BFV–BRST Analysis of the Classical and Quantum 
$q$-deformations of the $sl(2)$ Algebra

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Abstract

BFV–BRST charge for $q$-deformed algebras is not unique. Different constructions of it in the classical as well as in the quantum phase space for the $q$-deformed algebra $sl_q(2)$ are discussed. Moreover, deformation of the phase space without deforming the generators of $sl(2)$ is considered. $h$-$q$-deformation of the phase space is shown to yield the Witten’s second deformation. To study the BFV–BRST cohomology problem when both the quantum phase space and the group are deformed, a two parameter deformation of $sl(2)$ is proposed, and its BFV-BRST charge is given.
1 Introduction

Gauge symmetries of a lagrangian manifest themselves as first-class constraints in the hamiltonian framework. These constraints can be employed in constructing a fermionic, nilpotent operator, known as Batalin–Fradkin–Vilkovisky–Becchi–Rouet–Stora–Tyutin (BFV–BRST) charge [1], after quantizing the related phase space and introducing ghost variables (fields). Although ghost variables are an artifact of quantization procedure, they can be incorporated into classical mechanics by endowing classical phase space with the generalized Poisson brackets. Hence it appears that one can establish a BFV–BRST charge and study its cohomology either in quantum or in classical framework [2]. In constrained Hamiltonian systems BFV–BRST cohomology classes are equated with physical states. Therefore one can utilize BFV–BRST charge to define a gauge theory.

First class constraints are in involution on the constraint surface, and in Yang–Mills theories they constitute a Lie algebra after quantization (in the absence of anomalies). This suggests to study the $q$-deformed algebras [3]–[7] in a similar manner, to elucidate their structure and to extract some clues useful in formulating $q$-deformed gauge theories. Obviously, quantization ($\hbar$-deformation) and $q$-deformations are distinct [8],[9]. Hence, BFV–BRST analysis of $q$-deformations can be accomplished either in classical or quantum framework.

There are some attempts to formulate a $q$-deformed gauge theory [8],[10], but a complete understanding is lacking. The study of the BFV–BRST structure of $q$-deformed systems in terms of quantum as well as classical mechanics can facilitate construction of the desired gauge theories.

When we deal with the Lie algebras or with the usual constrained systems, construction of BFV–BRST charge is unique up to canonical transformations. For $q$-deformed algebras it depends on the differential calculus adopted over the group or on the behaviour of constraints. In Refs.[11]–[12] two different possibilities are considered.

Properties of the $\hbar$-$q$-deformed algebras are well established, but to investigate the classical case (i.e. only $q$-deformation) there are some different methods [8],[11],[13],[14]. A powerful way of defining quantum mechanics as $\hbar$-deformation of the classical one is to utilize the Moyal $\star$ product [13]. Then $\star$ product can be used to acquire $q$-deformations of classical phase space. $\star$ product also establishes the distinction of $\hbar$- and $q$-deformations obviously.
Moreover, it is useful in introducing multiparameter deformations. In fact, in Section 2 this construction of “q-classical mechanics” is used to obtain a BFV–BRST charge for \( sl(2) \) generators. Moreover, there we deal with the phase space endowed with the usual Poisson brackets but a \( q \)-deformed “classical \( sl(2) \)” algebra.

In Section 3 we perform \( \hbar \)-deformation of the cases studied in Section 2. It is shown that a realization of BFV–BRST charge for \( U_q(SU(2)) \) yields a formulation of BRST cohomology similar to the usual case [10],[17]. When the phase space is \( \hbar-q \)-deformed the usual \( sl(2) \) generators lead to the Witten’s second deformation [3]. In Section 4 a two parameter deformation of \( sl(2) \) is introduced to study the BFV-BRST cohomology problem when both the phase space and the algebra are deformed, and the related BFV-BRST charge is given.

## 2 Classical BFV–BRST Charges

We deal with a 1-d system (the usual time coordinate), and \( \mathbb{R}^2 \) phase space. In terms of the phase space variables \( (p, x) \), satisfying the usual Poisson brackets

\[
\{p, x\} = 1,
\]

the “classical \( sl(2) \)” algebra

\[
\{H^0, X^0_\pm\} = \pm 2X^0_\pm, \quad \{X^0_+, X^0_-\} = H^0, \quad \text{(1)}
\]

can be realized if the generators are taken to be

\[
H^0 = 2px, \quad X^0_+ = -\sqrt{2}x, \quad X^0_- = \frac{1}{\sqrt{2}}p^2x. \quad \text{(2)}
\]

We consider “\( q \)-classical systems” defined as

1) Poisson brackets are standard, nevertheless the “classical \( q \)-deformed algebra \( sl_q(2) \)” is functionally realized in \( C^\infty(\mathbb{R}^2) \).

2) The phase space is endowed with \( q \)-deformed Poisson brackets, but the generators are as in (2).

1) In the phase space endowed with the usual Poisson brackets a functional realization of the “classical \( sl_q(2) \)”

\[
\{H, X_\pm\} = \pm 2X_\pm, \quad \{X_+, X_-\} = \frac{q^H - q^{-H}}{q - q^{-1}} \equiv [H]_q, \quad \text{(3)}
\]
can be achieved in terms of

\[ H = 2px, \quad X_+ = -\frac{1}{2}x, \quad X_- = \frac{1 + \cosh(2\alpha px)}{x\alpha \sinh\alpha}, \tag{4} \]

where \( q \equiv e^{\alpha} \).

Let us introduce some ghost variables by enlarging the classical phase space endowed it with a generalized Poisson bracket structure, to write a BFV–BRST charge. How many ghost fields are needed? In \cite{12} three ghost variables (and their momenta) are used demanding that the related \( q\hbar \)-deformed BFV–BRST charge would be a polynomial in \( q^H \). Although this is quite plausible (in comultiplication of \( sl_q(2) \), \( q^H \) appears), it is not the unique choice: the form (or the number) of constraints will dictate the number of ghost variables. In Ref.\cite{12} it is assumed that there are three constraints behaving as \( X_\pm \), and \([H]_q\). But a priori one does not know the structure of the constraints. There may be different choices: in Ref.\cite{8} a candidate for a \( q \)-deformed gauge theory is shown to possess infinite gauge field components (hence infinite constraints) depending on the representation of the universal covering algebra. Another deformation of the BRST algebra is given in Ref.\cite{15} where the number of the ghost fields is 4. Let us deal with the cases where there are three ghost variables, but the assumed constraint structures are different from Ref.\cite{12}.

After choosing three ghost variables and their momenta, we should also define generalized Poisson brackets of them. This depends on the conditions which we require that BFV–BRST charge satisfies.

To assume that the constraints behave as \( X_\pm \), and \( H \), seems to be the simplest choice. By using

\[
\frac{e^{\alpha H} - e^{-\alpha H}}{e^\alpha - e^{-\alpha}} = \frac{\alpha}{e^\alpha - e^{-\alpha}} \sum_{k=0}^{\infty} \frac{(\alpha H)^{2k}}{(2k + 1)!} \]
\[
= \frac{\alpha}{e^\alpha - e^{-\alpha}} \prod_{k=1}^{\infty} \left( 1 + \frac{\alpha^2 H^2}{k^2 \pi^2} \right) \]
\[
\equiv H f(H, q), \tag{5} \]

and introducing the fermionic (ghost) variables \((c^i, \pi_i)\), \(i, j = 0, +, -\), which satisfy the usual generalized Poisson brackets

\[
\{\pi_i, c^j\} = \delta^j_i, \quad \{\pi_i, \pi_j\} = 0, \quad \{c^i, c^j\} = 0, \tag{6} \]
one can write the classical BFV–BRST charge as
\[
\Omega_1 = c^+ X_+ + c^- X_- + \frac{1}{\sqrt{2}} e^0 H - \sqrt{2} f(q, H) c^+ c^- \pi_0 + \sqrt{2} e^0 \pi_+ - \sqrt{2} e^0 \pi_-
\]
(7)
The generalized Poisson brackets are
\[
\{ f, g \} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} + \frac{\partial f}{\partial \pi_i} \frac{\partial g}{\partial c^i} + \frac{\partial f}{\partial c^i} \frac{\partial g}{\partial \pi_i}.
\]
One can easily observe that \( \Omega_1 \) satisfies the classical nilpotency relation
\[
\{ \Omega_1, \Omega_1 \} = 0.
\]
We suppose that the generalized Poisson brackets of the ghosts are non-deformed due to the fact that we did not deform the original phase space. But the ghost variables are associated with the gauge (group) generators, so that deforming their Poisson brackets, even if the original phase space is not deformed, is not ruled out.

Another possibility is to suppose that the constraints behave as \( X_\pm \), and \( [H^2]_q \). The choice where \( [H^2]_q \) is replaced by \( [H]_q \) seems more natural because in the coproduct of \( sl_q(2) \), \( q^H \) appears. Nevertheless, in the following section we show that our choice possesses more similarities with the usual BFV–BRST analysis. The related classical BFV–BRST charge satisfying
\[
\{ \Omega_2, \Omega_2 \} = 0,
\]
is given by
\[
\Omega_2 = X_+ c^+ + X_- c^- + (q + q^{-1})^{1/2} \left[ H \right]_q e^0 - (q + q^{-1})^{-1/2} \left( \frac{q^H - q^{-H}}{q^{H/2} - q^{-H/2}} \right) \pi_0 e^0 e^- \\
+ \frac{ln q}{(q - q^{-1})} (q + q^{-1})^{1/2} (q^{H/2} + q^{-H/2}) (\pi_+ e^0 e^- - \pi_- e^0 e^-) \\
- ln^2 q (q + q^{-1})^{1/2} \left[ H \right]_q \pi_+ \pi_- e^0 e^0 e^0.
\]
(8)

2) As announced before a \( \ast \) product approach is preferred to \( q \)-deform the phase space (we follow Ref. [14]).
Attach a two dimensional internal space parametrized by $\xi$, and $\rho$, to each point of the phase space by defining

$$x_\xi = x e^{i\gamma\xi}, \quad p_\rho = p e^{i\gamma\rho}. \quad (9)$$

Then define a $\star$ product of any functions $f$ and $g$ as

$$f \star \gamma g \equiv \sum_{n=0}^{\infty} \left( -\frac{\gamma}{2} \right)^n \frac{n!}{n!} \sum_{k=0}^{n} \binom{n}{k} \left( -1 \right)^k \left( \partial_{\xi}^{n-k} \partial_{\rho}^k f \right) \left( \partial_{\rho}^{n-k} \partial_{\xi}^k g \right). \quad (10)$$

This $\star$ product is associative and can be used to define the $q$-deformed Poisson brackets

$$\{ f, g \}_{\gamma} = -2 \frac{f \star \gamma g - g \star \gamma f}{xp \ln q} = -2 \sum_{n=0}^{\infty} \frac{(-\gamma/2)^{2n+1} 2n+1}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \left( -1 \right)^k \left( \partial_{\xi}^{2n+1-k} \partial_{\rho}^k f \right) \left( \partial_{\rho}^{2n+1-k} \partial_{\xi}^k g \right), \quad (11)$$

where $q \equiv \exp(\gamma^3)$. Let us deal with the functional realization of classical $sl(2)$, given in (4) by replacing $x \rightarrow x_\xi$, $p \rightarrow p_\rho$:

$$H_{\gamma} = 2p_\rho \star \gamma x_\xi, \quad X_+ = -\sqrt{2}x_\xi, \quad X_- = \frac{1}{\sqrt{2}}p_\rho \star \gamma p_\rho \star x_\xi. \quad (12)$$

These satisfy the following $q$-deformed Poisson brackets in the limit $\rho, \xi \rightarrow 0$,

$$\{ H, X_\pm \}_{\gamma} = \pm 2a(q) X_\pm, \{ X_+, X_- \}_{\gamma} = (a(q)/2)(q^{1/2} + q^{-1/2})H,$$

where

$$a(q) = \frac{1 - q^{-1}}{\ln q}.$$

It is a Lie algebra in terms of the new brackets, thus we obviously need three ghost variables and their momenta for the BFV–BRST analysis. The generalized Poisson brackets of the fermionic ghost variables should be deformed or not? In the $q$-$\hbar$-deformed case there is somehow a natural answer to this, but at the classical level it is completely arbitrary. Hence we suppose that they satisfy the usual conditions

$$c^i c^j = -c^j c^i, \quad \pi_i \pi_j = -\pi_j \pi_i, \quad i \neq j. \quad (12)$$
and
\[\{c^i, \pi_j\}^\gamma = \delta^i_j.\] (13)

Then, the generalized $q$-deformed Poisson brackets are
\[\{f, g\}^\gamma \equiv -2\frac{f \star \gamma g - (-)^{\epsilon(f)\epsilon(g)} g \star \gamma f}{xp ln q} + \frac{\partial f}{\partial \pi_i} \partial c^i - (-)^{\epsilon(f)\epsilon(g)} \frac{\partial g}{\partial \pi_i} \partial c^i,\] (14)

where $\epsilon(f)$ indicates the ghost number:
\[\epsilon(c^i) = -\epsilon(\pi_i) = 1, \quad \epsilon(fg) = \epsilon(f) + \epsilon(g).\]

Hence we write the BFV–BRST charge as
\[\Omega_3 = Hc^0 + X_+ c^+ + X_- c^- - a(q)\pi_+ c^0 c^+ + a(q)\pi_- c^0 c^- - (a(q)/2)(q^{1/2} + q^{-1/2})\pi_0 c^+ c^- ,\] (15)

which satisfies
\[\{\Omega_3, \Omega_3\}^\gamma = 0.\]

In the next section we show that in $\bar{\hbar}$-$q$-deformed case, it is natural to keep (13) but deform (12). If we adopt a similar deformation in the classical case a BFV–BRST charge which possesses terms linear in the generators as in (15), will not exist.

3 Quantization

When we deal with the non-deformed phase space, there is no difference between introducing the $\hbar$-deformation in terms of the Moyal brackets or the canonical quantization as far as the purposes of this section are considered. If we drop $\star$ in the former formulation, both of the them will yield the following fundamental commutators
\[[p, x] = -i\hbar.\] (16)

Of course, when (16) is considered as Moyal brackets $p$ and $x$ are classical variables, but they are operators in terms of the canonical quantization.

After an appropriate rescaling of the generators, (1) becomes the usual $sl(2)$ algebra and (3) reads
\[[H, X_\pm] = \pm 2X_\pm, \quad [X_+, X_-] = [H]_q.\] (17)
The ghost fields, then, satisfy

\[ [\pi_i, c^j] = -i\hbar \delta^j_i, \]

where \([f, g] = fg - (-)^{e(f)e(g)}gf\). For simplicity we rescale the phase space variables such that

\[ [p, x] = 1, \quad [\pi_i, c^j] = \delta^j_i. \]

Under the \(\hbar\)-deformation \(\Omega_1 \rightarrow Q_1\) which is in the same form but satisfying

\[ Q_2 = 0, \]

(18)

The BFV–BRST charge for 

\[ U_q(SU(2)), \quad (17), \]

\[ Q_2, \]

satisfying

\[ Q_2^2 = 0, \]

when the constraints are supposed to behave like \(X_\pm\), and \(H\), is not any more given similar to (8), but

\[
Q_2 = X_+ c^+ + X_- c^- + (q + q^{-1})^{1/2} \left[ \tfrac{H}{2} \right] c^0 - (q + q^{-1})^{-1/2} \left( q^H - q^{-H} \right) \left( \tfrac{H}{2} \right) \pi_0 c^+ c^-
\]

\[ + \frac{(q + q^{-1})^{1/2}}{q^{1/2} - q^{-1/2}} \{ (q^{(H+1)/2} + q^{-(H+1)/2}) \pi_+ c^+ c^0 - (q^{(H-1)/2} + q^{-(H+1)/2}) \pi_- c^- c^0 \} \]

\[ - \frac{(q + q^{-1})^{1/2}}{(q^{1/2} + q^{-1/2})^2} (q - q^{-1})^2 \left[ \tfrac{H}{2} \right] \pi_+ \pi_- c^- c^0. \]

(19)

To obtain the physical states or the solution of the BRST cohomology, let us consider the space of the states

\[ \Psi(c) = \sum_{l=0}^{3} \frac{1}{l!} c^i_{1} \cdots c^i_{l} \Psi_{i_1 \cdots i_l}^{(l)}. \]

The \(\Psi^{(l)}\) coefficients are some complex functions on the space where the constraints or the generators act. Action of \(\pi_i\) on the states is

\[ (\pi_i \Psi)^{(l)}_{i_1 \cdots i_l} = \Psi^{(l+1)}_{i_1 i_2 \cdots i_l}; \quad l = 0, 1, 2. \]

When one deals with a Lie algebra the coefficients \(\Psi^{(l)}_{i_1 \cdots i_l}\), can be considered as \(l\)-forms on the algebra, and the indices are raised or lowered by the Cartan metric of the algebra. Thus one can introduce the scalar product

\[ \langle \Phi, \Psi \rangle = \sum_{l=0}^{3} \frac{1}{l!} \Phi^{(l)}_{i_1 \cdots i_l} \Psi^{(l)}_{i_1 \cdots i_l}, \]

(20)

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which is positive definite. With respect to this product
\[ c^i = \pi_i. \]  

\( Q^\dagger \) obtained from the BFV-BRST charge \( Q \) of the Lie group is also nilpotent. When we deal with \( SU(2) \) and demand that \([Q, Q^\dagger]\) is a generalization of the quadratic Casimir of \( SU(2) \), in the basis we adopted the scalar product should also yield
\[ X^\dagger_\pm = X_\mp, \ H^\dagger = H. \]  

In the case where we assume that the constraints behave like \( X_\pm, H \), the conjugation defined by (21)-(22) leads to \( Q_1^\dagger \) which is nilpotent. Unfortunately, when the constraints are supposed to behave like \( X_\pm, [H/2], Q_2^\dagger \) obtained from (19) is not nilpotent. This is due to the fact that in the former case BFV-BRST charge is insensible to the ordering of ghost variables, but in the latter a change in the ordering of ghost variables would create some terms which spoil the nilpotency condition.

To overcome this difficulty let us introduce the following positive definite scalar product
\[ (\Phi^*, \Psi) = \sum_{l=0}^{3} \frac{1}{l!} \Phi^*_{i i_1 i_2} g^{i i_1 i_2} \cdot g^{i_1 i_2} \Psi_{i_1 i_2}, \]  

where \( g^{00} = g^{+-} = g^{-+} = 1 \), and \( \Phi^*_{i i_1 i_2} \) is the complex conjugate of \( \Phi^{(l)}_{i i_1 i_2} \). With respect to this product the conjugate of the generators and the ghost variables are
\[ X^*_{\pm} = X_\mp, \ H^* = H, \ c^{0*} = \pi_0, \ c^{+*} = \pi_+, \ c^{-*} = \pi_-, \]  

where \((f^*)^* = f\). Conjugation of \( Q_1 \) yields the following co-BFV-BRST charge
\[ Q_1^* = \pi_- X_+ + \pi_+ X_- + \frac{1}{\sqrt{2}} \pi_0 H - \sqrt{2} f(q, H) \pi_+ \pi_- c^0 + \sqrt{2} \pi_- \pi_0 c^- - \sqrt{2} \pi_+ \pi_0 c^+, \]  

which is nilpotent and \([Q_1, Q_1^*]\) is a generalization of the quadratic Casimir of \( SU(2) \). This justifies the choice of the normalization factors of the terms linear in the ghost variables.
In terms of the conjugation given in (24), the co-BFV–BRST charge derived from \(Q\) is

\[
Q^*_2 = X_+ \pi_- + X_- \pi_+ + (q + q^{-1})^{1/2} \left[ \frac{H}{2} \right]_q \pi_0 - (q + q^{-1})^{-1/2} \frac{q^H - q^{-H}}{q^{H/2} - q^{-H/2}} c^0 \pi_- \pi_+ + \frac{(q + q^{-1})^{1/2}}{q^{1/2} - q^{-1/2}} \{(q^{(H+1)/2} + q^{-(H+1)/2})c^- \pi_0 - (q^{(H-1)/2} + q^{-(H-1)/2})c^+ \pi_0 \} - \frac{(q + q^{-1})^{1/2}}{(q^{1/2} + q^{-1/2})^2 (q - q^{-1})^2} \left[ \frac{H}{2} \right]_q c^- c^+ \pi^- \pi^0_0.
\]

This charge as in the usual case, can be used to define

\[
\{Q_2, Q^*_2\}|_{\pi = c = 0} = C_q, \quad (27)
\]

where \(C_q\) is the quadratic Casimir of \(U_q(SU(2))\) [19, 20]:

\[
C_q = X_+ X_+ + \left[ \frac{H}{2} \right]_q \left[ \frac{H + 2}{2} \right]_q = \frac{1}{2} \left( X_+ X_+ + X_- X_- + (q + q^{-1}) \left[ \frac{H}{2} \right]_q \left[ \frac{H}{2} \right]_q \right).
\]

Hence by using the positive definite scalar product (23), the physical states can be identified with the states \(\omega\) satisfying

\[
(Q + Q^*)^2 \omega = 0, \quad Q\omega = 0, \quad Q^*\omega = 0,
\]

where \(Q\) and \(Q^*\) are given either by (19) and (23) or by (19) and (24). At zero ghost number the cohomology classes given by (19) include the ones found in Ref. [12], and the states \(\omega_2\) satisfying \(Q_2 \omega_2 = 0\), contain the singlets of \(U_q(SU(2))\). Although at zero ghost number \(Q_1 \omega_1 = 0\) yields the states \(\omega_1\), which are singlets of \(SU(2)\), by including ghost number one sector the other states of \(U_q(SU(2))\) can be obtained.

If we \(\hbar\)-deform the phase space after the \(q\)-deformation we get [14]

\[
x \ast_{\gamma \hbar} p - qp \ast_{\gamma \hbar} x = -i\hbar q^{1/2}. \quad (28)
\]

The \(\ast_{\gamma \hbar}\) product is defined as

\[
f(x,p) \ast_{\gamma \hbar} g(x,p) = f(x_\xi, p_\rho) \sum_{n=0}^{\infty} \frac{(-\gamma/2)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-)^k (\partial_\xi^n - k \partial_\rho^k)(\partial_\rho^{n-k}\partial_\xi^k)
\sum_{m=0}^{\infty} \frac{(-i\hbar/2)^m}{m!} \sum_{l=0}^{m} \binom{m}{l} (-)^l (\partial_{x}^{m-l}\partial_{p}^l)(\partial_{p}^{m-l}\partial_{x}^l) \ g(x_\xi, p_\rho)|_{\xi = \rho = 0},
\]
where in the sums the first two derivatives act on the left and the others on the right hand side.

For our purposes, once the $\hbar$-deformation is achieved we can forget about the $\star_{\gamma\hbar}$ and also set $\hbar = 1$, to obtain

$$xp - qpx = -iq^{1/2}. \quad (29)$$

By keeping the form of the generators as in (2) we get

$$HX_+ - q^{-1}X_+H = -i2q^{1/2}X_+, \quad (30)$$
$$HX_+ - q^{-1}X_+H = i2q^{-1/2}X_+, \quad (31)$$
$$X_+X_+ - q^2X_-X_+ = (-i/2)(q^{1/2} + q^{3/2})H. \quad (32)$$

After rescaling as

$$X_\pm \to iq^{-1/2}\sqrt{(q^{1/2} + q^{3/2})}, \quad H \to i2H$$
and setting

$$q = r^2,$$

one can see that the relations given in (30)-(32) read

$$r^{-1}HX_+ - rX_-H = -X_-, \quad (33)$$
$$rHX_+ - r^{-1}X_+H = X_+, \quad (33)$$
$$r^{-2}X_+X_+ - r^2X_-X_+ = H,$$

in which we recognize the Witten's second deformation [3].

When we $q$-$\hbar$-deform the phase space a natural requirement for the BFV–BRST charge is to demand that the anticommutation of the terms which are linear in ghost variables and the generators, with themselves generates the algebra. For the deformed algebra (33), this condition leads to

$$c^i c^j = -\nu^{ij} c^i c^j; \quad \nu^{ij} = 0, \nu^{0+} = r^2, \nu^{-0} = r^2, \nu^{-+} = r^4. \quad (34)$$

These relations could also be obtained by demanding that $c^i$ behave like one forms on $sl_q(2)$ [4].

Although one can also deform the commutators, it is not necessary. In fact, we deal with the ghost variables satisfying

$$[\pi_i, c^i] = \delta^i_i, c^2 = \pi_i^2 = 0. \quad (35)$$
Now the associativity leads to
\[ \pi_i \pi_j = -\nu^{ij} \pi_j \pi_i, \quad \pi_i c_j = -\nu^{ji} c_j \pi_i. \]

Hence the BFV–BRST charge which satisfies \( Q_3^2 = 0 \), is
\[ Q_3 = H c^0 + X_+ c^+ + X_- c^- - r(\pi_+ c^+ c^0 + \pi_- c^0 c^-) - r^2 \pi_0 c^+ c^- - (r - r^{-1}) \pi_- \pi_+ c^0 c^+ c^- \]  \( (36) \)

One can observe that the choice \((34)-(35)\) follows if we require that \( Q_3 \) behaves like the exterior derivative, so that \( \{ Q_3, c^i \} \) coincide with the Cartan-Maurer structure equations on \( sl_q(2) \).

To find solution of the cohomology of \( Q_3 \) one should define a state space endowed with a scalar product, and introduce the co-BFV-BRST charge. A choice is given in Ref.[11]. The choice should be dictated by the desired physical content of the gauge theory. This is still obscure, so that the issue of defining a scalar product and co-BFV-BRST charge is not discussed here.

4 Two Parameter Deformation

Recently, attention is paid to multiparameter deformations of groups [21]. Because of the fact that the requirements are different, not all of these deformations fulfill the condition of being an Hopf algebra.

In the procedure which we follow, obviously, one can \( q \)-deform the \( \hbar \)-deformed phase space as well as the \( \hbar \)-deformed algebra with different parameters. In this section \( q \)-deformation of the quantum phase space is supposed to be realized as given in Ref.[4], which is known to be equivalent to the Witten’s deformation \((33), [20] \). If we demand to obtain one of the deformations at some special values of the deformation parameters, it is quite natural to consider the following two parameter deformation of \( sl(2) \),

\[ \mu^2 H X_+ - \frac{1}{\mu^2} X_+ H = (1 + \mu^2) X_+, \]
\[ \mu^2 X_- H - \frac{1}{\mu^2} H X_- = (1 + \mu^2) X_-, \]  \( (37) \)
\[ \frac{1}{\mu} X_+ X_- - \mu X_- X_+ = [H]_q. \]
To keep the resemblance with $\mu = 1$ and $q = 1$ cases we introduce the ghost fields satisfying

\[ [\bar{\eta}_i, \eta^j] = \delta^j_i, \quad (38) \]

\[ \eta^i \eta^j = l_{ij} \eta^j \eta^i, \quad \bar{\eta}_i \bar{\eta}_j = l_{ij} \bar{\eta}_j \bar{\eta}_i, \quad \eta^i \bar{\eta}_j = l_{ji} \eta^i \bar{\eta}_j, \quad (39) \]

where

\[ l_{0+} = \mu^4, \quad l_{0-} = \mu^4, \quad l_{++} = \mu^2, \quad l_{ii} = 0. \]

The BFV–BRST charge which leads to the one given in Ref. [12] for $\mu = 1$ and to the one given in Ref. [11] for $q = 1$, moreover satisfying $Q^2 = 0$, is

\[ Q = X_+ \eta^- + X_- \eta^+ + [H]_{q} \eta^0 - \mu \bar{\eta}_0 \eta^+ \eta^- - F(H) \bar{\eta}_+ c^+ c^0 
   + G(H) \bar{\eta}_- c^0 c^- - (G(H) + F(H)) \bar{\eta}_- \bar{\eta}_+ c^+ c^0 c^- , \quad (40) \]

where

\[ F(H) = \frac{\mu^2}{q - q^{-1}}[\mu^2(q^H - q^{-H}) - \mu^2(q^a - q^{-a})], \]

\[ G(H) = \frac{\mu^2}{q - q^{-1}}[\mu^2(q^H - q^{-H}) - \mu^2(q^b - q^{-b})], \]

\[ a = \mu^2(\mu^2 H + \mu^2 + 1), \]

\[ b = \mu^2(\mu^2 H - \mu^2 - 1). \]

To find solutions of the $Q$-cohomology, one should introduce a state space and a scalar product. Obviously, there are different choices and as it is mentioned above, it is closely related to the desired physical properties of the system.

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