ON TRACE THEOREMS AND POINCARÉ INEQUALITY
FOR ONE-DIMENSIONAL SOBOLEV SPACES

BIENVENIDO BARRAZA MARTÍNEZ, JONATHAN GONZÁLEZ OSPINO,
AND JAIRO HERNÁNDEZ MONZÓN

Abstract. In these notes, we present versions of trace theorems for Sobolev spaces over an interval in the real line, and also a one-dimensional version of the well-known Poincaré inequality.

1. Introduction

The aim of these notes is to provide particular results relative to traces of functions in Sobolev spaces as well as a generalized version of Poincaré’s inequality, all in one dimension, that can be cited in academic works where such results are needed for the analysis of transmission problems that model the dynamics of coupled structures in one dimension. Of course, such results already exist in the classical literature on Sobolev spaces (see e.g. [1, 2]), but mostly in multidimensional form, and we believe that it may be practical to have such concrete one-dimensional results at hand.

Throughout this notes let \( a \) and \( b \) be real numbers with \( a < b \) and \( \lambda := b - a \). For \( m \in \mathbb{N} \) and \( 1 \leq p < \infty \) let \( H^{m,p}(a,b) \) be defined as the completion of \( C^m([a,b]) \) with respect to the norm

\[
\|u\|_{m,p} := \left( \sum_{j=0}^{m} \int_a^b |u^{(j)}(x)|^p \, dx \right)^{1/p},
\]

where \( C^m([a,b]) \) denotes the set of all complex valued \( m \) times continuously differentiable functions in \([a,b]\), and \( u^{(j)} \) is the derivative of order \( j \) of \( u \).

The elements belonging to \( H^{m,p}(a,b) \) are identified with functions in \( L^p(a,b) \) whose derivatives up to order \( m \) also belong to \( L^p(a,b) \) (see [1, Chapter 3]).

As usual, if \( p = 2 \) we write \( H^m(a,b) \) instead \( H^{m,2}(a,b) \) and we will write also \( \| \cdot \|_m \) instead \( \| \cdot \|_{m,2} \).

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2. Trace theorems for Sobolev spaces in one dimension

Next we will present versions, in the one dimensional case, of the well known trace theorems for Sobolev spaces $H^m(a,b)$, with $m \in \mathbb{N}$, endowed with its usual norm. For this, let us at first make precise some concepts.

**Definition 2.1.** We define the space $L^2(\{a,b\})$ as the space of all functions $\varphi : \{a,b\} \to \mathbb{C}$ endowed with the norm given by

$$
\|\varphi\|_{L^2(\{a,b\})} := \left( |\varphi(a)|^2 + |\varphi(b)|^2 \right)^{1/2}.
$$

**Proposition 2.2.** $(L^2(\{a,b\}), \| \cdot \|_{L^2(\{a,b\})})$ is a Banach space.

Proof. Let $(\varphi_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $L^2(\{a,b\})$. Then $(\varphi_n(a))_{n \in \mathbb{N}}$ and $(\varphi_n(b))_{n \in \mathbb{N}}$ are Cauchy sequences in $\mathbb{C}$. Therefore, there exist $\alpha, \beta \in \mathbb{C}$ with $\varphi_n(a) \to \alpha$ and $\varphi_n(b) \to \beta$ in $\mathbb{C}$, when $n \to \infty$. Now we define $\varphi : \{a,b\} \to \mathbb{C}$ by $\varphi(a) := \alpha$ and $\varphi(b) := \beta$. Then we have $\|\varphi_n - \varphi\|_{L^2(\{a,b\})} \to 0$ when $n \to \infty$. □

Now we can establish the following trace theorem.

**Theorem 2.3 (Trace theorem).** The operator

$$
(2.1)
$$

$$
u \mapsto u|_{\{a,b\}} : C^1([a,b]) \to L^2(\{a,b\})
$$

admits a unique extension to a bounded linear operator

$$T_0 : H^1(a,b) \to L^2(\{a,b\}).
$$

Proof. For $u \in C^1([a,b])$ and $a \leq \tau \leq b$ we have

$$u(a) = -\int_a^\tau u'(s)ds + u(\tau).
$$

Then, due to triangular and Cauchy-Schwarz inequalities, we obtain

$$
(2.2)
|u(a)| \leq \int_a^b |u'(s)|ds + |u(\tau)| \leq \sqrt{\lambda} \left( \int_a^b |u'(s)|^2ds \right)^{1/2} + |u(\tau)|.
$$

Therefore Young inequality yields

$$
|u(a)\|^2 \leq 2\lambda \int_a^b |u'(s)|^2ds + 2|u(\tau)|^2,
$$

$a \leq \tau \leq b$.

Now, by integration with respect to $\tau$ in $[a,b]$ in the last inequality, we have

$$
\lambda |u(a)|^2 \leq 2\lambda \int_a^b |u'(s)|^2ds + 2 \int_a^b |u(\tau)|^2 d\tau
$$

and then

$$
(2.3)
|u(a)|^2 \leq 2\lambda \int_a^b |u'(s)|^2ds + \frac{2}{\lambda} \int_a^b |u(\tau)|^2 d\tau.
$$

Noting that $u(b) = \int_a^b u'(s)ds + u(\tau)$ for $a \leq \tau \leq b$, we obtain similarly to (2.2) that

$$
|u(b)| \leq \sqrt{\lambda} \left( \int_a^b |u'(s)|^2ds \right)^{1/2} + |u(\tau)|
$$
and therefore we obtain again
\begin{equation}
|u(b)|^2 \leq 2\lambda \int_a^b |u'(s)|^2 \, ds + \frac{2}{\lambda} \int_a^b |u(\tau)|^2 \, d\tau.
\end{equation}

Summing (2.3) and (2.4) we have
\begin{equation}
|u(a)|^2 + |u(b)|^2 \leq 4\lambda \int_a^b |u'(s)|^2 \, ds + \frac{4}{\lambda} \int_a^b |u(\tau)|^2 \, d\tau
\end{equation}
and therefore
\begin{equation}
\|u\|_{L^2(a,b)} \leq C\lambda u\|_{H^1(a,b)}
\end{equation}
for all \(u \in C^1([a,b])\), where, in this case\(^1\), \(\lambda := \left(\max\left\{4\lambda, \frac{4}{\lambda}\right\}\right)^{1/2}\).

Since \(C^1([a,b])\) is dense in \(H^1(a,b)\) (indeed \(H^1(a,b)\) is the completion of \(C^1([a,b])\) with respect to the usual norm in \(H^1(a,b)\)), there exists a unique linear extension \(T_0 : H^1(a,b) \to L^2([a,b])\) of (2.1) from \(C^1([a,b])\) to \(H^1(a,b)\), satisfying
\begin{equation}
\|T_0 u\|_{L^2([a,b])} \leq C\lambda \|u\|_{H^1(a,b)} \quad (u \in H^1(a,b)).
\end{equation}
Then \(T_0 : H^1(a,b) \to L^2([a,b])\) is a bounded linear operator and \(T_0 u = u|_{[a,b]}\) for all \(u \in C^1([a,b])\).

\[\square\]

Remark 2.4. The operator \(T_0 : H^1(a,b) \to L^2([a,b])\) is called Trace operator of order zero. Usually this operator is also denoted \(\gamma_0\).

This last result can be generalized to traces of higher order.

**Theorem 2.5** (Higher order traces). Let \(m \in \mathbb{N}, m > 1\). The operator
\begin{equation}
u \mapsto (u|_{[a,b]}, u'|_{[a,b]}, \cdots, u^{(m-1)}|_{[a,b]} : C^m([a,b]) \to \prod_{j=0}^{m-1} L^2([a,b])
\end{equation}

admits a unique extension to a bounded linear operator
\[T_{m-1} : H^m(a,b) \to \prod_{j=0}^{m-1} L^2([a,b])\]
with \(T_{m-1} u := (\gamma_0 u, \gamma_1 u, \cdots, \gamma_{m-1} u)\), where \(\gamma_j u = u^{(j)}|_{[a,b]}\) for \(j = 0, 1, \ldots, m-1\), and \(u \in C^m([a,b])\). Here \(\prod_{j=0}^{m-1} L^2([a,b])\) is the (cartesian) product of \(m\) copies of \(L^2([a,b])\), endowed with the usual product topology. \(T_{m-1}\) is called trace operator of order \(m - 1\).

\[\text{1From now on, } C\lambda \text{ will denotes several positive constants, which will only depend on } \lambda.\]
Proof. Let \( u \in C^m([a, b]) \). Analogously to (2.5), for \( k \in \{0, 1, \ldots, m - 1\} \) we have
\[
|u^{(k)}(a)|^2 + |u^{(k)}(b)|^2 \leq C^2 \left( \int_a^b |u^{(k+1)}(t)|^2 \, dt + \int_a^b |u^{(k)}(\tau)|^2 \, d\tau \right).
\]
We recall that \( C^2 \) denotes several constants, which depend only on \( \lambda \). Then, adding these inequalities, we get
\[
\sum_{k=0}^{m-1} \left( |u^{(k)}(a)|^2 + |u^{(k)}(b)|^2 \right) \leq C^2 \sum_{j=0}^{m-1} \int_a^b |u^{(j)}(t)|^2 \, dt.
\]
That is,
\[
\sum_{k=0}^{m-1} \|u^{(k)}\|_{L^2([a, b])}^2 \leq C^2 \sum_{j=0}^{m-1} \|u^{(j)}\|_{L^2([a, b])}^2.
\]
Therefore,
\[
\left\| (u|_{\{a, b\}}, u'|_{\{a, b\}}, \ldots, u^{(m-1)}|_{\{a, b\}}) \right\|_{\prod_{j=0}^{m-1} L^2([a, b])} \leq \left( \sum_{k=0}^{m-1} \|u^{(k)}\|_{L^2([a, b])}^2 \right)^{1/2} \leq C^2 \left( \sum_{j=0}^{m-1} \|u^{(j)}\|_{L^2([a, b])}^2 \right)^{1/2} = C^2 \|u\|_{H^m([a, b])},
\]
for all \( u \in C^m([a, b]) \). The estimates above shows that
\[
(2.9) \quad \|T_{m-1}u\|_{\prod_{j=0}^{m-1} L^2([a, b])} \leq C^2 \|u\|_{H^m([a, b])} \quad (u \in C^m([a, b])).
\]
Since \( C^m([a, b]) \) is dense in \( H^m(a, b) \) \( (H^m(a, b) \) is the completion of \( C^m([a, b]) \) with respect to the usual norm in \( H^m([a, b]) \), there exists a unique operator \( T_{m-1} : H^m(a, b) \to \prod_{j=0}^{m-1} L^2([a, b]) \) which extends (2.8) from \( C^m([a, b]) \) to \( H^m(a, b) \), satisfying
\[
\|T_{m-1}u\|_{\prod_{j=0}^{m-1} L^2([a, b])} \leq C^2 \|u\|_{H^m(a, b)} \quad (u \in H^m(a, b)).
\]
If we define \( T_{m-1}u := (\gamma_0 u, \gamma_1 u, \ldots, \gamma_{m-1} u) \) for \( u \in H^m(a, b) \), then we have, as extension of (2.8), that \( \gamma_j u = u^{(j)}|_{\{a, b\}} \) for \( j = 0, 1, \ldots, m-1 \), and \( u \in C^m([a, b]) \). \( \square \)

**Remark 2.6.** Since \( m > (m-1) + 1 \), due to the Sobolev imbedding Theorem, we have \( H^m(a, b) \hookrightarrow C^{m-1}([a, b]) \). Therefore, for \( u \in H^m(a, b) \) we can take its \( C^{m-1} \) representative and then, the traces \( u|_{\{a, b\}}, u'|_{\{a, b\}}, \ldots, u^{(m-1)}|_{\{a, b\}} \), exist in the classical sense. That is, it holds also
\[
T_{m-1}u = (u|_{\{a, b\}}, u'|_{\{a, b\}}, \ldots, u^{(m-1)}|_{\{a, b\}}).
\]
for all $u \in H^m(a,b)$.

3. **Poincare inequality in one dimension**

In this section we show a generalized version of the well known Poincare inequality in the case of dimension one.

**Theorem 3.1** (Poincare inequality). There exists a positive constant $C_P$ such that for $x_0 \in \{a, b\}$ it holds

$$||u||_{L^2(a,b)} \leq C_P \left(||u'||_{L^2(a,b)} + |T_0 u(x_0)|\right)$$

for all $u \in H^1(a,b)$, where $T_0 : H^1(a,b) \to L^2(\{a, b\})$ is the trace operator of order zero given in Theorem 2.3.

**Proof.** Without loss of generality let us suppose that $x_0 = a$ (the proof for $x_0 = b$ is similar). Then, for $u \in C^1([a, b])$ and $a \leq x \leq b$, we have

$$u(x) = \int_a^x u'(t) \, dt + u(a).$$

Therefore, applying triangular and Cauchy-Schwarz inequalities, we obtain

$$|u(x)| \leq \int_a^x |u'(t)| \, dt + |u(a)|
\leq \int_a^b |u'(t)| \, dt + |u(a)|
\leq \sqrt{\lambda} \left(\int_a^b |u'(t)|^2 \, dt\right)^{1/2} + |u(a)|.$$  

Then, Young inequality yields

$$|u(x)|^2 \leq 2\lambda \int_a^b |u'(t)|^2 \, dt + 2|u(a)|^2, \quad a \leq x \leq b.$$  

We integrate both sides of the last inequality with respect to $x$ over the interval $[a, b]$ and obtain

$$\int_a^b |u(x)|^2 \, dx \leq 2\lambda^2 \int_a^b |u'(t)|^2 \, dt + 2\lambda |u(a)|^2.$$  

That is, we have

$$||u||_{L^2(a,b)}^2 \leq 2\lambda^2 ||u'||_{L^2(a,b)}^2 + 2\lambda |u(a)|^2$$

for all $u \in C^1([a, b])$.

Now, let $u \in H^1(a, b)$ and $(u_n)_{n \in \mathbb{N}}$ a sequence of functions belonging to $C^1([a, b])$ such that $u_n \to u$ in $H^1(a, b)$ when $n \to \infty$. In virtue of (3.2), the inequality

$$||u_n||_{L^2(a,b)}^2 \leq 2\lambda^2 ||u'_n||_{L^2(a,b)}^2 + 2\lambda |u_n(a)|^2$$

holds for all $n \in \mathbb{N}$. Due to the convergence of the sequence $(u_n)_{n \in \mathbb{N}}$ to $u$ in $H^1(a, b)$, it follows clearly that

$$||u_n||_{L^2(a,b)}^2 \to ||u||_{L^2(a,b)}^2 \quad \text{and} \quad ||u'_n||_{L^2(a,b)}^2 \to ||u'||_{L^2(a,b)}^2.$$
when $n \to \infty$. For the remaining term in (3.3), note that the Trace theorem (inequality (2.7)) implies
\[
|u_n(a)| - |T_0u(a)| \leq |T_0u_n(a) - T_0u(a)|
= |T_0(u_n - u)(a)|
\leq \|T_0(u_n - u)\|_{L^2(a,b)}
\leq C\|u_n - u\|_{H^1(a,b)} \quad n \to \infty \to 0.
\]
Then, $|u_n(a)| \quad n \to \infty \to |T_0u(a)|$.

Taking $n \to \infty$ in (3.3), we obtain
\[
\|u\|_{L^2(a,b)}^2 \leq 2\lambda^2\|u'\|_{L^2(a,b)}^2 + 2\lambda|T_0u(a)|^2.
\]
It follows that
\[
\|u\|_{L^2(a,b)}^2 \leq 2\lambda^2\|u'\|_{L^2(a,b)}^2 + 2\lambda|T_0u(a)|^2
\leq 2\max \{\lambda, \lambda^2\} \left(\|u'\|_{L^2(a,b)}^2 + |T_0u(a)|^2\right)
\leq 2\max \{\lambda, \lambda^2\} \left(\|u'\|_{L^2(a,b)} + |T_0u(a)|\right)^2
\]
for all $u \in H^1(a,b)$. Taking square root in the last inequality, we obtain (3.1) for $x_0 = a$, with $C_P := \sqrt{2\max \{\lambda, \lambda^2\}^{1/2}} = \sqrt{2\max \{\sqrt{\lambda}, \lambda\}}$. With similar calculations we obtain (3.1) for $x_0 = b$, with the same $C_P$. \hfill \Box

Remark 3.2. The Poincaré inequality (3.1) leads to the so-called Friederichs inequality (see [2], Theorem 1.9.) in dimension one:

\[
\|u\|_{H^1(a,b)} \leq \text{const} \left(\|u'\|_{L^2(a,b)}^2 + |T_0u(x_0)|^2\right)^{1/2} \quad (u \in H^1(a,b)),
\]
which in turn implies the equivalence between the usual norm $\| \cdot \|_{H^1(a,b)}$ and the norm in $H^1(a,b)$ given by
\[
u \mapsto \left(\|u'\|_{L^2(a,b)}^2 + |T_0u(x_0)|^2\right)^{1/2} \quad (u \in H^1(a,b)).
\]

Referencias

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B. Barraza Martínez, Universidad del Norte, Departamento de Matemáticas y Estadística, Barranquilla, Colombia
Email address: bbarraza@uninorte.edu.co

J. González Ospino, Universidad del Norte, Departamento de Matemáticas y Estadística, Barranquilla, Colombia
Email address: gjonathan@uninorte.edu.co

J. Hernández Monzón, Universidad del Norte, Departamento de Matemáticas y Estadística, Barranquilla, Colombia
Email address: jahernan@uninorte.edu.co