OSOM: A Simultaneously Optimal Algorithm for Multi-Armed and Linear Contextual Bandits

Niladri S. Chatterji  Vidya Muthukumar  Peter L. Bartlett
{chatterji,vidya.muthukumar,peter}@berkeley.edu

University of California, Berkeley

Abstract

We consider the stochastic linear (multi-armed) contextual bandit problem with the possibility of hidden simple multi-armed bandit structure in which the rewards are independent of the contextual information. Algorithms that are designed solely for one of the regimes are known to be sub-optimal for their alternate regime. We design a single computationally efficient algorithm that simultaneously obtains problem-dependent optimal regret rates in the simple multi-armed bandit regime and minimax optimal regret rates in the linear contextual bandit regime, without knowing a priori which of the two models generates the rewards. These results are proved under the condition of stochasticity of contextual information over multiple rounds. Our results should be viewed as a step towards principled data-dependent policy class selection for contextual bandits.

1 Introduction

The contextual bandit paradigm involves sequential decision-making settings in which we repeatedly pick one out of $K$ actions (or "arms") in the presence of contextual side information. Algorithms for this problem usually involve policies that map the contextual information to a chosen action, and the reward feedback is typically limited in the sense that it is only obtained for the action that was chosen. The goal is to maximize the total reward over several $n$ rounds of decision-making, and the performance of an online algorithm is typically measured in terms of regret with respect to the best policy within some policy class $\Pi$ that is fixed a priori. Applications of this paradigm include advertisement placement/web article recommendation [Li+10; Aga+16], clinical trials and mobile health-care [Woo79; TM17].

The contextual bandit problem can be thought of as an online supervised learning problem (over policies mapping contexts to actions) with limited information feedback, and so the optimal regret bounds scale like $O(\sqrt{Kn \log |\Pi|})$, a natural measure of the sample complexity of the policy class [Auc+02; MS09; Bey+11]. These are typically achieved by algorithms that are inefficient (linear in the size of the policy class). Much of the research in contextual bandits has tackled computational efficiency [LZ08; Aga+14; RS16; SKS16; Syr+16; FK18]: do there exist computationally efficient algorithms that achieve the optimal regret guarantee? A question that has received relatively less attention involves the choice of policy class itself. Even for a fixed regret-minimizing algorithm, the choice of policy class is critical to maximize the overall reward of the algorithm. As can be seen in applications of contextual bandits models for article recommendation [Li+10], the choice is often made in hindsight, and more complex policy classes are used if the algorithm is run for more rounds. Quantitative understanding of how to do this is still lacking, and intuitively, we should expect the optimal choice of policy class to not be static. Ideally, we could design adaptive contextual bandit algorithms that would initially use simple policies, and switch over to more complex ones as more data is obtained.

Theoretically, what this means is that the regret bounds derived for a contextual bandit algorithm are only meaningful for rewards that are generated by a policy within the policy class to which the algorithm is tailored. If the rewards are derived from a "more complex" policy outside the policy class, even the optimal policy may neglect obvious patterns and obtain a very low reward. If the
rewards are derived from a policy that is expressible by a much smaller class, the regret that is accumulated is unnecessary. Let us view this through the lens of the simplest possible example: the standard linear contextual bandits [Chu+11] paradigm, where we can choose one out of $K$ arms and rewards are generated according to the process
\[ r_{i,t} = \mu_i + \langle \theta^*, \alpha_{i,t} \rangle + \eta_{i,t} \text{ for all } i \in [K], \]
where $\mu_i$ represents a “bias” of arm $i$, $\theta^* \in \mathbb{R}^d$ represents the linear parameter of the model (which is shared across all arms), $\alpha_{i,t} \in \mathbb{R}^d$ represents the contextual information and $\{\eta_{i,t}\}_{i=1}^n$ represents noise in the reward observations. It is well-known that variants of linear upper confidence bound algorithms like LinUCB [Chu+11] and OFUL [APS11] succeed at most $O((\sqrt{d} + \sqrt{K}) \sqrt{n})$ regret with respect to the optimal linear policy. However, setting $\theta^* = 0$ yields the important case of the reward distribution being independent from the contextual information. Here, a simple upper confidence bound algorithm like UCB [ACF02] would yield the optimal $O(\log n)$ regret bound, which does not depend on the dimension of the contexts $d$. Thus, we pay substantial extra regret by using the algorithm meant for linear contextual bandits on such instances with much simpler structure. On the other hand, upper confidence bounds that ignore the contextual information will not guarantee any control on the policy regret: it can even be linear. It is natural to desire a single approach that adapts to the inherent complexity of the reward-generating model and obtains the optimal regret bound as if this complexity was known in hindsight. Specifically, this paper seeks an answer to the following question:

*Does there exist a single algorithm that simultaneously achieves the $O(\log n)$ regret rate on simple multi-armed bandit instances and the $O((\sqrt{d} + \sqrt{K}) \sqrt{n})$ regret rate on linear contextual bandit instances?*

1.1 Our contributions

We answer the question of simultaneously optimal regret rates in the multi-armed (“simple”) bandit regime and the linear contextual (“complex”) bandit regime affirmatively under the condition that the contexts are generated from a stochastic process that yields covariates that are not ill-conditioned. Our algorithm, OSOM (for Optimistic Selection of Models), essentially exploits the best policy (simply the best arm) that is learned under the assumption of the simple reward model - while conducting a sequential statistical test for the presence of additional complexity in the model, and particularly whether ignoring this additional complexity would lead to substantial regret. This is a simple statistical principle that could conceivably be generalized to arbitrary policy classes that are nested: we will see that the OSOM algorithm critically exploits the nested structure of the simple bandit model within the linear contextual model.

1.2 Related work

The contextual bandit paradigm was first considered by Woodroofe (1979) to model clinical trials. Since then it has been studied intensely both theoretically and empirically in many different application areas under many different pseudonyms. We point the reader to [TM17] for an extensive survey of the contextual bandits history and literature.

Treating policies as experts (EXP4 [Aue+02]) with careful control on the exploration distribution led to the optimal regret bounds of $O(\sqrt{Kn \log |\Pi|})$ in a number of settings. From an efficiency point of view (where efficiency is defined with respect to an arg-max-oracle that is able to compute the best greedy policy in hindsight), the first approach conceived was the epoch-greedy approach [LZ08], that suffers a sub-optimal dependence of $n^{2/3}$ in the regret. More recently, “randomized-UCB” style approaches [Aga+14] have been conceived that retain the optimal regret guarantee with $\tilde{O}(\sqrt{n})$ calls to the arg-max-oracle. This question of computational efficiency has

---

1. This is the model that was described in [Chu+11]. It is worth noting that more complex variants of this model with a separate $\theta^*$ for every $i \in [K]$ have also been empirically evaluated [Li+10].

2. Guarantees for OFUL were established under slightly different constraints on $\theta^*$ and the context vectors which led to a regret bound of $\tilde{O}((d + K) \sqrt{n})$. We show in Lemma 6 that a slight variant of OFUL has its regret bounded by $\tilde{O}((\sqrt{d} + \sqrt{K}) \sqrt{n})$ in our setting.

3. The $\tilde{O}()$ notation hides poly-logarithmic factors.
generated a lot of research interest [RS16; SKS16; Syr+16; FK18]. The problem of policy class selection itself has received less attention in the research community, and how this is done in practice in a statistically sound manner remains unclear. An application of linear contextual bandits was to personalized article recommendation using hand-crafted features of users [Li+10]: two classes of linear contextual bandit models with varying levels of complexity were compared to simple (multi-armed) bandit algorithms in terms of overall reward (which in this application represented the click-through rate of ads). A striking observation was that the more complex models won out when the algorithm was run for a longer period of time (e.g: 1 day as opposed to half a day). Surveys on contextual bandits as applied to mobile health-care [TM17] have expressed a desire for algorithms that adapt their choice of policy class according to the amount of information they have received (e.g. the number of rounds). At a high level, we seek a theoretically principled way of doing this.

Perhaps the most relevant work to online policy class selection involves significant attempts to corral a band of $M$ base bandit algorithms into a meta-bandit framework [Aga+17]. The idea is to bound the regret of the meta-algorithm in terms of the regret of the best base algorithm in hindsight. (This is clearly useful for policy class selection that we study here – by coralling together an algorithm designed for the linear model and one for the simple multi-armed bandits model.) The Corral framework is very general and can be applied to any set of base algorithms, whether efficient or not. This generality is attractive, but it is not the optimal choice of computationally efficient algorithm for the multi-armed-vs-linear-contextual bandit problem for a couple of reasons.

1. It is not clear what (if any) choice of base algorithms would lead to a computationally efficient algorithm that is also statistically optimal in a minimax sense simultaneously for both problems.

2. The meta-algorithm framework uses an experts algorithm (in particular, mirror descent with log-barrier regularizer and importance weighting on the base algorithms) to choose which base algorithm to play in each round. Thus, it is impossible to expect the instance-optimal regret rate of $O(\log n)$ on the simple bandit instance. More generally, the Corral framework will not yield instance-optimal rates on any policy class.

The Corral framework highlights the principal difficulty in contextual bandit model selection that can be thought of as an even finer exploration-exploitation tradeoff: algorithms (designed for particular model classes) that fall out of favor in initial rounds could be picked very rarely and the information required to truly perform model selection may be absent even after many rounds of play. Corral tackles this difficulty using the log-barrier regularizer for the meta-algorithm as a natural form of heightened exploration [Fos+16], together with clever learning rate schedules. Related recent work [Kri+19] adapts to the unknown Lipschitz constant of the optimal policy (function from context to recommended action) in the stochastic contextual bandit problem with an abstract policy class and continuous action space.

Our stylistic approach to the model selection problem is a little different, as we focus on the much more specific case of 2 models: the simple multi-armed bandit model and the linear contextual bandit model. We encounter a similar difficulty and obtain striking clarity on the extent of this difficulty owing to the simplicity of the models. On the other hand, we observe that commonly encountered sequences of contexts can help us carefully navigate the finer exploration-exploitation tradeoff when the model classes are nested.

Our algorithm (OSOM) utilizes a simple “best-of-both-worlds” principle: exploit the possible simple reward structure in the model until (unless) there is significant statistical evidence for the presence of complex reward structure that would incur substantial complex policy regret if not exploited. This algorithmic framework is inspired by the initial “best-of-both-worlds” results for stochastic and adversarial multi-armed bandits; in particular, the “Stochastic and Adversarial Optimal” (SAO) algorithm [BS12] (although the details of the phases of the algorithm and the statistical test are very different). In that framework, instances that are not stochastic (and could be thought of as “adversarial”) are not always detected as such by the test. The test is designed in

\[ \text{On our much simpler instance of bandit-vs-linear-bandit, we do obtain instance-optimal rates for at least the simple bandit model.} \]

\[ \text{An undesirable side effect of using the log-barrier regularizer is a dependence on } \sqrt{M} \text{ as opposed to } \sqrt{\log M} \text{ in the regret bound, where } M \text{ is the number of policy classes.} \]
an elegant manner such that the regret is optimally bounded on instances that are not detected as adversarial, even if an algorithm meant for stochastic rewards is used. Our test to distinguish between simple and complex instances shares this flavor – in fact, all theoretically complex instances ($\theta^* \neq 0$) are not detected as such.

Also related are results on contextual bandits with similarity information on the contexts, which automatically encodes a potentially easier learning problem [Sli14]. The main novelty in these results involves adapting to such similarity online.

Technically, our proofs leverage the most recent set of theoretical results on regret bounds for linear bandits [APS11], which can easily be applied to the linear contextual bandit model, and sophisticated self-normalized concentration bounds for our estimates of both the bias terms $\mu_i$ and the parameter vector $\theta^*$. For the latter, we find that the Matrix Freedman inequality [Oli09; Tro11] is particularly useful.

1.3 Problem Statement

At the beginning of each round $t \in [n]$, the learner is required to choose one of $K$ arms and gets a reward associated with that arm. To help make this choice the learner is handed a context vector from a distribution such that the best one can hope to do in this setting in the worst case is 0. Several algorithms like upper confidence bounds (UCB) in the stochastic case ([LR85]) tell us that the best one can hope to do in this setting in the worst case is $\mathbb{E}[R^*_n] = \Omega(\sum \log(n)/\Delta_i)$. Several algorithms like upper confidence bounds (UCB) [ACF02] and minimax-optimal strategies in the stochastic case (MOSS) [AB10; DP16] achieve this lower bound up to logarithmic (and constant) factors.

Simple Model: Under the simple multi-armed bandit model, the mean rewards of $K$ arms are fixed and are not a function of the contexts. That is, at each round

$$g_{i,t} = \mu_i + \eta_{i,t}, \quad \forall i \in [K]$$

where $\mu_i \in [-1, 1], \{\eta_{i,t}\}_{t=1}^K$ are identical, independent, zero mean, $\sigma$-sub-Gaussian noise (defined below). Let the arm with the highest reward have mean $\mu^*$ and be indexed by $i^*$. The benchmark that the algorithm hopes to compete against is the pseudo-regret (henceforth regret for brevity),

$$R_n^i := n\mu^* - \sum_{s=1}^n \mu_{A_s}.$$  

Define the gap as the difference in the mean rewards of the best arm compared to the mean reward of the $i^\text{th}$ arm, that is, $\Delta_i := \mu^* - \mu_i$. Previous literature on multi-armed bandits [LR85] tells us that the best one can hope to do in this setting in the worst case is $\mathbb{E}[R^*_n] = \Omega(\sum \log(n)/\Delta_i)$.

Complex Model: In this model the mean reward of each arm is a linear function of the contexts (linear contextual bandits). We work with the following stochastic assumptions on the context vectors. Each of these contexts vectors $\alpha_{i,t} \in \mathbb{E}^d_2(1)$ and are drawn independent of the past from a distribution such that $\alpha_{i,t}$ is independent of $\{\alpha_{j,t}\}_{j \neq i}$ and, $\forall i \in [K]$ and $\forall t \in [n]$,

$$\mathbb{E}_{t-1}[\alpha_{i,t}] := \mathbb{E}\left[\alpha_{i,t} | \eta_{j,s}, \alpha_{j,s} \right]_{j \in [K], s \in [t-1]} = 0,$$

$$\mathbb{E}_{t-1}[\alpha_{i,t} \alpha_{i,t}^\top] := \mathbb{E}\left[\alpha_{i,t} \alpha_{i,t}^\top | \eta_{j,s}, \alpha_{j,s} \right]_{j \in [K], s \in [t-1]} = \Sigma_c \succeq \rho_{\min} \cdot I. \quad (1)$$

That is the conditional mean of the context vectors are 0 and the co-variance matrix has its minimum eigenvalue bounded below by $\rho_{\min}$.

In this complex model, we assume there exists an underlying linear predictor $\theta^* \in \mathbb{R}^d$ and biases $[\mu_1, \ldots, \mu_K] \in \mathbb{R}^K$ of the $K$ arms, such that the mean rewards of the arms are affine functions of the contexts, i.e.,

$$g_{i,t} = \mu_i + \langle \theta^*, \alpha_{i,t} \rangle + \eta_{i,t}.$$  

We impose compactness constraints on the parameters: in particular, we have $\mu_i \in [-1, 1], \theta^* \in \mathbb{E}^d_2(1)$. Further, the noise $\{\eta_{i,t}\}_{t=1}^n$ are identical, independent, zero mean, and $\sigma$-sub-Gaussian.
Clearly, simple model instances (which are parameterized only by the biases \([\mu_1, \ldots, \mu_K] \in \mathbb{R}^K\)) can be expressed as complex model instances by setting \(\theta^* = 0\).

At each round define \(\kappa_t = \arg\max_{s \in \{1, \ldots, K\}} \{\mu_s + \langle \theta^*, \alpha_{s,t} \rangle\}\) to be the best arm at round \(t\). Here, we define pseudo-regret with respect to the optimal policy under the generative linear model:

\[
R^*_n := \sum_{s=1}^n [\mu_s + \langle \theta^*, \alpha_{s,s} \rangle - \mu_{\hat{A}_s} - \langle \theta^*, \alpha_{\hat{A}_s,s} \rangle].
\]

As noted above, past literature on this problem yielded algorithms like LinUCB \([\text{Chu}+11]\) and OFUL \([\text{APS11}]\) that only suffer from the minimax regret of \(\tilde{O}((\sqrt{d} + \sqrt{K})\sqrt{n})\). As we will see in the simulations, these algorithms incur the dependence on the dimension in the regret bound even for simple instances.

### 2 Construction of Confidence Sets

In our algorithm, which is presented subsequently at the end of round \(t\), we build an upper confidence estimate for each arm. Let \(T_i(t) := \sum_{s=1}^t \mathbb{I}[A_s = i]\) be the number of times arm \(i\) was pulled and \(\hat{g}_{i,t} := \sum_{s=1}^t g_{s,i} \mathbb{I}[A_s = i] / T_i(t)\) be the average reward of that arm at the end of round \(t\). For each arm we define the upper confidence estimate as follows,

\[
\hat{\mu}_{i,t} := \hat{g}_{i,t} + \sigma \sqrt{\frac{1 + T_i(t)}{T_i^2(t)} \left( 1 + 2 \log \left( \frac{K(1 + T_i(t))^{1/2}}{\delta} \right) \right)},
\]

(2)

Lemma 6 in \([\text{APS11}]\) (restated below as Lemma 1 here) uses a refined self-normalized martingale concentration inequality to bound \(|\mu_i - \hat{g}_{i,t}|\) across all arms and all rounds.

**Lemma 1.** Under the simple model, with probability at least \(1 - \delta\) we have, \(\forall i \in \{1, \ldots, K\}, \forall t \geq 0,

\[
|\mu_i - \hat{g}_{i,t}| \leq \sigma \sqrt{\frac{1 + T_i(t)}{T_i^2(t)} \left( 1 + 2 \log \left( \frac{K(1 + T_i(t))^{1/2}}{\delta} \right) \right)}.
\]

For any round \(t > K\), let \(\hat{\theta}_t\) be the \(\ell^2\)-regularized least-squares estimate of \(\theta^*\), which we define explicitly below.

\[
\hat{\theta}_t = (\alpha_{K+1,t}^\top \alpha_{K+1:t} + I)^{-1} \alpha_{K+1,t}^\top G_{K+1:t},
\]

(3)

where \(\alpha_{K+1:t}\) is the matrix whose rows are the context vectors selected from round \(K + 1\) up until round \(t\): \(\alpha_{K+1:t} = [\alpha_{K+1,K}, \ldots, \alpha_{A_t,K+1}, \ldots, \alpha_{A_t,K+1,t}]^\top\). Here we are regressing on the rewards seen to estimate \(\theta^*\), while using the bias estimates \(\hat{\mu}_{i,t-1}\) obtained by our upper confidence estimates defined in Eq. (2).

**Lemma 2.** Let \(\hat{\theta}_t\) be defined as in Eq. (3). Then, with probability at least \(1 - 3\delta\) we have that for all \(t > K\), \(\theta^*\) lies in the set

\[
C^c_t := \left\{ \theta \in \mathbb{R}^d : ||\theta - \hat{\theta}_t||_2 \leq K_\delta(t,n) \right\},
\]

(4)

where \(K_\delta(t,n) = \tilde{O}(\sigma \sqrt{d \cdot n})\) is defined in Eq. (11).

We present a proof of this lemma in Appendix A.
Algorithm 1: OSOM - Optimistic Selection Of Models

1 for $t = 1, \ldots, K$ do
2 Play arm $t$ and receive reward $g_{i,t}$,  \hspace{1em} (Play each arm at least once.)
3 for $t = K + 1, \ldots, n$ do
4 Current Model $\leftarrow$ ‘Simple’
5 Simple Model Estimate:
   \hspace{1em} $i_t \in \text{argmax}_{i \in \{1, \ldots, K\}} \{\tilde{\mu}_{i,t-1}\}$ \hspace{1em} (5)
6 Complex Model Estimate:
   \hspace{1em} $j_t, \tilde{\theta}_t \in \text{argmax}_{i \in \{1, \ldots, K\}, \theta \in C_{t-1}} \{\tilde{\mu}_{i,t-1} + \langle \alpha_{i,t}, \theta \rangle\}$, \hspace{1em} $C_{t-1}$ defined in (4). \hspace{1em} (6)
7 if Current Model = ‘Simple’ and $t > K + 1$ then
8 \hspace{1em} Check the condition:
9 \hspace{2em} $\sum_{s=K+1}^{t-1} \left\{\tilde{\mu}_{j_s,s-1} + \langle \alpha_{j_s,s}, \tilde{\theta}_s \rangle - g_{i_s,s}\right\} \leq W_{\delta}(t-1, n)$, \hspace{1em} $W_{\delta}(t, n)$ defined in (12). \hspace{1em} (7)
10 \hspace{1em} if violated then set Current Model $\leftarrow$ ‘Complex’.
11 if Current Model = ‘Simple’: Play arm $i_t$ and receive reward $g_{i_t,t}$.
12 Else if Current Model = ‘Complex’: Play arm $j_t$ and receive $g_{j_t,t}$.
13 Update $\{\tilde{\mu}_{i,t}\}_{t=1}^{K}$ and $C_t$.

3 Algorithm and Main Result

The intuition behind Algorithm 1 is straightforward. The algorithm starts off by using the simple model estimate of the recommended action, i.e., $i_t$; until it has reason to believe that there is a benefit from switching to the complex model estimates. If the rewards are truly coming from the simple model, or from a complex model that is well approximated by a simple multi-armed bandit model, then Condition 7 will not be violated and the regret shall continue to be bounded under either model. However, if Condition 7 is violated then algorithm switches to the complex estimates $- j_t$ for the remaining rounds. The condition is designed using the function $W_{\delta}(t, n)$ which is of the order $\tilde{O}(\sigma(\sqrt{d} + \sqrt{K})\sqrt{t})$. This corresponds to the additional regret incurred when we attempt to estimate the extra parameter $- \tilde{\theta}_t \in \mathbb{R}^d$.

Our main theorem optimally bounds the regret of OSOM under either of the two reward-generating models.

Theorem 3. With probability at least $1 - 9\delta$, we obtain the following upper bounds on regret for the algorithm OSOM (Algorithm 1):

(a) Under the Simple Model $R_n^s \leq \sigma \cdot \sum_{i: \Delta_i > 0} \left[3\Delta_i + \frac{16}{\sqrt{\Delta_i}} \log \left(\frac{2K}{\Delta_i}\right)\right]$.

(b) Under the Complex Model $R_n^c \leq 4(K + 1) + 3W_{\delta}(n, n) = \tilde{O} \left\{\sigma(\sqrt{d} + \sqrt{K})\sqrt{n}\right\}$,

where $W_{\delta}(n, n)$ is defined in Eq. (12).

Notice that Theorem 3 establishes regret bounds on the algorithm OSOM which are minimax optimal under both simple model and the complex model up to logarithmic factors. In fact, under the simple model we are able to obtain problem-dependent regret rates. Note that the above regret bound is with high probability and also implies a bound in expectation by setting $\delta = \Omega(\log(n)/n)$ and using Markov’s inequality.
4 Analysis

To prove Theorem 3, we need to show that the regret of OSOM is bounded under either underlying model. In Lemma 4 we demonstrate that whenever the rewards are generated under the simple model, Condition 7 is not violated with high probability. This ensures that when the data is generated from the simple model, we have that the Boolean variable Current Model = ‘Simple’ throughout the run of the algorithm. Thus, the regret is automatically equal to the regret incurred by the UCB algorithm, which is meant for simple model instances.

On the other hand, when the data is generated according to the complex model, we demonstrate (in Lemma 5) that the regret remains appropriately bounded if Condition 7 is not violated. If the condition gets violated at a certain round, we switch to the estimates of the complex model, i.e. $j_t$. This corresponds to a variant of the algorithm OFUL, which is meant for complex instances. Thus, the regret remains bounded in the subsequent rounds under this event as well (formally proved in Lemma 6).

We define below several functions which will be used throughout the proof. These arise naturally by applying the concentration inequalities on terms that appear while controlling the regret.

$$\tau_{\min}(\delta, n) := \left(\frac{16}{n'_{\min}} + \frac{8}{3n_{\min}}\right) \log \left(\frac{2dn}{\delta}\right).$$  \hspace{1cm} (8)

$$\Upsilon_\delta(t, n) := \left(\frac{20}{3} + \frac{10\sigma^2}{3} \sqrt{1 + 2 \log \left(\frac{2Kn^2}{\delta}\right)}\right) \left[\log \left(\frac{2dn}{\delta}\right) + \frac{1}{\delta} \sqrt{\log \left(\frac{2dn}{\delta}\right) + \log^2 \left(\frac{2dn}{\delta}\right)}\right].$$ \hspace{1cm} (9)

$$\mathcal{M}_\delta(t) := \sqrt{2\sigma^2 \left(\frac{d}{2} \log \left(1 + \frac{t}{d}\right) + \log \left(\frac{t}{\delta}\right)\right)} + 1.$$ \hspace{1cm} (10)

$$\mathcal{K}_\delta(t, n) := \begin{cases} \mathcal{M}_\delta(t) + \Upsilon_\delta(t, n), & \text{if } K < t \leq K + \tau_{\min}(\delta, n), \\ \frac{\mathcal{M}_\delta(t)}{\sqrt{1 + \rho_{\min}(t - K)/2}} + \frac{\Upsilon_\delta(t, n)}{1 + \rho_{\min}(t - K)/2}, & \text{if } K + \tau_{\min}(\delta, n) < t. \end{cases}$$ \hspace{1cm} (11)

$$\mathcal{W}_\delta(t, n) := 2 \sum_{s=K+1}^{t} \mathcal{K}_\delta(s - 1, n) + \sigma \sqrt{1 + \frac{t}{2} \log \left(\frac{1}{\delta}\right)} + \sigma \sqrt{1 + 2 \log \left(\frac{K^{t/2}}{\delta}\right)} + \sqrt{Kt}.$$ \hspace{1cm} (12)

Given the definitions above, it is straightforward to verify that $\mathcal{W}_\delta(t, n) = \tilde{O} \left(\sigma (\sqrt{d} + \sqrt{K}) \sqrt{t}\right)$.

Additionally, we define several statistical events that will be useful in proofs of the lemmas that follow.

$$\mathcal{E}_1 := \left\{\sum_{s=K+1}^{t} \eta_{s, t} \leq \sigma \sqrt{\frac{t}{2} \log \left(\frac{1}{\delta}\right)}, \forall t \in \{K + 2, \ldots, n\}\right\},$$ \hspace{1cm} (13a)

$$\mathcal{E}_2 := \left\{|\mu_t - g_t| \leq \sigma \frac{1 + T_t(s)}{T_t(s)} \left(1 + 2 \log \left(\frac{K(1 + T_t(s))^{1/2}}{\delta}\right)\right), \forall t \in [K] \text{ and } s \in [n]\right\},$$ \hspace{1cm} (13b)

$$\mathcal{E}_3 := \left\{|\tilde{\theta}_t - \theta^*|_2 \leq \mathcal{K}_\delta(t, n), \forall t \in \{K + 1, \ldots, n\}\right\}.$$ \hspace{1cm} (13c)

Event $\mathcal{E}_1$ represents control on the fluctuations due to noise: applying Theorem 9 in the one-dimensional case with $V = 1$ and $Y_s = 1$, we get $\mathbb{P}(\mathcal{E}_1) \leq \delta$ for all $t \geq 0$. Event $\mathcal{E}_2$ represents control on the fluctuations of the empirical estimate of the biases $[\mu_1, \ldots, \mu_K]$ around their true values; by Lemma 1 we have $\mathbb{P}(\mathcal{E}_2) \leq \delta$. Finally, event $\mathcal{E}_3$ represents control on the fluctuations of the empirical estimate of the parameter vector $\theta^*$ around its true value: by Lemma 2, we have $\mathbb{P}(\mathcal{E}_3) \leq 3\delta$. We define the desired event $\mathcal{E} := \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$ as the intersection of these three events. The union bound gives us $\mathbb{P}(\mathcal{E}^c) \leq 5\delta$. For the rest of the proof, we condition on the event $\mathcal{E}$.

4.1 Regret under the Simple Model

The following lemma establishes that under the simple model, Condition 7 is not violated with high probability.
Lemma 4. Assume that rewards are generated under the simple model. Then, with probability at least $1 - 5\delta$, we have

$$
\sum_{s=K+1}^{t-1} \left[ \tilde{\mu}_{j,s-1} + \langle \alpha_{j,s}, \tilde{\theta}_s \rangle \right] - \sum_{s=K+1}^{t-1} g_{i,s} < W_8(t-1, n), \quad \forall t \in \{K+2, \ldots, n\}.
$$

Proof Under the simple model, we have the model for the rewards is

$$
\tilde{\mu}_{j,s} = \langle \alpha_{j,s}, \tilde{\theta}_s \rangle
$$

The first term

$\sum_{s=K+1}^{t-1} \left[ \tilde{\mu}_{j,s-1} + \langle \alpha_{j,s}, \tilde{\theta}_s \rangle \right]$

corresponds to the difference between the true mean reward $\mu_i$ and simple estimate of the mean reward $\tilde{\mu}_{i,s-1}$, which is controlled under the event $\mathcal{E}_1$. The second term $\Gamma_{sim1}$ corresponds to the difference between the true mean reward $\mu_i$ and simple estimate of the mean reward $\tilde{\mu}_{i,s-1}$, which is controlled under the event $\mathcal{E}_2$. The third term $\Gamma_{sim2}$ is the difference between the mean rewards prescribed by the simple estimate and complex estimate $\tilde{\mu}_{i,s-1}$ and $\hat{\mu}_{i,s-1}$ respectively. Finally, the last term $\Gamma_{lin}$ is only a function of the estimated linear predictor (and since the true predictor is $\theta^* = 0$, this term is controlled by even $\mathcal{E}_3$).

Step (i) (Bound on $\Gamma_{no}$): Under the event $\mathcal{E}_1$, we have

$$
\Gamma_{no} \leq \sigma \sqrt{\frac{t}{2 \log \left( \frac{1}{\delta} \right)}}.
$$

Step (ii) (Bound on $\Gamma_{sim1}$): By the definition of $\tilde{\mu}_{i,s-1}$ we have,

$$
\Gamma_{sim1} = \sum_{s=K+1}^{t-1} \tilde{\mu}_{i,s-1} - \mu_i
$$

$$
\leq 2\sigma \sum_{s=K+1}^{t-1} \frac{1 + T_i(s-1)}{T_i^2(s-1)} \left( 1 + 2 \log \frac{K(1 + T_i(s-1))^{1/2}}{\delta} \right)
$$

$$
\leq 2\sigma \sum_{s=K+1}^{t-1} \frac{1 + T_i(s-1)}{T_i^2(s-1)} \left( 1 + 2 \log \left( \frac{K(t-1)^{1/2}}{\delta} \right) \right)
$$

$$
= \left[ 2\sigma \sqrt{1 + 2 \log \left( \frac{K(t-1)^{1/2}}{\delta} \right)} \right] \sum_{i=1}^{K} T_i(t-2) \sqrt{\frac{1}{r^2}}
$$

$$
\leq 2\sigma \sqrt{1 + 2 \log \left( \frac{K(t-1)^{1/2}}{\delta} \right)} \sum_{i=1}^{K} T_i(t-2) \sqrt{\frac{1}{r^2}}
$$

$$
= 2\sigma \sqrt{1 + 2 \log \left( \frac{K(t-1)^{1/2}}{\delta} \right)} \sum_{i=1}^{K} T_i(t-2) \sqrt{\frac{1}{r^2}}
$$

$$
\leq 2\sigma \sqrt{1 + 2 \log \left( \frac{K(t-1)^{1/2}}{\delta} \right)} \sum_{i=1}^{K} T_i(t-2)
$$

$$
\leq 2\sigma \sqrt{1 + 2 \log \left( \frac{K(t-1)^{1/2}}{\delta} \right)} \sqrt{K(t-1)},
$$

8
where (i) follows under the event $\mathcal{E}_2$, (ii) follows as
\[
2 \sum_{r=1}^{T_r(t-2)} \sqrt{\frac{t}{r}} \leq 2 \int_0^{T_r(t-2)} \sqrt{\frac{t}{r}} \, dr \leq \sqrt{T_r(t-2)},
\]
and (iii) follows by Jensen’s inequality and the fact that $\sum_{i=1}^{K} T_i(t-2) = t - 2 < t - 1$.

**Step (iii) (Bound on $\Gamma_{\text{sim}2}$):** Eq. (5), which shows the optimality of arm $i_s$, tells us that $\tilde{\mu}_{i,s-1} \geq \tilde{\mu}_{j,s-1}$ for all $s$. Therefore $\Gamma_{\text{sim}2} \leq 0$.

**Step (iv) (Bound on $\Gamma_{\text{lin}}$):** By the Cauchy-Schwartz inequality, the constraint $\|\alpha_{i,t}\|_2 \leq 1$ and the triangle inequality, we get
\[
\Gamma_{\text{lin}} = \sum_{s=K+1}^{t-1} (\alpha_{j,s}, \tilde{\theta}_s) \leq \sum_{s=K+1}^{t-1} \|\alpha_{j,s}\|_2 \|\tilde{\theta}_s\|_2 \leq \sum_{s=K+1}^{t-1} \|\hat{\theta}_s - \theta^*\|_2 \leq \sum_{s=K+1}^{t-1} \|\hat{\theta}_s - \hat{\theta}_{s-1}\|_2 \leq 2 \sum_{s=K+1}^{t-1} K_\delta(s - 1, n),
\]
where $K_\delta(s - 1, n)$ is defined in Eq. (11).

Combining the bounds on $\Gamma_{\text{no}}, \Gamma_{\text{sim}1}, \Gamma_{\text{sim}2}$ and $\Gamma_{\text{lin}}$ and by the definition of $W_\delta(t-1, n)$, we have
\[
\sum_{s=K+1}^{t-1} \left[ \tilde{\mu}_{j,s-1} + (\alpha_{j,s}, \tilde{\theta}_s) \right] - \sum_{s=K+1}^{t-1} g_{i,s} \leq W_\delta(t-1, n),
\]
which completes the proof.

**Proof** [Proof of Part (a) of Theorem 3] We have established above that Condition 7 is not violated with probability at least $1 - 5\delta$ under the simple model by the lemma above. Conditioned on this event, OSOM plays according to the simple model estimate, $i_t$, for all rounds. Invoking Theorem 7 in [APS11] gives us that with probability at least $1 - \delta$, $R_n^* \leq \sum_{t : \Delta_t > 0} 3\Delta_t + (16/\Delta_t) \log(2K/\Delta_\delta)$. Applying the union bound over these two events gives this regret bound with probability at least $1 - 6\delta$.

**4.2 Regret under the Complex Model**

The bound on the regret under the complex model follows by establishing two facts. First, when Condition 7 is not violated, we demonstrate in Lemma 5 that the regret is appropriately bounded. Second, if the condition does get violated, say at round $\tau_s$, our algorithm OSOM chooses arms according to the complex model estimates ‘$j_t$’ for $t \in [\tau_s, \ldots, n]$. In Lemma 6, we show that the regret remains bounded in this case as well.

We start with the first case by stating and proving Lemma 5.

**Lemma 5.** Consider $t \in \{K + 1, \ldots, n\}$. Let Condition 7 not be violated up until round $t + 1$, i.e.
\[
\sum_{s=K+1}^{t} \left\{ \tilde{\mu}_{j,s-1} + (\alpha_{j,s}, \tilde{\theta}_s) \right\} - g_{i,s} \leq W_\delta(t, n).
\]

Then, we have
\[
R_n^c \leq 2K + 2W_\delta(t, n)
\]
with probability at least $1 - 5\delta$.

**Proof** Since we have already conditioned on the event $\mathcal{E}$, we can assume that events $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ hold. Note that if Condition 7 is not violated up to round $t$ then we have that $A_s = i_s$ for all
$s \leq t$. Using the definition of $R^c_t$, we get

$$R^c_t = \sum_{s=1}^{t} [\mu_{s} + \langle \theta^*, \alpha_{s,s} \rangle - \mu_{s} - \langle \theta^*, \alpha_{s,s} \rangle]$$

$$\leq 4K + \sum_{s=K+1}^{t} [\mu_{s} + \langle \theta^*, \alpha_{s,s} \rangle - \mu_{s} - \langle \theta^*, \alpha_{s,s} \rangle]$$

$$= 4K + \sum_{s=K+1}^{t} (\mu_{s} + \langle \theta^*, \alpha_{s,s} \rangle - g_{s,s}) + \sum_{s=K+1}^{t} (g_{s,s} - \mu_{s} - \langle \theta^*, \alpha_{s,s} \rangle)$$

$$= 4K + \sum_{s=K+1}^{t} \left( \mu_{s} + \langle \theta^*, \alpha_{s,s} \rangle - \tilde{\mu}_{s-1} - \langle \tilde{\theta}_s, \alpha_{s,s} \rangle \right)$$

$$+ \sum_{s=K+1}^{t} \left( \tilde{\mu}_{s-1} + \langle \tilde{\theta}_s, \alpha_{s,s} \rangle - g_{s,s} \right) + \sum_{s=K+1}^{t} \eta_{s,s}$$

$$\leq 4K + W_3(t,n) + \sum_{s=K+1}^{t} \left( \mu_{s} + \langle \theta^*, \alpha_{s,s} \rangle - \tilde{\mu}_{s-1} - \langle \tilde{\theta}_s, \alpha_{s,s} \rangle \right) + \sum_{s=K+1}^{t} \eta_{s,s}$$

$$= \Gamma_{lin}$$

where $4K$ is the maximum possible regret incurred in the first $K$ rounds under the complex model. By the definition of $E_1$, we get $\Gamma_{no} \leq \sigma \sqrt{((1 + t)/2)\log(1/\delta)}$. Next, let us control $\Gamma_{lin}$. We have

$$\Gamma_{lin} = \sum_{s=K+1}^{t} \left( \mu_{s} + \langle \theta^*, \alpha_{s,s} \rangle - \tilde{\mu}_{s-1} - \langle \tilde{\theta}_s, \alpha_{s,s} \rangle \right)$$

$$= \sum_{s=K+1}^{t} \left( \mu_{s} + \langle \theta^*, \alpha_{s,s} \rangle - \tilde{\mu}_{s-1} - \langle \tilde{\theta}_s, \alpha_{s,s} \rangle \right)$$

$$+ \sum_{s=K+1}^{t} \left( \tilde{\mu}_{s-1} + \langle \tilde{\theta}_s, \alpha_{s,s} \rangle - \tilde{\mu}_{s-1} - \langle \tilde{\theta}_s, \alpha_{s,s} \rangle \right),$$

where the non-positivity of the second term is because of the optimality of arm $j_s$ as expressed in Eq. (6). Hence, we have

$$\Gamma_{lin} \leq \sum_{s=K+1}^{t} \left( \mu_{s} + \langle \theta^*, \alpha_{s,s} \rangle - \tilde{\mu}_{s-1} - \langle \tilde{\theta}_s, \alpha_{s,s} \rangle \right)$$

$$= \sum_{s=K+1}^{t} \mu_{s} - \tilde{\mu}_{s-1} + \sum_{s=K+1}^{t} \langle \alpha_{s,s}, \theta^* - \tilde{\theta}_s \rangle$$

$$\leq \sum_{s=K+1}^{t} \mu_{s} - \tilde{\mu}_{s-1} + \sum_{s=K+1}^{t} ||\alpha_{s,s}||_2 ||\theta^* - \tilde{\theta}_s||_2$$

$$\leq \sum_{s=K+1}^{t} \mu_{s} - \tilde{\mu}_{s-1} + \sum_{s=K+1}^{t} ||\theta^* - \tilde{\theta}_s||_2,$$

where the last two inequalities follow from the Cauchy-Schwarz inequality and the constraint $||\alpha_{s,s}||_2 \leq 1$ respectively. Under the event $E_2$, we have $\mu_{s} - \tilde{\mu}_{s-1} \leq 0$. Also, by the definition of $\tilde{\theta}_s$ and under event $E_3$, we have

$$\Gamma_{lin} \leq 2 \sum_{s=K+1}^{t} K_d(s-1, n).$$
Combining these bounds, we get
\[ R^c_t \leq 4K + W_0(t, n) + \sigma \sqrt{\frac{1 + t}{2} \log \left( \frac{1}{\delta} \right)} + 2 \sum_{s=K+1}^{t} K_3(s-1, n) \leq 4K + 2W_0(t, n) \]
under the assumption that event \( \mathcal{E} \) holds. Since we already showed that \( P(\mathcal{E}) \geq 1 - 5\delta \), our proof is complete.

Now, we move on to the second case. The next lemma shows that if Condition 7 was violated at round \( \tau_* \) (which is, in general, a random variable), then playing the complex model estimates \( j_s \) for all \( s \geq \tau_* \) keeps the regret bounded in subsequent rounds.

**Lemma 6.** If Condition 7 is violated at round \( \tau_* \) that is,
\[ \sum_{s=K+1}^{\tau_*-1} \{ \hat{\mu}_{j_s,s-1} + (\alpha_{j_s,s}, \hat{\theta}_s) - g_{i,s} \} > W_3(\tau_* - 1, n). \]
Then with probability at least \( 1 - 4\delta \) we have,
\[ R^c_{\tau_*:n} := \sum_{s=\tau_*}^{t} [\mu_{j_s} + \langle \theta^*, \alpha_{j_s,s} \rangle - \mu_{j_s} - \langle \theta^*, \alpha_{j_s,s} \rangle] \leq 2W_3(n, n). \]

**Proof.** For this proof, we only need events \( \mathcal{E}_2 \) and \( \mathcal{E}_3 \) to simultaneously hold. We define the event \( \mathcal{E}' := \mathcal{E}_2 \cap \mathcal{E}_3 \). Again, by the union bound we have \( P(\mathcal{E}') \leq 4\delta \). For the rest of this proof we assume the event \( \mathcal{E}' \).

If Condition 7 is violated at round \( \tau_* \), then we have \( A_s = j_s \) for all rounds \( s \geq \tau_* \). Thus,
\[ R^c_{\tau_*:n} = \sum_{s=\tau_*}^{n} [\mu_{j_s} + \langle \theta^*, \alpha_{j_s,s} \rangle - \mu_{j_s} - \langle \theta^*, \alpha_{j_s,s} \rangle] \]
\[ = \sum_{s=\tau_*}^{n} [\mu_{j_s} + \langle \theta^*, \alpha_{j_s,s} \rangle - \hat{\mu}_{j_s,s-1} - \langle \hat{\theta}_s, \alpha_{j_s,s} \rangle] + \sum_{s=\tau_*}^{n} [\hat{\mu}_{j_s,s-1} + \langle \hat{\theta}_s, \alpha_{j_s,s} \rangle - \mu_{j_s} - \langle \theta^*, \alpha_{j_s,s} \rangle] \]
\[ = \sum_{s=\tau_*}^{n} [\mu_{j_s} + \langle \theta^*, \alpha_{j_s,s} \rangle - \hat{\mu}_{j_s,s-1} - \langle \hat{\theta}_s, \alpha_{j_s,s} \rangle] \]
\[ + \sum_{s=\tau_*}^{n} [\hat{\mu}_{j_s,s-1} + \langle \hat{\theta}_s, \alpha_{j_s,s} \rangle - \mu_{j_s} - \langle \theta^*, \alpha_{j_s,s} \rangle] \]
\[ \leq 0 \]
\[ + \sum_{s=\tau_*}^{n} [\hat{\mu}_{j_s,s-1} + \langle \hat{\theta}_s, \alpha_{j_s,s} \rangle - \mu_{j_s} - \langle \theta^*, \alpha_{j_s,s} \rangle], \]
where the second term is non-positive by the optimality of arm \( j_s \) as expressed in Eq. (6). Under the event \( \mathcal{E}_2 \), we have \( \mu_{i,s-1} - \hat{\mu}_{i,s-1} \leq 0 \) for all \( s > 0 \) and \( i \in [K] \). Therefore, we get
\[ R^c_{\tau_*:n} \leq \sum_{s=\tau_*}^{n} [\langle \theta^* - \hat{\theta}_s, \alpha_{j_s,s} \rangle + \langle \hat{\theta}_s - \theta^*, \alpha_{j_s,s} \rangle] + \sum_{s=\tau_*}^{n} [\hat{\mu}_{j_s,s-1} - \mu_{j_s}] \]
\[ \leq \sum_{s=\tau_*}^{n} \| \theta^* - \hat{\theta}_s \| \| \alpha_{j,s,s} \| + \| \alpha_{j,s,s} \| + \sum_{s=\tau_*}^{n} [\hat{\mu}_{j_s,s-1} - \mu_{j_s}] \]
\[ \leq 2 \sum_{s=\tau_*}^{n} \| \theta^* - \hat{\theta}_s \| + \sum_{s=\tau_*}^{n} [\hat{\mu}_{j_s,s-1} - \mu_{j_s}], \]
\[ = \Gamma_{\text{lin}} + \Gamma_{\text{lin}}, \]
where the inequalities follow by two applications of the Cauchy-Schwarz inequality and the constraint \( \| \alpha_{j,s} \| \leq 1 \). First we control \( \Gamma_{\text{lin}} \). Under the event \( \mathcal{E}_3 \) we have
\[ \Gamma_{\text{lin}} = 2 \sum_{s=\tau_*}^{n} \| \theta^* - \hat{\theta}_s \| \leq 4 \sum_{s=\tau_*}^{n} K_3(s-1, n). \]
Next, we control the term $\Gamma_{bias}$. By the definition of $\tilde{\mu}_{j,s-1}$, we have

$$
\Gamma_{bias} = \sum_{s=1}^{n} [\tilde{\mu}_{j,s-1} - \mu_{j,s}] \leq 2\sigma \sum_{s=1}^{n} \sqrt{1 + T_i(s-1) \left( 1 + 2 \log \left( \frac{K(1 + T_i(s-1))^{1/2}}{\delta} \right) \right)}
$$

$$
\leq 2\sigma \sum_{s=1}^{n} \sqrt{1 + T_i(s-1) \left( 1 + 2 \log \left( \frac{Kn^{1/2}}{\delta} \right) \right)}
$$

$$
\leq \left[ 2\sigma \left( 1 + 2 \log \left( \frac{K(n^{1/2})}{\delta} \right) \right) \right] \sum_{i=1}^{K} T_i(n-1) \sum_{r=1}^{T_i(n-1)} \sqrt{1 + \frac{r}{r^2}}
$$

$$
\leq \left[ 2\sigma \left( 1 + 2 \log \left( \frac{Kn^{1/2}}{\delta} \right) \right) \right] \sum_{i=1}^{K} \sum_{r=1}^{T_i(n-1)} 2\sqrt{\frac{1}{r}}
$$

$$
\leq \left[ 2\sigma \left( 1 + 2 \log \left( \frac{Kn^{1/2}}{\delta} \right) \right) \right] \sum_{i=1}^{K} \sqrt{T_i(n-1)}
$$

$$
\leq \left( 5 \right) \left[ 2\sigma \left( 1 + 2 \log \left( \frac{Kn^{1/2}}{\delta} \right) \right) \right] \sqrt{Kn},
$$

where (i) follows by Jensen’s inequality and the fact that $\sum_{i=1}^{K} T_i(n-1) = n-1 < n$. The rest of the inequalities can be verified by some simple algebra. Combining the bounds on the respective terms, we get

$$
R_{\tau,n}^c \leq 4 \sum_{s=1}^{n} K_\delta(s-1,n) + \left[ 2\sigma \left( 1 + 2 \log \left( \frac{Kn^{1/2}}{\delta} \right) \right) \right] \sqrt{Kn} \leq 2\mathcal{W}_3(n,n),
$$

which completes the proof. 

Armed with these two lemmas, we are now ready to complete the proof of Part (b) of Theorem 3, and bound the regret under the complex model.

Proof [Proof of Part (b) of Theorem 3] We recap the two cases below.

Case 1: Assume that Condition 7 is never violated throughout the run of the algorithm. Then by Lemma 5 we have

$$
R_n^c \leq 4K + \mathcal{W}_3(n,n)
$$

with probability at least $1 - 5\delta$.

Case 2: Assume that Condition 7 is violated in round $\tau_s < n$. We know by Lemma 5 that

$$
R_{\tau_s-2}^c \leq 4K + \mathcal{W}_3(n,n)
$$

with probability at least $1 - 5\delta$. Also, by Lemma 6, we have

$$
R_{\tau_s,n}^c := \sum_{s=\tau_s}^{n} [\mu_{\kappa,s} + \langle \theta^{*}, \alpha_{\kappa,s} \rangle - \mu_{\kappa} - \langle \theta^{*}, \alpha_{\kappa,s} \rangle] \leq 2\mathcal{W}_3(n,n)
$$

with probability at least $1 - 4\delta$. We can decompose the cumulative regret up to round $n$ as follows:

$$
R_n^c \leq R_{\tau_s-1}^c + R_{\tau_s,n}^c + 4,
$$

where $R_{\tau_s,n}^c$ denotes the regret of the algorithm starting from round $\tau^*$ up to round $n$ and the 4 appears as it is the maximum regret that could be incurred in round $\tau_s$ by the algorithm under the complex model. By taking a union bound and using the decomposition of the regret above, we get

$$
R_n^c \leq 4(K + 1) + 3\mathcal{W}_3(n,n),
$$

with probability at least $1 - 9\delta$. 



5 Experiments

To experimentally verify our claims, we ran our model-selecting algorithm, OSOM, on both simple and complex instances. We compared its performance to that of UCB (which is optimal up to logarithmic factors under the simple model) and OFUL (which is minimax optimal under the complex model). The data was generated synthetically with the number of arms $K = 5$, and the dimension of $\theta^\ast, d = 50$. The mean rewards of the arms $\mu_i \sim \text{Unif}(-1, 1)$, were drawn independently from a uniform distribution, and the context vectors $\alpha_{i,t}$ were drawn independently from the uniform distribution over the sphere. The noise $\eta_{i,t} \sim \mathcal{N}(0, 1)$ was drawn from a 1-dimensional Gaussian with unit variance. Under the simple model $\theta^\ast = 0$, while under the complex model $\theta^\ast$ was also drawn from the uniform distribution over the unit sphere in $d$-dimensions. In both the experiments we average over 50 runs over 300 rounds to estimate the expected regret incurred. The realizations of the problem were drawn independently for each run of each algorithm.

When data is generated according to the simple model ($\theta^\ast = 0$), we see that OSOM and UCB suffer regret that is sub-linear, and is significantly lower than the regret suffered by OFUL whose regret is also sub-linear but pays for the additional variance of estimating a more complex model. While when the data is generated from the complex model ($\|\theta^\ast\|_2 = 1$) the regret suffered by UCB is linear as it does not identify and estimate the linear structure of the mean rewards. Here, the regret suffered by both OFUL and OSOM is sub-linear and almost identical.

6 Discussion

We were able to successfully obtain minimax-optimal rates in both regimes under suitable stochastic conditions on the contextual information. This is a natural step to understanding data-dependent model selection for contextual bandits. A number of exciting directions remain open for future work.

- We crucially relied on the linear structure of the rewards to obtain our regret bounds. It is conceivable that this linearity is not essential, and that these algorithmic ideas could be generalized to arbitrary nested models.

- Another interesting direction would be to investigate bounds on overall reward when the data is generated from a richer model that is not from a linear model or a simple bandit model, but can be reasonably approximated by it.
• Our guarantees here are under a stochastic assumption on both the rewards and the distribution of the contexts. It would be interesting to understand whether these assumptions can be loosened, or if there exist fundamental limitations to model-selecting under bandit feedback in adversarial settings.

Acknowledgements

The authors would like to thank Kush Bhatia, Akshay Krishnamurthy and Anant Sahai for helpful discussions. We gratefully acknowledge the support of the NSF through grants AST-1444078, IIS-1619362 and ECCS-1343398, and to ML4Wireless center member companies. This work was done in part while the authors were visiting the Simons Institute for the Theory of Computing.

References

[AB10] Jean-Yves Audibert and Sébastien Bubeck. “Regret bounds and minimax policies under partial monitoring”. In: Journal of Machine Learning Research 11.Oct (2010), pp. 2785–2836 (Cited on page 4).

[ACF02] Peter Auer, Nicolo Cesa-Bianchi, and Paul Fischer. “Finite-time analysis of the multiarmed bandit problem”. In: Machine Learning 47.2-3 (2002), pp. 235–256 (Cited on pages 2, 4).

[Aga+14] Alekh Agarwal, Daniel Hsu, Satyen Kale, John Langford, Lihong Li, and Robert E Schapire. “Taming the monster: A fast and simple algorithm for contextual bandits”. In: International Conference on Machine Learning. 2014, pp. 1638–1646 (Cited on pages 1, 2).

[Aga+16] Alekh Agarwal, Sarah Bird, Markus Cozowicz, Luong Hoang, John Langford, Stephen Lee, Jiaji Li, Dan Melamed, Gal Oshri, Oswaldo Ribas, Siddhartha Sen, and Alex Slivkins. “Making contextual decisions with low technical debt”. In: arXiv preprint arXiv:1606.03966 (2016) (Cited on page 1).

[Aga+17] Alekh Agarwal, Haipeng Luo, Behnam Neyshabur, and Robert E Schapire. “Corralling a Band of Bandit Algorithms”. In: Conference on Learning Theory. 2017, pp. 12–38 (Cited on page 3).

[APS11] Yasin Abbasi-Yadkori, Dávid Pál, and Csaba Szepesvári. “Improved algorithms for linear stochastic bandits”. In: Advances in Neural Information Processing Systems. 2011, pp. 2312–2320 (Cited on pages 2, 4, 5, 9, 19).

[Aue+02] Peter Auer, Nicolo Cesa-Bianchi, Yoav Freund, and Robert E Schapire. “The non-stochastic multi-armed bandit problem”. In: SIAM Journal on Computing 32.1 (2002), pp. 48–77 (Cited on pages 1, 2).

[Bey+11] Alina Beygelzimer, John Langford, Lihong Li, Lev Reyzin, and Robert E Schapire. “Contextual bandit algorithms with supervised learning guarantees”. In: Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics. 2011, pp. 19–26 (Cited on page 1).

[BS12] Sébastien Bubeck and Aleksandrs Slivkins. “The best of both worlds: Stochastic and adversarial bandits”. In: Conference on Learning Theory. 2012, pp. 42–1 (Cited on page 3).

[Chu+11] Wei Chu, Lihong Li, Lev Reyzin, and Robert E Schapire. “Contextual bandits with linear payoff functions”. In: Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics. 2011, pp. 208–214 (Cited on pages 2, 5).

[DP16] Rémy Degenne and Vianney Perchet. “Anytime optimal algorithms in stochastic multi-armed bandits”. In: International Conference on Machine Learning. 2016, pp. 1587–1595 (Cited on page 4).

[FK18] Dylan J Foster and Akshay Krishnamurthy. “Contextual bandits with surrogate losses: Margin bounds and efficient algorithms”. In: Advances in Neural Information Processing Systems. 2018, pp. 2621–2632 (Cited on pages 1, 3).
Dylan J Foster, Zhiyuan Li, Thodoris Lykouris, Karthik Sridharan, and Eva Tardos. “Learning in games: Robustness of fast convergence”. In: Advances in Neural Information Processing Systems. 2016, pp. 4734–4742 (Cited on page 3).

Akshay Krishnamurthy, John Langford, Aleksandrs Slivkins, and Chicheng Zhang. “Contextual Bandits with Continuous Actions: Smoothing, Zooming, and Adapting”. In: arXiv preprint arXiv:1902.01520 (2019) (Cited on page 3).

Lihong Li, Wei Chu, John Langford, and Robert E Schapire. “A contextual-bandit approach to personalized news article recommendation”. In: Proceedings of the 19th International conference on World Wide Web. ACM. 2010, pp. 661–670 (Cited on pages 1–3).

T.L Lai and Herbert Robbins. “Asymptotically Efficient Adaptive Allocation Rules”. In: Adv. Appl. Math. 6.1 (Mar. 1985), pp. 4–22. ISSN: 0196-8858. DOI: 10.1016/0196-8858(85)90002-8. URL: http://dx.doi.org/10.1016/0196-8858(85)90002-8 (Cited on page 4).

Tor Lattimore and Csaba Szepesvári. Bandit Algorithms. Cambridge University Press (preprint), 2019 (Cited on page 19).

John Langford and Tong Zhang. “The epoch-greedy algorithm for multi-armed bandits with side information”. In: Advances in Neural Information Processing Systems. 2008, pp. 817–824 (Cited on pages 1, 2).

H Brendan McMahan and Matthew Streeter. “Tighter bounds for multi-armed bandits with expert advice”. In: (2009) (Cited on page 1).

Roberto Imbuzeiro Oliveira. “Concentration of the adjacency matrix and of the Laplacian in random graphs with independent edges”. In: arXiv preprint arXiv:0911.0600 (2009) (Cited on page 4).

Alexander Rakhlin and Karthik Sridharan. “BISTRO: An efficient relaxation-based method for contextual bandits”. In: Proceedings of the 33rd International Conference on International Conference on Machine Learning-Volume 48. JMLR. org. 2016, pp. 1977–1985 (Cited on pages 1, 3).

Vasilis Syrgkanis, Akshay Krishnamurthy, and Robert E Schapire. “Efficient algorithms for adversarial contextual learning”. In: International Conference on Machine Learning. 2016, pp. 2159–2168 (Cited on pages 1, 3).

Aleksandrs Slivkins. “Contextual bandits with similarity information”. In: The Journal of Machine Learning Research 15.1 (2014), pp. 2533–2568 (Cited on page 4).

Vasilis Syrgkanis, Haipeng Luo, Akshay Krishnamurthy, and Robert E Schapire. “Improved regret bounds for oracle-based adversarial contextual bandits”. In: Advances in Neural Information Processing Systems. 2016, pp. 3135–3143 (Cited on pages 1, 3).

Ambuj Tewari and Susan A Murphy. “From ads to interventions: Contextual bandits in mobile health”. In: Mobile Health. Springer, 2017, pp. 495–517 (Cited on pages 1–3).

Joel Tropp. “Freedman’s inequality for matrix martingales”. In: Electronic Communications in Probability 16 (2011), pp. 262–270 (Cited on pages 4, 19).

Michael Woodroofe. “A one-armed bandit problem with a concomitant variable”. In: Journal of the American Statistical Association 74.368 (1979), pp. 799–806 (Cited on pages 1, 2).
Appendix

A Omitted Proof Details

We recall Lemma 2, which is an error bound on the ridge regression estimate $\hat{\theta}_t$, and present a proof below.

**Lemma 2.** Let $\hat{\theta}_t$ be defined as in Eq. (3). Then, with probability at least $1 - 3\delta$ we have that for all $t > K$, $\theta^*$ lies in the set

$$C^*_t := \{ \theta \in \mathbb{R}^d : \| \theta - \hat{\theta}_t \|_2 \leq K_0(t, n) \}, \quad (4)$$

where $K_0(t, n) = \tilde{O}(\sigma \sqrt{d \cdot n})$ is defined in Eq. (11).

**Proof.** To unclutter notation, let $\alpha = \alpha_{K+1:t}, G = G_{K+1:t}$. Further, define $\eta = [\eta_{A_{K+1}, K+1}, \ldots, \eta_{A_t}]^T$, $\mu = [\mu_{A_{K+1}}, \ldots, \mu_{A_t}]^T$ and $\tilde{\mu} = [\tilde{\mu}_{A_{K+1}, K}, \ldots, \tilde{\mu}_{A_t}]^T$. By the definition of $\hat{\theta}_t$, we have

$$\hat{\theta}_t = (\alpha^T \alpha + I)^{-1} \alpha^T G$$

$$= (\alpha^T \alpha + I)^{-1} \alpha^T (\alpha \theta^* + (\mu - \tilde{\mu} + \eta))$$

$$= \theta^* - (\alpha^T \alpha + I)^{-1} \theta^* + (\alpha^T \alpha + I)^{-1} \alpha^T (\mu - \tilde{\mu}) + (\alpha^T \alpha + I)^{-1} \alpha^T \eta.$$ 

Now, let us define $V_t := \alpha^T \alpha + I$. Then, for any vector $w \in \mathbb{R}^d$ (whose choice we will specify shortly), we get

$$w^T(\hat{\theta}_t - \theta^*) = -w^T V_t^{-1} \theta^* + w^T V_t^{-1} \alpha^T (\mu - \tilde{\mu}) + w^T V_t^{-1} \alpha^T \eta$$

$$= -w^T V_t^{-1/2} V_t^{-1/2} \theta^* + w^T V_t^{-1/2} V_t^{-1/2} \alpha^T (\mu - \tilde{\mu}) + w^T V_t^{-1/2} V_t^{-1/2} \alpha^T \eta.$$ 

By the Cauchy-Schwartz inequality, we have

$$\left| w^T(\hat{\theta}_t - \theta^*) \right| \leq \| w \|_{V_t^{-1}} \left( \| \alpha^T \eta \|_{V_t^{-1}} + \| \theta^* \|_{V_t^{-1}} + \| \alpha^T (\mu - \tilde{\mu}) \|_{V_t^{-1}} \right),$$

$$\leq \| w \|_{V_t^{-1}} \left( \| \alpha^T \eta \|_{V_t^{-1}} + \| \alpha^T (\mu - \tilde{\mu}) \|_{V_t^{-1}} + 1 \right), \quad (14)$$

where the second step follows as $\| \theta^* \|_{V_t^{-1}} \leq \sqrt{(1/\gamma_{\min}(V_t))} \cdot \| \theta^* \|_2 \leq 1$. We now define three events $E_4, E_5$ and $E_6$ below:

$$E_4 := \left\{ \| \alpha^T \eta \|_{V_t^{-1}} \leq \sqrt{2\sigma^2 \log \left( \frac{\det(V_t)^{1/2}}{\delta} \right)} \right\},$$

$$E_5 := \left\{ N_t := \left\| \sum_{s=K+1}^{t} \alpha_{A_{s, t}} (\mu_{A_{s, t}} - \tilde{\mu}_{A_{s, t}}) \right\|_2 \leq \Upsilon_\delta(t, n), \forall t \in \{K + 1, \ldots, n\} \right\},$$

$$E_6 := \left\{ \gamma_{\min}(V_t) \geq 1 + \rho_{\min}(t - K)/2, \forall t \in \{K + \tau_{\min}(\delta, n), \ldots, n\} \right\}.$$ 

Define the event $E'' := E_4 \cap E_5 \cap E_6$. By Theorem 9 with $V = I$ we have, $P(E''_t) \leq \delta$, by Lemma 8 we have $P(E'_t) \leq \delta$ and Lemma 7 tells us that $(P(E'_t) \leq \delta$. Therefore by a union bound $P(E'') \leq 3\delta$. For the rest of the proof, we assume the event $E''$. Hence, we get

$$\| \alpha^T \eta \|_{V_t^{-1}} \leq \sqrt{2\sigma^2 \log \left( \frac{\det(V_t)^{1/2}}{\delta} \right)} \leq \sqrt{2\sigma^2 \left( \frac{d}{2} \log \left( 1 + \frac{t}{d} \right) + \log \left( \frac{1}{\delta} \right) \right)}, \quad (15)$$

where $(i)$ follows by the technical Lemma 11. For the other term, we have

$$\| \alpha^T (\mu - \tilde{\mu}) \|_{V_t^{-1}} \leq \frac{N_t}{\sqrt{\gamma_{\min}(V_t)}} \leq \frac{\Upsilon_\delta(t, n)}{\sqrt{\gamma_{\min}(V_t)}}, \quad (16)$$

16
Under $\mathcal{E}_0$, we have

$$
\|\alpha^\top(\mu - \hat{\mu})\|_{V_1^{-1}} \leq \begin{cases} 
  \frac{\mathcal{T}_0(t, n)}{\sqrt{1 + \rho_{\min}(t - K)/2}}, & \text{if } \tau_{\min}(\delta) \geq t - K > 0, \\
  \frac{\mathcal{Y}_0(t, n)}{\sqrt{1 + \rho_{\min}(t - K)/2}}, & \text{if } t - K > \tau_{\min}(\delta).
\end{cases}
$$

(17)

Choosing $w = V_1^{-1}(\hat{\theta}_t - \theta^*)$ and plugging in the upper bounds established in Eq. (15) and Eq. (17) into Eq. (14), we get

$$
\|\hat{\theta}_t - \theta^*\|_{V_1} \leq \begin{cases} 
  \mathcal{M}_0(t) + \mathcal{Y}_0(t, n), & \text{if } \tau_{\min}(\delta, n) \geq t - K > 0, \\
  \mathcal{M}_0(t) + \frac{\mathcal{Y}_0(t, n)}{\sqrt{1 + \rho_{\min}(t - K)/2}}, & \text{if } t - K > \tau_{\min}(\delta, n),
\end{cases}
$$

Recall the definition of $\mathcal{M}_0(t)$ in Eq. (10). Using the fact that $\|\hat{\theta}_t - \theta^*\|_2 \leq (1/\sqrt{\gamma_{\min}(V_1)})\|\hat{\theta}_t - \theta^*\|_{V_1}$ along with the event $\mathcal{E}_0$, we get

$$
\|\hat{\theta}_t - \theta^*\|_2 \leq \begin{cases} 
  \mathcal{M}_0(t) + \mathcal{Y}_0(t, n), & \text{if } \tau_{\min}(\delta, n) \geq t - K > 0, \\
  \mathcal{M}_0(t) + \frac{\mathcal{Y}_0(t, n)}{\sqrt{1 + \rho_{\min}(t - K)/2}}, & \text{if } t - K > \tau_{\min}(\delta, n),
\end{cases}
$$

where the last equality is by the definition of $\mathcal{K}_0(t, n)$ in Eq. (11).

Now, we establish a couple of concentration inequalities on quantities of interest in the proof of Lemma 2: these constitute Lemmas 7 and 8.

**Lemma 7.** Define the matrix $M_t$ as

$$
M_t := I + \sum_{s=1}^t \alpha_{A_s,s}A_{A_s,s}^\top.
$$

Then, with probability at least $1 - \delta$, we have

$$
\gamma_{\min}(M_t) \geq 1 + \frac{\rho_{\min}t}{2}
$$

for all $\tau_{\min}(\delta, n) \leq t \leq n$.

**Proof** Note that by the definition of $M_t$, we have $\gamma_{\min}(M_t) = 1 + \gamma_{\min}\left(\sum_{s=1}^t \alpha_{A_s,s}A_{A_s,s}^\top\right)$. By the assumption on the distribution of the contexts as specified in Eq. (1), we have $E_{s-1}[\alpha_{A_s,s}A_{A_s,s}^\top] = \Sigma_c \geq \rho_{\min}I$. Consider the matrix martingale defined by

$$
Z_t := \sum_{s=1}^t [\alpha_{A_s,s}A_{A_s,s}^\top - \Sigma_c] \text{ for } t = 1, 2, \ldots
$$

with $Z_0 = 0$ and the corresponding martingale difference sequence $Y_s := Z_s - Z_{s-1}$ for $s \in \{1, 2, \ldots\}$. As $\|\alpha_{A_s,s}\|_2 \leq 1$ and $\|\Sigma_c\|_{op} = \|E_{s-1}[\alpha_{A_s,s}A_{A_s,s}^\top]\|_{op} \leq 1$, we have

$$
\|Y_s\|_{op} = \|\alpha_{A_s,s}A_{A_s,s}^\top - \Sigma_c\|_{op} \leq 2.
$$

We also have,

$$
\|E_{s-1}[Y_sY_s^\top]\|_{op} = \|E_{s-1}[Y_s^\top Y_s]\|_{op} = \|E_{s-1}[\{\alpha_{A_s,s}A_{A_s,s}^\top - \Sigma_c\}(\alpha_{A_s,s}A_{A_s,s}^\top - \Sigma_c)]\|_{op} \leq \|E_{s-1}[\{A_{A_s,s}^\top A_{A_s,s} - \Sigma_c^2\}]\|_{op} \leq 2.
$$

By applying the Matrix Freedman inequality (Theorem 10 in Appendix B) with $R = 2$, $\omega_2 = 2t$ and $u = \rho_{\min}t/2$, we get that if $t \geq (16/\rho_{\min} + 8/(3\rho_{\min})) \log(2dn/\delta)$, then

$$
\mathbb{P}\left\{ \left\| \sum_{s=1}^t \alpha_{A_s,s}A_{A_s,s}^\top - t \cdot \Sigma_c \right\|_{op} \geq \frac{\rho_{\min}t}{2} \right\} \leq \frac{\delta}{n}.
$$
This implies that
\[ \gamma_{\text{min}} \left( \sum_{s=1}^{t} \alpha_{A_{s}, s} \alpha_{A_{s}, s}^{T} \right) \geq \frac{\rho_{\text{min}}}{2} \]
for a given \( t \in \{ \tau_{\text{min}}(\delta, n), \ldots, n \} \) with probability at least \( 1 - \delta/n \). Taking a union bound over all \( t \in \{ \tau_{\text{min}}(\delta, n), \ldots, n \} \) yields the desired claim.

**Lemma 8.** Define the vector \( N_t := \sum_{s=K+1}^{t} \alpha_{i_{s}, s} (\mu_{i_{s}} - \tilde{\mu}_{i_{s}, s-1}) \). For all \( K < t \leq n \) we have,
\[ \|N_t\|_2 \leq \Upsilon_\delta(t, n), \]
with probability at least \( 1 - \delta \).

**Proof** Consider \( K < t \leq n \). Note that \( \tilde{\mu}_{i_{s}, s-1} \) is a function of \( g_{i_{1}, 1}, \ldots, g_{i_{s-1}, s-1} \) and \( i_1, \ldots, i_{s-1} \). Also, the simple model estimate \( \hat{i}_{s} \) is just a function of \( g_{i_{1}, 1}, \ldots, g_{i_{s-1}, s-1} \) and \( A_{1}, \ldots, A_{s-1} \).

Therefore, we have
\[ \mathbb{E}_{s-1}[\alpha_{i_{s}, s} (\mu_{i_{s}} - \tilde{\mu}_{i_{s}, s-1})] = (\mu_{i_{s}} - \tilde{\mu}_{i_{s}, s-1}) \mathbb{E}_{s-1}[\alpha_{i_{s}, s}] = 0 \]
for all \( s \in \{ K+1, \ldots, t \} \), as \( \alpha_{i_{s}, s} \) is assumed to be drawn from a distribution with zero (conditional) mean. Recall that \( \mu_{i_{s}} \in [-1, 1] \). By the definition \( \tilde{\mu}_{i_{s}, s-1} \), we have
\[ \hat{\mu}_{i_{s}, s-1} = \sum_{r=1}^{s-1} g_{i_{r}, r} \frac{[A_{r} = i_{r}]}{T_{i_{r}}(s-1)} + \sigma \sqrt{\frac{1 + T_{i_{s}}(s-1)}{T_{i_{s}}(s-1)}} \sqrt{\frac{1 + 2 \log \left( \frac{K(1+T_{i_{s}}(s-1))^{1/2}}{\delta} \right)}{2}} \leq 2 + 2 \log \left( \frac{K(1+n)}{\delta} \right) =: \mathcal{P}_n, \quad \forall s \in \{1, \ldots, n\}. \]

Define a martingale \( Z_{t-K} := N_t \) and the martingale difference sequence \( Y_s := Z_s - Z_{s-1} \). Then we have, for any \( s \in \{ K+1, \ldots, t \} \),
\[ \|Y_{s-K}\|_{op} = \|Y_{s-K}\|_2 \leq \|\alpha_{i_{s}, s} (\mu_{i_{s}} - \tilde{\mu}_{i_{s}, s-1})\|_2 \leq \|\alpha_{i_{s}, s}\|_2 \|\mu_{i_{s}} - \tilde{\mu}_{i_{s}, s-1}\| \leq |\mu_{i_{s}} - \tilde{\mu}_{i_{s}, s-1}| \leq \mathcal{P}_n. \]

We also have
\[ \left\| \mathbb{E}_{s-1} \left[ \alpha_{i_{s}, s} \alpha_{i_{s}, s}^T (\mu_{i_{s}} - \tilde{\mu}_{i_{s}, s-1})^2 \right] \right\|_{op} \leq \mathcal{P}_n^2 \|\Sigma_c\|_{op} \leq \mathcal{P}_n^2, \]
and
\[ \left\| \mathbb{E}_{s-1} \left[ \alpha_{i_{s}, s}^T \alpha_{i_{s}, s} (\mu_{i_{s}} - \tilde{\mu}_{i_{s}, s-1})^2 \right] \right\|_{op} \leq \mathcal{P}_n^2 \|\alpha_{i_{s}, s}\|^2 \leq \mathcal{P}_n^2. \]

Invoking Theorem 10 with \( R = \mathcal{P}_n \) and \( \omega^2 = \mathcal{P}_n^2(t - K) \), we get
\[ \mathbb{P} \left\{ \|N_t\|_2 \geq \frac{\mathcal{P}_n}{3} \log \left( \frac{2dn}{\delta} \right) + \frac{\mathcal{P}_n}{3} \sqrt{18(t-K) \log \left( \frac{2dn}{\delta} \right) + \log^2 \left( \frac{2dn}{\delta} \right)} \right\} \leq \frac{\delta}{n}. \]

From the definition of \( \Upsilon_{\delta}(t, n) \) in Eq. (9) and applying the union bound over all \( t \in \{ K+1, \ldots, n \} \), we get
\[ \mathbb{P} \{ \exists t \in \{ K+1, \ldots, n \} : \|N_t\|_2 \geq \Upsilon_{\delta}(t, n) \} \leq \delta. \]

This completes the proof.
B Concentration Inequalities and Technical Results

In this section we state technical concentration inequalities that are useful in our proofs. We start by defining notation specific to this section.

Let \( \{ \mathcal{F}_t \}_t \) be a filtration. Let \( \{ \xi_t \}_t \) be a real-valued stochastic process such that \( \xi_t \) is \( \mathcal{F}_t \)-measurable and \( \xi_t \) is conditionally \( \sigma \)-sub-Gaussian. Let \( \{ Y_t \}_t \) be an \( \mathbb{R}^d \)-valued stochastic process such that \( Y_t \) is \( \mathcal{F}_{t-1} \)-measurable. Assume that \( V \) is a \( d \times d \) positive definite matrix. For any \( t > 0 \) define

\[
V_t := V + \sum_{s=1}^{t} Y_s Y_s^\top, \quad S_t := \sum_{s=1}^{t} \xi_s Y_s.
\]

With this setup in place, the following is a re-statement of Theorem 1 of Abbasi-Yadkori, Pál, and Szepesvári [APS11], which is essentially a self-normalized concentration inequality.

**Theorem 9.** For any \( \delta > 0 \), we have

\[
S_t^\top V_t^{-1} S_t = \|S_t\|_{V_t^{-1}}^2 \leq 2\sigma^2 \log \left( \frac{\det(V_t)^{1/2} \det(V_t)^{-1/2}}{\delta} \right)
\]

with probability at least \( 1 - \delta \) for all \( t \geq 0 \).

Next we state a version of the Matrix Freedman Inequality due to Tropp [Tropp11, Corollary 1.3] that we use multiple times in our arguments. Define a matrix martingale as a sequence \( \{ Z_s : s = 0, 1, \ldots \} \) such that \( Z_0 = 0 \) and

\[
\mathbb{E}[Z_s|\mathcal{F}_{s-1}] = Z_{s-1} \quad \text{and} \quad \mathbb{E}[\|Z_s\|_{op}] \leq \infty, \quad \text{for } s = 1, \ldots.
\]

Also define the martingale difference sequence \( X_s := Z_s - Z_{s-1} \).

**Theorem 10.** Consider a matrix martingale \( \{ Z_s : s = 0, 1, \ldots \} \) whose values are matrices with dimension \( d_1 \times d_2 \), and let \( \{ X_s : s = 0, 1, \ldots \} \) be the martingale difference sequence. Assume that the difference sequence is almost surely uniformly bounded, i.e.,

\[
\|X_s\|_{op} \leq R \quad \text{a.s.} \quad \text{for } s = 1, 2, \ldots
\]

Define two predictable quadratic variation processes of the martingale:

\[
W_{col,t} := \sum_{s=1}^{t} \mathbb{E}[X_s X_s^\top | \mathcal{F}_{s-1}] \quad \text{and} \quad W_{row,t} := \sum_{s=1}^{t} \mathbb{E}[X_s^\top X_s | \mathcal{F}_{s-1}]
\]

for \( t = 1, 2, \ldots \)

Then for all \( u \geq 0 \) and \( \omega^2 > 0 \), we have

\[
\mathbb{P} \left\{ \exists t \geq 0 : \|Z_t\|_{op} \geq u \text{ and } \max \{ \|W_{col,t}\|_{op}, \|W_{row,t}\|_{op} \} \leq \omega^2 \right\} \leq (d_1 + d_2) \exp \left( -\frac{u^2/2}{\omega^2 + Ru/3} \right).
\]

The final technical result we recap characterizes the growth of the determinant of the matrix \( V_n \), and is useful in constructing our confidence sets for the estimate of \( \theta^* \). This result is a restatement of Lemma 19.1 in the pre-print [LS19].

**Lemma 11.** Let \( V_0 \in \mathbb{R}^{d \times d} \) be a positive definite matrix and \( z_1, \ldots, z_n \in \mathbb{R}^d \) be a sequence of vectors with \( \|z_t\|_2 \leq L < \infty \) for all \( t \in [n] \). Further, let \( v_0 := \text{tr}(V_0) \) and \( V_n := V_0 + \sum_{s=1}^{n} z_s z_s^\top \). Then, we have

\[
\log \left( \frac{\det(V_n)}{\det(V_0)} \right) \leq d \log \left( \frac{v_0 + nL^2}{d \det^{1/d}(V_0)} \right).
\]