ILL-POSEDNESS OF FREE BOUNDARY PROBLEM OF THE INCOMPRESSIBLE IDEAL MHD

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ABSTRACT. In the present paper, we show the ill-posedness of the free boundary problem of the incompressible ideal magnetohydrodynamics (MHD) equations in two spatial dimensions for any positive vacuum permeability $\mu_0$, in Sobolev spaces. The analysis is uniform for any $\mu_0 > 0$.

1. Introduction

This paper is concerned with the ill-posedness of the following free boundary problem for the incompressible ideal MHD equations

\[
\begin{align*}
\partial_t \upsilon + \upsilon \cdot \nabla \upsilon + \nabla q &= \frac{1}{\mu_0} \hat{H} \cdot \nabla \hat{H}, \quad \text{in } \Omega_t, \\
\partial_t \hat{H} + \upsilon \cdot \nabla \hat{H} &= \hat{H} \cdot \nabla \upsilon, \quad \text{in } \Omega_t, \\
\text{div } \upsilon &= 0, \quad \text{div } \hat{H} = 0, \quad \text{in } \Omega_t, \\
\partial_t \Gamma(t) &= \upsilon \cdot N, \quad \hat{H} \cdot N = 0, \quad q = 0, \quad \text{on } \Gamma_t, \\
v(0, x) &= v_0(x), \quad \hat{H}(0, x) = \hat{H}_0(x), \quad x \in \Omega := \Omega_0,
\end{align*}
\]

where $\upsilon$ is the velocity field, $\hat{H}$ is the magnetic field, $q$ is the total pressure and $\mu_0 > 0$ is the vacuum permeability; $N$ is the unit outward normal vector on the boundary $\Gamma_t$, $\Omega_0$ is the initial bounded domain.

Without the magnetic field, i.e. $\hat{H} = 0$ in (1.1), the problem (1.1) reduces to the free boundary problem of incompressible Euler equations, for which the local-in-time well-posedness in Sobolev spaces was obtained first in [27, 28] for the irrotational case. Extensions including the case without irrotationality assumption have been made in [1, 2, 4, 5, 9, 10, 13, 14, 15, 16, 21] and [29]. The Taylor sign condition, $\nabla_N p < 0$ on $\partial \Omega_t$, plays a crucial role in establishing the above well-posedness results for Euler equations, where $p$ is the fluid pressure for Euler equations. It was found by Ebin [6] that the free boundary problem for incompressible Euler equations is ill-posed when the Taylor sign condition fails. A natural question for MHD free boundary problem is that if this is still the case or the magnetic field has some stabilizing effect, as discussed for the current-vortex sheet problem of MHD in [3, 23, 25]. A stability condition,

\[
\nabla_N q < 0, \quad \text{on } \partial \Omega_t
\]

was identified in [9] for the MHD free boundary problem (1.1) under which the a priori estimates are derived. The present paper is concerned with what will happen if this condition is violated.

In 2-spatial dimension case, there is a particular steady solution to (1.1) with pure rotation: For $t \geq 0$, $\Omega_t = \{ x \in \mathbb{R}^2 : |x| < 1 \} = B_1(0)$, the unit disk, $\upsilon(t, x) = (-x_2, x_1) = x^\perp$, $\hat{H}(t, x) = bv(t, x)$ (for $x = (x_1, x_2)$ and $b \in \mathbb{R}$). In this case, $q$ is solved by the following Dirichlet problem:

\[
\Delta q = 2 \left( 1 - \frac{b^2}{\mu_0} \right), \quad \text{in } B_1(0); \quad q = 0, \quad \text{on } \partial B_1(0).
\]
For this solution, it is easy to verify that, on \(\partial B_1(0)\),
\[
\nabla_N q = \left( \frac{b^2}{\mu_0} - 1 \right) (v \cdot \nabla v) \cdot x = 1 - \frac{b^2}{\mu_0},
\]
by noting that \(N = x\) on \(\partial B_1(0)\). Therefore, the Taylor sign condition (1.2) fails in this case when \(\mu_0 \geq b^2\). A natural question arises: For \(\mu_0 \geq b^2\), is the free boundary problem (1.1) still well-posed? We prove in this paper that this is not the case when \(\mu_0 \gg b^2\) in the following sense:

we can construct a family of initial data for the problem (1.1) which converges to the above particular steady rotation solution. However, as long as \(t > 0\), the family of solutions to (1.1) with those initial data diverges in some Sobolev \(H^2\)-norm for \(\mu \geq 2\).

It should be also noted that our analysis is uniform in \(\mu_0 > 0\).

Our construction is strongly motivated by that of Ebin [6] for incompressible Euler equations. This approach involves showing that the linearized problem about the particular steady rotation solution has solutions which exhibit rapid exponential growth and the actual nonlinear problem behaves like the linearized one.

There are few results for the posedness of the free boundary problem of incompressible ideal MHD equations. A priori estimates for this problem were derived in [9] with a bounded initial domain homeomorphic to a ball, provided that the size of the magnetic field to be invariant on the free boundary. A local existence result was proved in [7] for the initial flat domain. The plasma-vacuum system was investigated in [8] where the a priori estimates were derived. For the special case where the magnetic field is zero on the free boundary and in vacuum, the local existence and uniqueness of the free boundary problem of incompressible viscous-diffusive MHD flow in three-dimensional space with infinite depth setting was proved in [11] where also a local unique solution was obtained for the free boundary MHD without kinetic viscosity and magnetic diffusivity via zero kinetic viscosity-magnetic diffusivity limit. For the incompressible viscous MHD equations, a free boundary problem in a simply connected domain of \(\mathbb{R}^3\) was studied by a linearization technique and the construction of a sequence of successive approximations in [22] with an irrotational condition for magnetic fields in a part of the domain. The well-posedness of the linearized plasma-vacuum interface problem in incompressible ideal MHD was studied in [17] in an unbounded plasma domain. For other related results of MHD equations with free boundaries or interfaces, one may refer to [3,12,20,24–26].

2. Lagrangian Description

For simplicity, we denote \(H = \mu_0^{-1/2} \tilde{H}\). Then, the problem (1.1) reduces to the case \(\mu_0 = 1\), i.e.,
\[
\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla q &= H \cdot \nabla H, & \text{in } \Omega_t, \\
\partial_t H + v \cdot \nabla H &= H \cdot \nabla v, & \text{in } \Omega_t, \\
\text{div } v &= 0, & \text{in } \Omega_t, \\
\partial_t \Gamma(t) &= v \cdot N, & H \cdot N = 0, & q = 0, & \text{on } \Gamma_t, \\
v(0,x) &= v_0(x), & H(0,x) &= H_0(x), & x & \in \Omega := \Omega_0.
\end{align*}
\]

We transform the system (2.1) into Lagrangian variables (e.g. [18]). We denote \(\eta = \eta(t,a) = (\eta^1, \eta^2)\) the position of a fluid element or parcel at time \(t\) with \(a = (a^1, a^2)\) being the fluid element label, defined to be the position of the fluid element at the initial time, \(a = \eta(0,a)\), but this is not necessarily always the case. Denote \(\Omega_t\) the domain occupied by the fluid at time \(t\), then \(\eta : \Omega_0 \rightarrow \Omega_t\) is assumed to be 1-1 and onto, at each fixed time \(t\).
Let $\partial \eta^i / \partial a^j =: \eta^i_j$ be the deformation matrix, its Jacobian determinant $J := \det(\eta^i_j)$ is given by

$$J = \frac{1}{2} \varepsilon_{k,j} \varepsilon^{i,l} \eta^k_i \eta^l_j,$$

where $\varepsilon_{ij} = \varepsilon^{ij}$ is the two-dimensional unit, purely antisymmetric, Levi-Civita tensor density. In this notation,

$$d\eta = J da,$$

and components of an area form map according to

$$(dS(q))_i = J a^j_i (dS(a))_j,$$

where $J a^j_i$ is the transpose of the cofactor matrix of $\eta^i_j$, given by

$$J a^j_i = \varepsilon_{ik} \varepsilon^{jl} \eta^k_i \eta^l_j.$$

Clearly, $\dot{\eta}(t, a) = v(t, x)$, where $\dot{\eta}(t, a) = \frac{\partial \eta(t, a)}{\partial t}$. The label of the element will given by $a = \eta^{-1}(t, x) =: a(t, x)$. Thus, the Eulerian velocity field is given by

$$v(t, x) = \dot{\eta}(t, a)|_{a = a(t, x)}.$$

For an incompressible fluid, $J = 1$. Thus, we can invert $\eta = \eta(t, a)$ to obtain $a = a(t, \eta)$,

$$\eta^i_k a^k_j = \delta^i_j,$$

where $a^k_i = \partial a^k / \partial \eta^i$ (repeated indices are summed), and the components of the Eulerian gradient are given by

$$\frac{\partial}{\partial x^k} = a^i_k \frac{\partial}{\partial a^i}|_{a = a(t, x)}.$$

Thus, for $x = \eta(t, a)$ and any function $f(t, a) = \tilde{f}(t, x) = \tilde{f}(t, \eta(t, a))$,

$$\tilde{f}|_{a = a(t, x)} = \frac{\partial \tilde{f}}{\partial t} + \dot{\eta}(t, a) \frac{\partial \tilde{f}}{\partial x^i}|_{a = a(t, x)} = \frac{\partial \tilde{f}}{\partial t} + v \cdot \nabla \tilde{f}(t, x).$$

For ideal MHD, a magnetic field, $H_0(a)$, can be attached to a fluid element, and then the frozen flux condition yields $H \cdot dS(x) = H_0 \cdot dS(a)$, and from (2.3) we obtain

$$H^i = \eta^i_j H^j_0,$$

which can be also obtained from the equation (2.1b). In fact, we have by (2.1b)

$$\dot{H}^i = H^k a^i_k \dot{v}^j_\ell = H^k a^i_k \dot{\eta}^j_\ell = -H^k \dot{a}^i_k \eta^j_\ell,$$

due to

$$\dot{\eta}^i_k a^k_\ell = -\eta^i_k \dot{a}^k_\ell,$$

from (2.4), and by (2.4) again

$$\dot{H}^i a^j_\ell = -H^k \dot{a}^i_k \eta^j_\ell a^j_\ell = -H^k \dot{a}^i_k,$$

and then

$$\frac{d}{dt} (H^i a^j_\ell) = 0,$$

which yields the desired identity (2.5).

It follows from (2.1a), (2.1c) and $q = 0$ on $\Gamma_t = \eta(\Gamma)$, that

$$\begin{cases}
\Delta q = \text{tr}(DH)^2 - \text{tr}(DV)^2, & \text{in } \Omega_t, \\
q = 0, & \text{on } \Gamma_t,
\end{cases}$$

(2.6a)
where $\text{tr}(Dv)^2 := \partial_i v^i \partial_j v^j$ and $\text{tr}(DH)^2 := \partial_i H^j \partial_j H^i$ are the trace of the square of the matrices of differentiations of $v$ and $H$, respectively. This elliptic Dirichlet boundary value problem admits a unique solution, so $\Delta^{-1}$ is well-defined. Here $\Delta^{-1}g = f$ in a domain means $\Delta f = g$ in this domain and $f = 0$ on the boundary of this domain. Hence, 

$$q = \Delta^{-1}(\text{tr}(DH)^2 - \text{tr}(Dv)^2).$$

(2.7)

Therefore, (2.1a) can be rewritten as 

$$\ddot{\eta} = (\nabla(\Delta^{-1}(\text{tr}(Dv)^2 - \text{tr}(DH)^2))) \circ \eta + (H \cdot \nabla H) \circ \eta. \tag{2.8}$$

Note that $v(t, x) = \dot{\eta}(t, a(t, x))$ and (2.5), so (2.8) is of the form 

$$\ddot{\eta} = Z(\eta, \dot{\eta}). \tag{2.9}$$

Then we study the initial value problem for (2.9) with the initial data $\eta(0) = \eta_0$ and $\dot{\eta}(0)$.

3. An example

As in [6], we consider a disc of MHD fluid spinning at constant angular velocity. Let $\Omega$ be the unit disc in $\mathbb{R}^2$. For convenience, we identify points in $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$. Then $\eta(0) = \eta_0$ is the inclusion of $\Omega$ in $\mathbb{C}$. We choose $\dot{\eta}(0)$ to be a $\pi/2$ rotation. Thus, for $z \in \Omega$ being a complex variable, $\eta(0, z) = z$ and $\dot{\eta}(0, z) = iz$. Since $H_0 \cdot \nabla\eta = 0$ on the boundary, we can take $H_0(\eta(z)) = ibz$ for some constant $b \in \mathbb{R}$ and $|b| < 1$.

As we will demonstrate, the solution to (2.8) with the above initial data is $\eta(t, z) = e^{bt}z$ and $H(t, z) = ibz$. We have $\dot{\eta}(t, z) = i\eta(t, z) = ie^{bt}z$, and $a(t, z) = e^{-bt}z$. Thus, $v(t, z) = \dot{\eta}(t, a(t, z)) = ie^{bt}a(t, z) = ie^{bt}e^{-bt}z = iz$. By using real variables, we find that $D\eta = e^{bt}I$ where $I$ is the $2 \times 2$ unit matrix $(\delta_{ij})$. Thus, by (2.5), we have $H(t, z) = ibz$. Moreover, $v(x_1, x_2) = (-x_2, x_1)$ and $H(x_1, x_2) = b(-x_2, x_1)$, both satisfy the divergence-free condition, and we get the matrices 

$$Dv = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad DH = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}.$$

Thus, $\text{tr}(Dv)^2 = -2$ and $\text{tr}(DH)^2 = -2b^2$, and the Dirichlet problem is solved by 

$$q = (b^2 - 1)\Delta^{-1}(-2) = \frac{1-x_1^2-x_2^2}{2}(b^2 - 1).$$

Then, $\nabla q = (1 - b^2)(x_1, x_2) = (1 - b^2)z$. It is easy to check that $H \cdot \nabla H = -b^2z$ by using real variables.

Since $\dot{\eta}(t, z) = i\eta(t, z)$ and $\dot{\eta}(t, z) = -\eta(t, z)$, we get 

$$\ddot{\eta}(t, z) + \nabla q(t, \eta(t, z)) = (H \cdot \nabla H)(\eta(t, z)),$$

namely, $\eta(t, z)$ satisfies (2.8). In this example we note that, both the velocity $v(t, z) = iz$ and the magnetic field $H(t, z) = ibz$ are independent of the time $t$.

4. Reformulation in Lagrangian variables

We first recall some operators defined in [6]. The operator $R_\eta : C^\infty(\Omega) \to C^\infty(\Omega)$ defined by $R_\eta(f) = f \circ \eta = f(\eta)$. It is obvious that the inverse of $R_\eta$ is $R_{\eta^{-1}}$ given by $R_{\eta^{-1}}(g) = g \circ \eta^{-1} = g(\eta^{-1})$ where $\eta^{-1}(t, x) = a(t, x)$.

For a differential operator $P$, we set $P_\eta := R_\eta P R_{\eta^{-1}}$. For examples, the operator $D_\eta = R_\eta D R_{\eta^{-1}} : C^\infty(\Omega) \to C^\infty(\Omega)$, where $D$ is the total derivative, and the operator $\nabla_\eta = \nabla_\eta \Delta_\eta R_{\eta^{-1}}$ where $\nabla$ is the gradient. Define 

$$K(\eta) = R_\eta \Delta R_{\eta^{-1}} : C^\infty_0(\Omega) \to C^\infty(\Omega).$$
where \( C_0^\infty(\Omega) = \{ f \in C_0^\infty(\Omega) : f|_{\partial\Omega} = 0 \} \). Namely, \( K(\eta) = \Delta\eta \). \( K(\eta) \) is invertible because so does \( \Delta \). In fact,

\[
K(\eta)^{-1} = R\eta \Delta^{-1}R\eta^{-1}.
\]

Since \( v = \dot{\eta} \circ \eta^{-1} \), we have

\[
DV \circ \eta = R\eta DV = R\eta D(\dot{\eta} \circ \eta^{-1}) = R\eta DR\eta^{-1} \dot{\eta} = D_{\eta} \dot{\eta}.
\]

Similarly, from [2.5], it follows

\[
\partial_t H(t, x) = \partial_t (\eta^i_k(t, a(t, x)) H^k_0(x)) = \eta^i_{k,l}(t, a(t, x)) a^l_j(t, x) H^k_0(x) + \eta^i_k(t, a(t, x)) H^k_0 \partial_x^i(x),
\]

\[
\text{tr}(DH)^2 = (\eta^i_{k,l}(t, a(t, x)) a^l_j(t, x) H^k_0(x) + \eta^i_k(t, a(t, x)) H^k_0 \partial_x^i(x)) \cdot (\eta^i_{m,n}(t, a(t, x)) a^m_j(t, x) H^m_0(x) + \eta^i_m(t, a(t, x)) H^m_0 \partial_x^i(x))
\]

\[
= \eta^i_{k,l}(t, a(t, x)) a^l_j(t, x) H^k_0(x) \eta^i_{m,n}(t, a(t, x)) a^m_j(t, x) H^m_0(x)
+ 2 \eta^i_{k,l}(t, a(t, x)) H^k_0(x) \partial_x^i(x)
+ \eta^i_k(t, a(t, x)) H^k_0(x) \eta^i_m(t, a(t, x)) H^m_0(x)
\]

\[
= \text{tr}(\eta^i_{k,l} \circ \eta^{-1})(\eta^{-1})^l_m H^k_0)^2 + \text{tr}(\eta^i_{k,l} \circ \eta^{-1})(H^k_0 H^l_0),
\]

and

\[
(H \cdot \nabla H)_k = \delta_{ki} H^i_j H^j_l = \delta_{ki} \eta^l_{m,n}(H^m_0 a^l_j H^n_0)_{,l} = \delta_{ki} H^i_0 \eta^l_{m,n} a^l_j H^n_0_{,l}
\]

\[
= \delta_{ki} H^i_0 (\eta^l_{m,n} a^l_j H^n_0 + \eta^l_n H^n_0_{,l})
\]

\[
= \delta_{ki} (\eta^l_{m,n} \circ \eta^{-1}) (\eta^{-1})^l_m H^n_0 H^n_0 + \delta_{ki} (\eta^l_n \circ \eta^{-1}) H^n_0 H^n_0.
\]

Then, from (2.8), we get

\[
Z(\eta, \dot{\eta}) = (\nabla (\Delta^{-1} R\eta^{-1} (DV \circ \eta))) \circ \eta
\]

\[
= (\nabla \Delta^{-1} \text{tr}(\text{tr}(\eta^i_{k,l} \circ \eta^{-1})(\eta^{-1})^l_m H^k_0)^2) \circ \eta
\]

\[
= (\nabla \Delta^{-1} \text{tr}(\eta^i_{k,l} \circ \eta^{-1})(H^k_0 H^l_0),) \circ \eta
\]

\[
= (\nabla \Delta^{-1} \text{tr}(\eta^i_{k,l} \circ \eta^{-1})(\eta^{-1})^l_m H^n_0 H^n_0) \circ \eta
\]

\[
= (\nabla \Delta^{-1} \text{tr}(\nabla \eta \cdot \eta^{-1})(\eta^{-1})^l_m H^n_0 H^n_0) \circ \eta
\]

Hence, (2.9) can be written as

\[
\dot{\eta} = \nabla \eta \nabla (\Delta^{-1} \Delta^{-1} R\eta^{-1} (DV \circ \eta)) - \nabla \eta \nabla (\Delta^{-1} \Delta^{-1} R\eta^{-1} (DV \circ \eta)) - \nabla \eta \nabla (\eta^i_{k,l} \circ \eta^{-1})(\eta^{-1})^l_m H^n_0 H^n_0) + \eta_{nm} \text{tr}(\eta^{-1})^l_m H^n_0 H^n_0)
\]

(4.1)
5. Linearization of the Equation

We consider now a family of solutions to (4.11) parametrized by \( s \), call it \( \zeta(t, s) \). Assume that \( \zeta \) is differentiable in \( s \) and let

\[
w(t) = \partial_s \zeta(t, s)|_{s=0},
\]

the tangent of \( \zeta(t, s) \) at \( s = 0 \). Denote \( \zeta(t, 0) \) by \( \zeta(t) \). Then \( w(t) \) satisfies

\[
\dot{w}(t) = \partial_s \zeta(t, s)|_{s=0} = \partial_s Z(\zeta(t, s), \zeta(t, s))|_{s=0}
\]

\[
= DZ(\zeta(t, s), \dot{\zeta}(t, s))(\partial_s \zeta(t, s), \partial_s \dot{\zeta}(t, s))|_{s=0}
\]

\[
= DZ(\zeta(t), \dot{\zeta}(t))(w(t), \dot{w}(t)).
\]  

(5.1)

In order to compute (5.1) more explicitly, we set \( u = \Re \zeta^{-1} w \), \( w = u \circ \zeta \). Clearly \( K(\zeta)^{-1} \) is inverse to \( K(\zeta) \) if the domain of \( K(\zeta) \) is \( C^\infty(\Omega) \). If this domain is enlarged to \( C^\infty(\Omega) \), \( K(\zeta)^{-1} \) is only a right inverse. That is \( K(\zeta)K(\zeta)^{-1} = I \), but

\[
K(\zeta)^{-1} K(\zeta) = I - \mathcal{H}(\zeta),
\]

where \( \mathcal{H}(\zeta) \) is defined as follows. For \( \zeta = Id \), the identity, \( \mathcal{H}(\zeta) \) projects a function onto its harmonic part. Thus, if \( \mathcal{H}(Id)f = g \) then \( f = g \) on \( \partial \Omega \) and \( \Delta g = 0 = K(Id)g \). For arbitrary \( \zeta, \mathcal{H}(\zeta)f = g \) if \( f = g \) on \( \partial \Omega \) and \( K(\zeta)g = 0 \).

Next, we recall some results on the commutators of operators, denoted by \([\cdot, \cdot]\), proved in [6].

**Lemma 5.1.** 1) \( \partial_s \nabla \zeta(t, s)|_{s=0} = \Re(\zeta(t))[u \cdot \nabla, \nabla]\Re(\zeta(t))^{-1} = [u \cdot \nabla, \nabla]|_{\zeta(t)} \), where \( u \cdot \nabla \) means the derivative in direction \( u \).

2) \( \partial_s K(\zeta(t, s))|_{s=0} = [u \cdot \nabla, \Delta]|_{\zeta(t)} \).

3) \( \partial_s K(\zeta(t, s))^{-1}|_{s=0} = ([u \cdot \nabla, \Delta^{-1}] - \mathcal{H}(u \cdot \nabla)\Delta^{-1})|_{\zeta(t)} \).

As in [6] (4.11)], we have

\[
\partial_s \text{tr}(D^2 \zeta) = 2\text{tr}
\]

\[
(\partial_s \nabla \zeta)^2|_{s=0} - (\partial_s \Delta \zeta)|_{s=0}.
\]

Since

\[
\partial_s (\zeta^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0)))
\]

\[
= \zeta^{i,j}_k \partial_s \Re(\zeta(\zeta^{-1}_i J^j H^0_0)) + \zeta^{i,j}_k \partial_s (\zeta^{-1}_l J^j H^0_0 |_{\zeta(t, s)}) + \zeta^{i,j}_k (\zeta^{-1}_l J^j H^0_0 |_{\zeta(t, s)}) \partial_s H^0_0 |_{\zeta(t, s)}
\]

\[
= \zeta^{i,j}_k \partial_s \Re(\zeta(\zeta^{-1}_i J^j H^0_0)) - \zeta^{i,j}_k (\zeta^{-1}_l J^j H^0_0 |_{\zeta(t, s)}) \partial_s \zeta^{n,m}_m
\]

\[
+ \zeta^{i,j}_k (\zeta^{-1}_l J^j H^0_0 |_{\zeta(t, s)}) \partial_s \zeta^{n,m}_m,
\]

it follows that

\[
\partial_s (\zeta^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0)))|_{s=0}
\]

\[
=w^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0)) - \zeta^{i,j}_k (\zeta^{-1}_l J^j H^0_0 |_{\zeta(t, s)}) \partial_s \zeta^{n,m}_m H^0_0 |_{\zeta(t, s)} + \zeta^{i,j}_k (\zeta^{-1}_l J^j H^0_0 |_{\zeta(t, s)}) \partial_s \zeta^{n,m}_m w^{n,m}_m
\]

and

\[
\partial_s \text{tr}(\zeta^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0)))|_{s=0}
\]

\[
= 2\zeta^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0))(w^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0)) - \zeta^{i,j}_k (\zeta^{-1}_l J^j H^0_0 |_{\zeta(t, s)}) \partial_s \zeta^{n,m}_m H^0_0 |_{\zeta(t, s)}
\]

\[
+ \zeta^{i,j}_k (\zeta^{-1}_l J^j H^0_0 |_{\zeta(t, s)}) \partial_s \zeta^{n,m}_m w^{n,m}_m)
\]

\[
= 2\zeta^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0)) w^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0)) - 2\zeta^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0 |_{\zeta(t, s)}) \partial_s \zeta^{n,m}_m H^0_0 |_{\zeta(t, s)}
\]

\[
+ 2\zeta^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0 |_{\zeta(t, s)}) \partial_s \zeta^{n,m}_m H^0_0 |_{\zeta(t, s)}) \partial_s \zeta^{n,m}_m H^0_0 |_{\zeta(t, s)}).
\]

Due to

\[
\partial_s (\zeta^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0))|_{s=0} = \partial_s \zeta^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0))|_{s=0} + \zeta^{i,j}_k \Re(\zeta(\zeta^{-1}_i J^j H^0_0)) \partial_s \zeta^{n,m}_m |_{s=0}
\]
we get
\[ \partial_s \text{tr}(\zeta_{ik}^j \mathcal{R}_\zeta(H_0^k H_0^l),i)|_{s=0} = w^j_{ikl} \mathcal{R}_\zeta(H_0^k H_0^l),i + \zeta_{kl}^j \mathcal{R}_\zeta(H_0^k H_0^l),i,m u^m). \]

Since
\[ \partial_s(\zeta_{ik}^j \mathcal{R}_\zeta H_0^k)|_{s=0} = \partial_s \zeta_{ik}^j \mathcal{R}_\zeta H_0^k|_{s=0} + \zeta_{kl}^j \mathcal{R}_\zeta(H_0^k u^l), \]

it yields
\[ \partial_s \text{tr}(\zeta_{ik}^j \mathcal{R}_\zeta H_0^k)^2|_{s=0} = 2 \zeta_{ik}^j w^j_{ikl} \mathcal{R}_\zeta(H_0^k H_0^l) + 2 \zeta_{ik}^j \zeta_{kl}^j \mathcal{R}_\zeta(H_0^k H_0^l u^l). \]

Now considering \((4.1)\) as an equation in \(\zeta(t,s)\) and applying \(\partial_s|_{s=0}\) to obtain
\[ \dot{w} = (\partial_s \nabla \zeta_{(t,s)}(\nabla \zeta(t,s))(2 \zeta_{ik}^j \mathcal{R}_\zeta H_0^k)^2 + \nabla \zeta([u \cdot \nabla, \Delta^{-1}] - \mathcal{H}(u \cdot \nabla) \Delta^{-1}) \zeta \text{tr}(D \zeta) - \mathcal{H}(u \cdot \nabla) \Delta^{-1}) \zeta \text{tr}(D \zeta) + (D \zeta)^{-1} \dot{w}(D \zeta)^{-1} \zeta \text{tr}(D \zeta)) \]

It yields from Lemma 5.1
\[ \dot{w} = [u \cdot \nabla, \nabla] \zeta \Delta^{-1} \zeta \text{tr}(D \zeta)^2 + \nabla \zeta([u \cdot \nabla, \Delta^{-1}] - \mathcal{H}(u \cdot \nabla) \Delta^{-1}) \zeta \text{tr}(D \zeta) - \mathcal{H}(u \cdot \nabla) \Delta^{-1}) \zeta \text{tr}(D \zeta) + (D \zeta)^{-1} \dot{w}(D \zeta)^{-1} \zeta \text{tr}(D \zeta) \]

\[ \dot{w} = [u \cdot \nabla, \nabla] \zeta \Delta^{-1} \zeta \text{tr}(D \zeta)^2 + \nabla \zeta([u \cdot \nabla, \Delta^{-1}] - \mathcal{H}(u \cdot \nabla) \Delta^{-1}) \zeta \text{tr}(D \zeta) - \mathcal{H}(u \cdot \nabla) \Delta^{-1}) \zeta \text{tr}(D \zeta) + (D \zeta)^{-1} \dot{w}(D \zeta)^{-1} \zeta \text{tr}(D \zeta) \]

\[ \dot{w} = [u \cdot \nabla, \nabla] \zeta \Delta^{-1} \zeta \text{tr}(D \zeta)^2 + \nabla \zeta([u \cdot \nabla, \Delta^{-1}] - \mathcal{H}(u \cdot \nabla) \Delta^{-1}) \zeta \text{tr}(D \zeta) - \mathcal{H}(u \cdot \nabla) \Delta^{-1}) \zeta \text{tr}(D \zeta) + (D \zeta)^{-1} \dot{w}(D \zeta)^{-1} \zeta \text{tr}(D \zeta) \]
\[ - [u \cdot \nabla, \nabla \Delta^{-1}]_\zeta \text{tr}(\zeta_{,kl} \mathcal{R}_\zeta ((\zeta^{-1})^j_i H_0^k))^2 \]
\[ + \nabla \zeta (\mathcal{H}(u \cdot \nabla) \Delta^{-1})_\zeta \text{tr}(\zeta_{,kl} \mathcal{R}_\zeta ((\zeta^{-1})^j_i H_0^k))^2 \]
\[ - \nabla \zeta \Delta^{-1}(2\zeta_{,k'}^j \mathcal{R}_\zeta ((\zeta^{-1})^j_i H_0^k) \psi^l_{,kl} \mathcal{R}_\zeta ((\zeta^{-1})^l_j H_0^k)) \]
\[ + \nabla \zeta \Delta^{-1}(2\zeta_{,k'}^j \mathcal{R}_\zeta ((\zeta^{-1})^j_i H_0^k) \zeta^{l'}_l) \]
\[ - \nabla \zeta \Delta^{-1}(2\zeta_{,k'}^j \mathcal{R}_\zeta ((\zeta^{-1})^j_i H_0^k) \zeta^{l'}_l) \]
\[ - [u \cdot \nabla, \nabla \Delta^{-1}]_\zeta (\zeta_{,kl} \mathcal{R}_\zeta (H_0^k H_0^l)_j) + \nabla \zeta (\mathcal{H}(u \cdot \nabla) \Delta^{-1})_\zeta (\zeta_{,kl} \mathcal{R}_\zeta (H_0^k H_0^l)_j) \]
\[ - \nabla \zeta \Delta^{-1}(w^l_{,kl} \mathcal{R}_\zeta (H_0^k H_0^l)_i) - \nabla \zeta \Delta^{-1}(\zeta_{,kl} \mathcal{R}_\zeta ((H_0^k H_0^l)_i, i m u^m)) \]
\[ - [u \cdot \nabla, \nabla \Delta^{-1}]_\zeta (\zeta_{,kl} \mathcal{R}_\zeta (H_0^k H_0^l)_i) + \nabla \zeta (\mathcal{H}(u \cdot \nabla) \Delta^{-1})_\zeta (\zeta_{,kl} \mathcal{R}_\zeta (H_0^k H_0^l)_i) \]
\[ - \nabla \zeta \Delta^{-1}(2\zeta_{,k'}^j \mathcal{R}_\zeta ((\zeta^{-1})^j_i H_0^k)) - \nabla \zeta \Delta^{-1}(2\zeta_{,k'}^j \mathcal{R}_\zeta ((\zeta^{-1})^j_i H_0^k)) \]
\[ + \zeta_{,nm} \mathcal{R}_\zeta ((\zeta^{-1})^n_j H_0^k) H_0^l + \zeta_{,n} \mathcal{R}_\zeta (H_0^m H_0^l). \quad (5.3) \]

For our choice of \( \eta(t) \) in place of \( \zeta(t) \) we can compute explicitly some terms in the linearized equation. To do so, let us return to complex variables for convenience. We know that \( D\eta(t, z) = e^{it} \) for \( \eta \) given in section 3 so that \( (D\eta)^{-1} D\eta = i \) and \( \text{tr}((-D\eta \circ \eta)(-1)) = \text{div} u = 0 \). Also, \( \text{tr}(D\eta)^2 = \text{tr}((Du)^2) \circ \eta = -2 \). Due to \( \eta_{,j}^2 = e^{it} \delta_{,j}^i \), \( (-\eta_{,j}^2)^2 = e^{it} \delta_{,j}^i \) and \( H_{0,i l} = (l - n) b (l, n \in \{1, 2\}) \). Therefore, any second-order spatial derivatives of both \( \eta, \eta_{,j}^2 \) and \( H_0 \) are zeros. Hence \( \text{tr}(\eta_{,j}^2 \mathcal{R}_\eta (H_0^k H_0^l)_i) = -2b^2 e^{2it} \) and \( \eta_{,n} \mathcal{R}_\eta (H_{0,i l}^n H_0^l) = b^2 e^{it} \). Therefore, it turns out from (5.5)

\[ \dot{\bar{w}} = (1 - b^2) \left( [u \cdot \nabla, \nabla \Delta^{-1}]_\eta(-2) - \nabla \eta (\mathcal{H}(u \cdot \nabla) \Delta^{-1})_\eta(-2) \right) \]
\[ + \nabla \eta \Delta^{-1}(2\text{tr} \left( (Du \circ \eta + i e^{-it} D\eta) \right)) \]
\[ - \nabla \eta \Delta^{-1}(w^l_{,kl} \mathcal{R}_\eta (H_0^k H_0^l)_i) \]
\[ - e^{it} \nabla \eta \Delta^{-1}(2w^l_{,kl} \mathcal{R}_\eta (H_0^k H_0^l)_i) \]
\[ + b^2 e^{it} \eta. \quad (5.4) \]

Now we restrict our attention to special solutions of (5.4). Assume that \( w = \nabla f(t) \circ \eta \) where \( f(t) \) is a harmonic function on \( \Omega \). Thus, \( u = \nabla f \) a harmonic gradient. Then \( Dw = D\eta (D\nabla f) \circ \eta \) and \( \text{tr}(i e^{-it} D\dot{w}) = 0 \) as similar as in [6] Eq. (4.16)], so (5.4b) vanishes. We also have

\[ w_{,k}^l = \partial_t \partial_j f(t, \eta(t, x)) \eta_{,k}^j(t, x) = e^{it} \partial_k \partial_j f(t, \eta(t, x)), \]
\[ w_{,k}^l = e^{it} \partial_m \partial_k \partial_l f(t, \eta(t, x)) \eta_{,m}^l(t, x) = e^{it} \partial_k \partial_m \partial_l f(t, \eta(t, x)), \]

and then

\[ w_{,k}^l \mathcal{R}_\eta (H_0^k H_0^l)_i = b^2 e^{it} \sum_{i,k=1}^{2} \partial_l \partial_k \partial_l f(t, \eta(t, x))(i - k) \]
\[ = b^2 e^{it} \sum_{i,k=1}^{2} x^l \partial_l \partial_k \partial_l f(t, \eta(t, x))(i - k) \]
\[ = 0, \quad (5.5) \]

and

\[ w_{,j}^l \mathcal{R}_\eta (H_0^j H_0^k)_i = b^2 e^{it} \sum_{i,j,k=1}^{2} \partial_k \partial_j f(t, \eta(t, x))(j - i)(i - k) \]
If we assume the initial conditions
\[ w = C \]
where
\[ \eta \]
Thus, both (5.4c) and (5.4d) vanish. Hence, similar to the discussion in [6],
\[ \Delta f \circ \eta = 0. \]  
(5.6)
where \( x \cdot \nabla \) is the radial derivative and commutes with composition with the rotation \( \eta \). Now, we recall a useful lemma.

**Lemma 5.2** ([6] Lemma 4.19). If \( f \) is harmonic, and \( \eta(t, z) = e^{it}z \), then there exists a harmonic function \( g \) such that \( \nabla f \circ \eta = \nabla g \).

Since \( f \) is harmonic, by Lemma 5.2 there exists a harmonic function \( g \) such that \( \nabla f \circ \eta = \nabla g \). Thus, (5.7) can be written as
\[ \nabla g = (1 - b^2)x \cdot \nabla (\nabla g) + b^2e^{2it}z. \]  
(5.8)
Define \( A \) by \( A \nabla g = (x \cdot \nabla)\nabla g \). Let \( g(z) = \text{Re} \ z^n \ (n \geq 1) \), then \( A \nabla g = nz^{n-1} \). Then \( A \nabla g = n z \cdot \nabla z^{n-1} = (n-1)z^{n-1} = (n-1)\nabla g \). Similarly, if \( g(z) = \text{Re} \ iz^n \ (n \geq 1) \), then \( A \nabla g = niz^{n-1} = (n-1)\nabla g \). Note that \( E := \{nz^{n-1}, iz^{n-1}\}_{n=1}^\infty \) form a basis of the set of Harmonic gradients on \( \Omega \). So \( A \) has this set as a complete set of eigenfunctions and has double eigenvalues \( 0, 1, 2, \ldots \). We seek the solutions by separated variables of the form \( g(t, z) = \sigma(t)h(z) \) for \( h \in E \), so that (5.8) can be written as
\[ \dot{\sigma}(t) = (1 - b^2)(n-1)\sigma(t) + \frac{b^2e^{2it}z}{h(z)}. \]  
(5.9)
Let \( B = \sqrt{(1 - b^2)(n-1)} \). The usual solution to (5.9) can be written as
\[
\sigma(t) = C_1e^{Bt} + C_2e^{-Bt} + \int_0^t \frac{e^{Bs}e^{-Bt} - e^{Bt}e^{-Bs}}{Be^{Bs}e^{-Bt} - Be^{Bt}e^{-Bs}}b^2e^{2is}z/h(z)ds
\]
\[
= C_1e^{Bt} + C_2e^{-Bt} - \frac{b^2z}{2Bh(z)} \int_0^t (e^{Bs}e^{-Bt} - e^{Bt}e^{-Bs})e^{2is}ds
\]
\[
= C_1e^{Bt} + C_2e^{-Bt} - \frac{b^2z}{2Bh(z)} \left( \frac{e^{2it} - e^{-Bt}}{B + 2i} + \frac{e^{2it} - e^{-Bt}}{B - 2i} \right)
\]
\[
= C_1e^{Bt} + C_2e^{-Bt} - \frac{b^2z}{2Bh(z)} \left( \frac{2B}{B^2 + 4} e^{2it} - \frac{e^{-Bt}}{B - 2i} - \frac{e^{Bt}}{B + 2i} \right).
\]
Hence, we get the solution of (5.8) of the form
\[
w_n(t, z) = C_1e^{Bt}nz^{n-1} + C_2e^{-Bt}nz^{n-1} - \frac{b^2z}{2B} \left( \frac{2B}{B^2 + 4} e^{2it} - \frac{e^{-Bt}}{B + 2i} - \frac{e^{Bt}}{B - 2i} \right).
\]
If we assume the initial conditions \( w_n(0) = 0 \) and \( \dot{w}_n(0) = e^{-n/4}z^n \), then
\[ C_1 + C_2 = 0 \] and \( (C_1 - C_2)Bn^{n-1} = e^{-n/4}z^n \), i.e., \( C_1 = -C_2 = e^{-n/4}z/(2Bn) \). Hence, for \( n \geq 2 \),
\[
w_n(t) = \frac{1}{\sqrt{(1 - b^2)(n-1)}} e^{-n/4} \sinh(\sqrt{(1 - b^2)(n-1)}t)z^n
\]
\[
+ \frac{b^2z}{(1 - b^2)(n-1) + 4} \left( \cosh(\sqrt{(1 - b^2)(n-1)}t) - e^{2it} \right).
\]
\[ \frac{2b^2iz \sinh\left(\sqrt{(1-b^2)(n-1)t}\right)}{\sqrt{(1-b^2)(n-1)}((1-b^2)(n-1) + 4)}, \quad (5.10) \]

is a sequence of solutions to (5.4) with initial conditions \( w_n(0) = 0 \) and \( \dot{w}_n(0) = e^{-n^{1/4}}z^n \). When \( b^2 < 1 \), this sequence is as useful as for the Euler equation, discussed in [6], because the initial data go to zero in \( C^\infty(\Omega) \), but for any \( t > 0 \), in view of the exponential growth of \( \sinh \) and \( \cosh \), \( \{w_n(t)\}_{n=2}^{\infty} \) is an unbounded sequence in \( C^\infty(\Omega) \).

6. **Discontinuous dependence on initial data**

In this section, we will show that the problem (2.8) is ill-posed.

6.1. **Construction of the sequences of initial data and solutions.** Let \( \eta(t,z) = e^{\mu z}, \) the solution to (2.8) given in section 3, and set \( \zeta_n(0,z) = \eta(0,z) = z \) and \( \dot{\zeta}_n(0,z) = \eta(0,z) + e^{-n^{1/4}}z^n \). Then, \( (\zeta_n(0,z), \dot{\zeta}_n(0,z)) \to (\eta(0,z), \eta(0,z)) \) in \( C^\infty(\Omega) \times C^\infty(\Omega) \) as \( n \to \infty \). Assuming that there exists some positive \( T \) such that for all \( n \), \( \zeta_n(t) \) is the unique solution of (2.8) for \( 0 \leq t \leq T \), the goal is to show that \( \zeta_n(t) \) does not converge to \( \eta(t) \), not in \( C^\infty(\Omega) \) for any positive \( t \leq T \).

For this, we define \( y_n(t) = \zeta_n(t) - \eta(t) \).

We will show that \( y_n(t) \) grows like \( w_n(t) \) of (5.10) and thus, \( \lim_{n \to \infty}(\zeta_n(t) - \eta(t)) \) is not zero for \( t > 0 \).

Since both \( \eta(t) \) and \( \zeta_n(t) \) satisfy (2.9), we have

\[
\begin{align*}
\ddot{\zeta} - \ddot{\eta} &= Z(\zeta, \dot{\zeta}) - Z(\eta, \dot{\eta}) = \int_0^1 Z_{j}(\zeta(s), \dot{\zeta}(s))(\zeta - \eta, \dot{\zeta} - \dot{\eta})ds \\
&= \int_0^1 \frac{DZ(\zeta(s), \dot{\zeta}(s))(\zeta - \eta, \dot{\zeta} - \dot{\eta})ds}{Z_{j}(\zeta(s), \dot{\zeta}(s))},
\end{align*}
\]

where \( \zeta(s) = \zeta + s(\eta - \zeta) \). Hereafter we suppress the subscript “n” in \( \zeta \) for simplicity. Thus, by the mean value theorem of integrals,

\[
\begin{align*}
\ddot{y}(t) &= \int_0^1 Z_{j}(\zeta(s), \dot{\zeta}(s))(y, \dot{y})ds \\
&= \int_0^1 (Z_{j}(\zeta(s), \dot{\zeta}(s))(y, \dot{y})^2 - Z_{j}(\eta, \dot{\eta})(y, \dot{y})^2)ds \\
&= \int_0^1 (1 - s) \left( \int_0^s Z_{jk}(\zeta(\sigma), \dot{\zeta}(\sigma))(y, \dot{y})^k d\sigma \right) (y, \dot{y})^2ds \\
&= :DZ(\eta, \dot{\eta})(y, \dot{y}) + \int_0^1 (1 - s) \left( \int_0^s D^2Z(\zeta(\sigma), \dot{\zeta}(\sigma))(y, \dot{y})(y, \dot{y})d\sigma \right) ds, \quad (6.1)
\end{align*}
\]

since \( \zeta(s) - \eta = \zeta + s(\eta - \zeta) - \eta = (1 - s)(\zeta - \eta) = (1 - s)y \) for \( s \in [0,1] \) and \( (1 - \sigma')(\zeta(s) + \sigma'\eta = (1 - \sigma')(1 - s)\zeta + s\eta) + \sigma'\eta = (1 - s)(1 - \sigma')\zeta + s(1 - s)\sigma'\eta = (1 - \sigma')\zeta + s\eta = \zeta(\sigma), \) where \( \sigma = s + (1 - s)\sigma' \) and \( 1 - \sigma = s(1 - s)\sigma' \) for \( \sigma' \in [0,1] \) and \( \sigma \in [0,1] \).

Denote \( H^s(\Omega) \) the Sobolev spaces for any real number \( s \) with inner product \( \langle \cdot, \cdot \rangle_s \) and corresponding norm \( \| \cdot \|_s \). For a nonnegative integer \( s \),

\[
(f, g)_s = \sum_{k=0}^{s} \int_{\Omega} \langle D^k f, D^k g \rangle dx,
\]

with the usual extensions to other \( s \).
Proposition 6.1. Let \( s \geq 1 \) and \( H_0 \in H^{s+2} \). Then,
\[
\|D^2 Z(\zeta, \dot{\zeta}, ((y, \dot{y}), (y, \dot{y})))\|_s \leq C\|(y, \dot{y})\|_s^{(2s-1)/2s} \|D^{s+1}(y, \dot{y})\|_0^{(2s+1)/2s} + C\|(y, \dot{y})\|_s \|D^{s+1}(y, \dot{y})\|_0 + Cb^2\|y\|_{s+2},
\]
where \( C \) is uniform for all \((\zeta, \dot{\zeta})\) in a \( H^{s+3} \) neighborhood of the curve \((\eta(t), \dot{\eta}(t))\).

**Proof.** We compute \( DZ(\eta, \dot{\eta})(y, \dot{y}) \) first. Let \( y = \partial_s \zeta(t, s)|_{s=0} \) and \( u = y \circ \zeta^{-1} \) for convenience. The \( (D\zeta)^{-1} = D(\zeta^{-1}) \circ \zeta \), so
\[
(D\zeta)^{-1}D\dot{\zeta} = D(\zeta^{-1})^i \circ \zeta D\dot{\zeta} = \Re \zeta(D(\zeta^{-1})^i)(D\zeta \circ \zeta^{-1}) = \Re \zeta(D(\zeta^{-1}) = D\zeta
\]
and similarly \( (D\zeta)^{-1}D\dot{y} = D\zeta \dot{y} \). Moreover,
\[
[u \cdot \nabla, D]_{\zeta} \dot{\zeta} = \Re \zeta[u \cdot \nabla, D](\dot{\zeta} \circ \zeta^{-1}) = \Re \zeta(Du \cdot \nabla(\dot{\zeta} \circ \zeta^{-1})) = -\Re \zeta(Du \cdot \nabla \zeta^{-1}) = -\Re \zeta(Du \cdot \nabla \zeta^{-1})
\]
and similarly \( (D\zeta)^{-1}D\dot{y} = D\zeta \dot{y} \).

Thus, similar to (5.3), one has
\[
DZ(\zeta, \dot{\zeta}, (y, \dot{y})) = ((u \cdot \nabla, \nabla \Delta^{-1} - \nabla(H(u \cdot \nabla)\Delta^{-1})) \zeta \tr(D\zeta)^2 + 2\nabla\zeta \Delta^{-1} \tr(D\zeta D\zeta \dot{y} + D\zeta \zeta \zeta \tr(D\zeta)(u \cdot \nabla)\zeta)) \cdot \zeta \tr(D\zeta)^2
\]
\[
- [u \cdot \nabla, \Delta^{-1}] \zeta \tr(\zeta^j \Re \zeta((\zeta^{-1})^j H^k)) \cdot \zeta \tr(D\zeta)^2
\]
\[
+ \nabla\zeta(H(u \cdot \nabla)\Delta^{-1}) \zeta \tr(\zeta^j \Re \zeta((\zeta^{-1})^j H^k)) \cdot \zeta \tr(D\zeta)^2
\]
\[
- \zeta \Delta^{-1}(2u^i \Re \zeta((\zeta^{-1})^i H^k) y^j \Re \zeta((\zeta^{-1})^j H^k)) \cdot \zeta \tr(D\zeta)^2
\]
\[
+ \nabla\zeta \Delta^{-1}(2u^i \Re \zeta((\zeta^{-1})^i H^k) \zeta \Re \zeta((\zeta^{-1})^i H^k)) \cdot \zeta \tr(D\zeta)^2
\]
\[
- \zeta \Delta^{-1}(2u^i \Re \zeta((\zeta^{-1})^i H^k) \Re \zeta((\zeta^{-1})^i H^k)) \cdot \zeta \tr(D\zeta)^2
\]
\[
- [u \cdot \nabla, \Delta^{-1}] \zeta \Re \zeta((H^k H^l) \zeta) \cdot \zeta \tr(D\zeta)^2
\]
\[
+ \nabla\zeta(H(u \cdot \nabla)\Delta^{-1}) \zeta \Re \zeta((H^k H^l) \zeta) \cdot \zeta \tr(D\zeta)^2
\]
\[
- \zeta \Delta^{-1}(y^j \Re \zeta((H^k H^l) \zeta)) - \zeta \Delta^{-1}(\zeta^j \Re \zeta((H^k H^l) \zeta)) \cdot \zeta \tr(D\zeta)^2
\]
\[
- [u \cdot \nabla, \Delta^{-1}] \zeta \Re \zeta((H^k H^l) \zeta) \cdot \zeta \tr(D\zeta)^2
\]
\[
+ \nabla\zeta(H(u \cdot \nabla)\Delta^{-1}) \zeta \Re \zeta((H^k H^l) \zeta) \cdot \zeta \tr(D\zeta)^2
\]
\[
- \zeta \Delta^{-1}(2u^i \Re \zeta((H^k H^l) \zeta)) \cdot \zeta \tr(D\zeta)^2
\]
\[
- \zeta \Delta^{-1}(2u^i \Re \zeta((H^k H^l) \zeta)) \cdot \zeta \tr(D\zeta)^2
\]
\[
+ \zeta \Re \zeta((H^k H^l) \zeta) + \zeta \Re \zeta((H^k H^l) \zeta). \quad (6.2)
\]

For any function \( f \) on \( \Omega \), \( \nabla^j f = \Re \zeta \partial_j f(\zeta^{-1}) = \Re \zeta \partial_j f(\zeta^{-1}) \partial\zeta(\zeta^{-1})^j = (D(\zeta^{-1}) \circ \zeta \nabla f) \cdot \zeta \)
and \( \frac{\partial}{\partial \zeta^j} f = \frac{\partial}{\partial \zeta^j} f(\zeta^{-1}(\zeta)) = \partial_j f \frac{\partial}{\partial \zeta^j} (\zeta^{-1}(\zeta)) = (D(\zeta^{-1}) \circ \zeta \nabla f) \cdot \zeta \), i.e., \( \nabla^j f \equiv \frac{\partial}{\partial \zeta^j} f \). Thus,
\[
D^2 Z(\dot{\zeta}, \zeta)((y, \dot{y}), (y, \dot{y})) = \nabla^j DZ(\zeta, \dot{\zeta}, (y, \dot{y})) y_j + \frac{\partial}{\partial \zeta^j} DZ(\zeta, \dot{\zeta}, (y, \dot{y})) \dot{y}_j.
\]

Clearly,
\[
\frac{\partial}{\partial \zeta^j} DZ(\zeta, \dot{\zeta}, (y, \dot{y})) \dot{y}_j = 0
\]
due to the form of $DZ(\zeta, \dot{\zeta})(y, \dot{y})$ in (6.2), where all $\dot{\zeta}$'s appear in the form of derivatives like $D_\zeta \dot{\zeta}$. So only the first term needs to be considered.

\[
y_j \nabla_\zeta^2 DZ(\zeta, \dot{\zeta})(y, \dot{y}) = D_y DZ(\zeta, \dot{\zeta})(y, \dot{y})
\]

\[
= D_y \left[ (u \cdot \nabla, \nabla \Delta^{-1}) - \nabla (H(u \cdot \nabla) \Delta^{-1}) \right] \zeta \text{tr}(D_\zeta \dot{\zeta})^2
\]

\[
+ \left( (u \cdot \nabla, \nabla \Delta^{-1}) - \nabla (H(u \cdot \nabla) \Delta^{-1}) \right) \zeta D_y \text{tr}(D_\zeta \dot{\zeta})^2
\]

\[
+ 2D_y (\nabla \Delta \zeta) \text{tr} \left( D_\zeta \dot{\zeta} D_\zeta \dot{y} + D_\zeta \dot{\zeta} \left[ (u \cdot \nabla, D_\zeta \dot{\zeta}) \right] \right)
\]

\[
+ 2 \nabla \Delta \zeta \text{tr} \left( D_\zeta \dot{\zeta} D_\zeta \dot{y} + D_\zeta \dot{\zeta} \left[ (u \cdot \nabla, D_\zeta \dot{\zeta}) \right] \right)
\]

\[
- D_y \left( (u \cdot \nabla, \nabla \Delta^{-1}) - \nabla (H(u \cdot \nabla) \Delta^{-1}) \right) \zeta \text{tr}(\zeta_m \Re \zeta((\zeta^{-1})^i_s H^k_0)^2)
\]

\[
- \left( (u \cdot \nabla, \nabla \Delta^{-1}) - \nabla (H(u \cdot \nabla) \Delta^{-1}) \right) \zeta \text{tr}(\zeta_m \Re \zeta((\zeta^{-1})^i_s H^k_0)^2)
\]

\[
- D_y (\nabla \Delta \zeta) (2\zeta^i_{kl} y^m_{kl} \Re \zeta((\zeta^{-1})^i_s H^k_0) y^m_{kl} \Re \zeta((\zeta^{-1})^i_s H^k_0))
\]

\[
- \nabla \Delta \zeta \Delta \zeta \text{tr}(2\zeta^i_{kl} y^m_{kl} \Re \zeta((\zeta^{-1})^i_s H^k_0))
\]

\[
+ D_y (\nabla \Delta \zeta) (2\zeta^i_{kl} y^m_{kl} \Re \zeta((\zeta^{-1})^i_s H^k_0))
\]

\[
+ \nabla \Delta \zeta \Delta \zeta \text{tr}(2\zeta^i_{kl} y^m_{kl} \Re \zeta((\zeta^{-1})^i_s H^k_0))
\]

\[
- D_y \left( (u \cdot \nabla, \nabla \Delta^{-1}) - \nabla (H(u \cdot \nabla) \Delta^{-1}) \right) \zeta \text{tr}(\zeta_m \Re \zeta((\zeta^{-1})^i_s H^k_0,H^l_{0,m,u} u^m))
\]

\[
- (\zeta_m \Re \zeta((\zeta^{-1})^i_s H^k_0,H^l_{0,m,u} u^m))
\]

\[
- D_y \left( (u \cdot \nabla, \nabla \Delta^{-1}) - \nabla (H(u \cdot \nabla) \Delta^{-1}) \right) \zeta \text{tr}(\zeta_m \Re \zeta((\zeta^{-1})^i_s H^k_0,H^l_{0,m,u} u^m))
\]

\[
- (\zeta_m \Re \zeta((\zeta^{-1})^i_s H^k_0,H^l_{0,m,u} u^m))
\]

\[
- D_y (\nabla \Delta \zeta) (2\zeta^i_{kl} y^m_{kl} \Re \zeta((\zeta^{-1})^i_s H^k_0))
\]

\[
- \nabla \Delta \zeta \Delta \zeta \text{tr}(2\zeta^i_{kl} y^m_{kl} \Re \zeta((\zeta^{-1})^i_s H^k_0))
\]

\[
- D_y \left( (u \cdot \nabla, \nabla \Delta^{-1}) - \nabla (H(u \cdot \nabla) \Delta^{-1}) \right) \zeta \text{tr}(\zeta_m \Re \zeta((\zeta^{-1})^i_s H^k_0))
\]

\[
- (\zeta_m \Re \zeta((\zeta^{-1})^i_s H^k_0))
\]

\[
+ D_y (\zeta_m \Re \zeta((\zeta^{-1})^i_s H^k_0))
\]

\[
+ \zeta_m \Re \zeta((\zeta^{-1})^i_s H^k_0)
\]

Since $y = u \circ \zeta$, it follows that $y(u \circ \zeta) = \partial_y \zeta \zeta(t, s)|_{s=0}$ and

\[
\partial_y K(\zeta(t, s))|_{s=0} = -\nabla \zeta K(\zeta(t, s)) \partial_y \zeta(t, s)|_{s=0} = y_j \nabla \zeta K(\zeta) = D_y \Delta \zeta,
\]

which yields from $H_\zeta = I - \Delta^{-1} \Delta \zeta$ and the identity 2) in Lemma 5.1 that

\[
D_y H_\zeta = D_y (I - \Delta^{-1} \Delta \zeta)
\]

\[
= - (D_y \Delta^{-1} \Delta \zeta - \Delta^{-1} (D_y \Delta \zeta))
\]

\[
= \Delta^{-1} (D_y \Delta \zeta) \Delta^{-1} \Delta \zeta - \Delta^{-1} (D_y \Delta \zeta)
\]
Similarly, we have from the identity 1) in Lemma 5.1

\[ D_y (u \cdot \nabla) \zeta = y_j \nabla^j (u \cdot \nabla) \zeta = \nabla_i^j (u \cdot \nabla) \zeta \partial_\zeta (t, s) \bigg|_{s=0} = \partial_s (u \cdot \nabla) \zeta (t, s) \bigg|_{s=0} = \partial_s (u \cdot \nabla) \zeta (t, s) \bigg|_{s=0}
\]

\[ = u_{k,j} \zeta (t, s) \partial_s \zeta^l (t, s) \nabla^k_{\zeta (t, s)} \bigg|_{s=0} + u (\zeta (t, s)) \partial_s \nabla \zeta_{(t, s)} \bigg|_{s=0} = u^l (\zeta) u_{k,j} (\zeta) \nabla^k \zeta
\]

\[ = u^l (\zeta) u_{k,j} (\zeta) \nabla^k \zeta + u_k (\zeta) u_j (\zeta) \nabla \zeta \nabla \zeta - u^k (\zeta) u_{j,k} (\zeta) \nabla \zeta
\]

\[ = 0.
\]

Hence, it follows from the identity 3) in Lemma 5.1 that,

\[ D_y [u \cdot \nabla, \nabla \Delta^{-1}] \zeta
\]

\[ = D_y (u \cdot \nabla \nabla \Delta^{-1}) \zeta - D_y (\nabla \Delta^{-1} (u \cdot \nabla)) \zeta
\]

\[ = (u \cdot \nabla) \zeta (D_y \nabla \Delta^{-1} + \nabla \zeta (D_y \Delta^{-1}))
\]

\[ - ((D_y \nabla \Delta^{-1} + \nabla \zeta (D_y \Delta^{-1})) (u \cdot \nabla) \zeta
\]

\[ = [u \cdot \nabla, [u \cdot \nabla, \nabla \Delta^{-1}]] \zeta - [u \cdot \nabla, \nabla \mathcal{H} (u \cdot \nabla) \Delta^{-1}] \zeta.
\]

Therefore, we obtain, from (6.4)-(6.7), that

\[ (6.3a) = 2 \left( [u \cdot \nabla, [u \cdot \nabla, \nabla \Delta^{-1}]] \zeta - [u \cdot \nabla, \nabla \mathcal{H} (u \cdot \nabla) \Delta^{-1}] \zeta
\]

\[ - ([u \cdot \nabla, \nabla \mathcal{H} (u \cdot \nabla) \Delta^{-1}] + (\nabla \Delta^{-1} [u \cdot \nabla, \Delta] \mathcal{H} (u \cdot \nabla) \Delta^{-1}] \zeta
\]

\[ + (\nabla \mathcal{H} (u \cdot \nabla) \Delta^{-1} [u \cdot \nabla, \Delta] \Delta^{-1} \zeta \right) \text{tr}(D \zeta) \Delta^{-1}.
\]

The other terms in (6.3) including the operator \( D_y (u \cdot \nabla, \nabla \Delta^{-1}) \mathcal{H} (u \cdot \nabla) \Delta^{-1} \zeta \), namely \( (6.3e), (6.3m) \), and \( (6.3h) \), have similar identities.

We have, from (6.2) and the analogue of (6.4), that

\[ (6.3d) = 2 \left( [u \cdot \nabla, \nabla \Delta^{-1}] - \nabla \mathcal{H} (u \cdot \nabla) \Delta^{-1} \zeta
\]

\[ \cdot \text{tr} \left( D \zeta \Delta \mathcal{H} + D \zeta \mathcal{H} \right)
\]

\[ = (6.8).
\]

By Lemma 5.1, one has

\[ (6.3c) = 2 \left( [u \cdot \nabla, \nabla \Delta^{-1}] - \nabla \mathcal{H} (u \cdot \nabla) \Delta^{-1} \zeta
\]

\[ \cdot \text{tr} \left( D \zeta \Delta \mathcal{H} + D \zeta \mathcal{H} \right) = (6.8).
\]

The other terms in (6.3) including the operator \( D_y (\nabla \Delta^{-1}) \zeta \), namely \( (6.3g), (6.3i), (6.3k), (6.3o), (6.3q), (6.3u) \), and \( (6.3w) \), have similar identities.

Since \( D_y \zeta_k = y_j \nabla^j \zeta_k = y_j \delta^j_k = y_k \), \( D_y \zeta = y \) and

\[ (6.3d) = 2 \nabla \Delta^{-1} \text{tr} \left( (D \zeta)^2 + [u \cdot \nabla, \Delta] \zeta D \zeta \mathcal{H} + D \zeta \mathcal{H} \right)
\]

\[ + ([u \cdot \nabla, \Delta] \zeta) \mathcal{H} + D \zeta \mathcal{H} \right)
\]

\[ + ([u \cdot \nabla, \Delta] \zeta) \mathcal{H} + D \zeta \mathcal{H} \right)
\]

\[ = (6.8).
\]
+ D_\zeta \hat{\zeta}(u \cdot \nabla) \zeta [u \cdot \nabla, D] \zeta [u \cdot \nabla, D] \zeta - D_\zeta (u \cdot \nabla) \zeta \hat{y})
\end{align*}

= 2 \nabla_\zeta \Delta_\zeta^{-1} \text{tr} \left( (D_\zeta \hat{y})^2 + [u \cdot \nabla, D] \zeta \hat{D}_\zeta \hat{y} + 2 D_\zeta \hat{\zeta} [u \cdot \nabla, D] \zeta \hat{y} \right.
\left. + ([u \cdot \nabla, D] \zeta \hat{\zeta})^2 + D_\zeta \hat{\zeta} [u \cdot \nabla, [u \cdot \nabla, D] \zeta \hat{\zeta} \right)
\zeta - D_\zeta (u \cdot \nabla) \zeta \hat{D}_\zeta \hat{y} .

Due to

\begin{align*}
D_y \zeta^m_{k,l}(t, \zeta^{-1}(t, \zeta)) = y^i \nabla_j \zeta^m_{k,l}(t, \zeta^{-1}(t, \zeta)) = y^i \zeta^m_{k,l}(t, \zeta^{-1}(t, \zeta))(\zeta^{-1})^l_j \circ \zeta
\end{align*}

we have

\begin{align*}
D_y D \zeta = D^2 \zeta (D \zeta)^{-1} y,
\end{align*}

and in general

\begin{align*}
D_y D^k \zeta = D^{k+1} \zeta (D \zeta)^{-1} y, \quad \forall k \in \mathbb{N} \cup \{0\}.
\end{align*}

Thus, we get for (6.3f)

\begin{align*}
D_y \text{tr}(\zeta^m_{k,l} R_\zeta ((\zeta^{-1})^l_1 H_0^k)^2) &= D_y (\zeta^m_{k,l} ((D \zeta)^{-1})_1^l H_0^k(\zeta)^{i,j} ((D \zeta)^{-1})_m^l H_0^k(\zeta))
\end{align*}

= 2 \zeta^m_{k,l} ((D \zeta)^{-1})_1^l y^i ((D \zeta)^{-1})_m^l H_0^k(\zeta) \zeta_{k,l}^i ((D \zeta)^{-1})_m^l H_0^k(\zeta)
- 2 \zeta^m_{k,l} ((D \zeta)^{-1})_1^l \zeta_{m,n}^i ((D \zeta)^{-1})_p^l y^i ((D \zeta)^{-1})_m^l H_0^k(\zeta) \zeta_{k,l}^i ((D \zeta)^{-1})_m^l H_0^k(\zeta)
+ 2 \zeta^m_{k,l} ((D \zeta)^{-1})_1^l H_0^k(\zeta) y^i \zeta_{k,l}^i ((D \zeta)^{-1})_m^l H_0^k(\zeta).

Generally, the derivative $D_y$ makes the target to increase one derivative with respect to $\zeta$ for $\zeta, y, u$ or $H_0$. So we omit the detailed derivation of $D_y$ terms in (6.3h), (6.3j), (6.3l), (6.3p), (6.3r), (6.3s), (6.3t), (6.3y) and (6.3y).

For any $s$, $\Delta^{-1} : H^s(\Omega) \to H^{s+2}(\Omega)$ and thus $H : H^s(\Omega) \to H^s(\Omega)$ is bounded.

Therefore $D^2 Z(\zeta, \hat{\zeta})$ is a bilinear operator in $(y, \hat{y})$ due to $y = u \circ \zeta$ with coefficients depending on at most three derivatives of $(\zeta, \hat{\zeta})$. Moreover, those terms excluding $H_0$ are the first order in $(y, \hat{y})$ times zeroth order in $(y, \hat{y})$, while the orders of derivatives of $y$ or $H_0$ in each term is at most 2 for those terms including $H_0$. Thus, by Sobolev’s embedding theorem (cf. [19]),

\begin{align*}
&\|D^2 Z(\zeta, \hat{\zeta})((y, \hat{y}), (y, \hat{y}))\|_1 \\
\leq & \|C(\zeta, \hat{\zeta})(y, \hat{y})(D_y, D\hat{y})\|_1 + b^2 \|C(\zeta, \hat{\zeta}, H_0)(y, D_y, D^2 y)\|_1 \\
\leq & \|C(\zeta, \hat{\zeta})(y, \hat{y})(D_y, D\hat{y})\|_1 + \|D C(\zeta, \hat{\zeta})(y, D_y, D\hat{y})\|_1 \\
+ & \|C(\zeta, \hat{\zeta})y([D_y, D\hat{y}])\|_0 + \|C(\zeta, \hat{\zeta})(y, D_y, D^2 y)\|_0 \\
+ & b^2 \|D C(\zeta, \hat{\zeta})(y, D_y, D^2 y)\|_0 + b^2 \|D C(\zeta, \hat{\zeta})(y, D_y, D^2 y)\|_0 \\
\leq & C \|D^{1/2}(y, \hat{y})\|_0 \|D^{3/2}(y, \hat{y})\|_0 + C \|D^{3/2}(y, \hat{y})\|_0 \\
& + C \|D^{1/2}(y, \hat{y})\|_0 \|D^{3/2}(y, \hat{y})\|_0 + C b^2 \sum_{k=0}^3 \|y\|_k \\
\leq & C \|y(y, \hat{y})\|_0 \|D^{1/2}(y, \hat{y})\|_0 \|D^{3/2}(y, \hat{y})\|_0 + C \|D(y, \hat{y})\|_0 \|D^{1/2}(y, \hat{y})\|_0 \\
& + C \|D(y, \hat{y})\|_0 \|D^{1/2}(y, \hat{y})\|_0 \|D^{3/2}(y, \hat{y})\|_0 + C b^2 \sum_{k=0}^3 \|y\|_k \\
\leq & C \|y(y, \hat{y})\|_1 \|D^2(y, \hat{y})\|_0 + C \|y(y, \hat{y})\|_1 \|D^2(y, \hat{y})\|_0 + C b^2 \|y\|_3 ,
\end{align*}
where $C(\cdots)$ is a smooth function of $(\zeta, \dot{\zeta})$ and their first three derivatives.

The image of $\eta(t)$ is a circle in $H^{s+3}(\Omega; \mathbb{R}^2)$ which is compact and since $H^s(\mathbb{R}^2)$ is an algebra for $s > 1$, it then follows

$$\|D^2 Z(\zeta, \dot{\zeta})((y, \dot{y}), (y, \dot{y}))\|_s \leq C(\zeta, \dot{\zeta})(y, \dot{y})(Dy, D\dot{y})\|_s + b^2 \|C(\zeta, \dot{\zeta}, H_0)(y, Dy, D^2 y)\|_s$$

$$\leq C\|y, \dot{y}\|_s\|D^{s+1}(y, \dot{y})\|_0 + C\|(\eta, \dot{\eta})\|^{{{(2s-1)}/2s}}\|D^{s+1}(y, \dot{y})\|_0^{(2s+1)/2s} + Cb^2\|y\|_{s+2}$$

Thus, the desired estimates follow for all $(\zeta, \dot{\zeta})$ near $(\eta(t), \dot{\eta}(t))$ in $H^{s+3}(\Omega; \mathbb{R}^2)$ with fixed $C$.

\[ \square \]

**Lemma 6.2.** Let $m > l \geq 1$ be an integer satisfying $\|D^{m+1} y\|_0 \leq C_m\|Dy\|_0$. Then, it holds

$$\|y\|_{l+1} \leq C\|y\|_1.$$

**Proof.** From Sobolev’s embedding theorem (cf. [19]), we have for any $0 \leq j < m$,

$$\|D^{j+1} y\|_0 \leq C_{j,m}\|D^{m+1} y\|^{j/m}_0\|Dy\|^{1-j/m}_0 \leq C_{j,m}\|Dy\|_0.$$

Therefore, for any $l < m$

$$\|Dy\|_1 = \sum_{j=0}^l\|D^{j+1} y\|_0 \leq \sum_{j=0}^l C_{j,m}\|Dy\|_0 \leq C_m\|Dy\|_0,$$

the desired estimate follows then. \[ \square \]

**Corollary 6.3.** Let $m > s$ be an integer satisfying $\|D^m y, \dot{y}\|_0 \leq C_m\|(y, \dot{y})\|_0$. Then, it holds

$$\|D^2 Z(\zeta, \dot{\zeta})((y, \dot{y}), (y, \dot{y}))\|_s \leq C\|(y, \dot{y})\|^{2-1/m}_0\|D^m (y, \dot{y})\|^{1/m}_0 + Cb^2\|y\|_{s+2}.$$

**Proof.** From Sobolev’s embedding theorem (cf. [19]), it follows that

$$\|D^2 Z(\zeta, \dot{\zeta})((y, \dot{y}), (y, \dot{y}))\|_s \leq C\left(\sum_{k=0}^{\text{max}(s,2)}\|D^m (y, \dot{y})\|^{k/m}_0\|(y, \dot{y})\|^{1-k/m}_0\right)$$

$$\cdot \left(\sum_{l=0}^s\|D^m (y, \dot{y})\|_0^{l/(m-1)}\|D(y, \dot{y})\|_0^{1-l/(m-1)}\right) + Cb^2\|y\|_{s+2}$$

$$\leq C\|(y, \dot{y})\|_0^{1 - \|D^m (y, \dot{y})\|_0^{(\text{max}(s,2)+1)/m}\|(y, \dot{y})\|_0^{-(\text{max}(s,2)+1)/m}}$$

$$\cdot \left(\sum_{l=0}^s\|D^m (y, \dot{y})\|_0^{l/(m-1)}\left(\|D^m (y, \dot{y})\|_0^{1/m}\|(y, \dot{y})\|_0^{1-1/m}\right)^{1-l/(m-1)}\right) + Cb^2\|y\|_{s+2}$$

$$\leq C\|(y, \dot{y})\|_0^{2-1/m} - \|D^m (y, \dot{y})\|_0^{(\text{max}(s,2)+1)/m}\|(y, \dot{y})\|_0^{-(\text{max}(s,2)+1)/m}}$$

$$\cdot \left(\sum_{l=0}^m\|D^m (y, \dot{y})\|_0^{l/m}\|(y, \dot{y})\|_0^{1-l/m}\right) + Cb^2\|y\|_{s+2}$$

$$\leq C\|(y, \dot{y})\|_0^{2-1/m}\|D^m (y, \dot{y})\|_0^{1/m}(1 - \|D^m (y, \dot{y})\|_0^{1/m}\|(y, \dot{y})\|_0^{1/m})^{-2}$$

$$(1 - \|D^m (y, \dot{y})\|_0^{(\text{max}(s,2)+1)/m}\|(y, \dot{y})\|_0^{-(\text{max}(s,2)+1)/m}})$$
provided that \( \|y\|_0 \leq C \|y\|_{s+2} \),

\[ C \|(y, \dot{y})\|^2_0 2^{-\frac{1}{m}} \leq C \|(y, \dot{y})\|^{\frac{1}{m}}_0 + C \|y\|_{s+2}, \]

6.2. Decomposition of solutions. We decompose \( y = y_n \) into three parts as that in \([6]\), each of which will be estimated separately.

Let \( q = y - \nabla \Delta^{-1} \text{div} y \), the divergence free part of \( y \), and \( h \) be a harmonic function satisfying

\[ \langle \nabla h, \nu \rangle = \langle q, \nu \rangle, \text{ on } \partial \Omega. \]

Define

\[ N = y - \nabla h. \]  

(6.9)

Note that, \( h \), as a harmonic function, is the real part of some holomorphic function \( \varphi(z) \). Therefore, it can be written as \( h = \text{Re} \sum_{j=0}^\infty a_j z^j \). Set \( g = \text{Re} \sum_{j=0}^{n-1} a_j z^j \) and \( f = h - g = \text{Re} \sum_{j=n}^\infty a_j z^j \). We then decompose \( y \) as

\[ y = \nabla f + \nabla g + N. \]  

(6.10)

Denote \( P_i, i = 1, 2, 3 \) the projection onto the \( i \)-th summand of \([6.10]\).

It is shown in \([6]\) that the summands of \([6.10]\) are orthogonal with respect to the \( L^2(\Omega) \) inner product.

Following that in \([6]\), the next step is to decompose equation \([6.1]\). Denote, for simplicity,

\[ Q = Q(y, \dot{y}) := \int_0^1 (1 - s) \left( \int_0^s D^2 Z(\zeta(\sigma), \dot{\zeta}(\sigma))((y, \dot{y}), (y, \dot{y}))d\sigma \right) ds, \]

and \( Q_i = P_i Q \) for \( i = 1, 2, 3 \). Whence,

\[ \ddot{y} = DZ(\eta, \dot{\eta})(y, \dot{y}) + Q. \]  

(6.11)

From \([5.4]\), it follows

\[ DZ(\eta, \dot{\eta})(y, \dot{y}) = (1 - b^2) \left( [u \cdot \nabla, \nabla \Delta^{-1}] \eta(-2) - \nabla \eta(\mathcal{F}(u \cdot \nabla) \Delta^{-1}) \eta(-2) \right) \]

\[ + 2 \nabla \eta \Delta^{-1} \text{tr} \left( D u \circ \eta + ie^{-it} D \dot{y} \right) - \nabla \eta \Delta^{-1} \eta_{y_{k^i}} \text{Re}_{\eta}(H_{0}^k H_{0}^k),_i \]

\[ = - (1 - b^2) y + \tilde{A} y + (\nabla \Delta^{-1}) \eta \text{tr} M + b^2 \mathcal{E} \eta, \]

where \( \tilde{A} y = (\nabla \mathcal{F})_{\eta}(\eta, \eta) \) which depends on \( \eta \), and

\[ M = 2 D_{N^i} N + 2ie^{-it} D \mathcal{N} - N^i_{k^l \text{Re}_{\eta}(H_{0}^k H_{0}^l),_i^j} - 2e^{it} N^i_{k^l \text{Re}_{\eta}(H_{0}^k H_{0}^l),_i^j}. \]  

(6.12)

In fact, by \([6.9]\), we have \( y = \nabla h + N \) with \( \Delta h = 0 \). In view of Lemma \([5.2]\) there exists a harmonic function \( \hat{h} \) such that \( \nabla\hat{h} \circ \eta = \nabla h \). Since \( \eta^{-1}(t, z) = e^{-it} z = e^{-2it} \eta \), we have

\[ \partial_t \hat{h}(z) = \partial_t \eta \hat{h}(e^{-2it} \eta(z)). \]

Thus,

\[ \partial_t \partial_t \hat{h}(z) = \partial_t \partial_m \hat{h}(e^{-2it} \eta) e^{-2it} \eta_{j^m} = e^{-it} \partial_t \partial_k \hat{h}(e^{-2it} \eta), \]

and

\[ \partial_t \partial_t \partial_t \hat{h}(z) = e^{-2it} \partial_t \partial_t \partial_t \hat{h}(e^{-2it} \eta). \]

Therefore, as in \([5.5]\) and \([5.6]\), it yields

\[ \sum_i \partial_t h_{\eta, k,i} \text{Re}_{\eta}(H_{0}^k H_{0}^k),_i = 0, \text{ and } \sum_j \partial_t h_{\eta, k,j} \text{Re}_{\eta}(H_{0}^k H_{0}^k, H_{0}^k) = 0. \]
Also, \( \text{tr}D(\nabla h(\eta)) = \text{tr}e^{it}D\nabla h(\eta) = 0 \) and \( \text{tr}(ie^{-it}D\nabla \hat{h}) = 0 \) due to \( \Delta h = 0 \).

Therefore,
\[
\dot{y} = (b^2 - 1)y + \dot{A}y + (\nabla \Delta^{-1})_\eta \text{tr} M + b^2 e^{it} \eta + Q.
\] (6.13)

Applying \( \mathbb{P}_3 \) to (6.13), noticing that \( \mathbb{P}_3 \dot{A}y = 0, \mathbb{P}_3(\nabla H)_\eta = 0 \) and \( (\mathbb{P}_1 + \mathbb{P}_2)(\nabla \Delta^{-1})_\eta = 0 \), we obtain, in view of (6.12),
\[
\dot{N} = (b^2 - 1)N + (\Delta \Delta^{-1})_\eta \text{tr} M + b^2 e^{it} \eta + Q_3
\]
\[=(b^2 - 1)N + 2(\nabla \Delta^{-1})_\eta \text{div} \eta N + 2ie^{-it}(\nabla \Delta^{-1})_\eta \text{tr}(D\dot{N}) + b^2 e^{it} \eta + Q_3
\]
\[-(\nabla \Delta^{-1})_\eta (N^i_k \mathbb{R}_\eta (H^k_0 H^i_0, l) - 2e^{it}(\nabla \Delta^{-1})_\eta (N^i_k \mathbb{R}_\eta (H^k_0 H^i_0, l)))
\]
\[=:(b^2 - 1)N + B_1 N + B_2 \dot{N} + b^2 e^{it} \eta + Q_3 + B_3 N + B_4 N,
\] (6.14)

where we have used \( \mathbb{P}_3 \eta = \eta \) since
\[
\nabla \Delta^{-1} \text{div} \eta = e^{it} \nabla \Delta^{-1} \text{div} (x_1, x_2) = e^{it} \nabla \Delta^{-1}(2) = e^{it}(x_1, x_2) = \eta.
\]

Applying \( \mathbb{P}_1 + \mathbb{P}_2 \) to (6.13) to obtain
\[
\tilde{\nabla} f + \tilde{\nabla} g = (b^2 - 1)(\nabla f + \nabla g) + \tilde{A}(\nabla f + \nabla g) + \tilde{A} \nabla N + Q_1 + Q_2.
\]

The terms \((b^2 - 1)(\nabla f + \nabla g)\) and \(\tilde{A}(\nabla f + \nabla g)\) were computed in (5.7) and (5.8), where they were written as \((1 - b^2)A(\nabla f + \nabla g) + b^2 e^{it} \eta \). Since \( A \tilde{z}^k = (k - 1) \tilde{z}^k \), \( A \) commutes with \( \mathbb{P}_j \) for \( j = 1 \) or 2. Denote \( \mathcal{A}_j = \mathbb{P}_j A \) for \( j = 1 \) or 2. Thus, in view of the fact \( \mathbb{P}_j \eta = \eta \) for \( j = 1 \) or 2 due to \( \mathbb{P}_3 \eta = \eta \), we get
\[
\tilde{\nabla} f = (1 - b^2)A \nabla f + \mathcal{A}_1 N + Q_1,
\]
\[
\tilde{\nabla} g = (1 - b^2)A \nabla g + \mathcal{A}_2 N + Q_2.
\] (6.15) (6.16)

Thus, (6.11) has been decomposed into the system (6.14), (6.15) and (6.16).

6.3. Estimates of \( \nabla f, \nabla g \) and \( N \). Let \( y_n = \nabla f_n + \nabla g_n + N_n \) be the sequence of solutions with the initial data \( y_n(0) = 0, \dot{y}_n(0) = e^{-n^{1/4}z^n} \). For any harmonic function \( h \) and any real \( s \), we introduce the equivalent form of the usual \( H^s \)-norm, i.e.,
\[
\|\nabla h\|_s = (A^s \nabla h, A^s \nabla h)^{1/2},
\] (6.17)

where \( A^s \) is defined by \( A^s \tilde{z}^k = (k - 1)^s \tilde{z}^k \). Then, it implies that for any \( s \),
\[
\|A \nabla g\|_s \leq (n - 1)\|\nabla g\|_s.
\] (6.18)

For \( \mu, \nu \geq 1 \) and \( \sigma \geq 2 \), denote
\[
E_{\mu,b}^\pm = \|\nabla f \pm \sqrt{1 - b^2} A^{1/2} \nabla f\|_\mu = \|A^\mu(\nabla f \pm \sqrt{1 - b^2} A^{1/2} \nabla f)\|_0^2.
\]
\[E_{\mu,b} = E_{\mu,b}^+ + E_{\mu,b}^-.
\]
\[F_\sigma = n\|N\|_\sigma^2 + \|\dot{N}\|_\sigma^2,
\]
\[G_\nu = \|\nabla g\|_\nu^2 + \|A^{1/2} \nabla g\|_\nu^2.
\]

It is follows from (6.17) the inequality
\[
E_{\mu,b}^\pm \geq n^{2(\mu - \nu)} E_{\nu,b}^\pm, \text{ for } \mu \geq \nu.
\] (6.19)

**Proposition 6.4.** Let \( \mu \geq 2 \). For sufficiently large \( n \), the set \( E_{\mu,b}^+ \geq E_{\mu,b}^- \), \( E_{\mu,b}^+ \geq \sqrt{n}F_{\mu+1}, \) and \( E_{\mu+1/2,b}^+ \geq \frac{2\sqrt{n}}{(2 - b^2)\nu^{3/4}} G_\nu \) is invariant under the evolution defined by (6.14), (6.15) and (6.16). Of course \( E_{\mu,b} \leq 2E_{\mu,b}^+ \).
Proof. From \([6.15]\),
\[
\dot{E}_{\mu,b}^\pm = 2(\nabla f + \sqrt{1-b^2}A^I\nabla f, \nabla f + \sqrt{1-b^2}A^I\nabla f)_\mu
\]
\[
= 2((1-b^2)A\nabla f + \dot{A}_1 N + Q_1 \pm \sqrt{1-b^2}A^I\nabla f, \dot{\nabla} f + \sqrt{1-b^2}A^I\nabla f)_\mu
\]
\[
= 2(\sqrt{1-b^2}A^I(\sqrt{1-b^2}A^I\nabla f + \hat{\nabla} f), \dot{\nabla} f + \sqrt{1-b^2}A^I\nabla f)_\mu
\]
\[
+ 2(\dot{A}_1 N + Q_1, \hat{\nabla} f + \sqrt{1-b^2}A^I\nabla f)_\mu
\]
\[= \pm 2\sqrt{1-b^2}E_{\mu+b}^\pm + 2(\dot{A}_1 N + Q_1, \nabla f + \sqrt{1-b^2}A^I\nabla f)_\mu. \tag{6.20}\]
Hence,
\[
\dot{E}_{\mu,b}^+-\dot{E}_{\mu,b}^- = 2\sqrt{1-b^2} \left( E_{\mu+b}^+ - E_{\mu+b}^- \right)
\]
\[
= 2(\dot{A}_1 N + Q_1, \nabla f + \sqrt{1-b^2}A^I\nabla f)_\mu
\]
\[
+ 2(\dot{A}_1 N + Q_1, \hat{\nabla} f + \sqrt{1-b^2}A^I\nabla f)_\mu
\]
\[\geq 2\sqrt{1-b^2}E_{\mu+b}^+ - 2\sqrt{E_{\mu,b}^+(K\|N\|_{\mu+1} + \|Q_1\|_\mu)}
\]
\[\geq 2\sqrt{1-b^2}E_{\mu+b}^+ - 2\sqrt{E_{\mu,b}(K\sqrt{F_{\mu+1} + \|Q_1\|_\mu})}
\]
\[\geq 2\sqrt{1-b^2}E_{\mu+b}^+ - 2Kn^{-5/4}E_{\mu+b}^+ - 2n^{-1/4}\sqrt{E_{\mu+b}^+\|Q_1\|_\mu}. \tag{6.21}\]
By Proposition \([6.1]\) and Lemma \([6.2]\),
\[
\|Q_1\|_\mu \leq C\epsilon^{(2\mu+1)/2\mu}(\|y\|_1)^{(2\mu-1)/2}\mu + C\epsilon^{(2\mu-1)/4}\mu + C\epsilon^{(2\mu-1)/4}\mu
\]
\[
\leq C\epsilon^{(2\mu-1)/4}\mu + C\epsilon^{(2\mu-1)/4}\mu + C\epsilon^{(2\mu-1)/4}\mu
\]
\[
\leq C\epsilon^{(2\mu-1)/4}\mu + C\epsilon^{(2\mu-1)/4}\mu + C\epsilon^{(2\mu-1)/4}\mu
\]
\[= C\epsilon\left(n^{3/2-2\mu}(E_{\mu+b}^+)^{(2\mu-1)/4}\mu + C\epsilon^{(2\mu-1)/4}\mu\right), \]
where \(\epsilon\) can be any sequence of positive constants such that \(\lim_{n \to \infty} \epsilon_n = 0\). So,
\[
\dot{E}_{\mu,b}^+-\dot{E}_{\mu,b}^- \geq 2(\sqrt{1-b^2} - Kn^{-5/4} - C(\epsilon\mu + b^2)\mu^{1/2}\mu)
\]
\[\geq 2\epsilon_n n^{-1/4} - 2\epsilon_n n^{-1/4} - 2\epsilon_n n^{-1/4} - 2\epsilon_n n^{-1/4}.
\]
which is positive for \(n\) large. Then, one obtains easily that \(E_{\mu,b} \leq 2E_{\mu,b}^+\).
In view of \([6.14]\), we have
\[
\dot{E}_\sigma = 2n(\dot{N}, N)_\sigma + 2(\ddot{N}, \dot{N})_\sigma
\]
\[
= 2(n - 1 + b^2)(N + B_1 N + B_2 N + b^2e^{3t}y + Q_3 + B_3 N + B_4 N)_\sigma
\]
\[
= 2(\dot{N}, B_1 N)_\sigma + 2(\dot{N}, B_2 N)_\sigma + 2(\dot{N}, N)_\sigma + 2(\dot{N}, B_3 N)_\sigma
\]
\[+ 2(\dot{N}, B_4 N)_\sigma + 2(\dot{N}, Q_3)_\sigma + 2b^2e^{2t}(\dot{N}, (x_1, x_2))_\sigma.
\]
It holds that
\[
|\langle \dot{N}, N \rangle_\sigma | = |\langle A^\sigma \dot{N}, A^\sigma N \rangle | \leq \|A^\sigma \dot{N}\|_0 \|A^\sigma N\|_0 = \frac{1}{2\sqrt{n}} \sqrt{\|\dot{N}\|_\sigma \|N\|_\sigma}
\]
\[
\leq \frac{1}{2\sqrt{n}} (\|\dot{N}\|_\sigma^2 + n \|N\|_\sigma^2) = \frac{1}{2\sqrt{n}} F_\sigma,
\]
whereby
\[ 2(n - 1 + b^2)|(|\hat{N}, N)|_\sigma| \leq \frac{(n - 1 + b^2)}{\sqrt{n}} F_\sigma \leq \sqrt{n - 1 + b^2} F_\sigma. \]

Note that the operator $\nabla \Delta^{-1} \text{div}$ is bounded in the $H^\sigma$-norm,
\[ 2(|\hat{N}, B_1N)|_\sigma| = 4(|A^\mu \hat{N}, A^\nu (\nabla \Delta^{-1})_\sigma \text{div} g\eta)| \leq 4C||\hat{N}|_\sigma||N|_\sigma| \leq Kn^{-1/2}F_\sigma. \]

Similarly, it holds that
\[ 2(|\hat{N}, B_2\tilde{N})|_\sigma| = 2(|\hat{N}, 2ie^{-it}(\nabla \Delta^{-1})_\gamma \text{tr}(D\hat{N}))|_\sigma| \leq 4C||\hat{N}|_\sigma|^2 \leq K F_\sigma, \]

and
\[ 2(|\hat{N}, B_4N)|_\sigma| = 4(|\hat{N}, (\nabla \Delta^{-1})_\gamma (N_{ij} R_{\eta} (H_{0,j}^i H_{0,i}^j))|_\sigma| \leq C b^2 n^{-1/2}F_\sigma, \]

since $H_{0,j} = (j - i)b$ for $i, j \in \{1, 2\}$ implies $H_{0,i}^i H_{0,i}^j = (j - i)(i - k)b^2$. Also,
\[ 2(|\hat{N}, B_3\tilde{N})|_\sigma| = 2(|\hat{N}, (\nabla \Delta^{-1})_\gamma (N_{ij} R_{\eta} (H_{0,j}^i H_{0,i}^j))|_\sigma| = 4b^2(i - k)|\hat{N}, (\nabla \Delta^{-1})_\gamma ((-x_2, x_1) \cdot \nabla N_{ik})|_\sigma| \leq C b^2 n^{-1/2}F_{\sigma + 1/2}. \]

Due to $A(x_1, x_2) = (x_1, x_2)$ and the boundedness of $\Omega$,
\[ |(\hat{N}, (x_1, x_2))|_\sigma| \leq C ||\hat{N}|_\sigma| \leq K F_\sigma^{1/2}. \]

Thus,
\[ \hat{F}_\sigma \leq (\sqrt{n - 1 + b^2} + K) F_\sigma + 2\sqrt{F_\sigma} (||Q_3||_\sigma + C b^2) + C b^2 n^{-1/2} F_{\sigma + 1/2}. \quad (6.22) \]

Since
\[ G_\nu = (A^\nu \nabla g, A^\nu \nabla g) + (A^\nu + 1/2 \nabla g, A^{\nu + 1/2} \nabla g), \]

and (6.16),
\[ \hat{G}_\nu = 2(A^\nu \nabla g, A^\nu \nabla g) + 2(A^\nu + 1/2 \nabla g, A^{\nu + 1/2} \nabla g) = 2((-b^2) A^{\nu + 1/2} \nabla g + A^\nu \tilde{A}_2 N + A^\nu Q_2, A^\nu \nabla g). \]

By (6.18), one has
\[ (A^{\nu + 1} \nabla g, A^\nu \nabla g) \leq ||A \nabla g||_\nu \nabla ||g||_\nu \leq \sqrt{n - 1} ||A^{1/2} \nabla g||_\nu ||\nabla g||_\nu \]
\[ \leq \frac{1}{2} \sqrt{n - 1} ||A^{1/2} \nabla g||^2_\nu + ||\nabla g||^2_\nu = \frac{1}{2} \sqrt{n - 1} G_\nu. \]

Noting that $\tilde{A}_2 N = P_2 \tilde{A} N = P_2 (\nabla H)_\eta (N, \eta)$, one has for some constant $K > 0$,
\[ ||\tilde{A}_2 N||_\nu \leq K ||N||_\nu + 1. \]

This implies
\[ \hat{G}_\nu \leq (2 - b^2) \sqrt{n - 1} G_\nu + 2K ||N||_{\nu + 1} ||\nabla g||_\nu + 2 ||Q_2||_\nu ||\nabla g||_\nu \]
\[ \leq (2 - b^2) \sqrt{n - 1} G_\nu + 2 \sqrt{G_\nu} (K \sqrt{F_{\nu + 1}/n} + ||Q_2||_\nu). \quad (6.23) \]

It follows from (6.20), (6.22), (6.19) and (6.18), that
\[ (\hat{E}_{\mu + b}^+ - \sqrt{n} \hat{F}_{\mu + 1}) \]
\[ \geq 2\sqrt{1 - b^2} E_{\mu + 1/2}^+ b + 2(\tilde{A}_1 N + Q_1, \nabla f + \sqrt{1 - b^2} A_2^2 \nabla f)_\mu \]
\[ - \sqrt{n} \left( (\sqrt{n - 1 + b^2} + K) F_{\mu + 1} + 2 \sqrt{F_{\mu + 1}} (||Q_3||_{\mu + 1} + C b^2) \right) - C b^2 F_{\mu + 2}^\perp \] (6.24)
\[
\begin{align*}
\forall \sqrt{1 - b^2 E_{\mu + \frac{1}{2}, b}^+} &- 2 \sqrt{E_{\mu, b}^+ (\| \hat{A} N \|_\mu + \| Q_1 \|_\mu) + \mathbb{E}_2^+} \\
\forall \sqrt{1 - b^2 E_{\mu + \frac{1}{2}, b}^+} &- 2 K \sqrt{E_{\mu, b}^+ \frac{F_{\mu + 1/n}}{n}} \\
- 2 C \sqrt{E_{\mu, b}^+ (\varepsilon_{n, \mu} + b^2)^2 (E_{1, b} + n^{5/4 - 2\mu} E_{\mu + \frac{1}{2}, b}^+)^{2(\mu - 1)/4}} \\
- \sqrt{n} (\sqrt{n - 1 + b^2 + K}) F_{\mu + 1} &- 2 \sqrt{n} \sqrt{F_{\mu + 1} \| Q_3 \|_{\mu + 1}} \\
- C b^2 (2 \sqrt{n F_{\mu + 1} + F_{\mu + \frac{1}{2}}}) \\
\forall \sqrt{1 - b^2 E_{\mu + \frac{1}{2}, b}^+} &- 2 K n^{-3/4} E_{\mu, b}^+ \\
- 2 C \sqrt{E_{\mu, b}^+ (2(\mu + 1)/2) (n^{2(1 - \mu)} E_{\mu, b} + n^{5/4 - 2\mu} E_{\mu + \frac{1}{2}, b}^+)^{(2(\mu - 1)/4} \\
- 2 C \sqrt{E_{\mu, b}^+ (\varepsilon_{n, \mu} + b^2)^2 (n^{2(1 - \mu)} E_{\mu, b} + n^{5/4 - 2\mu} E_{\mu + \frac{1}{2}, b}^+)^{1/2} \\
- (\sqrt{n - 1 + b^2 + K}) E_{\mu, b}^+ &- 2 n^{1/4} \sqrt{E_{\mu, b}^+ \| Q_3 \|_{\mu + 1}} \\
- C b^2 (2n^{1/4} \sqrt{E_{\mu, b}^+} + n^{-1/4} E_{\mu + \frac{1}{2}, b}^+) \\
\forall \sqrt{1 - b^2 E_{\mu + \frac{1}{2}, b}^+} &- (\sqrt{n - 1 + b^2 + K} + 2 K n^{-3/4} n^{-1/2} E_{\mu + \frac{1}{2}, b}^+) \\
- C \varepsilon_{n, \mu}^{(2(\mu + 1)/2) (n^{2(1 - \mu)} + n^{5/4 - 2\mu} E_{\mu + \frac{1}{2}, b}^+)^{(2(\mu - 1)/4} \\
- C \varepsilon_{n, \mu}^{(2(\mu + 1)/2) (n^{7/8 - 5/16} E_{\mu + \frac{1}{2}, b}^+)^{(2(\mu - 1)/4} \\
- C (\varepsilon_{n, \mu} + b^2) n^{3(4 - 3) / 4} E_{\mu + \frac{1}{2}, b}^+ &- 2 n^{1/4} \sqrt{E_{\mu, b}^+ \| Q_3 \|_{\mu + 1}} \\
- C b^2 (2 \sqrt{E_{\mu + \frac{1}{2}, b}^+} + n^{-1/4} (n - 1) / n^{1/4} E_{\mu + \frac{1}{2}, b}^+) &. \\
\end{align*} \]

Due to Proposition 6.1 and Lemma 6.2,

\[
\| Q_3 \|_{\mu + 1} \leq C \varepsilon_{n, \mu}^{(2(\mu + 3)/2(\mu + 1))} (y, \hat{y})_1^{(2(\mu + 1)/2(\mu + 1))} + C \varepsilon_{n, \mu} \| (y, \hat{y})_1^{(2(\mu + 1)/2(\mu + 1))} + C b^2 \| y_1 \\
\leq C \varepsilon_{n, \mu}^{(2(\mu + 3)/2(\mu + 1))} (E_{1, b} + F_2 + G_1)^{(2(\mu + 1)/2(\mu + 1))} \\
+ C (\varepsilon_{n, \mu} + b^2) (E_{1, b} + F_2 + G_1)^{1/2} \\
\leq C \varepsilon_{n, \mu}^{(2(\mu + 3)/2(\mu + 1))} (E_{1, b} + n^{5/4 - 2\mu} E_{\mu + \frac{1}{2}, b}^+)^{(2(\mu + 1)/2(\mu + 1))} \\
+ C (\varepsilon_{n, \mu} + b^2) (E_{1, b} + n^{5/4 - 2\mu} E_{\mu + \frac{1}{2}, b}^+)^{1/2}.
\]

Thus,

\[
\begin{align*}
2 n^{1/4} \sqrt{E_{\mu, b}^+ \| Q_3 \|_{\mu + 1}} &\leq C \varepsilon_{n, \mu}^{(2(\mu + 3)/2(\mu + 1))} n^{1/4} \sqrt{E_{\mu, b}^+ (E_{1, b} + n^{5/4 - 2\mu} E_{\mu + \frac{1}{2}, b}^+)^{(2(\mu + 1)/2(\mu + 1))} \\
+ C (\varepsilon_{n, \mu} + b^2) n^{1/4} \sqrt{E_{\mu, b}^+ (E_{1, b} + n^{5/4 - 2\mu} E_{\mu + \frac{1}{2}, b}^+)^{1/2} \\
\leq C \varepsilon_{n, \mu}^{(2(\mu + 3)/2(\mu + 1))} \sqrt{E_{\mu + \frac{1}{2}, b}^+ (n^{3/2 - 2\mu} E_{\mu + \frac{1}{2}, b}^+ + n^{5/4 - 2\mu} E_{\mu + \frac{1}{2}, b}^+)^{(2(\mu + 1)/2(\mu + 1))} \\
+ C (\varepsilon_{n, \mu} + b^2) \sqrt{E_{\mu + \frac{1}{2}, b}^+ (n^{3/2 - 2\mu} E_{\mu + \frac{1}{2}, b}^+ + n^{5/4 - 2\mu} E_{\mu + \frac{1}{2}, b}^+)^{1/2}}.
\end{align*}
\]
Similarly, using Proposition 6.1 and Lemma 6.2 to obtain
\[
\leq C_{\varepsilon_{n,\mu}}(\varepsilon_{n,\mu} + b^2)n^{(3-4\mu)/(2\mu+1)}E_{\mu+\frac{1}{2},b}^{+(4\mu-1)/2(4\mu+1)}
\]
\[
+ C(\varepsilon_{n,\mu} + b^2)n^{(3-4\mu)/4}E_{\mu+\frac{1}{2},b}^{+}.
\]
Hence, for large \( n \),
\[
\dot{E}_{\mu,b}^+ - \sqrt{n}\dot{F}_{\mu+1} > 0.
\]
From (6.20), (6.23), we get
\[
\dot{E}_{\mu+\frac{1}{2},b}^+ - \frac{2\sqrt{1 - b^2}}{(2 - b^2)}n^{3/4}\dot{G}_{\mu}
\]
\[
\geq 2\sqrt{1 - b^2}E_{\mu+\frac{1}{2},b}^+ + 2(\dot{A}_1 N + Q_1) + \sqrt{1 - b^2}A^\perp\nabla f
\]
\[
- 2\sqrt{1 - b^2}n^{3/4}\sqrt{n - 1}G_{\mu} - \frac{4\sqrt{1 - b^2}}{(2 - b^2)}n^{3/4}\sqrt{G_{\mu}(K\sqrt{F_{\mu+1}}/n + \|Q_2\|_{\mu})}
\]
\[
\geq 2\sqrt{1 - b^2}E_{\mu+\frac{1}{2},b}^+ - 2(\sqrt{\frac{F_{\mu+\frac{1}{2},b}}{n} + \|Q_1\|_{\mu+\frac{1}{4}}})
\]
\[
- 2\sqrt{1 - b^2}\sqrt{n - 1}E_{\mu+\frac{1}{2},b}^+ - 2n^{3/8}\sqrt{E_{\mu+\frac{1}{2},b}^+}(K\sqrt{F_{\mu+1}/n + \|Q_2\|_{\mu}})
\]
\[
\geq 2\sqrt{1 - b^2}E_{\mu+\frac{1}{2},b}^+ - 2n^{-3/4}E_{\mu+\frac{1}{2},b}^+ + \sqrt{E_{\mu+\frac{1}{2},b}^+}(Q_{\mu+\frac{1}{4}})
\]
\[
- 2\sqrt{1 - b^2}\sqrt{n - 1}E_{\mu+\frac{1}{2},b}^+ - 2n^{3/8}\sqrt{E_{\mu+\frac{1}{2},b}^+}(Kn^{-\frac{3}{4}}E_{\mu+\frac{1}{2},b}^+ + \|Q_2\|_{\mu})
\]
\[
\geq 2\sqrt{1 - b^2}E_{\mu+\frac{1}{2},b}^+ - 2n^{-5/4}E_{\mu+\frac{1}{2},b}^+ + n^{-1/4}\sqrt{E_{\mu+\frac{1}{2},b}^+}(Q_{\mu+\frac{1}{4}})
\]
\[
- 2\sqrt{1 - b^2}\sqrt{(n - 1)/n}E_{\mu+\frac{1}{2},b}^+ - 2n^{-3/8}\sqrt{E_{\mu+\frac{1}{2},b}^+}(Kn^{-5/4}E_{\mu+\frac{1}{2},b}^+ + \|Q_2\|_{\mu})
\]
\[
\geq 2\sqrt{1 - b^2}(1 - \sqrt{(n - 1)/n}) - 2n^{-5/4} - 2Kn^{-9/8}E_{\mu+\frac{1}{2},b}^+
\]
\[
- 2n^{-1/4}\sqrt{E_{\mu+\frac{1}{2},b}^+}||Q_1||_{\mu+\frac{1}{4}} - 2n^{-1/8}\sqrt{E_{\mu+\frac{1}{2},b}^+}||Q_2||_{\mu}.
\]  
(6.27)

Using Proposition 6.1 and Lemma 6.2 to obtain
\[
||Q_2||_{\mu} \leq C_{\varepsilon_{n,\mu}}(\varepsilon_{n,\mu} + b^2)(E_{1,b} + n^{-3/4}E_{\mu+\frac{1}{2},b}^+)^{(2\mu-1)/4\mu}
\]
\[
+ C(\varepsilon_{n,\mu} + b^2)(E_{1,b} + n^{-3/4}E_{\mu+\frac{1}{2},b}^+)^{1/2}
\]
\[
\leq C_{\varepsilon_{n,\mu}}(\varepsilon_{n,\mu} + b^2)(E_{1,b} + n^{3/4-2\mu}E_{\mu+\frac{1}{2},b}^+)^{(2\mu-1)/4\mu}
\]
\[
+ C(\varepsilon_{n,\mu} + b^2)(E_{1,b} + n^{3/4-2\mu}E_{\mu+\frac{1}{2},b}^+)^{1/2}
\]
\[
\leq C_{\varepsilon_{n,\mu}}n^{-\frac{(2\mu-1)^2}{4\mu}}(E_{\mu+\frac{1}{2},b}^+)^{(2\mu-1)/4\mu}
\]
\[
+ C(\varepsilon_{n,\mu} + b^2)n^{1/2-\mu}(E_{\mu+\frac{1}{2},b}^+)^{1/2}.
\]  
(6.28)

Similarly,
\[
||Q_1||_{\mu+\frac{1}{4}} \leq C_{\varepsilon_{n,\mu}}(E_{1,b} + n^{-3/4}E_{\mu+\frac{1}{2},b}^+)^{(4\mu-1)/2(4\mu+1)}
\]
\[
+ C(\varepsilon_{n,\mu} + b^2)(E_{1,b} + n^{-3/4}E_{\mu+\frac{1}{2},b}^+)^{1/2}
\]
\[
\leq C_{\varepsilon_{n,\mu}}(\varepsilon_{n,\mu} + b^2)n^{-\frac{(4\mu-1)^2}{4\mu}}(E_{\mu+\frac{1}{2},b}^+)^{(4\mu-1)/2(4\mu+1)}
\]
\[
+ C(\varepsilon_{n,\mu} + b^2)n^{1/2-\mu}(E_{\mu+\frac{1}{2},b}^+)^{1/2}.
\]  
(6.29)
Thus, combining (6.27), (6.28) and (6.29),
\[
\dot{E}_{\mu+\frac{1}{2},b}^+ - \frac{2\sqrt{1 - b^2}}{(2 - b^2)} n^{3/4} G_n \\
\geq (2\sqrt{1 - b^2}(1 - \sqrt{(n - 1)/n}) - 2n^{-5/4} - 2Kn^{-9/8}) E_{\mu+\frac{1}{2},b}^+ \\
- C\varepsilon_{n,\mu} n^{-1/4}(4\mu - 1/2) E_{\mu+\frac{1}{2},b}^+ (E_{\mu+\frac{1}{2},b}^+)^{4\mu/(4\mu + 1)} \\
- C\varepsilon_{n,\mu} n^{1/4}(4\mu - 1/2) E_{\mu+\frac{1}{2},b}^+ E_{\mu+\frac{1}{2},b}^+ \\
- C\varepsilon_{n,\mu} n^{1/4}(4\mu - 1/2) E_{\mu+\frac{1}{2},b}^+ (E_{\mu+\frac{1}{2},b}^+)^{4\mu/(4\mu + 1)} \\
\geq \frac{1}{n} \left[ \sqrt{1 - b^2} - n^{-1/4} - Kn^{-1/8} - C\varepsilon_{n,\mu} n^{1/4}(4\mu - 1/2) E_{\mu+\frac{1}{2},b}^+ \\
- C\varepsilon_{n,\mu} n^{1/4}(4\mu - 1/2) E_{\mu+\frac{1}{2},b}^+ (E_{\mu+\frac{1}{2},b}^+)^{4\mu/(4\mu + 1)} \\
- C\varepsilon_{n,\mu} n^{1/4}(4\mu - 1/2) E_{\mu+\frac{1}{2},b}^+ (E_{\mu+\frac{1}{2},b}^+)^{4\mu/(4\mu + 1)} \right],
\]
which is positive for large \(n\) provided \(\mu \geq 2\).

Therefore, we have proved the desired results. \(\square\)

**Theorem 6.5.** Let \(\mu \geq 2\) and \(|b| \ll 1\). For large \(n\), \(E_{\mu,b}^+(t) \geq E_{\mu,b}^+(0)e^{\sqrt{1 - b^2}\sqrt{\mu}}\) for \(E, F\) and \(G\) in the invariant set of Proposition 6.4.

**Proof.** It follows from (6.20) and similar arguments for (6.21), and (6.19) that
\[
\dot{E}_{\mu,b}^+ \geq \sqrt{1 - b^2} E_{\mu+\frac{1}{2},b}^+ \geq \sqrt{1 - b^2} n^{1/2} E_{\mu,b}^+,
\]
which yields the desired estimates. \(\square\)

### 6.4. Ill-posedness.

The ill-posedness theorem is the following:

**Theorem 6.6.** Suppose \(|b| \ll 1\). For the initial data \(\Omega_0 = \{z \in \mathbb{C}, |z| \leq 1\}\), \(\zeta_n(0) = z\), \(\zeta_n(0) = e^{-n^{1/4}} \bar{z} - iz\) (\(n \geq 2\)) and \(H_n(z) = ibz\) for \(z \in \Omega_0\), \(\zeta_n(t) (n \geq 2)\) be the solution to problem (4.1) in some time interval \([0, T]\). Then, for \(\mu \geq 2\), \(\|(y_n(t), \dot{y}_n(t))\|_\mu \to 0\) as \(n \to \infty\), but for any \(t > 0\),
\[
\|(y_n(t), \dot{y}_n(t))\|_\mu \to \infty, \quad n \to \infty,
\]
where
\[
y_n(t) = \zeta_n(t) - \eta(t),
\]
and \(\eta(t, z) = e^{it}z\) is a special solution to (4.1).

**Proof.** Decompose \(y_n\) into \(\nabla f + \nabla g + N\), then we define \(E_{\mu,b}^+, F_{\mu}\) and \(G_{\mu}\) which are in the invariant set of Proposition 6.4 at time \(t = 0\). Hence, \(E_{\mu,b}(t) \geq E_{\mu,b}^+(0)e^{\sqrt{1 - b^2}\sqrt{\mu}}\), by Theorem 6.5. However,
\[
E_{\mu,b}^+(0) = \int_{\Omega} |A^\mu(\nabla f + \nabla g + N)|^2 \mu_{\Omega} dx \\
= \int_{\Omega} |(n - 1)^\mu e^{-n^{1/4}} \bar{z} - iz|^2 dx \\
= \int_0^{2\pi} d\theta \int_0^1 e^{-2n^{1/4}(n - 1)^2\mu_{\theta}^2n} dr \\
= \frac{2\pi(n - 1)^{2\mu}}{2n + 1} e^{-2n^{1/4}}.
\]
Therefore, 
\[ E_{\mu,b}^+(t) \geq \frac{2\pi(n-1)^{2\mu}}{2n+1}e^{\sqrt{1-b^2/\sqrt{n+t}}(2n)^{1/4}}, \]
which tends to $\infty$ for any $t > 0$ as $n \to \infty$. Therefore, 
\[ \| (y_n(t), \dot{y}_n(t)) \|_\mu \to \infty \]
for $t > 0$. \hfill \Box

The above theorem shows the ill-posedness for that solutions $\zeta_n(t)$ would exist and converge to $\eta(t)$ on some interval $[0, T]$ if (4.1) were well posed.

The final remark is that the analysis of this paper is uniform in the vacuum permeability $\mu_0 > 0$ as long as we take $b^2 \ll \min(1, \mu_0)$ in the proofs.

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REFERENCES

[1] Alazard, T., Burq, N., Zuily, C.: On the Cauchy problem for gravity water waves. Invent. Math. 198(1), 71-163 (2014)
[2] Ambrose, D.M., Masmoudi, N.: The zero surface tension limit of two-dimensional water waves. Comm. Pure Appl. Math. 58(10), 1287-1315 (2005)
[3] Chen, G.-Q., Wang, Y.-G.: Existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics. Arch. Ration. Mech. Anal. 187(3), 369–408 (2008)
[4] Christodoulou, D., Lindblad, H.: On the motion of the free surface of a liquid. Comm. Pure Appl. Math. 53(12), 1536–1602 (2000)
[5] Coutand, D., Shkoller, S.: Well-posedness of the free-surface incompressible Euler equations with or without surface tension. J. Amer. Math. Soc. 20(3), 829–930 (2007).
[6] Ebin, D.G.: The equations of motion of a perfect fluid with free boundary are not well posed. Comm. Partial Differential Equations 12(10), 1175–1201 (1987)
[7] Gu, X., Wang, Y.: On the construction of solutions to the free-surface incompressible ideal magnetohydrodynamic equations, arXiv:1609.07013
[8] Hao, C.: On the motion of free interface in ideal incompressible MHD. Arch. Ration. Mech. Anal. 224(2), 515–553 (2017)
[9] Hao, C., Luo, T.: A priori estimates for free boundary problem of incompressible inviscid magnetohydrodynamic flows. Arch. Ration. Mech. Anal. 212(3), 805–847 (2014)
[10] Hao C., Wang, D.: A priori estimates for the free boundary problem of incompressible neo-Hookean elastodynamics, J. Differential Equations 261(1), 712–737 (2016)
[11] Lee, D.: Uniform estimate of viscous free-boundary magnetohydrodynamics with zero vacuum magnetic field. SIAM J. Math. Anal. 49(4), 2710–2789 (2017)
[12] Lee, D.: Zero kinetic viscosity-magnetic diffusivity limit of free boundary magnetohydrodynamics. arXiv:1409.5384
[13] Lannes, D.: Well-posedness of the water waves equations. J. Am. Math. Soc. 18(3), 605–54 (2005)
[14] Lindblad, H.: Well-posedness for the motion of an incompressible liquid with free surface boundary. Ann. of Math. (2) 162(1), 109–194 (2005)
[15] Lindblad, H., Nordgren, K.H.: A priori estimates for the motion of a self-gravitating incompressible liquid with free surface boundary. J. Hyperbolic Differ. Equ. 6(2), 407–432 (2009)
[16] Luo, T., Zeng, H.: On the free surface motion of highly subsonic heat-conducting inviscid flows. arXiv:1709.06925
[17] Morando, A., Trakhinin, Y., Trebeschi, P.: Well-posedness of the linearized plasma-vacuum interface problem in ideal incompressible MHD. Quart. Appl. Math. 72(3), 549–587 (2014)
[18] Morrison, P.J.: On hamiltonian and action principle formulations of plasma dynamics. AIP Conference Proceedings 1188(1), 329–344 (2009)
[19] Nirenberg, L.: On elliptic partial differential equations. Ann. Scuola Norm. Sup. Pisa 13(3), 115–162 (1959)
[20] Secchi, P., Trakhinin, Y.: Well-posedness of the plasma-vacuum interface problem. Nonlinearity 27(1), 105–169 (2014)
[21] Shatah, J., Zeng, C.: Geometry and a priori estimates for free boundary problems of the Euler equation. Comm. Pure Appl. Math. 61(5), 698–744 (2008)
[22] Padula, M., Solonnikov, V.A.: On the free boundary problem of magnetohydrodynamics. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 385, 135–186 (2010). Translation in J. Math. Sci. (N. Y.) 178(3), 313–344 (2011)
[23] Sun, Y., Wang, W., Zhang, Z.: Nonlinear stability of the current-vortex sheet to the incompressible MHD equations. Comm. Pure Appl. Math. 71(2), 356–403 (2018)
[24] Trakhinin, Y.: On well-posedness of the plasma-vacuum interface problem: the case of non-elliptic interface symbol. Commun. Pure Appl. Anal. 15(4), 1371–1399 (2016)
[25] Wang, Y.-G.; Yu, F.: Stabilization effect of magnetic fields on two-dimensional compressible current-vortex sheets. Arch. Ration. Mech. Anal. 208(2), 341–389 (2013)
[26] Wang, Y., Xin, Z.: Incompressible inviscid resistive MHD surface waves in 2D, arXiv:1801.04694
[27] Wu, S.: Well-posedness in Sobolev spaces of the full water wave problem in 2-D. Invent. Math. 130(1), 39–72 (1997)
[28] Wu, S.: Well-posedness in Sobolev spaces of the full water wave problem in 3-D. J. Amer. Math. Soc. 12(2), 445–495 (1999)
[29] Zhang, P., Zhang, Z.: On the free boundary problem of three-dimensional incompressible Euler equations. Comm. Pure Appl. Math. 61(7), 877–940 (2008)

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