Spacetime deployments parametrized by gravitational and electromagnetic fields

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Abstract. We consider spacetimes with measurements of conformally invariant physical properties. Then, applying the Pfaff theory for PDE to a particular conformally equivariant system of differential equations, we make explicit the dependence of any kind of function describing a “spacetime deployment”, on \( n(n+1) \) parametrizing functions, denoting by \( n \) the spacetime dimension. These functions, appearing in a linear differential Spencer sequence can be consistently ascribed to unified electromagnetic and gravitational fields, at any spacetime dimensions \( n \geq 4 \).

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1. Introduction: spacetime and navigation

1.1. Substrat and unfolded spacetimes

In the present article, most part of the matter is based on a particular type of gravity known under the spell of “conformal gravity”. Actually we will adopt throughout a specific Equivalence Principle which has recently been formulated by M. Ghins and T. Budden [13], and from which we think that the very conformal aspect of our approach proceeds. This so-called Punctual Equivalence Principle (PEP) can be stated as follows [13] p. 44 but with the notation $p_0$ instead of $p$:

**Punctual Equivalence Principle (PEP):** for all $p_0 \in \mathcal{M}$, special relativity holds at $p_0$ “in the restricted sense”.

This is a local definition on a spacetime manifold $\mathcal{M}$, and the “in the restricted sense” should be given the meaning [13] p. 43:

**Special relativity holds at $p_0$:** there exists a local chart $x^\mu$ of a neighbourhood of $p_0$ such that the fundamental dynamical and curvature-free special relativistic laws hold in their standard vectorial form in $x^\mu$ at $p_0$.

The coordinate maps $x^\mu$ ($\mu = 1, \ldots, \dim \mathcal{M}$) are local charts defined on a neighbourhood $U(p_0)$ of a given point $p_0$ in $\mathcal{M}$.

Indeed, a peculiar type of charts is selected, made out of those charts which, to a given $U(p_0)$, associate a neighbourhood of the origin of the vectorial space, $T_{p_0}\mathcal{M}$, tangent to $\mathcal{M}$ at $p_0$. This local, punctual approach can certainly be intuitively motivated different ways. Let us just mention that it seems intimately related to some of the practical aspects met in the situations of satellites navigation such as the GPS or GALILEO systems. In effect, each of the satellites belonging to a GPS constellation, so to say, realizes somehow this kind of a local punctual equivalence by registering other satellites own ephemeris and calendar data, as well as proper times given by embarked atomic clocks. In this respect, one can say that spatio-temporal charts are achieved “on board”, on some kind of “table of the charts” $T_{p_0}\mathcal{M}$, as one would say with the help of a navigation terminology broadly used in the GPS technologies. From the mathematical standpoint, it will therefore be postulated that these “on board” or “table of the charts” aspects, are those of a running vectorial tangent spacetime $T_{p_0}\mathcal{M}$.

One has indeed to realize a set of charts of the projective space defined on, and associated to each moving tangent spacetime. That is, in particular, and at a given fixed proper time, charts of the celestial sphere attached to a given observer at $p_0$. This, in turn, amounts to recognize that the charts are implicitly associated to a conformal geometry.

In order to stress the fundamental deployment aspect of our approach, the following terminology will be used throughout:

- $T_{p_0}\mathcal{M}$, the spacetime tangent to $\mathcal{M}$ at $p_0$, will be referred to as the underlying or substrat spacetime $\mathcal{S}$. 

• From this substrat spacetime, the unfolded spacetime $\mathcal{M}$ will be envisaged and referred, by means of the Punctual Equivalence Principle at $p_0 \in \mathcal{M}$.

It may be worth stressing that these moving tangent spacetimes, $T_{p_0}\mathcal{M}$, are those spaces in which the conformal physical measures will be achieved some way or other, with the help of rods, compass, clocks, recorders, etc. Eventually, all of the foregoing considerations may be adapted to a variety of pretty different situations as the most essential aspect will be the one of a spacetime manifold deployment ($\mathcal{M}$) from a substrat spacetime ($\mathcal{S}$). We build up the mathematical formalism corresponding to the deployment of a conformal geometric structure from an isometrical one.

1.2. To tie a spacetime ship with its environnement: the principle of equivalence

We assume the unfolded spacetime $\mathcal{M}$ to be of class $C^\infty$, of dimension $n \geq 4$ and locally connected. Let $p_0$ be a particular point in $\mathcal{M}$, $U(p_0)$ an open neighborhood of $p_0$ in $\mathcal{M}$, and $T_{p_0}\mathcal{M}$ its tangent space. The so-called “punctual” principle of equivalence we will be relying on (compatible with the usual ones from M. Ghins view; Private communication), states that it exists a local diffeomorphism, $\varphi_{p_0}$, we call the “equivalence map”, attached to $p_0$, and putting in a one-to-one correspondence the points $p \in U(p_0)$ with some vectors $\xi \in T_{p_0}\mathcal{M}$ in an open neighborhood of the origin of $T_{p_0}\mathcal{M}$:

$$\varphi_{p_0} : p \in U(p_0) \subset \mathcal{M} \rightarrow \xi \in T_{p_0}\mathcal{M}, \quad \varphi_{p_0}(p_0) = 0.$$  

We will see that this description is really well-suited since it is also strongly related to the mathematical tools to be used in the sequel, where any given point $p_0$ acquires a specific mathematical status.

Let us add that a more standard local equivalence principle would consist in considering $\mathcal{S}$ an Euclidean space $\mathbb{R}^n$ and the equivalence map $\varphi$ as a local chart of an atlas of $\mathcal{M}$ on an open neighborhood $U$, such that:

$$\varphi : p \in U \subset \mathcal{M} \rightarrow \xi \in \mathbb{R}^n \simeq \mathcal{S}.$$ 

Moreover, we assume that each of these two kinds of spacetimes is endowed with a metric field, denoted by $g$ for $\mathcal{M}$, and by $\omega$ with signature $(+, -, -, -)$ for $\mathcal{S}$.

Eventually, we make the general assumption that $\mathcal{S}$ has a constant Riemannian scalar curvature with value $n(n-1)k_0$ ($k_0 \in \mathbb{R}$), and then is “conformally flat”, i.e., the Weyl tensor is vanishing. This assumption, in view of the H. Weyl theorem [39], will ensure the integrability of the conformal Lie pseudogroup associated to $\omega$, and denoted here by $\Gamma_{\hat{G}}$.

Then if $\hat{f} \in \Gamma_{\hat{G}}$, $\hat{f}$ is a solution of the PDE system:

$$\hat{f}^*(\omega) = e^{2\alpha} \omega,$$  

with $\det(J(\hat{f})) \neq 0$, $J(\hat{f})$ the Jacobian of $\hat{f}$, and $\hat{f}^*$ its pull-back. Also, $\alpha$ is a function associated to, and varying with each $\hat{f}$. As a particular case, the set of diffeomorphisms $\hat{f}$ for which $\alpha = 0$ constitutes the so-called “Poincaré Lie pseudogroup $\Gamma_G$” or, equivalently,
the Lie pseudogroup of isometries. We denote by \( f \) the elements of \( \Gamma_G \). Contrarily to appearances, an element \( \hat{f} \) can’t be defined in a somewhat one-to-one correspondence, out of a given element \( f \) and a given function \( \alpha \). This can be obtained only if the metric field \( \omega \) on \( S \) satisfies a particular condition, that we will call “S-admissibility”, which will be defined and precised in the sequel. Moreover, if the latter condition is satisfied, then only the elements of a proper Lie sub-pseudogroup of \( \Gamma_G \), we denote by \( \Gamma_{\hat{H}} \), can admit such a decomposition, as will be demonstrated at the exterior differential forms level (see Theorem 3 below). This S-admissibility condition allows us to define a deployment in the sense of a deformation from the \( \Gamma_G \) pseudogroup to the \( \Gamma_{\hat{G}} \) one.

In fact, since both spacetimes \( \mathcal{M} \) and \( \mathcal{S} \) are locally diffeomorphic to \( \mathbb{R}^n \), the above alluded deployment relates the geometrical structures, i.e., their metric fields. Then we consider that the metric field \( \nu \equiv \phi_{p_0}^{-1}(g) \) on \( T_{p_0}\mathcal{M} \), is a deployment or a deformation of \( \omega \). The deployment or deformation terminology will accordingly be used in either cases of \( \mathcal{M} \) and \( \mathcal{S} \), or \( \nu \) and \( \omega \).

We have to focus on the fact that though spacetimes \( \mathcal{M} \) and \( \mathcal{S} \) are diffeomorphic in view of the equivalence map, their two respective metric fields are not, i.e., \( g \) is not a pull-back of \( \omega \). Nevertheless these two metric fields will be conformally equivariant. In some analogy with the decomposition for \( \hat{f} \), the metric field \( \nu \) will be described in terms of \( \omega \) and some other fields which will reveal to be thinkable in terms of unified electromagnetic and gravitational fields. In fact, the deformations in \( \Gamma_{\hat{H}} \) of the applications \( \hat{f} \in \Gamma_{\hat{G}} \) will define tetrads used to obtain \( \nu \) out of \( \omega \). Hence, as a result, the equivalence map will be also parametrized by electromagnetic and gravitational potentials.

This classical approach based on deformation theory, differs from the classical gauge one in general relativity [19]. Indeed the latter are developed out of a given gauge Lie group. But at first, they are not Lie pseudogroups, and in a second place, they are associated to Lie groups invariance of the tangent spaces (not the tangent fiber bundle but the fibers) at any fixed base point \( p_0 \). They can accordingly be regarded as isotropy Lie subgroups of the corresponding pseudogroups. For instance, in fixing the function \( \alpha \) to a constant, the set of applications \( \hat{f} \) becomes a Lie group and not a Lie pseudogroup. In that case the applications \( \hat{f} \) would depend on 15 real variables at \( n = 4 \), and no longer on a set of arbitrary functions, as we will be seen in the conformal pseudogroup case.

Also it is neither a Kaluza-Klein type theory nor is it based on a Riemann-Cartan geometry. In the present model, there is no torsion. It also completly differs from the H. Weyl unifications and the J.-M. Souriau approach [34, 38, 28]. Close approaches to ours, are developed on the one hand by M. O. Katanaev & I. V. Volovich [20], and on the other hand by H. Kleinert [21] and J.-F. Pommaret [32]. At lower dimensions, other general relativity models are investigated within similar approaches such as models of gravity in [8, 14, 15] for instance. In fact, our present work can somehow be viewed as an extension of the T. Fulton et al. approach and model [10], or, as a continuation of the original works of J. Haantjes [18].
Here below, we summarize the mathematical procedure and assumptions presented in the sequel and based on the previous discussion (we refer to definitions of involution i.e. integrability, symbols of differential equations, acyclicity and formal integrability such as those given in [6, 12, 16, 32, 35] for instance):

- The metric field $\omega$ is conformally equivariant.
- The Riemann scalar curvature $\rho_s$ associated to the metric field $\omega$, is assumed to be a constant, $n(n-1)k_0$, as a consequence of the constant Riemann curvature tensor assumption. And then, the Weyl tensor associated to $\omega$ is vanishing, i.e., we have a conformally flat structure.
- The system (1) of differential equations in $\hat{f}$, being non-integrable, will be supplied with an other system of equations, obtained from a prolongation procedure which will be stopped as the integrability conditions of the resulting complete system of partial differential equations is met.
- The covariant derivatives involved in the prolongation procedure will be assumed to be torsion free, i.e. we will make use of the Levi-Civita covariant derivatives.
- We will extract from the latter system of PDE, a subsystem, which will be called the “Generalized Haantjes-Schouten-Struik (GHSS) system” (see system [11, 12, 13] with $k_0 = 0$), defining completely the sub-pseudogroup $\Gamma_{\hat{H}}$ of those applications $\hat{f}$ which are strictly smooth deformations in $\Gamma_G$ of applications $f \in \Gamma_G$. This PDE subsystem will be satisfied by the functions $\alpha$. This is the core system of our model and to our knowledge it has never been really studied, or, at least, related to any unification model. The S-admissibility property appears at this step as a system of PDE satisfied by the metric field $\omega$; A system which strongly resembles an Einstein equation.
- By considering Taylor series solutions to the “GHSS system”, we will show how general solutions depend on a particular finite set of parametrizing functions.
- We show that this set of parametrizing functions is associated to a Spencer differential sequence [35], and that they can be identified with both electromagnetic and gravitational gauge potentials.
- We deduce the metric field $\nu \equiv \omega + \delta \omega$ of the infinitesimal smooth deformations, depending on the electromagnetic and gravitational gauge potentials, from which covariant derivatives would be deduced. The Newtonian limit will also be indicated as well as the meaning of the so-called “meshing assumption” [13] in the present context.

To finish, we indicate that the mathematical tools used for this unification finds its roots, first in the conformal Lie structure that has been extensively studied by H. Weyl [39], K. Yano [10], J. Gasqui [11], J. Gasqui & H. Goldschmidt [12] and J.-F. Pommaret [32]. Meanwhile, we only partially refer to some of these aspects since it mainly has to do with the general theory of Lie equations, and not exactly with the set of PDE we are concerned with. We essentially indicate, succintly, the cornerstones which are absolutely necessary for our explanations and descriptions of the present framework.
2. The conformal finite Lie equations of the substratum spacetime

First of all, and from the previous sections, we assume that the pseudogroup of relativity is no longer Poincaré but the conformal Lie pseudogroup. In particular, this means that no physical law changes occur, shifting from a given frame embedded in a gravitational field, to a uniformly accelerated relative isolated one [29, 30].

The conformal finite Lie equations are deduced from the conformal action on a local metric field \( \omega \) defining a pseudo-Riemannian structure on \( \mathbb{R}^n \simeq S \). We insist on the fact we do local studies, meaning that we consider local charts from open subsets of the latter manifolds into a common open subset of \( \mathbb{R}^n \). Hence by geometric objects or computations on \( \mathbb{R}^n \), we mean local geometric objects or computations on the manifolds \( M, TM \) and/or \( S \). Also, it is well-known that the mathematical results displayed below are independent of the dimension when the latter is greater or equal to 4 [12].

Let us consider \( \hat{f} \in \text{Diff}^\infty_{\text{loc}}(\mathbb{R}^n) \), the set of local diffeomorphisms of \( \mathbb{R}^n \) of class \( C^\infty \), and any function \( \alpha \in C^\infty(\mathbb{R}^n, \mathbb{R}) \). Then if \( \hat{f} \in \Gamma_{\hat{G}} \) (\( \Gamma_{\hat{G}} \) being the pseudogroup of local conformal bidifferential maps on \( \mathbb{R}^n \)), \( \hat{f} \) is a solution of the PDE system (1). In fact other PDE must be satisfied to completely define \( \Gamma_{\hat{G}} \) as will be seen in the sequel. Also, only the \( +e^{2\alpha} \) positive conformal factors are retained so as to preserve one orientation only on \( \mathbb{R}^n \), and we will accordingly restrict ourselves to those \( \hat{f} \) which preserve that orientation (we recall that \( \alpha \) is a varying function depending on each \( \hat{f} \) and consequently not fixed). We denote \( \tilde{\omega}_f \) the metric on \( \mathbb{R}^n \) such that by definition: \( \hat{f}^* (\omega) \equiv \tilde{\omega}_f \), and we agree on putting a tilde on each tensor or geometrical “object” relative to, or deduced from the metric \( \tilde{\omega}_f \).

Now, performing a first prolongation of the system (1), we deduce another second order system of PDE’s connecting the Levi-Civita covariant derivatives \( \nabla \) and \( \tilde{\nabla} \), respectively associated to \( \omega \) and \( \tilde{\omega}_f \). These new differential equations are (see for instance [11]) \( \forall X, Y \in T\mathbb{R}^n: \)

\[
\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X) Y + \alpha(Y) X - \omega(X, Y) \ast d\alpha,
\]

(2)

where \( d \) is the exterior differential and \( \ast d\alpha \) is the “\( \omega \)-dual” vector field of the 1-form \( d\alpha \) with respect to the metric \( \omega \), i.e., such that \( \forall X \in T \mathbb{R}^n: \)

\[
\omega(\ast d\alpha, X) = d\alpha(X) = < d\alpha | X >.
\]

(3)

Since the Weyl tensor \( \tau \) associated to \( \omega \) is vanishing, the Riemann tensor \( \rho \) can be rewritten \( \forall X, Y, Z, U \in C^\infty(T\mathbb{R}^n) \) as:

\[
\omega(U, \rho(X, Y) Z) = \frac{1}{(n-2)} \left\{ \omega(U) \sigma(Y, Z) - \omega(Y) \sigma(U, Z) + \omega(Y, Z) \sigma(U, X) - \omega(X) \sigma(Y, Z) \right\},
\]

(4)

where \( \sigma \) is defined by (see the tensor “L” in [40] up to a constant depending on \( n \))

\[
\sigma(X, Y) = \rho_{\text{nc}}(X, Y) - \frac{\rho_s}{2(n-1)} \omega(X, Y),
\]

(5)

where \( \rho_{\text{nc}} \) is the Ricci tensor and \( \rho_s \) is the Riemann scalar curvature. Consequently, the first order system of PDE in \( \hat{f} \) “connecting” \( \hat{\rho} \) and \( \rho \), can be rewritten as a first order
system of PDE concerning $\tilde{\sigma}$ and $\sigma$. Using the torsion free property of the Levi-Civita covariant derivatives and applying again the covariant derivative $\tilde{\nabla}$ on the relation (2), one obtains the following third order system of PDE (since $\alpha$ is depending on the first order derivatives of $\hat{f}$):

$$\hat{f}^*(\sigma)(X, Y) \equiv \tilde{\sigma}(X, Y) = \sigma(X, Y) + (n - 2)\left(\alpha(X)\alpha(Y) - \frac{1}{2}\omega(X, Y)\alpha(\alpha) - \mu(X, Y)\right),$$  \hspace{1cm} \text{(6)}$$

in which we have defined the symmetric tensor $\mu \in C^\infty(S^2\mathbb{R}^n)$ by:

$$\mu(X, Y) = \frac{1}{2}\left\{<\nabla_X(\alpha)|Y> + <\nabla_Y(\alpha)|X>\right\}.$$  \hspace{1cm} \text{(7)}$$

To go further, it is important again to notice that the relation (6) is directly related to a third order system of PDE, we will denote by (T), since it is deduced from a supplementary prolongation procedure applied to the second order system (2). Then it follows, from the well-known theorem of H. Weyl on the equivalence of conformal structures \cite{12, 39, 40}, and because of the Weyl tensor vanishing, that the systems of differential equations (1) and (2) when completed with the latter third order system (T), becomes an involutive system of order three. Let us stress again that $\alpha$ is merely defined by $\hat{f}$ and its first order derivatives, according to the relation (1).

Looking only at those applications $\hat{f}$ which are smooth deployments of applications $f$, this third order system of PDE must reduce to a particular system of PDE, associated to a conformal Lie sub-pseudogroup we denote by $\Gamma^{\hat{f}_{\mu}}$. Indeed, if $\alpha$ tends towards the zero function (with respect to the $C^2$-topology) then, the previous set of smoothly deformed applications $\hat{f}$ must tend towards the Poincaré Lie pseudogroup. But this condition is not satisfied by all of the applications $\hat{f}$ in the conformal Lie pseudogroup $\Gamma^{\hat{f}}$, since in full generality, the non-trivial third order system of PDE (T) would be kept at the zero $\alpha$ function limit: the sub-pseudogroup $\Gamma^{\hat{f}_{\mu}}$ would have to be defined by an involutive second order system of PDE which would tend towards the involutive system defining the Poincaré Lie pseudogroup.

The systems of differential equations (1) and (2) would be well suited to define partially this pseudogroup $\Gamma^{\hat{f}_{\mu}}$. The $n$-acyclicity property of $\Gamma^{\hat{f}}$ would be restored and borrowed, at the order two, from the Poincaré one, provided however that an, a priori, arbitrary input perturbative function $\alpha$ is given before, instead of being defined from an application $\hat{f}$ in accordance with the relation (11). But the formal integrability is obtained only if the tensor $\sigma$ satisfies one of the following equivalent relations (see formula (16.3) with definition (3.12) in \cite{12}):

$$\sigma = k_0\frac{(n - 2)}{2}\omega \iff \rho_{\text{ic}} = (n - 1)k_0\omega,$$  \hspace{1cm} \text{(8)}$$

deduced from relation (5), in order to avoid adding up the supplementary first order system of PDE (5) to (1), $\alpha$ being considered an input function. In fact, as it is well-known, the relations above are a consequences of the constant Riemann curvature tensor assumption, but they appear in a different way. Then, considering the system
(1) and (8), the system (6) reduces to a second order system of PDE concerning the input function $\alpha$ only, which is thus constrained, contrarily to what might have been expected, and such that:

$$\mu(X, Y) = \frac{1}{2} \left\{ \left[ k_0 (1 - e^{2\alpha}) - d\alpha (d\alpha) \right] \omega(X, Y) \right\} + d\alpha(X) d\alpha(Y).$$

(9)

Obviously, as can be easily verified, this is an involutive system of PDE, since it is a formally integrable system with an elliptic symbol (i.e. a vanishing symbol) of order two.

Thus, we have series of PDE deduced from (1) defining all the smooth deformations of the applications $f$ contained in the conformal Lie pseudogroup $\Gamma_{\hat{G}}$.

In addition, the metric field $\omega$ satisfies the relation (8) (which is an Einstein equation when the stress-energy tensor is proportional to the metric, or when the latter is vanishing but with a non-zero cosmological term). In that case, such a metric field $\omega$, or the substratum spacetime $S$ will be called $S$-admissible (with $S$ as “Substratum”), and this $S$-admissibility is assumed in the sequel.

Setting $\omega = \sum_{i,j=1}^{n} \omega_{ij}(x) \, dx^i \otimes dx^j$, in an orthonormal system of coordinates, then the PDE (1), (2) and (9) defining $\Gamma_{\hat{H}} \subset \Gamma_{\hat{G}}$ can be written, with $\det(J(\hat{f})) \neq 0$ and $i, j, k = 1, \cdots, n$ as:

$$\sum_{r,s=1}^{n} \omega_{rs}(\hat{f}) \hat{f}^r_i \hat{f}^s_j = e^{2\alpha} \omega_{ij},$$

(10a)

$$\hat{f}^k_{ij} + \sum_{r,s=1}^{n} \gamma^k_{rs}(\hat{f}) \hat{f}^r_i \hat{f}^s_j = \sum_{q=1}^{n} \hat{f}^k_q \left( \gamma^q_{ij} + \alpha_q \delta^q_j + \alpha_j \delta^q_i - \omega_{ij} \alpha_q \right),$$

(10b)

$$\mu_{ij} = \alpha_{ij} - \sum_{k=1}^{n} \alpha_k \gamma^k_{ij} = \frac{1}{2} \left\{ k_0 (1 - e^{2\alpha}) - \sum_{k=1}^{n} \alpha_k \alpha_k \right\} \omega_{ij} + \alpha_i \alpha_j,$$

(10c)

where $\delta^i_j$ is the Kronecker tensor, and where one denotes as usual $\hat{f}^i_j \equiv \partial \hat{f}^i / \partial x^j \equiv \partial_j \hat{f}^i$, etc . . . , $T_k = \sum_{h=1}^{n} T^h \omega_{hk}$ and $T^k = \sum_{h=1}^{n} T_h \omega^{hk}$ for any tensor $T$, where $\omega^{ij}$ is the inverse metric tensor, and $\gamma$ is the Riemann-Christoffel form associated to the $S$-admissible metric $\omega$. This is the set of our starting equations. It matters to notice that the (T) system is not included in the above set of PDE. Indeed, the latter being already involutive from order 2, this involves, by definition of involution, that the (T) system is redundant since all the applications $\hat{f} \in \Gamma_{\hat{H}}$, solutions of (10), will also be solutions of all the systems of PDE obtained by prolongation.

It is pertinent to notice that $\mu$ or, equivalently, the tensor $\hat{\alpha}_2 \equiv \{ \alpha_{ij}, i, j = 1, \cdots, n \}$ might be considered as an Abraham-Eötvös type tensor [1, 27] encountered in the Eötvös-Dicke experiments for the measurement of the stress-energy tensor of the gravitational potential.

At this point, it may also be worth making contact with some previous set of physical interpretations [17, 18, 29, 31, 33] (up to a constant for units and with $n = 4$) according to which the tensor $\mu$ is ascribed to the stress-energy tensor, $\hat{\alpha}_1$ to the gravity acceleration 4-vector, and $\alpha$ to the Newtonian potential of gravitation.
3. Functional dependence of the spacetime deployment

Let us now look for the formal series, solutions of the PDE system (10), assuming from now on, that the metric $\omega$ is analytic. We know these series will be convergent in a suitable open subset and thus will provide analytic solutions, since the analytic system is involutive and in particular elliptic, because of a vanishing symbol (see, in appendix 4 of [20], the Malgrange theorem for elliptic systems). Nevertheless, we need of course to know the Taylor coefficients. For instance, we can choose for the applications $\hat{f}$ and the functions $\alpha$ the following series at a point $x_0 \in \mathbb{R}^n$:

$$
\hat{f}^i(x) : S^i(x, x_0, \{\hat{a}\}) = \sum_{|J| \geq 0}^\infty \hat{a}^i_J (x - x_0)^J / |J|!, \quad (11)
$$

$$
\alpha(x) : s(x, x_0, \{c\}) = \sum_{|K| \geq 0}^\infty c_K (x - x_0)^K / |K|!,
$$

with $x \in U(x_0) \subset \mathbb{R}^n$ being a suitable open neighborhood of $x_0$ to insure the convergence of the series, $i = 1, \cdots, n$, $J$ and $K$ are multiple index notations such as $J = (j_1, \cdots, j_n)$, $K = (k_1, \cdots, k_n)$ with $|J| = \sum_{i=1}^n j_i$ and similar expressions for $|K|$. Likewise, $\{\hat{a}\}$ and $\{c\}$ are sets of Taylor coefficients, whereas the $\hat{a}^i_j$ and $c_K$ are real values and not functions of $x_0$, though of course, they can also be values of functions at $x_0$ (let us remark that we could consider instead the vector: $\xi = x - x_0 \in T_{\omega_0} \mathcal{M}$, which strengthens the equivalence principle we are using with $T_{\omega_0} \mathcal{M}$ as substratum spacetime $\mathcal{S}$).

Also we must add that usual partial derivatives will be used in these Taylor series determinations instead of covariant derivatives. The use of either of the two derivatives is thoroughly as discussed in the appendix, and shown to be basically equivalent.

3.1. The “Generalized Haantjes-Schouten-Struik system”

We call the “Generalized Haantjes-Schouten-Struik system” (GHSS system), the system of PDE (10) (see [10] [18] [33] for this system, but at $k_0 = 0$). It is from this set of PDE that gauge potentials and fields of interactions could occur. From the series $s$, at zero-th order one obtains the algebraic equations ($i, j = 1, \cdots, n$; $c_1 = \{c_1, \cdots, c_n\}$):

$$
c_{ij} = \frac{1}{2} \left\{ k_0(1 - e^{2x_0}) - \sum_{k,h=1}^n \omega^{kh}(x_0)c_h c_k \right\} \omega_{ij}(x_0) + c_i c_j + \sum_{k=1}^n c_k \gamma_{ij}^k
\equiv F_{ij}(x_0, c_0, c_1), \quad (12)
$$

and it follows that the $c_K$’s such that $|K| \geq 2$, will depend recursively only on $x_0$, $c_0$ and $c_1$. It is none but the least the meaning of formal integrability of so-called involutive systems. Hence the series for $\alpha$ can be written as a convergent series, $s(x, x_0, c_0, c_1)$, developed with respect to powers of $(x - x_0)$, $c_0$ and $c_1$. Let us notice that we can change or not the values at $x$ of the series $s$, by varying $x_0$, $c_0$ or $c_1$.

Let $J_1$ be the 1-jets affine bundle of the $C^\infty$ real valued functions on $\mathbb{R}^n$. Then, from the latter remark, it exists a subset associated to $c^1_0 \equiv (x_0, c_0, c_1) \in J_1$, we denote by
$S^1_c(c_0^1) \subset J_1$; the set of elements $(x_0', c_0', c_1') \in J_1$, such that there is an open neighborhood $U(c_0^1) \subset S^1_c(c_0^1)$, being projected on $\mathbb{R}^n$ in an open neighborhood of a given $x \in U(x_0)$, for which, for all $(x_0', c_0', c_1') \in U(c_0^1)$, then, $s(x, x_0, c_0, c_1) = s(x, x_0', c_0', c_1')$. Assuming that the variation $ds$ with respect to $x_0, c_0$ and $c_1$ is vanishing, at a given fixed $x$, is the subset $S^1_c(c_0^1)$ a submanifold of $J_1$? From $ds \equiv 0$ it follows that $(k = 1, \cdots, n)$:

$$
\sigma_0 \equiv dc_0 - \sum_{i=1}^n c_i \, dx_0^i = 0,
$$

$$
\sigma_k \equiv dc_k - \sum_{j=1}^n F_{kj}(x_0, c_0, c_1) \, dx_0^j = 0.
$$

We recognize a regular analytic Pfaff system, we denote by $P_c$, generated by the 1-forms $\sigma_0$ and $\sigma_k$, and the meaning of their vanishing is that the solutions $s$ for $\alpha$ do not change for such variations of $c_0, c_1$ and $x_0$. Also, as can be easily verified, the Pfaff system $P_c$ is integrable since the Fröbenius conditions of involution are satisfied, and all of the prolonged 1-forms $\sigma_K$ with $|K| \geq 2$, are linear combinations of these $n+1$ generating forms, thanks to the recursion property of formal integrability. Then the subset $S^1_c(c_0^1)$ of dimension $n$ containing a particular element $c_0^1 \equiv (x_0, c_0, c_1)$, is a submanifold of $J_1$. It is a particular leaf of, at least, a local foliation $\mathcal{F}_1$ on $J_1$ of codimension $n + 1$.

Then, since the system of PDE defined by the involutive Pfaff system $P_c$, namely the GHSS system, is elliptic (i.e., vanishing symbol in the present case) and formally integrable, one deduces that it exists on $J_1$, local analytic systems of coordinates $(\tau_0, \tau_1, \cdots, \tau_n)$ of the transverse submanifold of the foliation, such that each leaf $S^1_c(c_0^1)$ is an analytic submanifold for which $\tau_0 = \text{cst}$ and $\tau_i = \text{cst}$ ($i = 1, \cdots, n$). In other words, all the series $s(x, x_0', c_0', c_1')$ with $(x_0', c_0', c_1') \in S^1_c(c_0^1)$ are convergent and correspond to one single analytic solution $u(x, \tau_0, \mathbf{\tau}_1)$ ($\mathbf{\tau}_1 \equiv \{\tau_1, \cdots, \tau_n\}$), analytic with respect to $x$ as well as with respect to the $\tau$'s. This results from the $s$ continuous series convergent character, whatever the fixed set of given values $x$, $x_0$, $c_0$ and $c_1$. Thus, in full generality, considering the difference $s(x, x_0, c_0, c_1) - s(x, x_0', c_0', c_1')$, we have the relation:

$$
\sigma_0 \equiv dc_0 - \sum_{i=1}^n c_i \, dx_0^i = 0,
$$

with the $\tau$ parameters related by ($i = 1, \cdots, n$)

$$
\tau_i' - \tau_0 = \int_{c_0^1}^{c_1^1} \sigma_0, \quad \tau_i' - \tau_i = \int_{c_0^1}^{c_1^1} \sigma_i.
$$

Now, we consider the $c$'s as values of differential (i.e. $C^\infty$) functions $\rho: c_K = \rho_K(x_0)$, as expected for usual Taylor series coefficients, and defined on a starlike open neighborhood of $x_0$ (the integrals above define none but the least than a homotopy operator, and that “starlike” open subsets obviously mean simply connected open subsets). Roughly speaking, we make a pull-back on $\mathbb{R}^n$ by differentiable sections $\rho$, inducing a projection from the subbundle of projectable elements in $T^*J_1$ to $T^*\mathbb{R}^n \otimes_{\mathbb{R}} J_1$. Then, we set (with
\[ \rho_1 \equiv \{ \rho_1, \cdots, \rho_n \}, \quad \partial_0^i \equiv \partial / \partial x_0^i \] and no changes of notations for the pull-backs):

\[ \sigma_0 \equiv \sum_{i=1}^{n} (\partial_0^i \rho_0 - \rho_i) \, dx_0^i \equiv \sum_{i=1}^{n} A_i \, dx_0^i, \quad (15a) \]

\[ \sigma_i \equiv \sum_{j=1}^{n} (\partial_0^j \rho_i - F_{ij}(x_0, \rho_0, \rho_1)) \, dx_0^j \equiv \sum_{j=1}^{n} B_{j,i} \, dx_0^j, \quad (15b) \]

and it follows that the integrals (14) must be performed from \( x_0 \) to \( x'_0 \) in a starlike open neighborhood of \( x_0 \). In particular, if \( c_0^1 \) is an element of the “null” submanifold, or stratum, corresponding to the vanishing solution of the “GHSS system”, then the difference (13) involves that

\[ \alpha(x) \equiv s(x, x'_0, c'_0, c'_1) = u(x, \tau'_0, \tau'_1), \]

with

\[ \tau'_0 = \int_{x_0}^{x'_0} \sum_{i=1}^{n} A_i \, dx^i + \tau_0, \quad \tau'_i = \int_{x_0}^{x'_0} \sum_{j=1}^{n} B_{j,i} \, dx^j + \tau_i. \]

This result displays the functional dependencies of the \( \tau \) deformation parameters of the solutions of the “GHSS system”, with respect to the functions \( \rho_0 \) and \( \rho_1 \). These smooth infinitesimal deformations define the fields \( A \) and \( B \), i.e., \( n(n+1) \) potential functions if \( n = 4 \), which can also be considered as infinitesimal smooth deployments from “Poincaré solutions” of the system (10) at \( \alpha \equiv 0 \), to some “conformal solutions” whatever is \( \alpha \) satisfying the GHSS system.

Moreover the functions \( \rho \), and consequently the fields \( A \) and \( B \), must satisfy additional differential equations coming from the Fröbenius conditions of involution for the Pfaff system \( P_c \). More precisely, from the relations \( d\sigma_0 = \sum_{i=1}^{n} dx_0^i \wedge \sigma_i \), \( d\sigma_i = \sum_{j=1}^{n} dx_0^j \wedge \sigma_{ij} \) and

\[ \sigma_{ij} = c_i \sigma_j + c_j \sigma_i - \omega_{ij} \left\{ k_0 e^{2c_0} \sigma_0 + \sum_{k, h=1}^{n} \omega^{kh} c_h \sigma_k \right\} + \sum_{k=1}^{n} \gamma_{ij}^k \sigma_k \]

\[ \equiv \partial_{ij}(e_0^1, \sigma_J; |J| \leq 1), \quad (16) \]

one deduces a set of algebraic relations to be satisfied at \( x_0 \):

\[ J_{k,j,i} \equiv \omega_{ij} \left\{ k_0 e^{2c_0} A_k + \sum_{r, s=1}^{n} \omega^{rs} \rho_r B_{k,s} \right\} \]

\[ + \partial_0^j B_{k,i} - \rho_i B_{k,j} - \rho_j B_{k,i} - \sum_{s=1}^{n} \gamma_{ij}^s B_{k,s}, \quad (17a) \]

\[ I_{i,k} = I_{k,i} \equiv \partial_0^i A_i - B_{i,k} \], \quad \[ J_{k,j,i} = J_{j,k,i}. \quad (17b) \]

In these relations, the set of functions \((\rho_0, \rho_1)\) appears to be, \textit{a priori} only, a set of arbitrary differential functions. Finally, we deduce:
Theorem 1  All the analytic solutions of the involutive system of PDE (10c) can be written in a suitable starlike open neighborhood $U(x_0)$ of $x_0$ as

$$\alpha(x) \equiv u(x, \int_{x_0}^{x_0'} \sum_{i=1}^{n} A_i \, dx^i + \tau_0, \int_{x_0}^{x_0'} \sum_{j=1}^{n} B_{j,1} \, dx^{ij} + \tau_1, \ldots, \int_{x_0}^{x_0'} \sum_{j=1}^{n} B_{j,n} \, dx^{ij} + \tau_n),$$

with $u(x_0, \tau_0, \tau_1, \ldots, \tau_n) = 0$, $x_0' \in U(x_0)$, and where $u$ is a unique fixed analytic function depending on the $n(n+1)C^\infty$ integrable functions $A_i$ and $B_{j,k}$ defined by the relations (13). The integrals in $u$ are called the “potential of interactions”. Let us remark that we can set $x_0' \equiv v(x)$ if the gradient of $v$, i.e. $\nabla v$, is in the annihilator of the Pfaff system $P_c$ of 1-forms $\sigma$.

This Theorem making explicit the dependence of the GHSS system solutions on the $\mathcal{A}$ and $\mathcal{B}$ “gauge fields”, can be viewed as an illustration of the well-known Cartan-Kähler Theorem, and indicates also how the fields $\mathcal{A}$ and $\mathcal{B}$ “gauge” the geometrical deformations of the (isometries) Poincaré Lie pseudogroup.

Also, considering physical aspects, and defining $\mathcal{F}$ and $\mathcal{G}$ as being the respectively skew-symmetric and symmetric parts of the tensor of components $\partial_i \rho_j$, one deduces, from the symmetry properties of the relations (17), what we call the first set of differential equations associated to $\mathcal{S}$ at $x_0$:

$$\partial_i^0 \mathcal{F}_{jk} + \partial_j^0 \mathcal{F}_{ki} + \partial_k^0 \mathcal{F}_{ij} = 0,$$

$$2 \partial_j^0 \mathcal{G}_{ki} - \partial_i^0 \mathcal{G}_{kj} - \partial_k^0 \mathcal{G}_{ij} = \partial_i^0 \mathcal{F}_{jk} - \partial_j^0 \mathcal{F}_{ij},$$

with

$$\mathcal{F}_{ij} = \partial_i^0 \rho_j - \partial_j^0 \rho_i = \partial_i^0 \mathcal{A}_j - \partial_j^0 \mathcal{A}_i,$$

$$\mathcal{G}_{ij} = - (\partial_i^0 \rho_j + \partial_j^0 \rho_i) \equiv \partial_i^0 \mathcal{A}_j + \partial_j^0 \mathcal{A}_i \mod (\rho_0, \partial_i \rho_0).$$

The PDE (19a) with (20a) can be interpreted as the first set of Maxwell equations. In view of physical interpretations, we can easily compute the Euler-Lagrange equations of a conformally equivariant Lagrangian density

$$\mathcal{L}(x_0, \rho_0, \rho_1, \mathcal{A}, \mathcal{F}, \mathcal{G}) \, d^n x_0,$$

or more generally

$$\mathcal{L}(x_0, \rho_0, \rho_1, \mathcal{A}, \mathcal{B}, \mathcal{I}, \mathcal{J}) \, d^n x_0,$$

with $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{G}, \mathcal{I}$ and $\mathcal{J}$ satisfying the relations (15), (20) and (17). Doing so, we would obtain easily what could be dubbed the second set of differential equations associated to $\mathcal{S}$ at $x_0$, some aspects of which will be discussed in the last section.

Then to proceed further, a few well-known definitions are in order [4]. We call germ at $x_0$ of an application $f$, the class of $C^\infty$ applications $\tilde{f}$, for which it exist an open subset $U$ of $x_0$ such that $f/_{U} = \tilde{f}/_{U}$. We call ring on $U$ or local ring (if it contains
a unique maximal proper ideal) on \( U \) a set of germs of applications defined on \( U \), all satisfying possibly the same given “formulae” on a given subset of points in \( U \) (i.e., on a punctuated open subset \( U \)). The set of rings can be endowed with a so-called presheaf structure. In the sequel, we will not consider sheafs, since the compatibility condition, defining sheafs from presheafs, will not be used here because only local (on one given open subset) topological considerations will be relevant to our concern.

**Definition 1** We denote:

(i) \( \theta_\mathbb{R} \), the presheaf of rings of germs of the differential (i.e. \( C^\infty \)) functions defined on \( \mathbb{R}^n \),

(ii) \( J_1 \), the presheaf of \( \theta_\mathbb{R} \)-modules of germs of differential sections of \( J_1 \),

(iii) \( S^0_c \subset \theta_\mathbb{R} \), the presheaf of rings of germs of functions which are solutions with their first derivatives, of the “algebraic equations” GHSS (10c) taken at any given points \( x_0 \) in \( \mathbb{R}^n \), not simultaneously at each point in \( \mathbb{R}^n \) (see Remark 1 below),

(iv) \( S^1_c \subset J_1 \), projectable on \( J_2 \) (\( J_1 \simeq S^1_c \)), the embedding in \( J_2 \) of the presheaf of \( \theta_\mathbb{R} \)-modules of germs of differential sections of \( J_2 \), defined by the system (10a) of algebraic equations at any given points \( x_0 \in \mathbb{R}^n \) (not everywhere, as mentioned above),

(v) \( T^*\mathbb{R}^n \), the presheaf of \( \theta_\mathbb{R} \)-modules of germs of global 1-forms on \( \mathbb{R}^n \).

**Remark 1:** Through this set of definitions, we do not consider PDEs solutions, but instead, solutions of algebraic equations at any given point \( x_0 \). In this light, PDEs solutions are to be regarded as particular “coherent” subsheafs for which equations (10c) are satisfied everywhere in \( \mathbb{R}^n \), i.e., at \( x \neq x_0 \), and not solely at \( x_0 \). We insist that the algebraic equations (10c) do not concern solutions of a PDE system, but the values of second derivatives of functions at \( x_0 \), depending on those of first order at most at \( x_0 \), with no constraints between first and zero-th order values of these functions at \( x_0 \).

Then, considering the local diffeomorphisms

\[
(\wedge^k T^*\mathbb{R}^n \otimes_{\mathbb{R}} J_r)_{x_0} \simeq (\{x_0\} \otimes_{\mathbb{R}} J_r) \times (\wedge^k T^*_{x_0}\mathbb{R}^n \otimes_{\mathbb{R}} J_r)
\]

with \( 0 \leq k \leq n \) and \( r \geq 0 \), we set the definitions:

**Definition 2** We define the local operators:

(i) \( j_1 : (x_0, \rho_0) \in S^0_c \longrightarrow (x_0, \rho_0, \rho_1, \rho_2) \in S^1_c \) with \( \rho_1 = (\partial_1 \rho_0, \ldots, \partial_n \rho_0) \) and \( \rho_2 = (\partial^2_{11} \rho_0, \partial^2_{12} \rho_0, \ldots, \partial^2_{nn} \rho_0) \),

(ii) \( D_{1,c} : \rho_0^2 \equiv (x_0, \rho_0, \rho_1, \rho_2) \in S^1_c \longrightarrow (\rho_0^1, \sigma_0, \sigma_1, \ldots, \sigma_n) \in T^*\mathbb{R}^n \otimes_{\theta^c} J_1 \), with \( \mathcal{A}, \mathcal{B} \) and \( \rho_0^1 \equiv (x_0, \rho_0, \rho_1) \) satisfying relations (13), and \( P_c = \{\sigma_0, \sigma_1, \ldots, \sigma_n\} \) being a Pfaffian system of linearly independent regular 1-forms on \( J_1 \),

(iii) \( D_{2,c} : (\rho_0^1, \sigma_0, \sigma_1, \ldots, \sigma_n) \in T^*\mathbb{R}^n \otimes_{\theta^c} J_1 \longrightarrow (\rho_0^1, \zeta_0, \zeta_1, \ldots, \zeta_n) \in \wedge^2 T^*\mathbb{R}^n \otimes_{\theta^c} J_1 \),

with

\[
\zeta_0 = \sum_{i,j=1}^n \mathcal{I}_{i,j} dx^i_0 \wedge dx^j_0, \quad \zeta_k = \sum_{i,j=1}^n \mathcal{J}_{j,i,k} dx^i_0 \wedge dx^j_0,
\]
the functions \((x_0, \rho_0, \rho_1, \rho_2) \in S^1_c\) and the tensors \(I, J, A\) and \(B\) satisfying the relations \([17]\).

Then from all that precedes, we can deduce:

**Theorem 2** The differential sequence

\[
0 \longrightarrow S^0_c \xrightarrow{j_2} S^1_c \xrightarrow{D_{1,c}} T^*\mathbb{R}^n \otimes_{\theta_R} J_1 \xrightarrow{D_{2,c}} \wedge^2 T^*\mathbb{R}^n \otimes_{\theta_R} J_1,
\]

with the \(\mathbb{R}\)-linear local differential operators \(D_{1,c}\) and \(D_{2,c}\), is exact (where the first injectivity, namely \(j_1\), results from remark 1).

Remark 2: Before proceeding with the proof of this Theorem, a few comments are in order. The continuation “on the right” of the differential sequence above would require, in order to demonstrate the exactness, a generalization of the Fröbenius Theorem to \(p\)-forms with \(p \geq 2\), which, to our knowledge at least, is not available in full generality, no more as the concept of canonical contact \(p\)-forms representations. Indeed, the higher local differential operators \(D_{i \geq 3,c}\) would be non-linear, in contrary to the usual Spencer differential operators, because of the non-linearity of the GHSS system. We are faced to the same situation encountered in the Spencer sequences for Lie equations, these sequences being truncated at this same order two. The sequence above is a physical gauge sequence, for which we can make the following identification: \(T^*\mathbb{R}^n \otimes_{\theta_R} J_1\) is the space of the gauge potentials \(A\) and \(B\), whereas \(\wedge^2 T^*\mathbb{R}^n \otimes_{\theta_R} J_1\) is the space of the gauge strength fields \(I\) and \(J\).

This sequence is close to a kind of Spencer linear sequence \([35]\). It differs essentially in the tensorial product which is taken on \(\theta_R\) (because of the non-linearity of the “GHSS system”, inducing a \(\rho\) “dependence” of the various Pfaff forms) rather than on the \(\mathbb{R}\) field as is in the original linear Spencer theory \([35]\) (other developments have included the \(\theta_R\) case after this first Spencer original version). Also, since the system \(P_c\) is integrable, it is always, at least locally, diffeomorph to an integrable set of Cartan 1-forms in \(T^*\mathbb{R}^n \otimes_{\mathbb{R}} J_1\) associated to a particular finite Lie algebra \(g_c\) (of dimension greater or equal to \(n + 1\)), with corresponding Lie group \(G_c\) acting on the left on each leaf of the foliation \(\mathcal{F}_1\) \([2, 3, 37]\). It follows that the integrals in \([18]\) would define a deformation class in the first non-linear Spencer cohomology space of deformations of global sections from \(\mathbb{R}^n\) to a sheaf of Lie groups \(G_c\) \([22]\) (see also \([25]\), though within a different approach).

In addition, 1) in Theorem 1, the fonction \(u\) is defined with integrals associated to the definition of a homotopy operator of the differential sequence above \([7]\), and 2) in Theorem 2, the metric \(\omega\) is allowed to be of class \(C^\infty\), rather than analytic, as in Theorem 1, because formal properties only are considered.

**Proof of Theorem 2:** At \(S^1_c\) the sequence exactness is trivial and we may pass to the exactness of the differential sequence at \(T^*\mathbb{R}^n \otimes_{\theta_R} J_1\).

In a neighbourhood of an open set \(V(\mathcal{C}_0^1) \subset J_1\) of \(\mathcal{C}_0^1 \in J_1\), the condition \(D_{2,c}(\sigma) = 0\) implies the relations:

\[
d\tilde{\sigma}_0 = \sum_{i=1}^n dx_0^i \wedge \tilde{\sigma}_i,
\]

(22)
Spacetime deployment

\[ d\tilde{\sigma}_i = \sum_{j=1}^{n} dx_0^j \wedge \tilde{\sigma}_{ij} \]  

(23)

with:

\[ \tilde{\sigma}_{ij} = c_i \tilde{\sigma}_j + c_j \tilde{\sigma}_i - \omega_{ij} \left\{ k_0e^{2\zeta_0} + \sum_{k,h=1}^{n} \omega^{kh} c_h \tilde{\sigma}_k \right\} + \sum_{k=1}^{n} \gamma_{ij}^k \tilde{\sigma}_k, \]  

(24)

the “\(\tilde{\sigma}\)” 1-forms are defined above \(J_1\), and correspond to the 1-forms defined in a neighbourhood \(W(X_0) \subset \mathbb{R}^n\) at \(x_0 \in W(X_0)\): they are such that if \(p_1 : J_1 \rightarrow \mathbb{R}^n\) stands for the standard projection, then \(p_1(V(C^1_0)) = W(X_0), p_1(c_0) = x_0\) and \(p_1(C^1_1) = X_0\).

Regularity and linear independence, ensure the existence of a locally integrable manifold \(V_1\), with dimension \(n\), and of \(n + 1\) first integrals \(\{y_\nu\} (\nu = 0, 1, \ldots, n)\). Up to constants, the functions \(y_\nu\) can be choosen such that \(y_\nu(C^1_0) = 0\). Then, at \(C^1_0\), we have the relations:

\[ \tilde{\sigma}_\nu(C^1_0) \equiv dy_\nu/C^1_0, \]  

(25)

and in \(V(C^1_0)\), the relations

\[ \tilde{\sigma}_\nu = dy_\nu - \sum_{\mu \neq \nu} f_\nu^\mu(y)dy_\mu, \]  

(26)

with \(f_\nu^\mu(y) \rightarrow 0\) when \(y \rightarrow 0\), that is when \(c_0 \rightarrow C^1_0\).

These relations can also be defined on the presheaves of the \(J_1\) local sections. This is because it exists a \(C^1\)-mapping, say \(s\), from \(W(X_0)\) into \(V(C^1_0)\), such that \(s(W(X_0)) = U(C^1_0) \subset V(C^1_0)\) and \(s(x_0) = c_0\). And thus locally, one has \(V_1 \cap U(C^1_0) \simeq W(X_0)\). In the relations (26), it is therefore possible to take \(y_j \equiv c_j (j = 1, \ldots, n)\). Setting \(s^*(dy_j) \equiv dx_j^0\), and denoting by “\(\rho\)” the functions \(\rho_j(x_0) = c_j\) and \(\rho_0(x_0) = y_0\), associated to \(s\), we have immediately in particular \(\sigma_0 = s^*(\tilde{\sigma}_0)\), and \(\forall x_0:\)

\[ \sigma_0 \equiv d\rho_0 - \sum_{i=0}^{n} f_0^i(\rho) dx_i. \]  

(27)

We set \(\rho_i \equiv f_0^i(\rho)\). Now, from (27) and the pull-back of (22), one deduces that

\[ \sum_{i=1}^{n} dx_i^0 \wedge (d\rho_i - \sigma_i) = 0, \]  

(28)

and in particular:

\[ dx_0^1 \wedge dx_0^2 \wedge \ldots \wedge dx_0^n \wedge (d\rho_i - \sigma_i) = 0. \]  

(29)

Consequently

\[ d\rho_i - \sigma_i = \sum_{j=1}^{n} \rho_{i,j} dx_0^j \implies \sigma_i = d\rho_i - \sum_{j=1}^{n} \rho_{i,j} dx_0^j, \]  

(30)
that are alternatives to the pull-backs by $s$ of relations (26). We thus have,

$$\sigma_0 = d\rho_0 - \sum_{i=1}^{n} \rho_i \, dx_i^0,$$

$$\sigma_i = d\rho_i - \sum_{j=1}^{n} \rho_{i,j} \, dx_j^0.$$  

(31)

Now, out of (32) and the pull-backs of (23), one deduces also the relations,

$$\sum_{j=1}^{n} dx^j_0 \wedge d\rho_{i,j} = \sum_{j=1}^{n} dx^j_0 \wedge \sigma_{ij} \iff \sum_{j=1}^{n} dx^j_0 \wedge (d\rho_{i,j} - \sigma_{ij}) = 0.$$  

(33)

By the same procedure as above, we thus get:

$$d\rho_{i,j} - \sigma_{ij} = \sum_{j=1}^{n} \rho_{i,j,k} \, dx_k^0 \iff \sigma_{ij} = d\rho_{i,j} - \sum_{k=1}^{n} \rho_{i,j,k} \, dx_k^0.$$  

(34)

Moreover, in view of (28) and (32) we deduce the symmetries: $\rho_{i,j} = \rho_{j,i} \equiv \rho_{ij}$ and $\rho_{i,j,k} = \rho_{j,i,k} \equiv \rho_{ij,k}$. Then, considering the coefficients of a same basis differential form with $\rho_2 \equiv (\rho_{ij})$, $\rho_1 \equiv (\rho_i)$, and the system of algebraic equations for $\rho_0^2$ deduced from (22) and (23), we conclude that $\rho_0^2 \in S_1^c$. □

In order to know the effects on $\mathcal{M}$ of these infinitesimal deformations, we need to describe what are their incidences upon the objects acting primarily on $\mathbb{R}^n$, namely the applications $\hat{f}$.

Thus, we pass to the study of what we call the “ab system” of the PDE system (10).

3.2. The “ab system”

This system is defined by the first two sets of PDE (10a) and (10b). For this system of Lie equations, we will begin with recalling well-known results, but in the framework of the present context. Applying the same reasoning than in the previous subsection, and considering the series (11) for the $\hat{f}$, we first obtain the following results, which hold up to order two:

$$\sum_{r,s=1}^{n} \omega_{rs}(\hat{a}_0) \, \hat{a}_i^r \hat{a}_j^s = e^{2c_0} \omega_{ij}(x_0),$$

(35a)

$$\hat{a}_{ij}^k + \sum_{r,s=1}^{n} \gamma_{rs}^k(\hat{a}_0) \, \hat{a}_i^r \hat{a}_j^s =$$

$$\sum_{q=1}^{n} \hat{a}_k^q \left( \gamma_{ij}^q(x_0) + c_i \delta_j^q + c_j \delta_i^q - \omega_{ij}(x_0)c^q \right),$$  

(35b)

which clearly show that, fixing the $c$’s, $J_1(\mathbb{R}^n)$ is diffeomorphic to an embedded submanifold of the 2-jets affine bundle $J_2(\mathbb{R}^n)$ of the $C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ differentiable applications on $\mathbb{R}^n$. In second place, we get relations, from the (T) system (cf. section 2),
for the coefficients of order 3 that we write as \( \hat{a}_1 \equiv (\hat{a}_j^i); \hat{a}_2 \equiv (\hat{a}_{jk}^i), \ldots, \hat{a}_k \equiv (\hat{a}_{j_1 \ldots j_k}^i); \hat{a}_0^k \equiv (\hat{a}_0^i, \ldots, \hat{a}_k) \):

\[
\hat{a}_{jkh}^i \equiv \hat{A}_{jkh}^i(x_0, \hat{a}_0^2),
\]

where \( \hat{A}_{jkh}^i \) are algebraic functions, pointing out in this expression the independence on the \( \text{"c"} \) coefficients (as in the relations \( \text{(36a)} \), for instance, when \( c_0 \) is expressed in terms of the determinant of \( \hat{a}_1 \)). We denote by \( \hat{\Omega}_j^i \) the Pfaff 1-forms at \( x_0 \) and \( \{ \hat{a} \} \) (or at \( (x_0, \{ \hat{a} \}) \)):

\[
\hat{\Omega}_j^i \equiv \hat{d} \hat{a}_j^i - \sum_{k=1}^n \hat{a}_{j+1k}^i \hat{d}x_0^k,
\]

and setting the \( \hat{a} \)'s as values of functions \( \hat{\tau} \) depending on \( x_0 \) (in some way we make a pull-back on \( \mathbb{R}^n \)), we define the tensors \( \hat{\kappa} \) by:

\[
\hat{\Omega}_j^i \equiv \sum_{k=1}^n \left( \partial_k \hat{\tau}_j^i - \hat{\tau}_{j+1k}^i \right) \hat{d}x_0^k \equiv \sum_{k=1}^n \hat{\kappa}_{k,j}^i \hat{d}x_0^k.
\]

Then from the relations:

\[
e^{2\hat{c}_0} \omega^{ij}(\hat{a}_0) = \sum_{i,j=1}^{n} \omega^{ij}(x_0) \hat{a}_r^i \hat{a}_s^j, \quad \sum_{i=1}^{n} \gamma_i^k = \frac{1}{2} \sum_{i,j=1}^{n} \omega^{ij} \partial_k \omega_{ij},
\]

we deduce from \( \text{(39a)} \) with \( \hat{b} \equiv \hat{a}_1^{-1} \), that the \( \hat{\Omega}_j^i \) 1-forms satisfy, at \( (x_0, \hat{a}_0^2) \), the relations:

\[
\hat{H}_0(x_0, \hat{a}_0^1, \hat{\Omega}_j^i; |J| \leq 2) \equiv \sum_{i,j=1}^{n} \hat{b}_i^j \hat{\Omega}_j^i + \sum_{j,k=1}^{n} \gamma_{jk}^i (\hat{a}_0) \hat{\Omega}_k^i = n \sigma_0.
\]

Similar computations show that the 1-forms \( \sigma_i \) can be expressed as quite long relations, linear in the \( \hat{\Omega}_j^i (|J| \leq 2) \), with coefficients which are algebraic functions depending on the \( \hat{a}_K (|K| \leq 2) \), the derivatives of the metric and the Riemann-Christoffel symbols, all of them taken either at \( x_0 \) or \( \hat{a}_0 \). Then, we set:

\[
\sigma_i \equiv \hat{H}_i(x_0, \hat{a}_0^2, \hat{\Omega}_j^i; |I| \leq 2).
\]

From \( \text{(30)} \) the 1-forms \( \hat{\Omega}_{jkh}^i \) are also sums of 1-forms \( \hat{\Omega}_K^r \) (\( |K| \leq 2 \)) with the same kind of coefficients and not depending on the \( \sigma \)'s, and we write (without any more details since it is not necessary for our demonstration below):

\[
\hat{\Omega}_{jkh}^i \equiv \hat{K}_{jkh}^i(x_0, \hat{a}_0^2, \hat{\Omega}_K^r; |K| \leq 2),
\]

where \( \hat{K}_{jkh}^i \) are functions which are linear in the 1-forms \( \hat{\Omega}_K^r \).

Let us denote by \( \hat{P}_2 \subset J_2(\mathbb{R}^n) \) the set of elements \( (x_0, \hat{a}_0^2) \) satisfying the relations \( \text{(36)} \) whatever are the \( \text{"c"} \)'s. Then the Pfaff system we denote \( \hat{P}_2 \) over \( \hat{P}_2 \) and generated by the 1-forms \( \hat{\Omega}_K^r \in T^* \mathbb{R}^n \otimes J_2(\mathbb{R}^n) \) in \( \hat{\Omega}_j^i \) with \( |K| \leq 2 \), is locally integrable on every neighborhood \( U(x_0, \hat{a}_0^2) \subset J_2(\mathbb{R}^n) \), since at \( (x_0, \hat{a}_0^2) \) we have \( |J| \leq 2 \):

\[
d \hat{\Omega}_j^i - \sum_{k=1}^{n} \hat{d}x_0^k \wedge \hat{\Omega}_{j+1k}^i \equiv 0,
\]
together with (41).

Let us now consider the “Poincaré system” whose corresponding notations will be free of “hats”. We denote by $\Omega_j$ the Pfaff 1-forms corresponding to this system, i.e. the system defined by the PDE (10a) and (10b) with a vanishing function $\alpha$. The corresponding 1-forms “$\sigma$” are also vanishing everywhere on $\mathbb{R}^n$ and the $\Omega_j$ satisfy all of the previous relations, but with the $\sigma$’s cancelled out. Then it is easy to see that the $\Omega_j$ 1-forms ($|J| \geq 2$) are generated by the set of 1-forms $\Omega_K^i$ ($|K| \leq 1$); We have in particular

$$\Omega^k_{ij} = - \left\{ \sum_{r,s,h=1}^n (\hat{\gamma}^0_{r,s} \gamma^r_{h,s})(a_0)\Omega^h a_i^r a_j^s + \sum_{r,s=1}^n \gamma^k_{r,s}(a_0)\left[ a_i^r \Omega_j^s + a_j^s \Omega_i^r \right] \right\},$$

(43)

with $(x_0, \hat{a}_0^1 \equiv a_0^1) \in \mathcal{P}_1 \subset J_1(\mathbb{R}^n)$, $\mathcal{P}_1$ being the set of elements satisfying the relations (35a) with $c_0 = 0$. Similarly the Pfaff system we denote by $P_1$ over $\mathcal{P}_1$ and generated by the 1-forms $\Omega^1_K$ in (37), but free of hats and with $|K| \leq 1$, is locally integrable on every neighborhood $U(x_0, a_0^1) \subset \mathcal{P}_1$, since at the point $(x_0, a_0^1)$ we have the relations (42) with $|J| \leq 1$ together with relations (43).

Then, considering $J_1(\mathbb{R}^n)$ embedded in $J_2(\mathbb{R}^n)$, as well as $\mathcal{P}_1$ in $\hat{\mathcal{P}}_2$, and defining $\mathcal{P}_2 \subset \hat{\mathcal{P}}_2$ as the set of elements $(x_0, a_0^2)$ satisfying the relations (35) with $c_0 = c_1 = \ldots = c_n = 0$, we obtain the following theorem justifying the structure’s deformation point of view:

**Theorem 3** The sequence

$$0 \longrightarrow P_1 \overset{b_1}{\longrightarrow} \hat{\mathcal{P}}_2 \overset{e_1}{\longrightarrow} P_c \longrightarrow 0,$$

(44)

is a local exact splitted sequence over $\mathcal{P}_2$.

In this sequence a back-connection $b_1$ and a connection $c_1 : P_c \longrightarrow \hat{\mathcal{P}}_2$ are such that ($|J| \leq 2$):

$$\hat{\Omega}^i_J = \Omega^i_J + \chi^i_J(x_0, a_0^2) \sigma_0 + \sum_{k=1}^n \chi^i_{J,k}(x_0, a_0^2) \sigma_k \equiv \Omega^i_J + c_1^i(\sigma_K; |K| \leq 1),$$

(45)

with $(\Omega^i_{J,k}) = b_1(\Omega_j; |J| \leq 1)$ satisfying (38) for any given $\Omega^i_J$ with $|J| \leq 1$, and where the tensors $\chi$ are defined on $\mathcal{P}_2$. The maps $b_1$ and $c_1$ define the projective map $e_1$ if the tensors $\chi$ satisfy the relations:

$$\hat{H}_0(x_0, a_0^1, \chi^k_L; |L| \leq 1) = 0,$$

$$\hat{H}_i(x_0, a_0^2, \chi^k_L; |L| \leq 2) = 0, 1$$

(46a)

in order to preserve the relations (39) and (40), i.e. $e_1 \circ c_1 = id$.

4. The spacetime $\mathcal{M}$ unfolded by Gravitation and Electromagnetism

From now on, we consider the relations (45) with $|J| = 0$ and the $\hat{\Omega}^i_J$ as fields of “tetrads”. Then we get a metric $\nu$ for the “unfolded spacetime manifold $\mathcal{M}$” defined at $x_0$, and
corresponding to the metric $g$ at $p_0$ in $\mathcal{M}$:

$$\nu(x_0) = (\omega + \delta \omega)(x_0) = \sum_{i,j=1}^{n} \omega_{ij} \circ \tau(x_0) \hat{\Omega}^i_j(x_0) \otimes \hat{\Omega}^j_i(x_0),$$

$$\hat{\Omega}^i_j(x_0) = d^2 \hat{\tau}^i - \sum_{k=1}^{n} \hat{\tau}^i_k(x_0) dx_0^k \equiv \sum_{k=1}^{n} \hat{\kappa}^i_k(x_0) dx_0^k,$$

$$\hat{\kappa}^i_k(x_0) = \kappa^i_k(x_0) + \chi^i_k(x_0, \tau^2_0) A_k(x_0) + \sum_{h=1}^{n} \chi^{i,k}(x_0, \tau^2_0) B_{k,h}(x_0).$$

We consider the particular case for which the S-admissible metric $\omega$ is equal to $\text{diag}[+1, -1, \cdots, -1]$ (and thus $k_0 = 0$), the $\chi$’s are independent on $x_0$ and the $\tau$’s since in view of relations (14), the independence of the zero-th order $\chi$’s can be consistently assumed, and $\kappa^i_j = \delta^i_j$, i.e., the deformation of $\omega$ is only due to the tensors $A$ and $B$. Thus, one has the general relation between $\nu$ and $\omega$: $\nu = \omega + \text{linear and quadratic terms in } A \text{ and } B$, and from the metric $\nu(x_0)$, one can deduce the Riemann and Weyl curvature tensors of the “unfolded spacetime $\mathcal{M}$” at $x_0$.

Under these assumptions, we can also define the dual vector fields $\hat{\partial}$ such that at first order $\hat{\Omega}^i(\hat{\partial}_j) \simeq \delta^i_j$. We have, the relations:

$$\hat{\partial}_j = \sum_{q=1}^{n} \beta^q_j(x_0) \partial^0_q,$$

with

$$\beta^q_j(x_0) = \delta^q_j(x_0) - \chi^q_j(x_0, \tau^2_0) A_j(x_0) - \sum_{k=1}^{n} \chi^{q,k}(x_0, \tau^2_0) B_{j,k}(x_0).$$

In view of making easier computations for a relativistic action deduced from the metric tensor $\nu$, we consider this metric in a “weak fields limit”, where the metric $\nu$ is linear in the tensors $A$ and $B$, and where the quadratic terms are neglected. It follows that in a wide part of tensorial expressions, the derivatives $\hat{\partial}$ can be approximated by the $\partial^0$ ones. Furthermore, from relations (17) and taking into account the latter approximation, we have:

$$\partial^0_k A_k - \partial^0_k A_i - B_{k,i} = F_{ik}, \quad \partial^0_j B_{k,i} - \partial^0_k B_{j,i} \simeq 0, \quad J_{j,k,i} \simeq \partial^0_k B_{j,i};$$

since the functions $\rho$ take also small values in this assumed weak fields limit. We can therefore write $\nu_{ij} \simeq \omega_{ij} + \epsilon_{ij}$, where the coefficients $\epsilon_{ij}$ can be considered as small perturbations of the metric field $\omega$. Now, a most important point comes about when considering that this perturbation is (linearly) constructed out of $A$, $B$ and $\chi$ tensors, realizing, in view of relations (15) and (19), an explicit and non-trivial unification of the electromagnetic and gravitational aspects valid for all $n \geq 4$.

It is worth noticing that this feature is conserved by the full non-linearized expressions, though under a more complicated form, and that a numerical resolution for the $\nu$ metric field is certainly worth looking for.

This unification is most definitely at variance with the ones we are used to, based on superstring field theories, and in some sense just goes the other way round. Still it
Spacetime deployment opens over a wide range of quite unexpected interpretations and/or speculations, some of them to be quickly evoked shortly. We think that this formalism could also shed an interesting new light on enigmas or difficulties such as those listed in [24], for example.

Now, we can mention that from (21a), the tensor \( A \) satisfies the well-known system of PDE with the Lorentz gauge condition (\( \Box^0 \) being the d’Alembertian with respect to \( x_0 \)):

\[
\Box^0 A_i = \tilde{J}_i ,
\]

where \( \tilde{J} \) is an \( n \)-current. On the other hand, from a Lagrangian density of type (21b) such as (with uppering and lowering of indices operated by \( \omega \) at first order):

\[
L \equiv \sum_{j,k=1}^{n} B_{j,k} \tilde{K}^{j,k} + \frac{1}{2} \sum_{i,j,k=1}^{n} J_{j,[k,i]} J_{j,[k,i]}
\]

where \( J_{j,[k,i]} = J_{j,k,i} - J_{j,i,k} \), an analogous PDE system results:

\[
\Box^0 B_{j,i} = \tilde{K}_{j,i}
\]

with “gauge conditions”:

\[
\sum_{i=1}^{n} \partial^0_i B^{i,j} = 0 ,
\]

and a “generalized \( n \)-current” \( \tilde{K} \). Now, a most interesting aspect is that from the PDE systems (47) and (48), it is possible to calculate a metric field \( \nu \), and in the static case, a Newtonian potential, linearly depending on \( A \) and/or \( B \), and satisfying Poisson equations. This means that in our approach, the Newtonian limit is reached without any need for Einstein equations to be satisfied by the metric field \( \nu \), the latter being replaced by the Euler-Lagrange equations deduced from (21b) together with the first set of differential equations (19)!

Eventually, we will end up this section with a few speculations concerning the point base \( p_0 \) motion in a spacetime endowed with the metric field \( \nu \).

Let \( i \) be a differential map \( i : s \in [0, \ell] \subset \mathbb{R} \rightarrow i(s) = x_0 \in \mathbb{R}^n \simeq \mathcal{M} \), and \( U(s) \equiv di(s)/ds \), such as \( \nu(U,U) \equiv \|U\|^2 = 1 \). We define the relativistic action \( S_1 \) by:

\[
S_1 = \int_0^\ell \sqrt{\nu(U(s),U(s))} \, ds \equiv \int_0^\ell \sqrt{L_\nu} \, ds .
\]

We also take the tensors \( \chi \) as depending on \( s \) only. The Euler-Lagrange equations for the Lagrangian density \( \sqrt{L_\nu} \) are not independent because \( \sqrt{L_\nu} \) is a homogeneous function of degree 1, and thus satisfies an additional homogeneous differential equation. Then, it is well-known that the variational problem for \( S_1 \) is equivalent to consider the variation of the action \( S_2 \) defined by

\[
S_2 = \int_0^\ell \nu(U(s),U(s)) \, ds \equiv \int_0^\ell L_\nu \, ds ,
\]

where \( L_\nu \) is the Lagrangian density of type (21b) such as (with uppering and lowering of indices operated by \( \omega \) at first order).
but constrained by the condition $L_\nu = 1$. In this case, this shows that $L_\nu$ must be
considered, firstly, with an associated Lagrange multiplier, namely a mass, and secondly,
that the $L_\nu$ explicit expression with respect to $U$ will appear only in the variational
calculus. In the weak fields limit, we obtain:

$$L_\nu = \omega(U, U) + 2 \sum_{j,k=1}^n \omega_{kj} \chi^k U^j \sum_{i=1}^n A_i U^i$$
$$+ 2 \sum_{j,k,h=1}^n \chi^{k,h} \omega_{kj} U^j \sum_{i=1}^n U^i B_{i,h}.$$  (49)

From the latter relation, we can deduce a few physical consequences among others. On
the one hand, if we denote by ($h = 1, \ldots, n$)

$$C^h(\chi, U) \overset{\text{def.}}{=} \sum_{j,k=1}^n \omega_{kj} \chi^k U^j \equiv \zeta^h,$$  (50a)

$$C^0(\chi, U) \overset{\text{def.}}{=} \sum_{k,j=1}^n \omega_{kj} \chi^k U^j \equiv \zeta^0,$$  (50b)

then we recover in (49), up to some suitable constants, the Lagrangian density for
a particle (with charge $\zeta^0$), with the velocity $n$-vector $U$ ($\|U\|^2 = 1$), embedded in
an external electromagnetic field. But also from the relation (50b) we will find “a
generalized Thomas precession” if the tensor $(\chi^k)$, assumed to depend on $s$, in that
specific case only, but not on $x_0$, is ascribed (up to a suitable constant for units) to a
“polarization $n$-vector” [5, p. 270] “dressing” the particle (a spin for instance). Likewise,
the tensor $(\chi^{k,h})$ might be a polarization tensor of some matter, and again, the particle
would be “dressed” with this kind of polarization.

More generally, the Euler-Lagrange equations associated to $S_2$ would define a
system of “local” geodesic equations with Riemann-Christoffel symbols $\Gamma$ and such that
(with $\nu^i \simeq \omega^i$ at first order and recalling that the $\chi$’s are constants)

$$\frac{dU^r}{ds} = - \sum_{j,k=1}^n \Gamma^r_{jk} U^j U^k + \zeta^0 \sum_{i,k=1}^n \omega^{kr} F_{ki} U^i,$$  (51)

with

$$\Gamma^r_{jk}(x_0) = \frac{1}{2} \left\{ \chi^r \left( \partial_r^0 A_j + \partial_j^0 A_k \right) + \sum_{\ell=1}^n \chi^{r,\ell} \left( \partial_r^0 B_{k,\ell} + \partial_k^0 B_{j,\ell} \right) \right\}.$$

Let us indicate that we can compare (51) with the analogous equation (6.9′′) in [10] but
with different Riemann-Christoffel symbols.

Moreover $\mathcal{A}$ and $\mathcal{B}$ must satisfy the first and second sets of differential equations
associated to $\mathcal{M}$, i.e., the relations (19) and (20) but into which the derivatives $\partial^0_j$ are
substituted by the derivatives $\hat{\partial}_j$. Nevertheless in the weak fields limit these equations
reduce again to the equations (19) and (20).
The tensor $\Gamma$ would be associated to gravitational fields, also providing other physical interpretations for the tensors $\chi$. This tensor can also satisfy the so-called “meshing assumption” of Ghins and Budden [13]. In the present context, for $x_0$ restricted to a free-falling worldline $\mathcal{W}$, it involves in full generality that we must have the relation: $\Gamma^i_{jk}(x_0) = \gamma^i_{jk}(x_0)$. In particular, with the choice taken for $\omega$ we have $\Gamma^i_{jk}(x_0) = 0$. Then, as required by the latter assumption and quoting Ghins and Budden, we would deduce on $\mathcal{W}$ only that, indeed, “special relativistic laws [would] hold in [their] standard vectorial forms” with relations (19), (20) and equation (51), for an interacting particle in a non-necessarily locally flat spacetime. This meshing assumption is fundamental since it prevents one from considering the metric $\nu$ as a pull-back of $\omega$ in anyway, but rather like a deformation, as we did in the present paper. Indeed in this pull-back case, at any point $x_0$, the meshing condition would be always satisfied from the definition of a metric applied on vectors at $x_0$ (and would no longer be an assumption!), i.e., we would have everywhere only special relativity laws, even in a non-flat spacetime (see the deep Ghins and Budden paper).

Coming back to equations (51), the latter are deduced irrespective of the conditions (50) to be set to constants. Now, if the $\zeta$’s are constants and in the case of an explicit $s$ dependence of the $\chi$’s, not the one induced by $x_0 = i(s)$, this would lead to a modification of the action $S_2$ resulting from the introduction of Lagrange multipliers $\lambda_0$ and $\lambda_k (k = 1, \cdots, n)$ in the Lagrangian density definition. We would then define a new action of the type:

$$S_2 = \int_0^l \left\{ m\|U\|^2 + \sum_{i=1}^{n} \epsilon_{ij} U^j U^i - \sum_{k=0}^{n} \lambda_k c^k(\chi, U) \right\} ds.$$ 

The associated Euler-Lagrange equations would be analogous to (51), but with additional terms coming from the generalized Thomas precession previously evoked. Moreover, since we have the constraint $\|U\|^2 = 1$, we need a new Lagrange multiplier denoted by $m$.

Then the variational calculus would also lead to additional precession equations giving rise to torsion. In the present situation, torsion is not related to unification but to parallel transports on manifolds which is a well-known geometrical fact [9]. Hence, the existence of a precession phenomenon for a spin or polarization $n$-vector ($\chi^k$) would be correlated with the existence of linear ODE for a charged particle of charge $\zeta^0$ interacting with an electromagnetic field. Otherwise, without (50b) the ODE’s would be non-linear and there wouldn’t be any kind of precession of any spin or polarization $n$-vector.

Consequently the motion defined by the second term in the r.h.s. of (51) for a spinning charged particle would just be, in this model, a point of view resulting from an implicit separation of rotational and translational degrees of freedom achieved by the specialized (sensitive to particular subgroups of the symmetry group of motions) experimental apparatus in $T_{p_0}\mathcal{M}$. This separation would insure either some simplicity (i.e., linearity) or, since the measurements are achieved in $T_{p_0}\mathcal{M}$, that the equations of motion are associated (via some kind of projections inherent to implicit dynamical
Spacetime deployment

constraints, due to the experimental measurement process and apparatus, fixing, for instance, \( \zeta^0 \) to a constant) to linear representations of tangent actions of the Lorentz Lie group on \( T_{p_0} \mathcal{M} \). In the latter case, one could say, a somewhat provocative way of course, that special relativity invariance would have to be satisfied, \textit{as much as possible}, by physical laws. Note that this “reduction” to linearity can’t be done on the first summation in the r.h.s. of (51), which must be left quadratic contrarily to the second one, since the Riemann-Christoffel symbols can’t be defined covariantly (other arguments can be found in [10]).

To conclude, equations (51) would provide us with another interpretation of spin or polarization (the \( \chi \)’s) as an object allowing moving particles to generate effective spacetime deformations, as “wakes” for instance.

5. Conclusion

In the present article, we have been using the Pfaff systems theory and the Spencer theory of differential equations, to study the formal solutions of the conformal Lie system with respect to the Poincaré one. More precisely, we determined the difference between these two sets of formal solutions. We gave a description of a “relative” set of PDE, namely the “GHSS system”, which provides the basis of a deployment from the Poincaré Lie pseudogroup to a sub-pseudogroup of the conformal Lie pseudogroup. We studied these two systems of Lie equations because of their specific occurrence in physics, particularly in electromagnetism as well as in Einsteinian relativity.

Relying on this concept of deployment, we made the assumption that the unfolding is related to the existence of two kinds of spacetimes, namely, a substratum spacetime \( S \), from which the spacetime manifold \( \mathcal{M} \) is unfolded. We recall that not all of the given metrics on \( S \) are admissible so as to define, at least along the lines proposed in the present article, such a deployment of \( \mathcal{M} \) out of a substratum spacetime \( S \). In the case of a substratum spacetime \( S \) endowed with an appropriate \( S \)-admissible metric, allowing for unfolding, we assumed that \( S \) is equivariant with respect to the conformal and Poincaré pseudogroups, and set its Riemannian scalar curvature to a constant \( n(n - 1)k_0 \), and its Weyl tensor to zero. At this stage, the deployment evolution can be trivial or not depending on the occurrences of spacetime singularities (of the deformation potentials) parametrizing or dating what can be considered somehow as a kind of spacetime history. The deformation potentials are built out of a particular relative Spencer differential sequence associated to the “GHSS system”, and describing smooth deformations of \( S \). Then a “local” metric \( \nu \), defined on a moving tangent spacetime \( T_{p_0} \mathcal{M} \) to the unfolded spacetime \( \mathcal{M} \), is constructed out of the \( S \)-admissible substratum metric \( \omega \), and of the deformation potentials. An unification of two of the most fundamental aspects of our physical word come out realized with, we think, a number of new and interesting new lights shed on various issues of contemporary physics.

The tangent spacetimes dynamics are given by a system of PDE satisfied by their Lorentzian velocities \( n \)-vectors \( U \), exhibiting both classical electrodynamic and “local”
Since metrics are local) geodesic navigation in a spacetime endowed with gravitation.

Throughout, our approach has remained “classical” and quantization doesn’t seem to play much role. But quite on the contrary, we think that the formalism developed here could provide the basis of a new and deeper recasting of whole branches of physics, the quantic world included. This is of course because of the fundamental role played by symmetries at any scale, and not solely at the classical one. In this line of thinking, it is worth pointing out some recent reflexions of G. ’t Hooft about “Obstacles on the Way Towards Quantization of Space, Time and Matter” [36]. ’t Hooft’s theory of “Ontological States”, involved in his approach of “deterministic quantization”, strongly requires such a description of spacetime as a fluid, as well as its close relationship with the principle of coordinate invariance.

Appendix A. Appendix

It is interesting to remark that the Taylor coefficients could be defined, for any solution, from partial as well as covariant derivatives of the solution; The results remain essentially the same. Indeed, given a covariant derivative \( \tilde{\nabla} \) on \( T_{p_0}M \), the basis vectors \( e_i \) associated to the coordinates \( \xi^i = x^i - x^i_0 \), the notations \( \tilde{\nabla}^j \equiv (\tilde{\nabla})^j \) and \( \partial^j \equiv (\partial)^j \) for derivatives of order \( j \), and the monomial \( m \equiv k (x^1 - x^1_0)^{i_1} \ldots (x^n - x^n_0)^{i_n} \) (where \( k \) is an element of an \( \mathbb{R} \)-vector space of finite rank type), the following relation at \( x_0 \) holds true:

\[
\frac{1}{i_1! \ldots i_n!} \tilde{\nabla}_{e_1}^{i_1} \ldots \tilde{\nabla}_{e_n}^{i_n} (m) = \frac{1}{i_1! \ldots i_n!} \partial_1^{i_1} \ldots \partial_n^{i_n} (m) = k.
\]

This result is nothing but an example of a more general situation such as the one encountered in the Kumpera-Spencer property [22, p. 70] [23, p. 34] or in the Gasqui Lemma [11, Lemma 0.1] [12, Lemma 0.2]. This shows also the vectorial (not affine) feature of the symbol spaces, i.e., the space of elements \( k \) at any given order \( |I| = i_1 + \ldots + i_n \).

This property has also much to do with the so-called “Meshing Assumption” of Ghins and Budden exhibited here as a ground mathematical property, necessarily satisfied in the framework of our gravity approach.

Moreover, in full generality we will not put any restrictions on the kind of vectors \( \xi \), the former to be defined by some constraints involving covariant derivatives for instance. In that case, we would consider for example, other Taylor coefficients by substituting the \( c_{ij} - \sum_{\ell=1}^n \gamma_{ij}^{\ell} c_{\ell} \) for the \( c_{ij} \). And then, considering \( \alpha_i \) to be invariant along a geodesic curve associated to the basis vector \( e_i \), we would deduce \( (c_{ij} - \sum_{\ell=1}^n \gamma_{ij}^{\ell} c_{\ell}) \xi^i = 0 \). This would simplify the Taylor expansion. Nevertheless, the discussion in what follows would be complexified, setting a supplementary restrictive assumption on \( \alpha \) (for a detailed discussion on that point, one may see [31]).

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