SUBREGULARITY IN INFINITELY LABELED GENERATING TREES OF
RESTRICTED PERMUTATIONS

TOUFIK MANSOUR, REZA RASTEGAR, AND MARK SHATTUCK

Abstract. In this paper, we revisit the application of generating trees to the pattern avoid-
ance problem for permutations. In particular, we study this problem for certain general sets
of patterns and propose a new procedure leveraging the FinLabel algorithm and exploit-
ing the subregularities in the associated generating trees. We consider some general kinds
of generating trees for which the FinLabel algorithm fails to determine in a finite number
of iterations the generating function that enumerates the underlying class of permutations.
Our procedure provides a unified approach in these cases leading to a system of equations
satisfied by a certain finite set of generating functions which can be readily solved with the
aid of programming.

1. Introduction

The study of pattern avoidance in permutations has been an object of ongoing interest to
combinatorists over the past few decades. See, for example, the text [13] for a general review of
main results, techniques and directions. We seek a general procedure for enumerating broad
classes of pattern restricted permutations. Previously, an automatic approach to counting
members of an avoidance class via enumeration schemes was initiated by Zeilberger [24] and
later applied to a variety of problems (see [6, 7, 9, 17] and references contained therein). In
[8], further algorithms were found that derived functional equations for the generating
functions automatically which enabled polynomial-time enumeration for a set of consecutive
patterns. In [16], the more general problem of counting permutations according to the number
occurrences of a pattern was undertaken using an automatic approach and the problem for
patterns of length three was considered in detail.

Here, we revisit the classical avoidance problem for various sets of patterns and provide
a somewhat general procedure through an in-depth analysis of certain kinds of generating
trees. Recall that each node of a generating tree corresponds to a combinatorial object,
and the branch leading to a node encodes a particular choice made in the construction of
the object. Certain families of combinatorial objects admit a recursive description in terms
of generating trees [2, 3, 4, 11], which frequently leads to the enumeration of the object in
question, related explicit formulas and efficient random generation algorithms [1].

Generating trees were first utilized in the enumeration of subclasses of permutations by
West [22, 23] in the context of pattern avoidance and have been further exploited in closely
related problems [5]. Later in [20, 21], the generating tree idea was developed in the context of
restricted permutations and powerful algorithms were found that can automatically produce
the related generating functions, in particular, in the case when the associated generating
trees are of finite size. Herein, we demonstrate how to extend these algorithms applicable
only to finitely labeled generating trees to study several families in which they are infinitely labeled. This is achieved through a more in-depth understanding of a form of subregularity in the tree structure.

We put forth in this section a preliminary discussion following [20] and demonstrate how the enumeration of permutations avoiding a given set of patterns can be described in terms of counting paths within the corresponding generating tree. A few definitions are in order. Recall that a generating tree is a rooted, labeled tree such that the label of a node determines the labels of its children, if any. The nature of the labels of the nodes is immaterial and, as can be seen in our context, we use permutations to label the nodes. To specify a generating tree, it suffices to identify: (i) the label of the root, and (ii) a set of succession rules of the form

\[(l) \sim (l_1), (l_2), \ldots, (l_s)\]

describing how to label the nodes connected by the edges emanating from a node with label \(l\) using the labels \((l_1), (l_2), \ldots, (l_s)\). One often refers to the label on the left side of the rule as the parent and the labels on the right as the children, with these terms applying to the nodes themselves as well. For example, each node in an infinite complete binary tree has two children, hence it is enough to use only one label, say \((2)\), leading to the following description:

Root: \((2)\)
Rule: \((2) \rightarrow (2)(2)\).

We use the standard notation \(\mathbb{N}\) and \(\mathbb{C}\) to refer to the sets of natural and complex numbers, respectively. Let \([n] = \{1, \ldots, n\}\) for \(n \in \mathbb{N}\), with \([0] = \emptyset\). Also, for any word \(\sigma\) of length \(n \in \mathbb{N}\), let \(\sigma(i)\) represent its \(i\)-th entry for \(i \in [n]\). Similarly, for any matrix \(M, M(i,j)\) will denote its \((i,j)\)-th entry. A permutation of the set \([n]\) is any arrangement of the elements of \([n]\). We denote the set of all permutations of \([n]\) by \(S_n\) and let \(S := \cup_{j \geq 1} S_j\) be the set of all permutations of finite length. Throughout this paper, for any \(\pi \in S_n\), \(|\pi| := n\) refers to the length of the permutation \(\pi\). For \(\tau = \tau(1)\tau(2) \cdots \tau(k) \in S_k\) and \(\sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \in S_n\), we say that the permutation \(\sigma\) contains \(\tau\) as a pattern if there exist indices \(1 \leq i_1 < i_2 < \cdots < i_k \leq n\) such that \(\sigma(i_i) < \sigma(i_j)\) if and only if \(\tau(a) < \tau(b)\) for all \(1 \leq a, b \leq k\). Otherwise, it is said that \(\sigma\) avoids \(\tau\). We denote the set of all permutations in \(S_n\) that avoid the pattern \(\tau\) by \(S_n(\tau)\), and similarly, define \(S(\tau) := \cup_{j \geq 1} S_j(\tau)\) as the set of all permutations avoiding \(\tau\). More generally, for a set \(B \subseteq S\) of patterns, we use the notation \(S_n(B) := \cap_{\tau \in B} S_n(\tau)\) and \(S(B) := \cap_{\tau \in B} S(\tau)\) to refer to the set of permutations of a given length or of any length, respectively, avoiding all patterns in the set \(B\). Our interest here is to find the number of permutations in \(S_n(B)\), i.e., \(|S_n(B)|\), or equivalently to study the corresponding generating function

\[G_B(x) := \sum_{n \geq 1} |S_n(B)| x^n, \quad x \in \mathbb{C}.\]

To establish a useful connection between generating trees and the avoidance problem in permutations, we define a pattern-avoidance tree \(T(B)\) for a given set of patterns \(B\) as follows. The tree \(T(B)\) is understood to be empty if there is no permutation of arbitrary length avoiding the set \(B\). Otherwise, \(1 \notin B\) and the root can always be taken as 1, i.e., \(1 \in T(B)\). Starting with this root, the remainder of the tree \(T(B)\) can then be constructed in a recursive manner. To this end, we let the \(n\)-th level of the tree consist precisely of the elements of \(S_n(B)\) arranged in such a way that the parent of a permutation \(\pi := \pi(1) \cdots \pi(n) \in S_n(B)\) for which \(\pi(j) = n\) for some \(1 \leq j \leq n\) is the unique permutation \(\pi' := \pi'(1) \cdots \pi'(n-1) \in S_{n-1}(B)\)
where $\pi'(i) = \pi(i)$ for $i \in [j - 1]$ and $\pi'(i) = \pi(i + 1)$ for $i \in [j, n - 1]$. See Figure 1 for the first few levels of $\mathcal{T}(\{123\})$. A simple but important observation is that the size of $S_n(B)$ is equal to the number of nodes in the $n$-th level of $\mathcal{T}(B)$.

Hence, we focus on an understanding of the nature of this tree, more specifically, subregular structures contained within it. More precisely, let $\mathcal{T}(B; \pi)$ denote the subtree consisting of $\pi$ and its descendants in $\mathcal{T}(B)$. For any $1 \leq m < n \in \mathbb{N}$, we say that the node (labeled by) $\pi \in S_n(B)$ is reducible to the node $\pi' \in S_m(B)$ if the subtrees starting from $\pi$ and $\pi'$ are isomorphic, i.e., $\mathcal{T}(B; \pi) \cong \mathcal{T}(B; \pi')$. For instance, it is seen that $\mathcal{T}(\{123\}; 12) \cong \mathcal{T}(\{123\}; 1)$ and $\mathcal{T}(\{123\}; 312) \cong \mathcal{T}(\{123\}; 21)$. Suppose $t$ is the length of the longest pattern in $B$. Then, from [21], we have $\mathcal{T}(B; \pi) \cong \mathcal{T}(B; \pi')$ for $\pi, \pi' \in S(B)$ if and only if, for each $1 \leq j \leq t$, the number of nodes in the $j$-th level of subtree $\mathcal{T}(B; \pi)$ is equal to the number of nodes in the $j$-th level of subtree $\mathcal{T}(B; \pi')$.

Now, based on this subregularity concept, we form the tree denoted by $\mathcal{T}[B]$ which is an isomorphic copy of $\mathcal{T}(B)$ wherein the nodes belonging to the same irreducible class are labeled the same. Clearly, $\mathcal{T}[B]$ is a generating tree whose labels correspond exactly to the isomorphism classes of $\mathcal{L}[B] := \{\mathcal{T}(B; \pi) | \pi \in S(B)\}$. We let $\mathcal{R}[B]$ denote the set of succession rules for this generating tree. For instance, the first few levels of $\mathcal{T}[\{123\}]$ are given in Figure 2.

For any generating tree $\mathcal{T}[B]$, we define the directed graph $\mathcal{D}[B]$ whose vertices correspond to the set of all isomorphism classes of labels in $\mathcal{T}(B)$. An edge from the label $\alpha$ to the label $\beta$ exists if and only if the rule $\alpha \rightarrow \beta$ belongs to the set of succession rules $\mathcal{R}[B]$. For instance, the graph $\mathcal{D}[\{123\}]$ is depicted in Figure 3. Note that multiple edges occurring between $\alpha$ and $\beta$ corresponds to the case when $\beta$ arises more than once as a child of $\alpha$.
We equip the set $L[B]$ of all isomorphism classes of labels with the lexicographical ordering wherein each permutation of length $k$ appears before all permutations of length $k+1$ for all $k$. For example, the nodes of $D[\{123\}]$ are ordered as $1, 21, 321, \ldots$. We then define $M[B]$ for the graph $D[B]$ as the matrix whose entries are given by $M[B](v, w) = s$ for all $v, w \in D[B]$, where $s$ is the number of edges from $v$ to $w$. $M[B]$ is referred to as the transition matrix of the graph $D[B]$ and clearly has non-negative integral entries.

It is seen that the number of permutations in $S_n(B)$ is equal to the number of paths of length $n-1$ starting at the node 1 in the graph $D[B]$. Hence, the transfer-matrix method [19, Theorem 4.7.2] implies the generating function $G_B(x)$ is given by

$$G_B(x) = (1, 0, 0, \ldots) \sum_{n \geq 1} (M[B])^{n-1} x^n (1, 1, 1, \ldots)^t,$$

where $v^t$ denotes the transpose of the vector $v$. If the set of all isomorphism classes $\{T(B; \pi) | \pi \in S(B)\}$ is finite (i.e., if $M[B]$ is finite), then (1) implies the generating function $G_B(x)$ is rational and equal to

$$G_B(x) = x (1, 0, 0, \ldots) (I - x M[B])^{-1} (1, 1, 1, \ldots)^t.$$

In [14] and [21], it was shown that the set of isomorphism classes is finite if and only if $B$ contains both a child of an increasing permutation and a child of a decreasing permutation. Furthermore, a Maple package has been developed (described in [21] and available at [http://math.rutgers.edu/~vatter/](http://math.rutgers.edu/~vatter/)) that finds the generating function in this case. We will refer to this package (algorithm) as the FinLabel algorithm throughout this paper.

We close this section with a simple finite case example.

**Example 1.** Let $B = \{123, 43215\}$. Then the rules of $T[B]$ are given by

$$R[B] = \{1 \sim 1, 21\} \cup \{21 \sim 1, 21, 321\} \cup \{321 \sim 1, 21, 321, 321\},$$

with the root 1. Thus $D[B]$ is as presented in Figure 4.

The matrix $M[B]$ is given by

$$M[B] = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$
Hence, by (2), the generating function $G_B(x)$ is equal to
\[ x(1,0,0)(I-xM[B])^{-1}(1,1,1)^t = \frac{x(1-2x)}{(1-x)(1-3x)}, \]
as expected (see Theorem 3.1 in [10]).

However, when the matrix $M[B]$ is infinite, the evaluation of $G_B(x)$ can be an intricate task which we will focus on in this paper. In the next section, we develop an algorithm for computing $T[B]$, $D[B]$ and $M[B]$ after finitely many iterations which is applicable to cases when $M[B]$ is infinite. In the third section, we apply this algorithm together with a simple general enumerative result to deduce $G_B(x)$ for several classes of pattern sets $B$ for which $D[B]$ belongs to one of three general families of graphs.

2. Infinite size $M[B]$: subregularity structures

To study $M[B]$ of infinite size, we define $P_n(B;\pi)$ to be the number of nodes at the $n$-th level of $T(B;\pi)$. Let $F_\pi(x)$ be given by
\begin{equation}
F_\pi(x) := x^{\lvert \pi \rvert-1} \sum_{n=1}^{\infty} P_n(B;\pi)x^n.
\end{equation}
In words, $F_\pi(x)$ is the generating function that enumerates the paths beginning with the root $\pi$ of $T(B;\pi)$. Clearly, $G_B(x) = F_1(x)$. Note that $F_\pi(x)$ is analytic for all $\pi$ in some interval containing zero dependent upon $B$. For any rule $v \rightarrow w_1w_2\cdots w_s$, we have
\begin{equation}
P_n(B;v) = \begin{cases} 1, & \text{if } n = 1; \\ \sum_{j=1}^{s} P_{n-1}(B;w_j), & \text{if } n \geq 2. \end{cases}
\end{equation}
Therefore, substituting (4) into (3) implies
\begin{equation}
F_v(x) = x^{|v|} + x^{|v|-1} \sum_{n=2}^{\infty} \left( \sum_{j=1}^{s} P_{n-1}(B;w_j) \right) x^n
\end{equation}
\begin{equation}
= x^{|v|} + \sum_{j=1}^{s} x^{|v|+1-w_j}|F_{w_j}(x),
\end{equation}
where we exchange sums and apply definition (3) to $w_j$ in obtaining the second equality.

Our general approach will be to rewrite the set of succession rules as a set of equations of the form (5) and then obtain information on $F_1(x)$. We begin with a simple example to illustrate the general idea.
Example 2. Let $B = \{123, 132\}$. Here, the generating tree $\mathcal{T}[B]$ has the rules
\begin{align}
(6) \quad \mathcal{R}[B] := \{1 \sim 21, 12\} \\
(7) \quad \cup \{k(k-1) \cdots 21 \sim (12)^k, (k+1)k \cdots 21 \mid k \geq 2\} \\
(8) \quad \cup \{k(k-1) \cdots 312 \sim (12)^{k-2}, (k+1)k \cdots 312 \mid k \geq 2\},
\end{align}
with the ordered set
\[ \mathcal{L}[B] = \{1, 21, 12, 321, 312, 4321, 4312, \ldots\}, \]
and the infinite matrix $\mathcal{M}[B]$ given by
\[
\mathcal{M}[B] = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
& & & & & & & & & \vdots
\end{pmatrix}.
\]

Since $\mathcal{M}[B]$ is not finite, the FinLabel algorithm fails to count the members of $S_n(B)$. However, we can describe $\mathcal{M}[B]$ as an infinite system of equations as follows. Observe that, by \(6\), the rule \(7\) can be written as
\[
F_{k(k-1)\cdots1}(x) = x^k + kx^{k-1}F_{12}(x) + F_{(k+1)k\cdots1}(x), \quad k \geq 2,
\]
and
\[
F_{k(k-1)\cdots312}(x) = x^k + (k-2)x^{k-1}F_{12}(x) + F_{(k+1)k\cdots312}(x), \quad k \geq 2.
\]
Consider the set of equations \(9\) for all $k \geq 2$. By addition of the left and right sides of all the equations, and cancellation of like terms on both sides, we obtain
\[
F_{21}(x) = \frac{x^2}{1-x} + F_{12}(x) \sum_{k \geq 2} kx^{k-1}.
\]
In a similar manner, \(10\) implies
\[
F_{12}(x) = \frac{x^2}{1-x} + F_{12}(x) \sum_{k \geq 2} (k-2)x^{k-1}.
\]
Solving the system \(11\) and \(12\) for $F_{12}(x)$ and $F_{21}(x)$ yields
\[
F_{12}(x) = \frac{x^2(1-x)}{1-2x} \quad \text{and} \quad F_{21}(x) = \frac{x^2(1+x)}{1-2x}.
\]
Hence, we have
\[
G_B(x) = F_1(x) = x + F_{12}(x) + F_{21}(x) = \frac{x}{1-2x},
\]
which is in accordance with \(18\).
Algorithm 1 Calculating \( R[B] \) and \( D[B] \)

(\textbf{I}): \textbf{Input} Let 1 \( \not\in B \subset S \) be any set of patterns and \( D \geq 2 \).

(\textbf{II}): Let \( P = \{1\} \) and \( R_D = \emptyset \) as specified by the FinLabel algorithm given in [21]. Then run the FinLabel algorithm \( D \) iterations to update \( P \) and \( R_D \).

(\textbf{III}): Let \( R' := R_D[B] \). Construct one or more general rules from \( R \subseteq R' \) through use of Proposition \( \text{I} \) then remove \( R \) from \( R' \) and replace with general rule(s).

(\textbf{IV}): Using Proposition \( \text{I} \) and induction, we attempt to show \( R' = R[B] \). If successful, proceed to (\textbf{V}). Otherwise, increase \( D \) by one and return to (\textbf{II}).

(\textbf{V}): Find \( D[B] \) and the associated matrix \( M[B] \) for \( T[B] \).

We illustrate how the algorithm works with the following two examples.
Thus, proceeding similarly as in the derivation of (11), we obtain

\[ \mathcal{R}[B] = \{k(k-1)\cdots 312 \leadsto 12k^{-2}, (k + 1)k\cdots 312 \mid k \geq 2\} \]
\[ \cup \{k(k-1)\cdots 21 \leadsto 12k, (k + 1)k\cdots 21 \mid k \geq 1\} \]

By an induction argument with respect to \(k\), we have that the generating tree \(\mathcal{T}[B]\) is indeed given by the rules \(\mathcal{R}[B]\). The associated matrix \(\mathcal{M}[B]\) is given in Example 2.

**Example 3.** Let \(B = \{123, 132\}\) and \(D = 200\). Note that step (II) gives the set of rules \(\mathcal{R}_D = \{1 \leadsto 21, 12; 12 \leadsto 312; 21 \leadsto 12^2, 321; 312 \leadsto 12, 4312; 321 \leadsto 12^2, 4321; 4312 \leadsto 12^2, 54312; 4321 \leadsto 12^4, 54321; \ldots\}\). Then step (III) outputs

\[ \mathcal{R}[B] = \{k(k-1)\cdots 312 \leadsto 12k^{-2}, (k + 1)k\cdots 312 \mid k \geq 2\} \]
\[ \cup \{k(k-1)\cdots 21 \leadsto 12k, (k + 1)k\cdots 21 \mid k \geq 1\} \]

Likewise, the rule \(k(k-1)\cdots 312 \leadsto 12k^{-2}, (k + 1)k\cdots 312\) as

\[ F_{k(k-1)\cdots 312}(x) = x^k + (k - 3)x^{k-1}F_{12}(x) + F_{(k+1)k\cdots 312}(x), \quad k \geq 3. \]

Thus, proceeding similarly as in the derivation of (11), we obtain

\[ F_{132}(x) = \frac{x^3}{1 - x} + F_{12}(x) \sum_{k \geq 3} (k - 3)x^{k-1}. \]

Likewise, the rule \(k(k-1)\cdots 21 \leadsto 12^k, (k + 1)k\cdots 21\) yields

\[ F_{k(k-1)\cdots 21}(x) = x^k + kx^{k-1}F_{12}(x) + F_{(k+1)k\cdots 21}(x), \quad k \geq 1, \]

and hence

\[ F_1(x) = \frac{x}{1 - x} + F_{12}(x) \sum_{k \geq 1} kx^{k-1}. \]

By the last rule above and upon taking \(k = 1\) in the formula for \(F_{k(k-1)\cdots 1}(x)\), we have

\[ F_{12}(x) = x^2 + xF_{21} + F_{132}(x), \]
\[ F_1(x) = x + F_{12}(x) + F_{21}(x). \]

Thus, solving the system (13)-(16) for \(F_1(x)\) gives

\[ G_B(x) = F_1(x) = \frac{x}{1 - 2x - x^2}. \]

Let us say that \(a(x)\) is a rational linear combination of \(b_1(x), \ldots, b_s(x)\) if there exist rational functions \(c_j(x)\) such that \(a(x) = c_0(x) + \sum_{j=1}^{s} c_j(x)b_j(x)\). In order to systematically leverage Algorithm (1) we will need the following simple yet important result.

**Theorem 5.** Let \(1 \notin B\) be any set of patterns and \(m \geq 1\) be a natural number. Suppose that for any node \(\pi\) of \(\mathcal{D}[B]\) with \(|\pi| = m\), the generating function \(F_{\pi}(x)\) can be expressed as a rational linear combination of the \(F_{\pi'}(x)\) with \(\pi' \in \mathcal{D}[B]\) and \(|\pi'| \leq m\) such that \(c_j(0) = 0\) for all \(j\) in the corresponding coefficients \(c_j(x)\). Then \(G_B(x)\) is a rational generating function.
Proof. Since $1 \notin B$, there is a rule in $\mathcal{R}[B]$ with parent 1. Further, the result is apparent if $m = 1$, so we may assume $m \geq 2$. Define $R'$ to be the set of all rules in $\mathcal{R}[B]$ whose parents are of length at most $m - 1$. For any rule $v \sim w_1 w_2 \cdots w_s$ in $R'$, we rewrite it in the form of (5):

\begin{equation}
F_v(x) = x^{\pi} + \sum_{j=1}^{s} x^{\pi+1-|w_j|} F_{w_j}(x),
\end{equation}

where $|w_j| \leq m$ for any $1 \leq j \leq s$. By hypothesis, for each $\pi \in \mathcal{D}[B]$ with $|\pi| = m$, $F_\pi(x)$ is a rational linear combination of $F_v(x)$ with $|v| \leq m$. Therefore, combining these equations with those given in (17) for $|v| \leq m - 1$, one obtains a linear system of equations in the variables $F_v(x)$ where $|v| \leq m$. The associated coefficient matrix of this system has rational function coefficients and non-zero determinant since each main diagonal entry is of the form $1 - x f(x)$ for some rational $f(x)$, with each entry below the diagonal seen to be a multiple of $x$ (possibly zero). Indeed, the determinant is equal to 1 at $x = 0$, and hence by continuity, there exists some open interval containing zero for which the determinant is non-zero. Thus, Cramer’s rule implies that each component of the solution of the system (valid for all $x$ on the interval) is a rational function. In particular, $G_B(x) = F_1(x)$ is rational, as desired. \qed

3. Enumeration results for families of patterns

In this section, we use Algorithm 1 along with Theorem 5 to study several families of sets of patterns whose corresponding graphs are infinite and hence FinLabel is not applicable directly. In each of these cases, let $G = (V, E)$ be a directed graph with set of nodes $V \subset S$ and edge set $E \subset V \times V$.

3.1. Almost path-directed graphs. We say that $G$ is an almost path-directed graph if the set of nodes $V$ can be partitioned as $V := V' \cup W$, where $V' := \{v_0, v_1, \ldots \}$ and $|v_i| = i + m$ for all $i \geq 0$ with $m \geq 1$ fixed such that

- The length of any node $w$ in $W = V \setminus V'$ is at most $m$.
- $(v_j, v_{j+1}) \in E$ for all $j \geq 0$ and is not repeated.
- All other edges $(c, d) \in E$ may be repeated (with $c = d$ possible) and are such that $c$ or $d$ belongs to $W$ (possibly both).

In this context, $V'$ and $W$ will be referred to as the parameters of the almost path-directed graph $G$. Note that $(e_1, e_2) \in E$ with $e_1 \in W$ implies $e_2 \in W \cup \{v_0, v_1\}$ since $|e_1| \leq m$. In practice, we frequently have for each $m' > m$ that the node $v_{m'-m} \in V'$ is obtained from $v_0$ by replacing $m$ with either $m(m+1) \cdots m'$ or $m'(m'-1) \cdots m$. Furthermore, the result below is seen to apply more generally to any generating function which enumerates paths starting from the root and having $n - 1$ steps for $n \geq 1$ in an almost path-directed graph $G$ independent of whether or not $G$ arose in the context of pattern avoidance. A similar remark applies to the graphs discussed in the subsequent two sections.

When $B$ is a set of patterns whose directed graph $\mathcal{D}[B]$ is almost path-directed, we may apply Algorithm 1 together with the following result to ascertain the generating function.

Theorem 6. Let $1 \notin B \subset S$ be any set of patterns. Supposed $\mathcal{D}[B]$ is an almost path-directed graph with parameters $V'$ and $W$. If for each $w \in W$, the generating function

$$\sum_{j \geq 0} \mathcal{M}[B](v_j, w)x^j$$

is rational, then $G_B(x)$ is rational.
Proof. We first show that $F_{v_0}(x)$ is a rational linear combination of the $F_w(x)$ with $w \in W$ such that $c_j(0) = 0$ for each corresponding coefficient $c_j(x)$. To this goal, since $D[B]$ is almost path-directed, we have

$$F_{v_j}(x) = x^{v_0} + \sum_{w \in W} \mathcal{M}[B](v_j, w)x^{v_0} + j + 1 - |w|F_w(x) + F_{v_{j+1}}(x), \quad j \geq 0.$$  

Hence, by summing over $j \geq 0$ and using the fact that $F_{v_j}(x) \to 0$ as $j \to \infty$ for $x$ sufficiently close to zero, we obtain

$$F_{v_0}(x) = x^{v_0} + \sum_{w \in W} x^{v_0 + 1 - w} \left( \sum_{j \geq 0} \mathcal{M}[B](v_j, w)x^j \right) F_w(x).$$  

By assumption, each of the functions $\sum_{j \geq 0} \mathcal{M}[B](v_j, w)x^j$ is rational, and hence by (18), $F_{v_0}(x)$ is a rational linear combination of the $F_w(x)$ with $w \in W$ of the desired form and $W$ a finite set. By a similar argument (starting all sums from $j = 1$), the same holds for $F_{v_1}(x)$. Note that the set of all nodes of length at most $m$ is given by $W' = W \cup \{v_0\}$ since $D[B]$ is almost path-directed. Then $W'$ is seen to meet the conditions of Theorem 5 concerning nodes of length $m$, which implies the stated result. \hfill $\Box$

Example 7. Let $B = \{123, 312\}$. Then Algorithm II outputs

$$\mathcal{R}[B] = \{12 \leadsto 12\} \cup \{k(k-1) \cdots 1 \leadsto 12^k, (k+1)k \cdots 1 \mid k \geq 1\}.$$  

See Figure 5 for the schematic of the corresponding directed graph which is almost path-directed. One may verify that the parameters in this case are $V' = \{v_j := (j+2)(j+1) \cdots 1 \mid j \geq 0\}$ and $W = \{1, 12\}$. We also have that $\sum_{j \geq 0} \mathcal{M}[B](j+2)(j+1) \cdots 1, 12)x^j$ is the rational function $\sum_{j \geq 0} (j+2)x^j = \frac{2-x}{(1-x)^2}$ (with $\sum_{j \geq 0} \mathcal{M}[B](j+2)(j+1) \cdots 1, 1)x^j = 0$). Therefore, Theorem 6 implies that $G_B(x)$ is a rational generating function. Moreover, it allows us to calculate $G_B(x)$ by solving the following system:

$$F_1(x) = x + F_{12}(x) + F_{21}(x),$$
$$F_{12}(x) = x^2 + xF_{12}(x),$$
$$F_{21}(x) = \frac{x^2}{1-x} + \frac{x(2-x)}{(1-x)^2}F_{12}(x).$$

Hence,

$$G_B(x) = F_1(x) = \frac{x}{1-x} + \frac{x^2}{(1-x)^2}.$$  

Figure 5. Directed graph $D[\{123, 312\}]$
**Example 8.** Let $B = \{123, 2143\}$. Applying Algorithm 1 yields

$$R[B] = \{k(k - 1) \cdots 1 \sim 1^k, (k + 1)k \cdots 1 \mid k \geq 1\}.$$ 

Note that $\mathcal{D}[B]$ is an almost path-directed graph with parameters $V' = \{v_j = (j + 2)(j + 1) \cdots 1 \mid j \geq 0\}$ and $W = \{1\}$. Thus, Theorem 6 implies $G_B(x)$ is rational with associated linear system

$$F_1(x) = x + xF_1(x) + F_{21}(x),$$

$$F_{21}(x) = \frac{x^2}{1 - x} + \frac{x^2(2 - x)}{(1 - x)^2}F_1(x).$$

Hence,

$$G_B(x) = F_1(x) = \frac{x - x^2}{1 - 3x + x^2}.$$  

**Example 9.** Let $B = \{123, 312, 21543\}$. Then by Algorithm 1, we get

$$R[B] = \{1 \sim 12, 21\}$$

$$\cup \{12 \sim 12\}$$

$$\cup \{213 \sim 2143\}$$

$$\cup \{j(j - 1) \cdots 1 \sim 12, 213j^{-1}, (j + 1)j \cdots 1 \mid j \geq 2\}.$$ 

Hence, $\mathcal{D}[B]$ is almost path-directed with parameters $V' = \{v_j = (j + 3)(j + 2) \cdots 1 \mid j \geq 0\}$ and $W = \{1, 12, 21, 213\}$. We then have the following system of equations:

$$F_1(x) = x + xF_{12}(x) + F_{21}(x),$$

$$F_{12}(x) = x^2 + xF_{12}(x),$$

$$F_{213}(x) = x^3 + F_{2143}(x) = x^3 + x^4,$$

$$F_{321}(x) = \frac{x^3}{1 - x} + \frac{x^2}{1 - x}F_{12}(x) + F_{213}(x) \sum_{j \geq 3} (j - 1)x^j.$$ 

Hence, solving this system for $F_1(x)$ gives

$$G_B(x) = F_1(x) = \frac{x + x^3 + x^4}{(1 - x)^2}.$$ 

In general, $\mathcal{D}[B]$ is almost path-directed for all pattern sets $B = \{123, 312, 21k(k - 1) \cdots 3\}$ where $k \geq 4$.

### 3.2. Backward path-directed graphs.

We say $G$ is a **backward path-direct graph** if the set of nodes $V$ can be partitioned as $V := V' \cup W$, where $V' := \{v_0, v_1, \ldots\}$ and $|v_i| = i + m$ for all $i \geq 0$ with $m \geq 1$ fixed such that

- The length of any node $w$ in $W = V \setminus V'$ is at most $m$.
- For each $j \geq 0$, $(v_j, v_i) \in E$ for all $i = 0, 1, \ldots, j - 1, j + 1$, with all of these edges occurring once. Additionally, for some fixed integer $a \geq 0$, there are $a$ loops at the node $v_i$ for all $i \geq 0$.
- All other edges $(c, d) \in E$ may be repeated (with $c = d$ possible) and are such that $c$ or $d$ belongs to $W$ (possibly both).
In this context, $V'$, $W$ and $a$ are called the parameters of the backward path-directed graph $G$. In practice, we frequently have for each $m' > m$ that the node $v_{m'-m} \in V'$ is obtained from $v_0$ by replacing $m$ with either $m(m+1) \cdots m'$ or $m'(m'-1) \cdots m$. A basic example of a backward path-directed graph is $D[\{123\}]$, where the tree $T[\{123\}]$ is defined by the root 1 and the set of succession rules $k(k-1) \cdots 1 \Rightarrow 1, 21, \ldots, (k+1)k \cdots 1$.

**Example 10.** Let $B = \{1243, 1324, 1342, 1423, 1432, 2143, 2413, 2431, 3142, 4132\}$. Then the Algorithm describes the tree $T[B]$ by the following succession rules:

\begin{equation}
R[B] = \{1 \Rightarrow 12, 21\}
\end{equation}

$\cup \{12 \Rightarrow 21, 123, 132\}$

$\cup \{21 \Rightarrow 21, 213, 321\}$

$\cup \{123 \Rightarrow 21, 123\}$

$\cup \{213 \Rightarrow 21, 123\}$

$\cup \{j(j-1) \cdots 1 \Rightarrow 21, 213, 321, 4321, \ldots, (j+1)j \cdots 1 | j \geq 3\}$

One may verify that $D[B]$ is backward path-directed with parameters $V' = \{v_j = (j+3)(j+2) \cdots 1 | j \geq 0\}$, $W = \{1, 12, 21, 123, 132, 213\}$ and $a = 1$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{$D[\{1243, 1324, 1342, 1423, 1432, 2143, 2413, 2431, 3142, 4132\}]$}
\end{figure}

To determine $G_B(x)$ when $D[B]$ is backward path-directed, we may apply Algorithm to determine $T[B]$ and then employ the following result.

**Theorem 11.** Let $1 \notin B \subset S$ be any set of patterns. Suppose $D[B]$ is a backward path-directed graph with parameters $V'$, $W$ and $a \geq 0$. If $\sum_{j \geq 0} \mathcal{M}[B](v_j, w)x^j$ is a rational generating function for any $w \in W$, then $G_B(x)$ is a rational generating function of $x$ and

\begin{equation}
t_0 := \frac{1 + x - ax - \sqrt{(1 + x - ax)(1 - 3x - ax)}}{2x(1 + x - ax)}.
\end{equation}

**Proof.** We first determine an expression for the generating function $F_{v_0}(x)$. Since $D[B]$ is backward path-directed, formula yields

\begin{equation}
F_{v_j}(x) = x^{v_0}x^j + axF_{v_j}(x) + F_{v_{j+1}}(x) + x^{j+1}F_{v_0}(x) + \cdots + x^2F_{v_{j-1}}(x)
\end{equation}

\begin{equation}
+ \sum_{w \in W} \mathcal{M}[B](v_j, w)x^{v_0+j+1-|w|}F_w(x), \quad j \geq 0.
\end{equation}
Define $A(t) := \sum_{j \geq 0} F_{v_j}(x)t^j$. Multiplying both sides of (20) by $t^j$, and summing over all $j \geq 0$, we obtain

$$A(t) = \frac{x^{v_0}}{1 - xt} + \sum_{w \in W} \left( \sum_{j \geq 0} \mathcal{M}[B](v_j, w)x^{v_0 + 1 - |w|}t^j \right) F_w(x)$$

$$\quad + \frac{x^2t}{1 - xt} A(t) + axA(t) + \frac{A(t) - A(0)}{t},$$

which is equivalent to

$$\left(1 - \frac{x^2t}{1 - xt} - ax - \frac{1}{t}\right) A(t) = \frac{x^{v_0}t_0}{1 - xt} + \sum_{w \in W} \left( \sum_{j \geq 0} \mathcal{M}[B](v_j, w)x^{v_0 + 1 - |w|}t^j \right) F_w(x)$$

$$\quad - \frac{1}{t}A(0).$$

We apply the kernel method (see, e.g., [12]) to the last equation and take $t = t_0$, where $t_0$ satisfies $1 - \frac{x^2t_0}{1 - xt_0} - ax - \frac{1}{t_0} = 0$, to obtain

$$F_{v_0}(x) = A(0) = \frac{x^{v_0}t_0}{1 - xt_0} + t_0 \sum_{w \in W} \left( \sum_{j \geq 0} \mathcal{M}[B](v_j, w)x^{v_0 + 1 - |w|}(xt_0)^j \right) F_w(x),$$

with

$$t_0 = \frac{1 + x - ax - \sqrt{(1 + x - ax)(1 - 3x - ax)}}{2x(1 + x - ax)} = 1 + ax + (a^2 + 1)x^2 + \cdots.$$ 

By (21) and the assumed rationality of $\sum_{j \geq 0} \mathcal{M}[B](v_j, w)x^j$, we have that $F_{v_0}(x)$ is a linear combination of $F_v(x)$ for $w \in W$ whose coefficients are rational in $x$ and $t_0$ with $W$ a finite set. Upon considering $B(t) = \sum_{j \geq 1} F_{v_j}(x)t^{j-1}$ and finding $B(0) = F_{v_0}(x)$, one can show that $F_{v_0}(x)$ is a similar linear combination of $F_{v_j}(x)$ and the $F_w(x)$ with $w \in W$. Upon substituting out this expression for $F_{v_0}(x)$ as needed, it is seen that the nodes of length $m$ meet the conditions of Theorem 5 but where the coefficients $c_j(x)$ are now rational in $x$ and $t_0$. Thus, it follows that the generating function $F_1(x) = G_B(x)$ is rational in $x$ and $t_0$. □

In the following example, we elaborate on how to apply Theorem 11 to Example 10.

**Example 12.** It is easy to check that Theorem 11 applies to the case

$$B = \{1243, 1324, 1342, 1423, 1432, 2143, 2413, 2431, 3142, 4132\},$$

and hence $G_B(x)$ is rational in $x$ and $t_0 = C(x)$, where $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n \geq 0} C_nx^n$ and $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the $n$-th Catalan number. The set of succession rules given by (19) can be
written as the system of equations
\[
F_1(x) = x + F_{12}(x) + F_{21}(x),
F_{12}(x) = x^2 + x F_{21}(x) + F_{123}(x) + F_{132}(x),
F_{21}(x) = x^2 + x F_{21}(x) + F_{213}(x) + F_{321}(x),
F_{132}(x) = x^3,
F_{123}(x) = x^3 + x^2 F_{21}(x) + x F_{123}(x),
F_{213}(x) = x^3 + x^2 F_{21}(x) + x F_{123}(x),
F_{321}(x) = A(0).
\]

In addition, from \(\mathcal{M}[B](v_j, 21) = \mathcal{M}[B](v_j, 213) = 1\) for all \(j \geq 0\), have
\[
A(0) = \frac{x^3 t_0}{1 - x t_0} + \frac{x^2 t_0}{1 - x t_0} F_{21}(x) + \frac{x t_0}{1 - x t_0} F_{213}(x).
\]
Hence, by solving the above system, we obtain \(G_B(x) = F_1(x) = x^3 - 1 + C(x)\), as was shown in [15] Lemma 4.13.

Similarly, one may verify that Theorem [11] is applicable to the case
\[
B = \{1234, 1243, 1324, 1342, 1423, 2134, 2314, 2341, 3124, 3142\}.
\]
In particular, it yields \(G_B(x) = x^3 - 1 + C(x)\), as was also shown in [15] Lemma 4.13.

Remark: Note that we may allow for nodes in \(W\) in the definition above to have length greater than \(m\), provided it is required that \(W\) be finite where there are no edges \((a, b)\) such that \(a \in W\) with \(|a| > m\) and \(b = v_i\) for some \(i \geq 2\). The same can be said in the case when \(D[B]\) is almost path-directed.

3.3. Directed graph with \(\alpha\)-growing paths. For \(\alpha \geq 1\), suppose that the set \(V\) of nodes in \(D[B]\) for some \(B\) can be partitioned into sets \(V^{(j)}\) for \(1 \leq j \leq \alpha\) and \(W\), where \(V^{(j)} := \{v^{(j)}_i \mid i \geq 0\}\) and \(W\) is a finite set with \(W := \{w_1, \ldots, w_\ell\}\) for some \(\ell \geq 1\). Then we will say that \(V\) has \(\alpha\)-growing paths if the following conditions are satisfied:
- \(|v^{(1)}_1| = \cdots = |v^{(\alpha)}_i| = i + m\) for all \(i \geq 0\) where \(m \geq 1\) is fixed.
- The length of any node in \(W\) is at most \(m\).
- The edges that start with a member of \(\cup_{j=1}^{\alpha} V^{(j)}\) are dictated by the following succession rules:
  \[
  v_k^{(1)} \sim v_0^{(1)}, v_1^{(1)}, \ldots, v_{k-1}^{(1)}, v_k^{(1)}, (v_k^{(1)})^{r_1}, (v_k^{(1)})^{r_1, 1}, (v_k^{(1)})^{r_1, \ell},
  v_k^{(s)} \sim v_0^{(s)}, v_1^{(s)}, \ldots, v_{k-1}^{(s)}, v_k^{(s)}, (v_k^{(s)})^{r_s}, (v_k^{(s)})^{r_s, 1}, (v_k^{(s)})^{r_s, 2},
  \]
  \[
  (w_1)^{p_s, 1}, \ldots, (w_\ell)^{p_s, \ell}, \quad 1 \leq s' < s \leq \alpha - 1,
  v_k^{(\alpha)} \sim v_0^{(\alpha)}, v_1^{(\alpha)}, \ldots, v_{k-1}^{(\alpha)}, (v_k^{(\alpha)})^{r_\alpha}, (v_k^{(\alpha)})^{r_\alpha, 1}, \ldots, (v_k^{(\alpha)})^{r_\alpha, \alpha},
  (w_1)^{p_\alpha, 1}, \ldots, (w_\ell)^{p_\alpha, \ell}, \quad 1 \leq \alpha' < \alpha.
  \]
- All other edges start with a node in \(W\).

Here, the non-negative exponents \(r_i^a, r_i, r_{a,i}\) for \(i \in [\alpha], r_{i,1}, r_{i,2}\) for \(i \in [2, \alpha - 1]\) and \(p_{i,j}^{(k)}\) for \(k \geq 0, 1 \leq i \leq \alpha\) and \(1 \leq j \leq \ell\) are all assumed to be fixed. Further, the parameters \(\alpha'\) and \(s'\) in the penultimate condition above are also fixed with \(s'\) depending upon \(s > 1\). For each
$1 \leq j \leq \alpha$, it is often the case that the node $v_i^{(j)} \in V^{(j)}$ for $i > 0$ contains either the subword $m(m + 1) \cdots (m + i)$ or $(m + i)(m + i - 1) \cdots m$.

We now consider two examples of sets of patterns whose corresponding directed graphs have $\alpha$-growing paths.

Example 13. Let $B = \{1324, 1423, 2143, 2413, 3124, 3142, 3412, 4132, 4213, 4231, 4312\}$. Then, by Algorithm 1, $R[B]$ is given by

$$
R[B] = \{1 \leadsto 12, 21\}
\cup\{12 \leadsto 12, 132, 312\}
\cup\{21 \leadsto 213, 231, 321\}
\cup\{132 \leadsto 132^2\}
\cup\{a_k \leadsto a_4, \ldots, a_k, 213, \ k \geq 3\}
\cup\{b_k \leadsto a_4, \ldots, a_k, b_k, 132, 213, \ k \geq 3\}
\cup\{c_k \leadsto a_4, \ldots, a_k, a_{k+1}, b_{k+1}, c_{k+1}, 213, \ k \geq 3\},
$$

Figure 7. The label on the loop indicates that is repeated.
where
\[
\begin{align*}
    a_k &= (k-1)(k-2)(k-3)(k-4) \cdots 1, \\
    b_k &= (k-1)(k-2)(k-3) \cdots 1, \\
    c_k &= k(k-1) \cdots 1.
\end{align*}
\]

Here, \( D[B] \) (see Figure 7) has 3-growing paths.

**Example 14.** Let \( B = \{1243, 1324, 1342, 1423, 1432, 2143, 2413, 2431, 3142, 3412, 4132\} \). \( R[B] \) is given by
\[
\begin{align*}
    R[B] &= \{1 \rightsquigarrow 12, 21\} \\
    &\cup \{12 \rightsquigarrow 123, 132, 312\} \\
    &\cup \{21 \rightsquigarrow 213, 231, 321\} \\
    &\cup \{123 \rightsquigarrow 123^2\} \\
    &\cup \{213 \rightsquigarrow 123, 213\} \\
    &\cup \{312 \rightsquigarrow 123, 312\} \\
    &\cup \{a_k \rightsquigarrow a_3, \ldots, a_{k-1}, a_k^2, 213, k \geq 3\} \\
    &\cup \{b_k \rightsquigarrow a_3, \ldots, a_k, a_{k+1}, b_{k+1}, 213, k \geq 3\},
\end{align*}
\]

where
\[
\begin{align*}
    a_k &= (k-1)(k-2)(k-3) \cdots 1, \\
    b_k &= k(k-1) \cdots 1.
\end{align*}
\]

Here, \( D[B] \) (see Figure 8) has 2-growing paths.

*Figure 8.* The labels on four of the loops indicate their repetition as edges in the graph.
If $B$ is a set of patterns whose directed graph $\mathcal{D}[B]$ has $\alpha$-growing paths, we may apply Algorithm 1 and the following result to deduce the rationality of $G_B(x)$.

**Theorem 15.** Let $1 \not\in B \subset S$ be any set of patterns whose corresponding $\mathcal{D}[B]$ has $\alpha$-growing paths with $V^{(1)}, \ldots, V^{(\alpha)}, W$ as described above. If the generating function given by $\sum_{j \geq 0} \mathcal{M}[B](v^{(j)}_i, w)x^j$ is rational for each $i \in [\alpha]$ and $w \in W$, then $G_B(x)$ is rational.

**Proof.** We prove the statement in the case when $|\alpha| \geq 3$, as the adjustments required for the $\alpha = 1, 2$ cases will be apparent. Note that $\{v^{(1)}_1, v^{(2)}_1, \ldots, v^{(\alpha)}_1\}$ comprises the set of all nodes in $V$ of length $m + 1$. By Theorem 5, it suffices to show that $F_{v^{(i)}_k}(x)$ for each $i \in [\alpha]$ is a rational linear combination of the $F_v(x)$ for $v \in V$ with $|v| \leq m$ wherein the corresponding coefficients all vanish at $x = 0$. In order to aid in doing so, we define the generating function $A_i(t) = \sum_{k \geq 0} F_{v^{(i)}_k}(x)t^k$ for $1 \leq i \leq \alpha$. From the succession rules, we have

$$
(1 - r_1 \alpha - \frac{x^2 t}{1 - xt}) A_1(t) = \frac{x^{m+1}}{1 - xt} + \sum_{w \in W} x^{m+1-|w|} \left( \sum_{k \geq 0} \mathcal{M}[B](v^{(1)}_k, w)(xt)^k \right) F_w(x),
$$

$$
(1 - r_{s,2} \alpha) A_s(t) = \frac{x^{m+1}}{1 - xt} + \left( \frac{x^2 t}{1 - xt} + r_{s,1} \alpha \right) A_0(t) - r_{s,1} \frac{A_k(0)}{t} + \sum_{w \in W} x^{m+1-|w|} \left( \sum_{k \geq 0} \mathcal{M}[B](v^{(s)}_k, w)(xt)^k \right) F_w(x), \quad 2 \leq s \leq \alpha - 1,
$$

and

$$
(1 - r_{\alpha} \alpha') A_\alpha(t) = \frac{x^{m+1}}{1 - xt} + \left( \frac{x^2 t}{1 - xt} + r_{\alpha} \alpha \right) A_0(t) + \sum_{i=1}^{\alpha} r_{\alpha, i} \left( \frac{A_i(t) - A_0(t)}{t} \right) + \sum_{w \in W} x^{m+1-|w|} \left( \sum_{k \geq 0} \mathcal{M}[B](v^{(\alpha)}_k, w)(xt)^k \right) F_w(x).
$$

By (22), we have that $A_1(t)$ is a linear combination of the $F_w(x)$ for $w \in W$ whose coefficients $c_j(x, t)$ are rational in $x$ and $t$ and satisfy $c_j(0, t) = 0$ for all $j$ and each fixed $t$. Then by an induction argument using (23), we have that $A_s(t)$ for $2 \leq s \leq \alpha - 1$ also admits of such a form. Note that $r_{\alpha, i} \in \{0, 1\}$ for $i \in [\alpha]$ since there is at most one way to produce a certain offspring of length $k + 1$ from a parent of length $k$. If $r_{\alpha, \alpha} = 0$, then (24) implies, like in the prior cases, that $A_\alpha(t)$ admits of this form too. If $r_{\alpha, \alpha} = 1$, then taking $t = 1$ in (24), and solving for $A_\alpha(0)$, implies $A_\alpha(0)$ has the desired form and hence $A_\alpha(t)$ does as well. Thus, for each $i \in [\alpha]$, we have in particular that $F_{v^{(i)}_k}(x) = \frac{A_i(t) - A_0(t)}{t}$ at $t = 0$ is a rational linear combination of the $F_v(x)$ with $|v| \leq m$ of the form stated above, which completes the proof. \hfill \Box

**Example 16.** As shown in [15] using different techniques, there are exactly 10 sets of patterns of size 11 consisting of members of $S_4$ where the FinLabel algorithm fails to terminate in a
finite number of iterations, and they are given by

\[ B_1 = \{1324, 1423, 2143, 2413, 3124, 3142, 3412, 4132, 4123, 4231, 4312\}, \]
\[ B_2 = \{1324, 1423, 2143, 2413, 3124, 3142, 3412, 4132, 4123, 4231, 4312\}, \]
\[ B_3 = \{1324, 1423, 2143, 2413, 3124, 3142, 3412, 4132, 4123, 4231, 4312\}, \]
\[ B_4 = \{1324, 1423, 2143, 2413, 3124, 3142, 3412, 4132, 4123, 4231, 4312\}, \]
\[ B_5 = \{1324, 1423, 2143, 2314, 3124, 3124, 3412, 4123, 4132, 4231, 4312\}, \]
\[ B_6 = \{1243, 1324, 1342, 1423, 2143, 3142, 3142, 4132, 4132, 4312\}, \]
\[ B_7 = \{1324, 1423, 2143, 2413, 3124, 3142, 3412, 4132, 4123, 4312\}, \]
\[ B_8 = \{1324, 1423, 2143, 2314, 3124, 3142, 3412, 4123, 4132, 4312\}, \]
\[ B_9 = \{1243, 1324, 1342, 1423, 2143, 2431, 3142, 4132, 4132, 4312\}, \]
\[ B_{10} = \{1324, 1243, 1324, 1342, 1423, 2143, 2314, 2341, 3142, 4132, 4213\}. \]

The set \( B_1 \) was discussed in Example 13 where \( D[B_1] \) was seen to have 3-growing paths with \( V^{(1)} = \{(k-1)(k-2)(k-3)(k-4) \cdots 1 | k \geq 4\} \), \( V^{(2)} = \{(k-1)(k-2)(k-3) \cdots 1 | k \geq 4\} \), \( V^{(3)} = \{k(k-1) \cdots 1 | k \geq 4\} \) and \( W = \{1, 12, 21, 132, 213, 231, 312, 321\} \). Note that \( \sum_{v \in W} \sum_{j \geq 0} \mathcal{M}(B)[v_j(x), v]x^j \) equals \((1-x)^{-1}\) when \( s = 1, 3 \) and \( 2(1-x)^{-1}\) when \( s = 2 \). Hence, Theorem 15 implies \( G_{B_1}(x) \) is a rational generating function. As noted in the following table, all other cases also have \( \alpha \)-growing paths. One can readily show that Theorem 15 is applicable in each of these cases as well. Additionally, following the same process as the one given in Example 9, one can calculate \( G_{B_s}(x) \) in all cases, which we omit here for the sake of brevity.

| \( B_s \) | \( \alpha \)-growing paths |
|----------|----------------------|
| \( B_1, B_2, B_3, B_4, B_5, B_{10} \) | 3 |
| \( B_6, B_7 \) | 4 |
| \( B_8, B_9 \) | 2 |

We conclude by mentioning some further applications of the preceding results. Algorithm 1 together with Theorems 6 and 15 have been applied to many sets \( B \) consisting of members of \( S_3 \) where \( 3 \leq |B| \leq 12 \). In our study, we have shown that there are 48 sets of patterns of size ten for which the FinLabel algorithm fails to find \( G_B(x) \) in a finite number of iterations. From these cases, exactly 10 (resp. 19, 10, 2 and 3) cases have directed graphs with 2-growing (resp. 3-, 4-, 5- and 6-) paths. In other words, with the exception of four cases, Theorem 15 (and its proof) provides the solution to the problem of finding the generating function \( G_B(x) \) when \( |B| = 10 \). Interestingly enough, Theorem 11 yields \( G_B(x) \) for the four remaining cases, two of which were treated in Example 12.

References

[1] C. Banderier, M. Bousquet-Mélon, A. Denise, P. Flajolet, D. Gardy and D. Gouyou Beauchamps, Generating functions for generating trees, Discrete Math. 246:1-3 (2002), 29–55.

[2] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, A methodology for plane tree enumeration, Discrete Math. 180:1-3 (1998), 45–64.

[3] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, ECO: a methodology for the enumeration of combinatorial objects, J. Difference Equ. Appl. 5:4-5 (1999), 435–490.
[4] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, From Motzkin to Catalan permutations, *Discrete Math.* 217:1–3 (2000), 33–49.

[5] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, Permutations avoiding an increasing number of length-increasing forbidden subsequences, *Discrete Math. Theor. Comput. Sci.* 4:1 (2000), 31–44.

[6] A. M. Baxter, Algorithms for permutation statistics, Ph.D. thesis, Rutgers, 2011.

[7] A. M. Baxter, Refining enumeration schemes to count according to permutation statistics, *Electron. J. Combin.* 21:2 (2014), P2.50.

[8] A. Baxter, B. Nakamura and D. Zeilberger, Automatic generation of theorems and proofs on enumerating consecutive-Wilf classes, *Advances in Combinatorics: Waterloo Workshop in Computer Algebra*, W80, May 26–29, 2011.

[9] A. M. Baxter and L. K. Pudwell, Enumeration schemes for vincular patterns, *Discrete Math.* 312:10 (2012), 1699–1712.

[10] T. Chow and J. West, Forbidden subsequences and Chebyshev polynomials, *Discrete Math.* 204 (1999), 119–128.

[11] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., and M. Kleiman, The number of Baxter permutations, *J. Combin. Theory Ser. A* 24:3 (1978), 382–394.

[12] Q. Hou and T. Mansour, Kernel method and linear recurrence system, *J. Comput. Appl. Math.* 216(1) (2008), 227–242.

[13] S. Kitaev, *Patterns in Permutations and Words*, Monographs in Theoretical Computer Science, Springer, 2011.

[14] D. Kremer and W. C. Shiu, Finite transition matrices for permutations avoiding pairs of length four patterns, *Discrete Math.* 268:1-3 (2003), 171–183.

[15] T. Mansour and M. Schork, Wilf classification of subsets of four letter patterns, *J. Combin. Number Theory* 8:1 (2016), 1–129.

[16] J. Noonan and D. Zeilberger, The enumeration of permutations with a prescribed number of forbidden patterns, *Adv. in Appl. Math.* 17 (1996), 381–407.

[17] L. Pudwell, Enumeration schemes for words avoiding permutations, in *Permutation Patterns: London Math. Soc. Lecture Notes Ser.*, Vol. 376, Cambridge University Press (2010), 193–211.

[18] R. Simion and F. W. Schmidt, Restricted permutations, *European J. Combin.* 6 (1985), 383–406.

[19] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, Cambridge University Press, 1997.

[20] V. Vatter, Enumeration schemes for restricted permutations, *Combin., Probab. and Comput.* 17:1 (2005), 137–159.

[21] V. Vatter, Finitely labeled generating trees and restricted permutations, *J. Symbolic Comput.* 41 (2006), 559–572.

[22] J. West, Generating trees and the Catalan and Schröder numbers, *Discrete Math.* 146 (1996), 247–262.

[23] J. West, Generating trees and forbidden subsequences, *Discrete Math.* 157 (1996), 363–374.

[24] D. Zeilberger, Enumeration schemes, and more importantly, their automatic generation, *Ann. Comb.* 2 (1998), 185–195.

Department of Mathematics, University of Haifa, 3498838 Haifa, Israel
Email address: tmansour@univ.haifa.ac.il

Occidental Petroleum Corporation, Houston, TX 77046 and Departments of Mathematics and Engineering, University of Tulsa, OK 74104, USA
Email address: reza_rastegar2@oxy.com

Department of Mathematics, University of Tennessee, Knoxville, TN 37996, USA
Email address: shattuck@math.utk.edu