THE OSTROGRADSKY SERIES AND RELATED PROBABILITY MEASURES

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Abstract. We develop a metric and probabilistic theory for the Ostrogradsky representation of real numbers, i.e., the expansion of a real number $x$ in the following form:

$$x = \sum_{n} \frac{(-1)^{n-1}}{q_1 q_2 \ldots q_n} = \sum_{n} \frac{(-1)^{n-1}}{g_1 (g_1 + g_2) \ldots (g_1 + g_2 + \ldots + g_n)} \equiv \bar{O}^1(g_1, g_2, \ldots, g_n, \ldots),$$

where $q_{n+1} > q_n \in \mathbb{N}$, $g_1 = q_1$, $g_{k+1} = q_{k+1} - q_k$. We compare this representation with the corresponding one in terms of continued fractions.

We establish basic metric relations (equalities and inequalities for ratios of the length of cylindrical sets). We also compute the Lebesgue measure of subsets belonging to some classes of closed nowhere dense sets defined by characteristic properties of the $\bar{O}^1$-representation. In particular, the conditions for the set $C[\bar{O}^1, V]$, consisting of real numbers whose $\bar{O}^1$-symbols take values from the set $V \subset \mathbb{N}$, to be of zero resp. positive Lebesgue measure are found. For a random variable $\xi$ with independent $\bar{O}^1$-symbols $g_n(\xi)$ we prove the theorem establishing the purity of the distribution. In the case of singularity the conditions for such distributions to be of Cantor type are also found.

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Introduction

There are many different methods for the expansions and encodings (representations) of real numbers by using a finite as well as infinite alphabet $A$. The $s$-adic expansions, continued fractions, $f$-expansions, the Lüroth expansions etc. are widely used in mathematics (see, e.g., [16]). Each representation has its own specificity and it generates its “own geometry” and metric theory. To each representation there is associated system of cylindrical sets, which forms a system of partitions of the unit interval (real line). We also have a corresponding “coordinate system” (a union of conditions for the determination of the position of a point) which is a convenient tool for the description of a wide class of fractals in a simple formal way. From the ratios of the lengths of cylindrical sets the basic metric relations follow (in the form of equalities and inequalities) which are crucial for the development of the corresponding metric theory, i.e. a theory about measure (e.g., Jordan, Lebesgue, Hausdorff, Hausdorff-Billingsley,...) of sets of real numbers defined by characteristic properties of their digits in the corresponding representation.

Let $A$ be an alphabet of symbols for the representation of numbers in some fixed system of representation, let $\alpha_k(x)$ be the $k$-th symbol of the representation $\Delta_{\alpha_1, \alpha_2, ..., \alpha_n}$ of a real number $x \in [0, 1]$, let $N_i(x, n)$ be the number of the symbol “$i$” among the first $n$ symbols of the representation of $x$, $i \in A$, and let $\nu_i(x) = \lim_{n \to \infty} \frac{N_i(x, n)}{n}$ (supposed to exist).

The following sets are traditional objects for the investigations in the metric theory:

$E_f\left(\frac{k_1k_2 \cdots k_n}{c_1c_2 \cdots c_n}\right) = \{x : \alpha_k(x) = c_i \in A\}$,

$C[f, \{V_k\}] = \{x : \alpha_k(x) \in V_k \subset A\}$,

$M[f, \tau] = \left\{x : \nu_i(x) = \tau_i, \forall i \in A; \tau = (\tau_1, \tau_2, \ldots), \tau_i \geq 0, \sum_i \tau_i = 1\right\}$,

$T[f, \bar{\nu}] = \{x : \nu_i(x) \text{ does not exist for all symbols from } A\}$.

Let us remark that $f$ denotes the method of the representation of numbers ($f : L \to [0, 1]$, where $L = A \times A \times \cdots$). During recent years the interest into the latter set has been considerably increasing (see, e.g., [1, 2, 8, 9, 14]).

The presented paper devoted to the investigation of the expansion of real numbers in the first Ostrogradsky series (they were introduced by M. V. Ostrogradsky, a well known ukrainian mathematician who lived from 1801 to 1862). We shall also present the development of the corresponding metric and probabilistic theory. In this case the alphabet $A$ coincides with the set $\mathbb{N}$ of positive integers.

The expansion of $x$ of the form:

$$x = \frac{1}{q_1} - \frac{1}{q_1q_2} + \cdots + \frac{(-1)^{n-1}}{q_1q_2 \cdots q_n} + \cdots,$$

where $q_n$ are positive integers and $q_{n+1} \geq q_n$ for all $n$, is said to be the expansion of $x$ in the first Ostrogradsky series. The expansion of $x$ of the form:

$$x = \frac{1}{q_1} - \frac{1}{q_2} + \cdots + \frac{(-1)^{n-1}}{q_n} + \cdots,$$
where $q_n$ are positive integers and $q_{n+1} \geq q_n(q_n + 1)$ for all $n$, is said to be the expansion of $x$ in the second Ostrogradsky series. Each irrational number has a unique expansion of the form (1) or (2). Rational numbers have two finite different representations of the above form (see, e.g., [15]).

Equality (1) can be rewritten in the following way:

$$(3) \quad x = \frac{1}{g_1} - \frac{1}{g_1(g_1 + g_2)} + \cdots + \frac{(-1)^{n-1}}{g_1(g_1 + g_2) \cdots (g_1 + g_2 + \cdots + g_n)} + \cdots,$$

where $g_1 = q_1$, $g_{n+1} = q_{n+1} - q_n$ for any $n \in \mathbb{N}$. The expression (3) is said to be the $\bar{O}^1$-representation and the symbol $g_n = g_n(x)$ is said to be the $n$-th $\bar{O}^1$-symbol of $x$.

Shortly before his death, M. Ostrogradsky has proposed two algorithms for the representation of real numbers via alternating series of the form (1) and (2), but he did not publish any papers on this problems. Short Ostrogradsky’s remarks concerning the above representations have been found by E. Ya. Remez [15] in the hand-written fund of the Academy of Sciences of USSR. E. Ya. Remez has pointed out some similarities between the Ostrogradsky series and continued fractions. He also paid a great attention to the applications of the Ostrogradsky series for the numerical methods for solving algebraic equations. In the editorial comments to the book [5] B. Gnedenko has pointed out that there are no fundamental investigations of properties of the above mentioned representations. Analogous problems were studied by W. Sierpiński [17] and T. A. Pierce [10] independently. Some algorithms for the representation of real numbers in positive and alternating series were proposed in [17]. Two of these algorithms lead to the Ostrogradsky series (1) and (2). An algorithm also leading to the representation of irrational numbers in the form of the series (1) has been considered in [10].

There exists a series of papers devoted to the applications of the Ostrogradsky series. Let us mention some of them. Connections between the Ostrogradsky algorithms and the algorithm for the continued fractions have been established in [4]. This book contains also generalizations of the above algorithms. In the paper [6] different types of $p$-adic continued fractions have been constructed on the basis of $p$-adic analogs of Euclid and Ostrogradsky algorithms. Combining in a special way the algorithms of Engel and Ostrogradsky, the same author in the paper [7] has constructed an algorithm for the representation of real numbers via series which converge faster then the corresponding Engel’s and Ostrogradsky’s series. [18] is devoted to the investigation of the first Ostrogradsky algorithm and to the determination of the expectation of the random variables $(q_j + 1)^\nu$, $\nu \geq 0$ and

$$r_n = \sum_{j=n+1}^{\infty} \frac{(-1)^{j+1}}{q_1 q_2 \cdots q_j},$$

where $q_j = q_j(\alpha)$ are random variables depending on the random variable $\alpha$, uniformly distributed on the unit interval. In the same paper a generalization of the Ostrogradsky algorithms for approximations in Banach spaces has been proposed.

In the presented paper we study basic metric relations (equalities and inequalities for ratios of the length of corresponding cylinders) for the $\bar{O}^1$-representation of reals. In Section 2 we paid the main attention to the problem of the approximation of real numbers by partial sums of the Ostrogradsky series. We stress some similarities of the $\bar{O}^1$-representation
with the continued fraction representation. Recurrent formulas for the $\bar{O}^1$-convergents (analogos of the convergents for continued fractions) is also studied in this Section.

In Section 3 we prove basic metric relations of the $\bar{O}^1$-representation and compare them with the corresponding relations for continued fractions.

Sections 4 and 5 are the main ones of the paper. Section 4 is devoted to the study of the set $C[\bar{O}^1, \{V_k\}]$, consisting of real numbers whose k-th $\bar{O}^1$-symbols take values from the set $V_k \subset \mathbb{N}$. The central object of this section is the set $C[\bar{O}^1, V]$ which is a particular case of the previous one (for $V_n = V, \forall n \in \mathbb{N}$). Conditions for the set $C[\bar{O}^1, V]$ to be of zero resp. positive Lebesgue measure $\lambda$ are found. In particular, we prove that $\lambda(C) > 0$, if $V = \{m + 1, m + 2, \ldots\}$, where $m$ is an arbitrary positive integer. This fact stresses an essential difference between the metric theories of continued fractions and $\bar{O}^1$-representations.

In Section 5 we study the random variable $\xi$ with independent $\bar{O}^1$-symbols $\xi_k$. In particular, we prove that the random variable $\xi$ with independent $\bar{O}^1$-symbols is of pure type, i.e., it is either pure singular continuous, or pure absolutely continuous or pure atomic. On the basis of results of the previous Sections we study properties of the topological support of the random variable $\xi$. In the atomic case we completely describe the set of all atoms of the distribution. In the continuous case we give sufficient conditions for $\xi$ to be a singular continuous distribution of the Cantor type.

1. Representations of real numbers by the Ostrogradsky series

**Definition 1.** A finite or an infinite expression

\[
\sum_n \frac{(-1)^{n-1}}{q_1 q_2 \ldots q_n} = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \cdots,
\]

where $q_n$ are natural and $q_{n+1} > q_n$ for all $n$, is called the first Ostrogradsky series (in the sequel the Ostrogradsky series). The numbers $q_n$ are called the symbols of the Ostrogradsky series (4).

We denote the expression (4) briefly by

\[O^1(q_1, q_2, \ldots, q_n)\]

if it contains a finite number of terms, and we speak in this case of a finite Ostrogradsky series. We denote (4) by

\[O^1(q_1, q_2, \ldots, q_n, \ldots)\]

if it contains an infinite number of terms.

Every Ostrogradsky series is convergent and its sum belongs to $[0, 1]$.

**Theorem 1 ([15]).** Any real number $x \in (0, 1)$ can be represented in the form (4). If $x$ is irrational then the expression (4) is unique and it has an infinite number of terms. If $x$ is rational then it can be represented in the form (4) in the following different ways:

\[x = O^1(q_1, q_2, \ldots, q_{n-1}, q_n, q_n + 1) = O^1(q_1, q_2, \ldots, q_{n-1}, q_n + 1).\]
We can find the symbols of the Ostrogradsky series for a given number \( x \) using the following algorithm:

\[
1 = q_1 x + \alpha_1 \quad (0 \leq \alpha_1 < x), \\
1 = q_2 \alpha_1 + \alpha_2 \quad (0 \leq \alpha_2 < \alpha_1), \\
\ldots \\
1 = q_n \alpha_{n-1} + \alpha_n \quad (0 \leq \alpha_n < \alpha_{n-1}), \\
\ldots \\
\]

Let
\[
g_1 = q_1 \quad \text{and} \quad g_{n+1} = q_{n+1} - q_n \quad \text{for any} \ n \in \mathbb{N}.
\]

Then one can rewrite series (4) in the form

\[
\sum_n \frac{(-1)^n}{g_1(g_1 + g_2) \ldots (g_1 + g_2 + \ldots + g_n)} = \frac{1}{g_1} - \frac{1}{g_1(g_1 + g_2)} + \ldots.
\]

We denote the expression (6) by
\[
\tilde{O}_1(g_1, g_2, \ldots, g_n, \ldots).
\]

A representation of a number \( x \in (0, 1) \) by expression (6) is called the \( \tilde{O}_1 \)-representation.

The number \( g_n = g_n(x) \) is called \( n \)-th \( \tilde{O}_1 \)-symbol of the number \( x \).

**Definition.** The number
\[
\frac{A_k}{B_k} = \tilde{O}_1(q_1, q_2, \ldots, q_k) = \frac{1}{q_1} - \frac{1}{q_1q_2} + \ldots + \frac{(-1)^{k-1}}{q_1q_2 \ldots q_k}
\]
is called the convergent of order \( k \) of the Ostrogradsky series.

By using the method of mathematical induction, it is easy to prove that for any natural number \( k \) the following equalities hold:

\[
A_k = A_{k-1}q_k + (-1)^{k-1}, \\
B_k = B_{k-1}q_k = q_1q_2 \ldots q_k,
\]

\((A_0 = 0, B_0 = 1)\).

From the Leibniz theorem on the convergence of alternating series, it follows that the sequence of convergents of an even order increases and the sequence of convergents of an odd order decreases. Moreover, any convergent of odd order is greater than any convergent of even order.

2. Cylindrical sets and their properties

**Definition 2.** A set \( \tilde{O}_1^{[c_1c_2 \ldots c_m]} \) which is the closure of the set of all numbers \( x \in (0, 1) \), whose first \( m \) \( \tilde{O}_1 \)-symbols are equal to \( c_1, c_2, \ldots, c_m \) correspondingly, is said to be the cylindrical set (cylinder) of rank \( m \) with the base \( (c_1, c_2, \ldots, c_m) \).

Let us consider some basic properties of cylindrical sets.
1. \( \bar{O}_{[c_1...c_m]}^1 = [a, b] \), where
\[
\begin{align*}
    a &= \min \{ \bar{O}^1(c_1, \ldots, c_m), \bar{O}^1(c_1, \ldots, c_m + 1) \}, \\
    b &= \max \{ \bar{O}^1(c_1, \ldots, c_m), \bar{O}^1(c_1, \ldots, c_m + 1) \}.
\end{align*}
\]

Remark. We shall denote by \( \bar{O}_{(c_1...c_m)}^1 \) the interior part of the set \( \bar{O}_{[c_1...c_m]}^1 \).

2. \( \bar{O}_{[c_1...c_m]}^1 = \bigcup_{c=1}^{\infty} \bar{O}_{[c_1...c_m,c]}^1 \bigcup \bar{O}^1(c_1, c_2, \ldots, c_m) \), moreover
\[
\begin{align*}
    \sup \bar{O}_{[c_1...c_m,c]}^1 &= \inf \bar{O}_{[c_1...c_m(c+1)]}^1 \quad \text{if } m \text{ is odd}, \\
    \inf \bar{O}_{[c_1...c_m,c]}^1 &= \sup \bar{O}_{[c_1...c_m(c+1)]}^1 \quad \text{if } m \text{ is even},
\end{align*}
\]
and
\[
\bar{O}_{[c_1...c_m]}^1 \cap \bar{O}_{[c_1...c_m(c+1)]}^1 = \{ \bar{O}^1(c_1, c_2, \ldots, c_m, c + 1) \}.
\]

3. \( \bar{O}_{[c_1...c_m]}^1 = \bar{O}_{[s_1...s_k]}^1 \) if and only if \( m = k \) and \( c_i = s_i \) for all \( i = \overline{1,m} \).

4. \( \bar{O}_{[c_1...c_m]}^1 \subset \bar{O}_{[s_1...s_k]}^1 \) if and only if \( m \geq k \) and \( c_i = s_i \) for all \( i = \overline{1,k} \).

5. \( \bar{O}_{(c_1...c_m)}^1 \cap \bar{O}_{(s_1...s_k)}^1 = \emptyset \) if and only if there exists \( j \) such that \( c_j \neq s_j \).

6. The Lebesgue measure of the cylindrical set \( \bar{O}_{[c_1...c_m]}^1 \) is equal to
\[
\left| \bar{O}_{[c_1...c_m]}^1 \right| = \frac{1}{\sigma_1 \sigma_2 \ldots \sigma_m (\sigma_m + 1)},
\]
where \( \sigma_k = \sum_{i=1}^{k} c_i \).

Corollary 1. The cylindrical set \( \bar{O}_{[11...1]}^1 \) has the largest length among the cylindrical sets of rank \( m \), namely
\[
\left| \bar{O}_{[11...1]}^1 \right| = \frac{1}{(m+1)!}.
\]

Remark. There exist cylindrical sets of different ranks with the same lengths. For instance,
\[
\left| \bar{O}_{[1c]}^1 \right| = \left| \bar{O}_{[c+1]}^1 \right|, \quad \left| \bar{O}_{[1c2c3...c_m]}^1 \right| = \left| \bar{O}_{[(c_2+1)c_3...c_m]}^1 \right|.
\]

Corollary 2. For any given \( c \in \mathbb{N} \) and \( s \in \mathbb{N} \), the ratio
\[
\frac{\left| \bar{O}_{[c_1...c_m,c]}^1 \right|}{\left| \bar{O}_{[c_1...c_m]}^1 \right|} = \frac{(\sigma_m + c)(\sigma_m + c + 1)}{(\sigma_m + s)(\sigma_m + s + 1)}
\]
converges to 1, if \( \sigma_m = \sum_{i=1}^{m} c_i \) converges to \( +\infty \).
Corollary 3. The ratio
\[
\frac{|\bar{O}_{[c_1 \ldots c_m]}^1(c+1)|}{|\bar{O}_{[c_1 \ldots c_m]}^1|} = \frac{\sigma_m + c}{\sigma_m + c + 2} = 1 - \frac{2}{\sigma_m + c + 2}
\]
converges to 1 for \( m \to \infty \) (or even \( \sigma_m \to \infty \)) or \( c \to \infty \).

So, if \( \sigma_m \) is large enough, then the “weights” of two consecutive \( \bar{O}^1 \)-symbols \( c \) and \( c + 1 \) are “practically equal”.

3. Some metric problems and relations

Lemma 1. For any given \( s \in \mathbb{N} \), the ratio of lengths of cylindrical sets \( \bar{O}_{[c_1 \ldots c_m]}^1 \) and \( \bar{O}_{[c_1 \ldots c_m]}^1 \) satisfies the following equality
\[
\frac{|\bar{O}_{[c_1 \ldots c_m]}^1|}{|\bar{O}_{[c_1 \ldots c_m]}^1|} = \frac{a}{(a + s - 1)(a + s)} = f_s(a),
\]
where \( a = 1 + \sigma_m \). Moreover,
\[
f_s(a) \leq \frac{1}{2 \cdot (2s - 1)}
\]
and for \( m \geq s - 1 \)
\[
\frac{|\bar{O}_{[c_1 \ldots c_m]}^1|}{|\bar{O}_{[c_1 \ldots c_m]}^1|} \leq \frac{m + 1}{(m + s)(m + s + 1)}.
\]

Proof. Equality (7) follows directly from property 6 of cylindrical sets. Let us consider
\[
f_s(x) = \frac{x}{(x + s - 1)(x + s)}
\]
as a function of a real variable \( x, x \geq 1 \). This function increases on \( [1, \sqrt{(s-1)s}] \) and decreases on \( [\sqrt{(s-1)s}, +\infty) \). Since \( a \) takes only natural values, we have
\[
\max_{a \in \mathbb{N}} f_s(a) = f_s(s - 1) = f_s(s) = \frac{1}{2 \cdot (2s - 1)}.
\]
So, inequality (8) holds. The corresponding equality holds for \( a = s \) and for \( a = s - 1 \) (if it is possible, because \( a \geq m + 1 \) and it is impossible for \( m \geq s \)).

Function \( f_s(x) \) decreases on interval \( (s, +\infty) \). Hence \( f_s(a) \leq f_s(m+1) \), so inequality (9) holds.

Corollary. If \( c_1 + \cdots + c_m = s_1 + \cdots + s_k \) then
\[
\frac{|\bar{O}_{[c_1 \ldots c_m]}^1|}{|\bar{O}_{[c_1 \ldots c_m]}^1|} = \frac{|\bar{O}_{[s_1 \ldots s_k]}^1|}{|\bar{O}_{[s_1 \ldots s_k]}^1|}.
\]
Lemma 2. Let $\Delta_{c_1\ldots c_m}^{c.f.}$ be a cylindrical set generated by the continued fractions representation of real numbers. It is well known (see, e.g., [5]) that

$$\frac{|\Delta_{c_1\ldots c_m}^{c.f.}|}{|\Delta_{c_1\ldots c_m}|} = \frac{1}{s^2} \cdot \frac{1 + \frac{Q_{m-1}}{Q_m}}{\left(1 + \frac{Q_{m-1}}{sQ_m}\right) \left(1 + \frac{1}{s} + \frac{Q_{m-1}}{sQ_m}\right)},$$

where $Q_k$ is the denominator of the $k$-th convergent of the continued fraction $[c_1, c_1, \ldots, c_n, \ldots]$, i.e.,

$$Q_k = c_k Q_{k-1} + Q_{k-2} \quad \text{with} \quad Q_0 = 1, \quad Q_1 = a_1.$$

From the latter equality it follows that the following double inequality holds for any sequence $(c_1, \ldots, c_m)$ and for any $s \in \mathbb{N}$. For the $\overline{O}^1$-representation we have $f_s(a) \to 0$ ($a \to \infty$) and Lemma 1 shows the fundamental difference between metric relations in the representation of numbers by the first Ostrogradsky series and by continued fractions.

Lemma 2. Let $\overline{O}_{c_1\ldots c_m}^1$ be a fixed cylindrical set, then

$$\lambda \left( \bigcup_{s=1}^{k} \overline{O}_{c_1\ldots c_m}^1 \right) = \frac{k}{\sigma_m + k + 1} \left| \overline{O}_{c_1\ldots c_m}^1 \right|.$$

Proof. From the property 6 of cylindrical sets it follows that

$$\lambda \left( \bigcup_{s=1}^{k} \overline{O}_{c_1\ldots c_m}^1 \right) = \sum_{s=1}^{k} \left| \overline{O}_{c_1\ldots c_m}^1 \right| =$$

$$= \frac{1}{\sigma_1 \sigma_2 \ldots \sigma_m} \sum_{s=1}^{k} \frac{1}{(\sigma_m + s)(\sigma_m + s + 1)} =$$

$$= \frac{1}{\sigma_1 \sigma_2 \ldots \sigma_m} \left( \frac{1}{\sigma_m + 1} - \frac{1}{\sigma_m + k + 1} \right) =$$

$$= \frac{1}{\sigma_1 \sigma_2 \ldots \sigma_m (\sigma_m + 1)} \cdot \frac{k}{\sigma_m + k + 1} =$$

$$= \left| \overline{O}_{c_1\ldots c_m}^1 \right| \cdot \frac{k}{\sigma_m + k + 1},$$

which proves Lemma 2. \qed

Corollary 1. For any $k \in \mathbb{N}$ and for any sequence $(c_1, \ldots, c_m)$ the following inequality holds:

$$\frac{1}{\sigma_m + 2} \left| \overline{O}_{c_1\ldots c_m}^1 \right| \leq \lambda \left( \bigcup_{s=1}^{k} \overline{O}_{c_1\ldots c_m}^1 \right) \leq \frac{k}{m + k + 1} \left| \overline{O}_{c_1\ldots c_m}^1 \right|.$$

Remark. If $V \subset \mathbb{N}$, then it is evident that

$$\sum_{s \in V} \left| \overline{O}_{c_1\ldots c_m}^1 \right| = \left| \overline{O}_{c_1\ldots c_m}^1 \right| - \sum_{s \in \mathbb{N}\setminus V} \left| \overline{O}_{c_1\ldots c_m}^1 \right|.$$
Corollary 2. Let \( \tilde{O}_{[c_1 \ldots c_m]} \) be a fixed cylindrical set, then
\[
\lambda \left( \bigcup_{c=k+1}^{\infty} \tilde{O}_{[c_1 \ldots c_m]} \right) = \frac{\sigma_m + 1}{\sigma_m + k + 1} \left| \tilde{O}_{[c_1 \ldots c_m]} \right|.
\]

Corollary 3. For any \( k \in \mathbb{N} \) and for any sequence \( (c_1, \ldots, c_m) \) the following inequality holds:
\[
\frac{m + 1}{m + k + 1} \left| \tilde{O}_{[c_1 \ldots c_m]} \right| \leq \lambda \left( \bigcup_{c=k+1}^{\infty} \tilde{O}_{[c_1 \ldots c_m]} \right) \leq \frac{\sigma_m + 1}{\sigma_m + 2} \left| \tilde{O}_{[c_1 \ldots c_m]} \right|.
\]

Theorem 2. The Lebesgue measure of the set
\[
A_\sigma = \{ x : x = \tilde{O}(g_1(x), \ldots, g_m(x), \ldots), g_{m+1}(x) > g_1(x) + \cdots + g_m(x) \ \forall \ m \in \mathbb{N} \}
\]
is equal to 0.

Proof. Let
\[
L_k = \bigcup_{c_1 \in \mathbb{N}} \bigcup_{c_2 > \sigma_1} \bigcup_{c_k > \sigma_{k-1}} \tilde{O}_{[c_1 \ldots c_k]}.
\]
Then \( \lambda(L_1) = \sum_{c_1 \in \mathbb{N}} \left| \tilde{O}_{[c_1]} \right| = 1 \), and from Corollary 2 after Lemma 2 it follows that
\[
\lambda(L_2) = \sum_{c_1=1}^{\infty} \sum_{c_2=c_1+1}^{\infty} \left| \tilde{O}_{[c_1 c_2]} \right| = \sum_{c_1=1}^{\infty} \frac{c_1 + 1}{2c_1 + 1} \left| \tilde{O}_{[c_1]} \right| < \frac{2}{3} \sum_{c_1=1}^{\infty} \left| \tilde{O}_{[c_1]} \right| = \frac{2}{3} \lambda(L_1),
\]
since the function \( f(x) = \frac{x+1}{2x+1} \) decreases on \((1, +\infty)\).

Similarly,
\[
\lambda(L_{k+1}) = \sum_{c_1=1}^{\infty} \sum_{c_2=\sigma_1+1}^{\infty} \cdots \sum_{c_k=\sigma_{k-1}+1}^{\infty} \left| \tilde{O}_{[c_1 \ldots c_k]} \right| = \sum_{c_1=1}^{\infty} \sum_{c_2=\sigma_1+1}^{\infty} \cdots \sum_{c_k=\sigma_{k-1}+1}^{\infty} \frac{\sigma_k + 1}{2\sigma_k + 1} \left| \tilde{O}_{[c_1 \ldots c_k]} \right| < \frac{2}{3} \sum_{c_1=1}^{\infty} \sum_{c_2=\sigma_1+1}^{\infty} \cdots \sum_{c_k=\sigma_{k-1}+1}^{\infty} \left| \tilde{O}_{[c_1 \ldots c_k]} \right| = \frac{2}{3} \lambda(L_k).
\]

So,
\[
\lambda(L_{k+1}) < \frac{2}{3} \lambda(L_k)
\]
and we have
\[
\lambda(L_{k+1}) < \left( \frac{2}{3} \right)^k \lambda(L_1).
\]

From \( A_\sigma = \bigcap_{k=1}^{\infty} L_k \) it follows that
\[
\lambda(A_\sigma) = \lim_{k \to \infty} \lambda(L_{k+1}) = 0,
\]
which proves the Theorem. \( \square \)
Corollary. The Lebesgue measure of set \([0,1] \setminus A_\sigma\) is equal to 1. That is for Lebesgue almost all \(x:\)
\[g_{m+1}(x) \leq g_1(x) + \cdots + g_m(x),\]
for at least one natural \(m.\)

4. The set \(C[\bar{0}^1, \{V_n\}]\)

In this Section we shall study metric properties of the set \(C[\bar{0}^1, \{V_n\}]\), which is the closure of the set \(\{x : g_n(x) \in V_n, n \in \mathbb{N}\}\) consisting of the real numbers \(x \in [0,1]\) whose \(\bar{0}^1\)-symbols satisfy the condition
\[g_n(x) \in V_n,\]
where \(\{V_n\}\) is a fixed sequence of nonempty subsets of \(\mathbb{N}.\)

It is evident that
1. if \(V_n = \mathbb{N}\) for all \(n \in \mathbb{N}\), then \(C[\bar{0}^1, \{V_n\}] = [0,1],\)
2. if \(V_n = \mathbb{N}\) for all \(n > n_0\), then the set \(C[\bar{0}^1, \{V_n\}]\) is a union of segments.

We are interested only in the case where \(V_n \neq \mathbb{N}\) for an infinite number of \(n.\)

Let \(F_k = \left( \bigcup_{c_1 \in V_1} \cdots \bigcup_{c_k \in V_k} \bar{0}^1_{c_1\cdots c_k} \right)\), where \(\text{cl}\) stands for the closure.

Lemma 3. The set \(C[\bar{0}^1, \{V_n\}]\) can be represented in the form
\[C[\bar{0}^1, \{V_n\}] = \bigcap_{k=1}^{\infty} F_k.\]

It is a perfect set (that is a closed set without isolated points). If \(V_n \neq \mathbb{N}\) for an infinite number of \(n,\) then it is a nowhere dense set.

Proof. The irrational number \(x_0\) belongs to the set \(C[\bar{0}^1, \{V_n\}]\) if and only if for all natural \(k\) there exists a cylindrical set \(\bar{0}^1_{c_1\cdots c_k}\) of rank \(k\) containing \(x_0,\) and \(c_1 \in V_1, c_2 \in V_2, \ldots, c_k \in V_k.\) Let now \(y_0 \in C[\bar{0}^1, \{V_n\}]\) be a rational number. From the definition of the set \(C[\bar{0}^1, \{V_n\}]\) it follows that for any \(s \in \mathbb{N}\) the interval \((y_0 - \frac{1}{s}, y_0 + \frac{1}{s})\) contains an irrational number \(x_s \in C[\bar{0}^1, \{V_n\}]\). From what has already been proved, it follows that \(x_s \in \bigcap_{k=1}^{\infty} F_k.\) Since the latter set is closed and \(x_s \to y_0,\) we have \(y_0 \in \bigcap_{k=1}^{\infty} F_k.\)

The proof of the inverse inclusion is completely similar.

From
\[C[\bar{0}^1, \{V_n\}] = [0,1] \setminus \bigcup_{m=0}^{\infty} \bigcup_{\substack{c_1 \in V_1, \\ i_1, \ldots, i_m \in \mathbb{N} \setminus V_{m+1}}} \left( \bigcup_{c \in \mathbb{N} \setminus V_{m+1}} \bar{0}^1_{(c_1\cdots c_m)c} \cup A_{m+1} \right),\]
\[A_{m+1} = \bigcup_{\substack{i,j \in \mathbb{N} \setminus V_{m+1}, \\ i \neq j}} \left( \bar{0}^1_{(c_1\cdots c_m)i} \cap \bar{0}^1_{(c_1\cdots c_m)j} \right)\]

it follows that \(C[\bar{0}^1, \{V_n\}]\) is perfect.

If \(V_n \neq \mathbb{N}\) for an infinite number of \(n,\) then \(C[\bar{0}^1, \{V_n\}]\) is nowhere dense, because for any interval \((a, b)\) there exist an interval \(\bar{0}^1_{(c_1\cdots c_m)} \subset (a, b)\) and an interval \(\bar{0}^1_{(c_1\cdots c_{m+k})}\) such that \(\bar{0}^1_{(c_1\cdots c_{m+k})} \cap C[\bar{0}^1, \{V_n\}] = \emptyset,\) where \(c_{m+k} \in \mathbb{N} \setminus V_{n+k} \neq \emptyset.\)

\(\square\)
Corollary. The Lebesgue measure
\[ \lambda(C[\bar{O}^1, \{V_n\}]) \leq \sum_{c_1 \in V_1} \ldots \sum_{c_k \in V_k} \frac{1}{\sigma_1 \ldots \sigma_k(\sigma_k + 1)} = \lambda(F_k) \]
for any \( k \in \mathbb{N} \), and
\[ \lambda(C[\bar{O}^1, \{V_n\}]) = \lim_{k \to \infty} \lambda(F_k). \]
Let \( M_k \) be the union of all “admissible” cylinders of rank \( k \), i.e.,
\[ M_k = \bigcup_{c_1 \in V_1} \ldots \bigcup_{c_k \in V_k} \bar{O}^1_{[c_1c_2 \ldots c_k]}, \quad M_0 = [0, 1], \]
and
\[ \bar{M}_{k+1} := M_k \setminus M_{k+1}. \]
Then
\[ \lambda(M_k) = \sum_{c_1 \in V_1} \ldots \sum_{c_k \in V_k} \frac{1}{\sigma_1 \ldots \sigma_k(\sigma_k + 1)}, \]
\[ \lambda(\bar{M}_{k+1}) = \lambda\left( \bigcup_{c_1 \in V_1} \ldots \bigcup_{c_k \in V_k} \bigcup_{s \in V_{k+1}} \bar{O}^1_{[c_1c_2 \ldots c_k s]} \right) = \]
\[ = \sum_{c_1 \in V_1} \ldots \sum_{c_k \in V_k} \sum_{s \in V_{k+1}} \frac{1}{\sigma_1 \ldots \sigma_k(\sigma_k + s)(\sigma_k + s + 1)} = \]
\[ = \sum_{c_1 \in V_1} \ldots \sum_{c_k \in V_k} \left[ \frac{1}{\sigma_1 \ldots \sigma_k} \sum_{s \in V_{k+1}} \frac{1}{(\sigma_k + s)(\sigma_k + s + 1)} \right]. \]

Lemma 4. The Lebesgue measure of the set \( C[\bar{O}^1, \{V_n\}] \) is equal to 0 if and only if
\[ \sum_{k=1}^{\infty} \frac{\lambda(\bar{M}_{k+1})}{\lambda(M_k)} = +\infty. \]

Proof. Since \( F_k \setminus M_k \) is at most a countable set, we have:
\[ \lambda(C[\bar{O}^1, \{V_n\}]) = \lim_{k \to \infty} \lambda(M_{k+1}) = \lim_{k \to \infty} \frac{\lambda(M_{k+1})}{\lambda(M_k)} \cdot \frac{\lambda(M_k)}{\lambda(M_{k-1})} \cdot \ldots \cdot \frac{\lambda(M_1)}{\lambda(M_0)} = \]
\[ = \prod_{k=0}^{\infty} \frac{\lambda(M_{k+1})}{\lambda(M_k)} = \prod_{k=0}^{\infty} \frac{\lambda(M_k) - \lambda(\bar{M}_{k+1})}{\lambda(M_k)} = \]
\[ = \prod_{k=0}^{\infty} \left( 1 - \frac{\lambda(\bar{M}_{k+1})}{\lambda(M_k)} \right) = 0 \iff \sum_{k=1}^{\infty} \frac{\lambda(\bar{M}_{k+1})}{\lambda(M_k)} = +\infty, \]
since \( 0 < \frac{\lambda(M_{k+1})}{\lambda(M_k)} < 1 \). \qed

First of all we shall study the problem of the determination of the Lebesgue measure of the set \( C[\bar{O}^1, V] = C[\bar{O}^1, \{V_n\}] \) with \( V = V_n \), where \( V \) is a fixed proper subset of positive integers. The sets \( C[\bar{O}^1, V] \) with
\begin{enumerate}
  \item \( V = \{1, 2, \ldots, m\} \),
  \item \( V = \{m + 1, m + 2, \ldots\} \),
  \item \( V = \{1, 3, 5, \ldots\} \)
\end{enumerate}
are the most simple sets among \( C[\bar{O}^1, V] \).

Let us solve the first problem in more general setting.

**Theorem 3.** If the set \( V_k \) contains \( N_k \) symbols \((k \in \mathbb{N})\) and
\[
\lim_{k \to \infty} \frac{N_1 N_2 \ldots N_k}{(k + 1)!} = 0
\]
then the Lebesgue measure of the set \( C[\bar{O}^1, \{V_k\}] \) is equal to 0.

**Proof.** From the properties of cylindrical sets it follows that
\[
\lambda(M_k) = \sum_{v_i \in V_i, i = 1, \ldots, k} \left| \bar{O}^{1}_{v_1 v_2 \ldots v_k} \right| \leq \frac{N_1 N_2 \ldots N_k}{(k + 1)!}.
\]
From Lemma 3 and from the continuity of Lebesgue measure it follows that
\[
\lambda(C[\bar{O}^1, \{V_k\}]) = \lim_{k \to \infty} \lambda(M_k) \leq \lim_{k \to \infty} \frac{N_1 N_2 \ldots N_k}{(k + 1)!} = 0.
\]

**Corollary.** If \( N_k \leq m \) (for any \( k \in \mathbb{N} \)) for some fixed \( m \), then the Lebesgue measure of the set \( C[\bar{O}^1, \{V_k\}] \) is equal to 0.

**Theorem 4.** Let \( V_k = \{1, 2, \ldots, m_k\} \). If \( \sum_{k=1}^{\infty} \frac{1}{m_k} = +\infty \), then the Lebesgue measure of the set \( C[\bar{O}^1, \{V_k\}] \) is equal to 0.

**Proof.** Let \( \bar{O}^1_{[c_1 c_2 \ldots c_k]} \) be a fixed cylindrical set of rank \( k \). Then
\[
\sum_{c \notin V_{k+1}} \left| \bar{O}^1_{(c_1 c_2 \ldots c_k c)} \right| = \frac{1}{\sigma_1 \sigma_2 \ldots \sigma_k} \sum_{c=m_k+1}^{\infty} \frac{1}{(\sigma_k + c)(\sigma_k + c + 1)} = \frac{1}{\sigma_1 \sigma_2 \ldots \sigma_k (\sigma_k + m_k + 1)}.
\]
Since
\[
\frac{1}{\sigma_k + m_k + 1} > \frac{1}{(m_k + 1)(\sigma_k + 1)},
\]
we have
\[
\sum_{c \notin V_{k+1}} \left| \bar{O}^1_{(c_1 c_2 \ldots c_k c)} \right| > \frac{1}{m_k + 1} \cdot \left| \bar{O}^1_{[c_1 c_2 \ldots c_k]} \right|.
\]
Summing over all \( c_1 \in V_1, c_2 \in V_2, \ldots, c_k \in V_k \), we have
\[
\lambda(\bar{M}_{k+1}) > \frac{1}{m_k + 1} \lambda(M_k), \quad \text{i.e.,} \quad \frac{\lambda(\bar{M}_{k+1})}{\lambda(M_k)} > \frac{1}{m_k + 1}
\]
for any \( k \in \mathbb{N} \), and the statement of the Theorem follows directly from Lemma 4.

Let \( E \) be the set of all real numbers with bounded \( \bar{O}^1 \)-symbols, i.e., \( x \in E \) iff there exists a constant \( K_x \) such that \( g_k(x) \leq K_x \) for all \( k \in \mathbb{N} \).

**Theorem 5.** The Lebesgue measure of the set \( E \) of all real numbers \( x \in [0, 1] \) with bounded \( \bar{O}^1 \)-symbols is equal to 0.
Proof. For a given \( m \in \mathbb{N} \), let us consider the set \( E_m = \{ x : g_k(x) \leq m, \forall k \in \mathbb{N} \} \) of \( m \)-uniformly bounded symbols. It is not hard to see that \( E_m = C[\bar{O}_1, \{ V_k \}] \) with \( V_k = \{1, 2, \ldots, m\} \). From the latter Theorem it follows that \( \lambda(C[\bar{O}_1, \{ V_k \}]) = 0 \).

Since \( E = \bigcup_{m=1}^{\infty} E_m \) and \( \lambda(E_m) = 0 \), we have the desired conclusion. \( \Box \)

Corollary. For Lebesgue almost all real numbers \( x \in [0, 1] \) the following equality holds:

\[
\lim_{k \to \infty} g_k(x) = \infty.
\]

Let us now consider the case, where \( V_k = \{ v_k + 1, v_k + 2, \ldots \} \), and \( \{v_k\} \) is a fixed sequence of positive integers.

Lemma 5. Let \( \bar{O}_{[c_1\ldots c_n]} \) be a fixed cylindrical set or, if \( n = 0 \), the unit interval \([0, 1]\);
let \( \{v_k\} \) be a fixed sequence of positive integers, let \( V_k = \{ v_k + 1, v_k + 2, \ldots \} \), and let

\[
M_{k+1}^{c_1\ldots c_n} := M_{n+k} \cap \bar{O}_{[c_1\ldots c_n]} = \bigcup_{c_{n+1} > v_{n+1}} \cdots \bigcup_{c_{n+k} > v_{n+k}} \bar{O}_{[c_1\ldots c_{n+1} \ldots c_{n+k}]},
\]

\[
\bar{M}_{k+1}^{c_1\ldots c_n} = M_{k+1}^{c_1\ldots c_n} \setminus M_k^{c_1\ldots c_n} = \bigcup_{c_{n+1} > v_{n+1}} \cdots \bigcup_{c_{n+k} > v_{n+k}} \bigcup_{s=1}^{v_{n+k+1}} \bar{O}_{(c_1\ldots c_{n+k}s)}.
\]

Then

\[
\lambda(\bar{M}_{k+1}^{c_1\ldots c_n}) < \frac{1}{2} \cdot \frac{v_{n+k+1}}{v_{n+k}}.
\]

Proof. Let \( \bar{O}_{(c_1\ldots c_{n+k}s)} \) be a fixed cylindrical interval of rank \( n + k \). Then

\[
\sum_{s \in V_{n+k}} |\bar{O}_{(c_1\ldots c_{n+k}s)}| = \sum_{s=1}^{v_{n+k}} \frac{1}{\sigma_1 \ldots \sigma_{n+1} \ldots \sigma_{n+k}(\sigma_{n+k-1} + s)(\sigma_{n+k-1} + s + 1)} = \frac{1}{\sigma_1 \ldots \sigma_{n+1} \ldots \sigma_{n+k-1}} \left( \frac{1}{\sigma_{n+k-1} + 1} - \frac{1}{\sigma_{n+k-1} + v_{n+k} + 1} \right) = \frac{v_{n+k}}{\sigma_1 \ldots \sigma_{n+1} \ldots \sigma_{n+k-1}(\sigma_{n+k-1} + 1)(\sigma_{n+k-1} + v_{n+k} + 1)}.
\]
Let $\bar{O}_1^{n+k}$ be a fixed cylindrical interval of rank $n + k + 1$. Then

$$\sum_{c_n+k \in V_{n+k}, s \notin V_{n+k+1}} \left| \bar{O}_1^{c_1 \ldots c_n \ldots c_{n+k}} \right| =$$

$$= \sum_{c_n+k = v_{n+k}+1}^{\infty} \frac{1}{\sigma_1 \ldots \sigma_{n+k}} \left( \frac{1}{\sigma_{n+k} + 1} - \frac{1}{\sigma_{n+k} + v_{n+k+1} + 1} \right) =$$

$$= \frac{1}{\sigma_1 \ldots \sigma_{n+k}} \sum_{s = v_{n+k+1}}^{\infty} \left( \frac{1}{(\sigma_{n+k-1} + s)(\sigma_{n+k-1} + s + 1)} - \frac{1}{(\sigma_{n+k-1} + s)(\sigma_{n+k-1} + s + v_{n+k+1} + 1)} \right) =$$

$$= \frac{1}{\sigma_1 \ldots \sigma_{n+k-1}} \left( \frac{1}{\sigma_{n+k-1} + v_{n+k} + 1} - \frac{1}{1 + v_{n+k+1}} \sum_{i=1}^{v_{n+k+1}+1} \frac{1}{\sigma_{n+k-1} + v_{n+k} + i} \right) =$$

$$= \frac{\sigma_{n+k-1} + 1}{v_{n+k}} \left( 1 - \frac{1}{1 + v_{n+k+1}} \sum_{i=1}^{1+v_{n+k+1}} \frac{\sigma_{n+k-1} + v_{n+k} + 1}{\sigma_{n+k-1} + v_{n+k} + i} \right) =$$

$$= \frac{\sigma_{n+k-1} + 1}{v_{n+k}} \left( 1 - \frac{1}{1 + v_{n+k+1}} \sum_{i=1}^{1+v_{n+k+1}} \left( 1 - \frac{i - 1}{\sigma_{n+k-1} + v_{n+k} + i} \right) \right) =$$

$$= \frac{\sigma_{n+k-1} + 1}{v_{n+k}} \cdot \frac{1}{1 + v_{n+k+1}} \cdot \sum_{i=2}^{1+v_{n+k+1}} \frac{i - 1}{\sigma_{n+k-1} + v_{n+k} + i}.$$

Now let us estimate the following sum

$$\frac{1}{n_0 + 1} + \frac{2}{n_0 + 2} + \cdots + \frac{m_k}{n_0 + m_k},$$

where $n_0$ and $m_k > 1$ are natural numbers. Let

$$C_k := \frac{1}{n_0 + 1} + \frac{1}{n_0 + 2} + \cdots + \frac{1}{n_0 + m_k}$$
and let us consider the following “matrix”:

\[
\begin{pmatrix}
\frac{1}{n_0+1} & \frac{1}{n_0+2} & \frac{1}{n_0+3} & \cdots & \frac{1}{n_0+m_k} \\
\frac{1}{n_0+1} & \frac{1}{n_0+2} & \frac{1}{n_0+3} & \cdots & \frac{1}{n_0+m_k} \\
\frac{1}{n_0+1} & \frac{1}{n_0+2} & \frac{1}{n_0+3} & \cdots & \frac{1}{n_0+m_k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{n_0+1} & \frac{1}{n_0+2} & \frac{1}{n_0+3} & \cdots & \frac{1}{n_0+m_k}
\end{pmatrix}
\]

The sum of all addends over the whole “matrix” is equal to \(m_k \cdot C_k\).

The sum of all addends over the “main diagonal of the matrix” is equal to \(C_k\).

The sum of all elements standing above “the main diagonal” is less then the sum of all elements standing under “the main diagonal” (for any element above the “main diagonal” there exists the symmetrical element (under the “main diagonal”), which is greater then the initial one).

The sum of all elements standing outside the “main diagonal” is equal to \((m_k - 1) \cdot C_k\).

So, the sum of all elements standing above the “main diagonal” is equal to \(\frac{m_k - 1}{2} \cdot C_k\), and the sum of all elements above the “main diagonal” and over the “main diagonal” is equal to

\[
\frac{1}{n_0+1} + \frac{2}{n_0+2} + \cdots + \frac{m_k}{n_0+m_k} < \frac{m_k - 1}{2} \cdot C_k + C_k = \frac{m_k + 1}{2} \cdot C_k.
\]

Therefore,

\[
X_k = \frac{\sigma_{n+k-1} + 1}{v_{n+k}} \cdot \frac{1}{1 + v_{n+k+1}} \cdot \sum_{i=1}^{v_{n+k+1}} \frac{i}{(\sigma_{n+k-1} + v_{n+k} + 1) + i} < \frac{\sigma_{n+k-1} + 1}{v_{n+k}} \cdot \frac{1}{1 + v_{n+k+1}} \cdot \frac{v_{n+k+1} + 1}{2} \cdot \sum_{i=1}^{v_{n+k+1}} \frac{1}{\sigma_{n+k-1} + v_{n+k} + i + 1} = \frac{1}{2v_{n+k}} \sum_{i=1}^{v_{n+k+1}} \frac{\sigma_{n+k-1} + 1}{\sigma_{n+k-1} + v_{n+k} + i + 1} < \frac{1}{2} \cdot \frac{v_{n+k+1}}{v_{n+k}}.
\]

So, the inequality

\[
\sum_{c_{n+k} \in V_{n+k}} \sum_{sg V_{n+k+1}} \bar{O}_{(c_1 \cdots c_{n+k}, s)}^1 < \frac{1}{2} \cdot \frac{v_{n+k+1}}{v_{n+k}} \cdot \sum_{sg V_{n+k}} \bar{O}_{(c_1 \cdots c_{n+k}, s)}^1
\]

holds. Hence, summing over all \(c_{n+1} \in V_{n+1}, c_{n+2} \in V_{n+2}, \ldots, c_{n+k-1} \in V_{n+k-1}\), we have

\[
\lambda(M_{k+1}^{c_1 \cdots c_n}) < \frac{1}{2} \cdot \frac{v_{n+k+1}}{v_{n+k}} \cdot \lambda(M_k^{c_1 \cdots c_n})
\]

which proves the Lemma.

\[\square\]

**Corollary 1.** Let \(V_k = \{v_k + 1, v_k + 2, \ldots\}\), \(v_k \in \mathbb{N}\). Then

\[
\lambda(M_{k+1}) < \frac{1}{2} \cdot \frac{v_{k+1}}{v_k} \lambda(M_k).
\]
Corollary 2. Let $V_k = V = \{m + 1, m + 2, \ldots\}$, $m \in \mathbb{N}$. Then
\[
\lambda (\bar{M}_{k+1}^{c_1 \ldots c_n}) < \frac{1}{2} \lambda (\bar{M}_k^{c_1 \ldots c_n})
\]
for any natural number $k$ and any $c_1 \in V, \ldots, c_n \in V$, and, therefore,
\[
\lambda(\bar{M}_{k+1}) < \frac{1}{2} \lambda(\bar{M}_k).
\]

Theorem 6. Let $\{v_k\}$ be a fixed sequence of positive integers, and let
\[
V_k = \{v_k + 1, v_k + 2, \ldots\}.
\]
If there exists $k_0 \in \mathbb{N}$ such that
\[
\frac{v_{k+1}}{v_k} \leq C_0 < 2 \quad \text{for any } k > k_0,
\]
then the set $C[\bar{O}^1, \{V_k\}]$ is of positive Lebesgue measure.

Proof. Let $\bar{O}_{[c_1 \ldots c_n]}$ be any fixed cylindrical set with $n > k_0$ and $c_i \in V_i$. We shall prove that the set
\[
\Delta_{c_1 \ldots c_n} = C[\bar{O}^1, \{V_k\}] \cap \bar{O}_{[c_1 \ldots c_n]}
\]
has positive Lebesgue measure. To this aim let us consider a cylindrical set $\bar{O}_{[c_1 \ldots c_n c_{n+1}]}$, $c_{n+1} > v_{n+1}$, and the corresponding subset
\[
\Delta_{c_1 \ldots c_n c_{n+1}} = C[\bar{O}^1, \{V_k\}] \cap \bar{O}_{[c_1 \ldots c_n c_{n+1}]}.
\]
From Lemma 5 it follows that
\[
\lambda(\bar{M}_{k+1}^{c_1 \ldots c_n c_{n+1}}) < \frac{1}{2} \cdot \frac{v_{n+k+1}}{v_{n+k}} \cdot \lambda(\bar{M}_k^{c_1 \ldots c_n c_{n+1}}) \leq \frac{1}{2} \cdot C_0 \cdot \lambda(\bar{M}_k^{c_1 \ldots c_n c_{n+1}}) <
\]
\[
< \frac{1}{2} \cdot C_0 \cdot \frac{1}{2} \cdot \frac{v_{n+k}}{v_{n+k-1}} \cdot \lambda(\bar{M}_{k-1}^{c_1 \ldots c_n c_{n+1}}) \leq \left( \frac{C_0}{2} \right)^k \cdot \lambda(\bar{M}_1^{c_1 \ldots c_n c_{n+1}}) < \ldots
\]
for any $k \in \mathbb{N}$. Using Lemma 2, we have
\[
\lambda(\bar{M}_1^{c_1 \ldots c_n c_{n+1}}) = \sum_{s=1}^{v_{n+2}} |\bar{O}_{[c_1 \ldots c_n c_{n+1}s]}| = \frac{v_{n+2}}{\sigma_{n+1} + v_{n+2} + 1} \cdot |\bar{O}_{[c_1 \ldots c_n c_{n+1}]}|.
\]
So,
\[
\lambda(\Delta_{c_1 \ldots c_n c_{n+1}}) = |\bar{O}_{[c_1 \ldots c_n c_{n+1}]}| - \sum_{k=1}^{\infty} \lambda(\bar{M}_k^{c_1 \ldots c_n c_{n+1}}) >
\]
\[
> |\bar{O}_{[c_1 \ldots c_n c_{n+1}]}| - \sum_{k=1}^{\infty} \left( \frac{C_0}{2} \right)^{k-1} \cdot \lambda(\bar{M}_1^{c_1 \ldots c_n c_{n+1}}) =
\]
\[
= |\bar{O}_{[c_1 \ldots c_n c_{n+1}]}| \cdot \left( 1 - \frac{2}{2 - C_0} \cdot \frac{v_{n+2}}{\sigma_{n+1} + v_{n+2} + 1} \right).
\]
Since the numbers $c_1, \ldots, c_n, v_{n+2}, C_0$ are fixed, and $c_{n+1} > v_{n+1}$, there exists a number $c^* \in \mathbb{N}$ such that
\[
1 - \frac{2}{2 - C_0} \cdot \frac{v_{n+2}}{\sigma_{n+1} + v_{n+2} + 1} > 0
\]
Theorem 7. Let \( m \) be a fixed natural number and \( V = \mathbb{N} \setminus \{1, 2, \ldots, m\} \), then the set \( C[\bar{O}^1, V] \) is of positive Lebesgue measure and

\[
\lambda(C[\bar{O}^1, V]) > \frac{1}{(m+1)^2}.
\]

Proof. The first statement of the Theorem follows directly from the Theorem 6. Let us prove the second statement. To this aim we consider an arbitrary cylindrical set \( \bar{O}^{1}_{c_1c_2\ldots c_m} \) such that \( c_1 \in V, c_2 \in V, \ldots, c_m \in V \). From the Corollary 2 after Lemma 5 it follows that

\[
\lambda(M_{k+1}^{c_1c_2\ldots c_m} < \frac{1}{2^k} \lambda(M_1^{c_1c_2\ldots c_m}).
\]

So, we have

\[
\lambda(\Delta_{c_1c_2\ldots c_m}) = |\bar{O}^{1}_{c_1c_2\ldots c_m}| - \sum_{k=1}^{\infty} \lambda(M_k^{c_1c_2\ldots c_m}) >
\]

\[
|\bar{O}^{1}_{c_1c_2\ldots c_m}| - \lambda(M_1^{c_1c_2\ldots c_m}) \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} = |\bar{O}^{1}_{c_1c_2\ldots c_m}| - 2\lambda(M_1^{c_1c_2\ldots c_m}).
\]

Since

\[
\lambda(M_1^{c_1c_2\ldots c_m}) = \sum_{c=1}^{m} \bar{O}^{1}_{(c_1c_2\ldots c_m)} = \frac{m}{\sigma_m + m + 1} \cdot |\bar{O}^{1}_{c_1c_2\ldots c_m}| \leq \frac{m}{(m+1)^2} \cdot |\bar{O}^{1}_{c_1c_2\ldots c_m}|,
\]

it follows that

\[
\lambda(\Delta_{c_1c_2\ldots c_m}) > \frac{m^2 + 1}{(m+1)^2} \cdot |\bar{O}^{1}_{c_1c_2\ldots c_m}|.
\]

Now we shall estimate the Lebesgue measure of \( \bigcup_{c_1 \in V} \cdots \bigcup_{c_m \in V} |\bar{O}^{1}_{c_1c_2\ldots c_m}| : \)

\[
\sum_{c_1=m+1}^{\infty} \cdots \sum_{c_m=m+1}^{\infty} \frac{1}{\sigma_1 \sigma_2 \ldots \sigma_{m-1}(\sigma_{m-1} + c_m)(\sigma_{m-1} + c_m + 1)} =
\]

\[
= \sum_{c_1=m+1}^{\infty} \cdots \sum_{c_{m-1}=m+1}^{\infty} \frac{1}{\sigma_1 \sigma_2 \ldots \sigma_{m-1}(\sigma_{m-1} + m + 1)} >
\]

\[
> \sum_{c_1=m+1}^{\infty} \cdots \sum_{c_{m-1}=m+1}^{\infty} \frac{1}{\sigma_1 \sigma_2 \ldots \sigma_{m-2}(\sigma_{m-1} + m)(\sigma_{m-1} + m + 1)} =
\]

\[
= \cdots = \sum_{c_1=m+1}^{\infty} \frac{1}{\sigma_1(\sigma_1 + (m-1)m)} >
\]

\[
> \sum_{c_1=m+1}^{\infty} \frac{1}{(c_1 + (m-1)m)(c_1 + (m-1)m + 1)} = \frac{1}{m^2 + 1}.
\]
Since
\[ \lambda(C[\bar{O}^1, V]) = \sum_{c_1 \in V} \cdots \sum_{c_m \in V} \lambda(\Delta_{c_1 c_2 \cdots c_m}), \]
we have inequality (11).

**Corollary.** Let the sequence \( \{v_k\} \) be uniformly bounded (i.e., there exists a number \( D_0 \) such that \( v_k \leq D_0, \forall k \in \mathbb{N} \)). Then the set \( C[\bar{O}^1, \{V_k\}] \) is of positive Lebesgue measure.

Finally, let us consider the more general case where \( V_k = V = \mathbb{N} \setminus \{a_1, a_2, \ldots, a_n, \ldots\} \) and \( \{a_n\} \) is an arbitrary increasing sequence of positive integers.

**Theorem 8.** Let \( \{a_n\} \) be an increasing sequence of positive integers with \( a_{n+1} - a_n \leq d \) for some fixed natural number \( d \geq 2 \), and for any \( n \in \mathbb{N} \). If \( V_k = V = \mathbb{N} \setminus \{a_1, a_2, \ldots, a_n, \ldots\} \), then the Lebesgue measure of the set \( C[\bar{O}^1, V] \) is equal to 0.

**Proof.** Let us fix a cylindrical set \( \bar{O}^1_{[c_1 c_2 \cdots c_k]} \) and estimate the following sum
\[
\sum_{c \not\in V} |\bar{O}^1_{(c_1 c_2 \cdots c_k)}| = \frac{1}{\sigma_1 \sigma_2 \cdots \sigma_k} \sum_{n=1}^{\infty} \frac{1}{(\sigma_k + a_n)(\sigma_k + a_n + 1)} > \frac{1}{\sigma_1 \sigma_2 \cdots \sigma_k} \sum_{n=1}^{\infty} \frac{1}{(\sigma_k + a'_n)(\sigma_k + a'_n + d)} = \frac{1}{d} \frac{1}{\sigma_1 \sigma_2 \cdots \sigma_k (\sigma_k + a_1)},
\]
where \( a'_1 = a_1, a'_{n+1} = a'_n + d \geq a_{n+1} \) for all natural \( n \). Since
\[
\frac{1}{\sigma_k + a_1} \geq \frac{1}{a_1 (\sigma_k + 1)},
\]
we have
\[
\sum_{c \not\in V} |\bar{O}^1_{(c_1 c_2 \cdots c_k)}| > \frac{1}{a_1 d} \cdot |\bar{O}^1_{[c_1 c_2 \cdots c_k]}|.
\]
Summing over all \( c_1 \in V, c_2 \in V, \ldots, c_k \in V \), we have
\[
\lambda(M_{k+1}) > \frac{1}{a_1 d} \lambda(M_k), \quad \text{i.e.,} \quad \frac{\lambda(M_{k+1})}{\lambda(M_k)} > \frac{1}{a_1 d}
\]
for any \( k \in \mathbb{N} \), and the statement of the Theorem follows directly from Lemma 4.

**Corollary 1.** If \( V_k = V = \{b_1, b_2, \ldots, b_n, \ldots\} \) with \( b_{n+1} - b_n \geq 2 \), then the Lebesgue measure of the set \( C[\bar{O}^1, V] \) is equal to 0.

**Corollary 2.** If \( V = \{1, 3, 5, \ldots\} \) or \( V = \{2, 4, 6, \ldots\} \) then \( \lambda(C[\bar{O}^1, V]) = 0 \).

5. Random variables with independent \( \bar{O}^1 \)-symbols

Let us consider the following random variable
\[
\xi = \bar{O}^1(\xi_1, \xi_2, \ldots, \xi_k, \ldots) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\xi_1 (\xi_1 + \xi_2) \cdots (\xi_1 + \xi_2 + \cdots + \xi_n)},
\]
where \( \xi_k \) are independent random variables taking the values 1, 2, \ldots, \( m \), \ldots with probabilities \( p_{1k}, p_{2k}, \ldots, p_{mk}, \ldots \) correspondingly, \( p_{mk} \geq 0, \sum_{m=1}^{\infty} p_{mk} = 1 \).

Since the random variable \( \xi \) is a sum of an infinite number of terms, it can takes irrational values only.

Lemma 6 ([13]). The distribution function \( F_\xi \) of the random variable \( \xi \) is of the following form

\[
F_\xi(x) = \beta_1(x) + \sum_{k \geq 2} (-1)^{k-1} \beta_k(x) \prod_{i=1}^{k-1} p_{g_i(x)i}, \quad \text{if} \quad 0 < x \leq 1,
\]

where

\[
\beta_k(x) = 1 - \sum_{j=1}^{g_k(x)-1} p_{jk},
\]

and \( g_k(x) \) is the \( k \)-th \( \tilde{O}^1 \)-symbol of number \( x \), and the expression (12) has a finite resp. infinite number of terms according to rationality resp. irrationality of the number \( x \).

Theorem 9. The random variable \( \xi \) has a discrete distribution if and only if

\[
\prod_{k=1}^{\infty} \max_m \{p_{mk}\} > 0,
\]

and it is continuously distributed if and only if the infinite product in (13) diverges to 0.

Moreover, in the discrete case the atomic spectrum of the distribution of the random variable \( \xi \) consists of real numbers \( x \in [0,1] \) whose \( \tilde{O}^1 \)-representation differs from the \( \tilde{O}^1 \)-representation of

\[
x_0 = \tilde{O}^1(g_1', g_2', \ldots, g_k', \ldots) \quad \text{with} \quad p_{g_k'k} = \max_m \{p_{mk}\} \quad \text{for all} \quad k \in \mathbb{N},
\]

by at most a finite number of \( \tilde{O}^1 \)-symbols \( g_k(x) \) with \( p_{g_k(x)k} > 0 \).

Proof. If \( \xi \) has an atomic distribution, then there exists a point \( x \) such that

\[
P\{\xi = x\} = \prod_{k=1}^{\infty} p_{g_k(x)k} > 0.
\]

In such a case we have

\[
\prod_{k=1}^{\infty} \max_m \{p_{mk}\} \geq \prod_{k=1}^{\infty} p_{g_k(x)k} > 0.
\]

So, if the distribution of the random variable \( \xi \) has atoms, then (13) holds.

Let now (13) holds. Let us consider an arbitrary \( x \in [0,1] \) whose \( \tilde{O}^1 \)-representation differs from the \( \tilde{O}^1 \)-representation of the above \( x_0 \) by at most a finite number of \( \tilde{O}^1 \)-symbols \( g_k(x) \) with \( p_{g_k(x)k} > 0 \). It is evident that \( x \) is also an atom of the distribution \( \xi \).

We shall prove that \( \xi \) has a discrete distribution.

Let \( x^{(m)}_j = \tilde{O}^1(g_1, g_2, \ldots, g_m, g_{m+1}', \ldots, g_k', \ldots) \) be an arbitrary atom among all atoms whose \( \tilde{O}^1 \)-symbols coincide with the \( \tilde{O}^1 \)-symbols of \( x_0 \) starting from the \((m+1)\)-th symbol.
Then
\[
P\{\xi \in \left\{ x_j^{(m)} \right\} \} = \sum_{g_1: p_{g_1} > 0}^{\infty} \sum_{g_2: p_{g_2} > 0}^{\infty} \cdots \sum_{g_m: p_{g_m} > 0}^{\infty} \prod_{k=m+1}^{\infty} p_{g_k}(x_k) = \\
= \prod_{k=m+1}^{\infty} p_{g_k}(x_k).
\]

The set \( D = \bigcup_{m=1}^{\infty} \left\{ x_j^{(m)} \right\} \) is at most a countable set and
\[
P\{\xi \in D\} = \lim_{m \to \infty} P\{\xi \in \left\{ x_j^{(m)} \right\} \} = \lim_{m \to \infty} \prod_{k=m+1}^{\infty} p_{g_k} = 1.
\]

So, the random variable \( \xi \) is supported by an at most countable set and thus it is discretely distributed by definition.

\[\square\]

**Theorem 10.** The distribution of the random variable \( \xi \) is of pure type. It is either pure discrete or pure singular continuous or pure absolutely continuous.

**Proof.** Taking into account Theorem 9, it is sufficient to prove that in the continuous case the distribution of \( \xi \) is either pure singular or pure absolutely continuous.

Let \( x = \bar{O}^1(g_1(x), g_2(x), \ldots, g_n(x), \ldots) \) and let \( t_1, \ldots, t_n \) be fixed natural numbers. We shall set
\[
\Delta_{t_1\ldots t_n}(x) = \bar{O}^1(t_1, \ldots, t_n, g_{n+1}(x), g_{n+2}(x), \ldots)
\]
and for any set \( E \subset [0, 1] \) we shall set
\[
\Delta_{t_1\ldots t_n}(E) = \left\{ u : u = \Delta_{t_1\ldots t_n}(x), x \in E \right\},
\]
\[
T_n(E) = \bigcup_{t_1\ldots t_n} \Delta_{t_1\ldots t_n}(E), \quad T(E) = \bigcup_{n} T_n(E).
\]

Let us consider an event \( A = \{ \xi \in T(E) \} \). Since the random variables \( \xi_k \) are independent, the event \( A \) is residual. So, from the Kolmogorov 0–1 law it follows that either \( P(A) = 0 \) or \( P(A) = 1 \).

Since \( T(E) \supset E \), from the inequality \( P\{\xi \in E\} > 0 \) it follows that \( P\{\xi \in T(E)\} \geq P\{\xi \in E\} > 0 \), so \( P\{\xi \in T(E)\} = 1 \).

Only one of the following two cases can occur:

1. There exists a set \( E \) such that \( \lambda(E) = 0 \), but \( P\{\xi \in E\} > 0 \).
2. For any set \( E \) with \( \lambda(E) = 0 \) it follows that \( P\{\xi \in E\} = 0 \).

In the first case from equality \( \lambda(E) = 0 \) it follows that \( \lambda(T(E)) = 0 \), which implies that there exists a set \( T(E) \) such that \( \lambda(T(E)) = 0 \), but \( P\{\xi \in T(E)\} = 1 \), that is the distribution of \( \xi \) is pure singular by definition.

In the second case the distribution of the random variable \( \xi \) is absolutely continuous by definition. \( \square \)
Proof. It is well known that for any arbitrary random variable $\xi$, the topological support $S_\xi$ of the random variable $\xi$. These properties are completely determined by the infinite stochastic matrix $P_\xi = \|p_{ik}\|$, where the $k$-th column of the matrix corresponds to the distribution of the random variable $\xi_k$: $p_{ik} = P\{\xi_k = i\}$.

**Theorem 11.** The topological support $S_\xi$ of the random variable $\xi$ is a nowhere dense set if and only if the matrix $P_\xi$ contains an infinite number of columns having zero elements.

If the set $V_k(\xi) = \{i : p_{ik} > 0\}$ has one of the following properties:

1. $V_k(\xi)$ contains $N_k$ elements and $\lim_{k \to \infty} \frac{N_1N_2...N_k}{(k+1)} = 0$;
2. $V_k(\xi) = \{1, 2, \ldots, m_k\}$, and $\sum_{k=1}^{\infty} \frac{1}{m_k} = +\infty$;
3. $V_k(\xi) = V = \mathbb{N} \setminus \{a_1, a_2, \ldots, a_n, \ldots\}$, where $a_n$ is an arbitrary increasing sequence of positive integers with $a_{n+1} - a_n \leq d$ for some fixed $d \geq 2$ and for any $n \in \mathbb{N}$;

then the topological support of the random variable $\xi$ is of zero Lebesgue measure.

Proof. It is well known that for any arbitrary random variable $\eta$ with the distribution function $F_\eta$ the topological support $S_\eta$ coincides with the set

$$\{x : F_\eta(x + \varepsilon) - F_\eta(x - \varepsilon) > 0, \forall \varepsilon > 0\}.$$

Let us consider the set $\widetilde{C}[\tilde{O}^1, \{V_k(\xi)\}]$ with $V_k(\xi) = \{i : p_{ik} > 0\}$. If $x = \tilde{O}^1(g_1(x), g_2(x), \ldots, g_n(x), \ldots) \in \widetilde{C}[\tilde{O}^1, \{V_k(\xi)\}]$, then

$$P\{\xi \in \tilde{O}^1(g_1(x), g_2(x), g_n(x))\} = \prod_{k=1}^{n} p_{g_k(x)k} > 0,$$

for any $n \in \mathbb{N}$. So, $x \in S_\xi$.

If $x = \tilde{O}^1(g_1(x), g_2(x), \ldots, g_n(x), \ldots) \notin \widetilde{C}[\tilde{O}^1, \{V_k(\xi)\}]$, then there exists a number $n_0$ such that $g_{n_0}(x) \notin V_{n_0}(\xi)$. So, $p_{g_{n_0}(x)n_0} = 0$, and

$$P\{\xi \in \tilde{O}^1(g_1(x), g_2(x), g_{n_0}(x))\} = \prod_{k=1}^{n_0} p_{g_k(x)k} = 0.$$

Hence, $x \notin S_\xi$. Therefore, the topological support $S_\xi$ of the random variable $\xi$ coincides with the set $\widetilde{C}[\tilde{O}^1, \{V_k(\xi)\}]$.

If the matrix $P_\xi$ contains only a finite number of columns having zero elements (i.e., there exists a number $k_0$ such that $p_{ik} > 0$ for any $k > k_0$ and for any $i \in \mathbb{N}$), then the topological support $S_\xi$ completely contains any cylindrical set $\tilde{O}^1_{[c_1...c_k]}$ with $k > k_0$ and $c_i \in V_i$.

If the matrix $P_\xi$ contains an infinite number of columns having zero elements, then for any $n \in \mathbb{N}$ there exists a column $l_n > n$ and a number $s_n \in \mathbb{N}$, such that $p_{s_nl_n} = 0$. Therefore, for any cylindrical set $\tilde{O}^1_{[c_1c_2...c_n]}$ with $c_i \in V_i$ there exists a subset $\tilde{O}^1_{[c_1c_2...c_{n-1}s_n]}$ such that $\tilde{O}^1_{[c_1c_2...c_n...c_{n-1}s_n]} \cap \tilde{C}[\tilde{O}^1, \{V_k(\xi)\}] = \emptyset$. Hence, $S_\xi$ is a nowhere dense set.

If condition 1 (condition 2 resp. condition 3) of the Theorem holds, then, from the equality $S_\xi = \tilde{C}[\tilde{O}^1, \{V_k(\xi)\}]$ and Theorem 3 (Theorem 4 resp. Theorem 8) it follows that $\lambda(S_\xi) = 0$. \qed
Corollary. If
\[
\prod_{k=1}^{\infty} \max_m \{p_{mk}\} = 0
\]
and one of the conditions 1, 2, 3 of Theorem 11 holds, then the random variable \( \xi \) has a Cantor-type singular continuous distribution.

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