Meta-nilpotent knot invariants and symplectic automorphism groups of free nilpotent groups

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Abstract

We develop nilpotently $p$-localization of knot groups in terms of the (symplectic) automorphism groups of free nilpotent groups. We show that any map from the set of conjugacy classes of the outer automorphism groups yields a knot invariant. We also investigate the automorphism groups and compute the resulting knot invariants.

Keywords

Meta-nilpotent quotient, knot, symplectic representation, mapping class group, Milnor pairing

Subject Codes

57K10, 57K18, 20J06, 20F18, 20F28

1 Introduction

In this paper, we study the meta-nilpotence of a knot $K$ in an integral homology 3-sphere $M$ and suggest an approach to such knots regarding certain automorphism groups.

First, we will establish the terminology regarding nilpotent groups. For a group $G$ and subgroups $K, H \subset G$, let $[K, H]$ be the commutator subgroup generated by elements $hkh^{-1}k^{-1}$ with $h \in H, k \in K$. Define $G_1$ to be $[G, G]$, and inductively $G_{k+1}$ to be $[G, G_k]$. Namely, $G \supset G_1 \supset G_2 \supset \cdots$ is a lower central series. For any prime $p$ (possibly $p = 0$), we can define the localization of the nilpotent quotient $G/G_k$ and denote it by $G/G_k \otimes \mathbb{Z}(p)$; see Section 3 for details. If the abelianization of $G$ is $\mathbb{Z}$, we have the semi-direct product $[G, G] \rtimes \mathbb{Z}$; the restricted action of $\mathbb{Z}$ on $[G, G]_k$ ensures the semi-direct product $[G, G]/[G, G]_k \rtimes \mathbb{Z}$, which we call the $(k$-th$)$ meta-nilpotent quotient (of $G$).

In Section 3, we develop a nilpotent approach to studying knots as in attitudes of fibered spaces with a surface fiber. The correspondence between fibered knots and the associated monodromy is a complete invariant of fibered knots; however, some knots are fibered, but others are not (we discuss some properties of fiberedness in Section 2). As a generalization suitable for every knot, we will show (Theorem 3.2) that, for an appropriate prime $p$, the localization of the meta-nilpotent quotient of any knot group $\pi_K := \pi_1(M \setminus K)$ is isomorphic to that of a free group $F$, that is,

$$([\pi_K, \pi_K]/([\pi_K, \pi_K]) \otimes \mathbb{Z}(p)) \rtimes \mathbb{Z} \cong (F/F_k \otimes \mathbb{Z}(p)) \rtimes \mathbb{Z},$$

where the rank of $F$ is equal to the degree of the Alexander polynomial $\Delta_K$. If $p = 0$, the isomorphism is mentioned and studied in the viewpoint of “Fox pairings”; see [Tur].

Moreover, we completely classify the choices of the semi-direct product, up to conjugacy, as follows. By considering the projection of outer automorphism groups,

$$q_k : \text{Out}(F/F_k \otimes \mathbb{Z}(p)) \to \text{Out}(F/F_1 \otimes \mathbb{Z}(p)) = \text{GL}(\text{rank} F; \mathbb{Z}(p)),$$

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2An integral homology 3-sphere is a closed 3-manifold having integral homology of the 3-sphere.
we take the subgroup, SpOut($F/F \otimes \mathbb{Z}_{(p)}$), of the preimage of the symplectic group Sp(rank$F; \mathbb{Z}_{(p)}$). Then, we show (Theorem 3.2) that, when the action of $\mathbb{Z}$ in the semi-direct product is regarded as an automorphism in Out($F/F \otimes \mathbb{Z}_{(p)}$), it lies in SpOut($F/F \otimes \mathbb{Z}_{(p)}$) by Milnor duality [3]. Theorem 4.2 shows that the correspondence between a knot and the monodromy of the semi-direct product defines a map

$$\{\text{a knot } K \text{ in an integral homology 3-sphere with } \deg \Delta_K = \text{rank } F\} \rightarrow \frac{\text{SpOut}(F/F \otimes \mathbb{Z}_{(p)})}{\text{conjugation}}.$$  

(1)

In Section 5, we will see that the restriction on the set of fibered knots is a priori equal to the Dehn-Nielsen embedding. In other words, the construction of such monodromies is an extension of the correspondence between a fibered knot and the monodromy. Section 5 also discusses the image and surjectivity of the map (1), as a comparison with the Dehn-Nielsen embedding and homology cobordism; see Propositions 5.2 and 5.5.

Meanwhile, the map (1) implies that any map from the conjugacy classes of SpOut($F/F_k \otimes \mathbb{Z}_{(p)}$) is a knot invariant. To obtain such maps, we invoke the works on the Johnson homomorphisms of the automorphism group Aut($F/F_k$) and of the mapping class group, $\mathcal{M}_{g,1}$, of an orientable surface with one boundary. Roughly speaking, the Johnson homomorphism is a nilpotent study of Aut($F/F_k$) and of some homomorphism $\rho_k : \mathcal{M}_{g,1} \rightarrow \text{Aut}(F/F_k)$, together with the help of Sp-representations. Inspired by the works of Morita [1, 2, 3], in Section 6, we investigate the above group SpOut($F/F_k \otimes \mathbb{Z}_{(p)}$) in details, while in Section 7, we explicitly describe the group structure of SpOut($F/F_k \otimes \mathbb{Z}_{(p)}$) with $k = 2, 3$ in terms of Sp-representations. As a corollary, we will define some maps from the conjugacy classes of SpOut($F/F_k \otimes \mathbb{Z}_{(p)}$), when $k = 1, 2, 3$; see Section 8.

This paper is organized as follows. Section 2 reviews some properties of fibered knots. In Sections 3 and 4, we study nilpotent localization of knot groups and determine the uniqueness up to conjugacy. Section 5 makes a comparison with the Dehn-Nielsen embedding. In Sections 6 and 7, we investigate the automorphism groups Aut($F/F_k \otimes \mathbb{Z}_{(p)}$) and Out($F/F_k \otimes \mathbb{Z}_{(p)}$). Section 8 suggests some knot invariants obtained from the nilpotent localization.

**Conventional notation.** Every knot $K$ is understood to be smooth, oriented, and embedded in a rational homology 3-sphere $M$ as a circle, where $K$ is null-homologous in $H_1(M; \mathbb{Z})$. We denote by $\pi_K$ the knot group $\pi_1(M \setminus K)$, and by $\Delta_K \in \mathbb{Q}[t^{\pm 1}]$ the Alexander polynomial of $K$; see [L1] for the definition. By $p$ we mean a prime number (possibly $p = 0$). For a polynomial $f(t)$, let $l\text{-coeff}(t)$ be the leading coefficient.

Given groups $G, H$ and a homomorphism $\phi : G \rightarrow \text{Aut}(H)$, we can define the semi-direct product $H \rtimes \phi G$; when we should emphasize the existence of $\phi$, we will often denote $H \rtimes \phi G$ by $H \rtimes G$. If $G = \mathbb{Z}$, such a $H \rtimes G$ is sometimes called a mapping torus.
2 Fibered knots and monic Alexander polynomials

We will mainly focus on localized meta-nilpotent quotients of knot groups. Before going to the quotient, we will review fibered knots as a toy model. A knot \( K \subset M \) is said to be fibered if \( M \setminus K \) is the total space of a fiber bundle over \( S^1 \) whose fiber is given by a Seifert surface. It is worth noting that, if \( K \) is fibered, then \( \pi_1(M \setminus K) \cong F \rtimes \mathbb{Z} \) for some free group \( F \); in particular, the \( k \)-th meta-nilpotent quotient of \( \pi_1(M \setminus K) \) is isomorphic to \( F/F_k \rtimes \mathbb{Z} \).

Conversely, as is classically known, if \( K \) is not fibered, no free group \( F \) admits \( \pi_1(M \setminus K) \cong F \rtimes \mathbb{Z} \). However, as indicated in [Go §5], we can show that, under a certain condition, the meta-nilpotence of \( \pi_1(M \setminus K) \) is a mapping torus: More precisely,

**Proposition 2.1** (folklore). Let \( M \) be an integral homology 3-sphere. Let \( \pi_K \) be the knot group \( \pi_1(M \setminus K) \) of a knot \( K \subset M \). Suppose that the leading coefficient of \( \Delta_K \) is \( \pm 1 \), i.e., \( l\text{-coeff} \Delta_K = \pm 1 \).

Then, there exist a free group \( F \) of rank \( \deg \Delta_K \) and an isomorphism \( \tau : F \rightarrow F \) such that the \( k \)-th meta-nilpotent quotient of \( \pi_1(M \setminus K) \) is isomorphic to \( F/F_k \rtimes \tau \mathbb{Z} \) for any \( k \in \mathbb{N} \).

**Proof.** Rapaport [Ra] and Crowell [Cro] prove that, since \( \Delta_K \) is monic, \( H_1([\pi_K, \pi_K]; \mathbb{Z}) \) is finitely and freely generated and of rank \( \deg \Delta_K \), that is, \( H_1([\pi_K, \pi_K]; \mathbb{Z}) \cong \mathbb{Z}^{\deg \Delta_K} \). For \( m \geq 2 \), the homology \( H_m([\pi_K, \pi_K]; \mathbb{Z}) = 0 \) is widely known; see, e.g., [Cro]. Therefore, it follows from [Sta, Theorem 3.4] that \( [\pi_K, \pi_K]/([\pi_K, \pi_K])_k \) is isomorphic to \( F/F_k \) for some free group \( F \). Since the action of \( \mathbb{Z} \) on \( [\pi_K, \pi_K]/([\pi_K, \pi_K])_k \) can be identified with that on \( F/F_k \), we have \( [\pi_K, \pi_K]/([\pi_K, \pi_K])_k \rtimes \mathbb{Z} \cong F/F_k \rtimes \tau \mathbb{Z} \) for some \( \tau \), as required.

We end this section by remarking on the assumption that \( \Delta_K(t) \) is monic. As is known [Cro, Ra], if it is not so, then \( H_1([\pi_K, \pi_K]; \mathbb{Z}) \) is not finitely generated over \( \mathbb{Z} \); hence, the \( k \)-th meta-nilpotent quotient can never be \( F/F_k \rtimes \tau \mathbb{Z} \) for any free group \( F \) of rank \( < \infty \).

3 Meta-nilpotent quotients of knot groups

While \( \Delta_K(t) \) is required to be monic in Proposition 2.1, here, we give a similar theorem (Theorem 3.2) that is suitable to every knot. For this, we will discuss meta-nilpotent quotients.

First, let us briefly review the \( p \)-localization of nilpotent groups. Here, we refer the reader to [Hil]. For any group \( G \) and \( k \in \mathbb{N} \cup \{ \infty \} \), there are uniquely a group \( G/G_k \otimes \mathbb{Z}(p) \) and a homomorphism \( \kappa : G/G_k \rightarrow G/G_k \otimes \mathbb{Z}(p) \) satisfying that

(R1) For any \( m \in \mathbb{Z} \), the quotient homomorphism \( G_m/G_{m+1} \rightarrow (G/G_k \otimes \mathbb{Z}(p))_m/(G/G_k \otimes \mathbb{Z}(p))_{m+1} \) induced by \( \kappa \) is equal to the localization at \( p \) of the additive homomorphism \( G_m/G_{m+1} \rightarrow (G/G_{m+1})_p \).

(R2) The induced map \( \kappa_* : H_*(G/G_k; \mathbb{Z}) \rightarrow H_*(G/G_k \otimes \mathbb{Z}(p); \mathbb{Z}) \) on the group homology with coefficients \( \mathbb{Z} \) is the localization at \( p \).

The purpose of this section is to show the followings:
Proposition 3.1. Let $\tau : G \rightarrow G$ be an automorphism. Then, there exists a group isomorphism $\tau \otimes \mathbb{Z}_p : G/G_k \otimes \mathbb{Z}_p \rightarrow G/G_k \otimes \mathbb{Z}_p$ such that $(\tau \otimes \mathbb{Z}_p) \circ \kappa = \kappa \circ \tau$ and the quotient map
\[
(\tau \otimes \mathbb{Z}_p)_m : \frac{(G/G_k \otimes \mathbb{Z}_p)_m}{(G/G_k \otimes \mathbb{Z}_p)_{m+1}} \rightarrow \frac{(G/G_k \otimes \mathbb{Z}_p)_m}{(G/G_k \otimes \mathbb{Z}_p)_{m+1}}
\]
is the $p$-localization of the quotient $\overline{\tau} : G_m/G_{m+1} \rightarrow G_m/G_{m+1}$ for any $m \in \mathbb{N}$.

Furthermore, the existence of $\tau \otimes \mathbb{Z}_p$ is unique up to conjugacy.

Theorem 3.2. Let $K$ be a knot in an integral homology 3-sphere $M$, and let $F$ be the free group of rank $\deg \Delta_K = 2g$. Suppose $p$ and $l$-coefficients are relatively prime, say $p = 0$.

(i) For any $m \in \mathbb{N}$, there exist an automorphism $\tau : F/F_m \otimes \mathbb{Z}_p \rightarrow F/F_m \otimes \mathbb{Z}_p$ and a $\mathbb{Z}$-equivariant homomorphism $\psi_m : \pi_K \rightarrow (F/F_m \otimes \mathbb{Z}_p) \rtimes \tau \mathbb{Z}$, which gives rise to an isomorphism
\[
([\pi_K, \pi_K]/[\pi_K, \pi_K]_m) \otimes \mathbb{Z}_p \rtimes \mathbb{Z} \cong (F/F_m \otimes \mathbb{Z}_p) \rtimes \tau \mathbb{Z},
\]
and $\tau$ with $m = 1$ lies in the symplectic group $\text{Sp}(2g; \mathbb{Z}_p) \subset \text{GL}(2g; \mathbb{Z}_p)$. Here, $g = \deg \Delta_K / 2$, and we regard any automorphism $F/F_m \otimes \mathbb{Z}_p \rightarrow F/F_m \otimes \mathbb{Z}_p$ as an element of $\text{GL}(2g; \mathbb{Z}_p)$.

(ii) If $[\pi_K, \pi_K]$ is residually nilpotent or if $K$ is a pseudo-alternating knot in the sense of $\text{[MM]}$, then the inverse limit of the isomorphism in (i) is also an isomorphism.

Remark 3.3. According to $\text{[JJ, MM]}$ and references therein, we can find many knots such that $[\pi_K, \pi_K]$ are residually nilpotent. For example, We can do so if $K$ is fibered, or a two-bridge knot, or pseudo-alternating having $l$-coefficients equal to $q^l$ for some prime $q$.

Definition 3.4. Take a knot $K \subset M$. If $p$ and $l$-coefficients are coprime, we call such an automorphism $\tau : F/F_m \otimes \mathbb{Z}_p \rightarrow F/F_m \otimes \mathbb{Z}_p$ a (nilpotently) $p$-localized monodromy (of $K$). (Later, we discuss the choices of $\tau$; see $\text{[4]}$). If $p$ and $l$-coefficients are not coprime, we define a $p$-localized monodromy of $K$ by 1.

Before going to the proofs, let us review a relation between lifts of homomorphisms and second group homology (see Proposition 3.5 below). Let $\rho : \Gamma \rightarrow G$ be a group homomorphism, and consider the following group homomorphisms:
\[
\begin{array}{cccccc}
0 & \rightarrow & N & \overset{\kappa}{\rightarrow} & \tilde{\Gamma} & \overset{\rho}{\rightarrow} & \Gamma & \rightarrow & 1 \\
0 & \rightarrow & K & \overset{p}{\rightarrow} & \tilde{G} & \rightarrow & G & \rightarrow & 1
\end{array}
\]
(central extension)

Here, we suppose that the center of $\tilde{G}$ is equal to $K$. Let us discuss the existence of a lift $\tilde{\rho} : \tilde{\Gamma} \rightarrow \tilde{G}$ of $\rho$. Recall the delta map $\delta : H^1(N; K) \rightarrow H^2(\Gamma; K)$ in the five-term exact sequences of group cohomology. Since $H^2(G; K)$ is in 1:1 correspondence with the set of equivalent classes of central extensions of $G$ (see, e.g., $\text{[BT] Sections 3-4}$), we have the associated cohomology 2-class $\sigma_G \in H^2(G; K)$ with $\rho : \tilde{G} \rightarrow G$. Then, the following is true:
**Proposition 3.5** (See, e.g., [BT, Propositions 2.1.8 and 2.1.9]). In the above notation, the homomorphism \( \rho \) admits a lift \( \tilde{\rho} : \tilde{\Gamma} \to \tilde{G} \) if and only if there is a homomorphism \( \alpha : N \to K \) such that \( \delta(\alpha) = \rho^*(\sigma_G) \). Here, we identify \( H^1(N; K) \) with \( \text{Hom}(N, K) \).

Furthermore, suppose \( \kappa N \subset [\tilde{\Gamma}, \tilde{\Gamma}] \). Then, another such lift \( \tilde{\rho}' \) is conjugate to \( \rho \) if and only if \( \alpha = \alpha' \). Here, \( \alpha' \) is the associated homomorphism \( N \to K \) with \( \rho' \) satisfying \( \delta(\alpha') = (\rho')^*(\sigma_G) \).

**Proof of Proposition 3.1.** The proof is by induction on \( \delta \) such that \( \rho \) admits a lift \( \tilde{\rho} \) at \( \rho \).

Choose a genus- \( \infty \) Seifert surface of \( K \). Let us assume \( m > 2 \) and a homomorphism \( \tau \otimes \mathbb{Z}_p : G/G_m \otimes \mathbb{Z}_p \to G/G_m \otimes \mathbb{Z}_p \) satisfying the required condition. Let \( \alpha \) be the localization of the restriction of \( \tau \):

\[
\alpha = (\tau \otimes \mathbb{Z}_p)_* : (G_m/G_{m+1})_p = \frac{(G/G_k \otimes \mathbb{Z}_p)_m}{(G/G_k \otimes \mathbb{Z}_p)_m} \to \frac{(G/G_k \otimes \mathbb{Z}_p)_m}{(G/G_k \otimes \mathbb{Z}_p)_m}.
\]

By (R3) and using diagram chasing on the five-term exact sequences, we can easily check that \( \delta(\alpha) = \rho^*(\sigma_G) \). Therefore, applying the setting,

\[
\rho = \tau \otimes \mathbb{Z}_p, \quad G = \Gamma = G/G_m \otimes \mathbb{Z}_p, \quad \tilde{G} = \tilde{\Gamma} = G/G_{m+1} \otimes \mathbb{Z}_p,
\]

to Proposition 3.5, we have \( \tilde{\rho} : G/G_{m+1} \otimes \mathbb{Z}_p \to G/G_{m+1} \otimes \mathbb{Z}_p \). If \( \tilde{\rho} \) is replaced by \( \tau \otimes \mathbb{Z}_p \), it satisfies the required properties by construction. Moreover, by Proposition 3.5, the construction of \( \tilde{\rho} \) is unique up to conjugacy.

Turning now to the proof of Theorem 3.2, we need two lemmas. Since \( H_*(M \setminus K; \mathbb{Z}) \cong H_*(S^1; \mathbb{Z}) \) by Alexander duality, there is uniquely an epimorphism \( \pi_1(M \setminus K) \to \mathbb{Z} \) up to signs, and we can take the infinite cyclic covering space \( E^\infty_K \).

**Lemma 3.6** (cf. [Mi] Assertion 5). Suppose \( p \) and \( 1 - \text{coef}_{\Delta_K} \) are relatively prime. Then, the homology \( H_1(E^\infty_K; \mathbb{Z}_p) \) with coefficients \( \mathbb{Z}_p \) is isomorphic to \( (\mathbb{Z}_p)^{\text{deg}_{\Delta_K}} \).

**Proof.** Choose a genus- \( g \) Seifert surface of \( K \). According to [Li] Theorem 6.5, there is a finite presentation of \( H_1(E^\infty_K; \mathbb{Z}_p) \)

\[
(\mathbb{Z}_p[t^{\pm 1}])^{2g} \xrightarrow{A - tA^t} (\mathbb{Z}_p[t^{\pm 1}])^{2g} \to H_1(E^\infty_K; \mathbb{Z}_p) \to 0
\]

such that \( \det A - tA^t = \Delta_K \). Since \( (p, 1 - \text{coef}_{\Delta_K}(t)) = 1 \), \( \Delta_K \) may be monic. Therefore, \( H_1(E^\infty_K; \mathbb{Z}_p) \) is free and finitely generated over \( \mathbb{Z}_p \). After tensoring with \( \mathbb{Q} \), we can verify by using the elementary divisor theorem that the rank is determined by the degree of \( \Delta_K(t) \). Hence, \( H_1(E^\infty_K; \mathbb{Z}_p) \cong (\mathbb{Z}_p)^{\text{deg}_{\Delta_K}} \), as desired.

**Lemma 3.7** (Localization at \( p \) of Stallings theorem [Sta, Theorem 7.3]). Let \( f : G \to L \) be a group homomorphism such that \( H_1(f; \mathbb{Z}_p) \) is an isomorphism and \( H_2(f; \mathbb{Z}_p) \) is surjective. Assume that, for any \( m \in \mathbb{N} \), \( G_m/G_{m+1} \otimes \mathbb{Z}_p \) is finitely generated and free over \( \mathbb{Z}_p \). Then, \( f \) induces isomorphisms,

\[
(G_{k-1}/G_k) \otimes \mathbb{Z}_p \cong (L_{k-1}/L_k) \otimes \mathbb{Z}_p,
\]

and induces embeddings \( G/G_k \hookrightarrow L/L_k \) for any \( k \in \mathbb{N} \).
By Milnor pairing again, the second and third terms are regarded as an element of $\text{Sp}(2g; \mathbb{Z})$. Let $G$ denote $[\pi_K, \pi_K] = \pi_1(E_K^\infty)$. This is known as the Milnor pairing (see [Mil] Assertion 9 and Remark 1]), and there is a non-degenerate anti-symmetric bilinear map $\langle , \rangle : H^1(E_K^\infty; \mathbb{Z}_p) \times H^1(E_K^\infty; \mathbb{Z}_p) \to \mathbb{Z}_p$ satisfying $\langle ta, tb \rangle = \langle a, b \rangle$ for any $a, b \in H^1(E_K^\infty; \mathbb{Z}_p)$. This implies the existence of a symplectic basis on $H^1(E_K^\infty; \mathbb{Z}_p)$ with respect to $\langle , \rangle$. Namely, this $t$ can be regarded as an element of $\text{Sp}(2g; \mathbb{Z}_p)$. In addition, we claim $H_2(G; \mathbb{Z}_p) = 0$. To see this, consider the long exact sequence,

$$0 \to H_2(E_K^\infty; \mathbb{Z}_p) \to H_2(E_K^\infty, \partial E_K^\infty; \mathbb{Z}_p) \to H_1(\partial E_K^\infty; \mathbb{Z}_p) \xrightarrow{\text{(inclusion)}} H_1(E_K^\infty; \mathbb{Z}_p).$$

By Milnor pairing again, the second and third terms are $\mathbb{Z}_p$. The last map is zero since $H_1(\partial E_K^\infty; \mathbb{Z}_p) \cong \mathbb{Z}_p$ is generated by a longitude, which is bounded by a Seifert surface in $E_K^\infty$. Therefore, $H_2(E_K^\infty; \mathbb{Z}_p) = 0$. Since $\pi_1 E_K^\infty = G$ and there is an epimorphism $H_2(E_K^\infty; \mathbb{Z}_p) \to H_2(\pi_1(E_K^\infty); \mathbb{Z}_p)$, we have $H_2(G; \mathbb{Z}_p) = 0$, as desired.

Let us construct an isomorphism below (3). Choose a homomorphism $\phi : F \to G$ whose induced map on the first homology $H_1(\bullet, \mathbb{Z}_p)$ sends the generators of $F$ to the symplectic basis. Then, $\phi$ induces an isomorphism on $H_1(\bullet, \mathbb{Z}_p)$, which can be regarded as an element of $\text{Sp}(2g; \mathbb{Z}_p)$. Since $H_2(G; \mathbb{Z}_p) = 0$ and $H_1(G; \mathbb{Z}_p)$ is free by Lemma 3.6, it follows from Lemma 3.7 that $\phi$ induces $F/F_m \to G/G_m$, which ensures isomorphisms $F_k/F_{k+1} \otimes \mathbb{Z}_p \cong (G/G_k) \otimes \mathbb{Z}_p(k \geq 0)$ and

$$\Psi_k : F/F_k \otimes \mathbb{Z}_p \xrightarrow{\sim} (G/G_k) \otimes \mathbb{Z}_p,$$

for any $k \in \mathbb{N}$. By Proposition 3.1, the action of $\mathbb{Z}$ on $G$ yields that of $(G/G_k) \otimes \mathbb{Z}_p$, which gives an automorphism $\tau : F/F_k \otimes \mathbb{Z}_p \to F/F_k \otimes \mathbb{Z}_p$ via $\Psi_k$. Hence, by construction, the composite

$$\psi : \pi_K \xrightarrow{\text{proj}} (G/G_k) \times \mathbb{Z} \xrightarrow{\text{localization}} (G/G_k \otimes \mathbb{Z}_p) \times \mathbb{Z} \xrightarrow{\Psi_k^{-1} \times \text{id}_\mathbb{Z}} (F/F_k \otimes \mathbb{Z}_p) \times_\tau \mathbb{Z}$$

has the desired properties of (i).

To show (ii), it is enough to prove that $G \otimes \mathbb{Z}_p$ is residually nilpotent. If $\bigcap_{m \geq 0} G_m = 0$, the properties of the localization imply $\bigcap_{m \geq 0}(G \otimes \mathbb{Z}_p)_m = 0$. On the other hand, if $K$ is (pseudo-)alternating, Theorems 2.2 and 2.5 in [MM] immediately lead to $\bigcap_{m \geq 0}(G \otimes \mathbb{Z}_p)_m = 0$ as well. Hence, this completes the proof.

Remark 3.8. For a concrete knot $K \subset S^3$, the above proof implies that, to compute $\tau$ explicitly, we need to describe a symplectic basis of $H^1(E_K^\infty; \mathbb{Z}_p)$ explicitly. Fortunately, Ohkura [O] gives a list of matrix presentations of the Milnor pairings on $H^1(E_K^\infty; \mathbb{Z}_p)$ with $p = 0$, where $K$ is one of “quasi-Pretzel knots”. Thus, it is not so hard to concretely obtain 0-localized monodromies according to this list.
Remark 3.9. Furthermore, when \( p = 0 \), we can easily check that Theorem 3.2 holds for any knot \( K \) in any rational homology sphere \( M \). Since the proof is almost the same as the above proof, we omit the details.

4 Knot invariants from conjugacy classes of \( \text{Out}(F/F_k \otimes \mathbb{Z}(p)) \)

Here, we will discuss the choices of localized monodromies of a knot (Proposition 4.1) and suggest a procedure to obtain knot invariants (Theorem 4.2). In what follows, \( \text{Aut}(G) \) denotes the automorphism group of \( G \) and \( \text{Inn}(G) \) (resp. \( \text{Out}(G) \)) is the inner (resp. outer) automorphism group of \( G \). Here, \( \text{Out}(G) \) is defined to be \( \text{Aut}(G)/\text{Inn}(G) \).

Proposition 4.1. Let \( G \) be a group. Take \( \varphi, \psi \in \text{Aut}(G) \). We denote the classes in \( \text{Out}(G) \) by \( \varphi \) or \( \psi \). Assume that the first homology groups of \( G \times \varphi \mathbb{Z} \) and \( G \times \psi \mathbb{Z} \) are \( \mathbb{Z} \). Then, the following are equivalent:

(I) There is a group isomorphism \( \rho : G \times \varphi \mathbb{Z} \cong G \times \psi \mathbb{Z} \).

(II) The classes \( \varphi \) and \( \psi \) are conjugate in \( \text{Out}(G) \).

Proof. Suppose (I). By assumption, \( \rho(e, 1) = (g, 1) \) for some \( g \in G \). Thus, we have an automorphism \( \alpha \in \text{Aut}(G) \) such that \( \rho(x, 0) = (\alpha(x), 0) \) for any \( x \in G \). Notice the equality \( (e, 1)(x, 0)(e, 1)^{-1} = (\varphi(x), 0) \in G \times \varphi \mathbb{Z} \). The application of \( \rho \) to the left-hand side is computed as

\[
\rho((e, 1)(x, 0)(e, 1)^{-1}) = (g, 1)(\alpha(x), 0)(\psi^{-1}(g^{-1}), -1) = (g\psi(\alpha(x))g^{-1}, 0),
\]

while its application to the right side implies \( \rho((\varphi(x), 0)) = (\alpha(\varphi(x)), 0) \). Namely, \( g\psi(\alpha(x))g^{-1} = \alpha(\varphi(x)) \), which means (II).

Conversely, if we assume (II), we have \( g\psi(\alpha(x))g^{-1} = \alpha(\varphi(x)) \) for some \( g \in G \) and \( \alpha \in \text{Aut}(G) \). Then, the correspondence defined by \( (x, 0) \mapsto (\alpha(x), 0) \) and \( (e, 1) \mapsto (g, 1) \) gives rise to an isomorphism \( G \times \varphi \mathbb{Z} \cong G \times \psi \mathbb{Z} \), as desired. \( \square \)

Next, we will discuss symplecticness and the choices of localized monodromies. Let \( q_k : \text{Aut}(F/F_k \otimes \mathbb{Z}(p)) \to \text{Aut}(F/F_1 \otimes \mathbb{Z}(p)) = \text{GL}(2g; \mathbb{Z}(p)) \) be the epimorphism induced by the projection \( F/F_k \rightarrow F/F_1 \), \( \text{SpAut}(F/F_k \otimes \mathbb{Z}(p)) \) be the subgroup \( q_k^{-1}(\text{Sp}(2g; \mathbb{Z}(p))) \) and \( \text{SpOut}(F/F_k \otimes \mathbb{Z}(p)) \) be the subgroup of \( \text{Out}(F/F_k \otimes \mathbb{Z}(p)) \) generated by \( q_k^{-1}(\text{Sp}(2g; \mathbb{Z}(p))) \). For a knot \( K \) in \( M \), the localized monodromy \( \tau \) lies in \( \text{SpAut}(F/F_k \otimes \mathbb{Z}(p)) \) by Theorem 3.2. As a modification of Proposition 4.1, we will classify the choice of the monodromies.

Theorem 4.2. For a knot \( K \) in an integral homology 3-sphere \( M \), suppose that \( p \) and 1-coef\( \Delta_K(t) \) are relatively prime. Choose a \( p \)-localized monodromy \( \tau \in \text{SpAut}(F/F_k \otimes \mathbb{Z}(p)) \). Then, the conjugacy class of \( \tau \) in \( \text{SpOut}(F/F_k \otimes \mathbb{Z}(p)) \) depends only on the knot type of \( K \).

Proof. Let \( \psi_k, \psi_k' \) be \( p \)-localized monodromies of the knot \( K \). Recall the isomorphisms \( \Psi_k, \Psi_k' \) from [3], which are constructed from choices of the symplectic basis of \( F/F_1 \otimes \mathbb{Z}(p) \). Since
we show the following (see Section 6 for the proof): later, this induces an injective homomorphism \( \text{Aut}(\sim) \) of the localized meta-nilpotence of knots and fibered knots.

In other words, the choice of \( F/F_k \otimes \mathbb{Z}_{(p)} \) as a fixed group can be covered by conjugacy of \( \text{SpAut}(F/F_k \otimes \mathbb{Z}_{(p)}) \). Recall that \( H_1(\pi_1(M \setminus K); \mathbb{Z}) \cong \mathbb{Z} \) by Alexander duality. Hence, the proof of Proposition 4.1 implies that \( \psi_k \) and \( \psi_k' \) are conjugate in \( \text{SpOut}(F/F_k \otimes \mathbb{Z}_{(p)}) \).

In conclusion, we can use several algorithms to get knot invariants. As one of them, we reach the following definition:

**Definition 4.3.** Take a representation \( \rho : \text{SpOut}(F/F_k \otimes \mathbb{Z}_{(p)}) \to \text{GL}(V) \) for some finite-dimensional vector space \( V \). For a knot \( K \), we call the eigenvalue polynomial of \( \rho(\tau) \) the meta-nilpotent Alexander polynomial (associated with \( \rho \)).

**Example 4.4.** If \( k = 1 \) and \( V = H_1(E_K; \mathbb{Q}) \) and \( \rho \) is the identity, the eigenvalue polynomial of \( \rho(\phi) \) is the classical Alexander polynomial \( \Delta_K \). In fact, this \( \rho(\phi) \) can be regarded as a characteristic polynomial of \( H_1([\pi_K, \pi_K]; \mathbb{Q}) \), which is known to be \( \Delta_K \); see [Lic, Theorem 6.17].

**Remark 4.5.** Recall that the eigenvalue polynomial \( \lambda(t) \) of a symplectic matrix \( A \) satisfies a symmetry in the sense of \( \lambda(t^{-1}) = t^{-2rkA}\lambda(t) \). In particular, if \( \rho : \text{SpOut}(F/F_k \otimes \mathbb{Z}_{(p)}) \to \text{GL}(V) \) is a symplectic representation, the meta-nilpotent Alexander polynomial has a symmetry as well.

**Example 4.6.** The next example is drawn from [CK] (see also [Ka] for a Jacobi-diagrammatic construction). Given a Lie algebra \( g \) of finite dimension over \( \mathbb{Q} \), the authors considered a linearly equivalent relation \( \sim \) of the universal enveloping algebra \( U(g)^{\otimes 2g} \) and constructed a representation \( \text{Out}(F/F_k \otimes \mathbb{Q}) \to \text{End}(U(g)^{\otimes 2g})/\sim \). It is interesting to analyze knot invariants from such a viewpoint.

In summary, for such an invariant, it is important to concretely construct a representation \( \text{SpOut}(F/F_k \otimes \mathbb{Z}_{(p)}) \to \text{GL}(V) \). In this paper, we will give other ways to obtain knot invariants. As shown in Appendix [A] we can get knot invariants in terms of non-abelian group cohomology. We will give some knot invariants in the case of \( k \leq 3 \); see Section 8.

5 Comparison to Dehn-Nielsen embedding

As a summary of the results in the preceding sections, this section makes a comparison between the localized meta-nilpotence of knots and fibered knots.

For this, we need Proposition 5.1 below. Since \( F/F_k \) is known to be torsion-free nilpotent, Lemma 3.7 gives an embedding \( \iota : F/F_k \hookrightarrow F/F_k \otimes \mathbb{Z}_{(p)} \). Moreover, as will be shown in (9) later, this induces an injective homomorphism \( \text{Aut}(F/F_k) \hookrightarrow \text{Aut}(F/F_k \otimes \mathbb{Z}_{(p)}) \). Furthermore, we show the following (see Section 6 for the proof):

\[^{3}\text{Strictly speaking, the paper [CK] only defines a representation } \text{Out}(F) \to \text{End}(U(g)^{\otimes 2g})/\sim \text{; however, from the definition, if we replace } \sim \text{ by an appropriate equivalence, then the representation yields representations of } \text{Out}(\lim F/F_k \otimes \mathbb{Q}).\]
 Proposition 5.1. Let \( p = 0 \). The quotient of the injection onto the outer automorphism groups \( \iota_\ast : \text{Out}(F/F_k) \to \text{Out}(F/F_k \otimes \mathbb{Q}) \) is also injective.

Let us explain. Let \( \text{rk} F = 2g \), and \( \mathcal{M}_{g,1} \) be the mapping class group of the one-punctured surface \( \Sigma_{g,1} \). Recall the Dehn Nielsen embedding \( \mathcal{M}_{g,1} \hookrightarrow \text{SpOut}(F) \); see, e.g., [FM, Theorem 8.8]; as mentioned in [Mo2, Page 446], the inverse limit according to \( F/F_k \to F/F_{k-1} \) induces an injective homomorphism \( \text{SpOut}(F) \hookrightarrow \text{SpOut}(\lim F/F_k) \), where the injectivity is immediately obtained from the embedding \( F \hookrightarrow \lim F/F_k \). To summarize, by Proposition 5.1, we have the injections,

\[
\text{DN} : M_{g,1} \hookrightarrow \text{SpOut}(\lim_{\infty \leftarrow k}(F/F_k \otimes \mathbb{Z}(p))) \subset \text{Out}(\lim_{\infty \leftarrow k}(F/F_k \otimes \mathbb{Q})). \tag{4}
\]

We will make the comparison in the diagram below. Starting from a fibered knot \( K \) with monodromy \( \phi_K \in \mathcal{M}_{g,1} \), the monodromy \( \tau \) obtained in Theorem 3.2 is equal to \( \text{DN}(\phi_K) \) by the construction of \( \tau \). Let \( M \) be an integral homology sphere. The above discussion can be summarized in the form of the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M}_{g,1}/\text{conj.} & \xrightarrow{\text{DN embedding}} & \bigcup_{p \in \text{Spec} \mathbb{Z}} \text{SpOut}(\lim_{\infty \leftarrow k}(F/F_k \otimes \mathbb{Z}(p))) / \text{conj.} \\
\downarrow & & \downarrow \\
\{\text{fibered knots of genus } g\} & \xleftarrow{\text{inclusion}} & \{\text{knots in } M \text{ with } \deg \Delta_K = 2g\}. \\
\end{array}
\tag{5}
\]

Here, the vertical maps are the correspondences between a knot \( K \) and a (\( p \)-localized) monodromy \( \tau \). To conclude, the right vertical map is an extension of the correspondence between a fibered knot and the monodromy.

This diagram is suggestive. For example, it is natural to pose the question of injectivity and the image of the right vertical map. One can easily show that the left vertical map is injective; in particular, knot invariants using \( \tau \) can completely classify fibered knots. However, the right vertical map is not injective. For example, if \( K' \) is a knot with Alexander polynomial \( \Delta_{K'} = 1 \), the localized monodromies of \( K \) and of the connected sum \( K \# K' \) are equal by construction. However, it is a problem whether the restrictions on the sets of alternating knots or of \( \mathbb{Q} \)-homologically fibered knots are injective or not. Here, a knot \( K \subset M \) is said to be \( \mathbb{Q} \)-homologically fibered, if twice the Seifert genus of \( K \) is equal to \( \deg \Delta_K \). As is known (see, e.g., [MM]), every (pseudo-)alternating knot is \( \mathbb{Q} \)-homologically fibered; all knots with less than 12 crossings are \( \mathbb{Q} \)-homologically fibered, except \( 11_{n34}, 11_{n42}, 11_{n45}, 11_{n67}, 11_{n73}, 11_{n107}, 11_{n152} \); see [CL].

On the other hand, let us discuss a subgroup of \( \text{SpOut}(F/F_k \otimes \mathbb{Z}(p)) \), which should contain the image of the right vertical map, as an almost surjective map. What is the minimal subgroup of \( \text{SpOut}(\lim F/F_k \otimes \mathbb{Z}(p)) \) that contains the image of the right vertical map? This section gives partial answers as shown in Propositions 5.2 and 5.5 below.

To state the propositions, we need some terminology. Let \( F \) be the free group on \( 2g \) generators \( x_1, \ldots, x_{2g} \), identified with \( \pi_1(\Sigma_{g,1}) \), where the generator \( \zeta \) of \( \pi_1(\partial \Sigma_{g,1}) \cong \mathbb{Z} \) is regarded as \( [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] \). Then, the image of the Dehn Nielsen embedding is the
subgroup generated by elements \( \phi \in \text{Aut}(F) \) satisfying \( \phi(\zeta) = \zeta \). Using the same notation as in [GL] and [Mo2, Section 3], consider the subgroup of the form,

\[
\text{Aut}_0(F/F_k \otimes \mathbb{Z}_p) := \{ \phi \in \text{SpAut}(F/F_k \otimes \mathbb{Z}_p) \mid \phi(\iota_k(\zeta)) = \iota_k(\zeta) \},
\]

where \( \iota_k \) is the inclusion \( F/F_k \to F/F_k \otimes \mathbb{Z}_p \), and define

\[
\text{Aut}_0(F/F_k \otimes \mathbb{Z}_p) := p_{k+1}(\text{Aut}_0(F/F_{k+1} \otimes \mathbb{Z}_p)), \quad (6)
\]

where \( p_{k+1} \) is the projection \( \text{SpAut}(F/F_{k+1} \otimes \mathbb{Z}_p) \to \text{SpAut}(F/F_k \otimes \mathbb{Z}_p) \). Integrally speaking, we can define a subgroup \( \text{Aut}_0(F/F_k) \) of \( \text{Aut}_0(F/F_k \otimes \mathbb{Z}_p) \) in a similar way.

**Proposition 5.2.** Let \( K \subset M \) be a \( \mathbb{Q} \)-homologically fibered knot of genus \( g \). Then, there is a localized monodromy of \( K \), which lies in the subgroup \( \text{Aut}_0(F/F_k \otimes \mathbb{Z}_p) \).

We give the proofs in Appendix B.

Meanwhile, we discuss a realization of localized monodromies. Precisely,

**Problem 5.3.** Let \( \tau \) in the subgroup \( \text{Aut}_0(F/F_k \otimes \mathbb{Q}) \) satisfies that \( \text{id}_{F/F_1} - q_k(\tau) : F/F_1 \otimes \mathbb{Q} \to F/F_1 \otimes \mathbb{Q} \) is isomorphic. Then, is there a knot in a rational homology sphere such that the monodromy is equal to \( \tau \)?

Here, we can easily see that the bijectivity of \( \text{id}_{F/F_1} - q_k(\tau) \) is a necessary condition to require existence of a rational homology sphere.

First, in integral cases, we obtain a simple result:

**Proposition 5.4.** Let \( \tau \) in the subgroup \( \text{Aut}_0(F/F_k) \) satisfies that \( \text{id}_{F/F_1} - q_k(\tau) : F/F_1 \otimes \mathbb{Q} \to F/F_1 \otimes \mathbb{Q} \) is isomorphic. There is a knot in a rational homology sphere such that the monodromy is equal to \( \tau \).

Next, we discuss rational cases, and give partial answers to the above problem.

**Proposition 5.5.** Let \( p = 0 \).

\( (1) \) Let \( k = 1 \), and \( \tau \in \text{Aut}_0(F/F_1 \otimes \mathbb{Q}) = \text{Sp}(2g; \mathbb{Q}) \) satisfy that \( \text{id}_{F/F_1} - \tau : F/F_1 \otimes \mathbb{Q} \to F/F_1 \otimes \mathbb{Q} \) is isomorphic. There is a knot in a rational homology sphere such that the monodromy is \( \tau \).

\( (2) \) Let \( k > 1 \). For any \( \mathbb{Q} \)-homologically fibered knot in a rational homology sphere with the monodromy \( \tau \in \text{Aut}_0(F/F_k \otimes \mathbb{Q}) \) and any \( \eta \in \text{ISpAut}_0(F/F_k) \), there is a knot in a rational homology sphere with the monodromy \( \tau \eta \in \text{Aut}_0(F/F_k \otimes \mathbb{Q}) \).

We also give the proofs in Appendix C with a relation to homology cobordisms.

6 The automorphism groups \( \text{Aut}(F/F_k \otimes \mathbb{Z}_p) \) and \( \text{Out}(F/F_k \otimes \mathbb{Z}_p) \)

Before constructing knot invariants from Theorem 4.2, we will study the automorphism groups \( \text{SpAut}(F/F_k \otimes \mathbb{Z}_p) \) and \( \text{SpOut}(F/F_k \otimes \mathbb{Z}_p) \) in detail. Throughout this section, we will employ
the dual space. which is not central. To see a centrality, we define the subgroup \( I = \text{Hom}(N) \) induced by the projection \( N \) be summarized as the kernel of the projection \( \text{Aut}(N) \) is known to be a central extension. Hence, \( I \) is a central extension.

We will start with an observation by S. Morita [Mo1, §2] and review the sequences (7) and (8). Consider \( \text{Hom}(H, \mathcal{L}_{k+1}) \) as a \( \mathbb{Z} \)-module. For a homomorphism \( f : H \to \mathcal{L}_{k+1} \), define \( f : N_{k+1} \to N_{k+1} \) by \( \tilde{f}(\gamma) = f([\gamma])\gamma \), where \( \gamma \in N_{k+1} \) and \( [\gamma] \in H \) is the class of \( \gamma \) under the projection \( N_{k+1} \to N_1 = H \). Then, the correspondence \( f \mapsto \tilde{f} \) gives rise to a homomorphism \( \iota : \text{Hom}(H, \mathcal{L}_{k+1}) \to \text{Aut}(N_{k+1}) \). Furthermore, let \( g_* : \text{Aut}(N_{k+1}) \to \text{Aut}(N_k) \) be the map induced by the projection \( N_{k+1} \to N_k \). According to Proposition 2.6 in [Mo1], these maps can be summarized as

\[
0 \to \text{Hom}(H, \mathcal{L}_{k+1}) \xrightarrow{\iota} \text{Aut}(N_{k+1}) \xrightarrow{g_*} \text{Aut}(N_k) \to 1 \quad \text{(group extension),} \tag{7}
\]

which is not central. To see a centrality, we define the subgroup \( I(N_k) \) of \( \text{Aut}(N_k) \) to be the kernel of the projection \( \text{Aut}(N_k) \to \text{Aut}(N_1) = \text{GL}(H) \). Then, the restriction of (7)

\[
0 \to \text{Hom}(H, \mathcal{L}_{k+1}) \xrightarrow{\iota} I(N_{k+1}) \xrightarrow{g_*} I(N_k) \to 1 \tag{8}
\]

is known to be a central extension. Hence, \( I(N_k) \) is a nilpotent group. Here, we should note the fact that the projection \( \text{Aut}(N_k) \to \text{Aut}(N_1) = \text{GL}(H) \) does not split.

In addition, we will examine the localizations of the above sequences. Let \( I(N_k \otimes \mathbb{Z}_{(p)}) \) be the kernel of the projection \( \text{Aut}(N_k \otimes \mathbb{Z}_{(p)}) \to \text{GL}(H_{(p)}) \). Then, the localization of (8) yields a central extension,

\[
0 \to \text{Hom}(H_{(p)}, \mathcal{L}_{k+1} \otimes \mathbb{Z}_{(p)}) \xrightarrow{\iota \otimes \mathbb{Z}_{(p)}} I(N_{k+1} \otimes \mathbb{Z}_{(p)}) \xrightarrow{g_*} I(N_k \otimes \mathbb{Z}_{(p)}) \to 1. \tag{9}
\]

Furthermore, according to Proposition 2.6 in [Mo1], if \( p = 0 \), then the projection produces a split extension,

\[
0 \to I(N_k \otimes \mathbb{Z}_{(p)}) \to \text{Aut}(N_k \otimes \mathbb{Z}_{(p)}) \to \text{GL}(H_{(p)}) \to 1 \quad \text{(split extension),} \tag{10}
\]

and two such splittings are conjugates of each other (In Section 7 we will see that this extension splits even if \( k \leq 3 \) and \( p > 3 \)). In summary, as a symplectic restriction, \( \text{SpAut}(N_k \otimes \mathbb{Z}_{(p)}) \) is also a semi-direct product of the nilpotent group \( I(N_k \otimes \mathbb{Z}_{(p)}) \) and \( \text{Sp}(H_{(p)}) \). Since \( \text{Aut}(F/F_1) \cong \text{Out}(F/F_1) \), the outer \( \text{SpOut}(N_k \otimes \mathbb{Z}_{(p)}) \) is the semi-direct product of a nilpotent group and \( \text{Sp}(H_{(p)}) \).

Furthermore, when \( p = 0 \), concerning the subgroup \( \text{Aut}_0(N_k \otimes \mathbb{Q}) \) in (9), it has been shown [Mo2 Theorem 3.3] that the intersection \( \text{Hom}(H_{(0)}, \mathcal{L}_k \otimes \mathbb{Q}) \cap \text{Aut}_0(N_k \otimes \mathbb{Q}) \) is equal to the kernel of the bracketing:

\[
\text{Ker}([.] : H_{(0)} \otimes (\mathcal{L}_k \otimes \mathbb{Q}) \to \mathcal{L}_{k+1} \otimes \mathbb{Q}), \tag{11}
\]

where \( H_{(0)} \cong H_{(0)}^* \) by Poincaré duality, we identify \( \text{Hom}(H_{(0)}, \mathcal{L}_k \otimes \mathbb{Q}) \) with \( H_{(0)} \otimes (\mathcal{L}_k \otimes \mathbb{Q}) \). Since the bracketing is \( \text{Sp} \)-equivariant, the subgroup \( \text{Aut}_0(N_k \otimes \mathbb{Q}) \) turns out to be also a semi-direct product of a nilpotent group and \( \text{Sp}(H_{(0)}) \).
Next, we will examine the automorphism groups \( \text{Inn}(N_k \otimes \mathbb{Z}_{(p)}) \subset \text{Aut}(N_k \otimes \mathbb{Z}_{(p)}) \). Notice that the inner group \( \text{Inn}(N_k \otimes \mathbb{Z}_{(p)}) \) is isomorphic to \( N_k \otimes \mathbb{Z}_{(p)}/Z(N_k \otimes \mathbb{Z}_{(p)}) \cong N_{k-1} \otimes \mathbb{Z}_{(p)} \), where \( Z(G) \) means the center of a group \( G \). Further, the inclusion \( N_{k-1} \otimes \mathbb{Z}_{(p)} = \text{Inn}(N_k \otimes \mathbb{Z}_{(p)}) \subset \text{Aut}(N_k \otimes \mathbb{Z}_{(p)}) \) is described as follows: For \( b \in \mathcal{L}_k \), we define the homomorphism

\[
\lambda_k(b) : H \longrightarrow \mathcal{L}_{k+1}; \quad h \longmapsto [b, h],
\]

which the reader should keep in mind. Since \( \text{GL}(H) = \text{Aut}(F/F_1) \supset \text{Inn}(F/F_1) = 0 \), we have \( \text{Inn}(N_k \otimes \mathbb{Z}_{(p)}) \subset I(N_k \otimes \mathbb{Z}_{(p)}) \), which yields the following diagram:

\[
\begin{array}{c}
0 \longrightarrow \mathcal{L}_k \otimes \mathbb{Z}_{(p)} \longrightarrow N_k \otimes \mathbb{Z}_{(p)} \longrightarrow N_{k-1} \otimes \mathbb{Z}_{(p)} \longrightarrow 1 \\
\downarrow \lambda_k \otimes \mathbb{Z}_{(p)} & \downarrow \lambda_k \otimes \mathbb{Z}_{(p)} & \downarrow \lambda_k \otimes \mathbb{Z}_{(p)} & \\
0 \longrightarrow \text{Hom}(H_{(p)}, \mathcal{L}_{k+1} \otimes \mathbb{Z}_{(p)}) \longrightarrow I(N_{k+1} \otimes \mathbb{Z}_{(p)}) \longrightarrow I(N_k \otimes \mathbb{Z}_{(p)}) \longrightarrow 1 \\
\end{array}
\tag{12}
\]

This diagram is commutative by the definitions of \( \iota \) and \( \lambda_k \). Here, it is worth noting that \( \lambda_k \) (resp. \( \lambda_k \otimes \mathbb{Z}_{(p)} \)) is not equivariant under the action of \( \text{GL}(H) \) (resp. \( \text{GL}(H_{(p)}) \)). In addition, we will show an \( \text{Sp} \)-equivariance: precisely,

**Lemma 6.1.** Suppose \( \text{rk}F = 2g \) and \( p \neq 2 \). The inclusion \( \lambda_k \otimes \mathbb{Z}_{(p)} \) is equivariant under the action of \( \text{Sp}(H_{(p)}) \).

Finally, we are now in a position to give the proofs of Proposition 5.1 and Lemma 6.1. The hasty reader may skip reading them.

**Proof of Proposition 5.1.** The proof is by induction on \( k \). When \( k = 1 \), the statement obviously follows from the inclusion \( \text{GL}(H) = \text{Out}(F/F_1) \supset \text{Out}(F/F_1) \otimes \mathbb{Q} = \text{GL}(H_{(0)}) \). So, for \( k \geq 1 \), let us suppose that \( \text{Out}(N_k) \to \text{Out}(N_k \otimes \mathbb{Q}) \) is injective. By (12), we have the commutative diagram,

\[
\begin{array}{c}
0 \longrightarrow \text{Coker}(\lambda_k) \longrightarrow \text{Out}(N_{k+1}) \longrightarrow \text{Out}(N_k) \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
0 \longrightarrow \text{Coker}(\lambda_k \otimes \mathbb{Q}) \longrightarrow \text{Out}(N_{k+1} \otimes \mathbb{Q}) \longrightarrow \text{Out}(N_k \otimes \mathbb{Q}) \longrightarrow 0 \\
\end{array}
\tag{central extension}
\]

where the vertical maps are the localizations at 0. Therefore, if the induced map \( \text{Coker}(\lambda_k) \to \text{Coker}(\lambda_k \otimes \mathbb{Q}) \) is injective, by diagram chasing, the middle vertical map turns out to be injective, as required.

We will show injectivity on the cokernels. For this, it suffices to prove that the cokernel \( \text{Coker}(\lambda_k) \) is torsion-free. Let \( e_1, \ldots, e_n \) be a basis of \( H \). For \( j \leq n \), let us first consider the homomorphism \( \kappa_j : \mathcal{L}_k \to \mathcal{L}_{k+2} \) that takes \( h \) to \( [e_j, [e_j, h]] \). Let \( B_k \subset \mathcal{L}_k \) be the Hall basis. We claim that \( \text{Coker}(\kappa_j) \) is torsion-free. In fact, the set \( \{[e_j, [e_j, h]] \in \mathcal{L}_{k+2} \mid h \in B_k \} \) is a subset of \( B_{k+2} \) by definition. Furthermore, \( \{[e_j, [e_j, h]] \}_{1 \leq j \leq n, h \in B_k} \) is also a subset of the Hall basis; hence, from the claim above, the cokernel of \( \sum_{j=1}^n \kappa_j \) is torsion-free.
Meanwhile, the map \( \lambda_k : \mathcal{L}_k \to \text{Hom}(H, \mathcal{L}_{k+1}) \cong H^* \otimes \mathcal{L}_{k+1} \cong H \otimes \mathcal{L}_{k+1} \) can be regarded as the correspondence \( h \mapsto \sum_{i=1}^{n} e_i \otimes [e_i, h] \). Considering the bracketing \([,] : H \otimes \mathcal{L}_{k+1} \to \mathcal{L}_{k+2} ; \) which is known to be a surjection. We have (see, e.g., [Mo1, Mo2, Sa1])

\[
\begin{array}{c}
0 \\
\downarrow \lambda_k \\
\text{Ker}([,]) \\
\downarrow \sum_{j=1}^{n} \kappa_j \\
0 \\
\end{array} \xrightarrow{\text{id}} \mathcal{L}_k \xrightarrow{\lambda_k} \mathcal{L}_k \xrightarrow{\text{id}} 0 \\
\begin{array}{c}
H \otimes \mathcal{L}_{k+1} \\
\downarrow \sum_{j=1}^{n} \kappa_j \\
\text{Ker}([,]) \\
\downarrow \sum_{j=1}^{n} \kappa_j \\
0 \\
\end{array} \xrightarrow{\text{id}} \mathcal{L}_k \xrightarrow{\lambda_k} \mathcal{L}_k \xrightarrow{\text{id}} 0 \\
\begin{array}{c}
\end{array}
\]

(\text{exact})

which is commutative. Then, by the snake lemma and the previous claim, the cokernel of \( \lambda_k \) is also torsion-free, as desired.

\[ \square \]

\textbf{Proof of Lemma 6.1.} Let \( J \) be a \((2g \times 2g)\) matrix of the form \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Suppose that \( e_1, \ldots, e_{2g} \) is a symplectic basis of \( H(p) \) with respect to the intersection form \( \langle v, w \rangle = \langle v, Jw \rangle \) \((v, w \in H(p))\). For any \( h \in \text{Sp}(H(p)) \), it is sufficient to show \( -\lambda \cdot (\sum_{i=1}^{2g} e_i^* \otimes [e_i, \ell]) = \sum_{i=1}^{2g} e_i^* \otimes [e_i, h \cdot \ell] \). Since the symplectic group \( \text{Sp}(H(p)) \) is generated by transvections (see the Introduction of [Ch]), we may suppose \( a \in \mathbb{Z}(p), w \in H(p) \) such that \( h \cdot v = v + a\langle v, w \rangle w \) for any \( v \in H(p) \). Expand \( w \) as \( \sum_{j=1}^{2g} w_j e_j \) for some \( w_j \in \mathbb{Z}(p) \). Then, we have

\[
e_{2j}^* \cdot h = (e_{2j}, Jh) = e_{2j}^* + a\langle e_{2j-1}, w \rangle w J = e_{2j}^* - aw_{2j} Jw^*
\]

and \( (e_{2j-1} \cdot h)^* = e_{2j-1}^* - aw_{2j-1} (Jw)^* \). Therefore, we obtain the desired equality as follows:

\[
h \cdot (\sum_{i=1}^{2g} e_i^* \otimes [e_i, \ell]) = \sum_{i=1}^{2g} h \cdot e_i^* \otimes [h \cdot e_i, h \cdot \ell]
\]

\[
= \sum_{i=1}^{2g} (e_i^* - aw_i Jw^*) \otimes [e_i + a\langle e_i, w \rangle w, h \cdot \ell]
\]

\[
= \sum_{i=1}^{2g} (e_i^* \otimes [e_i, h \cdot \ell] - J w^* \otimes [aw_i e_i, h \cdot \ell]) + \sum_{i=1}^{g} a(e_{2i-1}^* \otimes w_{2i}[w, h \cdot \ell] - e_{2i}^* \otimes w_{2i-1}[w, h \cdot \ell])
\]

\[
+ \sum_{i=1}^{g} a^2 (w_{2i} J w^* \otimes w_{2i-1}[w, h \cdot \ell] - w_{2i-1} J w^* \otimes w_{2i}[w, h \cdot \ell])
\]

\[
= \sum_{i=1}^{2g} (e_i^* \otimes [e_i, h \cdot \ell]) - a J w^* \otimes [w, h \cdot \ell] + a J w^* \otimes [w, h \cdot \ell] + 0 = \sum_{i=1}^{2g} e_i^* \otimes [e_i, h \cdot \ell].
\]

\[ \square \]
7 Observations of $\text{SpOut}(N_k \otimes \mathbb{Z}_{(p)})$ with $k = 2, 3$

According to Theorem 4.2, it is important to analyze quantitatively the conjugacy classes of $\text{Out}(N_k \otimes \mathbb{Z}_{(p)})$ and $\text{Auto}(N_k \otimes \mathbb{Z}_{(p)})$.

As examples with $k = 2$ and $k = 3$, this section will analyze $\text{SpOut}(N_k \otimes \mathbb{Z}_{(p)})$ in detail. We will see that the Enomoto-Satoh trace [ES, Sa1] is applicable to our construction (Section 8.2).

Throughout this section, we suppose $p \neq 2$ and $p \neq 3$, and let $\text{GL}(H(p))$ act on $\text{Hom}(H(p), \mathcal{L}_k \otimes \mathbb{Z}_{(p)})$ naturally by

$$(Af)(u) := Af(A^{-1}u) \quad (A \in \text{GL}(H(p))),$$

for $f \in \text{Hom}(H(p), \mathcal{L}_k \otimes \mathbb{Z}_{(p)})$ and $u \in H(p)$.

7.1 Example 1: $\text{SpOut}(N_k \otimes \mathbb{Z}_{(p)})$ with $k = 2$

Before we describe $\text{Out}(N_2 \otimes \mathbb{Z}_{(p)})$, we will review an observation in [Mo1, Page 204]. Let $H(p) \wedge H(p)$ be the exterior square of $H(p)$. Let $G_2$ be $(H(p) \wedge H(p)) \times H(p)$ with the group operation,

$$(a, v) \cdot (b, w) := \left( \frac{1}{2}v \wedge w + a + b, v + w \right), \quad (v, w \in H(p), \ a, b \in H(p) \wedge H(p)).$$

Then, $N_2 \otimes \mathbb{Z}_{(p)} \cong G_2$ is known (see [Mo1]). Consider the action of $(f, A) \in \text{Hom}(H(p), \Lambda^2 H(p)) \ltimes \text{GL}(H(p))$ on $N_2 \otimes \mathbb{Z}_{(p)}$ defined by

$$(f, A)(\xi, u) := (2f(Au) + A\xi, Au) \quad ((\xi, u) \in N_3 \otimes \mathbb{Z}_{(p)}).$$

Then, we can easily verify that the homomorphism $\text{Hom}(H(p), \Lambda^2 H(p)) \ltimes \text{GL}(H(p)) \to \text{Aut}(N_2 \otimes \mathbb{Z}_{(p)})$ induced from this action is an isomorphism.

Next, we will give decompositions of $\text{Hom}(H(p), \Lambda^2 H(p))$, and discuss normal subgroups of $\text{Aut}(N_2 \otimes \mathbb{Z}_{(p)})$. Consider the Sp-equivariant inclusion $I_3 : \Lambda^3 H(p) \hookrightarrow \text{Hom}(H(p), \Lambda^2 H(p))$ given by

$$I_3(v \wedge w \wedge u)(z) = (z, w)u \wedge v + (z, u)v \wedge w + (z, v)w \wedge u,$$

where $v, w, z \in H(p)$ and the symbol $\langle, \rangle$ means the intersection form. Notice that $\Lambda^3 H(p)$ contains an Sp-submodule of the form $\{h \wedge (\sum_{i=1}^g e_{2i-1} \wedge e_{2i}) \mid h \in H(p)\}$ if $g > 1$. Furthermore, as a cokernel of $I_3$, let us consider the subspace of $H(p) \otimes \Lambda^2 H(p)$ generated by $u \otimes (v \wedge w) - v \otimes (w \wedge u)$ for $u, v, w \in H(p)$. That is,

$$t = \langle u \otimes (v \wedge w) - v \otimes (w \wedge u) \mid u, v, w \in H(p) \rangle \subset H(p) \otimes \Lambda^2 H(p).$$

$$\dim(t) = (8g^3 - 2g)/3$$

is known. In summary, by the symplectic Littlewood-Richardson rule (see [KT]) in the context of the Young tableau, we have a decomposition of Sp-representations:

$$\text{Hom}(H(p), \Lambda^2 H(p)) \cong H(p) \otimes \Lambda^2 H(p) \cong \Lambda^3 H(p) \oplus t \cong ([1^3] \oplus [1]) \oplus ([2, 1] \oplus [1]),$$

if $g > 1$. Recall from Lemma 6.1 that the image $\text{Im}(\lambda_2 \otimes \mathbb{Z}_{(p)})$ is an Sp-representation subspace; we have
Lemma 7.1. The image of $\lambda_2 \otimes \mathbb{Z}_p$ is contained in the subspace $t$. Namely, $\text{Im}(\lambda_2 \otimes \mathbb{Z}_p)$ is the last term $[1]$ in (15).

Proof. Let $e_1, \ldots, e_{2g} \in H(p)$ be a basis of $H(p)$. We can verify that the image of $\lambda_2$ is given by \( \{ f_b \in \text{Hom}(H(p), \Lambda^2 H(p)) | b \in H(p) \} \), where $f_b$ sends $a \in H(p)$ to $a \wedge b \in \Lambda^2 H(p) = \mathbb{L}_2 \otimes \mathbb{Z}_p$. Hence, the image $\text{Im}(\lambda_2 \otimes \mathbb{Z}_p) \subset H(p) \otimes \Lambda^2 H(p)$ has a basis $\sum_{j=1}^2 e_{2\ell-1} \otimes (e_{2\ell} \wedge e_j) + e_{2\ell} \otimes (e_{2\ell-1} \wedge e_j)$ with $1 \leq j \leq n$. By applying $u = e_{2\ell}$, $v = e_{2\ell-1}$, $w = e_j$ to (14), we can readily see that the basis is contained in $t$. Hence, we have $\text{Im}(\lambda_2 \otimes \mathbb{Z}_p) \subset t$, as required. \hfill \Box

Notice the classical fact (see, e.g., [DJ]) that the kernel $\text{Ker}([1])$ in (11) is isomorphic to $\cong [1^3] \oplus [1]$ and contains $\Lambda^3 H(p)$ in (15). To summarize, we determine $\text{SpOut}(N_2 \otimes \mathbb{Z}_p)$ as follows:

Proposition 7.2. If $g = 1$, then $\text{SpOut}(N_2 \otimes \mathbb{Z}_p)$ and $\text{SpOut}_0(N_2 \otimes \mathbb{Z}_p)$ are isomorphic to $\text{Sp}(H(p))$. If $g > 1$, there are group isomorphisms

$\text{SpOut}(N_2 \otimes \mathbb{Z}_p) \cong ([1^3] \oplus [1] \oplus [2, 1]) \times \text{Sp}(H(p))$, $\text{SpOut}_0(N_2 \otimes \mathbb{Z}_p) \cong ([1^3] \oplus [1]) \times \text{Sp}(H(p))$.

7.2 Example 2: $\text{SpOut}(N_k \otimes \mathbb{Z}_p)$ with $k = 3$

The structure of $\text{Aut}(N_3 \otimes \mathbb{Z}_p)$ when $p = 0$ is discussed in [Mo4]. However, even if $p > 4$, we will explicitly express the group structures of $N_3 \otimes \mathbb{Z}_p$ and $\text{Aut}(N_3 \otimes \mathbb{Z}_p)$ and make an observation about $\text{SpOut}(N_3 \otimes \mathbb{Z}_p)$.

Let us introduce a bilinear map $\nu : H(p) \times \Lambda^2 H(p) \to H(p) \otimes \Lambda^2 H(p)$ defined by

$$\nu(u, v \wedge w) = \frac{2}{3} u \otimes (v \wedge w) - \frac{1}{3} v \otimes (w \wedge u) - \frac{1}{3} w \otimes (u \wedge v).$$  \hfill (16)

Recall the subspace $t$ of $H(p) \otimes \Lambda^2 H(p)$ in (14). For $f \in \text{Hom}(H(p), \Lambda^2 H(p))$, let us define a linear map $\Upsilon_f : \Lambda^2 H(p) \to t$ by

$$\Upsilon_f(u \wedge v) := \nu(u, f(v)) - \nu(v, f(u)).$$  \hfill (17)

Then, we can define the group structure on $\text{Hom}(H(p), t) \times \text{Hom}(H(p), \Lambda^2 H(p)) \times \text{GL}(H(p))$ by

$$(F', f', A')(F, f, A) := (F' + A'F + \Upsilon_{f'}(A'f(\bullet)) - \Upsilon_{A'}f'(\bullet), f' + A'f, A').$$  \hfill (18)

Proposition 7.3. Let $L_3$ be the product $t \times \Lambda^2 H(p) \times H(p)$. For $(\alpha, a, v), (\beta, b, w) \in L_3$, we define the binary operation $(\alpha, a, v) \cdot (\beta, b, w)$ on $L_3$ by

$$(\alpha + \beta + \frac{1}{2} \nu(v, b) - \frac{1}{2} \nu(w, a) + \frac{1}{12} \nu(v, v \wedge w) + \frac{1}{12} \nu(w, w \wedge v), a + b + \frac{1}{2} v \wedge w, v + w).$$ \hfill (19)

Consider the action of the group $[18]$ on $L_3$ defined by

$$(F, f, A) \cdot (\alpha, a, v) := (2F(Av) + A\alpha + \Upsilon_f(2f(Av) + 2Aa), 2f(Av) + Aa, Av).$$ \hfill (20)

Then, the operation $[19]$ gives rise to a group structure in $L_3$, and the operation $[20]$ preserves the group operation. Furthermore, the group on $L_3$ is isomorphic to $N_3 \otimes \mathbb{Z}_p$, and the homomorphism

$$\mu : \text{Aut}(N_3 \otimes \mathbb{Z}_p) \longrightarrow \text{Hom}(H(p), t) \times \text{Hom}(H(p), \Lambda^2 H(p)) \times \text{GL}(H(p))$$
induced by \[20\] is a group isomorphism.

Proof. While it is a bit complicated to check the former part, it is elementary. So, we will leave it for the reader to verify. Instead, we will show
\[\text{N} \]

(The case of Example 7.5)

While it is a bit complicated to check the former part, it is elementary. So, we will
\[\text{Proof.}\]

induced by \[20\]

Richardson rule again, if \[\text{N} \]

\[\text{dim}(\text{H}) = 0 \]

and \[\text{dim}(\text{H}) = 3.\]

To summarize, we conclude that
\[\text{SpOut}(N_3 \otimes \mathbb{Z}(p)) \cong [2] \rtimes \text{Sp}(H(p)), \quad \text{SpOut}_0(N_3 \otimes \mathbb{Z}(p)) \cong \text{Sp}(H(p)).\]

Next, we will examine the image of \[\lambda_3 \otimes \mathbb{Z}(p)\] (Lemma 7.4 below). Thanks to the Littlewood-Richardson rule again, if \[g > 1\] and \[p \geq 7\], we have a decomposition of Sp-representations,
\[\text{Hom}(H(p), t) \cong H(p) \otimes t \cong ([1] \otimes [2, 1]) \oplus ([1] \otimes [1])\]

\[\cong ([2, 1, 1] \oplus [2^2] \oplus [3, 1] \oplus [2] \oplus [1^2]) \oplus ([2] \oplus [1^2] \oplus [0]).\] (21)

If \[g = 1\], then \[\text{Hom}(H(p), t) \cong [1] \otimes [1] \cong [0] \oplus [2].\]

Lemma 7.4. For \(s, t \leq 2g\), let \[\gamma_{s,t}\] be the linear map \(H(p) \to t\) which sends \[e_1\] to \[\nu(e_x, e_s \wedge e_t)\]. Let \[R_3\] be the subspace of \[\text{Hom}(H(p), t)\] spanned by \[\gamma_{s,t}\] with \(s < t \leq 2g\).

Then, \[R_3\] is equal to the image of \[\lambda_3\]. Furthermore, \[R_3\] is isomorphic to \([1^2] \oplus [0]\) as an Sp-representation and contains the image of the map
\[\text{Im}(\lambda_3 \otimes \mathbb{Z}(p)) \times \text{Im}(\lambda_3 \otimes \mathbb{Z}(p)) \to \text{Hom}(H(p), t); \quad (f, f') \mapsto \Upsilon_{f, f'}(f(\bullet)) - \Upsilon_f(f'(\bullet)).\]

In particular, the inner automorphism group \[\text{Inn}(N_3 \otimes \mathbb{Z}(p))\] corresponds to the subset
\[R_3 \times [1] \times \{\text{id}_{H(p)}\} \subset \text{Hom}(H(p), t) \times \text{Hom}(H(p), \Lambda^2 H(p)) \times \text{GL}(H(p)).\]

Proof. By definition, \[R_3\] is isomorphic to \[\Lambda^2 H(p) \cong [1^2] \oplus [0].\] For \(s \leq 2g\), let \[f_s\] be a linear map that sends \[e_i\] to \[e_i \wedge e_s\], as an element of \[\text{Im}(\lambda_3 \otimes \mathbb{Z}(p))\]. Then, \[\Upsilon_{f_t}(f_s(e_x)) - \Upsilon_{f_s}(f_t(e_x)) = \nu(e_x, e_s \wedge e_t)\] by direct calculation. Therefore, the linear map \[\lambda_3 : e_x \mapsto [e_x, [e_s, e_t]]\] is equal to \[\Upsilon_{f_t}(f_s(\bullet)) - \Upsilon_{f_s}(f_t(\bullet))\], as required. 

Example 7.5 (The case of \[g = 1\]). Next, we have that the representation space \([1] \otimes [2, 1]\) is zero, and \[\text{dim}(H_Q \otimes \mathcal{L}_3) = 4, \quad \text{dim(Im}(\lambda_3)) = 1.\] Hence, \[\text{SpAut}(N_3 \otimes \mathbb{Z}(p))\] subject to \[\text{Inn}(N_3 \otimes \mathbb{Z}(p))\] is isomorphic to \([2] \rtimes \text{Sp}(H(p))\). Notice that the kernel \[\text{Ker}([\cdot])\] is \([2]\) since \[\text{dim}(\mathcal{L}_3) = 3.\] To summarize, we conclude that
\[\text{SpOut}(N_3 \otimes \mathbb{Z}(p)) \cong [2] \rtimes \text{Sp}(H(p)), \quad \text{SpOut}_0(N_3 \otimes \mathbb{Z}(p)) \cong \text{Sp}(H(p)).\]
Example 7.6 (The case of \( g > 1 \)). Meanwhile, when \( g > 1 \), the kernel in (11) is known to be isomorphic to \([2^2] \oplus [1^2] \oplus [0]\) (see, e.g., [Sa2, page 33]). Therefore, we conclude that \( \text{SpOut}_0(N_3 \otimes \mathbb{Z}(p)) \) is isomorphic to the group on \( \{([2^2] \oplus [1^2] \oplus [0]) \times \{[1^3] \oplus [1]\}\} \rtimes \text{Sp}(H_{(p)})\).

7.3 Morita trace map and Enomoto-Satoh trace map

Let \( p = 0 \). Morita [Mo3] introduced a trace map from \( \text{Hom}(H_{(0)}, \mathcal{L}_k) \), which gives some obstructions of the Johnson homomorphism. Satoh [Sa1] defines a modified trace map from \( \text{Hom}(H_{(0)}, \mathcal{L}_k) \), which has been studied from the viewpoint of representations as a joint work with Enomoto [ES].

We will start by giving a brief review of the Enomoto-Satoh trace. Let \( \mathcal{L} \) be \( \bigoplus_{k \geq 0} \mathcal{L}_k \otimes \mathbb{Q} \), which is a Lie algebra induced by the bracketing. For each \( k \geq 1 \), let \( \Phi_k : H_{(0)}^* \otimes H_{(0)}^\otimes(k+1) \to H_{(0)}^\otimes k \) be the contraction map with respect to the first component, defined by

\[
x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^*(x_{j_1}) \cdot x_{j_2} \otimes \cdots \otimes x_{j_{k+1}}.
\]

Let \( \iota_k : \mathcal{L}_k \otimes \mathbb{Q} \to H_{(0)}^\otimes k \) be the natural map defined by \([X, Y] \mapsto X \otimes Y - Y \otimes X\), which gives rise to a Lie homomorphism \( \mathcal{L} \to \bigoplus_{k \geq 0} H_{(0)}^\otimes k \). This map is injective by the Poincaré-Birkhoff-Witt theorem. Let \( C_{2g}(k) \) be the quotient \( \mathbb{Q}\)-vector space of \( H_{(0)}^\otimes k \) by the action of the cyclic group \( \mathbb{Z}/k\mathbb{Z} \) on the components, that is,

\[
C_{2g}(k) := H_{(0)}^\otimes k / \langle a_1 \otimes a_2 \otimes \cdots \otimes a_k - a_2 \otimes a_3 \otimes \cdots \otimes a_k \otimes a_1 \mid a_i \in H_{(0)} \rangle.
\]

Then, the Enomoto-Satoh trace map is defined to be the composite map

\[
\text{Tr}_{ES}^k : H_{(0)}^* \otimes \mathcal{L}_{k+1} \xrightarrow{\text{id} \otimes \iota_{k+1}} H_{(0)}^* \otimes H_{(0)}^\otimes (k+1) \xrightarrow{\Phi_k} H_{(0)}^\otimes k \xrightarrow{\text{proj.}} C_{2g}(k).
\]

According to [Sa2, Section 7], the Morita trace map [Mo3] can be summarized as the composite map \( P_k \circ \text{Tr}_{ES}^k : H_{(0)}^* \otimes \mathcal{L}_{k+1} \to S^k H_{(0)} \), where \( S^k H_{(0)} \) is the \( k \)-fold symmetric tensor product of \( H_{(0)} \) and \( P_k \) is the natural projection \( \mathcal{C}_n(k) \to S^k H_{(0)} \). We should remark that the trace maps are \( \text{GL}(H_{(0)}) \)-equivariant by definition.

We present a lemma and construct the group homomorphism below [22].

Lemma 7.7. Let \( k \geq 2 \). The composite \( \text{Tr}_{ES}^k \circ (\lambda_k \otimes \text{id}_\mathbb{Q}) \) is zero.

This lemma is proven at the end of this subsection. To conclude, we have a \( \text{GL}(H_{(0)}) \)-equivariant Lie homomorphism:

\[
\bigoplus_{j \geq 2} \text{Tr}_j^ES : \text{Hom}(H_{(0)}, \mathcal{L}_{j+1}) \to \bigoplus_{j \geq 2} C_{2g}(j),
\]

where the lie algebra structure of the image is trivial. Then, by the same discussion as [Mo2, Section 3] on the Lie group-algebra correspondence, we have the group homomorphism

\[
\bigoplus_{j:2 \leq j < k} \tilde{\text{Tr}}_j^ES : I(N_k \otimes \mathbb{Q}) \rtimes \text{GL}(H_{(0)}) = \text{Aut}(N_k \otimes \mathbb{Q}) \to \left( \bigoplus_{j:2 \leq j < k} C_{2g}(j) \right) \rtimes \text{GL}(H_{(0)}).
\]
Here, we give some knot invariants from the viewpoints of the conjugacy classes of \( \text{SpOut}(N_k \otimes \mathbb{Q}) \). The direct sum induces the group homomorphism

\[
\bigoplus_{j: 2 \leq j < k} \widetilde{\text{Tr}}_j^{\text{ES}} : \text{SpOut}(N_k \otimes \mathbb{Q}) \longrightarrow \left( \bigoplus_{j: 2 \leq j < k} C_{2g}(j) \right) \rtimes \text{Sp}(H(0)).
\]

(22)

**Example 7.8** (The case of \( k = 3 \)). Notice that \( C_{2g}(2) \cong S^2H(0) \cong [2] \) by definition. Recall from Proposition 7.3, the isomorphism of \( L_3 \cong \mathfrak{t} \). Then, if \( g > 1 \), the ES trace \( \text{Hom}(H(0), \mathfrak{t}) \to [2] \) is the projection on the first component of [2], where we use the decomposition (21). On the other hand, if \( g = 1 \), the ES trace is the projection \( \text{Hom}(H(0), \mathfrak{t}) \cong [2] + [0] \to [2] \).

**Proof of Lemma 7.7.** Denote \( t_1(e_j) \) by \( X_j \), and \([e_j, [e_{j+1}, \ldots [e_{j+1}, e_{j+k}] \ldots]]\) by \( Y_s \). We can easily verify that \( L_k \otimes \mathbb{Q} \) is generated by elements of the form \([e_j, [e_{j+1}, \ldots [e_{j+1}, e_{j+k}] \ldots]]\) by induction on \( k \). Therefore, it is enough to show \( \text{Tr}^{\text{ES}}_k \left( \sum_{i=1}^{2g} e_i^* \otimes [e_i, Y_k] \right) = 0 \). We have

\[
\Phi_k \circ (\text{id} \otimes t_{k+1})(e_i^* \otimes [e_i, Y_k]) = 2g t_k(Y_k) + \sum_{i=1}^{2g} (\Phi_{k-1}(X_i^* \otimes t_k(Y_k))) \otimes X_i,
\]

by definition. Since \( t_k(Y_k) = 0 \in C_{2g}(k) \) is clear, we will show that the second term vanishes. We will omit writing the symbol \( \otimes \). For any \( s \leq k \), notice the equality,

\[
\sum_{i=1}^{2g} \Phi_{k-s+1}(X_i^* \otimes t_s(Y_s)) X_{j_{s-1}} \cdots X_{j_1} X_i =
\]

\[
= t_{s+1}(Y_{s+1}) X_{j_{s-1}} X_{j_{s-2}} \cdots X_{j_1} X_{j_s} - \sum_{i=1}^{2g} \Phi_{k-s}(X_i^* \otimes t_{s+1}(Y_{s+1})) X_{j_{s-1}} \cdots X_{j_1} X_i \in H^{\otimes k}(0).
\]

Therefore, by repeating this equality, the sum in (23) is computed as

\[
(-1)^k [X_{j_{k-1}}, X_{j_k}] X_{j_{k-2}} \cdots X_{j_1} + \sum_{s=1}^{k-2} (-1)^s t_{s+1}(Y_{s+1}) X_{j_{s-1}} \cdots X_{j_1} X_{j_s}.
\]

(24)

Here, by induction on \( m \) with \( m < k - 1 \), we can immediately verify the following:

\[
\sum_{s=1}^{m} (-1)^s t_{s+1}(Y_{s+1}) X_{j_{s-1}} \cdots X_{j_1} X_{j_s} = (-1)^{m+1} t_{m+1}(Y_{m+1}) X_{j_{m-1}} \cdots X_{j_1} X_{j_s} \in C_{2g}(k).
\]

Then, (24) is zero, since \( t_{k-1}(Y_{k-1}) = [X_{j_{k-1}}, X_{j_k}] \). Namely, the sum in (23) is zero, as required.

\[
\square
\]

8 Some meta-nilpotent invariants of knots

Here, we give some knot invariants from the viewpoints of the conjugacy classes of \( \text{SpOut}(F/F_k \otimes \mathbb{Z}_{(p)}) \). We say that two knots are \((k, p)\)-equivalent if the associated monodromies are conjugate.
in \( \text{SpOut}(F/F_k \otimes \mathbb{Z}_{(p)}) \). Throughout this section, we will denote the projection \( \text{SpOut}(F/F_k \otimes \mathbb{Z}_{(p)}) \rightarrow \text{SpOut}(F/F_1 \otimes \mathbb{Z}_{(p)}) \) by \( q_k \). Then, by \([7]\), we canonically have a surjection:

\[
\{\text{knot}\}/\{(k + 1, p)\text{-equivalent}\} \rightarrow \{\text{knot}\}/\{(k, p)\text{-equivalent}\}.
\]

Thus, it is natural to discuss \( \text{SpOut}(F/F_k \otimes \mathbb{Z}_{(p)}) \) with \( k \) in ascending order. Below, we examine the cases of \( k = 1, 2, 3 \).

### 8.1 The case of \( k = 1 \); conjugacy classes of \( \text{Sp}(H_{(g)}) \)

We begin by analyzing the case of \( k = 1 \), i.e., \( \text{Sp}(2g; \mathbb{Z}_{(p)}) \). In general, it is hard to completely classify the conjugacy classes of \( \text{Sp}(2g; \mathbb{Z}_{(p)}) \). However, it is not so hard to check whether two elements in \( \text{Sp}(2g; \mathbb{Z}_{(p)}) \) are conjugate or not; furthermore, there are class functions of \( \text{Sp}(2g; \mathbb{Z}_{(p)}) \). For example, there is the normal form theorem which uses the symplectic conjugate; see, e.g., \([\text{Gutt}]\). In this subsection, to obtain knot invariants with \( g = 1 \) as a special case, we will discuss the conjugacy classes of \( \text{Sp}(2; \mathbb{Z}_{(p)}) \) with \( p = 0 \) in detail.

#### 8.1.1 The case of \( g = 1 \) and \( p = 0 \)

Let \( \mathcal{O}(\mathbb{Q}^\times/\mathbb{Q}^\times)^2) \) be the set of all subsets of the quotient multiplicative group \( \mathbb{Q}^\times/\mathbb{Q}^\times)^2 \). According to \([\text{Ca}]\), we will construct a map \( \text{sgn} : \text{Sp}(2; \mathbb{Q})/\text{conj.} \rightarrow \mathcal{O}(\mathbb{Q}^\times/\mathbb{Q}^\times)^2) \).

We say that \( A \in \text{Sp}(2; \mathbb{Q}) \) is decomposable if the eigenvalues are distinct. Choose a decomposable \( A \in \text{Sp}(2; \mathbb{Q}) \). When the eigenvalues \( \lambda, \lambda^{-1} \) are contained in \( \mathbb{Q} \), we define \( \text{sgn}(A) = \mathbb{Q}^\times/\mathbb{Q}^\times)^2 \). Otherwise, let \( E \) be the quadratic field extension \( \mathbb{Q}(\lambda) \). Let \( \bar{\cdot} : E \rightarrow E \) be the conjugation as the generator of the Galois group \( \text{Gal}(E/\mathbb{Q}) \cong \mathbb{Z}/2 \). Let \( v = t(x, y) \in E^2 \) be the eigenvector with respect to \( \lambda \). Then, by setting \( \Xi = \begin{pmatrix} x & \bar{x} \\ y & \bar{y} \end{pmatrix} \), we have \( A = \Xi \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \Xi^{-1} \).

Then, we define

\[
\text{sgn}_{E/\mathbb{Q}}(A) = \det(\Xi)/(%(\lambda - \bar{\lambda}) \in E^\times)
\]

modulo the norm \( N_{E/\mathbb{Q}} := \{cc\mid E^\times \} \) and define

\[
\text{sgn}(A) := \text{the preimage in } \mathbb{Q}^\times/\mathbb{Q}^\times)^2 \text{ of } \text{sgn}_{E/\mathbb{Q}}(A).
\]

Then, we can easily show \( \text{sgn}(A) = \text{sgn}(gAg^{-1}) \) for any \( g \in \text{Sp}(2; \mathbb{Q}) \). It is known that the restriction of the pair of the trace map \( \text{Tr} \) and \( \text{sgn} \) on the sets of decomposable matrices is injective (see \([\text{Ca}, \text{Page } 8]\)).

**Example 8.1.** Ohkura \([\text{O}]\) gives a list of monodromies in \( \text{Sp}(2; \mathbb{Q}) \) of some knots, including 2-bridge knots of genus 1. For example, if \( K = 7_4 \) and \( K' = 9_2 \), the monodromies are \( A = \begin{pmatrix} 1 & 1/2 \\ -1/2 & 3/4 \end{pmatrix} \) and \( A' = \begin{pmatrix} 1 & 1/4 \\ -1 & 3/4 \end{pmatrix} \). We can easily check that they are not conjugate using \( \text{sgn} \).

Similarly, we can compute the signatures and distinguish some knots (e.g., \( 9_5 \) and \( 13_2 \)). In our experience, the map \( \text{sgn} \) seems to be a strong invariant for knots of genus one.
8.2 The case of $k = 2$; Class function from $\text{SpOut}(N_2 \otimes \mathbb{Z}(p))$

Next, we address the case of $k = 2$. Recall $\text{SpOut}(F/F_2 \otimes \mathbb{Z}(p)) \cong \text{Sp}(2; \mathbb{Z}(p))$ if $g = 1$; thus, we may suppose $g > 1$. To obtain a class function of $\text{SpOut}(F/F_2 \otimes \mathbb{Z}(p))$, we need the following lemma:

**Lemma 8.2.** Take a representation $\text{Sp}(2g; \mathbb{Z}(p)) \to \text{GL}(V)$ and the semi-direct product $V \rtimes \text{Sp}(2g; \mathbb{Z}(p))$. For $g \in \text{Sp}(2g; \mathbb{Z}(p))$, let $C(g)$ be the centralizer subgroup of $g$ and $V_{C(g)}$ be the coinvariant of $V$ subject to the action of $C(g)$. Then,
\[(b, h)^{-1}(a, g)(b, h) = ((1 - g)b + ha, h^{-1}gh) \in V \rtimes \text{Sp}(2g; \mathbb{Z}(p)).\]
In particular, the correspondence,
\[\Psi_2 : V \rtimes \text{Sp}(2g; \mathbb{Z}(p)) \longrightarrow V_{C(g)} \times (\text{Sp}(2g; \mathbb{Z}(p))/\text{conj}); \quad (a, g) \longmapsto (a, g),\]
is invariant with respect to conjugacy in $V \rtimes \text{Sp}(2g; \mathbb{Z}(p))$.

**Example 8.3.** Recall from Proposition 7.2 that $\text{SpOut}(F/F_2 \otimes \mathbb{Z}(p)) \cong ([1^3] \oplus [1] \oplus [2, 1]) \times \text{Sp}(2g; \mathbb{Z}(p))$. Using these isomorphisms to Lemma 8.2, we will compute some knot invariants.

For example, let us consider the knots $K = 8_{20}$ and $K' = 12n_{582}$, which are fibered of genus 2. The Alexander polynomial is $(1-t+t^2)^2$. Then, we can verify that the monodromies $\tau$ and $\tau'$ in $\text{Sp}(4; \mathbb{Z})$ are conjugate, and that, if $p = 0$ and $V = ([1^3] \oplus [1] \oplus [2, 1])$, then $V_{C(\tau)} \cong \mathbb{Q}$. By carefully checking the elements $\Psi_2(\tau)$ and $\Psi_2(\tau')$ in $V_{C(\tau)}$, it is not so hard to show that the monodromies $\tau$ and $\tau'$ in $V \rtimes \text{Sp}(4; \mathbb{Z})$ are not conjugate.

**Remark 8.4.** If $p = 0$ and zeros of the Alexander polynomial are distinct and not any root of unity, then $V_{C(\tau)}$ is isomorphic to zero in many cases, where $V = ([1^3] \oplus [1] \oplus [2, 1])$ as above. For example, by considering the set
\[C_{2g} := \left\{ A \in \text{Sp}(2g; \mathbb{Q}) \ \middle| \ \text{The eigenvalues } \lambda_1, \ldots, \lambda_{2g} \in \mathbb{C} \text{ satisfy } \lambda_i^{n_i} \neq \lambda_j^{n_j} \text{ for any } i \neq j, 1 \leq n_i \leq 3, 1 \leq n_j \leq 3 \right\},\]
which is closed under conjugacy, we can see that, if $\tau \in p_2^{-1}(C_{2g})$, then $V_{C(\tau)} \cong 0$. For example, if the crossing number of $K \subset S^3$ is $< 12$, $K$ is hyperbolic, and $\Delta_K$ is irreducible, then $\tau \in C_{2g}$. For knots with such Alexander polynomials, to non-trivial knot invariants from $\text{SpOut}(F/F_2 \otimes \mathbb{Z}(p))$, we should suppose $p > 0$.

Finally, let us briefly compare the invariants with an extension of the first Johnson homomorphism [Mo1]. Let $\mathcal{M}(2) \subset \mathcal{M}_{g,1}$ be the normal subgroup generated by all the Dehn twists along separating simple closed curves. Since $\mathcal{M}_{g,1}$ acts on $\pi_1(\Sigma_{g,1}) = F$, we have a group homomorphism $\rho_k : \mathcal{M}_{g,1} \to \text{Aut}(F) \overset{\text{proj}}{\longrightarrow} \text{Aut}(F/F_k)$. Let $k = 2$. Morita showed [Mo1] Theorem 4.8] that $\rho_2$ induces an embedding $\mathcal{M}_{g,1}/\mathcal{M}(2) \hookrightarrow 1/2 \Lambda^3 H \rtimes \text{GL}(H)$; furthermore, the restriction on the Torelli group is the first Johnson homomorphism and the image is of finite index. Thus, the difference between the invariants of fibered knots and non-fibered knots might able to be detected from finite information. For this reason, to obtain useful information from the invariants of a non-fibered knot, we shall suppose $p > 0$. 

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8.3 The case of \( k = 3 \); Symplectic quadratic forms from \( \text{SpOut}(N_3 \otimes \mathbb{Z}_p) \)

Using the description of \( \text{SpOut}(N_3 \otimes \mathbb{Z}_p) \) in Section 7.2, we will construct knot invariants of symplectic quadratic forms. Although we can describe the invariants for every knot, the description will be quite complicated; for simplicity, we will focus on the set \( C_{2g} \) in (25) and suppose \( p = 0 \), and construct a class function

\[
\rho : q_3^{-1}(C_{2g}) \to \{\text{isomorphism classes of symplectic quadratic forms}\}^{4g^2-3g}. \tag{26}
\]

Let \( D \) be the diagonal subgroup of \( \text{GL}(2g; \mathbb{C}) \), which is isomorphic to \((\mathbb{C}^\times)^{2g}\). For a representation \( V \) of \( D \), we mean the \( D \)-invariant part by \( V^D \). Then, we have the following:

**Lemma 8.5.** Recall the subspaces \( t \subset H_{(0)} \otimes \Lambda^2 H_{(0)} \) and \( R_3 \cong [1^2] \oplus [0] \) defined in Lemma 7.4. The dimensions of the invariant parts \( \text{Hom}(H_C, t \otimes \mathbb{C})^D \) and of \(([1^2] \oplus [0])^D \) are \( 4g^2 - 2g \) and \( g \), respectively. In particular, \( (\text{Hom}(H_C, t \otimes \mathbb{C})/R_3)^D \) is of dimension \( 4g^2 - 3g \).

For \( A \in C_{2g} \subset \text{Sp}(2g; \mathbb{Q}) \), let \( P_A \in \text{Sp}(2g; \mathbb{C}) \) be a matrix such that \( P_A A P_A^{-1} \) is the diagonal matrix \((\lambda_1) \oplus (\lambda_2) \oplus \cdots \oplus (\lambda_{2g})\), and let \( \pi : \text{Hom}(H_C, t \otimes \mathbb{C}) \to \text{Hom}(H_C, t \otimes \mathbb{C}/R_3)^D \) be the projection from the direct summand. As mentioned in Remark 8.4, \( 1 - A : \text{Hom}(H_{(p)}, \Lambda^2 H_{(p)}) \to \text{Hom}(H_{(p)}, \Lambda^2 H_{(p)}) \) is isomorphic. Then, we define a map, \( \rho^{\text{pre}} \), from the preimage \( q_3^{-1}(C_{2g}) \subset \text{SpOut}(N_3 \otimes \mathbb{Z}_p) \) by

\[
\rho^{\text{pre}} : q_3^{-1}(C_{2g}) \longrightarrow (\text{Hom}(H_C, t \otimes \mathbb{C})/R_3)^D;
\]

\[
(F, f, A) \longmapsto \pi(P_A(F(\bullet) - \Upsilon_f((1-A)^{-1}f(\bullet)) + \Upsilon_{(1-A)^{-1}f}(f(\bullet)))).
\]

It is not difficult to show that \( \rho^{\text{pre}} \) does not depend on the choice of \( P_A \), while \( \rho^{\text{pre}} \) is not invariant with respect to conjugacy.

To solve the non-invariance, consider the action of \( \text{Sp}(2g; \mathbb{Q}) \) on the sets of symplectic basis, \( SB_g \). Since this action is free and transitive, for \( B \in \text{Sp}(2g; \mathbb{Q}) \), we can uniquely choose the corresponding symplectic basis \( \mu(B) \). Then, \( \rho^{\text{pre}}(F, f, A) \) can be regarded as a map from \( \mu(A) \) to \((\text{Hom}(H_C, t \otimes \mathbb{C})/R_3)^D \); further, we can see that this map is a quadratic map by Proposition 7.3. Therefore, we have

\[
\rho : q_3^{-1}(C_{2g}) \longrightarrow \text{Map}(SB_g, (\text{Hom}(H_C, t \otimes \mathbb{C})/R_3)^D); (F, f, A) \longmapsto (\mu(A) \mapsto \rho^{\text{pre}}(F, f, A)).
\]

Accordingly, it is not hard to verify that the symplectic congruence classes of such quadratic forms are invariant with respect to conjugacy. To summarize

**Proposition 8.6.** The map (26) is a class function of \( \text{SpOut}(F/F_3 \otimes \mathbb{Z}_p) \).

Although the construction of \( \rho \) is a bit complicated, it is not hard to make a computer program to describe the map \( \rho \), when \( g = 2, 3 \). In fact, we can check that the symplectic quadratic forms are not trivial; since symplectic quadratic forms over \( \mathbb{C} \) are classified (see, e.g., [BHSS]), we obtain non-trivial quantitative information from the resulting quadratic forms. However, the resulting computations are hard to describe; we will omit the details.
A Knot invariants from non-abelian group cohomology

As a result of Theorem 4.2, we will construct knot invariants from non-abelian group cohomology, as a different approach from Section 8.

Let us give a brief review of non-abelian cohomology. Suppose that a group $A$ acts on a group $G$. The 0-th cohomology, $H^0(A; G)$, is defined by the invariant subgroup $G^A$. Next, consider the set of the maps $\varphi : A \to G$ satisfying $\varphi(gh) = \varphi(g)[g \cdot \varphi(h)]$, which are called derivations, and denote the set by $\text{Der}(A, G)$. Two derivations $\varphi, \varphi'$ are equivalent if there is a $g$ in $G$ such that $a \varphi'(a) = \varphi(a)(a \cdot g)$. The quotient set is written as $H^1(A, G)$, and it does not always canonically have a group structure when $A$ is non-abelian. A truncated long exact sequence is known (see, e.g., [SK]); specifically, let $K$ be an abelian group, and let $0 \to K \to G \to H \to 1$ be a short exact sequence of $A$-groups; then, there is an exact sequence of pointed sets,

$$0 \to K^A \to G^A \to H^A \xrightarrow{\delta} H^1(A, K) \to H^1(A, G) \to H^1(A, H) \to H^2(A, K). \tag{27}$$

Here, $\delta$ is defined as follows. For an $A$-invariant element $c \in H^A$, choose a lift $\bar{c} \in G$; then $\delta(c)$ is defined by a class of $\varphi_\bar{c} : A \to K$ which takes $a \to \bar{c}^{-1}(ac)$. Furthermore, since $K$ is abelian, $H^1(A; K)$ is identified with the coinvariant $K/\{k - a \cdot k\}_{k \in K, a \in A}$.

Next, in the case that $A = \mathbb{Z}$ and $G$ is nilpotent, we give a set-theoretic computation of $H^1(\mathbb{Z}, G)$. Notice the bijection from $\text{Der}(\mathbb{Z}, G)$ onto $G$, which sends $\varphi$ to $\varphi(1)$ by definition. Taking the lower central series $G \supset G_1 \supset \cdots$, we set up the central extension $0 \to G/G_{k+1} \to G/G_k \to 1$. Consider the quotient abelian group defined by

$$Q_k := \frac{G_k/G_{k+1}}{\{x \cdot y^{-1}(1 \cdot y) : x \in G_k/G_{k+1}, y \in G_k^1 \}}.$$ 

Since the second $H^2(\mathbb{Z}; G/G_{k+1})$ vanishes, by (27), we have a set-theoretic exact sequence of pointed sets:

$$0 \to Q_k \to H^1(\mathbb{Z}; G/G_{k+1}) \to H^1(\mathbb{Z}; G/G_k) \to 0.$$ 

To summarize, we inductively conclude:

**Proposition A.1.** There is a bijection

$$H^1(\mathbb{Z}; G/G_k) \simeq Q_1 \times Q_2 \times \cdots \times Q_{k-1}.$$ 

By Theorem 4.2, we immediately get a knot invariant:

**Proposition A.2.** Let $K$ be a knot. Recall the isomorphism 2 concerning the localized meta-nilpotent quotient of the knot group $\pi_K$. Regard the choice of the meridian in $\pi_K$ as a section $s : \mathbb{Z} \to (F/F_k \otimes \mathbb{Z}(p)) \times \mathbb{Z}$. Then, $\text{proj}_2 \circ s : \mathbb{Z} \to F/F_k \otimes \mathbb{Z}(p)$ is a derivation, and the class $[\text{proj}_2 \circ s]$ in $H^1(\mathbb{Z}; F/F_k \otimes \mathbb{Z}(p))$ depends only on the knot type of $K$.

**Example A.3.** Since the evaluation of the Alexander polynomial at 1 is 1, we have $H^*(\mathbb{Z}; F/F_1 \otimes \mathbb{Z}(p)) = 0$. Therefore, if $k = 2$, we have $Q_1 = \{0\}$ and that $H^1(\mathbb{Z}; F/F_2) \cong Q_2$ is the coinvariant $(F_1/F_2)_\mathbb{Z}$ by definition. When $p = 0$, using the main results in [11], we can compute
the rank of $H^1(\mathbb{Z}; F/F_2) \cong Q_2$. In fact, in [N1] Theorem 2.1, the author gives a formula for computing the invariant $(F_1/F_2 \otimes Q)^\mathbb{Z}$, where we should notice the existence of an isomorphism $(F_1/F_2 \otimes Q)^\mathbb{Z} \cong (F_1/F_2 \otimes Q)_\mathbb{Z}$.

In that paper, $(F_1/F_2 \otimes Q)^\mathbb{Z}$ is shown to be equal the center of $(F/F_2 \otimes Q) \rtimes \mathbb{Z}$. In conclusion, the capacity from the conjugacy classes of $\tau$ can mostly be detected by using the center of $(F/F_2 \otimes Q) \rtimes \mathbb{Z}$.

To study the invariant from $\text{Out}(F/F_2 \otimes Q)$, it is useful to use the study of the centers of $(F/F_2 \otimes Q) \rtimes \mathbb{Z}$’s. The author suggests a preliminary study in terms of quadratic forms (see Sections 3 and 4 in [N1]).

However, analyzing the case $k \geq 3$ remains to be done. In fact, it does not seem so easy to explicitly compute the coinvariant $(F_k/F_{k+1})_\mathbb{Z}$ and the delta map.

## B Proof of Proposition 5.2

Here, we give a proof of Proposition 5.2. The proof uses the terminology developed in Sections 2 and 5. We will often regard the complement $M \setminus K$ to be a 3-manifold obtained from $M$ by removing an open tubular neighborhood of $K$.

As a preparation, let us examine some fundamental groups. Choose a Seifert surface $S$ of genus $g$. Fix a meridian $m \in \pi_1(M \setminus K)$. Take a bicollar $S \times [-1, 1]$ of $S$ such that $S \times \{0\} = S$. Let $\iota_+: S \to M \setminus S$ be the embeddings whose images are $S \times \{\pm 1\}$. Take generating sets $W := \{u_1, \ldots, u_{2g}\}$ of $\pi_1 S$ and $X := \{x_1, \ldots, x_{2g}, x_{2g+1}, \ldots, x_{2g+k}\}$ of $\pi_1(M \setminus S)$ for some $k$. Here, according to the discussion in [Tr] Page 480, we can choose $X$ such that the linking number of the cycles represented by $u_i$ and $x_i$ is the Kronecker delta $\delta_{ij}$, while the remaining generators $x_{2g+1}, \ldots, x_{2g+k}$ lie in $[\pi_1(M \setminus S), \pi_1(M \setminus S)]$, and that $[x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$ represents a longitude of $\pi_1(M \setminus K)$. Set $y_i := (\iota_+)_{+}(u_i)$ and $z_i = (\iota_-)_{+}(u_i)$; a von Kampen argument yields a presentation of $\pi_1(M \setminus K)$:

\[
\langle m, x_1, \ldots, x_{2g}, x_{2g+1}, \ldots, x_{2g+k} \mid m^{-1}y_imz_i^{-1} (1 \leq i \leq 2g), r_{2g+1}, \ldots, r_{2g+t} \rangle,
\]

for some relators $r_{2g+1}, \ldots, r_{2g+t}$, which do not contain $m$ and are contained in the commutator subgroup $[\pi_1(M \setminus S), \pi_1(M \setminus S)]$. Let $p : E^\infty_K \to M \setminus K$ be the $\infty$-cyclic covering. Then, by using the Reidemeister-Schreier method, $\pi_1(E^\infty_K) = [\pi_K, \pi_K]$ is presented by

\[
\langle x_1^{(n)}, \ldots, x_{2g+k}^{(n)} \mid y_{i}^{(n)}(z_{i}^{(n+1)})^{-1} (1 \leq i \leq 2g, n \in \mathbb{Z}), r_{2g+1}, \ldots, r_{2g+t} \rangle,
\]

and the injection $p_* : \pi_1(E^\infty_K) \to \pi_1(M \setminus K)$ is represented by the correspondence $x_i^{(k)} \mapsto m^{-k}x_i^{(k)}m^k$.

**Proof of Proposition 5.2**. We will use the above notation. Suppose that $(p, 1, \text{coef} \Delta_K) = 1$. Consider a lift, $\tilde{S}$, of the Seifert surface $S$ in $E^\infty_K$. Since $I = [x_1^{(0)}, x_2^{(0)}] \cdots [x_{2g-1}^{(0)}, x_{2g}^{(0)}]$ can be regarded as a boundary loop in $\partial \tilde{S}$ or a longitude of $K$, this $I$ commutes with the monodromy on $\pi_1(E^\infty_K)$. Define $F$ to be the free group with a basis $\{x_{2i-1}^{(0)}, x_{2i}^{(0)}\}_{i=1}^g$. Then, the monodromy induces $\tau : F/F_k \otimes \mathbb{Z}(p) \to F/F_k \otimes \mathbb{Z}(p)$, which preserves $\iota_k(\zeta)$.
The remaining part of the proof is to show that \( \tau \) lies in \( \text{SpAut}(F/F_k \otimes \mathbb{Z}_{(p)}) \) for any \( m \). Notice that, by Lemma B.1 below, the inclusion \( i : \tilde{S} \hookrightarrow E_K^\infty \) induces an isomorphism

\[
i^* : H^*(E_K^\infty, \partial E_K^\infty; \mathbb{Z}_{(p)}) \longrightarrow H^*(\tilde{S}, \partial \tilde{S}; \mathbb{Z}_{(p)}).
\]

Therefore, \( \{x_{2i-1}^{(0)}, x_{2i}^{(0)}\}_{i=1}^g \) represents a symplectic basis of \( H^1(E_K^\infty; \mathbb{Z}_{(p)}) \). Hence, by following the proof of Theorem 3.2, we can show that \( \tau \) lies in \( \text{SpAut}(F/F_k \otimes \mathbb{Z}_{(p)}) \), as required. \( \square \)

**Lemma B.1** (folklore). If the genus of \( S \) is equal to \( \text{deg} \Delta_K/2 \), then the inclusion \( i : \tilde{S} \hookrightarrow E_K^\infty \) induces the isomorphism \( i_* : H_1(\tilde{S}; \mathbb{Z}_{(p)}) \rightarrow H_1(E_K^\infty; \mathbb{Z}_{(p)}) \) over \( \mathbb{Z}_{(p)} \).

**Proof.** It is enough to show the surjectivity of \( i_* \), since \( H_1(E_K^\infty; \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}^{2g} \) by the assumption \( 2g = \text{deg} \Delta_K \) and Lemma 3.6. Since \( x_{2g+1}, \ldots, x_{2g+k} \) lie in \([\pi_1(M \setminus S), \pi_1(M \setminus \Sigma)]\), we obtain from (29) a presentation of \( H_1(E_K^\infty; \mathbb{Z}_{(p)}) \):

\[
\langle x_1^{(n)}, \ldots, x_{2g}^{(n)} \mid y_i^{(n)}(z_i^{(n+1)})^{-1} \quad (1 \leq i \leq 2g, n \in \mathbb{Z}), r_{2g+1}, \ldots, r_{2g+t} \rangle.
\]

Thus, it suffices to show that this is generated by only \( x_1^{(0)}, \ldots, x_{2g}^{(0)} \), since \( i_* \) sends \( x_i \) to \( x_i^{(0)} \). Denote the tuple \( ([x_1^{(n)}], \ldots, [x_{2g}^{(n)}]) \in \mathbb{Z}^{2g} \subset \mathbb{Z}_{(p)}^{2g} \) by \( \bar{x}^{(n)} \). Then, the relation \( y_i^{(n)}(z_i^{(n+1)})^{-1} \) implies that some matrices \( Y \) and \( Z \in \text{Mat}(2g \times 2g; \mathbb{Z}) \) satisfy \( Y \bar{x}^{(n+1)} = Z \bar{x}^{(n)} \) in the above presentation. If either of \( Y \) and \( Z \) are not invertible over \( \mathbb{Z}_{(p)} \), then \( H_1(E_K^\infty; \mathbb{Z}_{(p)}) \) is of infinite rank. Thus, both \( Y \) and \( Z \) must be invertible. We have \( \bar{x}^{(n+1)} = Y^{-1}Z \bar{x}^{(n)} \), which means \( \bar{x}^{(n)} = (Y^{-1}Z)^n \bar{x}^{(0)} \). Hence, \( H_1(E_K^\infty; \mathbb{Z}_{(p)}) \) is generated by \( x_1^{(0)}, \ldots, x_{2g}^{(0)} \), as desired. \( \square \)

### C

**Proofs of Propositions 5.4 and 5.5**

We give the proofs of Propositions 5.4 and 5.5 from the viewpoints of homology cobordisms.

We review homology cobordisms [GL]. A **homology cobordism** \((N, i^+, i^-)\) over \( \Sigma_{g,1} \) consists of a compact oriented 3-manifold \( N \) with two embeddings \( i^+, i^- : \Sigma_{g,1} \hookrightarrow \partial N \) such that:

(i) \( i^+ \) is orientation-preserving and \( i^- \) is orientation-reversing,

(ii) \( \partial M = i^+(\Sigma_{g,1}) \cup i^-(\Sigma_{g,1}) \) and \( i^+(\Sigma_{g,1}) \cap i^-(\Sigma_{g,1}) = i^+(\partial \Sigma_{g,1}) = i^-(\partial \Sigma_{g,1}) \),

(iii) \( i^+|_{\partial \Sigma_{g,1}} = i^-|_{\partial \Sigma_{g,1}} \), and

(iv) \( i^+, i^- : H_*(\Sigma_{g,1}; \mathbb{Z}) \rightarrow H_*(M; \mathbb{Z}) \) are isomorphisms.

Similarly, we can define a **rational homology cobordism** by replacing (iv) with the condition that (iv') \( i^+, i^- : H_*(\Sigma_{g,1}; \mathbb{Q}) \rightarrow H_*(M; \mathbb{Q}) \) are isomorphisms. The maps \( i^+, i^- \) are called **markings**. The **closure**, \( \text{cl}(N, i^\pm) \), of \((N, i^\pm)\) is defined as a 3-manifold obtained by identifying \( i^+(\Sigma_{g,1}) \) and \( i^-(\Sigma_{g,1}) \).

Two homology cobordisms \((N, i^+, i^-) \) and \((N', j^+, j^-)\) over \( \Sigma_{g,1} \) are said to be **isomorphic** if there exists an orientation-preserving diffeomorphism \( f : N \cong N' \) satisfying \( j^+ = f \circ i^+ \) and \( j^- = f \circ i^- \). We denote by \( \mathcal{C}_{g,1} \) the set of all isomorphism classes of homology cobordisms over
\[ (N, i^+, i^-) \cdot (N', j^+, j^-) = (N \cup_{i^- \circ (j^+)^{-1}} N', i^+, j^-). \]

Similarly, we can define the monoid, \( C^Q_{g,1} \), of all isomorphism classes of rational homology cobordisms over \( \Sigma_{g,1} \), and an injection \( C^Q_{g,1} \hookrightarrow C^Q_{g,1} \).

According to [GL], we construct for every \( k \) a homomorphism \( \sigma_k : C_{g,1} \to \text{Aut}_0(F/F_k) \). Given \( (N, i^+, i^-) \in C_{g,1} \) consider the homomorphisms \( i^+_s : F \to \pi_1(N) \), where the base-point is taken in \( \partial(i^+(\Sigma_{g,1})) = \partial(i^-(\Sigma_{g,1})) \). Since \( i^\pm \) are homology isomorphisms, Stallings theorem implies that they induce isomorphisms \( i^+_k : F/F_k \to \pi_1(N)/\pi_1(N)_k \). We then define \( \sigma_k(N, i^\pm) = (i^-_k)^{-1} \circ (i^+_k) \), and can easily see that \( \sigma_k(N, i^\pm) \in \text{Aut}_0(F/F_k) \). The surjectivity of \( \sigma_k \) is known (see [GL] Theorem 3]). In a parallel way, we can rationally define a homomorphism \( \sigma_k^Q : C^Q_{g,1} \to \text{Aut}_0(F/F_k \otimes \mathbb{Q}) \). Moreover, we can show the following by the construction of \( \sigma_k \).

**Lemma C.1.** Given \( (N, i^+, i^-) \in C_{g,1} \), we suppose that \( \sigma_1(N, i^\pm) - \text{id}_{F/F_1} \) is isomorphic. Then, the closure \( \text{cl}(N, i^\pm) \) can be regarded as a knot \( K \) in an integral homology sphere, and the localized monodromy of \( K \) is equal to \( \sigma_k(N, i^\pm) \).

Similarly, if \( (N, i^+, i^-) \in C^Q_{g,1} \) admits the bijectivity of \( \sigma_1^Q(N, i^\pm) - \text{id}_{F/F_1 \otimes \mathbb{Q}} \), then the closure \( \text{cl}(N, i^\pm) \) can be regarded as a knot \( K \) in a rational homology sphere, and the localized monodromy of \( K \) is equal to \( \sigma_k(N, i^\pm) \).

**Proof of Proposition 5.4.** Let \( \tau \in \text{Aut}_0(F/F_k) \) satisfy that \( \text{id}_{F/F_1} - q_k(\tau) : F/F_1 \to F/F_1 \) is isomorphic. Thanks to the surjectivity of \( \sigma_k \), we have a homology cobordism \( (N, i^\pm) \) such that \( \sigma_k(N, i^\pm) = \tau \). Hence, the closure \( M \setminus K = \text{cl}(N, i^\pm) \) satisfies the required conditions.

Next, we turn to prove Proposition 5.5. For this, we need terminology. For a simple closed curve \( \gamma \) in \( \Sigma_{g,1} \), let \( T_\gamma \in \mathcal{M}_{g,1} \) be the Dehn-twist along \( \gamma \). For a ratio \( s/t \in \mathbb{Q} \), we define a transvection \( [T_\gamma]_{s,t} : H_1(\Sigma_{g,1}; \mathbb{Q}) \to H_1(\Sigma_{g,1}; \mathbb{Q}) \) to be the homomorphism that sends \( x \) to \( s/t(x, [\gamma]) + x \). According to [Noz] Figure 2.1, if \( g = 1 \), then there are a knot in the lens space \( L(s, t) \) and a Seifert surface \( S_{\gamma,s/t} \) such that the complement \( L(s, t) \setminus S_{\gamma,s/t} \) gives rise to a rational homology cobordism \( C_{\gamma,1,s/t} \) such that \( \sigma_1(C_{\gamma,s/t}) = [T_\gamma]_{s,t} \). Furthermore, if \( g > 1 \), the connected sum of \( C_{\gamma,1,s/t} \) and \( \Sigma_{g-1} \times [0, 1] \) provides a rational homology cobordism \( C_{\gamma,g,s/t} \).

**Proof of Proposition 5.5.** (1) Fix \( \tau \in \text{Sp}(2g; \mathbb{Q}) \) in assumption. Since the symplectic group \( \text{Sp}(2g; \mathbb{Q}) \) is generated by transvections (see the introduction of [Ch]), we can choose simple closed curves \( \gamma_1, \ldots, \gamma_n \) and integers \( s_1, \ldots, s_n, t_1, \ldots, t_n \) such that \( q_k(\tau) = [T_{\gamma_1}]_{s_1,t_1} \cdots [T_{\gamma_n}]_{s_n,t_n} \). We now define a homology cobordism \( N_\tau \) by the composite \( N_\tau := C_{\gamma,g,s_1/t_1} \cdots C_{\gamma,g,s_n/t_n} \); the closure gives a knot of a rational homology sphere.

(2) Let \( K \subset M \) be a \( \mathbb{Q} \)-homologically fibered knot with a Seifert surface \( S \) of genus \( g \). Then, the complement \( M \setminus S \) can be regarded as a rational homology cobordism. For \( \eta \in \text{ISpAut}_0(F/F_k) \), the above surjectivity of \( \sigma_k \) ensures a rational homology cobordism \( C_\eta \) satisfying \( \sigma_1(C_\eta) = \eta \). Therefore, the composite \( (M \setminus S) \cdot C_\eta \) takes a preimage satisfying the required condition.
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