FINITE VARIATIONS ON THE ISOPERIMETRIC PROBLEM

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Abstract. The isoperimetric problem asks for the maximum area of a region of given perimeter. It is natural to consider other measurements of a region, such as the diameter and width, and ask for the extreme value of one when another is fixed. The solution of these problems is known if the competing regions are general convex disks, however several of these problems are still open if the competing regions are polygons with at most a given number of sides. The present work surveys these problems.

Let $a$, $p$, $w$, and $d$ denote the area, perimeter, width and diameter of a convex disk. Fixing one of these four quantities, what is the infimum and the supremum of another one of them? Of course, fixing one quantity and asking for the supremum of another one is equivalent to the problem of fixing the second quantity and asking for the infimum of the first one. The solution of one half of the twelve problems arising this way is obvious: The answer is either zero or infinity. In the case of the six meaningful problems, we can ask for minima and maxima and for the convex disks attaining the optimum.

The isoperimetric problem asks for the convex disk of maximum area with given perimeter. Its solution is the circle, which was known already in Ancient Greece, although a mathematically rigorous proof was obtained only in the 19th century. The solution of the problem on the maximum width for a given diameter is obvious: The optimal sets are convex disks of constant width. Convex disks of constant width were also characterized by Blaschke [1915] as those among domains of a given width that have minimum perimeter, and by Rosenthal and Szász [1916] as those among with a given diameter, that have maximum perimeter. Bieberbach [1915] proved that among all domains of a fixed diameter the one of maximum area is the circular disk, and Pál [1921] proved that the minimum area of a convex disk with given width is attained by the regular triangle.

An interesting area of research is to consider these optimum problems restricted to polygons with at most a given number of sides. The isoperimetric problem for $n$-gons was solved centuries ago by Zenodorus (see [2005]): Implicitly assuming the existence of a solution, he proved that a regular $n$-gon has greater area than all other $n$-gons with the same perimeter. This is expressed in the inequality

$$p^2 \geq 4n a \tan \frac{\pi}{n}.$$
The theorem of Pál solves the problem of minimum area for a given width. Concerning the remaining four problems only partial results are known.

Reinhardt [1922] considered the problem of maximizing the perimeter of a convex \( n \)-gon with a given diameter and proved that the perimeter \( p \) of an \( n \)-gon of diameter \( d \) satisfies the inequality

\[
p \leq 2n \sin \frac{\pi}{2n}d.
\]

Equality is attained here if and only if \( n \) is not a power of 2. This result, together with the characterization of the case of equality, was rediscovered by Larman and Tamvakis [1984], Datta [1997] and A. Bezdek and Fodor [2000].

To describe the polygons for which equality is attained, we start with a convex polygon with an odd number of sides such that each vertex is at distance \( d \) from the endpoints of the opposite side. Replacing each side by a circular arc of radius \( d \) centered at the opposite vertex we obtain a Reuleaux polygon. If \( n \) is not a power of 2, the \( n \)-gons of diameter \( d \) with perimeter \( 2n \sin(\pi/2n) \) are inscribed in a Reuleaux polygon in such a way that every vertex of the Reuleaux polygon is a vertex of the polygon, and all sides of the polygon are of equal length. Such polygons are called by Audet, Hansen and Messine [2009a] clipped Reuleaux polygons, while Mossinghoff [2011] uses the term Reinhardt polygons for them.

Consider a Reuleaux polygon with \( m \) vertices. Its diagonals form an \( m \)-gon which for \( m > 3 \) is a star polygon. The sum of the angles of this polygon is \( \pi \), so in order that it can accommodate a clipped Reuleaux polygon its angles must be integer multiples of \( \pi/n \). The clipped Reuleaux polygons were studied by Gashkov [2007, 2013], Mossinghoff [2011] and Hare and Mossinghoff [2013, 2019]. For a given \( n \), there are clipped Reuleaux \( n \)-gons with \( k \)-fold rotational symmetry for some divisor \( k \) of \( n \). Besides these, called periodic by Mossinghoff, there may be some others, called sporadic. The latter name is misleading, since it turned out that the sporadic clipped Reuleaux polygons outnumber the periodic ones for almost all \( n \). Figure 1 shows a regular and a sporadic clipped Reuleaux polygon with 30 vertices.

Finding the maximum perimeter of an \( n \)-gon of given diameter when \( n \) is a power of 2 is difficult. Only the cases of the quadrangle and octagon are solved. The best quadrangle was determined by Tamvakis [1987] and rediscovered by Datta [1997]. The octagon’s case was settled by Audet, Hansen and Messine [2007a]. Tamvakis described a sequence of unit-diameter \( n \)-gons for \( n = 2^k \) whose perimeter exceeds that of the regular \( n \)-gon, and differs from the upper bound \( 2n \sin \frac{\pi}{2n} \) by \( O(n^{-4}) \).
By improved constructions the difference from the upper bound was reduced to $O(n^{-5})$ by Mossinghoff [2006a] and lately to $O(n^{-6})$ by Bingane [2021a].

Combining the inequality $p \leq 2n \sin \frac{\pi}{2n} d$ with the isoperimetric inequality $p^2 \geq 4n \tan \frac{\pi}{n} a$, Reinhardt [1922] obtained the inequality

$$a \leq \frac{n}{2} \cos \frac{\pi}{n} \tan \frac{\pi}{2n} d^2$$

with equality only for odd $n$ and regular $n$-gons. Thus, for odd $n$, among all $n$-gons of a given diameter the regular one has maximum area. Alternative proofs were given by Lenz [1956], Griffiths and Culpin [1975], and Gashkov [1985].

Reinhardt proved that for even $n \geq 6$, the optimal $n$-gon is never regular. Alternative proofs were given by Schäffer [1916], Audet, Hansen and Messine [2008] and Mossinghoff [2006a]. The latter author constructed a sequence of $n$-gons with unit diameter for even $n$ whose area exceeds the area of the unit-diameter regular $n$-gon by $O(n^{-2})$, and whose area differs from the maximum area of such $n$-gons by a term of at most $O(n^{-3})$. Bingane [2021b] improved Mossinghoff’s construction without improving on the order of difference from the maximum area.

The maximum area of a quadrangle of diameter $d$ is $d^2/2$. The diagonals of the optimal quadrangles are perpendicular and have length $d$. The case of the hexagon was solved by Graham [1975]. He confirmed the conjecture of Bieri [1961] that the non-regular hexagon shown in Figure 2 is the unique optimal solution. He also formulated a conjecture for all even $n \geq 6$, stating that for such $n$, every optimal $n$-gon’s diameter graph consists of an $(n-1)$-cycle with one additional edge emanating from one of the cycle’s vertices. Note that the conjecture leaves the geometric realization of the best polygon undetermined, subject to an optimization problem. Graham’s conjecture was confirmed for $n = 8$ by Audet, Hansen, Messine, and Xiong [2002], who also solved the corresponding optimization problem, thus determining the best octagon. Subsequently, Graham’s conjecture was confirmed in general by Foster and Szabo [2007]. The corresponding optimal polygons for $n = 10$ and $n = 12$ were determined by Henrion and Messine [2013].

![Figure 2](image)

Graham [1975] also asked for the solution of the higher-dimensional analogue of the problem: Which convex $d$-polytope with $n$ vertices and unit diameter has the largest volume? For $n = d + 1$ the solution is the regular simplex. Kind and Kleinschmidt [1976] solved the problem for $n = d + 2$ and described all the extremal polytopes. The case $n = d + 3$ was attacked by Klein and Wessler [2003], however their proof turned out to be incomplete (cf. Klein and Wessler [2005]), thus, this case is still open.
In the Russian journal for high school students Quant, Gashkov [1985] wrote a small article about the isoperimetric problem and its relatives. There he gave a proof of Reinhard’s theorem based on central symmetrization. He used the following facts about a convex \( n \)-gon \( P \) and its central symmetric image \( P^* = \frac{1}{2}(P - P) \): \( P^* \) is a convex polygon with at most \( m \leq 2n \) sides and the same width \( w \), diameter \( d \) and perimeter \( p \) as \( P \). The inradius of \( P^* \) is at least \( w \) and the circumradius of \( P^* \) is at least \( d \). It follows that
\[
2m \sin \frac{\pi}{2m} d \geq p \geq m \tan \frac{\pi}{m} w.
\]
Since \( m \leq 2n \), using the monotonicity of the functions \( x \sin \frac{1}{x} \) and \( x \tan \frac{1}{x} \), we get, on one hand, Reinhardt’s inequality for the maximum perimeter of an \( n \)-gon with given diameter, and on the other hand, the new inequality
\[
p \geq 2n \tan \frac{\pi}{2n} w.
\]
The combination of these inequalities yields the inequality
\[
w \leq \cos \frac{\pi}{2n} \frac{d}{w}
\]
between the width and diameter of a convex \( n \)-gon. Equality is attained in these inequalities for every \( n \geq 3 \) that has an odd factor by a clipped Reuleaux polygon.

Gashkov’s article remained unnoticed. The last inequality was rediscovered by A. Bezdek and Fodor [2000], and the inequality between perimeter and width by Audet, Hansen and Messine [2009a]. These authors also solved the case of the quadrangle for both problems. The octagon of a given diameter with maximum width was determined by Audet, Hansen, Messine and Ninin [2013].

Motivated by a question of Erdős, Vincze [1950] studied the problem of finding the maximum perimeter of an equilateral \( n \)-gon with given diameter. He solved the problem if \( n \) is not a power of 2. Of course, this case is an immediate consequence of Reinhardt’s theorem. However, Vincze’s argument works without the assumption that the sides have equal length, so it yields an alternative proof of Reinhardt’s theorem. The case that \( n \) is a power of 2 \( > 4 \) is of similar difficulty as the problem for general \( n \)-gons. The only case solved is the one for the octagon settled by Audet, Hansen, Messine and Perron [2014]. Mossinghoff [2008] constructed a sequence of equilateral \( n \)-gons with unit diameter for \( n = 2^k \), \( k \geq 4 \), and proved that their perimeter differs from the maximum perimeter of such \( n \)-gons by a term of at most \( O(n^{-4}) \). By constructing a differen sequence of polygons Bingane and Audet [2021a] further improved the lower bound for the optimum perimeter.

The question about the maximum area of an equilateral \( n \)-gon with given diameter \( d \) is solved for all \( n \): It is \( \frac{d^2}{2} \cos \frac{\pi}{n} \tan \frac{\pi}{2n} \), attained only for a regular \( n \)-gon. This follows from Reinhardt’s theorem for odd \( n \) and was proved by Audet [2017] for even \( n \). Bingane and Audet [2021b] determined the equilateral octagon of unit diameter with maximum width. They also provided a family of equilateral \( n \)-gons of unit diameter, for \( n = 2^s \) with \( s \geq 4 \), whose widths are within \( O(n^{-4}) \) of the maximum width. It appears that the question about the maximum width of an equilateral polygon with \( n = 2^s \) sides and a given perimeter has not been studied so far.

By restricting the class of competing polygons to equilateral polygons some problems with obvious solutions become interesting. The area, perimeter, and diameter of a general unit-width convex \( n \)-gon can be arbitrarily large. This is still the case
for an equilateral polygon with an even number of sides. However, these quantities are bounded for equilateral convex $n$-gons when the number of sides is odd. Audet and Ninin [2013] determined the maximal perimeter, diameter and area of an equilateral unit-width convex $n$-gon for every odd $n \geq 3$. The optimal polygon is the same for all three problems: For $n = 3$ it is an equilateral triangle of side length $\frac{2}{\sqrt{3}}$, and for $n = 2k + 1 \geq 5$ a trapezoid whose non-parallel sides have length equal to $\frac{2}{\sqrt{3}}$, and the parallel ones have length $m\frac{2}{\sqrt{3}}$ and $(m-1)\frac{2}{\sqrt{3}}$.

The papers by Mossinghoff [2006b] and Audet, Hansen and Messine [2007b, 2009b] contain nice surveys about variations of the isoperimetric problem for polygons.

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