Hydraulic Jump

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Abstract. We consider the issue of the mechanism of the hydraulic jump. Combining viscosity with the existence of a horizon in the inviscid limit which allows only one way passage of information gives a reasonable picture. Some recent experiments which seem to support such a point of view are discussed.

1. Introduction

Hydraulic jump, discussed in almost all text books [1, 2, 3, 4], is a reasonable ubiquitous phenomena occurring in kitchen sinks, spillway of dams, in tidal bores and so on. The basic phenomena consists of a fast flowing fluid on a substrate which after traversing a distance jumps by a certain amount and thereafter flows out as a reduced speed. The flow could be circular as shown in Fig. 1 (the kitchen sink) where it originates from a point (where the jet from the tap hits the sink) and flows radially out or it could be channel like originating along a line like element (Fig. 2). The basic question is that given the constant volumetric flow rate (to be denoted by $Q$) and the parameters of the fluid, is it possible to predict the position (radius) of the jump and the amount of jump [5, 6, 7, 8]. This has turned out to be an impossibly high requirement and there is no clear cut answer. In the early nineties, Bohr, Dimon and Putkaradze [9, 10] offered an extremely useful way of analyzing the jump. In effect, this treated the jump as a shock and recent experiments [11, 12] have shown the relevance of the shock quite clearly in the high viscosity, low surface tension regime.

A different twist was brought to this problem in the last decade by Schützhold and Unruh [13] who showed that for an irrotational flow in a channel, a small perturbation around the steady flow satisfies a Klein-Gordon equation with a Schwarzschild metric. Subsequently, the jump position became with the horizon of the Schwarzschild metric [14, 15] leading to an interesting physical picture where the shock and the horizon are combined with help of viscosity to lead to the formation of the jump [16, 17]. An interesting outcome of the inclusion of viscosity is an instability [17] where an incoming perturbation from outside the jump increases in amplitude as the horizon is approached.
Figure 1. Circular hydraulic jump in a kitchen sink.

Figure 2. An experimental set up to study one dimensional hydraulic jump.

2. Channel Flow
We consider a one dimensional channel flow to begin with where a fast moving fluid layer (velocity $v_1$, height $h_1$) suddenly jumps to height $h_2$ and flows with a lower velocity, $v_2$. We ask the question whether the laws of inviscid hydrodynamics would allow such a motion. Referring to Fig. 3 and using as a control volume, the region bounded by the dashed lines, we can write the mass conservation as (incompressible fluid),

$$v_1 h_1 = v_2 h_2 = \frac{Q}{L},$$  \hfill (1)

where $L$ is the channel width and $Q$ the volumetric flow rate. Newton’s law applied to the control volume would require the momentum flowing out per unit time, $L v_2^2 h_2 - L v_1^2 h_1$, to be equal to the force due to the pressure difference i.e. $\frac{1}{2} L \rho g (h_1^2 - h_2^2)$ and hence,

$$v_2^2 h_2 - v_1^2 h_1 = \frac{1}{2} g (h_1^2 - h_2^2).$$  \hfill (2)

With $h_2/h_1 = X$ and $F = v_1^2 / \rho g h_1$, the Froude number, Eqs. (1) and (2) yield,

$$2F = X(1 + x),$$  

or  

$$2X = -1 + \sqrt{1 + 8F}. \hfill (3)$$

It is possible to have $X \geq 1$ for $F \geq 1$. This implies that for the jump to occur the fluid velocity
has to be exceed \( \sqrt{gh} \), which is the speed of capillary waves in a thin fluid layer. This fact will become important as we go along. However, our approach so far cannot locate the position or calculate the magnitude of the jump. For that, we need to look at a more detailed picture. The steady state differential equations for incompressible fluid in a channel of width \( L \) are (assuming the width \( L \) is large enough to ignore motion in the \( y \)-direction),

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (4)
\]

\[
u \frac{\partial^2 u}{\partial z^2} = \frac{\partial P}{\partial x} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} + \nu \nabla^2 u. \quad (5)
\]

The constant volumetric flow rate is,

\[
Q = Lhu, \quad (6)
\]

where \( h(x) \) is the height of the fluid layer. The pressure \( P \) is clearly \( \Pi + (h(x) - z)\rho g \), where \( \Pi \) is the atmospheric pressure and the local coordinates is \((x, z)\). The primary variation is along \( z \)-direction and hence

\[
u \frac{\partial^2 u}{\partial z^2} = \frac{\partial P}{\partial x} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} + \nu \nabla^2 u. \quad (7)
\]

In eqs. (4), (6) and (7), we have three equations for the three unknowns \( u, w \) and \( h \). The boundary conditions are the so called no-slip conditions at \( z = 0 \), which means \( u(x, 0) = w(x, 0) = 0 \) for all \( x \) and stress free at the surface i.e. \( \nu(x, h(x)) = 0 \) and \( \partial u / \partial z = 0 \)

It is difficult to solve these equations with the given boundary conditions. Numerically a decent scheme for solving these equations have been devised by Dasgupta and Govindarajan [18] but even their scheme does not yield an answer for the amount of the jump. To get a hold on both the position and magnitude of the jump, we resort to an approximation introduced by Bohr et al.

The technique of reference [9], is to introduce variables which are averaged over the \( z \)-direction, i.e. we use \( \langle f(x, z) \rangle = f(x) \), where

\[
\langle f(x, z) \rangle = \frac{1}{h(x)} \int_0^h f(x, z) dz. \quad (8)
\]

Consequently, we average Eq. (7) over \( z \) and find

\[
\langle \frac{\partial u}{\partial x} \rangle + \langle \frac{\partial w}{\partial z} \rangle = \frac{\partial h}{\partial x} + \nu \langle \frac{\partial^2 u}{\partial z^2} \rangle. \quad (9)
\]
An integration by parts shows \( \langle w \frac{\partial u}{\partial x} \rangle = \langle u \frac{\partial w}{\partial x} \rangle \) on using the continuity equation. Further, 
\[
\frac{\partial^2 u}{\partial z^2}\bigg|_{z=0} \simeq -\frac{\langle w \rangle}{h^2},
\]
on using the stress free boundary condition. Thus,
\[
2 \left< u \frac{\partial u}{\partial x} \right> = -g \frac{dh}{dx} - \nu \frac{\langle u \rangle}{h^2}. \tag{10}
\]
We use a factorization approach to write,
\[
\left< u \frac{\partial u}{\partial x} \right> = \alpha \langle u \rangle \left< \frac{du}{dx} \right>, \tag{11}
\]
where \( \alpha \) is a number of \( \mathcal{O}(1) \). Noting that an averaging of Eq. (6) leads to \( Q = \langle u \rangle hL \), we find
\[
\left( \alpha \frac{Q^2}{L^2} - gh^3 \right) \frac{dh}{dx} = \frac{\nu Q}{L}. \tag{12}
\]
The above equation can easily be integrated to obtain \( h(x) \). It is obvious that a blind use of the solution, which can be written as
\[
\alpha \frac{Q^2}{L^2} h - g \frac{h^4}{4} = \frac{\nu Q}{L} x + C \tag{13}
\]
would lead to a double valued solution. This is a clear cut indication that there are two branches, which need to be connected by a shock. The anticipated form is shown in Fig. 4. Before proceeding, we would like to write Eq. (13) in a dimensionless form. This can be done in the following fashion. First, we write Eq. (13) as
\[
\alpha \frac{Q^2}{L^2} h - g h^4 \frac{h}{4} = \frac{\nu Q}{L} x + C \tag{14}
\]
where \( h_0 = (Q^2/L^2 g)^{1/3} \) has the dimension of length and provides the scaling for the height \( h \). The dimensionless height \( H = h/h_0 \) and hence we have
\[
\alpha H - 4 H^4 = \frac{\nu g^{1/3}}{(Q/L)^{5/3}} + \bar{C} = X + \bar{C} \tag{15}
\]
where \( X = x/x_0, \) \( x_0 = (Q/L)^{5/3} \nu^{-1} g^{-1/3} \) and \( \bar{C} \) is a dimensionless constant of integration. We need two branches and hence two boundary conditions. For the flow to the left of the shock (located at \( x_c \)) in Fig. 4, we can use \( H = 0 \) at \( X = 0 \), so that
\[
\alpha H - 4 H^4 = X \quad (X < x_c). \tag{16}
\]
For \( X > x_c \), we need to fix the flow at a distance. If \( H \to 0 \) at \( X = D \), then
\[
\frac{1}{4} H^4 - \alpha H = D - X \quad (X > x_c). \tag{17}
\]
We now plot \( H \) vs \( X \) from Eq. (17). This is the curve marked ‘a’ in Fig. 5. Across the shock (inviscid approximation good), mass and momentum balance imply Eqs. (1) and (2). For a
given \( X, H(X > X_c) \) and thus \( h_2 \) is found from Eq. (17) and the corresponding \( v_2 \) from Eq. (1) and then Eq. (2) yields,

\[
\frac{2v_2^2}{gh_2} = \frac{h_1}{h_2} \left( 1 + \frac{h_1}{h_2} \right),
\]

from which \( h_1/h_2 \) and hence \( h_1(x) \) is obtained. This \( h_1(x) \) is shown as the curve marked 'b' in Fig. 5. We now plot the solution \( h_1(x) \) which we find from Eq. (16) and this is the curve marked 'c' in Fig. 5. The point P, where the two curves 'b' and 'c' meet is the point where the solution to the governing equation of motion (Eq. (12)) gives the correct amount of discontinuity and hence P is the location of the shock. The \( h(x) \) that is physically relevant follows curve 'c' upto point P, then has a discontinuous jump and finally follows the curve marked 'a'. It is clear that the radius of the jump (i.e. the position corresponding to \( X_c \)) scales as \((Q/L)^{5/3}g^{-1/3}\) while the magnitude of the jump scales as \((Q/L)^{2/3}g^{-1/3}\). It is the magnitude of jump that is not addressed usually. The procedure of locating the shock gives the magnitude of the jump. The scaling property of the radius of the jump for the two dimensional flow (Fig. 1) was found by Bohr et al to be \( r_c \propto Q^{5/8}\nu^{-3/8}g^{-1/8} \). Thus we have,

One dimensional flow: \( r_c \propto \left( \frac{Q}{L} \right)^{5/3}\nu^{-1}g^{-1/3} \)

Two dimensional flow: \( r_c \propto Q^{5/8}\nu^{-3/8}g^{-1/8} \) \hspace{1cm} (19)

In the next section, we will take, in some sense, a more physically motivated look at the jump

and this will help get both the above relations from the same picture.

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**Figure 4.** Schematic diagram of shock.

**Figure 5.** Schematic diagram...
3. Circular Jump

For the circular jump (Fig. 1), the approximation carried out by Bohr et al had led to the steady state equations

\[
rvh(r) = \text{Const.} = \frac{Q}{2\pi} \tag{20}
\]

\[
v \frac{\partial v}{\partial r} = -g \frac{\partial h}{\partial r} - \nu \frac{v}{h^2} \tag{21}
\]

The procedure for analyzing the above as carried through by Bohr et al is actually very similar to locating the shock as described for the one dimensional flow given in Sec. 2. We assume that \(v_0(r)\) and \(h_0(r)\) are the steady state solutions that describe the hydraulic jump. We consider time dependent perturbation around this solution. To do so we need the dynamic equations and these are,

\[
\frac{\partial h}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(rvh(r)) = 0, \tag{22}
\]

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = -g \frac{\partial h}{\partial r} - \nu \frac{v}{h^2}. \tag{23}
\]

We can use these equations to discuss perturbations around the static flow \(h = h_0(r), v = v_0(r)\). This is best done in a linear stability analysis in terms of the variable,

\[f = rvh,\tag{24}\]

which is a constant \(f_0\) in the static limit. If we denote the variations around \(f_0, h_0, v_0\) by \(f', h', v'\) and restrict ourselves to terms which are linear in \(f', h'\) and \(v'\), then

\[f' = r(v_0h' + h_0v')\tag{25}\]

From Eq. (22),

\[\frac{\partial h'}{\partial t} = -\frac{1}{r} \frac{\partial f'}{\partial r} \tag{26}\]

and using this with Eq.( 25), we are led to

\[\frac{\partial v'}{\partial t} = \frac{1}{rh_0} \left( \frac{\partial f'}{\partial t} \right) + \frac{v_0}{r_0h_0} \left( \frac{\partial f'}{\partial r} \right) \tag{27}\]

The linearization of Eq.( 23) yields

\[\frac{\partial v'}{\partial t} + v_0 \frac{\partial v'}{\partial r} + v' \frac{\partial v_0}{\partial r} = -g \frac{\partial h'}{\partial r} - \nu \frac{v'}{h_0^2} + 2\nu \frac{v_0}{h_0^3} \tag{28}\]

taking the partial derivatives with respect to time in Eqs. (27) and (28) and equating the two expressions of \(\frac{\partial v'}{\partial t}\) so obtained leads after some manipulation to

\[\frac{\partial}{\partial t} \left[ v_0 \frac{\partial f'}{\partial t} \right] + \frac{\partial}{\partial r} \left[ v_0^2 \frac{\partial f'}{\partial r} \right] + \frac{\partial}{\partial \theta} \left[ v_0^2 \frac{\partial f'}{\partial \theta} \right] + \frac{\partial}{\partial r} \left[ v_0(v_0^2 - g h_0) \frac{\partial f'}{\partial r} \right] = -\nu \frac{v_0}{h_0^2} \left( \frac{\partial f'}{\partial t} + 3v_0 \frac{\partial f'}{\partial r} \right) \tag{29}\]

This result can be cast in a compact form

\[\partial_a \left( F_{\alpha \beta} \partial_{\beta} f' \right) = -\nu \frac{v_0}{h_0^2} \left( \frac{\partial f'}{\partial t} + 3v_0 \frac{\partial f'}{\partial r} \right) \tag{30}\]
where $F^{\alpha\beta}$ is the $2 \times 2$ symmetric matrix,

$$F^{\alpha\beta} = \begin{pmatrix} 1 & v_0^2 - v_0 g h_0 \\ v_0 & v_0^2 - g h_0 \end{pmatrix} \quad (31)$$

Let us focus on the inviscid limit, when the right hand side of Eq. (30) is zero and the perturbation $f'$ satisfies the wave equation

$$\Box^2 f' = \frac{1}{\sqrt{-g}} \partial_\alpha \left( \sqrt{-g} g^{\alpha\beta} \partial_\beta f' \right) = 0 \quad (32)$$

where we make the identification $F^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta}$ and $g = \text{Det} F^{\alpha\beta}$, so that near critical condition $v_0^2 = g h_0$, we can identify

$$g^{\alpha\beta} = \begin{pmatrix} 1 & v_0 \\ v_0 & v_0^2 - g h_0 \end{pmatrix} \quad (33)$$

This is a remarkable result. The perturbation around the static jump solution in the inviscid limit satisfies a Klein-Gordon equation for a massless particle in the Schwarzschild background if we identify the velocity of light with $\sqrt{g h_0}$. It should be noted that our derivation here following references [16] and [17] is different in character from that of the pioneering work of Schützhold and Unruh. It is specific to the jump problem and consequently able to throw light on the mechanism of the jump. An identical calculation holds for the one dimensional case as shown in [16].

The important thing to note is that the perturbation to the jump profile satisfies a wave equation in a background metric which has a horizon defined by $v_0^2 = g h_0$. Now the wave propagating in a shallow water of height $h$ is the usual capillary wave with the speed $\sqrt{g h}$. The horizon condition above implies that the flow speed of the fluid equals the wave speed. Let us combine the picture with what we saw in the last section. There is a fast flowing fluid on one side of the jump (the interior or the supercritical region) and then a slower flow on the other (exterior or subcritical region) side. The jump was modelled by fitting a shock. We can identify the shock with the horizon. In the interior of the jump the velocity exceeds the wave speed $\sqrt{g h}$ and hence the wave cannot propagate in that region. The waves carry information and hence there can be no information carried from outside the jump to the inside. This ties up with the picture of the horizon interpreted as a white hole - the information can only come out, it cannot go in. This is crucial to the jump formation. Everywhere interior to the jump $v_0 > \sqrt{g h_0}$, in spite of the fact that the fluid is slowing down due to viscosity (an ingredient not considered in the horizon calculation). The fast moving layers at small values of $r(< r_c)$ do not know that the fluid in front has slowed down (no information transmission) and when they reach $r = r_c$, they "jump" over the slowed down fluid causing a jump at $r = r_c$. There are two essential points in the jump:

(i) In the inviscid limit, the perturbation to the static flow live in a space with a Schwarzschild like metric with $c$ replaced by $\sqrt{g h}$ - the velocity of capillary waves. The metric leads to the existence of a horizon where $v = \sqrt{g h_0}$. This is the barrier for information transmission. No information crosses to the left of the barrier where $v > \sqrt{g h}$.

(ii) The jump is caused by the viscosity which slows down the flow from its origin to the horizon but due to (i) above, this information does not reach the fast moving flow. When at the "horizon" ($v = \sqrt{g h}$) there is sudden realization and thus there is an accumulation that occurs to the jump.

Clearly, the shock of the last section is the horizon of this and a clear experimental realization of this can be seen from the work of Jannes et al [12]. The formation of the Mach cone shown
in Fig. 6, taken from the reference [12] clearly shows that the jump occurs at \( v = \sqrt{gh} \) and \( v > \sqrt{gh} \) in the interior of the jump.

We now need to obtain a qualitative position of the jump that will be valid for any dimension. For this we need to know the viscous time scale. To do so, we write a possible solution for \( f_0 (r; t) \) as \( e^{-\lambda t} \phi(r) \) and use Eq.(29) to obtain

\[
\lambda^2 \phi - \lambda \left[ 2 \frac{\partial}{\partial r} (v_0 \phi) + \frac{\nu \phi}{h_0^2} \right] + \frac{1}{v_0} \left[ v_0 (v_0^2 - gh_0) \phi' \right] + 3\nu \frac{v_0 \phi'}{h_0^2} = 0.
\]

Taking a slowly varying \( \phi \), so that \( \phi' \) can be dropped in comparison and integrating over a region at the boundaries of which \( \phi \) vanishes (localized \( \phi(r) \)), the last two terms of Eq.34 are dropped and then (we use \( v_0 \phi \) to multiply before integrating)

\[
\lambda^2 \int v_0 \phi^2 dr - \lambda \nu \int \frac{v_0 \phi^2}{h_0^2} dr = \lambda \int \frac{\partial}{\partial r} (\phi v_0)^2 dr = 0,
\]

giving \( \lambda \simeq \frac{\nu}{h_0} \). (35)

if \( h_0 \) is slowly varying.

Thus we obtain the time scale in which viscosity slows down the velocity. This is consistent with what one obtains from Eq.(23) if only the viscous term is kept. That gives \( v \simeq -\nu v/h_0^2 \), showing \( v \approx v_0 e^{-\nu v/h_0^2} \) yielding Eq.(35). The viscous time scale is always \( \tau = h_0^2/\nu \). Now, we need the time scale for the flow to reach the jump point. In the inviscid limit, the only velocity scale is \( \sqrt{gh} \) and hence the time to the jump point is \( \tau' \approx r_c/\sqrt{gh} \). The jump in the above scenario occurs when \( \tau = \tau' \), i.e.

\[
\frac{r_c}{\sqrt{gh}} \approx \frac{h_0^2}{\nu} \quad \text{or} \quad r_c \approx g^{1/2} h_0^{5/2} \nu^{-1}.
\]

(36)

in a base flow in \( D \)-dimensions, the volumetric flow rate is

\[
Q = r^{D-1} v - 0 h_0.
\]

(37)

Evaluating every quantity at \( r = r_c \),

\[
Q = r_c^{D-1} \sqrt{gh_0 h_0} = r_c^{D-1} g^{1/2} h_0^{3/2}.
\]

(38)

Substituting for \( h_0 \) from Eq.(38) into Eq.(38) we finally get,

\[
Q = r_c^{D-1} g^{1/2} (r_c \nu g^{-1/2}) \approx r_c^{D-2/5} g^{1/5} \nu^{3/5}.
\]

(39)
For $D = 1$, we get

$$r_c \simeq Q^{5/3} g^{-1/3} \nu^{-1},$$

(40)

while for $D = 2$, we have

$$r_c \simeq Q^{5/8} g^{-1/8} \nu^{-3/8},$$

(41)

exactly in agreement with Eq.(19).

4. Viscosity Induced Instability

We return to Eq.(30), with a view to looking at travelling wave solutions which should certainly be the interesting ones. Writing

$$f' = Ae^{iS(r,t)} = Ae^{i(\omega t - \psi(r))}.$$  

(42)

Straightforward calculations yield

$$\omega^2 v_0 - 2\omega v_0^2 \psi' + v_0(v_0^2 - c^2)\psi'^2 + i\omega \partial_r v_0^2 - i\partial_r [v_0(v_0^2 - c^2)\psi'] = \frac{\nu v_0}{h_0} (i\omega + 3iv_0\psi')$$

(43)

In the above $c^2 = gh(r)$ and $\psi' = \partial_r \psi$. If we assume that $\psi(r)$ is primarily real i.e.

$$\psi(r) = \psi_R(r) + i\psi_I(r)$$

with $\psi_I \ll \psi_R$, then

$$\psi_R = \omega \int \frac{dr}{v_0 \pm c}$$

(44)

and

$$2 [\omega v_0^2 - v_0(v_0^2 - c^2)\psi_R'] \psi_I' = -\frac{\nu v_0}{h_0} (\omega + 3v_0\psi_R')$$

(45)

The real part is singular when $v_0 = c$ i.e. at the jump radius and since this is the dominant contribution, we will focus on $\psi_R' = \frac{\omega}{v_0 - c}$. This leads to

$$\psi_I' = -\frac{\nu}{2c h_0} \left[ 1 + \frac{3v_0}{v_0 - c} \right].$$

(46)

For $v_0 < c$, $\psi_R'$ is negative and hence we are discussing a wave that is travelling towards the jump radius. The growing part of $\psi_I'$ (see Eq.(46)) is positive at this point and hence the travelling wave gets magnified by the factor $e^{3v_0/2c h_0}$ which blows up for $c = v_0$. Hence the amplitude of the perturbation grows as one approaches the jump. There is a large amplitude of

Figure 7. xxxxx
the perturbation at the jump (this could be some kind of an analogy of the Miles instability) - certainly an instability triggered by viscosity. This dramatic effect of viscosity has been captured by Kate et al [19]. A hydraulic jump set up as shown in Fig. 1. Now one needs to perturb this flow from outside the jump. To do that a second tap is arranged setting up a second jump whose distance from the first is greater than the radius of the first. The flowing water from the second tap is a perturbation on the first, while the flow from the first is a boundary for the second. At the junction the viscosity makes the fluid pile up as exhibited in Eq.(46) and so clearly demonstrated in Fig. 7.

5. Conclusion
We thus arrive at a picture where the inviscid limit gives rise to a horizon across which the region external to the jump (slow moving) cannot communicate with the internal region (fast moving). Adding viscosity to this requires fast moving fluid to know that fluid to know that the fluid in front has slowed down. Since the communication cannot be transmitted, the knowledgs is not received till one is at the horizon and hence the jump as the fast moving fluid "climbs over" over the slowed down flow.

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