Synthetic Wavelength Holography: An Extension of Gabor’s Holographic Principle to Imaging with Scattered Wavefronts

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Scattering of light is an elementary process that makes it possible to see the objects around us. However, when light is scattered between object and observer (e.g., due to fog), it poses a serious problem to optical imaging. In this work, we demonstrate how to circumvent the deleterious effects of scattering by exploiting spectral correlations of scattered wavefronts. This allows us to extend the use of optical imaging to conditions where strong scattering would otherwise obstruct a clear view. Our method draws inspiration from Gabor’s attempts to improve the resolving power of electron microscopes. Gabor’s method was to record aberrated wavefronts at electron wavelengths, play this recording back at optical wavelengths, and finally perform an optical aberration correction. Similar to Gabor’s approach, we transfer the problem of aberration correction to a larger ‘Synthetic Wavelength’ by interpreting the wavefront distortion of scattered light as uncharacterized stochastic aberrations. We computationally mix speckle fields recorded at two closely spaced optical wavelengths $\lambda_1, \lambda_2$, and uncover object information by playing back the computationally assembled wavefront at a ‘Synthetic Wavelength’ $\Lambda = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}$. An attractive feature of our method is that it generalizes well to many different types of scattering. Moreover, we show that our method works at the fundamental limits of imaging performance in the presence of scattering. Our findings are applicable to a wide range of wave phenomena, opening up new avenues for imaging with scattered wavefronts.

Introduction

In his 1971 acceptance speech for the Nobel Prize in Physics, Denis Gabor spoke of the moment that led to his discovery of the holographic imaging principle:

"After pondering this problem for a long time, a solution suddenly dawned on me, one fine day at Easter 1947, ... Why not take a bad electron picture, but one which contains the whole information, and correct it by optical means? The electron microscope was to produce the ... interference pattern I called a 'hologram', from the Greek word 'holos' - the whole, because it contained the whole information. The hologram was then reconstructed with light, in an optical system which corrected the aberrations of the electron optics" [1].

Central to Gabor’s award-winning research were two innovative ideas. The first is the notion that an interferogram acquired at electron wavelengths provides a complete (‘whole’ or 3D) representation of atomic structure, warranting the designation of ‘hologram’. This notion of imaging using interferometric principles laid the foundations for a subsequent revolution in holography, using a variety of wave phenomena including electromagnetic radiation, acoustic waves, and others. Although Gabor’s original interpretation of holography was largely restricted to a single-wavelength, it has since been extended to accommodate multiple wavelengths, and in the process ushered a revolution in high-accuracy optical metrology [2–6].

The second innovation in Gabor’s pioneering work was an Analysis/Synthesis paradigm that combined wavefront acquisition (Analysis) at a smaller wavelength with wavefront correction/reconstruction (Synthesis) at a larger wavelength. Gabor utilized this idea to correct for uncompensated spherical aberration in his electron wavelength holograms, using optical lenses designed for visible wavelengths [7]. His optical aberration correction has since been replaced by digital wavefront correction, but the notion of holographic Analysis-and-Synthesis has endured and is used in this paper to provide deeper insights on fundamental limits of imaging.

Synthetic Wavelength Holography (SWH)

The present work builds on Gabor’s holographic principle with the specific goal of imaging under extensive scatter. The connection to Gabor’s Analysis/Synthesis paradigm is detailed below (and illustrated in Fig. 1):

- **Analysis**: we record optical wavefronts at two closely spaced wavelengths $\lambda_1$ and $\lambda_2$, each of which is susceptible to scattering. The physical process of scattering may be interpreted as an unknown randomized aberration. In each individual hologram $E(\lambda_1)$ or $E(\lambda_2)$, it irreversibly corrupts the phase of the optical field and destroys any information about the object.

- **Aberration Correction**: we exploit spectral correlations in the recorded optical fields to computationally assemble a ‘Synthetic Wavelength Hologram’ (SWH)
Fig. 1. Our method of ‘Synthetic Wavelength Holography’ is inspired by Gabor’s idea of Analysis, Synthesis, and Correction for improving the resolution of Electron Microscopes. Left: Gabor envisioned recording an electron wavefront with aberrated electron microscope optics (Analysis, wavelength $\lambda_1$), then reconstructing this electron image by playing the hologram back with an optical wavefront (Synthesis, wavelength $\lambda_{opt} > > \lambda_1$) while exploiting optical wavefront correction (Correction, wavelength $\lambda_{opt}$) [1]. Right: In Synthetic Wavelength Holography, we adopt Gabor’s initial idea to correct unknown wavefront aberrations $\Psi$ introduced when visible light is transported through scenes with strong scattering. We capture two holograms at two closely spaced wavelengths (Analysis, wavelengths $\lambda_1$ and $\lambda_2$) each showing random aberrations. By computationally beating the two signals together, we produce a low frequency ‘Synthetic Wavelength Hologram’ (Correction). The Synthetic Wavelength Hologram is not subject to aberrations and contains information on the order of a ‘Synthetic Wavelength’ $\Lambda$, which is the beat wavelength of $\lambda_1$ and $\lambda_2$. Similar to Gabor’s idea, the object is reconstructed by playing back the computationally corrected hologram with the much larger Synthetic Wavelength $\Lambda$ (Synthesis).

$E(\Lambda) = E(\lambda_1) E^*(\lambda_2)$, which contains a phase that is virtually impervious to the effects of scattering at the optical wavelengths $\lambda_1, \lambda_2$. It is demonstrated that the SWH encapsulates field information at a ‘Synthetic Wavelength’ (SWL) $\Lambda = \frac{\lambda_1 \cdot \lambda_2}{|\lambda_1 - \lambda_2|} > > \lambda_1, \lambda_2$.

- **Synthesis:** we digitally play back the SWH at the longer SWL $\Lambda$ to uncover object information that cannot be retrieved at the optical wavelength $\lambda_{1,2}$.

The principal distinction between Gabor’s original approach and the one proposed here, lies in the recording of holograms at multiple wavelengths, the computational compensation of unknown aberrations, and the digital replay of the recorded hologram.

**SWH and Fundamental Limits of Imaging with Scattered Wavefronts**

An important question is just how much information can be recovered by exploiting spectral correlations in the scattered optical fields? Adolf Lohmann was one of the first who applied information theoretical concepts to characterize optical systems [8–10]. Lohmann used the ‘Space-Bandwidth Product’ (SBP) to calculate the performance of an imaging system, then studied this performance as a function of important physical parameters (e.g., size, weight, and cost). This approach has been widely adopted to study a broad variety of imaging modalities [11–16] including holography [17, 18]. The SBP reflects a fundamental tradeoff between the Field-of-View (FOV) $W$ and the lateral resolution $\delta x^{-1}$ of an imaging system. The SBP of a hologram is defined as the product of the physical extent $D$ of the hologram and its spatial frequency bandwidth $2v_x$, which is limited by the highest resolvable frequency in the hologram. In accordance with the above definition the SBP may be also viewed as a measure of the number of resolvable spots in the reconstructed hologram, defined by $W/\delta x$. This yields the following expression for the SBP of a hologram (analog in $y$-direction):

$$SBP = \frac{W}{\delta x} = 2Dv_x$$  \hspace{1cm} (1)

The SBP is bounded by the highest spatial frequency in a propagating field as fundamentally limited by the inverse of the wavelength [19], so that $max(v_x) = \lambda^{-1}$. However, information at spatial frequencies within this band limit is preserved only when the wavefront aberrations are negligible. For wavefront aberrations in an optical imaging system, Lord Rayleigh [20] theorized that the maximum tolerable wavefront error $\Psi_{max}$ can not exceed one quarter of the optical wavelength. Rayleigh’s view has been repeatedly confirmed by optical designers, and commonly referred to as the ‘Rayleigh Quarter Wavelength Rule’ (RQWR) [19–21]:

$$\Psi_{max} \leq \frac{\lambda}{4}$$  \hspace{1cm} (2)

For a scattering process, the maximal wavefront error $\Psi_{max}$ represents the worst case Optical Path Difference (OPD) of the numerous scattered light paths that share a common source location, object location and detector pixel. In view of this definition, it is not surprising that the RQWR is violated by scattering processes at optical wavelengths, such as light bouncing off walls (height fluctuations $\sigma_h >> \lambda$), and light propagation through scattering media like fog or tissue (thickness $L > transport mean free path \ell^* >> \lambda$). The result is randomized interference which induces speckle-artifacts in the object wave that propagates toward the hologram. The
speckle irreversibly corrupts the phase captured in the hologram, making it impossible to reconstruct the object at the recording wavelength. The problem may be bypassed by recording fields at much longer wavelengths, such as ultrasound and Radar. However, the reflectance phenomenology at such wavelengths can be vastly different than at visible wavelengths, and spatially resolved detectors are currently most prevalent at optical wavelengths. The ability to simultaneously accommodate scatter and record reflectance phenomenology at visible shorter wavelengths has remained a longstanding problem in optical imaging.

In this work, we show that Gabor’s Analysis/Synthesis paradigm provides a blueprint for combining the best attributes of holographic imaging at shorter wavelengths with the immunity to scatter afforded by longer wavelengths. We demonstrate that computational mixing of speckle fields recorded at two closely spaced optical wavelengths $\lambda_1, \lambda_2$ preserves phase information at scales comparable to the SWL $\Lambda = \frac{\lambda_1 \lambda_2}{|\lambda_1 - \lambda_2|} \gg \lambda_1, \lambda_2$, provided that the SWL $\Lambda$ fulfills the RQWR requirement of Eq. 2:

$$\frac{1}{\Lambda} \geq \Psi_{\text{max}} \gg \frac{\lambda_1}{4}, \frac{\lambda_2}{4}$$  \hspace{1cm} (3)

The simplicity of this observation is remarkable given the mathematical complexity of analyzing spectral correlations for light scattered by a disordered medium. The existence of such correlations is well documented [22–34], albeit in the ensemble sense. Experiments describing spectral correlation for a single realization of disorder is available in [35–37]. The supplementary material puts forth mathematical arguments supporting the existence of RQWR (Eq. 3) for a single realization of a surface scattering process (see Section 1.6). It requires that the change in optical path length induced by a small change in the optical frequency is negligibly small for ray paths that share a common source location, object location and detector pixel [38–40]. The argument may be extended to accommodate volumetric disorder by adopting a diffusive approach to light propagation [33].

The relevance of the RQWR (Eq. 3) to imaging in the presence of scatter, emerges in its ability to define the smallest physical and also synthetic wavelength that is unaffected by scattering.

The synthetic wave, although a computational construct, has distinct characteristics that it shares with a physical wave at the respective wavelength $\Lambda$. For example, in the presence of scattering it starts to form speckle if the RQWR of Eq. 3 is violated. Consequently, we can relate the SBP of the SWH to the maximum spatial frequency:

$$\nu_x = \frac{1}{\Lambda} \leq \frac{1}{4 \Psi_{\text{max}}}$$  \hspace{1cm} (4)

Incorporating Eq. 4 into the definition of the SBP in Eq. 1 yields an upper bound on the the maximum SBP that can be achieved:

$$\text{SBP} = \frac{W}{\delta x} \leq \frac{D}{2 \Psi_{\text{max}}}$$  \hspace{1cm} (5)

Eq. 5 represents an uncertainty relation that is intrinsic to imaging with scattered wavefronts. It captures the tradeoff between the achievable FOV $W$ and lateral resolution $\delta x^{-1}$. At the same time, it represents an upper bound for the maximum SBP that can be attained by any scheme for imaging with scattered wavefronts following the laws of linear optics. Methods operating close to this limit are as good as physics allows and cannot be improved by the use of better sources, detectors or algorithms.

The SBP limit of Eq. 5 is directly linked to the severity of scatter. Increased scatter exaggerates the wavefront error $\Psi_{\text{max}}$ by increasing the statistical spread in light path length distributions. For surface scattering, it can be shown that the spread in path lengths is fundamentally limited by $2\sigma_h$, where $\sigma_h$ represents the RMS surface roughness (see supplementary material). For volumetric scattering, the spread in path lengths is given by $2L^2/\pi$, where $L$ denotes the thickness of the scattering medium and $\ell^*$ denotes the transport mean free path [31, 41] (the factor 2 accommodates round trip propagation through the scattering medium).

Due to the dependence of the SBP on the severity of the scatter, it is especially important that we are able to adapt to varying degrees of scatter. In our approach, this is realized by tuning the SWL so as to fulfill the RQWR, making it possible to operate at the fundamental SBP limit for a variety of scattering conditions (see experimental demonstrations in Figs. 3 and 4).

SWH has a broad range of applications including imaging through scattering and turbid media, imaging through obscurants such as fog and smoke, and ‘Non-Line-of-Sight’ (NLoS) imaging. However, the scale of wavefront error can vary substantially depending on the imaging task. For instance, the typical wavefront error $\Psi_{\text{max}}$ for the surface scattering processes in NLoS imaging is below 1 millimeter, whereas it can be several centimeters for imaging through tissue, and up to meters or even hundreds of meters, for imaging through fog (depending on the transport mean free path $\ell^*$ in the volumetric scatterer).

**SWH and NLoS Imaging**

Recent work in NLoS imaging has demonstrated impressive reconstructions of room sized scenes with down to a few centimeters resolution [42–46]. A subset of these techniques exploits source modulation at radio frequency (RF) timescales, with recent publications demonstrating the connection between RF-ToF measurements and wavefront reconstruction using ‘Phasor fields’ [44–50]. There is, however, a deep performance gap between the SBP that is achieved by these approaches and the fundamental SBP limit, that can be estimated by inserting $\Psi_{\text{max}} = 1 mm$ (an upper estimate for the maximal wavefront aberrations in NLoS imaging) in Eq. 5. The huge gap is caused by technical limitations of currently available electro-optic components such as limited source modulation rate ($< \text{GHz}$), and detector timing jitter ($> 50 \text{ps}$). In this work we provide a mathematical basis and experimental demonstration of how to overcome this technical limitation and achieve imaging performance close to the
 physical limit.

A second class of NLoS imaging techniques exploits spatial correlations in scattered light [40, 51–53]. These techniques are able to recover images of objects obscured from view at much higher resolution (∼100µm at 1 m standoff). However, in agreement with the fundamental limit expressed by Eq. 5, the gain in resolution comes at the price of an extremely limited field of view (< 2°), determined by the angular decorrelation of scattered light (‘memory effect’) [25]. Moreover, in practice, such approaches rely on non-linear phase-retrieval algorithms that make it difficult to achieve the SBP limit of Eq. 5 for general scenes. Furthermore, the limited FOV afforded by these techniques introduce significant challenges in using them to develop practical NLoS implementations.

In stark contrast to previous work in NLoS imaging, we present an approach that reaches the fundamental limit in SBP expressed by Eq. 5, while also providing wide FOV imaging capabilities. The experimental results presented in Figs. 3 - 5 confirms this. The resolving power of the proposed approach can be ascertained by identifying the spot size of the numerically propagated SWH. The theoretical value for the best possible resolving power is obtained by replacing the SWL \( \Lambda \) with its smallest value \( 4\Psi_{\text{max}}z/D \), as prescribed by the RWQR. This leads to:

\[
\delta x \approx \frac{\Lambda z}{D} \geq 4\Psi_{\text{max}} \frac{z}{D} ,
\]

where \( z \) is the standoff distance between the hologram and target planes and \( D \) is the diameter of the holographic detector. Equation 6 expresses an intuitive but striking relationship between the choice of SWL, the degree of scatter, and the highest resolution that can be achieved. It shows that our method can always achieve the best possible resolution for full field approaches by selecting the smallest SWL that satisfies the RQWR of Eq. 3. The experimental results shown in Figs. 3 - 5 demonstrate our ability to recover spatial detail on the obscured target at the physical resolution limit of Eq. 6.

Figure 2 shows our experimental setup for using SWH in NLoS applications. The diffuser designated ‘Virtual Source’ (VS) in the illustration of Fig. 2b scatters light that indirectly illuminates the target obscured from view. The wall to the left of VS intercepts the light scattered by the target. The light is collected by the aperture, which is shown in the bottom of the figure. The lensed fiber arrangement of Fig. 2e is used to launch the reference beam. The optical field emerging from the wall surface designated VD in Fig. 2b, is recorded by the interferometric imaging apparatus of (Interferometer setup in Fig. 2g) via snapshot acquisition. Due to the multiple scattering at rough surfaces, the recorded hologram at the optical wavelength, bears little resemblance to a holographic repre-
Fig. 3. Experimental results for NLoS measurements. a)-j) Imaging the character ‘N’ around the corner at five different SWLs. a)-e) Phase maps of synthetic holograms captured at the VD surface. f)-j) Respective reconstructions. The resolution of the reconstructions increases with decreasing SWL. However, the speckle-artifacts increase due to the decorrelation of the two optical fields at $\lambda_1$ and $\lambda_2$.

k)-p) Reconstruction of a point source around the corner for three different SWLs. k)-m) Phase maps of the synthetic holograms captured at the VD surface. n)-p) Reconstruction of the point source. As in classical optics, the diameter is linearly dependent on the wavelength (in this case the SWL). The experimental value is close to the theoretical expectation. For p), the point source is reconstructed with sub-mm precision.

Figures 3(f-j) demonstrate the ability to reconstruct the cutout of a small character ‘N’ (dimensions $15\text{mm} \times 20\text{mm}$) that is imaged ‘around the corner’. An image of the object is retrieved by digitally replaying the SWH with the SWL $\Lambda$ (see methods section). The resolution of the reconstruction $\delta x^{-1}$ increases with decreasing SWL in accordance with the expression $\delta x = \Lambda \ z / D$ where, $z$ is the propagation distance and $D$ is the size of the VD surface. Notice that the reconstruction exhibits speckle artifacts as the SWL approaches four times the estimated $\Psi_{\text{max}}$ (see columns for $\Lambda \leq 0.56\text{mm}$). A further reduction in the SWL violates the RWQR, resulting in degraded visibility of the obscured object (see Fig. 3j).

A second experiment described in Fig. 3 seeks to quantify the resolution of our NLoS imager. An exposed fiber connector positioned $z = 95\text{mm}$ behind the VD surface (see Fig. 2g) is used as a proxy for a point-source. Holograms at the VD surface acquired with multiple optical wavelengths are processed to recover a multitude of SWHs, each of which is backpropagated by $z = 95\text{mm}$, to obtain an estimate of the imager resolution. The experimentally observed point sizes, shown in Figures 3n-p, are consistent with theoretical predictions (red circles, calculated with VD diameter $D = 58\text{mm}$), and decrease with increasing SWL. For a SWL of $280\mu\text{m}$, we are able to reach sub-millimeter resolution on target. This represents large improvement over competing approaches [44, 45, 49, 50] with comparable angular fields of view.

SWH in Transmissive Scattering Regimes

The experiment shown in Fig. 4 verifies the stated fundamental limits of imaging with scattered wavefronts, disclosed in Eqs. 5 and 6. The experiment was conducted in transmission mode to highlight the versatility of our approach. The holographic reconstructions of the character ‘U’ (dimensions $15\text{mm} \times 20\text{mm}$) are displayed in Fig. 4d-g. The maximum...
wavefront aberration was computed to be $\Psi_{\text{max}} \approx 65 \mu m$, based on knowledge of system parameters such as the size of VD and VS, the grit size of the diffuser and the abrasive material used to sandblast the character. As the SWL approaches 260$\mu m$, we should expect to violate the RQWR and begin to notice speckle artifacts in the reconstructed image. This behavior is apparent in the reconstructions of Figure 4g, leading us to conclude that our SWH imager is able to operate at the fundamental SBP and resolution limits predicted by Eqs. 5 and 6 respectively.

In a fourth experiment, we demonstrate the ability of our SWH technique to image through volumetric scatter. To this end, we swap the transmissive diffuser used in the previous experiment with a 4mm thick milky acrylic plate. The evidence of degraded visibility due to volumetric scatter is apparent in the images of a checkerboard that is viewed through the acrylic plate and a reference 220 grit ground glass diffuser (Figure 4c). Despite pronounced scattering in the acrylic plate, we are able to reconstruct the character ‘U’ for SWLs exceeding 360$\mu m$, as shown in Figs. 4h-k. This suggests the ability to recover image information at visibility levels far below the perceptual threshold. However, a comparison of the reconstructions at the smallest achievable SWLs for the acrylic plate and the diffuser reveals only a marginal change in the maximum wavefront error. The discrepancy may be resolved by recognizing that visibility of ballistic light paths decays exponentially with the propagation distance (in accordance with Beer’s law [54]), whereas the resolution limit of Eq. 6 decays linearly with propagation distance.

Our proposed concept, i.e., exploiting spectral correlations in holographic imaging with scattered wavefronts, is by no means restricted to two wavelengths. The spectral diversity afforded using multiple illumination wavelengths is expected to yield an improvement in the longitudinal resolution, in much the same manner as Optical Coherence Tomography (OCT) [55–58] and White-Light Interferometry (WLI) [59, 60]. However, unlike OCT and WLI, we neither need to match the pathlengths nor the power in the two arms of our interferometric imager which leads to an approach that most closely resembles work by Erons et al. [61] in Fourier Synthesis Holography.

The improved longitudinal resolution afforded by the use of multiple SWLs is demonstrated in the final experiment in Fig 5. The objective is to computationally section a simple multi-planar scene consisting of two characters ‘N’ and ‘U’ (introduced in previous experiments) that are offset in

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Fig. 4. Experimental results for measurements through scattering media. a) Schematic setup. Instead of scattered from a wall, the light is now scattered in transmission. b) Imaged character ‘U’ with dimensions $\sim 15 \text{mm} \times 20 \text{mm}$. c) Scatters used in the imaging path: A 220 grit ground glass diffuser and a milky plastic acrylic plate of $\sim 4 \text{mm}$ thickness, both placed $\sim 1 \text{cm}$ over a checker pattern demonstrate the decay in visibility. d)-g) Reconstructions of measurements taken through the ground glass diffuser. h)-k) Reconstructions of measurements taken through the milky acrylic plate. The character can be reconstructed with impressive quality. The larger OPD in the acrylic plate leads to a faster decorrelation if the SWL is decreased.
Fig. 5. Depth separation of two hidden objects by creating a ‘synthetic pulse train’. a) Target, consisting of two characters with a longitudinal separation of 33 mm. b)-e) Reconstruction of the characters, using only $N = 1$ SWL ($\Lambda = 0.8$ mm). Due to the properties of holographic backpropagation, a separation of the characters in depth is not possible. f)-i) Reconstruction, calculated from coherent superposition of the backpropagated fields at $N = 1$ SWLs. Letters are separable. The pulse distance of the synthesized pulse train can be seen in (h) and (i).

Discussion and Conclusion

This paper has introduced the new method of ‘Synthetic Wavelength Holography’ that is inspired by Gabor’s original principle for wavefront-based Analysis, Synthesis, and Correction. We studied fundamental limits in imaging performance through densely scattering media, and provided experimental demonstration of SWH reconstructions. We used tunable lasers to demonstrate that our method is able to reach the physical limit of imaging performance for a broad range of scattering conditions, just by tuning the SWL to the lowest possible value that does not violate the RQWR. While the experiments in this paper were carried out with baseband frequencies in the optical domain (100s of THz), lock-in detection of our synthetic wavefront is performed at an RF modulation frequency (a few kHz, see methods section). This enables full-field SWH detection in a snapshot using state-of-the-art focal plane array cameras. The benefit of this approach has not been discussed in detail, due to our focus on SBP limits. If, however, it is desired to optimize the ‘Space-Time-Bandwidth-Product’ (STBP), or the Channel Capacity [12, 14], then the snapshot acquisition mode of our SWH implementation is of high value.

The example provided in Fig. 4 clearly demonstrates that our method is able to image through transmissive, scattering media, even when the visibility at the baseband frequency is extremely poor. The approach in its present form is best suited for imaging through thin scattering media ($L > \ell^*$), such as the shown acrylic plate. For thick media ($L \gg \ell^*$), the increased spread of light path lengths distributions (i.e. spread in travel times) severely limits the achievable SWL [62]. The problem may be mitigated by restricting attention to scattered light paths with a prescribed time of travel. This may have important implications for imaging through participating media such as fog, clouds, and rain, a problem of particular importance to Naval surveillance applications, geospatial imaging, climatology research, tissue imaging or imaging deeper into the brain. A specific embodiment of LiDAR that exploits frequency diversity within the detector integration time (FMCW LiDAR [63]), is perfectly suited for the task at hand. By combining the time-gating ability of FMCW LiDAR with the synthetic wavelength principle, it may be possible to see through meters of dense fog, using a synthetic wavelength that is smaller than 1 cm.

The SWH principle described in this paper digressed slightly from Gabor’s original principle because we focused on the problem of correcting unknown or random wavefront aberrations caused by the scattering of light. However, we also envision a scenario where SWH could be used to compensate for aberrations at the SWL, in a manner that is analo-
gous to the use of adaptive optics in astronomical telescopes. In this scenario, wavefront distortions relative to the SWL are measured using a separate wavefront sensing device observing a guide star (or some other known reference), then the aberrations present in a captured SWH image are corrected in post-processing. This would relax the Rayleigh Quarter Wave constraint for the SWL expressed in Eq. 3, provided that the wavefront aberration can be measured to within this tolerance.

Gabor’s initial demonstration of optical holography served as a launchpad for subsequent demonstrations of holography using other wave phenomena. We envision our initial demonstrations of optical SWH as a first step in demonstrating a more general solution to the problem of aberration corrected imaging using wavefronts of any physical nature. In particular, our method provides the greatest benefit when signal contrast at baseband frequencies is essential, yet the visibility of this contrast is effectively eliminated by scattering in a disordered medium. While we have demonstrated SWH with optical baseband frequencies in this paper, we envision that the same principle may also be applied using wavefront sensing of entirely different phenomena. For instance, we imagine that the same principle may also be applied using wavefronts of any physical nature. In particular, this is because the light at the two wavelengths traverses nearly identical ray paths and experiences nearly identical path length fluctuations. This assumption and observation forms the basis of our computational approach to accommodating scatter where we correlate the complex-valued fields to recover the SWH. It can be shown (see supplementary material) that the residual phase fluctuations in the SWH, given by $\phi(\Lambda) = \phi(\lambda_1) - \phi(\lambda_2)$, preserves phase variations at scales equal or larger than the SWL $\Lambda$, and is robust to speckle artifacts. However, the magnitude of the SWH, given by $|E(\Lambda)| = |E(\Lambda_1)| \cdot |E(\Lambda_2)|$, still exhibits speckle artifacts (see Fig. 1).

**Methods**

**Aberration correction by formation of a SWH.** The aberration correction step adopted in SWH draws inspiration from multi-wavelength interferometry on rough surfaces [22–24]. The process illustrated in the right half of Fig. 1 (purple box) involves recording speckle fields $E(\lambda_1), E(\lambda_2)$ at two closely spaced illumination wavelengths. Due to the stochastic nature of light scattering, the phase $\phi(\lambda_1), \phi(\lambda_2)$ of each field separately is completely randomized and bears no resemblance to the macroscopic structure of the object. However, the illumination beams at the two wavelengths originate from the same source position (such as from a single fiber) and the inhomogeneities in the scattering medium are quasi-static, then the fields incident on the detector are highly correlated. This is because the light at the two wavelengths traverses nearly identical ray paths and experiences nearly identical path length fluctuations. This assumption and observation forms the basis of our computational approach to accommodating scatter where we correlate the complex-valued fields to recover the SWH $E(\Lambda) = E(\Lambda_1)E^*(\lambda_2)$, with $\Lambda = \frac{\lambda_1 \lambda_2}{|\lambda_1 - \lambda_2|}$. It can be shown (see supplementary material) that the residual phase fluctuations in the SWH, given by $\phi(\Lambda) = \phi(\lambda_1) - \phi(\lambda_2)$, preserves phase variations at scales equal or larger than the SWL $\Lambda$, and is robust to speckle artifacts. However, the magnitude of the SWH, given by $|E(\Lambda)| = |E(\Lambda_1)| \cdot |E(\Lambda_2)|$, still exhibits speckle artifacts (see Fig. 1).

**Interferometer design and lock-in detection of the SWH.** The discussion on SBP limits in previous sections has implicitly assumed the availability of idealized sources and detectors. In practice, poor signal-to-background or signal-to-noise ratios, or both, can limit our ability to achieve the theoretical SBP. Interferometric approaches exploiting frequency heterodyning have particularly advantageous properties with respect to this problem. The principal benefit of adopting these approaches to record holograms is the ability to exploit the heterodyne gain [64] afforded by the use of a strong reference beam, whose baseband optical frequency is slightly detuned from the frequency of light in the object arm. The difference in frequency $\nu_m$ is chosen in the RF frequency range (3kHz for our experiments) and realized by using a cascade of acousto-optic or electro-optic modulators (AOM or EOM). Figs. 2(f,g) depict the two interferometer designs that we use to acquire the holograms at the two optical wavelengths. Each design is an adaptation of a Michelson Interferometer, and incorporates a small difference $\nu_m$ in the baseband frequency of light in the two arms of the interferometer. It is emphasized that the RF modulation frequency $\nu_m$ is fully decoupled from the choice of SWL (and therefore from the resolution of our method!), and can be chosen independent of the SWL.

A Lock-In Focal Plane Array (LI-FPA) [65] capable of synchronously demodulating the received irradiance at each detector pixel, is operated to detect the RF frequency $\nu_m$. The process directly yields the interferogram at the SWL $\Lambda$. The method avoids the need for time consuming raster
scanning as necessary in ToF-NLoS, and phase-shifting in holographic-NLoS. It also vastly improves the Signal-to-Background ratio of our measurements by suppressing the unmodulated ambient illumination. The Heliotis C3 LI-FPA [65] used in our experiments yields a 300 × 300 pix image per measurement. The exposure time of each measurement is \( t_{exp} = 23 \text{ms} \) corresponding to 70 cycles of the RF frequency \( \nu_m = 3\text{kHz} \). Two independently tunable narrow linewidth CW lasers (Toptica DFB pro 855nm) are used to illuminate and interrogate the scene. The center wavelength of each laser is 855nm, and the maximum tuning range is \( \sim 2.5\text{nm} \). This allows us to achieve SWLs \( \Lambda > 300\mu\text{m} \), corresponding to a beat frequencies \( > 1\text{THz} \).

The holograms in our proof-of-principle experiments were recorded using two specific heterodyne interferometer architectures: a Dual-Wavelength Heterodyne Interferometer (Fig. 2 g), and a Superheterodyne Interferometer (Fig. 2 f). The Dual-Wavelength Heterodyne Interferometer is preferred when light loss in the interferometer should be minimized, which is important for many NLoS applications. Light from the two lasers operating at \( \lambda_1, \lambda_2 \) are coupled together, before being split into the reference and sample arm. The reference arm is additionally modulated by \( \nu_m = 3\text{kHz} \), using a cascade of two fiber AOM’s. During acquisition, each laser is shuttered independently and the lock-in camera records the holograms by the two wavelengths, in a time-sequential manner. The LI-FPA provides two images: In-Phase (I) and Quadrature (Q), each of which represents the real and imaginary parts of the speckle fields incident on the image sensor. The expression for the I- and Q-images recorded by the LI-FPA for the wavelength \( \lambda_n \) is:

\[
\begin{align*}
I_I(\lambda_n) &= A_n \cos(\phi(\lambda_n)) \\
I_Q(\lambda_n) &= A_n \sin(\phi(\lambda_n)),
\end{align*}
\]

where \( A_n \) is the amplitude at \( \lambda_n \) and \( \phi(\lambda_n) \) is the difference in the phase of light in the object and reference arms. Please note that Eq. 7 omits any reference to spatial locations, in the interest of clarity.

Subsequently, the SWH \( E(\Lambda) \) is assembled as follows:

\[
E(\Lambda) = |I_I(\lambda_1) + i I_Q(\lambda_1)| \cdot |I_I(\lambda_2) + i I_Q(\lambda_2)|^* = A_1 A_2 \exp(i \phi(\lambda_1) - \phi(\lambda_2)) \tag{8}
\]

An attractive feature of the time-sequential approach to hologram acquisition described above is that it does not require the use of two tunable lasers. Identical results can be achieved with one laser that is tuned between the two measurements. Possible extensions include: one tunable and one fixed wavelength laser, and one fixed wavelength laser that is split in two arms, one of which includes an additional modulator.

Unfortunately, the simplicity of the time-sequential approach comes at the expense of increased sensitivity to object motion between measurements, and time-varying fluctuations in the environmental conditions. Increased robustness to these fluctuations is afforded by the Superheterodyne Interferometer design, wherein light from both lasers is used to simultaneously illuminate the target and scene. A possible realization is shown in Fig. 2 f: each laser beam is split into two arms, each of which is independently modulated with an AOM. The RF drive frequencies for AOM’s 1A and 1B are identically set to \( \nu_{AOM1} \), but include a phase offset \( \Delta \varphi_{AOM} \) that is user controlled. Light from the two AOM’s is combined and modulated with a third AOM (frequency \( \nu_{AOM2} \)), which produces the desired modulation frequency \( \nu_m = \nu_{AOM1} - \nu_{AOM2} = 3\text{kHz} \). The expression for the I- and Q-images (In-Phase and Quadrature) recorded by the LI-FPA are:

\[
\begin{align*}
I_I(\lambda_1, \lambda_2) &= A_1 \cos(\phi(\lambda_1) + \Delta \varphi_{AOM}) + A_2 \cos(\phi(\lambda_2)) \\
I_Q(\lambda_1, \lambda_2) &= A_1 \sin(\phi(\lambda_1) + \Delta \varphi_{AOM}) + A_2 \sin(\phi(\lambda_2))
\end{align*}
\]

The SWH \( E(\Lambda) \) is assembled by calculating:

\[
E(\Lambda) = |I_I^2 + I_Q^2| = A_1^2 + A_2^2 + A_1 A_2 \cos(\varphi(\lambda_1) - \varphi(\lambda_2) + \Delta \varphi_{AOM}) \tag{9}
\]

The synthetic phase map is recovered from the interferograms recorded with three or more phase shifts \( \varphi(\Lambda) \) introduced between measurements. It should be emphasized that the use of two tunable lasers is also not a pre-requisite for the approach. Identical results can be achieved with one fixed and one tuned laser, or similar combinations discussed above. The principal benefit of the Superheterodyne approach is the robustness to environmental fluctuations and object motion. However, it requires an additional AOM and fiber splitters that significantly reduce the available output power compared to the Dual Wavelength Heterodyne Interferometer discussed previously. The loss of power presents light throughput challenges for NLoS experiments that are intrinsically light starved.

In practice, there exists a trade-off between light throughput and robustness to environmental fluctuations, which depends on a multiple factors including stand-off distance, reflectivity of the involved surfaces, and laser power.

**Reference beam injection with reduced radiometric losses.** The reference beam required for interferometric sensing of the speckle fields at the optical wavelengths is directed towards the Lock-In FPA, as shown in Fig. 2 a. In one possible embodiment, a lensed fiber needle (WT&T Inc.) positioned in the front focal plane of the imaging optic (see Fig. 2 e) produces a near planar reference beam on the FPA. The use of a lensed fiber provides two distinct advantages over a beam-splitter: (1) the imaging optic can be
directly threaded to the camera (eliminates the need for inserting beam splitter between optic and sensor) and easily swapped during operation, and (2) improved light throughput (see Tab. 1).

| Light Loss in:       | Reference Beam | Sample Beam |
|----------------------|----------------|-------------|
| Lensed Fiber Needle  | ~ 30%          | ~ 0%        |
| 50/50 Beam Splitter  | ~ 50%          | ~ 50%       |

Table 1. Light loss at combination of reference and sample arm: Lensed fiber needle vs. conventional 50/50 beam splitter

Experimental setup and image formation in NLoS application. The experimental apparatus of Fig. 2 is used to demonstrate the ability of SWH to discern objects obscured from view, in this case a cutout of the character ‘N’ with dimensions ~ 20mm x 15mm. The size of the object was deliberately chosen to be smaller than the typical size of a resolution cell (~ 2cm) in competing wide-field NLoS approaches based on ToF sensing. The disadvantage when using such a small object is that it emits less light than the background. The problem is additionally compounded by the limited laser power in the object arm (about 30mW). In an effort to bypass these engineering limitations, we glued a thin sheet of silver foil to the sandblasted (280 grit) surface of the object ‘N’ and repeated the process for the VS surface. An image of the VS surface under ambient light (also representative for the surface of the object ‘N’) is included in Fig. 2c. In both cases, we ensured that the fields reflected by these materials are fully developed speckle patterns. The VD wall surface is constructed from a standard dry-wall panel that has been painted white (Beer Eggshell).

Our approach to NLoS imaging relies on the availability of an intermediary scattering surface (such as the wall in Fig. 2c) that serves to indirectly illuminate the obscured target and intercept the light scattered by the target. Accordingly, the intermediary surface may be viewed as a Virtualized Source (VS) of illumination and a Virtualized Detector (VD) for the obscured object.

Laser light from the physical source (at wavelengths $\lambda_1$ and $\lambda_2$) is directed towards the VS surface using a focusing optic. This light is scattered by the VS surface so as to illuminate the obscured object with a fully developed objective speckle pattern. A fraction of the light incident on the obscured object is redirected towards the VD surface. A second scattering event at the VD surface directs a tiny fraction of the object light towards the collection aperture, and subsequently the LI-FPA. The speckle fields impinging on the LI-FPA are synchronously demodulated to recover the real and imaginary parts of the holograms at the optical wavelengths $\lambda_1$ and $\lambda_2$. Each of these holograms is additionally subject to defraction due to the finite collection aperture. However, the diffraction effects are observed at optical wavelengths and have little impact on the SWL $\Lambda$.

If $\Lambda$ is chosen sufficiently large, the phase map at the SWL is robust to speckle and aliasing artifacts. This is unlike the case for holograms recorded at the optical wavelengths $\lambda_1$ and $\lambda_2$. The hidden object can be reconstructed by back-propagating the SWH, using a propagator (Free-Space propagator) at the SWL $\Lambda$.

Figure 3 includes the result of processing the NLoS measurements acquired using the experimental setup of Fig. 2. The measurements were captured at different SWLs ranging from $280\mu m$ to $2.6 mm$. Figure 3 shows five exemplary results for $\Lambda = 1.3 mm$, $\Lambda = 920 \mu m$, $\Lambda = 610 \mu m$, $\Lambda = 560 \mu m$ and $\Lambda = 440 \mu m$. The phase of the SWH associated with each SWL is shown in Fig. 3 a-e. The phasemaps have been low-pass filtered with kernel size $\approx \Lambda$ for better visualization.

As discussed previously, the reconstruction resolution improves with decreasing SWL. However, decreasing the SWL leads to an increased spectral decorrelation of the speckle fields at the two optical wavelengths. The decorrelation manifests as excessive phase fluctuations in the SWH, which in turn produces increased speckle artifacts in the reconstructed images. The problem can be mitigated (to an extent) by exploiting speckle diversity at the VS, specifically by averaging over multiple speckle realizations of the virtualized illumination. In our experiment, we realized the speckle diversity by small movements of the VS position. The image insets in Figure 3f-j represent the result of incoherent averaging (intensity-averaging) of the backpropagated images, for 5 different VS positions. The improvement in reconstruction quality comes at the expense of increased number of measurements, but not unlike competing ToF based NLoS approaches (e.g. > 20,000 VS positions are used in [44]). The distinction is that we need far fewer images. We conclude our discussion by observing that for static objects, the reconstruction quality may be further improved by increasing the number of VS positions used to realize speckle diversity.

Experimental setup and image formation for imaging through scattering media. The experimental apparatus of Fig. 4a is used to demonstrate the ability of SWH to image through scattering media. In a first experiment, we illuminate and image the character ‘U’ (see Fig. 4 b) through an optically rough ground glass diffuser (220 grit). The geometry is unlike other transmission mode experiments wherein the object is illuminated directly [66] or sandwiched between two diffusers. The current choice of geometry is deliberate and designed to mimic the imaging of a reflective target embedded in a scattering medium. Measurements were acquired for different SWLs ranging from $280\mu m$ to $2.6 mm$. Figures 4 d-g show four exemplary reconstructions for $\Lambda = 1.3 mm$, $\Lambda = 920 \mu m$, $\Lambda = 610 \mu m$, and $\Lambda = 280 \mu m$. In each instance, we incoherently averaged the reconstruction results for two VS positions. A comparison of the image insets in Figures 4 confirms the increased decorrelation for decreasing SWL. As discussed previously, the wavefront error for the diffuser is estimated to be $\Psi \approx 65 \mu m$, and the results for $\Lambda = 280 \mu m$ demonstrate performance close to the physical limit expressed by Eq. 5.

In a second experiment the ground glass diffuser within the imaging path is swapped with a milky acrylic plastic plate of ~ 4mm thickness. The acrylic plate exhibits pronounced multiple scattering, representative of imaging through volu-
metric scatter. Figure 4c compares the visibility of a checkerboard viewed through the 220 grit ground glass diffuser and the acrylic plastic plate. In both cases, the checkerboard is positioned 1 cm under the scattering plate and viewed under ambient illumination. It is evident from Figure 4c that the visibility of the checkerboard pattern is vastly diminished when viewed through the acrylic plate, whereas the pattern is still visible when viewed through the diffuser.

Figure 4h-k shows reconstruction results for the same character ‘U’ as imaged through the acrylic plate, for the same set of SWLs as the diffuser. In each instance, we incoherently averaged the reconstruction results for two VS positions. The character is reconstructed with high fidelity despite pronounced multiple scattering, suggesting the potential of SWH for imaging through volumetric scatter. A comparison of the image insets in Figures 4 confirms the diminished fidelity of imaging through volumetric scattering when compared to surface scatter.

We conclude our discussion by observing that the proposed approach for holographic imaging using scattered wavefronts can yield a higher resolution (smaller SWL) than competing approaches relying on TOF and fast detectors such as SPADs [44, 45, 47]. Additionally, the SWL can be adapted to accommodate different degrees of scatter. Furthermore, the notion of imaging using scattered wavefronts can be trivially extended to multiple wavelengths, mimicking interrogation by a pulsed source. Such unique attributes highlight the versatility of the SWH concept.

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Supplementary Material

Introduction

It is common knowledge in holography and interferometry that peak-to-valley excursions in the wavefront error (wave aberration) that are limited to $\frac{1}{4}$ of the optical wavelength (Rayleigh criterion) introduce minimal errors in the retrieved intensity and phase of an optical field. However, it is not apparent that computational mixing of scattered fields recorded at two closely spaced wavelengths $\lambda_1, \lambda_2$, each of which exhibits wavefront aberrations far in excess of $\frac{1}{4}$ of the optical wavelength, do indeed preserve phase information at scales smaller than the synthetic wavelength $\Lambda = \frac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2}$. The present document puts forth a mathematical basis for the aforementioned claim. The analysis accommodates scattering at optically rough surfaces, optical blurring in an imaging optic and sampling at the detector. The analysis while broad in scope, focuses on the application of imaging objects that are beyond the imager line-of-sight (LoS) and obscured from view. The canonical scene arrangement of Figure1 is used to introduce relevant concepts.

Our approach exploits the availability of an intermediary scattering surface (such as the wall in Figure1) in serving the dual purpose of illuminating objects obscured from view and intercepting the light scattered by the obscured objects. Accordingly, the intermediary surface may be viewed as a virtualized source of illumination and detection for the obscured objects. Herein, we seek to simultaneously recover a spatially resolved image of the obscured objects and their position within the hidden volume. To this end, we record the optical field emerging from the intermediary surface, using a holographic imaging apparatus (camera in Figure1). The recorded hologram, however, bears little resemblance to the macroscopic structure of the obscured object, due in large part to the randomzied dephasing of light following a reflection at the intermediary surface. Circumventing this problem without any knowledge of the scattering properties of the intermediary surface is a key contribution of our work. It is made possible by interrogating the hidden scene using two closely spaced optical frequencies and exploiting the spectral correlation in scattering at the intermediary surface.

The notion of exploiting spectral correlations in mitigating the deleterious effects of randomzied dephasing due to repeated scattering, has broad applications ranging from imaging through scattering/turbid media, imaging through tissue, imaging through fog/smoke, and imaging under

![Figure1](image-url): Canonical Scene arrangement for indirect imaging of objects obscured from view

...
brownout conditions. This document describes the mathematical principles underpinning the use of spectral field correlations in adapting scattering as degree of freedom from the standpoint of imaging.

In the interest of clarity, attention is restricted to the modeling the propagation of coherent light in the canonical scene of Figure 1, and the subsequent recovery of a hologram of the obscured object(s). The task of accurately modeling light propagation within the canonical scene is exceedingly difficult, due in large part to the multiplicity of scattering surfaces and numerous light bounces. The task is further complicated by factors such as multiple scattering at interfaces, shadowing, and Fresnel reflections. Incorporating these effects into a comprehensive model for imaging is mathematically intractable. Imposing specific restrictions and simplifying assumptions allows us to develop a mathematically tractable framework for light transport in the canonical scene. The list of restrictions and simplifying assumptions are enumerated below:

1. A linearly polarized narrow-linewidth tunable CW laser source with center frequency $\nu$ is used to interrogate the hidden scene by illuminating the Virtual Source surface.
2. Scalar diffraction is sufficient to model field transport through the scene.
3. The temporal fluctuations of the CW laser source are statistically uncorrelated with those of ambient light sources in the scene.
4. The principal contribution to the irradiance observed at the sensor is restricted to three bounce ray paths originating at the physical source, bouncing off the Virtual Source surface, the hidden object and the Virtual Detector surface, prior to terminating at the physical detector. The radiometric throughput of fourth and higher order bounces is assumed to be negligibly small, a fact borne out in experiments.
5. The coherence length of the source exceeds the cumulative length of all three-bounce ray originating at the physical source, traversing the hidden scene and terminating at the physical sensor. Consequently, the indirectly illuminated object can be expressed as a countably-finite collection of secondary point sources that are mutually coherent.
6. The propagation medium is free space and devoid of inhomogenities. The propagation distances exceed the spatial extent of the Virtual Source and the Virtual Detector.
7. Physical objects in the scene are optically rough at the scale of the optical wavelength of the CW laser source. We assume that path length variations induced by fluctuations in the surface height are the sole source of scattering, consistent with Goodman’s approach [1].
8. A diffraction limited optic is used to relay the image of the Virtual Detector surface onto an image sensor. The coordinate system used to develop the model is centered about entrance pupil of the imaging optic, $XY$ plane is aligned with image sensor and $Z$-axis is aligned with the optical axis
9. A lock-in sensor whose operation is described in [2], records interference of light scattered by obscured object and a planar reference beam. The lock-In camera (Lock-In sensor + imaging optics) independently acquires holograms of coherently illuminated scene/object
10. The reference beam envelope and phase does not change appreciably over the finite extent of a single detector pixel.
11. The illumination source subtends a small solid angle with respect to the obscured object. The spectral reflectance of source is unchanged for small change in optical frequency of illumination source.
12. The object albedo is unchanged for small change in optical frequency of illumination source.
13. The defocus error introduced by the microscopic roughness of the Virtual Detector surface is negligible.

**Key findings/claims**

1. A small change $\Delta \nu$ in the optical frequency of the CW source used to interrogate the hidden scene of Figure 1 imparts an additional spherical phase component to the field contribution of each obscured object point. The excess spherical phase encodes the position of the obscured object at the “synthetic” wavelength $c\Delta \nu^{-1}$ meters. (Eqs. (28),(29))

2. Computational mixing of holograms recorded at two closely spaced optical frequencies encapsulates field information at sub-millimeter synthetic wavelength scales that are insensitive to scattering at the optical frequency of interrogation. (Sections 1.5, 1.6)

3. Wavefront errors in the computational hologram are negligible if the change in optical path length stemming from a change $\Delta \nu$ in the optical frequency of interrogation is limited to $\frac{1}{4\Delta \nu}$ synthetic wavelength $c\Delta \nu^{-1}$ meters, for any ray path originating at the source and terminating at the sensor. (Section 1.6)

4. Optical holograms acquired at regularly spaced optical frequencies may be combined to computationally filter light paths with a prescribed round-trip distance from the physical source to the physical detector, so long as condition 3 holds for the largest frequency separation. (Section 2)

**Notation**

The mathematical analysis furnished in this document oftentimes involves multiple integrals and summations over spatial and time dimensions that can be continuous or discrete. In an effort to improve the clarity of the analysis we adopt the following convention in describing quantities of interest.

| Symbol | Description | Units |
|--------|-------------|-------|
| $x$ | Position vector of a point on the sensor plane | $\mathbb{R}^2$ meters |
| $d$ | Position vector associated with a point on the Virtual Detector | $\mathbb{R}^3$ meters |
| $s$ | Position vector associated with a point on the Virtual Source | $\mathbb{R}^3$ meters |
| $p$ | Position vector associated with the obscured object point | $\mathbb{R}^3$ meters |
| $\bar{v}$ | Optical frequency of the CW source used to interrogate the hidden scene | THz |
| $\bar{\omega} = 2\pi \bar{v}$ | Angular frequency of the CW source used to interrogate the hidden scene | radians |
| $t$ | Time | seconds |
| $I$ | Irradiance (real valued & non-negative) | $\mathbb{R}^2$ arbitrary units |
| $U$ | Optical field (complex-valued) | $\mathbb{C}^2$ arbitrary units |
| $\Delta$ | Sensor pixel pitch (in $\mu m$) | Microns |
| $F$ | Fill factor of sensor pixel ($0 < F \leq 1$) | $\mathbb{R}^+$ dimensionless |
**Key expressions**

Key expressions from the mathematical analysis are summarized in Figure 2. These include the expression for the sampled holograms at two closely optical frequencies, and a mathematical formulation of Claim-1.

Figure 2: Key mathematical expressions
In the interest of clarity, the task of modeling light transport in the canonical scene of Figure 1 is sub-divided into two tasks. The first of these tasks seeks to model the process of imaging the Virtual Detector surface including sampling at the image sensor. The second task models the physical propagation of light originating at the Virtual Source, bouncing off the obscured object and terminating at the Virtual Detector.

1.1 Imaging identity and optical blur

Without loss of generality, it is assumed that light emerging from the Virtual Detector surface is relayed to the Lock-In sensor using a well-corrected optic with focal length \( f \) and unity pupil magnification. The relation between a point on the Virtual Detector surface and its pixel projection on the image sensor satisfies the following geometric relation:

\[
x_1 = -z_d \frac{w_1}{w_3}, \quad x_2 = -z_d \frac{w_2}{w_3}
\]  

(1)

The term \( z_d \) represents the distance from the sensor plane to the exit pupil plane of the imaging optics. The negative sign accommodates image inversion. If the imager optical axis and the macroscopic normal vector to the Virtual Detector surface (disregarding change in local surface normal due to microscopic roughness of VD surface) are aligned, then the magnification \( z_d^{-1}w_3 \approx z_d^{-1}w_3 \), where \( w_3 \) is the macroscopic perpendicular distance from the VD surface to the entrance pupil plane of the imaging optics.

The diffraction limited optical blur associated with imaging the VD surface may be modeled as a paraxial blur [3] with image side numerical aperture \( D z_d^{-1} \), where \( D \) is the diameter of the exit pupil. The resulting amplitude PSF is disclosed in Eq.(2). The term \( P(u, v) \) represents the transmittance function of the exit pupil and modeled as an indicator function with diameter \( D \). The term \( f \) represents the focal length of the imaging optic. The depth \( w_3 \) of a point on the VD surface is measured with respect to the entrance pupil plane of the imaging optic.

\[
\begin{align*}
    h_{\text{blur}} (x \equiv [x_1] ; w & \equiv [w_1 w_2 w_3] ) \\
    &= \left( \frac{\psi}{\lambda} \frac{2}{z_d w_3} \exp \left( \frac{2 \pi \psi}{c} \left( z_d + w_3 + \frac{x_1^2 + x_2^2}{2 z_d} + \frac{w_1^2 + w_2^2}{2 w_3} \right) \right) \times \\
    &\int dudv \left( P(u, v) \exp \left( \frac{2 \pi \psi}{c} \frac{1}{z_d w_3} \left( u^2 + v^2 \right) \right) \exp \left( -\frac{2 \pi \psi}{c} \frac{[x_1 w_1 + w_3][x_2 + w_3 v]}{z_d w_3} \right) \right) \right)
\end{align*}
\]  

(2)

1.2 Recording optical fields using the Lock-In sensor

As is common practice in holography and interferometry, we acquire optical fields by recording the interference of the said field with a reference beam whose temporal fluctuations are highly correlated with the desired optical field. To this end, the field incident on the Lock-In sensor may be expressed as the superposition of the field contributions from the indirectly illuminated object and a planar reference beam, as shown below:

\[
U(x, t; \psi) = U_o(x; \psi) \exp(i \omega t) + U_r(x; \psi) \exp(i [\omega + \omega_h] t)
\]  

(3)

The term \( U_r(x; \psi) \) in Eq.(3) represents the baseband envelope of the reference field incident on the Lock-in sensor plane. The term \( U_o(x; \psi) \) represents the baseband envelope of the subjective speckle field due to the indirectly illuminated object/target. The angular frequency difference \( \omega_h \) between the two arms of the interferometer helps in isolating the desired field component from the corrupting influence of the zeroth
order and twin components that arise during square-law detection of the irradiance \(|U(x, t; \nu)|^2\). Insight into the process can be gleaned by examining the expression for the instantaneous irradiance incident on the Lock-In sensor plane, disclosed in Eq.(4).

\[
I(x, t; \nu) = |U(x, t; \nu)|^2
= |U_o(x; \nu)|^2 + |U_r(x; \nu)|^2 + 2\text{Real}[U_o(x; \nu)U_r^*(x; \nu) \exp(-i\omega_b t)]
= |U_o(x; \nu)|^2 + |U_r(x; \nu)|^2 + 2|U_r(x; \nu)| \times |U_o(x; \nu)| \cos(\phi_o(x; \nu) - \phi_r(x; \nu) - \omega_b t)
= I_o(x; \nu) + I_r(x; \nu) + 2\sqrt{I_o(x; \nu)} \sqrt{I_r(x; \nu)} \cos(\phi_o(x; \nu) - \phi_r(x; \nu) - \omega_b t)
\] (4)

The instantaneous intensity recorded by the \(n^{th}\) Lock-In pixel is obtained by integrating the irradiance \(I(x, t; \nu)\) incident on its active area. The corresponding expression is disclosed below:

\[
l[n_t; \nu] = \int_{(n-0.5F)\Delta}^{(n+0.5F)\Delta} dx \int_{(n-0.5F)\Delta}^{(n+0.5F)\Delta} dx \int_{(n-0.5F)\Delta}^{(n+0.5F)\Delta} dx \int_{(n-0.5F)\Delta}^{(n+0.5F)\Delta} dx \left\{ \delta\left[ I_o(x; \nu) + I_r(x; \nu) \right] + 2 \int_{(n-0.5F)\Delta}^{(n+0.5F)\Delta} dx \left[ \sqrt{I_o(x; \nu)} \sqrt{I_r(x; \nu)} \cos(\phi_o(x; \nu) - \phi_r(x; \nu) - \omega_b t) \right] \right\}
\] (5)

Following assumption-6, the envelope and phase of the reference beam does not change appreciably over the finite extent of a single detector pixel, so that:

\[
l_r(x; \nu) \approx l_r(n; \nu) \equiv l_r[n; \nu] \approx \cos(\phi_r(n; \nu)) \equiv \cos(\phi_r[n; \nu])
\] \(\forall x \in (n\Delta - 0.5F\Delta, n\Delta + 0.5F\Delta)\) (6)

As a result, the reference beam’s contribution to the integrated irradiance may be simplified as follows:

\[
\int_{(n-0.5F)\Delta}^{(n+0.5F)\Delta} dx \int_{(n-0.5F)\Delta}^{(n+0.5F)\Delta} dx \int_{(n-0.5F)\Delta}^{(n+0.5F)\Delta} dx \int_{(n-0.5F)\Delta}^{(n+0.5F)\Delta} dx \left\{ \delta\left[ I_o(x; \nu) + I_r(x; \nu) \right] \approx \delta\left[ I_o(n; \nu) \right] \approx l_o[n; \nu] \right\} \approx \Delta^2 \times l_r[n; \nu]
\] (7)

Incorporating Eqs.(7) & (6) into Eq.(5) yields the following expression for the instantaneous irradiance recorded by the \(n^{th}\) lock-in pixel:

\[
l[n_t; \nu] \approx \Delta^2 \times l_o(n; \nu) + \int_{(n-0.5F)\Delta}^{(n+0.5F)\Delta} dx \left[ \sqrt{I_o(x; \nu)} \sqrt{I_r(x; \nu)} \cos(\phi_o(x; \nu) - \phi_r(n; \nu) - \omega_b t) \right]
\] (8)

The first term in Eq.(8) represents the zeroth-order time-invariant irradiance contribution of the hologram recorded at \(\nu\). The second and third terms in Eq.(8) represents the time-varying irradiance contribution associated with the detector integration of the real & imaginary part of the subjective speckle field \(U_o(x; \nu)\), respectively. The temporal carrier associated with these terms has angular frequency \(\omega_b\) and phase \(\frac{\pi}{2} - \phi_r[n; \nu]\) and \(-\phi_r[n; \nu]\) respectively. The amplitude \(\sqrt{I_o[n; \nu]}\) of these terms describes the heterodyne gain arising from synchronous demodulation at \(\omega_b\). The integral over spatial locations in the second and third terms signify the real and imaginary parts of the phasor sum of the subjective speckle cells that can be accommodated within the \(n^{th}\) detector pixel. The integration accommodates signal fading arising from the summation of a disproportionately large number of statistically independent speckle cells [4].
Each pixel of the Lock-in sensor functions as a homodyne receiver that accumulates the result of demodulating the received irradiance with two local oscillator (LO) signals \(\sin(\omega_b t)\) and \(\cos(\omega_b t)\) that are in quadrature phase. The demodulation is restricted to \(N\) periods of the LO signal, and yields a sampled representation of the in-phase (real part) & quadrature (imaginary part) components of the complex-valued optical field incident on the Lock-In sensor. The expression for the in-phase component of the optical field recorded by the Lock-In sensor is furnished below:

\[
I_i[n; \nu] = \sum_{t=0}^{N\nu_b^{-1}} dt \cos(\omega_b t) I[n, t; \nu]
\]

\[
= \sum_{t=0}^{N\nu_b^{-1}} dt \cos(\omega_b t) \cos(\omega_b t) \int_{(n+0.5\Delta)}^{(n-0.5\Delta)} dx \left[ \sqrt{I_o(x; \nu)} \cos(\varphi_o(x; \nu) - \varphi_r[n; \nu]) \right]
\]

\[
= \frac{1}{2} \sum_{t=0}^{N\nu_b^{-1}} dt [1 + \cos(2\omega_b t)] \int_{(n+0.5\Delta)}^{(n-0.5\Delta)} dx \left[ \sqrt{I_o(x; \nu)} \cos(\varphi_o(x; \nu) - \varphi_r[n; \nu]) \right]
\]

\[
= N\nu_b^{-1} \sqrt{I_r[n; \nu]} \int_{(n-0.5\Delta)}^{(n+0.5\Delta)} dx \left[ \sqrt{I_o(x; \nu)} \cos(\varphi_o(x; \nu) - \varphi_r[n; \nu]) \right]
\]

(9)

The expression for the quadrature component of the optical field recorded by the Lock-In sensor is furnished below:

\[
I_q[n; \nu] = \sum_{t=0}^{N\nu_b^{-1}} dt \sin(\omega_b t) I[n, t; \nu]
\]

\[
= N\nu_b^{-1} \sqrt{I_r[n; \nu]} \int_{(n-0.5\Delta)}^{(n+0.5\Delta)} dx \left[ \sqrt{I_o(x; \nu)} \sin(\varphi_o(x; \nu) - \varphi_r[n; \nu]) \right]
\]

(10)

By combining the sampled in-phase and quadrature images, the sampled hologram at the optical wavelength may be reconstituted, as shown in Eq.(11).

\[
U_\Delta[n; \nu] = I_i[n; \nu] + \sqrt{-1}I_q[n; \nu]
\]

\[
= N\nu_b^{-1} \sqrt{I_r[n; \nu]} \int_{(n-0.5\Delta)}^{(n+0.5\Delta)} dx \left[ \sqrt{I_o(x; \nu)} \exp(i[\varphi_o(x; \nu) - \varphi_r[n; \nu]]) \right]
\]

\[
= N\nu_b^{-1} \sqrt{I_r[n; \nu]} \exp(-i\varphi_r[n; \nu]) \int_{(n-0.5\Delta)}^{(n+0.5\Delta)} dx \left[ \sqrt{I_o(x; \nu)} \exp(i\varphi_o(x; \nu)) \right]
\]

\[
= N\nu_b^{-1} U_r[n; \nu] \int_{(n-0.5\Delta)}^{(n+0.5\Delta)} dx U_o(x; \nu)
\]

(11)
We conclude our discussion on recording optical fields by enumerating the principal benefits of a Lock-In sensor, namely:

- The ability to detect a weak sinusoidal signal buried in strong background and noise (accommodate a wide range of ambient illumination levels)
- The heterodyne gain afforded by combination of background subtraction and synchronous demodulation of the received irradiance at each pixel (eliminates need for longer exposure times and optical stabilization of interferometer)
- Eliminating need for matching the optical power in the two interferometer arms
- Snapshot hologram acquisition with maximal utilization of the limited spatial bandwidth of the FPA, (no mechanical movement or temporal phase-shifting required for isolating the hologram term).

The interested reader is referred to [5],[6] for details on the mechanics of recoding holograms using heterodyne interferometry and the Lock-In sensor.

### 1.3 Light transport from the Virtual Source to the Virtual Detector

The discussion in Sections-1.1,1.2 restricted attention to the process of recording optical fields. The present section is devoted to the development of a mathematical model for propagating quasi-monochromatic scalar optical fields in the canonical scene arrangement of Figure1.

Our analysis begins with the observation that light scattered by the Virtual Source surface behaves as a partially coherent source of illumination for the obscured objects. Each point on the indirectly illuminated object may then be viewed as a secondary source of partially coherent light that directs spherical wavefronts towards the Virtual Detector surface. Each of these spherical wavefronts is additionally scattered by the Virtual Detector surface before being intercepted by the finite collection aperture of the imaging optic. The resulting spatial pattern is recorded by the image sensor, and exhibits a mottled appearance reminiscent of speckle [1].

The combined field contribution of the obscured object is obtained as a weighted superposition of the elementary phasors associated with light paths originating at a point \( s \) on the Virtual Source, bouncing off an obscured object point \( p \), and terminating at the Virtual Detector point \( w \), whose geometric projection onto the image sensor is \( x \). The final expression for the obscured object field \( \mathcal{U}_o(x;\overline{v}) \) incident on the Lock-In sensor is disclosed below:

\[
\mathcal{U}_o(x;\overline{v}) = \sqrt{v} \int \int \psi(s,p) \exp\left(\frac{2\pi i}{\lambda} \left\| s - p \right\| \right) \sqrt{B_o(p;\overline{v})} \psi(p,w) \exp\left(\frac{2\pi i}{\lambda} \left\| p - w \right\| \right) \sqrt{R_{vd}(w;\overline{v})} h_{deloc}(x;w;\overline{v}) \tag{12}
\]

The innermost integral of Eq.(12) which is colored in magenta represents the complex-valued illumination incident on the obscured object point \( p \), following scattering at the Virtual source. It encapsulates light transport from the physical source to the obscured object point. The term \( \mathcal{A}(s;\overline{v}) \) represents the spectral reflectance of the Virtual Source. It represents the combined influence of the real-valued albedo of the Virtual Source surface, and the complex-valued illumination beam incident on the Virtual Source surface. The term \( \sqrt{B_o(p;\overline{v})} \) represents the square-root of the real-valued albedo of the object point \( p \). It encapsulates reflection from an infinitesimally small area element \( dp \) in the immediate vicinity of the object point \( p \). The term \( \sqrt{R_{vd}(w;\overline{v})} \) represents the square-root of the real-valued albedo of a point \( w \) on the Virtual Detector.
surface. It encapsulates Lambertian reflection at the Virtual Detector surface. The term $h_{\text{blur}}$ represents the amplitude PSF of the imaging optic and encapsulates diffraction limited imaging under coherent illumination.

**Accommodating scattering**

The length of the ray paths $\|s - p\|$ encapsulate the microscopic roughness of the Virtual Source surface and the obscured object. The corresponding phase fluctuations given by $\frac{\nu}{\epsilon} \|s - p\|$ induce randomized dephasing of the spherical waves arriving at the obscured object point, manifesting as speckle illumination. In a similar fashion, the length of the ray paths $\|p - w\|$ encapsulate the microscopic roughness of the obscured object and the Virtual Detector surface. The corresponding phase fluctuations given by $2\pi \frac{\nu}{\epsilon} \|p - w\|$ induce randomized dephasing in the spherical waves arriving at the Virtual Detector surface, irreversibly corrupting the phase of the obscured object and the Virtual Detector surface. Phase fluctuations arising from scattering at the object are embedded in the path length calculations $\|s - p\|, \|p - w\|$.

**Connection to conventional imaging**

By decoupling ray paths from the Virtual Source $\rightarrow$ obscured object $\rightarrow$ Virtual Detector into its constituent paths namely: Virtual Source $\rightarrow$ obscured object, and obscured object $\rightarrow$ Virtual Detector, it is possible to recast the expression for the object field contribution at the Lock-In sensor in a mathematical form that closely resembles the standard space-variant imaging formulation [7]. In particular, the field contribution of the obscured object may be expressed as a coherent superposition of stochastic patterns that are each weighted by the real-valued albedo of the obscured object:

$$U_o(x; \nu) = \int dp \left[ U_{\text{ill}}(p; \nu) \sqrt{E_o(p; \nu)} h_{\text{vd}}(x; p, \nu) \right]$$

(13)

The definition of the various terms in Eq.(13) is furnished below:

$$U_{\text{ill}}(p; \nu) \equiv \left( \frac{\nu}{\epsilon} \right) \left| \int ds A(s; \nu) \frac{\exp \left( i2\pi \frac{\nu}{\epsilon} \|s - p\| \right)}{\|s - p\|} \psi(s, p) \right|$$

complex-valued speckle illumination incident on the object point $p$, due to scattering at the Virtual Source.

$$h_{\text{vd}}(x; p, \nu) \equiv \left( \frac{\nu}{\epsilon} \right) \left[ \int dw \left( \frac{\exp \left( i2\pi \frac{\nu}{\epsilon} \|p - w\| \right)}{\|p - w\|} \psi(p, w) \right) \sqrt{R_{\text{vd}}(w; \nu)} \right] h_{\text{blur}}(x; w, \nu)$$

Speckle point-spread-function (PSF) of light transport from the obscured object point $p$ to a point $x$ on the sensor plane, following a scattering event at the Virtual Detector.

The reformulation helps draw parallels between conventional imaging using physical sources/detectors and indirect imaging using virtualized sources/detectors. The term $U_{\text{ill}}(p; \nu)$ represents the incident illumination on the obscured object, while $h_{\text{vd}}(x; p, \nu)$ represents the PSF associated with light transport from the object to the sensor. The PSF $h_{\text{vd}}(x; p, \nu)$ represents the combined influence of scattering at the Virtual Detector relay wall and blurring intrinsic to optical imaging. As a result, the indirect imaging PSF $h_{\text{vd}}(x; p, \nu)$ is stochastic in character, and its structure depends on the unknown roughness profile of the Virtual Detector surface.
1.4 Expression for sampled hologram recorded by Lock-In sensor

At this point, we have the necessary ingredients to identify the expression for the sampled hologram recorded by the Lock-In sensor. It is obtained by incorporating the expression for the obscured object field $\mathcal{U}_o(x, \bar{v})$ identified in Eq. (13) into Eq. (11). The resulting expression is shown below:

$$
\mathcal{U}_{\Delta}(n; \bar{v}) = \frac{N}{\nu_{\bar{v}}} \mathcal{U}_{\nu_{\bar{v}}}[n; \bar{v} \Delta \nu] \left\{ \int_{(n-0.5)\Delta}^{(n+0.5)\Delta} dx \mathcal{U}_o(x; \bar{v} \Delta \nu) \right\}
$$

$$
= \frac{N}{\nu_{\bar{v}}} \mathcal{U}_{\nu_{\bar{v}}}[n; \bar{v} \Delta \nu] \left\{ \int_{(n-0.5)\Delta}^{(n+0.5)\Delta} dx \mathcal{U}_o(x; \bar{v} \Delta \nu) \int dw \int dp \sqrt{B_{\nu_{\bar{v}}}(p, \bar{v})} \int_{\text{IFOV}} \exp \left( \frac{ijm}{\nu_{\bar{v}}} \frac{||s - p||}{||s - p - w||} \cdot \mathcal{A}(x; \nu_{\bar{v}}) \psi_p \psi_{p(r)} \right) \sqrt{R_{\nu_{\bar{v}}}(w; \nu_{\bar{v}}) h_{\nu_{\bar{v}}}(x; p, \bar{v})} \right\}
$$

$$
= \frac{N}{\nu_{\bar{v}}} \mathcal{U}_{\nu_{\bar{v}}}[n; \bar{v} \Delta \nu] \left\{ \int dp \left( \mathcal{U}_{\nu_{\bar{v}}}(p; \bar{v}) \sqrt{B_{\nu_{\bar{v}}}(p, \bar{v})} \right) \int_{(n-0.5)\Delta}^{(n+0.5)\Delta} dx h_{\nu_{\bar{v}}}(x; p, \bar{v}) \right\}
$$

The innermost nested integrals in Eq. (14) represent the sum of elementary phasors associated with each ray path originating at the Virtual Source, bouncing off the obscured object and terminating at the $n^{th}$ Virtual Detector pixel. The spatial extent of the $n^{th}$ Virtual Detector pixel is determined by the geometric image of the $n^{th}$ physical detector pixel (IFOV).

The integral identity of Eq. (15) recasts the expression for the sampled hologram in a mathematical form that closely resembles the standard space-variant imaging formulation [7]. The term within blue brackets captures the effect of detector aliasing, and describes the number of speckle cells within the PSF $h_{\nu_{\bar{v}}}(x; p, \bar{v})$ that can be accommodated in a single detector pixel. This number should not be excessively large ($<10$) to avoid amplitude/signal fading due to coherent averaging of speckle amplitudes within a pixel [4].

It is also evident from Eq. (15) that the optical hologram recorded by the physical detector does not exhibit any deterministic relationship to the obscured object field $\mathcal{U}_{\nu_{\bar{v}}}(p; \bar{v}) \sqrt{B_{\nu_{\bar{v}}}(p, \bar{v})}$. This is due in large part to the stochastic character of the PSF $h_{\nu_{\bar{v}}}(x; p, \bar{v})$, and the additional randomization imposed by integration of the PSF within each sensor pixel. Fortunately, it is possible under restricted conditions, to recover a holographic representation of the obscured object by computational mixing of the sampled hologram at $\bar{v}$ with a second hologram acquired at a closely spaced frequency $\bar{v} + \Delta \nu$. The remainder of this section identifies the conditions under which a latent hologram of the obscured object may be recovered.

We begin by identifying the expression for the sampled hologram at the second optical frequency $\bar{v} + \Delta \nu$, and is disclosed below:

$$
\mathcal{U}_{\Delta}(n; \bar{v} + \Delta \nu) = \frac{N}{\nu_{\bar{v}}} \mathcal{U}_{\nu_{\bar{v}}}[n; \bar{v} + \Delta \nu] \left\{ \int_{(n-0.5)\Delta}^{(n+0.5)\Delta} dx \mathcal{U}_o(x; \bar{v} + \Delta \nu) \right\}
$$

$$
= \frac{N}{\nu_{\bar{v}}} \mathcal{U}_{\nu_{\bar{v}}}[n; \bar{v} + \Delta \nu] \left\{ \int dp \left( \mathcal{U}_{\nu_{\bar{v}}}(p; \bar{v} + \Delta \nu) \sqrt{B_{\nu_{\bar{v}}}(p; \bar{v} + \Delta \nu)} \right) \int_{(n-0.5)\Delta}^{(n+0.5)\Delta} dx h_{\nu_{\bar{v}}}(x; p, \bar{v} + \Delta \nu) \right\}
$$

The definition of the various terms in Eq. (16) mirrors Eq. (13), and is restated for the benefit of the reader:

$$
\mathcal{U}_{\nu_{\bar{v}}}(p; \bar{v} + \Delta \nu) \equiv \left( \frac{\bar{v} + \Delta \nu}{ic} \right) \left( \int ds \mathcal{A}(s; \bar{v} + \Delta \nu) \frac{\exp \left( \frac{2\pi \bar{v} + \Delta \nu}{c} \frac{||s - p||}{||s - p - w||} \right) \psi(s, p)}{||s - p||} \right)
$$
\[ h_{\text{bl}}(x; p, \bar{\nu} + \Delta \nu) \equiv \left( \frac{\bar{\nu} + \Delta \nu}{i c} \right) \int dw \left( \frac{\exp \left( i 2 \pi \frac{\bar{\nu} + \Delta \nu}{c} \| p - w \| \right)}{\| p - w \|} \psi(p, w) \sqrt{R_{\text{bl}}(w; \bar{\nu} + \Delta \nu) h_{\text{bl}}(x; w, \bar{\nu} + \Delta \nu)} \right) \]

Examination of the expressions for the sampled holograms disclosed in Eqs.(15),(16) fails to divulge any obvious relation between the detected fields at the two closely spaced optical frequencies \( \bar{\nu} \) and \( \bar{\nu} + \Delta \nu \). The relation between the sampled holograms may be elucidated by making simplifying assumptions regarding the indirect imaging geometry and the scattering properties of the Virtual Source Detector surfaces. These assumptions are enumerated below:

A1. \( \mathcal{A}(s; \bar{\nu} + \Delta \nu) = \mathcal{A}(s; \bar{\nu}) \)  
Spectral reflectance of Virtual Source surface is unchanged for small change in optical frequency

A2. \( B_o(p; \bar{\nu} + \Delta \nu) = B_o(p; \bar{\nu}) \)  
Object albedo is unchanged for small change in optical frequency

A3. \( R_{\text{vd}}(p; \bar{\nu} + \Delta \nu) = R_{\text{vd}}(p; \bar{\nu}) \)  
Virtual Detector albedo is unchanged for small change in optical frequency

A4. \( \mathcal{U}_r[n; \bar{\nu} + \Delta \nu] = \mathcal{U}_r[n; \bar{\nu}] \)  
Phase fluctuations induced by a small change in the optical frequency of the reference beam are negligibly small

A5. The Virtual Source subtends a small solid angle with respect to the object, so that a small change in the optical frequency of the CW source induces a proportional change in the optical path length (OPL) of ray paths \( \bar{s} \bar{p} \) originating at a point \( s \) on the Virtual Source and terminating at the object point \( p \). The excess phase induced by the change in OPL is given below:

\[ \Phi(\bar{s} \bar{p}; \Delta \nu) \equiv 2\pi \left( \frac{\bar{\nu} + \Delta \nu}{c} \right) \| s - p \| - \left( \frac{\bar{\nu}}{c} \right) \| s - p \| = 2\pi \left( \frac{\Delta \nu}{c} \right) \| s - p \| \]

\[ \approx 2\pi \left( \frac{\Delta \nu}{c} \right) \| \bar{s} - \bar{p} \| \left( 1 + \frac{1}{2} \| s - \bar{s} \|^2 + \frac{(s - \bar{s})^T (s - \bar{p})}{\| s - \bar{p} \|^2} \right) \]

\[ \approx 2\pi \left( \frac{\Delta \nu}{c} \right) \| s - p \| + \epsilon \]  
(17)

where \( \bar{s} \) is a fixed point on the Virtual Source surface, such as the centroid of the VS. The assumption of small solid-angle is evidenced in the binomial approximation of Eq.(17).

The first term in Eq.(18) is the phase associated with a spherical wave at the notional wavelength (or synthetic wavelength) \( c \Delta \nu^{-1} \) meters. It encodes the position of the obscured object from the vantage point of the Virtual Source. The second term \( \epsilon \) in Eq.(18) is the excess phase or wavefront aberration induced by a change in the optical frequency. The term is dominated by the phase component \( 2\pi \left( \frac{\Delta \nu}{c} \right) (s - s)^T (\frac{s - \bar{s}}{\| s - \bar{s} \|}) \). The wavefront error imparted by this component is negligible so long as \( |\epsilon| \ll 1 \) radian [3], for each ray-path \( \bar{s} \bar{p} \) originating at the Virtual Source and terminating at the obscured object point \( p \). As a result,

\[ |\epsilon| \ll 1 \implies \left( \frac{\Delta \nu}{c} \right) \left( \frac{s - \bar{s}}{\| s - \bar{s} \|} \right)^T \left( \frac{s - \bar{p}}{\| s - \bar{p} \|} \right) \ll \frac{1}{2\pi} \]

\[ \therefore \| s - s \| \left( \frac{s - \bar{s}}{\| s - \bar{s} \|} \right)^T \left( \frac{s - \bar{p}}{\| s - \bar{p} \|} \right) \ll \frac{c \Delta \nu^{-1}}{2\pi} \implies \| s - s \| < \frac{c \Delta \nu^{-1}}{2\pi} \]

\[ \| s - \bar{s} \| \approx \sqrt{\left( \frac{\Theta_{\text{vs}}}{2} \right)^2 + \sigma_n^2} \text{ so that } \sqrt{\left( \frac{\Theta_{\text{vs}}}{2} \right)^2 + \sigma_n^2} < \frac{c \Delta \nu^{-1}}{8} \]

(19)
represents the change in wavelength resulting from a change in the optical frequency of the CW source by \( \Delta \nu \) Hz.

The empirical bound of Eq.(20) divulges a tradeoff between the maximum permitted frequency shift \( \Delta \nu \), and the choice of beam waist and RMS roughness of the Virtual Source surface. For a Virtual Source surface with approximate knowledge of the RMS roughness, a larger frequency shift may only be admitted by reducing the size of the Virtual Source, a fact borne out in experiments.

A6. A small change \( \Delta \nu \) in the optical frequency of the CW illumination source induces longitudinal chromatic aberration in the diffraction limited optical blur at \( \nu \), and is expressed as follows:

\[
\begin{align*}
    h_{\text{blur}}(x) \equiv & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad w \equiv \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}; \quad \tilde{\nu} = \nu + \Delta \nu \\
    \approx & \left( \frac{\tilde{\nu} + \Delta \nu}{\nu} \right)^2 \exp \left( i \frac{2\pi \Delta \nu}{c} \left( z_d + \frac{[x_1^2 + x_2^2]}{2z_d} + \overline{w}_3 + \frac{[w_1^2 + w_2^2]}{2\overline{w}_3} \right) \right) h_{\text{blur}}(x; w, \nu)
\end{align*}
\]

The term \( \overline{w}_3 \) represents the mean distance from the VD surface to the entrance pupil plane of the imaging optic. The approximation is valid so long as the peak-valley height fluctuations in the Virtual Detector surface are smaller than \( \frac{\Delta \nu}{\nu} \) change in the wavelength of the interrogation source, namely \( c\Delta \nu \) meters. A formal proof is furnished in Section-0.

- **Finite extent of each detector pixel:** Over the finite extent of the \( n^{th} \) detector pixel, the quadratic phase variation \( \frac{2\pi \Delta \nu}{c} \|n\|^2 \), imparted by the longitudinal chromatic aberration in the optical blur of Eq.(21), can be approximated by the complex valued constant shown below:

\[
\frac{2\pi \Delta \nu}{c} \frac{\|n\|^2}{2z_d} \approx \frac{2\pi \Delta \nu}{c} \frac{\|n\|^2}{2z_d} \quad \forall x \in (n\Delta - 0.5F\Delta, n\Delta + 0.5F\Delta)
\]

- **Finite extent of the optical blur:** The finite spatial extent of the optical blur \( h_{\text{blur}} \) limits the set of Virtual Detector locations \( w \) \( (w_1, w_2, w_3) \) that contribute to the integrated field amplitude at the \( n^{th} \) detector pixel. These locations are confined to a small region in the vicinity of the geometric image of the center of the \( n^{th} \) detector pixel, as observed on the Virtual Detector surface. Thus, the position vector \( [w_1, w_2, w_3]^T \) associated with a point on the VD surface may instead be approximated by the vector \( \left(-z_d^{-1}\overline{w}_3n\Delta, \overline{w}_3\right)^T \), for all \( x \in (n\Delta - 0.5F\Delta, n\Delta + 0.5F\Delta) \). The scalar \( z_d^{-1}\overline{w}_3 \) in the approximation represents the transverse magnification associated with the imaging identity of Eq.(1), and \( \overline{w}_3 \) is the mean perpendicular distance from the VD surface to the entrance pupil plane of the imaging optics. As a consequence of this approximation, the quadratic phase variations \( \frac{2\pi \Delta \nu}{c} \|w_1^2 + w_2^2\| \), imparted by the longitudinal chromatic aberration in the optical blur of Eq.(21), can be approximated by the complex valued constant shown below:

\[
\frac{2\pi \Delta \nu}{c} \frac{[w_1^2 + w_2^2]}{2\overline{w}_3} \approx \frac{2\pi \Delta \nu}{c} \frac{\|z_d^{-1}\overline{w}_3n\Delta\|^2}{2\overline{w}_3} \quad \forall x \in (n\Delta - 0.5F\Delta, n\Delta + 0.5F\Delta)
\]

A7. Each Virtual Detector pixel subtends a small solid angle with respect to the object, so that change a small change in the optical frequency of the CW source induces a proportional change in the optical path length (OPL) of ray paths \( \overrightarrow{pw} \) originating at a point \( p \) on the obscured object and terminating at the point \( w \) on the Virtual Detector surface. The excess phase induced by the change in OPL is given below:
The primary Field propagation from Virtual Source to obscured object: A small change $\Delta \nu$ in the optical frequency of the CW source induces a proportional change in the OPL of ray paths $\bar{sp}$ originating at a point $s$ on the VS and terminating at the object point $p$. The change in OPL can be approximated as $\|s - p\|$. The change in OPL is proportional to $\Delta \nu$, and the resulting phase shift is proportional to $\frac{\Delta \nu}{c} \|p - w\|$. The phase shift can be approximated as $2\pi \left( \frac{\nu + \Delta \nu}{c} \right) \|p - w\| - 2\pi \left( \frac{\nu}{c} \right) \|p - w\| = 2\pi \left( \frac{\Delta \nu}{c} \right) \|p - w\|$. This approximation is valid when $\|p - w\|$ is much smaller than the wavelength $\lambda$ and the change in frequency $\Delta \nu$ is much smaller than the bandwidth $\nu$. The approximation simplifies the phase shift calculation and allows for a more accurate prediction of the phase change.
where $\bar{s}$ is a fixed point such as the centroid of the illuminated region that makes up the Virtual Source.

- **Field propagation from obscured object to Virtual Detector:** A small change $\Delta \nu$ in the optical frequency of the CW source induces a proportional change in the OPL of ray paths $\vec{w}\vec{w}$ originating at a point $p$ on the obscured object and terminating at the point $w$ on the Virtual Detector surface. The change can be approximated as $\|p - [-z_d^{-1}\vec{w}_3\bar{n}\Delta]\||$, where $[-z_d^{-1}\vec{w}_3\bar{n}\Delta]$ is the geometric center of the $n^{th}$ Virtual Detector pixel (geometric image of center of the $n^{th}$ detector pixel on the Virtual Detector surface).

- **Field propagation from Virtual Detector to image sensor:** A small change $\Delta \nu$ in the optical frequency of the CW source induces longitudinal chromatic aberration in the diffraction limited optical blur associated with imaging the Virtual Detector surface at $\vec{v}$. The phase fluctuations due to chromatic aberration may be approximated as $\frac{2\pi \Delta \nu}{c} \left( \frac{\|z_d^{-1}\vec{w}_3\bar{n}\Delta\|^2}{2\vec{w}_3} + \frac{\|\bar{n}\Delta\|^2}{2z_d} \right)$ where-in
  - $z_d^{-1}\vec{w}_3$ is the transverse magnification of the imager,
  - $\vec{w}_3$ is the mean perpendicular distance from the VD surface to the entrance pupil plane of the imaging optics,
  - $z_d$ is the distance from the sensor plane to the exit pupil plane of the imaging optics.

### 1.5 Redundant information in the digital holograms at optical frequencies $\vec{\nu}$ and $\vec{\nu} + \Delta \nu$

A crucial step in assembling the holographic description of the obscured objects is identifying redundant information in the complex-valued holograms recorded at the optical frequencies $\vec{\nu}$ and $\vec{\nu} + \Delta \nu$. To this end, we impose the restrictions $A1$ upon the expressions for the indirect illumination $U_{\|}(\vec{\nu})$ and the indirect imager PSF $h_{vd}$. The expressions relating the indirect illumination at the two optical frequencies is disclosed below:

$$U_{\|}(p; \vec{\nu} + \Delta \nu) = \left( \frac{\vec{\nu} + \Delta \nu}{\vec{\nu}} \right) \times \exp \left( i \frac{2\pi \Delta \nu}{c} \|s - p\| \right) \times \frac{\exp \left( i \frac{2\pi \Delta \nu}{c} \|s - p\| \right)}{\|s - p\|} \times U_{\|}(p; \vec{\nu})$$

where $\vec{\nu}$ is the optical frequency of the CW source and $\Delta \nu$ is the change in optical frequency.

It is evident from Eq.(28) that a small change in optical frequency of the CW source imparts a spherical phase to the field incident on the obscured object point $p$. The excess phase depends on the propagation distance from the centroid of the Virtual Source to the obscured object point, and the incremental change in the wavelength of the CW source (given by $c\Delta \nu^{-1}$ meters). The phase encodes the position of the obscured object from the vantage point of the Virtual Source. (Observation-1)

In a similar fashion, it can be shown (Eq.(29)) that a small change in the optical frequency imparts additional spherical phase to the indirect imager PSF associated with each obscured object point $p$. 

14
\[
\begin{align*}
    h_{\text{wd}}(x; p, \bar{v} + \Delta \nu) & \equiv \left(\frac{\bar{v} + \Delta \nu}{v}\right) \int dw \left(\frac{\exp(i2\pi \frac{\bar{v} + \Delta \nu}{c} \|p - w\|)}{\|p - w\|}\psi(p, w)\right) \sqrt{R_{\text{wd}}(w; \bar{v} + \Delta \nu)} h_{\text{blur}}(x; w, \bar{v} + \Delta \nu) \\
    & = \left(\frac{\bar{v} + \Delta \nu}{v}\right)^3 \left\{ \exp\left(i \frac{2\pi \Delta \nu}{c} \left[ \left\|p - \left[- \frac{z_d^{-1} \bar{w}_3 n \Delta}{\bar{w}_3}\right]\right\| + z_d + \frac{\|n \Delta\|^2}{2z_d} + \bar{w}_3 + \frac{\|z_d^{-1} \bar{w}_3 n \Delta\|^2}{2\bar{w}_3} \right) \right\} \left\{ \int dw \left(\frac{\exp(i2\pi \frac{\bar{v} + \Delta \nu}{c} \|p - w\|)}{\|p - w\|}\psi(p, w)\right) \sqrt{R_{\text{wd}}(w; \bar{v})} h_{\text{blur}}(x; w, \bar{v}) \right\} \\
    & = \left(\frac{\bar{v} + \Delta \nu}{v}\right)^3 \left\{ \exp\left(i \frac{2\pi \Delta \nu}{c} \left[ \left\|p - \left[- \frac{z_d^{-1} \bar{w}_3 n \Delta}{\bar{w}_3}\right]\right\| + z_d + \frac{\|n \Delta\|^2}{2z_d} + \bar{w}_3 + \frac{\|z_d^{-1} \bar{w}_3 n \Delta\|^2}{2\bar{w}_3} \right) \right\} h_{\text{wd}}(x; p, \bar{v}) \quad (29)
\end{align*}
\]

The additional phase contribution highlighted in yellow varies with the propagation distance from the obscured object point \( p \) to the center of the \( n^{th} \) Virtual Detector pixel. The excess phase contribution highlighted in blue varies with the length of the chief ray originating at the center of the \( n^{th} \) Virtual Detector pixel and terminating at the center of \( n^{th} \) sensor pixel.

A small change in the optical frequency of the CW source imparts a spherical phase to the impulse response of the indirect imager. The excess phase depends on the incremental change in the wavelength of the CW source (given by \( c\Delta \nu^{-1} \) meters), and the cumulative propagation distance from the obscured object point to the center of the \( n^{th} \) Virtual Detector and sensor pixels. (Observation-2)

Incorporating Eqs.(28)-(29) into Eq.(16) yields the following revised expression for the sampled hologram at the optical frequency \( \bar{v} + \Delta \nu \):

\[
\begin{align*}
    U_{\Delta}[n; \bar{v} + \Delta \nu] & = \left(\frac{\bar{v} + \Delta \nu}{v}\right)^N U_{\bar{v}}[n; \bar{v}] \exp(i\phi[n; \Delta \nu]) \left\{ dp \left(\exp(i\theta[n; p, \Delta \nu]) \left( U_{\text{all}}(p; \bar{v}) \sqrt{b_d(p; \bar{v})} \left( \int dx h_{\text{wd}}(x; p, \bar{v}) \right) \right) \right\} \quad (30)
\end{align*}
\]

The definition of the phasors \( \phi[n; \Delta \nu] \) and \( \theta[n; p, \Delta \nu] \) is furnished below:

\[
\begin{align*}
    \phi[n; \Delta \nu] & \equiv 2\pi \frac{\Delta \nu}{c} \left( z_d + \frac{\|n \Delta\|^2}{2z_d} + \bar{w}_3 + \frac{\|z_d^{-1} \bar{w}_3 n \Delta\|^2}{2\bar{w}_3} \right) \quad \text{scene-independent phasor field representing the change in OPL associated with light transport from} \ n^{th} \ \text{Virtual Detector pixel to the} \ n^{th} \ \text{sensor pixel.}
\end{align*}
\]

\[
\begin{align*}
    \theta[n; p, \Delta \nu] & \equiv 2\pi \frac{\Delta \nu}{c} \left( \|s - p\| + \|p - \left[- \frac{z_d^{-1} \bar{w}_3 n \Delta}{\bar{w}_3}\right]\| \right) \quad \text{scene dependent phasor field representing the change in the OPL of all ray paths originating at the Virtual Source, bouncing off the obscured object and terminating at the} \ n^{th} \ \text{Virtual Detector pixel.}
\end{align*}
\]

A small change \( \Delta \nu \) in the optical frequency of the CW source used to interrogate the hidden scene imparts an additional spherical phase to the field contribution of each obscured object point \( p \). The excess spherical phase encodes the position of the obscured object point \( p \) at the “synthetic” wavelength \( c\Delta \nu^{-1} \) meters. (Observation-3)

1.6 Recovering the latent hologram at the synthetic wavelength \( c\Delta \nu^{-1} \)

Observation-3 forms the basis of our claim (Claim-1) that a holographic description of the obscured object can be recovered despite scattering at the Virtual Source and Virtual Detector surfaces. As a first step
towards corroborating this claim, we seek to computationally mix the sampled holograms at \( \vec{v}, \vec{v} + \Delta \vec{v} \). The resulting expression is shown below:

\[
\mathcal{U}[n; \vec{v}] \times \mathcal{U}[n; \vec{v} + \Delta \vec{v}]
\]

\[
= \left( \frac{N}{\nu_0} \right)^2 \mathcal{U}[n; \vec{v}] \mathcal{U}^*[n; \vec{v} + \Delta \vec{v}] \times \left( \int_{(n-0.5)\Delta}^{(n+0.5)\Delta} \mathcal{U}(x; \vec{v}) \right) \times \left( \int_{(n-0.5)\Delta}^{(n+0.5)\Delta} \mathcal{U}^*(x; \vec{v} + \Delta \vec{v}) \right)
\]

\[
= \left( \frac{\vec{v} + \Delta \vec{v}}{\nu_0} \right)^4 \left( \frac{N}{\nu_0} \right)^2 I[n; \vec{v}] \exp(i\phi[n; \Delta \vec{v}]) \left\{ \int_{p \in \mathbb{P}} dp \left( |\mathcal{U}_{ ill}(p; \vec{v})|^2 B_0(p; \vec{v}) \right) \int_{(n-0.5)\Delta}^{(n+0.5)\Delta} \mathcal{U}^*(x; \vec{v} + \Delta \vec{v}) \right\} \exp(i\phi[n; \Delta \vec{v}]) \left\{ \int_{(n-0.5)\Delta}^{(n+0.5)\Delta} \mathcal{U}(x; \vec{v}) \right\}
\]

(31) yields the following expression for the computational hologram:

\[
\mathcal{U}[n; \vec{v}] \times \mathcal{U}^*[n; \vec{v} + \Delta \vec{v}]
\]

\[
= \left( \frac{\vec{v} + \Delta \vec{v}}{\nu_0} \right)^4 \left( \frac{N}{\nu_0} \right)^2 I[n; \vec{v}] \exp(i\phi[n; \Delta \vec{v}]) \left\{ \int_{p \in \mathbb{P}} dp \left( |\mathcal{U}_{ ill}(p; \vec{v})|^2 B_0(p; \vec{v}) \right) \int_{(n-0.5)\Delta}^{(n+0.5)\Delta} \mathcal{U}^*(x; \vec{v} + \Delta \vec{v}) \right\} \exp(i\phi[n; \Delta \vec{v}]) \left\{ \int_{(n-0.5)\Delta}^{(n+0.5)\Delta} \mathcal{U}(x; \vec{v}) \right\}
\]

Notice that the expression for the computational hologram is comprised of two terms. The first of these

The second term highlighted in gray encapsulates the spectral interference of field contributions of two

The exact expression for the latent hologram may be identified by incorporating the definition of the

The spherical phase factors embedded in Eq.(34) fully encode the position of the obscured object point \( p \) at

Consequently, numerical backpropagation of the latent hologram should permit reconstruction of the light distribution in the hidden volume. However, the ability to localize the obscured object point \( p \) is limited by the amplitude/strength of its contribution to the latent hologram, and determined by the following factors:

- Intensity of the indirect illumination contribution from the Virtual Source, given by \( |\mathcal{U}_{ ill}(p; \vec{v})|^2 \).
- Albedo of the obscured object \( B_0(p; \vec{v}) \).
- Number of speckle cells in the indirect imager PSF \( h_{ill}(x; p, \vec{v}) \) that can be accommodated within a single detector pixel. This number should not be large (< 100) to avoid amplitude/signal fading due to coherent averaging of speckle amplitudes within a detector pixel [4].
• Radiometric fall-off due to propagation from the Virtual Source to the obscured object point \( p \) (encapsulated in the definition of \( U_{\text{ill}}(p; \bar{v}) \)), and propagation from the obscured object point \( p \) to the \( n^{th} \) Virtual Detector pixel (encapsulated in the definition of \( h_{vd}(x; p, \bar{v}) \)).

In practice, the recovery of the latent hologram at the synthetic wavelength is complicated by the presence of the parasitic interference term in the computationally assembled hologram of Eq.(32). Insight into isolating the contributions of this term can be gleaned by examining the exact expression for the parasitic interference component, which is furnished below:

\[
U_{\text{parasitic}}[n; \Delta \nu] = \left( \frac{\bar{\nu} + \Delta \nu}{\nu} \right)^4 \left( \frac{N}{\nu_p} \right)^2 I_s[n; \bar{v}] \exp(i\phi[n; \Delta \nu]) \int_{p \in \mathbb{P}} dp \exp \left\{ i \frac{\Delta \nu}{c} \left( \| \mathbf{s} - p \| + \| p - \left[ -z_d^{-1} \bar{w}_3 n \mathbf{\Delta} \right] \right) \right\} (35)
\]

It is observed that the product of the indirect imager PSF’s for distinct object positions exhibits spatial fluctuations at a scale determined by the numerical aperture of the imaging optics, which is far smaller than the synthetic wavelength. Consequently, the contributions of the parasitic interference term can be suppressed by low-pass filtering the computationally assembled hologram \( U_{\Delta}[n, \bar{v}] \times U_{\Delta}[n, \bar{v} + \Delta \nu] \) with a cutoff frequency of \( \frac{\Delta \nu}{c} \) cycles/meter. The exact same behavior may alternatively be realized by numerical backpropagation of the computational hologram at the synthetic wavelength \( \Lambda = c \Delta \nu^{-1} \) meters.

**Sufficient condition for avoiding wavefront aberration in latent hologram**

The ability to recover a latent hologram of the obscured objects by computational mixing of optical holograms recorded at two closely spaced frequencies is predicated on the validity of approximations A5 and A7. Violation of the assumptions (Eqs.(20) and (27)) underlying these approximations induces wavefront aberrations in the latent hologram. Wavefront aberrations in the latent hologram may be avoided by summing up the constraints of Eqs.(20) and (27), namely:

\[
\text{Rayleigh criterion for wavefront reconstruction by computational mixing of scattered fields at two optical frequencies } \left( \sqrt{\frac{\Theta_{vs}^2}{2}} + \sigma_h^2 + \sqrt{\left( \frac{z_d^{-1} \bar{w}_3 F \Delta}{2} \right)^2 + \sigma_h^2} < \frac{c \Delta \nu^{-1}}{4} \right) (36)
\]

The term \( \Theta_{vs} \) represents the diameter of the beam incident on the Virtual Source surface, while \( z_d^{-1} \bar{w}_3 F \Delta \) represents the physical extent of the active area of a Virtual Detector pixel. The term \( \sigma_h \) represents the RMS roughness of the Virtual Source/Detector surfaces. It is worth noting that the right hand side of Eq.(36) mirrors the Rayleigh quarter wave criterion for the synthetic wavelength, in what can only be described as a serendipitous confluence of constraints.

The inequality of Eq.(36) also divulges a complex tradeoff between the synthetic wavelength \( \Lambda = c \Delta \nu^{-1} \) and the Indirect Imaging system parameters namely: the spatial extent of the Virtual Source, the size of a Virtual Detector pixel, and the RMS roughness \( \sigma_h \) of the Virtual Source/Detector surfaces. It is observed that smaller Virtual Source diameters, smaller Virtual Detector pixels and smoother Virtual Source/Detector surfaces permit the use of a smaller synthetic wavelength, a fact borne out in experiments.

Furthermore, the inequality of Eq.(36) may be recast to obtain a bound on the largest change in optical frequency \( \Delta \nu \), and thereby the smallest synthetic wavelength \( \Lambda = c \Delta \nu^{-1} \) that is free of wavefront aberration. The resulting expression is shown below:
The resolving power of the latent hologram is
\[ \Lambda \equiv c\Delta v^{-1} > 4 \left( \frac{\theta_{ps}}{2} \right)^2 + \sigma_h^2 + \left( \frac{z_d^{-1}w_3F\Delta}{2} \right)^2 + \sigma_h^2 \]  
(37)

It is evident from the above discussion that computational mixing of scattered optical field recorded at two closely spaced frequencies \( v_1 = \bar{v}, v_2 = (\bar{v} + \Delta v) \) preserves phase information at scales smaller than the difference frequency \( \Delta v \), provided the maximum change in path length induced by a change in the optical frequency of interrogation is smaller than the Rayleigh criterion \( \frac{c\Delta v^{-1}}{4} \) meters. 

\[ \text{(Observation-4)} \]

1.7 Resolution limits

In the absence of wavefront aberrations, the resolving power of the latent hologram is fundamentally limited by the synthetic wavelength \( \Lambda = c\Delta v^{-1} \) meters, where \( \Delta v \) is spacing between the optical frequencies used to interrogate the hidden volume. This limit stems from the inability to reproduce spatial detail exceeding \( \Lambda^{-1} = c^{-1}\Delta v \) cycles/meter, when replaying the latent hologram. In practice, the achievable lateral resolution is further limited by the spatial extent of the Virtual Detector, and scales inversely with the propagation distance. The behavior is fully consistent with established limits in classical holography [8]. The expression for the lateral resolution at a nominal backpropagation distance of \( Z \) meters from the Virtual Detector surface is given by \( \Lambda \left( \frac{Z}{M \times (z_d^{-1}w_3\Delta)} \right) \). The product term \( M\Delta \) represents the physical dimension of the image sensor, which is assumed to be square for simplicity. The product term \( z_d^{-1}w_3 \) represents the transverse magnification of the optics used to image the Virtual Detector surface. Consequently, \( M \times (z_d^{-1}w_3\Delta) \) represents the spatial extent of the Virtual Detector.

**Space-bandwidth product (SBP) of latent hologram**

The finite spatial extent of the latent hologram (given by \( M \times (z_d^{-1}w_3\Delta) \) meters) combined with the finite spatial frequency bandwidth (given by \( 2\Lambda^{-1} = 2c^{-1}\Delta v \) cycles/meter) of the latent hologram, imposes a hard limit on the complexity of obscured objects that can be faithfully recorded and reproduced. Their product is a measure of the number of degrees of freedom of the latent hologram, and is referred to as the Space-Bandwidth Product (SBP) [3],[9] in optics literature. The SBP of an optical signal is a measure of its information carrying capacity, and provides an upper bound on system performance. The SBP of the latent hologram is disclosed below:

\[ \text{SBP(} U_{\text{latent}}) \equiv \left( \left( M \times (z_d^{-1}w_3\Delta) \right) \right) \times (2\Lambda^{-1}) \]  
(38)

The product term \( M\Delta \) represents the physical dimension of the image sensor, which is assumed to be square for simplicity. The product term \( z_d^{-1}w_3 \) represents the transverse magnification of the optics used to image the Virtual Detector surface. The expression for SBP disclosed in Eq.(38), is strictly valid when the recovered latent hologram is devoid of any wavefront aberration at the synthetic wavelength. The expression for the smallest synthetic wavelength that is devoid of wavefront aberration was first disclosed in Eq.(37). sufficient condition for avoiding wavefront errors in the latent hologram. Incorporating the aforementioned result into the expression for the SBP, yields an upper-bound on the SBP of the latent hologram:

\[ \text{Upper-bound of } \text{SBP of Indirect Imaging using } \text{Synthetic Wavelength Holography} \]

\[ \text{SBP} \leq \frac{\left( M \times (z_d^{-1}w_3\Delta) \right)}{2 \left( \frac{\theta_{ps}}{2} \right)^2 + \sigma_h^2 + \left( \frac{z_d^{-1}w_3F\Delta}{2} \right)^2 + \sigma_h^2} \]  
(39)
The term $\varnothing_{vs}$ represents the diameter of the beam incident on the Virtual Source surface, while $z_d^{-1}\bar{w}_3\Delta$ and $z_d^{-1}\bar{w}_3F\Delta$ represents the active area and pixel pitch of a Virtual Detector pixel, respectively. The term $\sigma_h$ represents the RMS roughness of the Virtual Source/Detector surfaces. It is worth noting that the expression for SBP disclosed in Eq.(39), closely resembles Eq.(5) of the original manuscript. The term in the denominator may be viewed as a proxy for the peak-valley wavefront aberration at the synthetic wavelength $\Lambda$.

The upper bound on SBP for Synthetic Wavelength Holography disclosed in Eq.(39) is by no means the best that can be achieved from a theoretical standpoint. The asymptotic limit of the SBP for Indirect Imaging using Synthetic Wavelength Holography is obtained as $F \rightarrow 1$, $\varnothing_{vs} \rightarrow c\nu^{-1}$ and $z_d^{-1}\bar{w}_3F\Delta \rightarrow 0.5c\nu^{-1}$. These system parameters represent the absolute best that can achieved from the standpoint of sensor fill factor, spot diameter of the Virtual Source and recording fields with the highest resolution using classical optics. Under these restrictions, the RMS roughness of the intermediary surface $\sigma_h \gg \varnothing_{vs}, z_d^{-1}\bar{w}_3F\Delta$ so that the asymptotic SBP is given by:

$$\text{SBP} < (Mc\nu^{-1}) \left( \frac{1}{4\sigma_h} \right)$$  \hspace{1cm} (40)

Consequently, the information carrying capacity of Synthetic Wavelength Holography is fundamentally limited by the roughness of the intermediary scattering surfaces that are adapted to serve as the Virtual Source/Detector. A comparison of the definition of the Space-Bandwidth product, disclosed in Eq.(38), and the expression for the asymptotic SBP disclosed in Eq.(40), reveals a fundamental limit to the achievable resolution in Synthetic Wavelength Holography, and is given by $\Lambda > 8\sigma_h$.

Localisation accuracy of the latent hologram

In the absence of wavefront aberrations, the longitudinal resolution of the latent hologram is limited to

$$2\Lambda \left( \frac{z}{M \times (z_d^{-1}\bar{w}_3\Delta)} \right)^2,$$

and fully consistent with established limits in classical holography [8]. This limits our ability to precisely localize obscured objects within the hidden volume, and its impact is best illustrated in Figure-5 of the main manuscript. The traditional approach to tackling this problem is time-resolved holography using pulsed sources. Unfortunately, pulsed sources lack the temporal coherence needed to record holograms of objects obscured from view. The problem may be remedied by examining a computational approach to synthesizing light pulses using a multitude of optical frequencies, and is the topic of Section-2.

2. Optical sectioning using a multitude of regularly spaced frequencies

It is common knowledge that an ultrashort pulse train admits a Fourier series decomposition in the optical frequency domain. Herein we seek to computationally mimic the behavior by independently interrogating the hidden scene using a countably finite number of regularly spaced optical frequencies, and record the corresponding holograms. By computationally delaying the hologram recorded at each optical frequency, and accumulating the result across optical frequencies, it is possible to mimic interrogation of the hidden scene by a pulse train. The process is illustrated in Figure3, and is inspired by work in Fourier Synthesis Holography [10], and holographic laser radar [11],[12].

The periodicity of the pulse train is determined by the smallest separation between the optical frequencies, while the pulse duration is determined by the largest separation between the optical frequencies. It can be shown that $K$ optical frequencies spaced apart by $\Delta\nu$ Hz can be used to computationally synthesize a periodic optical pulse train, with pulse duration $\tau_p = (K\Delta\nu)^{-1}$ and repetition rate $\tau_{rep} = \Delta\nu^{-1}$ seconds. The shape of the pulse may be manipulated by appropriately weighting the holograms recorded at each optical frequency, prior to accumulation. In the simple case that the weights
are chosen uniformly, the pulse envelope resembles a sinc-like function. By additionally delaying the hologram at each optical frequency, it is possible to synthesize a delayed pulse train that preferentially selects ray paths from the Virtual Source to the Virtual Detector with a prescribed round-trip distance. The notion is illustrated in the gary box designated “Computational path-length filtering” in Figure 3.

Unfortunately, the computationally filtered hologram $U_{\text{filt}}[n]$ is still plagued by speckle at the optical frequency. The problem is remedied by computationally mixing the filtered hologram with the hologram recorded at the optical frequency $\bar{\nu}$. The expression for the resulting latent hologram closely resembles the expression for the two frequency latent hologram disclosed in Eq. (34). The distinction emerges in the additional attenuation introduced by the path length filter $g \left( \| \bar{s} - p \| + \| p - \left[ -z_d^{-1} \bar{w}_3 n_d \right] \| - D \right)$. This filter promotes constructive interference of light at the synthetic wavelength for select propagation distances, and promotes destructive interference at remaining propagation distances.
Figure 3: Optical sectioning using a multitude of regularly spaced frequencies
The periodicity of the synthesized pulse train introduces periodic ambiguities in the response of the path length filter. This fact is corroborated in the experimental results of Figure 5 furnished in the main manuscript.

The latent hologram assembled using the proposed approach is insensitive to scattering at the Virtual Source and Virtual Detector, features a lateral resolution limit of $c \left( \frac{(K-1) \Delta v}{2} \right)^{-1}$ meters and a longitudinal resolution of $2c(K\Delta v)^{-1}$ meters.

3. Relating the optical blur at $\bar{v} + \Delta v$ to the optical blur at $\bar{v}$

The optical blur associated with imaging the VD surface may be modeled as a paraxial blur [3] with image side numerical aperture $Dz_a^{-1}$, where $D$ is the exit pupil diameter. The resulting amplitude PSF was originally disclosed in Eq.(2), and repeated below for the benefit of the reader.

$$h_{\text{blur}} \left( x \equiv \frac{x_1}{x_2}; w \equiv \frac{w_1}{w_2}, \bar{v} \right) = \int dudv \left( P(u,v) \exp \left( \frac{2\pi\bar{v}}{c} \left[ x_1 + \frac{x_2^2}{2z_d} + \frac{w_1^2 + w_2^2}{2w_3} \right] \right) \right)$$

The term $P(u,v)$ represents the transmittance function of the exit pupil and modeled as an indicator function with diameter $D$. The term $f$ represents the focal length of the imaging optic. The depth $w_3$ of a point on the VD surface is measured with respect to the entrance pupil plane of the imaging optic.

The expression for the optical blur resulting from a small change in the optical frequency of the CW illumination source is disclosed below:

$$h_{\text{blur}} \left( x \equiv \frac{x_1}{x_2}; w \equiv \frac{w_1}{w_2}, \bar{v} + \Delta v \right) = \left( \begin{array}{c} \left( \frac{\bar{v} + \Delta v}{i} \right)^2 \frac{1}{z_d w_3} \exp \left( i2\pi \left( \frac{\bar{v} + \Delta v}{c} \right) \left[ z_d + w_3 + \frac{x_1^2 + x_2^2}{2z_d} + \frac{w_1^2 + w_2^2}{2w_3} \right] \right) \\
\int dudv \left[ P(u,v) \exp \left( i2\pi \left( \frac{\bar{v} + \Delta v}{c} \right) \left[ \frac{1}{z_d} + \frac{1}{w_3} - \frac{1}{f} \left( u^2 + v^2 \right) \right] \exp \left( -i2\pi \left( \frac{\bar{v} + \Delta v}{c} \right) \left[ \frac{x_1^2 + x_2^2}{2z_d} + \frac{w_1^2 + w_2^2}{2w_3} \right] \right) \right) \\
\int dudv \left[ \exp \left( i2\pi \left( \frac{\Delta v}{c} \right) \left[ \frac{1}{z_d} + \frac{1}{w_3} - \frac{1}{f} \left( u^2 + v^2 \right) \right] \exp \left( -i2\pi \left( \frac{\Delta v}{c} \right) \left[ \frac{x_1^2 + x_2^2}{2z_d} + \frac{w_1^2 + w_2^2}{2w_3} \right] \right) \right) \right) \end{array} \right)$$

(42)

**Observation-4:** The defocus aberration $\exp \left( i \frac{2\pi \Delta v}{c} \left[ \frac{x_1^2 + x_2^2}{2z_d} + \frac{w_1^2 + w_2^2}{2w_3} \right] \right)$ induced by a small change in the optical frequency can be omitted so long as $\left[ \frac{1}{z_d} + \frac{1}{w_3} - \frac{1}{f} \frac{\Delta v}{c} \right] \ll \frac{1}{2\pi}$. This implies that the defocus in waves $< \frac{1}{8\pi}$ the synthetic wavelength $c\Delta v^{-1}$ meters.

Incorporating the aforementioned constraint into Eq.(42) yields the revised expressions for optical blur disclosed in Eqs.(43)-(44):
\[
\begin{align*}
\int \frac{du dv}{2\pi} & \exp \left( \frac{iu \varphi}{c} \right) \exp \left( -\frac{1}{2f} (u^2 + v^2) \right) \left( \frac{\varphi}{c} + \frac{1}{2} \right) \frac{1}{x_0}
\end{align*}
\]

The Dirac-delta function in Eq.(44) is a restatement of the imaging identity: \( x_1 = -z_d w_1 w_3, x_2 = -z_d w_2 w_3 \) first disclosed in Eq.(1), and is intrinsic to the process of imaging the Virtual Detector surface. The integral highlighted in blue yields a function of the form \( g \left( \frac{\varphi}{c} \left( \frac{x_1}{z_d} \right) + \frac{w_1}{w_3} \right) \), which when convolved with the Dirac-delta function remains unchanged in functional form. This behavior can be attributed to the fact that the centroid of the optical blur spot \( h_{\text{blur}}(x; w, \varphi) \) also satisfies the imaging identity. In view of this relation, the Dirac-delta function can be ignored from subsequent analysis. Consequently,

\[
\begin{align*}
\int \frac{du dv}{2\pi} & \exp \left( \frac{iu \varphi}{c} \right) \exp \left( -\frac{1}{2f} (u^2 + v^2) \right) \left( \frac{\varphi}{c} + \frac{1}{2} \right) \frac{1}{x_0}
\end{align*}
\]

Further simplification is possible if it is assumed that the RMS roughness of the VD surface is much smaller than the mean depth of the Virtual Detector surface, so that

\[
\frac{1}{w_3} = \frac{1}{w_3} - \frac{\delta w_3}{w_3^2}
\]

where the depth \( w_3 \) of a point on the Virtual Detector surface is assumed to be a 2D random variable with mean value \( \bar{w}_3 \) and a zero-mean stochastic term \( \delta w_3 \) representing the surface height fluctuations. If additionally the peak height fluctuation in the VD surface \( \delta w_3 \) \( w_3 \) \( \frac{c(\Delta \nu)^{-1}}{8 \bar{w}_3} \), then the linear phase fluctuation \( \frac{2\pi \Delta \nu \delta w_3}{c \bar{w}_3^2} \) induced by the optical roughness of the Virtual Detector surface may be omitted from further consideration. Incorporating the above into Eqs.(45)-(46) yields the following simplified relation between the optical blurs at \( \varphi, \varphi + \Delta \nu \):

\[
\frac{1}{w_3} \approx \frac{1}{w_3} - \frac{\delta w_3}{w_3^2}
\]
\[ h_{\text{blur}} \left( x \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; w \equiv \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} , \nu + \Delta \nu \right) = \left( \frac{\nu + \Delta \nu}{\nu} \right)^2 \exp \left( i 2 \pi \frac{\Delta \nu}{c} \left[ z_d + \frac{[x_1^2 + x_2^2]}{2z_d} + \bar{w}_3 + \frac{[w_1^2 + w_2^2]}{2\bar{w}_3} \right] \right) h_{\text{blur}} \left( x \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; w \equiv \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} , \nu \right) \] (47)

This concludes the proof.

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