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Hidden Bottom Pentaquark in the SU(5) Version of the Flavor-Spin Model

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Abstract We generalize to five distinct flavors the flavor-spin hyperfine interaction introduced previously for four flavors and used in the study of \( uudc \) pentaquark. As a particular case here we study the lowest states of the pentaquark \( uudb \), of either positive or negative parity, in a constituent quark model with linear confinement and the presently extended hyperfine interaction. The positive parity states have one unit of angular momentum located in the subsystem of four quarks and are described by translationally invariant states of orbital permutation symmetry \([31]_O\) which requires the configuration \( s^3p \). The negative parity states are described by the configuration \( [34]_O \) of permutation symmetry \([34]_O\). We show that the lowest state has the quantum numbers \( J^P = \frac{1}{2}^+ \) or \( \frac{3}{2}^+ \) and \( I = \frac{1}{2} \) and is located below the \( \Sigma_b B \) threshold by \(-132\) MeV. We present a comparison between the spectra of \( uudc \) and \( uudb \) pentaquarks.

1 Introduction

The discovery of the hidden charm and strange hidden charm pentaquarks by the LHCb Collaboration has and is expected to have important consequences in hadron spectroscopy.

In 2015 the LHCb Collaboration reported the existence of two pentaquark-like resonances named \( P_c^+(4380) \) and \( P_c^+(4450) \) in the \( \Lambda_b^0 \to J/\psi pK^- \) decay [1]. Due to the \( J/\psi p \) component these structures were interpreted as hidden charm pentaquarks of flavor content \( uudc \).

In 2019 the LHCb Collaboration updated the analysis of the \( \Lambda_b^0 \to J/\psi pK^- \) decay [2] and reported the existence of three narrow structures named \( P_c^+(4312), P_c^+(4440) \) and \( P_c^+(4457) \) where \( P_c^+(4312) \) was entirely new. The other two resonances replaced the previous \( P_c^+(4450) \). The broad \( P_c^+(4380) \) resonance awaits confirmation.

In 2021 following the observation of hidden charm pentaquarks the LHCb Collaboration reported a new hadronic exotic state named \( \psi(4459) \) in the invariant mass distribution of the \( \Sigma_{b}^- \to J/\psi \Lambda K^- \) decay [3]. Because of the quark content of \( J/\psi \) and \( \Lambda \) this structure is a candidate for a strange hidden charm pentaquark of flavor content \( uudc \). Very recently the LHCb Collaboration detected another strange hidden charm pentaquark with preferred quantum numbers \( J^P = 1/2^- \) named \( P_{cs}(4338) \) [4].

The new name of \( P_{cs}^+(4312) \), recently proposed by the LHCb collaboration is \( P_{\psi^3}(4312) \) and the new name of \( P_{cs}^0(4459) \) is \( P_{\psi^2}(4459) \) [5].

The 2019 LHCb observation has triggered many theoretical interpretations. Although observed in the \( J/\psi p \) channel, the proximity of the mass of \( P_c^+(4312) \) and of the masses of \( P_c^+(4440) \) and \( P_c^+(4457) \) to the respective thresholds favored the molecular scenario [6–21].

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The 2015 LHCb pentaquarks have also been analyzed in compact pentaquark models based on the chromomagnetic interaction of the one gluon exchange model, with quark/antiquark correlations [22] or without correlations [23, 24]. In both cases the lowest state has negative parity.

The spectrum of positive and negative parity states of the $uudc\bar{c}$ pentaquarks has also been studied in a constituent quark model with an SU(4) flavor-spin hyperfine interaction [25] which is a generalization of the model of Ref. [26] based on SU(3). The merit of the flavor-spin (FS) model is that it reproduces the correct ordering of positive and negative parity states of both nonstrange and strange baryons [26–28] in contrast to the one gluon exchange (OGE) model. However, it cannot explain the hyperfine splitting in mesons, because it does not explicitly contain a quark-antiquark interaction.

An SU(4) classification of pentaquarks and its decomposition into SU(3) submultiplets, by selecting those with the charm quantum number $C = 0$, has been considered in Ref. [29] and several properties as mass spectrum, magnetic moments and photocouplings have been studied. Other approaches can be found in the review papers [30, 31].

Presently, the spin and parity of the narrow structures $P_c^+ (4312)$, $P_c^+ (4440)$ and $P_c^+ (4457)$ remains to be established experimentally.

In this context it is reasonable to expect a future observation of the bottom analogues of the hidden charm pentaquarks. An optimistic note is that the naming convention of a possible $uu\bar{d}\bar{b}$ pentaquark has already been listed in Ref. [5]. as $P_{u}^{uu\bar{d}\bar{b}}(mass)$. Here we explore the spectrum of the $uu\bar{d}\bar{b}$ pentaquark in a constituent quark model with a flavor-spin hyperfine interaction [25] which is now extended from SU(4) to SU(5).

Let us first recall a few previous studies for hidden bottom pentaquarks. In an early exploratory paper [32] hidden bottom pentaquarks have been studied as hadronic $\Sigma_b B$ molecules and a few narrow resonances were found around 11 GeV.

Subsequently the 2015 LHCb observation of the $P_c (4380)$ and $P_c (4450)$ pentaquarks stimulated interest in the study of hidden bottom pentaquarks. For example in Ref. [33] the mass of the hidden bottom $uu\bar{d}\bar{b}$ pentaquark was estimated around 10.8 GeV in a simplified phenomenological model.

In Ref. [34] properties of hidden bottom pentaquarks $qq\bar{q}\bar{b}$ ($q = u, d, s$) have been explored in the framework of a simple quark model with chromomagnetic interaction which gave satisfactory results for the 2015 LHCb $P_c$ resonances. The theoretical masses were compared to the $\Upsilon \rho$, $\Upsilon \Delta$ and $\Sigma_b B$ thresholds. A bound state with $I = 1/2$ and $J^P = 3/2^-$ was found. Pentaquarks with masses lower than the relevant hadronic molecules were found.

In Ref. [35] a chiral quark model which successfully explains meson and baryon phenomenology was applied to the hidden bottom sector to search for possible bound states with isospin $1/2$ and $3/2$. Several candidates were found for negative parity states. The lowest state $I(J^P) = 1/2(1/2^-)$, related to $\Sigma_b B$ channel, has a mass of 11,078 MeV.

In Ref. [36] hidden bottom pentaquarks were investigated as $\bar{B}^{(*)} \Lambda_b$ and $\bar{B}^{(*)} \Sigma_b$ molecules coupled to compact 5-quark states. It was found that the pion exchange interaction is strong enough to produce resonant and bound states.

In Ref. [25] the extension to SU(4) has been made in the spirit of Ref. [37] where, in addition to the Goldstone bosons of the hidden approximate chiral symmetry of QCD, the flavor exchange interaction was augmented by a phenomenological hyperfine flavor exchange of $D$, $D_s$ and $B$ mesons. The model provided a satisfactory description of heavy flavor baryons.

In this work we extend the flavor-spin model [25] from SU(4) to SU(5). In addition to the Goldstone bosons of the hidden approximate chiral symmetry of QCD, the flavor exchange interaction is augmented by an exchange of open charm $D$, $D_s$, open bottom $B$, $B_s$ and open charm-bottom $B_c$ mesons. For consistency with the SU(5) algebra the hidden-charm and the hidden-bottom mesons $\eta_c$ and $\eta_b$ were added too.

The paper is organized as follows. In Sect. 2 we introduce the model Hamiltonian and the two-body matrix elements of the FS interaction corresponding to SU(5), including those needed for studying the $uu\bar{d}\bar{b}$ pentaquark. Section 3 describes the orbital part of the four quark subsystem constructed to be translationally invariant both for positive and negative parity states. Sections 4, 5 and 6 summarize analytic formulas. Section 7 contains numerical results and a comparison with previous studies. The last section is devoted to conclusions. “Appendix A” is a reminder of useful group theory formulas for SU(n). “Appendix B” exhibits a variational solution for the baryon masses relevant for the present study. “Appendix C” reproduces the SU(5) generators $\lambda_i$ used in the calculation of the matrix elements of the flavor-spin interaction. “Appendix D” gives the explicit form of the flavor states of the $uu\bar{d}\bar{b}$ four quark subsystem in the Rutherford–Young–Yamanouchi basis.
2 The Hamiltonian

Here we closely follow the description of the model as presented in Ref. [25]. The Hamiltonian has the general form [26]

\[
H = \sum_i m_i + \sum_i \frac{\vec{p}_i^2}{2m_i} - \frac{1}{2} \sum_i m_i \text{ } + \sum_{i<j} V_{\text{conf}}(r_{ij}) + \sum_{i<j} V_x(r_{ij}),
\]

(1)

with \( m_i \) and \( \vec{p}_i \) denoting the quark masses and momenta respectively and \( r_{ij} \) the distance between the quarks \( i \) and \( j \). The Hamiltonian contains the internal kinetic energy and the linear confining interaction

\[
V_{\text{conf}}(r_{ij}) = -\frac{3}{8} \lambda_i^x \cdot \lambda_j^x C r_{ij}.
\]

(2)

The SU(5) extension of the hyperfine interaction \( V_x(r_{ij}) \) has the following form

\[
V_x(r_{ij}) = \left\{ \sum_{F=1}^{3} V_\pi(r_{ij}) \lambda_i^F \lambda_j^F + \sum_{F=4}^{7} V_K(r_{ij}) \lambda_i^F \lambda_j^F \\
+ V_\eta(r_{ij}) \lambda_i^8 \lambda_j^8 + V_{\eta'}(r_{ij}) \lambda_i^0 \lambda_j^0 \\
+ \sum_{F=9}^{12} V_D(r_{ij}) \lambda_i^F \lambda_j^F + \sum_{F=13}^{14} V_{Ds}(r_{ij}) \lambda_i^F \lambda_j^F \\
+ V_{\eta_b}(r_{ij}) \lambda_i^{15} \lambda_j^{15} \\
+ \sum_{F=16}^{19} V_B(r_{ij}) \lambda_i^F \lambda_j^F + \sum_{F=20}^{21} V_{Bs}(r_{ij}) \lambda_i^F \lambda_j^F + \sum_{F=22}^{23} V_{B_s}(r_{ij}) \lambda_i^F \lambda_j^F \\
+ V_{\eta_b'}(r_{ij}) \lambda_i^{24} \lambda_j^{24} \right\} \vec{\sigma}_i \cdot \vec{\sigma}_j,
\]

(3)

with the SU(5) generators \( \lambda_i^F \) \( (F = 1,2,\ldots,24) \) and \( \lambda_i^0 \) proportional to the unit matrix. In the SU(5) version the interaction (3) contains \( \gamma = \pi, K, \eta, D, D_s, \eta_b, B, B_s, B_s, \eta_b \) and \( \eta' \) meson-exchange terms. Every \( V_\gamma(r_{ij}) \) is a sum of two contributions: a Yukawa-type potential containing the mass \( \mu_\gamma \) of the exchanged meson and a short-range contribution of opposite sign, the role of which is crucial in baryon spectroscopy. For a given meson \( \gamma \) the meson exchange potential is

\[
V_\gamma(r) = \frac{g_\gamma^2}{4\pi} \frac{1}{12m_im_j} (\theta(r - r_0) \mu_\gamma^2 e^{-\mu_\gamma r} \\
- \frac{4}{\sqrt{\pi}} \alpha^3 \exp(-\alpha^2(r - r_0)^2)),
\]

(4)

where \( \mu_\gamma \) is the mass of the exchanged meson, \( m_i \) the quark mass, \( \frac{g_\gamma^2}{4\pi} \) the coupling constant and \( r_0 \) a phenomenological parameter defined in Ref. [27].

In our application to hidden bottom pentaquarks which contain \( u, d \) and \( b \) quarks we need a few parameters which are chosen as follows. For the light quarks we use the parameters of Ref. [27] to which we add the \( \mu_B \) mass and the coupling constant \( \frac{g_{Bd}}{4\pi} \). These are

\[
\frac{8^2\pi q}{4\pi} = \frac{g_{Bd}^2}{4\pi} = \frac{g_{Bd}^2}{4\pi} = 0.67, \quad \frac{g_{Bd}^2}{4\pi} = 1.206,
\]

\[
r_0 = 0.43 \text{ fm}, \quad \alpha = 2.91 \text{ fm}^{-1}, \quad C = 0.474 \text{ fm}^{-2},
\]

\[
\mu_\pi = 139 \text{ MeV}, \quad \mu_\eta = 547 \text{ MeV}, \quad \mu_{\eta'} = 958 \text{ MeV}, \quad \mu_B = 5279 \text{ MeV}.
\]
The meson masses correspond to the experimental values from the Particle Data Group [38].

The model of Ref. [27] has previously been used to study the stability of open flavor tetraquarks [39] and open flavor pentaquarks [40, 41]. Accordingly, for the quark masses we take the values determined variationally in Refs. [39, 40]

\[
m_u, d = 340 \text{ MeV}, \quad m_b = 4660 \text{ MeV}.
\]

They were adjusted to satisfactorily reproduce the average mass \( \bar{M} = (M + 3M^*)/4 = 5313 \text{ MeV} \) of the \( B \) and \( B^* \) mesons.

After integration in the flavor space the two-body matrix elements become products of the spin-spin operator and expressions depending on \( V_F(r) \). Then, in the complete SU(5) extension containing five flavors, we have

\[
V_{ij} = \bar{\sigma}_i \cdot \bar{\sigma}_j = \begin{cases} 
V_{\pi}^{uu} + \frac{1}{3} V_{\eta}^{uu} + \frac{1}{6} V_{\eta c}^{uu} + \frac{1}{10} V_{\eta b}^{uu}, & [2]_F, I = 1 \\
2V_{\pi}^{us} - \frac{2}{3} V_{\eta}^{us}, \quad 2V_{\pi}^{uc} - \frac{1}{2} V_{\eta c}^{uc}, \quad 2V_{\pi}^{ub} - \frac{2}{3} V_{\eta b}^{ub}, & [2]_F, I = \frac{1}{2} \\
2V_{D_s}^{sc} - \frac{1}{2} V_{\pi}^{sc}, \quad 2V_{B_s}^{sb} - \frac{2}{3} V_{\eta c}^{sb}, \quad 2V_{B_s}^{cb} - \frac{2}{3} V_{\eta b}^{cb}, & [2]_F, I = 0 \\
-2V_{D_s}^{sc} + \frac{1}{2} V_{\eta}^{sc}, \quad -2V_{B_s}^{sb} + \frac{2}{3} V_{\eta c}^{sb}, \quad -2V_{B_s}^{cb} + \frac{2}{3} V_{\eta b}^{cb}, & [11]_F, I = \frac{1}{2} \\
-3V_{\pi}^{uu} + \frac{1}{3} V_{\eta}^{uu} + \frac{1}{6} V_{\eta c}^{uu} + \frac{1}{10} V_{\eta b}^{uu}, & [11]_F, I = 0
\end{cases}
\]

where the quark pair \( ij \) is either in a symmetric \( [2]_F \) or an antisymmetric \( [11]_F \) flavor state and the isospin \( I \) is defined by the quark content. The upper index of \( V \) exhibits the flavor of the two quarks interchanging a meson specified by the lower index. In every sum/difference of Eq. (6) the upper index is the same for all terms. Note that the \( K, D, D_s, B, B_s \) and \( B_c \) meson exchanges contribute with a factor +2 for symmetric pairs and -2 for antisymmetric pairs. In each of these mesons the quark and the antiquark have different flavors.

The present study is devoted to hidden bottom pentaquark, the most expected to be searched for experimentally, which contains \( u, d \) and \( b \) quarks. In this case there are no \( K, D, D_s, B, B_s \) and \( B_c \) meson exchanges. Only the terms of Eq. (6) related to \( \pi, \eta, B, \eta_c \) and \( \eta_b \) exchanges contribute to the hyperfine interaction and in practice we ignore the contribution of \( \eta_c, \eta_b \) exchanges, because little \( uu \) and \( dd \) are expected in real \( \eta_c, \eta_b \). We recall that the scalar mesons \( \eta_c \) and \( \eta_b \) used in the theoretical derivation of the expressions of Table 1, based on the SU(5) algebra, are defined in “Appendix C”.

We can now present the contribution of the hyperfine interaction (3) to the pentaquark states which comes from the four-quark subsystem. The group theoretical structure of the states under consideration is specified in column 1 of Table 1 for each state by the partitions [F] for flavor, [S] for spin and [FS] for flavor-spin.

In calculating the matrix elements of the hyperfine interaction (3) the first step is to decouple the flavor and spin parts of the wave functions of partitions [F] \( \rightarrow \) \( S \). This is done by using Clebsch-Gordan coefficients (isoscalar factors) of the permutation group \( S_4 \) [42], which allowed to reduce the four-body to two-body matrix elements.

At this stage one needs the explicit form of the spin and flavor wave functions of the four-quark subsystem specified in column 1 of Table 1 for every partition [FS]. The spin wave functions are trivial and not given in the paper. The flavor wave functions were obtained by analogy to the flavor wave functions of \( uudc \) derived in Ref. [25]. Here they are given in the Rutherford–Young–Yamanouchi basis in “Appendix D”. In this basis the vectors are written as linear combinations of products of symmetric or antisymmetric two-quark pairs, useful to calculate the matrix elements of the hyperfine interaction using Eq. (5).

After lengthy calculations we have obtained the diagonal matrix elements of the flavor-spin interaction (3) for the four quark states presented in Table 1, every expression representing the contribution of six pairs of quarks. All off-diagonal matrix elements vanish identically.

It is useful to recall that in the exact SU(5) limit, the flavor-spin interaction takes the following form

\[
V_F = - C_F \sum_{i < j} \lambda_i^F \lambda_j^F \bar{\sigma}_i \cdot \bar{\sigma}_j,
\]

with \( C_F \) an equal strength constant for all pairs. To reproduce the exact SU(5) limit of the hyperfine interaction in Table 1 one has to take \( V_{\pi}^{uu} = V_{\eta}^{uu} = V_{\eta c}^{uu} = V_{\eta b}^{uu} = V_{\eta}^{ub} = V_{\eta c}^{ub} = V_{\eta b}^{ub} = -C_F \) and \( V_{\eta}^{uu} = V_{\eta}^{ub} = 0 \). One obtains
Table 1 The hyperfine interaction $V_x$, Eq. (3), integrated in the flavor-spin space, for the quark subsystem $uudb$ with $I = 1/2$

| State | $V_x$ |
|-------|--------|
| $[1] = ([31]_O [22]_F [22]_S [4]_F S)$ | $15 V_u - V_{uu} - 2V_{uu} - \frac{1}{2} V_{uu} - \frac{1}{10} V_{uu} + 12V_{ub} + \frac{6}{5} V_{ub} - 2V_{q'q}$ |
| $[2] = ([31]_O [31]_F [31]_S [4]_F S)$ | $3 V_u + V_{uu} + 2V_{uu} + \frac{1}{2} V_{uu} + \frac{1}{5} V_{uu} + 14V_{ub} + 2V_{ub} - \frac{10}{2} V_{q'q}$ |
| $[3] = ([4]_O [21]_F [21]_S [31]_F S)$ | $7V_u - \frac{5}{2} V_{uu} - \frac{14}{18} V_{uu} - \frac{7}{18} V_{uu} - \frac{2}{20} V_{uu} + \frac{22}{3} V_{ub} + \frac{22}{15} V_{ub} - \frac{22}{9} V_{q'q}$ |
| $[4] = ([4]_O [31]_F [22]_S [31]_F S)$ | $-\frac{5}{2} V_u - \frac{5}{2} V_{uu} - \frac{10}{18} V_{uu} - \frac{5}{18} V_{uu} - \frac{5}{20} V_{uu} + \frac{22}{3} V_{ub} + \frac{26}{15} V_{ub} - \frac{26}{9} V_{q'q}$ |

$V_{q'q}$ are defined in Eq. (6) where the upper index $q_d q_b$ indicates the flavor of the interacting quark pair

$-\frac{132}{5} C_{R'} - \frac{104}{5} C_R - \frac{72}{5} C_{R'}$ and $-\frac{32}{5} C_R$ respectively. These values can be checked with the formulas given in “Appendix A”. They suggest that the lowest state is $[1]$. We shall see that even with a broken SU(5) symmetry the lowest state of $uudb$ has positive parity because it acquires the largest attraction due to the FS interaction, similar to the $uudc\bar{c}$ hidden charm pentaquark [25] or the open charm $uudd\bar{c}$ pentaquark [41].

3 Orbital Space

As first introduced in Ref. [41] the orbital wave functions are defined in terms of four internal Jacobi coordinates for pentaquarks chosen a

\[
\begin{align*}
\vec{x} &= \vec{r}_1 - \vec{r}_2, \\
\vec{y} &= (\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3) / \sqrt{3}, \\
\vec{z} &= (\vec{r}_1 + \vec{r}_2 + \vec{r}_3 - 3\vec{r}_4) / \sqrt{6}, \\
\vec{t} &= (\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4) / \sqrt{10},
\end{align*}
\]

where 1, 2, 3 and 4 are the quarks and 5 the antiquark so that $t$ gives the distance between the antiquark and the center of mass coordinate of the four-quark subsystem.

3.1 $P = +1$ Pentaquarks

The parity of the pentaquark is given by $P = (-)^{\ell + 1}$, where $\ell$ is the orbital angular momentum. The lowest positive parity pentaquark contains one unit of orbital excitation in the four-quark subsystem and has the symmetry $[31]_O$. The method of constructing translationally invariant states of definite permutation symmetry containing a unit of angular momentum was first given in Ref. [41] and later revised in Ref. [25]. The three independent states denoted below by $\psi_i$, which define the basis vectors of the irreducible representation $[31]_O$ in terms of shell model states $\langle \vec{r} | n \ell m \rangle$ where $n = 0$, $\ell = 1$, are

\[
\begin{align*}
\psi_1 &= \langle \vec{x} | 000 \rangle \langle \vec{y} | 000 \rangle \langle \vec{z} | 010 \rangle, \\
\psi_2 &= \langle \vec{x} | 000 \rangle \langle \vec{y} | 010 \rangle \langle \vec{z} | 000 \rangle, \\
\psi_3 &= \langle \vec{x} | 010 \rangle \langle \vec{y} | 000 \rangle \langle \vec{z} | 000 \rangle.
\end{align*}
\]

The pentaquark orbital wave functions $\psi_i^5$ are obtained by multiplying each $\psi_i$ from above by the wave function $\langle \vec{r} | 000 \rangle$ which describes the relative motion between the four-quark subsystem and the antiquark $\vec{r}$. Assuming an exponential behavior we introduce two variational parameters, $a$ for the internal motion of the four-quark subsystem and $b$ for the relative motion between the subsystem $qqgb$ and $\vec{b}$. We explicitly have

\[
\begin{align*}
\psi_1^5 &= N \exp \left[ -\frac{a}{2} \left( x^2 + y^2 + z^2 \right) - \frac{b}{2} t^2 \right] \ z \ Y_{10} (\hat{z}), \quad (12) \\
\psi_2^5 &= N \exp \left[ -\frac{a}{2} \left( x^2 + y^2 + z^2 \right) - \frac{b}{2} t^2 \right] \ y \ Y_{10} (\hat{y}), \quad (13) \\
\psi_3^5 &= N \exp \left[ -\frac{a}{2} \left( x^2 + y^2 + z^2 \right) - \frac{b}{2} t^2 \right] \ x \ Y_{10} (\hat{x}), \quad (14)
\end{align*}
\]
where
\[ N = \frac{2^{3/2}a^{11/4}b^{3/4}}{3^{1/2}\pi^{5/2}}. \] (15)

3.2 \( P = -1 \) Pentaquarks

The orbital wave function of the lowest negative parity state described by the \( s^4 \) configuration of symmetry \([4]_O \) is defined as
\[ \phi_0 = N_0 \exp \left[ -\frac{a}{2}(x^2 + y^2 + z^2) - \frac{b}{2}t^2 \right], \] (16)
with
\[ N_0 = (\frac{a}{\pi})^{9/4}(\frac{b}{\pi})^{3/4}. \] (17)

4 Kinetic Energy

The kinetic energy \( T \) of the Hamiltonian (1) can be calculated analytically. For \( P = +1 \) states the expectation value of the kinetic energy is defined by the average over the three wave functions defined by Eqs. (12)–(14). One obtains
\[ \langle T \rangle = \frac{\hbar^2}{3} \left[ \langle \psi_1^5 | T | \psi_1^5 \rangle + \langle \psi_2^5 | T | \psi_2^5 \rangle + \langle \psi_3^5 | T | \psi_3^5 \rangle \right] \] (18)
with
\[ \frac{4}{\mu_1} = \frac{3}{m_q} + \frac{1}{m_\bar{Q}}, \] (19)
and
\[ \frac{5}{\mu_2} = \frac{1}{\mu_1} + \frac{4}{m_\bar{Q}}, \] (20)
where \( q = u, d \) and \( Q = b \) with masses defined by Eq. (5).

For \( P = -1 \) states there is no orbital excitation and the orbital wave function of the four-quark subsystem has the permutation symmetry \([4]_O \). In this case Eq. (16) gives
\[ \langle T \rangle = \hbar^2 \left( \frac{9}{2\mu_1} a + \frac{3}{2\mu_2} b \right), \] (21)
with \( \mu_1 \) and \( \mu_2 \) as above.

5 Confinement

By integrating in the color space, the expectation value of the confinement interaction (2) has the same form as that of the \( uudc\bar{c} \) system [25]
\[ \langle V_{confl} \rangle = \frac{C}{2} \left( 6 \langle r_{12} \rangle + 4 \langle r_{145} \rangle \right) \] (22)
where \( \langle r_{ij} \rangle \) is the interquark distance and the coefficients 6 and 4 account for the number of quark-quark and quark-antiquark pairs, respectively, but the expression for \( \langle r_{ij} \rangle \) depends on parity.
For $P = +1$ one has

$$
\langle r_{ij} \rangle = \frac{1}{3} \left[ \langle \psi_1^5 \mid r_{ij} \mid \psi_1^5 \rangle + \langle \psi_2^5 \mid r_{ij} \mid \psi_2^5 \rangle + \langle \psi_3^5 \mid r_{ij} \mid \psi_3^5 \rangle \right],
$$

(23)

where $i, j = 1, 2, 3, 4, 5 \ (i \neq j)$. An analytic evaluation gives

$$
\langle r_{12} \rangle = \frac{20}{9} \sqrt{\frac{1}{\pi a}},
$$

(24)

and

$$
\langle r_{45} \rangle = \frac{1}{3 \sqrt{2\pi}} \left[ 2 \sqrt{\frac{3}{a}} + \frac{5}{b} + \sqrt{5b} \left( \frac{1}{2a} + \frac{1}{b} \right) \right].
$$

(25)

For $P = -1$ there is no orbital excitation so that the four quarks are in the $s^4$ configuration of permutation symmetry [4]. The expectation value of the confinement interaction is given by Eq. (22) as well, with

$$
\langle r_{12} \rangle = \sqrt{\frac{4}{\pi a}},
$$

(26)

and

$$
\langle r_{45} \rangle = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{3}{a} + \frac{5}{b}}.
$$

(27)

### 6 Flavor-Spin Interaction

For integrating the expressions of Table 1 in the orbital space one has to decouple the orbital part of the wave function $\langle f \rangle_0$ from the part containing the other degrees of freedom by again using Clebsch-Gordan coefficients (isoscalar factors) of the permutation group $S_4$ [42] but this time related to the orbital and the flavor-spin space. The use of the permutation properties of the translationally invariant orbital wave functions is necessary at this stage.

Including the orbital space, it turns out that for positive parity states with one unit of orbital excitation the result is a linear combination of two-body matrix elements of type $\langle ss \mid V_{qaqb}^{aib} \mid ss \rangle \langle sp \mid V_{qbq}^{aib} \mid sp \rangle$ and $\langle sp \mid V_{qbq}^{aib} \mid ps \rangle$. For negative parity states the hyperfine interaction contains only two-body matrix elements of type $\langle ss \mid V_{qaqb}^{aib} \mid ss \rangle$. In every term $qaqb$ is a pair of quarks from Eq. (6).

### 7 Results and Discussion

The lowest part of the mass spectrum of the $uud\bar{b}\bar{b}$ pentaquark given by Hamiltonian of Sect. 2 has been calculated variationally with the wave functions described in Sect. 3, containing the parameters $a$ and $b$. The flavor-spin part of each wave function is a product of a four quarks subsystem state defined in Table 1 and the bottom antiquark wave function denoted by $\mid \bar{b} \rangle$. The total angular momentum is $\vec{J} = \vec{L} + \vec{S} + \vec{s}_Q$, with $\vec{L}$ and $\vec{S}$ the angular momentum and the spin of the four-quark cluster respectively and $\vec{s}_Q$ is the spin of the heavy antiquark. In numerical calculations the expressions of the hyperfine interaction of Table 1 are simplified. The contribution of $V_{\eta c}^{uu}, V_{\eta b}^{uu}$ and $V_{\eta b}^{ub}$ are neglected because little $u\bar{u}$ and $d\bar{d}$ are expected in $\eta_c$ and $\eta_b$. We have also neglected $V_{\eta c}^{ub}$ assuming a little $b\bar{b}$ component in $\eta'$. Thus, in the expressions of Table 1 we took

$$
V_{\eta c}^{uu} = V_{\eta b}^{uu} = V_{\eta b}^{ub} = V_{\eta b}^{ub} = 0.
$$

(28)

The numerical results are presented in Table 2. The eigenvalues of $\mid 1 \rangle \mid \bar{b} \rangle$ and $\mid 2 \rangle \mid \bar{b} \rangle$ states are degenerate for the allowed values of $J$ in each case. For $\mid 2 \rangle \mid \bar{b} \rangle$ the states with $J^P = 1/2^+$ and $3/2^+$ have multiplicity 2. One can see that the lowest state has positive parity like for $uudc\bar{c}$ [25] and also for the $uudd\bar{c}$ pentaquark studied some time ago [41].
gives a measure of the compactness of the four quark subsystem because the mean distance between the levels is smaller than the relative positions in the hidden charm spectrum. But the mass difference absolute value being dependent on the input for the quark masses, as it is known. One can notice that the pentaquarks with charm and bottom. It is relevant to look at the relative positions in a given spectrum, the on the SU(4) flavor-spin model. In Table 4 we recall the masses of the three common states calculated for pentaquark spectrum and compare it with the spectrum of hidden charm pentaquark studied in Ref. [25] based on the SU(4) flavor-spin model. In Table 4 we recall the masses of the three common states calculated for pentaquarks with charm and bottom. It is relevant to look at the relative positions in a given spectrum, the absolute value being dependent on the input for the quark masses, as it is known. One can notice that the spectrum of the hidden bottom pentaquark is more compressed, in the sense that the relative position of the levels is smaller than the relative positions in the hidden charm spectrum. But the mass difference [2] |B⟩ − [1] |B⟩ is 180 MeV for hidden charm and 172 MeV for hidden bottom pentaquarks, thus not much different.

As the number of observed heavy flavor baryons is increasing there is hope that more heavy flavor pentaquarks will be searched for. In the present model it will be useful to study the strange hidden-bottom states.

Table 2 Lowest positive and negative parity uuddcb pentaquarks of isospin \( I = \frac{1}{2} \) and symmetry structures [1], [2], [3] and [4] defined in Table 1.

| State | \( J^P \) | Variational parameters | Mass (MeV) | \( \langle r_{12} \rangle \) (fm) | \( \langle r_{45} \rangle \) (fm) |
|-------|-----------|------------------------|-----------|-----------------|-----------------|
| [1] | \( \frac{1}{2}^+ , \frac{3}{2}^+ \) | \( a = 1.900 \) \( b = 1.387 \) | 10961 | 0.910 | 0.950 |
| [2] | \( \frac{1}{2}^+ , \frac{3}{2}^+ , \frac{5}{2}^+ \) | \( a = 1.387 \) \( b = 1.387 \) | 11133 | 1.065 | 1.018 |
| [3] | \( \frac{1}{2}^- \) | \( a = 1.027 \) \( b = 1.387 \) | 11112 | 1.113 | 1.019 |
| [4] | \( \frac{1}{2}^- \) | \( a = 0.514 \) \( b = 1.387 \) | 11334 | 1.575 | 1.226 |

Column 1 gives the state, column 2 the parity and total angular momentum, column 3 and 4 the optimal variational parameters associated to the wave functions defined in Sect. 3, column 5 the calculated mass and columns 6 and 7 the relative distance between quarks/antiquarks.

Table 3 Partial contributions of the Hamiltonian of Sect. 2 to the calculated masses (MeV) of the pentaquarks given in Table 2.

| State | Parity | \( K.E. \) | \( V_{conf} \) | \( V_{\chi} \) |
|-------|--------|----------|--------------|--------------|
| [1] | +1 | 970 | 432 | −781 |
| [2] | +1 | 723 | 487 | −416 |
| [3] | −1 | 457 | 501 | −186 |
| [4] | −1 | 254 | 669 | 71 |

The optimal value of the parameter \( a \) varies with the state but \( b \) is the same for all states. The parameter \( a \) gives a measure of the compactness of the four quark subsystem because the mean distance between the quarks is inverse proportional to the square root of \( a \), see Eq. (26). Thus the excited state [4] is less compact than the others.

The detailed contribution of different parts of the Hamiltonian are given in Table 3. One can see that, although the kinetic energy of the lowest positive parity state [1] |B⟩ is about twice larger than that of the lowest negative parity state [3] |B⟩, the flavor-spin interaction overcomes this excess and generates a lower eigenvalue of 10,961 MeV for the [31]O state having an \( s^4p \) configuration than for [4]O having an \( s^4 \) configuration, the eigenvalue of which is 11,112 MeV. The contribution of the pion exchange is dominant in \( V_{\chi} \) as compared to the other meson exchanges. In particular for the state [1] |B⟩ the coefficient of the pion exchange contribution is 15 which makes this state to be the lowest one. The \( B^{-} \) meson exchange contributes only with −60 MeV to the total value of −781 MeV for [1] |B⟩ and with −25 MeV to the total value of −186 MeV for [3] |B⟩. These quantities are not large, but important for the position of the state relative to the threshold. Another interesting remark is that the contribution of \( V_{\chi} \) to the mass of the higher negative parity state [4] |B⟩ is repulsive. This is due to the fact that the dominant pion exchange in the expression of Table 1 corresponding to [4] |B⟩ has negative sign, contrary to the other three cases.

In Table 2 we also exhibited the distance \( \langle r_{12} \rangle \) between a pair of quarks and the distance \( \langle r_{45} \rangle \) between a quark and the antiquark. For all considered states they have small and comparable values which indicates that the pentaquark is rather compact. For the state [4] |B⟩ the distance \( \langle r_{12} \rangle \) is larger which can be explained by the larger contribution of \( V_{conf} \) and the repulsive contribution of \( V_{\chi} \) shown in Table 3.

Besides the level ordering it would be useful to present a few more general features of the hidden bottom pentaquark spectrum and compare it with the spectrum of hidden charm pentaquark studied in Ref. [25] based on the SU(4) flavor-spin model. In Table 4 we recall the masses of the three common states calculated for pentaquarks with charm and bottom. It is relevant to look at the relative positions in a given spectrum, the absolute value being dependent on the input for the quark masses, as it is known. One can notice that the spectrum of the hidden bottom pentaquark is more compressed, in the sense that the relative position of the levels is smaller than the relative positions in the hidden charm spectrum. But the mass difference [2] |B⟩ − [1] |B⟩ is 180 MeV for hidden charm and 172 MeV for hidden bottom pentaquarks, thus not much different.
Table 4 Pentaquark masses (MeV) denoted by $M(uud\bar{Q})\ (Q = c, b)$ having quark flavor structures defined in Table 1

| State $|\bar{Q}\rangle$ | $J^P$ | $M(uudc\bar{c})$ | $M(uudb\bar{b})$ |
|-----------------|--------|------------------|------------------|
| $|1\rangle|\bar{Q}\rangle$ | $\frac{1}{2}^+, \frac{3}{2}^+$ | 4273             | 10961            |
| $|2\rangle|\bar{Q}\rangle$ | $\frac{1}{2}^+, \frac{3}{2}^+, \frac{5}{2}^+$ | 4453             | 11133            |
| $|3\rangle|\bar{Q}\rangle$ | $\frac{1}{2}^+$ | 4487             | 11112            |

Column 1 gives the state, column 2 the spin and parity, column 3 from Ref. [25] and 4 the present results.

Table 5 Masses of ground state baryons with the flavor-spin interaction of Sect. 2

| Baryon | $I$ | $J^P$ | Calc. Mass (MeV) | $a$(fm$^{-2}$) | Exp.mass (MeV) |
|--------|-----|--------|------------------|----------------|---------------|
| $\Lambda_b$ | 0   | $\frac{1}{2}^+$ | 5585             | 2.080          | 5620          |
| $\Sigma_b$  | 0   | $\frac{1}{2}^+$ | 5747             | 1.284          | 5813          |
| $\Sigma_b^*$| 0   | $\frac{3}{2}^+$ | 5773             | 1.284          | 5832          |

Column 1 gives the baryon, column 2 the isospin, column 3 the spin and parity column 4 the calculated mass, column 5 the variational parameter and the last column the experimental mass.

8 Conclusions

The present work is a natural extension of that of Ref. [25] where the spectrum of the $uudc\bar{c}$ pentaquark has been analyzed using the SU(4) flavor-spin model. Here we have extended the model to SU(5) and applied it to study the lowest part of the $uudb\bar{b}$ pentaquark spectrum. The model provides an isospin dependence and an internal structure of pentaquarks contrary to the molecular scenario. For positive parity states the angular momentum is located in the internal motion of the four-quark subsystem and it turns out that the lowest state has positive parity, as in the case of the hidden charm pentaquark. In particular, we found that the coupling between the two negative parity states $|3\rangle|\bar{b}\rangle$ and $|4\rangle|\bar{b}\rangle$ vanish identically, although the spin part is the same. The lowest state has a mass of 10,961 MeV and is located below the experimental thresholds $\Sigma_b + B$ (11,093 MeV) and $\Sigma_b^* + B$ (11,112 MeV), thus it is stable against the corresponding strong decays. The lowest state is stable as well against the theoretical threshold $\Sigma_b + B$. We recall that the mass of the quark $b$, given in Eq. (5), has been obtained from fitting the calculated mass of the $B$ and $B^*$ mesons which are degenerate, to the average experimental mass which is 5312 MeV. Together with the calculated mass of $\Sigma_b$ (“Appendix Table 5”) one obtains 11,049 MeV, i.e. above the lowest state mass.

The masses obtained in our study fall in a range similar to those of the previous studies [32–36] devoted solely to negative parity states. We present results both for positive and negative parity states. The lowest state can have $J^P = 1/2^+$ or $3/2^+$ and is stable against the decay into $\Sigma_b + B$.

There is a common feature with that of Ref. [36], namely that the pion exchange turns out to be important to the binding.

Our work is closest to that of Ref. [35] where the Hamiltonian contains both a chromomagnetic interaction due to one gluon exchange (OGE) and a hyperfine spin-flavor interaction due to Goldstone boson exchange (light meson exchange). As a result the positive parity pentaquarks are unbound and the lowest state has negative parity. This could be due to the fact that chromomagnetic interaction dominates over the spin-flavor part.

There exists however another, quite popular scenario, where the lowest state has positive parity. This is the hadrocharmonium model which has been used in Ref. [43] to interpret the $P_c$ pentaquarks observed at LHCb [2]. The lightest of them, the pentaquark $P_c(4312)$, was interpreted as a bound state of $\chi_0(1P)$ and a nucleon with total quantum numbers $I = 1/2$ and $J^P = 1/2^+$.
Possible future observations of $uudb\bar{b}$ pentaquarks will be essential in discriminating between the existing models or inspire new developments.

We hope that in the future the present work could stimulate some interest in studying pentaquarks containing quarks/antiquarks of five distinct flavors in various models. To our knowledge some specific pentaquarks, namely $udsc\bar{b}$ and $udsb\bar{c}$ have been studied so far in the molecular scenario, based on the assumption that the $P^N_{\psi}$ are molecular states [44]. Using the chain of unitary groups $SU(5) \supset SU(4) \supset SU(3)$ one can systematically search for the most stable pentaquark multiplets.

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Author Contributions  
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Declarations

Competing interests  
The authors declare no competing interests.

Appendix A: Exact SU(5) Limit

The exact SU(5) limit is useful in checking the integration in the flavor space, made in Table 1. In this limit every expectation value of Table 1 reduces to the expectation value of Eq. (7) and one can use the following formula [29]

$$\left\langle \sum_{i<j} \lambda^F_i \cdot \lambda^F_j \tilde{\sigma}_i \cdot \tilde{\sigma}_j \right\rangle = 4C^{SU(2n)}_2 - 2C^{SU(n)}_2 - \frac{4}{n} C^{SU(2)}_2 - k \frac{3(n^2 - 1)}{n}$$  \hspace{1cm} (A1)

where $n$ is the number flavors and $k$ the number of quarks, here $n = 5$ and $k = 4$. $C^{SU(n)}_2$ is the Casimir operator eigenvalues of $SU(n)$ which can be derived from the expression [45]:

$$C^{SU(n)}_2 = \frac{1}{2} \left[ f'_1 (f'_1 + n - 1) + f'_2 (f'_2 + n - 3) + f'_3 (f'_3 + n - 5) \right. $$

$$+ f'_4 (f'_4 + n - 7) + \cdots + f'_{n-1} (f'_{n-1} - n + 3) \left] - \frac{1}{2n} \left( \sum_{i=1}^{n-1} f'_i \right)^2 \right.$$  \hspace{1cm} (A2)

where $f'_i = f_i - f_n$, for an irreducible representation given by the partition $[f_1, f_2, \ldots, f_n]$. Equation (A1) has been previously used for $n = 3$ and $k = 6$ in Ref. [45].

Appendix B: The Baryons

The masses of $\Lambda_b^*$, $\Sigma_b^*$ and $\Sigma_b^{*\prime}$ relevant for this study were calculated variationally with a radial wave function of the form $\phi \propto \exp[-\frac{a}{2} (x^2 + y^2)]$ with the variational parameter $a$ and the coordinates $x$ and $y$ defined by Eq. (8). We took $V_{uu} V_{uu} = V_{uu} V_{ub} = V_{ub} V_{ub} = 0$ like for pentaquarks. The results are indicated in Table 5 together with the experimental masses. These masses can be used to estimate theoretical thresholds consistent with the model, as it is done in the conclusions. One can see that the experimental $\Sigma_b^* - \Sigma_b$ splitting is better reproduced than that obtained from the phenomenological flavor-spin interaction of Ref. [37], where a $B$-meson exchange was simply assumed by analogy to $D$-meson exchange.

Appendix C: SU(5) Generators

Here we reproduce the $\lambda_i$ matrices which are the SU(5) generators in the fundamental representation of SU(5). Implementing them in Eq. (3) one can obtain the two-body matrix elements of Eq. (6) for each pair of quarks of a given flavor.
The first 15 matrices are an extension of the SU(4) generators [46] with one 0-row and one 0-column added. We have

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_9 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_{10} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \lambda_{11} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{12} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_{13} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \lambda_{14} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, & \lambda_{15} &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}.
\end{align*}
\]

The additional matrices are

\[
\begin{align*}
\lambda_{16} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{17} &= \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{18} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_{19} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{20} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, & \lambda_{21} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}, \\
\lambda_{22} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{23} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}, & \lambda_{24} &= \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}.
\end{align*}
\]

The definition of the scalar mesons introduced in the Hamiltonian

\[
\eta_c = \frac{1}{\sqrt{12}}(u\bar{u} + d\bar{d} + s\bar{s} - 3c\bar{c}), \quad \eta_b = \frac{1}{\sqrt{20}}(u\bar{u} + d\bar{d} + s\bar{s} + c\bar{c} - 4b\bar{b})
\]

are consistent with the $\lambda_i$ matrices.
Appendix D: Flavor Wave Functions

The flavor wave functions of the $uudb$ subsystem can be obtained by analogy to those of the $uudc$ subsystem, given in Ref. [25], because they have similar Young tableaux. They are basis vectors for a fixed irrep [$f$]. It is convenient to write each basis vector in the Rutherford–Young–Yamanouchi form [46] in terms of products of symmetric $\phi_{12}(q_2q_4) = (q_2q_4 + q_4q_2)/\sqrt{2}$ or antisymmetric $\phi_{11}(q_2q_4) = (q_2q_4 - q_4q_2)/\sqrt{2}$ states for the pairs 12 and 34. This allows a forward calculation of the flavor integrated matrix elements (6) and in addition one can easily read off the isospin of the corresponding wave function. We assume that the order of particles is 1234 and the position of the particles in the corresponding Young tableau is denoted by the Yamanouchi symbol $Y = (r_1, r_2, \ldots, r_i)$ where $r_i$ represents the row of the particle $i$.

For the irrep [22] there are two basis vectors

$$[[22]]_F^{2121} = \sqrt{\frac{1}{6}}[\sqrt{2}\phi_{12}(uu)\phi_{12}(db) + \sqrt{2}\phi_{12}(db)\phi_{12}(uu) - \phi_{12}(ud)\phi_{12}(ub) - \phi_{12}(ub)\phi_{12}(ud)],$$  

(D1)

and

$$[[22]]_F^{2111} = \sqrt{\frac{1}{6}}[\phi_{11}(ud)\phi_{12}(uu) + \phi_{11}(ub)\phi_{11}(ud)],$$  

(D2)

where (D2) obviously has isospin $I = 1/2$ which means that the pairs 12 and 34 in (D1) have to couple to $I = 1/2$ as well.

For the irrep [31] there are three basis vectors. The first reads

$$[[31]]_F^{1211} = \sqrt{\frac{1}{3}}[\phi_{11}(db)\phi_{12}(uu) + \sqrt{2}\phi_{11}(ub)\phi_{12}(ud) - \phi_{12}(db)\phi_{12}(uu)],$$  

(D3)

where the pair 12 is in an antisymmetric state and 34 in a symmetric state.

For the remaining two basis vectors the last two particles are in different rows in the corresponding Young tableaux. Then in the Rutherford–Young–Yamanouchi basis the vectors are constructed such that the last two particles are either in a symmetric state or an antisymmetric state. In the notation of Ref. [46] we have

$$[[31]]_F^{1211} = \sqrt{\frac{1}{6}}[\phi_{12}(uu)\phi_{12}(db) + \sqrt{2}\phi_{12}(ud)\phi_{12}(ub) - \sqrt{2}\phi_{12}(db)\phi_{12}(uu)],$$  

(D4)

where the pair 34 is in a symmetric state, and

$$[[31]]_F^{1211} = -\sqrt{\frac{1}{3}}[\phi_{12}(uu)\phi_{11}(db) + \sqrt{2}\phi_{12}(ud)\phi_{11}(ub)],$$  

(D5)

where the pair 34 is in an antisymmetric state.

In a similar way we obtain the three basis vectors for the irrep [211]. They are

$$[[211]]_F^{3211} = \frac{1}{2}[\phi_{12}(uu)\phi_{11}(db) - \phi_{12}(ud)\phi_{11}(ub) + \phi_{12}(ub)\phi_{11}(ad)],$$  

(D6)

$$[[211]]_F^{3211} = \frac{1}{2}[\phi_{11}(ud)\phi_{12}(ub) + \sqrt{2}\phi_{11}(db)\phi_{12}(uu) - \phi_{11}(ub)\phi_{12}(ad)],$$  

(D7)
and

\[
[211]_F \frac{\bar{1}321} = \sqrt{\frac{T}{2}} \begin{cases} \phi_{1[1]}(ud) \phi_{1[1]}(ub) \\ + \phi_{1[1]}(ub) \phi_{1[1]}(ud) \end{cases}.
\]

(D8)

The states (D6)–(D8) have \( l = 1/2 \). A tentative of extending the Goldstone boson exchange model to SU(5) in order to apply it to heavy baryons has been made in [47].

References

1. R. Aaij et al., [LHCb Collaboration], Observation of \( J/\psi p \) resonances consistent with pentaquark states in \( \Lambda_b^0 \rightarrow J/\psi K^- p \) Decays. Phys. Rev. Lett. 115, 072001 (2015)
2. R. Aaij et al., [LHCb Collaboration], Observation of a narrow pentaquark state, \( P_c(4312)^+ \), and of two-peak structure of the \( P_c(4450)^+ \). Phys. Rev. Lett. 122(22), 222001 (2019)
3. R. Aaij et al., [LHCb], Evidence of a \( J/\psi \Lambda \) structure and observation of excited \( \Xi^- \) states in the \( \Xi^-_b \rightarrow J/\psi \Lambda K^- \) decay. Sci. Bull. 66, 1278–1287 (2021), arXiv:2012.10380 [hep-ex]
4. C. Chen, E. Spadaro Norella, LHCb-PAPER-2022-031. https://indico.cern.ch/event/1176505
5. T. Gershon [LHCb], Exotic hadron naming convention. arXiv:2206.15233 [hep-ex]
6. Z.H. Guo, J.A. Oller, Anatomy of the newly observed hidden-charm pentaquark states: \( P_c(4312), P_c(4440) \) and \( P_c(4457) \). Phys. Lett. B 793, 144 (2019)
7. F.K. Guo, H.J. Jing, U.G. Meißner, S. Sakai, Isospin breaking decays as a diagnosis of the hadronic molecular structure of the \( P_c(4457) \). Phys. Rev. D 99(9), 091501 (2019)
8. C.J. Xiao, Y. Huang, Y.B. Dong, L.S. Geng, D.Y. Chen, Exploring the molecular scenario of \( P_c(4312), P_c(4440) \), and \( P_c(4457) \). Phys. Rev. D 100(1), 014022 (2019)
9. C.W. Xiao, J. Nieves, E. Oset, Heavy quark spin symmetric molecular states from \( \bar{D}^+(1)\Sigma_c^* \) and other coupled channels in the light of the recent LHCb pentaquarks. Phys. Rev. D 100(1), 014021 (2019)
10. Y. Shimizu, Y. Yamaguchi, M. Harada, Heavy quark spin multiplet structure of \( P_c(4312), P_c(4440) \), and \( P_c(4457) \). arXiv:1904.00587 [hep-ph]
11. Y.H. Lin, B.S. Zou, Strong decays of the latest LHCb pentaquark candidates in hadronic molecule pictures. Phys. Rev. D 100, 056005 (2019)
12. M.Z. Liu, Y.W. Pan, F.Z. Peng, M. Sánchez Sánchez, L.S. Geng, A. Hosaka, M. Pavyon Valderrama, Emergence of a complete heavy-quark spin symmetry multiplet: seven molecular pentaquarks in light of the latest LHCb analysis. Phys. Rev. Lett. 122, 242001 (2019)
13. L. Meng, B. Wang, G.J. Wang, S.L. Zhu, The hidden charm pentaquark states and \( \Sigma_c \bar{D}^{(*)} \) interaction in chiral perturbation theory. Phys. Rev. D 100, 014031 (2019)
14. Q. Wu, D.Y. Chen, Production of \( P_c \) states from \( \Lambda_b \) decay. Phys. Rev. D 100, 114002 (2019)
15. M. Pavyon Valderrama, One pion exchange and the quantum numbers of the \( P_c(4440) \) and \( P_c(4457) \) pentaquarks. Phys. Rev. D 100, 094028 (2019)
16. J.L. Du, Y. Barn, F.K. Guo, C. Hanhart, U.G. Meißner, J.A. Oller, Q. Wang, Interpretation of the LHCb \( P_c \) States as Hadronic Molecules and Hints of a Narrow \( P_c(4380) \). Phys. Rev. Lett. 124, 072001 (2020)
17. G.J. Wang, L.Y. Xiao, R. Chen, X.H. Liu, X. Liu, S.L. Zhu, Probing hidden-charm decay properties of \( P_c \) states in a molecular scenario. arXiv:1911.09613 [hep-ph]
18. H. Xu, Q. Li, C. H. Chang, G. L. Wang, Recently observed \( P_c \) as molecular states and possible mixture of \( P_c(4457) \). arXiv:2001.02980 [hep-ph]
19. H. Xu, C. H. Chang, G. L. Wang, Decay properties of \( P_c \) states through the Fierz rearrangement. arXiv:2001.09563 [hep-ph]
20. T.J. Burns, E.S. Swanson, Molecular Interpretation of the \( P_c(4440) \) and \( P_c(4457) \) States. Phys. Rev. D 100(11), 114033 (2019)
21. C. Fernandez-Ramirez et al., [JPAC Collaboration], Interpretation of the LHCb \( P_c(4312)^+ \) signal. Phys. Rev. Lett. 123, 092001 (2019)
22. A. Ali, A.Y. Parkhomenko, Interpretation of the narrow \( J/\psi p \) Peaks in \( \Lambda_b \rightarrow J/\psi p K^- \) decay in the compact diquark model. Phys. Lett. B 793, 365 (2019)
23. X.Z. Weng, X.L. Chen, W.Z. Deng, S.L. Zhu, Hidden-charm pentaquarks and \( P_c \) states. Phys. Rev. D 100, 016014 (2019)
24. J.B. Cheng, Y.R. Liu, \( P_c(4457)^+ \), \( P_c(4440)^+ \), and \( P_c(4312)^+ \): molecules or compact pentaquarks? Phys. Rev. D 100, 054002 (2019)
25. F. Stancu, Spectrum of the \( uudc \bar{c} \) hidden charm pentaquark with an SU(4) flavor-spin hyperfine interaction. Eur. Phys. J. C 79(11), 957 (2019)
26. L.Y. Glozman, D.O. Riska, The Spectrum of the nucleons and the strange hyperons and chiral dynamics. Phys. Rep. 268, 263 (1996)
27. L.Y. Glozman, Z. Papp, W. Plessas, Light baryons in a constituent quark model with chiral dynamics. Phys. Lett. B 381, 311 (1996)
28. L.Y. Glozman, Z. Papp, W. Plessas, K. Varga, R.F. Wagenbrunn, Light and strange baryons in a chiral constituent-quark model. Nucl. Phys. A 623, 90C (1997)
29. E. Ortiz-Pacheco, R. Bijker, C. Fernandez-Ramirez, Hidden charm pentaquarks: mass spectrum, magnetic moments, and photocouplings. J. Phys. G 46(6), 065104 (2019)
30. H.X. Chen, W. Chen, X. Liu, S.L. Zhu, The hidden-charm pentaquark and tetraquark states. Phys. Rep. 639, 1–121 (2016)
31. H.X. Chen, W. Chen, X. Liu, Y.R. Liu, S.L. Zhu, An updated review of the new hadron states. arXiv:2204.02649 [hep-ph]
32. J.J. Wu, L. Zhao, B.S. Zou, Prediction of super-heavy $N^*$ and $\Lambda^*$ resonances with hidden beauty. Phys. Lett. B 709, 70–76 (2012)
33. V. Kopeliovich, I. Potashnikova, Simple estimates of the masses of pentaquarks with hidden beauty or strangeness. Phys. Rev. D 93, 074012 (2016)
34. J. Wu, Y.R. Liu, K. Chen, X. Liu, S.L. Zhu, Hidden-charm pentaquarks and their hidden-bottom and $B_s$-like partner states. Phys. Rev. D 95(3), 034002 (2017)
35. G. Yang, J. Ping, J. Segovia, Hidden-bottom pentaquarks. Phys. Rev. D 99(1), 014035 (2019)
36. Y. Yamaguchi, A. Giachino, A. Hosaka, E. Santopinto, S. Takeuchi, M. Takizawa, Hidden-charm and bottom meson-baryon molecules coupled with five-quark states. Phys. Rev. D 96(11), 114031 (2017)
37. L.Y. Glozman, D.O. Riska, The Charm and bottom hyperons and chiral dynamics. Nucl. Phys. A 603, 326 (1996) Erratum: [Nucl. Phys. A 620 (1997) 510]
38. R.L. Workman [Particle Data Group], Review of particle physics. PTEP 2022, 083C01 (2022). https://doi.org/10.1093/ptep/ptac097
39. S. Pepin, F. Stancu, M. Genovese, J.M. Richard, Tetraquarks with color blind forces in chiral quark models. Phys. Lett. B 393, 119 (1997)
40. M. Genovese, J.M. Richard, F. Stancu, S. Pepin, Heavy flavor pentaquarks in a chiral constituent quark model. Phys. Lett. B 425, 171 (1998)
41. F. Stancu, Positive parity pentaquarks in a Goldstone boson exchange model. Phys. Rev. D 58, 111501 (1998)
42. F. Stancu, S. Pepin, Isoscalar factors of the permutation group. Few Body Syst. 26, 113 (1999)
43. M.I. Eides, V.Y. Petrov, M.V. Polyakov, New LHCb pentaquarks as hadrocharmonium states. Mod. Phys. Lett. A 35(18), 2050151 (2020)
44. F.Z. Peng, M.Z. Liu, Y.W. Pan, M. Sánchez Sánchez, M. Pavon Valderrama, Five-flavor pentaquarks and other light- and heavy-flavor symmetry partners of the LHCb hidden-charm pentaquarks. Nucl. Phys. B 983, 115936 (2022). arXiv:1907.05322 [hep-ph]
45. F. Stancu, S. Pepin, L.Y. Glozman, The Nucleon-nucleon interaction in a chiral constituent quark model. Phys. Rev. C 56, 2779 (1997)
46. F. Stancu, Group theory in subnuclear physics. Oxford Stud. Nucl. Phys. 19, 1 (1996)
47. J.P. Day, W. Plessas, Ki-Seok Choi, Universal constituent quark model for baryons. arXiv:1205.6918

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