QUANTITATIVE UNIQUE CONTINUATION FOR THE HEAT EQUATION WITH COULOMB POTENTIALS

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Abstract. In this paper, we establish a Hölder-type quantitative estimate of unique continuation for solutions to the heat equation with Coulomb potentials in either a bounded convex domain or a $C^2$-smooth bounded domain. The approach is based on the frequency function method, as well as some parabolic-type Hardy inequalities.

1. Introduction and main results. This paper is concerned with the quantitative unique continuation property for solutions to the heat equation with singular Coulomb potentials at the origin

$$
\begin{aligned}
\partial_t u - \Delta u - \frac{k}{|x|} u &= 0 \quad \text{in } \Omega \times (0, +\infty), \\
u &= 0 \quad \text{on } \partial\Omega \times (0, +\infty), \\
v(\cdot, 0) &\in L^2(\Omega),
\end{aligned}
$$

(1)

where $k \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded Lipschitz domain which contains the origin. It is well-known that, for each initial value $v(\cdot, 0) \in L^2(\Omega)$ and each $T > 0$, Equation (1) has a unique solution $v \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ (cf., e.g., [18], [16] or [29]).

Recall that the space-like strong unique continuation property for any solution $v$ to parabolic equations is as follows. For any point $x_0 \in \Omega$ and any time $t_0 > 0$, if $v(\cdot, t_0)$ vanishes of infinite order at the point $x_0$ (i.e.,

$$
\int_{B_r(x_0)} |v(x, t_0)|^2 \, dx = o(r^m) \quad \text{as } r \to 0^+,
$$

for any positive integer $m$), then $v(\cdot, t_0) \equiv 0$ in $\Omega$. Furthermore, if $v$ is zero on the lateral boundary $\partial\Omega \times (0, t_0)$, then $v \equiv 0$ in $\Omega \times (0, t_0)$ by the backward uniqueness property, see [8] or [17] for instance. In other words, when a solution of parabolic equations enjoys such a space-like strong unique continuation property, then it either vanishes in $\Omega$ or cannot vanishes of infinite order at any point in $\Omega$. This kind of

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unique continuation for solutions to general second order parabolic equations have been established in the works [3, 8, 9, 12, 17] and references therein.

Moreover, quantitative estimates of strong unique continuation for second order parabolic equations, such as the doubling property and the two-ball one-cylinder inequality, have been well understood (see, e.g., [9, 23, 24, 26]). We refer to [30] for a more extensive review on this subject. We also mention that the unique continuation property for stochastic parabolic equations has been recently studied in [19, 20, 35].

The aim of this paper is to establish the following quantitative unique continuation: Given a nonempty open subset $\omega$ of $\Omega$, there are positive constants $N = N(\Omega, \omega, k, n, T) \geq 1$ and $\alpha = \alpha(\Omega, \omega, k, n)$ with $\alpha \in (0, 1)$ such that for any solution $u$ to Equation (1) and for any $T \in (0, 1]$,

$$\|u(\cdot, T)\|_{L^2(\Omega)} \leq N \|u(\cdot, 0)\|_{L^2(\Omega)}^{1-\alpha}$$

for all $u(\cdot, 0) \in L^2(\Omega)$. (2)

This kind of Hölder-type quantitative estimate of unique continuation was first established in [23] for the heat equation with bounded potentials in a bounded convex domain. Later on, it has been extended in [26] to the case of bounded domain with a $C^2$-smooth boundary (see also [2, 24]). Using sharp analyticity estimates for solutions to general parabolic equations or systems with analytic coefficients, such a kind of quantitative estimate have been established in a series of recent works [1, 5, 6].

The purpose of this paper is twofold. Firstly, although the quantitative estimate (2) of unique continuation for the heat equation with $L^q(\Omega)$ potentials for any $q > n$ has been established in [24], however, it is still unknown so far for the heat equation with the Coulomb potential, since the Coulomb potential does not fall into the class of $L^q(\Omega)$ with some $q > n$.

Secondly, several applications for the above interpolation estimate (2) in Control Theory, such as impulse control, observability inequalities from measurable subsets, and bang-bang properties of optimal controls for parabolic equations, have been recently discussed in [1, 5, 6, 24, 25, 26, 31, 32, 33, 34, 36].

The main results of this paper are included in Theorems 1.1 and 1.2 below. In order to present the basic ideas in our strategy, the first one below is for a particular case that the bounded regular domain $\Omega$ has the convex structure and the interior observation region $\omega$ is a ball. Moreover, one can specify the explicit expression of the dependency of two constants appearing in (2) for this particular case.

**Theorem 1.1.** Let $\Omega$ be a bounded convex domain. Assume $x_0 \in \Omega$ and $r > 0$ to be such that $B_r(x_0) \subset \Omega$. Then, there exists a constant $N = N(k, n) \geq 1$ such that for any solution $u$ to Equation (1) and any $0 < T \leq 1$,

$$\int_{\Omega} |u(x, T)|^2 \, dx \leq 2 \left( \int_{B_r(x_0)} |u(x, T)|^2 \, dx \right)^{\alpha(r)} \left( Ne^{\frac{R_\Omega^2}{4T}} \int_{\Omega} |u(x, 0)|^2 \, dx \right)^{1-\alpha(r)}$$

with

$$\alpha(r) = \frac{1}{1 + \frac{32R_\Omega^2}{\tau^2} \frac{\mu^*}{r^2}}$$

where $\mu^* := (n - 2)^2/4$ and $R_\Omega$ is the diameter of $\Omega$.1

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1Note that $\mu^*$ is the best constant of the Hardy inequality.
Remark 1. The above theorem is also true for the case that $\Omega$ is star-shaped with respect to $x_0 \in \Omega$. The proof follows from our arguments with minor modifications.

Next, we turn to state the main result for the more general class of $C^2$-smooth domains.

Theorem 1.2. Let $\Omega$ be a bounded domain with a $C^2$-smooth boundary, and let $\omega \subset \Omega$ be a non-empty open subset. Then, there are constants $N = N(\Omega, \omega, k, n) \geq 1$ and $\alpha = \alpha(\Omega, \omega, k, n)$ with $\alpha \in (0, 1)$ such that for any solution $u$ to Equation (1) and any $0 < T \leq 1$,

$$\int_{\Omega} |u(x, T)|^2 \, dx \leq \left( \int_{\omega} |u(x, T)|^2 \, dx \right)^\alpha \left( Ne^\frac{N}{T} \int_{\Omega} |u(x, 0)|^2 \, dx \right)^{1-\alpha}. \quad (4)$$

The proofs of Theorems 1.1 and 1.2 are based on the weighted frequency function method and two parabolic-type Hardy inequalities. The frequency function method is well known in the studies of quantitative estimates of unique continuation for the second elliptic equations, such as the three-ball inequality and the doubling property (cf., e.g., [13], [14], [15] and references therein). The extension of this method to the parabolic equations is first made in [27], where the strong unique continuation property of parabolic equations in the whole space $\mathbb{R}^n$ was built up. Later on, Escauriaza, et al., developed this method to deduce quantitative estimates of unique continuation for general second order parabolic equations (see, e.g., [7], [8], [9], [12] and [23]).

Compared with the context in the earlier works [23, 24, 26], the new difficulty here is how to deal with the singular lower-order term. The novelty of this paper is to apply some parabolic-type Hardy inequalities (see Section 2.2 below) to overcome this difficulty.

The rest of this paper is organized as follows. In Section 2, we apply the global frequency function method to deduce the interpolation inequality (2) when $\Omega$ is a bounded and convex domain (i.e., Theorem 1.1). In Section 3, we show how to extend it to a $C^2$-smooth bounded domain by the localized frequency function method (i.e., Theorem 1.2). Finally, we conclude the paper with several comments in Section 4.

2. Global frequency function method: Proof of Theorem 1.1.

2.1. Monotonicity property of frequency function. For each $\lambda > 0$, let us set

$$G_\lambda(x, t) = (T - t + \lambda)^{-n/2} e^{-\frac{|x|^2}{4(T-t+\lambda)}}, \quad (x, t) \in \mathbb{R}^n \times [0, T],$$

which is the backward caloric function in $\mathbb{R}^n \times [0, T]$:

$$(\partial_t + \Delta)G_\lambda(x, t) = 0.$$

Given any $x_0 \in \Omega$, let us write

$$G_{\lambda,x_0}(x, t) = G_\lambda(x - x_0, t), \quad (x, t) \in \mathbb{R}^n \times [0, T], \lambda > 0.$$

Let $f \in L^2(\Omega \times (0, T))$ and let $u$ be a solution of

$$\begin{cases}
\partial_t u - \Delta u = f(x, t) & \text{in } \Omega \times (0, T), \\
u = 0 & \text{on } \partial \Omega \times (0, T), \\
u(\cdot, 0) \in L^2(\Omega). 
\end{cases} \quad (5)$$
Associated with each triplet \((u, f, x_0)\) (where \(f \in L^2(\Omega \times (0, T))\), \(u\) solves (5) and \(x_0 \in \Omega\)), we define the weighted frequency function

\[
N_\lambda(t) = \frac{I_\lambda(t)}{H_\lambda(t)}
\]

for all \(t \in \{t \in (0, T); H_\lambda(t) \neq 0\}\), where

\[
I_\lambda(t) = \int_\Omega |\nabla u(x, t)|^2 G_{\lambda, x_0}(x, t) \, dx, \quad t \in (0, T], \lambda > 0,
\]

\[
H_\lambda(t) = \int_\Omega |u(x, t)|^2 G_{\lambda, x_0}(x, t) \, dx, \quad t \in (0, T], \lambda > 0.
\]

We begin with the following lemma, which has been proved in [23]. For the sake of the completeness of the paper, we provide a probably simple proof here.

**Lemma 2.1.** Let \(f \in L^2(\Omega \times (0, T))\), \(u\) be a solution of (5) and \(x_0 \in \Omega\). Then the weighted frequency function associated with \((u, f, x_0)\) has the following properties:

(i). For each \(t \in (0, T]\) with \(u(\cdot, t) \neq 0\) in \(L^2(\Omega)\),

\[
N_\lambda(t) = -\frac{d}{dt} \log(H_\lambda(t)) + \frac{\int_\Omega u f G_{\lambda, x_0} \, dx}{\int_\Omega u^2 G_{\lambda, x_0} \, dx}.
\]

(ii). When \(\Omega\) is either convex or star-shaped with respect to \(x_0\),

\[
\frac{d}{dt} N_\lambda(t) \leq \frac{N_\lambda(t)}{T-t+\lambda} + \frac{\int_\Omega f^2 G_{\lambda, x_0} \, dx}{H_\lambda(t)}
\]

for each \(t \in (0, T]\) with \(u(\cdot, t) \neq 0\) in \(L^2(\Omega)\).

**Proof.** By the Green formula, we have

\[
\frac{d}{dt} H_\lambda(t) = \int_\Omega 2u \partial_t u G_{\lambda, x_0} + u^2 \partial_t G_{\lambda, x_0} \, dx
\]

\[
= 2 \int_\Omega u (\partial_t u - \Delta u) G_{\lambda, x_0} \, dx - 2 \int_\Omega |\nabla u|^2 G_{\lambda, x_0} \, dx
\]

\[
= 2 \int_\Omega u f G_{\lambda, x_0} \, dx - 2I_\lambda(t),
\]

which leads to (i).

We now prove (ii). By the integration by parts, it follows that

\[
\frac{d}{dt} H_\lambda(t) = 2 \int_\Omega (\partial_t u - \frac{\nabla u \cdot (x - x_0)}{2(T-t+\lambda)}) \frac{f}{2} G_{\lambda, x_0} \, dx + \int_\Omega f u G_{\lambda, x_0} \, dx,
\]

(6)

\[
I_\lambda(t) = -\int_\Omega (\partial_t u - \frac{\nabla u \cdot (x - x_0)}{2(T-t+\lambda)}) \frac{f}{2} G_{\lambda, x_0} \, dx + \frac{1}{2} \int_\Omega f u G_{\lambda, x_0} \, dx.
\]

(7)

Meanwhile,

\[
\frac{d}{dt} I_\lambda(t) = 2 \int_\Omega \nabla u \cdot \nabla (\partial_t u) G_{\lambda, x_0} \, dx + \int_\Omega \nabla(|\nabla u|^2) \cdot \nabla G_{\lambda, x_0} \, dx + \mathcal{A},
\]

(8)

where

\[
\mathcal{A} = \int_{\partial \Omega} \frac{(x - x_0) \cdot \nu_x}{2(T-t+\lambda)} \left( \frac{\partial u}{\partial \nu} \right)^2 G_{\lambda, x_0} \, d\sigma.
\]

Since

\[
\nabla(|\nabla u|^2) \cdot \nabla G_{\lambda, x_0} = \frac{-1}{T-t+\lambda} \left( \nabla u \cdot \nabla (\nabla u \cdot (x - x_0)) - |\nabla u|^2 \right) G_{\lambda, x_0},
\]

(9)
it follows from (8) and (9) that
\[
\frac{d}{dt} I_\lambda(t) = 2 \int_\Omega \nabla u \cdot \nabla \left( \partial_t u - \frac{\nabla u \cdot (x - x_0)}{2(T - t + \lambda)} \right) G_{\lambda,x_0} \, dx \\
+ \frac{I_\lambda(t)}{T - t + \lambda} + \mathcal{A}.
\] (10)

On the other hand, we have by the Green formula,
\[
\int_\Omega \nabla u \cdot \nabla \left( \partial_t u - \frac{\nabla u \cdot (x - x_0)}{2(T - t + \lambda)} \right) G_{\lambda,x_0} \, dx \\
= -\mathcal{A} - \int_\Omega \left( \partial_t u - \frac{\nabla u \cdot (x - x_0)}{2(T - t + \lambda)} \right) \left( \partial_t u - \frac{\nabla u \cdot (x - x_0)}{2(T - t + \lambda)} - f \right) G_{\lambda,x_0} \, dx.
\] (11)

From (10) and (11), it holds that
\[
\frac{d}{dt} I_\lambda(t) = -2 \int_\Omega \left( \partial_t u - \frac{\nabla u \cdot (x - x_0)}{2(T - t + \lambda)} - \frac{f}{2} \right)^2 G_{\lambda,x_0} \, dx + \frac{1}{2} \int_\Omega f^2 G_{\lambda,x_0} \, dx \\
+ \frac{I_\lambda(t)}{T - t + \lambda} - \mathcal{A}.
\] (12)

Finally, we obtain from (6), (7) and (12) that
\[
\frac{d}{dt} N_\lambda(t) = \frac{\frac{d}{dt} (I_\lambda(t)) H_\lambda(t) - I_\lambda(t) \frac{d}{dt} H_\lambda(t)}{H_\lambda^2(t)}
\]
\[
= \frac{N_\lambda(t)}{T - t + \lambda} + \frac{1}{2} \int_\Omega f^2 G_{\lambda,x_0} \, dx - \frac{\mathcal{A}}{H_\lambda(t)} - \frac{\left( \int_\Omega f G_{\lambda,x_0} \, dx \right)^2}{2H_\lambda^2(t)}
\]
\[
- \frac{2}{H_\lambda(t)} \int_\Omega \left( \partial_t u - \frac{\nabla u \cdot (x - x_0)}{2(T - t + \lambda)} - \frac{f}{2} \right)^2 G_{\lambda,x_0} \, dx
\]
\[
+ \frac{2}{H_\lambda^2(t)} \int_\Omega \left( \partial_t u - \frac{\nabla u \cdot (x - x_0)}{2(T - t + \lambda)} - \frac{f}{2} \right) u G_{\lambda,x_0} \, dx.
\]

Since \( \Omega \) is either convex or star-shaped with respect to \( x_0 \), it holds \( (x - x_0) \cdot \nu_x \geq 0 \) when \( x \in \partial \Omega \) (see for instance [10, pp. 515]). (Here, \( \nu_x \) is the outward normalized vector of \( \partial \Omega \) at \( x \).) Consequently, \( \mathcal{A} \geq 0 \). By the Cauchy-Schwartz inequality,
\[
\left[ \int_\Omega \left( \partial_t u - \frac{\nabla u \cdot (x - x_0)}{2(T - t + \lambda)} - \frac{f}{2} \right) u G_{\lambda,x_0} \, dx \right]^2
\]
\[
\leq \int_\Omega \left( \partial_t u - \frac{\nabla u \cdot (x - x_0)}{2(T - t + \lambda)} - \frac{f}{2} \right)^2 G_{\lambda,x_0} \, dx \times \int_\Omega u^2 G_{\lambda,x_0} \, dx,
\]

which leads to
\[
\frac{d}{dt} N_\lambda(t) \leq \frac{N_\lambda(t)}{T - t + \lambda} + \frac{\int_\Omega f^2 G_{\lambda,x_0} \, dx}{2H_\lambda(t)},
\]
i.e., (ii) stands.

2.2. **Two parabolic-type Hardy inequalities.** We next introduce the following two parabolic-type Hardy inequalities, which are crucial in the proof of main results.
Lemma 2.2. Let \( x_0 \in \Omega \) and \( \mu^* := (n - 2)^2/4, \ n \geq 3 \). Then for each \( \lambda > 0 \) and each \( \varphi \in H_0^2(\Omega) \), it holds that
\[
\frac{1}{16\lambda^2} \int_{\Omega} |x - x_0|^2 \varphi^2 e^{-|x-x_0|^2/4\lambda} \, dx \leq \int_{\Omega} \left( |\nabla \varphi|^2 - \frac{\mu^*}{|x|^2} \varphi^2 \right) e^{-|x-x_0|^2/4\lambda} \, dx + \frac{n}{4\lambda} \int_{\Omega} \varphi^2 e^{-|x-x_0|^2/4\lambda} \, dx.
\]  
(13)

Proof. Recall the well-known Hardy inequality (cf., e.g., [29]):
\[ \mu^* \int_{\Omega} \frac{g^2}{|x|^2} \, dx \leq \int_{\Omega} |\nabla g|^2 \, dx \quad \text{for all } g \in H_0^1(\Omega). \]
(14)

Given \( \varphi \in H_0^1(\Omega) \), \( \lambda > 0 \) and \( x_0 \in \Omega \), let \( g = \varphi e^{-|x-x_0|^2/8\lambda} \). Clearly, \( g \in H_0^1(\Omega) \) and
\[ \nabla g = \nabla \varphi e^{-|x-x_0|^2/8\lambda} - \frac{x-x_0}{4\lambda} \varphi e^{-|x-x_0|^2/8\lambda}. \]

By (14),
\[ \mu^* \int_{\Omega} \frac{\varphi^2}{|x|^2} e^{-|x-x_0|^2/4\lambda} \, dx \leq \int_{\Omega} |\nabla \varphi|^2 e^{-|x-x_0|^2/4\lambda} + \frac{|x-x_0|^2}{16\lambda^2} \varphi^2 e^{-|x-x_0|^2/4\lambda} \, dx \]
\[ - \frac{1}{2\lambda} \int_{\Omega} (x-x_0) \cdot \nabla \varphi e^{-|x-x_0|^2/4\lambda} \, dx. \]

By the integration by parts, we have
\[ \int_{\Omega} (x-x_0) \cdot \nabla \varphi e^{-|x-x_0|^2/4\lambda} \, dx = -\frac{1}{2} \int_{\Omega} \left( n - \frac{|x-x_0|^2}{2\lambda} \right) \varphi^2 e^{-|x-x_0|^2/4\lambda} \, dx. \]  
(15)

The last two inequalities imply (13). \( \Box \)

Lemma 2.3. Let \( x_0 \in \Omega \) and \( \mu^* \) be as above. Then for each \( m \geq 0 \) and each \( \gamma \in (0, 2) \), there exists a constant \( C = C(m, \gamma) > 0 \) such that when \( \lambda > 0 \) and \( \varphi \in H_0^1(\Omega) \),
\[
\frac{1}{16\lambda^2} \int_{\Omega} |x-x_0|^2 \varphi^2 e^{-|x-x_0|^2/4\lambda} \, dx \leq \int_{\Omega} \left[ |\nabla \varphi|^2 - \frac{\mu^*}{|x|^2} \varphi^2 - \frac{m}{|x|^\gamma} \varphi^2 \right] e^{-|x-x_0|^2/4\lambda} \, dx + \frac{n}{4\lambda} \int_{\Omega} \varphi^2 e^{-|x-x_0|^2/4\lambda} \, dx.
\]  
(16)

Proof. Given \( x_0 \in \Omega \), \( \lambda > 0 \) and \( \varphi \in H_0^1(\Omega) \), let \( z = \varphi e^{-|x-x_0|^2/8\lambda} \). Then,
\[ |\nabla z|^2 = |\nabla \varphi|^2 e^{-|x-x_0|^2/8\lambda} + \frac{|x-x_0|^2}{16\lambda^2} \varphi^2 e^{-|x-x_0|^2/8\lambda} \]
\[ - \frac{x-x_0}{2\lambda} \cdot \nabla \varphi e^{-|x-x_0|^2/8\lambda}. \]  
(17)

and
\[ |\nabla z|^2 = |\nabla \varphi|^2 e^{-|x-x_0|^2/8\lambda} + \frac{|x-x_0|^2}{16\lambda^2} \varphi^2 e^{-|x-x_0|^2/8\lambda} - \frac{(x-x_0) \cdot \nabla \varphi}{2\lambda} e^{-|x-x_0|^2/8\lambda}. \]  
(18)

This, together with (15), (17), (18), and the following improved version of Hardy inequality (cf., for instance, [29, (2.15)])
\[ m \int_{\Omega} \frac{z^2}{|x|^\gamma} \, dx \leq \int_{\Omega} \left[ |\nabla z|^2 - \frac{\mu^*}{|x|^2} z^2 \right] \, dx + C(m, \gamma) \int_{\Omega} z^2 \, dx, \quad \forall z \in H_0^1(\Omega), \]
leads to (16). \( \Box \)

Remark 2. Note that in the proofs of Lemmas 2.2 and 2.3 we do not use the convexity of the domain \( \Omega \). In fact, they are still valid whenever \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^n, n \geq 3 \).
2.3. The proof of Theorem 1.1. We first apply the above-mentioned three lemmas to obtain the following weighted estimate for solutions to Equation (1).

**Lemma 2.4.** Assume that the bounded domain $\Omega$ is convex. Let $x_0 \in \Omega$ and let $0 < \lambda \leq T \leq 1$. Then for each $r > 0$ with $B_r(x_0) \subset \Omega$ and each solution $u$ to Equation (1), it holds that

$$
\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4x^2}} dx \leq \int_{B_r(x_0)} |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4x^2}} dx
$$

$$
+ \frac{64\lambda}{r^2} e^{|x_0|^2} \left[ \log \int_\Omega |u(x,0)|^2 dx + C_1 \right] \int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4x^2}} dx, \quad (19)
$$

where

$$
C_1 = C_1(T, k, n, \mu^*, R_\Omega) = \frac{R_\Omega^2}{2T} + C(k) + \frac{n}{4}(1 + \frac{k^2}{\mu^*}) + n.
$$

Here and in the sequel, $R_\Omega$ is the diameter of $\Omega$, and $C(\cdot)$ denotes a positive constant depending only on what are enclosed in the brackets and it may change from line to line in the context.

**Proof.** We only need to show the desired estimate (19) for an arbitrarily fixed solution $u$ to Equation (1) with $u(\cdot, 0) \neq 0$ in $L^2(\Omega)$. Let $N_\lambda(\cdot)$ be the weighted frequency function associated with $(u, f, x_0)$ where $f = ku/|x|$. It follows from Lemma 2.5 in [24] that

$$
\int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4x^2}} dx \leq \int_{B_r(x_0)} |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4x^2}} dx
$$

$$
+ \frac{16\lambda}{r^2} \left[ \lambda N_\lambda(T) + \frac{n}{4} \right] \int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4x^2}} dx. \quad (20)
$$

Next, we apply Lemmas 2.1, 2.2 and 2.3 to deduce a bound for the quantity $\lambda N_\lambda(T)$ when $0 < \lambda \leq T \leq 1$. Since $u(\cdot, 0) \neq 0$ in $L^2(\Omega)$, by backward uniqueness it holds that $u(\cdot, t) \neq 0$ in $L^2(\Omega)$, for each $t \in [0, T]$. From (ii) of Lemma 2.1 (where $(u, f) = (u, ku/|x|)$), we have

$$
dt N_\lambda(t) \leq \frac{N_\lambda(t)}{T-t+\lambda} + \frac{k^2}{\mu^*} \int_\Omega \frac{u^2}{|x|^2} G_{\lambda,x_0} dx + \frac{k^2}{\mu^*} \int_\Omega u^2 G_{\lambda,x_0} dx, \quad t \in (0, T]. \quad (21)
$$

By Lemma 2.2 (where $\varphi = u$ and $\lambda = T-t+\lambda$ with $t \in (0, T)$), it follows that

$$
k^2 \int_\Omega \frac{u^2}{|x|^2} G_{\lambda,x_0} dx \leq \frac{k^2}{\mu^*} \int_\Omega |\nabla u|^2 G_{\lambda,x_0} dx + \frac{k^2}{\mu^*} \int_\Omega u^2 G_{\lambda,x_0} dx, \quad t \in (0, T].
$$

This, along with (21), indicates that

$$
dt N_\lambda(t) \leq \frac{N_\lambda(t)}{T-t+\lambda} + \frac{k^2}{\mu^*} N_\lambda(t) + \frac{nk^2}{4\mu^*} \frac{1}{T-t+\lambda}, \quad t \in (0, T].
$$

Consequently,

$$
dt [(T-t+\lambda)N_\lambda(t)] \leq \frac{k^2}{\mu^*} (T-t+\lambda) N_\lambda(t) + \frac{nk^2}{4\mu^*}, \quad t \in (0, T].
$$

This yields

$$
dt \left[ e^{-tk^2/\mu^*} (T-t+\lambda) N_\lambda(t) \right] \leq \frac{nk^2}{4\mu^*} e^{-tk^2/\mu^*}, \quad t \in (0, T].
$$
Thus, from which and the standard energy estimate (20), arrives at the desired estimate (19).

Integrating the above inequality from \( t \) to \( T \), we get that
\[
e^{-Tk^2/\mu^*} \lambda N_\lambda(T) \leq e^{-tk^2/\mu^*} (T - t + \lambda) N_\lambda(t) + \frac{nk^2}{4\mu^*} \int_t^T e^{-sk^2/\mu^*} \, ds, \quad t \in (0, T].
\]

Thus,
\[
e^{-Tk^2/\mu^*} \lambda N_\lambda(T) \leq (T + \lambda) N_\lambda(t) + \frac{nk^2}{4\mu^*}, \quad t \in (0, T].
\]

Integrating the above inequality with respect to \( t \) over \((0, T/2)\), we obtain that
\[
e^{-Tk^2/\mu^*} \lambda N_\lambda(T) \leq \frac{2(T + \lambda)}{T} \int_0^{T/2} N_\lambda(t) \, dt + \frac{nk^2}{4\mu^*}. \tag{22}
\]

On the other hand, by (i) of Lemma 2.1 (where \( (u, f) = (u, ku/|x|) \)), we see that

\[
N_\lambda(t) = -\frac{1}{2} \frac{d}{dt} \log (H_\lambda(t)) + \int_\Omega \frac{k}{|x|^2} u^2 G_{\lambda,x_0} \, dx \cdot H_\lambda(t). \tag{23}
\]

By Lemma 2.3 (where \( \varphi = u, \ m = 2|k|, \ gamma = 1 \) and \( \lambda = T - t + \lambda \)), it stands that

\[
2|k| \int_\Omega \frac{u^2}{|x|^2} G_{\lambda,x_0} \, dx \leq \int_\Omega |\nabla u|^2 G_{\lambda,x_0} \, dx + \left[\frac{n}{4(T - t + \lambda)} + C(k)\right] \int_\Omega u^2 G_{\lambda,x_0} \, dx.
\]

This, along with (23), implies that
\[
N_\lambda(t) \leq -\frac{d}{dt} \log (H_\lambda(t)) + \left[ C(k) + \frac{n}{4(T - t + \lambda)} \right].
\]

Integrating the above inequality over \((0, T/2)\), we get that
\[
\int_0^{T/2} N_\lambda(t) \, dt \leq \log \frac{H_\lambda(0)}{H_\lambda(T/2)} + \frac{T}{2} \left[ C(k) + \frac{n}{2T} \right]. \tag{24}
\]

Notice that
\[
\frac{H_\lambda(0)}{H_\lambda(T/2)} = \frac{(T + \lambda)^{-n/2} \int_\Omega |u(x, 0)|^2 e^{-\frac{|x - x_0|^2}{4(T + \lambda)}} \, dx}{(T + \lambda)^{-n/2} \int_\Omega |u(x, T/2)|^2 e^{-\frac{|x - x_0|^2}{4(T + \lambda)}} \, dx} \leq e^{\frac{nR_\lambda^2}{8T}} \frac{\int_\Omega |u(x, 0)|^2 \, dx}{\int_\Omega |u(x, T/2)|^2 \, dx}.
\]

From which and the standard energy estimate
\[
\int_\Omega |u(x, T)|^2 \, dx \leq e^{C(k)T} \int_\Omega |u(x, T/2)|^2 \, dx,
\]
we have
\[
\frac{H_\lambda(0)}{H_\lambda(T/2)} \leq e^{\frac{nR_\lambda^2}{8T} + C(k)} \frac{\int_\Omega |u(x, 0)|^2 \, dx}{\int_\Omega |u(x, T)|^2 \, dx}. \tag{25}
\]

Therefore, it follows from (22), (24) and (25) that when \( 0 < \lambda \leq T \leq 1 \)
\[
\lambda N_\lambda(T) \leq 4e^{\frac{k^2}{\mu^*}} \left[ \log \frac{\int_\Omega |u(x, 0)|^2 \, dx}{\int_\Omega |u(x, T)|^2 \, dx} + \frac{R_\lambda^2}{2T} + C(k) + \frac{n}{4} \left(1 + \frac{k^2}{4\mu^*} \right) \right].
\]

This, combined with (20), arrives at the desired estimate (19).

We end this section with the proof of Theorem 1.1.
The proof of Theorem 1.1. Let us now choose
\[
\lambda_0 = \frac{r^2}{128} e^{-\frac{r^2}{8}} \left[ \log \frac{\int_{\Omega} |u(x, 0)|^2 \, dx}{\int_{\Omega} |u(x, T)|^2 \, dx} + C_1 \right]^{-1},
\]
with the same constant $C_1$ given in Lemma 2.4. It is easy to check that $0 < \lambda_0 < T$. According to Lemma 2.4 (where $\lambda$ for all $r > \int_R$), it holds that
\[
\frac{1}{2} \int_{\Omega} |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda_0}} \, dx \leq \int_{B_r(x_0)} |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda_0}} \, dx.
\]
Which implies
\[
\int_{\Omega} |u(x, T)|^2 \, dx \leq 2e^{\frac{\lambda_0^2}{4}} \int_{B_r(x_0)} |u(x, T)|^2 \, dx.
\]
Notice that
\[
\kappa^2_0 e^{\frac{\lambda_0^2}{4}} = e^{-\frac{32\lambda_0^2}{128} \frac{k^2}{r^2}} \left[ \log \frac{\int_{\Omega} |u(x, 0)|^2 \, dx}{\int_{\Omega} |u(x, T)|^2 \, dx} + C_1 \right] = \left( e^{C_1 \frac{\int_{\Omega} |u(x, 0)|^2 \, dx}{\int_{\Omega} |u(x, T)|^2 \, dx}} \right)^{\frac{32\lambda_0^2}{128} \frac{k^2}{r^2}}.
\]
This, along with (26), leads to
\[
\int_{\Omega} |u(x, T)|^2 \, dx \leq 2 \left( e^{C_1 \frac{\int_{\Omega} |u(x, 0)|^2 \, dx}{\int_{\Omega} |u(x, T)|^2 \, dx}} \right)^{\frac{32\lambda_0^2}{128} \frac{k^2}{r^2}} \int_{B_r(x_0)} |u(x, T)|^2 \, dx.
\]
Which implies (3) and completes the proof.

Remark 3. By following the argument in [9] and the facts established above, one can easily obtain the following doubling property: For any $u(\cdot, 0) \in L^2(\Omega)$, there exists a constant $C$, independent of $r$ and $x_0$, such that
\[
\int_{B_2r(x_0)} |u(x, T)|^2 \, dx \leq C \int_{B_r(x_0)} |u(x, T)|^2 \, dx,
\]
for all $r > 0$ such that $B_2r(x_0) \subset \Omega$. From which one can derive the space-like strong unique continuation property for solutions to (1).

3. Local frequency function method: Proof of Theorem 1.2. We sketch the main idea of the proof of Theorem 1.2 as follows. First, we apply the frequency function method (as in Section 2) in a star-shaped sub-domain to deduce a localized version of (2), which means that the left hand side of (2) is replaced by the local energy of the solution at the time $T$. Then, by iterating the above-mentioned localized version and the standard argument of propagation of smallness, as well as a finite covering argument, we can conclude the desired inequality (2) when $\Omega$ is a bounded and $C^2$-smooth domain. Noting that any bounded and $C^2$-smooth domain can be covered by finite numbers of star-shaped domains (see, e.g., [1, Theorem 8]).

3.1. Backward estimates. We start with a version of locally backward energy estimate for solutions to Equation (1), which is similar to [26, Lemma 3].

Lemma 3.1. Let $x_0 \in \Omega$, $0 < T \leq 1$, $R \in (0, 1]$ and $\delta \in (0, 1]$. Then there exist constants $C(k) \geq 1$ and $C_1(k) \geq 1$ such that for any solution $u$ of Equation (1) with $u(\cdot, 0) = u_0 \in L^2(\Omega) \setminus \{0\}$,
\[
\|u_0\|_{L^2(\Omega)}^2 \leq C(k) e^{\frac{k^2}{80}} \|u(\cdot, t)\|_{L^2(\Omega \cap B_{1+\delta}(x_0) \cap R(x_0)\cap B_{1+\delta}(x_0))}^2, \text{ when } t \in [T - h_0, T]
\]
with $h_0$ given by the equality

$$h_0 = \frac{\delta^3 R^2}{8(1 + \delta)^2 \log \left[ \frac{C_0(k)}{8\pi R^2} C_2 \right] \frac{\|u_0\|_{L^2(\Omega)}}{\|u_0\|_{L^2(\Omega \cap B_R(x_0))}^2}}. \quad (28)$$

**Proof.** We carry out the proof into three steps.

**Step 1.** Choose a suitable multiplier.

Let $\psi \in C_0^\infty(\Omega \cap B_{(1+\delta)}R(x_0))$ be the cut-off function verifying

$$0 \leq \psi \leq 1 \text{ in } \Omega \cap B_{(1+\delta)}R(x_0), \quad \psi \equiv 1 \text{ in } \Omega \cap B_{(1+3\delta/4)}R(x_0) \text{ and } |\nabla \psi| \leq \frac{C}{\delta R},$$

for some generic constant $C \geq 1$ independent of $R$ and $\delta$. For $h > 0$ to be fixed later, multiplying by $e^{-\frac{|x-x_0|^2}{h}} \psi^2 u$ the first equation of Equation (1) and integrating the latter over $\Omega \cap B_{(1+\delta)}R(x_0)$, we have

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega \cap B_{(1+\delta)}R(x_0)} |\psi u|^2 e^{-\frac{|x-x_0|^2}{h}} \, dx \right) + \int_{\Omega \cap B_{(1+\delta)}R(x_0)} \nabla u \cdot \nabla (e^{-\frac{|x-x_0|^2}{h}} \psi^2 u) \, dx$$

$$- \int_{\Omega \cap B_{(1+\delta)}R(x_0)} \frac{k}{|x|} e^{-\frac{|x-x_0|^2}{h}} |\psi u|^2 \, dx = 0.$$

From

$$\nabla (e^{-\frac{|x-x_0|^2}{h}} \psi^2 u) = -\frac{2(x - x_0)}{h} e^{-\frac{|x-x_0|^2}{h}} \psi^2 u + 2e^{-\frac{|x-x_0|^2}{h}} \psi \nabla \psi u + e^{-\frac{|x-x_0|^2}{h}} \psi^2 \nabla u,$$

it holds that

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega \cap B_{(1+\delta)}R(x_0)} |\psi u|^2 e^{-\frac{|x-x_0|^2}{h}} \, dx \right)$$

$$+ \int_{\Omega \cap B_{(1+\delta)}R(x_0)} \left[ |\nabla (\psi u)|^2 - \frac{k}{|x|} |\psi u|^2 \right] e^{-\frac{|x-x_0|^2}{h}} \, dx$$

$$= \int_{\Omega \cap B_{(1+\delta)}R(x_0)} |\nabla \psi|^2 u^2 e^{-\frac{|x-x_0|^2}{h}} \, dx$$

$$+ \frac{1}{h} \int_{\Omega \cap B_{(1+\delta)}R(x_0)} (x - x_0) \cdot (\nabla (u^2)) \psi^2 e^{-\frac{|x-x_0|^2}{h}} \, dx.$$

By the integration by parts,

$$\int_{\Omega \cap B_{(1+\delta)}R(x_0)} (x - x_0) \cdot (\nabla (u^2)) \psi^2 e^{-\frac{|x-x_0|^2}{h}} \, dx.$$
These last two inequalities indicate that

\[
= - \int_{\Omega \cap B_{(1+\delta)} R(x_0)} \text{div} \left((x - x_0) \psi^2 e^{-\frac{|x-x_0|^2}{h}} u^2\right) dx
\]

\[
= - \int_{\Omega \cap B_{(1+\delta)} R(x_0)} u^2 \left(n \psi^2 + 2 \psi (x - x_0) \cdot \nabla \psi - \frac{2|x-x_0|^2}{h} \psi^2\right) e^{-\frac{|x-x_0|^2}{h}} dx
\]

\[
= - n \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |\psi u|^2 e^{-\frac{|x-x_0|^2}{h}} dx
\]

\[
- \int_{\Omega \cap B_{(1+\delta)} R(x_0)} 2 \psi u^2 \nabla \psi \cdot (x - x_0) e^{-\frac{|x-x_0|^2}{h}} dx
\]

\[
+ \frac{2}{h} \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |x - x_0|^2 \psi u e^{-\frac{|x-x_0|^2}{h}} dx.
\]

Hence,

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |\psi u|^2 e^{-\frac{|x-x_0|^2}{h}} dx \right)
\]

\[
+ \int_{\Omega \cap B_{(1+\delta)} R(x_0)} \left[|\nabla (\psi u)|^2 - \frac{k}{|x|} |\psi u|^2\right] e^{-\frac{|x-x_0|^2}{h}} dx
\]

\[
= \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |\nabla \psi|^2 u^2 e^{-\frac{|x-x_0|^2}{h}} dx - \frac{n}{h} \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |\psi u|^2 e^{-\frac{|x-x_0|^2}{h}} dx
\]

\[
- \frac{1}{h} \int_{\Omega \cap B_{(1+\delta)} R(x_0)} 2 \psi u^2 \nabla \psi \cdot (x - x_0) e^{-\frac{|x-x_0|^2}{h}} dx
\]

\[
+ \frac{2}{h^2} \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |x - x_0|^2 |\psi u| e^{-\frac{|x-x_0|^2}{h}} dx
\]

\[
\leq 2 \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |\nabla \psi|^2 u^2 e^{-\frac{|x-x_0|^2}{h}} dx - \frac{n}{h} \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |\psi u|^2 e^{-\frac{|x-x_0|^2}{h}} dx
\]

\[
+ \frac{3}{h^2} \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |x - x_0|^2 |\psi u|^2 e^{-\frac{|x-x_0|^2}{h}} dx.
\]

By Lemma 2.3 (where \( \varphi = \psi u, \lambda = h \) and \( \gamma = 1 \)), we have

\[
\int_{\Omega \cap B_{(1+\delta)} R(x_0)} \left[|\nabla (\psi u)|^2 - \frac{k}{|x|} |\psi u|^2\right] e^{-\frac{|x-x_0|^2}{h}} dx
\]

\[
\geq - \left[\frac{n}{h} + C(k)\right] \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |\psi u|^2 e^{-\frac{|x-x_0|^2}{h}} dx.
\]

These last two inequalities indicate that

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |\psi u|^2 e^{-\frac{|x-x_0|^2}{h}} dx \right)
\]

\[
\leq \frac{3}{h^2} \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |x - x_0|^2 |\psi u|^2 e^{-\frac{|x-x_0|^2}{h}} dx
\]

\[
+ 2 \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |\nabla \psi|^2 u^2 e^{-\frac{|x-x_0|^2}{h}} dx
\]

\[
+ C(k) \int_{\Omega \cap B_{(1+\delta)} R(x_0)} |\psi u|^2 e^{-\frac{|x-x_0|^2}{h}} dx.
\]
Which leads to
\[
\frac{d}{dt} \left( \int_{\Omega \cap B(1+\delta)R(z_0)} |\psi u|^2 e^{-|x-x_0|^2/h} \, dx \right) \\
\leq \frac{6(1 + \delta)^2 R^2}{h^2} + C(k) \int_{\Omega \cap B(1+\delta)R(z_0)} |\psi u|^2 e^{-|x-x_0|^2/h} \, dx \\
+ \frac{C}{\delta^2 R^2} \int_{\Omega \cap B(1+\delta)R(z_0) \setminus B(1+\frac{1}{8})R(z_0)} e^{-|x-x_0|^2/h} \, u^2 \, dx. 
\] 

(29)

**Step 2.** Derive a localized energy estimate for \( \int_{\Omega \cap B_R(z_0)} |u(x, T)|^2 \, dx \).

For the simplicity of writing, we set
\[
A = \frac{6(1 + \delta)^2 R^2}{h^2} + C(k), 
\]

(30)

\[
B = \frac{C}{\delta^2 R^2} e^{-\left(1+\frac{3}{8}\right)^2 R^2/h}, 
\]

(31)

where \(C(k)\) and \(C\) are two constants in (29), and
\[
D = \frac{\delta}{4(1 + \delta)^2}. 
\]

(32)

Clearly,
\[
0 < D < \frac{\delta}{8\delta} = \frac{1}{8}. 
\]

(33)

Therefore, by (29), it holds that
\[
\frac{d}{dt} \left( \int_{\Omega \cap B(1+\delta)R(z_0)} |\psi u|^2 e^{-|x-x_0|^2/h} \, dx \right) \leq A \int_{\Omega \cap B(1+\delta)R(z_0)} |\psi u|^2 e^{-|x-x_0|^2/h} \, dx + B e^{C(k)} \|u_0\|_{L^2(\Omega)}. 
\]

Then, we have
\[
\int_{\Omega \cap B(1+\delta)R(z_0)} e^{-|x-x_0|^2/h} |\psi u(x, T)|^2 \, dx \\
\leq e^{A(T-t)} \int_{\Omega \cap B(1+\delta)R(z_0)} e^{-|x-x_0|^2/h} |\psi u(x, t)|^2 \, dx + e^{A(T-t)} B e^{C(k)} \|u_0\|_{L^2(\Omega)}. 
\]

Which implies that when \(T - Dh \leq t \leq T\),
\[
\int_{\Omega \cap B(1+\delta)R(z_0)} e^{-|x-x_0|^2/h} |\psi u(x, T)|^2 \, dx \\
\leq e^{ADh} \int_{\Omega \cap B(1+\delta)R(z_0)} e^{-|x-x_0|^2/h} |\psi u(x, t)|^2 \, dx + e^{ADh} B e^{C(k)} \|u_0\|_{L^2(\Omega)}. 
\]

Since \(\psi(x) = 1\) and \(e^{-|x-x_0|^2/h} \geq e^{-R^2/h}, \ x \in \Omega \cap B_R(x_0)\), we have
\[
\int_{\Omega \cap B_R(z_0)} |u(x, T)|^2 \, dx \leq e^{ADh + \frac{h^2}{8}} \int_{\Omega \cap B(1+\delta)R(z_0)} e^{-|x-x_0|^2/h} |u(x, t)|^2 \, dx \\
+ e^{ADh + \frac{h^2}{8}} B e^{C(k)} \|u_0\|_{L^2(\Omega)}. 
\]

(34)
In view of (30), (31) and (32), it holds that
\[ ADh + \frac{R^2}{h} \leq \left(1 + \frac{3\delta}{2}\right) \frac{R^2}{h} + C(k) \] (35)
and
\[ e^{ADh + \frac{R^2}{h}} Be^{C(k)} \leq \frac{C_1(k)}{\delta^2 R^2} \exp \left\{ \frac{R^2}{h} \left[ 1 + \frac{3\delta}{2} - \left(1 + \frac{3\delta}{4}\right)^2 \right] \right\}, \] (36)
for some new constant \( C_1(k) \geq 1 \).

**Step 3.** Fix \( h > 0 \).

By (36), we have
\[ e^{ADh + \frac{R^2}{h}} Be^{C(k)} \leq e^{-\frac{\mu^2}{\pi}} \frac{C_1(k)}{\delta^2 R^2} \leq e^{-\frac{\mu^2}{\pi}} \frac{C_1(k)}{\delta^2 R^2}. \]

This, along with (34), derives that the inequality
\[
\int_{\Omega \cap B_R(x_0)} |u(x,t)|^2 dx \leq e^{ADh + \frac{R^2}{h}} \int_{\Omega \cap B_{(1+\delta)R}(x_0)} |u(x,t)|^2 dx
+ e^{-\frac{\mu^2}{\pi}} \frac{C_1(k)}{\delta^2 R^2} \|u_0\|^2_{L^2(\Omega)},
\]
holds when \( T - Dh \leq t \leq T \leq 1 \).

Notice that when \( T \leq 1 \), by the standard energy estimate, there exists a constant \( C \geq 1 \), depending on \( k \), such that
\[ \|u(\cdot, T)\|_{L^2(\Omega \cap B_R(x_0))} \leq \|u(\cdot, T)\|_{L^2(\Omega)} \leq C\|u_0\|^2_{L^2(\Omega)}. \] (38)

Now, we fix
\[ h = \frac{\delta^2 R^2 / 2}{\log \left\{ \frac{C_1(k)}{\delta^2 R^2} e^{\frac{C_1(k)}{\delta^2 R^2}} \|u(\cdot, T)\|^2_{L^2(\Omega \cap B_R(x_0))} \right\}}. \]
to be such that
\[ e^{-\frac{\mu^2}{\pi}} \frac{C_1(k)}{\delta^2 R^2} C\|u_0\|^2_{L^2(\Omega)} = e^{-\frac{1}{2}} \|u(\cdot, T)\|^2_{L^2(\Omega \cap B_R(x_0))}. \] (39)

Let \( h_0 = Dh \). Then \( h_0 \) satisfies \( h_0 \in (0, T) \). Indeed, it follows from (33) and (38) that
\[ 0 < h_0 \leq \frac{D\delta^2 R^2 / 2}{\log \left\{ \frac{C_1(k)}{\delta^2 R^2} e^{\frac{C_1(k)}{\delta^2 R^2}} \|u(\cdot, T)\|^2_{L^2(\Omega \cap B_R(x_0))} \right\}} \leq \frac{1}{16} \frac{T}{\log \left\{ e^{\frac{1}{2}} \right\}} = \frac{T}{16}. \]

Hence, it follows from (32), (35) and (37) that when \( t \in [T - h_0, T] \),
\[
(1 - e^{-\frac{1}{2}}) \|u(\cdot, T)\|^2_{L^2(\Omega \cap B_R(x_0))} \leq e^{ADh + \frac{R^2}{h}} \|u(\cdot, t)\|^2_{L^2(\Omega \cap B_{(1+\delta)R}(x_0))}
\leq C(k) e^{\frac{R^2}{h_0} \left(1 + \frac{3\delta}{4}\right)} \|u(\cdot, t)\|^2_{L^2(\Omega \cap B_{(1+\delta)R}(x_0))}. \] (40)

Because
\[ 1 - e^{-\frac{1}{2}} \geq \frac{1}{2}, \text{ when } 0 < T \leq 1, \]
\[ \frac{R^2}{h} \left(1 + \frac{3\delta}{2}\right) = \frac{DR^2}{h_0} \left(1 + \frac{3\delta}{2}\right) < \frac{1}{h_0}, \]
the desired estimate (27) is valid from (40) and (39). \( \square \)
3.2. The proof of Theorem 1.2. We first use Lemmas 2.1, 2.2, 2.3, 3.1 and the similar arguments as those in Lemma 2.4 to deduce the following localized version of interpolation inequality (2).

Lemma 3.2. Let $0 < r < R \leq 1$ and $\delta \in (0, 1]$. Suppose that $B_r(x_0) \subset \Omega$ and $\Omega \cap B_{(1+2\delta)R}(x_0)$ is star-shaped with center $x_0 \in \Omega$. Then, there exist two constants $N = N(\delta, R, k) \geq 1$ and $\theta = \theta(\delta, R, r, k)$ with $\theta \in (0, 1)$, such that for any solution $u$ to Equation (1) and any $0 < T \leq 1$,

$$
\int_{\Omega \cap B_R(x_0)} |u(x,T)|^2 \, dx 
\leq \left( Ne^{\frac{N}{2}} \int_{\Omega} |u(x,0)|^2 \, dx \right)^{\theta} \left( \int_{B_r(x_0)} |u(x,T)|^2 \, dx \right)^{1-\theta}.
$$

(41)

Proof. We only need to prove the desired estimate (41) for an arbitrarily fixed solution $u$ to Equation (1) with $u(\cdot,0) \not\equiv 0$ in $L^2(\Omega)$. Let $\psi \in C_0^\infty(\Omega \cap B_{R+2\delta R}(x_0))$ be the cut-off function such that $0 \leq \psi \leq 1$, \[ \psi = 1 \text{ in } \Omega \cap B_{R+2\delta R}(x_0), \quad |\nabla \psi|^2 + |\Delta \psi| \leq \frac{C}{\delta^2 R^2}, \]

where the generic constant $C$ is independent of $R$ and $\delta$. Let $z = \psi u$ and

$$
f = \partial_t z - \Delta z = \frac{k}{|x|^2} z - 2\nabla \psi \cdot \nabla u - \Delta \psi u. \quad (42)
$$

Associated with the triple $(z, f, x_0)$, we set for each $\lambda > 0$ and $t \in (0, T]$,

$$
H_\lambda(t) = \int_{\Omega \cap B_{(1+2\delta)R}(x_0)} z^2 G_{\lambda,x_0} \, dx
$$

and

$$
N_\lambda(t) = \frac{\int_{\Omega \cap B_{(1+2\delta)R}(x_0)} \nabla z|^2 G_{\lambda,x_0} \, dx}{\int_{\Omega \cap B_{(1+2\delta)R}(x_0)} z^2 G_{\lambda,x_0} \, dx}.
$$

Step 1. For each $t \in [T - D_1 h_0, T]$, where $h_0$ is given by (28) and $0 < D_1 \leq 1$ is to be fixed later, we obtain from Lemmas 2.2, 2.3 and the similar argument as in Lemma 2.4, as well as Lemma 3.1 (cf. [26, Lemma 4, Step 1]) that

$$
\frac{\int_{\Omega \cap B_{R+2\delta R}(x_0)} z f G_{\lambda,x_0} \, dx}{\int_{\Omega \cap B_{R+2\delta R}(x_0)} z^2 G_{\lambda,x_0} \, dx}
\leq \frac{1}{2} N_\lambda(t) + \left[ \frac{n}{8(T-t+\lambda)} + C(2k) \right] + \frac{C}{\delta^2 R^2} e^{-\frac{\mu^2 k^2}{(T-t+\lambda)^2}} \left( 1 + T^{-1/2} \right) e^\frac{2}{n} \quad (43)
$$

and

$$
\frac{\int_{\Omega \cap B_{R+2\delta R}(x_0)} f^2 G_{\lambda,x_0} \, dx}{\int_{\Omega \cap B_{R+2\delta R}(x_0)} z^2 G_{\lambda,x_0} \, dx}
\leq \frac{3k^2}{\mu^2} N_\lambda(t) + \frac{3nk^2}{4\mu^*(T-t+\lambda)} + \frac{C}{\delta^4 R^2} e^{-\frac{\mu^2 k^2}{(T-t+\lambda)^2}} \left( 1 + T^{-1/2} \right) e^\frac{2}{n}. \quad (44)
$$
Step 2. It follows from [24, Lemma 2.5] that
\[
\int_{B \cap B_{R+2\delta R}(x_0)} |z(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \leq \int_{B_r(x_0)} |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \\
+ \frac{16\lambda}{\mu^*} \left[ AL_\lambda(T) + \frac{n}{4} \right] \int_{B \cap B_{R+2\delta R}(x_0)} |z(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx.
\]

Step 3. By (44) and (ii) of Lemma 2.1 (where \( u = z \) and \( f \) is given by (42)), we have that for each \( t \in [T - D_1 h_0, T] \),
\[
\frac{d}{dt} N_\lambda(t) \leq \frac{N_\lambda(t)}{T - t + \lambda} \\
+ \frac{3k^2}{4\mu^*} N_\lambda(t) + \frac{3nk^2}{4\mu^*} (T - t + \lambda) N_\lambda(t) \\
+ \frac{C}{\delta^2 R^2} e^{-\frac{\mu^2}{4\lambda + \mu}} (1 + T^{-1/2}) e^{\frac{\lambda}{\mu^2}} (T + \lambda).
\]
Consequently,
\[
\frac{d}{dt} [(T - t + \lambda) N_\lambda(t)] \leq \frac{3k^2}{4\mu^*} (T - t + \lambda) N_\lambda(t) \\
+ \frac{3nk^2}{4\mu^*} + \frac{C}{\delta^2 R^2} e^{-\frac{\mu^2}{4\lambda + \mu}} (1 + T^{-1/2}) e^{\frac{\lambda}{\mu^2}} (T + \lambda).
\]
It follows from the above inequality that when \( t \in [T - D_1 h_0, T] \),
\[
e^{-3T k^2/\mu^*} AL_\lambda(T) \leq (T - t + \lambda) N_\lambda(t) \\
+ \frac{3nk^2}{4\mu^*} + \frac{C}{\delta^2 R^2} e^{-\frac{\mu^2}{4\lambda + \mu}} (1 + T^{-1/2}) e^{\frac{\lambda}{\mu^2}} (T + \lambda).
\]
Integrating the above inequality with respect to \( t \) over \( [T - D_1 h_0, T - D_1 h_0/2] \), we obtain that
\[
e^{-3T k^2/\mu^*} AL_\lambda(T) \leq \frac{2(D_1 h_0 + \lambda)}{D_1 h_0} \int_{T - D_1 h_0}^{T - D_1 h_0/2} N_\lambda(t) dt \\
+ \frac{3nk^2}{4\mu^*} + \frac{C}{\delta^2 R^2} e^{-\frac{\mu^2}{4\lambda + \mu}} (1 + T^{-1/2}) e^{\frac{\lambda}{\mu^2}} (T + \lambda). \tag{45}
\]
On the other hand, we see from (43) and (i) in Lemma 2.1 (where \( u = z \) and \( f \) is given by (42)) that
\[
N_\lambda(t) \leq -\frac{d}{dt} \log (H_\lambda(t)) + \left[ C(2k) + \frac{n}{4(T - t + \lambda)} \right] + \frac{C}{\delta^2} e^{-\frac{\mu^2}{4\lambda + \mu}} (1 + T^{-1/2}) e^{\frac{\lambda}{\mu^2}}.
\]
Integrating the above inequality over \( [T - D_1 h_0, T - D_1 h_0/2] \), we get that
\[
\int_{T - D_1 h_0}^{T - D_1 h_0/2} N_\lambda(t) dt \leq \log \frac{H_\lambda(T - D_1 h_0)}{H_\lambda(T - D_1 h_0/2)} + \frac{T}{2} \left[ C(k) + \frac{n}{2T} \right] \\
+ \frac{C}{\delta^2} e^{-\frac{\mu^2}{4\lambda + \mu}} (1 + T^{-1/2}) e^{\frac{\lambda}{\mu^2}}. \tag{46}
\]
Notice from Lemma 3.1 that
\[
\frac{H_\lambda(T - D_1 h_0)}{H_\lambda(T - D_1 h_0/2)} \leq e^{C(k) + \frac{(R + h_0)^2}{4\lambda} + \frac{T}{2\mu^2}}. \tag{47}
\]
Therefore, it follows from \((45), (46)\) and \((47)\) that when \(0 < \lambda \leq D_1 h_0\),
\[
\lambda N_\lambda(T) \leq 4e^{\frac{3n^2}{2}} \left[ \frac{(R + \delta R)^2}{2D_1 h_0} + \frac{2}{h_0} + C(k) + \frac{n}{2} \left( 1 + \frac{3k^2}{\mu^*} \right) + M \right]
\]
with
\[
M = \frac{C}{\delta^4 R^2} e^{-\frac{\delta^2 R^2}{2\delta h_0}} (1 + T^{-1/2}) e^{\frac{2}{h_0}} + \frac{C}{\delta^2} e^{-\frac{\delta^2 R^2}{2\delta h_0}} (1 + T^{-1/2}) e^{\frac{2}{h_0}}.
\]

**Step 4.** Let \(D_1 = \delta^2 R^2\) and \(\lambda = \varepsilon D_1 h_0\).

Here \(\varepsilon \in (0, 1)\) will be choose later. Then
\[
\frac{2}{h_0} - \frac{\delta^2 R^2}{D_1 h_0} + \lambda \leq 0,
\]
and consequently
\[
M \leq \frac{C}{\delta^4 R^2} (1 + T^{-1/2}).
\]

Therefore, by Step 3 and noticing that \(0 < h_0 < T \leq 1\), we have
\[
\frac{16\lambda}{\tau^2} \left[ \lambda N_\lambda(T) + \frac{n}{4} \right] \leq \varepsilon \frac{D_1}{\tau^2} e^{\frac{3k^2}{\mu^*} R} \left[ C(\mu^*, k, n) h_0 + D_1 h_0 \frac{C}{\delta^4 R^2} (1 + T^{-1/2}) + \frac{(R + \delta R)^2}{2D_1} \right]
\leq \varepsilon C(\delta, R, r, k, \mu^*).
\]

**Step 5.** Finally, we choose \(\varepsilon \in (0, 1)\), which is independent of \(T\), such that
\[
\frac{16\lambda}{\tau^2} \left[ \lambda N_\lambda(T) + \frac{n}{4} \right] \leq \varepsilon C(\delta, R, r, k, \mu^*) = \frac{1}{2}
\]

Hence, by Step 2, we get that
\[
\int_{\Omega \cap B_{R+2\delta R}(x_0)} |z(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\tau}} dx \leq 2 \int_{B_r(x_0)} |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\tau}} dx,
\]
which implies that
\[
\int_{\Omega \cap B_R(x_0)} |u(x,T)|^2 dx \leq e^{\frac{\mu^2}{\tau}} \int_{B_r(x_0)} |u(x,T)|^2 dx. \quad (48)
\]
Recall that
\[
\lambda = \varepsilon D_1 h_0 = \frac{C(r, R, \delta, k, \mu^*)}{\log \left[ \frac{C}{\delta^2 R^2} e^{\frac{3k^2}{\mu^*} R} \left[ \frac{\|u(\cdot,0)\|_{L^2(\Omega)}^2}{\|u(\cdot,T)\|_{L^2(\Omega \cap B_R(x_0))}^2} \right] \right]},
\]
Thereby,
\[
e^{\frac{\mu^2}{\tau}} = \left[ \frac{C}{\delta^2 R^2} e^{\frac{3k^2}{\mu^*} R} \left[ \frac{\|u(\cdot,0)\|_{L^2(\Omega)}^2}{\|u(\cdot,T)\|_{L^2(\Omega \cap B_R(x_0))}^2} \right] \right] C(r, R, \delta, k, \mu^*).
\]

Which, together with \((48)\), completes the proof. \(\Box\)

At the end, by successively making use of Lemma 3.2 and an argument of propagation of smallness (see, e.g., [26]), we have the following.
The proof of Theorem 1.2. Firstly, by Lemma 3.2 and constructing a sequence of balls chained along a curve, we claim that, for any compact sets \( K \) with non-empty interior in \( \Omega \), there are constants \( N_1 = N_1(K_1, K_2, k, n) \geq 1 \) and \( \alpha_1 = \alpha_1(K_1, K_2, k, n) \in (0, 1) \) such that

\[
\int_{K_1} |u(x, T)|^2 \, dx \leq \left( N_1 e^{\frac{N_2}{2}} \int_{\Omega} |u(x, 0)|^2 \, dx \right)^{1-\alpha_1} \left( \int_{K_2} |u(x, T)|^2 \, dx \right)^{\alpha_1}.
\]

Indeed, let

\[ K_1 \subset \bigcup_{i=1}^{p} B_r(x_i) \subset \Omega, \quad B_r(x_0) \subset K_2, \quad \tag{50} \]

and for each \( B_r(x_i) \) with \( 1 \leq i \leq p \), there exists a chain of balls \( B_r(x_i^j) \), \( 1 \leq j \leq n_i \), such that

\[
B_r(x_i^1) = B_r(x_i), \quad B_r(x_i^{n_i}) = B_r(x_0),
\]

\[
B_r(x_i^j) \subset B_{2r}(x_i^{j+1}) \subset \Omega, \quad 1 \leq j \leq n_i - 1,
\]

where \( 0 < r = r(\Omega, K_1, K_2) < 1/2 \) is a constant. By Lemma 3.2, we get that there are \( N_i = N_i(r, k, n) \geq 1 \) and \( \theta_i = \theta_i(r, k, n) \in (0, 1) \) such that

\[
\int_{B_{2r}(x_i^{j+1})} |u(x, T)|^2 \, dx \leq \left( N_i e^{\frac{N_i}{2}} \int_{\Omega} |u(x, 0)|^2 \, dx \right)^{\theta_i} \left( \int_{B_r(x_i^j)} |u(x, T)|^2 \, dx \right)^{1-\theta_i}.
\]

This, together with (51), implies that there are \( N_i = N_i(K_1, K_2, k, n) \geq 1 \) and \( \theta_i = \theta_i(K_1, K_2, k, n) \in (0, 1) \) such that

\[
\int_{B_r(x_i)} |u(x, T)|^2 \, dx \leq \left( N_i e^{\frac{N_i}{2}} \int_{\Omega} |u(x, 0)|^2 \, dx \right)^{\theta_i} \left( \int_{B_r(x_0)} |u(x, T)|^2 \, dx \right)^{1-\theta_i}.
\]

This, along with (50) leads to the estimate (49).

Secondly, since \( \partial \Omega \) is of class \( C^2 \), there are a finite set \( \Lambda \subset \Omega, \ 0 < \delta < 1 \) and a family of positive numbers \( 0 < r_x \leq 1, \ x \in \Lambda \), such that

\[
\partial \Omega \subset \bigcup_{x \in \Lambda} B_{r_x}(x) \quad \text{and} \quad B_{(1+2\delta)r_x}(x) \cap \Omega \text{ is star-shaped with center } x.
\]

Then we apply Lemma 3.2 with \( \Omega \cap B_{(1+2\delta)r_x}(x), x \in \Lambda \), and the same arguments as above to get that, when \( \Gamma \) is a neighborhood of \( \partial \Omega \) and \( K_3 \) is a compact set with non-empty interior in \( \Omega \), there are constants \( N_2 = N_2(\Gamma, K_3, k, n) \geq 1 \) and \( \alpha_2 = \alpha_2(\Gamma, K_3, k, n) \in (0, 1) \) such that

\[
\int_{\Gamma} |u(x, T)|^2 \, dx \leq \left( N_2 e^{\frac{N_2}{2}} \int_{\Omega} |u(x, 0)|^2 \, dx \right)^{1-\alpha_2} \left( \int_{K_3} |u(x, T)|^2 \, dx \right)^{\alpha_2}.
\]

Finally, we derive the desire estimate (4) from the previous two statements with \( \Omega \subset (\Gamma \cup K_1) \) and \( (K_2 \cup K_3) \subset \omega. \) \( \square \)
4. Conclusion and further comments. In the present paper, by adapting the frequency function method in [23, 26], we have derived a Hölder-type quantitative estimate of unique continuation at one point in time for any solution to the heat equation with singular Coulomb potentials in either a bounded Lipschitz domain or a bounded domain with a $C^2$-smooth boundary. Several remarks are given in order.

1. Applications in Control Theory. As addressed in the introduction, such a kind of quantitative estimate of unique continuation has been proved to be applicable in the subject of control theory in recent years (see e.g. [1, 5, 6, 24, 25, 31, 32, 33, 34, 36, 37]). In particular, it can be used to establish the null controllability from measurable subsets of positive measure, and to obtain the bang-bang property of time and norm optimal control problems for the heat equation with singular Coulomb potentials.

2. Variable Coefficients. In the main theorems of this paper, we established the quantitative estimate of unique continuation for the heat equation with a singular potential $k/|x|$, where $k$ is a constant. If we allow that $k$ is a bounded function depending on both space and time variables instead of being a constant, then the corresponding quantitative estimate for the heat equation with a singular potential $|x|^{-1}k(x,t)$ could also be obtained by using the same arguments as in the current paper with some minor modifications. The details are left to the interested reader. Notice that quantitative estimates of strong unique continuation for second order parabolic equations with bounded potentials have been obtained in [2, 9, 23, 24, 26]), by either the Carleman estimate method or the frequency function method. However, it is still an open question whether it could be improved for the general second-order parabolic equation with singular Coulomb potentials.

3. Inverse Square Potentials. Another question is whether one can still expect results as in Theorems 1.1 and 1.2 if the Coulomb potential in (1) is replaced by the inverse square potential $\mu/|x|^2$, which is known as the most singular lower-order potential from the viewpoint of well-posedness and unique continuation (cf., e.g., [29] and [13]). The strong unique continuation at the singularity point for the heat equation with inverse square potentials has been obtained in [27] and [11] (see also [21]). By Carleman estimates associated with the heat operator with inverse square potentials, the observability inequality from an observation region, which is away from the singularity, for the heat equation with inverse square potentials has also been obtained in [4] and [28]. Because of the strong singularity near the origin, however, our present methods do not allow us to derive the quantitative strong unique continuation estimate (2) for the heat equation with this kind of inverse square potential, although we dare to conjecture that it should be possible (see, e.g., [22]).

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