The effect of mass eccentricity upon tribological test results

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Abstract. The paper presents the estimation of the effects of mass eccentricity of a rotating geometrical axisymmetric body upon the experimental tribological results obtained through inclined plane methodology. The first part of the paper presents the steps for deducing the nonlinear ordinary differential equation for the motion of an axisymmetric body with mass eccentricity in pure rolling on an inclined plane. This nonlinear differential equation is numerically integrated and the effect of eccentricity upon the moving time is pointed out. The experimental aspects are presented in the second part of the work. A new designed device, used in accurately finding the moving time of the rolling body is presented. A solution for a rotor with rigorously controlled eccentricity is proposed.

1. Introduction
Friction is a frequently met phenomenon in machine construction since whenever two solid bodies are brought into contact the elements of the friction torsor occur in the contact region. Two major types of contact are distinctive, depending on the dimensions of the zones on the two contacting bodies: conform and non-conform contact [1]. Because a form of friction will oppose to each of the six simple motions that a free body may run, three rotations and three translations, it is expected that, for the case of non-conform or Hertzian contacts, the friction phenomenon is more complex than in the case of surface contacts.

For a Hertzian point contact, there are defined the tangent plane and the common normal in the theoretical initial contact point and the only annihilated motion is the translation along the common normal. The following types of friction correspond to the five simple relative motions between the two elements: sliding friction, which resists to the relative motion in the tangent plane, spinning friction torque, which opposes to the rotation about the normal and the rolling friction torque counters the relative rotation about a straight line from the tangent plane [2-8].

Technical literature presents specific methods for finding the characteristic tribological parameters for each of the three types of friction. For the characterization of the rolling friction, the rolling friction torque, depending on the normal force through the coefficient of rolling friction $\mu$, is sought after [9-15].

A recognized technique in finding the coefficient of rolling friction is the inclined plane method [16]. The principle of the method consists of acquiring the time necessary for a rotor to run a certain distance downwards on an inclined plane. The method is particularly expedite and does not require complex equipment, so it has low costs, but the main drawback is the fact that for fine accuracy it requires an inclination angle of the plane comparable to the ratio between the coefficient of rolling
friction and the rolling radius of the rotor $s/r$, meaning angles of about 1°. If this condition is not satisfied, then appreciable errors occur in the experimental tests carried out for finding the coefficient of rolling friction. The use of small tilting angles of the inclined plane makes the motion of the rotor along the plane be strongly influenced by the eccentricity of the mass center of the rotor with respect to the geometrical axis of symmetry. It results that accurate equilibrated, both statically and dynamically, rotors are required for tests. Next, it is presented the effect of the eccentricity of a rotor upon the experimental results obtained by the help of the inclined plane method and an approach of finding the coefficient of rolling friction using a rotor with controlled mass eccentricity.

2. Theoretical considerations

On the top of an inclined plane having the angle $\alpha$ with respect to the horizontal plane, figure 1, it is positioned the rotor with the axis of symmetry perpendicular to the steepest line of the plane. The assumption that the center of gravity of the rotor is placed at the distance $\xi$ from the geometrical axis of symmetry is made. The immobile reference frame $Oxyz$ is chosen as the axis $Ox$ must be along the plane and oriented as the trace of displacement of the rotor. The $Oz$ axis is parallel to the inclined plane and oriented in such a manner as by watching the rotation of the body from its positive sense, the rotation should be in positive sense. The $Oy$ axis completes the right triorthogonal system. The axes of the fix system have the versors $i$, $j$ and $k$.

![Figure 1. Principle of rotor on inclined plane](image)

A second system of coordinates is attached to the rotor, the $Ox'$ axis coincides to the axis of symmetry of the rotor, the $Ox'$ axis passes through the center of mass $G$ of the rotor and the $Oy'$ axis completes a right triorthogonal frame. At the initial moment, the $Ox'$ axis makes the angle $\phi_0$ with the $Ox$ axis.

Considering that from the initial moment to the current instant, the rotor revolved with the angle $\phi$ the versors of the mobile system $i'$, $j'$ and $k'$ are expressed with respect to the versors of the fixed system by the relations:
Accepting the assumption that pure rolling exists in the rotor-inclined plane contact leads to the mathematical relation:

\[ x_0 = R \varphi \]  

where \( x_0 \) is the distance run by the rotation axis of the rotor and \( R \) is the radius of the circle traced by the contact point with respect to the axis of the rotor. In order to find the law of motion of the rotor, the center of mass theorem and moment of momentum theorem with respect to the center of mass are applied [17-18]. The following forces are applied upon the rotor: the gravitational force \( G \) acting in the center of mass \( G \), the normal reaction \( N \) together to the tangential reaction \( T \) applied in the contact point and the rolling friction torque \( M_r \) parallel and opposite to the angular velocity. The center of mass theorem has the form:

\[ M \alpha_G = G + N + T \]  

and presents projections only on the axes from the plane of motion. The moment of momentum theorem, expressed as:

\[ J_c \ddot{\varphi} k = G C \times ( N + T ) - k M_r \ sgn \omega \]  

has projection only on the \( O_z \) axis. Assuming pure rolling condition fulfilled and based on relation (2), the position of the rotor is completely defined by the angle of rotation \( \varphi \).

In this case, the unknowns of the problem are: the angle of rotation \( \varphi \) and the values \( T, N \) and \( M_r \) of the reactions (tangential, normal and rolling friction torque). Therefore, an additional equation is required and it is obtained from the supposition that the rolling friction torque and the value of normal reaction are proportional, expressed by the relation:

\[ M_r = sN \]  

where \( s \) is the coefficient of rolling friction.

From figure 1, it can be observed that:

\[ r_G = x_0 i + \xi j = i [ R \varphi + \cos( \varphi + \varphi_0 )] + j \sin( \varphi + \varphi_0 ) \]  

The equation (6) is then projected on the axes of the fix frame and by twice derivations, the components of the accelerations of the center of mass are obtained:

\[ a_{Gx} = \ddot{\varphi} - \dot{\varphi}^2 \xi \cos( \varphi + \varphi_0 ) \]  
\[ a_{Gy} = \dot{\varphi} \xi \cos( \varphi + \varphi_0 ) - \dot{\varphi}^2 \xi \sin( \varphi + \varphi_0 ) + \]  

The projections of the equation (3) are:

\[ M \{ [ R - \sin( \varphi + \varphi_0 )] \ddot{\varphi} - \dot{\varphi}^2 \xi \cos( \varphi + \varphi_0 ) \} = Mg \sin \alpha - T \]  
\[ M \{ \dot{\varphi} \xi \cos( \varphi + \varphi_0 ) - \dot{\varphi}^2 \xi \sin( \varphi + \varphi_0 ) \} = Mg \cos \alpha + N \]  

From equation (8) there are obtained the expressions for \( N \) and \( T \) as functions of the law of motion:
\[ T = -\mathcal{M} \dot{\phi} + M \ddot{\phi} \xi \sin(\phi + \phi_0) + M \dot{\phi}^2 \xi \cos(\phi + \phi_0) + M g \sin \alpha \] (9)

\[ N = -\mathcal{M} \dot{\phi} \xi \cos(\phi + \phi_0) + M \dot{\phi}^2 \xi \sin(\phi + \phi_0) + M g \cos \alpha \]

The expression of the position vector of the contact point with respect to the center of mass is required in order to apply the moment of momentum theorem:

\[ \overrightarrow{GC} = r_C - r_G = -i \dot{\xi} \cos(\phi + \phi_0) + j[R - \sin(\phi + \phi_0) \dot{\xi}] \] (10)

By introducing the relation (4) in expression (10) and considering the relations (5) and (9), a nonlinear differential equation [19] with respect to \( \phi \) angle is obtained:

\[ \dot{\phi} = \frac{\frac{\mathcal{M}}{M} \dot{\phi}^2 \cos(\phi + \phi_0) + g \sin \alpha \mathcal{J} R + \dot{\xi} g \cos(\phi + \phi_0 + \alpha) - s[\dot{\xi} \dot{\phi}^2 \sin(\phi + \phi_0) + g \cos(\alpha) J]}{R^2 + \frac{\dot{\xi}}{M} + \dot{\xi}^2 - 2 \dot{\xi} \sin(\phi + \phi_0) - s \dot{\xi} \cos(\phi + \phi_0)} \] (11)

A numerical algorithm is necessary for integration of equation (11). The Runge-Kutta 4 algorithm [20-21] was the option for the present work. The equation (11) was numerically integrated for random values of the parameters involved and the results are presented in figures 2-5. The integration was done for a value of \( \alpha \) angle slightly superior to the value that ensures the static equilibrium of the rotor and for \( \phi_0 = 0 \). The variation in time of the angle \( \phi \) is presented in figure 2 and the variation of the angular velocity of the rotor is shown in figure 3. Figure 2 exemplifies the identification by a numerical manner of the moment when the distance run by the center of the rotor attains a prescribed value \( \mathcal{D} \). From figure 3, it can be be observed that after launching, the rotor takes a pseudo-resting state and after that starts the motion again. For a launching with an angle \( \phi_0 = \pi \) it is noticed that the plots in figures 2-3 take the form of periodical motions and this confirms the impossibility that the rotor runs downwards the plane. The variation of the ratio \( T/N \) is represented in figure 4 aiming to verify the completion of pure rolling condition \( |T| < \mu N \). Since for the steel-steel pair the dynamic sliding friction coefficient \( \mu \) has values greater than 0.2, it is concluded that the pure rolling condition is totally satisfied. The variation of time of motion for a sequence of increasing values of the angle \( \phi_0 \) is represented in polar coordinated in figure 5. The two plots correspond to the cases when the rolling friction is omitted and when it is considered for a value of the coefficient of rolling friction \( s = 2 \cdot 10^{-5} m \), respectively.

**Figure 2.** Finding the time when an imposed distance is reached

**Figure 3.** Angular velocity variation with pseudo-resting effect
Figure 4. Confirmation of pure rolling condition

Figure 5. Time vs angular position of eccentricity: with and without rolling friction

An important conclusion to be applied results from figure 5: the maximum time for downward rolling corresponds for the value $\phi_0 = 180^\circ$ for both considered situations, with or without rolling friction.

3. Experimental set-up. Design and function

The experimental device is presented in figure 6. An aluminum plate is positioned on a horizontal surface and the tilting angle is accomplished by introducing beneath it, at one end, stripes of steel of controlled thickness. The length of the plate is 1000 mm and the strip thickness is 2 mm, thus it results an angular increment $2/1000 = 7^\circ$. A prismatic body with two copper plated cable boards attached to it is placed on top of the aluminum plate. At the lower end of the plate there are fixed two leafs of aluminum foil, insulated by a paper sheet from the main plate.

The rotor is constructed from two identical bearing balls fastened together with silicon adhesive. In the position of launching the rotor is placed in the upper end of the plate and each ball comes into contact with a corresponding cable board. Thus, the rotor closes an electrical circuit with a small lamp that is turned off when the rotor starts moving. When the rotor reaches the lower end of the plate, the balls make contact each one with a tin-foil and a second electrical circuit with another small lamp is turned on. The device permits finding precisely the running time, during which a revolution body completes a distance along a tilted plane of known angle.

Figure 6. Experimental set-up
For time measurements, the motion of the body was filmed and the instants when the two electrical circuits are opened and closed, respectively, are identified by the frame-analysis of the movie.

The value of the eccentricity $\xi$ and its position with respect to an attached frame -the angle $\varphi_0$, must be stated. The angle can be stipulated based on figure 5: a circular stamp, with equidistant angular directions being traced, is applied with the center on the axis of symmetry of the rotor. Launching are made and for each traced radius starting the motion from the same position and a plot similar to the one from figure 5 is obtained and it is remarked that the value $\varphi_0 = \pi$ corresponds to all the maximum running-down times. The value of the parameter $\xi$ must be found and to this end, the moment of inertia $J_\xi$, the mass of the rotor $M$ and rolling friction coefficient $s$ are given.

The equation (11) is integrated for a preset value of the parameter $\xi$, considering $\varphi_0 = \pi$ as found above and $\varphi_0 = 0$ opposite to the case $\varphi_0 = \pi$ and finding the difference $\Delta t$ between the running times corresponding to the two values of the parameter $\varphi_0$. The procedure is repeated for a sequence of values and the dependency $\Delta t = \Delta t(\xi)$ is obtained in table format:

$$\Delta t = \Delta t(\xi) \quad (12)$$

The curve (12) is plotted, next, the value $\Delta t_{exp}$ for the actual rotor is established and the value of eccentricity corresponding to $\Delta t_{exp}$ is established on the graph.

4. Exemplification of methodology

For the illustration of the method, a rotor is required that has well-known both position and eccentricity value. A first solution is presented in figure 7: two identical bearing balls are threaded by electro-erosion and jointed by a threaded rod.
are cumulated and an appreciable total error concerning the poison of the center of mass of the rotor occurs. Therefore, another variant is proposed as in figure 8, where the rotor consists in two identical bearing balls, having the diameter \(2R = 41.28 \text{mm}\), rigidly connected. A smaller ball of diameter \(2r = 9 \text{mm}\) is rigidly attached, tangent to both balls. The rolling bodies are very precisely manufactured, therefore the center of mass of the body from figure 8 is positioned in the plane obtained by the three centers of mass of the spheres (figure 9) and in the symmetry plane normal to the axis of the initial rotor. The radius of the eccentricity \(\xi\) is the parameter sought after:

\[
\xi = \frac{my}{2M + m} = \frac{m\sqrt{(R + r)^2 - R^2}}{2M + m} = 7.4 \cdot 10^{-5} \text{m}
\]

(13)

where \(2M\) is the mass of the rotor made by the two balls, \(m\) is the mass of the eccentric ball, \(y\) is the position of the centre of eccentric ball with respect to the axis of the initial rotor.

The central moment of inertia \(J_z\) of the final rotor from figure 8 is found with respect to the axis of rotation, using the Steiner’s theorem [18] and has the value:

\[
J_z = 9.854 \cdot 10^{-5} \text{kgm}^2
\]

(14)

The ratio between the moment of inertia of the final rotor and the initial rotor (two balls) is:

\[
\frac{J_z}{2J_1} = 1.006
\]

(15)

and it fully justifies the done approximation, namely, the moment of inertia of the actual rotor is considered as the moment of inertia computed with respect to the axis of rotation.

\[
\begin{align*}
\Delta \theta &\quad \Delta \theta_{exp} \\
\Delta \theta_{exp} &\quad \Delta \theta
\end{align*}
\]

Figure 10. Experimental finding of the eccentricity of the rotor

From the experimental device, \(\alpha = 0.4^\circ\) is used in the integration of equation (11) and then the plots \(\xi = \xi(\Delta t)\) for \(s = 0\) and \(s = 20 \cdot 10^{-6} \text{m}\) are traced. For the rotor from figure 8, three launchings for \(\phi_0 = 0\) and \(\phi_0 = \pi\), respectively, were done and \(\Delta t_{exp} = 0.45 \text{sec}\) was found as mean values; after that, on the trace \(\xi = \xi(\Delta t)\), the corresponding value \(\xi_{exp}\) was established, as seen in figure 10. From figure 10, it results \(\xi = 55 \mu\text{m}\) for the case of rolling in the absence of rolling friction and \(\xi = 43 \mu\text{m}\) for the case of a coefficient of rolling friction \(s = 20 \mu\text{m}\).
To resume the methodology, the equation of motion depends on the position angle of the rotor, on time and on the eccentricity. The inertial characteristics of the rotor can be precisely found (with the remark that the moment of inertia of the rotor is practically insensitive to the change of the position of the mass center, for the given attached ball). Integrating the equation of motion for different values of the coefficient of rolling friction and eccentricity, a net of curves is traced and the dependence eccentricity versus maximum rolling time difference for different initial position of the rotor is obtained graphically. A series of launchings are performed with the experimental rotor and the maximum difference for the running time is found. If the position of the center of mass is accurately known, the point of coordinates \( (\Delta \xi_{\text{exp}}, \xi_{\text{exp}}) \) is found on the mesh of curves and, thus, the coefficient of rolling friction is obtained from the curve passing through this point. In return, when the coefficient of rolling friction is well known, the method can be used in accurate establishing of the position of center of mass.

5. Conclusions
The paper aims two key objectives. The first one intends to prove the effect of the mass eccentricity of a rotor upon the experimental result concerning the coefficient of rolling friction found through the inclined plane method. The second one consists in illustrating how a rotor with controlled mass eccentricity can be employed in establishing the coefficient of rolling friction.

In the first part of the work, a rotor with geometric axi-symmetry, but with mass eccentricity, is considered and the differential equation of motion of the rotor on an inclined plane is deduced for different initial launching positions.

Next, an experimental device is presented which allows for accurately finding the running time of a rotor downward on the inclined plane for different initial launching positions. In order to test the precision of the method the construction of a rotor with rigorously controlled mass eccentricity is proposed.

Finally, using a chart traced based on the integration of the equation of the rotor motion, which presents the dependency between the extreme values of the running time versus eccentricity the value of the coefficient of rolling friction is obtained.

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