High frequency perturbation formulas for the effect of small inhomogeneities

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Abstract

We derive asymptotic expressions for the scattering effects of small inhomogeneities in the context of the two dimensional Helmholtz equation. Asymptotic refers to the limit as the inhomogeneity size tends to zero. One novelty is that these asymptotic formulas also apply to large frequency, not just finite or zero frequency, as has been considered previously. The derived formulas are compared to numerically simulated solutions of the Helmholtz equation.

1 Introduction

In this presentation we shall derive asymptotic expressions for the “shape” of an electromagnetic field scattered by small inhomogeneities. For simplicity we restrict our attention to the “transverse magnetic” situation in which case the scalar electric field satisfies a two dimensional Helmholtz equation. We take the incident wave to be a planar wave with direction of propagation $\eta$. For simplicity we consider only one inhomogeneity, and we assume that the permeability and the permittivity both equal 1 outside the inhomogeneity. Disregarding multiple scattering effects, our results immediately extend to several inhomogeneities. The object of study is the solution $u_\varepsilon$ to the equation

$$\nabla \cdot \left( \frac{1}{\mu_\varepsilon} \nabla u_\varepsilon \right) + k^2 q_\varepsilon u_\varepsilon = 0 \quad \text{in} \ \mathbb{R}^2,$$

for which the “backscattered” part $u_\varepsilon^{(s)}(y) = u_\varepsilon(y) - u^{(inc)}(y) = u_\varepsilon(y) - e^{ik\eta \cdot y}$ satisfies the “outgoing” radiation condition

$$\frac{\partial}{\partial r} u_\varepsilon^{(s)} - iku_\varepsilon^{(s)} = o(r^{-1/2}) \quad \text{as} \ r \to \infty.$$
The permeability, $\mu_\epsilon$, and the permittivity, $q_\epsilon$, are given by

$$\mu_\epsilon(y) = \begin{cases} 
\mu & \text{for } y \in \epsilon D \\
1 & \text{for } y \in \mathbb{R}^2 \setminus \epsilon D 
\end{cases}, \quad q_\epsilon(y) = \begin{cases} 
qu & \text{for } y \in \epsilon D \\
1 & \text{for } y \in \mathbb{R}^2 \setminus \epsilon D 
\end{cases},$$

where $D$ is a smooth, bounded domain of $\mathbb{R}^2$ (containing the origin, say) and $\epsilon$ is small. The constant $\mu$ is real and positive, the constant $q = q_{re} + i q_{im}$ has a positive real part $q_{re}$, and a non-negative imaginary part, $q_{im}$, representing the rescaled conductivity of the inhomogeneity. Frequently one writes $q_{im} = \sigma / k$, where the physical conductivity $\sigma$ is considered independent of $k$.

The goal of this presentation is to derive asymptotic formulas for the “backscattered” part of the solution $u_\epsilon^{(s)}(y) = u_\epsilon(y) - e^{ik\eta \cdot y}$, for $y$ bounded, and bounded away from $\epsilon D$. The term asymptotic refers to the limit $\epsilon \to 0$. The novelty is that these asymptotic formulas are valid also for large frequency $k$, not just finite or zero frequency, as has been considered previously [3], [17], and [1]. In future work we shall discuss the effective use of these formulas as a tool to determine the location and certain geometric properties of the inhomogeneity(ies). As an illustration of the effect of changes in frequency consider Figure 1. This figure displays an $L^2$ norm, $\|u_\epsilon^{(s)}\|_{L^2(\{r=2\})} = \left( \frac{1}{2\pi} \int_{\theta=0}^\pi |u_\epsilon^{(s)}|^2 \, d\theta \right)^{1/2}$, of the backscattered part of the solution as a function of frequency, $k$, for three different size, small circular inhomogeneities. The inhomogeneities are centered at the origin, and the radii are $0.01$, $4 \times 0.001$, and $0.001$. The coefficients $\mu$ and $q$, the permeability and permittivity inside the inhomogeneity, are $2$ and $2 + 2i$ (independently of $k$) respectively. The $L^2$ norm of $u_\epsilon^{(s)}$ is computed on the circle of radius $2$. As is evident, the asymptotic behaviour as $\epsilon \to 0$ is very different for small $k$ and large $k$. Consistent with the left frame in Figure 1, and as will be explained in the next section, the asymptotic “size” of $u_\epsilon^{(s)}$ for small (fixed) $k$ is of the order $\epsilon^2$. In the next section we also briefly explain the asymptotics for $k$ of the order $\epsilon^{-1}$. However, it is the asymptotic behaviour for very large $k$ that is the main
focus of this presentation, and which will be analyzed in sections 3.1 and 3.2. In accordance with the right frame in Figure 1 we should, in the very high frequency regime, expect an asymptotic “size” of the order $\sqrt{\epsilon}$. Such behaviour would be consistent with the rigorous norm estimates of the scattered field derived in [6].

As $\epsilon \to 0$, we characterize the low frequency asymptotic regime by: $\tilde{k} = k\epsilon \to 0$, the moderate frequency regime by: $\tilde{k} \to \lambda_0$, with $0 < \lambda_0 < \infty$, and the high frequency regime by: $\tilde{k} \to \infty$. Note that low frequency here means low relative frequency, i.e., the situation when the wavelength is large relative to the diameter of the scatterer. Our common approach to deriving asymptotic formulas for $u^{(s)}_\epsilon$ for these three separate regimes is to first define the function

$$v_k(y) = u_\epsilon(\epsilon y)$$

which satisfies the equivalent of the equations (1)-(2), with $\epsilon D$ replaced by $D$ and $k$ replaced by $\tilde{k}$. We then perform a formal asymptotic analysis in $\tilde{k}$ for this rescaled problem.

For low frequency—that is, $k$ of the order $o(\epsilon^{-1})$—our approach yields an asymptotic formula identical to that already derived in [17] and [1] (and essentially the same as a formula derived in [7], [8]) for the case of fixed frequency, namely

$$u_\epsilon(y) \approx u^{(inc)}(y) + \epsilon^2 \nabla_x \Phi_k(0, y) \cdot \left(1 - \frac{1}{\mu}\right) M \nabla u^{(inc)}(0) + \epsilon^2 k^2 \Phi_k(0, y)(q - 1) |D| u^{(inc)}(0) \quad (3)$$

for $y$ bounded and bounded away from $\epsilon D$. Here $\Phi_k$ is the (outgoing) free space Green’s function for the Helmholtz operator $\Delta + k^2$, and $M$ is a polarization tensor determined by the shape of $D$. Scattering in the regime where the wavelength of the incident wave is of a larger order than the size of the scatterer is known as Rayleigh scattering [8]. It is important to distinguish the small volume perturbation studied here from a small amplitude perturbation. If the perturbation were a small amplitude perturbation, meaning if $k^2q$ were close to $k^2$ and $\mu$ were close to 1, and if $\epsilon$ were not small, then a relevant perturbation formula would be the Born approximation. We have the integral representation formula

$$u_\epsilon(y) = u^{(inc)}(y) + k^2(q - 1) \int_{\epsilon D} u_\epsilon(x) \Phi_k(x, y) \, dx + \left(1 - \frac{1}{\mu}\right) \int_{\epsilon D} \nabla u_\epsilon(x) \nabla_x \Phi_k(x, y) \, dx$$

as can be verified by straightforward integration by parts. The Born approximation results when $u^{(inc)}$ is substituted for $u_\epsilon$ in the above integrals. There are no polarization effects. In contrast, to arrive at the Rayleigh approximation (3) the Lebesgue measure in the first integral is replaced by the point mass $\epsilon^2 |D| \delta_{x=0}$ – in the second integral the Lebesgue measure is also replaced by a point mass,
Figure 2: As Figure 1, but the inhomogeneities are non-conducting.

but the gradient $\nabla u_\epsilon$ is replaced by the polarized gradient $\epsilon^2 M \nabla u^{(inc)}_k$, which incorporates parts of the geometry of the scatterer.

The case of high frequency is, loosely speaking, the case of a small volume but very high amplitude perturbation. For this asymptotic regime, our approach is to first construct approximations to $v^{(s)}_k$ and $\frac{\partial}{\partial n} v^{(s)}_k$ on $\partial D$ by way of the technique of geometric optics ([10] and [14]). We then insert these approximations into a Green’s integral representation of $u^{(s)}_\epsilon$, exterior to $\epsilon D$, and perform a stationary phase analysis of this integral. We note that our approach has some similarity to that studied in [2], but is different in the sense that our goal is an explicit representation formula, not a stable numerical method. The representation formula we find shares some similarities with a formula derived in [13] for the so-called filtered scattering amplitude in a three-dimensional setting.

At this point it is relevant to note that the very smooth behaviour in the “tail” of the graphs in the right frame in Figure 1 is related to the “absorbing” character of the inhomogeneity (due to its positive conductivity). Figure 2 is the analogue of the right frame of Figure 1, only this time the permittivity of the inhomogeneity equals 2, not $2 + 2i$. The “tail” now oscillates with changing frequency (due to the presence of multiple scattering effects from the “back-side” of the inhomogeneity) though the average behaviour at high $k$ still suggests an asymptotic behaviour of the order $\sqrt{\epsilon}$ as $\epsilon \to 0$. The behaviour at small to moderate frequency is unaffected by the vanishing of the conductivity. This figure does suggest that appropriate frequency averaging may be advantageous when processing “broad-band” data originating from non-conducting inhomogeneities. We hope to be able to very effectively use the change in asymptotic behaviour of $u^{(s)}_\epsilon$ to estimate the size of multiple inhomogeneities. With that in mind we also display in Figure 3 the (modulus) of the average $\frac{1}{2\pi} \int_{r=2} u^{(s)}_\epsilon \ d\theta$ as a function of $k$ for two of the circular inhomogeneities from Figure 1 (of radius .01 and .001). The location of the extremum of this average quantity seemingly yields a reasonable estimate of $\epsilon^{-1}$. 
Figure 3: Averages on \( \{ r = 2 \} \) for two of the inhomogeneities from Figure 1.

2 Small to moderate frequency

After replacing the variable \( y \) by \( \epsilon y \), and introducing \( \tilde{k} = k\epsilon \), we obtain the function

\[
v_{\epsilon}(y) := u_\epsilon(\epsilon y),
\]

which satisfies the equivalent of the equations (1)-(2), with \( \epsilon D \) replaced by \( D \), and \( k \) replaced by \( \tilde{k} \). Since the incident wave is given by \( u^{(\text{inc})}(y) = e^{i\tilde{k}\eta \cdot y} \), we may decompose \( v_{\tilde{k}} \) as

\[
v_{\tilde{k}}(y) = \begin{cases} v_{\tilde{k}}^{(\text{tr})}(y) & \text{for } y \in D \\ v_{\tilde{k}}^{(s)}(y) + e^{i\tilde{k}\eta \cdot y} & \text{for } y \in \mathbb{R}^2 \setminus \overline{D} \end{cases}
\]

with \( v_{\tilde{k}}^{(\text{tr})} \) and \( v_{\tilde{k}}^{(s)} \) satisfying

\[
(\Delta + \tilde{k}^2 \mu q) v_{\tilde{k}}^{(\text{tr})} = 0 \quad \text{in} \quad D, \quad (\Delta + \tilde{k}^2) v_{\tilde{k}}^{(s)} = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D},
\]

and the transmission conditions

\[
v_{\tilde{k}}^{(\text{tr})}(y) = v_{\tilde{k}}^{(s)}(y) + e^{i\tilde{k}\eta \cdot y} \quad \text{for } y \in \partial D,
\]

\[
\frac{1}{\mu} \frac{\partial}{\partial n} v_{\tilde{k}}^{(\text{tr})}(y) = \frac{\partial}{\partial n} v_{\tilde{k}}^{(s)}(y) + i\tilde{k}\eta \cdot n e^{i\tilde{k}\eta \cdot y} \quad \text{for } y \in \partial D,
\]

plus the outgoing radiation condition

\[
\frac{\partial}{\partial r} v_{\tilde{k}}^{(s)}(r) - i\tilde{k} v_{\tilde{k}}^{(s)}(r) = o(r^{-1/2}) \quad \text{as } r \to \infty.
\]

It is easy to see that \( v_{\tilde{k}} \) depends continuously on \( 0 < \tilde{k} < \infty \). Furthermore, if \( w = (w_1, w_2) \), where \( w_j \) denotes the solution to

\[
\Delta w_j = 0 \quad \text{in} \quad D, \quad \Delta w_j = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D},
\]
with \[ w_j^- = w_j^+ , \quad \text{and} \quad \frac{1}{\mu} \partial_\mathbf{n} w_j^- = \frac{\partial}{\partial \mathbf{n}} w_j^+ \quad \text{on} \quad \partial D , \]

and

\[ w_j(y) - y_j \to 0 \quad \text{as} \quad |y| \to \infty , \]

then, by formal asymptotics we get, for any \( L < \infty , \)

\[ v^{(tr)}_k(y) = 1 + i\tilde{k}^2 \quad \text{for} \quad y \in D , \quad \text{and} \]

\[ v^{(s)}_k(y) = i\tilde{k} \quad \text{satisfying the outgoing radiation condition} \]

\[ \Phi_k(x, y) \text{ be the solution to} \]

\[ (\Delta + \tilde{k}^2) \Phi_k(\cdot, y) = -\delta_y \quad \text{in} \quad \mathbb{R}^2 , \]

satisfying the outgoing radiation condition

\[ \frac{\partial}{\partial r} \Phi_k(\cdot, y) - i\tilde{k} \Phi_k(\cdot, y) = o(r^{-1/2}) \quad \text{as} \quad r \to \infty . \]

It is well known that

\[ \Phi_k(x, y) = \frac{i}{4} H^0_1(\tilde{k}|x - y|) , \]

where \( H^0_1 \) denotes the zero'th order Hankel function of type one. For \( y \) strictly outside \( D \) we may now write

\[ v^{(s)}_k(y) = \int_{\mathbb{R}^2} (\Delta + \tilde{k}^2) \Phi_k(x, y) v^{(s)}_k(x) \, dx \]

\[ = \int_{\partial D} \frac{\partial}{\partial \mathbf{n}_x} \Phi_k(x, y) v^{(s)}_k(x) \, d\sigma_x - \int_{\partial D} \Phi_k(x, y) \frac{\partial}{\partial \mathbf{n}_x} v^{(s)}_k(x) \, d\sigma_x , \]

and so, by use of (6) and integration by parts

\[ u^{(s)}_k(y) = v^{(s)}_k(y/e) \]

\[ = \int_{\partial D} \frac{\partial}{\partial \mathbf{n}_x} \Phi_k(x, y/e) v^{(s)}_k(x) \, d\sigma_x - \int_{\partial D} \Phi_k(x, y/e) \frac{\partial}{\partial \mathbf{n}_x} v^{(s)}_k(x) \, d\sigma_x \]

\[ = \int_{\partial D} \frac{\partial}{\partial \mathbf{n}_x} \Phi_k(x, y/e) v^{(s)}_k(x) \, d\sigma_x \]

\[ - \int_{\partial D} \Phi_k(x, y/e) \left( \frac{1}{\mu} \frac{\partial}{\partial \mathbf{n}_x} v^{(tr)}_k(x) - \frac{\partial}{\partial \mathbf{n}_x} e^{ik\theta} \right) \, d\sigma_x \]

\[ = \int_{\partial D} \frac{\partial}{\partial \mathbf{n}_x} \Phi_k(x, y/e) v^{(s)}_k(x) \, d\sigma_x \]

\[ - \int_{\partial D} \Phi_k(x, y/e) \left( \frac{1}{\mu} \frac{\partial}{\partial \mathbf{n}_x} v^{(tr)}_k(x) - e^{ik\theta} \right) \, d\sigma_x \]

\[ - \tilde{k}^2 \int_D \Phi_k(x, y/e) \left( \frac{1}{\mu} - q \right) v^{(tr)}_k(x) \, dx . \]
Now consider first the case when $\tilde{k} = k \epsilon \to 0$ (and $k > k_0 > 0$). We then have, for $x \in \partial D$, and for $y$ bounded, and bounded away from zero

$$\Phi_{\tilde{k}}(x, y/\epsilon) = \frac{i}{4} H^{(1)}_0(\tilde{k}|x - y/\epsilon|)$$

$$\approx \frac{i}{4} H^{(1)}_0(k|y|) = \Phi_k(0, y),$$

and

$$\frac{\partial}{\partial n_x} \Phi_{\tilde{k}}(x, y/\epsilon) = \frac{i\tilde{k}}{4} \frac{y - x}{|y - x|} \cdot n_x H^{(1)}_1 \left( \frac{1}{\epsilon} |y - x| \right)$$

$$\approx \frac{i\tilde{k}}{4} \frac{y}{|y|} \cdot n_x H^{(1)}_1(k|y|)$$

$$= \epsilon \nabla_x \Phi_k(0, y) \cdot n_x.$$

By insertion of these asymptotic identities into (9), and use of the identities (7)-(8) for $v^{(tr)}_k$ and $v^{(s)}_k$, we obtain

$$u^{(s)}_\epsilon(y) \approx i\epsilon^2 k \nabla_x \Phi_k(0, y) \cdot \int_{\partial D} n_x \eta \cdot (w - x) \, d\sigma_x$$

$$-i\epsilon^2 k \nabla_x \Phi_k(0, y) \cdot \int_{\partial D} n_x \eta \cdot \left( \frac{1}{\mu} w - x \right) \, d\sigma_x$$

$$- \int_{\partial D} \frac{\partial}{\partial n_x} \Phi_{\tilde{k}}(x, y/\epsilon) \left( \frac{1}{\mu} - 1 \right) \, d\sigma_x$$

$$- \tilde{k}^2 \Phi_k(0, y) \left( \frac{1}{\mu - q} \right) |D|$$

$$= i\epsilon^2 k \nabla_x \Phi_k(0, y) \cdot \left( 1 - \frac{1}{\mu} \right) \int_{\partial D} n_x \eta \cdot w \, d\sigma_x$$

$$+ \tilde{k}^2 \Phi_k(0, y)(q - 1)|D| \ ,$$

(10)

The $2 \times 2$ matrix with entries

$$\int_D \frac{\partial}{\partial x_i} w_j \, dx$$

is exactly the polarization tensor $M$ from [17] (see also [3]), and so in summary we have

$$u_\epsilon(y) \approx u^{(inc)}_\epsilon(y) + \epsilon^2 \nabla_x \Phi_k(0, y) \cdot \left( 1 - \frac{1}{\mu} \right) M \nabla u^{(inc)}(0)$$

$$+ \epsilon^2 \tilde{k}^2 \Phi_k(0, y)(q - 1)|D|u^{(inc)}(0) \ ,$$

(11)

for $y$ bounded and bounded away from $\epsilon D$. This formally extends the validity of the identical asymptotic formula already derived in [17] and [1] (for a fixed
frequency \( k \) to the more general case of \( k \epsilon \to 0 \) (i.e. \( k \) of the order \( o(\epsilon^{-1}) \)). The formula derived in [7], [8] (for a fixed frequency \( k \)) is very similar, except that Jones employs the so-called content matrix in place of the polarization tensor \( M \).

We now give a brief analysis of the limiting behaviour of \( u_\epsilon(y) \) as \( \epsilon \to 0 \), and \( \tilde{k} = k \epsilon \to \lambda_0 \), with \( 0 < \lambda_0 < \infty \). For that purpose we return to the first identity in (9), and we use the fact that under the present circumstances

\[
\phi_k(x, y/\epsilon) = \frac{i}{4} H_0^{(1)}(\tilde{k}|x - y/\epsilon|) \\
\approx \frac{i}{4} H_0^{(1)}(k|y| - \lambda_0 \frac{y}{|y|} \cdot x) \\
\approx \sqrt{\epsilon} \sqrt{\frac{1}{8\pi\lambda_0|y|}} e^{i(k|y| - \lambda_0 \frac{y}{|y|} \cdot x + \frac{\pi}{4})}
\]

and

\[
\frac{\partial}{\partial n_x} \phi_k(x, y/\epsilon) = \frac{i\tilde{k}}{4} \frac{y - x}{|x - x|} \cdot n_x H_1^{(1)}(\tilde{k}|x - y/\epsilon|) \\
\approx \frac{i\tilde{k}}{4} \frac{y}{|y|} n_x H_1^{(1)}(k|y| - \lambda_0 \frac{y}{|y|} \cdot x) \\
\approx \sqrt{\epsilon} \frac{y}{|y|} \cdot n_x \sqrt{\frac{\lambda_0}{8\pi|y|}} e^{i(k|y| - \lambda_0 \frac{y}{|y|} \cdot x - \frac{\pi}{4})}
\]

These relations, together with the fact that \( v_k^{(s)} \) is continuous with respect to \( \tilde{k} \), lead to

\[
u_k^{(s)}(y) = \int_{\partial D} \frac{\partial}{\partial n_x} \phi_k(x, y/\epsilon) v_k^{(s)}(x) \ d\sigma_x - \int_{\partial D} \phi_k(x, y/\epsilon) \frac{\partial}{\partial n_x} v_k^{(s)}(x) \ d\sigma_x \\
\approx \sqrt{\epsilon} \sqrt{\frac{1}{8\pi|y|}} \sqrt{\lambda_0} \int_{\partial D} \left( \frac{y}{|y|} \cdot n_x v_k^{(s)}(x) - i \frac{\partial}{\partial n_x} v_k^{(s)}(x) \right) e^{-i\lambda_0 \frac{y}{|y|} \cdot x} \ d\sigma_x.
\]

As already discussed in the introduction, the formula (10) (or(11)) yields an asymptotic scattered field of the magnitude \( \epsilon^2 \), for fixed \( k \), whereas the formula (12) yields an asymptotic scattered field of the magnitude \( \sqrt{\epsilon} \), for \( k \) of the order \( \epsilon^{-1} \). However, it is interesting to note the continuous transition between the size of the expressions in these two formulas, when we gradually admit larger frequencies. If we take into account \( k \) when estimating the size of (10) (or(11)), then we obtain an estimate of the order \( \epsilon^2 k^{3/2} \), since \( \nabla_x \Phi_k(0, y) \) is of the order \( \sqrt{k} \) and \( \Phi_k(0, y) \) is of the order \( k^{-1/2} \). By taking \( k = \epsilon^{-\alpha} \), \( 0 < \alpha < 1 \) we thus obtain \( \epsilon^2 \sqrt{k}^\alpha \), which “transitions” nicely between \( \epsilon^2 \) and \( \sqrt{\epsilon} \) as \( \alpha \) goes from 0 to 1.
Figure 4: $\|u_\varepsilon^{(s)}\|_{L^2(r=2)}$ as a function of frequency for three different hard circular inhomogeneities.

**Remark 1.** In what we have discussed so far we have not included inhomogeneities in the form of so-called hard scatterers. By this notion we refer to the case when, of the two equations in (4), only the one in $\mathbb{R}^2 \setminus D$ is satisfied, and the transmission conditions (5)-(6) are replaced by the boundary condition $u_\tilde{k}^{(s)}(y) + e^{i\tilde{k}Ry} = 0$ on $\partial D$. In the framework of (4) and (5)-(6) this may, for a fixed $\tilde{k}$, for example be obtained by taking negative values for $q$, and considering the limiting process $\mu \to 0$, and $\mu q \to c < 0$. This limiting process carries over immediately to the moderate frequency approximation (12). As far as the low frequency approximation (10) goes the situation is a little bit more complicated: the first term has a finite limit (of order $\varepsilon^2 k^3/2$) as $\mu \to 0$ (cf. the limit analysis of the polarization tensor performed in [3]) whereas the second term becomes unbounded as $q \to -\infty$. Thus the asymptotic magnitude of $u_\varepsilon^{(s)}$ as $\tilde{k} = k\varepsilon \to 0$ (with $k > k_0 > 0$) is not of the order $\varepsilon^2 k^{3/2}$. Instead the analogous approximate statement for a hard scatterer becomes

$$u_\varepsilon^{(s)}(y) \approx [\log(\varepsilon k)]^{-1} \Phi_k(0,y) 2\pi u^{(inc)}(0),$$

where $\Phi_k(0,y)$ is the limit of the polarization tensor for a hard scatterer.

In Figure 4 we display the $L^2$ norm of $u_\varepsilon^{(s)}$ on $r = 2$ for the case of a hard scatterer in the shape of an $\varepsilon$ disk centered at the origin ($\varepsilon = 0.01, 4 \times 0.001$, and 0.001). For large $k$ this $L^2$ norm behaves the same way as for an absorbing, transmitting inhomogeneity, but the behaviour for small fixed $k$ is entirely different (in agreement with (13)). A sharp transition appears before $k$ reaches the order of $\varepsilon^{-1}$. 
We now proceed to study the case of very high frequency. One way to do this would be to study the “limiting behaviour” of the asymptotic formula (12) as \( \lambda_0 \to \infty \). Indeed, the principal formula we derive could be found by exactly that approach, and corresponds to the regime \( \tilde{k} = k\epsilon \to \infty \), but \( k\epsilon^2 \to 0 \). However, to better understand these limitations as well as the changes that would be required in the context of even higher frequencies, we return to the original problem involving \( u_\epsilon \) and \( v_\tilde{k} \). To recapitulate: the physical, scalar electric field is given by \( u_\epsilon(y) = v_\tilde{k}(y/\epsilon) \), where \( v_\tilde{k} \) may be decomposed as

\[
v_\tilde{k}(y) = \begin{cases} v^{(tr)}_\tilde{k}(y) & \text{for } y \in D \\ v^{(s)}_\tilde{k}(y) + e^{i\tilde{k}\cdot y} & \text{for } y \in \mathbb{R}^2 \setminus \overline{D} \end{cases},
\]

and the transmitted and the scattered fields satisfy

\[
(\Delta + \tilde{k}^2 \mu q) v^{(tr)}_\tilde{k} = 0 \quad \text{in } D ,
(\Delta + \tilde{k}^2) v^{(s)}_\tilde{k} = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D} ,
\]

with the transmission conditions

\[
\begin{align*}
v^{(tr)}_\tilde{k}(y) &= v^{(s)}_\tilde{k}(y) + e^{i\tilde{k}\cdot y} & \text{for } y \in \partial D , \\
\frac{1}{\mu} \frac{\partial}{\partial n} v^{(tr)}_\tilde{k}(y) &= \frac{\partial}{\partial n} v^{(s)}_\tilde{k}(y) + i\tilde{k}\cdot y & \text{for } y \in \partial D ,
\end{align*}
\]

and the outgoing radiation condition

\[
\frac{\partial}{\partial r} v^{(s)}_\tilde{k} - i\tilde{k} v^{(s)}_\tilde{k} = o(r^{-1/2}) \quad \text{as } r \to \infty .
\]

Our approach, in order to approximate \( u_\epsilon(y) \), is to first construct (geometric optics) approximations to \( v^{(s)}_\tilde{k} \) and \( \frac{\partial}{\partial n} v^{(s)}_\tilde{k} \) on \( \partial D \), then insert these approximations into the Green’s formula

\[
u^{(s)}_\epsilon(y) = \int_{\partial D} \frac{\partial}{\partial n_x} \Phi(\tilde{k})(x, y/\epsilon) v^{(s)}_\tilde{k}(x) \, d\sigma_x - \int_{\partial D} \Phi(\tilde{k})(x, y/\epsilon) \frac{\partial}{\partial n} v^{(s)}_\tilde{k}(x) \, d\sigma_x
\]

and then perform a “stationary phase” analysis to arrive at the corresponding high frequency, small \( \epsilon \) asymptotics. At this point it is necessary to assume that the domain \( D \), representing the shape of the inhomogeneity, is convex. We note that it remains an open, interesting problem to try to find ansätze for the boundary data that are better than the geometric optics approximations (as we shall see, this is particularly relevant for transmitting, non-conducting inhomogeneities). In this connection we find that the work on “on surface radiation conditions” [9] and the work on impedance boundary conditions [12] may be relevant.
3.1 Geometric optics asymptotics for the boundary traces

In order to derive approximations to \( v_k \) and \( \frac{\partial}{\partial n} v_k \) on \( \partial D \) we use the well known technique of geometric optics (see [10] and [14]). The very basis of this technique is to introduce an ansatz of the form

\[
v^{(tr)}_k(y) = A^{tr}_k(y) e^{ik\phi_{tr}(y)} , \quad \text{and} \quad v^{(s)}_k(y) = A_s(y) e^{ik\phi_s(y)} ,
\]

to insert this into the equations (14), (15), and (16), and then to perform a formal asymptotic expansion in powers of \( \tilde{k}^{-1} \). For the zeroth order terms one gets

\[
\nabla \phi^{0}_{tr} \cdot \nabla \phi^{0}_{tr} = \mu q \quad \text{in} \quad D \quad , \quad \nabla \phi^{0}_{s} \cdot \nabla \phi^{0}_{s} = 1 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D} ,
\]

the so-called Eikonal Equations, with

\[
\phi^{0}_{tr}(y) = \phi^{0}_s(y) = \eta \cdot y , \quad A^{0}_{tr}(y) = A^{0}_s(y) + 1 \quad y \in \partial D \quad ,
\]

and

\[
\frac{1}{\mu} A^{0}_{tr} \frac{\partial \phi^{0}_{tr}}{\partial n} = A^{0}_s \frac{\partial \phi^{0}_{s}}{\partial n} + \eta \cdot n \quad \text{on} \quad \partial D .
\]

These latter transmission conditions may also be reformulated as

\[
\phi^{0}_{tr}(y) = \phi^{0}_s(y) = \eta \cdot y , \quad A^{0}_{tr}(y) = A^{0}_s(y) + 1 \quad y \in \partial D \quad ,
\]

and

\[
A^{0}_s = \frac{\mu \eta \cdot n - \frac{\partial \phi^{0}_{s}}{\partial n} \pm \sqrt{\mu q - (\eta \cdot \tau)^2}}{\frac{\partial \phi^{0}_{s}}{\partial n}} \quad \text{on} \quad \partial D .
\]

The amplitudes \( A^{0}_{tr} \) and \( A^{0}_s \) then satisfy the transport equations

\[
\nabla A^{0}_{tr} \cdot \nabla \phi^{0}_{tr} + \frac{1}{2} A^{0}_{tr} \Delta \phi^{0}_{tr} = 0 \quad \text{and} \quad \nabla A^{0}_s \cdot \nabla \phi^{0}_{s} + \frac{1}{2} A^{0}_s \Delta \phi^{0}_{s} = 0 \quad ,
\]

in \( D \), and \( \mathbb{R}^2 \setminus \overline{D} \), respectively. A combination of (18) and the first identity in (19) immediately gives

\[
\left( \frac{\partial \phi^{0}_{tr}}{\partial n} \right)^2 = \mu q - (\eta \cdot \tau)^2 \quad , \quad \text{and} \quad \left( \frac{\partial \phi^{0}_{s}}{\partial n} \right)^2 = 1 - (\eta \cdot \tau)^2 = (\eta \cdot n)^2 .
\]

We thus conclude that

\[
\frac{\partial \phi^{0}_{s}}{\partial n} = \pm \eta \cdot n , \quad \text{and} \quad \frac{\partial \phi^{0}_{tr}}{\partial n} = \pm \sqrt{\mu q - (\eta \cdot \tau)^2} , \quad \text{on} \quad \partial D .
\]

Let us first consider \( \frac{\partial \phi^{0}_{s}}{\partial n} \): the above identity together with the fact that we want a continuous function \( \frac{\partial \phi^{0}_{s}}{\partial n} \) (and the convexity of the domain \( D \)) implies that there are four possibilities, namely

\[
\frac{\partial \phi^{0}_{s}}{\partial n}|_{\partial D} = -\eta \cdot n , \quad \frac{\partial \phi^{0}_{s}}{\partial n}|_{\partial D} = \eta \cdot n , \quad \frac{\partial \phi^{0}_{s}}{\partial n}|_{\partial D} = -|\eta \cdot n| , \quad \text{or} \quad \frac{\partial \phi^{0}_{s}}{\partial n}|_{\partial D} = |\eta \cdot n| .
\]
There are several ways to decide which of these four possibilities is physically correct. One way is by means of a principle of “limiting absorption”: suppose the permittivity outside $D$ is not 1, but $1 + i\delta$, for some $\delta > 0$. In that case the second equation in (21) gets replaced by

$$\left( \frac{\partial \phi^0_s}{\partial n} \right)^2 = 1 + i\delta - (\eta \cdot \tau)^2 = i\delta + (\eta \cdot n)^2 .$$

Of the two solutions to this

$$\frac{\partial \phi^0_s}{\partial n} \bigg|_{\partial D} = \sqrt{i\delta + (\eta \cdot n)^2}$$

has a positive imaginary part, whereas the other solution has a negative imaginary part. A negative imaginary part gives an ansatz $A_s(y) e^{i\tilde{k}\phi^0_s(y)}$ that grows exponentially as one goes from $\partial D$ into $IR^2 \setminus \mathcal{D}$ – we discard this as physically unstable, and thus pick (22). As $\delta$ approaches 0 (as the absorption vanishes) this solution converges to

$$\frac{\partial \phi^0_s}{\partial n} \bigg|_{\partial D} = |\eta \cdot n| ,$$

which we thus select as the physically relevant solution among the four possible. It should not be surprising that a principle of “limiting absorption” can be used to find the right approximate solution – after all, it is well know that such a principle may be used to select that (exact) solution which satisfies the appropriate outgoing radiation condition [11], [16]. As already indicated there are other ways to eliminate the three “wrong” choices among the four listed above – ways that are more directly linked to the “satisfaction” of the outgoing radiation condition. For example $\frac{\partial \phi^0_s}{\partial n} \big|_{\partial D} = \eta \cdot n$ would yield $\phi^0_s(y) = \eta \cdot y$, which is in clear violation of the outgoing radiation condition. A simple integration by parts yields

$$\int_{|y| = R} \tilde{\pi}^{(s)}_k \frac{\partial v^{(s)}_k}{\partial r} d\sigma - \int_{\partial D} \tilde{\pi}^{(s)}_k \frac{\partial v^{(s)}_k}{\partial n} d\sigma = \int_{BR \setminus \mathcal{D}} \left( |\nabla v^{(s)}_k|^2 - \tilde{k}^2 |v^{(s)}_k|^2 \right) dy ,$$

and so

$$\Im \left( \int_{\partial D} \tilde{\pi}^{(s)}_k \frac{\partial v^{(s)}_k}{\partial n} d\sigma \right) = \Im \left( \lim_{R \to \infty} \int_{|y| = R} \tilde{\pi}^{(s)}_k \frac{\partial v^{(s)}_k}{\partial r} d\sigma \right)$$

$$= \lim_{R \to \infty} \int_{|y| = R} |v^{(s)}_k|^2 d\sigma \geq 0 .$$

Our formal asymptotics indicate that

$$\tilde{k}^{-1} \int_{\partial D} \tilde{\pi}^{(s)}_k \frac{\partial v^{(s)}_k}{\partial n} d\sigma \quad \text{approaches} \quad i \int_{\partial D} |A^0_s|^2 \frac{\partial \phi^0_s}{\partial n} d\sigma ,$$

and so...
as \( \tilde{k} \to \infty \), and by a combination with (24) we now conclude that

\[
\int_{\partial D} |A^0_s|^2 \frac{\partial \phi^0}{\partial n} \, d\sigma \geq 0 .
\]

This effectively rules out that \( \frac{\partial \phi^0}{\partial n} \big|_{\partial D} \) equals \(-|\eta \cdot n|\). Finally we note that with the choice \(-\eta \cdot n\) for \( \frac{\partial \phi^0}{\partial n} \big|_{\partial D} \) we would get a set of characteristics emanating from \( \partial D \) (and defined by the vectorfield \( \nabla \phi^0_s \)) that mutually intersect. For these reasons we are again left with the single possibility (23). This choice is also consistent with the reflection principle of classical optics.

When it comes to the choice of sign (in front of the square root) for \( \partial v^{(s)}_k / \partial n \) we observe that when \( \mu q - (\eta \cdot \tau)^2 \) is not in \( \mathbb{R}^+ \cup \{0\} \), then \( \sqrt{\mu q - (\eta \cdot \tau)^2} \) has a positive imaginary part, and hence

\[
\frac{\partial \phi^0_{tr}}{\partial n} = -\sqrt{\mu q - (\eta \cdot \tau)^2}
\]

(25)

corresponds to an ansatz \( A^0_{tr} e^{i \phi^0_{tr}} \) which decays exponentially as one goes from the boundary \( \partial D \) towards the inside of \( D \). The other sign leads to an exponentially increasing approximate solution, and is therefore eliminated. By continuity (as before) we maintain the same sign when \( \mu q - (\eta \cdot \tau)^2 \) lies in \( \mathbb{R}^+ \cup \{0\} \). This choice is also consistent with the so-called Snell’s Law of Refraction. In summary:

**The Geometric Optics Approach** yields the following high frequency approximations to \( v^{(s)}_k \) and \( \partial v^{(s)}_k / \partial n \) on \( \partial D \). As \( \tilde{k} \to \infty \)

\[
v^{(s)}_k \big|_{\partial D} \approx A^0_s e^{i \eta \cdot y} , \quad \text{and} \quad \frac{\partial v^{(s)}_k}{\partial n} \big|_{\partial D} \approx i \tilde{k} |\eta \cdot n| A^0_s e^{i \eta \cdot y} ,
\]

(26)

with

\[
A^0_s(y) = \frac{-\mu \eta \cdot n + \sqrt{\mu q - (\eta \cdot \tau)^2}}{\mu |\eta \cdot n| + \sqrt{\mu q - (\eta \cdot \tau)^2}} .
\]

(27)

As already noted the permittivity \( q \) is given by \( q = q_{re} + i q_{im} (= q_{re} + i \frac{\sigma}{\tilde{k}}) \), where the \( q_{re} > 0 \) and \( \sigma \geq 0 \) are the real permittivity and the conductivity, respectively. We expect the geometric optics approach to lead to very good approximations for large \( \tilde{k} \) when \( \sigma \) is positive (and \( \sigma / q_{re} \) is at least of the order of \( k \)) whereas we expect the quality of the approximations to deteriorate for smaller \( \sigma \). In Figure 5 we compare \( v^{(s)}_k \) and its geometric optics approximation for \( \tilde{k} = 100 \) \( (k = 10^6, \epsilon = 10^{-4}) \) and \( D = \) the unit disk. We take \( \eta = (0,1) \), and \( \mu = 2, q = 2 + 2i \). The left frame shows the real part of geometric optics approximation (the imaginary part looks similar) the right frame shows the modulus of the difference between \( v^{(s)}_k \) and its geometric optics approximation. In Figure 6 we show in the the same plot the real parts of the geometric optics approximation
Figure 5: Comparison between $v^{(s)}$ and its geometric optics approximation on the surface of a conducting circular inhomogeneity.

Figure 6: The real part of $v^{(s)}$ (in dashed line) and its geometric optics approximation (in solid line) on the right half of the surface of a non-conducting circular inhomogeneity.

and $v^{(s)}_k$, only now for $\mu = 2$ and $q = 2$. The graphs in all these figures are shown only on half of the boundary of $D$ (from the southpole $= -\pi/2$ counterclockwise to the northpole $= \pi/2$) – since both the solution and the geometric optics approximation are entirely symmetric. The deterioration in the approximation visible in Figure 6 may be attributed to signals that are “back-scattered” when the (non-absorbed) transmitted wave hits the “rear” boundary of $\epsilon D$.

3.2 Green’s formula and stationary phase

We now proceed to derive an approximation to $u^{(s)}_k$ based on the boundary data approximations we have just obtained. For that purpose we need the representa-
\[ u_{e}^{(s)}(y) = \int_{\partial D} \frac{\partial}{\partial n_{x}} \Phi_{k}(x,y/e)v_{k}^{(s)}(x) \, d\sigma_{x} - \int_{\partial D} \Phi_{k}(x,y/e) \frac{\partial}{\partial n_{x}} v_{k}^{(s)}(x) \, d\sigma_{x}. \]

The functions \( \Phi_{k}(x,y/e) \) and \( \frac{\partial}{\partial n_{x}} \Phi_{k}(x,y/e) \) have the asymptotics

\[
\Phi_{k}(x,y/e) = \frac{i}{4} H_{0}^{(1)}(\tilde{k} | x - y/e |) = \left( 8 \pi \tilde{k} | x - y/e | \right)^{-1/2} e^{i(\tilde{k} | x - y/e | + \pi/4)} (1 + O([\tilde{k} | x - y/e |]^{-1}))
\]

\[
\frac{\partial}{\partial n_{x}} \Phi_{k}(x,y/e) = \frac{i \tilde{k} y - x}{4 \left| x - y/e \right|} \cdot n_{x} H_{1}^{(1)}(\tilde{k} | x - y/e |) = \tilde{k} \frac{y - x}{\left| x - y/e \right|} \cdot n_{x} \left( 8 \pi \tilde{k} | x - y/e | \right)^{-1/2} e^{i(\tilde{k} | x - y/e | - \pi/4)} \times (1 + O([\tilde{k} | x - y/e |]^{-1})).
\]

For \( x \in \partial D \) and for \( y \) that are bounded a finite distance away from \( \epsilon D \) (0 < \( r_{0} < |y| < R_{0} \)) we thus get the asymptotic formulas

\[
\Phi_{k}(x,y/e) = \frac{1}{\sqrt{k} \sqrt{8 \pi |y|}} e^{i \left( \tilde{k} | x - y/e | + \pi/4 \right)} (1 + O(\epsilon + k^{-1})) \quad (28)
\]

\[
\frac{\partial}{\partial n_{x}} \Phi_{k}(x,y/e) = \sqrt{\tilde{k} |y|} \cdot n_{x} \frac{1}{\sqrt{8 \pi |y|}} e^{i \left( \tilde{k} | x - y/e | - \pi/4 \right)} (1 + O(\epsilon + k^{-1})) \quad (29)
\]

and

\[
\frac{\partial}{\partial n_{x}} \Phi_{k}(x,y/e) = \epsilon \sqrt{\tilde{k} |y|} \cdot n_{x} \frac{1}{\sqrt{8 \pi |y|}} e^{i \left( \tilde{k} | x - y/e | - \pi/4 \right)} (1 + O(\epsilon + k^{-1} + ke^{2})) \quad (29)
\]

as \( \epsilon \to 0, k \to \infty \), and \( ke^{2} \to 0 \) (but \( \tilde{k} = k \to \infty \)). Here we have used that

\[
\tilde{k} | x - y/e | = \tilde{k} \left( \frac{|y|}{\epsilon} - \frac{y \cdot x}{|y|} \right) + O(ke^{2}) = k|y| - \tilde{k} \frac{y \cdot x}{|y|} + O(ke^{2}).
\]

By substitution of the formulas (28)–(29) and the geometric optics approximations (26) into the above representation formula for \( u_{e}^{(s)} \) we now obtain the following approximation

\[
u_{e}^{(s)}(y) \approx \epsilon \sqrt{k} e^{-i \frac{\pi}{4}} \frac{e^{i \tilde{k} |y|}}{\sqrt{8 \pi |y|}} \cdot \int_{\partial D} n_{x} A_{k}^{0}(x) e^{i \left( \eta \cdot \frac{y}{|y|} \right) \cdot x} \, d\sigma_{x}
\]

\[
- i \epsilon \sqrt{k} e^{i \frac{\pi}{4}} \frac{e^{i \tilde{k} |y|}}{\sqrt{8 \pi |y|}} \int_{\partial D} |\eta \cdot n_{x}| A_{k}^{0}(x) e^{i \left( \eta \cdot \frac{y}{|y|} \right) \cdot x} \, d\sigma_{x}
\]

\[
= \epsilon \sqrt{k} e^{-i \frac{\pi}{4}} \frac{e^{i \tilde{k} |y|}}{\sqrt{8 \pi |y|}} \int_{\partial D} \left( \frac{y \cdot n_{x} + |\eta \cdot n_{x}|}{|y|} \right) A_{k}^{0}(x) e^{i \left( \eta \cdot \frac{y}{|y|} \right) \cdot x} \, d\sigma_{x}
\]
with $A_*^0$ given by (27). We expect this formula to yield a good (pointwise) approximation to the scattered field for a conducting inhomogeneity. The approximation will deteriorate as $y/|y|$ approaches $\eta$, but should be excellent in the whole back-scattered region, and in significant parts of the forward scattered region. For an inhomogeneity representing a “hard” scatterer (the “limiting” case when $A_*^0 = -1$) we also expect a quite reasonable approximation. However, for general non-conducting inhomogeneities the formula (30) will not lead to a very good pointwise approximation due to the already mentioned deficiencies of the geometric optics approach. As already mentioned our approximation is predicated on the assumption that $\tilde{k} = k\epsilon \to \infty$, $\epsilon \to 0$, and $k\epsilon^2 \to 0$. If $\tilde{k}$ was even larger, say for instance $k\epsilon^2 \to \infty$, $\epsilon \to 0$, and $k\epsilon^3 \to 0$, then we should allow one more term in the expansion of $\tilde{k}|x - \frac{y}{\epsilon}|$

$$\tilde{k}|x - \frac{y}{\epsilon}| = k|y| - \frac{k\epsilon \cdot x}{|y|} + \frac{1}{2}k\epsilon^2\frac{|x|^2|y|^2 - (x \cdot y)^2}{|y|^3} + O(k\epsilon^3) .$$

This will change the phase function in the integral in (30) from $\left(\eta - \frac{y}{|y|}\right) \cdot x$ to $\left(\eta - \frac{y}{|y|}\right) \cdot x + \frac{1}{2} \frac{|x|^2|y|^2 - (x \cdot y)^2}{|y|^3}$ and so the effect will be mostly restricted to $\frac{y}{|y|}$ near $\eta$ — furthermore it will not significantly change the amplitude of the approximation to the scattered wave (which will remain of the order $\sqrt{\tau}$) but only its phase. In order to further develop the formula (30) we proceed to study oscillatory (boundary) integrals with the phase function $\psi_\eta(x, y) = \left(\eta - \frac{y}{|y|}\right) \cdot x$.

We first prove

**Lemma 1.** Let $D$ be a bounded, convex, smooth domain. Let $y$ be a nonzero vector, and let $\eta$ be a unit vector, with $\eta - \frac{y}{|y|} \neq 0$. The function $x \to \psi_\eta(x, y) = \left(\eta - \frac{y}{|y|}\right) \cdot x$ has exactly two stationary points on $\partial D$. These two points, $x_1$ and $x_2$, are characterized by

$$\psi_\eta(x_1, y) = \min_{x \in \partial D} \psi_\eta(x, y) \quad \text{and} \quad \psi_\eta(x_2, y) = \max_{x \in \partial D} \psi_\eta(x, y) .$$

Alternatively $x_1$ and $x_2$ may be characterized by

$$n_{x_1} = -\frac{\eta - \frac{y}{|y|}}{\eta - \frac{y}{|y|}} \quad \text{and} \quad n_{x_2} = \frac{\eta - \frac{y}{|y|}}{\eta - \frac{y}{|y|}} ,$$

where $n_x$ denotes the outward normal to $D$ at the point $x \in \partial D$.

**Proof.** This result follows immediately from the convexity of $D$, and the fact that stationary points for $x \to \psi_\eta(x, y)$ on $\partial D$ are points at which $\frac{\partial}{\partial \tau_x} \psi_\eta(x, y) = \left(\eta - \frac{y}{|y|}\right) \cdot \tau_x = 0$, i.e., points at which $n_x = \pm \left(\eta - \frac{y}{|y|}\right) / \left|\eta - \frac{y}{|y|}\right|$.

It will for later use be convenient to have evaluated a certain density at the stationary points $x_1$ and $x_2$.
Lemma 2. Let $D$, $y$, $\eta$, $\psi$, and $x_1$, $x_2$ be as in Lemma 1, and let $A_0^s$ be given by (27). By $0 < \theta < 2\pi$ we denote the angle of (counterclockwise) rotation between $\eta$ and $y/|y|$. Then

$$
\left( \frac{y}{|y|} \cdot n_{x_1} + |\eta \cdot n_{x_1}| \right) A_0^s(x_1) = 2 \sin(\theta/2) \frac{\mu \sin(\theta/2) - \sqrt{\mu q - 1 + (\sin(\theta/2))^2}}{\mu \sin(\theta/2) + \sqrt{\mu q - 1 + (\sin(\theta/2))^2}} ,
$$

and

$$
\left( \frac{y}{|y|} \cdot n_{x_2} + |\eta \cdot n_{x_2}| \right) A_0^s(x_2) = 0 .
$$

Proof. A simple calculation yields

$$
\left( \frac{y}{|y|} \cdot n_{x_1} + |\eta \cdot n_{x_1}| \right) = -\frac{y}{|y|} \cdot \frac{\eta - \frac{y}{|y|}}{|\eta - \frac{y}{|y|}|} + \eta \cdot \frac{\eta - \frac{y}{|y|}}{|\eta - \frac{y}{|y|}|} = \left| \eta - \frac{y}{|y|} \right| = \sqrt{2(1 - \cos \theta)} = 2 \sin(\theta/2) .
$$

Similarly we obtain

$$
\left( \frac{y}{|y|} \cdot n_{x_2} + |\eta \cdot n_{x_2}| \right) = \frac{y}{|y|} \cdot \frac{\eta - \frac{y}{|y|}}{|\eta - \frac{y}{|y|}|} + \eta \cdot \frac{\eta - \frac{y}{|y|}}{|\eta - \frac{y}{|y|}|} = 0 .
$$

We may also calculate

$$
A_0^s(x_1) = -\frac{\mu \eta \cdot n_{x_1} + \sqrt{\mu q - (\eta \cdot \tau_{x_1})^2}}{\mu |\eta \cdot n_{x_1}| + \sqrt{\mu q - (\eta \cdot \tau_{x_1})^2}} = \frac{\mu \sin(\theta/2) - \sqrt{\mu q - 1 + (\sin(\theta/2))^2}}{\mu \sin(\theta/2) + \sqrt{\mu q - 1 + (\sin(\theta/2))^2}} ,
$$

and

$$
A_0^s(x_2) = -\frac{\mu \eta \cdot n_{x_2} + \sqrt{\mu q - (\eta \cdot \tau_{x_2})^2}}{\mu |\eta \cdot n_{x_2}| + \sqrt{\mu q - (\eta \cdot \tau_{x_2})^2}} = \frac{-\mu \eta \cdot n_{x_2} + \sqrt{\mu q - (\eta \cdot \tau_{x_2})^2}}{\mu |\eta \cdot n_{x_2}| + \sqrt{\mu q - (\eta \cdot \tau_{x_2})^2}} = -1 .
$$

A combination of these four identities immediately leads to the formulas of this lemma. □

By application of this lemma and the technique referred to as Stationary Phase we now obtain the following asymptotic result
Lemma 3. Let $D$, $y$, $\eta$ and $\psi_{\eta}(x,y) = \left(\eta - \frac{y}{|y|}\right) \cdot x$ be as in Lemma 1. Let $x_1$ and $x_2$ denote the arg-min and the arg-max of $\psi_{\eta}(\cdot, y)$ on $\partial D$, respectively. Suppose $K(x_1)$ and $K(x_2)$ are both positive, where $K(x)$ denotes the curvature of $\partial D$ at the point $x$. By $0 < \theta < 2\pi$ we denote the angle of (counterclockwise) rotation between $\eta$ and $y/|y|$. For any continuous, piecewise $C^1$ function, $a$, on $\partial D$

$$\int_{\partial D} a(x)e^{i\tilde{k}\psi_{\eta}(x,y)} \, d\sigma_x = \frac{\sqrt{\pi}}{\sqrt{k} \sin(\theta/2)} \left( \frac{e^{i\tilde{k}}}{\sqrt{K(x_1)}} e^{i\tilde{k}\psi_{\eta}(x_1,y)} a(x_1) + \frac{e^{-i\tilde{k}}}{\sqrt{K(x_2)}} e^{i\tilde{k}\psi_{\eta}(x_2,y)} a(x_2) \right) + o\left(\frac{1}{\sqrt{k}}\right),$$

as $\tilde{k} \to \infty$.

Proof. On a segment $\Gamma$ of $\partial D$, along which $|\frac{\partial}{\partial s}\psi_{\eta}(x,y)| > c > 0$, it is quite easy (by integration by parts) to see that

$$\int_{\Gamma} a(x)e^{i\tilde{k}\psi_{\eta}(x,y)} \, d\sigma_x = O(\tilde{k}^{-1}).$$

In order to verify the asymptotic formula of this lemma it thus suffices to consider the contribution from (arbitrarily small) segments around the two critical points $x_1$ and $x_2$. Let $\gamma_{1,\delta}$ be a neighborhood of $x_1$ given by

$$\gamma_{1,\delta} = \partial D \cap \{x : \psi_{\eta}(x,y) < \psi_{\eta}(x_1,y) + \delta\},$$

for some $\delta > 0$. Let $s$ denote the (signed) curvelength along $\gamma_{1,\delta}$, with $s = 0$ corresponding to $x_1$, and $s$ being positive for points to the right of $x_1$ (when facing $x_1$ from outside $D$). Let $x(s) \in \gamma_{1,\delta}$ denote the point corresponding to curvelength $s$. We then have

$$\int_{\gamma_{1,\delta}} a(x)e^{i\tilde{k}\psi_{\eta}(x,y)} \, d\sigma_x = \int_{\gamma_{1,\delta}^+} a(x)e^{i\tilde{k}\psi_{\eta}(x,y)} \, d\sigma_x + \int_{\gamma_{1,\delta}^-} a(x)e^{i\tilde{k}\psi_{\eta}(x,y)} \, d\sigma_x,$$

with

$$\gamma_{1,\delta}^+ = \partial D \cap \{x(s) : \psi_{\eta}(x(s),y) < \psi_{\eta}(x_1,y) + \delta\}, \quad \text{and} \quad s > 0,$$

and

$$\gamma_{1,\delta}^- = \partial D \cap \{x(s) : \psi_{\eta}(x(s),y) < \psi_{\eta}(x_1,y) + \delta\}, \quad \text{and} \quad s < 0.$$
it follows that
\[ t = \left( \eta - \frac{y}{|y|} \right) \cdot x_1 + \left( \eta - \frac{y}{|y|} \right) \cdot x''(0) \frac{1}{2} s^2 + O(s^3) \]
\[ = \psi_\eta(x_1, y) + \left| \eta - \frac{y}{|y|} \right| K(x_1) \frac{1}{2} s^2 + O(s^3) \]
\[ = \psi_\eta(x_1, y) + \sin(\theta/2) K(x_1) s^2 + O(s^3) \]
where \( K(x_1) \) denotes the curvature of \( \partial D \) at \( x_1 \). Hence
\[ s = \left( \frac{t - \psi_\eta(x_1, y)}{\sin(\theta/2) K(x_1)} \right)^{1/2} + O(t - \psi_\eta(x_1, y)) \]

On \( \gamma_{1,\delta}^+ \) the change of variable from \( x \) to \( t \) is thus associated with
\[ d\sigma_x = \left| \frac{dt}{ds} \right|^{-1} dt = \frac{1}{2 \sin(\theta/2) K(x_1) s} (1 + O(s)) \frac{dt}{\sqrt{t - \psi_\eta(x_1, y)}} \]

We may now calculate
\[ \int_{\gamma_{1,\delta}^+} a(x) e^{ik\psi_\eta(x,y)} d\sigma_x \]
\[ = \frac{1}{2 \sqrt{\sin(\theta/2) K(x_1) s}} \int_{\psi_\eta(x_1, y)}^{\psi_\eta(x_1, y) + \delta} \tilde{a}(t) e^{ikt} \frac{dt}{\sqrt{t - \psi_\eta(x_1, y)}} \]
\[ = \frac{\sqrt{\pi}}{2 \sqrt{\sin(\theta/2) K(x_1)} \sqrt{k}} e^{ik\psi_\eta(x_1,y)} a(x_1) + o(\sqrt{k}^{-1}) \]

In the second equality we used a standard result for oscillatory integrals, cf. Theorem 13.1 in [15]. We may similarly calculate
\[ \int_{\gamma_{1,\delta}^-} a(x) e^{ik\psi_\eta(x,y)} d\sigma_x = \frac{\sqrt{\pi}}{2 \sin(\theta/2) K(x_1) \sqrt{k}} e^{ik\psi_\eta(x_1,y)} a(x_1) + o(\sqrt{k}^{-1}) \]
so that
\[ \int_{\gamma_{1,\delta}^-} a(x) e^{ik\psi_\eta(x,y)} d\sigma_x = \frac{\sqrt{\pi}}{2 \sin(\theta/2) K(x_1) \sqrt{k}} e^{ik\psi_\eta(x_1,y)} a(x_1) + o(\sqrt{k}^{-1}) \]

For a similar small neighborhood around \( x_2 \)
\[ \gamma_{2,\delta} = \partial D \cap \{ x : \psi_\eta(x,y) > \psi_\eta(x_2,y) - \delta \} \]
we obtain that
\[
\int_{\gamma_{2,\alpha}} a(x)e^{ik\psi_\eta(x,y)} \, d\sigma_x = \frac{\sqrt{\pi}}{\sqrt{\sin(\theta/2)K(x_2)}} e^{-i\frac{x}{k}} e^{ik\psi_\eta(x_2,y)} a(x_2) + o(\sqrt{k}^{-1}).
\]

These two identities immediately lead to asymptotic statement of this lemma. □

By a combination of (30) with Lemma 2 and Lemma 3 we arrive at the following

**Approximation Assertion:** Let \( u^{(s)}_\epsilon(y) \) denote the scattered field arising from an incident plane wave \( u^{(inc)}(y) = e^{ik\eta y} \) and a convex, conducting scatterer \( \epsilon D \) with permeability \( \mu > 0 \) and permittivity \( q (= q_{re} + i\frac{q_{im}}{\mu} \) with \( q_{re} > 0 \), and \( \sigma > 0 \). Suppose \( K(x) > 0 \) for all \( x \in \partial D \), where \( K(\cdot) \) denotes the curvature of \( \partial D \). Let \( \psi_\nu(x,y) = (\eta - \frac{\nu}{|y|}) \cdot x \) and let \( x_1 \) denote the arg-min of \( \psi_\nu(x,y) \) (i.e., \( x_1 \) is the solution to \( \psi_\nu(x_1,y) = \min_{x \in \partial D} \psi_\nu(x,y) \)). Let \( 0 < \theta < 2\pi \) denote the angle of (counterclockwise) rotation between \( \eta \) and \( y/|y| \). The preceeding analysis leads us to believe that a good approximation to \( u^{(s)}_\epsilon(y) \), for \( y \) that are bounded and bounded away from \( \epsilon D \), is given by

\[
u^{(s)}_\epsilon(y) \approx \frac{\sqrt{\epsilon}}{\sqrt{2}} \sqrt{\frac{\sin(\theta/2)}{\sin(\theta/2)}} \frac{\mu \sin(\theta/2) - \sqrt{\mu q - 1 + (\sin(\theta/2))^2} e^{ik\psi_\nu(x_1,y)} e^{ik|y|}}{\mu \sin(\theta/2) + \sqrt{\mu q - 1 + (\sin(\theta/2))^2}} \sqrt{K(x_1)} \frac{1}{\sqrt{|y|}}
\]

(Remark 2. If \( D \) is the unit disk then \( K(x_1) = 1 \), \( x_1 = n_{x_1} \), and

\[
\psi_\nu(x_1,y) = -\frac{\eta - y}{|y|} = -2\sin(\theta/2).
\]

The approximate formula in the above assertion thus simplifies to

\[
u^{(s)}_\epsilon(y) \approx \frac{\sqrt{\epsilon}}{\sqrt{2}} \sqrt{\frac{\sin(\theta/2)}{\sin(\theta/2)}} \frac{\mu \sin(\theta/2) - \sqrt{\mu q - 1 + (\sin(\theta/2))^2} e^{ik\psi_\nu(x_1,y)} e^{ik|y|}}{\mu \sin(\theta/2) + \sqrt{\mu q - 1 + (\sin(\theta/2))^2}} \sqrt{K(x_1)} \frac{e^{ik|y|}}{\sqrt{|y|}}.
\]

For a so-called hard scatterer (e.g., formally obtained in the limit as \( \mu \to 0 \) and \( \mu q \to c < 0 \)) the formula (32) becomes

\[
u^{(s)}_\epsilon(y) \approx -\frac{\sqrt{\epsilon}}{\sqrt{2}} \sqrt{\frac{\sin(\theta/2)}{\sin(\theta/2)}} e^{ik\psi_\nu(x_1,y)} e^{ik|y|} \sqrt{K(x_1)} \frac{e^{ik|y|}}{\sqrt{|y|}},
\]

and so for a “hard” scatterer in the shape of an \( \epsilon \) disk

\[
u^{(s)}_\epsilon(y) \approx -\frac{\sqrt{\epsilon}}{\sqrt{2}} \sqrt{\frac{\sin(\theta/2)}{\sin(\theta/2)}} e^{ik|y|} \sqrt{K(x_1)} \frac{e^{ik|y|}}{\sqrt{|y|}}.
\]
Finally we note that if the inhomogeneity was centered at $z_0$, i.e., if instead of $\epsilon D$ the inhomogeneity was represented by $z_0 + \epsilon D$, then $y$ would be replaced by $y - z_0$, $\theta$ would be replaced by the angle between $\eta$ and $y - z_0/|y - z_0|$, and the approximate expression would be multiplied by $u^{(inc)}(z_0) = e^{ik\eta \cdot z_0}$ in (32) and all of the other formulas above.

Figure 7 illustrates the approximation properties of the formula (33) for a circular inhomogeneity, centered at the origin, and for various values of $k$ and $\epsilon$, and $\mu = 2$, $q = 2 + 2i$ (independently of $k$). The left frames show the real part of the formula (33), the right frames show the modulus of the difference between $u^{(s)}$ and the formula (33). The incident field has $\eta = (0,1)$. The data is taken on $r = 2$, but only shown on the right half of this circle, from the southpole = $-\pi/2$ counterclockwise to the northpole = $\pi/2$ (for obvious reasons of symmetry). The values of $k$ and $\epsilon$ are from top to bottom: $k = 100$, $\epsilon = 0.1$, $k = 10^6$, $\epsilon = 0.01$, and $k = 10^6$, $\epsilon = 10^{-4}$. Of these three cases only really the last one falls in the asymptotic regime where (33) is valid ($\epsilon$ is small, $\tilde{k} = k\epsilon$ is large, and $k\epsilon^2$ is small) – not surprisingly it is also the case in which we obtain the best approximation: the formula (33) is an excellent approximation to the whole backscattered field, and even to a significant portion of the forward scattered field. Figure 8 illustrates the approximation properties of the hard scatterer version of formula (33) (i.e., (34)) for $k = 10^6$ and a circular inhomogeneity, centered at the origin, of radius $\epsilon = 10^{-4}$ (the right asymptotic regime). As before the plots are evaluated at $r = 2$. The approximation properties are clearly very similar to those of a conducting inhomogeneity.

Remark 3. Note that

$$e^{i k x_0 (x_1,y)} e^{i k|y|} = e^{i k y \cdot x_1} e^{i k\left(|y| - c x_1 \cdot \frac{y}{|y|}\right)}$$

$$= u^{(inc)}(x_1) e^{i k|y| - c x_1\left(1 + O(k\epsilon^2)\right)} .$$

Since we assume $k\epsilon^2$ is small, it may appear more natural to use, in place of (32), the following approximation.

$$u^{(s)}(y) \approx \frac{\sqrt{2}}{\sqrt{|y|}} \frac{\mu \sin(\theta/2)}{\mu \sin(\theta/2) + \sqrt{|y - c x_1|}} u^{(inc)}(x_1) .$$

Let $v^{(inc)}(y)$ be a more general incident wave (i.e., a solution to $(\Delta + k^2)v = 0$ in $\mathbb{R}^2$) that may be expressed as a superposition of planar waves. Define

$$a(s) = \sum_{l=-\infty}^{\infty} a_l e^{-ils} , \quad \text{with} \quad a_l = \frac{1}{J_l(k\epsilon)} \frac{1}{2\pi} \int_0^{2\pi} v^{(inc)}(\eta(s)) e^{ils} \, ds ,$$

and $\eta(s) = (\cos s, \sin s)$. We then have

$$v^{(inc)}(y) = \frac{1}{2\pi} \int_0^{2\pi} a(s) e^{ik\eta(s) \cdot y} \, ds .$$
Figure 7: The formula (33) and its “error” for various values of $k$ and $\epsilon$ for a conducting circular inhomogeneity, centered at the origin. The plots are evaluated on the right half of $r = 2$. 
Figure 8: As the last two frames in Figure 7, but for a “hard” scatterer.

Figure 9: $u_{s}^{(s)}$ (in dashed line) and the approximation (33) (in solid line) at $r = 2$ for a non-conducting, circular inhomogeneity.

It is very natural to surmise that a good approximation to the scattered wave associated with $v_{\text{inc}}$ is obtained by a similar “weighted averaging” of the approximate formula (32).

Figure 9 shows $u_{s}^{(s)}$ (in dashed line) and the approximation (33) (in solid line) on the right half of the circle $\{r = 2\}$. We have taken $k = 10^6$ and $\epsilon = 10^{-4}$ (and a circular inhomogeneity, centered at the origin) but the inhomogeneity is non-conducting ($\mu = 2$, $q = 2$). It is evident that (33) does not represent a reasonable pointwise approximation.

Finally we show some computations with the approximate formula (32) for cases other than disks. Figure 10 shows the rescaled ellipse of aspect ratio 1:2, and indicates three incident “directions” $\eta_j$, $j = 1, 2, 3$. We take $k = 10^6$ and $\epsilon = 10^{-4}$ and “center” the ellipse at the origin. Figure 11 shows the approximate scattered field (32) at $r = 2$. The field is shown on the entire circle, $\pi/2$ being the northpole, $3\pi/2$ being the southpole. In all cases $\mu = 2$ and $q = 2 + 2i$ inside the ellipse.

As far as the “accuracy” of the geometric optics approximation (26)-(27) is concerned it is possible to prove that for a “hard” scatterer the (appropriately
Figure 10: Diagram of ellipse orientation and incident directions.

Figure 11: The approximate scattered fields for an ellipse. The field is evaluated at $r = 2$. The left frame corresponds to incident direction $\eta_1$, the center frame to $\eta_2$, and the right frame to $\eta_3$. 
(rescaled) distance between these approximations and the corresponding traces of the scattered part of the exact solution tends to zero as $k \to \infty$ in any space $H^{-\frac{1}{2}-\delta}(\partial D)$ [5]. We suspect a similar statement holds in the case of a conducting inhomogeneity. Unfortunately, due to the oscillatory character of the Green’s function appearing in the representation formula (17) error estimates of this kind do not immediately lead to any useful statements about the error inherent in our approximation (32).

4 Conclusions

In this presentation we have shown how the small volume asymptotic formulas for the perturbed (or scattered) field, derived for 0 frequency in [3], and for fixed, nonzero frequency in [17] and [1], really remain valid for frequency, $k$, of order $o(\epsilon^{-1})$.

We have also shown how these perturbation formulas may be viewed as special cases of a formula valid for $k$ of the order $\epsilon^{-1}$.

Finally we have derived small volume perturbation formulas that apply to very large frequency ($k$ of magnitude larger than $\epsilon^{-1}$) and we have tested these extensively numerically. Our high frequency formulas appear to be extremely accurate for conducting, transmitting scatterers and for hard scatterers, whereas their accuracy is insufficient for non-conducting, transmitting scatterers.

We are convinced that a combination of these formulas may be used very effectively for the inverse problem of determining information about the location and the shape of small convex inhomogeneities based on measurements of the scattered field, at a broad band of frequencies.

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