We investigate the universal inequalities relating the $\alpha$-Rényi entropies of the marginals of a multipartite quantum state. This is in analogy to the same question for the Shannon and von Neumann entropies ($\alpha = 1$), which are known to satisfy several non-trivial inequalities such as strong subadditivity. Somewhat surprisingly, we find for $0 < \alpha < 1$ that the only inequality is non-negativity: in other words, any collection of non-negative numbers assigned to the non-empty subsets of $n$ parties can be arbitrarily well approximated by the $\alpha$-entropies of the $2^n - 1$ marginals of a quantum state. For $\alpha > 1$, we show analogously that there are no non-trivial homogeneous (in particular, no linear) inequalities. On the other hand, it is known that there are further, nonlinear and indeed non-homogeneous, inequalities delimiting the $\alpha$-entropies of a general quantum state. Finally, we also treat the case of Rényi entropies restricted to classical states (i.e. probability distributions), which, in addition to non-negativity, are also subject to monotonicity. For $\alpha \neq 0, 1$, we show that this is the only other homogeneous relation.

1. Prologue

The von Neumann entropy $S(\rho) = -\text{Tr} \rho \log \rho$ of a quantum state $\rho$ is a key notion in quantum information
theory [1] as well as in statistical physics [2]. It is furthermore the canonical measure of entanglement for bipartite pure states [3]. In many cases, the relative magnitude of the entropy of the reduced states of different subsystems is important. Thus, for example, for a tripartite state $\rho_{ABC}$, one can compute $S(\rho_A)$, $S(\rho_B)$, and so on. For any positive number $a$, one can find a quantum state such that $S(\rho_A) = a$, for example. However, for a fixed quantum state, there are inequalities between the values of the entropies of the reduced states of the subsystems.

There are essentially two such unconstrained inequalities known (up to permutation of the parties): strong subadditivity and weak monotonicity [4,5],

$$\begin{align*}
S(\rho_{ABC}) + S(\rho_B) &\leq S(\rho_{AB}) + S(\rho_{BC}), \\
S(\rho_A) + S(\rho_B) &\leq S(\rho_{AC}) + S(\rho_{BC}).
\end{align*}$$

(1.1)

There are no other constraints for up to three parties, but the analogous statement is a major open problem for larger $n$ [5–7]. Indeed, we anticipate that the question, in general, may be very complicated. For example, the analogous question for classical (Shannon) entropies has been much studied and, in this case, for $n \geq 4$, there are infinitely many independent (linear) inequalities known [8–10]. All these inequalities and more discovered by Makarychev et al. [11] and Dougherty et al. [12,13] might also very well hold for the von Neumann entropy. Indeed, it is known that the von Neumann entropy satisfies some constrained inequalities that are counterparts of known classical constrained inequalities [7].

For a state of $n$ parties, there are $2^n - 1$ non-trivial von Neumann entropies, one corresponding to each non-empty subset of parties. Thus, the existence of these inequalities means that given a set of $2^n - 1$ positive numbers there will, in general, be no quantum state whose reduced state entropies have these values.

In this paper, we consider the analogous questions for quantum Rényi entropies, also called $\alpha$-entropies [14], of a quantum state $\rho$ (given as a unit trace density operator on a suitable Hilbert space),

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \text{Tr} \rho^\alpha,$$

for $0 \leq \alpha \leq \infty$. (We note that the case of $\alpha = 1$ is the von Neumann entropy.)

We will show that, very surprisingly, the case of Rényi entropies for $\alpha \neq 1$ is much different from that for the von Neumann entropy.

For $0 < \alpha < 1$, we will show (in theorem 3.3) that the only inequality is non-negativity $S_\alpha(\rho) \geq 0$. In other words, any collection of non-negative numbers assigned to the non-empty subsets of $n$ parties can be arbitrarily well approximated by the $\alpha$-entropies of the $2^n - 1$ marginals of a quantum state.

For $\alpha > 1$, we will prove that there are no linear (or indeed homogeneous) inequalities. We show (in theorem 4.3) that, given any vector $v$ of $2^n - 1$ positive numbers, it may or may not be the case that this is the vector of entropies of the $2^n - 1$ marginals of a quantum state; however, it is arbitrarily well approximated by a positive multiple of the $\alpha$-entropies of the $2^n - 1$ marginals of a quantum state. On the other hand (contrary to the case $0 < \alpha < 1$), there are other, nonlinear, inequalities delimiting the set of possible entropy vectors; one such inequality was proved by Audenaert [15], which we recall in §4.

Finally, we show (in §5) that in the classical case the only homogeneous inequalities are non-negativity and monotonicity (under the inclusion of subsets of parties), for all $\alpha \neq 0, 1$.

2. The Rényi entropy

The definition in (1.2) is clearly well defined, and continuous in the state as well as in $\alpha$, for $\alpha \in (0, \infty) \setminus \{1\}$. For $\alpha = 0, 1, \infty$, the function is defined by taking a limit, yielding

$$S_\alpha(\rho) = \log \text{rank} \rho,$$
\[
S_1(\rho) = S(\rho) := -\text{Tr } \rho \log \rho \text{ (von Neumann entropy)}
\]
and
\[
S_\infty(\rho) = -\log \|\rho\|,
\]
where \(\|\cdot\|\) denotes the operator norm, i.e. \(\|\rho\|\) is the largest eigenvalue of \(\rho\).

By their definition, all of the quantum Rényi entropies depend only on the spectrum of \(\rho\), which we can think of as a probability distribution \(P\). In this sense, the above formulae generalize the notion introduced by Rényi [14] in his axiomatic investigation of information measures for random variables and their distributions, following Shannon’s example [16]. This approach has generated a lot of subsequent activity [17].

It is easy to see that for states \(\rho \geq 0\), \(\text{Tr } \rho = 1\), all \(S_a(\rho) \geq 0\), with equality if and only if \(\rho\) is pure, i.e. a rank-one projector \(\rho = |\phi\rangle \langle \phi|\). Furthermore, for fixed \(\rho\), the function \(\alpha \mapsto S_a(\rho)\) is monotonically non-increasing [14]. Many other useful, interesting and curious mathematical properties of the Rényi entropies are known [17].

Rényi entropies and, more generally, Rényi relative entropies and the corresponding channel capacities play an important role in classical as well as quantum information theory. The Rényi quantities with parameter \(\alpha \in (0, 1)\) are related to the so-called direct domain of information theoretic problems. They can be used to quantify the trade-off between the rates of the two types of error probabilities in binary state discrimination [18–22], which, in turn, yields a trade-off relation between the error rate and the compression rate in state compression (see [18] for the classical case; the quantum case is completely analogous). The related capacities quantify the trade-off between the error rate and the coding rate for classical information transmission [18,19], and can be used to obtain lower bounds on the single-shot classical capacities [23]. The Rényi quantities with parameter \(\alpha > 1\) are related to converse problems. They can be used to quantify the trade-off between the rates of the type I error and the type II success probability in binary state discrimination [18,24,25], as well as the trade-off between the rate of the success probability and the compression rate in state compression [18]. The related capacity formulae give bounds on the success rate for coding rates above the Holevo capacity [18,26,27], and can be used to give upper bounds on the single-shot classical capacities of quantum channels [28]. In addition, Rényi entropies feature prominently in the theory of bipartite pure state transformations by local operations and classical communication: only recently, it was shown [29] that the monotonicity of the Rényi entropies of the reduced states for \(\alpha > 1\) is both necessary and sufficient for catalytic transformations (whereas unassisted transformations are long known to be characterized by majorization [30,31]). In Hayden & Winter [32], Rényi entropies (essentially \(\alpha = 0\) and \(\alpha = \infty\)) were used to put bounds on the classical communication required for a given transformation. And, finally, Rényi entropies were used to put lower bounds on the communication complexity of certain distributed computation problems [33,34].

While the von Neumann entropy can be obtained as the limit of the Rényi entropies for \(\alpha \to 1\), and hence it can be considered as one particular member of this parametric family of entropies, its basic properties sharply distinguish it from all other members of the family. Indeed, while the von Neumann entropy is strongly subadditive, the other Rényi entropies with \(\alpha \in (0, +\infty) \setminus \{1\}\) are not even subadditive.

To illustrate the consequences of this difference, we mention the problem of entropy asymptotics on spin chains. Given a translation-invariant state \(\rho\) on an infinite spin chain, subadditivity of entropy ensures the existence of the limit \(s(\rho) := \lim_{n \to +\infty}(1/n)S(\rho|_{[1,n]})\), where \(\rho|_{[1,n]}\) is the restriction of \(\rho\) to any \(n\) consecutive sites, and \(s(\rho)\) gives the ultimate compression rate for an ergodic \(\rho\) [35]. More refined knowledge about the decay of error for rates below \(s(\rho)\) can be obtained using the method developed by Hiai et al. [21]; for this, however, one has to show the
existence of the regularized Rényi entropies \( S_\alpha(\rho) := \lim_{n \to +\infty} (1/n) S_\alpha(\rho) \) for every \( \alpha \in (0, 1) \). Owing to the lack of subadditivity, the existence of this limit is not at all straightforward, and is actually only known for some special classes of states \([21,36,37]\).

When \( \rho \) is pure, the block entropies \( S_\alpha(\rho) \) are used to measure the entanglement between the block \([1, n]\) and the rest of the chain, and the scaling of these entropies is closely related to the presence or absence of criticality in the system \([38,39]\). It follows from strong subadditivity that the entanglement entropy \( S(\rho) \) is a monotone increasing function of the block size \([40]\). This is no longer true when the entanglement is measured by some Rényi entropy; a counterexample with oscillating block Rényi entropies for \( \alpha > 2 \) was found in Giampaolo et al. \([41]\). It is not known, however, whether such oscillating behaviour can happen for Rényi entropies with parameter \( \alpha \) arbitrarily close to 1.

In view of the above examples, it is natural to ask whether there are other universal inequalities between the Rényi entropies of the subsystems of a multi-partite quantum system, and this is what we are going to investigate in the following.

To fix notation, we shall concern ourselves with \( n \)-partite quantum systems with generic tensor product Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \). Within the discussion, we usually consider \( n \) and \( \alpha \) to be fixed, but the local systems are unconstrained, i.e. we do not impose limits on the dimension of the \( \mathcal{H}_i \). For a state \( \rho \) on \( \mathcal{H} \), we have the reduced states \( \rho_i = \text{Tr}_{\mathcal{H} \setminus i} \rho \), with the partial trace over all parties in the complement \( F = [n] \setminus I \), and we shall consider them and their entropies all at once, for all non-empty subsets of \([n] = \{1, \ldots , n\}\). The power set and the power set without the empty set we denote as follows:

\[
P([n]) := \{ I \subset [n] \}
\]
and

\[
P_0([n]) := \{ I \subset [n] : I \neq \emptyset \} = P([n]) \setminus \{ \emptyset \}.
\]

Here and henceforth, we use the notation \( A \subset B \) in the sense that \( A \) is a subset of \( B \) which might, as a special case, be equal to \( B \), whereas to indicate that \( A \) is a proper subset of \( B \) we write \( A \subsetneq B \).

We are interested in the universal relations obeyed by the \( \alpha \)-entropies of a general \( n \)-party state \( \rho \). For instance, by definition, clearly

\[
S_\alpha(\rho_i) \geq 0
\]
for all subsets \( I \subset [n] \).

Note that via the usual diagonal matrix representation we can view a probability distribution of \( n \) discrete random variables as a quantum state, and, conversely, states which are diagonal in a tensor product basis of an \( n \)-party system can be identified with a classical \( n \)-party probability distribution, and hence we will call states of that form classical. In this case, there is another inequality,

\[
S_\alpha(\rho_i) \leq S_\alpha(\rho)
\]
for \( I \subset I \), i.e. monotonicity of the entropy function with respect to subset inclusion.

These examples motivate the introduction of the set of all entropic vectors,

\[
\Sigma_\alpha := \{(S_\alpha(\rho))_{\rho} : \rho \text{ state} \} \subset \mathbb{R}^{P_0([n])},
\]
and the same for the classical case

\[
P_\alpha := \{(S_\alpha(\rho))_{\rho} : \rho \text{ classical state} \} = \{(H_\alpha(P))_{P} : P \text{ prob. distr.} \} \subset \mathbb{R}^{P_0([n])}.
\]

In fact, as we tend to consider \( \leq \)-type inequalities between continuous functions of the coordinates, it makes sense to focus on the topological closures \( \Sigma_\alpha, P_\alpha \subset \mathbb{R}^{P_0([n])} \). The universal inequalities we are looking for are the constraints describing the geometric shape of these sets.

The above examples of known inequalities are homogeneous, indeed linear, relations. (By a homogeneous inequality, we mean an inequality of the form \( f(v) \geq 0 \), \( v \in \Sigma_\alpha \), where \( f \) is a homogeneous function on \( \mathbb{R}^{P_0([n])} \), i.e. there exists a \( d \in \mathbb{R} \) such that \( f(\lambda v) = \lambda^d f(v) \) holds for every
\( \lambda \in \mathbb{R}_{\geq 0} \) and \( \mathbf{v} \in \mathbb{R}^{\mathcal{P}[n]} \). That it is meaningful to look for such relations is motivated by the observation that all Rényi entropies are extensive, i.e.

\[
S_{\alpha}(\rho \otimes \sigma) = S_{\alpha}(\rho) + S_{\alpha}(\sigma).
\]  

(2.1)

And, because this is true for all subset reduced states simultaneously, we have, for non-negative integers \( k \) and \( \ell \),

\[
k\Sigma_{\alpha}^{n} + \ell \Sigma_{\alpha}^{n} \subseteq \Sigma_{\alpha}^{n}, \quad k\Gamma_{\alpha}^{n} + \ell \Gamma_{\alpha}^{n} \subseteq \Gamma_{\alpha}^{n},
\]

and likewise for the respective closures. If this held for non-negative reals, then it would mean that the corresponding set is a convex cone. This is, indeed, known for \( \alpha = 1 \) [5,42], but not true for \( \alpha > 1 \) (see below).

3. \( 0 < \alpha < 1 \)

Here, \( \alpha \) is fixed in the interval \((0, 1)\). We start off with a simple classical construction. For \( I \in \mathcal{P}_{\alpha}[n] \), let \( \delta_{I} \) denote the corresponding basis vector in \( \mathbb{R}^{\mathcal{P}_{\alpha}[n]} \). That is, \( \delta_{I} \) is the vector in \( \mathbb{R}^{\mathcal{P}_{\alpha}[n]} \) with 1 for the entry \( I \) and 0 for every other entry \( J \in \mathbb{R}^{\mathcal{P}_{\alpha}[n]} \), \( J \neq I \).

**Lemma 3.1.** For any \( s > 0 \), the vector \( s \delta_{[n]} \in \mathbb{R}^{\mathcal{P}_{\alpha}[n]_{\geq 0}} \) is approximately \( \alpha \)-entropic, i.e. \( s \delta_{[n]} \in \Sigma_{\alpha}^{n} \). In fact, this vector can be approximated arbitrarily well by classical states.

**Proof.** For integers \( M_{1}, \ldots, M_{n} \) consider ‘local’ alphabets \( \mathcal{X}_{i} := \{0\} \cup \{M_{i}\} \) and define distributions \( P_{t;\mathcal{X}_{i}} \) \((0 \leq t \leq 1)\) on the Cartesian product \( \mathcal{X}_{1} \times \cdots \times \mathcal{X}_{n} \) as follows:

\[
P_{t;\mathcal{X}_{i}}(x_{1}, \ldots, x_{n}) := \begin{cases} 
1 - t, & \text{if } x_{1} = \cdots = x_{n} = 0, \\
t, & \text{if } x_{1}, \ldots, x_{n} \neq 0, \\
0, & \text{otherwise}.
\end{cases}
\]

The marginals on \( \mathcal{X}_{i} = \bigtimes_{i \in I} \mathcal{X}_{i} \) for a subset \( I \subset \{n\} \), are easy to construct: they are given precisely by \( P_{t;\mathcal{X}_{i}}(x_{i} : i \in I) \). The corresponding quantum state and its marginals hence are

\[
\rho = \sum_{x_{1}, \ldots, x_{n}} P_{t;\mathcal{X}_{i}}(x_{1}, \ldots, x_{n})|x_{1}\rangle \langle x_{1}| \otimes \cdots \otimes |x_{n}\rangle \langle x_{n}|.
\]

With this, the Rényi entropies are straightforward to compute,

\[
S_{\alpha}(\rho_{t}) = \frac{1}{1 - \alpha} \log((1 - t)^{\alpha} + t^{\alpha}M_{I}^{1 - \alpha}),
\]

with \( M_{I} = \prod_{i \in I} M_{i} \).

Now, we can set

\[
t := \left( \frac{2^{\alpha(1 - \alpha)} - 1}{M_{I}^{1 - \alpha}} \right)^{1/\alpha}
\]

(3.1)

so that \( 0 < t < 1 \) for sufficiently large \( M_{[n]} \). Then, in the limit \( \min\{M_{1}, \ldots, M_{n}\} \to \infty \),

\[
S_{\alpha}(\rho_{[n]}) = \frac{1}{1 - \alpha} \log((1 - t)^{\alpha} + 2^{\alpha(1 - \alpha)} - 1) \to s,
\]

because \( t \to 0 \). On the other hand, for \( I \subseteq [n] \),

\[
S_{\alpha}(\rho_{I}) = \frac{1}{1 - \alpha} \log \left( (1 - t)^{\alpha} + \frac{2^{\alpha(1 - \alpha)} - 1}{M_{[n],I}^{1 - \alpha}} \right) \to 0,
\]

because \( M_{I} \to \infty \).
Proposition 3.2. For any \( I \subset [n] \), where \( I \neq \emptyset \), and \( s > 0 \), the vector \( s\delta_I \in \mathbb{R}_{\geq 0}^{\mathcal{P}_{n}} \) is approximately \( \alpha \)-entropic, i.e. \( s\delta_I \in \overline{\mathcal{S}}_{\alpha} \).

Proof. It is sufficient to show that, for any \( \epsilon > 0 \), there exist local systems \( \mathcal{H}_1, \ldots, \mathcal{H}_n, \mathcal{H}_{n+1} \) and a pure state \( \rho = |\psi\rangle\langle\psi| \) on \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \otimes \mathcal{H}_{n+1} \) with

\[
s - \epsilon \leq S_\alpha(\rho_I) \leq s + \epsilon \quad \text{and} \quad S_\alpha(\rho_I) \leq \epsilon \text{ if } J \subset [n], \ J \neq I.
\]

If \( I = [n] \), then we just use the \((n+1)\)st party to purify the classical state from lemma 3.1. If \( |I| = 1 \), then we likewise take the classical state of lemma 3.1 on the \( n \)-party system \([n+1] \setminus I\) and purify it using the system \( I \). Thus, from now on, we may assume that \( k = |I| \) and \( \ell = |I^c| = n + 1 - k \) are both \( \geq 2 \). The idea is that, for integer \( M \), the distributions \( P_{t;[M';i\in I]} \) on the systems \( I \), and \( P_{t;[M';j\in I^c]} \) on the systems \( I^c = [n+1] \setminus I \), both with the same \( t \) given by

\[
t = \left( \frac{2^{s(1-\alpha)} - 1}{M^{k(1-\alpha)}} \right)^{1/\alpha},
\]

have the same non-zero probabilities, just arranged differently. In other words, the corresponding classical states are isospectral; hence, we may view them as reduced states of an \((n+1)\)-party pure state.

In detail, we may without loss of generality relabel the systems such that \( I = \{1, \ldots, k\} \) and \( I^c = \{k + 1, \ldots, k + \ell = n + 1\} \). For every \( i \in \{1, \ldots, k\} \) and \( j \in \{1, \ldots, \ell\} \), let \( \mathcal{K}_{ij} \) be an \( M \)-dimensional Hilbert space with an orthonormal basis \( |k\rangle_i : k = 1, \ldots, M \) and define

\[
\mathcal{H}_i := \mathbb{C}|0\rangle_i \oplus \bigotimes_{j=1}^{\ell} \mathcal{K}_{ij}, \quad \mathcal{H}_{k+j} := \mathbb{C}|0\rangle_{k+j} \oplus \bigotimes_{i=1}^{k} \mathcal{K}_{ij},
\]

where \( |0\rangle_i \) are unit vectors. For a \( k \times \ell \) matrix \( x \in [M]^{k \times \ell} \) and \( i \in \{1, \ldots, k\} \), \( j \in \{1, \ldots, \ell\} \), let

\[
|x^{(0)}\rangle := \bigotimes_{j=1}^{\ell} |x^{(0)}_j\rangle_i \in \mathcal{H}_i, \quad |x^{(k+j)}\rangle := \bigotimes_{i=1}^{k} |x^{(0)}_j\rangle_i \in \mathcal{H}_{k+j}
\]

and

\[
|\psi\rangle := \sqrt{1-t} \bigotimes_{i=1}^{n+1} |0\rangle_i + \sqrt{\frac{t}{M^{k\ell}}} \sum_{x \in [M]^k \ell} |x^{(1)}\rangle \cdots |x^{(k)}\rangle |x^{(k+1)}\rangle \cdots |x^{(n+1)}\rangle.
\]

(Here, \( |\chi\rangle \langle \gamma| \) stands for \( |\chi\rangle \otimes |\gamma\rangle \).)

Let \( \rho := \rho_{[n]} = \text{Tr}_{n+1} |\psi\rangle\langle\psi| \). The crucial property of this definition is that every party \( i \in I \) and \( j \in I^c \) have a coordinate in common, namely \( x^{(0)}_i \in [M] \). One can easily see that \( \rho_I \) is a classical state of the type studied in lemma 3.1, and the same calculation as in lemma 3.1 shows that

\[
S_\alpha(\rho_I) = \frac{1}{1-\alpha} \log((1-t)^{\alpha} + 2^{s(1-\alpha)} - 1) \to s,
\]

as \( M \to \infty \). If \( J \in \mathcal{P}_{\emptyset}^{[n]} \) is different from \( I \), then there are \( i \in I \) and \( j \in I^c = [n+1] \setminus I \), such that either \( i, j \in J \) or \( i, j \in J^c \). The second case has entropy equivalent to the first, because we may just go to the complementary set. In the first case, we have

\[
\rho_J = (1-t) \bigotimes_{i \in J} |0\rangle_i \langle 0|_i + t\sigma,
\]

where \( \sigma \) is supported on a space which contains each \( \mathcal{K}_{ij} \) at most once if the \( i \)th or the \((k+j)\)th system has been traced out, and twice otherwise. Hence,

\[
\sigma = \left( \bigotimes_{i \in J \cap I} \bigotimes_{j \in [n] \setminus I} |\psi_{ij}\rangle \langle \psi_{ij}| \right) \otimes \sigma',
\]
where

\[ |\psi_{ij}\rangle = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} |k\rangle_{ij} |k\rangle_{ij} \in K_{ij} \otimes K_{ij}, \]

and \( \sigma' \) is a density operator supported on a space of dimension at most \( M^{k-1} \). Hence, \( S_\alpha(\sigma) = S_\alpha(\sigma') \leq \log M^{k-1} \), or, equivalently, \( \text{Tr} \sigma^\alpha \leq M^{(k-1)(1-\alpha)} \). This yields

\[
S_\alpha(\rho_j) \leq \frac{1}{1-\alpha} \log((1 - t)^\alpha + M^{k-1}(1-\alpha)t^\alpha)
= \frac{1}{1-\alpha} \log \left( (1 - t)^\alpha + \frac{1}{M^{1-\alpha}}(2^{(1-\alpha)} - 1) \right) \to 0,
\]
as \( M \to \infty \).

**Theorem 3.3.** Every element \( v \in \mathbb{R}^{P_\delta[\mathbb{N}]} \) is approximately \( \alpha \)-entropic. In other words, there are no non-trivial inequalities constraining the Rényi entropies (with fixed \( \alpha < 1 \)) of a multi-party state: the only restriction is non-negativity,

\[ \Sigma_\alpha = [\mathbb{R}^{P_\delta[\mathbb{N}]}]. \]

**Proof.** Using proposition 3.2, this is quite obvious: observe that \( v = \sum_{i \in P_\delta[\mathbb{N}]} v_i \delta_{IJ} \), and that, for each subset \( I \subset [n] \) and every \( \epsilon > 0 \), we can find a state \( \rho^{(I)} \) on an \( n \)-party Hilbert space \( \mathcal{H}_1^{(I)} \otimes \ldots \otimes \mathcal{H}_n^{(I)} \) such that its entropies approximate \( v_i \delta_{IJ} \) well, i.e.

\[
|S_\alpha(\rho_j^{(I)}) - v_i \delta_{IJ}| \leq \epsilon \quad \text{for all } I, J \subset [n]. \tag{3.2}
\]

Here, \( \delta_{IJ} \) stands for the Kronecker delta, i.e. \( \delta_{IJ} := 1 \) if \( I = J \) and \( \delta_{IJ} := 0 \), otherwise.

For each \( i \in [n] \), let \( \mathcal{H}_i := \bigotimes_{j \in P_\delta[\mathbb{N}]} \mathcal{H}_j^{(I)} \) be the local Hilbert space of the \( i \)th party. Then, \( \rho := \bigotimes_{i \in P_\delta[\mathbb{N}]} \rho_i^{(I)} \) defines a state on the \( n \)-party Hilbert space \( \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n \), with the state of any subset \( J \) of parties given by \( \rho_J = \bigotimes_{i \in P_\delta[\mathbb{N}]} \rho_i^{(I)} \). Owing to the extensivity (2.1) of the Rényi entropies, we have \( S_\alpha(\rho) = \sum_{i \in P_\delta[\mathbb{N}]} S_\alpha(\rho_i^{(I)}) \), and (3.2) yields

\[
|S_\alpha(\rho_j) - v_i| = \left| \sum_{i \in P_\delta[\mathbb{N}]} (S_\alpha(\rho_j^{(I)}) - v_i \delta_{IJ}) \right| \leq 2^{n-1} \epsilon \quad \text{for all } I \subset [n].
\]

Because \( \epsilon \) can be chosen arbitrarily small, we are done.

**Remark 3.4.** From theorem 3.3, we can see that \( \Sigma_\alpha \) is not a closed set (assuming \( n \geq 2 \)). Indeed, we found that, for any \( I \subset [n] \), the ray \( \mathbb{R} \delta_{I} \) is in \( \Sigma_\alpha \). However, it is easy to see that, except for the origin, none of its points \( \delta_I \) can be an element of \( \Sigma_\alpha \).

For, otherwise, there would be a state \( \rho \) with \( S_\alpha(\rho_I) = s \) and all other \( S_\alpha(\rho_j) = 0 \). Now, if \( |I| \geq 2 \), say \( I = \{i_1, i_2, \ldots, I\} \), then \( S_\alpha(\rho_I) = 0 \) implies that all single-party marginals \( \rho_i \) are pure, meaning that \( \rho = \rho_1 \otimes \cdots \otimes \rho_n \) is also pure. Hence, we would have \( S(\rho) = 0 \) as well. If, on the other hand, \( I = \{i\} \), then we may choose \( j \notin I \) and reason similarly that \( \rho_j = 0 \) is a pure state; hence, \( \rho_{\{i,j\}} = \rho_i \otimes \rho_j \) and so \( S(\rho_{\{i,j\}}) = S(\rho_i) + S(\rho_j) = s \neq 0 \), obtaining a contradiction again.

### 4. \( 1 < \alpha \leq \infty \)

As in §3, we start with the basic construction to attain entropy vectors arbitrarily close to the coordinate axes. Throughout the section, \( 1 < \alpha \leq \infty \) is fixed.

**Lemma 4.1.** For all \( s > 0 \), there is a vector \( s \delta_{[n]} + O(1) \in \Sigma_\alpha \). To be precise, there exists a constant \( C \), which may be chosen as \( C = (1/(1 - 1/\alpha)) + \log n \), and classical states with

\[
s \leq S_\alpha(\rho) \leq s + C \quad \text{and} \quad S_\alpha(\rho_I) \leq C \text{ for } I \neq [n].
\]

In particular, \( \delta_{[n]} \in \overline{\mathbb{R}_{\geq 0} \Sigma_\alpha} \).
Proof. The following argument is presented for \( \alpha < \infty \); to obtain the claims, in the case \( \alpha = \infty \), one simply takes the limit.

For an integer \( M \), consider ‘local’ alphabets \( \mathcal{X}_i := \{0\} \cup [M] \) and define distributions \( Q_{[n]:R} \) on the Cartesian product \( \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \) as follows:

\[
Q_{[n]:R}(x_1, \ldots, x_n) := \begin{cases} \frac{1}{n} R(x_i), & \text{if } x_i \neq 0 \text{ and } x_j = 0 \forall j \neq i, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( R \) is an arbitrary probability distribution on \([M]\).

For the corresponding classical state \( \rho_{[n]} \), it is straightforward to verify that

\[
S_\alpha(\rho_{[n]}) = \frac{1}{1 - \alpha} \log \left( n \sum_{x=1}^M \left( \frac{1}{n} R(x) \right)^\alpha \right) = \log n + H_{\alpha}(R).
\]

On the other hand, the marginal state \( \rho_I \) for any \( I \subseteq [n] \) has an eigenvalue \( \lambda \geq 1/n \); hence,

\[
S_\alpha(\rho_I) \leq \frac{1}{1 - \alpha} \log \lambda^\alpha \leq \frac{1}{1 - \alpha} \log \left( \frac{1}{n} \right)^\alpha = C,
\]

and we are done, because we can choose \( M \) large enough to accommodate a distribution \( R \) on \([M]\) with \( H_{\alpha}(R) = s \).

Proposition 4.2. For all \( s > 0 \) and \( I \in \mathcal{P}_n \), there is a vector \( s\delta_I + O(1) \in \Sigma_s^\alpha \). To be precise, there exists a constant \( C \), which may be chosen as \( C = (1/(1 - 1/\alpha)) \log(|I|/(n + 1 - |I|)) \), and states with

\[
s \leq S_\alpha(\rho_I) \leq s + C \quad \text{and} \quad S_\alpha(\rho_I) \leq C \text{ for } I \neq I.
\]

In particular, \( \delta_I \in \mathbb{R}_{\geq 0} \Sigma_s^\alpha \).

Proof. If \( I = [n] \), this is lemma 4.1, but we shall present a direct quantum construction, of a pure state on \( n + 1 \) parties; hence, view \( I \) as a subset of \([n+1]\). Pick a distribution \( R \) on some finite alphabet \([M]\) with \( H_{\alpha}(R) = s \) and fix a purification of \( R \), which we identify with the quantum state \( R = \sum_x R(x)|x\rangle \langle x| = \sum_{x=1}^M \sqrt{R(x)} |x\rangle \langle x| \).

Now, construct the following \((n + 1)\)-party pure state vector:

\[
|\psi\rangle := \sqrt{\frac{1}{|I| |I^c|}} \bigoplus_{i \in I, j \in I^c} |\mu\rangle_{ij} \otimes |ij\rangle^{\otimes |n+1|},
\]

where \( I^c = [n+1] \setminus I \), and \( \rho := \text{Tr}_{n+1} |\psi\rangle \langle \psi| \). However, our reasoning will be based on the pure state \( |\psi\rangle \langle \psi| \). Above, the direct sum means that we take direct sums of the local Hilbert spaces, which we indicate by the label ‘ij’ attached to each local system, whereas \( |\mu\rangle_{ij} \) is the state \( |\mu\rangle \) shared between parties \( i \) and \( j \).

It is straightforward to check that

\[
\rho_I = \frac{1}{|I| |I^c|} \bigoplus_{i \in I, j \in I^c} R_i \otimes |ij\rangle \langle ij|^{\otimes I},
\]

where \( R_i \) denotes the mixed state \( R \) held by party \( i \). Hence, \( S_{\alpha}(I) = s + \log(|I| |I^c|) \). On the other hand, if \( J \subset [n] \) with \( J \neq I \), then there exist \( i \in I \) and \( j \in I^c \) such that either both \( i, j \in J \) or both \( i, j \in I^c \). Thus, \( \rho_I \) has a direct sum component \( |\mu\rangle_{ij} \langle ij|^{\otimes I} \) or \( |ij\rangle \langle ij|^{\otimes I} \), respectively, and as a consequence an eigenvalue \( \geq 1/|I| |I^c| \); hence,

\[
S_\alpha(I) \leq \frac{1}{1 - \alpha} \log \left( \frac{1}{|I| |I^c|} \right)^\alpha = C,
\]

and we are done.
Theorem 4.3. For every element $v \in \mathbb{R}_{\geq 0}^{P_\alpha[n]}$, there is a vector $v + O(1) \in \Sigma_\alpha^n$. To be precise, there exists a constant $C$, which may be chosen as $C = (1/(1 - 1/\alpha))(\log(n + 1))^{2^{n+1}}$, and states with

$$|S_\alpha(\rho_I) - v_I| \leq C \quad \text{for all } I \subset [n].$$

In other words, there are no non-trivial homogeneous inequalities constraining the Rényi entropies (with fixed $\alpha > 1$) of a multi-party state: the only restriction is non-negativity,

$$\mathbb{R}_{\geq 0}^{\Sigma_\alpha^n} = \mathbb{R}_{\geq 0}^{P_\alpha[n]}.$$

Proof. Using proposition 4.2, this is trivial: $v = \sum_{I \in \mathcal{P}_\alpha[n]} v_I \delta_I$, and for each subset $I \subset [n]$, we can find an $n$-party state $\rho^{(I)}$ such that its entropies approximate $v_I \delta_I$, i.e.

$$|S_\alpha(\rho^{(I)}_I) - v_I \delta_{I,J}| \leq \frac{1}{1 - 1/\alpha} \log n$$

for all $I, J \subset [n]$.

Letting $\rho := \bigotimes_{I \in \mathcal{P}_\alpha[n]} \rho^{(I)}$, we are done. \hfill \blacksquare

Remark 4.4. From theorem 4.3, we can see that $\mathbb{R}_{\geq 0}^{\Sigma_\alpha^n}$ is not a closed set (assuming $n \geq 2$).

This is argued in the same way as in remark 3.4 for the case $\alpha < 1$.

Theorem 4.3 looks very similar to theorem 3.3 for $0 < \alpha < 1$. However, although there we could conclude that there are no non-trivial inequalities whatsoever for the Rényi entropies of a multi-party state, here we get only that there cannot be any further homogeneous inequalities apart from non-negativity.

That this is the most we can hope to obtain follows from the observation that there are other, nonlinear and non-homogeneous, inequalities constraining the entropy vectors. In fact, $\Sigma_\alpha^n$ is not a cone at all for $\alpha > 1$!

An example of such an inequality was presented by Audenaert [15]: the $\alpha$-Schatten norm $\|\rho\|_\alpha = (\text{Tr} |\rho|^{\alpha})^{1/\alpha}$ (the operator norm $\|\rho\|_\infty = \|\rho\|$ is obtained in the limit $\alpha \to \infty$) is related to the $\alpha$-entropy by

$$S_\alpha(\rho) = \frac{\alpha}{1 - \alpha} \log \|\rho\|_\alpha, \quad S_\infty(\rho) = - \log \|\rho\|_\infty,$$

and satisfies

$$\|\rho_A\|_\alpha + \|\rho_B\|_\alpha \leq 1 + \|\rho_{AB}\|_\alpha$$

for arbitrary bipartite state $\rho_{AB}$.

The following is a strengthening of Audenaert’s inequality.

Proposition 4.5. Let $\rho_{AB}$ be a bipartite state and $1 < \alpha \leq \infty$. Define

$$M_\alpha := \max \left\{ \left( \frac{\|\rho_A\|_\infty}{\|\rho_A\|_\alpha} \right)^{\alpha-1}, \left( \frac{\|\rho_B\|_\infty}{\|\rho_B\|_\alpha} \right)^{\alpha-1} \right\} \quad \text{for } \alpha < \infty,$$

as well as $M_\infty := \lim_{\alpha \to +\infty} M_\alpha = \max\{1/m_A, 1/m_B\}$, where $m_A$ and $m_B$ are the multiplicities of $\|\rho_A\|_\infty$ and $\|\rho_B\|_\infty$ as an eigenvalue of $\rho_A$ and $\rho_B$, respectively.

Then,

$$\|\rho_A\|_\alpha + \|\rho_B\|_\alpha \leq \min \left\{ \kappa + \frac{1}{\kappa} \|\rho_{AB}\|_\alpha : M_\alpha \leq \kappa \right\} \leq \frac{2\sqrt{\|\rho_{AB}\|_\alpha}}{\sqrt{\|\rho_{AB}\|_\alpha}} \quad \text{if } M_\alpha \leq \sqrt{\|\rho_{AB}\|_\alpha}$$

$$= M_\alpha + \frac{1}{M_\alpha} \|\rho_{AB}\|_\alpha, \quad \text{if } \sqrt{\|\rho_{AB}\|_\alpha} \leq M_\alpha \leq 1 + \|\rho_{AB}\|_\alpha \quad \text{if } 1 + \|\rho_{AB}\|_\alpha \leq M_\alpha$$

The last inequality holds with equality if and only if at least one of $\rho_A$, $\rho_B$ or $\rho_{AB}$ is a pure state. Moreover, we have $\|\rho_A\|_\alpha + \|\rho_B\|_\alpha = 1 + \|\rho_{AB}\|_\alpha$ if and only if $\rho_A$ or $\rho_B$ is a pure state.
Proof. We follow Audenaert’s proof [15] with a slight modification. Let \( \rho_A = \sum_{i=1}^{d} \lambda_i |e_i\rangle \langle e_i| \) and \( \rho_B = \sum_{j=1}^{d} \eta_j |f_j\rangle \langle f_j| \) be eigen-decompositions such that the \( \lambda_i \)'s and the \( \eta_j \)'s are arranged in a decreasing order. For \( \alpha = \infty \), define

\[
X := \sum_{i=1}^{m_A} \frac{1}{m_A} |e_i\rangle \langle e_i|, \quad Y := \sum_{i=1}^{m_B} \frac{1}{m_B} |f_i\rangle \langle f_i|,
\]

and let \( \beta := 1 \). For \( \alpha < \infty \), define

\[
X := \sum_{i=1}^{\infty} x_i |e_i\rangle \langle e_i|, \quad x_i := \frac{\lambda_i^{\alpha-1}}{\|\rho_A\|^{\alpha-1}_2}, \quad Y := \sum_{j=1}^{\infty} y_j |f_j\rangle \langle f_j|, \quad y_j := \frac{\eta_j^{\alpha-1}}{\|\rho_B\|^{\alpha-1}_2},
\]

and let \( \beta \) be such that \( 1/\alpha + 1/\beta = 1 \). Let \( x \) and \( y \) be the vectors formed of the \( x_i \)'s and \( y_j \)'s, respectively. Then, \( \|X\|_\beta = \|Y\|_\beta = \|x\|_\beta = \|y\|_\beta = 1 \) and \( \|\rho_A\|_\alpha = \text{Tr} X \rho_A \) and \( \|\rho_B\|_\alpha = \text{Tr} Y \rho_B \). Hence, we have, for any real number \( \kappa \), that

\[
\|\rho_A\|_\alpha + \|\rho_B\|_\alpha = \text{Tr}(X \otimes I_B + I_A \otimes Y) \rho AB = \kappa + \text{Tr}(X \otimes I_B + I_A \otimes Y - \kappa I_A \otimes I_B) \rho AB \\
\leq \kappa + \text{Tr}(X \otimes I_B + I_A \otimes Y - \kappa I_A \otimes I_B) + \rho AB = \kappa + \text{Tr} Z_\kappa \rho AB,
\]

where \( Z_\kappa := (X \otimes I_B + I_A \otimes Y - \kappa I_A \otimes I_B)_+ \) is the positive part.

Consider now the function \( a \mapsto f_\kappa(a) := \left( \sum_j (y_j + a - \kappa)^{1/\beta}_+ \right)^{1/\beta} = \|y + a - \kappa\|_\beta \). This function is convex, \( f_\kappa(\kappa) = \|y\|_\beta = 1 \), and \( f_\kappa(0) = 0 \) if we assume that \( \kappa \geq \max_j y_j = \|y\|_\infty \). Hence, under this assumption, \( f_\kappa(a) \leq a/\kappa \) for every \( 0 \leq a \leq \kappa \). Thus, if \( \kappa \geq \|x\|_\infty \), then

\[
\|Z_\kappa\|_\beta = \|(X \otimes I_B + I_A \otimes Y - \kappa I_A \otimes I_B)_+\|_\beta = \sum_{i,j=1}^{d} (x_i + y_j - \kappa)^{1/\beta}_+ = \sum_{i=1}^{d} f_\kappa(x_i)^{1/\beta}_+ \\
\leq \sum_{i=1}^{d} \left( \frac{x_i}{\kappa} \right)^{\alpha} = \frac{\|x\|_\beta^{\alpha}}{\kappa^{\alpha}} = \frac{1}{\kappa^{\beta}},
\]

i.e. \( \|Z_\kappa\|_\beta \leq 1/\kappa \). Owing to Hölder’s inequality, \( \text{Tr} Z_\kappa \rho AB \leq \|Z_\kappa\|_\beta \|\rho AB\|_\alpha \leq \|\rho AB\|_\alpha /\kappa \). Combined with (4.4), this yields

\[
\|\rho_A\|_\alpha + \|\rho_B\|_\alpha \leq \kappa + \frac{1}{\kappa} \|\rho AB\|_\alpha := g(\kappa).
\]

Because this is true for every \( \kappa \geq \max \{\|x\|_\infty, \|y\|_\infty\} = M_\alpha \), we have proved equation (4.1).

It is easy to see that \( g \) is strictly convex and it has a global minimum at \( \sqrt{\|\rho AB\|_\alpha} \leq 1 \) with a minimum value of \( 2\sqrt{\|\rho AB\|_\alpha} \). In particular, \( g \) is strictly decreasing on the interval \( (0, \sqrt{\|\rho AB\|_\alpha}) \) and strictly increasing on \( [\sqrt{\|\rho AB\|_\alpha}, 1] \), and hence we obtain equation (4.2). The inequality (4.3) is obvious.

Using the above properties of \( g \), we have equality in (4.3) if and only if \( \max\{M_\alpha, \sqrt{\|\rho AB\|_\alpha}\} = 1 \). Obviously, \( \sqrt{\|\rho AB\|_\alpha} = 1 \) if and only if \( \rho AB \) is a pure state, and it is easy to see that \( M_\alpha = 1 \) if and only if \( \rho_A \) or \( \rho_B \) is pure.

If \( \rho_A \) is a pure state, then \( \rho AB = \rho_A \otimes \rho_B \) and \( 1 + \|\rho AB\|_\alpha = 1 + \|\rho_A\|_\alpha \|\rho_B\|_\alpha = 1 + \|\rho_B\|_\alpha = \|\rho A\|_\alpha + \|\rho B\|_\alpha \), and a completely similar argument works if \( \rho_B \) is pure. On the other hand, if \( \|\rho_A\|_\alpha + \|\rho_B\|_\alpha = 1 + \|\rho AB\|_\alpha \), then equality has to hold in (4.3), and hence \( \rho_A, \rho_B \) or \( \rho AB \) has to be pure. If \( \rho AB \) is pure but \( \rho_A \) is not then \( \|\rho A\|_\alpha = \|\rho B\|_\alpha < 1 \) and hence \( \|\rho_A\|_\alpha + \|\rho_B\|_\alpha < 2 = 1 + \|\rho AB\|_\alpha \). This proves the last assertion about the equality case.

It appears that, even for two parties, no description of \( \Sigma^2_\alpha \) or \( \overline{\Sigma}^2_\alpha \) is known. Neither is it known which other inequalities constrain the latter.
5. Classical case

As remarked in the Introduction, if restricted to classical states \( \rho \), then the Rényi entropies are monotonic, i.e.

\[
I \subset J \Rightarrow S_\alpha(\rho_I) \leq S_\alpha(\rho_J).
\]

(5.1)

(More generally, this holds for separable states, thanks to the majorization result of Nielsen & Kempe [43].) In this section, we denote the set of \( \alpha \)-entropic vectors of a generic distribution of \( n \) random variables by \( F^n_\alpha \). In other words, this is a subset of \( \Sigma^n_n \), with the restriction that the underlying states are classical.

The extremal rays of the convex cone \( MO^n \) described by non-negativity and equation (5.1)—which thus contains \( F^n_\alpha \)—are easy to describe in combinatorial language [44]: they are precisely the rays spanned by the indicator functions

\[
i_\mathcal{U} : I \mapsto \begin{cases} 1, & \text{if } I \in \mathcal{U}, \\ 0, & \text{otherwise} \end{cases}
\]

of a non-empty set family \( \mathcal{U} \subset \mathcal{P}_\emptyset[n] \) with the property that \( J \supset I \in \mathcal{U} \) implies \( J \in \mathcal{U} \) (hence always \( [n] \in \mathcal{U} \)). Such set families are known in combinatorics as ‘upsets’ (or sometimes ‘ideals’).

Some of the simplest upsets are generated by a single element,

\[
\uparrow J = \{ I \in \mathcal{P}_\emptyset[n] : J \subset I \}.
\]

These have the property that the unique minimal element of the family is \( J \). Note also that an upset contains, with each element \( J \), the entire \( \uparrow J \). This means that every upset \( \mathcal{U} \) can be written

\[
\mathcal{U} = \bigcup_{J \in \mathcal{L}} \uparrow J
\]

with \( \mathcal{L} \) the set of minimal elements of \( \mathcal{U} \).

For instance, for \( n = 2 \), there are four upsets and clearly all four associated rays are attainable (whole ray for \( \alpha < 1 \), sufficiently long dilution for \( \alpha > 1 \)).

Next, we show that this is the only difference from the quantum case, at least as long as we are looking only for homogeneous inequalities. Namely, the only homogeneous inequalities obeyed by the classical \( \alpha \)-entropies are non-negativity and monotonicity.

**Theorem 5.1.** Let \( 0 < \alpha < 1 \). For any upset \( \mathcal{U} \subset \mathcal{P}_\emptyset[n] \) and all \( s > 0 \), there is a vector \( s i_\mathcal{U} + O(1) \in F^n_\alpha \).

To be precise, there exists a probability distribution \( P \) with

\[
s \leq H_\alpha(P_I) \leq s + \frac{\log |\mathcal{L}| + 1}{1 - \alpha}
\]

for \( I \in \mathcal{U} \),

\[
H_\alpha(P_I) \leq \frac{\log |\mathcal{L}| + 1}{1 - \alpha}
\]

for \( I \notin \mathcal{U} \).

In particular, for \( s \to \infty \) we obtain \( i_\mathcal{U} \in \mathbb{R}_{\geq 0} F^n_\alpha \). As a consequence, \( \mathbb{R}_{\geq 0} F^n_\alpha = MO^n \).

**Proof.** Let \( \mathcal{U} = \bigcup_{J \in \mathcal{L}} \uparrow J \), which can be achieved by choosing \( \mathcal{L} \) to be the minimal elements of \( \mathcal{U} \). For each \( i \in [n] \), let the local alphabet \( X_i \) be of the form

\[
X_i \uparrow i = \bigcup_{J \in \mathcal{L}} X_i \uparrow J,
\]

where \( \bigcup \uparrow \) denotes disjoint union, i.e. \( X_i \uparrow J \) and \( X_i \uparrow J' \) are disjoint if \( J \neq J' \). For each \( J \in \mathcal{L} \), let \( P_{i_J}([M_i : i \in J]) \) be a probability distribution on \( \times_{i \in J} X_i \) defined as in lemma 3.1, with \( i_\alpha M_i^{1 - \alpha} = 2^s (1 - \alpha) \), where \( s' := s + (\alpha/(1 - \alpha)) \log |\mathcal{L}| \). Let \( Q_J := P_{i_J([M_i : i \in J])} \otimes \delta_J \), where \( \delta_J \) is the trivial probability distribution on the single-element set \( \times_{i \in J} X_i \). Note that the supports of \( Q_J \) and \( Q_J' \) are disjoint for \( J \neq J' \). We claim that

\[
P := \frac{1}{|\mathcal{L}|} \sum_{J \in \mathcal{L}} Q_J = \frac{1}{|\mathcal{L}|} \sum_{J \in \mathcal{L}} P_{i_J([M_i : i \in J])} \otimes \delta_J
\]

has the desired properties for large enough \( M_0 := \min_i M_i \).
Indeed, it is easy to see that, for any $\emptyset \neq I \subset [n],$

\[
H_\alpha(P_I) = \frac{1}{1 - \alpha} \log \left( \frac{1}{|L|^\alpha} \left( \sum_{|\mathcal{I} \cap \mathcal{J} = \emptyset} 1 + \sum_{\mathcal{J} \neq \emptyset} ((1 - t)J^\alpha + \sum_{\emptyset \neq \mathcal{J} \subseteq \mathcal{I}} \left( (1 - t)J^\alpha + \frac{2^{\gamma(1-\alpha)}}{M_{\mathcal{J},\emptyset}^{1-\alpha}} \right) \right) \right)
\]

\[
= \frac{1}{1 - \alpha} \log \left( \frac{1}{|L|^\alpha} \left( \sum_{|\mathcal{I} \cap \mathcal{J} = \emptyset} 1 + \sum_{\mathcal{J} \neq \emptyset} (1 - t)J^\alpha + \frac{1}{|L|^\alpha} \sum_{\emptyset \neq \mathcal{J} \subseteq \mathcal{I}} 2^{\gamma(1-\alpha)} + \frac{1}{|L|^\alpha} \sum_{\emptyset \neq \mathcal{J} \subseteq \mathcal{I}} \frac{2^{\gamma(1-\alpha)}}{M_{\mathcal{J},\emptyset}^{1-\alpha}} \right) \right)
\]

\[\xrightarrow{M_0 \to +\infty} \frac{1}{1 - \alpha} \log \left( \frac{|L|^{1-\alpha} + 2^{\gamma(1-\alpha)}|\{J \in \mathcal{L} : J \subset I\}|}{|L|^\alpha} \right) \]

\[= \frac{1}{1 - \alpha} \log(|L|^{1-\alpha} + 2^{\gamma(1-\alpha)}|\{J \in \mathcal{L} : J \subset I\}|), \tag{5.2}\]

where, in all summations over $J$ above, $J \in \mathcal{L}$ is implicit. Now, if $I \in \mathcal{U}$, then there exists a $J \in \mathcal{L}$ such that $J \subset I$, and hence the expression in (5.2) can be lower bounded by $s$, and upper bounded by $(1/(1 - \alpha)) \log(|L|(1 + 2^{\gamma(1-\alpha)})) < s + (\log |L| + 1)/(1 - \alpha).$ On the other hand, if $I \not\in \mathcal{U}$ then $|\{J \in \mathcal{L} : J \subset I\}| = 0$ and hence the expression in (5.2) is equal to $\log |\mathcal{L}|.$

**Theorem 5.2.** Let $1 < \alpha \leq \infty$. For any upset $\mathcal{U} \subset \mathcal{P}_\emptyset[n]$ and $s > 0$, there is a vector $s i_\mathcal{U} \in O(1) F^\alpha_n$. To be precise, there exists a constant $C$, which may be chosen as $C = 1/(1 - 1/\alpha) \log(2^n)$ if $\mathcal{U}$ is generated by $k$ elements ($k < 2^n$ always), and a probability distribution $Q$ with

\[s \leq H_\alpha(Q_I) \leq s + C \quad \text{for } I \in \mathcal{U}, \quad H_\alpha(Q_I) \leq C \quad \text{for } I \not\in \mathcal{U}.\]

In particular, for $s \to \infty$, we obtain $i_\mathcal{U} \in \mathbb{R}_{\geq 0} F^\alpha_n.$ As a consequence, $\mathbb{R}_{\geq 0} F^\alpha_n = M \mathcal{O}^n.$

**Proof.** Let $M$ be the smallest natural number such that $s \leq 1 + \log M$. We take as our building blocks the distributions $Q_{[n]:M}$ from lemma 4.1 and its proof, for the sake of simplicity with uniform $R$ on $[M]$. Furthermore, define the following uniform distribution on the diagonal of $[M]^n$:

\[\Delta_M := \frac{1}{M} \sum_{x=1}^M \delta_x \otimes \delta_x \otimes \cdots \otimes \delta_x.\]

Now, for upset $\mathcal{U} = \bigcup_{I \in \mathcal{L}} I$, let

\[Q := \frac{1}{2} \Delta_M \oplus \frac{1}{2} \bigotimes_{J \in \mathcal{L}} (Q_{J,M} \otimes \delta_F),\]

where $\delta_F$ as before refers to a generic point mass for parties $F = [n] \setminus J$, the product over $J \in \mathcal{L}$ implies Cartesian products of the local alphabets, and the direct sum likewise a direct sum of the local alphabets.

The first term in the direct sum makes sure that in each marginal the largest probability value occurring is at least $\geq 1/2M$, with multiplicity $M$, not allowing the Rényi entropy of any subset to become larger than $\log M + 1/(1 - 1/\alpha).$ Turning to the second term, note that, in the tensor product over $J \in \mathcal{L}$, the distributions are designed such that, for $J \subset I$, the distribution $(Q_{J,M} \otimes \delta_F)_I$ is uniform on $|J|M$ elements, whereas for $J \not\subset I$ it has at least one value $\geq 1/|J|$. Thus,

\[H_\alpha(Q_I) \begin{cases} \leq \frac{1}{1 - 1/\alpha} \log(2^n), & \text{for } I \not\in \mathcal{U}, \\ \geq 1 + \log M, & \text{for } I \in \mathcal{U}, \end{cases}\]

and we are done.

**6. Epilogue**

We have carried out an analysis of the inequalities obeyed by quantum Rényi entropies in multipartite systems, in analogy to the very deep ongoing programme for the von Neumann entropy.
In the quantum case, our findings can be summarized concisely as saying that apart from trivial non-negativity of individual entropies there are no inequalities obeyed by the Rényi $\alpha$-entropies of a multi-partite state, when $0 < \alpha < 1$. For $1 < \alpha \leq \infty$, there are no other homogeneous inequalities, but the set of attainable entropic vectors is not a cone, meaning that there are further, non-homogeneous inequalities. In the classical case (and more broadly that of separable quantum states), there is furthermore monotonicity in the sense that a smaller subset of parties cannot have larger entropy, and we could show similarly that this is the only homogeneous inequality for all $\alpha \neq 1$. It is curious to contrast this with the limit $\alpha = 1$, the von Neumann entropy, which is subject to subadditivity and strong subadditivity, as well as triangle inequality and weak monotonicity, all crucial relations for the development of statistical mechanics and quantum information theory. The classical case [45] has even more inequalities, owing to Zhang & Yeung [8,9] and subsequent works [10–13].

We did not discuss the other limit $\alpha = 0$, for which the Rényi entropy is the logarithm of the rank of the density operator, which indeed behaves rather differently from the other $\alpha$-entropies: for one thing, it takes only discrete values in the logarithm of integers, and it is discontinuous. Furthermore, it is easy to see that it obeys subadditivity

$$S_0(\rho_{I\cup J}) \leq S_0(\rho_I) + S_0(\rho_J),$$

and it is unknown which other inequalities (whether homogeneous, linear or other) it satisfies. Note, however, that it definitely does not satisfy strong subadditivity (J. Cadney 2012, personal communication).

We leave a few other open questions and further directions for future investigations: for instance, we like to know all necessary non-homogeneous inequalities to describe the $\Sigma_n^\alpha$. Note that the classical/separable sets $I_n^\alpha$ for $\alpha > 1$ are not cones either, but what about $0 < \alpha < 1$? Finally, and perhaps most interestingly in the light of recovering what rich structure is known for $\alpha = 1$, can we extend the present investigation to relations between Rényi entropies for different $\alpha$? For example, it is well known that, for $\alpha < \beta$, $S_\alpha(\rho) \geq S_\beta(\rho)$, or $S_\alpha(\rho_{AB}) \leq S_\alpha(\rho_B) + S_0(\rho_B)$, but we do not know which other inequalities, if any, exist.

In a different direction, one could consider suitably defined conditional Rényi entropies [46–48] and ask about the inequalities relating them. Unlike the von Neumann entropy case, where the conditional entropy $H(A|B)$ equals a difference of plain entropies $H(AB) - H(B)$, the Rényi case is based on independent definitions. Hence, the absence of inequalities for the Rényi entropies does not carry over to conditional Rényi entropies, and indeed the common definitions usually obey the monotonicity (or data processing) property $H_\alpha(A|BC) \leq H_\alpha(A|B)$.

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