Study on possible molecular states composed of $\Lambda_c \bar{D}$ ($\Lambda_b B$) and $\Sigma_c \bar{D}$ ($\Sigma_b B$) within the Bethe-Salpeter framework

Hong-Wei Ke$^1$ *, Mei Li$^1$, Xiao-Hai Liu$^1$ † and Xue-Qian Li$^2$‡,

$^1$ School of Science, Tianjin University, Tianjin 300072, China
$^2$ School of Physics, Nankai University, Tianjin 300071, China

Abstract

$P_c(4312)$ observed by LHCb collaboration is confirmed as a pentaquark and its structure, production and decay behaviors attract great attention of theorists and experimentalists. Since its mass is very close to sum of $\Sigma_c$ and $\bar{D}$ masses, it is naturally tempted to be considered as a molecular state composed of $\Sigma_c$ and $\bar{D}$. Moreover, $P_c(4312)$ is observed in the channel with $J/\psi p$ final state, requiring isospin conservation $P_c(4312)$ should be an isospin-1/2 eigenstate. In literature, several groups used various models to estimate its spectrum. We are going to systematically study the pentaquarks within the framework of the Bethe-Salpeter equation, thus $P_c(4312)$ is an excellent target because of the available data. We re-calculate the spectrum of $P_c(4312)$ in terms of the Bethe-Salpeter equations and further study its decay modes. Some predictions on other possible pentaquark states which can be tested in the future experiments, are made.

PACS numbers: 12.39.Mk, 12.40.-y, 14.40.Nd

* khw020056@tju.edu.cn
† xiaohai.liu@tju.edu.cn
‡ lixq@nankai.edu.cn
I. INTRODUCTION

Due to the innovation of experimental techniques and facilities as well as the advances in theory of recent years several exotic states have been experimentally observed and theoretically confirmed. Indeed, more constituents would cause more ambiguities, unlike the simplest $qq$ for mesons and $qqq$ for baryons. The inner structures of the exotic states are still not clear yet, those discoveries stir up large numbers of discussions\(^1\). Indeed the theoretical exploration is crucial for getting a better understanding on the quark model and obtaining valuable information about non-perturbative physics. Definitely, to complete the theoretical job achieving more accurate data would compose the key.

Some hidden charm or bottom states were measured in two-meson final states\(^2\)–\(^9\). They are regarded as tetraquark states or meson-meson molecular states. In 2003 a baryon was measured by LEPS\(^12\) which was conjectured as a pentaquark, however later the allegation was negated by further more accurate experiments. Breaking the frustration on existence of pentaquark which was predicted by Gell-Mann in his first paper on quark model, the LHCb collaboration reported two pentaquark states observed in $\Lambda_b$ decays where peaks appear at the $J/\psi p$ final states\(^13\).

Recently another narrow pentaquark state $P_c(4312)$\(^14\) has also been observed in the $J/\psi p$ mass spectrum. Its mass and width are $4311.9 \pm 0.7^{+6.8}_{-0.6}$ MeV and $9.8 \pm 2.7^{+3.7}_{-4.5}$ MeV respectively. Since its mass is very close to the sum of $\Sigma_c$ and $\bar{D}$ masses, it is natural to regard it as a molecular state of $\Sigma_c \bar{D}$\(^15\)–\(^23\). Furthermore its width is rather wide in accord with the property of molecular states, so the phenomenon further supports the proposal of its molecular structure. Some other theorists conjecture $P_c(4312)$ as a compact pentaquark\(^23, 24\) instead. In Ref.\(^25\) the authors think the interaction between $\Sigma_c$ and $\bar{D}$ is too weak to bind them into a bound state. It is worth of deeper explorations about whether the molecule picture is reasonable. In this work we will calculate the mass spectrum of $P_c(4312)$ based on the assumption that it is a stable bound state of $\Sigma_c$ and $\bar{D}$. Additionally we also study other possible bound states of $\Lambda_c \bar{D}$, $\Lambda_b B$ and $\Sigma_b B$ and see if they can be formed.

We will employ the Bethe-Salpeter (B-S) equation to study the possible bound state which consists of a baryon and a meson. The B-S equation is a relativistic equation to deal with the bound state and established on the basis of quantum field theory\(^26\). Initially, people use the B-S equation to study the bound state of two fermions\(^27, 28\) and the system of one-fermion-one-boson\(^29\). In Ref.\(^30, 31\) the authors employed the Bethe-Salpeter equation to study the $K\bar{K}$ or $BK$ molecular state and their decays. With the approach we studied the molecular state of $B\pi$\(^32\), $D^{(*)}D^{(*)}$ and $B^{(*)}B^{(*)}$\(^33\). Recently the approach is extended to explore double charmed baryons\(^34, 35\) and pentaquarks which are assumed to be two-body bound systems. In Ref.\(^36\) the authors studied possible bound states of $\Lambda$ ($\Sigma$) and $\bar{K}$. In this work we will a similar approach to study the possible bound states of $\Sigma_c \bar{D}$, $\Lambda_c \bar{D}$, $\Lambda_b B$ and $\Sigma_b B$.

At present pentaquark states $P_c(4312)$, $P_c(4380)$, $P_c(4440)$ and $P_c(4457)$ have been measured in decays of $\Lambda_b$ where the pentaquark states peak up at the invariant mass spectrum of $J/\psi p$, so their isospin is $\frac{1}{2}$ because of isospin conservation. Thus we require that the
two hadron constituents reside in an isospin eigenstate. Instead, for the $\Lambda_c \bar{D}$ (as well $\Lambda_b B$) system its isospin must be $\frac{1}{2}$ but the $\Sigma_c \bar{D}$ (or $\Sigma_b B$) system may reside in either isospin $\frac{1}{2}$ or $\frac{3}{2}$. Certainly, for a bound system with spin-parity $\frac{1}{2}^-$ the two constituents are in the $S$-state.

For carrying on our calculation the interactions between two constituents are needed. According to the quantum field theory two particles interact via exchanging certain mediate particles. Since two constituents in a pentaquark are color-singlet hadrons the exchanged particles are some light hadrons such as $\rho$ or (and) $\omega$ etc..

The effective interaction deduced from the chiral lagrangian can be written as $\mathcal{L} = \mathcal{L}_{MMV} + \alpha \mathcal{L}_{BBV} + \ldots$ where $\mathcal{L}_{BBV}$ and $\mathcal{L}_{MMV}$ are for baryon and meson parts respectively (See Appendix A) and the ellipsis represents other possible parts. It is noted that in the effective lagrangian a free phase factor $\alpha$ exists which is not determined by any fundamental theory. Principally, it can be a complex phase, but in this work, we just require it to be either 1 or -1. The kernel between a baryon and a meson is proportional to the hadronic matrix element $K = \langle P | T(\mathcal{L}_{MMV}\mathcal{L}_{BBV}) | P \rangle$ where $|P\rangle$ is a spin-isospin eigensate corresponding to the required pentaquark which is a molecular state composed by a charmed baryon $B$ and a charmed meson $M$. Just as Dyson noticed [37], a factor (he was discussing interaction between two electrons in comparison with that between an electron and a positron) is crucial for the physics, since it can reverse a repulsive interaction into an attractive one. Generally, it is a classical phenomenon, but also applies to quantum field theory. This effect results in divergence of perturbation of QED, however, Dyson also pointed that it does not affect the phenomenological application as long as the physical input is set. Similarly, in our case, the free phase factor $\alpha$ can only be determined by experiments. Since its value would determine if the interaction (potential) is attractive or repulsive, so that is playing a key role for a definite spin-isospin composition for the pentaquark state. Namely the factor would determine if the spin-isospin structure is stable. Actually, we do not prior set the value of $\alpha$, but let the experiment decide.

With the effective interactions we derive the kernel and obtain the corresponding B-S equation. In our calculation we include two different cases corresponding to $\alpha = 1$ and -1. In the case I $\alpha$ takes 1 i.e. the kernel is calculated using the effective interaction presented in the appendix A directly while in the case II $\alpha$ takes -1 i.e. a minus sign is added into the kernel obtained in the case I.

With a reasonable parameter set, the B-S equation is solved. For a spin-isospin eigenstate, if the equation does not possess a solution, then we conclude that the corresponding bound state should not exist in the nature, by contraries, a solution of the B-S equation implies the bound state being formed. At the same time the B-S wave function is obtained and we are able to use the corresponding formula for calculating the rates of strong decay $P_c(4312) \rightarrow$ proton + $\nu$ (vector) which can be compared with the data.

This paper is organized as follows: after this introduction we will derive the B-S equations related to possible bound states composed of a baryon and a meson and the formula for its strong decay. Then in section III we will solve the B-S equation numerically and present our results. Section IV is devoted to a brief summary.
II. THE BOUND STATES OF $\Lambda_c\bar{D}$ AND $\Sigma_c\bar{D}$

Since the newly found pentaquarks $P_c(4312)$, $P_c(4380)$, $P_c(4440)$ and $P_c(4457)$ are all hadrons containing hidden charms(or hidden bottoms), so we will focus on the molecular structures composed of one charmed (bottomed) baryon and an anti-charmed(anti-bottomed) meson. Concretely, in this paper we will study $\Lambda_c\bar{D}$, $\Sigma_c\bar{D}$, $\Lambda_b\bar{B}$ and $\Sigma_b\bar{B}$ systems whose spin-parity is $\frac{1}{2}^-$ i.e. the spatial wave function is in $S$–wave. In this section as an example we only formulate the corresponding quantities for $\Lambda_c\bar{D}$ and $\Sigma_c\bar{D}$ systems. These formulas can be equally applied to $\Lambda_b\bar{B}$ and $\Sigma_b\bar{B}$ systems.

A. The isospin states of $\Lambda_c\bar{D}$ and $\Sigma_c\bar{D}$

The isospin structure of the possible bound state of $\Lambda_c\bar{D}$ is

$$|\frac{1}{2}, \frac{1}{2}\rangle = |\Lambda_c\bar{D}^0\rangle,$$

(1)

We will use $P_{c(\frac{1}{2}, \frac{1}{2})}$ to denote this resonance.

Instead, the possible bound states of $\Sigma_c\bar{D}$ should be in three isospin assignments i.e. $|I, I_3\rangle$ are $|\frac{1}{2}, \pm \frac{1}{2}\rangle$, $|\frac{3}{2}, \pm \frac{1}{2}\rangle$ and $|\frac{3}{2}, \pm \frac{3}{2}\rangle$. Let us work out the explicit isospin states

$$|\frac{1}{2}, \frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|\Sigma_c^{++}\bar{D}^-\rangle - \sqrt{\frac{1}{3}}|\Sigma_c^{+}\bar{D}^0\rangle,$$

(2)

$$|\frac{3}{2}, \frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|\Sigma_c^{++}\bar{D}^-\rangle + \sqrt{\frac{2}{3}}|\Sigma_c^{+}\bar{D}^0\rangle,$$

(3)

and

$$|\frac{3}{2}, \frac{3}{2}\rangle = |\Sigma_c^{++}\bar{D}^0\rangle.$$

(4)

While the states $|\frac{1}{2}, -\frac{1}{2}\rangle$, $|\frac{3}{2}, -\frac{1}{2}\rangle$ and $|\frac{3}{2}, -\frac{3}{2}\rangle$ are just the charge conjugate states of $|\frac{1}{2}, \frac{1}{2}\rangle$, $|\frac{3}{2}, \frac{1}{2}\rangle$ and $|\frac{3}{2}, \frac{3}{2}\rangle$, therefore their hadronic properties are the same. We use $P_{c(\frac{1}{2}, -\frac{1}{2})}$, $P_{c(\frac{3}{2}, -\frac{1}{2})}$...
and $P_c(\frac{1}{2}, \frac{3}{2})$ to denote the three isospin states of $\Sigma_c \bar{D}$: $|\frac{1}{2}, \frac{1}{2}\rangle$, $|\frac{3}{2}, \frac{1}{2}\rangle$ and $|\frac{3}{2}, \frac{3}{2}\rangle$ respectively for latter discussions.

In order to discuss the Isospin factors in the B-S equation we define the fields of baryons and mesons in the expressions [36]:

\[
\begin{align*}
\mathcal{B}_1(x) &= \int \frac{d^4q}{(2\pi)^4\sqrt{2B^{++}}}(a_{B^-}e^{-iqx} + a_{B^+}^{\dagger}e^{iqx}), \\
\mathcal{B}_2(x) &= \int \frac{d^4q}{(2\pi)^4\sqrt{2B^+}}(a_{B^-}e^{-iqx} + a_{B^+}^{\dagger}e^{iqx}), \\
\mathcal{M}_1(x) &= \int \frac{d^4q}{(2\pi)^4\sqrt{2M^+}}(a_{M^+}e^{-iqx} + a_{M^-}^{\dagger}e^{iqx}), \\
\mathcal{M}_2(x) &= \int \frac{d^4q}{(2\pi)^4\sqrt{2M^0}}(a_{M^0}e^{-iqx} + a_{M^{\dagger}}^{\dagger}e^{iqx}),
\end{align*}
\]

(5)

where $\mathcal{B}$ represents $\Lambda_c$ or $\Sigma_c$ and $\mathcal{M}$ denotes $\bar{D}$.

**B. The Bethe-Salpeter (B-S) equation for $\frac{1}{2}^-$ molecular state**

In the effective theory a meson and a baryon can interact by exchanging mesons. The Feynman diagram at the leading order are depicted in Fig. 1. The relative and total momenta of the bound state in the equations are defined as

\[
p = \eta_2 p_1 - \eta_1 p_2, \quad q = \eta_2 q_1 - \eta_1 q_2, \quad P = p_1 + p_2 = q_1 + q_2,
\]

(6)

where $p$ and $q$ are the relative momenta at the two sides of the effective vertex, $p_1$ ($q_1$) and $p_2$ ($q_2$) are those momenta of the constituents, $P$ is the total momentum of the bound state, $k$ is the momentum of the exchanged meson, $\eta_i = m_i/(m_1 + m_2)$ and $m_i (i = 1, 2)$ is the mass of the $i$-th constituent meson.

The bound state composed of a baryon and a meson can be written as

\[
\chi_P(x_1, x_2) = \langle 0 | T \mathcal{B}(x_1) \mathcal{M}(x_2) | P \rangle.
\]

(7)

The B-S wave function is a Fourier transformation of that in momentum space

\[
\chi_P(x_1, x_2) = e^{-iPX} \int \frac{d^4q}{(2\pi)^4} \chi_P(p).
\]

(8)

In the so-called ladder approximation the corresponding B-S equation was deduced in earlier references as

\[
\chi_P(p) = S_B(p_1) \int \frac{d^4q}{(2\pi)^4} K(P, p, q) \chi_p(q) S_M(p_2),
\]

(9)

where $S_B(p_1)$ is the propagator of the baryon ($\Lambda_c$ or $\Sigma_c$), $S_M(p_2)$ is that of the meson ($\bar{D}$) and $K(P, p, q)$ is the kernel which can be obtained by calculating the Feynman diagram in Fig.
1. For the later convenience the relative momentum $p$ is decomposed into the longitudinal $p_l (≡ p · v)$ and transverse projection $p_{\mu} (≡ p^\mu - p_l v^\mu) = (0, p_T)$ according to the momentum of the bound state $P (v = \frac{P}{M})$.

$$S_B(\lambda_1 P + p) = \frac{i[(\eta_1 M + p_l)\delta + p_k + m_1]}{(\eta_1 M + p_l + \omega_l - i\epsilon)(\lambda_1 M + p_l - \omega_l + i\epsilon)}, \quad (10)$$

$$S_M(\lambda_2 P - p) = \frac{i}{(\eta_2 M - p_l + \omega_2 - i\epsilon)(\eta_2 M - p_l - \omega_2 + i\epsilon)}, \quad (11)$$

where $M$ is the total energy of the bound state, $E_i = \sqrt{p_l^2 + m_i^2}$ and $m_1 (m_2)$ is the mass of baryon (meson).

By the Feynman diagram the kernel $K(\bar{P}, p, q)$ is written as

$$K(\bar{P}, p, q) = -C_{I, I_z} g_{M, MV} g_{BBV} (\gamma^\alpha - \frac{\kappa_{BB\rho}}{2m_B} \sigma^{\alpha\beta} k_\beta)(p_2 + q_2)\Delta_{\alpha\mu}(k, m_V) F^2(k), \quad (12)$$

where $m_V$ is the mass of the exchanged meson, $g_{M, MV}, g_{BBV}$ and $\kappa_{BB\rho}$ are the concerned coupling constants, $C_{I, I_z}$ is the isospin coefficient which is given in Appendix B. Apparently the contribution of the tensor term is much smaller than that of the first term, thus we can be ignored it in practical computations. Indeed, a numerical estimate verifies this allegation.

Since the constituents of the molecule (meson and baryon) are not point particles, a form factor at each effective vertex should be introduced. The form factor is suggested by many researchers is of the form:

$$F(k, m_V^2) = \frac{\Lambda^2 - m_V^2}{\Lambda^2 + k^2}, \quad k = p - p', \quad (13)$$

where $\Lambda$ is a cutoff parameter. Since the form factor is not derived from a fundamental principle, the concerned cutoff parameter is neither determined theoretically, thus until now we know little about the cutoff parameter $\Lambda$. In some Refs. $[38-41]$ the form factor is parameterized as $\Lambda_{QCD} + m_V$ with $\Lambda_{QCD} = 220$ MeV and the dimensionless parameter $\lambda$ being of order of unit. We will employ the expression $\Lambda = \lambda \Lambda_{QCD} + m_V$ in our calculation.

The three-dimension B-S wave function is obtained after integrating over $p_l$

$$\chi_P(p_l) = \int \frac{dp_l}{2\pi} \chi_P(p). \quad (14)$$

For the $S$-wave system, the spatial wave function can be easily derived $[34-36]$

$$\chi_P(p_l) = [f_1(|P_T|) + f_2(|P_T|) p_k] u(v, s), \quad (15)$$

where $f_1(|p_l|)$ and $f_2(|p_l|)$ are the radial wave functions, $u(v, s)$, $v$ and $s$ are the spinor, velocity and total spin of the pentaquark.
Multiplying $\int \frac{dp_t}{2\pi}$ on the both sides of Eq. (9), integrating over $p_t$ and $q_t$, substituting Eqs. (12), (15) into Eq.(9) and using the so-called covariant instantaneous approximation $q_t = p_t$ we obtain

$$f_1(|p_T|) + f_2(|p_T|)p_T u(v, s) = -\int \frac{dp_t}{(2\pi)^3} \int \frac{dq_T}{(2\pi)^3} \left\{ \kappa \left[ (\eta_2 M - p_t)q_T - p_t q_T \right] \right\} F^2(k, m_B) f_1(|q_T|) + f_2(|q_T|)q_T u(v, s).$$

(16)

Now let us fix the expressions of $f_1(|p_T|)$ and $f_2(|p_T|)$. Multiplying $u(v, s)$ on both sides of Eq.(17), we get an expression which only contains $f_1$ whereas multiplying $u(v, s)p_T$ the expression for $f_2$ is obtained, then taking a trace, the resultant formulae are

$$f_1(|p_T|) = -\int \frac{dp_t}{(2\pi)^3} \int \frac{dq_T}{(2\pi)^3} \left\{ \kappa \left[ (\eta_2 M - p_t)q_T - p_t q_T \right] \right\} F^2(k, m_B) f_1(|q_T|) + f_2(|q_T|)q_T u(v, s).$$

(17)

$$f_2(|p_T|) = -\int \frac{dp_t}{(2\pi)^3} \int \frac{dq_T}{(2\pi)^3} \left\{ \kappa \left[ (\eta_2 M - p_t)q_T - p_t q_T \right] \right\} F^2(k, m_B) f_1(|q_T|) + f_2(|q_T|)q_T u(v, s).$$

(18)

To extract $f_1(|p_T|)$ and $f_2(|p_T|)$ from the above equations, instead of the procedure adopted in earlier works, we multiply $u(v, s)$ from the right side of the equation and sum over the spin projections of $u(v, s)$, then taking a trace of the modified equation, the job is done. The advantage of this procedure is to keep the equation of motion $\psi u(v, s) = u(v, s)$. 

7
Now we perform an integral over \( p_i \) on the right side of Eqs. (17) and (18) where four poles at \(-\eta_1 M - \omega_1 + i\epsilon, -\eta_1 M + \omega_1 - i\epsilon, \eta_2 M + \omega_2 - i\epsilon\) and \(-\eta_2 M - \omega_2 + i\epsilon\) exist. By choosing an appropriate contour (17) and (18) we calculate the residuals at \( p_i = -\eta_1 M - \omega_1 + i\epsilon \) and \( p_i = \eta_2 M - \omega_2 + i\epsilon \). The coupled equations after the contour integrations are collected in appendix (Eqs. (C1) and (C2)). Then one can carry out the azimuthal integration and reduce Eqs. (C1) and (C2) to one dimensional integral equations

\[
\begin{align*}
  f_1(|\mathbf{p}_T|) &= \int d|\mathbf{q}_T| [A_{11}(|\mathbf{q}_T|, |\mathbf{p}_T|) f_1(|\mathbf{q}_T|) + A_{12}(|\mathbf{q}_T|, |\mathbf{p}_T|) f_2(|\mathbf{q}_T|)] \\
  f_2(|\mathbf{p}_T|) &= \int d|\mathbf{q}_T| [A_{21}(|\mathbf{q}_T|, |\mathbf{p}_T|) f_1(|\mathbf{q}_T|) + A_{22}(|\mathbf{q}_T|, |\mathbf{p}_T|) f_2(|\mathbf{q}_T|)]
\end{align*}
\]  

(19)

where \( A_{11}, A_{12}, A_{21} \) and \( A_{22} \) are presented in Appendix (see Eqs. (C6), (C7), (C8) and (C9)).

C. The normalization condition for the B-S wave function

The normalization condition for the B-S wave function of a bound state is [30, 34]

\[
i \int \frac{d^4 p d^4 q}{(2\pi)^8} \bar{\chi}_P(p) \frac{\partial}{\partial P_0} [I(P, p, q) + K(P, p, q)] \chi_P(q) = 1,
\]

(20)

where \( P_0 \) is the energy of the bound state and the spinor relation \( \sum_s u(v, s)\bar{u}(v, s) = \frac{k+1}{2} \) is used. \( I(P, p, q) \) is the reciprocal of the four-point propagator

\[
I(P, p, q) = \frac{\delta^4(p - q)}{(2\pi)^4} [S_B(p_1)]^{-1} [S_M(p_2)]^{-1}.
\]

(21)

For the molecular states composed of two mesons the second term in the normalization condition is several orders smaller than the first term [32, 33], thus we have all reasons to believe that the rule also applies to the case where the molecule is composed of a baryon and a meson, consequently the term \( \frac{\partial}{\partial P_0} K(P, p, q) \) can be ignored and then

\[
- \int \frac{d^4 p}{(2\pi)^4} \bar{\chi}_P(p) \eta_1 \psi[S_M(p_2)]^{-1} \chi_P(q) - \int \frac{d^4 p}{(2\pi)^4} \bar{\chi}_P(p) 2\eta_2 p_2 \cdot v [S_B(p_1)]^{-1} \chi_P(q) = 1.
\]

(22)

Let us define the transverse projections of the B-S wave function as follows:

\[
\begin{align*}
  \alpha_P(p) &= -i[S_B(p_1)]^{-1} \chi_P(q) [S_M(p_2)]^{-1}, \\
  \beta_P(p) &= -i[S_M(p_2)]^{-1} \bar{\chi}_P(q) [S_B(p_1)]^{-1},
\end{align*}
\]

(23)

the normalization condition is

\[
\begin{align*}
  &- \int \frac{d^4 p}{(2\pi)^4} \text{Tr}[\alpha_P(p)\beta_P(p)S_B(p_1)\eta_1\psi S_B(p_1)S_M(p_2)] \\
  &- \int \frac{d^4 p}{(2\pi)^4} \text{Tr}[\alpha_P(p)\beta_P(p)2\eta_2 p_2 \cdot v S_B(p_1)S_M(p_1)S_M(p_1)S_M(p_1)] = 1.
\end{align*}
\]

(24)
Substituting the expression \( \chi_P(p) \) (Eq. (9)) into the Eqs. (23) under the covariant instantaneous approximation one can obtain the expressions of \( \alpha_P(p) \) and \( \beta_P(p) \), for example

\[
\alpha_P(p) = - \frac{d^3q_T}{(2\pi)^3} C_{i,1/2} g_{M,MV} g_{BB} \left\{ \frac{2(\eta_2 M - p_l)\eta - p_k - q_l - (p_k - q_l)(p_T^2 - q_T^2)/m_v^2}{-(p_T - q_T)^2 - m_v^2} \right\}
\]

\[
\kappa[2(\eta_2 M - p_l)\eta - p_k - q_l] \frac{2(\eta_2 M - p_l)\eta - p_k - q_l - (p_k - q_l)[2(\eta_2 M - p_l)\eta - p_k - q_l]}{4m_B[-(p_T - q_T)^2 - m_v^2]}
\}

\]

\[
I_I, I_I, \kappa^2 \left\{ \frac{d^2}{(2\pi)^2} \right\}
\]

\[
F^2(k, m_V) \left\{ f_1(|q_T|) + f_2(|q_T|)q_0 \right] u(v, s).
\]

and \( \alpha_P(p) \) and \( \beta_P(p) \) can be parameterized into

\[
\alpha_P(p) = [h_1(|p_T|)] + h_2(|p_T|)p_k \left\{ \frac{d^3q_T}{(2\pi)^3} \right\}
\]

\[
\beta_P(p) = \tilde{u}(v, s)[h_1(|p_T|)] + h_2(|p_T|)p_k
\]

with

\[
h_1(|p_T|) = - \frac{d^3q_T}{(2\pi)^3} \cdot \left\{ \frac{2f_1(q_T)(M\eta_2 - p_l) + f_2(|q_T|)[q_T^2 + p_T \cdot q_T + (p_T^2 - q_T^2)/m_v^2]}{-(p_T - q_T)^2 - m_v^2} \right\}
\]

\[
+ \frac{4\kappa f_2(|q_T|)(M\eta_2 - p_l)[p_T \cdot q_T - q_T^2]}{4m_B[-(p_T - q_T)^2 - m_v^2]} \}
\]

\[
h_2(|p_T|) = - \frac{d^3q_T}{(2\pi)^3} \cdot \left\{ \frac{f_1(q_T)(M\eta_2 - p_l)[p_T \cdot q_T - q_T^2]}{-(p_T - q_T)^2 - m_v^2} \right\}
\]

\[
+ \frac{4\kappa f_2(|q_T|)[p_T^2 q_T^2 - p_T \cdot q_T^2 + f_1(q_T)q_T^2]}{4m_B[-(p_T - q_T)^2 - m_v^2]} \}
\]

\[
C_{i,1/2} g_{M,MV} g_{BB} F^2(k, m_V).
\]

Substituting Eqs. (10), (11) and equation group (20) into Eq. (21) we obtain

\[
i \int \frac{d^4p}{(2\pi)^4} \left\{ h_1^2(m_1^2 + p_1^2 + p_T^2 + 2Mp_1\eta_1 + M^2\eta_1^2 + 2m_1(p_1 + M\eta_1)] + h_2^2(p_T^2) + 2Mp_1\eta_1 + M^2\eta_1^2 - 2m_1(p_1 + M\eta_1) \right\}
\]

\[
/ \{[\eta_1(M + p_1)] - \omega_1^2 + i\epsilon]^2[(\lambda_1 M - p_1)^2 - \omega_1^2 + i\epsilon] \}
\]

\[
i \int \frac{d^4p}{(2\pi)^4} \left\{ h_1^2(m_1 + p_1 + M\eta_1) - 2h_1h_2 p_T^2 + h_2^2 p_T^2(p_1 - m_1 + M\eta_1)]
\]

\[
/ \{[(\eta_1 M + p_1)] - \omega_1^2 + i\epsilon]^2[(\lambda_1 M - p_1)^2 - \omega_1^2 + i\epsilon] \} = 1.
\]

After the contour integration on \( p_l \) and the azimuthal integration the normalization condition can be calculated numerically and the values of \( f_1(|p_T|) \) and \( f_2(|p_T|) \) are fixed at the same time.

D. the decay of \( P_c \rightarrow V + \)proton

Now we investigate the strong decays of \( P_c \) in terms of the framework formulated above.

9
FIG. 2: the decay of $P_c$ by exchanging mesons.

The amplitudes corresponding to the two diagrams in Fig. 2 are,

$$A_a = C_I g_{B'B} g_{DDV} \int \frac{d^4 p}{(2\pi)^4} \bar{U}_B \gamma^5 \chi_P(p)(k - p_2)_\nu \epsilon^\nu \frac{1}{k^2 - M_D^2} F^2(k, m_D),$$  \hspace{1cm} (29)

$$A_b = 2C_I g_{B'B^*} g_{DD^*V} \int \frac{d^4 p}{(2\pi)^4} \bar{U}_B' (\gamma^\sigma - \frac{\kappa_{BBD^*}}{2m_B} \sigma^{\sigma\omega} k_\omega) \chi_P(p) \epsilon^{\alpha\beta\mu\nu} k_\mu q_2, \epsilon_{\nu}$$  \hspace{1cm} (30)

$$\frac{g_{\sigma\beta} - k_\beta k_\sigma / M_{D^*}^2}{k^2 - M_{D^*}^2} F^2(k, m_{D^*}),$$  \hspace{1cm} (31)

where $C_I$ is the isospin coefficient of the transition, $k = p - (\eta_2 q_1 - \eta_1 q_2)$, $B$ denotes the charmed baryon in the molecular state: $\Sigma_c$ or $\Lambda_c$, $\epsilon$ is the polarization vector of $V$ and $B'$ represents proton. We still take the approximation $k_0 = 0$ to carry out the calculation.

The total amplitude is

$$A = A_a + A_b = \bar{u}_B \gamma^5 g_1 \gamma^\mu + i g_2 \sigma^{\mu\nu} q_2, \epsilon^\nu + i g_3 \gamma^\mu \epsilon^{\mu\alpha\beta} P_\alpha q_2, \epsilon_\beta u(v) \epsilon^\mu.$$  \hspace{1cm} (32)

The factors $g_1$, $g_2$ and $g_3$ can be extracted from the expressions of $A_1$ and $A_2$.

Then the partial width is expressed

$$d\Gamma = \frac{1}{32\pi^2} |A|^2 \frac{|g_2|^2}{M^2} d\Omega.$$  \hspace{1cm} (33)

III. NUMERICAL RESULTS

A. the numerical results

In order to solve the B-S equation numerically some parameters are needed. The mass $m_{\Lambda_c}$, $m_{\Sigma_c}$, $m_D$, $m_\omega$, $m_\rho$ are taken from the databook[42]. Following Ref.[43, 44], we set the coupling constants $g_{DD_\omega} = g_{DD_\rho} = 3.02$, $g_{\Lambda_c\omega} = 8.125$, $g_{\Sigma_c\omega} = g_{\Sigma_c\rho} = 7.475$, $f_{\Sigma_c\omega} = 9.9125$, $f_{\Sigma_c\rho} = 9.9125$.

With these parameters and the corresponding isospin factors a complete B-S equation (the coupled equations [19]) is established. These coupled equations are complicated integral
equations, thus to numerically solve them, the standard way is to discretize them, namely we would convert them into algebraic equations. Concretely, we set a reasonable finite range for \(|p_T|\) and \(|q_T|\), and let the variables take \(n\) \((n=129\) in our calculation\) discrete values \(Q_1, Q_2, ..., Q_n\) which distribute with equal gap from \(Q_1=0.001\) GeV to \(Q_n=2\) GeV. The gap between two adjacent values is \(\Delta |p_T|=(1.999/128)\) GeV. For clarity, we let \(n\) values of \(f_1(|p_T|)\) and \(n\) values of \(f_2(|p_T|)\) constitute a column matrix with \(2n\) rows and the \(2n\) elements \(f_1(|q_T|), f_2(|q_T|)\) construct another column matrix residing on the right side of the equation as shown below. The column matrix composed of \(f_1(|p_T|)\) and \(f_2(|p_T|)\) is associated with the right column matrix of \(f_1(|q_T|)\) and \(f_2(|q_T|)\) by a \(2n \times 2n\) matrix whose elements are the coefficients given in Eq. \([19]\). The standard way to treat the equation is to let \(|p_T|\) and \(|q_T|\) take the same sequential values \(Q_1, Q_2, ..., Q_n\) for discretizing the integral equation.

\[
\begin{pmatrix}
  f_1(Q_1) \\
  f_1(Q_{129}) \\
  f_2(Q_1) \\
  f_2(Q_{129})
\end{pmatrix}
= A(\Delta E, \lambda)
\begin{pmatrix}
  f_1(Q_1) \\
  f_1(Q_{129}) \\
  f_2(Q_1) \\
  f_2(Q_{129})
\end{pmatrix}.
\]

As a matter of fact, it is a homogeneous linear equation group. If it possesses non-trivial solutions, the necessary and sufficient condition is the coefficient determinant to be zero. In our case, it is \(|A(\Delta E, \lambda) - I| = 0\). By calculating the determinant of \(|A(\Delta E, \lambda) - I|\) \((I\) is the unit matrix) where \(A(\Delta E, \lambda)\) is a function of the binding energy \(\Delta E = m_1 + m_2 - M\) and parameter \(\lambda\). Our strategy is following: we arbitrarily vary \(\Delta E\) within a possible range, by requiring \(|A(\Delta E, \lambda) - I| = 0\), we obtain a corresponding \(\lambda\). In Ref.\([38]\) \(\lambda\) was fixed to be 3. In our earlier paper\([41]\) we change the value of \(\lambda\) from 1 to 3 to explore possible dependence of the results on it, it seems that a value of \(\lambda\) within the range of 0 ~ 4 is reasonable for forming a bound state of two hadrons. Consequently, if the obtained \(\lambda\) is much beyond the range, we would conclude that the resonance cannot exist.

To get the wavefunction \(T(f_1(Q_1), f_1(Q-2), ..., f_2(Q_1) ... f_2(Q_{129}))\), we adopt a special method. Namely, we suppose a matrix equation \((A(\Delta E, \lambda)_{ij})(f(i)) = \beta(f(i)))\) it is an eigenequation, in terms of the standard software, we can find all the possible “eigenvalues” \(\beta\), among them only \(\beta = 1\) is the solution we expect, then the corresponding wavefunction is gained which just is the solution of the B-S equation.

For \(|A(\Delta E, \lambda) - I| = 0\), inputting some binding energies, we would check whether we can obtain reasonable values for \(\lambda\). If yes, we substitute the values of \(\lambda\) and the binding energy into the matrix equation to obtain the B-S wavefunctions.

With this strategy, let us investigate the molecular structure of \(\Lambda_c\) and \(D\) as well as that of \(\Sigma_c\) and \(\bar{D}\).

If the exchanging particles are limited to light vector meson, only \(\omega\) and \(\rho\) can be exchanged between charmed baryons and \(D\). Of course, exchanging two \(\rho\) mesons between \(\Lambda_c\) \((\Sigma_c)\) and \(\bar{D}\) can also induce a potential, but it undergoes a loop suppression, therefore, we do not consider that contribution.
1. case I

As aforementioned, in the chiral lagrangian \( \mathcal{L} = \mathcal{L}_{MMV} + \alpha \mathcal{L}_{BBV} \) there is a free phase factor \( \alpha \) which could be either +1 or -1. In the case I, we set \( \alpha = +1 \).

As the first trial, let us study a simple compound, namely we explore the possible bound states of \( \Lambda_c \) and \( \bar{D} \) which is an \( I = \frac{1}{2} \) state, therefore only \( \omega \) can be exchanged between \( \Lambda_c \) and \( \bar{D} \). We find that there is no solution for the B-S equation, therefore we would conclude that the interaction induced by the single \( \omega \) exchange is repulsive.

With the same procedure, we study a molecule composed of \( \Sigma_c \) and \( \bar{D} \) whose isospin could be either \( \frac{1}{2} \) or \( \frac{3}{2} \) and \( C_{1,2} = 1 \). Since \( P_c(4312) \) is observed in the \( J/\psi p \) portal, it is confirmed to be a state of \( I = \frac{1}{2} \). In this case both \( \omega \) and \( \rho \) exchanges between the two ingredients are allowed. The isospin factor for the \( \rho \) exchange is \( -2 \), namely plays an opposite role to the \( \omega \) exchange. We try to solve the equation \( |A(\Delta E, \Lambda) - I| = 0 \) for some chosen \( \Delta E \) and find a solution for \( \Sigma_c \bar{D} \) with the quantum number \( I(J) = \frac{1}{2}(\frac{1}{2}) \) where the factor \( \lambda \) can span a large range.

The result indicates that although, the \( \omega \) exchange contributes a repulsive interaction, for \( \Sigma_c \bar{D} \) molecule, the total interaction can be attractive due to the \( \rho \) exchange. Numerically, the obtained values of \( \lambda \) and corresponding \( \Delta E \) for \( \Sigma_c \bar{D} \) system are presented in Tab. I. Our numerical computation also confirms that the tensor coupling in the \( \mathcal{L}_{BBV} \) has little effect on the results. For example setting \( \Delta E = 8 \text{ MeV} \) one can fix \( \lambda = 3.77 \text{ GeV} \) and \( 3.88 \text{ GeV} \) with and without the tensor contribution the obtained wave functions are very close to each other so we can ignore the tensor coupling in the vertex \( \mathcal{L}_{BBV} \). Apparently when \( \Delta E \) is very small the obtained \( \lambda \) is smaller than \( 4 \text{ GeV} \) so \( \Sigma_c \) and \( \bar{D} \) should form a weak bound state.

At present the pentaquark \( P_c(4312) \) was experimentally observed in \( \Lambda_b \rightarrow J/\psi pK \) portal, which is peaked at the invariant mass spectrum of \( J/\psi p \) and has the invariant mass of about 4312 MeV. Apparently its isospin is \( \frac{1}{2} \), thus the majority of authors \[15-19, 22, 23\] regarded this pentaquark as a bound state of \( \Sigma_c \) and \( \bar{D} \).

Using the normalized wave functions the transition \( P_c(\frac{1}{2}, \frac{1}{2}) \rightarrow J/\psi + p \) is calculable. The form factors defined in Eq. (32) with the coupling constants are evaluated: \( g_{BB'D'} = 2.7, g_{BB'^D} = 3.0, g_{DD'\psi} = 7.4, g_{DD'^\psi} = 2.5 \text{ GeV}^{-1} \)[45]. We obtain \( g_1 = 0.396 \text{ GeV}, g_2 = 0.270, g_3 = 0.00632 \text{ GeV}^{-1} \) and the decay width \( \Gamma[P_c(I = \frac{3}{2}, I_z = \frac{1}{2}) \rightarrow \rho^0 p] = 3.90 \text{ MeV} \). If the binding energy is \( 20 \text{ MeV} \), \( g_1 = 0.412 \text{ GeV}, g_2 = 0.282, g_3 = 0.00923 \text{ GeV}^{-1} \) can be obtained and the decay width \( \Gamma[P_c(\frac{3}{2}, \frac{1}{2}) \rightarrow J/\psi p] = 3.66 \text{ MeV} \). We notice our results are close to that of Ref. [17, 22], but the results given in Ref. [21] are 1-3 orders smaller than ours where different ultraviolet regulators were employed.

Following our observation given above, for the state with \( I = \frac{3}{2} \) the isospin factor is \( 1 \) for exchanging either \( \omega \) or \( \rho \), therefore the total interaction is repulsive, it means that \( \Sigma_c \) and \( \bar{D} \) cannot form a bound state with \( I = \frac{3}{2} \).
TABLE I: the cut off parameter $\lambda$ and the corresponding binding energy $\Delta E$ for the bound state $\Sigma_c \bar{D}$ with $I = \frac{1}{2}$ and $I_z = \frac{1}{2}$ 

| $\Delta E$ (MeV) | 2  | 8  | 20 | 30 | 40 |
|------------------|-----|-----|-----|-----|-----|
| $\lambda$       | 3.31 | 3.88 | 4.58 | 5.04 | 5.44 |

FIG. 3: The normalized wave function $f_1(|p_T|)$ and $f_2(|p_T|)$ for $P_{c(\frac{1}{2}\downarrow)}$

2. case II

Just for our theoretical interest, we take an alternative phase factor rather than that in the case I. Here we adopt the phase factor $\alpha$ to be $-1$. As the sign of the concerned coupling may change the whole physical picture, we would like to see what consequences can be induced. When one changes the situation from an electron-electron system to an electron-positron system, the interaction converts from repulsive to attractive. As Dyson noted, the well-known classical physical phenomenon can be manifested in quantum electrodynamics. We should consider that similar situation can also show up at QCD and related fields. Now we deliberately change the relative phase factor in the lagrangian, then the behavior of the interaction immediately changes. The bound states predicted in the case I would disappear while some other possible pentaquarks which were sentenced to death in case I would survive. Let us analyze the change.

In fact, the relative sign between the coupling $L_{BBV}$ and $L_{PPV}$ is not determined by a fundamental principle, so that we just keep it as a free phase to be fixed by experimental measurements. When the sign of $L_{BBV}$ or that of $L_{PPV}$ is changed, the sign of the transition
TABLE II: the cut off parameter \( \Lambda \) and the corresponding binding energy \( \Delta E \) for the bound state \( \Lambda_c \bar{D} \)

| \( \Delta E \) MeV | 6   | 10  | 20  | 30  | 40  |
|------------------|-----|-----|-----|-----|-----|
| \( \lambda \) GeV | 3.95| 4.25| 4.836| 5.32| 5.74|

TABLE III: the cut off parameter \( \Lambda \) and the corresponding binding energy \( \Delta E \) for the bound state \( \Sigma_c \bar{D} \) with \( I = \frac{3}{2} \)

| \( \Delta E \) MeV | 10  | 20  | 30  | 40  | 50  |
|------------------|-----|-----|-----|-----|-----|
| \( \lambda \)    | 2.17| 2.43| 2.634| 2.80| 2.96|

matrix would change simultaneously, and a repulsive interaction can convert into attractive.

For \( \Lambda_c \bar{D} \) system when a negative sign is added, we solve the equation again. At this time we can find a solution which satisfies the equation, i.e. can exist in nature. The obtained values of \( \lambda \) and corresponding \( \Delta E \) are presented in Tab. II. From the values in the table II one can find \( \lambda \) is close to 4 when the binding energy \( \Delta E \) is also small. Thus we can expect a weak bound state to exist in the nature, even though not very stable.

For \( \Sigma_c \) and \( D \) system with \( I = \frac{3}{2} \) the total interaction turns to be repulsive now because the isospin coefficient \( C_{\frac{1}{2}, \frac{1}{2}} \) is \(-2\) for exchanging \( \rho \) which means the interaction to be repulsive whereas exchanging \( \omega \) it is +1 (attractive). Now, we try to solve the equation \( |A(\Delta E, \lambda) - I| = 0 \) for some \( \Delta E \) and find that no \( \lambda \) satisfies the equation. For the state with \( I = \frac{3}{2} \) the isospin coefficient \( C_{\frac{3}{2}, \frac{1}{2}} \) is 1 for exchanging \( \rho \) and \( \omega \), so the force is attractive. The obtained values of \( \lambda \) and corresponding \( \Delta E \) for the \( \Sigma_c \) and \( D \) system are presented in Tab. III. Apparently within a certain range of \( \Delta E \) the obtained \( \lambda \) locates within a reasonable range, a bound state of \( \Sigma_c \) and \( D \) system with \( I = \frac{3}{2} \) is formed. Since the conservation of isospin \( P_{c(\frac{3}{2}, \frac{1}{2})} \) and \( P_{c(\frac{1}{2}, \frac{1}{2})} \) can decay into \( \rho + p \). Supposing the binding energy is 40 MeV, with the coupling constants \( g_{BS\Sigma} = 3.0, g_{BS\Sigma} \rho^* = 2.7, g_{DD\rho} = 2.8 \text{ GeV}^{-1} \), we obtain \( g_1 = 0.222 \text{ GeV, } g_2 = 1.043, g_3 = 0.00390 \text{ GeV}^{-1} \) and the decay width \( \Gamma[P_{c(\frac{3}{2}, \frac{1}{2})} \to \rho^0 p] = 16.1 \text{ MeV} \).

The two different cases induce different physical consequences. According to the present knowledge, only experiment can make a judgement. The available data about \( P_c(4312) \) seems to support the case I. However, if in new experiments a pentaquark is observed from the invariant mass spectrum of \( pp \) the case II would be favored.

B. predictions about pentaquark \( P_b \)

The isospin of the \( \Lambda_b B^+ \) system is

\[
\left| \frac{1}{2}, \frac{1}{2} \right> = \left| \Lambda_b B^+ \right>
\]

The isospin of the \( \Sigma_b B \) system can be also \( \left| \frac{1}{2}, \pm \frac{1}{2} \right>, \left| \frac{3}{2}, \pm \frac{1}{2} \right>, \left| \frac{3}{2}, \pm \frac{3}{2} \right> \). Let us work on
the cut off parameter $\Lambda$ and the corresponding binding energy $\Delta E$ for the bound state $\Sigma_bB$ with $I = \frac{1}{2}$ and $I_z = \frac{1}{2}$

| $\Delta E$ | 10  | 20  | 30  | 40  | 50  |
|------------|-----|-----|-----|-----|-----|
| $\lambda$  | 2.13| 2.51| 2.82| 3.09| 3.35|

the isospin states

$$|\frac{1}{2}, \frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|\Sigma_0^+ B^0\rangle - \sqrt{\frac{1}{3}}|\Sigma_b^0 B^+\rangle,$$

(35)

$$|\frac{3}{2}, \frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|\Sigma_0^+ B^0\rangle + \sqrt{\frac{2}{3}}|\Sigma_b^0 B^+\rangle,$$

(36)

and

$$|\frac{3}{2}, \frac{3}{2}\rangle = |\Sigma_b^0 B^+\rangle.$$  

(37)

Since the phase convention in case I of last subsection can lead to results which are consistent with data, we employ that to explore the $\Lambda_bB$, and $\Sigma_bB$ systems. Using the masses of $\Lambda_b$, $\Sigma_b$ and $B$ presented in Ref.[42] and other parameter are listed in previous sections, we solve those B-S equations. It is found that only the equation for the $\Sigma_bB$ system with $I = \frac{1}{2}$ has a solution. The binding energies and corresponding $\lambda$s are presented in Tab. IV. That implies that the bound state with $I = \frac{1}{2}$ can exist in the nature. Under the heavy quark symmetry we suppose the couplings unchanged when $b$-hadrons replace $c$-hadrons. We study the transition $P_{b(\frac{3}{2}, \frac{1}{2})} \rightarrow \Upsilon p$ and obtain $g_1 = 0.00346$ GeV, $g_2 = 0.252$, $g_3 = 0.0000911$ GeV$^{-1}$ and predict the decay width $\Gamma [P_{b(\frac{3}{2}, \frac{1}{2})} \rightarrow \Upsilon p] = 0.690$ keV when the binding energy is 10 MeV. If $\Delta E = 20$ MeV the decay width $\Gamma [P_{b(\frac{3}{2}, \frac{1}{2})} \rightarrow \Upsilon p] = 1.09$ keV and $g_1 = 0.00435$ GeV, $g_2 = 0.318$, $g_3 = 0.000149$ GeV$^{-1}$.

IV. CONCLUSION AND DISCUSSION

Within the B-S framework we explore several bound states which are composed of a baryon and a meson. Their total spin and parity is $\frac{1}{2}$, i.e. the the orbital angular momentum $L = 0$ ($S$-wave). We try to solve the B-S equation for getting possible spatial wave functions for $\Lambda_c \bar{D}$, $\Sigma_c \bar{D}$, $\Lambda_bB$ and $\Sigma_bB$ systems. If the B-S equation for a supposed molecular structure has a stable solution, we would conclude that the concerned pentaquark could exist in the nature, oppositely, no-solution means the supposed pentaquark cannot appear as a resonance or the molecular state is not an appropriate structure. The criteria can apply for a check of the pentaquark states which have already been or will be experimentally observed. In this scenario, the two constituents interact by exchanging light vector mesons. For the $\Lambda_c \bar{D}$
(Λ_b B) system only ω is the exchanged mediate meson, while for the Σ_c D system (Σ_b B) both ω and ρ contribute. The chiral interaction determines if those molecular states can be formed.

For $\frac{1}{2}^-$ baryon (S-wave), the B-S wave function possesses two scalar functions $f_1(|\mathbf{p}_T|)$ and $f_2(|\mathbf{p}_T|)$ which should be solved numerically. Discretizing the integral equations, we simplify the B-S equation into two coupled algebraic equations about $f_1(|\mathbf{p}_T|)$ and $f_2(|\mathbf{p}_T|)$.

As $|\mathbf{p}_T|$ $(i = 1, 2)$ takes $n$ discrete values the two coupled equations are converted into a matrix equation which can be easily solved numerically in terms of available softwares. When all known parameters are input there still is one undetermined parameter λ. Our strategy is inputting binding energies within a range and then fixing λ by solving the matrix equation. If λ is located in a reasonable range one can expect the bound state to exist. In case I we find the B-S equation of the state Λ_c D system has no solution for λ when the binding energy takes experimentally allowed values. For the Σ_c D system there are three isospin eigenstates. Due to the isospin factors, the B-S equations for $P_c(\frac{1}{2}^+, \frac{1}{2}^+)$, $P_c(\frac{1}{2}^+, \frac{3}{2}^+)$ and $P_c(\frac{3}{2}^+, \frac{3}{2}^-)$ are set. In case I we find the equation for $|\frac{1}{2}^+, \frac{1}{2}^+\rangle$ has a solution for λ falling into a reasonable range. It means that $P_c(4312)$ maybe is a molecular state of Σ_c D. The decay width of $P_c(\frac{3}{2}^+, \frac{3}{2}^-) \rightarrow J/\psi p$ is calculated within this framework and we obtain it as about 3.66 MeV. By comparison, In case II we get opposite results: $P_c(\frac{3}{2}^+, \frac{3}{2}^-)$, $P_c(\frac{1}{2}^+, \frac{1}{2}^+)$ and $P_c(\frac{3}{2}^+, \frac{3}{2}^-)$ are possible bound states but $P_c(\frac{3}{2}^+, \frac{3}{2}^-)$ are not.

It is also noted that $P_c(\frac{3}{2}^+, \frac{3}{2}^-)$ and $P_c(\frac{3}{2}^+, \frac{3}{2}^-)$ may decay to ρp and the decay width is about 16 MeV when the binding energy is about 40 MeV. We suggest experimentalists to look for the final product of ρp in the Λ_b decays which would help to confirm several possible bound states. Moreover, it would offer valuable knowledge for understanding the effective interactions and the structure of $P_c(4312)$ and others.

In terms of the present data the case I seems to possess the right convention by which we study Λ_b B and Σ_b B systems. The B-S equations of Σ_b B and Λ_b B are solved. The bound state $P_6(\frac{1}{2}^+, \frac{1}{2}^+)$ is still a promising pentaquark state. The partial width $P_6(\frac{1}{2}^+, \frac{1}{2}^+) \rightarrow \Upsilon p$ is about 1.06 keV at $\Delta E = 20$ MeV.

Within the B-S framework, we systematically investigate the molecular structure of pentaquarks. We pay a special attention to $P_c(4312)$ because it is experimentally well measured. From that study, we have accumulated valuable knowledge on probable molecular structure of pentaquarks which can be applied to the future research. Definitely, the discovery of pentaquarks opens a window for understanding the quark model established by Gell-Mann and several other predecessors. Deeper study on their structure and concerned effective interaction which binds the ingredients to form a molecule would greatly enrich our theoretical asset. So we will continue to do research along the line.

Acknowledgement

This work is supported by the National Natural Science Foundation of China (NNSFC) under the contract No. 11375128. We author would like to thank Prof. Bing-Song Zou, Prof. Xiang Liu, Prof. Yu-Ming Wang and Dr. Zhen-Yuan Wang for their suggestions and
useful discussions.

[1] H. X. Chen, W. Chen, X. Liu, Y. R. Liu and S. L. Zhu, Rept. Prog. Phys. 80, no. 7, 076201 (2017) doi:10.1088/1361-6633/aa6420 [arXiv:1609.08928 [hep-ph]].
[2] S. K. Choi et al. [Belle Collaboration], Phys. Rev. Lett. 91, 262001 (2003) [arXiv:hep-ex/0309032].
[3] K. Abe et al. [Belle Collaboration], Phys. Rev. Lett. 98, 082001 (2007) [arXiv:hep-ex/0507019].
[4] S. K. Choi et al. [Belle Collaboration], Phys. Rev. Lett. 94, 182002 (2005).
[5] S. K. Choi et al. [BELLE Collaboration], Phys. Rev. Lett. 100, 142001 (2008) [arXiv:0708.1790 [hep-ex]].
[6] B. Aubert et al. [BaBar Collaboration], Phys. Rev. Lett. 95, 142001 (2005) doi:10.1103/PhysRevLett.95.142001 [hep-ex/0506081].
[7] M. Ablikim et al. [BESIII Collaboration], Phys. Rev. Lett. 112, 132001 (2014) [arXiv:1308.2760 [hep-ex]].
[8] M. Ablikim et al. [BESIII Collaboration], Phys. Rev. Lett. 111, 242001 (2013) [arXiv:1309.1896 [hep-ex]].
[9] M. Ablikim et al. [BESIII Collaboration], Phys. Rev. Lett. 110, 252001 (2013) [arXiv:1303.5949 [hep-ex]].
[10] Z. Q. Liu et al. [Belle Collaboration], Phys. Rev. Lett. 110, 252002 (2013) [arXiv:1304.0121 [hep-ex]].
[11] I. Adachi [Belle Collaboration], arXiv:1105.4583 [hep-ex].
[12] T. Nakano et al. [LEPS Collaboration], Phys. Rev. Lett. 91, 012002 (2003) doi:10.1103/PhysRevLett.91.012002 [hep-ex/0301020].
[13] R. Aaij et al. [LHCb Collaboration], Phys. Rev. Lett. 115, 072001 (2015) doi:10.1103/PhysRevLett.115.072001 [arXiv:1507.03414 [hep-ex]].
[14] R. Aaij et al. [LHCb Collaboration], Phys. Rev. Lett. 122, no. 22, 222001 (2019) doi:10.1103/PhysRevLett.122.222001 [arXiv:1904.03947 [hep-ex]].
[15] H. X. Chen, W. Chen and S. L. Zhu, arXiv:1903.11001 [hep-ph].
[16] J. He, Eur. Phys. J. C 79, no. 5, 393 (2019) doi:10.1140/epjc/s10052-019-6906-1 [arXiv:1903.11872 [hep-ph]].
[17] C. J. Xiao, Y. Huang, Y. B. Dong, L. S. Geng and D. Y. Chen, Phys. Rev. D 100, no. 1, 014022 (2019) doi:10.1103/PhysRevD.100.014022 [arXiv:1904.00872 [hep-ph]].
[18] J. R. Zhang, arXiv:1904.10711 [hep-ph].
[19] Z. G. Wang and X. Wang, arXiv:1907.04582 [hep-ph].
[20] R. Chen, Z. F. Sun, X. Liu and S. L. Zhu, Phys. Rev. D 100, no. 1, 011502 (2019) doi:10.1103/PhysRevD.100.011502 [arXiv:1903.11013 [hep-ph]].
[21] Y. H. Lin and B. S. Zou, arXiv:1908.05309 [hep-ph].
[22] Y. J. Xu, C. Y. Cui, Y. L. Liu and M. Q. Huang, arXiv:1907.05097 [hep-ph].
[23] Z. G. Wang, arXiv:1905.02892 [hep-ph].
[24] J. B. Cheng and Y. R. Liu, arXiv:1905.08605 [hep-ph].
[25] C. Fernández-Ramírez et al. [JPAC Collaboration], arXiv:1904.10021 [hep-ph].
[26] E. E. Salpeter, Phys. Rev. 87, 328 (1952). doi:10.1103/PhysRev.87.328
[27] C. H. Chang, J. K. Chen, X. Q. Li and G. L. Wang, Commun. Theor. Phys. 43, 113 (2005) doi:10.1088/0253-6102/43/1/023 hep-ph/0406050.
[28] C. H. Chang, C. S. Kim and G. L. Wang, Phys. Lett. B 623, 218 (2005) doi:10.1016/j.physletb.2005.07.059 hep-ph/0505205.
[29] X. H. Guo, A. W. Thomas and A. G. Williams, Phys. Rev. D 59, 116007 (1999) doi:10.1103/PhysRevD.59.116007 hep-ph/9805331.
[30] X. H. Guo and X. H. Wu, Phys. Rev. D 76 (2007) 056004 arXiv:0704.3105 [hep-ph].
[31] G. Q. Feng, Z. X. Xie and X. H. Guo, Phys. Rev. D 83 (2011) 016003.
[32] H. W. Ke and X. Q. Li, Eur. Phys. J. C 78, no. 5, 364 (2018) doi:10.1140/epjc/s10052-018-5834-9 arXiv:1801.00675 [hep-ph].
[33] H. W. Ke, X. Q. Li, Y. L. Shi, G. L. Wang and X. H. Yuan, JHEP 1204, 056 (2012) doi:10.1007/JHEP04(2012)056 arXiv:1202.2178 [hep-ph].
[34] M.-H. Weng, X.-H. Guo and A. W. Thomas, Phys. Rev. D 83, 056006 (2011) doi:10.1103/PhysRevD.83.056006 arXiv:1012.0082 [hep-ph].
[35] Q. Li, C. H. Chang, S. X. Qin and G. L. Wang, arXiv:1903.02282 [hep-ph].
[36] Z. Y. Wang, J. J. Qi, X. H. Guo and J. Xu, arXiv:1901.04474 [hep-ph].
[37] F. J. Dyson, Phys. Rev. 85, 631 (1952). doi:10.1103/PhysRev.85.631
[38] C. Meng and K. T. Chao, Phys. Rev. D 77, 074003 (2008) doi:10.1103/PhysRevD.77.074003 arXiv:0712.3595 [hep-ph].
[39] H. Y. Cheng, C. K. Chua and A. Soni, Phys. Rev. D 71, 014030 (2005) doi:10.1103/PhysRevD.71.014030 hep-ph/0409317.
[40] X. Liu, B. Zhang and S. L. Zhu, Phys. Lett. B 645, 185 (2007) doi:10.1016/j.physletb.2006.12.031 hep-ph/0610278.
[41] H. W. Ke, X. Q. Li and X. Liu, Phys. Rev. D 82, 054030 (2010) doi:10.1103/PhysRevD.82.054030 arXiv:1006.1437 [hep-ph].
[42] M. Tanabashi et al. [Particle Data Group], Phys. Rev. D 98, no. 3, 030001 (2018). doi:10.1103/PhysRevD.98.030001
[43] D. Ronchen et al., Eur. Phys. J. A 49, 44 (2013) doi:10.1140/epja/i2013-13044-5 arXiv:1211.6998 [nucl-th].
[44] C. W. Shen, D. Rönchen, U. G. Meißner and B. S. Zou, Chin. Phys. C 42, no. 2, 023106 (2018) doi:10.1088/1674-1137/42/2/023106 arXiv:1710.03885 [hep-ph].
[45] C. W. Shen, F. K. Guo, J. J. Xie and B. S. Zou, Nucl. Phys. A 954, 393 (2016) doi:10.1016/j.nuclphysa.2016.04.034 arXiv:1603.04672 [hep-ph].
[46] J. He, Phys. Rev. D 95, no. 7, 074031 (2017) doi:10.1103/PhysRevD.95.074031 arXiv:1701.03738 [hep-ph].
Appendix A: The effective interactions

The effective interactions can be found in [43, 45, 46]

\[ \mathcal{L}_{PP\rho} = ig_{PP\rho} \phi_P \rho^\mu \cdot \tau \partial_\mu \phi_P + \text{c.c.,} \] (A1)

\[ \mathcal{L}_{PPV} = ig_{PPV} \phi(x)_P \rho \phi(x)_P \phi(x)_P^\dagger + \text{c.c.,} \] (A2)

\[ \mathcal{L}_{BB\rho} = -g_{BB\rho} \bar{\psi}_B (\gamma^\mu - \frac{K_{BB\rho}}{2m_B} \sigma^{\mu\nu} \partial_\nu) \rho^\mu \cdot \tau \psi_B, \] (A3)

\[ \mathcal{L}_{BB\omega} = -g_{BB\omega} \bar{\psi}_B (\gamma^\mu - \frac{K_{BB\omega}}{2m_B} \sigma^{\mu\nu} \partial_\nu) \omega^\mu \psi_B, \] (A4)

\[ \mathcal{L}_{VVP} = ig_{VVP} \varepsilon_{\mu\nu\alpha\beta} \partial^\mu \phi_V(x)_P \partial_\nu \phi_V(x)_P \phi_P(x) + \text{c.c.,} \] (A5)

\[ \mathcal{L}_{BB\phi} = ig_{BB\phi} \bar{\psi}_B \gamma^5 \psi_B, \] (A6)

where c.c. is the complex conjugate term, \( \tau \) is Pauli matrix for \( I = \frac{1}{2} \) and \( \tau_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \),

\[ \tau_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \] and \( \tau_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \) for \( I = 1 \). when \( \phi_P = \begin{pmatrix} D_0 \\ D_+ \end{pmatrix} \) and \( \psi_B = \begin{pmatrix} \Sigma_{++}^0 \\ \Sigma_{+}^+ \\ \Sigma_{0}^0 \\ \Sigma_{-}^0 \end{pmatrix} \), the effective interactions are consistent with those in Ref. [20].

Appendix B: The isospin factors in the kernel

To gain the characteristic hadronic property of the pentaquark, one needs to project the bound states on the vacuum via the field operators \( \mathcal{B}_1, \mathcal{B}_2, \mathcal{M}_1 \) and \( \mathcal{M}_2 \) and

\[ \langle 0 | T \mathcal{B}_1(x_1) \mathcal{M}_j(x_2) | P \rangle_{I, I_3} = C_{(I, I_3)}^{ij} \chi_P^I(x_1, x_2), \] (B1)

where \( \chi_P^I(x_1, x_2) \) is the B-S wave function for the bound state with isospin \( I \). The isospin coefficients \( C_{(\frac{1}{2}, \frac{1}{2})}^{22} \) for \( \Lambda_c D \) bound state is 1, the isospin coefficients for \( \Sigma_c D \) bound states are

\[ C_{(\frac{1}{2}, \frac{1}{2})}^{11} = \sqrt{\frac{2}{3}}, \quad C_{(\frac{1}{2}, \frac{1}{2})}^{22} = -\sqrt{\frac{1}{3}}, \quad C_{(\frac{1}{2}, \frac{1}{2})}^{11} = \sqrt{\frac{2}{3}}, \quad C_{(\frac{1}{2}, \frac{1}{2})}^{22} = \sqrt{\frac{1}{3}}, \quad C_{(\frac{1}{2}, \frac{1}{2})}^{12} = 1. \] (B2)

Then corresponding B-S equation was deduced in Ref. [36] as

\[ C_{(I, I_3)}^{ij} \chi_P^I(p) = S_B(p_\perp) \int \frac{d^4 q}{(2\pi)^3} K_{ij, lk}^{\perp}(P, p, q) C_{(I, I_3)}^{lk} \chi_P^I(q) S_M(p_\perp), \] (B3)

where \( K_{ij, lk}^{\perp}(P, p, q) \) is still the kernel and its superscripts \( ij \) and \( lk \) denote the initial and final components.
For $\Lambda_cD$

$$
\chi_P(p) = S_B(p_1) \int \frac{d^4q}{(2\pi)^3} K^{22,22}_\chi(p_2) S_M(p_2),
$$
(B4)

For $I = \frac{3}{2}, I_z = \frac{1}{2}$ state $\Sigma_c\bar{D}$ if the components are $\Sigma^\pm D^+$

$$
\chi_P(p) = S_B(p_1) \int \frac{d^4q}{(2\pi)^3} (-K^{11,11}_{\chi} - \frac{1}{\sqrt{2}} K^{11,22}_{\chi})\chi_p(q) S_M(p_2),
$$
(B5)

if the components are $\Sigma^+_c\bar{D}^0$

$$
\chi_P(p) = S_B(p_1) \int \frac{d^4q}{(2\pi)^3} (K^{22,22}_\chi - \sqrt{2} K^{22,11}_\chi)\chi_p(q) S_M(p_2),
$$
(B6)

so

$$
\chi_P(p) = S_B(p_1) \int \frac{d^4q}{(2\pi)^3} (-\frac{2}{3} K^{11,11}_{\chi} - \frac{\sqrt{2}}{3} K^{11,22}_{\chi} + \frac{1}{3} K^{22,22}_{\chi} - \frac{\sqrt{2}}{3} K^{22,11}_{\chi})\chi_p(q) S_M(p_2),
$$
(B7)

For $I = \frac{3}{2}, I_z = \frac{1}{2}$ $\Sigma_c\bar{D}$ state if the components are $\Sigma^\pm D^-$

$$
\chi_P(p) = S_B(p_1) \int \frac{d^4q}{(2\pi)^3} (-K^{11,11}_{\chi} + \frac{1}{\sqrt{2}} K^{11,22}_{\chi})\chi_p(q) S_M(p_2),
$$
(B8)

if the components are $\Sigma^+_c\bar{D}^0$

$$
\chi_P(p) = S_B(p_1) \int \frac{d^4q}{(2\pi)^3} (K^{22,22}_\chi + \sqrt{2} K^{22,11}_\chi)\chi_p(q) S_M(p_2),
$$
(B9)

so

$$
\chi_P(p) = S_B(p_1) \int \frac{d^4q}{(2\pi)^3} (-\frac{1}{3} K^{11,11}_{\chi} + \frac{\sqrt{2}}{3} K^{11,22}_{\chi} + \frac{2}{3} K^{22,22}_{\chi} + \frac{\sqrt{2}}{3} K^{22,11}_{\chi})\chi_p(q) S_M(p_2),
$$
(B10)

The sign “−” before $K^{11,11}$ in Eq. (B5) and (B8) comes from the interactions in Appendix A. For $\Lambda_c\bar{D}$ state the two components interact only by exchanging $\omega$. However $\omega$ and $\rho$ can contribute to the $\Sigma_c\bar{D}$ state. One also has $K^{11,11}(\omega) = K^{22,22}(\omega), K^{11,22}(\omega) = K^{22,11}(\omega) = 0, K^{11,22}(\rho) = K^{22,11}(\rho) = \sqrt{2} K^{11,11}(\rho)$ and $K^{22,22}(\rho) = 0$. In the Eqs. (B7) and (B10) $K^{11,11}, K^{11,22}$ and $K^{22,11}$ can be changed into $K^{11,11}$ and then the coefficient of $K^{11,11}$ is just the isospin factor $C_{I,I_z}^L: C_{\frac{3}{2}, \frac{3}{2}}^L = 1, -2$ for $\omega$ and $\rho$, $C_{\frac{3}{2}, \frac{3}{2}}^L = C_{\frac{1}{2}, \frac{1}{2}}^L = 1, 1$ for $\omega$ and $\rho$.

**Appendix C:** The coupled equation of $f_1(|p_T|)$ and $f_2(|p_T|)$ after integrating over $p_t$ and some formulas for azimuthal integration

$$
f_1(|p_T|) = -\int \frac{d^3q_T}{(2\pi)^3} \frac{C_{I,I_z} g_{MMV} g_{BRV} F^2(k, m_V)}{2\omega_1 (M + \omega_1 + \omega_2) (M + \omega_1 - \omega_2)}
$$
\[
\begin{align*}
\{ \p_T \cdot \q_T + \p_T^2 + 2(m_1 - \omega_1)(M + \omega_1) + (\p_T^2 - \p_T \cdot \q_T)(\p_T^2 - \q_T^2)/m_V^2 \} f_1(|\p_T|) + \\
\frac{(\p_T^2 - \q_T^2)(\p_T \cdot \q_T - \q_T^2)(m_1 - \omega_1)}{m_V^2} + (m_1 - \omega_1)(\p_T \cdot \q_T + \q_T^2) + 2(M + \omega_1)\p_T \cdot \q_T \\
- \frac{\kappa}{4m_B} - 4[\p_T \cdot \q_T^2 - \p_T^2 \q_T^2 + (m_1 - \omega_1)(\p_T \cdot \q_T - \q_T^2)(\omega_1 + M)] f_2(|\q_T|) + \\
\frac{\kappa}{4m_B} - 4[\p_T \cdot \q_T^2 - \p_T^2 \q_T^2 + (m_1 - \omega_2 + M)(q_T^2 - \p_T \cdot \q_T)(-\omega_2)] f_2(|\q_T|) + \\
\frac{\kappa}{4m_B} - 4[\p_T \cdot \q_T^2 - \p_T^2 \q_T^2 + (m_1 - \omega_1 + M)(\p_T \cdot \q_T - \q_T^2) - 2\omega_1\p_T \cdot \q_T] f_2(|\q_T|)
\end{align*}
\]
- \frac{\kappa}{4m_B} \frac{4(p_T \cdot q_T - p_T^2)\omega_2(M - m_1 - \omega_2)}{-(p_T - q_T)^2 - m_V^2} f_1(|q_T|). \quad (C2)

Since \(d^3q_T = q_T^2 \sin(\theta) d\theta |q_T| d\theta d\phi\) and \(p_T \cdot q_T = |p_T||q_T|\cos(\theta)\) one can carry out the azimuthal integration for Eqs. (C1) and (C2) analytically. Some useful integrations are defined as follow:

\[ J_0 \equiv \int_0^\pi \sin(\theta) d\theta \frac{1}{-(p_T - q_T)^2 - m_V^2} \left[ \frac{\Lambda^2 - m_V^2}{\Lambda^2 - (p_T - q_T)^2} \right]^2 \]
\[ = \int_0^\pi \sin(\theta) d\theta \frac{p_T \cdot q_T}{-(p_T - q_T)^2 - m_V^2} \left[ \frac{\Lambda^2 - m_V^2}{\Lambda^2 - (p_T - q_T)^2} \right]^2 \]
\[ = \int_0^\pi \sin(\theta) d\theta \frac{p_T}{(m_V^2 - \Lambda^2)(p_T^2 + q_T^2 + \Lambda^2)} \]
\[ + \frac{1}{2|p_T||q_T|} \left\{ \ln \left( \frac{|p_T| + |q_T|)^2 + \Lambda^2}{(|p_T| - |q_T|)^2 + \Lambda^2} \right) - \ln \left( \frac{|p_T| + |q_T|)^2 + m_V^2}{(|p_T| - |q_T|)^2 + m_V^2} \right) \right\}, \quad (C3) \]

\[ J_1 \equiv \int_0^\pi \sin(\theta) d\theta \frac{|p_T| \cdot q_T}{-(p_T - q_T)^2 - m_V^2} \left[ \frac{\Lambda^2 - m_V^2}{\Lambda^2 - (p_T - q_T)^2} \right]^2 \]
\[ = \int_0^\pi \frac{|p_T| \cdot q_T}{-(p_T - q_T)^2 - m_V^2} \left[ \frac{\Lambda^2 - m_V^2}{\Lambda^2 - (p_T - q_T)^2} \right]^2 \]
\[ = \int_0^\pi \frac{|p_T| \cdot q_T}{(m_V^2 - \Lambda^2)(p_T^2 + q_T^2 + \Lambda^2)} \] 
\[ + \frac{1}{4|p_T||q_T|} \left\{ \ln \left( \frac{|p_T| + |q_T|)^2 + \Lambda^2}{(|p_T| - |q_T|)^2 + \Lambda^2} \right) - \ln \left( \frac{|p_T| + |q_T|)^2 + m_V^2}{(|p_T| - |q_T|)^2 + m_V^2} \right) \right\}, \quad (C4) \]

\[ J_2 \equiv \int_0^\pi \sin(\theta) d\theta \frac{(p_T \cdot q_T)^2}{-(p_T - q_T)^2 - m_V^2} \left[ \frac{\Lambda^2 - m_V^2}{\Lambda^2 - (p_T - q_T)^2} \right]^2 \]
\[ = \int_0^\pi \frac{(p_T \cdot q_T)^2}{(m_V^2 - \Lambda^2)(p_T^2 + q_T^2 + \Lambda^2)} \] 
\[ + \frac{1}{8|p_T||q_T|} \left\{ \ln \left( \frac{|p_T| + |q_T|)^2 + \Lambda^2}{(|p_T| - |q_T|)^2 + \Lambda^2} \right) - \ln \left( \frac{|p_T| + |q_T|)^2 + m_V^2}{(|p_T| - |q_T|)^2 + m_V^2} \right) \right\}, \quad (C5) \]

\[ A_{11}(p_T, q_T) = \frac{-q_T^2}{(2\pi)^2 2\omega_1 (M + \omega_1 + \omega_2)(M + \omega_1 - \omega_2)} \]
\[ \{ [J_1 + p_T^2 J_0 + 2(m_1 - \omega_1)(M + \omega_1)J_0 + (p_T^2 J_0 - J_1)(p_T^2 - q_T^2)/m_V^2] f_1(|q_T|) \} \]
\[
- \frac{k}{4m_B} [4(p_T^2 J_0 - J_1)(M \eta_2 + \omega_1)] f_1(|q_T|) + \\
\frac{q_T^2}{C_{1,1,1,1g,MMVgSSV}} \left\{ \frac{(p_T^2 - q_T^2)(J_1 - q_T^2 J_0)(m_1 - \omega_2)}{m_v^2} + (m_1 - \omega_1)(J_1 + q_T^2 J_0) + 2(M + \omega_1)J_1 \right\} f_2(|q_T|) + \\
\frac{C_{1,1,1,1g,MMVgSSV}}{(2\pi)^2 2\omega_2(M + \omega_1 - \omega_2)(M - \omega_1 - \omega_2)} \left\{ \frac{(p_T^2 - q_T^2)(J_1 - q_T^2 J_0)(m_1 - \omega_2 + M)}{m_v^2} + (m_1 + M - \omega_2)(J_1 + q_T^2 J_0) + 2\omega_2 J_1 \right\} f_2(|q_T|)
\]

\[A_{12}(p_T, q_T) = \frac{q_T^2}{(2\pi)^2 2\omega_1(M + \omega_1 + \omega_2)(M + \omega_1 - \omega_2)} \left\{ \frac{(p_T^2 - q_T^2)(J_1 - q_T^2 J_0)(m_1 - \omega_1)}{m_v^2} + (m_1 - \omega_1)(J_1 + q_T^2 J_0) + 2(M + \omega_1)J_1 \right\} f_2(|q_T|) + \\
\frac{k}{m_B} [J_2 - p_T^2 q_T^2 J_0 + (m_1 - \omega_1)(J_1 - q_T^2 J_0)(\omega_1 + M)] f_2(|q_T|) + \\
\frac{q_T^2}{C_{1,1,1,1g,MMVgSSV}} \left\{ \frac{(p_T^2 - q_T^2)(J_1 - q_T^2 J_0)(m_1 - \omega_2 + M)}{m_v^2} + (m_1 + M - \omega_2)(J_1 + q_T^2 J_0) + 2\omega_2 J_1 \right\} f_2(|q_T|)\]

\[A_{21}(p_T, q_T) p_T^2 = \frac{q_T^2}{(2\pi)^2 2\omega_1(M + \omega_1 + \omega_2)(M + \omega_1 - \omega_2)} \left\{ \frac{(p_T^2 - q_T^2)(J_1 - p_T^2 J_0)(-m_1 - \omega_1)}{m_v^2} + (m_1 - \omega_1)(J_1 + p_T^2 J_0) + 2M p_T^2 J_0 + 2\omega_1 J_1 \right\} f_1(|q_T|) - \\
\frac{4k}{m_B} \frac{4(J_1 - p_T^2 J_0)(-\omega_1 - M)(m_1 + \omega_1)}{-(p_T - q_T)^2 - m_v^2} f_1(|q_T|) + \\
\frac{q_T^2}{C_{1,1,1,1g,MMVgSSV}} \left\{ \frac{(p_T^2 - q_T^2)(J_1 - p_T^2 J_0)(M - m_1 - \omega_2)}{m_v^2} + (m_1 - M - \omega_2)(J_1 + p_T^2 J_0) + 2\omega_2 J_1 \right\} f_1(|q_T|) + \\
\frac{C_{1,1,1,1g,MMVgSSV}}{(2\pi)^2 2\omega_2(M + \omega_1 - \omega_2)(M - \omega_1 - \omega_2)} \left\{ \frac{(p_T^2 - q_T^2)(J_1 - p_T^2 J_0)(M - m_1 - \omega_2)}{m_v^2} + (m_1 - M - \omega_2)(J_1 + p_T^2 J_0) + 2\omega_2 J_1 \right\} f_1(|q_T|) - \\
\frac{4k}{m_B} [4(J_1 - p_T^2 J_0)\omega_2(M - m_1 - \omega_2)] f_1(|q_T|)\]

\[A_{22}(p_T, q_T) p_T^2 = \frac{q_T^2}{(2\pi)^2 2\omega_1(M + \omega_1 + \omega_2)(M + \omega_1 - \omega_2)} \left\{ \frac{-(p_T^2 - q_T^2)(J_1 + q_T^2 J_0)}{m_v^2} + 2J_1(m_1 + \omega_1) + p_T^2 q_T^2 J_0 - J_1 \right\} f_2(|q_T|) + \\
\frac{k}{4m_B} \left\{ \frac{4(M + \omega_1) p_T^2 J_1 - (m_1 + \omega_1)J_2 + p_T^2 q_T^2 (m_1 - M) J_0}{m_v^2} \right\} f_2(|q_T|)\]
\[ + \frac{q_T^2}{(2\pi)^2 \frac{C_{1.1}}{2\omega_2(M + \omega_1 - \omega_2)(M - \omega_1 - \omega_2)}} \frac{g_{MM}g_{BB}}{2} \times \]

\[ \left\{ \left[ -p_T^2(J_1 + q_T^2J_0) + 2(m_1 + \omega_2 - M)\omega_2J_1 + p_T^2(q_T^2J_0 - J_1) \frac{(p_T^2 - q_T^2)}{m_v^2} \right] f_2(|q_T|) + \right. \]

\[ - \frac{\kappa}{4m_B} 4[\omega_2 p_T^2 J_1 - (m_1 + \omega_2 - M) J_2 + p_T^2 q_T^2 (m_1 - M) J_0] f_2(|q_T|) \}. \quad (C9) \]