Transformation groups of certain flat affine manifolds

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Abstract
In this paper we characterize the group of affine transformations of a flat affine simply connected manifold whose developing map is a diffeomorphism. This is proved by making use of some simple facts about homeomorphisms of $\mathbb{R}^n$ preserving open connected sets. We show some examples where the characterization is useful.

Keywords Flat affine manifolds · Flat affine Lie groups · Group of affine transformations · Étale affine representations · Left symmetric products

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1 Introduction
In what follows $M$ is a connected real n-dimensional manifold without boundary and $\nabla$ a linear connection on $M$. The torsion and curvature tensors of a connection $\nabla$ are defined as

\[ T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \]
\[ K_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \]
for any $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the space of smooth vector fields on $M$. If the curvature and torsion tensors of $\nabla$ are both null, the connection is called flat affine and the pair $(M, \nabla)$ is called a flat affine manifold. An affine transformation of $(M, \nabla)$ (or of $M$ relative to $\nabla$) is a diffeomorphism $F$ of $M$ verifying the condition $F^* \nabla_X Y = \nabla_{F_*X} F_*Y$, for all $X, Y \in \mathfrak{X}(M)$. The set of affine transformations of $(M, \nabla)$, denoted by $\text{Aff}(M, \nabla)$, is a group under composition and is called the group of affine transformations of $(M, \nabla)$ (or of $M$ relative to $\nabla$). This group endowed with the open-compact topology is a Lie group (see [4] page 229).

The usual connection $\nabla^0$ in $\mathbb{R}^n$ is defined, in local coordinates, by $\nabla^0_X Y = \sum_{i=1}^n X(f_i) \frac{\partial}{\partial x^i}$ when $Y = \sum_{i=1}^n f_i \frac{\partial}{\partial x^i}$. The group $\text{Aff}(\mathbb{R}^n, \nabla^0)$ agrees with the group of classical affine transformations $\text{Aff}(\mathbb{R}^n)$, that is, diffeomorphisms of $\mathbb{R}^n$ obtained as a composition of a linear transformation and a translation, that is, diffeomorphisms defined by $w \mapsto T(w) + v$ (by linear transformation we mean a linear isomorphism). We will also use the matrix notation $[A \ 0 \ v']$ with $A$ the matrix of the linear transformation and $v'$ the coordinate vector of the translation under some basis.

An infinitesimal affine transformation of $(M, \nabla)$ is a smooth vector field $X$ on $M$ whose local 1-parameter groups $\phi_t$ are local affine transformations of $(M, \nabla)$. We will denote by $\mathfrak{a}(M, \nabla)$ the real vector space of infinitesimal affine transformations of $(M, \nabla)$. An element $X$ of $\mathfrak{X}(M)$ belongs to $\mathfrak{a}(M, \nabla)$ if and only if it verifies

$$\mathcal{L}_X \circ \nabla_Y - \nabla_Y \circ \mathcal{L}_X = \nabla_{[X,Y]}, \text{ for all } Y \in \mathfrak{X}(M),$$

where $\mathcal{L}_X$ denotes the Lie derivative. If the connection is flat affine, the previous equation gives that $X \in \mathfrak{a}(M, \nabla)$ if and only if

$$\nabla_{\nabla_Y Z} X = \nabla_Y \nabla_Z X, \text{ for all } Y, Z \in \mathfrak{X}(M).$$

(1)

The vector subspace $\mathfrak{aff}(M, \nabla)$ of $\mathfrak{a}(M, \nabla)$ whose elements are complete, with the usual bracket of vector fields, is the Lie algebra of the group $\text{Aff}(M, \nabla)$ (see [4]). In this paper we use a simple fact about groups of homeomorphisms to determine the group of affine transformations in some particular open sets of $\mathbb{R}^n$.

Let $p : \tilde{M} \to M$ denote the universal covering map of a real $n$-dimensional flat affine connected manifold $(M, \nabla)$. The pullback $\tilde{\nabla}$ of $\nabla$ by $p$ determines a flat affine structure on $\tilde{M}$ and $p$ is an affine map. Moreover, the group $\pi_1(M)$ of deck transformations acts on $\tilde{M}$ by affine transformations. There exists an affine immersion $D : (\tilde{M}, \tilde{\nabla}) \to (\mathbb{R}^n, \nabla^0)$, called the developing map of $(M, \nabla)$, and a group homomorphism $A : \text{Aff}(\tilde{M}, \tilde{\nabla}) \to \text{Aff}(\mathbb{R}^n, \nabla^0)$ so that the following diagram commutes

(2)
The map $D$ is called the developing map and it was introduced by Ehresman (see [3]). In particular for every $\gamma \in \pi_1(M)$, we have $D \circ \gamma = h(\gamma) \circ D$ with $h(\gamma) := A(\gamma)$. The map $h$ is also a group homomorphism called the holonomy representation of $(M, \nabla)$.

If $M = G$ is an $n$-dimensional Lie group, a connection $\nabla$ is called left invariant if left multiplications, i.e., maps $L_g$ of $G$, with $g \in G$, defined by $L_g(h) = gh$, are affine transformations. Left invariant connections are usually denoted as $\nabla^+$. A Lie group $G$ endowed of a flat affine and left invariant connection $\nabla^+$ is called a flat affine Lie group. It is known that $(G, \nabla^+)$ is a flat affine Lie group if and only if there exists a homomorphism $\rho : \hat{G} \rightarrow \text{Aff}(\mathbb{R}^n)$ whose corresponding action leaves an open orbit with discrete isotropy, see [5] and [6]. Such representations are called affine étale representations. The open orbit turns out to be the image of the developing map $D : \hat{G} \rightarrow \mathbb{R}^n$ and $D$ is a covering map of $\mathcal{O}$. To have a left invariant linear connection $\nabla^+$ on a Lie group $G$ is equivalent to have a bilinear product on $\mathfrak{g} = \text{Lie}(G)$ given by $X \cdot Y = (\nabla^+X, Y)_\epsilon$, where $\epsilon$ denotes the neutral element of $G$ and $X^+, Y^+$ are the left invariant vector fields on $G$ determined respectively by $X$ and $Y$. When the bilinear product is given, the connection is defined by $\nabla_{X+}Y^+ = (X \cdot Y)^+$ forcing the conditions $\nabla fX+Y^+ = f\nabla X+Y^+$ and $\nabla X+fY^+ = X^+(f)Y^+ + f\nabla X+Y^+$. That $\nabla^+$ is torsion free is equivalent to the condition

$$[X, Y] = X \cdot Y - Y \cdot X$$

(3)

and the connection is flat if and only if $[X, Y] \cdot Z = X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z)$. Finally, the connection $\nabla^+$ is flat affine if and only if the bilinear product is left symmetric, that is,

$$(X \cdot Y) \cdot Z - (Y \cdot X) \cdot Z = X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z).$$

In this case, the algebra $(\mathfrak{g}, \cdot)$ is called a left symmetric algebra (see [6]). It is also known that the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ is endowed of an associative product compatible with the Lie bracket if and only if the left invariant connection $\nabla^+$ is bi-invariant. That is, if right multiplications $R_g : G \rightarrow G$ defined by $R_g(h) = hg$, for any $g$ in $G$, are also affine transformations of $G$ relative to $\nabla^+$ (see [6]).

2 Transformations preserving connected open sets

In this section we show that, for connected open sets of $\mathbb{R}^n$, homeomorphisms of $\mathbb{R}^n$ preserve the open set if and only if they send one point of the set into the set and preserve its boundary. We present some examples where the result is useful.

Lemma 1 If $\mathcal{O}$ is an open set of $\mathbb{R}^n$ with boundary $\partial \mathcal{O}$ then

$$\{T \in \text{Homeo}(\mathbb{R}^n) \mid T(\mathcal{O}) = \mathcal{O}\} \subseteq \{T \in \text{Homeo}(\mathbb{R}^n) \mid T(\partial \mathcal{O}) = \partial \mathcal{O}\},$$

where Homeo($\mathbb{R}^n$) is the group of homeomorphisms of $\mathbb{R}^n$. 

$\$
Proof First notice that \( \mathcal{A} = \{ T \in \text{Homeo}(\mathbb{R}^n) \mid T(\partial \mathcal{O}) = \partial \mathcal{O} \} \) and \( \{ T \in \text{Homeo}(\mathbb{R}^n) \mid T(\partial \mathcal{O}) = \partial \mathcal{O} \} \) are groups. Now, take \( T \in \text{Homeo}(\mathbb{R}^n) \) so that \( T(\mathcal{O}) = \mathcal{O} \) and let \( x \in \partial \mathcal{O} \). Choose a convergent sequence \( (x_n) \subseteq \mathcal{O} \) with \( x = \lim_{n \to \infty} x_n \). Hence \( T(x) = \lim_{n \to \infty} T(x_n) \) and therefore \( T(x) \in \overline{\mathcal{O}} \). However \( T(x) \notin \mathcal{O} \), otherwise since \( T^{-1} \in \mathcal{A} \), we would have that \( x = T^{-1} \circ T(x) \in \mathcal{O} \). It follows that \( T(\partial \mathcal{O}) \subseteq \partial \mathcal{O} \). As \( T^{-1} \in \mathcal{A} \), the previous argument shows that \( T^{-1}(\partial \mathcal{O}) \subseteq \partial \mathcal{O} \) and by applying \( T \), the other inclusion follows. \( \square \)

Lemma 2 If \( \mathcal{O} \) is a connected open set of \( \mathbb{R}^n \), then we have

\[
\{ T \in \text{Homeo}(\mathbb{R}^n) \mid T(\mathcal{O}) = \mathcal{O} \}
= \{ T \in \text{Homeo}(\mathbb{R}^n) \mid T(\partial \mathcal{O}) = \partial \mathcal{O}, T(p) \in \mathcal{O} \text{ for some } p \in \mathcal{O} \}
\]

Proof The first inclusion follows from the previous lemma. For the other inclusion, let \( T \in \{ T \in \text{Homeo}(\mathbb{R}^n) \mid T(\partial \mathcal{O}) = \partial \mathcal{O}, T(p) \in \mathcal{O} \text{ for some } p \in \mathcal{O} \} \) and let \( q \in \mathcal{O} \). Since \( \mathcal{O} \) is connected, it is path connected, so there exists a continuous path \( s \) from \( q \) to \( p \) totally contained in \( \mathcal{O} \). Hence \( T \circ s \) is a continuous path from \( T(q) \) to \( T(p) \) and we claim that \( T(q) \in \mathcal{O} \). Otherwise, as \( T(p) \in \mathcal{O} \) there should exist a point \( q' \) on the path \( s \) so that \( T(q') \in \partial \mathcal{O} \). But since \( T^{-1} \) preserves \( \partial \mathcal{O} \), we would have that \( q' \in \partial \mathcal{O} \) contradicting the choice of \( s \). This proves that \( T(\mathcal{O}) \subseteq \mathcal{O} \).

A similar argument considering a continuous path from \( q \) to \( T(p) \), shows that \( T^{-1}(q) \in \mathcal{O} \) whenever \( q \in \mathcal{O} \). That is, \( T^{-1}(\mathcal{O}) \subseteq \mathcal{O} \). By applying \( T \), we get that \( \mathcal{O} \subseteq T(\mathcal{O}) \). \( \square \)

Remark 1 An easy verification shows that the previous lemmas hold true replacing the group \( \text{Homeo}(\mathbb{R}^n) \) by \( \text{Aff}(\mathbb{R}^n) \). In the sequel, we will apply these results only to affine transformations of \( \mathbb{R}^n \).

Now let us recall the following important known result that will be useful in the rest of this work. It is a consequence of Theorem 6.1 in the page 252 of [4].

Lemma 3 Let \( \mathcal{O} \) be a connected open subset of \( \mathbb{R}^n \) endowed with the connection \( \nabla \) induced by the usual connection \( \nabla^0 \), i.e., the usual affine structure of \( \mathbb{R}^n \). Any affine transformation of \( \mathcal{O} \) preserving \( \nabla \) extends to a classical affine transformation of \( \mathbb{R}^n \).

The following result is an application of Lemma 2. First recall that an affine frame on the affine space \( \mathbb{R}^n \) is a set of \( n + 1 \) points \( p_0, p_1, \ldots, p_n \) so that the vectors \( \overrightarrow{p_0p_1}, \ldots, \overrightarrow{p_0p_n} \) form a linear basis of \( \mathbb{R}^n \) seen as a vector space.

Lemma 4 Let \( p_0, p_1, \ldots, p_n \) be an affine frame on the \( n \)-dimensional affine space \( \mathbb{R}^n \) and consider the manifold \( M_i = \mathbb{R}^n \setminus \{ p_0, \ldots, p_{i-1} \} \) endowed with the connection \( \nabla^i \) given by the restriction to \( M_i \) of the usual connection \( \nabla^0 \). Then for \( 0 \leq i < n \) we have

\[
\text{Aff}(M_i + 1, \nabla^{i+1})_0 \cong (\mathbb{R}^{n-i})^i \rtimes_{\theta} \text{GL}(\mathbb{R}^{n-i})^+
\]
where \( \theta(A)(w_1, \ldots, w_i) = (Aw_1, \ldots, Aw_i) \). The index 0 is to denote the connected component of the group \( \text{Aff}(M_{i+1}, \nabla^{i+1}) \) containing the unit element and the + denotes the group of orientation preserving linear transformations of \( \mathbb{R}^{n-i} \).

Moreover, the group \( \text{Aff}(M_{n+1}, \nabla^{n+1}) \) is discrete and isomorphic to the symmetric group \( S_{n+1} \), the group of permutations of \( n + 1 \) elements.

**Proof** For \( k = 1, \ldots, n \), set \( v_k = \overrightarrow{p_0 p_k} \) and consider the basis \( \beta = \{v_1, \ldots, v_n\} \) of \( \mathbb{R}^n \). By lemmas 2 and 3, the group \( \text{Aff}(M_1, \nabla^1) \) is given by affine transformations of \( \mathbb{R}^n \) fixing the point \( p_0 \). Hence, any \( T \in \text{Aff}(M_1, \nabla^1) \) is determined by its linear part. That is, \( \text{Aff}(M_1, \nabla^1) \) is isomorphic to \( GL(\mathbb{R}^n) \), the group of linear transformations of \( \mathbb{R}^n \).

It is easy to verify that, in the space of affine transformations fixing a set of two points \( \{q_1, q_2\} \), there is no continuous path from one transformation fixing both points to a transformation permuting them. This observation and Lemma 2, imply that the connected component \( \text{Aff}(M_{i+1}, \nabla^{i+1})_0 \) containing the unit of the group \( \text{Aff}(M_{i+1}, \nabla^{i+1}) \) is isomorphic to the group of affine transformations of \( \mathbb{R}^n \) fixing the points \( p_0, \ldots, p_i \) whose linear part is orientation preserving. Since every element of \( \text{Aff}(M_{i+1}, \nabla^{i+1})_0 \) fixes \( p_0 \), every \( T \in \text{Aff}(M_{i+1}, \nabla^{i+1})_0 \) is determined by its linear part \( L \) and this linear transformation fixes the vectors \( v_1, \ldots, v_i \). Thus, the matrix of \( L \) with respect to the basis \( \beta \) is of the form \( [L]_\beta = \begin{bmatrix} I & B \\ 0 & A \end{bmatrix} \), with \( I \) the identity matrix of size \( i \times i \) and \( A \) of size \( (n-i) \times (n-i) \). Hence, for \( i \leq n \), the group \( \text{Aff}(M_{i+1}, \nabla^{i+1})_0 \) is isomorphic to the matrix Lie group \( \begin{bmatrix} I & B \\ 0 & A \end{bmatrix} \mid \det A > 0 \}. \) This group is also isomorphic to the Lie group \( \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \mid \det A > 0 \}. \) Now, for \( i < n \), this group is isomorphic to the group \( \left( \mathbb{R}^{n-i} \right)^i \times \theta \) \( GL(\mathbb{R}^{n-i})^+ \) where \( \theta(F)(w_1, \ldots, w_i) = (Fw_1, \ldots, Fw_i) \). It is easy to check than an isomorphism is given by the map sending an element \( (w_1, \ldots, w_i, F) \in \left( \mathbb{R}^{n-i} \right)^i \times \theta \) \( GL(\mathbb{R}^{n-i})^+ \) to the matrix \( \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \) where \( A \) is the matrix of \( F \) relative to a fixed basis \( \gamma \) of \( \mathbb{R}^n \) and \( B \) the matrix whose column entries are the coordinates vectors \( w_1, \ldots, w_i \) relative to \( \gamma \).

Finally, from Lemmas 2 and 3, we have that the elements of the group \( \text{Aff}(M_{n+1}, \nabla^{n+1}) \) are all affine transformations of \( \mathbb{R}^n \) permuting the set of points \( p_0, \ldots, p_n \), and since an affine transformation is uniquely determined by its values on a frame (see [2]), it follows that \( \text{Aff}(M_{n+1}, \nabla^{n+1}) \) is isomorphic to \( S_{n+1} \).

**Remark 2** For \( i = 0, \ldots, n \), the Lie algebra of the group \( \text{Aff}(M_{i+1}, \nabla^{i+1}) \) is isomorphic to the algebra of squared matrices of the form \( \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \) with \( A \) and \( B \) of sizes \( (n-i) \times (n-i) \) and \( (n-i) \times i \), respectively. Notice also that this algebra is endowed with an associative product compatible with its Lie bracket, namely, the usual product of matrices.

**Example 1** Consider the open set of \( \mathbb{R}^n \) given by \( O = \{(x_1, x_2, \ldots, x_n) \mid x_1 > 0\} \). According to Lemma 2, to determine the elements in \( \text{Aff}(\mathbb{R}^n) \) preserving \( O \), we need to find all affine transformations fixing its boundary, i.e., \( T(\partial O) = \partial O \). That is, we
need conditions so that the following equation holds true

\[
\begin{bmatrix}
a_{11} & \cdots & a_{1n} & a_{1,n+1} \\
\vdots & \ddots & \vdots & \vdots \\
a_{n1} & \cdots & a_{nn} & a_{n+1,n+1}
\end{bmatrix}
\begin{bmatrix}
0 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
x_2' \\
\vdots \\
x_n'
\end{bmatrix}.
\]

A simple computation gives that \( a_{1j} = 0 \) para \( 1 < j \leq n + 1 \). We also need to make sure that \( T(p) \in O \) for some \( p \in O \), and this happens when \( a_{11} > 0 \). Therefore, the group of transformations preserving \( O \) is locally isomorphic to the group of affine transformation of the form

\[
A = \begin{cases}
\begin{bmatrix}
a_{11} & 0 & \cdots & 0 & 0 \\
a_{21} & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n,n+1}
\end{bmatrix} & \text{A invertible and } a_{11} > 0
\end{cases}
\]

where the first \( n \) columns give the linear part and the last column is the translation part.

More generally, if \( O_i = \{(x_1, x_2, \ldots, x_n) \mid x_1 > 0, x_2 > 0, \ldots, x_i > 0\} \) and \( \nabla^i \) the connection \( \nabla^0 \) in \( O_i \), with \( 1 \leq i \leq n \), by Lemma 3 we have that the connected component \( \text{Aff}(O_i, \nabla^i)_0 \) of the group of transformations \( \text{Aff}(O_i, \nabla^i) \) preserving this orbit \( O_i \) is given by affine transformations of the form

\[
B = \begin{cases}
\begin{bmatrix}
a_{11} & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{i+1,1} & \cdots & a_{i+1,i} & a_{i+1,i+1} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
a_n & \cdots & a_{ni} & a_{n,i+1} & \cdots & a_{nn} & a_{n,n+1}
\end{bmatrix} & \text{B invertible and } a_{11}, \ldots, a_{ii} > 0
\end{cases}
\]  

(4)

where the first \( n \) columns denote the linear part and the last column the translation part.

For orbits as in the previous example we have.

**Lemma 5** If \( O_i \) and \( \nabla^i \) are as in the previous example, the group \( \text{Aff}(O_i, \nabla^i) \) is locally isomorphic to

\[
\text{Aff}(O_i, \nabla^i)_0 = \left( (\mathbb{R}^{n-i})^i \times_{\theta_1} \text{Aff}(\mathbb{R}^{n-i}) \right)^{\text{op}} \times_{\theta_2} (\mathbb{R}^{>0})^i \]  

(5)

where the super-index \( \text{op} \) denotes the opposite Lie group and \( \theta_1 \) and \( \theta_2 \) are, respectively, the action of \( \text{Aff}(\mathbb{R}^{n-i}) \) on \( (\mathbb{R}^{n-i})^i \) and the action of \( (\mathbb{R}^{>0})^i \) on \( (\mathbb{R}^{n-i})^i \times_{\theta_1} \text{Aff}(\mathbb{R}^{n-i}) \) defined by \( \theta_1(A, w)(v_1, \ldots, v_i) = (Av_1, \ldots, Av_i) \) and \( \theta_2(\lambda_1, \ldots, \lambda_i)(v_1, \ldots, v_i, A, w) = (\lambda_1 v_1, \ldots, \lambda_i v_i, A, w) \). Moreover, its Lie algebra is associative.
Proof An easy calculation shows that the map between the groups given in (4) and in the right hand side of (5) defined by 
\[
\begin{bmatrix}
D \\
A & B & w
\end{bmatrix}
\mapsto
((A, B, w), D)
\]
is an isomorphism, where the matrix on the left denotes an element on the group given in (4) with B invertible and D a diagonal matrix with positive entries.

It follows that the Lie algebra \( \text{aff}(\mathcal{O}_i, \nabla^i) \) is isomorphic to the algebra of matrices of the form
\[
\begin{bmatrix}
D & 0 & 0 \\
A & B & w \\
0 & 0 & 0
\end{bmatrix}
\]
with \( D \) diagonal. Hence it is an associative algebra with the usual product of matrices.

\[\square\]

3 Applications and examples

Given a manifold \( M \) endowed of a linear connection, for any \( p \in M \), there exists a neighborhood \( N_p \) of 0 \( \in T_p M \) so that for any \( X_p \in N_p \), the geodesic \( \gamma \) with initial condition \((p, X_p)\) is defined in an open interval \((-\epsilon, \epsilon)\) with \( \epsilon > 1 \). The map
\[
\exp_p^{\nabla} : T_p M \to M \\
X_p \mapsto \gamma_1
\]
is called the exponential map relative to \( \nabla \).

As an application of the results in the previous section, we get the following.

**Theorem 6** Let \((M, \nabla)\) be a flat affine simply connected manifold. If the domain of the exponential map relative to \( \nabla \) at some \( p \in M \) is a convex subset of \( T_p M \), then the group of affine transformations \( \text{Aff}(M, \nabla) \) is locally isomorphic to \( \{ T \in \text{Aff}(\mathbb{R}^n) \mid T(\mathcal{O}) = \mathcal{O} \} \), with \( \mathcal{O} \) the image under the developing map. It is also locally isomorphic to
\[
\{ T \in \text{Aff}(\mathbb{R}^n) \mid T(\partial \mathcal{O}) = \partial \mathcal{O}, \ T(p) \in \mathcal{O} \text{ for some } p \in \mathcal{O} \}.
\]
More generally, the conclusion is true when the developing map \( D : M \to \mathcal{O} \) is a diffeomorphism onto \( \mathcal{O} \).

**Proof** As \( M \) is simply connected and the domain of the exponential is convex, the developing map \( D : (M, \nabla) \to (\mathcal{O}, \nabla^0) \) is an affine diffeomorphism (see [5], see also [9], page 151), hence the result follows from the commutativity of the diagram (2).

\[\square\]

**Example 2** In \( \mathbb{R}^2 \) there are, up to isomorphism, six flat affine connections whose group of affine transformations act transitively on \( \mathbb{R}^2 \) (see [1]). The connections are left
invariant determined by the following affine étale representations

\[
\rho_1(a, b) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_2(a, b) = \begin{bmatrix} 1 & b & a + \frac{1}{2}b^2 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_3(a, b) = \begin{bmatrix} 1 & 0 & a \\ 0 & e^b & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

\[
\rho_4(a, b) = \begin{bmatrix} e^a & be^a & 0 \\ 0 & e^a & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

\[
\rho_5(a, b) = \begin{bmatrix} e^a & 0 & 0 \\ 0 & e^b & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_6(a, b) = e^a \begin{bmatrix} \cos b & \sin b & 0 \\ -\sin b & \cos b & 0 \end{bmatrix}
\]

where the first two columns of the images give the linear part and the third column, the translation part. The corresponding developing maps are defined by

\[
D_1(x, y) = (x, y), \quad D_2(x, y) = \left( x + \frac{y^2}{2}, y \right), \quad D_3(x, y) = (x, e^y),
\]

\[
D_4(x, y) = (e^x, ye^x), \quad D_5(x, y) = (e^x, e^y), \quad D_6(x, y) = (e^x \cos y, e^x \sin y).
\]

Hence the orbit determined by the action corresponding to the representations \(\rho_1\) and \(\rho_2\) is the whole plane, the orbit determined by \(\rho_3\) and \(\rho_4\) is the upper half plane \(\{(x, y) \mid y > 0\}\), the representation \(\rho_5\) leaves the first quadrant \(\{(x, y) \mid x, y > 0\}\) as the open orbit and for the action \(\rho_6\), the orbit is the punctured plane \(\{(x, y) \mid (x, y) \neq (0, 0)\}\). As the developing maps \(D_1\) to \(D_5\) are diffeomorphisms onto the respective orbit, by denoting by \(\nabla^i, i = 1, \ldots, 6\) the left invariant connection on \(\mathbb{R}^2\) determined by \(\rho_i\), from Lemma 2 and Theorem 6, we get

\[
\text{Aff}(\mathbb{R}^2, \nabla^1)_0 = \text{Aff}(\mathbb{R}^2, \nabla^2)_0 \cong \text{Aff}(\mathbb{R}^2, \nabla^0)
\]

\[
\text{Aff}(\mathbb{R}^2, \nabla^3)_0 = \text{Aff}(\mathbb{R}^2, \nabla^4)_0 \cong \left\{ \begin{bmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid a \neq 0, d > 0 \right\}
\]

\[
\text{Aff}(\mathbb{R}^2, \nabla^5)_0 \cong \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b > 0 \right\} \cong (\mathbb{R}^+)^2
\]

Although Theorem 6 does not apply in the last case, it is known that \(\text{Aff}(\mathbb{R}^2, \nabla^6)_0 \cong GL_2(\mathbb{R})^+\) (see [8]), where the + is used to denote linear transformations preserving orientation.

**Example 3** Consider the group \(G = \text{Aff}(\mathbb{R})_0\) given by classical affine orientation preserving transformations of \(\mathbb{R}\), known as the group of affine motions of the line (orientation preserving). This group is isomorphic to \(\mathbb{R}^+ \times \mathbb{R}\) with the product \((a, b)(c, d) = (ac, ad + b)\). Its Lie algebra is isomorphic to \(\mathfrak{g} = \text{Lie}(G) \cong \mathbb{R}e_1 \oplus \mathbb{R}e_2\) with the bracket \([e_1, e_2] = e_2\). Up to isomorphism, there are two families and four exceptional left symmetric products compatible with the bracket on \(\mathfrak{g}\) (see [7]).
where $\alpha$ is a real parameter. Denoting by $F_{i,(\alpha)}$ and $E_j$ the corresponding developing maps determined by the products $F_i(\alpha)$ and $E_j$, respectively, for $i = 1, 2$ and $j = 1, \ldots, 4$ one can check that

$$F_{1,(\alpha)}(x, y) = \left(\frac{1}{\alpha} x^\alpha, y\right) \text{ for } \alpha \neq 0, \quad F_{1,(0)}(x, y) = (\ln x, y)$$

$$F_{2,(\alpha)}(x, y) = \left(\frac{1}{\alpha} x^\alpha, x^\alpha y\right) \text{ for } \alpha \neq \{0, -1\} \quad F_{2,(-1)}(x, y) = \left(-\frac{1}{x}, \frac{y}{x}\right)$$

$$E_1(x, y) = (x, 1 + x + y + x \ln x) \quad E_2(x, y) = \left(-\frac{1}{x}, \frac{1}{x} + \frac{y}{x} + \ln x - 1\right)$$

$$E_3(x, y) = \left(\frac{1}{2} (x^2 + y^2 - 1), y\right) \quad E_4(x, y) = \left(\frac{1}{2} (x^2 - y^2 - 1), y\right)$$

It is easy to check that the image of $F_{1,(0)}$ is the whole plane, the image under the maps $F_{1,(\alpha)}$, $F_{2,(\alpha)}$, $E_1$, and $E_2$ is the right half plane $\mathcal{O} = \{(x, y) \mid x > 0\}$, and the image under the maps $E_3$ and $E_4$ is the interior of the parabola $x = \frac{1}{2} (y^2 - 1)$. It is also easy to see that these developing maps are diffeomorphisms, hence by applying Lemma 2, Lemma 3, Theorem 6, and using Example 1 we get

$$\text{Aff}(G, \nabla^+_1(0)) \cong \text{Aff}(\mathbb{R}^2)$$

$$\text{Aff}(G, \nabla^+_i(\alpha)) \cong \text{Aff}(G, \nabla^+_i) \cong \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a > 0 \text{ and } c \neq 0 \right\}, \quad i = 1, 2 \text{ and } \alpha \neq 0$$

$$\text{Aff}(G, \nabla^+_3) \cong \left\{ \begin{bmatrix} a^2 / 2 & ab (a^2 + b^2 - 1) / 2 \\ 0 & a \end{bmatrix} \mid a > 0 \right\}$$

$$\text{Aff}(G, \nabla^+_4) \cong \left\{ \begin{bmatrix} a^2 / 2 & -ab (a^2 - b^2 - 1) / 2 \\ 0 & a \end{bmatrix} \mid a > 0 \right\}$$

where $\nabla^+_i(\alpha)$ and $\nabla^+_j$ denote the corresponding left invariant flat affine connections on $G$ determined by the products $F_i(\alpha)$ and $E_j$, respectively, for $i = 1, 2$ and $j = 1, \ldots, 4$.

We finish this work with the following.

**Corollary 7** If $(M, \nabla)$ is a flat affine simply connected manifold so that the developing map is an isomorphism onto $\mathcal{O}_i = \{(x_1, \ldots, x_n) \mid x_1 > 0, \ldots, x_i > 0\}$ or $\mathcal{O}_i$ equal to the space $\mathbb{R}^n$ with $i$ holes, $0 \leq i \leq n$, then the group $\text{Aff}(M, \nabla)$ admits a flat affine bi-invariant connection.
**Proof** From Remark 2 and Lemma 5 the Lie algebra $\text{aff}(O_i, \nabla^i)$ of the group $\text{Aff}(O_i, \nabla^i)$ admits an associative product compatible with the Lie bracket. Hence, it determines a flat affine bi-invariant connection on $O_i$ (see [6]). Since Theorem 6 implies that the group $\text{Aff}(M, \nabla)$ is locally isomorphic to $\text{Aff}(O_i, \nabla^i)$, we get that $\text{Aff}(M, \nabla)$ admits a flat affine bi-invariant connection. $\square$

**Example 4** If $\nabla$ is any of the connections $\nabla^1$ to $\nabla^5$ on $\mathbb{R}^2$ given in Example 2, the group of affine transformations $\text{Aff}(\mathbb{R}^2, \nabla^i)$ admits a flat affine bi-invariant connection. The group $\text{Aff}(\mathbb{R}^2, \nabla^6)$ also admits a flat bi-invariant connection as it is locally isomorphic to $GL_2(\mathbb{R}^2)^+$ and its Lie algebra, $gl_2(\mathbb{R}^2)$, admits an associative product compatible with the Lie bracket. The corollary also applies to the group $\text{Aff}(\text{Aff}(\mathbb{R})_0, \nabla^+)$ of affine transformations of $G = \text{Aff}(\mathbb{R})_0$ preserving any left invariant connection $\nabla^+$ on $G$.

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**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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