Correlation functions of boundary field theory from bulk Green’s functions and phases in the boundary theory

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Abstract

In the context of the bulk-boundary correspondence we study the correlation functions arising on a boundary for different types of boundary conditions. The most general condition is the mixed one interpolating between the Neumann and Dirichlet conditions. We obtain the general expressions for the correlators on a boundary in terms of Green’s function in the bulk for the Dirichlet, Neumann and mixed boundary conditions and establish the relations between the correlation functions. As an instructive example we explicitly obtain the boundary correlators corresponding to the mixed condition on a plane boundary $R^d$ of a domain in flat space $R^{d+1}$. The phases of the boundary theory with correlators of the Neumann and Dirichlet types are determined. The boundary correlation functions on sphere $S^d$ are calculated for the Dirichlet and Neumann conditions in two important cases: when sphere is a boundary of a domain in flat space $R^{d+1}$ and when it is a boundary at infinity of Anti-De Sitter space $AdS_{d+1}$. For massless in the bulk theory the Neumann correlator on the boundary of AdS space is shown to have universal logarithmic behavior in all AdS spaces. In the massive case it is found to be finite at the coinciding points. We argue that the Neumann correlator may have a dual two-dimensional description. The structure of the correlators obtained, their conformal nature and some recurrent relations are analyzed. We identify the Dirichlet and Neumann phases living on the boundary of AdS space and discuss their evolution when the location of the boundary changes from infinity to the center of the AdS space.

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1 Introduction

It was recently proposed in [2], [3] that there is a correspondence between theories defined in the bulk and on the boundary. Particularly, the quantum correlation functions of the boundary theory are expressed in terms of classical Green’s functions in the bulk. This correspondence was demonstrated in [2], [3] for Anti-de Sitter (AdS) space the spatial infinity of which plays the role of the boundary. Green’s function in the bulk arises in the boundary value problem with the Dirichlet condition at infinity. The consideration of [2], [3] is motivated by the suggestion made in [1] that the large $N$ limit of a superconformal Yang-Mill theory with gauge group $SU(N)$ in $d$ dimensions is governed by supergravity on the product space $AdS_{d+1} \times \Sigma$ ($\Sigma$ is a compact manifold, often it is sphere $S^{d+1}$). Thus, one may hope to understand important features of QCD by just solving the supergravity on the AdS space. Particularly, this may establish the long time suspected [4] underlying string theory description of QCD in four dimensions since the supergravity under consideration is what arises in the low-energy limit of type IIB superstring compactified on $AdS_5 \times \Sigma$. The works [1]-[3] initiated a flow of papers developing further the bulk-boundary (usually referred as CFT/AdS) correspondence [5]-[16].

The classical and quantum fields on AdS spaces are studied for long time [17], [18]. This study is, in particular, relevant to the physics of three-dimensional (BTZ) black holes (see [19] and references therein) and to the extreme limit of higher-dimensional black holes [20]. That the AdS space has a boundary at infinity means that some condition should be imposed there. In the context of supergravity the analysis made in [18] shows that there are two types of conditions which preserve the supersymmetry: when one fixes the field function $u$ (the Dirichlet type condition) or its normal derivative $\partial_n u$ (the Neumann type condition) at infinity of AdS space.

In the present paper we investigate in a systematic way the correlation functions arising on a boundary for different types of boundary conditions. One of the starting points of our study is a simple observation that the bulk-boundary correspondence is not a feature of AdS spaces only but is a general phenomenon. In the Euclidean version it arises in an elliptic boundary value problem on arbitrary manifold with a boundary. Different boundary conditions describe different theories on the boundary or, better to say, different phases of the boundary theory. The most general condition one may impose is the mixed one when the combination $(\partial_n u - hu)$ is fixed on the boundary. Changing $h$ from zero to infinity it interpolates between the Neumann and Dirichlet conditions. Thus, in terms of the coupling $h$ the Neumann phase appears on the boundary in the weak coupling regime while the Dirichlet phase corresponds to the strong coupling. Not for any manifold the elliptic boundary problem for the mixed condition can be solved explicitly. As a simple example when it can be done we consider a domain of flat space $R^{d+1}$ with plane boundary $R^d$. The boundary correlators corresponding to the mixed boundary condition interpolate (when $h$ varies) between the correlators arising in the Neumann and Dirichlet problems. For a finite $h$ both phases (with the Neumann and Dirichlet type correlators) present on the boundary. When either boundary or manifold itself (or both) is curved the general picture remains the same though the solving the elliptic problem becomes technnically more difficult. Therefore, one can solve the Neumann and Dirichlet problems first and then apply the dimensional arguments to get the structure of the correlation functions corresponding to the mixed boundary condition in the regimes of weak and strong coupling $h$. Proceeding this way, we calculate the Neumann and Dirichlet correlators arising on
sphere $S^d$ in two important cases: when $S^d$ is a boundary of a domain in flat space $R^{d+1}$ and when it is a boundary at infinity of space $AdS_{d+1}$. These cases are actually related. They are the limiting points in the family of boundary problems on a domain of AdS space when the boundary is shrinking from infinity to the center of the space.

Our results for the Dirichlet phase at infinity of AdS space are in agreement with the consideration of [2], [3]. What concerns the Neumann phase, we find that it has quite remarkable feature: in the massless case the corresponding boundary correlation function is logarithmic for all AdS spaces. We argue that there is a dual two-dimensional description of this phase. In the context of the Yang-Mills theory on the boundary of AdS space the Neumann phase may describe that regime of the theory where the string (two-dimensional) nature of QCD manifests the most. The mass in the bulk plays the role of a regulator in the Neumann phase: the corresponding correlator becomes finite at the coinciding points.

This paper is organized as follows. In the next Section we give a general consideration of the boundary-bulk correspondence and obtain the expressions for the correlators on the boundary in terms of Green’s function in the bulk for the Dirichlet, Neumann and mixed boundary conditions. We establish also the relations between Green’s functions corresponding to these conditions. The boundary value problem with the mixed condition on the plane boundary $R^d$ of a domain of flat space $R^{d+1}$ is explicitly solved and the corresponding boundary correlators are analyzed in Section 3. In Sections 4 and 5 we study the correlation functions (of the Dirichlet and Neumann type) arising on sphere $S^d$ considered as a boundary of a domain in flat space $R^{d+1}$ and space $AdS_{d+1}$ respectively. The structure of the correlators obtained, their conformal nature and some recurrent relations are analyzed in Section 6. In Section 7 we discuss the Dirichlet and Neumann phases living on the boundary of AdS space and discuss their deformation in the limit when the location of the boundary changes from infinity to the center of the AdS space. The presence of black hole inside AdS space is shown to affect the boundary theory in this limit. Throughout the paper we consider only Euclidean version of the boundary-bulk correspondence.

### 2 Preliminary

Our starting point is the action

$$W = \frac{1}{2} \int_\mathcal{M} (\nabla u)^2$$

(2.1)

for the scalar field $u$ on Euclidean manifold $\mathcal{M}$ with boundary $\mathcal{B}$. Varying (2.1) with respect to $u$ we arrive at the Laplace equation

$$\Box u = 0 \ .$$

(2.2)

The solving of this elliptic problem requires imposing on the function $u$ some condition on the boundary $\mathcal{B}$. The minimal condition which can be imposed is determined by considering the term $\delta W_\mathcal{B} = \int_\mathcal{B} \partial_n u \delta u$ arising on the boundary under variation of the action (2.1), $\partial_n = n^\mu \partial_\mu$ is derivative with respect to outer normal $n$ to the boundary $\mathcal{B}$. The term $\delta W_\mathcal{B}$ vanishes in two cases: i) if $\partial_n u = 0$ on $\mathcal{B}$ or ii) value of $u$ is fixed on $\mathcal{B}$, particularly we may put $u|_\mathcal{B} = 0$. In fact, either of these conditions is necessary to impose
in order to have the Laplace operator \( \Box \) self-adjoint on the manifold \( \mathcal{M} \). Considering perturbation of these conditions we find

\[
\partial_n u|_B = g \tag{2.3}
\]

and

\[
u|_B = f \tag{2.4}
\]

which are known respectively as the Neumann and Dirichlet boundary value problems, \( g \) and \( f \) are some functions on the boundary. The function \( g \) is not arbitrary. It must satisfy the condition

\[
\int_B g = 0 \tag{2.5}
\]

The standard way to solve an elliptic boundary-value problem is to apply Green’s formula

\[
u(M) = \int_B (G(M, P)\partial_n u(P) - \partial_n G(M, P)u(P))d\Sigma_P , \tag{2.6}
\]

where \( G(M, P) \) is source function (Green’s function) defined as a solution of the equation (2.2) which has a singularity at the point \( M = P \). When point \( M \) approaches point \( P \) we have

\[
G(M, P) \approx \frac{1}{2\pi} \ln \frac{1}{\sigma(M, P)} , \text{ if } \text{dim}(\mathcal{M}) = 2
\]

\[
G(M, P) \approx \frac{1}{(d - 1)\Sigma_d} \frac{1}{\sigma^{d-1}(M, P)} , \text{ if } \text{dim}(\mathcal{M}) = d + 1 > 2 \tag{2.7}
\]

where \( \sigma(M, P) \) is geodesical distance between the points, \( \Sigma_d = \int_{S^d} 1 = \frac{2\pi^{d/2}}{\Gamma(d/2)} \) is area of \( d \)-dimensional sphere of unit radius.

Additionally on should impose the boundary condition

\[
G(M, P)|_{P\in B} = 0 \tag{2.8}
\]

for the Dirichlet problem and

\[
\partial_n G(M, P)|_{P\in B} = 0 \tag{2.9}
\]

for the Neumann problem. Then the solution of the elliptic boundary value problem takes simple integral form:

\[
u_D = -\int_B \partial_n G_D(M, P)f(P)d\Sigma_P
\]

\[
u_N = \int_B G_N(M, P)g(P)d\Sigma_P + \text{constant} \tag{2.10}
\]

where \( D(N) \) refers to the Dirichlet (Neumann) case respectively. In the Neumann case the solution is determined up an irrelevant constant. Values of the harmonic function \( u \)
inside the manifold $\mathcal{M}$, thus, are completely determined by the boundary conditions on $\mathcal{B}$.

This purely classical and elementary (actually, taken from the students text books [21], [22]) consideration of the boundary value problem occurs to be important in the quantum theory. Indeed, a quantum state of field $u$ on manifold $\mathcal{M}$ with boundary $\mathcal{B}$ is determined by fixing the boundary condition (2.3) or (2.4) and considering the functional integral over all $u$ approaching the fixed values at the boundary:

$$
\Psi_D[f, \mathcal{B}] = \int_{u|_{\mathcal{B}}=f} D u e^{-W[u]}
$$

$$
\Psi_N[g, \mathcal{B}] = \int_{\partial_n|_{\mathcal{B}}=g} D u e^{-W[u]}
$$

Since we have different types of the boundary conditions there are different quantum states $\Psi_D$ and $\Psi_N$.

For a free field described by the action (2.1) it is easy to find how the quantum state $\Psi_N(D)$ depends on the condition on the boundary. Indeed, arbitrary field $u$ in the integral (2.11) defying D(N)-state can be represented in the form

$$
u = u_{D(N)} + u^q_{D(N)}
$$

where $u_{D(N)}$ is classical solution (2.10) and $u^q_{D(N)}$ is a “quantum” field with zero boundary condition:

$$
\partial_n u^q_{N|\mathcal{B}} = 0, \quad u^q_{D|\mathcal{B}} = 0
$$

These conditions are what is necessary to impose on the boundary in order to the Laplace operator $\Box_D(\Box_N)$ be self-adjoint. The functional integration in (2.11) then can be performed and the result reads as follows

$$
\Psi_D[f, \mathcal{B}] = e^{-W[u_{D}]\Box_D} det^{-1/2}\Box_D
$$

$$
\Psi_N[g, \mathcal{B}] = e^{-W[u_N]\Box_N} det^{-1/2}\Box_N
$$

The dependence of the quantum state on the values of the quantum field on the boundary, thus, is given by the classical action functional (2.1) considered on the classical solution (2.10):

$$
W = \frac{1}{2} \int_B u \partial_n u \ d\Sigma
$$

$$
W[u_N] = \frac{1}{2} \int_B \int g(P) G_N(P, P') g(P') d\Sigma_P d\Sigma'_P
$$

$$
W[u_D] = -\frac{1}{2} \int_B \int f(P) \partial_{n_P} \partial_{n_P} G_N(P, P') f(P') d\Sigma_P d\Sigma'_P
$$

Note that due to the condition (2.5) the adding of arbitrary constant to $G_N$ does not change value of the functional $W[u_N]$ in (2.13).

Since the quantum state (2.12) is a functional of values on the boundary $\mathcal{B}$ it is quite natural to relate it with some field theory on $\mathcal{B}$. The proposal of [2], [3] is to identify (2.12) with the expectation value

$$
\Psi_D[f, \mathcal{B}] = < e^{-\int_B f^D} >_B
$$
of an operator $O_D$ arising in this boundary field theory. Generalizing this proposal for
the Neumann condition we have

$$\Psi_N[g, \mathcal{B}] = e^{-\int_B g \partial^2} .$$

(2.15)

The functions $f$ and $g$ play the role of the external source for the theory on the boundary
$\mathcal{B}$. Note, that the right hand side of the equation (2.15) does not change under shift the
operator $O_N$ on a constant: $O_N \to O_N + \text{constant}$. This is due to the condition (2.5) for
the Neumann source $g$. Variation with respect to the source standardly gives correlation
functions

$$< O_D(P)O_D(P') > = -\frac{1}{2} \partial_n \partial_n G_D(P, P')$$

$$< O_N(P)O_N(P') > = \frac{1}{2} G_N(P, P') ,$$

(2.16)

$P$ and $P'$ lying on the boundary $\mathcal{B}$, which in our case are completely determined by the
bulk Green’s functions.

An interesting generalization of the above construction is to consider the so-called
third boundary value problem (or Robin type condition):

$$\left( \partial_n u - hu \right) |_{\mathcal{B}} = g ,$$

(2.17)

where $h$ can in principle be some function on $\mathcal{B}$ but in the simplest case it is just a
constant. It is easy to see that the condition (2.17) is intermediate between the Dirichlet
and Neumann conditions. Indeed, in the limit $h \to 0$ eq. (2.17) is just the Neumann
condition while in the limit $h \to \infty$ we arrive at the Dirichlet boundary condition provided
that $g = -hf$.

In order to accomplish the condition (2.17) in the action principle we need add to (2.1)
some boundary term describing “boundary interaction”. The total action is

$$W^{(h)} = \frac{1}{2} \int_M (\nabla u)^2 - \frac{1}{2} \int_B hu^2 .$$

(2.18)

Variation of (2.18) with respect to $u$ gives us the “minimal” boundary condition
$(\partial_n u - hu) |_{\mathcal{B}} = 0$. It determines $h$-dependent self-adjoint extension $\Box^{(h)}$ of the Laplace
operator on $M$.

Defying the corresponding Green’s function $G^{(h)}$ with the boundary condition

$$\left( \partial_n G^{(h)}(P, P') - h G^{(h)}(P, P') \right) |_{P \in \mathcal{B}} = 0$$

(2.19)

and applying Green's formula (2.6) we obtain the solution of the third boundary value problem (2.17)

$$u^{(h)} = \int_B G^{(h)} g .$$

(2.20)

The corresponding quantum state is defined as follows

$$\Psi_{(h)}[g, \mathcal{B}] = e^{-W^{(h)}[u^{(h)}]} det^{-1/2} \Box^{(h)} ,$$

(2.21)

where

$$W^{(h)}[u^{(h)}] = \frac{1}{2} \int_B u^{(h)} g$$

$$= \frac{1}{2} \int_B \int_B g(P) G^{(h)}(P, P') g(P') \sum d\Sigma \sum d\Sigma' .$$

(2.22)
We see that the parameter \( h \) can be viewed as strength of the boundary interaction in (2.18). The importance of the states (2.21) is that they give us a flow between \( D \)- and \( N \)-states. Particularly, considering the boundary operator \( O(h)(P) \) such that

\[
\Psi(h)[g, B] = e^{-\int gO(h)} > B
\]

we expect that the corresponding correlation functions

\[
< O(h)(P)O(h)(P') > = \frac{1}{2} G(h)(P, P')
\]  

form a one-parametric family interpolating between \( N \)- and \( D \)-correlators.

Interestingly, the Neumann correlator arises in the weak coupling limit \( (h \to 0) \) while the Dirichlet one appears in the strong coupling regime \( (h \to \infty) \). Indeed, taking the limit \( h \to 0 \) in (2.19) we obtain the Neumann Green’s function \( G^{(h\to 0)}(P, P') = G_N(P, P') \) and the action (2.22) coincides with the Neumann expression \( W[u_N] \) (2.13). On the other hand, we get the Dirichlet condition when taking the limit \( h \to \infty \) in (2.19), so we have \( G^{(h\to \infty)}(P, P') = G_D(P, P') \). Moreover, applying the normal derivative \( \partial_n \) to eq.(2.19) and considering \( P' \) lying on the boundary as well we obtain the relation

\[
G(h)(P, P')|_{P, P' \in B} = \frac{1}{h^2} \partial_n \partial_n' G^{(h)}(P, P')|_{P, P' \in B} .
\]

Therefore, in the limit \( h \to \infty \) we find that

\[
G^{(h\to \infty)}(P, P')|_{P, P' \in B} = \frac{1}{h^2} \partial_n \partial_n' G_D(P, P')|_{P, P' \in B}
\]

and the action (2.22) becomes minus the Dirichlet action \( W[u_D] \) (2.13) if we substitute \( g(P) = -hf(P) \). Hence, the correlator (2.23) is the Neumann correlator for \( h \to 0 \) and is minus the Dirichlet correlator for \( h \to \infty \).

Not for any manifold \( \mathcal{M} \) the \( h \)-problem can be solved explicitly. In the next section we give the explicit solution of the problem when \( \mathcal{M} \) is flat space and its boundary \( B \) is a plane. However, it is always useful to remember that \( N \)- and \( D \)-problems are just limiting cases of more general class of problems.

It is worth noting that the boundary condition of the type (2.17) arises in the important case of the scalar field coupled non-minimally to curvature \( \mathcal{R} \) of \( \mathcal{M} \):

\[
W = \frac{1}{2} \int_{\mathcal{M}} ((\nabla u)^2 - \xi \mathcal{R} u^2) - \int_{\mathcal{B}} \xi k u^2 ,
\]

where \( k \) is extrinsic curvature of the boundary \( \mathcal{B} \). The adding of the boundary term in (2.24) is necessary to have the well-defined variation of \( W \) (2.24) with respect to metric \( \mathcal{g} \). The natural boundary condition arising from (2.24) is the following

\[
(\partial_n u - 2\xi k u) |_{\mathcal{B}} = g
\]

Note that even if manifold \( \mathcal{M} \) is flat (\( \mathcal{R} = 0 \)) the non-minimal coupling in (2.24) still manifests if the boundary \( \mathcal{B} \) has non-trivial extrinsic curvature. Note also that by adding a boundary term \( a \int g\partial_n u \) to (2.24) one can change the relative coefficient in the left hand side of eq.(2.23).
3 Flat space with plane boundary, Dirichlet-Neumann duality

In this Section we consider the case when the manifold \( M \) and its boundary \( B \) are \( \mathbb{R}^{d+1} \) and \( \mathbb{R}^d \) respectively. In this case the consideration given in the previous Section is realized explicitly.

Let \((x_1, ..., x_d, z)\) be coordinates in the space \( \mathbb{R}^{d+1} \). We consider only a part of \( \mathbb{R}^{d+1} \) defined by condition \( z \geq 0 \). So, the boundary \( B = \mathbb{R}^d \) is the plane \( z = 0 \) and the normal derivative is \( \partial_n = -\partial_z \). The fundamental solution of the Laplace equation in \( \mathbb{R}^{d+1} \) is the function

\[
g_d \left( \rho^2 + (z - z')^2 \right) = \frac{1}{(d-1) \Sigma_d} \frac{1}{\rho^2 + (z - z')^2} , \\
g_d \left( \rho^2 + (z - z')^2 \right) = -\frac{1}{4\pi} \ln \frac{1}{\rho^2 + (z - z')^2} ,
\]

where \( \rho = \sqrt{(x_1 - x'_1)^2 + ... + (x_d - x'_d)^2} \) is the distance between the points on the plane \( B \). It is easy to construct Green’s function in both the Dirichlet and Neumann boundary value problems

\[
G^{(d)}_{D(N)} = g_d \left( \rho^2 + (z - z')^2 \right) \pm g_d \left( \rho^2 + (z + z')^2 \right) ,
\]

where sign \((-\) stands for the Dirichlet problem and \((+) is for the Neumann problem. The boundary conditions

\[
G_D |_{z=0} = 0 , \quad \partial_z G_N |_{z=0} = 0
\]

are obviously satisfied.

Green’s function for the Robin type condition (2.19) can be found explicitly [24] as well. Let us search it in the form

\[
G^{(h)} = G_N + v ,
\]

where \( v = v(x_1 - x'_1, ..., x_d - x'_d, z + z') \). Then from the boundary condition

\[
\left( \partial_z G^{(h)} - h G^{(h)} \right) |_{z=0} = 0
\]

we find the differential equation for the function \( v = v(x_1 - x'_1, ..., x_d - x'_d, \xi) \)

\[
\partial_\xi v - h \, v = 2h g_d(\rho^2 + \xi^2) .
\]

The solution reads

\[
v = -2h \int_{\xi}^{+\infty} e^{-hs} g_d(\rho^2 + s^2) ds ,
\]

where the constant of integration has been chosen in such a way that \( v \) goes to zero when \( \xi \) goes to infinity. It is easy to check that the function \( v \) is indeed a solution of the Laplace equation. Green’s function for the mixed type boundary value problem then reads

\[
G^{(h)}_d = g_d \left( \rho^2 + (z - z')^2 \right) + g_d \left( \rho^2 + (z + z')^2 \right) - 2h \int_{(z + z')}^{+\infty} e^{-hs} g_d(\rho^2 + s^2) ds
\]

(3.2)
It is straightforward to check that in the limit \( h \to 0 \) Green’s function \( G(h) \) indeed goes to the Neumann Green’s function \( G_N \) (3.1) as it was anticipated in the previous Section. On the other hand in the limit of large positive \( h \) we have

\[
\int_{(z+z')}^{+\infty} e^{-hs} g_d \left( \rho^2 + s^2 \right) \simeq g_d \left( \rho^2 + (z+z')^2 \right) \int_{(z+z')}^{+\infty} e^{-hs} ds
\]

\[
= \frac{1}{h} e^{-h(z+z')} g_d \left( \rho^2 + (z+z')^2 \right)
\]

and the Dirichlet Green’s function \( G_D \) arises in (3.2). This is also in accord with the expectations of the previous Section.

The action functional calculated on the classical solution, thus, takes the form (2.22) with Green’s function \( G(h) \) in the form (3.2). We find the correlator induced on the boundary \( B \)

\[
<\mathcal{O}(h)(P)\mathcal{O}(h)(P')> = K^{d>1}_N = \frac{1}{2} G(h)(P, P') ,
\]

\[
K^{d>1}_N = \frac{1}{(d-1) \Sigma_d} \left( \frac{1}{\rho^{d-1}} - h \int_0^{+\infty} \frac{e^{-hs}}{(\rho^2 + s^2)^{d+1}} ds \right) ,
\]

\[
K^{d=1}_N = \frac{1}{4\pi} \left( -\ln \rho^2 + h \int_0^{+\infty} e^{-hs} \ln(\rho^2 + s^2) ds \right) ,
\]

where \( \rho \) is the distance between points \( P \) and \( P' \) on \( R^d \). Another representation for the correlators (3.3) is the following

\[
K^{d}(h) = -\frac{1}{\Sigma_d} \partial_h \left( \int_0^{+\infty} \frac{e^{-hs}}{(\rho^2 + s^2)^{d+1}} ds \right) .
\]

In the weak coupling limit \( h \to 0 \) the correlators \( K^{d}(h) \) go to the ones arising on the boundary \( B \) in the Neumann problem

\[
K^{d}_{N} = g_d(\rho^2) ,
\]

\[
K^{d>1}_{N} = \frac{1}{(d-1) \Sigma_d} \frac{1}{\rho^{d-1}} , \quad K^{d=1}_{N} = \frac{1}{2\pi} \ln \rho .
\]

In the strong coupling limit \( h \to +\infty \) we have

\[
\int_0^{+\infty} \frac{e^{-hs}}{(\rho^2 + s^2)^{d+1}} \simeq \frac{1}{h} \frac{1}{\rho^{d+1}} - \frac{(d-1)}{h^3} \frac{1}{\rho^{d+1}} ,
\]

\[
\int_0^{+\infty} e^{-hs} \ln(\rho^2 + s^2) ds \simeq \frac{1}{h} \ln \rho^2 + \frac{2}{h^3 \rho^2}
\]

and the correlators (3.3) behave as

\[
K^{d}(h) \simeq -\frac{1}{h^2} K_D , \quad K^{d>1}(h) = \frac{1}{\Sigma_d} \frac{1}{\rho^{d+1}} .
\]

where \( K_D = -\frac{1}{2} \partial_z \partial_{z'} G_D |_{z=z'=0} \) is the correlator arising on the boundary in the Dirichlet problem with Green’s function \( G_D \) (3.1). So in the limit of large \( h \) the boundary operator
The correlators (3.3) possess a series of the divergent terms $\rho h$ in the region $(\rho h)^{-1}$ by the Neumann correlator (3.6) arising in the free regime. In the opposite case, when the expansion (3.7), (3.8) is valid for $(\rho h)^{-1} < 1$ the weak coupling regime is restored. In this regime an operators are some coefficients. For $(d - 1)$ even only even powers of $\rho$ can appear in the series (3.7) so in this case the coefficients $b^{(d)}_{2k}$ vanish. In a few particular cases the series (3.7) reads

$$K^d_{(h)} \simeq \frac{1}{(d - 1) \Sigma_d} \left( \frac{1}{\rho^{d-1}} + \frac{b_0^{(d)} h}{\rho^{d-2}} + \ldots + \frac{b_k^{(d)} h^k}{\rho^{d-k-1}} + \ldots + b_{d-1}^{(d)} h^{d-1} \ln(\rho h) \right), \quad (3.7)$$

where $\{b_k^{(d)}\}$ are some coefficients. For $(d - 1)$ even only even powers of $\rho$ can appear in the series (3.7) so in this case the coefficients $b^{(d)}_{2k}$ vanish. In a few particular cases the series (3.7) reads

$$K^1_{(h)} \simeq -\frac{1}{4\pi} \ln \rho^2,$$

$$K^2_{(h)} \simeq \frac{1}{\Sigma_2} \left( \frac{1}{\rho} + h \ln(\rho h) \right),$$

$$K^3_{(h)} \simeq \frac{1}{2\Sigma_3} \left( \frac{1}{\rho^2} - h^2 \ln(\rho h) \right),$$

$$K^4_{(h)} \simeq \frac{1}{3\Sigma_4} \left( \frac{1}{\rho^3} - \frac{h}{\rho^2} + \frac{h^3}{2} \ln(\rho h) \right),$$

$$K^5_{(h)} \simeq \frac{1}{4\Sigma_5} \left( \frac{1}{\rho^4} - \frac{h}{2\rho^2} + \frac{h^4}{6} \ln(\rho h) \right),$$

$$K^6_{(h)} \simeq \frac{1}{5\Sigma_6} \left( \frac{1}{\rho^5} - \frac{2h}{3\rho^4} - \frac{h^3}{6\rho^2} + \frac{h^4}{9}\rho + \frac{h^5}{24} \ln(\rho h) \right). \quad (3.8)$$

The expansion (3.7), (3.8) is valid for $(\rho h) << 1$. The leading contribution is given by the Neumann correlator (3.6) arising in the free regime. In the opposite case, when $(\rho h) >> 1$, the correlators (3.3) tend to the strong coupling expression (3.6). We see that in the region $(\rho h) << 1$ the weak coupling regime is restored. In this regime an operators $O^{(h)}(P)$ can be viewed as a (perturbative in $h$) composite of operators of dimensions $\frac{1}{(d-k-1)} - \frac{1}{2}$. The contribution of each such operator to the correlator (3.7), (3.8) comes with weight $h^k$. In the region $(\rho h) >> 1$ we have a strong coupling (Dirichlet) phase. The parameter $h$ characterizes the size of the region separating the free (Neumann) and the strong coupling (Dirichlet) phases.

The correlators (3.3) being considered as functions of the geodesical distance $\rho$ possess interesting recurrent relations. Consider a set of operators $\Delta_n = \rho^{-n-1} \partial_\rho (\rho^{n-1} \partial_\rho)$. $\Delta_n$ is radial part of the Laplace operator in $d$ dimensions. A combination of these operators is a first-order differential operator: $\Delta_n - \Delta_{n-k} = \frac{d}{\rho} \partial_\rho$. It is easy to see using the property $\Sigma_{d+2} = \frac{2\pi}{d+1} \Sigma_d$ that

$$\rho^{-1} \partial_\rho K^{(d)}_{(h)} = -2\pi K^{d-2}_{(h)} \quad d \geq 1. \quad (3.9)$$
This relation is independent of \( h \). In the limit of strong coupling we find that \( K^{d}_{(h)} \approx \frac{2\pi}{\rho} K^{d+2}_{h=0} = -\frac{1}{\rho^2} K^{d}_{D} \) and we have the relations

\[
\rho^{-1} \partial_{\rho} K^{d}_{N} = K^{d+2}_{D} , \quad K^{d}_{D} = -2\pi K^{d+2}_{N}
\]

which establish the duality between the Neumann and Dirichlet problems on flat space.

The correlators \( K^{d}_{(h)}(\rho) \) are defined on \( d \)-dimensional space \( \mathbb{R}^{d} \). However, one can consider them on space of arbitrary dimension. More precisely, on a space \( C = \mathbb{R}^{n} \) one can define all tower of correlators \( \{ K^{d}_{(h)}(\rho), \ d \geq 1 \} \) where \( \rho \) is the distance measured on \( C = \mathbb{R}^{n} \). It can be done in the following way. Consider two points \( P \) and \( P' \) on \( C = \mathbb{R}^{n} \) and a space \( M = \mathbb{R}^{d+1} \) which boundary \( B = \partial M \) intersects the space \( C = \mathbb{R}^{n} \) in such a way that \( P \) and \( P' \) are lying on \( B \). Then a field theory on \( M \) induces correlator \( K^{d}_{(h)} \) for any pair of points on \( B \) and particularly for the points \( P \) and \( P' \). Using now the group of symmetry of the space \( C \) and moving \( M \) along \( C \) one can define the correlator \( K^{d}_{(h)}(\rho) \) for any pair of points on \( C \). This offers an interesting interpretation for the correlation function \( (3.7) \). It arises for a pair of points of the space \( B = \mathbb{R}^{d} \) lying on intersection of \( B \) with boundaries of the set of planes \( \{ P^{k} = \mathbb{R}^{d-k+1}, \ k = 0, ..., d-1 \} \) with the Neumann boundary problem considered on each plane. Then each plane \( P^{k} \) induces \( N \)-correlator \( \sim \frac{1}{\rho^{d-k+1}} \) for this pair of points, and the total correlator \( (3.7) \) is just a sum over all these contributions.

We finish this Section with a brief comment. Consider the following integral

\[
\int d^{d}x \int d^{d}y f(x) \frac{1}{|x-y|^{n}} f(y) .
\]

It is well-defined for \( n < d \) and is divergent when \( x \to y \) for \( n \geq d \). The integrals of this type appear in our boundary action. It follows that the boundary action \( W^{(h)} = \int_{B} g(P) K^{d}_{(h)}(P, P') g(P') \) is finite for any finite \( h \) since the leading divergence in the correlator \( K^{d}_{(h)} \) is \( \frac{1}{\rho^{d-k+1}} \). On the other hand, the correlation function \( K^{d}_{D} \) in the Dirichlet phase behaves as \( \frac{1}{\rho^{d+k-1}} \) and the action \( W^{D} = \int_{B} \int f(P) K^{d}_{D}(P, P') f(P') \) should be regularized. The simplest way to do this is to consider the action

\[
W^{D}_{\text{reg}} = \int_{B} \int f(P) K^{d}_{D}(P, P') f(P') - \int_{B} \int f(P) K^{d}_{D}(P, P') f(P) \quad (3.10)
\]

which is obviously finite. The last term in \( (3.10) \) can be re-written as follows

\[
\tilde{h} \int_{B} f^{2}(P) , \quad \tilde{h} = -c \int_{0}^{+\infty} r^{-2} dr ,
\]

where \( c \) is a (positive) constant. We see that the counter-term to be added to \( W^{D} \) takes the form of local boundary term as in \( (2.13) \). Alternatively, one may use analyticity method \( [27] \) to regularize the Dirichlet action. It involves the identity \( \int_{0}^{+\infty} r^{3} dr = 0 \) which is proved by analyticity in \( \lambda \). A regularization of the correlation functions has been also discussed in \( [3], [8] \).
4 Flat space with spherical boundary

In this Section we study the boundary correlation functions described in Section 2 when manifold $\mathcal{M}$ is a domain of $(d + 1)$-dimensional flat space $\mathbb{R}^{d+1}$ with boundary $\mathcal{B}$ being $d$-dimensional sphere $S^d$ of radius $R$. We start with analysis of the simplest $d = 1$ case.

4.1 Dirichlet problem on $\mathbb{R}^2$ with boundary $S^1$

Let $P$ and $M$ be points on $\mathbb{R}^2$ with polar coordinates $(\rho_0, \phi')$ and $(\rho, \phi)$ respectively. We consider the domain inside the circle $S^1$ of radius $R$, i.e. $\rho_0, \rho \leq R$. Green’s function takes the form

$$ G(P, M) = \frac{1}{2\pi} \ln \frac{1}{r(P, M)} + v(P, M) , \quad (4.1) $$

where $r(P, M) = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi')}$ is distance between the points $M$ and $P$ and $v(P, M)$ is harmonic function which is regular everywhere in the domain.

$v(P, M)$ is determined by the Dirichlet condition $G(P, M) = 0$ if $P$ lies on $\mathcal{B}$. In order to construct $v(P, M)$ consider point $M^*$ with coordinates $(\rho^* = \frac{R^2}{\rho}, \phi)$. It is conjugate to the point $M$.

The distance $r_1$ between $P$ and $M^*$ satisfies the relation

$$ r_1^2 = \frac{R^2}{\rho^2} \left( r^2 + \rho^2 \left( \frac{\rho_0^2}{R^2} - 1 \right) + R^2 - \rho^2 \right) . \quad (4.2) $$

If $P$ lies on $S^1$ ($\rho_0 = R$) then we have $r_1 = \frac{R\rho}{\rho}$. This implies that the Dirichlet Green’s function $G_D$ takes the well-known form

$$ G_D = \frac{1}{2\pi} \left( \ln \frac{1}{r} - \ln \frac{R}{\rho r_1} \right) . \quad (4.3) $$

It is easy to find that

$$ (\partial_n r)_{\rho_0=R} = \frac{R^2 + r^2 - \rho^2}{2Rr} , \quad (\partial_n r_1)_{\rho_0=R} = \frac{\rho^2 + r^2 - R^2}{2\rho r} \quad (4.4) $$

and the normal derivative of $G_D$ is

$$ \partial_n G_D|_{\rho_0=R} = -\frac{1}{2\pi R} \frac{(R^2 - \rho^2)}{r^2} . $$

Thus, the solution of the Dirichlet problem $u|_{\rho_0=R} = f(\phi)$ takes the form (see eq.(2.10))

$$ u_D(\rho, \phi) = \frac{1}{2\pi R} \int_0^{2\pi} \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho \cos(\phi - \phi')} f(\phi') Rd\phi' . \quad (4.5) $$

This formula is known as Poisson’s integral for a circle [21].

Calculating the second normal derivative of $G_D$ on the boundary we obtain according to eq.(2.16) the correlation function on the boundary $\mathcal{B} = S^1$

$$ < \mathcal{O}_D(\phi) \mathcal{O}_D(\phi') > = K_D(\phi, \phi') = -\frac{1}{8\pi R^2 \sin^2(\frac{\phi - \phi'}{2})} . \quad (4.6) $$

This is in agreement with that the boundary operator $\mathcal{O}_D$ has dimension 1.
4.2 Dirichlet problem on $R^{d+1}$ with boundary $S^d$

The above two-dimensional result can be easily generalized for the Dirichlet problem on a domain of $(d + 1)$-dimensional space $R^{d+1}$ with spherical boundary $S^d$. The metric on $R^{d+1}$ takes the form

$$ds^2 = d\rho^2 + \rho^2 d\omega^2,$$

where $d\omega^2$ is the metric on the $d$-dimensional unit sphere. In the spherical coordinate system with origin in the center of the sphere consider two points $P$ and $M$ with coordinates $(\rho_0, \theta_i')$ and $(\rho, \theta_i)$ respectively, $\theta_i$, $i = 1, \ldots, d$ are angle coordinates on the sphere $S^d$. The distance $r$ between them is found from the relation

$$r^2 = \rho_0^2 + \rho^2 - 2\rho\rho_0 \cos \gamma,$$

where $\gamma$ ($0 \leq \gamma \leq \pi$) is the geodesical distance measured on the $d$-dimensional unit sphere.

Green’s function satisfying the Dirichlet boundary condition on $S^d$ takes the form

$$G_D = \frac{1}{(d-1)\Sigma_d} \frac{1}{r^{d-1}} - \left( \frac{R}{\rho} \right)^{d-1} \frac{1}{r_1^{d-1}} ,$$

(4.7)

where $r_1$ is the distance between the points $P$ and $M^*$ defined in the same way as in the previous subsection. Note that $r$ and $r_1$ satisfy the same relations (4.2) and (4.4) as in two-dimensional case. Proceeding in the same way as in Section 4.1 we find a generalization of Poisson’s integral (4.5) in higher dimensions

$$u_D(\rho, \theta_i) = \frac{1}{\Sigma_d R} \int_{S^d} \frac{(R^2 - \rho^2)}{(R^2 + \rho^2 - 2R\rho \cos \gamma)^{d+1}} f(\theta') R^d d\mu(\theta) ,$$

(4.8)

where $R^d d\mu(\theta)$ is measure on $S^d$, $\gamma$ is the geodesical distance on the unit sphere between points with coordinates $\theta$ and $\theta'$. The correlation function defined on the boundary $S^d$ then reads

$$K_D(\theta, \theta') = -\frac{1}{\Sigma_d (4R^2 \sin^2 \frac{\gamma}{2})^{d+1}} .$$

(4.9)

It is a higher-dimensional generalization of the expression (4.6).

4.3 Massive case and the limit $R \to 0$

If the field $u$ is massive then we have the equation

$$(\Box - m^2)u = 0$$

(4.10)

instead of the Laplace equation (2.2). In $(d + 1)$-dimensional flat space $\mathcal{M} = R^{d+1}$ the fundamental solution of this equation is found to take the form

$$g_d(m, r) = \frac{1}{(2\pi)^{d+1}} \frac{m}{r} K_{d+1}(mr) ,$$

(4.11)

where $r$ is the distance between points $P$ and $M$ in $R^{d+1}$. For even $d$ the expression (4.11) is particularly simple taking into account that the modified Bessel function $K_\nu$ is

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!(2z)^k}$$

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for integer $n$.

The expression (4.11) can be obtained as a result of applying the formula

$$g_d(r) = \int_0^{+\infty} ds K_{ld+1}(s, r) ,$$

where $K_{ld+1}(s, r)$ is the heat kernel function on $R^{ld}$ defined as solution of the equation

$$\partial_s K = (\Box - m^2) K .$$

In $R^{ld}$ the heat kernel is known to take the form

$$K_{ld+1}(s, r) = \frac{1}{(4\pi s)^{d+1/2}} e^{-m^2 s} e^{-\frac{r^2}{4s}} .$$

(4.12)

Green’s function for the operator (4.10) with the Dirichlet condition on sphere can not be found using the method of images so useful in the massless case. It is, however, possible to construct an auxiliary Green’s function $\tilde{G}_D(P, P')$ which vanishes when both points $P$ and $P'$ lie on the boundary

$$\tilde{G}_D(m, r) = g_d(m, r) - g_d\left(\frac{R}{\rho} m, \frac{R}{\rho} r_1\right) ,$$

(4.13)

$\rho$ and $r_1$ are defined as in eq.(4.7). Near the boundary the true Green’s function $G_D(P, P') \simeq \tilde{G}_D(P, P') + \alpha_0(\gamma) + \alpha_1(\gamma)(R - \rho) + \alpha_1'(\gamma)(R - \rho')$. Therefore, the knowledge of $\tilde{G}_D(P, P')$ is enough for our goals.

Using the eq.(4.4) and the identity $x \partial_a K_\nu(x) - \nu K_\nu u(x) = -x K_{\nu+1}(x)$ for the modified Bessel functions and calculating the normal derivatives of the expression (4.13) we find the boundary correlation function

$$K_D^d(m, \gamma) = -\frac{1}{(2\pi)^{d+1/2}} \frac{m}{2R \sin \frac{\gamma}{2}} K_{d+1}\left(2mR \sin \frac{\gamma}{2}\right)$$

$$+ \frac{1}{(2\pi)^{d+1/2}} \frac{m^4}{4R^2} \frac{m}{2R \sin \frac{\gamma}{2}} K_{d+1}\left(2mR \sin \frac{\gamma}{2}\right)$$

(4.14)

corresponding to the massive theory (4.10) in the bulk. The first term in (4.14) is the leading one for small $\gamma$. It takes the same form as for plane boundary. The second term in (4.14) collects the effects of curvature of the boundary. For small $\gamma$ it is proportional to $\frac{m^4}{R^2}(R \sin \frac{\gamma}{2})^{3-d}$ and leads to new divergent terms only for $d > 3$.

It is of interest to consider the limit $R \to 0$ in the expression (4.14). Then the Bessel function can be approximated by its value at small values of argument. The leading term is given by the correlator (4.9) of the massless case. This is not surprising because the eq.(4.10) is effectively massless for small values of the radial coordinate. However, there also appears the whole series of the subleading divergent terms when we take the limit of small radius $R$ in (4.14). These terms are proportional to powers of mass $m$. In order to accomplish the limit $R \to 0$ we introduce an operator $\hat{O}_D(P)$ having the same dimension $\frac{d+1}{2}$ as the operator $\tilde{O}_D(P)$. For small $R$ we have $O_D(P) \simeq \left(\frac{R}{a}\right)^{-\frac{d+1}{2}} \hat{O}_D(P)$ where $a$ is some (finite) scale. The limit $R \to 0$ for the correlation function of the operators $\hat{O}_D(P)$ is well defined

$$< \hat{O}_D(P) \hat{O}_D(P') >_{R \to 0} = \frac{1}{\Sigma_d(2a \sin \frac{\gamma}{2})^{d+1}}$$

(4.15)
and coincides with the the correlation function \((1.9)\) on sphere \(S_d\) of radius \(a\) in the massless case.

**4.4 Dirichlet problem on \(R^{d+1} \times \Sigma\)**

The results of Section 4.3 are useful when we consider the Dirichlet problem on a domain of a product space \(R^{d+1} \times \Sigma\) with boundary \(\mathcal{B} = S^d \times \Sigma\), where \(\Sigma\) is compact manifold with coordinates \(\{\chi\}\). Let the functions \(\{Y_n(\chi)\}\) form the orthonormal basis of eigenfunctions of the Laplace operator \(\Box_\Sigma\) considered on the compact space \(\Sigma\), i.e. we have

\[
\Box_\Sigma Y_n = -\lambda_n^2 Y_n
\]

\[
\int_\Sigma Y_n(\chi)Y_m(\chi)d\mu(\chi) = \delta_{n,m} . \tag{4.16}
\]

The field function \(u\) being considered on \(R^{d+1} \times \Sigma\) expands with respect to this basis

\[
u(P) = \sum_n Y_n(\chi)u_n ,
\]

where \(\{u_n\}\) are functions on \(R^{d+1}\) satisfying the equation \((4.10)\) with “mass” \(m = \lambda_n\).

The Dirichlet boundary condition consists in fixing the infinite set of functions \(\{f_n\}\):

\[
u|_\mathcal{B} = f(\chi, \theta) = \sum_n Y_n(\chi)f_n(\theta) ,
\]

where \(\{\theta\}\) are coordinates on \(S^d\).

Green’s function expands with respect to the basis \(\{Y_n\}\) as follows

\[
G(P, P') = \sum_n Y_n(\chi)Y_n(\chi')G_n \tag{4.17}
\]

where \(G_n\) is Green’s function for the operator \((4.9)\) on \(R^{d+1}\) with mass \(m = \lambda_n\).

Considering on the boundary the source term

\[
\int_\mathcal{B} f(\chi, \theta)\mathcal{O}_D(\chi, \theta)d\mu(\chi)d\mu(\theta)
\]

we obtain the correlation function for the boundary operator \(\mathcal{O}_D(\chi, \theta)\)

\[
< \mathcal{O}_D(\chi, \theta)\mathcal{O}_D(\chi', \theta') >= K_{S^d \times \Sigma}(\chi, \theta, \chi', \theta') = \sum_n Y_n(\chi)Y_n(\chi')K_D(\lambda_n, \gamma) , \tag{4.18}
\]

where \(K_D(\lambda, \gamma)\) is the correlator \((4.14)\). In general it is rather complicated function. We are, however, interested in considering the limit of \((4.18)\) when radius \(R\) of the sphere \(S^d\) goes to zero. In this limit the function \(K_D(\lambda_n, \gamma)\) becomes independent of \(\lambda_n\). Introducing new operators \(\tilde{\mathcal{O}}_D(\chi, \theta)\) such that \(\mathcal{O}_D(P) \simeq (\frac{R}{a})^{-\frac{d+1}{2}} \tilde{\mathcal{O}}_D(P)\) for small \(R\), we obtain the following remarkable factorization

\[
\tilde{K}_{S^d \times \Sigma}(\chi, \theta, \chi', \theta') = \delta_\Sigma(\chi - \chi')K_{S^d}(\theta, \theta') , \tag{4.19}
\]

where \(K_{S^d}(\theta, \theta')\) is the correlator \((4.9), (4.15)\) arising on sphere \(S^d\) of radius \(a\). The correlator \((4.19)\) describes motion of a quantum particle which does not propagate along the component \(\Sigma\) of the space \(S^d \times \Sigma\).
4.5 Neumann problem on $R^2$
Solving the elliptic problem with the Neumann condition imposed on spherical boundary
we need to modify the definition of Green’s function. More precisely, Green’s function
$G_N(P, P')$ of the Neumann problem should be a solution of the equation
\[ \Box G_N(P, P') = C \] (4.20)
subject to the Neumann boundary condition
\[ \partial_n G_N|_{P \in B} = 0 \] (4.21)
The right hand side of the equation (4.20) is a constant $C$. Therefore, we still can
apply Green’s formula (2.6) and obtain the expression (2.10) for the solution
$u_N(P)$ of the Neumann problem taking into account that $u_N(P)$ is determined up to an irrelevant
constant. In fact, the constant $C$ in (4.20) is not arbitrary and is determined by the
requirement of consistency of the boundary condition (4.21). In two dimensions we find
that $C = \frac{1}{\pi R^2}$ and Green’s function takes the form
\[ G_N(P, P') = \frac{\rho_0^2}{4\pi R^2} + \frac{1}{2\pi} \left( \ln \frac{1}{r} + \ln \frac{R}{\rho r_1} \right) \] , (4.22)
where $(\rho_0, \phi')$, $(\rho, \phi)$ are coordinates of the points $P$ and $P'$ respectively. Definitions of $r$
and $r_1$ are the same as in Section 4.1. By means of simple calculations using eq.(4.4) one shows that the condition (4.21) indeed fulfills for Green’s function (4.22). Considering
(4.22) when both points lie on the boundary $B$ ($\rho_0 = \rho = R$) we obtain according to
(2.16) the Neumann correlation function on $B = S^1$
\[ K_N(\phi, \phi') = -\frac{1}{4\pi} \ln \sin^2 \frac{\phi - \phi'}{2} \] , (4.23)
where we neglected all irrelevant constants.

4.6 Neumann problem on $R^{d+1}$, $d \geq 2$
In higher dimensions we search the solution of the equation (1.21) in the form [22]
\[ G_N(P, P') = \alpha^{-1} \rho_0^2 + \frac{1}{(d-1) \Sigma_d} + v(P, P') \] (4.24)
where $v(P, P')$ is harmonic function, $\Box v = 0$, which is regular everywhere in the domain. Note that in $R^{d+1}$ the function $\rho_0^2$ is a solution of the equation $\Box \rho_0^2 = 2(d+1)$. The condition (4.21) is consistent if we put $\alpha = 2(d+1)\Sigma_{d+1}$ in (4.24) and $C = 2(d+1)\alpha^{-1}$ in (4.20).
In 3-dimensional case Green’s function with the Neumann condition on $S^2$ takes the following form [23]
\[ G_N = \frac{\rho_0^2}{8\pi R^3} + \frac{1}{4\pi r} + \frac{1}{4\pi r_1} \frac{R}{\rho} + w \] , (4.25)
where
\[ w = -\frac{1}{4\pi R} \ln \left( r_1 + \frac{R^2}{\rho} - \rho_0 \cos \gamma \right) \] (4.26)
is harmonic function, \( \Box w = 0 \). The calculation of normal derivatives

\[
\partial_n \rho_0^2 \big|_{\rho_0 = R} = 2R, \quad \partial_n w \big|_{\rho_0 = R} = -\frac{1}{4\pi R} \left( \frac{1}{R} - \frac{1}{r} \right)
\]

\[
\partial_n \left( \frac{1}{4\pi R} + \frac{1}{4\pi R_1} \frac{R}{\rho} \right) \big|_{\rho_0 = R} = -\frac{1}{4\pi R}
\]

shows that the condition (4.21) fulfills for Green’s function (4.25).

The boundary correlation function corresponding, in accord with (2.16), to Green’s function (4.25) reads

\[
K^d_{\Sigma} = \frac{1}{2} G_N(P, P')|_{P, P' \in B}
= \frac{1}{8\pi R \sin \frac{\gamma}{2}} - \frac{1}{8\pi R} \ln \sin \frac{\gamma}{2},
\]

(4.27)

where \( \gamma (0 \leq \gamma \leq \pi) \) is geodesical distance between \( P \) and \( P' \) on the unit two-dimensional sphere. If the spherical coordinates of the points \( P \) and \( P' \) are \( (\phi, \theta) \) and \( (\phi', \theta') \) respectively then \( \gamma \) is found from the relation

\[
\sin^2 \frac{\gamma}{2} = \sin^2 \left( \frac{\theta - \theta'}{2} \right) + \sin \theta \sin \theta' \sin^2 \left( \frac{\phi - \phi'}{2} \right).
\]

(4.28)

The first term in (4.27) is what we could anticipate basing on the analysis of the Neumann problem on plane boundary (see eq.(3.5)). Surprisingly, we obtain also the logarithmic term in the correlator (4.27). It arises due to curvature of the boundary and could not be anticipated just from the analysis of the plane boundary problem.

The generalization of the result (4.27) to higher dimensions, \( d > 2 \), is technically difficult. As a conjecture we, however, propose that the general structure of the Neumann correlation function for \( d \geq 3 \) includes the series

\[
K^d_{\Sigma} = \frac{1}{(d-1) \Sigma_d} \left( \frac{1}{\sigma^{d-1}} + \frac{c_1^{(d)} R^{-1}}{\sigma^{d-2}} + \ldots + \frac{c_k^{(d)} R^{-k}}{\sigma^{d-k-1}} + \ldots + c_{d-1}^{(d)} R^{1-d} \ln \sigma \right)
\]

(4.29)

with respect to variable \( \sigma = 2R \sin \frac{\gamma}{2} \), \( \gamma \) is geodesical distance measured on unit sphere \( S^d \) and \( c_k^{(d)} \) are constant coefficients. Note that all terms in the series are proportional to \( R^{1-d} \). The expression (4.29) is worth comparing with the Eq.(3.7). It seems that the behavior (4.29), (3.7) is typical for the correlation function in the free (Neumann) phase. In both cases the expected term \( \sim \frac{1}{\sigma^{d-1}} \) gets modified by a series with respect to dimensional parameter (additional to the geodesical distance \( \sigma \)) which is \( h \) in Eq.(3.7) and \( R \) in (4.29).

In the massive case the arguments similar to that of Section 4.3 say that taking the limit \( R \to 0 \) one gets the massless Neumann correlators (4.27), (4.29) provided that we rescaled the Neumann boundary operators: \( \mathcal{O}_N = (\frac{2}{a})^{-d+1} \mathcal{O}_{\Sigma} \). As a consequence of this, the correlator arising in the Neumann problem on a product space \( R^{d+1} \times \Sigma \) (\( \Sigma \) is a compact manifold) possess in the limit of small \( R \) (\( R \) is radius of the boundary in \( R^{d+1} \))

the factorization

\[
\tilde{K}_N[S^d \times \Sigma] = \delta_{\Sigma} K_N[S^d]
\]

(4.30)

similar to what we had for the Dirichlet case (4.13).
Anti-de Sitter (AdS) space with boundary at infinity

5.1 Dirichlet problem on AdS

General analysis of AdS spaces we start with the consideration of two-dimensional AdS space $H^2$. It is space of constant negative curvature $-\frac{2}{l^2}$ described by the metric
\[ ds^2 = l^2(dx^2 + \sinh^2 x d\phi^2) , \]
where the coordinates $(x, \phi)$ run in the limits $0 \leq \phi \leq 2\pi$, $0 \leq x \leq +\infty$. Boundary $B$ of the space $H^2$ is a circle $S^1$ lying at the infinite value of the coordinate $x$. Considering the Laplace equation on $H^2$ we impose the Dirichlet boundary condition $u|_{x \to +\infty} = f(\phi)$ on the field function $u$ at infinity.

Green’s function which satisfies the Dirichlet condition $G_{H^2}(x, x', \phi, \phi') = 0$ at infinity $(x \to \infty)$ takes the following form
\[ G_{H^2} = -\frac{1}{2\pi} \ln \tanh \frac{\sigma}{2l} , \]
where $\sigma$ is geodesical distance between points $(x, \phi)$ and $(x', \phi')$ on $H^2$, it can be found from the relation
\[ \cosh \frac{\sigma}{l} = \cosh x \cosh x' - \sinh x \sinh x' \cos(\phi - \phi') , \]
which is a hyperbolic analog of the relation (4.28) for the geodesical distance on 2-sphere.

Fixing the point $(x', \phi')$ consider the limit when the second point $(x, \phi)$ goes to infinity $(x \to \infty)$. In this limit we have $G_{H^2} = \frac{1}{\pi} e^{-\frac{x}{l}}$. On the other hand, we find from the relation (5.3) that
\[ e^{\frac{x}{l}} \simeq e^x \Delta(x', \phi - \phi') , \quad \Delta(x', \phi - \phi') = \cosh(x') - \sinh x' \cos(\phi - \phi') . \]
Hence Green’s function for $x \to +\infty$ reads
\[ G_{H^2} = \frac{1}{\pi} e^{-x} \frac{\Delta}{\Delta} . \]
Normal derivative $\partial_n = l^{-1} \partial_x$ of this expression
\[ \partial_n G_{H^2} = -\frac{1}{\pi l} e^{-x} \frac{\Delta}{\Delta} \]
is exponentially decreasing for $x$ going to infinity.

The measure $d\mu = l \sinh x d\phi$ arising on the boundary $B$ contains the exponentially growing factor. Therefore, applying the general formula (2.10) we find the solution of the elliptic problem on $H^2$ with Dirichlet boundary condition at infinity in the form
\[ u_D(x', \phi') = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\phi) d\phi}{\cosh(x') - \sinh x' \cos(\phi - \phi')} \]
that is a hyperbolic version of Poisson’s integral (4.5). Note, that (5.7) gives us the solution of the Laplace equation on the whole space $H^2$. 

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Putting $x' = 0$ in (5.7) we find that $u(0, \phi') = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi$ for value of the harmonic function $u$ in the center of the space $H^2$. It is exactly the same value which we get in the case of flat disk (putting $\rho_0 = 0$ in (4.3)). On the other hand, considering $x' \to +\infty$ in (5.7) we find that the kernel in the integral (5.7) reproduces $\delta$-function $\delta(\phi - \phi')$ what is in agreement with the imposed boundary condition.

Taking the limit $x' \to +\infty$ in (5.6) and calculating the normal derivative $\partial_n' = l^{-1} \partial_{x'}$ we find that

$$\partial_n' \partial_n G_{H^2} = \frac{1}{\pi l^2} \frac{e^{-l(x+x')}}{\sin^2(\frac{\phi-\phi'}{2})}$$

and the action $W[u_D]$ (2.13) reads

$$W[u_D] = -\frac{1}{8\pi} \int_0^{2\pi} \int_0^{2\pi} f(\phi)\frac{1}{\sin^2(\frac{\phi-\phi'}{2})} f(\phi') d\phi d\phi' .$$

This results in the correlation function

$$K_D(\phi, \phi') = -\frac{1}{8\pi d^2} \frac{1}{\sin^2(\frac{\phi-\phi'}{2})}$$

(5.8)

arising on the boundary $B = S^1$ with measure $d\phi$. It exactly reproduces the correlation function (4.6) appearing in the Dirichlet problem on flat disk $D^2$. It is as expected since spaces $H^2$ and $D^2$ are conformally related.

These results can be generalized for higher-dimensional AdS spaces. The $(d+1)$-dimensional hyperbolic space $H^{d+1}$ is described by the metric

$$ds^2 = l^2(dx^2 + \sinh^2 x d\omega^2) ,$$

(5.9)

where $d\omega^2$ is metric of $d$-dimensional unit sphere $S^d$, set of angles $\{\theta_i, i = 1, \ldots, d\}$ being coordinates on $S^d$. The geodesical distance $\sigma$ between two points $(x, \theta)$ and $(x', \theta')$ on $H^{d+1}$ can be found from the relation

$$\cosh \frac{\sigma}{l} = \cosh x \cosh x' - \sinh x \sinh x' \cos \gamma$$

(5.10)

analogous to eq.(5.3). $\gamma$ is geodesical distance measured on unit sphere $S^d$. Solution of the Laplace equation $\Box u = 0$ on $H^{d+1}$ for large values of $x$ behaves as follows [3]

$$u(x, \theta) \simeq u_0(\theta) + u_1(\theta) e^{-l x} ,$$

(5.11)

where $u_0$ and $u_1$ are some functions on $S^d$. Demanding that $u(x, \theta)$ is regular at $x = 0$ one finds that only one of these functions is independent.

The Dirichlet boundary condition at infinity reads

$$u(x, \theta)|_{x \to \infty} = f(\theta) .$$

As in two-dimensional case we fix point $(x', \theta')$ and consider the other point $(x, \theta)$ approaching infinity. Green’s function in $H^{d+1}$ space vanishing at infinity has the following integral representation

$$G_{H^{d+1}} = \frac{l^{d-1}}{\sum_d} \int_0^{+\infty} \frac{dx}{\sinh^d x} .$$

(5.12)
We, however, need to know only behavior of Green’s function for large values of $\sigma$

$$G_{H^{d+1}} \simeq \frac{2^d}{d\Sigma_d} \frac{1}{l^{d-1}} e^{-d\sigma} . \quad (5.13)$$

We find from (5.10) the expression

$$e^{\frac{x}{2}} \simeq e^x \Delta(x', \theta, \theta') , \quad \Delta(x', \theta, \theta') = \cosh x' - \sinh x' \cos \gamma , \quad (5.14)$$

valid for large value of $x$, $\gamma$ is geodesical distance on $S^d$ between points with coordinates $\theta$ and $\theta'$. Equation (5.14) is a higher-dimensional generalization of eq.(5.4). Combining (5.13) and (5.14) we get Green’s function for large values of $x$

$$G_{H^{d+1}} \simeq \frac{2^d}{d\Sigma_d} \frac{1}{l^{d-1}} \Delta d e^{-dx} . \quad (5.15)$$

We see from (5.11) and (5.15) that $(\partial_n G u) \sim e^{-dx} + O(e^{-2dx})$ and $(\partial_n u G) \sim O(e^{-2dx})$. Measure $d\mu = l^d \sinh^d x d\mu(\theta)$ induced on the boundary $\mathcal{B}$ grows as $e^{dx}$ for large $x$. Therefore we conclude that the first term in Green’s formula (2.6) is negligible for large $x$ and the solution of the Laplace equation for the Dirichlet boundary problem is indeed given by the expression (2.10). After simple computation we get

$$u_D(x', \theta') = -\frac{1}{\Sigma_d} \int_{S^d} \frac{f(\theta)}{\Delta^d(x', \theta, \theta')} d\mu(\theta) \quad (5.16)$$

for the solution and

$$W_D = \frac{d}{2 \Sigma_d} l^{d-1} \int_{S^d} \int_{S^d} f(\theta) \frac{1}{(\sin \frac{\gamma}{2})^{2d}} f(\theta') d\mu(\theta) d\mu(\theta') \quad (5.17)$$

for the boundary action.

Interpreting the kernel in the integral (5.17) as a correlation function for operators $O_D(\theta)$ consider the boundary source term $\int_{\mathcal{B}} O_D f d\mu$. It is finite if the operator $O_D \sim e^{-dx}$ for large $x$. This is an indication that $O_D$ should have dimension $d$. Therefore, the correct source term should be $l^{\frac{d-1}{2}} \int_{\mathcal{B}} O_D f d\mu$ and the correlation function for $O_D$ reads

$$< O_D(\theta) O_D(\theta') > = \frac{d}{2 \Sigma_d} \frac{1}{a^{2d}} \frac{1}{(\sin \frac{\gamma}{2})^{2d}} , \quad (5.18)$$

where we introduced a scale $a$ defying measure on the boundary $S^d$ as $a^d d\mu(\theta)$, the scale $a$ drops out in (5.17) and its value is completely irrelevant. The consideration of this section is in agreement with the results obtained in [2], [3].

5.2 Neumann problem on $AdS_{d+1}$

As in the previous subsection we start with the consideration of two-dimensional case. The Neumann condition at infinity of the space $H^2$ is formulated as follows

$$\left( \partial_n u - g(\phi)e^{-x} \right) |_{x \rightarrow \infty} = 0 , \quad (5.19)$$

where $g(\phi)$ is a function on $\mathcal{B} = S^1$ which satisfies the condition (2.5) $\int_{\mathcal{B}} g(\theta) d\mu(\phi) = 0$. 

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We search Green’s function for the Neumann problem on $H^2$ by analogy with the flat case described in Section 4.6 as a solution of the inhomogeneous Laplace equation
\[ \Box G_N = -\frac{1}{2\pi l^2} \] (5.20)
satisfying the boundary condition
\[ \partial_n G_N |_{x \to \infty} = c + O(e^{-2x}) \] (5.21)
where $c$ is some constant.

The solution of the equation (5.20) can be found explicitly
\[ G_N = -\frac{1}{2\pi} \ln \sinh \frac{\sigma}{l} . \] (5.22)
Expanding this expression for large $\sigma$ and taking into account (5.4) one gets
\[ G_N \simeq -\frac{1}{2\pi} x - \frac{1}{2\pi} \ln \Delta + \frac{1}{2\pi} e^{-2x} \Delta . \] (5.23)
We see that Green’s function (5.22)-(5.23) satisfies the condition (5.21) with $c = -\frac{1}{2\pi} l$.

The solution of the Neumann problem then is given by the expression
\[ u_N(x', \theta') = \int_B G_N \partial_n u = -\frac{1}{4\pi} \int_0^{2\pi} g(\phi) \ln (\cosh x' - \sinh x' \cos(\phi - \phi')) d\phi . \] (5.24)
This results in the boundary action
\[ W[u_N] = -\frac{1}{16\pi} \int_0^{2\pi} \int_0^{2\pi} g(\phi) \ln \sin^2 \left( \frac{\phi - \phi'}{2} \right) g(\phi') d\phi d\phi' \] (5.25)
and the correlation function
\[ K_N(\phi, \phi') = -\frac{1}{16\pi} \ln \sin^2 \left( \frac{\phi - \phi'}{2} \right) \] (5.26)
what is identical (up to a constant factor) to the Neumann correlation function (4.23) appearing on the boundary of two-dimensional flat disk $D^2$. It is due to the conformal equivalence of spaces $H^2$ and $D^2$. The difference in the factor can be removed by the rescaling the function $g(\phi)$ and/or re-defining the radius of the circle $S^1$.

The solving the Neumann problem on higher-dimensional AdS spaces goes in a similar manner. The Neumann condition at infinity of the space $H^{d+1}$ can be imposed as follows
\[ \left( \partial_n u - g(\theta) e^{-dx} \right)_{x \to \infty} = 0 , \] (5.27)
where $g(\theta)$ is a function on $B = S^d$, $\int_B g(\theta) d\mu(\theta) = 0$. Green’s function $G_N(x, x', \theta, \theta')$ for the Neumann problem should satisfy the condition
\[ \partial_n G_N |_{x \to \infty} = c + o(e^{-dx}) , \] (5.28)
where $c$ is a constant and $o(z)$ is defined as $z^{-1}o(z) \to 0$ if $z \to 0$. The asymptote (5.28) is necessary to have \( \int_B \partial_\theta G_N u = \text{const.} \). Then the solution of the Neumann problem is given by the expression (2.10). From the condition (5.28) one finds the form of the function $G_N$

\[
G_N(x, x', \theta, \theta') = c\sigma + o(e^{-d x}) ,
\]

where $\sigma \simeq x + l\ln \Delta(x', \theta')$ (see eq.(5.14)). This leads us to the expression for the solution of the Neumann problem on $H^{d+1}$

\[
u_N(x', \theta') = cl \int_B \ln \Delta(x', \theta', \theta') g(\theta) d\mu(\theta) + \text{constant} ,
\]

where $\Delta(x', \theta, \theta')$ takes the form (5.14). The above arguments leading to eq.(5.30) are not rigorous. We, however, may check by direct computation that that the function (5.30) is a solution of the Laplace equation $\Box u_N = 0$ on the whole space $H^{d+1}$. It is seen from the fact that on AdS space $H^{d+1}$ the function $\ln \Delta(x', \theta, \theta')$ is a solution of the inhomogeneous Laplace equation

\[
\Box \ln \Delta(x', \theta, \theta') = \frac{d}{l^2} .
\]

Therefore, acting the Laplace operator $\Box$ on the function (5.30) and using the condition (2.3) we find that $\Box u_N = 0$.

It is also easy to check that (5.30) satisfies the condition (5.27). For large $x'$ and for $\theta' \simeq \theta$ we have for the normal derivative of (5.30)

\[
l^{-1} \partial_{x'} u_N(x', \theta') \simeq 4c e^{-d x'} \int_B \frac{e^{-(2-d)x'} |\theta - \theta'|^2 + 2e^{-2x'} g(\theta) d\mu(\theta)} ,
\]

where $|\theta - \theta'|^2 = \sum_{i=1}^{d} |\theta_i - \theta'_i|^2$. It approaches for large $x'$ the condition (5.27) provided that we exploit (for $k = 1$) the limit

\[
\frac{e^{2k-d}}{(\Delta^2 + 2e^2)^k} \to \alpha_k \delta^d(\theta - \theta')
\]

when $\epsilon \to 0$, where the $\delta$-function is defined by the condition $\int_{\mathbb{S}^d} \delta^d(\theta - \theta') d\mu(\theta) = 1$ and $\alpha_k$ is some numerical coefficient (actually, the constant $c$ in (5.28) is determined by the condition $4c\alpha_1 = 1$).

Inserting the solution (5.30) into the boundary action $W[u_N] = \frac{1}{2} \int_B u_N \partial_\theta u_N$ we obtain that

\[
W[u_N] = 2^{-l\delta+1} \int_{\mathbb{S}^d} g(\theta) \ln \sin^2 \frac{\gamma}{2} g(\theta') d\mu(\theta) d\mu(\theta') .
\]

Thus, we find that the Neumann correlation function (defined as second derivative of $W[u_N]$ with respect to $g(\theta)$)

\[
< \mathcal{O}_N(\theta) \mathcal{O}_N(\theta') > = K_N(\theta, \theta') = 2^{l\delta+1} \ln \sin^2 \frac{\gamma}{2}
\]

has universal for all AdS spaces logarithmic behavior. The correlation function (5.33) appears to be natural if we analyze the boundary source term $\int_B \mathcal{O}_N \partial_\theta u d\mu$. It is finite if

\[
\Box = l^{-2} \frac{1}{\sin^2 x'} \partial_{x'} \sin^d x' \partial_{x'} + l^{-2} \frac{1}{\sin^2 x'} \sin^{-1} \gamma \partial_{\theta'} \sin^{d-1} \theta' \partial_{\theta'}.
\]

In the appropriate coordinate system the Laplace operator takes the form

\[
\Box = l^{-2} \frac{1}{\sin^4 x'} \partial_{x'} \sin^d x' \partial_{x'} + l^{-2} \frac{1}{\sin^4 x'} \sin^{-1} \gamma \partial_{\theta'} \sin^{d-1} \theta' \partial_{\theta'}.
\]
the operator $O_N$ approaches finite value for large $x$, i.e. $O_N$ should have dimension 0 and (5.33) is the correlation function of operators of such dimension.

5.3 Dirichlet and Neumann problems on $AdS_{d+1}$ for massive field

The solution of the Laplace equation with mass (4.10) behaves at infinity of AdS space as follows [3]

$$u(x, \theta) = u_0(\theta)e^{k_+_x} + u_1(\theta)e^{-(k_++d)x}, \quad (5.34)$$

where $k_+ = 2^{-1}(\sqrt{d^2 + 4m^2} - d)$ and $-(k_++d)$ are two roots of the equation $k(d+k) = m^2$. Only one of the functions $u_0(\theta)$ and $u_1(\theta)$ is independent. Fixing one of them on the boundary $S^d$ of $AdS_{d+1}$ we should be able to determine the other one. We have the Dirichlet boundary problem when the function $u_0(\theta)$ is fixed at infinity. In the other case, fixing the function $u_1(\theta)$ we have the Neumann problem.

It is easy to see that the boundary action $W_B = \frac{1}{2} \int_B u \partial_n u$ considered on a function of the form (5.34) diverges

$$W_B \simeq \frac{l^{d-1}}{2^{d+1}} \int_{S^d} u_0^2 k_+ e^{(2k_++d)x} d\mu(\theta)$$

for large $x$ and needs to be regularized. A natural way of doing this is just to subtract the contribution of the leading term in (5.34)

$$W_{reg} = W_B[u] - W_B[u_0e^{k_+x}] \quad (5.35)$$

The regularized action then

$$W_{reg} = -\frac{dl^{d-1}}{2^{d+1}} \int_{S^d} u_0(\theta) u_1(\theta) d\mu(\theta) \quad (5.36)$$

is finite. Note, that the way we define $W_{reg}$ is similar to the subtraction procedure one usually applies to make finite the gravitational action [23]. One subtracts the contribution of the asymptotically dominant part (typically it is that of flat space-time) of the metric considering it as a background. This is exactly what we are doing in (5.33).

In order to calculate the regularized boundary action (5.36) we have to find (for each type of boundary condition) the functional relation between $u_0(\theta)$ and $u_1(\theta)$. We start with the considering the Dirichlet problem when $u_0(\theta)$ is fixed at infinity. The corresponding Green’s function $G_D$ is a solution of the equation $(\Box - m^2)G_D = 0$ and for large $x$ behaves as follows

$$G_D \simeq ce^{-(k_++d)x} = e^{-\frac{(k_++d)}{\Delta k_+ + d}}. \quad (5.37)$$

It is important to note that the solution of the Dirichlet problem in this case is not given by the equation (2.10) since the term $(G_D \partial_n u)$ does not vanish at infinity. We have to use Green’s formula (2.4). The solution then reads

$$u_D(x', \theta') = \frac{cl^{d-1}}{2^d(2k_+ + d)} \int_{S^d} u_0(\theta)d\mu(\theta). \quad (5.38)$$

The kernel $\frac{1}{\Delta k_++d}$ of the integral (5.38) approaches (see equation (5.31)) $\alpha k_+ e^{k_+x}\delta^d(\theta - \theta')$ when $\gamma \rightarrow 0$ ($\theta \rightarrow \theta'$) and is $e^{-(k_+ +d)x}(\sin^2 \frac{\gamma}{2})^{-(k_+ +d)}$ for $\gamma \neq 0$. The singular part of the kernel after the integration gives the term $u_0(\theta)e^{k_+x}$ in (5.38) what is in agreement with
the imposed Dirichlet condition. On the other hand, the finite part of the kernel gives the functional relation

$$u_1(\theta') = \frac{c l^{d-1}}{2^d} (2k_+ + d) \int_{S^d} \frac{u_0(\theta)}{(\sin^2 \frac{\theta}{2})^{k_+ + d}} d\mu(\theta) .$$  \hspace{1cm} (5.39)$$

It should be noted that the separation made in the kernel $\frac{1}{\Delta^{k_+ + \sigma}}$ on singular (when $\gamma \to 0$) and finite parts is not well-defined since the “finite” part proportional to $\frac{1}{(\sin^2 \frac{\theta}{2})^{k_+ + \sigma}}$ diverges when $\gamma \to 0$. In fact, the finite part is defined up to the term $a \delta^d(\theta - \theta')$ with some (divergent) coefficient $a$. It can be chosen to regularize the kernel $\frac{1}{(\sin^2 \frac{\theta}{2})^{k_+ + \sigma}}$. Below (and earlier in (5.17), (5.18)) namely this definition of the kernel should be meant.

Mathematically strict consideration of the divergent kernels and their regularization can be found in [27], in the present context it was done in [7], [8].

Below the Neumann problem we fix the function $u_1(\theta)$ at the infinity of the space $H^{d+1}$. Green’s formula (2.6) says us that

$$u_N = -\int_B \left( (\partial_n G_N - \frac{k_+}{l} G_N) u_0 e^{(d+k_+)} + (\partial_n G_N + \frac{k_+ + d}{l} G_N) u_1 e^{-k_+} \right) .$$

It gives us the solution of the Neumann problem if the solution $u_N$ in the bulk is determined only by the function $u_1$ fixed on the boundary. It is so if Green’s function $G_N$ satisfies the conditions

$$\partial_n G_N - \frac{k_+}{l} G_N = o(e^{-(d+k_+)x})$$

$$\partial_n G_N + \frac{k_+ + d}{l} G_N = constant \ e^{k_+ x} + o(e^{k_+ x}) .$$

The function which satisfies these conditions is

$$G_N = \frac{c_1 l}{k_+} e^{k_+ x} + o(e^{-(d+k_+)x}) ,$$  \hspace{1cm} (5.42)
where $c_1$ is a constant and we include the factor $\frac{1}{k_+}$ in order to have the correspondence with the massless case: $G_N \simeq c_1 \sigma + \text{const}$ when $k_+ \to 0$.

Using (5.42) we find

$$u_N(x', \theta') = -c_1 \frac{d^{d-1}}{2^d} (2k_+ + d) \frac{1}{k_+} \int_{S^d} \Delta^+ (x', \theta) u_1(\theta) d\mu(\theta)$$  \hspace{1cm} (5.43)

for the solution of the Neumann problem.

That the function (5.43) is a solution of the equation $(\Box - m^2) u = 0$ on $H^{d+1}$ follows from the fact that the function $\Delta^k(x', \theta, \theta')$ for arbitrary $k$ satisfies the equation

$$\Box \Delta^k = k(k+d) \Delta^k .$$  \hspace{1cm} (5.44)

Thus, for $k = k_+$ we find from (5.44) that the kernel $\Delta^k$ arising in (5.43) is a solution of the Laplace equation with mass $m$. Another solution of the massive field equation corresponds to the value $k = -(k_+ + d)$ and is the kernel $\Delta^{-k_+}$ arising in the Dirichlet problem.

If $\theta \to \theta'$ we have $\Delta^k(x', \theta, \theta') \to e^{-(k_+ + d)x} \alpha_k \delta^d(\theta - \theta')$ what is obtained from (5.31) by analytical continuation to negative $k$. The integral (5.43) then reproduces the term $u_1(\theta) e^{-(k_+ + d)x}$ as it should be by the imposed boundary condition. The regular part of the kernel $\Delta^k$ in (5.43) gives us the functional relation

$$u_0(\theta') = -c_1 \frac{d^{d-1}}{2^d} (2k_+ + d) \frac{1}{k_+} \int_{S^d} (\sin^2 \frac{\gamma}{2}) \Delta^k u_1(\theta) d\mu(\theta)$$  \hspace{1cm} (5.45)

between the functions $u_0$ and $u_1$. Substituting (5.45) into the regularized action (5.37) and taking the variational derivatives with respect to $u_1(\theta)$ we find

$$K_N(\theta, \theta') = -c_1 \frac{d^{2d-2}}{2^{2d+1}} (2k_+ + d) \frac{1}{k_+} (\sin^2 \frac{\gamma}{2})^{k_+}$$  \hspace{1cm} (5.46)

for the Neumann correlation function. We see that due to the mass in the bulk the Neumann operator $O_N$ on the boundary $S^d$ of the space $AdS_{d+1}$ acquires negative anomalous dimension $-k_+$. As a consequence of this the Neumann correlation function (5.46) vanishes when the points $\theta$ and $\theta'$ coincide. However, if $2k_+$ is not integer then $n$-order derivative of (5.46) diverges at $\theta' = \theta$ for $n > 2k_+$. This means that $K_N(\theta, \theta')$ (5.46) is not analytical at $\theta' = \theta$. We see that mass $m$ plays the role of regulator for the Neumann correlator (5.33) and makes it finite at the coinciding points. In the bulk mass typically improves the infra-red (IR) behavior of the theory while the short distance divergences of correlators are of UV nature. So, what we find for the Neumann correlator seems to be another manifestation of the relation between IR regime in the bulk and UV regime on the boundary noted in [28].

6 Hierarchy of correlators

Summarizing the results of Sections 4 and 5 we have found that the correlators (both the Neumann and Dirichlet ones) arising on sphere $S^d$ (considered either as a boundary of a
domain of flat space $R^{d+1}$ or Anti-de Sitter space $H^{d+1}$ are functions of the geodesical distance $\gamma$ measured on $S^d$ and are constructed by means of the following basic elements

$$K_0 = -\ln \sin^2 \frac{\gamma}{2}, \quad K_n = \frac{1}{\sin^n \frac{\gamma}{2}}, \quad n > 0.$$  \hspace{1cm} (6.1)

The functions (6.1) can be defined on sphere of arbitrary dimension $d$. The geodesical distance $\gamma$ then is the azimuthal angle $\theta$ between the points $P$ and $P'$ on $S^d$. However, only on sphere of certain dimension an element (6.1) can be considered as a correlator of free fields. In order to make this statement more precise we consider the set of differential operators $\Delta_n = \frac{1}{\sin^n \theta} \partial_\theta (\sin^n \theta \partial_\theta)$. $\Delta_n$ is the “azimuthal” part of the Laplace operator on $(n+1)$-dimensional sphere $S^{n+1}$. We have the following set of recurrent relations for the elements $K_n(\theta)$ (6.1)

$$\Delta_{n+2}K_0 = -\frac{n}{2}K_2 + (n + 1)$$

$$\left(\Delta_{n+2} - \frac{1}{4}m(2n - m + 2)\right)K_m = \frac{1}{4}m(m - n)K_{m+2}, \quad m \neq 0$$ \hspace{1cm} (6.2)

Note that the combination $\Delta_{n+1} - \Delta_n = \cot \theta \partial_\theta$ is the first order differential operator. The elements (6.1) satisfy also the first order differential recurrent relations

$$\cot \theta \partial_\theta K_0 = -\frac{1}{4}K_2 + \frac{1}{2}$$

$$\left(\cot \theta \partial_\theta - \frac{n}{2}\right)K_n = -\frac{n}{4}K_{n+2},$$ \hspace{1cm} (6.3)

which are analogous to the relations (6.9) for the correlators arising on plane boundary.

We will say that $K$ is primary correlator on $S^d$ if $K \sim < \phi(P)\phi(P') >$ where $\phi(P)$ is a field on $S^d$ satisfying free (second order) field equation. In other words, the primary correlator is the one on which the recurrent sequence (6.2) stops for certain $n$ and $m$ and does not produce a new correlator. Non-primary correlators are correlators of composite operators built from the free fields. We say that correlator $K$ is a descendant if it is obtained by differentiating a primary correlator. One can see from the relations (6.2) that a correlator $K_n$ is primary on $(n + 2)$-dimensional sphere $S^{n+2}$. Indeed, we find from (6.2) that

$$\Delta_2K_0 = 1$$

$$\left(\Delta_{n+2} - \frac{n(n + 2)}{4}\right)K_n = 0.$$ \hspace{1cm} (6.4)

It is easy to recognize the conformal nature of the operators in (6.4). Therefore, $K_n$ is correlator of conformal field $\phi(x)$ on sphere $S^{n+2}$ satisfying the conformal field equation

$$\Delta_2\phi = c \mathcal{R}, \quad n = 0$$

$$\left(\Delta_{n+2} - \frac{n}{4(n + 1)}\mathcal{R}\right)\phi = 0, \quad n > 0$$ \hspace{1cm} (6.5)

where $c$ is two-dimensional central charge and the scalar curvature $\mathcal{R}$ of $(n+2)$-dimensional unit sphere $S^{n+2}$ is $(n+2)(n+1)$. There are two different points of view on the correlation functions under consideration. For example, the Neumann correlator (4.29) of the
boundary operators $O_N$ can be viewed as entirely arisen on sphere $S^d$ with only component $K_{d-2}$ being a primary correlator. In the dual picture the operator $O_N$ is considered to be a composite of the free field operators $O[S^m]$ having support on sphere $S^m$. The correlator (4.29) arises then for points $P$ and $P'$ lying on the intersection of the set of spheres $S^{d+1}, S^d, ..., S^{d-m}, ..., S^2$ with a component $K_{d-m-2}$ being the primary correlator on sphere $S^{d-m}$.

As another illustration of this let us consider the correlator $K_0$. It arises as the Neumann correlator on the circle $S^1$ being boundary of two-dimensional disk $D^2$. On the other hand, $S^1$ can be considered as a meridian on 2d sphere $S^2$ and $K_0$ is the primary correlator of conformal scalar field on $S^2$ described by the action

$$W[S^2] = \int_{S^2} \left( \frac{1}{2} (\nabla \phi)^2 - c R \phi \right).$$

The correlator $K_2$ arises on $S^1$ in the Dirichlet problem on $D^2$. On $S^2$ it is interpreted as a descendant of $K_0$ since we have from (1.2) that $-2\partial_\theta^2 K_0(\theta) = K_2(\theta)$. On the other hand, $K_2(\theta)$ is the primary conformal correlator on four-dimensional sphere $S^4$.

As we saw in the previous Section, the correlator $K_0$ also arises on sphere $S^d$ in the Neumann problem on Anti-de Sitter space $AdS_{d+1}$ and the present analysis shows that it has natural two-dimensional description. The Dirichlet problem on $AdS_{d+1}$ produces the correlator $K_{2d}$ on the boundary $S^d$. It is primary on $(2d + 2)$-dimensional space $S^{2d+2}$.

In the context of the works [1], [2], [3] the five-dimensional AdS space is of interest. The Dirichlet correlator arising on 4-dimensional boundary $S^4$ is $K_8$. On $S^4$ it is a descendant of the primary correlator $K_2$. At the same time it has a dual 10-dimensional description on $S^{10}$ as a primary conformal field correlator.

### 7 Phases on the boundary of AdS space

In this Section we want to point out on the existence of various phases of the field theory living on the boundary of Anti-de Sitter space. These phases may arise in essentially two different ways.

#### 7.1 Neumann and Dirichlet phases

In the first way phases arise on the boundary of AdS space when we impose the mixed boundary condition similar to what we considered in Section 2 for flat space. This condition should be consistent with the asymptotic behavior (5.34) of a solution of the massive Laplace equation in AdS space. In order to formulate it we introduce the field $\tilde{u} = u e^{-k_+ x}$ with the asymptotic behavior

$$\tilde{u} \simeq u_0(\theta) + u_1(\theta) e^{-(2k_+ + d)}$$

at infinity. In terms of the field $\tilde{u}$ the third type boundary condition can be imposed as follows

$$\partial_\theta \tilde{u} + h e^{-(2k_+ + d)x} \tilde{u} = g(\theta) e^{-(2k_+ + d)x}$$

(7.1)

or, equivalently,

$$h u_0(\theta) - (2k_+ + d) l^{-1} u_1(\theta) = g(\theta).$$
As it is seen from (7.1) the coupling $h$ being considered as a function on the boundary has dimension $(2k_+ + d)$. The dimensional analysis occurs to be useful in finding the asymptotic behavior of the corresponding correlation function $K(h)$ arising on the boundary. We have

$$K(h) \simeq \frac{1}{k_+} \sigma^{k_+} + h \sigma^{4k_+ + d} + O(h^2)$$

(7.2)

for small $h$ and

$$K(h) \simeq \frac{1}{h^2} \sigma^{2d+2k_+} + O\left(\frac{1}{h^3}\right),$$

(7.3)

for large $h$, where $\sigma = a \sin \frac{\gamma}{2}$ and we omitted the numerical coefficients. The equations (7.2)-(7.3) mean that $K(h) \simeq K_N$ for small $h$ and $K(h) \simeq \frac{1}{h^2} K_D$ for large $h$ in agreement with the consideration of Section 2. In the massless case ($k_+ \rightarrow 0$) the term $\frac{1}{k_+} \sigma^{k_+}$ in (7.2) should be replaced by $\ln \sigma$. The picture of phases arising on the boundary of AdS space is similar to that we had for flat space in Section 3. The only essential difference is that the coupling $h$ in AdS space has higher scaling dimension $(2k_+ + d)$ compared to the flat space (where $h$ has dimension 1). For a finite value $h$ there are two phases of the boundary system. In the regime $(h \sigma^{d+2k_+}) << 1$ the free (Neumann) phase is restored and the correlation function behaves as (7.2). On the other hand, for $(h \sigma^{d+2k_+}) >> 1$ we have the Dirichlet (strong coupling) phase. Thus, the coupling $h$ brings some natural scale on the boundary so that the boundary theory at a finite $h$ is not conformal, though, it interpolates between the conformal phases.

It should be noted that the Neumann phase arising on the boundary of AdS space is quite different from what we had in flat space. In the massive case the Neumann correlator (7.2) does not diverge at the coinciding points. Though, for arbitrary mass $(k_+)$ it is non-analytical at the point $\sigma = 0$ in the sense that higher order derivatives of $K_N(\sigma)$ become divergent there. When mass is zero ($k_+ = 0$) the Neumann correlator $K_N \sim \ln \sigma$ is logarithmic. Remarkably, it is so in all AdS spaces independently of the dimension.

We stress that the correlators arising in the Neumann phase well behave at the coinciding points. In the massless case the divergence is logarithmic there and, therefore, is integrable. So, integrals involving $K_N$ are finite. In the massive case the situation even better since the Neumann correlator is out of divergence at all. In the Dirichlet phase the correlators diverge as $\sigma^{-2d}$ in massless case and as $\sigma^{-2(d+k_+)}$ in massive case. These divergences are not integrable. Therefore, we need to use some regularization in order to give sense to the integrals involving $K_D$. Tracing these divergences to the UV behavior of the boundary theory one could say that the theory is finite in the Neumann phase and renormalizable in the Dirichlet phase. Note, that the presence of mass in the bulk improves the behavior of the Neumann correlator and makes worse the behavior of the Dirichlet one.

The Neumann correlator arising at the infinity of AdS space takes the same form as the correlator arising in the Neumann problem on the flat disk $D^2$ with boundary $S^1$. The circle $S^1$ can be considered as lying on the sphere $S^d$ (boundary of AdS space) and joining the points $P$ and $P'$. The geodesical distance $\gamma$ between the points $P$ and $P'$ then is the angle $|\phi - \phi'|$ measured on the circle $S^1$. Two bulk theories, one is living on $AdS_{d+1}$ and another on $D^2$, produce the same correlator on the circle. On the other hand, the corresponding boundary theories (living on $S^1$ and $S^d$ respectively) both have dual description as a conformal theory (6.6) on two-dimensional sphere $S^2$. This fact, in
particular, proposes that the Neumann phase on the boundary of AdS space may have infinite-dimensional underlying conformal group of symmetry. It is in contrast with the Dirichlet phase whose conformal symmetry is finite-dimensional.

In the construction considered in Refs. 1, 2, 3 the Anti-de Sitter space is AdS$_5$ and the four-dimensional theory on the boundary is identified with the large $N$ limit of a super Yang-Mill theory with gauge group $SU(N)$. The scalar Dirichlet boundary operator then should have dimension 4 and can be constructed from the Yang-Mills field $A_{\mu}$: $O_D = Tr(F_{\mu\nu}F^{\mu\nu})$. In the Neumann phase the boundary operator should have dimension 0. It seems that no such operator can be constructed from local combinations of $A_{\mu}$ and its derivatives. A possible candidate for the boundary operator in the Neumann phase is the non-local one

$$ O_N[P] \sim Tr P \exp \int_{C_P} A_{\mu} dx^\mu , $$

where $C_P$ is a circle shrinking to the point $P$. It is also an interesting question as what meaning in terms of the Yang-Mills theory may have the boundary coupling $h$.

### 7.2 Infinity-horizon transition in AdS like space

In general, the location of the boundary in the AdS space can be arbitrary. One may put it at $x = x_B$ and consider the domain $0 \leq x \leq x_B$ of the AdS space. When the boundary moves across AdS space this induces some automorphism in the theory living on the boundary. In some cases this automorphism is trivial and the boundary field theory does not depend on the choice of the boundary. This is the case in three dimensions when the theory in the bulk is the Chern-Simons theory. It induces a conformal (Wess-Zumino) theory on the boundary which remains the same no matter where one chooses the boundary. Note, that this is a feature of this concrete model which is mainly due to the absence in the bulk of the propagating degrees of freedom in the Chern-Simons theory. In general, the moving the boundary may lead to drastic deformation of the field theory on the boundary. In the case under consideration, the theory in the bulk is described by the Laplace equation. Varying $x_B$, the location of the boundary, we find that there are two limiting cases: when the boundary $B$ approaches infinity ($x_B \to \infty$, $B = B_\infty$) and when it shrinks ($x_B \to 0$, $B = B_0$) to the center of the AdS space, in both cases the boundary is topologically sphere. In the later case the domain $0 \leq x \leq x_B$ is well approximated by domain $0 \leq r \leq R$ of flat space $R^{d+1}$ with sphere $S^d$ of radius $R = lx_B$ as a boundary. The classical field theory on such domain and the correlation functions arising on its boundary $S^d$ were considered in Section 4. The theory on the boundary $B$ of the AdS space is completely characterized by the scaling dimensions of the operators $O_D$ (in the Dirichlet problem) and $O_N$ (in the Neumann problem). When the boundary is at infinity the dimension of $O_N$ is $-k_+$ (in the massless case it is 0) while the dimension of $O_D$ is $2(d + k_+)$. On the opposite end, when the boundary stays arbitrary close to the center ($x = 0$) of the AdS space we have dimensions $(d - 1)$ and $(d + 1)$ respectively for the operators $O_N$ and $O_D$. The limit $x_B \to 0$ is correctly accomplished if we deal with the rescaled operators $\tilde{O}_N$ and $\tilde{O}_D$ defined as $\tilde{O}_N = (\frac{2^{d+1}}{a})^{(d-1)}O_N$ and $\tilde{O}_D = (\frac{2^{d+1}}{a})^{(d+1)}O_D$ where $a$ is an arbitrary (finite) scale parameter playing the role of the radius of the sphere $S^d$. One then finds that the scaling dimensions of the operators $\tilde{O}_N$ and $\tilde{O}_D$ are independent of the mass $m$ and the correlation functions are effectively massless. The Neumann correlation function is presumably given by the series $1/24$ with respect to the variable $\sigma = 2a \sin^{2} \frac{\sigma}{2}$. It includes the logarithmic term $\ln \sigma$. It happens that only this
term survives (in the massless case) when the boundary moves to the infinity where the Neumann correlator \((5.33)\) has universal logarithmic behavior.

The theory arising on \(B_{\infty}\) is not much sensitive to deformations made deep inside the AdS instanton. It is not so for the theory living on the boundary \(B_0\) because the topology of \(B_0\) may change drastically under some deformations. An interesting and important way to deform the AdS space is to put a black hole inside. The complete manifold remains asymptotically AdS space. The topology of the manifold near the center, however, is modified by the presence of the black hole horizon. It is that of the product space \(D^2 \times \Sigma\), where \(D^2\) is two-dimensional disk and \(\Sigma = S^{d-1}\) is the horizon surface. So that the boundary \(B^h_0\) is no more a sphere \(S^d\) but the product space \(B^h_0 = S^1 \times \Sigma\). According to \((4.19)\) and \((4.30)\) the correlation functions arising on \(B^h_0\) factorize \(K_{B^h_0} = \delta_{\Sigma} K_{S^1}\), where \(\delta_{\Sigma}\) is delta-function on \(\Sigma\) and \(K_{S^1}\) is the circle correlation function arising on \(S^1\) considered as a boundary of two-dimensional disk \(D^2\). The averaged correlation function obtained by integration over \(\Sigma\), \(\int_\Sigma \int K_{B^h_0} = A_\Sigma K_{S^1}\), is proportional to the horizon area \(A_\Sigma\). On \(B^h_0\) the boundary operators \(\tilde{O}_N\) and \(\tilde{O}_D\) have the effective scaling dimensions 0 and 1 respectively. The Neumann correlator then

\[
K_{S^1} = K_N(\phi, \phi') = -\frac{1}{4\pi} \ln \sin^2 \frac{\phi - \phi'}{2}
\]

(7.4)

is purely logarithmic and is identical (after identifying the circle \(S^1\) with geodesic joining the points \(P\) and \(P'\) on \(B_{\infty} = S^d\)) to the Neumann correlator on \(B_{\infty}\). So, in the case of the black hole the Neumann phase remains unperturbed in the transition from the infinity to the horizon. As we discussed previously, the correlation function \(K_N(\phi, \phi')\) has natural two-dimensional description as a correlator of free conformal fields on sphere \(S^2\) (the circle \(S^1\) is considered as a meridian on \(S^2\)). Therefore, the Neumann phase arising on the boundary \(B^h_0\) has dual conformal field theory description on the space \(S^2\). On the other hand, the correlation function

\[
K_{S^1} = K_D(\phi, \phi') = -\frac{1}{8\pi a^2 \sin^2 (\frac{\phi - \phi'}{2})}
\]

(7.5)

arising on \(B^h_0\) in the Dirichlet phase has a description as a primary correlator of conformal field theory on 4-dimensional sphere \(S^4\). So, in the Dirichlet case the boundary field theory on \(B^h_0\) is dual to the conformal field theory on \(S^4\). It is interesting to note that in four dimensions the correlator of the form \((7.5)\) appears also as a bulk correlation function of free quantum fields \(< \varphi(P)\varphi(P') >\) when both \(P\) and \(P'\) lie on the horizon \(\Sigma = S^2\), \(\theta\) in this case is the azimuthal angle on \(\Sigma\). The bulk correlation function in this case is Green’s function calculated in the Hartle-Hawking state. When at least one of the points lies on the horizon it can be found exactly as was demonstrated by Frolov \([26]\) long ago for a charged rotating black hole. It would be nice to understand better the coincidence of two correlation functions.

We see, thus, that the Dirichlet phase on the boundary deforms essentially in the transition from infinity to the horizon in AdS like space. This manifests in the scaling dimension of the Dirichlet boundary operator (that changes from value \(d\) at infinity to 1 at the horizon) and in the behavior of the corresponding correlation function. The Neumann phase is, however, more stable under this transition. The scaling dimension 0 of the Neumann boundary operator and the logarithmic behavior of the correlator (it is
so up to the factor $\delta_\Sigma$ at the horizon) remain the same. The boundary near black hole horizon, $\mathcal{B}_h^0 = S^1 \times \Sigma$, has structure typical for the Euclidean description of a thermal field theory (the circumference of $S^1$ being the inverse temperature). Therefore, in terms of the boundary theory this transition seems to correspond to the high temperature regime. However, a more detail investigation is required.

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