Robust low-rank tensor regression via clipping and Huber loss

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Abstract

In this paper, we construct a parameter estimation framework under robust low-rank tensor regression based on the truncation method and Huber loss, and study robust low-rank tensor regression model under random noise with only finite second-order moment. Through the gradient descent method, our proposed Huber-type robust estimator is theoretically optimal in two aspects: (1) our statistical error rate is nearly the same as the optimal upper bound deduced by the traditional least squares method under sub-Gaussian error; (2) the sample complexity of recovering the tensor parameter is also optimal. Extensive numerical experiments show the robustness of our estimator, and the utilization of truncation and Huber loss is beneficial to improve the stability and statistical effectiveness of the proposed algorithm which is superior to the least squares method. Meanwhile, the phenomenon of phase transition in the convergence rate of the proposed robust estimator is confirmed through simulation. Furthermore, we apply this estimation technique to image compression, which demonstrates that our method is more effective.

Keywords: Tensor regression; Huber loss; Truncation; Heavy tails; Robust estimation

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1 Introduction

In recent years, the tail-robustness issue in parameter regression estimation has become an interesting research highlight. The important reason of gathering a lot of attention is that many well-used statistical methods and algorithms based on the sub-Gaussian assumption suffer from heavy-tailed distribution and the real-world datasets, especially in the financial area, often exhibit heavy tails. Therefore, some effective approaches to adequately estimate regression parameter with the heavy-tailed noise have been proposed by numerous literature. One of the popular ways is to substitute the traditional square loss with some robust loss functions such as absolute loss, quantile loss, Huber loss (Huber (1964)[9]) and Cauchy loss. This type of robust technique was originally aimed at achieving the outlier-robustness. Recently, Fan et al. (2017)[4] first employed the Huber loss into linear regression problem to investigate the robustness against heavy-tailness of the regression error. Their theoretical result unveils that under only finite second order moment condition on the noise, the proposed robust estimator has the same optimal rate as the case of sub-Gaussian tails via carefully tuning the robustification parameter of Huber loss. Further, Sun et al. (2020)[19] revisited this adaptive Huber regression with only bounded \((1 + \epsilon)^{-th}\) moment noise, and found a tight phase transition phenomenon for the convergence rate of the regression parameter. Wang et al. (2021)[22] proposed a data-driven approach for Huber-type robust mean estimation and linear regression such that the robustification parameter can efficiently be tuned. There are also a broad range of other literature using the adaptive Huber loss to study regression problems with heavy-tailed errors. See, for example, Zhou et al (2018)[26] and Luo and Gao et al. (2022)[14] for linear regression, Fan et al. (2019)[6] for large-scale multiple testing, Shen et al. (2022)[18] for matrix recovery and Fan et al. (2022)[7] for neural networks.

Another convenient and effective way to tame heavy-tailed data is to directly clip the samples, which has become widely used. Fan et al. (2021)[5] proposed a shrinkage principle for low-rank matrix recovery with heavy-tailed data. Specifically, they truncated large heavy-tailed responses or covariates and used the clipped data in the least-squares method. Their theoretical results show that, under mild moment constraints, the robust estimator achieves a similar statistical error rate as in the case with sub-Gaussian tails. Subsequently, Zhu and Zhou (2020)[27] adopted this robust methodology in generalized linear models and obtained good theoretical and numerical results. Inspired by Fan et al. (2021)[5], Wang and Tsay (2023)[21] and Liu and Zhang (2021)[12] studied high-dimensional vector autoregression with heavy-tailed time series data. Both of their robust estimation procedures required clipping the data vector.

As more and more datasets appear in the form of tensors, conventional methods for studying regression problems based on vector and matrix-valued data are inadequate. Therefore, significant developments have been made in tensor regression. For example, Lu et al. (2020)[13] proposed
an estimation procedure to estimate the tensor parameter in high-dimensional quantile regression. To further fill the gap in robust estimation under tensor regression models and make progress in overcoming heavy-tailed errors from a statistical perspective, this paper investigates tensor regression models with heavy-tailed errors that have only finite second-order moments. By designing a robust gradient descent algorithm, we obtain a Huber-type robust estimator, which theoretically achieves the same optimal estimation error as in the case with Gaussian noise, as shown in Han et al. (2022)[8]. We also optimize the sample complexity through careful proofs under mild constraints. The magnitudes of the robustification parameters used to control the bias and tail robustness are specified with respect to the dimensionality, rank, sample size, and finite second-order moments. Furthermore, to accommodate asymmetric additive errors, we generalize the Huber loss to a broader class of loss functions. Numerical simulations demonstrate that our estimator outperforms that of Han et al. (2022)[8] under both homogeneous and heteroscedastic models. Finally, we apply our method to image compression. Compared with the original algorithm, the robust Huber loss-based algorithm shows a significant improvement in recovery performance.

The remainder of the paper is organized as follows: Section 2 gives the mathematical notation and definitions used in this paper. Section 3 presents our Huber-type parameter estimator for tensor regression with heavy-tailed errors and gives an upper bound in theory. Section 4 shows numerical experiments and empirical analysis to illustrate the validity of the methods in this paper. Section 5 gives the summary and discussion of this paper. Finally, the proofs of theorems are shown in Appendix A.

2 Notation and definitions

For any positive integer $n$, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$. Uppercase letters denote matrices, while calligraphic letters denote 3-order tensors. For two matrices $X, Y \in \mathbb{R}^{p_1 \times p_2}$ and two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{p_1 \times p_2 \times p_3}$, $\langle X, Y \rangle := \text{tr}(X^T Y)$ and $\langle \mathcal{X}, \mathcal{Y} \rangle := \sum_{i,j,k} \mathcal{X}_{(i,j,k)} \mathcal{Y}_{(i,j,k)}$. The Frobenius norms of $X$ and $\mathcal{X}$ are defined as $\|X\|_F = \sqrt{\sum_{i,j} X_{(i,j)}^2}$ and $\|\mathcal{X}\|_F = \sqrt{\sum_{i,j,k} \mathcal{X}_{(i,j,k)}^2}$, respectively. The nuclear and spectral norm are defined as $\|A\|_* = \text{tr} \left( \sqrt{A^T A} \right)$ and $\|A\|_{\text{op}} = \sqrt{\lambda_{\text{max}}(A^T A)}$. The unit Euclidean sphere on $d$-dimensional space is denoted by $S^{d-1}$. $O^{p \times q} := \{ U \in \mathbb{R}^{p \times q} : U^T U = I_q \}$.

As a higher-order generalization of principal component analysis, the Tucker decomposition of a tensor is presented below:

**Definition 1. (Tensor Tucker decomposition)** For a given tensor $A \in \mathbb{R}^{p_1 \times p_2 \times p_3}$, if there exist a tensor $S \in \mathbb{R}^{r_1 \times r_2 \times r_3}$ and a matrix $U_i \in O^{p_i \times r_i}$ where $r_i < p_i, i \in [3]$ such that $A = S \times_1 U_1 \times_2 U_2 \times_3 U_3$, then $A$ is said to be the rank-$(r_1, r_2, r_3)$ tensor, and $S$ is called the kernel tensor. For convenience, denote $A = S \times_1 U_1 \times_2 U_2 \times_3 U_3 =: [S; U_1, U_2, U_3]$. 


where \((S \times_1 U_1)_{i_1i_2i_3} := \sum_{j=1}^{r_1} S_{j_1j_2j_3} (U_1)_{i_1j}\) \(S \times_2 U_2\) and \(S \times_3 U_3\) for \(U_2 \in \mathbb{R}^{p_2 \times r_2}, U_3 \in \mathbb{R}^{p_3 \times r_3}\) are defined in a similar way.

Figure 1 shows a schematic diagram of the tensor Tucker decomposition.

The matricization operator \(M_k : \mathbb{R}^{p_1 \times p_2 \times p_3} \rightarrow \mathbb{R}^{p_k \times (p_{k+1}p_{k+2})}\) is defined as
\[
[M_k(A)]_{i_{k},i_{k+1}+p_{k+1}(i_{k+2}-1)} = A'_{ki_{k+1}i_{k+2}} \quad \text{for all} \quad A \in \mathbb{R}^{p_1 \times p_2 \times p_3},
\]
where \(k + 1\) and \(k + 2\) are computed with module 3. The low-rank tensor can be Tucker decomposed by the following higher-order singular value decomposition algorithm proposed by Lathauwer et al. (2000) [3].

**Algorithm 1:** High order singular value decomposition (HOSVD)

**Input:** rank-\((r_1, r_2, r_3)\) tensor \(Y \in \mathbb{R}^{p_1 \times p_2 \times p_3}\).

\[U_k = \text{SVD}_{r_k} (M_k(Y)), k = 1, 2, 3\]

\[S = [Y; U_1^\top, U_2^\top, U_3^\top]\]

**Output:** \((S, U_1, U_2, U_3)\).

Let \(\kappa = \bar{\lambda}/\bar{\Lambda}\) where \(\bar{\lambda} := \max \left\{ \|M_k(A^*)\|_{\text{op}} : k \in [3] \right\}\) and \(\bar{\Lambda} := \min \left\{ \sigma_{r_k} (M_k(A^*)) : k \in [3] \right\}\), \(\bar{p} = \max\{p_k : k \in [k]\}\) and \(\bar{r} = \max\{r_k : k \in [k]\}\). For a differentiable function \(f : \mathbb{R}^{p_1 \times p_2 \times p_3} \rightarrow \mathbb{R}\), we write \(\nabla f\) as its gradient function. Given two sequences \(\{a_n\}_{n=1}^{\infty}\) and \(\{b_n\}_{n=1}^{\infty}\), we use the notation \(a_n \simeq b_n\) if \(b_n \lesssim a_n \lesssim b_n\) where \(a_n \lesssim b_n\) means that there exists a positive constant \(C\) such that \(a_n \leq Cb_n\) for all \(n\).

**Definition 2.** (Huber (1964)[9] loss). The Huber loss \(\ell_{\varpi}(\cdot)\) is defined as
\[
\ell_{\varpi}(x) = \begin{cases} 
  x^2/2, & \text{if } |x| \leq \varpi; \\
  \varpi|x| - \varpi^2/2, & \text{if } |x| > \varpi,
\end{cases}
\]
where \(\varpi\) is a robustification parameter which strikes a balance between the induced bias and tail robustness.

From the above definition, we obtain the following truncation function by deriving the Huber loss:

**Definition 3.** (Clipping function). Let \(\psi_\tau(\cdot)\) be a clipping function defined by

\[
\psi_\tau(x) = (|x| \land \tau) \text{sign}(x), \quad x \in \mathbb{R},
\]

where \(\tau\) is a robustification parameter.

Figure 2 shows the image of Huber loss, squared loss and truncation function.

![Figure 2: Illustration of Huber loss, square loss and truncation function.](image)

### 3 Main result

#### 3.1 Model and theoretical results

Suppose \(n\) i.i.d. samples \(\{y_i, X_i\}_{i=1}^n\) are generated from the following tensor regression:

\[
y_i = \langle X_i, \mathcal{A}^* \rangle + \varepsilon_i \quad \text{with} \quad \mathbb{E}(\varepsilon_i \mid X_i) = 0 \quad \text{and} \quad M = \sqrt[3]{\mathbb{E}(\varepsilon_i^2 | X_i)^{\frac{1}{3}} \mathcal{A}_i^k)} < \infty, \quad (1)
\]

where \(\mathcal{A}^* \in \mathbb{R}^{p_1 \times p_2 \times p_3}\) is a rank-\((r_1, r_2, r_3)\) tensor with \(r_k \ll p_k\), to be estimated and \(\{X_i\}_{i=1}^n\) are tensor covariates. Since the noise only possesses mild polynomial moment, the traditional least square method is not efficient at estimating the tensor parameter \(\mathcal{A}^*\) and sensitive to outliers and heavy-tailed data. In order to handle heavy-tailed noises, we propose to utilize Huber loss in
and define the empirical loss function $L_{\pi}(A) = \frac{1}{n} \sum_{i=1}^{n} \ell_{\pi}(y_i - \langle x_i, A \rangle)$. We solve the following optimization problem to obtain a low-rank estimator for $A^*$ under the heavy-tailed setting:

$$
\left( \hat{S}, \hat{U}_1, \hat{U}_2, \hat{U}_3 \right) = \arg\min_{S, U_1, U_2, U_3} \left\{ L_{\pi} \left( \|S; U_1, U_2, U_3\| \right) + \frac{a}{2} \sum_{k=1}^{3} \left\| U_k^T U_k - b^2 I_n \right\|_F^2 \right\},
$$

where $a, b > 0$ are tuning parameters. The regular term $\frac{a}{2} \sum_{k=1}^{3} \left\| U_k^T U_k - b^2 I_n \right\|_F^2$ ensures that in the process of using gradient descent method, $\{U_k\}_{k=1}^{3}$ are nonsingular without changing the minimum point. Before we employ the gradient descent method to get an excellent approximation of $\left[ \hat{S}; \hat{U}_1, \hat{U}_2, \hat{U}_3 \right]$, a reliable initialization is necessary. Han et al. (2022)\cite{8} constructed an unbiased estimator $\frac{1}{n} \sum_{i=1}^{n} y_i x_i$ based on the normality assumption for noise, but our noise distribution possesses heavy-tail. Therefore, the response variable $y_i$ needs to be trimmed via the clipping function defined in Definition 3.

By the above method, we use the following robust gradient descent algorithm to obtain our Huber-type robust estimator.

### Algorithm 2: The Robust Gradient Descent Algorithm

**Input:** $\{y_i, x_i\}_{i=1}^{n}$, robustification parameters $\tau$ and $\omega$, tuning parameters $a$ and $b$, step size $\eta$, the number of iteration $T_{\text{max}}$.

\[
\tilde{A} = \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(y_i) x_i
\]

$\left( \tilde{S}, \tilde{U}_1, \tilde{U}_2, \tilde{U}_3 \right) = \text{HOSVD} \left( \tilde{A} \right)$  (HOSVD in Algorithm 1)

Let $U_k^{(0)} = b U_k$ for $k \in [3]$ and $S^{(0)} = \tilde{S}/b^3$.

**for** $T = 0, 1, 2, \ldots, T_{\text{max}} - 1$ **do**

**for** each $k \in [3]$, we compute the following equation

$$
U_k^{(T+1)} = U_k^{(T)} - \eta \left( \nabla_{U_k} L_{\pi} \left( S^{(T)}, U_1^{(T)}, U_2^{(T)}, U_3^{(T)} \right) \right) + a U_k^{(T)} \left( U_k^{(T)\top} U_k^{(T)} - b^2 I \right).
$$

Then $S^{(T+1)} = S^{(T)} - \eta \nabla_S L_{\pi} \left( S^{(T)}, U_1^{(T)}, U_2^{(T)}, U_3^{(T)} \right)$.

**end**

**Output:** $A^{(T_{\text{max}})} = S^{(T_{\text{max}})} \times_1 U_1^{(T_{\text{max}})} \times_2 U_2^{(T_{\text{max}})} \times_3 U_3^{(T_{\text{max}})}$.

There are two main differences in Algorithm 2 compared with Han et al. (2022)\cite{8} such that it possesses the robustness against heavy-tailness of noise:

1. The initializer $\tilde{A}$ contains a robustification parameter $\tau$ which clips the response $\{y_i\}_{i=1}^{n}$. The clipping technique has been successfully applied by Fan et al. (2021)\cite{5} and Zhu and Zhou (2020)\cite{27}. Some specific benefits of doing so are stated in Remark 1.

2. We adopt Huber loss rather than the ordinary least squares considered by Han et al. (2022)\cite{8}.

The statistical guarantee of $A^{(T_{\text{max}})}$ is presented as follows.
Theorem 1. Assume that the following conditions hold:

(1) Let all entries of $X$ be i.i.d. sampled from sub-Gaussian distribution with mean-zero, variance-one and $\|X(i,k,l)\|_{\psi_2} \leq K < \infty$. Assume that $\{X_i\}_{i=1}^n$ are i. i. d. copies of $X$ and $\{y_i\}_{i=1}^n$ are i.i.d. from $\mathcal{N}$.

(2) Denote $\bar{\lambda} := \max\left\{\|M_k \mathcal{A}^*\|_{op} : k \in [3]\right\}$ and $\lambda := \min \{\sigma_{r_k} (M_k (\mathcal{A}^*)) : k \in [3]\}$, where $\bar{\rho} = \max\{p_k : k \in [3]\}$ and $\tilde{r} = \max\{r_k : k \in [3]\}$. There exist some positive constants $\{c_i\}_{i=1}^3$ such that $\bar{\rho} \geq c_1 \tilde{r}$, $M \leq c_2 \|\mathcal{A}^*\|_2$, $\lambda \geq (\sqrt{\bar{\rho}} \log(\bar{\rho}))^{\frac{1}{2}}$ and $\lambda \geq c_3$.

If $\omega \asymp_{K,k} (Mn/df)^{\frac{1}{2}}$, $\tau \asymp_{K,k} (Rn/df)^{\frac{1}{2}}$, $b \asymp \bar{\lambda}^{1/4}$ and $a \asymp \bar{\lambda}^{-\frac{1}{2}}$ are chosen, then for $\forall t > \log(13)$, there exist positive constants $c_0, c_1, c_2$ and $\{C_i\}_{i=1}^5$ such that as long as $\eta > c_0 \lambda^{-\frac{1}{2}}$,

$$n > C_1 \max \left\{ R^2 \bar{\rho}^2 \tilde{r}, R \kappa^2 \bar{\rho} \sqrt{\tilde{r}}, \kappa^2 \sqrt{\tilde{r}} \right\},$$

$$T_{\max} > C_2 \log \left( \frac{n \lambda}{M \sqrt{df} \kappa} \right),$$

with probability at least $1 - 6 \exp(-c_1 \bar{\rho}) - 4 \exp(df \log(13) - t) - C_3 T_{\max} \exp(-C_4 df) - (T_{\max} + 4) e^{-\bar{\rho}^2 \tilde{r}} - 4 \bar{\rho}^{-c_2}$, we have

$$\left\| \mathcal{A}^{(T_{\max})} - \mathcal{A}^* \right\|_F < C_5 \kappa^2 \sqrt{M \sqrt{df} / n \left( \sqrt{t} + t \right)},$$

where $df := r_1 r_2 r_3 + \sum_{i=1}^3 p_i r_i$, $\kappa = \bar{\lambda} / \lambda$ and $R := \mathbb{E}[y_i^2]$.

Remark 1. From Theorem 1 it follows that under the bounded second-order moments condition, our proposed estimator realizes the minmax optimal rate of convergence established by Han et al. (2022)[8]. On the other hand, the sample complexity $n \geq \bar{\rho}^2 \tilde{r}$ is also optimal if $R$ is smaller than some fixed constant.

Remark 2. In the presence of heavy-tailed noises, the initializer $\tilde{A}$ offers some key virtues in both theoretical and numerical aspects. At first, after appropriate truncation, $\tilde{A}$ has a non-asymptotic upper bound with an exponential-type exception probability. Therefore, it is a good initialization and well reflects the parameter tensor $\mathcal{A}^*$, which can provide a firm foundation for rank-$(r_1, r_2, r_3)$ estimation. Besides, $\tilde{A}$ can shorten steps of iterative procedure $T_{\max}$ and enhance the stability of Algorithm 2.

Remark 3. The constraints on $\tilde{r}$ in condition (2) is easy to be satisfied. For example, if $\bar{\rho} = 10^4$, we only need $\tilde{r} \geq 10$. Note that the scales of $\tau$ and $\omega$ are related to the rank-$(r_1, r_2, r_3)$, but in practice, the rank is often unknown. Therefore, in order to overcome this drawback, the estimate of $\{r_k\}_{k=1}^3$ can be obtained by substituting the robust initial value $\tilde{A}$ into the rank selection method of Han et al. (2022)[8]:

$$\tilde{r}_k = \max \left\{ r : \sigma_r (M_k (\tilde{A})) \geq c \delta_k \right\}, \quad k = 1, 2, 3,$$

where $\delta_k := \text{Median} \left\{ \sigma_1 (M_k (\tilde{A})), \ldots, \sigma_{p_k} (M_k (\tilde{A})) \right\}$ and $c > 0$ are threshold levels.
Remark 4. Without considering the theoretical upper bound on the initial value $\tilde{A}$, from the proof of Theorem 1, we can further obtain that when the error term has only finite $(1 + \delta)$-order moments, i.e., for $\delta > 0$, $M_\delta := \sqrt[2\delta]{\mathbb{E}(|\varepsilon_i|^{1+\delta}|X_i|^k)} < \infty$, if we choose $\omega \approx_{K,k} \left( M_\delta^{1/k} n/df \right)^{\frac{1}{1+\delta}}$, then we have

$$\|A^{(T_{\text{max}})} - A^\dagger\|_F \lesssim \kappa^2 M_\delta \left( \frac{df}{n} \right)^{\frac{1}{1+\delta}}$$

with high probability holds.

It can be seen that the estimator has a smooth phase transition when $0 < \delta < 1$, which is similar to the adaptive Huber linear regression established by Sun et al. (2020) [19]. The simulation experiments in next section confirm the phenomenon.

The following corollary clarifies that as the sample size increases, we do not need the additional constraint to be added to the rank $\tau$.

Corollary 1. Without the constraint condition $\tau \geq (\sqrt{p \log(\bar{p})})^\frac{3}{2}$, the conclusion of Theorem 1 would still hold as long as $\tau \approx_{K,k} \|A^\dagger\|_F (n/df)^{\frac{3}{2}}$ and $n \gtrsim \max \left\{ \kappa^4 \bar{p}^2 \log(\bar{p}), \kappa^4 \bar{p}^2 \bar{r}^3, \kappa^6 \bar{p} \bar{r}^2 \right\}$ and are chosen.

### 3.2 Generalization of the loss function

Since different robust loss functions have different results when dealing with different types of heavy-tailed errors, in this subsection we generalize Huber loss to asymmetric loss functions (Man et al. (2024) [15]) which only need to satisfy the following assumptions:

**Assumption 1.** Let $\ell_\omega(x)$ fulfill the following conditions:

1. $\ell_\omega(x) = \omega^2 \ell_1(x/\omega)$, where $\ell_1 : \mathbb{R} \rightarrow [0, \infty)$;
2. $\ell_1'(0) = 0$, for $\forall x \in \mathbb{R}$, $|\ell_1'(x)| \leq \min (c_1, |x|)$ and $|\ell_1'(x) - x| \leq c_2 x^2$;
3. $\ell_1'(0) = 1$ and for $\forall |x| \leq c_3$, $\ell_1''(x) \geq c_4$, where $\{c_i\}_{i=1}^4$ is a positive constant.

Assumption 1 contains a variety of loss functions such as Huber loss, Tukey’s biweight loss $\ell_\omega(x) = \begin{cases} \frac{\omega^2}{6} \left(1 - \left(1 - \frac{x^2}{\omega^2}\right)^3\right), & \text{if } |x| \leq \omega, \\ \frac{\omega^2}{6}, & \text{if } |x| > \omega \end{cases}$, and Cauchy loss $\ell_\omega(x) = \frac{\omega^2}{2} \log \left(1 + \frac{x^2}{\omega^2}\right)$. Using a robust approach similar to the previous section, define the empirical loss as $L_\omega(A) = \frac{1}{n} \sum_{i=1}^n \ell_\omega(y_i - \langle A_i, \mathbf{A} \rangle)$ and for the estimation of the initial values, we use the derivative function satisfying Assumption 1 to truncate the response variable and obtain $\tilde{A} = \frac{1}{n} \sum_{i=1}^n \ell_\omega'(y_i - \langle A_i, \mathbf{A} \rangle)$. Based on this, the following theorem shows that the estimates produced by Algorithm 2 are still optimal in terms of the statistical error and sample complexity.

**Theorem 2.** Under Assumption 1 and the conditions of Theorem 1, the following conclusion holds: For $\forall t > \log(13)$, with probability at least $1 - 6 \exp(-c\bar{p}) - 4 \exp(df(\log(13) - t)) - C_2 T_{\text{max}} \exp(-C_3 df) - \ldots$
\[(T_{\text{max}} + 4)e^{-\bar{p}^2 r} - 4\bar{p}^{-c}, \text{ we have}\]
\[
\left\| A^{(T_{\text{max}})} - A^* \right\|_F < C_4\kappa^2 \sqrt{Mdf/n} \left(\sqrt{t} + t \right).
\]

4 Simulation analysis and application

4.1 Numerical experiments

In this section, we illustrate the statistical performance of Algorithm 2. Let \( r_1 = r_2 = r_3 = r \) and \( p_1 = p_2 = p_3 = p \) to facilitate simulation. The parameter tensor \( A^* = [S^*; U_1^*, U_2^*, U_3^*] \) is constructed by the following two steps:

1. \( S^* = \lambda S / \min \{ \sigma_r(M_k(S)) : k \in [3] \} \) where each entry of \( S \) is drawn from \( \mathcal{N}(0, 1) \) and \( \lambda \) is a specified constant.

2. For each \( k \in [3] \), \( U_k^* \) is top \( r \) eigenvectors of \( p \)-dimensional sample covariance matrix from 100 i.i.d. standard Gaussian random vectors. Tensor covariates \( \{X_i\}_{i=1}^n \) have i.i.d. entries which are valued from \( \mathcal{N}(0, 1) \).

We compare Algorithm 2 with the algorithm of Han et al. (2022) and abbreviate them as RGD and GD, respectively. The results are shown in Figure 3 and Table 1. The experimental results are based on 200 independent repetitions. From Figure 3 with error bars, two algorithms possess similar statistical effects after 300 iterations, when the random noise follows standard normal distribution. However, when the noise obeys a scaled \( t \) distribution with 2.1 degrees of freedom, RGD performs significantly better than GD. Table 1 presents parameter values in the experiments and shows that the initial value \( \hat{A} \) via using the truncation method outperforms the original estimator in terms of mean and standard deviation. There is a certain probability that the algorithm GD will fail to converge in 200 independently repeated experiments. The results in Figure 3 are calculated after eliminating these non-converging data, which show that our algorithm is more robust.
Figure 3: Comparison of statistical performance between RGD and GD. T and the error represent the number of iterations and $\|A^{(T)} - \hat{A}\|_F$ respectively.

| λ  | η     | (a, b) | (τ, ω)            | $\hat{A}$     | Number of failures | Algorithms |
|----|--------|--------|-------------------|---------------|-------------------|------------|
| 5  | $1 \times 10^{-3}$ | (5, 1) | $(10, 3)\sqrt{\frac{n}{df}}$ | 40.04 (5.18) | 0                 | RGD        |
|    |        |        | $+\infty$         | 48.92 (10.53) | 3                 | GD         |
| 5  | $2 \times 10^{-3}$ | (5, 1) | $(10, 5)\sqrt{\frac{n}{df}}$ | 31.84 (1.93) | 0                 | RGD        |
|    |        |        | $+\infty$         | 55.06 (7.49)  | 14                | GD         |
| 5  | $1 \times 10^{-3}$ | (5, 1) | $(15, 5)\sqrt{\frac{n}{df}}$ | 46.47 (5.65) | 0                 | RGD        |
|    |        |        | $+\infty$         | 64.47 (14.24) | 4                 | GD         |
| 8  | $8 \times 10^{-4}$ | (5, 1) | $(20, 8)\sqrt{\frac{n}{df}}$ | 61.30 (3.94) | 0                 | RGD        |
|    |        |        | $+\infty$         | 78.83 (19.01) |                  | GD         |

Table 1: Parameter values and initial values in Algorithm 2 for Figure 1 and the statistical performance of the initializer $\hat{A}$.
In the above simulations we consider homogeneous model, i.e., the error distribution is independent of the covariance. In the next experiments, we consider evaluating the performance of RGD under the heteroscedastic model:

\[ y_i = \langle X_i, A^* \rangle + 5c^{-1} (\langle X_i, A^* \rangle)^2 \varepsilon_i, \]

where the constant \( c = \sqrt{3} \| A^* \|^2_F \) such that \( \mathbb{E} \left[ c^{-2} (\langle X_i, A^* \rangle)^4 \right] = 1 \). In order to model the various shapes of the error distribution, we consider the following three scenarios:

(a) \( t_{2.1} \) distribution  
(b) Pareto distribution  
(c) LogNormal distribution

The composition of the tensor parameter \( A^* \) is the same as the case of the homogeneous model and considers \( r = 3 \) and \( p = 13 \). We choose \( \tau = 10\sqrt{n/df}, \varpi = 5\sqrt{n/df} \) and \( \eta = 10^{-1} \). Based on 200 independently repeated experiments, Tables 2 and 3 show that our method outperforms the least squares method regardless for asymmetric heavy-tailed error.

| Case | \( n \) | 500 | 1000 | 1500 | 3000 | Method |
|------|------|-----|-----|-----|-----|-------|
| (a)  | 5.6289(1.762) | 2.5250 (0.617) | 1.9901(0.471) | 1.4469(0.126) | RGD |
|      | 10.5183(5.459) | 5.8554(3.227) | 4.3039(1.805) | 2.9340(0.830) | GD |
| (b)  | 4.7553(2.107) | 2.1107(0.767) | 1.4920(0.294) | 1.0559(0.088) | RGD |
|      | 7.8253(5.238) | 4.6653(3.478) | 3.6750(2.737) | 2.2123(1.162) | GD |
| (c)  | 5.3948(1.812) | 2.5135 (0.653) | 1.8884(0.332) | 1.3708(0.127) | RGD |
|      | 9.0962(4.277) | 4.7237(2.025) | 3.6291(1.178) | 2.4001(0.475) | GD |

Table 2: Simulation results under the heteroskedasticity model. (Standard deviations are in parentheses)

| Case | \( n \) | 500 | 1000 | 1500 | 3000 | Method |
|------|------|-----|-----|-----|-----|-------|
| (a)  | 0 | 0 | 0 | 0 | RGD |
|      | 17 | 10 | 12 | 10 | GD |
| (b)  | 0 | 0 | 0 | 0 | RGD |
|      | 11 | 3 | 11 | 8 | GD |
| (c)  | 0 | 0 | 0 | 0 | RGD |
|      | 10 | 10 | 10 | 9 | GD |

Table 3: The number of times for the algorithm fails to converge in Table 2.
4.2 Phase transition phenomenon

According to Remark 4, the resulting estimator has the theoretical rate of order \( C(M_\delta, \kappa) \times \left( \frac{df}{n} \right)^{\min\left\{ \frac{1}{2}, \frac{\nu}{\nu+2} \right\}} \) under Frobenius norm, where \( C(M_\delta, \kappa) \) is a positive constant depending only on \( M_\delta \) and \( \kappa \). This means that when \( \delta < 1 \), the statistical error will increase sharply. To authenticate this phase transition phenomenon, we choose scaled Student’s \( t_\nu \)-distributions with degrees of freedom \( \nu \in \{1.01, 1.1, 1.2, \ldots, 2.9, 3\} \) as the error distribution. Take \( \delta = \nu - 1.01 \) in the truncation and robustification parameters level when \( \nu \leq 2 \), otherwise \( \delta = 1 \). The statistical behavior of the adaptive Huber estimator is shown in Figure 2. For simplicity, the constant \( C(M_\delta, \kappa) \) in the theoretical bound is set as the fixed constant 13.4 for (a) and 13.2 for (b). The empirically fitted curves in Figure 2 match the theoretical curves very well when \( \delta \leq 1 \). While \( \delta > 1 \), the statistical error still decreases gradually with \( \delta \) increasing, which adheres to the theory and intuitive expectation. Specifically, it is general knowledge that \( t_\nu \to N(0,1) \) a.s. as \( \nu \to \infty \) and \( M = \mathbb{E}[t_\nu^2] = \frac{\nu}{\nu-2} \to 1 \). When the order of the moment grows, \( C(M, \kappa) \) will drop and the tail of the noise distribution is lighter. Therefore, the proposed estimator achieves a better statistical performance instinctively than the case of the relatively small \( \delta \).

4.3 Image compression

Since a two-dimensional image can be considered as a matrix, one approach to image compression is to approximate it by singular value decomposition using a low-rank matrix. For example, Recht et al. (2010)\[17\] and Wakin et al. (2006)\[24\]. In this section, we show the application of our algorithm in image restoration, which is used to verify the feasibility and effectiveness of the proposed method.
Firstly, we consider the matrix version of (1), i.e., the matrix compressed sense model:

\[ y_i = \langle X_i, A^* \rangle + \varepsilon_i, \]

where \( X_i \in \mathbb{R}^{d_1 \times d_2} \) are covariate, \( A^* \in \mathbb{R}^{d_1 \times d_2} \) is a low-rank matrix parameter and \( \text{rank}(A^*) = r \).

Write \( A^* \) in the form of singular value decomposition denoted as \( A^* = U^* S^* V^* \top \) and obtain the low-rank estimate of \( A^* \) by solving the following optimization problem:

\[
\left( \hat{S}, \hat{U}, \hat{V} \right) = \arg \min_{S,U,V} \mathcal{L}_\infty \left( USV^\top \right) + \frac{a}{2} \left( \| U^\top U - I_r \|_F^2 + \| V^\top V - I_r \|_F^2 \right).
\]

By the matrix version of Algorithm 2 and Theorem 1, it can be obtained that when \( n \gtrsim R^2 \bar{p} r \), after enough iterations \( T_{\text{max}} \), we have with high probability

\[
\| A^{(T_{\text{max}})} - A^* \|_F \lesssim \kappa^2 \sqrt{M \bar{p} r / n},
\]

where \( R := \mathbb{E} \left[ y_i^2 \right], \kappa := \| A^* \|_{\text{op}} / \sigma_r(A^*), \bar{p} := p_1 \lor p_2 \) and \( M := \sqrt{\mathbb{E} \left( \mathbb{E} (\varepsilon_i^2 \mid X_i)^k \right)} < \infty \). This is the same error upper bound established by Negahban and Wainwright (2011) in sub-Gaussian noise. Next, we use each of three \( 43 \times 53 \) dimensional 0-1 matrices (Figure 1 of Kong et al. (2020)) as the parameter matrices of (1). As showed in the first row of images in Figure 5, we denote them as \( \{ A_i^* \}_{i \in [3]} \) respectively. Each term of the covariates is chosen to i.i.d follow \( \mathcal{N}(0,1) \). For the random error term, we consider \( t \) distribution, 2-fold \( t \) distribution and 4-fold \( t \) distribution with 2.1 degrees of freedom in each of the three experiments.

Table 4 shows the simulation results based on 200 repeated experiments, demonstrating that the proposed robust estimator has better statistical performance than the benchmark in terms of mean and standard deviation. We randomly select a dataset from the 200 experiments and plot the estimators of \( \{ A_i^* \}_{i \in [3]} \). These images are shown in Figure 5 where the images in the second row are the images reconstructed by the estimator proposed in this paper; The third row is the image reconstructed by the original least squares method. Figure 5 illustrates that our robust estimator outperforms the conventional least squares estimation.

Secondly, we confirm the superiority of the algorithm under heavy-tailed noise by combining each channel of the four cigarette package images (Golden Dragon, Yellow Crane Tower, Hatamen and Hongtashan), each of which has the resolution of \( 110 \times 70 \) into a third-order tensor with appropriately scaling as a parameter of the model (i.e., \( p_1 = 110, p_2 = 70, p_3 = 12 \)) (shown in Figure 6).

Each term of the covariates is chosen to be i.i.d. following \( \mathcal{N}(0,1) \), and the random error term \( \varepsilon_i \) follows a 5-fold \( t \)-distribution with 2.1 degrees of freedom. The sample size is \( n = 25000 \). We choose \( r_1 = 60, r_2 = 40, r_3 = 12 \) as the rank of the recovered tensor and the parameters are selected as \( a = 5, b = 1 \) and \( \bar{w} = 100 \sqrt{n/d_f} \). The error \( \| A^{(T_{\text{max}})} - A^* \|_F \) obtained after \( T_{\text{max}} = 500 \) iterations is 15.1514. The parameter pictures are recovered as shown in Figure 7. The results show that
the original image can be compressed and recovered better by Algorithm 2. For the least squares method, the algorithm will not converge.

Figure 5: Images constructed by randomly selecting an experimental result from Table 4.

| Method | $A_1^*$   | $A_2^*$   | $A_3^*$   |
|--------|-----------|-----------|-----------|
| RGD    | 0.6045(0.0295) | 0.9159(0.0127) | 0.6647(0.0073) |
| GD     | 1.1600(0.8077)  | 1.2499(0.1514)  | 0.8682(0.0187)  |
| $n$    | 5000      | 10000     | 10000     |

Table 4: Comparison of the relative estimation errors of two methods.
5 Discussion

This paper proposes a robust parameter estimation framework for low-rank tensor regression model in the context of heavy-tailed errors, and theoretically establishes optimal convergence guarantee for only finite second-order moment noise. We also conduct simulations under both homogeneous and heterogeneous models, demonstrating that the proposed robust estimator outperforms traditional methods in both scenarios. Finally, the method is applied to image compression, yielding significant results. Furthermore, while this study focuses on parameter estimation for third-order tensor regression, our algorithm and results can also be extended to $d$-order low-rank tensor regression models. Specifically,

$$
(\hat{S}, \hat{U}_1, \hat{U}_2, \ldots, \hat{U}_d) = \arg\min_{S, U_1, U_2, \ldots, U_d} \left\{ \mathcal{L}_\omega (\|S; U_1, U_2, \ldots, U_d\|) + \frac{\alpha}{2} \sum_{k=1}^{d} \|U_k^T U_k - b^2 I_r\|_F^2 \right\}.
$$

Under conditions of Theorem 1 by choosing $\omega \asymp_{K,k} (Mn/df)^{\frac{1}{2}}$ and $\tau \asymp_{K,k} (Rn/df)^{\frac{1}{2}}$, after a sufficiently large number of iterations and sample sizes, the robust estimator yielded by Algorithm 2
has the error upper bound of
\[ \| A^{(\text{max})} - A^* \|_F \lesssim \kappa^2 \sqrt{M \text{df}/n} \]
with high probability, where \( \text{df} := \bar{p}^2 + \bar{r}^2 \), \( \bar{p} := \max \{ p_k : k \in [d] \} \) and \( \bar{r} := \max \{ r_k : k \in [d] \} \), and \( \kappa := \max \{ \| \mathcal{M}_k (A^*) \|_{\text{op}} : k \in [d] \} / \min \{ \sigma_{r_k} (\mathcal{M}_k (A^*)) : k \in [d] \} \). On the other hand, our improved algorithm can also be used in the low SNR case (i.e., the case where \( \| A^* \|_F / \mathbb{E} [\varepsilon_i^2] \) is low).

Due to space limitation, we do not continue the simulation demonstration in this paper. There are still many areas in this paper that can be further improved and studied. The robust parameter \( \varpi \) of Huber loss in the paper can be determined through cross-validation, while Wang et al. (2021) [22] proposed a data-driven adaptive equation to determine the magnitude of the robust parameter in high-dimensional sparse linear regression model. Whether a similar adaptive method for solving the robustification parameter can be designed for low-rank tensor regression model is a question for future research. On the other hand, the assumption in Theorem 1 that each covariate term is i.i.d. is difficult to satisfy in practice. Whether this condition can be weakened is a very worthwhile research question.

### A Appendix

#### A.1 Proof of Theorem 1

**Proof.** For the convenience of the proof, we use the following mathematical notation: Since the loss function \( L_{\varpi} (A) = \frac{1}{n} \sum_{i=1}^{n} \ell_{\varpi} (y_i - \langle X_i, A \rangle) \), denote \( X : \mathbb{R}^{p_1 \times p_2 \times p_3} \rightarrow \mathbb{R}^n \) as the linear operator such that \( [X(A)]_i = \langle X_i, A \rangle \) and \( X^* \) are denoted as the adjoint operator of \( X \):
\[
X^*(x) = \frac{1}{n} \sum_{j=1}^{n} x_j X_j, \quad x \in \mathbb{R}^n.
\]

Then equation (1) can be rewritten as \( y = \mathcal{X} (A^*) + \varepsilon \in \mathbb{R}^n \), \( \varepsilon := (\varepsilon_1, \ldots, \varepsilon_n)^\top \) and the gradient of the loss function \( \nabla L_{\varpi} (A) \) has the analytic form \( \mathcal{X}^* (\psi_{\varpi} (y - \mathcal{X}(A))) \). For each iteration \( t = 0, 1, \ldots, T_{\text{max}} \), define the measurement error
\[
E^{(t)} := \min_{R_k \in G_{k \times r_k}} \left\{ \sum_{k=1}^{3} \| U_k^{(t)} - U_k^* R_k \|_F^2 + \| S^{(t)} - [S^*; R_1^T, R_2^T, R_3^T] \|_F^2 \right\}.
\]

With the above notation, we proceed to prove the theorem in the following two steps:

**Step (1):** \( E^{(0)} \approx \frac{1}{\sqrt{\kappa^2 \text{df}}} \);

**Step (2):** The loss function \( L_{\varpi} (A^{(T)}) \) satisfies the constrained strong convexity condition with high probability within the closed set centered at \( A^* \).
where the last inequality is derived from Stirling’s inequality and $z$ where $z$ follows from Proposition 1 of Cai and Zhang (2018) [2]. We first find the $c$ and $\tau$.

By Bernstein’s inequality, we obtain that there exists an absolute constant $\delta$ such that $E\frac{1}{n}\sum_{i=1}^{n}\psi_i(y_i) z_i - E \psi_i(y_i) z_i ≤ \epsilon \sqrt{\frac{k}{n}} τ$. Therefore, since $\sqrt{\frac{1}{n}} \sup_{\epsilon > 0} E[z_i^2] ≤ \epsilon \sqrt{k} K$ for all $k ≥ 1$ and $\|\psi\|_2 = K$, we have for $q ≥ 2$,

$$E |\psi_i(y_i)z_i|^q ≤ q^{q-2}E|y_i^2 z_i^q| ≤ 2 \cdot q^{q-2} \left(E|z_i|^2|z_i^q| + E|\langle \epsilon, \epsilon \rangle|^{q} |z_i^q| \right)$$

$$≤ 2q^{q-2} \left(\epsilon \sqrt{E|z_i|^2}^{k-1} + \epsilon \sqrt{E|\langle \epsilon, \epsilon \rangle|^{q} |z_i^q|} \right)$$

$$≤ 2q^{q-2} \left(M \left(\frac{K^2 q k}{k-1} \right)^{q/2} + 4K^2\|\epsilon\|_F^2 \left(K \sqrt{2q} \right)^q \right) ≤ 2q!M \left(\frac{c k \sqrt{q} k}{k-1} \right)^{q/2} \epsilon,$$

where the last inequality is derived from Stirling’s inequality and $\|\epsilon\|_F^2 ≤ E|\langle \epsilon, \epsilon \rangle|^{q} ≤ E|y_i^2| ≤ R$.

By Bernstein’s inequality, we obtain that there exists an absolute constant $C$ depending on $K, k$ and $c$ such that

$$P \left(\frac{1}{n} \sum_{i=1}^{n}\psi_i(y_i) z_i - E \psi_i(y_i) z_i ≤ \frac{c \tau}{n} \right) ≥ 1 - 2e^{-t}.$$ (3)

For the second term, we have

$$E|\psi_i(y_i)z_i - E[y_i z_i]| = E|\psi_i[|y_i| > \tau] z_i | ≤ \sqrt{E|y_i z_i|^2 P(|y_i| > \tau)}$$

$$≤ \sqrt{E|y_i|^2 E|z_i|^2} / \tau ≤ 2 \sqrt{E|\epsilon|^2 + E|\langle \epsilon, A \rangle|^{2} z_i^2 R^2 / \tau} ≤ K, k, c R / \tau.$$ (4)
Therefore, by combining (3) with (1), there is a constant $C_1$ depending only on $K, k$ and $c$ such that
\[
P \left( \left| \langle \tilde{A} - A^* , T \rangle \right| \leq C_1 \sqrt{\frac{Rt}{n}} + C_1 \frac{\tau t}{n} + C_1 R/\tau \right) \geq 1 - 2e^{-t}.
\]

By following the $1/4$-net argument in Lemma E.5 of Han et al. (2022)[8], we can obtain that with probability at least $1 - 2 \exp(\text{df}(\log(13) - t))$,
\[
\sup_{T \in \mathbb{R}^{p_1 \times p_2 \times p_3}, \|T\|_F \leq 1 \atop \text{rank}(T) \leq (r_1, r_2, r_3)} \left\langle \tilde{A} - A^* , T \right\rangle \leq C \sqrt{\frac{\text{Mdf} \times t}{n}} + C \frac{\text{df} \times t}{n} + C_1 M/\tau
\]
where $C_2$ is an absolute constant. By choosing $\tau \asymp_{K,k} (Mn/\text{df})^{1/2}$ and $t > \log(13)$, the above inequality changes into
\[
P \left( \sup_{T \in \mathbb{R}^{p_1 \times p_2 \times p_3}, \|T\|_F \leq 1 \atop \text{rank}(T) \leq (r_1, r_2, r_3)} \left\langle \tilde{A} - A^* , T \right\rangle \leq C_3 \sqrt{\frac{\text{Rdf} \times t}{n}} \right) \geq 1 - 2 \exp(C_4 \text{df}).
\]

Next, we solve for the upper bound of $T_1$. In the proof of Theorem 4 in Zhang et al. (2019)[25], via applying Lemma 2 and 3, we obtain that there exists a positive constant $c < 1$ such that with probability at least $1 - 2e^{-c\text{p}_1} - 4e^{-t}$, the following inequality holds:
\[
\frac{\sigma_{r_k}(U_k^\top \mathcal{M}_k(\tilde{A}))}{{\sigma_{r_k}^2(U_k^\top \mathcal{M}_k(\tilde{A}))}^{1/2}} \leq (1 - c)\sigma_{r_1}(\mathcal{M}_k(A^*)) + a_1 \sqrt{\frac{p_2p_3}{n}} a_1 \sqrt{\frac{p_1}{n}}
\]
\[
\leq (1 - c)\sigma_{r_1}(\mathcal{M}_k(A^*)) + a_2 \sqrt{\frac{p_2p_3}{n}} \leq a_3 \left( p_2p_3 + C\sqrt{p_1p_2p_3} \right)
\]
\[
\leq (1 - c)\sigma_{r_1}(\mathcal{M}_k(A^*)) + 2\tau^2 \sqrt{\frac{2\text{d} f}{n}} p_2p_3/n - C\frac{E[\sigma_1^2] + \tau^2\sqrt{2\text{d} f}}{n} \sqrt{p_1p_2p_3}
\]
\[
\leq (1 - c)\sigma_{r_1}(\mathcal{M}_k(A^*)) \left( \|A^*\|_F \sqrt{p_1/n} + \sqrt{\frac{2\text{d} f}{n}} \|A^*\|_F \sqrt{p_1/\text{df}} \right) + \|A^*\|_F \sqrt{\frac{p^{3/2}}{n}} + \|A^*\|_F \sqrt{2\text{df}^3/n},
\]
where $a_1 := \sqrt{2\text{d} f} + \tau^2 \sqrt{\frac{2\text{d} f}{n}}, a_2 := \sqrt{E[\phi_+(\eta_k)^2]} - \tau^2 \sqrt{\frac{2\text{d} f}{n}}$ and $a_3 := \frac{E[\phi_+(\eta_k)^2] + \tau^2 \sqrt{2\text{d} f}}{n}$. Therefore, combining (6) with (5), by the union bound, we obtain that when $t = c \log(\bar{p})$, with probability at least $1 - 4\bar{p}^{-c} - 6\exp(\log(\bar{p})) - 2 \exp(C_5 \text{df})$ such that
\[
\left\| A^{(0)} - A^* \right\|_F \leq C_4 \|A^*\|_F \sqrt{\frac{\text{df}}{n}} + C_4 \bar{A} \sqrt{\text{df}} \|A^*\|_F
\]
\[
\times (1 - c)\sigma_{r_1}(\mathcal{M}_k(A^*)) \left( \sqrt{\frac{2\log(\bar{p})}{n}} \sqrt{\bar{p}/\text{df}} \right) + \|A^*\|_F \sqrt{\frac{p^{3/2}}{n}} + \|A^*\|_F \sqrt{\frac{2\log(\bar{p})}{n}}^{3/2}.
\]
Because of the inequalities \((x + y)^2 \leq 2x^2 + 2y^2\) and \(n \geq \max \{ Rn^2 \tilde{p}^3 \sqrt{r}, \kappa^6 \tilde{p}^2 \}\), by Lemma E.2 of Han et al. (2022), we have

\[
E^{(0)} \leq 11\kappa^2 b^{-6} \left\| A^{(0)} - A^* \right\|_F^2 \\
\lesssim \kappa^2 b^{-6} \chi^2 \left( \frac{\|A\|_p^3 p_{11} n}{\sigma^2_{p_1}(M_k(A^*))} + \frac{\|A^*\|_p^3 \bar{p}^3}{\sigma_{p_1}^3(M_k(A^*))} \right) \lesssim \frac{1}{\kappa^3/2}.
\]

Proof of step (2): We generalize Lemma 4 of Sun et al. (2020) to present that the restricted strong convexity condition is satisfied for the Huber loss under the set \(C(R) := \{ A \in \mathbb{R}^{p_1 \times p_2 \times p_3} : \|A - A^*\|_F \leq R, \text{rank}(A - A^*) \leq 2(r_1, r_2, r_3) \}\) with high probability.

**Lemma 1.** Suppose that all entries of \(X_i\) are i.i.d. sampled from sub-Gaussian distribution with mean-zero and variance-one. Then for all \(A \in C(R)\), as long as \(\kappa \geq \max \{ (4M)^2, 4c_1^2 R \}\) and \(n \geq \kappa^2 (2R^3/F + t)\), we have that

\[
P\left( (\nabla L_{\pi}(A) - \nabla L_{\pi}(A^*), A - A^*) \geq \frac{4}{5} \|A - A^*\|_F^2 \right) \geq 1 - e^{-t},
\]

where \(c_1 \geq \sup_{\|\mathcal{V}\|_F \leq 1} (\mathbb{E}(|\mathcal{V}, X_i)|^4)^{1/4}\) is a positive constant.

**Proof.** The difference with the proof of Lemma 4 in Sun et al. (2020) is that we denote

\[
Z_A := \frac{\tau}{2Rn} \sum_{i=1}^n \eta(A, A - A^*) \|A - A^*\|_F,
\]

where \(G_i \sim N(0, 1)\) and are independent with \(X_i\). Because \(G_i X_{i(j,k,l)}\) is a sub-exponential random variable with \(\|G_i X_{i(j,k,l)}\|_{\psi_1} = K\), by Theorem 2.5 of Boucheron, Lugosi and Massart (2013), we have that

\[
\mathbb{E} \left( \left|\frac{1}{n} \sum_{i=1}^n G_i X_{i(j,k,l)} \right|^2 \right) \lesssim \frac{K^2}{n} \quad \text{and} \quad \mathbb{E} \left( \left|\frac{1}{n} \sum_{i=1}^n G_i X_{i(j,k,l)} \right|^4 \right) \lesssim \frac{K^4}{n^2}.
\]

For \(\mathcal{A}_i \in \mathbb{R}^{p_1 \times p_2 \times 3}\), denote \(\overline{\mathcal{M}} \left( \frac{1}{n} \sum_{i=1}^n G_i X_i \right) = \left[ A_{1j} \downarrow A_{2j} \downarrow \cdots \downarrow A_{lj} \right] \in \mathbb{R}^{p_1 \times \sqrt{3p_2}}\), where

\[
\left[ A_{1j} \downarrow A_{2j} \downarrow \cdots \downarrow A_{lj} \right] = \left[ \mathcal{M}_1 \left( \frac{1}{n} \sum_{i=1}^n G_i X_i \right) \downarrow 0_{p_1 \times 2(3p_2 - p_3)} \right],
\]

Then, by Latala (2005), it follows that

\[
\mathbb{E} \left( \left\| \overline{\mathcal{M}} \left( \frac{1}{n} \sum_{i=1}^n G_i X_i \right) \right\|_{op} \right) \lesssim \kappa \sqrt{\frac{p_2 [\sqrt{3p_2}]}{n}} + \sqrt{\frac{p_1 [\sqrt{3p_2}]}{n}} + \sqrt{\frac{p_1 [\sqrt{3p_2}]^2 p_3}{n^2}} \lesssim \kappa \sqrt{\frac{\tilde{p}^3/2}{n}}. \tag{7}
\]
Because of rank \((\mathcal{M}(A - A^*)) \leq 2r_1\), we get
\[
E \left\{ \sup_{A \in \mathcal{C}(R')} \frac{\varpi}{2R} \mathbb{E} \left\{ \sup_{A \in \mathcal{C}(R')} \frac{\mathcal{M} \left( \frac{1}{n} \sum_{i=1}^{n} G_i^* X_i \right) \cdot \mathcal{M}(A - A^*)}{\|A - A^*\|_F} \right\} \right\}
\leq \frac{\varpi}{R} \mathbb{E} \left\{ \sup_{A \in \mathcal{C}(R')} \mathcal{M} \left( \frac{1}{n} \sum_{i=1}^{n} G_i^* X_i \right) \right\} \|\mathcal{M}(A - A^*)\|_{\text{op}} \|A - A^*\|_F
\leq \frac{\varpi}{R} \mathbb{E} \left\{ \mathcal{M} \left( \frac{1}{n} \sum_{i=1}^{n} G_i^* X_i \right) \right\} \sqrt{r_1} \approx \kappa \frac{\varpi}{R} \sqrt{p^{3/2} \frac{\varpi}{n}},
\]
where the first and second inequalities come from \((A, B) \leq \|A\|_{\text{op}} \|B\|_*\), where \(A\) and \(B\) are any two matrices of the same size, and \(\|A\|_* \leq \sqrt{\text{rank}(A)\|A\|_F}\).

Based on Lemma 1 and \(\varpi \approx K_k (M n/d)^{1/2}\), we can get that at \(T\)-th iteration, \(A^{(T)} \in \mathcal{C}(\varpi)\) and (D.12) in the proof of Theorem 4 in Han et al. (2022) is satisfied with high probability.

With the above two steps as a basis, by \(A^{(T)} \in \mathcal{C}(R)\), by Lemma E.6 of Han et al. (2022), we obtain that as long as \(n \geq df\), for any rank-(\(r_1, r_1, r_3\)) tensor,
\[
\left\langle X', \nabla \mathcal{L} (A^{(T)}) - \nabla \mathcal{L} (A^*) \right\rangle = \left\langle \frac{1}{n} X'(X')^*, \psi (X (A^{(T)}) - y) - \psi (X (A^*)) \right\rangle
\leq \frac{1}{n} \|X'(X')\|_2 \|\psi (X (A^{(T)}) - y) - \psi (X (A^*))\|_2
\leq \frac{11}{9\sqrt{n}} \|X'\|_F \left\|X (A^{(T)}) - X (A^*)\right\|_2
\leq \frac{3}{2} \|X'\|_F \left\|X (A^{(T)}) - X (A^*)\right\|_F
\]
with probability at least \(1 - C_{10} \exp(-C_{11} df)\). Finally, by following the lines of the proof of Theorem 4.2 in Han et al. (2022), we obtain that
\[
\|A^{(T+1)} - A^*\|_F^2 \leq C_{12} \left( \kappa^4 \xi^2 + \left( 1 - \frac{\eta_0}{1000\kappa^2} \right)^{T+1} \kappa^2 \|A^{(0)} - A^*\|_F^2 \right),
\]
where \(\xi := \sup_{T \in \mathbb{R}_{p_1 \times p_2 \times p_3, \|T\|_F \leq 1, \text{rank}(T) \leq (r_1, r_2, r_3)}} \left\|\nabla \mathcal{L} (\varepsilon_i, T)\right\|\) and \(\eta_0 < c\), and \(c\) is a small positive constant.

Therefore, as long as \(T_{\text{max}} \geq \log \left( \frac{\|A^{(0)} - A^*\|_F}{\kappa^2 \frac{2^2}{\xi} \eta_0} \right) / \log \left( \frac{2^{2 \xi}}{C_{11} df} \right)\), we have \(\|A^{(T_{\text{max}})} - A^*\|_F \leq \kappa^2 \xi\).

By the proof of Theorem 2, it follows that
\[
P \left( \|A^{(T_{\text{max}})} - A^*\|_F \leq \kappa^2 \left( \frac{Md \varepsilon}{n} \right)^{1/2} \left( \sqrt{t} + t \right) \right) \geq 1 - 2 \exp (df (\log(13) - t)).
\]

A.2 Proof of Theorem 2

**Proof.** Since
\[
\xi := \sup_{T \in \mathbb{R}_{p_1 \times p_2 \times p_3, \|T\|_F \leq 1, \text{rank}(T) \leq (r_1, r_2, r_3)}} \left\|\nabla \mathcal{L} (\varepsilon_i, T)\right\| = \sup_{T \in \mathbb{R}_{p_1 \times p_2 \times p_3, \|T\|_F \leq 1, \text{rank}(T) \leq (r_1, r_2, r_3)}} \left\|\frac{1}{n} \sum_{i=1}^{n} \psi (\varepsilon_i) X_i, T\right\|
\]

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and $\mathbb{E} [\xi_i X_{i(j,k,l)}] = \mathbb{E} [X_{i(j,k,l)} \mathbb{E} [\xi_i | X_{i(j,k,l)}]] = 0$ for $(j, k, l) \in [p_1] \times [p_2] \times [p_3]$, we have

$$\frac{1}{n} \sum_{i=1}^{n} \psi_{\omega}(\xi_i) \sum_{(j,k,l)} X_{i(j,k,l)} T_{(j,k,l)} \leq \left| \mathbb{E} \left[ \psi_{\omega}(\xi_i) z_i \right] - \mathbb{E} [\xi_i z_i] \right| + \frac{1}{n} \sum_{i=1}^{n} \psi_{\omega}(\xi_i) z_i - \mathbb{E} [\psi_{\omega}(\xi_i) z_i].$$

where $z_i := \sum_{(j,k,l)} X_{i(j,k,l)} T_{(j,k,l)}$. For the first term, since $|\ell'(x) - x| \leq x^2$, we have

$$|\mathbb{E} [\psi_{\omega}(\xi_i) z_i] - \mathbb{E} [\xi_i z_i]| \leq \omega \left| \mathbb{E} \{ |\ell'(\xi_i/\omega) - \xi_i/\omega| z_i \} \right| \leq \omega^{-1} \left| \mathbb{E} [\xi_i^2 z_i] \right| \leq \omega^{-1} M K \sqrt{\frac{k}{k-1}}.$$

For the second term, since $|\psi_{\omega}(\xi_i)| = |\omega \ell' \xi_i| \leq \min \{ c_1 \omega, |\xi_i| \}$, then for any $q \geq 2$,

$$\mathbb{E} \left| \psi_{\omega}(\xi_i) z_i \right|^q \leq (c_1 \omega)^{q-2} \mathbb{E} |\xi_i|^{2q} \leq (c_1 \omega)^{q-2} \sqrt{\mathbb{E} \left( \mathbb{E} [\xi_i^2 | X_i] \right)^k} \left( \mathbb{E} |z_i^{2k+1}| \right)^{\frac{k-1}{k}} \leq (c_1 \omega)^{q-2} M \left( \frac{K^2 q \omega^2}{k-1} \right)^{\frac{q}{2}} \leq q! M \left( e K c_1 \omega \sqrt{\frac{2k}{k-1}} \right)^{q-2}.$$

Therefore, by following the proof in [1, 5], we obtain that

$$P \left( \xi \leq C_5 \left( \frac{M d f}{n} \right)^{\frac{1}{2}} \left( \sqrt{t} + t \right) \right) \geq 1 - 2 \exp \left( df \log(13) - t \right).$$

\[ \square \]

**Lemma 2.** Consider $\{\eta_i\}_{i=1}^n$ are i.i.d. with $\mathbb{E}[\eta_i^2] \leq M$ and $\{X_{i}\}_{i=1}^n$ are n i.i.d. random $d_1 \times d_2$ matrices whose entries $X_{i(j,k)}$ are i.i.d. sub-gaussian random variables with $\|X_{i(j,k)}\|_2 = K$. Denote $A = \frac{1}{n} \sum_{i=1}^{n} \psi_{\tau}(\eta_i) X_i$ where $\tau$ is given in Theorem 1. Then there exist positive constants $C$ and $c$ depending only on $K$ such that

$$P \left( \sigma_{\text{max}}^2(A) \leq \frac{\mathbb{E}[\psi_{\tau}(\eta_i)^2] + \tau^2 \sqrt{\frac{2M}{n}} (d_1 + C \sqrt{d_2 + \sqrt{2t}})^2}{n} \right) \geq 1 - 2e^{-ct},$$

$$P \left( \sigma_{\text{min}}^2(A) \geq \frac{\mathbb{E}[\psi_{\tau}(\eta_i)^2] - \tau^2 \sqrt{\frac{2M}{n}} (d_1 - C \sqrt{d_2 - \sqrt{2t}})^2}{n} \right) \geq 1 - 2e^{-ct}.$$

**Proof.** This lemma is proved in a similar manner to the proof of Lemma 6 in Zhang et al. (2020) [25]. Since $\|\frac{1}{n} \sum_{i=1}^{n} \psi_{\tau}(\eta_i) X_{i(j,k)}\|_2 \leq K \sqrt{\sum_{i=1}^{n} \psi_{\tau}(\eta_i)^2 / n}$ for given $\{\psi_{\tau}(\eta_i)\}_{i=1}^n$, by Theorem 5.39 in Vershynin (2010) [20], we obtain that

$$P \left( \sigma_{\text{max}}^2(A) \geq \frac{\sum_{i=1}^{n} \psi_{\tau}(\eta_i)^2}{n^2} (d_1 + C \sqrt{d_2 + \sqrt{2t}})^2 \right) \left\{ \{\psi_{\tau}(\eta_i)\}_{i=1}^n \right\} \leq e^{-ct},$$

$$P \left( \sigma_{\text{min}}^2(A) \leq \frac{\sum_{i=1}^{n} \psi_{\tau}(\eta_i)^2}{n^2} (d_1 - C \sqrt{d_2 - \sqrt{2t}})^2 \right) \left\{ \{\psi_{\tau}(\eta_i)\}_{i=1}^n \right\} \leq e^{-ct}.$$
For $\frac{1}{n} \sum_{i=1}^{n} \psi_\tau(\eta_i)^2$, by Hoeffding’s inequality, we can get that
\[
P \left( \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(\eta_i)^2 - \mathbb{E}[\psi_\tau(\eta_i)^2] \geq \tau^2 \frac{2t}{n} \right) \leq e^{-t}, \quad P \left( \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(\eta_i)^2 - \mathbb{E}[\psi_\tau(\eta_i)^2] \leq -\tau^2 \frac{2t}{n} \right) \leq e^{-t}.
\]
Therefore,
\[
P \left( \sigma_{\text{max}}^2(A) \leq \frac{\mathbb{E}[\psi_\tau(\eta_i)^2]}{n} + \frac{\tau^2 \sqrt{\frac{2t}{n}}}{n} \left( \sqrt{d_1} + C \sqrt{d_2} + \sqrt{2t} \right)^2 \right) \geq P \left( \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(\eta_i)^2 \leq \mathbb{E}[\psi_\tau(\eta_i)^2] + \tau^2 \sqrt{\frac{2t}{n}} \right) \times P \left( \sigma_{\text{max}}^2(A) \leq \frac{\mathbb{E}[\psi_\tau(\eta_i)^2]}{n} + \frac{\tau^2 \sqrt{\frac{2t}{n}}}{n} \left( \sqrt{d_1} + C \sqrt{d_2} + \sqrt{2t} \right)^2 \mid \{\psi_\tau(\eta_i)\}_{i=1}^{n} \right) \geq 1 - 2e^{-ct},
\]
\[
P \left( \sigma_{\text{min}}^2(A) \geq \frac{\mathbb{E}[\psi_\tau(\eta_i)^2]}{n} - \frac{\tau^2 \sqrt{\frac{2t}{n}}}{n} \left( \sqrt{d_1} - C \sqrt{d_2} - \sqrt{2t} \right)^2 \right) \geq P \left( \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(\eta_i)^2 \geq \mathbb{E}[\psi_\tau(\eta_i)^2] - \tau^2 \sqrt{\frac{2t}{n}} \right) \times P \left( \sigma_{\text{min}}^2(A) \geq \frac{\mathbb{E}[\psi_\tau(\eta_i)^2]}{n} - \frac{\tau^2 \sqrt{\frac{2t}{n}}}{n} \left( \sqrt{d_1} - C \sqrt{d_2} - \sqrt{2t} \right)^2 \mid \{\psi_\tau(\eta_i)\}_{i=1}^{n} \right) \leq 1 - 2e^{-ct}.
\]

Lemma 3. Consider $\{X_i\}_{i=1}^{n}$ are n i.i.d. random $d_1 \times d_2$ matrices whose entries are i.i.d. sub-gaussian random variables and $\{y_i\}_{i=1}^{n}$ are from $\mathcal{N}$. Then there exists a constant $C$ such that with probability at least $1 - 2e^{(d_1 + d_2)(\log(t) - t)}$,
\[
\left\| \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(y_i)X_i - \mathbb{E}[y_iX_i] \right\|_{op} \leq C \left( \frac{df}{Mn} \right)^{\frac{1}{2}} \left( \sqrt{(d_1 + d_2)t/df} + (d_1 + d_2)t/df + 1 \right).
\]

Proof. By the triangle inequality,
\[
\left\| \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(y_i)X_i - \mathbb{E}[y_iX_i] \right\|_{op} \leq \left\| \mathbb{E}[\psi_\tau(y_i)X_i] - \mathbb{E}[y_iX_i] \right\|_{op} + \left\| \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(y_i)X_i - \mathbb{E}[\psi_\tau(y_i)X_i] \right\|_{op}.
\]
Denote $z_i := u^\top X_i v$ where $u \in S^{d_1-1}$ and $v \in S^{d_2-1}$. $\frac{1}{n} \sum_{i=1}^{n} \psi_\tau(y_i)u^\top X_i v = \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(y_i)z_i$. Since $z_i$ is sub-Gaussian random variable with $\|z_i\|_{\psi_2} = K$, by using the similar treatment with [4], we obtain that for some constants $C_1$ and $C_2$ depending only on $K, k$, and $c$,
\[
\left\| \mathbb{E}[\psi_\tau(y_i)X_i] - \mathbb{E}[y_iX_i] \right\|_{op} = \sup_{u,v} \mathbb{E}[\psi_\tau(y_i)z_i] - \mathbb{E}[y_i z_i] \leq C_1 \left( M^{1/k} + c^2 \|A^\ast\|^2_k \right) / \tau \leq C_2 \left( M + c^2 \|A^\ast\|^2_k \right) \left( \frac{df}{Mn} \right)^{\frac{1}{2}}.
\]
For the second term, according to [2] and [3], there exists an absolute constant $C_3$ depending on $K, k$ and $c$ such that
\[
P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(y_i)z_i - \mathbb{E}[\psi_\tau(y_i)z_i] \right| \leq C_3 \left( \frac{M}{n} + C_3 \frac{\tau}{n} \right) \right) \geq 1 - 2e^{-t}.
\]
Let $N_{d_1}^{d_1}$ and $N_{d_2}^{d_2}$ be $\frac{1}{3}$-nets of $S_{d_1}^{d_1-1}$ and $S_{d_2}^{d_2-1}$ respectively, where $|N_{d_1}^{d_1}| \leq 7^{d_1}$ and $|N_{d_2}^{d_2}| \leq 7^{d_2}$. There exist $u_1 \in N_{d_1}^{d_1}$ and $v_1 \in N_{d_2}^{d_2}$ such that $\|u - u_1\|_2 \leq 1/3$ and $\|v - v_1\|_2 \leq 1/3$. Thus, for any matrix $A \in \mathbb{R}^{d_1 \times d_2}$, we have

$$u^\top Av = u_1^\top Av_1 + (u - u_1)^\top Av_1 + u_1^\top A(v - v_1) + (u - u_1)^\top A(v - v_1) \leq \sup_{u \in N_{d_1}^{d_1}, v \in N_{d_2}^{d_2}} u^\top Av + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{9}\right) \sup_{u \in S_{d_1}^{d_1-1}, v \in S_{d_2}^{d_2-1}} u^\top Av.$$ 

Thus, $\|A\|_{\text{op}} \leq \frac{9}{2} \sup_{u \in N_{d_1}^{d_1}, v \in N_{d_2}^{d_2}} u^\top Av$. Therefore, the following inequality holds:

$$\frac{1}{n} \sum_{i=1}^{n} \psi_\tau(y_i)X_i - \mathbb{E}[\psi_\tau(y_i)X_i] \leq \max_{u \in N_{d_1}^{d_1}, v \in N_{d_2}^{d_2}} \left| \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(y_i)z_i - \mathbb{E}\psi_\tau(y_i)z_i \right|.$$ 

By using the union bound for all $u \in N_{d_1}^{d_1}$ and $v \in N_{d_2}^{d_2}$ and (6), we have with probability at least $1 - 2e^{(d_1+d_2)(\log(7)-t)}$,

$$\frac{1}{n} \sum_{i=1}^{n} \psi_\tau(y_i)X_i - \mathbb{E}[\psi_\tau(y_i)X_i] \leq C_4 \sqrt{\frac{M(d_1 + d_2)t}{n}} + C_4 \frac{\tau(d_1 + d_2)t}{n} \leq C_5 M \left(\frac{df}{n}\right)^{\frac{1}{2}} \left(\sqrt{\frac{(d_1 + d_2)t}{df}} + \frac{d_1 + d_2}{df}\right).$$

Therefore, the conclusion holds by combining (8) with (6). 

\[\square\]

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