Quantum Logic as Classical Logic*

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Abstract

We propose a semantic representation of the standard quantum logic QL within a classical, normal modal logic, and this via a lattice-embedding of orthomodular lattices into Boolean algebras with one modal operator. Thus our classical logic is a completion of the quantum logic QL. In other words, we refute Birkhoff and von Neumann’s classic thesis that the logic (the formal character) of Quantum Mechanics would be non-classical as well as Putnam’s thesis that quantum logic (of his kind) would be the correct logic for propositional inference in general. The propositional logic of Quantum Mechanics is modal but classical, and the correct logic for propositional inference need not have an extroverted quantum character. One normal necessity modality □ suffices to capture the subjectivity of observation in quantum experiments, and this thanks to its failure to distribute over classical disjunction. The key to our result is the translation of quantum negation as classical negation of observability.

Keywords: Boolean Algebras with Operators (BAOs); algebraic, normal modal logic; lattice representation theory; ordered algebraic structures; orthomodular lattices; quantum logic; quantum structures.

1 Introduction

The idea for this paper originates in an observation that I made in the plenum of a talk delivered by Newton da Costa at the 4th World Congress and School on Universal Logic in Rio de Janeiro in 2013. The observation is about a presentation of a quantum experiment that has been put forward as a counter-example to the suitability of classical logic for reasoning about quantum phenomena and as a motivation for alternative logics such as quantum logics, over and over again. The presentation usually involves three statements, say P, Q, and R, each one being about an elementary quantum phenomenon produced by the experiment,
but such that

\[ P \text{ is observed to be true and } (Q \text{ or } R) \text{ is observed to be true}. \quad (1) \]

\[ \square P \text{ is true} \quad \square (Q \lor R) \text{ is true} \]

\[ \square P \land \square (Q \lor R) \text{ is true} \]

Notice that the observation of the truth of a disjunction does not imply the observation of the truth of one of its disjuncts. That is, \( \square (Q \lor R) \rightarrow (\square Q \lor \square R) \) is not a valid principle. This is an essential uncertainty. (On the other hand, the converse \( (\square Q \lor \square R) \rightarrow \square (Q \lor R) \) is a valid principle.) Hence, and in fact,

\[ \neg \square (P \land Q) \land \neg \square (P \land R) \text{ is true} \quad \text{(or: } \neg (\square (P \land Q) \lor \square (P \land R)) \text{ is true}) \quad (2) \]

The quantum mechanical details need not concern us here. However, what does need concern us here is that the presentation of the experiment concludes that \( (P \text{ and } (Q \text{ or } R)) \text{ is true but not } ((P \text{ and } Q) \text{ or } (P \text{ and } R)). \)

That is,

“\((P \text{ and } (Q \text{ or } R))\)” and “\(((P \text{ and } Q) \text{ or } (P \text{ and } R))\)” are not equivalent. (3)

Apparently, the distributivity of classical conjunction and disjunction fails! Whence arises the motivation for special quantum conjunction and disjunction.

Now, my observation is that the experiment—though classic—is not well presented, that is, the formalisation of the experimental observations is unfortunate. The point is that the fact of observing the fact \( P \) and the fact \((Q \text{ or } R)\) should be made explicit in the formalisation too, for example as \( \square P \land \square (Q \lor R) \). Hence in the experiment, \( \square P \land \square (Q \lor R) \) is true \((1)\) but not \( \square (P \land Q) \lor \square (P \land R) \) \((2)\). That is, \( (\square P \land \square (Q \lor R)) \rightarrow (\square (P \land Q) \lor \square (P \land R)) \) is false. On the other hand, the converse \( (\square (P \land Q) \lor \square (P \land R)) \rightarrow (\square P \land \square (Q \lor R)) \) is true, because:

\[ (\square (P \land Q) \lor \square (P \land R)) \rightarrow (\square (P \land Q) \lor (P \land R)) \text{ is true} \]

\[ \leftrightarrow \square(P \land (Q \lor R)) \text{ is true} \]

\[ \leftrightarrow (\square P \land \square (Q \lor R)) \text{ is true} \]

(As noticed above, \( \square \) distributes over \( \land \) in both directions, but over \( \lor \) only in one direction.) Thus, and in close correspondence with \((3)\),

\[ (\square P \land \square (Q \lor R)) \leftrightarrow (\square (P \land Q) \lor \square (P \land R)) \text{ is false}. \quad (4) \]

Hence, if we make explicit the fact of observing facts (for example by means of a modal operator \( \square \)) then we do not need to introduce the special-purpose formalism of Quantum Logic with special and possibly counter-intuitive quantum operators to account for quantum phenomena (due to the apparent failure of classical conjunction to distribute over classical disjunction), but can get by with intuitive classical (Boolean) logic at the small price of adding a single, classical modal operator \( \square \). This operator is characterised by the following two axioms K (normality) and BQ plus the single deduction rule N (normality):
1. □(A → B) → (□A → □B)  (Kripke’s law, K)
2. □◊A ↔ A, where ◊ := ¬□¬  (BQ)
3. from A infer □A  (necessitation rule, N)

The laws of Boolean logic plus the three modal laws K, BQ, and N, which are embodied in our classical modal logic BQ, suffice to model the logical essence of Quantum Mechanics as captured by the standard quantum logic QL [7, Page 32]. To appreciate the smallness of the price of understanding BQ, consider the price of understanding QL. This standard quantum logic is characterised by the following 12 axioms plus one deduction rule [7, Page 32]:

1. (A ≡ B) →₀ ((B ≡ C) →₀ (A ≡ C))
2. (A ≡ B) →₀ (¬A ≡ ¬B)
3. (A ≡ B) →₀ ((A ∧ C) ≡ (B ∧ C))
4. (A ∧ B) ≡ (B ∧ A)
5. (A ∧ (B ∧ C)) ≡ ((A ∧ B) ∧ C)
6. (A ∧ (A ∨ B)) ≡ A
7. (¬A ∧ A) ≡ ((¬A ∧ A) ∧ B)
8. A ≡ ¬¬A
9. ¬(A ∨ B) ≡ (¬A ∧ ¬B)  (De-Morgan law)
10. (A ≡ B) ≡ (B ≡ A)
11. (A ≡ B) →₀ (A →₀ B)
12. (A →₀ B) →₃ (A →₃ (A →₃ B))
13. from A and A →₃ B infer B

with the three abbreviations:

\[
A →₀ B := \sim A ∨ B \\
A →₃ B := (\sim A ∧ B) ∨ (\sim A ∧ \sim B) ∨ (A ∧ (\sim A ∧ B)) \\
A ≡ B := (A ∧ B) ∨ (\sim A ∧ \sim B)
\]

and ∧, ∨, and ¬ symbolising quantum conjunction, quantum disjunction, and quantum negation, respectively (my choice, to distinguish the quantum operators clearly from their classical counterparts). Observe that this axiom system for QL employs two notions of implication, symbolised as →₀ and →₃, as well as one notion of equivalence, symbolised as ≡, but that is defined in terms of neither notion of implication! Proving the adequacy of this axiom system for the standard quantum structures of orthomodular lattices must have been
a real *tour de force*, which must be appreciated as such. Fortunately, we do not need to understand nor use this axiom system in order to understand the logical essence of Quantum Mechanics. All we need to understand is that QL and its corresponding, standard quantum-structure semantics of orthomodular lattices satisfy the De-Morgan law, and thus quantum disjunction is definable in terms of quantum negation and quantum conjunction. Hence in essence, the culprit for the failure of quantum conjunction to distribute over quantum disjunction boils down to quantum negation! We thus must find a translation of quantum negation in terms of the modal operator $\Box$ and Boolean operators. The translation that we have found and shall now present and explicate is to

\[
\text{translate quantum negation } \sim \text{ as } \neg \Box.
\]

That is, we translate *quantum negation as classical negation of observability*. Recall from classical normal modal logic that $\neg \Box$ ("not necessarily") is the same as $\Diamond \neg$ ("possibly not"). Hence, the classical negation of observability is classically equivalent to the possibility of observing classical negation. Thus, we can also view *quantum negation as the possibility of observing classical negation*.

2 The technical details of the translation

We shall carry out our translation from QL into BQ semantically, by producing a *lattice-embedding* from the standard orthomodular-lattice (OML-)model of QL into the standard Boolean-algebra-with-one-operator (BAO-)model of BQ \[13\]. So, not only logicians but also mathematicians will be able to appreciate our result. For simplicity, we will reuse some of the symbols for the (syntactic) operators of QL from the introduction for their corresponding (semantic) operators of the OML-model of QL, and only use different symbols for the (syntactic) operators of BQ and their corresponding (semantic) operators of our BAO. Both algebraic models involve a set of subsets of a set of states as carrier together with algebraic operations on this carrier set. So, let $S$ designate our *state space*, that is, the set of all possible worlds, points, or states, $H(S)$ the set of subsets of $S$ that is algebraically closed under the OML-operations for the carrier of the OML-model, and $P(S)$ the powerset of $S$ for the carrier of our BAO-model.

Then, translate the *orthomodular* (and thus De Morgan) *lattice* \[7\]

\[
\text{OML} := \langle H(S), 0, \land, \lor, 1, \perp, \equiv \rangle
\]

on $S$ to a corresponding (inclusion-ordered, complete) *Boolean (powerset) algebra (lattice)*

\[
\text{BAO} := \langle P(S), \emptyset, \cap, \cup, S, \sim, \langle R \rangle, \subseteq \rangle
\]

with one operator \(\langle R \rangle : 2^S \to 2^S\)

\[
\langle R \rangle(S) := \{ s \in S | \text{there is } s' \in S \text{ such that } s R s' \text{ and } s' \in S \}
\]

for an—only for now—arbitrary (see the end of this section) accessibility relation \(R \subseteq S \times S \) \[13\], that is, a binary relation on $S$ of no particularity, and

\[1\] Accessibility relations are at the heart of Kripke-semantics for modal logics \[4\].
with dual operator \([R] : 2^S \to 2^S\) \[12\] Definition 3.8.2

\[ [R](S) := \{ s \in S \mid \text{for all } s' \in S, \text{if } s \ R s' \text{ then } s' \in S \} \]

by means of an injective mapping \(\rho : \mathcal{OML} \hookrightarrow \mathcal{BAO}\) such that

\[
\rho(H^\perp) = \sim \rho(H) \quad (\sim\text{-homomorphism}) \\
\rho(H \triangleleft H') = \rho(H) \cap \rho(H') \quad \text{(meet homomorphism)}
\]

where \(\sim := \langle R \rangle \circ \perp\). This is the semantic analog of the syntactic translation \(\sim := \Diamond \neg\) mentioned in the introduction. The operator \([R]\) is the semantic analog of the modality \(\square\), and is related to \(R\) as asserted by the following fact.

**Fact 1** (\[12\] Exercise 3.65).

\(s \ R s'\) if and only if for all \(S \subseteq S\), \(s \in [R](S)\) implies \(s' \in S\)

As opposed to the operator \([R]\), its dual \(\langle R \rangle\) does distribute over set union \(\cup\), as asserted by the following well-known, but to our development crucial fact.

**Fact 2** (Property of \(\langle R \rangle\)).

\[\langle R \rangle(S \cup S') = \langle R \rangle(S) \cup \langle R \rangle(S')\]

Now, recall the following laws of orthomodular lattices:

- **De Morgan:**
  \[H \bowtie H' = (H^\perp \land H'^\perp)^\perp\]

- **orthocomplementarity:**
  - involution: \(H^{\perp\perp} = H\)
  - disjointness: \(H \land H^\perp = 0\)
  - exhaustiveness: \(H \bowtie H^\perp = 1\)
  - antitonicity: \(H \preccurlyeq H' \Rightarrow H'^\perp \preccurlyeq H^\perp\)

- **orthomodularity (OM):**
  \[H \preccurlyeq H' \iff H = H \bowtie H' \iff H' = H \bowtie (H' \bowtie H^\perp) \quad \text{(OM)}\]

**Proposition 1.** The complete lattice \(\mathcal{BAO}\) is a completion \[6\] Definition 7.36] of the lattice (and thus partially ordered set) \(\mathcal{OML}\) via the order-embedding \(\rho\), that is,

for all \(H, H' \in \mathcal{H}(S)\), \(H \preccurlyeq H'\) if and only if \(\rho(H) \subseteq \rho(H')\).
Proposition 2 (Properties of \(\sim\)).

1. \(\sim\sim\rho(H) = \rho(H)\) (involutive interaction with itself)

2. \(\sim(S \cap S') = \sim S \cup \sim S'\) (De-Morgan interaction with meet and join)

   (thus \(\sim(\rho(H) \cap \rho(H')) = \sim\rho(H) \cup \sim\rho(H')\))

3. \(\sim(\rho(H) \cup \rho(H')) = \sim\rho(H) \cap \sim\rho(H')\)

4. \(\rho(H) \cap \sim\rho(H) = \rho(0)\)

5. \(\rho(H) \cup \sim\rho(H) = \rho(1)\)

6. \((H \preccurlyeq H' \text{ or } \rho(H) \subseteq \rho(H'))\) implies
   
   (a) \(\sim\rho(H') \subseteq \sim\rho(H)\) and
   
   (b) \(\rho(H') = \rho(H) \cup (\rho(H') \cap \sim\rho(H))\)

7. \(\sim\rho(0) = \rho(1)\)

8. \(\sim\rho(1) = \rho(0)\)

Proof. For (1), let \(H \in \text{H(S)}\) and recall that \(H^{\perp \perp} = H\). Thus \(\rho(H^{\perp \perp}) = \rho(H)\). Hence \(\sim\sim\rho(H) = \rho(H)\). For (2), let \(S, S' \in \text{P(S)}\) and consider:

\[
\sim(S \cap S') = \sim \left( \overline{S} \cup \overline{S'} \right) \quad \text{([Boolean] De Morgan)}
\]

\[
= \left( \langle R \circ \tau \circ \tau \rangle \left( \overline{S} \cup \overline{S'} \right) \right) \quad \text{(definition)}
\]

\[
= \langle R \rangle \left( \overline{S} \cup \overline{S'} \right) \quad \text{([Boolean] involution)}
\]

\[
= \langle R \rangle \left( \overline{S} \right) \cup \langle R \rangle \left( \overline{S'} \right) \quad \text{(Fact 2)}
\]

\[
= \left( \langle R \circ \tau \rangle(S) \cup \left( \langle R \circ \tau \rangle(S') \right) \right) \quad \text{(definition)}
\]

\[
= \sim S \cup \sim S' \quad \text{(definition)}
\]

For (3), let \(H, H' \in \text{H(S)}\) and consider:

\[
\sim(\rho(H) \cup \rho(H')) = \sim(\sim\rho(H)) \cup \sim(\sim\rho(H')) \quad \text{(1)}
\]

\[
= \sim\sim(\rho(H) \cap \sim\rho(H')) \quad \text{(2)}
\]

\[
= \sim\rho(H) \cap \sim\rho(H') \quad \text{(1)}
\]
For (4) and (5), let $H \in H(S)$. For (4), recall that $H \perp H^{\perp} = 0$. Thus $\rho(H \perp H^{\perp}) = \rho(0)$. Hence, $\rho(H) \cap \rho(H^{\perp}) = \rho(0)$ and then $\rho(H) \cap \sim \rho(H) = \rho(0)$. For (5), recall that $H \Join H^{\perp} = 1$. Hence:

$$\rho(1) = \rho(H \Join H^{\perp})$$

$$= \rho \left( \left( H^{\perp} \Join H^{\perp} \right)^{\perp} \right) \quad \text{(definition)}$$

$$= \rho \left( \left( H^{\perp} \Join H^{\perp} \right)^{\perp} \right) \quad \text{(involution)}$$

$$= \sim \rho(H^{\perp} \Join H) \quad \text{(\sim-homomorphism)}$$

$$= \sim (\rho(H^{\perp}) \cap \rho(H)) \quad \text{(meet-homomorphism)}$$

$$= \sim (\rho(H^{\perp}) \cup \sim \rho(H)) \quad \text{(2)}$$

$$= \sim \rho(H) \cup \sim \rho(H) \quad \text{(\sim-homomorphism)}$$

$$= \rho(H) \cup \sim \rho(H) \quad \text{(1)}$$

For (6.a) and (6.b), let $H, H' \in H(S)$ and suppose that $H \preceq H'$ or $\rho(H) \subseteq \rho(H')$. For (6.a), first suppose that $H \preceq H'$. Thus $H^{\perp} \preceq H^{\perp}$ by antitonicity. Hence $\rho(H^{\perp}) \subseteq \rho(H)$ by Proposition \[.\] Hence $\sim \rho(H') \subseteq \sim \rho(H)$ by $\sim$-homomorphism. Now suppose for (6.a) that $\rho(H) \subseteq \rho(H')$. Hence $H \preceq H'$ by Proposition \[.\] and proceed like in the first case. For (6.b), first suppose that $H \preceq H'$. Thus $H' = H \Join (H' \Join H^{\perp})$ by orthomodularity. Hence:

$$\rho(H') = \rho(H \Join (H' \Join H^{\perp}))$$

$$= \rho \left( \left( H^{\perp} \Join (H' \Join H^{\perp}) \right)^{\perp} \right) \quad \text{(definition)}$$

$$= \sim \rho \left( H^{\perp} \Join (H' \Join H^{\perp}) \right) \quad \text{(\sim-homomorphism)}$$

$$= \sim \rho \left( \rho(H^{\perp}) \cap \rho \left( (H' \Join H^{\perp}) \right) \right) \quad \text{(meet homomorphism)}$$

$$= \sim \rho(H^{\perp}) \cup \sim \rho \left( (H' \Join H^{\perp}) \right) \quad \text{(2)}$$

$$= \sim \rho(H) \cup \sim \rho(H) \quad \text{(\sim-homomorphism)}$$

$$= \rho(H) \cup \rho(H') \quad \text{(1)}$$

$$= \rho(H) \cup \rho(H') \cap \sim \rho(H) \quad \text{(meet-homomorphism)}$$

$$= \sim \rho(H) \cup \sim \rho(H) \quad \text{(\sim-homomorphism)}$$

Now suppose for (6.b) that $\rho(H) \subseteq \rho(H')$. Hence $H \preceq H'$ by Proposition \[.\] and proceed like in the first case. For (7), consider (4). Hence:

$$\sim \rho(0) = \sim (\rho(H) \cap \sim \rho(H))$$

$$= \sim \rho(H) \cup \sim \rho(H) \quad \text{(2)}$$

$$= \sim \rho(H) \cup \rho(H) \quad \text{(1)}$$

$$= \rho(1) \quad \text{(5)}$$

7
For (8), consider (7). Thus \( \sim \rho(0) = \sim \rho(1) \). Hence \( \rho(0) = \sim \rho(1) \) by (1).

In spite of Proposition 2 having only the status of a proposition, its proof actually contains more information than the proof of the following (main) theorem.

**Theorem 1** (Representation Theorem for Orthomodular Lattices). The structure \( \mathcal{S} := \langle \{ \rho(H) \mid H \in H(S) \}, \rho(0), \cap, \cup, \rho(1), \sim, \subseteq \rangle \) is a sublattice of sets of the powerset lattice \( \mathcal{B} \mathcal{A} \mathcal{O} \) that is isomorphic to \( \mathcal{O} \mathcal{M} \mathcal{L} \) via the lattice-embedding \( \rho \), that is, \( \rho \) is a bijection between \( \mathcal{O} \mathcal{M} \mathcal{L} \) and \( \mathcal{S} \), and preserves the structure:

\[
\begin{align*}
\rho(0) &= \rho(0) \\
\rho(H \land H') &= \rho(H) \cap \rho(H') \\
\rho(H \lor H') &= \rho(H) \cup \rho(H') \\
\rho(1) &= \rho(1) \\
\rho(H^\perp) &= \sim \rho(H)
\end{align*}
\]

*Proof.* By definition, \( \rho \) is an injection from \( \mathcal{O} \mathcal{M} \mathcal{L} \) into \( \mathcal{B} \mathcal{A} \mathcal{O} \) and thus also into \( \mathcal{S} \), and preserves the orthocomplement \( \perp \) and the orthomodular meet \( \land \). By definition of \( \mathcal{S} \), \( \rho \) is also a surjection from \( \mathcal{O} \mathcal{M} \mathcal{L} \) onto \( \mathcal{S} \), and preserves also the orthomodular bounds 0 and 1. For the preservation of the orthomodular join \( \lor \) consider that:

\[
\begin{align*}
\rho(H \lor H') &= \rho \left( \left( H^\perp \land H'^\perp \right)^\perp \right) \quad \text{(definition)} \\
&= \sim \rho(H^\perp \land H'^\perp) \quad \text{(\( \sim \)-homomorphism)} \\
&= \sim (\rho(H^\perp) \cap \rho(H'^\perp)) \quad \text{(meet homomorphism)} \\
&= \sim \rho(H^\perp) \cup \sim \rho(H'^\perp) \quad \text{(Proposition 2.2)} \\
&= \sim \sim \rho(H) \cup \sim \sim \rho(H') \quad \text{(\( \sim \)-homomorphism)} \\
&= \rho(H) \cup \rho(H') \quad \text{(Proposition 2.1)}
\end{align*}
\]

Let us take stock, and record which properties of the accessibility relation \( R \) were actually required to prove our theorem. Observe that only Proposition 2.1 and 2.2 require such properties. The proof of Proposition 2.2 requires Fact 2 and only that one as such a property. Less obviously, because somehow hidden in plain sight, Proposition 2.1 itself is actually another such property. It stipulates that for all \( S \in \{ \rho(H) \mid H \in H(S) \} \),

\[
S = ((R) \circ \tau \circ (R) \circ \tau)(S)
\]
we shall call it the $Q$-property of $R$, its corresponding axiom $\Box \Diamond A \rightarrow A$ the $Q$-axiom, and the resulting normal modal logic the logic $BQ$ ($= K+B+Q$).

In the following corollary, we apply $\rho$ tacitly on QL-formulas $A$ rather than on their semantic counterparts $H$ (their Lindenbaum-Tarski algebra quotient).

**Corollary 1.** Syntactically, $\rho$ is a linear-time reduction from QL to $BQ$:

$$A \in QL \text{ if and only if } \rho(A) \in BQ.$$  

**Proof.** Assuming that any quantum disjunction in a given QL-formula $A$ has been expanded by its definition in terms of quantum negation and quantum conjunction, we just substitute any occurrence of the quantum-negation symbol $\sim$ in $A$ with $\neg \Box$, or with $\Diamond \neg$, in order to obtain the corresponding $BQ$-formula. When performed on $A$ represented as a (linear) string of symbols, this substitution procedure obviously takes linear time (no back-tracking required). 

### 3 Conclusion

We have demonstrated that Quantum Logic (QL) is a fragment of the classical normal modal logic $BQ$. In other words, we have refuted Birkhoff and von Neumann’s classic thesis that the logic (the formal character) of Quantum Mechanics would be non-classical [1] as well as Putnam’s thesis that quantum logic (of his kind) would be the correct logic for propositional inference in general [11].

The propositional logic of Quantum Mechanics has turned out to be modal but classical, and the correct logic for propositional inference need not have an extroverted quantum character. The philosophical key to our result has been to internalise observability into our (logical) system (by means of a normal necessity modality), which in some sense is what Quantum Mechanics has always told us to do. With that, the mystery of the failure of classical conjunction to distribute over classical disjunction has dissolved and an elementary-logical solution for this (weak) paradox has emerged. (Other paradoxes of Quantum Mechanics may subsequently dissolve too.) In a formal sense, we have reduced Quantum Logic (QL) to Classical Logic, within a simple modal logic ($BQ$).

First-order extensions (with quantifiers) as well as dynamic and fixpoint extensions can now be worked out (see for example [5], [10], and [4], respectively).

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