Photon Berry phases, Instantons, Quantum chaos and quantum analog of Kolmogorov-Arnold-Moser (KAM) theorem in Dicke models

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(Dated: March 8, 2019)

The quantum analog of Lyapunov exponent has been discussed in the Sachdev-Ye-Kitaev (SYK) model and its various generalizations. Here we investigate possible quantum analog of Kolmogorov-Arnold-Moser (KAM) theorem in the $U(1)/\mathbb{Z}_2$ Dicke model which contains both the rotating wave (RW) term $g$ and the counter-RW term $g'$ at a finite $N$. We first study its energy spectrum by the analytical $1/J$ expansion, supplemented by the non-perturbative instanton method. Then we evaluate its energy level statistic (ELS) at a given parity sector by Exact diagonalization (ED) at any $0 < \beta = g'/g < 1$. We establish an intimate relation between the KAM theorem and the evolution of the scattering states and the emergence of bound states as the ratio $\beta$ increases. We stress the important roles played by the Berry phase and instantons in the establishment of the quantum analogue of the KAM theorem to the $U(1)/\mathbb{Z}_2$ Dicke model. Experimental implications in cavity QED systems such as cold atoms inside an optical cavity or superconducting qubits in side a microwave cavity are also discussed.

In classical chaos, the Lyapunov exponent was used to characterize the exponential growth of two classical trajectories when there are just a tiny difference in the initial conditions. The classical concept of Lyapunov exponent can be extended to its quantum analog which can be used to characterize the exponential growth of two initially commuting operators in the early time under the evolution of a quantum chaotic Hamiltonian \(1-3\). There are recent flurry of research activities to extract the Lyapunov exponent \(\lambda_L\) on Sachdev-Ye-Kitaev (SYK) models, various variants and its possible dual asymptotic AdS\(_2\) bulk quantum black holes through evaluating Out of time ordered correlation (OTOC) functions \(4,13\).

From a different perspective, Quantum chaos can also be characterized by the system’s energy level statistics (ELS) and level-level correlations encoded in the spectral form factor (SFF) through Random Matrix Theory (RMT) \(16,17\). The ELS and SFF are always evaluated at a finite but sufficiently large system. The ELS of SYK can be described by the Wigner-Dyson (WD) distributions in a $N(\text{mod}8)$ way \(16,17\): $N = 2, 6$ Gaussian unitary ensembles (GUE); $N = 0$ Gaussian orthogonal ensemble (GOE), $N = 4$ Gaussian symplectic ensemble (GSE). The quantum chaos in the SYK models are due to the quenched disorders. However, it inspired a new class of clean models called (colored or un-colored) Tensor (Gurau-Witten) model \(18,20\) which share similar quantum chaotic properties as the SYK at least in the large $N$ limit.

The quantum chaos in quantum optics also shows up in another class of clean models called $Z_2$ Dicke models \(21\). In the thermodynamic limit $N = \infty$, as the atom-photon coupling strength increases above a critical value, it displays a normal to a superradiant phase transition. However, the system becomes non-integrable at any finite $N$. By studying its ELS at a given parity sector by ED at several large finite size $N \geq 10$, the authors \(21\) found that it is Poissonian $P_{w}(s) = e^{-s}$ in the normal regime, but becomes WD distribution in the GOE $P_{w}(s) = \frac{2}{\pi}s e^{-\frac{s^2}{2}}$ in the superradiant regime. This fact suggests that the onset of the quantum chaos of the quantum ELS through a chaotic to non-chaotic transition (CNCT) may be closely related to the normal to super-radiant phase transition at a finite $N$. In \(26\), we studied the $U(1)$ Dicke model \(22,23\) by a $1/N$ expansion. Obviously, due to the $U(1)$ symmetry, it is integrable and its ELS satisfies Poissonian distribution inside the superradiant phase at a finite $N$. In \(31,33\), we investigated the $U(1)/\mathbb{Z}_2$ Dicke model from both its $1/J$ expansion and its $Z_2$ limit by a strong coupling expansion. The $J - U(1)/\mathbb{Z}_2$ Dicke model contains both the rotating wave (RW) term $g$ and the counter-RW term $g'$ at a finite $N$. It includes the four well known standard quantum optics model in cavity QED such as Rabi \(27\), Dicke \(24\), Jaynes-Cummings (JC) \(28\) and Tavis-Cummings (TC) \(30\) model as its various special limits. It would be interesting to study the quantum chaos in the $U(1)/\mathbb{Z}_2$ Dicke model and its possible connection to the normal to the super-radiant quantum phase transition.

In classical chaos, the classical Kolmogorov-Arnold-Moser (KAM) theorem states that if an integrable Hamiltonian $H_0$ is disturbed by a small perturbation $\Delta H$, which makes the total Hamiltonian $H = H_0 + \Delta H$ non-integrable. If the two conditions are satisfied: (a) $\Delta H$ is sufficiently small (b) the frequencies $\omega_i$ of $H_0$ are incommensurate, then the system remains quasi-integrable.
Just like the quantum analog of Lyapunov exponent can be studied in the context of the SYK models \[^{4,15}\], it is important to explore the quantum analog of KAM theorem. Despite some previous efforts, the quantum analogue of the KAM theorem remains elusive. In this work, by using the \(1/J\) expansion, non-perturbative instanton method, ED and random matrix theory, we explore possible quantum analog of KAM theorem in the \(U(1)/Z_2\) Dicke models \[^{37}\]. These methods are complementary to the strong coupling expansion used by the authors in \[^{33}\] to study the same model in its dual \(Z_2/U(1)\) representation.

Our results are presented in Fig.1-4. When \(0 < \beta < \beta_{U(1)} \sim 0.35\), the super-radiant phase at \(N = \infty\) splits into the two regimes at a finite \(N\): the \(U(1)\) and quantum tunneling (QT) regime \[^{34}\]. In the \(U(1)\) regime, we perform a (non-)degenerate perturbation to evaluate the energy spectrum. It is the Berry phase which leads to the level crossings between the even and odd parity, therefore the alternating parities on the ground state and excited states. In the QT regime, by the WKB method, we find the emergences of bound states one by one as the interaction strength increases, then investigate a new class of quantum tunneling processes through the instantons between the two bound states in the compact photon phase subject to the Berry phase. It is the Berry phase interference effects in the instanton tunneling event which leads to Schrodinger Cats oscillating with even and odd parities in both ground and higher energy bound states. We also illuminate a duality relation between the eigenenergies in the \(U(1)\) regime and those in the QT regime. This duality may be used to explain why the ELS is the same (namely, remains Poissonian) in both regimes. However, when \(\beta_{U(1)} < \beta < 1\), the \(U(1)\) regime disappears, so does the duality relation, the system directly crossovers from the normal to the QT regime \[^{33}\]. Then we study the energy level statistic by ED \[^{21}\] at a given \(0 < \beta < 1\) at a given parity sector. We find that the existence of the \(U(1)\) regime when \(0 < \beta < \beta_{U(1)}\) implies the validity of the quantum analogue of KAM theorem, therefore the energy level statistics remains Poissonian through the whole normal/\(U(1)/QT\) regime. We stress the crucial roles of the Berry phase and instantons in the establishment of the quantum analogue of the KAM theorem to the \(J - U(1)/Z_2\) Dicke model. Possible intrinsic connections between the onset of quantum chaos characterized by the RMT and the onset of superradiant phase transition characterized by Renormalization group (RG) are discussed. Experimental implications in detecting quantum chaos in cavity QED systems are also briefly discussed.

**RESULTS**

1. The \(1/J\) expansion in the superradiant phase.

\[ H_{U(1)/Z_2} = \omega_a a^\dagger a + \omega_b b^\dagger b + \frac{g}{\sqrt{2J}}(a^\dagger J_- + aJ_+) \]

\[ + \frac{g'}{\sqrt{2J}}(a^\dagger J_+ + aJ_-) \quad (1) \]

where \(\omega_a, \omega_b\) are the energy of the cavity photon and the two atomic levels respectively, \(g = \sqrt{N}g, g' = \sqrt{N}g'\); \(N = 2J\) are the collective photon-atom rotating wave (RW) coupling and counter-rotating wave (CRW) term respectively. If \(\beta = 0\), Eqn (1) reduces to the \(U(1)\) Dicke model \[^{22,26,31}\] with the \(U(1)\) symmetry \(a \to ae^{i\theta}, \sigma^- \to \sigma^- e^{i\theta}\) leading to the conserved quantity \(P = a^\dagger a + J_z\). The CRW \(g'\) term breaks the \(U(1)\) to the \(Z_2\) symmetry \(a \to -a, \sigma^- \to -\sigma^-\) with the conserved parity operator \(\Pi = e^{i\pi(a^\dagger a + J_z)}\). If \(\beta = 1\), it becomes the
$Z_2$ Dicke model [21,33]. If $\beta = \infty$, it can be mapped to the static version of Landau-Zener (LZ) model [35]. In this work, we fix the ratio to be $0 < g'/g = \beta < 1$. The other case with $1 < \beta < \infty$ need a different treatment and will be discussed in a separate publication [36,37].

Following [31], inside the super-radiant phase, it is convenient to write both the photon and atom in the polar coordinates $a = \sqrt{\lambda^2 + \delta \rho_a e^{i\phi_a}}, b = \sqrt{\lambda^2 + \delta \rho_b e^{i\phi_b}}$. When performing the controlled $1/J$ expansion, we keep the terms to the order of $\sim j, \sim 1$ and $\sim 1/j$, but drop orders of $1/j^2$ or higher. We first minimize the ground state energy at the order $j$ and found the saddle point values of $\lambda_a$ and $\lambda_b$:

$$\lambda_a = \frac{g + g'}{\omega_a} \sqrt{\frac{j}{2}(1 - \mu^2)}, \quad \lambda_b = \sqrt{\frac{j}{2}(1 - \mu)} \quad (2)$$

where $\mu = \omega_a \omega_b/(g + g')^2$. In the superradiant phase $g + g' > g'_c = \sqrt{\omega_a \omega_b}$. In the normal phase $g + g' < g'_c$, one gets back to the normal phase $\lambda_a = \lambda_b = 0$. At a fixed $\beta$, the QCP happens at

$$g_c = \frac{\sqrt{\omega_a \omega_b}}{1 + \beta} \quad (3)$$

Well inside the superradiant phase, $\lambda_a^2 \sim \lambda_b^2 \sim j$, it is convenient to introduce the $\pm$ modes: $\pm = (\theta_a \pm \theta_b)/2, \delta \rho \pm = \delta \rho_a \pm \delta \rho_b, \lambda^2 = \lambda_a^2 \pm \lambda_b^2$. The Berry phase in the $+$ sector [26,51,32] can be defined as $\lambda^2 = P + \alpha$ where $P = 1, 2, \cdots$ is the closest integer to the $\lambda^2$, so $-1/2 < \alpha < 1/2$. Due to the large gap in the $\theta$ sector when $0 < \beta < 1$, it is justified to drop the Berry phase in the $-$ sector. After shifting $\theta_+ \rightarrow \theta_+ + \pi/2$, we reach the Hamiltonian to the order of $1/j$:

$$\mathcal{H}[\delta \rho \pm, \theta]_\pm = \frac{D}{2}(\delta \rho^2 - \alpha)^2 + D_- [\delta \rho^2 - \gamma (\delta \rho^2 - \alpha)]^2 + 4\omega_a \lambda^2 [\frac{1}{1 + \beta} \sin^2 \theta_+ + \frac{\beta}{1 + \beta} \sin^2 \theta_+] \quad (4)$$

where $D = \frac{2\omega_a (g + g')^2}{E_H}$ is the phase diffusion constant in the $+$ sector, $D_- = E_H^2/16\omega_a \omega_b$ with $E_H^2 = (\omega_a + \omega_b)^2 + 4(g + g')^2 \lambda^2 / N$. The $\gamma = \frac{\omega_a^2}{E_H^2}(1 - \frac{(g + g')^2}{\omega_a^2})$ is the coupling between the $+$ and $-$ sector.

When $0 < \beta < 1$, for most purposes, it maybe justified to neglect the quantum fluctuations of the $\theta_-$ mode, namely, by setting $\theta_-$ at its classical value $\theta_-$ at $0$, Eqn. (4) is simplified to:

$$\mathcal{H}_+[\delta \rho_+, \theta_+] = \frac{D}{2}(\delta \rho^2 - \alpha)^2 + 2\omega_a \lambda^2 [\frac{2\beta}{1 + \beta} \sin^2 \theta_+] \quad (5)$$

Deep inside the superradiant phase $4\lambda^2 \frac{\beta}{1 + \beta} \gg 1$ in Fig.1, one can identify the approximate atomic mode:

$$\omega^2_{\theta_0} = 4\omega_a \lambda^2 [\frac{2\beta}{1 + \beta} D = \frac{4}{E_H^2} \frac{\beta}{1 + \beta} [(g + g')^2 - g'_c] \quad (6)$$

which is the pseudo-Goldstone mode due to the CRW $g'$ term [31,32].

Note that for sufficiently small $\beta < 1$, the condition to reach the quantum tunneling (QT) regime $4\lambda^2 \frac{\beta}{1 + \beta} \gg 1$ is more stringent than the $1/J$ expansion condition $\lambda^2 \gg 1$ in the superradiant phase. So there exists a window between the two conditions which is the $U(1)$ regime in Fig.1. This fact will be confirmed by more detailed analysis in the following.

2. $U(1)$ regime and the formation of consecutive bound states in the quantum tunneling (QT) regime. As the potential in the $\theta_+$ sector in Eqn.5 gets deeper and deeper, there are consecutive bound states formations at $g_{c1} < g_{c2} \cdots$. The QT regime in Fig.1 is signated by the first appearance of the bound state at $g = g_{c1}$ after which there are consecutive appear-
ances of more bound states at higher energies leading to the “atomic” energy scale $e_a$ (Fig 2a). The regime $g_c < g < g_{c1}$ is the $U(1)$ regime in Fig 1.

One can estimate all these $g_c < g_{c1} < g_{c2} \cdots$ by using the Bohr-Sommerfeld quantization condition for a smooth potential $\int_0^\beta pd\theta = (n + 1/2)\pi h$, $n = 0, 1, 2, \cdots$ where $p = \sqrt{2m(E_{n+1} - V(\theta))}$ and $E_{n+1} = E_1, E_2, \cdots$ is the $(n + 1) - th$ bound state energy in Fig 2a. From Eqn 5, we can see that $m = \frac{1}{D}, V(\theta) = \omega_\alpha^2 \lambda^2_{\alpha} \pi / 1 + \beta (1 - \cos 2\theta_\alpha)$ and the $a$ and $b$ are the two end points shown in Fig 2b1. We find that the bound state emerges one by one at

$$\frac{\omega_{-\alpha}}{D} = (n + 1/2)\pi / 2h, \ n = 0, 1, 2, \cdots$$

(7)

By setting $n = 0$, one can see when $\omega_{-\alpha}/D < \pi/2h$, there is no bound state. This is the $U(1)$ regime in Fig 1. Substituting the expression for the phase diffusion constant $D$ in Eqn 4 and the atomic mode $\omega_{-\alpha}$ in Eqn 4 leads to the condition for the $U(1)$ regime:

$$4\lambda_{\alpha}^2 \sqrt{\frac{\beta}{1 + \beta}} < \frac{\pi}{4}$$

(8)

Note that applying the $1/J$ expansion in the superradiant phase requires $\lambda_{\alpha}^2 \gg 1$. So for sufficiently small $g'/g = \beta$, there is an appreciable $U(1)$ regime $g_c < g < g_{c1}$ before the quantum tunneling (QT) regime in Fig 1.

The emerging of the bound states one by one after the $U(1)$ regime is shown in Fig 1b and Fig 2. As $g \rightarrow \infty$, $D \rightarrow 0$, while $\omega_{-\alpha} \rightarrow \sqrt{2\omega_a} \sqrt{2\pi / 1 + \beta}$, so the left hand side of Eqn 7 diverges, there are infinite number of bound states shown in Fig 1b. Obviously, due to the phase wandering in the $U(1)$ regime (Fig 2b1), the Berry phase $\alpha$ in Eqn 5 plays important roles. Naively, the bound state in the QT regime is either localized around $\theta_+ = 0$ or $\theta_+ = \pi$, so the Berry phase plays no roles, so it may be dropped. However, as to be shown in the following sections, it does play very important roles in the quantum tunneling (QT) process due to instantons between the two bound states shown in Fig 2b2.

3. The parity oscillations due to Instantons subject to the Berry phase in the $U(1)$ regime. In this $U(1)$ regime, the second term in Eqn 5 breaks the $U(1)$ symmetry to $Z_2$ symmetry, the Goldstone mode at $N = \infty$ simply becomes the pseudo-Goldstone mode in Eq 4. The total excitation $P$ is not conserved anymore and is replaced by the conserved parity $\Pi = (-1)^P$, the energy levels are only grouped into even and odd parities in Fig 1. One can treat the second term in Eqn 5 perturbatively either by degenerate perturbation expansion at $\alpha = 0$, or a non-degenerate one at $\alpha \neq 0$.

As shown in 31, at a given sector $P$, there are $P$ crossings at $\alpha = 0$ with $m = 0, \pm 1, \cdots, \pm P$. A first order degenerate perturbation 31 at $m = \pm 1$ leads to the even-odd splitting at $\alpha = 0$ in Fig 1b to be $R_1 = \frac{\omega_a^2 \lambda_{\alpha}^2}{2} \frac{2\beta}{1 + \beta}$. The even-odd gap in the first bubble in Fig 1b is

$$\Delta_{m=\pm 1, U(1)}(\alpha = 0) = \frac{D}{2} - \frac{\omega_a^2 \lambda_{\alpha}^2}{2} \frac{2\beta}{1 + \beta}$$

(9)

which decreases as $g$ increases. Using Eqn 6 one can rewrite $\Delta = \frac{D}{2} (1 - \frac{\omega_a^2}{D})^2 > 0$.

For general $\pm m$ (with $|m| \leq P$ in a given sector $P$), one needs $m - th$ order degenerate perturbation calculation 31 to find the splitting at $\pm m$ to be $R_m \sim (\frac{\omega_a^2 \lambda_{\alpha}^2}{D})^m, m = 0, 1, ..., P$. Setting $R_0 = 0$, then $m - th$ even-odd gap at $\alpha = 0$ in the $m - th$ bubble in Fig 1b is:

$$\Delta_{m, U(1)}(\alpha = 0) = D(m + 1/2) - R_{m+1} - R_m$$

(10)

Setting $m = 0$ recovers Eqn 9.

Note that the degenerate pair $(m, -m - 1)$ at the edge at $\alpha = \pm 1/2$ have opposite parity, so will not be mixed in any order of perturbations 31. So despite there could be a slight shift in the crossing point between the two opposite parities at $(m, -m - 1)$, the crossing between the pair stays, so

$$\Delta_{m, U(1)}(\alpha = \pm 1/2) = 0$$

(11)

Eqn 10 and 11 can be contrasted with Eqn 14 and 15 in the QT regime which will be evaluated in the following section.

4. The parity oscillations due to Instantons subject to the Berry phase in the QT regime. At any finite $N$, there is a QT due to instantons between the two bound states shown in Fig 2 which is a non-perturbative beyond the $1/J$ expansion. The instanton solution for a Sine-Gordon model was well known 38. From Eqn 3 we can find the classical instanton solution connecting the two minima from $\theta_+ = 0$ or $\theta_+ = \pi$: $\theta_+(\tau) = 2\tan^{-1}e^{\omega_a^2(\tau - \tau_0)}$ where $\tau_0$ is the center of the instanton. Its asymptotic form as $\tau \rightarrow \infty$ is $\theta_+(\tau \rightarrow \infty) \rightarrow \pi - 2e^{-\omega_a^2(\tau - \tau_0)}$. The corresponding classical instanton action is:

$$S_0 = \frac{2\omega_{-\alpha}}{D}$$

(12)

The instanton problems in the three well known systems (1) a double well potential (DWP) in a $\phi^4$ theory (2) periodic potential problem (PPP) (3) a particle on a circle (POC) are well documented in 38. The tunneling problem in the present problem is related, but different than all the three systems in the following important ways: (1) the potential $V(\theta) = 2 \omega_a \lambda_{\alpha}^2 \pi / 1 + \beta (1 - \cos 2\theta_\alpha)$ in Eqn 5 is a periodic potential in $\theta_\alpha$. In this regard, it is different than the $\phi^4$ theory, but similar to PPP. (2) The $\theta_\alpha$ is a compact angle confined in $0 < \theta_\alpha < 2\pi$. In this regard, it is different than the PPP, but similar to POC. (3) There are two minima inside the range $0 < \theta_\alpha < 2\pi$.
instead of just one. In this regard, it is different than the POC, but similar to the $\phi^4$ theory. (4) Furthermore, it is also important to consider the effects of Berry phase which change the action of instanton to $S_{\text{int}} = S_0 + i\alpha \pi$, that of anti-instanton to $S_{\text{int}} = S_0 - i\alpha \pi$ ( Fig.2-2 ) where $-1/2 < \alpha < 1/2$ is the Berry phase in Eqn.6. So the present quantum tunneling problem is a new class one.

Taking into account the main differences of the present QT problem from the DWP, PPP and POC studied per- 18viously [38], especially the crucial effects of the Berry phase, we can evaluate the transition amplitude from

\[ |e\rangle_{0, SC} = \frac{A}{\sqrt{2}} |\theta_+ = 0\rangle + |\theta_+ = \pi\rangle, \]

\[ |o\rangle_{0, SC} = \frac{A}{\sqrt{2}} |\theta_+ = 0\rangle - |\theta_+ = \pi\rangle, \] 

(13)

where the overlapping coefficient is $A^2 = |\langle x = 0 | n = 0 \rangle|^2 = (\frac{\pi}{8N})^{1/2}$. They have the energy $E_{e/o} = \frac{\hbar}{\pi} + \Delta_0/2$ with the splitting between them given by:

\[ \Delta_0(\alpha) = 8\omega_{-0} \cos\alpha \pi \left( \frac{\omega_{-0}}{\pi D} \right)^{1/2} e^{-\frac{2\pi}{\Delta_0}} \]

\[ \sim (\cos\alpha \pi) \sqrt{N} e^{-\pi N} \] 

(14)

which decreases as $g$ increases.

One can see that it is the Berry phase which leads to the oscillation of the gap and the parity of the ground state. It vanishes at the two end points $\alpha = \pm 1/2$ and reaches maximum at the middle $\alpha = 0$. Note that the Berry phase $\alpha$ is defined [26, 31, 32] at a given sector $P$. So Eqn.13 has a background parity $\Pi = (-1)^P$. So there is an infinite number of oscillating parities in Eqn.14 as $g$ increases.

Because the Berry phase effects remain in the given $P$ sector, extending the results in Ref. [34], we find the splitting in the $n$-th excited bound states ($n = 0, 1$ in Fig.2-2):

\[ \Delta_n(\alpha) = \frac{1}{n!} \left( \frac{8\omega_{-0}}{D} \right)^n \Delta_0 \] 

(15)

where $n = 0, 1, 2, \cdots$ with the corresponding $n$-the Schrodinger "Cat" state with even/odd parity and the energy $E_{e/o,n} = (n + \frac{1}{2})\hbar\omega_{-0} + \Delta_n/2$. Eqn.14 in the QT regime can be contrasted with Eqn.10 and 11 in the $U(1)$ regime.

One can see the higher the bound state in the Fig.2 the larger the splitting is. These results are completely consistent with those achieved from the strong coupling expansion in [33] after identifying $n \sim l$. For example, there is an extra oscillating sign $(-1)^l$ in Eqn.5 in [33] achieved from the strong coupling expansion, which is crucial to reconcile the results achieved from the two independent approaches.

The $U(1)$ regime and the QT regime can be theoretically well distinguished by its qualitatively different features ( Fig.3): When $0 < \beta < \beta_{U(1)}$, the even-odd oscillations happen before the formation of the bound states at $g_{c1}$. In the normal regime, there is no oscillations. In the $U(1)$ regime before $g_{c1}$, the oscillations are due to the scattering states (open bubbles). In the QT regime after $g_{c1}$, the oscillations are due to the bound states (shaded bubbles). The ELS stays Possian. There is no CNCT as $g$ increases. As $\beta$ increases, the $U(1)$ regime shrinks from (a) to (b). (c) where $\beta = \beta_{U(1)} \sim 0.35$, the first oscillation happens at the same time as the formation of the bound state at $g_{c1}$. The $U(1)$ regime just disappears. (d) and (e) where $\beta_{U(1)} < \beta \leq 1$, it happens after the first formation of the bound state at $g_{c1}$. The $U(1)$ regime does not exist anymore. When $\beta = 1$, the oscillation was pushed to infinity, so there is no oscillations at all after the formation of the bound states. There is a CNCT near $g_e$ where the ELS changes from the Possian to GOE. Compare to Fig.4.

![FIG. 3. (Color online) The evolution of the two lowest energy with even and odd parity. (a) and (b) where $0 < \beta < \beta_{U(1)} \sim 0.35$, the first oscillation happens before the first formation of the bound state at $g_{c1}$.

The energy level statistics and the chaotic to non-chaotic transitions (CNCT). Here we study]
so it becomes regular too instead of being random. bound states in Fig.1 which push out all the scattering state s, random [43]. When up to 1, the system just tends to be regular instead of being corresponding to Fig.3a-e. There is a CNCT tuned by can only be tested by specific calculations.

Practically, one need to perform a specific ED to test if operators (See footnote Ref.31 in [33]). Obviously, which takes complex conjugate on the atomic spin can define the anti-unitary Time reversal symmetry T as it is at the QCP or inside the superradiant phase g/g

The energy level statistics at a given \( 0 < \beta < 1 \). One can define the anti-unitary Time reversal symmetry \( T = K \) which takes complex conjugate on the atomic spin operators (See footnote Ref.31 in [33]). Obviously, \( T^2 = 1 \), so just from symmetry point of view, it is in GOE. Practically, one need to perform a specific ED to test if a specific Hamiltonian indeed satisfies GOE. If the KAM theorem applies is out of the symmetry classification and can only be tested by specific calculations.

At the \( U(1) \) limit \( g' = 0 \) (\( \beta = 0 \)), the Dicke model is integrable. So the energy level always satisfies the Poisson distribution \( P_p(s) = e^{-s} \). At the \( Z_2 \) limit \( \beta = 1 \), the system becomes non-integrable at any finite \( N \), the energy level statistics (ELS) is Possionian in the normal regime, but becomes Wigner-Dyson (WD) distribution in the Grand orthogonal ensemble (GOE) \( P_w(s) = \frac{1}{2\pi} e^{-\frac{s^2}{4}} \) in the superradiant regime [21]. This fact suggests that the emergence of the quantum chaos or the changing of ELS thorough a CNCT at any finite \( N \) may be related to the normal to super-radiant phase transition. Now we will study the energy level statistic by ED [21] at a given \( 0 < \beta < 1 \) at a given parity sector, then compare with the analytical results on the low energy level evolution from the normal, \( U(1) \) regime to the QT regime in Fig.1 and Fig3 to explore their possible intrinsic connections.

The nearest-neighbor energy level spacings \( s_n = E_n - E_{n-1} \) where we labeled all the energy levels in the descending orders \( E_0 < E_1 < E_2 < \cdots E_n < \cdots \). By ED, we drew the diagram of \( s_n \) versus \( n \) for tunable cutoff from \( n = 400 \) to \( n = 4000 \) at various \( \beta = 1,0.8,0.5,0.2,0.1 \). After determining the optimal energy level spacing \( ds \) which can distinguish the two level spacing most efficiently, we also evaluated the energy level statistics \( P(s) \) on \( s = \bar{s}n/s \) where \( \bar{s} \) is the mean value of \( \{s_n\} \) at these \( \beta \) values [41].

To get ride of the dependence on the local density of states, it is convenient to look at the distribution of the ratio of two adjacent energy level spacings \( \frac{r_n}{\bar{s}} = \frac{s_n}{s_{n+1}} \) which distributes around 1. Then \( P_p(s) \) and \( P_w(s) \) leads to \( p_p(r) = \frac{1}{(1+r)^2} \) and \( p_w(r) = \frac{1}{2\pi} \frac{r}{(1+r^2)^{1+\frac{1}{2}}}, \beta = 1, Z = 8/27 \) respectively. We computed the distribution of the logarithmic ratio \( P(\ln r) = p(r) r \). Because \( p(\ln r) \) dr is symmetric under \( r \leftrightarrow 1/r \), one may confine \( 0 < r < 1 \) and double the possibility density \( p(\bar{r}) = 2p(r) \). Therefore, the above two distributions have two different expected values of \( \bar{r} = \min\{r, 1/r\} \):

\[
\langle \bar{r} \rangle_p = \int_0^1 2p_p(r) dr = 2\ln 2 - 1, \\
\langle \bar{r} \rangle_w = \int_0^1 2p_w(\beta = 1, r) dr = 4 - 2\sqrt{3} \tag{16}
\]

We plot \( \langle \bar{r} \rangle \) vs \( \ln \beta \) in Fig4a at a fixed \( g/g_c = 3,2,1 \) which is at the QCP (namely, along the dashed line in Fig3) or inside the superradiant phase. It shows that there is a CNCT at \( \ln \beta \approx \pm 1 \), so the CNCT tuned by \( \beta \) happens near \( \beta \approx 0.37 \). As shown in Fig4b, there is a CNCT near \( g/g_c = 1 \) tuned by \( g \) at a fixed \( \beta_{U(1)} < \beta < 1 \) but none when \( 0 < \beta < \beta_{U(1)} \).

6. The Berry phase effects lead to the quantum analogue of the Kolmogorov-Arnold-Moser (KAM) theorem. Here we explore a possible quantum analogue of the classical KAM theorem in the context of \( J - U(1)/Z_2 \) Dicke model. The \( U(1) \) Dicke model at \( \beta = 0 \) is integrable. If ignoring the Berry phase \( \alpha \), its eigen-energy would be commensurate. So it is the frustration due to the Berry phase effects which make its eigen-energies become in-commensurate [24,31] except at \( \alpha = 0, \pm 1/2 \) which have zero measures anyway, so the quantum analogue of the KAM theorem may apply.

As shown in Fig4c, at a given \( \beta \), if there is one (in fact, only one) CNCT tuned by \( g/g_c \), it happens near the QCP \( g/g_c = 1 \). So one may just focus on how the ELS changes at \( g/g_c = 1 \) as \( \beta \) decreases (along the dashed line in Fig3). The ELS in (e) must be GOE, remains GOE in (d), must be Possionan in (a) and remain to be Possionan in (b), so the CNCT must happen at \( \beta = \beta_{U(1)} \) where the \( U(1) \) regime just disappears in (c).

We make the following statement on the quantum analogue of the KAM theorem (1) The existence of the \( U(1) \) regime between the normal phase and the QT regime when \( 0 < \beta < \beta_{U(1)} \) indicates that the quantum KAM theorem apply, the ELS remains Possionian throughout
the normal, $U(1)$ and the QT regime. In fact, as shown below, there is a duality relation between the eigen-energies in the $U(1)$ regime and those in the QT regime. (2) The absence of the $U(1)$ regime when $\beta_{U(1)} < \beta < 1$ indicates that the quantum KAM theorem fails, the ELS changes from Possianion in the normal to Wigner-Dyson in the QT regime. Of course, the duality relation also disappears. This conclusion is expected to hold when $1 < \beta < \infty$ when one moves from the $Z_2$ limit at $\beta = 1$ to the LZ limit at $\beta = \infty$. This conclusion is in sharp contrast to the claims made in Ref. [37]. The main reason is that Ref. [37] is a purely numerical data without any physical insights from analytical calculations.

The ELS in Fig.1 (see also Fig.5a) must be Possionian. The bound states will appear one by one in the QT regime in Fig.1-e. However, because their numbers are finite anyway, so will not affect the ELS which involves infinite number of energy levels. So the ELS stays Possionian from the $U(1)$ to the QT regime in Fig.1. In fact, there are some illuminating duality in the quantum numbers characterizing the energy spectrum when $0 < \beta < \beta_{U(1)}$. In the $(1)$ regime in Fig.1 and 3a,b, it is convenient to use the complete eigenstates $|l\rangle_m$ where the Landau level index $l = 0, 1, ..., N$ ( $N + 1$ Landau levels ) denotes the high energy Higgs type of excitation, the magnetic number $m = -P, -P + 1, ..., \infty$ (no upper bounds) denotes the low energy atomic-Goldstone mode. In the QT regime in Fig.1 and 3a,b, it is convenient to use a different set of complete eigenstates $|l\rangle_m|j\rangle_m$ where the Landau level index $l = 0, 1, ..., 2j + 1 = N + 1$ denotes the high energy optical mode. So the Landau level index and the magnetic number exchanges their roles from the $(1)$ to the QT regime when $0 < \beta < \beta_{U(1)}$. This duality relation between the two sets of eigenstates implies that the ELS remains the same from the $(1)$ to the QT regime, namely, remains to be Possionian. Of course, when $\beta_{U(1)} < \beta < 1$, the $(1)$ regime disappears and so does the duality relations.

One may also understand the quantum analog of the KAM theorem from a dual point of view, namely from the chaotic $Z_2$ limit and investigate how the chaotic behaviours change from non-chaotic as a perturbation acts on. At $\beta = 1$, the super-radiant phase becomes the QT regime at any finite $N$ (Fig.13). As $g$ decreases, the CNCT transition happens near $g/g_c = 1$. When not too far away from the $Z_2$ limit $\beta_{U(1)} < \beta < 1$, the super-radiant phase still becomes the QT regime at any finite $N$ (Fig.31). As $g$ decreases, the CNCT transition still happens at $g/g_c = 1$. However, when $\beta$ decreases to $0 < \beta < \beta_{U(1)}$ (Fig.3a,b), the super-radiant phase splits into the QT regime and the $U(1)$ regime. The CNCT disappears, the ELS remains Possionian.

DISCUSSIONS.

In this work, we only focused on the the ELS of the bulk spectrum. In fact, when doing the ELS, one can even just throw away the low energy bound states which are finite number anyway. So the bulk states simply reflect the high energy scattering states. It would be interesting to just focus on the edge spectrum which can reflect the low energy bound states. So in a future publication, we may study the ELS of the edge spectrum to reflect the bound state evolution in Fig.3.

As said in the introduction, in addition to the ELS in the RMT classifications, the CNCT may also be diagnosed from the spectral form factor (SFF) in the RMT. The SFF at $\beta_T = 0$ may also be a useful diagnostic tool for the CNCT. A slope-dip-ramp-plateau structure was considered to be evidence for quantum chaotic behaviours. This feature may disappear when $0 < \beta < \beta_{U(1)}$. While the Lyapunov exponent at an early time is a completely different way to characterize the quantum chaos. So it may also be interesting to investigate the KAM from the OTOC perspective. For the $U(1)/Z_2$ Dicke models, fixing at a given $g/g_c \geq 1$ (at the QCP or the superradiant phase in Fig.3 and Fig.4a), we expect that the Lyapunov exponent $\lambda_L$ at the infinite temperature $\beta_T = 0$ becomes maximum at the $Z_2$ limit $\beta = 1$, then decreases as $\beta$ becomes smaller than 1, becomes zero at $\beta_{U(1)}$, remains to be zero when $0 < \beta < \beta_{U(1)}$. These results will be presented in a separate publication [36].

The Quantum analog of KAM theorem at a generic $0 < \beta < 1$ in the $U(1)/Z_2$ Dicke model may be contrasted with the QPT from the normal to the super-radiant phase. In the $U(1)/Z_2$ Dicke model, any $g' > 0$ breaks explicitly the $U(1)$ symmetry of the integrable $U(1)$ Dicke model and is clearly a relevant perturbation. There is a line of fixed point at $g_c = \frac{\sqrt{g_0}}{1 + g_0}$ tuning from the $U(1)$ limit at $\beta = 0$ to the $Z_2$ limit at $\beta = 1$. QPT only involves the changing of the ground states and the low energy excitations, while the high energy states are irrelevant. Strictly speaking, it happens only in the thermodynamic limit $N \to \infty$ when symmetry broken can happen, there is no QPT in any finite systems. While the ELS in RMT involves all the energy levels at a finite but large enough $N$. In a recent preprint [46], we studied quantum chaos and quantum analog of Kolmogorov-Arnold-Moser (KAM) theorem in hybrid Sachdev-Ye-Kitaev (SYK) models and contrasted with those in the $U(1)/Z_2$ Dicke model studied in this paper. We found that the similar phenomena also appear in the hybrid SYK models [46]. So despite the absence of QPT in the present cases of the $U(1)/Z_2$ Dicke model and hybrid SYK models, there could still be a CNCT.

It was well established that in terms of symmetries and the space dimension, the Renormalization Group (RG) (including the DMRG, MPS and tensor network) can be used to classify many body phases and phase transitions [44,45]. The RG focus on either infra-red (IR) behaviours of the system which are determined by the
ground state and low energy excitations. The RG is also intimately connected to General Relativity (GR) through the holographic principle. The 10-fold way in RMT can be classified just by a few global discrete symmetries. It covers all the energy levels of the system in the Hilbert space and can also be used to characterize the CNCT. Possible deep connections between the onset of quantum chaos characterized by the RMT and the onset of quantum phase transition characterized by the RG need to be explored further. The $U(1)/Z_2$ Dicke models and the hybrid SYK models (or the GW tensor models) may supply new platforms to investigate possible relations between the two dramatically different classification schemes.

Finally, we comment on the experimental realizations of the $U(1)/Z_2$ Dicke model. Due to recent tremendous advances in technologies, the $U(1)/Z_2$ Dicke model has been realized in at least two kinds of cavity QED systems (1) a BEC atoms inside an ultrahigh-finesse optical cavity [47–51] and (2) superconducting qubits inside a microwave circuit cavity [52–55] or quantum dots inside a semi-conductor microcavity [56]. All the results achieved in this work should be detected in these systems by various standard quantum optics techniques such as fluorescence spectrum, phase sensitive homodyne detections and Hanbury-Brown-Twiss (HBT) type of experiments respectively [57–59]. Our work shows that the cavity QED systems may provide experimentally accessible systems to investigate quantum chaos and quantum information scramblings in strong light-matter interacting systems.

Acknowledgements

JY acknowledge AFOSR FA9550-16-1-0412 for supports. CLZ’s work has been supported by National Keystonfe Basic Research Program (973 Program) under Grant No. 2007CB310408, No. 2006CB302901 and by the Funding Project for Academic Human Resources Development in Institutions of Higher Learning Under the Jurisdiction of Beijing Municipality.

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So when collecting the ELS of the U(1) Dicke model, we take all the energy levels at all possible different \gamma which fit the Possianain distribution. If confining to a fixed \gamma sub-Hilbert space, we can not get any sensible ELS. This is in constrast to the way in [15] to collect the ELS of complex SYK at a given \gamma = \sum c_i^*c_i, also the way at the hybrid SYK models to be presented in [46]. Of course, any \gamma \neq 0 in Eq.1 breaks the U(1) symmetry, only the parity \gamma = e^{-iQ} becomes a conserved quantity. So we collect the ELS at a given parity sector. For all the cases from the U(1) to the Z2 limit, there is always a regular regime when \gamma/g < 0.15.

It seems Fig.1 enjoys a emergent symmetry \beta \leftrightarrow 1/\beta. This possible emergent symmetry in ELS will be explored from both the U(1) Dicke model side 0 < \beta < 1 and the Landau-Zener model side 1 < \beta < \infty in Ref. [36].

Being regular means the energy levels show a regular behaviours, so the ELS satisfies neither Possion nor WD. Being random means the energy levels show random distribution, so the ELS satisfies WD in RMT. 

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