EINSTEIN-LIKE DOUBLY WARPED PRODUCT MANIFOLDS

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Abstract. In this paper, it is proved that the factor manifolds $M_i$, $i = 1, 2$ of a doubly warped product manifold $M = f_1 M_1 \times f_2 M_2$ acquire the Einstein-like class type $A, B, P, I \cap B, I \cap A, \text{or } A \cap B$ of $M$ by imposing a sufficient condition on the warping functions in each case. As an application, Einstein-like doubly warped product spacetimes of type $A, B$ or $P$ are considered.

1. An introduction

Einstein-like manifolds are natural extension of Einstein manifolds. This concept is deeply studied by Alfred Gray in [25] and Arthur Besse in [5, Chapter 16] (see also [29,31]). Gray presented an interesting decomposition of the covariant derivative of the Ricci tensor to orthogonal classes of Einstein-like manifolds. The same decomposition is discussed extensively in [5, Chapter 16]. Einstein-like manifolds admitting different curvature conditions were considered by G. Calvaruso in [9,12]. Einstein-like manifolds of dimension 3 are studied in [17,18] whereas of dimension 4 are considered in [12]. Projective spaces and spheres furnished with class $A$ or class $B$ Einstein-like metrics were classified in [35]. An interesting study in [31] shows Einstein-like Generalized Robertson-Walker space-times are perfect fluid space-times except one class of Gray’s decomposition which is not restricted. Sufficient conditions on manifolds in this class to be a perfect fluid are derived in [15]. The concept of Einstein manifolds has many other generalizations such as Ricci soliton, $(m, \rho)$–quasi-Einstein manifolds [17], generalized quasi-Einstein manifolds [28], mixed quasi-Einstein manifolds [40] and $m$–quasi-Einstein manifolds [27].

Doubly warped products is a generalization of singly warped products introduced by Bishop and O’Neil in 1969. The geometric properties of doubly warped product manifolds have been investigated by many authors such as pseudo-convexity in [3], hyperbolicity in [2], harmonic Weyl conformal curvature tensor in [21], conformal flatness in [23,24], geodesic completeness in [41], and doubly warped product submanifolds in [20,33,34,36]. Doubly warped space-times are widely used as exact solutions of Einstein’s field equations. Recently, a non-existence result of non-trivial compact Einstein doubly warped product manifolds is considered in [26]. This result is a negative answer to the Besse’s question about the existence of non-trivial compact Einstein warped product manifolds.

Inspired by these studies and by a recent study of Einstein-like metrics on warped product manifolds(see [30]), we investigated Einstein-like doubly warped product manifolds as well as Einstein-like doubly warped space-times. Sufficient conditions

2010 Mathematics Subject Classification. Primary 53C21; Secondary 53C50, 53C80.

Keywords and phrases. Einstein-like manifolds, doubly warped manifolds, Gray decomposition, doubly warped space-times.
on the warping functions that guarantee the factor manifolds inheritance of the
Einstein-like class type \( A, B, P, I \oplus A, I \oplus B \) or \( A \oplus B \). One can recall the results of \([30]\) on singly warped product manifolds by imposing one of the warping functions
as constant. Finally, doubly warped space-times of Classes \( A, B, \) and \( P \) as an
application.

2. Preliminaries

The (pseudo-)Riemannian product manifold \( M = M_1 \times M_2 \) of two (pseudo-
)Riemannian manifolds \( (M_i, g_i, D_i), i = 1, 2, \) equipped with the metric tensor
\[
g = (f_2 \circ \pi_2)^2 \pi_1^* (g_1) \oplus (f_1 \circ \pi_1)^2 \pi_2^* (g_2),
\]
where the functions \( f_i : M_i \rightarrow (0, \infty), i = 1, 2 \) is called a doubly warped product
manifold and is denoted by \( M = f_2 M_1 \times f_1 M_2 \). The natural projection maps of
the Cartesian product \( M_1 \times M_2 \) onto \( M_i \) are \( \pi_i : M_1 \times M_2 \rightarrow M_i \), where \( i = 1, 2 \). The functions \( f_i \) are called the warping functions of the warped product manifold
\( M_i \) and \( \ast \) denotes the pull-back operator on tensors. In particular, if for example
\( f_2 = 1 \), then \( M = M_1 \times f_1 M_2 \) is called a singly warped product manifold (see \([18,41]\)
for doubly warped products and \([6,10,19,32,38,39]\) for singly warped products).
Throughout this work, all tensors on factor manifolds are identified with their lifts
to the product manifold and consequently we use the same notation for both of
them. For example, we use the same notation for a function \( f_i \) on \( M_i \) and for its
lift \((f_i \circ \pi_i) \) on \( M \). Let \((M, g, D)\) be a pseudo-Riemannian doubly warped product
manifold of two pseudo-Riemannian \((M_i, g_i, D_i), i = 1, 2\) with dimensions \( n_i \),
where \( n = n_1 + n_2 \) and \( \text{Ric} \), \( \text{Ric}^i \) be the Ricci curvature tensors on \( M, M^i \) respectively.
The gradient and Laplacian of \( f_i \) on \( M_i \) are denoted by \( \nabla^i f_i \) and \( \Delta^i f_i \) whereas
\( f_i^o = f_i \Delta^i f_i + (n_j - 1) g_i (\nabla^i f_i, \nabla^i f_i), i \neq j \). From now and on, we will use the indices \( i \) and \( j \) to denote the geometric objects of the factor manifolds \( M_i \) and \( M_j \).
Example, \( \text{Ric}^i \) is the Ricci tensor of \((M_i, g_i)\).

The Levi-Civita connection \( D \) on \( M = f_2 M_1 \times f_1 M_2 \) is given by
\[
D_{X_i} X_j = X_i (\ln f_i) X_j + X_j (\ln f_j) X_i,
\]
\[
D_{X_i} Y_j = D^i_{X_i} Y_j - \frac{f_2^2}{f_1^2} g_i (X_i, Y_i) \nabla^j (\ln f_j),
\]
where \( i \neq j \) and \( X_i, Y_i \in \mathfrak{X} (M_i) \). Then the Ricci curvature tensor \( \text{Ric} \) on \( M \) is
given by
\[
\text{Ric} (X_i, Y_j) = \text{Ric}^i (X_i, Y_i) - \frac{n_j}{f_1} H^{f_i} (X_i, Y_i) - \frac{f_2}{f_1^2} g_i (X_i, Y_i),
\]
\[
\text{Ric} (X_i, Y_j) = (n - 2) X_i (\ln f_i) Y_j (\ln f_j),
\]
where \( i \neq j \) and \( X_i, Y_i, Z_i \in \mathfrak{X} (M_i) \). The reader is referred to \([13,14,22]\) for some
studies of curvature conditions on warped product manifolds.

3. Einstein-like doubly warped product manifolds

This section is devoted to the study of Einstein-like doubly warped product mani-
folds of Class \( A, B \) or Class \( P \). The proofs are too lengthy and consequently
are moved to the Appendix. For simplicity, let us define the \((0, 2)\) tensors \(\mathcal{F}^i\) as follows

\[
\mathcal{F}^i (X_i, Y_i) = \frac{n_i}{f_i} H^i (X_i, Y_i),
\]

where \(i, j = 1, 2, i \neq j\) and \(X_i \in \mathfrak{X}(M_i)\).

### 3.1. Einstein-like manifolds in Class \(\mathcal{A}\)

A pseudo-Riemannian manifold \((M, g)\) admitting a cyclic parallel Ricci tensor, that is,

\[
(D_X \text{Ric}) (Y, Z) + (D_Y \text{Ric}) (Z, X) + (D_Z \text{Ric}) (X, Y) = 0,
\]

for any vector fields \(X, Y, Z \in \mathfrak{X}(M)\) are called Einstein-like of class \(\mathcal{A}\). It is noted that the above identifying condition of this class is equivalent to

\[
(D_X \text{Ric}) (X, X) = 0,
\]

for any vector field \(X \in \mathfrak{X}(M)\). The inheritance property of Class \(\mathcal{A}\) is controlled by the following result.

**Theorem 1.** Let \(M = f_2 M_1 \times f_1 M_2\) be an Einstein-like doubly warped product manifold of Class \(\mathcal{A}\). Then \((M_1, g_1)\) is an Einstein-like manifold of Class \(\mathcal{A}\) if and only if

\[
(D_X \text{Ric}) (X, X) = (D_X \text{Ric}) (X, X) - 2 \text{Ric} (D_X X, X).
\]

Thus, for the special case where \(X = X_i\) lands on one factor, one may get

\[
0 = (D_X \text{Ric}) (X_i, X_i)
\]

\[
= X_i \left( \text{Ric}^i (X_i, X_i) - \mathcal{F}^i (X_i, X_i) \right)
\]

\[
-2 \text{Ric}^i (D_X^i X_i, X_i) + 2 \mathcal{F}^i (D_X^i X_i, X_i) + 2 \frac{2}{f_i} g_i (D_X^i X_i, X_i)
\]

\[
+2 (n - 2) \frac{1}{f_i^2} \left( \nabla^j f_j \right) (f_j) X_i (f_j) g_i (X_i, X_i).
\]

Thus, after lengthy computations, it is

\[
0 = (D_X \text{Ric}^i) (X_i, X_i) - (D_X \mathcal{F}^i) (X_i, X_i)
\]

\[
+ \frac{2}{f_i} X_i (f_i) g_i (X_i, X_i) \left[f_j^i + (n - 2) \left( \nabla^j f_j \right) (f_j) \right].
\]

These equations complete the proof. \(\square\)

For a singly warped product manifold, one can recall the following result.

**Corollary 1.** Let \(M = M_1 \times f_1 M_2\) be an Einstein-like singly warped product manifold of Class \(\mathcal{A}\). Then \((M_1, g_1)\) is an Einstein-like manifold of Class \(\mathcal{A}\) if and only if \(\mathcal{F}^i\) is cyclic parallel. Moreover, \((M_2, g_2)\) is an Einstein-like manifold of Class \(\mathcal{A}\).
3.2. **Einstein-like manifolds in Class \( \mathcal{B} \).** If the Ricci tensor is a Codazzi tensor, i.e.,
\[
(D_X \text{Ric}) (Y, Z) = (D_Y \text{Ric}) (X, Z),
\]
then \((M, g)\) is called Einstein-like of class \( \mathcal{B} \). This condition is equivalent to one of the following conditions: (i) The Riemann tensor is harmonic, or (ii) The Weyl conformal tensor is harmonic and the scalar curvature is constant. The factor manifolds gain the Einstein-like Class \( \mathcal{B} \) according to the following result.

**Theorem 2.** Let \( M = f_2 M_1 \times f_1 M_2 \) be an Einstein-like doubly warped product manifold of Class \( \mathcal{B} \). Then \((M_i, g_i)\) is an Einstein-like manifold of Class \( \mathcal{B} \) if and only if
\[
(D^i_{X}, \mathcal{F}^i) (Y_i, Z_i) = (D^i_{Y_i} \mathcal{F}^i) (X_i, Z_i)
+ \frac{1}{f_i^2} X_i (f_i) g_i (Y_i, Z_i) \left( 2 f_j^o - (n - 2) \left( \nabla^j f_j \right) f_j \right)
- \frac{1}{f_i^2} Y_i (f_i) g_i (X_i, Z_i) \left( 2 f_j^o - (n - 2) \left( \nabla^j f_j \right) f_j \right),
\]
where \(i, j = 1, 2, i \neq j\) and \(X_i, Y_i, Z_i \in \mathfrak{X} (M_i)\).

**Proof.** Let us define the deviation tensor \( B (X, Y, Z) \) as follows
\[
B (X, Y) Z = (D_X \text{Ric}) (Y, Z) - (D_Y \text{Ric}) (X, Z).
\]
There are three different cases. Let us consider the first case, that is,
\[
B (X_i, Y_i, Z_i) = (D_X \text{Ric}) (Y_i, Z_i) - (D_Y \text{Ric}) (X_i, Z_i).
\]
It is enough to find \((D_X \text{Ric}) (Y_i, Z_i)\) as
\[
(D_X \text{Ric}) (Y_i, Z_i) = X_i \left( \text{Ric}^i (Y_i, Z_i) \right) - X_i \left( \mathcal{F}^i (Y_i, Z_i) \right) - f_j^o X_i \left( \left( \frac{1}{f_i^2} g_i (Y_i, Z_i) \right) \right)
- \text{Ric}^i \left( D^i_X Y_i, Z_i \right) + \mathcal{F}^i \left( D^i_X Y_i, Z_i \right) + \frac{f_j^o}{f_i^2} g_i \left( D^i_X Y_i, Z_i \right)
- \text{Ric}^i \left( Y_i, D^i_X Z_i \right) + \mathcal{F}^i \left( Y_i, D^i_X Z_i \right) + \frac{f_j^o}{f_i^2} g_i \left( Y_i, D^i_X Z_i \right)
+ (n - 2) \frac{1}{f_i^2} g_i \left( X_i, Y_i \right) \nabla^j f_j (f_j) Z_i (f_i)
+ (n - 2) \frac{1}{f_i^2} g_i \left( X_i, Z_i \right) \nabla^j f_j (f_j) Y_i (f_i).
\]
Simplifying this expression, it is
\[
(D_X \text{Ric}) (Y_i, Z_i) = \left( D^i_X \text{Ric}^i \right) (Y_i, Z_i) - \left( D^i_X \mathcal{F}^i \right) (Y_i, Z_i) + 2 \frac{f_j^o}{f_i^2} X_i (f_i) g_i (Y_i, Z_i)
+ (n - 2) \frac{1}{f_i^2} g_i \left( X_i, Y_i \right) \nabla^j f_j (f_j) Z_i (f_i)
+ (n - 2) \frac{1}{f_i^2} g_i \left( X_i, Z_i \right) \nabla^j f_j (f_j) Y_i (f_i).
\]
By exchanging \(X_i\) and \(Y_i\) in the last equation and substitution in Equation (3.1), one gets the deviation tensor. For Einstein-like manifolds of Class \( \mathcal{B} \), the deviation tensor vanishes from which the result hold. \( \square \)
The constancy of one of the warping function will simplify the above result as follows.

**Corollary 2.** Let $M = M_1 \times f_1 M_2$ be an Einstein-like singly warped product manifold of Class $\mathcal{B}$. Then $(M_1, g_1)$ is an Einstein-like manifold of Class $\mathcal{B}$ if and only if
\[
(D_{X_i} \mathcal{F}^1)(Y_i, Z_i) = (D_{Y_i} \mathcal{F}^1)(X_i, Z_i),
\]
where $X_i, Y_i, Z_i \in \mathfrak{X}(M_1)$. $(M_2, g_2)$ is Einstein-like of Class $\mathcal{B}$.

3.3. **Einstein-like manifolds in Class $\mathcal{P}$**. Manifolds having a parallel Ricci curvature tensor, i.e.,
\[
(D_X \text{Ric})(Y, Z) = 0,
\]
are called Einstein-like of class $\mathcal{P}$. The Ricci tensor in this case is parallel and manifolds admitting a parallel Ricci tensor are called Ricci symmetric manifolds.

**Theorem 3.** Let $M = f_2 M_1 \times f_1 M_2$ be an Einstein-like doubly warped product manifold of Class $\mathcal{P}$. Then $(M_i, g_i)$ is an Einstein-like manifold of Class $\mathcal{P}$ if and only if
\[
(D_{X_i} \text{Ric}^i)(Y_i, Z_i) = \frac{n-2}{f_i} [g_i(X_i, Y_i) Z_i(f_i) + g_i(X_i, Z_i) Y_i(f_i)] \Delta^i f_j f_j + 2 \frac{f_i}{f_i} X_i(f_i) g_i(Y_i, Z_i),
\]
where $i, j = 1, 2, i \neq j$ and $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$.

**Proof.** Let $M = f_2 M_1 \times f_1 M_2$ be a Ricci symmetric doubly warped product manifold, that is,
\[
0 = (D_X \text{Ric})(Y, Z).
\]
Equation (3.2) infers
\[
(D_X \text{Ric})(Y_i, Z_i) = (D_{X_i} \text{Ric}^i)(Y_i, Z_i) - (D_{X_i} \mathcal{F}^i)(Y_i, Z_i) + 2 \frac{f_i}{f_i} X_i(f_i) g_i(Y_i, Z_i)
\]
\[+ (n-2) \frac{1}{f_i} g_i(X_i, Y_i) \Delta^j f_j f_i Z_i(f_i)
\]
\[+ (n-2) \frac{1}{f_i} g_i(X_i, Z_i) \Delta^j f_j f_j Y_i(f_i).
\]
Thus, having a parallel Ricci tensor implies
\[
(D_{X_i} \text{Ric}^i)(Y_i, Z_i) = (D_{X_i} \mathcal{F}^i)(Y_i, Z_i) - 2 \frac{f_i}{f_i} X_i(f_i) g_i(Y_i, Z_i)
\]
\[- \frac{n-2}{f_i} [g_i(X_i, Y_i) Z_i(f_i) + g_i(X_i, Z_i) Y_i(f_i)] \Delta^j f_j f_j.
\]
This equation completes the proof.

We end this section by recalling the above result for singly warped product manifolds.

**Corollary 3.** Let $M = f_2 M_1 \times f_1 M_2$ be an Einstein-like doubly warped product manifold of Class $\mathcal{P}$. Then $(M_1, g_1)$ is an Einstein-like manifold of Class $\mathcal{P}$ if and only if
\[
(D_{X_1} \mathcal{F}^1)(Y_1, Z_1) = 0,
\]
where $X_1, Y_1, Z_1 \in \mathfrak{X}(M_1)$. Also, $(M_2, g_2)$ is Einstein-like of Class $\mathcal{P}$. 
The divergence of the Weyl tensor is given by [5]

\[ \nabla_{\gamma} \left[ R_{\alpha\beta} - \frac{R}{2(n-1)} g_{\alpha\beta} \right] = \nabla_{\alpha} \left[ R_{\gamma\beta} - \frac{R}{2(n-1)} g_{\gamma\beta} \right], \]

that is, the tensor \( R_{\alpha\beta} - \frac{R}{2(n-1)} g_{\alpha\beta} \) is a Codazzi tensor. This condition is equivalent to

\[ \nabla_{\varepsilon} C_{\alpha\beta\gamma} = 0, \]

where \( C \) is the Weyl conformal curvature tensor, i.e., \( M \) has a harmonic Weyl tensor. Let \( g_{\beta\gamma} = \varphi^2 \bar{g}_{\beta\gamma} \) be a conformal change of on a manifold \( M \). It is well known that the Weyl tensor \( C_{\alpha\beta\gamma} \) is invariant, that is, \( \bar{C}_{\alpha\beta\gamma} = C_{\alpha\beta\gamma} \) however \( C_{\alpha\beta\gamma\varepsilon} = \varphi^2 \bar{C}_{\alpha\beta\gamma\varepsilon} \).

The divergence of the Weyl tensor is given by [5]

\[ \nabla_{\varepsilon} C_{\alpha\beta\gamma} = \nabla_{\varepsilon} \bar{C}_{\alpha\beta\gamma} - \frac{n-3}{\varphi^2} (\nabla_{\varepsilon} \varphi) \bar{C}_{\alpha\beta\gamma}. \]

The doubly warped product metric may be rewritten as follows

\[ g = f_1^2 f_2^2 (f_1^{-2} g_1 + f_2^{-2} g_2) = f_1^2 f_2^2 (\bar{g}_1 + \bar{g}_2) = f_1^2 f_2^2 \bar{g} \]

where \( g_i = f_i^2 \bar{g}_i \) and \( \bar{g} = \bar{g}_1 + \bar{g}_2 \). The doubly warped product manifold \( (M, g) \) has harmonic Weyl tensor if and only

\[ \nabla_{\varepsilon} \bar{C}_{\alpha\beta\gamma} = \nabla_{\varepsilon} (f_1 f_2) \bar{C}_{\alpha\beta\gamma} = 0, \]

having a harmonic Weyl tensor is equivalent to the condition

\[ 0 = \bar{T}_{\alpha\beta\gamma} \]

\[ = \nabla_{\beta} R_{\alpha\beta} - \nabla_{\alpha} R_{\beta\alpha} - \frac{1}{2(n-1)} \left[ (\nabla_{\gamma} \bar{R}) \bar{g}_{\alpha\beta} - (\nabla_{\alpha} \bar{R}) \bar{g}_{\beta\alpha} \right], \]

where \( \bar{T} \) is the Cotton tensor. The metric \( \bar{g} \) splits as \( \bar{g} = \bar{g}_1 + \bar{g}_2 \) and consequently the divergence of the Cotton tensor \( \bar{T} \) splits on the factor manifolds \( (M_i, \bar{g}_i) \) as

\[ 0 = \bar{T}_{\alpha\beta\gamma}^i + \frac{n_2}{2(n-1)(n_1-1)} \left[ (\nabla_{\gamma} \bar{R}^i)_{\alpha\beta} - (\nabla_{\alpha} \bar{R}^i) \bar{g}_{\beta\alpha} \right]. \]

In this case, \( \nabla_{\gamma} \bar{R}^i \bar{g}_{\alpha\beta} = 0 \), that is, \( \bar{R}^i \) is constant if and only if the cotton tensor \( \bar{T}^i \) on the doubly warped factor manifolds \( (M^i, \bar{g}_i) \) vanishes i.e.

\[ \nabla_{\varepsilon} \bar{T}^i_{\alpha\beta\gamma} = 0. \]

The Weyl tensors \( \bar{C}^i \) on doubly warped product factor manifolds \( (M_i, g_i) \) satisfy

\[ 0 = \nabla_{\varepsilon} \bar{C}^i_{\alpha\beta\gamma} \]

\[ = \nabla_{\varepsilon} C^i_{\alpha\beta\gamma} + \frac{n_i - 3}{f_i} (\nabla_{\varepsilon} f_i) C^i_{\alpha\beta\gamma}. \]

It is time now to write the following result.
Theorem 4. Let \( M = f_2 \ M_1 \times f_1 \ M_2 \) be an Einstein-like doubly warped product manifold of Class \( \mathcal{I} \oplus \mathcal{A} \). Assume that \( \nabla (f_1 f_2) \ C_{\alpha \beta \gamma, \tau} = 0 \) and the conformal change \((M_i, f_i^{-2} g_i)\) has a constant scalar curvature. Then \((M_i, g_i)\) is an Einstein-like manifold of Class \( \mathcal{I} \oplus \mathcal{B} \) if and only if \( (\nabla^i f_i) \ C_{\alpha \beta \gamma, \tau}^{i} = 0 \) for each \( i = 1, 2 \).

\[ \text{Proof.} \] Assume that \( M = f_2 \ M_1 \times f_1 \ M_2 \) be an Einstein-like doubly warped product manifold of Class \( \mathcal{I} \oplus \mathcal{A} \). Then \((M_i, g_i)\) is an Einstein-like manifold of Class \( \mathcal{I} \oplus \mathcal{A} \) if and only if

\[
(D_{X, i} \mathcal{F}^i) (X_i, X_i) = \frac{2}{f_i} f_i (f_i) g_i (X_i, X_i) [f_i^2 + (n - 2) (\nabla^j f_j) (f_j)] - \frac{2}{n + 2} \left( D_X R - \frac{n + 2}{n + 2} D_{X, i} R^i \right) g_i (X_i, X_i)
\]

Using equation (3.2), it is

\[
0 = (D_X) \left( \text{Ric} - \frac{2R}{n + 2} g \right) (X, X)
\]

\[
= (D_X \text{Ric}) (X, X) - \frac{2}{n + 2} g (X, X) D_X R
\]

and consequently, one has

\[
0 = (D_{X, i} \text{Ric}^i) (X_i, X_i) - (D_{X, i} \mathcal{F}^i) (X_i, X_i)
\]

\[
\frac{2}{f_i} f_i (f_i) g_i (X_i, X_i) [f_i^2 + (n - 2) (\nabla^j f_j) (f_j)]
\]

\[
- \frac{2}{n + 2} (D_{X, i} R) g_i (X_i, X_i)
\]

which completes the proof. \( \Box \)
3.6. Einstein-like manifolds in Class $\mathcal{A} \oplus \mathcal{B}$. This class is identified by having a constant scalar curvature. Let $M = f_1, M_1 \times f_2, M_2$ be an Einstein-like doubly warped product manifold of Class $\mathcal{A} \oplus \mathcal{B}$, that is, the scalar curvature $R$ of $M$ is constant, say $c$. The use of Equation 7 in [21] implies that $M_i$ is of Class $\mathcal{A} \oplus \mathcal{B}$ if there is a constant $c_i$ such that

$$\frac{c_i}{f_j^2} + \frac{c_j}{f_i^2} - \frac{n_i(n_i - 1)}{f_j^2} \Delta_j f_j - \frac{n_j(n_j - 1)}{f_i^2} \Delta_i f_i - \frac{2n_i}{f_j} F_j - \frac{2n_j}{f_i} F_i = c,$$

where $F_i = g^{ij} \nabla^i \nabla^j f_i$.

4. On doubly warped product space-times

Let $\tilde{M} = f I \times \sigma M$ be a doubly warped product space-time furnished with the metric tensor $\tilde{g} = - f^2 dt^2 \oplus \sigma^2 g$ where $(M, g)$ is a Riemannian manifold, $f : M \to (0, \infty)$ and $\sigma : I \to (0, \infty)$ are smooth functions. Then the Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$ is given by

$$\tilde{\nabla}_{\partial_t} \partial_t = \frac{f}{\sigma^2} \nabla f,$$

$$\tilde{\nabla}_{\partial_t} X = D_X \partial_t = \frac{\dot{g}}{g} X + \frac{1}{f} X (f) \partial_t,$$

$$\tilde{\nabla}_X Y = D_X Y - \frac{\sigma \dot{\sigma}}{f^2} g(X, Y) \partial_t,$$

where $X, Y \in \mathfrak{X}(M)$. Then the Ricci curvature tensor $\tilde{\text{Ric}}$ on $\tilde{M}$ is given by

$$\tilde{\text{Ric}}(\partial_t, \partial_t) = \frac{n}{\sigma^2} \frac{\dot{\sigma}}{\sigma} + \frac{f^2}{\sigma^2}$$

$$\tilde{\text{Ric}}(X, Y) = \text{Ric}(X, Y) - \frac{1}{f^2} H^f (X, Y) - \frac{\sigma \dot{\sigma}}{f^2} g(X, Y)$$

$$\tilde{\text{Ric}}(\partial_t, X) = (n - 1) \frac{\dot{\sigma}}{\sigma} X (\ln f)$$

where $X, Y, Z \in \mathfrak{X}(M)$. The reader is referred to [18] and [37] and references therein for the definition and physical significance of doubly warped space-times.

**Theorem 6.** Let $\tilde{M} = f I \times \sigma M$ be an Einstein-like doubly warped product space-time of Class $\mathcal{A}$. Then $(M, g)$ is an Einstein-like manifold of Class $\mathcal{A}$ if and only if

$$(D_X \mathcal{F})(X, X) = \left((n - 1) \frac{\dot{\sigma}^2 + \sigma^2}{\sigma^2} \right) \frac{2}{f^2} X (f) g(X, X).$$

**Theorem 7.** Let $\tilde{M} = f I \times \sigma M$ be an Einstein-like doubly warped product space-time of Class $\mathcal{B}$. Then $(M, g)$ is an Einstein-like manifold of Class $\mathcal{B}$ if and only if

$$(D_X \mathcal{F})(Y, Z) = (D_Y \mathcal{F})(X, Z) + (2\sigma^2 - (n - 1) \dot{\sigma}^2) \frac{1}{f^2} X (f) g(Y, Z)$$

$$- (2\sigma^2 + (n - 1) \dot{\sigma}^2) \frac{1}{f^2} Y (f) g(X, Z).$$
Theorem 8. Let $\tilde{M} = I \times_{\sigma} M$ be an Einstein-like doubly warped product space-time of Class $P$. Then $(\tilde{M}, g)$ is an Einstein-like manifold of Class $P$ if and only if

$$(Dx\mathcal{F}) (Y, Z) = 2\frac{\sigma}{f^3} X(f) g(Y, Z) + \frac{\sigma^2}{f^3} (n - 1) (g(X, Z) Y(f) + g(X, Y) Z(f)).$$

References

[1] Agaoka, Yoshio, In-Bae Kim, Byung Hak Kim, and Dae Jin Yeom. "On doubly warped product manifolds." Memoirs of the Faculty of Integrated Arts and Sciences, Hiroshima University. IV, Science reports: studies of fundamental and environmental sciences 24 (1998): 1-10.

[2] Allison, Dean Edwin. "Lorentzian warped products and static space-times." PhD diss., University of Missouri–Columbia, 1985.

[3] Allison, Dean. "Pseudoconcaveity in Lorentzian doubly warped products." Geometriae Dedicata 39, no. 2 (1991): 223-227.

[4] Berndt, Jürgen. "Three-dimensional Einstein-like manifolds." Differential Geometry and its Applications 2, no. 4 (1992): 385-397.

[5] Besse, Arthur L. Einstein manifolds. Springer Science & Business Media, 2007.

[6] Bishop, Richard L., and Barrett O'Neill. "Manifolds of negative curvature." Transactions of the American Mathematical Society 145 (1969): 1-49.

[7] Boeckx, E. Einstein like semisymmetric spaces, Archiv. Math. 29 (1992) 235–240.

[8] Bueken, Peter, and Lieven Vanhecke. "Three- and four-dimensional Einstein-like manifolds and homogeneity." Geometriae Dedicata 75, no. 2 (1999): 123-136.

[9] Calvaruso, Giovanni. "Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds." Geometriae Dedicata 127, no. 1 (2007): 99-119.

[10] Calvaruso, Giovanni, and Barbara De Leo. "Curvature properties of four-dimensional generalized symmetric spaces." Journal of Geometry 90, no. 1-2 (2008): 30-46.

[11] Calvaruso, Giovanni. "Einstein-like curvature homogeneous Lorentzian three-manifolds." Results in Mathematics 55, no. 3-4 (2009): 295.

[12] Calvaruso, Giovanni. "Riemannian 3–metrics with a generic Codazzi Ricci tensor." Geometriae Dedicata 151, no. 1 (2011): 259-267.

[13] Chojnacka-Dulas, J., R. Deszcz, M. Głogowska, and M. Prvanović. "On warped product manifolds satisfying some curvature conditions." Journal of Geometry and Physics 74 (2013): 328-341.

[14] De, Uday Chand, Cengizhan Murathan, and Cihan Özgür. "Pseudo symmetric and pseudo Ricci symmetric warped product manifolds." Communications of the Korean Mathematical Society 25, no. 4 (2010): 615-621.

[15] De, Uday Chand, and Sameh Shenawy. "On Generalized Quasi-Einstein GRW Space-Times." International Journal of Geometric Methods in Modern Physics, to appear.

[16] U. C. De, Sameh Shenawy and Bulent Unal, Sequential Warped Products: Curvature and Killing Vector Fields, Filomat, to appear.

[17] Altay Demirbag, Sezgin, and Sinem Guler. "Rigidity of $(\varepsilon, \rho)$-quasi Einstein manifolds." Mathematische Nachrichten 290, no. 14-15 (2017): 2100-2110.

[18] El-Sayied, H. K., Sameh Shenawy, and Noha Syied. "Conformal vector fields on doubly warped product manifolds and applications." Advances in Mathematical Physics 2016 (2016), Article ID: 6508309, 11 pp.

[19] El-Sayied, H. K., Sameh Shenawy, and Noha Syied. "On symmetries of generalized Robertson-Walker spacetimes and applications." Journal of Dynamical Systems and Geometric Theories 15, no. 1 (2017): 51-69.

[20] Faghfouri, Morteza, and Ayyoub Majidi. "On doubly warped product immersions." Journal of Geometry 106, no. 2 (2015): 243-254.

[21] Gebarowski, Andrzej. "Doubly warped products with harmonic Weyl conformal curvature tensor." In Colloquium Mathematicae, vol. 67, no. 1, pp. 73-89. 1994.

[22] Gebarowski, A. N. D. R. Z. E. J. "On nearly conformally symmetric warped product spaces." Soochow J. Math 20, no. 1 (1994): 61-75.

[23] Gebarowski, A. "On conformally flat doubly warped products." Soochow J. Math 21 (1995): 125-129.
[24] Gebarowski, Andrzej. "On conformally recurrent doubly warped products." Tensor. New series 57, no. 2 (1996): 192-196.
[25] Gray, Alfred. "Einstein-like manifolds which are not Einstein." Geometriae dedicata 7, no. 3 (1978): 259-280.
[26] Gupta, Punam. "On compact Einstein doubly warped product manifolds." Tamkang Journal of Mathematics 49, no. 4 (2018): 267-275.
[27] Hu, Zejun, Dehe Li, and Shujie Zhai. "On generalized $\varepsilon$–quasi-Einstein manifolds with constant Ricci curvatures." Journal of Mathematical Analysis and Applications 446, no. 1 (2017): 843-851.
[28] Huang, Guangyue, and Fanqi Zeng. "A note on gradient generalized quasi-Einstein manifolds." Journal of Geometry 106, no. 2 (2015): 297-311.
[29] Mantica, Carlo Alberto, and Luca Guido Molinari. Riemann compatible tensors, In Colloquium Mathematicum, vol. 128, pp. 197-210. Instytut Matematyczny Polskiej Akademii Nauk, 2012.
[30] Mantica, Carlo Alberto, and Sameh Shenawy. "Einstein-like warped product manifolds." International Journal of Geometric Methods in Modern Physics 14, no. 11 (2017): 1750166.
[31] Mantica, Carlo Alberto, Luca Guido Molinari, Young Jin Suh, and Sameh Shenawy. Perfect-Fluid, Generalized Robertson-Walker Space-times, and Gray’s Decomposition, Journal of Mathematical Physics, to appear.
[32] B. O’Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press Limited, London, 1983.
[33] Olteanu, Andreea. "A general inequality for doubly warped product submanifolds." Mathematical Journal of Okayama University 52, no. 1 (2010).
[34] Olteanu, Andreea. "Doubly warped products in $S$–space forms." Romanian Journal of Mathematics and Computer Science 4, no. 1 (2014): 111-124.
[35] Peng, ChiaKuei, and Chao Qian. "Homogeneous Einstein-like metrics on spheres and projective spaces." Differential Geometry and Its Applications 44 (2016): 63-76.
[36] Perktas, Selcen Yuksel, and Erol Kılıc. "Biharmonic maps between doubly warped product manifolds." Balkan Journal of Geometry and Its Applications 15, no. 2 (2010): 1591-170.
[37] Ramos, M. P. M., E. G. L. R. Vaz, and J. Carot. "Double warped space-times." Journal of Mathematical Physics 44, no. 10 (2003): 4839-4865.
[38] Sameh Shenawy and B. Unal. 2–Killing vector fields on warped product manifolds, International Journal of Mathematics, 26(2015), no. 8, 1550065(17 pages).
[39] Sameh Shenawy and B. Unal. The $W_2$–curvature tensor on warped product manifolds and applications, International Journal of Geometric methods in Mathematical Physics, 13(2016), no. 7, 1650099 (14 pages).
[40] Suh, Young Jin, Pradip Majhi, and Uday Chand De. "On mixed quasi-Einstein spacetimes." FILOMAT 32, no. 8 (2018).
[41] Unal, Bulent. "Doubly warped products." Differential Geometry and Its Applications 15, no. 3 (2001): 253-263.
[42] Zaeim, Amirhesam, and Ali Haji-Badali. "Einstein-like pseudo-Riemannian homogeneous manifolds of dimension four." Mediterranean Journal of Mathematics 13, no. 5 (2016): 3455-3468.