VALLEY METHOD VERSUS
INSTANTON-INDUCED EFFECTIVE
LAGRANGIAN UP TO $(E/E_{\text{sphaleron}})^{8/3}$.

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Abstract

We compare the two most popular approaches to the problem of instanton-
antiinstanton interaction at high energies - the valley method and the effective-
Lagrangian approach - and use them to calculate the next-to-next-to-leading
term in the expansion of "holy grail" function determining the cross section
with baryon number violation in the Standard Model.

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1. Introduction

In the last three years there was a surge of interest in instanton-induced processes leading to baryon number violation (BNV) in the standard model. Although the very existence of instanton-induced baryon (and lepton) number violation was known from the pioneering work of t’Hooft [1] these processes were considered to be of only academic interest since the predicted probabilities all seemed to be of the order $\exp(-16\pi^2/g_w^2) \sim 10^{-170}$. This scale is set by the square of the Gamow factor corresponding to instanton tunneling for a potential barrier of height $E_{\text{spha}} \sim m_W/\alpha_W \sim 10$ TeV ($m_W$ is the $W$ boson mass and the subscript $\text{spha}$ stands for ‘sphaleron’ which is the classical configuration leading over the top of the barrier).

The situation changed, when Ringwald [2] suggested that the barrier penetration probability could be strongly enhanced for collisions of particles with energies comparable to the barrier height. It is even hoped that BNV might be observable at SSC. Up to now this suggestion could be neither confirmed nor rejected (although there are arguments both pro [3,4] and contra[5-7]) since no technique is known to reliably calculate instanton-induced processes at sphaleron energies. What has been done (and what we continue to do in this contribution) is to calculate these cross sections at small energies $\ll E_{\text{spha}}$. In doing so one hopes that some way can be found to extract from the results also the high energy behaviour (similar to e.g. the summation of leading logs in perturbation theory). At small energies $E \ll E_{\text{spha}}$ the BNV cross section turns out to be determined by the instanton-antiinstanton interaction at large separations which is given by the well-known dipole-dipole formula [8]. Indeed, due to the optical theorem the BNV cross section is determined by the imaginary part of the forward scattering amplitude in the background of the instanton-antiinstanton ($I\bar{I}$) configuration. The relevant exponential term is

$$\sigma_{\text{BNV}} \sim \text{Im} \int d\rho_1 d\rho_2 dR du \exp \left( E R_0 - \frac{16\pi^2}{g^2} \right.$$  
$$+ \frac{32\pi^2}{g^2 R^6} (4(u \cdot R)^2 - R^2) \rho_1^2 \rho_2^2 - \pi^2 v^2 (\rho_1^2 + \rho_2^2) \right)$$  

(1.1)

where $\rho_1$ and $\rho_2$ are the $I$ and $\bar{I}$ sizes, $R$ is the separation and $u$ is the 4-vector determining the SU(2) matrix of relative $I\bar{I}$ orientation in isospin space. The
origin of the exponential terms in eq.(1) is the following:

(i) ER\textsubscript{0} comes from the initial (and final) particles (the Euclidean calculation of the $I\bar{I}$ contribution should be performed at imaginary energy in order to obtain the necessary imaginary part, see refs. [9,10]).

(ii) The second term is twice the classical action of an instanton (which is exactly the WKB suppression factor discussed above).

(iii) The third term is the dipole-dipole $I\bar{I}$ interaction potential.

(iv) The last term is the classical action due to the Higgs component of the instanton (it is this term which makes the $\rho$ integrals convergent unlike in QCD).

The integral in eq.(1.1) is dominated by the saddle point

$$\vec{R}_* = 0 \ , \ \ u^*_\mu \parallel R^\mu \ , \ \ R_{0*} = \frac{\epsilon^{1/3} \sqrt{6}}{m_W} \ , \ \ \rho_{*1} = \rho_{*2} = \sqrt{\frac{3}{2}} \frac{\epsilon^{2/3}}{m_W}$$ (1.2)

such that with exponential accuracy

$$\sigma_{BNV} \sim \exp\left(-\frac{16\pi^2}{g_W^2} (1 - \frac{9}{8} \epsilon^{4/3})\right)$$ (1.3)

where we have used the standard notation $\epsilon = E/E_0$, $E_0 = \sqrt{6} m_W / \alpha_W \sim 17$ TeV. This is the result of ref. [2] (but with the correct numerical coefficient [9,11]) which lead to so much enthusiasm since the Gamow factor is cancelled exactly at SSC energies! Unfortunately eq.(1.3) is only valid at low energies $E \ll E_{\text{sph}}$ and has to be modified at higher energies. It is convenient to introduce the so-called ‘holy grail’ function $F(\epsilon)$ determining the BNV cross section with exponential accuracy

$$\sigma_{BNV} \sim \exp\left(-\frac{16\pi^2}{g_W^2} F(\epsilon)\right)$$ (1.4)

(the form of eq.(1.4) is fixed by dimensional considerations [9]). Then eq.(1.3) gives the first two terms in the expansion of $F(\epsilon)$ for small $\epsilon$. Up to now one additional term, proportional to $\epsilon^2$ had been calculated and in this paper we determine the fourth term

$$F(\epsilon) = 1 - \frac{9}{8} \epsilon^{4/3} + \frac{9}{16} \epsilon^2 - \frac{3}{32} (4 - 3 \frac{m_H^2}{m_W^2}) \epsilon^{8/3} \ln \frac{1}{\epsilon} + O(\epsilon^{8/3} \cdot \text{const})$$ (1.5)
where $m_H$ is the Higgs mass which we consider to be of order $m_W$. The expansion in powers of $\epsilon^{2/3}$ reflects an expansion of the $\bar{I}I$ interaction potential in powers of $\rho^2/R^2_s (= \epsilon^{2/3}/4, \text{see eq.(1.2)}$. Of course, the main goal is to find the holy grail function at large $\epsilon \geq 1$ and see whether it comes close to zero as advocated in refs. [3,4] or is bounded from below by $\frac{1}{2}$ (the one instanton action) as suggested in refs. [6,7]. In the former case BNV might be observable at SSC while in the latter case it stays exponentially small. However, without a breakthrough in the calculations of $F(\epsilon)$ at $\epsilon \sim 1$ one can only attack this problem by calculating more terms at small $\epsilon$ and extrapolating these results to higher and higher energies.

At small $\epsilon$ we face the familiar situation that the $\bar{I}I$ separation is much greater than the instanton sizes ($\rho^2/R^2_s = \frac{1}{4} \epsilon^{2/3}/4$, see eq.(1.2)). There is a number of approaches to describe the instanton-antiinstanton interaction at large separations and all were applied to calculate the $\epsilon^{4/3}$ term in the expansion of $F(\epsilon)$ in eq.(1.5). It was found first in ref.[2] by direct summation of $2 \to N$ BNV-amplitudes in the instanton background. Soon after this it was realized that the optical theorem relates the $\epsilon^{4/3}$ term to the $\bar{I}I$ dipole-dipole interaction at large distances [11]. The connection between these two calculations can be explained most easily using the instanton-induced effective Lagrangean [11,12].

\[
\begin{align*}
\text{L}_{\text{eff}} &= \int dx \int \frac{d\rho}{\rho^5} \int du \ d(\rho) \ \exp(-2\pi^2 \rho^2 \vec{\phi}(x) \phi(x)) \\
&\left\{ \exp\left(\frac{2\pi^2 i}{g} \rho^2 Tr\{\sigma_\alpha \bar{\sigma}_\beta \ G_{\alpha\beta}(x)\}\right) + \exp\left(\frac{2\pi^2 i}{g} \rho^2 Tr\{u \bar{\sigma}_\alpha \sigma_\beta \bar{u} \ G_{\alpha\beta}(x)\}\right) \right\} \tag{1.6}
\end{align*}
\]

where the first term in braces correspond to instanton and the second to $I$. Here $\rho$ and $x$ are size and center of would-be (anti)instanton, $u \equiv u_\mu \sigma_\mu$ (respectively $\bar{u} \equiv u_\mu \bar{\sigma}_\mu$) is the matrix of relative $\bar{I}I$ orientation, and $d(\rho)$ is the usual instanton density given by the $\exp(-\frac{8\pi^2}{g^2_W})$ times quantum determinant in the instanton background. We use the four-dimensional Pauli matrices $\sigma_\mu = (1, -i \vec{\sigma}), \ \bar{\sigma}_\mu = (1, i \vec{\sigma})$ related to the t’Hooft $\eta$-symbols by

\[
\sigma_\mu \bar{\sigma}_\nu = \delta_{\mu\nu} + i \eta_{\mu\nu}^a \tau^a, \quad \bar{\sigma}_\mu \sigma_\nu = \delta_{\mu\nu} + i \eta_{\mu\nu}^a \tau^a \tag{1.7}
\]

In principle, the effective Lagrangean contains an infinite series of operators with growing dimension (see section 4). However, up to order $\epsilon^{8/3} \ln \epsilon$ we shall need only the terms given in eq.(1.6).
Using this Lagrangean, the instanton (in the singular gauge) can be depicted as a local vertex from which an arbitrary number of W bosons and Higgs particles can emerge. Every W boson provides a factor $\rho^2/g$ and every Higgs a factor $\rho$ (or $\rho^2 v$ after the usual shift $\phi \rightarrow \phi + v/\sqrt{2}$). After contracting the W’s emitted by the effective vertex (6) with the usual perturbative vertices the Green functions in the instanton background (and the instanton field itself) are obtained as power series in $\rho^2$. This is illustrated in Fig.1a for the classical field and in Fig.1b for the Green function. (Gauge fields are depicted as wavy lines, Higgs fields as plain ones.) Note that due to cancellations between the factors $g$ and $\rho^2/g$ each loop adds a factor $\rho^2$ rather than $g^2$. The dipole-dipole $\bar{I}I$ interaction is then given by the sum of diagrams shown in Fig.2. This sum corresponds to the exponentiation of the first non-trivial diagram which can be easily calculated giving the dipole-dipole interaction potential

$$\frac{2\pi i}{g} \rho_1^2 \text{Tr} \left\{ u \sigma_\alpha \sigma_\beta \bar{u} \tau^a \right\} \left( \frac{2\pi i}{g} \rho_2^2 \text{Tr} \left\{ \sigma_\mu \bar{\sigma}_\nu \tau^b \right\} \frac{G^{a \alpha}_\nu(R)G^{b \mu}_\rho(0)}{2} \right)$$

$$= \frac{32\pi^2}{g^2} \rho_1^2 \rho_2^2 \frac{(u \cdot R)^2}{R^2} - 1 + O(m_W^2 R^2)$$

(1.8)

The BNV cross section in eq.(1.1) is obtained by continuing to Minkowski energies and taking the imaginary part in the saddle point (1.2). On the other hand, this imaginary part can be taken at the level of the diagrams in Fig.2. Since in this order in $\epsilon$ the only effect of incoming particles is to provide energy we obtain exactly the same series of $2 \rightarrow N$ cross sections summed up in ref.[2].

The next-to-leading term in the expansion (1.5) $\sim \epsilon^2$ was first calculated in ref.[10] with the valley method [13-15]. Using the approximate conformal invariance of the pure gauge theory at tree level it could be shown that the gauge part of the $\bar{I}I$ interaction potential depends only on the conformal parameter

$$\zeta = \frac{R^2 + \rho_1^2 + \rho_2^2}{\rho_1 \rho_2}$$

(1.9)

Since $U^\text{gauge}_{\text{int}}(\zeta)$ is expanded in powers of $\zeta^2$ the dipole-dipole term $\left(32\pi^2/g^2 \zeta^2 \right)(4 \cos^2 \phi - 1)$ contributes to the next-to-leading power in $\rho^2/R^2$. After adding the simple part of $U_{\text{int}}$ which is due to the Higgs field it gives the $\epsilon^2$ term in the expansion of $F(\epsilon)$. Later this result was reproduced by direct calculations...
of the amplitudes in the instanton background [16-18] which in the language of an effective Lagrangean correspond to the diagrams shown in Fig.3. The first of the diagrams (Fig.3a) gives the term proportional to \( \rho^6/(g^2 R^6) \) in the expansion of the gauge part of \( U_{\text{int}} \). It coincides with the one obtained from the conformal valley. The second diagram (Fig.3b) gives the mass correction to the dipole-dipole term which is of order \( (m^2 \rho^4/(g^2 R^6)) \) (see eq.(1.7)) and the last diagram describes the part of the \( \bar{I}I \) potential which is due to Higgs exchange and proportional to \( v^2 \rho^4/R^2 \). For the saddle point (2) all these diagrams contribute to the order \( \epsilon^2 \) and their sum reproduces the valley result.

As to the last term in eq.(1.5) which is proportional to \( \epsilon^{8/3} \ln \epsilon \), there exists a calculation [19] for the gauge part of \( U_{\text{int}} \) \((\sim (\rho^8/g^2 R^8 \ln(R/\rho)) \) using the effective Lagrangean. In this calculation many two-loop diagrams had to be summed. All of them are listed in section 5, Fig. 4a shows typical examples. It should be emphasized, that unlike for lower orders (up to \( \rho^6/R^6 \)) in this order the effective Lagrangean result [19] and the conformal valley result [14,15] for the gauge part of \( U_{\text{int}} \) do not coincide. The reason is that in the conformal valley approach, starting at order \( \rho^8/R^8 \), the contribution from the gauge part of \( U_{\text{int}} \) depends on the specific choice for the valley, a dependence which is canceled by the Higgs part of \( U_{\text{int}} \) (see below). Similarly for the effective Lagrangean approach, the gauge part of \( U_{\text{int}} \) should be supplemented by a \( \epsilon^{8/3} \) contribution from diagrams containing Higgs bosons (an example is shown in Fig. 4b) and from mass corrections to the diagrams in Fig.3a. If all contributions are properly included both approaches give the same \( \epsilon^{8/3} \ln \epsilon \) term.

It is very important that the term proportional to \( \epsilon^{8/3} \) is the last one in equation (1.5) which is known to be determined exclusively by the \( \bar{I}I \) interaction. Starting at order \( \epsilon^{10/3} \) the situation becomes unclear. From this order on hard-hard corrections due to exchanges between the initial or final particles enter the game. It was argued in ref.[20] (see, however, the recent ref. [21]) that these corrections could exponentiate such that \( F(\epsilon) \) would depend on the initial and final states and not on \( U_{\text{int}} \) alone. Then \( F(\epsilon) \) could depend on the specific BNV process.

For the same reason the \( \epsilon^{8/3} \) term is the last one obtainable by analytic continuation of an Euclidian calculation for the forward scattering amplitude in the \( \bar{I}I \) background. The reason is that the intermediate states with and without BNV lead to two discontinuities (in energy) of the hard-hard cor-
rections (see Fig. 5a and 5b). Both contribute to the imaginary part of the analytically continued euclidean diagram, but only Fig.5a contributes to the BNV cross section (a detailed discussion can be found in ref.[22]). Up to order $\epsilon^{8/3}$ the corresponding diagrams contain no hard-hard corrections and thus have only the one discontinuity (of the type shown in Fig.5a) which can be obtained from the imaginary part of the continued Euclidian diagrams.

2. Valley method

The saddle-point gaussian approximation is the usual technique to calculate functional integrals in weak coupling theories. The valley method [13] constitute a generalization to cases in which physically relevant ‘approximate solutions’ can be given. The case of an instanton-antiinstanton pair at large separation is a typical example. To illustrate the idea we consider quantum mechanics in a double-well potential with the instanton being the simple kink

$$\phi_I = \frac{1}{2}(\text{th} \frac{\alpha - t}{2} + 1) \quad \phi_{\bar{I}} = \frac{1}{2}(\text{th} \frac{t + \alpha}{2} + 1) \quad (2.1)$$

describing the tunneling between the two minima. We want to calculate the non-perturbative part of the vacuum energy

$$Z = N^{-1} \int D\Phi \exp \left\{ -\frac{1}{g^2} \int dt \frac{1}{2}[(\dot{\Phi})^2 + \Phi^2(1 - \Phi^2)] \right\} \quad (2.2)$$

which at small $g^2$ is dominated by the $I\bar{I}$ configurations [24]. For infinitely large separations the $I\bar{I}$ configuration is just the sum of two kinks. It obeys the classical field equations and possesses two zero modes corresponding to the independent translations of both instantons. The zero modes can be re-diagonalized in such a way that one of them describes the trivial translations of the complete $I\bar{I}$ configuration and the other changes in the instanton separation. For large but finite separations (as compared to the instanton size which we chose as 1 for this toy model) the second one becomes a quasizero mode: the action varies slowly along this direction in functional space but grows rapidly in orthogonal directions, which correspond to changes in the instanton profile and are associated with normal modes. As a landscape in functional space this looks like a steep canyon with the course of the valley corresponding to the quasizero mode (see Fig.6).
In order to integrate over the field configurations close to the $\bar{I}I$ valley one has to perform the following steps: (i) determine the course of the valley in the functional space, (ii) perform the Gaussian integrations in the directions orthogonal to this valley, and (iii) carry out the final integration along the valley. In step (i) one determines the valley as the trajectory in functional space which minimizes the action. For any direction orthogonal to the valley the constraint
\[(\Phi - \Phi_v, \omega(\alpha) \frac{\partial \Phi_v}{\partial \alpha}) = 0 \quad (2.3)\]
must be fulfilled, where $(f, g)$ denotes the usual scalar product of functions $\int dt f(t)g(t)$ and $\omega(\alpha, t)$ is a suitable weight function. Applying the standard technique of Lagrange multipliers this leads to the following classical constraints (the ‘valley equations’ [13])
\[\frac{\delta S_v}{\delta \Phi(t)} \bigg|_{\Phi = \Phi_v} = \chi(\alpha) \omega(\alpha, t) \frac{\partial \Phi_v(\alpha, t)}{\partial \alpha} \quad (2.4)\]
where $\chi(\alpha)$ is the Lagrange multiplier. This equation has to be solved with the boundary condition that for infinite $\alpha \Phi_v$ approaches the field of an infinitely separated $\bar{I}I$ pair.
\[\Phi_v(\alpha, t) \rightarrow \Phi_I(t - \alpha) + \Phi_I(t + \alpha) - 1 \quad (2.5)\]
As the valley action increases monotonously with $\alpha$
\[\frac{\partial S_v(\Phi(\alpha))}{\partial \alpha} = \chi(\alpha) \left( \frac{\partial \Phi_v}{\partial \alpha}, \omega \frac{\partial \Phi_v}{\partial \alpha} \right) \geq 0 \quad (2.6)\]
we can conclude that $\alpha = 0$ corresponds to the classical perturbative vacuum and that $S$ reaches a finite value for large $\alpha$ only if $\chi(\alpha) \to 0$, i.e. for a classical solution (see eq. (2.4)). Generally speaking the valley always interpolates between two classical solutions, in our case the $\bar{I}I$ configuration and the vacuum.

In order to integrate over the orthogonal Gaussian modes (step (ii)) we use the standard Faddeev-Popov trick and insert
\[1 = -\int d\alpha \delta(\Phi - \Phi_v, \omega \Phi_v') \{(\Phi_v, \omega \Phi_v') - (\Phi - \Phi_v, (\omega \Phi_v')')\} \quad (2.7)\]
with $\Phi_v' \equiv d\Phi_v/d\alpha$. (Strictly speaking an additional $\delta$-function, $\delta((\Phi - \Phi_v, d\Phi_v/dt))$ is needed to exclude total translations from the integral (zero mode). This adds some technical complexity without changing the arguments [13].) Then we shift $\Phi$ to $\Phi + \Phi_v$ and expand the action in powers of $\Phi$

$$S(\Phi + \Phi_v) = S(\Phi_v) + (\Phi, J_v) + \frac{1}{2}(\Phi, \Box_v \Phi) + O(\Phi^3) \quad (2.8)$$

where $J_v = \delta S/\delta \Phi_v$ and $\Box_v = -\partial^2 + 1 - 6\Phi_v + 6\Phi_v^2$ is the operator of the second derivative of the action.

Now comes the central point: the linear term $(\Phi, J_v)$ in the expansion (2.8) vanishes due to the factor $\delta(\Phi, \Phi_v')$ in the integrand and the valley equation (2.3).

Thus $\Phi_v$ enters the functional integral like a classical solution (for which $J = 0$).

$$N^{-1} \int d\alpha \left( \Phi_v, \omega \Phi_v' \right) e^{-\frac{1}{g^2}S(\Phi_v)} \int D\Phi \delta(\Phi, \omega \Phi_v') e^{-\frac{1}{2g^2}(\Phi, \Box_v \Phi)} \left( 1 + O(g^2) \right) \quad (2.9)$$

All effects $\sim 1/g^2$ originate from the classical action. Quantum corrections come from the terms of order $\Phi^3$ in eq.(2.8) and from the collective coordinate Jacobian (2.7).

Finally in step (iii) the explicit integration over the valley parameter $\alpha$ must be performed. In this case it gives the non-perturbative part of the vacuum energy, see ref. [13].

The crucial point of the whole procedure is the vanishing of the linear term $(\Phi, J)$, which will occur for any weight function $\omega(\alpha, t)$. Hence, at first sight, any valley starting from infinitely separated instantons and antiinstantons seems appropriate. The fact that some choices are worse than others shows up in the size of the quantum corrections. A ‘good’ valley should minimize them. The standard recipe to find such a valley is the following: start from infinitely separated instantons and follow the direction of the negative quasizero mode of the operator $\Box_v$ (see the discussion in ref.[22]).

$$\Phi_v \xrightarrow{\alpha \to \infty} \frac{1}{2} \text{th} \frac{t + \alpha}{2} - \frac{1}{2} \text{th} \frac{t - \alpha}{2}$$

$$\Phi_v' \xrightarrow{\alpha \to \infty} \Phi_\infty \sim \text{ch}^{-2} \frac{t + \alpha}{2} + \text{ch}^{-2} \frac{t - \alpha}{2} \quad (2.10)$$

All valley satisfying eq.(2.10) are appropriate to calculate the nonperturbative part of the vacuum energy. (Of course, the final answer obtained after
integrating over the valley parameter $\alpha$ in (2.9) is the same for all valleys.) The simplest choice for such a valley is the sum of the kinks
\[
\Phi_v = \frac{1}{2} \tanh \frac{t + \alpha}{2} - \frac{1}{2} \tanh \frac{t - \alpha}{2}
\] (2.11)
which trivially satisfies the condition (2.10) and obeys the valley equation (2.4) with the weight function
\[
\omega(\alpha, t) = \frac{e^{\alpha \sinh(\alpha)}}{4 \cosh(t) \cosh(\alpha)} + 1 (2.12)
\]
The corresponding Lagrange multiplier is
\[
\chi(\alpha) = \frac{12}{\zeta^2}, \quad \zeta = e^\alpha (2.13)
\]
and the valley action equals
\[
S_v \equiv S(\Phi_v) = \frac{6\zeta^2 - 14}{(\zeta - 1/\zeta)^2} - \frac{17}{3} + \left[ \frac{(5/\zeta - \zeta)(1/\zeta + 1/\zeta)^2}{(\zeta - 1/\zeta)^3} + 1 \right] \ln \zeta (2.14)
\]
For $\alpha = 0$, $\zeta = 1$ this expression gives zero (perturbative vacuum) and with increasing $\alpha$ it approaches $1/3$ which is twice the instanton action. Here $1/\zeta$ is the small parameter corresponding to an expansion of the $II$ interaction at large separations. (The conformal transformation will turn $\zeta$ into the expression (1.9).) The leading terms in the expansion of the valley action are
\[
S_v = \frac{1}{3} - \frac{2}{\zeta} + \frac{12}{\zeta^4} \ln \zeta + ... (2.15)
\]
In the integral (2.9) $\alpha$ is typically of the order $\ln(-g^2)$ (The sign of $g^2$ has to be changed in order to extract the nonperturbative part of the partition function (2.9), see ref.[23].) such that the third term $\sim \zeta^{-4} \ln \zeta$ (and higher ones) in eq.(2.14) mix with the quantum corrections $O(g^2)$. If there were an extra parameter dominating $\zeta$ (e.g. the energy for the calculation of the correlator $\langle \Phi(E)\Phi(-E) \rangle$) the expansions in $g^2$ and $\zeta$ would be independent. The latter is the case for BNV processes where $\zeta \sim (E/E_{sph})^{2/3}$.

If one is only interested in the first terms of the expansion (2.15), say up to order $\zeta^{-2k}$, it is sufficient to fulfill the valley equation (2.4) up to the order $\zeta^{-k-1}$.
\[
J_v = \chi\omega \frac{\partial \Phi_v}{\partial \alpha} + O(\zeta^{-k-1}) (2.16)
\]
To show this let us estimate the contribution of the linear term of equation (2.8), namely \((\Phi, J_v)\), to the action \(S_v\) in the exponent of (2.9). We obtain

\[
e^{-S_v g^2 \int D\Phi \delta(\Phi, \omega) e^{-g^2 (\Phi, \square_v \Phi)}} = e^{-S_v g^2 + \frac{1}{2g^2} (J_v, GJ_v) \int D\Phi \delta(\Phi, \omega) e^{-g^2 (\Phi, \square_v \Phi)}},
\]

where \(G\) is the Green function of the operator \(\square_v\) with the constraint

\[
G(t_1, t_2) = N^{-1} \int D\Phi \delta(\Phi, \omega) \frac{1}{g^2} \Phi(t_1) \Phi(t_2) e^{-g^2 (\Phi, \square_v \Phi)} = (t_1 | \frac{1}{\square_v} | t_2)
\]

(2.17)

(Here we have used the Schwinger notation for Green functions \(G(x, y) = (x|1 \overline{\square}|y), (x|y) = \delta^4(x - y)\). Now it is easy to see that the additional term in the exponent has the order \(\zeta^{-2k}\) since

\[
(J_v, GJ_v) = ((J - \chi \omega \Phi'_v), G(J - \chi \omega \Phi'_v)) \sim \frac{1}{\lambda_-} ((J - \chi \omega \Phi'_v), \Phi_-)^2 \sim \zeta^{-2k}
\]

(2.19)

and the eigenvalue corresponding to the negative quasizero mode is \(\sim \zeta^{-2}\) (see Ref.[13]). In practice it is often more convenient to check directly the condition (2.19) for a given valley than to verify the valley equation (2.16). Starting from the double-well valley (2.11) it is easy to construct a suitable valley for massless gauge theories (QCD). Due to the conformal invariance of QCD at the tree level it is possible to construct a whole family of \(\overline{I} I\) configurations with finite separation from a spherically symmetric configurations with separation zero [14]. It is known, that for this spherical ansatz (and for collinear gauge orientations) QCD is equivalent at tree level to ordinary double-well quantum mechanics (specified by eq. (2.2)). For \(A_\mu(x) = -\frac{i}{g} (\sigma_\mu \bar{x} - x_\mu) x^{-2} \Phi(t, \alpha)\)

(2.20)

with \(t = \ln x^2/\rho^2\) the QCD action coincides with the simple quantum mechanical expression (2.2) up to an overall factor \(48\pi^2/g^2\). Using the quantum-mechanical valley (2.11) we obtain thus the gauge field in the form

\[
A_\mu(x)^v = -\frac{i}{g} (\sigma_\mu \bar{x} - x_\mu) \left( \frac{\rho^2/\zeta}{x^2 + \rho^2/\zeta} - \frac{\rho^2\zeta}{x^2 + \rho^2\zeta} \right)
\]

(2.21)
which coincides with the sum of one instanton field in the regular gauge with radius $\rho \sqrt{\zeta}$ and one antiinstanton field in the singular gauge with radius $\rho/\sqrt{\zeta}$ up to a gauge transformation with the matrix $\bar{x}/\sqrt{x^2}$. This gauge field obeys the valley equation

$$\frac{\delta S}{\delta A_\mu(x)}\bigg|_{A=A_v} = \chi(\zeta)\omega(x, \zeta)P_{\mu\nu}^\perp \zeta \frac{dA^\nu_v(x, \zeta)}{d\zeta}$$  \hspace{1cm} (2.22)$$

where $P_{\mu\nu}^\perp = \delta_{\mu\nu} - \mathcal{D}_{v,\mu}(1/D_v^2)\mathcal{D}_{v,\nu}$ is a projector ensuring the decoupling of the gauge and non-gauge constraints [14,15] (recall that the valley equation is a constraint classical equation). The action of this projector on fields of the type (2.20) is trivial since $D_{\mu}A_\mu = 0$. Therefore we shall omit $P_{\mu\nu}^\perp$ in what follows (cf. refs. [14,15]). The weightfunction $\omega(x, \zeta)$ is simply (2.11) taken at $t = \ln x^2/\rho^2$. To obtain the $II$ valley configuration for arbitrary sizes $\rho_1, \rho_2$ and separations $R$ one has to perform the translation $x \rightarrow x - x_0$, the inversion

$$(x - a)_\mu \rightarrow \frac{r^2}{(x - a)^2}(x - a)_\mu$$  \hspace{1cm} (2.23)$$

and a gauge transformation with the matrix $x(\bar{x} - \bar{R})R/\sqrt{x^2(x - R)^2R^2}$. The parameters $\rho_1, \rho_2$ and $R$ are then related to those of the initial configuration (2.21) and of the canonical transformation (2.23) as follows.

$$\rho_1 = \frac{r^2\sqrt{\zeta}}{(x_0 - a)^2 + \rho_2^2/\zeta}, \quad \rho_2 = \frac{r^2\rho/\sqrt{\zeta}}{(x_0 - a)^2 + \rho_2^2/\zeta},$$

$$R = (x_0 - a)\left(\frac{r^2}{(x_0 - a)^2 + \rho_2^2/\zeta} - \frac{r^2}{(x_0 - a)^2 + \rho_2^2/\zeta}\right)$$  \hspace{1cm} (2.24)$$

After some algebra one obtains the final answer for the gauge valley in the form

$$A^v_\mu = A_\mu^I + A_\mu^f + B_\mu$$  \hspace{1cm} (2.25)$$

where

$$A_\mu^I = -\frac{i}{g} \sigma_\mu \bar{x} - x_\mu \rho_1^2, \quad A_\mu^f = -\frac{i}{g} \frac{R(\bar{\sigma}_\mu(x - R) - (x - R)_\mu)\bar{R}}{R^2(x - R)^4\Pi_2} \rho_2^2,$$  \hspace{1cm} (2.26)$$
$B_\mu = \frac{i}{g} \frac{\rho_1 \rho_2}{\zeta} \left\{ \frac{x(\bar{x} - \bar{R})\sigma_\mu \bar{x}}{x^4(x - R)^2 \Pi_1} - \frac{R(\bar{x} - \bar{R})\sigma_\mu \bar{x}(x - R)\bar{R}}{R^2 x^2(x - R)^4 \Pi_2} \right. \\
+ \frac{\sigma_\mu \bar{R}}{x^2(x - R)^2 \Pi_1 \Pi_2} + \rho_1^2 \left( 1 - \frac{\rho_2}{\zeta \rho_1} \right) \frac{\sigma_\mu \bar{x}}{x^4(x - R)^2 \Pi_1 \Pi_2} \\
+ \rho_2^2 \left( 1 - \frac{\rho_1}{\zeta \rho_2} \right) \frac{R\sigma_\mu(x - R)\bar{R}}{R^2 x^2(x - R)^4 \Pi_1 \Pi_2} \\
- \frac{\rho_1 \rho_2}{\zeta} \frac{R(\bar{x} - \bar{R})\sigma_\mu \bar{x}}{x^4(x - R)^4 \Pi_1 \Pi_2} - (\text{trace}) \right\} \quad (2.27)$

where $\Pi_1 = 1 + \rho_1^2/x^2$ and $\Pi_2 = 1 + \rho_2^2/(x - R)^2$. ‘$O - (\text{trace})$’ means the traceless part of $O$. (Strictly speaking one has to subtract $\frac{1}{4} \text{Tr} O$.) Thus the valley field is a sum of instanton and anti-instanton (in a singular gauge) with relative orientation collinear with $R$ (the maximal attractive orientation) plus a small additional field proportional to $1/\zeta$. (The form of the valley used in ref.[22] differs from 2.25) by a gauge transformation with the matrix $x(x - a)(x - b)(x - R)/\sqrt{x^2(x - a)^2(x - b)^2(x - R)^2}$, where $b = R + (x_0 - a)^2/(x_0 - a)^2$. The valley of ref.[15] is obtained by a slightly more complicated gauge transformation.

The valley equation for this configuration has the usual form

$$-D_\mu G_{\mu \alpha} = 2\chi(\zeta)\omega(x, \zeta)\zeta \frac{\partial A_\alpha}{\partial \zeta} \quad (2.28)$$

where [16]

$$\omega(x, \zeta) = \frac{\zeta^2 - 1}{(x^2 + \rho_1^2)^2 \rho_1^2 + ((x - R)^2 + \rho_2^2)^2 \rho_2^2} \quad . \quad (2.29)$$

$\chi(\zeta)$ is given by eq.(2.13) and $\zeta$ depends on the new variables $\rho_1$, $\rho_2$, and $R$ according to

$$\zeta = \frac{R^2 + \rho_1^2 + \rho_2^2}{2 \rho_1 \rho_2} + \sqrt{\left( \frac{R^2 + \rho_1^2 + \rho_2^2}{2 \rho_1 \rho_2} \right)^2 - 1} + \frac{R^2 + \rho_1^2 + \rho_2^2}{\rho_1 \rho_2} \quad (2.30)$$

The classical action of this configuration coincides, of course, with eq.(2.11)

$$S = \frac{48\pi^2}{g^2} \left\{ \frac{6\zeta^2 - 14}{(\zeta - 1/\zeta)^2} - \frac{17}{3} + \left[ \frac{(5/\zeta - \zeta)(\zeta + 1/\zeta)^2}{(\zeta - 1/\zeta)^3} + 1 \right] \ln \zeta \right\} \\
\sim \frac{16\pi^2}{g^2} \left( 1 - \frac{6}{\zeta^2} + \frac{36}{\zeta^4} \ln \zeta + ... \right) \quad (2.31)$$
where the dots stand for non-logarithmic contributions \( \sim \zeta^{-4} \) as well as for other higher terms. It is worth noting that the argument of the running coupling constant \( g(\mu) \) in (2.31) could be taken as \( \mu = \rho_1 \rho_2 \) with our accuracy (since at large \( R \) it should reproduce \( 8\pi^2/g^2(\rho_1) + 8\pi^2/g^2(\rho_2) \) corresponding to independent instantons).

3. The \( \epsilon^{8/3} \ln \epsilon \) term for gauge theories with Higgs field in the valley approach

First let us reproduce the \( \epsilon^2 \) term in the expansion of the holy grail function (1.5). To this end we calculate the BNV part of the forward scattering amplitude of \( W \) bosons

\[
A(p, k) = N^{-1} \int \mathcal{D}A \mathcal{D}\phi \mathcal{D}\bar{\phi} \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S} A_\mu^a(p) A_b^\nu(k) A_\mu^a(-p) A_b^\nu(-k) \quad (3.1)
\]

by the valley method \( (A_\mu(p) \equiv \int dx A_\mu(x) \exp(ipx)) \). We shall consider the conventional model without hypercharge

\[
L = \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - \sum_{k=1}^{12} \bar{\psi}_k i \not{\nabla} \psi_k + |\nabla \phi|^2 + \lambda(|\phi|^2 - v^2/2)^2 \quad (3.2)
\]

As we shall see below we can neglect particle masses at order \( \epsilon^2 \) and use the \( II \) valley in the form (2.26). In addition we introduce

\[
\phi_v = \phi_1 \phi_2 \frac{ve}{\sqrt{2}}, \quad \bar{\phi}_v = \phi_1 \phi_2 \frac{v\bar{e}}{\sqrt{2}}, \quad \phi_1 = \frac{1}{\sqrt{\Pi_1}}, \quad \phi_2 = \frac{1}{\sqrt{\Pi_2}} \quad (3.3)
\]

where \( e \) is the unit vector which we chose to be \( (1, 0) \) (so \( \bar{e} = (1, 0) \)). (Recall that \( \phi_1 \frac{ve}{\sqrt{2}} \) is the Higgs component of the instanton satisfying the equation \( \nabla_I^2 \phi_1 = 0 \) and similarly for \( \bar{I} \).) The fermion component of the valley could also be taken into account, e.g. as the product of zero modes corresponding to \( I \) and \( \bar{I} \) (cf. ref.[24]). However, to calculate \( F(\epsilon) \) this is not necessary as the fermions affect only the preexponential factor.

The calculation of the amplitude (3.1) with the valley method proceeds as for our example, the double-well vacuum energy. We insert the necessary
δ-functions to exclude the quasizero modes from the integral (in order to have the valley equation in the gauge sector one of these constraints should be $\delta(A_\mu - A_\mu^v, \omega\partial A_\mu^v/\partial \zeta)$), make the shift $A \rightarrow A_v + A_q$, $\phi \rightarrow \phi_v + \phi_q$, and $\psi \rightarrow \psi_v + \psi_q$ (as mentioned above fermions do not affect the exponential such that the last transformation will not be done explicitly), and expand the action in powers of $A_q$ and $\phi_q$. As we shall demonstrate below, due to the valley equation in the gauge sector, the $I\bar{I}$ configuration (3.3) is the approximate valley for the gauge-Higgs model (3.2) up to order $\epsilon^2$. Since we neglect the hard-hard and hard-soft corrections (see the discussion in sect. 1) we can insert in eq.(3.1) just the Fourier transform of the valley field (2.26). With exponential accuracy of order $\epsilon^2$ the answer has the form:

$$A(p, k) \sim N^{-1} \int d\rho_1 \ d\rho_2 \ dR \ A^v(\rho_1) A^\phi(\rho_2) A^v(-\rho_1) A^\phi(-\rho_2) e^{-S_v(\rho_1, \rho_2, R)}$$

(3.4)

where

$$A^v(\rho_1) \big|_{p^2 \rightarrow 0} = \frac{1}{2\pi^2} \frac{2\pi^2}{g} \left( \rho_1^2(\sigma_\mu \tilde{b} - p_\mu) + \rho_2^2 R(\tilde{b}_\mu p - p_\mu) R e^{ipR} R^2 + O(\rho^2/\zeta) \right)$$

(3.5)

$$S_v \equiv S(\phi_v, A_v) = \frac{16\pi^2}{g^2} \left( 1 - 6\frac{\rho_1^2 p_2^2}{R^4} + 12\frac{\rho_1^2 \rho_2^2}{R^6}(\rho_1^2 + \rho_2^2) \right) + \pi^2 \rho_2^2 + 2\pi^2 v^2 \frac{\rho_1^2 \rho_2^2}{R^2} + O\left( \frac{\rho^8}{g^2 R^8}, \frac{v^2 \rho_2}{R^4}, \lambda v^4 \right)$$

(3.6)

The first term in eq.(3.6) comes from the gauge part of the action while the second term comes from the $|\nabla \phi|^2$ term (see ref.[10] and eq.(3.42) below). The contribution of the $\lambda(|\phi|^2 - v^2/2)^2$ term is of order $\lambda v^4 \rho^4$ and therefore exceeds our accuracy (see below). The BNV cross section is obtained from (3.4) by analytic continuation to Minkowski energies $E = p_0 + k_0$.

$$\sigma_{BNV} \sim \text{Im} \int d\rho_1 d\rho_2 dR \ \exp(ER_0 - S_v(\rho_1, \rho_2, R))$$

(3.7)

In next to leading order we do not need to account for the shift of the saddle point (1.2). Thus we can insert the saddle values for $\rho_1$, $\rho_2$, and $R$ into the integral (3.7) which gives the $\epsilon^2$ term in the expansion of the holy grail function (1.5).
Last but not least, we should prove that the configuration (3.3) is indeed an approximate valley in $\epsilon^2$ accuracy. As discussed in the previous section, the simplest way to prove this is by estimating the possible additions to $S_\nu$ due to the linear terms $(A_\mu^a, \delta S/\delta A_\mu^a)$, $(\bar{\phi}^a, \delta S/\delta \bar{\phi}^a)$, and $(\delta S/\delta \phi^a, \dot{\phi})$, and make sure that they are of higher order. After Gaussian integration of the linear terms we have (similarly to (2.17))

$$S_\nu \rightarrow S_\nu - \frac{1}{2} \left( J_\mu^a \left| (\Box^{-1})_{\mu\nu} \right| J_\nu^b \right) - \frac{1}{2} \left( \bar{J} \left| (\Box^{-1}) \right| J \right)$$  \hspace{1em} (3.8)

with $J_\mu^a = \delta S/\delta A_\mu^a$, $\bar{J} = \delta S/\delta \bar{\phi}$, and $J = \delta S/\delta \phi$. Here $\Box^{-1}$ is the Green function of the operator $\Box \equiv \delta^2 S/\delta \bar{\phi}\delta \phi = -\nabla^2 + 4\lambda \bar{\phi}\phi$ and $(\Box^{-1})_{\mu\nu}$ is the constraint Green function of the operator $\Box_{\mu\nu} \equiv \delta^2 S/\delta A_\mu^a \delta A_\nu^b = -D^2 \delta_{\mu\nu} + 2iG_{\mu\nu} + \frac{1}{2} g^2 \bar{\phi}\phi$ (in the background Feynman gauge). One of the constraints is given by the valley equation in the gauge sector of the theory and the others could be taken as linear combinations of the derivatives of $A_\mu^a$ with respect to other valley parameters which are orthogonal to $\zeta$ (at large separations $2\frac{d\rho_2}{d\rho_1} \approx -\rho_1 \frac{d}{d\rho_1} - \rho_2 \frac{d}{d\rho_2}$). Also, the Green functions in the background of weakly interacting $I$ and $\bar{I}$ are given by the cluster expansion (see e.g. ref [26]):

$$\frac{1}{\Box} = \frac{1}{-\partial^2 + m_H^2} + \left( \frac{1}{\Box_I} - \frac{1}{-\partial^2 + m_H^2} \right) + \left( \frac{1}{\Box_I} - \frac{1}{-\partial^2 + m_H^2} \right) + ...$$  \hspace{1em} (3.9)

$$\left( \frac{1}{\Box} \right)_{\mu\nu} = \frac{\delta_{\mu\nu} \delta_{ab}}{-\partial^2 + m_W^2} + \left( \left( \frac{1}{\Box_I} \right)_{\mu\nu} - \frac{\delta_{\mu\nu} \delta_{ab}}{-\partial^2 + m_W^2} \right) + \left( \left( \frac{1}{\Box_I} \right)_{\mu\nu} - \frac{\delta_{\mu\nu} \delta_{ab}}{-\partial^2 + m_W^2} \right) + ...$$  \hspace{1em} (3.10)

where the dots stand for the higher terms in the expansion which are $\sim \rho^2/R^2$. Here $(-\partial^2 + m^2)^{-1}$ are the bare propagators for the $W$ and Higgs particles ($m_W = g\nu/2$, $m_H = v\sqrt{2\lambda}$) and $(1/\Box_I)_{\mu\nu}$, $1/\Box_I$ are the corresponding propagators in the field of a single instanton (similarly for $\bar{I}$). Since the measure $\omega(x, \zeta)$ is approximately $1/\rho_1^2$ near the instanton (at $x^2 \sim \rho_1^2$) and $1/\rho_2^2$ near the antinstanton (at $(x - R)^2 \sim \rho_2^2$) the Green functions $(1/\Box_I)_{\mu\nu}$, and $(1/\Box_I)_{\mu\nu}$ could be taken as the BCCL propagators [27] with the restrictions.
proportional to the pure zero modes. (We shall not, however, need the explicit form of these BCCL propagators.)

Now let us estimate the additional term in (3.8) at the saddle point values for $\rho$ and $R$ (1.2). The first functional derivatives are

$$J_\alpha^a \equiv \frac{\delta S}{\delta A_\alpha^a} = -D_\mu G_{\mu\alpha}^a + ig\tilde{\phi}\{t^a, \nabla_\alpha\} \phi \quad (3.11)$$

$$D_\mu G_{\mu\alpha}^w = \frac{24i}{g} \frac{\rho_1^2 \rho_2^2}{R^2} \frac{x\bar{\sigma}_\alpha(x - R)\bar{R} - (\text{trace})}{x^4(x - R)^4 \Pi_1^2 \Pi_2^2} + \ldots \quad (3.12)$$

$$t^a \cdot ig\tilde{\phi}_v (t^a \nabla_v^\alpha + \nabla_v^\alpha t^a) \phi_v = -\frac{igv^2}{4} \left( \rho_1^2 (\sigma_\mu \bar{x} - x_\mu) \right) \frac{\rho_2^2 R(\bar{\sigma}_\mu (x - R) - (x - R)_\mu)\bar{R}}{R^2(x - R)^4 \Pi_1^2 \Pi_2^2} + \ldots \quad (3.13)$$

and

$$J \equiv \frac{\delta S}{\delta \phi} = -\nabla^2 \phi + 2\lambda(\tilde{\phi}\phi - v^2/2)\phi \quad , \quad \bar{J} \equiv \frac{\delta S}{\delta \bar{\phi}} = -\bar{\phi} \bar{\nabla}^2 + 2\lambda(\bar{\phi}\phi - v^2/2)\bar{\phi} \quad (3.14)$$

$$(\nabla^2\phi)_v = \frac{4ve}{\sqrt{2\Pi_1 \Pi_2}} \frac{\rho_1^2 \rho_2^2}{R^2 x^4(x - R)^4 \Pi_1 \Pi_2} \left\{ 2x \cdot R \cdot (x - R) \cdot R + 2(\rho_1^2 + \rho_2^2) (x^2 - (x \cdot R)^2/R^2) - (x\bar{R} - x \cdot R)(R^2 + x^2 + (x - R)^2) \right\} + \ldots \quad (3.15)$$

$$2\lambda(\tilde{\phi}_v \phi_v - v^2/2)\phi_v = -\frac{\lambda v^3 e}{\sqrt{2\Pi_1 \Pi_2}} \left( \frac{\rho_1^2 x}{x^2} + \frac{\rho_2^2}{(x - R)^2} \right) + \ldots \quad (3.16)$$

and similarly for $J$ (as usual dots stand for higher orders in $\rho^2/R^2$). The double integrals in (3.8) are of the type $\int dx dy J(x)G(x, y)J(y)$ which encompass three characteristic regions of integration: (i) $x, y \sim \rho_1$ (or $x - R, y - R \sim \rho_2$), (ii) $x, y \sim R$, and (iii) $x, y \sim 1/m \gg R$. We first consider the contribution to eq.(3.8) coming from the first region. Here

$$J_\mu \equiv \frac{i}{g} \frac{\rho_1 \bar{x} - x_\mu}{x^4 \Pi_1^2} \left( 24 - \frac{g^2 v^2}{2 \rho_1} \right) \quad (3.17)$$
\[ J = \frac{8}{\zeta^2} \frac{xR}{x^4\Pi_1} \frac{ve}{\sqrt{2\Pi_1}} \]  

(3.18)

and the Green functions (3.9) and (3.10) are the BCCL propagators in the instanton background. From dimensional considerations it follows

\[
(J_{\mu} | (\frac{1}{\Box})_{\mu\nu} | J_{\nu}) \sim \frac{1}{g^2} \left( 24\zeta^{-2} - 2m_w^2\rho_1^2 \right) \sim \frac{1}{g^2} \epsilon^{8/3} 
\]

\[
(\bar{J} | (\frac{1}{\Box})_J | J) \sim v^2R^2/\zeta^4 \sim \frac{1}{g^2} \epsilon^{10/3} 
\]

(3.19) (3.20)

In fact the expression (3.19) is also \( \sim \epsilon^{10/3}g^{-2} \) since \( J_{\mu} \) is proportional to \( \rho_1 \partial A_{\mu}/\partial \rho_1 \) which is one of the instanton zero modes while \( (1/\Box)_{\mu\nu} \) is the BCCL propagator orthogonal to the zero modes (and the dilatation mode \( \partial A/\partial \rho \) in particular). Let us consider next the contribution from region (iii) where the Green functions (3.9) and (3.10) are bare propagators.

\[ J_{\mu} = -\frac{i}{g} \frac{m_W^2}{x^4} \left[ \rho_1^2(\sigma_{\mu} \bar{x} - x_{\mu}) + \rho_2^2 R(\bar{\sigma}_{\mu} x - x_{\mu}) \bar{R} \right] + \ldots 
\]

(3.21)

\[ J = -\frac{m_H^2}{2} \frac{\rho_1^2 + \rho_2^2}{x^2} \frac{ve}{\sqrt{2}} + \ldots 
\]

(3.22)

Again dimensional considerations give

\[
(J_{\mu}^a | \frac{1}{-\partial^2 + m_W^2} | J_{\mu}^a) \sim \frac{1}{g^2} m^4w^4 \rho^4 \sim \frac{1}{g^2} \epsilon^{8/3} 
\]

\[
(J | \frac{1}{-\partial^2 + m_H^2} | J) \sim m^4_H \rho^4 v^2 \cdot \frac{1}{m^2_H} \sim \frac{1}{g^2} \epsilon^{8/3} 
\]

(3.23) (3.24)

Similarly, it is easy to verify that the contributions from the region \( x, y \sim R \), where the propagators (3.9) and (3.10) are bare and \( J_{\mu} \) and \( J \) are given by (3.11)-(3.16), is also \( \frac{1}{g^2} \epsilon^{8/3} \). This proves that (3.3) is a valley solution up to order \( \epsilon^2 \) but not in higher orders as the linear terms give then additional contributions.

However, one can modify the configuration (3.3) such that it becomes a valley at least up to the order \( \epsilon^{8/3} \ln\epsilon \) (the deviations are of order \( \epsilon^{8/3} \)). Since all the contributions to eq. (3.8) \( \sim \epsilon^{8/3} \) come from the terms (3.23) and (3.24) and are proportional to \( m_W^2 \) and \( m_H^2 \) the valley has to be modified such

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that it accounts for the gauge boson and Higgs masses. This implies that the valley configuration should decrease exponentially at large \( x^2 \gg R^2 \) with the Higgs and boson masses setting the scale. In other words, the improved valley has to be constructed from constraint instantons (see ref.\[28\]) rather than from ordinary ones. For the accuracy we aim at it is sufficient to modify the valley (3.3) in the following manner

\[
A^\nu = A^I_\mu + A^{\bar{I}}_\mu + B_\mu, \quad \phi_v = \phi_1 \phi_2 \frac{ve}{\sqrt{2}}, \quad \bar{\phi}_v = \phi_1 \phi_2 \frac{\bar{v}\bar{e}}{\sqrt{2}}, \quad (3.25)
\]

where \( I \) and \( \bar{I} \) are now constraint instantons \[28\]:

\[
A^I_\mu = -\frac{i}{g} \rho^2 \sigma_\mu \bar{x} - x_\mu \frac{\Theta(R^2 - x^2)}{x^4} - \Theta(x^2 - R^2)G'_W(x^2) \quad (3.26)
\]

\[
A^{\bar{I}}_\mu = -\frac{i}{g} \rho^2 R(\bar{\sigma}_\mu (x - R) - (x - R)_\mu) \bar{R} \frac{\Theta(R^2 - (x - R)^2)}{(x - R)^4} - \Theta((x - R)^2 - R^2)G'_W((x - R)^2) \quad (3.27)
\]

\[
\phi_1 = \frac{\Theta(R^2 - x^2)}{\sqrt{\Pi_1}} + \Theta(x^2 - R^2)\sqrt{1 - \rho^2 \Pi_1^{-1}G_H(x^2)} \quad (3.28)
\]

\[
\phi_1 = \frac{\Theta(R^2 - (x - R)^2)}{\sqrt{\Pi_2}} + \Theta((x - R)^2 - R^2)\sqrt{1 - \rho^2 \Pi_2^{-1}G_H((x - R)^2)} \quad (3.29)
\]

Here \( G_W \) and \( G_H \) are the bare propagators with masses \( m_W \) and \( m_H \) respectively:

\[
G_W(x^2) = \int \frac{dp}{4\pi^2} \frac{e^{-ipx}}{m_W^2 + p^2}, \quad G_H(x^2) = \int \frac{dp}{4\pi^2} \frac{e^{-ipx}}{m_H^2 + p^2} \quad (3.30)
\]

and \( G'_W(x^2) \) is the derivative of \( G_W(x^2) \) with respect to \( x^2 \). The field \( B_\mu \) should also be modified to become exponentially decreasing but since \( B_\mu \) itself is small \( (\sim \rho^2/R^2) \) its mass dependence is not essential at our accuracy. For vanishing masses the new valley (3.25) obviously reduces to the configuration (3.3). For large \( x^2 \), on the other hand, it corresponds to the emission of massive particles. Let us demonstrate now that for the improved configuration all additional contributions to \( S_v \) (due to linear terms) are of higher order than \( \epsilon^{8/3} \ln \epsilon \).
Logarithmic contributions of the order $\epsilon^{8/3}\ln \epsilon$ can come from two regions of integration in $x$ and $y$: (1) $R^2 \gg x^2, y^2 \gg \rho_1^2$ (or $R^2 \gg (x-R)^2, (y-R)^2 \gg \rho_2^2$) and (2) $m^{-2} \gg x^2, y^2 \gg R^2$ (as usual we assume that $m_H$ is of the same order of magnitude as $m_W$). In the first region the valley configuration coincides with the massless valley (3.3), $J_\mu$ and $J$ are given by eqs. (3.17) and (3.18) and the Green functions (3.9) and (3.10) are the BCCL propagators in the $I(\vec{f})$ background. Strictly speaking in region (1) we should use the large $x^2$ asymptotics of these expressions. Since the BCCL propagator contains explicitly the logarithmic term $\sim \Phi_0(x)\Phi_0(y) \ln(R^2/\rho^2)$ it looks as if the region $x^2, y^2 \sim \rho^2$ were also essential. However, the contribution to $(J_\mu(\square^{-1})_{\nu},J_\nu)$ coming from this region vanishes because $J_\mu$ is proportional to $\partial A^I_\mu/\partial \rho$ which is one of the constraints of the BCCL propagator. For the same reason no $\epsilon^{8/3} \cdot const$ term arises from the region $x^2, y^2 \sim \rho^2$. It could, however, result from the region $x^2, y^2 \sim R^2$. The term $(\square^{-1}J)$ does not contain any such contribution $\sim \epsilon^{8/3}$ (see eq.(3.20)) and is therefore unimportant.

In the second region ($m_H^{-2} \gg x^2, y^2 \gg R^2$) all fields decrease exponentially, so

$$J_\mu^v = -(\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu)A_\nu^v + t^v \frac{g^2 v^2}{2} \{ t^\alpha, A_\mu \} e = -(m_w^2 - g^2 v^2/4)A_\mu^v \tag{3.31}$$

$$J = -\partial^2 \phi_v + 2\lambda \left( \tilde{\phi}_v \phi_v - \frac{v^2}{2} \right) \phi_v = -(m_H^2 - 2\lambda v^2)(\phi_v - \frac{ve}{\sqrt{2}})$$

$$J = -(m_H^2 - 2\lambda v^2)(\tilde{\phi}_v - \frac{ve}{\sqrt{2}}) \tag{3.32}$$

where $A_\mu^v$ and $\phi_v$ are given by the asymptotics of eq. (3.25) at large $x^2$. Since in this region the Green functions (3.9) and (3.10) reduce to the bare propagators we obtain

$$\left( J_{\mu}^{va} \mid \frac{1}{\partial^2 + m_W^2} \mid J_{\mu}^{va} \right) = - \left( m_W^2 - g^2 v^2/4 \right)^2 \frac{12(\rho_1^2 + \rho_2^2)}{g^2} \int \frac{dp}{m_W + p^2} \tag{3.33}$$

$$\simeq - \left( m_W^2 - g^2 v^2/4 \right)^2 \frac{12\pi^2(\rho_1^2 + \rho_2^2)}{g^2} \ln \frac{1}{m_W R^2}$$

$$\left( \bar{J} \mid \frac{1}{\partial^2 + m_H^2} \mid J \right) = (m_H^2 - 2\lambda v^2)^2 \frac{v^2(\rho_1^2 + \rho_2^2)}{8} \int \frac{dp}{m_H^2 + p^2} \tag{3.34}$$

$$= (m_H^2 - 2\lambda v^2)^2 \frac{\pi^2 v^2(\rho_1^2 + \rho_2^2)}{8m_H^2} \sim \frac{m_W^2 m_H^2}{g^2} \rho^2$$

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We see now that for the properly chosen mass \( m_W^2 = g^2 v^2 / 4 \) the logarithmic contribution in eq. (3.33) vanishes. As eq. (3.34) does not contain any such term from the beginning we conclude that the configuration (3.25) is a proper valley up to \( \epsilon^{8/3} \cdot \text{const} \) terms.

Finally, in order to find the \( \epsilon^{8/3} \ln \epsilon \) contribution to the holy grail function (1.4) we have to calculate the action of the valley configuration (3.25) at the saddle point values of \( \rho_1, \rho_2 \) and \( R \). (If we were interested in the \( \epsilon^{8/3} \cdot \text{const} \) terms we would also have to account for the shift of the saddle point due to the next-to-leading terms \( \sim g^{-2} \rho_6 R^{-6} \) and \( v^2 \rho_4 R^{-2} \) (see eq. (3.6)). It is convenient to expand the action of the valley configuration (3.25) as a power series in \( m_W^2 \) and \( m_H^2 \) since we need only the first few terms of this expansion (see below). Let us start with the gauge part of the action \( S_g = \int d^4x G_{\mu\nu} G^{\mu\nu} \).

\[
S_g^v = S_g \bigg|_{m_W^2 = 0} - m_W^2 \left( \frac{\partial A_{\mu}^{va}}{\partial m_W^2}, D_\mu G^{\mu a} \right) \bigg|_{m_W^2 = 0} + \frac{m_W^4}{2} \left\{ \left( \frac{\partial^2 A_{\mu}^{va}}{\partial (m_W^2)^2}, -D_\mu G^{\mu a} \right) + \left( \frac{\partial A_{\mu}^{va}}{\partial m_W^2}, -D_\mu \delta_{\mu \nu} + 2iG_{\mu \nu} \right) \chi_{ab} \frac{\partial A_{\nu}^{vb}}{\partial m_W^2} \right\} \bigg|_{m_W^2 = 0} + \ldots \quad (3.35)
\]

The first term is the action of the massless valley up to order \( \rho_1^2 \ln \frac{R^2}{\rho_1^2} \).

\[
S_g^v \bigg|_{m_W^2 = 0} = \frac{16\pi^2}{g^2} \left( 1 - 6 \frac{\rho_1^2 \rho_2^2}{R^4} + 12 \frac{\rho_1^4 + \rho_2^4}{R^8} \rho_1^2 \rho_2^2 + 36 \frac{\rho_1^4 \rho_2^4}{R^8} \ln \frac{R^2}{\rho_1 \rho_2} + \ldots \right) \quad (3.36)
\]

where the last term gives in the saddle point a \( \epsilon^{8/3} \ln \epsilon \) contribution to the holy grail function.

We shall discuss next the other terms in (3.35). All the higher terms in the \( m_W^n \) expansion contribute only in higher orders. This is easy to see, as the dimension of \( m_W^n \) is always balanced by a factor proportional to \( \rho_{1,2}^n \sim \epsilon^{2n/3} \). Since

\[
\frac{\partial A_{\nu}}{\partial m_W^2} \bigg|_{m_W^2 = 0} = \frac{i \rho_1^2 \sigma_{\mu} \bar{\epsilon} - x_\mu}{4g} \frac{x^2 \Pi}{x^2 - R^2} \Theta(x^2 - R^2)
\]

\[
+ \frac{i \rho_2^2 R(\bar{\sigma}_\mu (x - R) - (x - R)_{\mu})}{4g} \frac{R}{R^2(x - R)^2 \Pi} \Theta((x - R)^2 - R^2) \quad (3.37)
\]

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we obtain
\[
m^2_W \left( \frac{\partial A^{va}_\mu}{\partial m^2_W}, D_\mu G^{va}_{\mu\alpha} \right) = -\frac{72 m^2_W}{g^2} \frac{\rho^2 \rho^2_2}{R^2} \int dx \left\{ \rho^2_1 (x - R) \cdot R \Theta(x^2 - R^2) + \rho^2_2 x \cdot R \Theta((x - R)^2 - R^2) \right\} x^{-4} (x - R)^{-4}
\sim \frac{m^2_W \rho^2 \rho^2_2}{g^2 R^4} (\rho^2_1 + \rho^2_2) \sim \epsilon^{8/3}
\] (3.38)

Similarly we can show that the first term of the \(m^4_W\) contribution in eq.(3.35) is of higher order
\[
-m^4_W \left( \frac{\partial^2 A^{va}_\mu}{\partial (m^2_W)^2}, D_\mu G^{va}_{\mu\alpha} \right) \sim m^4_W \frac{\rho^2 \rho^2_2}{R^2} (\rho^2_1 + \rho^2_2) \sim \epsilon^{10/3}
\] (3.39)
and we are left with the second term of this contribution. Since all fields vanish exponentially at large \(x^2\) we can replace the covariant derivatives by the ordinary ones.

\[
\frac{m^4_W}{2} \left( \frac{\partial A^{va}_\mu}{\partial m^2_W}, -\frac{\partial^2 A^{va}_\mu}{\partial m^2_W} \right) = m^4_W \frac{3 \rho^4_1}{2 g^2} \int dx \left\{ \frac{3 \rho^4_1}{x^4} \Theta(x^2 - R^2)
+ \frac{3 \rho^4_2}{(x - R)^4} \Theta((x - R)^2 - R^2) - \frac{2 \rho^2 \rho^2_2}{R^2 x^2 (x - R)^2} \left( \frac{1}{x^2} + \frac{1}{(x - R)^2} \right)
(4 x \cdot R (x - R) \cdot R - R^2 x \cdot (x - R)) \Theta(x^2 - R^2) \Theta((x - R)^2 - R^2) \right\}
\sim \frac{3 \pi m^4_W}{2 g^2} (\rho^2_1 + \rho^2_2) \ln \frac{1}{m^2_W R^2} \sim \epsilon^{8/3} \ln \epsilon
\] (3.40)

(The upper limit of the logarithmic integral is \(1/m^2_W\) and the lower one is \(R^2\) due to the \(\Theta\)-function.) This term contributes to the holy grail function in the order we are interested in.

Next we consider the gauge-Higgs part of the action \(S_{gH} = \int dx |\nabla \Phi|^2\) and expand it in powers of \(m^2_W\) and \(m^2_H\).

\[
S_{gH} = S_{gH}|_{m_H,m_W=0} + m^2_W ig \left( \frac{\partial A^n_\mu}{\partial m^2_W}, \bar{\phi} \{t^n, \nabla_\mu\} \phi \right)|_{m_W,m_H=0}^v
\]
\[
- m^2_H \left\{ \left( \frac{\partial \bar{\phi}}{\partial m^2_H}, \nabla^2 \phi \right) + \left( \frac{\partial \phi}{\partial m^2_H}, \nabla^2 \bar{\phi} \right) \right\}|_{m_W,m_H=0}^v + ...
\] (3.41)
The best way to calculate the first term is to use the explicit expression (3.15) for $\nabla^2 \Phi$ for the massless valley. We have

$$S_{hH} = \pi^2 v^2 (\rho_1^2 + \rho_2^2) - 4v^2 \frac{\rho_1^2 \rho_2^2}{R^2} \int dx \frac{x \cdot R (x - R) \cdot R}{x^4(x - R)^4 \Pi_1^2 \Pi_2^2} \quad (3.42)$$

$$= \pi^2 v^2 (\rho_1^2 + \rho_2^2) + 2\pi^2 v^2 \frac{\rho_1^2 \rho_2^2}{R^2} - 6\pi^2 v^2 \frac{\rho_1^2 \rho_2^2}{R^4} \left( \frac{\rho_1^2}{\Pi_1^2} + \frac{\rho_2^2}{\Pi_2^2} \right)$$

From the three terms in the braces on the r.h.s. of eq.(3.15) only the first contributes. The second is of higher order in $\epsilon$ and the third one vanishes after integration over $x$. The first term on the r.h.s. of eq.(3.42) is the surface term due to partial integration: $\int dx |\nabla \phi_v|^2 = \pi v^2 (\rho_1^2 + \rho_2^2) - \int dx \phi_v \nabla^2 \phi_v$.

Next, let us address the $m_H^2$ term on the r.h.s. of eq.(3.41). With (3.13) and (3.37) we easily obtain the same integral as in eq.(3.40) (but with different coefficient).

$$m_W^2 \frac{3\rho_1^4}{x^4} \Theta((x - R)^2 - R^2) + \frac{3\rho_1^4}{(x - R)^4} \Theta((x - R)^2 - R^2) + \ldots) = -\frac{3\pi^2 v^2}{4} m_W^2 v^2 (\rho_1^4 + \rho_2^4) \ln \frac{1}{m_W^2 R^2} \quad (3.43)$$

The $m_H^2$ term in eq.(3.41) does not contribute at our accuracy. By inserting the explicit form of $\nabla^2 \Phi_v$ (3.15) one can convince oneself that it only contributes in the order $\rho_1^6 R^{-2} \sim \epsilon^{10/3}$.

Finally we come to the last part of the action, namely the Higgs self-interaction.

$$S_H = \int dx \lambda(\phi_v, \phi_v - v^2/2) = \frac{\lambda v^4}{4} \int dx \left\{ \frac{\rho_1^2}{\Pi_1^2} \Theta(R^2 - x^2)^2 x^{-4} \right.$$  
$$+ \Theta(x^2 - R^2) G_H^2(x^2) \bigg]$$  
$$+ \frac{\rho_2^4}{\Pi_2^2} \Theta((x - R)^2) (x - R)^{-4} + \Theta((x - R)^2 - R^2) G_H^2((x - R)^2) \bigg] +$$  
$$+ 2\rho_1^2 \rho_2^2 G_H(x^2) G_H((x - R)^2) \Theta(x^2 - R^2) \Theta((x - R)^2 - R^2) \bigg\}$$  
$$= \frac{\lambda v^4 \pi^2}{4} \left( \rho_1^4 \ln \frac{1}{m_H^2 \rho_1} + \rho_2^4 \ln \frac{1}{m_H^2 \rho_2} + \rho_1^2 \rho_2^2 \ln \frac{1}{m_H^2 R^2} \right) \quad (3.44)$$
Combining now all our results, namely (3.6), (3.36), (3.40), (3.42), (3.43), and (3.44) we obtain the final answer for the action of our massive valley (3.25).

\[
S_v = \frac{16\pi^2}{g^2} \left\{ 1 - 6\frac{\rho_1^3\rho_2^3}{R^4} + 12\frac{\rho_1^2\rho_2^2}{R^6}(\rho_1^2 + \rho_2^2) + 36\frac{\rho_1^4\rho_2^4}{R^8} \ln \frac{R^2}{\rho_1\rho_2} 
\right.
\]
\[
+ \frac{3m_W^4}{32}(\rho_1^4 + \rho_2^4) \ln \frac{1}{m_W^2 R^2} \right\}
\]
\[
+ \pi^2 v^2 \left\{ \rho_1^2 + \rho_2^2 + 2\frac{\rho_1^2\rho_2^2}{R^2} - 6\frac{\rho_1^2\rho_2^2}{R^4} \left( \rho_1^2 \ln \frac{R^2}{\rho_1^2} + \rho_2^2 \ln \frac{R^2}{\rho_2^2} \right) 
\right.
\]
\[
- \frac{3m_W^2}{4}(\rho_1^4 + \rho_2^4) \ln \frac{1}{m_W^2 R^2} \right\}
\]
\[
+ \frac{\lambda\pi^2 v^4}{4} \left\{ \rho_1^4 \ln \frac{1}{m_H^2\rho_1^2} + \rho_2^4 \ln \frac{1}{m_H^2\rho_2^2} 
\right.
\]
\[
+ 2\rho_1^2\rho_2^2 \ln \frac{1}{m_H^2 R^2} \right\} + ...
\]

Substituting now the saddle point values (1.2) for \(\rho_1, \rho_2, \) and \(R\) we obtain the expansion (1.5) of the holy grail function. (Recall that we assume \(m_H\) to be of order \(m_W\) so we do not distinguish between \(\ln m_H^2\) and \(\ln m_W^2\). The mass difference enters only at order \(\epsilon^{8/3}\), const.)

4. Effective Lagrangean for instanton-induced interactions.

As we discussed in the Introduction, the basic assumption of the effective Lagrangean approach is that the instanton-induced processes are given by the ordinary perturbative diagrams and that additional multiparticle vertices originate from the effective Lagrangean

\[
L_{\text{eff}} = L_{\text{eff}}^I + L_{\text{eff}}^{\bar{I}}
\]

\[
L_{\text{eff}}^{(I)}(z) = \int \frac{d\rho}{\rho^3} d(\rho) L_\psi^{(I)}(\rho, u) \exp \left( -L_\psi^{(I)}(z) \right)
\]

where \(d(\rho) \sim \exp(-\frac{8\pi^2}{g_{\psi}^2})\) is the usual instanton density[1]. Here \(L_\psi^{(I)}\) is the
t'Hooft effective Lagrangean for fermions in the (anti)instanton field [1]

\[ L_\psi^I = (4\pi^2 \rho^3)^6 \prod_{k=1}^6 (\psi^{k+6}, u_0 \bar{u}_I \epsilon)(\epsilon u_I \bar{u}_0, \psi^k) + ... \]

\[ L_\bar{\psi}^I = (4\pi^2 \rho^3)^6 \prod_{k=1}^6 (\bar{\psi}_k, u_I \bar{\epsilon})(\epsilon u_\bar{I}, \bar{\psi}^{k+6}) + ... \]  

(4.2)

and \( L^{(I)} \) is the instanton-induced effective Lagrangian for bosons [11, 12]

\[ L^I = -\frac{2\pi^2}{g} \rho^2 I_T \mathcal{R}\{u_0 \bar{u}_I \sigma_\alpha \bar{\sigma}_\beta u_I \bar{u}_0 G_{\alpha\beta}\} + 2\pi^2 \rho^2 \phi \phi + ... \]

\[ L^{\bar{I}} = -\frac{2\pi^2}{g} \rho^2 \bar{I}_T \mathcal{R}\{u_I \bar{u}_0 \sigma_\alpha \bar{\sigma}_\beta \bar{u}_I \bar{G}_{\alpha\beta}\} + 2\pi^2 \rho^2 \phi \phi + ... \]  

(4.3)

\( (u_0 \) is an arbitrary unit vector which drops from the final results for physical amplitudes). As usual \( (\epsilon u) \) denotes \( \epsilon_{\alpha\beta} u^\beta \gamma \) etc. The ellipsis stand for operators of higher dimensions, multiplied by additional powers of \( \rho \). (Some of the next to leading terms \( \sim \rho^4 \) are given below, see eq.(4.26).) Eqs. (4.2) and (4.3) are infinite series of local operators with increasing dimension, multiplied by growing powers of \( \rho \).

The effective Lagrangean (4.1) added to the ordinary one reproduces the instanton-induced effects. More precisely, if we start from the Lagrangean

\[ L = \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + |\nabla \phi|^2 + \lambda (|\phi|^2 - v^2/2)^2 + L_{\text{eff}}^I + L_{\text{eff}}^{\bar{I}} \]  

(4.4)

and expand up to the m-th power in \( L_{\text{eff}}^I \) and n-th power in \( L_{\text{eff}}^{\bar{I}} \), sum up the corresponding perturbative diagrams with m \( I \) -type vertices and n \( \bar{I} \) -type vertices (each of them couples 12 fermions and an arbitrary number of W’s and H’s), we should reproduce the answer for the original amplitude calculated with a background of m instantons and n antiinstantons. To illustrate this equivalence let us consider the simplest example, namely just one instanton and let us neglect for the moment the W and H masses. One instanton effects are described by the first term in the power expansion of \( \int dz L_{\text{eff}}^I(z) \). Thus \( \langle A_\mu(x)e^{-L_{\text{eff}}(z)} \rangle \) should reproduce an instanton field with size \( \rho \), center \( z \) and orientation matrix \( u \bar{u}_0 \) (in the singular gauge):

\[ \langle A_\mu(x) \exp \left( \frac{2\pi i}{g} \rho^2 \mathcal{R}\{u_0 \bar{u}_\mu \bar{\sigma}_\alpha \sigma_\beta u_\mu \bar{u}_0 G_{\alpha\beta}(z)\} \right) \rangle = \]

\[ -\frac{i}{g} u_0 \bar{u}(\sigma_\mu \bar{\Delta} - \Delta_\mu) u\bar{u}_0 \frac{\rho^2}{\Delta^2 (\Delta^2 + \rho^2)} , \quad \Delta = x - z \]  

(4.5)
Here by $\langle O \rangle$ we denote averaging $O$ with $\exp(-S)$, $S$ being the ordinary action (3.1). Let us verify (4.5) by expanding its l.h.s. in powers of $\rho^2 G_{\mu\nu}^a$. The first term of the expansion is simply (see Fig.7a)

$$A_\mu(x)G_{\alpha\beta}(z)\frac{2\pi i}{g} \rho^2 T r\{u_0 \bar{u} \sigma_\alpha \sigma_\beta u \bar{u} a\} = \frac{i}{g} \rho^2 u_0 \bar{u} (\sigma_\mu \bar{\Delta} - \Delta_\mu) u \bar{u} \Delta^{-4} \quad (4.6)$$

which gives the asymptotics of the instanton field at $\Delta^2 \gg \rho^2$. The second term in the expansion of the l.h.s. of eq.(4.5) corresponds to the diagram shown in Fig.7b. The calculation of this diagram gives the second term in the expansion of the instanton field in powers of $\rho^2/\Delta^2$, namely $i g \rho^4 u_0 \bar{u} (\sigma_\mu \bar{\Delta} - \Delta_\mu) u \bar{u} \Delta^{-6}$. The third term corresponds to the two diagrams in Fig7c. It can be demonstrated that the logarithmic contributions $\sim (\ln \Delta)^2 \Delta^{-8} \rho^6$ from these diagrams cancel and the result coincides with the third term in the expansion of the instanton field (4.5) in powers of $\rho^2/\Delta^2$. In general a series of diagrams of the type shown in Fig.7 describes the perturbative solution of the classical equation $D_\mu G_{\mu\nu} = 0$ using the asymptotics (4.6) as first iteration. Since every extra factor of $g$ is compensated by an extra factor $\rho^2/g$ coming from the emission of an additional gauge boson by the instanton this perturbative expansion is in powers of $\rho^2$ rather than $g$.

The situation for the scalar instanton component is quite similar - it is reproduced by $\langle \phi(x) e^{-L_I(z)} \rangle$ as a power series in $\rho^2/\Delta^2$ corresponding to the diagrams shown in Fig. 8. Note that after the shift $\phi \rightarrow \phi + (v/\sqrt{2}) e$ we have

$$L' = -\frac{2\pi i}{g} \rho^2 T r\{u_0 \bar{u} \sigma_\alpha \sigma_\beta u \bar{u} a\} + 2\pi^2 \rho^2 \phi \phi + \pi^2 v^2 \rho^2 \quad (4.7)$$

and similarly for $\bar{L}$. The asymptotical scalar field comes from the diagram in Fig. 8b (the trivial term $(v/\sqrt{2}) e$ can be depicted as in Fig. 8a).

$$\phi = \frac{v}{\sqrt{2}} e - 2\pi^2 \rho^2 \phi(x) \phi(z) e \frac{v}{\sqrt{2}} e = \frac{v}{\sqrt{2}} e \left(1 - \frac{\rho^2}{2\Delta^2}\right) \quad (4.8)$$

The further diagrams of the series in Fig.8 give subsequent terms in the expansion of the scalar component of the instanton field in powers of $\rho^2/\Delta^2$

$$\langle \phi(x) e^{-L_I(z)} \rangle = \frac{v}{\sqrt{2}} e \frac{e}{\sqrt{1 + \rho^2/\Delta^2}} = \frac{v}{\sqrt{2}} e \left(1 - \frac{\rho^2}{2\Delta^2} + \frac{3\rho^4}{8\Delta^4} + \ldots\right) \quad (4.9)$$

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This series corresponds to the perturbative solution of the equation $\nabla^2 \phi = 0$ starting from (4.8) as the first iteration, expressing the field $A^I_\mu$ as power series in $\rho^2 / x^2$.

It should be mentioned that for massive particles (e.g. if the massive propagator is inserted in Fig. 6-8) the classical fields $\langle A_\mu(x)e^{-L_I(z)} \rangle$ and $\langle \phi(x)e^{-L_I(z)} \rangle$ become the exponentially decreasing configurations of the constraint-instanton type [28]. The expansions (4.5) and (4.9) are then only valid for $x^2 \ll m^2$ while at large $x$ every term in the expansion is multiplied by a function of $m^2 x^2$ determined by the corresponding diagram. For example, the first nontrivial terms $\sim \rho^2$ are proportional to the bare massive propagators with $m_W$ and $m_H$, and at this order the classical fields $\langle A_\mu(x)e^{-L_I(z)} \rangle$ and $\langle \phi(x)e^{-L_I(z)} \rangle$ coincide with our valley configurations (3.26) and (3.28) (at $x^2 > R^2$).

It is also instructive to analyse how the Green functions in the instanton background are obtained within the effective Lagrangean approach. For simplicity we shall consider the scalar propagator in the background of a single instanton with center $z = 0$ and orientation matrix $u\bar{u}_0 = 1$. The explicit form of this propagator is [28] (here we again neglect $m_W$ and $m_H$):

$$G(x, y) = \frac{1 + \frac{\rho^2 x\bar{y}}{x^2 y^2}}{4\pi^2(x - y)^2 \sqrt{\Pi_x \Pi_y}} = \frac{1}{4\pi^2(x - y)^2} + \frac{\rho^2 (2x\bar{y} - x^2 - y^2)}{8\pi^2 x^2 y^2 (x - y)^2}$$

$$+ \frac{\rho^4}{32\pi^2} \left\{ \frac{1}{(x - y)^2} \left[ \left( \frac{1}{x^2} - \frac{1}{y^2} \right)^2 - \frac{4(x\bar{y} - x \cdot y)}{x^2 y^2} \left( \frac{1}{x^2} - \frac{1}{y^2} \right) \right] + \frac{2}{x^2 y^2} \left( \frac{1}{x^2} - \frac{1}{y^2} \right) \right\} + ...$$

(4.10)

where $\Pi_x = 1 + \rho^2 x^{-2}$, $\Pi_y = 1 + \rho^2 y^{-2}$ and the ellipsis stands for higher order terms in $\rho^2$. On the other hand, using the effective Lagrangean this propagator can be represented as

$$\langle \phi(x)\bar{\phi}(y) \exp -L_I(0) \rangle$$

or, in detail, by the sum of diagrams in Fig.9 corresponding to an expansion of (4.11) in powers of $L'$ (i.e. in powers of $\rho^2$, see eq.(4.7)). Apart from the bare propagator (shown in Fig. 9a) the first non-trivial term $\sim \rho^2$ comes from the two diagrams in Fig. 9b. The first diagram is generated by the $\rho^2 G$
term in the effective Lagrangean (4.7). The calculation yields

$$\frac{\rho^2(x\dot{y} - x \cdot y)}{4\pi^2 x^2 y^2 (x - y)^2} \quad (4.12)$$

(To verify this result it is easiest to differentiate (4.12) to amputate one leg of the first diagram in Fig.9b, i.e. to calculate \( \partial^2 / \partial x_\alpha \partial x_\alpha \) of eq.(4.12)). The second contact-type diagram is obtained when we take the last term in the effective Lagrangean (4.7) \( \sim \rho^2 \ddot{\phi} \phi \). The result is

$$- \frac{\rho^2}{8\pi^2 x^2 y^2} \quad (4.13)$$

and it is easy to see that the sum of these two diagrams reproduces the second term in the expansion (4.10). Further diagrams (Fig.9c etc.) describe the perturbative solution for the Green function, i.e. of the equation \( \nabla^2 G(x, y) = \delta^4(x - y) \) (leading to a power series in \( \rho^2 \) rather than \( g \) as discussed above). Contributions of these diagrams to \( G(x, y) \) reproduce the higher order terms (in \( \rho^2 \)) (4.10). In order to reproduce the scalar propagator (4.10) we need only the first four terms in the effective Lagrangean (4.7) \( \sim \rho^2 \), the higher operators \( \sim \rho^4 \) etc do not contribute.

The situation is more subtle for the \( W \) propagator

$$\langle A_\mu(x)A_\nu(y) \exp -L_I(0) \rangle \quad (4.14)$$

since the boson Green function in the instanton background does not exist due to zero modes of the operator \( \square_I \). In its place we use therefore the constraint Green function satisfying the equation

$$\square_{\mu\nu} G_{\alpha\nu}(x, y) = \delta_{\mu\nu} - \sum f^{(k)}_{\mu}(y) \quad (4.15)$$

where \( \Phi^{0(k)}(y) \) are the zero modes and \( f^{(k)}_{\mu} \) the constraints (see ref. [30]). Therefore, starting from the order \( \sim \rho^4 \) the second term in eq.(4.15) enters the game. It appears that the structure of the higher terms in the effective Lagrangean (4.3) \( \sim \rho^4 G_{\mu\nu}^2 \) and higher) should be correlated with eq.(4.15). As to the \( \rho^2 \) part of the \( W \) boson propagator, it could be verified to be reproduced by the diagrams in Fig. 10, at least on the mass shell (see ref.[16-18]).
Finally, let us discuss the next-to-leading terms in the effective Lagrangean \(4.3\) (with logarithmic accuracy). We start from the operator \(\rho^4(\bar{\phi}\phi)^2\). The coefficient in front of this operator can be obtained by comparison of the four-point diagrams in Fig.11 calculated in the instanton background with the analogous result in the effective Lagrangean approach. (One should not take into account the disconnected diagram in Fig.12a since it contributes only to the second term \((\rho^2\bar{\phi}\phi)^2\) of the expansion of the operator \(\exp(-2\pi^2 \rho^2\bar{\phi}\phi)\), see Fig.12b). Let us start with the diagram in Fig.11a where we can use the explicit expressions for the Green functions \(4.10\) and consider it at relatively large separations \(\rho^2 \ll x_i^2 \ll \mu^{-2}\). The logarithmic contribution to this diagram comes from the region of large \(z\) such as \(\rho^2 \ll z^2 \ll \mu^{-2}\) where \(\mu\) is the normalization point of the effective Lagrangean serving as IR-cutoff for the \(z\) integration (we consider \(\mu^2 \ll m^2\)). At that large separations one can use the asymptotic expansion \(4.10\) (for massless propagators since \(x_i, z \ll \mu^{-1}\)) keeping only the first three terms \(\sim \rho^6, \rho^2, \text{ and } \rho^4\). As we discussed above they correspond to the diagrams in Fig.9 a,b, and c respectively. The corresponding diagrams for the four-point Green function are shown in Fig. 13. It is easy to see that only the diagrams in Fig. 13a and b give contributions \(\sim \rho^4 \ln \rho^2\) (times the four tails corresponding to outgoing particles). After some combinatorics one obtains the \(\rho^4(\bar{\phi}\phi)^2\) term in the effective Lagrangean in the form

\[-3\lambda \rho^4 \pi^2 (\bar{\phi}\phi)^2 \ln(\rho^2 \mu^2) \quad (4.16)\]

where 3 is the sum of two 3/2’s coming from Fig. 13a and 13b.

It is convenient to calculate the coefficients in front of effective Lagrangean operators prior to the shift \(\phi \to \phi + ve/\sqrt{2}\) as thus one does not have to trace how the combinations \(\phi + ve/\sqrt{2}\) arise in the effective Lagrangean approach. In this case we have to take into account the negative vertex \(-\lambda v^2(\bar{\phi}\phi)\), it leads to a negative contribution \(-\lambda \rho^4 v^2(\bar{\phi}\phi)\). The corresponding coefficient is obtained from a comparison of the two-point diagrams in Fig.14. Again, in order to obtain the logarithmic part of the diagram in Fig.14a it is sufficient to keep the first three terms of the asymptotic expansion \(4.10\). The corresponding diagrams are shown in Fig. 15 and the result is

\[\pi^2 \lambda \rho^4 v^2(\bar{\phi}\phi) \ln(\rho^2 \mu^2) \quad (4.17)\]

with 1 being the sum of 1/4 coming from the diagram in Fig. 15a and 3/4 from Fig. 15b. (There are also diagrams similar to Fig. 13c-g which do not give the logarithmic contribution).
It is instructive to demonstrate how the coefficient (4.16) can be calculated (with logarithmic accuracy) directly within the effective Lagrangian approach. Indeed, the diagrams in Fig.13 which we did calculate are the perturbative diagrams describing the mixing of the operators

\[
G_I = -\frac{2\pi^2 i}{g} \frac{2}{\rho_I^2} \operatorname{Tr} \{ u_0 \bar{u}_I \sigma_\alpha \bar{\sigma}_\beta u_I \bar{u}_0 G_{\alpha \beta} \} \\
H_I = + 2\pi^2 \rho_I^2 \bar{\phi} \phi
\]

with the operator \( H^2 \sim \rho^4(\bar{\phi} \phi)^2 \). (This type of mixing is familiar, e.g. in the treatment of perturbative gluon corrections to weak decays, see e.g. ref.[32]). We have

\[
\frac{1}{2} (H_I)^{\mu_2^2} \times (H_I)^{\mu_2^2} \rightarrow \frac{1}{2} (H_I)^{\mu_1^2} \times (H_I)^{\mu_1^2} - \frac{3\lambda}{8\pi^2} (\ln \frac{\mu_2^2}{\mu_1^2}) H_I^2
\]

from the diagram in Fig. 13a and

\[
\frac{1}{2} (G_I)^{\mu_2^2} \times (G_I)^{\mu_2^2} \rightarrow \frac{1}{2} (G_I)^{\mu_1^2} \times (G_I)^{\mu_1^2} - \frac{3\lambda}{8\pi^2} (\ln \frac{\mu_2^2}{\mu_1^2}) H_I^2
\]

from the diagram in Fig. 13b (here 1/2 is the combinatorics factor). Note that we do not consider here the one-loop corrections to a single operator \( H_I \) (or \( G_I \)) since they correspond to the disconnected diagrams of the Fig.12 type and hence have nothing to do with the \( \rho^4 \) term of the effective Lagrangian. Now, since at \( \mu_2^2 \sim \rho^2 \) there are no logarithmic contributions to the coefficient in front of \( \rho^4(\bar{\phi} \phi)^2 \) we reobtain eq. (4.16) as a result of the evolution of the first two operators \( G_I \) and \( H_I \) in the effective Lagrangian (4.3) from the UV-cutoff \( \mu_2^2 = \rho^2 \) to the normalization point of the effective Lagrangian \( \mu_1 = \mu \).

Similarly, since the constant \( ( \text{in the non-shifted Lagrangian (3.2))} \lambda v^2 \) carries dimension there will be the mixing

\[
\frac{1}{2} (H_I)^{\mu_2^2} \times (H_I)^{\mu_2^2} \rightarrow \frac{1}{2} (H_I)^{\mu_1^2} \times (H_I)^{\mu_1^2} + \frac{\lambda}{8\pi^2} v^2 \rho^2 (\ln \frac{\mu_2^2}{\mu_1^2}) H_I
\]

coming from the diagram in Fig.15a and

\[
\frac{1}{2} (G_I)^{\mu_2^2} \times (G_I)^{\mu_2^2} \rightarrow \frac{1}{2} (G_I)^{\mu_1^2} \times (G_I)^{\mu_1^2} + \frac{3\lambda}{8\pi^2} v^2 \rho^2 (\ln \frac{\mu_2^2}{\mu_1^2}) H_I
\]
coming from Fig. 15b. Again, taking $\mu_2^2 = \rho^{-2}$ as the initial point of the evolution we reobtain eq. (4.17).

Now let us turn to the coefficient in front of $(\bar{\phi}\phi)^2$ proportional to $g^2$ which comes from the diagram in Fig. 11b. Again, if we consider this diagram at large $x_i^2 \beta \rho^2$ (but $\ll m^{-2}$) we can leave only the $\rho^0, \rho^2,$ and $\rho^4$ terms in the expansion of propagators in the instanton background. Unfortunately, the explicit form of the $\rho^4$ term in the expansion of the gluon propagator is unknown (and depends on the constraint, see the discussion above). But with our accuracy we do not need it since the logarithmic contribution comes only from the diagram in Fig. 16a and the corresponding term in the effective Lagrangian is

$$\frac{3}{8}g^2\rho^4\pi^2(\bar{\phi}\phi)^2 \ln(\rho^2\mu^2) \quad (4.23)$$

It corresponds to the same type of mixing of the two gauge operators with $\bar{\phi}\phi$ as in the eq. (4.20).

The method just described allows us also to determine the coefficients in front of the remaining operators of order $\rho^4$, namely the scalar-gauge operator $\sim \hat{\phi}G\phi$ and the gauge ones $\sim G \ast G$, as a result of the evolution of the operators $G_i$ and $H_I$ from $\mu_2^2 = \rho^{-2}$ to $\mu_1^2 = \mu^2$. (There exists also the operator $(\nabla \bar{\phi})(\nabla \phi)$ of the same dimension which, however, does not contribute to $U_{\text{int}}$ in the order $\epsilon^{8/3}$). After simple but somewhat lengthy calculations one obtains the mixing in the form

$$\frac{1}{2}(G_I)^{\mu_2^2} \times (G_I)^{\mu_2^1} \rightarrow \frac{1}{2}(G_I)^{\mu_2^1} \times (G_I)^{\mu_2^1} + \frac{g^2}{4\pi^2}(\ln\frac{\mu_2^2}{\mu_1^2})(O_I + \frac{5}{4}P_I) \quad (4.24)$$

and

$$\frac{1}{2}(H_I)^{\mu_2^2} \times (G_I)^{\mu_2^1} \rightarrow$$

$$\frac{1}{2}(H_I)^{\mu_2^1} \times (G_I)^{\mu_2^1} - \frac{g^2}{16\pi^2}(\ln\frac{\mu_2^2}{\mu_1^2})P_I \quad (4.25)$$

where

$$O_I = \frac{2\pi^4\rho^4}{g^2}(G_{\alpha \beta}^a G_{\alpha \beta}^a + 2G_{\alpha \beta}^a \tilde{G}_{\alpha \beta}^a)$$

$$- \text{Tr}\{u_0 \bar{u}_I \sigma_\alpha \bar{\sigma}_\beta u_I \bar{u}_0 G_{\beta \gamma} \} \text{Tr}\{u_0 \bar{u}_I \sigma_\beta \bar{\sigma}_\gamma u_I \bar{u}_0 G_{\alpha \gamma} \})$$

$$P_I = \frac{2\pi^4\rho^4}{g}i\bar{\phi}\{u_0 \bar{u}_I \sigma_\mu \bar{\sigma}_\nu u_I \bar{u}_0, G_{\mu \nu} \}\phi$$

(4.26)
We have used here the gauge-invariant external-field technique (see e.g. ref.[32]). In terms of usual perturbative diagrams, the mixing in eq.(4.24) with the gauge operator $O_I$ and the scalar-gauge operator $P_I$ come from the diagrams in Figs.17 and 18 respectively and the mixing (4.25) is depicted in Fig.19. Again, evolving the $\rho^2 G$ and $\rho^2 \bar{\phi}\phi$ operators from $\mu_2^2 = \rho^{-2}$ (where the coefficients in front of the gauge-operators contain no logarithmic terms) to $\mu_1^2 = \mu^2$ we obtain the corresponding contribution to the effective Lagrangian in the form

$$\frac{g^2}{4\pi^2}(\ln \rho^2 \mu^2)(O_I + P_I) \quad (4.27)$$

To find the non-logarithmic term one should really calculate the four-particle amplitudes in the instanton background (and use the exact constrained Green function instead of the first few terms of its asymptotic expansion) but at logarithmic accuracy we could avoid this terrifying perspective. The final form of the effective Lagrangean up to $\rho^4 \ln \rho^2$ terms is

$$L' = G_I + H_I + \frac{g^2}{4\pi^2}(\ln \rho^2 \mu^2)(O_I + P_I + \frac{3}{8} H_I^2) + \frac{\lambda}{4\pi^2}(\ln \rho^2 \mu^2)(-3H_I^2 + 2\pi^2 v^2 \rho^2 H_I) + O(\rho^4 \cdot \text{const}) \quad (4.28)$$

In order to find the corresponding Lagrangean after spontaneous symmetry breaking we shift the fields according to $\phi \rightarrow \phi + ve/\sqrt{2}$ ($\bar{\phi} \rightarrow \bar{\phi} + \bar{e}v/\sqrt{2}$), and obtain

$$L' = G_I + H_I + V_I + \pi^2 v^2 \rho^2 + \frac{g^2}{4\pi^2} \ln \rho^2 \mu^2 (O_I + P_I + Q + E_I) + \left( \frac{3}{8} (H_I^2 + 2H_I V_I + V_I^2 + 2\pi^2 v^2 \rho^2 H_I + 2\pi^2 v^2 \rho^2 V_I + \pi^2 v^4 \rho^4) \right) + \frac{\lambda}{4\pi^2} \ln \rho^2 \mu^2 (-3H_I^2 - 6H_I V_I - 3V_I^2 - 4\pi^2 v^2 \rho^2 H_I + 4\pi^2 v^2 \rho^2 V_I - \pi^2 v^4 \rho^4) + O(\rho^4 \cdot \text{const}) \quad (4.29)$$

where we used the notations

$$V_I = \sqrt{2} \pi^2 \rho^2 v(\bar{\phi}e + \bar{e}\phi)$$

$$Q_I = -\frac{g}{2} \pi^4 \rho^4 \nu \{u_0 \bar{u}_I \sigma_\mu \sigma_\nu u_I \bar{u}_0, G_{\mu\nu}\} e + \bar{e} \{u_0 \bar{u}_I \sigma_\mu \sigma_\nu u_I \bar{u}_0, G_{\mu\nu}\} \phi$$

$$E_I = -\frac{g}{2} \pi^4 \rho^4 \nu^2 \{u_0 \bar{u}_I \sigma_\mu \sigma_\nu u_I \bar{u}_0, G_{\mu\nu}\} e$$

(4.30)
The answer for the antiinstanton effective Lagrangian $L_{\bar{I}}$ is obtained by the substitution $u\bar{u}_0 \rightarrow \bar{u}_I$ and $u_0 \bar{u} \rightarrow u_I$ (which implies also changing of all $\sigma$’s into $\bar{\sigma}$’s and vice versa, e.g. $G_{\tilde{I}} = -\frac{2\pi^2 i}{g^3} \rho_1^2 \text{Tr}\{u_I\bar{\sigma}_\alpha\sigma_\beta\bar{u}_I G_{\alpha\beta}\}$).

In the next section we shall use this effective Lagrangian to reproduce the valley result for the $\epsilon^{8/3} \ln \epsilon$ term of the holy grail function.

5. Effective Lagrangean calculation of the holy grail function

In the effective Lagrangean approach the amplitude for forward scattering in the $I\bar{I}$ background (3.1) can be written as

$$A(p, k) = \int \frac{d\rho_1}{\rho_1^3} d(\rho_1) \int \frac{d\rho_2}{\rho_2^3} d(\rho_2) \int dR \int du \langle A^a_\mu(p) A^b_\nu(k) A^a_\mu(-p) A^b_\nu(-k) L^I_\psi(0) \exp(-L_I(0)) L^I_{\bar{\psi}}(R) \exp(-L_{\bar{I}}(R)) \rangle$$

(5.1)

where $L_I$ and $L_{\bar{I}}$ are given in eq. (4.28) and we choose $u_0 = u_I$ so $u \equiv u_I$ will be the matrix of relative $I\bar{I}$ orientation. The coefficients in front of the effective Lagrangean operators which come from the integration over high momenta and can therefore be calculated prior or after the shift $\phi \rightarrow \phi + ve/\sqrt{2}$. In contrast the matrix elements of the effective Lagrangean we are considering in this section are determined by the region of small momenta and they have to be calculated for the physical, massive theory, i.e. after the shift $\phi \rightarrow \phi + ve/\sqrt{2}$, in order to avoid infrared divergences. As discussed in section 1 we can neglect hard-hard and hard-soft corrections at the level of accuracy we are interested in. Then, $A_\mu(p)$ is given by the large-distance asymptotics of the sum of $I$ and $\bar{I}$ fields (see eq.(3.5)) and the expression (5.1) reduces to

$$A(p, k) \equiv \int d\rho_1 d\rho_2 dR du d(\rho_1) d(\rho_2) \exp(iER_0) \exp(-L_I(0)) \exp(-L_{\bar{I}}(R))$$

(5.2)

with exponential accuracy. The BNV cross section is given by the discontinuity of this amplitude continued to imaginary energies (cf. eq.(3.7)) :

$$\sigma_{BNV} \equiv \text{Im} \int d\rho_1 d\rho_2 dR du d(\rho_1) d(\rho_2) \exp(RE_0) \exp(-L_I(0)) \exp(-L_{\bar{I}}(R))$$

(5.3)

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It is convenient to separate corrections to the instanton density given by the disconnected contributions to the correlator in eq. (5.3) from the $\bar{I}I$ interaction. We have
\[
\langle \exp(-L_I(0)) \exp(-L_I(R)) \rangle = \exp\left(-S_{I_H}^I - S_{\bar{I}_H}^I + U_{\text{int}}(\rho_1, \rho_2, R, u)\right) \tag{5.4}
\]
where $S_{I_H}^I$ and $S_{\bar{I}_H}^I$ are the additional contributions to the instanton action $8\pi^2/g^2$ due to the Higgs condensate and $U_{\text{int}}$ is the interaction potential.

Using the standard virial expansion we obtain
\[
S_{I_H}^I = \langle L_I \rangle - \frac{1}{2} \left(\langle L_I^2 \rangle - \langle L_I \rangle^2\right) + \frac{1}{6} \left(\langle L_I^3 \rangle - 3\langle L_I \rangle^2\langle L_I \rangle + 2\langle L_I \rangle^3\right) + \ldots \tag{5.5}
\]
(similarly for $\bar{I}$) and
\[
U_{\text{int}} = \langle L_I L_I \rangle - \langle L_I \rangle \langle L_I \rangle - \frac{1}{2} \left(\langle L_I^2 L_I \rangle + \langle L_I L_I^2 \rangle - \langle L_I \rangle^2\langle L_I \rangle - \langle L_I^2 \rangle \langle L_I \rangle\right) - 2 \left[\langle L_I \rangle^2 - \langle L_I \rangle \langle L_I \rangle\right] - \frac{1}{2} \left(\langle L_I^2 \rangle - \langle L_I \rangle^2\right) + \ldots \tag{5.6}
\]
Since each $L_I$ contains at least one power of $\rho^2$ we have to keep only the first few terms of the virial expansions (5.5) and (5.6).

Let us start with the disconnected contributions of (5.4) corresponding to the corrections to the instanton density $d(\rho) \sim e^{-8\pi^2/g^2}$ coming from $S_{I_H}^I$ and $S_{\bar{I}_H}^I$ (see eq. (5.5)). The first term of the r.h.s. of eq.(5.5) is obtained simply by inspection of eq.(4.28):
\[
\langle L_I \rangle = \pi^2 v^2 \rho_1^2 - \frac{\pi^2 \lambda v^4}{4} \rho_1^4 \ln(\mu^2 \rho_1^2) + \frac{3\pi^2 g^2 v^4}{32} \rho_1^4 \ln(\mu^2 \rho_1^2) \tag{5.7}
\]
The second and third term correspond to the diagrams in Fig. 20a and 20b respectively. The logarithmic contributions come from the loop momenta $\rho^{-2} \gg p^2 \gg \mu^2$. The second term on the r.h.s. of eq. (5.5) is
\[
-\frac{1}{2} \left(\langle L_I^2 \rangle - \langle L_I \rangle^2\right) = -\frac{1}{2} \langle V_I^2 \rangle - \frac{1}{2} \langle G_I^2 \rangle = -2\pi^4 v^2 \rho_1^4 \int \frac{dp}{(2\pi)^2} \left\{ \frac{1}{p^2 + m_H^2} \right\}

- \frac{24\pi^4 \rho_1^4}{g^2} \int \frac{dp}{(2\pi)^2} \left\{ \frac{p^2}{p^2 + m_W^2} \right\} \tag{5.8}
\]
Subtracting, as usual, the quadratic (and higher) ultraviolet divergences, we obtain at the logarithmic accuracy we are interested in
\[
-\frac{1}{2} \left(\langle L_I^2 \rangle - \langle L_I \rangle^2\right) = \frac{\pi^2 v^2 m_H^2 \rho_1^4}{8} \ln\frac{\mu^2}{m_H^2} - \frac{3\pi^2 m_W^4 \rho_1^4}{2g^2} \ln\frac{\mu^2}{m_W^2} \tag{5.9}
\]
Combining eq.(5.7) and (5.9) we see that the normalization point $\mu$ drops out, as it should be, and the final result reads:

$$S_{IH} = \pi^2 v^2 \rho_1^2 - \frac{\pi^2 m_H^2 v^2}{8} \rho_1^4 \ln(m_H^2 \rho_1^2) + \frac{3\pi^2 m_W^4}{2g^2} \rho_1^4 \ln(m_W^2 \rho_1^2) \quad (5.10)$$

The higher terms in the expansion (5.5) contain extra powers of $\rho^2 m^2$ such that we can disregard them. $S_{\bar{H}}$ is obtained from $S_{IH}$ by simply substituting $\rho_1 \rightarrow \rho_2$.

It is worth noting that the logarithmic contribution to eq.(5.8) obtained by expanding the propagators in powers of $m_H^2$ and $m_W^2$ can be depicted by the same diagrams as those of Fig. 20, in which case the vertices denote $m_H^2$ and $m_W^2$ mass insertions. Then, the normalization point $\mu$ has to be interpreted as boundary between the high momentum region, contributing to the coefficient functions in front of the effective Lagrangean operators (see eqs. (4.28) and (5.7)), and the low momentum region, contributing to the matrix elements of these operators (see eq.(5.9)).

Let us turn now to the interaction between $I$ and $\bar{I}$. For our accuracy the expansion (5.6) takes the form (there is also a contribution $\sim \langle G_I^2(0) V_I(R) \rangle$ of order $\epsilon^{8/3}$ but without $\ln \epsilon$):

$$U_{\text{int}} = \langle G_I(0) G_I(R) \rangle + \langle V_I(0) V_I(R) \rangle - \frac{1}{2} \left( \langle G_I(0) G_I(R) \rangle + \langle G_I(0) G_I^2(R) \rangle \right)$$
$$- \left( \langle G_I(0) G_I(R) V_I(R) \rangle + \langle G_I(0) V_I(0) G_I(R) \rangle \right)$$
$$+ \frac{1}{4} \left( \langle G_I^2(0) G_I^2(R) \rangle - \langle G_I^2(0) \rangle \langle G_I^2(R) \rangle \right)$$
$$+ \frac{g^2}{4\pi^2} \ln(\rho_1^2 \mu^2) \left( - \frac{1}{2} \langle O_I(0) G_I^2(R) \rangle + \langle E_I(0) G_I(R) \rangle \right)$$
$$+ \frac{g^2}{4\pi^2} \ln(\rho_2^2 \mu^2) \left( - \frac{1}{2} \langle G_I^2(0) O_I(R) \rangle + \langle G_I(0) E_I(R) \rangle \right) \quad (5.11)$$

The first term corresponding to the diagram shown in Fig. 21a is simply

$$\langle G_I(0) G_I(R) \rangle = - \frac{16\pi^4}{g^2} \int \frac{dq}{(2\pi)^4} e^{-iqR} \frac{4(q \cdot u)^2 - q^2}{q^2 + m_W^2} = \frac{32\pi^2 \rho_1^2 \rho_2^2}{g^2 R^4} \left( \frac{(u \cdot R)^2}{R^2} - 1 \right)$$
$$\left\{ 1 - \frac{m_W^2 R^2}{8} + \frac{m_W^4 R^4}{64} + O(m_W^6 R^6) \right\} \quad (5.12)$$
The first two terms in braces have the order $\epsilon^{4/3}$ and $\epsilon^2$ in the saddle point (1.2). (The first term is in fact the dipole-dipole interaction (1.8).) The third term contains no logarithmic contribution and therefore exceeds our accuracy. Similarly (see Fig. 21b),

$$
\langle V_I(0)V_I(R) \rangle = 4\pi^2 v^4 \rho_1^2 \rho_2^2 \int \frac{dq}{(2\pi)^4} e^{-iqR} \frac{1}{q^2 + m_H^2} \\
= \frac{\pi^2 v^2 \rho_1^2 \rho_2^2}{R^2} \left( 1 - \frac{m_H^2 R^2}{4} \ln(m_H^2 R^2) + O(m_H^4 R^4) \right)
$$

(5.13)

where the first term is $\sim \epsilon^2$ and the second $\epsilon^{8/3} \ln \epsilon$.

The third term on the r.h.s. of eq. (5.11) comes from the diagrams shown in Fig. 22. The graph in Fig. 22a contributes

$$
\langle G_I^2(0)G_I(R) \rangle = \frac{16\pi^2 g^2 \rho_1^2 \rho_2^2 (\rho_1^2 + \rho_2^2)}{(2\pi)^4} \int \frac{dq}{q^2 + m_W^2} \int \frac{dp}{(2\pi)^4} \int \frac{dp}{(2\pi)^4} e^{-iqR} \\
\frac{16(q \cdot u) ((p \cdot u)(q \cdot (2p-q)) - (p \cdot (2p-q))(q \cdot u)) + 8(p^2 q^2 - (p \cdot q)^2)}{(p^2 + m_W^2)((q-p)^2 + m_W^2)}
$$

(5.14)

Expanding the numerators in powers of $m_W^2$ we obtain

$$
\frac{2\pi^2}{g^2} \rho_1^2 \rho_2^2 (\rho_1^2 + \rho_2^2) \int \frac{dq}{(2\pi)^4} e^{-iqR} \left\{ \frac{\Gamma(2 - d/2)}{(q^2)^{2-d/2}} - \frac{1}{2 - \frac{d}{2}} \right\} \\
\left( \frac{4(u \cdot R)^2}{q^2} - 1 \right) \left( q^2 + 5m_W^2 + O(m_W^4/q^2) \right)
$$

(5.15)

where the second term in braces is the counterterm added in the $\overline{MS}$ scheme (see e.g. ref.[29]). Thus

$$
- \frac{1}{2} \left( \langle G_I^2(0)G_I(R) \rangle + \langle G_I(0)G_I^2(R) \rangle \right) = \frac{64\pi^2}{g^2} \left( \frac{4(R \cdot u)^2}{q^2} - 1 \right) \\
\frac{\rho_1^2 \rho_2^2 (\rho_1^2 + \rho_2^2)}{R^6} \left( 1 + \frac{5}{16} m_W^2 R^2 \ln(R^2 \mu^2) + O(m_W^2 R^2) \right)
$$

(5.16)

where we have added the contribution of the diagram in Fig. 22b (due to the commutator term in $G_{\mu\nu}$) which is

$$
- \frac{16\pi^2}{g^2} \frac{\rho_1^2 \rho_2^2 (\rho_1^2 + \rho_2^2)}{R^6} \left( \frac{4(R \cdot u)^2}{q^2} - 1 \right) \left( 1 + O(m_W^2 R^2) \right)
$$

(5.17)
The factor 5/16 in eq.(5.17) is the sum of 3/8 coming from the mass insertion shown in Fig. 23a and -1/16 coming from Fig. 23b. The first term in braces in eq.(5.16) is of order $\epsilon^2$ and the second of order $\epsilon^{8/3} \ln \epsilon$. Another term of this order is

$$- \langle G_1(0) G_i(R)V_i(R) \rangle - \langle G_1(0)V_i(0)G_i(R) \rangle =$$

$$-v^2 \rho_1^2 \rho_2^2 \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right) \ln(R^2 \mu^2)$$

(5.18)

generated by Fig. 24.

Similarly to the case of $S_H^I$ considered above, the logarithmic $\mu$ dependence of the matrix elements of the operator-correlators from the r.h.s. of eq.(5.11) has to be cancelled by the $\ln \mu^2$ terms coming from the coefficient functions in front of the operators in the effective Lagrangean. For example, for the two correlators just considered the relevant operator is

$$\left( \frac{g^2}{4\pi^2} \right) \ln(\rho_1^2 \mu^2) E$$

and we obtain (see Fig. 25)

$$\frac{g^2}{4\pi^2} \ln(\mu^2 \rho_1^2) \langle E_1(0)G_i(R) \rangle + \frac{g^2}{4\pi^2} \ln(\mu^2 \rho_2^2) \langle G_i(0)E_i(R) \rangle$$

$$= \frac{4\pi^2}{g^2} \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right) \rho_1^2 \rho_2^2 \left( \rho_1^2 \ln(\mu^2 \rho_1^2) + \rho_2^2 \ln(\mu^2 \rho_2^2) \right)$$

(5.19)

Combining eqs.(5.17-19) gives the contribution to $U_{\text{int}}$ in the form

$$\frac{64\pi^2}{g^2} \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right) \left\{ \frac{\rho_1^2 \rho_2^2 (\rho_1^2 + \rho_2^2)}{R^2} \right\}$$

$$+ \frac{m_W^2 \rho_1^2 \rho_2^2}{4R^4} \left( \rho_1^2 \ln(R^2/\rho_1^2) + \rho_2^2 \ln(R^2/\rho_2^2) \right)$$

(5.20)

which in fact does not depend on $\mu$. Again, it is instructive to note that we calculated both times the same diagrams (Fig. 23a, 23b, and 24) with the loop momenta $\mu^2 \gg p^2 \gg R^2$ ascribed to matrix elements of correlators (eq. (5.17) and (5.18)) and momenta $\rho^2 \gg p^2 \gg \mu^2$ to the coefficient in front of the operator $E$ ($=P$, see diagrams in Fig. 18 and 19).

The last term of order $\epsilon^{8/3} \ln \epsilon$ is represented by the correlator $\langle G_1^2(0)G_2^2(R) \rangle$ (see Fig.26). The explicit calculation of this correlator is rather tedious (see Ref.[19]) but with logarithmic accuracy the answer can be easily restored from eq.(4.24) since we know that the logarithmic dependence on $\mu$ in this
correlator should be canceled with the $\ln(\mu^2 \rho^2)$ term in the coefficient function in front of the operator $O_I$ in the effective Lagrangian. The result is

$$-\frac{1}{2} \left( \langle G_I^2(0) G_I^2(R) \rangle - \langle G_I^2(0) \rangle \langle G_I^2(R) \rangle \right)$$

\[(5.21)\]

$$\frac{-64\pi^2}{g^2} \rho_1^4 \rho_2^4 \left( 6 + 2 \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right) \right)^2 - 2 \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right) \ln(R^2 \mu^2)$$

(it is worth noting that similar correlator but with $G_I^2(R)$ corresponding to the interaction of the two instantons vanishes as one should expect from general considerations). The result eq. (5.21) coincides with the calculation in ref. [19] for the $I$ and $\bar{I}$ with maximal attractive orientations which only contribute to $F(\epsilon)$, but for arbitrary orientations it disagrees with the answer in ref. [19] unfortunately. We have (see Fig. 27)

$$-\frac{g^2}{8\pi^2} \ln(\mu^2 \rho_1^2) \langle O_I(0) G_I^2(R) \rangle - \frac{g^2}{8\pi^2} \ln(\mu^2 \rho_2^2) \langle G_I^2(0) O_I(R) \rangle$$

\[(5.22)\]

$$= \frac{64\pi^2}{g^2} \rho_1^4 \rho_2^4 \left( 6 + 2 \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right) \right)^2 - 2 \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right) \ln(\rho_1 \rho_2 \mu^2)$$

So the contribution to $U_{\text{int}}$ takes the form

$$-\frac{64\pi^2}{g^2} \rho_1^4 \rho_2^4 \left( 6 + 2 \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right) \right)^2 - 2 \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right) \ln(R^2)$$

\[(5.23)\]

Note that by calculating the coefficient in front of the two-gluon or $(O_I)$ in the effective Lagrangean we, in fact, reproduced the result obtained in ref. [19] by a hard two-loop calculation.

Thus the final form of $U_{\text{int}}$ (up to $\epsilon^{8/3} \ln \epsilon$) is

$$U_{\text{int}} = \frac{32\pi^2}{g^2} \rho_1^2 \rho_2^2 \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right)$$

$$-\frac{64\pi^2}{g^2} \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right) \rho_1^2 \rho_2^2 \left( \frac{\rho_1^2 + \rho_2^2}{R^2} + \frac{m_W^2 R^2}{16} \right)$$

$$+ \frac{\pi^2 v^2 R^2}{R^2} - \frac{64\pi^2 \rho_1^4 \rho_2^4}{g^2} \left( 6 + 2 \left( \frac{4(R \cdot u)^2}{R^2} - 1 \right) \right)$$

38
The first two terms in $U_{\text{int}}$ for the pure gauge sector reproduce the first two terms of the expansion of the conformal expression (2.31) but the third deviates from it. This means that a calculation of $U_{\text{int}}$ using the effective Lagrangean corresponds to using a different valley than that in eq.(2.26). However, the final result for the holy grail function $F(\epsilon)$ is the same in both cases. Indeed we have (see eq. (5.4))

\[
\begin{align*}
-2\left(\frac{4(R \cdot u)^2}{R^2} - 1\right) \ln\frac{R^2}{\rho_1 \rho_2} & - \frac{16\pi^2}{g^2} \left(\frac{4(R \cdot u)^2}{R^2} - 1\right) \frac{m_W^2 \rho_1^2 \rho_2^2}{R^4} \left(\frac{\rho_1^2 \ln \frac{R^2}{\rho_1^2} + \rho_2^2 \ln \frac{R^2}{\rho_2^2}}{R^2} \right) \\
- \frac{\pi^2 v^2}{4} m_H^2 \rho_1^2 \rho_2^2 \ln(m_H^2 R^2)
\end{align*}
\]

The first two terms in $U_{\text{int}}$ for the pure gauge sector reproduce the first two terms of the expansion of the conformal expression (2.31) but the third deviates from it. This means that a calculation of $U_{\text{int}}$ using the effective Lagrangean corresponds to using a different valley than that in eq.(2.26). However, the final result for the holy grail function $F(\epsilon)$ is the same in both cases. Indeed we have (see eq. (5.4))

\[
\begin{align*}
- S_I^H - S_I^\bar{H} + U_{\text{int}} &= -\pi^2 v^2 (\rho_1^2 + \rho_2^2) + \frac{\pi^2 v^2 \rho_1^2 \rho_2^2}{R^2} \\
&+ \frac{32\pi^2 \rho_1^2 \rho_2^2}{g^2 R^4} \left(\frac{4(R \cdot u)^2}{R^2} - 1\right) - \frac{64\pi^2}{g^2} \left(\frac{4(R \cdot u)^2}{R^2} - 1\right) \frac{\rho_1^2 \rho_2^2}{R^4} \left(\frac{\rho_1^2 + \rho_2^2}{R^2} + \frac{m_W^2 R^2}{16}\right) \\
&- \frac{\pi^2 v^2 m_H^2}{8} \left(\rho_1^4 \ln(m_H^2 \rho_1^2) + \rho_2^4 \ln(m_H^2 \rho_2^2)\right) + \frac{3\pi^2 m_W^4}{2 g^2} \left(\rho_1^4 \ln(m_W^2 \rho_1^2) + \rho_2^4 \ln(m_W^2 \rho_2^2)\right) \\
&- \frac{64\pi^2 \rho_1^4 \rho_2^4}{g^2 R^8} \left(6 + 2 \left(\frac{4(R \cdot u)^2}{R^2} - 1\right)^2 - 2 \left(\frac{4(R \cdot u)^2}{R^2} - 1\right)\right) \ln\frac{R^2}{\rho_1 \rho_2} \\
&- \frac{16\pi^2}{g^2} \left(\frac{4(R \cdot u)^2}{R^2} - 1\right) \frac{m_W^2 \rho_1^2 \rho_2^2}{R^4} \left(\frac{\rho_1^2 \ln \frac{R^2}{\rho_1^2} + \rho_2^2 \ln \frac{R^2}{\rho_2^2}}{R^2} \right) \\
&- \frac{\pi^2 v^2}{4} m_H^2 \rho_1^2 \rho_2^2 \ln(m_H^2 R^2)
\end{align*}
\]

Evaluating this expression at the saddle point (1.2) one reproduces the valley result (1.5) although the explicit form of equ.(5.24) and (3.45) differ.
6. Conclusions

We have calculated the $\epsilon^{8/3} \ln \epsilon$ term of the holy grail function with two methods, namely the valley method and the effective Lagrangean approach. Though the final result for both methods is the same they differ completely in the way it is obtained. The effective Lagrangean approach is more pictorial and also it gives us an opportunity to deal with the multi-instanton amplitudes in a simple way: just expand several times in powers of $L_{I}^{eff}$ and $L_{\bar{I}}^{eff}$ and calculate the obtained Feynman diagrams with additional multi-W (and multi-Higgs) vertices. On the other hand, the valley method enables us to use the (tree-level) conformal invariance in a pure gauge sector which gives the expansion of $U_{int}^{g}$ in powers of conformal parameter $1/\xi = \rho_1 \rho_2 / (R^2 + \rho_1^2 + \rho_2^2)$ instead of reconstructing it from the expansion in powers of $\rho_1^2 / R^2$ and $\rho_2^2 / R^2$. Also, the valley method saves us from calculating the two-loop diagrams in Fig.26 (at a price of more complex contributions in the gauge-Higgs sector).

Of course, the question one really would like to answer is how these instanton-induced cross sections behave at SSC energies. Unfortunately, as we mentioned above, in order to answer this question we have to continue the expansion of $F(\epsilon)$ in $\epsilon$ which implies that we must take into account not only the $II$ potential $U_{int}$ but the hard-hard and hard-soft quantum corrections as well. Also, this should be done in the Minkowski space due to the reasons discussed in the Introduction. The continuation of the effective-Lagrangian approach to the Minkowski space is quite direct - one simply should take care of the trivial $i$’s and signs according to the general rules of Wick rotation. (In fact, it is the most simple way to understand the instanton-induced amplitudes in the Minkowski space). The valley method can also be modified to meet our purposes. One can write down the functional integral directly for the cross sections with BNV in the final state (at a price of doubling of the number of fields). The valleys for this double-set functional integral determine the cross sections with BNV in the leading semiclassical approximation, see ref.[32]. However, as we mentioned above, at large energies the quantum corrections are also essential and one faces the problem of determining the high-energy behavior of the propagators in the background of these valley fields. The similar problem within the effective-Lagrangian approach corresponds to the summation of the diagrams of the type shown in Fig.10 but
with both $I$ and $I'$ effective vertices taken into account. We hope to return to these questions in further publications.

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Figure Captions:

Fig. 1: Perturbative diagrams reproducing the classical instanton fields \((a)\) and the propagator in the instanton background \((b)\) in the effective-Lagrangian approach. Small open circle denotes the instanton-induced effective vertex \(\exp\left(\frac{2\pi i}{g} \rho^2 Tr\{\sigma_\alpha \bar{\sigma}_\beta G_{\alpha\beta}(x)\}\right)\).

Fig. 2: Perturbative diagrams for \(I\bar{I}\) interaction in the effective-Lagrangian approach. Small full circle denotes the antiinstanton effective vertex \(\exp\left(\frac{2\pi i}{g} \rho^2 Tr\{u\bar{\sigma}_\alpha \sigma_\beta \bar{u} G_{\alpha\beta}(x)\}\right)\).

Fig. 3: The \(\epsilon^2\) contributions to the instanton-antiinstanton interaction due to an additional W (Fig. 1a), mass insertions (Fig. 1b) and Higgs exchange (Fig. 1c).

Fig. 4: Two examples of \(\epsilon^8/3 \ln \epsilon\) contributions to the instanton-antiinstanton interaction.

Fig. 5: Two discontinuities of the hard-hard correction corresponding to the cross section with BNV \((a)\) and without \((b)\).

Fig. 6: Illustration of the valley configuration.

Fig. 7: Large-distance expansion of the instanton field in eq. (4.5) in terms of perturbative diagrams induced by effective Lagrangian.

Fig. 8: Expansion of the scalar component of the instanton. A cross on the end of a scalar line denotes the Higgs condensate.

Fig. 9: Scalar propagator in the instanton background as a sum of the perturbative diagrams in the effective-Lagrangian approach.

Fig. 10: Perturbative diagrams for the W boson propagator in the field of an instanton.
Fig.11: Quartic scalar Green function in the instanton background - connected diagrams.

Fig.12: Disconnected part of the quartic Green function coming from the square of scalar propagator in the instanton background.

Fig.13: Quartic scalar Green function in the effective-Lagrangian approach (a and b diagrams represent the logarithmic mixing of the operators $H_I$ and $G_I$ with the four-Higgs operator $\rho^4(\bar{\phi}\phi)^2$)

Fig.14: Additional negative contribution to the Green function of the non-shifted scalar field in the instanton background due to the vertex $-\lambda v^2\bar{\phi}\phi$.

Fig.15: The same additional term in the effective-Lagrangian approach (only the logarithmic diagrams corresponding to mixing (4.21) and (4.22) are depicted).

Fig.16: The leading logarithmic perturbative diagram for the contribution of the type of Fig.11b to the quartic scalar Green function in the effective-Lagrangian approach (it corresponds to the mixing of $G_I$ with the four-Higgs operator).

Fig.17: One-loop diagrams for mixing of the operator $G_I$ with the two-gluon operators $\sim \rho^4 G * G$.

Fig.18: Mixing of $G_I$ with the scalar-gluon operator $\sim \rho^4\bar{\phi}G\phi$.

Fig.19: Diagram for the mixing (4.25).

Fig.20: Leading diagrams for the corrections to the instanton density $\sim m^2_H v^2 \rho^4 (a)$ and $\sim \frac{1}{g_W^2} m_{W^*}^4 \rho^4 (b)$ in the effective-Lagrangian approach.

Fig.21: The leading contributions to $I\bar{I}$ interaction due to exchange by W or Higgs boson given by the correlators $\langle G_I(0)G_I(R) \rangle (a)$ and $\langle V_I(0)V_I(R) \rangle$
Fig. 22: The next-to-leading graphs given by the correlator \(G_I(0)^2 G_I(R)\).

Fig. 23: The logarithmic contributions to \(U_{\text{int}} \sim \epsilon^{8/3} \ln \epsilon\) coming from expanding the propagators in Fig. 22 in powers of \(m_W\).

Fig. 24: The \(\sim \epsilon^{8/3} \ln \epsilon\) contribution to \(U_{\text{int}}\) coming from the correlator \(G_I(0)V_I(0)G_I(R)\).

Fig. 25: The diagram for the correlator \(E_I(0)G_I(R)\). (The coefficient \(\frac{g^2}{4\pi^2} \ln(\rho_2^2 \mu^2)\) in front of this correlator corresponds to the region of large momenta \(p^2 \gg \rho^2 \gg \mu^2\) in the Feynman graphs shown in Figs. 23 and 24).

Fig. 26: The two-loop diagrams for the \(\frac{\pi^2 \rho_1^4 \rho_2^4}{4\pi^2} \ln \left(\frac{p^2}{\rho_1 \rho_2}\right)\) part of \(U_{\text{int}}\) given by the correlator \(G_I^2(0)G_I^2(R)\).

Fig. 27: The first-order diagrams for the correlators \(O_I(0)G_I^2(R)\) and \(G_I^2(0)O_I(R)\).