FINITE SYMMETRIC INTEGRAL TENSOR CATEGORIES WITH
THE CHEVALLEY PROPERTY

WITH AN APPENDIX BY KEVIN COULEMBIER AND PAVEL ETINGOF

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ABSTRACT. We prove that every finite symmetric integral tensor category \( C \) with the Chevalley property over an algebraically closed field \( k \) of characteristic \( p > 2 \) admits a symmetric fiber functor to the category of supervector spaces. This proves Ostrik’s conjecture [O, Conjecture 1.3] in this case. Equivalently, we prove that there exists a unique finite supergroup scheme \( G \) over \( k \) and a grouplike element \( \epsilon \in kG \) of order \( \leq 2 \), whose action by conjugation on \( G \) coincides with the parity automorphism of \( G \), such that \( C \) is symmetric tensor equivalent to \( \text{Rep}(G, \epsilon) \). In particular, when \( C \) is unipotent, the functor lands in \( \text{Vec} \), so \( C \) is symmetric tensor equivalent to \( \text{Rep}(U) \) for a unique finite unipotent group scheme \( U \) over \( k \). We apply our result and the results of [Ge] to classify certain finite dimensional triangular Hopf algebras with the Chevalley property over \( k \) (e.g., local), in group scheme-theoretical terms. Finally, we compute the Sweedler cohomology of restricted enveloping algebras over an algebraically closed field \( k \) of characteristic \( p > 0 \), classify associators for their duals, and study finite dimensional (not necessarily triangular) local quasi-Hopf algebras and finite (not necessarily symmetric) unipotent tensor categories over an algebraically closed field \( k \) of characteristic \( p > 0 \).

The appendix by K. Coulembier and P. Etingof gives another proof of the above classification results using the recent paper [C], and, more generally, shows that the maximal Tannakian and super-Tannakian subcategory of a symmetric tensor category over a field of characteristic \( \neq 2 \) is always a Serre subcategory.

1. INTRODUCTION

This paper is motivated by the problem of classifying finite symmetric tensor categories \( C \) over an algebraically closed field \( k \) of characteristic \( p > 0 \). This problem was recently solved in the semisimple case by Ostrik [O], who proved that any symmetric fusion category \( C \) over \( k \) admits a symmetric fiber functor to the Verlinde category \( \text{Ver}_p \). In other words, Ostrik proved that \( C \) is symmetric tensor equivalent to the category \( \text{Rep}_0(G) \) of a unique finite group scheme \( G \) in \( \text{Ver}_p \), with a homomorphism \( \pi_1(C) \to G \) such that the adjoint action of \( \pi_1(C) \) on \( G \) is canonical, where \( \text{Rep}_0(G) \) is the symmetric tensor category of representations of \( G \) whose pullback to \( \pi_1(C) \) is canonical.

The most natural family of non-semisimple finite symmetric tensor categories \( C \) over \( k \) to start with is the one consisting of such categories \( C \) in which the Frobenius-Perron dimensions of objects are integers (i.e., \( C \) is integral). This is, in a sense,

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the easiest case, since by a result of [EO], such a category is the representation category of a finite dimensional triangular quasi-Hopf algebra over \( k \). One is thus led naturally to the problem of classifying finite-dimensional triangular quasi-Hopf algebras over \( k \). Like for Hopf algebras, the easiest non-semisimple finite dimensional triangular quasi-Hopf algebras to understand are those which have the Chevalley property. Recall that a quasi-Hopf algebra \( H \) has the Chevalley property if the tensor product of every two simple \( H \)-modules is semisimple. For example, every basic (e.g., local) quasi-Hopf algebra has the Chevalley property. The objective of this paper is to classify finite dimensional triangular quasi-Hopf algebras over \( k \) with the Chevalley property.

It is shown in [EO, Proposition 2.17] that a finite dimensional local quasi-Hopf algebra over \( \mathbb{C} \) is 1-dimensional. On the other hand, there do exist non-trivial finite dimensional triangular local quasi-Hopf algebras over a field \( k \) of characteristic \( p > 0 \). For example, if \( G \) is a finite unipotent group scheme over \( k \) then its group algebra \( kG \), equipped with the \( R \)-matrix \( 1 \otimes 1 \), is a finite dimensional triangular local cocommutative Hopf algebra over \( k \).

In categorical terms, the objective of this paper is to classify finite symmetric integral tensor categories \( C \) over \( k \) which have the Chevalley property. Recall that a tensor category \( C \) has the Chevalley property if the tensor product of every two objects of \( C \) is semisimple. Namely, we prove the following theorem.

**Theorem 1.1.** Every finite symmetric integral tensor category \( C \) with the Chevalley property over an algebraically closed field \( k \) of characteristic \( p > 2 \) admits a symmetric fiber functor to the category \( sVec \) of supervector spaces. Thus, there exists a unique finite supergroup scheme \( G \) over \( k \) and a grouplike element \( \epsilon \in kG \) of order \( \leq 2 \), whose action by conjugation on \( G \) coincides with the parity automorphism of \( G \), such that \( C \) is symmetric tensor equivalent to \( \text{Rep}(G, \epsilon) \).

**Remark 1.2.** (1) The proof of Theorem 1.1 uses [O, Theorem 1.1] and [EOV, Theorem 8.1], and the method we use in the proof is motivated by Drinfeld’s paper [D, Propositions 3.5, 3.6].

(2) Theorem 1.1 implies Ostrik’s conjecture [O, Conjecture 1.3] for finite symmetric integral tensor categories \( C \) with the Chevalley property over an algebraically closed field \( k \) of characteristic \( p > 2 \). Namely, such categories \( C \) admit a symmetric fiber functor to \( sVec \subset \text{Ver}_p \). However, there even exist fusion symmetric tensor categories over \( k \) which are not integral, so in particular do not admit a symmetric fiber functor to \( sVec \) (unlike in characteristic 0 [EG4]).

If \( C \) is unipotent (i.e., the unit object of \( C \) is its unique simple object) then it is integral and has the Chevalley property. Thus, specializing to unipotent symmetric tensor categories, we have the following result.

**Corollary 1.3.** Every finite symmetric unipotent tensor category \( C \) over an algebraically closed field \( k \) of characteristic \( p > 2 \) admits a symmetric fiber functor to \( \text{Vec} \). Thus, there exists a unique finite unipotent group scheme \( U \) over \( k \) such that \( C \) is symmetric tensor equivalent to \( \text{Rep}(U) \).

**Remark 1.4.** (1) Corollary 1.3 gives another proof of [EHO, Proposition 3.6] for finite dimensional Hopf algebras.

(2) Unlike the proof of Theorem 1.1, the proof of Corollary 1.3 does not require [O, Theorem 1.1] and [EOV, Theorem 8.1].
The organization of the paper is as follows. In Section 2 we recall some necessary stuff about quasi-Hopf algebras, finite tensor categories, finite (super)group schemes, and cohomology. In particular, we compute the cohomology ring of the coordinate algebra of a finite connected group scheme in a functorial form, generalizing a result from [FN]. Section 3 is devoted to the proof of Theorem 1.1 and Corollary 1.3. In particular, in Theorem 3.1 we classify finite dimensional triangular quasi-Hopf algebras with the Chevalley property over an algebraically closed field $k$ of characteristic $p > 2$ (up to pseudotwist equivalence). Section 4 is devoted to the classification of certain finite dimensional triangular Hopf algebras with the Chevalley property (e.g., local) in characteristic $p > 2$, in group scheme-theoretical terms (see Theorem 4.2). In Section 5 we compute the Sweedler cohomology groups $H_{Sw}^i(u(g))$, $i \neq 2$, of restricted enveloping algebras $u(g)$ over an algebraically closed field $k$ of characteristic $p > 0$ (see Theorem 5.4 and Corollary 5.6), and classify associators for their duals $u(g)^*$ (see Corollary 5.14 and Theorem 5.17). Finally, in Section 6 we consider finite dimensional (not necessarily triangular) local quasi-Hopf algebras and finite (not necessarily symmetric) unipotent tensor categories over an algebraically closed field $k$ of characteristic $p > 0$ (see Theorems 6.1 and 6.2).

The paper also contains an appendix by K. Coulembier and P. Etingof. This appendix gives another proof of Theorem 1.1 and Corollary 1.3 using the recent paper [C], and, more generally, shows that the maximal Tannakian and super-Tannakian subcategory of a symmetric tensor category over a field of characteristic $\neq 2$ is always a Serre subcategory.

Remark 1.5. In a separate publication we plan to do the following:

1. Discuss the classification of finite symmetric integral tensor categories with the Chevalley property over an algebraically closed field of characteristic 2.
2. Compute the Sweedler cohomology group $H_{Sw}^2(u(g))$ of restricted enveloping algebras $u(g)$ over an algebraically closed field $k$ of characteristic $p > 0$.
3. Classify finite dimensional triangular Hopf algebras with the Chevalley property over an algebraically closed field $k$ of characteristic $p > 0$.

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2. Preliminaries

All constructions in this paper are done over an algebraically closed field $k$ of characteristic $p > 0$.

2.1. Quasi-Hopf algebras. Recall [D] that a quasi-Hopf algebra over $k$ is an algebra $H$ equipped with a comultiplication $\Delta$ which is associative up to conjugation by an invertible element $\Phi \in H^{\otimes 3}$ (called the associator), a counit $\varepsilon$, and an antipode $S$ together with two distinguished elements $\alpha, \beta \in H$, satisfying certain axioms. In particular, we have

\begin{align*}
(1) \quad (\text{id} \otimes \text{id} \otimes \Delta)(\Phi)(\Delta \otimes \text{id} \otimes \text{id})(\Phi) &= (1 \otimes \Phi)(\text{id} \otimes \Delta \otimes \text{id})(\Phi)(\Phi \otimes 1),
\end{align*}

\begin{align*}
(2) \quad (\varepsilon \otimes \text{id} \otimes \text{id})(\Phi) &= (\text{id} \otimes \varepsilon \otimes \text{id})(\Phi) = (\text{id} \otimes \text{id} \otimes \varepsilon)(\Phi) = 1
\end{align*}
and
\[ \Phi(\Delta \otimes \text{id})(\Delta(h))\Phi^{-1} = (\text{id} \otimes \Delta)(\Delta(h)), \quad h \in H. \]

A triangular quasi-Hopf algebra is a quasi-Hopf algebra \( H \) equipped with an invertible element \( R \in H \otimes^2 \), satisfying
\[ \Delta^{\text{op}}(\cdot) = R\Delta(\cdot)R^{-1}, \]
\[ R^{-1} = R_{21}, \]
\[ (\Delta \otimes \text{id})(R) = \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123}, \]
and
\[ (\text{id} \otimes \Delta)(R) = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{123} \Phi_{123}^{-1}. \]

It is known that the pair \((R, \Phi)\) satisfies the following quantum Yang-Baxter equation:
\[ R_{12} \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123} = \Phi_{321} R_{23} \Phi_{213}^{-1} \Phi_{123} R_{12}. \]

An invertible element \( J \in H \otimes H \) satisfying \((\epsilon \otimes \text{id})(J) = (\text{id} \otimes \epsilon)(J) = 1\) is called a pseudotwist for \( H \). Using a pseudotwist \( J \) for \( H \), one can define a new quasi-Hopf algebra structure \( H^J = (H, \Delta^J, \epsilon, \Phi^J, S^J, \beta^J, \alpha^J, 1) \) on the algebra \( H \). In particular,
\[ \Delta^J(h) = J^{-1} \Delta(h) J, \quad h \in H, \]
and
\[ \Phi^J := (1 \otimes J^{-1})(\text{id} \otimes \Delta)(J^{-1})\Phi(\Delta \otimes \text{id})(J)(J \otimes 1). \]

Moreover, if \( H \) has a triangular \( R \)-matrix \( R \) then
\[ R^J := J_{21}^{-1} RJ \]
is a triangular \( R \)-matrix for \( H^J \).

We will denote the pseudotwist equivalence class of \( \Phi \) by \([\Phi]\). We will also say that two quasi-Hopf algebras \( H \) and \( K \) are pseudotwist equivalent if \( K \) and \( H^J \) are isomorphic as quasi-Hopf algebras for some pseudotwist \( J \) for \( H \). In this situation \( \text{Rep}(H) \) and \( \text{Rep}(K) \) are tensor equivalent. If \( H \) is pseudotwist equivalent to a Hopf algebra, we say that the associator of \( H \) is trivial.

A twist for a Hopf algebra \( H \) is a pseudotwist \( J \) satisfying
\[ (\text{id} \otimes \Delta)(J)(1 \otimes J) = (\Delta \otimes \text{id})(J)(J \otimes 1). \]

2.2. Finite tensor categories. Recall that a finite rigid tensor category \( \mathcal{C} \) over \( k \) is a rigid tensor category over \( k \) with finitely many simple objects and enough projectives such that the unit object \( 1 \) is simple (see \([\text{EGNO}])\). Let \( \text{Irr}(\mathcal{C}) \) denote the set of isomorphism classes of simple objects of \( \mathcal{C} \). Then to each object \( X \in \mathcal{C} \) there is attached a non-negative number \( \text{FPdim}(X) \), called the Frobenius-Perron (FP) dimension of \( X \) (namely, it is the largest non-negative eigenvalue of the operator of multiplication by \( X \) in the Grothendieck ring of \( \mathcal{C} \)), and the Frobenius-Perron (FP) dimension of \( \mathcal{C} \) is defined by
\[ \text{FPdim}(\mathcal{C}) := \sum_{X \in \text{Irr}(\mathcal{C})} \text{FPdim}(X)\text{FPdim}(P(X)), \]
where \( P(X) \) denotes the projective cover of \( X \) (see [EGNO]). For example, if \( \mathcal{C} \cong \text{Rep}(H) \), \( H \) a finite dimensional quasi-Hopf algebra, then FPdim\((X) = \text{dim}(X)\) and FPdim\((\mathcal{C}) = \text{dim}(H)\); in particular the FP-dimensions of objects are integers. Moreover, by [EO, Proposition 2.6], \( \mathcal{C} \) is equivalent to \( \text{Rep}(H) \). \( H \) a finite dimensional quasi-Hopf algebra, if and only if \( \mathcal{C} \) is integral (i.e., the FP-dimensions of its objects are integers).

2.3. Finite dimensional algebras. Let \( H \) be a finite dimensional algebra over \( k \), and let \( I := \text{Rad}(H) \) be its Jacobson radical. Then we have a filtration on \( H \) by powers of \( I \), so one can consider the associated graded algebra \( \text{gr}(H) = \bigoplus_{r \geq 0} H[r] \), \( H[r] := I^r/I^{r+1} \) \( (I^0 := H) \). By a standard result (see, e.g., [EG1, Lemma 2.2]), \( \text{gr}(H) \) is generated by \( H[0] \) and \( H[1] \). If moreover \( H \) is a quasi-Hopf algebra and \( I \) is a quasi-Hopf ideal of \( H \) (i.e., \( H \) has the Chevalley property), then the associated graded algebra \( \text{gr}(H) \) has a natural structure of a graded quasi-Hopf algebra.

We will need the following lemma.

Lemma 2.1. Let \( H \) be a finite dimensional algebra over \( k \), and let \( I := \text{Rad}(H) \). Then the following hold:

1. If \( e \in H/I \) is an idempotent, then \( e \) has a unique lift to an idempotent \( \bar{e} \in H \) (up to conjugacy by elements of \( 1+I \)).
2. Assume \( p > 2 \). Then for every \( h \in I \), there exists a unique element \( a \in 1+I \) such that \( a^2 = 1+h \).

Proof. (1) See, e.g., [IRT, Proposition 9.1.1].

(2) Since \( h \) is nilpotent, \( h^i = 0 \) for some \( i \). Also, for every integer \( i \), the number of times \( p \) appears in the numerator of

\[
\binom{1/2}{i} = \frac{1}{2} \cdot \frac{1}{2-1} \cdots \frac{1}{2-i+1}
\]

equals at least the number of times it appears in the denominator. Hence, we see that \( a := \sum_{i \geq 0} \binom{1/2}{i} h^i \) is well defined, and it is clear that \( a^2 = 1+h \).

The uniqueness of \( a \) follows by an easy induction. Namely, suppose \( a \) is unique modulo \( I^n \) and we have two solutions \( a, a' \) modulo \( I^{n+1} \). Then \( a' = a + b \) for some \( b \in I^n \) and \( (a+b)^2 = a^2 \), so \( ab + ba + b^2 = 0 \). Hence, \( 2b = 0 \) modulo \( I^{n+1} \), so \( b = 0 \) modulo \( I^{n+1} \), i.e., \( a' = a \) modulo \( I^{n+1} \), as desired.

For every \( h \in I \), the unique element \( a \) from Lemma 2.1 is denoted by \((1+h)^{1/2}\).

2.4. Finite group schemes. An affine group scheme \( G \) over \( k \) is called finite if its representing Hopf algebra (= coordinate Hopf algebra) \( \mathcal{O}(G) \) is finite dimensional. In this case, \( kG := \mathcal{O}(G)^* \) is a finite dimensional cocommutative Hopf algebra, which is called the group algebra of \( G \). A finite group scheme \( G \) is called constant if its representing Hopf algebra \( \mathcal{O}(G) \) is the Hopf algebra of functions on some finite abstract group with values in \( k \), and is called étale if \( \mathcal{O}(G) \) is semisimple. Since \( k \) is algebraically closed, it is known that \( G \) is étale if and only if it is a constant group scheme [Wall 6.4].

Let \( G \) be a finite commutative group scheme over \( k \), i.e., \( \mathcal{O}(G) \) is a finite dimensional commutative and cocommutative Hopf algebra. In this case, \( kG \) is also a

\[
1/2 = a_0 + a_1 p + a_2 p^2 + \cdots \text{ be the } p\text{-adic representation of the } p\text{-adic integer } 1/2. \text{ Then } a = (1+h)^{a_0} (1+h^p)^{a_1} (1+h^{p^2})^{a_2} \cdots .
\]
finite dimensional commutative and cocommutative Hopf algebra, so it represents a finite commutative group scheme $G^D$ over $k$, which is called the Cartier dual of $G$. For example, the Cartier dual of the group scheme $G = \mathbb{G}_m$ (the Frobenius kernel of the additive group $\mathbb{G}_a$) is $\mathbb{G}_m$, while the Cartier dual of $\mu_2 := \mathbb{G}_{m,1}$ (the Frobenius kernel of the multiplicative group $\mathbb{G}_m$) is $\mathbb{Z}/2\mathbb{Z}$.

A finite group scheme $G$ is called connected if $O(G)$ is a local Hopf algebra. Equivalently, $G$ is connected if $\text{Rep}(O(G))$ is a finite unipotent tensor category over $k$.

A finite group scheme $U$ is called unipotent if $O(U)$ is a connected Hopf algebra (i.e., $O(U)$ has a unique grouplike element). Equivalently, $U$ is unipotent if its group algebra $kU = O(U)^*$ is a local cocommutative Hopf algebra. Thus, $\text{Rep}(U)$ is a finite unipotent tensor category over $k$. Every finite commutative unipotent group scheme $U$ over $k$ decomposes into a direct product $U = U^o \times P$, where $U^o$ is a connected commutative unipotent group scheme (i.e., the identity component of $U$) and $P$ is a constant $p$-group (see, e.g., [Wat 6.8, p.52]).

2.5. Finite supergroup schemes. Assume $p > 2$. Recall that a finite supergroup scheme $G$ over $k$ is a finite group scheme in the category $s\text{Vec}$ of supervector spaces. Let $O(G)$ be the coordinate Hopf algebra. Then $G_0$ be the even part of $G$. We have $g_0 = \text{Lie}(G_0)$ is the Lie algebra of $G_0$. It is known that $O(G) = O(G_0) \otimes \Lambda g_0^*$ as superalgebras (see, e.g., [Ma Section 6.1]).

Let $G$ be a finite supergroup scheme over $k$, let $\epsilon \in G(k)$ be a grouplike element of $kG$ of order $\leq 2$ acting on $O(G)$ by the parity automorphism, and let $R_\epsilon := \frac{1}{2}(1 \otimes 1 + 1 \otimes \epsilon + \epsilon \otimes 1 - \epsilon \otimes \epsilon)$. Recall that $\text{Rep}(G, \epsilon)$ is the category of representations of $G$ on finite dimensional supervector spaces on which $\epsilon$ acts by parity, equipped with its standard tensor structure and the symmetric structure determined by $R_\epsilon$.

2.6. Cohomology of products of $O_p,r$.s. Let $O_{p,r}$ be the $r$th Frobenius kernel of $\mathbb{G}_a$. Then $k[x]/(x^{p^r})$, where $x$ is a primitive element, is the coordinate Hopf algebra of $O_{p,r}$.

Let $G := \prod_{i=1}^n O_{p,r_i}$ and let $A := O(G) = k[x_1, \ldots, x_n]/(x_1^{p^{r_1}}, \ldots, x_n^{p^{r_n}})$ be its coordinate Hopf algebra. Then $a := H^1(G, k) = P(A)$ is the $(\sum_{i=1}^n r_i)$-dimensional vector space of primitive elements in $A$ with basis $\{x_1^p \mid 1 \leq j \leq n, 0 \leq i \leq r_j - 1\}$. Let $\xi(f) := \sum_{i=1}^{p-1} \frac{1}{i} \left(\frac{p-1}{i-1}\right) f^i \otimes f^{p-i}, \quad f \in A.$ Then $\xi$ defines a twisted linear injective homomorphism $\tilde{\xi} : H^1(G, k) \to H^2(G, k)$ (see, e.g., [DG II, §3, 4.6]). Let $a^{(1)}$ be the Frobenius twist of $a$. Identifying $a^{(1)}$ with $\text{Im}(\tilde{\xi})$, and writing $S^1a^{(1)}$ for $Sa^{(1)}$, where we put every element of $S^1a^{(1)}$ in

\[ \text{Im}(\tilde{\xi}) = a^p f \text{ for every } a \in k \text{ and } f \in A. \]
degree 2i, we have
\[ H^i(G, k) = \wedge^i a \otimes \delta a^{(1)}, \quad p > 2, \]
and
\[ H^0(G, k) = \delta a, \quad p = 2. \]
(For more details see, e.g., [J, Chapter 4].)

In particular, we have the following proposition.

Proposition 2.2.
\[ H^3(G, k) \cong \wedge^3 a \oplus (a \otimes a^{(1)}), \quad p > 2, \]
and
\[ H^3(G, k) \cong S^3 a, \quad p = 2. \]

We will use Proposition 2.2 in Example 5.17 below.

2.7. Cohomology of products of \( \alpha_{p,r}'s. \)

Let \( A := \mathcal{O}(\alpha_{p,r}) = k[x]/(x^{p^r}) \) be the coordinate Hopf algebra of \( \alpha_{p,r} \). Let
\[ \{ x^{i*} \mid 0 \leq i \leq p^r - 1 \} \]
be the basis of \( A^* = \mathcal{O}(\alpha_{p,r}) \) dual to the basis \( \{ x^i \mid 0 \leq i \leq p^r - 1 \} \) of \( A \). We have \( A^* = \bigotimes_{i=0}^{p^r-1} \left( k[x^{i*}]/(x^{(i*+i)^p}) \right) \) as an algebra, with comultiplication map determined by
\[ \Delta(x^{i*}) = \sum_{j=0}^{i} x^{j*} \otimes x^{(i-j)*} \]
for every \( 0 \leq i \leq p^r - 1 \). In particular, the group \( G(A^*) \) of grouplike elements of \( A^* \) is 1-dimensional, and \( b := P(A^*) \) is spanned by \( x^* \).

Set
\[ \beta := \sum_{j=1}^{p^r-1} x^{j*} \otimes x^{(p^r-j)*} \quad \text{and} \quad \gamma := x^* \otimes \beta. \]

It is straightforward to verify that \( \beta \) and \( \gamma \) are Hochschild algebra 2-cocycle and 3-cocycle of \( A \), respectively. Also, since \( \beta \in A^* \otimes A^* \) has degree \( p^r \) it follows that \( \beta \), and hence also \( \gamma \), are non-trivial cocycles.

Let \( H^*(A, k) \) be the Hochschild cohomology of the algebra \( A \) with coefficients in the trivial \( A \)-bimodule \( k \). Note that by definition, we have \( H^*(A, k) = H^*(\alpha_{p,r}, k) \).

It is known that \( H^1(A, k) = \text{Der}_k(A, k) \) (see, e.g., [Wi, Lemma 9.2.1]), hence \( H^1(A, k) = b \) is 1-dimensional. It is also known that \( H^{2i+1}(A, k) \cong H^1(A, k) \) and \( H^{2i+2}(A, k) \cong H^2(A, k) \) for every \( i \geq 0 \). This can be seen by writing a 2-periodic resolution of \( k \) by free \( A \)-modules of rank 1, with even differentials being multiplication by \( x \) and odd ones by \( x^{p^r-1} \) (see, e.g., [Wi, Exercise 9.1.4]). In particular, \( H^2(A, k) \) and \( H^3(A, k) \) are 1-dimensional, spanned by \( [\beta] \) and \( [\gamma] \), respectively. (See also [FN, Remark 3.6].) Furthermore, we have the following result (for a proof see, e.g., [FN, Proposition 3.5]).
Proposition 2.3. The following hold:

1. If $p > 2$, or $p = 2$ and $r > 1$, then $H^\bullet(O(\alpha_{p,r}), k) = H^\bullet(\alpha^D_{p,r}, k)$ is a free supercommutative algebra generated by $[x^*]$ of degree 1 so, $[x^*]^2 = 0$ and $[\beta]$ of degree 2. In particular, $H^2(O(\alpha_{p,r}), k) = H^2(\alpha^D_{p,r}, k)$ is spanned by $[\gamma] = [\beta][x^*]$.

2. If $p = 2$ then $H^\bullet(O(\alpha_{p,r}), k) = H^\bullet(\alpha^D_{p,r}, k)$ is a free commutative algebra generated by $[x^*]$ of degree 1. In this case, we have $[\beta] = [x^*]^2 \neq 0$ and $[\gamma] = [\beta][x^*] = [x^*]^3$. \hfill $\square$

2.8. Cohomology of coordinate algebras of finite connected group schemes.

Next, let $G$ be any finite connected group scheme over $k$. Then we have an algebra isomorphism

\begin{equation}
\mathcal{O}(G) = k[x_1, \ldots, x_n]/ \left( x_1^{p^1}, \ldots, x_n^{p^n} \right)
\end{equation}

(see, e.g., [Wal 14.4, p. 112]). Thus, by the Künneth formula, we have an isomorphism of graded commutative algebras

\begin{equation}
H^\bullet(\mathcal{O}(G), k) \cong \bigotimes_{i=1}^n H^\bullet \left( k[x_i]/ \left( x_i^{p^i} \right), k \right).
\end{equation}

We would now like to give a functorial (i.e., coordinate-independent) formulation of this statement. Namely, we have the following generalization of [FN Proposition 3.5].

Let $\mathfrak{g}$ be the Lie algebra of $G$. Thus any $z \in \mathfrak{g}$ defines a derivation of $\mathcal{O}(G)$ into its augmentation module. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}(G)$ and $r$ be the smallest integer such that $\mathfrak{m}^r = 0$. Let $J_i$ be the ideal of $f \in \mathcal{O}(G)$ such that $f^p = 0$. Thus $\mathfrak{m} = J_r \supset J_{r-1} \supset \cdots \supset J_0 = 0$. Let $F^i\mathfrak{g}$ be the Lie subalgebra of $z \in \mathfrak{g}$ such that $z(J_i) = 0$. Thus $\mathfrak{g} = F^0\mathfrak{g} \supset F^1\mathfrak{g} \supset \cdots \supset F^r\mathfrak{g} = 0$. Let $\text{gr}(\mathfrak{g})_i := F^{i-1}\mathfrak{g}/F^i\mathfrak{g}$, $1 \leq i \leq r$.

Set $\tilde{\mathfrak{g}} := ((J_1/J_1m))^*(1)$, where the superscript here and below denotes the Frobenius twist. Note that $V_0 := J_1/J_1m$ has a descending filtration by subspaces

$V_i := (J_{i+1}/(J_{i+1}m + J_i))^{p^{i-1}},$

i.e., $V_0 \supset V_1 \supset \cdots \supset V_r = 0$. Thus $\tilde{\mathfrak{g}}$ has an ascending filtration

$0 = W_0 \subset W_1 \subset \cdots \subset W_r = \tilde{\mathfrak{g}},$

where $W_i := (V_i)^{(1)}$. Note that $W_i/W_{i-1} = \text{gr}(\mathfrak{g})_{i}^{(1)}$. So $\text{gr}(\tilde{\mathfrak{g}}) = \bigoplus_{i=1}^r \text{gr}(\mathfrak{g})_{i}^{(1)}$. Thus $\tilde{\mathfrak{g}}$ has the same composition factors as $\mathfrak{g}$, except they are Frobenius twisted and occur in opposite order.

Finally, let $\phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be the twisted linear map given by the composition $\mathfrak{g} \to \text{gr}(\mathfrak{g})_1 = \text{gr}(\mathfrak{g})_1^{(1)} \hookrightarrow \tilde{\mathfrak{g}}$.

Proposition 2.4. The following hold:

1. If $p > 2$ then we have a canonical isomorphism of graded algebras

$H^\bullet(\mathcal{O}(G), k) \cong \wedge \mathfrak{g} \otimes S'(\tilde{\mathfrak{g}}).$

2. If $p = 2$ then we have a canonical isomorphism of graded algebras

$H^\bullet(\mathcal{O}(G), k) \cong S\mathfrak{g} \otimes S'(\tilde{\mathfrak{g}})/(z^2 − \phi(z), z \in \mathfrak{g}).$
Proof. Let us construct a canonical linear map \( \eta : \bar{\mathfrak{g}} \to H^2(O(G), k) \). To this end let \( Y \subset G(k[t]/t^{p^r}) \) be the set of points \( y \) such that \( y(J_i) \subset (t^{p^r}) \) for all \( i \). Thus each \( y \in Y \) defines a linear map \( c_y : J_1 \to k \) sending \( f \in J_1 \) to the coefficient of \( t^{p^r} \) in \( f(y) = y(f) \). It is clear that \( c_y(J_1 \mathfrak{m}) = 0 \), so \( c_y \) descends to a linear map \( J_1/J_1 \mathfrak{m} \to k \). In other words, we can view \( c_y \) as an element of \( \bar{\mathfrak{g}} \). Thus we obtain a canonical map \( c : Y \to \bar{\mathfrak{g}} \). It is easy to see using the explicit presentation of \( O(G) \) given by (2.16) that \( c \) is surjective.

For \( y \in Y \), let us try to lift the homomorphism \( y : O(G) \to k[t]/t^{p^r} \) to a homomorphism \( O(G) \to k[t]/t^{p^r+1} \). This runs into an obstruction in \( H^2(O(G), k) \). Namely, let \( \bar{y} \) be any lift of \( y \) just as a linear map. Then \( \bar{y}(ab) - \bar{y}(a)\bar{y}(b) = t^{p^r+1}\omega(a, b) \), where \( \omega \in Z^2(O(G), k) \), and the cohomology class of \( \omega \) is independent of the choice of the lifting. Thus we obtain a well-defined map \( \theta : Y \to H^2(O(G), k) \), given by \( \theta(y) = [\omega] \).

It is easy to check using (2.16) that the map \( \theta \) factors through \( c \), i.e., there exists a (necessarily unique) map \( \eta : \bar{\mathfrak{g}} \to H^2(O(G), k) \) such that \( \theta = \eta \circ c \). Moreover, it is easy to see that \( \eta \) is linear. Thus \( \eta \) is a desired linear map \( \bar{\mathfrak{g}} \to H^2(O(G), k) \).

There is also an obvious canonical inclusion \( i : \mathfrak{g} \hookrightarrow H^1(O(G), k) \). If \( p > 2 \), the maps \( i \) and \( \eta \) define a graded algebra map \( \xi : \wedge \mathfrak{g} \otimes S'(\mathfrak{g}) \to H^*(O(G), k) \). It is easy to check using (2.16) that \( \xi \) is an isomorphism, which proves (1). If \( p = 2 \), we similarly have a graded map \( \xi : S\mathfrak{g} \otimes S'(\mathfrak{g}) \to H^*(O(G), k) \), which annihilates \( z^2 - \phi(z) \) for any \( z \in \mathfrak{g} \), so descends to a graded map \( \xi : \bar{\mathfrak{g}} \otimes S'(\mathfrak{g})/(z^2 - \phi(z), z \in \mathfrak{g}) \to H^*(O(G), k) \), which is again checked to be an isomorphism using (2.16). This proves (2). \( \Box \)

Remark 2.5. 1. The group structure of \( G \) is irrelevant in Proposition 2.4; only its scheme structure (i.e., the algebra structure of \( O(G) \)) matters. In other words, the description of the cohomology ring of \( O(G) \) in Proposition 2.4 is functorial even in the category of schemes, where there are more morphisms than in the category of group schemes.

2. The filtration on \( \bar{\mathfrak{g}} \) does not split canonically, even in the category of group schemes. Indeed, take \( G = \alpha_p \times \alpha_p \), i.e., \( O(G) = k[x, y]/(x^{p^r}, y^{p^r}) \), where \( x \) and \( y \) are primitive. Then \( J_1/J_1 \mathfrak{m} \) is spanned by \( x \) and \( y^{p^r} \). We have an automorphism of \( G \) given by \( x \mapsto x + y^{p^r}, y \mapsto y \). It is clear that this automorphism acts unipotently with respect to the filtration but nontrivially on \( J_1/J_1 \mathfrak{m} \), which shows that the filtration on \( J_1/J_1 \mathfrak{m} \) and hence on \( \bar{\mathfrak{g}} \) does not split canonically.

3. Note that if all \( r_i = 1 \) then Proposition 2.4 reduces to (2.13) - (2.14).

Set \( x_t^* := (x_t^*)^*, 1 \leq t \leq n, 0 \leq l \leq r_t - 1 \). For every \( 1 \leq i, j \leq n \), let

\[(2.18) \beta_j := \sum_{l=1}^{p^r - 1} x_j^{p^r - l} \otimes x_j^* \] and \( \gamma_{ij} := x_t^* \otimes \beta_j \).

Corollary 2.6. Let \( G \) be a finite connected group scheme over \( k \) with Lie algebra \( \mathfrak{g} \) as above. Then we have a short exact sequence

\[0 \to \mathfrak{g} \otimes \bar{\mathfrak{g}} \to H^3(O(G), k) \to \wedge^3 \mathfrak{g} \to 0,\]

which canonically splits for \( p > 2 \). Moreover, the set

\[
\{ [\gamma_{ij}] \mid 1 \leq i, j \leq n \} \cup \{ [x_t^* \otimes x_j^* \otimes x_t^*] \mid 1 \leq i < j < l \leq n \}
\]

forms a linear basis for \( H^3(O(G), k) \).

Proof. Follows from Proposition 2.4. \( \Box \)
3. The proof of Theorem 1.1

In the next two sections assume \( k \) has characteristic \( p > 2 \), unless otherwise explicitly specified.

By [EO Theorem 2.6], \( C \) is symmetric tensor equivalent to \( \text{Rep}(H, R, \Phi) \) for some finite dimensional triangular quasi-Hopf algebra \((H, R, \Phi)\) over \( k \) with the Chevalley property. Thus, we have to prove the following theorem.

**Theorem 3.1.** Let \((H, R, \Phi)\) be a finite dimensional triangular quasi-Hopf algebra over \( k \) with the Chevalley property. Then there exists a pseudotwist \( F \) for \( H \otimes H \) such that

\[
(H, R, \Phi)^F = (H^F, R_u, 1),
\]

where \( u \in H^F \) is a grouplike element of order \( \leq 2 \).

We will prove Theorem 3.1 in several steps.

3.1. \( \text{gr}(H) \). Let \((H, R, \Phi)\) be a finite dimensional triangular quasi-Hopf algebra over \( k \) with the Chevalley property. Let \( I := \text{Rad}(H) \) be the Jacobson radical of \( H \). Since \( I \) is a quasi-Hopf ideal of \( H \), the associated graded algebra \( \text{gr}(H) = \bigoplus_{r \geq 0} H[r] \) has a natural structure of a graded triangular quasi-Hopf algebra with some \( R \)-matrix \( R_0 \in H[0] \otimes H[0] \) and associator \( \Phi_0 \in H[0] \otimes H[0] \) (see 2.3).

**Proposition 3.2.** The following hold:

1. \( H[0] \) is semisimple.
2. \((H[0], R_0, \Phi_0)\) is a triangular quasi-Hopf subalgebra of \((\text{gr}(H), R_0, \Phi_0)\).
3. \( \text{Rep}(H[0], R_0, \Phi_0) \) is symmetric tensor equivalent to \( \text{Rep}(G, \epsilon) \) for some finite semisimple group scheme \( G \) over \( k \).
4. \((\text{gr}(H), R_0, \Phi_0)\) is pseudotwist equivalent to a graded triangular Hopf algebra with \( R \)-matrix \( R_\epsilon \), whose degree 0-component is \((kG, R_\epsilon, 1)\)

**Proof.** Parts (1) and (2) are clear.

Since \( \text{Rep}(H[0], R_0, \Phi_0) \) is an integral symmetric fusion category, Part (3) follows from [EOV Theorem 8.1] (which in turn relies on [O Theorem 1.1]).

By Part (3), there exists a pseudotwist \( J \) for \( H[0] \) such that

\[
(H[0], R_0, \Phi_0)^J \cong (kG, R_\epsilon, 1).
\]

Hence, pseudotwisting \( \text{gr}(H) \) using \( J \), we have

\[
(\text{gr}(H), R_0, \Phi_0)^J \cong (\text{gr}(H)^J, R_\epsilon, 1),
\]

and since \( J \) has degree 0, we have \( \text{gr}(H)^J = \left( \bigoplus_{r \geq 0} H[r] \right)^J \), as claimed in Part (4). \( \square \)

**Corollary 3.3.** Let \((H, R, \Phi)\) be a finite dimensional triangular quasi-Hopf algebra over \( k \) with the Chevalley property. Then \( \text{gr}(H) \) is pseudotwist equivalent to \( kG \) for some finite supergroup scheme \( G \) over \( k \) containing \( G \) as a closed subgroup scheme. \( \square \)

**Remark 3.4.** (1) By Nagata’s theorem (see [A, p.223]), we have \( G = \Gamma \ltimes P^D \), where \( \Gamma \) is a finite group of order coprime to \( p \) and \( P \) is a finite abelian \( p \)-group. Hence, we have \( kG = k\Gamma \ltimes \mathcal{O}(P) \). Thus, \( \epsilon \in \Gamma \) is a central element of order \( \leq 2 \) acting trivially on \( \mathcal{O}(P) \).

(2) Note that \( G \) is an ordinary group scheme if and only if \( \epsilon \) is central in \( \text{gr}(H) \). Hence, if \( \epsilon = 1 \) then \( G \) is an ordinary group scheme.
3.2. Trivializing $R$. By Corollary 3.3 we can assume that $\text{gr}(H) = kG$ as superalgebras.

Pick a lift $u$ of $\epsilon$ to $H$ such that $u^2 = 1$. Note that by Lemma 2.1(1), $u$ is unique up to conjugation by $1 + \text{Rad}(H)$. Set

$$R_u := \frac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u).$$

**Proposition 3.5.** There exists a pseudotwist $J$ for $H \otimes 2$, projecting to 1 in $H[0] \otimes 2$, such that

$$(H, R, \Phi) = (H^J, R_u, \Phi^J),$$

and $J^{-1}\Delta(u)J = u \otimes u$.

**Proof.** We need to show that there exists a pseudotwist $J$ in $H \otimes 2$, projecting to 1 in $H[0] \otimes 2$, such that $J_{u}^{-1}RJ = R_u$ and $J^{-1}\Delta(u)J = u \otimes u$. Let us first find $J$ satisfying the first condition. This condition is equivalent to the identity $J^{-1}\sigma RJ = \sigma R_u$ in $k[\mathbb{Z} / 2\mathbb{Z}] \rtimes H^\otimes 2$, where $\sigma$ is the generator of $\mathbb{Z} / 2\mathbb{Z}$.

Applying Lemma 2.1(1) to $A := k[\mathbb{Z} / 2\mathbb{Z}] \rtimes H^\otimes 2$ and $e := (\sigma R_u + 1 \otimes 1)/2$, we get that $\tilde{e}_1 := (\sigma R + 1 \otimes 1)/2$ and $\tilde{e}_2 := (\sigma R_u + 1 \otimes 1)/2$ are two liftings of $e$, so there exists $L \in A$, projecting to 1 in $k[\mathbb{Z} / 2\mathbb{Z}] \rtimes H[0] \otimes 2$, such that $\tilde{e}_1 L = L \tilde{e}_2$, hence $\sigma RL = LR_u$. Writing $L = J + \sigma K$, where $J, K \in H^\otimes 2$, we find that $\sigma RJ = JR_u$ and $J$ projects to 1 in $H[0] \otimes 2$, as desired.

Thus, we may (and will) assume without loss of generality that $R = R_u$. It remains to find a pseudotwist $J$ projecting to 1 in $H[0] \otimes 2$ such that $J$ commutes with $\sigma R_u$ and $J^{-1}\Delta(u)J = u \otimes u$. To this end, note that $\tilde{e}_1 := (\Delta(u) + 1 \otimes 1)/2$ and $\tilde{e}_2 := (u \otimes u + 1 \otimes 1)/2$ are both liftings of the idempotent $e := (\epsilon \otimes \epsilon + 1 \otimes 1)/2$.

Thus Lemma 2.1(1) applied to the centralizer $A'$ of $\sigma R_u$ in $A$ yields an element $L \in A$ projecting to 1 in $k[\mathbb{Z} / 2\mathbb{Z}] \rtimes H[0] \otimes 2$ and commuting with $\sigma R_u$ such that $\Delta(u)L = L(u \otimes u)$. Writing $L = J + \sigma K$, where $J, K \in H^\otimes 2$, we find that $\Delta(u)J = J(u \otimes u)$ and $J$ commutes with $\sigma R_u$ and projects to 1 in $H[0] \otimes 2$, as desired. \hfill $\Box$

**Remark 3.6.** If $\epsilon = 1$ then $u = 1$, and we may take $J := R_{21}^{1/2} = R_{21}^{1/2}$ (which is a well-defined invertible element in $H \otimes H$ by Lemma 2.1(2)). Indeed, it is straightforward to verify that we have

$$J_{21}^{-1}RJ = R_{21}^{1/2}R_{21}^{1/2} = R_{21}R = 1.$$

**Remark 3.7.** By [V] Section 1.5, the assumption that $p \neq 2$ in Proposition 3.5 is essential, even when $\Phi = 1$. Namely, it is shown there that the tensor category $\text{Rep}(\alpha_2)$, equipped with the symmetric structure determined by the $R$-matrix $R := 1 \otimes 1 + x \otimes x$, does not admit a symmetric fiber functor to $\text{Vec}_2 = \text{Vec}$.

3.3. Trivializing $\Phi$. Let $(H, R_u, \Phi)$ be a finite dimensional triangular quasi-Hopf algebra over $k$ with the Chevalley property, where $u \in H$ satisfies $\Delta(u) = u \otimes u$, $u^2 = 1$ and projects to $\epsilon$ in $H[0]$. Let $m, \varepsilon$ denote the multiplication and counit maps of $O(G)$.

If $\Phi \neq 1$, consider $\Phi - 1$. If it has degree $r$ then let $\phi$ be its projection to $\text{gr}(H) \otimes^B r$. Then $\phi$ is even.

For every permutation $(i_1, i_2, i_3)$ of $(123)$, we will use $\phi_{i_1i_2i_3}$ to denote the 3-tensor obtained by permuting the components of $\phi$ accordingly, multiplied by $\pm 1$ according to the sign rule.
Lemma 3.8. The following hold:

(1) \[ \phi \circ (id \otimes id \otimes m) + \phi \circ (m \otimes id \otimes id) = \epsilon \otimes \phi + \phi \circ (id \otimes m \otimes id) + \phi \otimes \epsilon \]

and

\[ \phi \circ (id \otimes id \otimes 1) = \phi \circ (1 \otimes id \otimes id) = \phi \circ (id \otimes 1 \otimes id) = \epsilon \otimes \epsilon, \]

i.e., \( \phi \in Z^3(O(G),k) \) is an even normalized Hochschild 3-cocycle of \( O(G) \) with coefficients in the trivial module \( k \).

(2) \( \text{Alt}(\phi) := \phi_{312} - \phi_{132} + \phi_{123} - \phi_{231} - \phi_{321} = 0. \)

(3) \( \phi_{123} + \phi_{321} = 0. \)

Proof. (1) Follows from (2.1) and (2.2) in a straightforward manner.

(2) Follows from (2.8) in a straightforward manner.

(3) Using (2.6), (2.7) and the identity \( \Delta(u) = u \otimes u \), it is straightforward to verify that

\[ \phi_{312} - \phi_{132} + \phi_{123} = 0 \]

and

\[ -\phi_{231} + \phi_{213} - \phi_{123} = 0. \]

Therefore, by using Part (2), we get

\[ 0 = \phi_{312} - \phi_{132} + \phi_{123} - (-\phi_{231} + \phi_{213} - \phi_{123}) \]

\[ = \text{Alt}(\phi) + \phi_{321} + \phi_{123} \]

\[ = \phi_{123} + \phi_{321}, \]

as claimed. \( \square \)

Let \( G_0 \) be the even part of \( G \), and let \( g = g_0 \oplus g_1 \) be the Lie superalgebra of \( G \), where \( g_0 = \text{Lie}(G_0) \) is the Lie algebra of \( G_0 \) (see 2.4). Since \( O(G) = O(G_0) \otimes \wedge g_1^* \) as superalgebras, it follows from the Künneth formula that

\[ H^*(O(G),k) = H^*(O(G_0^*),k) \otimes H^*(\wedge g_1^*,k) = H^*(O(G_0^*),k) \otimes S^* g_1, \]

where \( G_0^* \) is the identity component of \( G_0 \). In particular, we have

\[ H^3(O(G),k) \]

\[ = H^3(O(G_0^*),k) \oplus (H^2(O(G_0^*),k) \otimes g_1) \oplus (H^1(O(G_0^*),k) \otimes S^2 g_1) \oplus S^3 g_1, \]

\[ = H^3(O(G_0^*),k) \oplus (H^2(O(G_0^*),k) \otimes g_1) \oplus (g_0 \otimes S^2 g_1) \oplus S^3 g_1. \]

Thus by Corollary 2.6, the even part \( H^3(O(G),k)_{\text{even}} \) of \( H^3(O(G),k) \) is given by

\[ H^3(O(G),k)_{\text{even}} = H^3(O(G_0^*),k) \oplus (g_0 \otimes S^2 g_1) \]

\[ = (g_0 \otimes g_0) \oplus \wedge^3 g_0 \oplus (g_0 \otimes S^2 g_1). \]

Proposition 3.9. The 3-cocycle \( \phi \) is a coboundary.
Remark 3.4 imply the following special case.

Let \( \phi \in Z^3(\mathcal{G}, k) \) in the following form:

\[
\phi = \sum_{i,j=1}^{n} a_{ij} \gamma_{ij} + \sum_{1 \leq i < j < l \leq n} b_{ijl}(x_i^* \otimes x_j^* \otimes x_l^*) + \sum_{1 \leq j \leq n} c_{ijk}(x_i^* \otimes \tau_j \otimes \tau_k) + df,
\]

for some \( a_{ij}, b_{ijl}, c_{ijk} \in k \) and even \( f \in kG^{\otimes 2} \). Since we have \( |\gamma_{ij}| = [(\gamma_{ij})_{31}], [x_i^* \otimes x_j^* \otimes x_l^*] = -[x_i^* \otimes x_j^* \otimes x_l^*] \) and \( [x_i^* \otimes \tau_j \otimes \tau_k] = -[\tau_k \otimes \tau_j \otimes x_i^*] \), it follows that

\[
\phi_{321} = \sum_{i,j=1}^{n} a_{ij} \gamma_{ij} - \sum_{1 \leq i < j < l \leq n} b_{ijl}(x_i^* \otimes x_j^* \otimes x_l^*) - \sum_{1 \leq j < k \leq m} c_{ijk}(x_i^* \otimes \tau_j \otimes \tau_k) + dg,
\]

for some \( g \in kG^{\otimes 2} \). Since \( \phi_{123} + \phi_{321} = 0 \) by Lemma 3.8 we have

\[
0 = \phi_{123} + \phi_{321} = \sum_{i,j=1}^{n} 2a_{ij} \gamma_{ij} + d(f + g),
\]

which implies that \( \sum_{i,j=1}^{n} 2a_{ij} \gamma_{ij} = 0 \). But \( p \neq 2 \), hence \( a_{ij} = 0 \) for every \( i, j \). Thus, we have

\[
\phi = \sum_{1 \leq i < j < l \leq n} b_{ijl}(x_i^* \otimes x_j^* \otimes x_l^*) + \sum_{1 \leq j \leq n} c_{ijk}(x_i^* \otimes \tau_j \otimes \tau_k) + df.
\]

Now by Lemma 3.8 \( \text{Alt}(\phi) = 0 \), and since \( kG \) is super cocommutative we have \( \text{Alt}(df) = 0 \). Hence by (3.3),

\[
0 = \text{Alt}(\phi) = \sum_{1 \leq i < j < l \leq n} b_{ijl}\text{Alt}(x_i^* \otimes x_j^* \otimes x_l^*) + \sum_{1 \leq j \leq n} c_{ijk}\text{Alt}(x_i^* \otimes \tau_j \otimes \tau_k).
\]

Finally since the tensors \( \text{Alt}(x_i^* \otimes x_j^* \otimes x_l^*) \), \( \text{Alt}(x_i^* \otimes \tau_j \otimes \tau_k) \) are linearly independent, it follows that \( b_{ijl} = c_{ijk} = 0 \) for every \( i, j, l, k \). Thus \( \phi = df \) is a coboundary, as claimed.

\[\square\]

3.4. The proof of Theorem 3.1. Let \( \Gamma := \mathcal{G}_0/\mathcal{G}_0^0 \). Then \( \Gamma \) is a finite constant group with order not divisible by \( p \). Since we have \( \mathcal{G}_0 = \Gamma \ltimes \mathcal{G}_0^0 \), it follows that \( \mathcal{O}(\mathcal{G}_0) = \mathcal{O}(\Gamma) \ltimes \mathcal{O}(\mathcal{G}_0^0) \) as algebras.

By Proposition 3.9 we have \( \phi = df \) for some even \( f \in (\mathcal{O}(\mathcal{G})^*)^{\otimes 2} \) with the same degree as \( \phi \). Since \( \phi_{123} + \phi_{321} = 0 \), it follows that \( \phi = d(f_{231}) \). Thus, replacing \( f = f_{12} + (f_{12} + f_{231})/2 \) (this is possible as \( p > 2 \)), we see that \( \phi = df \) for some even and symmetric (in the super sense) element \( f \in (\mathcal{O}(\mathcal{G})^*)^{\otimes 2} \) with the same degree as \( \phi \), i.e., \( f \) commutes with \( \sigma \mathcal{R}_u \) and \( \epsilon \otimes \epsilon \).

Choose an even symmetric lift \( \tilde{f} \) of \( f \) to \( H^{\otimes 2} \), i.e., \( \tilde{f} \) commutes with \( \sigma \mathcal{R}_u \) and \( u \otimes u \) (it is clear that such a lift exists). Then it follows from the above that the pseudotwist \( F := 1 + \tilde{f} \) is even and symmetric, so \( (H, \mathcal{R}_u, \Phi)^F = (H^F, \mathcal{R}_u, \Phi^F) \), and the pseudotwisted associator \( \Phi^F \) is equal to 1+ terms of degree \( \geq r+1 \). Also, the condition \( \Delta(u) = u \otimes u \) is preserved. By continuing this procedure, we will come to a situation where \( (H^F, \mathcal{R}_u, \Phi)^F = (H^F, \mathcal{R}_u, 1) \) for some pseudotwist \( F \in H^{\otimes 2} \), as desired.

\[\square\]

Since local quasi-Hopf algebras have the Chevalley property, Theorem 3.1 and Remark 3.3 imply the following special case.
Theorem 3.10. Let \((H, R, \Phi)\) be a finite dimensional local triangular quasi-Hopf algebra over \(k\). Then \((H, R, \Phi)\) is pseudotwist equivalent to a triangular cocommutative Hopf algebra with \(R\)-matrix \(1 \otimes 1\). \(\square\)

Remark 3.11. Note that Theorem 3.10 is equivalent to Corollary 1.3.

The following corollary is known in the Hopf case even without the symmetry assumption (see Remark 6.5 below).

Corollary 3.12. Let \(C\) be a finite symmetric unipotent tensor category over \(k\) such that \(\text{FPdim}(C) = p\). Then \(C\) is symmetric tensor equivalent to either \(\text{Rep}(\mathbb{Z}/p\mathbb{Z})\) or \(\text{Rep}(\alpha_p)\).

Proof. Follows immediately from Theorem 1.3. (See also Corollary 6.4 below.) \(\square\)

4. Triangular Hopf Algebras with the Chevalley Property

We first observe the following consequence of Theorem 3.1.

Corollary 4.1. Let \((H, R)\) be a finite dimensional triangular Hopf algebra with the Chevalley property over \(k\). Then there exists a finite supergroup scheme \(G\) over \(k\) such that \((H, R)\) is twist equivalent to \((kG, R, 1)\). If moreover \(H\) is local, then there exists a finite unipotent group scheme \(U\) over \(k\) such that \((H, R)\) is twist equivalent to \((kU, 1 \otimes 1)\).

Proof. Applying Theorem 3.1 to \((H, R, 1)\) yields the existence of a pseudotwist \(J\) for \(H\) such that \((H, R, 1^J) = (H^J, R, 1^J)\). In particular, we have \(1^J = 1\), which is equivalent to \(J\) being a twist. \(\square\)

Next we observe that Corollary 4.1 together with \cite[Corollary 6.3 & Proposition 6.7]{Ge} imply the classification of certain finite dimensional triangular Hopf algebras with the Chevalley property over \(k\).

More precisely, let \(\mathcal{T}\) be the set of all finite dimensional triangular Hopf algebras \(H\) with the Chevalley property over \(k\), such that \(\text{gr}(H)\) is the group algebra of a finite group scheme over \(k\) (see Corollary 3.3). For example, every finite dimensional triangular local Hopf algebra belongs to \(\mathcal{T}\).

Let \(L\) be any finite group scheme over \(k\). Recall that a twist \(J\) for \(kL\) is called minimal if the triangular Hopf algebra \((kL, J^{-1}2^1)\) is minimal \cite{R1}, i.e., if the left (right) tensorands of \(J^{-1}2^1\) span \(kL\).

Recall also that a 2-cocycle \(\psi : L \times L \to \mathbb{G}_m\) (equivalently, a twist \(\psi\) for \(O(L)\)) is called non-degenerate if the category \(\text{Corep}(O(L)_{\psi})\) of finite dimensional comodules over \(O(L)_{\psi}\) is equivalent to \(\text{Vec}\) (i.e., the coalgebra \(O(L)_{\psi}\) obtained from \(O(L)\) by twisting its comultiplication on one side is simple).

We are now ready to state the following classification result.

Theorem 4.2. The following three sets are in canonical bijection with each other:

1. Isomorphism classes of triangular Hopf algebras \(H\) in \(\mathcal{T}\).
2. Conjugacy classes of triples \((G, L, J)\), where \(G\) is a finite group scheme over \(k\), \(L\) is a closed group subscheme of \(G\), and \(J\) is a minimal twist for \(kL\).
3. Conjugacy classes of triples \((G, L, \psi)\), where \(G\) is a finite group scheme over \(k\), \(L\) is a closed group subscheme of \(G\), and \(\psi\) is a non-degenerate 2-cocycle on \(L\) with coefficients in \(\mathbb{G}_m\).

Remark 4.3. The following hold:
(1) The correspondence between (1) and (2) in Theorem 4.2 is given by $H = kG^J$. A non-degenerate 2-cocycle $\psi$ on $L$ as in Theorem 4.2(3) determines a module category over $\text{Rep}(G)$ of rank 1, i.e., a tensor structure on the forgetful functor $\text{Rep}(G) \to \text{Vec}$, thus a twist $J$ for $kG$ supported on $L$.

(2) Theorem 4.2 is the analog of [EG4, Theorem 5.1] for finite dimensional triangular Hopf algebras in $T$ (e.g., finite dimensional triangular local Hopf algebras).

(3) Corollary 4.1 may be used to extend Theorem 4.2 to arbitrary finite dimensional triangular Hopf algebras with the Chevalley property over $k$, once an extension of [Ge, Corollary 6.3 & Proposition 6.7] to the supercase is available. We plan to achieve this in a future publication.

5. Sweedler cohomology of restricted enveloping algebras and associators for their duals

5.1. Truncated exponentials and logarithms. Let $A$ be a finite dimensional local commutative algebra over $k$, and let $I$ be its maximal ideal. Assume that $x^p = 0$ for all $x \in I$.

**Proposition 5.1.** The following hold:

1. Let $n \geq 2$. Then there is a unique homomorphism of unipotent algebraic groups

$$E : (I^\otimes n, +) \to (1 + I^\otimes n, \times)$$

such that for decomposable $T$ we have

$$E(T) = \sum_{j=0}^{p-1} \frac{T^j}{j!}.$$ (1)

2. Let $n \geq 2$. Then the homomorphism $E$ is an isomorphism, whose inverse $L := E^{-1}$ satisfies

$$L(S) = \sum_{j=1}^{p-1} (-1)^{j-1} \frac{(S - 1)^j}{j!}$$

for every $S = E(T)$ with $T$ decomposable.

3. Let $n \geq 3$, and let $A_i := I^\otimes(i-1) \otimes A \otimes I^\otimes(n-i)$ for every $1 \leq i \leq n$. Then $E$ and $L$ can be extended to homomorphisms

$$E : A_1 + \cdots + A_n \to (1 + A_1) \cdots (1 + A_n)$$

and

$$L : (1 + A_1) \cdots (1 + A_n) \to A_1 + \cdots + A_n,$$

which are inverse to each other.

**Proof.** (1) Recall that $I^\otimes n$ is the free abelian group generated by decomposable tensors $T$ modulo the relations $a \otimes (b_1 + b_2) \otimes c = a \otimes b_1 \otimes c + a \otimes b_2 \otimes c$, where $a \in I^\otimes(i-1)$, $b_1, b_2 \in I$ and $c \in I^\otimes(n-i)$ are decomposable, $1 \leq i \leq n$. So our job is to show that

$$E(a \otimes (b_1 + b_2) \otimes c) = E(a \otimes b_1 \otimes c)E(a \otimes b_2 \otimes c).$$

Here by decomposable we mean a tensor of the form $a_1 \otimes \cdots \otimes a_n$, $a_i \in I$. 

Indeed, we have
\[ E(a \otimes (b_1 + b_2) \otimes c) = \sum_{j=0}^{p-1} \frac{(a \otimes (b_1 + b_2) \otimes c)^j}{j!} \]
\[ = \sum_{j=0}^{p-1} \sum_{l=0}^{j} \frac{a^j \otimes b_1^l b_2^{j-l} \otimes c^j}{j!} = \sum_{j=0}^{p-1} \sum_{l=0}^{j} \frac{a^j \otimes b_1^l b_2^{j-l} \otimes c^j}{(j-l)!l!} \]
\[ = \sum_{l=0}^{p-1} \sum_{i=0}^{p-1-l} \frac{a^{i+l} \otimes b_1^l b_2^i \otimes c^{i+l}}{i!} = \sum_{i=0}^{p-1} \sum_{l=0}^{p-1-i} \frac{(a \otimes b_1 \otimes c)^l (a \otimes b_2 \otimes c)^i}{i!} \]
\[ = E(a \otimes b_1 \otimes c) E(a \otimes b_2 \otimes c), \]
as desired. (Note that the equation before the last one is justified by our assumption that \( x^p = 0 \) for all \( x \in I \).)

(2) We have to show that
\[ \sum_{j=1}^{p-1} (-1)^{j-1} \frac{(E(T) - 1)^j}{j} = T \]
for every decomposable \( T \). To prove this, it suffices to check it in the ring \( k[T]/(T^p) \).

But for this it is enough to check this identity in \( \mathbb{Z}[1/(p-1)!][T]/(T^p) \) and then specialize to \( k[T]/(T^p) \) by modding out by \( p \). But for this, in turn, it suffices to establish the identity in \( \mathbb{Q}[T]/(T^p) \) (as \( \mathbb{Z}[1/(p-1)!][T]/(T^p) \subset \mathbb{Q}[T]/(T^p) \)). Finally, in \( \mathbb{Q}[T]/(T^p) \) this identity follows by truncation of the corresponding identity in \( \mathbb{Q}[[T]] \) for the usual (non-truncated) \( \text{Exp} \) and \( \text{Log} \).

Finally, it is clear that \( E \) is an isomorphism since \( \text{gr}(E) = \text{id} \).

(3) Since \( A = k \oplus I \), we can identify \( A_i \) with \( I \otimes (n-1) \oplus I \otimes n \) for every \( 1 \leq i \leq n \), and hence extend \( E \) to \( A_i \) in an obvious way (as \( n \geq 3 \)). It is easy to see that we obtain a natural isomorphism \( E : A_1 \rightarrow 1 + A_i \), which means that we have an isomorphism \( E : A_1 + \cdots + A_n \rightarrow (1 + A_1) \cdots (1 + A_n) \), as desired. The proof for \( L \) is similar. \( \Box \)

**Remark 5.2.**

1. Proposition 5.1 is false for \( n = 1 \), as \( E(a + b) \neq E(a)E(b) \) if \( a, b \in I \) and \( E \) is defined by the above formula. Moreover the groups \( I \) and \( 1 + I \) are in general not isomorphic. For example, if \( p = 2 \) and \( A = k[x]/x^3 \) then \( I = k^2 \) with usual addition, while \( 1 + I = k^2 \) with composition
\[ (a_1, b_1) \ast (a_2, b_2) = (a_1 + a_2, b_1 + b_2 + a_1a_2), \]
which are not isomorphic.

Similarly, the assumption \( n \geq 3 \) is crucial in the extension of \( E \) to \( A_1 + \cdots + A_n \).

2. If \( T \) is not decomposable then \( E(T) \) is not given by the above formula, in general. For example, if \( p = 2 \) then one has \( E(T) = 1 + T \) for decomposable \( T \), however
\[ E(T + U) = (1 + T)(1 + U) = 1 + T + U + TU, \]
which is in general not equal to \( 1 + T + U \) (where \( T, U \) are decomposable).

**Corollary 5.3.** Let \( V \) be a vector space over \( k \), and let \( A \) be the Hopf algebra \( SV/V^p \). Then we have a natural group homomorphism \( E : V^\otimes n \rightarrow Z^n(A^*, \mathbb{G}_m) \) for every \( n > 1 \), where \( Z^n \) is the space of \( n \)-cocycles.
Proof. Since the maximal ideal I of A is generated by V, we have \( I^p = 0 \). Thus by Proposition 5.1 we have a group homomorphism \( E : V^\otimes n \to 1 + I^\otimes n \) for every \( n \geq 1 \). Since \( V = P(A) \), it follows that every element \( v \) in \( V^\otimes n \), viewed as an element in \( \text{Hom}((A^\otimes n)^*, k) \), is a Hochschild \( n \)-cocycle of \( A^* \) with coefficients in \( k \). Thus, using Proposition 5.1 it is straightforward to verify that \( E(v) \) belongs to \( Z^n(A^*, \mathbb{G}_m) \) for every \( n > 1 \), as desired. \( \square \)

5.2. Sweedler cohomology of \( u(g) \). We will now use Proposition 5.1 to compute the Sweedler cohomology \([S2]\) of restricted enveloping algebras.

Let \( g \) be a finite dimensional restricted \( p \)-Lie algebra over \( k \), and let \( \Gamma \) be the finite group scheme over \( k \) such that \( \mathcal{O}(\Gamma) = u(g)^* \). Then \( \mathcal{O}(\Gamma) \) is a local commutative algebra such that \( x^p = 0 \) for every \( x \) in its augmentation ideal \( I \).

Recall that by definition, we have

\[
H^n(\Gamma, \mathbb{G}_m) = H^n_{Sw}(u(g)),
\]

where \( H^n_{Sw}(u(g)) := H^n_{Sw}(u(g), k) \) is the \( n \)-th Sweedler cohomology group of the cocommutative Hopf algebra \( u(g) = k^\Gamma \) with coefficients in the trivial \( u(g) \)-module algebra \( k \). \([S2]\). Thus, the following result gives the group \( H^n_{Sw}(u(g)) \) in terms of the much better understood group \( H^n(\Gamma, \mathbb{G}_a) \).

Recall \([S2]\) Theorem 4.3] that the Sweedler cohomology of usual and normalized cochains in positive degree is the same. Thus, in computing Sweedler cohomology and Hochschild cohomology, we may use only normalized cochains.

**Theorem 5.4.** Let \( \Gamma \) be as above. Then the following hold:

1. \( H^0(\Gamma, \mathbb{G}_m) = k^* \) and \( H^1(\Gamma, \mathbb{G}_m) = \text{Hom}(\Gamma, \mathbb{G}_m) \).
2. For every \( n \geq 2 \), the assignment\( E : Z^n(\Gamma, \mathbb{G}_m) \to Z^n(\Gamma, \mathbb{G}_m), \phi \mapsto E(\phi), \)
   is a group isomorphism, whose inverse is given by
   \[
   L : Z^n(\Gamma, \mathbb{G}_m) \to Z^n(\Gamma, \mathbb{G}_m), \Phi \mapsto L(\Phi).
   \]
3. For every \( n \geq 3 \), the assignment\
   \[
   E : H^n(\Gamma, \mathbb{G}_a) \to H^n(\Gamma, \mathbb{G}_m), [\phi] \mapsto [E(\phi)],
   \]
   is a well defined group isomorphism, whose inverse is given by
   \[
   L : H^n(\Gamma, \mathbb{G}_m) \to H^n(\Gamma, \mathbb{G}_a), [\Phi] \mapsto [L(\Phi)].
   \]

Proof. (1) Follows from the definitions.

(2) For simplicity we will prove the claim for \( n = 2 \), the proof for \( n > 2 \) being similar. Let \( \phi \) be an element in \( Z^2(\Gamma, \mathbb{G}_a) \). Since \( x^p = 0 \) for every \( x \) in the augmentation ideal \( I \) of \( \mathcal{O}(\Gamma) \) it follows from Proposition 5.1 that \( \Phi := E(\phi) \) is a well-defined invertible element of \( \mathcal{O}(\Gamma)^\otimes 2 \). We have to show that \( \Phi \) belongs to \( Z^2(\Gamma, \mathbb{G}_m) \).

Indeed, since

\[
(id \otimes \Delta)(\phi) + 1 \otimes \phi = (\Delta \otimes id)(\phi) + \phi \otimes 1,
\]

we have

\[
E((id \otimes \Delta)(\phi) + 1 \otimes \phi) = E((\Delta \otimes id)(\phi) + \phi \otimes 1),
\]

Hence by Proposition 5.1, we have

\[
(\text{id} \otimes \Delta)(\phi)E(1 \otimes \phi) = E((\Delta \otimes id)(\phi))E(\phi \otimes 1).
\]

(5.1)
Now it is straightforward to verify that

\[ E(1 \otimes \phi) = 1 \otimes \Phi, \quad E(\phi \otimes 1) = \Phi \otimes 1, \]

and

\[ E((\text{id} \otimes \Delta)(\phi)) = (\text{id} \otimes \Delta)(\Phi), \quad E((\Delta \otimes \text{id})(\phi)) = (\Delta \otimes \text{id})(\Phi). \]

Therefore, \( (5.1) \) is equivalent to

\[ (\text{id} \otimes \Delta)(\Phi)(1 \otimes \Phi) = (\Delta \otimes \text{id})(\Phi)(\Phi \otimes 1). \]

Thus \( \Phi \) belongs to \( Z^2(\Gamma, G_m) \), as desired.

The proof about \( L \) is similar.

Finally, \( E \) and \( L \) are inverse to each other by Proposition \( 5.1 \).

(3) Let \( \phi \) be an element in \( Z^n(\Gamma, G_a) \), and let \( \Phi := E(\phi) \). We have to show that \( E(df) = d(E(f)) \) for every \( f \in I \otimes (n-1) \). Then we will have

\[ E(\phi + df) = E(\phi)E(df) = \Phi d(E(f)), \]

as desired.

Indeed, \( (5.2) \) follows from our assumption that \( n-1 > 1 \) and the commutativity of \( O(\Gamma) \). For simplicity we will prove it for \( n = 3 \). The proof for \( n > 3 \) is similar.

So let \( f \in I \otimes 2 \), and set \( F := E(f) \). Then, using Proposition \( 5.1 \) we have

\[ E(df) = E((\text{id} \otimes \Delta)(f) + 1 \otimes f - (\Delta \otimes \text{id})(f) - f \otimes 1) = (\text{id} \otimes \Delta)(F)(1 \otimes F)(\Delta \otimes \text{id})(F)^{-1}(F^{-1} \otimes 1) = d(F), \]

as required.

Thus, \( E \) is a well-defined group homomorphism, as desired.

The proof that \( L \) is a well-defined group homomorphism is similar.

Finally, \( E \) and \( L \) are inverse to each other by Proposition \( 5.1 \). \( \square \)

**Remark 5.5.**

(1) The proof of Theorem \( 5.4 \)(2),(3) is similar to Sweedler’s proof for usual enveloping algebras. In fact, in characteristic \( p > 0 \) Sweedler uses truncated exponentials [S2, Theorem 4.3].

(2) The proof of Theorem \( 5.4 \) works for any Hopf algebra quotient \( H \) of \( U(\mathfrak{g}) \).

(3) Note that \( H^1(\Gamma, G_m) \not\cong H^1(\Gamma, G_a) \).

(4) Theorem \( 5.4 \)(3) fails for \( n = 2 \) since it may happen that \( E(df) \neq d(E(f)) \) for \( f \in I \) (see Remarks \( 5.2 \) \( 5.4 \)).

Let \( G := \mathbb{Z}/p\mathbb{Z} \). Recall that \( \mu_p = G^D \), i.e., \( O(\mu_p) = O(G)^* = kG \). We have \( O(G) = u(\mathfrak{g}) \), where \( \mathfrak{g} \) is the abelian restricted \( p \)-Lie algebra with basis \( x \) over \( k \) such that \( x^p = x \). Hence, Theorem \( 5.4 \) applies to \( G := \mathbb{Z}/p\mathbb{Z} = \mu_p \).

**Corollary 5.6.** Let \( B := O(\mathbb{Z}/p\mathbb{Z}) \) be the Hopf algebra of functions on \( \mathbb{Z}/p\mathbb{Z} \) with values in \( k \). Then the Sweedler cohomology of \( B \) is as follows: \( H^1(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z} \), and \( H^i(\mathbb{Z}/p\mathbb{Z}) = 0 \) for every \( i \geq 2 \).

**Proof.** Since \( (\mathbb{Z}/p\mathbb{Z})^D \) is connected, \( H^1(\mathbb{Z}/p\mathbb{Z}) = \text{Hom}(\mathbb{Z}/p\mathbb{Z})^D, G_m) = \mathbb{Z}/p\mathbb{Z} \).

Since \( H^2(\mathbb{Z}/p\mathbb{Z}) \) is the group of gauge equivalence classes of twists for \( \mathbb{Z}/p\mathbb{Z} \), the claim that \( H^2(\mathbb{Z}/p\mathbb{Z}) = 0 \) follows from [Ge, Corollary 6.9].

Finally, by Theorem \( 5.4 \) \( H^i(\mu_p, G_a) \cong H^i(\mu_p, G_m) \) for every \( i \geq 3 \). But it is well known that \( H^i(\mu_p, G_m) = 0 \) for every \( i \geq 1 \) (since \( \mu_p \) is a diagonalizable group scheme). \( \square \)
Proposition 5.7. Let \( G \) be a finite abelian \( p \)-group and \( B := \mathcal{O}(G) \) the Hopf algebra of functions on \( G \) with values in \( k \). Then the Sweedler cohomology of \( B \) is as follows: \( H^1_{\text{Sweedler}}(B) = G \) and \( H^i_{\text{Sweedler}}(B) = 0 \) for \( i \geq 2 \).

Proof. The first statement is clear, since \( H^1_{\text{Sweedler}}(B) = \text{Hom}(G^D, \mathbb{G}_m) = G \) (as \( G^D \) is connected).

To prove the second statement, we interpret \( H^i_{\text{Sweedler}}(B) \) as \( H^i(G^D, \mathbb{G}_m) \). We have \(|G| = p^n\). The proof is by induction in \( n \), starting with the base case \( n = 1 \), which has already been settled in Corollary 5.6.

Let \( K \) be a subgroup of index \( p \) in \( G \), and consider the Lyndon-Hochschild-Serre sequence for the cohomology \( H^n(G^D, \mathbb{G}_m) \) attached to the short exact sequence

\[ 1 \to (G/K)^D \to G^D \to K^D \to 1. \]

The \( E_2 \) page of this spectral sequence consists of the groups

\[ H^r(K^D, H^q((G/K)^D, \mathbb{G}_m)). \]

So it suffices to show that these groups vanish when \( r + q = i \geq 2 \). To this end, note that \( H^q((G/K)^D, \mathbb{G}_m) = H^q(\mu_p, \mathbb{G}_m) = 0 \) for \( q \geq 2 \) by Corollary 5.6. Thus it remains to show that

\[ H^i(K^D, H^0((G/K)^D, \mathbb{G}_m)) = 0 \] and \( H^{i-1}(K^D, H^1((G/K)^D, \mathbb{G}_m)) = 0 \).

The first group equals \( H^i(K^D, \mathbb{G}_m) \), which vanishes by the induction assumption. The second group is \( H^{i-1}(K^D, G/K) = H^{i-1}(K^D, \mathbb{Z}/p\mathbb{Z}) \), which vanishes for \( i \geq 2 \) since \( K^D \) is connected and \( G/K \) is discrete.

The proof of the proposition is complete. \( \square \)

Corollary 5.8. The group algebra \( kG \) of any finite abelian \( p \)-group \( G \) does not admit non-trivial twists or associators. \( \square \)

Remark 5.9. Proposition 5.7 was proved by Guillot for \( G := (\mathbb{Z}/2\mathbb{Z})^r \), \( r \geq 1 \), and \( p = 2 \) [Gu Theorem 1.1]. Note that, for \( \mathbb{Z}/2\mathbb{Z} \) with generator \( g \), we have

\[ E(d(1 + g)) = 1 + (1 + g) \otimes (1 + g) = 1 = d(E(1 + g)). \]

Let \( G := \prod_{j=1}^n \alpha_{p, r_j} \). Then \( \mathcal{O}(G) = u(g) \), where \( g \) is an \( \sum_{j=1}^n r_j \)-dimensional abelian restricted \( p \)-Lie algebra over \( k \) equipped with the \( p \)-associative power as the \( p \)-operator. Hence, Theorem 5.4 applies to \( \Gamma := G^D \).

Corollary 5.10. Let \( G := \prod_{j=1}^n \alpha_{p, r_j} \). Then for every \( i \geq 3 \), we have

\[ H^i_{\text{Sweedler}}(\mathcal{O}(G)) \cong H^i(G^D, \mathbb{G}_a). \]

More explicitly, let \( a := \text{Lie}(G^D) \) and let \( \tilde{a} \) be the filtered space associated to \( G^D \) as in Proposition 2.4. Then the following hold for every \( i \geq 3 \):

1. If \( p > 2 \), then \( H^i_{\text{Sweedler}}(\mathcal{O}(G)) \) is isomorphic to the \( i \)-th component of the graded supercommutative algebra \( \land a \otimes S'(\tilde{a}) \) equipped with its standard grading (see Proposition 2.4(1)).

2. If \( p = 2 \), then \( H^i_{\text{Sweedler}}(\mathcal{O}(G)) \) is isomorphic to the \( i \)-th component of the graded commutative algebra \( S a \otimes S'(\tilde{a})/(z^2 = \phi(z), z \in a) \) equipped with its standard grading (see Proposition 2.4(2)).

Proof. Follows from the above and Proposition 2.4. \( \square \)
Proposition 5.11. Let $\mathfrak{g}$ be a finite dimensional commutative $p$-Lie algebra over $k$, equipped with the zero $p$-power map. Then $H^2_{Sw}(u(\mathfrak{g})) \cong \wedge^2 \mathfrak{g}^*$. In particular, $H^2_{Sw}(u(\mathfrak{g})) \not\cong H^2(u(\mathfrak{g}), k)$ for $\mathfrak{g} \neq 0$.

Proof. By [Ge, Proposition 6.8], computation of $H^2_{Sw}(u(\mathfrak{g}))$ is equivalent to the classification of twists for $u(\mathfrak{g})^*$ up to gauge equivalence. Let $J$ be such a twist. If $J \neq 1$, let $r$ be the lowest degree part of $J - 1$ (say, it has degree $n$). Then $r$ is a 2-cocycle of $u(\mathfrak{g})$ with coefficients in $k$. Recall that $H^2(u(\mathfrak{g}), k) = \wedge^2 \mathfrak{g}^* \oplus (\mathfrak{g}^*)^{(1)}$ and the map $\xi$ (see Subsection 2.6). Thus $r$ can be written as

$$r = \Delta(x) - x \otimes 1 - 1 \otimes x + s + \xi(y),$$

where $x \in u(\mathfrak{g})^*[n]$, $s \in \wedge^2 \mathfrak{g}^*$, $y \in \mathfrak{g}^*$, and $s$ can be nonzero only for $n = 2$, while $y$ can be nonzero only for $n = p$. Thus by gauge transformation by $(1 + x)E(y)$, where $E(y) = \sum_{m=0}^{p-1} \frac{y^m}{m!}$, we can bring $J$ to a form in which the degree part of $J - 1$ is $s$. Hence $J' := JE(-\tilde{s})$, where $\tilde{s}$ is a preimage of $s$ in $\mathfrak{g}^* \otimes \mathfrak{g}^*$, is a twist such that $J' - 1$ has degree at least $n + 1$. (For the statement that $E(-\tilde{s})$ is a twist see also [Ge, Example 6.13].)

This argument shows that any twist $J$ is gauge equivalent to $E(\tilde{s})$ for some $\tilde{s}$. Also, by looking at the degree 2 part, it is clear that $E(\tilde{s})$ is equivalent to $E(s')$ if and only if $s = s'$, so the assignment $s \mapsto E(\tilde{s})$ is an isomorphism between $\wedge^2 \mathfrak{g}^*$ and $H^2_{Sw}(u(\mathfrak{g}))$, as desired. \hfill \Box

Remark 5.12. By [Ge, Corollary 6.10], the twist $E(\tilde{s})$ is minimal if and only if $s$ is non-degenerate.

5.3.Associators for $u(\mathfrak{g})^*$. Let $G$ be a finite commutative group scheme over $k$. Then the following lemma follows from the definitions in a straightforward manner.

Lemma 5.13. There is a bijection between $H^3(G^D, \mathbb{G}_m)$ and the group of pseudotwist equivalence classes of associators for $kG = \mathcal{O}(G^D)$. \hfill \Box

Thus, by Lemma 5.13 every normalized 3-cocycle $\Phi$ in $Z^3(G^D, \mathbb{G}_m)$ can be viewed as an associator for $kG$. We will denote the associated (cocommutative and commutative) quasi-Hopf algebra by $(kG, \Phi)$. Note that $(kG, \Phi)$ is pseudotwist equivalent to a Hopf algebra (i.e., to $kG$) if and only if $\Phi$ is a coboundary.

Corollary 5.14. Let $\mathfrak{g}$ be a finite dimensional restricted $p$-Lie algebra over $k$, and let $\Gamma$ be the finite group scheme over $k$ such that $\mathcal{O}(\Gamma) = u(\mathfrak{g})^*$. Then every associator for $u(\mathfrak{g})^*$ is of the form $E(\phi)$ for a unique $\phi \in Z^3(\Gamma, \mathbb{G}_a)$. \hfill \Box

Proof. Follows immediately from Theorem 5.4.

Example 5.15. By Corollary 5.8 the group algebra $kG$ of any finite abelian $p$-group $G$ does not admit non-trivial associators.

Example 5.16. Let $G := \prod_{i=1}^n \alpha_{p, r_i}$. By Corollary 5.10 every associator for $kG$ is of the form $E(\phi)$ for a unique $\phi \in Z^3(G^D, \mathbb{G}_a)$. In particular, every associator for $k\alpha_p^n$ is of the form $E(\phi)$ for a unique $\phi \in Z^3(\mathbb{G}_a, \mathbb{G}_a)$ since $(\alpha_p^n)^D = \alpha_p^n$.

Note that Proposition 2.3 and Example 5.10 (together with the Künneth formula) imply an explicit classification of associators for $k[\prod_{i=1}^n \alpha_{p, r_i}]$. In the next theorem we illustrate it with the simplest example.
Let
\[(5.3) \quad \Phi := 1 + \sum_{i=1}^{p-1} \frac{1}{i} \left( \frac{p-1}{i-1} \right) x \otimes x^i \otimes x^{p-i} \in k\alpha_p^\otimes 3.\]

**Theorem 5.17.** If \(\Phi'\) is a non-trivial associator for \(k\alpha_p\), then \((k\alpha_p, \Phi')\) is pseudotwist equivalent to \((k\alpha_p, \Phi)\). In particular, \((k\alpha_p, \Phi)\) is a local (cocommutative and commutative) quasi-Hopf algebra of dimension \(p\) over \(k\), which is not pseudotwist equivalent to a Hopf algebra.

**Proof.** Let \(O(\alpha_p) = k[\alpha]/(\alpha^p)\). By \((2.13)\), \(H^3(\alpha_p, k) \cong a \otimes a^{(1)}\) if \(p > 2\) (as \(\wedge^3 a = 0\), since \(a\) is 1-dimensional), and by \((2.14)\), \(H^3(\alpha_2, k) \cong S^3 a\). In particular, since \(a\) is 1-dimensional, we see that \(H^3(\alpha_p, k)\) is 1-dimensional for every \(p\). It follows that the class \(\phi\) forms a basis for \(H^3(\alpha_p, k)\), where
\[\phi := x \otimes \bar{x}(x) = \sum_{i=1}^{p-1} \frac{1}{i} \left( \frac{p-1}{i-1} \right) x \otimes x^i \otimes x^{p-i}.\]

Now, since \((x^i \otimes x^{p-i})(x^j \otimes x^{p-j}) = 0\) for every \(1 \leq i, j \leq p - 1\), it follows from Theorem \((5.3)\) that
\[\Phi_s := E(s(x \otimes \bar{x}(x))) = \prod_{i=1}^{p-1} E \left( \frac{s}{i} \left( \frac{p-1}{i-1} \right) x \otimes x^i \otimes x^{p-i} \right) = 1 + \sum_{i=1}^{p-1} \left( \frac{s}{i} \left( \frac{p-1}{i-1} \right) x \otimes x^i \otimes x^{p-i} \right)\]
is an associator for \(k\alpha_p\) for every \(s \in k\), and every associator of \(k\alpha_p\) is pseudotwist equivalent to one of this form.

Finally, observe that \(\Phi_0 = 1\), and \((k\alpha_p, \Phi) \cong (k\alpha_p, \Phi_s)\) for every \(0 \neq s \in k\). Indeed, the map \((k\alpha_p, \Phi) \rightarrow (k\alpha_p, \Phi_s)\) given by \(x \mapsto \mu x\), where \(\mu \in k\) satisfies \(\mu^{p+1} = s\), is an isomorphism of quasi-Hopf algebras.

We are done. \(\square\)

### 6. Finite unipotent tensor categories

In this section we consider finite dimensional (not necessarily triangular) local quasi-Hopf algebras and finite (not necessarily symmetric) unipotent tensor categories over \(k\).

**Theorem 6.1.** Let \(H\) be a finite dimensional local quasi-Hopf algebra over \(k\). Then the following hold:

1. There exists a unique finite connected unipotent group scheme \(U\) over \(k\) that has a \(G_m\)-action such that \(\text{gr}(H) \cong kU\) as Hopf algebras. In particular, \(\text{gr}(H)\) is a cocommutative local Hopf algebra generated by primitive elements, i.e., a quotient of an enveloping algebra.

2. \(\dim(H) = |U|\), hence \(\dim(H)\) is a power of \(p\).

**Proof.** Clearly \(I := \text{Rad}(H)\) is a quasi-Hopf ideal of \(H\), so the associated graded algebra \(\text{gr}(H) = \bigoplus_r H[r]\) is a local quasi-Hopf algebra. Since the associator of \(\text{gr}(H)\) lives in \(H[0]^\otimes 3\), it is equal to 1. Thus, \(\text{gr}(H)\) is a radically graded local Hopf algebra. Since \(H[1]\) consists of primitive elements and by \((2.23)\), \(H[1]\) generates \(\text{gr}(H)\), it follows that \(\text{gr}(H)\) is cocommutative. Therefore \(\text{gr}(H) \cong kU\) is the group algebra of some finite connected (as \(\text{gr}(H)^* \cong O(U)\) has a unique grouplike element, by locality of \(kU\)) unipotent group scheme \(U\). \(\square\)
Theorem 6.2. Let $\mathcal{C}$ be a finite unipotent tensor category over $k$. Then the following hold:

1. $\mathcal{C}$ is tensor equivalent to the representation category $\text{Rep}(H)$ of a finite dimensional local quasi-Hopf algebra $H$ over $k$.

2. $\text{FPdim}(\mathcal{C})$ is a power of $p$.

Proof. (1) Since the unit object is the unique simple object in $\mathcal{C}$, $\mathcal{C}$ is integral. Therefore, by [EO, Proposition 2.6], $\mathcal{C}$ is tensor equivalent to the representation category $\text{Rep}(H)$ of a finite dimensional quasi-Hopf algebra $H$ over $k$, and since $\mathcal{C}$ is unipotent, $H$ is local.

(2) Follows from Part (1) and Theorem 6.1.

Recall the associator $\Phi$ for $k\alpha_p$ given in (5.3).

Theorem 6.3. Let $H$ be a local quasi-Hopf algebra over $k$ of dimension $p$. Then $H$ is pseudotwist equivalent to either the Hopf algebra $k[\mathbb{Z}/p\mathbb{Z}]$, the Hopf algebra $k\alpha_p$, or the quasi-Hopf algebra $(k\alpha_p, \Phi)$.

Proof. By Theorem 6.1, $\text{gr}(H) = kU$ for a connected unipotent group scheme $U$ of order $p$. Thus $U = \alpha_p$, and we can identify $\text{gr}(H)$ with $k[x]/(x^p)$ as graded vector spaces. Since $H$ is generated by $x$ in degree 1, it is clear that $H = k[x]/(x^p)$ as algebras. Thus, $H$ is a local commutative Hopf algebra, equipped with an invertible element in $H^{\otimes 3}$ satisfying (2.1), (2.2) (as (2.9) is satisfied automatically).

Now, if $H$ has $p$ distinct grouplike elements then $H = k[\mathbb{Z}/p\mathbb{Z}]$, so the claim follows from Example 5.15.

Otherwise, $H$ is connected, hence must have a non-trivial primitive element, which implies that $H = k[x]/(x^p)$ as Hopf algebras, so the claim follows from Theorem 5.17.

Corollary 6.4. Let $\mathcal{C}$ be a unipotent tensor category over $k$ of FP-dimension $p$. Then $\mathcal{C}$ is tensor equivalent to one of the following three non-equivalent tensor categories: $\text{Rep}(\mathbb{Z}/p\mathbb{Z})$, $\text{Rep}(\alpha_p)$, or $\text{Rep}(\alpha_p, \Phi)$.

Proof. Follows from Theorems 6.2, 6.3.

Remark 6.5. Connected Hopf algebras of dimension $p^2$ and $p^3$ over $k$ were classified in [Wan1, NWW], and those with large abelian primitive spaces were classified in [Wan2]. Also, it was recently proved in [NW] that the only Hopf algebras of dimension $p$ over $k$ are $k[\mathbb{Z}/p\mathbb{Z}]$, $O(\mathbb{Z}/p\mathbb{Z})$ and $k\alpha_p$.

7. Appendix: Maximal Tannakian and super-Tannakian subcategories of symmetric tensor categories

By Kevin Coulembier and Pavel Etingof

Using cotriangular coquasi-Hopf algebras, Theorem 6.1 and Corollary 6.3 can be extended to the case when the category $\mathcal{C}$ is not necessarily finite, and it is only required that the semisimple part of $\mathcal{C}$ is an integral fusion category. Namely, we have the following theorem.

Theorem 7.1. Let $\mathcal{C}$ be a symmetric tensor category over an algebraically closed field $k$ of characteristic $p > 2$ with the Chevalley property, such that its semisimple part $\mathcal{C}_0$ is an integral fusion category. Then $\mathcal{C}$ admits a symmetric fiber functor to...
the category $sVec$ of supervector spaces. Thus, there exists a unique affine super-
group scheme $G$ over $k$ and a grouplike element $\varepsilon \in kG$ of order $\leq 2$, whose action
by conjugation on $O(G)$ is the parity automorphism, such that $C$ is symmetric ten-
sor equivalent to $\text{Rep}(G, \varepsilon)$. In particular, if $C$ is unipotent then there is a unipotent
affine group scheme $G$ such that $C \cong \text{Rep}(G)$. 

This theorem also follows from the combination of [EOV, Theorem 8.1], [E
Corollary 3.5] and [C, Proposition 6.2.2].

Since it is of independent interest, we will prove the following pro-
position, which is an improvement of [C, Proposition 6.2.2], and from which we will derive The-
orem 7.1. Furthermore, we make the proof more self-contained by providing an
alternative for the use of [E, Corollary 3.5].

From now on, we let $k$ be an algebraically closed field. Recall that a symmetric
tensor category over $k$ is called (super-)Tannakian if it admits a tensor functor to
the category of finite dimensional (super)vector spaces over $k$.

**Proposition 7.2.** The following hold:

1. Let $C$ be a symmetric tensor category over $k$. Then $C$ has a unique maximal
Tannakian subcategory $C_T$ and, if $\text{char}(k) \neq 2$, a unique maximal super-
Tannakian subcategory $C_T$.

2. If $\text{char}(k) \neq 2$ then $C_T, C_T^+$ are Serre subcategories of $C$.

The proposition will be proved in Subsection 7.2.

**Remark 7.3.** Note that Proposition 7.2(2) fails for $\text{char}(k) = 2$ (for $C_T^+$), see
Remark 3.7 above and [V, Subsection 1.5].

**Proof of Theorem 7.1.** As explained in Section 4 above, by [EOV, Theorem 8.1],
the subcategory $C_0 \subset C$ is super-Tannakian. Thus the maximal super-Tannakian
subcategory $C_T$ of $C$ contains all simple objects of $C$ and is a Serre subcategory
by Proposition 7.2(ii), which implies that $C_T = C$, i.e., $C$ is super-Tannakian, as
claimed.

7.1. **Extending quasi-fiber functors.** Let $A$ be an artinian category over $k$, see
[EGNO §1.8]. It follows easily that each simple object in $A$ has a projective cover
in the pro-completion $\text{Pro}A$. We denote by $\mathcal{P}(A)$ the full subcategory of $\text{Pro}A$ of
products of such projective covers in which each cover appears only finitely many
times. In this case, the product is also a categorical coproduct and therefore objects
of $\mathcal{P}(A)$ are projective.

**Lemma 7.4.** The following hold:

1. The assignment $P \mapsto \text{Hom}(P, -)$ yields an equivalence between $\mathcal{P}(A)^{op}$ and
the category of $k$-linear exact functors $A \to \text{Vec}$.

2. Let $V$ be semisimple artinian. For a $k$-linear exact functor $F : A \to V$
we have the induced homomorphism $\text{Gr}(F) : \text{Gr}(A) \to \text{Gr}(V)$ between the
Grothendieck groups. The map $F \mapsto \text{Gr}(F)$ is surjective onto the set of
group homomorphisms $\text{Gr}(A) \to \text{Gr}(V)$ which have non-negative coeffi-
cients with respect to the bases labelled by simple objects. The fibers of
the map are precisely the isomorphism classes of functors.

3. Let $A$ and $V$ be as in (2), with $A_0$ an abelian subcategory of $A$ containing
all simple objects. Restriction is a dense (essentially surjective) functor
from the category of $k$-linear exact functors $A \to V$ to the corresponding
category of functors $A_0 \to V$. Furthermore, when we consider the functor categories with same objects but only isomorphisms, this restriction functor is full.

Proof. Consider an exact linear functor $A \to \text{Vec}$. It follows from the artinian nature of $A$ that the functor is representable by an object in $\mathcal{P}(A)$. Part (1) thus follows from the Yoneda lemma.

We prove part (2) for the special case $V = \text{Vec}$, the general case then follows from this. By Part (1) it suffices to observe that $P \mapsto \dim \text{Hom}(P, -)$ can be interpreted as a bijection between the set of isomorphism classes of objects in $\mathcal{P}(A)$ and maps from the set of simple objects in $C$ to $\mathbb{Z}_+$. Finally we prove part (3). For $P$ in $\mathcal{P}(A)$ we denote by $P_0$ its maximal quotient contained in $\text{Pro} A_0$. In particular, $P$ is the projective cover of $P_0$ in $\text{Pro} A$. Restriction of exact linear functors from $A$ to $A_0$ corresponds, using the equivalence in part (1), to mapping $\text{Hom}(P, -)$ to $\text{Hom}(P_0, -)$. It follows from the defining properties of projective objects that the functor $P \mapsto P_0$ is full and dense between the respective categories of projective objects and isomorphisms. □

For functors $F : C \to C'$ between tensor categories, we will be interested in isomorphisms

$$J_{XY} : F(X) \otimes F(Y) \cong F(X \otimes Y), \quad \text{for all } X, Y \in C,$$

natural in both variables. Following [EGNO, Definition 5.1.1], for tensor categories $C, V$ with $V$ semisimple, a quasi-tensor functor $F : C \to V$ is a $k$-linear, exact and faithful functor with $F(1) \cong 1$, equipped with $J$ as in (7.1).

**Proposition 7.5.** Consider a tensor category $C$ with a tensor subcategory $C_0$ containing all simple objects of $C$, and a semisimple tensor category $V$. Then each quasi-tensor functor $(F_0, J_0)$ from $C_0$ to $V$ extends to a quasi-tensor functor $(F, J)$ from $C$ to $V$.

Proof. That $F_0$ extends to a linear exact functor $F$ follows from Lemma 7.4(3). Faithfulness and $F(1) \cong 1$ are automatically inherited from $F_0$.

Now we will use Deligne’s tensor product $A \boxtimes A'$ of artinian categories $A$ and $A'$, see e.g. [EGNO, §1.11]. By construction, $A \boxtimes A'$ is again an artinian category and we have a bilinear bifunctor $A \times A' \to A \boxtimes A'$ which induces equivalences between the category of exact $k$-linear functors $A \boxtimes A' \to V$ and the category of biexact bifunctors $A \times A' \to V$. It thus follows that we can identify $J$ as in (7.1) with a natural isomorphism between the two functors $C \boxtimes C \to V$ which correspond to

$$C \times C \xrightarrow{F \times F} V \times V \xrightarrow{\otimes} V \quad \text{and} \quad C \times C \xrightarrow{\otimes} C \xrightarrow{F} V.$$

Applying this to $J_0$ yields an isomorphism between the two functors on $C_0 \boxtimes C_0$ induced from $F_0$. That the isomorphism corresponding to $J_0$ lifts to the functors on $C \boxtimes C$ follows again from Lemma 7.4(3). □

**Remark 7.6.** Consider tensor categories $C, V$, with $V$ semisimple. For an exact functor $F : C \to V$ there exists $J$ as in (7.1) if and only if $\text{Gr}(F) : \text{Gr}(C) \to \text{Gr}(V)$ is a ring homomorphism. This can be proved using the techniques from the proof of Proposition 7.5.

Recall the notion of locally semisimple tensor categories from [C, Section 3],
Corollary 7.7. Assume $\text{char}(k) \neq 2$ and let $\mathcal{C}$ be a tensor category over $k$ with a Tannakian or super-Tannakian subcategory $\mathcal{C}_0$ which contains all simple objects. Then $\mathcal{C}$ is locally semisimple.

Proof. Consider the setting of Proposition 7.5 with $\mathcal{V} = \text{Vec}$ or $\mathcal{V} = \text{sVec}$. It suffices to show that when $(F_0, J_0)$ is a tensor functor, and hence a (super) fiber functor, $\mathcal{C}$ is locally semisimple. Consider an arbitrary object $Y$ in $\mathcal{C}$ with a Loewy filtration. Hence $\text{gr}Y$ is in $\mathcal{C}_0$. It suffices to show that the canonical epimorphism 

$$S_n(\text{gr}Y) \rightarrow \text{gr}(S_nY)$$

between symmetric powers is always an isomorphism.

First assume that $\mathcal{C}_0$ is Tannakian. Assume the contrary, i.e., that there exists $Y \in \mathcal{C}$ which does not yield an isomorphism. Since $F: \mathcal{C} \rightarrow \text{Vec}$ from Proposition 7.5 is faithful, this means that

$$H_0(S_n, \text{gr}M) \rightarrow \text{gr}H_0(S_n, M),$$

with $M := F(Y \otimes^n)$, is not an isomorphism. Here $M$ is a (filtered) $k[S_n]$-module through

$$S_n \rightarrow \text{End}(Y \otimes^n) \rightarrow \text{End}_k(M)$$

and, since $F_0$ is symmetric monoidal, the graded module $\text{gr}M$ is isomorphic to $V \otimes^n$, for the vector space $V := F(\text{gr}Y)$. Consequently $\text{gr}M$ is a direct sum of permutation modules. From Shapiro's lemma and Mackey’s theorem, it follows that there are no first extensions between permutation modules when $\text{char}(k) \neq 2$. Hence $\text{Ext}^1_{S_n}(\text{gr}M, \text{gr}M) = 0$, which means that $\text{gr}M \cong M$. The latter isomorphism contradicts the fact that the dimension of their spaces of coinvariants would differ.

Now assume that $\mathcal{C}_0$ is super-Tannakian. We can then proceed as in the previous paragraph to find a filtered $S_n$-representation $M$ in $\text{sVec}$. Now $\text{gr}M$ will be of the form $V \otimes^n$ for a super vector space $V$. Hence $M$ is a direct sum of permutation modules and sign-twisted permutation modules. If $\text{char}(k) \notin \{2, 3\}$ it follows as before that such representations have no self-extensions and therefore $\text{gr}M \cong M$. The latter isomorphism contradicts the fact that the dimension of their spaces of coinvariants would differ.

Now assume that $p = 3$, one can calculate directly that $\text{Ext}^1_{S_3}(V \otimes^3, V \otimes^3) = 0$, for any $V \in \text{sVec}$, where the extension is taken in the category of super representations of $S_3$. The latter decomposes into the direct sum of two copies of the ordinary category $\text{Rep}(S_3)$. We can therefore conclude as above that $S^3\text{gr}Y \rightarrow \text{gr}S^3Y$ is always an isomorphism. That this is sufficient for $\mathcal{C}$ to be locally semisimple follows from [C, Theorem C].

□

7.2. Proof of Proposition [7.2]. We start with the following lemma, which appears in the Tannakian case as [B, Proposition 1] and can be proved similarly in the super-Tannakian case.

Lemma 7.8. Let $\mathcal{D}, \mathcal{E}$ be symmetric tensor categories over $k$ and $F: \mathcal{D} \rightarrow \mathcal{E}$ a surjective symmetric tensor functor (i.e., every object of $\mathcal{E}$ is a subquotient of $F(X)$ for some $X \in \mathcal{D}$). If $\mathcal{D}$ is finitely tensor-generated and (super-)Tannakian, then so is $\mathcal{E}$.

Remark 7.9. [B, Proposition 1] addresses a more general case when $\mathcal{E}$ and $F$ are not necessarily symmetric, and assumes that $F$ is a central functor. It applies to our situation because a symmetric monoidal functor is automatically central. Also, in our setting we do not need [B, Lemma 3], which is needed in [B, Proposition 1] because it is not assumed there that $\mathcal{E}$ is rigid.
Let $\mathcal{C}_T^+ \subset \mathcal{C}$ be the full subcategory of $X \in \mathcal{C}$ which tensor generate a Tannakian subcategory $\mathcal{C}(X) \subset \mathcal{C}$. By definition, $\mathcal{C}_T^+$ is closed under subquotients. Also, if $X, Y \in \mathcal{C}_T^+$ then we have a surjective symmetric tensor functor $\mathcal{C}(X) \boxtimes \mathcal{C}(Y) \to \mathcal{C}(X \otimes Y)$, where $\boxtimes$ is the Deligne tensor product. Hence by Lemma 7.8, $\mathcal{C}(X \oplus Y)$ is Tannakian, i.e., $X \oplus Y \in \mathcal{C}_T^+$ (as $X \otimes Y$ is a direct summand in $(X \oplus Y) \otimes 2$). Thus, $\mathcal{C}_T^+$ is a tensor subcategory of $\mathcal{C}$, in which every object generates a Tannakian subcategory. Hence by a result of Deligne (see [C, A.2.3 and A.4.1]), $\mathcal{C}_T^+$ is Tannakian. By definition, $\mathcal{C}_T^+$ contains every Tannakian subcategory of $\mathcal{C}$, so $\mathcal{C}_T^+$ is the unique maximal Tannakian subcategory of $\mathcal{C}$.

Similarly, for char($k$) $\neq 2$ the full subcategory $\mathcal{C}_T \subset \mathcal{C}$ of $X \in \mathcal{C}$ which tensor generate a super-Tannakian subcategory $\mathcal{C}(X) \subset \mathcal{C}$ is the unique maximal super-Tannakian subcategory of $\mathcal{C}$.

(2) Let $\mathcal{C}_T^0$ be the smallest Serre subcategory of $\mathcal{C}$ containing $\mathcal{C}_T$. By Corollary 7.7, the tensor category $\mathcal{C}_T^0$ is locally semisimple. Hence by [C, Proposition 6.2.2], the maximal super-Tannakian subcategory of $\mathcal{C}_T^0$ (which is clearly $\mathcal{C}_T$) is a Serre subcategory and hence $\mathcal{C}_T = \mathcal{C}_T^0$.

The proof that $\mathcal{C}_T^+$ is a Serre subcategory of $\mathcal{C}$ is similar, using [C, Proposition 6.1.2].

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