = \rho \}, \quad Q = \{ v + se : v \in X^- , s \geq 0, \|v + se\| \leq r \}. \]

Then there exist a constant \( c \in [\kappa, \sup \varphi(Q)] \) and a sequence \( \{u_n\} \subset X \) satisfying

\[ \varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0. \]
Let \( A = -\frac{d^2}{dt^2} + L \). Then \( A \) is self-adjoint in \( L^2(\mathbb{R}, \mathbb{R}^N) \) with domain \( \mathcal{D}(A) = H^2(\mathbb{R}, \mathbb{R}^N) \) (see [11, Theorem 4.26]). Let \( \{\mathcal{E}(\lambda) : -\infty \leq \lambda \leq +\infty\} \) and \( |A| \) be the spectral family and the absolute value of \( A \), respectively, and let \( |A|^{1/2} \) be the square root of \( |A| \). Set \( \mathcal{U} = \text{id} - \mathcal{E}(0) - \mathcal{E}(0) \). Then \( \mathcal{U} \) commutes with \( A \), \( |A| \) and \( |A|^{1/2} \), and \( A = \mathcal{U}|A| \) is the polar decomposition of \( A \) (see [10, Theorem IV 3.3]).

Let

\[
E = \mathcal{D}(|A|^{1/2}), \quad E^- = \mathcal{E}(0-)E, \quad E^+ = [\text{id} - \mathcal{E}(0)]E.
\]

For any \( u \in E \), it is easy to see that \( u = u^- + u^+ \), where

\[
u^- := \mathcal{E}(0-)u \in E^-, \quad u^+ := [\text{id} - \mathcal{E}(0)]u \in E^+
\]

and

\[
\mathcal{A}u = -|A|u, \quad \forall \ u \in E^-; \quad \mathcal{A}u = |A|u, \quad \forall \ u \in E^+ \cap \mathcal{D}(A).
\]

Define an inner product

\[
(u, v) = \left( |A|^{1/2}u, |A|^{1/2}v \right)_{L^2}, \quad u, v \in E
\]

and the corresponding norm

\[
\|u\| = \left\| |A|^{1/2}u \right\|_2, \quad u \in E,
\]

where \((\cdot, \cdot)_{L^2}\) denotes the usual inner product in \( L^2(\mathbb{R}, \mathbb{R}^N) \) and \( \|\cdot\|_s \) stands for the usual norm in \( L^s(\mathbb{R}, \mathbb{R}^N) \). Since \( E = H^1(\mathbb{R}, \mathbb{R}^N) \) with equivalent norms under \( L \), \( E \) is continuously embedded in \( L^s(\mathbb{R}, \mathbb{R}^N) \) for all \( 2 \leq s \leq \infty \), i.e., for all \( 2 \leq s \leq \infty \), there exists \( \gamma_s > 0 \) such that

\[
\|u\|_s \leq \gamma_s \|u\|, \quad \forall \ u \in E.
\]

In addition, one has the decomposition \( E = E^- \oplus E^+ \) orthogonal with respect to both \((\cdot, \cdot)_{L^2}\) and \((\cdot, \cdot)\).

In view of (2.3) and (2.5), we have

\[
\int_{\mathbb{R}} \left[ |\dot{u}|^2 + (L(t)u, u) \right] dt = \|u^+\|^2 - \|u^-\|^2.
\]

From (1.3) and (2.7), one has

\[
\Phi(u) = \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}} W(t, u) dt, \quad \forall \ u = u^- + u^+ \in E,
\]

\[
\langle \Phi'(u), v \rangle = \int_{\mathbb{R}} [\dot{u} \cdot \dot{v} + (L(t)u) \cdot v] dt - \int_{\mathbb{R}} \nabla W(t, u) \cdot v dt, \quad \forall \ u, v \in E
\]

and

\[
\langle \Phi'(u), u \rangle = \|u^+\|^2 - \|u^-\|^2 - \int_{\mathbb{R}} \nabla W(t, u) \cdot u dt, \quad \forall \ u = u^- + u^+ \in E.
\]

Let

\[
\Psi(u) = \int_{\mathbb{R}} W(t, u) dt.
\]

Then we can prove the following lemma by a standard argument.

**Lemma 2.2.** Assume that (W1) and (W2) hold. Then \( \Psi \) is nonnegative, weakly sequentially lower semi-continuous, and \( \Phi' \) is weakly sequentially continuous.
3. Proof of Theorem 1.1. The following lemmas will be useful in the proof of Theorem 1.1.

**Lemma 3.1.** Assume that (L0), (L2), (W1) and (W2) hold. Then there exists a constant $\rho > 0$ such that $\alpha := \inf \Phi(S^+_\rho) > 0$, where $S^+_\rho = \{u \in E^+ : \|u\| = \rho\}$.

The proof is standard, so we omit it.

**Lemma 3.2.** Assume that (L0), (L2), (W1), (W2) and (W3) hold. Let $e \in E^+$ with $\|e\| = 1$. Then there is $\tau > 0$ such that $\sup \Phi(\partial Q) \leq 0$ for $r \geq \tau$, where

$$Q = \{w + se : w \in E^-, s \geq 0, \|w + se\| \leq r\}. \quad (3.1)$$

**Proof.** Since $\Phi(w) \leq 0$ for all $w \in E^-$, we only need to show that $\Phi(w + se) \to -\infty$ as $\|w + se\| \to \infty$. Arguing indirectly, assume that there exists a sequence $\{w_n + s_n e\} \subset E^- \oplus \mathbb{R}$ with $\|w_n + s_n e\| \to \infty$ such that $\Phi(w_n + s_n e) \geq 0$ for all $n \in \mathbb{N}$. Set $v_n = (w_n + s_n e)/\|w_n + s_n e\| = v^-_n + \tau_n e$, then $\|v^-_n + \tau_n e\| = 1$. Passing to a subsequence, we may assume that $\tau_n \to \bar{\tau}$, $v^-_n \to v^-$ a.e. in $\mathbb{R}$. Hence,

$$0 \leq \frac{\Phi(w_n + s_n e)}{\|w_n + s_n e\|^2} = \frac{\tau_n^2}{2}\|e\|^2 - \frac{1}{2}\|v^-_n\|^2 - \int_\mathbb{R} \frac{W(t, w_n + s_n e)}{\|w_n + s_n e\|^2} dt. \quad (3.2)$$

If $\bar{\tau} = 0$, then it follows from (3.2) that

$$0 \leq \frac{1}{2}\|v^-_n\|^2 + \int_\mathbb{R} \frac{W(t, w_n + s_n e)}{\|w_n + s_n e\|^2} dt \leq \frac{\tau_n^2}{2}\|e\|^2 \to 0,$$

which yields $\|v^-_n\| \to 0$, and so $1 = \|v^-_n + \tau_n e\|^2 \to 0$, a contradiction.

If $\bar{\tau} \neq 0$, then it follows from (3.2) and (W3) that

$$0 \leq \limsup_{n \to \infty} \left[ \frac{\tau_n^2}{2}\|e\|^2 - \frac{1}{2}\|v^-_n\|^2 - \int_\mathbb{R} \frac{W(t, w_n + s_n e)}{\|w_n + s_n e\|^2} dt \right]$$

$$\leq \frac{\bar{\tau}^2}{2}\|e\|^2 - \liminf_{n \to \infty} \int_\mathbb{R} \frac{W(t, w_n + s_n e)}{\|w_n + s_n e\|^2} |v^-_n + \tau_n e|^2 dt$$

$$\leq \frac{\bar{\tau}^2}{2}\|e\|^2 - \liminf_{n \to \infty} \int_\mathbb{R} \frac{W(t, w_n + s_n e)}{\|w_n + s_n e\|^2} |v^-_n + \tau_n e|^2 dt$$

$$= -\infty,$$

a contradiction again. \hfill \square

**Lemma 3.3.** Assume that (L0), (L2), (W1), (W2) and (W4) hold. Then any sequence $\{u_n\} \subset E$ satisfying

$$\Phi(u_n) \to c > 0, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0 \quad (3.3)$$

is bounded in $E$.

**Proof.** To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$ and $\|v_n\|_p \leq \gamma_p \|v_n\| = \gamma_p$ for $2 \leq p \leq \infty$. Observe that

$$c + o(1) = \Phi(u_n) - \frac{1}{2} \langle \Phi'(u_n), u_n \rangle = \int_\mathbb{R} \tilde{W}(t, u_n) dt. \quad (3.4)$$
Since \( \lim_{s \to \infty} g(s) = \infty \), there exists \( M > 0 \) such that
\[
g(s) \geq 16(c+1)\gamma^2 \Omega, \quad \forall \ s \geq M. \tag{3.5}
\]
Let
\[
\Omega_n = \left\{ t \in \mathbb{R} : \frac{|\nabla W(t, u_n)|}{|u_n|} \leq \frac{1}{4\gamma^2} \right\}
\quad \text{and} \quad
\Theta_n = \{ t \in \mathbb{R} : |u_n| \geq M \}. \tag{3.6}
\]
Hence, it follows from (3.6) that
\[
\int_{\Omega_n} \frac{|\nabla W(t, u_n)|}{|u_n|} |v_n| \, dt = \int_{\Omega_n} \frac{|\nabla W(t, u_n)|}{|u_n|} |v_n|^2 \, dt \\
\leq \frac{1}{4\gamma^2} \int_{\Omega_n} |v_n|^2 \, dt \\
\leq \frac{1}{4\gamma^2} \|v_n\|^2 \leq \frac{1}{4}. \tag{3.7}
\]
From (W4), (3.4), (3.5) and (3.6), one has
\[
\int_{(\mathbb{R} \setminus \Omega_n) \cap \Theta_n} \frac{|\nabla W(t, u_n)|}{|u_n|} |v_n| \, dt = \int_{(\mathbb{R} \setminus \Omega_n) \cap \Theta_n} \frac{|\nabla W(t, u_n)|}{|u_n|} |v_n|^2 \, dt \\
\leq |v_n|^2 \int_{(\mathbb{R} \setminus \Omega_n) \cap \Theta_n} \frac{1}{g(|u_n|)} \widetilde{W}(t, u_n) \, dt \\
\leq \frac{1}{8(1+c)} \int_{(\mathbb{R} \setminus \Omega_n) \cap \Theta_n} \widetilde{W}(t, u_n) \, dt \\
\leq \frac{1}{8} + o(1) \tag{3.8}
\]
and
\[
\int_{(\mathbb{R} \setminus \Omega_n) \cap (\mathbb{R} \setminus \Theta_n)} \frac{|\nabla W(t, u_n)|}{|u_n|} |v_n| \, dt = \frac{\|v_n\|_\infty}{\|u_n\|} \int_{(\mathbb{R} \setminus \Omega_n) \cap (\mathbb{R} \setminus \Theta_n)} \frac{|u_n|}{g(|u_n|)} \widetilde{W}(t, u_n) \, dt \\
\leq \frac{C_1}{\|u_n\|} \int_{(\mathbb{R} \setminus \Omega_n) \cap (\mathbb{R} \setminus \Theta_n)} \widetilde{W}(t, u_n) \, dt \\
= \frac{C_1(c + o(1))}{\|u_n\|} = o(1). \tag{3.9}
\]
Now, combining (3.3) with (3.7)-(3.9) implies
\[
1 + o(1) = \frac{\|u_n\|^2 - \langle \Phi(u_n), u_n \rangle}{\|u_n\|^2} \\
= \frac{1}{\|u_n\|} \int_{\mathbb{R}^N} \nabla W(t, u_n) \cdot v_n \, dt \\
\leq \frac{1}{\|u_n\|} \int_{\mathbb{R}^N} |\nabla W(t, u_n)| |v_n| \, dt \\
= \int_{\Omega_n} \frac{|\nabla W(t, u_n)|}{|u_n|} |v_n| \, dt + \int_{(\mathbb{R} \setminus \Omega_n) \cap \Theta_n} \frac{|\nabla W(t, u_n)|}{|u_n|} |v_n| \, dt \\
+ \int_{(\mathbb{R} \setminus \Omega_n) \cap (\mathbb{R} \setminus \Theta_n)} \frac{|\nabla W(t, u_n)|}{|u_n|} |v_n| \, dt \\
\leq \frac{3}{8} + o(1). \tag{3.10}
\]
This contradiction implies that \( \{u_n\} \) is bounded in \( E \). \( \Box \)
Proof of Theorem 1.1. In view of Lemmas 2.1, 2.2, 3.1 and 3.2, there exist \( c \geq \alpha \) and a sequence \( \{u_n\} \subset E \) satisfying
\[
\Phi(u_n) \to c > 0, \quad \|\Phi'(u_n)(1 + \|u_n\|)\| \to 0, \quad (3.11)
\]
By Lemma 3.2, \( \|u_n\| \) is bounded. The rest of the proof is standard, so we omit it.

4. Proof of Theorem 1.2. To prove Theorem 1.2, we need several lemmas following below.

Lemma 4.1. Assume that (L0), (L2), (W1), (W2) and (W5) hold. Then:
\[
\Phi(u) \geq \Phi(\theta u + w) + \frac{1}{2}\|w\|^2 + \frac{1 - \theta^2}{2} \langle \Phi'(u), u \rangle - \theta \langle \Phi'(u), w \rangle, \quad \forall \theta \geq 0, \ u \in E, \ w \in E^-.
\]
(4.1)

Proof. By (2.8), (2.9), (2.10) and (W5), one has
\[
\Phi(u) - \Phi(\theta u + w) = \frac{1}{2}\|w\|^2 + \frac{1 - \theta^2}{2} \langle \Phi'(u), w \rangle - \theta \langle \Phi'(u), w \rangle = \int_\mathbb{R} [W(t, u) - W(t, \theta u + w)] dt
\]
\[
\geq \frac{1}{2}\|w\|^2 + \frac{1 - \theta^2}{2} \langle \Phi'(u), u \rangle - \theta \langle \Phi'(u), w \rangle, \quad \forall \theta \geq 0, \ u \in E, \ w \in E^-.
\]
This shows that (4.1) holds.

From Lemma 4.1, we have the following two corollaries.

Corollary 4.2. Assume that (L0), (L2), (W1), (W2) and (W5) hold. Then for any
\[ u \in \mathcal{M} \] one has:
\[
\Phi(u) \geq \Phi(\theta u + w), \quad \forall \theta \geq 0, \ w \in E^-.
\]
(4.2)

Corollary 4.3. Assume that (L0), (L2), (W1), (W2) and (W5) hold. Then:
\[
\Phi(u) \geq \frac{\theta^2}{2}\|u\|^2 + \frac{1 - \theta^2}{2} \langle \Phi'(u), u \rangle + \frac{\theta^2}{2} \langle \Phi'(u), u \rangle - \int_{\mathbb{R}^N} W(t, \theta u^+ + \theta u^-) dt, \quad \forall u \in E, \ \theta \geq 0.
\]
(4.3)

Combining Lemma 3.1 and Corollary 4.2 with the argument used in [23] to prove Lemma 3.6, we can establish the following statement.

Lemma 4.4. Assume that (L0), (L2), (W1), (W2) and (W5) hold. Then:
\begin{enumerate}
\item[(i)] there exists \( \rho > 0 \) such that
\[ m := \inf_{\mathcal{M}} \Phi \geq \kappa := \inf \{ \Phi(u) : u \in E^+, \|u\| = \rho \} > 0. \]
\item[(ii)] \( \|u^+\| \geq \max \{ \|u^-\|, \sqrt{2m} \} \) for all \( u \in \mathcal{M} \).
\end{enumerate}

Following the argument used in the proof of Lemma 3.8 from [23], it is easy to show the following lemma.
Lemma 4.5. Assume that (L0), (L2), (W1), (W2), (W3) and (W5) hold. Then there exist a constant \( c_* \in [\kappa, m] \) and a sequence \( \{u_n\} \subset E \) satisfying
\[
\Phi(u_n) \to c_*, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{4.4}
\]

Lemma 4.6 Assume that (L0), (L2), (W1), (W2), (W3) and (W5) hold. Then any sequence \( \{u_n\} \subset E \) satisfying
\[
\Phi(u_n) \to c \geq 0, \quad \langle \Phi'(u_n), u_n^\pm \rangle \to 0 \tag{4.5}
\]
is bounded in \( E \).

Proof. To prove the boundedness of \( \{u_n\} \), arguing by contradiction, suppose that \( \|u_n\| \to \infty \). Let \( v_n = u_n/\|u_n\| \). Then \( \|v_n\| = 1 \). By (2.6), \( \|v_n\|_2 \leq \gamma_2 \). If
\[
\delta := \limsup \sup_{n \to \infty} \sup_{s \in \mathbb{R}} \int_{s-T}^{s+T} |v_n^r|^2 dt = 0,
\]
then by Lions’ concentration compactness principle [31, Lemma 1.21], \( v_n^\pm \to 0 \) in \( L^*([R, R^N]) \) for \( s \in (2, \infty) \). Fix \( R > [2(1 + c)]^{1/2} \). By virtue of (W1) and (W2), for \( \varepsilon = 1/4(R\gamma_2)^2 > 0 \), there exists \( r \in (0, R\gamma_2) \) such that
\[
W(t, x) \leq \frac{1}{4(R\gamma_2)^2} |x|^2, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \ |x| \leq r. \tag{4.6}
\]
Let
\[
\beta := \max \left\{ \frac{W(t, x)}{|x|^3} : t \in [0, T], r \leq |x| \leq R\gamma_2 \right\}. \tag{4.7}
\]
Then \( 0 \leq \beta < +\infty \). Hence, it follows from (4.6) and (4.7) that
\[
W(t, x) \leq \frac{1}{4(R\gamma_2)^2} |x|^2 + \beta |x|^3, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \ |x| \leq R\gamma_2. \tag{4.8}
\]
Combining (4.8) with \( \|v_n^r\|_\infty \leq \gamma_\infty\|v_n^\pm\| \leq \gamma_\infty \), one has
\[
\limsup_{n \to \infty} \int_{\mathbb{R}} W(t, Ru_n^r/\|u_n\|) dt = \limsup_{n \to \infty} \int_{\mathbb{R}} W(t, Ru_n^+\|u_n\|) dt
\leq \frac{1}{4\gamma_2^2} \limsup_{n \to \infty} \int_{\mathbb{R}} |v_n^r|^2 dt + R^2 \beta \limsup_{n \to \infty} \int_{\mathbb{R}} |v_n^+|^3 dt
\leq \frac{1}{4}. \tag{4.9}
\]
Let \( \theta_n = R/\|u_n\| \). Hence, by virtue of (4.3), (4.5) and (4.9), one has
\[
c + o(1) = \Phi(u_n)
\geq \frac{\theta_n^2}{2} \|u_n\|^2 - \int_{\mathbb{R}} W(t, \theta_n u_n^+) dt + \frac{1 - \theta_n^2}{2} \langle \Phi'(u_n), u_n \rangle + \theta_n^2 \langle \Phi'(u_n), u_n^- \rangle
= \frac{R^2}{2} - \int_{\mathbb{R}} W(t, Ru_n^+/\|u_n\|) dt
+ (1 - \frac{R^2}{2\|u_n\|^2}) \langle \Phi'(u_n), u_n \rangle + \frac{R^2}{\|u_n\|^2} \langle \Phi'(u_n), u_n^- \rangle
= \frac{R^2}{2} - \int_{\mathbb{R}} W(t, Ru_n^+/\|u_n\|) dt + o(1)
\geq \frac{R^2}{2} - \frac{1}{4} + o(1) > \frac{3}{4} + c + o(1).
This contradiction shows that \( \delta > 0 \). We may assume that there exists \( k_n \in \mathbb{Z} \) such that \( \int_{(k_n-2)T}^{(k_n+2)T} |v_n^+|^2 \, dt > \frac{\delta}{2} \). Let \( w_n(t) = v_n(t + k_n T) \). Then

\[
\int_{-2T}^{2T} |w_n^+|^2 \, dt > \frac{\delta}{2}.
\]

(4.10)

Now, we define \( \tilde{u}_n(t) = u_n(t + k_n T) \). Then \( \tilde{u}_n / \|u_n\| = w_n \) and \( \|w_n\| = 1 \). Passing to a subsequence, we have \( w_n \to w \) in \( E \), \( w_n \to w \) in \( L^2_{\text{loc}}(\mathbb{R}) \) and \( w_n \to w \) a.e. in \( \mathbb{R} \). Obviously, (4.10) implies that \( w \neq 0 \). Hence, it follows from (4.5), (W3) and Fatou’s lemma that

\[
0 = \lim_{n \to \infty} c + o(1) / \|u_n\|^2 = \lim_{n \to \infty} \Phi(u_n) / \|u_n\|^2
= \lim_{n \to \infty} \left[ \frac{1}{2} (\|v_n^+\|^2 - \|v_n^-\|^2) - \int_{\mathbb{R}} W(t, u_n) \, dx \right]
= \lim_{n \to \infty} \left[ \frac{1}{2} (\|v_n^+\|^2 - \|v_n^-\|^2) - \int_{\mathbb{R}} W(t + k_n T, \tilde{u}_n) |w_n|^2 \, dx \right]
\leq \frac{1}{2} \lim inf_{n \to \infty} \int_{\mathbb{R}} W(t, \tilde{u}_n) |w_n|^2 \, dx
= -\infty,
\]

which is a contradiction. Thus \( \{u_n\} \) is bounded in \( E \).

\[ \square \]

**Lemma 4.7.** ([24, Lemma 2.3]) Assume that \( h \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and for any \( x \in \mathbb{R} \), the function \( t \mapsto h(x, t) \) is nondecreasing on \( \mathbb{R} \) and \( h(x, 0) = 0 \). Then the following inequality holds:

\[
\left( 1 - \frac{\theta^2}{2} \tau - \theta \sigma \right) h(x, \tau) \tau \geq \int_{\theta \tau + \sigma}^{\tau} h(x, s) |s| \, ds, \quad \forall \, \theta \geq 0, \ \tau, \ \sigma \in \mathbb{R}.
\]

(4.11)

Following the proofs of Lemmas 3.14 and 3.15 from [23], we can prove the following two lemmas.

**Lemma 4.8.** Assume that \( W(t, x) = \int_0^{\left| \zeta \cdot x \right|} g(t, s) \, ds \), where \( \zeta \in \mathbb{R}^N \setminus \{0\} \) and \( g \in \mathcal{N}D \). Then \( W \) satisfies (W1), (W2) and (W5).

**Proof.** It is easy to see that \( W \) satisfies (W1) and (W2). Next, we show that \( W \) also satisfies (W5). Let \( g(t, s) = 0 \) for \( s < 0 \). Note that

\[
|\theta a + b| \geq \theta |a| + \frac{ab}{|a|}, \quad \forall \ a, b \in \mathbb{R}.
\]

(4.12)

For any \( x \in \mathbb{R} \), it follows from (4.11) and (4.12) that

\[
\left[ 1 - \frac{\theta^2}{2} \nabla W(t, x) \cdot x - \theta \nabla W(t, x) \cdot y + W(t, \theta x + y) - W(t, x) \right]
\geq \left[ 1 - \frac{\theta^2}{2} (\zeta \cdot x)^2 - \theta (\zeta \cdot x) (\zeta \cdot y) \right] g(t, |\zeta \cdot x|) \int_{|\theta(\zeta \cdot x) + (\zeta \cdot y)|}^{\left| \zeta \cdot x \right|} g(t, s) \, ds
\geq \left[ \frac{1}{2} (\zeta \cdot x)^2 - \theta (\zeta \cdot x) (\zeta \cdot y) \right] g(t, |\zeta \cdot x|)
- \int_{\theta|\zeta \cdot x| + (\zeta \cdot x)(\zeta \cdot y)/|\zeta \cdot x|}^{\left| \zeta \cdot x \right|} g(t, s) |s| \, ds \geq 0, \quad \forall \ x, y \in \mathbb{R}^N.
\]
This shows that (W5) holds.

Lemma 4.9. Assume that \( W(t, x) = \int_{0}^{(Ax)\cdot x} h(t, s) ds \), where \( A \in \mathbb{R}^{N \times N} \) is symmetric and positive definite and \( h \in \mathcal{N} \mathcal{D} \). Then \( W \) satisfies (W1), (W2) and (W5).

Proof. By virtue of Lemma 4.7, one has
\[
\frac{1 - \theta^2}{2} \sqrt{(Ax) \cdot x} - \frac{\theta (Ax) \cdot y}{\sqrt{(Ax) \cdot x}} h \left( x, \sqrt{(Ax) \cdot x} \right) \sqrt{(Ax) \cdot x} 
\geq \int_{\theta \sqrt{(Ax) x + (Ax) y} / \sqrt{(Ax) x}}^{\sqrt{(Ax) x}} h(t, \tau) |\tau| d\tau, \quad \forall \ \theta \geq 0, \ x, y \in \mathbb{R}^N. \quad (4.13)
\]

It is easy to verify that
\[
(Ax) \cdot y \leq \sqrt{(Ax) \cdot x} \sqrt{(Ay) \cdot y}, \quad \forall \ \ x, y \in \mathbb{R}^N,
\]
which together with (4.13) implies that
\[
\frac{1 - \theta^2}{2} \nabla W(t, x) \cdot x - \theta \nabla W(t, x) \cdot y + W(t, \theta x + y) - W(t, x) 
= \frac{1 - \theta^2}{2} h \left( x, \sqrt{(Ax) \cdot x} \right) (Ax) \cdot x - \theta \left( (Ax) \cdot y \right) h \left( x, \sqrt{(Ax) \cdot x} \right) 
+ \int_{\theta \sqrt{(Ax) x + (Ax) y} / \sqrt{(Ax) x}}^{\sqrt{(Ax) x}} h(t, \tau) |\tau| d\tau 
= \left[ \frac{1 - \theta^2}{2} \sqrt{(Ax) \cdot x} - \frac{\theta \left( (Ax) \cdot y \right)}{\sqrt{(Ax) \cdot x}} \right] h \left( x, \sqrt{(Ax) \cdot x} \right) \sqrt{(Ax) \cdot x} 
+ \int_{\sqrt{(Ax) x + (Ax) y} / \sqrt{(Ax) x}}^{\theta \sqrt{(Ax) x + (Ax) y} / \sqrt{(Ax) x}} h(t, \tau) |\tau| d\tau 
\geq \left[ \frac{1 - \theta^2}{2} \sqrt{(Ax) \cdot x} - \frac{\theta \left( (Ax) \cdot y \right)}{\sqrt{(Ax) \cdot x}} \right] h \left( x, \sqrt{(Ax) \cdot x} \right) \sqrt{(Ax) \cdot x} 
+ \int_{\sqrt{(Ax) x + (Ax) y} / \sqrt{(Ax) x}}^{\theta \sqrt{(Ax) x + (Ax) y} / \sqrt{(Ax) x}} h(t, \tau) |\tau| d\tau 
\geq 0.
\]

This shows that (W5) holds.

Proof of Theorem 1.2. In view of Lemmas 2.1, 2.2, 4.5 and 4.6, there exists a bounded sequence \( \{ u_n \} \subset E \) satisfying (4.4). The rest of the proof is standard (see the proof of Theorem 1.2 in [24]), and so we omit it.

Using Lemmas 4.8 and 4.9, it is easy to verify that \( W(t, x) \) in (W6) satisfies (W5). Therefore, Corollary 1.3 follows from Theorem 1.2 immediately.

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