On O–X mode conversion in 2D inhomogeneous plasma with a sheared magnetic field

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Received 31 August 2009, in final form 3 December 2009
Published 18 February 2010
Online at stacks.iop.org/PPCF/52/035008

Abstract
The conversion of an ordinary wave to an extraordinary wave in a 2D inhomogeneous slab model of the plasma confined by a sheared magnetic field is studied analytically.

1. Introduction
The linear conversion of waves in nonuniform media is a common phenomenon in physics and has been studied in fields as diverse as RF heating of fusion plasmas, ionospheric physics and so on. The particular case of the linear mode conversion, namely, the conversion of an ordinary (O) wave to an extraordinary (X) wave at electron cyclotron frequency being a primary consideration for an auxiliary electron heating and current drive in fusion devices, was intensively studied for plasmas with a cold plasma dielectric tensor. A quantitatively accurate description of this problem was given attention a couple of decades ago within both the WKB approximation and the full-wave analysis for a slab model of the 1D inhomogeneous plasma with zero magnetic field shear [1–4]. Having been renewed by considerable success in experimental realization of the O–X mode conversion scheme in Wendelstein 7-AS (W7-AS) [5], the problem was examined in a number of papers [6–12] within the 2D plasma model and in [13] within the 3D plasma model by analysis of a set of the reduced wave equations valid in the vicinity of the O–X mode conversion layer. In all these papers the curvature of the magnetic flux surfaces was neglected under an assumption of high localization of the conversion region and magnetic field was considered to be shearless with the straight magnetic field lines. No effort was mounted to describe the O–X mode conversion in the more realistic case of the 2D inhomogeneous plasma in a sheared magnetic field except possibly in paper [14], where the effect of the shear on the efficiency of the O–X mode conversion was studied within the 1D inhomogeneous slab model of plasma. In this work we are concerned with the analysis of the O–X mode conversion problem for the 2D inhomogeneous plasma with the sheared magnetic field.
2. Basic equations

In this paper we consider plasmas with 2D inhomogeneity that assumes the surfaces \( n = \text{const} \) and \( B = \text{const} \) being not approximately parallel and the sheared magnetic field. Assuming high localization of the conversion layer, we neglect the curvature of the magnetic flux surfaces and treat a problem in the simplest slab geometry accounting for plasma axisymmetry, expecting that the results obtained within this simple model hold in toroidal plasmas.

Let us start with the introduction of a Cartesian coordinate system \((x, y, z)\) with its origin located at the O-mode cut-off surface and coinciding with the center of an incident beam of the monochromatic ordinary waves. The coordinates \( x, y \) and \( z \) being scaled in \( c/\omega \), where \( \omega \) is a frequency of the wave and \( c \) stands for the speed of light, imitate the flux function, the poloidal angle and the toroidal angle, respectively. The magnetic field in these Cartesian coordinates in the conversion region can be considered to be the sum of two parts

\[
\mathbf{B} = B_y \mathbf{e}_y + B_z \mathbf{e}_z.
\]

The component \( B_y \) produced by a sheet current in the \( z \)-direction is a function of \( x \), \( B_y = B_y(x) \), and \( B_z \), which is due to external currents, depends on the coordinate \( R \) (see figure 1(a)) or is a function of \( x \) and \( y \), \( B_z = B_z(x, y) \). Being defined in such a way \( \mathbf{B} \) obeys the constraint \( \mathbf{\nabla} \cdot \mathbf{B} = 0 \) automatically. In this study we assume an artificial current profile for which \( B_y \) is zero at the O-mode cut-off surface. The magnetic field is shown in figure 1(b) and is directed along the \( z \)-direction at \( x = 0 \) and inclined towards the \( y \)-direction at \( x \neq 0 \). This model of the magnetic field allows us to simplify the analysis clarifying the role of the magnetic field shear in the 2D O–X mode conversion problem. Using the Taylor expansion of the magnetic field components at the origin of the coordinate system \((x, y, z)\), we represent \( B_y \) and \( B_z \) as

\[
B_y \simeq B_0 G x, \quad B_z \simeq B_0 (1 + x/L_{bx} + y/L_{by}),
\]

where \( B_0 = B_0|_{x,y=0} = \text{const} \), \( G = \partial_x B_y|_{x,y=0}/B_0 = \text{const} \), \( L_{bx}^{-1} = \partial_x B_z|_{x,y=0}/B_0 = \text{const} \), \( L_{by}^{-1} = \partial_y B_z|_{x,y=0}/B_0 = \text{const} \), \( \partial_x = \partial/\partial x \), \( \partial_y = \partial/\partial y \). In the case of the circular magnetic surfaces (see figure 1(a)) we can estimate these parameters as \( G \sim q/R_0 \ll 1 \), where \( q = R_0/B_0 \cdot \partial_x B_y|_{x,y=0} \sim 1 \), \( R_0 \) being the major radius of a tokamak scaled in \( c/\omega \) is \( \gg 1 \), \( L_{bx}^{-1} \sim \cos(\theta_0)/R_0 \ll 1 \) and \( L_{by}^{-1} \sim \sin(\theta_0)/R_0 \ll 1 \), where \( \theta_0 \) is a poloidal angle of the incident beam at \( x = 0 \). We then use the Taylor expansion procedure for the density at the origin of the coordinate system \((x, y, z)\)

\[
n(x) \simeq n_0 (1 + x/L_n),
\]

Figure 1. (a) Cut-off surfaces \( \varepsilon_{\parallel} = 0 \) and \( \varepsilon_{\perp} = n_{\text{res}}^2 = (1 + \omega/\omega_{ce}|x,y=0)^{-1} \), in a poloidal cross-section of a tokamak. The ordinary wave incidents on the region of the O–X mode conversion. The local Cartesian coordinate system \((x, y, z)\) is shown; (b) a slab model representation of the sheared magnetic field, \( \varepsilon_{\parallel} \mathbf{e}_z \) at \( x = 0 \), \( \varepsilon_{\parallel} \) is not parallel to \( \mathbf{e}_z \) at \( x \neq 0 \).
The model) depends on the magnetic field line and the modulus of magnetic field, $\epsilon$, while all other terms are of order unity. Thus, an angle between the surface $\epsilon = 0$ and the surface $\epsilon - n_{res}^2 = 0$. An evanescent layer is shaded.

At the same approximation the modulus of magnetic field, $B = |\mathbf{B}|$, is a function of $x$ and $y$

$$B(x, y) \simeq B_0(1 + x/L_{bx} + y/L_{by}) + O(\epsilon^2).$$

As the magnetic field lines are straight at first approximation in the poloidal cross-section $\theta(x) = \arctan (B_y/B_x) \simeq Gx \sim O(\epsilon)$.

An expansion of the components of the cold dielectric tensor

$$\varepsilon_s(x, y) = 1 - \frac{\omega_{pe}^2(x)}{\omega(\omega + \omega_{ce}(x, y))}, \quad \varepsilon_s(x) = 1 - \frac{\omega_{pe}^2(x)}{\omega^2},$$

where $\omega_{pe}$ is the electron Langmuir frequency, at the origin of the coordinate system $(x, y, z)$ in powers of $\epsilon$ gives

$$\varepsilon_s \simeq \varepsilon_s|_{x,y=0} + \partial_x \varepsilon_s|_{x,y=0} \cdot x + \partial_y \varepsilon_s|_{x,y=0} \cdot y + O(\epsilon^2)$$

$$= n_{res}^2 - |\nabla \varepsilon_s| (\cos \varphi \cdot x - \sin \varphi \cdot y) + O(\epsilon^2),$$

where $\varepsilon_s|_{x,y=0} = n_{res}^2 \equiv 1/(1 + \omega/\omega_{ce}(x,y)), \varepsilon_s|_{x=0} = 0, |\nabla \varepsilon_s| = ((\partial_x \varepsilon_s)^2 + (\partial_y \varepsilon_s)^2)^{1/2}$, $s = (+, |)$ and $\varphi = \arctan (\partial_x \varepsilon_s/\partial_y \varepsilon_s)$ coincides with an angle between two surfaces $\varepsilon_s - n_{res}^2 = 0$ and $\varepsilon_s = 0$ in the poloidal cross-section at $x = 0$ as shown in figure 2(b).

We first note that the WKB approximation is applicable for description of the interacting waves except a narrow localized conversion layer in the vicinity of the O-mode cut-off surface.
where the full-wave analysis connecting the incoming and outgoing WKB waves is needed. The WKB approach can give us not only the boundary conditions but also some hints on how to reduce the full-wave equations in the conversion layer. The dispersion curves of the O and X modes are adequately described by the cold plasma model. These curves are coupled if the local wave-vector $\mathbf{n}$ with components $n_x = n \cdot \mathbf{e}_x$, $n_y = n \cdot (\mathbf{e}_y \times \mathbf{e}_z)$, $n_z = n \cdot \mathbf{e}_z$ has at the O-mode cut-off surface the only parallel component equal to $n_{\text{res}} = (1 + \omega/\omega_{ce})^{-1/2}$ where $\omega_{ce}$ is the electron cyclotron frequency at $x = 0$, $y = 0$. In other words in this case the perfect (with 100% efficiency) O–X mode conversion occurs. Because $\mathbf{e}_\parallel \parallel \mathbf{e}_z$ and $n_\parallel = n_z$ at the O-mode cut-off surface ($x = 0$), we seek the electric field of the incident wave, $E_m$, within the O–X mode conversion layer as

$$E_m = E(x, y, z) \exp(i n_{\text{res}} z),$$

where $E(x, y, z)$ is an amplitude varying slowly. This means that the terms proportional to $n_{\text{res}}$ are the zero order in $\epsilon$, $\partial_\parallel E$, $\partial_\parallel E$ and $\partial_\parallel E$ are the first order in $\epsilon$ and $(\partial_{x,x}, \partial_{x,z}, \partial_{y,y}, \partial_{y,z}) \cdot E$ are the second order in $\epsilon$. The derivatives $\partial_\parallel$ and $\partial_\perp$ acting on $E_m$ give $\partial_\parallel E_m \approx \exp(i n_{\text{res}} z)(i \partial_{x,x} E + \partial_{x,z} E + O(\epsilon^2))$ and $\partial_\perp E_m \approx \exp(i n_{\text{res}} z)(\partial_{y,y} E + \partial_{y,z} E + O(\epsilon^2))$. It should be mentioned here that the second term in the expression of $\partial_\parallel E_m$ appears due to the magnetic field shear.

Then, we substitute the dielectric tensor components and the derivatives $\partial_\parallel$ and $\partial_\perp$ after the above expansion procedures over $\epsilon$ into the system of Maxwell’s equations for $E_m$ and obtain the following set of equations for the electric field components $E = (E_x, E_y, E_z)$:

$$n_{\text{res}} D_x E_x - i(\nabla \times E)(x \cos \varphi - y \sin \varphi) - 2i n_{\text{res}} \partial_z E_x = 0,$$

$$i2|\nabla E_x| x E_z - n_{\text{res}} D_z E_x = 0,$$

$$E_\perp = 0,$$

where $E_\parallel = E_x \pm i E_y$, $D_\parallel = \partial_\parallel \pm n_{\text{res}} G x$, $D_\perp = \partial_\perp \pm i \partial_z$. As the terms $\sim \epsilon^0$ are canceled, all the terms constituting equation (1) are of the order $\epsilon$. In deriving (1) we have in mind that these equations are stated for $E_x = E_x + i E_y$, $E_y = E \cdot (\mathbf{e}_y \times \mathbf{e}_z)$ and $E_z = E \cdot \mathbf{e}_z$, which are closely equal to $E_x + i E_y$ and $E_z$ in the conversion layer (note, $E_\parallel \equiv E_x$ and $E_\perp \equiv E_z$ at $x = 0$). As the only first-order terms in $\epsilon$ have been kept in (1), we neglected the differences between $E_\parallel$ and $E_\perp$, $E_x$ and $E_z$, resulting in the effects being higher order of vanishing ($\sim \epsilon^2$). This is our basic system. It contains the term $n_{\text{res}} G x$ originated due to the magnetic field shear. This term was missed in the system of equations studied in [9]. As the operators $D_\parallel$ and $D_\perp$ are non-commuting, $D_\parallel D_\perp \neq D_\perp D_\parallel$, (1) lost symmetry inherent in the case of the magnetic field with zero shear $G = 0$ (in this case $\partial_\parallel \partial_{\perp} \equiv \partial_{\perp} \partial_\parallel$). This was at first disturbing by virtue of the fact that no separation of variables and no existence of eigenvalues are permitted in the case of the sheared magnetic field. Fortunately, this initial impression is misleading as will be shown in the forthcoming sections. With the use of the procedure proposed in [8] we will obtain the second-order partial differential equation instead of system (1). Then, applying an integral representation of the Laplace integral type as in [9] of unknown function, we will separate variables and obtain eigenvalues and eigenfunctions of this partial differential equation.

3. Analysis of system (1)

Following a procedure proposed in [8] opens a way of overcoming underlying mathematical difficulties and permits us to separate variables in (1). As we assume an axisymmetric plasma, we can seek a solution of (1) in the form $\sim \exp(i n_z z)$, where $n_z = \text{const}$. Introducing new
where Cairns and Lashmore-Davies [14]. To separate variables in (6) we introduce that yields

\[ \tilde{x}, \tilde{y} = x, (y - 2n_{res}n_z/\sin \varphi/|\nabla \varphi|) \cdot (2|\nabla \varphi|/|\nabla \varphi|)|n_{res}^{1/2} \]

we read (1) as

\[ iD_xa + (\tilde{x}\cos \varphi - \tilde{y}\sin \varphi)ib = 0, \]
\[ iD_yb + \tilde{x} \cdot a = 0, \]

where \( D_k = \partial_k \pm g\tilde{x} \) and \( \partial_k = \partial/\partial x \pm i\beta/\partial \tilde{y} \). In what follows we omit the tilde above \( x \) and \( y \). Let us use a transform of \( a(x, y) \rightarrow a(x, q) = L[a(x, y)] \) and \( b(x, y) \rightarrow b(x, q) = L[b(x, y)] \) where

\[ \hat{L}[\cdot] = \exp \left( -i\frac{g}{\sin \varphi}q^2 \right) \int_{-\infty}^{\infty} dy \exp \left( -iqy - i\frac{\sin \varphi}{4g}y^2 \right)[\cdot] \]

We introduce \( k = -2g/\sin \varphi \cdot q \) and seek \( a(x, k), b(x, k) \) in the form

\[ a(x, k) = \hat{\partial}_k - f, \]
\[ b(x, k) = \hat{\beta} - \hat{\partial}_k + (g\beta + i)x \cdot f, \]

where \( \hat{\partial}_k = \partial_k \pm i\beta_k, \beta = (\xi - ig) \exp(\varphi), \xi = (\cos \varphi - g^2 - i\sin \varphi)^{1/2} = |\xi| \exp(-i\psi), |\xi| = (1 - 2g^2 \cos \varphi + g^4)^{1/2}, 2\psi = \arctan[\sin \varphi/(\cos \varphi - g^2)]. \) For more details of the procedure developed in [8] and permitting us to derive (4) and (5), see appendix A. Using (4) and (5) we derived a partial differential equation of second order for a function \( f \) instead of system (3) of two partial differential equations of first order for \( a \) and \( b \)

\[ \hat{\partial}_k^2f + \hat{\beta}_k^2f + ((\cos \varphi - g^2)x^2 - \sin \varphi \cdot xk + i|\xi| \exp(-i\psi))f = 0. \]

This equation deserves a few comments. For zero shear, \( g = 0, (6) \) coincides with the equation analyzed in [8]. For the sheared magnetic field we estimate the parameter \( g \) as \( g \sim L_n/L_0 \ll 1 \), where \( L_n \) is a scale on which the density profile at the O-mode cut-off surface changes and \( L_0 \) is a minor radius. Henceforth, we keep in mind that the coefficient \( \cos \varphi - g^2 \) at \( x^2 \) in (6) is positive. At \( \varphi = 0 \) we return to the equation treated within the 1D O–X mode conversion problem in the sheared magnetic field. The first to analyze this case by the WKB analysis were Cairns and Lashmore-Davies [14]. To separate variables in (6) we introduce

\[ u = \sqrt{|\xi|}(\cos \varphi \cdot x - \sin \varphi \cdot k), \]
\[ v = \sqrt{|\xi|}(\sin \varphi \cdot x + \cos \varphi \cdot k) \]

that yields

\[ \hat{\partial}_k^2f + \hat{\beta}_k^2f + (\cos \varphi^2u^2 - \sin \varphi^2v^2 + i \exp(-i\psi))f = 0. \]

A particular solution of (7) is

\[ f(u, v) = \sum_{p=0}^{\infty} c_p D_{ivp/\pi}(u)\phi_p(v), \]

where \( v_p = \pi |\tan \varphi(p + 1 - \text{sign}(\varphi))/2, D_{ivp/\pi}(u) = D_{ivp/\pi}(\sqrt{2} \cos \varphi \exp(\text{i}\varphi/4)u) \) is the parabolic cylinder function [15], \( \phi_p(v) = \phi_p(\sqrt{\sin |\varphi|}v) \) are the Hermitian polynomials

\[ \phi_p(v) = (2^p\sqrt{\pi}p!)^{-1/2} \exp(-v^2/2)H_p(v) \]
possessing a property \( \int \phi_p(v) \phi_k(v) \, dv = \delta_{pk} \). Substituting (8) in (5), we obtain
\[
a(x,k) = (\exp(i\psi)|\xi| + ig) \cdot R - (\exp(i\psi)|\xi| - ig) \cdot I,
\]
where
\[
I = \sqrt{|\xi|} \sum_{p=0}^{\infty} c_p D_{\gamma_p/\pi}(u) (\partial_v + \sin \psi \cdot v) \phi_p(v),
\]
\[
R = \sqrt{|\xi|} \sum_{p=0}^{\infty} c_p \phi_p(v) (-i\partial_u + \cos \psi \cdot u) D_{\gamma_{p-1}/\pi}(u).
\]
Using properties of \( D_{\gamma_p/\pi}(u) \), \( \partial_u D_{\gamma_p/\pi}(u) \), \( \phi_p(v) \) and \( \partial_v \phi_p(v) \) [15], we can represent (10) and (11) as
\[
I = \sqrt{\sin|\psi|} \sum_{p=0}^{\infty} C_p D_{\gamma_{p+1}/\pi}(v) \phi_p(v),
\]
\[
R = \sqrt{i \cos|\psi|} \sum_{p=0}^{\infty} C_p D_{\gamma_{p-1}/\pi}(v) \Lambda_{p-1}(v),
\]
where
\[
\gamma_p = \pi \sec \psi(p + \frac{1}{2}(1 + \text{sign}(\psi))).
\]
The term \( I \) contains a sum of eigenmodes \( I = \sum_p I_p \) localized in the \( v \)-direction and propagating in the positive direction along \( u \):
\[
I_p \sim \exp(3\gamma_p/4 - i\cos \psi/2 \cdot u^2 + i\gamma_p/\pi \ln(\sqrt{2 \cos \psi u})),
\]
at \( u \to -\infty \),
\[
I_p \sim \exp(-\gamma_p/4 - i\cos \psi/2 \cdot u^2 + i\gamma_p/\pi \ln(\sqrt{2 \cos \psi u})),
\]
at \( u \to \infty \) [15]. Therefore, each of these terms \( I_p \) corresponds to the incident wave at \( u \to -\infty \) and to the converted wave at \( u \to \infty \). The term \( R \) also contains the infinite sum of eigenmodes \( R = \sum_p R_p \) localized in the \( v \)-direction and propagating in the negative direction along \( u \):
\[
R_p \sim \frac{\sqrt{2\pi}}{\Gamma(1 - i\gamma_p/\pi)} \exp(\gamma_p/4 + i\cos \psi/2 \cdot u^2 - i\gamma_p/\pi \ln(\sqrt{2 \cos \psi u})),
\]
at \( u \to -\infty \) and
\[
R_p \sim O(1/|u|),
\]
at \( u \to \infty \), with \( \Gamma \) being the Gamma function. Thus, the terms proportional to \( D_{\gamma_{p-1}/\pi}(u) \) correspond to the reflected waves. The coefficients \( C_p \) in (12) can be chosen in such a way as to fit the beam of the incident WKB ordinary waves, \( A(x, y) \exp(i\psi, \xi) \), in the WKB region.

As far as (12) is an expansion in orthogonal functions \( \phi_p \), we can define \( C_p \) as follows:
\[
A(x, q) = \hat{L}(A(x, y)),
\]
\[
C_p = \frac{1}{D_{\gamma_p/\pi}(u_b)} \int_{-\infty}^{\infty} dv \cdot \phi_p(v) A(x, q)|_{u_b},
\]
where \( u_b \) is an arbitrary point, which tends to \( -\infty \). As \( A(u_b, v), q(u_b, v)|_{u_b \to -\infty} \sim \exp(-i\cos \psi/2 \cdot v^2)A(v) \) (see appendix B) and \( D_{\gamma_p/\pi}|_{u_b \to -\infty} \sim \exp(-i\cos \psi/2 \cdot u_b^2) \), \( C_p \) is independent of \( u_b \).
Figure 3. The conversion coefficient $T_p$ (16) corresponding to a separate $p$ term in the sum (12) versus the parameter of the magnetic field shear $g$ at $\varphi = +30^\circ$.

4. Conversion coefficients

Since $I$ corresponds to the incident wave at $u < 0$ and to the converted wave at $u > 0$ we can derive the conversion coefficient for the O–X mode conversion using asymptotic expression of $D_{\varphi_p/\pi}(u)$ [15] entering (12). Let us introduce the conversion coefficient ($T_p$) corresponding to a separate $p$ term in the sum (12) over the Hermitian polynomials

$$T_p = \exp\left(-2\pi(p + \mu)\tan|\psi|\right), \quad \mu = 1 \text{ at } \varphi > 0, \quad \mu = 0 \text{ at } \varphi < 0,$$

(16)

$$\tan|\psi| = \left(\sqrt{1 + \sin\varphi^2/\left(\cos\varphi - g^2\right)^2} - 1\right)^{1/2}.$$

As in the case of the magnetic field with zero shear, the conversion coefficient for the fixed $p$ mode (16) has asymmetry, considered first in [7], with respect to the sign of the angle $\varphi$ between two surfaces $\epsilon_\parallel = 0$ and $\epsilon_\perp - n_{\text{res}} = 0$ ($\mu = 1$ at $\varphi > 0$, $\mu = 0$ at $\varphi < 0$). For the fixed $n_z$, the conversion coefficient $T$ for the O–X mode conversion can be expressed as

$$T = \frac{\sum_{p=0}^{\infty} T_p|C_p|^2}{\sum_{p=0}^{\infty}|C_p|^2}. \quad (17)$$

We illustrate this by an example. Let us assume that $\varphi < 0$ and the wave structure $A(x, y)$ is defined in the vicinity of the O–X mode conversion layer by the following expression:

$$A(x, y) = \exp\left(\frac{-i\sin\varphi}{2g} \left(\frac{y - \tan\psi \cdot x}{2} - i\frac{|\xi|}{\cos\psi} \frac{x^2}{2}\right)\right) \times \exp\left(-\frac{(|\sin\psi|/2g) \cdot (y - \tan\psi \cdot x) - |\xi| |\sin\psi| \cdot x^2}{2(\lambda_x^2 + i\lambda_z^2)}\right),$$

where $\lambda_x^2 = |\xi| \sin\psi (\cos\psi)^2$ and $\lambda_z^2 = |\xi| \cos\psi (\sin\psi)^2 - |\sin\varphi|/(2g)$ and $|\xi|, \psi$ are defined after equation (5). We note that this structure has a form of two-dimensional Gaussian distribution of the general type with astigmatism and phase modulation. Using (15)–(17) gives $T = T_0$. The above distribution of $A(x, y)$ at $g = 0$ is familiar as an optimal beam (see [7–12]) for which the O–X mode conversion is perfect, $T_0 = 1$ at $g = 0, \varphi < 0$, and the incident beam
of the O-waves experiences no reflection from the O–X mode conversion layer. It is not the case for an arbitrary \( g \neq 0 \), as illustrated in figure 3.

For the magnetic field with zero shear \( g = 0 \) transform (4) yields \( A(x, y) = \hat{L}[A(x, y)] \) and the conversion coefficient of the fixed mode \( p \) is performed as in [8]

\[
T_p = \exp \left(-2\pi \tan \left(\frac{\phi}{2} (p + \mu)\right)\right).
\]

At \( \phi = 0 \) with the use of Mehler’s formula [15] we return to the conversion coefficient

\[
T = \frac{\int dq |\tilde{A}(q)|^2 \exp \left(-2\pi q^2/(1 - g^2)^{3/2}\right)}{\int dq |\tilde{A}(q)|^2},
\]

(18)

\[
\tilde{A}(q) = \int dy \cdot \tilde{A}(y) \exp (-iqy)
\]

derived first by Cairns and Lashmore–Davies in [14] by the WKB analysis. We recall that \( q \) and \( g \) are scaled according to (2) and \( n_z = 0 \) is assumed. Analyzing (4), (9), (12) and (14) we can summarize that the magnetic field shear degrades the efficiency of the O–X mode conversion. This degradation is illustrated by figure 3 where the partial conversion coefficient \( T_p \) versus \( g \) is given. An important feature is an increase in the degradation with increasing mode number. Although the parameter \( g \) for the typical experimental conditions is small, \( g \ll 1 \), we can expect a remarkable effect of the magnetic field shear on the efficiency of the O–X mode conversion for the modes with \( p \gg 1 \). For example, under conditions of electron Bernstein waves (EBWs) heating experiment in MAST \( (\omega/(2\pi) = 60 \text{ GHz}, \omega^2/\omega_{pe}^2|_{x=x_{cut-off}} \approx 25, G = 0.5 \text{ m}^{-1}, \text{L-mode, } L_n = 0.05 \text{ m}) \): \( g \approx G \cdot L_n \omega^2/\omega_{pe}^2|_{x=x_{cut-off}} = 0.4 \). For the Globus-M tokamak \( (\omega/(2\pi) = 20 \text{ GHz}, \omega^2/\omega_{pe}^2|_{x=x_{cut-off}} \approx 0.3, G = 1.3 \text{ m}^{-1}, \text{L-mode, } L_n = 0.1 \text{ m}) \): \( g \approx G \cdot L_n \omega^2/\omega_{pe}^2|_{x=x_{cut-off}} = 0.04 \). This means the effect of the magnetic field shear affects the efficiency of the O–X mode conversion rather more in MAST (where \( \omega^2/\omega_{pe}^2|_{x=x_{cut-off}} \gg 1 \)) than in Globus-M tokamak.

5. Conclusion

The paper offers a new insight into the 2D O–X mode conversion problem by considering plasmas confined by the sheared magnetic field. Ignoring the curvature of the magnetic flux surfaces under an assumption of high localization of the conversion region and assuming the straight magnetic field lines we have studied the problem in the simplest slab geometry accounting for plasma axisymmetry. Nevertheless, the results being obtained are expected to hold in more realistic toroidal plasmas. The resulting expressions for the electric field components (9), (12) and (13) have been derived that permit finding the conversion coefficient (16) explicitly. As has been demonstrated, the magnetic field shear degrades the efficiency of the O–X mode conversion. For the usual experimental setup this degradation appears to be remarkable. Finally, extending the model to the case when the magnetic field direction is not parallel to the toroidal direction at the O-mode cut-off surface, which is more relevant to tokamak physics, needs to be done. This issue will be studied in a future paper.

Acknowledgments

This work was supported by the RFBR (grant 10-02-00887, 07-02-92162-CNRS, 09-02-00453).
Appendix A. Derivation of the differential equation (6)

In the main text we considered the set of equations (3) describing the O–X mode conversion in the plasma with the sheared magnetic field. Here, we show that this set of partial differential equations is equivalent to a second-order differential equation, which we found using a procedure developed in [8]. We seek $a(x, y)$ and $b(x, y)$ in the form

$$a(x, y) = \exp(\sigma) \delta_+ F,$$

$$b(x, y) = \beta \exp(\sigma) \delta_- F,$$

where $F$ is an unknown function, $\sigma = \sigma_{xx} x^2/2 + \sigma_{xy} x y + \sigma_{yy} y^2/2$, $\sigma_{xx}$, $\sigma_{xy}$, $\sigma_{yy}$ and $\beta$ are coefficients to be chosen. Substituting (A.1) into (3) yields

$$\partial_+ \partial_- F + \partial_- \partial_+ F - g x \partial_- F - i \beta (x \cos \varphi - y \sin \varphi) \partial_+ F = 0,$$

$$\partial_+ \partial_- F + \partial_- \partial_+ F + g x \partial_- F - i x / \beta \cdot \partial_- F = 0.$$  

The equations constituting (A.2) are identical if the coefficients at $\partial_+ F$ and $\partial_- F$ in each of them are equal.

$$\partial_- F + g x = -i \beta (x \cos \varphi - y \sin \varphi),$$

$$\partial_+ F - g x = -i x / \beta.$$  

The system of ordinary equations has a solution:

$$\sigma_{xx} = -i / (2 \beta) - i \beta \cos \varphi / 2,$$

$$\sigma_{xy} = i \beta \sin \varphi / 2,$$

$$\sigma_{yy} = -\beta \sin \varphi / 2,$$

$$\beta = (\xi - i g) \exp(i \psi),$$

where $\xi = (\cos \varphi - g^2 - i \sin \varphi)^{1/2}$. Upon introducing $F = \exp(-\sigma) \cdot f$ and substituting (A.1) with (A.4) into any one of (A.2), we obtain the second-order partial differential equation for $f$ instead of system (3) of two partial differential equations of the first order for $a$ and $b$

$$\partial_+^2 f + \partial_-^2 f - i 2 g x \partial_- f + ((\cos \varphi - g^2) x^2 - \sin \varphi x y + i |\xi| \exp(-i \psi)) f = 0,$$  

where

$$|\xi| = (1 - 2 g^2 \cos \varphi + g^4)^{1/4} = \text{const},$$

$$2 \psi = \arctan \left[ \sin \varphi / (\cos \varphi - g^2) \right] = \text{const}.$$  

At $g = 0$ the expression for $2 \psi$ reduces to the expression derived in [9] $2 \psi = \varphi$ (or using the notation of [9] $2 \psi = 2 \arctan(L_b/L_n - \sqrt{1 + L_n^2/L_b^2})$). We begin by considering a solution of (A.5) in the form

$$f(x, y) = f_1(x, y) \cdot \exp \left( \frac{i \sin \varphi}{4 g} y^2 \right),$$

where $f_1(x, y)$ is to be found. Then, we make the Laplace transform of $f_1(x, y)$:

$$f_1(x, y) = \int_C \frac{dq}{2\pi} f_1(x, q) \exp(i q y).$$

The path of integration in the $q$ plane is such that the integrand vanishes rapidly at the ends of the contour or at infinity [16]. As far as $f_{1 | q \to \pm \infty} \sim \exp(-\text{const} \cdot q^2)$ (see (12) and (13)) and the integrand has both no branching points and no poles, $C$ can be chosen along the real axis of $q$. Finally, we use a substitution

$$f_1(x, q) = f(x, q) \cdot \exp \left( \frac{q}{\sin \varphi} \frac{y^2}{4 g} \right).$$
This sequence of mathematical operations is revealed in the following transform of $f(x, q)$:

$$f(x, q) = \hat{L}[f(x, y)], (f(x, y) = \hat{L}^*[f(x, q)],$$

where

$$\hat{L}[\cdot] = \exp\left(-i\frac{g}{\sin \varphi} q^2\right) \int_{-\infty}^{\infty} dy \exp\left(-iqy - i\frac{\sin \varphi}{4g} y^2\right) [\cdot]. \quad (A.10)$$

$$\hat{L}^*[\cdot] = \exp\left(i\frac{\sin \varphi}{4g} y^2\right) \int_{-\infty}^{\infty} dq \exp\left(iqy + i\frac{g}{\sin \varphi} q^2\right) [\cdot].$$

As shown in appendix B, where the ray representation of the wave field is given, transform (A.10) is intrinsic to the class of the partial differential equations such as (A.5). Substituting $f(x, y) = \hat{L}^*[f(x, q)]$ to (A.1) and (A.5), we finally get

$$a(x, k) = \hat{a}_+ f = ((g - i\beta \cos \varphi)x + i\beta \sin \varphi k + 2\beta g \hat{\partial}_k) \cdot f, \quad (B.1)$$

$$b(x, q) = \hat{b}_+ f = (g\beta + i)x \cdot f,$$

where $a(x, k) = \hat{L}[a(x, y)], b(x, q) = \hat{L}[b(x, y)], \hat{\partial}_k = \partial_x \pm i \partial_y, k = -2g/\sin \varphi \cdot q$, and

$$\partial_x^2 f + \partial_k^2 f + ((\cos \varphi - g^2)x^2 - \sin \varphi \cdot xk + i|\xi| \exp(-i\psi)) f = 0 \quad (A.12)$$

allowing separation of the variables.

**Appendix B. Ray Hamiltonian dynamics**

In this section we analyze the ray representation of the wave field

$$a_1(x, y) = a(x, y) \exp(-i\sin \varphi/(4g) \cdot y^2), \quad (B.1)$$

$$b_1(x, y) = b(x, y) \exp(-i\sin \varphi/(4g) \cdot y^2)$$

in four dimensional phase space $\mathbf{r} = (x, y), \mathbf{n} = (\kappa, q)$, which is governed by Hamilton’s equations:

$$\frac{dr}{ds} = -\frac{\partial D}{\partial n} \left| \frac{\partial D}{\partial n} \right|^{-1}, \quad \frac{dn}{ds} = \frac{\partial D}{\partial r} \left| \frac{\partial D}{\partial n} \right|^{-1}, \quad (B.2)$$

where

$$D = \kappa^2 + (q - gx + \sin \varphi/2g) y^2 - (x^2 \cos \varphi - xy \sin \varphi) = 0 \quad (B.3)$$

is the local dispersion relation for (3) with (B.1) and $s$ denotes the orbit parameter. Introducing the generalized coordinates $x, Y = \sin \varphi/(2g) \cdot (y + 2g/\sin \varphi \cdot q)$, $\kappa$ and $k$ (we recall here that $k = -2g/\sin \varphi \cdot q$), we can represent (B.3) as

$$D = \kappa^2 + Y^2 - x^2 (\cos \varphi - g^2) + xk \sin \varphi = 0. \quad (B.4)$$

Note that the generalized coordinate $Y$ is intrinsic to (B.3). As a consequence of this, an additional phase $g/\sin \varphi \cdot q^2$ in (A.9) appears in the full-wave analysis. Since the general picture of the ray behavior in the two-dimensional subspace $x - k$ is of interest, we reparametrize $s \rightarrow \tau$, where $d\tau = ds|\partial D/\partial n|^{-1}$. Substituting (B.4) to (B.2) gives

$$\frac{dx}{d\tau} = -2\kappa, \quad \frac{dk}{d\tau} = -2x(\cos \varphi - g^2) + k \sin \varphi, \quad \frac{dY}{d\tau} = -x \sin \varphi, \quad \frac{dk}{d\tau} = 2Y.$$  

Ray equations with the Hamiltonian function (B.4) are equivalent to two linear second-order ordinary differential equations

$$\frac{d^2x}{d\tau^2} = 4x(\cos \varphi - g^2) - 2k \sin \varphi, \quad (B.5)$$

$$\frac{d^2k}{d\tau^2} = -2x \sin \varphi.$"
The equations constituting (B.5) are coupled with \( \sin \phi \) being the coupled coefficient. One can introduce the normal coordinates as in (10) that reduces (B.5) to
\[
\frac{d^2 u}{d\tau^2} = 4 \cos \psi \cdot u, \quad (B.6)
\]
\[
\frac{d^2 v}{d\tau^2} = -4 \sin \psi \cdot v.
\]
We can write immediately their solutions as
\[
u \sim C_1 u \exp (\nu \tau) + C_2 u \exp (-\nu \tau), \quad (B.7)
\]
\[
v \sim C_1 v \exp (i\omega \tau) + C_2 v \exp (-i\omega \tau),
\]
where \( C_{st} = \text{const}, s = (1, 2), t = (u, v) \) and \( \nu = 2 |\cos \psi|^{1/2}, \omega = 2 |\sin \psi|^{1/2} \). Expressions (B.7), which show the projections of Hamiltonian rays on the \( u, v \) plane (or \( x, q \) plane), deserve a few comments. We can note the oscillatory behavior along the \( v \)-direction, while the \( u \)-motion displays the influence of a retarding force. The full-wave solution corresponding to this kind of ray behavior is a product of the function \( \tilde{A}(v) \), confined in the \( v \)-direction and presented as a sum of the eigenmodes, namely, the Hermitian polynomials, and a superposition of the linear independent parabolic cylinder functions (see [17]). Thus, we can represent the WKB beam of the ordinary waves entering the O–X mode conversion layer, where the expansion procedure of the components of the dielectric tensor is even valid, in the form
\[
(a, b) \simeq \exp \left( \frac{i}{4g} \cdot qy \right) \int dq \exp \left( i \cdot \frac{g \cos \psi}{2} \cdot u^2 + iqy \right) \tilde{A}(v), \quad (B.8)
\]
where \( a = u(x, q), v = v(x, q) \) and \( \tilde{A}(v) \) is a distribution of the beam in the \( v \)-direction.

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