Lp BOUNDS FOR A MAXIMAL DYADIC SUM OPERATOR

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Abstract. The authors prove Lp bounds in the range 1 < p < ∞ for a maximal dyadic sum operator on \( \mathbb{R}^n \). This maximal operator provides a discrete multidimensional model of Carleson’s operator. Its boundedness is obtained by a simple twist of the proof of Carleson’s theorem given by Lacey and Thiele adapted in higher dimensions. In dimension one, the Lp boundedness of this maximal dyadic sum implies in particular an alternative proof of Hunt’s extension of Carleson’s theorem on almost everywhere convergence of Fourier integrals.

1. The Carleson-Hunt theorem

A celebrated theorem of Carleson states that the Fourier series of a square-integrable function on the circle converges almost everywhere to the function. Hunt extended this theorem to Lp functions for 1 < p < ∞. Alternative proofs of Carleson’s theorem were provided by C. Fefferman and by Lacey and Thiele. The last authors proved the theorem on the line, i.e. they showed that for \( f \) in \( L^2(\mathbb{R}) \) the sequence of functions

\[ S_N(f)(x) = \int_{|\xi| \leq N} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \]

converges to \( f(x) \) for almost all \( x \in \mathbb{R} \) as \( N \to \infty \). This result was obtained as a consequence of the boundedness of the maximal operator

\[ C(f) = \sup_{N > 0} |S_N(f)| \]

from \( L^2(\mathbb{R}) \) into \( L^{2,\infty}(\mathbb{R}) \). In view of the transference theorem of Kenig and Tomas the above result is equivalent to the analogous theorem for Fourier series on the circle. Lacey and Thiele have also obtained a proof of Hunt’s theorem by adapting the techniques in to the \( L^p \) case but this proof is rather complicated compared with the relatively short and elegant proof they gave for \( p = 2 \).

Investigating higher dimensional analogues, Pramanik and Terwilleger recently adapted the proof of Carleson’s theorem by Lacey and Thiele to prove weak type \((2,2)\) bounds for a discrete maximal operator on \( \mathbb{R}^n \) similar to the one which arises in the aforementioned proof. After a certain averaging procedure, this result provides an alternative proof of Sjölin’s theorem on the weak \( L^2 \) boundedness of

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maximally modulated Calderón-Zygmund operators on $\mathbf{R}^n$. The purpose of this note is to extend the result of Pramanik and Terwilleger \cite{8} to the range $1 < p < \infty$ via a variation of the $L^2 \to L^{2,\infty}$ case. Particularly in dimension 1, the theorem below yields a new proof of Hunt’s theorem (i.e. the $L^p$ boundedness of $\mathcal{C}$ for $1 < p < \infty$) using a variation of the proof of Lacey and Thiele \cite{6}.

2. Reduction to two estimates

We use the notation introduced in \cite{8} and expanded in \cite{8}. A tile in $\mathbf{R}^n \times \mathbf{R}^n$ is a product of dyadic cubes of the form

$$\prod_{j=1}^n I_j \equiv \prod_{j=1}^n [m_j 2^k, (m_j + 1)2^k),$$

where $k$ and $m_j$ are integers for all $j = 1, 2, \ldots, n$. We denote a tile by $s = I_s \times \omega_s$, where $|I_s| |\omega_s| = 1$. The cube $I_s$ will be called the time projection of $s$ and $\omega_s$ the frequency projection of $s$. For a tile $s$ with $\omega_s = \omega_s^1 \times \omega_s^2 \times \ldots \times \omega_s^n$, we can divide each dyadic interval $\omega_s^j$ into two intervals of the form

$$\omega_s^j = (\omega_s^j \cap (-\infty, c(\omega_s^j)) \cup (\omega_s^j \cap [c(\omega_s^j), \infty))$$

for $j = 1, 2, \ldots, n$. Then $\omega_s$ can be decomposed into $2^n$ subcubes formed from all combinations of cross products of these half intervals. We number these subcubes using the lexicographical order on the centers and denote the subcubes by $\omega_s(i)$ for $i = 1, 2, \ldots, 2^n$. A tile $s$ is then the union of $2^n$ semi-tiles given by $s(i) = I_s \times \omega_s(i)$ for $i = 1, 2, \ldots, 2^n$.

We let $\phi$ be a Schwartz function such that $\hat{\phi}$ is real, nonnegative, and supported in the cube $[-1/10, 1/10]^n$. Define

$$\phi_s(x) = |I_s|^{-\frac{1}{n}} \phi \left( \frac{x - c(I_s)}{|I_s|^\frac{1}{n}} \right) e^{2\pi i c(\omega_s(1)) \cdot x},$$

where $c(J)$ is the center of a cube $J$. As in \cite{8} and \cite{8}, we will consider the dyadic sum operator

$$\mathcal{D}_r(f) = \sum_{s \in D} \langle f, \phi_s \rangle (\chi_{\omega_s(r)} \circ N) \phi_s,$$

where $2 \leq r \leq 2^n$ is a fixed integer, $N : \mathbf{R}^n \to \mathbf{R}^n$ is a fixed measurable function, $D$ is a set of tiles, and $\langle f, g \rangle$ is the complex inner product $\int_{\mathbf{R}^n} f(x) \bar{g}(x) \, dx$.

The following theorem is the main result of this article.

**Theorem 1.** Let $1 < p < \infty$. Then there is a constant $C_{n,p}$ independent of the measurable function $N$, of the set $D$, and of $r$ such that for all $f \in L^p(\mathbf{R}^n)$ we have

$$\|\mathcal{D}_r(f)\|_{L^p(\mathbf{R}^n)} \leq C_{n,p} \|f\|_{L^p(\mathbf{R}^n)}.$$  \tag{1}

In one-dimension, using the averaging procedure introduced in \cite{8}, it follows that the norm estimate \cite{8} implies

$$\|\mathcal{C}(f)\|_{L^p} \leq C_p \|f\|_{L^p},$$
which is the Carleson-Hunt theorem. Using the Marcinkiewicz interpolation theorem \[11\] and the restricted weak type reduction of Stein and Weiss \[12\], estimate (1) will be a consequence of the restricted weak type estimate

\[ \|D_r(\chi_F)\|_{L^p(\mathbb{R}^n)} \leq C_{n,p}|F|^{\frac{1}{p}}, \quad 1 < p < \infty \]

which is supposed to hold for all n-dimensional sets \( F \) of finite measure. But to show that a function \( g \) lies in \( L^{p,\infty} \), it suffices to show that for every measurable set \( E \) of finite measure, there is a subset \( E' \) of \( E \) which satisfies \( |E'| \geq \frac{1}{2}|E| \) and also

\[ \left| \int_{E'} g(x) \, dx \right| \leq A |E|^{\frac{p-1}{p}}; \]

this implies that \( \|g\|_{L^{p,\infty}(\mathbb{R}^n)} \leq c_p A \), where \( c_p \) is a constant that depends only on \( p \).

Let \( C(n,q) \) be the weak type \((q,q)\) operator norm for the Hardy-Littlewood maximal operator. Given a set \( E \) of finite measure we set

\[ \Omega = \left\{ M(\chi_F) > \left( \frac{|F|}{|E|} \right)^{\frac{1}{q}} C(n,q) \right\}, \]

where we choose \( q \) so that \( p < q \leq \infty \) if \( |F| > |E| \) and \( 1 \leq q < p \) if \( |F| \leq |E| \). Note that in the first case the set \( \Omega \) is empty. Using the \( L^q \) to \( L^{q,\infty} \) boundedness of the Hardy-Littlewood maximal operator, we have \( |\Omega| \leq \frac{1}{2}|E| \) and hence \( |E'| \geq \frac{1}{2}|E| \).

Thus estimate (2) will follow from

\[ \left| \int_{E'} D_r(\chi_F)(x) \, dx \right| \leq C_{n,p}|E|^{\frac{p-1}{p}} |F|^{\frac{1}{p}}, \]

where \( C_{n,p} \) depends only on \( p \) and dimension \( n \). The required estimate (3) will then be a consequence of the following two estimates:

\[ \left| \int_{E'} \sum_{s \in D, I_s \subseteq \Omega} \langle \chi_F, \phi_s \rangle (\chi_{\omega_s(r)} \circ N) \phi_s(x) \, dx \right| \leq C_{n,p,q}|E|^{\frac{p-1}{p}} |F|^{\frac{1}{p}}, \]

and

\[ \sum_{s \in D, I_s \not\subseteq \Omega} |\langle \chi_F, \phi_s \rangle | |\langle \chi_{E' \cap N^{-1} \omega_s(r)}, \phi_s \rangle| \leq C_{n,p,q}|E|^{\frac{p-1}{p}} |F|^{\frac{1}{p}}. \]

3. The proof of estimate (4)

Following \[11\], we denote by \( I(D) \) the dyadic grid which consists of all the time projections of tiles in \( D \). For each dyadic cube \( J \) in \( I(D) \) we define

\[ D_J := \{ s \in D : I_s = J \} \]

and a function

\[ \psi_J(x) := \left| J \right|^{-\frac{1}{2}} \left( 1 + \frac{|x - c(J)|}{\left| J \right|^{\frac{1}{n}}} \right)^{-\gamma}, \]

where \( \gamma \) is a large integer to be chosen shortly. For each \( k = 0, 1, 2, \ldots \) we introduce families

\[ F_k = \{ J \in I(D) : 2^k J \subseteq \Omega, 2^{k+1} J \notin \Omega \}. \]
We may assume $|F| \leq |E|$, otherwise the set $\Omega$ is empty and (4) is trivial.

We begin by controlling the left hand side of (4) by

$$
\sum_{J \in I(D)} \left| \sum_{s \in D(J)} \int_{E'} \langle \chi_F | \phi_s \rangle \chi_{\omega_s}(x) (N(x)) \phi_s(x) \, dx \right|
$$

(6)

$$
\leq \sum_{k=0}^{\infty} \sum_{J \in I(D)} \sum_{J' \in \mathcal{F}_k} \int_{E'} \sum_{s \in D(J')} \langle \chi_F | \phi_s \rangle \chi_{\omega_s(r)}(N(x)) \phi_s(x) \, dx
$$

Using the fact that the function $M(\chi_F)^{\frac{1}{2}}$ is an $A_1$ weight with $A_1$-constant bounded above by a quantity independent of $F$, it is easy to find a constant $C_0 < \infty$ such that for each $k = 0, 1, \ldots$ and $J \in \mathcal{F}_k$ we have

$$
\langle \chi_F, \psi_J \rangle \leq |J|^\frac{1}{2} \inf_J M(\chi_F) \leq |J|^\frac{1}{2} C_0^k \inf_{2^{k+1}J} M(\chi_F) \leq C(n, q)2^{\frac{k}{2}} C_0^k |J|^{\frac{1}{2}} \left( \frac{|F|}{|E|} \right)^{\frac{1}{2}}
$$

(7)

since $2^{k+1}J$ meets the complement of $\Omega$. For $J \in \mathcal{F}_k$ one also has that $E' \cap 2^k J = \emptyset$ and hence

$$
\int_{E'} \psi_J(y) \, dy \leq \int_{(2^k J)^c} \psi_J(y) \, dy \leq |J|^\frac{1}{2} C_\gamma 2^{-k \gamma}.
$$

(8)

Next we note that for each $J \in I(D)$ and $x \in \mathbb{R}^n$ there is at most one $s = s_x \in D_J$ such that $N(x) \in \omega_{s_x(r)}$. Using this observation along with (7) and (8) we can therefore estimate the expression on the right in (4) as follows

$$
\leq \sum_{k=0}^{\infty} \sum_{J \in I(D)} \sum_{J' \in \mathcal{F}_k} \int_{E'} \langle \chi_F | \phi_s \rangle \chi_{\omega_s(r)}(N(x)) \phi_s(x) \, dx
$$

$$
\leq C \sum_{k=0}^{\infty} \sum_{J \in I(D)} \sum_{J' \in \mathcal{F}_k} \int_{E'} \langle \chi_F, \psi_J \rangle \psi_J(x) \, dx
$$

$$
\leq C \left( \frac{|F|}{|E|} \right)^{\frac{1}{2}} \sum_{k=0}^{\infty} C_0^k \sum_{J' \in \mathcal{F}_k} |J|^{\frac{1}{2}} \int_{E'} \psi_J(x) \, dx
$$

(9)

$$
\leq C \left( \frac{|F|}{|E|} \right)^{\frac{1}{2}} \sum_{k=0}^{\infty} (C_0 2^{-\gamma})^k \sum_{J \in \mathcal{F}_k} |J|
$$

and at this point we pick $\gamma$ so that $C_0 2^{-\gamma} < 1$. It remains to control $\sum_{J \in \mathcal{F}_k} |J|$ for each nonnegative integer $k$. In doing this we let $\mathcal{F}_k^*$ be all elements of $\mathcal{F}_k$ which are maximal under inclusion. Then we observe that if $J \in \mathcal{F}_k^*$ and $J' \in \mathcal{F}_k$ satisfy $J' \subseteq J$ then $\text{dist} (J', J^c) = 0$ (otherwise $2J'$ would be contained in $J$ and thus $2^{k+1}J' \subseteq 2^k J \subseteq \Omega$.) But for any fixed $J$ in $\mathcal{F}_k^*$ and any scale $m$, all the cubes $J'$ in $J' \in \mathcal{F}_k$ of sidelength $2^m$ that touch $J$ are concentrated near the boundary of $J$ and have total measure at most $2^m \cdot 2^n (|J|^{\frac{1}{2}})^{n-1}$. Summing over all integers $m$ with
$2^m \leq |J|^\frac{1}{d}$, we obtain a bound which is at most a multiple of $|J|$. We conclude that

$$\sum_{J \in \mathcal{F}_k} |J| = \sum_{J \in \mathcal{F}^*_L} \sum_{J' \in \mathcal{F}_k \cap J} |J'| \leq \sum_{J \in \mathcal{F}_k^*} c_n |J| \leq c_n |\Omega|$$

since elements of $\mathcal{F}_k^*$ are disjoint and contained in $\Omega$. Inserting this estimate in (3) and using that the Hardy-Littlewood maximal operator is of weak type $(1,1)$, we obtain the required bound

$$C( \frac{|F|}{|E|} )^{\frac{1}{q}} |\Omega| \leq C' |F| \leq C' |E|^{\frac{q-1}{p}} |F|^{\frac{1}{q}}$$

for the expression on the right in (3) and hence for the expression on the left in (4).

4. The proof of estimate (3)

In proving estimate (3) we may assume that $\frac{1}{2} \leq |E| \leq 1$ by a simple scaling argument. (The scaling changes the sets $D$, $\Omega$, and the measurable function $N$ but note that the final constants are independent of these quantities.) In addition all constants in the sequel are allowed to depend on $n$ and $p$ as described above. We may also assume that the set $D$ is finite. Note that under the normalization of the set $E$, our choice of $q$ is as follows: $1 \leq q < p$ if $|F| \leq c_0$ and $p < q \leq \infty$ when $|F| > c_0$ where $c_0$ is a fixed number in the interval $(\frac{1}{2}, 1)$, (in fact $c_0 = |E|$).

We recall that a finite set of tiles $T$ is called a tree if there exists a tile $t \in T$ such that all $s \in T$ satisfy $s < t$ (which means $I_s \subset I_t$ and $\omega_t \subset \omega_s$.) In this case we call $t$ the top of $T$ and we denote it by $\top(T)$. A tree $T$ is called an $r$-tree if

$$\omega_{t(T)(r)} \subset \omega_{s(r)}$$

for all $s \in T$. For a finite set of tiles $Q$ we define the energy of a nonzero function $f$ with respect to $Q$ by

$$E(f; Q) = \frac{1}{\|f\|_{L^2(\mathbb{R}^n)}} \sup_T \left( \frac{1}{|I_t(T)|} \sum_{s \in T} |\langle f, \phi_s \rangle|^2 \right)^{\frac{1}{2}},$$

where the supremum is taken over all $r$-trees $T$ contained in $Q$. We also define the mass of a set of tiles $Q$ by

$$M(Q) = \sup_{s \in Q} \sup_{u \in Q} \int_{E \cap \mathbb{R}^n \setminus \omega_{u|s}} \frac{|I_u|^{-1}}{\left(1 + \frac{|x - c(I_u)|}{|I_u|^1/2} \right)^{n}} dx.$$
Lemma 1. There is a constant $C$ such that for all measurable sets $F$ and all finite set of tiles $P$ which satisfy $I_s \notin \Omega$ for all $s \in P$, we have

$$\mathcal{E}(\chi_F; P) \leq C|F|^{\frac{1}{q} - \frac{1}{2}}$$

Note that this gives us decay no matter if $|F|$ is large or small due to the choice of $q$ (the reader is reminded that if $|F| \leq c_0$ then $q \in [1, p)$ while if $|F| \geq c_0$ then $q \in (p, \infty]$). We also recall the estimate below from [8].

Lemma 2. There is a finite constant $C_1$ such that for all trees $T$, all $f \in L^2(\mathbb{R}^n)$, and all measurable sets $E'$ with $|E'| \leq 1$ we have

$$\sum_{s \in T} \left| \langle f, \phi_s \rangle \langle \chi_{E' \cap N^{-1}[\omega_{s(r)}]}, \phi_s \rangle \right| \leq C_1 |I_t(T)| \mathcal{E}(f; T) \mathcal{M}(T) \|f\|_{L^2(\mathbb{R}^n)}.$$  \hspace{1cm} (10)

Given the sequence of sets $P_j$ as above, we use (a), (b), (e), the observation that the mass is always bounded by 1, and Lemmata 1 and 2 to obtain

$$\sum_{s \in P} \left| \langle \chi_F, \phi_s \rangle \langle \chi_{E' \cap N^{-1}[\omega_{s(r)}]}, \phi_s \rangle \right|$$

$$= \sum_j \sum_{s \in P_j} \left| \langle \chi_F, \phi_s \rangle \langle \chi_{E' \cap N^{-1}[\omega_{s(r)}]}, \phi_s \rangle \right|$$

$$\leq \sum_j \sum_k \sum_{s \in T_{jk}} \left| \langle \chi_F, \phi_s \rangle \langle \chi_{E' \cap N^{-1}[\omega_{s(r)}]}, \phi_s \rangle \right|$$

$$\leq C_1 \sum_j \sum_k |I_{t(T_{jk})}| \mathcal{E}(T_{jk}) \mathcal{M}(T_{jk}) \|f\|_{L^2(\mathbb{R}^n)}$$

$$\leq C_1 |F|^{\frac{1}{2}} \sum_j \sum_k |I_{t(T_{jk})}| \min(2^{(j+1)n}, C|F|^{\frac{1}{q} - \frac{1}{2}}) \min(1, 2^{(2j+2)n})$$

$$\leq C'|F|^{\frac{1}{2}} \sum_j 2^{-2jn} \min(2^{jn}, |F|^{\frac{1}{q} - \frac{1}{2}}) \min(1, 2^{jn})$$

$$\leq C''|F|^{\frac{1}{2}} (1 + \log |F|^{\frac{1}{2} - \frac{1}{q}})$$

$$\leq C''' \min(1, |F|) (1 + \log |F|)$$

$$\leq C_p |F|^{\frac{1}{2}}$$
for all $1 < p < \infty$. We observe that the choice of $q$ was made to deal with the logarithmic presence in the estimate above. Had we taken $q = p$ throughout, we would have obtained the sought estimates with the extra factor of $1 + \log |F|$.

Looking at the penultimate inequality above, we note that we have actually obtained a stronger estimate than the one claimed in (1). Rescaling the set $E$ and taking $q$ to be either 1 or $\infty$, we have actually proved that for every measurable set $E$ of finite measure, there is a subset $E'$ of $E$ such that for all measurable sets $F$ of finite measure we have

$$\left| \int_{E'} D_r(\chi_F) \, dx \right| \leq C |E| \min\left(1, \frac{|F|}{|E|}\right) \left(1 + \log \frac{|F|}{|E|}\right).$$

This will be of use to us in section 6.

5. THE PROOF OF LEMMA 1

It remains to prove Lemma 1. Because of our normalization of the set $E$ we may assume that $\Omega = \{M(\chi_F) > c |F|^\frac{1}{2}\}$ for some $c > 0$. Fix an $r$-tree $T$ contained in $P$ and let $I_t = I_{(T)}$ be the time projection of its top.

We write the function $\chi_F$ as $\chi_{F \cap 3I_t} + \chi_{F \cap (3I_t)^c}$. We begin by observing that for $s$ in $P$ one has

$$\langle \chi_{F \cap (3I_t)^c}, \phi_s \rangle \leq \frac{C \gamma |I_s|^{\frac{1}{2}}}{\left(1 + \frac{\text{dist}(3I_t^c, cI_s)}{|I_s|^{\frac{1}{2}}}\right)^{\gamma}} \leq C \gamma |I_s|^{\frac{1}{2}} |F|^\frac{1}{2} \left(\frac{|I_s|}{|I_t|}\right)^{\gamma},$$

since $I_s$ meets the complement of $\Omega$ for every $s \in P$. Square this inequality and sum over all $s$ in $T$ to obtain

$$\sum_{s \in T} |\langle \chi_{F \cap (3I_t)^c}, \phi_s \rangle|^2 \leq C |I_t| |F|^\frac{2}{3},$$

where the last estimate follows by placing the $I_s$'s into groups $G_m$ of cardinality at most $2^{mn}$ so that each element of $G_m$ has size $2^{-mn}|I_t|$.

We now turn to the corresponding estimate for the function $\chi_{F \cap 3I_t}$. At this point it will be convenient to distinguish the case $|F| > c_0$ from the case $|F| \leq c_0$. In the case $|F| > c_0$ the set $\Omega$ is empty and therefore

$$\sum_{s \in T} |\langle \chi_{F \cap 3I_t}, \phi_s \rangle|^2 \leq C \|\chi_{F \cap 3I_t}\|_{L^2}^2 \leq C |I_t| |F|^\frac{2}{3},$$

where the first estimate follows from the Bessel inequality (13), which holds on any $r$-tree $T$; the reader may consult [2] or prove it directly.

We therefore concentrate on the case $|F| \leq c_0$. In proving Lemma 1 we may assume that there exists a point $x_0 \in I_t$ such that $M(\chi_F)(x_0) \leq c |F|^\frac{1}{2}$, otherwise there is nothing to prove. We may also assume that the center of $\omega_{(T)}$ is zero, i.e. $c(\omega_{(T)}) = 0$, otherwise we may work with a suitable modulation of the function $\chi_{F \cap 3I_t}$ in the Calderón-Zygmund decomposition below.
We write the set $\Omega = \{ M(\chi_F) > c |F|^{\frac{1}{7}} \}$ as a disjoint union of dyadic cubes $J'_\ell$ such that the dyadic parent $\overset{\sim}{J}'_\ell$ of $J'_\ell$ is not contained in $\Omega$ and therefore

$$|F \cap J'_\ell| \leq |F \cap \overset{\sim}{J}'_\ell| \leq 2c |F|^{\frac{1}{7}} |J'_{\ell}|.$$  

Now some of these dyadic cubes may have size larger than or equal to $|I_\ell|$. Let $J'_\ell$ be such a cube. Then we split $J'_\ell$ in $\lfloor \frac{|J'_\ell|}{|I_\ell|} \rfloor$ cubes $J'_{\ell,m}$ each of size exactly $|I_\ell|$. Since there is an $x_0 \in I_\ell$ with $M(\chi_F)(x_0) \leq c |F|^{\frac{1}{7}}$, it follows that

$$|F \cap J'_{\ell,m}| \leq 2c |F|^{\frac{1}{7}} |I_\ell| \left( 1 + \frac{\text{dist}(I_\ell, J'_{\ell,m})}{|I_\ell|^{\frac{1}{7}}} \right)^n. \tag{11}$$

We now have a new collection of dyadic cubes $\{J_k\}_k$ contained in $\Omega$ consisting of all the previous $J'_\ell$ when $|J'_\ell| < |I_\ell|$ and the $J'_{\ell,m}$’s when $|J'_{\ell,m}| \geq |I_\ell|$. In view of the construction we have

$$|F \cap J_k| \leq \begin{cases} 2c |F|^{\frac{1}{7}} |J_k| & \text{when } |J_k| < |I_\ell| \\ 2c |F|^{\frac{1}{7}} |J_k| \left( 1 + \frac{\text{dist}(I_\ell, J_k)}{|I_\ell|} \right)^n & \text{when } |J_k| = |I_\ell| \end{cases} \tag{12}$$

for all $k$. We now define the “bad functions”

$$b_k = \chi_{J_k \cap 3I_\ell \cap F} - \frac{|J_k \cap 3I_\ell \cap F|}{|J_k|} \chi_{J_k}$$

which are supported in $J_k$, have mean value zero, and they satisfy

$$\|b_k\|_{L^1(\mathbb{R}^n)} \leq 2c |F|^{\frac{1}{7}} |J_k| \left( 1 + \frac{\text{dist}(I_\ell, J_k)}{|I_\ell|} \right)^n.$$

We also set

$$g = \chi_{F \cap 3I_\ell} - \sum_k b_k$$

the “good function” of the above Calderón-Zygmund decomposition. We check that $\|g\|_{L^\infty(\mathbb{R}^n)} \leq C |F|^{\frac{1}{7}}$. Indeed, for $x$ in $J_k$ we have

$$g(x) = \frac{|F \cap 3I_\ell \cap J_k|}{|J_k|} \leq \begin{cases} \frac{|F \cap J_k|}{|J_k|} & \text{when } |J_k| < |I_\ell| \\ \frac{|F \cap 3I_\ell|}{|I_\ell|} & \text{when } |J_k| = |I_\ell| \end{cases}$$

and both of the above are at most a multiple of $|F|^{\frac{1}{7}}$; the latter is because there is an $x_0 \in I_\ell$ with $M(\chi_F)(x_0) \leq c |F|^{\frac{1}{7}}$. Also for $x \in (\cup_k J_k)^c = \Omega^c$, $g(x) = \chi_{F \cap 3I_\ell}(x)$ which is at most $M(\chi_F)(x) \leq c |F|^{\frac{1}{7}}$. We conclude that $\|g\|_{L^\infty(\mathbb{R}^n)} \leq C |F|^{\frac{1}{7}}$. Moreover

$$\|g\|_{L^1(\mathbb{R}^n)} \leq \sum_k \int_{J_k} \frac{|F \cap 3I_\ell \cap J_k|}{|J_k|} \, dx + \|\chi_{F \cap 3I_\ell}\|_{L^1(\mathbb{R}^n)} \leq C |F \cap 3I_\ell| \leq C |F|^{\frac{1}{7}} |I_\ell|$$
since the $J_k$ are disjoint. It follows that
$$
\|g\|_{L^2(\mathbb{R}^n)} \leq C \|F\|^{\frac{1}{2}} |F|^{\frac{1}{2n}} \|I_t\|^{\frac{1}{2}} = C |F|^{\frac{1}{n}} |I_t|^{\frac{1}{2}}.
$$
Using the simple Bessel inequality
\begin{equation}
\sum_{s \in T} |\langle g, \phi_s \rangle|^2 \leq C \|g\|^2_{L^2(\mathbb{R}^n)}
\end{equation}
we obtain the required conclusion for the function $g$.

For a fixed $s \in P$ and $J_k$ we will denote by
$$
d(k, s) = \text{dist} (J_k, I_s).
$$
Then we have the following estimate for all $s$ and $k$:
\begin{equation}
|\langle b_k, \phi_s \rangle| \leq C_\gamma |F|^{\frac{1}{n}} |J_k| \left(1 + \frac{d(k, t)}{|I_t|^{\frac{1}{n}}} \right)^n \frac{|J_k| |I_s|^{-\frac{3}{2}}}{(1 + d(k, s)/|I_s|^{\frac{1}{n}})^{\gamma + n}} \leq C_\gamma |F|^{\frac{1}{n}} |J_k|^2 |I_s|^{-\frac{3}{2}} \left(1 + \frac{d(k, s)}{|I_s|^{\frac{1}{n}}} \right)\gamma
\end{equation}
since $1 + \frac{d(k, t)}{|I_t|^{\frac{1}{n}}} \leq 1 + \frac{d(k, s)}{|I_s|^{\frac{1}{n}}}$.

We also have the estimate
\begin{equation}
|\langle b_k, \phi_s \rangle| \leq C_\gamma |F|^{\frac{1}{n}} |I_s|^{\frac{1}{2}} \left(1 + \frac{d(k, s)}{|I_s|^{\frac{1}{n}}} \right)^\gamma.
\end{equation}

To prove (14) we use the fact that the center of $\omega_l(T) = 0$ (which implies that $\phi'_s$ obeys size estimates similar to $|I_s|^{-1} |\phi_s|$) and the mean value property of $b_k$ to obtain
$$
|\langle b_k, \phi_s \rangle| = \left| \int_{J_k} b_k(y) (\phi_s(y) - \phi_s(c(J_k))) \, dy \right| \leq \|b_k\|_{L^1} |J_k| \sup_{\xi \in J_k} \frac{C_\gamma |I_s|^{-\frac{3}{2}}}{(1 + \frac{|\xi - c(I_k)|}{|I_k|^{\frac{1}{n}}})^{\gamma}}.
$$
To prove estimate (15) we note that
$$
|\langle b_k, \phi_s \rangle| \leq C_\gamma |I_s|^\frac{1}{2} \left( \inf_{I_s} M(b_k) \right) \frac{1}{(1 + \frac{d(k, s)}{|I_s|^{\frac{1}{n}}})^\gamma}
$$
and that
$$
M(b_k) \leq M(\chi_F) + \frac{|F \cap 3I_k \cap J_k|}{|J_k|} M(\chi_{J_k})
$$
and since $I_s \not\subset \Omega$ we have $\inf_{I_s} M(\chi_F) \leq c |F|^{\frac{1}{n}}$ while the second term in the sum above was observed earlier to be at most $C |F|^{\frac{1}{n}}$.

Finally we have the estimate
\begin{equation}
|\langle b_k, \phi_s \rangle| \leq C_\gamma |F|^{\frac{1}{n}} |J_k| |I_s|^{-\frac{1}{2}} \left(1 + \frac{d(k, s)}{|I_s|^{\frac{1}{n}}} \right)^\gamma
\end{equation}
which follows by taking the geometric mean of (14) and (15).

Now for a fixed $s \in P$ we may have either $J_k \subseteq I_s$ or $J_k \cap I_s = \emptyset$ (since $I_s$ is not contained in $\Omega$.) Therefore for fixed $s \in P$ there are only three possibilities for $J_k$:
(a) $J_k \subseteq 3I_s$
(b) $J_k \cap 3I_s = \emptyset$
(c) $J_k \cap I_s = \emptyset$, $J_k \cap 3I_s \neq \emptyset$, and $J_k \not\subset 3I_s$.

Observe that case (c) is equivalent to the following statement:
(c) $J_k \cap I_s = \emptyset$, $d(k, s) = 0$, and $|J_k| \geq 2^n|I_s|$.

Let us start with case (c). Note that for each $I_s$ there exists at most $2^n - 1$ choices of $J_k$ with the above properties. Thus for each $s$ in the sum below we can pick one $J_k(s)$ at a cost of $2^n - 1$, which is harmless. Also note that since $d(k, s) = 0$ and $|J_k| \geq 2^n|I_s|$, we must have that $I_s \subset 2J_k$. But $I_s \subset I_t$ and $|J_k| \leq |I_t|$ implies that $J_k \subset 3I_t$. Now for a given $J_k$ and a fixed scale $m \geq 1$, there are at most $2^m \times (\# \text{ of sides}) + 2^n$ possibilities of $I_s$ such that $2^{-mn}|J_k| = |I_s|$ and $d(k, s) = 0$. Using (13) we obtain

$$\sum_{s \in T} \left| \sum_{k: J_k \text{ as in (c)}} \langle b_k, \phi_s \rangle \right|^2 \leq (2^n - 1)^2 \sum_{s \in T} \left| \langle b_{k(s)}, \phi_s \rangle \right|^2 \leq C_n |F|^\frac{q}{2} \sum_{s \in T \text{ for which } \exists J_k \text{ as in (c)}} |I_s| \leq C_n |F|^\frac{q}{2} \sum_{s \in T} \sum_{m \geq 1} 2^{-mn} |J_k(s)| \leq C_n |F|^\frac{q}{2} \sum_{m \geq 1} (2^m \times (\# \text{ of sides}) + 2^n) 2^{-mn} \sum_k |J_k| \leq C_n |F|^\frac{q}{2} |I_t|,$$

where we have used the disjointness of the $J_k$’s. This finishes case (c).

We now consider case (a). Using (14) we can write

$$\left( \sum_{s \in T} \left| \sum_{k: J_k \text{ as in (a)}} \langle b_k, \phi_s \rangle \right|^2 \right)^{\frac{q}{2}} \leq C_\gamma |F|^\frac{q}{2} \left( \sum_{s \in T} \left| \sum_{k: J_k \subset 3I_s} |J_k|^\frac{q}{2} \frac{|J_k|^2}{|I_s|^2} \right| \right)^{\frac{q}{2}}$$

and we control the expression inside the parenthesis above by

$$\sum_{s \in T} \left( \sum_{k: J_k \subset 3I_s} |J_k|^3 \right) \left( \sum_{k: J_k \subset 3I_s} \frac{|J_k|}{|I_s|^3} \right) \leq \sum_{k: J_k \subset 3I_s} |J_k|^3 \sum_{J_k \subset 3I_s} \frac{1}{|I_s|^2}$$

in view of the Cauchy-Schwarz inequality and of the fact that the dyadic cubes $J_k$ are disjoint and contained in $3I_s$. Finally note that the last sum above adds up to at most $C_n |J_k|^{-2}$ since for every dyadic cube $J_k$ there exist at most $2^n + 1 + (\# \text{ of sides})$ dyadic cubes of a given size whose triples contain it. The required estimate $C_{n, \gamma} |F|^\frac{q}{2} |I_t|^\frac{1}{2}$ now follows.

Finally we deal with case (b) which is the most difficult case. We split the set of $k$ into two subsets, those for which $J_k \subset 3I_t$ and those for which $J_k \not\subset 3I_t$, (recall $|J_k| \leq |I_t|$.) Whenever $J_k \not\subset 3I_t$ we have $d(k, s) \approx d(k, t)$. In this case we use
Minkowski’s inequality below and estimate (16) with $\gamma > n$ to obtain the estimate

\[
\left( \sum_{s \in T} \left| \sum_{k: J_k \subseteq 3I_s} \langle b_k, \phi_s \rangle \right|^2 \right)^{\frac{1}{2}} \leq \sum_{k: J_k \subseteq 3I_s} \left( \sum_{s \in T} \left| \langle b_k, \phi_s \rangle \right|^2 \right)^{\frac{1}{2}}
\]

\[
\leq C_\gamma |F|^{\frac{1}{2}} \sum_{k: J_k \subseteq 3I_s} |J_k| \left( \sum_{s \in T} \frac{|I_s|^{\frac{n}{n-1}}}{d(k, s)^{2\gamma}} \right)^{\frac{1}{2}}
\]

\[
\leq C_\gamma |F|^{\frac{1}{2}} \sum_{k: J_k \subseteq 3I_s} \frac{|J_k|}{d(k, t)^{\gamma}} \left( \sum_{s \in T} |I_s|^{\frac{n}{n-1}} \right)^{\frac{1}{2}}
\]

\[
\leq C_\gamma |F|^{\frac{1}{2}} |I_t|^{\frac{n}{n-1} - \frac{1}{2}} \sum_{k: J_k \subseteq 3I_t} \frac{|J_k|}{d(k, t)^{\gamma}} \sum_{l=1}^{\infty} \sum_{t: d(k, t) \approx 2^l |I_t|^{\frac{1}{n}}} \frac{|J_k|}{(2^l |I_t|^{\frac{1}{n}})^{\gamma}}.
\]

But note that all the $J_k$ with $d(k, t) \approx 2^l |I_t|^{\frac{1}{n}}$ are contained in $2^{l+2} |I_t|$ and since they are disjoint we can estimate the last sum above by $C2^{|m|} |I_t| (2^l |I_t|^{\frac{1}{n}})^{-\gamma}$. The required estimate $C_\gamma |F|^{\frac{1}{2}} |I_t|^{\frac{1}{2}}$ now follows.

Next we consider the sum below in which we use estimate (14)

\[
\left( \sum_{s \in T} \left| \sum_{k: J_k \subseteq 3I_s, J_k \cap 3^m I_s = \emptyset, |J_k| \leq |I_s|} \langle b_k, \phi_s \rangle \right|^2 \right)^{\frac{1}{2}}
\]

\[
\leq C_\gamma |F|^{\frac{1}{2}} \left( \sum_{s \in T} \left| \sum_{k: J_k \subseteq 3I_s, J_k \cap 3^m I_s = \emptyset, |J_k| \leq |I_s|} |J_k|^2 |I_s|^{-\frac{3}{2}} \left( \frac{|I_s|^{\frac{1}{n}}}{d(k, s)} \right)^{\gamma} \right|^2 \right)^{\frac{1}{2}}
\]

\[
\leq C_\gamma |F|^{\frac{1}{2}} \left( \sum_{s \in T} \left| \sum_{k: J_k \subseteq 3I_s, J_k \cap 3^m I_s = \emptyset, |J_k| \leq |I_s|} \frac{|J_k|^3}{|I_s|^2} \left( \frac{|I_s|^{\frac{1}{n}}}{d(k, s)} \right)^{\gamma} \left( \sum_{k: J_k \subseteq 3I_s, J_k \cap 3^m I_s = \emptyset, |J_k| \leq |I_s|} \frac{|J_k|}{|I_s|^{\frac{1}{n}}} \right)^{-\gamma} \right)^{\frac{1}{2}}.
\]

(17) \[
\leq C_\gamma |F|^{\frac{1}{2}} \left( \sum_{s \in T} \left| \sum_{k: J_k \subseteq 3I_s, J_k \cap 3^m I_s = \emptyset, |J_k| \leq |I_s|} \frac{|J_k|^3}{|I_s|^2} \left( \frac{|I_s|^{\frac{1}{n}}}{d(k, s)} \right)^{\gamma} \left( \sum_{k: J_k \subseteq 3I_s, J_k \cap 3^m I_s = \emptyset, |J_k| \leq |I_s|} \frac{|J_k|}{|I_s|^{\frac{1}{n}}} \right)^{-\gamma} \right)^{\frac{1}{2}}.
\]

The second sum above can be estimated by

\[
\sum_{k: J_k \subseteq 3I_s, J_k \cap 3^m I_s = \emptyset, |J_k| \leq |I_s|} \int_{J_k} \left( \frac{|x - c(I_s)|}{|I_s|^{\frac{1}{n}}} \right)^{-\gamma} dx \leq \int_{(3I_s)^c} \left( \frac{|x - c(I_s)|}{|I_s|^{\frac{1}{n}}} \right)^{-\gamma} dx = C_\gamma.
\]
Putting this estimate into (17), we have

\[
\leq C_\gamma |F| \frac{1}{\gamma} \left\{ \sum_{s \in T} \sum_{k: J_k \subseteq 3I_t, J_k \cap 3I_s = \emptyset} |J_k|^3 |I_s|^{-2} \left( \frac{|I_s|^{2 \gamma}}{(d(k, s))^\gamma} \right)^{\frac{1}{2}} \right\}
\]

\[
\leq C_\gamma |F| \frac{1}{\gamma} \left\{ \sum_{k: J_k \subseteq 3I_t, J_k \cap 3I_s = \emptyset} |J_k|^3 \sum_{m \geq \log |J_k| \over n} 2^{-2mn} \sum_{s \in T} \left( \frac{d(k, s)}{2m} \right)^{-\gamma} \right\}^{\frac{1}{2}}
\]

\[
\leq C_\gamma |F| \frac{1}{\gamma} \left\{ \sum_{k: J_k \subseteq 3I_t, J_k \cap 3I_s = \emptyset} |J_k|^3 \sum_{m \geq \log |J_k| \over n} 2^{-2mn} \right\}^{\frac{1}{2}}
\]

\[
\leq C_\gamma |F| \frac{1}{\gamma} \left\{ |J_k|^3 |J_k|^{-2} \right\}^{\frac{1}{2}}
\]

\[
\leq C_\gamma |F| \frac{1}{\gamma} |I_s|^{\frac{2}{\gamma}}.
\]

There is also the subcase of case (b) in which $|J_k| \geq |I_s|$. Here we have the two special subcases: $I_s \cap 3J_k = \emptyset$ and $I_s \subseteq 3J_k = \emptyset$. We begin with the first of these special subcases in which we use estimate (13). We have

\[
\left( \sum_{s \in T} \left| \sum_{k: J_k \subseteq 3I_t, J_k \cap 3J_s = \emptyset} \langle b_k, \phi_s \rangle \right|^2 \right)^{\frac{1}{2}}
\]

\[
\leq C_\gamma |F| \left( \sum_{s \in T} \left| \sum_{k: J_k \subseteq 3I_t, J_k \cap 3J_s = \emptyset} |I_s|^{\frac{2}{2}} \frac{|I_s|^{\frac{2}{\gamma}}}{d(k, s)^{\gamma}} \right|^2 \right)^{\frac{1}{2}}
\]

\[
\leq C_\gamma |F| \left( \sum_{s \in T} \left[ \sum_{k: J_k \subseteq 3I_t, J_k \cap 3J_s = \emptyset} \frac{|I_s|^2}{|J_k|} \frac{|I_s|^{\frac{2}{\gamma}}}{d(k, s)^{\gamma}} \right] \left[ \sum_{k: J_k \subseteq 3I_t, J_k \cap 3J_s = \emptyset} \frac{|J_k|}{|I_s|} \frac{|I_s|^{\frac{2}{\gamma}}}{d(k, s)^{\gamma}} \right] \right)^{\frac{1}{2}}.
\]
Since $I_s \cap 3J_3 = \emptyset$ we have that $d(k, s) \approx |x - c(I_s)|$ for every $x \in J_k$. Therefore the second term inside square brackets above satisfies

$$\sum_{k: J_k \subseteq 3I_t \atop J_k \cap 3I_s = \emptyset \atop |J_k| > |I_s| \atop I_s \cap 3J_k = \emptyset} \frac{|J_k|}{|I_s|^2} \frac{|I_s|^{\frac{2}{n}}}{d(k, s)^\gamma} \leq \sum_k \int_{J_k} \left( \frac{|x - c(I_s)|}{|I_s|^\frac{1}{n}} \right)^{-\gamma} dx \leq C_{\gamma}.$$

Putting this estimate into (19), we obtain

$$C_{\gamma} |F|^{\frac{1}{\gamma}} \left( \sum_{s \in T} \sum_{k: J_k \subseteq 3I_t \atop J_k \cap 3I_s = \emptyset \atop |J_k| > |I_s| \atop I_s \cap 3J_k = \emptyset} |I_s|^2 \frac{|I_s|^{\frac{2}{n}}}{|J_k| \frac{1}{d(k, s)^\gamma}} \right)^{\frac{1}{2}} \leq \sum_{s \in T} \sum_{k: J_k \subseteq 3I_t \atop J_k \cap 3I_s = \emptyset \atop |J_k| > |I_s| \atop I_s \cap 3J_k = \emptyset} \frac{|I_s|^{\frac{2}{n}}}{d(k, s)^\gamma} \leq C_{\gamma} |F|^{\frac{1}{\gamma}} \left( \sum_{s \in T} \sum_{k: J_k \subseteq 3I_t \atop J_k \cap 3I_s = \emptyset \atop |J_k| > |I_s| \atop I_s \cap 3J_k = \emptyset} |I_s|^2 \frac{|I_s|^{\frac{2}{n}}}{|J_k| \frac{1}{d(k, s)^\gamma}} \right)^{\frac{1}{2}}.$$

Since the last sum above is at most a constant (18) satisfies the estimate $C_{\gamma} |F|^{\frac{1}{\gamma}} |I_t|^{\frac{1}{\gamma}}$.

Finally there is the subcase of case (b) in which $|J_k| \geq |I_s|$ and $I_s \subseteq 3J_k = \emptyset$. Here again we use estimate (15). We have

$$(20) \quad \left( \sum_{s \in T} \sum_{k: J_k \subseteq 3I_t \atop J_k \cap 3I_s = \emptyset \atop |J_k| > |I_s| \atop I_s \cap 3J_k} |b_k, \phi_s|^2 \right)^{\frac{1}{2}} \leq C_{\gamma} |F|^{\frac{1}{\gamma}} \left( \sum_{s \in T} \sum_{k: J_k \subseteq 3I_t \atop J_k \cap 3I_s = \emptyset \atop |J_k| > |I_s| \atop I_s \cap 3J_k} \frac{|I_s|^{\frac{2}{n}}}{d(k, s)^\gamma} \right)^{\frac{1}{2}}.$$

Let us make some observations. For a fixed $s$ there exists at most finitely many $J_k$’s contained in $3I_t$ with size at least $|I_s|$. Consider the following sets for $\alpha \in \{0, 1, 2, \ldots\}$,

$$J^\alpha := \{ J_k \text{ as in the sum above} : 2^\alpha |I_s|^{\frac{1}{n}} \leq d(k, s) < 2^{\alpha + 1} |I_s|^{\frac{1}{n}} \}.$$

We would like to know that for all $\alpha$ the cardinality of $J^\alpha$ is bounded by a fixed constant depending only on dimension. This would allow us to work with a single cube $J^\alpha(s)$ from each set at the cost of a constant in the sum below. Fix $\alpha \in \{0, 1, 2, \ldots\}$ and note that $I_s \subseteq 3J_k$ and $d(k, s) > 2^\alpha |I_s|^{\frac{1}{n}}$ implies that $|J_k| > 2^{2\alpha} |I_s|$. It is clear that the cardinality of $J^\alpha$ would be largest if we had $|J_k| = 2^{\alpha + 1} |I_s|$ for all $J_k \in J^\alpha$. Then the cube of size $7^{\alpha} 2^{2\alpha} |I_s|$ centered at $I_s$ would contain all elements of $J_k$. This bounds the number of such elements by $(\frac{7}{2})^{\alpha}$.
Then using the Cauchy-Schwarz inequality we obtain
\[
\left| \sum_{k: J_k \subseteq 3I_t} \frac{|I_s|^{\frac{\gamma}{n}}}{d(k, s)} \right|^2 \leq \left( \frac{7}{2} \right)^{2n} \sum_{\alpha=1}^{\infty} \frac{|I_s|^{\frac{\gamma}{n}}}{\text{dist}(J_\alpha(s), I_s)^{\frac{\gamma}{2}}} \frac{1}{2^{\frac{\gamma}{2}}} \right)^2
\]
\[
\leq C_n \sum_{\alpha=1}^{\infty} \frac{|I_s|^{\frac{\gamma}{n}}}{\text{dist}(J_\alpha(s), I_s)^{\gamma}}
\]
\[
\leq C_n \sum_{k: J_k \subseteq 3I_t} \frac{|I_s|^{\frac{\gamma}{n}}}{d(k, s)^{\gamma}}
\]

Putting this estimate into the right hand side of (20), the estimate $C_{n, \gamma} |F|^\frac{\gamma}{n} |I_s|^\frac{\gamma}{2}$ now follows as in the previous case. This concludes the proof of Lemma 1.

6. APPLICATIONS

We conclude by discussing some applications. We show how one can strengthen the results of the previous sections to obtain distributional estimates for the function $D_r(\chi_F)$ similar to those in the paper of Sjölin [10].

We showed in section 4 that for any measurable set $E$ there is a set $E'$ of at least half the measure of $E$ such that
\[
\left| \int_{E'} D_r(\chi_F) \, dx \right| \leq C \min(|E|, |F|) \left( 1 + \left| \log \frac{|F|}{|E|} \right| \right)
\]
for some constant $C$ depending only on the dimension. For $\lambda > 0$ we define
\[
E_\lambda = \{|D_r(\chi_F)| > \lambda\}
\]
and also
\[
E_\lambda^1 = \{\text{Re } D_r(\chi_F) > \lambda\} \quad E_\lambda^2 = \{\text{Re } D_r(\chi_F) < -\lambda\}
\]
\[
E_\lambda^3 = \{\text{Im } D_r(\chi_F) > \lambda\} \quad E_\lambda^4 = \{\text{Im } D_r(\chi_F) < -\lambda\}.
\]

We apply (21) to each set $E_\lambda^j$ to obtain
\[
\lambda |E_\lambda^j| \leq \min(|E_\lambda^j|, |F|) \left( 1 + \left| \log \frac{|F|}{|E_\lambda^j|} \right| \right).
\]

Using this fact in combination with the easy observation that for $a > 1$
\[
\frac{a}{\log a} \leq \frac{1}{\lambda} \implies a \leq \frac{10}{\lambda} \log \left( \frac{1}{\lambda} \right),
\]
to obtain that
\[
|E_\lambda^j| \leq C'^\lambda |F| \begin{cases} \frac{1}{\lambda} \log \left( \frac{1}{\lambda} \right) & \text{when } \lambda < \frac{1}{2} \\ e^{-c\lambda} & \text{when } \lambda \geq \frac{1}{2}. \end{cases}
\]

Since $|E_2 \cap \lambda| \leq \sum_{j=1}^{4} |E_\lambda^j|$ we conclude a similar estimate for $E_\lambda$. 
Next we obtain similar distributional estimates for maximally modulated singular integrals $M$ such as the maximally modulated Hilbert transform (i.e. Carleson’s operator) or the maximally modulated Riesz transforms

$$M(f)(x) = \sup_{\xi \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} e^{2\pi i \xi \cdot y} f(y) \, dy \right|.$$ 

To achieve this in the one dimensional setting, one applies an averaging argument similar to that in [6] to both terms of estimate (21) to recover a similar estimate with the Carleson operator. For more general homogeneous singular integrals with sufficiently smooth kernels, one applies the averaging argument to suitable modifications of the operators $D_r$ as in [8]. Then one obtains a version of estimate (21) in which $D_r(\chi_F)$ is replaced by $M(\chi_F)$. The same procedure as above then yields the distributional estimate

$$\left| \{ |M(\chi_F)| > \lambda \} \right| \leq C'_n |F| \begin{cases} \frac{1}{\lambda} \log \left( \frac{1}{\lambda} \right) & \text{when } \lambda < \frac{1}{2} \\ e^{-c\lambda} & \text{when } \lambda \geq \frac{1}{2} \end{cases}$$

which recovers Lemma 1.2 in [10]. It should be noted that the corresponding estimate

$$\left| \{ |D_r(\chi_F)| > \lambda \} \right| \leq C_n |F| \begin{cases} \frac{1}{\lambda} \log \left( \frac{1}{\lambda} \right) & \text{when } \lambda < \frac{1}{2} \\ e^{-c\lambda} & \text{when } \lambda \geq \frac{1}{2} \end{cases}$$

obtained here for $D_r$ is stronger as it concerns an “unaveraged version” of all the aforementioned maximally modulated singular integrals $M$.

Using the idea employed in Sjölin [9] we can obtain the following result as a consequence of (22). Let $B$ be a ball in $\mathbb{R}^n$.

**Proposition 1.** (i) If $\int_B |f(x)| \log^+ |f(x)| \log^+ \log^+ |f(x)| \, dx < \infty$, then $D_r(f)$ is finite a.e. on $B$.

(ii) If $\int_B |f(x)| (\log^+ |f(x)|) dx < \infty$, then $D_r(f)$ is integrable over $B$.

(iii) For all $\lambda > 0$ we have

$$\left| \{ x \in \mathbb{R}^n : |D_r(f)(x)| > \lambda \} \right| \leq C e^{-c\lambda/\|f\|_{L^\infty}}$$

where $C,c$ only depend on the dimension (in particular they are independent of the measurable function $N : \mathbb{R}^n \to \mathbb{R}^n$.)

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