The Betti Number of the Independence Complex of Ternary Graphs

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Overview

1. Introduction

2. Topological Prerequisites

3. Proof of Main Theorem
- $f_G = \sum (-1)^{|A|}$ over all independent sets $A$.
- $I(G)$: the simplicial complex whose faces are the independent sets of $V(G)$.
- $\tilde{b}_i(I(G)) = dim\tilde{H}_i(I(G))$: the $i$-th reduced Betti number of $I(G)$.
- $b(G) = \sum_i \tilde{b}_i(I(G))$.
- Homological fact: $\chi_{I(G)} = 1 + \sum_i (-1)^i \tilde{b}_i(G) = \sum (-1)^{|A|-1}$, sum over all the non-empty independent sets in $G$.
- $f_G = \sum_{i=0}^{\infty} (-1)^{i+1} \tilde{b}_i(G)$
- $|f_G| \leq b(G)$
- A graph is *ternary* if it has no induced cycle of length divisible by three.
Introduction

Theorem (Chudnovsky, Scott, Seymour and Spirkl, 2020)
If $G$ is a graph with no induced cycle of length divisible by three, then $|f_G| \leq 1$.

Theorem (Hehui Wu and Wentao Zhang, 2021)
If $G$ is a graph with no induced cycle of length divisible by three, then $b(G) \leq 1$.

Theorem (Engstrom, 2020)
If $G$ is a graph without cycles of length divisible by three, then $I(G)$ is contractible or homotopy equivalent to a sphere.

Conjecture (Engstrom, 2020)
If $G$ is a graph without induced cycles of length divisible by three, then $I(G)$ is contractible or homotopy equivalent to a sphere.
Some Notation

- Given a graph $G$, $X$ is an independent set of $G$ and $Y$ is a vertex set disjoint from $X$.
- $G(X | Y)$ is the subgraph induced by $V(G) - N[X] - Y$.
- Use $I(X | Y)$, $b(X | Y)$ and $\tilde{b}_i(X | Y)$ for Simplicity.

\[
I(X | Y) = I(G(X | Y))
\]
\[
b(X | Y) = b(G(X | Y))
\]
\[
\tilde{b}_i(X | Y) = \tilde{b}_i(I(G(X | Y)))
\]
Mayer-Vietoris Sequence

Let $K'$ and $K''$ be subcomplexes such that $K = K' \cup K''$ and let $L = K' \cap K''$. There is an exact sequence of reduced homology groups called the Mayer-Vietoris sequence.

\[ \cdots \to \tilde{H}_i(L) \xrightarrow{\lambda_i} \tilde{H}_i(K') \oplus \tilde{H}_i(K'') \to \tilde{H}_i(K) \to \tilde{H}_{i-1}(L) \xrightarrow{\lambda_{i-1}} \tilde{H}_{i-1}(K') \oplus \tilde{H}_{i-1}(K'') \to \cdots \to \tilde{H}_0(K) \to 0 \]

Short exact sequence:

\[ 0 \to \text{cok } \lambda_i \to \tilde{H}_i(K) \to \ker \lambda_{i-1} \to 0 \]
• \( N_i = \ker \lambda_i, \, \beta(N_i) \) is its dimension.

\[
\beta_i(K) = \beta(\cok \lambda_i) + \beta(\ker \lambda_{i-1})
\]

\[
= \beta(\tilde{H}_i(K') \oplus \tilde{H}_i(K''))/\text{im} \lambda_i) + \beta(N_{i-1})
\]

\[
= \beta_i(K') + \beta_i(K'') - \beta_i(L) + \beta(N_i) + \beta(N_{i-1})
\]  (1)

• Suppose \( v \) is a vertex of \( G \), take \( K = I(G), \, K' = I(G - v), \, K'' = I(G - N(v)) \) and \( L = I_G(v \mid \emptyset) \).

• If \( H \) has an isolated vertex, then \( b(H) = 0 \).

\[
\tilde{b}_i(G) = \tilde{b}_i(\emptyset \mid v) - \tilde{b}_i(v \mid \emptyset) + \beta(N_i) + \beta(N_{i-1}), \quad \forall i.
\]  (2)

\[
\beta(N_i) \leq \tilde{b}_i(v \mid \emptyset), \quad \forall i.
\]  (3)
Proof of Main Theorem

**Theorem**

If \( b(G) \geq 2 \) and \( b(H) \leq 1 \) for every induced subgraph \( H \) of \( G \), then \( G = C_{3k} \) for some integer \( k \).

- \( b(X \mid Y) = 0 \) or 1 if \( X \cup Y \neq \emptyset \).
- If \( b(H) = 1 \) for some graph \( H \), let \( d(H) \) be the dimension of the reduced Betti number taking value 1. \( d(H) = i \) if \( \tilde{b}_i(H) = 1 \) and \( \tilde{b}_j(H) = 0 \) for \( j \neq i \).
- If \( b(H) = 0 \), let \( d(H) \) be ‘*’.
- If \( X \) is not independent, let \( d(X \mid Y) = * \).
- If \( H \) is null graph, let \( d(H) = -1 \).
Lemma 1

For any disjoint vertex set $X$ and $Y$ in $G$ with $X \cup Y \neq \emptyset$ and a vertex $v$ not in $X$ or $Y$, the triple $(d(X \mid Y), d(v), d(v))$ fits into one of the following four patterns: $(k, *, k)$, $(*, *, *)$, $(*, k, k)$ and $(k + 1, k, *)$ for some integer $k$. 

\[
\begin{array}{cccccc}
(X, Y) & k & * & * & k + 1 \\
(X \cup \{v\} \mid Y) & X \mid Y \cup \{v\} & * & k & * & k & k & * \\
\end{array}
\]
Lemma 1

For any disjoint vertex set $X$ and $Y$ in $G$ with $X \cup Y \neq \emptyset$ and a vertex $v$ not in $X$ or $Y$, the triple $(d(X \mid Y), d(v), d(v))$ fits into one of the following four patterns: $(k, *, k)$, $(*, *, *)$, $(*, k, k)$ and $(k + 1, k, *)$ for some integer $k$. 

\[
\begin{array}{c|c|c|c|c}
(X, Y) & k & * & * & k + 1 \\
(X \cup \{v\} \mid Y) & * & k & * & * \\
(X \mid Y \cup \{v\}) & * & k & * & k \\
\end{array}
\]
Lemma 2

Suppose $X, Y$ are vertex set of $G$ with $d(X \mid Y) = k$ for some integer $k$. If $v_1, v_2$ are two vertices not in $X \cup Y$ with $d(v_1) = k - 1$ and $d(v_2) = \ast$, then $d(v_1, v_2) = \ast$. 
Lemma 3

There is some \( k \geq 0 \) such that \( \tilde{b}_k(G) = 2 \) and \( \tilde{b}_i(G) = 0 \) for all \( i \neq k \). Furthermore, for every vertex \( v \), \( d(v \mid \emptyset) = k - 1 \) and \( d(\emptyset \mid v) = k \).
Proof of Main Theorem

Lemma 3

There is some \( k \geq 0 \) such that \( \tilde{b}_k(G) = 2 \) and \( \tilde{b}_i(G) = 0 \) for all \( i \neq k \). Furthermore, for every vertex \( v \), \( d(v | \emptyset) = k - 1 \) and \( d(\emptyset | v) = k \).

\[
\begin{align*}
(|X|, |Y|): & & (|X|, |Y|): & & d(X | Y): \\
(1, 0) & & (0, 1) & & k - 1 & & / \\
(2, 0) & & (1, 1) & & (0, 2) & & k - 2 & & * & & / \\
(3, 0) & & (2, 1) & & (1, 2) & & (0, 3) & & k - 3 & & * & & * & & / \\
& & \vdots & & & & \vdots & & & & \vdots & & & & \\
(t, 0) & & (t-1, 1) & & \cdots & & \cdots & & (1, t-1) & & (0, t) & & k - t & & \cdots & & \cdots & & * & & / \\
\end{align*}
\]
**Proof of Main Theorem**

**Lemma 3**

There is some $k \geq 0$ such that $\tilde{b}_k(G) = 2$ and $\tilde{b}_i(G) = 0$ for all $i \neq k$. Furthermore, for every vertex $v$, $d(v | \emptyset) = k - 1$ and $d(\emptyset | v) = k$.

\[
\begin{align*}
(|X|, |Y|): & \quad (1, 0) \quad (0, 1) \\
& \quad (2, 0) \quad (1, 1) \quad (0, 2) \\
& \quad (3, 0) \quad (2, 1) \quad (1, 2) \quad (0, 3) \\
& \quad \vdots \\
& \quad (t, 0) \quad (t-1, 1) \quad \cdots \quad \cdots \quad (1, t-1) \quad (0, t) \\
\end{align*}
\]

\[
\begin{align*}
d(X | Y): & \quad k-1 \quad / \\
& \quad k-2 \quad * \quad / \\
& \quad k-3 \quad * \quad * \quad / \\
& \quad \vdots \\
& \quad k-t \quad * \quad \cdots \quad \cdots \quad * \quad / 
\end{align*}
\]
Proof of Main Theorem

Construct a new graph $H$ on $V(G)$ such that $u, v$ are adjacent if and only if $d(u, v \mid \emptyset) = k - 2$.

**Proposition 4**

In $H$, any two vertices $u, v$ satisfies

1. If $u \sim v$ in $H$, then $u \sim v$ in $G$. That is, $E(G) \cap E(H) = \emptyset$.
2. If $u \sim v$ in $H$, then $d(u, v \mid \emptyset) = k - 2$, $d(u \mid v) = d(v \mid u) = \ast$, and $d(\emptyset \mid u, v) = k$,
3. If $u \nabla v$ in $H$, then $d(u, v \mid \emptyset) = d(\emptyset \mid u, v) = \ast$, and $d(u \mid v) = d(v \mid u) = k - 1$. 
Proof of Main Theorem

Lemma 5

Every component $C$ of $H$ is a complete graph. Furthermore, for any disjoint subsets $X$ and $Y$ of $V(C)$ with $X \cup Y \neq \emptyset$, we have

$$d(X \mid Y) = \begin{cases} 
  k - |X|, & Y = \emptyset, \\
  *, & X, Y \neq \emptyset, \\
  k, & X = \emptyset.
\end{cases}$$
Proof of Main Theorem

Lemma 5

Every component $C$ of $H$ is a complete graph. Furthermore, for any disjoint subsets $X$ and $Y$ of $V(C)$ with $X \cup Y \neq \emptyset$, we have

$$d(X \mid Y) = \begin{cases} k - |X|, & Y = \emptyset, \\ *, & X, Y \neq \emptyset, \\ k, & X = \emptyset. \end{cases}$$

$$(|X|, |Y|):$$

$$\begin{array}{ccc} (1, 0) & (0, 1) \\ (2, 0) & (1, 1) & (0, 2) \\ (3, 0) & (2, 1) & (1, 2) & (0, 3) \\ \vdots \\ (t, 0) & (t-1,1) & \cdots & \cdots & (1,t-1) & (0, t) \end{array}$$

$d(X, Y):$

$$\begin{array}{ccc} k-1 & k \\ k-2 & * & k \\ k-3 & * & * & k \\ \vdots \\ k-t & * & \cdots & \cdots & * & k \end{array}$$
Claim 6
There does not exist a vertex \( v \) with all neighbors in \( G \) located in one component of \( H \).
Lemma 7

There do not exist two edges \( v_1v_2, v_3v_4 \) in \( G \), with \( v_1, v_2, v_3, v_4 \) located in four distinct components of \( H \).

\[
\begin{align*}
(|X|, |Y|): & \quad (|X|, |Y|): \\
(1, 0) & \quad (0, 1) \\
(2, 0) & \quad (1, 1) \quad (0, 2) \\
(3, 0) & \quad (2, 1) \quad (1, 2) \quad (0, 3) \\
(4, 0) & \quad (3, 1) \quad (2, 2) \quad (1, 3) \quad (0, 4) \\
\end{align*}
\]

\[
\begin{align*}
d(X \mid Y): & \quad d(X \mid Y): \\
& \quad k-1 \quad k \\
& \quad * \quad k-1 \quad * \\
& \quad * \quad * \quad k-1 \quad k-1 \\
& \quad * \quad * \quad * \quad ?
\end{align*}
\]
Proof of Main Theorem

Theorem

If \( b(G) \geq 2 \) and \( b(H) \leq 1 \) for every induced subgraph \( H \) of \( G \), then \( G = C_{3k} \) for some integer \( k \).
The End