Towards New Membrane Flow from de Wit-Nicolai Construction

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Abstract

The internal 4-form field strengths with 7-dimensional indices have been constructed by de Wit and Nicolai in 1986. They are determined by the following six quantities: the 56-bein of 4-dimensional $\mathcal{N} = 8$ gauged supergravity, the Killing vectors on the round seven-sphere, the covariant derivative acting on these Killing vectors, the warp factor, the field strengths with 4-dimensional indices and the 7-dimensional metric.

In this paper, by projecting out the remaining mixed 4-form field strengths in an $SU(8)$ tensor that appears in the variation of spin $\frac{1}{2}$ fermionic sector, we also write down them explicitly in terms of some of the above quantities. For the known critical points, the $\mathcal{N} = 8 SO(8)$ point and the nonsupersymmetric $SO(7)^+$ point, we reproduce the corresponding 11-dimensional uplifts by computing the full nonlinear expressions. Moreover, we find out the 11-dimensional lift of the nonsupersymmetric $SO(7)^+$ invariant flow. We decode their implicit formula for the first time and the present work will provide how to obtain the new supersymmetric or nonsupersymmetric membrane flows in 11-dimensions.

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1 Introduction

The truncation of 11-dimensional supergravity on the seven-sphere to the massless sector is equivalent to the 4-dimensional gauged $\mathcal{N} = 8$ supergravity [1]. This enables us to write down the full nonlinear metric ansatz directly from the vacuum expectation values of the scalar and pseudoscalar fields of 4-dimensional gauged $\mathcal{N} = 8$ supergravity [2], together with both warp factor and the Killing vectors on the seven-sphere. The 7-dimensional inverse metric is generated from the $SU(3)$-singlet vacuum expectation values of 4-dimensional gauged $\mathcal{N} = 8$ supergravity [3]. Although this metric is written in terms of the rectangular coordinates, the standard metric in terms of 7-dimensional global coordinates is recovered. For the 4-form field strengths, some of the components of the full nonlinear ansatz are found by de Wit and Nicolai [1] but the remaining ones of the 4-form components, where the four indices contain both the internal 7-dimensional indices and 4-dimensional indices, are not known so far. There exist some previous works [4, 5] where the full nonlinear metric ansatz is used but the 4-form ansatz is not used (the 4-forms are determined by brute force) because there exist only partial informations on these 4-form field strengths and it is difficult to decode their implicit formula for practical use.

In this paper, we reexamine the work of de Wit and Nicolai [1] and would like to see whether the full nonlinear ansatz for 4-form field strengths provide a master equation for the 11-dimensional solution. The situation when we deal with the 4-forms is more complicated than the one with the metric because the $SU(8)$ covariance for the theory requires the five-fold product of the ‘generalized’ vielbein $e^m_{ij}$ where $i, j$ are $SU(8)$ indices and $m$ is the 7-dimensional curved index. For the full nonlinear metric ansatz, the two-fold product of them is needed. This is the reason why the explicit computations for the 4-forms completely are not known so far during last 25 years. The supersymmetric flow solutions for $SU(3) \times U(1)_R$ invariant flow [6] and $G_2$ invariant flow [7] are found by taking the appropriate 4-forms ansatz via the symmetry of the theory rather than using the formula of [1]. Of course, the full nonlinear metric ansatz [2] are used here.

For the two-fold product of generalized vielbeins in the full nonlinear metric, it is nontrivial to write down the rectangular coordinates in terms of the 7-dimensional curved coordinates or frame coordinates but it is straightforward to express the 7-dimensional inverse metric in terms of the rectangular coordinates, as a first step. For the 4-forms, the data from 4-dimensional gauged $\mathcal{N} = 8$ supergravity goes into the generalized vielbein. The five copies among these generalized vielbeins with an appropriate $SU(8)$ indices make the explicit computation complicated. As we multiply them successively, the expressions are getting more involved. For the
time being, we consider and focus on the simplest cases where the 4-dimensional data looks very simple: $SO(8)$ critical point, $SO(7)^+$ critical point and $SO(7)^+$ invariant flow. Note that there exist three basic representations of $SO(8)$, the vector $8_v$, the left handed spinor $8_-$ and the right handed spinor $8_+$. The embeddings of $SO(7)^\pm$ in $SO(8)$ for representation $8_v$ are the same but those for the representation $8_\pm$ are different from those for the representation $8_v$. For the $SO(7)^+$ critical point, the scalar has nonzero vacuum expectation values while for the $SO(7)^-$ critical point, the pseudo scalar has nonzero vacuum expectation values.

In section 2, we review the main results of the de Wit and Nicolai’s construction, and obtain the mixed 4-form field strengths newly.

In section 3, we apply the formula of section 2 to the two critical points and the nonsupersymmetric flow and find out the corresponding 11-dimensional solutions. Some of them are previously known.

In section 4, we summarize the present work and comment on the future directions.

In the Appendices, we present the detailed expressions for three cases in section 3 and describe the supersymmetry checking for the 11-dimensional $SO(7)^+$ flow solution.

2 de Wit-Nicolai construction

In this section, we describe the de Wit and Nicolai construction which provides the full nonlinear ansatz for the 4-form field strengths. Later, we continue to apply their construction to the mixed 4-forms which is necessary for the 11-dimensional uplift of 4-dimensional domain wall solutions.

2.1 The 4-form field strengths

In this section, we describe the relevant parts for the full nonlinear ansatz [1] for the 11-dimensional 4-form field strengths [8] with internal 7-dimensional indices in terms of the data of 4-dimensional gauged $\mathcal{N} = 8$ supergravity [9, 10]. For those who are interested in the details of this construction, we refer to the original paper [1] by de Wit and Nicolai.

The variation of spin-1 field of 11-dimensional supergravity contains the generalized vielbein that has the coefficient function, which depends on only 4-dimensional space-time, in front of the Killing vector on the round seven-sphere. It turns out, from $\mathcal{N} = 8$ transformation rule for vector field, that this coefficient function can be written as the following four-dimensional quantity [1]

$$w_{ij}^{IJ}(x) \equiv u_{ij}^{IJ}(x) + v_{ijIJ}(x),$$

(2.1)
where $u_{ij}^{IJ}$ and $v_{ijIJ}$ fields are $28 \times 28$ matrices of $\mathcal{N} = 8$ gauged supergravity that depend on the 4-dimensional curved space-time $x^\mu$. Here the $SU(8)$ indices $[ij]$ are antisymmetrized and the $SO(8)$ indices $[IJ]$ are also antisymmetrized. We will use these properties all the times. All the indices run from 1 to 8. The complex conjugation of $w_{ij}^{IJ}$ can be obtained by raising or lowering the indices. That is, $(w_{ij}^{IJ})^* = w_{ij}^{ij} = (u_{ij}^{IJ})^* + (v_{ijIJ})^* = u_{ij}^{ij} + v_{ijIJ}$ from the 4-dimensional gauged $\mathcal{N} = 8$ supergravity [9]. The explicit expressions for these 28-beins, in $SU(3)$-singlet sector, in terms of four supergravity fields are given in [11]. By restricting these to constants further, one gets $SO(8)$ critical point and $SO(7)^+$ critical point. For the $SO(7)^+$ flow, one has a single supergravity field which depends on the radial coordinate of $AdS_4$ space.

How does one determine the Killing vector? On the seven-sphere, there exist eight scalar fields, $X^A (A = 1, \cdots , 8)$ satisfying some constraints [12, 2, 13]. Using the $\Gamma$ matrices that are $SO(8)$ generators [14, 7], the Killing vectors on the unit ‘round’ $S^7$, that depend on the 7-dimensional curved space $y^m$ via $X^A$, are given by [2]

$$\mathring{K}_m^{IJ} (y) = \frac{1}{2} (\Gamma^{IJ})_{AB} \left( X^A \partial_m X^B - X^B \partial_m X^A \right).$$

(2.2)

The two Killing vectors, $\mathring{K}_m^{IJ}$ and $\mathring{K}_m^{AB}$, are related to each other by triality where $\mathring{K}_m^{IJ} = (\Gamma^{IJ})_{AB} \mathring{K}_m^{AB}$. The 28 Killing vectors on seven-sphere can be expressed via the Killing spinors satisfying the eight Killing spinor equations [13]. The 7-dimensional coordinates $y^m = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta)$ are related to the $\mathbb{R}^8$ coordinates $X^A (y^m)$ that are as follows [7]:

$$X^1(y^m) = \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \sin \theta,$$

$$X^2(y^m) = \cos \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \sin \theta,$$

$$X^3(y^m) = \cos \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \sin \theta,$$

$$X^4(y^m) = \cos \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \sin \theta,$$

$$X^5(y^m) = \cos \theta_4 \sin \theta_5 \sin \theta_6 \sin \theta,$$

$$X^6(y^m) = \cos \theta_5 \sin \theta_6 \sin \theta,$$

$$X^7(y^m) = \cos \theta,$$

$$X^8(y^m) = \cos \theta_6 \sin \theta.$$

(2.3)

We denote the 11-th coordinate as the angle $\theta$. Sometimes the Killing vectors can be written in terms of rectangular coordinates $X^A$‘s only by multiplying the transformation matrix between $X^A$ and $y^m$ into (2.2), as in [3]($\partial_m$ goes into $\partial_C$). Note $\sum_{A=1}^8 (X^A)^2 = 1$. For the round seven-sphere with radius $L$, one should add $L$ in front of (2.2).
Then what is the generalized vielbein we mentioned before? Let us take the contraction of $SO(8)$ indices present in (2.1) and (2.2) with upper index $m$ of Killing vector as follows [1]:

$$e^m_{ij}(x, y) \equiv w_{ij}^{IJ}(x) \hat{K}^{mIJ}(y). \quad (2.4)$$

Here the upper index $n$ of Killing vector can be lowered via the ‘round’ seven-sphere metric $\hat{g}_{mn}$ as $\hat{K}^{IJ}_{m} = \hat{g}^{mn} \hat{K}^{IJ}_{n}$. Similarly, one has $\hat{K}^{mIJ} = \hat{g}^{mn} \hat{K}^{IJ}_{n}$ with its inverse metric $\hat{g}^{mn}$.

The $y$-dependence in (2.4) arises through the Killing vector while 4-dimensional data comes from $w_{ij}^{IJ}(x)$ we defined in (2.1). This generalized vielbein satisfies the $SU(8)$ covariant “Clifford property” [15]

$$e^m_{ij} e^{njk} + e^m_{ij} e^{mjk} = \frac{1}{4} \delta^k_i \delta^m_j \Delta^{-1} g^{mn}, \quad (2.5)$$

where the numerical factor 8 depends on the normalization of Killing vector we use in (2.2). In the convention of [1], this coefficient becomes 2 rather than 8. This determines the 7-dimensional metric $g_{mn}(x, y)$. Furthermore, the condition (2.5) is not satisfied all the time for given Killing vector (2.2) and one should find out the correct Killing vectors which will satisfy (2.5), using the $SO(8)$ invariance for the rectangular coordinates $X^A$. The warp factor is defined by

$$\Delta(x, y) \equiv \sqrt{\frac{\det g_{mn}(x, y)}{\det \hat{g}_{mn}(y)}}. \quad (2.6)$$

The last equation in (2.5) comes from the nonlinear metric ansatz developed by [2]. Note the presence of extra minus sign there due to the antisymmetric property of the $SU(8)$ indices of generalized vielbein (2.4).

The supersymmetry transformation of spin $\frac{1}{2}$ or $\frac{3}{2}$ fermionic sector [15, 16] contains $A^{ABCD}_m(x, y)$ self-dual tensor with “curved” $SU(8)$ indices $A, \cdots, D$ and curved 7-dimensional index $m$. Its “flat” version with flat $SU(8)$ indices $i, j, \cdots$ obtained by both multiplying the product of four Killing spinors $\eta^i_A \eta^j_B \eta^k_C \eta^l_D$ and contracting those curved indices, is given by

$$A_{ijkl} = \frac{4}{7} m_7 \hat{K}^{KL}_{m} (v_{ijLM} u_{kM}^{kl} - u_{ij}^{kL} u_{LM}^{klKM}) - \frac{3}{28} \hat{D}_m \hat{K}^{KL}_{n} \hat{K}^{MN}_{n} (u_{ij}^{kL} u_{LM}^{kl} - v_{ij}^{km} u_{LM}^{kMN}), \quad (2.7)$$

where the covariant derivative $\hat{D}_m$ contains the affine connection as well as the ordinary partial derivative [13]. The relative coefficients $\frac{1}{7}$ and $\frac{3}{28}$ were fixed completely by 1) solving the generalized vielbein postulates(which generalize the usual vielbein postulate of Riemannian geometry to the complex geometry) and 2) requiring that the T-tensor identified from 11-dimensional supergravity also become $y$-independent [1]. The $|m_7|$ is the inverse radius of
seven-sphere. This tensor will play the crucial role for the full nonlinear 4-forms ansatz together with the generalized vielbein. Note that the Killing vectors in the second term are contracted each other with 7-dimensional index and one can lower the upper index \( n \) by using the round metric as before. The \( x \)-dependence arises via \( u, v \) 28-beins and \( y \)-dependence appears in the Killing vectors and the covariant derivative acting on them.

On the other hand, the above \( A_{ijkl}^n \) tensor can be written in terms of 4-forms with internal 7-dimensional indices explicitly [15, 16]. By \( SU(8) \) invariance, one can take a particular \( SU(8) \) rotation as in [15]. Through the generalized vielbein postulate given in [15, 16], we multiply a five-fold product of the generalized vielbein \( e_m^{ij} \) (2.4) into (2.7) in order to preserve the \( SU(8) \) covariance. Using the various \( \Gamma \) matrix properties in [16], one obtains the nonlinear expression for the field strength given by [1]

\[
\frac{4}{i} f g_{np} \delta^m_q + \frac{1}{2} F_{npq} = \frac{i}{480} \sqrt{2} \Delta^4 \sqrt{\hat{g}} \varepsilon_{pqrstuv} e_m^{ij} \left( e_r^e s^e t^e v^e e_r^e s^e t^e v^e \right) A_{ijkl}^n, \tag{2.8}
\]

where the field strengths with 4-dimensional flat indices \( \alpha, \ldots, \delta \) appear in

\[
f \equiv \frac{1}{24i} \varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta} = \frac{1}{24i\Delta} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}. \tag{2.9}
\]

Note that there exists a typo in [1] and the factor \( \sqrt{\hat{g}} \) should be in the numerator not denominator (we fix here). The \( \hat{g} \) is the determinant of the round metric of seven-sphere. In the right hand side of (2.8), one sees the \( SU(8) \) covariance in the product of between the five generalized vielbeins and \( A_{ijkl}^n \) tensor. Moreover, the remaining \( SU(8) \) indices are contracted with those of generalized vielbien and then this leads to the \( SU(8) \) invariance. It is amazing that the right hand side can decompose into the two tensors of the left hand side. One can rewrite (2.8) by lowering the upper index with 7-dimensional metric as follows:

\[
\frac{4}{i} f (g_{np} g_{mq} - g_{nq} g_{mp}) + F_{mnpq} = \frac{i}{480} \sqrt{2} \Delta^4 \sqrt{\hat{g}} \eta_{pqrstuv} g_{mn} e_m^{ij} \left( e_r^e s^e t^e u^e \right) A_{ijkl}^n, \tag{2.10}
\]

where the 7-dimensional eta tensors with lower indices \( \eta_{pqrstuv} \) are purely numerical. We use a simplified notation for the \( kl \)-element in the right hand side and the explicit components for five-fold product are given in (2.8) where there exists a complex conjugation between the upper indices and lower ones for \( SU(8) \) we described in (2.1). For the three cases we are considering in this paper, the 28-beins are real and there is no difference between the upper and lower indices appearing in the generalized vielbeins. Since the 7-dimensional metric we will use contains the factor \( \sqrt{\Delta} \), the warp factor-dependence of left hand side of (2.10) disappears when we use the second expression for \( f \) in (2.9).

Therefore, the internal 4-form field strengths \( F_{mnpq} \) with 7-dimensional indices are determined by the following six quantities:
1) the 28-beins $u, v$ that appear in (2.1), (2.4) and (2.7),
2) the Killing vectors (2.2) that are present in (2.4) and (2.7) on the round seven-sphere,
3) the covariant derivative acting on these Killing vectors via (2.7),
4) the warp factor (2.6),
5) the field strengths with 4-dimensional flat indices (2.9) and
6) the 7-dimensional metric.

Now we compute all these quantities appearing in the right hand side (2.10) and compare the resulting expressions with the left hand side of (2.10). Then one can read off the correct informations on the 4-forms which will be the 11-dimensional solutions in the background we are interested in. Since we already expressed the Killing vectors in terms of 7-dimensional curved coordinates $y^m$ rather than the rectangular coordinates $X^A$, the results will do not contain the rectangular coordinates and we do not have to do extra works, contrary to the case of full nonlinear metric ansatz as in [3].

### 2.2 The mixed 4-form field strengths

What about the other components for the 4-form field strengths? For example, mixed 4-form field strengths with some internal indices and some non-internal indices. Although these are mentioned in [1] at the end of paper, the explicit expressions are not known so far. The full nonlinear expressions for the remaining 4-form field strengths (for the 4-forms $F_{\mu \nu mn}$ with two internal indices we will describe at the end of this subsection) can be obtained by projecting out the appropriate components in $A^{ijkl}_{\mu}$ using the four-dimensional results for these, as done exactly in [1]. The supersymmetry transformation of spin $1/2$ fermion sector has $A^{ijkl}_{\mu}$ tensor which is fully antisymmetric and self-dual in the indices $i, j, k, l$ and it is given by

$$A^{ijkl}_{\mu} = -\frac{1}{48} \sqrt{2} e^\alpha_\mu \varepsilon^{\alpha \beta \gamma \delta} F_{\alpha \beta \gamma \delta} \Gamma^b_{[ij} \Gamma^a_{kl]} + \frac{1}{8} \sqrt{2} e^\alpha_\mu F_{abc\alpha} \Gamma^a_{[ij} \Gamma^b_{kl]} + \cdots,$$  (2.11)

where the abbreviated part of (2.11) contains the term $\Gamma^a_{[ij}, \Gamma^b_{kl]}$. When we compute the six generalized vielbein and $A^{ijkl}_{\mu}$ tensor as in (2.8), one should use the explicit form for the generalized vielbein. For example, the equation (2.19) of [1]. Then there exist six Gamma matrices. Each Gamma matrix from each generalized vielbein. The five of them can be reduced to two from the identity given in [16]. Totally, one has three Gamma matrices $\Gamma^c_{ij}, \Gamma^d_{kl}$ from six generalized vielbeins. Then by combining these two factors, one can check that the quantity $\Gamma^a_{[ij} \Gamma^b_{kl]} \Gamma^c_{ij} \Gamma^d_{kl}$ vanishes identically. This feature is the same as the one in (2.8).

One can project out the mixed 4-forms $F_{\alpha \beta \gamma \delta}$ and $F_{abc\alpha}$ in $SU(8)$ invariant way as before.
This leads to the following expression

\[ e^m_{ij} \left( \epsilon^{[m} e^p e^q e^r e^{s]} \right)_{kl} A^{ijkl}_\mu = i \sqrt{2} \Delta^{-4} \frac{\varepsilon^{npqrstuv}}{\sqrt{g}} \left( \frac{1}{3 \Delta} \varepsilon_{\mu
u\rho\sigma} F^{\nu\rho\sigma}_{[t} \delta^m_{u]} + 2 F^m_{\mu tu} \right), \]  

(2.12)

where we use some Gamma matrix identities again. Let us emphasize that in the original paper [1], they used the fact that the identity \( \Gamma^{npqrs}_{CD} = i 2 \varepsilon^{npqrstuv} (\Gamma^{tu}_{CD}) \) holds where \( g \) is the determinant of the 7-dimensional metric. Via vielbeins \( e^m_{\mu} \) and \( e^a_{\mu} \), the 4-forms \( F_{a\beta\gamma\delta} \) and \( F_{abc\alpha} \) given in (2.11) are changed into \( F_{\mu\nu\rho\sigma} \) and \( F_{\mu npq} \) (2.12) respectively. In order to extract the 4-form field strengths, one can further simplify (2.12), by inverting it, as

\[
\frac{1}{3 \Delta} \varepsilon_{\mu
u\rho\sigma} \left( F^{\nu\rho\sigma}_{p} g_{qm} - F^{\nu\rho\sigma}_{q} g_{pm} \right) = 2 F_{\mu npq} 
= \frac{i}{480} \sqrt{2} \Delta^{4} \sqrt{g} \eta_{pqrsuv} \eta_{mn'} e'^{m}_{ij} \left( e' e^a e^b e^c \right)_{kl} A^{ijkl}_\mu. \]  

(2.13)

Once we figure out the equation (2.10), it is straightforward to compute this quantity also. For example, the \( kl \)-component of five-fold product of generalized vielbein, where the explicit structure of indices are given in (2.8), appears in (2.13) again and we do not have to compute this repeatedly. Note that for both \( G_2 \) invariant flow and \( SU(3) \times U(1)_R \) flow, it is known that the 4-forms appearing in the left hand side of (2.13) occur naturally. On the other hand, the \( A^{ijkl}_\mu(x) \) tensor in (2.13) appears in the scalar kinetic terms of 4-dimensional \( \mathcal{N} = 8 \) gauged supergravity and it is given by

\[ A^{ijkl}_\mu(x) = -2 \sqrt{2} \left( u^{ij}_{IJ} \partial_\mu v^{kIJ} - v^{ijIJ} \partial_\mu u^{kl}_{IJ} \right). \]  

(2.14)

In order to compute this tensor one has to know the \( x \)-dependence on \( u, v \) 28-beins. Since we are interested in the domain wall solutions, one should have the first order differential equations between the supergravity fields. These are found in [11] for \( SU(3) \)-singlet sector. For \( SO(7)^+ \) flow case we consider in this paper, the corresponding first order differential equations are found in [17].

Therefore, the mixed 4-form field strengths \( F_{a\beta\gamma\delta} \) (or \( F_{\mu\nu\rho\sigma} \) with world indices) and \( F_{a b c d} \) (or \( F_{\mu npq} \) with world indices) are determined by the following four quantities:

1) the 28-beins \( u, v \) that appear in (2.1), (2.4) and (2.14),
2) the Killing vectors (2.2) that are present in (2.4) and (2.7) on the round seven-sphere,
3) the warp factor (2.6) and
4) the 7-dimensional metric.

So far, we have considered the 4-forms, \( F_{\mu\nu\rho\sigma} \) with no internal indices and \( F_{\mu npq} \) with all the internal indices in (2.10), \( F_{\mu\nu\rho\sigma} \) with one internal index and \( F_{\mu npq} \) with three internal...
indices in (2.13). What happens for $F_{\mu \nu mn}$ with two internal indices? According to the result of [1], there exists a relation $(u_{ij}^{IJ} + v_{ij}^{IJ}) F_{\mu \nu}^{-ij} = [F_{\mu \nu}^{IJ}]_-$ where the $SO(8)$ field strength $F_{\mu \nu}^{IJ}$ is $F_{\mu \nu}^{IJ} = \partial_{\mu} A_{\nu}^{IJ} - \partial_{\nu} A_{\mu}^{IJ} - 2g A_{\mu}^{K[I} A_{\nu]K}$ and $[F_{\mu \nu}^{IJ}]_-$ is the anti-self dual part of this field strength. If there are no gauge fields in the 4-dimensional $\mathcal{N} = 8$ gauged supergravity (the action consists of scalar and gravity part), then the above relation implies $F_{\mu \nu}^{-ij} = 0$ (or $F_{\alpha \beta}^{-ij} = 0$). Furthermore, using the two spinors one can contract the indices $i, j$ and this leads to $C_{\alpha \beta}^{-AB}$ which is contained in $F_{\mu \nu mn}$. See the equation (8.6) of [16] where the multiple product of $e_{AB}^m$ tensor (sechsfundfzigbein) are contracted with $C_{\alpha \beta}^{-AB}$. In other words, $F_{\mu \nu mn} \simeq \sqrt{g} \eta_{mpqrst} e_{BC}^p e_{CD}^q e_{EF}^r e_{FA}^s \Delta^2 C_{\mu \nu}^{-AB}$. Therefore, for the 11-dimensional background with domain wall we are considering, there exists no $F_{\mu \nu mn}$ with two internal indices.

However, in the context of AdS/CMT where the gauge fields of 4-dimensional $\mathcal{N} = 8$ gauged supergravity play an important role [18], it is necessary to obtain nonzero $F_{\mu \nu mn}$ with two internal indices. See the relevant work by [19, 20] where the supergravity theory is not realized by 4-dimensional gauged $\mathcal{N} = 8$ supergravity but the 4-forms with two internal indices and two from 4-dimensional indices are nonvanishing due to the nonzero 2-form field strength along the 2-dimensions inside the 4-dimensions. It is an open problem to find out nontrivial $F_{\mu \nu mn}$ with two internal indices in this background in the context of 4-dimensional gauged $\mathcal{N} = 8$ supergravity.

3 The eleven-dimensional solutions

In this section, at first, we compute the right hand side of (2.10) for the known critical points ($SO(8)$ and $SO(7)^+$) by collecting (2.4), (2.7) with 11-dimensional metric explicitly and compare them with the left hand side of (2.10), for given 7-dimensional metric. In other words, the quantities $f, g_{mn}$ and $F_{mnpq}$ are known for the critical points. What we are doing newly is to calculate the right hand side of (2.10) based on the six quantities we mentioned before and to check whether the full nonlinear ansatz is right or not for consistency check. This is never done before and we will present the details in this section.

Later, we will consider membrane flow solution connecting between the $SO(8)$ critical point and the $SO(7)^+$ critical point. More precisely the flow solution contains $SO(8)$ critical point but does not contain $SO(7)^+$ critical point. One should use the equation (2.13) also as well as (2.10). The full nonlinear expressions for the 4-form field strengths (2.10) and mixed-form field strengths (2.13) will provide how to obtain the new supersymmetric (or non-supersymmetric) membrane flows in 11-dimensions, once the 4-dimensional RG flow equations where the supergravity fields vary with the radial coordinate of $AdS_4$ space are known.
3.1 The $\mathcal{N} = 8$ $SO(8)$ critical point

Let us consider the $SO(8)$ critical point. The verification for this critical point was done in [1] already and it is a good exercise to check this first. For the round seven-sphere metric

$$g_{mn} = \begin{pmatrix}
    s_\theta^2 & s_\theta^2 & s_\theta^2 & s_\theta^2 & s_\theta^2 & 0 & 0 & 0 & 0 \\
    0 & s_\theta^2 & s_\theta^2 & s_\theta^2 & s_\theta^2 & 0 & 0 & 0 & 0 \\
    0 & 0 & s_\theta^2 & s_\theta^2 & s_\theta^2 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & s_\theta^2 & s_\theta^2 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & s_\theta^2 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & s_\theta^2 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & s_\theta^2 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & s_\theta^2 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} = \hat{g}_{mn}, \quad (3.1)
$$

one writes down the square root of determinant as

$$\sqrt{\hat{g}} = s_\theta^6 s_2 s_3^2 s_4^2 s_5^2 s_6^5, \quad s_\theta \equiv \sin \theta, \quad s_i \equiv \sin \theta_i. \quad (3.2)$$

From the definition of (2.6), the warp factor $\Delta$ becomes 1 since $g_{mn} = \hat{g}_{mn}$. The generalized vielbein $e^m_{ij}$ is given by (2.4) with $w_{ij}^{IJ} = \delta_{ij}^{IJ}$ and the Killing vector can be obtained from (2.2) and (2.3) explicitly. The $A_{ijkl}^m$ tensor is given by (2.7) where the first term vanishes because $\nu_{ij}^{KL} = 0$ for $SO(8)$ critical point. In order to compute the second term of (2.7), it is better to use some property of Killing vector when acting on the covariant derivative. That is, one can rewrite as

$$\hat{D}_m \hat{K}^{nKL} \hat{K}^{MN}_n = \hat{D}_m \hat{g}^{mn'} \hat{K}^{KL}_{n'} \hat{K}^{MN}_n + \hat{g}^{mn'} \hat{D}_m \hat{K}^{KL}_{n'} \hat{K}^{MN}_n + \hat{g}^{mn'} \hat{K}^{KL}_{n'} (\hat{D}_m \hat{K}^{MN}_n)$$

with the metric of round seven-sphere in terms of three terms. The first term vanishes and the remaining terms can be simplified further. Then this becomes

$$\hat{D}_m \hat{K}^{nKL} \hat{K}^{MN}_n = m_7 \left( \hat{g}^{mn'} \hat{K}^{KL}_{m'} \hat{K}^{MN}_n + \hat{g}^{mn'} \hat{K}^{KL}_{n'} \hat{K}^{MN}_m + \hat{g}^{mn'} \hat{K}^{KL}_{n'} \hat{K}^{MN}_{M'} \hat{K}^{N'}_{M'} \right), \quad (3.3)$$

where we used the relation (A.3) of [1] together with Killing spinor equations and Killing vectors expressed in terms of the Killing spinors of [1] and $\hat{D}_m \hat{g}^{mn'} = 0$. Of course, there appears the minus sign in (3.3) for the skew-whiffing or orientation reversal of seven-manifold [13].

By substituting all of these (3.1), (3.2) including (3.3) into the right hand side of (2.10), one arrives at the final nonzero components and they are given in the Appendix A explicitly. By computing the left hand side of (2.10) with the condition $g_{mn} = \hat{g}_{mn}$ (3.1), one concludes that the following relations for the 4-form field (2.9) and the internal ones should hold

$$f = 3\sqrt{2} m_7, \quad F_{mnpq} = 0, \quad (3.4)$$

which was observed in [1] also. This is well-known Freund-Rubin solution for round seven-sphere compatification [21]. In the appropriate normalization, (3.4) implies that the 4-form
along the membrane is given by $F_{1234} = 3m_7 e^{3A(r)}$ in the background

$$ds^2_{11} = \Delta^{-1} \left( dr^2 + e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu \right) + ds_7^2,$$

where the 3-dimensional metric is $\eta_{\mu\nu} = (-, +, +)$, the radial coordinate is transverse to the domain wall, and the scale factor $A(r)$ in (3.5) behaves linearly in $r(\equiv x^i)$ at UV and IR regions. The warp factor is defined as (2.6). Of course, at the $SO(8)$ critical point, the 28 beins $u$ is constant and $v$ is equal to zero and (2.14) also vanishes. From (2.13), there are no mixed 4-forms.

### 3.2 The nonsupersymmetric $SO(7)^+\ critical point$

Let us describe the next nontrivial example. If one uses the previous rectangular coordinates (2.3), then the Clifford property (2.5) does not satisfy. For this critical point, one should transform the rectangular coordinates (2.3) using transformation matrix $R$ [3] as follows:

$$\tilde{X} = R^{-1} X,$$

where the $8 \times 8$ orthogonal matrix $R$ in (3.6) is given by

$$R = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}.$$  

(3.7)

The reason for this is due to the fact that the generalized vielbein for $SO(7)^+$ critical point should also satisfy the Clifford property (2.5). This is a useful check whether one has the right choice for the Killing vectors. Originally, the presence of $R$ in (3.7) was necessary in order to obtain the standard Kahler form from the inverse metric for $SU(4)^-\ critical point$ in the context of full nonlinear metric ansatz [2].

The 7-dimensional metric is given by

$$g_{mn} = \sqrt{\Delta a} \begin{pmatrix}
s_6^2 s_3^2 s_4^2 s_5^2 s_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & s_6^2 s_3^2 s_4^2 s_5^2 s_6^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & s_6^2 s_4^2 s_5^2 s_6^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s_6^2 s_4^2 s_5^2 s_6^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s_6^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s_6^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s_6^2
\end{pmatrix},$$

(3.8)

$$\text{where the 3-dimensional metric is } \eta_{\mu\nu} = (-, +, +), \text{ the radial coordinate is transverse to the domain wall, and the scale factor } A(r) \text{ in (3.5) behaves linearly in } r(\equiv x^i) \text{ at UV and IR regions. The warp factor is defined as (2.6). Of course, at the } SO(8) \text{ critical point, the 28 beins } u \text{ is constant and } v \text{ is equal to zero and (2.14) also vanishes. From (2.13), there are no mixed 4-forms.}$$
where the deformed norm in (3.8) is given by
\[ \xi^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta. \] (3.9)
One can also express this 7-dimensional metric in terms of rectangular coordinates (2.3) and the eccentricity of 7-dimensional ellipsoid depends on \((a, b)\). The warp factor (2.6), together with (3.9), becomes
\[ \Delta = a^{-1} \xi^{-\frac{4}{3}}. \] (3.10)
Of course, for round seven-sphere, we have \(a = b = 1\) (and \(\xi^2 = 1\)) and the metric (3.8) becomes the round metric (3.1). The geometric parameters \((a, b)\) in the 7-dimensional ellipsoid can be identified with the AdS\(_4\) supergravity fields. How does one see the SO\(_{(7)}\) symmetry? By writing the 7-dimensional warped ellipsoid as
\[ ds^2_7 = \sqrt{\Delta} a \left( \frac{\xi^2}{a^2} d\theta^2 + \sin^2 \theta d\Omega^2_6 \right), \]
one sees that the metric contains six-sphere whose isometry is nothing but SO(7). Here the SO\(_{(7)}^+\) invariant critical point fixes the AdS\(_4\) supergravity fields as follows:
\[ a = 5^{\frac{4}{12}}, \quad b = 5^{-\frac{4}{12}}. \] (3.11)
The scalar potential is a function of two supergravity fields and the SO\(_{(7)}^+\) symmetry further restricts to them. The derivative of scalar potential with respect to a single supergravity field vanishes at the critical point. However, the derivative of superpotential at the critical point does not vanish. The warp factor (3.10) becomes \(\Delta = \frac{5^{\frac{4}{12}}}{(3+2 \cos 2\theta)^{\frac{7}{12}}} \) by substituting (3.11) into (3.10) and (3.9). In next subsection, we will consider the case where AdS\(_4\) supergravity fields \((a, b)\) vary with the radius \(r\) of AdS\(_4\) space in the 11-dimensional background (3.5). The SO\(_{(7)}^-\) critical point corresponds to \(a = b = \frac{\sqrt{5}}{2}\). Due to the \(\xi^2 = 1\) (3.9), there is no deformation in round seven-sphere except the overall factor (3.8). Since the 28-bein \(v\) is imaginary along the SO\(_{(7)}^-\) flow [17], the generalized vielbein has imaginary part as well as real part and this makes the computations complicated.

Now we are ready to compute the right hand side of the equation (2.10). It is known in [11] that the 28-beins are given in terms of AdS\(_4\) supergravity fields (3.11). The Killing vector is given by (2.2) where \(X\) is replaced by \(\tilde{X}\) in (3.6). The warp factor is given by (3.10) with (3.9) and (3.11). Finally, the 7-dimensional metric is given by (3.8). One can plug these data into the right hand side of (2.10) and it turns out that there exists a mismatch. It does not provide the known 4-forms: nonzero constant \(f\) with vanishing \(F_{mnpq}\).

How does one resolve this problem? We have to look at what we have done so far again. In order to do that, let us introduce two real constants in front of each term of \(A^i_{mijkl}\) in (2.7)
$k_1$ and $k_2$ as follows:

$$
\tilde{A}_{ijkl}(x, y) = k_1 \frac{4}{7} m_7 \tilde{K}_m^{KL} (v_{ijkl}^{LM} u_{LM}^{kl} - u_{LM}^{i} v_{KL}^{kl})
- k_2 \frac{3}{28} D_m \tilde{K}_n^{[KL} \tilde{K}_n^{MN]} (u_{ijkl}^{LM} - v_{ijkl}^{KL})
$$

(3.12)

We want to see whether we can fix these constants by requiring that the equation (2.10) should satisfy for $SO(7)\pm$ critical point. Therefore, we compute the right hand side of (2.10) where $\tilde{A}_{ijkl}$ (3.12) is used. It turns out that the two constants can be fixed as follows:

$$
k_1 = \frac{1}{2}, \quad k_2 = -\frac{1}{3}.
$$

(3.13)

Unfortunately, there are extra minus sign in the second term of (2.7) and the numerical factors are not equal to each other. Are the numerical factors $\frac{4}{7}$ and $\frac{3}{28}$ in (2.7) wrong?

How does one understand this behavior? Let us first consider the sign problem. The minus sign in $k_2$ can be understood from the Killing spinor equations. The way of appearance of $m_7$ in (3.3) comes from the Killing spinor equations. The third choice for Killing vector is to take the same Killing vector (2.2) with (3.6) but with spinors satisfying the Killing spinor equations of the opposite sign [2]. This is equivalent to send the siebenbein $\overset{\circ}{e}^a_m$ to $-\overset{\circ}{e}^a_m$. Then the right hand side of (3.3) has an extra minus sign. Let us consider the different relative factors. Now then how does one understand the relative coefficient $\frac{2}{3}$ between $k_1$ and $k_2$ in (3.13)? One way to see this is to introduce two real parameters as follows:

$$
g_{mn} \to l_2^2 g_{mn}, \quad \tilde{K}_m^{IJ} \to l_2^2 \tilde{K}_m^{IJ}.
$$

(3.14)

According to (3.14), one knows how the inverse metric $g^{mn}$, the Killing vector with upper index $\overset{\circ}{K}^{mIJ}$, the generalized vielbein $e^m_{ij}$, and $e_{mi}$ transform. Then one can easily check that under the condition $l_2^2 l_1^2 = \frac{2}{3}$, the tensor (3.12) will provide the correct result. In other words, by using the equation (2.10) with modified tensor (3.12) with (3.13), the right hand side is summarized in the Appendix B. Then by reading off the left hand side, one gets it turns out that

$$
f = \sqrt{2} m_7, \quad F_{mnpq} = 0.
$$

(3.15)

This is a solution for ellipsoidal deformation of the 7-manifold [22]. The solution (3.15) corresponds to the nonzero component of 4-form as $F_{1234} = 5^\frac{4}{7} m_7 e^{3A(r)}$ in the background (3.5). What happens for $a = b = 1$ limit? It is easy to see that it reproduces the result of subsection 3.1 as we expect. We will consider what happens if we turn on certain supergravity
field in the $AdS_4$ supergravity (i.e., $a(r)$) where it approaches to zero in the UV and develops a nontrivial profile as a function of $r$ as one goes to the IR next subsection \(^2\).

For convenience, we present the explicit expressions for the generalized vielbeins in the Appendix C where the $a(r)$-dependence appears and this holds for the three cases in this paper. For $a(r) = 1$, the expressions in the Appendix C will give those in the subsection 3.1 and for $a(r) = 5\frac{1}{4}$, those correspond to the ones in the subsection 3.2 and finally for the general $a(r)$ with domain wall condition, the equation in the Appendix C will provide the generalized vielbein in next subsection.

### 3.3 The nonsupersymmetric \(SO(7)^+\) invariant flow

So far, we have described two cases, \(SO(8)\) critical point and \(SO(7)^+\) critical point. Now let us consider more general case. The supergravity fields vary with 4-dimensional space-time $x$. In particular, we are interested in the domain wall solutions. The first order differential equations from \(SO(8)\) to \(SO(7)^+\) are written as [17]

\[
\frac{da(r)}{dr} = -\frac{1}{2L}\sqrt{a(r)}(a(r)^4 - 1), \quad \frac{dA(r)}{dr} = \frac{1}{4L}\frac{a(r)^4 + 7}{\sqrt{a(r)}}.
\]

(3.16)

At the supersymmetric \(SO(8)\) critical point (i.e., $a(r) = 1$), the right hand side of first equation (3.16) vanishes while at the nonsupersymmetric \(SO(7)^+\) critical point (i.e., $a(r) = 5\frac{1}{4}$) those quantity does not vanish. The \(SO(8)\) gauge coupling constant $g$ of 4-dimensional gauged $\mathcal{N} = 8$ supergravity is replaced with $\sqrt{2}L$ ($= \sqrt{2}m_7$).

The 11-dimensional bosonic field equations are [8]

\[
R^N_M = \frac{1}{3} F_{MPQR} F^{NPQR} - \frac{1}{36} \delta^N_M F_{PQRS} F^{PQRS},
\]

\[
\nabla_M F^{MNPQ} = -\frac{1}{576} E \epsilon^{NPQRSTU VWXY} F_{RSTU} F_{VWXY},
\]

(3.17)

for given 11-dimensional metric (3.5) and (3.8) with $L^2$ factor and 4-form field strengths. The covariant derivative $\nabla_M$ on $F^{MNPQ}$ in (3.17) is given by $E^{-1} \partial_M (EF^{MNPQ})$ together

\(^2\)Recently, in [23], the most general solution of the generalized vielbein postulate (3.2) of [1] (corresponding to (2.14) of [23]), by adding a homogeneous term which does not affect the $T$ tensor of 4-dimensional gauged supergravity theory, is found. They also presented the complete expression for the flux. Let us describe how our results can fit their flux lift formulae. According to their (6.73) of [23], for $\alpha^m_n = -\frac{4}{3} \delta^m_n$, one sees that the coefficient from the first two terms in the right hand side becomes $\frac{2}{3}$ and the coefficient from the remaining terms becomes $\frac{2}{3}$. By adding these correction terms to (2.7), one has $\frac{1}{3} + \frac{2}{3} = \frac{2}{3}$ and $-\frac{2}{4} + \frac{2}{3} = -\frac{1}{2}$ respectively. This is consistent with the results (3.13) with (3.12) because the former is equal to $k_1 \frac{2}{3} (= \frac{2}{3})$ and the latter is equal to $-k_2 \frac{1}{2} (= \frac{1}{2})$ except the minus sign. As we described before, this sign problem can be resolved by using the Killing spinor equations of the opposite sign. We also have checked the equation (2.28) of [23] for \(SO(7)^+\) case and the internal 4-form flux $F_{mnpq}$ vanishes as expected in (3.15).
with elfbein determinant $E \equiv \sqrt{-g_{11}}$. The epsilon tensor $\epsilon_{NPQRSTUVWXY}$ with lower indices is purely numerical. Imposing the $r$-dependence to the vacuum expectation value $a(r)$, the 11-dimensional metric (3.5) generates the Ricci tensor components [7]. Applying the RG flow equations (3.16), all the $r$-derivatives in the Ricci tensor components can be replaced with polynomial of $a(r)$.

Now we want to obtain 11-dimensional solution satisfying (3.17) under the RG flow equations. Let us first consider the mixed 4-form field strengths using the equation (2.13). For the $G_2$ invariant sector, we have relations $c(r) = a(r)$ and $d(r) = b(r)$ [3]. Further constraint $b(r) = \frac{1}{a(r)}$ gives the $SO(7)^+$ invariant flow where the original field given in [24] is related to $a(r) = e^{\frac{M(r)}{2}}$. The parametrization for the $SU(3)$-singlet space [24, 25] contains the complex self-dual tensor describing 35 scalars and 35 pseudo scalars of 4-dimensional gauged $\mathcal{N} = 8$ supergravity. The supergravity fields reduce to two by $G_2$ invariance and these two further reduce to one by $SO(7)^+$ symmetry.

Let us consider the equation (2.13). Due to the domain wall solution (3.16), the nontrivial solution of (2.13) appears only when the $\mu$ index is equal to $r = x^4$. For other case($\mu = 1, 2, 3$), the right hand side of (2.13) vanishes and this implies that the 4-forms in the left hand side should be equal to zero. For fixed $\mu = 4$, there exist three free indices $m, p, q$. Then one can compute the right hand side explicitly. Then it turns out the nonzero contributions arise when $(m, p, q) = (1, 1, 7), (2, 2, 7), (3, 3, 7), (4, 4, 7), (5, 5, 7), (6, 6, 7)$. These are given in the (D.1) of the Appendix D. This implies that the antisymmetric 4-form $F_{\mu npq}$ vanishes because two of indices are equal from the right hand side of (2.13). So the remaining 4-forms in the left hand side can be read off directly. By rescaling the $A^{ijkl}_\mu$ tensor by $\frac{i\sqrt{2}}{g}$, one obtains the nonzero mixed 4-form with the indices $(\mu, \nu, \rho, m) = (1, 2, 3, 11)$

$$F_{12311}(r, \theta) = -\frac{e^{3A(r)}}{2\sqrt{a(r)}} \left[ a(r)^4 - 1 \right] \sin 2\theta. \quad (3.18)$$

At $SO(8)$ critical point($a(r) = 1$), this vanishes. At the $SO(7)^+$ critical point($a(r) = 5^\frac{1}{4}$), the above $F_{12311}$ (3.18) does not vanish. This implies that the nonsupersymmetric $SO(7)^+$ invariant flow solution does not include the previous $SO(7)^+$ critical point solution where there are no mixed 4-forms according to (3.15). This feature looks different from the ones for supersymmetric flow cases where 11-dimensional flow solutions contain either $SU(3) \times U(1)_R$ critical point or $G_2$ critical point at the IR fixed point. The reason comes from the domain wall solutions in (3.16). Since the right hand side of the first equation does not vanish at the $SO(7)^+$ critical point, this nonzero effect will go into the expression (2.14) and eventually the equation (2.13). However, the supersymmetric cases give the vanishings of derivatives of
supergravity fields with respect to the $r$ at the critical point. The overall factor $[a(r)^4 - 1]$ in (3.18) comes from the tensor (2.14) which has the supergravity derivative with respect to $r$. This derivative is replaced by the first equation of (3.16) and the right hand side of it has this overall factor.

It is not obvious to see the $\theta$-dependence of (3.18) from the (2.13) because we do not see any this dependence from the 7-dimensional metric, the generalized vielbein, the tensor (2.14) or the five-fold product of generalized vielbein. Only after all the summations over the contracted indices are completed, the $\theta$-dependence occurs.

Let us move the 4-forms with internal space indices and the 4-forms with the membrane indices. As we have done for the $SO(7)^+$ critical point case, by using the equation (2.10) with modified tensor (3.12) with (3.13) along the RG flow, the right hand side is summarized in the Appendix D. Therefore, the 4-forms do not change, compared with the $SO(7)^+$ critical point case and they are given in (3.15). It seems that the vanishing of $F_{mnlpq}$ is reasonable but the constant $f = \sqrt{2}m_7$ is not what we want to have because it does not tell us any $r$-dependence on the 4-form. See the results in (D.2) of the Appendix D. In order to generalize the ansatz to the flow solution also as well as the critical points solutions we have described so far, one has to introduce some $(r, \theta)$ dependent factor in the first terms of (2.10). This extra piece can be determined by Einstein equation. Or the other possibility comes from the presence of the inverse of this extra piece as an overall factor in the right hand side of (2.10). An immediate question arises. If we make an replacement by $m_7 \rightarrow \tilde{m}_7(r, \theta)$ on the full linear ansatz of [1], can we see the inverse of extra piece from $A_{ijkl}$ tensor automatically? Maybe the known supersymmetric critical point and flow solutions will help for us to analyze for the above ambiguity on whether the extra structure should appear in the 4-form in the left hand side or the right hand side of (2.10) because we will have further information on the nonzero internal 4-forms and this will provide some implication behind the ansatz (2.10).

One can substitute the 4-form (3.18) into the Einstein equation (3.17). The Ricci component $R^{11}_{11}$ provides the product of $F_{1234}$ and $F_{12311}$. Then one can obtain the 4-form $F_{1234}$ explicitly and it is

$$F_{1234}(r, \theta) = \frac{e^{3A(r)}}{2L a(r)} \left[ -a(r)^6 \cos^2 \theta + a(r)^4 (4 + 3 \cos 2\theta) + 5 \sin^2 \theta \right]$$

$$= \frac{e^{3A(r)}}{2L a(r)} \left[ a(r)^{\frac{1}{2}} \Delta(r, \theta)^{-\frac{3}{4}} (5 - a(r)^4) + 2a(r)^4 \right].$$

(3.19)

Due to the factor $(5 - a(r)^4)$, at the $SO(7)^+$ critical point, this 4-form (3.19) reduces to the one considered in (3.15) because the first two terms vanish and the remaining term becomes $F_{1234} = \frac{\Delta^{\frac{3}{4}}}{L} e^{3A(r)}$ in subsection 3.2. Of course, at $SO(8)$ critical point, this reduces to the
previous $F_{1234} = \frac{3}{L} e^{3A(r)}$ in subsection 3.1. For convenience, we present the warp factor (3.10) here

$$\Delta(r, \theta) = \frac{2^3 a(r)^{\frac{3}{2}}}{[1 + a(r)^4 + (-1 + a(r)^4) \cos 2\theta]^{\frac{3}{4}}}.$$  \hspace{1cm} (3.20)

The 11-dimensional Bianchi identity $\partial_{[M} F_{NPQR]} = 0$ can be checked explicitly. From the particular component of $\partial_{[11} F_{1234]} = 0$, one obtains the relation $\partial_{\theta} F_{1234} = \partial_{r} F_{12311}$ by substituting the two results (3.18) and (3.19). One can easily check that the other components of Bianchi identity give the trivial result. We have an extension of the Freund-Rubin compactification and the 3-form gauge field with 3-dimensional membrane indices looks like $A_{123}(r, \theta) \sim e^{3A(r)} \tilde{W}(r, \theta)$ where $\tilde{W}(r, \theta)$ is so-called geometric superpotential. This quantity is 11-dimensional lift of 4-dimensional superpotential in the sense that the former becomes the latter for particular fixed internal angle. Note the $\theta$-dependence on the geometric superpotential here. One can check that there are no other components of 3-form gauge fields having other directions due to the presence of nonzero 4-forms (3.19) and (3.18).

For the the Maxwell equations in (3.17), one can also check the two 4-forms (3.18) and (3.19) satisfy the second equation of (3.17). In particular, the $(npq), (np11)$ and $(np4)$-components of Maxwell equations are trivially satisfied. In other words, the left hand side and the right hand side are identically zero. For the component $(\nu\rho\sigma)$ of Maxwell equations, the left hand side can be written as

$$\partial_{\nu} \left[ \Delta(r, \theta)^3 \sin^6 \theta e^{-3A(r)} F_{1234}(r, \theta) \right] + \partial_{\theta} \left[ g_{44} g^{1111} \Delta(r, \theta)^3 \sin^6 \theta e^{-3A(r)} F_{12311}(r, \theta) \right].$$ \hspace{1cm} (3.21)

Now one substitutes the explicit expressions for the nonzero 4-forms (3.19) and (3.18) into this expression (3.21) and it leads to vanish via (3.20) and (3.16), together with (3.5) and (3.8). On the other hands, the right hand side of $(\nu\rho\sigma)$ component of Maxwell equations also gives zero because there are no internal 4-form field strengths.

We also checked that the other way to compute (2.10) is given the equation (7.8) of [1] and those computations also give the consistent results. For the computation $e^m_{ij} A^{ijkl}_{n}$ in (2.10), one can use the $SU(8)$ covariant derivative with $SU(8)$ connection which is known. Using $D_n e^m_{ij} = \tilde{D}_n e^m_{ij} + B^k_{n[i} e^m_{j]k}$, one can read off the above $SU(8)$ covariant quantity in different way. The $SU(8)$ connection is related to $B_{mij}^{kl}$ tensor through $B_{mij}^{kl} = \delta_{[i}^{[k} B_{m]}^{j]l}$ which is given in [1] explicitly as follows:

$$B_{mij}^{kl} = \frac{4}{m_7} \hat{K}_{m}^{iJ} (u_{ij}^{JK} u_{1K}^{kl} - v_{ijJK} v^{1k1K}) + \frac{3}{28} \tilde{D}_m \hat{K}^{iJ} (u_{ij}^{[IJ} v^{kl1K]} - v_{ijIJ} u_{1K}^{kl}).$$ \hspace{1cm} (3.22)
From (3.22), one gets the $SU(8)$ connection and computes the covariant derivative acting on the generalized vielbeins. At the $SO(8)$ critical point, the first term of (3.22) survives since $v^{ijKL}$ vanishes.

Other way to check that the 11-dimensional solution by (3.18) and (3.19) is correct is to consider the previous 11-dimensional solution for supersymmetric $G_2$ invariant flow. Since the group $SO(7)$ has its subgroup $G_2$, at least the 11-dimensional solution for $SO(7)^+$ invariant flow preserves the $G_2$ symmetry in the 11-dimensional metric and the 4-forms. As observed previously, the metric has an isometry of $SO(7)$. The coefficient functions appearing in the 4-forms of $G_2$ invariant flow occur in the 4-forms with 4-dimensional indices, 4-forms with three internal indices, and 4-forms with four internal indices. By examining these more closely, one realizes that many of these coefficient functions contain $[a(r)b(r) - 1]$ factor. Therefore, as soon as we impose the $SO(7)^+$ constraint $b(r) = \frac{1}{a(r)}$ into these coefficient functions, they vanish. What remains for the 4-forms is exactly the components of $F_{1234}^{AI}$ and $F_{1235}^{AI}$ in [7, 26] where the 5-th direction is the direction of $\theta$. We observe that they are the same as the ones (3.19) and (3.18) exactly.

For the maximally supersymmetric $SO(8)$ limit on the $SO(7)^+$ invariant flow, the result reproduces the one in subsection 3.1 while the nonsupersymmetric $SO(7)^+$ limit on the same flow does not give the result of subsection 3.2. This is one of the reasons why we analyze the subsection 3.2 separately. As long as the supergravity fields do not vary with respect to the radial coordinate, then we can go to the subsections 3.1 or 3.2. As they vary, this subsection holds along the RG flow which contains the $SO(8)$ critical point.

We explicitly computed the 11-dimensional supersymmetry for the RG flow in the Appendix E. There exists no supersymmetry except the $SO(8)$ critical point which has a maximal supersymmetry by solving (E.6).

4 Conclusions and outlook

Using the de Wit and Nicolai’s formula (2.10), we computed the right hand side explicitly for three cases 1) $SO(8)$ critical point, 2) $SO(7)^+$ critical point and 3) $SO(7)^+$ flow. For our simple Killing vectors and 7-dimensional metric, one should use the modified tensor given by (3.12). For the last case, we should also consider a new formula (2.13) which appears in this paper for the first time.

So far we have considered only some part of membrane flows. The known supersymmetric membrane flows are given by $G_2$ invariant flow and $SU(3) \times U(1)_R$ invariant flow [6, 27]. These are 11-dimensional lifts of 4-dimensional domain wall solutions in [27, 28]. One should also
observe these flows by using the methods in this paper based on (2.10) and (2.13). We expect to have the nonzero 4-form field strengths $F_{mnpq}$ with internal indices. This will provide how the two tensors appearing in the left hand side of (2.10) decompose nontrivially. The main difficulty comes from the equation (2.10). Is there any simple way to compute this efficiently? Maybe it is helpful to use the 8-dimensional description for the internal space given in [3] rather than 7-dimensional description.

Moreover, there should be $SO(7)^-$ invariant flow which contains the solution of [29] and $SU(4)^-$ invariant flow which should contain the solution of [30] similarly. Both of them are nonsupersymmetric. In principle, there will be no problem for the former although it is rather involved. However, for the latter, it is not known how to construct the domain wall solution yet.

Eventually, one needs to understand the 11-dimensional lift of the whole $SU(3)$ invariant flow which cover all of these supersymmetric or nonsupersymmetric flows. The present work will give some hints how to obtain the nontrivial 4-form field strengths. The main input is the 28-beins characterized by four supergravity fields, the construction of Killing vectors and the 11-dimensional metric. It would be an interesting open problem to find out this explicitly.

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Appendix A  The $SO(8)$ critical point

The nonzero components where $m, n, p, q = 1, 2, \cdots, 7$ of the right hand side of (2.10) for $SO(8)$ critical point can be summarized by

\[
\begin{align*}
[12][12] &= -\frac{12}{7} i \sqrt{2} s_6^2 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[13][13] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[14][14] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[15][15] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[16][16] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[17][17] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[23][23] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[24][24] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[25][25] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[26][26] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[27][27] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[34][34] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[35][35] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[36][36] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[37][37] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[45][45] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[46][46] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[47][47] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[55][55] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[56][56] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[57][57] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7, \\
[67][67] &= -\frac{12}{7} i \sqrt{2} s_2^4 s_2^2 s_3^4 s_5^4 s_6^4 m_7. \\
\end{align*}
\]

The antisymmetric notation for $[12][12]$ has the components of 1221, 2112 and 2121 also and the first two are the same as 1212 with minus sign and the last one is the same as 1212.

Appendix B  The $SO(7)^+$ critical point

The nonzero components of the right hand side of (2.10) together with (3.12) for $SO(7)^+$ critical point can be summarized by

\[
\begin{align*}
[12][12] &= -\frac{2i\sqrt{2} \cdot 5^4}{7} s_6^4 s_2^4 s_3^4 s_5^4 s_6^4 m_7, \\
[13][13] &= -\frac{2i\sqrt{2} \cdot 5^4}{7} s_6^4 s_2^4 s_3^4 s_5^4 s_6^4 m_7, \\
[14][14] &= -\frac{2i\sqrt{2} \cdot 5^4}{7} s_6^4 s_2^4 s_3^4 s_5^4 s_6^4 m_7, \\
[15][15] &= -\frac{2i\sqrt{2} \cdot 5^4}{7} s_6^4 s_2^4 s_3^4 s_5^4 s_6^4 m_7, \\
[16][16] &= -\frac{2i\sqrt{2} \cdot 5^4}{7} s_6^4 s_2^4 s_3^4 s_5^4 s_6^4 m_7, \\
[17][17] &= -\frac{2i\sqrt{2} (3 + 2c_{20})}{7 \cdot 5^4} s_6^4 s_2^4 s_3^4 s_5^4 s_6^4 m_7, \\
[23][23] &= -\frac{2i\sqrt{2} \cdot 5^4}{7} s_6^4 s_2^4 s_3^4 s_5^4 s_6^4 m_7, \\
[24][24] &= -\frac{2i\sqrt{2} \cdot 5^4}{7} s_6^4 s_2^4 s_3^4 s_5^4 s_6^4 m_7, \\
\end{align*}
\]
Let us present the generalized vielbeins of the metric has

\[ \xi^i = \begin{pmatrix} \xi_1^i & \xi_2^i & \xi_3^i & \xi_4^i & \xi_5^i & \xi_6^i & \xi_7^i \end{pmatrix} \]

Furthermore, the metric is given by (3.8) where the (7, 7) component of the metric has \( \xi^2 \) dependence.

**Appendix C  The generalized vielbeins**

Let us present the generalized vielbeins \( e_{ij}^m = e^{mij} \) as follows. For \( m = 1 \), there are

\[
\begin{align*}
\epsilon_{12}^1 &= -\frac{2i\sqrt{2} \cdot 5^1}{7} s_\theta^2 s_2 s_5^2 s_6 m_7, & \epsilon_{16}^1 &= -\frac{2i\sqrt{2} \cdot 5^1}{7} s_\theta^4 s_4 s_5^2 s_6 m_7, \\
\epsilon_{13}^1 &= \frac{2i\sqrt{2} \cdot (3 + 2 c_{2g})}{7 \cdot 5^1} s_\theta^2 s_2 s_4 s_5 s_6 m_7, & \epsilon_{17}^1 &= -\frac{2i\sqrt{2} \cdot \xi^2}{7} s_\theta^2 s_4 s_5^2 s_6 m_7, \\
\epsilon_{14}^1 &= \frac{2i\sqrt{2} \cdot (3 + 2 c_{2g})}{7 \cdot 5^1} s_\theta^2 s_2 s_4 s_5 s_6 m_7, & \epsilon_{18}^1 &= \frac{2i\sqrt{2} \cdot (3 + 2 c_{2g})}{7 \cdot 5^1} s_\theta^2 s_4 s_5^2 s_6 m_7, \\
\epsilon_{15}^1 &= \frac{2i\sqrt{2} \cdot (3 + 2 c_{2g})}{7 \cdot 5^1} s_\theta^2 s_2 s_4 s_5 s_6 m_7, & \epsilon_{19}^1 &= \frac{2i\sqrt{2} \cdot (3 + 2 c_{2g})}{7 \cdot 5^1} s_\theta^2 s_4 s_5^2 s_6 m_7, \\
\epsilon_{16}^1 &= \frac{2i\sqrt{2} \cdot (3 + 2 c_{2g})}{7 \cdot 5^1} s_\theta^2 s_2 s_4 s_5 s_6 m_7, & \epsilon_{20}^1 &= \frac{2i\sqrt{2} \cdot (3 + 2 c_{2g})}{7 \cdot 5^1} s_\theta^2 s_4 s_5^2 s_6 m_7, \\
\end{align*}
\]

The expressions having the index 7 contain the quantity \( (3 + 2 \cos 2\theta) \) which is proportional to \( \xi^2 \) from (3.9) and (3.10). Furthermore, the metric is given by (3.8) where the (7, 7) component of the metric has \( \xi^2 \) dependence.
Appendix D  The $SO(7)^+$ invariant flow

Moreover, the nonzero quantities of the right hand side of (2.13) for fixed $\mu = r$ are given by

\[
\begin{align*}
[117] &= -\frac{1024 i a(r)^{\frac{1}{2}} [a(r)^4 - 1]}{[1 + a(r)^4 + (-1 + a(r)^4)c_{20}]} c_\theta s^3 s^2 s^2 s^2 s^2 s^2 m_7, \\
[227] &= -\frac{1024 i a(r)^{\frac{1}{2}} [a(r)^4 - 1]}{[1 + a(r)^4 + (-1 + a(r)^4)c_{26}]} c_\theta s^3 s^2 s^2 s^2 s^2 m_7, \\
[337] &= -\frac{1024 i a(r)^{\frac{1}{2}} [a(r)^4 - 1]}{[1 + a(r)^4 + (-1 + a(r)^4)c_{20}]} c_\theta s^3 s^2 s^2 s^2 s^2 m_7, \\
[447] &= -\frac{1024 i a(r)^{\frac{1}{2}} [a(r)^4 - 1]}{[1 + a(r)^4 + (-1 + a(r)^4)c_{20}]} c_\theta s^3 s^2 s^2 m_7, \\
[557] &= -\frac{1024 i a(r)^{\frac{1}{2}} [a(r)^4 - 1]}{[1 + a(r)^4 + (-1 + a(r)^4)c_{20}]} c_\theta s^3 s^2 m_7, \\
[667] &= -\frac{1024 i a(r)^{\frac{1}{2}} [a(r)^4 - 1]}{[1 + a(r)^4 + (-1 + a(r)^4)c_{20}]} c_\theta s^3 m_7.
\end{align*}
\]

Note that the overall factor $\sin \theta \cos \theta$ appears in these expressions and this plays the role of $\theta$-dependence in the 4-form (3.18).

The nonzero components of the right hand side of (2.10) where the equation (3.12) is used for $SO(7)^+$ invariant flow can be summarized by

\[
\begin{align*}
[12][12] &= -\frac{2i \sqrt{2} \cdot a(r)}{7} s^4 s^2 s^3 s^4 s^4 s^4 m_7, & [13][13] &= -\frac{2i \sqrt{2} \cdot a(r)}{7} s^4 s^2 s^3 s^4 s^5 s^6 m_7,
\end{align*}
\]

For other values for $m$, the generalized vielbeins we do not present (for simplicity) here can be constructed from (2.1), (2.2), (2.3), and (2.4). The indices $i, j$ in the generalized vielbein are antisymmetric.
\[14\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_2^2 s_3^2 s_4^2 s_5^4 s_7 m_7\), \[15\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_2^2 s_3^2 s_5^4 s_6^4 m_7\), 
\[16\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_2^2 s_3^2 s_4^2 s_5^2 s_7 m_7\), 
\[17\] = \(-\frac{2i\sqrt{2} [1 + a(r)^4 + (-1 + a(r)^4)c_{2\theta}]}{7 \cdot a(r)^3} s_4^2 s_2^2 s_3^2 s_4^2 s_5^2 s_6^2 m_7\), 
\[23\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_3^4 s_4^4 s_6^4 m_7\), \[24\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_3^4 s_4^4 s_6^4 m_7\), 
\[25\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_3^2 s_4^2 s_5^2 s_6^4 m_7\), \[26\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_3^2 s_4^2 s_5^2 s_6^4 m_7\), 
\[27\] = \(-\frac{2i\sqrt{2} [1 + a(r)^4 + (-1 + a(r)^4)c_{2\theta}]}{7 \cdot a(r)^3} s_4^2 s_2^2 s_3^2 s_4^2 s_5^2 s_6^2 m_7\), 
\[34\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_3^2 s_4^4 s_5^4 m_7\), 
\[35\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_3^2 s_4^2 s_5^2 s_6^4 m_7\), \[36\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_3^2 s_5^2 s_6^2 m_7\), 
\[37\] = \(-\frac{2i\sqrt{2} [1 + a(r)^4 + (-1 + a(r)^4)c_{2\theta}]}{7 \cdot a(r)^3} s_4^2 s_3^2 s_5^2 s_6^2 m_7\), 
\[45\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_5^4 s_6^4 m_7\), \[46\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_5^2 s_6^2 m_7\), 
\[47\] = \(-\frac{2i\sqrt{2} [1 + a(r)^4 + (-1 + a(r)^4)c_{2\theta}]}{7 \cdot a(r)^3} s_4^2 s_5^2 s_6^2 m_7\), 
\[56\] = \(-\frac{2i\sqrt{2} \cdot a(r)}{7} s_4^2 s_5^2 s_6^2 m_7\), \[57\] = \(-\frac{2i\sqrt{2} [1 + a(r)^4 + (-1 + a(r)^4)c_{2\theta}]}{7 \cdot a(r)^3} s_4^2 s_5^2 s_6^2 m_7\), 
\[67\] = \(-\frac{2i\sqrt{2} [1 + a(r)^4 + (-1 + a(r)^4)c_{2\theta}]}{7 \cdot a(r)^3} s_4^2 s_6^2 m_7\). (D.2)

Of course, at the critical point condition (3.11), the above results (D.2) reduces to the ones in (B.1). For the SO(8) critical point (a(r) = 1), this (3.11) becomes the ones in (A.1) except the overall factor.

**Appendix E  The supersymmetry transformation for 11-dimensional solutions**

The supersymmetry transformation rule [13] of the gravitino of 11-dimensional supergravity becomes in a purely bosonic background

\[\delta \Psi_M = \mathcal{D}_M \epsilon,\]
where \( \epsilon \) is an anticommuting parameter and

\[
\mathcal{D}_M = D_M - \frac{i}{144} \left( \Gamma_M^{NPQR} - 8\delta_M^N \Gamma^{PQR} \right) F_{NPQR}, \quad D_M = \partial_M - \frac{1}{4} \omega_M^{AB} \Gamma_{AB}. \tag{E.1}
\]

Sometimes the numerical factor in front of the 4-forms in (E.1) is different due to the different normalization of 4-forms in the 11-dimensional Einstein-Maxwell equations. The number of supersymmetries preserved by an 11-dimensional background depends on the number of covariantly constant spinors \( \mathcal{D}_M \epsilon = 0 \) called Killing spinors. The Killing spinors satisfy the integrability condition

\[
[\mathcal{D}_M, \mathcal{D}_N] \epsilon = 0. \tag{E.2}
\]

This is a necessary but not sufficient condition for the existence of Killing spinors.

Let us look at this condition (E.2) closely. The condition for zero torsion leads to the fact that the spin connection is given by

\[
\omega_{MAB} = \frac{1}{2} (-\Omega_{MAB} + \Omega_{ABM} - \Omega_{BMA}), \tag{E.3}
\]

where \( \Omega_{MN}^A = -2\partial_M e^A_N \). The first term of (E.3) can be obtained from the vielbein, the \( \Omega_{MN}^A \) and the metric \( \eta_{AB} = \text{diag}(-1, 1, \cdots, 1) = \eta^{AB} \) and the last two terms of (E.3) can be written in terms of the first term as follows:

\[
\Omega_{MAB} = e^N_A \Omega_{MN}^C \eta_{CB}, \quad \Omega_{ABM} = e^N_A \Omega_{NBC} e^C_M, \quad \Omega_{BMA} = e^N_B \Omega_{NCA} e^C_M. \tag{E.4}
\]

Then from (E.4), one gets the final expression for (E.3). In (E.1), the spin connection can be written in terms of (E.3) \( \omega_{MAB}^{\text{frame}} = \omega_{MCD} \eta^{CA} \eta^{DB} \). We want to write (E.1) in the frame basis (tangent space indices) for simplicity of (E.1) (Gamma matrix with world indices are complicated but those with tangent indices are constant) and we need to have the relation \( \omega^{BC}_A = e^M_A \omega^{BC}_M \). Let us decompose (E.1) into the \( \partial_M \) term and the other. By multiplying the vielbein \( e^M_A \), the latter can be written as

\[
\bar{\omega}_A \equiv -\frac{1}{4} \omega_A^{BC} \Gamma_{BC} - \frac{i}{144} \left( \Gamma_A^{BCDE} - 8\delta_A^B \Gamma^{CDE} \right) F_{BCDE}, \tag{E.5}
\]

where as usual, the 4-form in frame basis is related to the one in coordinate basis: \( F_{ABCD} = e^M_A e^P_B e^Q_C e^R_D F_{MNPQ} \). The Gamma matrix is given in [14, 7]. By multiplying \( e^M_A e^N_B \) into (E.2), one gets, by carefully reorganizing it, the following integrability condition (Note 11-dimensional spacetime coordinates are decomposed into as follows: \( z^M = (x^\mu, y^m) \))

\[
\left( e^M_A \frac{\partial \bar{\omega}_B}{\partial z^M} - e^M_A \frac{\partial e^N_B}{\partial z^M} e^C_N \bar{\omega}_C - e^N_B \frac{\partial \bar{\omega}_A}{\partial z^N} + e^M_B \frac{\partial e^N_A}{\partial z^M} e^C_N \bar{\omega}_C + [\bar{\omega}_A, \bar{\omega}_B] \right) \epsilon = 0. \tag{E.6}
\]
Now we are ready to compute (E.6) with (E.5) and vielbein which can be obtained from 11-dimensional metric (3.5) and (3.8). For example, one can compute the 32×32 matrix elements of \([\mathcal{D}_1, \mathcal{D}_5]\) in (E.6). The nonzero expressions of them are summarized by the following matrix elements (1, 2), (1, 18), (3, 4), (3, 20), (6, 5), (6, 21), (8, 7), (8, 23), (9, 26), (10, 9), (11, 28), (12, 11), (13, 14), (14, 29), (15, 16), (16, 31), (18, 1), (18, 17), (20, 3), (20, 19), (21, 6), (21, 22), (23, 8), (23, 24), (25, 26), (26, 9), (27, 28), (28, 11), (29, 14), (30, 29), (31, 16), (32, 31), which have the nonzero value
\[
-\frac{[a(r)^4 - 1][a(r)^4 + 5]}{36 L^2 a(r)^\frac{7}{2} [a(r)^4 c_\theta^2 + s_\theta^2]^{\frac{5}{2}}}.
\]
(E.7)
The transpose elements have same nonzero values except minus sign. After substituting (E.7) into (E.6) for 32 components of \(\epsilon\), the half of them are fixed. Now we move on the matrix elements of \([\mathcal{D}_4, \mathcal{D}_5]\) in (E.6). The nonzero expressions of them are summarized by the following matrix elements (1, 2), (2, 17), (3, 4), (4, 19), (5, 22), (6, 5), (7, 24), (8, 7), (9, 10), (9, 26), (11, 12), (11, 28), (14, 13), (14, 29), (16, 15), (16, 31), (17, 18), (18, 1), (19, 20), (20, 3), (21, 6), (22, 21), (23, 8), (24, 23), (25, 10), (25, 26), (27, 12), (27, 28), (30, 13), (30, 29), (32, 15), (32, 31), which have the nonzero value
\[
-\frac{[a(r)^4 - 1] a(r)^{10} [6a(r)^8 c_\theta^4 - 2a(r)^4 c_\theta^2(-16 + c_{20}) + (-11 + c_{20}s_\theta^2)]}{72 L^2 [a(r)^4 c_\theta^2 + s_\theta^2]^{\frac{5}{2}}}.
\]
(E.8)
Furthermore, the following nonzero matrix elements (1, 23), (2, 24), (3, 21), (4, 22), (5, 3), (6, 4), (7, 1), (8, 2), (13, 11), (13, 27), (14, 12), (14, 28), (15, 9), (15, 25), (16, 10), (16, 26), (17, 7), (18, 8), (19, 5), (20, 6), (21, 19), (22, 20), (23, 17), (24, 18), (29, 11), (29, 27), (30, 12), (30, 28), (31, 9), (31, 25), (32, 10), (32, 26), have the value
\[
-\frac{[a(r)^4 - 1] a(r)^{10} [a(r)^8 c_\theta^4 - a(r)^4 (22 + 3c_{20}) - 5s_\theta^2 s_{20}]] s_{20}}{72 L^2 [a(r)^4 c_\theta^2 + s_\theta^2]^{\frac{5}{2}}}.
\]
(E.9)
After substituting (E.8) and (E.9) into (E.6) for 32 components of \(\epsilon\), the remaining independent parameters are vanishing. Therefore the supersymmetry is completely broken. That is, along the RG flow, there is no supersymmetry except the \(SO(8)\) critical point. As expected, at the \(SO(8)\) critical point \((a(r) = 1)\), one sees that the matrix elements (E.7), (E.8) and (E.9) vanish and moreover, all the other matrix elements we do not present here vanish and therefore, there exists the maximal supersymmetries. At the \(SO(7)^+\) critical point, one can go back the equation (3.15) and check the above supersymmetry condition and this gives the nonsupersymmetric case.
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