Fast Rates for Contextual Linear Optimization

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Incorporating side observations in decision making can reduce uncertainty and boost performance, but it also requires us tackle a potentially complex predictive relationship. While one may use off-the-shelf machine learning methods to separately learn a predictive model and plug it in, a variety of recent methods instead integrate estimation and optimization by fitting the model to directly optimize downstream decision performance. Surprisingly, in the case of contextual linear optimization, we show that the naïve plug-in approach actually achieves regret convergence rates that are significantly faster than methods that directly optimize downstream decision performance. We show this by leveraging the fact that specific problem instances do not have arbitrarily bad near-dual-degeneracy. While there are other pros and cons to consider as we discuss and illustrate numerically, our results highlight a nuanced landscape for the enterprise to integrate estimation and optimization. Our results are overall positive for practice: predictive models are easy and fast to train using existing tools, simple to interpret, and, as we show, lead to decisions that perform very well.

Key words: Contextual stochastic optimization, Personalized decision making, Estimate then optimize

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1. Introduction

A central tenet of machine learning is the use of rich feature data to reduce uncertainty in an unknown variable of interest, whether it is the content of an image, medical outcomes, or future stock price. Recent work in data-driven optimization has highlighted the potential for rich features to similarly reduce uncertainty in decision-making problems with uncertain objectives and thus improve resulting decisions’ performance (Bertsimas and Kallus 2020, Chen et al. 2021, Diao and Sen 2020, Donti et al. 2019, El Balghiti et al. 2019, Elmachtoub and Grigas 2021, Estes and Richard 2019, Ho and Hanasusanto 2019, Ho-Nguyen and Kilinc-Karzan 2020, Kallus and Mao 2020, Loke et al. 2020, Notz and Pibernik 2021, Vahn and Rudin 2019). For decision-making problems modeled by linear optimization with uncertain coefficients, this is captured by the contextual linear optimization (CLO) problem, defined as follows:

\[ \pi^*(x) \in \mathcal{Z}^*(x) = \arg \min_{z \in \mathcal{Z}} f^*(x)^T z, \quad f^*(x) = \mathbb{E}[Y \mid X = x], \quad \mathcal{Z} = \{ z \in \mathbb{R}^d : Az \leq b \}. \] (1)

Here, \( X \in \mathbb{R}^p \) represents the contextual features, \( z \in \mathcal{Z} \subseteq \mathbb{R}^d \) linearly-constrained decisions, and \( Y \in \mathbb{R}^d \) the random coefficients. Examples of CLO are vehicle routing with uncertain travel times,
portfolio optimization with uncertain security returns, and supply chain management with uncertain shipment costs. In each case, \( X \) represents anything that we can observe before making a decision \( z \) that can help reduce uncertainty in the random coefficients \( Y \), such as recent traffic or market trends. The decision policy \( \pi^*(x) \) optimizes the conditional expected costs, given the observation \( X = x \). (We reserve \( X, Y \) for random variables and \( x, y \) for their values.) We assume throughout that \( Z \) is a polytope (\( \sup_{z \in Z} \|z\| \leq B \)) and \( Y \) is bounded (without loss of generality, \( Y \in \mathcal{Y} = \{y : \|y\| \leq 1\} \)), and we let \( Z^\perp \) denote the set of extreme points of \( Z \).

Nominally, we only do better by taking features \( X \) into consideration when making decisions:

\[
\min_{z \in Z} \mathbb{E}[Y^\top z] \geq \mathbb{E}[\min_{z \in Z} \mathbb{E}[Y^\top z \mid X]] = \mathbb{E}[f^*(X)^\top \pi^*(X)],
\]

and the more \( Y \)-uncertainty explained by \( X \) the larger the gap. That is, at least if we knew the true conditional expectation function \( f^* \). In practice, we do not, and we only have data \( \mathcal{D} = \{(X_1, Y_1), \ldots, (X_n, Y_n)\} \), which we assume consist of \( n \) independent draws of \( (X, Y) \). The task is then to use these data to come up with a well-performing data-driven policy \( \hat{\pi}(x) \) for the decision we will make when observing \( X = x \), namely one having low average regret:

\[
\text{Regret}(\hat{\pi}) = \mathbb{E}_\mathcal{D} \mathbb{E}_X [f^*(X)^\top (\hat{\pi}(X) - \pi^*(X))],
\]  

where we marginalize both over new features \( X \) and over the sampling of the data \( \mathcal{D} \) (i.e., over \( \hat{\pi} \)).

One approach is the naïve plug-in method, also known as “estimate and then optimize” (ETO). Since \( f^* \) is the regression of \( Y \) on \( X \), we can estimate it using a variety of off-the-shelf methods, whether parametric regression such as ordinary least squares or generalized linear models, nonparametric regression such as \( k \)-nearest neighbors or local polynomial regression, or machine learning methods such as random forests or neural networks. Given an estimate \( \hat{f} \) of \( f^* \), we can construct the induced policy \( \pi_{\hat{f}} \), where for any generic \( f : \mathbb{R}^p \to \mathbb{R}^d \) we define the plug-\( f \)-in policy

\[
\pi_{\hat{f}}(x) \in \arg \min_{z \in Z} f(x)^\top z.
\]

Notice that given \( f \), \( \pi_{\hat{f}} \) need not be unique; we restrict to choices \( \pi_{\hat{f}}(x) \in Z^\perp \) that break ties arbitrarily but consistently (i.e., by some ordering over \( Z^\perp \)). Notice also that \( \pi_{\hat{f}}(x) \in Z^* \) (\( x \)). Given a hypothesis class \( \mathcal{F} \subseteq [\mathbb{R}^p \to \mathcal{Y}] \) for \( f^* \), we can for example choose \( \hat{f} \) by least-squares regression:

\[
\hat{f}_{\mathcal{F}} \in \arg \min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \|Y_i - f(X_i)\|^2.
\]  

We let \( \hat{\pi}^{\text{ETO}}_{\hat{f}_{\mathcal{F}}} = \pi_{\hat{f}_{\mathcal{F}}} \) be the ETO policy corresponding to least-squares regression over \( \mathcal{F} \). ETO has appealing practical benefits. It is easily implemented using tried-and-true, off-the-shelf, potentially flexible prediction methods. More crucially, it easily adapts to decision support, which is often the
reality for quantitative decision-making tools: rather than a blackbox prescription, it provides a
decision maker with a prediction that she may judge and eventually use as she sees fit.

Nonetheless, a criticism of this approach is that Eq. (4) uses the wrong loss function as it does not
consider the impact of \( \hat{f} \) on the downstream performance of the policy \( \pi_f \) and in a sense ignores the
decision-making problem. The alternative empirical risk minimization (ERM) method directly
minimizes an empirical estimate of the average costs of a policy; given a policy class \( \Pi \subseteq [\mathbb{R}^p \rightarrow \mathcal{Z}] \),

\[
\hat{\pi}_{\Pi}^{\text{ERM}} \in \arg \min_{\pi \in \Pi} \frac{1}{n} \sum_{i=1}^{n} Y_i^\top \pi(X_i). \tag{5}
\]

In particular, a hypothesis class \( \mathcal{F} \) induces the plug-in policy class \( \Pi_{\mathcal{F}} = \{ \pi_f : f \in \mathcal{F} \} \), and ERM
over \( \Pi_{\mathcal{F}} \) corresponds to optimizing the empirical risk of \( \pi_f \) over choices \( f \in \mathcal{F} \), yielding a different
criterion from Eq. (4) for choosing \( f \in \mathcal{F} \). We call this the induced ERM (IERM) method, which
thus integrates the estimation and optimization aspects of the problem into one, sometimes referred
to as end-to-end estimation. We let \( \hat{\pi}_{\mathcal{F}}^{\text{IERM}} = \hat{\pi}_{\Pi_{\mathcal{F}}}^{\text{ERM}} \) denote the IERM policy induced by \( \mathcal{F} \).

Although the latter IERM approach appears to much more correctly and directly deal with the
decision-making problem of interest, in this paper we demonstrate a surprising fact:

*Estimate-and-then-optimize approaches can have much faster regret-convergence rates.*

To theoretically characterize this phenomenon, we develop regret bounds for ETO and IERM when
\( f^* \in \mathcal{F} \). Without further assumptions beyond such well-specification (which is necessary for any
hope of vanishing regret), we show that the regret convergence rate \( 1/\sqrt{n} \) reigns. However, appropri-
ately limiting how degenerate an instance can be uncovers faster rates and a divergence between
ETO and IERM favoring ETO. This can be attributed to ETO leveraging structure in \( \mathcal{F} \) compared to IERM using only what is implied about \( \Pi_{\mathcal{F}} \). Numerical examples corroborate our theory’s
predictions and demonstrate the conclusions extend to flexible/nonparametric specifications, while
highlighting the benefits of IERM for simple/interpretable models that are bound to be misspec-
ifed. We provide a detailed discussion on how this fits into the larger practical considerations of
choosing between ETO and end-to-end methods such as IERM for developing decision-making and
decision-support systems.

1.1. Background and Relevant Literature

*Contextual linear and stochastic optimization.* The IERM problem is generally nonconvex in
\( f \in \mathcal{F} \). For this reason \[ Elmachtoub and Grigas \] \( [2021] \) develop a convex surrogate loss they call
SPO+, which they show is Fisher consistent under certain regularity conditions in that if \( f^* \in \mathcal{F} \)
then the solution to the convex surrogate problem solves the nonconvex IERM problem. \[ El Balghiti et al. \] \( [2019] \) prove an \( O(\log(|\mathcal{Z}|n)/\sqrt{n}) \) regret bound for IERM when \( \mathcal{F} \) is linear functions.
Both [El Balghiti et al. (2019), Elmachtoub and Grigas (2021)] advocate for the integrated IERM approach to CLO, referring to it as *smart* in comparison to the naïve ETO method.

CLO is a special case of the more general contextual stochastic optimization (CSO) problem, 
\[ \pi^*(x) \in \arg\min_{\pi \in \mathcal{Z}} \mathbb{E}[c(z; Y) \mid X = x]. \]
Bertsimas and Kallus (2020) study ETO approaches to CSO where the distribution of \( Y \mid X = x \) is estimated by a re-weighted empirical distribution of \( Y_i \), for which they establish asymptotic optimality. Diao and Sen (2020) study stochastic gradient descent approaches to solving the resulting problems. Ho and Hanasusanto (2019) propose to add variance regularization to this ETO rule to account for errors in this estimate. Bertsimas and Kallus (2020) additionally study ERM approaches to CSO and provide generic regret bounds (see their Appendix EC.1). Notz and Pibernik (2021) apply these bounds to reproducing kernel Hilbert spaces (RKHS) in a capacity planning application. Vahn and Rudin (2019) study ERM with a sparse linear model for the newsvendor problem. Kallus and Mao (2020) construct forest policies for CSO by using optimization perturbation analysis to approximate the generally intractable problem of ERM for CSO over trees; they also prove asymptotic optimality. Many other works that study CSO generally advocate for *end-to-end* solutions that integrate or harmonize estimation and optimization (Donti et al. 2017, Estes and Richard 2019, Ho-Nguyen and Kılınç-Karzan 2020, Loke et al. 2020).

**Classification.** Classification is a specific case of CLO with \( Y \in \{-1, 1\} \) and \( Z = [-1, 1] \). Then \( \frac{1}{n} \text{Regret}(\hat{\pi}) = \mathbb{P}(Y \neq \hat{\pi}(X)) - \mathbb{P}(Y \neq \pi^*(X)) \) is the excess error rate. Bartlett et al. (2005), Koltchinskii et al. (2006), Massart and Nédélec (2006), Tsybakov (2004), Vapnik and Chervonenkis (1974) among others study regret and generalization bounds for ERM approaches, convexifications, and related approaches. Our work is partly inspired by Audibert and Tsybakov (2007), who compared such ERM classification approaches to methods that estimate \( \mathbb{P}(Y = 1 \mid X) \) and then classify by thresholding at \( 1/2 \) and showed that these can enjoy fast regret convergence rates under a noise condition (also known as margin) that quantifies the concentration of \( \mathbb{P}(Y = 1 \mid X) \) near \( 1/2 \). In contrast to Audibert and Tsybakov (2007), we study fast rates for the more general CLO problem as our aim is to shed light on data-driven optimization, we use complexity notions that allow direct comparison of ETO and IERM (rather than ERM) using the same hypothesis class (while entropy conditions for ERM and plug-in used by Audibert and Tsybakov 2007 are incomparable), and we provide lower bounds that rigorously show the gap between IERM and ETO for any given polytope (the lower bounds of Audibert and Tsybakov 2007 only apply to Hölder-smooth functions and classification and they show the optimality of plug-in methods rather than the *sub*optimality of ERM). Similar noise or margin conditions have also been used in contextual bandits (Bastani and Bayati 2020, Goldenshluger and Zeevi 2013, Hu et al. 2020, Perchet and Rigollet 2013, Rigollet and Zeevi 2010). Our condition is similar to these but adapted to CLO.
1.2. A Simple Example

We start with a simple illustrative example. Consider univariate decisions, \( Z = [-1, 1] \), univariate features, \( X \sim \text{Unif}[-1, 1] \), and a simple linear relationship, \( f^*(X) = X, \ Y - f^*(X) \sim \mathcal{N}(0, \sigma^2) \). Let us default to \( z = -1 \) under ties. Then, given a hypothesis set \( \mathcal{F} = \{ f_\theta(x) = x - \theta : \theta \in [-1, 1] \} \), we have \( \pi_{f_\theta}(x) = 2 \mathbb{I}[x \leq \theta] - 1 \). Let us default to smaller \( \theta \) under ties. We can compute \( \mathbb{E}_X [f^*(X)^\top(\pi_{f_\theta}(X) - \pi^*(X))] = \frac{1}{2} \theta^2 \). We can also immediately see that \( \hat{\pi}_{\text{ETO}}^\ast \mathcal{F}(x) = 2 \mathbb{I}[x \leq \hat{\theta}_{\text{OLS}}] - 1 \) where \( \hat{\theta}_{\text{OLS}} = \frac{1}{n} \sum_{i=1}^n X_i - \frac{1}{n} \sum_{i=1}^n Y_i \sim \mathcal{N}(0, \sigma^2/n) \). Thus, \( \text{Regret}(\hat{\pi}_{\text{ETO}}^\ast \mathcal{F}) = \frac{\sigma^2}{2n} \).

Unfortunately, \( \hat{\pi}_{\text{IERM}}^\ast \mathcal{F} \) and its regret is harder to compute. We can instead study it empirically. Figure 1 displays results for 500 replications for each of \( n = 32, 38, 45, \ldots, 2048 \) with \( \sigma^2 = 1 \). The plot is shown on a log-log scale with linear trend fits. The slope for ETO is \(-1.02\) and for IERM is \(-0.669\). (We also plot IERM where we choose the midpoint of the argmin set for \( \theta \) rather than left endpoint to show not much changes. In the special case of \( \sigma^2 = 0 \), we can actually analytically derive \( \text{Regret}(\hat{\pi}_{\text{IERM}}^\ast \mathcal{F}) = \Theta(1/n^2) \), infinitely slower than \( \text{Regret}(\hat{\pi}_{\text{ETO}}^\ast \mathcal{F}) = 0 \); see Appendix E.1.)

The first thing to note is that both slopes are steeper than the usual \( 1/\sqrt{n} \) convergence rate (i.e., \(-0.5 \) slope), such as is derived in El Balghiti et al. (2019). This suggests the usual theory does not correctly predict the behavior in practice. The second thing to note is that the slope for ETO is steeper than for IERM, with an apparent rate of convergence of \( n^{-1} \) as compared to \( n^{-2/3} \). While ETO is leveraging all the information about \( \mathcal{F} \), IERM is only leveraging what is implied about \( \Pi_{\mathcal{F}} \), so it cannot, for example, distinguish between \( \theta \) values lying between two consecutive observations of \( X \). Our fast (noise-dependent) rates will exactly predict this divergent regret behavior. Note this very simple example is only aimed to illustrate this convergence phenomenon and need not be representative of real problems, which we explore further in Sections 4 and 5.

2. Slow (Noise-Independent) Rates

Our aim is to obtain regret bounds in terms of primitive quantities that are common to both the ETO and IERM approaches. To compare them, we will consider implications of our general results for the case of a correctly specified hypothesis class \( \mathcal{F} \) with bounded complexity. One standard notion of the complexity for scalar-valued functions \( \mathcal{F} \subseteq [\mathbb{R}^p \to \mathbb{R}] \) is the VC-subgraph dimension.
No commonly accepted notions appear to exist for vector-valued classes of functions. Here we define and use an apparently new, natural extension of VC-subgraph dimension.

**Definition 1.** The VC-linear-subgraph dimension of a class of functions $\mathcal{F} \subseteq [\mathbb{R}^p \to \mathbb{R}^d]$ is the VC dimension of the sets $\mathcal{F}^o = \{(x, \beta, t) : \beta^\top f(x) \leq t \} : f \in \mathcal{F}$ in $\mathbb{R}^{p+d+1}$, that is, the largest integer $\nu$ for which there exist $x_1, \ldots, x_\nu \in \mathbb{R}^p$, $\beta_1, \ldots, \beta_\nu \in \mathbb{R}^d$, $t_1, \ldots, t_\nu \in \mathbb{R}$ such that

$$\{(\llbracket \beta_1^\top f(x_1) \leq t_1 \rrbracket, \ldots, \llbracket \beta_\nu^\top f(x_\nu) \leq t_\nu \rrbracket) : f \in \mathcal{F}\} = \{0, 1\}^\nu.$$

Our standing assumption will be that $f^* \in \mathcal{F}$ where $\mathcal{F}$ has bounded VC-linear-subgraph dimension. (In Appendices C.4 and D we study other functions classes, including RKHS and Hölder functions.)

**Assumption 1 (Hypothesis class).** $f^* \in \mathcal{F}$, $\mathcal{F}$ has VC-linear-subgraph dimension at most $\nu$.

**Example 1 (Vector-valued linear functions).** Suppose $\mathcal{F} \subseteq \{Wx : W \in \mathbb{R}^{d \times p}\}$. (Note we can always pre-transform $x$.) Since $\beta^\top f(x) = \text{vec}(W)^\top \text{vec}(\beta x^\top)$, the VC-linear-subgraph dimension of $\mathcal{F}$ is at most the usual VC-subgraph dimension of $\{v \leftrightarrow w^\top v : w \in \mathbb{R}^d\}$, which is $dp$.

**Example 2 (Trees).** Suppose $\mathcal{F}$ consists of all binary trees of depth at most $D$, where each internal node queries “$w^\top x \leq \theta$?” for a choice of $w \in \mathbb{R}^p$, $\beta_0 \in \mathbb{R}$ for each internal node, splitting left if true and right otherwise, and each leaf node assigns the output $v$ to $x$ that reach it, for any choice of $v \in \mathbb{R}^d$ for each leaf node. (In particular, this is a superset of restricting $w$ to be a vector of all zeros except for a single one so that the splits are axis-aligned.) Then, $\mathcal{F}^o$ is contained in the disjunction over leaf nodes of the classes of sets representable by a leaf, which is the conjunction over internal nodes’ half-spaces on the path to the leaf and over the final query of $\beta^\top v \leq t$. Since there are at most $2^D$ leaf nodes and at most $D$ internal nodes on the path to each, applying Van Der Vaart and Wellner (2009) Theorem 1.1 twice, the VC dimension of $\mathcal{F}^o$ is at most $22(D^2p + Dd)2^D \log(8D)$.

### 2.1. Slow Rates for ERM and IERM

We first establish a generalization result for generic ERM for CLO and then apply it to IERM.

**Definition 2.** The Natarajan dimension of a class of functions $\mathcal{G} \subseteq [\mathbb{R}^p \to \mathcal{S}]$ with co-domain $\mathcal{S}$ is the largest integer $\eta$ for which there exist $x_1, \ldots, x_\eta \in \mathbb{R}^p$, $s_1 \neq s'_1, \ldots, s_\eta \neq s'_\eta \in \mathcal{S}$ such that

$$\{(\llbracket g(x_1) = s_1 \rrbracket, \ldots, \llbracket g(x_\eta) = s_\eta \rrbracket) : g \in \mathcal{G}, g(x_1) \in \{s_1, s'_1\}, \ldots, g(x_\eta) \in \{s_\eta, s'_\eta\}\} = \{0, 1\}^\eta.$$

**Theorem 1.** Suppose $\Pi \subseteq [\mathbb{R}^p \to \mathcal{Z}]$ has Natarajan dimension at most $\eta$. Then, for a universal constant $C$, with probability at least $1 - \delta$,

$$\sup_{\pi \in \Pi} \frac{1}{n} \sum_{i=1}^n Y_i^\top \pi(X_i) - \mathbb{E}_X [f^*(X)^\top \pi(X)] \leq CB \sqrt{\frac{\eta \log(|Z| + 1) \log(5/\delta)}{n}}.$$

(6)
Equation (6) also implies that the in-class excess loss, \( \inf_{\pi \in \Pi} E_X [f^*(X)^\top (\hat{\pi}^\text{ERM}(X) - \pi(X))] \), is bounded by twice the right-hand side of Eq. (6). Note El Balghiti et al. (2019) prove a similar result to Theorem 1 but with an additional suboptimal dependence on \( \sqrt{\log(n)} \).

To study IERM, we next relate VC-linear-subgraph dimension to Natarajan dimension.

**Theorem 2.** The VC-linear-subgraph dimension of \( \mathcal{F} \) bounds the Natarajan dimension of \( \Pi_{\mathcal{F}} \).

**Corollary 1.** Suppose Assumption 1 holds. Then, for a universal constant \( C \),

\[
\text{Regret}(\hat{\pi}^\text{IERM}_{\mathcal{F}}) \leq CB\sqrt{\frac{\nu \log(|Z\angle| + 1)}{n}}.
\]

We can in fact show that the rate in Theorem 1 is optimal in \( n \) and \( \eta \) by showing any algorithm must suffer at least this rate on some example. When \( Z = [-1, 1] \) our result reduces to that of Devroye and Lugosi (1995) for binary classification, but we tackle CLO with any polytope \( Z \).

**Theorem 3.** Fix any polytope \( Z \). Fix any \( \Pi \subseteq [\mathbb{R}^p \to Z\angle] \) with Natarajan dimension at least \( \eta \). Fix any algorithm mapping \( D \mapsto \hat{\pi} \in \Pi \). Then there exists a distribution \( \mathbb{P} \) on \((X, Y) \in \mathbb{R}^p \times \mathcal{Y}\) satisfying \( \pi^* \in \Pi \) such that for any \( n \geq 4\eta \), when \( D \sim \mathbb{P}^n \), we have

\[
\text{Regret}(\hat{\pi}) \geq \frac{\rho(Z)}{2e^4} \sqrt{\frac{\eta}{n}},
\]

where \( \rho(Z) = \inf_{z \in Z\angle, z' \in \text{conv}(Z\angle \setminus \{z\})} \|z - z'\| \) (which is positive by definition).

In general Theorem 3 also shows that the rate in Corollary 1 is optimal in \( n \) when we only assume \( \pi^* \in \Pi_{\mathcal{F}} \), but not necessarily in \( \nu \), since Theorem 2 is only an upper bound. In many cases, however, it can be very tight. In Example 1 we upper bounded the VC-linear-subgraph dimension of \( \mathcal{F} \) by \( dp \), while Corollary 29.8 of Shalev-Shwartz and Ben-David (2014) shows the Natarajan dimension of \( \Pi_{\mathcal{F}} \) is at least \((d - 1)(p - 1)\) when \( Z \) is the simplex, so the gap is very small.

### 2.2. Slow Rates for ETO

We next establish comparable rates for ETO. The following is immediate from Cauchy-Schwartz.

**Theorem 4.** Let \( \hat{f} \) be given. Then,

\[
\text{Regret}(\hat{\pi}_f) \leq 2B\mathbb{E}_D\mathbb{E}_X \|f^*(X) - \hat{f}(X)\|.
\]

To study ETO under Assumption 1 we next establish a convergence rate for \( \hat{f}_\mathcal{F} \) to plug in above.

**Theorem 5.** Suppose Assumption 1 holds and that \( \mathcal{F} \) is star shaped at \( f^* \), i.e., \((1 - \lambda)f + \lambda f^* \in \mathcal{F} \) for any \( f \in \mathcal{F} \), \( \lambda \in [0, 1] \). Then, there exist positive universal constants \( C_0, C_1, C_2 > 0 \) such that, for any \( \delta \leq (nd + 1)^{-C_0} \), with probability at least \( 1 - C_1\delta \),

\[
\mathbb{E}_X \|\hat{f}_\mathcal{F}(X) - f^*(X)\| \leq C_2 \sqrt{\frac{\nu \log(1/\delta)}{n}}.
\]
In Appendix C, we prove a novel finite-sample guarantee for least squares with vector-valued response over a general function class $F$, which is of independent interest (relying on existing results for scalar-valued response leads to suboptimal dependence on $d$). Theorem 5 is its application to the VC-linear-subgraph case. The star shape assumption is purely technical but, while it holds for Example 1, it does not for Example 2. We can avoid it by replacing $F$ with $F = \{(1-\lambda)f + \lambda f': f, f' \in F, \lambda \in [0,1]\}$ in Eq. (4) (for Example 2, we even have $F = F + F$), which does not affect the result, only the universal constants. We omit this because least squares over $F$ is not so standard.

Corollary 2. Suppose the assumptions of Theorem 5 hold. Then, for a universal constant $C$,

$$\text{Regret}(\hat{\pi}_{ETO}^F) \leq CB \sqrt{\frac{\nu \log(nd+1)}{n}}.$$  

We can remove the term $\log(nd+1)$ in the specific case of Example 1 (see Corollary 5 in Appendix C.4). Since $\log(|Z^\perp| + 1)$ is generally of order $d$ (Barvinok 2013, Henk et al. 2018), the $d$-dependence above may be better than in Corollary 1 even for general VC-linear-subgraph classes.

Note Corollaries 1 and 2 uniquely enable us to compare ETO and IERM using the same primitive complexity measure. In contrast, complexity measures like bounded metric entropy or Rademacher complexity on $F$ may not provide similar control on the complexity of $\Pi_F$. The slow rates for IERM and ETO are nonetheless the same (up to polylogs), suggesting no differentiation between the two. Studying finer instance characteristics beyond specification reveals the differentiation.

3. Fast (Noise-Dependent) Rates

We next show that much faster rates actually occur in any one instance. To establish this, we characterize the noise in an instance as the level of near-dual-degeneracy (multiplicity of solutions).

Assumption 2 (Noise condition). Let $\Delta(x) = \inf_{z \in Z^\perp(x)} f^*(x)^\top z - \inf_{z \in Z} f^*(x)^\top z$ if $Z^\perp(x) \neq Z$ and otherwise $\Delta(x) = 0$. Assume for some $\alpha, \gamma \geq 0$,

$$\mathbb{P}_X (0 < \Delta(X) \leq \delta) \leq (\gamma \delta/B)^\alpha \ \forall \delta > 0. \quad (7)$$

Assumption 2 controls the mass of $\Delta(X)$ near (but not at) zero. It always holds for $\alpha = 0$ (with $\gamma = 1$). If $\Delta(X) \geq B/\gamma$ is bounded away from zero (akin to strict separation assumptions in Foster et al. 2020, Massart and Nédélec 2006) then Assumption 2 holds for $\alpha \to \infty$. Generically, for any one instance, Assumption 2 holds for some $\alpha \in (0, \infty)$. E.g., if $X$ has a bounded density and $f^*(x)$ has a Jacobian that is uniformly nonsingular (or, if $f^*(x)$ is linear) then Assumption 2 holds with $\alpha = 1$. In particular, the example in Section 1.2 has $\Delta(X) = |X| \sim \text{Unif}[0,1]$ and hence $\alpha = 1$. 


3.1. Fast Rates for ERM and IERM

Under Assumption 2 we can obtain a faster rate both for generic ERM and specifically for IERM.

**Theorem 6.** Suppose Assumption 2 holds, $P(\|Z^*(X)\| > 1) = 0$, $\Pi \subseteq [\mathbb{R}^p \rightarrow Z^\perp]$ has Natarajan dimension at most $\eta$, and $\pi^* \in \Pi$. Then, for a constant $C(\alpha, \gamma)$ depending only on $\alpha, \gamma$,

$$\text{Regret}(\hat{\pi}_{\Pi}^{\text{ERM}}) \leq C(\alpha, \gamma)B\left(\frac{\eta \log(|Z^\perp| + 1) \log(n + 1)}{n}\right)^{\frac{1+\alpha}{2+\alpha}}.$$ 

Whenever $\alpha > 0$, this is faster than the noise-independent rate (Theorem 1). $P(\|Z^*(X)\| > 1) = 0$ requires that, in addition to nice near-dual-degeneracy, we almost never have exact dual degeneracy.

**Corollary 3.** Suppose Assumptions 1 and 2 hold and $P(\|Z^*(X)\| > 1) = 0$. Then,

$$\text{Regret}(\hat{\pi}_{\mathcal{F}}^{\text{IERM}}) \leq C(\alpha, \gamma)B\left(\nu \log(|Z^\perp| + 1) \log(n + 1)\right)^{\frac{1+\alpha}{2+\alpha}}.$$ 

Notice that with $\alpha = 1$, this exactly recovers the rate behavior observed empirically in Section 1.2.

We next show that the rate in $n$ in Theorem 6 and Corollary 3 (and in $\eta$ in the former) is in fact optimal (up to polylogs) under Assumption 2 when we only rely on well specification of the policy.

**Theorem 7.** Fix any $\alpha \geq 0$. Fix any polytope $Z$. Fix any $\Pi \subseteq [\mathbb{R}^p \rightarrow Z^\perp]$ with Natarajan dimension at least $\eta$. Fix any algorithm mapping $D \mapsto \hat{\pi} \in \Pi$. Then there exists a distribution $P$ on $(X, Y) \in \mathbb{R}^p \times \mathcal{Y}$ satisfying $\pi^* \in \Pi$ and Assumption 2 with the given $\alpha$ and $\gamma = B/\rho(Z)$ such that for any $n \geq 2^{2+\alpha}(\eta - 1)$, when $D \sim P^n$, we have

$$\text{Regret}(\hat{\pi}) \geq \frac{\rho(Z)}{2e^4}\left(\eta - 1 \frac{1}{n}\right)^{\frac{1+\alpha}{2+\alpha}}.$$ 

3.2. Fast Rates for ETO

We next show the noise-level-specific rate for ETO is even faster, sometimes much faster. While Theorems 6 and 7 are tight if we only leverage information about the policy class, leveraging the information on $\mathcal{F}$ itself, as ETO does, can break that barrier and lead to better performance.

**Theorem 8.** Suppose Assumption 2 holds and, for universal constants $C_1, C_2$ and a sequence $a_n$, $\hat{f}$ satisfies that, for any $\delta > 0$ and almost all $x$, $P(\|\hat{f}(x) - f^*(x)\| \geq \delta) \leq C_1 \exp(-C_2 a_n \delta^2)$. Then, for a constant $C(\alpha, \gamma)$ depending only on $\alpha, \gamma$,

$$\text{Regret}(\pi_{\hat{f}}) \leq C(\alpha, \gamma)B a_n^{\frac{1+\alpha}{2+\alpha}}.$$ 

While Theorem 4 requires $\hat{f}$ to have good average error, Theorem 8 requires $\hat{f}$ to have a pointwise tail bound on error with rate $a_n$. This is generally stronger but holds for a variety of estimators.
For example, if \( \hat{f} \) is given by, e.g., a generalized linear model then we can obtain \( a_n = n^{\text{McCullagh and Nelder}[1989]} \), which together with Theorem 8 leads to an even better regret rate of \( n^{-\frac{1+\alpha}{2}} \).

While such point-wise rates generally hold when \( \hat{f} \) is parametric, VC-linear-subgraph dimension only characterizes average error so a comparison based on it requires we also make a recoverability assumption to study pointwise error (see also Foster et al. [2020], Hanneke [2011]). In Appendix B we show Assumption 3 generally holds for Examples 1 and 2 (Propositions 1 and 2).

**Assumption 3 (Recovery).** There exists \( \kappa \) such that for all \( f \in \mathcal{F} \) and almost all \( x \),

\[
\| f(x) - f^*(x) \|^2 \leq \kappa \mathbb{E}[\| f(X) - f^*(X) \|^2]
\]

**Corollary 4.** Suppose Assumptions 1 to 3 hold and \( \mathcal{F} \) is star shaped at \( f^* \). Then,

\[
\text{Regret}(\hat{\pi}_{ETo}^{\mathcal{F}}) \leq C(\alpha, \gamma) B \kappa^{1+\alpha} \left( \frac{\nu \log(nd+1)}{n} \right)^{\frac{1+\alpha}{2}}.
\]

With \( \alpha = 1 \), this exactly recovers the rate behavior observed in Section 1.2. We can also remove the \( \log(nd+1) \) term in the case of Example 1 (see Corollary 5 in Appendix C.4). Compared to Corollary 3 we see the regret rate’s exponent in \( n \) is faster by a factor of \( 1 + \frac{\alpha}{2} \). This can be attributed to using all the information on \( \mathcal{F} \) rather than just what is implied about \( \Pi_{\mathcal{F}} \).

### 3.3. Fast Rates for Nonparametric ETO

Assumption 1 is akin to a parametric restriction, but ETO can easily be applied using any flexible nonparametric or machine learning regression. For some such methods we can also establish theoretical results (with correct \( d \)-dependence, compared to relying on existing results for regression). If, instead of Assumption 1, we assume that \( f^* \) is \( \beta \)-smooth (roughly meaning it has \( \beta \) derivatives), then we show in Appendix D how to construct an estimator \( \hat{f} \) satisfying the point-wise condition in Theorem 4 with \( a_n = n^{\frac{\beta}{2d}} / d \) and without a recovery assumption. This leads to a regret rate of \( n^{-\frac{\beta(1+\alpha)}{2d+\beta}} \) for ETO. While slower than the rate in Corollary 4 the restriction on \( f^* \) is nonparametric, and the rate can still be arbitrarily fast as either \( \alpha \) or \( \beta \) grow. In Appendix C.4 we also analyze estimates \( \hat{f} \) based on kernel ridge regression, which we also deploy in experiments in Section 5.

### 4. Considerations for Choosing Separated vs Integrated Approaches

We next provide some perspective on our results and on their implications. We frame this discussion as a comparison between IERM and ETO approaches to CLO along several aspects.

**Regret rates.** Section 2 shows that the noise-level-agnostic regret rates for IERM and ETO have the same \( n^{-1/2} \)-rate (albeit, the ETO rate may also have better \( d \)-dependence). But this hides the fact that specific problem instances do not actually have arbitrarily bad near-degeneracy, i.e., they satisfy Assumption 2 for some \( \alpha > 0 \). When we restrict how bad the near-degeneracy can be,
we obtained fast rates in Section 3. In this regime, we showed that ETO can actually have much better regret rates than IERM. It is important to emphasize that, since specific instances do satisfy Assumption 2, this regime truly captures how these methods actually behave in practice in specific problems. Therefore, in terms of regret rates, this shows a clear preference for ETO approaches.

**Specification.** Our theory focused on the well-specified setting, that is, $f^* \in \mathcal{F}$. When this fails, convergence of the regret of $\hat{\pi}$ to $\pi^*$ to zero is essentially hopeless for any method that focuses only on $\mathcal{F}$. ERM, nonetheless, can still provide best-in-class guarantees: regret to the best policy in $\Pi$ still converges to zero. For induced policies, $\pi_f$, this means IERM gets best-in-class guarantees over $\Pi_F$, while ETO may not. Given the fragility of correct specification if $\mathcal{F}$ is too simple, the ability to achieve best-in-class performance is important and may be the primary reason one might prefer (I)ERM to ETO. Nonetheless, if $\mathcal{F}$ is not well-specified, it begs the question why use IERM rather than ERM directly over some policy class $\Pi$. The benefit of using $\Pi_F$ may be that it provides an easy way to construct a reasonable policy class that respects the decision constraints, $\mathcal{Z}$.

**BYOB (Bring Your Own Blackbox).** While IERM is necessarily given by optimizing over $\mathcal{F}$ and is therefore specified by $\mathcal{F}$, ETO accommodates any regression method as a blackbox, not just least squares. This is perhaps most important in view of specification: many flexible regression methods, including local polynomial or gradient boosting regression, do not take the form of minimization over $\mathcal{F}$. (See Section 3.3 regarding guarantees for the former.)

**Interpretability.** ETO has the benefit of an interpretable output: rather than just having a black box spitting out a decision with no explanation, our output has a clear interpretation as a prediction of $Y$. We can therefore probe this prediction and understand more broadly what other implications it has, such as what happens if we changed our constraints $\mathcal{Z}$ and other counterfactuals. This is absolutely crucial in decision-support applications, which are the most common in practice.

If we care about model explainability – understanding how inputs lead to outputs – it may be preferable to focus on simple models like shallow trees. For these, which are likely not well-specified, IERM has the benefit of at least ensuring best-in-class performance (Elmachtoub et al. 2020).

**Computational tractability.** Another important consideration is tractability. For ETO, this reduces to learning $f$, and both classic and modern prediction methods are often tractable and built to scale. On the other hand, IERM is nonconvex and may be hard to optimize. This is exactly the motivation of Elmachtoub and Grigas (2021), which develop a convex relaxation. However, it is only consistent if $\mathcal{F}$ is well-specified, in which case we expect ETO has better performance.
Contextual stochastic optimization. While we focused on CLO, a question is what do our results suggest for CSO generally. CSO with a finite feasible set (or, set of possibly-optimal solutions), $Z = \{z^{(1)}, \ldots, z^{(K)}\}$, is immediately reducible to CLO by replacing $Z$ with the $K$-simplex and $Y$ with $(c(z^{(1)}; Y), \ldots, c(z^{(K)}; Y))$. Then, our results still apply. Continuous CSO may require a different analysis to account for a non-discrete notion of a noise condition. In either the continuous or finite setting, however, ETO would entail learning a high-dimensional object, being the conditional distribution of $Y \mid X = x$ (or, rather, the conditional expectations $E[c(z; Y) \mid X = x]$ for every $z \in Z$, whether infinite or finite and big). While certainly methods for this exist, if $Z$ has reasonable dimensions, a purely-policy-based approach, such as ERM or IERM, might be more practical. For example, Kallus and Mao (2020) show that directly targeting the downstream optimization problem when training random forests significantly improves forest-based approaches to CSO. This is contrast to the CLO case, where both the decision policy and relevant prediction function have the same dimension, both being functions $\mathbb{R}^p \to \mathbb{R}^d$.

5. Experiments

We next demonstrate these considerations in an experiment. We consider the stochastic shortest path problem shown in Fig. 2a. We aim to go from $s$ to $t$ on a $5 \times 5$ grid, and the cost of traveling on edge $j$ is $Y_j$. There are $d = 40$ edges, and $Z$ is given by standard flow preservation constraints, with a source of $+1$ at $s$ and a sink of $-1$ at $t$. We consider covariates with $p = 5$ dimensions and $f^*(x)$ being a degree-5 polynomial in $x$, as we detail in Appendix E.2.

Ideally we would like to compare ETO to IERM. However, IERM involves a difficult optimization problem that cannot feasibly be solved in practice. We therefore employ the SPO+ loss proposed by Elmachtoub and Grigas (2021), which is a convex surrogate for IERM’s objective function. Like IERM, this is still an end-to-end method that integrates estimation and optimization, standing in stark contrast to ETO, which completely separates the two steps. We consider three different hypothesis classes $F$ for each of ETO (using least-squares regression, $\hat{f}_F$) and SPO+:

- Correct linear: $F$ is as in Example 1 with $\phi(x)$ a 31-dimensional basis of monomials spanning $f^*$. This represents the unrealistic ideal where we have a perfectly-specified parametric model.
- Wrong linear: $F$ is as in Example 1 with $\phi(x) = x \in \mathbb{R}^5$. This represents the realistic setting where parametric models are misspecified.
- Kernel: $F$ is the RKHS with Gaussian kernel, $K(x, x') = \exp(-\rho \|x - x'\|^2)$. This represents the realistic setting of using flexible, nonparametric models.

We employ a ridge penalty in each of the above and choose $\rho$ and this penalty by validation. We use Gurobi to solve the SPO+ optimization problem, except for the RKHS case where due to the heavy computational burden of this we must instead use stochastic gradient descent (SGD)
(a) The CLO instance is a stochastic shortest path problem. We need to go from \( s \) to \( t \). The random cost of an edge \( j \) is \( Y_j \in \mathbb{R} \). Whether we choose to proceed along an edge \( j \) is \( z_j \in \{0,1\} \).

(b) The regret of different methods, relative to average minimal cost. Shaded regions represent 95% confidence intervals.

**Figure 2** Comparing ETO and SPO+ with well-specified, misspecified, and nonparametric hypotheses.

For \( n \) larger than 500. See details in Appendix E.2. By averaging over 50 replications of \( \mathcal{D} \), we estimate relative regret, 

\[
\frac{\mathbb{E}_P\mathbb{E}_X \left[ f^*(X)^\top (\hat{\pi}(X) - \pi^*(X)) \right]}{\mathbb{E}_P\mathbb{E}_X \left[ f^*(X)\pi^*(X) \right]},
\]

for each method and each \( n = 50, 100, \ldots, 1000 \), shown in Fig. 2b with shaded bands for plus/minus one standard error.

Although the theoretical results in Sections 2 and 3 do not directly apply to SPO+, our experimental results support our overall insights. With correctly specified models, the ETO method can achieve better performance than end-to-end methods that integrate estimation with optimization (see circle markers for “Correct linear”). However, from a practical lens, perfectly specified linear models are not realistic. For misspecified linear models, our experiments illustrate how end-to-end methods can account for misspecification to obtain best-in-class performance, beating the corresponding misspecified ETO method (see square markers for “Wrong linear”). At the same time, we see that such best-in-class performance may sometimes still be bad in an absolute sense. Using more flexible models can sometimes close this gap. The kernel model (triangle markers) is still misspecified in the sense that the RKHS does not contain the true regression function and can only approximate it using functions of growing RKHS norm. When using such a flexible model, we observe that ETO achieves regret converging to zero with performance just slightly worse than the correctly-specified case, while end-to-end methods have higher regret. Therefore, even though end-to-end methods handle decision-making problems more directly, our experiments demonstrate that the more straightforward ETO approach can be better even in decision-problem performance.
6. Concluding Remarks

In this paper we studied the regret convergence rates for two approaches to CLO: the naïve, optimization-ignorant ETO and the end-to-end, optimization-aware IERM. We arrived at a surprising fact: the convergence rate for ETO is orders faster than for IERM, despite its ignoring the downstream effects of estimation. We reviewed various reasons for preferring either approach. This highlights a nuanced landscape for the enterprise to integrate estimation and optimization. The practical implications, nonetheless, are positive: relying on regression as a plug-in is easy and fast to run using existing tools, simple to interpret as predictions of uncertain variables, and as our results show it provides downstream decisions with very good performance. Beyond providing new insights with practical implications, we hope our work inspires closer investigation of the statistical behavior of data-driven and end-to-end optimization in other settings. Section 4 points out non-linear CSO as one interesting setting; other settings requiring attention include partial feedback (observe $Y^TZ$, not $Y$, for historical $Z$), sequential/dynamic optimization problems, and online learning. The unique structure of constrained optimization problems brings up new algorithmic and statistical questions, and the right approach is not always immediately clear, as we showed here for CLO.

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Supplemental Material for
Fast Rates for Contextual Linear Optimization

Appendix A: Proofs

A.1. Preliminaries and Definitions
For any integer $q$, we let $[q] = \{1, \ldots, q\}$.

Throughout the following we define
$$
\Psi(t) = \frac{1}{5} \exp(t^2).
$$
Notice that whenever $\mathbb{E}\Psi(|W|/w) \leq 1$ for some random variable $W$, we have by Markov’s inequality that
$$
\mathbb{P}(|W| > t) \leq \frac{1}{5} \exp(-t^2/w^2),
$$
where $q$ is understood from context (will either be $n$ or $nd$, depending on the case). That is, $\mathbb{E}_\sigma$ denotes an expectation over $q$ independent and identically distributed Rademacher random variables independent of all else, in which we marginalize over nothing else (e.g., the data is treated as fixed).

Given a set $S \subseteq \mathbb{R}^q$ we let $D(\epsilon, S)$ be the $\epsilon$-packing number, or the maximal number of elements in $S$ that can be taken so that no two are $\epsilon$ close to one another in Euclidean distance, and $N(\epsilon, S)$ be the $\epsilon$-covering number, or the minimal number of $\mathbb{R}^q$ elements (not necessarily in $S$) needed so that every element of $S$ is at least $\epsilon$ close to one of these in Euclidean distance. It is immediate (see Wainwright 2019, Lemma 5.5) that
$$
N(\epsilon, S) \leq D(\epsilon, S) \leq N(\epsilon/2, S).
$$

The Natarajan dimension of a set $T \subseteq S^q$ (in contrast to a class of functions as in Definition 2) is the largest integer $\eta$ for which there exists $i_1, \ldots, i_\eta \in \{1, \ldots, q\}$ and $s_1 \neq s_1', \ldots, s_\eta \neq s_\eta' \in S$ such that
$$
\{(\mathbb{I}[t_{i_1} = s_1], \ldots, \mathbb{I}[t_{i_\eta} = s_\eta]) : t \in T, t_{i_1} \in \{s_1, s_1'\}, \ldots, t_{i_\eta} \in \{s_\eta, s_\eta'\}\} = \{0, 1\}^\eta.
$$

Thus, the Natarajan dimension of a function class $G \subseteq [\mathbb{R}^p \to S]$ is exactly the largest possible Natarajan dimension of $\{(g(x_1), \ldots, g(x_n)) : g \in G\}$ for $x_1, \ldots, x_n \in \mathbb{R}^p$.

When $S \subseteq \mathbb{R}$, the pseudo-dimension of $T$ (also known as its VC-index or VC-subgraph dimension) is the largest integer $\nu$ for which there exists $i_1, \ldots, i_\nu \in \{1, \ldots, q\}$ and $s_1, \ldots, s_\nu \in S$ such that
$$
\{(\mathbb{I}[t_{i_1} \leq s_1], \ldots, \mathbb{I}[t_{i_\nu} \leq s_\nu]) : t \in T\} = \{0, 1\}^\nu.
$$

The pseudo-dimension (or VC-index or VC-subgraph dimension) of a function class $G \subseteq [\mathbb{R}^p \to S]$ is the largest possible pseudo-dimension of $\{(g(x_1), \ldots, g(x_n)) : g \in G\}$ for $x_1, \ldots, x_n \in \mathbb{R}^p$. Notice that our Definition $\mathbb{I}$ is equivalent to the pseudo-dimension of $\{(\beta, x) \mapsto \beta^T f(x) : f \in F\}$. 

A.2. Slow Rates for ERM and IERM (Section 2.1)

Proof of Theorem 1

Let

\[ h_i(\pi) = Y_i^\top \pi(X_i), \quad h(\pi) = (h_1(\pi), \ldots, h_n(\pi)), \quad H = \{ h(\pi) : \pi \in \Pi \}, \]

\[ L_n(\pi) = \frac{1}{n} \sum_{i=1}^{n} h_i(\pi), \quad L(\pi) = E h_1(\pi) = E[Y^\top \pi(X)] = E[f^\top(X)^\top \pi(X)]. \]

Notice that all of these but \( L(\pi) \) are random objects as they depend on the data.

By [Pollard 1990] Theorem 2.2, for any convex, increasing \( \Phi \),

\[
\mathbb{E} \Phi \left( \sup_{\pi \in \Pi \setminus H} |L_n(\pi) - L(\pi)| \right) \leq \mathbb{E} \Phi \left( \frac{2}{n} \sup_{h \in H} |\langle \sigma, h \rangle| \right). \tag{11}
\]

Notice that \( \sup_{\pi \in \Pi} |h_i(\pi)| \leq B \) for each \( i \). By [Pollard 1990] Theorem 3.5,

\[
\mathbb{E} \Phi \left( \frac{1}{n} \sup_{h \in H} |\langle \sigma, h \rangle| \right) \leq 1, \quad \text{where} \quad J = \frac{2}{n} \int_{0}^{2B \sqrt{\epsilon \sqrt{n}}} \sqrt{\log D(\epsilon, H)} d\epsilon. \tag{12}
\]

Let \( V \) denote the pseudo-dimension of \( H \). By Eq. (10) and [Van Der Vaart and Wellner 1996] Theorem 2.6.7, there exists a universal constant \( K_0 \) such that

\[
D(2B \sqrt{n} \epsilon, H) \leq N(2B \sqrt{n} \epsilon, H) \leq K_0 (V + 1)(16 \epsilon)^{V + 1} \left( \frac{2}{\epsilon} \right)^{2V}.
\]

Therefore,

\[
J = \frac{18B}{\sqrt{n}} \int_{0}^{1} \sqrt{\log D(2B \sqrt{n} \epsilon, H)} d\epsilon \\
\leq \frac{18B}{\sqrt{n}} \int_{0}^{1} \sqrt{\log K_0 + \log (V + 1) + (V + 1) \log (16 \epsilon) + 2V \log 2 - 2V \log \epsilon} \sqrt{\epsilon} d\epsilon \\
\leq 18B \int_{0}^{1} \sqrt{\log K_0 + 3 \log 2 + 2 \log (16 \epsilon) - 2 \log \epsilon} \sqrt{\epsilon} d\epsilon \\
= C' B \sqrt{\frac{V}{n}},
\]

where \( C' = 18 \int_{0}^{1} \sqrt{\log K_0 + 3 \log 2 + 2 \log (16 \epsilon) - 2 \log \epsilon} d\epsilon < \infty \) is some universal constant.

Combining Eqs. (8), (11) and (12),

\[
\Pr(\sup_{\pi \in H} |L_n(\pi) - L(\pi)| > t) \leq 5 \exp(-nt^2/(VB^2 C'^2)).
\]

We now proceed to bound \( V \). Note that \( h_i(\pi) \) can only take values in the multiset \( S_i = \{ Y_i^T z : z \in \mathcal{Z}^- \} \), which has cardinality \( |\mathcal{Z}^-| \). Let \( R_i(\pi) \in \{ |\mathcal{Z}^-| \} \) be the rank of \( h_i(\pi) \) in \( S_i \), where we give ties equal rank, \( R(\pi) = (R_1(\pi), \ldots, R_n(\pi)) \), and \( \bar{H} = \{ R(\pi) : \pi \in \Pi \} \subseteq \{ |\mathcal{Z}^-| \}^n \). Then, the pseudo-dimension \( \bar{H} \) is the same as that of \( H \), i.e., \( V \), and the Natarajan dimension of \( \bar{H} \) is the same as the Natarajan dimension of \( H \), which is at most \( \eta \) by assumption. By Theorem 10 and Corollary 6 of [Ben-David et al. 1995],

\[
V \leq 5\eta \log(|\mathcal{Z}^-| + 1),
\]

completing the proof. \( \square \)
Proof of Theorem 3 Suppose there exist \( x_1, \ldots, x_m \in \mathbb{R}^p \), \( z_1 \neq z'_1, \ldots, z_m \neq z'_m \in \mathcal{Z}^c \) such that for any \( I \subset \{1, \ldots, m\} \), some \( \pi \in \Pi_F \) satisfies that

\[
\pi(x_i) = z_i \quad \forall i \in I, \quad \pi(x_i) = z'_i \quad \forall i \notin I.
\]

For each pair \( z_i, z'_i \), let \( z_i \) be the one first in the tie-breaking preference ordering. This must then necessarily mean that there exists some \( f \in F \) such that

\[
f(x_i) \top z_i \leq f(x_i) \top z'_i \quad \forall i \in I, \quad f(x_i) \top z_i > f(x_i) \top z'_i \quad \forall i \notin I.
\]

Equivalently, letting \( \beta_i = z_i - z'_i \) and \( t_i = 0 \),

\[
\{ (\beta_i \top f(x_i) \leq t_i) \}_{i=1}^m : f \in F \} = \{0,1\}^m,
\]

which must mean that \( m \leq \nu \).

Proof of Corollary 7 Using the definitions of \( L_n, L \) from the proof of Theorem 1 and by optimality of \( \tilde{\pi}^{\text{ERM}}_F \) for \( L_n \) and of \( \pi^* \) for \( L \), we have

\[
L \left( \tilde{\pi}^{\text{ERM}}_F \right) \leq L_n \left( \tilde{\pi}^{\text{ERM}}_F \right) + \sup_{\pi \in \Pi_F} |L(\pi) - L_n(\pi)|
\]

\[
\leq L_n \left( \pi^* \right) + \sup_{\pi \in \Pi_F} |L(\pi) - L_n(\pi)|
\]

\[
\leq L \left( \pi^* \right) + 2 \sup_{\pi \in \Pi_F} |L(\pi) - L_n(\pi)|.
\]

Applying Theorems 1 and 2 we have

\[
\mathbb{P} \left( L \left( \tilde{\pi}^{\text{ERM}}_F \right) - L(\pi^*) > t \right) \leq 5 \exp(-n(t/2)^2/(CB\nu \log |\mathcal{Z}^c| + 1)).
\]

Integrating over \( t \) from 0 to \( \infty \), we obtain for another universal constant \( C' \) that

\[
\mathbb{E} \left[ L \left( \tilde{\pi}^{\text{ERM}}_F \right) - L(\pi^*) \right] \leq C' B \sqrt{\frac{\nu \log |\mathcal{Z}^c| + 1}{n}}.
\]

Iterated expectations reveal that the left-hand side is equal to \( \text{Regret}(\tilde{\pi}^{\text{ERM}}_F) \), completing the proof.

Proof of Theorem 3 To prove the theorem, we will construct a collection of distributions \( \mathbb{P} \) such that the average regret among them satisfies the lower bound; thus, at least one will satisfy the lower bound.

First we will make some preliminary constructions. For any \( z \in \mathcal{Z}^c \), let \( \bar{z} \) be the projection of \( z \) onto \( \text{conv}(\mathcal{Z}^c \setminus \{z\}) \), and define

\[
v(z) = \frac{\bar{z} - z}{\|\bar{z} - z\|}.
\]

We have

\[
\min_{z' \in \mathcal{Z}^c \setminus z \neq z} v(z) \top (z' - z) = \min_{z' \in \mathcal{Z}^c \setminus z \neq z} v(z) \top (\bar{z} - z) + v(z) \top (z' - \bar{z}) \geq \rho(\mathcal{Z}),
\]

where the last inequality comes from the definition of \( \rho(\mathcal{Z}) \) and the fact that \( v(z) \perp (\bar{z} - z') \) due to projection.

Since \( \Pi \) has Natarajan dimension at least \( \eta \), there exist \( x_1, \ldots, x_\eta \in \mathbb{R}^p, z^{(0)}_1 \neq z^{(1)}_1, \ldots, z^{(0)}_\eta \neq z^{(1)}_\eta \in \mathcal{Z}^c \) such that, for every \( b = (b_1, \ldots, b_\eta) \in \{0,1\}^\eta \), there is a \( \pi_b \in \Pi \) such that \( \pi_b(x_i) = z^{(b_i)}_i \) for \( i = 1, \ldots, \eta \).
We now construct a distribution \( P_b \) for each \( b \in \{0, 1\}^\eta \). For the marginal distribution of \( X \), we always put equal mass \( 1/\eta \) at each \( x_i, i = 1, \ldots, \eta \). We next construct the conditional distribution of \( Y \mid X = x_i \). For each \( i \in [\eta] \), let

\[
\begin{align*}
u_{0i} &= \frac{\zeta(\nu(\eta(0)) + \nu(\eta(1))) + \nu(\eta(0)) - \nu(\eta(1))}{2}, \\
u_{1i} &= \frac{\zeta(\nu(\eta(0)) + \nu(\eta(1))) + \nu(\eta(1)) - \nu(\eta(0))}{2},
\end{align*}
\]

We now construct the conditional distribution of \( Y \mid X = x_i \). Set \( \zeta = \sqrt{\eta/m} \) and note \( \zeta \in [0, 1/2] \) by assumption. If \( b_i = 0 \), we let

\[
Y = \begin{cases}
u_{0i}, & \text{with probability } (1 + \zeta)/2, \\
u_{1i}, & \text{with probability } (1 - \zeta)/2,
\end{cases}
\]

and if \( b_i = 1 \), we let

\[
Y = \begin{cases}
u_{1i}, & \text{with probability } (1 + \zeta)/2, \\
u_{0i}, & \text{with probability } (1 - \zeta)/2.
\end{cases}
\]

Since \( \|\nu(\eta(0))\| = \|\nu(\eta(1))\| = 1 \) by definition, triangle inequality yields \( \|\nu_{0i}\| \leq 1, \|\nu_{1i}\| \leq 1 \), and hence the above distribution is on \( \mathcal{Y} \). We then have that,

\[
\mathbb{E}_{P_b}[Y \mid X = x_i] = f_b(x_i) = \zeta \nu(\eta^{(b_i)}).
\]

By Eq. (13), the optimal decision at \( x_i \), is uniquely \( \nu^{(b_i)} \). That is, the optimal policy is \( \pi_b \), which is in \( \Pi \), as was desired.

For \( \hat{\pi} \in \Pi \), define \( \hat{b} \in \{0, 1\}^\eta \) to be a binary vector whose \( i \)th element is \( \hat{b}_i = \mathbb{I}\{\hat{\pi}(x_i) = \nu^{(1)}\} \). Consider a prior on \( b \) such that \( b_1, \ldots, b_\eta \) are i.i.d. and \( b_1 \sim \text{Ber}(1/2) \). Let \( \text{Regret}_b(\hat{\pi}) \) denote the regret when the data is drawn from \( P_b \), the regret satisfies the following inequalities:

\[
\sup_{b \in \{0, 1\}^\eta} \text{Regret}_b(\hat{\pi}) = \sup_{b \in \{0, 1\}^\eta} \mathbb{E}_{P_b} \mathbb{E}_X [f_b(X)^\top (\hat{\pi}(X) - \pi_b(X))]
\]
\[
\geq \mathbb{E}_b \mathbb{E}_{P_\hat{b}} \mathbb{E}_X [f_b(X)^\top (\hat{\pi}(X) - \pi_b(X))]
\]
\[
= \mathbb{E}_b \mathbb{E}_{P_\hat{b}} \mathbb{E}_X [f_b(X)^\top (\hat{\pi}(X) - \pi_b(X)) \mathbb{I}[\hat{\pi}(X) \neq \pi_b(X)]]
\]
\[
\geq \zeta \rho(\mathcal{Z}) \mathbb{E}_b \mathbb{E}_{P_\hat{b}} \mathbb{E}_X [\mathbb{I}[\hat{\pi}(x_i) \neq \pi_b(x_i)]]
\]
\[
\geq \zeta \rho(\mathcal{Z}) \sum_{i=1}^{\eta} \mathbb{E}_b \mathbb{E}_{P_\hat{b}} [\mathbb{I}[b_i \neq \hat{b}_i]],
\]

where the second inequality comes from Eq. (13), and the third inequality comes from the fact that \( b_i \neq \hat{b}_i \) implies \( \hat{\pi}(x_i) \neq \pi_b(x_i) \).

The term \( \mathbb{E}_b \mathbb{E}_{P_\hat{b}} [\mathbb{I}[b_i \neq \hat{b}_i]] \) above is the Bayes risk of the algorithm \( D \mapsto \hat{b} \), with respect to the loss function being the misclassification error of the random bit \( b_i \). We will now lower bound this by computing the minimum Bayes risk. Letting \( \hat{P} \) denote the distribution of \( (b, D) \) when we draw \( b_i \) as iid Bernoulli as
above and then data as \( (D \mid b) \sim P^n_b \). Then \( \tilde{P}(b_i = 1 \mid D) \) is the posterior probability that \( b_i = 1 \) and the minimum Bayes risk is simply \( \min \left\{ \tilde{P}(b_i = 1 \mid D), 1 - \tilde{P}(b_i = 1 \mid D) \right\} \). We conclude that

\[
\zeta \rho(Z) \frac{\eta}{\eta} \sum_{i=1}^{\eta} E_{b_i} E_{\tilde{b}_i} \left[ \mathbb{I}_{b_i \neq \tilde{b}_i} \right] \geq \zeta \rho(Z) \frac{\eta}{\eta} \sum_{i=1}^{\eta} E_{\tilde{b}_i} \left[ \min \left\{ \tilde{P}(b_i = 1 \mid D), 1 - \tilde{P}(b_i = 1 \mid D) \right\} \right].
\]

We proceed to calculate the latter. For \( i \in [\eta] \), let

\[
N_i^0 = \sum_{j=1}^{\eta} \mathbb{I}_{X_j = x_i, Y_j = u_{0i}}, \quad N_i^1 = \sum_{j=1}^{\eta} \mathbb{I}_{X_j = x_i, Y_j = u_{1i}}.
\]

We can then write the posterior distribution as

\[
\tilde{P}(b_i = 1 \mid D) = \frac{\left( \frac{1 + \xi}{2} \right)^{N_i^1} \left( \frac{1 - \xi}{2} \right)^{N_i^0}}{\left( \frac{1 + \xi}{2} \right)^{N_i^1 + N_i^0} + \left( \frac{1 + \xi}{2} \right)^{N_i^0} \left( \frac{1 - \xi}{2} \right)^{N_i^1}}.
\]

Hence,

\[
\min \left\{ \tilde{P}(b_i = 1 \mid D), 1 - \tilde{P}(b_i = 1 \mid D) \right\} = \frac{\min \left\{ \left( \frac{1 + \xi}{2} \right)^{N_i^1} \left( \frac{1 - \xi}{2} \right)^{N_i^0}, \left( \frac{1 + \xi}{2} \right)^{N_i^0} \left( \frac{1 - \xi}{2} \right)^{N_i^1} \right\}}{\left( \frac{1 + \xi}{2} \right)^{N_i^1 + N_i^0} + \left( \frac{1 + \xi}{2} \right)^{N_i^0} \left( \frac{1 - \xi}{2} \right)^{N_i^1}}
\]

\[
= \frac{1}{1 + \left( \frac{1 + \xi}{1 - \xi} \right)^{|N_i^1 - N_i^0|}}.
\]

Therefore,

\[
\zeta \rho(Z) \frac{\eta}{\eta} \sum_{i=1}^{\eta} E_{\tilde{b}_i} \left[ \min \left\{ \tilde{P}(b_i = 1 \mid D), 1 - \tilde{P}(b_i = 1 \mid D) \right\} \right] = \zeta \rho(Z) \frac{\eta}{\eta} \sum_{i=1}^{\eta} E_{\tilde{b}_i} \left[ \left( 1 + \left( \frac{1 + \xi}{1 - \xi} \right)^{|N_i^1 - N_i^0|} \right)^{-1} \right]
\]

\[
\geq \zeta \rho(Z) \frac{2\eta}{\eta} \sum_{i=1}^{\eta} E_{\tilde{b}_i} \left[ \left( 1 + \frac{\xi}{1 - \xi} \right)^{-|N_i^1 - N_i^0|} \right]
\]

\[
\geq \zeta \rho(Z) \frac{2\eta}{\eta} \sum_{i=1}^{\eta} \left( 1 + \frac{\xi}{1 - \xi} \right)^{-|N_i^1 - N_i^0|},
\]

where the first inequality is due to the fact that \( \frac{1 + \xi}{1 - \xi} \geq 1 \) and the second inequality follows from Jensen’s inequality. Given our symmetric prior distribution on \( b \), the marginal distribution of \( Y_i \) given \( X_j = x_i \) is \( \tilde{P}(Y_j = u_{0i} \mid X_j = x_i) = \tilde{P}(Y_j = u_{1i} \mid X_j = x_i) = 1/2 \). Thus, letting Bin \((k, 1/2)\) represent a binomial random variable with parameters \( k \) and 1/2, we have

\[
E_{\tilde{b}_i} |N_i^1 - N_i^0| = \sum_{k=0}^{\eta} \binom{n}{k} \left( \frac{1}{\eta} \right)^k \left( 1 - \frac{1}{\eta} \right)^{n-k} \mathbb{E} \left[ 2 \text{Bin} \left( k, \frac{1}{2} \right) - k \right]
\]

\[
\leq \sum_{k=0}^{\eta} \binom{n}{k} \left( \frac{1}{\eta} \right)^k \left( 1 - \frac{1}{\eta} \right)^{n-k} \sqrt{\mathbb{E} \left[ (2 \text{Bin} \left( k, \frac{1}{2} \right) - k)^2 \right]} \sqrt{k}
\]

\[
= \sum_{k=0}^{\eta} \binom{n}{k} \left( \frac{1}{\eta} \right)^k \left( 1 - \frac{1}{\eta} \right)^{n-k} \sqrt{k}
\]

\[
= \sqrt{\binom{n}{\frac{1}{\eta}}}
\]

\[
\leq \sqrt{\frac{\pi}{\eta}}.
\]
where the two inequalities follow from the Cauchy-Schwarz inequality.

Putting our calculations together, we get

\[
\sup_{b \in \{0, 1\}^n} \text{Regret}_b(\hat{\pi}) \geq \frac{\zeta \rho(Z)}{2} \exp \left( -\frac{2\zeta}{1-\zeta} \sqrt{\frac{n}{\eta}} \right),
\]

where the second inequality follows from the fact that \(1 + x \leq e^x\) for any \(x \in \mathbb{R}\). Finally, plugging in \(\zeta = \sqrt{\eta/n}\) and noting that \(\zeta \leq 1/2\) since \(n \geq 4\eta\), we have

\[
\frac{\zeta \rho(Z)}{2} \exp \left( -\frac{2\zeta}{1-\zeta} \sqrt{\frac{n}{\eta}} \right) = \rho(Z) \sqrt{\frac{\eta}{n}} \exp \left( -\frac{2}{1-\zeta} \sqrt{\frac{n}{\eta}} \right),
\]

as desired. \(\square\)

A.3. Slow Rates for ETO (Section 2.2)

The proof of Theorem 5 is very involved and is therefore relegated to its own Appendix C.

**Proof of Theorem 4**

By optimality of \(\pi\) with respect to \(\hat{f}\), we have that

\[
\text{Regret}(\hat{\pi}) = E \left[ f^*(X)^\top (\hat{\pi}(X) - \pi^*(X)) \right] \leq E \left[ f^*(X)^\top \pi_j(X) - \hat{f}(X)^\top \pi_j(X) + \hat{f}(X)^\top \pi^*(X) - f^*(X)^\top \pi^*(X) \right] \leq 2B E \left[ \|f^*(X) - \hat{f}(X)\| \right].
\]

\(\square\)

**Proof of Corollary 2**

The result follows by integrating the tail bound from Theorem 5 to bound the expected error and invoking Theorem 4. \(\square\)

A.4. Fast Rates for ERM and IERM (Section 3.1)

A.4.1. Preliminaries and Definitions.

For any policy \(\pi, \pi' \in [\mathbb{R}^p \to \mathcal{Z}]\), define

\[
d(\pi, \pi') = \frac{1}{B} E_X [f^*(X)^\top (\pi'(X) - \pi(X))],
\]

\[
d_\Delta(\pi, \pi') = P_X(\pi(X) \neq \pi'(X)).
\]

In this section, we let \(E_P\) be the expectation with respect to \(P_{X,Y}\), \(E_D\) the expectation with respect to the sampling of data \(D\), and \(E_n\) the expectation with respect to the empirical distribution. Moreover, for any function \(h(x, y)\), we define \(\|h\|_{L_2(\rho)} = \sqrt{E_P[h^2(X, Y)]}\).

A.4.2. Supporting Lemmas. We first show that \(d\) and \(d_\Delta\) have the following relationship:

**Lemma 1.** Suppose Assumption 3 holds and \(P(\|Z^*(X)\| > 1) = 0\). Then

\[
d(\pi^*, \pi) \leq 2d_\Delta(\pi^*, \pi),
\]

\[
d_\Delta(\pi^*, \pi) \leq c_1 d(\pi^*, \pi) \gamma^\alpha,
\]

where \(c_1 = (\alpha \gamma^\alpha)^{-\frac{\alpha}{1-\alpha}} (\alpha + 1)^\gamma\).
Proof of Lemma [2] First of all,
\[
d(\pi^*, \pi) = \frac{1}{B} \mathbb{E}_X [f^*(X)^T (\pi(X) - \pi^*(X))] \mathbb{I}[\pi(X) \neq \pi^*(X)] \\
\leq 2 \mathbb{P}_X (\pi(X) \neq \pi^*(X)) \\
= 2d_\Delta (\pi^*, \pi).
\]

Now we prove the second statement. For any \( t > 0 \), we have
\[
d(\pi^*, \pi) = \frac{1}{B} \mathbb{E}_X [f^*(X)^T (\pi(X) - \pi^*(X))] \mathbb{I}[\pi(X) \neq \pi^*(X)] \\
\geq \frac{1}{B} \mathbb{E}_X [f^*(X)^T (\pi(X) - \pi^*(X))] \mathbb{I}[\pi(X) \neq \pi^*(X), \Delta(X) > tB] \\
\geq t \mathbb{P}_X (\pi(X) \neq \pi^*(X), \Delta(X) > tB) \\
= t[d_\Delta (\pi^*, \pi) - \mathbb{P}_X (\pi(X) \neq \pi^*(X), \Delta(X) \leq tB)] \\
\geq t[d_\Delta (\pi^*, \pi) - \mathbb{P}_X (\Delta(X) \leq tB)] \\
\geq t[d_\Delta (\pi^*, \pi) - \gamma^\alpha t^\alpha].
\]

If we take \( t = ((\alpha + 1) \gamma^\alpha)^{-1/\alpha} [d_\Delta (\pi^*, \pi)]^{1/\alpha} \), we have
\[
d(\pi^*, \pi) \geq \alpha \gamma^\alpha \frac{((\alpha + 1) \gamma^\alpha)^{-1/\alpha} d_\Delta (\pi^*, \pi)^{\alpha+1}}{\alpha}.
\]

Therefore,
\[
d_\Delta (\pi^*, \pi) \leq \frac{\alpha \gamma^\alpha}{\alpha+1} ((\alpha + 1) \gamma^\alpha d(\pi^*, \pi)^{\alpha+1}).
\]

We will also need the following concentration inequality due to Bousquet [2002].

**Lemma 2.** Let \( \mathcal{H} \) be a countable family of measurable functions such that \( \sup_{h \in \mathcal{H}} E_P(h^2) \leq \delta^2 \) and \( \sup_{h \in \mathcal{H}} \|h\|_\infty \leq \bar{H} \) for some constants \( \delta \) and \( \bar{H} \). Let \( S = \sup_{h \in \mathcal{H}} (E_n(h) - E_P(h)) \). Then for every \( t > 0 \),
\[
\mathbb{P} \left( S - E(S) \geq \sqrt{\frac{2(\delta^2 + 4H^2E(S))}{n} t + \frac{2\bar{H}t}{3n}} \right) \leq \exp(-t).
\]

Note the restriction on countable \( \mathcal{H} \). If an uncountable \( \mathcal{H} \), however, satisfies \( \sup_{h \in \mathcal{H}} E_P(h^2) \leq \delta^2 \) and \( \sup_{h \in \mathcal{H}} \|h\|_\infty \leq \bar{H} \) and is separable with respect to \( \max \{L_2(P), L_2(P_n)\} \) then we can just take a dense countable subset, apply Lemma [2] and obtain the result for the uncountable \( \mathcal{H} \), since the random variable \( S \) would be unchanged. In particular, we have the above separability if \( \mathcal{H} \) has finite packing numbers with respect to \( L_2(Q) \) for any \( Q \) because we can simply take the union of \( (1/k) \)-maximal-packings with respect to \( L_2((P_n + P)/2) \) for \( k = 1, 2, \ldots \) (note we take the packings and not coverings to ensure the points are inside the set). In the below, our set \( \mathcal{H} \) has a finite pseudo-dimension and therefore finite packing numbers with respect to \( L_2(Q) \) for any \( Q \) (Van Der Vaart and Wellner [1996] Theorem 2.6.7).

Finally, the following lemma bounds the mean of a supremum of a centered empirical process indexed by functions with bounded \( L_2(P) \) norm.
Lemma 3. Suppose $\Pi \subseteq [\mathbb{R}^p \rightarrow Z]$ has Natarajan dimension at most $\eta$. Define a class of functions indexed by $\pi \in \Pi$:

$$H_\delta = \{ h(X, Y; \pi) = \frac{1}{B} (Y^T \pi^*(X) - Y^T \pi(X)) : \pi \in \Pi, \|h\|_{L^2(P)} \leq \delta \}.$$ 

There exists a universal constant $C_0$ such that for any $n \geq \frac{20C_0^2 \log(\|Z\|^2 + 1) \log(n+1)}{\delta^2}$,

$$\mathbb{E}_\delta[\sup_{h \in H_\delta} (\mathbb{E}_n(h) - \mathbb{E}_\rho(h))] \leq (1 + \sqrt{2})C_0 \sqrt{\frac{5\eta \log(\|Z\|^2 + 1) \log(n+1)}{n}} \delta.$$

Proof of Lemma 3. Fix $(X_1, Y_1), \ldots, (X_n, Y_n)$. Define $h(\pi) = (h(X_1, Y_1; \pi), \ldots, h(X_n, Y_n; \pi))$ and $H_\delta = \{ h(\pi) : h(\cdot; \pi) \in H_\delta \} \subseteq \mathbb{R}^n$. Let $V$ denote the pseudo-dimension of $H_\delta$. Let $\delta_n = \frac{1}{\sqrt{n}} \sup_{h \in H_\delta} \|h^*\|$ and $H_\delta$ be the envelope of $H_\delta$. We have $\|H_\delta\| \leq n\delta_n$. By Pollard [1990] Theorem 2.2,

$$\mathbb{E}_\delta[\sup_{h \in H_\delta} (\mathbb{E}_n(h) - \mathbb{E}_\rho(h))] \leq \mathbb{E}_\delta\mathbb{E}_\sigma \left[ \frac{2}{\sup_{h \in H_\delta}} |\langle \sigma, h \rangle| \right].$$

By Pollard [1990] Theorem 3.5,

$$\mathbb{E}_\sigma\sqrt{\frac{1}{n} \mathbb{E}_n(|\langle \sigma, h \rangle|)} \leq 1,$$

where $J = \frac{9}{n} \int_0^{\sqrt{\pi} \delta_n} \sqrt{\log D(\epsilon, H_\delta)} d\epsilon. \tag{17}$

By Eq. (10) and Van Der Vaart and Wellner [1996] Theorem 2.6.7), there exists a universal constant $K_0$ such that

$$D(\sqrt{n\delta_n} x, H_\delta) \leq N \left( \frac{1}{2} \frac{\sqrt{n\delta_n}}{n} \right) \delta_n$$

$$\leq N \left( \frac{x}{2\sqrt{n}} \right) \|H_\delta\|,$$

$$\leq K_0 (V+1)(16e)^{V+1} \left( \frac{2\sqrt{n}}{x} \right)^{2V}.$$

Therefore,

$$J = \frac{9}{\sqrt{n}} \int_0^1 \delta_n \sqrt{\log D(\sqrt{n\delta_n} x, H_\delta)} dx$$

$$\leq \frac{9}{\sqrt{n}} \int_0^1 \delta_n \sqrt{\log K_0 + \log(V+1) + (V+1) \log(16e) + V \log n + 2V \log 2 - 2V \log x} dx$$

$$\leq 9 \int_0^1 \sqrt{2 \log K_0 + 4 + 4 \log(16e) - 4 \log x} \sqrt{\frac{V \log(n+1)}{n}} \delta_n$$

$$= C_0 \frac{\sqrt{V \log(n+1)}}{10} \delta_n,$$

where $C_0 = 90 \int_0^1 \sqrt{2 \log K_0 + 4 + 4 \log(16e) - 4 \log x} dx < \infty$. By Eq. (9),

$$\mathbb{E}_\sigma\left[ \frac{1}{n} \sup_{h \in H_\delta} |\langle \sigma, h \rangle| \right] \leq \frac{C_0}{2} \sqrt{\frac{V \log(n+1)}{n}} \delta_n,$$

and combining Eqs. (16) and (17) we get

$$\mathbb{E}_\delta[\sup_{h \in H_\delta} (\mathbb{E}_n(h) - \mathbb{E}_\rho(h))] \leq C_0 \sqrt{\frac{V \log(n+1)}{n} \mathbb{E}_\delta(\delta_n)}$$

$$= C_0 \sqrt{\frac{V \log(n+1)}{n} \mathbb{E}_\delta \left( \sup_{h \in H_\delta} (\mathbb{E}_n(h^2))^{1/2} \right)}$$

$$\leq C_0 \sqrt{\frac{V \log(n+1)}{n} \left( \mathbb{E}_\delta \left( \sup_{h \in H_\delta} (\mathbb{E}_n(h^2)) \right) \right)^{1/2}}. \tag{18}$$
Note that \( \mathbb{E}_n(h^2) \) can be bounded by
\[
\mathbb{E}_n(h^2) = \mathbb{E}_n(h^2 - \mathbb{E}_P(h^2)) + \mathbb{E}_P(h^2)
\]
\[
= \mathbb{E}_n((h - \|h\|_{L^2(P)})(h + \|h\|_{L^2(P)})) + \|h\|_{L^2(P)}^2
\]
\[
\leq 4 \mathbb{E}_n(h - \|h\|_{L^2(P)}) + \delta^2
\]
\[
\leq 4 \mathbb{E}_n(h - \mathbb{E}_P(h)) + \delta^2.
\]
Combining with Eq. (18) we get
\[
\mathbb{E}_P[\sup_{h \in \mathcal{H}_s} (\mathbb{E}_n(h) - \mathbb{E}_P(h))] \leq C_0 \sqrt{\frac{V \log(n+1)}{n}} \sqrt{4 \mathbb{E}_P[\sup_{h \in \mathcal{H}_s} (\mathbb{E}_n(h) - \mathbb{E}_P(h))] + \delta^2}.
\]
Solving this inequality for \( \mathbb{E}_P[\sup_{h \in \mathcal{H}_s} (\mathbb{E}_n(h) - \mathbb{E}_P(h))] \) we get
\[
\mathbb{E}_P[\sup_{h \in \mathcal{H}_s} (\mathbb{E}_n(h) - \mathbb{E}_P(h))] \leq 2C_0 \sqrt{\frac{V \log(n+1)}{n}} \left( \sqrt{V \log(n+1)} + \frac{\delta^2}{4C_0^2} \right).
\]
When \( \frac{V \log(n+1)}{n} \leq \frac{\delta^2}{4C_0^2} \), i.e., when \( n \geq \frac{4C_0^2 V \log(n+1)}{\delta^2} \), we have
\[
\mathbb{E}_P[\sup_{h \in \mathcal{H}_s} (\mathbb{E}_n(h) - \mathbb{E}_P(h))] \leq (1 + \sqrt{2}) C_0 \sqrt{\frac{V \log(n+1)}{n}} \delta.
\]
Finally, by similar arguments as in the proof of Theorem 1
\[
V \leq 5\eta \log \left( \left| \mathcal{Z}^c \right|^2 + 1 \right),
\]
completing the proof. \( \square \)

### A.4.3. Proof of Theorem 6 and Corollary 3

**Proof of Theorem 6** Note that
\[
d(\pi, \pi') = \frac{1}{B} \mathbb{E}_P[Y^\top (\pi'(X) - \pi(X))],
\]
and define
\[
\mathcal{H}_\Pi = \{ h(X, Y; \pi) = \frac{1}{B} (Y^\top \pi^*(X) - Y^\top \pi(X)) : \pi \in \Pi \}.
\]
Because \( \|Y\| \leq 1 \), \( \sup_{z \in \mathbb{R}} \|z\| \leq B \), \( \mathcal{H}_\Pi \) has envelope 2. Besides, we can write \( d(\pi^*, \pi) = -\mathbb{E}_P(h(X, Y; \pi)) \), and we know that \( -\mathbb{E}_P(h) \geq 0 \) for all \( h \in \mathcal{H}_\Pi \). Moreover, we can define the sample analogue of \( d(\pi, \pi') \) as
\[
d_n(\pi, \pi') = \frac{1}{nB} \sum_{i=1}^{n} Y_i^\top (\pi'(X_i) - \pi(X_i)).
\]
Let \( a = \sqrt{\kappa t \epsilon_n} \) with \( \kappa \geq 1, t \geq 1, \) and \( \epsilon_n > 0 \), where \( t \geq 1 \) is arbitrary, \( \kappa \) is a constant that we choose later, and \( \epsilon_n \) is a sequence indexed by sample size \( n \) whose proper choice will be discussed in a later step. Define
\[
\mathbb{V}_a = \sup_{h \in \mathcal{H}_\Pi} \left( \frac{\mathbb{E}_n(h) - \mathbb{E}_P(h)}{-\mathbb{E}_P(h) + a^2} \right) = \sup_{h \in \mathcal{H}_\Pi} \left( \mathbb{E}_n \left( \frac{h}{-\mathbb{E}_P(h) + a^2} \right) - \mathbb{E}_P \left( \frac{h}{-\mathbb{E}_P(h) + a^2} \right) \right).
\]
By definition \( \hat{\pi}_\Pi^{ERM} = \arg \min_{\pi \in \Pi} \frac{1}{n} \sum_{i=1}^{n} Y_i^\top \pi(X_i) \). Since \( \pi^* \in \Pi \), \( d_n(\pi^*, \hat{\pi}_\Pi^{ERM}) \leq 0 \) and
\[
d(\pi^*, \hat{\pi}_\Pi^{ERM}) \leq d(\pi^*, \hat{\pi}_\Pi^{ERM}) - d_n(\pi^*, \hat{\pi}_\Pi^{ERM})
\]
\[
= \mathbb{E}_n(h(X, Y; \hat{\pi}_\Pi^{ERM})) - \mathbb{E}_P(h(X, Y; \hat{\pi}_\Pi^{ERM}))
\]
\[
\leq \mathbb{V}_a[d(\pi^*, \hat{\pi}_\Pi^{ERM}) + a^2].
\]
On the event $V_a < 1/2$, we have $d(\pi^*, \hat{\pi}^{ERM}_\Pi) < a^2$ holds, which implies

$$
P(d(\pi^*, \hat{\pi}^{ERM}_\Pi) \geq a^2) \leq P(V_a \geq 1/2).$$

(19)

In what follows, we aim to prove that $P(V_a \geq 1/2) \leq \exp(-t)$.

First of all, note that for all $h \in \mathcal{H}_\Pi$,

$$\mathbb{E}_P\left(h \left( -\mathbb{E}_P(h) + a^2 \right)^2 \right) \leq \frac{4d_\Delta(\pi^*, \pi)}{(-\mathbb{E}_P(h) + a^2)^2} \leq 4c_1 \frac{\left(-\mathbb{E}_P(h)\right)^{2\frac{1}{1\gamma \pi}}}{\left(-\mathbb{E}_P(h) + a^2\right)^2} \leq 4c_1 \sup_{\epsilon \geq 0} \frac{\epsilon^{2\frac{1}{1\gamma \pi}}}{(\epsilon^2 + a^2)^2} \leq 4c_1 \frac{1}{a^2} \sup_{\epsilon \geq 0} \frac{\epsilon^{2\frac{1}{1\gamma \pi}}}{\epsilon \vee a} \leq 4c_1 a^{2\frac{1}{1\gamma \pi}}$$.  

where $c_1$ is the constant in Lemma 1. Moreover,

$$\sup_{h \in \mathcal{H}_\Pi} \left\| \frac{h}{-\mathbb{E}_P(h) + a^2} \right\|_\infty \leq \frac{2}{a^2}.$$

By Lemma 2

$$P \left( V_a \leq \mathbb{E}(V_a) + \sqrt{8(c_1 a^{2\frac{1}{1\gamma \pi}})^2 + 2\mathbb{E}(V_a)t} + \frac{4t}{3a^2 n} \right) \geq 1 - \exp(-t).$$

(20)

We now aim to prove an upper bound on $\mathbb{E}(V_a)$. Let $r > 1$ be arbitrary and partition $\mathcal{H}_\Pi$ by $\mathcal{H}_0, \mathcal{H}_1, \ldots$ where $\mathcal{H}_0 = \{ h \in \mathcal{H}_\Pi : -\mathbb{E}_P(h) \leq a^2 \}$ and $\mathcal{H}_j = \{ h \in \mathcal{H}_\Pi : r^{2(j-1)}a^2 < -\mathbb{E}_P(h) \leq r^{2j}a^2 \}$ for $j \geq 1$. Then,

$$V_a \leq \sup_{h \in \mathcal{H}_0} \mathbb{E}_n(h) - \mathbb{E}_P(h) + a^2) + \sum_{j \geq 1} \left( 1 + r^{2(j-1)} \right)^{-1} \sup_{h \in \mathcal{H}_j} \mathbb{E}_n(h) - \mathbb{E}_P(h) \right) \leq \frac{1}{a^2} \left[ \sup_{h \in \mathcal{H}_0} \mathbb{E}_n(h) - \mathbb{E}_P(h) \right] + \sum_{j \geq 1} \left( 1 + r^{2(j-1)} \right)^{-1} \sup_{h \in \mathcal{H}_j} \mathbb{E}_n(h) - \mathbb{E}_P(h) \right) \leq \frac{1}{a^2} \left[ \sup_{h \in \mathcal{H}_0} \mathbb{E}_n(h) - \mathbb{E}_P(h) \right] + \sum_{j \geq 1} \left( 1 + r^{2(j-1)} \right)^{-1} \sup_{h \in \mathcal{H}_j} \mathbb{E}_n(h) - \mathbb{E}_P(h) \right) \right].$$

(21)

By Lemma 1

$$||h||_{L_2(P)}^2 = \mathbb{E}_P(h^2) \leq 4d_\Delta(\pi^*, \pi) \leq 4c_1 \left[ -\mathbb{E}_P(h) \right]^{2\frac{1}{1\gamma \pi}}$$

so we know that $-\mathbb{E}_P(h) \leq r^{2j}a^2$ implies $||h||_{L_2(P)} \leq 2c_1^{1/2} r^{2\frac{1}{1\gamma \pi}} a^{\frac{1}{1\gamma \pi}}$. Thus, Eq. (21) can be further bounded by

$$V_a \leq \frac{1}{a^2} \left[ \sup_{||h||_{L_2(P)} \leq 2c_1^{1/2} r^{2\frac{1}{1\gamma \pi}} a^{\frac{1}{1\gamma \pi}}} \mathbb{E}_n(h) - \mathbb{E}_P(h) \right] + \sum_{j \geq 1} \left( 1 + r^{2(j-1)} \right)^{-1} \sup_{||h||_{L_2(P)} \leq 2c_1^{1/2} r^{2\frac{1}{1\gamma \pi}} a^{\frac{1}{1\gamma \pi}}} \mathbb{E}_n(h) - \mathbb{E}_P(h) \right).$$

For the rest of the proof, we let $\bar{V} = 5n\log(|Z^c|^2 + 1)$ for notational simplicity. By Lemma 3

\[
\mathbb{E}_D[V_a] \leq 2(1 + \sqrt{2})C_0c_1^{1/2} \frac{\sqrt{V \log(n+1)}}{n} a^{\frac{\alpha}{1+n}} \leq \left[ 1 + \sum_{j \geq 1} (1 + r^{2(j-1)})^{-1} r^{-\frac{\alpha}{1+n}} \right] \frac{2C_0c_1^{1/2} \sqrt{V \log(n+1)}}{n} a^{\frac{\alpha}{1+n}} \leq c_2 \frac{\sqrt{V \log(n+1)}}{n} a^{\frac{\alpha}{1+n}} - 2
\]

for

\[
n \geq \frac{4C_0^2 \bar{V} \log(n+1)}{c_1 a^{\frac{\alpha}{1+n}}} \iff a \geq \left( \frac{4C_0^2 \bar{V} \log(n+1)}{c_1} \right)^{\frac{1+n}{\alpha}} - \left( \frac{V \log(n+1)}{n} \right)^{\frac{1+n}{\alpha}}.
\]

where $c_2 = 2(1 + \sqrt{2})C_0c_1^{1/2}\left( \frac{\bar{V} \log(n+1)}{1 + r^{-\frac{\alpha}{1+n}}} \right) \sqrt{1}$ Plugging this back into Eq. (20) we get with probability at least $1 - \exp(-t)$

\[
V_a \leq c_2 \sqrt{\frac{V \log(n+1)}{n} a^{\frac{\alpha}{1+n}} - 2} + \sqrt{\frac{8}{a^2 n} \left( \frac{c_1 a^{\frac{2\alpha}{1+n}}}{4c_2^2} + 2c_2 \frac{\sqrt{V \log(n+1)}}{n} a^{\frac{\alpha}{1+n}} - 2 \right) t} + \frac{4t}{3a^2 n}.
\]

Choose $\epsilon_n$ to be

\[
\epsilon_n = \left( c_2 \sqrt{\frac{V \log(n+1)}{n}} \right)^{\frac{1+n}{\alpha}}.
\]

Note that the right hand side of Eq. (23) is decreasing in $a$ and $a \geq \epsilon_n$ by construction. Thus, if $\epsilon_n$ satisfies

\[
\epsilon_n \geq \left( \frac{4C_0^2 \bar{V} \log(n+1)}{c_1} \right)^{\frac{1+n}{\alpha}} \leq a \geq \left( \frac{V \log(n+1)}{n} \right)^{\frac{1+n}{\alpha}} \iff n \geq c_2^{-\alpha} \left( \frac{4C_0^2 \bar{V} \log(n+1)}{c_1} \right)^{\frac{2+n}{\alpha}} V \log(n+1),
\]

we can substitute $\epsilon_n$ for $a$ to bound the right hand side of Eq. (23). Note that

\[
c_2 \sqrt{\frac{V \log(n+1)}{n} a^{\frac{\alpha}{1+n}} - 2} \leq \frac{\epsilon_n}{a} = \frac{1}{\sqrt{k} \epsilon_n} \leq \frac{1}{\sqrt{k}},
\]

\[
a^{\frac{2\alpha}{1+n}} - 2 \leq \epsilon_n^{\frac{2\alpha}{1+n}} = \left( \epsilon_n^{\frac{2\alpha}{1+n}} \right)^2 \leq \epsilon_n^2 \leq c_2^{-2} V^{-1} n c_n^2,
\]

\[
n c_n^2 = c_2^{-\alpha} \left( \frac{V \log(n+1)}{n} \right)^{\frac{1+n}{\alpha}} n^{\frac{1}{\alpha}} \geq 1.
\]

Therefore, with probability at least $1 - \exp(-t)$ we have

\[
V_a \leq \frac{1}{\sqrt{k} \epsilon_n} + \sqrt{\frac{8 \left( c_1 a^{-2} \bar{V}^{-1} n c_n^2 + 2 \right)}{n k c_n^2}} + \frac{4}{3n k c_n^2} \leq \frac{1}{\sqrt{k} \epsilon_n} + \sqrt{\frac{8 \left( c_1 a^{-2} + 2 \right)}{k}} + \frac{4}{3k}.
\]

By choosing $k$ large enough we can make the right hand side of Eq. (24) less than $1/2$, and we can conclude that

\[
P(V_a < \frac{1}{2}) \geq 1 - \exp(-t).
\]

Combining with Eq. (19) we get for all $t \geq 1$,

\[
P(d(\pi^*, \hat{\pi}^{ERM}) \geq k t c_n^2) \leq \exp(-t).
\]
Thus, when \( n \geq c_2^{-\alpha} \left( \frac{4C^2_n}{c_1} \right)^{\frac{2+\alpha}{2+\alpha}} \bar{V} \log(n+1), \)

\[
\mathbb{E}_P[d(\pi^*, \hat{\pi}_{ERM}^P)] = \int_0^\infty \mathbb{P}(d(\pi^*, \hat{\pi}_{ERM}^P) \geq t')dt' \\
\leq k \epsilon_n^2 + \int_{k \epsilon_n^2}^\infty \mathbb{P}(d(\pi^*, \hat{\pi}_{ERM}^P) \geq t')dt' \\
\leq (1 + e^{-1})k \epsilon_n^2 \\
\leq (1 + e^{-1})k \epsilon_n^2 \left( \frac{\bar{V} \log(n+1)}{n} \right)^{\frac{1+\alpha}{2+\alpha}}.
\]

On the other hand, when \( n < c_2^{-\alpha} \left( \frac{4C^2_n}{c_1} \right)^{\frac{2+\alpha}{2+\alpha}} \bar{V} \log(n+1), \) it is trivially true that

\[
\mathbb{E}_P[d(\pi^*, \hat{\pi}_{ERM}^P)] \leq 2 \leq 2c_2^{-\alpha} \left( \frac{4C^2_n}{c_1} \right)^{\frac{1+\alpha}{2+\alpha}} \left( \frac{\bar{V} \log(n+1)}{n} \right)^{\frac{1+\alpha}{2+\alpha}}.
\]

Finally, by noting that \( \text{Regret}(\hat{\pi}_{ERM}^P) = B \mathbb{E}_P[d(\pi^*, \hat{\pi}_{ERM}^P)] \) we complete the proof.

**Proof of Corollary 3** Corollary 3 follows directly from Assumption 1 and Theorems 2 and 6.

**A.4.4. Proof of Theorem 7**

**Proof of Theorem 7** Similarly to the proof of Theorem 3 we will construct a collection of distributions and lower bound the average regret among them. This time, our distributions will additionally satisfy Assumption 2.

Again, we make some preliminary constructions. For any \( z \in \mathcal{Z}, \) let again \( \bar{z} \) denote the projection of \( z \) onto \( \text{conv}(\mathcal{Z}\setminus\{z\}) \), and define

\[
w(z) = \frac{\rho(\mathcal{Z})}{||z - \bar{z}||^2}(\bar{z} - z).
\]

By definition of \( \rho(\mathcal{Z}) \), we have \( ||w(z)|| \leq 1 \). Moreover, like before, we have

\[
\min_{z' \in \mathcal{Z}\setminus\{z\}} w(z')^\top (z' - \bar{z}) = \min_{z' \in \mathcal{Z}\setminus\{z\}} w(z)^\top (\bar{z} - z) + w(z)^\top (z' - \bar{z}) = \rho(\mathcal{Z}). \tag{25}
\]

As before, since \( \Pi \) has Natarajan dimension at least \( \eta \), there exist \( x_1, \ldots, x_\eta \in \mathbb{R}^p, z_1^{(0)} \neq z_1^{(1)}, \ldots, z_\eta^{(0)} \neq z_\eta^{(1)} \in \mathcal{Z} \) such that, for every \( \mathbf{b} = (b_1, \ldots, b_\eta) \in \{0, 1\}^\eta \), there is a \( \pi'_b \in \Pi \) such that \( \pi'_b(x_i) = z_i^{(b_i)} \) for \( i = 1, \ldots, \eta \).

We now construct a distribution \( \mathbb{P}_b \) for each \( \mathbf{b} \in \{0, 1\}^{\eta-1} \) (notice there are half as many distributions as in the proof of Theorem 3). Set \( \zeta = \left( \frac{\eta-1}{n} \right)^{\frac{1}{\eta-1}} \) and note that \( \zeta \in [0, 1/2] \) by assumption. For the marginal distribution of \( X \), we set

\[
\mathbb{P}_b(X = x_i) = \frac{\zeta^\alpha}{\eta - 1} \text{ for } i \in [\eta - 1], \text{ and}
\]

\[
\mathbb{P}_b(X = x_\eta) = 1 - \zeta^\alpha.
\]

We next construct the conditional distribution of \( Y \mid X = x_i \). For \( i \in [\eta - 1] \), let

\[
u_{0i} = \frac{\zeta(w(z_i^{(0)}) + w(z_i^{(1)})) + w(z_i^{(0)}) - w(z_i^{(1)})}{2},
\]

\[
u_{1i} = \frac{\zeta(w(z_i^{(0)}) + w(z_i^{(1)})) + w(z_i^{(1)}) - w(z_i^{(0)})}{2}.
\]
Finally, for construct the following conditional distribution of $Y \mid X = x_i$ for $i \in [\eta - 1]$: if $b_i = 0$, we let

$$Y = \begin{cases} u_{0i}, & \text{with probability } (1 + \zeta)/2, \\ u_{1i}, & \text{with probability } (1 - \zeta)/2, \end{cases}$$

and if $b_i = 1$, we let

$$Y = \begin{cases} u_{0i}, & \text{with probability } (1 + \zeta)/2, \\ u_{1i}, & \text{with probability } (1 - \zeta)/2. \end{cases}$$

We then have that for $i \in [\eta - 1]$,

$$\mathbb{E}_{\pi_0}[Y \mid X = x_i] = f_b(x_i) = \zeta w(z_i^{(b)}) = \zeta \alpha.$$  

Finally, for $i = \eta$, we simply set $P_b(Y = w(z_i^{(b)})) = 1$. Then $\mathbb{E}_{\pi_0}[Y \mid X = x_\eta] = f_b(x_\eta) = w(z_\eta^{(b)})$.

By Eq. (25), for $i \in [\eta - 1]$, the optimal decision at $x_i$ is $z_i^{(b)}$, and the optimal decision at $x_\eta$ is $z_\eta^{(b)}$. In other words, the optimal policy is $\pi(b_1, \ldots, b_{\eta - 1}, b_\eta)$ when $b \in \{0, 1\}^{\eta - 1}$.

Moreover, under every $P_b$, by Eq. (25), we have that $\Delta(x_i) = \zeta \rho(\mathcal{Z})$ for $i \in [\eta - 1]$ and $\Delta(x_\eta) = \rho(\mathcal{Z})$. Therefore, we have

$$\mathbb{P}_b(0 < \Delta(X) \leq \delta) = \begin{cases} 0, & \text{for } \delta \in [0, \zeta \rho(\mathcal{Z})], \\ \zeta^\alpha, & \text{for } \delta \in [\zeta \rho(\mathcal{Z}), \rho(\mathcal{Z})], \\ 1, & \text{for } \delta \in [\rho(\mathcal{Z}), \infty). \end{cases}$$

Thus, Assumption 2 is satisfied with $\alpha$ and $\gamma = B/\rho(\mathcal{Z})$, as desired.

For $\hat{\pi} \in \Pi$, define $b \in \{0, 1\}^{\eta - 1}$ to be a binary vector whose $i$th element is $\hat{b}_i = \mathbb{I}\{\hat{\pi}(x_i) = z_i^{(b)}\}$. Consider a prior on $b$ such that $b_1, \ldots, b_{\eta - 1}$ are i.i.d. and $b_1 \sim \text{Ber}(1/2)$, and as before let $\tilde{\mathbb{P}}$ denote the joint distribution of $(b, \mathcal{D})$ under this prior. Letting $\text{Regret}_b(\hat{\pi})$ denote the regret when the data is drawn from $\mathbb{P}_b$, the regret satisfies the following inequalities:

$$\sup_{b \in \{0, 1\}^{\eta - 1}} \text{Regret}_b(\hat{\pi}) = \sup_{b \in \{0, 1\}^{\eta - 1}} \mathbb{E}_{\mathbb{P}_b} \mathbb{E}_X \left[ f_b(X)^\top (\hat{\pi}(X) - \pi_b(X)) \right]$$

$$\geq \mathbb{E}_{\mathbb{P}_b} \mathbb{E}_X \left[ f_b(X)^\top (\hat{\pi}(X) - \pi_b(X)) \right]$$

$$\geq \zeta \rho(\mathcal{Z}) \mathbb{E}_{\mathbb{P}_b} \mathbb{E}_X (\hat{\pi}(X) \neq \pi_b(X))$$

$$= \zeta \rho(\mathcal{Z}) \left( \frac{\zeta^\alpha}{\eta - 1} \sum_{i=1}^{\eta - 1} \mathbb{E}_{\mathbb{P}_b} \mathbb{P}_X (\hat{\pi}(x_i) \neq \pi_b(x_i)) \right) + (1 - \zeta^\alpha) \mathbb{E}_{\mathbb{P}_b} \mathbb{P}_X (\hat{\pi}(x_\eta) \neq \pi_b(x_\eta))$$

$$\geq \frac{\zeta^{\alpha + 1} \rho(\mathcal{Z})}{\eta - 1} \sum_{i=1}^{\eta - 1} \mathbb{E}_{\mathbb{P}_b} \mathbb{I} \{\hat{\pi}(x_i) \neq \pi_b(x_i)\}$$

$$\geq \frac{\zeta^{\alpha + 1} \rho(\mathcal{Z})}{\eta - 1} \sum_{i=1}^{\eta - 1} \mathbb{E}_{\mathbb{P}_b} \mathbb{P}_X (\hat{b}_i \neq \hat{b}_i)$$

$$\geq \frac{\zeta^{\alpha + 1} \rho(\mathcal{Z})}{\eta - 1} \sum_{i=1}^{\eta - 1} \mathbb{E}_{\mathbb{P}_b} \left[ \min \{\tilde{\mathbb{P}}(b_i = 1 \mid \mathcal{D}), 1 - \tilde{\mathbb{P}}(b_i = 1 \mid \mathcal{D})\} \right]$$

$$\geq \frac{\zeta^{\alpha + 1} \rho(\mathcal{Z})}{2} \exp \left( -\frac{2\zeta}{1 - \zeta} \sqrt{\frac{\eta \zeta^\alpha}{\eta - 1}} \right),$$
where the second inequality comes from the fact that \( \min_{i \in [n]} \Delta(x_i) \geq \zeta \rho(Z) \), the third inequality follows from \((1 - \zeta^r) \zeta \rho(Z) E_b E_b^T \{ \hat{\pi}(x_i) \neq \pi_b(x_i) \}\) being non-negative, the fourth inequality comes from the fact that \( b_i \neq \hat{b}_i \) implies \( \hat{\pi}(x_i) \neq \pi_b(x_i) \), the fifth inequality follows from the same reasoning as in obtaining Eq. (14), and the sixth inequality follows from the same reasoning as in obtaining Eq. (15).

Finally, plugging in \( \zeta = \left( \frac{2 - 1}{n} \right)^{\frac{1}{2 \alpha}} \) and recalling \( \zeta \leq 1/2 \) by assumption that \( n \geq 2^{2\alpha}(\eta - 1) \), we have
\[
\frac{\zeta^{\alpha+1} \rho(Z)}{2} \exp \left( -\frac{2 \zeta}{1 - \zeta} \sqrt{\frac{n n^{\alpha}}{c}} \right) = \frac{\rho(Z)}{2} \left( \eta - 1 \right)^{\frac{1 + \alpha}{2 \alpha}} \exp \left( -\frac{2}{1 - \zeta} \right) \geq \frac{\rho(Z)}{2e^4} \left( \eta - 1 \right)^{\frac{1 + \alpha}{2 \alpha}},
\]
which concludes the proof.

\( \Box \)

A.5. Fast Rates for ETO (Section 3.2)

**Proof of Theorem 3.** By optimality of \( \pi_f \) with respect to \( \hat{f} \), we have that
\[
f^*(X)^T (\pi_f(X) - \pi_f^*(X)) \leq f^*(X)^T \hat{f}(X) - \pi_f(X) + \hat{f}(X)^T \pi_f^*(X) - f^*(X)^T \pi_f^*(X) \leq 2B \| f^*(X) - \hat{f}(X) \|.
\]
Thus, fixing \( \delta > 0 \) and peeling on \( \| f(X) - \hat{f}(X) \| \), we obtain
\[
\text{Regret}(\pi_f) = \mathbb{E}[f^*(X)^T (\pi_f(X) - \pi_f^*(X)) | f^*(X)^T (\pi_f(X) - \pi_f^*(X)) > 0] \leq 2B \mathbb{E}[\| f^*(X) - \hat{f}(X) \| I\{ f^*(X)^T (\pi_f(X) - \pi_f^*(X)) > 0 \}]
= 2B \mathbb{E}[\| f^*(X) - \hat{f}(X) \| I\{ f^*(X)^T (\pi_f(X) - \pi_f^*(X)) > 0, 0 < \| f(X) - \hat{f}(X) \| \leq \delta \}]
+ 2B \sum_{r=1}^{\infty} \mathbb{E}[\| f^*(X) - \hat{f}(X) \| I\{ f^*(X)^T (\pi_f(X) - \pi_f^*(X)) > 0, 2^{-r-1} \delta < \| f^*(X) - \hat{f}(X) \| \leq 2^r \delta \}]
\leq 2B \delta \mathbb{P}(f^*(X)^T (\pi_f(X) - \pi_f^*(X)) > 0, 0 < \| f^*(X) - \hat{f}(X) \| \leq \delta)
+ B \delta \sum_{r=1}^{\infty} 2^{r+1} \mathbb{P}(f^*(X)^T (\pi_f(X) - \pi_f^*(X)) > 0, 2^{-r-1} \delta < \| f^*(X) - \hat{f}(X) \| \leq 2^r \delta)
\leq 2B \delta \mathbb{P}(0 \leq \Delta(X) \leq 2B \delta) + B \delta \sum_{r=1}^{\infty} 2^{r+1} \mathbb{P}(\| f^*(X) - \hat{f}(X) \| > 2^{-r-1} \delta, 0 < \Delta(X) \leq 2^{r+1} B \delta),
\]
where the very last inequality is due to the implication
\[
f^*(X)^T (\pi_f(X) - \pi_f^*(X)) > 0, \| f^*(X) - \hat{f}(X) \| \leq 2^r \delta \implies 0 < f^*(X)^T (\pi_f(X) - \pi_f^*(X)) \leq 2^{r+1} B \delta
\implies 0 < \Delta(X) \leq 2^{r+1} B \delta,
\]
since \( \pi_f(x) \in Z^c \) is always an extreme point, for any \( f \) and \( x \).

Therefore, iterating expectations with respect to \( X \), we have
\[
\text{Regret}(\pi_f) \leq 2B \delta \mathbb{P}(0 \leq \Delta(X) \leq 2B \delta) + B \delta \sum_{r=1}^{\infty} 2^{r+1} \mathbb{E} \left[ \mathbb{P}(\| f^*(X) - \hat{f}(X) \| > 2^{-r-1} \delta | X) I\{0 < \Delta(X) \leq 2^{r+1} B \delta\} \right]
\leq 2B \delta \mathbb{P}(0 < \Delta(X) \leq 2B \delta) + C_1 B \delta \sum_{r=1}^{\infty} 2^{r+1} \exp(-C_2 a_n(2^{-r+1} \delta)^2) \mathbb{P}(0 < \Delta(X) \leq 2^{r+1} B \delta)
\leq B \gamma^\alpha (2\delta)^{\alpha+1} + B \gamma^\alpha C_1 (\delta)^{\alpha+1} \sum_{r=1}^{\infty} 2^{(r+1)(\alpha+1)} \exp(-C_2 a_n(2^{-r-1} \delta)^2).
\]
If we take \( \delta = a_n^{-1/2} \), we get
\[
\text{Regret}(\pi_f) \leq 2^{\alpha+1} \gamma^\alpha B \left[ 1 + C_1 \sum_{r=1}^{\infty} 2^{r(\alpha+1)} \exp(-C_2(2^{2(r-1)}) \right] a_n^{-(\alpha+1)/2}.
\]
\( \Box \)
Proof of Corollary 4 When Assumptions 1 and 3 hold, by Theorem 8 and Lemmas 11 and 12 there are universal constants \((c_0, c_1, c_2)\) such that for any \(\delta \geq c_0 \sqrt{\frac{\nu \log(nd+1)}{n}}\) and almost all \(x\),

\[
\mathbb{P}(||f(x) - f^*(x)|| \geq \kappa \delta) \leq c_1 e^{-c_2 \nu \delta^2}.
\]

Equivalently, there are universal constants \((c_1, c_2)\) such that for any \(\delta > 0\) and almost all \(x\),

\[
\mathbb{P}(||f(x) - f^*(x)|| \geq \delta) \leq c_1 e^{-c_2 \frac{n \log(nd+1)}{n}} \delta^2.
\]

By Theorem 8,

\[
\text{Regret}(\hat{\pi}_x^{\text{ETO}}) \leq C(\alpha, \gamma, B) \kappa^{1+\alpha} \left(\frac{\nu \log(nd+1)}{n}\right)^{\frac{3+\alpha}{2}}.
\]

\(\square\)

Appendix B: Verifying Assumption 3 (Recovery)

Proposition 1. Suppose \(\mathcal{F}\) is as in Example 1, \(\phi(X)\) has nonsingular covariance, and \(\|\phi(X)\| \leq B'\). Then Assumption 3 is satisfied.

Proof. Let \(\Sigma\) denote the covariance of \(\phi(X)\), \(\sigma_{\min} > 0\) its smallest eigenvalue, and \(f^*(x) = W^* \phi(x)\). Then, for any \(f(x) = W \phi(x)\),

\[
\mathbb{E}_X \|f(X) - f^*(X)\|^2 = \sum_{j=1}^{d} \mathbb{E}_X (W_j^\top J \phi(X) - W_j^* X)^2 = \sum_{j=1}^{d} (W_j - W_j^*)^\top \Sigma (W_j - W_j^*),
\]

while for almost all \(x\), \(\|\phi(x)\| \leq B'\), and so,

\[
\|f(x) - f^*(x)\|^2 = \sum_{j=1}^{d} \left( (W_j - W_j^*)^\top \phi(x) \right)^2 \leq \|\phi(x)\| \sum_{j=1}^{d} (W_j - W_j^*)^\top (W_j - W_j^*)
\]

\[
\leq \frac{B'}{\sigma_{\min}} \sum_{j=1}^{d} (W_j - W_j^*)^\top \Sigma (W_j - W_j^*),
\]

showing Assumption 3 holds with \(\kappa = B'/\sigma_{\min}\). \(\square\)

Proposition 2. Suppose \(\mathcal{F}\) is as in Example 2 where interior nodes queries \("w^\top x \leq \theta?\"\) are restricted to \(w\) being a canonical basis vectors and \(\theta \in \{1/\ell, \ldots, 1-1/\ell\}\), \(X \in [0, 1]^d\), and \(X\) has a density bounded below by \(\mu_{\min}\). Then Assumption 3 is satisfied.

Proof. Fix \(f \in \mathcal{F}\) and \(x\). Consider the intersection \(S\) of the regions defined by leaves \(x\) falls into in \(f\) and in \(f^*\). Note \(S\) has volume at least \(v_{\min} = (1/\ell)^{2D}\). Then,

\[
||f(x) - f^*(x)||^2 = \mathbb{E}[||f(X) - f^*(X)||^2 | X \in S] 
\]

\[
\leq \mathbb{E}[||f(X) - f^*(X)||^2] / (v_{\min} \mu_{\min}),
\]

completing the proof. \(\square\)

Appendix C: Finite-Sample Guarantees for Nonparametric Least Squares with Vector-Valued Response

In this section we prove Theorem 5. In particular, we prove a generic result for vector-valued nonparametric least squares, which may be of general interest, and then apply it to the VC-linear-subgraph case.
C.1. Preliminaries and Definitions

For any $\mathcal{F} \subseteq [\mathbb{R}^p \rightarrow \mathcal{Y}]$, let $\mathcal{F}^* = \mathcal{F} - f^*$. When $f^* \in \mathcal{F}$, we have $f^* \in \arg \min_{f \in \mathcal{F}} \mathbb{E}[||Y - f(X)||^2]$, where every element of this argmin is in fact equal to $f^*$ at almost all $x$.

A set $\mathcal{S}$ is star shaped if $\lambda s \in \mathcal{S}$ for any $\lambda \in [0, 1], s \in \mathcal{S}$. Thus, that $\mathcal{F}$ is star shaped at $f^*$ is equivalent to $\mathcal{F}^*$ being star shaped.

Define

$$w_i = Y_i - f^*(X_i) \in \mathbb{R}^d,$$

and note we have $||w_i|| \leq 2$. Since the samples $\{(X_i, Y_i)\}_{i=1}^n$ are i.i.d, $w_1, \ldots, w_n$ are independent.

Given a function $h = (h_1, \ldots, h_d): \mathcal{X} \rightarrow \mathbb{R}^d$ and a probability distribution $\mathbb{P}$ on $\mathcal{X}$, define the $L_2(\mathbb{P})$-norm:

$$||h||_2 = \sqrt{\mathbb{E}||h(X)||^2} = \sqrt{\mathbb{E} \sum_{j=1}^{d} h_j^2(X)}.$$

Given samples $\{X_1, \ldots, X_n\}$, define the empirical $L_2$ norm:

$$||h||_n = \sqrt{\frac{1}{n} \sum_{i=1}^{n} ||h(X_i)||^2} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{d} h_j^2(X_i)}.$$  

Define the localized $w$-complexity

$$G_n(\delta; H) = \mathbb{E}_w \left[ \sup_{h \in H, ||h||_n \leq \delta} \left| \frac{1}{n} \sum_{i=1}^{n} w_i^\top h(X_i) \right| \right],$$

where the expectation $\mathbb{E}_w$ is only over $w_1, \ldots, w_n$, i.e., over $(Y_1, \ldots, Y_n) | (X_1, \ldots, X_n)$. Define the localized empirical Rademacher complexity

$$\tilde{R}_n(\delta; H) = \mathbb{E}_\sigma \left[ \sup_{h \in H, ||h||_n \leq \delta} \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{d} \sigma_{ij} h_j(X_i) \right| \right],$$

and the localized population Rademacher complexity

$$\bar{R}_n(\delta; H) = \mathbb{E}_{\sigma, X} \left[ \sup_{h \in H, ||h||_2 \leq \delta} \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{d} \sigma_{ij} h_j(X_i) \right| \right],$$

where $\{\sigma_{ij}\}_{i \in [n], j \in [d]}$ are i.i.d Rademacher variables (equiprobably $\pm 1$).

C.2. Generic Convergence Result.

We will next prove the following generic convergence result for nonparametric least-squares with vector-valued response for a general function class $\mathcal{F} \subseteq [\mathbb{R}^p \rightarrow \mathcal{Y}]$.

**Theorem 9.** Suppose $\mathcal{F}^*$ is star-shaped. Let $\delta_n$ be any positive solution to $\tilde{R}_n(\delta; \mathcal{F}^*) \leq \frac{\delta}{\sqrt{2}}$, and $\epsilon_n$ be any positive solution to $\mathbb{E}_\sigma \left[ \frac{\epsilon_n(\epsilon_n; \mathcal{F}^*)}{\epsilon} \right] \leq \epsilon$ (note here $\epsilon_n$ is a random variable that depends on $(X_i)_{i=1}^n$). There are universal positive constants $(c_0, c_1, c_2)$ such that

$$\mathbb{P}(||\hat{f}_x - f^*||_2^2 \geq c_0(\epsilon_n^2 + \delta_n^2)) \leq c_1 e^{-c_2 n \delta_n^2}.$$
C.2.1. Supporting Lemmas. We first prove a lemma that shows the functions \( \delta \mapsto \frac{\underline{G}_n(\delta; H)}{\delta} \) and \( \delta \mapsto \frac{\overline{R}_n(\delta; H)}{\delta} \) are non-increasing, which will be used repeatedly in the rest of the proof.

**Lemma 4.** For any star-shaped function class \( \mathcal{H} \subseteq [X \to \mathbb{R}^d] \), the functions \( \delta \mapsto \frac{\underline{G}_n(\delta; H)}{\delta} \) and \( \delta \mapsto \frac{\overline{R}_n(\delta; H)}{\delta} \) are non-increasing on the interval \((0, \infty)\). Consequently, for any constant \( c > 0 \), the inequalities \( \frac{\underline{G}_n(\delta; H)}{\delta} \leq c\delta \) and \( \frac{\overline{R}_n(\delta; H)}{\delta} \leq c\delta \) have a smallest positive solution.

**Proof of Lemma 4.** Given \( 0 < \delta < t \) and any function \( h \in \mathcal{H} \) with \( ||h||_n \leq t \), we can define the rescaled function \( \tilde{h} = \frac{\delta}{t} h \) such that \( ||\tilde{h}||_n \leq \delta \). Moreover, since \( \delta \leq t \), the star-shaped assumption guarantees that \( \tilde{h} \in \mathcal{H} \). Therefore,

\[
\frac{\delta}{t} \underline{G}_n(t; \mathcal{H}) = \mathbb{E}_w \left[ \sup_{h \in \mathcal{H}, ||h||_n \leq \delta} \frac{1}{n} \sum_{i=1}^n w_i^\top \left( \frac{\delta}{t} h(X_i) \right) \right]
= \mathbb{E}_w \left[ \sup_{\tilde{h} = \frac{\delta}{t} h \in \mathcal{H}, ||\tilde{h}||_n \leq \delta} \frac{1}{n} \sum_{i=1}^n w_i^\top \tilde{h}(X_i) \right]
\leq \underline{G}_n(\delta; \mathcal{H}).
\]

The proof for \( \frac{\overline{R}_n(\delta; H)}{\delta} \) is symmetric. \( \square \)

We now prove a technical lemma (Lemma 5) that will be used to establish our main result. Lemmas 5 and 6 below are, in turn, supporting lemmas used to prove Lemma 7.

For any non-negative random variable \( Z \geq 0 \), define the entropy

\[
\mathbb{H}(Z) = \mathbb{E}[Z \log Z] - \mathbb{E}[Z] \log \mathbb{E}[Z].
\]

**Lemma 5.** Let \( X \in \mathbb{R}^d \) be a random variable such that \( ||X|| \leq b \). Then for any convex and Lipschitz function \( g : \mathbb{R}^d \to \mathbb{R} \), we have

\[
\mathbb{H}(e^{\lambda g(X)}) \leq 4b^2 \lambda^2 \mathbb{E}[||\nabla g(X)||^2 e^{\lambda g(X)}] \quad \text{for all } \lambda > 0,
\]

where \( \nabla g(x) \) is the gradient (which is defined almost everywhere for convex Lipschitz functions).

**Proof of Lemma 5.** Let \( Y \) be an independently copy of \( X \). By definition of entropy,

\[
\mathbb{H}(e^{\lambda g(X)}) = \mathbb{E}_X [\lambda g(X)e^{\lambda g(X)}] - \mathbb{E}_X [e^{\lambda g(X)}] \log (\mathbb{E}_Y [e^{\lambda g(Y)}])
\leq \mathbb{E}_X [\lambda g(X)e^{\lambda g(X)}] - \mathbb{E}_{X,Y} [\lambda g(Y)]
\leq \frac{1}{2} \lambda \mathbb{E}[(e^{\lambda g(X)} - e^{\lambda g(Y)})(g(X) - g(Y))] = \lambda \mathbb{E}[(e^{\lambda g(X)} - e^{\lambda g(Y)})(g(X) - g(Y))1\{g(X) \geq g(Y)\}],
\]

where the inequality follows from Jensen’s, and the last step follows from symmetry of \( X \) and \( Y \). By convexity of the exponential, \( e^s - e^t \leq e^s(s-t) \) for all \( s, t \in \mathbb{R} \), which implies \( (e^s - e^t)(s-t)1\{s \geq t\} \leq e^s(s-t)^2 1\{s \geq t\} \). Therefore,

\[
\mathbb{H}(e^{\lambda g(X)}) \leq \lambda^2 \mathbb{E}[e^{\lambda g(X)}(g(X) - g(Y))^2 1\{g(X) \geq g(Y)\}].
\]

Since \( g \) is convex and Lipschitz, we have \( g(x) - g(y) \leq \langle \nabla g(x), x - y \rangle \), and hence, for \( g(x) \geq g(y) \) and \( ||x||, ||y|| \leq b \),

\[
(g(x) - g(y))^2 \leq ||\nabla g(x)||^2 ||x - y||^2 \leq 4b^2 ||\nabla g(x)||^2.
\]

Combining the pieces yields the claim. \( \square \)
Given a function $f : \mathbb{R}^{nd} \to \mathbb{R}$, an index $k \in [n]$, and a vector $x_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \mathbb{R}^{(n-1)d}$ where $x_i \in \mathbb{R}^d$, we define the conditional entropy in coordinate $k$ via

$$\mathbb{H}(e^{\lambda f_k(X_k)} | x_{-k}) = \mathbb{H}(e^{\lambda f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n)})$$

where $f_k : \mathbb{R}^d \to \mathbb{R}$ is the function $x_k \mapsto f(x_1, \ldots, x_k, \ldots, x_n)$.

**Lemma 6.** Let $f : \mathbb{R}^{nd} \to \mathbb{R}$, and let $\{X_k\}_{k=1}^n$ be independent $d$-dimensional random variables. Then

$$\mathbb{H}(e^{\lambda f(X_1, \ldots, X_n)}) \leq \mathbb{E}\left[\sum_{k=1}^n \mathbb{H}(e^{\lambda f_k(X_k)} | X_{-k})\right] \text{ for all } \lambda > 0.$$

**Proof of Lemma 6** By [Wainwright (2019) Eq. (3.24)],

$$\mathbb{H}(e^{\lambda f(X)}) = \sup_g \left\{ \mathbb{E}[g(X)e^{\lambda f(X)}] \mid \mathbb{E}[e^{g(X)}] \leq 1 \right\}. \quad (26)$$

For each $j \in [n]$, define $X_j = (X_j, \ldots, X_n)$. Let $g$ be any function that satisfies $\mathbb{E}[e^{g(X)}] \leq 1$. We can define a sequence of functions $\{g^1, \ldots, g^n\}$ via

$$g^1(X_1, \ldots, X_n) = g(X) - \log \mathbb{E}[e^{g(X)} | X_2^n]$$

and

$$g^k(X_k, \ldots, X_n) = \log \frac{\mathbb{E}[e^{g(X)} | X_k^n]}{\mathbb{E}[e^{g(X)} | X_{k+1}]} \text{ for } k = 2, \ldots, n.$$ 

By construction,

$$\sum_{k=1}^n g^k(X_k, \ldots, X_n) = g(X) - \log \mathbb{E}[e^{g(X)}] \geq g(X)$$

and $\mathbb{E}[\exp(g^k(X_k, \ldots, X_n)) | X_{k+1}^{n}] = 1$. Therefore,

$$\mathbb{E}[g(X)e^{\lambda f(X)}] \leq \sum_{k=1}^n \mathbb{E}[g^k(X_k, \ldots, X_n)e^{\lambda f(X)}]$$

$$= \sum_{k=1}^n \mathbb{E}_{X_{-k}}[\mathbb{E}_{X_k}[g^k(X_k, \ldots, X_n)e^{\lambda f(X)} | X_{-k}]]$$

$$\leq \sum_{k=1}^n \mathbb{E}_{X_{-k}}[\mathbb{H}(e^{\lambda f_k(X_k)} | X_{-k})],$$

where the last inequality follows from Eq. (26). Since $g$ is arbitrary, taking the supremum over the left-hand side and combining with Eq. (26) yield the claim. \hfill \Box

**Lemma 7.** Let $\{X_i\}_{i=1}^n$ be independent $d$-dimensional random vectors satisfying $||X_i|| \leq b$ for all $i$, and let $f : \mathbb{R}^{nd} \to \mathbb{R}$ be convex and $L$-Lipschitz with respect to the Euclidean norm. Then, for all $\delta > 0$,

$$\mathbb{P}(f(X) \geq \mathbb{E}[f(X)] + \delta) \leq \exp\left(-\frac{\delta^2}{16L^2b^2}\right).$$

**Proof of Lemma 7** For any $k \in [n]$ and fixed vector $x_{-k} \in \mathbb{R}^d$, our assumption implies that $f_k$ is convex, and hence Lemma 5 implies that, for all $\lambda > 0$,

$$\mathbb{H}(e^{\lambda f_k(X_k)} | x_{-k}) \leq 4b^2\lambda^2\mathbb{E}[||\nabla f_k(X_k)||^2e^{\lambda f_k(X_k)} | x_{-k}].$$

Combined with Lemma 6 we find that

$$\mathbb{H}(e^{\lambda f(X)}) \leq 4b^2\lambda^2\mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^d \left(\frac{\partial f(X)}{\partial x_{ij}}\right)^2 e^{\lambda f(X)}\right].$$

Since $f$ is Lipschitz, we know $\sum_{i=1}^n \sum_{j=1}^d \left(\frac{\partial f(X)}{\partial x_{ij}}\right)^2 \leq L^2$ almost surely. The conclusion then follows from [Wainwright (2019) Proposition 3.2]. \hfill \Box
C.2.2. Controlling \( \| \hat{f}_x - f^* \|_n \).

In this section, we show that for any given samples, \( \| \hat{f}_x - f^* \|_n \) can be well-bounded with high probability (Lemma 9). Lemma 8 is a supporting lemma that is used to prove Lemma 9.

**Lemma 8.** Fix sample points \( \{x_i\}_{i=1}^n \). Let \( \mathcal{H} \subseteq [\mathcal{X} \rightarrow \mathbb{R}^d] \) be a star-shaped function class, and let \( \delta_n > 0 \) satisfy \( \frac{\mathcal{G}_n(\hat{\delta}_n; \mathcal{H})}{\delta_n} \leq \delta \). For any \( u \geq \delta_n \), define

\[
A(u) = \{ \exists h \in \mathcal{H} \cap \{\|h\|_n \geq u\} \mid \frac{1}{n} \sum_{i=1}^n w_i^\top h(x_i) \geq 2\|h\|_n u \}.
\]

We have

\[
\mathbb{P}_w(A(u)) \leq e^{-\frac{nu^2}{\delta_n}}.
\]

**Proof of Lemma 8** Suppose there exists some \( h \in \mathcal{H} \) with \( \|h\|_n \geq u \) such that

\[
\frac{1}{n} \sum_{i=1}^n w_i^\top h(x_i) \geq 2\|h\|_n u.
\]

Let \( \tilde{h} = \frac{u}{\|h\|_n} h \), and we have \( \|\tilde{h}\|_n = u \). Since \( h \in \mathcal{H} \) and \( \frac{u}{\|h\|_n} \in (0, 1] \), the star-shaped assumption implies that \( \tilde{h} \in \mathcal{H} \). Therefore, \( A(u) \) implies that there exists a function \( \tilde{h} \in \mathcal{H} \) with \( \|\tilde{h}\|_n = u \) such that

\[
\frac{1}{n} \sum_{i=1}^n w_i^\top \tilde{h}(x_i) = \frac{u}{\|h\|_n} \sum_{i=1}^n w_i^\top h(x_i) \geq 2u^2.
\]

Thus, define \( Z_n(u) = \sup_{\tilde{h} \in \mathcal{H}, \|\tilde{h}\|_n \leq u} \frac{1}{n} \sum_{i=1}^n w_i^\top \tilde{h}(x_i) \), and we get

\[
\mathbb{P}_w(A(u)) \leq \mathbb{P}_w(Z_n(u) \geq 2u^2).
\]

Let us view \( Z_n(u) \) as a function of \( (w_1, \ldots, w_n) \). It is convex since it is the maximum of a collection of linear functions. We now prove that it is Lipschitz. For another vector \( w' \in \mathbb{R}^d \), define \( Z_n'(u) = \sup_{\tilde{h} \in \mathcal{H}, \|\tilde{h}\|_n \leq u} \frac{1}{n} \sum_{i=1}^n (w_i' - w_i)^\top \tilde{h}(x_i) \). For any \( \tilde{h} \in \mathcal{H} \) with \( \|\tilde{h}\|_n \leq u \), we have

\[
\frac{1}{n} \sum_{i=1}^n w_i^\top \tilde{h}(x_i) - Z_n'(u) = \frac{1}{n} \sum_{i=1}^n w_i^\top \tilde{h}(x_i) - \sup_{\tilde{h}' \in \mathcal{H}, \|\tilde{h}'\|_n \leq u} \frac{1}{n} \sum_{i=1}^n (w_i' - w_i)^\top \tilde{h}(x_i)
\]

\[
\leq \frac{1}{n} \sum_{i=1}^n (w_i - w_i')^\top \tilde{h}(x_i)
\]

\[
\leq \frac{1}{\sqrt{n}} \|w - w'\| \|\tilde{h}\|_n
\]

\[
\leq \frac{u}{\sqrt{n}} \|w - w'\|,
\]

and taking suprema yields that \( Z_n(u) - Z_n'(u) \leq \frac{u}{\sqrt{n}} \|w - w'\| \). Similarly, we can show that \( Z_n'(u) - Z_n(u) \leq \frac{u}{\sqrt{n}} \|w - w'\| \), so \( Z_n(u) \) is Lipschitz with constant at most \( \frac{u}{\sqrt{n}} \). By Lemma 7

\[
\mathbb{P}_w(Z_n(u) \geq \mathbb{E}_w(Z_n(u)) + u^2) \leq e^{-\frac{nu^2}{\delta_n}}.
\]

Finally,

\[
\mathbb{E}_w(Z_n(u)) \leq \mathcal{G}_n(u; \mathcal{H}) \leq u \frac{\mathcal{G}_n(\delta_n; \mathcal{H})}{\delta_n} \leq u\delta_n \leq u^2,
\]

where the first inequality follows from Lemma 4 and the second follows from the definition of \( \delta_n \). Thus,

\[
\mathbb{P}_w(Z_n(u) \geq 2u^2) \leq e^{-\frac{nu^2}{\delta_n}}.
\]

\( \square \)
LEMMA 9. Fix sample points \( \{x_i\}_{i=1}^n \). Suppose \( \mathcal{F}^* \) is star-shaped, and let \( \delta_n \) be any positive solution to \( \frac{2w(x_i,f^*)}{\delta} \leq \delta \). Then for any \( t \geq \delta_n \),

\[
P_w[||\hat{f}_n - f^*||_n^2 \geq 16t\delta_n] \leq e^{-\frac{nt\delta_n}{16}}.
\]

Proof of Lemma 9. By definition,

\[
\frac{1}{2n} \sum_{i=1}^{n} ||Y_i - \hat{f}_n(x_i)||^2 \leq \frac{1}{2n} \sum_{i=1}^{n} ||Y_i - f^*(x_i)||^2.
\]

Recall that \( Y_i = f^*(x_i) + w_i \), so we have

\[
\frac{1}{2}||\hat{f}_n - f^*||_n^2 \leq \frac{1}{n} \sum_{i=1}^{n} w_i^\top (\hat{f}(x_i) - f^*(x_i)),
\]  

(27)

Apply Lemma 8 with \( \mathcal{H} = \mathcal{F}^* \) and \( u = \sqrt{t\delta_n} \) for some \( t \geq \delta_n \), we get

\[
P_w(\mathcal{A}(\sqrt{t\delta_n})) \geq 1 - e^{-\frac{nt\delta_n}{16}}. 
\]

Let us now condition on \( \mathcal{A}(\sqrt{t\delta_n}) \). If \( ||\hat{f}_n - f^*||_n < \sqrt{t\delta_n} \), it is obvious that \( ||\hat{f}_n - f^*||_n^2 < 16t\delta_n \). Otherwise, if \( ||\hat{f}_n - f^*||_n \geq \sqrt{t\delta_n} \), Eq. (27) implies that

\[
||\hat{f}_n - f^*||_n^2 \leq \frac{2}{n} \sum_{i=1}^{n} w_i^\top (\hat{f}(x_i) - f^*(x_i)) < 4||\hat{f}_n - f^*||_n \sqrt{t\delta_n},
\]

or equivalently \( ||\hat{f}_n - f^*||_n^2 < 16t\delta_n \). Therefore,

\[
P_w[||\hat{f}_n - f^*||_n^2 \geq 16t\delta_n] \leq P_w(\mathcal{A}(\sqrt{t\delta_n})) \leq e^{-\frac{nt\delta_n}{16}}.
\]

\[\blacksquare\]

We next state a lemma that controls the deviations in the random variable \( |||h|||_2^2 - ||h||_2^2 \), when measured in a uniform sense over a function class \( \mathcal{H} \).

LEMMA 10. Given a star-shaped function class \( \mathcal{H} \) with \( \sup_{h \in \mathcal{H}} \sup_{x} ||h(x)|| \leq b \). Let \( \delta_n \) be any positive solution of the inequality

\[
\mathcal{R}_n(\delta; \mathcal{H}) \leq \frac{\delta^2}{16b}.
\]

Then for any \( t \geq \delta_n \), we have

\[
|||h|||_2^2 - ||h||_2^2 | \leq \frac{1}{2} ||h||_2^2 + \frac{1}{2} t^2 \quad \text{for all } h \in \mathcal{H}
\]  

(28)

with probability at least \( 1 - 2e^{-\frac{Cn\delta_n^2}{100}} \), where \( C \) is a universal constant.

Proof of Lemma 10. Define

\[
Z'_n = \sup_{h \in B_2(t; \mathcal{H})} |||h|||_2^2 - ||h||_2^2 |, \quad \text{where } B_2(t; \mathcal{H}) = \{ h \in \mathcal{H} | ||h||_2 \leq t \}.
\]

Let \( \mathcal{E} \) denote the event that Eq. (28) is violated, and \( \mathcal{A}_0 = \{ Z'_n \geq t^2/2 \} \).

We first prove that \( \mathcal{E} \subseteq \mathcal{A}_0 \). We divide the analysis into two cases. First, if there exists some function with \( ||h||_2 \leq t \) that violates Eq. (28), then we must have \( Z'_n \geq |||h|||_2^2 - ||h||_2^2 | > \frac{1}{2} t^2 \). Otherwise, if Eq. (28) is
violated by some function with $||h||_2 > t$, we can define the rescaled function $\tilde{h} = \frac{t}{||h||_2}h$ so that $||\tilde{h}||_2 = t$. By the star-shaped assumption, $\tilde{h} \in \mathcal{H}$, so $Z'_n \geq ||\tilde{h}'||_2^2 - ||\tilde{h}||_2^2 \geq \frac{t^2}{||h||_2^2} (||h||_2^2 - ||h||_2^2) > \frac{1}{2} t^2$.

We now control event $\mathcal{A}_0$, where we need to control the tail behavior of $Z'_n$.

We first control $\mathbb{E}[Z'_n]$. Note that

$$||h(x)||^2 - ||h'(x)||^2 \leq ||h(x) - h'(x)||((||h(x)|| + ||h'(x)||) \leq 2b||h(x) - h'(x)||.$$ 

Therefore,

$$\begin{align*}
\mathbb{E}[Z'_n] & \leq 2\mathbb{E}\left[ \sup_{h \in B_2(t; \mathcal{H})} \left| \frac{1}{n} \sum_{i=1}^{n} \sigma_i ||h(X_i)||^2 \right| \right] \\
& \leq 4\sqrt{2b}\mathbb{E}\left[ \sup_{h \in B_2(t; \mathcal{H})} \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{d} \sigma_{ij} h_j(X_i) \right| \right] \\
& = 4\sqrt{2b}\mathcal{R}_n(t; \mathcal{H}),
\end{align*}$$

where the first inequality follows from a standard symmetrization argument (cf. Theorem 2.2 of Pollard (1990)), and the second inequality follows from Corollary 1 of Maurer (2016). Since $\mathcal{H}$ is star-shaped and $t \geq \delta_n$, by Lemma 4

$$\frac{\mathcal{R}_n(t; \mathcal{H})}{t} \leq \frac{\mathcal{R}_n(\delta_n; \mathcal{H})}{\delta_n} \leq \frac{\delta_n}{16b},$$

where the last inequality follows from our definition of $\delta_n$. Thus, we conclude that $\mathbb{E}[Z'_n] \leq \frac{\sqrt{2}}{4} t^2$.

Next, we establish a tail bound of $Z'_n$ above $\mathbb{E}[Z'_n]$. Let $g(x) = ||h(x)||^2 - \mathbb{E}||h(X)||^2$. Since $\sup_x ||h(x)|| \leq b$ for any $h \in \mathcal{H}$, we have $||g||_{\infty} \leq b^2$, and moreover

$$\mathbb{E}g^2(X) \leq \mathbb{E}||h(X)||^4 \leq b^2 \mathbb{E}||h(X)||^2 \leq b^2 t^2,$$

using the fact that $h \in B_2(t; \mathcal{H})$. By Talagrand’s inequality Wainwright (2019, Theorem 3.27), there exists a universal constant $C$ such that

$$\mathbb{P}(Z'_n \geq \mathbb{E}[Z'_n] + \frac{1}{7} t^2) \leq 2 \exp(-C \frac{nt^2}{b^2}).$$

We thus conclude the proof by observing that $\mathbb{E}[Z'_n] + \frac{1}{7} t^2 \leq \frac{4}{7} t^2$. \hfill \boxed

C.2.3. Proof of the generic convergence result. Equipped with Lemmas 9 and 10 we are now prepared to prove Theorem 4.

Proof of Theorem 4. First of all, note that $\sup_{f, -f \in \mathcal{F}^*} \sup_{x} ||f(x) - f^*(x)|| \leq 2$.

When $\delta_n \geq \epsilon_n$, we have $\frac{\delta_n}{\epsilon_n} \leq \delta_n$, and by Lemma 9

$$\mathbb{P}_w(\frac{||\hat{f}_x - f^*||^2}{\epsilon_n^4} \geq 16\delta_n^2) \leq e^{-\frac{\epsilon_n^4}{16}}.$$ 

On the other hand, Lemma 10 implies that

$$\mathbb{P}(||\hat{f}_x - f^*||^2 \geq 2 ||\hat{f}_x - f^*||^2_d + \delta_n^2) \leq 2e^{-Cn\delta_n^2/4}.$$ 

Therefore,

$$\mathbb{P}(||\hat{f}_x - f^*||^2 \geq 31\delta_n^2, \delta_n \geq \epsilon_n) \leq 3e^{-n\delta_n^2/(64+4/C)}.$$
We now assume that $\mathcal{A} = \{\delta_n < \epsilon_n\}$ holds. Define $\mathcal{E} = \{\|\hat{f}_x - f^*\|^2 \geq 32\epsilon_n^2 + \delta_n^2\}$, and $\mathcal{B} = \{\|\hat{f}_x - f^*\|^2 \leq 16\epsilon_n^2\}$. It suffices to bound

$$\mathbb{P}(\mathcal{E} \cap \mathcal{A}) \leq \mathbb{P}(\mathcal{E} \cap \mathcal{B}) + \mathbb{P}(\mathcal{A} \cap \mathcal{B}^c).$$

By Lemma 10,

$$\mathbb{P}(\mathcal{E} \cap \mathcal{B}) \leq \mathbb{P}(\|\hat{f}_x - f^*\|^2 \geq 2\|\hat{f}_x - f^*\|^2 + \delta_n^2) \leq 2e^{-Cn\delta_n^2/4}.$$

By Lemma 9,

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}^c) \leq \mathbb{E}[e^{-\frac{n\delta_n^2}{2}} 1\{\mathcal{A}\}] \leq e^{-\frac{n\delta_n^2}{4}}.$$

Putting together the pieces yields the claim. \hfill \square

**C.3. Application to VC-Linear-Subgraph Case**

To prove Theorem 5 we next apply Theorem 9 to the case of a VC-linear-subgraph class of functions. To do this, the key step is to compute the critical radii, $\epsilon_n, \delta_n$.

**C.3.1. Computing the Critical Radii.**

**Lemma 11.** Suppose Assumption 1 holds and $\mathcal{F}^*$ is star-shaped. Let $\delta_n^*$ and $\epsilon_n^*$ be the smallest positive solutions to the inequalities $\bar{\mathcal{R}}_n(\delta; \mathcal{F}^*) \leq \frac{\delta_n^*}{4}$ and $\mathcal{G}_n(\epsilon; \mathcal{F}^*) \leq \epsilon^2$, respectively. Then there is a universal constant $C$ such that

$$\mathbb{P}(\delta_n^* \leq C\sqrt{\frac{\nu \log(nd + 1)}{n}}) = 1, \quad \mathbb{P}(\epsilon_n^* \leq C\sqrt{\frac{\nu \log(nd + 1)}{n}}) = 1.$$

*Proof of Lemma 11* Let $g(f) = (f_1(X_1), f_2(X_1), \ldots, f_d(X_n)) = (e_1^\top f(X_1), e_2^\top f(X_1), \ldots, e_d^\top f(X_n)) \in \mathbb{R}^{nd}$, where $e_j$ is the $j$th canonical basis vector, and $\mathcal{S} = \{g(f) : f \in \mathcal{F}^*, \| f \|_{\mathcal{F}} \leq \delta\}$. Note that $\| s \| \leq \sqrt{n} \delta$ for all $s \in \mathcal{S}$. By Pollard [1990] Theorem 3.5,

$$E_s \Psi \left( \frac{1}{J} \sup_{f \in \mathcal{F}^* : \| f \|_{\mathcal{F}} \leq \delta} \sum_{i=1}^n \sum_{j=1}^d \sigma_{ij} f_j(X_i) \right) \leq 1, \quad \text{where } J = 9 \int_0^{\sqrt{n} \delta} \sqrt{\log D(t, \mathcal{S})} dt,$$

so by Eq. [9],

$$\bar{\mathcal{R}}_n(\delta; \mathcal{F}^*) \leq \frac{5}{n} J.$$

Treat $(e_1, X_1), (e_2, X_1), \ldots, (e_d, X_n)$ as $nd$ data points. By (a vector version of) Van Der Vaart and Wellner [1996] Lemma 2.6.18 (v), $\mathcal{F}'' = \{(\beta, x) \mapsto \beta^\top f(x) : f \in \mathcal{F}^*, \| f \|_{\mathcal{F}} \leq \delta\}$ has VC-subgraph dimension at most $\nu$ per Assumption 1. Note that $\sqrt{n} \delta$ is the envelope of $\mathcal{F}''$ on $(e_1, X_1), (e_2, X_1), \ldots, (e_d, X_n)$. Applying Theorem 2.6.7 of Van Der Vaart and Wellner [1996] gives

$$D(\sqrt{n} \delta, \mathcal{S}) \leq C(\nu + 1)(16\epsilon)^{\nu + 1} \left( \frac{4nd}{\nu t^2} \right)^{\nu}$$

for a universal constant $C$. We therefore obtain that for a (different) universal constant $C$

$$\bar{\mathcal{R}}_n(\delta; \mathcal{F}^*) \leq \frac{C}{32} \sqrt{\frac{\nu \log(nd + 1)}{n}} \delta.$$
Thus, for any samples \( \{X_i\}_{i=1}^n \), any \( \delta_n \geq C \sqrt{\frac{\log(nd+1)}{n}} \) is a valid solution to \( \bar{R}_n(\delta;\mathcal{F}^*) \leq \frac{\delta^2_n}{2} \), which implies the first conclusion.

Now let us focus on \( \hat{\delta}^*_n \). Define \( G_f = \sum_{i=1}^n w_i f(X_i) \). Since \( w_i (f(X_i) - f'(X_i)) \leq 2||f(X_i) - f'(X_i)|| \), it is \( 2||f(X_i) - f'(X_i)|| \)-sub-Gaussian. Moreover, \( w_i \) are independent, so we know that \( G_f - G_{f'} \) is \( 2\sqrt{n}||f - f'||_1 \)-sub-Gaussian. By Theorem 5.22 of Wainwright (2019),

\[
\mathcal{G}_n(\epsilon;\mathcal{F}^*) \leq \frac{64}{\epsilon} \int_0^{2\sqrt{n}} \sqrt{\log N(t,\mathcal{S})} dt.
\]

The rest of the proof is similar as before, and we omit the details here. \( \square \)

**Lemma 12.** Suppose Assumption \( \square \) holds and \( \mathcal{F}^* \) is star-shaped. Let \( \delta^*_n \) be the smallest positive solution to the inequality \( \bar{R}_n(\delta;\mathcal{F}^*) \leq \frac{\delta^2_n}{2} \). For \( nd \geq 2 \), there is a universal constant \( C \) such that

\[
\delta^*_n \leq C \sqrt{\frac{\log(nd+1)}{n}}.
\]

**Proof of Lemma 12.** In what follows, we write \( \bar{R}_n(\delta;\mathcal{F}^*) \) as \( \bar{R}_n(\delta) \) and \( \bar{R}_n(\delta;\mathcal{F}^*) \) as \( \bar{R}_n(\delta) \).

Let \( \hat{\delta}^*_n \) be the smallest positive solutions to the inequality \( \bar{R}_n(\delta;\mathcal{F}^*) \leq \frac{\delta^2_n}{2} \). We first show that there are universal constants \( c_1, c_2 \) such that

\[
\mathbb{P}(\frac{\delta^*_n}{5} \leq \hat{\delta}^*_n \leq 3\delta^*_n) \geq 1 - c_1 e^{-c_2 n (\delta^*_n)^2 / \log(2n+1)}.
\]  

(29)

For each \( t > 0 \), define the random variable

\[
\bar{Z}_n(t) = \mathbb{E}\left[ \sup_{f \in \mathcal{F}^*, ||f||_2 \leq 1} \left| \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^d \sigma_{ij} f_j(X_i) \right| \right]
\]

so that \( \bar{R}_n(t) = \mathbb{E}_X[\bar{Z}_n(t)] \) by construction. Define the events

\[
\mathcal{E}_0(t) = \{ |\bar{Z}_n(t) - \bar{R}_n(t)| \leq \frac{\delta^*_n t}{112} \} \quad \text{and} \quad \mathcal{E}_1 = \{ \sup_{f \in \mathcal{F}^*} \frac{||f||_2}{||f||_1} \leq 1 + \delta^*_n \}.
\]

Conditioned on \( \mathcal{E}_1 \), we have for all \( f \in \mathcal{F}^* \),

\[
||f||_n \leq \sqrt{\frac{3}{2}} ||f||_2^2 + \frac{1}{2} (\delta^*_n)^2 \leq 2||f||_2 + \frac{1}{2} (\delta^*_n)^2 \quad \text{and} \quad ||f||_2 \leq \sqrt{2} (\bar{f}^*_n + (\delta^*_n)^2) \leq 2||f||_n + \delta^*_n.
\]

As a result, conditioned on \( \mathcal{E}_1 \),

\[
\bar{Z}_n(t) \leq \mathbb{E}\left[ \sup_{f \in \mathcal{F}^*, ||f||_1 \leq 2t + \delta^*_n} \left| \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^d \sigma_{ij} f_j(X_i) \right| \right] = \bar{R}_n(2t + \delta^*_n)
\]  

(30)

and

\[
\bar{R}_n(t) \leq \bar{Z}_n(2t + \delta^*_n).
\]  

(31)

Let us consider the upper bound in Eq. (29) first. Conditioned on \( \mathcal{E}_0(7\delta^*_n) \) and \( \mathcal{E}_1 \), we have

\[
\bar{R}(3\delta^*_n) \leq \bar{Z}_n(7\delta^*_n) \leq \bar{R}_n(7\delta^*_n) + \frac{7}{112} (\delta^*_n)^2,
\]

where the first inequality follows from Eq. (31), and the second follows from \( \mathcal{E}_0(7\delta^*_n) \). By Lemma 4. \( \bar{R}_n(7\delta^*_n) \leq 7\bar{R}_n(\delta^*_n) \leq 7(\delta^*_n)^2 \). Thus, \( \bar{R}(3\delta^*_n) \leq \frac{(3\delta^*_n)^2}{42} \), and we have \( \delta^*_n \leq 3\delta^*_n \).
Now let us look at the lower bound in Eq. (29). Conditioned on \( \mathcal{E}_0(\delta_n^*) \), \( \mathcal{E}_0(7\delta_n^*) \) and \( \mathcal{E}_1 \), we have
\[
\frac{(\delta_n^*)^2}{32} = \mathcal{R}_n(\delta_n^*) \leq \hat{Z}_n(\delta_n^*) + \frac{1}{112}(\delta_n^*)^2 \leq \hat{R}_n(3\delta_n^*) + \frac{1}{112}(\delta_n^*)^2 \leq \frac{3\delta_n^* \hat{R}_n}{32} + \frac{1}{112}(\delta_n^*)^2,
\]
where the first inequality follows from \( \mathcal{E}_0(\delta_n^*) \), the second follows from Eq. (30), and the third follows from the fact that \( \hat{\delta}_n \leq 3\delta_n^* \) and Lemma 4. Rearranging yields that \( \frac{1}{2}\delta_n \leq \hat{\delta}_n^* \).

Till now we have shown that
\[
\mathbb{P}(\frac{\delta_n^*}{3} \leq \hat{\delta}_n^* \leq 3\delta_n^*) \geq \mathbb{P}(\mathcal{E}_0(\delta_n^*) \cap \mathcal{E}_0(7\delta_n^*) \cap \mathcal{E}_1).
\]

Lemma 10 implies that \( \mathbb{P}(\mathcal{E}_1) \leq c_1 e^{-c_2 n(\delta_n^*)^2} \). Moreover, let
\[
\hat{Z}_n^*(t) = \mathbb{E}_n \left[ \sup_{f \in \mathcal{F}, \|f\|_2 \leq t} \frac{1}{n} \sum_{j \in [n]} \sum_{j = 1}^d \sigma_{ij} f_j(X_i) \right],
\]
and we have
\[
0 \leq \hat{Z}_n(t) - \hat{Z}_n^*(t) \leq \mathbb{E}_n \left[ \sup_{f \in \mathcal{F}, \|f\|_2 \leq t} \frac{1}{n} \sum_{j = 1}^d \sigma_{kj} f_j(X_k) \right] \leq \sqrt{\frac{\nu \log(d+1)}{n}}.
\]
where the last inequality follows from a standard chaining argument. Thus, by Boucheron et al. (2003, Theorem 15) and noticing the fact that \( \mathcal{R}(\alpha \delta_n^*) \geq \mathcal{R}(\delta_n^*) = \frac{(\delta_n^*)^2}{32} \) for any \( \alpha \geq 1 \), we have
\[
\mathbb{P}(\mathcal{E}_0(7\delta_n^*)) \leq c_1 e^{-c_2 n(\delta_n^*)^2} \quad \text{and} \quad \mathbb{P}(\mathcal{E}_0(\delta_n^*)) \leq c_1 e^{-c_2 n(\delta_n^*)^2},
\]
so Eq. (29) follows.

By Lemma 11 \( \mathbb{P}(\hat{\delta}_n^* \leq C_0 \sqrt{\frac{\nu \log(n d + 1)}{n}}) = 1 \) for some universal \( C_0 \). Let \( C > 5C_0 \) be a constant such that \( c_1 \exp(-c_2C^2) < 1 \). If \( \delta_n^* > C \sqrt{\frac{\nu \log(n d + 1)}{n}} \), by Eq. (29) we have \( \mathbb{P}(\hat{\delta}_n^* > C \sqrt{\frac{\nu \log(n d + 1)}{n}}) > 0 \), which leads to contradiction. Thus, \( \delta_n^* \leq C \sqrt{\frac{\nu \log(n d + 1)}{n}} \).

C.3.2. Proof of Theorem 5

Proof of Theorem 5 By Theorem 6 and Lemmas 11 and 12, there exists universal constant \( (c_0, c_1, c_2) \) such that for any \( \delta \geq c_0 \sqrt{\frac{\nu \log(n d + 1)}{n}} \),
\[
\mathbb{P}(\|\hat{f}_n - f^*\|_2 \geq \delta) \leq c_1 e^{-c_2 n \delta^2},
\]
and our conclusion follows.

C.4. Vector-Valued Reproducing Kernel Hilbert Spaces

In this section we develop an analogue of Theorem 5 to the case of vector-valued RKHS instead of a VC-linear-subgraph class. We proceed by computing the critical radii and applying our Theorem 6. Then, in Section C.4.1 we apply this to Example 1 in order to avoid the suboptimal logarithmic term one obtains by instead relying on its VC-linear-subgraph dimension and applying Corollaries 2 and 4.

A (multivariate) positive semidefinite kernel is a function \( K : \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^{d \times d} \) such that \( K(x, x') = K(x', x) \) is symmetric and for all \( m \in \mathbb{N}, x_1, \ldots, x_m \in \mathbb{R}^p, v_1, \ldots, v_m \in \mathbb{R}^d \) we have \( \sum_{i=1}^m \sum_{j=1}^m v_i^\top K(x_i, x_j) v_j \geq 0 \). We then consider the space span\((\mathcal{K}(x, \cdot) : x \in \mathbb{R}^p, v \in \mathbb{R}^d)\) \( \subset \mathbb{R}^p \rightarrow \mathbb{R}^d \) endowed with the inner product \( \langle \sum_{i=1}^m \mathcal{K}(x_i, \cdot) v_i, \sum_{j=1}^{m'} \mathcal{K}(x_j', \cdot) v'_j \rangle = \sum_{i=1}^m \sum_{j=1}^{m'} v_i^\top K(x_i, x_j') v'_j \). Using the norm \( \|f\|_K = \langle f, f \rangle \), this has
a unique completion to a Hilbert space, which we call \( \mathcal{H}_K \). This Hilbert space has the property that for every \( f \in \mathcal{H}_K, x \in \mathbb{R}^p, v \in \mathbb{R}^q \) we have \( v^\top f(x) = \langle K(x, \cdot), v \rangle \), known as the representer property. That is, \( (x' \mapsto K(x, x') v) \in \mathcal{H}_K \) is the Riesz representer of the linear operator \( f \mapsto v^\top f(x) \), which is bounded.

Given these definitions, we now consider the hypothesis class given by an \( R \)-radius ball in \( \mathcal{H}_K \):

\[
\mathcal{F} = \{ f \in \mathcal{H}_K : \| f \|_K \leq R \}.
\]

We also allow \( R = \infty \), in which case we set \( \mathcal{F} = \mathcal{H}_K \). A prominent example of a multivariate kernel is a diagonal kernel: given a usual univariate positive semidefinite kernel, \( K' : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R} \), we let \( K(x, x') = K'(x, x') I_{d \times d} \), e.g., \( K'(x, x') = \exp(-\|x - x'\|^2 / \sigma^2) \) or \( K'(x, x') = x^\top x' \). Then, applying Theorem 9 we obtain the first statement. The second statement is given by integrating the tail bound and applying Theorem 4. The third statement is given by Corollary 5.

\[\begin{align*}
\text{Consequently, for a universal constant } C, \\
\text{Regret}(\hat{\pi}^{\text{ETO}}_X) &\leq C B \sqrt{\frac{d \nu}{n}}. \\
\end{align*}\]

And, if Assumptions 2 and 3 hold, then for a constant \( C(\alpha, \gamma, B) \) depending only on \( \alpha, \gamma, B \),

\[\begin{align*}
\text{Regret}(\hat{\pi}^{\text{ETO}}_X) &\leq C(\alpha, \gamma, B) \left( \frac{d \nu}{n} \right)^{\frac{1+\alpha}{2}}. \\
\end{align*}\]

**Proof** The operators \( \hat{T}_K \) and \( T_K \) have each rank at most \( d \nu \) because their nonzero eigenvalues are respectively given by the duplicating \( d \) times the nonzero eigenvalues of the matrices \( \frac{1}{n} \sum_{i=1}^n \phi(X_i) \phi(X_i)^\top \) and \( \mathbb{E} \phi(X) \phi(X)^\top \). Therefore, Lemma 13 gives \( G_n(\delta; \mathcal{F}^*) \leq 4 \sqrt{2} \delta \sqrt{\frac{d \nu}{n}} \) and \( \mathcal{R}(\delta; \mathcal{F}^*) \leq 2 \sqrt{2} \delta \sqrt{\frac{d \nu}{n}} \). Note that \( \mathcal{F} \) is convex so it is also star-shaped. Then, applying Theorem 8 we obtain the first statement. The second statement is given by integrating the tail bound and applying Theorem 11. The third statement is given by applying Theorem 8 \( \square \).
C.4.2. Proof of Lemma 13

The proof adapts arguments from Mendelson (2002) to the vector-valued case.

Proof of Lemma 13 We first argue $\hat{T}_K$ has at most $nd$ nonzero eigenvalues. Define the matrix $K \in \mathbb{R}^{(n \times d) \times (n \times d)}$ given by $K_{(i,k),(j,l)} = \frac{1}{n} (K(X_i, X_j))_{kl}$. If $f \in \mathcal{H}$ is an eigenfunction of $\hat{T}_K$ with eigenvalue $\lambda$, then for any $i$ we have $\lambda f(X_i) = (\hat{T}_K f)(X_i) = \frac{1}{n} \sum_{j=1}^{nd} K(X_i, X_j) f(X_j)$. Letting $v \in \mathbb{R}^{n \times d}$ be given by $v_k = f_k(X_i)$, this means that $K v = \lambda v$. So, either $v = 0$, which means that $\hat{T}_K f = 0$ and hence $\lambda = 0$, or $v$ is an eigenvector of $K$ with eigenvalue $\lambda$. Of course, $K$ has at most $nd$ eigenvalues.

Let $\varphi_1, \varphi_2, \ldots$ be an orthonormal basis of $\mathcal{H}$ such that $\hat{T}_K \varphi_i = \hat{\lambda}_i \varphi_i$ for $i = 1, \ldots, nd$ and $\hat{T}_K \varphi_i = 0$ for $i > nd$. Fix $0 \leq h \leq nd$. Consider $f \in \mathcal{F}$ (i.e., $\|f\|_{\mathcal{H}} \leq \delta$) with $\|f\|_n \leq \delta$. Then we have, $\delta^2 \geq \frac{1}{n} \sum_{i=1}^{n} \|f(X_i)\|^2 = \langle f, \hat{T}_K f \rangle = \sum_{i=1}^{nd} \hat{\lambda}_i \langle f, \varphi_i \rangle^2 \geq \sum_{i=1}^{h} \hat{\lambda}_i \langle f, \varphi_i \rangle^2$.

Therefore,

\[ \sum_{i=1}^{n} w_i^\top f(X_i) = \sum_{i=1}^{n} \langle f, K(X_i, \cdot) w_i \rangle \]

(32)

\[ \leq \delta \left( \sum_{j=h+1}^{n} \frac{1}{\lambda_j} \left( \sum_{i=1}^{n} \langle K(X_i, \cdot) w_i, \varphi_j \rangle \right)^2 \right)^{1/2} \]

(36)

\[ + R \left( \sum_{j=h+1}^{n} \left( \sum_{i=1}^{n} \langle K(X_i, \cdot) w_i, \varphi_j \rangle \right)^2 \right)^{1/2} \]  

(37)

Next note that, since $\mathbb{E}[w_i w_i^\top] \preceq 4I_{d \times d}$,

\[ \frac{1}{n} \mathbb{E}_w \left( \sum_{i=1}^{n} K(X_i, \cdot) w_i, \varphi_k \right)^2 = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi_k(X_j)^\top \mathbb{E}_w [w_i w_i^\top] \varphi_k(X_j) \]

\[ \leq \frac{4}{n} \sum_{i=1}^{n} \|\varphi_k(X_i)\|^2 = 4 \langle \varphi_k, \hat{T}_K \varphi_k \rangle = 4 \hat{\lambda}_k. \]

We conclude from $\sqrt{a} + \sqrt{b} \leq \sqrt{2} \sqrt{a + b}$ that

\[ G_n(\delta; \mathcal{F}) \leq \frac{2 \sqrt{2} \sqrt{n}}{\sqrt{n}} \left( \sum_{j=1}^{nd} \min\{\delta^2, R^2 \hat{\lambda}_j\} \right)^{1/2}. \]

Repeating the same argument using Mercer’s theorem for $\mathcal{T}_K$ in $L_2(\mathcal{P})$ (which may have more than $nd$ nonzero eigenvalues) and using Rademacher variables instead of $w_i$, noting that $\mathbb{E}[\sigma_i \sigma_i^\top] = I_{d \times d}$, we find that

\[ \tilde{R}(\delta; \mathcal{F}) \leq \frac{\sqrt{2} \sqrt{n}}{\sqrt{n}} \left( \sum_{j=1}^{\infty} \min\{\delta^2, R^2 \hat{\lambda}_j\} \right)^{1/2}. \]

Noting that $G_n(\delta; \mathcal{F}^c) \leq 2G_n(\delta; \mathcal{F})$ and $\tilde{R}(\delta; \mathcal{F}^c) \leq 2\tilde{R}(\delta; \mathcal{F})$ completes the proof. \qed
Appendix D: Convergence Rates for Vector-Valued Local Polynomial Regression

In this section we provide rates for vector-valued regression assuming Hölder smoothness and using local polynomial regression. Our arguments are largely based on those of [Audibert and Tsybakov (2007), Stone (1982)] but avoid the bad \(d\)-dependence one would get by naively invoking their results for each response component. To do this we leverage a vector Bernstein concentration inequality (Minsker (2017)).

Fix \(\beta > 0\) and define \(|\beta| = \sup\{j \in \mathbb{Z} : j < |\beta|\}\) as the largest integer strictly smaller than \(\beta\) (slightly differently than the usual floor function). For any \(x\) and any \(|\beta|\)-times continuously differentiable real-valued function \(g : \mathbb{R}^p \to \mathbb{R}\), define the Taylor expansion of \(g\) at \(x\) as

\[
g_x(x') = \sum_{|s| \leq |\beta|} \frac{(x'-x)^s}{s!} D^s g(x).
\]

We say that \(g : \mathbb{R}^p \to \mathbb{R}\) is \((\beta, L, \mathbb{R}^p)\)-Hölder if it is \(|\beta|\)-times continuously differentiable and satisfies

\[
|g(x') - g_x(x')| \leq L \|x - x'\|^\beta \quad \forall x, x' \in \mathbb{R}^p.
\]

We say that \(g : \mathbb{R}^p \to \mathbb{R}^d\) is \((\beta, L, \mathbb{R}^p \to \mathbb{R}^d)\)-Hölder smooth if each component, \(g(i) : x \mapsto (g(x))_i\), is \((\beta, L, \mathbb{R}^p)\)-Hölder. We also write \(g_x = (g_x^{(1)}, \ldots, g_x^{(d)})\).

To estimate a vector-valued Hölder smooth function, we will use a local polynomial estimator. Given a kernel \(K(u)\) satisfying (examples include the uniform, Gaussian, and Epanechnikov kernels)

\[
\exists c > 0 : K(x) \geq c 1\{|x| \leq c\} \quad \forall x \in \mathbb{R}^p,
\]

\[
\int_{\mathbb{R}^p} K(u) du = 1,
\]

\[
\sup_{u \in \mathbb{R}^p} (1 + ||u||^{2\beta}) K(u) < \infty,
\]

\[
\int_{\mathbb{R}^p} (1 + ||u||^{2\beta}) K(u) du < \infty,
\]

\[
\int_{\mathbb{R}^p} (1 + ||u||^{4\beta}) K^2(u) du < \infty,
\]

and a bandwidth \(h > 0\), the estimator at \(x\) is defined as

\[
\hat{f}_n^{LP}(x) = \hat{\vartheta}(0, \ldots, 0) \quad \hat{\vartheta} \in \arg \min_{\vartheta \in \mathbb{R}^{p \times M}} \sum_{i=1}^n K\left(\frac{X_i - x}{h}\right) \left|Y_i - \sum_{|s| \leq |\beta|} \vartheta_s(X_i - x)^s\right|^2,
\]

where \(M\) is the cardinality of the set \(\{s \in \mathbb{Z}_+^d : |s| \leq |\beta|\}\) and, for each \(|s| \leq |\beta|\), \(\vartheta_s\) refers to the corresponding column of \(\vartheta \in \mathbb{R}^{d \times M}\). In case of multiple minimizers \(\hat{\vartheta}\) in the argmin, we just set \(\hat{f}_n^{LP}(x) = 0 \in \mathbb{R}^d\). Finally, we define \(\hat{f}_n^*(x)\) to be the projection of \(\hat{f}_n^{LP}(x)\) onto the unit ball \(B_d(0, 1)\).

We can then prove the following:

**Theorem 10.** Suppose \(f^*\) is \((\beta, L, \mathbb{R}^p \to \mathbb{R}^d)\)-Hölder. Suppose the distribution of \(X\) has a density with a compact support \(\mathcal{X}\) on which it is bounded in \([\mu_{\text{min}}, \mu_{\text{max}}] \subseteq (0, \infty)\). Suppose moreover that for some \(c_0, r_0 > 0\), \(\text{Leb}(\mathcal{X} \cap B(x,r)) \geq c_0 \text{Leb}(B(x,r)) \forall 0 < r \leq r_0, x \in \mathcal{X}\), where \(\text{Leb}[\cdot]\) is the Lebesgue measure. Set \(h = n^{-1/(2\beta + p)}\).

Then, there exists \(C_1, C_2 > 0\) depending only on \(p, \beta, L, \mu_{\text{min}}, \mu_{\text{max}}, c_0, r_0\) such that for all \(\delta > 0, n \geq 1\), and almost all \(x\),

\[
\mathbb{P}\left(\left\|\hat{f}_n^*(x) - f(x)\right\| \geq \delta\right) \leq C_1 \exp\left(-C_2 n^{2\beta/(2\beta + p)} \delta^2/d\right).
\]
D.1. Proof of Theorem 10

We first make some convenient definitions. Define the vector $U(u) = (u^s)_{|s| \leq |\beta|} \in \mathbb{R}^M$. Let $h = n^{-1/(2\beta + p)}$ be a bandwidth, and define the matrix $V \in \mathbb{R}^{d \times M}$, where for each $|s| \leq |\beta|$, the corresponding column of $V$ is

$$V_i = \sum_{i=1}^{n} (X_i - x)^s K \left( \frac{X_i - x}{h} \right) Y_i, \quad i \in \mathbb{R}^d,$$

and the matrix $Q = (Q_{s_1,s_2})_{|s_1|,|s_2| \leq |\beta|}$, where

$$Q_{s_1,s_2} = \sum_{i=1}^{n} (X_i - x)^{s_1+s_2} K \left( \frac{X_i - x}{h} \right),$$

and the matrix $\bar{B} = (\bar{B}_{s_1,s_2})_{|s_1|,|s_2| \leq |\beta|}$, where

$$\bar{B}_{s_1,s_2} = \frac{1}{nh^p} \sum_{i=1}^{n} \left( \frac{X_i - x}{h} \right)^{s_1+s_2} K \left( \frac{X_i - x}{h} \right).$$

It is easy to derive from Audibert and Tsybakov (2007, Proposition 2.1) that $\hat{f}_{LP}^n(x)$ can be expressed as

$$\hat{f}_{LP}^n(x) = VQ^{-1}U(0)$$

if $Q$ is positive definite, and $\hat{f}_{LP}^n(x) = 0$ otherwise.

**Proof of Theorem 10**

First of all, let $S$ denote the class of all compact subsets of $B(0,c)$ having Lebesgue measure $c_0v_d c^p$, and we define

$$\mu_0 = \frac{1}{2} c \mu_{\min} \min_{||w|| = 1, \Sigma \in S} \int_{S} \left( \sum_{|s| \leq |\beta|} w_s u^s \right)^2 \, du > 0,$$

where $c$ is the constant in Eq. (38). By the same arguments as in Audibert and Tsybakov (2007) Proof of Theorem 3.2),

$$\mathbb{P}(\lambda_{\min}(\bar{B}) \leq \mu_0) \leq 2M^2 \exp(-Cnh^p).$$

Since

$$\mathbb{P}(|\hat{f}_{LP}^n(x) - f(x)| \geq \delta) \leq \mathbb{P}(\lambda_{\min}(\bar{B}) \leq \mu_0) + \mathbb{P}(|\hat{f}_{LP}^n(x) - f(x)| \geq \delta, \lambda_{\min}(\bar{B}) > \mu_0),$$

we aim to control the second term in the rest of our proof.

Recall that we can write

$$\hat{f}_{LP}^n(x) = VQ^{-1}U(0).$$

Define the matrix $Z = (Z_{s,i})_{|s| \leq |\beta|, 1 \leq i \leq n}$ with elements

$$Z_{s,i} = (X_i - x)^s \sqrt{K \left( \frac{X_i - x}{h} \right)}.$$

Denote the $s$th row of $Z$ as $Z_s$, and we introduce

$$Z^{(s)} = \sum_{|s| \leq |\beta|} \frac{f^{(s)}(x)}{x^s} Z_s \in \mathbb{R}^{d \times n}.$$
Since $Q = ZZ^T$, we get
\[ \forall |s| \leq |\beta|, \quad Z_s Z^T Q^{-1} U(0) = \mathbb{I} \{ s = (0, \ldots, 0) \}, \]
hence $Z(f)Z^T Q^{-1} U(0) = f(x)$. Thus, we can write
\[
\tilde{f}_{n}^{LP}(x) - f(x) = (V - Z(f)Z^T)Q^{-1} U(0) = a\tilde{B}^{-1} U(0),
\]
where $a = \frac{1}{nh^p} (V - Z(f)Z^T)H \in \mathbb{R}^{d \times M}$ and $H$ is the diagonal matrix $H = (H_{s, s})_{|s| \leq |\beta|}$ with $H_{s, s} = h^{-s_1} \mathbb{I} \{ s_1 = s_2 \}$. For $\lambda_{\text{min}}(\tilde{B}) > \mu_0$,
\[
\| \tilde{f}_{n}^{LP}(x) - f(x) \| \leq \| a\tilde{B}^{-1} \| \| a \| \| f \| \leq \mu_0^{-1} M_{\text{max}} \| a_s \|,
\]
where $a_s$ is the $s$-th column of $a$ given by
\[
a_s = \frac{1}{nh^p} \sum_{i=1}^{n} \left( \frac{X_i - x}{h} \right)^s K \left( \frac{X_i - x}{h} \right) (Y_i - f_s(X_i)).
\]
Define
\[
T_i^{(s, 1)} = \frac{1}{h^p} \left( \frac{X_i - x}{h} \right)^s K \left( \frac{X_i - x}{h} \right) (Y_i - f(X_i)),
\]
\[
T_i^{(s, 2)} = \frac{1}{h^p} \left( \frac{X_i - x}{h} \right)^s K \left( \frac{X_i - x}{h} \right) (f(X_i) - f_s(X_i)).
\]
We have
\[
\| a_s \| \leq \frac{1}{n} \sum_{i=1}^{n} T_i^{(s, 1)} + \frac{1}{n} \sum_{i=1}^{n} (T_i^{(s, 2)} - ET_i^{(s, 2)}) + \| ET_i^{(s, 2)} \|,
\]
Define
\[
\kappa_1 = \sup_{u \in \mathbb{R}^p} (1 + \| u \|^{2\beta}) K(u),
\]
\[
\kappa_2 = \mu_{\text{max}} \int_{\mathbb{R}^p} (1 + \| u \|^{4\beta}) K^2(u) du,
\]
\[
\kappa_3 = \mu_{\text{max}} \int_{\mathbb{R}^p} (1 + \| u \|^{2\beta}) K(u) du.
\]
Note that $ET_i^{(s, 1)} = 0$, $\| T_i^{(s, 1)} \| \leq 2\kappa_1 h^{-p}$, and
\[
\mathbb{E} \| T_i^{(s, 2)} \|^2 \leq 4h^{-p} \mu_{\text{max}} \int_{\mathbb{R}^p} u^{2\beta} K^2(u) du \leq 4\kappa_2 h^{-p},
\]
\[
\| T_i^{(s, 2)} - ET_i^{(s, 2)} \| \leq \sqrt{d}L\kappa_1 h^{\beta-p} + \sqrt{d}L\kappa_3 h^{\beta} \leq \sqrt{d}L(\kappa_1 + \kappa_3) h^{\beta-p},
\]
\[
\mathbb{E} \| T_i^{(s, 2)} - ET_i^{(s, 2)} \|^2 \leq \mathbb{E} \| T_i^{(s, 2)} \|^2 \leq dL^2 h^{2\beta-p} \mu_{\text{max}} \int_{\mathbb{R}^p} \| u \|^{2\beta+2\beta} K^2(u) du \leq dL^2 \kappa_2^{2\beta-p}.
\]
Recall that $h = n^{-1/(2\beta+\beta_p)}$. By Minzker [2017] Corollary 4.1, for $\epsilon_1 \geq \frac{1}{3} (\kappa_1 + \sqrt{\kappa_1^2 + 36\kappa_2}) h^{\beta}$,
\[
\mathbb{P} \left( \left\| \frac{1}{n} \sum_{i=1}^{n} T_i^{(s, 1)} \right\| \geq \epsilon_1 \right) \leq 28 \exp \left( -\frac{nh^p \epsilon_1^2}{8\kappa_2 + 4\kappa_1 \epsilon_1/3} \right),
\]
and for $\epsilon_2 \geq \frac{\sqrt{d}L(\kappa_1 + \kappa_3 + \sqrt{(\kappa_1 + \kappa_3)^2 + 36\kappa_2}) h^{\beta}}{6}$,
\[
\mathbb{P} \left( \left\| \frac{1}{n} \sum_{i=1}^{n} (T_i^{(s, 2)} - ET_i^{(s, 2)}) \right\| \geq \epsilon_2 \right) \leq 28 \exp \left( -\frac{nh^p \epsilon_2^2/2}{dL^2 \kappa_2 + \sqrt{dL(\kappa_1 + \kappa_3)} \epsilon_2/3} \right).
Since also

\[ \left\| \mathcal{E}_i^{(s,2)} \right\| \leq \sqrt{\beta} h^\beta \mu_{\text{max}} \int_{\mathbb{R}^p} \|u\|^{\beta+\epsilon} K(u)\,du \leq \sqrt{\beta} L_k h^\beta, \]

we get that when \(2 \geq \delta \geq M\mu_0^{-1}\left(3\sqrt{\Delta L k_3} \lor (\kappa_1 + \sqrt{\kappa_1^2 + 36k_2}) \lor \frac{\sqrt{2L}}{2}(\kappa_1 + \kappa_3 + \sqrt{(\kappa_1 + \kappa_3)^2 + 36k_2})\right) h^\beta,\)

\[ \mathbb{P}\left(\|a_i\| \geq \frac{\mu_0 \delta}{M}\right) \leq \mathbb{P}\left(\left\| \frac{1}{n} \sum_{i=1}^{n} T_i^{(s,1)} \right\| \geq \frac{\mu_0 \delta}{3M} \right) + \mathbb{P}\left(\left\| \frac{1}{n} \sum_{i=1}^{n} (T_i^{(s,2)} - \mathcal{E}_i^{(s,2)}) \right\| \geq \frac{\mu_0 \delta}{3M} \right) \leq 56 \exp(-Ch^p\delta^2/d). \]

Recall that \(\hat{f}_n^*(x)\) is the projection onto \(B_2(0,1)\). Combined with Eqs. (43) to (45) we get when \(\delta \geq M\mu_0^{-1}\left(3\sqrt{\Delta L k_3} \lor (\kappa_1 + \sqrt{\kappa_1^2 + 36k_2}) \lor \frac{\sqrt{2L}}{2}(\kappa_1 + \kappa_3 + \sqrt{(\kappa_1 + \kappa_3)^2 + 36k_2})\right) h^\beta,\)

\[ \mathbb{P}\left(\left\| \hat{f}_n^*(x) - f(x) \right\| \geq \delta \right) \leq C_1 \exp\left(-C_2 n h^p \delta^2/d\right) = C_1 \exp\left(-C_2 n^{2\beta/(2\beta+p)} \delta^2/d\right). \]

When \( \delta < M\mu_0^{-1}\left(3\sqrt{\Delta L k_3} \lor (\kappa_1 + \sqrt{\kappa_1^2 + 36k_2}) \lor \frac{\sqrt{2L}}{2}(\kappa_1 + \kappa_3 + \sqrt{(\kappa_1 + \kappa_3)^2 + 36k_2})\right) h^\beta,\)

exp\left(-C_2 n^{2\beta/(2\beta+p)} \delta^2/d\right) is lowered bounded by a constant independent of \(n\) and \(d\), so we know the inequality essentially holds for all \(\delta > 0\) (with possibly modified constants \(C_1\)).

\[ \square \]

**Appendix E:** Omitted Details

**E.1. Details for IERM under \(\sigma^2 = 0\) in Section [1,2]**

If \(\sigma^2 = 0\) then \(Y_i = X_i\) and therefore the set of IERM solutions is arg\(\min_{\theta \in [-1,1]} \frac{1}{n} \sum_{i=1}^{n} X_i(1 - 2I[X_i \leq \theta]) = \{\max\{X_i : X_i < 0\}, \min\{X_i : X_i > 0\}\},\) where we define \(\max(\emptyset) = -1\) and \(\min(\emptyset) = 1\). Thus, we have that \(\hat{\pi}_F^{\text{ERM}} = \pi^\star_{\text{ERM}}\) where \(\hat{\theta}_{\text{ERM}} = \max(\{X_i : X_i < 0\})\). Let \(n_- = \{X_i : X_i < 0\} \sim \text{Bin}(n, \frac{1}{2})\). Then, \(\mathbb{E}[\hat{\theta}_{\text{ERM}}^2 | n_-] = f_{\text{erm}}^{(1)} \mathbb{P}(\hat{\theta}_{\text{ERM}} < -\sqrt{u} | n_-) du = f_{\text{erm}}^{(1)} (1 - \sqrt{u})^{-n_-} du = 2/(2 + 3n_- + n_-^2).\) Notice this works even when \(n_- = 0\).

We conclude that \(\text{Regret}(\hat{\pi}_F^{\text{ERM}}) = \mathbb{E}[1/(2 + 3n_- + n_-^2)] = (4 - (3 + n)/(n + 2))/((n + 1)(n + 2)) = \Theta(1/n^2).\)

**E.2. Details for the experiment in Section [5]**

**Data-Generating Process.** Here we specify the distribution from which we draw \((X,Y)\). Recall \(Y\) has \(d = 40\) dimensions. We consider covariates \(X\) with \(p = 5\) dimensions, with the data generated as follows. We let \(X \sim \mathcal{N}(0,I_p)\) be drawn from the standard multivariate normal distribution. We then set \(Y = \text{diag}(\epsilon)(W\phi(X) + 3)\), where \(\phi(x) \in \mathbb{R}^{31}\) consists of all features and all products of any number of distinct features \((i.e., \phi(x) = (\prod_{j=1}^{5} k_j : k_j \in \{0,1\}, 1 \leq \sum_{j=1}^{5} k_j \leq 5) = (x_1, \ldots, x_5, x_1x_2, x_1x_3, \ldots, x_1x_2x_3, \ldots, x_1x_2x_3x_4x_5), W \in \mathbb{R}^{40 \times 31}\) is a fixed coefficient matrix, and \(\epsilon \sim \text{Unif}[3/4, 5/4]^d\) is a multiplicative noise. Note \(f^*(x) = W\phi(x) + 3\) is a degree-5 polynomial in \(x\). To fix some matrix \(W\) we draw its entries independently at random from \(\text{Unif}[0,1]\). We do this just once, with a fixed random seed of 10, so that that \(W\) is a fixed matrix. For each replication of the experiment, we then draw a training dataset of size \(n\) from this distribution of \((X,Y)\).

**Methods.** As detailed in Section [5] we consider 6 methods: ETO using least-squares and SPO+, each using three different hypothesis classes \(\mathcal{F}\). We employ a ridge penalty with parameter \(\lambda\) in each of these cases, i.e., \(\lambda\) times the squared sum of linear coefficients for both linear settings and \(\lambda\) times the RKHS norm of \(f\) in the kernel setting. In the kernel setting, there is an additional parameter, \(\rho\), known as the length-scale of the Gaussian kernel. We choose \(\lambda\) (and also \(\rho\) in the kernel setting) by validation. We use an
independent validation dataset of size \( n \). For ETO, we focus on least squares: we choose the parameters that result in minimal squared error on the validation data. For SPO+, we focus on the decision-problem and we choose the parameters that result in minimal average decision costs on the validation data (i.e., the IERM cost function on the validation set). This validation scheme is in line with ETO doing the regression step as a completely separate procedure that disregards the optimization problem and IERM integrating the steps and directly targeting decision costs. In the linear settings we search over \( \lambda \in \{0, \frac{1}{10}, \frac{1}{10^2}, \frac{1}{10^3}, 1, \ldots, 100\} \). In the RKHS settings, we search over \( \lambda \in \{\frac{1}{10^3}, \frac{1}{10^2}, \frac{1}{10^3}, \frac{1}{10^4}, 1, \ldots, 100\} \) and \( \rho \in \{0.01, 0.1, 0.5, 1, 2\} \) (we drop \( \lambda = 0 \) in the RKHS case as it leads to an ill-posed solution). We solve SPO+ using the formulation in Section 5.1 of [Elmachtoub and Grigas (2021)] and Gurobi 9.1.1. Because this is extremely slow for the RKHS case, we can only do so up to \( n = 500 \), and for larger \( n \), we use the stochastic gradient descent (SGD) approach in Appendix C of [Elmachtoub and Grigas (2021)](https://github.com/paulgrigas/SmartPredictThenOptimize). We follow the their accompanied implementation setting the batch size to 10, number of iterations to 1000, and the step size to \( \frac{1}{\sqrt{t + 1}} \) for the \( t \)th SGD iteration. In Fig. 2b, we show that the results of this SGD approach closely track the reformulation approach for \( n \leq 500 \). We solve the ridge-penalized least squares using the python library scikit-learn.

**Results.** For each of \( n = 50, 100, \ldots, 1000 \), we run 50 replications of the experiment. Using a test data set of 10000 draws of just \( X \), we then compute the sample averages of \( \mathbb{E}_X [f^*(X)\pi^*(X)] \) and of \( \mathbb{E}_X [f^*(X)^\top \hat{\pi}(X)] \) for each policy \( \hat{\pi} \) resulting from one of the 6 methods and for each replication. For any \( x \) and \( f \), we compute \( \pi_f(x) \) using Gurobi 9.1.1. Recall \( \pi^*(X) = \pi_{f^*}(X) \) so that this is also applied to computing \( \pi^*(X) \). Finally, by computing averages over replications, we estimate the relative regret for \( n \) and for each method \( \hat{\pi} \), being \( \mathbb{E}_D \mathbb{E}_X [f^*(X)^\top (\hat{\pi}(X) - \pi^*(X))] / \mathbb{E}_D \mathbb{E}_X [f^*(X)\pi^*(X)] \).