We consider a one-dimensional system consisting of electrons with short-ranged repulsive interactions and coupled to small-momentum transfer acoustic phonons. The interacting electrons are bosonized and described in terms of a Luttinger liquid which allows to calculate exactly the one- and two-electron Green function. For non-interacting electrons, the coupling to phonons induces a singularity at the Fermi surface which is analogous to that encountered for electrons with an instantaneous attractive interaction. The exponents which determine the presence of singlet/triplet superconducting pairing fluctuations, and spin/charge density wave fluctuations are strongly affected by the presence of the Wentzel–Bardeen singularity, resulting in the favoring of superconducting fluctuations. For the Hubbard model the equivalent of a phase diagram is established, as a function of: the electron–phonon coupling, the electron filling factor, and the on-site repulsion between electrons. The Wentzel–Bardeen singularity can be reached for arbitrary values of the electron–phonon coupling constant by varying the filling factor. This provides an effective mechanism to push the system from the antiferromagnetic into the metallic phase, and finally into the superconducting phase as the electron filling factor is increased towards half filling.

PACS Numbers: 72.10.-d; 05.30.Fk;73.20.Dx

I. INTRODUCTION

In the light of recent technological advances in nanostructure fabrication, there has been a renewed interest in the properties of one-dimensional electron systems. A two-dimensional electron gas created in a GaAs/AlGaAs heterostructure can be confined in its lateral dimension by means of metallic gates, yielding to a good approximation a one-dimensional wire. Up to recently, the theoretical study of quasi-one-dimensional mesoscopic systems has focused mostly on non-interacting electrons. In the present work, we will consider the properties of a system of electrons in one dimension interacting via a short ranged repulsive potential, which in turn interacts with phonons through a deformation potential coupling. The coupling to the phonons gives rise to a retarded attractive electron–electron interaction which opens the possibility for superconducting fluctuations. On the other hand, the instantaneous repulsion between electrons may be so strong that only spin density wave (SDW) and charge density wave (CDW) fluctuations can survive. In addition, the type of fluctuations which dominate will depend crucially on the electron filling factor. We will show that the interplay between the phonon–mediated attractive interaction and the instantaneous repulsive interaction can be studied systematically in one dimension, and thus obtaining the equivalent of a phase diagram for the fluctuations.

Aside from the experimental motivations mentioned above, one-dimensional models can provide useful an-
swers: they are in general easier to handle than their higher dimensional analogues, for which a full treatment of the correlation effects is often difficult. While the one-dimensionality strictly rules out the possibility of long range order, it is still legitimate to ask what type of ordering fluctuations exist in such systems, such as singlet and triplet pairing fluctuations, or spin density wave and charge density wave fluctuations. This question was addressed for a purely instantaneous interaction between electrons in the seventies, and later on for non-interacting electrons coupled to phonons. More recently, the attention has focused on interacting electrons coupled to phonons.

The interest in the latter systems is by no means new: static properties of 1D interacting electron systems coupled to low energy acoustic phonons were considered four decades ago by Wentzel and Bardeen as a possible candidate for the theory of superconductivity. The electrons were described in terms of Tomonaga bosons, and it was shown that for a critical value of the electron–phonon coupling constant, the system becomes unstable and acquires a negative compressibility. The thermodynamic quantities for the electron-phonon system (within the Einstein model) were studied near this singular point, indicating the presence of a phase transition: the divergence of the specific heat is accompanied by a collapse of the system induced by the strong electron–phonon interaction. We shall refer to this point as the Wentzel–Bardeen singularity in the course of this work. Nevertheless, the Wentzel–Bardeen singularity was not taken too seriously in the context of superconductivity, as the (strong) coupling needed to trigger this dramatic behavior was judged to have unphysical values.

More recently, a system similar to that of ref. was examined, with the conclusion that the coupling to low energy acoustic phonons can be neglected in most systems because the effects associated with the phonons contain the small parameter $c/v_F$, where $c$ is the speed of sound and $v_F$ is the Fermi velocity. As a result, this type of coupling ceased to receive much attention, and the interest shifted towards higher momentum processes where phonons transfer (or backscatter) an electron from one side of the Fermi surface to the other side: phonon processes. These processes are understood to be at the origin of the Peierls instability. Nevertheless, in the present work, we shall come back to the case of low energy acoustic phonons, and will point out that for strongly correlated electron systems, this type of coupling can have spectacular effects as one approaches the Wenztel–Bardeen singularity. For strongly correlated systems, the effects associated with the phonons do not depend on the small parameter $c/v_F$: rather, the Fermi velocity $v_F$ is replaced by the charge velocity $u_\rho$ associated with low energy particle–hole excitations close to the Fermi surface. In contrast to $v_F$, $u_\rho$ can depend strongly on the electron filling factor: for the Hubbard model, $u_\rho$ vanishes at half filling whereas $v_F$ does not. Near half filling, the decay of $u_\rho$ is most dramatic for low values of the on site repul-
sion parameter $U$, i.e. the system is extremely sensitive to the filling factor. As a consequence, the effects associated with these low energy phonons can no longer be dismissed as a small correction. Below, we will argue that: 1) the Wentzel–Bardeen singularity can be reached for arbitrary non-zero value of the electron–phonon coupling constant as one approaches half filling in the Hubbard model. 2) Near the Wentzel–Bardeen singularity, CDW and SDW fluctuations are strongly suppressed, and the system is first pushed towards an intermediate (metallic) phase, and eventually into the superconducting phase.

The low energy properties of a one–dimensional electron system can be treated rigorously using the so–called bosonization technique. This approach has been applied recently to a variety of problems in mesoscopic physics: the transport properties of quasi–one–dimensional interacting electrons systems \[16,17\], the persistent current of a one–dimensional ring of interacting fermions \[18,19\] among others. In the Luttinger liquid picture, the field operator describing the electrons are characterized by a charge field and a spin field, which refer to the two different types of elementary excitations which are present in the system at low temperatures. In the Luttinger liquid approximation, these excitations are totally decoupled, and each is characterized by a separate velocity. Because of this, charge and spin excitations can be localized at different points in space. For the particular type of interaction considered here, the coupling to the phonons occurs only with the charge degrees of freedom. The collective excitations describing the charge–phonon dynamics are then characterized by two distinct velocities, which depend on the charge velocity, the sound velocity, and the electron–phonon coupling. This property is reminiscent of the phenomenon of hybridization. As a consequence, the electronic system coupled to phonons is characterized by three distinct velocities.

The first part of this paper will be concerned with the calculation of the single–electron Green function where the phonon part is integrated out. The corresponding Green function for interacting spinless fermions coupled to phonons has been calculated in the early eighties within perturbation theory, assuming a linear spectrum near the Fermi surface \[23\], and the results were expanded in the ratio $c/v_F$. Here, we shall use a functional integral approach, where the phonons can be explicitly traced out of the problem. This is the equivalent of summing up all phonon diagrams of the perturbation series \[23\]. The results for the Green function can in turn be used to determine the the momentum distribution function of the electrons, which is of the Fermi–Dirac type for a free electron gas. However, both the instantaneous interaction and the phonon–mediated retarded interaction contribute to drive the system away from the free electron behavior. This is in analogy with the early work on bosonization which showed that for one–dimensional electron systems, the momentum distribution function departs from the Fermi-Dirac distribution for arbitrarily small coupling \[24\].
In a second part, we will focus on the many body properties of the system. Two–particle correlation functions in Matsubara representation can be computed with the same method, yielding the long range and long (imaginary) time behavior of the singlet and triplet pairing correlation function, as well as of the the SDW and CDW correlation functions. For an uncorrelated fermion system in the limit of vanishing temperature, arbitrarily small phonon coupling leads to a divergence of the Fourier transform of the pairing correlation function at low momentum and low frequency, which is precisely the signature for superconducting fluctuations [2]. This picture will be modified when we include repulsive interactions. For an interacting electron system with a short range repulsive potential between electrons and with a filling factor between zero and half filling, but not close to these values, the onset of superconducting fluctuations correspond in general to a finite value of the electron–phonon coupling constant. This threshold value of the phonon coupling is directly related to the parameters of the electronic model which is considered. As an example, we study the Hubbard model, where the Luttinger liquid parameters can be computed exactly. This will allow us to study the instabilities of the system for filling factors close to half filling, and therefore to study the sensibility to the Wentzel–Bardeen singularity.

Finally, we can apply our results to study the behavior of two coupled electron system or alternatively a electron system coupled to a hole system, with a local interaction. The density–density coupling has then precisely the same form as for electrons coupled to low energy phonons, and consequently there is the possibility that one system may induce superconducting fluctuations in the other. While it is possible that the inclusion of $2k_F$ phonons could modify the behavior of the electron–phonon system, for two coupled electron systems, there is no such possibility: the Fermi velocities of each mode are fixed and distinct from each other, and there is no cancellation of the fast varying components of the density operator. Only the slow varying component of these operators has to be taken into account, and our model is “exact” in this sense.

The paper is organized as follows: in Sec. II, the Hamiltonian of the coupled electron–phonon system is described, and a brief review of the bosonization technique is presented. In Sec. III, we introduce the functional integral formalism to calculate the thermodynamic quantities. Sec. IV is devoted to the calculation the single particle Green function. The ordering correlation functions are considered in Sec. V. We illustrate our results with the Hubbard model in Sec. VI. The relevance of our results for coupled electron chains are discussed in Sec. VII. We conclude in Sec. VIII.
II. DESCRIPTION OF THE MODEL

The starting point of our approach is a system of interacting electrons on a lattice coupled to acoustic phonons in one dimension, with periodic boundary conditions. The Hamiltonian describing the acoustic phonons is given by:

\[ H_p = \frac{1}{2} \int dx \left[ \zeta^{-1} \Pi_d^2 + \zeta c^2 (\partial_x d)^2 \right], \tag{2.1} \]

where \( d \) is the phonon field operator, \( \Pi_d \) is its canonical conjugate, \( \zeta \) is the mass density of the lattice and \( c \) is the speed of sound.

Let \( \psi_s \) be the field operator for fermions with spin \( s \). In the deformation potential approximation, the electron–phonon coupling is given by [25]:

\[ H_{el-p} = g \sqrt{\frac{\pi}{2}} \sum_s \int dx \psi_s^\dagger \psi_s \partial_x d, \tag{2.2} \]

where \( g \) is the electron–phonon coupling constant (the numerical factor \( \sqrt{\pi/2} \) is introduced to simplify later algebra). As mentioned in the introduction, alternative choices for the phonon interaction are possible, as is the case when one studies conducting polymers [26,14] or molecular crystals [15]. However, for mesoscopic systems such as GaAs heterostructures in the metallic regime and at low temperatures \( T < 1K \), the acoustic phonons which are coupled by the deformation potential to the electronic degrees of freedom constitute a realistic model of the lowest energy modes of the system.

We now turn to the interacting electron system. It has been shown [27] that the low energy and long wave length behavior of a system of spinless fermions with short range interaction can be described by a continuum Hamiltonian expressed in terms of bosonic fields. A similar treatment exists for fermions with spin [3,4]. We restrict ourselves to correlated electron models with nearest neighbor hopping and short–ranged repulsion/attraction. The field operator describing electrons with spin \( s = \pm 1/2 (= \uparrow, \downarrow) \) is decomposed into a right and left moving part:

\[ \psi_s(x) = e^{ikF x} \psi_{s+}(x) + e^{-ikF x} \psi_{s-}(x), \tag{2.3} \]

The right (left) moving field operator \( \psi_{s+} (\psi_{s-}) \) is then expressed as an exponential of bosonic fields:

\[ \psi_{s\pm}(x) = \frac{1}{\xi L} \exp(i\sqrt{\pi}(\pm \varphi_s(x) - \theta_s(x))) \), \tag{2.4} \]

where \( \theta_s(x) = \int^x dx' \Pi_s(x') \), and the bosonic field \( \Pi_s \) is the canonical conjugate of \( \varphi_s \). We then introduce the bosonic fields describing the charge and spin fields:

\[ \varphi_\rho = \frac{1}{\sqrt{2}} (\varphi_{\uparrow} + \varphi_{\downarrow}), \tag{2.5a} \]

\[ \varphi_\sigma = \frac{1}{\sqrt{2}} (\varphi_{\uparrow} - \varphi_{\downarrow}), \tag{2.5b} \]
and their respective canonical conjugates $\Pi_\rho$ and $\Pi_\sigma$. With these definitions the electron density operator becomes [28]:

$$\sum_s \psi_s^\dagger \psi_s = \sqrt{\frac{2}{\pi}} \partial_x \varphi_\rho,$$  \hspace{1cm} (2.6)

where we have kept only the slow spatial variations of the density operator. Expressed in terms of the variables in Eqs. (2.5a) and (2.5b), and their canonical conjugate, the Hamiltonian describing the interacting electron separates into a charge and spin contribution. Each contribution is of the sine–Gordon type. Schulz [29,30] has argued that these Hamiltonians can be replaced by “free”, quadratic Hamiltonians, provided that the parameters of each contributions are properly renormalized by the interactions. The quadratic Hamiltonian of Ref. [29] is our starting point:

$$H_{el} = H_\rho + H_\sigma,$$  \hspace{1cm} (2.7)

with

$$H_\nu = \frac{1}{2} \int dx \left[ u_\nu K_\nu \Pi_\nu^2 + \frac{u_\nu}{K_\nu} (\partial_x \varphi_\nu)^2 \right],$$  \hspace{1cm} (2.8)

with $\nu = \rho, \sigma$. The free electron gas limit corresponds to $K_\rho = K_\sigma = 1$ and $u_\rho = u_\sigma = v_F$, the Fermi velocity of non–interacting electrons. Note that we have omitted an additional cosine term [29] in the spin channel: this marginally irrelevant contribution will be discussed briefly in Sec. V B.

For the Hubbard model with on–site repulsion, the justification for this “free” Hamiltonian [29] is based on the fact that the large $U$ and the small $U$ limit represent the same phase of the model, and there are no singular points between these two limits. Consequently the interaction renormalize to the “fixed point” $K_\sigma = 1$. The remaining parameters $K_\rho$, $u_\rho$ and $u_\sigma$ of the Hamiltonian in Eq. (2.7) can be obtained by solving numerically the equations describing the ground [31] and excited states [32], as will be shown later in Sec. VI. For other, more general models where an exact solution is not available, the parameters $u_\rho$, $u_\sigma$, $K_\rho$, $K_\sigma$ can be obtained, for example, by finite size diagonalization for small systems.

The electron–phonon coupling originates from the electrostatic interaction between the electrons and the atoms of the solid. Consequently, the phonons couple only to $H_\rho$, since it is the total charge density from up and down spins that matters.

### III. LAGRANGIAN FORMULATION

To calculate thermodynamic quantities, we use a Lagrangian formulation [33] where the partition function is represented as a functional integral over the phonon (ph) displacement field and the bosonic fields associated with
the charge and spin degrees of freedom. For example, if we wish to calculate the thermodynamic average of a functional $f[\hat{\phi}^\nu, \hat{\Pi}^\nu]$ which depends on the conjugate operators $\hat{\phi}^\nu$ and $\hat{\Pi}^\nu$, with $\nu = \rho, \sigma$, we can express this average in terms of a functional integral as follows

$$<: f[\hat{\phi}^\nu, \hat{\Pi}^\nu] :> = Z^{-1} \int D\hat{\phi}^\nu D\hat{\Pi}^\nu DdD\Pi_d \exp\{-S_E[\phi^\nu, \Pi^\nu, d, \Pi_d]\} \times f[\phi^\nu, \Pi^\nu],$$

(3.1)

where $Z$ is the partition function. The symbol $:\hat{f}:$ denotes the normal ordered form of $\hat{f}$ with the momentum operators brought to the left of their conjugate field operators. We shall see below that this type of normal ordering only leads to oscillatory phase factors and does not affect the asymptotic behavior (in the thermodynamic limit) which is our main interest. These oscillatory terms, however, are relevant for finite size effects [18]. The Euclidean action $S_E$ can be decomposed as follows:

$$S_E[\phi^\nu, \Pi^\nu, d, \Pi_d] = \int_0^\beta d\tau \int_0^L dx [L_{\text{ph}} + L_{\rho} + L_{\sigma} + L_{\rho-\text{ph}}].$$

(3.2)

Here, $\tau$ is the imaginary time coordinate ($\beta^{-1} = k_B T$). The corresponding Lagrangian densities for the phonons, the charge and spin degrees of freedom, and the coupling term are defined by:

$$L_{\text{ph}} = \frac{1}{2k} \Pi_d^2 + \frac{\zeta^2}{2} (\partial_x d)^2 - i\Pi_d \partial_x d$$

(3.3a)

$$L_{\nu} = \frac{2K}{2} \Pi^\nu + \frac{u_{\nu}}{2K} (\partial_x \phi^\nu)^2 - i\Pi^\nu \partial_x \phi^\nu$$

(3.3b)

$$L_{\rho-\text{ph}} = g(\partial_x d)(\partial_x \phi^\rho),$$

(3.3c)

with $\nu = \rho, \sigma$. Note that the Lagrangian density describing the system is quadratic in the fields, and the functional integral of Eq. (3.1) will be performed exactly. This is a direct consequence of the bosonization method, as the fermion density is linear in $\partial_x \phi^\rho$ (see Eq. (2.6)).

In principle, the functional integrals of Eqs. (3.1) implicitly contains summations over the winding numbers associated with the zero-modes of the fields [18]. These zero-modes, however, shall be ignored here as we are calculating only bulk quantities, but they should be included in the calculation of any finite size effects. Consequently, we shall omit in the Fourier sums below the $k = 0$ contribution (this constraint becomes irrelevant in the thermodynamic limit).

The cross terms $-i\Pi_d \partial_x d$ and $-i\Pi_d \partial_x \phi^\nu$ in Eqs. (3.3a) and (3.3c) can be eliminated if we redefine the canonical conjugates as follows:

$$\Pi_d \rightarrow \Pi_d - i\zeta \partial_x d$$

$$\Pi^\nu \rightarrow \Pi^\nu - i\partial_x \phi^\nu / u_{\nu} K^\nu$$

The Lagrangian densities for the phonon, charge, and spin degrees of freedom then become:
\[
L_{\mu} = \frac{1}{2\zeta} \Pi_{\mu}^2 + \frac{\zeta c^2}{2}(\partial_x d)^2 + \frac{\zeta}{2}(\partial_t d)^2
\]

(3.4)

\[
L_{\nu} = \frac{u_{\nu} K_{\nu}}{2} \Pi_{\nu}^2 + \frac{u_{\nu}}{2K_{\nu}}(\partial_x \varphi_{\nu})^2 + \frac{1}{2u_{\nu} K_{\nu}}(\partial_t \varphi_{\nu})^2.
\]

(3.5)

Later on, in the calculation of the Green function and the correlation functions, we will have to keep in mind this shift in the canonical conjugate \(\Pi_{\nu}\).

As we shall later be concerned with the calculation of one and two particle Green function for electrons only, we can integrate out the phonon degrees of freedom at this point. We must therefore perform the following Gaussian integral:

\[
I_1[\varphi_{\rho}] = \int Dd \exp \left[ -\int d\mathbf{x} \left( \frac{\zeta}{2} d[-\partial_t^2 - c^2 \partial_x^2]d \\
- gd\partial_x^2 \varphi_{\rho} \right) \right],
\]

(3.6)

where \(d\mathbf{x} = dx d\tau\). Denoting by \([-\partial_t^2 - c^2 \partial_x^2]^{-1}\) the inverse of the operator acting on the phonon field on the right hand side of Eq. (3.6), the functional integration over the phonon field gives:

\[
I_1[\varphi_{\rho}] = c_1 \exp \left[ \frac{g^2}{2\zeta} \int d\mathbf{x} d\mathbf{x}' (\partial_x^2 \varphi_{\rho})(\mathbf{x})[-\partial_t^2 - c^2 \partial_x^2]^{-1}(\mathbf{x}, \mathbf{x}') (\partial_x^2 \varphi_{\rho})(\mathbf{x}') \right].
\]

(3.7)

where \(c_1 = [\det(-\partial_t^2 - c^2 \partial_x^2)]^{-1/2}\). We can calculate the propagator in Eq. (3.7) by going to a Fourier representation for the charge field thereby assuming periodic boundary conditions in space and imaginary time:

\[
\varphi_{\rho}(\mathbf{x}) = \frac{1}{\beta L} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \varphi_{\rho}(\mathbf{k}),
\]

(3.8)

with \(\mathbf{k} = (k, \omega), \mathbf{x} = (x, \tau)\) and \(\mathbf{k} \cdot \mathbf{x} = kx + \omega t\), and where the \(k = 0\) term corresponding to the zero-mode is to be omitted henceforth. We then obtain the contribution for the effective action for the charge degrees of freedom:

\[
\bar{S}[\varphi_{\rho}] = \frac{1}{2\beta L} \sum_{\mathbf{k}} D_{\rho}(\mathbf{k}) |\varphi_{\rho}(\mathbf{k})|^2,
\]

(3.9)

where we have introduced the inverse propagator for the charge field \(\varphi_{\rho}\):

\[
D_{\rho}(\mathbf{k}) = \frac{1}{K_{\rho} u_{\rho}} \left( \frac{\omega^2 + u_{\rho}^2 k^2}{\omega^2 + c^2 k^2} \right),
\]

(3.10)

where \(b = g \sqrt{K_{\rho} u_{\rho}/\hbar \zeta}\) (with \(\hbar\)'s restored) is a constant which has the dimension of a velocity squared. The first two terms of Eq. (3.10) represent the contribution of the uncoupled charge field. The coupling to phonons therefore creates an additional quadratic contribution to the action, which contains all retardation effects: a collective excitation of the interacting electron system may
create a phonon at time $t$ and reabsorb it at a later time $t'$. We note that $k^2/D_\rho(k)$ is the effective charge density propagator in Fourier space. This retarded interaction is always attractive, and will be responsible for the formation of Cooper pairs. We also note that this retarded interaction term enters with a minus sign irrespective of the sign of the coupling constant $g$. As a result, an instability arises at zero-frequency and when $u_\rho$ approaches $b/c$ (from above) because $D_\rho$ then approaches zero, signaling an instability in the charge density propagator, $k^2/D_\rho(k)$, towards long wave-length density fluctuations. This instability has been known for a long time and has first been recognized by Wentzel [10] and later by Bardeen [11], its consequences for the thermodynamic (static) properties of a non-interacting system have been discussed in Ref. [22]. This instability will be of great importance in our following discussion of the dynamic properties of the Luttinger liquid, in particular with respect to its various possible phases characterized by the retarded response functions.

Had we treated the electrons with conventional fermion operators the integration of the phonon variables would have resulted in an effective interaction which is quartic in the fermion operators. Moreover, in problems where a single particle is coupled to a phonon bath [34], the “influence functional” which results from the functional integration of the bath variables is in general quite complicated. The particle is then coupled to an effective potential which contains “memory effects” [35,36], and one has to take recourse to approximation methods to understand how the coupling to the phonon bath affects the single particle motion. This is not the case here: a one-dimensional electron system (with or without interactions) coupled to phonons can be treated exactly if one uses the bosonization description for the charge and spin fields.

**IV. SINGLE PARTICLE PROPERTIES**

In this section we discuss the effects of the electron–phonon interaction on the single-particle properties. In particular, we show that an arbitrarily small coupling to phonons immediately drives the system into a non-Fermi liquid state, even in the absence of electron interactions.

**A. Green function**

We proceed to calculate the single–particle Green function:

$$G_s(x, \tau) = - \langle T \psi_s(x, \tau) \psi_s^\dagger(0, 0) \rangle \quad (4.1)$$

where $T$ is the imaginary time ordering operator. Using the decomposition of the fermion operators into right and left moving components (Eq. [2.4], the Green function becomes:
\[ G_s(x, \tau) = - \sum_{\alpha, \alpha' = \pm} e^{i k_F x} < T \psi_{s\alpha}(x, \tau) \psi^\dagger_{s\alpha'}(0, 0) > , \]

(4.2)

where \( \psi_{s\alpha}(x, \tau) \) is expressed in terms of bosonic variables as in Eq. (2.4). The diagonal contribution \( \alpha = \alpha' \) turns out to be the dominant one. We thus need to calculate the expression

\[ < T \psi_{s\alpha}(x, \tau) \psi^\dagger_{s\alpha}(0, 0) > = \frac{1}{\epsilon L} < e^{-i \sqrt{\frac{\pi}{2}} \left( \theta_{s\rho}(x, \tau) - \theta_{s\rho}(0, 0) \right)} e^{i \alpha \sqrt{\frac{\pi}{2}} \left( \phi_{s\rho}(x, \tau) - \phi_{s\rho}(0, 0) \right)} > \]

(4.3)

with the charge and spin fields introduced in Eqs. (2.5a) and (2.5b), and with \( s_{\rho} = 1 \) and \( s_{\sigma} = s \). The last line on the r.h.s. of Eq. (4.3) is a c–number which arises from the normal ordering procedure, and, moreover, contains the information about the time ordering of the Fermi operators:

\[ g_n(x, \tau) = \pm \prod_{\nu = \rho, \sigma} e^{\alpha \pi \left[ \theta_{\nu}(0, 0), \phi_{\nu}(x, \tau) \right]/2} , \quad \text{for} \quad \tau > 0 \]

\[ g_n(x, \tau) = - \prod_{\nu = \rho, \sigma} e^{\alpha \pi \left[ \theta_{\nu}(x, \tau), \phi_{\nu}(0, 0) \right]/2} , \quad \text{for} \quad \tau < 0 . \]

(4.4)

The normal ordering contribution, Eq. (4.3), involves the commutators between \( \theta_{\nu} \) and \( \phi_{\nu} \) evaluated at unequal times. The calculation of these commutators is given in Appendix A, where also the explicit time dependence of the field operators is derived. The normal ordering contribution then becomes:

\[ g_n(x, \tau) = \text{sgn}(\tau) \prod_{\nu = \rho, \sigma} e^{-\alpha \pi \text{sgn}(\tau) \left[ \theta_{\nu}(x, \tau), \phi_{\nu}(0, 0) \right]/2} \]

\[ = \text{sgn}(\tau) e^{\alpha \pi \text{sgn}(\tau)/L} , \quad x \neq 0 , \]

(4.5)

where the explicit form of the commutator is given in Eq. (A8a,A8b). For later use we also give the limiting result for \( \tau \to 0^\pm \),

\[ g_n(x, 0^\pm) = \pm \left( \frac{1 + \bar{z}(x)}{1 + z(x)} \right)^{\pm \alpha/2} , \]

(4.6)

where \( z(x) = \exp(2\pi i x/L - \epsilon) (\epsilon L \text{ is a short– distance cutoff of the order of the lattice constant}) \). Eq. (4.6) is valid for arbitrary \( x \).

Next we compute the expectation value in Eq. (4.3). Going to the Fourier representation,

\[ \Pi_{\nu}(x) = \frac{1}{\beta L} \sum_{k} e^{i k \cdot x} \Pi_{\nu}(k) , \]

(4.7)

we perform the integral over the canonical momenta \( \Pi_{\rho}, \Pi_{\sigma} \) with the result:
\[ < \exp \left[ - i s_\nu \sqrt{\frac{\pi}{2}} \left( \int dx' \Pi_\nu(x', \tau) - \int dx' \Pi_\nu(x', 0) \right) \right] > \]
\[ = \exp \left[ - \frac{\pi}{4 \beta L u_\nu K_\nu} \sum_{k, \omega} \left| e^{i k \cdot x} - 1 \right|^2 \frac{1}{k^2} \right]. \tag{4.8} \]

Note that the exponent on the r.h.s. of Eq. (4.8) contains a negative infinite constant due to the summation over Matsubara frequencies, however, as we shall see below, this contribution is exactly cancelled when integrating out the field \( \varphi_\nu \).

The functional integral over \( \varphi_\nu \) gives the contribution:

\[ I_\nu(x) = < \exp \left[ \alpha s_\nu i \sqrt{\frac{\pi}{2}} \left( \varphi_\nu(x, \tau) - \varphi_\nu(0, 0) \right) \right. \]
\[ + \left. \sqrt{\frac{\pi}{2 u_\nu K_\nu}} \left( \int dx' \partial_\tau \varphi_\nu(x', \tau) - \int dx' \partial_\tau \varphi_\nu(x', 0) \right) \right] > , \tag{4.9} \]

where the last term in the exponential comes from the shift in the conjugate momentum as discussed in Sec. III.

The propagator for the spin field \( \varphi_\sigma \) is given by

\[ D_\sigma(k) = K_\sigma^{-1}(u_\sigma^{-1}\omega^2 + u_\sigma k^2). \tag{4.10} \]

Using this result and Eq. (3.10) we obtain for \( I_\nu \)

\[ I_\nu(x) = \exp \left[ \frac{\pi}{4 \beta L} \sum_k D_\nu^{-1}(k) \right. \]
\[ \times \left( \left| e^{i k \cdot x} - 1 \right|^2 \left( \frac{\omega^2}{k^2 u_\nu^2 K_\nu^2} - 1 \right) \right. \]
\[ \left. + \frac{4 i \alpha s_\nu \omega}{k u_\nu K_\nu} \cos(k \cdot x) \right]. \tag{4.11} \]

The first term in the exponent of Eq. (4.11) diverges due to the sum over Matsubara frequencies. This term is positive, and the divergent part is cancelled by the contribution of Eq. (4.8). To see this it is useful to write:

\[ \frac{\omega^2}{k^2 D_\nu(k)} = \frac{\omega^2 - u_\nu K_\nu D_\nu(k)}{k^2 D_\nu(k)} + \frac{u_\nu K_\nu}{k^2}. \tag{4.12} \]

When inserted in Eq. (4.11), the second term on the right hand side cancels exactly the contribution of Eq. (4.8).

The thermodynamic average in Eq. (4.3) then becomes:

\[ < T \psi_\nu^\dagger(x) \psi_\nu(0) > = \frac{1}{\epsilon L} \prod_{\nu = \rho, \sigma} \exp \left[ - \frac{\pi}{4 \beta L} \sum_k D_\nu^{-1}(k) \left( \left| e^{i k \cdot x} - 1 \right|^2 \left( 1 + \frac{u_\nu K_\nu D_\nu(k) - \omega^2}{u_\nu^2 K_\nu^2 k^2} \right) \right) \right]. \]
\[ + \frac{4i\omega s_P}{u_v k_P} \sin(kx) \sin(\omega \tau) ] \]
\[ \times g_n(x) . \]  

Note that the exponent in Eq. (4.13) has a real and imaginary part coming from both the thermodynamic average and the normal ordering procedure. This implies that the Green function has an oscillatory component and decays with a power law as expected. The frequency and momentum sums in Eq. (4.13) are performed in Appendix B. The result is:

\[
<T\psi_{s\alpha}(x)\psi_{s\alpha}^\dagger(0) >= \frac{1}{\epsilon L} \exp \left[ - \frac{1}{4} \sum_{\beta=\pm} \left( \frac{C_\beta}{K_\sigma} + \frac{K_\sigma u_\rho}{v_\beta} F_\beta \right) \ln \left| 1 - z(x + iv_\beta |\tau|) \right| / \epsilon \right]
\]
\[ - \frac{1}{4} \left( K_\sigma + 1 \right) \ln \left| 1 - z(x + iu_\sigma |\tau|) \right| / \epsilon \]
\[ - \text{sgn}(\tau) \frac{\alpha s}{4} \ln \frac{1 - \bar{z}(x + iu_\sigma |\tau|)}{1 - z(x + iu_\sigma |\tau|)} \]
\[ - \text{sgn}(\tau) \frac{\alpha}{4} \sum_{\beta=\pm} F_\beta \ln \frac{1 - \bar{z}(x + iv_\beta |\tau|)}{1 - z(x + iv_\beta |\tau|)} \]
\[ \times g_n(x) . \]  

We have introduced effective velocities induced by the phonon coupling:

\[ v_\pm^2 = \frac{1}{2} (u_\rho^2 + c^2) \mp \frac{1}{2} \sqrt{(u_\rho^2 - c^2)^2 + 4b^2} , \]  

as well as the abbreviations:

\[
C_\pm = \frac{u_\rho v_\pm^2 - c^2 + b^2/u_\rho^2}{v_\pm^2 - v_\pm^2} \quad (4.16a)
\]
\[
F_\pm = \frac{v_\pm^2 - c^2}{v_\pm^2 - v_\pm^2} . \]  

The thermodynamic limit corresponds to taking \(|x + iv\tau|/L << 1\), which in turn implies \(1 - z(y) \to -i2\pi y/L + \epsilon\). We can now give the final expression for the Green function in the thermodynamic limit using Eq. (4.14a) and Eq. (4.2):

\[
g_s(x, \tau) = \frac{1}{\epsilon L} \left| \frac{\epsilon L/2\pi}{x + iu_\sigma \tau} \right|^{K_\sigma/4 + 1/4K_\sigma}
\]
\[ \times \prod_{\beta=\pm} \left| \frac{\epsilon L/2\pi}{x + iv_\beta \tau} \right|^{K_\rho u_\rho F_\beta/4v_\beta + C_\beta/4K_\rho}
\]
\[ \times \sum_{\alpha=\pm} e^{i\alpha k_F x} \left( \frac{s\alpha x + iu_\sigma |\tau|}{|x + iu_\sigma |\tau|} \right)^{\text{sgn}(\tau)/2}
\]
\[ \times \prod_{\gamma=\pm} \left( \frac{s\gamma x + iv_\gamma |\tau|}{|x + iv_\gamma |\tau|} \right)^{\text{sgn}(\tau)F_\gamma/2} g_n(x, \tau) . \]  

Near \(x = 0\) one has to add \(\epsilon L/2\pi\) to \(v|\tau|\). At large distances and/or times \(g_n\) becomes unity and the power
law decay of the Green function becomes then (in obvious notation)

\[ G_s(x) \propto |x, \tau|^{-1-\delta} , \quad (4.18) \]

with

\[ \delta = \frac{K_{\sigma}}{4} + \frac{1}{4K_{\rho}} - 1 + \frac{B}{4K_{\rho}} + \frac{AK_{\rho}}{4}, \quad (4.19) \]

and the electron–phonon parameters A and B are defined by

\[ A(u_{\rho}, c, b) \equiv \sum_{\beta=\pm} \frac{u_{\rho}}{v_{\beta}} F_{\beta} = \frac{u_{\rho}}{v_+ + v_-} \left( 1 + \frac{c^2}{v_+ v_-} \right) \quad (4.20a) \]

\[ B(u_{\rho}, c, b) \equiv \sum_{\beta=\pm} C_{\beta} = \frac{u_{\rho}}{v_+ + v_-} \left( 1 + \frac{v_+ v_-}{u_{\rho}^2} \right) \quad (4.20b) \]

These exponents play an important role in the following discussion of the ordering fluctuations. For later use we only note here their limiting behavior. Defining a critical charge velocity by the equality \( b/u_{\rho} c = 1 \), or alternatively:

\[ \frac{u_{\rho}^*}{K_{\rho}^*} = \frac{g^2}{\hbar \zeta c^2} , \quad (4.21) \]

we see from Eq. (4.13) that \( v_+^2 \) tends to zero, and \( v_-^2 \) to \( u_{\rho}^* + c^2 \), as \( u_{\rho} \) approaches this critical velocity \( u_{\rho}^* \) from above. As a consequence, the exponent \( A \) diverges to infinity and \( B \) decreases to the finite value \( \sqrt{u_{\rho}^2 + c^2} \) as \( u_{\rho} \to u_{\rho}^* \). For \( u_{\rho} < u_{\rho}^* \) the velocities become complex and the model becomes unphysical. Thus we must require that \( u_{\rho}/K_{\rho} \geq u_{\rho}^*/K_{\rho}^* \), or equivalently that \( b/(cu_{\rho}) \leq 1 \), the equality sign defines the Wentzel–Bardeen singularity \[37\]. We emphasize that this singular behavior is a non-perturbative effect of the electron-phonon coupling and originates from the instability of the effective electron propagator as pointed out after Eq. (3.10). We also note here that the limits \( g \to 0 \) and \( u_{\rho} \to u_{\rho}^* \) do not commute.

In the limit of vanishing phonon coupling constant, the charge and spin propagator have an identical structure, and \( A, B \to 1 \). The result is:

\[ \lim_{g \to 0} G_s(x) \propto |x, \tau|^{-K_{\rho}/4 - 1/4K_{\rho} - K_{\sigma}/4 - 1/4K_{\sigma}} . \quad (4.22) \]

In particular, for non–interacting electrons \( K_{\rho} = K_{\sigma} = 1 \), we recover the \( 1/x \) power law dependence of the Green function of the free electron gas.

To illustrate our results with a specific model of interacting electrons, we plot the quantity \( \delta \) for the Hubbard model in Fig. 1a and 1b. The Luttinger liquid parameters \( K_{\rho}, u_{\rho} \) are determined numerically later on in Sec. V4.
We first plot \( \delta \) for a quarter filled band as a function of \( U \) in Fig. 1a, for several values of the phonon coupling parameter. For small phonon coupling, \( \delta \) increases monotonically with increasing \( U \). As the phonon coupling is further increased, \( \delta \) acquires a minimum.

Next, we choose \( U = 2 \) and vary the filling factor from zero to half filling in Fig. 1b. \( \delta \) increases dramatically when these two extremes are reached, as the correlation effects between electrons dominate the physics in both cases. In particular, \( \delta \to \infty \) at the Wentzel–Bardeen singularity.

**B. Momentum distribution function**

We now examine the effect of the phonon coupling on the momentum distribution function which is given by

\[
N_\alpha (k) \equiv \frac{1}{2} \sum_s \int_{-\infty}^{+\infty} dx \ e^{i(k-\alpha k_F)x} \langle \psi_{s\alpha}^\dagger(x,0)\psi_{s\alpha}(0) \rangle .
\]

(4.23)

Next, we insert the result of Eqs. (4.18) and (4.19) at \( \tau = 0^- \), use that in the thermodynamic limit \( g_n(x,0^\pm) = \pm i\alpha |x + i\alpha|/(x + i\alpha) \) (see Eq. (4.6)), and find (setting \( a_o = \epsilon L/2\pi \)),

\[
N_\alpha (k) = -\alpha \int_{-\infty}^{+\infty} \frac{dx}{2\pi i} \frac{e^{i(k-\alpha k_F)x}}{|x + i\alpha a_o|^\delta} a_o^\delta .
\]

(4.24)

As pointed out in Ref. [39], for a system with \( \delta > 0 \), the exponential in the integrand of Eq. (4.24) is not necessary any more to insure the convergence of the integral. For \( k \approx \alpha k_F \), we may omit this oscillatory factor, and obtain the distribution function evaluated at \( k = \alpha k_F \)

\[
N_\alpha (\alpha k_F) = \int_0^\infty \frac{dy}{\pi} (1 + y^2)^{-1+\delta/2} ,
\]

\[
= \frac{\Gamma(1/2 + \delta/2)}{2\sqrt{\pi} \Gamma(1 + \delta/2)} ,
\]

(4.25)

where \( \Gamma(x) \) denotes the Gamma function. In the limit \( \delta \to 0 \), \( N_\alpha (\alpha k_F) = 1/2 \), which differs from the Fermi step function obtained for the free system at \( \delta = 0 \). The momentum distribution function has a finite value at the Fermi wave vector, but its derivatives are all singular at that point (see below). This result was previously found in Ref. [39] for a system of interacting fermions but without phonons. The fact that the momentum distribution function has no jump at the Fermi surface for interacting fermions expresses the failure of the Fermi liquid quasiparticle picture to capture the low temperature behavior of interacting one–dimensional systems, which has been known for the Luttinger model since the pioneering work of Mattis and Lieb [24].

We can further determine the behavior of the momentum distribution function around \( k = k_F \) by noting that
\[ N_+(k) = \frac{1}{\pi} \int_0^\infty dx \frac{\cos(a_0 x (k - k_F))}{(x^2 + 1)^{1+\delta/2}} \cdot (1 + \mathcal{O}(\sqrt{|k - k_F|})). \]

(4.26)

Evaluating the tabulated integral we then find
\[ N(k \simeq k_F) \simeq N_+(k_F) - \text{csgn}(k - k_F)|k - k_F|^\delta, \]

(4.27)

with \( c = \alpha^n \delta^{-1/2} \Gamma(-1 - \delta)/\Gamma(-\delta/2) \). Eq. (4.27) exhibits the singular dependence of the momentum distribution function at \( k = k_F \). Note that for the pure electron system (\( g \rightarrow 0 \)), taking \( K_{\sigma} = 1 \), we recover the exponent of the momentum distribution function given in Ref. [28], namely:
\[ \delta \rightarrow \frac{K_{\rho}}{4} + \frac{1}{4K_{\rho}} - \frac{1}{2}. \]

(4.28)

C. Free electron gas coupled to phonons

In Eq. (4.19), both the effect of the interaction between electrons and the effect of the coupling with the phonons are included. In this section, we specialize to the case of free electrons coupled to phonons in order to clarify the role played by the phonons. In this case, \( \delta \) takes the form:
\[ \delta = A/4 + B/4 - 1/2. \]

(4.29)

It is convenient to rewrite \( A \) in the following way:
\[ A = B + \frac{b^2}{u_\rho v_+ v_- (v_+ + v_-)}. \]

(4.30)

Noting that
\[ B^2 = 1 - b^2/[u_\rho (v_+ + v_-)]^2 \leq 1, \]

(4.31)

we calculate
\[ (A + B)^2 = 4 + \left( \frac{b}{v_+ v_- (v_+ + v_-)} \right)^2 \left( \frac{b^2}{u_\rho^2} + 4v_+ v_- \right). \]

(4.32)

With Eq. (4.29), and \( A, B \geq 0 \) this implies that \( \delta > 0 \), for \( g \neq 0 \). If we start from a free Fermi gas, and couple it to phonons, the system is driven away from the free Fermi behavior and the Fermi distribution function is totally destroyed for an arbitrarily small phonon coupling strength \( g \).

V. ORDERING FLUCTUATIONS IN ONE–DIMENSION

In this section, we determine which type of fluctuations dominate in an interacting electron gas coupled to phonons.
A. Correlation functions

The tendency of the system towards ordering manifests itself in divergent correlation functions, which describe the response of the system to external perturbations. Thus, to characterize the type of long-range order (collective state), which might be realized in higher dimensions, we follow standard practice \cite{3,4,29} and consider response functions of the type

\[ R(x,t) = -i\Theta(t) < [O^\dagger(x,t),O] >, \]  

(5.1)

where the charge-density wave (CDW) response is generated by the operator, \( O_{CDW} = [\psi^\dagger_{+\uparrow}\psi_{-\uparrow} + \psi^\dagger_{+\downarrow}\psi_{-\downarrow}] / 2 \), the transverse spin-density wave (SDW) response by \( O_{SDW} = \psi^\dagger_{+s}\psi_{-s} \), singlet pairing (SS) by \( O_{SS} = \psi^\dagger_{+s}\psi^\dagger_{-s} \), and triplet pairing (TS) by \( O_{TS} = \psi^\dagger_{+s}\psi^\dagger_{-s} \).

The corresponding two-particle Green functions in Matsubara representation read then:

\[ N(x,\tau) = -ie^{i2kFx} < T\psi^\dagger_{-\uparrow}(x,\tau)\psi_{+\uparrow}(x,\tau)\psi^\dagger_{+\downarrow}\psi_{-\downarrow} > \]  

(5.2a)

\[ \chi(x,\tau) = -ie^{i2kFx} < T\psi^\dagger_{-\downarrow}(x,\tau)\psi_{+\uparrow}(x,\tau)\psi^\dagger_{+\downarrow}\psi_{-\uparrow} > \]  

(5.2b)

\[ \Delta_s(x,\tau) = -i < T\psi_{-\uparrow}(x,\tau)\psi_{+\uparrow}(x,\tau)\psi^\dagger_{+\downarrow}\psi^\dagger_{-\downarrow} > \]  

(5.2c)

\[ \Delta_t(x,\tau) = -i < T\psi_{-\downarrow}(x,\tau)\psi_{+\uparrow}(x,\tau)\psi^\dagger_{+\uparrow}\psi^\dagger_{+\downarrow} >, \]  

(5.2d)

with the notation \( \psi_{\pm s} \) defined in Eq. (2.4). \( N \) represents the correlation function associated with the CDW state, which for example occurs when the nearest neighbor repulsion is large compared to the other parameters of the model. \( \chi \) represents the transverse SDW response function, which describes the onset of antiferromagnetic ordering. \( \Delta_s \) (\( \Delta_t \)) gives the probability amplitude for a singlet (triplet) Cooper pair emitted at the origin of space and time to be annihilated at \( x \) and \( \tau \), and thus characterizes the superconducting fluctuations. To simplify notation we have retained only a representative term in the correlation functions, and this term will determine the correct small frequency/momentum behavior of the whole correlation function.

Using the bosonized version of \( \psi_{\pm s} \) in conjunction with the decomposition into charge and spin of Eqs. (2.5a) and (2.5b), we obtain (for \( \tau > 0 \)):

\[ N(x,\tau) = -ie^{i2kFx} \left( \frac{\pi}{(\epsilon L)^2} \right) \prod\limits_{\nu=\rho,\sigma} \exp \left[ i\sqrt{2\pi}(\varphi_{\nu}(x,\tau) - \varphi_{\nu}(0,0)) \right] > \]  

(5.3a)

\[ \chi(x,\tau) = -ie^{i2kFx} \left( \frac{\pi}{(\epsilon L)^2} \right) \exp \left[ -i\sqrt{2\pi} \left( \int_0^x dx' \Pi_{\rho}(x',\tau) - \int_0^x dx' \Pi_{\sigma}(x',0) \right) \right] \exp \left[ i\sqrt{2\pi}(\varphi_{\rho}(x,\tau) - \varphi_{\rho}(0,0)) \right] > \]  

(5.3b)

\[ \Delta_s(x,\tau) = \frac{-i}{(\epsilon L)^2} \exp \left[ -i\sqrt{2\pi} \left( \int_0^x dx' \Pi_{\rho}(x',\tau) - \int_0^x dx' \Pi_{\rho}(x',0) \right) \right] \]  

(5.3c)

\[ \Delta_t(x,\tau) = \frac{-i}{(\epsilon L)^2} \exp \left[ -i\sqrt{2\pi} \left( \int_0^x dx' \Pi_{\rho}(x',\tau) - \int_0^x dx' \Pi_{\rho}(x',0) \right) \right] \]  

(5.3d)
\[ \Delta_l(x, \tau) = \frac{-i}{(\epsilon L)^2} \exp \left[ -i \sqrt{2\pi} \left( \int dx' \Pi_\rho(x', \tau) - \int^0 dx' \Pi_\rho(x', 0) \right) \right] \exp \left[ -i \sqrt{2\pi} \left( \int dx' \Pi_\sigma(x', \tau) - \int^0 dx' \Pi_\sigma(x', 0) \right) \right] >, \] 

(5.3c)

We therefore omit the details of this calculation. Also, because we are only interested in the long distance and long time properties of the correlation functions, we have ignored the oscillatory terms arising from the normal ordering procedure. The results are:

\[ N(x, \tau) = -\frac{i e^{i 2k F x}}{(\epsilon L)^2} \exp \left[ -K_\sigma \sum_{\beta=\pm} \frac{u_\rho}{v_\beta} F_\beta \ln \left| 1 - z(x + iv_\beta \tau) \right| / \epsilon \right. \]

\[ \left. -K_\sigma \ln \left| 1 - z(x + iv_\sigma \tau) \right| / \epsilon \right] \] 

(5.4a)

\[ \chi(x, \tau) = -\frac{i e^{i 2k F x}}{(\epsilon L)^2} \exp \left[ -K_\rho \sum_{\beta=\pm} \frac{u_\rho}{v_\beta} F_\beta \ln \left| 1 - z(x + iv_\beta \tau) \right| / \epsilon \right. \]

\[ \left. -\frac{1}{K_\sigma} \ln \left| 1 - z(x + iv_\sigma \tau) \right| / \epsilon \right] \] 

(5.4b)

\[ \Delta_s(x, \tau) = -\frac{i}{(\epsilon L)^2} \exp \left[ -\frac{1}{K_\rho} \sum_{\beta=\pm} C_\beta \ln \left| 1 - z(x + iv_\beta \tau) \right| / \epsilon \right. \]

\[ \left. -K_\sigma \ln \left| 1 - z(x + iv_\sigma \tau) \right| / \epsilon \right] \] 

(5.4c)

\[ \Delta_l(x, \tau) = -\frac{i}{(\epsilon L)^2} \exp \left[ -\frac{1}{K_\rho} \sum_{\beta=\pm} C_\beta \ln \left| 1 - z(x + iv_\beta \tau) \right| / \epsilon \right. \]

\[ \left. -\frac{1}{K_\sigma} \ln \left| 1 - z(x + iv_\sigma \tau) \right| / \epsilon \right] \] 

(5.4d)

with \( C_\pm \) and \( F_\pm \) defined in Eqs. (4.16a) and (4.16b).

The thermodynamic limit is then taken to obtain the power law behavior of the correction functions:

\[ N(x, \tau) \propto e^{i 2k F x} \prod_{\beta=\pm} \left| x + iv_\beta \tau \right|^{-u_\rho K_\rho / v_\beta} \left| x + iv_\sigma \tau \right|^{-K_\sigma} \] 

(5.5a)

\[ \chi(x, \tau) \propto e^{i 2k F x} \prod_{\beta=\pm} \left| x + iv_\beta \tau \right|^{-u_\rho K_\rho / v_\beta} \left| x + iv_\sigma \tau \right|^{-1 / K_\sigma} \] 

(5.5b)

\[ \Delta_s(x, \tau) \propto \prod_{\beta=\pm} \left| x + iv_\beta \tau \right|^{-C_\beta / K_\rho} \left| x + iv_\sigma \tau \right|^{-K_\sigma} \] 

(5.5c)

\[ \Delta_l(x, \tau) \propto \prod_{\beta=\pm} \left| x + iv_\beta \tau \right|^{-C_\beta / K_\rho} \left| x + iv_\sigma \tau \right|^{-1 / K_\sigma} \] 

(5.5d)
B. Divergence criteria

The indication that a given type of fluctuations dominates in the system is provided by the divergence of the Fourier transform of the corresponding correlation function at low frequencies and small momentum relative to \( q = 2k_F \) (\( q = 0 \)) for \( N \) and \( \chi(\Delta_s \text{ and } \Delta_t) \).

Inspecting Eqs. (5.5a) to (5.5d), we notice that such a divergence can only originate from the power law behavior of the correlation functions at large distances and imaginary times. Whether or not there is a divergence then depends on the exponents which were calculated in the previous section. In particular, we note that

\[
|x + iv_{\text{max}}\tau|^{-\gamma_k} \leq |C_k(x, \tau)| \leq |x + iv_{\text{min}}\tau|^{-\gamma_k},
\]

(5.6a)

where \( C_k \) is given by the corresponding rhs of Eqs. (5.5a) to (5.5d), \( \gamma_k \) by the associated sum of exponents, and \( v_{\text{max/min}} = \max/\min\{v_+, v_-, u_\sigma\} \). From this, we obtain immediately the divergence criteria for the Fourier transforms of the correlation functions. The condition for CDW fluctuations to occur is then, from Eq. (5.5a):

\[
AK_{\rho} + K_\sigma \leq 2 . \quad \text{(CDW)} \tag{5.7}
\]

From Eq. (5.5b), SDW fluctuations occur when

\[
AK_{\rho} + \frac{1}{K_\sigma} \leq 2 . \quad \text{(SDW)} \tag{5.8}
\]

From Eq. (5.5c), singlet Cooper pairs fluctuations are present in the system when:

\[
\frac{B}{K_\rho} + K_\sigma \leq 2 . \quad \text{(singlet)} \tag{5.9}
\]

Finally, we will have triplet superconducting fluctuations when:

\[
\frac{B}{K_\rho} + \frac{1}{K_\sigma} \leq 2 . \quad \text{(triplet)} \tag{5.10}
\]

For models with interactions which do not depend on the spin (\( SU(2) \) symmetry), such as the Hubbard model, \( K_\sigma = 1 \), and the criteria for CDW and for SDW order (for singlet and for triplet pairing) are a priori the same. However we expect SDW and triplet Cooper pairs to dominate: CDW order will not occur because SDW order builds up before the longer range type of fluctuations have time to settle in the system [3]. Similarly, triplet superconductivity will dominate over singlet superconductivity. This is not noticeable in the criteria of Eqs. (5.7) to (5.10) because additional logarithmic corrections have been neglected here [40]. For the electronic models we are considering in this paper, these corrections arise from marginally irrelevant operators in the spin channel, which effectively generate longer tails for the SDW (triplet) correlation function than for the CDW (singlet) correlation functions. These operators have been omitted in Eq. (2.8) for simplicity. If both divergence criteria for
spin waves and triplet Cooper pairs are met for a given set of parameters, the correlation function “which diverges the most” determines the phase of the system. Note that the criteria of Eqs. (5.7)–(5.10) are valid for any interacting electron system with a finite range interaction: what remains to be done is to study the dependence of $K_\rho$ and $u_\rho$ on the interaction parameter(s) of the electron Hamiltonian and the filling factor for the electrons. Note that for repulsive interactions, we have $K_\rho < 1$.

At $K_\rho = K_\sigma = 1$, the criteria for SDW order and triplet pairing take the simple form:

$$A \leq 1 \quad \text{(SDW)} \quad (5.11a)$$
$$B \leq 1 \quad \text{(Cooper pairs)} \quad (5.11b)$$

from the results of Sec. IV C and noting that

$$A^2 = \frac{1 - b^2/[c(v_+ + v_-)]^2}{1 - b^2/(cu_\rho)^2}, \quad (5.12)$$

with $u_\rho < v_+ + v_-$ and $c > 0$, we conclude that $A > 1$, and Eq. (5.11a) is never satisfied, while Eq. (5.11b) is always satisfied according to Eq. (4.31). The Cooper instability is always present for a system of non-interacting electrons, regardless of the magnitude and sign of the phonon coupling constant.

When no phonons are present in the system, $A = B = 1$, and a system of electrons with repulsive on–site interaction always has $K_\rho < 1$, which implies from the inequality (5.8) that SDW (CDW) fluctuations dominate over superconducting fluctuations, as expected.

A check of the above results can be obtained by considering non–interacting electrons ($K_\rho = K_\sigma = 1$) coupled to phonons in the limit where the speed of sound is much larger than the Fermi velocity. If we set $c \to \infty$, keeping $b/cu_\rho$ finite ($0 < b/cu_\rho < 1$), the propagator of Eq. (3.10) becomes:

$$D_\rho(k) \simeq (K_\rho u_\rho)^{-1}(\omega^2 + u_\rho^2 k^2 - b^2 k^2/c^2 + c^{-2}O(\omega^2 k^2)). \quad (5.13)$$

This propagator now corresponds to a system of electrons (with no phonons) with an attractive interaction without retardation effects, as in the original Cooper problem. We should therefore be able to reproduce the result of Luther and Peschel for spinless fermions [2], which predicts strong pairing fluctuations for the attractive case. In the limit $c \to \infty$, we obtain from Eqs. (4.20a) and (4.20b) the result that $A = (1-(b/cu_\rho)^2)^{-1/2}$ and $B = (1-(b/cu_\rho)^2)^{1/2}$. The pairing correlation function for triplet pairing diverges, according to Eq. (5.10), but the criterion for SDW order is not met. The attractive interaction induces strong pairing fluctuations, in analogy with the Cooper problem and the result of Luther and Peschel for spinless fermions.
VI. HUBBARD MODEL

In this section, we shall consider the effect of phonons on the correlation function exponents derived in the previous sections, for specific interacting electron models. Most of the discussion will consider a Hubbard model with on–site repulsion parameter $U$:

$$H_{el} = -t \sum_{i,s} (c^\dagger_i c_{i+1,s} + c^\dagger_{i+1,s} c_i) + U \sum_i n_{i\uparrow} n_{i\downarrow}, \quad (6.1)$$

with $t$ the hopping matrix element. We shall also consider the extended Hubbard model with nearest neighbor repulsion parameter $V$ in the limit $U \to \infty$, where analytical expressions for $u_\rho$ and $K_\rho$ can be found at quarter filling.

A. Weak interactions

We first consider the on–site parameter as a perturbation. The charge velocity is then given by the Fermi velocity:

$$u_\rho = v_F = 2t \sin(n\pi/2), \quad (6.2)$$

for a system which has $n$ electrons per site (half filling corresponds to $n = 1$). The shift in energy (per site) due to the on–site repulsion term in Eq.(6.1) is then $\Delta E = Un^2/4$. One therefore deduces the first order expression for $K_\rho$:

$$K_\rho = 1 - U/\pi v_F + O(U^2). \quad (6.3)$$

Inserting the latter expression in the criteria for SDW order and triplet pairing, we obtain:

$$\frac{2U}{\pi v_F} > \frac{b^2(c^2 + 2v_+ v_-)}{v_F^2 v_F^2 (v_+ + v_-)^2} \quad \text{(SDW)} \quad (6.4a)$$

$$\frac{2U}{\pi v_F} < \frac{b^2}{v_F^2(v_+ + v_-)^2}, \quad \text{(Cooper pairs)} \quad (6.4b)$$

with $v_\pm$ given by Eq. (4.15). Eqs. (6.4a) and (6.4b) displays the effects mentioned in the Introduction: the coupling to phonons will induce Cooper pairs if the coupling constant is large enough to overcome the instantaneous repulsion between the electrons. Similarly, in order to have SDW fluctuations, the on–site repulsion parameter must be large enough to overcome the retarded electron attractive interaction mediated by the phonons. For sufficiently small phonon coupling $v_+ \to c$ and $v_- \to v_F$, the above inequalities are mutually exclusive: there is a region in parameter space:

$$\frac{b^2}{v_F^2(v_F + c)^2} < \frac{2U}{\pi v_F} < \frac{b^2(1 + 2v_F/c)}{v_F^2(v_F + c)^2} \quad (6.5)$$

which separates the SDW order and the triplet pairing phase.
Note that the perturbation expansion breaks down in both limits $n \to 0$ and $n \to 1$. In these two cases, we shall see in Sec. VI B that $K_ρ \to 0.5$, which signals that the correlation effects are important. Near $n = 1$, Umklapp scattering becomes relevant, and the system feels the proximity of the metal-insulator transition.

We plot the phase diagram of the system for $U = 0.01$ in Fig. 2, using the perturbative result of Eq. (6.3), as a function of the filling factor $n$ and the phonon coupling parameter $b/c_ρ$. We choose $c \sim \epsilon Lt/\hbar$, which is reasonable for heterostructure systems where $c \simeq 5 \times 10^5 [\text{cm/s}]$ and $v_F \simeq 10^6$. The superconducting phase is separated from the SDW phase by a metallic region, as suggested by the small coupling result of Eq. (6.5).

B. Hubbard model at arbitrary filling factor

To determine the Luttinger liquid parameters $K_ρ$ and $u_ρ$ for arbitrary $U$ and arbitrary filling factor $n$, we follow the method proposed by Ref. [29]. We calculate the ground state energy by solving numerically the integral equation of Lieb and Wu [31,43]:

$$f(k) = \frac{1}{2\pi} + \frac{4}{u} \cos(k) \int_{-Q}^{Q} dk' R\left(\frac{4}{u}(\sin(k) - \sin(k'))\right) f(k') ,$$

(6.6)

where $u = U/t$, and $f(k)$ is a distribution function evaluated at pseudo–momentum $k$. $k$ take values in the interval $[-Q, Q]$, and $Q < \pi$ depends on the filling factor through the relation $\int_{-Q}^{Q} dk f(k) = n$. The function $R(x)$ is the cosine Fourier transform of $1/(1 + e^x)$ evaluated at $x/2$. With these notations, the ground state energy is given by:

$$E_G = -2t \int_{-Q}^{Q} dk \cos(k) f(k) .$$

(6.7)

A first relation between $K_ρ$ and $u_ρ$ is obtained by noting [29,30]:

$$\frac{\partial^2 E_G}{\partial n^2} = \frac{\pi u_ρ}{2K_ρ} .$$

(6.8)

The velocity of the charge excitations is obtained from the particle (or hole) excitation spectrum, which was calculated by Coll [32]. $u_ρ$ corresponds to the group velocity of particles (holes) in the limit of small momentum:

$$u_ρ = \lim_{k \to 0} \frac{\Delta ε(k)}{\Delta p(k)} = \left(2 \sin(Q) - \mu \int_{-Q}^{Q} dk g(k) - 2t \int_{-Q}^{Q} dk \cos(k) g(k)\right) / 2\pi f(Q) ,$$

(6.9)

with $\Delta ε(k)$ the particle (hole) excitation energy, $\Delta p(k)$ its associated momentum and $\mu$ the chemical potential. The function $g(k)$ in Eq. (6.9) is the derivative of the
particle (hole) excitation distribution function in pseudo momentum space. It is the solution of the integral equation:

\[
\begin{align*}
g(k) &= -\frac{16}{u^2} \cos(k) \cos(Q) R'(4u \cos(Q) - \sin(k)) \\
&+ \frac{4}{u} \cos(k) \int_{-Q}^{Q} dk' R\left(\frac{4}{u} \cos(k) - \sin(k')\right) g(k') ,
\end{align*}
\]

(6.10)

where \(R'(x)\) denotes the derivative of \(R(x)\).

Once the parameters \(u_\rho\) and \(K_\rho\) are specified, we can determine in what region of parameter space the criterion (5.10) and (5.8) are satisfied. The speed of sound is assumed to have the same value as in Sec. VI A. We choose to represent the phonon coupling with the dimensionless parameter \(b/cu_\rho\). We emphasize that:

\[
\frac{b}{cu_\rho} \equiv \frac{g}{c} \sqrt{\frac{K_\rho}{\hbar \zeta u_\rho}}
\]

(6.11)

is an effective electron–phonon parameter, which, strictly speaking, depends on the filling factor. Nevertheless, it allows us to display the results in an obvious fashion.

First we consider the case of \(b/cu_\rho\) strictly smaller than unity (i.e. excluding the Wentzel–Bardeen singularity). First we vary the filling factor \(n\) and the phonon coupling for fixed \(U\). Our results are illustrated in Figs. (3a)–(3c), for \(U/t = 0.7, U/t = 0.3, U/t = 0.1\). The regions where SDW fluctuations dominate, or alternatively, superconducting fluctuations dominate, are separated by a metallic region, as was proposed in Sec. VI A. For small phonon coupling, SDW fluctuations dominate for arbitrary filling factor. As the coupling is increased, we cross the metallic region first near quarter filling, and eventually we reach the regime of superconducting fluctuations. For low filling factors and near half filling, correlation effects dominate the picture, and superconducting fluctuations cannot be sustained, even at large phonon coupling. By comparing Fig. (3a) to Fig. (3b) and Fig. (3c), we observe that the region of parameter space where superconducting fluctuations dominate shrinks as \(U\) is increased. This demonstrates the interplay between the instantaneous repulsion and the phonon mediated attraction for determining the phase of the system. We warn the reader that due to some limitations in numerical accuracy associated with the sharp variations of the charge velocity in the immediate neighborhood of \(n = 1\), Figs. 3a to 3c seem to indicate that the SDW phase dominates in this limit. This is not the case, for as we shall see in Sec. VI C, antiferromagnetic fluctuations are always suppressed for \(b/cu_\rho = 1\).

Next, we choose the filling factor at quarter filling \((n = 1/2)\), and vary \(U\) and the phonon coupling, see Fig. 4. Again, the metallic region separates the superconducting phase from the SDW phase. At low \(U (U/t < 0.1)\), \(K_\rho \approx 1\), and the Cooper instability pushes the system in
the superconducting phase. As $U$ is further increased, the phase boundaries appear, and the phonon–mediated attractive interaction is quickly overcome by the instantaneous repulsion between electrons. Beyond $U/t = 2$, the SDW fluctuations dominate for all values of the phonon coupling.

C. The Wentzel–Bardeen singularity

In the preceding section, we plotted the phase diagram of the system as a function of the filling factor and the on-site repulsion versus the phonon coupling parameter $b/u_c \rho$. The maximal value for this latter parameter in Figs. 3 and 4 is 1, and corresponds to the Wentzel–Bardeen singularity where the compressibility of the system becomes negative. We now show that for the Hubbard model, this singular point is accessible by varying the filling factor and discuss the physics in this region of parameter space.

We first reexpress the phonon coupling parameter in terms of the bare phonon coupling constant, the charge velocity, and the parameter $K_\rho$, as is apparent in Eq. (6.11). Next, we plot the ratio $u_\rho/K_\rho$ in Fig. 3 for several values of $U$, as a function of the filling factor. $u_\rho/K_\rho$ vanishes both for zero filling and half filling, and in the latter case, the variation of $u_\rho/K_\rho$ is most dramatic for low $U$. If we now consider the phonon coupling constant to be fixed at an arbitrary non–zero value, we can get arbitrarily close to $b/u_c \rho = 1$ by, for example, increasing the filling factor towards $n = 1$. As pointed out in Sec. IV A, when $u_\rho/K_\rho$ approaches from above the critical value $u^*_\rho/K^*_\rho$ defined in Eq. (4.21), the quantity $A$ which appears in the Green function exponent and in the SDW and CDW correlation function exponents diverges, and $B$ decreases to a finite value $B^* \equiv u^*_\rho/(u^*_{\rho} + c^2)^{1/2}$. Consequently, SDW (CDW) fluctuations are suppressed as we approach the Wentzel–Bardeen singularity, and the system is pushed towards the triplet superconducting phase, via the metallic phase depicted in Figs. 3 and 4. Given the fact that $B^* \propto g^2$ and $1/2 \leq K_\rho \leq 1$ the superconductivity criterion, $B \leq K_\rho$, can be met for sufficiently small $g \neq 0$, by approaching the Wentzel–Bardeen singularity. The system can be driven in the superconducting phase by approaching half filling.

D. Extended Hubbard model in the limit $U \to \infty$

We now add a nearest neighbor interaction term:

$$V \sum_i n_i n_{i+1}, \quad (6.12)$$

to the electronic Hamiltonian of Eq. (6.1) and let $U \to \infty$. Each site can be occupied at most by one electron, so that the system reduces effectively to spinless fermions with the substitution $k_F \to 2k_F$. At half filling (which
corresponds to \( n = 1/2 \) here as the Fermi momentum is \( 2k_F \), the spinless fermion Hamiltonian with nearest neighbor interaction can be mapped to the Hamiltonian of the XXZ model for a spin chain. The latter Hamiltonian can be solved exactly with the Bethe ansatz [41]. For this reason, it is possible [30,42] to get analytic expressions for the parameters \( K_\rho \) and \( u_\rho \):

\[
K_\rho = (2 + 4 \sin^{-1}(v)/\pi)^{-1} \tag{6.13a}
\]

\[
u_\rho = \pi t \sqrt{1 - v^2 / \cos^{-1} v} , \tag{6.13b}
\]

for \( v \equiv V/2|t| < 1 \). The case \( v > 1 \) represents a dimerized insulating phase [29]. One can therefore study the behavior of the correlation function exponents as the insulating phase is approached.

The phase diagram is plotted in Fig. 6. Obviously, the \( U \to \infty \) limit pushes the system towards the SDW phase, but at relatively low \( V/2t \), a strong coupling to phonons \( b/cu_\rho \approx 0.94 \) destroys the SDW fluctuations. As \( V/2t \) increases from 0 to 1, the effects of the phonons become less pronounced, and the metallic region gradually disappears.

**VII. APPLICATION TO ELECTRON WAVE GUIDES**

The results described in the previous section can be applied in a different context. The contribution of the Hamiltonian in Sec. II which describes the coupling between the charge and phonon fields can alternatively represent the coupling between two charge fields associated with two interacting quasi one–dimensional electron strings corresponding to two transverse modes in an electron wave guide.

The question of the stability of coupled one–dimensional interacting electrons systems has been the focus of recent interest [30,44,45]. Ref. [44] argues that the coupled system is stable, in the sense that it stays in the Luttinger liquid phase, as long as the hopping matrix element between the two chains remains below a critical value. Alternatively, a renormalization group argument proposed in Ref. [30] seems to imply that the coupled system is unstable for an arbitrarily small coupling. Both Ref. [30] and Ref. [44] assume a transverse hopping matrix element as the mechanism for destabilisation.

We reexamine this question from a slightly different starting point, considering the electrostatic effects between the two chains. We will show below that a threshold coupling will induce superconducting fluctuations in one of the chains, and in both chains as the coupling is further increased. After all, if the two electron chains can be associated with two orthogonal transverse states, it is reasonable to assume that in the absence of impurities, the (screened) Coulomb interaction between the two chains is the basic interaction mechanism which breaks the orthogonality between the two transverse states. In
this situation, Eq. \(2.2\) represents a local density–
density interaction:

\[
V_{1-2} = \pi g \sum_{s,s' = \uparrow, \downarrow} \int dx \psi_{1s}^\dagger(x)\psi_{1s}(x)\psi_{2s'}^\dagger(x)\psi_{2s'}(x) = g\pi \frac{2}{2} \int dx (\partial_x \varphi_{\rho i})(\partial_x \varphi_{\rho j}),
\]  

(7.1)

where \(\psi_{is}\) is the electron field operator associated with
an electron on chain \(i\) with spin \(s\), and \(\varphi_{\rho i}\) is the charge bosonic field of the same chain. Note that the last equality of Eq. \(7.1\) follows from the fact that the two transverse modes have distinct Fermi velocities. Hence the fast varying components of the density operators in the first line of Eq. \(7.1\) do not compensate each other as in the coupling between electrons and \(2k_F\) phonons. In the case of quasi–one–dimensional systems created in a GaAs/AlGaAs heterostructure, the assumption of a short range interaction can be justified because the surrounding metallic gates screen all interactions. The interaction potential of Eq. \(7.1\) is precisely the same form of coupling which was recently used \[17\] to study the transport properties of arrays of one–dimensional chains. This coupling describes two effects: first, the fact that the local “site” energy of, say, the first wire \((1)\) is raised (or lowered) in the presence of a nearby density perturbation in the second wire \((2)\). Secondly, Eq. \(7.1\) allows electrons from 1 and 2 to exchange as a pair.

The remaining part of the Luttinger liquid Hamiltonian which describes the uncoupled system is then:

\[
H_{el} = H_{\rho 1} + H_{\sigma 1} + H_{\rho 2} + H_{\sigma 2},
\]  

(7.2)

with, for \(\nu = \rho, \sigma\) and \(i = 1, 2\)

\[
H_{\nu i} = \frac{1}{2} \int dx \left[ u_{\nu i} K_{\nu i} \Pi_{\nu i}^2 + \frac{u_{\nu i}}{K_{\nu i}} (\partial_x \varphi_{\nu i})^2 \right],
\]  

(7.3)

with \(\Pi_{\nu i}\) the canonical conjugate of \(\varphi_{\nu i}\). In general, the charge velocities associated with the two electron chains can be different.

Using the results of Sec. \[\Pi\], the effective propagator describing the evolution in a given chain \(i\) becomes:

\[
D_{\rho i}(k) = (K_{\rho i} u_{\rho i})^{-1} \left( \omega^2 + u_{\rho i}^2 k^2 - \frac{b^2 k^4}{\omega^2 + u_{\rho j}^2 k^2} \right),
\]  

(7.4)

with \(i \neq j\) and \(b^2 = g^2 K_{\rho 1} K_{\rho 2} u_{\rho 1} u_{\rho 2} \hbar^2\).

The Green function exponent associated with electrons from chain \(i\) then reads:

\[
1 + \delta_i = \frac{K_{\sigma i}}{4} + \frac{1}{4K_{\sigma i}} + \frac{B_i}{4K_{\rho i}} + \frac{K_{\rho i} A_{ij}}{4}.
\]  

(7.5)

where
A_{ij} = \frac{u_{pi}}{v_+ + v_-} \left( 1 + \frac{u_{pj}^2}{v_+ v_-} \right) \quad (7.6a)

B_i = \frac{u_{pi}}{v_+ + v_-} \left( 1 + \frac{v_+ v_-}{u_{pi}^2} \right) \quad (7.6b)

v_\pm = \frac{1}{2} \left( u_{pj1}^2 + u_{pj2}^2 \right) \pm \frac{1}{2} \sqrt{\left( u_{pj1}^2 - u_{pj2}^2 \right)^2 + 4b^2} \quad (7.6c)

The coupling between the two chains will favor superconducting fluctuations in each chain, in analogy with the phonon case. The correlations functions which provide the signature for these fluctuations are read directly from Eqs. (5.2c) and (5.2d), with all fermions operators belonging to the same chain.

The criterion indicating strong superconducting fluctuations in a given chain \( i \) is then translated from Eqs. (5.9) and (5.10):

\[ \frac{B_i}{K_{\rho i}} + K_{\sigma i} \leq 2 \quad \text{(singlet)} \quad (7.7a) \]

\[ \frac{B_i}{K_{\rho i}} + \frac{1}{K_{\sigma i}} \leq 2 \quad \text{(triplet)} \quad (7.7b) \]

It is important to note that this effect does not depend on whether the interaction between the electron chains is positive or negative. As a result, the strong superconducting fluctuations induced by the interchain coupling could be observed in wires where only two channels propagate, or alternatively in one-dimensional coupled electron–hole systems. We can also determine from our analysis of the phonon case that the chain with the lower charge velocity will be the first to go superconducting.

For completeness, it should be mentioned that there are two additional correlation functions describing superconducting fluctuations for this particular system. These occur when a singlet or triplet Cooper pair is formed with one electron from chain 1 and one electron from chain 2. We shall ignore this additional superconductivity mechanism here, because the inequalities (7.7a) and (7.7b) are in general satisfied for a lower threshold phonon coupling.

For specificity, we consider the case of two Hubbard chains coupled by the interaction of Eq. (7.1): we refer the reader to Sec. VII, where the dimensionless coupling parameter has to be replaced by \( b/\rho_{12} \). Away from zero filling and half filling, we see that for a reasonably small \( U/t \), superconducting fluctuations will be induced above a threshold coupling. Our result therefore seems to be in qualitative agreement with Ref. [44], where the two chains, if initially in the Luttinger liquid regime, will remain in this regime below the threshold coupling. Also, note that in analogy to the electron–phonon case, the Wentzel–Bardeen singularity can be exploited to drive the system in the superconducting phase.

\section{VIII. SUMMARY AND CONCLUSION}

This work has focused on the low temperature properties of a system of interacting electrons coupled to
phonons in one dimension. The low energy properties of interacting electrons can then be described with the Luttinger liquid picture introduced by Haldane \cite{27} for spinless fermions, and later on generalized by Schulz \cite{29} for electrons with spin. When such a system is coupled to low energy phonons with a deformation potential coupling, the phonons can be integrated out, allowing for an exact calculation of the correlation function exponents as a function of the Luttinger liquid parameters, the phonon coupling constant, and the sound velocity.

We first considered single particle properties, calculating the Green function “dressed” by the phonon modes. For a gas of free electrons, the coupling to phonons alone is sufficient to introduce a singularity of the momentum distribution function at the Fermi surface, as is the case for fermions in the presence of an instantaneous, attractive interaction: the power law decay of the Green function at large distances has an exponent $1 + \delta$ which is always larger than 1. For increasing, but small phonon coupling, the repulsive interaction further accentuates this effect. However, beyond a threshold phonon coupling, $\delta$ has a minimum as a function of $U$. This indicates that for strong enough phonon coupling, at low $U$, the instantaneous repulsion and the retarded attractive interaction work in opposite ways.

Next, we considered the many body aspects of the system. In two and three dimensions it is understood that the parameters of the electron Hamiltonian can drive the system in a definite “phase” at vanishing temperature, such as a SDW or CDW phase. In the presence of phonons, a natural question is to ask whether the retarded attraction mediated by the phonons can flip the system into a triplet superconducting phase. In 2D and 3D, the interplay between retarded attractive interaction and instantaneous repulsive interaction cannot be accounted for in a precise matter. After all, the BCS theory of superconductivity \cite{46} assumes from the start an attractive interaction without retardation effects. The interplay between attractive retarded interaction and instantaneous repulsive interaction can be resolved exactly in one dimension. A set of inequalities was derived, which determine which type of fluctuations are present in the system.

To illustrate our method, we applied our results to the Hubbard model, where the Luttinger liquid parameters can be determined through a numerical computation, for arbitrary filling factor and on-site repulsion. The SDW fluctuations dominate at large $U$ and near zero and half filling, where correlation effects between electrons are known to be important. The region in parameter space where triplet superconducting fluctuations dominate is separated from the region of SDW fluctuations by a metallic region. This constitutes a novelty, as previous attempts \cite{3,4} to draw a phase diagram for purely instantaneous interactions predicted sharp transitions between the SDW (CDW) and superconducting regions. The retarded interaction mediated by the phonons is responsible for this new phase. For a fixed $U$, we found that as
the phonon coupling is increased, superconducting fluctuations are most likely to occur near quarter filling, or away from half filling and zero filling where correlation effects are most important. A natural extension of our investigation would be to study other models of interacting electrons with nearest neighbor interaction, such as the extended Hubbard model, where, however, it is necessary to rely on numerical estimates for the Luttinger liquid parameters.

We noted that by varying the electron filling factor, the Wentzel–Bardeen singularity can be approached for arbitrary values of the electron–phonon coupling. This singularity was previously understood to be the point where the coupling is so strong that the system acquires a negative compressibility. Nevertheless, for correlated electrons systems with on site repulsion such as the Hubbard model, the singularity is reached by decreasing the charge velocity associated with the particle–hole excitations. Near this point, SDW (CDW) correlations are totally suppressed, and the system is pushed through the metallic phase towards the superconducting phase. We finally note here that this behavior is reminiscent of a characteristic feature of high temperature superconductivity materials: the increase of doping (and thus change of filling factor) changes the phase of the material from antiferromagnetic to metallic and finally to superconducting (at low enough temperatures). Since there is some indication that the 2D layers of these new materials can be described in terms of Luttinger liquids, it is not unreasonable to conjecture that the mechanism discussed here could be of relevance for these materials. However, this is an open question which deserves further investigation.

Next, we argued that our results can be applied to study the (in)stability of two electrons (or one electron–hole) modes in an electron wave guide, coupled with a short range density–density interaction. For specific Hamiltonians describing the interaction between electrons within the chains, we can determine the threshold coupling where the chains become superconducting. Alternatively, the transition to the superconducting regime could be observed by varying the electron density, exploiting once again the WB singularity. This could be probed by studying the periodicity of the persistent current of a two channel mesoscopic ring as a function of electron density. Such rings with few transverse channels can now be fabricated using GaAs/AlGaAs heterostructures with metallic gates, as in the remarkable experiment of Mailly et al. The electron density can be varied by means of an electrostatic gate. The flux causes a twist in the boundary conditions, but does not affect the bulk properties, such as the correlation function exponents. In addition, the mutual inductance of the two transverse modes can be neglected because its effects are proportional to the small ratio $\alpha v_F/c_l$, with $\alpha \sim 137$ the fine structure constant and $c_l$ the velocity of light. In the metallic regime, the power spectrum has a peak associated with the flux quantum $\phi_0 = h c/e$, whereas in the super-
conducting case, this peak should give place to a peak at the superconducting flux quantum $\phi_0/2$. Finally, the study of the stability of two coupled transverse channels could in principle be generalized to an arbitrary number of channels, as it was recently shown that the correlation function exponents for the multicomponent Tomonaga–Luttinger model can be obtained formally [48] using conformal invariance arguments.

There are still challenging problems to be addressed. First, we have neglected throughout this paper the coupling between the fast oscillating part of the electron operator to the $2k_F$ phonon mode. This coupling is known to be at the origin of the Peierls instability, but is absent for the case of the coupled electron modes in the wave guide. For electron–phonon systems, it would be interesting to study whether the dramatic effects triggered by the Wentzel–Bardeen singularity can survive this additional interaction. This question can be answered in the case where the electron bandwidth is comparable to the phonon cutoff, where the retardation effects associated with the $2k_F$ phonon processes can be neglected, and this additional coupling amounts simply to a reduction of the on site repulsion parameter. The Wentzel–Bardeen singularity survives the $2k_F$ phonon processes in this case. In most physical systems, however, the phonon cutoff is smaller than the electron cutoff, and these retardation effects have to be taken into account [7,8].

Second, the present treatment uses exact results for the isolated Hubbard model, and applies it to a system which is coupled to phonons. It would be desirable to have a self–consistent treatment, where the effects of the correlated electrons on the lattice are also described [49].

ACKNOWLEDGMENTS

We thank C.P. Enz, A. J. Leggett, and S. Trugman for useful discussions. The work of D.L. is supported by NSERC of Canada.

APPENDIX A: TIME EVOLUTION

In order to calculate the commutator in Eq. (4.4), we need to find the explicit time dependence of the operators $\varphi_\rho$, $\varphi_\sigma$, $\Pi_\rho$ and $\Pi_\sigma$. The evolution in real time is determined by the following equation of motions ($\hbar = 1$):

\begin{align}
\partial_t \varphi_\nu &= K_\nu u_\nu \Pi_\nu , \quad \nu = \rho, \sigma \\
\partial_t \Pi_\sigma &= \frac{u_\sigma}{K_\sigma} \partial_x^2 \varphi_\sigma \\
\partial_t \Pi_\rho &= \frac{u_\rho}{K_\rho} \partial_x^2 \varphi_\rho + g \partial_x^2 d \\
\partial_t d &= \zeta^{-1} \Pi_d \\
\partial_t \Pi_d &= \zeta c^2 \partial_x^2 d + g \partial_x^2 \varphi_\rho .
\end{align}

(A1a) (A1b) (A1c) (A1d) (A1e)
Assuming for the above quantities a dependence in time and space of the form $\exp(ikx - i\omega t)$ we obtain the dispersion relations:

$$\omega^2_{\rho,\pm} = v_{\pm}^2 k^2$$  \hspace{1cm} (A2a)

$$\omega^2_{\sigma} = a^2 k^2$$  \hspace{1cm} (A2b)

for the charge and spin fields. The hybrid velocities $v_{\pm}$ are defined in Eq. (4.15).

At $t = 0$, the fields can be expressed as a Fourier series:

$$\varphi_{\rho}(x,0) = \sum_k \left( \frac{1}{2L|k|} \right)^{1/2} e^{ikx} (a_k^\dagger + a_{-k})$$  \hspace{1cm} (A3a)

$$\Pi_{\rho}(x,0) = i \sum_k \left( \frac{|k|}{2L} \right)^{1/2} e^{ikx} (a_k^\dagger - a_{-k})$$  \hspace{1cm} (A3b)

$$\varphi_{\sigma}(x,0) = \sum_k \left( \frac{1}{2L|k|} \right)^{1/2} e^{ikx} (b_k^\dagger + b_{-k})$$  \hspace{1cm} (A3c)

$$\Pi_{\sigma}(x,0) = i \sum_k \left( \frac{|k|}{2L} \right)^{1/2} e^{ikx} (b_k^\dagger - b_{-k})$$  \hspace{1cm} (A3d)

$$d(x,0) = \sum_k \left( \frac{\hbar}{2Lc|k|\zeta} \right)^{1/2} e^{ikx} (c_k^\dagger + c_{-k})$$  \hspace{1cm} (A3e)

$$\Pi_{\sigma}(x,0) = i \sum_k \left( \frac{\hbar c|k|\zeta}{2L} \right)^{1/2} e^{ikx} (c_k^\dagger - c_{-k})$$  \hspace{1cm} (A3f)

where $a_k$ ($b_k$) is an annihilation operator characterizing a charge (spin) excitation at momentum $k$, and $c_k$ is a phonon annihilation operator. These expressions are implicitly regularized by the large momentum cutoff, $\exp(-\epsilon |k|L/2\pi)$, and furthermore the $k = 0$ mode is excluded in the sum.

Eq. (A2a) indicates that the time dependence of the charge fields should be of the form:

$$\varphi_{\rho}(x,t) = \sum_k \left( \frac{1}{2L|k|} \right)^{1/2} e^{ikx} e^{-\beta kv_{\pm}^2 t}$$  \hspace{1cm} (A4a)

$$\Pi_{\rho}(x,t) = \sum_k \left( \frac{|k|}{2L} \right)^{1/2} e^{ikx} e^{-\beta kv_{\pm}^2 t}$$  \hspace{1cm} (A4b)

Similarly, the time dependence of the spin fields is determined from Eq. (A2b):

$$\varphi_{\sigma}(x,t) = \sum_k \left( \frac{1}{2L|k|} \right)^{1/2} e^{ikx} \left[ C_+ e^{-ikv_{\sigma}^2 t} + C_- e^{ikv_{\sigma}^2 t} \right]$$  \hspace{1cm} (A5a)
The operators $A_{\pm \pm}$, $B_{\pm \pm}$, $C_{\pm}$ and $D_{\pm}$ are determined by expressing the initial values $\varphi_{\nu}(x,0)$, $\partial_t \varphi_{\nu}(x,0)$, $\partial^2_t \varphi_{\nu}(x,0)$, $\partial^2_{xx} \varphi_{\nu}(x,0)$, in terms of spatial derivatives via Eqs. (A1a)- (A1d), and similarly for $\Pi_{\nu}$. We then obtain:

\[ A_{\alpha \beta}(k) = \frac{1}{2(v^2_\alpha - v^2_\beta)} \left( u^2_\rho - v^2_\alpha \right) \left[ a^\dagger_k \left( 1 - \frac{\beta \rho K_\rho}{v_\alpha} \text{sgn}(k) \right) \right. \\
+ a_{-k} \left( 1 + \frac{\beta \rho K_\rho}{v_\alpha} \text{sgn}(k) \right) \\
\left. + iK_\rho u_\rho g(c\zeta)^{-1/2} \left[ c_k \left( -1 + \frac{\beta c}{v_\alpha} \right) \right. \right. \\
\left. \left. \left. + c^\dagger_{-k} \left( 1 + \frac{\beta c}{v_\alpha} \text{sgn}(k) \right) \right) \right] \]  

\[ B_{\alpha \beta}(k) = \frac{1}{2(v^2_\alpha - v^2_\beta)} \left( ia^\dagger_k \left( u^2_\rho - v^2_\alpha \right) - \frac{\beta}{K_\rho v_\alpha u_\rho} \left( u^4_\rho - u^2_\rho v^2_\alpha + b^2 \text{sgn}(k) \right) \right. \\
+ ia_{-k} \left( -u^2_\rho + v^2_\alpha - \frac{\beta}{K_\rho v_\alpha u_\rho} \left( u^4_\rho - u^2_\rho v^2_\alpha + b^2 \text{sgn}(k) \right) \right) \\
\left. + g(c\zeta)^{-1/2} \left[ c_k \left( -c - \frac{\beta}{v_\alpha} \left( u^2_\rho + c^2 - v^2_\alpha \right) \text{sgn}(k) \right) \right. \right. \\
\left. \left. \left. \left. + c^\dagger_{-k} \left( -c + \frac{\beta}{v_\alpha} \left( u^2_\rho + c^2 - v^2_\alpha \right) \text{sgn}(k) \right) \right) \right) \right] \]  

\[ C_{\alpha}(k) = \left( \frac{1 - \alpha K_\sigma \text{sgn}(k)}{2} \right) b^\dagger_k + \left( \frac{1 + \alpha K_\sigma \text{sgn}(k)}{2} \right) b_{-k} \]  

\[ D_{\alpha}(k) = i \left( \frac{1 - \alpha \text{sgn}(k)}{2K_\sigma} \right) b^\dagger_k + \left( \frac{1 + \alpha \text{sgn}(k)}{2K_\sigma} \right) b_{-k} \]

with $\alpha, \beta = \pm$. These operators satisfy the commutation relations:

\[ [B_{\alpha \beta}(k), A_{\gamma \delta}(k')] = -i \delta_{k,k'} \delta_{\alpha,\beta} \delta_{\gamma,\delta} \frac{u^2 - v^2}{v^2_\alpha - v^2_\beta} \]  

(A7a)

\[ [D_{\alpha}(k), C_{\beta}(k')] = -i \delta_{k,k'} \delta_{\alpha,\beta} \]  

(A7b)

These relations are used in turn to calculate the commutators which appear in the normal ordered products of Sec. IV A with the substitution $t \rightarrow i\tau$ for the thermodynamic quantities:

\[ [\theta_\rho(x,\tau), \varphi_\rho(0,0)] = -\frac{1}{4\pi} \left( \frac{u^2_\rho - v^2_\rho}{v^2_\rho + v^2_\tau} \ln \frac{1 - \tilde{z}(x + v_+ i\tau)}{1 - z(x + v_+ i\tau)} \right. \\
\left. + \frac{u^2_\rho - v^2_\rho}{v^2_\tau + v^2_\rho} \ln \frac{1 - \tilde{z}(x - v_+ i\tau)}{1 - z(x - v_+ i\tau)} \right) \]  

(A8a)

\[ [\theta_\sigma(x,\tau), \varphi_\sigma(0,0)] = -\frac{1}{4\pi} \ln \left( \frac{1 - \tilde{z}(x - u_\sigma i\tau)}{1 - z(x - u_\sigma i\tau)} \right) \]  

(A8b)
Noting that \( [\theta_\nu(x, \tau), \varphi_\nu(0, 0)] = [\theta_\nu(0, 0), \varphi_\nu(-x, -\tau)] \), we obtain the explicit form for \( g_n \). If in addition \( x \neq 0 \), we may put \( \epsilon = 0 \) and use that \( \tilde{z}(x + \nu \tau) = 1/z(x - \nu \tau) \), which then leads to the second equality in Eq. (4.5) after some simple manipulations.

**APPENDIX B: FOURIER SUMS**

In this Appendix we compute the Fourier sums which appear in Eq. (4.13) in the limit \( T \to 0 \). For that purpose it is convenient to rewrite the quantities to be summed over in terms of irreducible fractions expressed as functions of \( \omega^2 \) and \( k^2 \):

\[
D_{\rho}^{-1}(k) = u_\rho K_\rho \left( \frac{c^2 - v_+^2}{v_+^2 - v_-^2} \left( \frac{1}{\omega^2 + v_+^2 k^2} - \frac{1}{\omega^2 + v_-^2 k^2} \right) + \frac{1}{\omega^2 + v_+^2 k^2} \right) \tag{B1a}
\]

\[
\frac{u_\rho K_\rho D_{\rho}(k) - \omega^2}{K_\rho u_\rho k^2 D_{\rho}(k)} = \frac{c^2 - b^2/v_+^2 - v_-^2}{v_+^2 - v_-^2} \left( \frac{1}{\omega^2 + v_+^2 k^2} - \frac{1}{\omega^2 + v_-^2 k^2} \right) + \frac{1}{\omega^2 + v_+^2 k^2}, \tag{B1b}
\]

where the velocities \( v_\pm \) are defined in Eq. (4.13).

With the above decompositions, the sums over Matsubara frequencies in Eq. (4.13) yield, in the \( T \to 0 \) limit:

\[
\frac{1}{\beta} \sum_\omega D_{\rho}^{-1}(k) = \frac{u_\rho K_\rho}{2|k|(v_+ + v_-)} \left( 1 + \frac{c^2}{v_+ v_-} \right) \tag{B2a}
\]

\[
\frac{1}{\beta} \sum_\omega D_{\sigma}^{-1}(k) = \frac{K_\sigma}{2|k|} \tag{B2b}
\]

\[
\frac{1}{\beta} \sum_\omega \omega \sin(\omega \tau) D_{\rho}(k) = \text{sgn}(\tau) \frac{u_\rho K_\rho}{2} \left( \frac{v_+^2 - c^2}{v_+^2 - v_-^2} e^{-v_- |k\tau|} + \frac{v_-^2 - c^2}{v_+^2 - v_-^2} e^{-v_+ |k\tau|} \right) \tag{B2c}
\]

\[
\frac{1}{\beta} \sum_\omega \omega \cos(\omega \tau) D_{\sigma}(k) = \text{sgn}(\tau) \frac{u_\rho K_\rho}{2} e^{-u_\rho |k\tau|} \tag{B2d}
\]

\[
\frac{1}{\beta} \sum_\omega \cos(\omega \tau) D_{\rho}(k) = \frac{K_\rho}{2|k|} \left( \frac{u_\rho v_+^2 - c^2}{v_+^2 - v_-^2} e^{-v_- |k\tau|} + \frac{u_\rho v_-^2 - c^2}{v_+^2 - v_-^2} e^{-v_+ |k\tau|} \right) \tag{B2e}
\]

\[
\frac{1}{\beta} \sum_\omega \cos(\omega \tau) D_{\sigma}(k) = \frac{K_\sigma}{2|k|} e^{-u_\sigma |k\tau|}, \tag{B2f}
\]

where \( \text{sgn}(\tau) = 1, 0, -1 \), if \( \tau >,=, < 0 \). Finally, the summations over momentum which occur in the calculation of the Green function exponent are:

\[
\frac{\pi}{L} \sum_\frac{1}{k} = -\ln \epsilon \tag{B3a}
\]

\[
\frac{\pi}{L} \sum_\frac{\cos(kx)e^{-\nu |k\tau|}}{|k|} = -\ln |1 - z(x + i\nu |\tau|)| \tag{B3b}
\]
\[
\frac{\pi}{L} \sum_k \frac{\sin(kx)e^{-v|k\tau|}}{k} = \frac{1}{2i} \ln \frac{1 - \bar{z}(x + iv|\tau|)}{1 - \bar{z}(x + iv|\tau|)}, \quad (B3c)
\]

where \(z(x) = \exp(2\pi i x/L - \epsilon)\), and where again we have implicitly used the large momentum cut-off, \(\exp(-\epsilon|k|L/2\pi)\), and omitted the \(k = 0\) term.

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FIG. 1. Green function exponent for the Hubbard model: a) at fixed filling factor, as a function of $u = U/t$ for $b/cu_\rho = 0$ (solid line), $b/cu_\rho = 0.2$ (dotted line), $b/cu_\rho = 0.4$ (dashed line) and $b/cu_\rho = 0.6$ (dashed–dotted line). b) for fixed $U$, as a function of filling factor and for the same values of $b/cu_\rho$ as in Fig. 1a.

FIG. 2. Phase diagram for $U = 0.01t$, for low filling factor $n < 0.2$, as a function of the phonon coupling $b/cu_\rho$, using the perturbative result of the inequalities (6.4a) and (6.4b). SDW, M, and SC refer to the spin density waves, metallic, and triplet superconducting regions.

FIG. 3. Phase diagram of the Hubbard model coupled to phonons, as a function of filling factor and phonon coupling constant $b/cu_\omega$: a) $U/t = 0.7$. b) $U/t = 0.3$. c) $U/t = 0.1$. We use the same labels as in Fig. 2a.
FIG. 4. Phase diagram of the Hubbard model coupled to phonons at quarter filling \((n = 1/2)\), as a function of \(U/t\) and \(b/cu_\rho\).

FIG. 5. Plot of \(u_\rho/K_\rho\) as a function of the filling factor \(n\). \(u_\rho\) is measured in units of \(v_F(n = 1)/2 = at\). From top to bottom: \(U/t = 16, 8, 4, 2\). Note the abrupt change for small values of \(U\) as one approaches half filling.

FIG. 6. Phase diagram for the extended Hubbard model in the limit \(U \to \infty\), as a function of the nearest neighbor interaction \(V/2t\) and the phonon coupling constant \(b/cu_\rho\).