A Formula for the Specialization of Skew Schur Functions*

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\textbf{Abstract.} We give a formula for \(s_{\lambda/\mu}(1, q, q^2, \ldots)/s_{\lambda}(1, q, q^2, \ldots)\), which generalizes a result of Okounkov and Olshanski about \(f_{\lambda/\mu}/f_{\lambda}\).

\textit{Keywords:} skew Schur function, \(q\)-analogue, jeu de taquin

1. Introduction

For the notation and terminology below on symmetric functions, see Stanley [6] or Macdonald [4]. Let \(\mu\) be a partition of some nonnegative integer. A reverse tableau of shape \(\mu\) is an array of positive integers of shape \(\mu\) which is weakly decreasing in rows and strictly decreasing in columns. Let \(RT(\mu, n)\) be the set of all reverse tableaux of shape \(\mu\) whose entries belong to \(\{1, 2, \ldots, n\}\).

Recall that \(f_{\lambda}\) and \(f_{\lambda/\mu}\) denote the number of SYT (standard Young tableaux) of shape \(\lambda\) and \(\lambda/\mu\) respectively, and \(l(\mu)\) denotes the length of \(\mu\). Okounkov and Olshanski [5, (0.14) and (0.18)] give the following surprising formula.

\textbf{Theorem 1.1.} Let \(\lambda \vdash m, \mu \vdash k\) with \(\mu \subseteq \lambda\) and \(n \in \mathbb{N}\) such that \(l(\mu) \leq l(\lambda) \leq n\). Then

\[
\frac{(m)_k f_{\lambda/\mu}}{f_{\lambda}} = \sum_{T \in RT(\mu, n)} \prod_{u \in \mu} (\lambda_{T(u)} - c(u));
\]

where \(c(u)\) and \(T(u)\) are the content and entry of the square \(u\) respectively, and \((m)_k = m(m-1) \cdots (m-k+1)\).

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In this paper, we generalize the above result to a $q$-analogue. Our main result is the following.

**Theorem 1.2.** Let $\lambda$ and $\mu$ be partitions with $\mu \subseteq \lambda$ and $n \in \mathbb{N}$ such that $l(\mu) \leq l(\lambda) \leq n$. Then

$$
\frac{s_{\lambda/\mu}(1, q, q^2, \ldots)}{s_\lambda(1, q, q^2, \ldots)} = \sum_{T \in \mathcal{R}(\mu, n)} \prod_{u \in \mu} q^{1-T(u)} \left(1 - q^{\lambda_{T(u)} - c(u)}\right),
$$

where the right-hand side is defined to be 1 when $\mu$ is the empty partition.

2. **Proof of the Main Result**

For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we define $[n] = 1 - q^n$ and denote by $(n \mid k)$ the $k$th falling $q$-factorial power, i.e.,

$$(n \mid k) = \begin{cases} [n][n-1]\cdots[n-k+1], & \text{if } k = 1, 2, \ldots, \\ 1, & \text{if } k = 0. \end{cases}$$

In particular, we use $[k]!$ to denote $(k \mid k)$, and $\frac{[n]!}{[k]![n-k]!}$ for $n \geq k$.

Let $\lambda$ and $\mu$ be partitions with $\mu \subseteq \lambda$ and $n \in \mathbb{N}$ such that $l(\mu) \leq l(\lambda) \leq n$. We define

$$t_{\lambda/\mu, n}(q) = s_{\lambda/\mu}(1, q, q^2, \ldots) \prod_{u \in \lambda/\mu} [n + c(u)]. \quad (2.1)$$

The following lemma is given in [6, Exer. 102, p. 551 and Lem. 7.21.1].

**Lemma 2.1.** Let $\lambda$ and $\mu$ be partitions with $\mu \subseteq \lambda$ and $n \in \mathbb{N}$ such that $l(\mu) \leq l(\lambda) \leq n$. Then we have

(a) $t_{\lambda/\mu, n}(q) = \det \left[ \frac{\lambda_i + n - i}{\lambda_i - \mu_j - i + j} \right]_{i,j=1}^n$,

(b) $\prod_{u \in \lambda} [n + c(u)] = \prod_{i=1}^n \frac{[\nu_i]!}{[n - i]!}$, where $\nu_i = \lambda_i + n - i$.

**Lemma 2.2.** Let $\lambda$ and $\mu$ be partitions with $\mu \subseteq \lambda$ and $n \in \mathbb{N}$ such that $l(\mu) \leq l(\lambda) \leq n$. Then we have

$$
\frac{s_{\lambda/\mu}(1, q, q^2, \ldots)}{s_\lambda(1, q, q^2, \ldots)} = \frac{\det[(\lambda_i + n - i \mid \mu_j + n - j)]_{i,j=1}^n}{\det[(\lambda_i + n - i \mid n - j)]_{i,j=1}^n}. \quad (2.2)
$$
Proof. By Lemma 2.1, we have

\[
\frac{s_{\lambda/\mu}(1, q, q^2, \ldots)}{s_{\lambda}(1, q, q^2, \ldots)} = \frac{t_{\lambda/\mu,n}(q)}{t_{\lambda,n}(q)} \prod_{u \in \mu} [n + c(u)]
\]

\[
= \frac{\det \left[ \left[ \frac{\lambda_i + n - i}{\lambda_i - \mu_j - i + j} \right]_{i,j=1}^n \right]}{\det \left[ \left[ \frac{\lambda_i + n - i}{\lambda_i - i + j} \right]_{i,j=1}^n \right]} \prod_{j=1}^n [\mu_j + n - j]!
\]

\[
= \frac{\det \left[ \left[ \frac{\lambda_i + n - i}{\lambda_i - \mu_j - i + j} \right] \right]_{i,j=1}^n [\mu_j + n - j]!}{\det \left[ \left[ \frac{\lambda_i + n - i}{\lambda_i - i + j} \right] \right]_{i,j=1}^n [n - j]!}
\]

\[
= \frac{\det \left[ (\lambda_i + n - i \mid n - j) \right]_{i,j=1}^n}{\det \left[ (\lambda_i + n - i \mid n - j) \right]_{i,j=1}^n}
\]

We first consider the denominator of the right-hand side of (2.2).

Lemma 2.3. We have

\[
\det \left[ (\lambda_i + n - i \mid n - j) \right]_{i,j=1}^n = \left( \prod_{i=1}^n q^{(i-1)\lambda_i} \right) \prod_{1 \leq i < j \leq n} [\lambda_i - \lambda_j - i + j]. \tag{2.3}
\]

Proof. For \( j = 1, \ldots, n - 1 \), we subtract from the \( j \)th column of the determinant on the left-hand side the \((j + 1)\)th column, multiplied by \( [\lambda_1 + j] \). Then for all \( j < n \), the \((i, j)\)th entry becomes

\[
(\lambda_i + n - i \mid n - j - 1)([\lambda_i + j + 1 - i] - [\lambda_1 + j]). \tag{2.4}
\]

In particular, the \((1, j)\)th entry becomes 0 for \( j < n \). Therefore, the determinant on the left-hand side becomes

\[
\left( \prod_{i=2}^n [\lambda_1 - \lambda_i - 1 + i] q^{\lambda_i} \right) \det \left[ (\lambda_{i+1} + n - i - 1 \mid n - j - 1) \right]_{i,j=1}^{n-1},
\]

and then the result follows by induction.

The following lemma is almost the same as [5, Lemma 2.1], just lifted to the \( q \)-analogue.

Lemma 2.4. Let \( x, y \in \mathbb{Z} \) with \( x + 1 \neq y \) and \( k \in \mathbb{N} \). Then we have

\[
\frac{(y \mid k + 1) - (x + 1 \mid k + 1)}{-q^y + q^{y+1}} = \sum_{l=0}^k q^{-l}(y \mid l)(x - l \mid k - l).
\]
Proof. We have
\[
(-q^y + q^{x+1}) \sum_{l=0}^{k} q^{-l} (y | l) (x - l | k - l)
\]
\[
= \sum_{l=0}^{k} \left( (-q^{y-l} + q^{x+1-l}) \right) (y | l) (x - l | k - l)
\]
\[
= \sum_{l=0}^{k} (y | l) (x - l | k - l) [y - l] - \sum_{l=0}^{k} (y | l) (x - l | k - l) [x + 1 - l]
\]
\[
= \sum_{l=1}^{k+1} (y | l) (x - l + 1 | k - l + 1) - \sum_{l=0}^{k} (y | l) (x - l + 1 | k - l + 1)
\]
\[
= (y | k + 1) - (x + 1 | k + 1).
\]

For two partitions \( \mu \) and \( \nu \), we write \( \mu \succeq \nu \) if \( \mu_i \geq \nu_i \geq \mu_{i+1} \) for all \( i \in \mathbb{N} \), or equivalently \( \nu \) is obtained from \( \mu \) by removing a horizontal strip. Thus given a reverse tableau \( T \in RT(\mu, n) \), we can regard it as a sequence
\[
\mu = \mu^{(1)} \succeq \mu^{(2)} \succeq \cdots \succeq \mu^{(n+1)} = \emptyset,
\]
where \( \mu^{(i)} \) is the shape of the reverse tableau consisting of entries of \( T \) not less than \( i \).

Let \( \mu / \nu \) be a skew diagram. We define
\[
(x | \mu / \nu) = \prod_{u \in \mu / \nu} [x - c(u)]. \tag{2.5}
\]
This is a generalization of the falling \( q \)-factorial powers. Now we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.2, it is equivalent to prove that
\[
\frac{\det((\lambda_i + n - i | \mu_j + n - j))_{i,j=1}^{n}}{\det((\lambda_i + n - i | n - j))_{i,j=1}^{n}} = \sum_{T \in RT(\mu, n)} \prod_{u \in \mu} q^{1-T(u)} [\lambda_{T(u)} - c(u)]. \tag{2.6}
\]
Since Lemma 2.2 still holds when \( \mu \not\preceq \lambda \), in which case both sides of (2.2) are equal to 0, we just assume \( l(\mu) \leq l(\lambda) \leq n \) in (2.6). The proof of (2.6) is by induction on \( n \). The case \( n = 0 \) is trivial, which is equivalent to the statement \( \frac{1}{1} = 1 \). For the induction step \( (n > 0) \), it suffices to prove that
\[
\frac{\det((\lambda_i + n - i | \mu_j + n - j))_{i,j=1}^{n}}{\det((\lambda_i + n - i | n - j))_{i,j=1}^{n}} = \sum_{\nu \leq \mu \atop l(\nu) < n} q^{-|\nu|} (\lambda_1 | \mu / \nu) \frac{\det \left[ (\lambda_i + n - 1 - i | \nu_j + n - 1 - j) \right]_{i,j=1}^{n-1}}{\det \left[ (\lambda_i + n - 1 - i | n - 1 - j) \right]_{i,j=1}^{n-1}}. \tag{2.7}
\]
where \( \lambda^\dagger \) denotes the partition obtained from \( \lambda \) by removing \( \lambda_1 \).

To see the sufficiency, let \( T^\dagger \) be the reverse tableau obtained from a given \( T \in \text{RT}(\mu, n) \) by removing all entries equal to 1 and decreasing remaining entries by 1. Let \( v \) be the shape of \( T^\dagger \). Then we have \( v \preceq \mu \) and \( l(v) < n \). On the other hand, given partitions \( v \) and \( \mu \) with \( v \preceq \mu \) and \( l(v) < n \), then for \( T^\dagger \in \text{RT}(v, n - 1) \), we can uniquely recover \( T \in \text{RT}(\mu, n) \) from \( T^\dagger \) in a reverse way. Thus for a fixed \( v \) with \( v \preceq \mu \) and \( l(v) < n \), we have

\[
\sum_{T \in \text{RT}(\mu, n)} \prod_{u \in \mu} q^{1 - T(u)} [\lambda_T(u) - c(u)]
\]

(2.8)

\[
= \sum_{T^\dagger \in \text{RT}(v, n - 1)} (\lambda_1 \mid \mu / v) \prod_{u \in v} q^{1 - T^\dagger(u)} [\lambda_{T^\dagger}(u) - c(u)]
\]

\[
= q^{-|v|} (\lambda_1 \mid \mu / v) \sum_{T^\dagger \in \text{RT}(v, n - 1)} \prod_{u \in v} q^{1 - T^\dagger(u)} [\lambda_{T^\dagger}(u) - c(u)]
\]

\[
= q^{-|v|} (\lambda_1 \mid \mu / v) \frac{\det \left( \lambda_{j}^{\dagger} + n - 1 - i \mid v_j + n - 1 - j \right)_{i, j=1}^{n-1}}{\det \left( \lambda_{j}^{\dagger} + n - 1 - i \mid n - 1 - j \right)_{i, j=1}^{n-1}},
\]

(2.9)

where the last equality follows from induction hypothesis. By summing (2.8) and (2.9) respectively over all partitions \( v \) with \( v \preceq \mu \) and \( l(v) < n \), we then obtain (2.6) from (2.7).

Consider the numerator of the right-hand side of (2.7),

\[
\det \left( \lambda_{i} + n - 1 - i \mid \mu_{j} + n - j \right).
\]

(2.10)

For \( j = 1, 2, \ldots, n - 1 \), we subtract from the \( j \)th column of (2.10) the \((j + 1)\)th column, multiplied by \((\lambda_1 - \mu_{j+1} + j \mid \mu_{j} - \mu_{j+1} + 1)\). Then for all \( j < n \), the \((i, j)\)th entry of (2.10) becomes

\[
(\lambda_i + n - i \mid \mu_{j+1} + n - j - 1)((\lambda_i - \mu_{j+1} + j + 1 - i \mid \mu_{j} - \mu_{j+1} + 1)
\]

\[
- (\lambda_1 - \mu_{j+1} + j \mid \mu_{j} - \mu_{j+1} + 1)).
\]

(2.11)

In particular, the first row of (2.10) becomes

\[
(0, \ldots, 0, (\lambda_1 + n - 1 \mid \mu_{n})).
\]

We can now apply Lemma 2.4, where we set

\[
x = \lambda_i - \mu_{j+1} + j - 1, \quad k = \mu_{j} - \mu_{j+1}, \quad y = \lambda_i - \mu_{j+1} + j + 1 - i.
\]

Then (2.11) becomes

\[
-(\lambda_i + n - i \mid \mu_{j+1} + n - j - 1)(\lambda_i - \lambda_i + i - 1)q^{\lambda_i - \mu_{j+1} + j + 1 - i}
\]

\[
\cdot \sum_{l=0}^{\mu_{j} - \mu_{j+1}} q^{-l}(y \mid l)(x - l \mid k - l).
\]

(2.12)
Let \( v_j = l + \mu_{j+1} \). Since

\[
q^v - q^{x+1} = q^{\lambda_i - \mu_{j+1} + j + 1 - i} \cdot [\lambda_i - \lambda_i + i - 1],
\]

\[
(x - \lambda_k - l) = (\lambda_i - v_j + j - 1 | \mu_j - v_j),
\]

\[
(y - l) = (\lambda_i - \mu_{j+1} + j + 1 - i | v_j - \mu_{j+1}),
\]

\[
(\lambda_i + n - i | \mu_{j+1} + n - j - 1)(y - l) = (\lambda_i + n - i | v_j + n - j - 1),
\]

(2.12) becomes

\[
- [\lambda_i - \lambda_i + i - 1]q^{\lambda_i - \mu_{j+1} + j + 1 - i} \sum_{v_j = \mu_{j+1}}^{\mu_j} q^{\lambda_i - v_j}(\lambda_i - v_j + j - 1 | \mu_j - v_j)
\cdot (\lambda_i + n - i | v_j + n - j - 1).
\]

Expand the determinant (2.10) by the first row,

\[
(\lambda_i + n - 1 | \mu_n) \det \left[ [\lambda_i - \lambda_i + 1 + i]q^{\lambda_i + 1 + j - i} \sum_{v_j = \mu_{j+1}}^{\mu_j} q^{-v_j} \right]_{i,j=1}^{n-1}
\]

(2.13)

For any chosen value of \( v_j \) (1 \( \leq j \leq n - 1 \)) in the range from \( \mu_{j+1} \) to \( \mu_j \), \( v = (v_1, \ldots, v_{n-1}) \) is a partition, and we have \( v \preceq \mu \). Furthermore, when \( v_j \) (1 \( \leq j \leq n - 1 \)) ranges from \( \mu_{j+1} \) to \( \mu_j \), \( v \) ranges over all partitions with \( v \preceq \mu \) and \( l(v) < n \). Therefore, (2.13) equals

\[
(\lambda_i + n - 1 | \mu_n) \sum_{v \preceq \mu \atop l(v) < n} \det \left[ [\lambda_i - \lambda_i + 1 + i]q^{\lambda_i + 1 + j - i - v_j} \right]_{i,j=1}^{n-1}
\]

(2.14)

For the determinant in (2.14), we can extract \( [\lambda_i - \lambda_i + 1 + i]q^{\lambda_i + 1 - i} \) from the \( i \)th row and extract \( q^{-v_j}(\lambda_i - v_j + j - 1 | \mu_j - v_j) \) from the \( j \)th column by multilinearity for \( 1 \leq i, j \leq n - 1 \). Then (2.14), which is equal to (2.10), becomes

\[
\left( \prod_{i=1}^{n-1} [\lambda_i - \lambda_i + 1 + i]q^{\lambda_i + 1} \right) \sum_{v \preceq \mu \atop l(v) < n} \left( \prod_{j=1}^{n-1} q^{-v_j} \right) (\lambda_i | \mu / v)
\]

(2.15)

\[
\cdot \det[(\lambda_i + n - i - 1 | v_j + n - j - 1)]_{i,j=1}^{n-1}.
\]
On the other hand, by Lemma 2.3 we have
\[
\det[(\lambda_i + n - i | n - j)]_{i,j=1}^{n-1} = \prod_{i=1}^{n-1} (|\lambda_i - \lambda_{i+1} + i| q^{\lambda_{i+1}}) \det[(\lambda_{i+1} + n - i - 1 | n - j - 1)]_{i,j=1}^{n-1}.
\]

Combining (2.15) and (2.16) together, we then obtain (2.7), which implies Theorem 1.2.

Theorem 1.1 can be recovered from Theorem 1.2 by setting \( q = 1 \). To show that, we need the following result given in [6, Prop. 7.19.11].

**Lemma 2.5.** Let \( |\lambda/\mu| = m \). Then
\[
s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_T q^{\text{maj}(T)} \frac{[m]}{[m]!},
\]
where \( T \) ranges over all SYTs of shape \( \lambda/\mu \), and \( \text{maj}(T) \) is the major index of \( T \).

**Proof of Theorem 1.1:** Divide both sides of (1.2) by \( (1 - q)^{|\mu|} \) and then set \( q = 1 \). Then the right-hand side of (1.2) becomes
\[
\sum_{T \in RT(\mu, n)} \prod_{u \in \mu} (\lambda_T(u) - c(u)).
\]
Since
\[
\sum_T q^{\text{maj}(T)} \bigg|_{q=1} = f_{\lambda/\mu}^\lambda,
\]
when \( T \) ranges over all SYTs of shape \( \lambda/\mu \), we know by Lemma 2.5 that
\[
\frac{s_{\lambda/\mu}(1, q, q^2, \ldots)}{(1 - q)^{|\mu|} s_{\lambda}(1, q, q^2, \ldots)} \bigg|_{q=1} = \frac{[m]! \sum_{T_1} q^{\text{maj}(T_1)}}{(1 - q)^{|\mu|} [m - k]! \sum_{T_2} q^{\text{maj}(T_2)}} \bigg|_{q=1}
\]
\[
= \frac{(m)_k f_{\lambda/\mu}^\lambda}{f_{\lambda}^\lambda},
\]
where \( T_1 \) and \( T_2 \) range over all partitions of shape \( \lambda/\mu \) and \( \lambda \), respectively. Combining (2.17) and (2.18) together, we then obtain Theorem 1.1.

**Corollary 2.6.** The rational function
\[
\frac{s_{\lambda/\mu}(1, q, q^2, \ldots)}{(1 - q)^{|\mu|} s_{\lambda}(1, q, q^2, \ldots)}
\]
is a Laurent polynomial in \( q \) with nonnegative integer coefficients.
Proof. Given \( T \in \text{RT}(\mu, n) \), if \( \lambda_{T(u)} < c(u_0) \) for some \( u_0 \in \mu \), then
\[
\prod_{u \in \mu} \left[ \lambda_{T(u)} - c(u) \right] = 0. \tag{2.19}
\]

In fact, while \( u \) moves from right to left along rows of \( T \), \( \lambda_{T(u)} \) is weakly decreasing, and \( c(u) \) is decreasing by 1 in each step. Let \( u_1 \) be the leftmost square in the row containing \( u_0 \). Since \( \lambda_{T(u_1)} \geq c(u_1) \) and \( \lambda_{T(u_0)} < c(u_0) \), we have \( \lambda_{T(u_2)} = c(u_2) \) for some square \( u_2 \), which implies Equation (2.19).

On the other hand, by Theorem 1.2 we have
\[
\frac{s_{\lambda/\mu} (1, q, q^2, \ldots)}{(1 - q)^{|\mu|} s_{\lambda} (1, q, q^2, \ldots)} = \sum_{T \in \text{RT}(\mu, n)} \prod_{u \in \mu} q^{1 - T(u)} \cdot \frac{\lambda_{T(u)} - c(u)}{1 - q}.
\]

Then the result follows after omitting the sum terms that equal to 0 on the right-hand side. \( \blacksquare \)

For the special case when \( \mu = 1 \), we give a simple formula for \( s_{\lambda/1}(1 - q)s_{\lambda} \) in Corollary 2.7 below. Before giving a combinatorial proof of this result, we first introduce some notation.

The acronym SSYT stands for a semistandard Young tableau where 0 is allowed as a part. \textit{Jeu de taquin} (jdt) is a kind of transformation between skew tableaux obtained by moving entries around, such that the property of being a tableau is preserved. For example, given a tableau \( T \) of shape \( \lambda \), we first delete the entry \( T(i, j) \) for some box \( (i, j) \). If \( T(i, j - 1) > T(i - 1, j) \), we then move \( T(i, j - 1) \) to box \( (i, j) \); otherwise, we move \( T(i - 1, j) \) to \( (i, j) \). Continuing this moving process, we eventually obtain a tableau of shape \( \lambda/1 \). On the other hand, given a tableau of shape \( \lambda/1 \), we can regard \( (0, 0) \) as an empty box. By moving entries in a reverse way, we then get a tableau of shape \( \lambda \) with an empty box after every step. For more information about jdt, readers can refer to [6, Ch. 7, App. I].

The following result was first obtained by Kerov [2, Thm. 1 and (2.2)] (after sending \( q \mapsto q^{-1} \)) and by Garsia and Haiman [1, (I.15), Thm. 2.3] (setting \( t = q^{-1} \)) by algebraic reasoning. For further information see [3, p. 9].

Corollary 2.7. We have
\[
\frac{s_{\lambda/1} (1, q, q^2, \ldots)}{(1 - q)s_{\lambda} (1, q, q^2, \ldots)} = \sum_{u \in \lambda} q^{c(u)}. \tag{2.20}
\]

Proof. We define two sets in the following way:
\[
T_{\lambda/1} = \{(T, k) \mid T \text{ is an SSYT of shape } \lambda/1, \text{ and } k \in \mathbb{N}\},
\]
\[
T_{\lambda} = \{(T, u) \mid T \text{ is an SSYT of shape } \lambda, \text{ and } u \in \lambda\}.
\]

Since we can rewrite (2.20) as
\[
s_{\lambda/1} (1, q, q^2, \ldots) \cdot \sum_{i \geq 0} q^i = s_{\lambda} (1, q, q^2, \ldots) \cdot \sum_{u \in \lambda} q^{c(u)},
\]
it suffices to prove that there is a bijection $\varphi: T_{\lambda} \to T_{\lambda/1}$, say $\varphi(T, u) = (T_{\varphi}, k)$, such that $|T| + c(u) = |T_{\varphi}| + k$, where $|T|$ and $|T_{\varphi}|$ denote the sum of the entries in $T$ and $T_{\varphi}$ respectively.

We define $\varphi$ in the following way. Given $(T, u) \in T_{\lambda}$, let $k = T(u) + c(u)$. To obtain $T_{\varphi}$, we first delete the entry $T(u)$ from $T$, and then carry out the jdt operation. Since $T$ is an SSYT, we have $k \geq 0$, and thus the definition is reasonable.

On the other hand, given $(T_{\varphi}, k) \in T_{\lambda/1}$, we carry out the jdt operation to $T_{\varphi}$ step-by-step in the reverse way. Denote by $u_t$ the empty box and $T_t$ the tableau obtained after $t$ steps. If we get an SSYT by filling $u_t$ with $k - c(u_t)$ in $T_t$, then we call $u_t$ a nice box.

We first show that a nice box exists. For the sake of discussion, if $(i, j)$ is not a box of a tableau $T$, then we define $T(i, j) = -\infty$ if $i < 0$ or $j < 0$, and $T(i, j) = +\infty$ if $i \geq 0$ and $j \geq 0$. The existence is obvious if the initial empty box $u_0$ is nice. If there is no integer $t$ such that $k - c(u_t)$ is less than the adjacent entry left or not greater than the adjacent entry above in $T_t$, then by filling the last empty box $u_{t_0}$ with $k - c(u_{t_0})$, we get an SSYT, which implies that $u_{t_0}$ is a nice box. Otherwise, let $t$ be the smallest integer such that $k - c(u_t)$ is less than the adjacent entry left or not greater than the adjacent entry above. Then we claim that $u_{t-1}$ is a nice box. Assume that $u_t = (i, j)$. Since $T_{t-1}$ and $T_t$ satisfy the conditions of SSYT except for the empty box, we have $T_{t-1}(i-1, j) \leq T_{t-1}(i, j) \leq T_{t-1}(i + 1, j)$ if $u_{t-1} = (i-1, j)$, and $T_{t-1}(i-1, j) < T_{t-1}(i, j) < T_{t-1}(i + 1, j)$ if $u_{t-1} = (i, j-1)$. In the first case, we have $k - c(u_t) \leq T_{t-1}(i, j)$, thus $k - c(u_{t-1}) = k - c(u_t) - 1 \leq T_{t-1}(i, j) \leq T_{t-1}(i - 1, j)$ in the latter one, we have $k - c(u_t) < T_{t-1}(i, j)$, so $k - c(u_{t-1}) = k - c(u_t) + 1 \leq T_{t-1}(i, j) < T_{t-1}(i + 1, j)$.

By assumption, $k - c(u_{t-1})$ is not less than the entries left and greater than the entry above in $T_{t-1}$. Therefore, we get an SSYT by filling $u_{t-1}$ with $k - c(u_{t-1})$ in $T_{t-1}$ in both cases, which completes the proof of the existence.

Next we show the uniqueness of the nice box. Let $u = (i, j)$ be the first nice box and let $T$ be the corresponding SSYT. If there exists another nice box $u' = (i', j')$, and $T'$ is the corresponding SSYT, then we have $i' \geq i$ and $j' \geq j$. Since $T'$ is an SSYT, we must have $T'(i', j') \geq T'(i, j) + i' - i$. Since $T$ is an SSYT, we have $T'(i, j) > k + i - j$ when $j' = j$, and $T'(i, j) \geq k + i - j$ when $j' > j$. In either case we get a contradiction, since $T'(i', j') = k + i' - j'$ by the definition of $T'$.

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