THE THERMODYNAMIC EQUIVALENCE AT WORK: 
TRANSFORMING FERMI SYSTEMS INTO BOSE 
SYSTEMS

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Abstract. We generalize the method introduced in J. Phys. A: Math. Gen. 35, 7255 (2002) based on the concept of thermodynamic equivalence and we transform a Fermi system of general density of states into a thermodynamically equivalent Bose system. This consists of mapping configurations of fermions from the original system onto configurations of bosons, the initial and final configurations having the same energy above the many-body ground state energy. In this way we obtain two systems of particles of different exclusion statistics, but which have the same entropies—and therefore identical canonical thermodynamic properties. This method enables one in general to calculate the system properties in either of the bosonic and fermionic representations. We check the method here in microscopic detail by calculating the equilibrium particle distributions in the two representations using the entropy maximization at fixed particle number and fixed “fermionic” and “bosonic” energies, respectively. Analytical calculations seem difficult to do, but we check the results numerically and we find them equal within the numerical accuracy.

Key words: Quantum statistics, ensemble equivalence, thermodynamic equivalence

1 INTRODUCTION

The heat capacity of ideal systems of constant density of single particle states (DOS) and fixed number of particles is independent of the exclusion statistics of the constituent particles [1, 2, 3, 4, 5, 6, 7, 8]. Moreover, integrating the ratio between the heat capacity, $C$, and the temperature, $T$, with respect to the temperature, we obtain the entropy, $S = \int_0^T \frac{C}{T'}dT'$. So $S$, expressed as a function of $T$, particle number $N$, and all the other extensive parameters of the system (let us denote them by $X$), is also independent of the exclusion statistics. Based on this property, systems of equal, constant, DOS have been called thermodynamically equivalent [6].

The thermodynamic equivalence relation can be extended from the set of systems of constant DOS to the set of all physical systems [8]. To define the thermodynamic equivalence from a more general perspective, let me split the internal energy of the system as $U \equiv U_{gs} + U_{B}$, where $U_{gs}$ is the energy of the
system at zero temperature and \( U_B \) is the excitation or Bose energy. Then two systems of entropies \( S_1(U_1, N, X) \) and \( S_2(U_2, N, X) \) are thermodynamically equivalent if
\[
S_1(U_B + U_{g.s.,1}(N, X), N, X) = S_2(U_B + U_{g.s.,2}(N, X), N, X).
\]
The thermodynamic equivalence splits the set of all physical systems into equivalence classes—like the classes which include the ideal systems of the same, constant DOS [8, 9].

The equality of the entropies of two systems with the same \( U_B \) and \((N,X)\) implies that the number of microstates with the same excitation energy is the same in the two systems for any values of \( U_B \) and extensive parameters \((N,X)\). This was shown explicitly to be true for systems of the same, constant DOS, by mapping configurations of particles of arbitrary exclusion statistics onto configurations of bosons of the same \( U_B \) [5]. Vice-versa, the mapping of configurations of ideal bosons into configurations of ideal fermions puts into correspondence the bosons on the ground state (the Bose-Einstein condensate) with a degenerate subsystem of fermions at the bottom of the single particle spectrum, which was called the Fermi condensate [8, 9]. The configurations of fermions which contain the Fermi condensate have higher probability than the “standard” Fermi distribution, as it was shown in Ref. [9], contrary to what was generally believed. In interacting systems, the existence of the Fermi condensate may modify so dramatically the weights of different configurations of particles that it may lead to first order phase transitions [8, 10].

In this paper we generalize the method of Ref. [8] and we show how to map a Fermi system of general DOS to a thermodynamically equivalent Bose system. The technique may be extended (although not in a trivial way) to any transformation between Bose, Haldane, and Fermi systems and, to distinguish it from other transformations, it will be called exclusion statistics transformation (EST).

In Section 2.1 we calculate the quasiparticle energies and the density of states of the Bose gas. If the Fermi DOS is not constant, the Bose quasiparticle energies depend on the population of the other quasiparticle levels; in other words, the bosons are interacting. This effect is discussed in Section 3.

By EST, the particle population of the energy levels in the Bose system is determined by the particle population in the Fermi system and vice-versa. On the other hand, once we set the rules for calculating the DOS and the interaction between the particles in the Bose gas, the particle populations in the two systems (Bose and Fermi) can be calculated independently, by entropy maximization. In this way it is shown that one can make calculations in either of the systems and transfer the results to the other one.

Last section is reserved for the discussion of the results and conclusions.
2 THE TRANSFORMATION FROM FERMI TO BOSE SYSTEMS

Concretely, at microscopic scale the EST from a Fermi to a Bose system may be imagined as follows: we picture the fermions as forming groups of “close packed” particles that occupy completely certain intervals along the single particle energy axis (no empty single particle states in the intervals) and we associate to each of these groups an energy level in the equivalent Bose system (see figure 1). The groups are labeled with $i = 0, 1, \ldots$, starting from the bottom of the energy axis and going up. So, the group $i$ of $N_i$ fermions is associated to the level $\epsilon_{B,i}$ of $N_i$ bosons, in such a way that $\epsilon_{B,0} = 0$, $\epsilon_{B,i} \leq \epsilon_{B,j}$ if $i < j$, and the total energy, $E_B$, of the new system is the same as the excitation energy of the original Fermi system—the excitation energy of the Fermi system being the difference between the energy of the system, $E_F$, and the lowest energy attainable at the given particle number, $U_{gs}$ (in other words, $U_{gs}$ is the energy of the system at zero K).

To distinguish between the energies of individual configurations and the (average) internal energy of the system in given thermodynamic conditions, we denote by $E_F$ the energy of a configuration of fermions (the Fermi energy will be denoted by $\epsilon_F$) and by $E_B(= E_F - U_{gs})$ the energy of a configuration of bosons. Since for a given excitation energy, $E_B$, to each configuration of fermions it corresponds a configuration of bosons and vice-versa, then the entropy of the Fermi system, $S_F$, of internal energy $U = U_{gs} + E_B$, is equal to the entropy of the Bose system of internal energy $U_B = E_B [S_F(U_{gs} + U_B, N, \ldots) = S_B(U_B, N, \ldots)]$, i.e. the two systems are constructed to be thermodynamically equivalent. With these definitions we have $T^{-1} = \partial S_B / \partial E_B = \partial S_F / \partial E_F$ and $\mu_B / T = -\partial S_B / \partial N = (\mu - \epsilon_F) / T$, where $\mu / T = -\partial S_F / \partial N$ and $\epsilon_F \equiv \epsilon_F$. 

Figure 1: The illustration of the basic idea of exclusion statistics transformation (EST).
In figure 1 we see also how the Fermi condensate [9] becomes Bose-Einstein condensate in the Bose system [8, 11].

2.1 BOSONIC SINGLE PARTICLE ENERGIES

Let’s now calculate the bosonic energy levels. We shall use $\epsilon$ and $\epsilon_B$ to denote the single particle energies in the Fermi and Bose systems, respectively. If $m$ is the number of particles in the system, the particle number dependent Fermi energy is denoted by $\epsilon_F(m)$. Moreover, in both (Bose and Fermi) systems we number the single particle states form zero to infinity, starting from the lowest energy level and going up-wards; the order in which we number the degenerate levels is not important. At zero temperature, the fermions occupy all the $N$ states, from zero to the Fermi energy, $\epsilon_F(N)$. This is the ground state (g.s.) of the Fermi system and is put into correspondence with the ground state of the Bose system, $\epsilon_{B,0}$, like in figure 2 (a).

The correspondence between the Fermi and the Bose energy levels is done recursively, starting from the lowest energies, as it is depicted in Figs. 2 (b), (c), and (d). If the lowest hole in the Fermi system is created by lifting a particle from the level $i$, of energy $\epsilon_i(< \epsilon_F(N))$ to the level $N + 1$, then in the Bose system $i - 1$ particles are left on the g.s. and $N - i + 1$ particles are lifted on the first excited level, $\epsilon_{B,1}$ [figure 2 (b)]. The excitation energy in both systems is the same, so we define

$$\epsilon_{B,1} \equiv \frac{\epsilon_F(N + 1) - \epsilon_i}{N - i + 1}.$$  

The recursion procedure continues upwards; let’s say that we arrived at the level $\epsilon_j$. We assume that we lifted $k$ particles from the energy levels below $\epsilon_j$, so we created already $k$ quasiparticle energy levels in the Bose system. At least from $\epsilon_j$, upwards, all the Fermi levels are occupied up to $\epsilon_F(N + k)$ (i.e. the total number of particles, $N$, plus the $k$ particles that have been lifted above the Fermi energy) and all these particles stay on the highest occupied energy level in the Bose system, $\epsilon_{B,k}$. Let us now lift the $n$ particles from the energy levels $j + 1, \ldots, j + n$, onto $N + k + 1, \ldots, N + k + n$. This corresponds in the Bose system to $n - 1$ free states, say $\epsilon_{B,k+1}, \ldots, \epsilon_{B,k+n-1}$, and an $n$th state, $\epsilon_{B,k+n}$, with a population of $N - (j - k)$ particles. The difference between the Bose levels which are successively occupied, $\epsilon_{B,k}$ and $\epsilon_{B,k+n}$, is

$$\epsilon_{B,k+n} - \epsilon_{B,k} = \sum_{l=1}^{n} [\epsilon_F(N + k + l) - \epsilon_{j+l}] \approx n \cdot \frac{[\epsilon_F(N + k) - \epsilon_j]}{N + k - j} \quad (1)$$

(see figure 2).

The procedure continues upwards until we reach the highest energy particles in the Fermi system.


2.2 CONTINUOUS LIMIT

To discuss the thermodynamic limit, we need to go from the discrete case to the (quasi)continuous limit. For this, we denote the fermionic and the bosonic DOSs by $\sigma(\epsilon)$ and $\sigma_B(\epsilon_B)$, respectively. We assume that in the Fermi system $N_0$ particles are condensed [9], so there are no holes in the energy interval from 0 to $\epsilon_F(N_0)$. Above $\epsilon_F(N_0)$ we divide the single particle energy axis into microscopic intervals, numbered $i = 1, \ldots, \infty$. Then the interval $i$ of length $\delta\epsilon_i$, contains $d_i = \sigma(\epsilon_i)\delta\epsilon_i$ single particle states. The population of the interval $\delta\epsilon_i$ is denoted as $\delta n_i \equiv f(\epsilon_i)d_i$ and we assume that $f(\epsilon_i) < 1$ for any $i \geq 1$. After EST, $N_0$ represents the number of particles on the Bose g.s., $\epsilon_{B,0}$, and the intervals $\delta\epsilon_i$ transform into the intervals $\delta\epsilon_{B,i}$. The number of bosonic states in each of these intervals is $d_{B,i} = d_i - \delta n_i + 1 \approx d_i - \delta n_i$ (for large enough $d_{B,i}$). Although it has no relevance in the continuous limit, note that because of an overlap between successive Bose intervals, the expression $d_{B,i} = d_i - \delta n_i$, leads to a better counting of total number of bosonic states (the same argument is in [12]). Applying equation (1), $\delta\epsilon_{B,i}$ is calculated by
exciting \([1 - f(\epsilon_i)]d_i\) fermions from the interval \(\delta \epsilon_i\), which leads to

\[
\delta \epsilon_{B,i} = (d_i - \delta n_i) \cdot \frac{\epsilon_F [N + 1 + \sum_{j=1}^{i}(d_j - \delta n_j)] - \epsilon_i}{N - N_0 - \sum_{j=1}^{i} \delta n_j}.
\]

From equation (2) we obtain the Bose DOS,

\[
\sigma_B(\epsilon_{B,i}) = \frac{d_{B,i}}{\delta \epsilon_{B,i}} = \frac{N - N_0 - \sum_{j=1}^{i} \delta n_j}{\epsilon_F [N + 1 + \sum_{j=1}^{i}(d_j - \delta n_j)] - \epsilon_i}.
\]

and the Bose population (the number of particles divided by the number of states),

\[
b(\epsilon_{B,i}) = \frac{\delta n_i}{d_i - \delta n_i} = \frac{1}{f^{-1}[\epsilon_i(\epsilon_{B,i})]} - 1.
\]

Changing the summations into integrals, equation (3) becomes

\[
\sigma_B(\epsilon_B) = \frac{N - N_0 - \int_0^{\epsilon_B} b(\epsilon_B')\sigma_B(\epsilon_B')d\epsilon_B'}{\epsilon_F [N + 1 + \int_0^{\epsilon_B} \sigma_B(\epsilon_B')d\epsilon_B'] - \epsilon(\epsilon_B)},
\]

where \(\epsilon(\epsilon_B)\) is the single particle energy in the original Fermi systems that corresponds to the Bose energy \(\epsilon_B\).

To calculate \(\epsilon(\epsilon_B)\), note that the number of Fermi states in the small energy interval \(\delta \epsilon\) which corresponds to the Bose energy interval \(\delta \epsilon_B\) is equal to the number of particles in the interval, plus the number of bosonic energy levels:

\[
d(\delta \epsilon) = [b(\epsilon_B) + 1] \sigma_B(\epsilon_B) \delta \epsilon_B.
\]

Integrating over all the Bose energy levels, we get the expression

\[
\epsilon(\epsilon_B) \equiv \epsilon_F \left\{ N_0 + \int_0^{\epsilon_B} \sigma_B(\epsilon)[b(\epsilon) + 1]d\epsilon \right\}.
\]

The total Bose energy,

\[
E_B = \int_0^{\infty} d\epsilon_B \sigma_B(\epsilon_B) \epsilon_B b(\epsilon_B),
\]

is nothing but the excitation energy of the system, in agreement also with the definitions given in [8].

### 3 Interaction in the Bose System

In the general case, the Bose system obtained after the EST is not an ideal system. A change in the population of Bose “single particle” states, say \(\delta b(\epsilon_B)\), changes all the single particle energies and, further, \(\sigma_B\). To see this, let us insert \(\xi\) particles in the interval \(i\) of the Fermi system. By doing this, we also
expand the interval by $\xi$ states, so that the number of Bose states in any of the
intervals would not change (see figure 3). All the quantities that are affected
by this insertion of quasiparticles will bear a superscript which refers to the
energy interval $(i)$, or directly to the energy levels $[\epsilon_j \text{ (Fermi)}$ or $\epsilon_{B,j} \text{ (Bose)}]$ where the insertion took place. Applying equation (2), we calculate the new energy interval
\begin{align}
\delta \epsilon_{B,k}^i &= \epsilon_{B,k}^i - \epsilon_{B,k-1}^i;
\end{align}

\begin{align}
\delta \epsilon_{B,k}^i &= \left\{ \epsilon_F [N + 1 + \xi + \sum_{j=1}^{k} (d_j - \delta n_j)] - \epsilon_F \left[ N_0 + \xi + \sum_{j=1}^{k} d_j \right] \right\} \\
&\quad \times \frac{d_k - \delta n_k}{N - N_0 - \sum_{j=0}^{k} \delta n_j}, \text{ if } i < k,
\end{align}

or

\begin{align}
\delta \epsilon_{B,k}^i &= \left\{ \epsilon_F [N + 1 + \xi + \sum_{j=1}^{k} (d_j - \delta n_j)] - \epsilon_F \left[ N_0 + \sum_{j=1}^{k} d_j \right] \right\} \\
&\quad \times \frac{d_k - \delta n_k}{N + \xi - N_0 - \sum_{j=0}^{k} \delta n_j}, \text{ if } i \geq k.
\end{align}

If $\xi \ll \delta n_i$, then

\begin{align}
\delta \epsilon_{B,k}^i &\approx \delta \epsilon_{B,k} + \xi \frac{d - \delta n_k}{N - N_0 - \sum_{j=0}^{k} \delta n_j} \left\{ \frac{d \epsilon_F}{dN} [N + 1 + \sum_{j=1}^{k} (d_j - \delta n_j)] \right\}
\end{align}
\begin{equation}
-\frac{d\epsilon_F}{dN}[N_0 + \sum_{j=1}^{k} d_j], \quad \text{if } i < k \tag{10}
\end{equation}

or

\begin{equation}
\delta\epsilon_{B,k}^i = \delta\epsilon_{B,k} - \frac{\delta\epsilon_{B,k}}{N - N_0 - \sum_{j=0}^{k} \delta n_j} + \frac{\xi}{N - N_0 - \sum_{j=0}^{k} \delta n_j}
\times \frac{d\epsilon_F}{dN}[N + 1 + \sum_{j=1}^{k} (d - \delta n_k)], \quad \text{if } i \geq k. \tag{11}
\end{equation}

If in the Bose system we add up the energies of all particles, we obtain

\[ E_\xi = E_B + \xi \epsilon_{B,i} + E_I \]

where \( E_B \) is given by equation (7) and \( E_I \propto \xi \) is the interaction term. We may interpret \( \epsilon_B \) as the quasiparticle energy, and write it as

\[ \epsilon_B = \epsilon_{BF} + \sum_{\epsilon_{BF}'} v(\epsilon_{BF}, \epsilon_{BF}') \delta n(\epsilon_{BF}'), \tag{12} \]

where \( \epsilon_{BF} \) is the energy of the “free particles”, while \( v \) is the interaction potential. Note that \( v(\epsilon_{BF}, \epsilon_{BF}') \) depends on \( n(\epsilon_{BF}') \), for any value of \( \epsilon_{BF}' \).

### 3.1 Calculation of the Interaction Potential

The Bose quasiparticle energies and the interaction potential \( v \) from equation (12) may be calculated by adding the microscopic energy intervals given by the equations (10) and (11). Transforming the summations into integrals and using the equations (3), (5) and (6) we obtain

\begin{align*}
\epsilon_{B,i}^{\text{BF}}(\xi) &= \epsilon_B - \xi \int_0^{\epsilon_{\text{BF}}} \frac{d\epsilon_B}{N - N_0 - \int_0^{\epsilon_{\text{BF}}} b(\epsilon) \sigma_B(\epsilon) d\epsilon} \\
&\quad + \xi \int_{\epsilon_{\text{BF}}}^{\epsilon_B} \frac{d\epsilon_F}{dN} \frac{[N + 1 + \int_0^{\epsilon_B} \sigma_B(\epsilon) d\epsilon] b(\epsilon) \sigma_B(\epsilon) d\epsilon}{[N + 1 + \int_0^{\epsilon_B} \sigma_B(\epsilon) d\epsilon]} \\
&= \epsilon_B - \xi \int_0^{\epsilon_B} \frac{1 - \sigma_B(\epsilon_B) \frac{d\epsilon_F}{dN} [N + 1 + \int_0^{\epsilon_B} \sigma_B(\epsilon) d\epsilon]}{N - N_0 - \int_0^{\epsilon_B} b(\epsilon) \sigma_B(\epsilon) d\epsilon} d\epsilon_B, \tag{13}
\end{align*}

for \( \epsilon_{B,i} \geq \epsilon_B \)

and

\begin{equation}
\epsilon_{B,i}^{\text{BF}}(\xi) = \epsilon_B - \xi \int_0^{\epsilon_{\text{BF}}} \frac{1 - \sigma_B(\epsilon_B) \frac{d\epsilon_F}{dN} [N + 1 + \int_0^{\epsilon_B} \sigma_B(\epsilon) d\epsilon]}{N - N_0 - \int_0^{\epsilon_B} b(\epsilon) \sigma_B(\epsilon) d\epsilon} d\epsilon_B \tag{14}
\end{equation}
In terms of $\epsilon$, and using the shorthand notation $\epsilon_0 \equiv \epsilon_F(N_0)$, the equations (13) and (14) become

$$
\epsilon_B^i(\xi) = \left[ N - N_0 - \int_{\epsilon_0}^{\epsilon_1} d\epsilon_2 \sigma(\epsilon_2) f(\epsilon_2) \right]^{-1}
\left[ N - N_0 - \int_{\epsilon_0}^{\epsilon_1} d\epsilon_2 \sigma(\epsilon_2) f(\epsilon_2) \right]^{-1}
- \frac{\sigma^{-1} \left[ \epsilon_F \left( N + \int_{\epsilon_0}^{\epsilon_1} d\epsilon_2 \sigma(\epsilon_2)(1 - f(\epsilon_2)) \right) \right]}{\epsilon_F \left[ N + \int_{\epsilon_0}^{\epsilon_1} d\epsilon_2 \sigma(\epsilon_2)(1 - f(\epsilon_2)) \right] - \epsilon_1}
$$
for $\epsilon_i \geq \epsilon(\epsilon_B)$, \hspace{1cm} (15)

and

$$
\epsilon_B^i(\xi) = \left[ N - N_0 - \int_{\epsilon_0}^{\epsilon_1} d\epsilon_2 \sigma(\epsilon_2) f(\epsilon_2) \right]^{-1}
\left[ N - N_0 - \int_{\epsilon_0}^{\epsilon_1} d\epsilon_2 \sigma(\epsilon_2) f(\epsilon_2) \right]^{-1}
- \frac{\sigma^{-1} \left[ \epsilon_F \left( N + \int_{\epsilon_0}^{\epsilon_1} d\epsilon_2 \sigma(\epsilon_2)(1 - f(\epsilon_2)) \right) \right] + \xi \int_{\epsilon_i}^{\epsilon_B} d\epsilon_1 \sigma(\epsilon_1)(1 - f(\epsilon_1))}{N - N_0 - \int_{\epsilon_0}^{\epsilon_1} d\epsilon_2 \sigma(\epsilon_2) f(\epsilon_2)}
$$
$$
\times \frac{\sigma^{-1} \left[ \epsilon_F \left( N + \int_{\epsilon_0}^{\epsilon_1} d\epsilon_2 \sigma(\epsilon_2)(1 - f(\epsilon_2)) \right) \right] - \sigma^{-1}(\epsilon_1)}{N - N_0 - \int_{\epsilon_0}^{\epsilon_1} d\epsilon_2 \sigma(\epsilon_2) f(\epsilon_2)}
$$
for $\epsilon_i \leq \epsilon(\epsilon_B)$, \hspace{1cm} (16)

respectively. The terms proportional to $\xi$ in equations (13), (14), (15), and (16) represent the interaction term $v(\epsilon_B, \epsilon_{B,i})$ of equation (12).

### 3.2 Bose Density of States

The changes of the single particle energy levels produce changes of the Bose DOS, $\sigma_B(\epsilon_B)$. The new DOS is obtained by using eqs. (10) and (11):

$$
\sigma_B^i(\epsilon_B, k) = \frac{d - \delta n_k}{\delta \epsilon_{B,k}} \approx \sigma_B(\epsilon_B, k) \cdot \left( 1 - \xi \frac{\sigma_B(\epsilon_B, k)}{N - N_0 - \sum_{j=1}^{k} \delta n_j} \right)
$$
$$
\times \left\{ \frac{d \sigma_F}{dN} [N + 1 + \sum_{j=1}^{k} (d_j - \delta n_j)] - \frac{d \sigma_F}{dN} [N_0 + \sum_{j=1}^{k} d_j] \right\}
$$
\hspace{1cm}, if $i < k$, \hspace{1cm} (17)
and

$$\sigma_B(\epsilon_{B,k}) = \frac{d - \delta n_k}{\delta \epsilon_{B,k}} \approx \sigma_B(\epsilon_{B,k}) \cdot \left( 1 + \frac{\xi}{N - N_0 - \sum_{j=0}^{k} \delta n_j} \right)$$

(18)

$$- \xi \frac{\sigma_B(\epsilon_{B,k})}{N - N_0 - \sum_{j=1}^{k} \delta n_j} \frac{dF}{dN} [N + 1 + \sum_{j=1}^{k} (d - \delta n_j)] \right), \text{ if } i \geq k.$$  Writing equations (13), (14), (15) and (16) in the format

$$\epsilon_{B,k}^i = \epsilon_{B,k} + \epsilon_{B,k}^{B,i} \cdot \frac{\partial \epsilon_{B,k}}{\partial \epsilon_{B,k}^{B,i}},$$

(19)

and using equations (17) and (18) we obtain

$$\sigma_B^{\epsilon_{B,i}}(\epsilon_{B,k} + \epsilon_{B,k}^{B,i} \cdot \frac{\partial \epsilon_{B,k}}{\partial \epsilon_{B,k}^{B,i}}) = \sigma_B(\epsilon_{B,k}) \cdot \left( 1 - \frac{\xi \epsilon_{B,i}}{N - N_0 - \int_{0}^{\epsilon_{B,k}} b(\epsilon_{B})\sigma_B(\epsilon_{B})d\epsilon_{B}} \right)$$

$$\times \left\{ \frac{dF}{dN} \left|_{N+1+\int_{0}^{\epsilon_{B,k}} \sigma_B(\epsilon_{B})d\epsilon_{B}} - \frac{dF}{dN} \left|_{N_0+\int_{0}^{\epsilon_{B,k}} \sigma_B(\epsilon_{B})d\epsilon_{B}} \right\} \right\}, \text{ if } \epsilon_{B,i} < \epsilon_{B,k},$$

(20)

and

$$\sigma_B^{\epsilon_{B,i}}(\epsilon_{B,k} + \epsilon_{B,k}^{B,i} \cdot \frac{\partial \epsilon_{B,k}}{\partial \epsilon_{B,k}^{B,i}}) = \sigma_B(\epsilon_{B,k}) \cdot \left( 1 + \frac{\xi \epsilon_{B,i}}{N - N_0 - \int_{0}^{\epsilon_{B,k}} b(\epsilon_{B})\sigma_B(\epsilon_{B})d\epsilon_{B}} \right)$$

$$\times \left\{ 1 - \sigma_B(\epsilon_{B,k}) \frac{dF}{dN} \left|_{N+1+\int_{0}^{\epsilon_{B,k}} \sigma_B(\epsilon_{B})d\epsilon_{B}} \right\} \right\}, \text{ if } \epsilon_{B,i} \geq \epsilon_{B,k}. \quad (21)$$

Writing the integrals in terms of \( \epsilon \), Eqs. (20) and (21) become

$$\sigma_B^{\epsilon_{B,i}}(\epsilon_{B,k} + \epsilon_{B,k}^{B,i} \cdot \frac{\partial \epsilon_{B,k}}{\partial \epsilon_{B,k}^{B,i}}) = \sigma_B(\epsilon_{B,k}) \cdot \left( 1 - \frac{\xi \epsilon_{B,i} \sigma_B(\epsilon_{B,k})}{N - N_0 - \int_{0}^{\epsilon_{B,k}} \epsilon \sigma(\epsilon) f(\epsilon) d\epsilon} \right)$$

$$\times \left\{ \sigma^{-1} \left[ \epsilon \left( N + \int_{0}^{\epsilon_k} d\epsilon \sigma(\epsilon)(1 - f(\epsilon)) \right) - \sigma^{-1}(\epsilon_k) \right] \right\} \right\}, \text{ if } \epsilon_{B,i} < \epsilon_{B,k}$$

(22)

and

$$\sigma_B^{\epsilon_{B,i}}(\epsilon_{B,k} + \epsilon_{B,k}^{B,i} \cdot \frac{\partial \epsilon_{B,k}}{\partial \epsilon_{B,k}^{B,i}}) = \sigma_B(\epsilon_{B,k}) \cdot \left( 1 + \frac{\xi \epsilon_{B,i}}{N - N_0 - \int_{0}^{\epsilon_{B,k}} \epsilon \sigma(\epsilon) f(\epsilon) d\epsilon} \right)$$

$$\times \left\{ 1 - \frac{\sigma_B(\epsilon_{B,k})}{\sigma \left[ \epsilon \left( N + \int_{0}^{\epsilon_k} d\epsilon \sigma(\epsilon)(1 - f(\epsilon)) \right) \right] \right\}, \text{ if } \epsilon_{B,i} \geq \epsilon_{B,k}. \quad (23)$$
4 FINDING THE EQUILIBRIUM DISTRIBUTION

The Bose and the Fermi systems are thermodynamically equivalent by construction. Therefore, the equilibrium particle distribution in the Fermi gas, if transformed by (4), should give us the equilibrium particle distribution in the Bose gas and knowing this, we can calculate all the microscopic parameters of the Bose system and the quasiparticle-quasiparticle interactions like in sections 2.2 and 3.1, determining in this way all the properties of the gas. The Fermi equilibrium (i.e. most probable) particle distribution is the well known

\[ f(\epsilon) = \frac{\exp(\beta \epsilon - \beta \mu) + 1}{\exp(\beta \epsilon - \beta \mu) + 1} \]

(where \( \beta \equiv (k_B T)^{-1} \)), and this gives us by (4), the bosonic population

\[ b(\epsilon_B) = \exp[\beta \mu - \beta \epsilon(\epsilon_B)] \] (24)

which can be readily used in the calculations.

Nevertheless, we have still another method to calculate the equilibrium particle distribution in the Bose gas. In sections 2 and 3 we defined self-consistently the quasiparticle energy levels in the Bose gas and the quasiparticle-quasiparticle interaction, so we can use this as a starting point to build the Bose entropy, \( S_B \), and canonical partition function, \( Z_B \). Then the equilibrium particle distribution in the Bose gas should be the one that maximizes \( Z_B \) at constant \( N \) and \( U_B \). The Bose and the Fermi systems are thermodynamically equivalent, so the canonical partition function of the Fermi system, \( Z_F \), is identical to \( Z_B \) by construction, and indeed, the distribution that maximizes one should correspond (by an EST transformation 4) to the distribution that maximized the other, unless the canonical and grandcanonical ensembles in at least one of the systems are not equivalent.

To check this equivalence, let me calculate \( S_B \) and maximize it at constant \( U_B \) and \( N \). (Note that constant \( U_B \) and \( N \) is equivalent to constant \( U \) and \( N \).) So let again \( \delta \epsilon_{B,k} \) be the microscopic energy intervals along the \( \epsilon_B \) axis, each containing \( d_{B,k} \) states. The total number of configurations available in one interval is

\[ w_{B,k} = \frac{(\delta n_k + d_{B,k})!}{\delta n_k! d_{B,k}!} \] (25)

Summing up the contributions of all the intervals and applying the Stirling approximation, \( \log m! \approx m \log(m/e) \), we obtain the Bose entropy:

\[ S_B/k_B = \int_0^\infty d\epsilon_B \sigma_B(\epsilon_B) \{[1 + b(\epsilon_B)] \log[1 + b(\epsilon_B)] - b(\epsilon_B) \log[b(\epsilon_B)] \} \] (26)

To find the equilibrium configuration, we have to maximize \( S_B \) as a functional of \( b \), at fixed \( E_B \) and \( N \). For this we introduce the Lagrange multipliers \( \beta_B \)
and $\beta_B\mu_B$, corresponding to $E_B$ and $N$, respectively, and we solve
\[
\frac{\delta(S/k_B - \beta_B E_B + \beta_B\mu_B N)}{\delta b} = 0, \tag{27}
\]
where, as shown in section 2, from thermodynamic equivalence arguments we have $\beta_B = \beta$ and $\mu_B = \mu - \epsilon_F$ (see also Ref. [8]). The functional derivative $\delta E_B/\delta b(\epsilon_B)$ is not simply $\epsilon_B$. To calculate it we rewrite Eq. (12) as
\[
\epsilon_B^{\delta b} = \epsilon_B + \int_0^\infty d\epsilon_B' \sigma_B(\epsilon_B, \epsilon_B')\delta b(\epsilon_B') \tag{28}
\]
where $\epsilon_B$ is the quasiparticle energy at equilibrium configuration and $\delta b(\epsilon_B)$ is the deviation of the population from the equilibrium configuration, $b(\epsilon_B)$. Note that both, $b(\epsilon_B)$ and $\delta b(\epsilon_B)$ are given as functions of the energy before the insertion of particles, $\epsilon_B$. The interaction term is calculated in Section 3.1.

Using (28) we calculate the energy
\[
E_B' = \int_0^\infty d\epsilon_B\epsilon_B^{\delta b}\sigma_B(\epsilon_B) + \delta \sigma_B(\epsilon_B)|b(\epsilon_B) + \delta b(\epsilon_B)| \tag{29}
\]
\[
= \int_0^\infty d\epsilon_B \left[ \epsilon_B + \int_0^\infty d\epsilon_B' \sigma_B(\epsilon_B, \epsilon_B')\delta b(\epsilon_B') \right] \sigma_B(\epsilon_B)|b(\epsilon_B) + \delta b(\epsilon_B)|,
\]
where we used the equation $d\epsilon_B^{\delta b}/d\epsilon_B = \sigma_B(\epsilon_B)|\sigma_B(\epsilon_B^{\delta b}) + \delta \sigma_B(\epsilon_B^{\delta b})|$ and the obvious notation $b(\epsilon_B) \equiv b[\epsilon_B(\epsilon_B^{\delta b})]$ in the first line. From (29), and keeping only the first order in $\delta b$, we obtain
\[
E_B' = E_B + \int_0^\infty d\epsilon_B \epsilon_B\sigma_B(\epsilon_B)\delta b(\epsilon_B) \tag{30}
+ \int_0^\infty d\epsilon_B \int_0^\infty d\epsilon_B' b(\epsilon_B)\sigma_B(\epsilon_B)\sigma_B(\epsilon_B')v(\epsilon_B, \epsilon_B')\delta b(\epsilon_B').
\]
The first integral in equation (30), let’s call it $E_{qp}$, is the typical contribution of the added quasiparticle energies to the total energy of the system, whereas the double integral (say $E_{\mu}$) is due to the collective change of quasiparticle energies and can not be eliminated by a redefinition of the ground state energy, as e.g. in the Fermi liquid theory. So $E_{\mu}$ will produce an effective quasiparticle energy
\[
\bar{\epsilon}_B(\epsilon_B) = \epsilon_B + \int_0^\infty d\epsilon_B' b(\epsilon_B)\sigma_B(\epsilon_B')v(\epsilon_B, \epsilon_B') \equiv \epsilon_B + \bar{\epsilon}_B(\epsilon_B). \tag{31}
\]
Since $\delta S/\delta b|_{\epsilon_B} = k_B\sigma_B(\epsilon_B)\log[1 + b^{-1}(\epsilon_B)]$ and $\delta N/\delta b|_{\epsilon_B} = \sigma_B(\epsilon_B)$, plugging (30) into equation (27), one obtains the following equation for the equilibrium distribution:
\[
\log[1 + b^{-1}(\epsilon_B)] - \beta\bar{\epsilon}_B(\epsilon_B) + \beta\mu_B = 0. \tag{32}
\]
Equation (32) leads to the typical Bose quasiparticle level population,

\[ b(\epsilon_B) = \left[ \exp(\beta \bar{\epsilon}_B(\epsilon_B) - \beta \mu_B) - 1 \right]^{-1}. \]  

(33)

If neither the Fermi, nor the resultant Bose system is pathologic, then the expressions (24) and (33) should give the same result, eventually with some corrections which vanish in the thermodynamic limit.

5 COMPARISON OF THE BOSE AND FERMI DESCRIPTIONS

To make concrete calculations and check if Eqs. (24) and (33) give the same results, we assume that the Fermi DOS has the general form \( \sigma(\epsilon) = Cs \), where \( C \) and \( s \) are constants and we calculate the Bose energies, DOS, and chemical potential by using the Fermi distribution, \( f(\epsilon) = \left[ \exp(\beta \epsilon - \beta \mu) + 1 \right]^{-1} \), in the formulas of Section 2.2. Using these results and the expressions for \( v(\epsilon'_B, \epsilon_B) \) calculated in Section 3, we can compare Eqs. (24) and (33).

5.1 CONSTANT DENSITY OF STATES

In Ref. [8] it is proven that the Bose and Fermi descriptions (obtained by EST) of systems with constant DOS are equivalent under canonical conditions. Therefore we start by using such systems to test the formalism introduced above. If the density of states is constant (\( \sigma \equiv C \)) then \( \epsilon_F(n) = n/C \) and by using (6) to rewrite \( \epsilon_i \) of equation (3) as

\[ \epsilon_i = C^{-1} \cdot \left[ N_0 + 1 + \sum_{j=1}^{i} d_i \right], \]

the Bose DOS becomes [8]

\[ \sigma_B(\epsilon_{B,i}) \equiv C = \sigma. \]  

(34)

Moreover, equation (2) becomes

\[ \delta\epsilon_{B,i} = C^{-1} \cdot (d_i - \delta n_i), \]  

(35)

which, in the continuous limit leads to the Bose energies

\[ \epsilon_B = \int_0^\infty [1 - f(\epsilon_1)]d\epsilon_1 = k_B T \log \left[ \frac{1 + e^{\beta(\epsilon - \mu)}}{1 + e^{-\beta \mu}} \right]. \]  

(36)

From equations (10) and (11) follows that \( v \equiv 0 \), so the Bose gas is also noninteracting, with \( \mu_B = \mu - \epsilon_F \) [8] and \( \tilde{\epsilon}_B = 0 \).
To test the Bose-Fermi description equivalence we invert equation (36) to get
\[ e^\beta(\epsilon - \mu) = e^{\beta B} \left[ 1 + e^{-\mu B} \right] - 1, \quad (37) \]
which, if plugged into (24), gives
\[ b = \left[ e^{\beta B} \left[ 1 + e^{-\mu B} \right] - 1 \right]^{-1}. \quad (38) \]
But for a Fermi gas of constant DOS we have the identity (from equation 4, Ref. \[8\])
\[ \exp(\beta \mu) = \exp(\beta \epsilon_F) - 1, \quad (39) \]
so
\[ 1 + e^{-\mu} = 1 + [\exp(\beta \epsilon_F) - 1]^{-1} = \exp[\beta(\epsilon_F - \mu)] \equiv \exp[-\beta \epsilon_B]. \quad (40) \]
Plugging (37) into (40) we obtain equation (33),
\[ b = \exp[\beta(\epsilon - \mu)] = \left( \exp\{\beta[\epsilon_B(\epsilon) - \mu_B]\} - 1 \right)^{-1}, \quad (41) \]
which proves the equivalence between the Bose and Fermi descriptions of the gas with constant DOS.

The equivalence of the Bose and Fermi descriptions does not prove the canonical-grandcanonical ensemble equivalence. The ensemble inequivalence in condensed Bose systems have been extensively studied (see \[13, 14\] and references therein); from the EST perspective, the fluctuations of the Bose and Fermi energy level populations for systems of constant DOS have been compared in Ref. \[11\].

5.2 NON-CONSTANT DENSITY OF STATES

Again, if we take \( f(\epsilon) = [\exp(\beta \epsilon - \beta \mu) + 1]^{-1} \), then \( N_0 = 0 \) (no “condensation”). If \( \sigma(\epsilon) \equiv C\epsilon^s \) with \( s \neq 0 \), then
\[ N = \int_0^{\epsilon_F} C\epsilon^s \, d\epsilon = \frac{C\epsilon_F^{s+1}}{s+1}, \quad (42) \]
and
\[ \sigma_B(\epsilon_B(\epsilon)) = \frac{N - \int_0^\epsilon \frac{\sigma(\epsilon) \, d\epsilon}{1 + \epsilon^{s+1}}}{\left[ \frac{s+1}{C} \right]^{1/(s+1)} \cdot \left[ N + \int_0^\epsilon \frac{\sigma(\epsilon) \, d\epsilon}{1 + e^{-\beta(\epsilon - \mu)}} \right]^{1/(s+1)} - \epsilon}, \quad (43) \]
where again
\[ b(\epsilon_B) = e^{-\beta[\epsilon_B(\epsilon) - \mu]}. \quad (44) \]
Using the equations (42-44) we rewrite the equation (2) as

\[
\text{de}_B = d\epsilon \cdot \frac{Ce^\epsilon}{1 + e^{-\beta(\epsilon - \mu)}} \cdot \left[ N + \int_0^\epsilon \frac{\sigma(\epsilon) d\epsilon}{1 + e^{-\beta(\epsilon - \mu)}} \right]^{1/(s+1)} - \epsilon,
\]

(45)
to obtain

\[
\epsilon_B = \int_0^\epsilon \left\{ \frac{d\epsilon_1 C\epsilon_1^s}{1 + e^{-\beta(\epsilon_1 - \mu)}} \cdot \left[ N + \int_0^{\epsilon_1} \frac{\sigma(\epsilon_2) d\epsilon_2}{1 + e^{-\beta(\epsilon_2 - \mu)}} \right]^{1/(s+1)} - \epsilon_1 \right\}.
\]

(46)

For the two cases, \( \epsilon' \geq \epsilon \) and \( \epsilon' < \epsilon \), the interaction potential \( v \) is

\[
v(\epsilon, \epsilon' \geq \epsilon) = -\int_0^\epsilon d\epsilon_1 \frac{1}{\sigma_B(\epsilon_B(\epsilon_1))} \frac{Ce^\epsilon}{1 + e^{-\beta(\epsilon_1 - \mu)}} \times \left[ N - \int_0^{\epsilon_1} d\epsilon_2 \frac{Ce^{\epsilon_2}}{1 + e^{\beta(\epsilon_2 - \mu)}} \right]^{-1} - C^{-1} \left( \epsilon_F^{\epsilon + 1} \right)
\]

\[
+ \int_0^{\epsilon_1} d\epsilon_2 \frac{Ce^{\epsilon_2}}{1 + e^{\beta(\epsilon_2 - \mu)}} \left[ N + \int_0^{\epsilon_1} d\epsilon_2 \frac{Ce^{\epsilon_2}}{1 + e^{\beta(\epsilon_2 - \mu)}} \right]^{-1}
\]

\[
= -\int_0^\epsilon d\epsilon_1 \frac{1}{\sigma_B(\epsilon_B(\epsilon_1))} \frac{Ce^\epsilon}{1 + e^{-\beta(\epsilon_1 - \mu)}} \left[ N - \int_0^{\epsilon_1} d\epsilon_2 \frac{Ce^{\epsilon_2}}{1 + e^{\beta(\epsilon_2 - \mu)}} \right]^{-1} \times (s + 1) \left[ N + \int_0^{\epsilon_1} d\epsilon_2 \frac{Ce^{\epsilon_2}}{1 + e^{\beta(\epsilon_2 - \mu)}} \right]^{-1}
\]

\[
\times \left[ N + \int_0^{\epsilon_1} d\epsilon_2 \frac{Ce^{\epsilon_2}}{1 + e^{\beta(\epsilon_2 - \mu)}} \right]^{-1}
\]

(47)

and

\[
v(\epsilon, \epsilon' < \epsilon) = -\int_0^\epsilon d\epsilon_1 \frac{1}{\sigma_B(\epsilon_B(\epsilon_1))} \frac{Ce^\epsilon}{1 + e^{-\beta(\epsilon_1 - \mu)}} \times \left[ N - \int_0^{\epsilon_1} d\epsilon_2 \frac{Ce^{\epsilon_2}}{1 + e^{\beta(\epsilon_2 - \mu)}} \right]^{-1} \left( s + 1 \right) \left[ N + \int_0^{\epsilon_1} d\epsilon_2 \frac{Ce^{\epsilon_2}}{1 + e^{\beta(\epsilon_2 - \mu)}} \right]^{-1}
\]

\[
- \epsilon_1 C^\frac{\epsilon_1}{\gamma + 1} (s + 1)^{\frac{\epsilon_1}{\gamma + 1}} \left[ N + \int_0^{\epsilon_1} d\epsilon_2 \frac{Ce^{\epsilon_2}}{1 + e^{\beta(\epsilon_2 - \mu)}} \right]^{-1}
\]

\[
+ \int_0^{(\epsilon - \epsilon_1)_+} d\epsilon_1 \frac{Ce^\epsilon}{1 + e^{-\beta(\epsilon_1 - \mu)}} \times \left[ C^\frac{\epsilon_1}{\gamma + 1} (s + 1)^{\frac{\epsilon_1}{\gamma + 1}} \left( N + \int_0^{\epsilon_1} d\epsilon_2 \frac{\sigma(\epsilon_2)}{[1 + e^{-\beta(\epsilon_2 - \mu)}]} \right) \right]^{-1} \times \left( Ce_1^{\epsilon_1} \right)^{-1}
\]

(48)
5.2.1 Dimensionless expressions

To identify the relevant parameters of the gas, and for the convenience of the calculations, let us isolate dimensionless quantities from the equations (42-46). Hereafter we shall use extensively the notations

\[ x \equiv \beta \epsilon \]

\[ x' \equiv \beta \epsilon' \]

\[ y \equiv \beta \mu \]

and also their Bose correspondents

\[ x_B \equiv \beta \epsilon_B \]

\[ x'_B \equiv \beta \epsilon'_B \]

A quantity that appears in almost all the expressions is the integral over Fermi distribution:

\[
Fi(T, C, \epsilon, \mu, s) \equiv \int_0^\epsilon \frac{C e^{\frac{\epsilon}{\beta}} d\epsilon}{e^{\frac{\epsilon}{\beta} + 1}} = C(k_B T)^{s+1} \int_0^x \frac{x d\epsilon}{e^{\frac{x}{\beta} + 1}}
\]

related to the integral over the hole density,

\[
Fi'(T, C, \epsilon, \mu, s) \equiv \int_0^\epsilon \frac{C e^{\frac{\epsilon'}{\beta}} d\epsilon}{e^{\frac{\epsilon'}{\beta} + 1}} = C(k_B T)^{s+1} \int_0^{x'} \frac{x' d\epsilon'}{e^{\frac{x'}{\beta} + 1}} - \left[F_i(x, y, s)\right]'
\]

Both, \(Fi\) and \(Fi'\) are defined for \(s > -1\). Another quantity of interest is the “reduced” particle number

\[
N_r \equiv \frac{(s+1)N}{C(k_B T)^{s+1}} = \left(\frac{\epsilon_F}{k_B T}\right)^{s+1}
\]

With the definitions, the density of states becomes

\[
\sigma_B(\epsilon_B) = \frac{C(k_B T)^s}{(s+1)} \frac{[N_r - (s+1)Fi(x, y, s)]}{[N_r + (s+1)Fi'(x, y, s)]^{1/(s+1)} - x}
\]

and the Bose energy may be written as

\[
\epsilon_B = (s+1)k_B T \int_0^x \frac{d\epsilon x^s}{1 + e^{-\frac{x}{\beta} + y}} \cdot \sigma_B^{-1}(N_r(y, s), x_1, y, s) = (k_B T)\epsilon_B(x, y, s).
\]

The interaction term \(v(\epsilon_B, \epsilon_B')\) has two different expressions, depending on the sign of \(\epsilon_B - \epsilon_B'\), or equivalently, \(\epsilon - \epsilon'\):

\[
v(\epsilon, \epsilon' \geq \epsilon) = -\frac{s+1}{C(k_B T)^s} \int_0^x \frac{d\epsilon x^s}{(1 + e^{-\frac{x}{\beta} + y})\sigma_B} \left\{ \frac{s+1}{N_r + (s+1)Fi(x_1, y, s)} \right\}
\]

\[
\left[ N_r + (s+1)Fi'(x_1, y, s) - x_1 \left[ N_r + (s+1)Fi'(x_1, y, s) \right]^{1/(s+1)} \right]
\]

\[
\left/ (s+1) \right.
\]

16
and

\[
v(\epsilon, \epsilon') < \epsilon = -\frac{s+1}{C(k_BT)^s} \int_0^{x'} \frac{dx_1 x_1^s}{(1 + e^{-x_1+y})} \sigma_{Br}\left\{ \frac{s+1}{N_r - (s+1)Fir(x_1, y, s)} \right\}
- \frac{1}{N_r + (s+1)Fir'(x_1, y, s)} \int_0^{x'} \frac{dx_1}{1 + e^{-x_1+y}} \cdot [N_r + (s+1)Fir'(x_1, y, s)]^{s/(s+1)}
+ \frac{s+1}{C(k_BT)^s} \int_{x'}^x \frac{dx_1}{1 + e^{-x_1+y}} \cdot [N_r + (s+1)Fir'(x_1, y, s)]^{s/(s+1)} - x_1^s.
\]

(54)

### 5.2.2 Low energy expressions for \( s \neq 0 \)

All the formulas in Section 5.2.1 are very difficult to calculate numerically. Therefore, low energy asymptotic expressions can be very useful in calculations. To find the low energy behavior of the functions involved, we calculate the Taylor expansions of the dimensionless quantities around \( \epsilon = 0 \) (\( x = 0 \)).

We start with the integrals over Fermi distributions,

\[
Fir(x, y, s) = \int_0^x \frac{x_1^s dx_1}{e^{x_1-y}+1} \approx \frac{1}{e^{-y}+1} \cdot \frac{x^{s+1}}{s+1} - \frac{1}{(e^{-y}+1) \cdot (e^y+1)} \cdot \frac{x^{s+2}}{s+2}
\]

(55)

and

\[
Fir'(x, y, s) = \int_0^x \frac{x_1^{s+1} dx_1}{e^{x_1-y}+1} \approx \frac{1}{e^y+1} \cdot \frac{x^{s+1}}{s+1} + \frac{1}{(e^{-y}+1) \cdot (e^y+1) \cdot s+2}
\]

\[
= \frac{x^{s+1}}{s+1} - Fir(x, y, s),
\]

(56)

for \( x \ll 1 \). Using Eqs. (55) and (56) we get asymptotic expressions for all the other variables. The Bose DOS is

\[
\sigma_{Br}(x, y, s) \approx \frac{N_r^{-1/(s+1)} \cdot \left[ N_r - \frac{x^{s+1}}{e^{-y}+1} \cdot \frac{e^{-y}}{N_r} \right]^{s/(s+1)} \cdot x^{s+1} \cdot \frac{N_r^{-1/(s+1)} \cdot e^{-y}}{N_r} \cdot \left[ 1 + \frac{e^{-y}}{x} \right]}{N_r^{s/(s+1)}} + \theta(x) \cdot \frac{N_r^{s/(s+1)} - \theta(-s) \cdot x^{s+1} \cdot \frac{N_r^{-1/(s+1)} \cdot e^{-y}}{N_r} \cdot \left[ 1 + \frac{e^{-y}}{x} \right] \cdot x^{s+1}}{N_r^{s/(s+1)}}.
\]

(57)

where \( \theta(x) \) is the Heaviside step function. At \( x = x_B = 0 \), \( \sigma_{Br} = N_r^{s/(s+1)} \), so it attains a finite value, even if \( \sigma \) is either zero or infinite. \( N_r^{s/(s+1)} \) is the average density of the Fermi gas, from zero to \( \epsilon_F \). If \( s > 0 \), \( \sigma_{Br} \) is increasing with \( x \), while for \( s < 0 \), \( \sigma_{Br} \) is decreasing. This is a general result, since already from equation (3) one can observe that \( \sigma_B \) is monotonic for monotonic \( \sigma \).
Bose energies are:

\[
\epsilon_B \approx (s + 1)k_B T \int_0^x \frac{dx_1 x_1^s}{1 + e^{-x_1+y}} \left[ N_r + \frac{x_1^{s+1}}{e^{y+1}} \right]^{1/(s+1)} - x_1
\]

\[
= \frac{k_B T N_r^{-s/(s+1)}}{e^y + 1} \left\{ x^{s+1} \theta(s) \cdot x^{s+2} s + 1 \frac{1}{s+2} \left( \frac{1}{N_r^{1/(s+1)}} - \frac{1}{e^{-y+1}} \right) \right. \\
+ \left. \frac{\theta(-s)}{2} e^{2s+2} \left( 1 + e^{-y} \right) \right\}.
\]

In the lowest order, for both, \(s < 0\) and \(s > 0\), \(x_B\) have the same expression:

\[
x_B \equiv \frac{\epsilon_B}{k_B T} \approx \frac{N_r^{-s/(s+1)}}{e^y + 1} \cdot x^{s+1}
\]

By eliminating \(x\) from Eqs. (59) and (57) we obtain the asymptotic expression of \(\sigma_B(\epsilon_B)\),

\[
\sigma_B(x, y, s) \approx N_r^{s/(s+1)} + \theta(s) \cdot N_r^{(s+1)^2} (e^y + 1)^{1/(s+1)} x^{1/(s+1)}
\]

\[
- \theta(-s) N_r^{x^{s+2}} e^y \cdot \left[ \frac{s}{1 + e^{-y}} \right] x_B.
\]

So \(\sigma_B(x, y, s)\) decreases linearly with \(x_B\) for \(s < 0\), while for \(s > 0\) it has an infinite positive slope at \(x_B = 0\).

The expressions for the interaction potential are more complicated. Let us start with \(x' \geq x\). After some calculations we arrive to the following results:

\[
v(x, x' \geq x) = -\frac{s + 1}{C(k_B T)^s} \cdot \frac{N_r^{2s+1}}{e^y + 1} \left[ s x^{s+1} + x^{s+2} \frac{1}{s+1} \left( \frac{s}{1 + e^{-y}} - \frac{s + 1}{N_r^{s+1}} \right) \right],
\]

for \(s > 0\)

and

\[
v(x, x' \geq x) = -\frac{1}{C(k_B T)^s} \cdot \frac{N_r^{2s+1}}{e^y + 1} \left\{ s x^{s+1} + \frac{2s + 1}{2(s+1)} \left( 1 + \frac{e^{-y}}{s+1} \right) \right. \\
\left. \times \frac{x^{2s+2}}{N_r(e^{-y}+1)} \right\}, \text{ for } s < 0.
\]

Obviously, \(v(x, x' \geq x) \to 0\) as \(x \to 0\), since \(\epsilon_B = 0\) is the ground state for any configuration.
Figure 4: The Bose quasiparticle energy, $\epsilon_B$, vs. $\epsilon$, for (a) 1D ($s = -1/2$) and (b) 3D ($s = 1/2$) systems, and the Bose DOS, $\sigma_B(\epsilon_B)$ for the same systems. In all the cases we take $\epsilon_F = 10k_B T_0$, where $T_0$ is a scaling temperature. The four curves in each plot correspond to four values of $T$: $0.1T_0$ (solid line), $T_0$ (dotted line), $2T_0$ (dashed line), and $4T_0$ (dash-dot line).

For $x > x'$,

$$v(x,x' < x) = \frac{s + 1}{N_r N_r} \frac{N_r^{s+1}}{e^y + 1} \left[ (x-x') + \frac{x^2 - (x')^2}{2(1 + e^{-y})} - \frac{x^{s+1}}{s+1} \right] + (x')^{s+1}, \quad \text{for } s > 0,$$

and

$$v(x,x' < x) = \frac{N_r^{s+1}}{2(s+1)N_r(1 + e^{-y})} \left[ x^{s+1} - (x')^{s+1} - (x - x') \cdot (s + 1) \right] \times \frac{2s + 1}{2s + 1} \frac{(x')^{2s+2}}{N_r(e^{-y} + 1)},$$

for $s < 0$.

In both equations (61) and (62) only the two lowest orders are correct, whichever these are, depending on $s$. Again, note that $v(x,x' < x) \to 0$ as $x \to 0$, for any $s$.

In Figs. (a) and (b) it is shown the dependence of $\epsilon_B$ on $\epsilon$ for 1D and 3D gases and four different temperatures. The dependence of $\sigma_B$ on $\epsilon_B$ is
Figure 5: For a 1D Fermi gas \( s = -1/2 \): (a) The population of the bosonic energy levels, \( b_F[\tilde{\epsilon}_B(\epsilon)] \) and \( b_B[\tilde{\epsilon}_B(\epsilon)] \), for \( y = \mu/(k_B T) = 10 \) (1) and \( y = \mu/(k_B T) = 20 \) (2). For both values of \( y \), the curves corresponding to \( b_F \) and \( b_B \) are indistinguishable. (b) The bosonic effective quasiparticle energy, \( \tilde{\epsilon}_B(\epsilon) \), for the same values of \( y \) as in (a).

shown in figure 4 (c) and (d) for the same gases and temperatures. The four curves in each of the plots correspond to four different temperatures, namely \( 0.1T_0, T_0, 2T_0, \) and \( 4T_0 \), where \( T_0 \) is a scaling temperature and \( \epsilon_F \) is chosen so that \( \epsilon_F = 10k_B T_0 \).

5.3 THE EQUIVALENCE OF THE BOSE AND FERMI DESCRIPTIONS

Having now the bosonic picture of our Fermi system, built based on the Fermi distribution, we can analyze the bosonic ensemble as in Section 4. Using the results of Section 5.2 we calculate \( b \) by both formulas (24) and (33) and compare the results. To avoid confusion, we denote by \( b_F \) and \( b_B \) the populations given by equations (24) and (33), respectively. Since the two systems are thermodynamically equivalent, \( b_F \) and \( b_B \) should be identical in the thermodynamic limit. This identity appears to be difficult to prove analytically, so we calculated numerically \( \tilde{\epsilon}_B(\epsilon) \) and with this, \( b_B \), which we then compared with \( b_F \). As one can see in figure 5, \( b_F[\tilde{\epsilon}_B(\epsilon)] \) and \( b_B[\tilde{\epsilon}_B(\epsilon)] \) are equal within the numerical accuracy, for any \( \mu/(k_B T) \). (Note: we plotted the numerical results for \( s = -1/2 \) because for this value of \( s \) the results are much more accurate and the multiple integrals involved converge for a wider range of chemical potentials and single particle energy levels.)
6 CONCLUSIONS

In this paper it is introduced the basic idea and results of what have been called exclusion statistics transformation (EST). The EST from Fermi to Bose systems is done in the following way: the fermions are pictured as piles of particles that occupy completely intervals along the single particle energy axis (no empty single particle states in each of the intervals), which are then rotated into horizontal position by associating to all of them a single “Bose” energy (see figure 1). Using EST we transformed an ideal Fermi gas into a Bose gas with the same excitation spectrum. Having the same excitations, the two gases are thermodynamically equivalent by construction [6], i.e. if the Fermi gas has entropy $S_F$, internal energy $U_F$ and particle number $N$, then the Bose gas has the same entropy at any temperature as the Fermi gas, $S_B = S_F$. The internal energy of the Bose gas is $U_B = U_F - U_{g.s}$, where $U_{g.s.}$ is the energy of the Fermi system at zero temperature.

Having constructed in this way the Bose gas, one can calculate all the thermodynamic quantities of the Bose and Fermi gases independently, by maximizing the partition functions with respect to the quasiparticle levels populations. Based on the thermodynamic equivalence, all the canonical properties of the two gases should be identical and, moreover, applying EST, the Fermi distribution should transform into the Bose distribution and vice-versa. We checked this identity numerically, by calculating the Bose and Fermi populations. The populations transform indeed into each-other by EST within the numerical accuracy. So the thermodynamic equivalence is not applicable only to a very special class of systems, namely ideal systems with the same, constant, DOS, but should be regarded as a very general concept. We have now a method to transform gases of a given exclusion statistics into equivalent gases of some other exclusion statistics.

Although this is not discussed here, at low temperatures the Bose system condenses and the Bose condensate is related by EST to the particles in the Fermi system that stay on the lowest energy levels and occupy completely an energy interval. By analogy to the Bose system, these particles form the so called Fermi condensate [9]. Using this analogy, the Fermi system may be described as consisting of a condensate of, say $N_0$ particles, and with $N - N_0$ particles above the condensate. The condensate forms in interacting systems Ref. [8, 10], but also in ideal systems [9]. At low temperatures, when the condensate lies close the the Fermi energy, the single particle spectrum of the “thermally active” $N - N_0$ particles may be approximated as constant and the EST becomes very simple (in any number of dimensions), since it reduces to the EST in systems of constant DOS [8].

It is well known from Bose systems that the properties of the condensate are strongly influenced by the dimensionality of the system (see for example
but the collective properties of the condensate, which certainly deserve investigation, were omitted in this paper.

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