DISCRETE STOCHASTIC APPROXIMATIONS OF THE MUMFORD-SHAH FUNCTIONAL

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Abstract. We propose a Γ-convergent discrete approximation of the Mumford-Shah functional. The discrete functionals act on functions defined on stationary stochastic lattices and take into account general finite differences through a non-convex potential. In this setting the geometry of the lattice strongly influences the anisotropy of the limit functional. Thus we can use statistically isotropic lattices and stochastic homogenization techniques to approximate the vectorial Mumford-Shah functional in any dimension.

1. Introduction

The Mumford-Shah functional has its origin in image segmentation problems [38]. Given a rectangle (or more generally a bounded domain $D \subset \mathbb{R}^2$) and a function $g : D \to \mathbb{R}$ representing the gray level of an image, one aims to minimize the functional

$$MS(u, K) = \int_{D \setminus K} |\nabla u|^2 \, dx + \beta \mathcal{H}^1(K) + \gamma \int_D |u - g|^2 \, dx.$$  \hspace{1cm} (1.1)

Here $K$ is the union of a finite number of points and a finite set of smooth arcs joining these points with no other intersections. The function $u$ is supposed to be differentiable on $D \setminus K$, but may have discontinuities on $K$. Then the pair $(u, K)$ is an approximation of the image. $K$ represents the sharp edges in the image, while the smooth part $u$ yields a cartoon-like total image since it rules out fine textures away from $K$.

Existence and regularity of minimizing pairs $(u, K)$ is far from being trivial. In [29] it was shown that there exists a minimizing pair $(u, K)$ among all closed sets $K$ and $u \in C^1(D \setminus K)$ provided $g \in L^\infty(D)$. As commonly done in variational problems, one first has to enlarge the set of competitors in order to obtain compactness of minimizing sequences. This leads to the nowadays well-known formulation of the Mumford-Shah functional for $SBV$-functions (see Section 2.1): Given $u \in SBV(D)$, the Mumford-Shah functional takes the form

$$MS(u) = \int_D |\nabla u|^2 \, dx + \beta \mathcal{H}^1(S_u) + \gamma \int_D |u - g|^2 \, dx.$$  \hspace{1cm} (1.2)

Here $S_u$ denotes the discontinuity set of $u$. The closed set $K$ then can be recovered setting $K = \overline{S_u}$ since minimizers have an essentially closed discontinuity set (see [29, Lemma 5.2]). However it is still unknown if $K$ can be taken as a finite union of regular arcs. We refer the interested reader to the recent survey articles [30, 37] for more known regularity results for minimizers.

Besides the regularity of minimizers, there is the natural question how to minimize the Mumford-Shah functional [12] in practice. A very popular approach is given by the Ambrosio-Tortorelli approximation [10], where the surface term is replaced by a Modica-Mortola-type approximation with an additional variable. More precisely, given a small parameter $\varepsilon > 0$ and $0 < \eta_\varepsilon << \varepsilon$ one defines an elliptic approximation $\mathcal{AT}_\varepsilon : W^{1,2}(D) \times W^{1,2}(D) \to [0, +\infty]$ by

$$\mathcal{AT}_\varepsilon(u, v) = \int_D (\eta_\varepsilon + v^2)|\nabla u|^2 \, dx + \frac{\beta}{2} \int_D \varepsilon |\nabla v|^2 + \frac{1}{\varepsilon} (v - 1)^2 \, dx + \gamma \int_D |u - g|^2 \, dx.$$  

In [10] it is shown that the family $\mathcal{AT}_\varepsilon$ approximates the Mumford-Shah functional (1.2) in the sense of Γ-convergence (we refer to the monographs [14, 26] for details on this type of convergence). In particular,
up to subsequences, the \( u \)-component of any global minimizer \((u_\varepsilon, v_\varepsilon)\) of \(\mathcal{A}_\varepsilon\) converges to a global minimizer of \(\mathcal{MS}\). This approach was recently extended to second order penalizations, that means to replace the gradient \(\nabla v\) either by the Hessian or the Laplacian with boundary conditions on \(v\) and scale them by \(\varepsilon^3\) (see [13] for more details or [14] for an anisotropic version).

Instead of introducing a second variable, Braides and Dal Maso constructed non-local approximations. In [14] they showed that the sequence of functionals \(\mathcal{NL}_\varepsilon : W^{1,2}(D) \to [0, +\infty)\) defined by

\[
\mathcal{NL}_\varepsilon(u) = \frac{1}{\varepsilon} \int_D f(\varepsilon \int_{B_\varepsilon(x) \cap D} |\nabla u(y)|^2 \, dy + \gamma \int_D |u - g|^2 \, dx
\]

\(\Gamma\)-converges to \(\mathcal{MS}\) provided \(f\) is continuous, increasing and satisfies

\[
\lim_{t \to 0} \frac{f_0(t)}{t} = 1, \quad \lim_{t \to +\infty} f(t) = f_\infty < \infty.
\]

In this case it turns out that \(\beta = 2f_\infty\).

Note that both approximations are defined on more regular, but still infinite-dimensional spaces. Hence one has to perform a discretization to numerically solve the minimization problems for the approximating functionals. Moreover \(\varepsilon\) should be taken very small in order to obtain almost sharp interfaces. However in dimension one there exists a direct approximation based on finite differences. In that case the small parameter \(\varepsilon\) represents the mesh size of a one-dimensional grid \(\varepsilon \mathbb{Z}\). Given a function \(u : \varepsilon \mathbb{Z} \cap (0, 1) \to \mathbb{R}\) we define the functional \(F_\varepsilon\) by

\[
F_\varepsilon(u) = \frac{[1/\varepsilon] - 2}{\varepsilon} \min \left\{ \frac{u(\varepsilon(i + 1)) - u(\varepsilon i)}{\varepsilon}, \beta \right\} + \gamma \sum_{i=1}^{[1/\varepsilon] - 1} \varepsilon |u(\varepsilon i) - g_\varepsilon(\varepsilon i)|^2,
\]

where \(g_\varepsilon\) is a suitable discretized version of \(g \in L^\infty\). The proof of convergence to the one-dimensional version of the Mumford-Shah functional can be found for example in [14] Chapter 8.3. The functional \(F_\varepsilon\) above has a natural extension to higher dimensions. Indeed, given \(u : \varepsilon \mathbb{Z}^d \cap D \to \mathbb{R}\) one sets

\[
F_\varepsilon(u) = \frac{1}{2} \sum_{\varepsilon i, \varepsilon j \in \varepsilon \mathbb{Z}^d \cap D, |i - j| = 1} \varepsilon^{-1} \min \left\{ \frac{|u(\varepsilon i) - u(\varepsilon j)|}{\varepsilon}, \beta \right\} + \gamma \sum_{\varepsilon i \in \varepsilon \mathbb{Z}^d \cap D} \varepsilon^d |u(\varepsilon i) - g_\varepsilon(\varepsilon i)|^2.
\]

However in higher dimensions the anisotropy of the lattice \(\mathbb{Z}^d\) leads to anisotropic surface integrals. For \(d = 2\) Chambolle proved in [20] that the functionals \(F_\varepsilon\) \(\Gamma\)-converge to an anisotropic version of the Mumford-Shah functional given by

\[
F(u) = \int_D |\nabla u|^2 \, dx + \beta \int_{S_u} |\nu_u|_{1} \, dH^1 + \gamma \int_D |u - g|^2 \, dx,
\]

where \(|\cdot|_{1}\) denotes the \(l_1\)-norm of the normal vector \(\nu_u\) at \(x \in S_u\) (see Section 2.1 for precise definitions). With the results obtained in this paper one can extend this result to any dimension.

In order to avoid the anisotropy, there have been found two approaches: On the one hand, inspired by the nonlocal approximation of the Mumford-Shah functional studied in [35], one can consider long-range interactions in [14] instead of only nearest neighbors. This has been analyzed in [21] and indeed anisotropy can be reduced but the functional to be minimized gets more complex as the number of interactions grows. On the other hand Chambolle and Dal Maso considered functionals defined on piecewise affine functions with respect to a whole class of two-dimensional triangulations. More precisely, let \(\mathcal{T}_\varepsilon(D, \theta)\) be the set of all finite triangulations containing \(D\) such that for each triangle the inner angles are at least \(\theta\) and the side lengths are between \(\varepsilon\) and \(w(\varepsilon)\), where \(w(\varepsilon) \geq 6\varepsilon\) satisfies \(\lim_{\varepsilon \to 0} w(\varepsilon) = 0\). Denote by \(V_\varepsilon(D, \theta)\) the set of all continuous functions that are piecewise affine with respect to some \(T \in \mathcal{T}_\varepsilon(D, \theta)\). In [22] it is shown that there exists \(0 < \theta_0 < 60^\circ\) such that for all \(0 < \theta < \theta_0\) the functionals \(F_{\varepsilon, \theta}\) defined on \(V_{\varepsilon}(D, \theta)\) by

\[
F_{\varepsilon, \theta}(u) = \frac{1}{\varepsilon} \int_D f(\varepsilon |\nabla u|^2) \, dx + \gamma \int_D |u - g|^2 \, dx
\]

\(\Gamma\)-converge to the Mumford-Shah functional when \(f\) satisfies (1.3). In this case it holds that \(\beta = f_\infty \sin(\theta)\). We remark that the triangulation is not fixed, so it is part of the minimization problem to find the optimal
Given the probabilistic assumptions above, the result is quite robust. For example, the same limit (with surely (a.s.) $\Gamma$-converge in $L^1$) stems mostly from the lattice, we replace the periodic lattice by so-called stochastic lattices. As a first consequence of our analysis, which is described more in detail below, we can identify the $\Gamma$-limit of the functionals $\tilde{F}_{\varepsilon,g}$ in dimension two. In particular we show that it differs from the Mumford-Shah functional due to an anisotropic surface integral (the last point can verified in any dimension; see Remark 14). Motivated by the fast algorithm for discrete approximations presented in [40], we then construct a random family of discrete functionals for that we can prove $\Gamma$-convergence towards the Mumford-Shah functional. The basic idea is simple: Since the anisotropy in the $\Gamma$-limit of the family of functionals in [4] stems mostly from the lattice, we replace $\mathbb{Z}^d$ by a more isotropic point set. Since there exist no isotropic, countable sets, we need to go beyond deterministic models and consider realizations of random point sets. Those have the flexibility to be isotropic at least in distribution. We consider random, countable point sets $\mathcal{L}(\omega) \subset \mathbb{R}^d$ that satisfy the following geometric constraints:

(i) There exists $R > 0$ such that $\text{dist}(x, \mathcal{L}(\omega)) < R$ for all $x \in \mathbb{R}^d$;
(ii) There exists $r > 0$ such that $\text{dist}(x, \mathcal{L}(\omega) \setminus \{x\}) \geq r$ for all $x \in \mathcal{L}(\omega)$.

Given a small parameter $\varepsilon > 0$ (again representing a kind of mesh-size) and $q > 1$, one possible approximation is given by the random functional defined on functions $u : \varepsilon \mathcal{L}(\omega) \cap D \to \mathbb{R}^m$ by

$$F_{\varepsilon,g}(\omega)(u) = \sum_{(x,y) \in \mathcal{N}(\omega)} \varepsilon^{-d-1} \int_D \left( u(x) - u(y) \right)^2 \ dx + \sum_{x \in \varepsilon \mathcal{L}(\omega) \cap D} \varepsilon^d |u(\varepsilon x) - g_\varepsilon(\varepsilon x)|^q, \quad (1.6)$$

where $\mathcal{N}(\omega)$ denotes the set of Voronoi neighbors (see Definition 2.6), $f$ is a function satisfying (1.3) and $g_\varepsilon(\cdot)$ is a suitable discretization of some $g \in L^q(D, \mathbb{R}^m)$. We require in addition that the random variable $\mathcal{L}$ is stationary and isotropic, that means $\mathcal{L}$ and $R\mathcal{L} + z$ have the same statistics for all $z \in \mathbb{Z}^d$ and all $R \in SO(d)$. If the shift operation is realized by an ergodic group action and $g_\varepsilon(\omega) \to g$ in $L^q(D, \mathbb{R}^m)$, our main result states that there exist three constants $c_1, c_2, c_3$ such that the functionals $F_{\varepsilon,g}(\omega)$ almost surely (a.s.) $\Gamma$-converge in $L^1(D, \mathbb{R}^m)$ to the deterministic functional $F_g$ defined by

$$F_g(u) = \begin{cases} c_1 \int_D |\nabla u|^2 \ dx + c_2 \mathcal{H}^{d-1}(S_u) + c_3 \int_D |u - g|^q \ dx & \text{if } u \in L^q(D, \mathbb{R}^m) \cap GSBV^2(D, \mathbb{R}^m), \\ +\infty & \text{otherwise}. \end{cases}$$

Given the probabilistic assumptions above, the result is quite robust. For example, the same limit (with different constant $c_2$) can be proven for the random version of (1.5) given by

$$\tilde{F}_{\varepsilon,g}(\omega) = \sum_{x \in D} \varepsilon^{-d-1} \int_{(x,y) \in \mathcal{N}(\omega)} \varepsilon^d \left| u(x) - u(y) \right|^2 \ dx + \sum_{x \in \varepsilon \mathcal{L}(\omega) \cap D} \varepsilon^d |u(\varepsilon x) - g_\varepsilon(\varepsilon x)|^q. \quad (1.5)$$

Some remarks are in order:
(i) A point process that satisfies all our assumptions is given by the random parking process $[34]$. 
(ii) The coefficients $c_i$ are not explicit but are derived from three abstract homogenization formulas. However for fixed $L$ one can still tune them since $c_1$ and $c_2$ are proportional to $f'(0)$ and $f'_\infty$ respectively, while for $c_3$ we can multiply the second term in (1.0) by some factor.
(iii) We will prove the convergence for more general finite differences. Those require more technical notation that we want to avoid in this introduction. Hence we restrict the description of our analysis below to pairwise interactions.
(iv) In the proof we will identify $u$ with a piecewise constant function on Voronoi cells. However this is not needed for minimizing the functional $F_{*,g}(\omega)$. In particular one only has to determine the Voronoi neighbors, but no volume of cells or piecewise affine interpolations on Delaunay triangulations.
(v) We prove that global minimizers of $F_{*,g}$ convergence to minimizers of $F_g$ in $L^q(D,\mathbb{R}^m)$. This is not the natural compactness to be expected from finite energy sequences.
(vi) Still the discrete functional is non-convex which cannot be avoided since the $\Gamma$-limit of any convex functionals remains convex.
(vii) We put no restriction on the underlying dimension. While our arguments could be simplified in the case $d = 2$ using Delaunay triangulations, we prefer to prove the convergence in full generality. As a side remark, the treatment of vector-valued $u$ is quite straightforward. In our setting those models correspond to color images.
(viii) A different randomization of the functional [17] has been considered in another context in [17].

This paper treats a random choice between the potential $f(s) = \min\{s,1\}$ and $\tilde{f}(s) = s$. However isotropy of the limit functional remained an open problem.

We now give a short overview of the paper and explain briefly the steps to prove our main approximation theorem. Section 2 is divided into three parts. At first we recall the necessary function spaces that we use to define our approximating functionals $F_{*,g}(\omega)$. We also establish a doubling property and a Poincaré inequality for stochastic lattices. This will be an important tool since in dimensions larger than two we cannot use piecewise affine interpolations to embed the problem in a continuous setting. In the final part we define localized versions of the functionals $F_{*,g}(\omega)$ given by

$$F_{*,g}(\omega)(u) = \sum_{(x,y) \in N(\omega)} \epsilon^{d-1} f \left( \epsilon \frac{|u(x) - u(y)|}{\epsilon} \right).$$

For the sake of generality we take a general exponent $p > 1$. The localized versions of these functionals will be the main objects to be studied in the subsequent sections.

Section 3 contains the two main steps of the proof. As a first step, assuming only the geometric properties of a single realization $L(\omega)$, we prove in Theorem [3.1] that (up to subsequences) the $\Gamma$-limit of the family $F_{*,g}(\omega)$ always has the form of a free discontinuity functional, that means it is finite only on $GSBV^p(D,\mathbb{R}^m)$, where it can be written as

$$F(\omega)(u) = \int_D h(x,\nabla u) \, dx + \int_{S_n} g(x,\nu_u) \, d\mathcal{H}^{d-1}. \quad (1.7)$$

This result is obtained by standard techniques combining the abstract methods of $\Gamma$-convergence with an integral representation theorem. In the second part we give characterizations of the functions $h$ and $g$ in formula (1.7). We argue that, along the chosen subsequence, the gradient part of the functional $F(\omega)$ coincides with the $\Gamma$-limit of the convex functionals

$$E_{*,g}(\omega)(u) = f'(0) \sum_{(x,y) \in N(\omega)} \epsilon^{d} \left| \frac{u(x) - u(y)}{\epsilon} \right|^p,$$
while the surface part of the functional $F(\omega)$ is given by the $\Gamma$-limit of the Ising-type energies defined on functions $u : \varepsilon L(\omega) \to \{\pm 1\}$ by

$$I_\varepsilon(\omega)(u) = \frac{f_{\infty}}{2} \sum_{(x,y) \in \mathcal{N}(\omega)} \varepsilon^{d-1} |u(\varepsilon x) - u(\varepsilon y)|,$$

The proof of these two characterizations is the most delicate step. Even though similar results have been obtained in a continuum setting (see [16, 23, 19]), we cannot use interpolation and copy the argument. This has several reasons: On the one hand, in dimensions larger than two piecewise affine interpolations on Delaunay tessellations might be degenerate due to very flat tetrahedrons. On the other hand, even in two dimensions Voronoi cells can have very short boundary sides, so that the discrete functional overestimates the length of interfaces. Moreover, fine constructions based on geometric measure theory may be incompatible with the prescribed lattice structure. Thus our arguments, which are nevertheless inspired by the continuum case, need to use only the discrete environment as long as possible.

In Section 4 we address the issue that the limit might exist only up to subsequences, be inhomogeneous and anisotropic. The first two issues can be solved by requiring that the random variable $\mathcal{L}$ is stationary with respect to integer shifts. Under this additional assumption we show in Theorem 4.4 that the $\Gamma$-limit of $F_\varepsilon(\omega)$ exists a.s. and is of the form

$$F(\omega)(u) = \int_D h_\omega(\nabla u) \, dx + \int_{S_u} g_\omega(\nu_u) \, d\mathcal{H}^{d-1},$$

where the functions $h_\omega$ and $g_\omega$ are deterministic under an additional ergodicity assumption. With the characterizations proven in the previous section, this theorem is a straightforward consequence of the results on discrete-to-continuum stochastic homogenization for elastic and Ising-type energies obtained in the two papers [1, 5], respectively. In Theorem 4.2 we prove the $\Gamma$-convergence when we add the fidelity term to the functional $F_\varepsilon(\omega)$, while in Corollary 4 we establish $L^p$-convergence of global minimizers. Since our results also apply in the periodic setting, we can then identify the $\Gamma$-limit of the sequence $F_\varepsilon$ in the case $d = 2$ (see Corollary 2). Theorem 1.1 contains the announced approximation result for the Mumford-Shah functional in the case $p = 2$.

The appendix contains a technical argument how to choose $\Gamma$-converging diagonal sequences when the functionals are not equicoercive.

2. SETTING OF THE PROBLEM AND PRELIMINARY RESULTS

We first introduce some notation that will be used in this paper. Given a measurable set $B \subset \mathbb{R}^d$ we let denote by $|B|$ its $d$-dimensional Lebesgue measure, while more generally $\mathcal{H}^k(A)$ stands for the $k$-dimensional Hausdorff measure. We denote by $1_B$ the characteristic function of $B$. If $B$ is finite, $\#B$ means its cardinality. Given an open set $O \subset \mathbb{R}^d$, we denote by $\mathcal{A}(O)$ the family of all bounded, open subsets of $O$, while $\mathcal{A}(O)$ stands for the bounded, open subsets with Lipschitz boundary. For $x \in \mathbb{R}^d$ or $y \in \mathbb{R}^m$ we denote by $|x|$ and $|y|$ the Euclidean norm. Given a matrix $\xi \in \mathbb{R}^{m \times d}$ we let $|\xi|$ be its Frobenius norm. As usual we denote $B_\rho(x_0)$ the open ball with radius $\rho$ centered at $x_0 \in \mathbb{R}^d$. We simply write $B_\rho$ when $x_0 = 0$. Given $\nu \in S^{d-1}$ we let $\nu_1 = \nu, \nu_2, \nu_3$ be an orthonormal basis of $\mathbb{R}^d$ and we define the cube $Q_\nu$ as

$$Q_\nu = \{ z \in \mathbb{R}^d ; |\langle z, \nu_i \rangle| < \frac{1}{2} \},$$

where the brackets $\langle \cdot \rangle$ denote the scalar product. Given $x_0 \in \mathbb{R}^d$ and $\rho > 0$ we set $Q_\nu(x_0, \rho) = x_0 + \rho Q_\nu$. The notation $\text{co}(x_1, \ldots, x_d)$ means the convex hull of finitely many points in $\mathbb{R}^d$. We will use $\|u\|_{L^p(A)}$ for the $L^p(A, \mathbb{R}^m)$ norm. There should be no confusion about the dimension $m$. The symbol $\otimes$ stands for the outer product of vectors, that is, for any $a \in \mathbb{R}^m$, $b \in \mathbb{R}^d$ we have $a \otimes b \in \mathbb{R}^{m \times d}$ with $(a \otimes b)_{ij} := a_i b_j$. Finally, the letter $C$ stands for a generic positive constant that may change every time it appears.
2.1. Functions of bounded variation. We first recall the basic properties of the function spaces we are going to use in this paper. We refer to [6, 9, 27] for more details.

Let $O \subset \mathbb{R}^d$ be an open set. A function $u \in L^1(O)$ is a function of bounded variation, if there exists a finite vector-valued Radon measure $Du$ on $O$ such that for any $\varphi \in C^\infty_c(O, \mathbb{R}^d)$ it holds that

$$
\int_O u \text{div} \varphi \, dx = - \int_O \langle \varphi, Du \rangle.
$$

In this case we write $u \in BV(O)$ and $Du$ is the distributional derivative of $u$. A function $u \in L^1(O, \mathbb{R}^m)$ belongs to $BV(O, \mathbb{R}^m)$ if every component belongs to $BV(O)$. In this case $Du$ denotes the matrix-valued Radon measure consisting of the distributional derivatives of each component.

A Borel-function $u : O \rightarrow \mathbb{R}^m$ is said to have an approximate limit at $x \in O$ whenever there exists $\varepsilon > 0$,

$$
\lim_{\rho \to 0} \frac{1}{\rho^d} \left| \{ y \in B_\rho(x) : |u(y) - z| > \varepsilon \} \right| = 0.
$$

The approximate limit is unique whenever it exists. We use the standard notation $\text{ap lim}_{y \to x} u(y) = z$ and let $S_u \subset O$ be the set, where $u$ has no approximate limit. Let $x \in O \setminus S_u$ and $\tilde{u}(x) := \text{ap lim}_{y \to x} u(y)$. Then $u$ is said to be approximately differentiable at $x$, if there exists a matrix $\nabla u(x) \in \mathbb{R}^{m \times d}$ such that

$$
\text{ap lim}_{y \to x} \frac{u(y) - \tilde{u}(x) - \nabla u(x)(y-x)}{|y-x|} = 0.
$$

The matrix $\nabla u(x)$ is unique whenever it exists. For the moment consider now $u \in BV(O, \mathbb{R}^m)$. Then it is known that $S_u$ is countably $(d-1)$-rectifiable. We next introduce approximate jump points: Given $x \in O$ and $\nu \in S^{d-1}$ we set

$$
\begin{align*}
B^+_\rho(x, \nu) &= \{ y \in B_\rho(x) : \langle y-x, \nu \rangle > 0 \}, \\
B^-_\rho(x, \nu) &= \{ y \in B_\rho(x) : \langle y-x, \nu \rangle < 0 \}.
\end{align*}
$$

We say that $x \in O$ is an approximate jump point of $u$ if there exist $a \neq b \in \mathbb{R}^m$ and $\nu \in S^{d-1}$ such that

$$
\lim_{\rho \to 0} \frac{1}{\rho^d} \int_{B^+_\rho(x, \nu)} |u(y) - a| \, dy = \lim_{\rho \to 0} \frac{1}{\rho^d} \int_{B^-_\rho(x, \nu)} |u(y) - b| \, dy = 0.
$$

Note that the triplet $(a, b, \nu)$ is uniquely determined up to the change to $(b, a, -\nu)$. We denote it by $(u^+(x), u^-(x), \nu_u(x))$. Write $J_u$ for the set of approximate jump points of $u$. Then the triplet $(u^+, u^-, \nu_u)$ can be chosen as a Borel function on the Borel set $J_u$ and it can be shown that $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$. It follows that, for all $\varepsilon > 0$,

$$
\lim_{\rho \to 0} \frac{1}{\rho^d} \left| \{ y \in B_\rho^\pm(x, \nu_u(x)) : |u(y) - u^\pm(x)| > \varepsilon \} \right| = 0. \tag{2.1}
$$

Moreover, the approximate differential $\nabla u$ exists almost everywhere (a.e.). Denoting the density of the absolutely continuous part of $Du$ with respect to the Lebesgue measure by $D^a u$, it holds that $D^a u(x) = \nabla u(x)$ a.e. and we can decompose $Du$ as

$$
Du(B) = \int_B \nabla u \, dx + \int_{J_u \cap B} (u^+(x) - u^-(x)) \otimes \nu_u(x) \, d\mathcal{H}^{d-1} + D^c u(B),
$$

where $D^c u$ is the so-called Cantor part of $Du$.

The space of special functions of bounded variation is defined as the set of those $u \in BV(O, \mathbb{R}^m)$ such that $D^c u = 0$. We write $u \in SBV(O, \mathbb{R}^m)$. Given $p \in (1, +\infty)$, we define $SBV^p(O, \mathbb{R}^m) \subset SBV(O, \mathbb{R}^m)$ as the set of those functions such that $\nabla u \in L^p(O, \mathbb{R}^{m \times d})$ and $\mathcal{H}^{d-1}(S_u) < +\infty$. Due to a lack of compactness in many free discontinuity problems, we have to enlarge this class. We say that a Borel-function $u : O \rightarrow \mathbb{R}^m$ is a generalized special function of bounded variation, if $\varphi \circ u \in SBV_{\text{loc}}(O, \mathbb{R}^m)$ for every function $\varphi \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ such that $\nabla \varphi$ has compact support and write $u \in GS\text{BV}(O, \mathbb{R}^m)$. In this case, the approximate differential $\nabla u(x)$ still exists a.e. and $S_u$ is countably $(d-1)$-rectifiable.
Finally, we set $GSBV^p(O, \mathbb{R}^m)$ as those functions $u \in GSBV(O, \mathbb{R}^m)$ such that $\nabla u \in L^p(O, \mathbb{R}^{m \times d})$ and $\mathcal{H}^{d-1}(S_u) < +\infty$.

As shown in [27 Section 2], the set $GSBV^p(O, \mathbb{R}^m)$ is a vector space and, if $\varphi \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ is such that $\nabla \varphi$ has compact support, then $\varphi \circ u \in GSBV^p(O, \mathbb{R}^m)$. Moreover, given $u \in GSBV^p(O, \mathbb{R}^m)$, one can define a Borel-function $\nu_u : S_u \rightarrow S^{d-1}$ and two Borel-functions $u^+, u^- : S_u \rightarrow \mathbb{R}^m$ such that (2.1) still holds for $\mathcal{H}^{d-1}$-a.e. $x \in S_u$.

For our analysis we make use of a special family of smooth truncations as in [19]. To this end, consider a function $\theta \in C_\infty(\mathbb{R})$ such that $\theta(t) = t$ for all $|t| \leq 1$, $\theta(t) = 0$ for $t \geq 3$ and $\|\theta\|_{\infty} \leq 1$. We define the function $\varphi \in C_\infty^\prime(\mathbb{R}, \mathbb{R}^m)$ by

$$\varphi(u) = \begin{cases} \theta(|u|) \frac{u}{|u|} & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases}$$

As shown at the beginning of [19 Section 4] the function $\varphi$ is 1-Lipschitz. Given $k > 0$ we further set $\varphi_k(u) = k\varphi(\frac{u}{k})$ which is still 1-Lipschitz. Then we have the following approximation result:

**Lemma 2.1.** Let $u \in GSBV^p(D, \mathbb{R}^m) \cap L^1(D, \mathbb{R}^m)$ and let $k > 0$. Defining the truncation $T_k u = \varphi_k(u)$, the function $T_k u$ belongs to $GSBV^p(D, \mathbb{R}^m) \cap L^\infty(D, \mathbb{R}^m)$ and

1. $T_k u \rightarrow u$ a.e. and in $L^1(D, \mathbb{R}^m)$ when $k \rightarrow +\infty$,
2. $\nabla T_k u(x) = \nabla \varphi_k(u(x)) \nabla u(x)$ for a.e. $x \in D$,
3. $S_{T_k u} \subset S_u$, $\lim_{k \rightarrow +\infty} \mathcal{H}^{d-1}(S_{T_k u}) = \mathcal{H}^{d-1}(S_u)$ and $\nu_{T_k u} = \pm \nu_{T_k u} \mathcal{H}^{d-1}$-a.e. on $S_{T_k u}$.

**Proof.** (i) follows by dominated convergence. (ii) is a consequence of [6 Proposition 1.2 (i)]. The inclusion in (iii) is shown in [3 Proposition 1.1 (iii)], while the convergence is a consequence of the inclusion and lower semicontinuity [3 Theorem 3.7]. The orientation property holds by the construction of the normal fields in [6 Proposition 1.3].

Although we consider functionals which have as domain the large space $GSBV^p$, we will make use of the following integral representation result [12 Theorem 1].

**Theorem 2.2.** Let $F : GSBV^p(D, \mathbb{R}^m) \times A(D) \rightarrow [0, +\infty)$ satisfy for every $(u, A) \in GSBV^p(D, \mathbb{R}^m) \times A(D)$ the following hypotheses:

1. $F(u, \cdot)$ is the restriction to $A(D)$ of a Radon measure;
2. $F(u, A) = F(v, A)$ whenever $u = v$ a.e. on $A \in A(D)$;
3. $F(\cdot, A)$ is $L^1(D, \mathbb{R}^m)$-lower semicontinuous;
4. there exists $c > 0$ such that

$$\frac{1}{c} \left( \int_A |\nabla u|^p \, dx + \int_{S_u \cap A} (1 + |u^+ - u^-|) \, d\mathcal{H}^{d-1} \right) \leq F(u, A) \leq c \left( \int_A (1 + |\nabla u|^p) + \int_{S_u \cap A} (1 + |u^+ - u^-|) \, d\mathcal{H}^{d-1} \right).$$

Then for every $u \in GSBV^p(D, \mathbb{R}^m)$ and $A \in A(D)$

$$F(u, A) = \int_A h(x, u, \nabla u) \, dx + \int_{S_u \cap A} g(x, u^+, u^-, \nu_u) \, d\mathcal{H}^{d-1},$$

with

$$h(x, u_0, \xi) = \limsup_{\rho \rightarrow 0} \frac{m(u_0 + \xi(x - x_0), Q_x(x_0, \rho))}{\rho^d},$$

$$g(x_0, a, b, \nu) = \limsup_{\rho \rightarrow 0} \frac{m(u_{a,b}^{x_0, \nu}, Q_v(x_0, \rho))}{\rho^{d-1}},$$

where, for given $x_0 \in D$, $\nu \in S^{d-1}$ and $a, b \in \mathbb{R}^m$,

$$u_{a,b}^{x_0, \nu} := \begin{cases} a & \text{if } \langle x - x_0, \nu \rangle > 0, \\ b & \text{otherwise}. \end{cases}$$
and for any \((v, A) \in SBV^p(D, \mathbb{R}^m) \times \mathcal{A}(D)\) we set
\[
m(v, A) = \inf \{ \mathcal{F}(u, A) : u \in SBV^p(A, \mathbb{R}^m), \; u = v \text{ in a neighbourhood of } \partial A \}.
\]

2.2. **Stochastic lattices.** We next introduce the random point sets that build the basis for our discretization methods. Throughout this paper we let \(\Omega\) be a probability space with a complete \(\sigma\)-algebra \(\mathcal{F}\) and probability measure \(\mathbb{P}\). We call a random variable \(\mathcal{L} : \Omega \to (\mathbb{R}^d)^N\) a stochastic lattice. The following definition essentially forbids clustering of points as well as arbitrary big empty regions in the space.

**Definition 2.3.** Let \(\mathcal{L}\) be a stochastic lattice. \(\mathcal{L}\) is called admissible if there exist \(R > r > 0\) such that the following two conditions hold a.s.:
\begin{enumerate}[(i)]
  \item \(\text{dist}(x, \mathcal{L}(\omega)) < R\) for all \(x \in \mathbb{R}^d\);
  \item \(\text{dist}(x, \mathcal{L}(\omega)\backslash\{x\}) \geq r\) for all \(x \in \mathcal{L}(\omega)\).
\end{enumerate}

**Remark 1.** We also make use of the associated Voronoi tessellation \(\mathcal{V}(\omega) = \{\mathcal{C}(x)\}_{x \in \mathcal{L}(\omega)}\), where the Voronoi cells are defined as
\[\mathcal{C}(x) = \{z \in \mathbb{R}^d : |z - x| \leq |z - y| \text{ for all } y \in \mathcal{L}(\omega)\}.
\]
If \(\mathcal{L}(\omega)\) is admissible, then [5, Lemma 2.3] yields the inclusions \(B_{\delta}(x) \subset \mathcal{C}(x) \subset B_R(x)\).

Next we introduce the random point sets that build the basis for our discretization methods.

**Definition 2.4.** We say that a family \(\{\tau_z\}_{z \in \mathbb{Z}^d}, \tau_z : \Omega \to \Omega\), is an additive group action on \(\Omega\) if
\[\tau_{z_1 + z_2} = \tau_{z_2} \circ \tau_{z_1} \text{ for all } z_1, z_2 \in \mathbb{Z}^d.
\]
Such an additive group action is called measure preserving if
\[\mathbb{P}(\tau_z B) = \mathbb{P}(B) \quad \text{for all } B \in \mathcal{F}, \; z \in \mathbb{Z}^d.
\]
Moreover \(\{\tau_z\}_{z \in \mathbb{Z}^d}\) is called ergodic if, in addition, for all \(B \in \mathcal{F}\) we have
\[(\tau_z(B) = B \quad \forall z \in \mathbb{Z}^d) \quad \Rightarrow \quad \mathbb{P}(B) \in \{0, 1\}.
\]

In terms of a stochastic lattice the probabilistic properties read as follows:

**Definition 2.5.** A stochastic lattice \(\mathcal{L}\) is said to be stationary if there exists a measure preserving group action \(\{\tau_z\}_{z \in \mathbb{Z}^d}\) on \(\Omega\) such that for all \(z \in \mathbb{Z}^d\)
\[\mathcal{L} \circ \tau_z = \mathcal{L} + z.
\]
If in addition \(\{\tau_z\}_{z \in \mathbb{Z}^d}\) is ergodic, then \(\mathcal{L}\) is called ergodic, too.

We call \(\mathcal{L}\) isotropic, if for every \(R \in SO(d)\) there exists a measure preserving function \(\tau'_R : \Omega \to \Omega\) such that
\[\mathcal{L} \circ \tau'_R = R\mathcal{L}.
\]

In order to define gradient-like structures, we equip a stochastic lattice with a set of directed edges. We summarize the necessary properties in a separate definition:

**Definition 2.6.** Let \(\mathcal{L}\) be a stochastic lattice and \(\mathcal{E} \subset \mathbb{L}^2\). We call \(\mathcal{E}\) admissible edges if for all \(i, j \in \mathbb{N}\) the set \(\{\omega \in \Omega : (\mathcal{L}(\omega)_i, \mathcal{L}(\omega)_j) \in \mathcal{E}(\omega)\}\) is \(\mathcal{F}\)-measurable and
\begin{enumerate}[(i)]
  \item there exists \(M > R\) such that a.s.
    \[\sup\{|x - y| : (x, y) \in \mathcal{E}(\omega)\} < M;\] (2.2)
  \item the nearest neighbors defined by \(\mathcal{N}(\omega) := \{(x, y) \in \mathcal{L}(\omega)^2 : \mathcal{H}^{d-1}(\mathcal{C}(x) \cap \mathcal{C}(y)) \in (0, +\infty)\}\) are contained in \(\mathcal{E}(\omega)\) up to symmetrizing, that means
    \[\mathcal{N}(\omega) \subset \mathcal{E}(\omega) \cup \{(y, x) \in \mathcal{L}(\omega)^2 : (x, y) \in \mathcal{E}(\omega)\};\] (2.3)
\end{enumerate}
If \(\mathcal{L}\) is stationary or isotropic, we say that the edges \(\mathcal{E}\) are stationary or isotropic if \(\mathcal{E} \circ \tau_z = \mathcal{E} + (z, z)\) for all \(z \in \mathbb{Z}^d\) or \(\mathcal{E} \circ \tau'_R = R\mathcal{E}\) for all \(R \in SO(d)\).
Remark 2. In the proof of [5, Lemma A.2] it is shown, that the choice $\mathcal{E}(\omega) = \mathcal{N}(\omega)$ satisfies the measurability assumption. Then one can add for example non-nearest neighbors by selecting them on based on a maximal distance.

Having introduced the random framework, we will encounter it again only from Section 4 on. To save notation, we will drop the dependence on $\omega$ for some quantities. If so, we tacitly assume that we have a realization satisfying properties (i) and (ii) of both Definitions 2.3 and 2.6.

Up to enlarging $M$, by Remark 1 we may assume
\[
\sup_{x \in \mathcal{L}(\omega)} \# \{y \in \mathcal{L} : (x, y) \in \mathcal{E} \text{ or } (y, x) \in \mathcal{E} \} \leq M. \tag{2.4}
\]

It will be convenient to view a stochastic lattice also as an undirected graph $G = (V, E)$ with vertices $V = \mathcal{L}$ and edges $E = \mathcal{E} \cup \{(y, x) \in \mathcal{L}(\omega)^2 : (x, y) \in \mathcal{E}\}$. We say that $P$ is a path of length $n$ if $P = (x_0, \ldots, x_n)$ with $(x_{i-1}, x_i) \in E$ for all $i = 1, \ldots, n$. Given $x, y \in V$, we define the graph distance as
\[
d_G(x, y) = \inf \{\text{length of a path } P \text{ such that } x, y \in P\}.
\]

We denote by $B_G(x, \eta) = \{y \in V : d_G(x, y) \leq \eta\}$ the closed ball with radius $\eta$ with respect to the graph metric. In the next lemma we establish a doubling property of the counting measure and a weak Poincaré inequality that allow us to relate Lipschitz continuity to discrete maximal functions. Given $\varepsilon > 0$ and $u : \varepsilon V \to \mathbb{R}^m$ we define the length of its edge gradient $|\nabla_{e, \varepsilon} u| : \varepsilon V \to \mathbb{R}$ by
\[
|\nabla_{e, \varepsilon} u|(\varepsilon x) = \sum_{(x, y) \in E} \left| \frac{u(x) - u(y)}{\varepsilon} \right|.
\]

Lemma 2.7. Let $G = (V, E)$ be a graph associated to an admissible stochastic lattice. Then there exists a constant $C = C(r, R, M) > 0$ such that for all $x \in V$, $\eta > 0$ and $u : V \to \mathbb{R}^m$
\[(i) \ #B_G(x, 2\eta) \leq C \#B_G(x, \eta), \]
\[(ii) \ \sum_{y \in B_G(x, \eta)} |u(y) - u_{B_G(x, \eta)}| \leq C\eta \sum_{y \in B_G(x, C\eta)} |\nabla_{e, 1} u|(y), \]

where the average $u_{B_G(x, \eta)}$ is defined as
\[
u_{B_G(x, \eta)} = \frac{1}{\#B_G(x, \eta)} \sum_{y \in B_G(x, \eta)} u(y).
\]

Proof. (i): We may assume that $\eta \geq \frac{r}{2}$. Our aim is to compare the graph-metric with the Euclidean one. Given $x, y \in V$ and an optimal path $P = (x_0 = x, x_1, \ldots, x_n = y)$, by (2.2) we have
\[
|d_G(x, y)| \leq M d_G(x, y).
\]

On the other hand, for $0 < \delta \ll 1$ consider the collection of segments
\[
G_{\delta}(x, y) = \{z + \lambda(y - x) : z \in B_{\delta}(x), 0 \leq \lambda \leq 1\}.
\]

We argue that there exists a segment $g* = \{z_0 + \lambda(y - x) : 0 \leq \lambda \leq 1\} \subset G_{\delta}(x, y)$ that does not intersect any Voronoi facet of the tessellation $\mathcal{V}(\omega)$ of dimension less than $d - 1$. Indeed, assume by contradiction that the claim is false for all $z_0 \in B_{\delta}(x)$. By Remark 1 we find finitely many Voronoi facets of dimension less than $d - 1$ whose projection onto the hyperplane containing $x$ and orthogonal to $y - x$ covers a $d - 1$-dimensional set. Since projections onto hyperplanes are Lipschitz continuous, we obtain a contradiction. Hence due to (2.3), for $\delta$ small enough we can construct a path connecting $x, y$ by suitably numbering the set
\[
P(x, y) = \{z \in V : C(z) \cap g* \neq \emptyset\}. \tag{2.6}
\]

Note that by Remark 1 and a covering argument it holds that
\[
\#P(x, y) \leq |B_{\varepsilon}(0)|^{-1}(|x - y| + 2R)(2R)^{d-1} \leq |B_{\varepsilon}(0)|^{-1} \frac{2(2R)^d}{r} |x - y| =: C_{r, R} |x - y|, \tag{2.7}
\]
whence it follows that
\[ d_G(x, y) \leq \#P(x, y) \leq C_{r, R}|x - y|. \tag{2.8} \]
Using again Remark \[1\] (2.5) and (2.8) we deduce that for \( \eta \geq \frac{1}{2} \)
\[ \#B_G(x, 2\eta) \leq \#(L(\omega) \cap B_{2\eta}(x)) \leq |B_2|^{-1}|B_{4\eta}(x)| \]
as well as for any \( \rho > 0 \)
\[ |B_G(x, \rho)| \leq |B_R(0)| \#(L(\omega) \cap B_{2\rho}(x)) \leq |B_R(0)| \#B_G(x, 2C_{r, R}\rho). \tag{2.9} \]
The claim now follows from the scaling properties of the Lebesgue measure by choosing \( \rho = \frac{\eta}{2C_{r, R}} \).
(ii) We can assume that \( \eta \geq 1 \). First note that due to the triangle inequality we have
\[ \sum_{y \in B_G(x, \eta)} |u(y) - u_B(x, \eta)| \leq \frac{1}{\#B_G(x, \eta)} \sum_{y, z \in B_G(x, \eta)} |u(y) - u(z)|. \tag{2.10} \]
Now fix \( y, z \in B_G(x, \eta) \) and consider the path \( P(y, z) = (x_0, \ldots, x_n) \) constructed in (2.6). Then
\[ |u(y) - u(z)| \leq \sum_{i=1}^n |u(x_{i-1}) - u(x_i)|. \tag{2.11} \]
The triangle inequality combined with (2.7) and (2.5) implies that for all \( x_i \in P(y, z) \)
\[ d_G(x_i, x) \leq d_G(x, y) + d_G(y, x) \leq \#P(y, z) + \eta \leq C_{r, R}|y - z| + \eta \leq C_{r, R}(|y - x| + |x - z|) + \eta \leq (2MC_{r, R} + 1)\eta. \]
Setting \( C = 2MC_{r, R} + 1 \), we deduce \( P(y, z) \subset B_G(x, C\eta) \). On the other hand, given an edge \( (x', y') \in E \) with \( x', y' \in B_G(x, C\eta) \), we denote by
\[ N(x', y') = \#\{(y, z) \in B_G(x, \eta) \times B_G(x, \eta) : x', y' \in P(y, z)\} \]
the number of pairs \((y, z)\) such that this edge is contained in the path constructed in (2.6). As a consequence of (2.9), (2.10), (2.11) and (2.4) we have
\[ \sum_{y \in B_G(x, \eta)} |u(y) - u_B(x, \eta)| \leq \frac{1}{\#B_G(x, \eta)} \sum_{(x', y') \in E} N(x', y')|u(x') - u(y')| \]
\[ \leq C \sup_{x', y' \in B_G(x, C\eta)} \frac{N(x', y')}{\eta^d} \sum_{y \in B_G(x, \eta)} |\nabla_{e, 1} u|(y). \tag{2.12} \]
It remains to prove a suitable upper bound for \( N(x', y') \). Since \( x', y' \in B_G(x, C\eta) \), it follows by (2.5) that \( x', y' \in B_{2MC\eta}(x) \) and therefore
\[ N(x', y') \leq \#\{(y, z) \in B_{2MC\eta}(x') \times B_{2MC\eta}(x') : x', y' \in P(y, z)\}, \]
where we used that \( C \geq 1 \). Consider then the boundary-like set \( \Gamma = \{b \in L(\omega) : C(b) \cap \partial B_{2MC\eta}(x') \neq \emptyset\} \) and for each \( b \in \Gamma \) let us define the cylinder-type set
\[ Z(b, x') = \{a + \lambda(x' - b) : \lambda \in [0, 2], a \in B_{bR}(b)\}. \]
For the moment fix any \( y, z \in B_{2MC\eta}(x') \) such that \( x', y' \in P(y, z) \). Then by the construction of \( P(y, z) \) there exists a point \( x_* \in \{y + \lambda(z - y) : \lambda \in [0, 1]\} \) such that \( |x_* - x'| \leq 2R \). Without loss of generality assume that \( |y - x_*| \leq |z - x_*| \) and denote by \( p \) the unique point \( p \in \{x_* + \lambda(z - x') : \lambda \geq 0\} \cap \partial B_{2MC\eta}(x') \).
Then we find \( b \in \Gamma \) with \( |p - b| \leq R \). We claim that \( y, z \in Z(b, x') \). For \( z \) this follows upon writing \( z = p + \lambda_z(x' - p) \) with \( \lambda_z \in [0, 1] \) and choosing \( a = \lambda_z b + (1 - \lambda_z)p \) and \( \lambda = \lambda_z \) in the definition of \( Z(b, x') \). For \( y \), recall that we assume \( |y - x_*| \leq |z - x_*| \), so that for some \( \lambda_y \in [1, 2] \) we can write \( y = z + \lambda_y(x_* - z) = p + \lambda_z(x' - p) + \lambda_y(x_* - p - \lambda_z(x' - p)) \).
Setting \( \lambda = \lambda_z + \lambda_y - \lambda_z\lambda_y \in [0, 2] \) and using the ansatz \( a = b + \xi \) in the definition of \( Z(b, x') \), we find that
\[ \xi = (p - b)(1 - \lambda) + \lambda_y(x_* - x'), \]
so that $|\xi| \leq 5R$ and we conclude that $y \in Z(b, x')$ as well. This proves that
\begin{equation}
N(x', y') \leq 2(\#\Gamma) \left( \sup_{b \in \Gamma} \#(\mathcal{L}(\omega) \cap Z(b, x')) \right)^2.
\end{equation}

Now we use again Remark 1 combined with a covering argument and the fact that $\eta \geq 1$ to find
\[ \#\Gamma \leq |B_{\infty}(0)|^{-1} \left( |B_{2MC\eta + R}(x')| - |B_{2MC\eta - R}(x')| \right) \leq C \left( (2MC\eta + R)^d - (2MC\eta - R)^d \right) \leq C\eta^{d-1}. \]

With the same technique we obtain the bound
\[ \#(\mathcal{L}(\omega) \cap Z(b, x')) \leq 2|B_{\infty}(0)|^{-1}(12R)^{d-1}(|x' - b| + 12R) \leq C\eta. \]

Combining the last two estimates with (2.12) and (2.13) we conclude the proof. \hfill \Box

**Remark 3.** The doubling property and the Poincaré inequality play an important role in the analysis of PDEs on graphs. For our purposes we will use the following consequence: Given $\eta > 0$ and a function $v : \varepsilon \mathcal{L}(\omega) \to \mathbb{R}$ we define the maximal function $M_{\eta}^v : \varepsilon \mathcal{L}(\omega) \to \mathbb{R}$ as
\begin{equation}
M_{\eta}^v(x) = \sup_{0 < \varepsilon \leq \eta} \left( \frac{1}{\# B_G(\xi, \varepsilon)} \sum_{y \in B_G(\xi, \varepsilon)} |v(y)| \right).
\end{equation}

Then, assuming the doubling property, the Poincaré inequality is even equivalent to the estimate
\begin{equation}
\left| \frac{u(\varepsilon x) - u(\varepsilon y)}{\varepsilon} \right| \leq C d_G(x, y) \left( M_{\varepsilon}^{\nabla e, e \varepsilon u}(x) + M_{\varepsilon}^{\nabla e, e \varepsilon u}(y) \right),
\end{equation}
where $C$ is a constant independent of $u : \varepsilon \mathcal{L}(\omega) \to \mathbb{R}^m$ and $x, y \in \mathcal{L}(\omega)$. For $\varepsilon, m = 1$ this fact can be found in a much more general context in [30, Lemma 5.15]. For $\varepsilon > 0$ and $m = 1$ it follows by applying the inequality to $v : \mathcal{L}(\omega) \to \mathbb{R}$ defined as $v(x) = \varepsilon^{-1}u(\varepsilon x)$ upon noticing that $M_{\varepsilon}^{\nabla e, e \varepsilon u}(x) = M_{\varepsilon}^{\nabla e, e \varepsilon u}(\varepsilon x)$. In particular the constant $C$ is independent of $\varepsilon$. For $m \geq 2$ the inequality remains true arguing for each component and increasing the constant $C$ by at most a factor of $m$.

2.3. A generalized weak-membrane energy. In order to discretize vectorial Mumford-Shah-type functionals we basically follow the approach used on periodic lattices for the scalar case. However we go beyond pairwise interactions. Due to the possibly non-ordered edges $E(\omega)$ this requires some notation. For $M \in \mathbb{N}$ satisfying (2.1) we denote by
\[ \mathcal{P}_+(M) = \{ p : [0, +\infty) \to \mathbb{N}_0 : \# p^{-1}(\mathbb{N}) < +\infty, \sum_{v \in p^{-1}(\mathbb{N})} p(v) \leq M \} \]
the set of all multisets over $[0, +\infty)$ with at most $M$ elements. Note that if $p_1, \ldots, p_k \in [0, +\infty)$ with $k \leq M$, then up to permutation we can identify these points with a unique $p \in \mathcal{P}_+(M)$ setting $p(p_i) = \#\{ j : p_j = p_i \}$ for all $1 \leq i \leq k$ and zero elsewhere. In this sense we sometimes use the more common notation $p = \{ p_i \}_b$, where the $b$ indicates a badge where elements can occur several times in contrast to ordinary sets. For $p \in \mathcal{P}_+(M)$ we set $\| p \|_1 = \sum_{v \in p^{-1}(\mathbb{N})} p(v)$. In this paper we fix a bounded function $f : \mathcal{P}_+(M) \to [0, +\infty)$ satisfying the following two structural assumptions: There exists $0 < \alpha < +\infty$ such that
\begin{equation}
\lim_{\| p \|_1 \to 0} \frac{f(p)}{\| p \|_1} = \alpha
\end{equation}
and $f$ is monotone increasing in the following sense: For all $p, p' \in [0, +\infty)^M$ with $p_i \leq p'_i$ for all $1 \leq i \leq M$ and for any $1 \leq k \leq M$ we have
\begin{equation}
f(\{ p_1, \ldots, p_k \}_b) \leq f(\{ p'_1, \ldots, p'_k \}_b).
\end{equation}
Finally, we assume that $f$ is lower semicontinuous in the sense that for all sequences $(p^n)_{n} \subset [0, +\infty)^M$ converging to some $p \in [0, +\infty)^M$ and for all $1 \leq k \leq M$ it holds that
\begin{equation}
f(\{ p_1, \ldots, p_k \}_b) \leq \liminf_{n \to +\infty} f(\{ p^n_1, \ldots, p^n_k \}_b).
\end{equation}
Remark 4. Note that from the boundedness of $f$, the monotonicity assumption and the property \[ (2.16) \]

$$c_f \min\{\|p\|_1, 1\} \leq f(p) \leq C_f \min\{\|p\|_1, 1\}. \tag{2.19}$$

Moreover, again by boundedness and monotonicity, for every $1 \leq l \leq k \leq M$ there exists the limit

$$\beta(l, k) = \lim_{N \to +\infty} f(l \mathbb{1}_N + (k - l) \mathbb{1}_0) > 0. \tag{2.20}$$

In order to build a discrete approximation of a continuous functional we scale a stochastic lattice by a small parameter $\varepsilon > 0$. Let us fix a reference set $D \subset \mathbb{R}^d$, which we assume to be a bounded Lipschitz domain, and a growth exponent $p \in (1, +\infty)$. Given $u : \varepsilon \mathcal{L}(\omega) \to \mathbb{R}^m$, an open set $A \in \mathcal{A}(D)$ and $\eta > 0$, we define the function $\eta|\nabla_{\omega, \varepsilon}|^p(u, A) : \varepsilon \mathcal{L}(\omega) \to \mathcal{P}_+(M)$ by

$$\eta|\nabla_{\omega, \varepsilon}|^p(u, A)(\varepsilon x) = \left\{ \eta \left| \frac{u(\varepsilon x) - u(\varepsilon y)}{\varepsilon} \right|^p : (x, y) \in \mathcal{E}(\omega), \varepsilon x, \varepsilon y \in A \right\}. $$

Then we define the localized discrete functionals (which we also call energy) as

$$F_\varepsilon(\omega)(u, A) = \sum_{\varepsilon x \in \varepsilon \mathcal{L}(\omega) \cap A} \varepsilon^{d-1} f(\varepsilon|\nabla_{\omega, \varepsilon}|^p(u, A)(\varepsilon x)).$$

Remark 5. We chose the abstract framework above for two reasons:

(i) We take directed edges to define $\eta|\nabla_{\omega, \varepsilon}|^p(u, A)(\varepsilon x)$ in order to include the functional \[ (1.5) \]

(ii) We define the function $f$ on multisets to handle pairwise and non-pairwise gradients simultaneously. For pairwise interactions, we set $f(p) = \sum_{v \in p^{-1}(\mathbb{N})} p(v)f_0(v)$ with $f_0$ satisfying \[ (1.3) \]. The other example of the introduction is given by $f(p) = f_0(\|p\|_1)$.

As we aim at using the abstract theory of $\Gamma$-convergence, we will identify a discrete variable with its piecewise constant interpolation on the Voronoi cells, that means with functions of the class

$$\mathcal{P}C_\omega^\varepsilon = \{ u : \mathbb{R}^d \to \mathbb{R}^m : u|_{\varepsilon \mathcal{L}(x)} \text{ is constant for all } x \in \mathcal{L}(\omega) \}.$$

Without relabeling we extend the functional to $F_\varepsilon(\omega) : L^1(D, \mathbb{R}^m) \times \mathcal{A}(D) \to [0, +\infty]$ by

$$F_\varepsilon(\omega)(u, A) = \begin{cases} F_\varepsilon(\omega)(u, A) & \text{if } u \in \mathcal{P}C_\omega^\varepsilon, \\ +\infty & \text{otherwise}. \end{cases} \tag{2.21}$$

Given $F_\varepsilon(\omega)$, we let $F'(\omega), F''(\omega) : L^1(D, \mathbb{R}^m) \times \mathcal{A}(D) \to [0, +\infty]$ be the corresponding $\Gamma$-lim inf and $\Gamma$-lim sup defined as

$$F'(\omega)(u, A) = \liminf_{\varepsilon \to 0} \inf_{u_\varepsilon} F_\varepsilon(\omega)(u_\varepsilon, A) : u_\varepsilon \to u \text{ in } L^1(D, \mathbb{R}^m),$$

$$F''(\omega)(u, A) = \limsup_{\varepsilon \to 0} \sup_{u_\varepsilon} F_\varepsilon(\omega)(u_\varepsilon, A) : u_\varepsilon \to u \text{ in } L^1(D, \mathbb{R}^m).$$

It is well-known that both functionals are $L^1(D, \mathbb{R}^m)$-lower semicontinuous.

Remark 6. We note that for any $u \in L^1(D, \mathbb{R}^m)$ there exists a sequence $u_\varepsilon \in \mathcal{P}C_\omega^\varepsilon$ such that $u_\varepsilon \to u$ in $L^1(D, \mathbb{R}^m)$. For $u \in C_\varepsilon(D, \mathbb{R}^m)$ this follows from Remark 11. In the general case one can use a density argument and construct suitable diagonal sequences.

Remark 7. We work in the space $L^1$ due to the applications we have in mind. A priori there is no equicoercivity in this space and therefore it seems more natural to define the $\Gamma$-limit for example with respect to convergence in measure. However, as shown in Lemma 3.3, this can be circumvented through a fidelity term that is part of the Mumford-Shah functional anyway.

Our final preliminary observation concerns the behavior of the discrete functional and possible $\Gamma$-limits under truncations.
Lemma 2.8. Let \( u_\varepsilon \in \mathcal{PC}_\varepsilon^\omega \). For any \( k > 0 \) set \( T_k u_\varepsilon \) as in Lemma 2.7. Then, for any \( A \in \mathcal{A}(D) \), it holds that \( F_\varepsilon(\omega)(T_k u_\varepsilon, A) \leq F_\varepsilon(\omega)(u_\varepsilon, A) \). In particular, whenever \( u \in L^\infty(D, \mathbb{R}^m) \) we can compute \( F'(\omega)(u, A) \) and \( F''(\omega)(u, A) \) considering sequences \( u_\varepsilon \in \mathcal{PC}_\varepsilon^\omega \) such that \( |u_\varepsilon(\varepsilon x)| \leq 3\|u\|_\infty \) for all \( x \in \mathcal{L}(\omega) \). Moreover, for all \( u \in L^1(D, \mathbb{R}^m) \) we have

\[
\lim_{k \to +\infty} F'(\omega)(T_k u, A) = F'(\omega)(u, A),
\]

\[
\lim_{k \to +\infty} F''(\omega)(T_k u, A) = F''(\omega)(u, A).
\]

Proof. For the estimate at the discrete level, it suffices to combine the fact that \( |T_k u_\varepsilon(\varepsilon x) - T_k u_\varepsilon(\varepsilon y)| \leq |u_\varepsilon(\varepsilon x) - u_\varepsilon(\varepsilon y)| \) for all \( x, y \in \mathcal{L}(\omega) \) with the monotonicity assumption (2.17). In order to restrict the class of approximating sequences, we use the first estimate and the fact that any truncated sequence \( T_k u_\varepsilon \) with \( k = \|u\|_\infty \) still converges to \( u \) in \( L^1(D, \mathbb{R}^m) \). The continuity property at the limit follows from \( L^1(D, \mathbb{R}^m) \)-lower semicontinuity of both functionals and the fact that the discrete upper estimate is conserved in the limit. \( \square \)

3. Integral representation and separation of scales

In this section we establish a representation result in the spirit of Theorem 2.2 but on the larger space \( \text{GSBV}^p \). More precisely we prove the following:

Theorem 3.1. Given any sequence \( \varepsilon \to 0 \) there exists a subsequence \( \varepsilon_n \) such that for all \( A \in \mathcal{A}^R(D) \) the functionals \( F_{\varepsilon_n}(\omega)(\cdot, A) \) \( \Gamma \)-converge in the \( L^1(D, \mathbb{R}^m) \)-topology to a functional \( F(\omega)(\cdot, A) : L^1(D, \mathbb{R}^m) \to [0, +\infty) \) that is finite only for \( A \in \mathcal{A}^R(D) \) such that \( u \in \text{GSBV}^p(A, \mathbb{R}^m) \). For such \((u, A)\) it can be written as

\[
F(\omega)(u, A) = \int_A h(x, \nabla u(x)) \, dx + \int_{S_u \cap A} g(x, u^+, u^-, \nu) \, d\mathcal{H}^{d-1}.
\]

We postpone the prove of Theorem 3.1 to the very end of Section 3 because it requires several steps that we split for the sake of a clear presentation. First we use Theorem 2.2 to represent (up to subsequences) the \( \Gamma \)-limit on \( \text{SBV}^p \). In a second part we study asymptotic formulas for the integrands \( h, g \) which allow to conclude the proof.

3.1. Integral representation on \( \text{SBV}^p \). As a preliminary step, we discuss the structure of possible \( \Gamma \)-limits on the smaller space \( \text{SBV}^p \).

Proposition 1. Given any sequence \( \varepsilon \to 0 \) there exists a subsequence \( \varepsilon_n \) such that for all \( A \in \mathcal{A}^R(D) \) the functionals \( F_{\varepsilon_n}(\omega)(\cdot, A) \) \( \Gamma \)-converge in the \( L^1(D, \mathbb{R}^m) \)-topology to a functional \( F(\omega)(\cdot, A) : L^1(D, \mathbb{R}^m) \to [0, +\infty] \). If \( u \in \text{SBV}^p(A, \mathbb{R}^m) \) then \( F(\omega)(u, A) \) can be written as

\[
F(\omega)(u, A) = \int_A h(x, \nabla u) \, dx + \int_{S_u \cap A} g(x, u^+, u^-, \nu) \, d\mathcal{H}^{d-1}.
\]

Before we prove this result we derive the properties necessary to apply Theorem 2.2 to a slightly modified functional. We start with the locality property.

Lemma 3.2. Let \( A \in \mathcal{A}^R(D) \). If \( u, v \in L^1(D, \mathbb{R}^m) \) and \( u = v \) a.e. on \( A \), then \( F''(\omega)(u, A) = F''(\omega)(v, A) \).

Proof. Let \( u_\varepsilon \) and \( v_\varepsilon \) be converging to \( u \) and \( v \) in \( L^1(D, \mathbb{R}^m) \), respectively and take \( u_\varepsilon \) such that

\[
\limsup_{\varepsilon \to 0} F_\varepsilon(\omega)(u_\varepsilon, A) = F''(\omega)(u, A).
\]

By Remark 3 we may assume that \( u_\varepsilon, v_\varepsilon \in \mathcal{PC}_\varepsilon^\omega \). Set \( \tilde{u}_\varepsilon \in \mathcal{PC}_\varepsilon^\omega \) as

\[
\tilde{u}_\varepsilon(\varepsilon x) = \mathbf{1}_A(\varepsilon x) u_\varepsilon(\varepsilon x) + (1 - \mathbf{1}_A(\varepsilon x)) v_\varepsilon(\varepsilon x).
\]

Since \( |\partial A| = 0 \) and \( u_\varepsilon \) and \( v_\varepsilon \) are equiintegrable, it follows that \( \tilde{u}_\varepsilon \to v \) in \( L^1(D, \mathbb{R}^m) \). Then by definition

\[
F''(\omega)(v, A) \leq \limsup_{\varepsilon \to 0} F_\varepsilon(\omega)(\tilde{u}_\varepsilon, A) = \limsup_{\varepsilon \to 0} F_\varepsilon(\omega)(u_\varepsilon, A) = F''(\omega)(u, A).
\]

Exchanging the roles of \( u \) and \( v \) we conclude the proof. \( \square \)
The next two lemmata provide lower and upper bounds for the limit functionals. However the boundedness of the discrete density $f$ rules out the lower bound needed for Theorem 2.2. The lemma below also includes an equicoercivity property under an additional equiintegrability assumption.

**Lemma 3.3.** Assume that $A \in A^0(D)$ and $u_\varepsilon \in \mathcal{PC}^\omega_x$ are such that

$$\sup_{\varepsilon > 0} F_\varepsilon(\omega)(u_\varepsilon, A) < +\infty.$$ 

If $u_\varepsilon$ is equiintegrable, then there exists a subsequence (not relabeled) such that $u_\varepsilon \rightarrow u$ in $L^1(A, \mathbb{R}^m)$ for some $u \in L^1(A, \mathbb{R}^m) \cap GSBV^p(A, \mathbb{R}^m)$. Moreover we have the estimate

$$\frac{1}{c} \left( \int_A |\nabla u| p \, dx + H^{d-1}(S_u \cap A) \right) \leq F'(\omega)(u, A)$$

for some constant $c > 0$ independent of $\omega, A$ and $u$.

**Proof.** We first construct a suitable function $v_\varepsilon \in SBV^p(A, \mathbb{R}^m)$ that is asymptotically close to $u_\varepsilon$. Given a triangulation $\mathcal{T}_d$ of the cube $[0,1]^d$ we define a triangulation of $\mathbb{R}^d$ via

$$\mathcal{T} = \{ T = z + T_d : z \in \mathbb{Z}^d, T_d \in \mathcal{T}_d \}.$$ 

We may assume that $\text{diam}(T) < R$ for all $T \in \mathcal{T}$. Then we define $\varphi_\varepsilon$ as a continuous piecewise affine interpolation on $\varepsilon \mathcal{T}$ in the following way: For each $z \in \mathbb{Z}^d$ we choose by a deterministic rule a point $x(z) \in \mathcal{L}(\omega)$ such that $z \in \mathcal{L}(x)$ and set

$$\varphi_\varepsilon(\varepsilon z) = u_\varepsilon(\varepsilon x(z)).$$

Next we decompose the scaled lattice $\varepsilon \mathcal{L}(\omega) = L_{0,\varepsilon} \cup L_{1,\varepsilon}$, where

$$L_{0,\varepsilon} = \{ \varepsilon x \in \varepsilon \mathcal{L}(\omega) : \|\varepsilon |\nabla_x \varphi| P(u, A)(\varepsilon x)\|_1 \leq 1 \}.$$ 

Let us also group the simplices overlapping with $A$ according to

$$\mathcal{T}_1 = \{ T \in \mathcal{T} : T \cap A \neq \emptyset, \inf_{z \in T} \text{dist}(z, L_{1,\varepsilon}) > 6R\varepsilon \text{ and } \inf_{z \in T} \text{dist}(z, \partial A) > 8R\varepsilon \},$$

$$\mathcal{T}_2 = \{ T \in \mathcal{T} : T \cap A \neq \emptyset, \inf_{z \in T} \text{dist}(z, \partial A) \leq 8R\varepsilon \},$$

$$\mathcal{T}_3 = \{ T \in \mathcal{T} : T \cap A \neq \emptyset, \inf_{z \in T} \text{dist}(z, L_{1,\varepsilon}) \leq 6R\varepsilon \text{ and } \inf_{z \in T} \text{dist}(z, \partial A) > 8R\varepsilon \}.$$ 

Given $\varphi_\varepsilon$ we define $v_\varepsilon$ on the interior of each simplex $\varepsilon T \in \varepsilon \mathcal{T}$ setting

$$v_{\varepsilon|\varepsilon T} = \begin{cases} \varphi_{\varepsilon|\varepsilon T} & \text{if } T \in \mathcal{T}_1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $v_\varepsilon \in SBV^p(A, \mathbb{R}^m)$. We start estimating the difference of $u_\varepsilon$ and $v_\varepsilon$ on $A$. Consider a simplex $\varepsilon T$ with $T \in \mathcal{T}_2$. Then $\varepsilon T \subset \partial A + B_{9R\varepsilon}(0)$. Since $\partial A$ is a Lipschitz boundary it admits a $(d-1)$-dimensional Minkowski content. Hence there exists a constant $C = C_R > 0$ such that for all $\varepsilon$ small enough we have

$$|\{ z \in A : \text{dist}(z, \partial A) \leq 9R\varepsilon \} | \leq C\varepsilon^{d-1}(\partial A) \varepsilon. \quad (3.1)$$

Next, if $T \subset \mathcal{T}_3$, then there exists $\varepsilon x \in L_{1,\varepsilon} \cap A$ such that $\varepsilon T \subset B_{8R\varepsilon}(\varepsilon x)$. From (2.19) we deduce

$$\left| \bigcup_{T \in \mathcal{T}_3} \varepsilon T \cap A \right| \leq C\varepsilon^d \# \{ \varepsilon x \in \varepsilon \mathcal{L}(\omega) \cap A : \|\varepsilon |\nabla_x \varphi| P(u, A)(\varepsilon x)\|_1 > 1 \} \leq C\varepsilon F_\varepsilon(\omega)(u_\varepsilon, A) \leq C\varepsilon. \quad (3.2)$$

Finally, if $z \in \varepsilon T \cap A$ for some $T \in \mathcal{T}_1$, then by definition $\text{dist}(z, L_{1,\varepsilon}) > 6R\varepsilon$ and $\text{dist}(z, \partial A) > 8R\varepsilon$. Let us write $T = co(z_0, \ldots, z^d) \in \mathcal{T}$ and choose $x \in \mathcal{L}(\omega)$ such that $z \in \mathcal{L}(x) \cap T$. Then, except for a null set where $u_\varepsilon$ is not well-defined,

$$|v_\varepsilon(z) - u_\varepsilon(z)| \leq \sum_{i=0}^d \lambda_i^2 |\varphi_\varepsilon(\varepsilon z^i) - u_\varepsilon(\varepsilon x^i)| \leq \sum_{y \in \mathcal{L}(\omega) \cap B_{8\varepsilon}(x)} |u_\varepsilon(\varepsilon y) - u_\varepsilon(\varepsilon x)|. \quad (3.3)$$
Given \( y \in \mathcal{L}(\omega) \cap B_{3R}(x) \), we construct the path \( P(y, x) = \{ y = x_0, x_1, \ldots, x_n = x \} \) of nearest neighbours between \( x \) and \( y \) as in (2.6). Since \( |x - y| \leq 3R \), by construction of the path with parameter \( \delta < R \) we have

\[
|x_i - \varepsilon^{-1}z| \leq |\text{dist}(x_i, g_x) + |y - x| + \delta + |x - \varepsilon^{-1}z| \leq R + 3R + \delta + R < 6R.
\]

In particular we conclude that \( \varepsilon x_i \in L_{0, \varepsilon} \cap A \) for all \( 0 \leq i \leq n \). Using (2.3), the definition of the set \( L_{0, \varepsilon} \) implies that \( |u_{\varepsilon}(\varepsilon x_i) - u_{\varepsilon}(\varepsilon x_{i+1})| \leq \varepsilon^{-1}\delta \) for all \( 0 \leq i \leq n - 1 \). By Remark 2 the number of nearest neighbours in \( B_{6R}(\varepsilon^{-1}z) \) is uniformly bounded, so that (3.3) implies the bound

\[
|v_{\varepsilon}(z) - u_{\varepsilon}(z)| \leq C\varepsilon^{-1}\delta.
\]

Since we have set \( \varepsilon \in (T_2 \cup T_3) \) and \( u_{\varepsilon} \) is equiintegrable by assumption, we conclude from (3.1), (3.2) and (3.4) that

\[
\lim_{\varepsilon \to 0} \|v_{\varepsilon} - u_{\varepsilon}\|_{L^1(A)} = 0.
\]

Moreover, the sequence \( v_{\varepsilon} \) is still equiintegrable. Thus it suffices to show the convergence for the sequence \( v_{\varepsilon} \). We start estimating its jump set. We have at most two contributions. One is given by

\[
\mathcal{H}^{d-1}(\bigcup_{T \in T_3} \partial \varepsilon T \cap A) \leq \sum_{T \in T_3} C\varepsilon^{d-1} \leq CF_{\varepsilon}(\omega)(u_{\varepsilon}, A) \leq C,
\]

where the second inequality follows by the same reasoning used in the lines preceding (3.2). The second contribution comes from jumps along edges of simplices in \( T_2 \). For those we have, again for small \( \varepsilon \) enough, the estimate

\[
\mathcal{H}^{d-1}(S_{\varepsilon^1} \cap A) \leq \mathcal{H}^{d-1}(\bigcup_{T \in T_2} \partial \varepsilon T \cap A) \leq C. \tag{3.6}
\]

To estimate the gradient it suffices to consider simplices \( T = co(\varepsilon z^1, \ldots, \varepsilon z^d) \in T_1 \). Write any basis vector \( e_k \) as \( e_k = \sum_{i=1}^d \lambda_i(z^i - z^0) \). Due to the periodic triangulation \( T \) the coefficients \( \lambda_i \) are equibounded with respect to the simplices. Take \( x \in \mathcal{L}(\omega) \) such that \( v_{\varepsilon}(\varepsilon z^0) = u_{\varepsilon}(\varepsilon x) \). Since \( v_{\varepsilon} \) is affine on \( \varepsilon T \) we have

\[
|\partial_k v_{\varepsilon}(\varepsilon x)| = \left| \sum_{i=1}^d \lambda_i \frac{v_{\varepsilon}(\varepsilon x^i) - v_{\varepsilon}(\varepsilon z^0)}{\varepsilon} \right| \leq C \sum_{i=1}^d \left| \frac{v_{\varepsilon}(\varepsilon x^i) - v_{\varepsilon}(\varepsilon z^0)}{\varepsilon} \right| \leq C \sum_{y \in \mathcal{L}(\omega) \cap B_{3R}(x)} \varepsilon^{-1} |u_{\varepsilon}(\varepsilon y) - u_{\varepsilon}(\varepsilon x)| \leq C \sum_{(x_1, x_2) \in \mathcal{N}(\omega)} \varepsilon^{-1} |u_{\varepsilon}(\varepsilon x_1) - u_{\varepsilon}(\varepsilon x_2)|,
\]

where the last inequality follows by the same reasoning we used for proving (3.4). Now observe that if \( T \in T_1 \) and \( |x_i - x| < 5R \), then \( \varepsilon x_1, \varepsilon x_2 \in L_{0, \varepsilon} \cap A \). Thus taking the \( p \)-th power of the above estimate and using (2.3) and (2.19) we obtain

\[
|\nabla v_{\varepsilon}(\varepsilon x)|^p \leq C \sum_{(x_1, x_2) \in \mathcal{N}(\omega)} \varepsilon^{-p} |u_{\varepsilon}(\varepsilon x_1) - u_{\varepsilon}(\varepsilon x_2)|^p \leq C \varepsilon^{-1} \sum_{x \in \mathcal{L}(\omega) \cap B_{3R}(x)} \|\varepsilon |\nabla v_{\varepsilon}|^p (u, A)(\varepsilon x_i)\|_1 \leq C \varepsilon^{-1} \sum_{x \in \mathcal{L}(\omega) \cap B_{3R}(x)} f(\varepsilon |\nabla v_{\varepsilon}|^p (u, A)(\varepsilon x_i)).
\]

Now we sum the last estimate over all \( T \in T_1 \). Since \( T \subset B_{7R}(x_i) \) we count each lattice point \( x_i \) at most \( C \) times and conclude

\[
\int_A |\nabla v_{\varepsilon}(z)|^p dz \leq C \sum_{x \in \mathcal{L}(\omega) \cap A} \varepsilon^{d-1} f(\varepsilon |\nabla v_{\varepsilon}|^p (u, A)(\varepsilon x_i)) = CF_{\varepsilon}(\omega)(u_{\varepsilon}, A) \leq C. \tag{3.7}
\]
By [3.6] and [3.7], the compactness theorem for $GSBV(A, \mathbb{R}^m)$-functions [3] Theorem 2.2] implies that, up to subsequences, $u_\varepsilon \to u$ in $GSBV(A, \mathbb{R}^m)$ in measure and by equiintegrability also in $L^1(A, \mathbb{R}^m)$. Moreover \( \nabla u_\varepsilon \to \nabla u \) in $L^p(A, \mathbb{R}^{m \times d})$ and from lower semicontinuity [1] Theorem 3.7] we deduce

\[
\int_A |\nabla u|^p \, dz + H^{d-1}(S_u \cap A) \leq C \liminf_{\varepsilon \to 0} F_\varepsilon(\omega)(u_\varepsilon, A) + CH^{d-1}(\partial A) \leq C.
\]

Thus by definition $u \in GSBV^p(A)$ which finishes the proof of compactness. In order to prove the lower bound, note that by the argument above we have for any open set $A' \subset \subset A$ the inequality

\[
\int_{A'} |\nabla u|^p \, dz + H^{d-1}(S_u \cap A') \leq C \liminf_{\varepsilon \to 0} F_\varepsilon(\omega)(u_\varepsilon, A).
\]

By the definition of $F'(\omega)$ and the arbitrariness of $A'$ we obtain the desired estimate. $\square$

We next show an upper bound for $F''(\omega)$.

**Lemma 3.4.** Let $u \in L^1(D, \mathbb{R}^m)$. There exists a constant $c > 0$ independent of $\omega$ and $u$ such that for all $A \in \mathcal{A}^D(D)$ with $u \in GSBV^p(A, \mathbb{R}^m)$ it holds that

\[
F''(\omega)(u, A) \leq c \left( \int_A |\nabla u|^p \, dx + H^{d-1}(S_u \cap A) \right)
\]

**Proof.** Take any ball $B_L$ such that $D \subset B_L$. For the moment let us assume that $u \in SBV^p(B_L)$ is such that

(i) $H^{d-1}(\overline{S_u \setminus B_L}) = 0$,
(ii) $S_u$ is the intersection of $B_L$ with a finite number of pairwise disjoint $(d-1)$-simplices,
(iii) $u \in W^{k, \infty}(B_L \setminus \overline{S_u}, \mathbb{R}^m)$ for all $k \in \mathbb{N}$.

We define an admissible sequence to bound $F''(\omega)(u|_D, A)$ setting

\[
u_\varepsilon(\varepsilon x) = \begin{cases} u(\varepsilon x) & \text{if } \varepsilon x \in B_L, \\ 0 & \text{otherwise.} \end{cases}
\]

Using properties (ii) and (iii) from above it follows by Remark [1] that $u_\varepsilon \to u|_D$ in $L^1(D, \mathbb{R}^m)$. To bound the energy, consider first the case that $\varepsilon x \in \varepsilon \mathcal{C}(\omega) \cap A$ is such that $\text{dist}(\varepsilon x, \overline{S_u}) \geq 3M\varepsilon$. Then for all $y \in \mathcal{L}(\omega)$ with $(x, y) \in \mathcal{E}(\omega)$ we have by Jensen’s inequality and the regularity of $u$

\[
\left| \frac{u_\varepsilon(\varepsilon x) - u_\varepsilon(\varepsilon y)}{\varepsilon} \right|^p = \left| \int_0^1 \nabla u(\varepsilon x + s\varepsilon(y - x))(y - x) \, ds \right|^p \leq |y - x|^p |\nabla \int_0^1 (\varepsilon x + s\varepsilon(y - x))| |\nabla u(\varepsilon x + s\varepsilon(y - x))| \, ds.
\]

Integrating both sides over $\varepsilon \mathcal{C}(x)$ we infer from Fubini’s theorem and Remark [1] the bound

\[
C\varepsilon \left| u_\varepsilon(\varepsilon x) - u_\varepsilon(\varepsilon y) \right|^p \leq C \left( \int_{\varepsilon \mathcal{C}(x)} \int_0^1 |\nabla u(\varepsilon x + s\varepsilon(y - x))| \, ds \, dz \right)^p
\]

\[
\leq C \left( \int_{\varepsilon \mathcal{C}(x)} \int_0^1 |\nabla u(z + s\varepsilon(y - x))| \, ds \, dz + c_u \int_{\varepsilon \mathcal{C}(x)} |z - \varepsilon x|^p \, dz \right)
\]

\[
\leq C \left( \int_0^1 \int_{\varepsilon \mathcal{C}(x) + s(y - x))} |\nabla u(z)| \, dz \, ds + c_u \varepsilon^{d+p} \right),
\]

(3.8)

where $c_u$ denotes the $L^\infty$-norm of $D^2 u$ on $B_L \setminus \overline{S_u}$. Here we used that, by Remark [1] it holds that

\[
t(\varepsilon x + s\varepsilon(y - x)) + (1 - t)(z + s\varepsilon(y - x)) \in B_{2M\varepsilon}(\varepsilon x) \subset B_L \setminus \overline{S_u},
\]

for all $z \in \varepsilon \mathcal{C}(x)$ and $s, t \in [0, 1]$, so that $c_u$ indeed provides Lipschitz estimates for $\nabla u$. Due to (2.19) we have $f(p) \leq C_f \|p\|_1$, so that (2.34), (3.8) and (3.9) imply

\[
\varepsilon^{d-1} f(\varepsilon |\nabla \omega, \varepsilon|^p(u, A)(\varepsilon x) \leq C \left( \int_{B_{2M\varepsilon}(\varepsilon x)} |\nabla u(z)|^p \, dz + c_u \varepsilon^{d+p} \right).
\]

(3.10)
Now fix a set $A' \in A_1^R(\mathbb{R}^d)$ such that $A \subset A'$. For $\varepsilon$ small enough Remark 1 allows to control the number of remaining lattice points by
\[
\varepsilon^{-d} \# \{ \varepsilon x \in \varepsilon L(\omega) \cap A : \text{dist}(\varepsilon x, \varepsilon L(\omega)) < 3M \varepsilon \} \leq \frac{\| (S_u \cap A') + B_{4M\varepsilon}(0) \|}{\varepsilon \| B_1(0) \|}.
\]
Recall that $\overline{S_u}$ is the intersection of $B_L$ with a finite union of pairwise disjoint $(d-1)$-simplices, so that $\overline{S_u} \cap \overline{A'}$ admits a $(d-1)$-dimensional Minkowski content. Hence, letting $\varepsilon \to 0$, it holds that
\[
\lim_{\varepsilon \to 0} \frac{\| (S_u \cap A') + B_{4M\varepsilon}(0) \|}{\varepsilon \| B_1(0) \|} \leq C \mathcal{H}^{d-1}(\overline{S_u} \cap \overline{A'}) = \mathcal{H}^{d-1}(S_u \cap \overline{A'}),
\]
where we used assumption (i) in the second identity. Since $f$ is bounded, by (3.10) and the bound above we conclude that
\[
\lim_{\varepsilon \to 0} F_\varepsilon(\omega)(u_\varepsilon, A) \leq \lim_{\varepsilon \to 0} \sum_{\varepsilon x \in \varepsilon L(\omega) \cap A} C \left( \int_{B_{1M\varepsilon}(\varepsilon x)} |\nabla u(z)|^p \, dz + c_0 \varepsilon^{d+p} \right) + C \mathcal{H}^{d-1}(S_u \cap \overline{A'})
\]
\[
\leq C \int_{A'} |\nabla u(z)|^p \, dz + C \mathcal{H}^{d-1}(S_u \cap \overline{A'}).
\]
Letting $A' \downarrow \overline{A}$ in this estimate yields by definition of $F''(\omega)$
\[
F''(\omega)(u|_D, A) \leq C \int_A |\nabla u(z)|^p \, dz + C \mathcal{H}^{d-1}(S_u \cap \overline{A}).
\]
(3.11)
From this estimate we can now prove the general case by density. First we assume that $u \in SBV^p(A, \mathbb{R}^m) \cap L^\infty(A, \mathbb{R}^m)$. Note that due to the Lipschitz regularity of $\partial A$ we can use a local reflection argument to extend $u$ to a function $\tilde{u} \in SBV^p(B_L, \mathbb{R}^m) \cap L^\infty(B_L, \mathbb{R}^m)$ such that $\mathcal{H}^{d-1}(S_u \cap \partial A) = 0$. By [24, Theorem 3.1] applied to the large set $B_L$ we find a sequence $u_n \in SBV^p(B_L, \mathbb{R}^m)$ fulfilling assumptions (i)-(iii) of the first part such that $u_n \to \tilde{u}$ in $L^1(B_L, \mathbb{R}^m)$, $\nabla u_n \to \nabla \tilde{u}$ in $L^p(B_L, \mathbb{R}^{m \times d})$ and $\lim_{n} \mathcal{H}^{d-1}(S_u \cap \overline{A}) \leq \mathcal{H}^{d-1}(S_u \cap \overline{A}) = \mathcal{H}^{d-1}(S_u \cap \overline{A})$. From Lemma 6.2 lower semicontinuity of $F''(\omega)(\cdot, A)$ and (3.11) we deduce
\[
F''(u, A) = F''(u|_D, A) \leq \liminf_{n} F''(\omega)(u_n|_D, A) \leq C \int_A |\nabla u(z)|^p \, dz + C \mathcal{H}^{d-1}(S_u \cap \overline{A}).
\]
It remains to remove the $L^\infty$-bound. Hence, given any $u \in GSBV^p(A, \mathbb{R}^m) \cap L^1(D, \mathbb{R}^m)$, we consider the truncated sequence $u_k \in SBV^p(A, \mathbb{R}^m) \cap L^\infty(A, \mathbb{R}^m)$. Then $u_k \to u$ in $L^1(D, \mathbb{R}^m)$ and, like in the previous reasoning, the claim follows by lower semicontinuity, Lemma 2.2 and the estimate established for bounded functions. \qed

The following technical lemma establishes an almost subadditivity of the set function $A \mapsto F''(\omega)(u, A)$.

**Proposition 2.** Let $A, B \in A_1^R(D)$. Moreover let $A' \in A_1^R(D)$ be such that $A' \subset A$. Then, for all $u \in L^1(D, \mathbb{R}^m)$,
\[
F''(\omega)(u, A' \cup B) \leq F''(\omega)(u, A) + F''(\omega)(u, B).
\]

**Proof.** We can assume that $F''(u, A)$ and $F''(u, B)$ are both finite. Since $A' \cup B \in \mathcal{A}(D)$, Lemma 2.8 allows us to reduce the proof to the case $u \in L^\infty(D, \mathbb{R}^m)$. Let $u_\varepsilon, v_\varepsilon \in \mathcal{P}_{\varepsilon}^\omega$ both converge to $u$ in $L^1(D, \mathbb{R}^m)$ such that
\[
\lim_{\varepsilon \to 0} F_\varepsilon(\omega)(u_\varepsilon, A) = F''(\omega)(u, A), \quad \lim_{\varepsilon \to 0} F_\varepsilon(\omega)(v_\varepsilon, B) = F''(\omega)(u, B).
\]
(3.12)
By Lemma 2.8 we may assume that $\|u_\varepsilon\|_\infty, \|v_\varepsilon\|_\infty \leq 3\|u\|_\infty$, so that both sequences actually converge to $u$ in $L^p(D, \mathbb{R}^m)$. Fix $h \leq \text{dist}(A', A'' \cup B)$ and $N \in \mathbb{N}$. For $i = 1, \ldots, N$ we define the sets
\[
A_i := \{ x \in A : \text{dist}(x, A') < \frac{i h}{2N} \}.
\]
Let $0 \leq \varphi_i \leq 1$ be a cut-off function between the sets $A_i$ and $A_{i+1}$, that means $\varphi_i = 1$ on $A_i$ and $\varphi_i = 0$ on $\mathbb{R}^d \setminus A_{i+1}$. We may assume that $\|\nabla \varphi_i\|_\infty \leq \frac{AN}{N}$. Then define $w^i_\varepsilon \in \mathcal{PC}_n^\varepsilon$ by

$$w^i_\varepsilon(\varepsilon x) = \varphi_i(\varepsilon x)u_\varepsilon(\varepsilon x) + (1 - \varphi_i(\varepsilon x))v_\varepsilon(\varepsilon x).$$

Note that for fixed $i \in \{1, \ldots, N\}$ it holds that $w^i_\varepsilon \to u$ in $L^p(D, \mathbb{R}^m)$. We define the layer-like set

$$S^i_\varepsilon := \{ x \in A' \cup B : \text{dist}(x, A_{i+1} \setminus A_{i-1}) < 3M\varepsilon \}.$$

Then by definition of the localized functional, for $\varepsilon$ small enough we can decompose $F_\varepsilon(\omega)(w^i_\varepsilon, A' \cup B)$ via

$$F_\varepsilon(\omega)(w^i_\varepsilon, A' \cup B) \leq F_\varepsilon(\omega)(u_\varepsilon, A_i) + F_\varepsilon(\omega)(v_\varepsilon, B \setminus A_{i+1}) + F_\varepsilon(\omega)(w^i_\varepsilon, S^i_\varepsilon) \leq F_\varepsilon(\omega)(u_\varepsilon, A) + F_\varepsilon(\omega)(v_\varepsilon, B) + F_\varepsilon(\omega)(w^i_\varepsilon, S^i_\varepsilon).$$

We show that the last term is negligible. This will be done by averaging. Observe that

$$w^i_\varepsilon(y) - w^i_\varepsilon(\varepsilon x) = \varphi_i(y)(u_\varepsilon(y) - u_\varepsilon(\varepsilon x)) + (1 - \varphi_i(y))(v_\varepsilon(y) - v_\varepsilon(\varepsilon x)) + \varphi_i(\varepsilon y - \varphi_i(\varepsilon x))(u_\varepsilon(\varepsilon x) - v_\varepsilon(\varepsilon x))$$

for all $x, y \in \mathcal{L}(\omega)$. Applying the convexity inequality $(a + b + c)^p \leq 3^{p-1}(a^p + b^p + c^p)$ and the mean value theorem for $\varphi_i$, we obtain for all $(x, y) \in \mathcal{E}(\omega)$ the bound

$$\varepsilon |w^i_\varepsilon(y) - w^i_\varepsilon(\varepsilon x)|^p \leq 3^{p-1} \varepsilon |u_\varepsilon(y) - u_\varepsilon(\varepsilon x)|^p + 3^{p-1} \varepsilon |v_\varepsilon(y) - v_\varepsilon(\varepsilon x)|^p + \frac{(12MN)^p}{3M^p} \varepsilon |u_\varepsilon(\varepsilon x) - v_\varepsilon(\varepsilon x)|^p.$$

Summing this estimate over all $\varepsilon y \in \varepsilon \mathcal{L}(\omega) \cap S^i_\varepsilon$ with $(x, y) \in \mathcal{E}(\omega)$ we infer

$$\|\varepsilon |\nabla_{\omega,\varepsilon}|^p(w^i_\varepsilon, S^i_\varepsilon)(\varepsilon x)\|_1 \leq 3^{p-1} \|\varepsilon |\nabla_{\omega,\varepsilon}|^p(u_\varepsilon, S^i_\varepsilon)(\varepsilon x)\|_1 + 3^{p-1} \|\varepsilon |\nabla_{\omega,\varepsilon}|^p(v_\varepsilon, S^i_\varepsilon)(\varepsilon x)\|_1 + C N^p \varepsilon |u_\varepsilon(\varepsilon x) - v_\varepsilon(\varepsilon x)|^p. \quad (3.14)$$

Having in mind this inequality we subdivide the lattice points according to

$$I^0_{i,t} = \{ \varepsilon x \in \varepsilon \mathcal{L}(\omega) \cap S^i_\varepsilon : \|\varepsilon |\nabla_{\omega,\varepsilon}|^p(u_\varepsilon, S^i_\varepsilon)(\varepsilon x)\|_1 + \|\varepsilon |\nabla_{\omega,\varepsilon}|^p(v_\varepsilon, S^i_\varepsilon)(\varepsilon x)\|_1 \leq 3^{-p} \},$$

$$I^1_{i,t} = (\varepsilon \mathcal{L}(\omega) \cap S^i_\varepsilon) \setminus I^0_{i,t}(\omega).$$

Since $f(p) \leq C_f \|p\|_1$, the estimate [3.14] and the lower bound in [2.19] imply for $\varepsilon x \in I^0_{i,t}$ that

$$\varepsilon^{d-1} f(\varepsilon |\nabla_{\omega,\varepsilon}|^p(w^i_\varepsilon, S^i_\varepsilon)(\varepsilon x)) \leq C \varepsilon^{d-1} f(\varepsilon |\nabla_{\omega,\varepsilon}|^p(u_\varepsilon, S^i_\varepsilon)(\varepsilon x)) + C \varepsilon^{d-1} f(\varepsilon |\nabla_{\omega,\varepsilon}|^p(v_\varepsilon, S^i_\varepsilon)(\varepsilon x)) + C N^p \varepsilon^d |u_\varepsilon(\varepsilon x) - v_\varepsilon(\varepsilon x)|^p. \quad (3.15)$$

On the other hand, if $\varepsilon x \in I^1_{i,t}$, we can use [2.19] and the elementary inequality $\min\{x + y, 1\} \leq \min\{x, 1\} + \min\{y, 1\}$ to deduce

$$\varepsilon^{d-1} f(\varepsilon |\nabla_{\omega,\varepsilon}|^p(w^i_\varepsilon, S^i_\varepsilon)(\varepsilon x)) \leq C \varepsilon^{d-1} f(\varepsilon |\nabla_{\omega,\varepsilon}|^p(u_\varepsilon, S^i_\varepsilon)(\varepsilon x)) + C \varepsilon^{d-1} f(\varepsilon |\nabla_{\omega,\varepsilon}|^p(v_\varepsilon, S^i_\varepsilon)(\varepsilon x)). \quad (3.16)$$

Summing [3.15] and [3.16] yields

$$F_\varepsilon(\omega)(w^i_\varepsilon, S^i_\varepsilon) \leq C \left(F_\varepsilon(\omega)(u_\varepsilon, S^i_\varepsilon) + F_\varepsilon(\omega)(v_\varepsilon, S^i_\varepsilon)\right) + C N^p \sum_{\varepsilon x \in \varepsilon \mathcal{L}(\omega) \cap S^i_\varepsilon} \varepsilon^d |u_\varepsilon(\varepsilon x) - v_\varepsilon(\varepsilon x)|^p.$$

For $\varepsilon$ small enough we have $S^i_\varepsilon \cap S^j_\varepsilon = \emptyset$ for $|i - j| \geq 3$. Moreover, $S^i_\varepsilon \subset A \cap B$ for $i \geq 2$ as well as $S^i_\varepsilon \subset A$ with a uniform distance to $\partial A$. Thus averaging the last inequality and applying [3.12] yields

$$\frac{1}{N - 1} \sum_{i=2}^{N} \frac{1}{N - 1} \sum_{i=2}^{N} \frac{1}{N - 1} \sum_{i=2}^{N} F_\varepsilon(\omega)(w^i_\varepsilon, S^i_\varepsilon) \leq C \left(\frac{1}{N} \left(F_\varepsilon(\omega)(u_\varepsilon, A) + F_\varepsilon(\omega)(v_\varepsilon, B)\right) + C N^{p-1} \sum_{\varepsilon c(x) \subset A} \varepsilon^d |u_\varepsilon(\varepsilon x) - v_\varepsilon(\varepsilon x)|^p \right) \leq C \left(\frac{1}{N} + C N^{p-1} \sum_{\varepsilon c(x) \subset A} \varepsilon^d |u_\varepsilon(\varepsilon x) - v_\varepsilon(\varepsilon x)|^p \right).$$
Since we have \( u_\varepsilon - v_\varepsilon \to 0 \) also in \( L^p(D, \mathbb{R}^m) \), the last term vanishes when \( \varepsilon \to 0 \). For every \( \varepsilon > 0 \) let \( \varepsilon \in \{2, \ldots, N\} \) be such that

\[
F_\varepsilon(\omega)(w_\varepsilon, S_\varepsilon) \leq \frac{1}{N - 1} \sum_{i=2}^N F_\varepsilon(\omega)(w_\varepsilon, S_\varepsilon^i) \leq C \frac{N^{n-1}}{N} \| u_\varepsilon - v_\varepsilon \|^2_{L^p(D)}
\]  

(3.17)

and set \( w_\varepsilon := w_\varepsilon^1 \). Note that \( w_\varepsilon \) still converges to \( u \) strongly in \( L^1(D, \mathbb{R}^m) \). Hence, using (3.12), (3.13) and (3.17), we conclude that

\[
F''(\omega)(u, A') \leq \limsup_{\varepsilon \to 0} F_\varepsilon(\omega)(w_\varepsilon, A' \cup B) \leq F''(\omega)(u, A) + F''(\omega)(u, B) + \frac{C}{N}.
\]

The claim now follows by letting \( N \to +\infty \).

For the sake of completeness we include here more or less standard consequences of the estimates we proved so far.

**Lemma 3.5.** Let \( u \in L^1(D) \). Then the following properties hold true:

1. \( F''(\omega)(u, A) = \sup_{A' \subset A} F''(\omega)(u, A') \);
2. \( F''(\omega)(u, A \cup B) \geq F''(\omega)(u, A) + F''(\omega)(u, B) \) for all \( A, B \in \mathcal{A}(D) \) such that \( A \cap B = \emptyset \).

**Proof.** (i): It suffices to prove one inequality since \( A \mapsto F''(\omega)(u, A) \) is monotone with respect to set inclusion. For given \( k \in \mathbb{N} \) define \( A_k = \{ x \in A : \text{dist}(x, \partial A) > 2^{-k}\} \). Then for all \( k \) large enough we have that \( A_k, A \setminus A_k \in \mathcal{A}^D(D) \) (see [31, Lemma 2.2]). Let us first treat the case \( u \not\in \text{GSBV}^p(A, \mathbb{R}^m) \). Then it is enough to prove that \( \limsup F''(\omega)(u, A_k) = +\infty \). Assume by contradiction that this sequence is bounded. Then, for each \( k \) we have \( u \in \text{GSBV}^p(A_k, \mathbb{R}^m) \) by Lemma 3.3 and thus \( u \in \text{GSBV}^p(A, \mathbb{R}^m) \). Since the measure of the jump set and the \( L^p \)-norm of the gradient are equibounded in \( k \), we easily reach the contradiction \( u \in \text{GSBV}^p(A, \mathbb{R}^m) \). Now assume that \( u \in \text{GSBV}^p(A, \mathbb{R}^m) \). Note that \( A = A_k \cup A \setminus A_{k+2} \) so that by Lemma 3.4 and Proposition 2 we have

\[
F''(\omega)(u, A) \leq F''(\omega)(u, A_{k+1}) + C \left( \|\nabla u\|_{L^p(A \setminus A_{k+2})}^p + \mathcal{H}^{d-1}(S_u \cap (A \setminus A_{k+2})) \right)
\]

\[
\leq \sup_{A' \subset A} F''(\omega)(u, A') + C \left( \|\nabla u\|_{L^p(A \setminus A_{k+2})}^p + \mathcal{H}^{d-1}(S_u \cap (A \setminus A_{k+2})) \right).
\]

Letting \( k \to +\infty \) we obtain the claim since \( u \in \text{GSBV}^p(A, \mathbb{R}^m) \).

(ii): The second inequality holds at the discrete level and thus is conserved at the limit since we take the limit inf which itself is superadditive.

**Proof of Proposition 2** Given a sequence \( \varepsilon \to 0^+ \), Lemma 3.5(i) allows us to use the compactness property of \( \Gamma \)-convergence on separable metric spaces (see [13, Proposition 1.42]) to construct a subsequence \( \varepsilon_n \) such that

\[
\Gamma- \lim_{n} F_{\varepsilon_n}(\omega)(u, A) =: \tilde{F}(\omega)(u, A)
\]

exists for every \( (u, A) \in L^1(D, \mathbb{R}^m) \times \mathcal{A}^D(D) \). We extend \( \tilde{F}(\omega)(u, \cdot) \) to \( \mathcal{A}(D) \) setting

\[
F(\omega)(u, A) := \sup \{ \tilde{F}(\omega)(u, A') : A' \subset A, A' \in \mathcal{A}^D(D) \}.
\]

By Lemma 3.5 this functional indeed extends \( \tilde{F}(\omega)(u, \cdot) \). In order to apply Theorem 2.2 we have to slightly modify the functional. Given \( \eta > 0 \), for any \( u \in \text{SBV}^p(D, \mathbb{R}^m) \) and \( A \in \mathcal{A}(D) \) we define the auxiliary functional \( \mathcal{F}_\eta : \text{SBV}^p(D, \mathbb{R}^m) \times \mathcal{A}(D) \to [0, +\infty) \) as

\[
\mathcal{F}_\eta(u, A) = F(\omega)(u, A) + \eta \int_{S_u \cap A} |u^+ - u^-| \, d\mathcal{H}^{d-1}.
\]

(3.18)

We argue that \( \mathcal{F}_\eta \) satisfies the assumptions of Theorem 2.2.

(i): We verify the De Giorgi-Letta criterion (see [31, Theorem 1.62]). Clearly \( A \mapsto \mathcal{F}_\eta(u, A) \) is a nonnegative, increasing and inner regular set function with \( \mathcal{F}_\eta(u, \emptyset) = 0 \). Moreover, from Lemma 3.5(ii) it follows that \( \mathcal{F}_\eta(u, A \cup B) \geq \mathcal{F}_\eta(u, A) + \mathcal{F}_\eta(u, B) \) whenever \( A \cap B = \emptyset \). In order to prove subadditivity, let \( A, B \in \mathcal{A}(D) \) and consider \( S \subset A \cup B \) such that \( S \in \mathcal{A}^D(D) \). Let us define the set \( A_k = \{ x \in A : \).
The claim follows by the arbitrariness of convergence to a neighborhood of \( \partial Q \). By compactness we find an index \( k_0 \) such that \( \overline{S} \subset A_{k_0} \cup B_{k_0} \). Next we regularize the sets \( A_{2k_0}, B_{2k_0} \) and \( A_{4k_0}, B_{4k_0} \) by standard methods to find further sets \( A_0, A_1, B_0, B_1 \in A^R(D) \) such that \( \overline{S} \subset A_0 \cup B_0 \) and \( A_0 \subset A_1 \subset A \) and \( B_0 \subset B_1 \subset B \). Then by Proposition 2

\[
\tilde{F}(\omega)(u, S) \leq \Gamma\text{-lim sup}_{n} F_{\varepsilon_n}(\omega)(u, A_0 \cup B_0) \leq \tilde{F}(\omega)(u, A_1) + \tilde{F}(\omega)(u, B_1) \leq F(\omega)(u, A) + F(\omega)(u, B).
\]

Taking the supremum over such \( S \) yields subadditivity of \( A \mapsto F(\omega)(u, A) \). The corresponding property for the perturbation term is straightforward. Thus the De Giorgi-Letta criterion applies and we infer that \( F_\eta(u, \cdot) \) is the trace of a Borel measure. Since this Borel measure is finite on \( D \) by Lemma 3.4, it is a Radon measure.

(ii)+(iii): The locality property follows from Lemma 3.2 and the definition of \( F(\omega)(u, A) \) by inner approximation as well as locality of the perturbation term. By the properties of \( \Gamma \)-limits we know that \( \tilde{F}(\omega)(\cdot, A) \) is \( L^1(D, \mathbb{R}^m) \)-lower semicontinuous and so is \( F(\omega)(\cdot, A) \) as the supremum of lower semicontinuous functionals. \( L^1(D, \mathbb{R}^m) \)-lower semicontinuity of the perturbation term along sequences such that \( F(\omega)(u_n, A) \) remains bounded follows from the bounds established in Lemma 3.3 that still hold for \( F(\omega)(u, A) \). Indeed those bounds yield stronger compactness so that we can combine [3] Theorems 2.2 and 3.7 to conclude lower semicontinuity. Hence \( F(\cdot, A) \) is lower semicontinuous as the sum of (finite) lower semicontinuous functionals.

(iv): The bounds follow from Lemmata 3.3 and 3.4 which still hold for \( F(\omega) \) in place of \( \tilde{F}(\omega) \), and the definition of the perturbation term.

From Theorem 2.2 we deduce that there exists functions \( h_\eta, g_\eta \) given by asymptotic minimization formulas such that, for all \( u \in SBV^p(D, \mathbb{R}^m) \) and \( A \in A(D) \),

\[
F_\eta(u, A) = \int_A h_\eta(x, u, \nabla u) \, dx + \int_{S_u \cap A} g_\eta(x, u^+, u^-, \nu_u) \, d\mathcal{H}^{d-1} \tag{3.19}
\]

and therefore, using locality, we deduce that, for every \( u \in L^1(D, \mathbb{R}^m) \) and every \( A \in A^R(D) \) such that \( u \in SBV^p(A, \mathbb{R}^m) \) we have

\[
\Gamma\text{-lim}_{n} F_{\varepsilon_n}(\omega)(u, A) = F(\omega)(u, A) = \int_A h_\eta(x, u, \nabla u) \, dx + \int_{S_u \cap A} (g_\eta(x, u^+, u^-, \nu_u) - \eta |u^+ - u^-|) \, d\mathcal{H}^{d-1}.
\]

It remains to show that the function \( h_\eta \) is independent of the \( u \)-variable. We prove that for any \( x_0 \in D, a_0, a_1 \in \mathbb{R}^m \) and \( \xi_0 \in \mathbb{R}^{m \times d} \) it holds that \( h_\eta(x_0, a_1, \xi_0) \leq h(x_0, a_0, \xi_0) \). To this end fix \( \delta > 0 \). By Theorem 2.2 for all \( \rho > 0 \) small enough there exists \( \nu_\rho \in SBV^p(Q_\nu(x_0, \rho), \mathbb{R}^m) \) such that \( \nu_\rho = a_0 + \xi(-x_0) \) in a neighborhood of \( \partial Q_\nu(x_0, \rho) \) and

\[
\rho^{-d} F_\eta(\nu_\rho, Q_\nu(x_0, \rho)) \leq f_\eta(x_0, a_0, \xi_0) + \delta.
\]

Let \( u_\eta \in \mathcal{PC}_{\nu_\rho}^\omega \) be a recovery sequence for \( \nu_\rho \). Then the function \( v_\nu \in \mathcal{PC}_{\nu_\rho}^\omega \) defined by \( v_\nu = u_\eta + (a_1 - a_0) \) converges to \( v_\rho := u_\rho + (a_1 - a_0) \). Moreover we have \( F_{\varepsilon_n}(\omega)(u_n, Q_\nu(x_0, \rho)) = F_{\varepsilon_n}(\omega)(v_n, Q_\nu(x_0, \rho)) \). Thus by the defining formula of Theorem 2.2 and \( \Gamma \)-convergence we infer

\[
f_\eta(x_0, a_1, \xi_0) \leq \limsup_{\rho \to 0} \rho^{-d} F_\eta(\nu_\rho, Q_\nu(x_0, \rho)) \leq \limsup_{\rho \to 0} \rho^{-d} \left( \liminf_{n} F_{\varepsilon_n}(\omega)(v_n, Q_\nu(x_0, \rho)) + \eta \int_{S_{v_\rho} \cap Q_\nu(x_0, \rho)} |v_\rho^+ - v_\rho^-| \, d\mathcal{H}^{d-1} \right)
\]

\[
= \limsup_{\rho \to 0} \rho^{-d} \left( \liminf_{n} F_{\varepsilon_n}(\omega)(u_n, Q_\nu(x_0, \rho)) + \eta \int_{S_{u_\rho} \cap Q_\nu(x_0, \rho)} |u_\rho^+ - u_\rho^-| \, d\mathcal{H}^{d-1} \right)
\]

\[
\leq f_\eta(x_0, a_0, \xi_0) + \delta.
\]

The claim follows by the arbitrariness of \( \delta > 0 \).
3.2. Separation of bulk and surface effects. In this section we prove Theorem 3.1. To this end we derive asymptotic formulas for the integrands given by Proposition 3.1. We show that for the surface integrand $g(x, a, b, \nu)$ we can consider the discrete functional restricted to functions taking only the two values $a, b$ and conclude that $g$ is independent of the values $a$ and $b$. For the integrand $h(x, \xi)$ we argue that it can be computed using the discrete functional with $f$ replaced by its linearization at 0. While for proving Theorem 3.1 the formula for $h$ is not important, it will be crucial in Section 4 for proving stochastic homogenization results.

3.2.1. The surface term. In order to reveal the precise structure of the integrand $g$ over the jump set, we study the asymptotic minimization problems given by Theorem 2.2 and its connection to boundary value problems for the discrete functionals $F_\varepsilon(\omega)$. Prescribing boundary conditions in our discrete setting requires some definitions. Given a set $A \in A^R(D)$ and $\delta > 0$, we set

$$\partial_\delta A := \{ x \in \mathbb{R}^d : \text{dist}(x, \partial A) \leq \delta \}. \quad (3.20)$$

For a pointwise well-defined function $g \in L^\infty_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^m)$, we define the set of discrete functions taking boundary value $g$ on $\partial_\delta A$ as

$$\mathcal{PC}^\varepsilon_{\varepsilon, \delta}(g, A) = \{ u \in \mathcal{PC}_\varepsilon : u(\varepsilon x) = g(\varepsilon x) \text{ for all } \varepsilon x \in \varepsilon \mathcal{L}(\omega) \cap \partial_\delta A \}. \quad (3.21)$$

Using a similar notation as in Theorem 2.2 we define the quantities

$$m^\delta_\varepsilon(\omega)(g, A) = \inf \{ F_\varepsilon(\omega)(v, A) : v \in \mathcal{PC}_\varepsilon^{\varepsilon, \delta}(g, A) \},$$

$$m(\omega)(g, A) = \inf \{ F(\omega)(v, A) : v \in \text{SBV}^p(A, \mathbb{R}^m), v = g \text{ in a neighbourhood of } \partial A \},$$

where the limit functional $F(\omega)$ is given (up to subsequences) by Proposition 3.1. At first we prove that the integrands of the functional $F(\omega)(u, D)$ can indeed be recovered from the values of $m(\omega)(\cdot, Q_\nu(x_0, \rho))$.

**Lemma 3.6.** Let $\varepsilon_n$ and $F(\omega)(u, A)$ be as in Proposition 3.1. Given $x_0 \in D$, it holds that

$$h(x_0, \xi) = \limsup_{\rho \to 0} \rho^{-d} m(\omega)(\xi(x_0, Q_\nu(x_0, \rho)) \quad \text{for all } \xi \in \mathbb{R}^{m \times d},$$

$$g(x_0, a, b, \nu) = \limsup_{\rho \to 0} \rho^{-d} m(\omega)(u_{x_0, \nu}^{a, b}, Q_\nu(x_0, \rho)) \quad \text{for all } a, b \in \mathbb{R}^m, \nu \in S^{d-1}.$$

**Proof.** Using the notation of the proof of Proposition 3.1 we know that $\mathcal{F}_\eta$ given by (3.18) satisfies the assumptions of Theorem 2.2. In particular we have

$$g_\eta(x_0, a, b, \nu) = \limsup_{\rho \to 0} \rho^{-d} \inf \{ \mathcal{F}_\eta(v, Q_\nu(x_0, \rho)) : v = u_{x_0, \nu}^{a, b} \text{ in a neighbourhood of } \partial Q_\nu(x_0, \rho) \},$$

and the mapping $\eta \mapsto g_\eta(x_0, a, b, \nu)$ is increasing and nonnegative on $(0, +\infty)$. Hence there exists $\tilde{g}(x_0, a, b, \nu) = \lim_{\eta \to 0} g_\eta(x_0, a, b, \nu)$. By the same reasoning there exists $\tilde{f}(x_0, \xi) = \lim_{\eta \to 0} f_\eta(x_0, \xi)$, while from monotone convergence we deduce that

$$F(\omega)(u, A) = \int_A \tilde{f}(x, \nabla u) \, dx + \int_{\partial_\varepsilon \cap A} \tilde{g}(x, u^+, u^-, \nu_u) \, dH^{d-1}$$

for every $A \in A^R(D)$ and every $u \in L^1(D, \mathbb{R}^m)$ such that $u \in \text{SBV}^p(A, \mathbb{R}^m)$. Up to replacing $f, g$ by equivalent integrands we can assume that $f = \tilde{f}$ and $g = \tilde{g}$. Moreover, since $F(\omega)(u, A) \leq \mathcal{F}_\eta(u, A)$, we know that

$$\limsup_{\rho \to 0} \rho^{-d} m(\omega)(u_{x_0, \nu}^{a, b}, Q_\nu(x_0, \rho)) \leq \lim_{\eta \to 0} g_\eta(x_0, a, b, \nu) = g(x_0, a, b, \nu).$$

In order to show the reverse inequality, we note that Lemma 2.8 implies a very weak maximum principle for $F(\omega)$: there exists $u_\rho \in \text{SBV}^p(Q_\nu(x_0, \rho), \mathbb{R}^m)$ admissible for the definition of $m(\omega)(u_{x_0, \nu}^{a, b}, Q_\nu(x_0, \rho))$ and satisfying $\|u\|_{\infty} \leq 3 \max\{|a|, |b|\}$ such that

$$\rho^{-d} F(\omega)(u_\rho, Q_\nu(x_0, \rho)) \leq m(\omega)(u_{x_0, \nu}^{a, b}, Q_\nu(x_0, \rho)) + \rho.$$
Then clearly \(|u^+_\varepsilon - u^-_\varepsilon| \leq 3|a| + 3|b|\) for \(H^{d-1}\)-a.e. \(x \in S_{u_\varepsilon}\). With the lower bound of Lemma 3.3 we obtain

\[
g_\eta(x_0, a, b, \nu) \leq \limsup_{\rho \to 0} \rho^{1-d} \left(F(\omega)(u_\rho, Q_\nu(x_0, \rho)) + 3\eta(|a| + |b|)H^{d-1}(S_{u_\rho} \cap Q_\nu(x_0, \rho))\right)
\]

\[
\leq \limsup_{\rho \to 0} \rho^{1-d} \left(m(\omega)(u_{x_0, \nu}^\varepsilon, Q_\nu(x_0, \rho)) + C\eta(|a| + |b|)F(\omega)(u_\rho, Q_\nu(x_0, \rho))\right)
\]

\[
\leq (1 + C\eta(|a| + |b|)) \limsup_{\rho \to 0} \rho^{1-d} m(\omega)(u_{x_0, \nu}^\varepsilon, Q_\nu(x_0, \rho)).
\]

Since the term on the right hand side is bounded, we conclude by taking the limit as \(\eta \to 0\). The proof for \(f\) is the same except that we can choose \(u_\rho\) even such that \(||u_\rho||_\infty \leq C|x|\), so that there is no need to let \(\eta \to 0\) at the end. \(\square\)

Next we have to find a connection from the minimum values to the discrete functional. We restrict the class of admissible boundary data to functions \(g \in SBV^p(D, \mathbb{R}^m) \cap L^\infty(D, \mathbb{R}^m)\) such that, setting \(g_\varepsilon \in \mathcal{PC}_\varepsilon^\omega\) as \(g_\varepsilon(\varepsilon x) = g(\varepsilon x)\), it holds that

\[
\limsup_{\varepsilon \to 0} \int_B |\nabla g(\varepsilon x)|^p dx + C\mathcal{H}^{d-1}(S_g \cap B),
\]

\[
g_\varepsilon \to g \text{ in } L^1(D, \mathbb{R}^m), \quad \mathcal{H}^{d-1}(S_g \cap \partial A) = 0
\]

for some \(C > 0\) uniformly for \(B \in \mathcal{A}^R(D)\). In particular, as seen in the proof of Lemma 3.4 we allow for piecewise smooth functions with polyhedral set that has no mass on \(\partial A\). We have the following convergence result.

**Lemma 3.7.** Let \(\varepsilon_n\) and \(F(\omega)\) be as in Proposition 2. Then, for any \(A \in \mathcal{A}^R(D)\) and \(g\) as in (3.22), it holds that

\[
\lim_{\delta \to 0} \liminf_{n} m^\delta_{\varepsilon_n}(\omega)(g, A) = \lim_{\delta \to 0} \limsup_{n} m^\delta_{\varepsilon_n}(\omega)(g, A) = m(\omega)(g, A).
\]

**Proof.** First note that by monotonicity the limits when \(\delta \to 0\) exist. Moreover, by the first assumption in (3.22) we have that \(m^\delta(\omega)(g, A)\) is equibounded. For any \(n \in \mathbb{N}\) let \(u_n \in \mathcal{PC}_{\varepsilon_n, \delta}(g, A)\) be such that \(m^\delta_{\varepsilon_n}(\omega)(g, A) = F_{\varepsilon_n}(\omega)(u_n, A)\). Since \(g \in L^\infty\) we can apply Lemma 2.8 and assume without loss of generality that \(||u_n(\varepsilon x)\| \leq 3\|g\|\) for all \(x \in \mathcal{L}(\omega)\). By Lemma 3.3 we know that, up to a subsequence (not relabeled), \(u_n \to u\) in \(L^1(A, \mathbb{R}^m)\). Using Remark 1 and again (3.22) it follows that \(u = g\) on \(\partial \delta A\).

Note that \(u \in L^\infty(A, \mathbb{R}^m)\), which implies \(u \in SBV^p(A, \mathbb{R}^m)\). Up to extension we can assume that \(u\) is admissible in the infimum problem defining \(m(\omega)(g, A)\) and Proposition 1 yields

\[
m(\omega)(g, A) \leq F(\omega)(u, A) \leq \liminf_{n} F_{\varepsilon_n}(\omega)(u_n, A) \leq \liminf_{n} m^\delta_{\varepsilon_n}(\omega)(g, A).
\]

As \(\delta\) was arbitrary, we conclude that \(m(\omega)(g, A) \leq \lim_{\delta \to 0} \liminf_{n} m^\delta_{\varepsilon_n}(\omega)(g, A)\).

In order to prove the remaining inequality, given \(\theta > 0\) we let \(u \in SBV^p(A, \mathbb{R}^m)\) be such that \(u = g\) in a neighbourhood of \(\partial A\) and \(F(\omega)(u, A) \leq m(\omega)(g, A) + \theta\). Take \(u_n \in \mathcal{PC}_{\varepsilon_n}^\omega\) converging to \(u\) in \(L^1(D, \mathbb{R}^m)\) and satisfying

\[
\limsup_{n} F_{\varepsilon_n}(\omega)(u_n, A) = F(\omega)(u, A).
\]

We will modify \(u_n\) such that it fulfills the discrete boundary conditions. The argument follows the proof of Proposition 2. Due to the boundary conditions of \(u\), there exist sets \(A' \subset A \subset A'' \subset A\) such that \(A', A'' \in \mathcal{A}^R(D)\) and

\[
u \setminus \partial A' \quad \text{and} \quad (3.24)
\]

Fix \(N \in \mathbb{N}\). For \(h \leq \text{dist}(A', \partial A'')\) and \(i \in \{1, \ldots, N\}\) we define the sets

\[
A_i = \{x \in A : \text{dist}(x, A') < \frac{h}{2N}\}.
\]

Let \(\varphi_i\) be a cut-off function between the sets \(A_i\) and \(A_{i+1}\) with \(||\nabla \varphi_i||_\infty \leq \frac{4N}{h}\) and define \(u_n^i \in \mathcal{PC}_{\varepsilon_n}^\omega\) by

\[
u \setminus \varphi_i(\varepsilon x) u_n(\varepsilon x) + (1 - \varphi_i(\varepsilon x))g(\varepsilon x).
\]
Using the same arguments as in the proof of Proposition 2 we conclude that
\[ F_n(\omega)(u^i_n, A) \leq F_n(\omega)(u_n, A) + F_n(\omega)(g_{\varepsilon_n}, A\setminus A') + F_n(\omega)(u^i_n, S^i_n). \] (3.25)

Using the same arguments as in the proof of Proposition 2 we conclude that
\[ F_n(\omega)(u^i_n, S^i_n) \leq C \sum_{i=2}^{N} F_n(\omega)(u^i_n, S^i_n) + C N P^{-h - p} \sum_{\varepsilon_n \in F(\omega) \cap S^i_n} n (u_n(\varepsilon x) - g_{\varepsilon_n}(\varepsilon x))^p \]

Again by construction \( S_n^i \cap S_j^j = 0 \) for \( |i - j| \geq 3 \) and \( S_n^i \subset A \setminus A' \) for \( i \geq 2 \). Averaging the previous inequality and using (3.22) and (3.26) yields
\[ \frac{1}{N - 1} \sum_{i=2}^{N} F_n(\omega)(u^i_n, S^i_n) \leq \frac{C}{N} + C N P^{-1} h^{-p} \| u_n - g_{\varepsilon_n} \|_{L^p(A \setminus A')}^2. \] (3.26)

By equiboundedness, properties (3.22) and (3.24) imply that \( u_n - g_{\varepsilon_n} \to 0 \) in \( L^p(A \setminus A', \mathbb{R}^m) \). For every \( n \) we choose \( n \in \{2, \ldots, N\} \) such that
\[ F_n(\omega)(u^i_n, S^i_n) \leq \frac{C}{N - 1} + C N P^{-1} h^{-p} \| u_n - g_{\varepsilon_n} \|_{L^p(A \setminus A')}^2. \] (3.26)

Note that \( u^i_n \) still converges to \( u \) in \( L^1(D, \mathbb{R}^m) \). Moreover, \( u^i_n(\varepsilon_n x) = g(\varepsilon_n x) \) for all \( \varepsilon_n x \in F(\omega) \cap A \setminus A' \). Hence \( u^i_n \in PC_{\varepsilon_n}(g, A) \) for all \( \delta > 0 \) small enough. From (3.25), (3.26) and (3.27) we obtain
\[ \limsup_{n} m^\delta_{\varepsilon_n}(\omega)(g, A) \leq \limsup_{n} F_n(\omega)(u^i_n, A) \leq F_n(\omega)(u, A) + \limsup_{n} F_n(\omega)(g_{\varepsilon_n}, A\setminus A') + \frac{C}{N}, \]

where we used (3.22) with \( B = A\setminus A' \). As \( \theta \) is arbitrary, the claim now follows letting first \( \delta \to 0 \), then \( N \to +\infty \) and finally \( A' \uparrow A \). \( \square \)

In view of Lemmata 3.6 and 3.7 we can further characterize the surface integrals of possible \( \Gamma \)-limits of the family \( F_n(\omega) \) by investigating the quantities \( m^\delta(\omega)(a, b, Q, \varepsilon x, \varepsilon y) \). To this end we define the class of interfaces via
\[ S_{\varepsilon_n}(\omega)^{(a, b, \varepsilon x, \varepsilon y, Q, \varepsilon x, \varepsilon y)} = \{ v \in PC_{\varepsilon_n}((a, b, \varepsilon x, \varepsilon y, Q, \varepsilon x, \varepsilon y)) : u(\varepsilon x, \varepsilon y) \in (a, b) \}. \]

We have the following important result.

**Proposition 3.** Let \( \varepsilon_n \to 0 \). Then, for all \( x_0 \in D \), all \( a, b \in \mathbb{R}^m \) and all \( \omega \in S^{d-1} \) it holds that
\[ \limsup_{\rho \to 0} \rho^{-d} \liminf_{\delta \to 0} \left( \inf \{ F_n(\omega)(u, Q, (x_0, \rho)) : u \in S_{\varepsilon_n}(\omega)^{(a, b, \varepsilon x, \varepsilon y, Q, \varepsilon x, \varepsilon y)} \} \right) \]
\[ \leq \limsup_{\rho \to 0} \rho^{-d} \liminf_{\rho \to 0} \left( m^\delta_{\varepsilon_n}(\omega)(u^i_n, Q, (x_0, \rho)) \right). \]

**Proof.** To reduce notation, we set \( Q_{\rho} := Q(\varepsilon x, \varepsilon y) \) and write \( \varepsilon \) instead of \( \varepsilon_n \). If \( a = b \) then both sides are zero. Thus assume \( a \neq b \). Fix \( u_\varepsilon \in PC_{\varepsilon_n}((a, b, \varepsilon x, \varepsilon y, Q_{\rho})) \) such that
\[ F_n(\omega)(u_\varepsilon, Q_{\rho}) \leq F_n(\omega)(u^i_n, Q_{\rho}) \leq C \rho^{d-1}, \] (3.27)
where the last inequality is an easy consequence of Remark 1 and the boundedness of the discrete density \( f \) (provided \( \varepsilon \) is small enough). We construct a sequence \( v_\varepsilon \in S_{\varepsilon_n}(\omega)^{(a, b, \varepsilon x, \varepsilon y, Q_{\rho})} \) that has almost the same energy. First, given \( \theta > 0 \), due to the monotonicity (2.17) it is possible to choose \( L_\theta \geq 1 \) such that
\[ |\beta(k, k) - f(p)| < \theta f(p) \quad \forall p \in P_+(M) \] (3.28)
for all \( 1 \leq k \leq M \), where \( \beta(k, k) \) is given by (2.20). We fix \( L_\theta \) from now on and consider the set of edges
\[ \mathcal{J}_{\varepsilon_n} = \{(x, y) \in E(\omega) : \varepsilon x, \varepsilon y \in Q_{\rho} \text{ and } \varepsilon^{1-p}|u_\varepsilon(\varepsilon x) - u_\varepsilon(\varepsilon y)| \geq L_\theta \}. \]
Denote by \( u_i^j \) the \( i^{th} \) component of \( u_x \). If \( a_i = b_i \) we set \( v_i^j(\varepsilon x) = a_i \) for all \( x \in \mathcal{L}(\omega) \). Otherwise, we suppose \( a_i < b_i \). The remaining case is similar up to exchanging the roles of \( a_i \) and \( b_i \). For \( t \in \mathbb{R} \) we define
\[
E^1(t) := \{ x \in \varepsilon \mathcal{L}(\omega) \cap Q_\rho : u_i^j(\varepsilon x) > t \}.
\]
To shorten notation, we also introduce the set
\[
R^1(t) = \{ (x, y) \in \mathcal{E}(\omega) : \varepsilon x \in Q_\rho \cap E^1(t), \varepsilon y \in Q_\rho \setminus E^1(t) \text{ or vice versa} \}.
\]
Observe that for \( (x, y) \in \mathcal{E}(\omega) \) with \( \varepsilon x, \varepsilon y \in Q_\rho \) we have \( (x, y) \in R^1(t) \) if and only if \( t \in [u_i^j(\varepsilon y), u_i^j(\varepsilon x)] \). Hence for such \( x, y \) the following coarea-type estimate holds true:
\[
\int_{a_i}^{b_i} \mathbf{1}_{\{ (x, y) \in R^1(t) \}} dt \leq \| u_x(\varepsilon x) - u_x(\varepsilon y) \|.
\]
Summing this estimate, we infer from (2.4) and Hölder’s inequality that
\[
\int_{a_i}^{b_i} \varepsilon^{d-1} \#(R^1_x(t)) \, dt \leq \sum_{(x, y) \in \mathcal{E}(\omega) \setminus \mathcal{J}_{u_x}} \varepsilon^{d-1} |u_x(\varepsilon x) - u_x(\varepsilon y)|
\leq C \varepsilon^{d-d-n} \#(\varepsilon \mathcal{L}(\omega) \cap Q_\rho) \leq C \left( \sum_{(x, y) \in \mathcal{E}(\omega) \setminus \mathcal{J}_{u_x}} \varepsilon^d |u_i^j(\varepsilon x) - u_i^j(\varepsilon y)|^p \right)^{\frac{1}{p}}.
\]
In order to estimate the last sum, recall that \( L_\theta \geq 1 \). Thus the definition of the set \( \mathcal{J}_{u_x} \), (2.4) and assumption (2.19) imply for \( \varepsilon \in \varepsilon \mathcal{L}(\omega) \cap Q_\rho \) the uniform bound
\[
\sum_{(x, y) \in \mathcal{E}(\omega) \setminus \mathcal{J}_{u_x}} \varepsilon |u_i^j(\varepsilon x) - u_i^j(\varepsilon y)|^p \leq C L_\theta \min \left\{ \varepsilon \sum_{(x, y) \in \mathcal{E}(\omega) \setminus \mathcal{J}_{u_x}} |u_i^j(\varepsilon x) - u_i^j(\varepsilon y)|^p, 1 \right\}
\leq C L_\theta \min \{ \| \nabla_{x, \varepsilon} \varepsilon p(\varepsilon x, Q_\rho)(\varepsilon x) \|_1, 1 \} \leq C L_\theta f(\varepsilon \| \nabla_{x, \varepsilon} \varepsilon p(\varepsilon x, Q_\rho)(\varepsilon x) \|). \]
Moreover, for \( \varepsilon = \varepsilon(\rho) \) small enough the cardinality term can be bound by \( C(\rho \varepsilon^{-d})^d \), so that
\[
\int_{a_i}^{b_i} \varepsilon^{d-1} \#(R^1_x(t)) \, dt \leq C \rho^\frac{d-n}{p} (L_\theta F_\varepsilon(\omega)(u_x, Q_\rho))^\frac{1}{p} \leq C L_\theta \rho^\frac{d-n}{p},
\]
where we applied (3.27) in the second inequality. Hence there exists \( t^*_x \in (a_i, b_i) \) such that
\[
\varepsilon^{d-1} \#(R^1_x(t)) \leq C |a_i - b_i|^{-1} L_\theta \rho^\frac{d-n}{p}. \tag{3.29}
\]
Now we define \( v_i^j \) by its values on \( \varepsilon \mathcal{L}(\omega) \) setting
\[
v_i^j(\varepsilon x) = \begin{cases} a_i & \text{if } u_x(\varepsilon x) \leq t^*_x, \\
 b_i & \text{if } u_x(\varepsilon x) > t^*_x. \end{cases}
\]
Due to the fact that \( t^*_x \in (a_i, b_i) \) it follows from the boundary conditions imposed on \( u_x \) that the vectorial function \( v_x \) satisfies \( v_x(\varepsilon x) = u_{x|a_i}^{a_i, b_i}(\varepsilon x) \) for all \( \varepsilon x \in \varepsilon \mathcal{L}(\omega) \cap \partial_3 Q_\rho \), so that \( v_x \in \mathcal{S}_{x|a_i}^{a_i, b_i}(\varepsilon x, Q_\rho) \). It remains to estimate the energy difference. Let \( \varepsilon x \in \varepsilon \mathcal{L}(\omega) \cap Q_\rho \) be such that \( \| \nabla_{x, \varepsilon} \varepsilon p(v_x, Q_\rho)(\varepsilon x) \|_1 \neq 0 \). We distinguish two exhaustive cases: Either \( (x, y) \in \mathcal{J}_{u_x} \) for all \( \varepsilon y \in \varepsilon \mathcal{L}(\omega) \cap Q_\rho \) with \( (x, y) \in \mathcal{E}(\omega) \), so that (2.17) and (3.25) imply
\[
f(\varepsilon \| \nabla_{x, \varepsilon} p(v_x, Q_\rho)(\varepsilon x) \|) \leq (1 + \theta) f(\varepsilon \| \nabla_{x, \varepsilon} p(u_x, Q_\rho)(\varepsilon x) \|) \tag{3.30}
\]
or there exists \( y \in \mathcal{L}(\omega) \) with \( (x, y) \in \mathcal{R}_x^1(t^*_x) \setminus \mathcal{J}_{u_x} \). In this case we can use the cardinality estimate to bound the number of such \( x \). Since \( f \) is bounded by assumption, we deduce from (3.30) that
\[
\rho^{1-d} F_\rho(\varepsilon x, Q_\rho) \leq (1 + \theta) \rho^{1-d} F_\rho(u_x, Q_\rho) + C \rho \sum_{i:a_i \neq b_i} |a_i - b_i|^{-1} L_\theta \rho^\frac{d-n}{p}.
\]
Taking the appropriate infimum on each side, then letting \( \varepsilon \to 0 \) before \( \delta \to 0 \) and \( \rho \to 0 \), we conclude the proof as \( \theta > 0 \) was arbitrary. \( \square \)
Now we are in a position to establish the integral representation on $GSBV^p(A, \mathbb{R}^m)$ with a surface integrand independent of the traces.

**Proof of Theorem 3.4.** Due to Proposition 3.1 and the lower bound in Lemma 3.3 it remains to characterize the functional $F(\omega)(u, A)$ for $A \in \mathcal{A}^E(D)$ and $u \in L^1(D, \mathbb{R}^m)$ with $u \in GSBV^p(A, \mathbb{R}^m) \cap SBV^p(A, \mathbb{R}^m)$. Moreover we have to show that the surface integrand $g(x, a, b, \nu)$ is independent of the values of $a, b$ provided $a \neq b$. We start proving the latter. Since Proposition 3.7 actually yields an equality, we deduce from Lemmata 3.6 and 3.7 that

$$g(x_0, a, b, \nu) = \lim_{\rho \to 0} \rho^{1-d} \lim_{\delta \to 0} \inf_n \left\{ F_{\varepsilon_n}(\omega)(u, Q_{\nu}(x_0, \rho)) : u \in S_{\varepsilon_n, \delta}^\omega(u^{a,b}_{x_0, \nu}, Q_{\nu}(x_0, \rho)) \right\}. $$

For any $u \in S_{\varepsilon_n, \delta}^\omega(u^{a,b}_{x_0, \nu}, Q_{\nu}(x_0, \rho))$ we define (with a slight abuse of notation) the scalar function $v \in S_{\varepsilon_n, \delta}^\omega(u^{a,b}_{x_0, \nu}, Q_{\nu}(x_0, \rho))$ by

$$v(\varepsilon x) = \begin{cases} -1 & \text{if } u(\varepsilon x) = a, \\ 1 & \text{otherwise} \end{cases}$$

and, given $A \in \mathcal{A}^E(D)$, we introduce the multi-body Ising-type energy $I_\varepsilon(\omega)(\cdot, A) : \{ v : \varepsilon \mathcal{L}(\omega) \to \{ \pm 1 \} \} \to \mathbb{R}$ defined as

$$I_\varepsilon(\omega)(v, A) = \sum_{\varepsilon x \in \varepsilon \mathcal{L}(\omega) \cap A} \varepsilon^{d-1} \beta \left( \sum_{\varepsilon y \in \varepsilon \mathcal{L}(\omega) \cap A \setminus \{ x, y \} \in \mathcal{E}(\omega)} \frac{1}{2} |v(\varepsilon x) - v(\varepsilon y)|, \# \{ \varepsilon y \in \varepsilon \mathcal{L}(\omega) \cap A : (x, y) \in \mathcal{E}(\omega) \} \right) \beta(\varepsilon, \varepsilon - \varepsilon - 1),$$

(3.31)

where the function $\beta$ is given by (2.20) and $\beta(0, 0) := 0$ for all $1 \leq k \leq M$. Then by monotonicity (2.17) it holds that

$$I_{\varepsilon_n}(\omega)(v, Q_{\nu}(x_0, \rho)) \geq F_{\varepsilon_n}(\omega)(u, Q_{\nu}(x_0, \rho)) \geq \min_{1 \leq l \leq k \leq M} \left\{ \frac{f(p)}{\beta(l, k)} : p = l \mathbb{1}_{\{ u-b_{\varepsilon} \} \} + (k - l) \mathbb{1}_{\{ 0 \}} \right\} I_{\varepsilon_n}(\omega)(v, Q_{\nu}(x_0, \rho)).$$

Hence the definition of $\beta(l, k)$ in (2.20) yields

$$g(x_0, a, b, \nu) = \lim_{\rho \to 0} \rho^{1-d} \lim_{\delta \to 0} \inf_n \left\{ F_{\varepsilon_n}(\omega)(v, Q_{\nu}(x_0, \rho)) : v \in S_{\varepsilon_n, \delta}^\omega(u^{a,b}_{x_0, \nu}, Q_{\nu}(x_0, \rho)) \right\}. $$

(3.32)

This formula is independent of $a, b$, so that (again with a slight abuse of notation) we can define $g(x, \nu) := g(x, -1, 1, \nu)$ and deduce from Proposition 3.1 that for any $A \in \mathcal{A}^E(D)$ such that $u \in SBV^p(A, \mathbb{R}^m)$

$$F(\omega)(u, A) = \int_A h(x, \nabla u) \, dx + \int_{S_\nu \cap A} g(x, \nu_u) \, dH^{d-1}. $$

Now we extend this formula to the case when $u \in GSBV^p(A, \mathbb{R}^m)$. To this end we apply the usual truncation argument. Given $k > 0$ we have that $T_k u \in SBV^p(A, \mathbb{R}^m)$. By Lemma 2.8 it holds that $F(\omega)(u, A) = \lim_{k \to +\infty} F(\omega)(T_k u, A)$. In order pass to the limit in the integrand formula, we use Lemma 3.1 and the symmetry $g(x, \nu_u) = g(x, -\nu_u)$, which imply

$$F(\omega)(T_k, A) = \int_{A \cap \{ u \leq k \}} h(x, \nabla u) \, dx + \int_{A \cap \{ u > k \}} h(x, \nabla T_k u) \, dx + \int_{S_\nu \cap A \setminus T_k u} g(x, \nu_u) \, dH^{d-1}. $$

Since $S_\nu = \bigcup_k S_{T_k u} \cup N$ with $H^{d-1}(N) = 0$, we can pass to the limit by monotone convergence. To this end note that by Lemmata 3.3 and 3.4 the integrand in the second term can be controlled by $C|\nabla u|^p$. This concludes the proof. □

**3.2.2. The gradient term.** We conclude this section with an analogue result of Proposition 3.8 for the function $h(x, \xi)$. However in this case we consider recovery sequences for affine functions instead of minimizing problems with boundary conditions since this approach makes the analysis less complicated. We are interested in restricting the recovery sequences to sequences which have an equiintegrable discrete gradient. Again we proceed in several steps.

Even though it is just a technical detail, we prove first that the function $h$ is a Carathéodory function, which is necessary to conclude its quasiconvexity.
Lemma 3.8. Let $h : D \times \mathbb{R}^{m \times d} \to [0, + \infty)$ be given by Lemma 3.6 Then, for every $\xi \in \mathbb{R}^{m \times d}$ the map $x \mapsto h(x, \xi)$ is measurable and, for every $x \in D$, the map $\xi \mapsto h(x, \xi)$ is continuous.

Proof. Denoting by $u_{x_0, \xi}$ the affine function $u_{x_0, \xi}(x) = \xi(x - x_0)$, it follows from the proof of Theorem 2.2 in [12] that the function $x \mapsto \partial h(x, \xi)$ defined in the proof of Proposition 2.2 is the Radon-Nikodym derivative of the measure $\mathcal{F}_u(u_{x_0, \xi}, \cdot)$ with respect to the Lebesgue-measure. Hence it is measurable and so is $x \mapsto h(x, \xi)$ as the pointwise limit. In order to prove continuity in $\xi$, we use the characterization given in Lemma 3.6. Let us write

$\nu = \partial h(u_{x_0, \xi}) \cdot 1_{B \times \mathbb{R}^{m \times d}}$ for fixed $\xi \in \mathbb{R}^{m \times d}$.

Then $\mathcal{H}^{d-1}(S_{u_2} \cap (Q(1 + 2\eta)^\frac{1}{2} \mathbb{R}^m)) = 0$ and $u_2 = u_{x_0, \xi_2}$ in a neighbourhood of $\partial Q_0$ (extended to the whole space) we define $u_2 \in \mathcal{SBV}^p(Q_0, \mathbb{R}^m)$ such that $u_1 = u_{x_0, \xi_1}$ in a neighbourhood of $\partial Q_0$.

Proof. It follows from the definition of $F(\omega)$ that

$$\rho^{-d} F(\omega)(u_{x_0, \xi}, B_\rho(x_0)) = \mathcal{F}_u(u_{x_0, \xi}, B_\rho(x_0)) \int_{B_\rho(x_0)} h(z, \xi) \, dz.$$
\[ \int_{B_\rho(x_0)} |h(x, \xi_n) - h(x, \xi)| \, dx \leq C_\xi |\xi - \xi_n| \rho^d. \]

Thus, for \( x_0 \in D \setminus N \) we can pass to the limit in \( n \) on both sides in (3.33) finishing the proof. \(\Box\)

Now we analyze the limit functional \( F(\omega)(u_{x_0, \xi}, B_\rho(x_0)) \) by studying recovery sequences. We compare them to discrete energies with standard \( p \)-growth on multiple scales. To this end we introduce the auxiliary functional \( E_\varepsilon(\omega)(\cdot, A) : PC_\varepsilon \to [0, +\infty] \) defined by

\[ E_\varepsilon(\omega)(u, A) = \alpha \sum_{(x, y) \in E(\omega) \cap (\varepsilon x, \varepsilon y) \subseteq A} \varepsilon^d |\frac{u(\varepsilon x) - u(\varepsilon y)}{\varepsilon}|^p, \quad (3.34) \]

where \( \alpha \) is given by (2.16). Setting this functional to \( +\infty \) for \( u \in L^p(D, \mathbb{R}^m) \setminus PC_\varepsilon \), the following result was proven in [4] Theorem 3.3.

**Theorem 3.10.** For every sequence \( \varepsilon \to 0 \) we can find a subsequence \( \varepsilon_j \) such that for every \( A \in \mathcal{A}^R(D) \) the functionals \( E_{\varepsilon_j}(\omega)(\cdot, A) \) \( \Gamma(L^p(D, \mathbb{R}^m)) \)-converge to a functional \( E(\omega) : L^p(D, \mathbb{R}^m) \times \mathcal{A}^R(D) \to [0, +\infty] \) that is finite only for \( u \in W^{1,p}(A, \mathbb{R}^m) \), where it takes the form

\[ E(\omega)(u, A) = \int_A q(x, \nabla u) \, dx \]

for some nonnegative Carathéodory-function \( q \) that is quasiconvex in the second variable for a.e. \( x \in D \) and satisfies the growth conditions

\[ \frac{1}{C'} |\xi|^p - C \leq q(x, \xi) \leq C(|\xi|^p + 1). \]

**Remark 8.** Equicoercivity for the functionals \( E_\varepsilon(\omega)(\cdot, A) \) holds in the presence of boundary conditions. However, we need it in the case of an a priori \( L^p(A, \mathbb{R}^m) \)-bound. This follows from Lemma 3.3 since by (2.19) we have \( f(P) \leq C f\|P\|_1 \), so that \( E_\varepsilon(\omega)(u, A) \leq C E_\varepsilon(\omega)(u, A) \). This estimate implies that the functionals are equicoercive in \( L^1(A, \mathbb{R}^m) \) (actually in any \( L^q(A, \mathbb{R}^m) \) for \( q < p \)). Moreover, the \( \Gamma \)-limit remains the same when we compute it with respect to the \( L^1(A, \mathbb{R}^m) \)-convergence. Indeed, whenever \( u_j \in PC_{\varepsilon_j} \) is converging in \( L^1(A, \mathbb{R}^m) \) to some \( u \in L^p(A, \mathbb{R}^m) \), first extend this \( u \) to 0 on \( D \setminus A \). Then, given \( k > 0 \) we consider the truncation \( T_ku_j \) that we set also to 0 on the set \( \varepsilon_j \mathcal{L}(\omega) \setminus A. \) Since \( A \in \mathcal{A}^R(D) \) we conclude by boundedness that \( T_ku_j \to T_ku \) in \( L^p(D, \mathbb{R}^m) \). Then by \( \Gamma \)-convergence in \( L^p(D, \mathbb{R}^m) \) and decrease by truncation we obtain

\[ \liminf_j E_{\varepsilon_j}(\omega)(u_{\varepsilon_j}, A) \geq \liminf_j E_{\varepsilon_j}(\omega)(T_ku_{\varepsilon_j}, A) \geq \int_A q(x, \nabla T_ku) \, dx. \]

Finally we obtain the lower bound for \( k \to +\infty \) by dominated convergence (recall Lemma 2.1).

The next auxiliary result is an immediate consequence of a change of variables. Therefore we omit its proof.

**Lemma 3.11.** Let \( \varepsilon_n \) and \( E(\omega) \) be as above. For \( \rho > 0 \) and \( x_0 \in D \) such that \( B_\rho(x_0) \subset D \), define the functional \( G_{\varepsilon_n, \rho}(x_0, \omega) : L^p(B_1, \mathbb{R}^m) \to [0, +\infty] \) to be finite only for \( u : \frac{\varepsilon_n}{\rho}(\mathcal{L}(\omega) - \frac{\varepsilon_n}{\rho}) \to \mathbb{R}^m \) with value

\[ G_{\varepsilon_n, \rho}(x_0, \omega)(u) = \alpha \sum_{(x, y) \in E(\omega) \setminus 2\varepsilon_n} \left( \frac{\varepsilon_n}{\rho} \right)^d \frac{|u(\frac{x+\varepsilon_n}{\rho}) - u(\frac{y+\varepsilon_n}{\rho})|^p}{\varepsilon_n \rho^d}. \]

1\(^1\)Interactions via a random edge set \( \mathcal{E}(\omega) \) are not covered by the results in [3]. However, the proof works also for general finite range interactions with \( p \)-growth and coercive nearest neighbor interactions. Due to (2.30) those assumptions are fulfilled. The additional structure of the edges in [3] is used only to prove homogenization.
Then \( G_{\epsilon, \rho}(x_0, \omega) \) Γ-converges with respect to the \( L^p(B_1, \mathbb{R}^m) \)-topology to the functional \( E_\rho(x_0, \omega) : L^p(B_1, \mathbb{R}^m) \to [0, +\infty] \) which is finite only on \( W^{1,p}(B_1, \mathbb{R}^m) \), where it is given by

\[
E_\rho(x_0, \omega)(u) = \int_{B_1} q(x_0 + \rho y, \nabla u(y)) \, dy.
\]

**Remark 9.** By Remark 8 we can compute the Γ-limit equivalently with respect to the \( L^1(B_1, \mathbb{R}^m) \)-topology, for which equicoercivity holds provided we have an a priori \( L^p(B_1, \mathbb{R}^m) \)-bound.

We will compare our discrete energies to the functionals \( G_{\epsilon, \rho}(x_0, \omega) \) in the limit when both \( \epsilon \to 0 \) and \( \rho \to 0 \) at the same time, but at different scales. Therefore we first identify the Γ-limit of the functionals \( E_\rho(x_0, \omega)(u, A) \) when \( \rho \to 0 \). This is already contained in [33, Lemma 2.1] for scalar problems. Here we provide a short proof in the vectorial case. We essentially follow the lines of [26, Theorem 5.14] up to some necessary modifications.

**Lemma 3.12.** Let \( E_\rho(x_0, \omega) \) be a functional as in Lemma 3.11. Then there exists a null set \( N \subset D \) such that for all \( x_0 \in D \setminus N \) it holds that

\[
\Gamma\text{-}\lim_{\rho \to 0} E_\rho(x_0, \omega)(u) = \begin{cases} 
\int_{B_1} q(x_0, \nabla u(y)) \, dy & \text{if } u \in W^{1,p}(B_1, \mathbb{R}^m), \\
+\infty & \text{otherwise},
\end{cases}
\]

where the Γ-limit is computed with respect to \( L^p(B_1, \mathbb{R}^m) \).

**Proof.** By coercivity, the Γ-liminf can be finite only on \( W^{1,p}(B_1, \mathbb{R}^m) \). Let \( N' \subset D \) be the null set where the function \( \xi \to q(x, \xi) \) is not quasiconvex or does not fulfill the bound \( \frac{1}{p} |\xi|^p - C \leq q(x, \xi) \leq C(|\xi|^p + 1) \). We redefine \( q(x, \xi) = |\xi|^p \) for all \( x \in N' \) not changing the functional \( E_\rho(x_0, \omega) \). Then, according to [24, Proposition 2.32], there exists a constant \( C_q \) such that

\[
|q(x, \xi) - q(x, \zeta)| \leq C_q (1 + |\xi|^{p-1} + |\zeta|^{p-1}) |\xi - \zeta|^p
\]

for all \( x \in D \) and all \( \xi, \zeta \in \mathbb{R}^{m \times d} \). Moreover, by Lebesgue’s differentiation theorem, for every \( \xi \in \mathbb{R}^{m \times d} \) there exists a null set \( N_\xi \) such that for every \( x_0 \in D \setminus N_\xi \) it holds that

\[
\lim_{\rho \to 0} \int_{B_1} |q(x_0 + \rho y, \xi) - q(x_0, \xi)| \, dy = \lim_{\rho \to 0} \frac{1}{\rho^d} \int_{B_\rho(x_0)} |q(z, \xi) - q(x_0, \xi)| \, dz = 0.
\]

Let us set \( N = N' \cup \bigcup_{\xi \in Q} N_\xi \), fix \( x_0 \in D \setminus N \) and consider a sequence \( \rho_j \to 0 \). We first prove that, along a suitable subsequence, for each \( \xi \in \mathbb{R}^{m \times d} \) the functions \( y \mapsto q_j(y, \xi) = q(x_0 + \rho_j y, \xi) \) converge a.e. on \( B_1 \) to \( q(x_0, \xi) \). Indeed, due to (3.35) the convergence is at least in \( L^1(B_1) \) for every \( \xi \in \mathbb{Q}^{m \times d} \).

Thus we find a (ξ-dependent) subsequence such that \( q_{j_k}(\cdot, \xi) \to q(x_0, \xi) \) a.e. on \( B_1 \). Enumerating the \( \xi \in \mathbb{Q}^{m \times d} \) we can ensure that the subsequences are nested. Then, by a diagonal argument, we find a common subsequence \( j_k \) such that \( q_{j_k}(\cdot, \xi) \to q(x_0, \xi) \) a.e. on \( B_1 \) for every \( \xi \in \mathbb{Q}^{m \times d} \). Using (3.35) we can extend the convergence to all \( \xi \in \mathbb{R}^{m \times d} \).

We now prove the Γ-convergence along this subsequence. By dominated convergence the existence of a recovery sequence is provided through the pointwise limit. In order to prove the lower bound, we can assume that \( u_j \to u \) in \( L^p(B_1, \mathbb{R}^m) \) and that \( u_j \) is bounded in \( W^{1,p}(B_1, \mathbb{R}^m) \). By [32, Lemma 1.2] there exists a subsequence (not relabelled) and another sequence \( z_j \in W^{1,p}(B_1, \mathbb{R}^m) \) such that \( |\nabla z_j|^p \) is equiintegrable and

\[
\lim_{n} \left| \left\{ z_j \neq u_j \text{ or } \nabla z_j \neq \nabla u_j \right\} \right| = 0.
\]

Note that the above property implies that \( z_j \) is also bounded in \( W^{1,p}(B_1, \mathbb{R}^m) \). Otherwise, along a subsequence, \( w_j = \frac{z_j}{\|z_j\|_{L^p}} \) converges in \( W^{1,p}(B_1, \mathbb{R}^m) \) to a non-zero constant, but the sequence \( v_j = \frac{u_j}{\|z_j\|_{L^p}} \) converges to 0 strongly in \( W^{1,p}(B_1, \mathbb{R}^m) \). This contradicts (3.37) which holds also for \( w_j \) and \( v_j \).

We deduce from (3.37) that \( z_j \to u \) in \( W^{1,p}(B_1, \mathbb{R}^m) \). Moreover, equiintegrability of \( |\nabla z_j|^p \), the bound \( 0 \leq q(x, \xi) \leq C(|\xi|^p + 1) \) and (3.37) imply that

\[
\liminf_{j} E_{\rho_j}(x_0, \omega)(u_j) \geq \liminf_{j} E_{\rho_j}(x_0, \omega)(z_j).
\]
Next, for any $\delta > 0$ we find again by equiintegrability and the upper bound on $q$ a number $\eta_\delta > 0$ such that for any measurable set $G \subset B_1$ with $|G| \leq \eta_\delta$ it holds
\[
\sup_j \int_G q(x_0, \nabla z_j(y)) \, dy < \delta.
\] (3.39)

Then we choose $t_\delta > 0$ such that $\{\{|\nabla z_j| > t_\delta\} \leq \eta_\delta$ and set $K_\delta = C_q(1 + 2t_\delta^{p-1})$ with $C_q$ given by (3.35).

By a compactness argument we find $\xi_1, \ldots, \xi_N \in \mathbb{R}^{m \times d}$ such that $|\xi_i| < t_\delta$ and
\[
\{\xi \in \mathbb{R}^{m \times d} : |\xi| \leq t_\delta\} \subset \bigcup_{i=1}^N \{\xi \in \mathbb{R}^{m \times d} : |\xi - \xi_i| < \frac{\delta}{K_\delta}\}.
\]

Applying Egorov’s theorem we obtain a set $G \subset B_1$ with $|G| \leq \eta_\delta$ such that the sequences $q_j(y, \xi_i)$ converge uniformly to $q(x_0, \xi)$ on $B_1 \setminus G$. Hence there exists $j_\delta \in \mathbb{N}$ such that for all $j \geq j_\delta$, all $i$ and all $y \in B_1 \setminus G$ we have $|q_j(y, \xi_i) - q(x_0, \xi_i)| < \delta$. Using (3.35) and the triangle inequality we obtain
\[
|q_j(y, \xi) - q(x_0, \xi)| \leq 2C_q(1 + 2t_\delta^{p-1}) \min_{i} |\xi_i - \xi| + \delta \leq 3\delta
\]
for all $y \in B_1 \setminus G$, all $\xi \in \mathbb{R}^{m \times d}$ with $|\xi| \leq t_\delta$ and all $j \geq j_\delta$. Thus we infer that for those $j$
\[
E_{\rho_j}(x_0, \omega)(z_j) \geq \int_{\{\{|\nabla z_j| \leq t_\delta\} \setminus G} q_j(y, \nabla z_j(y)) \, dy \geq \int_{\{\{|\nabla z_j| \leq t_\delta\} \setminus G} q(x_0, \nabla z_j(y)) \, dy - 3|B_1|\delta
\]
\[
\geq \int_{B_1} q(x_0, \nabla z_j(y)) \, dy - (3|B_1| + 2)\delta,
\]
where we used (3.39) and the definition of $t_\delta$. By quasiconvexity and the growth conditions on $q$ the right hand side is lower semicontinuous with respect to weak convergence in $W^{1,p}(B_1, \mathbb{R}^m)$. Hence (3.38) implies
\[
\liminf_j E_{\rho_j}(x_0, \omega)(u_j) \geq \int_{B_1} q(x_0, \nabla u(y)) \, dy - C\delta.
\]
By the arbitrariness of $\delta$ we obtain the lower bound. Since the limit functional is independent of any subsequence, we established the full $\Gamma$-convergence result.

**Remark 10.** Taking the same null set $N \subset D$ as in Lemma 3.12 the convergence (3.36) holds for every $x_0 \in D \setminus N$ and all $\xi \in \mathbb{R}^{d}$ again by (3.35).

In order to compare our discrete functionals with $E_{\varepsilon}(\omega)$ we have to use a diagonal sequence $\varepsilon = \varepsilon(\rho)$. In light of the previous lemma we would like to recover the function $q(x_0, \xi)$. However in general there is no metric characterizing the $\Gamma$-convergence when there is no equicoercivity, so that diagonal arguments are not always available. Therefore we provide an explicit construction in the appendix similar to the continuum setting treated in [28]. With that at hand we can indeed derive the following result.

**Lemma 3.13.** Under the assumptions of Lemma 3.11 let $x_0 \in D \setminus N$ where $N$ is given by Lemma 3.12. For every $\rho$ there exists $\varepsilon(\rho) > 0$ such that whenever we chose $\varepsilon_n(\rho) \leq \varepsilon(\rho)$ it holds
\[
\Gamma(L^p(B_1, \mathbb{R}^m)) \text{-} \lim_{\rho \to 0} \lim_{n \to 0} G_{\varepsilon_n(\rho)}(x_0, \omega)(u) = \begin{cases} \int_{B_1} q(x_0, \nabla u(y)) \, dy & \text{if } u \in W^{1,p}(B_1, \mathbb{R}^m), \\ +\infty & \text{otherwise}. \end{cases}
\]

**Proof.** This result is an immediate consequence of Lemma 3.11 and Remark 10 which allow to combine Lemma 3.12 and a diagonal argument with respect to the metric $\varepsilon$ constructed in the appendix.

Now we are in a position to compare our two discrete functionals.

**Proposition 4.** Let $\varepsilon$ and $F(\omega)$ be as in Theorem 3.7. Then for a.e. $x_0 \in D$ and every $\xi \in \mathbb{R}^{m \times d}$ it holds that
\[
|B_1| h(x_0, \xi) = \lim_{\rho \to 0} \rho^{-d} F(\omega)(u_{x_0, \xi} B_\rho(x_0)) = |B_1| q(x_0, \xi),
\]
where $q$ is an (equivalent) integrand given by the $\Gamma$-limit of $E_{\varepsilon}(\omega)(u, D)$, which in particular exists.
Applying Vitali's covering lemma on separable metric spaces we find a (finite) collection of disjoint sets. Let us fix $x_0 \in D$ satisfying the first equality and such that Lemmata 3.12 and 3.13 hold. Choose $0 < \rho_0 < \eta$. Now consider $0 < \rho < \rho_0$. By discrete superadditivity and $\Gamma$-convergence on $\mathcal{A}_R(D)$ it holds that

$$\limsup_{\varepsilon_n \to 0} F_{\varepsilon_n}(\omega)(u_{\varepsilon_n}, B_{\rho}(x_0)) \leq \lim_{\varepsilon_n \to 0} F_{\varepsilon_n}(\omega)(u_{\varepsilon_n}, B_{\rho_0}(x_0)) - \liminf_{\varepsilon_n \to 0} F_{\varepsilon_n}(\omega)(u_{\varepsilon_n}, B_{\rho_0}(x_0) \setminus B_{\rho}(x_0))$$

$$\leq F(\omega)(u_{x_0,\xi}, B_{\rho_0}(x_0)) - F(\omega)(u_{x_0,\xi}, B_{\rho_0}(x_0) \setminus B_{\rho}(x_0))$$

$$= F(\omega)(u_{x_0,\xi}, B_{\rho}(x_0)).$$ (3.40)

where in the last equality we used that $u_{x_0,\xi}$ is a Sobolev function so that its limit energy does not charge $\partial B_{\rho}(x_0)$. (3.40) shows that $u_{\varepsilon_n}$ is also a recovery sequence on each $B_{\rho}(x_0)$ for $0 < \rho < \rho_0$. Next we introduce a constant whose appearance will become clear later in the proof. Choose $k > 1$ satisfying

$$\overline{C} C_{r,R} M + |\xi| \leq k,$$

where $\overline{C}$ and $C_{r,R}$ are given by (2.14) and (2.8), respectively. Note that $|u_{x_0,\xi}| \leq |\xi|/\rho$ on $B_{\rho}(x_0)$. Hence by Lemma 2.8 the truncated function $T_{k\rho}u_{\varepsilon_n}$ is also a recovery sequence on $B_{\rho}(x_0)$. Now consider a sequence $\rho_j \to 0$. For any $\rho = \rho_j \in (0, (3MK^2)^{-1}\rho_0)$ we choose $\varepsilon_\rho = \varepsilon_n(\rho) \leq \min\{\rho^{p+3}, \rho^2\}$ non-decreasing in $\rho$, satisfying Lemma 3.13 and such that

$$F_{\varepsilon_\rho}(\omega)(T_{k\rho}u_{\varepsilon_\rho}, B_{3MK^2}\rho) \leq C|\xi|^p \rho^d,$$

$$\int_{B_{\rho}(x_0)} |T_{k\rho}u_{\varepsilon_\rho} - u_{x_0,\xi}|^p \, dx \leq \rho^{p+d+1}. $$ (3.41)

The first estimate is realizable due to Lemma 2.8 and the fact that $u_{\varepsilon_n}$ is a recovery sequence also on $B_{3MK^2}\rho(x_0)$. Moreover we require that

$$\lim_{\rho \to 0} \rho^{-d} F(\omega)(u_{x_0,\xi}, B_{\rho}(x_0)) = \lim_{\rho \to 0} \rho^{-d} F_{\varepsilon_\rho}(\omega)(T_{k\rho}u_{\varepsilon_\rho}, B_{\rho}(x_0)).$$ (3.42)

Our analysis now relies on several modifications of the sequence $T_{k\rho}u_{\varepsilon_n}$ and a rescaling to $B_1$.

**Step 1** Construction of Lipschitz competitors

The following argument is well-known for continuum functionals and we adapt it carefully to the discrete setting. Let us set $v_\rho : \mathcal{L}(\omega) \to \mathbb{R}^m$ via $v_\rho(x) = \varepsilon_\rho^{-1} T_{k\rho}u_{\varepsilon_\rho}(\varepsilon_\rho x)$. Then, for given $\Lambda > 0$, we define the sets

$$R_{\rho}^\Lambda := \{ x \in \mathcal{L}(\omega) \cap \varepsilon_\rho^{-1} B_{MK^2}(x_0) : M_{k^2 \rho \varepsilon_\rho}^1 |\nabla e_1 v_\rho|(x) \leq \Lambda \},$$

$$S_{\rho}^\Lambda := \{ x \in \mathcal{L}(\omega) : |\nabla e_1 v_\rho|(x) \geq \frac{\Lambda}{2} \},$$

where $M_{\rho}^1$ denotes the discrete maximal function operator defined in (2.14) and the length of the gradient is with respect to the undiscretized edges. At first we estimate the cardinality of $(\mathcal{L}(\omega) \cap \varepsilon_\rho^{-1} B_{MK^2}(x_0)) \setminus R_{\rho}^\Lambda$. To this end, we note that for every $x \in (\mathcal{L}(\omega) \cap \varepsilon_\rho^{-1} B_{MK^2}(x_0)) \setminus R_{\rho}^\Lambda$ there exists by definition a number $0 < \eta_x \leq k^2 \rho \varepsilon_\rho^{-1}$ such that

$$\Lambda \# B_G(x, \eta_x) < \sum_{y \in B_G(x, \eta_x)} |\nabla e_1 v_\rho|(y) =: |\nabla v_\rho|(B_G(x, \eta_x)).$$

Applying Vitali's covering lemma on separable metric spaces we find a (finite) collection of disjoint $B_G(x, \eta_i)$ with $x_i \in (\mathcal{L}(\omega) \cap \varepsilon_\rho^{-1} B_{MK^2}(x_0)) \setminus R_{\rho}^\Lambda$ satisfying the above inequality and

$$(\mathcal{L}(\omega) \cap \varepsilon_\rho^{-1} B_{MK^2}(x_0)) \setminus R_{\rho}^\Lambda \subset \bigcup_i B_G(x_i, 5\eta_i).$$
Since the balls are disjoint we conclude that
\[ \Lambda \# \left( \bigcup_i B_G(x_i, \eta_k) \right) < |\nabla_1 v_p| \left( \bigcup_i B_G(x_i, \eta_k) \right) \leq |\nabla_1 v_p| \left( \bigcup_i B_G(x_i, \eta_k) \cap S^A_p \right) + \frac{\Lambda}{2} \# \left( \bigcup_i B_G(x_i, \eta_k) \right), \]
where we used the definition of $S^A_p$ in the last estimate. Rearranging terms we obtain
\[ \# \left( \bigcup_i B_G(x_i, \eta_k) \right) < \frac{2}{\Lambda} |\nabla_1 v_p| \left( \bigcup_i B_G(x_i, \eta_k) \cap S^A_p \right). \tag{3.43} \]
To save notation, for $x \in \mathcal{L}(\omega)$ we set $E(x) = \{ y \in \mathcal{L}(\omega) : (x, y) \in \mathcal{E}(\omega) \}$. Moreover define
\[ \mathcal{J}_p = \{ x \in \mathcal{L}(\omega) : \varepsilon_p |\nabla_1 v_p|^p(x) \geq 1 \}. \]
Note that for any $x \in \mathcal{J}_p$ there exists a point $y \in E(x)$ with $\varepsilon_p |v_p(x) - v_p(y)|^p \geq \frac{1}{M \rho}$. Hence the growth condition (2.19) implies the inequality
\[ 1 \leq C \left( f(\varepsilon_p |\nabla_1 v_p|^p(v_p, \mathbb{R}^d)) + f(\varepsilon_p |\nabla_1 v_p|^p(v_p, \mathbb{R}^d)(y)) \right). \tag{3.44} \]
In order to control the location of such $y$, observe that $B_G(x_i, \eta_k) \subset B_M \rho, (x_i)$ by (3.5), which in turn implies that
\[ \bigcup_i x \in B_G(x_i, \eta_k) E(x) \subset \varepsilon_p^{-1} B_M^2 \rho (x_0). \tag{3.45} \]
Here we also used (2.2) and that $M^2 \rho \geq M \varepsilon_p$. Now we sum the estimate (3.44) over $x$. Note that due to (2.3) every term can appear at most $M + 1$ times. By definition of $v_p$ we obtain
\[ \varepsilon_p^{-1} \# \left( \bigcup_i B_G(x_i, \eta_k) \cap \mathcal{J}_p \right) \leq CF_{\varepsilon_p}(\omega)(T_{k \rho} u_{\varepsilon_p}, B_{3M^2 \rho}) \leq C|\xi|^p \rho^d, \]
where we applied the first bound in (3.41). By truncation we further know that $|\nabla_1 v_p|(x) \leq C \varepsilon_p^{-1} \rho$ for all $x \in \mathcal{L}(\omega)$, so that
\[ |\nabla_1 v_p| \left( \bigcup_i B_G(x_i, \eta_k) \cap \mathcal{J}_p \right) \leq C|\xi|^p \left( \frac{\rho}{\varepsilon_p} \right)^d. \tag{3.46} \]
On the other hand, Hölder’s inequality yields
\[ |\nabla_1 v_p| \left( \bigcup_i B_G(x_i, \eta_k) \cap S^A_p \setminus \mathcal{J}_p \right) \leq \# \left( \bigcup_i B_G(x_i, \eta_k) \cap S^A_p \setminus \mathcal{J}_p \right) \left( \sum_{x \in \bigcup_i B_G(x_i, \eta_k) \setminus \mathcal{J}_p} |\nabla_1 v_p|^p(x) \right)^{\frac{1}{p}}. \tag{3.47} \]
We now have to pass from the undirected gradient to the directed version: For $x \notin \mathcal{J}_p$ and $y \in E(x)$ it holds in particular that $\varepsilon_p |v_p(x) - v_p(y)|^p \leq 1$. Hence we infer from the bound (2.19) that for $x \in \bigcup_i B_G(x_i, \eta_k) \setminus \mathcal{J}_p$
\[ |\nabla_1 v_p|^p(x) \leq C \sum_{y \in E(x)} |v_p(x) - v_p(y)|^p = C \varepsilon_p \sum_{y \in E(x)} \min\{ \varepsilon_p |v_p(x) - v_p(y)|^p, 1 \} \]
\[ \leq C \varepsilon_p \sum_{y \in E(x)} f(\varepsilon_p |\nabla_1 v_p|^p(v_p, \varepsilon_p^{-1} B_{3M^2 \rho}(x_0))(x)) + f(\varepsilon_p |\nabla_1 v_p|^p(v_p, \varepsilon_p^{-1} B_{3M^2 \rho}(x_0))(y)), \]
where we used again (3.45). Again we can sum this estimate and by (2.4) each term is counted at most $2M$ times. Thus in combination with the first estimate in (3.41) we have
\[ \left( \sum_{x \in \bigcup_i B_G(x_i, \eta_k) \setminus \mathcal{J}_p} |\nabla_1 v_p|^p(x) \right)^{\frac{1}{p}} \leq C \varepsilon_p^{\frac{1}{p}} \left( F_{\varepsilon_p}(\omega)(T_{k \rho} u_{\varepsilon_p}, B_{3M^2 \rho}(x_0)) \right)^{\frac{1}{p}} \leq \left( \frac{\rho}{\varepsilon_p} \right)^{\frac{1}{p}} C|\xi|. \tag{3.48} \]
Combining this estimate with (3.47) leads to
\[ |\nabla_1 v_p| \left( \bigcup_i B_G(x_i, \eta_k) \cap S^A_p \setminus \mathcal{J}_p \right) \leq C \# \left( \bigcup_i B_G(x_i, \eta_k) \cap S^A_p \setminus \mathcal{J}_p \right) \varepsilon_p^{\frac{1}{p}} \left( \frac{\rho}{\varepsilon_p} \right)^{\frac{1}{p}} |\xi|. \]
In order to bound the cardinality term, note that by definition of $S^\Lambda_p$ and (3.48) it holds that
\[
\#\left(\bigcup_i B_G(x_i, \eta_i) \cap S^\Lambda_p \setminus J_p\right) \left(\frac{\Lambda}{2}\right)^p \leq \sum_{x \in \bigcup_i B_G(x_i, \eta_i) \setminus J_p} |\nabla_{\epsilon,1}v_p|^p(x) \leq C|\xi|^p \left(\frac{\rho}{\varepsilon_p}\right)^d.
\]
Plugging this estimate into the previous one yields
\[
|\nabla_1 v_p|\left(\bigcup_i B_G(x_i, \eta_i) \cap S^\Lambda_p \setminus J_p\right) \leq C \Lambda^{1-p} |\xi|^p \left(\frac{\rho}{\varepsilon_p}\right)^d.
\]
Applying twice the doubling property established in Lemma 2.17 and combining (3.33), (3.46) and (3.49) we get
\[
\#(\mathcal{L}(\omega) \cap \varepsilon_p^{-1} B_{MKp}(x_0)) \setminus R^\Lambda_p \leq \#(\bigcup_i B_G(x_i, 5\eta_i)) \leq C\#(\bigcup_i B_G(x_i, \eta_i)) \leq C|\xi|^p \left(\frac{\rho}{\varepsilon_p}\right)^d \left(\rho \Lambda^{1-1+\Lambda^{-p}}\right).
\]
Now we choose $\Lambda = \Lambda_p$ as $\Lambda^{p-1} = \rho^{-1}$, so that the last inequality can be written as
\[
\#(\mathcal{L}(\omega) \cap \varepsilon_p^{-1} B_{MKp}(x_0)) \setminus R^\Lambda_p \leq C|\xi|^p \left(\frac{\rho}{\varepsilon_p}\right)^d \Lambda_p^{-p}. \tag{3.50}
\]
With this choice of $\Lambda_p$, we now construct the Lipschitz competitor. First observe that for any $x, y \in \mathcal{L}(\omega) \cap \varepsilon_p^{-1} B_{MKp}(x_0)$ the definition of $k$ yields
\[
|v_p(x) - v_p(y)| \leq 2\mathcal{C} C_{r,R} \Lambda_p |x - y| \leq 2\mathcal{C} C_{r,R} M k \rho \varepsilon_p^{-1} \leq k^2 \rho \varepsilon_p^{-1},
\]
so that (2.5) and (2.8) imply for any $x, y \in R^\Lambda_p$ the Lipschitz estimate
\[
|v_p(x) - v_p(y)| \leq 2\mathcal{C} C_{r,R} \Lambda_p |x - y| \leq k \Lambda_p |x - y|.
\]
Using Kirszbraun’s extension theorem we find a Lipschitz function $\tilde{v}_p : \mathcal{L}(\omega) \to \mathbb{R}^m$ with Lipschitz constant $k \Lambda_p$ that agrees with $v_p$ on $R^\Lambda_p$. Moreover, by truncation we can additionally assume that $\|\tilde{v}_p\|_\infty \leq \varepsilon_p^{-1} k^3 \rho$.

**Step 2** From Lipschitz continuity to equiintegrability of discrete gradients

In order to further modify the function $\tilde{v}_\varepsilon$ constructed in the first step, it is convenient to rescale it onto $B_1$. To this end, we introduce some notation. We set $\sigma_p = \frac{x_0}{\varepsilon_p}$ and $\mathcal{L}(\omega)_p := \mathcal{L}(\omega) - \frac{x_0}{\varepsilon_p}$. For any $z \in \mathcal{L}(\omega)_p$ we further denote by $E_p(z) = E(z + \frac{x_0}{\varepsilon_p}) - \frac{x_0}{\varepsilon_p}$ the set of undirected adjacent points. In the gradient notation we will replace $e$ by $e_0$ and $\omega$ by $\omega_0$, respectively. Define $u^\varrho : \sigma_p \mathcal{L}(\omega)_p \to \mathbb{R}^m$ via
\[
u^\varrho(\sigma_p z) = \sigma_p \tilde{v}_p(z + \frac{x_0}{\varepsilon_p}).
\]
By the properties of $\tilde{v}_p$ established in the first step, the function $u^\varrho$ satisfies the following properties:

(i) $\|u^\varrho\|_\infty \leq 3k$;
(ii) $|u^\varrho(\sigma_p x) - u^\varrho(\sigma_p y)| \leq k \Lambda_p |x - y|$ for all $x, y \in \mathcal{L}(\omega)_p$;
(iii) $u^\varrho(\sigma_p x, x) = \sigma_p v_p(x + \frac{x_0}{\varepsilon_p}) = \rho^{-1} T_{k(\varepsilon_p)} u_{\varrho_p}(x + x_0)$ for all $x \in R^\Lambda_p - \frac{x_0}{\varepsilon_p}$.

We define a modified gradient length of $u^\varrho$ as a scalar-valued function on $\sigma_p \mathcal{L}(\omega)_p \cap B_{MK}$ as
\[
V^\varrho(\sigma_p x) = |\nabla_{e_0, \sigma_p} u^\varrho|(\sigma_p x).
\]
and extend it to 0 for $\sigma_p x \in \sigma_p \mathcal{L}(\omega)_p \setminus B_{MK}$. As usually, by piecewise constant interpolation on the Voronoi cells $\{\sigma_p C_p(x)\}$ of $\sigma_p \mathcal{L}(\omega)_p$ we can view $V^\varrho$ as an element of $L^p(\mathbb{R}^d)$. Observe that due to (ii) we have the bound $|V^\varrho(\sigma_p x)| \leq C \Lambda_p$. Combining this bound with (iii) and a change of variables we derive the norm
Then by Remark 1 and (3.51) we have

By definition of the maximal function operator it holds that \( \varepsilon |\nabla \varepsilon| |V^\rho| \leq \varepsilon \Lambda^p \leq 1 \) for \( x \in R_{\varepsilon}^\rho \), so that we can use (3.50) and the same reasoning as for (3.48) to continue the estimate via

Hence the sequence \( V^\rho \) is bounded in \( L^p(\mathbb{R}^d) \). With a slight abuse of notation we denote by \( M_{c}^\rho V^\rho \) the discrete maximal function of \( V^\rho \) with respect to the transformed graph \( G^\rho = (\sigma^\rho \mathcal{L}(\omega)_\rho, \sigma^\rho \bigcup_{x \in \mathcal{L}(\omega)_\rho} E^\rho(x)) \) (see Definition (2.14)). We claim that \( M_{c}^\rho V^\rho \) is also a bounded sequence in \( L^p(\mathbb{R}^d) \). To this end, we show that the discrete maximal function can be pointwise controlled by the standard Hardy-Littlewood maximal function. Indeed, by (3.54) and Remark 1 we have for any \( \sigma > 0 \) and any function \( v : \sigma \mathcal{L}(\omega)_\rho \rightarrow \mathbb{R} \) the estimate

Thus boundedness of the Hardy-Littlewood maximal function operator on \( L^p(\mathbb{R}^d) \) (see [31 Theorem 2.91]) implies that

For \( l > 0 \) we introduce the scalar truncation operator \( \delta_l : \mathbb{R} \rightarrow \mathbb{R} \) setting

Applying [31 Lemma 2.31], we know that there exists a subsequence \( \rho_j \) (not relabeled) and an increasing sequence of positive integers \( l_j \rightarrow +\infty \) such that, setting \( \sigma_j = \sigma_{\rho_j} \), the sequence \( \delta_{l_j} \circ (M_{c}^\rho V^\rho)|^p \) is equiintegrable on \( \mathbb{R}^d \). We need to modify one more time the sequence \( u^\rho \). Define

Then by Remark 1 and (3.51) we have

Next we bound \( M_{c}^\rho V^\rho \) on the set \( R_j \). Given \( y \in E_{\rho_j}(x) \) for some \( x \in R_j \) and \( \eta \geq 2\sigma_j \), it holds that

Hence it follows from applying twice the doubling property proven in Lemma 2.7 (i) that there exists a uniform constant \( C \) such that

\[
\sup_{\eta > 0} \left( \frac{1}{\# B_{G^\rho_j}(y, \eta \sigma_j)} \sum_{z \in B_{G^\rho_j}(y, \eta \sigma_j)} |V^\rho_j(\sigma_j z)| \right) \leq C \left( M_{c}^\rho V^\rho_j(\sigma_j x) \right) \leq C l_j.
\]
If $\eta < 2\sigma_j$, then by Remark 2.2 and (2.2) we still have (upon increasing $C$) the inequality and inclusion

$$\frac{1}{C} \# B_{G_{\rho_j}}(x,2) \leq \# B_{G_{\rho_j}}(y, \frac{\eta}{\sigma_j}) \quad \text{and} \quad B_{G_{\rho_j}}(y, \frac{\eta}{\sigma_j}) \subset B_{G_{\rho_j}}(x,2),$$

so that for $y \in R_j$ we conclude the estimate

$$\mathcal{M}_{\infty}^e V_{(\sigma_j, y)} \leq C_{\delta_j}.$$  \hspace{1cm} (3.53)

Again we want to use (2.15) and Kirszbraun's extension theorem. To this end, note that for all $x, y \in R_j \cap \sigma_j^{-1}B_3$ and $z \in B_{G_{\rho_j}}(x, \mathbb{S}d_{G_{\rho_j}}(x, y))$ it holds $|\sigma_j z| < 3 + \frac{k}{M} \leq Mk$ (recall $k \geq 1$ and $M \geq 4$). As $V_{(\sigma_j, y)}$ agrees with $|\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|$ on $B_{MK}$, it follows that

$$\mathcal{M}_{\sigma_j}^e \mathcal{L}_{e_{\sigma_j, u_{y_{(\sigma_j, y)}}}}^e \mathcal{M}_{\sigma_j}^e \mathcal{L}_{e_{\sigma_j, u_{y_{(\sigma_j, y)}}}}^e |\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}| = \mathcal{M}_{\sigma_j}^e \mathcal{L}_{e_{\sigma_j, u_{y_{(\sigma_j, y)}}}}^e \mathcal{M}_{\sigma_j}^e \mathcal{L}_{e_{\sigma_j, u_{y_{(\sigma_j, y)}}}}^e |\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}| \leq \mathcal{M}_{\sigma_j}^e \mathcal{L}_{e_{\sigma_j, u_{y_{(\sigma_j, y)}}}}^e \mathcal{M}_{\sigma_j}^e \mathcal{L}_{e_{\sigma_j, u_{y_{(\sigma_j, y)}}}}^e \mathcal{M}_{\sigma_j}^e \mathcal{L}_{e_{\sigma_j, u_{y_{(\sigma_j, y)}}}}^e |\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}| \leq C_{\delta_j}.$$  \hspace{1cm} (3.53)

Due to the above estimate we find again a sequence of Lipschitz functions $w_j : \sigma_j \mathbb{L}(\omega)_{\rho_j} \rightarrow \mathbb{R}^m$ that agrees with $u_{y_{(\sigma_j, y)}}$ on the large set $\sigma_j R_j \cap B_3$ and with Lipschitz constant bounded by $C_{\delta_j}$. We claim that $|\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|$ is equiintegrable on $B_2$. To verify this assertion, we observe that by definition of $R_j$ we have that $|\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}| = |\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|$ on $\sigma_j R_j' \cap B_2$, so that for $\sigma_j x \in \sigma_j R_j' \cap B_2$ it holds that

$$|\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|^p(\sigma_j x) = |\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|^p(\sigma_j x) = |\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|^p(\sigma_j x) \leq |\mathcal{M}_{\sigma_j}^e V_{\sigma_j x}|^p(\sigma_j x) = (\delta_j \circ |\mathcal{M}_{\sigma_j}^e V_{\sigma_j x}|)^p,$$

while on $\sigma_j (\mathbb{L}(\omega)_{\rho_j} \setminus R_j')$ the bound on the Lipschitz constant implies

$$|\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|^p(\sigma_j x) \leq C_{\delta_j} = (\delta_j \circ |\mathcal{M}_{\sigma_j}^e V_{\sigma_j x}|)^p.$$

Hence equiintegrability on $B_2$ transfers from $(\delta_j \circ |\mathcal{M}_{\sigma_j}^e V_{\sigma_j x}|)^p$ to $|\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|^p$. Finally, by a truncation argument based on the operator $\mathcal{T}_k$ we can still assume that $\|w_j\|_{L^\infty} \leq 9k$.

**Step 3** Conclusion of the first inequality

With the subsequence constructed in Step 2 we now prove one of the inequalities that we claim in the second inequality. First we derive an estimate on the $L^p$-norm of the sequence $w_j - u_{0, \xi}$ on the set $B_1$. We set $U_j = \sigma_j R_j' \cap \sigma_j (\rho_j \setminus \frac{x}{\rho_j})$. Then by construction $w_j(\sigma_j x) = \rho_j^{-1} T_{k_{\rho_j}} u_{\rho_j}(x_0 + \varepsilon_j x)$ for all $\sigma_j x \in B_2 \cup U_j$. By (3.41), (3.50), (3.52) and the $L^\infty$-bound on $w_j$, for $j$ large enough a change of variables yields

$$\|w_j - u_{0, \xi}\|^p_{L^p(B_1)} \leq C \rho_j^{-p-d} \int_{B_{\rho_j}(x_0)} |T_{k_{\rho_j}} u_{\rho_j} - u_{x_0, \xi}|^p dz + C \left( \sigma_j^d \#(\mathbb{L}(\omega)_{\rho_j} \setminus R_j') + \sigma_j^d \#(\mathbb{L}(\omega) \cap \varepsilon_j^{-1} B_{MK_{\rho_j}}(x_0) \setminus R_j') \right) \leq C \rho_j + \frac{C}{\rho_j} + C \|\xi\|^p_{\Lambda_{\rho_j}}.$$

In particular $w_j \rightarrow u_{0, \xi}$ in $L^p(B_1, \mathbb{R}^m)$. Now we turn to the energy estimates. Fix $\eta > 0$. As shown in the previous estimate it holds

$$\lim \left\| \sum_{\sigma_j x \in B_1 \setminus U_j, x \in \mathbb{L}(\omega)_{\rho_j}} \sigma_j \mathcal{C}_{\rho_j}(x) \right\| \leq \lim \left( C \sigma_j^d \#(B_2 \cap \sigma_j \mathbb{L}(\omega)_{\rho_j} \setminus U_j) \right) = \lim \left( \frac{1}{\rho_j} + \|\xi\|^p_{\Lambda_{\rho_j}} \right) = 0.$$

Hence Remark 2.2 and equiintegrability of $|\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|^p$ on $B_2$ imply that there exists $j_\eta$ such that for $j \geq j_\eta$

$$\sum_{\sigma_j x \in B_1 \setminus U_j, x \in \mathbb{L}(\omega)_{\rho_j}} \sigma_j^d |\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|^p(\sigma_j x) \leq C \sum_{\sigma_j x \in B_1 \setminus U_j, x \in \mathbb{L}(\omega)_{\rho_j}} \int_{\sigma_j \mathcal{C}_{\rho_j}(x)} |\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|^p(\sigma_j x) dz \leq \eta.  \hspace{1cm} (3.54)$$

Next for $t > 0$ we introduce the sets

$$A_j(t) = \{ \sigma_j x \in \sigma_j \mathbb{L}(\omega)_{\rho_j} \cap B_1 : |\nabla e_{\sigma_j, u_{y_{(\sigma_j, y)}}}|^p(\sigma_j x) > t \}.$$
Again due to the equiintegrability established in Step 2 there exists $t_{\eta} > 0$ such that for $j \geq j_{\eta}$

$$
\sum_{\sigma_j \in \mathcal{E}_j(t_{\eta})} \sigma_j^d |\nabla_{\epsilon_0, \sigma_j} w_j|^p(\sigma_j x) \leq C \int_{B_2 \cap \{ |\nabla_{\sigma_j} w_j|^p \geq \epsilon_{\eta} \}} |\nabla_{\epsilon_0, \sigma_j} w_j|^p(z) \, dz \leq \eta. \tag{3.55}
$$

Moreover, if $\epsilon_j x \in (x_0 + \rho_j (B_1 \cap U_j \setminus A_j(t_{\eta})))$, then

$$
\| \| \epsilon_j \| \nabla_{\omega, \epsilon_j} |^p(T_{\rho_j} u_{\epsilon_j}, B_{\rho_j}(x_0))(\epsilon_j x) \| 1 \leq \epsilon_j \| \nabla_{\epsilon_0, \epsilon_j} w_j|^p(\sigma_j x - \rho_j^{-1} x_0) \leq \epsilon_{\rho_j} t_{\eta}. \tag{3.56}
$$

The right hand side converges to zero. Thus after enlarging $j_{\eta}$ assumption (2.10) yields

$$
f(\epsilon_j, \nabla_{\omega, \epsilon_j} |^p(T_{\rho_j} u_{\epsilon_j}, B_{\rho_j}(x_0))(\epsilon_j x)) \geq (1 - \eta) \alpha \epsilon_j \| \nabla_{\omega, \epsilon_j} |^p(T_{\rho_j} u_{\epsilon_j}, B_{\rho_j}(x_0))(\epsilon_j x) \| 1
$$

for all $j \geq j_{\eta}$. For the remaining points we can use (3.54) and (3.55), so that from a change of variables we deduce the lower bound

$$
\rho_j^{-d} F_{\epsilon_j}(\omega)(T_{\rho_j} u_{\epsilon_j}, B_{\rho_j}(x_0)) \geq (1 - \eta) \alpha \sum_{\sigma_j \in \mathcal{E}(\omega)_{\rho_j}} \sigma_j^{d-1} \| \nabla_{\omega_0, \sigma_j} |^p(w_j, B_1)(\sigma_j x) \| 1 - 2 \eta
$$

with the functional $G_{\omega, \rho}(x_0, \omega)$ defined in Lemma 3.11. Since we have chosen $x_0$ and $\epsilon_\rho$ such that Lemma 6.13 holds, we deduce from (3.12), $\Gamma$-convergence and the convergence $w_j \to u_{0, \xi}$ in $L^p(B_1, \mathbb{R}^m)$ that

$$
\lim_{\rho \to 0} \rho^{-d} F(\omega)(u_{x_0, \xi}, B_{\rho}(x_0)) = \lim_{j \to \infty} \rho_j^{-d} F_{\epsilon_j}(\omega)(T_{\rho_j} u_{\epsilon_j}, B_{\rho_j}(x_0)) \geq \liminf_j (1 - \eta) \| G_{\omega, \rho}(x_0, \omega)(w_j) - 2 \eta \geq (1 - \eta) \int_{B_1} q(x, \xi) \, dz - 2 \eta = (1 - \eta) \| B_1 |q(x, \xi)| - 2 \eta.
$$

Since $\eta > 0$ was arbitrary, we conclude that for a.e. $x_0 \in D$ and all $\xi \in \mathbb{R}^{m \times d}$

$$
\lim_{\rho \to 0} \rho^{-d} F(\omega)(u_{x_0, \xi}, B_{\rho}(x_0)) \geq |B_1 |q(x, \xi)|. \tag{3.57}
$$

Observe that the exceptional set depends on the sequence chosen at the beginning.

**Step 4** Conclusion of the second inequality

To prove the reverse inequality in (3.57) we take a sequence $u_{\epsilon_n} \in \mathcal{P}C_{\epsilon_n}^\omega$ converging to $u_{x_0, \xi}$ in $L^p(D, \mathbb{R}^m)$ and such that

$$
\lim_{\epsilon_n \to 0} E_{\epsilon_n}(\omega)(u_{\epsilon_n}, B_{\rho_0}(x_0)).
$$

As in (3.40) one can show that $u_{\epsilon_n}$ is a recovery sequence on all balls $B_{\rho}(x_0)$ with $0 < \rho < \rho_0$. Again we consider the truncated functions $T_{\rho_k} u_{\epsilon_n}$ that still form a recovery sequence on $B_{\rho}(x_0)$. We transform it to a function $v^\rho_n : \mathbb{R}^d \to \mathbb{R}^m$ defined via

$$
v^\rho_n(\mathbb{R}\frac{\epsilon_n x}{\rho}) = \rho^{-1} T_{\rho_k} u_{\epsilon_n}(\mathbb{R}\frac{\epsilon_n x + x_0}{\rho}),
$$

where we set $\mathcal{L}(\omega)_n = \mathcal{L}(\omega) - \frac{\epsilon_n x}{\rho}$. Similar, for $z \in \mathcal{L}(\omega)_n$, the undirected adjacent points are denoted by $E_n(z) = E(z + \frac{\epsilon_n x}{\rho}) - \frac{\epsilon_n x}{\rho}$. As in Step 2 we define the scalar-valued function $V^\rho_n$ on $\mathbb{R}^d \mathcal{L}(\omega)_n \cap B_{M_k}$ setting

$$
V^\rho_n(\mathbb{R}\frac{\epsilon_n x}{\rho}) = |\nabla_{\epsilon_0, \mathbb{R}\frac{\epsilon_n x}{\rho}} v^\rho_n(\mathbb{R}\frac{\epsilon_n x}{\rho})|
$$

and extend it to 0 on $\mathbb{R}^d \mathcal{L}(\omega)_n \setminus B_{M_k}$. Then we have that

$$
\limsup_n \|V^\rho_n\|_{L^p(\mathbb{R}^d)} \leq C \rho^{-d} \limsup_n E(\omega)(T_{\rho_k} u_{\epsilon_n}, B_{2\rho M_k}(x_0)) = C \rho^{-d} E(\omega)(u_{x_0, \xi}, B_{2\rho M_k}(x_0)) \leq C|\xi|^p,
$$

so that the sequence is bounded in $L^p(\mathbb{R}^d)$. Repeating the arguments from Step 2, for fixed $\rho$ we construct a subsequence $\epsilon_{j_j} = \epsilon_{j_j}$, a sequence $w^\rho_{j_j}$ and the sets $R^\rho_{j_j} \subset R_{j_j}$ such that $w^\rho_{j_j}$ agrees with $v^\rho_{j_j}$ on $\mathbb{R}^d \setminus R^\rho_{j_j}$, the sequence $|\nabla_{\epsilon_0, \epsilon_{j_j}} w^\rho_{j_j}|$ is equiintegrable on $B_2$ and the sets $R^\rho_{j_j} \subset \mathcal{L}(\omega)_{j_j}$ satisfy

$$
\lim_j \# \mathcal{L}(\omega)_{j_j} \setminus R^\rho_{j_j} = 0. \tag{3.58}
$$
By truncation we may further assume that \( \|u_j^\rho\|_\infty \leq 3k \). Let us fix again \( \eta > 0 \) and define for \( t > 0 \) the sets \( \mathcal{A}_t^{\rho}(x) \) as

\[
\mathcal{A}_t^{\rho}(x) := \left\{ \frac{\varepsilon_j}{\rho} x + \frac{\varepsilon_j}{\rho} \mathcal{L}(\omega)_{x, t} \cap B_1 : \left| \nabla \varepsilon_j \mathcal{F}_{\rho}^{\rho}(\frac{\varepsilon_j}{\rho} x) \right| > t \right\}.
\]

We choose \( t_\eta > 0 \) (possibly depending on \( \rho \)) such that, having in mind the inequality \( f(p) \leq C \|f\|_1 \), for \( j \) large enough it holds

\[
\rho^{-d} \sum_{x \in \mathcal{A}_t^{\rho}(t_\eta)} \varepsilon_j^{-d+1} f \left( \varepsilon_j \left| \nabla \varepsilon_j \mathcal{F}_{\rho}^{\rho}(\frac{\varepsilon_j}{\rho} x) \right| \right) \leq C \int_{B_2 \setminus \{ \left| \nabla \varepsilon_j \mathcal{F}_{\rho}^{\rho}(\frac{\varepsilon_j}{\rho} x) \right| > t_\eta \}} \left| \nabla \varepsilon_j \mathcal{F}_{\rho}^{\rho}(\frac{\varepsilon_j}{\rho} x) \right| dz \leq \eta t_\eta \eta^d.
\]

where we used the fact that \( \left| \nabla \varepsilon_j \mathcal{F}_{\rho}^{\rho}(\frac{\varepsilon_j}{\rho} x) \right| \) is equiintegrable on \( B_2 \). Using the same estimates combined with \( 3.58 \), we find \( j_\eta \in \mathbb{N} \) such that for all \( j \geq j_\eta \) we have

\[
\rho^{-d} \sum_{x \in \mathcal{L}(\omega)_{x, \frac{\varepsilon_j}{\rho} \mathcal{F}_{\rho}^{\rho}(t_\eta)}} \varepsilon_j^{-d+1} f \left( \varepsilon_j \left| \nabla \varepsilon_j \mathcal{F}_{\rho}^{\rho}(\frac{\varepsilon_j}{\rho} x) \right| \right) \leq \eta.
\]

Let us define the function \( u_j^\rho : \varepsilon_j \mathcal{L}(\omega) \to \mathbb{R}^m \) setting

\[
u_j^\rho (\varepsilon_j x) = \rho u_j^\rho \left( \frac{\varepsilon_j}{\rho} (x - \frac{x_0}{\varepsilon_j}) \right).
\]

Then, similar to \( 3.59 \), the growth assumption \( 2.10 \), \( 3.59 \), \( 3.60 \) and twice a change of variables imply for large enough \( j \) the estimate

\[
\rho^{-d} E_{\varepsilon_j}(\omega)(T_{u_j^\rho}(u_j^\rho, B_\rho(x_0))) \geq \rho^{-d} \sum_{x \in \mathcal{R}_j \setminus \mathcal{A}_t^{\rho}(t_\eta)} \varepsilon_j^{-d+1} \left| \nabla \varepsilon_j \mathcal{F}_{\rho}^{\rho}(\frac{\varepsilon_j}{\rho} x) \right| \| f \|_1 \geq \rho^{-d} (1 - \eta) \sum_{x \in \mathcal{R}_j \setminus \mathcal{A}_t^{\rho}(t_\eta)} \varepsilon_j^{-d+1} f \left( \varepsilon_j \left| \nabla \varepsilon_j \mathcal{F}_{\rho}^{\rho}(\frac{\varepsilon_j}{\rho} x) \right| \right)
\]

Suppose we have shown that \( u_j^\rho \to u_{x_0, \xi} \in L^1(B_\rho(x_0), \mathbb{R}^m) \). Then, by a modification on \( \varepsilon_j \mathcal{L}(\omega) \setminus B_\rho(x_0) \) not affecting the energy, we can assume that the convergence holds in \( L^1(D, \mathbb{R}^m) \) and therefore we deduce from \( \Gamma \)-convergence that

\[
\rho^{-d} E(\omega)(u_{x_0, \xi}, B_\rho(x_0)) = \lim_{j \to \infty} \rho^{-d} E_{\varepsilon_j}(\omega)(T_{u_j^\rho}(u_j^\rho, B_\rho(x_0))) \geq \rho^{-d} (1 - \eta) F(\omega)(u_{x_0, \xi}, B_\rho(x_0)) - 2\eta
\]

By the arbitrariness of \( \eta \) this yields

\[
\rho^{-d} E(\omega)(u_{x_0, \xi}, B_\rho(x_0)) \geq \rho^{-d} F(\omega)(u_{x_0, \xi}, B_\rho(x_0)).
\]

In view of Remark \( 10 \) we conclude from the above inequality that

\[
|B_1| q(x_0, \xi) = \lim_{\rho \to 0} \rho^{-d} E(\omega)(u_{x_0, \xi}, B_\rho(x_0)) \geq \lim_{\rho \to 0} \rho^{-d} F(\omega)(u_{x_0, \xi}, B_\rho(x_0)).
\]

Combined with \( 3.57 \), this estimate yields the claim along the chosen subsequence. In the general case, we obtain that along any subsequence of \( \varepsilon_n \) the \( \Gamma \)-limit of \( E_{\varepsilon}(\omega) \) is uniquely defined through the integrand \( h(x, \xi) \), so that the \( \Gamma \)-limit along the sequence \( \varepsilon_n \) indeed exists by the Urysohn-property of \( \Gamma \)-convergence, although the integrand might differ on a negligible set depending on the subsequence.

It remains to prove the \( L^1 \)-convergence of \( u_j^\rho \) on \( B_\rho(x_0) \). To this end, observe that by construction \( u_j^\rho(\varepsilon_j x) = T_{u_j^\rho}(\varepsilon_j x) \) for all \( x \in \mathcal{L}(\omega) \setminus \left( \mathcal{R}_j + \frac{\varepsilon_j}{\rho} \right) \). Since \( u_j^\rho \) is uniformly bounded, the convergence follows from \( 3.58 \) and the convergence \( T_{u_j^\rho}(\varepsilon_j x) \to u_{x_0, \xi} \) in \( L^1(B_\rho(x_0), \mathbb{R}^m) \).

\[\square\]

**Remark 11.** As shown in Step 1 of the proof of \( 4 \) Theorem 2 \( , \) for a.e. \( x_0 \in D \) and every \( \xi \in \mathbb{R}^{m \times d} \) we have the asymptotic formula

\[
q(x_0, \xi) = \lim_{\rho \to 0} \rho^{-d} \lim_{j \to +\infty} \left\{ E_{\varepsilon_j}(\omega)(v, Q_{\varepsilon_j}(x_0, \rho)) : v \in \mathcal{P}_{\varepsilon_j, M_{\varepsilon_j}} u_{x_0, \xi}, Q_{\varepsilon_j}(x_0, \rho) \right\}.
\]
4. Stochastic homogenization

While for the representation result of Theorem 3.1 we did only use the geometric assumption on the lattice, we now prove a full $\Gamma$-convergence result incorporating the additional stationarity condition. Our approach heavily depends on the asymptotic formulas derived in the previous section.

**Theorem 4.1.** Assume that $\mathcal{L}$ is a stationary stochastic lattice with admissible stationary edges in the sense of Definitions 2.3 and 2.6. Then $\mathbb{P}$-a.s. the functionals $F_\varepsilon(\omega)$ $\Gamma$-converge in the $L^1(D,\mathbb{R}^m)$-topology to a functional $F(\omega) : L^1(D,\mathbb{R}^m) \to [0,\infty]$ with domain $L^1(D,\mathbb{R}^m) \cap GSBV^p(D,\mathbb{R}^m)$, where it can be written as

$$F(\omega)(u) = \int_D h_\omega(\nabla u) \, dx + \int_{S_u} g_\omega(\nu_u) \, dH^{d-1}$$

for some convex and $\tau$-invariant function $h_\omega$ and some convex, one-homogeneous and $\tau$-invariant function $g_\omega$. In particular, $f_\omega$ and $g_\omega$ are deterministic provided $\mathcal{L}$ is ergodic.

**Proof.** By Theorem 3.1 we already know that for $\mathbb{P}$-a.e. $w \in \Omega$ and any sequence $\varepsilon \to 0$ we can find a (possibly $\omega$-dependent) subsequence $\varepsilon_j$ such that

$$\Gamma\hbox{-}\lim_{j} F_{\varepsilon_j}(\omega)(u) = \begin{cases} \int_D h(\omega)(x,\nabla u) \, dx + \int_{S_u} g(\omega)(x,\nu_u) \, dH^{d-1} & \text{if } u \in GSBV^p(D,\mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

We argue that, up to set of probability zero, the functions $g(\omega)$ and $h(\omega)$ do depend neither on $x$ nor on the subsequence $\varepsilon_j$. This shows the full $\Gamma$-convergence result and then it remains to establish the properties of the integrands.

Let the auxiliary functionals $I_\varepsilon(\omega)$ and $E_\varepsilon(\omega)$ be defined as in 3.31 and 3.34, respectively. From Proposition 4 and Remark 11 we know that

$$h(\omega)(x_0,\xi) = \lim_{\rho \to 0} \rho^{-d} \lim_{j \to +\infty} \left( \inf \left\{ E_{\varepsilon_j}(\omega)(v,Q_{\varepsilon_j}(x_0,\rho)) : v \in PC_{\varepsilon_j,M_{\varepsilon_j}}(u_{x_0,\xi},Q_{\varepsilon_j}(x_0,\rho)) \right\} \right)$$

for a.e. $x_0 \in D$ and every $\xi \in \mathbb{R}^{m \times d}$. Now we can exploit the stochastic homogenization results for discrete gradient-type energies. Following word by word the proof of 4 Theorem 2 one can show that for a set of full probability there exists a function $h_\omega : \mathbb{R}^{m \times d} \to [0,\infty]$ such that

$$h_\omega(\xi) = \lim_{\varepsilon \to 0} \rho^{-d} \inf \left\{ E_\varepsilon(\omega)(v,Q_{\varepsilon}(x_0,\rho)) : v \in PC_{\varepsilon,M_\varepsilon}(u_{x_0,\xi},Q_{\varepsilon}(x_0,\rho)) \right\}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, for a.e. $x_0 \in D$, all $\rho > 0$ and every $\xi \in \mathbb{R}^{m \times d}$. In particular this shows that for those values we have $h(\omega)(x_0,\xi) = h_\omega(\xi)$. Moreover $h_\omega$ is $\tau$-invariant since it is given by the ergodic theorem.

Next we argue similar for the surface integral. Recall that by formula 3.32, for every $x_0 \in D$ and all $\nu \in S^{d-1}$ we have

$$g_\omega(\nu) = \lim_{\rho \to 0} \rho^{-1-d} \liminf_{\delta \to 0} \left( \inf \left\{ I_{\varepsilon_j}(\omega)(\nu,Q_{\varepsilon_j}(x_0,\rho)) : \nu \in S_{\varepsilon_j,\delta}(u_{x_0,\nu}^{-1,1},Q_{\varepsilon_j}(x_0,\rho)) \right\} \right).$$

Here we appeal to the more recent results on stochastic homogenization for spin systems. Repeating the rather lengthy argument for 5 Theorem 5.5], one can show that for $\mathbb{P}$-a.e. $w \in \Omega$, every $x_0 \in D$, all $\rho > 0$ and for every $\nu \in S^{d-1}$ there exists the $\tau$-invariant limit

$$g_\omega(\nu) = \lim_{\varepsilon \to 0} \rho^{-1-d} \inf \left\{ I_\varepsilon(\omega)(\nu,Q_{\varepsilon}(x_0,\rho)) : \nu \in S_{\varepsilon,2M_\varepsilon}(u_{x_0,\nu}^{-1,1},Q_{\varepsilon}(x_0,\rho)) \right\}.$$

Note that in the formula above we have a microscopic discrete boundary of width $2M\varepsilon$, where $2M$ is a bound on the maximal diameter of the set where the discrete gradient is computed (see also 5 Remark 4.3 (i))]. By monotonicity it holds that $g_\omega(\nu)(x_0,\nu) \geq g_\omega(\nu)$. In order to prove the reverse inequality we have to pass from a microscopic width $2M\varepsilon$ to $\delta > 0$. To this end, let $2\delta < \rho$. Then one can extend any

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2Here the stationarity and measurability of the edges becomes important. These properties are satisfied in 4 by the structure of the functional. With our assumptions on the edges the proof remains unchanged.

3In contrast to 3 the functional $I_\varepsilon(\omega)$ may contain non-pairwise interactions. This does not affect the argument up to enlarging the width of the discrete boundary, so we decided not to repeat the proof. Again here stationarity and measurability of the edges is important to define stationary stochastic processes for the ergodic theorem.
function \( v \in \mathcal{S}^{w,2}_{\varepsilon,\delta}(u_{x_0,v}^{-1,1}, Q_{\varepsilon}(x_0, \rho - 2\delta)) \) to a function \( \hat{v} \in \mathcal{S}^{w}_{\varepsilon,\delta}(u_{x_0,v}^{-1,1}, Q_{\varepsilon}(x_0, \rho)) \) by setting \( \hat{v} = u_{x_0,v}^{-1,1} \) on \( \varepsilon \mathcal{L}(\omega) \setminus Q_{\varepsilon}(x_0, \rho - 2\delta) \). Using the structure of the boundary conditions, for \( \varepsilon \) small enough we infer the estimate

\[
I_{\varepsilon}(\hat{v}, Q_{\varepsilon}(x_0, \rho)) \leq I_{\varepsilon}(v, Q_{\varepsilon}(x_0, \rho - 2\delta)) + C\delta \rho^{d-2}.
\]

Passing to the corresponding infimum on both sides and taking limits in the correct order yields

\[
g(\omega)(x_0, \nu) \leq \lim_{\nu \to 0} \lim_{\delta \to 0} \left( \frac{\rho}{\rho - 2\delta} \right)^{1-d} (g_{\omega}(\nu) + C\delta \rho^{-1}) = g_{\omega}(\nu).
\]

Thus we proved that, for \( \mathbb{P}\text{-a.e. } \omega \in \Omega \),

\[
\Gamma \lim_{\varepsilon \to 0} F_{\varepsilon}(\omega)(u) = \begin{cases} \int_D h_{\omega}(\nabla u) \, dx + \int_{S_u} g_{\omega}(\nu_u) \, dH^{d-1} & \text{if } u \in \text{GSBV}(D, \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}
\]

for some \( \tau \)-invariant functions \( h_{\omega} \) and \( g_{\omega} \). Convexity of \( h_{\omega} \) follows from the fact that the \( \Gamma \)-limit of the sequence of convex functionals \( E_{\varepsilon}(\omega) \) is again convex, whereas convexity of the one-homogeneous extension of \( g_{\omega} \) follows from standard \( L^1 \)-lower-semicontinuity results for functionals defined on sets of finite perimeter (see for instance [7, Theorem 3.1]). \( \square \)

4.1. **Convergence of minimizers.** Now we add a discrete fidelity term to our approximating functional \( F_{\varepsilon}(\omega) \) which will approximate the continuum fidelity term that penalizes the distance to the measured image.

In order to define the discrete approximation, we consider a discrete measurement of some continuum function. More precisely, throughout this section we fix an exponent \( q > 1 \) and consider a sequence \( g_{\varepsilon}(\omega) : \varepsilon \mathcal{L}(\omega) \to \mathbb{R}^m \), for which we assume that there exists \( g \in L^q(D, \mathbb{R}^m) \) such that \( \mathbb{P}\text{-a.s.} \)

\[
g_{\varepsilon}(\omega) \to g \quad \text{in } L^q(D, \mathbb{R}^m).
\]

**Remark 12.** For every given \( g \in L^q(D, \mathbb{R}^m) \), we find a sequence with this approximation property by first extending \( g = 0 \) on \( \mathbb{R}^d \setminus D \) and then setting

\[
g_{\varepsilon}(x) = \frac{1}{\varepsilon C(x)} \int_{\varepsilon C(x)} g(z) \, dz.
\]

To see this, we may assume that \( m = 1 \). It is a consequence of Lebesgue’s differentiation theorem that \( g_{\varepsilon} \to g \text{ a.e.} \). Since \( g_{\varepsilon} \) is bounded in modulus by the maximal function of \( q \) (which belongs itself to \( L^q(\mathbb{R}^d) \)), we deduce \( \text{[22]} \) from dominated convergence.

With \( g_{\varepsilon}(\omega) \) at hand, we introduce the approximation setting \( F_{\varepsilon,g}(\omega) : L^1(D, \mathbb{R}^m) \to [0, +\infty] \) as

\[
F_{\varepsilon,g}(\omega)(u) = \begin{cases} F_{\varepsilon}(\omega)(u) + \sum_{x \in \mathcal{L}(\omega) \cap D} \varepsilon^d |u(\varepsilon x) - g_{\varepsilon}(\varepsilon x)|^q & \text{if } u \in \mathcal{PC}_\varepsilon(\omega), \\ +\infty & \text{otherwise,} \end{cases}
\]

where \( F_{\varepsilon}(\omega) \) is defined in \( \text{[22]} \). Note that we chose a discrete fidelity term not depending on the measure of the Voronoi cells or other continuum quantities. The identification of the \( \Gamma \)-limit for this new sequence is contained in the following theorem.

**Theorem 4.2.** Let \( q \in (1, +\infty) \) and \( g_{\varepsilon}(\omega) \) satisfy \( \text{[4,7]} \). Under the assumptions of Theorem \( \text{[4,7]} \) there exists a random constant \( \gamma(\omega) \) such that \( \mathbb{P}\text{-a.s.} \) the functionals \( F_{\varepsilon,g}(\omega) \) \( \Gamma \)-converge with respect to the \( L^1(D, \mathbb{R}^m) \)-topology to the functional \( F_g(\omega) : L^1(D, \mathbb{R}^m) \to [0, +\infty] \) with domain \( L^q(D, \mathbb{R}^m) \cap \text{GSBV}^p(D, \mathbb{R}^m) \), where it is defined by

\[
F_g(\omega)(u) = \int_D h_{\omega}(\nabla u) \, dx + \int_{S_u} g_{\omega}(\nu_u) \, dH^{d-1} + \gamma(\omega) \int_D |u - g|^q \, dx,
\]

with the functions \( h_{\omega} \) and \( g_{\omega} \) given by Theorem \( \text{[4,7]} \). If \( \mathcal{L} \) is ergodic, then \( \gamma(\omega) \) is deterministic.
Proof. We first construct a candidate for the constant \( \gamma(\omega) \). Define the sequence of nonnegative equi-bounded functions \( c_\varepsilon(\omega) \in L^\infty(D) \) by
\[
c_\varepsilon(\omega)(z) = \sum_{x \in L(\omega)} \frac{1}{|C(x)|} \mathbf{1}_{C(x)}(z).
\]

(4.3)

We apply the ergodic theorem in order to establish weak*-convergence of \( c_\varepsilon(\omega) \). To this end, we introduce the family of half-open boxes with integer vertices as \( \mathcal{I} := \{(a, b) : a, b \in \mathbb{Z}^d, a_i < b_i \text{ for all } i\} \) and define the rescaled integral averages \( c : \mathcal{I} \to L^1(\Omega) \) by
\[
c(I, \omega) = \int_I \sum_{x \in L(\omega)} \frac{1}{|C(x)|} \mathbf{1}_{C(x)}(z) \, dz = \sum_{x \in L(\omega)} \frac{|C(x) \cap I|}{|C(x)|}.
\]

The following three properties can be verified:

(i) \( 0 \leq c(I, \omega) \leq |I| \) for all \( I \in \mathcal{I} \),

(ii) When \( I = \bigcup I_i \in \mathcal{I} \) with finitely many, pairwise disjoint \( I_i \in \mathcal{I} \), then \( c(I, \omega) = \sum_i c(I_i, \omega) \),

(iii) \( c(I, \tau_z \omega) = c(I - z, \omega) \) for all \( z \in \mathbb{Z}^d \).

Moreover, arguing as in [23, Lemma A.1], one can show that \( \omega \mapsto c(I, \omega) \) is \( \mathcal{F} \)-measurable. Hence we can apply the multi-parameter additive ergodic theorem (see [1] for the stronger super/subadditive version) and conclude that there exists a positive function \( \gamma : \Omega \to \mathbb{R} \) such that \( \mathbb{P}\)-a.s. and for all \( I \in \mathcal{I} \)
\[
\gamma(\omega) = \lim_{n \to +\infty} \frac{c(nI, \omega)}{|nI|}.
\]

It is straightforward to extend this result to all sequences \( t_n \to +\infty \) and then to all cubes in \( \mathbb{R}^d \) by a continuity argument. Now we identify the weak*-limit of \( c_\varepsilon(\omega) \). By usual density arguments it is enough to compute averages on cubes. Let \( Q \subset D \) be a cube. By a change of variables we obtain
\[
\int_Q c_\varepsilon(\omega)(z) \, dz = \varepsilon^d c(Q/\varepsilon, \omega) \to \gamma(\omega)|Q|,
\]
whence \( c_\varepsilon(\omega) \xrightarrow{\ast} \gamma(\omega) \) in \( L^\infty(D) \).

Next we prove the lower bound for \( \Gamma \)-convergence. Passing to a subsequence, for the \( \liminf \)-inequality it suffices to consider \( u \in L^1(D, \mathbb{R}^m) \) and a sequence \( u_\varepsilon \in \mathcal{PC}_\varepsilon^v \) such that \( u_\varepsilon \to u \) in \( L^1(D, \mathbb{R}^m) \) and
\[
\liminf_{\varepsilon \to 0} F_{\varepsilon, g}(u_\varepsilon) = \lim_{\varepsilon \to 0} F_{\varepsilon, g}(u)(u_\varepsilon) \leq C < +\infty.
\]

(4.4)

Without affecting the convergence properties or the functional value we redefine \( g_\varepsilon(\omega)(\varepsilon x) = u_\varepsilon(\varepsilon x) = 0 \) for all \( \varepsilon x \in \varepsilon L(\omega) \setminus D \). Then by Remark [1] we have
\[
\|u_\varepsilon - g_\varepsilon(\omega)\|_{L^q(D)} \leq C \sum_{\varepsilon x \in \varepsilon L(\omega) \cap D} \varepsilon^d |u_\varepsilon(\varepsilon x) - g_\varepsilon(\omega)(\varepsilon x)|^q \leq C,
\]
which in combination with (4.2) implies that \( u_\varepsilon \) is bounded in \( L^q(D, \mathbb{R}^m) \). Thus we obtain \( u \in L^q(D, \mathbb{R}^m) \), while Theorem [4.3] and (4.4) yield \( u \in GSBV^q(D, \mathbb{R}^m) \). Moreover, for any \( 1 \leq r < q \) we deduce the following convergence properties:
\[
u_\varepsilon \to u \quad \text{in } L^r(D, \mathbb{R}^m), \quad u_\varepsilon \to u \quad \text{in } L^q(D, \mathbb{R}^m).
\]

(4.5)

Observe that by the definition of the function \( c_\varepsilon(\omega) \) it holds that
\[
\sum_{\varepsilon x \in \varepsilon L(\omega) \cap D} \varepsilon^d |u_\varepsilon(\varepsilon x) - g_\varepsilon(\omega)(\varepsilon x)|^q \geq \int_D |c_\varepsilon(\omega)(z)| |u_\varepsilon(z) - g_\varepsilon(\omega)(z)|^q \, dz.
\]

Due to (4.2) and (4.3), the sequence \( u_\varepsilon - g_\varepsilon(\omega) \) converges to \( u - g \) in \( L^r(D, \mathbb{R}^m) \) for any \( 1 \leq r < q \). The lower semicontinuity result for pairs of weak-strong convergent sequences in [31, Theorem 7.5] and the
\( \Gamma \)-convergence proven in Theorem 4.1 imply
\[
\liminf_{\varepsilon \to 0} F_{\varepsilon,g}(\omega)(u_\varepsilon) \geq \liminf_{\varepsilon \to 0} F_{\varepsilon}(\omega)(u_\varepsilon, D) + \liminf_{\varepsilon \to 0} \sum_{\varepsilon x \in \mathcal{L}(\omega) \cap D} \varepsilon^d |u_\varepsilon(\varepsilon x) - g_\varepsilon(\omega)(\varepsilon x)|^q \\
\geq F(\omega)(u) + \gamma(\omega) \int_D |u - g|^q \, dz,
\] (4.6)
where we used that \( \gamma(\omega) \geq 0 \) to avoid the modulus. This finishes the proof of the lower bound.

For the upper bound, we consider \( u \in L^q(D, \mathbb{R}^m) \cap GSBV^p(D, \mathbb{R}^m) \). Note that for such \( u \) we can equivalently consider the \( \Gamma \)-limit with respect to \( L^q(D, \mathbb{R}^m) \)-convergence. Indeed, by Lemma 2.8 this is true for all truncated functions \( T_k u \) with \( k > 0 \) and by lower semicontinuity with respect to \( L^q \)-convergence and again Lemma 2.8 we obtain
\[
\Gamma(L^q(D)) \text{-} \limsup_{\varepsilon \to 0} F_{\varepsilon}(\omega)(u) \leq \liminf_{k \to +\infty} \left( \Gamma(L^q(D)) \text{-} \limsup_{\varepsilon \to 0} F_{\varepsilon}(\omega)(T_k u) \right) \leq \liminf_{k \to +\infty} F(\omega)(T_k u) = F(\omega)(u).
\]
Thus we find a sequence \( u_\varepsilon \in \mathcal{PC}_x^\omega \) such that \( u_\varepsilon \to u \) in \( L^q(D, \mathbb{R}^m) \) and
\[
\lim_{\varepsilon \to 0} F_{\varepsilon}(\omega)(u_\varepsilon, D) = F(\omega)(u).
\] (4.7)
Since \( D \) has Lipschitz boundary, it satisfies an interior cone condition and therefore we find a value \( c_D > 0 \) such that, whenever \( \varepsilon x \in D \), we have
\[
|\varepsilon C(x) \cap D| \geq c_D \varepsilon^d.
\]
Setting \( D_\varepsilon = \{ x \in D : \text{dist}(z, \partial D) \leq 2R\varepsilon \} \), we deduce from the above estimate that
\[
\sum_{\varepsilon x \in \mathcal{L}(\omega) \cap D} \varepsilon^d |u_\varepsilon(\varepsilon x) - g_\varepsilon(\omega)(\varepsilon x)|^q \leq \int_D c_\varepsilon(\omega)(z)|u_\varepsilon(z) - g_\varepsilon(\omega)(z)|^q \, dz \\
+ C \int_{D_\varepsilon} |u_\varepsilon(z) - g_\varepsilon(\omega)(z)|^q \, dz.
\]
The last term vanishes when \( \varepsilon \to 0 \) since the sequence \( |u_\varepsilon - g_\varepsilon|^q \) is equiintegrable on \( D \). Note that by its product structure the sequence \( c_\varepsilon(\omega)|u_\varepsilon - g_\varepsilon|^q \) converges weakly in \( L^1(D, \mathbb{R}^m) \) to \( \gamma(\omega)|u - g|^q \).

Therefore the last inequality implies
\[
\limsup_{\varepsilon \to 0} \sum_{\varepsilon x \in \mathcal{L}(\omega) \cap D} \varepsilon^d |u_\varepsilon(\varepsilon x) - g_\varepsilon(\omega)(\varepsilon x)|^p \leq \gamma(\omega) \int_D |u(z) - g(z)|^p \, dz.
\]
Combined with (4.7) we obtain the upper bound.

We conclude this section by an improved convergence result for global minimizers in \( L^q \) which is due to the special structure of our functionals.

**Corollary 1.** Under the assumptions of Theorem 4.2 for a set of full probability the following holds true:
For each \( \varepsilon > 0 \) there exists a global minimizer \( \hat{u}_\varepsilon \in \mathcal{PC}_x^\omega \) of the functional \( F_{\varepsilon,g}(\omega) \). Moreover, if \( u_\varepsilon \in \mathcal{PC}_x^\omega \) is any sequence such that
\[
\lim_{\varepsilon \to 0} \left( F_{\varepsilon,g}(\omega)(u_\varepsilon) - \inf_{u \in \mathcal{PC}_x^\omega} F_{\varepsilon,g}(\omega)(u) \right) = 0,
\]
then it is compact in \( L^q(D, \mathbb{R}^m) \) and each cluster point as \( \varepsilon \to 0 \) is a global minimizer of \( F_g(\omega) \).

For the proof we exploit the notion of biting convergence, which we recall here for reader’s convenience.

**Definition 4.3.** Let \( u_n \in L^1(D) \) be such that sup \( \|u_n\|_{L^1(D)} < +\infty \). We say that \( u_n \) converges weakly to \( u \in L^1(D) \) in the biting sense and write \( u_n \overset{b}{\rightharpoonup} u \), if there exists a decreasing sequence \( E_j \subseteq D \) of measurable sets such that \( |E_j| \to 0 \) and \( u_n \rightharpoonup u \) in \( L^1(D \setminus E_j) \) for all \( j \in \mathbb{N} \).

**Remark 13.** Note that if \( u_n \overset{b}{\rightharpoonup} u \) and \( u_n \rightharpoonup v \) a.e., then \( u = v \). This is a consequence of the uniqueness of the biting limit and equiintegrability of \( L^1 \)-weakly convergent sequences.
Proof of Corollary 1. Existence of minimizers for fixed $\varepsilon$ follows from $L^\infty$-coercivity of the fidelity term in $F_{\varepsilon,g}(\omega)$ and the lower semicontinuity assumption \([2.15]\). The remaining statement is the fundamental property of $\Gamma$-convergence except that we have to prove compactness in $L^q(D,\mathbb{R}^m)$. As shown for the lower bound in Theorem \([3.2]\) any sequence $u_\varepsilon$ as in the statement is bounded in $L^q(D,\mathbb{R}^m)$ and therefore Lemma \([3.3]\) yields that, up to subsequences, $u_\varepsilon \to u$ in $L^1(D,\mathbb{R}^m)$ for some $u \in L^q(D,\mathbb{R}^m)$. Clearly $u_\varepsilon$ is a recovery sequence for this $u$, so that $F_g(\omega)(u) = \lim_{\varepsilon \to 0} F_{\varepsilon,g}(\omega)(u_\varepsilon)$.

Repeating the reasoning for \([4.6]\) we conclude from the above limit that
\[
\lim_{\varepsilon \to 0} \int_D c_\varepsilon(\omega)(z) |u_\varepsilon(z) - g_\varepsilon(\omega)(z)|^q \, dz = \int_D \gamma(\omega)|u(z) - g(z)|^q \, dz, \tag{4.8}
\]
where $c_\varepsilon(\omega)$ is defined in \([3.3]\). We now consider the nonnegative sequence $a_\varepsilon := |u_\varepsilon - g_\varepsilon(\omega)|^q$. By \([4.6]\) and the qualitative lower bound $c_\varepsilon(\omega)(z) \geq c$ it is bounded in $L^1(D)$. By the biting lemma (see \([31]\) Lemma 2.63) and Remark \([3]\) we find a subsequence (not relabeled) such that $a_\varepsilon \to |u - g|^q$. Taking the same sets $E_j$ as for the biting convergence of $a_\varepsilon$ we can prove that the product $c_\varepsilon(\omega)a_\varepsilon$ converges in the biting sense to $\gamma(\omega)|u - g|^q$. Indeed, on $D \setminus E_j$ the sequence $a_\varepsilon$ is equiintegrable by the Dunford-Pettis theorem and thus strongly convergent in $L^1(D \setminus E_j)$. Then by the usual product rules we obtain $c_\varepsilon(\omega)a_\varepsilon \to \gamma(\omega)|u - g|^q$ in $L^1(D \setminus E_j)$. Now we use that $c_\varepsilon(\omega)a_\varepsilon$ is nonnegative. By \([4.6]\) and \([31]\) Proposition 2.67 this yields that $c_\varepsilon(\omega)a_\varepsilon \to \gamma(\omega)|u - g|^q$ even in $L^1(D)$. In particular the sequence $c_\varepsilon(\omega)a_\varepsilon$ and thus also $a_\varepsilon$ is equiintegrable on $D$. By Vitali’s convergence theorem we obtain that
\[
\lim_{\varepsilon \to 0} \|u_\varepsilon - g_\varepsilon\|_{L^q(D)} = \|u - g\|_{L^q(D)},
\]
which implies by uniform convexity of $L^q(D,\mathbb{R}^m)$ and \([4.2]\) that $u_\varepsilon \to u$ in $L^q(D,\mathbb{R}^m)$ as claimed. \(\square\)

4.2. The $\Gamma$-limit for forward differences on $\mathbb{Z}^2$. After we established rather abstract results, we now use them to analyze the asymptotic behavior of the discrete functional proposed in \([40]\). For this functional we take the deterministic square lattice $\mathcal{L} = \mathbb{Z}^2$ and consider the edges which yield standard forward differences, that is $\mathcal{E}_{FD} = \{(x, x + e_i) : x \in \mathbb{Z}^2, i = 1, 2\}$, where $e_i$ denotes the standard basis in $\mathbb{R}^2$. The discrete approximation is then defined for functions $u : \varepsilon\mathbb{Z}^2 \to \mathbb{R}^m$ and after rescaling it reads as
\[
\tilde{F}(u, A) = \sum_{\varepsilon x \in \varepsilon \mathbb{Z}^2 \cap A} \varepsilon \min\{\alpha \varepsilon^{-1}[(u(\varepsilon x + \varepsilon e_1) - u(\varepsilon x))^2 + |u(\varepsilon x + \varepsilon e_2) - u(\varepsilon x)|^2], 1\}. \tag{4.9}
\]
Clearly the set of edges satisfies \([2.3]\) as the nearest neighbors with respect to the Voronoi tessellation are given by all $x, y \in \mathbb{Z}^2$ such that $|x - y| = 1$. Moreover $\tilde{F}$ has the required structure which can be seen by setting $f(p) = \min\{\alpha\|p\|_1, 1\}$. In \([40]\) Cremers and Strelkalovich conjectured that $\tilde{F}$ approximates the Mumford-Shah functional. With the results proven in the previous sections we can identify the $\Gamma$-limit of $\tilde{F}$, which differs from the Mumford-Shah functional due to an anisotropic surface integral.

Corollary 2. The functionals $\tilde{F}_\varepsilon$ defined in \([4.9]\) $\Gamma$-converge with respect to the $L^1(D,\mathbb{R}^m)$-topology to the functional $F : L^1(D,\mathbb{R}^m) \cap GSBV^2(D,\mathbb{R}^m)$, where it is given by
\[
F(u) = \alpha \int_D |\nabla u|^2 + \int_{S_u} g_0(\nu) \, d\mathcal{H}^1,
\]
with the function $g_0 : \mathbb{R}^2 \to [0, +\infty)$ defined by
\[
g_0(\nu) = \begin{cases} |\nu_1| + |\nu_2| & \text{if } \nu_1 \cdot \nu_2 < 0, \\ \max\{|\nu_1|, |\nu_2|\} & \text{if } \nu_1 \cdot \nu_2 \geq 0. \end{cases}
\]

Proof. As outlined above we can apply Theorem \([1.1]\) to the sequence $\tilde{F}_\varepsilon$. Moreover, from Proposition \([4]\) we deduce that the function $h$ in Theorem \([1.1]\) is given by the density of the $\Gamma$-limit of the sequence $E_\varepsilon$
defined in (3.34). With our choice of \( f \) and \( E_{FD} \) the functional \( E_\varepsilon \) reads as

\[
E_\varepsilon(u) = \alpha \sum_{\varepsilon x, \varepsilon y \in \mathbb{Z}^2 \cap D \atop |x-y|=1} \frac{\varepsilon^2}{2} \left| \frac{u(\varepsilon x) - u(\varepsilon y)}{\varepsilon} \right|^2.
\]

By [3, Remark 5.3] in this case it holds that

\[
\Gamma(L^p(D)) - \lim_{\varepsilon \to 0} E_\varepsilon(u) = \alpha \int_D |\nabla u|^2 \, dx
\]

for all \( u \in W^{1,2}(D, \mathbb{R}^m) \), so that \( \tilde{h}(\xi) = \alpha |\xi|^2 \) and it remains to identify the surface integrand \( g \) to be \( g_0 \).

By convexity, the function \( g \) is continuous, so it suffices to treat the case \( \nu_1 \cdot \nu_2 \neq 0 \). As a preliminary step note that the functional \( I_\varepsilon \) defined in (3.31) here takes the form

\[
I_\varepsilon(u, A) = \frac{1}{2} \sum_{\varepsilon \in \mathbb{Z}^2 \cap A} \varepsilon \max\{|u(\varepsilon x + \varepsilon e_1) - u(\varepsilon y)| : i \in \{1, 2\} \text{ with } \varepsilon x + \varepsilon e_i \in A\},
\]

where \( u : \varepsilon \mathbb{Z}^2 \to \{\pm 1\} \).

By (4.1.1) we know that

\[
g(\nu) = \lim_{\varepsilon \to 0} \inf \left\{ I_\varepsilon(v, Q_\nu(0, 1)) : v \in S_{\varepsilon, 4\varepsilon}(u_{0, \nu}^{-1, 1}, Q_\nu(0, 1)) \right\}, \tag{4.10}
\]

Consider for fixed \( \varepsilon << 1 \) any function \( v \in S_{\varepsilon, 4\varepsilon}(u_{0, \nu}^{-1, 1}, Q_\nu(0, 1)) \). We locally construct a function \( \hat{v} \in BV(Q_\nu(0, 1), \{\pm 1\}) \) as follows: On \( Q_{\varepsilon_1}(\varepsilon x, \varepsilon) \) with \( x \in \mathbb{Z}^2 \) such that \( Q_{\varepsilon_1}(\varepsilon x, \varepsilon) \subset Q_\nu(0, t) \) we set

\[
\hat{v}(y) = \begin{cases} 
  v(\varepsilon x) & \text{if } \prod_{i=1}^{2} |v(\varepsilon x + \varepsilon e_i) - v(\varepsilon x)| = 0, \\
  v(\varepsilon x) & \text{if } \prod_{i=1}^{2} |v(\varepsilon x + \varepsilon e_i) - v(\varepsilon x)| \neq 0 \text{ and } \langle y - \varepsilon x, \varepsilon_1 + e_2 \rangle \leq 0, \\
  v(\varepsilon x + \varepsilon e_1) & \text{if } \prod_{i=1}^{2} |v(\varepsilon x + \varepsilon e_i) - v(\varepsilon x)| \neq 0 \text{ and } \langle y - \varepsilon x, \varepsilon_1 + e_2 \rangle > 0,
\end{cases}
\]

while we define \( \hat{v} = u_{0, \nu}^{-1, 1} \) on all cubes \( Q_{\varepsilon_2}(\varepsilon x, \varepsilon) \) with \( Q_{\varepsilon_2}(\varepsilon x, \varepsilon) \cap \partial Q_\nu(0, 1) \neq \emptyset \). The latter implies \( \hat{v} = u_{0, \nu}^{-1, 1} \) on \( \partial Q_\nu(0, 1) \) in the sense of traces. Moreover we modified the jump set away from the boundary in such a way that it contains diagonal lines of length \( \sqrt{2} \) instead of corners formed by the upper and the right hand side of a cube. Setting \( Q_\varepsilon := \{ y \in Q_\nu(0, 1) : \text{dist}(y, \partial Q_\nu(0, 1)) > 2\varepsilon \} \), the above construction and the boundary conditions on \( v \) imply

\[
I_\varepsilon(v, Q_\nu(0, 1)) \geq \int_{S_{\varepsilon, \nu} \cap Q_\nu} g_0(\nu_\varepsilon) \, d\mathcal{H}^1 \geq \int_{S_{\varepsilon, \nu} \cap Q_\nu(0, 1)} \! g_0(\nu_\varepsilon) \, d\mathcal{H}^1 - C\varepsilon. \tag{4.11}
\]

Observe that \( g_0 \) is convex. Hence the functional on the left hand side is \( BV \)-elliptic in the sense that

\[
\int_{S_{\varepsilon, \nu} \cap Q_\nu(0, 1)} g_0(\nu_\varepsilon) \, d\mathcal{H}^1 \geq g(\nu)
\]

for all \( \hat{v} \in BV(Q_\nu(0, 1), \{\pm 1\}) \) such that \( \hat{v} = u_{0, \nu}^{-1, 1} \) on \( \partial Q_\nu(0, 1) \) in the sense of traces (see [8] for more details). Since \( v \in S_{\varepsilon, 4\varepsilon}(u_{0, \nu}^{-1, 1}, Q_\nu(0, 1)) \) was arbitrary, we conclude from (4.10) and (4.11) that \( g(\nu) \geq g_0(\nu) \).

In order to prove an upper bound, first note that

\[
I_\varepsilon(u, A) \leq \frac{1}{4} \sum_{\varepsilon x, \varepsilon y \in \mathbb{Z}^2 \cap A \atop |x-y|=1} \varepsilon |u(\varepsilon x) - u(\varepsilon y)|. \tag{4.12}
\]

The \( \Gamma(L^1(D)) \)-limit of the right hand side of (4.12) is well-known. It is finite only on \( BV(D, \{\pm 1\}) \) and takes the form \( \int_{S_{\varepsilon, \nu}} |\nu| \, d\mathcal{H}^1 \) (see [2]). By comparison we obtain \( g(\nu) \leq |\nu_1| + |\nu_2| \). This finishes the proof in the case \( \nu_1 \cdot \nu_2 < 0 \). If \( \nu_1 \cdot \nu_2 > 0 \) denote by \( i_0 \) the index such that \( |\nu_{i_0}| = \max\{|\nu_1|, |\nu_2|\} \) and set \( i_1 = \{1, 2\} \setminus \{i_0\} \). We define a candidate for the minimum problem in (4.10) setting \( u_{\nu}(\varepsilon x) = u_{0, \nu_1}^{-1, 1}(\varepsilon x) \) for all \( x \in \mathbb{Z}^2 \). By definition it satisfies the correct boundary conditions. A straightforward analysis shows...
that for any \( x \in \mathbb{Z}^2 \) with \( u_\nu(\varepsilon x) \neq u_\nu(\varepsilon x + \varepsilon e_{i_0}) \) we have \( u_\nu(\varepsilon x) \neq u_\nu(\varepsilon x + \varepsilon e_{i_0}) \). Thus it suffices to count just the interactions along the direction \( e_{i_0} \). Those can be bounded by \( \varepsilon^{-1} |\nu_{i_0}| + C \), so that

\[
I_\varepsilon(u_\nu, Q_\nu(0,1)) \leq |\nu_{i_0}| + C \varepsilon = \max\{|\nu_1|,|\nu_2|\} + C \varepsilon.
\]

From \([4,10]\) we conclude that \( g(\nu) \leq g_0(\nu) \) which finishes the proof.

**Remark 14.** For the \( d \)-dimensional version of \( \tilde{F}_\varepsilon \) (defined in the introduction), one still has the existence of the \( \Gamma \)-limit with an anisotropic surface integrand. To see the latter, one first shows that \( g(\varepsilon 1) = 1 \). Then for the vector \( \nu_0 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \) the discretization of \( u_{\nu_0}^{-1,1} \) yields an upper bound \( g(\nu_0) \leq \frac{1}{\sqrt{2}} \) (actually equality holds). The precise \( \Gamma \)-limit in higher dimensions is beyond the scope of this paper.

### 4.3. From random parking to the Mumford-Shah functional

In this final part we use our general results to finally provide a discretization of the vector-valued Mumford-Shah functional. To this end, we need to take our parameters \( p = q = 2 \). However, it may be of interest to obtain also other exponents for the fidelity term and therefore we consider general \( q > 1 \) and just fix \( p = 2 \) and focus on the isotropy issue. To this end, we suggest to take as stochastic lattice the so-called random parking process. For the precise geometric construction of this process by suitably choosing projected points of a homogeneous Poisson point process in dimension \( d + 1 \), we refer the reader to \([34]\). Here we recall that the random parking process defines a stochastic lattice \( L_{RP} \) that is admissible, stationary, ergodic and, most important for our applications, isotropic in the sense of Definition \([2,3]\). Moreover, we can choose for instance \( \mathcal{E}(\omega) = \mathcal{N}(\omega) \) to obtain stationary and isotropic edges. We prove our result for general stochastic lattices satisfying these assumptions. Note that the following theorem covers in particular the two functionals presented in the introduction.

**Theorem 4.4.** Fix \( p = 2 \) and let \( q \in (1, +\infty) \) and \( g_2(\omega) \) satisfy \([4,2]\). Assume that \( L \) is an admissible stochastic lattice that is stationary, ergodic and isotropic with admissible stationary and isotropic edges. Then \( \mathbb{P} \)-a.s. the functionals \( F_{\varepsilon,\nu}(\omega) \) \( \Gamma \)-converge with respect to the \( L^1(D, \mathbb{R}^m) \)-topology to the functional \( F_g : L^1(D, \mathbb{R}^m) \to [0, +\infty] \) with domain \( L^q(D, \mathbb{R}^m) \cap \text{GSBV}^2(D, \mathbb{R}^m) \), where it is defined by

\[
F_g(u) = c_1 \int_D |\nabla u|^2 \, dx + c_2 H^{d-1}(S_u) + c_3 \int_D |u - g|^q \, dx
\]

for some \( c_1, c_2, c_3 > 0 \).

**Proof.** Due to ergodicity and Theorem \([4,2]\) it only remains to show that \( h(\xi) = c_1 |\xi|^2 \) and \( g(\nu) = c_2 \) for some constants \( c_1, c_2 > 0 \). Observe that the function \( h \) is also the density of the \( \Gamma \)-limit of the functionals \( F_{\varepsilon}(\omega) \) defined in \([5,34]\). For \( p = 2 \) these are non-negative quadratic forms, so we deduce from \([26, Theorem 11.1]\) that \( h \) is a non-negative quadratic form, too. We write it explicitly as

\[
h(\xi) = \sum_{i,k=1}^m \sum_{j,l=1}^d h_{ijkl} \xi_i \xi_k \xi_j \xi_l,
\]

where the coefficients satisfy the symmetry condition \( h_{ijkl} = h_{klij} \). Since the discrete functional is invariant under orthogonal transformations \( u \mapsto Qu \) it holds that \( h(Q\xi) = h(\xi) \) for all \( \xi \in \mathbb{R}^{m \times d} \) and \( Q \in O(m) \). Moreover, reasoning exactly as for the case \( m = d \) treated in \([4, Theorem 9]\) one can further show that ergodicity and isotropy imply \( h(\xi R) = h(\xi) \) for all \( \xi \in \mathbb{R}^{m \times d} \) and all \( R \in SO(d) \). We argue that \( h \) depends only on the singular values. To this end, we consider a generic matrix \( \xi \in \mathbb{R}^{m \times d} \) and any singular value decomposition \( \xi = Q\Sigma V^T \) with orthogonal matrices \( Q \in O(m) \) and \( V \in O(d) \) and a diagonal matrix \( \Sigma \in \mathbb{R}^{m \times d} \). If \( V \in SO(d) \) then \( h(\xi) = h(\Sigma) \). Otherwise we replace \( V \) by a rotation observing that

\[
A = Q P_m^{1,2} P_d^{1,2} \Sigma P_d^{1,2} P_m^{1,2} V^T,
\]

where we denote by \( P_n^{i,j} \) the \( n \times n \)-matrix which differs from the identity by exchanging the \( i \)th and the \( j \)th column. In this case \( h(\xi) = h(P_m^{1,2} \Sigma P_d^{1,2}) \) since the matrix \( P_d^{1,2} V^T \) belongs to \( SO(d) \). Set \( l = \min\{d, m\} \)
and write the singular values as $\lambda(\xi) \in \mathbb{R}^l$ with non-negative coefficients. We conclude that there exists a permutation $P(\xi) \in \{I, P_1^{1,2}\}$ such that
\[
h(\xi) = \sum_{i,k=1}^l h_{iikk}(P(\xi)\lambda(\xi))_i(P(\xi)\lambda(\xi))_k.
\]
Thus it is enough to characterize the coefficients $h_{iikk}$. We now test several diagonal matrices $\xi$. To simplify notation, given $v \in \mathbb{R}^l$ we denote by diag$(v) \in \mathbb{R}^{m \times d}$ the diagonal matrix with elements $v$. As a first step, note that we can find $Q \in O(m)$ and $R \in SO(d)$ such that diag$(e_1) = Q \text{diag}(-e_j) R$. Thus by invariance $h_{i111} = h_{jjjj}$ for all $i,j = 1, \ldots, l$. Now consider $i \neq k$. We argue that $h_{iikk} = 0$. To this end we use that there exists a matrix $Q \in O(m)$ such that diag$(e_i + e_k) = Q \text{diag}(e_i - e_k)$, which yields again by invariance that
\[
h_{i111} + h_{kkkk} + h_{ikk1} + h_{kk1k} - h_{1ikk} - h_{kki1}.
\]
By the symmetry condition on the coefficients of $h$ we obtain $h_{iikk} = 0$. Setting $c_1 = h_{1111}$ we conclude that
\[
h(\xi) = c_1 \sum_{i=1}^l \lambda_i(\xi)^2 = c_1|\xi|^2.
\]
It follows from Lemma 3.3 that $c_1 > 0$.

We now turn to the surface integrand $g$ and prove that $g(R\nu) = g(\nu)$ for all $R \in SO(d)$. Since $g$ is deterministic by ergodicity, we can take expectations in the asymptotic formula (4.1). Since $\tau_R^t$ is measure preserving, dominated convergence and a change of variables yield
\[
g(R\nu) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\Omega} \inf \{ F_\epsilon(\omega)(v, Q_{R\nu}(0, t)) : u \in S_{\epsilon,2M\epsilon}(u_{0,R\nu}(0, t)) \} \, d\mathbb{P}(\omega)
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\Omega} \inf \{ F_\epsilon(\tau_R^t \omega)(v, Q_{\nu}(0, t)) : v \in S_{\epsilon,2M\epsilon}(u_{0,\nu}(0, t)) \} \, d\mathbb{P}(\omega)
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\Omega} \inf \{ F_\epsilon(\omega')(v, tQ_{\nu}(0, t)) : v \in S_{\epsilon,2M\epsilon}(u_{0,\nu}(0, t)) \} \, d\mathbb{P}(\omega') = g(\nu),
\]
where we used from the first to the second line that by isotropy of $\mathcal{L}$ and $\mathcal{E}$ the discrete functional satisfies $F_\epsilon(\tau_R \omega)(u, Q_{R\nu}(0, t)) = F_\epsilon(\omega)(u \circ R, Q_{\nu}(0, t))$ for every $R \in SO(d)$ and every $t > 0$. We finish the proof setting $c_2 = g(\epsilon_1)$ since Lemma 3.3 implies $c_2 > 0$.

Remark 15. In the scalar case $m = 1$, the statement of Theorem 4.4 holds for every $p > 1$. Indeed, by [26, Proposition 11.6] the function $h$ has to be $p$-homogeneous and by stochastic isotropy it is constant on $S^{p-1}$. Hence $h(\xi) = c_1|\xi|^p$ for some $c_1 > 0$.

Combining Theorem 4.4 and Corollary 1, we obtain the discretization of the Mumford-Shah functional along with the convergence of minimizers. In practice it is of course impossible to create the stochastic lattice on the whole space but one has to take a finite particle approximation. Moreover the minimization of the discrete functionals $F_\epsilon(\omega)$ is still nontrivial due to non-convexity. However first numerical tests have shown promising results and we will further investigate our approach.

**Appendix A.**

In this appendix we provide a suitable framework to perform diagonal arguments along $\Gamma$-converging discrete energies. We need this step since the general theory [26, Chapter 10] to construct a metric for $\Gamma$-convergence requires an $L^p$-coercive lower bound for the discrete energies up to the boundary.

Given any function $G : L^p(B_1, \mathbb{R}^m) \to [0, +\infty]$ not identically $+\infty$ we define its Moreau-Yosida approximation for $\gamma > 0$ as
\[
G^\gamma(u) = \inf_{v \in \text{Lip}(B_1, \mathbb{R}^m)} \left( G(v) + \gamma \|u - v\|^p_{L^p(B_1)} \right).
\]
Since \( p > 1 \), the functional \( G^\gamma \) is locally Lipschitz-continuous on \( L^p(B_1, \mathbb{R}^m) \) (see \cite[Theorem 9.15]{26}). Let \( \{w_k\}_{k \in \mathbb{N}} \) be a dense subset of \( L^p(B_1, \mathbb{R}^m) \) containing 0. Given two lower semicontinuous functions \( G, H : L^p(B_1, \mathbb{R}^m) \to [0, +\infty) \) not identically \( +\infty \) we define their distance setting

\[
\vartheta(G, H) = \sum_{i, k \in \mathbb{N}} \frac{1}{g_{i, k}} |\arctan(G^i(w_k)) - \arctan(H^i(w_k))|.
\]

Note that on lower semicontinuous functions \( \vartheta \) is indeed a distance, since \( \vartheta(G, H) = 0 \) implies by local Lipschitz continuity that \( G^i = H^i \) for all \( i \in \mathbb{N} \). Letting \( i \to +\infty \) it follows by lower semicontinuity that \( G = H \) (see \cite[Remark 9.11]{26}). In order to state our result we need further notation: Let \( g : B_1 \times \mathbb{R}^{m \times d} \to [0, +\infty) \) be a Carathéodory-function such that \( \xi \mapsto g(x, \xi) \) is quasiconvex for a.e. \( x \in B_1 \) and

\[
\frac{1}{\gamma} |\xi|^\gamma - C \leq g(x, \xi) \leq C(|\xi|^\gamma + 1).
\]

Define the functional \( E^\gamma_g : L^p(B_1, \mathbb{R}^m) \to [0, +\infty] \) by setting

\[
E^\gamma_g(u) = \begin{cases}
\int_{B_1} g(x, \nabla u(x)) \, dx & \text{if } u \in W^{1,p}(B_1, \mathbb{R}^m), \\
+\infty & \text{otherwise}.
\end{cases}
\]

Then we have the following result.

**Lemma A.1.** Consider a sequence \( \varepsilon_j \to 0 \) and let \( x_0 \in D \) and \( \rho_j > 0 \) be such that \( B_{\rho_j}(x_0) \subset D \). Let further \( g : B_1 \times \mathbb{R}^{m \times d} \to [0, +\infty) \) be a Carathéodory-function as above. Define \( G_{\varepsilon_j, \rho_j}(x_0, \omega) \) as in Lemma 8.11 Then the following are equivalent:

(i) \( \Gamma(L^p(B_1, \mathbb{R}^m)) \cup \text{lim}_j G_{\varepsilon_j, \rho_j}(x_0, \omega) = E^\gamma_g \),

(ii) \( \text{lim}_j \vartheta(G_{\varepsilon_j, \rho_j}(x_0, \omega), E^\gamma_g) = 0 \).

**Proof.** (ii) \( \Rightarrow \) (i): First note that both functionals are lower semicontinuous on \( L^p(B_1, \mathbb{R}^m) \) and not identically \( +\infty \). Assumption (ii) implies that

\[
\lim_j G_{\varepsilon_j, \rho_j}(x_0, \omega)(w_k) = E^\gamma_g(w_k)
\]

for all \( i, k \in \mathbb{N} \). Since \( 0 \in \{w_k\} \) and \( G_{\varepsilon_j, \rho_j}(x_0, \omega)(0) = 0 \), we deduce from \cite[Theorem 9.15]{26} that the sequence \( G_{\varepsilon_j, \rho_j}(x_0, \omega) \) is locally equicontinuous, so that the convergence extends to \( L^p(B_1, \mathbb{R}^m) \). The claim then follows from a general characterization of \( \Gamma \)-convergence (see, for example, \cite[Theorem 9.5]{26}) when we let \( j \to +\infty \). (i) \( \Rightarrow \) (ii): Clearly \( G_{\varepsilon_j, \rho_j}(x_0, \omega)(u) \leq \gamma \| u \|_{L^p(B_1)} \), so that given a sequence \( u_j \) such that

\[
G_{\varepsilon_j, \rho_j}(x_0, \omega)(u_j) + \gamma \| u_j - u \|_{L^p(B_1)} \leq G_{\varepsilon_j, \rho_j}(x_0, \omega)(u) + \frac{1}{j}
\]

we conclude by Remark 8 that there exists \( v \in L^p(B_1, \mathbb{R}^m) \) such that, up to subsequences, \( u_j \to v \) in \( L^1(B_1, \mathbb{R}^m) \) and additionally \( u_j \to v \) in \( L^p(B_1, \mathbb{R}^m) \). From Remark 9 and weak lower semicontinuity of the \( L^p \)-norm we infer

\[
E^\gamma_g(v) + \gamma \| v - u \|_{L^p(B_1)}^p \leq \liminf_j \sup G_{\varepsilon_j, \rho_j}(x_0, \omega)(u_j).
\]

On the other hand, given any \( \tilde{u} \in L^p(B_1, \mathbb{R}^m) \) let us consider a sequence \( \tilde{u}_j \to \tilde{u} \) in \( L^p(B_1, \mathbb{R}^m) \) such that \( \sup_j G_{\varepsilon_j, \rho_j}(x_0, \omega)(\tilde{u}_j) \leq E^\gamma_g(\tilde{u}) \). Then by the definition of the Moreau-Yosida transformation

\[
\limsup_j G_{\varepsilon_j, \rho_j}(x_0, \omega)(u) \leq \limsup_j \sup G_{\varepsilon_j, \rho_j}(x_0, \omega)(\tilde{u}_j) + \gamma \| \tilde{u}_j - u \|_{L^p(B_1)}^p \leq E^\gamma_g(\tilde{u}) + \gamma \| \tilde{u} - u \|_{L^p(B_1)}^p.
\]

Combined with the previous inequality we obtain that \( E^\gamma_g(v) = E^\gamma_g(v) + \gamma \| v - u \|_{L^p(B_1)}^p \), and by setting \( \tilde{u} = v \) we showed that

\[
\lim_j G_{\varepsilon_j, \rho_j}(x_0, \omega)(u) = E^\gamma_g(u)
\]

for all \( u \in L^p(B_1, \mathbb{R}^m) \) and all \( \gamma > 0 \). This property implies (ii) by the definition of the metric. \( \square \)
Remark 16. The equivalence of Lemma A.1 remains valid if we consider two functionals $\mathcal{E}_g$ and $\mathcal{E}_j$ provided the Carathéodory functions $g_j$, $g$ satisfy the growth conditions uniformly in $j$. In this case the proof is even simpler since the Moreau-Yosida transformations are equicoercive in $L^p(B_1, \mathbb{R}^m)$ due to the Sobolev embedding.

Acknowledgement. MR acknowledges financial support from the European Research Council under the European Community’s Seventh Framework Programme (FP7/2014-2019 Grant Agreement QUANTOM 335410).

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