Projection Theorems using Effective Dimension

Don Stull

INRIA
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Fractal geometry studies irregular sets, which cannot be investigated using the usual tools.

\(^1\)Wikipedia.org
Fractal geometry studies irregular sets, which cannot be investigated using the usual tools.

- von Koch snowflake is small with respect to area (zero area)
- Yet it is large with respect to length (infinite length)

1Wikipedia.org
Fractal Geometry and Fractal Dimensions

Fractal geometry uses various notions of fractal dimension to study the size of irregular sets.
- Hausdorff dimension
- Packing dimension

Fractal dimensions generalize the classical notions of dimension so that sets can have non-integral dimension.

The fractal dimension of a line is 1, the dimension of a plane is 2, etc.

The Hausdorff (and packing) dimension of the von Koch snowflake is $\frac{\ln 4}{\ln 3}$.

Fractal dimensions give a fine grained notion of size of small (in terms of measure) sets.

Fractal geometry has become important in a number of different fields.

Fractal geometry uses techniques from many areas of mathematics.
- Combinatorics, classical geometry, Fourier analysis,...
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Kolmogorov Complexity

Definition

Fix a universal Turing machine $U$. Let $u$ be a finite binary string. The Kolmogorov complexity of $u$ is

$$K(u) = \min\{|\pi| \mid \pi \in \{0, 1\}^*, \text{ and } U(\pi) = u\}.$$
Fix a universal Turing machine $U$. Let $u$ be a finite binary string. The *Kolmogorov complexity of $u$* is

$$K(u) = \min\{|\pi| \mid \pi \in \{0, 1\}^*, \text{ and } U(\pi) = u\}.$$ 

- The choice of universal TM is irrelevant.
- Can be extended to $\mathbb{N}$ and $\mathbb{Q}$ in a natural way.
- Can be relativized to an oracle $A \subseteq \mathbb{N}$, written as $K^A(u)$. 

D. M. Stull (INRIA)
Kolmogorov Complexity in Euclidean Space

Definition

Let \( n, r \in \mathbb{N} \), and \( x \in \mathbb{R}^n \). The \textit{Kolmogorov complexity of} \( x \) \textit{at precision} \( r \) is

\[
K_r(x) = \min\{ K(q) \mid q \in B_{2^{-r}}(x) \cap \mathbb{Q}^n \},
\]

where \( B_{2^{-r}}(x) \) is the ball of radius \( 2^{-r} \) around \( x \).
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where $B_{2^{-r}}(x)$ is the ball of radius $2^{-r}$ around $x$.

For our purposes today, we may define

$$K_r(x) = K(u),$$

where $u = x \upharpoonright r$ is the first $nr$ bits in the binary representation of $x$. 

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Effective Dimensions of Points

Definition (Mayordomo ’03)
Let \( n \in \mathbb{N} \), and \( x \in \mathbb{R}^n \).
The (effective Hausdorff) dimension of \( x \) is
\[
\dim(x) = \lim\inf_{r \to \infty} \frac{K_r(x)}{r}.
\]

Definition (Athreya et al. ’07, Lutz and Mayordomo ’08)
Let \( n \in \mathbb{N} \), and \( x \in \mathbb{R}^n \). The (effective) strong dimension of \( x \) is
\[
\text{Dim}(x) = \lim\sup_{r \to \infty} \frac{K_r(x)}{r}.
\]

The effective dimensions of a point \( x \) measure the density of algorithmic information in \( x \).
The Point-to-Set Principle

**Theorem (J. Lutz and N. Lutz, ’16)**

For every set $E \subseteq \mathbb{R}^n$,

\[
\dim_H(E) = \min \sup_{A \subseteq \mathbb{N}, x \in S} \dim^A(x), \text{ and }
\]

\[
\dim_P(E) = \min \sup_{A \subseteq \mathbb{N}, x \in S} \dim^A(x).
\]

- The Hausdorff dimension of a set is characterized by the dimension of the points in the set.

- Allows us to use computability to answer questions in fractal geometry.
Some Successes

The point to set principle has been successfully applied to several interesting problems in Fractal Geometry.
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- Jack and Neil Lutz reproved Davies’ theorem showing that every Kakeya set in $\mathbb{R}^2$ has Hausdorff dimension 2.
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- Jack and Neil Lutz reproved Davies’ theorem showing that every Kakeya set in $\mathbb{R}^2$ has Hausdorff dimension 2.

- Neil Lutz showed that the intersection bound holds for every subsets $A, B \subseteq \mathbb{R}^n$ holds. For every $A, B$ and almost every point $z$,
  \[ \dim_H(A \cap (B + z)) \leq \max\{0, \dim_H(A \times B) - n\}. \]
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  \[ \dim_H(A \cap (B + z)) \leq \max\{0, \dim_H(A \times B) - n\}. \]

- N. Lutz and S. improved theorem of Molter and Rela on the lower bounds on the Hausdorff dimensions of Furstenberg sets.
If $E$ is big, is it true that $\text{proj}_\theta E$ is big?

A projection is Lipschitz continuous, so $\dim(H(\text{proj}_\theta E)) \leq \min\{\dim(H(E)), 1\}$.

Known that there are sets $E$ such that this inequality is strict.

2 kenneth falconer, sixty years of fractal projections
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- Known that there are sets $E$ such that this inequality is strict.

\[\text{Kenneth Falconer, Sixty years of fractal projections}\]
Theorem (Marstrand '54)

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_H(E) = s$. Then for almost every $
theta \in (0, 2\pi)$,

$$\dim_H(\text{proj}_\theta E) = \min\{s, 1\}.$$
Theorem (Marstrand ’54)

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- Mattila generalized this to arbitrary $n$.
- This theorem is now recognized as a fundamental theorem of fractal geometry.
- Active area of research investigating the projections specific classes of fractal sets.
- Davies has shown that, assuming the Continuum Hypothesis, there are sets for which this theorem does not hold.
Our Results

We use algorithmic information theory to reprove Marstrands theorem, and prove two new results on the fractal dimension of projections.

**Theorem (N. Lutz and S. ’17)**

Let $E \subseteq \mathbb{R}^2$ be any set with $\dim_H(E) = \dim_P(E) = s$. Then for almost every $\theta \in (0, 2\pi)$,

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Let $E \subseteq \mathbb{R}^2$ be any set with $\dim_H(E) = s$. Then for almost every $\theta \in (0, 2\pi)$,

$$\dim_P(\text{proj}_\theta E) \geq \min\{s, 1\}.$$
Overview of Proof

Our goal is to give lower bounds on the fractal dimension of the projection of a set $E$ onto the line at angle $\theta$.

- We will first focus on the effective dimension of projected points.
- We will use the point-to-set principle to connect this to our goal.
Overview of Proof

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- To prove lower bounds of the effective dimension of a point, we will prove lower bounds of the complexity of the point at every precision.
Overview of Proof

Our goal is to give lower bounds on the fractal dimension of the projection of a set $E$ onto the line at angle $\theta$.

- We will first focus on the effective dimension of projected points. We will use the point-to-set principle to connect this to our goal.

- To prove lower bounds of the effective dimension of a point, we will prove lower bounds of the complexity of the point at every precision.

- It will suffice to show that, for sufficiently nice $z \in \mathbb{R}^n$ and angle $\theta$, $K_r^\theta(z) \leq K_r(\text{proj}_\theta(z))$. 

Our goal is to show that we can compute our original point $z$ given the projected point $\text{proj}_\theta(z)$,

$$K_r^\theta(z) \leq K_r(\text{proj}_\theta(z)).$$

How can decide which $z$ is the correct point? There are infinitely many of them.
Suppose that the following conditions are satisfied.

1. The complexity of $z$, $K_r(z)$, is small.

2. For every point $w$ such that $\text{proj}_\theta(z) = \text{proj}_\theta(w)$, either
   - the complexity of $w$, $K_r(w)$, is large, or
   - $w$ is close to $z$.
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Then we can compute (an approximation of) $z$ given (an approximation of) the projected point $\text{proj}_\theta(z)$, with some small number of bits, i.e.

$$K_r^\theta(z) \preccurlyeq K_r(\text{proj}_\theta(z)).$$
Let $z \in \mathbb{R}^2$, $\theta \in (0, 2\pi)$, $A \subseteq \mathbb{N}$, and $r \in \mathbb{N}$. Assume the following are satisfied.

1. For every $t \leq r$, $K_t^z(\theta) \geq t - O(\log(t))$.
2. $K_r^{A,\theta}(z) \geq K_r(z)$.

Then,

$$K_r^{A,\theta}(\text{proj}_\theta(z)) \geq K_r(z).$$

Intuitively, this theorem states that if
- the complexity of $\theta$ is maximal, and
- the oracle $A$ and angle $\theta$ do not affect the complexity of $z$,

then we can ensure that the sufficient conditions of the previous slide are satisfied.
**Theorem (Marstrand '54)**

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_H(E) = s$. Then for almost every $\theta \in (0, 2\pi)$,

$$\dim_H(\text{proj}_\theta E) = \min\{s, 1\}.$$ 

By the point to set principle, it suffices to show that, for almost every $\theta$, for every oracle $A \subseteq \mathbb{N}$, and every $\epsilon > 0$, there is a point $z \in E$ such that

$$\dim^A(\text{proj}_\theta z) \geq \min\{s, 1\} - \epsilon.$$
New Proof of Marstrands Theorem

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- To use the bridging theorem, we want to pick a \( \theta \) which has maximal complexity.
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- To use the bridging theorem, we want to pick a $\theta$ which has maximal complexity.
- Then, for any $A \subseteq \mathbb{N}$ and $\epsilon > 0$, we need to pick a $z$ so that $(A, \theta)$ does not affect the complexity of $z$.
  - This is the tricky part.
Theorem (Hitchcock ’03)

Let $E \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{N}$ be such that $E$ is a $\Sigma^0_2$ set relative to $B$. Then

$$\dim_H(E) = \sup_{x \in E} \dim^B(x).$$

This restricted version allows us to eliminate a quantifier (the choice of oracle).

Standard arguments show that if $E$ is analytic, then there is a subset $F \subseteq E$ such that $\dim_H(F) = \dim_H(E)$, and for some oracle $B$, $F$ is $\Sigma^0_2$ relative to $B$.

For any such $F$, and any $\theta$, $\text{proj}_\theta F$ is $\Sigma^0_2$ relative to $(B, \theta)$. 
Using Restricted Point-to-Set Principle

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- This restricted version allows us to eliminate a quantifier (the choice of oracle).
- Standard arguments show that if \( E \) is analytic, then there is a subset \( F \subseteq E \) such that
  - \( \dim_H(F) = \dim_H(E) \), and
  - For some oracle \( B \), \( F \) is \( \Sigma^0_2 \) relative to \( A \).
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Let $E \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{N}$ be such that $E$ is a $\Sigma^0_2$ set relative to $B$. Then

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- This restricted version allows us to eliminate a quantifier (the choice of oracle).

- Standard arguments show that if $E$ is analytic, then there is a subset $F \subseteq E$ such that
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- For any such $F$, and any $\theta$, $\text{proj}_\theta F$ is $\Sigma^0_2$ relative to $(B, \theta)$. 

Let $F \subseteq E$ as in the previous slide, and $B \subseteq \mathbb{N}$ such that $F$ is $\Sigma^2_0$ relative to $B$. It suffices to show that, for almost every $\theta$ and every $\epsilon > 0$, there is a point $z \in E$ such that

$$\dim^{B,\theta}(\proj_{\theta} z) \geq \min\{s, 1\} - \epsilon.$$
New Proof of Marstrands Theorem

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- **First** pick \( z_1, z_2, \ldots \): Using the point to set principle, choose \( z_n \) such that \( \dim^B(z_n) \geq s - 1/n \).
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- **First** pick $z_1, z_2, \ldots$: Using the point to set principle, choose $z_n$ such that $\dim^B(z_n) \geq s - 1/n$.
- For almost every $\theta$,
  - For every $n$, $\dim^{B,z_n}(\theta) = 1$ (standard argument), and
  - For every $n$ and $r$, $K^{B,\theta}_r(z_n) = K_r^B(z_n)$ (by a theorem of Calude and Zimand).
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- **First** pick $z_1, z_2, \ldots$: Using the point to set principle, choose $z_n$ such that $\dim^B(z_n) \geq s - 1/n$.
- For almost every $\theta$,
  - For every $n$, $\dim^{B,z_n}(\theta) = 1$ (standard argument), and
  - For every $n$ and $r$, $K_{r}^{B,\theta}(z_n) = K_r^B(z_n)$ (by a theorem of Calude and Zimand).
- Then the conditions of our bridging theorem are satisfied for all sufficiently large $r$, and therefore

$$\dim^{B,\theta}(\text{proj}_\theta(z_n)) = \liminf_{r \to \infty} \frac{K_{r}^{B,\theta}(\text{proj}_\theta(z_n))}{r} \geq \liminf_{r \to \infty} \frac{K_r^B(z_n)}{r} \geq \min\{s, 1\} - \epsilon.$$
Theorem (N. Lutz and S. ’17)

Let \( E \subseteq \mathbb{R}^n \) be any set with \( \dim_H(E) = \dim_P(E) = s \). Then for almost every \( \theta \in (0, 2\pi) \),

\[
\dim_H(\text{proj}_\theta E) = \min\{s, 1\}.
\]

By the point to set principle, it suffices to show that, for almost every \( \theta \), for every oracle \( A \subseteq \mathbb{N} \), and \( \epsilon > 0 \), there is a point \( z \in E \) such that

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- To use the bridging theorem, we want to pick a $\theta$ such that $\dim(\theta) = 1$.
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- To use the bridging theorem, we want to pick a $\theta$ such that $\dim(\theta) = 1$.
  - Almost every $\theta$ satisfies this property.
- Then, for any $A \subseteq \mathbb{N}$ and $\epsilon > 0$, we need to pick a $z$ such that $(A, \theta)$ does not affect the complexity of $z$.
  - The assumption that $\dim_H(E) = \dim_P(E)$ allows us to do this without needing the existence of nice subsets of $E$. 
**Theorem (N. Lutz and S. ’17)**

Let \( E \subseteq \mathbb{R}^n \) be any set with \( \dim_H(E) = s \). Then for almost every \( \theta \in (0, 2\pi) \),

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\dim_P(\operatorname{proj}_\theta E) \geq \min\{s, 1\}.
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By the point to set principle, it suffices to show that, for almost every \( \theta \), for every oracle \( A \subseteq \mathbb{N} \), and \( \epsilon > 0 \), there is a point \( z \in E \) such that

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Pick a $\theta$ which has maximal complexity.
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$$\text{Dim}^A(\text{proj}_\theta z) \geq \min\{s, 1\} - \epsilon.$$ 

- Pick a $\theta$ which has maximal complexity.
- Then, for any $A \subseteq \mathbb{N}$ and $\epsilon > 0$, we need to pick a $z$ so that $(A, \theta)$ does not affect the complexity of $z$, $K_r(z)$, for infinitely many $r$.
  - This follows by the point to set principle.
Thank you!