ERROR ESTIMATES FOR APPROXIMATIONS OF NONHOMOGENEOUS NONLINEAR UNIFORMLY ELLIPTIC EQUATIONS

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Abstract. We obtain an error estimate between viscosity solutions and \(\delta\)-viscosity solutions of nonhomogeneous fully nonlinear uniformly elliptic equations. The main assumption, besides uniform ellipticity, is that the nonlinearity is Lipschitz-continuous in space with linear growth in the Hessian. We also establish a rate of convergence for monotone and consistent finite difference approximation schemes for such equations.

1. Introduction

We prove an estimate between viscosity solutions and \(\delta\)-viscosity solutions of the boundary value problem

\[
\begin{aligned}
F(D^2 u, x) &= f(x) & \text{in } U \subset \mathbb{R}^n, \\
u &= g & \text{on } \partial U,
\end{aligned}
\]

where \(F\) is uniformly elliptic (see (F1) below) and Lipschitz-continuous in space with linear growth in the Hessian (see (F2) below). As a consequence, we find a rate of convergence for monotone and consistent finite difference approximations to (1). Both results generalize the work of Caffarelli and Souganidis in [6] and [7], who consider either inhomogeneous equations or equations with separated dependence on the space variable and on the Hessian.

The nonlinearity \(F\) is a continuous function on \(S_n \times U\), where \(S_n\) is the set of \(n \times n\) real symmetric matrices endowed with the usual order and norm (for \(X \in S_n\), \(||X||| = \sup_{|v|=1} |Xv|\)). We make the following assumptions:

(F1) \(F\) is uniformly elliptic, which means there exist constants \(0 < \lambda \leq \Lambda\) such that for all \(x \in U\), any \(X \in S_n\), and for all \(Y \geq 0\),

\[
\lambda ||Y|| \leq F(X + Y, x) - F(X, x) \leq \Lambda ||Y||;
\]

and,

(F2) there exists a positive constant \(\kappa\) such that for all \(x, y \in U\) and all \(X \in S_n\),

\[
|F(X, x) - F(X, y)| \leq \kappa |x - y|(||X|| + 1).
\]

An example of an equation satisfying our assumptions is the Isaacs equation

\[
F(D^2 u, x) = \sup_{\alpha, \beta} \inf_{\alpha, \beta} L^{\alpha, \beta} u(x),
\]

where, for each \(\alpha\) and \(\beta\) in some index sets, the operator \(L^{\alpha, \beta}\) is given by

\[
L^{\alpha, \beta} u(x) = \sum_{i,j=1}^{n} a^{\alpha, \beta}_{ij}(x) \partial^2_{ij} u(x) + f^{\alpha, \beta}(x),
\]

Date: May 11, 2014.
and is uniformly elliptic with uniformly Lipschitz coefficients, which means there exists a $\kappa$ such that for all $x, y \in U$ and for all $\alpha, \beta, i$ and $j$,

$$|a_{ij}^{\alpha,\beta}(x) - a_{ij}^{\alpha,\beta}(y)| \leq \kappa|x - y|$$

and

$$|f^{\alpha,\beta}(x) - f^{\alpha,\beta}(y)| \leq \kappa|x - y|.$$  

The Isaacs equation arises in the study of stochastic differential games. We do not give further details about the Isaacs equation and refer the reader to Section 1 of Crandall, Ishii and Lions’ [8] for a list of references.

We also assume:

(U1) $U$ is a bounded subset of $\mathbb{R}^n$ with regular boundary,

(G1) $f \in C^{0,1}(U)$, and

(G2) $g \in C^{1,\gamma}(\partial U)$ for some $\gamma \in (0, 1]$.

The main result is a comparison between solutions and $\delta$-viscosity solutions (briefly, $\delta$-solutions) of (1). The definition of $\delta$-solutions is given in Section 2. Next we present a statement of our main result that has been simplified for the introduction; the full statement is in Section 6.

Theorem 1.1. Assume (U1), (F1), (F2), (G1), (G2). Let $u$ be a viscosity solution of (1) and assume that $\{v_{\delta}\}_{\delta \geq 0}$ is a family of $\delta$-solutions of (1) that satisfy, for all $\delta > 0$,

$$||v_{\delta}||_{C^{0,\eta}(U)} \leq M$$

and

$$v_{\delta} = u \text{ on } \partial U.$$  

There exist positive constants $\bar{\delta}$, $\bar{\alpha}$ and $\bar{c}$ such that, for any $\delta \leq \bar{\delta}$,

$$\sup_U |u - v_{\delta}| \leq \bar{c}\delta^\bar{\alpha}.$$  

The notion of $\delta$-solutions was introduced in [6] to establish error estimates for finite difference schemes for fully nonlinear uniformly elliptic equations of the form $F(D^2 u) = f(x)$. An error estimate between solutions and $\delta$-solutions of such equations was established in [7]. A key step in the proof of the error estimates was a regularity result ([6, Theorem A]), which says that, outside of sets of small measure, solutions of $F(D^2 u) = f(x)$ have second-order expansions with controlled error. The proof of this regularity theorem relies on the equation being homogeneous – differentiating $F(D^2 u) = f(x)$ implies that the derivatives $u_{x_i}$ solve the linear uniformly elliptic equation $\text{tr}(DF \cdot D^2 u_{x_i}) = f_{x_i}$; therefore, a known estimate that gives first-order expansions on large sets (see Chapter 7 of Caffarelli and Cabre [5]) applies to $u_{x_i}$, and from this the estimates on $u$ are deduced.

The main challenge in the $x$-dependent case is that this extra regularity result is not known. Because differentiating the $x$-dependent equation $F(D^2 u, x) = f(x)$ does not imply anything useful about the derivatives of $u$, we cannot hope to replicate the proof of [6, Theorem A] for non-homogeneous equations. Instead, we find a way to “localize” the equation and obtain an approximation of $u$ that is regular enough to compare to the $\delta$-solution $v_{\delta}$. We include a detailed outline of the proof of Theorem 1.1 in Section 6.

Remark 1.2. The assumption (F3) on the nonlinearity $F$ may be weakened. In fact, there exists a universal constant $\alpha$ such that if $\beta > 1 - \alpha$, our results hold for any $F$ that satisfies

$$|F(X, x) - F(X, y)| \leq \kappa (1 + ||X||)|x - y|^\beta.$$
This requirement on $\beta$ comes from the proof of Proposition 3.1. For simplicity, we will only work with the case $\beta = 1$ (in other words, we assume that $F$ satisfies \(F_2\)).

**Remark 1.3.** We often assume
\[
F(0, x) = 0 \text{ for all } x \in U.
\]
This is not a restrictive assumption: the equation $F(D^2 u, x) = f(x)$ is equivalent to $F(D^2 u, x) - F(0, x) = f(x) - F(0, x)$, and the nonlinearity $F(M, x) = F(M, x) - F(0, x)$ satisfies \(F_2\).

We also study finite difference approximations to \(1\). We write the finite difference approximations as
\[
\begin{cases}
F_h[v_h](x) = 0 & \text{in } U_h, \\
v_h = g & \text{on } \partial U_h,
\end{cases}
\]
where $U_h = U \cap h\mathbb{Z}^n$ is the mesh of discretization and $F_h$ is the finite difference operator. We assume:
\(\langle F_h 1\rangle\) if $v_h^1$ and $v_h^2$ satisfy $F_h[v_h^1](x) = 0$ in $U_h$ and $v_h^1 \leq v_h^2$ on $\partial U_h$, then $v_h^1 \leq v_h^2$ on $U_h$; and
\(\langle F_h 2\rangle\) there exists a positive constant $K$ such that for all $\phi \in C^2(U)$,
\[|F_h[\phi](x) - (F(D^2 \phi, x) - f(x))| \leq K(1 + ||D^2 \phi||_{L^\infty(U)})h \text{ in } U.
\]
Schemes that satisfy \(\langle F_h 1\rangle\) and \(\langle F_h 2\rangle\) are said to be, respectively, **monotone** and **consistent with an error estimate for** $F$.

We have simplified our notation here in order to state our main result; all the details about approximation schemes and the precise statement of Theorem 1.4 are given in Sections 7 and 8.

**Theorem 1.4.** Assume \(\langle U \rangle\), \(\langle F_1 \rangle\), \(\langle F_2 \rangle\), \(\langle G_1 \rangle\), and \(\langle G_2 \rangle\). Assume that $F_h$ is a monotone scheme that is consistent with an error estimate for $F$. Assume that $u$ is the viscosity solution of \(1\) and that $v_h$ satisfies \(3\). There exist positive constants $\overline{c}$, $\bar{\alpha}$ and $\bar{h}$ such that for all $h \leq \bar{h}$,
\[
\sup_{U_h} |u - v_h| \leq \overline{c} h^{\bar{\alpha}}.
\]

The convergence of monotone and consistent approximations of fully nonlinear second order PDE was first established by Barles and Souganidis [3]. Kuo and Trudinger [15, 16] later studied the existence of monotone and consistent approximations for nonlinear equations and the regularity of the approximate solutions $u_h$. They showed that if $F$ is uniformly elliptic, then there exists a monotone finite difference scheme $F_h$ that is consistent with $F$, and that the approximate solutions $v_h$ are in $C^{0,\eta}$. However, obtaining an error estimate remained an open problem.

The first error estimates for approximation schemes were established by Krylov in [12] and [13] for equations that are either convex or concave, but possibly degenerate. Krylov used stochastic control methods that apply in the convex or concave case, but not in the general setting. Barles and Jakobsen in [4, 5] improved Krylov’s error estimates for convex or concave equations. In [14] Krylov improved the error estimate to be of order $h^{1/2}$, but still in the convex/concave case. In addition, Jakobsen [10, 11] and Bonnans, Maroso, and Zidani [4] established error estimates for special equations or for special dimensions. The first error estimate for general nonlinear equations that are neither convex nor concave was obtained by Caffarelli and Souganidis in [6]. Their result holds for equations $F$ that do not depend on $x$.

To our knowledge, Theorem 1.4 is the first error estimate for general nonlinear uniformly elliptic equations that are neither convex nor concave and are not homogeneous. In particular, this is the first error estimate for approximations of the Isaacs equation. To prove Theorem 1.4
we show that an appropriate regularization of the solution of (3) is a \( \delta \)-solution of (1), where \( \delta \) depends on \( h \) (see Proposition 7.2). This allows us to essentially deduce Theorem 1.1 from our estimate in Theorem 1.1.

Our paper is structured as follows. In Section 2 we give the definition of \( \delta \)-solutions and state several known results about the regularity of viscosity solutions of (1). Section 3 is devoted to establishing an estimate between the solution \( u \) of (1) and solutions of the equation with “frozen coefficients” on small balls; mainly, solutions of \( F(D^2 \bar{u}, x_0) = f(x_0) \) in \( B_r(x_0) \) for small \( r \) (Proposition 3.1). In Section 4 we study perturbations of the equation (1) and prove an estimate between \( u \) and solutions of the perturbed equations (Proposition 4.1). In Section 5 we prove an elementary lemma that plays an important role in the proofs of Theorems 1.1 and 1.3. The full statement and the proof of Theorem 1.1 in Section 6. The first part of Section 6 is an outline the proof of Theorem 1.1. Then we formulate and prove an important technical lemma. Finally, we put everything together and give the proof of Theorem 1.1. Section 7 is devoted to introducing the necessary notation and stating known results about approximation schemes. In Proposition 7.2 we show that certain regularizations of the approximate solutions \( v_h \) are \( \delta \)-solutions of (1). In Section 8 we give the precise statement and proof of Theorem 1.4. In Appendix A we state several known results related to the comparison principle for viscosity solutions. In Appendix B we recall the technique of regularization by inf- and sup-convolution, which plays a key role in our arguments.

2. Preliminaries

In this section we establish notation, give the definition of \( \delta \)-solutions, and recall some known regularity results for solutions of uniformly elliptic equations.

2.1. Notation. We denote open balls in \( \mathbb{R}^n \) by

\[
B_r(x_0) = B(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| < r \},
\]

and we often write \( B_r \) to mean \( B_r(0) \). We denote the diameter of \( U \subset \mathbb{R}^n \) by \( \text{diam} U \). A paraboloid \( P(x) \) is a polynomial in \( x_1, ..., x_n \) of degree 2. We say that a paraboloid \( P \) is of opening \( M \) if

\[
P(x) = l(x) + M \frac{|x|^2}{2},
\]

where \( l \) is an affine function and \( M \) is a constant.

For \( u \in C(U) \), we say \( D^2 u(x) \geq M \) (resp. \( D^2 u \leq M \)) in the sense of distributions if there exists a paraboloid \( P \) of opening \( M \) such that \( u(x) = P(x) \) and, for all \( y \in B_r(x) \) for some \( r \),

\[
u(y) \geq P(y) \text{ (resp. } u(y) \leq P(y)).
\]

Throughout the paper we say a constant is universal if it is positive and depends only on \( n \), \( \lambda \), and \( \Lambda \).

2.2. Notions of solution. We consider solutions of (1) in the viscosity sense; see [8] for an introduction to the theory of viscosity solutions. Throughout, we say “solution” to mean “viscosity solution.”

Definition 2.1. We say that \( v \) is a \( \delta \)-subsolution (respectively, \( \delta \)-supersolution) of (1) if, for any \( x \) such that \( B_\delta(x) \subset U \), any paraboloid \( P \) with \( P(x) = v(x) \) and \( P(y) \geq v(y) \) (respectively, \( P(y) \leq v(y) \)) for all \( y \) in \( B_\delta(x) \) satisfies

\[
F(D^2 P, x) \geq 0 \text{ (respectively, } F(D^2 P, x) \leq 0).
\]

We say that \( v \) is a \( \delta \)-solution if \( v \) is both a \( \delta \)-subsolution and a \( \delta \)-supersolution.
From the definition, it is clear that a viscosity solution of (1) is a $\delta$-solution of (1) for any $\delta > 0$. The difference from the definition of viscosity solution is that for $u \in C(U)$ to be a $\delta$-supersolution (resp. $\delta$-subsolution), any test paraboloid must stay below (resp. above) $u$ on a set of fixed size.

2.3. Several known results. We recall that the concave envelope of a function $u \in C(B_r)$ is defined as

$$\Gamma_u(x) = \inf \{ l(x) : l \geq u \text{ in } B_r \text{ and } l \text{ is affine} \}.$$ 

The following fact is a key step in the proof of the well-known Alexander-Bakelman-Pucci (ABP) estimate, and it will play a central role in our arguments. It is Lemma 3.5 of [5], modified slightly for our setting.

Proposition 2.2. Assume $u \in C(B_r)$ is such that $u \leq 0$ on $\partial B_r$. Assume that there exists a constant $K$ such that $D^2 u(x) \geq K$ in the sense of distributions for all $x \in B_r$. There exists a universal constant $C$ such that

$$\sup_{B_r} u \leq Cr \left( \int_{\{u=\Gamma_u\}} |\det D^2 \Gamma_u| \right)^{\frac{1}{n}},$$

where $\Gamma_u$ is the concave envelope of $u^+$ in $B_{2r}$. Moreover, $\Gamma_u$ is twice differentiable almost everywhere.

Next we provide the statements of several known regularity results, for which we introduce the adimensional $C^{1,\alpha}$ norm, denoted by $\| \cdot \|_{C^{1,\alpha}(\overline{B_r})}$:

$$\| u \|_{C^{1,\alpha}(\overline{B_r})} = \| u \|_{L^\infty(\overline{B_r})} + r \| Du \|_{L^\infty(\overline{B_r})} + r^{1+\alpha} \| Du \|_{C^{0,\alpha}(\overline{B_r})}.$$ 

We need the following rescaled version of the interior $C^{1,\alpha}$ estimate [5, Corollary 5.7].

Proposition 2.3 (Interior estimate). Assume (F1). There exist universal constants $\alpha$ and $C$ such that if $u$ is a viscosity solution of $F(D^2 u) = 0$ in $B_r$, then $u \in C^{1,\alpha}(\overline{B_{r/2}})$ and

$$\| u \|_{C^{1,\alpha}(\overline{B_{r/2}})} \leq C(\| u \|_{L^\infty(\overline{B_r})} + |F(0)|).$$

In addition to the interior $C^{1,\alpha}$ interior estimate, we need the following global $C^{1,\alpha}$ estimate (Winter [17, Theorem 3.1, Proposition 4.1]).

Proposition 2.4 (Global estimate). Assume (F1), (F2), (G1), (G2) and $F(0, x) \equiv 0$. There exists a universal constant $\alpha$ and a positive constant $C$ that depends on $n$, $\lambda$, $\Lambda$, $\kappa$, diam$U$ and the regularity of $\partial U$, such that if $u$ is the viscosity solution of

$$\begin{cases}
F(D^2 u, x) = f(x) & \text{in } U, \\
u = g & \text{on } \partial U,
\end{cases}$$

then $u \in C^{1,\alpha}(U)$ and

$$\| u \|_{C^{1,\alpha}(U)} \leq C(\| f \|_{L^\infty(U)} + \| g \|_{C^{1,\gamma}(\partial U)}).$$

3. A local estimate between the solution of (11) and solutions of (11) with fixed coefficients.

In this section we establish the following result, which is an important step in the proof of Theorem 1.1.
Proposition 3.1. Assume (U1), (F1), (F2), (G1), (G2) and F(0, x) ≡ 0. Let u be the viscosity solution of (1). There exist a universal constant α, a positive constant \( r_0 = r_0(\lambda, \Lambda, n, \kappa) \) and a positive constant C that depends on \( \lambda, \Lambda, n, \kappa, \|f\|_{C^{0,1}(U)}, \|g\|_{C^{1,\gamma}(\partial U)}, \text{diam}U \) and the regularity of \( \partial U \) such that if, for \( r < r_0 \) and for \( x_0 \) such that \( B_r(x_0) \subset U \), \( \tilde{u} \) is the viscosity solution of

\[
\begin{cases}
F(D^2\tilde{u}, x_0) = f(x_0) & \text{in } B_r(x_0), \\
\tilde{u} = u & \text{on } \partial B_r(x_0),
\end{cases}
\]

then

\[||\tilde{u} - u||_{L^\infty(B_r(x_0))} \leq Cr^{2+\alpha}.\]

Two of our arguments – the proofs of Proposition 3.1 and Proposition 4.1, which we present in the next section – are similar to the proof of the comparison principle for uniformly elliptic equations of Ishii and Lions’ [9, Theorem III.1]. At the heart of the proofs of both propositions is the following lemma, which combines Theorem 3.2 of [8] and Lemma III.1 of [9]. For the convenience of the reader, we give the statements of Theorem 3.2 of [8] and Lemma III.1 of [9] in Appendix A.

Lemma 3.2. There exists a constant \( C(n) \) that depends only on \( n \) such that the following holds. Assume \( V \) is an open subset of \( \mathbb{R}^n \) and \( u, v \in C(V) \) are viscosity solutions of \( F(D^2u, x) = f(x) \) and \( G(D^2v, x) = g(x) \) in \( V \). Suppose that \( (x_a, y_a) \in V \times V \) is a local maximum of

\[u(x) - v(y) - \frac{\alpha}{2}|x-y|^2.\]

In addition, assume that there exist \( s, t \in \mathbb{R} \) with

\[t \leq \frac{\lambda}{2C(n)},\]

and such that for any \( M, N \in S_n \) with \( M \leq N \),

\[F(M, x_a) - G(N, y_a) \leq t\|M\| + s - \lambda\|N - M\|.\]

Then,

\[f(x_a) - g(y_a) - s \leq \frac{t^2C^2(n)\alpha}{2\lambda}.\]

Proof. We take \( C(n) \) to be the constant from Lemma B.4.

Since \( (x_a, y_a) \) is an interior maximum, Theorem B.9 implies that there exist \( X, Y \in S_n \) such that

\[-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},\]

(6)

\[F(X, x_a) \geq f(x_a),\]

and

\[G(Y, y_a) \leq g(y_a).\]

Subtracting B.3 from (7), we find

\[f(x_a) - g(y_a) \leq F(X, x_a) - G(Y, y_a).\]

We point out that the matrix inequality B.6 implies \( X \leq Y \). By assumption B.5, we therefore have

\[f(x_a) - g(x_a) \leq \|X\| + s - \lambda\|Y - X\|.\]

In addition, Lemma B.4 and B.6 imply

\[\|X\| \leq C(n) \left\{ a^{1/2}\|X - Y\|^{1/2} + \|X - Y\| \right\}.\]

(9)
Therefore, we find
\[ f(x_a) - g(x_a) - s \leq tC(n) \left\{ a^{1/2} ||X - Y||^{1/2} + ||X - Y|| \right\} - \lambda ||X - Y|| \]
\[ = ||X - Y|| (tC(n) - \lambda) + tC(n)a^{1/2}||X - Y||^{1/2} \]
\[ \leq -\frac{\lambda}{2} ||X - Y|| + tC(n)a^{1/2}||X - Y||^{1/2}, \]
where the last inequality follows from assumption (4). Thus, we have that the right-hand side is a quadratic polynomial in \( z = ||X - Y||^{1/2} \), with negative leading coefficient. Therefore, we obtain
\[ f(x_a) - g(x_a) - s \leq \frac{\lambda}{2} z^2 + tC(n) a^{1/2} z \]
as claimed. \( \square \)

For the proof of Proposition 3.1, we first rescale to a ball of radius 1, where we double variables and then apply Lemma 3.2. We will need to keep careful track of all the parameters once we double variables. For this, we need the following lemma. Its proof is elementary and is provided in Appendix A.

**Lemma 3.3.** Suppose \( v, w \in C^{0,1}(V) \) with \( v = w \) on \( \partial V \), and let \( a > 0 \). Then
\[
\sup_{(\partial V \times V) \cup (V \times \partial V)} (v(x) - w(y) - \frac{\partial v}{\partial x} x - y)^2 \leq 2(||Dv||_{L^\infty(V)}^2 + ||Dw||_{L^\infty(V)}^2) a^{-1};
\]
and, if \( (x_a, y_a) \in V \times V \) is a point at which the supremum above is achieved, then
\[
|x_a - y_a| \leq 2a^{-1} \min \{||Du||_{L^\infty(V)}, ||Dv||_{L^\infty(V)}\}.
\]

We proceed with:

**Proof of Proposition 3.1** We take \( \alpha \) to be the exponent given by Proposition 2.3 and \( C(n) \) to be the constant from Lemma 3.2. We define the constants \( r_0 \) and \( C_0 \) by
\[
r_0 = \min \left\{ \frac{\lambda}{2\kappa C(n)}, 1 \right\}
\]
and
\[
C_0 = \frac{2C(n)^2\kappa^2}{\lambda}.
\]

We take \( r \leq r_0 \). We will prove
\[
\sup_{B_r} (u - \bar{u}) \leq Cr^{2+\alpha} (r^2 + r^{2-2\alpha}) ||f||^2_{C^{0,1}(U)} + ||u||^2_{C^{1,0}(U)};
\]
the other side of the estimate is analogous. We define the constant \( C_1 \) by
\[
C_1 = C_0 + r^{1-\alpha} ||Df||_{L^\infty(U)} + r^{1-\alpha} \kappa.
\]
We introduce a perturbation \( \bar{u} \) of \( u \); we define \( \bar{u} \) to be the solution of
\[
\begin{cases}
F(D^2\bar{u}, x_0) = f(x_0) - C_1 r^\alpha & \text{in } B_r(x_0), \\
\bar{u} = u & \text{on } \partial B_r(x_0).
\end{cases}
\]
Next, for \( y \in B_1 \), we define the rescaled functions
\[
u^r(y) = \frac{u(yr + x_0) - u(x_0) - Du(x_0) \cdot (yr)}{r^{1+\alpha}}
\]
and
\[
\bar{u}^r(y) = \frac{\bar{u}(yr + x_0) - u(x_0) - Du(x_0) \cdot (yr)}{r^{1+\alpha}}.
\]
We denote by $\tilde{F}$ the rescaled linearity,
\[
\tilde{F}(X, y) = r^{1-\alpha}F(r^{\alpha-1}X, ry + x_0),
\]
and define $\tilde{f}(y) = r^{1-\alpha}f(ry + x_0)$. The nonlinearity $\tilde{F}$ is uniformly elliptic with the same ellipticity constants as $F$. Moreover, for any $y_1$ and $y_2$ in $B_1$, we have
\[
|\tilde{F}(X, y_1) - \tilde{F}(X, y_2)| \leq \kappa (r||X|| + r^{2-\alpha}). \tag{11}
\]
In addition, we have
\[
\|D\tilde{f}\|_{L^\infty(B_1)} = r^{2-\alpha}\|Df\|_{L^\infty(B,(x_0))} \leq r^{2-\alpha}\|Df\|_{L^\infty(U)}.
\]
The definitions of $\tilde{F}$, $\tilde{f}$, $u^r$ and $\bar{u}^r$ imply that $u^r$ is a solution of
\[
\tilde{F}(D^2u^r, y) = \tilde{f}(y) \text{ in } B_1,
\]
and that $\bar{u}^r(x)$ is a solution of
\[
\begin{cases}
\tilde{F}(D^2\bar{u}^r, 0) = \tilde{f}(0) - C_1r & \text{in } B_1, \\
\bar{u}^r = u^r & \text{on } \partial B_1.
\end{cases} \tag{12}
\]
We claim
\[
\sup_{B_1}(u^r - \bar{u}^r) \leq 4(||D\bar{u}^r||_{L^\infty(B_1)}^2 + ||Du^r||_{L^\infty(B_1)}^2)r, \tag{13}
\]
which we will prove by contradiction. To this end, we assume \ref{13} is not true.

We will apply Lemma \[5,2\] with $a = r^{-1}$ and with $\tilde{F}(M, x)$, $\tilde{F}(M, 0)$, $\tilde{f}$, and $\tilde{f}(0) - C_1r$ instead of $F$, $G$, $f$, and $g$, respectively. For this, we double variables and consider
\[
\sup_{x,y \in B_1} \left( u^r(x) - \bar{u}^r(y) - \frac{r^{-1}}{2}|x-y|^2 \right).
\]
Denote by $(x_r, y_r)$ a point in $B_1 \times B_1$ where the supremum is achieved. Since we assume that \ref{13} doesn’t hold, we must have
\[
\sup_{x,y \in B_1} \left( u^r(x) - \bar{u}^r(y) - \frac{r^{-1}}{2}|x-y|^2 \right) \geq \sup_{B_1}(u^r - \bar{u}^r) > 4(||D\bar{u}^r||_{L^\infty(B_1)}^2 + ||Du^r||_{L^\infty(B_1)}^2)r.
\]
Thus, by Lemma \[3,3\] applied with $a = r^{-1}$, $v = u^r$ and $w = \bar{u}^r$, we find $(x_r, y_r) \in B_1 \times B_1$.

Next, we verify the remaining hypotheses of Lemma \[3,2\] mainly \ref{5} and \ref{4}: assume $M, N \in S_n$ with $M \leq N$. Then, by \ref{11} and the ellipticity of $\tilde{F}$, we find
\[
\tilde{F}(M, x_r) - \tilde{F}(N, 0) \leq \tilde{F}(M, 0) + \kappa r||X|| + \kappa r^{2-\alpha} - \tilde{F}(N, 0) \\
\leq \kappa ||X|| + \kappa r^{2-\alpha} - \lambda ||N - M||.
\]
Our choice of $r \leq r_0$ implies that if we take $t = \kappa r$, then $t$ satisfies \ref{4}. Thus we have that the assumptions of Lemma \[3,2\] are satisfied, so we find
\[
\tilde{f}(x_a) - (\tilde{f}(0) - C_1r) - \kappa r^{2-\alpha} \leq \frac{\kappa^2 r C^2(n)}{2\lambda}.
\]
Since we have $\|D\tilde{f}\|_{L^\infty(B_1)} \leq r^{2-\alpha}\|Df\|_{L^\infty(U)}$ and $x_r \in B_1$, we obtain
\[
\tilde{f}(x_a) - \tilde{f}(0) + C_1r - \kappa r^{2-\alpha} \geq C_1r - r^{2-\alpha}\|Df\|_{L^\infty(U)} - \kappa r^{2-\alpha}.
\]
From the definition of $C_0$ and $C_1$ and the previous inequality, we find
\[
\tilde{f}(x_a) - \tilde{f}(0) + C_1r - \kappa r^{2-\alpha} \geq C_0r.
\]
Thus we conclude
\[
C_0r \leq \frac{C(n)^2\kappa^2}{\lambda} r,
\]
which is impossible, since we chose $C_0 = 2^{\frac{C(n)2^{n^2}}{\lambda}}$.

Thus, (13) must hold. It is now left to bound $\|Du^r\|_{L^\infty(B_1)}$ and $\|Du^r\|_{L^\infty(B_1)}$. From the definition of $u^r$, we see

$$\|u^r\|_{C^{1,\alpha}(B_1)} = \|u\|_{C^{1,\alpha}(B_r(\bar{x}_0))} \leq \|u\|_{C^{1,\alpha}(U)} \leq C_2,$$

where the second inequality follows from Proposition 2.4 and $C_2$ depends on $\lambda$, $\Lambda$, $n$, $\kappa$, $\|f\|_{L^\infty(U)}$, $\|g\|_{C^{1,\alpha}(\partial U)}$, $\text{diam}(\partial U)$ and the regularity of $\partial U$. Since $\bar{u}^r$ is the solution of (12), Proposition 2.4 applied to $\bar{u}^r$ in $B_1$ implies that there exists $C > 0$ that depends on $n, \lambda, \Lambda$ and $\kappa$ such that

$$\|\bar{u}^r\|_{C^{1,\alpha}(B_1)} \leq C(\|\bar{u}\|_{L^\infty(B_1)} + C_1 r + \|u^r\|_{C^{1,\alpha}(B_1)}) = C(r^{1-\alpha}\|f\|_{L^\infty(B_r(\bar{x}_0))} + C_1 r + C_2),$$

where the equality follows since $\|\bar{u}\|_{L^\infty(B_1)} = r^{1-\alpha}\|f\|_{L^\infty(B_r(\bar{x}_0))}$ and from the previous bound on $\|u^r\|_{C^{1,\alpha}(B_1)}$. Therefore, we have

$$\|D\bar{u}^r\|_{L^\infty(B_1)}^2 + \|Du^r\|_{L^\infty(B_1)}^2 \leq C_3,$$

where $C_3$ depends on $\lambda, \Lambda, n, \kappa, \|f\|_{C^{0,1}(U)}, \|g\|_{C^{1,\alpha}(\partial U)}$, $\text{diam}(\partial U)$ and the regularity of $\partial U$. We use this to bound the right-hand side of (13) and obtain

$$\sup_{B_1}(u^r - \bar{u}^r) \leq C_3 r.$$

Thus,

$$\sup_{B_r}(u - \bar{u}) \leq C_3 r^{2+\alpha}. \tag{14}$$

Finally, Lemma 9.2 implies

$$\sup_{B_r}(\bar{u} - \bar{u}) \leq \frac{C_1 r^{2+\alpha}}{2\lambda}.$$
4. Perturbations of the equation

In this section we introduce two perturbations of the equation (1) and prove an estimate between $u$ and solutions of the perturbed equations (Proposition 4.1). Our use of these perturbations is inspired by [13, Theorem 2.1], a proof of the existence of $C^{2,\alpha}$ approximate solutions for convex equations, which is a key step in Krylov’s analysis of the convex/concave case.

For a nonlinearity $F(X, x) \in C(S_n \times U)$, we define
\[ F_\varepsilon(X, x) = \inf_{y \in B_s(x)^c \cap U} F(X, y) \quad \text{and} \quad F^\varepsilon(x, x) = \sup_{y \in B_s(x)^c \cap U} F(X, y). \]

We observe that if $F$ satisfies (H1) and (H2), then so do $F_\varepsilon$ and $F^\varepsilon$. For $f \in C(U)$, we similarly define
\[ f_\varepsilon(x) = \inf_{y \in B_s(x)^c \cap U} f(y) \quad \text{and} \quad f^\varepsilon(x) = \sup_{y \in B_s(x)^c \cap U} f(y). \]

**Proposition 4.1.** Assume that $u$ and $u_\varepsilon$ are the viscosity solution of, respectively, (7) and
\[ \begin{cases} F_\varepsilon(D^2 u_\varepsilon, x) = f^\varepsilon(x) & \text{in } U, \\ u_\varepsilon = g & \text{on } \partial U. \end{cases} \]

There exist positive constants $\varepsilon_0$ and $C$ that depend on $n$, $\Lambda$, $\kappa$, $\|g\|_{C^{1,\gamma}(\partial U)}$, $\|f\|_{C^{0,1}(U)}$, $\text{diam}U$ and the regularity of $\partial U$ such that for $\varepsilon \leq \varepsilon_0$,
\[ \|u - u_\varepsilon\|_{L^\infty(U)} \leq C \varepsilon. \]

The same statement holds for $F^\varepsilon$ and $f_\varepsilon$ instead of $F_\varepsilon$ and $f^\varepsilon$.

The proof of Proposition 4.1 is similar to the proof Proposition 3.1—the key is the application of Lemma 3.2.

**Proof of Proposition 4.1.** Since $u_\varepsilon$ is a subsolution of (1), we have $u_\varepsilon \leq u$ for all $\varepsilon$.

By Proposition 2.4, there exists a constant $C$ that depends on $\Lambda$, $\kappa$, $\text{diam}U$ and the regularity of $\partial U$ such that $\|Du\|_{L^\infty(U)} \leq C(\|g\|_{C^{1,\gamma}(\partial U)} + \|f\|_{L^\infty(U)})$. We define
\[ \varepsilon_0 = \frac{\lambda}{2 \kappa C(n)(2C(\|g\|_{C^{1,\gamma}(\partial U)} + \|f\|_{L^\infty(U)}) + 1)} \]

so that
\[ \varepsilon_0 \leq \frac{\lambda}{2 \kappa C(n)(2\|Du\|_{L^\infty(U)} + 1)}. \]

Here $C(n)$ is the constant from Lemma 3.2. (We are being careful with the constants to make sure that $\varepsilon_0$ does not depend on $u$ itself). We fix $\varepsilon \leq \varepsilon_0$ and define
\[ \theta_0 = \frac{\kappa^2(2\|Du\|_{L^\infty(U)} + 1)}{\lambda} \]

and
\[ \theta = \theta_0 + \varepsilon(\kappa + \|Df\|_{L^\infty(U)})(2\|Du\|_{L^\infty(U)} + 1). \]

We define $\tilde{u}_\varepsilon$, a perturbation of $u_\varepsilon$, as the viscosity solution of
\[ \begin{cases} F_\varepsilon(D^2 \tilde{u}_\varepsilon, x) = -\theta & \text{in } U, \\ \tilde{u}_\varepsilon = g & \text{on } \partial U. \end{cases} \]

The claim is
\[ \sup_U (u - \tilde{u}_\varepsilon) > 4(\|D\tilde{u}_\varepsilon\|_{L^\infty(U)}^2 + \|Du\|_{L^\infty(U)}^2) \varepsilon, \]

which we will prove by contradiction. To this end, assume that (17) does not hold.
We will apply Lemma 3.2 with $a = \varepsilon^{-1}$, and $F_\varepsilon$ and $(f^\varepsilon(x) - \theta)$ instead of $G$ and $g(x)$. We double variables and consider

$$\sup_{x,y \in U} \left( u(x) - \tilde{u}_\varepsilon(y) - \varepsilon^{-1} \left| \frac{x - y} {2} \right|^2 \right).$$

Let $(x_\varepsilon, y_\varepsilon)$ be a point where the supremum is achieved. Since we assume (17) does not hold, Lemma 3.3 implies that $x_\varepsilon$ and $y_\varepsilon$ lie in the interior of $U$.

It is left to verify the remaining hypotheses of Lemma 3.2: let $M, N \in \mathcal{S}_n$ be such that $M \leq N$ and let $y_\varepsilon^*$ be a point where the supremum is achieved in the definition of $F_\varepsilon(N, y_\varepsilon^*)$. We have

$$F(M, x_\varepsilon) - F_\varepsilon(N, y_\varepsilon) = F(M, x_\varepsilon) - F(N, y_\varepsilon^*)$$

$$\leq F(M, y_\varepsilon^*) + \kappa(||M|| + 1)|x_\varepsilon - y_\varepsilon^*| - F(N, y_\varepsilon^*)$$

$$\leq \kappa(||M|| + 1)|x_\varepsilon - y_\varepsilon^*| - \lambda||N - M||,$$

where the first and second inequalities follow since $F$ satisfies (F2) and (F1), respectively. According to Lemma 3.3 and the definition of $y_\varepsilon^*$, we have

$$|x_\varepsilon - y_\varepsilon^*| \leq |x_\varepsilon - y_\varepsilon^*| + |y_\varepsilon - y_\varepsilon^*| \leq 2\varepsilon ||Du||_{L^\infty(U)} + \varepsilon = \varepsilon (2 ||Du||_{L^\infty(U)} + 1),$$

so if we let $t = \varepsilon \kappa (2 ||Du||_{L^\infty(U)} + 1)$, we find

$$F(M, x_\varepsilon) - F_\varepsilon(N, y_\varepsilon) \leq t||X|| + \kappa |x_\varepsilon - y_\varepsilon^*| - \lambda ||N - M||.$$

And, by our choice of $\varepsilon \leq \varepsilon_0$, we have that $t$ satisfies (4). Thus, the hypotheses (4) and (1) of Lemma 3.2 are satisfied, and therefore we obtain

$$f(x_\varepsilon) - (f^\varepsilon(y_\varepsilon) - \theta) - \kappa |x_\varepsilon - y_\varepsilon^*| \leq \varepsilon \kappa (2 ||Du||_{L^\infty(U)} + 1)^2.$$

Next we bound the left-hand side of (18) from below. Let $y_\varepsilon^+$ be a point where the supremum is achieved in the definition of $f^\varepsilon(y_\varepsilon)$. Since $f$ is Lipschitz, we find

$$f(x_\varepsilon) - (f^\varepsilon(y_\varepsilon) - \theta) - \kappa |x_\varepsilon - y_\varepsilon^*| = f(x_\varepsilon) - f(y_\varepsilon^+) + \theta - \kappa |x_\varepsilon - y_\varepsilon^*|$$

$$\geq - ||Df||_{L^\infty(U)} |x_\varepsilon - y_\varepsilon^+| + \theta - \kappa |x_\varepsilon - y_\varepsilon^*|.$$

According to Lemma 3.3 and the definition of $y_\varepsilon^+$, we have

$$|x_\varepsilon - y_\varepsilon^+| \leq |x_\varepsilon - y_\varepsilon^*| + |y_\varepsilon - y_\varepsilon^+| \leq 2\varepsilon ||Du||_{L^\infty(U)} + \varepsilon = \varepsilon (2 ||Du||_{L^\infty(U)} + 1),$$

so together with the previous inequality and the estimate on $|x_\varepsilon - y_\varepsilon^*|$, we find

$$f(x_\varepsilon) - (f^\varepsilon(y_\varepsilon) - \theta) - \kappa |x_\varepsilon - y_\varepsilon^*| \geq \theta - \varepsilon ||Df||_{L^\infty(U)} + \kappa (2 ||Du||_{L^\infty(U)} + 1) = \theta_0,$$

where the equality follows from our choice of $\theta$ and $\theta_0$. We use this to bound the left-hand side of (18) and obtain

$$\theta_0 \leq \varepsilon \kappa (2 ||Du||_{L^\infty(U)} + 1)^2,$$

which contradicts our choice of $\theta_0$. Therefore (17) holds. Together with Proposition (2.4), the bound (17) implies that there exists a constant $C$ that depends on $n, \lambda, \kappa$, $||g||_{C^1,\gamma(U)}$, $diam U$ and the regularity of $\partial U$ such that

$$\sup_{U} u - \tilde{u}_\varepsilon \leq C \varepsilon.$$

Lemma 9.2 implies

$$\sup_{U} (\tilde{u}_\varepsilon - u_\varepsilon) \leq \frac{(diam U)^2 \theta} {2\lambda}.$$
From (19) and (20) we conclude
\[ \sup_U u - u \leq \sup_U u \leq \varepsilon C, \]
where \( C \) depends on \( n, \lambda, \kappa, \| Df \|_{L^\infty(U)}, \| g \|_{C^{1,\gamma}(U)}, \text{ diam } U \) and the regularity of \( \partial U \).

\[ \square \]

5. An elementary lemma

In this section we establish an elementary lemma that plays an important role in the proof of our main result.

**Lemma 5.1.** Assume \( w \in C^{0,\eta}(U) \) is such that \( w \leq 0 \) on \( \partial U \). Then for any \( m \) such that \( m \leq \sup_U w \), there exists \( x_0 \in U \) with
\[ \tag{21} d(x_0, \partial U) \geq \left( \frac{m}{2|w|_{C^{0,\eta}(U)}} \right)^{1/\eta} \]
and an affine function \( l(x) \) with \( l(x_0) = w(x_0) \) and
\[ w(x) \leq l(x) - \frac{m}{2(\text{diam } U)^2} |x - x_0|^2 \]
for all \( x \in U \).

**Proof.** We denote \( R = \text{diam } U \). We fix some \( y \in U \). The function
\[ x \mapsto -\frac{m}{2R^2} |x - y|^2 - w(x) \]
achieves its minimum on \( U \) at some \( x_0 \in \bar{U} \). (We point out that \( x_0 \) is a point where \( w \) is touched from above by a concave paraboloid of opening \( \frac{m}{2R^2} \).) Thus, for all \( x \in U \), we have
\[ w(x_0) \geq w(x) + \frac{m}{2R^2} |x - y|^2 - \frac{m}{2R^2} |x_0 - y|^2 \geq w(x) - \frac{m}{2R^2} |x_0 - y|^2. \]
We take the supremum over \( x \in U \) of both sides to see
\[ w(x_0) \geq \sup_{x \in U} w(x) - \frac{m}{2R^2} |x_0 - y|^2 \geq m - \frac{m}{2} = \frac{m}{2}. \]
Since \( w \in C^{0,\eta}(U) \) and \( w \leq 0 \) on \( \partial U \), \( w(x_0) \leq d(x_0, \partial U)^{\eta} |w|_{C^{0,\eta}(U)} \). Together with the previous inequality, this implies (21).

Finally, we have that for all \( x \in U \),
\[ w(x) \leq w(x_0) + \frac{m}{2R^2} (|x_0 - y|^2 - |x - y|^2) \]
\[ = w(x_0) + \frac{m}{2R^2} (|x_0 - x|^2 - |x - x_0, x_0 - y| >) \]
\[ = l(x) - \frac{m}{2R^2} |x_0 - x|^2. \]
\[ \square \]
6. Proof of Theorem 6.1

Here is the precise statement of our main result:

**Theorem 6.1.** Assume \((U_1), (F_1), (F_2), (G_1), (G_2)\) and \(F(0, x) \equiv 0\). Assume \(u\) is a viscosity solution of (1) and assume that \(\{v_\delta\}_{\delta \geq 0}\) is a family of \(\delta\)-supersolutions (respectively, subsolutions) of (1) with

\[
\|v_\delta\|_{C^0, \eta(U)} \leq M
\]

and, for all \(\delta\),

\[
\sup_{\partial U} (u - v_\delta) \leq 0 \quad (\text{respectively, } \inf_{\partial U} (u - v_\delta) \geq 0).
\]

There exists a constant \(\bar{\delta} > 0\) such that for any \(\delta \leq \bar{\delta}\),

\[
\sup_U (u - v_\delta) \leq \bar{c}\delta^{\bar{\alpha}} \quad (\text{respectively, } \inf_U (u - v_\delta) \geq -\bar{c}\delta^{\bar{\alpha}}).
\]

The constant \(\bar{\alpha}\) depends on \(\eta, n, \lambda, \Lambda\); \(\bar{c}\) depends on \(n, \lambda, \kappa, M, ||f||_{C^{0,1}(U)}, ||g||_{C^{1,\gamma}(\partial U)}\), \(\text{diam} U\) and the regularity of \(\partial U\); and \(\bar{\delta}\) depends on \(n, \lambda, \kappa, M, ||f||_{C^{0,1}(U)}\) and \(||g||_{C^{1,\gamma}(\partial U)}\).

In this section, we first give a basic outline of the proof of Theorem 6.1. We also briefly point out the differences between the basic outline and the actual proof. Then we state and prove an important lemma. Finally, we put everything together and give the complete proof of Theorem 6.1.

6.1. Outline of the proof of Theorem 6.1. We outline the proof of the bound on \(m \equiv \sup u - v_\delta\); the proof of the bound on \(\sup v_\delta - u\) is similar. By Lemma 5.1 there exists a point \(x_0 \in U\) that is away from \(\partial U\) and a concave paraboloid of opening \(Cm\) that touches \(u - v_\delta\) from above at \(x_0\). Let us assume, for the sake of simplicity, that the affine function \(l(x)\) given by Lemma 5.1 is in fact \(l(x) \equiv u(x_0) - v_\delta(x_0)\). (See Figure 1.) This implies that, for any \(r\),

\[
\sup_{\partial B_r(x_0)} (u - v_\delta) \leq (u - v_\delta)(x_0) - Cm r^2.
\]

We therefore have a bound on \(m\) in terms of how much \(u - v_\delta\) changes on the small ball \(B_r(x_0)\):

\[
Cm r^2 \leq (u - v_\delta)(x_0) - \sup_{\partial B_r(x_0)} (u - v_\delta).
\]
Next, we consider the solution of the equation on $B_r(x_0)$, but with "frozen" coefficients: let $\bar{u}$ be the solution of
\[
\begin{cases}
F(D^2\bar{u}, x_0) = f(x_0) & \text{in } B_r(x_0), \\
\bar{u} = v & \text{on } \partial B_r(x_0).
\end{cases}
\]
By Proposition 3.1 and by (22),
\[
(23) \quad Cmr^2 \leq (\bar{u} - v_\delta)(x_0) - \sup_{\partial B_r(x_0)} (\bar{u} - v_\delta) + Cr^{2+\alpha}.
\]
Since $\bar{u}$ is the solution to a homogeneous equation, the regularity result of Caffarelli and Souganidis implies that $\bar{u}$ has second order expansions with controlled error on large portions of $B_r(x_0)$. As in [6], this extra regularity of $\bar{u}$ allows us to compare $\bar{u}$ and $v_\delta$ on $B_r(x_0)$ and conclude that
\[
\sup_{B_r(x_0)} (\bar{u} - v_\delta) - \sup_{\partial B_r(x_0)} (\bar{u} - v_\delta) \leq r^2 \delta^\alpha.
\]
(This argument is Lemma 6.2, which we state and prove in the next subsection.) Using the above bound in equation (23) yields
\[
m \leq C\delta^\alpha + Cr^\alpha.
\]
Choosing $r \leq \delta$ implies the desired estimate.

The main difference between the outline and the actual proof is that we need to work with inf- and sup- convolutions $v_\delta^{\theta,-}$ and $u_\delta^{\theta,+}$. (Inf- and sup- convolutions are a standard tool in the theory of viscosity solutions; the details are provided in the appendix.) Because we need take inf- and sup- convolutions inside of the small ball $B_r(x_0)$, the radius $r$ has to be bigger than the parameter $\theta$ of the inf- and sup- convolutions. This restriction leads to problems. To get around them, we introduce the perturbations $F_\varepsilon$ and $F_{\varepsilon}$ of the equation itself.

We replace $u$ with $u_\varepsilon$, the solution of (15), for some $\varepsilon \leq \delta^\alpha$. The error we make is of size $C\varepsilon \leq C\delta^\alpha$, which does not affect the final estimate. Next, we “freeze the coefficients” of $F_\varepsilon$: we consider the solution $\bar{u}_\varepsilon$ of
\[
(24) \quad \begin{cases}
F_\varepsilon(D^2\bar{u}_\varepsilon, x_0) = f_\varepsilon(x_0) & \text{in } B_r(x_0), \\
\bar{u}_\varepsilon = u_\varepsilon & \text{on } \partial B_r(x_0).
\end{cases}
\]
Then we proceed as explained in the outline. We do this for $\varepsilon >> r$, so that the equation with frozen coefficients “sees” outside of $B_r(x_0)$. This detail is extremely important and allows us to regularize using the inf- and sup- convolutions and complete the argument.

We will use the regularity estimate of [6] for inf- and sup- convolutions of solutions to a uniformly elliptic equation with fixed coefficients. We state this result in the appendix (see Proposition 10.3).

6.2. An important lemma. For ease of notation, we define the constants
\[
\zeta = \frac{\sigma}{4(3\sigma+n)}, \quad \theta = \frac{\zeta}{4(2-\alpha)}, \quad \alpha_1 = \frac{\zeta}{4}.
\]
Here $\sigma$ is the exponent from Proposition 10.3 and $\alpha$ is the exponent from Proposition 2.4. So, $\zeta$, $\theta$, and $\alpha_1$ all depend only on $n$, $\lambda$ and $\Lambda$.

We also define the function $\omega : \mathbb{R}^+ \to [0, 1]$ by
\[
\omega(c) = \min \left\{ 1, \left( \frac{1}{4(1+2c)} \right)^{\frac{1}{1-\theta}} \right\}.
\]
We point out that since $\zeta - 2\theta \geq \zeta/2 > 0$, we have
\[
t \leq \omega(c) \Rightarrow 2t^\zeta \theta^{-\theta}(1+2c) \leq \frac{t^\theta}{2}.
\]
Lemma 6.2. Assume \((f_1), (f_2), \text{ and } F(0, x) \equiv 0\). Fix two positive constants \(C_1\) and \(\nu\). Let \(\delta \leq \omega(C_1), r = \delta^0\), and \(\varepsilon = \nu + \delta^0 = \nu + r\). Fix \(x_0 \in U_\delta^r\).

(1): Assume \(v\) is a twice-differentiable almost everywhere \(\delta\)-supersolution of
\[
F_\varepsilon(D^2v, x) = f^\varepsilon(x) \text{ in } B_{2r}(x_0)
\]
and \(\tilde{u}_\varepsilon\) is a viscosity solution of
\[
F_\varepsilon(D^2\tilde{u}_\varepsilon, x_0) = f^\varepsilon(x) \text{ in } B_{2r}(x_0)
\]
that satisfy
\[
\begin{align*}
(A1) \quad ||\tilde{u}_\varepsilon||_{C^{1,\alpha}(B_r(x_0))} & \leq C_1, \\
(A2) \quad D^2v & \leq \delta^{-\zeta}I, \text{ and} \\
(A3) \quad \sup_{\partial B_{r/2}(x_0)} \tilde{u}_\varepsilon - v & \leq 0.
\end{align*}
\]
There exists a constant \(\tilde{c} = \tilde{c}(C_1, n, \lambda, \Lambda) > 0\) and a universal constant \(\tilde{\alpha} > 0\) such that
\[
\sup_{B_{r/2}(x_0)} \tilde{u}_\varepsilon - v \leq \tilde{c}\delta^\tilde{\alpha}r^2.
\]

(2): Assume \(w\) is a twice-differentiable almost everywhere \(\delta\)-subsolution of \(F^\varepsilon(D^2w, x) = f_\varepsilon(x)\) in \(B_{2r}(x_0)\) and \(\bar{u}_\varepsilon\) a solution of \(F^\varepsilon(D^2\bar{u}_\varepsilon, x_0) = f_\varepsilon(x)\) in \(B_{2r}(x_0)\) that satisfy \(||\bar{u}_\varepsilon||_{C^{1,\alpha}(B_r(x_0))} \leq C_1\), \(D^2w \geq -\delta^{-\zeta}I\), and
\[
\sup_{\partial B_{r/2}(x_0)} w - \bar{u}_\varepsilon \leq 0.
\]
Then, for the same constants \(\tilde{c}\) and \(\tilde{\alpha}\) as in (1) of this lemma,
\[
\sup_{B_{r/2}(x_0)} w - \bar{u}_\varepsilon \leq \tilde{c}\delta^\tilde{\alpha}r^2.
\]

We postpone the proof of this lemma until subsection 6.4 and continue with the proof of Theorem 6.1.

6.3. Proof of Theorem 6.1

Proof. We will give the proof when \(v_\delta\) is a \(\delta\)-supersolution; the other case is similar. Let \(C_1\) and \(r_0\) be the constants from Corollary 3.4, \(\varepsilon_0\) the constant from Proposition 4.1, and \(\zeta\) and \(\theta\) as given in the beginning of Section 6.2. We define the constant \(\delta\) by
\[
\delta = \min\left\{\left(\frac{\varepsilon_0}{4M^{1/2} + 1}\right)^{\frac{\theta}{2}}, \omega(C_1), r_0^{\frac{\theta}{2}}\right\}.
\]
For \(\delta \leq \delta\), denote \(\nu = 4||v_\delta||_{L^\infty(U)}^{1/2} \delta^{\zeta/2}, \varepsilon = r + \frac{\nu - \varepsilon_0}{2}\) and \(r = \delta^0\). Observe that since \(\delta \leq \delta\), our choices of constants imply that \(\varepsilon \leq \varepsilon_0\) and \(r \leq r_0\).

We introduce \(u_\varepsilon\), the viscosity solution of
\[
\begin{cases}
F_\varepsilon(D^2u_\varepsilon, x) = f^\varepsilon(x) & \text{in } U, \\
u_\varepsilon = g & \text{on } \partial U.
\end{cases}
\]
We regularize \(v_\delta\) by inf-convolution and define \(v^-\) by
\[
v^-(x) = \inf_{y \in U} \left\{v_\delta(y) + \frac{|x - y|^2}{2\delta^\zeta}\right\}.
\]
By Proposition 10.1, \(v^-\) is a \(\delta\)-supersolution of \(F_\varepsilon(D^2v, x) = f^\varepsilon(x)\) in \(U_\delta^r\).

We use the definition of \(U_\delta^r\), the properties of inf-convolutions and the bound
\[
||u_\varepsilon||_{C^{1,\alpha}(U)} \leq \mathcal{C}(||f||_{L^\infty(U)} + ||g||_{C^{1,\gamma}(\partial U)})
\]
to obtain the estimate
\[
\sup_{\partial U_\delta^C} (u_v - v^-) \leq \sup_{\partial U} (u - v_0) + c_1 \delta \frac{c_r}{\delta + r} = c_1 \delta \frac{c_r}{\delta^{n/2}},
\]
where \(c_1\) depends on \(n, \lambda, \Lambda, \kappa, \text{diam} U, \|f\|_{L^\infty(U)},\) the regularity of \(\partial U, M,\) and \(\|g\|_{C^{1,\gamma}(\partial U)}\).

We set
\[
m = \sup_{x \in U_\delta^C} u_v(x) - v^-(x) - c_1 \delta \frac{c_r}{\delta^{n/2}}.
\]
We will prove \(m \leq c_2 \delta^{\alpha_1}\), for
\[
\alpha_1 = \min \{\theta \eta, \tilde{\alpha}, \theta \alpha\}
\]
and
\[
c_2 = \max \left\{ 4[v_0 + u_v]_{C^{0,1}(U_\delta^C)}, 16R^2(\hat{c} + 2C_1) \right\},
\]
where \(\tilde{\alpha}\) and \(\hat{c}\) are the constants given in Lemma 6.2 and \(R = \text{diam} U\). We proceed by contradiction and assume
\[
m \geq c_2 \delta^{\alpha_1}.
\]
Lemma 5.1 with \(U_\delta^C\) instead of \(U\) and \(u_v(x) - v^-(x) - c_1 \delta \frac{c_r}{\delta^{n/2}}\) instead of \(w\) implies that there exists \(x_0 \in U_\delta^C\) with
\[
d(x, \partial U_\delta^C) \geq 2\delta \frac{c_r}{\delta^{n/2}} \geq 2\delta^9 = 2r,
\]
and an affine function \(l(x)\) such that
\[
l(x_0) = u_v(x_0) - v^-(x_0)
\]
and for all \(x \in U_\delta^C\),
\[
u_v(x) - v^-(x) \leq l(x) - \frac{m}{2R^2} |x - x_0|^2.
\]

Next we “freeze the coefficients” of \(F\) and define \(\tilde{u}_v\) as the solution of
\[
\begin{cases}
F_v(D^2 \tilde{u}_v; x_0) = f^c(x_0) & \text{in } B_{2r}(x_0) \\
\tilde{u}_v = u_v & \text{on } \partial B_{2r}(x_0).
\end{cases}
\]
Because \(F\) and \(B_{2r}(x_0)\) satisfy the assumptions of Corollary 3.4 and we took \(C_1\) to be the constant from Corollary 3.4, we find
\[
\|\tilde{u}_v - u_v\|_{L^\infty(B_{2r}(x_0))} \leq C_1 r^{2+\alpha}
\]
and
\[
\|\tilde{u}_v\|_{C^{1,\alpha}(B_r(x_0))} \leq C_1.
\]
We remark that \(C_1\) depends on \(n, \lambda, \Lambda, \kappa, \text{diam} U, \|f\|_{C^{0,1}(U)},\) the regularity of \(\partial U\) and \(\|g\|_{C^{1,\gamma}(\partial U)}\).

To place ourselves exactly into the situation of Lemma 5.2, we modify \(v^-\) by an affine function, and define \(v\) by
\[
v(x) = v^-(x) + l(x) - \frac{m}{8R^2} x^2 + C_1 r^{2+\alpha}.
\]
By (31), the definition of \(v\), and (30), we have, for all \(x \in \partial B_{r/2}(x_0),\)
\[
\tilde{u}_v(x) - v(x) \leq u_v(x) + C_1 r^{2+\alpha} - v(x)
\]
\[
= u_v(x) + C_1 r^{2+\alpha} - v^-(x) + l(x) - \frac{m}{8R^2} x^2 + C_1 r^{2+\alpha})
\]
\[
= u_v(x) - v^-(x) - l(x) + \frac{m}{8R^2} x^2 \leq 0.
\]
We have shown that $v$ and $\tilde{u}_\varepsilon$ satisfy the assumptions of Lemma 6.2 with our choices of $\nu$ and $\varepsilon$. Therefore,

\begin{equation}
\sup_{B_{r/2}(x_0)} \tilde{u}_\varepsilon - v \leq \bar{c}\delta^g r^2.
\end{equation}

We will now show that this bound and \[29\] lead to a contradiction. By \[31\], the definition of $v$, and \[29\], we have

\begin{align*}
\tilde{u}_\varepsilon(x_0) - v(x_0) &\geq u_\varepsilon(x_0) - C_1 r^{2+\alpha} - v(x_0) \\
&\geq u_\varepsilon(x_0) - C_1 r^{2+\alpha} - (v^{-}(x_0) + l(x_0)) - \frac{m}{8R^2} r^2 + C_1 r^{2+\alpha} \\
&\geq u_\varepsilon(x_0) - v^{-}(x_0) - l(x_0) + \frac{m}{8R^2} r^2 - 2C_1 r^{2+\alpha} \\
&= \frac{m}{8R^2} r^2 - 2C_1 r^{2+\alpha}.
\end{align*}

Rearranging and then using the bound \[33\], we find

\begin{equation}
m \leq 8R^2 (r^{-2}(\tilde{u}_\varepsilon(x_0) - v(x_0)) + 2C_1 r^{\alpha}) \leq 8R^2 (\bar{c}\delta^g + 2C_1 \delta^g \alpha) \leq \frac{c_2}{2} \delta^{\alpha_1}.
\end{equation}

But this contradicts \[28\]; therefore,

\begin{equation}
\sup_{x \in U_1^g} (u_\varepsilon(x) - v^{-}(x)) - c_1 \delta^g r^{\alpha_1} \leq c_2 \delta^{\alpha_1}.
\end{equation}

Since $||u||_{C^{1,\gamma}(U)} \leq C(||g||_{C^{1,\alpha}(\partial U)} + ||f||_{L^\infty(U)})$, and we have $[v_\delta]_{C^{0,\gamma}(U)} \leq M$ and $u \leq v_\delta$ on $\partial U$ by assumption, we obtain

\begin{equation}
\sup_{U} (u - v_\delta) \leq \sup_{U \setminus U_1^g} (u - v_\delta) \leq \sup_{U \setminus U_1^g} (u - v_\delta) + c_3 \delta^g,
\end{equation}

where $\gamma = \min \left\{ \frac{\alpha}{2 - \eta}, \eta, \alpha \right\}$ and $c_3$ depends on $n, \lambda, \Lambda, \kappa, M, ||f||_{L^\infty(U)}, ||g||_{C^{1,\gamma}(\partial U)}$, $\text{diam}U$ and the regularity of $\partial U$. By Proposition 4.1 and the bound \[34\], we have

\begin{equation}
\sup_{U_1^g} u - v^{-} \leq \sup_{U_1^g} (u_\varepsilon - v^{-}) + C\varepsilon \leq c_1 \delta^g + c_2 \delta^{\alpha_1} + C\varepsilon.
\end{equation}

Thus we conclude

\begin{equation}
\sup_{U} u - v_\delta \leq \bar{c}\delta^g,
\end{equation}

where $\bar{c}$ depends on $n, \lambda, \kappa, M, ||f||_{C^{\gamma,1}(U)}, ||g||_{C^{1,\gamma}(\partial U)}$, $\text{diam}U$ and the regularity of $\partial U$. \hfill \Box

6.4. Proof of Lemma 6.2. Before proceeding with the proof of Lemma 6.2 we need one more lemma, which we now state. We recall the notation of Proposition 10.3

\begin{equation}
B_{\rho}(x_i) = B(x_i, \rho - 2\delta^g ||D\tilde{u}_\varepsilon||_{L^\infty(B_{\rho}(x_i))}).
\end{equation}

Lemma 6.3. Under the assumptions of Lemma 6.2 let $C$ be a subset of $B_{r/2}(x_0)$ with

\begin{equation}
|C| \geq c_1 \delta^{n(\alpha_1 + \gamma)} r^n.
\end{equation}

There exist universal constants $C$ and $\sigma$ such that for

\begin{equation}
\rho = 2\delta^g (1 + 2 ||D\tilde{u}_\varepsilon||_{L^\infty(B_{\rho}(x_0))})
\end{equation}

and

\begin{equation}
t \leq \left( \frac{C \rho^{n-\sigma} ||D\tilde{u}_\varepsilon||_{L^\infty(B_{\rho}(x_0))}}{c_1 \delta^{n(\alpha_1 + \gamma)}} \right)^{\frac{1}{\sigma}},
\end{equation}

we have

\begin{equation}
\sup_{B_\rho(x_i)} \tilde{u}_\varepsilon \leq C (1 + 2 ||D\tilde{u}_\varepsilon||_{L^\infty(B_{\rho}(x_0))})^\sigma r^n.
\end{equation}
there exists \( x_I \in B_{r/2}(x_0) \), a point \( x \in C \cap B_{p/2}(x_I) \) and a paraboloid \( P \) such that \( P(x) = 0 \), \( F_e(D^2P,x_0) = f^e(x_0) \), and for all \( y \in B_{p/2}(x_I) \),

\[
\tilde{u}_e^+(y) \geq \tilde{u}_e^+(x) + P(y) - C t \rho^{-1}|x-y|^3.
\]

We proceed by contradiction and assume \( \tilde{u} = \tilde{u}_e^+ \) on \( B_{r/2}(x_0) \),

\[
\tilde{u}(x) = \tilde{u}_e^+(x) - \delta^{1/4} \left( \frac{x^2}{4} - |x-x_0|^2 \right).
\]

Assumption (A1) of this lemma implies \( ||\tilde{u}_e^+||_{L^\infty(B_r(x_0))} \leq C_1 \), so, by the properties of sup-convolution (see Proposition 10.1), we have

\[
\sup_{\partial B_{r/2}(x_0)} (\tilde{u}_e^+ - v) \leq \sup_{\partial B_{r/2}(x_0)} (\tilde{u}_e^+ - v) + 2\delta^{\zeta/2} ||\tilde{u}_e^+||_{L^\infty(B_r(x_0))} \leq 2\delta^{\zeta/2} C_1^{1/2}.
\]

Since \( \tilde{u} = \tilde{u}_e^+ \) on \( \partial B_{r/2}(x_0) \), we find

\[
\sup_{\partial B_{r/2}(x_0)} (\tilde{u} - v) = \sup_{\partial B_{r/2}(x_0)} (\tilde{u}_e^+ - v) \leq 2C_1^{1/2}\delta^{\zeta/2}.
\]

Proposition 10.1 and the assumption (A2) imply \( D^2(\tilde{u} - v)(x) \geq -2\delta^{-\zeta}I \) in the sense of distributions for all \( x \in B_r(x_0) \). According to Proposition 2.2, we therefore have

\[
\sup_{B_{r/2}(x_0)} (\tilde{u} - v - 2C_1^{1/2}\delta^{\zeta/2} I) \leq C r \left( \int_C \det D^2\tilde{\Gamma} \right)^{1/n},
\]

where \( C \) is a universal constant, \( \tilde{\Gamma} \) is the concave envelope of \( (\tilde{u} - v - 2C_1^{1/2}\delta^{\zeta/2})^+ \) on \( B_r(x_0) \), and

\[
C = \{ x \in B_{r/2}(x_0) : \tilde{\Gamma}(x) = \tilde{u}(x) - v(x) - 2C_1^{1/2}\delta^{\zeta/2} \}
\]

is the contact set of \( (\tilde{u} - v - 2C_1^{1/2}\delta^{\zeta/2})^+ \) with \( \tilde{\Gamma}(x) \). Moreover, \( \tilde{\Gamma} \) is twice differentiable almost everywhere on \( B_r(x_0) \). Since we have \( D^2(\tilde{u} - v) \geq -2\delta^{-\zeta}I \) in the sense of distributions, we find that for almost every \( x \in C \),

\[
D^2\tilde{\Gamma}(x) \geq -2\delta^{\zeta}I.
\]

Since \( \tilde{\Gamma} \) is concave, we have \( |\det D^2\tilde{\Gamma}(x)| \leq 2\delta^{-n\zeta} \) for almost every \( x \in C \). Therefore,

\[
\sup_{B_{r/2}(x_0)} (\tilde{u} - v - 2C_1^{1/2}\delta^{\zeta/2}) \leq C r \left( \int_C \delta^{-n\zeta} \right)^{1/n},
\]

We proceed by contradiction and assume

\[
\sup_{B_{r/2}(x_0)} (\tilde{u} - v - 2C_1^{1/2}\delta^{\zeta/2}) > c_1^{1/n} \delta^{\alpha_1} r^2,
\]
where $c_1 = c_1(n, \lambda, \Lambda, \kappa, C_1)$ will be specified later. Together with (38), our assumption (39) implies
\begin{equation}
|C| \geq c_1s^{n(\alpha_1 + \zeta)}r^n.
\end{equation}

By Lemma 6.3, there exists a point $x_1 \in B_{r/2}(x_0)$, a point $x \in C \cap B_{r/2}(x_0) \cap B_{\rho/2}^\delta(x_1)$ and a paraboloid $P$ that satisfies $P(x) = 0$, (36), and (37). An upper bound for the parameter $\rho$ is given in the statement of Lemma 6.3.

Because $x \in C$, there exists an affine function $l(y)$ such that for all $y \in B_{r/2}(x_0)$,
\[ u(y) - v(y) \leq l(y) \]
with equality holding at $x$.

Thus, for all $y \in B_{r/2}(x_0)$,
\[ v(y) \geq u(y) - l(y) = \tilde{u}^+_z(y) - \delta^{1/4} \left( \frac{r^2}{4} - |y - x_0|^2 \right) - l(y) \]
with equality holding at $x$. The parameter $t$ is given explicitly in the statement of Lemma 6.3 in (35).

Since $x \in B_{\rho/2}^\delta(x_1)$ and $\delta < \delta^* < \rho/2$, we have $B_\delta(x) \subset B_{\rho/2}^\delta(x_1)$. Thus, for $y \in B_\delta(x)$, we have
\[ v(y) \geq P(y) - C\delta \rho^{-1} |x - y|^3 + \delta^{1/4} |x - y|^2 + \tilde{l}(y) \]
with equality holding at $x$. We've found that the paraboloid $P(y) + (\delta^* - C\delta \rho^{-1}) |x - y|^2$ touches $v$ from below at $x$ on a ball of radius $\delta$. Because $v$ is a $\delta$-supersolution of (25), we therefore obtain
\[ F_v(D^2P + (\delta^{1/4} - C\delta \rho^{-1}) I, x) \leq f^v(x). \]

Let $x^*$ be a point at which the infimum is achieved in the definition of $F_v(D^2P + (\delta^{1/4} - C\delta \rho^{-1}) I, x)$ and let $x^+$ be a point at which the supremum is achieved in the definition of $f^v(x)$. Then
\begin{equation}
|x - x^*| \leq \nu, \quad |x - x^+| \leq \nu,
\end{equation}
and
\[ F(D^2P + (\delta^{1/4} - C\delta \rho^{-1}) I, x^+) \leq f(x^+). \]

Since $F$ is uniformly elliptic, the above implies
\begin{equation}
F(D^2P, x^*) - C\delta \rho^{-1} \leq f(x^*) - \delta^{1/4}.
\end{equation}

We subtract (42) from (36) to obtain
\begin{equation}
f^v(x) - f(x^+) + \delta^{1/4} \leq F_v(D^2P, x_0) - F(D^2P, x^*) + C\delta \rho^{-1}.
\end{equation}

By (41) and our choice of $\varepsilon$, we have
\[ |x^* - x_0| \leq |x^* - x| + |x - x_0| \leq \nu + r = \varepsilon, \]
so that $F_v(D^2P, x_0) \leq F(D^2P, x^*)$. Similarly, $|x^+ - x_0| \leq \varepsilon$, so $f^v(x) - f(x^+) \geq 0$. Therefore (43) becomes
\begin{equation}
\delta^{1/4} \leq C\delta \rho^{-1}.
\end{equation}

We remark that if had worked only with $\tilde{u}^+_z$ and not with the perturbation $u$, we would have 0 on the right-hand side of (44), instead of the strictly negative quantity $-\delta^{1/4}$. This detail is crucial to our argument, as this strictness is what allows us to obtain the needed contradiction.
We will now use the choices of $\rho$ and $t$ to bound the right-hand side of (44). We first recall the bound on $t$: 

$$
t \leq \left( \frac{C \rho^{-\sigma} \|D\tilde{u}_x\|_{L^\infty(B_r(x_0))}^\sigma}{c_1 \delta^\alpha \rho^{\alpha+1}} \right)^{\frac{1}{\sigma}}.
$$

$$
= Cc_1^{-\frac{1}{\sigma}} \rho^{-1} \|D\tilde{u}_x\|_{L^\infty(B_r(x_0))} \delta^{-\frac{1}{\sigma}} \zeta^{-\frac{1}{\sigma} \rho}.
$$

By assumption (A1) we have $\|D\tilde{u}_x\|_{L^\infty(B_r(x_0))} \leq C_1 r^{-1}$, and since we allow the constant $C$ to depend on $C_1$, we find 

$$
t \leq Cc_1^{-\frac{1}{\sigma}} \rho^{-1} \delta^{-\frac{1}{\sigma}} \zeta^{-\frac{1}{\sigma} \rho} r^{-1}.
$$

Therefore, 

$$
Ct \delta \rho^{-1} \leq Cc_1^{-\frac{1}{\sigma}} \rho^{-2} \delta^{1-\frac{1}{\sigma} \rho} r^{-1}.
$$

Since $\rho > \delta^\zeta$ and $r = \delta^\theta > \delta^\zeta$, we have 

$$
Cc_1^{-\frac{1}{\sigma}} \rho^{-2} \delta^{1-\frac{1}{\sigma} \rho} r^{-1} \leq Cc_1^{-\frac{1}{\sigma}} \delta^{1-\zeta (\frac{\sigma+1}{\sigma})} r^{-1}.
$$

Together with (44), the previous two inequalities imply 

$$
\delta^{1/4} \leq C t \delta \rho^{-1} \leq Cc_1^{1/\sigma} \delta^{1-\zeta (\frac{\sigma+1}{\sigma})} \leq \delta^{1/2},
$$

where the last inequality follows from choosing $c_1 = C^{-\sigma}$ and our previous choices of $\zeta$ and $\alpha_1$. We have obtained the desired contradiction. Therefore, 

$$
\sup_{B_{r/2}(x_0)} (\tilde{u} - v - 2 \sigma_1^{1/2} \zeta^{1/2}) \leq C_{c_1}^{1/n} \delta^{\alpha_1} r^2.
$$

Our choices of $r$ and $\alpha_1$ are such that $\alpha_1 + 2 \theta \geq \zeta/2$, so that $\delta^{1/2} \leq \delta^{\alpha_1} + 2 \theta = \delta^{\alpha_1} r^2$. Thus we obtain 

$$
(45) \sup_{B_{r/2}(x_0)} (\tilde{u} - v) \leq (2C_1^{1/2} + c_1^{1/n}) \delta^{\alpha_1} r^2.
$$

For $x \in B_{r/2}(x_0)$, we have 

$$
\tilde{u}_x(x) \leq \tilde{u}_x^+ (x) \leq \tilde{u}(x) + \delta^{1/2} r^2.
$$

Together with the bound (45) this implies: 

$$
\sup_{B_{r/2}(x_0)} (\tilde{u}_x - v) \leq \sup_{B_{r/2}(x_0)} (\tilde{u}_x^+ - v) \leq \sup_{B_{r/2}(x_0)} (\tilde{u} + \delta^{1/2} r^2 - v) \leq (2C_1 + c_1^{1/n}) \delta^{\alpha_1} r^2 + \delta^{1/2} r^2.
$$

We take $\tilde{\alpha} = \min \{\alpha_1, 1/4\}$ and $\tilde{c} = 2C_1 + c_1^{1/n} + 1$ to conclude. \hfill \Box

To complete the proof of Theorem 6.1, it left only to prove Lemma 6.3. Lemma 6.3 follows from Proposition 10.13 by a covering argument, which we now describe. We cover $B_{r/2}(x_0)$ by balls $B_{\rho/2}(x_i)$. By Proposition 10.13, $\tilde{u}_x$ has second order expansions with controlled error on large portions of each of the $B_{\rho/2}(x_i)$. We refer to such points as being in the “good set” of $\tilde{u}_x$. We will use the lower bound on $|C|$ to show that there is a point $x$ that is both in the good set of $\tilde{u}_x$ and in $C$.

**Proof of Lemma 6.3** The collection $\left\{ B_{\rho/2}(x_i) : x_i \in B_{r/2}(x_0) \right\}$ covers $B_{r/2}(x_0)$; we seek to extract a finite subcover. Since the radius of each ball $B_{\rho/2}(x_i)$ depends on $x_i$, we first show that there exists a lower bound on these radii that is uniform in $x_i$; we claim that for $x_i \in B_{r/2}(x_0)$, 

$$
B(x_i, \delta^c) \subset B_{\rho/2}(x_i).
$$


We now proceed with verifying (46). By assumption (A1), we have $||Du||_{L^\infty(B_r(x_0))} \leq C_1 r^{-1}$. We use this bound in the definition of $\rho$ to find

$$\rho = 2\delta^5 (1 + 2||D\tilde{u}_\varepsilon||_{L^\infty(B_r(x_0))}) \leq 2\delta^5 (1 + 2C_1 r^{-1}) \leq 2\delta^5 r^{-1} (1 + 2C_1) = 2\delta^5 - 2\delta^5 (1 + 2C_1).$$

Since we assumed $\delta \leq \omega(C_1)$, the definition of $\omega$ implies

$$2\delta^5 - 2\delta^5 (1 + 2C_1) \leq \frac{\delta^\theta}{2} = \frac{r}{2}.$$ 

Thus we have $\rho \leq r/2$. Therefore, for any $x_i \in B_{r/2}(x_0)$, we have

$$B_{\rho}(x_i) \subset B_{r/2}(x_i) \subset B_r(x_0)$$

and so

$$||D\tilde{u}_\varepsilon||_{L^\infty(B_{\rho}(x_i))} \leq ||D\tilde{u}_\varepsilon||_{L^\infty(B_r(x_0))}.$$ 

Therefore,

$$\frac{\rho}{2} - 2\delta^5 ||D\tilde{u}_\varepsilon||_{L^\infty(B_{\rho/2}(x_i))} \geq \frac{\rho}{2} - 2\delta^5 ||D\tilde{u}_\varepsilon||_{L^\infty(B_{r/2}(x))} = \delta^\zeta.$$ 

Thus (46) holds. Therefore, there exists a finite collection $\{B_{\rho/2}(x_1) : x_1, \ldots, x_L \in B_{r/2}(x_0)\}$ that covers $B_{r/2}(x_0)$, where $L = C\rho^n \delta^{-n} \zeta$, and $C$ is a universal constant. Since $\rho \leq C\delta^\zeta$, we have $L \leq C\rho^n \delta^{-n}$.

There must be one ball $B_{\rho/2}(x_1)$ such that

$$|C \cap B^{\delta^k}_{\rho/2}(x_1)| \geq \frac{|C \cap B_{r/2}(x_0)|}{L} \geq \frac{c_2 \rho^n (\alpha + \zeta)}{C \rho^n \delta^{-n}} = c_2 \rho^n (\alpha + \zeta).$$

Here the second inequality follows from the bound (40) and $C_2$ is universal.

It is clear that $B_{\rho/2}(x_1) \subset B_{\rho}(x_1)$, so (47) implies

$$B_{\rho}(x_1) \subset B_{r}(x_0),$$

and so $\tilde{u}_\varepsilon^+$ is a viscosity solution of (26) in $B_{\rho}(x_1)$. Therefore, $\tilde{u}_\varepsilon^+$ satisfies the hypotheses of Proposition 10.3 in $B_{\rho}^\delta(x_1)$ with $F_\varepsilon$ instead of $F$ and with constant right-hand side. Thus there exist sets $A_i^+$ (the "good sets" of $\tilde{u}_\varepsilon^+$) that satisfy

$$|B_{\rho/2}(x_1) \setminus A_i^+| \leq C_3 \rho^n \sigma ||D\tilde{u}_\varepsilon||_{L^\infty(B_{\rho}(x_1))} t^{-\sigma},$$

where $C_3$ and $\sigma$ depend only on $n$, $\lambda$, and $\Lambda$. We let

$$t = \left(\frac{2C_3 \rho^n \sigma ||D\tilde{u}_\varepsilon||_{L^\infty(B_{\rho}(x_1))}}{C_2 c_1 (\alpha + \zeta)}\right)^{\frac{1}{\sigma}},$$

so that

$$C_3 \rho^n \sigma ||Du||_{L^\infty(B_{\rho}(x_1))} t^{-\sigma} = \frac{1}{2} C_2 c_1 \delta^\zeta (\alpha + \zeta) \rho^n \leq \frac{1}{2} |C \cap B_{\rho/2}(x_1)|.$$ 

Thus,

$$|B_{\rho/2}(x_1) \setminus A_i^+| < |C \cap B_{\rho/2}(x_1)|.$$ 

Therefore, there exists a point $x \in C \cap B_{\rho/2}(x_1) \cap A_i$ and a paraboloid $P$ with $P(x) = 0$ and such that (36) and (37) hold. Moreover, (48) implies the upper bound (34) on $t$. □
7. Approximation schemes

We now present our result on monotone finite difference approximations to (1). First, we introduce the necessary notation and discuss our assumptions. In the next section we give the full statement of Theorem 1.4 and its proof. We follow the notation of [6], [15], and [16]. Our mesh of discretization is

\[ E = h\mathbb{Z}^n = \{ mh : m \in \mathbb{Z}^n \}, \]

the integer mesh of size \( h \). A general difference operator is written as

\[ F_h[u](x) = F_h(Tu(x), u(x), x), \]

where

\[ Tu(x) = \{ u(x + y) : y \in E \setminus \{ 0 \} \} \]

is the set of translates of \( u \). We always assume that \( F_h[u](x) \) is independent of \( u(x + y) \) for any \( y \) such that \( |y| > Nh \). We define

\[ Y_N = \{ y \in E : 0 < |y| \leq Nh \}. \]

The standard second-order difference operators \( \delta_y^2 \) are

\[ \delta_y^2 u(x) = \frac{u(x - y) - 2u(x) + u(x + y)}{|y|^2} \]

and

\[ \delta^2 u(x) = \{ \delta_y^2 u(x) : y \in Y_N \}. \]

We consider finite difference operators \( F_h \) of the form

\[ F_h[u](x) = \mathcal{F}(\delta^2 u(x), u(x), x), \]

where

\[ \mathcal{F} : \mathbb{R}^{Y_N} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}. \]

We assume that the operators are monotone, which means they satisfy:

- \((F_h1)\) for all \( x \) in \( U \), \( z, \tau \in \mathbb{R} \), and \( q, \eta \in \mathbb{R}^{Y_N} \) such that \( 0 \leq \eta_y \leq \tau \) for all \( y \in Y_N \),

\[ F_h(q + \eta, z, \tau) \geq F_h(q, z, \tau) \geq F_h(q + \eta, z, x). \]

This definition of a monotone operator is equivalent to the one given in the introduction.

We say that the family of difference operators \( \{ F_h \}_{0 \leq h \leq h_0} \) (also called a difference scheme) is consistent with \( F \) in \( U \) if for each \( \phi \in C^2(U) \),

\[ F_h[\phi](x) \to (F(D^2\phi, x) - f(x)) \text{ in } C(U) \text{ as } h \to 0. \]

In [16], it was shown that if \( F(X, x) \) is elliptic and continuous in \( x \), then there exists a difference scheme \( \{ F_h \} \) that is consistent with \( F \).

In order to obtain an error estimate, we need to quantify the above limit. As in [6], we make the following assumption:

- \((F_h2)\) there exists a positive constant \( K \) such that for all \( \phi \in C^3(U) \),

\[ |F_h[\phi](x) - (F(D^2\phi, x) - f(x))| \leq K(1 + ||D^3\phi||_{L^\infty(U)})h \text{ for all } x \in U \cap E. \]

Schemes that satisfy \((F_h2)\) are said to be consistent with an error estimate for \( F \).

We divide \( U \subset \mathbb{R}^n \) into interior and boundary points relative to an operator \( F_h \). Mainly, we let

\[ U_h = U \cap h\mathbb{Z}^n \]

be the mesh points inside \( U \) and

\[ U^i_h = \{ x \in U_h : d(x, \partial U) > Nh \} \]
be the interior points of $U_h$. We observe that $F_h[u](x)$, for any $x \in U_h^i$, depends only on the values of $u$ in $U$. We define the boundary points of $U_h$ by

$$U_h^b = U_h \setminus U_h^i.$$  

For a mesh function $u : U_h \to \mathbb{R}$ and for $V \subset U$ we define

$$||u||_{L^\infty(V)} = \sup_{x \in V \cap E} |u(x)|,$$

$$[u]_{C^{0,\eta}(V)} = \sup_{x,y \in V \cap E} \frac{|u(x) - u(y)|}{|x - y|^{\eta}},$$

and

$$||u||_{C^{0,\eta}(V)} = ||u||_{L^\infty(V)} + [u]_{C^{0,\eta}(V)}.$$

Given $g$ satisfying (C2), we consider the boundary value problem

$$\begin{cases}
F_h[v_h](x) = 0 & \text{in } U_h^i \\
v_h = g & \text{on } U_h^b.
\end{cases}$$

It is shown in [16] and [15] that (49) has a unique solution $v_h$ and that $v_h$ is uniformly Hölder continuous. We summarize these results:

**Theorem 7.1.** Assume $(U1), (F_h1), (F_h2)$ and (C2). There exists a unique solution $v_h$ of (49). Moreover, there exist constants $\eta$ and $C$ that depend $n$, $\lambda$, $\Lambda$, $||g||_{C^{1,\gamma}(\partial U)}$ and the regularity of $\partial U$ such that for every $h \in (0,1),$

$$||v_h||_{C^{0,\eta}(U_h)} \leq C.$$  

### 7.1. Inf and sup convolutions of mesh functions

The sup and inf convolutions of a mesh function $v_h$ are defined, for any $x \in U$, by

$$v_h^+, (x) = \sup_{y \in U_h} \left\{ v_h(y) - \frac{|x - y|^2}{2\theta} \right\} \quad \text{and} \quad v_h^-, (x) = \inf_{y \in U_h} \left\{ v_h(y) + \frac{|x - y|^2}{2\theta} \right\}.$$  

In the appendix we summarize the basic properties of inf- and sup- convolutions of mesh functions (see Propositions 10.1 and 10.2).

Given $h, \delta$ and $\theta$, we define

$$U_{\delta,\theta}^h = \{ x \in U \mid d(x, \partial U) > 4\theta^{1/2} ||v_h||_{L^\infty(U_h)}^{1/2} + \sqrt{\eta} h + \delta \}.$$  

It is a classical fact of viscosity theory that if $u \in C(U)$ is the viscosity solution of $F(D^2 u) = 0$ in $U$, then the sup-convolution of $u$ is a subsolution of the same equation (see Proposition 10.1). In the following proposition, we establish a similar relationship between solutions $v_h$ of (49) and $\delta$-solutions of (49).

**Proposition 7.2.** Assume that $F_h$ is a monotone scheme that is consistent with an error estimate for $F$ with constant $K$. Suppose $v_h$ is a solution of (49) in $U$. Then $v_h^+, \theta$ is a $\delta$-subsolution of

$$F^\nu(D^2 v, x) = f_\nu(x) - K h \text{ in } U_{\delta,\theta}^h,$$

and $v_h^-, \theta$ is a $\delta$-supersolution of

$$F^\nu(D^2 v, x) = f_\nu(x) + K h \text{ in } U_{\delta,\theta}^h,$$

with $\delta = Nh$ and $\nu = 4\theta^{1/2} ||u_h||_{L^\infty(U_h)}^{1/2} + \sqrt{\eta} h$.
Proof. We will show that \( v_h^{\theta,+} \) is a \( \delta \)-subsolution; the other part of the proof is very similar. Let \( x \in U_{h,\delta}^\theta \) and let \( P \) be a quadratic polynomial with
\[
P(x) = v_h^{\theta,+}(x)
\]
and
\[
P(y) \geq v_h^{\theta,+}(y)
\]
for every \( y \in B_{h,N}(x) \). Let \( x^* \) be a point where the supremum is achieved in the definition of \( v_h^{\theta,+}(x) \). Then, for any \( y \in Y_N \), we have
\[
\delta_y^2 P(x) = \frac{1}{|y|^2} (P(x+y) - 2P(x) + P(x-y))
\]
\[
\geq \frac{1}{|y|^2} (v_h^{\theta,+}(x+y) - 2v_h^{\theta,+}(x) + v_h^{\theta,+}(x-y))
\]
\[
\geq \frac{1}{|y|^2} \left( v_h(x^* + y) - \frac{|x-x^*|^2}{2\theta} - 2 \left( v_h(x^*) - \frac{|x-x^*|^2}{2\theta} \right) + \frac{|x-x^*|^2}{2\theta} \right)
\]
\[
= \delta_y^2 v_h(x^*).
\]
(It is exactly here that it is important that \( P \) stays above \( v_h^{\theta,+} \) on all of \( B_{h,N}(x) \).) Since \( P \) is a quadratic polynomial, we have \( \delta_y^2 P(x) = \delta_y^2 P(x^*) \). We use the monotonicity of \( F_h \) and the previous computation to obtain
\[
F_h[P](x^*) \geq F_h[v_h](x^*).  \tag{50}
\]
By the properties of sup-convolutions, (see item (ii) of Proposition 10.4), we find
\[
|x - x^*| \leq 4 \|v_h\|_{L^\infty(U_h)}^{1/2} \theta^{1/2} + \sqrt{\eta} h.  \tag{51}
\]
Since \( x \in U_{h,\delta}^\theta \), the definitions of \( U_{h,\delta}^\theta \) and \( U_h^\theta \), together with the previous bound, imply
\[
(B_{h,N}(x^*) \cap E) \subset U_h^\theta.
\]
Because \( v_h \) is a solution of (49) in \( U_h^\theta \), we have \( F_h[v_h](x^*) \geq 0 \). Together with (50), this implies
\[
F_h[P](x^*) \geq 0.
\]
Since \( F_h \) is consistent with an error estimate for \( F \), we obtain
\[
0 \leq F_h[P](x^*) \leq F(D^2 P, x^*) - f(x^*) + Kh \leq F^\nu(D^2 P, x) - f^\nu(x) + Kh,
\]
where the last inequality follows from (51) and our choice of \( \nu \).
\[\square\]

8. Proof of Theorem 1.4

Here is the precise statement of Theorem 1.4.

Theorem 8.1. Assume \((I), (II), (III), (IV), (V) and F(0, x) \equiv 0\). Assume that \( \{F_h\} \) is a monotone scheme that is consistent with an error estimate for \( F \) with constant \( K \). Assume that \( u \) is the viscosity solution of (1) and that \( v_h \in C^{0,\alpha}(U) \) is the solution of (3). There exist constants \( \bar{c} = \bar{c}(n, \lambda, \Lambda, \eta), \hat{h} = \hat{h}(n, \lambda, \Lambda, \kappa, \|f\|_{C^0(U)}, \|g\|_{C^1(U)}), \) and \( \bar{c} = \tilde{c}(n, \lambda, \Lambda, \kappa, \|f\|_{C^0(U)}, \|g\|_{C^1(U)}, U) \), such that for \( h \leq \hat{h} \),
\[
\sup_{U_h} \|u - v_h\| \leq \tilde{c} h \bar{c}.  \tag{52}
\]

The proof is very similar to the proof of Theorem 1.4 – we use Propositions 8.1 and 4.1 to reduce the situation to Lemma 6.2.
Proof. We will give the proof of the bound
\[ \sup_{U_h} u - v_h \leq \tilde{c} h^\alpha, \]
the proof of the other side of the estimate is similar.

We denote \( M = \sup_{h} ||v_h||_{C^{0,}\Lambda(U)} \), which is finite by Theorem 7.1. We take \( C_1 \) to be the constant from Corollary 5.1. \( r_0 \) the constant from Proposition 5.1. \( \varepsilon_0 \) the constant from Proposition 4.1, and \( \zeta \) and \( \theta \) as given in the beginning of Section 6.2. We define
\[ \bar{\eta} = \frac{1}{N} \min \left\{ \left( \frac{\varepsilon_0}{4M^{1/2} + 1} \right)^{\frac{1}{2}}, \omega(C_1), r_0 \right\}. \]
For \( h \leq \bar{\eta} \), we denote \( \delta = Nh, \nu = 4\theta^{1/2}||v_h||_{L^\infty(U_h)} + \sqrt{n} N \delta, \varepsilon = \nu + \delta^\theta \) and \( r = \delta^\theta \). Since we took \( h \leq \bar{\eta} \), our choices of the various constants imply that \( \varepsilon \leq \varepsilon_0 \) and \( r \leq r_0 \).

We regularize \( v_h \) by inf-convolution and define \( v^- \) by
\[ v^-(x) = \inf_{y \in U_h} \left\{ v(y) + \frac{|x-y|^2}{2\delta^\theta} \right\}. \]
By Proposition 7.2, \( v^- \) is a \( \delta \)-supersolution of
\[ F_{\nu}(D^2v, x) = f'(x) + Kh \text{ in } U^0_{h,\delta}. \]
We introduce \( u_\varepsilon \), the viscosity solution of
\[ \begin{cases} F_{\nu}(D^2u_\varepsilon, x) = f'(x) & \text{in } U \\ u_\varepsilon = g & \text{on } \partial U. \end{cases} \]
We use the definition of \( U^0_{\delta} \), the properties of inf-convolutions and the bound
\[ ||u_\varepsilon||_{C^{1,\gamma}(U)} \leq C_1(||g||_{C^{1,\gamma}(\partial U)} + ||f||_{L^\infty(U)}) \]
to obtain the estimate
\[ \sup_{\partial U^0_{\delta}} (u_\varepsilon - v^-) \leq \sup_{U_h} (u - v_h) + c_1 \delta^\gamma \eta = c_1 \delta^\gamma \eta, \]
where \( c_1 \) depends on \( n, \lambda, \Lambda, \kappa, \text{diam}U, \) the regularity of \( \partial U, M, ||f||_{L^\infty(U)} \) and \( ||g||_{C^{1,\gamma}(\partial U)} \).
We set
\[ m = \sup_{x \in U_{\delta,h}^0} (u_\varepsilon(x) - v^-(x) - c_1 \delta^\gamma \eta). \]
We will prove \( m \leq c_2 \delta^{\alpha_1} \), for
\[ \alpha_1 = \min \{ \theta \eta, \hat{\alpha}, \theta \alpha \} \]
and
\[ c_2 = \max \left\{ 4(M + [u_\varepsilon]_{C^{0,}\Lambda(U^0_{\delta,h}^0)}), 16R^2(\tilde{c} + 2C_1), \frac{2R^2K}{\Lambda} \right\}, \]
where \( \hat{\alpha} \) and \( \tilde{c} \) are the constants given in Lemma 6.2 and \( R = \text{diam}U \). We proceed by contradiction and assume
\[ (53) \]
Lemma 5.1 with \( U^0_{\delta,h} \) instead of \( U \) and \( u_\varepsilon(x) - v^-(x) - c_1 \delta^\gamma \eta \) instead of \( w \) implies that there exists \( x_0 \in U^0_{\delta,h} \) with
\[ d(x, \partial U^0_{\delta,h}) \geq 2\delta^\frac{\alpha_1}{2} \geq 2\delta^\theta = 2r, \]
and an affine function \( l(x) \) such that
\[ l(x_0) = u_\varepsilon(x_0) - v^-(x_0) \]
and for all \( x \in U_{\delta,h}^\ell \),
\[
(55) \quad u_\varepsilon(x) - v^-(x) \leq l(x) - \frac{m}{2R^2} |x - x_0|^2.
\]

Next we “freeze the coefficients” of \( F_\varepsilon \) and define \( \tilde{u}_\varepsilon \) to be the solution of
\[
\begin{align*}
F_\varepsilon(D^2\tilde{u}_\varepsilon, x_0) &= 0 & \text{in } B_{2r}(x_0) \\
\tilde{u}_\varepsilon &= u_\varepsilon & \text{on } \partial B_{2r}(x_0).
\end{align*}
\]

Because \( F_\varepsilon \) and \( B_{2r}(x_0) \) satisfy the assumptions of Corollary 3.3 and \( C_1 \) is the constant from Lemma 3.2, we find
\[
(56) \quad \|\tilde{u}_\varepsilon\|_{C^{1,\alpha}(B_r(x_0))} \leq C_1
\]
and
\[
(57) \quad \|\tilde{u}_\varepsilon - u_\varepsilon\|_{L^\infty(B_{2r}(x_0))} \leq C_1 r^{2+\alpha}.
\]

To place ourselves exactly into the situation of Lemma 6.2, we perturb \( v^- \) by an affine function and a small quadratic and define
\[
v(x) = v^-(x) + l(x) - \frac{m}{8R^2} r^2 + C_1 r^{2+\alpha} - \frac{K\varepsilon}{2\lambda} (|x - x_0|^2 - \frac{r^2}{4}).
\]

By the ellipticity of \( F_\nu \), we obtain
\[
F_\nu(D^2v, x) = F_\nu(D^2v^- - K\varepsilon\lambda^{-1}I, x) \leq f^\nu(x).
\]

By (57), the definition of \( v \), and (55), we have, for all \( x \in \partial B_{r/2}(x_0) \),
\[
\tilde{u}_\varepsilon(x) - v(x) \leq u_\varepsilon(x) + C_1 r^{2+\alpha} - (v^-(x) + l(x) - \frac{m}{8R^2} r^2 + C_1 r^{2+\alpha})
\]
\[
= u_\varepsilon(x) - v^-(x) - l(x) + \frac{m}{8R^2} r^2 \leq 0.
\]

We have shown that \( v \) and \( \tilde{u}_\varepsilon \) satisfy the assumptions of Lemma 6.2 with our choices of \( \nu \) and \( \varepsilon \). Therefore,
\[
(58) \quad \sup_{B_{r/2}(x_0)} \tilde{u}_\varepsilon - v \leq \tilde{c}\delta^\alpha r^2.
\]

We will now show that this bound and (54) lead to a contradiction. By (57), the definition of \( v \), and (54), we find
\[
\tilde{u}_\varepsilon(x_0) - v(x_0) \geq u_\varepsilon(x_0) - C_1 r^{2+\alpha} - v(x_0)
\]
\[
= u_\varepsilon(x_0) - C_1 r^{2+\alpha} - (v^-(x_0) + l(x_0) - \frac{m}{8R^2} r^2 + C_1 r^{2+\alpha} - \frac{K\varepsilon}{2\lambda} (|x - x_0|^2 - \frac{r^2}{4}))
\]
\[
\geq u_\varepsilon(x_0) - v^-(x_0) - l(x_0) + \frac{m}{8R^2} r^2 - 2C_1 r^{2+\alpha} - \frac{K\varepsilon r^2}{8\lambda}
\]
\[
= \frac{m}{8R^2} r^2 - 2C_1 r^{2+\alpha} - \frac{K\varepsilon r^2}{8\lambda}.
\]

Rearranging and using the bound (58) gives
\[
m \leq 8R^2 (r^{-2}(\tilde{u}_\varepsilon(x_0) - v(x_0)) + 2C_1 r^\alpha + \frac{K\varepsilon r^2}{8\lambda}) \leq 8R^2 (c_2 \delta^\alpha + 2C_1 r^\alpha + \frac{K\varepsilon r^2}{8\lambda}) \leq c_2 \delta^\alpha.
\]

But this contradicts (53); therefore,
\[
(59) \quad \sup_{x \in U_{\delta,h}^\ell} u_\varepsilon(x) - v^-(x) - c_1 \delta^{\frac{\alpha}{\alpha+1}} \leq c_2 \delta^\alpha.
\]
Since \( \|u\|_{C^{1,\gamma}(U)} \leq C\|g\|_{C^{1,\gamma}(\partial U)} + \|f\|_{L^{\infty}(U)} \), \( [v_h]_{C^{0,\gamma}(U_h)} \leq M \) and \( u \leq v_h \) on \( \partial U \), we obtain
\[
\sup_{U_h} u - v_h \leq \sup_{U_h^E} (u - v_h) + \sup_{E \cap U \cup U_h^E} (u - v_h) \leq \sup(u - v^-) + c_3 \gamma,
\]
where \( \gamma = \min \left\{ \frac{\alpha}{2\bar{\gamma}}, \eta, \alpha \right\} \) and \( c_3 \) depends on \( n, \lambda, \Lambda, \kappa, M, K, \|f\|_{L^{\infty}(U)}, \|g\|_{C^{1,\gamma}(\partial U)}, \text{diam } U \) and the regularity of \( \partial U \). By Proposition 4.1 and the bound (59), we find
\[
\sup_{U_{\delta,h}^E} u - v^- \leq \sup(u_c - v^-) + C\varepsilon \leq c_1 \bar{\gamma} + c_2 \gamma + C\varepsilon.
\]
Thus we conclude
\[
\sup_{U_h} u - v_h \leq \tilde{c}\bar{\gamma},
\]
where \( \tilde{c} \) depends on \( n, \lambda, \text{ and } \Lambda; \) and \( \bar{\gamma} \) depends on \( n, \lambda, \Lambda, \kappa, M, K, \|f\|_{C^{0,1}(U)}, \|g\|_{C^{1,\gamma}(\partial U)}, \text{diam } U \) and the regularity of \( \partial U \).

\[\square\]

9. Appendix A

In this section we recall the comparison principle for viscosity solutions ([8, Theorem 3.3]) and several related results.

**Proposition 9.1** (Comparison for viscosity solutions). Assume (H1). If \( u, v \in C(U) \) are, respectively, a subsolution and supersolution of \( F(D^2 u, x) = f(x) \) in \( U \) with \( u \leq v \) on \( \partial U \), then \( u \leq v \) in \( U \).

We also use the following lemma, which follows from Theorem 9.1 by a barrier argument.

**Lemma 9.2.** Assume (H1) and \( V \subset \mathbb{R}^n \) satisfies (U1). Assume that \( u \) is a solution of \( F(D^2 u, x) = f(x) \) in \( V \) and \( \bar{u} \) is a solution of
\[
\begin{cases}
F(D^2 \bar{u}, x) = f(x) + c & \text{in } V, \\
u = g & \text{on } \partial V,
\end{cases}
\]
for a positive constant \( c \). Then for all \( x \in V \) we have
\[
\bar{u}(x) \leq u(x) \leq \bar{u}(x) + \frac{c}{2\lambda} (\text{diam } V)^2.
\]

**Proof.** Since \( c > 0, \bar{u} \) is a subsolution of \( F(D^2 u, x) = f(x) \), so \( \bar{u} \leq u \) on \( V \) by Theorem 9.1.

We denote \( R = \text{diam } V \), so there exists \( x_0 \) such that \( V \subset B_R(x_0) \). For \( x \in V \) we define
\[
w(x) = \bar{u}(x) - \frac{c}{\lambda} \left( \frac{|x - x_0|^2}{2} - R^2 \right).
\]
If \( x \in \partial V \), then \( w(x) \geq \bar{u}(x) \). And, since \( F \) is uniformly elliptic, we have
\[
F(D^2 w, x) = F(D^2 \bar{u} - \frac{c}{\lambda} I, x) \leq F(D^2 \bar{u}, x) - c = f(x).
\]
Therefore, \( w \) is a supersolution of \( F(D^2 u, x) = f(x) \) on \( V \), so according to Theorem 9.1 we find that for all \( x \in V \),
\[
u(x) \leq w(x) \leq \bar{u}(x) + \frac{cR^2}{2\lambda}.
\]

\[\square\]

We state [8, Theorem 3.2], modified for our setting. This deep result was instrumental in establishing comparison for viscosity solutions; we use it in the proofs of Proposition 4.1 and Proposition 4.4.
Theorem 9.3. Suppose that $u, v \in C(U)$ are viscosity solutions of $F(D^2 u, x) = f(x)$ and $G(D^2 v, x) = g(x)$ in $U$. Suppose that $(x_a, y_a) \in U \times U$ is a local maximum of

$$u(x) - v(y) - \frac{a}{2} |x - y|^2.$$ 

Then there exist matrices $X$ and $Y$ that satisfy

$$(60) \quad - 3a \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{pmatrix} \leq \begin{pmatrix} X & 0 & 0 \\ 0 & -I & -Y \\ 0 & Y & I \end{pmatrix} \leq 3a \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{pmatrix},$$

and $F(X, x_a) = f(x_a)$, $G(Y, y_a) = g(y_a)$.

Together with Theorem 9.3, we use the following lemma of [9, Lemma III.1].

Lemma 9.4. There is a constant $C(n)$ such that if $(X, Y)$ are $n \times n$ matrices that satisfy (60) for some constant $a$, then

$$||X||, ||Y|| \leq C(n) \left\{ a^{1/2} ||X - Y||^{1/2} + ||X - Y|| \right\}.$$ 

Now we give the proof of Lemma 9.3.

Proof of Lemma 9.3. For any $x \in V$, we have

$$\sup_{y \in V} \left( v(x) - w(y) - \frac{a}{2} |x - y|^2 \right) = v(x) - \inf_{y \in V} \left( w(y) + \frac{a}{2} |x - y|^2 \right) \leq v(x) - w(x) + 2 ||Dw||_{L^\infty(V)} a^{-1},$$

where the inequality follows from the properties of inf-convolutions. Since $v = w$ on the boundary of $V$, we find

$$\sup_{y \in B_1, x \in \partial V} \left( v(x) - w(y) - \frac{a}{2} |x - y|^2 \right) \leq 2 ||Dw||_{L^\infty(V)} a^{-1}.$$ 

Similarly, if $y \in \partial V$, then

$$\sup_{x \in V, y \in \partial V} \left( v(x) - w(y) - \frac{a}{2} |x - y|^2 \right) \leq 2 ||Dv||_{L^\infty(V)} a^{-1}.$$ 

These two bounds imply the first claim of the lemma. We now proceed to give the proof of the second claim. By the definition of $(x_a, y_a)$ as a point at which the supremum is achieved, we have, for any $(x, y) \in V \times V$,

$$u(x_a) - v(y_a) - \frac{a}{2} |x_a - y_a|^2 \geq u(x) - v(y) - \frac{a}{2} |x - y|^2,$$

so in particular, this inequality holds with $(x, y) = (x_a, y_a)$. This implies

$$u(x_a) - \frac{a}{2} |x_a - y_a|^2 \geq u(y_a),$$

so we find

$$\frac{a}{2} |x_a - y_a|^2 \leq u(x_a) - u(y_a) \leq ||Du||_{L^\infty(V)} |x_a - y_a|,$$

from which we easily conclude $|x_a - y_a| \leq 2a^{-1} ||Du||_{L^\infty(V)}$. We find $|x_a - y_a| \leq 2a^{-1} ||Dv||_{L^\infty(V)}$ in a similar way. □
10. Appendix B

This section is dedicated to recalling the definitions and important properties of inf- and sup-convolutions, including the regularity theorem [6, Proposition 1.2] of Caffarelli and Souganidis that is crucial to our arguments.

The sup-convolution and inf-convolution of a function \( v \in C(U) \) and for \( \theta > 0 \) are defined as

\[
\begin{align*}
\inf_{\theta, y} u^\theta_+(x) &= \inf_{y \in U} \left\{ u(y) - \frac{|x - y|^2}{2\theta} \right\}, \\
\sup_{\theta, y} u^\theta_-(x) &= \sup_{y \in U} \left\{ u(y) + \frac{|x - y|^2}{2\theta} \right\}. 
\end{align*}
\]

For given \( \theta \) and \( \delta \), we define the set

\[
U_\delta = \{ x \in U : d(x, \partial U) > 2\theta^{1/2}||u||_{L^\infty(U)} + \delta \}.
\]

Next we summarize some basic properties of inf and sup convolutions. We refer to [7, Proposition 5.3] for its proof.

**Proposition 10.1.** Assume \( u \in C(U) \).

1. If \( x^* \) denotes a point where the supremum (resp. infimum) is achieved in the definition of \( u^\theta_+(x) \) (resp. \( u^\theta_-(x) \)), then

\[
|x - x^*| \leq 2\theta^{1/2}||u||_{L^\infty(U)}.
\]

   If, additionally, \( u \in C^{0,1}(U) \), then

\[
|x - x^*| \leq 2\theta ||Du||_{L^\infty(U)}.
\]

2. In the sense of distributions, \( D^2u^\theta_+(x) \geq -\theta^{-1}I \) and \( D^2u^\theta_-(x) \leq \theta^{-1}I \) for all \( x \in U \). Moreover, \( u^+_\theta \) and \( u^-_\theta \) are twice differentiable almost everywhere in \( U \).

3. If \( u \in C^{0,1}(U) \), then for all \( x \in U \),

\[
0 \leq (u^+_\theta - u)(x) \leq 2||Du||_{L^\infty(U)}\theta, \text{ and } 0 \leq (u^-_\theta - u)(x) \leq 2||Du||_{L^\infty(U)}\theta.
\]

4. Define \( \nu = 4\theta^{1/2}||u||_{L^\infty(U)}^{1/2} \). If \( u \) is a subsolution of \( (1) \) in \( U \), then \( u^\theta_+ \) is a subsolution of

\[
F_u(D^2u, x) = f_u(x) \text{ in } U_\delta^\theta;
\]

if \( u \) is a supersolution of \( (1) \) in \( U \), then \( u^\theta_- \) is a supersolution of

\[
F_u(D^2u, x) = f_u(x) \text{ in } U_\delta^\theta.
\]

(The perturbed equations \( F^\nu \) and \( F_\nu \), and \( f_\nu \) and \( f^\nu \), are defined in the beginning of Section 4.)

An analogous result to item 4 of Proposition 10.1 holds for \( \delta \)-solutions of \( (1) \):

**Proposition 10.2.** Assume that \( v \in C(U) \). Let \( \nu = 4\theta^{1/2}||v||_{L^\infty(U)}^{1/2} \). If \( v \) is a \( \delta \)-subsolution of \( (1) \) in \( U \), then \( v^\theta_+ \) is a \( \delta \)-subsolution of

\[
F^\nu(D^2v, x) = f_v(x) \text{ in } U_\delta^\theta;
\]

if \( v \) is a \( \delta \)-supersolution of \( (1) \) in \( U \), then \( v^\theta_- \) is a \( \delta \)-supersolution of

\[
F^\nu(D^2v, x) = f_v(x) \text{ in } U_\delta^\theta.
\]

We omit the proof; it is very similar to that of item 4 of Proposition 10.1, which may be found in [7, Proposition 5.3].

Next we state the regularity result [6, Proposition 1.2]. It is a result about the regularity of inf- and sup-convolutions of solutions to a uniformly elliptic equation with fixed coefficients.
Proposition 10.3. Assume \( (\forall x_0) \), \( f \in C^{0,1}(U) \) and fix some \( x_0 \) and \( \hat{x} \) in \( U \). Let \( w \in C^{0,1}(B_{\rho}(\hat{x})) \) be a viscosity solution of
\[
F(D^2w, x_0) = f(x) \text{ in } B_{\rho}(\hat{x}).
\]
We denote \( B_{\rho}^\theta(\hat{x}) = B(\hat{x}, \rho - 2|Dw||_{L^\infty(B_{\rho}(\hat{x}))}) \).

There exist universal constants \( \sigma \), \( t_0 \) and \( C \) such that for any \( t > t_0 \) there exists an open set \( A_t^+ \subset B_{\rho}^\theta(\hat{x}) \) (respectively, \( A_t^- \subset B_{\rho}^\theta(\hat{x}) \)) such that
\[
|B_{\rho}^\theta(\hat{x}) \setminus A_t^+| \leq C\rho^{n-\sigma}(|Du||_{L^\infty(B_{\rho}(\hat{x}))} + |Df||_{L^\infty(B_{\rho}(\hat{x}))})t^{-\sigma}
\]
and for all \( x \in A_t^+ \cap B_{\rho/2}(\hat{x}) \) there exists a quadratic polynomial \( P \) such that
\[
F(D^2P, x_0) = 0
\]
and, for all \( y \in B_{\rho}^\theta(\hat{x}) \),
\[
w^{\theta, +}(y) \geq w^{\theta, +}(x) + P(y) - Cr^{-1}t|y - x|^3
\]
(respectively, \( w^{\theta, -}(y) \leq w^{\theta, -}(x) + P(y) + Cr^{-1}t|y - x|^3 \)).

10.1. Inf and sup convolutions of mesh functions. We summarize some basic properties of inf and sup convolutions of mesh functions.

Proposition 10.4. Assume \( v \in C^{0,0}(U_h) \).

1. If \( x^* \in U_h \) denotes a point where the supremum (resp. infimum) is achieved in the definition of \( v_h^{\theta, +}(x) \) (resp. \( v_h^{\theta, -}(x) \)), then
\[
|x - x^*| \leq 4||vh||_{L^\infty(U_h)}^{1/2} + \sqrt{\eta}h.
\]
2. In the sense of distributions, \( D^2v^{\theta, +}(x) \geq \theta^{-1}I \) and \( D^2v^{\theta, -}(x) \leq \theta^{-1}I \) for all \( x \in U \).
Moreover, \( v^{\theta, +} \) and \( v^{\theta, -} \) are twice differentiable almost everywhere in \( U \).
3. There exists a constant \( C \) that depends on \( ||v||_{C^{0,\sigma}(U)} \) such that for all \( x \in U_h \),
\[
0 \leq (v^{\theta, +} - v)(x) \leq C\theta^{\frac{n}{2-\sigma}} \text{ and }
0 \leq (v - v^{\theta, -})(x) \leq C\theta^{\frac{n}{2-\sigma}}.
\]

Proof of Proposition 10.4. For \( x \in U \), we denote by \( x_h \) an element of the mesh that is closest to \( x \). Note \( |x - x_h| \leq \sqrt{\eta}h \).

Let \( x^* \in U_h \) be a point where the supremum is achieved in the definition of \( v_h^{\theta, +}(x) \). Then
\[
v_h(x^*) - \frac{|x - x^*|^2}{2\theta} \geq v_h(x_h) - \frac{|x_h - x|^2}{2\theta}.
\]
Therefore,
\[
\frac{|x - x^*|^2}{2\theta} \leq v_h(x^*) - v_h(x_h) + \frac{\eta h^2}{2\theta} \leq 2||v_h||_{L^\infty(U_h)} + \frac{\eta h^2}{2\theta},
\]
which easily implies the desired bound. The proof for \( v_h^{\theta, -}(x) \) is very similar. \( \square \)

We refer the reader to [10, Proposition 2.3] for the proof of the rest of Proposition 10.4.

Acknowledgements

The author thanks her thesis advisor, Professor Takis Souganidis, for suggesting this problem and for his guidance and encouragement.
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