On quasipolynomial multicut-mimicking networks and kernelization of multiway cut problems

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Abstract

We show the existence of an exact mimicking network of \( k^{O(\log k)} \) edges for minimum multicuts over a set of terminals in an undirected graph, where \( k \) is the total capacity of the terminals. Furthermore, if \textsc{Small Set Expansion} has an approximation algorithm with a ratio slightly better than \( \Theta(\log n) \), then a mimicking network of quasipolynomial size can be computed in polynomial time. As a consequence of the latter, several problems would have quasipolynomial kernels, including \textsc{Edge Multiway Cut}, \textsc{Group Feedback Edge Set} for an arbitrary group, \textsc{0-Extension} for integer-weighted metrics, and \textsc{Edge Multicut} parameterized by the solution and the number of cut requests. The result works via a combination of the matroid-based irrelevant edge approach used in the kernel for \textsc{s-Multiway Cut} with a recursive decomposition and sparsification of the graph along sparse cuts. The main technical contribution is a matroid-based marking procedure that we can show will mark all non-irrelevant edges, assuming that the graph is sufficiently densely connected.

This is the first progress on the kernelization of \textsc{Multiway Cut} problems since the kernel for \textsc{s-Multiway Cut} for constant value of \( s \) (Kratsch and Wahlström, FOCS 2012).

1 Introduction

Graph separation questions are home to some of the most intriguing open questions in theoretical computer science. In approximation algorithms, the well-known unique games conjecture (UGC) has been central to the area for close to two decades, and is closely related to graph separation problems. Even more directly, the small set expansion hypothesis, proposed by Raghavendra and Steurer [31], roughly states that it is NP-hard to approximate the \textsc{Small Set Expansion} problem (SSE) up to a constant factor, where SSE is the problem of finding a small-sized set in a graph with minimum expansion. (More precise statements are given in Section 2.2). Despite significant research, the best result available in polynomial time is an \( O(\log n) \)-approximation due to Räcke [30].

Another interesting notion from parameterized complexity is kernelization. Informally, a kernelization algorithm is a procedure that takes an input of a parameterized, usually NP-hard problem and reduces it to an equivalent instance of size bounded in the parameter, e.g., by discarding irrelevant parts of the input or transforming some part of the input into a smaller object with equivalent behaviour. For the seminal Nemhauser-Trotter theorem on the half-integrality of \textsc{Vertex Cover} [28] implies that an instance of \textsc{Vertex Cover} can be reduced to have at most \( 2k \) vertices, where \( k \) is the bound on the solution size. On the flip side, Fortnow and Santhanam [12] and Bodlaender et al. [3] gave a framework to exclude the existence of a kernel of any polynomial size, under a standard complexity-theoretic conjecture. An extensive collection of upper and lower bounds for kernelization exists (see, e.g., the recent book of Fomin et al. [11]), but a handful of central “hard questions” remain unanswered. One of the most notorious is \textsc{Multiway Cut}.

Let \( G = (V, E) \) be a graph and \( T \subseteq V \) a set of terminals in \( G \). An (edge) multiway cut for \( T \) in \( G \) is a set of edges \( X \subseteq E \) such that no two terminals are connected in \( G - X \). And \textsc{Multiway Cut} is the problem of finding a multiway cut of at most \( k \) edges. The problem is FPT [25] and NP-hard for

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Using methods from matroid theory, Kratsch and Wahlström [17] were able to show that if $|T| \leq s$, then Multiway Cut has a kernel with $O(k^{s+1})$ vertices, hence the problem has a polynomial kernel for every constant $s$. However, if $|T|$ is unbounded, the only known size bound for a kernel is $2^{O(k)}$, following from the FPT algorithm [23], and the question of whether Multiway Cut has a polynomial kernel in the general case is completely open.

We show a connection between kernelization of Multiway Cut-type problems and approximation algorithms for Small Set Expansion. Specifically, we show the existence of a kind of mimicking network for the problem, of size quasipolynomial in $k$; and if SSE has approximation algorithms slightly better than current state of the art, then it can be computed in polynomial time and Multiway Cut has a quasipolynomial kernel.

1.1 Mimicking networks and multiway cut sparsifiers

Although kernelization is most commonly described in terms of polynomial-time preprocessing as above, there is also a clear connection with succinct information representation. For example, consider a graph $G = (V, E)$ with a set of $k$ terminals $T \subseteq V$. The pair $(G, T)$ is referred to as a terminal network. A mimicking network for $(G, T)$ is a graph $G' = (V', E')$ with $T \subseteq V'$ such that for any sets $A, B \subseteq T$, the min-cut between $A$ and $B$ in $G$ and $G'$ have the same value. A mimicking network of size bounded in $k$ always exists, but the size of $G'$ can be significant. The best known general upper bound is double-exponential in $k$ [13, 15], and there is an exponential lower bound [20]. Better bounds are known for special graph classes, but even for planar graphs the best possible general bound has $2^{\omega(k)}$ vertices [20, 15] (see also recent improvements by Krauthgamer and Rika [19]).

A related notion is cut sparsifiers, which solve the same task up to some approximation factor $q \geq 1$ [27, 22], typically $q = \omega(1)$ in the general case. We focus on mimicking networks; see Krauthgamer and Rika [19] for an overview of cut sparsifiers.

However, if we include the capacity of the set of terminals in the bound (and if edges have integer capacity), then significantly stronger results are possible. Chuzhoy [4] showed that if the total capacity of $T$ is $cap_G(T) = \sum_{t \in T} d(t) = k$, then there exists an $O(1)$-approximate cut sparsifier of size $O(k^3)$. Kratsch and Wahlström [17] sharpened this to an exact mimicking network with $O(k^3)$ edges, which furthermore can be computed in randomized polynomial time. This is particularly remarkable given that the network has to replicate the exact cut-value for exponentially many pairs $(A, B)$. The network can be constructed via contractions on $G$. This built on an earlier result that used linear representations of matroids to encode the sizes of all $(A, B)$-min cuts into an object using $O(k^3)$ bits of space [18], although this earlier version did not produce an explicit graph, i.e., not a mimicking network.

These results had significant consequences for kernelization. The succinct representation in [18] was used to produce a (randomized) polynomial kernel for the Odd Cycle Transversal problem, thereby solving a notorious open problem in parameterized complexity [18]; and the mimicking network of [17] brought further (randomized) polynomial kernels for a range of problems, in particular including Almost 2-SAT, i.e., the problem of satisfying all but at most $k$ clauses of a given 2-CNF formula.

Similar methods are relevant for the question of separating a set of terminals into more than two parts. Let $(G, T)$ be a terminal network, and let $T = T_1 \cup \ldots \cup T_s$ be a partition of $T$. A multiway cut for $T$ is a set of edges $X \subseteq E(G)$ such that $G - X$ contains no path between any pair of terminals $t \in T_i$ and $t' \in T_j$ for $t, t' \notin X$ and $i \neq j$. Let us define a multicut-mimicking network for $(G, T)$ as a terminal network $(G', T)$ where $T \subseteq V(G')$ and for every partition $T = T_1 \cup \ldots \cup T_s$ of $T$, the size of a minimum multiway cut for $T$ is identical in $G$ and $G'$. (The term multicut-mimicking, as opposed to multiway cut-mimicking, is justified; see Section 2.1) The minimum size of a multicut-mimicking network, in terms of $k = cap_G(T)$, appears to lie at the core of the difficulty of the question of a polynomial kernelization of Multiway Cut. The kernel for $s$-Multiway Cut mentioned above builds on the computation of a mimicking network of size $O(k^{s+1})$ for partitions of $T$ into at most $s$ parts [17]. The kernel for $s$-Multiway Cut then essentially follows from considering the partition $T = \{t_1\} \cup \ldots \cup \{t_s\}$ of a set $T$ of $|T| = s$ terminals (along with known reduction rules bounding $cap_G(T)$). We are not aware of any non-trivial lower bounds on the size of a multicut-mimicking network in terms of $k$; it seems completely consistent that every terminal network $(G, T)$ would have a multicut-mimicking network of size poly($k$).
In this paper, we show that any terminal network \((G, T)\) with \(\text{cap}_G(T) = k\) admits a multicut-mimicking network \((G', T)\) where \(|V(G')| = kO(\log k)\), and furthermore, such a network could be computed in randomized polynomial time, given a polynomial number of queries to a sufficiently good approximation algorithm for a graph separation problem similar to Small Set Expansion. We also see a tradeoff between the quality of the approximation algorithm and the size of \((G', T)\). In particular, if Small Set Expansion has an approximation algorithm with a ratio of \(\alpha(n, k) = \log^{1-\varepsilon} n \cdot \log^{O(1)} k\) for some \(\varepsilon > 0\), where \(k\) is the number of edges cut in the optimal solution, then \((G', T)\) can be computed efficiently, with \(|V(G')|\) being quasipolynomial in \(k\). Whereas such an algorithm goes beyond the bounds of what is currently known (namely, a ratio of \(O(\log n)\) due to Räcke [30], improved for certain regimes by Bansal et al. [2]), it is certainly not excluded by any established hardness conjecture (to our knowledge). We also consider the existence result very interesting in its own right. The results strongly suggest the existence of a quasipolynomial kernel for Edge Multiway Cut. We leave open the question of existence of a poly(k)-sized multicut-mimicking network in general.

Flow sparsifiers. Finally, similarly to cut sparsifiers, there is a notion of a flow sparsifier of a terminal network \((G, T)\). Here the goal is to approximately preserve the minimum congestion for any multicommodity flow on \((G, T)\). For some results on achievable bounds for flow sparsifiers, see [1, 9]. However, the notion is incomparable to multicut-mimicking networks, because even an exact flow sparsifier would be subject to the corresponding multicut-mimicking separation gap, which is \(\Theta(\log k)\) in the worst case [13].

1.2 Our results

More formally, we have the following.

**Theorem 1.** Let \(A\) be an approximation algorithm for Small Set Expansion with an approximation ratio of \(\alpha(n, k) = O(\log^{1-\varepsilon} n \log^d k)\), where \(\varepsilon > 0\), \(d = O(1)\), and \(k\) is the number of edges cut in the optimal solution. Let \((G, T)\) be a terminal network with \(\text{cap}_G(T) = k\). Then there is a set \(Z \subseteq E(G)\) with \(|Z| = kO(\alpha(n, k) \log k)\) such that for every partition \(T = T_1 \cup \ldots \cup T_s\) of \(T\), there is a minimum multiway cut \(X\) for \(T\) such that \(X \subseteq Z\). Furthermore, \(Z\) can be computed in randomized polynomial time using calls to \(A\).

The precise requirement for the approximation algorithm is slightly relaxed from the above. We refer to the precise algorithm we need as a sublogarithmic terminal expansion tester; see Def. [1]. Simplifying the statement a bit gives us the following.

**Corollary 1.** Let \((G, T)\) be a terminal network with \(\text{cap}_G(T) = k\). The following holds.

1. There is a multicut-mimicking network for \((G, T)\) with \(kO(\log k)\) edges.

2. If there is a sublogarithmic terminal expansion tester – in particular, if Small Set Expansion has an approximation ratio as in Theorem [1] – then a multicut-mimicking network of size quasipolynomial in \(k\) can be computed in randomized polynomial time.

This would give us the following sampling of conditional breakthrough results in kernelization. We refer to previous kernelization work [17, 52] for the necessary definitions.

**Corollary 2.** If there is a sublogarithmic terminal expansion tester, then the following problems have randomized quasipolynomial kernels.

1. **Edge Multiway Cut** parameterized by solution size.
2. **Edge Multicut** parameterized by the solution size and the number of cut requests.
3. **Group Feedback Edge Set** parameterized by solution size, for any group.
4. **Subset Feedback Edge Set** with undeletable edges, parameterized by solution size.
5. **0-Extension** for integer-weighted graphs, parameterized by solution cost.
Preliminaries. A parameterized problem is a decision problem where inputs are given as pairs \( I = (X, k) \), where \( k \) is the parameter. A polynomial kernelization is a polynomial-time procedure that maps an instance \( (X, k) \) to an instance \( (X', k') \) where \( (X, k) \) is positive if and only if \( (X', k') \) is positive, and \(|X'|, k' \leq g(k) \) for some function \( g(k) \) referred to as the size of the kernel. A problem has a polynomial kernel if it has a kernel where \( g(k) = k^{O(1)} \). We extend this to discuss quasipolynomial kernels, which is the case that \( g(k) = k^{\log^{O(1)} k} \).

We use standard terminology from graph theory and parameterized complexity; see, e.g., [5,11] for references.

2 Terminal separation notions

For a graph \( G = (V, E) \) and sets \( A, B \subseteq V \), we let \( E_G(A, B) = \{ uv \in E \mid u \in A, v \in B \} \). As shorthand for \( S \subseteq V \) we also write \( E(S) = E(S, S), \delta_G(S) = E_G(S, V \setminus S) \), and \( \delta_G(S) = |\delta_G(S)| \). The total capacity of a set of vertices \( S \) in a graph \( G \) is

\[
\text{cap}_G(S) := \sum_{v \in S} d(v).
\]

In all cases, we may omit the index \( G \) if understood from context.

2.1 Multicut-mimicking networks

Let \( G = (V, E) \) be a graph and \( T \subseteq V \) a set of terminals with \( \text{cap}_G(T) = k \). An (edge) multiway cut for \( T \) in \( G \) is a set of edges \( X \subseteq E \) such that no two vertices in \( T \) are connected in \( G - X \). More generally, let \( T = \{ T_1, \ldots, T_t \} \) be a partition of \( T \). Then an (edge) multiway cut for \( T \) in \( G \) is a set of edges \( X \subseteq E \) such that in \( G - X \) every connected component contains terminals from at most one part of \( T \). Hence a multiway cut for \( (G,T) \) is equivalent to a multiway cut for \( (G, \{ \{ t \} \mid t \in T \} \). Furthermore, let \( R \subseteq \left(\frac{1}{2}\right)^T \) be a set of pairs over \( T \), referred to as cut requests. A multi-cut for \( R \) in \( G \) is a set of edges \( X \subseteq E \) such that every connected component in \( G - X \) contains at most one member of every pair \( \{u, v\} \in R \). A minimum multi-cut for \( R \) in \( G \) is a multi-cut for \( R \) in \( G \) of minimum cardinality. Similarly, a minimum multiway cut for \( T \) in \( G \) is a multiway cut for \( T \) in \( G \) of minimum cardinality.

We define a multicut-mimicking network for \( T \) in \( G \) as a graph \( G' = (V', E') \) such that \( T \subseteq V' \) and such that for every set of cut requests \( R \subseteq \left(\frac{1}{2}\right)^T \), the size of a minimum multi-cut for \( R \) is equal in \( G \) and in \( G' \). We observe that this is equivalent to preserving the sizes of all multiway cuts over partitions of \( T \).

Proposition 1. A graph \( G' \) with \( T \subseteq V(G') \) is a multicut-mimicking network for \( T \) in \( G \) if and only if, for every partition \( T \) of \( T \), the size of a minimum multiway cut for \( T \) is equal in \( G \) and in \( G' \).

Proof. It is clear that the condition is necessary, since for any partition \( T \) of \( T \) we could form the set \( R \) of all pairs over \( T \) which lie in distinct parts of \( T \), and a multi-cut for \( R \) is then necessarily a multiway cut for \( T \). To see that the condition is also sufficient, consider an arbitrary set of cut requests \( R \subseteq \left(\frac{1}{2}\right)^T \) and let \( X \) be a minimum multi-cut for \( (G, R) \). Let \( \mathcal{T} \) be the partition of \( T \) in \( G - X \) according to connected components. Then \( X \) is a multi-cut for \( T \) in \( G \), and any multi-way cut for \( T \) is also a multi-cut for \( R \). Hence the size of a minimum multi-cut for \( R \) is precisely the size of a minimum multi-cut for \( T \). □

As a slightly sharper notion, a multi-cut-covering set for \( (G, T) \) is a set \( Z \subseteq E(G) \) such that for every set of cut requests \( R \subseteq \left(\frac{1}{2}\right)^T \), there is a minimum multi-cut \( X \) for \( R \) in \( G \) such that \( X \subseteq Z \). Note that a multi-cut-covering set \( Z \) is essentially equivalent to a multicut-mimicking network formed by contraction (contracting all edges of \( E(G) \setminus Z \)). Our main result in this paper is the existence of a multi-cut-covering set of size quasipolynomial in \( k = \cap(T) \) in any undirected graph \( G \). Furthermore, such a set can be computed in polynomial time, subject to the existence of certain approximation algorithms that we will make precise later in this section. The term is a generalization of a cut-covering set, used in previous work [17].
2.2 Graph separation algorithms

The central technical approximation assumption needed in this paper is the following. For a graph $G$ with a set of terminals $T$, define the $T$-capacity of $S$ in $G$ as

$$\text{cap}_T(S) = \text{cap}_G(T \cap S) + \delta_G(S).$$

Then we define the following notion.

**Definition 1** (Sublogarithmic terminal expansion tester). Let $(G, T)$ be a terminal network with $\text{cap}_G(T) = k$. A terminal polynomial expansion tester (with approximation ratio $\alpha$) is a (possibly randomized) algorithm that, given as input $(G, T)$ and an integer $c \in \mathbb{N}$, with $c = \Omega(\log k)$, does one of the following.

1. Either returns a set $S \subseteq V$ such that $N_G(S) \neq V(G)$ and $|S| > \text{cap}_T(S)^c$,

2. or guarantees that for every set $S$ with $\emptyset \subset (S \cap T) \subset T$ and $|S| \leq |V(G)|/2$ we have $\text{cap}_T(S) \geq |S|^{1/c}/\alpha$.

A sublogarithmic terminal expansion tester is a terminal polynomial expansion tester with an approximation ratio $\alpha = O(\log^{1-\varepsilon} n \log^{O(1)} k)$ for some $\varepsilon > 0$. We say that $(G, T)$ is $(\alpha, c)$-dense if case 2 above applies, i.e., for every set $S$ with $S \cap T \notin \{\emptyset, T\}$ and $|S| \leq |V(G)|/2$ we have $\text{cap}_T(S) \geq |S|^{1/c}/\alpha$.

The conditions can be relaxed somewhat. It is sufficient if the algorithm works with parameters $c = \Omega(\alpha \log k)$. It is also possible to put a lower bound on the size of sets $S$ for which the guarantee needs to apply. However, these relaxed assumptions do not seem to make a difference for any algorithms we are aware of for the problem.

We note that such an algorithm would follow from a slightly improved approximation algorithm for SMALL SET EXPANSION. Let $G = (V, E)$ be a graph and $S \subseteq V$ a set of vertices. The edge expansion of $S$ is

$$\Phi(S) := \frac{\delta(S)}{|S|}.$$ 

For a real number $\rho \in (0, 1/2]$, one also defines the small set expansion

$$\Phi_\rho(G) := \min_{S \subseteq V, |S| \leq n \rho} \Phi(S).$$

In particular, for a value $s \in [n/2]$, $\Phi_{s/n}(G)$ denotes the worst (i.e., minimum) expansion among subsets of $G$ of size at most $s$. A sufficiently good approximation algorithm for SMALL SET EXPANSION implies a sublogarithmic terminal expansion tester, as follows.

**Lemma 1.** Assume that SMALL SET EXPANSION has a bicriteria approximation algorithm that in input $(G, \rho)$ returns a set $S$ with $|S| \leq \beta n$ and $\Phi(S) \leq \alpha \cdot \Phi_\rho$, for some $\alpha, \beta \geq 1$. If $\alpha \beta = O(\log^{1-\varepsilon} n \log^{O(1)} n \cdot \Phi_{s/n})$, for some $\varepsilon > 0$, then there is a sublogarithmic terminal expansion tester with ratio $\Theta(\alpha \beta)$ (with $n \cdot \Phi_\rho$ replaced by $k$).

**Proof.** Let $\alpha' = 2\alpha\beta$. Assume that $(G, T)$ is not $(\alpha', c)$-dense for some parameters $\alpha$ and $c$, and let $S \subseteq V$ be a set witnessing this, i.e., $S \cap T \neq \emptyset$, $N[S] \neq V(G)$, and $\text{cap}_T(S) < |S|^{1/c}/\alpha'$. We argue that the set $S \setminus T$ is also a legal return value for the algorithm. Indeed, note

$$\text{cap}_T(S \setminus T) = \delta(S \setminus T) \leq \delta(S) + \cap_G(T \cap S) = \text{cap}_T(S).$$

We also have $|S| > (\alpha' \text{cap}_T(S))^c \geq (\alpha' \text{cap}_T(S \setminus T))^c$. Now, recall that MINIMUM BISECTION is FPT parameterized by the solution value (i.e., the number of edges cut by an optimal solution), with the fastest FPT algorithm running in time $O(2^{O(k \log \log n)})$ for parameter $p$ [4]. Hence we can in polynomial time check for a bisection with $p = O(\log n / \log \log n)$ edges, and by replacing a vertex with a suitably large clique we can also check for a set $S'$ of cardinality $s$ with $\delta(S') \leq p$. Hence in the remaining case we assume $\text{cap}_T(S) \geq \delta(S) \geq \Omega(\log n / \log \log n)$. Furthermore, we may assume $c = \Omega(\log k)$. Hence

$$|S| \geq (\alpha' \text{cap}_T(S))^c \geq (\log n / \log \log n)^{\log k} = k^{\Omega(\log \log n)},$$

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and the difference in size between $|S|$ and $|S \setminus T| \geq |S| - k$ is negligible. Hence
\[
\Phi(S \setminus T) = \frac{\delta(S \setminus T)}{|S \setminus T|} \leq \frac{\text{cap}_T(S)}{(1 - o(1))|S|} \leq (1 + o(1))(1/\alpha')|S|^{1/c-1}.
\]

Now attach a large clique to every terminal in $G$, forming a graph $G'$, and call an approximation algorithm for SMALL SET EXPANSION with a parameter of $\rho = |S|/|V(G')|$. Assume that the algorithm returns a set $S' \subseteq V(G')$. Then $S' \cap T = \emptyset$ and $\text{cap}_T(S') = \delta(S')$. Furthermore $|S'| \leq \beta|S|$ and $\Phi(S') \leq o(1)|S|$. Then
\[
\text{cap}_T(S') = \delta(S') = |S'|\Phi(S') \leq \beta|S|\Phi(S') < \beta(1 + o(1))|S|^{1/c} \leq (\beta^{1/c}/2)(1 + o(1))|S|^{1/c-1}.
\]

Since $c = \Omega(\log k)$ and $\beta = o(k)$, we have $\beta^{1/c}(1 + o(1)) = (1 + o(1))$ and this factor is asymptotically defeated by the constant 2. Hence for large enough $k$ and $n$, we have $|S'| > \text{cap}_T(S')^c$ and $S'$ is a valid return value for the algorithm. By repeating the above for all target sizes $|S| = |V(G')|$ from 1 to $|V(G)|$, we can be sure to identify such a set $S'$ if one exists.

Existing approximation algorithms do not meet this threshold; the best known results are an $O(\log n)$-approximation due to Räcke [30] and a bicriteria algorithm of Bansal et al. [2] which achieves a ratio of $O(\sqrt{\log n \log(1/\rho)})$. Unfortunately, the latter improvement is insufficient to make the analysis in the next section work. However, it seems clear that no existing hardness conjecture could possibly rule out the existence of such an algorithm. Furthermore, testing for $(\alpha, c)$-denseness when $c = \Omega(\alpha \log k)$ corresponds to looking for significantly worse expanding sets than the regime usually focused on in the approximation literature. Hence we proceed with conditional results in the rest of the paper.

3 Multicut-covering sets

We now present the main result of the paper, namely the existence of quasipolynomial multicut-mimicking networks for terminal networks $(G, T)$, and the conditional efficient computability of such objects given a sublogarithmic terminal expansion tester.

At a high level, the process works through recursive decomposition of the graph $G$ across very sparse cuts, treating each piece $G[S]$ of the recursion as a new instance of multicut-covering set computation, where the edges of $\partial(S)$ are considered as additional terminals. The process repeatedly finds a single edge $e \in E(G)$ with a guarantee that for every set of cut requests $R \subseteq \frac{\alpha}{2}$ there is a minimum multicut $X$ for $R$ in $G$ such that $e \notin X$. We may then contract the edge $e$ and repeat the process. Thus the end product is a multicut-mimicking network, and the edges that survive until the end of the process form a multicut-covering set.

In somewhat more detail, the process uses a novel variant of the representative sets approach, which was previously used in the kernel for $s$-MULTIWAY CUT [17]. Refer to an edge $e$ as essential for $R$, for some $R \subseteq \frac{\alpha}{2}$, if every minimum multicut for $R$ in $G$ contains $e$, and essential for $(G, T)$ if it is essential for $R$ for some $R \subseteq \frac{\alpha}{2}$. We use a representative sets approach to return a set of at most $k^c$ edges which is guaranteed to contain every essential edge, if $(G, T)$ is already $(\alpha, c)$-dense, for an appropriate value $c = \Theta(\alpha \log k)$. On the other hand, if $(G, T)$ is not $(\alpha, c)$-dense, then (by careful choice of parameters) we can identify a cut through $G$ which is sufficiently sparse that we can reduce the size of one side of this cut via a recursive call. This gives a tradeoff between the size of the resulting multicut-covering set and the denseness-guarantee we may assume through the approximation algorithm. The threshold for feasibility for this tradeoff is the existence of a sublogarithmic terminal expansion tester.

3.1 Recursive replacement

We now present the recursive decomposition step in detail. Let $(G, T)$ be a terminal network with $\text{cap}_G(T) = k$. For a set $S \subseteq V$, we define the graph
\[
G_S = G[N_G[S]] - E(S),
\]
i.e., $G_S$ equals the graph $G[S]$ with the edges of $\partial(S)$ added back in. We also denote
\[
T(S) = (T \cap S) \cup N_G(S).
\]
as the terminals of $S$. Under these definitions, the $T$-capacity of $S$ in $G$ has two equivalent definitions as

$$\text{cap}_T(S) = \text{cap}_{G/e}(T(S)) = \text{cap}_G(T \cap S) + \delta_G(S).$$

The recursive instance at $S$ consists of the terminal network $(G_S, T(S))$. This is the basis of our recursive replacement procedure. Indeed, we show the following. Note that we consider $E(G_S) \subseteq E(G)$ in the following.

**Lemma 2.** Let $(G_S, T(S))$ be the recursive instance at $S \subseteq V(G)$. Let $Z_S$ be a multicut-covering set for $(G_S, T(S))$ and let $e \in E(G_S) \setminus Z_S$. Then $e$ is not essential for $(G, T)$.

**Proof.** By Prop. 1, it is sufficient to consider partitions $\mathcal{T}$ of $T$ and minimum multiway cuts $X$ for $\mathcal{T}$. Let $\mathcal{T}$ be some partition of $T$, and let $e$ be a minimum multiway cut for $\mathcal{T}$ in $G$. Let $\mathcal{T}'$ be the partition of $T(S)$ induced by the connected components of $G - X$ and $X_S = X \cap E(G_S)$. Then $X_S$ is a multiway cut for $\mathcal{T}'$ in $G_S$. Indeed, any path $P$ in $G_S - X_S$ between distinct parts of $\mathcal{T}'$ also exists in $G - X$. If $\mathcal{T}'$ consists of a single part, then we have $X_S = \emptyset$, as otherwise either $X$ contains an edge $uv$ whose both endpoints lie in the same connected component of $G - X$, or $G - X$ contains a connected component with no terminals, both of which contradict that $X$ is of minimum cardinality. Otherwise, by assumption there is a minimum multiway cut $X'_S$ for $\mathcal{T}'$ in $G_S$ such that $e \notin X'$. We claim that $X' := (X \setminus X_S) \cup X'_S$ is a minimum multiway cut for $\mathcal{T}$ in $G$. Note that $|X'| \leq |X|$, hence it remains to show that $X'$ is a multiway cut. Assume for a contradiction that $G - X'$ contains a path $P$ connecting different parts of $\mathcal{T}$, and consider the partition of $P$ into subpaths induced by splitting at every vertex of $T(S)$ that $P$ intersects. Note that every such subpath is either contained in $E(G_S)$ or disjoint from $E(G_S)$, and by assumption at least one such subpath is contained in $E(G_S)$, as otherwise $P$ uses only edges also present in $G - X$. But every such subpath goes between two vertices of $T(S)$ which lie in the same connected component of $G - X$ by definition of $\mathcal{T}'$. Thus every such subpath starts and ends in a single connected component of $G - X$, contradicting that $P$ starts and ends in different components. Therefore $X'$ is a minimum multiway cut for $\mathcal{T}$ in $G$. Since $e \notin X'$ we are done.

Let us also briefly note the formal correctness of contracting a non-essential edge.

**Proposition 2.** Let $e \in E(G)$ be a non-essential edge. Then for every $X \subseteq E(G)$ with $e \notin X$, and every partition $\mathcal{T}$ of $T$, $X$ is a multiway cut for $\mathcal{T}$ in $G$ if and only if it is a multiway cut for $\mathcal{T}$ in $G/e$. Furthermore, $G/e$ is a multicut-mimicking network for $(G, T)$, and any multicut-covering set $Z \subseteq E(G/e)$ for $(G/e, T)$ is also multicut-covering for $(G, T)$.

**Proof.** The first part is clear, since the contraction of an edge in $G - X$ does not change the structure of the connected components. Since $e$ is non-essential, by assumption there exists such an optimal $X$ with $e \notin X$ for every partition $\mathcal{T}$, hence $(G/e, T)$ is a multicut-mimicking network. It also follows that an optimal solution for $G$ always exists in $E(G/e)$, hence a solution-covering set for $(G/e, T)$ is also solution-covering for $(G, T)$.

The process now works as follows. Recall that $(G, T)$ is $(\alpha, c)$-dense if $\text{cap}_T(S) \geq |S|^{1/c}/\alpha$ for every set $S$ with $S \cap T \neq \emptyset$ and $|S| \leq |V|/2$. The main technical result is a marking process that marks all essential edges for $(G, T)$ on the condition that $(G, T)$ is $(\alpha, c)$-dense, and which marks at most $k^2$ edges in total. In such a case, we are clearly allowed to select and contract any unmarked edge of $G$. Now, assume that $(G, T)$ is not $(\alpha, c)$-dense. Then by definition there exists a set $S \subseteq V$ such that $\text{cap}_T(S) < |S|^{1/c}/\alpha$. If we can detect a set $S$ such that $\text{cap}_T(S) < |S|^{1/c}$, then we can recursively compute a multicut-covering set $Z_S$ for $(G_S, T(S))$, consisting of at most $\cap_T(S) < |S|$ edges. By the above, we may again select any single edge $e \in E(G_S) \setminus Z_S$ and contract $e$ in $G$. In either case, we replace $G$ by a strictly smaller graph until $|E(G)| \leq k^2$, at which point we are done.

The two ingredients in the above are thus the marking process for $(\alpha, c)$-dense graphs, which we present next, and the ability to distinguish the two cases, which is precisely the assumption of the existence of a sublogarithmic terminal expansion tester.

### 3.2 The $(\alpha, c)$-dense case

Let us now focus on the marking procedure. Let a terminal network $(G, T)$ with $\text{cap}_G(T) = k$ and an integer $c$ be given, and assume that $c = \Omega(\alpha \log k)$ for some $\alpha$. We show a process that marks a set
of at most \(k^c\) edges that contains every essential edge, assuming that \((G, T)\) is \((\alpha, c)\)-dense. (A more precise bound on the relationship between \(c\) and \(\alpha\) is given later, but the constant factors involved are not important to our main result.)

We will prove the following result. The proof takes up the rest of the subsection.

**Lemma 3.** Assume that \((G, T)\) is \((\alpha, c)\)-dense where \(c = \Omega(\alpha \log k)\). A multicut-covering set \(Z \subseteq E(G)\) of size less than \(k^c\) can be computed in randomized polynomial time.

The basis is the following. If \((G, T)\) is \((\alpha, c)\)-dense then for every partition \(T\) of \(T\), every minimum multiway cut \(X\) for \(T\), and every connected component \(H\) of \(G - X\) except possibly the largest one, it holds that \(\text{cap}(V(H)) \geq |V(H)|^{1/c}/\alpha\). We also have

\[
\sum_{H \in G - X} \text{cap}(V(H)) = \text{cap}_G(T) + 2|X| < 3k,
\]

where the sum ranges over connected components \(H\). This implies restrictions on the possible sizes of components of \(G - X\), which will help in the marking process (as we shall see). Essentially, if too many components are too large, then the above sum will exceed \(3k\) and we can conclude non-optimality of the corresponding multiway cut.

Finally, let us eliminate a silly edge case to assume \(c \leq k\).

**Lemma 4.** If \(c > k\) then a multicut-covering set of at most \(k^c\) edges can be marked deterministically.

**Proof.** If \(k^c \geq m\), simply return \(E(G)\). Otherwise, iterate over every partition \(T\) of \(T\) (numbering at most \(k^k\)), compute for every partition \(T\) of \(T\) a minimum multiway cut in time \(O^*(2^k)\) using the algorithm of Cygan et al. [7], and return the union of all solutions. The running time is discounted against a polynomial in the total input size, and the number of edges returned is at most \(k \cdot k^k = k^{k+1} \leq k^c\).

### 3.2.1 Matroid constructions

Before we show the marking procedure, we need some additional preliminaries. We refer to Oxley [29] and Marx [26] for further relevant background on matroids.

A **matroid** is a pair \(M = (E, I)\) where \(I \subseteq 2^E\) is the independent sets of \(M\), subject to the axioms

1. \(\emptyset \in I\);
2. if \(B \in I\) and \(A \subseteq B\) then \(A \in I\); and
3. if \(A, B \in I\) with \(|B| > |A|\) then there exists an element \(x \in B \setminus A\) such that \(A + x \in I\).

The **rank** of \(M\) is the size of a maximum independent set. A basis is a maximum independent set of \(M\).

Let \(A\) be a matrix, and let \(E\) label the columns of \(A\). The **column matroid** of \(A\) is the matroid \(M = (E, \mathcal{I})\) where \(S \in \mathcal{I}\) for \(S \subseteq E\) if and only if the columns indexed by \(S\) are linearly independent. A matrix \(A\) represents a matroid \(M\) if \(M\) is isomorphic to the column matroid of \(A\). We refer to \(A\) as a **linear representation** of \(M\).

We need three classes of basic matroid. For a set \(E\), the uniform matroid over \(E\) of rank \(r\) is the matroid

\[
\mathcal{U}(E, r) := (E, \{S \subseteq E \mid |S| \leq r\}).
\]

Uniform matroids are representable over any sufficiently large field.

The second class is a truncated graphic matroid. Given a graph \(G = (E, V)\), the **graphic matroid** of \(G\) is the matroid \(M(G) = (E, \mathcal{I})\) where a set \(F \subseteq E\) is independent if and only if \(F\) is the edge set of a forest in \(G\). Graphic matroids can be deterministically represented over all fields. The \(r\)-truncation of a matroid \(M = (E, \mathcal{I})\) for some \(r \in \mathbb{N}\) is the matroid \(M' = (E, \mathcal{I}')\) where \(S \in \mathcal{I}'\) if and only if \(S \in \mathcal{I}\) and \(|S| \leq r\). Given a linear representation of \(M\), over some field \(\mathbb{F}\), a truncation of \(M\) can be computed in randomized polynomial time, possibly by moving to an extension field of \(\mathbb{F}\) [29]. There are also methods for doing this deterministically [23], but the basic randomized form will suffice for us.

The final class is more involved. Let \(D = (V, A)\) be a directed graph and \(S \subseteq V\) a set of source vertices. A set \(T \subseteq V\) is linked to \(S\) in \(D\) if there are \(|T|\) pairwise vertex-disjoint paths starting in \(S\) and ending in \(T\). Let \(U \subseteq V\). Then

\[
M(D, S, U) = (U, \{T \subseteq U \mid T \text{ is linked to } S \text{ in } D\})
\]

defines a matroid over \(U\), referred to as a ** gammoid**. Note that by Menger’s theorem, a set \(T\) is dependent in \(M\) if and only if there is a \((S,T)\)-vertex cut in \(D\) of cardinality less than \(|T|\) (where the cut is allowed
to overlap \(S\) and \(T\). Like uniform matroids, gammoids are representable over any sufficiently large field, and a representation can be computed in randomized polynomial time \([23, 20]\). We will work over a variant of gammoids we refer to as a **edge-cut gammoid**, which are defined as gammoids, except in terms of edge cuts instead of vertex cuts. Informally, for a graph \(G = (V, E)\) and a set of source vertices \(S \subseteq V\), the edge-cut gammoid of \((G, S)\) is a matroid on a ground set of edges, where a set \(F\) of edges is independent if and only if it can be linked to \(S\) via pairwise edge-disjoint paths. However, we also need to introduce the “edge version” of sink-only copies of vertices, as used in previous work \([17\). That is, we introduce a second set \(E' = \{e' \mid e \in E\}\) containing copies of edges \(e \in E\) which can only be used as the endpoints of linkages, not as initial or intermediate edges.

More formally, for a graph \(G = (V, E)\) and a set of source vertices \(S \subseteq E\) we perform the following transformation.

1. Subdivide every edge \(e \in E\) by a new vertex \(z_e\).
2. Let \(p = \text{cap}_G(S)\). Inflate every vertex \(v \in V\) into a twin class of \(p + 1\) vertices (but do not inflate vertices \(z_e\) introduced in the previous step).
3. Replace every edge \(uv\) in the resulting graph by the two directed edges \((u, v), (v, u)\), creating a directed graph.
4. For every edge \(e = uv \in E\), introduce a further new vertex \(z'_e\), and create directed edges \((u_i, z'_{e_i})\) and \((v_j, z'_{e_j})\) for every copy \(u_i, v_j\) in \(D_G\) of the vertices \(u, v\) of \(G\).

Slightly abusing notation, we let \(E\) refer to the vertices \(z_e\) in \(D_G\), and we let \(E'\) refer to the vertices \(z'_e\) in \(D_G\). The **edge-cut gammoid** of \((G, S)\) is the gammoid \((D_G, \partial(S), E \cup E')\). Let us observe the resulting notion of independence.

**Proposition 3.** Let \(G = (V, E)\) and \(S \subseteq V\) be given. Let \(M = (E \cup E', \mathcal{I})\) be the edge-cut gammoid of \((G, S)\). Let \(X \subseteq E \cup E'\) be given, and let \(F = (X \cap E) \cup \{e \mid e' \in F \cap E'\}\). Then \(X\) is independent in \(M\) if and only if there exists a set \(\mathcal{P}\) of \(|X|\) paths linking \(X\) to \(S\), where paths are pairwise edge-disjoint except that if \(\{e, e'\} \subseteq X\) for some edge \(e\), then two distinct paths in \(\mathcal{P}\) end in \(e\).

We let \(U(E, p)\) denote the uniform matroid of rank \(p\) on ground set \(E(G)\), \(M_G(p)\) the \(p\)-truncated graphic matroid of \(G\), and \(M(T)\) the edge-cut gammoid of \((G, T)\).

If \(M_1 = (E_1, I_1)\) and \(M_2 = (E_2, I_2)\) are two matroids with \(E_1 \cap E_2 = \emptyset\), then their **disjoint union** is the matroid

\[
M_1 \uplus M_2 = (E_1 \cup E_2, \{I_1 \cup I_2 \mid I_1 \in I_1, I_2 \in I_2\}).
\]

If \(M_1\) and \(M_2\) are represented by matrices \(A_1\) and \(A_2\) over the same field, then \(M_1 \uplus M_2\) is represented by the matrix

\[
A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}
\]

We will define matroids \(M\) as the disjoint union over several copies of the base matroids \(M(T)\) and \(U(E, p)\) defined above. In such a case, we refer to the individual base matroids making up \(M\) as the **layers** of \(M\).

**Representative sets.** Our main technical tool is the representative sets lemma, due to Lovász \([23]\) and Marx \([26]\). This result has been important in FPT algorithms \([20, 10]\) and has been central to the previous kernelization algorithms for cut problems, including variants of **Multiway Cut** \([17\). We also introduce some further notions.

**Definition 2.** Let \(M = (E, \mathcal{I})\) be a matroid and \(X, Y \in \mathcal{I}\). We say that \(Y\) extends \(X\) in \(M\) if \(r(X \cup Y) = |X| + |Y|\), or equivalently, if \(X \cap Y = \emptyset\) and \(X \cup Y \in \mathcal{I}\). Furthermore, let \(c = O(1)\) be a constant and let \(\mathcal{Y} \subseteq \binom{E}{c}\). We say that a set \(\hat{Y} \subseteq \mathcal{Y}\) represents \(\mathcal{Y}\) in \(M\) if the following holds: For every \(X \in \mathcal{I}\) for which there exists some \(Y \in \mathcal{Y}\) such that \(Y\) extends \(X\) in \(M\), then there exists some \(Y' \in \hat{Y}\) such that \(Y'\) extends \(X\) in \(M\).

The representative sets lemma now says the following.
Lemma 5 (representative sets lemma \cite{24,26}). Let $M = (E, \mathcal{I})$ be a linear matroid represented by a matrix $A$ of rank $r + s$, and let $\mathcal{Y} \subseteq \binom{E}{r}$ be a collection of independent sets of $M$, where $s = O(1)$. In time polynomial in the size of $A$ and the size of $\mathcal{Y}$, we can compute a set $\hat{\mathcal{Y}} \subseteq \mathcal{Y}$ of size at most $\binom{r + s}{s}$ which represents $\mathcal{Y}$ in $M$.

We use the following product form of the representative sets lemma, with stronger specialized bounds. Note that the corresponding bound from Lemma 5 would be $\Theta((r_1 + \ldots + r_c)^c)$, which can be significantly larger when the layers of $M$ have different rank.

Lemma 6 (\cite{17}). Let $M = (E, \mathcal{I})$ be a linear matroid, given as the disjoint union of $c$ matroids $M_i = (E_i, \mathcal{I}_i)$, where $M_i$ has rank $r_i$. Let $\mathcal{Y} \subseteq \binom{E}{r}$ be such that every set $Y \in \mathcal{Y}$ contains precisely one member in each layer $M_i$ of $M$. Then the representative set $\hat{\mathcal{Y}} \subseteq \mathcal{Y}$ computed by the representative sets lemma will have $|\hat{\mathcal{Y}}| \leq \prod_{i=1}^{c} r_i$.

3.2.2 The marking process

We are now ready to present the marking process.

Let $r = c - 2$. We define a process that marks edges of $G$ in $r$ passes, where each pass is a call to the representative sets lemma with a different matroid construction. Specifically, for each $i \in [r]$, define the following. The matroid $M_i$ is the disjoint union of $i$ copies of the edge-cut gammoid $M(T)$, one copy of $M_G(k^{r-i})$, and one copy of $U(E, k)$, where for $i = r$ we simply skip the copy of $M_G(k^{r})$. We refer to the first $i$ layers in $M_i$ as the gammoid layers and the remaining as the additional layers. Note that a linear representation of $M_i$ over some common field $\mathbb{F}$ can be computed in randomized polynomial time, since every layer of $M_i$ can be represented over any sufficiently large field.

For each edge $e \in E$, let $t_i(e)$ be the set that contains a copy of $z_e^i$ in every gammoid layer, and a copy of $e$ in every additional layer. Let

$$E_i = \{t_i(e) \mid e \in E\}.$$ 

For each pass $i \in [r]$, we compute a representative set $\hat{E}_i \subseteq E_i$ in the matroid $M_i$, and let $Z_i \subseteq E$ be the set of edges represented in $\hat{E}_i$. Let $Z = Z_1 \cup \ldots \cup Z_r$. We consider an edge $e \in E$ marked if $e \in Z$. We finish the description by observing the bound on the number of marked edges.

Lemma 7. The total number of marked edges is at most $rk^{r+1} < k^c$.

Proof. By the product form of the representative sets lemma, $|Z_i| \leq k^{r+1}$ for every $i \in [r]$. \qed

Our main correctness condition for the marking is as follows. Consider a partition $T$ of $T$ and a corresponding minimum multiway cut $X \subseteq E$. Note that $|X| \leq k$ since $E(T, V)$ is a multiway cut for every partition. Say that $X$ is $p$-way plus $q$ if the $p$ largest connected components of $G \setminus X$ together cover all but $q$ of the vertices. Say that $X$ is covered if all edges essential for $T$ are marked. We then have the following.

Lemma 8. If $X$ is $p$-way plus $k^{r-p}$ for $p \in [r]$, then $X$ is covered in pass $p$ above.

Proof. Let $e \in E$ be an edge which is essential for $T$. Let $V(G) = V_1 \cup \ldots \cup V_d$ be the partition of $V(G)$ according to connected components of $G \setminus X$, where $|V_1| \geq \ldots \geq |V_d|$. Let $T = \{T_1, \ldots, T_d\}$ where $T_i = T \cap V_i$ for $i \in [d]$. Finally, define an independent set $F$ in $M_p$ as follows. In the $i$:th gammoid layer, $i \in [p]$, $F$ contains copies of vertices $z_e$ from the edges of $\partial(T_i) \cup X$. In the graphic matroid layer, if any, $F$ contains a spanning forest for components $V_{p+1}$ through $V_d$. In the final layer, $F$ contains the edges of $X - e$. We claim that $t_p(f)$ extends $F$ if and only if $f = e$.

For the easier direction, we note that $t_p(f)$ cannot extend $F$ if $e \neq f$. If $f \in E(V_i)$ for some $i \leq p$, then $f$ fails to extend $F$ in layer $i$. If $f \in E(V_i)$ for $i > p$, then $e$ fails to extend $F$ in the graphic matroid layer. Finally, if $f \in X$ then $e$ fails to extend $F$ in the uniform matroid layer. Hence it remains to show that $t_p(e)$ extends $F$.

For the gammoid layers, this works precisely as in \cite{17}. As noted in \cite{17} (Prop. 1), whether a sink-only copy $v'$ extends a set $U$ in a gammoid $(D, S)$ depends on whether the original copy $v$ is contained in the $(S, U)$-min cut closest to $S$. Here, including $\partial(T_i)$ in $F$ in layer $i$ effectively turns this condition into a cut between $X$ and $\partial(T \setminus T_i)$. Hence if $v'$ does not extend $X \cup \partial(T_i)$, then there is a min-cut $X_2$ between

10
X and \( \partial(T \setminus T_i) \) that is closer to \( \partial(T \setminus T_i) \) than X, and \( e \notin X_2 \). This contradicts that \( e \) is essential for \( \mathcal{T} \).

For the additional layer, the statement is trivial. Hence \( t_e(f) \) extends \( F \) if and only if \( f = e \), as promised, and \( e \in \mathbb{Z}_p \). \( \square \)

### 3.2.3 Correctness

We now argue that if \((G, T)\) is \((\alpha, c)\)-dense for \( c = \Theta(\alpha \log k) \) then every partition of \( T \) has a minimum \( k \)-way plus \( k^{r-\alpha} \) for some \( p \in [r] \). For this, assume for a contradiction that for some partition \( \mathcal{T} \) of \( T \) the minimum multiway cut \( X \) of \( \mathcal{T} \) is not covered in any of the above passes. We will derive that \( |X| > k \), contradicting that \( X \) is minimum. Assume that \( G - X \) has \( p \) components, and let \( n_1 \geq \ldots \geq n_p \) be the number of vertices in each component, sorted by size. The converse to Lemma 8 is the following.

**Corollary 3.** If \( X \) is not covered, then for every \( i \in [r] \) it holds that \( \sum_{j=i+1}^{p} n_j > k^{r-i} \).

For \( i \in [r] \), let us write \( n_{\geq i} = \sum_{j=i}^{p} n_j \). Hence for each \( i \in [r] \), \( n_{\geq i+1} > k^{r-i} \).

Now, refer as previously to the vertex sets of the connected components of \( G - X \) in order as \( V_1, \ldots, V_p \), where \( |V_i| = n_i, i \in [p] \). By the density assumption, for every \( i \geq 2 \),

\[
\text{cap}_r(V_i) \geq n_i^{1/c}/\alpha.
\]

On the other hand, as previously noted, if \( X \) is minimum we have

\[
\sum_{i=1}^{p} \text{cap}_r(V_i) = \sum_{i=1}^{p} (\text{cap}_G(T \cap V_i) + \delta(V_i)) = k + 2|X| \leq 3k.
\]

It now remains to estimate the value of the following system:

\[
\begin{align*}
\min \quad & \sum_{i=2}^{p} n_i^{1/c}/\alpha \\
\text{s.t.} \quad & \sum_{j=i+1}^{p} n_j > k^{r-i} \quad \forall i \in [r] \\
& \sum_{i=1}^{p} n_i = n \\
& n_1 \geq \ldots \geq n_p \geq 0
\end{align*}
\]

If we can determine that this value is greater than \( 3k \), then we will have derived a contradiction, showing that the cut \( X \) is covered. This is somewhat intricate, but not very difficult.

### 3.2.4 Bounding the number of edges

We now show the following, to wrap up the correctness.

**Lemma 9.** There is an \( e = \Theta(\alpha \log k) \) such that the following holds: If \((G, T)\) is \((\alpha, c)\)-dense, and if \( X \) is a multiway cut for some partition \( \mathcal{T} \) of \( T \) such that \( X \) is not covered, then \( |X| > k \).

We show that for some \( e = \Theta(\alpha \log k) \), \( 3.2.3 \) takes a value greater than \( 3k \), showing that \( X \) is non-optimal.

**Distribution of component sizes.** We claim that the worst possible distribution of values \( n_i \) (i.e., that sequence which minimizes the value of the above system) is when it is maximally skewed, i.e.,

\[
n_i = \begin{cases} 
n - k^{r-1} & i = 1, \\
k^{r-i} - k^{r-1} & 2 \leq i < p, \\
1 & i = p = r + 1.
\end{cases}
\]

Indeed, let \( i, j \in [p] \), \( 1 < i < j \) and \( n_i > n_j \). Since \( x^{1/c} \) is a strictly concave function in \( x \), we have

\[
(n_i - 1)^{1/c} + (n_j + 1)^{1/c} \geq n_i^{1/c} + n_j^{1/c},
\]
with equality only if \( n_i = n_j + 1 \). Conversely,
\[
(n_i + 1)^{1/c} + (n_j - 1)^{1/c} < n_i^{1/c} + n_j^{1/c}
\]
assuming \( n_j > 1 \). Note that \( \alpha \) is treated like constant (being fixed by \( n \) and \( k \)), and does not affect these conclusions. Similarly, since \( n_1 \) does not contribute to the value, it is clear that in the worst case, \( n_1 \) is as large as possible. Thus the worst case is when the component sizes are maximally skewed, so that for \( r \geq i \geq 2 \) we have \( n_{i-1} = k^{r-i+1} + 1 \) and \( n_{i+1} = 1 \), where we can clearly ignore the +1 as lower-order terms. We also get \( p = r + 1 \).

**Edge-count guarantee.** Next, we need an asymptotic bound on the value of \( \sum_{i=2}^{r+1} n_i^{1/c} / \alpha \). It can be readily verified that replacing \( n_i = n_{i-1} - n_{i+1} = k^{r-i+1} - k^{r-1} \) by simply \( n_i = k^{r-i+1} \) affects only a lower-order term of the guarantee. Thus we bound
\[
\sum_{i=2}^{r+1} (k^{r-i+1})^{1/c} = (1/\alpha) \sum_{j=0}^{r-1} (k^{1/c})^j
\]
\[
= (1/\alpha) \cdot \frac{1 - k^{r/c}}{1 - k^{1/c}}
\]
\[
= (1/\alpha) \cdot \frac{k^{1-2/c} - 1}{k^{1/c} - 1},
\]
using \( p = r + 1, r = c - 2 \), and recalling the formula for a geometric sum. To estimate this, let us write \( c = d \alpha \log k \), \( d = \Omega(1) \) and set \( x = k^{1/c} \). In the case that \( \alpha = O(1) \) (for example, for use in a pure existence proof) then \( x = 2^{1/(d\alpha)} = O(1) \) and the above bound is
\[
(1/\alpha) \cdot \frac{1}{x^2(x - 1)} \cdot k - o(k) = \Theta(k),
\]
with a constant factor that grows unboundedly as \( x \to 1 \). Clearly, there exists a value \( d = O(1) \) such that the bound is greater than \( 3k \).

Otherwise \( k^{1/c} = 1 + o(1) \) and we write \( x = 1 + z \), where thus \( z = o(1) \). We have
\[
x^{c} = k = (1 + z)^c = ((1 + z)^{1/z})^{cz} = (e - o(1))^{cz},
\]
hence \((cz)(1 - o(1)) = \ln k \) and
\[
z = (1 + o(1))(\ln k)/c.
\]
Thus the denominator above is \( \Theta(\ln k/c) = O(1/\alpha) \). We also note
\[
k^{1-2/c} - 1 = \frac{k}{k^{2/c} - 1} = \frac{k}{1 + o(1)} - 1 = (1 - o(1))k.
\]
Our bound becomes
\[
\frac{c}{\alpha \ln k} (1 - o(1))k.
\]
A value of \( c = 4\alpha \ln k \) thus suffices to conclude that, asymptotically, the summation achieves a value greater than \( 3k \) and, consequently, any minimum \( T \)-cut \( X \) that is not covered needs to contain more than \( k \) edges, contradicting it being minimum. Hence using a value of \( c = \Omega(\alpha \log k) \) suffices for the marking procedure to mark all essential edges for \((\alpha, c)\)-dense instances. This finishes the case of \((\alpha, c)\)-dense inputs and proves Lemma \( \Box \).

### 3.3 Completing the result

By the above, every terminal network \((G, T)\) that is \((\alpha, c)\)-dense for some \( c = \Theta(\alpha \log k) \) has a multicut-covering set of at most \( k^{c} \) edges, which can be computed in randomized polynomial time. We extend the result to any \((G, T)\), using a sublogarithmic terminal expansion tester.
Theorem 2 (Theorem 1 restated). Let $A$ be a sublogarithmic terminal expansion tester with ratio $\alpha(n, k)$. Let $(G, T)$ be a terminal network with $\text{cap}_G(T) = k$. There is a multicut-covering set $Z \subseteq E(G)$ with $|Z| \leq k^\mathcal{O}(\alpha(n, k) \log k)$, which furthermore can be computed in randomized polynomial time using calls to $A$.

Proof. Set $c = \Theta(\alpha \log k)$ as in Lemma 3. If $|E(G)| < k^c$ then return $Z = E(G)$, otherwise call $A$ on $(G, T, c)$. If $(G, T)$ is $(\alpha, c)$-dense, then Lemma 3 applies and we are done. Otherwise, let $S \subseteq V(G)$ be the set returned by $A$, and let $k_S = \text{cap}_T(S)$. Let $(G_S, T(S))$ be the recursive instance at $S$, and note that $|V(G_S)| = |N_G[S]| < |V|$ and $|S| > (\alpha k_S)^c$ by definition of $A$. We may now proceed by induction on $|V|$ and assume that we can compute a multicut-covering set $Z_S \subseteq E(G_S)$ of size $|Z_S| < k_S^c$. To eliminate a corner case, if there is a vertex $v \in V(G_S)$ with $v \notin T(S)$ and $d_{G_S}(v) \leq 2$, then delete $v$ if $v$ is a leaf, otherwise contract one edge incident with $v$. Note that since $v \notin T(S)$ we have $d_{G_S}(v) = d_{G_S}(v)$ and $v \notin T$, hence these reduction rules are clearly correct. If this rule does not apply, there must be some edge $e \in E(G_S) \setminus Z_S$, and by construction $e$ corresponds directly to an edge in $G$. Hence by Prop. 2 we may contract $e$ in $G$ and repeat. This yields a graph $G'$ with $|V(G')| < |V|$, hence by induction we can create a multicut-covering set $Z$ for $G'$, which is also a multicut-covering set of $G$ Prop. 2. Hence we can compute a multicut-covering set $Z$ with $|Z| < k^c$.

We observe the following consequences.

Corollary 4. Let $(G, T)$ be a terminal network with $\text{cap}_G(T) = k$. The following holds.

1. There is a multicut-mimicking network for $(G, T)$ with $k^\mathcal{O}(\log k)$ edges.

2. If there is a sublogarithmic terminal expansion tester – in particular, if Small Set Expansion has an approximation ratio as in Theorem 2 – then a multicut-mimicking network of size quasipolynomial in $k$ can be computed in randomized polynomial time.

Proof. The first is immediate using $\alpha(n, k) = 1$. For the second, all that remains is to clean up the value $|Z|$. For this, let $\alpha(n, k) \leq \log^{1-\varepsilon} n \log^d k$ and $c = \beta \log k$, for some bounded values $b, d$, and first assume that $|Z| \geq |V(G)| = n$. Then

\[
|Z| < k^\beta \log k \Rightarrow \\
\log n < \beta \log^2 k \Rightarrow \\
\log n < b \log^{1-\varepsilon} n \log^{d+2} k \Rightarrow \\
\log \varepsilon n < b \log^{d+2} k \Rightarrow \\
\log n < (b \log^2 k)^{1/\varepsilon},
\]

hence $|Z| \leq k^{\mathcal{O}(1)} k$, as promised. Otherwise, we contract all edges not present in $Z$ and compute a new multicut-covering set $Z'$ for the new system $(G', T)$. Eventually, this process halts, and at this point we will have a multicut-covering set $Z$ with $|Z| \leq k^{\mathcal{O}(1)} k$ for some graph $G''$ created by contractions from $G$, and by Prop. 2 this set $Z$ is also a multicut-covering set for $(G, T)$.

3.4 Kernelization extensions and consequences

As noted, we get the following consequences.

Corollary 5. If there is a sublogarithmic terminal expansion tester, then the following problems have randomized quasipolynomial kernels.

1. Edge Multiway Cut parameterized by solution size.

2. Edge Multicut parameterized by the solution size and the number of cut requests.

3. Group Feedback Edge Set parameterized by solution size, for any group.

4. Subset Feedback Edge Set with undeletable edges, parameterized by solution size.
5. 0-Extension for integer-weighted graphs, parameterized by solution cost.

Proof. For Edge Multiway Cut, let \((G, T, k)\) be an input. Known reduction rules can reduce the instance so that \(\text{cap}_G(T) \leq 2k\) \cite{17}. From this point, the (conditional) kernel follows.

For Edge Multicut, let the input be \((G, \{(s_1, t_1), \ldots, (s_r, t_r)\}, p)\) and let \(k = p + r\). Create a set of \(2r\) vertices \(T = \{s'_1, \ldots, s'_r\} \cup \{t'_1, \ldots, t'_r\}\) and \(p + 1\) subdivided parallel edges between \(s'_i, s_i\) and between \(t'_i, t_i\), for each \(i \in [r]\). Let \(G'\) be the new graph. We claim that \(I = (G, \{(s_1, t_1), \ldots, (s_r, t_r)\}, p)\) is a positive instance if and only if \(I' = (G', \{(s'_1, t'_1), \ldots, (s'_r, t'_r)\}, p)\) is. Indeed, any multicut for \(I\) is a multicut for \(I'\), and every multicut for \(I'\) containing at most \(p\) edges leaves all new terminals \(s'_1, t'_r\) connected to the old terminals \(s_i, t_i\) and is hence a multicut for \(I\). Furthermore \(\text{cap}_{G'}(T) = (p + 1)r = O(k)\). Now it suffices to compute a multicut-covering set \(Z\) for \((G', T)\) and contract any edge in \(E(G') \setminus Z\).

For Group Feedback Edge Set (GFES), we follow the approach of \cite{17}. The input to GFES is a tuple \((G, \phi, k)\), where \(\phi\) is a direction-dependent labelling of the edges of \(G\) from some multiplicative group \(\Gamma\), such that for every \(uv \in E\), \(\phi(uv) = \phi(vu)^{-1}\) (where the inverse is the group inverse). The goal is to remove \(k\) edges such that for every \(uv \in E(G)\) of \(G\), there is a group element \(\lambda\) with \(\lambda(u) = \lambda(u) \cdot \phi(uv)\). We will not need any assumptions about how the group elements are represented; we only need to assume that \(\phi(e)\) equals the group identity or not. Refer to a cycle as unbalanced if this does not hold. We first note that GFES has an \(O(\log k)\)-approximation. Indeed, GFES reduces easily to Group Feedback Vertex Set, which in turn is a special case of the meta-problem Biased Graph Cleaning \cite{33}. Lee and Wahlström \cite{21} showed that Biased Graph Cleaning admits an \(O(\log k)\)-approximation, using an oracle for testing whether cycles are unbalanced. Let \(X_0\) be an approximate solution with \(|X_0| = O(k \log k)\). Let \(T = V(X_0)\) be the endpoints of \(X_0\). By assumption, \(G - X_0\) admits an assignment \(\lambda\) with \(V(G) \rightarrow \Gamma\) as above, and such an assignment \(\lambda\) can be computed by starting with an arbitrary value from one vertex of each connected component. We now follow \cite{17} in untangling the group labels, so that every edge except those in \(X_0\) receive the identity label by \(\phi\). The solution to GFES now clearly corresponds to a multicut cut for some unknown partition \(T\) of \(G\), hence the multicut-mimicking network can be used for kernelization.

Subset Feedback Edge Set with undeletable edges, parameterized by solution size, is covered by the previous case, since it is a special case of GFES. Indeed, let \(S \subseteq E(G)\) be the special edges. We use labels \(\phi\) from the group \(GF(2)^k\) where every edge is labelled by \(\phi\) by identity except the edges of \(S\), which flip one bit of the group element each. It is now easy to see that a cycle is balanced if and only if it contains no edge from \(S\). It is furthermore easy to see that we can implement undeletable edges by creating parallel (subdivided) copies of edges, using the same group labels.

For 0-Extension we look slightly more carefully at the proof of Lemma \cite{8} We follow Reidel and Wahlström \cite{32}. As noted in \cite{32}, the operation of pushing, or replacing \(\partial(\lambda^{-1}(x))\) for some \(x \in D\), by some farthest min-cut towards the terminals of the instance, preserves optimal solutions \(\lambda\) if applied to any one label \(x \in D\). This can be used to bound \(\text{cap}_G(T)\) in terms of the solution bound \(k\), by computing min-cuts from each terminal to the other terminals. Furthermore, let \(\lambda\) be an optimal solution and let \(X\) be the edges of non-zero cost under \(\lambda\). Assume \(|X| \leq k\), as otherwise the instance is negative. Let \(e\) be an edge that has non-zero cost under \(\gamma\) for every optimal solution \(\gamma\). Then, as in \cite{32}, the edge \(e\) is resistant against pushing, and following the proof of Lemma \cite{8} \(e\) extends \(F\) in every gammoid layer. The rest of the proof now goes through, with the only minor adjustment being that if some component of \(G - X\) contains no terminal and is counted among the largest, then \(F\) contains only \(X\) in that layer, rather than \(X \cup \partial(T)\). Hence, even though the solution \(\lambda\) does not correspond to a direct minimum multicut cut of a partition of the terminals, we still find that if \(X\) acts like a “\(p\)-way plus few”-partition of \(V(G)\), then every essential edge of \(X\) is marked. The rest of the proof uses no further properties of multicut cuts or multiconnectedness, hence the result holds. \(\square\)

Finally, as in \cite{32}, the latter result extends to “0-Extension sparsifiers” which hold independent of the choice of metric. Let us briefly recall some details. An instance of 0-Extension can be defined as a terminal network \((G, T)\), a metric \(\mu: D \times D \rightarrow \mathbb{R}^+\) for some label set \(D\), and a partial labelling \(\tau: T \rightarrow D\). The goal is to find \(\lambda: V(G) \rightarrow D\) extending \(\tau\), to minimize the cost \(\sum_{uv \in E(G)} \mu(\lambda(u), \lambda(v))\). We note that the “kernel” in the previous result can be constructed without needing access to \(\mu\) or \(\tau\), i.e., it is valid for every metric \(\mu\) and every partial labelling \(\tau\).

Theorem 3. Let \(G = (V, E)\) be an undirected, unweighted graph and \(T \subseteq V\) a set of terminals, \(|T| = r\). For any integer \(p \in \mathbb{N}\), let \(k = p + r\); there exists a set \(Z \subseteq E\) with \(|Z| = kO(\log k)\) such that the following
holds. For any metric \( \mu : D \times D \to \mathbb{R}^+ \) and any labelling \( \tau : T \to D \) extending \( \tau \) where \( \lambda(u) \neq \lambda(v) \) for at most \( p \) edges \( uv \in E \), then there exists such a labelling \( \lambda \), of minimum cost among all such labellings, such that \( \lambda(u) = \lambda(v) \) for every edge \( uv \in E \setminus Z \).

Proof. The kernelization observation used in Corollary 5 for 0-EXTENSION uses no information about the metric or the concrete partial assignment on the terminal set whatsoever, but only uses that the set of all edges \( X = \{ uv \in E(G) \mid \lambda(u) \neq \lambda(v) \} \) forms a partition of \( V(G) \) where “essential” edges can be assumed to be stable under pushing. Since the actual kernelization algorithm does not take \( \tau \) or \( \mu \) into consideration, it is valid for any choice. The only consideration is the step from parameter \( k = p + r \) to \( \text{cap}_G(T) \), which can be handled as for EDGE MULTICUT by using a set of virtual terminals connected to \( T \) using \( p + 1 \) parallel subdivided edges.

4 Discussion

We defined the notion of a multicut-mimicking network, and showed that every terminal network \((G, T)\) with \( k = \text{cap}_G(T) \) admits one of size \( k^{O(\log k)} \), which furthermore may be computable in randomized polynomial time, subject to the precise approximation guarantees available for a restricted variant of SMALL SET EXPANSION. The mimicking network can be constructed via contractions on \( G \), i.e., it simply consists of a set of edges which form a multicut-covering set. As a consequence of such a result, a range of parameterized problems, starting from EDGE MULTIWAY CUT, would have quasipolynomial kernels. Unfortunately, the approximation guarantee required for this latter result appears to go just below the range of available guarantees from the literature.

A natural question is whether an appropriate approximation algorithm can be constructed. We note that an approximation ratio of \( \text{polylog}(k) \) for SMALL SET EXPANSION is sufficient, where \( k = \delta(S) \) is the number of edges cut in the optimal solution \( S \). We are not aware of approximation ratios in term of this parameter having been investigated. Also note that it is sufficient if the approximation algorithm has a running time quasipolynomial in \( k \) (but polynomial in \( n \)).

Another question is whether the existence of a polynomial-sized multicut-mimicking network can be established. Can such a result be excluded, even for the apparently more demanding situation of sparsifiers for 0-EXTENSION instances (as in Theorem 3)? We also have not investigated the vertex-deletion versions of these problems, which seem likely to bring significant additional difficulty (if such a generalization is possible).

In either case, the existence of such a multicut-covering set appears to rule out any possibility of a lower bound against the kernelizability of EDGE MULTIWAY CUT for any size larger than quasipolynomial, given the nature of the lower bound results against kernelization. We hope, therefore (but dare not explicitly conjecture) that EDGE MULTIWAY CUT and related problems have quasipolynomial (randomized) kernels or better, unconditionally.

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