ADMISSIBLE FUNCTION SPACES FOR WEIGHTED SOBOLEV INEQUALITIES

T. V. ANOOP, NIRJAN BISWAS, AND UJJAL DAS

Abstract. Let $k, N \in \mathbb{N}$ with $1 \leq k \leq N$ and let $\Omega = \Omega_1 \times \Omega_2$ be an open set in $\mathbb{R}^k \times \mathbb{R}^{N-k}$. For $p \in (1, \infty)$ and $q \in (0, \infty)$, we consider the following weighted Sobolev type inequality:

$$
\int_{\Omega} |g_1(y)||g_2(z)||u(y,z)|^p \, dydz \leq C \left( \int_{\Omega} |\nabla u(y,z)|^p \, dydz \right)^{\frac{q}{p}}, \quad \forall u \in C^1_c(\Omega),
$$

(0.1)

for some $C > 0$. Depending on the values of $N, k, p, q$ we have identified various pairs of Lorentz spaces, Lorentz-Zygmund spaces and weighted Lebesgue spaces for $(g_1, g_2)$ so that (0.1) holds. Furthermore, we give a sufficient condition on $g_1, g_2$ so that the best constant in (0.1) is attained in the Beppo-Levi space $D^{1,p}_0(\Omega)$-the completion of $C^1_c(\Omega)$ with respect to $\|\nabla u\|_{L^p(\Omega)}$.

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1. Introduction and the Main Results

Let $k, N \in \mathbb{N}$ be such that $1 \leq k \leq N$. For an open set $\Omega$ in $\mathbb{R}^N$ and $g \in L^1_{loc}(\Omega)$, we assume the following:

- $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1$ and $\Omega_2$ are open sets in $\mathbb{R}^k$ and $\mathbb{R}^{N-k}$ respectively,
- $g(x) = g_1(y)g_2(z)$, $x := (y, z) \in \Omega_1 \times \Omega_2$, where $g_1 \in L^1_{loc}(\Omega_1)$ and $g_2 \in L^1_{loc}(\Omega_2)$,
- If $k = N$, then $\Omega = \Omega_1$ and $g = g_1$.

Let $p \in (1, \infty)$ and $q \in (0, \infty)$. For $\Omega$ and $g$ as given in (A), we look for sufficient conditions on $g_1, g_2$ so that the following weighted Sobolev type inequality holds:

$$\int_{\Omega} |g(x)||u(x)|^q \, dx \leq C \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall u \in C^1_c(\Omega),$$

for some $C > 0$.

**Definition 1.1 (\((p, q)\)-Hardy potential).**

1. A weight function $g \in L^1_{loc}(\Omega)$ satisfying (1.1) is said to be a $(p, q)$-Hardy potential. The set of all $(p, q)$-Hardy potentials is denoted by $\mathcal{H}_{p,q}(\Omega)$, i.e.,

$$\mathcal{H}_{p,q}(\Omega) = \{ g \in L^1_{loc}(\Omega) : g \text{ is a } (p, q)\text{-Hardy potential} \}.$$

2. If $g$ is of the form $g(x) = g_1(y)g_2(z)$ for some $g_1$ and $g_2$, then we say $g$ is a cylindrical potential. If $g$ is not a cylindrical potential, then we say $g$ is a non-cylindrical potential.

It is not difficult to produce examples of weight functions $g$ in $\mathcal{H}_{p,q}(\Omega)$. For example, if $\Omega_1$ or $\Omega_2$ is bounded in one direction, then the Poincaré inequality shows that

$$L^\infty(\Omega) \subset \mathcal{H}_{p,p}(\Omega).$$

For $N > p$, let $p^* := \frac{Np}{N-p}$. Then for $\Omega \subset \mathbb{R}^N$ with $N > p$, the Sobolev inequality

$$\int_{\Omega} |u(x)|^{p^*} \, dx \leq C \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{p^*}{p}}, \quad \forall u \in C^1_c(\Omega)$$

ensures $L^\infty(\Omega) \subset \mathcal{H}_{p,p^*}(\Omega)$. Furthermore, using the duality of the Lebesgue spaces together with the Hölder’s inequality will give $L^\frac{N}{p}(\Omega) \subset \mathcal{H}_{p,p}(\Omega)$. In the literature, there are many existing results that provide various sufficient conditions for $g$ to be a $(p, q)$-Hardy potential. Before we discuss some of them, we introduce two functions that will be appearing more frequently in this manuscript.

**$\alpha(p, q)$:** For any $N, p, q$, we define

$$\alpha(p, q) := \frac{Np}{N(p-q) + qp}.$$
Notice that, $\alpha(p,p) = \frac{N}{p}$ and for $N > p$,

$$\alpha(p,q) = \frac{p^*}{p^* - q} = \left(\frac{p^*}{q}\right)', \ \forall q \in (0,p^*].$$

**$P^*(s)$:** For $N > p$ and $s \in [0,p]$, we define

$$P^*(s) := \frac{p(N-s)}{N-p}.$$ 

Observe that, $P^*(0) = p^*$, $P^*(p) = p$, and for $N > p$,

$$\alpha(p,P^*(s)) = \frac{N}{s}, \quad P^*(\frac{N}{\alpha(p,q)}) = q.$$

### 1.1. Various sufficient conditions for $(p,q)$-Hardy Potentials.

Now we discuss various sufficient conditions for a $(p,q)$-Hardy Potential available in the literature.

**Fefferman-Phong type conditions:** In [25], for $V \in L^{1}_{loc}(\mathbb{R}^N)$ with $V \leq 0$, Fefferman-Phong estimated the lowest bound of the Schrodinger operator $-\Delta + V$. Their result ensures that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx + \int_{\mathbb{R}^N} V(x)|u(x)|^2 \, dx \geq -CE_{big} \int_{\mathbb{R}^N} |u(x)|^2 \, dx, \ \forall u \in C^1_c(\mathbb{R}^N),$$

where

$$E_{big} := \sup_Q \left[ \left( \frac{1}{|Q|} \int_Q |V(x)|^s \, dx \right)^{\frac{1}{s}} - c \text{ diam}(Q)^{-2} \right]$$

with $c > 0$ and $s > 1$.

where $Q$ ranges over cubes in $\mathbb{R}^N$ with sides parallel to the axes, $|Q|$ and diam$(Q)$ are respectively the measure and the diameter of $Q$. Later, in the definition of $E_{big}$, Fefferman [24, Theorem 5] replaced the cubes with the balls as below:

$$E_{big} := \sup_{\{x \in \mathbb{R}^N, r > 0\}} \left[ \left( \frac{1}{|B_r(x)|} \int_{B_r(x)} |V(x)|^s \, dx \right)^{\frac{1}{s}} - cr^{-2} \right]$$

with $c > 0$ and $s > 1$.

Now if there exists $c > 0$, and $s > 1$ such that

$$\left( \frac{1}{|Q|} \int_Q |V(x)|^s \, dx \right)^{\frac{1}{s}} \leq c \text{ diam}(Q)^{-2}, \ \forall Q, \quad (1.3)$$

or

$$\left( \frac{1}{|B_r(x)|} \int_{B_r(x)} |V(x)|^s \, dx \right)^{\frac{1}{s}} \leq cr^{-2}, \ \forall x \in \mathbb{R}^N, \forall r > 0, \quad (1.4)$$

then $E_{big} \leq 0$ and hence

$$\int_{\mathbb{R}^N} V(x)|u(x)|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx, \ \forall u \in C^\infty_c(\mathbb{R}^N).$$

In particular, if for some $c > 0$ and $s > 1$, $V$ satisfies (1.3) or (1.4), then by applying the result of Fefferman-Phong to $-\Delta - |V|$ we get $V \in H_{2,2}(\mathbb{R}^N)$.

Next, we see Fefferman-Phong type conditions for general $p$ and $q$ via weighted inequalities for the fractional integrals. For $N \geq 3$ and $u \in C^\infty_c(\mathbb{R}^N)$, the Newtonian potential $\Gamma * (\Delta u)$ of $\Delta u$ coincide
with \( u \) ([60, Theorem 2, Pg-147]), where \( \Gamma \) is the Fundamental solution of the Laplacian. Thus using the integration by parts we get
\[
u(x) = \frac{1}{N \omega(N(2 - N))} \int_{\mathbb{R}^N} \frac{\Delta u(y)}{|x - y|^{N-2}} \, dy = \frac{1}{N \omega(N)} \int_{\mathbb{R}^N} \frac{(x - y) \cdot \nabla u(y)}{|x - y|^N} \, dy.
\]
Therefore,
\[
|u(x)| \leq \frac{1}{N \omega(N)} \int_{\mathbb{R}^N} \frac{|
abla u(y)|}{|x - y|^{N-1}} \, dy = \frac{1}{N \omega(N)} I_{1}(|\nabla u|),
\]
where \( I_{\gamma} \) is the Riesz potential operator defined as
\[
I_{\gamma}(f)(x) := \int_{\mathbb{R}^N} \frac{|f(y)|}{|x - y|^{N-\gamma}} \, dy; \quad \gamma \in (0, N).
\]
From (1.5) it is evident that (1.1) holds, if the following weighted inequality holds:
\[
\int_{\mathbb{R}^N} |I_{1}(f)(x)|^p |g(x)| \, dx \leq C \left( \int_{\mathbb{R}^N} f(x)^p \, dx \right)^{\frac{2}{p}}, \quad \forall f \in C_c(\mathbb{R}^N), f \geq 0.
\]
(1.6)
For \( N \geq 3, 1 < p \leq q < \infty \), many authors provided various sufficient conditions on \( g, h \) and \( \gamma \) so that the following weighted inequality for the fractional integral holds:
\[
\int_{\mathbb{R}^N} |I_{\gamma}(f(x)|^q |g(x)| \, dx \leq C \left( \int_{\mathbb{R}^N} f(x)^q |h(x)| \, dx \right)^{\frac{2}{q}}, \quad \forall f \in C_c^1(\mathbb{R}^N), f \geq 0.
\]
(1.7)
For example, see [52] \( (p = q = 2 \text{ and } h \equiv 1) \), [20] \( (p = q) \), [36, 43, 55] \( (p \leq q) \). In particular, for \( \gamma = 1 \) and \( h \equiv 1 \), their results provide examples of \( (p, q) \)-Hardy potentials. In [56, Theorem 1(A)], Sawyer-Wheeden have shown that, if there exist \( s > 0 \) and \( c > 0 \) such that
\[
|Q|^\frac{1}{p} + \frac{1}{q} - \frac{1}{p} \left( \frac{1}{|Q|} \int_Q |g(x)|^s \, dx \right)^{\frac{1}{s}} \left( \frac{1}{|Q|} \int_Q |h(x)|^{(1-p')s} \, dx \right)^{\frac{1}{p'}} \leq c, \quad \forall Q,
\]
(1.8)
then (1.7) holds. In particular, for \( \gamma = 1 \) and \( h \equiv 1 \) the above condition reads as
\[
|Q|^\frac{1}{p} + \frac{1}{q} \left( \int_Q |g(x)|^s \, dx \right)^{\frac{1}{s}} \leq c, \quad \forall Q.
\]
(1.9)
Thus \( g \) satisfying (1.9) lies in \( H_{p,q}(\mathbb{R}^N) \). Notice that, for \( p = q = 2 \), (1.9) coincides with the Fefferman-Phong condition (1.3). For recent developments concerning the weighted Sobolev inequalities and Fefferman-Phong type conditions, we refer to [45, 59] and the references therein.

(ii) **Bessel’s pair:** Let \( \Omega = B_R(0) \) with \( 0 < R \leq \infty \), and let \( g, h \) be two positive, radial, \( C^1 \) functions on \( \Omega \). A pair \( (g, h) \) is called Bessel pair if \( (g(r)r')' + h(r)r = 0 \) has a positive solution on \( (0, R) \). In [28], the authors showed that, if \( (p, r^{N-1}g, r^{N-1}h) \) is a Bessel pair with \( \int_0^R \frac{dr}{r g(r)} = \infty \) and \( \int_0^R r^{N-1}h(r)dr < \infty \), then the following inequality holds:
\[
\int_{\Omega} |g(x)||u(x)|^2 \, dx \leq C \int_{\Omega} |h(x)||\nabla u(x)|^2 \, dx, \quad \forall u \in C_c^1(\Omega).
\]
(1.10)
For further improvements in this direction, we refer to [38, 39] and the references therein.

(iii) **Maz’ya’s capacity conditions:** Using \( p \)-capacity, Maz’ya has provided a necessary and sufficient condition (Theorem 8.5 of [48]) on \( g \) so that (1.1) holds. Let us recall that, for \( F \subset \subset \mathbb{R}^N \), the \( p \)-capacity of \( F \) with respect to \( \Omega \) is defined as
\[
\text{Cap}_p(F, \Omega) = \inf \left\{ \int_{\Omega} |\nabla u|^p : u \in C_c^1(\Omega), u \geq 1 \text{ on } F \right\}.
\]
For $1 < p \leq q < \infty$, Maz’ya proved that $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$ if and only if

$$\|g\|_{\mathcal{H}_{p,q}} := \sup_{F \subset \mathbb{R}^N} \left\{ \frac{\int_F |g|}{[\text{Cap}_p(F)]^{\frac{p}{q}}} \right\} < \infty.$$ 

It is easy to see that $\mathcal{H}_{p,q}(\Omega) = \{ g \in L^1_{\text{loc}}(\Omega) : \|g\|_{\mathcal{H}_{p,q}} < \infty \} ; 1 < p \leq q < \infty$ and $\|g\|_{\mathcal{H}_{p,q}}$ defines a Banach function space norm on $\mathcal{H}_{p,q}(\Omega)$.

1.2. More admissible spaces of $(p, q)$-Hardy potentials. The results mentioned in the previous subsection assumes $k = N$ and $q \geq p$ or assumes $k = N$ and $g$ is radial. In this article, we allow the cases $0 < q < p$ and $1 \leq k \leq N$. In these cases, depending on the values of $N, k, p, q$ and the geometry of $\Omega$, we provide various classes of function spaces that lie in $\mathcal{H}_{p,q}(\Omega)$ mainly using two different techniques:

(i) symmetrization, (ii) polar decomposition.

(i) The symmetrization method relies on the classical inequalities concerning symmetrization such as Pólya-Szegö inequality, Hardy-Littlewood inequality and the Muckenhoupt condition [49] for the one-dimensional weighted Hardy inequalities.

(ii) The polar decomposition method is based on the use of the fundamental theorem of integration for various functions and the Hölder’s inequality for various conjugate pairs and conjugate triplets.

One can also identify some admissible function spaces for $(p, q)$-Hardy potentials using the embedding of the Beppo-Levi space $\mathcal{D}^{1,p}_0(\Omega)$-the completion of $C^1(\Omega)$ with respect to the the norm $(\int_\Omega |\nabla u(x)|^p \, dx)^{\frac{1}{p}}$ (if it is a well defined function space). For example, the Lorentz-Sobolev embedding and Moser-Trudinger embedding of $\mathcal{D}^{1,p}_0(\Omega)$ provide certain Lebesgue spaces, Orlicz spaces, and Lorentz spaces that lie in $\mathcal{H}_{p,q}(\Omega)$ (Theorem 1.3). Notice that, if $\Omega$ is bounded in one direction, then $\mathcal{D}^{1,p}_0(\Omega)$ coincides with the classical Sobolev space $W^{1,p}_0(\Omega)$. Unfortunately, for $N \leq p$, $\mathcal{D}^{1,p}_0(\mathbb{R}^N)$ is not a function space. In fact, Hörmander-Lions in [33] showed that $\mathcal{D}^{1,2}_0(\mathbb{R}^2)$ contains objects that do not belong to even in the space of distributions. To ensure that $\mathcal{D}^{1,p}_0(\Omega)$ is a well defined function space, we need to make some restrictions on $\Omega$. For $r \geq 0$, a open ball and a closed ball centred at $x$ with the radius $r$ are denoted by $B_r(x)$ and $B_r[x]$ respectively. Henceforth, we make the following assumptions on the open set $\Omega$:

$$\begin{align*}
\text{N} > p &: \quad \Omega \text{ is any open set in } \mathbb{R}^N, \\
\text{N} = p &: \quad \Omega \subset \mathbb{R}^N \setminus B_a[x] \text{ for some } x \in \mathbb{R}^N, \text{ with } a > 0, \\
\text{N} < p &: \quad \Omega \subset \mathbb{R}^N \setminus \{x\} \text{ for some } x \in \mathbb{R}^N.
\end{align*}$$

For $N \leq p$ and for $\Omega$ as given in (B), then we will show that $\mathcal{D}^{1,p}_0(\Omega)$ is always a well defined function space and it is continuously embedded in $L^p_{\text{loc}}(\Omega)$ (Corollary 1.7 and see also [8, Corollary 2.4]).

1.2.1. The $(p, q)$-Hardy potentials $(k = N)$. In this case, we have $\Omega = \Omega_1$ and $g = g_1$. If $N > p$, then using (1.2) and the Hölder’s inequality, it is easy to see that, for each $q \in (0, p^*)$,

$$L^{a(p,q)}(\Omega) \subset \mathcal{H}_{p,q}(\Omega).$$

Furthermore, the classical Hardy-Sobolev inequality

$$\int_{\Omega} \frac{|u(x)|^p}{|x|^p} \, dx \leq \left( \frac{p}{N-p} \right)^p \int_{\Omega} |\nabla u(x)|^p \, dx, \quad \forall u \in C^1_c(\Omega) \quad (1.11)$$

ensures that the Hardy potential $\frac{1}{|x|^p}$ belongs to $\mathcal{H}_{p,p}(\Omega)$. Notice that, if $\Omega$ contains the origin, then $\frac{1}{|x|^p}$ does not lie in any Lebesgue space. The inequality (1.11) has been improved by adding lower order radial weights to $\frac{1}{|x|^p}$, for example, see [2, 16, 26, 29] and the references therein. Indeed, all these improved Hardy-Sobolev inequalities provide examples of radial weights in $\mathcal{H}_{p,p}(\Omega)$. The improvements of (1.11) involving the distance functions are available in [38, 39, 40]. Many authors are also interested to extend (1.11) by considering more general class of weight functions in place of $\frac{1}{|x|^p}$. The following version of Caffarelli-Kohn-Nirenberg inequality [18] extends (1.11) for $q \in [p, p^*]:$

$$
\int_{\mathbb{R}^N} \frac{|u(x)|^q}{|x|^{\frac{N}{p} - \frac{N}{p'}}} \, dx \leq C \left( \int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall u \in C^1_c(\mathbb{R}^N). \tag{1.12}
$$

Thus $g(x) = |x|^{-\frac{N}{p} - \frac{N}{p'}} \in \mathcal{H}_{p,p}(\mathbb{R}^N)$ for $q \in [p, p^*].$

**Remark 1.2.** For $q \in [p, p^*]$, set $s = \frac{N}{\alpha(p,q)}$. Then (1.12) takes the following form:

$$
\int_{\mathbb{R}^N} \frac{|u(x)|^{\frac{p^*(s)}{p} + s}}{|x|^{s}} \, dx \leq C \left( \int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx \right)^{\frac{\frac{p^*(s)}{p} + s}{p}}, \quad \forall u \in C^1_c(\mathbb{R}^N),
$$

with $s \in [0, p]$. The above inequality is also known as the classical Caffarelli-Kohn-Nirenberg inequality. In [30, Lemma 3.1], the authors have shown that the conditions $q = P^*(s)$ and $s \in [0, p]$ are necessary for $|x|^{-s} \in \mathcal{H}_{p,q}(\mathbb{R}^N)$.

Observe that, $g = |x|^{-\frac{N}{\alpha(p,q)}}$ lies in the Lorentz space $L^{\alpha(p,q),\infty}(\Omega)$ (see Example 2.7). In [61], for $p = 2$ and $N > 2$, using the Lorentz-Sobolev embedding the authors have shown that $L^{\alpha(2,q),\infty}(\Omega) \subset \mathcal{H}_{p,q}(\Omega)$ for $q \in [2, 2^*]$. For $N = p$ and $\Omega = B_1(0)$, Edmunds-Triebel in [23] obtained an analogue of (1.11), namely:

$$
\int_{\Omega} \frac{|u(x)|^N}{(|x|(|\log(|x|)|) \left( |\log(|x|)| \right)^N} \, dx \leq \left( \frac{N}{N - 1} \right)^N \int_{\Omega} |\nabla u(x)|^N \, dx, \quad \forall u \in C^1(\Omega). \tag{1.13}
$$

Thus $(|x|(|\log(|x|)|)^{-N} \in \mathcal{H}_{N,N}(B_1(0))$. It is not hard to verify that $g = (|x|(|\log(|x|)|)^{-N}$ lies in the Lorentz-Zygmund space $L^{1,\infty,N}(B_1(0))$ (see Example 2.7). Indeed, following the same treatment as in [5], one can show that $L^{1,\infty,N}(\Omega) \subset \mathcal{H}_{N,N}(\Omega)$. Our first theorem improve all these results to general $N, p$ and also to a bigger range for $q$.

**Theorem 1.3.** Let $\Omega$ be an open set in $\mathbb{R}^N$. Let $\gamma = \frac{p}{p-q}$ for $q \in (0, p)$ and $\gamma = \infty$ for $q \geq p$.

(i) Let $N > p$. Then

$$
X := L^{\alpha(p,q),\gamma}(\Omega) \subset \mathcal{H}_{p,q}(\Omega), \quad \forall q \in \mathbb{R}^*.
$$

(ii) Let $N = p$ and $\Omega$ be bounded. Then

$$
X := \left\{ L^{1,\gamma,p}(\Omega), \quad q \in (0,1) \cup [p, \infty); \quad L^{1,\gamma,1}(\Omega), \quad q \in [1, p), \right\} \subset \mathcal{H}_{p,q}(\Omega).
$$

(iii) Let $N < p$ and $\Omega$ be bounded in one direction. Then

$$
X := L^{1}(\Omega) \subset \mathcal{H}_{p,q}(\Omega), \quad \forall q \in [0, \infty).
$$

Furthermore, for $g \in X$, there exists $C = C(N,p,q) > 0$ so that

$$
\int_{\Omega} |g(x)||u(x)|^q \, dx \leq C\|g\|_X \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall u \in C^1(\Omega). \tag{1.14}
$$
Our proof for the above theorem is based on the embeddings of $D^{1,p}_0(\Omega)$ into various function spaces with respect to the values of $N$ and $p$. For $N > p$, in [51] O’Neil proved that $D^{1,p}_0(\Omega)$ is embedded in the Lorentz space $L^{p',p}(\mathbb{R}^N)$. For $N = p$ and $\Omega$ bounded, $D^{1,p}_0(\Omega)$ is embedded in the Lorentz-Zygmund space $L^{\infty,N;-1}(\Omega)$, proved independently by Hansson [31] ($N = p = 2$), Brezis-Wainger [17, Lemma 1] (for $N = p$). For $N < p$ and $\Omega$ bounded in one direction, $D^{1,p}_0(\Omega)$ coincides with $W^{1,p}_0(\Omega)$ and hence embedded into the Lebesgue space $L^\infty(\Omega)$. In the appendix, we give simple alternate proofs for all these embeddings (Theorem A.4) using the Muckenhoupt condition for the one-dimensional weighted Hardy inequalities and certain classical inequalities such as Pólya-Szegö inequality, Hardy-Littlewood inequality. We also prove that, if $g$ is radial, radially decreasing, then for $g$ to be a $(p,q)$-Hardy potential, it is necessary that $g$ lies in the spaces given in the above theorem (see Proposition 7.5).

Next we produce another class of $(p,q)$-Hardy potentials on certain symmetric open sets via the polar decomposition method.

**Definition 1.4 (The sectorial sets).** Let $1 \leq k \leq N$ and $S$ be an open subset of $\mathbb{S}^{k-1}$ and $a, b \in [0, \infty)$ with $a < b$. Then consider the open set

$$\Omega_{a,b,S} = \text{int}\left( \left\{ x \in \mathbb{R}^k : a \leq |x| < b, \frac{x}{|x|} \in S \text{ if } x \neq 0 \right\} \right),$$

where $\text{int}(A)$ denotes the interior of a set $A$.

Notice that, $0 \in \Omega_{a,b,S}$, only if $a = 0$, $S = \mathbb{S}^{k-1}$. If $S_1 = \{x = (x_1,\ldots,x_k) \in \mathbb{S}^{k-1} : x_1 > 0\}$, then $\Omega_{0,\infty,S_1}$ becomes the half space $\mathbb{R}^k_+ = \{x \in \mathbb{R}^k : x_1 > 0\}$. Next we associate a radial function to a $L^1_{\text{loc}}(\Omega_{a,b,S})$ function via the notion of radial majorant.

**Definition 1.5 (The radial majorant).** For $f \in L^1_{\text{loc}}(\Omega_{a,b,S})$, we define the radial majorant of $f$ as below:

$$\tilde{f}(r) = \text{ess sup}\{ |f(r\omega)| : \omega \in S \}, \quad r \in (a,b),$$

where the essential supremum is taken with respect to the $(k-1)$-dimensional surface measure.

Notice that, $\tilde{f}(r)$ is finite a.e. in $\Omega_{a,b,S}$ ([27, Theorem 2.49]) and for a radial function $f$, $f(x) = \tilde{f}(|x|)$. Moreover, every function defined on $\Omega_{a,b,S}$ is dominated by its radial majorant.

In [8], for $\Omega_{1,\infty,S^{N-1}} (= B_1^N \text{the exterior of the unit ball centred at the origin})$ and for $q = p$, authors have considered class of weight functions that are dominated by radial functions. They have shown that, if $g$ is dominated by a radial function $w$ and $w \in L^1((1,\infty),r^{p-1})$ ([8, Theorem 1.1]), then $g \in \mathcal{H}_{p,p}(B_1^N)$. Observe that, the radial majorant $\tilde{g} \leq w$ and hence the same result is true, if $\tilde{g} \in L^1((1,\infty),r^{p-1})$. In this article, we extend this result for $q \in [0,p^*]$ and for the general sectorial sets.

**Theorem 1.6.** For $S \subset \mathbb{S}^{N-1}$ and $a, b \in [0,\infty]$ with $a < b$, let $\Omega = \Omega_{a,b,S}$ be the sectorial set as given in (1.15) and let $\tilde{g}$ be the radial majorant of $g \in L^1_{\text{loc}}(\Omega)$. For $q \in (0,\infty)$, let

$$X := \begin{cases} 
L^1((a,b),r^{N-1} \log^{N-1}(\frac{r}{a})), & p = N; \\
L^1((a,b),r^{N-1}(\log^{N}(\frac{r}{a})))^\frac{1}{N}, & p \neq N.
\end{cases}$$

(i) $N > p$: For $q \in (0,p^*]$, let $\tilde{g} \in X$ and in addition $\tilde{g}$ be strictly decreasing for $q \in [P^*(1),p^*]$. Then $g \in \mathcal{H}_{p,q}(\Omega)$.

(ii) $N = p$: For $q \in (0,p]$, let $\tilde{g} \in X$ and $a > 0$. Then $g \in \mathcal{H}_{p,q}(\Omega)$. 


(iii) $N \prec p$ : For $q \in (0, p]$, let $\tilde{g} \in X$ and $0 \notin \Omega$. Then $g \in \mathcal{H}_{p,q}(\Omega)$.

Furthermore, there exists $C = C(N, p, q) > 0$ so that

$$
\int_{\Omega} |g(x)||u(x)|^q \, dx \leq C \|\tilde{g}\|_X \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \ \forall u \in C^1_c(\Omega).
$$

As the immediate consequences of the above theorem, for $\Omega$ as given in Theorem 1.6, we have the well definedness of the Beppo-Levi space $\mathcal{D}^{1,p}_0(\Omega)$ for $N \leq p$. Recall, $B_r[x]$ is the closed ball centred at $x$ with the radius $r$.

**Corollary 1.7.** Let $\Omega = \mathbb{R}^N \setminus B_a[0]$ with $a > 0$ if $N = p$ and $a = 0$ if $N < p$. Then $\mathcal{D}^{1,p}_0(\Omega)$ is continuously embedded in $W^{1,p}_{loc}(\Omega)$, i.e., for every compact set $K$ in $\Omega$, there exists $C = C(K, p) > 0$ such that

$$
\int_K (|u(x)|^p + |\nabla u(x)|^p) \, dx \leq C \int_{\Omega} |\nabla u(x)|^p \, dx, \ \forall u \in \mathcal{D}^{1,p}_0(\Omega).
$$

1.2.2. **The cylindrical $(p, q)$-Hardy potentials** ($1 \leq k < N$). The weight functions provided by Theorem 1.3 and Theorem 1.6 do not exhaust the entire $\mathcal{H}_{p,q}(\Omega)$. In [11], for $N > p$, $q \in [p, p^*]$ and $\frac{N}{\alpha(p,q)} < k \leq N$ (equivalently, $q \in (P^*(k), p^*)$ if $k \leq p$, and $q \in [p, p^*]$ if $k > p$) Badiale-Tarantello obtained the following cylindrical version of the C-K-N inequality for $\Omega = \mathbb{R}^N$ and $k \geq 2$:

$$
\int_{\Omega} \frac{|u(x)|^q}{|y|^{\alpha(p,q)}} \, dx \leq C \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \ \forall u \in C^1(\Omega).
$$

(1.17)

In [10, 11, 54], using the above inequality, it has been proved that, if $r^{\frac{N}{\alpha(p,q)}} \phi(r) \in L^\infty((0, \infty))$, then the cylindrical weights $g(x) = \phi(|y|) \in \mathcal{H}_{p,q}(\mathbb{R}^N)$. In this article, we consider more general class of domains and the weight functions of the form given in (A) and also allow $k = 1$. For brevity, we only consider the case $N > p$. First we extend (1.17) for more general sectorial sets.

**Theorem 1.8.** Let $p \in (1, N)$ and $1 \leq k \leq N$. For $S \subset \mathbb{S}^{k-1}$ and $a, b \in [0, \infty]$ with $a < b$, let $\Omega = \Omega_{a, b, S} \times \mathbb{R}^{N-k}$.

(i) Then $|y|^{-\frac{N}{\alpha(p,q)}} \in \mathcal{H}_{p,q}(\Omega)$ for $q \in \begin{cases} [p, p^*], & k > p; \\ (P^*(k), p^*], & k \leq p. \end{cases}$

(ii) If $0 \notin \Omega_1$ and $k < p$, then $|y|^{-\frac{N}{\alpha(p,q)}} \in \mathcal{H}_{p,q}(\Omega)$ for $q \in [p, P^*(k)]$.

**Remark 1.9.** (i) If $a = 0$ and $b = \infty$, then for $|y|^{-s}$ to be in $\mathcal{H}_{p,q}(\Omega)$, it is necessary that $s = \frac{N}{\alpha(p,q)}$, one can see this by considering the scaling of a function. On the other hand, if $a > 0$ then Theorem 1.8 holds for $|y|^{-s}$ with $s \geq \frac{N}{\alpha(p,q)}$, and if $b < \infty$, then Theorem 1.8 holds for $|y|^{-s}$ with $s \leq \frac{N}{\alpha(p,q)}$. Since $|x|^{-s} \leq |y|^{-s}$, the restriction $p \leq q \leq p^*$ is also necessary for (1.17) (see also Remark 1.2).

(ii) In [40, Theorem 1.1], authors extended (1.17) by replacing $|y|$ (which is the distance of $x$ from $\mathbb{R}^{N-k}$) with distance function $\delta_E$ from general closed set $E$ in $\mathbb{R}^N$ and also allow the case $k = 1$. More precisely, for $q \in [p, p^*]$, they have established

$$
\int_{\mathbb{R}^N} \frac{|u(x)|^q}{\delta_E(x)^{\frac{N}{\alpha(p,q)}}} \, dx \leq C \left( \int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{\frac{q}{p}}, \ \forall u \in C^1_c(\mathbb{R}^N),
$$

if and only if, the Assouad dimension of $E$ is strictly less than $\frac{Np}{q}$. In particular, for $E = \mathbb{R}^{N-k}$, the Assouad dimension of $E$ is $N - k$ and $\delta_E(x) = |y|$ and hence, (1.17) holds if and only if $N - k < \frac{Np}{q}$. 


Thus for (1.17) to hold, we must have $q > P^*(k)$ as given in part (i) of the above theorem. In part (ii), we addresses the complementary case: $q \in [p, P^*(k)]$ for $k < p$ on a sectorial set with a hole at the origin.

(iii) For $q \in (0, P^*(k)]$, $|y|^{-\frac{N}{\alpha(p,q)}}$ is not locally integrable on any open set in $\mathbb{R}^k$ that contains the origin. On the other hand, if $\Omega_1$ does not contain the origin and $k < p$, then the above theorem ensures that $|y|^{-\frac{N}{\alpha(p,q)}} \in H_{p,q}(\Omega)$ for $q \in (p, p^*)$. In particular, by taking $k = 1, S = \{1\}, a = 0$, and $b = \infty$, we obtain (1.17) for $\Omega = \mathbb{R}_+^N$ and for $q \in [p, p^*)$.

The following corollary is immediate from the above theorem.

**Corollary 1.10.** Let $\Omega, q$ be as given in Theorem 1.8. Let $g(x) = g_1(y)$ be such that $\tilde{g}_1 \in L^\infty((a, b), r^{-\frac{N}{\alpha(p,q)}})$. Then the same conclusions of Theorem 1.8 hold for $g$ in place of $|y|^{-\frac{N}{\alpha(p,q)}}$.

Next, we consider the case in which both $g_1$ and $g_2$ are in certain Lorentz spaces. Indeed, the product of any two functions from the Lorentz spaces need not be a $(p, q)$-Hardy potential (see Example 7.2). Here we provide a two-parameter family of compatible pairs of Lorentz spaces so that the product of functions from these Lorentz spaces always give rise to a $(p, q)$-Hardy potential.

**Theorem 1.11.** Let $p \in (1, N)$ and let $t \in [0, 1]$. Let $k > p$ if $t > 0$, and $N - k > p$ if $t < 1$. Let $\Omega$ and $g$ be as given in (A).

(i) For $s, t \in [0, 1]$ with $st < 1$, let $(g_1, g_2) \in X_1 \times X_2 := L_k^{(1-st)p} \times L^\frac{1}{st}(\Omega_1) \times L^\frac{1}{st}(\Omega_2)$. Then $g \in H_{p,q}(\Omega)$ for $q = (1 - st)p$.

(ii) For $s, t \in [0, 1]$, let $(g_1, g_2) \in X_1 \times X_2 := L_k^{(1-st)p} \times L^\frac{N-k}{1-t} \infty(\Omega_1) \times L^\frac{N-k}{1-t} \infty(\Omega_2)$. Then $g \in H_{p,q}(\Omega)$ for $q = (1 - st)p + st p^*$.

Furthermore,

$$\int_\Omega |g(x)||u(x)|^q \, dx \leq C \|g_1\|_{X_1} \|g_2\|_{X_2} \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^\frac{q}{p}, \quad \forall u \in C^1_c(\Omega),$$

for some $C = C(N, k, p, q) > 0$.

In the following theorem, we consider the compatible pairs of $(g_1, g_2)$ from the weighted Lebesugue spaces. For $q \in (0, p^*)$, we set

$$\beta(p, q) := \frac{p^* - N'p}{p^* - N'q}.$$

**Theorem 1.12.** Let $p \in (1, N)$ and $k < N$. For $S \subset \mathbb{S}^{N-k-1}$ and $a, b \in (0, \infty)$ with $a < b$, let $\Omega = \Omega_{a,b,S} \times \mathbb{R}^{N-k}$ and $g$ be as given in (A). If $k \neq p$ and

$$\begin{cases}
(\tilde{g}_1, \tilde{g}_2) \in X_1 \times X_2 := \left\{
L^1((a, b), r^{-\frac{(p-k)q}{p} + k - 1}) \times L^\frac{p}{p-q}((0, \infty), r^N), \quad q \in (0, p);
L^\beta((a, b), r^{p-1}) \times L^\frac{\beta(p,q)}{M(p,q)}((0, \infty)), \quad q \in [p, P^*(1));
L^\infty((a, b)) \times L^\frac{\beta(p,q)}{M(p,q)}((0, \infty)), \quad q \in [P^*(1), p^*)
\right.\end{cases}$$

then $g \in H_{p,q}(\Omega)$. Furthermore,

$$\int_\Omega |g(x)||u(x)|^q \, dx \leq C \|\tilde{g}_1\|_{X_1} \|\tilde{g}_2\|_{X_2} \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^\frac{q}{p}, \quad \forall u \in C^1_c(\Omega),$$

where $C = C(N, k, p, q) > 0$. 

Having obtained a large class of cylindrical and non-cylindrical \((p, q)\)-Hardy potentials, next we consider the existence of solution for the Euler-Cauchy equation associated to (1.1).

1.3. The existence of solution. For \(g \in \mathcal{H}_{p,q}(\Omega)\) with \(g \geq 0\), let \(B_q(g)\) be the best constant in (1.1). Then

\[
\frac{1}{B_q(g)} = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p}{(\int_{\Omega} g|u|^q)^{\frac{p}{q}}}: u \in \mathcal{D}_0^{1,p}(\Omega) \setminus \{0\} \right\} = \inf \left\{ \int_{\Omega} |\nabla u|^p : u \in N_g \right\},
\]

where \(N_g = \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_{\Omega} g|u|^q = 1 \right\}\). If \(\frac{1}{B_q(g)}\) is attained for some \(u \in N_g\), then one can verify that for \(q > 1\), \(u\) satisfies the following equation:

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v = \frac{1}{B_q(g)} \int_{\Omega} g|u|^{q-2}uv, \quad \forall v \in \mathcal{D}_0^{1,p}(\Omega).
\]

In other words, for \(\lambda = \frac{1}{B_q(g)}\), \(u\) solves the following nonlinear partial differential equation weakly:

\[
-\Delta_p u = \lambda g(x)|u|^{q-2}u, \quad u \in \mathcal{D}_0^{1,p}(\Omega),
\]

where \(\Delta_p\) is the \(p\)-Laplace operator defined as \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\). Observe that, for \(q \neq p\), \(v = (\lambda B_q(g))^{-\frac{1}{q-p}} u\) solves the above equation for any \(\lambda > 0\). For \(q = p\), (1.19) is a nonlinear eigenvalue problem and a non-zero solution exists only for certain \(\lambda\) that are precisely the eigenvalues of (1.19).

The above partial differential equation appears in many important problems in mathematics as well as in physics. For example, radial \(g\) in the Matukuma’s models for the dynamics of globular cluster of stars [41, 50, 62], cylindrical potentials in the study of dynamics of galaxies [13, 21], scalar curvature problem [42], the weighted eigenvalue problems (1.19) for \(q = p\). One of the sufficient conditions that ensure the best constant \(B_q(g)\) is attained in \(\mathcal{D}_0^{1,p}(\Omega)\) is the compactness of the following map:

\[
G_q(u) := \int_{\Omega} g|u|^q, \quad u \in \mathcal{D}_0^{1,p}(\Omega).
\]

Indeed, for \(g \in \mathcal{H}_{p,q}(\Omega)\), from (1.1) it is clear that \(G_q\) is continuous.

For \(q = p\), in the context of studying weighted eigenvalue problems, many authors considered \(g\) in various Lebesgue and Lorentz spaces so that the map \(G_q\) is compact. For example, for \(g \in L^\gamma(\Omega)\), see [46] \((N > p = 2, \gamma > \frac{N}{2})\), [3] \((N > p = 2, \gamma = \frac{N}{2})\), [4, 58] \((N > p, \gamma = \frac{N}{p})\). For \(g \in L^{\frac{N}{\gamma}}(\Omega)\), see [61] \((N > p, \gamma < \infty)\), [9] \((N > p = 2, \gamma = \infty)\), [3] \((N = p = 2)\). In [8], the authors obtained the compactness of \(G_p\) for \(g\) dominated by a certain radial function. For \(q \neq p\), there are few results where the compactness of \(G_q\) is proved. For example, for \(p \in (1, N)\) and \(q \in (0, p^*)\), \(g \in L^{\infty}(\Omega) \cap L^{\alpha(p,q)}(\Omega)\) [63], for \(q \in [2, p^*)\), \(g \in L^{\alpha(2,q),\gamma}(\Omega)\) with \(1 \leq \gamma < \infty\) [61] and \(g\) in the closure of \(C_c^\infty(\Omega)\) in \(L^{\alpha(2,q),\infty}(\Omega)\) [9]. In this article, we state certain general assumptions on \(g\) that ensures the compactness of \(G_q\) and unify all the above compactness results.

For \(i = 1, 2\), let \(X_i(\Omega_i)\) be Banach (function) space containing \(C_c^\infty(\Omega_i)\). We define \(\mathcal{F}_{X_i} := \overline{C_c^\infty(\Omega_i)}^{X_i}\).

**Theorem 1.13.** Let \(\Omega\) and \(g\) be as given in (A) and (B) and let \(g \geq 0\). For \(i = 1, 2\), let \(g_i \in \mathcal{F}_{X_i}\) and the following inequality holds:

\[
\int_{\Omega} g(x)|u(x)|^q \, dx \leq C\|g_1\|_{X_1}\|g_2\|_{X_2} \left(\int_{\Omega} |\nabla u(x)|^p \, dx\right)^{\frac{q}{p}}, \quad \forall u \in C_c^1(\Omega).
\]
Then the map \( G_q = \int_{\Omega} |g|u|^q \) is compact on \( D_0^{1,p}(\Omega) \) for \( q \in (0, \delta) \), where \( \delta = p^* \) (if \( N > p \)), and \( \delta = \infty \) (if \( N \leq p \)). Moreover, for \( q \in (1, \delta) \), (1.19) admits a non-negative solution in \( D_0^{1,p}(\Omega) \).

**Remark 1.14.** (i) For \( q < p \), \( C_c^\infty(\Omega_i) \) is dense in the function spaces \( X_i \) as considered in Theorem 1.3 and Theorem 1.11. Thus by the above theorem, for \( g_i \in X_i \), the map \( G_q \) is compact. For \( q \geq p \), the map \( G_q \) is compact for \( g_i \) in \( F_{X_i} \), a proper closed subspace of the respective space \( X_i \). As a consequence, Theorem 1.13 together with Theorem 1.3 extends the compactness results of [9, 61] to \( q \in (0, p^*) \), [5] to \( q \in (0, \infty) \).

(ii) Since \( C_c^\infty((a, b)) \) is dense in \( X \) given in Theorem 1.6, for \( \tilde{g} \in X \) using Theorem 1.13 the map \( G_q \) is compact. For \( q \in (0, P^*(1)) \), \( C_c^\infty((a, b)) \) is dense in \( X_i \) given in Theorem 1.12, and hence for \( (\tilde{g}_1, \tilde{g}_2) \in X_1 \times X_2 \), \( G_q \) is compact.

(iii) Some of the non-compact cases for \( p = q \) are discussed in [6] using the variant of concentration compactness lemma and Maz’ya’s capacity conditions.

The rest of this article is organized as follows. In Section 2, we briefly discuss symmetrization and recall the Lorentz and Lorentz-Zygmund spaces. In Section 3, we prove some important properties of \( H_{p,q}(\Omega) \) that are required in subsequent sections. Section 4, Section 5, and Section 6 contain the proofs of theorems 1.3-1.13. In Section 7, we discuss some examples and the necessary conditions. In Appendix, we prove some results on Lorentz and Lorentz-Zygmund spaces and present alternative proofs of certain classical embeddings.

## 2. Preliminaries

In this section, we briefly describe the symmetrization and the one dimensional decreasing rearrangements. Using this, we define the Lorentz and Lorentz-Zygmund spaces and list some of their properties.

Firstly, we list some of the notations and conventions we used in this article:

- \( \frac{1}{0} = \infty \).
- \( \omega_N := \frac{\pi^N}{\Gamma(\frac{N}{2} + 1)} \) is the measure of a unit ball in \( \mathbb{R}^N \).
- For \( q \in [1, \infty) \), \( q' \) denote the conjugate of \( q \), i.e., \( \frac{1}{q} + \frac{1}{q'} = 1 \).
- For \( a, b, c \in [1, \infty] \) we say \( (a, b, c) \) is a conjugate triple, if \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 \), and we say \( (a, b) \) is a conjugate pair if \( (a, b, \infty) \) is a conjugate triple.
- If \( a \in [0, 1] \) and \( b \in [1, \infty) \), then \( \left( \frac{1}{a}, \frac{1}{1-a} \right) \) and \( (b, \frac{b}{b+1}) \) are conjugate pairs. If \( a, b \in [0, 1] \), then \( \left( \frac{1}{a}, \frac{1}{b}, \frac{1}{1-a-b} \right) \) is a conjugate triple.
- For \( u : \mathbb{R}^N \rightarrow \mathbb{R} \) and \( z \in \mathbb{R}^{N-k} \), the \( z \)-section of \( u \) is denoted by \( u_z \) i.e., \( u_z(y) = u(y, z) \) \( \forall y \in \mathbb{R}^k \). Similarly, the \( y \)-section of \( u \) is denoted by \( u_y \) i.e., \( u_y(z) = u(y, z) \), \( \forall z \in \mathbb{R}^{N-k} \).
- For \( u \in C^1(\mathbb{R}^N) \) and \( z \in \mathbb{R}^{N-k} \), \( \nabla_y u(y, z) := \nabla_y u_z(y) = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_k} \right) \).

### 2.1. Symmetrization

Let \( \Omega \subset \mathbb{R}^N \) be an open set and \( \mathcal{M}(\Omega) \) be the set of all extended real valued Lebesgue measurable functions that are finite a.e. in \( \Omega \). Given a function \( f \in \mathcal{M}(\Omega) \) and for \( s > 0 \), we define \( E_f(s) = \{ x \in \Omega : |f(x)| > s \} \). The distribution function \( \mu_f \) of \( f \) is defined as \( \mu_f(s) = |E_f(s)| \), where \( | \cdot | \) denotes the Lebesgue measure in \( \mathbb{R}^N \). We define the one dimensional decreasing rearrangement \( f^* \) of \( f \) as

\[
  f^*(t) = \inf \{ s > 0 : \mu_f(s) < t \}, \quad t > 0.
\]
The map $f \mapsto f^*$ is not sub-additive. However, we obtain a sub-additive function from $f^*$, namely the maximal function $f^{**}$ of $f^*$, defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(\tau) \, d\tau, \quad t > 0.$$ 

Next, we explicitly compute the rearrangement of certain class of functions.

**Remark 2.1.** Let $g$ be a non-negative strictly decreasing function on $\mathbb{R}^+ \cup \{0\}$. Let $f(x) := g(|x|)$ for $x \in \mathbb{R}^N$. Then for $s \in \text{Range}(g)$,

$$E_f(s) = \{x \in \mathbb{R}^N : g(|x|) > s\} = \{x \in \mathbb{R}^N : |x| < g^{-1}(s)\},$$

and $\mu_f(s) = \omega_N(g^{-1}(s))^N$. Hence for $t > 0$,

$$f^*(t) = \inf \{s > 0 : \mu_f(s) \leq t\} = \inf \{s \in (\inf g, \sup g) : \mu_f(s) \leq t\} \leq \inf \{s \in \text{Range}(g) : \omega_N(g^{-1}(s))^N \leq t\} = \inf \{s \in \text{Range}(g) : s \geq g\left(\omega_N^{-\frac{1}{p}} t^{\frac{1}{q}}\right)\} = g\left(\omega_N^{-\frac{1}{p}} t^{\frac{1}{q}}\right).$$

Further, notice that if $g$ is onto, then we have $f^*(t) = g(\omega_N^{-\frac{1}{p}} t^{\frac{1}{q}})$.

Now we state two important inequalities concerning symmetrization. For more details we refer to the books [32, 53].

**Proposition 2.2.** Let $N \geq 2$.

(a) **Hardy-Littlewood inequality:** Let $f$ and $g$ be nonnegative measurable functions. Then

$$\int_{\Omega} f(x) g(x) \, dx \leq \int_0^{\|\Omega\|} f^*(t) g^*(t) \, dt.$$ 

(b) **Pólya-Szegő inequality:** Let $u \in D_0^{1,p}(\mathbb{R}^N)$. Then

$$N^p \omega_N^{\frac{p}{N}} \int_0^\infty s^{p - \frac{p}{q}} |u^*(s)|^p \, ds \leq \int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx.$$ 

The following inequality is due to Maz’ja [47, Lemma 1, Pg-49].

**Proposition 2.3.** Let $q \geq 1$. Then, for any measurable function $f : \mathbb{R}^N \mapsto \mathbb{R}$ the following inequality holds

$$\int_0^\infty f^*(t)^q \, dt^q \leq \left(\int_0^\infty f^*(t) \, dt\right)^q.$$ 

2.2. **The Lorentz and Lorentz-Zygmund spaces.** The Lorentz spaces are two parameter family of function spaces introduced by Lorentz in [44] that refine the classical Lebesgue spaces. For more details on the Lorentz spaces, we refer to [1, 22].

Let $\Omega$ be an open set in $\mathbb{R}^N$ and $f \in M(\Omega)$. For $(p, q) \in (0, \infty) \times (0, \infty]$ we consider the following quantity:

$$|f|_{p,q} := \left\| t^{\frac{1}{p} - \frac{1}{q}} f^*(t) \right\|_{L^q((0,\infty))} = \begin{cases} \left( \int_0^\infty \left(t^{\frac{1}{p} - \frac{1}{q}} f^*(t)\right)^q \, dt\right)^{\frac{1}{q}}, & q < \infty; \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), & q = \infty. \end{cases} \quad (2.1)$$
The Lorentz space \( L^{p,q}(\Omega) \) is defined as
\[
L^{p,q}(\Omega) := \{ f \in \mathcal{M}(\Omega) : |f|_{p,q} < \infty \},
\]
where \(|f|_{p,q}\) is a complete quasi norm on \( L^{p,q}(\Omega) \). For \((p, q) \in (1, \infty) \times (0, \infty)\),
\[
|f|_{p,q} := \left( \frac{1}{p} \int |f|^p \right)^{\frac{1}{p}} \left( \frac{1}{q} \int |f|^q \right)^{\frac{1}{q}}.
\]
is a norm on \( L^{p,q}(\Omega) \) and it is equivalent to \(|f|_{p,q}\) [22, Lemma 3.4.6]. Note that \( L^{p,p}(\Omega) = L^p(\Omega) \) for \( p \in (0, \infty) \) and \( L^{p,\infty}(\Omega) \) coincides with the weak-\(L^p\) space := \( \{ f \in \mathcal{M}(\Omega) : \sup_{s>0} s(\alpha f(s))^{\frac{1}{p}} < \infty \} \).

**Remark 2.4.** For \( 0 < s < p \) and \( f \in L^{p,\infty}(\mathbb{R}^N) \), we consider the following quantity:
\[
|||f|||_{p,\infty} := \sup_{E \subset \mathbb{R}^N, |E| < \infty} |E|^{\frac{1}{p} - \frac{1}{s}} \left( \int_E |f(x)|^p \, dx \right)^{\frac{1}{s}}.
\]
Then \(|f|_{p,\infty} \leq |||f|||_{p,\infty} \leq \left( \frac{p}{p-s} \right)^{\frac{1}{s}} ||f||_{p,\infty} \) (see [19, Theorem 5.18]).

In the following proposition we list some properties of the Lorentz spaces.

**Proposition 2.5.** Let \( p, q, \tilde{p}, \tilde{q} \in [1, \infty] \).

(i) For \( \alpha > 0 \), \(|\alpha|_{p} = |\alpha|_{p,\tilde{p},\tilde{q}}\).

(ii) Generalized Hölder inequality: Let \( f \in L^{p_1,q_1}(\Omega) \) and \( g \in L^{p_2,q_2}(\Omega) \), where \((p_i, q_i) \in (1, \infty) \times [1, \infty)\) for \( i = 1, 2 \). If \((p, q)\) be such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \), then
\[
||fg||_{p,q} \leq C ||f||_{p_1,q_1} ||g||_{p_2,q_2},
\]
where \( C = C(p) > 0 \) is a constant such that \( C = 1 \), if \( p = 1 \) and \( C = p' \), if \( p > 1 \).

(iii) If \( q \leq \tilde{q} \), then \( L^{p,q}(\Omega) \hookrightarrow L^{\tilde{p},\tilde{q}}(\Omega) \), i.e., there exists a constant \( C > 0 \) such that
\[
||f||_{p,q} \leq C ||f||_{\tilde{p},\tilde{q}}, \quad \forall f \in L^{p,q}(\Omega).
\]

(iv) If \( \tilde{p} < p \), then \( L^{p,q}(\Omega) \hookrightarrow L^{\tilde{p},\tilde{q}}_{\text{loc}}(\Omega) \).

**Proof.** Proof of (i) directly follows using the definition of the Lorentz space. Proof of (ii) follows using [34, Theorem 4.5]. For the proof of (iii) and (iv), see [22, Proposition 3.4.3 and Proposition 3.4.4].

The Lorentz-Zygmund spaces are three parameter family of function spaces that refine the Lorentz spaces. For more information on Lorentz-Zygmund spaces, we refer to [12, 23]. Let \( \Omega \subset \mathbb{R}^N \) be a bounded open set and let \( l_1(t) = \log(\frac{|t|}{\Omega}) \). Given a function \( f \in \mathcal{M}(\Omega) \) and for \((p, q, \alpha) \in (0, \infty) \times (0, \infty) \times \mathbb{R} \), we consider the following quantity:
\[
|f|_{p,q,\alpha} := \left( \frac{1}{p} \int_{\Omega} \left( \frac{1}{q} \int_{\Omega} \left( l_1(t)^\alpha f^*(t) \right)^q \, dt \right)^{\frac{1}{q}} \right)^{\frac{1}{p}}\left( \sup_{0 < \epsilon < |\Omega|} \frac{1}{\epsilon^\alpha} l_1(t)^\alpha f^*(t) \right), \quad 0 < q < \infty;
\]
\[
|f|_{p,q,\alpha} := \left( \frac{1}{p} \int_{\Omega} \left( \frac{1}{q} \int_{\Omega} \left( l_1(t)^\alpha f^*(t) \right)^q \, dt \right)^{\frac{1}{q}} \right)^{\frac{1}{p}}\left( \sup_{0 < \epsilon < |\Omega|} \frac{1}{\epsilon^\alpha} l_1(t)^\alpha f^*(t) \right), \quad q = \infty.
\]

Then the Lorentz-Zygmund space \( L^{p,q,\alpha}(\Omega) \) is defined as
\[
L^{p,q,\alpha}(\Omega) := \{ f \in \mathcal{M}(\Omega) : |f|_{p,q,\alpha} < \infty \},
\]
Proposition 3.1. Let $\Omega$ be an open set in $\mathbb{R}^N$ with $N \geq 1$. For $t \in [0, 1]$, define $g_t(x) = |g_1(x)|^t |g_2(x)|^{1-t}$.

(i) For $q_1, q_2 \in (0, \infty)$, let $g_1 \in \mathcal{H}_{p,q_1}(\Omega)$ and $g_2 \in \mathcal{H}_{p,q_2}(\Omega)$. Then $g_t \in \mathcal{H}_{p,q}(\Omega)$ with $q = t q_1 + (1-t) q_2$.

Proof. This assertion immediately follows using the definition of Lorentz-Zygmund spaces.

(ii) and (iii) Proof follows using [12, Theorem 9.1 and Theorem 9.3].

Next we give some examples of functions that lie in certain Lorentz and Lorentz-Zygmund spaces.

Example 2.7. (i) For $0 < d < N$, consider $g(t) = t^{-d}, t \in (0, \infty)$ and $f(x) = g(|x|), x \in \mathbb{R}^N$. Since $g$ is strictly decreasing and onto, by Remark 2.1, $f^*(t) = g(\frac{\omega_N}{s} \frac{N}{t}) = (\frac{\omega_N}{s} \frac{N}{t})^{\frac{d}{N}}$. Consequently,

$$f^*(t) = \frac{1}{t} \int_0^t \left( \frac{\omega_N}{s} \right)^{\frac{d}{N}} ds = \frac{N}{N-d} \left( \frac{\omega_N}{t} \right)^\frac{d}{N}. $$

Therefore, $f \in L^{\frac{d}{N}, \infty}(\mathbb{R}^N)$ and $\|f\|_{\frac{d}{N}, \infty} = \frac{N \omega_N^d}{N-d}$.

(ii) Consider $g(t) = t^{-N}(\log(e(\frac{t}{R_1})))^{-N}, t \in (0, R_1)$ with $R_1 = \text{Re}^{\frac{1}{N}}$ and $f(x) = g(|x|), x \in B_{R_1}(0)$. One can verify that $g$ is strictly decreasing and onto. Then using Remark 2.1, we obtain

$$f^*(t) = \frac{\omega_N}{t} \left( \log(e \frac{\omega_N R_1}{t}) \right)^{-N} \leq \frac{\omega_N}{t} \left( \log(e \frac{\omega_N R_1}{t}) \right)^{-N}. $$

Hence

$$|f|_{1, \infty; N} = \sup_{0 < t < |B_{R_1}(0)|} t \left( \log(e \frac{|B_{R_1}(0)|}{t}) \right)^N f^*(t) \leq \omega_N.$$

Therefore, $f \in L^{1, \infty; N}(B_{R_1}(0))$.

3. The space $\mathcal{H}_{p,q}(\Omega)$

In this section, we provide various methods for constructing functions in $\mathcal{H}_{p,q}(\Omega)$. We also discuss some properties and inclusion relations of the function space $\mathcal{H}_{p,q}(\Omega)$, which will be used frequently in the subsequent sections.

Proposition 2.6. Let $p, q, \tilde{q} \in [1, \infty]$ and $\alpha, \beta \in (-\infty, \infty)$.

(i) Let $p, q \in (1, \infty], \alpha \in \mathbb{R}$, and $\gamma > 0$. Then there exists $C > 0$ such that $\|f^\gamma\|_{p, q, \alpha} \leq C \|f\|_{p, q, \alpha}^\gamma, \\forall f \in L^{p,q,\alpha}(\Omega)$.

(ii) If $\tilde{p} > p$, then $L^{\tilde{p},\beta}(\Omega) \hookrightarrow L^{p,\alpha}(\Omega)$, i.e., there exists $C > 0$ such that

$$\|f\|_{p, q, \alpha} \leq C \|f\|_{\tilde{p}, \beta}, \\forall f \in L^{p,q,\alpha}(\Omega).$$

(iii) If either $q \leq \tilde{q}$ and $\alpha \geq \beta$ or, $q > \tilde{q}$ and $\alpha + \frac{1}{q} > \beta + \frac{1}{\tilde{q}}$, then $L^{p,\alpha}(\Omega) \hookrightarrow L^{\tilde{p},\beta}(\Omega)$.

Proof. (i) This assertion immediately follows using the definition of Lorentz-Zygmund spaces.

(ii) and (iii) Proof follows using [12, Theorem 9.1 and Theorem 9.3].

where $|f|_{p,q,\alpha}$ is the quasi norm on $L^{p,q,\alpha}(\Omega)$, and for $(p, q, \alpha) \in (1, \infty) \times [1, \infty] \times \mathbb{R}$,

$$\|f\|_{p,q,\alpha} = \left\|\left| \frac{1}{t^{\frac{1}{p}-\frac{1}{q}}} l_1(t)^{\alpha} f^*(t) \right|_{L^q((0,|\Omega|))} \right\| (2.3)$$

is a norm in $L^{p,q,\alpha}(\Omega)$ equivalent to $|f|_{p,q,\alpha}$ [12, Corollary 8.2]. In the following proposition we list some important properties of the Lorentz-Zygmund spaces.
(ii) Let $g_1 \in \mathcal{H}_{p,q}(\Omega)$ and $g_2 \in L^1(\Omega)$. Then for $t \in (0,1]$, $g_t \in \mathcal{H}_{p,q}(\Omega)$ with $q = tp$.

Proof. (i) For $t \in [0,1]$, let $q = tq_1 + (1-t)q_2$. For $t = 0,1$, clearly, $g_t \in \mathcal{H}_{p,q}(\Omega)$. For $t \in (0,1)$, for $u \in C^1_c(\Omega)$, we use the Hölder’s inequality to the conjugate pair $(\frac{1}{t},\frac{1}{1-t})$ and (1.1) for $g_1$ and $g_2$ to obtain the following inequalities:

$$
\int_\Omega |g_t(x)||u(x)|^q \, dx = \int_\Omega |g_1(x)|^t|g_2(x)|^{1-t}|u(x)|^q |u(x)|^{(1-t)q_2} \, dx
\leq \left( \int_\Omega |g_1(x)||u(x)|^{q_1} \, dx \right)^t \left( \int_\Omega |g_2(x)||u(x)|^{q_2} \, dx \right)^{1-t}
\leq C \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^\frac{q_1}{p} \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^\frac{q_2}{p} = C \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^\frac{q}{p}.
$$

(ii) For $t \in (0,1]$, let $q = tp$. For $t = 1$, clearly, $g_t = |g_1| \in \mathcal{H}_{p,q}(\Omega)$. Let $t \in (0,1)$. For $u \in C^1_c(\Omega)$, we use the Hölder’s inequality and (1.1) for $g_1$ to obtain the following:

$$
\int_\Omega |g_t(x)||u(x)|^q \, dx = \int_\Omega |g_1(x)|^t|g_2(x)|^{1-t}|u(x)|^p \, dx
\leq \left( \int_\Omega |g_1(x)||u(x)|^p \, dx \right)^t \left( \int_\Omega |g_2(x)||u(x)|^p \, dx \right)^{1-t}
\leq C \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^\frac{q}{p}.
$$

\[\Box\]

Remark 3.2. (i) By taking $g = g_1 = g_2$ in the above proposition, we obtain the following inclusions:

$$
\mathcal{H}_{p,q_1}(\Omega) \cap \mathcal{H}_{p,q_2}(\Omega) \subset \mathcal{H}_{p,q}(\Omega), \quad \forall q \in [q_1, q_2].
$$

In particular,

$$
\mathcal{H}_{p,q}(\Omega) \supset \begin{cases}
\mathcal{H}_{p,p}(\Omega) \cap L^\infty(\Omega), & q \in [p, p^*]; \\
\mathcal{H}_{p,p}(\Omega) \cap L^1(\Omega), & q \in (0, p].
\end{cases}
$$

(ii) By the Sobolev inequality, $1 \in \mathcal{H}_{p,p^*}(\Omega)$. Thus by Proposition 3.1, for $g_1 \in \mathcal{H}_{p,p}(\Omega)$ we get $g(x) = |g_1(x)|^{\frac{p}{p^*}} \in \mathcal{H}_{p,q}(\Omega)$, for $q \in [p, p^*]$.

Proposition 3.3. Let $\Omega$ and $g$ be as given in (A). For $q \in (0,p]$, let $g_1 \in \mathcal{H}_{p,q}(\Omega_1)$ and $g_2 \in L^{\frac{p}{p-q}}(\Omega_2)$. Then $g(x) = g_1(y)g_2(z) \in \mathcal{H}_{p,q}(\Omega)$.

Proof. Let $q \in (0,p]$ and $u \in C^1_c(\Omega)$. Since $g_1 \in \mathcal{H}_{p,q}(\Omega_1)$ and $|\nabla_y u(y,z)| \leq |\nabla u(y,z)|$, we easily obtain

$$
\int_\Omega |g_1(y)||g_2(z)||u(y,z)|^q \, dydz \leq C \int_{\Omega_2} |g_2(z)| \left( \int_{\Omega_1} |\nabla_y u(y,z)|^p \, dy \right)^{\frac{q}{p}} \, dz
\leq C \int_{\Omega_2} |g_2(z)| \left( \int_{\Omega_1} |\nabla u(y,z)|^p \, dydz \right)^{\frac{q}{p}} \leq C \left( \int_{\Omega} |\nabla u(y,z)|^p \, dydz \right)^{\frac{q}{p}},
$$

where the last inequality follows from the Hölder’s inequality applied to the functions $|g_2(z)|$ and $f(z) := (\int_{\Omega_1} |\nabla u(y,z)|^p \, dy)^{\frac{q}{p}}$. Thus $g \in \mathcal{H}_{p,q}(\Omega)$. \[\Box\]

Remark 3.4. Let $q \in (0,p]$, and $\Omega_2$ be bounded if $q < p$. Then $g_2 = 1 \in L^{\frac{p}{p-q}}(\Omega_2)$ for every $q \in (0,p]$. For $q \in (0,p)$ and $g_1 \in \mathcal{H}_{p,q}(\Omega_1)$, define $g(x) = g_1(y)$. Then by the above proposition, $g \in \mathcal{H}_{p,q}(\Omega)$. 

For $\Omega = \Omega_1 \times \Omega_2$ with $\Omega \subseteq \mathbb{R}^k$ and $\Omega_2 \subseteq \mathbb{R}^{N-k}$, and for $u \in C^1_c(\Omega)$, $u_z : \Omega_2 \to \mathbb{R}$ is the $z$-section of $u$, defined as $u_z(y) := u(y, z)$. In the following proposition, we provide a sufficient condition for $g$ to be in $H_{p,q}(\Omega)$ on certain symmetric domains.

**Proposition 3.5.** Let $p \in (1, N)$ and $1 \leq k \leq N$. For $S \subseteq S^{k-1}$ and $a,b \in (0, \infty]$, let $\Omega = \Omega_{a,b,S} \times \mathbb{R}^{N-k}$, $g$ be as given in (A). Assume that

(i) $\tilde{g}_1 \in X_1 := L^1((a,b), r^{k-1}h(r))$, for some measurable function $h : (a,b) \mapsto (0, \infty)$, and

(ii) for some $\delta > 0$ and $\gamma \geq 0$ with $0 < \gamma + \delta \leq 1$,

\[
\tilde{g}_2 \in X_2 := \begin{cases} 
L^{1/(\gamma + \delta)}((0, \infty), r^{N-k-1}), & \text{for } \gamma + \delta < 1; \\
L^\infty((0, \infty)), & \text{for } \gamma + \delta = 1.
\end{cases}
\]

(iii) $C = C(k, p, q) > 0$ be such that for each $z \in \mathbb{R}^{N-k}$, $r \in (a, b), \omega \in S$, the following inequality holds:

\[
|u_z(r\omega)|^q \leq Ch(r) \left( \int_a^b \tau^{k-1}|u_z(\tau \omega)|^p \, d\tau \right)^{\gamma} \left( \int_a^b \tau^{k-1}|\nabla_y u_z(\tau \omega)|^p \, d\tau \right)^{\delta}, \quad (3.4)
\]

\[
\forall u \in C^1_c(\Omega).
\]

Then $g = |g_1g_2| \in H_{p,q}(\Omega)$ with $q = \delta p + \gamma p^*$, and

\[
\int_{\Omega} |g(x)||u(x)|^q \, dx \leq C\|\tilde{g}_1\|_{X_1}\|\tilde{g}_2\|_{X_2} \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall u \in C^1_c(\Omega),
\]

where $C = C(N, k, p, q) > 0$.

**Proof.** We consider two cases separately.

**Case 1:** $\gamma + \delta \in (0, 1)$. Let $\gamma > 0$. By noting that $(\frac{1}{\gamma}, \frac{1}{\delta}, \frac{1}{1-\gamma-\delta})$ is a conjugate triple, we integrate both sides of (3.4) over $S$ and apply the generalized Hölder’s inequality to get

\[
\int_S |u_z(r\omega)|^q \, dS \leq Ch(r) \left( \int_S dS \right)^{1-\gamma} \left( \int_S \int_a^b \tau^{k-1}|u_z(\tau \omega)|^p \, d\tau dS \right)^{\gamma} \left( \int_S \int_a^b \tau^{k-1}|\nabla_y u_z(\tau \omega)|^p \, d\tau dS \right)^{\delta}, \quad (3.5)
\]

Multiply the above inequality by $r^{k-1}\tilde{g}_1(r)$ and integrate over $(a, b)$ to get

\[
\int_{\Omega_{a,b,S}} \tilde{g}_1(|y|)|u_z(y)|^q \, dy \leq C \left( \int_a^b \tilde{g}_1(r)r^{k-1}h(r) \, dr \right)^{\gamma} \left( \int_{\Omega_{a,b,S}} |u_z(y)|^p \, dy \right)^{\delta},
\]

where $C = C(N, k, p, q) > 0$. Now multiply both sides by $\tilde{g}_2(|z|)$, integrate over $\mathbb{R}^{N-k}$ and then apply the Hölder’s inequality, so that the above inequality and the Fubini’s theorem gives

\[
\int_\Omega \tilde{g}_1(|y|)\tilde{g}_2(|z|)|u(y,z)|^q \, dx \leq C \left( \int_{\mathbb{R}^{N-k}} \tilde{g}_2(|z|)^{\frac{1}{1-(\gamma+\delta)}} \, dz \right)^{1/(\gamma+\delta)} \left( \int_a^b \tilde{g}_1(r)r^{k-1}h(r) \, dr \right)^{\gamma} \left( \int_{\Omega} |u(y,z)|^p \, dx \right)^{\delta}.
\]
Further, using the embedding of $D_0^{1,p}(\Omega)$ into $L^p(\Omega)$,
\[
\int_{\Omega} |g(x)||u(x)|^q \, dx \leq C \left( \int_{0}^{\infty} \tilde{g}_2(r) r^{1-(\gamma+\delta)} r^{N-k-1} \, dr \right)^{1-(\gamma+\delta)} \left( \int_{a}^{b} \tilde{g}_1(r) r^{k-1} h(r) \, dr \right) \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\delta+\gamma \frac{p}{p'}} , \quad \forall u \in C_c^1(\Omega).
\]
Therefore, $g \in \mathcal{H}_{p,q}(\Omega)$. For $\gamma = 0$, by noting that $(\infty, \frac{1}{\delta}, \frac{1}{1-\delta})$ is a conjugate triple, from the above calculations we also obtain $g \in \mathcal{H}_{p,q}(\Omega)$.

$\gamma + \delta = 1$: In this case we have $\tilde{g}_2 \in L^\infty((0, \infty))$ and $(\frac{1}{\gamma}, 1, \infty)$ is a conjugate triple. Then proof follows along the same lines except that in (3.5) (there is no term with the exponent $1 - \delta - \gamma$). \hfill \Box

The following proposition is analogous to Proposition 3.5, where we exchange the role of $g_1$ and $g_2$.

**Proposition 3.6.** Let $p \in (1, N)$ and $1 \leq k \leq N$. For $S \subset \mathbb{S}^{k-1}$ and $a, b \in (0, \infty]$, let $\Omega = \Omega_{a,b,S} \times \mathbb{R}^{N-k}$, $g$ be as given in (A). Assume that

(i) for some $\delta > 0$ and $\gamma \geq 0$ with $0 < \gamma + \delta \leq 1$, \\
$\tilde{g}_1 \in X_1 := \begin{cases} L^{1-(\gamma+\delta)}((a,b), r^{k-1}), & \text{for } \gamma + \delta < 1; \\ L^\infty((a,b)), & \text{for } \gamma + \delta = 1, \end{cases}$

and $\tilde{g}_2 \in X_2 := L^1((0, \infty), r^{N-k-1} h(r))$, for some measurable function $h : (0, \infty) \mapsto (0, \infty)$.

(ii) $C = C(N, k, p, q) > 0$ be such that for each $y \in \Omega_{a,b,S}$ and $(r, \omega) \in (0, \infty) \times \mathbb{S}^{N-k-1}$ the following inequality holds:

\[
|u_y(r\omega)|^q \leq C h(r) \left( \int_{0}^{\infty} \tau^{N-k-1} |u_y(\tau\omega)|^p \, d\tau \right)^{\gamma} \left( \int_{0}^{\infty} \tau^{N-k-1} |\nabla_y u_y(\tau\omega)|^p \, d\tau \right)^{\delta}, \\
\forall u \in C_c^1(\Omega).
\]

Then $g = |g_1g_2| \in \mathcal{H}_{p,q}(\Omega)$ with $q = \delta p + \gamma p^*$, and

\[
\int_{\Omega} |g(x)||u(x)|^q \, dx \leq C \|\tilde{g}_1\|_{X_1} \|\tilde{g}_2\|_{X_2} \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\delta + \gamma \frac{p}{p'}} , \quad \forall u \in C_c^1(\Omega),
\]

where $C = C(N, k, p, q) > 0$.

**Proof.** Proof follows from the similar set of arguments as given in the proof of Proposition 3.5. \hfill \Box

4. The $(p, q)$-Hardy potentials ($k = N$)

This section considers the case $k = N$ and identifies various Lorentz spaces and weighted Lebesgue spaces in $\mathcal{H}_{p,q}(\Omega)$. This section contains the proof of Theorem 1.3 and Theorem 1.6. The well-definedness of $D_0^{1,p}(\Omega)$ for the case $N \leq p$ is also discussed in this section. First, we prove Theorem 1.3.

**Proof of Theorem 1.3:** (i) and (ii): For $N \geq p$ and $q \in (0, p^*]$ (if $N > p$), $q \in (0, \infty)$ (if $N = p$), let $g, X$ be as given in Theorem 1.3. Then using Proposition A.2 and Proposition A.3, there exists $C = C(N, p, q) > 0$ such that

\[
\int_{0}^{\infty} g^*(\tau) u^*(\tau)^q \, d\tau \leq C \|g\|_{X} \left( \int_{0}^{\infty} \tau^{p-q} \|
abla u^*(\tau)\|^p \, d\tau \right)^{\frac{q}{p'}} , \quad \forall u \in C_c^1(\Omega).
\]
Now using the Hardy-Littlewood and Pólya-Szegö inequality (Proposition 2.2), we conclude that $g \in H_{p,q}(\Omega)$ and
\[
\int_{\Omega} |g(x)||u(x)|^q \, dx \leq C\|g\|_X \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{2}{p}} , \quad \forall u \in C^1_c(\Omega).
\]

(iii) Let $N < p$, $q \in (0, \infty)$, $\Omega$ be an one-sided bounded domain and $g \in L^1(\Omega)$. Then
\[
\int_{\Omega} |g(x)||u(x)|^q \, dx \leq \|u\|_{L^q(\Omega)} \int_{\Omega} |g(x)| \, dx, \quad \forall u \in C^1_c(\Omega).
\]

Now using the embedding $D_{0}^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ (for $N < p$), $g \in H_{p,q}(\Omega)$.

**Remark 4.1.**

(i) Let $N > p$. Notice that, $\alpha(p,q) \leq \frac{p}{p-q}$, for $q \in (0,p)$, and $\alpha(p,q) \leq \infty$, for $q \in [p,p^*]$. Therefore, as a consequence of Theorem 1.3 and the inclusions of the Lorentz spaces ((iii) of Proposition 2.5) we obtain the following Lebesgue spaces in $H_{p,q}(\Omega)$:
\[
L^{\alpha(p,q)}(\Omega) \subset H_{p,q}(\Omega), \quad \text{for} \ q \in (0,p^*].
\]

(ii) Using the inclusions of Lorentz spaces, we have
\[
H_{p,q}(\Omega) \supset \begin{cases} 
L^{\alpha(p,q),s}(\Omega), & \text{for} \ q \in (0,p), s \in \left[1, \frac{p}{p-q}\right]^\ast; \\
L^{\alpha(p,q),s}(\Omega), & \text{for} \ q \in [p,p^*], s \in \left[1, \infty\right].
\end{cases}
\]

(iii) If $\Omega$ is a bounded domain and $q \in (0,p^*]$, then we have
\[
L^{r,q}(\Omega) \subset H_{p,q}(\Omega), \quad r \in [\alpha(p,q), \infty], s \in \left[1, \infty\right].
\]

In particular, $L^r(\Omega) \subset H_{p,q}(\Omega)$ for $r \in [\alpha(p,q), \infty]$.

Now we identify various weighted Lebesgue spaces in $H_{p,q}(\Omega)$. Recall that, for $S \subset \mathbb{S}^{N-1}$ and $a, b \in [0, \infty]$ with $a < b$, we set $\Omega_{a,b,S} = \text{int} \left\{ x \in \mathbb{R}^N : a \leq |x| < b, \frac{x}{|x|} \in S \text{ if } x \neq 0 \right\}$.

**Proof of Theorem 1.6:**

(i) Let $\Omega = \Omega_{a,b,S}$ and $u \in C^1_c(\Omega)$. For $\tau \in (a,b)$ and $\omega \in S$, using the polar decomposition we define $\varphi(\tau) = u(\tau \omega)$. We consider three separate cases.

\textbf{q \in (0,p)}: Using the fundamental theorem of integration, we write
\[
\varphi(r) = - \int_r^b \varphi'(\tau) \, d\tau = - \int_r^b \varphi'(\tau) \tau^{\frac{N-1}{p}} \tau^{-\frac{1}{p}} \, d\tau.
\]

By the Hölder’s inequality,
\[
|\varphi(r)| \leq \left( \int_r^b \tau^{\frac{1-N}{p}} \, d\tau \right)^{\frac{p}{p-1}} \left( \int_r^b \tau^{N-1} |\varphi'(\tau)|^p \, d\tau \right)^{\frac{1}{p}}.
\]

The above inequality yields
\[
|\varphi(r)|^q \leq \left( \frac{p-1}{N-p} \right)^{\frac{q}{p}} \frac{r^{(p-N)q}}{p} \left( \int_r^b \tau^{N-1} |\varphi'(\tau)|^p \, d\tau \right)^{\frac{q}{p}}.
\]

Set $C = \left( \frac{p-1}{N-p} \right)^{\frac{q}{p}}$. Now $\varphi'(\tau) = \nabla u(\tau \omega) \cdot \omega$. Hence for each $\omega \in S$,
\[
|u(r\omega)|^q \leq C r^{\frac{(p-N)q}{p}} \left( \int_a^b \tau^{N-1} |\nabla u(\tau \omega)|^p \, d\tau \right)^{\frac{q}{p}}, \quad \forall u \in C^1_c(\Omega).
\]
Thus for \( q \in (0, p) \) and \( \bar{g} \in L^1((a, b), r^{N/(p-1)}) \), by taking \( \gamma = 0, \delta = \frac{2}{p} \), and \( h(r) = r^{(p-N)/p} \) one can verify that all the assumptions of Proposition 3.5 are satisfied. Therefore, \( g \in \mathcal{H}_{p,q}(\Omega) \) for \( q \in (0, p) \).

**q \in [p, P^* (1)]**: In this case, for \( q \geq p \) using the fundamental theorem of integration,

\[
|\varphi(r)|^q = - \int_r^b \frac{d}{d\tau}|\varphi(\tau)|^q \, d\tau = -q \int_r^b |\varphi(\tau)|^{q-1} \varphi'(\tau) \, d\tau
\]

Using the generalized Hölder’s inequality we estimate the right hand side of the above identity as

\[
|\varphi(r)|^q \leq q \left( \int_r^b \tau^d p_1 \, d\tau \right)^{\frac{1}{d_1}} \left( \int_r^b \tau^{N-1} |\varphi(\tau)|(q-1)p_2 \, d\tau \right)^{\frac{1}{d_2}} \left( \int_r^b \tau^{N-1} |\varphi'(\tau)|^p \, d\tau \right)^{\frac{1}{d_3}},
\]  

where \( (p, p_1, p_2) \) is a conjugate triple and \( d = (1 - N)(\frac{1}{p} + \frac{1}{p_2}) \). We set \( p_2 = \frac{p}{q-1} \) and hence \( p_1 = \frac{Np}{N(p-q)+qp-p} \). Now the first integral of (4.2) can be estimated as

\[
\int_r^b \tau^d p_1 \, d\tau \leq \frac{r^{dp_1+1}}{dp_1+1},
\]

where \( dp_1 + 1 = \frac{qp_1(p-N)}{p} < 0 \). Set \( C = -\frac{1}{dp_1+1} \). Thus for each \( \omega \in S \), (4.2) yields

\[
|u(r\omega)|^q \leq C q r^{\frac{d(p-N)}{p}} \left( \int_a^b \tau^{N-1} |u(\tau\omega)|^{q'p} \, d\tau \right)^{\frac{1}{d_2}} \left( \int_a^b \tau^{N-1} |\nabla u(\tau\omega)|^p \, d\tau \right)^{\frac{1}{d_3}},
\]  

\( \forall u \in C^1_c(\Omega) \). For \( q \in [p, P^* (1)] \) and \( \bar{g} \in L^1((a, b), r^{N/(p-1)}) \), by taking \( \gamma = \frac{1}{p_2}, \delta = \frac{1}{p} \), and \( h(r) = r^{(p-N)/p} \), one can verify that all the assumptions of Proposition 3.5 are satisfied. Therefore, \( g \in \mathcal{H}_{p,q}(\Omega) \) for \( q \in [p, P^* (1)] \).

**q \in [P^* (1), p^*]**: Let \( \bar{g} \) be strictly decreasing. Then for \( u \in C^\infty_c(\Omega) \), using the Hardy-Littlewood inequality (Proposition 2.2) and Remark 2.1, we get

\[
\int_\Omega \bar{g}(|x|)|u(x)|^q \, dx \leq \int_0^{\Omega} \bar{g}(\omega_N^{\frac{1}{p}} r^{\frac{1}{p}}) u^*(r)^q \, dr.
\]  

For \( u \in C^\infty_c(\Omega) \), by Pólya-Szegö inequality (Proposition 2.2) \( u^* \in W^{1,p}((0, |\Omega|)) \) and hence \( u^* \) is absolutely continuous. Thus using the fundamental theorem of integration, we write

\[
u^*(r)^q = -q \int_r^\infty u^*(\tau)^{q-1} u^{*'}(\tau) \, d\tau = -q \int_r^\infty u^*(\tau)^{q-1} u^{*'}(\tau) \tau^{N-1} \tau^{-\frac{1}{p}} \, d\tau.
\]

Therefore, by the Hölder’s inequality,

\[
u^*(r)^q \leq q \left( \int_r^\infty \tau^{N-1} \tau^{-\frac{1}{p}} u^*(\tau)^{q-1} \, d\tau \right)^{\frac{1}{d_2}} \left( \int_r^\infty \tau^{N-1} \tau^{-\frac{1}{p}} |u^{*'}(\tau)|^p \, d\tau \right)^{\frac{1}{d_3}}.
\]  

Notice that, for \( q \geq P^* (1) \), \( \frac{(q-1)p'}{p} \geq 1 \) and also \( \frac{1-N}{N} - \frac{q-1}{p} + \frac{1}{p} = \frac{1}{\alpha(p,q)} - 1 \). Now we estimate the first integral of (4.5) using Proposition 2.3 as shown below:

\[
\int_r^\infty u^*(\tau)^{q-1} \tau^{-\frac{1}{p}} \, d\tau = \int_r^\infty \tau^{\frac{1-N}{N} p' - \frac{(q-1)p'}{p}} + 1 \left( u^*(\tau)^p \right)^{\frac{q-1}{p}} \, d\tau \leq r^{\frac{1-N}{N} p' - \frac{(q-1)p'}{p}} \left( \int_0^\infty u^*(\tau)^p \, d\tau \right)^{\frac{q-1}{p}}.
\]
Hence using (4.5) we obtain
\[ u^*(r)^q \leq C \left( \int_0^\infty u^*(\tau)^p \, d\tau \right)^{\frac{q-1}{p}} \left( \int_0^\infty \tau^{\frac{N-1}{p}} |u^{*'}(\tau)|^p \, d\tau \right)^{\frac{1}{p}} , \]
where \( C = C(N,p,q) > 0 \). Next multiply the above inequality by \( \tilde{g}(\omega_N^{-\frac{1}{N}} r^{-\frac{1}{N}}) \) and integrate over \((0, \infty)\) to get
\[ \int_0^\infty \tilde{g}(\omega_N^{-\frac{1}{N}} r^{-\frac{1}{N}}) u^*(r)^q \, dr \leq \frac{C}{N \omega_N^{-\frac{1}{N}}} \left( \int_0^\infty \tilde{g}(\omega_N^{-\frac{1}{N}} r^{-\frac{1}{N}}) r^{\frac{1}{p} - \frac{1}{p}} \, dr \right) \times \left( \int_0^\infty u^*(\tau)^p \, d\tau \right)^{\frac{q-1}{p}} \left( \int_0^\infty \tau^{\frac{N-1}{p}} |u^{*'}(\tau)|^p \, d\tau \right)^{\frac{1}{p}} . \]
By change of variable,
\[ \int_0^\infty \tilde{g}(\omega_N^{-\frac{1}{N}} r^{-\frac{1}{N}}) r^{\frac{1}{p} - \frac{1}{p}} \, dr = \omega_N^{-\frac{1}{N}} \int_0^\infty \tau^{\frac{N-1}{p}} \tilde{g}(\tau) \, d\tau . \]
Therefore, using the Pólya-Szegö inequality (Proposition 2.2) and (4.4), we obtain that
\[ \int_\Omega \tilde{g}(|x|) |u(x)|^q \, dx \leq C \left( \int_0^\infty r^{\frac{N}{p} - \frac{1}{p}} \tilde{g}(r) \, dr \right) \left( \int_\Omega |u(x)|^p \, dx \right)^{\frac{q-1}{p}} \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}} , \]
for some \( C = C(N,p,q) > 0 \). Further, the embedding \( D_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \) gives
\[ \int_\Omega \tilde{g}(|x|) |u(x)|^q \, dx \leq C \left( \int_0^\infty r^{\frac{N}{p} - \frac{1}{p}} \tilde{g}(r) \, dr \right) \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}} , \quad \forall u \in C_c(\Omega) . \]
Therefore, \( g \in H_{p,q}(\Omega) \) for \( q \in [P^*(1), p^*] \).

(ii) and (iii): Let \( \Omega = \Omega_{a,b,S} \) with \( a > 0 \) if \( N = p \) and \( a \geq 0 \) if \( N < p \). For \( \tau \in (a,b) \) and \( \omega \in S \), we define \( \varphi(\tau) = u(\tau \omega) \). Then we express
\[ \varphi(r) = \int_a^r \varphi'(\tau) \, d\tau = \int_a^r \varphi'(\tau) \tau^{\frac{N-1}{p}} \tau^{\frac{1}{p}} \, d\tau . \]
Thus,
\[ |\varphi(r)| \leq \left( \int_a^r \tau^{\frac{N-1}{p}} \, d\tau \right)^{\frac{1}{p}} \left( \int_a^r \tau^{N-1} |\varphi'(\tau)|^p \, d\tau \right)^{\frac{1}{p}} , \]
and for each \( \omega \in S \),
\[ |u(r \omega)|^q \leq \begin{cases} \left( \frac{p-1}{p} - \frac{N}{p} \right) \tau^{\frac{p-N}{p}} \left( \int_a^b \tau^{N-1} |\nabla u(\tau \omega)|^p \, d\tau \right)^{\frac{q}{p}} , & \text{if } N < p ; \\ \left( \log \left( \frac{r}{a} \right) \right) \tau^{\frac{N-1}{p}} \left( \int_a^b \tau^{N-1} |\nabla u(\tau \omega)|^p \, d\tau \right)^{\frac{q}{p}} , & \text{if } N = p , \end{cases} \]
\( \forall u \in C_c(\Omega) \). Clearly, by taking \( h(r) = r^{\frac{p-N}{p}} \) (if \( N < p \), \( \log \left( \frac{r}{a} \right) \) (if \( N = p \), \( \gamma = 0 \), and \( \delta = \frac{q}{p} \), all the assumptions of Proposition 3.5 are satisfied. Therefore, \( g \in H_{p,q}(\Omega) \) for \( q \in (0,p] \).

\textbf{Remark 4.2.} (i) Suppose \( \Omega_0 \subset B_a[x_0]^c \) for some \( x_0 \in \mathbb{R}^N \) where \( a > 0 \) for \( N = p \), and \( a = 0 \) for \( N < p \). Let \( g_0 \in L^p_0(\Omega_0) \). Take \( \Omega = -x_0 + \Omega_0 \) and \( g(x) = g_0(x + x_0) \) for \( x \in \Omega \). Now, if the zero extension of \( g \) to \( B_a[0]^c \) satisfies one of the assumptions of the above theorem, then it is easy to see that \( g_0 \in H_{p,q}(\Omega_0) \). \( \square \)
(ii) For a measurable function $h : \Omega_{a,b,S} \mapsto \mathbb{R}$ with $\tilde{h}(r) \leq \tilde{g}(r)$, $\forall r \in (a, b)$, where $\tilde{g} : (a, b) \mapsto \mathbb{R}$ as in Theorem 1.6, we get $h \in \mathcal{H}_{p,q}(\Omega_{a,b,S})$. In this way, for $q \in [p^*(1), p^r]$, we can relax the strictly monotonicity (decreasing) of $\tilde{h}$.

**Proof of Corollary 1.7:** Let $N \leq p$ and $\Omega = \mathbb{R}^N \setminus B_a[0]$. We choose

$$w(r) := \begin{cases} (r^{N+1} \log(\frac{r}{a})^{(N-1)})^{-1}, & \text{for } N = p, a > 0; \\ (1 + r)^{-p}, & \text{for } N < p, a = 0. \end{cases}$$

It is easy to see that $w \in L^1((a, \infty), (r \log(\frac{r}{a}))^{N-1})$ (if $N = p$) and $w \in L^1((0, \infty), r^{p-1})$ (if $N < p$).

Therefore, by Theorem 1.6,

$$\int_K |u(x)|^p \, dx \leq \left( \frac{1}{\inf_{x \in K} w(|x|)} \right) \int_\Omega |u(x)|^p \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx, \quad \forall u \in D_{0}^{1,p}(\Omega),$$

where $K \subset \Omega$ is compact and $C = C(N, p, K) > 0$. Further, $|\nabla u| \in L^p(\Omega)$ for $u \in D_{0}^{1,p}(\Omega)$. Hence from the above inequality we get $D_{0}^{1,p}(\Omega) \subset W_{loc}^{1,p}(\Omega)$, and

$$\int_K (|u(x)|^p + |\nabla u(x)|^p) \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx, \quad \forall u \in D_{0}^{1,p}(\Omega),$$

where $C = C(N, p, K) > 0$. \hfill $\Box$

5. The cylindrical $(p, q)$-Hardy potentials

In this section, we identify product of functions from certain Lorentz spaces and weighted Lebesgue spaces in $\mathcal{H}_{p,q}(\Omega_1 \times \Omega_2)$, where $\Omega_1 \subset \mathbb{R}^k$ and $\Omega_2 \subset \mathbb{R}^{N-k}$. This section contains the proof of Theorem 1.8, Theorem 1.11, and Theorem 1.12.

**Proof of Theorem 1.8:** For $S \subset S^{k-1}$, let $\Omega_1 = \Omega_{a,b,S}$, and $\Omega, g$ be as in (A). For $u \in C^1_c(\Omega)$, as before, we let $u_z(y) = u(y, z)$, $\forall y \in \Omega_{a,b,S}$. For a fixed $\omega \in S$, define $\varphi(r) = u_z(r\omega)$ for $r \in (a, b)$ and $\varphi(r) = 0$ for $0 \leq r \leq a$ and for $r \geq b$.

(i) Our proof follows the similar arguments as in the proof of [11, Theorem 2.1]. By fundamental theorem of integration, for $q > 1$ and $r \in (a, b)$, we can write

$$|\varphi(r)|^q = - \int_1^\infty \frac{d}{d\lambda} |\varphi(\lambda r)|^q \, d\lambda = q \int_1^\infty |\varphi(\lambda r)|^{q-1} \varphi'(\lambda r) r \, d\lambda.$$ 

Now $\varphi'(r) = \nabla_y u_z(r\omega) \cdot \omega$. Thus for each $\omega \in S$ and $z \in \mathbb{R}^{N-k}$,

$$|u_z(r\omega)|^q = -q \int_1^\infty |u_z(\lambda r\omega)|^{q-1} \nabla_y u_z(\lambda r\omega) \cdot \omega r \, d\lambda.$$ 

Next we multiply both sides of the above inequality by $r^{k-1-s}$ and integrate over $\Omega_2 \times S \times (a, b)$ to obtain

$$\int_\Omega \frac{|u(x)|^q}{|y|^s} \, dx \leq q \int_1^\infty d\lambda \int_{\Omega_2} dz \int_S r^{k-1-s} |u(\lambda r\omega, z)|^{q-1} |\nabla_y u(\lambda r\omega, z)| r \, dr$$

$$= q \int_1^\infty d\lambda \int_{\Omega_2} dz \int_S r^{k-1-s} |u(\lambda r\omega, z)|^{q-1} |\nabla_y u(\lambda r\omega, z)| \frac{d\rho}{\lambda}$$

$$\leq q \int_1^\infty \frac{d\lambda}{\lambda^{k+1-s}} \int_{\Omega_2} dz \int_S r^{k-1-s} |u(\lambda r\omega, z)|^{q-1} |\nabla_y u(\lambda r\omega, z)| \, d\rho.$$
where the last inequality holds since \( \lambda > 1 \) and \( u(r\omega, z) = 0 \) for \( r \geq b \). Observe that,

\[
\int_{1}^{\infty} \frac{d\lambda}{\lambda^{k+1-s}} < \infty \iff s < k \iff \frac{N(p-q) + qp}{p} < k \iff q > \frac{p(N-k)}{N-p} = P^*(k). \tag{5.1}
\]

Thus for \( q > P^*(k) \) and \( s = \frac{N}{\alpha(p,q)} \), we get

\[
\int_{\Omega} \frac{|u(x)|^q}{|y|^s} \, dx \leq C \int_{\Omega} \frac{1}{|y|^{s-1}} |u(y,z)|^{q-1} |\nabla u(y,z)| \, dx. \tag{5.2}
\]

Next we estimate the right hand side of (5.2) for \( q \in [p, P^*(1)] \) i.e., for \( s \in [1, p] \). If \( s \in (1, p) \), then one can verify that \((\frac{s-1}{s}, \frac{sp}{p-s}, p)\) is a conjugate triple and

\[
\left( \frac{q}{s} - 1 \right) \frac{sp}{p-s} = \frac{q - \frac{N}{\alpha(p,q)}}{p - \frac{N}{\alpha(p,q)}} = \frac{Nq - Np}{p^2 - Np + Nq - pq} = \frac{Np}{N-p}.
\]

Now using the Hölder’s inequality we get

\[
\int_{\Omega} \frac{1}{|y|^{s-1}} |u(y,z)|^{q-1} |\nabla u(y,z)| \, dx = \int_{\Omega} \frac{1}{|y|^{s-1}} |u(x)|^{(\frac{s-1}{s})^{q-1} + \frac{2}{s-1} - 1} |\nabla u(x)| \, dx
\]

\[
\leq \left( \int_{\Omega} \frac{|u(x)|^q}{|y|^s} \, dx \right)^{\frac{s-1}{s}} \left( \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{2}{s-1}} \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq C \left( \int_{\Omega} |u(x)|^q \, dx \right)^{\frac{s-1}{s}} \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{2}{p}}, \tag{5.3}
\]

where the last inequality follows since \((\frac{p-s}{p})^{\frac{p}{p-s}} + \frac{1}{p} = \frac{N-s}{(N-p)s} = \frac{q}{ps}\). If \( s = 1 \) or \( s = p \), then the above conjugate triple become \((\infty, p', p)\) or \((p', \infty, p)\) and the similar estimates as above yields (5.3) for \( s = 1, p \). Further, if \( s < k \) (equivalently, if \( q > P^*(k) \)), then (5.2) yields

\[
\int_{\Omega} \frac{|u(x)|^q}{|y|^{\frac{N}{\alpha(p,q)}}} \, dx \leq C \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall u \in C^1_c(\Omega). \tag{5.4}
\]

Therefore,

\[
|y|^{-\frac{N}{\alpha(p,q)}} \in \mathcal{H}_{p,q}(\Omega), \quad q \in [p, P^*(1)] \text{ with } q > P^*(k). \tag{5.5}
\]

Next we consider \( q \in (P^*(1), P^*) \). In this case one can write \( q = tp^*(1) + (1-t)p^* \) for some \( t \in (0, 1) \). Therefore,

\[
t = \frac{q - p^*}{P^*(1) - p^*} = \frac{(N-p)q - Np}{-p} = \frac{N}{\alpha(p,q)} = s.
\]

Now we apply the Hölder’s inequality and use (5.4) for \( q = P^*(1) \) to get

\[
\int_{\Omega} \frac{|u(x)|^q}{|y|^s} \, dx \leq \left( \int_{\Omega} \frac{|u(x)|^{P^*(1)}}{|y|^s} \, dx \right)^{s} \left( \int_{\Omega} |u(x)|^p \, dx \right)^{1-s} \leq C \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{2}{p}},
\]

\( \forall u \in C^1_c(\Omega) \). For \( q = p^* \), the above inequality follows from the Sobolev inequality. Therefore,

\[
|y|^{-\frac{N}{\alpha(p,q)}} \in \mathcal{H}_{p,q}(\Omega), \quad q \in (P^*(1), p^*) \tag{5.6}.
\]

Now by (5.5) and (5.6), we conclude that

\[
|y|^{-\frac{N}{\alpha(p,q)}} \in \mathcal{H}_{p,q}(\Omega), \quad \text{for } q \in \left\{ \begin{array}{ll}
[p, p^*], & k > p; \\
(P^*(k), p^*], & k \leq p.
\end{array} \right.
\]
(ii) Let \( \Omega = \Omega_{a,b,S} \times \Omega_2 \) where \( 0 \notin \Omega_{a,b,S} \). Let \( u \in C_c^1(\Omega) \) and \( \varphi, s \) be as in (i). Then for \( q > 1 \) we write
\[
|\varphi(r)|^q = \int_0^1 \frac{d}{\lambda}|\varphi(\lambda r)|^q d\lambda = q \int_0^1 |\varphi(\lambda r)|^{q-1} \varphi'(\lambda r) r d\lambda.
\]
Thus for each \( \omega \in S \) and \( z \in \mathbb{R}^{N-k} \),
\[
|u_z(y)|^q = q \int_0^1 |u_z(\lambda r \omega)|^{q-1} \nabla_y u_z(\lambda r \omega) \cdot \omega r d\lambda.
\]

Multiply the above inequality \( r^{k-1-s} \) and integrate over \( \Omega \) to get
\[
\int_{\Omega} \frac{|u(x)|^q}{|y|^s} \, dx \leq q \int_0^1 \frac{d\lambda}{\lambda} \int_{\Omega_2} dz \int_S dS \int_a^b r^{k-1-s} |u(\lambda r \omega, z)|^{q-1} |\nabla_y u(\lambda r \omega, z)| r \, dr
\]
\[
= q \int_0^1 \frac{d\lambda}{\lambda} \int_{\Omega_2} dz \int_S dS \int_{\lambda a}^{\lambda b} \left( \frac{\rho}{\lambda} \right)^{-k-1} \rho^{q-1} |\nabla_y u(\rho \omega, z)| \rho \, d\rho,
\]
where the last inequality holds if \( \lambda < 1 \) and \( u(\rho \omega, z) = 0 \) for \( r \leq a \). Notice that
\[
\int_0^1 \frac{d\lambda}{\lambda^{k+1-s}} < \infty \iff s > k \iff q < P^*(k).
\]

Thus for \( q < P^*(k) \) the above inequality yields
\[
\int_{\Omega} \frac{|u(x)|^q}{|y|^s} \, dx \leq C \int_{\Omega} \frac{1}{|y|^{s-1}} |u(x)|^{q-1} |\nabla u(x)| \, dx.
\]

Now, for \( q \in [p, P^*(k)] \) we estimate the right hand side of the above inequality as before. Clearly, in this range of \( q \) we have \( s \in (k, p] \). Moreover, since \( k \geq 1 \), we also have \( s \in (1, p] \) and hence following the arguments that gives (5.4), we obtain
\[
\int_{\Omega} \frac{|u(x)|^q}{|y|^s} \, dx \leq C \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall u \in C_c^1(\Omega). \tag{5.7}
\]

Therefore, \( |y|^{-\frac{N}{\alpha_{p,q}}} \in H_{p,q}(\Omega) \) for \( q \in [p, P^*(k)] \). Next we consider \( q = P^*(k) \). In this case, we have \( \frac{N}{\alpha_{p,q}} = k \) and \( \frac{p(q-k)}{p-k} = P^* \). For \( k < p \), and \( u \in C_c^1(\Omega) \), we apply the Hölder’s inequality with the conjugate pair \( (\frac{p}{k}, \frac{p}{p-k}) \) to get
\[
\int_{\Omega} \frac{|u(x)|^q}{|y|^s} \, dx = \int_{\Omega} \frac{|u(x)|^k}{|y|^s} \frac{|u(x)|^{q-k}}{dx} \leq \left( \int_{\Omega} \frac{|u(x)|^p}{|y|^s} \, dx \right)^{\frac{k}{p}} \left( \int_{\Omega} |u(x)|^{q-k} \, dx \right)^{\frac{q-k}{p}}.
\]

Further, using (5.7), \( |y|^{-p} \in H_{p,p}(\Omega) \). Therefore, using the embedding \( D_0^{k,p}(\Omega) \hookrightarrow L^{P^*}(\Omega) \), we get the right hand side of the above inequality is lesser than \( \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{q}{p}} \). Thus, \( |y|^{-\frac{N}{\alpha_{p,q}}} \in H_{p,q}(\Omega) \) for \( q = P^*(k) \).

**Proof of Corollary 1.10:** Let \( g(x) = g_1(y) \) and \( \tilde{g}_1 \in L^\infty((a,b), r^{N \frac{N}{\alpha_{p,q}}}) \). If \( |y|^{-\frac{N}{\alpha_{p,q}}} \in H_{p,q}(\Omega) \), then
\[
\int_{\Omega} |g(x)||u(x)|^q \, dx \leq \text{ess sup}_{y \in \Omega_1} \tilde{g}_1(\|y\|) \|y|^{-\frac{N}{\alpha_{p,q}}} \int_{\Omega} \frac{|u(x)|^q}{|y|^s} \, dx
\]
\[
\leq C \|\tilde{g}_1\|_{L^\infty((a,b), r^{N \frac{N}{\alpha_{p,q}}})} \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall u \in C_c^1(\Omega),
\]
for some $C = C(N, k, p, q) > 0$, i.e., $g \in H_{p,q}(\Omega)$. Thus the same conclusions of Theorem 1.8 hold for $g$ in place of $|y|^{-\frac{\alpha}{p(q)}}$.

Next we proceed to prove Theorem 1.11. For proving Theorem 1.11 we require the following proposition:

**Proposition 5.1.** Let $p \in (1, N)$. Let $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_1$ and $\Omega_2$ be any two open sets in $\mathbb{R}^k$ and $\mathbb{R}^{N-k}$ respectively. Let $\Omega_1$ be bounded if $k \leq p$, and $\Omega$ be bounded for $q \in (0, p)$. Let $t \in [0, 1]$ and

$$g_1 \in X := \begin{cases} L^{\frac{k}{t}}(\Omega_1), & \text{if } k > p; \\ L^{\frac{k}{t},\infty,t(k-1)}(\Omega_1), & \text{if } k = p; \\ L^1(\Omega_1), & \text{if } k < p. \end{cases}$$

Then for $q = tp + (1 - t)p^*$ or $q = tp$ with $t > 0$, $g(x) = g_1(y) \in H_{p,q}(\Omega)$. Moreover,

$$\int_{\Omega}|g_1(y)||u(x)|^q \, dx \leq C\|g_1\|_X \left(\int_\Omega |\nabla u(x)|^p \, dx\right)^{\frac{q}{p}}, \quad \forall u \in C^1_c(\Omega),$$

for some $C = C(N, k, p, q) > 0$.

**Proof.** Let $t \in [0, 1]$ and $q = tp + (1 - t)p^*$. For $t = 0$, we have $q = p^*$ and $g_1 \in L^\infty(\Omega_1)$. Hence the proof follows from the Sobolev embedding. For $t \in (0, 1]$, and $u \in C^1_c(\Omega)$, we apply the Hölder’s inequality to get

$$\int_{\Omega}|g_1(y)||u(x)|^q \, dx \leq \left(\int_{\Omega}|g_1(y)|^\frac{q}{t}|u(x)|^p \, dx\right)^{rac{t}{q}} \left(\int_{\Omega}|u(x)|^{p^*} \, dx\right)^{1-t}. \quad (5.8)$$

Using (i) of Proposition 2.5 and Proposition 2.6,

$$|g_1|^\frac{1}{t} \in Y = \begin{cases} L_\infty^{\frac{k}{t}}(\Omega_1), & \text{if } k > p; \\ L^{1,\infty,k-1}(\Omega_1), & \text{if } k = p; \\ L^1(\Omega_1), & \text{if } k < p, \end{cases}$$

and $\|g_1|^{\frac{1}{t}}\|_Y \leq C\|g_1\|_X$ for some $C > 0$. Consequently, by Theorem 1.3, it follows that $|g_1|^\frac{1}{t} \in H_{p,p}(\Omega_1)$ and

$$\int_{\Omega_2}\int_{\Omega_1}|g_1(y)|^\frac{1}{t}|u(y,z)|^p \, dydz \leq C\|g_1|^{\frac{1}{t}}\|_Y \int_{\Omega_2}\int_{\Omega_1}|
abla_y u(y,z)|^p \, dydz$$

$$\leq C\|g_1\|_X \int_{\Omega_2}\int_{\Omega_1}|
abla u(y,z)|^p \, dydz,$$

where $C = C(k, p, q) > 0$. Therefore, using the embeddings of $D_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ in (5.8), we get

$$\int_{\Omega}|g_1(y)||u(x)|^q \, dx \leq C\|g_1\|_X \left(\int_{\Omega}|
abla u(x)|^p \, dx\right)^{t+\frac{(1-t)p^*}{p}} = C\|g_1\|_X \left(\int_{\Omega}|
abla u(x)|^p \, dx\right)^{\frac{q}{p}},$$

for some $C = C(N, k, p, q) > 0$. For $q = tp$ with $t \in (0, 1]$, using the boundedness of $\Omega$, the result follows along the same line except that in (5.8) (there is no term with the exponent $1 - t$). \hfill \Box

**Remark 5.2.** Let $t \in [0, 1]$ be such that $q = tp + (1 - t)p^*$. Then

$$\frac{k}{tp} = \frac{k}{p} \left(\frac{p^*}{p^*} - \frac{p}{p^*} - q\right) = k\alpha(p,q)\left(\frac{1}{p} - \frac{1}{p^*}\right) = \frac{\alpha(p,q)k}{N}.$$
Further, from (i) of Example 2.7, \( g_1(y) = |y|^{-\frac{\alpha}{(p,q)}} \in L^{\frac{\alpha}{(p,q)}}(\mathbb{R}^k) \), for \( k > \frac{N}{\alpha(p,q)} \). Moreover, it is easy to see that for \( q \geq p, p \geq \frac{N}{\alpha(p,q)} \). Thus for \( k > p \) and \( q \in [p, p^*] \), from the above proposition we get the following generalized cylindrical C-K-N inequality:

\[
\int_{\mathbb{R}^N} |g_1(y)||u(x)|^q \, dx \leq C\|g_1\|^{\frac{k}{(p,q)}\frac{1}{p}} \left( \int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall \, u \in C^1_c(\mathbb{R}^N).
\]

**Proof of Theorem 1.11:** We consider two cases.

1. **\( q \in (0, p) \):** Let \( s, t \in (0,1) \). Let \( k > p, g_1 \in L^{\frac{k}{(p,q)s}p + \frac{1}{st}}(\Omega) \) and \( g_2 \in L^{\frac{1}{st}}(\Omega_2) \). By part (i) of Theorem 1.3, we have \( g_1 \in \mathcal{H}_{p,q}(\Omega_1) \) with \( q = (1-st)p \). Hence Proposition 3.3 infers \( |g_1g_2| = g \in \mathcal{H}_{p,q}(\Omega) \). Moreover, using (1.14) and (3.3), we obtain

\[
\int_{\Omega} |g(x)||u(y,z)|^q \, dy \, dz \leq C\|g_1\|^{\frac{1}{(p,q)s}p + \frac{1}{st}} \int_{\Omega_2} |g_2(z)| \left( \int_{\Omega_1} |\nabla u(y,z)|^p \, dy \right)^{\frac{q}{p}} \, dz \\
\leq C\|g_1\|^{\frac{1}{(p,q)s}p + \frac{1}{st}} \int_{\Omega_1} |\nabla u(y,z)|^p \, dy \right)^{\frac{q}{p}}, \quad \forall \, u \in C^1_c(\Omega),
\]

where \( C = C(k, p, q) > 0 \).

2. **\( q \in [p, p^*] \):** Let \( s \in [0,1] \) and write \( q = (1-s)p + sp^* \). If \( t = 0 \), then we have \( q = p, N - k > p, g_1 \in L^\infty(\Omega_1) \) and \( g_2 \in L^{\frac{k}{p}p}N - k\,\infty(\Omega_2) \). By interchanging the roles of \( \Omega_1 \) and \( \Omega_2 \), we obtain from Proposition 5.1 that

\[
\int_{\Omega} |g_1(y)||g_2(z)||u(x)|^p \, dx \leq C\|g_1\|^{\frac{k}{(1-s)p}p} \int_{\Omega_1} |\nabla u(x)|^p \, dx, \quad \forall \, u \in C^1_c(\Omega).
\]

For \( t = 1 \), we have \( k > p, q = (1-s)p + sp^* \), \( g_1 \in L^{\frac{k}{(1-s)p}p}N - k\,\infty(\Omega_1) \) and \( g_2 \in L^\infty(\Omega_2) \). Thus, again by Proposition 5.1,

\[
\int_{\Omega} |g_1(y)||g_2(z)||u(x)|^q \, dx \leq C\|g_1\|^{\frac{k}{(1-s)p}p} \int_{\Omega_1} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall \, u \in C^1_c(\Omega).
\]

Next we consider the case \( t \in (0,1) \) and \( q = (1-s-t)p + stp^* \). In this case, we have both \( k > p \) and \( N - k > p, g_1 \in L^{\frac{k}{(1-s-t)p}p}N - k\,\infty(\Omega_1) \) and \( g_2 \in L^{\frac{k}{(1-s-t)p}p}N - k\,\infty(\Omega_2) \). Then \( |g_1|^{\frac{1}{t}} \in L^{\frac{k}{(1-s-t)p}p}N - k\,\infty(\Omega_1) \) and \( |g_2|^{\frac{1}{t}} \in L^{\frac{k}{(1-s-t)p}p}N - k\,\infty(\Omega_2) \). Hence by Proposition 5.1, we obtain that \( |g_1|^{\frac{1}{t}} \in \mathcal{H}_{p,q}(\Omega) \) with \( q = (1-s-t)p + (1-(1-s))p^* = (1-s)p + sp^* \) and \( |g_2|^{\frac{1}{t}} \in \mathcal{H}_{p,q}(\Omega) \). Therefore, (i) of Proposition 3.1 assures that \( q = t \left( q_1 + (1-t)q_2 \right) = t((1-s)p + sp^*) + (1-t)p = (1-st)p + stp^* \).

Moreover, using the Hölder’s inequality and Proposition 5.1,

\[
\int_{\Omega} |g_1(g_2(z)||u(x)|^q \, dx \leq \left( \int_{\Omega} |g_1(y)|^q |u(x)|^q \, dx \right) \left( \int_{\Omega} |g_2(z)|^p |u(x)|^p \, dx \right)^{\frac{q}{p}} \leq C\|g_1\|^{\frac{k}{(1-s-t)p}p} \int_{\Omega_1} |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall \, u \in C^1_c(\Omega),
\]

where \( C = C(N, k, p, q) > 0 \).

Now we prove Theorem 1.12.

**Proof of Theorem 1.12:** Let \( \Omega = \Omega_{a,b,S} \times \mathbb{R}^{N-k} \) and \( \tilde{g}_1, \tilde{g}_2 \) be as in Theorem 1.12. Let \( u \in C^1_c(\Omega) \).

For a fixed \( \omega \in S \) and \( \tau \in (a, b) \), let \( \varphi(\tau) = u_\tau(\tau\omega) \).
\( q \in (0, p] \): We express
\[
|\varphi'(r)| = \begin{cases} 
\left( \int_a^r \varphi'(\tau) \tau^{\frac{k-1}{p} - \frac{1}{p'}} d\tau \right)^{\frac{1}{p'}} \left( \int_a^r \tau^{\frac{1}{p'} - \frac{1}{p}} d\tau \right)^{\frac{1}{p}}, & \text{if } k < p; \\
-\left( \int_r^b \varphi'(\tau) \tau^{\frac{k-1}{p} - \frac{1}{p'}} d\tau \right)^{\frac{1}{p'}} \left( \int_r^b \tau^{\frac{1}{p'} - \frac{1}{p}} d\tau \right)^{\frac{1}{p}}, & \text{if } k > p.
\end{cases}
\]

As \( \varphi'(\tau) = \nabla_y u_z(\tau \omega) \cdot \omega \) for each \( \omega \in S \) and \( z \in \mathbb{R}^{N-k} \), we get
\[
|u_z(r\omega)|^q \leq \begin{cases} 
\left( \frac{p-1}{p-k} \right)^{\frac{q}{p}} r^{\frac{p-kq}{p}} \left( \int_a^r \tau^{\frac{k-1}{p'} - \frac{1}{p}} d\tau \right)^{\frac{q}{p}} \left( \int_a^r \tau^{\frac{1}{p'} - \frac{1}{p}} d\tau \right)^{\frac{q}{p}}, & \text{if } k < p; \\
\left( \frac{p-1}{k-p} \right)^{\frac{q}{p}} r^{\frac{p-kq}{p}} \left( \int_r^b \tau^{\frac{k-1}{p} - \frac{1}{p'}} d\tau \right)^{\frac{q}{p}} \left( \int_r^b \tau^{\frac{1}{p'} - \frac{1}{p}} d\tau \right)^{\frac{q}{p}}, & \text{if } k > p,
\end{cases}
\] (5.9)

\( \forall u \in C^1_c(\Omega) \). Thus for \( q \in (0, p] \), by taking \( \gamma = 0, \delta = \frac{q}{p} \), and \( h(r) = r^{\frac{p-kq}{p}} \) one can verify that the assumptions of Proposition 3.5 are satisfied. Therefore, \( g \in \mathcal{H}_{p,q}(\Omega) \) for \( q \in (0, p] \).

\( q \in (p, P^*(1)] \): We write \( q = tp + (1-t)P^*(1) \) for some \( t \in [0, 1] \). First, we consider \( t = 0 \) (i.e., \( q = P^*(1) \)), \( \tilde{g}_1 \in L^\infty((a,b)) \) and \( \tilde{g}_2 \in L^1((0,\infty)) \). Let \( u_y(z) = u(y,z) \) be the \( y \)-section of \( u \). For a fixed \( \omega \in S^{N-k-1} \), set \( \varphi(\tau) = u_y(\tau \omega) \) where \( \tau \in (0, \infty) \). Then we make the following estimate:
\[
|\varphi(\tau)|^{P^*(1)} = -P^*(1) \int_\tau^\infty |\varphi(1)|^{P^*(1)-1} \varphi(1) \tau^{N-k-\frac{1}{p}} \tau^{k-N} d\tau \\
\leq P^*(1) \left( \int_\tau^\infty \tau^{k-N-\frac{1}{p}} |\varphi(1)|^{P^*(1)-1} d\tau \right)^{\frac{1}{P^*(1)}} \left( \int_\tau^\infty \tau^{N-k-1} |\varphi'(1)|^{P^*(1)} d\tau \right)^{\frac{1}{P^*(1)}} \\
\leq P^*(1)r^{1+k-N} \left( \int_0^\tau \tau^{N-k-1} |\varphi(1)|^{P^*(1)} d\tau \right)^{\frac{1}{P^*(1)}} \left( \int_0^\tau \tau^{N-k-1} |\varphi'(1)|^{P^*(1)} d\tau \right)^{\frac{1}{P^*(1)}}.
\]

From the above inequality, for each \( \omega \in S^{N-k-1} \) and \( y \in \Omega_{a,b,S} \), we get
\[
|u_y(r\omega)|^{P^*(1)} \leq P^*(1)r^{1+k-N} \left( \int_0^\infty \tau^{N-k-1} |u_y(\tau \omega)|^{P^*(1)} d\tau \right)^{\frac{1}{P^*(1)}} \left( \int_0^\infty \tau^{N-k-1} |\nabla_x u_y(\tau \omega)|^{P^*(1)} d\tau \right)^{\frac{1}{P^*(1)}}, \quad \forall u \in C^1_c(\Omega).
\]

For \( q = P^*(1) \), \( \tilde{g}_1 \in L^\infty((a,b)) \) and \( \tilde{g}_2 \in L^1((0,\infty)) \), by taking \( \gamma = \frac{1}{p}, \delta = \frac{1}{p} \), and \( h(r) = r^{1+k-N} \), the assumptions of Proposition 3.6 are satisfied. Therefore, \( g = |g_1g_2| \in \mathcal{H}_{p,P^*(1)}(\Omega) \) and
\[
\int_\Omega |g_1(z)||g_2(z)||u(x)|^{P^*(1)} dx \leq C\|g_1\|_{L^\infty((a,b))} \|g_2\|_{L^1((0,\infty))} \left( \int_\Omega \nabla u(x)^p dx \right)^{\frac{P^*(1)}{p}}, \quad (5.10)
\]

\( \forall u \in C^1_c(\Omega) \) and for some \( C = C(N,k,p,q) > 0 \). For \( q = p \), we have
\[
\int_\Omega |g_1(z)||g_2(z)||u(x)|^p dx \leq C\|g_1\|_{L^1((a,b),r^{p-1})} \|g_2\|_{L^\infty((0,\infty))} \left( \int_\Omega \nabla u(x)^p dx \right), \quad (5.11)
\]

\( \forall u \in C^1_c(\Omega) \) and for some \( C = C(N,k,p) > 0 \). Now we consider \( t \in (0,1), \tilde{g}_1 \in L^{(p,q)}((a,b),r^{p-1}) \) and \( \tilde{g}_2 \in L^{(p,q)-1}((0,\infty)) \). Since
\[
t = \frac{q - P^*(1)}{p - P^*(1)} = \frac{p^* - N'q}{p^* - N'p} = \frac{1}{\beta(p,q)},
\]
we have \( g_1^\frac{1}{t} \in L^1((a, b), r^{p-1}) \) and \( g_2^{\frac{1}{1-t}} \in L^1((0, \infty)) \). Hence from (5.11) and (5.10), we get \( |g_1|^\frac{1}{t} \in \mathcal{H}_{p,p}(\Omega) \) and \( |g_2|^{\frac{1}{1-t}} \in \mathcal{H}_{p,P^*}(1) \). Therefore, using (i) of Proposition 3.1, we conclude that \( g = [g_1g_2] \in \mathcal{H}_{p,q}(\Omega) \) with \( q = tp + (1-t)P^*(1) \). Moreover, there exists \( C = C(N, k, p, q) > 0 \) such that

\[
\int_\Omega |g(x)||u(x)|^q \, dx \leq \left( \int_\Omega |g_1(y)|^\frac{t}{t} |u(x)|^p \, dx \right)^t \left( \int_\Omega |g_2(z)|^{\frac{1}{1-t}} |u(x)|^{P^*(1)} \, dx \right)^{1-t}
\]

\[
\leq C \left\| g_1 \right\|^{\frac{t}{t}}_{L^1((a, b), r^{p-1})} \left\| g_2 \right\|^{\frac{1}{1-t}}_{L^1((0, \infty))} \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall \ u \in C^1_c(\Omega).
\]

\( q \in (P^*(1), p^*) \): Let \( q = tP^*(1) + (1-t)p^* \) for some \( t \in [0, 1] \). Then \( t = \frac{N}{\alpha(p,q)} \). For \( t \in [0, 1) \), \( g_1 \in L^\infty((0, \infty)) \) and \( g_2 \in L^{\alpha(p,q)}(a, b) \), using the Hölder’s inequality, (5.10) and the embedding \( \mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega) \), we similarly get \( g = [g_1g_2] \in \mathcal{H}_{p,q}(\Omega) \) for \( q \in (P^*(1), p^*) \). We also get

\[
\int_\Omega |g(x)||u(x)|^q \, dx \leq C \left\| g_1 \right\|_{L^\infty((0, \infty))} \left\| g_2 \right\|_{L^{\alpha(p,q)}((a, b))} \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall \ u \in C^1_c(\Omega),
\]

for some \( C = C(N, k, p, q) > 0 \).

6. Compactness and the Existence of Solution

In this section, we prove the compactness of \( G_q \) for \( q \) as given in Theorem 1.13 and then show the existence of solutions to the problem (1.19). First, we prove the following compactness result:

**Proposition 6.1.** Let \( \Omega \) be as given in (B). Then \( \mathcal{D}^{1,p}_0(\Omega) \) is compactly embedded into \( L^q_{loc}(\Omega) \) for \( q \in (0, \delta) \), where \( \delta = p^* \) (if \( N > p \)), and \( \delta = \infty \) (if \( N \leq p \)).

**Proof.** For \( u \in \mathcal{D}^{1,p}_0(\Omega) \), \( |\nabla u| \in L^p(\Omega) \). If \( N > p \), then using the Sobolev embedding, we also obtain \( u \in \mathcal{D}^p_{loc}(\Omega) \). In particular, for each \( K \) compact set in \( \Omega \), there exists \( C = C(N, p, K) > 0 \) such that

\[
\int_K (|u(x)|^p + |\nabla u(x)|^p) \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx, \quad \forall \ u \in \mathcal{D}^{1,p}_0(\Omega).
\]

(6.1)

Thus \( \mathcal{D}^{1,p}_0(\Omega) \hookrightarrow W^{1,p}_{loc}(\Omega) \). If \( N \leq p \), then using Corollary 1.7, \( \mathcal{D}^{1,p}_0(\Omega) \hookrightarrow W^{1,p}_{loc}(\Omega) \). Further, by Rellich-Kondrachov compactness theorem,

\[
W^{1,p}_{loc}(\Omega) \hookrightarrow L^q_{loc}(\Omega) \text{ compactly for } q \in \begin{cases} [1, p^*], & \text{for } N > p; \\ [1, \infty), & \text{for } N \leq p. \end{cases}
\]

(6.2)

For \( q \leq 1 \), \( u \in \mathcal{D}^1_{loc}(\Omega) \), using the Hölder’s inequality with the conjugate pair \( \left( \frac{1}{q}, \frac{1}{1-q} \right) \), we get

\[
\int_K |u(x)|^q \, dx \leq \left( \int_K |u(x)| \, dx \right)^q |K|^{1-q},
\]

for every compact set \( K \) in \( \Omega \). Therefore, \( \mathcal{D}^1_{loc}(\Omega) \hookrightarrow L^q_{loc}(\Omega) \) and hence by (6.2), we conclude that \( W^{1,p}_{loc}(\Omega) \hookrightarrow L^q_{loc}(\Omega) \) compactly for \( q \in (0, 1) \) as well. This completes the proof.

Indeed, the above proposition shows that the map \( G_q \) is compact on \( \mathcal{D}^{1,p}_0(\Omega) \) for \( g = \chi_K \), where \( \chi_K \) is the characteristic function of \( K \) in \( \Omega \). However, in the following lemma, we prove the compactness of \( G_q \) for \( g \) in a more general class of weight functions.
Lemma 6.2. Let $p \in (1, \infty)$. For $i = 1, 2$, let $g_i \in \mathcal{C}_c^\infty(\Omega_i)^{X_i}$ and (1.20) holds. Then the map

$$G_q(u) = \int_\Omega |g||u|^q, \quad \forall u \in D_0^{1,p}(\Omega),$$

is compact on $D_0^{1,p}(\Omega)$ for $q \in (0, \delta)$, where $\delta = p^*$ (if $N > p$), and $\delta = \infty$ (if $N \leq p$).

Proof. Let $u_n \rightharpoonup u$ in $D_0^{1,p}(\Omega)$ and let $\epsilon > 0$ be given. Set $M = \sup\{|||\nabla u_n||_p^p + |||\nabla u||_p^p\}$. For $g_i \in \mathcal{C}_c^\infty(\Omega_i), (i = 1, 2)$, we split $g_i = g_{e_i} + (g_i - g_{e_i})$ where $g_{e_i} \in \mathcal{C}_c^\infty(\Omega_i)$ such that $\|g_i - g_{e_i}\|_{X_i} < \frac{\epsilon}{M}$.

Then we write

$$
\int_\Omega g_1 g_2 (|u_n|^q - |u|^q) \leq \int_\Omega |g_1 - g_{e_1}||g_2||(|u_n|^q - |u|^q) + \int_\Omega |g_{e_1}||g_2 - g_{e_2}||(|u_n|^q - |u|^q)|
\quad + \int_\Omega |g_{e_1}||g_{e_2}||(|u_n|^q - |u|^q)|. \tag{6.3}
$$

We estimate the first two integrals in the right hand side of (6.3), using (1.20) as

$$
\int_\Omega (|g_1 - g_{e_1}||g_2| + |g_{e_1}||g_2 - g_{e_2}|)(|u_n|^q - |u|^q)
\leq C (\|g_1 - g_{e_1}\|_{X_1} g_2 \|g_2 - g_{e_2}\|_{X_2} + \|g_{e_1}\|_{X_1} g_2 g_{e_2}\|_{X_2} \|\nabla u_n\|_p^p + \|\nabla u\|_p^p). \tag{6.4}
$$

Further, using Proposition 6.1, there exists $n_1 \in \mathbb{N}$ such that

$$
\int_\Omega g_1 g_2 (|u_n|^q - |u|^q) = \int_{K_1 \times K_2} g_{e_1} g_{e_2} (|u_n|^q - |u|^q) < \epsilon, \quad \forall n \geq n_1,
$$

where $K_i \subset \Omega_i$ is the compact support of $g_{e_i}$. Therefore, from (6.3) and (6.4),

$$
\int_\Omega |g||(|u_n|^q - |u|^q)| = \int_\Omega |g_1 g_2||(|u_n|^q - |u|^q)| < C\epsilon, \quad \forall n \geq n_1.
$$

Thus, $G_q(u_n) \rightharpoonup G_q(u)$ as $n \to \infty$.

Proof of Theorem 1.13: The compactness of $G_q$ follows from Lemma 6.2. Now for $q > 1$, we show the existence of non-negative solution to the problem (1.19). Recall that

$$
\frac{1}{B_q(g)} = \inf \{ J(u) : u \in N_g \} = \inf \{ R(u) : u \in D_0^{1,p}(\Omega) \setminus \{0\} \}, \tag{6.5}
$$

where $J(u) = \int_\Omega |\nabla u|^p, N_g = \{ u \in D_0^{1,p}(\Omega) : G_q(u) = 1 \}$, and $R(u) = (\int_\Omega g|u|^q)^\frac{p}{q} - \frac{1}{q} \int_\Omega |\nabla u|^p$. Let $(u_n)$ be a minimizing sequence for $J$ on the set $N_g$. By the coercivity of $J$, the sequence $(u_n)$ is bounded in $D_0^{1,p}(\Omega)$ and hence admits a subsequence $(u_{n_k})$ such that $u_{n_k} \rightharpoonup u_1$ in $D_0^{1,p}(\Omega)$. Now using the compactness of $G_q$, we have $u_1 \in N_g$. Further, the weak lowersemicontinuity of the norm $|||\nabla (\cdot)||_p$ gives

$$
\frac{1}{B_q(g)} = \lim_{k \to \infty} \int_\Omega |\nabla u_{n_k}|^p \geq \int_\Omega |\nabla u_1|^p \geq \frac{1}{B_q(g)}.
$$

Therefore, $\frac{1}{B_q(g)}$ is attained and $J$ admits a minimizer $u_1$ over $N_g$. Moreover, from (6.5), $u_1$ also minimizes $R$ over $D_0^{1,p}(\Omega) \setminus \{0\}$, and hence $\langle R'(u_1), v \rangle = 0$ for $v \in D_0^{1,p}(\Omega)$. Therefore, as $u_1 \in N_g$ and $q > 1$, we obtain

$$
\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla v = \frac{1}{B_q(g)} \int_\Omega g|u_1|^{q-2} u_1 v, \quad \forall v \in D_0^{1,p}(\Omega). \tag{6.6}
$$

Since $|u_1| \in D_0^{1,p}(\Omega)$ and $R(|u_1|) = R(u_1)$, we easily see that $|u_1|$ is a non-negative solution of (1.19). \qed
Remark 6.3. Let $N > p$. For $v \in C_c^\infty(\Omega)$ with $v \geq 0$, we have
\[ \int |\nabla(|u_1|)|^{p-2} \nabla(|u_1|) \cdot \nabla v = \frac{1}{B_q(g)} \int_{\Omega} g |u_1|^{q-1} v \geq 0. \]

Then for $q \in [p, p^*)$, $|u_1| \in \mathcal{D}_0^{1,p}(\Omega)$ satisfies all the assumptions of Proposition [35, Proposition 3.2]. Therefore, by the strong maximum principle, $|u_1| > 0$ a.e. in $\Omega$.

The following proposition shows that $G_q$ is not compact for the cylindrical weight $g(x) = |y|^{-\frac{N}{\alpha(p,q)}}$.

**Proposition 6.4.** Let $q \in [p, p^*)$ and $0 \in \overline{\Omega}$. Then the map $G_q$ is not compact for $g(x) = |y|^{-\frac{N}{\alpha(p,q)}}$.

**Proof.** Let $q \in [p, p^*)$, $s = \frac{N}{\alpha(p,q)}$ and $0 \in \overline{\Omega}$. Then using the Maz’ja’s criteria [47, Section 2.4.2, Theorem 1, page 130], one can verify that the map $G_q$ is not compact for $|x|^{-s}$. Now suppose, $G_q$ is compact for $|y|^{-s}$. Then for a sequence $u_n \rightarrow u$ in $\mathcal{D}_0^{1,p}(\Omega)$, $\int_{\Omega} \frac{|u_n|^q}{|y|^s} \rightarrow \int_{\Omega} \frac{|u|^q}{|y|^s}$, as $n \rightarrow \infty$. By Brézis-Lieb lemma [15],
\[ \lim_{n \rightarrow \infty} \left| \int_{\Omega} \frac{|u_n|^q}{|y|^s} - \int_{\Omega} \frac{|u|^q}{|y|^s} - \int_{\Omega} \frac{|u_n - u|^q}{|y|^s} \right| = 0. \]

Therefore, $\int_{\Omega} \frac{|u_n| - |u|^q}{|y|^s} \rightarrow 0$, as $n \rightarrow \infty$. Further, using $|x|^{-s} \leq |y|^{-s}$, we get $\int_{\Omega} \frac{|u_n| - |u|^q}{|x|^s} \rightarrow 0$ as $n \rightarrow \infty$, a contradiction. Therefore, $G_q$ is not compact for $|y|^{-s}$. \qed

**Remark 6.5.** For $q \in [p, p^*)$ and $g(x) = |y|^{-\frac{N}{\alpha(p,q)}}$, using the concentration compactness principles, the existence of the solutions of (1.19) is obtained on $\Omega = \mathbb{R}^N$ [11, Remark 2.6].

7. Examples and concluding remarks

In this section, we provide examples to show that the functions spaces given by Theorem 1.3, Theorem 1.6 and Theorem 1.12 are mutually independent. We also prove the necessary conditions. 

**Example 7.1.** (i) The spaces $L^{\alpha(p,q),r}(\mathbb{R}^N)$ and $L^1((0, \infty), r^{-\frac{N}{\alpha(p,q)-1}})$ are not comparable. For $N > p$ and $q \in [p, p^*)$, consider the following functions on $\mathbb{R}^N$:
\[ g_1(x) = |x|^{-\frac{N}{\alpha(p,q)}} \quad \text{and} \quad g_2(x) = (|x| + 1)^{-\frac{N}{\alpha(p,q)-1}}. \]

Then $g_1 \in L^{\alpha(p,q),\infty}(\mathbb{R}^N)$ and $g_2 \notin L^{\alpha(p,q),\infty}(\mathbb{R}^N)$. Further, $\tilde{g}_1 \notin L^1((0, \infty), r^{-\frac{N}{\alpha(p,q)-1}})$ and $\tilde{g}_2 \in L^1((0, \infty), r^{-\frac{N}{\alpha(p,q)-1}})$.

(ii) The function spaces provided by Theorem 1.6 and Theorem 1.12 are independent. For instance, Theorem 1.12 provides weight functions on the domain $\Omega = (\mathbb{R}^2 \setminus B_1(0)) \times \mathbb{R}$, for which Theorem 1.6 is not applicable. On the other hand, consider the following function:
\[ g(x) = \begin{cases} |x|^{-\frac{N}{2}}, & |x| \leq 1; \\ 0, & \text{otherwise}. \end{cases} \]

Since $\tilde{g} \in L^1((0, \infty))$, we can apply Theorem 1.6 to show $g \in \mathcal{H}_{p,P^*(1)}(\mathbb{R}^N)$. Although, Theorem 1.12 (for $q = P^*(1)$) is not applicable as $\tilde{g} \notin L^\infty((0, \infty))$.

Next we see that not all products of $g_1, g_2$ give rise to $(p,q)$-Hardy potentials.
Example 7.2. For \( q \in (0, P^*(k)) \), we consider \( g_1(y) = |y|^{-\frac{N}{\alpha(p,q)}}, y \in \mathbb{R}^k \). Hence \( g_1^s(t) = \left( \frac{t}{\omega_k} \right)^{-\frac{N}{\alpha(p,q)}} \) and \( |g_1|^s_{\alpha(p,q)k,\infty} = \omega_k^{-\frac{N}{\alpha(p,q)}} \). Therefore, \( g_1 \in L^{\alpha(p,q)}(\mathbb{R}^k) \). Since \( g_1 \) is not locally integrable on \( \mathbb{R}^k \), \( g(x) = g_1(y)g_2(z) \notin \mathcal{H}_{p,q}(\mathbb{R}^N) \) for \( q \in (0, P^*(k)) \) and for any non-zero \( g_2 \in L^1_{loc}(\mathbb{R}^{N-k}) \).

In the following remark, we describe the connection between Fefferman-Phong type conditions and Theorem 1.3.

Remark 7.3. (i) For \( N > p \) and \( q \in [p, p^*] \), every weight function in \( L^{\alpha(p,q)}(\mathbb{R}^N) \) satisfies (1.9) (by Remark 2.4) and hence \( \mathcal{L}^{\alpha(p,q)}(\mathbb{R}^N) \subset \mathcal{H}_{p,q}(\mathbb{R}^N) \). This gives an alternate proof for Theorem 1.3 -(i) without involving the embedding of \( D_0^{1,p}(\mathbb{R}^N) \).

(ii) We claim that for \( N > p \) and \( q > p^* \), if \( g \in \mathcal{L}^s_{loc}(\mathbb{R}^N) \) satisfies (1.9), then \( g \equiv 0 \). We choose \( x_0 \in \mathbb{R}^N \) and consider a sequence \( (Q_n) \) of cubes centred at \( x_0 \) and \( |Q_n| \to 0 \) as \( n \to \infty \). Since \( g \) satisfies (1.9),

\[
\lim_{n \to \infty} |Q_n|^s_{\alpha(p,q)} \left( \frac{1}{|Q_n|} \int_{Q_n} |g(x)|^s \, dx \right) \leq c_1, 
\]

where \( c_1 = C_1(p,q) > 0 \). By the Lebesgue-Besicovitch differentiation theorem,

\[
\lim_{n \to \infty} \frac{1}{|Q_n|} \int_{Q_n} |g(x)|^s \, dx = |g(x_0)|^s. 
\]

Moreover, since \( \alpha(p,q) < 0 \), we have \( |Q_n|^s_{\alpha(p,q)} \to \infty \), as \( n \to \infty \). Thus, (7.1) ensures that \( g(x_0) = 0 \).

Since \( x_0 \in \mathbb{R}^N \) is arbitrary, we get \( g = 0 \) a.e. in \( \mathbb{R}^N \).

(iii) Let \( N \leq p \leq q \), \( \Omega \) be bounded and \( g \in \mathcal{L}^s_{loc}(\Omega) \). Let \( g_{ext} \) be the zero extension of \( g \) outside \( \Omega \), such that \( g_{ext} \) satisfies (1.9). Choose \( Q_0 \supseteq \Omega \). Then

\[
|Q_0|^s_{\alpha(p,q)}^{-1} \int_{Q_0} |g_{ext}(x)|^s \, dx \leq \sup_{Q \subset \mathbb{R}^N : |Q| < \infty} |Q|^s_{\alpha(p,q)}^{-1} \int_Q |g_{ext}(x)|^s \, dx \leq c_1. 
\]

Consequently, \( g \in \mathcal{L}^s(\Omega) \). We would like to point out that \( \mathcal{L}^s(\Omega) \subseteq L^{1,\infty,\Phi}(\Omega) \subseteq L^1(\Omega) \) (by (ii) and (iii) of Proposition 2.6). Thus, for a bounded domain, Theorem 1.3 -(ii), (iii) gives a bigger class of weight functions than the Sawyer’s condition (1.9).

Remark 7.4. The weight functions provided in this article do not exhaust the entire \( \mathcal{H}_{p,q}(\Omega) \). For example, consider the weight functions of the form

\[
g(x) = g_1(y)g_2(z)g_3(w), \quad x = (y, z, w) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^{N-k-l},
\]

where \( 1 \leq k, l \leq \mathbb{N} \); \( k + l \leq N \). In general, corresponding to \( s_i \in \{1, 2, \ldots, N\} \) with \( 1 \leq i \leq m \leq N \) and \( \sum_{i=1}^m s_i = N \), consider the weight functions of the form

\[
g(x) = \prod_{i=1}^m g_i(y_i), \quad x = (y_1, y_2 \ldots y_m) \in \prod_{i=1}^m \mathbb{R}^{s_i}.
\]

One can provide conditions on \( g_1, g_2, \ldots, g_m \) so that \( g \) is in \( \mathcal{H}_{p,q}(\Omega) \).

7.1. The necessary conditions. In the following proposition we show under certain conditions on \( g \) that the spaces mentioned in Proposition A.2 and Proposition A.3 are necessary for \( g \) to satisfy (1.1). A similar result for the Hardy-Rellich inequalities is obtained in [7].
Proposition 7.5. Let $\Omega = B_R(0)$ with
\[ R \in \begin{cases} (0, \infty], & \text{if } N > p; \\ (0, \infty), & \text{if } N = p. \end{cases} \]
and \[ q \in \begin{cases} [p, p^*], & \text{if } N > p; \\ (p, \infty), & \text{if } N = p. \end{cases} \]

If $g \in \mathcal{H}_{p,q}(\Omega)$ is radial, radially decreasing, then
\[ g \in X := \begin{cases} L^{\alpha(p,q),\infty}(\Omega), & \text{if } N > p; \\ L^{1,\infty}(\Omega), & \text{if } N = p. \end{cases} \]

Proof. $N > p$: Let $g \in \mathcal{H}_{p,q}(\Omega)$ be a radial and radially decreasing. For each \( r \in (0, R) \), consider the following function:
\[ u_r(x) = \begin{cases} r - |x|, & \text{for } |x| \leq r; \\ 0, & \text{for } |x| \geq r. \end{cases} \]

Clearly,
\[ \nabla u_r(x) = \begin{cases} \frac{x}{|x|}, & \text{for } |x| \leq r; \\ 0, & \text{for } |x| \geq r. \end{cases} \]
and
\[ \int_{\Omega} |\nabla u_r(x)|^q \, dx = \int_{B_r} dx = \omega_N r^N. \quad (7.2) \]

Thus for each \( r \in (0, R) \), \( u_r \in \mathcal{D}_{0}^{1,p}(\Omega) \). Furthermore, since \( g \in \mathcal{H}_{p,q}(\Omega) \),
\[ \int_{\Omega} g(x)|u_r(x)|^q \, dx \leq C \left( \int_{\Omega} |\nabla u_r(x)|^p \, dx \right)^{\frac{q}{p}}, \quad \forall r \in (0, R). \quad (7.3) \]

Since \( g \) is radial and radially decreasing, we estimate the left hand side of the above inequality as below:
\[ \int_{\Omega} g(x)|u_r(x)|^q \, dx \geq \int_{B_r} g(|x|)|u_r(x)|^q \, dx \geq \left( r - \frac{r}{2} \right)^q \int_{B_\frac{r}{2}} g(|x|) \, dx \]
\[ = \left( \frac{r}{2} \right)^q \int_0^{\omega_N \left( \frac{r}{2} \right)^N} g^*(s) \, ds. \]

Therefore, from (7.2) and (7.3) we obtain
\[ \left( \frac{r}{2} \right)^q \int_0^{\omega_N \left( \frac{r}{2} \right)^N} g^*(s) \, ds \leq C r^{\frac{\alpha}{q - p}}. \]

Now by setting \( \omega_N \left( \frac{r}{2} \right)^N = t \) and since \( 0 < r < R \) is arbitrary, we conclude that
\[ \sup_{t \in (0, \frac{|\Omega|}{2N})} t^{\frac{1}{\alpha(p,q)}} g^{**}(t) \leq C. \]

Moreover, \( t^{\frac{1}{\alpha(p,q)}} g^{**}(t) \) is bounded on \( (\frac{|\Omega|}{2N}, |\Omega|) \). Therefore, \( g \) must belong to \( L^{\alpha(p,q),\infty}(\Omega) \).

$N = p$: Let \( R < \infty \) and \( q \in [N, \infty) \). For each \( r \in (0, R) \), we consider the following function:
\[ u_r(x) = \begin{cases} \log \left( \frac{|x|}{r} \right), & \text{for } |x| \leq r; \\ \log \left( \frac{|x|}{r^2} \right), & \text{for } |x| \geq r. \end{cases} \]

Clearly,
\[ \nabla u_r(x) = \begin{cases} 0, & \text{for } |x| \leq r; \\ -\frac{x}{|x|^2}, & \text{for } |x| \geq r, \end{cases} \]
and
\[ \int_{\Omega} |\nabla u_r(x)|^N \, dx = \omega_N \int_{r}^{R} \frac{dt}{t} = \omega_N \log \left( \frac{R}{r} \right). \quad (7.4) \]
Thus for each \( r \in (0, R) \), \( u_r \in D_0^{1,N}(\Omega) \). Furthermore, since \( g \in \mathcal{H}_{p,q}(\Omega) \),
\[
\int_{\Omega} g(x)|u_r(x)|^q \, dx \leq C \left( \int_{\Omega} |\nabla u_r(x)|^N \, dx \right)^{\frac{q}{N}}, \quad \forall r \in (0, R).
\] (7.5)
Since \( g \) is radial and radially decreasing, we estimate the left hand side of the above inequality as below:
\[
\int_{\Omega} g(x)|u_r(x)|^q \, dx \geq \left( \log \left( \frac{R}{r} \right) \right)^q \int_{B_r} g(|x|) \, dx = \left( \log \left( \frac{R}{r} \right) \right)^q \int_0^{\omega_N\lambda^{-N}} g^*(s) \, ds.
\]
Now using (7.4) and (7.5) we obtain
\[
\int_0^{\omega_N\lambda^{-N}} g^*(s) \, ds \leq C \left( \log \left( \frac{R}{r} \right) \right)^{\frac{q}{\gamma}}.
\]
By setting \( \omega_N\lambda^{-N} = t \) and since \( 0 < r < R \) is arbitrary, we conclude that
\[
\sup_{t \in (0,|\Omega|]} t \left( \log \left( \frac{|\Omega|}{t} \right) \right)^{\frac{q}{\gamma}} g^*(t) \leq C.
\]
Therefore, \( g \in L^{1,\infty, \frac{q}{\gamma}}(\Omega) \). \( \square \)

**APPENDIX A.**

In this section, we provide various Lorentz and Lorentz-Zygmund spaces in \( \mathcal{H}_{p,q}(\Omega) \). Then we supply alternative proofs for the Lorentz-Sobolev and Brezis-Wainger embeddings. First, we state a sufficient condition for the one-dimensional weighted Hardy inequalities due to Muckenhoupt in [49, Theorem 2] (for \( q = p \)), also see, [14, Theorem 2] (for \( q \geq p \)), [47, 57] (for \( 0 < q < p \)). For further readings on these inequalities, we refer to [37, Chapter 5].

**Proposition A.1** (Muckenhoupt condition). For \( b \in (0, \infty) \), let \( v, w \) be non-negative measurable functions on \((0, b)\) with \( w > 0 \). Let \( p \in (1, \infty), q \in (0, \infty), \) and \( \gamma = \frac{pq}{p-q} \).

(i) If \( 0 < q < 1 \) and
\[
A_1 = \left( \int_0^b \left( \int_0^s v(t) \, dt \right)^{\frac{q}{p}} \left( \int_s^b w(t)^{1-p'} \, dt \right)^{\frac{q}{p'}} v(s) \, ds \right)^{\frac{1}{q}} < \infty,
\]
then
\[
\left( \int_0^b \left| \int_s^b f(t) \, dt \right|^q v(s) \, ds \right)^{\frac{1}{q}} \leq (p')^{\frac{1}{q'}} q^\gamma A_1 \left( \int_0^b |f(s)|^p w(s) \, ds \right)^{\frac{1}{p}} \tag{A.1}
\]
holds for any measurable function \( f \) on \((0, b)\).

(ii) If \( 1 \leq q < p < \infty \) and
\[
A_2 = \left( \int_0^b \left( \int_0^s v(t) \, dt \right)^{\frac{q}{p}} \left( \int_s^b w(t)^{1-p'} \, dt \right)^{\frac{q}{p'}} w(s)^{1-p'} \, ds \right)^{\frac{1}{q'}} < \infty,
\]
then
\[
\left( \int_0^b \left| \int_s^b f(t) \, dt \right|^q v(s) \, ds \right)^{\frac{1}{q}} \leq (p')^{\frac{1}{q'}} q^\gamma A_2 \left( \int_0^b |f(s)|^p w(s) \, ds \right)^{\frac{1}{p}} \tag{A.2}
\]
holds for any measurable function \( f \) on \((0, b)\).
(iii) If \( 1 \leq p \leq q < \infty \) and

\[
A_3 = \sup_{0 \leq t \leq b} \left( \int_0^t v(s) \, ds \right)^{\frac{1}{q}} \left( \int_0^b w(s)^{1-p'} \, ds \right)^{\frac{1}{p'}} < \infty,
\]

then

\[
\left( \int_0^b \left| \int_s^b f(t) \, dt \right|^{q} v(s) \, ds \right)^{\frac{1}{q}} \leq \left( p' \right)^{\frac{1}{p'}} \left( p' \right)^{\frac{1}{p'}} A_3 \left( \int_0^b |f(s)|^{p} w(s) \, ds \right)^{\frac{1}{p}} \tag{A.3}
\]

holds for any measurable function \( f \) on \( (0, b) \).

**Proposition A.2.** Let \( N > p \) and

\[
X := \left\{ L^{\alpha(p,q)}_{\frac{N-1}{p-q}}(\Omega), \quad \text{for } q \in (0, p); \right. \\
\left. L^{\alpha(p,q)}_{\infty} (\Omega), \quad \text{for } q \in [p, p^*]. \right. 
\]

If \( g \in X \), then there exists \( C = C(N, p, q) > 0 \) such that

\[
\int_{\Omega} g^*(t) u^*(t)^q \, dt \leq C \| g \|_X \left( \int_0^b t^{p-rac{q}{p-q}} |u^*(t)|^p \, dt \right)^{\frac{2}{p}}, \quad \forall u \in C^1_c(\Omega). \tag{A.4}
\]

**Proof.** In Proposition A.1 we set \( f = u^*, v = g^* \) and \( w(t) = t^{p-\frac{q}{p-q}} \). Then we calculate

\[
\int_0^s v(t) \, dt = s (g^*(s)) \quad \text{and} \quad \int_0^{|\Omega|} w(t)^{1-p'} \, dt \leq \frac{N (p-1) \frac{p-q}{p-N}}{N - p}.
\]

Let \( C_1(N, p) = \frac{N (p-1) \frac{p-q}{p-N}}{N - p} \). We consider three cases.

\( q \in (0, 1) \): In this case,

\[
A_1^q = \int_0^{|\Omega|} \left( \int_0^s v(t) \, dt \right)^{\frac{q}{2}} \left( \int_s^{|\Omega|} w(t)^{1-p'} \, dt \right)^{\frac{q}{p'}} v(s) \, ds \\
\leq C_1^q \int_0^{|\Omega|} (sg^{**}(s))^{\frac{q}{2}} \frac{1}{s^{\frac{q (p-N)}{p}} g^*(s)} \, ds \leq C_1^q \int_0^{|\Omega|} (g^{**}(s))^{\frac{q}{2}} \, ds,
\]

where \( \gamma = \frac{qp}{p-q} \). Therefore,

\[
A_1 \leq C_1^q \left( \int_0^{|\Omega|} s^{\frac{q (p-N)}{p}} (g^{**}(s))^{\frac{1}{p-q}} \, ds \right)^{\frac{p-q}{qp}} = C_1^q \| g \|_{\alpha(p,q), \frac{p-q}{p}}.
\]

Thus for \( g \in L^{\alpha(p,q), \frac{p-q}{p-q}}(\Omega) \), the Muckenhoupt condition ((i) of Proposition A.1) is satisfied. Therefore, using (A.1) we obtain

\[
\int_{\Omega} g^*(t) u^*(t)^q \, dt \leq C \| g \|_{\alpha(p,q), \frac{p-q}{p-q}} \left( \int_0^{|\Omega|} t^{(p-\frac{q}{p-q})} |u^*(t)|^p \, dt \right)^{\frac{2}{p}},
\]

for some \( C = C(N, p, q) > 0 \).

\( q \in [1, p) \): In this case, we calculate

\[
A_2^q = \int_0^{|\Omega|} \left( \int_0^s v(t) \, dt \right)^{\frac{q}{2}} \left( \int_s^{|\Omega|} w(t)^{1-p'} \, dt \right)^{\frac{q}{p'}} w(s)^{1-p'} \, ds \\
\leq C_1^q \int_0^{|\Omega|} (sg^{**}(s))^{\frac{q}{2}} \frac{1}{s^{\frac{q (p-N)}{p}} - p(N-1)} \, ds,
\]
where \( \gamma = \frac{qp}{p-q} \). Moreover,

\[
\gamma + \frac{1}{N(p-1)} \left( \frac{\gamma(p-N)}{q^p} - p(N-1) \right) = \frac{qp}{N(p-q)}.
\]

Therefore, using (A.5), \( A_2 \leq C_1^\frac{1}{q^p} \|g\|^{\frac{1}{p^p}}_{\alpha(p,q),\frac{p}{q}} \). Thus for \( g \in L^{\alpha(p,q),\frac{p}{q}}(\Omega) \), the Muckenhoupt condition ((iii) of Proposition A.1) is satisfied, and hence using (A.2) we obtain (A.4).

\( q \in [p, p^*] \): In this case,

\[
A_3 = \sup_{s \in (0, [\Omega])} \left( \int_0^s v(t) \, dt \right) \frac{1}{q} \left( \int_0^{[\Omega]} w(t)^{1-p'} \, dt \right)^{\frac{1}{p'}} \\
\leq C_1 \sup_{s \in (0, [\Omega])} (sg^{**}(s))^\frac{1}{q} s^{\frac{p-N}{Np}} = C_1 \left( \sup_{s \in (0, [\Omega])} g^{**}(s)s^{\frac{N(p-q)+qp}{Np}} \right)^{\frac{1}{q}} = C_1 \|g\|^{\frac{1}{\alpha(p,q),\infty}}.
\]

Now for \( g \in L^{\alpha(p,q),\infty}(\Omega) \) using (A.3) we obtain (A.4). \( \square \)

**Proposition A.3.** Let \( N = p \) and \( \Omega \) be bounded. Let

\[
X := \left\{ \begin{array}{ll}
L^{1,\infty} N^{-\frac{q}{N}}(\Omega), & \text{for } q \in (0, 1); \\
L^{1,\infty} N^{-q} q-1(\Omega), & \text{for } q \in [1, N); \\
L^{1,\infty} N^q(\Omega), & \text{for } q \in [N, \infty). 
\end{array} \right.
\]

If \( g \in X \), then there exists \( C = C(N, q) > 0 \) such that

\[
\int_0^{[\Omega]} g^*(t)u^*(t)^q \, dt \leq C\|g\|_X \left( \int_0^{[\Omega]} t^{N-1}|u^*(t)|^N \, dt \right)^{\frac{1}{q}}, \quad \forall u \in C^1_c(\Omega). \tag{A.6}
\]

**Proof.** We only consider the case where \( q \in [N, \infty) \). For the other cases proof follows using similar set of arguments. As before, we set \( f = u^*, v = g^* \) and \( w(t) = t^{N-1} \). We see that \( \int_0^{[\Omega]} w(t)^{1-N'} \, dt \leq \log\left( \frac{e[\Omega]}{s} \right) \), and compute

\[
A_3 = \sup_{s \in (0, [\Omega])} \left( \int_0^s v(t) \, dt \right) \frac{1}{q} \left( \int_0^{[\Omega]} w(t)^{1-N'} \, dt \right)^{\frac{1}{p'}} \\
\leq \sup_{s \in (0, [\Omega])} (sg^{**}(s))^\frac{1}{q} \left( \log\left( \frac{e[\Omega]}{s} \right) \right)^{\frac{1}{p'}} = \|g\|^{\frac{1}{\alpha(p,q),N^q}},
\]

Thus for \( g \in L^{1,\infty} N^q(\Omega) \), the Muckenhoupt condition ((iii) of Proposition A.1) is satisfied. Therefore, using (A.3) we obtain (A.6). \( \square \)

In the following theorem we provide simple alternate proofs for the Lorentz-Sobolev embedding \( (N > p) \) and the Brezis-Wainger embedding \( (N = p) \).

**Theorem A.4.** Let \( \Omega \) be an open set in \( \mathbb{R}^N \) and \( p \in (1, N] \).

(i) **The Lorentz-Sobolev embedding:** Let \( N > p \). Then \( D_0^{1,p}(\Omega) \hookrightarrow L^{p^*} (\Omega) \), i.e., there exists \( C = C(N, p) > 0 \) such that

\[
\|u\|_{p^*} \leq C\|u\|_{D_0^{1,p}(\Omega)}, \quad \forall u \in D_0^{1,p}(\Omega). \tag{A.7}
\]
(ii) **The Brezis-Wainger embedding:** Let $N = p$ and $\Omega$ be bounded. Then $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{\infty,p;-1}(\Omega)$, i.e., there exists $C = C(p) > 0$ such that
\[
\|u\|_{\infty,p;-1} \leq C\|u\|_{\mathcal{D}_0^{1,p}(\Omega)}, \quad \forall u \in \mathcal{D}_0^{1,p}(\Omega).
\] (A.8)

**Proof.** (i) For $g = |x|^{-p} \in L^{\infty,p}(\mathbb{R}^N)$, from (i) of Example 2.7, we have $g^*(t) = \left(\frac{\omega N}{N-p}\right)^\frac{p}{N}$ and $\|g\|_{\frac{N}{p},\infty} = \frac{N\omega N}{N-p}$. Then using (A.4),
\[
\int_0^\infty t^{-\frac{p}{N}} u(t)^p \, dt \leq C \int_0^\infty t^{-\frac{p}{N}} |u^*(t)|^p \, dt, \quad \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N),
\]
where $C = C(N,p) > 0$. Notice that the left hand side of the above inequality is $\|u\|_{p^*,p}$ and it is equivalent to $\|u\|_{p^*,p}$. Thus from the Pólya-Szegö inequality (Proposition 2.2), we obtain
\[
\|u\|_{p^*,p} \leq C\|u\|_{\mathcal{D}_0^{1,p}(\mathbb{R}^N)}, \quad \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N).
\]
For $u \in \mathcal{D}_0^{1,p}(\Omega)$, where $\Omega$ is an open set in $\mathbb{R}^N$, one can consider the zero extension of $u$ to $\mathbb{R}^N$ to get (A.7).

(ii) Let $\Omega = B_R(0)$ where $R > 0$. We consider the function
\[
g(x) = \begin{cases} 
|x|^{-p} \left( \log \left( \frac{R}{|x|} \right)^p \right)^{-p}, & \text{for } x \in B_{R_1}(0); \\
(p|x|)^{-p}, & \text{for } x \in B_R(0) \setminus B_{R_1}(0),
\end{cases}
\]
where $R_1 = R e^{-\frac{1}{1-p}}$. Since $g$ is radial and radially decreasing on $B_R(0)$, using Example 2.7,
\[
g^*(t) = \begin{cases} 
\left( \log \left( \frac{\omega N}{t} \right) \right)^{-p}, & \text{for } t \in (0,|B_{R_1}(0)|); \\
\frac{\omega N}{p}\frac{\omega N}{t}, & \text{for } t \in ([B_{R_1}(0),|B_R(0)|]).
\end{cases}
\]
From (ii) of Example 2.7, $g \in L^{1,\infty,p}(B_{R_0}(0))$. Hence using (ii) of Proposition 2.6 and (ii) of Theorem 1.3, we conclude that $g \in \mathcal{H}_{p,p}^{\infty}(B_{R_0}(0))$. Moreover, from (A.6) and the Pólya-Szegö inequality, there exists $C = C(p) > 0$ such that
\[
\int_{|B_R(0)|} |\frac{u^*(t)}{\log(\frac{|B_{R_0}(0)|}{t})}|^p \, dt
t
\leq \int_{0}^{\frac{|B_{R_1}(0)|}{|B_R(0)|}} \left( \frac{u^*(t)}{\log(\frac{|B_{R_1}(0)|}{t})} \right)^p \, dt + \int_{\frac{|B_R(0)|}{|B_{R_1}(0)|}}^{\frac{|B_{R_0}(0)|}{|B_R(0)|}} \left( \frac{u^*(t)}{t} \right)^p \, dt
t
\leq C \int_{B_{R_1}(0)} g^*(t) |u^*(t)|^p \, dt
\leq C \int_{B_R(0)} \|
\n\|u\|_{\infty,p,-1} \leq C\|u\|_{\mathcal{D}_0^{1,p}(B_R(0))}, \quad \forall u \in \mathcal{D}_0^{1,p}(B_R(0)),
\]
where the first inequality holds since $(\log(\frac{|B_{R_1}(0)|}{t}))^{-1} \leq 1$ for $t \leq |B_R(0)|$. Notice that, the left hand side of the above inequality is $\|u\|_{\infty,p,-1}$ (equivalent to $\|u\|_{\infty,p,-1}$). Therefore,
\[
\|u\|_{\infty,p,-1} \leq C\|u\|_{\mathcal{D}_0^{1,p}(B_R(0))}, \quad \forall u \in \mathcal{D}_0^{1,p}(B_R(0)).
\]
Furthermore, every bounded open set $\Omega$ is contained in $B_R(0)$ for some $R > 0$. Thus the extension by zero to $B_R(0)$ together with above inequality yields (A.8). \qed
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References

[1] D. R. Adams. A sharp inequality of J. Moser for higher order derivatives. *Ann. of Math. (2)*, 128(2):385–398, 1988. doi:10.2307/1971445.
[2] Adimurthi, N. Chaudhuri, and M. Ramaswamy. An improved Hardy-Sobolev inequality and its application. *Proc. Amer. Math. Soc.*, 130(2):489–505, 2002. doi:10.1090/S0002-9939-01-06132-9.
[3] W. Allegretto. Principal eigenvalues for indefinite-weight elliptic problems in $\mathbb{R}^n$. *Proc. Amer. Math. Soc.*, 116(3):701–706, 1992. doi:10.2307/2159436.
[4] W. Allegretto and Y. X. Huang. Eigenvalues of the indefinite-weight $p$-Laplacian in weighted spaces. *Funkcial. Ekvac.*, 38(2):233–242, 1995. URL: http://www.math.kobe-u.ac.jp/~fe/xml/mr1355326.xml.
[5] T. V. Anoop. A note on generalized Hardy-Sobolev inequalities. *Int. J. Anal.*, pages Art. ID 784398, 9, 2013. doi:10.1155/2013/784398.
[6] T. V. Anoop and U. Das. The compactness and the concentration compactness via $p$-capacity. *Annali di Matematica Pura ed Applicata (1923 -)*, 2021. doi:10.1007/s10231-021-01098-2.
[7] T. V. Anoop, U. Das, and A. Sarkar. On the generalized Hardy-Rellich inequalities. *Proc. Roy. Soc. Edinburgh Sect. A*, 150(2):897–919, 2020. doi:10.1017/prm.2018.128.
[8] T. V. Anoop, P. Drábek, and S. Sasi. Weighted quasilinear eigenvalue problems in exterior domains. *Calc. Var. Partial Differential Equations*, 53(3-4):961–975, 2015. doi:10.1007/s00526-014-0773-2.
[9] T. V. Anoop, M. Lucia, and M. Ramaswamy. Eigenvalue problems with weights in Lorentz spaces. *Calc. Var. Partial Differential Equations*, 36(3):355–376, 2009. doi:10.1007/s00526-009-0232-7.
[10] M. Badiale and E. Serra. Critical nonlinear elliptic equations with singularities and cylindrical symmetry. *Rev. Mat. Iberoamericana*, 20(1):33–66, 2004. doi:10.4171/RMI/379.
[11] M. Badiale and G. Tarantello. A Sobolev-Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics. *Arch. Ration. Mech. Anal.*, 163(4):259–293, 2002. doi:10.1007/s002050200201.
[12] C. Bennett and K. Rudnick. On Lorentz-Zygmund spaces. *Dissertationes Math. (Rozprawy Mat.)*, 175:67, 1980.
[13] G. Bertin. *Dynamics of galaxies*. Cambridge University Press, Cambridge, 2000.
[14] J. S. Bradley. Hardy inequalities with mixed norms. *Canad. Math. Bull.*, 21(4):405–408, 1978. doi:10.4153/CMB-1978-071-7.
[15] H. Brézis and E. Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, 88(3):486–490, 1983. doi:10.2307/2044999.
[16] H. Brézis and J. L. Vázquez. Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Complut. Madrid*, 10(2):443–469, 1997.
[17] H. Brézis and S. Wainger. A note on limiting cases of Sobolev embeddings and convolution inequalities. *Comm. Partial Differential Equations*, 5(7):773–789, 1980. doi:10.1080/03605308008820154.
[18] L. Caffarelli, R. Kohn, and L. Nirenberg. First order interpolation inequalities with weights. *Compositio Math.*, 53(3):259–275, 1984. URL: http://www.numdam.org/item?id=CM_1984__53_3_259_0.
[19] R. E. Castillo and H. Rafeiro. *An introductory course in Lebesque spaces*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, [Cham], 2016. doi:10.1007/978-3-319-30034-4.
[20] S. Chanillo and R. L. Wheeden. L$p$estimates for fractional integrals and sobolev inequalities with applications to schrödinger operators. *Communications in Partial Differential Equations*, 10(9):1077–1116, 1985. doi:10.1080/03605308508820401.
[21] L. Ciotti. Dynamical models in astrophysics. *Lecture Notes, Scuola Normale Superiore, Pisa*, 2001.
[47] V. G. Maz’ja. *Sobolev spaces*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. doi:10.1007/978-3-662-09922-3. 12, 29, 32

[48] V. Maz’ya. Lectures on isoperimetric and isocapacitary inequalities in the theory of sobolev spaces. *Contemp. Math.*, 338:307–340, 01 2003. doi:10.1090/comm/338/06078. 4

[49] B. Muckenhoupt. Hardy’s inequality with weights. *Studia Math.*, 44:31–38, 1972. doi:10.4064/sm-44-1-31-38. 5, 32

[50] E. S. Noussair and C. A. Swanson. Solutions of Matukuma’s equation with finite total mass. *Indiana Univ. Math. J.*, 38(3):557–561, 1989. doi:10.1512/iiumj.1989.38.38026. 10

[51] R. O’Neil. Convolution operators and $L(p, q)$ spaces. *Duke Math. J.*, 30:129–142, 1963. URL: http://projecteuclid.org/euclid.dmj/1077374532. 7

[52] C. Pérez. Two weighted norm inequalities for Riesz potentials and uniform $L^p$-weighted Sobolev inequalities. *Indiana Univ. Math. J.*, 39(1):31–44, 1990. doi:10.1512/iiumj.1990.39.39004. 4

[53] G. Pólya and G. Szegö. *Isoperimetric Inequalities in Mathematical Physics*. Annals of Mathematics Studies, no. 27. Princeton University Press, Princeton, N. J., 1951. 12

[54] K. Sandeep. On a noncompact minimization problem of Hardy-Sobolev type. *Adv. Nonlinear Stud.*, 2(1):81–91, 2002. doi:10.1515/ans-2002-0106. 8

[55] E. Sawyer. A characterization of two weight norm inequalities for fractional and Poisson integrals. *Trans. Amer. Math. Soc.*, 308(2):533–545, 1988. doi:10.2307/2001090. 4

[56] E. Sawyer and R. L. Wheeden. Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Amer. J. Math.*, 114(4):813–874, 1992. doi:10.2307/2374799. 4

[57] G. J. Sinnamon. Weighted Hardy and Opial-type inequalities. *J. Math. Anal. Appl.*, 160(2):434–445, 1991. doi:10.1016/0022-247X(91)90316-R. 32

[58] A. Szulkin and M. Willem. Eigenvalue problems with indefinite weight. *Studia Math.*, 135(2):191–201, 1999. 10

[59] H. Tanaka. Two-weight norm inequalities for product fractional integral operators. *Bull. Sci. Math.*, 166:102940, 18, 2021. doi:10.1016/j.bulsci.2020.102940. 4

[60] H. Triebel. *Higher analysis*. Hochschulbücher für Mathematik. [University Books for Mathematics]. Johann Ambrosius Barth Verlag GmbH, Leipzig, 1992. Translated from the German by Bernhardt Simon [Bernhard Simon] and revised by the author. 4

[61] N. Visciglia. A note about the generalized Hardy-Sobolev inequality with potential in $L^{p,d}(\mathbb{R}^n)$. *Calc. Var. Partial Differential Equations*, 24(2):167–184, 2005. doi:10.1007/s00526-004-0319-0. 6, 10, 11

[62] E. Yanagida and S. Yotsutani. Global structure of positive solutions to equations of Matukuma type. *Arch. Rational Mech. Anal.*, 134(3):199–226, 1996. doi:10.1007/BF00379534. 10

[63] L. S. Yu. Nonlinear $p$-Laplacian problems on unbounded domains. *Proc. Amer. Math. Soc.*, 115(4):1037–1045, 1992. doi:10.2307/2159352. 10