ON EXAMPLES OF DIFFERENCE OPERATORS FOR \{0,1\}-VALUED FUNCTIONS OVER FINITE SETS

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1. Introduction and basic definitions

Recently V. I. Arnold have formulated a geometrical concept of monads and apply it to the study of difference operators on the sets of \{0,1\}-valued sequences of length \(n\). In [1]–[4] he made first steps in the study of this subject and formulated many nice questions. In [5] A. Garber showed an algorithm that gives a description of the combinatorial structure of monads for difference operators and answered many questions of V. I. Arnold. In the present note we show particular examples of these monads and indicate one question arising here.

The author is grateful to V. I. Arnold for useful remarks and discussions and Mathematisch Instituut of Universiteit Leiden for the hospitality and excellent working conditions.

A monad by V. I. Arnold is a map of a finite set into itself. Suppose \(M : S \to S\) is an arbitrary monad. It is naturally to associate an oriented graph to the monad \(M\). Its vertices coincide with elements of \(S\), and the set of its edges is the set of ordered pairs \((x,M(x))\). We denote such graph by \(G(M)\) The idea of V. I. Arnold is to study the combinatorial geometry of graphs for monads. He proposed to start with one important example.

Consider any positive integer \(n\), and take the set \(A_n = \{1,\ldots,n\}\). Denote by \(F_2(A_n)\) the vector space of \(\mathbb{Z}_2\)-valued functions on \(A_n\). Consider a “differential” difference operation \(\Delta\), defined as follows:

\[
(\Delta f)(x) = \begin{cases} 
  f(x) + f(x+1), & \text{if } x \neq n \\
  f(n) + f(1), & \text{if } x = n 
\end{cases}
\]

On Figure 1 we show an example of a monad for the difference operator for the set \(A_6\).

![Figure 1](image-url)  

Figure 1. An example of a monad \(G(\Delta)\) for the case of \(A_6\).
2. A FEW WORDS ABOUT ARNOLD’S COMPLEXITY

Let us briefly describe the concept of Arnold’s functional complexity. Further we will need the additivity property of $\Delta$:

$$\Delta(f + g) = \Delta(f) + \Delta(g).$$

The simplest functions on the set $A_n$ are polynomials. The set of all solutions of the functional equation $\Delta^k(f) = 0$ is called the set of polynomials of degree less than $k$, we denote it by $Pol_{k-1}$. Suppose $f \in Pol_k$ and $f \notin Pol_l$ for $l = 0, \ldots, k-1$, then $f$ is called a polynomial of degree $k$. Denote by $Pol(A_n)$ the set of all polynomials on $A_n$.

Actually, the set $Pol(A_n)$ is a vector space. If $n = 2^l m$ where $m$ is odd, then the space $Pol(A_n)$ is $2^l$-dimensional and contains $2^{2^l}$ elements. In particular, if $n = 2^l$, then $\mathcal{F}_2(A_n) = Pol(A_n)$.

Now we define another set of nice functions. Consider the functional equation $\Delta(f) = f$. The set of all solutions of such equation is called the set of special rational exponential polynomials of orders that divide $k$, we denote it by $Exp_k$. Let $Exp_0 = \{0\}$. Suppose $f \in Exp_k$ and $f \notin Exp_l$ for $l = 0, \ldots, k-1$, then $f$ is called a special rational exponential polynomial of order $k$, or exp-polynomial for short. Denote by $Exp(A_n)$ the set of all exp-polynomials on $A_n$. Note that the set $Exp(A_n)$ is a vector space.

**Proposition 2.1.** Any function $f$ of $\mathcal{F}_2(A_n)$ can be uniquely written in the form $f = p + r$, where $p$ is a polynomial and $r$ is an exp-polynomial, or in other worlds

$$\mathcal{F}_2(A_n) = Pol(A_n) \oplus Exp(A_n).$$

**Definition 2.2.** Consider an arbitrary function of $\mathcal{F}_2(A_n)$. Let $f = p + r$, where $p$ is a polynomial and $r$ is an exp-polynomial. We say that degree of $f$ is degree of $p$ and denote it by $\deg(f)$. We say that order of $f$ is order of $r$ and denote it by $\ord(f)$.

V. I. Arnold proposed the following definition the notion of functional complexity.

**Definition 2.3.** A function $f_1$ is said to be more complicated (in the sense of Arnold) than $f_2$ if either $\ord(f_1) > \ord(f_2)$, or $\ord(f_1) = \ord(f_2)$ and $\deg(f_1) > \deg(f_2)$.

If $\ord(f_1) = \ord(f_2)$ and $\deg(f_1) = \deg(f_2)$ then the functions $f_1$ and $f_2$ are said to be of the same complexity (in the sense of Arnold).

**Remark 2.4.** All the above easily can be generalized to the case of $\mathbb{Z}_N$-valued functions for an arbitrary positive integer $N > 2$.

3. SOME EXAMPLES OF $G(\Delta)$

3.1. A list of examples. Suppose $n = 2^l m$. The set of polynomials is a 2-valent symmetric tree of $r = 2^{2^l}$ elements. Denote this tree by $T_r$. Denote also a cycle of $s$ elements by $O_s$.

Each connected component of the graph $G(\Delta)$ contains a cycle. Denote its length by $s$. To each vertex of the cycle it is attached a tree equivalent to $T_r$ (as on Fig. 1 for the case $n = 6$). Denote such component by $O_s* T_r$.

Let us enumerate connected components for the graphs of the sets $\mathcal{F}_2(A_n)$ where $n \leq 25$. Expression $k(O_s * T_r)$ means that there are $k$ components of the type $O_s * T_r$. The graph $G(\Delta)$ for the set $\mathcal{F}_2(A_n)$ contains

- in the case of $n = 1$: $O_1* T_2$;
- in the case of $n = 2$: $O_1* T_4$;
- in the case of $n = 3$: $O_1* T_2$, $O_3* T_2$;
in the case of \( n = 4 \): \( O_1 \ast T_{16} \);
in the case of \( n = 5 \): \( O_1 \ast T_2, O_{15} \ast T_2 \);
in the case of \( n = 6 \): \( O_1 \ast T_4, O_3 \ast T_4, 2(O_6 \ast T_4) \);
in the case of \( n = 7 \): \( O_1 \ast T_2, 9(O_7 \ast T_2) \);
in the case of \( n = 8 \): \( O_1 \ast T_{256} \);
in the case of \( n = 9 \): \( O_1 \ast T_2, O_3 \ast T_2, 4(O_{63} \ast T_2) \);
in the case of \( n = 10 \): \( O_1 \ast T_4, O_{15} \ast T_4, 8(O_{30} \ast T_4) \);
in the case of \( n = 11 \): \( O_1 \ast T_2, 3(O_{341} \ast T_2) \);
in the case of \( n = 12 \): \( O_1 \ast T_{16}, O_3 \ast T_{16}, 2(O_6 \ast T_{16}), 20(O_{12} \ast T_{16}) \);
in the case of \( n = 13 \): \( O_1 \ast T_2, 5(O_{819} \ast T_2) \);
in the case of \( n = 14 \): \( O_1 \ast T_4, 9(O_7 \ast T_4), 288(O_{14} \ast T_4) \);
in the case of \( n = 15 \): \( O_1 \ast T_2, O_3 \ast T_2, 30(O_5 \ast T_2), 1082(O_{15} \ast T_2) \);
in the case of \( n = 16 \): \( O_1 \ast T_{216} \);
in the case of \( n = 17 \): \( O_1 \ast T_2, 51(O_{85} \ast T_2), 240(O_{255} \ast T_2) \);
in the case of \( n = 18 \): \( O_1 \ast T_4, O_3 \ast T_4, 2(O_6 \ast T_4), 4(O_{63} \ast T_4), 518(O_{126} \ast T_4) \);
in the case of \( n = 19 \): \( O_1 \ast T_2, 27(O_{970} \ast T_2) \);
in the case of \( n = 20 \): \( O_1 \ast T_{16}, O_{15} \ast T_{16}, 8(O_{30} \ast T_{16}), 1088(O_{60} \ast T_{16}) \);
in the case of \( n = 21 \): \( O_1 \ast T_2, O_3 \ast T_2, 9(O_7 \ast T_2), 9(O_{21} \ast T_2), 16640(O_{63} \ast T_2) \);
in the case of \( n = 22 \): \( O_1 \ast T_4, 3(O_{341} \ast T_4), 1536(O_{682} \ast T_4) \);
in the case of \( n = 23 \): \( O_1 \ast T_2, 2049(O_{2047} \ast T_2) \);
in the case of \( n = 24 \): \( O_1 \ast T_{256}, O_3 \ast T_{256}, 2(O_6 \ast T_{256}), 20(O_{12} \ast T_{256}), 2720(O_{24} \ast T_{256}) \);
in the case of \( n = 25 \): \( O_1 \ast T_2, O_{15} \ast T_2, 656(O_{25575} \ast T_2) \).

Denote by \( s(n) \) the order of the maximal possible length of cycles for the \( n \)-elements sequences. Actually the listed examples gives the negative answer to the following question of V. I. Arnold: \textit{is it true that} \((s(n)/n)+1\text{ is some power of } 2\)? It is not true, for example, for \( n = 23 \) where \( s(23) = 2047 \). Here \( s(23) \) is \( 11 \)-1 itself.

Denote by \( |n| \) the set of connected components of graphs \( G(\Delta) \), corresponding to the set \( F_2(A_n) \). The work \cite{5} of A. Garber immediately implies the following identities:

\begin{itemize}
  \item[a)] \( 3 \cdot 2^m \) = \( \{ O_1 \ast T_{22^m}, O_3 \ast T_{22^m} \} \cup \left\{ \frac{3^{2^k}}{3 \cdot 2^k \cdot 2^{2^k}}(O_{3 \cdot 2^k} \ast T_{22^m}) \middle| k = 2, \ldots, m \right\} \);
  \item[b)] \( 5 \cdot 2^m \) = \( \{ O_1 \ast T_{22^m}, O_{3,5} \ast T_{22^m} \} \cup \left\{ \frac{5^{2^k}}{5 \cdot 2^k \cdot 2^{2^k}}(O_{3,5 \cdot 2^k} \ast T_{22^m}) \middle| k = 2, \ldots, m \right\} \);
  \item[c)] \( 7 \cdot 2^m \) = \( \{ O_1 \ast T_{22^m}, 9(O_7 \ast T_{22^m}) \} \cup \left\{ \frac{7^{2^k}}{7 \cdot 2^k \cdot 2^{2^k}}(O_{7 \cdot 2^k} \ast T_{22^m}) \middle| k = 2, \ldots, m \right\} \);
  \item[d)] \( 11 \cdot 2^m \) = \( \{ O_1 \ast T_{22^m}, 3(O_{31,11} \ast T_{22^m}) \} \cup \left\{ \frac{11^{2^k}}{3 \cdot 11 \cdot 2^k \cdot 2^{2^k}}(O_{31,11 \cdot 2^k} \ast T_{22^m}) \middle| k = 2, \ldots, m \right\} \);
\end{itemize}

\( \ldots \)

3.2. \textbf{Particular case of } \( \delta \)-\textbf{functions.} Let us now study the structure of the piece-wise connected components of the graph \( G \) containing so-called \( \delta \)-\textbf{function}. Denote by \( \delta_k \) the following function of \( F_2(A_n) \):

\[ \delta_k(x) = \begin{cases} 
0, & \text{if } x \neq k \\
1, & \text{if } x = k 
\end{cases} \]

In \cite{5} A. Garber showed that the order of \( \delta_k \) coincide with \( s(n) \). So the piece-wise connected component of the graph \( G \) containing \( \delta_k \) is \( O_{s(k)} \ast T_{2^k} \), and it does not depend on the choice of \( k \). We now write down the values of \( s(n) \) for \( n \leq 50 \) in the following list.
There is a regularity in this sequence for primes \( n > 2 \). We use the following notation. Denote by \( \gamma_2(n) \) the minimal solution \( t \) of the equation \( 2^t \equiv 1 \pmod{n} \). Then

\[
\begin{align*}
\text{Table 1: } & n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
\text{s(n)} & 1 & 1 & 3 & 1 & 15 & 6 & 7 & 1 & 63 & 30 & 341 & 12 & 819 & 14 & 15 & 1 & 255 & 126 & 9709 & 60
\end{align*}
\]

\[
\begin{align*}
\text{Table 2: } & n & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 \\
\text{s(n)} & 63 & 682 & 2047 & 24 & 2557 & 1638 & 13797 & 28 & 475107 & 30 & 31 & 1 & 1023 & 510 & 4095 & 8388607
\end{align*}
\]

\[
\begin{align*}
\text{Table 3: } & n & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 \\
\text{s(n)} & 252 & 3233097 & 19418 & 4095 & 126 & 5461 & 1364 & 4095 & 4095 & 8388607
\end{align*}
\]

\[
\begin{align*}
\text{Table 4: } & n & 48 & 49 & 50 \\
\text{s(n)} & 48 & 2097151 & 51150
\end{align*}
\]

Denote by \( q(n) \) the following function

\[
q(n) := \begin{cases} 
n(2^{\frac{\gamma_2(n)}{2}} - 1), & \text{if } \gamma_2(n) \text{ is even} \\n2^{\gamma_2(n)} - 1, & \text{if } \gamma_2(n) \text{ is odd} \end{cases}
\]

Note that for all observed primes (except 37) we have \( s(n) = q(n) \).

**Problem 1.** Study the behaviour of the maximal length of the cycle. How often does it coincide with \( q(n) \)? Is it true that \( q(n) \) is always divisible by \( s(n) \)?

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