Deformations of Strongly Pseudoconvex Domains

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Abstract: We show that two smoothly bounded, strongly pseudoconvex domains which are diffeomorphic may be smoothly deformed into each other, with all intermediate domains being strongly pseudoconvex. This result relates to Lempert’s ideas about Kobayashi extremal discs, and also has intrinsic interest.

0 Introduction

Ever since the solution of the Levi problem in the 1940s and 1950s, it has been a matter of central importance to understand the geometry of pseudoconvex domains. This investigation has many aspects, including topological features and analytic features.

One question that has received little attention is that of deforming one pseudoconvex domain into another. This matter is subtle. A form of the question comes up at the end of Lempert’s seminal paper [LEM], where he deforms strongly convex domains. Of course that is a relatively easy matter, but it begs the question (if one wants to generalize Lempert’s results to strongly pseudoconvex domains) of deforming more general types of domains. See [KRA2] for an investigation of this type of generalization.

In the present paper we explore such deformation ideas, and we prove a positive deformation result for smoothly bounded, strongly pseudoconvex domains. One interesting aspect of our approach is that we make decisive use of the Fornæss embedding theorem [FOR]. This may be the first actual application of Fornæss’s theorem. Of course Fornæss’s theorem applies only to strongly pseudoconvex domains, and there is no hope of adapting the techniques presented

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here to a more general class of domains. It is plausible that various natural classes of Levi geometry should be preserved under smooth deformation, but the techniques for proving a very general result do not seem to be available at this time.

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1 Principal Results

Our main theorem is this:

**Theorem 1** Let \( \Omega_{-1}, \Omega_1 \) be smoothly bounded, strongly pseudoconvex domains in \( \mathbb{C}^n \). Write \( \Omega_j = \{ z \in \mathbb{C}^n : \rho_j(z) < 0 \}, j = -1, 1, \) where \( \rho_j \) is a defining function for \( \Omega_j \) (see [KRA1] for this concept). Assume that \( \Omega_{-1} \) and \( \Omega_1 \) are diffeomorphic.

Then there is a smooth deformation \( \rho(z, t) \) on \( \mathbb{C}^n \times [-1, 1] \) so that, for each \( t \in [-1, 1] \), the function \( \rho_t(z) \equiv \rho(z, t) \) is a smooth defining function. Also \( \rho(z, 0) = \rho_{-1}(z) \) and \( \rho(z, 1) = \rho_1(z) \). Moreover, for each \( t \in [-1, 1] \), the domain \( \Omega_t \equiv \{ z \in \mathbb{C}^n : \rho_t(z) < 0 \} \) is smoothly bounded and strongly pseudoconvex. Finally, there is a \( c > 0 \) so that all the eigenvalues of the Levi form at any boundary point of any \( \Omega_t \) are not less than \( c \).

It should be stressed that we do not conclude that the domains \( \Omega_{-1} \) and \( \Omega_1 \) are biholomorphic. This is impossible in the strongest sense—see [GRK] and [BSW]. We are only setting up a smooth deformation of one domain into the other. What is interesting about the proof is that we simultaneously Fornæss-embed the two domains \( \Omega_{-1} \) and \( \Omega_1 \) into a single strongly convex domain \( W_j \), perform the deformation in \( W_j \), and then pull it back. Along the way, an uncountably infinite family of \( \partial \) problems must be solved.

2 Proof of the Theorem

According to the Fornæss embedding theorem [FOR], there is, for \( j = -1, 1 \), a strongly convex domain \( W_j \subset \mathbb{C}^{N_j} \) (with, in general \( N_j >> n \)), a neighborhood \( U_j \) of \( \Omega_j \), and a univalent holomorphic mapping \( \Phi_j : U_j \to \mathbb{C}^{N_j} \), with \( \Phi_j(\Omega_j) \subset W_j \), \( \Phi_j(\Omega_j) \subset \mathbb{C}^{N_j} \setminus \partial W_j \), and so that \( \Phi_j(\Omega_j) \) is transversal to \( \partial W_j \) where they meet.

We may assume that \( N_j = N_{-1} \). Let \( N = 2N_1 + 1 \). Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be an even, strictly concave function so that

1. \( \varphi(-1) = 0; \)
2. \( \varphi(1) = 0; \)
3. \( \varphi(0) = 1; \)
4. \( \varphi \) is strictly decreasing to \( -\infty \) on \( (0, \infty) \);
(4) $-\varphi''(x) \geq K$ for some large positive $K > 0$ and all $x \in \mathbb{R}$.

Also let $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function satisfying

1. $\lambda$ is smooth;
2. $0 \leq \lambda(x) \leq 1$ for all $x$;
3. $\lambda(x) \equiv 0$ when $x \leq -1$;
4. $\lambda(x) \equiv 1$ when $x \geq 1$;
5. $\lambda$ is monotone increasing for $-1 < x < 1$;

Let $\mu_{-1}$ be a defining function for $W_{-1}$ and $\mu_1$ a defining function for $W_1$.

If $z \in \mathbb{C}^N$, then let us write

$$z = (z_1, z_2, \ldots, z_N) = (z'; z^*) ,$$

where

$$z' = (z_1, z_2, \ldots, z_{N_1})$$

and

$$z^* = (z_{N_1+1}, z_{N_1+2}, \ldots, z_N) .$$

Now we consider in $\mathbb{C}^N$ the domain with defining function

$$\rho(z) = (1 - \lambda(\text{Re } z_N))\mu_{-1}(z') + \lambda(\text{Re } z_N)\mu_1(z') - \varphi(|z^*|^2) .$$

We set

$$W = \{ z \in \mathbb{C}^N : \rho(z) < 0 \} .$$

We claim that, in a natural sense, $\Omega_{-1}$ and $\Omega_1$ are Fornæss-embedded into $W$. Let us see why.

First, $\rho$ is strictly convex. The pure second derivatives of $\rho$ in $z'$ are clearly under control and positive. The pure second derivatives in $z^*$ are controlled by $-\varphi''$ provided that $K$ is large enough. Also the mixed second derivatives are controlled by $-\varphi''$ provided that $K$ is large enough.

Second, $\rho(z'; (0, 0, \ldots, -1 + 0i)) = \mu_{-1}(z')$ so that $\{ z : z^* = (0, 0, \ldots, -1 + 0i), \rho(z) < 0 \}$ is simply a copy of $W_{-1}$. And $\rho(z'; (0, 0, \ldots, 1 + 0i)) = \mu_1(z')$, so that $\{ z : z^* = (0, 0, \ldots, 1 + 0i), \rho(z) < 0 \}$ is a copy of $W_1$. Thus

$$\tilde{\Phi}_{-1}(w) \equiv (\Phi_{-1}(w); (0, 0, \ldots, -1 + 0i))$$

embeds $\Omega_{-1}$ into $W$. And

$$\tilde{\Phi}_1(w) \equiv (\Phi_1(w); (0, 0, \ldots, 1 + 0i))$$

embeds $\Omega_1$ into $W$. As a result, we have both $\Omega_{-1}$ and $\Omega_1$ embedded, in the fashion of the Fornæss embedding theorem, into the single strongly convex domain $W$. 

3
Assume without loss of generality that 0 lies in $W$. Now we may define, for $z \in \mathbb{C}^n$, and $-1 \leq t \leq 1$, a smooth function $\omega(z, t)$ so that $\omega(z, -1)$ is a defining function for $W \cap \{z : z^* = (0, 0, \ldots, -1 + 0i)\}$, $\omega(z, 1)$ is a defining function for $W \cap \{z : z^* = (0, 0, \ldots, 1 + 0i)\}$, and, in general, $\omega(z, t)$ is a defining function for $W \cap \{z : z^* = (0, 0, \ldots, t + 0i)\}$, $-1 \leq t \leq 1$. Let $S_t = \{z \in W : z^* = (0, 0, \ldots, t + 0i)\}$. Then, for $t$ near $-1$, we may map $S_{-1}$ to $S_t$ by sending a point $z$ in $S_{-1}$ that is distance $\delta$ from the boundary to a point $\tilde{z}_t$ in $S_t$ that is distance $\delta$ from the boundary and so that the line through the origin and $\tilde{z}_t$ has the same projection to the set $\{z^* = 0\}$ as the line through the origin and $z$. This is a smooth mapping for $z$ near the boundary, and we may easily interpolate it to a smooth mapping on all of $S_{-1}$. Call the mapping $\pi_t$. Then, if $\tilde{\Phi}_{-1}$ is the Fornæss embedding of $\Omega_{-1}$ into $S_{-1}$ and $\tilde{\Phi}_{-1}^{-1}$ its inverse defined on the image of $\tilde{\Phi}_{-1}$, then we have a pseudo-inverse-embedding $\tilde{\Phi}_{-1}^{-1} \circ \pi_t^{-1}$ of $S_t$ into $\mathbb{C}^n$. We call this a pseudo-inverse-embedding because it will not be holomorphic. But we can solve the $\overline{\partial}$ equation (see [HEN] and [SIU])

$$\overline{\partial}h = -\overline{\partial}\left[\tilde{\Phi}_{-1}^{-1} \circ \pi_t^{-1}\right]$$

to find a function $h$ (which will have small $C^2$ norm because $\overline{\partial}\pi_t^{-1}$ has small $C^2$ norm) so that $\tau_t \equiv \tilde{\Phi}_{-1}^{-1} \circ \pi_t^{-1} + h$ is a holomorphic embedding of $S_t$ into $\mathbb{C}^n$. We think of the image of $\tau_t$ as a perturbation $\Omega_t$ of $\Omega_{-1}$. We can continue to incrementally increase the value of $t$ and create additional deformations of $\Omega_{-1}$ as $t$ increases. Note here that it is propitious to use the Henkin solution of the $\overline{\partial}$ problem—see [HEN]. For it is known [GRK] to provide smoothly varying solutions when the data is varied smoothly.

At the same time, we could begin at $\Omega_1$ and deform in the same fashion, decreasing values of $t$. This incrementally creates deformations of $\Omega_1$. When the values of $t$ from above and the values of $t$ from below meet (say at $t = 0$), they must of course have data sets $S_t$ that are close together in the smooth topology. So the deformations constructed in $\mathbb{C}^n$ will also be close together. In sum, the two streams of deformed domains give rise to a deformation of $\Omega_{-1}$ to $\Omega_1$. Finally, it is clear by inspection that the eigenvalues of the Levi form for $\Omega_t$ are pullbacks of eigenvalues of the Levi form for $W$. So the eigenvalues of the Levi form for all the $\Omega_t$ are uniformly bounded from 0.

### 3 Concluding Remarks

It would be a matter of some interest to solve the problem treated here in the algebraic category: If $\Omega_{-1}$ and $\Omega_1$ are bounded domains in $\mathbb{C}^n$ with strongly pseudoconvex, algebraic boundaries, then can one be deformed into the other with all intermediate domains being algebraic and strongly pseudoconvex? Likewise, one would like to solve this problem in the real analytic category: If $\Omega_{-1}$ and $\Omega_1$ are bounded domains in $\mathbb{C}^n$ with strongly pseudoconvex, real analytic boundaries, then can one be deformed into the other with all intermediate do-
mains being real analytic and strongly pseudoconvex? Unfortunately the methods of the present paper cannot be applied to either of these problems. Our constructions are strictly real-variable in nature. The methods used in Grauert’s solution of the Levi problem, and in his embedding theorem for real analytic manifolds (see [GRA]), may be useful in studying some of these new questions.

It also would be worthwhile to study these deformation questions for other types of domains, such as pseudoconvex, finite type domains. Certainly the Fornæss embedding theorem is not true for such domains, and we do not know how to attack such a problem at this time.

We hope to explore these new questions in a future paper.
REFERENCES

[BSW] D. Burns, S. Shnider, and R. O. Wells, On deformations of strictly pseudoconvex domains, *Invent. Math.* 46(1978), 237–253.

[FOR] J. E. Fornæss, Strictly pseudoconvex domains in convex domains, *Am. J. Math.* 98(1976), 529–569.

[GRA] H. Grauert, On Levi’s problem and the imbedding of real-analytic manifolds, *Ann. of Math.* 68(1958), 460–472.

[GRK] R. E. Greene and S. G. Krantz, Deformation of complex structures, estimates for the $\partial$ equation, and stability of the Bergman kernel, *Adv. Math.* 43(1982), 1–86.

[HEN] G. M. Henkin, Integral representations of functions holomorphic in strictly pseudoconvex domains and applications to the $\partial$ problem, *Mat. Sb.* 82(124), 300-308 (1970); *Math. U.S.S.R. Sb.* 11(1970), 273-281.

[KRA1] S. G. Krantz, *Function Theory of Several Complex Variables*, 2nd ed., American Mathematical Society, Providence, RI, 2001.

[KRA2] S. G. Krantz, The Kobayashi metric, extremal discs, and biholomorphic mappings, *Complex Variables and Elliptic Equations*, to appear.

[LEM] L. Lempert, La metrique Kobayashi et la representation des domaines sur la boule, *Bull. Soc. Math. France* 109(1981), 427–474.

[SII] Y.-T. Siu, The $\overline{\partial}$ problem with uniform bounds on derivatives, *Math. Ann.* 207(1974), 163–176.

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