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Review

Nonlinear control of infection spread based on a deterministic SEIR model

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ABSTRACT

In this study, a mathematical model (SEIR model) with a restriction parameter is used to explore the
dynamic of the COVID-19 pandemic. This work presents a nonlinear and robust control algorithm based
on variable structure control (VSC) to control the transmission of coronavirus disease (COVID-19). The VSC
algorithm is a control gain switching technique in which is necessary to define a switching surface.
Three switching surfaces are proposed based on rules that depend on: (i) exposed and infected population,
(ii) susceptible and infected population, and (iii) susceptible and total population. In case (iii) a model-
based state estimator is presented based on the extended Kalman filter (EKF) and the estimator is used
in combination with the VSC. Numerical results demonstrate that the proposed control strategies have
the ability to flatten the infection curve. In addition, the simulations show that the success of lowering
and flattening the epidemic peak is strongly dependent on the chosen switching surfaces. A comparison
between the VSC and sliding mode control (SMC) is presented showing that the VSC control can provide
better performance taking into account two aspects: time duration of pandemic and the flattened curve
peak with respect to SMC.

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1. Introduction

It can be said that the world face right now the biggest pand-
emic of its history. The World Health Organization (WHO) gave
the name COVID-19 to the disease caused by the new coronavirus
(SARS-CoV-2). The patients affected by COVID-19 have a pecu-
liar clinical characteristic because there are individuals who are
asymptomatic while others have serious conditions of the disease.
It can be said that the collapse of the health system that was ob-
served in several countries of the world is as serious as the disease
itself. For example, in Brazil, the number of deaths by COVID-19 to
date, March 29, 2021, is 312,206 with 12,534,638 confirmed cases
[1]. In the absence of vaccines, it is believed that the best way to
combat the spread of the disease is through preventive measure.
Several pandemics have been witnessed by humanity throughout
history. Some of the most serious are Bubonic Plague (sec. XIV)
and the Spanish Flu (sec. XX). It is estimated that Bubonic plague
killed between a third and half of the entire population [2], while
the Spanish flu has killed 20 to 40 million people [3].

Mathematical models are useful tools for investigating the dy-
namics and spread of diseases. Several models have been used to
study the dynamics of infectious diseases such as influenza A [4],
Zika [5], Ebola [6], SARS [7,8] MERS [9,10] and COVID-19 [11–18]. In
this work, the SEIR model changed by an additional control term is
used. The control term represents preventive measures indicating
Government action, like social distancing, quarantine, restriction of
public movement, masks, etc. In the absence of vaccines, prevent-
ive measures have been taken to flatten the epidemic curve, i.e.,
the objective of it is delaying and reducing the height of the curve
peak of the infected population, and, thus aiding the health sys-
tem. This hypothetical situation is illustrated in Fig. 1.

This paper deals with an epidemic model developed to inves-
tigate a possible control of the COVID-19 pandemic, which allows
the implementation of epidemic control measures through prevent-
ive measures. A nonlinear control problem by variable structure
using the deterministic model known as SEIR is formulated. The
preventive measures are the mechanism used to vary the structure
of the control, ensuring that the number of infected people does
not increase. The goal of epidemic control is flattening the epi-
demic curve. Three switching surfaces are proposed based on rules
that depend on: (i) exposed and infected population, (ii) suscepti-
bile and infected population, and (iii) susceptible and total popula-
tion. In particular, in case (iii), it is necessary to estimate the state
variables of the SEIR model. For this purpose, an extended Kalman
filter (EKF) is used. Numerical results show the capability of the
control procedure to avoid high spikes of infections. In addition,
the simulations show that the success of lowering and flattening

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the epidemic peak is strongly dependent on the chosen switching surfaces.

2. Disease dynamics - SEIR model

Deterministic mathematical models are important tools to characterize the dynamics of epidemics, and in this way, it can help to assist in the formulation of public health policies during infectious disease outbreaks. The SEIR mathematical compartmental model to forecast the evolution of the COVID-19 outbreak is used. The SEIR model describes the temporal evolution of the Susceptible (individuals who have not been contaminated and does not have resistance to the virus), Exposed (who were contaminated but are in the incubation period and are not yet contagious), Infectious (who are able to transmit disease and symptomatic individuals which are already starting to exhibit the first symptoms) and Removed (isolated people that can be healed, hospitalized or killed). The SEIR compartmental deterministic model is given by follows set of ordinary differential equations

\[ \begin{align*}
\frac{dS}{dt} &= -\frac{\beta}{N} SI \\
\frac{dE}{dt} &= \frac{\beta}{N} SI - \omega E \\
\frac{dI}{dt} &= \omega E - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{align*} \]  

(1)

where \( \beta = \frac{R_0}{\text{inc}} \), \( \omega = \frac{1}{\text{inc}} \) and \( \gamma = \frac{1}{\text{inc}} \) and \( N = S + E + I + R \) is the total population. The rate at which susceptibles are converted into exposed (\( \beta \)) depends both on infection period (\( T_{\text{inc}} \)) and the epidemiological parameter \( R_0 \) known as basic reproductive number of the infection system. In practice, this is the most important parameter since it describes the average number of secondary infections caused by an infectious individual. In the case of COVID-19, in it is estimated \( R_0 \) between 1.6 and 3.0 [19]. In addition, \( \omega \) is the rate at which exposed are converted into infected, \( \gamma \) is the rate at which those infected are converted into recovered, and \( T_{\text{inc}} \) is the incubation period.

Besides, the numerical integration of the system of ordinary differential Eq. (1), two new measures will be defined, namely: the number of newly infected individuals daily (\( I_d \)) and the total number of infected individuals (\( I_t \)). The \( I_d \) is defined as follows

\[ I_{d_k} = \int_{t_k}^{t_{k+1}} \beta SI \, dt, \quad k = 0, \ldots, f - 1, \text{ com } f \in \mathbb{N} \]  

(2)

It should be noted that the period of 1 day is defined as \( \Delta t = t_{k+1} - t_k \), so the first day is given \( \Delta t = t_1 - t_0 \), the second day as \( \Delta t = t_2 - t_1 \) and so on, up to \( \Delta t = t_f - t_{f-1} \), where \( t_f \) which corresponds to what is considered to be the ending of the period of study. The \( I_t \) is defined as

\[ I_t = E + I \]  

(3)

The simulations were performed by using the fourth-order Runge-Kutta method with a fixed step and reference parameter values from the literature are used to simulate the outbreak of COVID-19: \( R_0 = 2.2 \) [20], \( T_{\text{inc}} = 6.4 \) days [21], \( T_{\text{inf}} = 3 \) or 7 days [22]. Two possible scenarios for the outbreak of COVID-19 are discussed. Scenario 1 is defined as taking into account a short infectious period (3 days) and scenario 2 defines that the disease have a longer duration of infectiousness (7 days). In all simulation, the initial conditions (i.e.) required to compute the evolution of the system is a 4-ple of numbers, defined as \( (S_0, E_0, I_0, R_0) = (999, 0, 1, 0) \).

Fig. 2 depicts the time series obtained by numerical integration of the Eq. (1), for scenario 1 (Fig. 2a) and scenario 2 (Fig. 2c). Fig. 2b and d provides a zoom of Fig. 2a and c, respectively. For different values of \( T_{\text{inf}} \) we obtain different values for \( \beta \) and \( \gamma \). Thus;

- In the first scenario if \( T_{\text{inf}} = 3 \) then \( \beta = 0.7333 \) and \( \gamma = 0.333 \);
- In the second scenario if \( T_{\text{inf}} = 7 \) then \( \beta = 0.3143 \) and \( \gamma = 0.1429 \).

The value of \( \beta \) from scenario 1 is higher than from scenario 2, thus, the rate at which susceptibles become exposed individuals is faster. As can be seen from Fig. 2b and d, the peak of the Exposed individuals, in scenario 1, is reached on day 60, whereas, in scenario 2, it is observed on day 87. As can be seen from scenario 1, the peak of 6% of the infected individuals was reached after 63 days since the disease started to spread, while in scenario 2, the peak of approximately 10% of the population is reached 94 days after the onset of the disease. On the other hand, the rate at which infected individuals recover in scenario 1 is faster because the value of \( \gamma \) for scenario 1 is higher than scenario 2. As can be seen, the number of Infected is close to 1 after 126 days (scenario 1) whereas, in scenario 2, this value was obtained on day 189 (see Fig. 2d).

Fig. 3 shows scenarios 1 and 2 for \( I_d \) (Fig. 3a) and \( I_t \) (Fig. 3b). As expected, in scenario 1, \( I_d \) and \( I_t \) increases faster than scenario 2. There are fewer daily infections in scenario 2, meaning that the duration of the pandemic remains for much longer periods than scenario 1. At \( T_{\text{inf}} = 3 \) the peak is reached on day 50, while for \( T_{\text{inf}} = 7 \) the peak is obtained on day 75 (see Fig. 3a). As can be seen from Fig. 3b, peaks of equal amplitude are obtained to \( I_t \). However, one important difference, as already mentioned, is that for \( T_{\text{inf}} = 7 \) the infection is slower and more lasting. The numerical study shows that the speed of the infection spread is very significant for studies of epidemics.

So far, it has been observed that the qualitative behavior of the two scenarios presented is similar. Since the interest is to control the infection spread and so that the simulation does not get repetitive, here will be considered only scenario 1 because the speed of infection spread of scenario 1 is faster than the second one.

3. Epidemic control strategies

The impact of preventive measures on the infection spread is analyzed by considering the SEIR model. Preventive measures, or in
Fig. 2. Number of individuals: (a) Scenario 1, (b) zoom of (a), (c) Scenario 2 and (d) zoom of (c).

Fig. 3. Time-series of (a) $I_e$ and (b) $I_t$. 
the context of this paper, control action are restrictive government measures that are proposed to contain a pandemic, such as social distancing, quarantine, restriction of public movement, masks, and other efforts necessary to mitigate the spread of COVID-19. In order to incorporate the control action, the SEIR model is rewritten with an additional control term \((1 - \sigma)\) following the formulation used by Souza et al. [11], Boldog et al. [23]

\[
\begin{align*}
\frac{dS}{dt} &= -\frac{\beta}{N} SI(1 - \sigma) \\
\frac{dE}{dt} &= \frac{\beta}{N} SI(1 - \sigma) - \omega E \\
\frac{dI}{dt} &= \omega E - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{align*}
\]

(4)

where \(I\) is the number of individuals.

In this case, the parameter \(\sigma\) can be associated with the intensity of the restrictions imposed by the control policies adopted in order to reduce the spread of infection. For instance, Souza et al. [11], suggest that \(\sigma\) can represent the fraction of the infected individuals subjected to isolation.

The objective of epidemic/pandemic control is to ensure that the number of infected individuals does not increase which means decreasing the infection spread and death rate. The SEIR system can be controlled by the variable structure controller (VSC). The control law for systems with variable structure is defined as

\[
u(x, t) = \begin{cases} 
  u^+(x, t), & \text{if } S(x) > 0 \\
  u^-(x, t), & \text{if } S(x) < 0
\end{cases}
\]

where \(x\) is the state vector and \(S(x)\) is a scalar function. The function \(S(x) = 0\) defines a sliding surface in the state space to which the trajectories of the system must be directed. If for a certain \(t_0\), the response of the system is above the sliding surface \(S(x)\), then the control law \(u^+(x, t)\) lead it in the direction of \(S(x)\). When passing through the surface, the control is switched to \(u^-(x, t)\), bringing the response back to \(S(x)\), hence the trajectory is trapped in. In this work, three different functions \(S(x)\) are proposed and evaluated; these functions are defined by (i) number of infected individuals, (ii) number of newly infected, (iii) number total infected individuals.

3.1. Control based on the number of infected individuals

From Eq. (4), the time derivative of infected individuals is given by

\[
\frac{dI}{dt} = \omega E - \gamma I
\]

(5)

Hence,

- \(\frac{dI}{dt} > 0 \Rightarrow \alpha(E,I) > 0\), thus, the number of people infected is growing. Therefore, the acceleration of infections is observed.
- \(\frac{dI}{dt} < 0 \Rightarrow \alpha(E,I) < 0\), thus, the number of people infected is decreasing, so the slowing down of infections is observed.

where \(\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}\) is given by \(\alpha(E,I) = \frac{\alpha}{\sigma} - \frac{1}{I}\).

The VSC using the switching function \(\alpha(E,I)\) is introduced. Hence, the control law can be written as

\[
\sigma = \begin{cases} 
  \sigma^+, & \text{if } \alpha > 0 \\
  \sigma^-, & \text{if } \alpha < 0
\end{cases}
\]

(6)

A physical interpretation of the control action defined by Eq. (6) is follows: instead of keeping the control action \(\sigma\) at a constant level (aggressive or mild) throughout the disease period, according to this control law presented, the Government can either intensify or relax the restrictions like allowing essential services/ restricted freedom of movement, etc., to the people.

It is worth remembering that \(\sigma \in [0, 1]\) and therefore there are infinite possibilities for \(\sigma^+\) and \(\sigma^-\). Fig. 4a shows the behaviour of infected individuals and Fig. 4b shows the values of \(\sigma\) both as a function of time. It can be seen from Fig. 4 that as the \(\sigma\) increases the flattening the curve associated with more restrictive preventive measures is observed. This is a natural consequence of the SEIR model because the effective contact rate is reduced (that is, \(1 - \sigma \beta\) is reduced) when \(\sigma\) increases. The behavior of \(\sigma\) is shown in Fig. 4b. It should be noted that there is a moment when the value of \(\sigma\) changes many times in a short period of time, and this is associated with signal changes on the \(\sigma\) surface. As a consequence of this switching, the behaviour of infected individuals is affected, which defines the appearance of the plateaus (curve flattening).

Based on the previous discussion, if nothing is done to put an end to the transmission (i.e., \(\sigma = 0\) - base case), the peak of 6% of \(I\) is reached after 63 days since COVID-19 disease started to spread. By taking \(\sigma^+ = 0.1\) (in red), let us note that the peak of infected drops to 4.7% after a further 9 days with regard to the base case. Increasing the control value to \(\sigma^- = 0.2\) (in blue) leads to the appearance of a small plateau on the epidemic curve lasting 4 days. In addition, the peak of 3.5% of infected is reached after 21 days with regards to the base case. For \(\sigma^+ = 0.3\) (in green) the plateau lasts 12 days, and the peak of 2.3% of infected starts 40 days after the peak of the base case. For \(\sigma^- = 0.4\) (in gray) the plateau lasts 34 days with a peak of approximately 1% of the population and for \(\sigma^+ = 0.45\) (in cyan) the plateau lasts 47 days with peak of less than 1% of the population.

In a real case, the control action is not so instantaneous, i.e., may be necessary to wait a few days to see the expected result. To this end, a new control action is proposed. The procedure is this: the data of infected individuals should be collected day by day, a control decision should be taken, the outcome of this action is awaited for a period, and the control action can be maintained or modified depending on the values of \(I\). With this idea in mind, Fig. 5 shows the control action, which is applied every 7 days, that is, some control action is initiated, and it remains the same for 7 days. After that, the value of \(\alpha\) is evaluated, and a new action (or not) will be taken. Here, the value of \(\alpha^+\) is fixed, and \(\alpha^-\) changes. The simulation is initialized with \(\alpha = 0\) (without control) and only after 7 days an appropriate action will be carried out. As expected, in both cases, the number of infected after 7 days increased. Thus, we may assume that \(\sigma = \sigma^+\). When the surface \(\alpha\) is close to zero the value of \(\alpha\) decreases to \(\sigma^-\), that is, the restriction is alleviated, but the value of \(I\) will increase in the next 7 days. Therefore, 7 days after this procedure, a more severe restriction is required, and this process is repeated until controlling the spread of infections. Whether that happens only milder isolation values are adopted (\(\sigma = \sigma^-\)).

Fig. 6 shows results related to control action where each action must be taken every 14 days (Fig. 6a) and every 21 days (Fig. 6b). It should be noted that, in both cases, the peak of infected individuals is lower when \(\sigma^+\) and \(\sigma^-\) are close to each other, that is, even when \(\alpha(E,I) < 0 \) rigorous control measures were adopted. As can be seen from Figs. 5 and 6, with the relaxing of restriction, additional peaks (more prominent) are observed. Numerical investigations suggest that preventive measures should be taken to reduce the number of infected individuals over time. The best scenario is the one in which the relaxation of restrictions have been undertaken gradually, even if the number of infected people decreases. It is also clear from the present results that the longer it takes to start an appropriate control action more individuals will become infected, thus, there is an overload in hospitals, more people could die, and so on.
Fig. 4. (a) Time-series of the controlled system and (b) gain of the controller.

Fig. 5. (a) Time-series of the controlled system and (b) gain of the controller.

Fig. 6. Time-series with the control actions being taken: (a) every 14 days and (b) every 21 days.
3.2. Control based on the number of newly infected individuals daily

To track the number of new infections daily the new measure is defined as

$$p_{\text{new}} = M \int_{t_k}^{t_{k-1}} S(1 - \sigma) \, dt, \quad k = 0, \ldots, f - 1, \text{ com } f \in \mathbb{N} \quad (7)$$

To investigate whether the number of infected people increased or decreased after day, the following function is proposed

$$\Delta p_{\text{new}}^M = M_{\text{new}}^{f} - M_{\text{new}}^{t_k}, \quad M = 1, \ldots, t_f - 1 \quad (8)$$

where $M_{\text{new}}^t$ is given by Eq. (7). It should be noted that from Eq. (8) we obtain: (i) if $\Delta p_{\text{new}}^M > 0$ then the number of infected people is increasing day after day, and thus we have an acceleration of the spreading disease, (ii) if $\Delta p_{\text{new}}^M < 0$ then the epidemic curve is decreasing indicating that we are controlling COVID-19 outbreak.

As described in the previous section, it is necessary to wait a few days before making a control decision, thus, taking Eq. (8) and taking a control action every 7 days, we are analyzing the number of new infected on the first day and on the seventh day, thus, data of the day-2 until day-6 will be ignored. In order to prevent this from happening, it is defined as the switching surface the average of new cases diagnosed in n-days, i.e., we will compare the average of the number of new infections every n-day and thus check if the average is increasing or decreasing over time.

The average of new cases per period of n-days is defined as follows

$$\Delta \lambda_{\text{new}} = \frac{1}{n} \sum_{j=1}^{n} \Delta \lambda_{\text{new}}^{(j)}, \quad i = 0, n, 2n, \ldots \quad (9)$$

where $n$ represents the period of time of n-days, and the control law is then defined as follows

$$\sigma = \begin{cases} \sigma^+, & \text{if } \Delta \lambda_{\text{new}} > 0 \\ \sigma^-, & \text{if } \Delta \lambda_{\text{new}} < 0 \end{cases} \quad (10)$$

Fig. 7 depicts a comparison between the results of control strategy using the switching surface given $\Delta \lambda_{\text{new}}$ and $\sigma$. The solid lines show the results of switching surface $\Delta \lambda_{\text{new}}$, while the dashed lines show the results of $\sigma$ (previous case). In all figures, the control action is taken every 7 days. Time-series for $\sigma^+ = 0.4$ and $\sigma^- = 0.1$ are shown in Fig. 7a. For these values, the spread of the infection is kept under control in both cases, with peaks close to each other. However, a difference is observed, especially in the upward epidemic curve because, in the first 7 days there is a reduction of infected (this is typical of the SEIR model) which means that $\Delta \lambda_{\text{new}} < 0$, thus, the public health interventions can be relaxed. 

Thereby, the control begins to be implemented after 14 days. With this property of the model, the upward curve of the infected is faster if the switching surface with an average of the new infected were applied, however, more switching occurs between $\sigma^+$ and $\sigma^-$ (see Fig. 7d), which implies that the descent curve of infected is faster. This same situation occurs when we consider $\sigma^+ = 0.4$ and $\sigma^- = 0.2$ (see Fig. 7b) and $\sigma^+ = 0.4$ and $\sigma^- = 0.3$ (see Fig. 7c).

Fig. 8 shows the control actions being taken every 14 days (see Fig. 6a) and every 21 days (see Fig. 8b). In this case, as the control decision occurs over a slightly longer period of time, the responses at the beginning are identical to those observed in the previous case (see Fig. 6), that is, the adopted control strategy happen basically at the same time, and are independent of the switching surface adopted. However, the control interventions applied every 14 days, lead to a decrease in the infected curve when we consider $\Delta \lambda_{\text{new}}$, and this is due to the fact that the switching is more spaced. In Fig. 8b, the answers are identical to Fig. 6b.

3.3. Control based on the number of total infected individuals

The total number of infected without the control action was described in the previous Eq. (3). On the other hand, the total number of infected taking into account the preventive measure (control term) is defined as

$$I_{\sigma} = E + I \quad (11)$$

thus, the time derivative of $I_{\sigma}$ is given by

$$\frac{dI_{\sigma}}{dt} = \frac{dE}{dt} + \frac{dI}{dt} = \beta \frac{I}{N} (1 - \sigma) - \gamma I \quad (12)$$

Hence

- $\frac{dI_{\sigma}}{dt} > 0 \Leftrightarrow \Lambda > 0$, thus, the epidemic curve is growing, so the acceleration of spreading disease is observed.
- $\frac{dI_{\sigma}}{dt} < 0 \Leftrightarrow \Lambda < 0$, thus, the epidemic curve is decreasing, so the slowing down of spreading disease is observed.

where $\Lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the switching surface defined as $\Lambda(\lambda, \sigma) = \lambda - \sigma$, with $\lambda = 1 - \frac{m}{N}$. Note that $\Lambda$ dependent on $N = S + I + E + R$ (total population), $S$ (susceptible individuals), $\sigma$ (control parameter) and $\theta = \frac{\gamma}{\lambda}$.

It is important to observe that the switching surface $\Lambda$ depends on $\sigma$, that is, the numerical implementation of the proposed approach to finding the value of $\sigma$ requires the knowledge of its own value. To address this issue, an additional switching condition is added. The controller design is separated into two steps. The first one is used to find the parameter $\tilde{\sigma}$ which is calculated from VSC using the switching surface that will be defined. Once obtained $\tilde{\sigma}$, a second switching based on Eq. (12) is used to validate $\sigma$.

The VSC feedforward controller computes the $\tilde{\sigma}$ and the VSC feedback controller adjusts the control input ($\sigma$) to suppress the epidemic spreading. The controller uses an extended Kalman filter (EKF) to estimate the $\lambda$-parameter. The control system is shown schematically in Fig. 9. So, the switching surface of the feedback controller is here slightly rewritten as

$$\Lambda = \lambda - \tilde{\sigma} \quad (13)$$

and the control input is written as

$$\sigma = \begin{cases} \eta, & \text{if } \Lambda > 0 \\ \tilde{\sigma}, & \text{if } \Lambda < 0 \end{cases} \quad (14)$$

where $\eta$ will be defined later.

The Kalman filter addresses the general problem of trying to estimate the discrete-time states of a system represented by a stochastic model of the form

$$x_k = f(x_{k-1}, u_k, w_{k-1}) \quad (15)$$

where $u_k$ is the control term and $w_k$ is the random variable of the process with zero mean and a known covariance matrix Q. The measurement is given by

$$z_k = h(x_k, v_k) \quad (16)$$

where $v_k$ is the random variable of the measurement with zero mean and a known covariance matrix R. Besides, $w_k$ and $v_k$ are assumed to be independent of each other.

The Kalman filter addresses the general problem of trying to estimate the discrete-time states of a system represented by a stochastic model. The extended Kalman filter (EKF) is used whether the system to be estimated is non-linear. Such a procedure is very useful in situations where it is desirable that all model state variables are available. EKF is based on linearization on the average of the current estimate error and covariance. The EKF algorithm [24] can be written as
Fig. 7. Time-series of the controlled system (a), (b) and (c); gain of the controller (d), (e) and (f).

Fig. 8. Time-series with the control actions being taken: (a) every 14 days and (b) every 21 days.

Predicted state ($\hat{x}_k^-$) and covariance ($P_k^-$) estimate:

$$\hat{x}_k^- = f(\hat{x}_{k-1}, u_k, 0)$$
$$P_k^- = A_k P_{k-1} A_k^T + W_k Q_k W_k^T$$  \(17\)

Kalman Gain:

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + V_k R_k V_k^T)^{-1}$$  \(18\)

Updated state ($\hat{x}_k$) and covariance ($P_k$) estimate:

$$\hat{x}_k = \hat{x}_k^- + K_k (z_k - h(\hat{x}_k^-, 0))$$
$$P_k = (I - K_k H_k) P_k^-$$  \(19\)
where \( A, W, H, \) and \( V \) are Jacobian matrices defined as

\[
A = \frac{\partial f_{11}}{\partial x_{11}}(\hat{x}_{k-1}, u_k, 0), \quad W = \frac{\partial f_{11}}{\partial w_{11}}(\hat{x}_{k-1}, u_k, 0),
\]

\[
H = \frac{\partial h_{11}}{\partial x_{11}}(\hat{x}_k, 0), \quad V = \frac{\partial h_{11}}{\partial v_{11}}(\hat{x}_k, 0)
\]

(20)

The uncertainty is added to the system model through initial state error covariance \( P_0 = I_{4 \times 4} \) and the initial measurement noise covariance \( Q = 10^{-3} I_{4 \times 4} \) and \( R = 0.1 \). In order to examine the performance of the EKF on the dynamic response of the system, a comparison with the SEIR model has been made. Fig. 10a shows the comparison between the EKF predicted and the SEIR model for the infectious state variable without control action. Fig. 10b presents the estimation error for the same variable. As it is seen very good results are obtained when this state variable is estimated by using an EKF approach.

In the attempt to predict the \( \eta \) parameter (see Eq. (14)), it will be assumed that \( \eta = \lambda + \epsilon \). In this case, we adopted \( \hat{\sigma} = 0.4 \) (maximum value) or 0.1 (minimum value), and the switching surface is defined by Eq. (6). Fig. 11 shows the time-series of the controlled system using the EKF where control actions being taken every 7 days and \( \epsilon = 0.01 \) was assumed. Fig. 11a shows the time-series of \( l \), obtained when the control rule (see Fig. 11b) is applied to the system. Note that the increase of \( \sigma \) can promote a better performance related to control the number of infected. Hence, this preventive measure is so restrictive that the infectious disease does not spread, however, in order for it to work properly, the preventive measure should remain for a very long time. Thus, we conclude that this measure would be totally impracticable and counterproductive.

In order to find the proper parameter setting of \( \sigma \), a new version of the control input is considered. Therefore

\[
\sigma = \begin{cases} 
\frac{a \hat{\sigma} + b \lambda}{\bar{\sigma}} & \text{if} \quad \Lambda > 0 \\
\frac{a \hat{\sigma} - b \lambda}{\bar{\sigma}} & \text{if} \quad \Lambda < 0 
\end{cases}
\]

(21)

where \( a > 0 \) and \( b > 0 \) are the weighting parameters.

Fig. 12 shows the time-series obtained for various combinations of parameters where the control actions being taken every 7 days. In all cases, were used the switching surface defined by Eq. (6) to compute \( \hat{\sigma} \). Here, the response is assumed to be \( \hat{\sigma} = 0.4 \)
or \( \hat{\sigma} = 0.1 \). The maximum value of \( I \), as expected, decreases when the control is more restrictive, i.e., if the value of \( b \) increases, then parameter \( \lambda \) becomes more significant, which ensures that the intensity of restrictions associated with control policies reduces the infection. On the other hand, it is to be remarked that the more restrictive is the control action, the longer the period of time over which the COVID-19 pandemic extends. Fig. 12a shows the variation of control with respect to time for some combinations of the parameters \( a \) and \( b \). The first aspect that emerges from the comparison with the other simulations is that due to the assumption of linear approximation made in Eq. (21), the control action starts with a more restrictive value and decays slowly with time until abrupt switches occur.

In order to make a comparison among different measures proposed here, the duration of the pandemic will be taken into consideration. In the two first cases, the number of infected is close to zero at 300 days. This is obtained by taking \( a = 0.8 \) and \( b = 0.2 \). The response of the SEIR model taking into account Eq. (21) is illustrated in Fig. 13, where the solid and dashed black curves are the same presented in Fig. 7, and the green curve corresponds to control based on the number of total infected individuals. It turns out that the time-series are quite similar to the previous cases, with differences in terms of the extreme values of the infected individuals, which is the lowest amplitude than all others. In addition, these peaks present a delay, as compared to previous investigations. This indicates an interesting scenario with a low number of infections. However, the value of \( I \approx 0 \) is reached when \( t \approx 346 \) days, therefore, 46 days following the end of the infection when compared to the previous cases. The control signal time history is presented in Fig. 13d, e, and f.

Fig. 14 shows the time-series of the controlled system when the control actions are applied every 14 days (see Fig. 14a) or every 21 days (see Fig. 14b) for different values of \( \sigma^+ \) and \( \sigma^- \). The results show that this approach is effective in control infection spread.

### 3.4. Sliding mode control

One method which has been used very successfully to control infectious diseases is sliding mode control (SMC) [25–28]. In this study, the SMC developed to control the COVID-19 pandemic by Rohith and Devika [28] is used. The method proposed in [28] considers the state variable \( E \) to defined a sliding surface, but, to remain close to the context of this paper, the sliding surface is modified in order to include the state variable \( I \).

The number of total infected individuals \( k_\sigma \) (see Eq. (11)) is used to obtain the sliding surface. Therefore, the sliding surface, \( \zeta \) is defined as

\[
\zeta = \rho (k_\sigma - I_0^t)
\]  

Fig. 11. (a) Time-series of \( I \) and (b) gain of the controller.

Fig. 12. (a) Time-series of \( I \) and (b) gain of the controller.
where $I_{d\sigma}$ is the desired value of $I_{\sigma}$ and $\rho > 0$ is the slope of the sliding surface.

The sliding surface is obtained employing a procedure based on the constant rate reaching law (CRRL) [28] given in the form

$$\dot{\zeta} = -\kappa \text{sign}(\zeta)$$  \hspace{1cm} (23)

where $\kappa$ represent the controller gain in CRRL structure.

Using Eq. (22), the CRRL structure (Eq. (23)) can be rewritten as

$$\rho (I_{\sigma} - I_{d\sigma}) = -\kappa \text{sign}(\zeta)$$  \hspace{1cm} (24)

Assuming that $I_{d\sigma}$ is constant, its derivative with respect to time is $\dot{I}_{d\sigma} = 0$. Substituting Eq. (12) into (24) leads to

$$\rho \left(\frac{B}{N} \text{Si}(1 - \sigma) - \gamma I\right) = -\kappa \text{sign}(\zeta)$$  \hspace{1cm} (25)
which gives the Governmental control action, $\sigma$, as

$$\sigma = \frac{N}{\rho}{\hat{\beta}}{\hat{S}}I \left( \rho{\hat{\beta}}{\hat{S}}I - \gamma I \right) + \kappa \text{sign}(\xi)$$

(26)

4. Comparative analysis

Comparative analysis evaluates the performance of the SMC and the VSC based on the number of total infected individuals (see Eq. (21)) comparing the efficacy of each one to flatten the peak of the outbreak curve of COVID-19. Before starting the comparative analysis, let us define the parameters of the SMC control, namely, $\kappa = 1$ and $\rho = 1$.

Initially, it is considered the value of $l_{\rho \sigma}$ (limit on the number of infected individuals), equal to 3% of the total population. Fig. 15 shows the comparison of SMC and VSC based on $l_{\rho \sigma}$. It should be noticed that both procedures are capable to reduce the infection spread. As can be seen from Fig. 15b, the control action (at the beginning) is less aggressive in the SMC, and an amplified peak is obtained (see Fig. 15a). On the other hand, the control of VSC is more aggressive (see Fig. 15b), therefore, the amplitude of peak decrease considerably, but as a consequence of this restriction, the effective duration of the epidemic increases (see Fig. 15a).

The major problem in achieving an amplified peak of infected individuals is that one part of the infected will need hospitalization causing concern because of an overload in the health system. Thus, taking a decision the governments should have a clear understanding of the capacity of the health system to provide care to the population.

Fig. 16 shows the time-series response for $l_{\rho \sigma}$ equal to 2% of the total population. As it can be seen, the action of SMC is more aggressive as shown in Fig. 16b, so there was a decrease in the amplitude of the peak, but there is an extension of the duration of infection (see Fig. 16a). In addition, a large quantity of switching occurs of values of $\sigma$ is observed. This alternation of $\sigma$ makes it difficult to establish the role of any Government intervention and that is exactly what this work tries to avoid, that is, the VSC designed here provides the appropriate control action without that the control action suffer substantial and abrupt alterations. Thus, the action based on VSC control that is applied every 7 days (14 or 21 days) is feasible from the Government’s point of view.

Now, it is assumed that $l_{\rho \sigma}$ is equal to 1.3% of the total population, and control results are presented in Fig. 17. The Government control ($\sigma$) is more aggressive, and therefore, there is a reduction in the number of infected cases, however, Fig. 17b shows that $\sigma$ changes constantly making Government action challenging. This more aggressive action reflects an attenuation in the peak of infected but a longer duration of the pandemic is observed.
5. Conclusions

The influence of the containment restrictions parameter on the infection dynamics of the COVID-19 pandemic was studied. The SEIR model proposed in [11,23] was adopted to taking into account Government measures represented by $\beta$ to reduce the disease spread. Variable structure control is designed for suppressing spikes during preventive measures and this control strategy is tested through simulations. Simulation results for three controllers are presented. Variable Structure Control based on the number of infected individuals is effective in order to suppress spikes of infected individuals, but some prominent spikes can be observed when the easing of the restriction occurs. Variable Structure Control based on the number of newly infected individuals daily also suppresses spikes of infected individuals tends to be more effective at eliminating these prominent spikes. Variable Structure Control based on the number of total infected individuals with an estimator, based on the Extended Kalman Filter, is presented. This strategy is more effective in order to reduce the magnitude and delaying the peak of the COVID-19 outbreak, however, due to the easing or not of the preventive measures, prominent spikes are more frequent than in other strategies.

Numerical simulations show satisfactory results for the three measures proposed in order to suppress spikes of infections. Furthermore, the study of the restrictions parameter influence on the system responses revealed that lowering and flattening of the epidemic peak is affected by the period between the control decisions. This means that the longer the government time wait to impose preventive measures the more serious the effects can be.

Based on the simulations both methods demonstrate efficiency and the capability to avoid high spikes of infections, that is, SMC and VSC are effective to perform reduce the spread of the epidemic. The SMC approach presents good performance to reduce the duration of infection. However, it should be pointed out that the amplitude of peaks increases considerably. On the other hand, the VSC control can provide better performance taking into account two aspects: time duration of pandemic and the flattened curve peak with respect to SMC.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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