TIME DECAY FOR SCHRÖDINGER EQUATION WITH
ROUGH POTENTIALS

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Abstract. We obtain certain time decay and regularity estimates
for 3D Schrödinger equation with a potential in the Kato class by
using Besov spaces associated with Schrödinger operators.

1. Introduction

The Schrödinger equation \( iu_t = -\Delta u \) describes the waves of a free
particle in a non-relativistic setting. It is physically important to con-
sider a perturbed dispersive system in the presence of interaction be-
tween fields.

Let \( H = -\Delta + V \), where \( \Delta \) is the Laplacian and \( V \) is a real-valued
function on \( \mathbb{R}^n \). In this note we are concerned with the time decay of
Schrödinger equation with a potential
\[
   iu_t = Hu,
   \quad u(x, 0) = u_0,
\]
where the solution is given by \( u(x, t) = e^{-itH}u_0 \). For simple exposition
we consider the three dimensional case for \( V \) in the Kato class \([9, 4]\). Recall that \( V \) is said to be in the
\( K_n, n \geq 3 \) provided
\[
   \lim_{\delta \to 0^+} \sup_{x \in \mathbb{R}^n} \int_{|x-y|<\delta} \frac{|V(y)|}{|x-y|^{n-2}} dy = 0.
\]
Throughout this article we assume that \( V = V_+ - V_-, \ V_\pm \geq 0 \) so that
\( V_+ \in K_{n,loc} \) and \( V_- \in K_n \), where \( V \in K_{n,loc} \) if and only if \( V \chi_B \in K_n \)
for any characteristic function \( \chi_B \) of the balls \( B \) centered at 0 in \( \mathbb{R}^n \).

We seek to find minimal smoothness condition on the initial da-
ta \( u_0 = f \) so that \( u(x, t) \) has certain global time decay and regularity
estimates. The idea is to combine the results of Jensen-Nakamura and
Rodnianski-Schlag \([4, 7]\) for short and long time decay by using Besov
space method.

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Paley theory.
In [1, 4, 3, 6, 13] several authors introduced and studied the Besov spaces and Triebel-Lizorkin spaces associated with $H$. Let $\{\varphi_j\}_{j=0}^{\infty} \subset C_0^\infty(\mathbb{R})$ be a dyadic system satisfying

(i) $\text{supp} \varphi_0 \subset \{ x : |x| \leq 1 \}$, $\text{supp} \varphi_j \subset \{ x : 2^{j-2} \leq |x| \leq 2^j \}$, $j \geq 1$,

(ii) $|\varphi_j^{(k)}(x)| \leq c_k 2^{-kj}$, $\forall k \geq 0, j \geq 0$,

(iii) $\sum_{j=0}^\infty |\varphi_j(x)| = 1$, $\forall x$.

Let $\alpha \in \mathbb{R}$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. The (inhomogeneous) Besov space associated with $H$, denoted by $B^{\alpha,q}_p(H)$, is defined to be the completion of $S(\mathbb{R}^n)$, the Schwartz class, with respect to the norm

$$\|f\|_{B^{\alpha,q}_p(H)} = \left( \sum_{j=0}^\infty 2^{j\alpha q} \|\varphi_j(H)f\|_{L^p}^q \right)^{1/q}.$$ 

Similarly, the (inhomogeneous) Triebel-Lizorkin space associated with $H$, denoted by $F^{\alpha,q}_p(H)$, $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ is defined by the norm

$$\|f\|_{F^{\alpha,q}_p(H)} = \left( \sum_{j=0}^\infty 2^{j\alpha q} |\varphi_j(H)f|^q \right)^{1/q} \|f\|_{L^p}.$$ 

The main result is the following theorem. Let $\|V\|_K$ denote the Kato norm

$$\|V\|_K := \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x - y|} \, dy.$$ 

Let $\beta := \beta(p) = n(\frac{1}{p} - \frac{1}{2})$ be the critical exponent.

**Theorem 1.1.** Let $1 \leq p \leq 2$. Suppose $V \in K_n$, $n = 3$ so that $\|V\|_K < 4\pi$ and

$$\int_{\mathbb{R}^6} \frac{|V(x)||V(y)|}{|x - y|^2} \, dx \, dy < (4\pi)^2.$$ 

The following statements hold. a) If $0 < t \leq 1$, then

$$\|e^{-itH}f\|_{p'} \lesssim \|f\|_{p'} + t^\beta \|f\|_{B^{2\beta,1}_p(H)}.$$ 

b) If in addition, $|\partial_x^\alpha V(x)| \leq c_\alpha$, $|\alpha| \leq 2n$, $n = 3$, then for all $t > 0$

$$\|e^{-itH}f\|_{L^{p'}} \lesssim \langle t \rangle^{-\frac{n}{2} - \frac{1}{2}} \|f\|_{B^{2\beta,1}_p(H)}.$$ 

where $p' = p/(p - 1)$ is the conjugate of $p$ and $\langle t \rangle = (1 + t^2)^{1/2}$.

**Remark 1.2.** The short time estimate in [4] is an improvement upon [4] since we only demand smoothness order being $\beta$ rather than $2\beta$. 
It is well known that if \( V \) satisfies (1), then \( \sigma(H) = \sigma(H_{ac}) = [0, \infty) \). Note that by Hardy-Littlewood-Sobolev inequality, \( V \in L^{3/2} \) implies the finiteness of the L.H.S of (1). Moreover, \( V \in L^{3/2+} \cap L^{3/2-} \) implies \( \|V\|_{K} < \infty \) [3 Lemma 4.3]. In particular, if \( \|V\|_{L^{3/2+} \cap L^{3/2-}} \) is sufficiently small, then the conditions of Theorem 1.1 (a) are satisfied.

The proof of the main theorem is a careful modification of that of the one dimensional result for a special potential in [6]. For short time we obtain (2) by modifying the proof of [4, Theorem 4.6]. The long time estimates simply follows from the \( L^{p} \to L^{p'} \) estimates for \( e^{-itH} \), \( 1 \leq p \leq 2 \), a result of [7, Theorem 2.6], and the embedding \( B^{\alpha, q}_{p}(H) \hookrightarrow L^{p}, \epsilon > 0, 1 \leq p, q \leq \infty \).

Note that from the definitions of \( B(H) \) and \( F(H) \) spaces we have

\[
B^{\alpha, \min(p,q)}_{p}(H) \hookrightarrow F^{\alpha, q}_{p}(H) \hookrightarrow B^{\alpha, \max(p,q)}_{p}(H)
\]
for \( 1 \leq p < \infty, 1 \leq q \leq \infty \), where \( \hookrightarrow \) means continuous embedding.

2. Proof of Theorem 1.1

The following lemma is proved in [4, Theorem 3, Remark 2.2].

**Lemma 2.1.** ([4]) Let \( 1 \leq p \leq \infty \). Suppose \( V \in K_{n}, n = 3 \) and \( \phi \in C_{0}^{\infty}(\mathbb{R}) \). Then there exists a constant \( c > 0 \) independent of \( \theta \in (0, 1] \) so that

\[
\|\phi(\theta H)e^{-itH}f\|_{p} \leq c(t)^{\beta}\|f\|_{p}.
\]

**Remark 2.2.** We can also give a simple proof of this lemma based on the fact that the heat kernel of \( H \) satisfies an upper Gaussian bound in short time. The interested reader is referred to [13] and [4, 9].

The long time decay has been studied quite extensively under a variety of conditions on \( V \) [5, 7, 8, 11, 12]. The following \( L^{p} \to L^{p'} \) estimates follow via interpolation between the \( L^{2} \) conservation and the \( L^{1} \to L^{\infty} \) estimate for \( e^{-itH} \) that was proved in [7, Theorem 2.6].

**Lemma 2.3.** Let \( 1 \leq p \leq 2 \). Suppose \( \|V\|_{K} < 4\pi \) and

\[
\int_{\mathbb{R}^{n}} \frac{|V(x)||V(y)|}{|x-y|^{2}}dxdy < (4\pi)^{2}.
\]

Then \( \|e^{-itH}f\|_{L^{p'}} \lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})}\|f\|_{L^{p}} \).

2.4. Proof of Theorem 1.1 (a) Let \( 0 < t \leq 1 \). Let \( \{\varphi_{j}\}_{j=0}^{\infty} \) be a smooth dyadic system as given in Section 1. For \( f \in S \) we write

\[
e^{-itH}f = \sum_{2jt \leq 1} \varphi_{j}(H)e^{-itH}f + \sum_{2jt > 1} \varphi_{j}(H)e^{-itH}f.
\]
According to Lemma 2.1 if \( j \geq j_i := \lceil -\log_2 t \rceil + 1 \),
\[
\| \varphi_j(H)e^{-itH}f \|_{p'} \leq c(t2^j)^\beta \| \varphi_j(H)f \|_{p'}
\]
where we noted that \( \varphi_j(H) = \psi_j(H) \varphi_j(H) \), \( \psi_j = \psi(2^{-j}x) \) if taking \( \psi \in C_0^\infty \) so that \( \psi(x) \equiv 1 \) on \([ -1, -\frac{1}{4} ] \cup [ \frac{1}{4}, 1 ] \). It follows that
\[
\sum_{2^j > 1} \| \varphi_j(H)e^{-itH}f \|_{p'} \leq c(t2^j)^\beta \sum_{2^j > 1} 2^{j\beta} \| \varphi_j(H)f \|_{p'}.
\]
For the first term in the R.H.S. of (5), similarly we have by applying Lemma 2.1 again,
\[
\| \sum_{2^j \leq 1} \varphi_j(H)e^{-itH}f \|_{p'} \leq c(t2^j)^\beta \| \eta(2^{-j}H)f \|_{p'} \leq c\|f\|_{p'}
\]
where we take \( \eta \in C_0^\infty \) with \( \eta(x) \equiv 1 \) on \([ -1, 1 ] \) so that \( \eta(2^{-j}H) \sum_{2^j \leq 1} \varphi_j(H) = \sum_{2^j \leq 1} \varphi_j(H) \). Therefore we obtain that if \( 0 < t \leq 1 \),
\[
\| e^{-itH}f \|_{p'} \lesssim \| f \|_{p'} + t^\beta \| f \|_{B_\nu^{\beta,1}(H)},
\]
which proves part (a).

(b) Inequality (3) holds for \( t > 1 \) in virtue of Lemma 2.3 and the remarks below Theorem 1.1. For \( 0 < t \leq 1 \), (3) follows from the Besov embedding \( B_\nu^{2\beta,1}(H) \hookrightarrow B_\nu^{\beta,1}(H) \), which is valid because of the condition \( |\partial_x V(x)| \leq c_\alpha, |\alpha| \leq 2n \); cf. e.g. [10, 13].

**Remark 2.5.** It seems from the proof that the smoothness order \( 2\beta \) in (3) is optimal for the initial data \( f \).

**Remark 2.6.** If working a little harder, we can show that
\[
\| e^{-itH}f \|_{L^{p'}} \lesssim \langle t \rangle^{-n \frac{1}{p'} - \frac{1}{2}} \| f \|_{B_\nu^{\beta,2}(H)},
\]
if assuming the upper Gaussian bound for the gradient of heat kernel of \( H \) in short time, in addition to the conditions in Theorem 1.1 (a). The proof of (4) is based on the embedding \( B_\nu^{0,2}(H) \hookrightarrow B_\nu^{0,2}(H) = L^{p'}, \ p' \geq 2 \) which follows from a deeper result by applying the gradient estimates for \( e^{-itH} \); see [13] and [2].

**Corollary 2.7.** Let \( 1 \leq p \leq 2 \), \( \alpha \in \mathbb{R} \) and \( \beta = \beta(p) \). Suppose \( V \) satisfies the same conditions as in Theorem 1.1 (b). The following estimates hold.
(a) If \( 1 \leq q \leq \infty \), then
\[
\| e^{-itH}f \|_{B_\nu^{\beta,q}(H)} \lesssim \langle t \rangle^{-n \frac{1}{p'} - \frac{1}{2}} \| f \|_{B_\nu^{\beta+2\beta,q}(H)},
\]
where we noted that \( \varphi_j(H) = \psi_j(H) \varphi_j(H) \), \( \psi_j = \psi(2^{-j}x) \) if taking \( \psi \in C_0^\infty \) so that \( \psi(x) \equiv 1 \) on \([ -1, -\frac{1}{4} ] \cup [ \frac{1}{4}, 1 ] \).
b) If $1 \leq q \leq p$, then

\begin{equation}
\|e^{-itH}f\|_{F^{\alpha,q}_{p}(H)} \lesssim \langle t \rangle^{-n(\frac{1}{p} - \frac{1}{2})}\|f\|_{B^{\alpha,2\beta,q}_{p}(H)}.
\end{equation}

Proof. Substituting $\varphi_{j}(H)f$ for $f$ in (3) we obtain

\begin{align*}
\|\varphi_{j}(H)e^{-itH}f\|_{L^{p'}} & \lesssim \langle t \rangle^{-n(\frac{1}{p} - \frac{1}{2})}\|\varphi_{j}(H)f\|_{B^{2\beta,1}_{p}(H)} \\
& \approx \langle t \rangle^{-n(\frac{1}{p} - \frac{1}{2})}2^{2\beta j}\|\varphi_{j}(H)f\|_{L^{p}}
\end{align*}

where we used $\|\varphi_{j}(H)g\|_{p} \leq c\|g\|_{p}$ by applying Lemma 2.1 with $\theta = 2^{-j}$ and $t = 0$. Now multiplying $2^{\alpha j}$ and taking $\ell^{q}$ norms in the above inequality gives (7). The estimate in (8) follows from the embedding $B^{\alpha,q}_{p}(H) \hookrightarrow F^{\alpha,q}_{p}(H)$ if $q \leq p$, according to (4). \qed

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