1. \( L(s, \text{Ind}_H^G(\phi), \pi) \) for local fields

Due to time pressure (see footnote 1) this essay, which is part of a series consisting of a book [6] and ([7], [8], [9], [10]), has a number of speculative constructions, which I do not have time to provide the rigourous details. The main constructions appear in §1 and §2, there is a more extensive version [11] containing several other sections, being largely for reference.

Let \( G \) be a (usually connected) reductive algebraic group defined over a global field \( F \). Therefore \( F \) is an algebraic number field or a function field in one variable over a finite field.

Often I shall be concerned with the points of \( G \) over some local field given the completion of \( F \) at a non-Archimedean prime of \( F \). In this case, now writing \( F \) for its local completion, suppose that \( K \) is a (usually finite) Galois extension of \( F \) and that \( G \) is a quasi-split group over \( F \) which splits over \( K \).

I am going to use (doing my best to given page and line references for the terminology) the notation and conventions of [2]. That is a rather tall order for the reader, should one exist, but the notation in [2] is very technical and elaborate and is explained at more length than I can manage in the time available\(^1\).

Let \( K/F \) be a Galois extension of non-Archimedean local fields with Galois group \( \text{Gal}(K/F) \).

The main construction of this essay will be in the context of the following result.

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\(^1\)A deteriorating health problem over which I am not in control [4].
Theorem 1.1. ([2])

There is a Chevalley lattice in the Lie algebra of $G$ whose stabiliser $U_K$ is invariant under $\text{Gal}(K/F)$. $U_K$ is self-normalising. Moreover, $G_K = B_K U_K$, $H^1(\text{Gal}(K/F); U_K) = \{1\}$ and $H^1(\text{Gal}(K/F); B_K \cap U_K) = \{1\}$. If we choose two such Chevalley lattices with stabilisers $U_K$ and $U'_K$ then $U_K$ and $U'_K$ are conjugate in $G_K$.

The notation for Theorem 1.1 is given on ([2] pp.29 final paragraph). Suppose that $K/F$ is a (not necessarily unramified for this essay) extension of local fields and $G$ is a quasi-split group over $F$ which splits over $K$. Let $B$ be a Borel subgroup of $G$ and $T$ a Cartan subgroup of $B$ both of which are defined over $F$. Let $v$ be the valuation on $K$. It is a homomorphism from $K^*$ whose kernel is the group of units. $O_K^*$. If $t \in T$ let $v(t) \in \hat{L}$ be defined by $<\lambda, v(t)> = v(\lambda(t))$ for all $\lambda \in L$, the group of rational characters of $T$ ([2] p.22, line -6).

The dual group $\hat{G}$ of $G$ and the Galois action on $\text{Gal}(K/F)$ on it are defined in ([2] p.22 §2 to p.26 line 10) and, as hinted at above, the notation is quite involved but the constructions are straightforward enough. This material enables one to define ([2] p.26 line 10) $\hat{G}_F$, which is the semi-direct product of the Galois group with $G$.

Suppose that $\rho$ is a complex analytic representation of the semi-direct product $\hat{G}_F$ ([2] p.34 line 1) and that $\pi$ is an irreducible unitary representation of $G_F$ on $H$ whose restriction to $U_F$ contains the trivial representation (i.e. $H^{U_F}$ is one-dimensional).

If $C_c(G_F, U_F)$ is the Hecke algebra ([2] p. 30 line -7) of compactly supported functions $f$ such that $f(ug) = f(g) = f(ug)$ for all $u \in U_F, g \in G_F$. There is a representation of $C_c(G_F, U_F)$ on $H$ whose restriction to $U_F$ contains the trivial representation (i.e. $H^{U_F}$ is one-dimensional).

Therefore $\rho(t\sigma_F)$ makes sense, up to conjugate, and Langlands defines the 2-variable $L$-function in this case by the formula ([2] p.34 line 4)

$$L(s, \rho, \pi) = \frac{1}{\det(1 - \rho(t\sigma_F)|\pi_F|^s)}$$

where $\pi_F$ generates the maximal ideal of $O_F$.

Note that the "det" in Langlands definition above is legitimatised in the sense of ([1] §Theorem 3.1). A semi-simple element $a$ in the socle of a Banach algebra has an associated determinant, $\det(1 - a)$ given by the formula of ([1] §Theorem 3.1). The Banach algebra involved can be taken to be any completion containing $\rho(t\sigma_F)$.

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2In [2] $K/F$ is taken to be an unramified extension of local fields, which is sufficient but immaterial.
The same legitimisation, this time taking place in the $n \times n$ matrix ring with entries of the type $\rho(t_{i,j}\sigma_F)$ (see next paragraph) will be required in the explanation of the definition of

$$L(s, \text{Ind}_{H}^F(\phi), \pi)$$

where $H$ is a subgroup of the semi-direct product $\hat{G}_F$ containing the semi-direct product of $\mathbb{Z}(\hat{G}_K)$ with $\text{Gal}(K/F)$, modulo which it is compact open, and $\phi$ is a continuous complex-values character on $H$.

Firstly I believe that the example of (§4) is typical and that $\text{Ind}_{H}^F(\phi)^U$ is finite-dimensional. Choosing a basis $v_1, \ldots, v_n$ gives, by Langlands construction an $n \times n$ “matrix” $M$ of examples $t_{i,j}\sigma_F$ in the semi-direct product $\hat{G}_F$ on which the effect of making different choices is to conjugate the elementwise - by an element of $\hat{G}_F$. Let $\rho^{IF}$ denote the representation of $\hat{G}_F$ given by the subspace of $\rho$ fixed by the semi-direct product of the Galois group with the decomposition group of $i_F$ of $\pi_F$.

Modulo the legitimisation of ”det” my definition if the $L$-function is given by

$$L(s, \text{Ind}_{H}^F(\phi), \pi) = \frac{1}{\text{det}(1-\rho^{IF}(M)|\pi_F|^s)}.$$

I expect this definition to be well-defined, to be bi-multiplicative in the variables $\rho$ and $\pi$ and whose adelic Euler product should enjoy the sort of inductivity properties which are satisfied by the Artin L-function which are recapitulated in §§6-10 of a more extensive version of this essay [11].

2. Expectations of $L(s, \rho, \pi)$ via monomial resolutions

For $\rho$ and $\pi$ admissible representations in the local field situation of §1 suppose that

$$\ldots \rightarrow M_i \rightarrow M_{i-1} \ldots \rightarrow M_0 \rightarrow V \rightarrow 0$$

is a monomial resolution ([6], [7] §§8-10) then we are entitled, from §1, to a 2-variable $L$-function $L(s, M_i, \pi)$ defined as the product of the 2-variable $L$-functions $L(s, \text{Ind}_{H_{i,j}}^F(\phi), \pi)$ such that $M_i = \bigoplus_j \text{Ind}_{H_{i,j}}^F(\phi)$ in the monomial category.

When the representations are complex the monomial resolution is of “finite type” - a consequence of monomial resolutions of finite dimensional representations of finite groups being actually finite [6]. As a consequence I expect this definition of $L(s, \rho, \pi)$ to coincide with the definition of ([2] pp.29-34) and to enjoy all the analytic properties of that example.

In ( [7] §4) the notion of $\mathcal{M}_{\text{cmc}, \phi}(G)$-admissibility is introduced. It is particularly interesting in the di-p-adic situation (where the local field is $p$-adic and the representations are defined over the algebraic closure of $\mathbb{Q}_p$).
In this context the monomial resolution is not necessarily of finite type, as far as I know at the moment, nevertheless I am still optimistic about the following conjecture.

**Conjecture 2.1.**

The multiplicative Euler characteristic of 2-variable $L$-functions $L(s, M_i, \pi)$ defines a well-defined and analytically well-behaved $L$-function $L(s, \rho, \pi)$.

When our Galois extension $K/F$ is an extension of global fields, it is explained in [6] how to use the adelic Tensor Product Theorem ([6] Theorem 1.21) to define adelic monomial resolutions of automorphic representations. In this situation there is a simple reduction to the case in which the finite Galois group is soluble, which is based on the fact that the subgroups involved in the monomial resolution of any finite-dimensional of a finite group over any algebraically closed field of characteristic zero may be taken to be $M$-groups ([5] Proposition 2.1.17 p.30).

Recall that a finite group $G$ is nilpotent if and only if it has a lower central series

$$
\{1\} = Z_0 \triangleleft Z_1 = Z(G) \triangleleft Z_2 \triangleleft \ldots \triangleleft Z_n = G
$$

exists such that $Z_{i+1}/Z_i = Z(G/Z_i)$ for all $i$. In particular nilpotent groups are $M$-groups - each irreducible representation is induced from a 1-dimensional character of a subgroup. Since nilpotent groups are the product of their Sylow $p$-subgroups a subgroup of a nilpotent group is again nilpotent - but not so for $M$-groups.

An $M$-group is soluble [3]. The derived series of $G$ is

$$
G^0 = G, G^1 = [G^0, G^0] = [G, G], \text{the commutator subgroup of } G
$$

and $G^n = [G^{n-1}, G^{n-1}]$ for $n = 1, 2, 3, \ldots$. A soluble group $G$ is one for which $G^n = \{1\}$ for some $n$. Subgroups of soluble groups are soluble since $J < G$ implies $[J, J] < [G, G]$. We begin by writing the trivial one-dimensional representation of the Galois group as a sum of induced monomial representations $\text{Ind}_H^G(\phi)$ with $H$ an $M$-subgroup of $G$. We can inflate this relation to the semi-direct product and multiply the monomial resolution, as a complex of representation, with this formula for 1.

Recall also that the product in $R_+(G)$ is given by a double coset formula ([5] p.68, Exercise 2.5.7)

$$
(K, \phi)^G \cdot (H, \psi)^G = \sum_{w \in K \setminus G/H} (w^{-1}Kw \cap H, w^*(\phi)\psi)^G.
$$

Suppose, abbreviating $\text{Gal}(K/F)$ to $\text{Gal}$, we have a Galois semi-direct product $\text{Gal} \propto G$ with $(K, \phi)^{\text{Gal} \times G} \in R_+(\text{Gal} \propto G)$ and if $(H, \psi)^{\text{Gal}}$ and $\lambda : \text{Gal} \propto G$
\( G \rightarrow \text{Gal} \) is the projection then \((\lambda^{-1}(H), \psi \lambda)_{\text{Gal} \times G}\). Therefore

\[
(K, \phi)_{\text{Gal} \times G} \cdot (\lambda^{-1}(H), \psi \lambda)_{\text{Gal} \times G} = \sum_{w \in K \setminus \text{Gal} \times G/H \times G} (w^{-1}Kw \cap H, w^*(\phi)\psi)^G.
\]

If \( w = (z, g) \in \text{Gal} \times G \) and \((y, g_1) \in H \times G \) then \(((z, g)(y, g_1)) = (zy, gz(g_1))\) so we may take \( w = (z, 1) \in K \bigcap \text{Gal} \setminus \text{Gal}/H \) and \( w^{-1}Kw \bigcap H \propto G \subset H \propto G \).

This procedure reduced the Galois semi-direct products to ones involving solution Galois groups.

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