On Synchronization, Persistence and Seasonality in some Spatially Inhomogeneous Models in Epidemics and Ecology

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(December 17, 2013)

Recent studies in ecology and epidemiology indicate that it is important to include spatial heterogeneity, synchronization and seasonality in the theoretical models. In this work, spatial heterogeneity is introduced via coupled map lattices (CML) and partial differential equations. Stability and persistence of some realistic CML are discussed. Chaos control, synchronization and persistence are studied for some CML. Some applications in population biology and ecology are given. A simple method for finding the sufficient conditions for the existence of periodic solutions for differential equations with periodic coefficients is given. This will simplify the study of seasonality.

Key words: Spatial heterogeneity, Synchronization, Persistence, Seasonality, Differential equations with periodic coefficients.

I. INTRODUCTION

Recently it has been argued\cite{10,15} that synchronizing local epidemics increases the probability of its global fade out or eradication through careful vaccination program. Similarly synchrony between subpopulations may be an important element in species extinction. Therefore studying spatial synchrony, chaos control to achieve synchrony, and persistence in spatial systems is an important task. Also it was shown\cite{10,15} that seasonality is an important factor that should be included. This work is an attempt to model some of these factors simultaneously.

Chaos is an interesting topic with applications in many fields\cite{13,19}. Most interests are to study the time evolution of systems, while in many realistic systems spatial effects should be included. Hence both cellular automata (CA)\cite{4} and spatiotemporal chaos (STC) has arisen\cite{14,9}. In some cases they generalize differential equations. Typically STC arises in systems with large sizes and in systems consisting of many different components. These two conditions are satisfied in many biological and economic systems, hence studying STC in these systems is important.

The concept of coupled map lattices (CML) was introduced by Kaneko\cite{14} as a simple model with the essential features of STC. It can also be used to approximate partial differential equations. Hence it will be used here to model some biological and economic systems. In other cases partial differential equations will be used directly to model spatial heterogeneity.

The paper is organized as follows: In section 2, some applicable CML models are given. A CML for a monopoly of a company producing different products is studied. Also we have studied some problems in population biology like prey-predator, migrating salmon in a river and Schistosomiasis. Chaos control and chaos synchronization in CML are discussed in section 3. In section 4, persistence in CML is studied. Section 5 is devoted for developing a method to find the sufficient conditions for the existence of periodic solutions for differential equations with periodic coefficients. Some conclusions are given in section 6.

II. SOME APPLICABLE CML

In this section, some applicable CML are presented, then chaos control and synchronization for CML are studied. We begin by a model which is relevant to both economy and biology. Consider a monopoly model where a company produces quantity \(q(t)\) of a goods, then the profit function is given by

\[ \Pi(t) = q(t)[a - bq(t) - c], \]  \hspace{1cm} (1)

where \(a, b\) are some constants and \(c\) is the cost per unit of the product. The quantity produced in the next time step is expected to be in proportion to the profit i.e. \(q(t + 1) = \alpha \Pi(t)\), \(\alpha\) is a proportionality constant. Rescaling \(q(t)\), one gets the following dynamical system
\[ \dot{q}(t + 1) = r \tilde{q}(t)[1 - \tilde{q}(t)] \]  

(2)

where

\[ \tilde{q}(t) = \frac{b}{a - c}q(t), \quad r = \alpha(a - c). \]

Now assume that the company produces \( n \) different products \( \tilde{q}_i, i = 1, 2, \ldots, n \) (or it has different production sites), then the system (2) is replaced by the following CML

\[ q(i, t + 1) = \Pi(i, t) + D[\Pi(i + 1, t) + \Pi(i - 1, t) - 2\Pi(i, t)], \]

(3)

\[ \Pi(i, t) = r(i)q(i, t)[1 - q(i, t)], \]

where the tilde has been dropped for simplicity and \( D \) is a constant representing diffusion of the profits between different components of the company. Here \( D \) is considered to be small. It is straightforward to find the steady states of Eq. (3)

\[ q(i) \approx q_i^0 + Dq_i^1, \quad q_i^0 = 1 - \frac{1}{r(i)}, \quad q_i^1 = \frac{q_{i-1}^0 + q_{i+1}^0 - 2q_i^0}{r(i) - 1}. \]  

(4)

To determine the chaotic behavior of the system, local Lyapunov exponents \( \lambda_i, i = 1, 2, \ldots, n \) should be calculated. It is important to recall that the system exhibits STC if \( D_L \) is proportional to the volume of the system, where

\[ D_L = K + \frac{1}{|\lambda_{K+1}|} \sum_{i=1}^{K} \lambda_i, \quad \sum_{i=1}^{K} \lambda_i > 0 \text{ and } \sum_{i=1}^{K+1} \lambda_i < 0. \]  

(5)

In 1-dimensional systems this is equivalent to the condition that the number of positive Lyapunov exponents is proportional to the size of the chain. For \( D \ll 1 \), local Lyapunov exponents are given by

\[ \lambda_i \approx \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \ln |(1 - 2D)r(i)(1 - 2q(i, t))|. \]  

(6)

It is clear that the effect of diffusion is to stabilize the system. This may be interesting economically.

Consider single species living in patches \( i = 1, 2, \ldots, n \) with small number diffusing between the neighboring patches, then the corresponding CML is

\[ q_i^{t+1} = r_i q_i^t (1 - q_i^t) + D (q_{i-1}^t + q_{i+1}^t - 2q_i^t) \]  

(7)

The nonzero steady states are given by Eq. (4), and Lyapunov exponents are given by Eq. (6). Again diffusion has a stabilizing effect. This confirms that, in general, spatial diversity stabilizes the biological system.

A more realistic formulation for CML (Eqs. (3), (7)) which takes into account the fact that interactions have finite propagation speed is [21]

\[ q_i^{t+1} = r_i q_i^t (1 - q_i^t) + D (q_{i-1}^{t-1} + q_{i+1}^{t-1} - 2q_i^t) \]

The steady states are not affected by the delay.

Applying the above analysis to the 2-dimensional Lotka-Volterra predator \((y)\), prey \((x)\) CML [20]

\[ x_{i+1}(i, j) = \mu x_i(i, j)[1 - x_i(i, j) - y_i(i, j)] + D_1[x_{i+1}(i, j) + x_{i-1}(i, j) + x_{i}(i, j + 1) + x_{i}(i, j - 1) - 4x_{i}(i, j)] + y_{i+1}(i, j) = \beta x_i(i, j)y_i(i, j) + \]

\[ D_2[y_{i+1}(i, j) + y_{i-1}(i, j) + y_{i}(i, j + 1) + y_{i}(i, j - 1) - 4y_{i}(i, j)], \]

(8)

where \( \mu, \beta, D_1 \) and \( D_2 \) are some positive constants. The coexistence steady state \((x \neq 0, y \neq 0)\) are

\[ x(i, j) = \frac{1}{\beta}, \quad y(i, j) = 1 - \frac{1}{\beta} - \frac{1}{\mu}. \]

The largest local Lyapunov exponent is
\[ \lambda(i, j) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \ln |f_i(i, j)|, \]
\[ f_i(i, j) = \frac{1}{2} \left[ U_i(i, j) + \sqrt{U_i^2(i, j) - 4 \beta \alpha(x_i(i, j) + 1) y_t(i, j)} \right], \]
\[ U_i(i, j) = \mu (1 - 2x_i(i, j)) + \beta x_i(i, j) - 4 (D_1 + D_2). \]

**Proposition 1:** If \( 1 \leq r(i) \leq 4 \) \( \forall i = 1, 2, \ldots, n \), then the interval \([0, 1]\) is a local trapping region for the systems (3) and (7).

**Proof.** If \( |q_i^j| \leq 1 \), then
\[ |q_{i+1}^j| \leq (1 - 2D) |\Pi_t^j| + D |\Pi_{t+1}^j + \Pi_{t-1}^j| \leq \frac{r_i(1 - 2D)}{4} + \frac{D(r_{i-1} + r_{i+1})}{4} \leq 1. \]

Also using the results of [1], then it is direct to prove the following proposition.

**Proposition 2:** If \( D \) is sufficient small and \( r(i) > 4 \), \( \forall i = 1, 2, \ldots, n \), then the symbolic dynamics of the systems (3) and (7) is approximated by the direct product of the symbolic dynamic of the local subsystems.

An interesting application of CML to a realistic ecological problem is the problem of migrating Salmon in a river [8]. Typically the predators occupy fixed positions and their number is approximately constant, hence the system can be modelled by the following equations
\[ x_i^t = \frac{\gamma y^t}{1 + x_i^t} + x_i^{t-1}, \]
(10)
where \( x_i^t \) is the number of salmon at site \( i \) and time \( t \), \( y^t \) is the number of predator at site \( i \) (assumed to be time independent), and \( \gamma \) is a positive constant. The number of predated salmon in the \( i \)-th patch is \( \gamma y^t/(1 + x_i^t) \), hence it decreases as the patch size \( (x_i^t) \) increases. This agrees with observations. The steady state solution of (10) is
\[ x^t = \frac{1}{2} \prod_{k=1}^{i} \left[ \sqrt{4x^{k-1} + (1 + \gamma y^k)^2} - (1 + \gamma y^k) \right]. \]
(11)
To study the stability of Eq. (11), one imposes a small perturbation \( x_i^t = x^t + \epsilon_i^t \). Then it is straightforward to obtain that \( \epsilon_i^t = \sum_{k=1}^{i} [-\gamma y^k/(1 + x^k)]^t \). Since, typically, \( x^k \gg y^k \), then we have:

**Proposition 3:** For all finite values of \( n \), the steady states (11) are stable.

Now CML for an epidemic model is constructed. Consider a certain population, according to the health of each individual, the population is classified to susceptible (S), infected (I) or recovered (removal) (R). Assuming that only infectives (and infectives) diffuse, then a CML for the susceptible-infected-removed-susceptible model [11] (SIRS) can be written as
\[ s_{i+1}^j = s_i^j - \beta s_i^j I_i^j + \alpha R_i^j, \quad R_{i+1}^j = R_i^j + \gamma I_i^j - \alpha R_i^j, \]
\[ I_{i+1}^j = I_i^j + \beta s_i^j I_i^j - \gamma I_i^j + D(I_i^{j+1} + I_i^{j-1} - 2I_i^j), \]
(12)
where \( \alpha \), \( \beta \) and \( \gamma \) are positive constants.

For the realistic case of rabies, the CML model can be written in the following form:
\[ s_{i+1}^j = s_i^j - \beta s_i^j I_i^j, \quad I_{i+1}^j = I_i^j + \beta s_i^j I_i^j - \gamma I_i^j + D(I_i^{j+1} + I_i^{j-1} - 2I_i^j). \]
(13)
The largest local Lyapunov exponent is given by
\[ \lambda_i = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \ln \left| \frac{a + \sqrt{a^2 - 4b}}{2} \right|, \]
where
\[ a = 2 - \gamma - 2D - \beta I_t + \beta s_t, \quad b = 1 - 2D - \gamma - \beta I_t(1 - 2D - \gamma) + \beta s_t. \]

Now the CML of schistosomiasis is discussed. Schistosomiasis is one of the major communicable diseases of public health. It has socio-economic importance in many developing countries. Moreover, it has been observed that infection is concentrated in certain regions (centres), hence the dynamics of the disease depends on the interaction between infected persons at these regions. This can be represented as CML. We begin by studying the dynamics at one centre, then later introduce the coupling between them. Field observations have indicated that there are two periods (May-June and October-November), where infection of humans by cercaria is highly probable. This makes difference equations more suitable to model schistosomiasis with one time step approximately equal six months. The basic assumptions are:

(i) Human population grows exponentially.

(ii) Total snail population is approximately constant.

(iii) Infected snails do not reproduce.

Thus the local equations can be written as

\[ H_{t+1} = \alpha H_t - \beta P_t, \quad P_{t+1} = \gamma P_t(1 - \frac{P_t}{d}) + \lambda C_t H_t, \quad C_{t+1} = \mu P_t - \nu C_t, \]  

where \( H_t \) is the human population at time \( t \), \( P_t \) is the parasite (adult worm) population at time \( t \), \( C_t \) is the cercaria population at time \( t \), and \( \alpha, \beta, \gamma, \lambda, \mu, \nu, d \) are positive constants. To keep in touch with observations, we will not scale away any of them. The steady states of this system are

\[ H = \frac{\beta P}{\alpha - 1}, \quad C = \frac{\mu P}{\nu + 1}, \]  

where \( \alpha > 1, \quad 0 < \gamma < 1 \). These conditions agree with observations, since populations in developing countries are increasing. The condition \( 0 < \gamma < 1 \) means that parasite population increases only through infections by new cercaria.

The stability analysis depends on the eigenvalues of the Jacobian matrix \( A \)

\[ A = \begin{bmatrix} \alpha & -\beta & 0 \\ \lambda C & \gamma(1 - \frac{2P}{d}) & \lambda H \\ 0 & \mu & -\nu \end{bmatrix}. \]

The zero solution is unstable, while the nonzero solution is stable if the following conditions are satisfied:

\[ \frac{\nu+1)(\alpha-1)^2}{\beta\mu \nu} \beta \lambda \mu \left(\frac{\nu+1)(\alpha-1)^2}{\beta\mu \nu} \beta \lambda \mu \right) > 0, \]

\[ \left| \alpha^2 \gamma \nu \left(1 - \frac{2P}{d} \right) + \frac{\alpha \beta \lambda \mu \nu}{\nu+1} \right| < 1. \]  

The first inequality is the familiar threshold condition.

There are two ways to resist schistosomiasis namely treatment and snail fighting. Both decrease the parameter \( \lambda \) in Eq. (14). If the treatment is perfect, then \( \lambda \to 0, \quad P = C = 0, \quad \alpha \to 1 \) and \( H \) is constant. However it is known that no treatment is perfect, hence the maximum allowed value for \( \lambda \) is

\[ \lambda \leq \frac{\nu(\nu+1)(\alpha-1)}{\beta\mu d}. \]

Allowing the infected persons to diffuse, one gets the following CML

\[ H_{t+1}^i = \alpha H_t^i - \beta P_t^i + D(H_{t-1}^i + H_{t+1}^i - 2H_t^i), \]

\[ P_{t+1}^i = \gamma P_t^i(1 - \frac{P_t^i}{d}) + \lambda C_t^i H_t^i, \]

\[ C_{t+1}^i = \mu P_t^i - \nu C_t^i. \]

In the limit \( D \ll 1 \), the steady states are given by Eq. (15). From the above discussions, we conclude the following:

1. Human population grows exponentially.
2. Total snail population is approximately constant.
3. Infected snails do not reproduce.

Thus the local equations can be written as

\[ H_{t+1} = \alpha H_t - \beta P_t, \quad P_{t+1} = \gamma P_t(1 - \frac{P_t}{d}) + \lambda C_t H_t, \quad C_{t+1} = \mu P_t - \nu C_t, \]  

where \( H_t \) is the human population at time \( t \), \( P_t \) is the parasite (adult worm) population at time \( t \), \( C_t \) is the cercaria population at time \( t \), and \( \alpha, \beta, \gamma, \lambda, \mu, \nu, d \) are positive constants. To keep in touch with observations, we will not scale away any of them. The steady states of this system are

\[ H = \frac{\beta P}{\alpha - 1}, \quad C = \frac{\mu P}{\nu + 1}, \]  

where \( \alpha > 1, \quad 0 < \gamma < 1 \). These conditions agree with observations, since populations in developing countries are increasing. The condition \( 0 < \gamma < 1 \) means that parasite population increases only through infections by new cercaria.

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The zero solution is unstable, while the nonzero solution is stable if the following conditions are satisfied:

\[ \frac{\nu+1)(\alpha-1)^2}{\beta\mu \nu} \beta \lambda \mu \left(\frac{\nu+1)(\alpha-1)^2}{\beta\mu \nu} \beta \lambda \mu \right) > 0, \]

\[ \left| \alpha^2 \gamma \nu \left(1 - \frac{2P}{d} \right) + \frac{\alpha \beta \lambda \mu \nu}{\nu+1} \right| < 1. \]  

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There are two ways to resist schistosomiasis namely treatment and snail fighting. Both decrease the parameter \( \lambda \) in Eq. (14). If the treatment is perfect, then \( \lambda \to 0, \quad P = C = 0, \quad \alpha \to 1 \) and \( H \) is constant. However it is known that no treatment is perfect, hence the maximum allowed value for \( \lambda \) is

\[ \lambda \leq \frac{\nu(\nu+1)(\alpha-1)}{\beta\mu d}. \]

Allowing the infected persons to diffuse, one gets the following CML

\[ H_{t+1}^i = \alpha H_t^i - \beta P_t^i + D(H_{t-1}^i + H_{t+1}^i - 2H_t^i), \]

\[ P_{t+1}^i = \gamma P_t^i(1 - \frac{P_t^i}{d}) + \lambda C_t^i H_t^i, \]

\[ C_{t+1}^i = \mu P_t^i - \nu C_t^i. \]

In the limit \( D \ll 1 \), the steady states are given by Eq. (15). From the above discussions, we conclude the following:
(i) Treatment of most infected persons should be done simultaneously to prevent reinfection through the diffusion between sites. This agrees with the idea of synchronization mentioned before.

(ii) Snail fighting should be done as efficient as possible, since they are crucial in the disease spread \((\gamma < 1)\).

(iii) More than one drug (with independent mode of action) should be used alternatively to avoid drug resistance.

An alternative model to the ones used so far to study spatial effects in epidemics is given by \[15\]

\[
\frac{ds_i}{dt} = \mu - \mu s_i - \sum_{j=1}^{n} \beta_{ij} I_j s_i,
\]

\[
\frac{dI_i}{dt} = -(\mu + \gamma)I_i + \sum_{j=1}^{n} \beta_{ij} I_j s_i,
\]

where \(s_i(I_i)\) is the fraction of susceptible (infected and infective) individuals in the \(i\)-th patch, \(i = 1, 2, ..., n\), and \(\gamma, \mu\) are constants, \(\beta_{ij}\) are a measure that an infective from the \(j\)-th patch will infect a susceptible from the \(i\)-th patch. The total population in each patch is assumed to be constant. It is realistic to assume that

\[
\beta_{ii} \gg \beta_{ij}, \; i \neq j.
\]

Expanding \(s_i = s_i^0 + s_i^1, I_i = I_i^0 + I_i^1\) and linearizing in \(\beta_{ij}, \; i \neq j, s_i^1\) and \(I_i^1\), one gets the following equilibrium solutions

\[
s_i^0 = \frac{\mu + \gamma}{\beta_{ii}} < 1, \quad I_i^0 = \mu \frac{\beta_{ii} - \mu - \gamma}{\beta_{ii}(\mu + \gamma)},
\]

\[
s_i^1 = -s_i^0 \frac{\sum_{j \neq i} \beta_{ij} I_j^0}{\mu + \beta_{ii} I_i^0 - \mu s_i^0 \beta_{ii}/(\mu + \gamma)}, \quad I_i^1 = -\mu \frac{s_i^1}{\mu + \gamma}
\]

It is clear that if the patches are different (i.e. \(\beta_{ii}\) changes significantly with \(i\)), then the equilibrium solutions are asynchronous. Using the assumption that \(\beta_{ii} \gg \beta_{ij}, \; i \neq j\), the approximated solution becomes

\[
s_i = 1, \; I_i = 0 \; \forall i = 1, 2, ..., n.
\]

This solution is asymptotically stable if the following determinants are positive

\[
\Delta_2 = \begin{vmatrix} a_1 & 1 & \cdots & 0 \\ a_3 & a_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_n \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & \vdots \\ a_5 & a_4 & a_3 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & a_n \end{vmatrix}, \quad \ldots \Delta_n = \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & \vdots \\ a_5 & a_4 & a_3 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & a_n \end{vmatrix},
\]

where

\[
a_p = \sum_{i_1 \neq i_2 \neq \cdots \neq i_p} (-1)^p \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{p-2}} (\alpha_{i_1} \alpha_{i_2} - \beta_{i_{p-1}i_p} \beta_{i_{p-1}i_{p-1}}),
\]

\[
\alpha_i = \beta_{ii} - \mu - \gamma.
\]

In this work, terms were kept to the second order in \(\beta_{ij}, \; i \neq j\).

### III. CHAOS CONTROL AND CHAOS SYNCHRONIZATION IN CML

In the previous sections, it has been shown that chaos exists in some realistic CML. In some cases chaos is not an acceptable phenomena, hence chaos control is required. Chaos control in ordinary systems has been discussed in several references e.g. \[16\]. To control chaos in CML, we propose to increase the coupling constant \(D\). Using numerical simulations we found that in a logistic CML (Eq. (3)) with \(n = 6\), such that only one of them with
$r(i) = 2$ (the other five oscillators had $r(j) = 4$) was sufficient to control the chaos of the whole CML provided that $0.5 \geq D \geq 0.493$.

In some cases it is not required to control chaos, but to synchronize it \[17,8\]. In the work of Codreanu and Savici [8], they synchronized chaos of two identical logistic oscillators using a third one. Their work has a drawback that there is no bound on the used control. In many cases this is not realistic. Here their work is generalized to CML and bounded control is used to synchronize chaos. Consider the logistic CML (Eq. (3)), with $r(i) = 3.7 + 0.3 \ast \text{rnd}$, where \text{rnd} is a uniformly distributed random number between zero and one. Thus all oscillators are chaotic but different. The symmetric case can be studied similarly. Assume that it is required to synchronize the CML to the a certain oscillator (without loss of generality call it the $l$-th oscillator). Then to each oscillator $i \neq l$ apply the following control

$$u_{i}^{t+1} = \begin{cases} u & \text{if } |u| < u_{\max}, \\ u_{\max} & \text{if } u > u_{\max}, \\ -u_{\max} & \text{if } u < -u_{\max}, \end{cases}$$

where

$$u = -r(i)q_{i}^{t} + r(i)(q_{i}^{t})^{2} + r(l)q_{l}^{t} - r(l)(q_{l}^{t})^{2} + \frac{1}{2}(q_{i}^{t} - q_{l}^{t}),$$

and $u_{\max}$ is the maximum allowed control. We found numerically that if $n = 10$, then $u_{\max} = 0.09$ is sufficient to synchronize the CML. Furthermore the coupling constant $D$ was irrelevant to the value of $u_{\max}$, which is contrary to the case of chaos control.

**Proposition 4**: There is bounded $u_{\max}$, such that the control (18) will synchronize the CML (3).

**Proof.** Since $q_{i}^{t}$ in Eq. (3) is bounded, then $\exists q_{\max} < \infty$, such that $q_{i}^{t} \leq q_{\max}$. Choosing $u_{\max} = q_{\max}$, then

$$u_{i}^{t+1} = -r(i)q_{i}^{t} + r(i)(q_{i}^{t})^{2} + r(l)q_{l}^{t} - r(l)(q_{l}^{t})^{2} + \frac{1}{2}(q_{i}^{t} - q_{l}^{t}) \text{ if } |u_{i}^{t+1}| < u_{\max}.$$

Thus the controlled deviation

$$\Delta_{i}^{t+1} = q_{i}^{t} - q_{l}^{t} + u_{i}^{t},$$

satisfies

$$\Delta_{i}^{t+1} = \frac{1}{2}\Delta_{i}^{t} \text{ i.e. } \Delta_{i}^{t} \to 0.$$

In practice, simulations have shown that the required $u_{\max} \ll q_{\max}$. The previous study is applicable to any CML.

**IV. PERSISTENCE IN SOME SPATIALLY INHOMOGENEOUS SYSTEMS**

Typically a 1-dimension CML is given by

$$\theta_{j}^{t+1} = (1 - D)\theta_{j}^{t} + \frac{D}{2}[\theta_{j+1}^{t} + \theta_{j-1}^{t}] + f(\theta_{j}^{t}),$$

or

$$\theta_{j}^{t+1} = (1 - D)f(\theta_{j}^{t}) + \frac{D}{2}[f(\theta_{j-1}^{t}) + f(\theta_{j+1}^{t})],$$

where $t = 1, 2, \ldots, j = 1, 2, \ldots, n$.

**Definition 1**: A dynamical system is persistent if $\forall x(0) \in \text{int}(S)$, then $\lim_{t \to \infty} \inf x_{i}(t) > 0 \ \forall i = 1, 2, \ldots, n$. A CML is persistent if all its components are persistent.
An open question is what is the effect of diffusion on persistence?

**Proposition 5:** (i) If \( \exists \varepsilon > 0 \) such that \( \forall \theta \in [0, \varepsilon] \), then \( f'(\theta) \geq 0 \), and \( (1 - D + f'(\theta)) \geq 0 \), then the CML (19) is persistent.

(ii) If \( f(\theta) \geq 0 \), \( \forall \theta \geq 0 \) and \( \exists \varepsilon > 0 \), such that \( \theta \in [0, \varepsilon] \Rightarrow f'(\theta) \geq \eta > 0 \), and \( \eta(1 - D) \geq 1 \), then the CML (20) is persistent.

**Proof.** (i) \( \forall \theta_j^{t} \in (0, \varepsilon] \), using mean value theorem, then \( \exists \theta_j^{t} \in [0, \theta_j^{t}] \), such that

\[
\theta_j^{t+1} = (1 - D)\theta_j^{t} + \frac{D}{2}[\theta_j^{t+1} + \theta_j^{t-1}] + f'(\theta_j^{t})\theta_j^{t} \geq (1 - D)\theta_j^{t} + \frac{D}{2}[\theta_j^{t+1} + \theta_j^{t-1}].
\]

Hence

\[
\theta_j^{t+1} \geq \phi_j^{t+1} : (1 - D)\theta_j^{t} + \frac{D}{2}[\theta_j^{t+1} + \theta_j^{t-1}].
\]

The right hand side is a circulant matrix whose eigenvalues are known, and its maximum eigenvalue is unity. Thus

\[
\phi_j^{t+1} = \phi_j^{0} = \theta_j^{0} \Rightarrow \theta_j^{t+1} \geq \theta_j^{t} > 0.
\]

(ii) For CML (20), the conditions of the proposition imply \( \theta_j^{t+1} \geq (1 - D)f(\theta_j^{t}) \). If \( \theta_j^{t} \in (0, \varepsilon] \), then using mean value theorem one gets (so long as \( \theta_j^{s} \in (0, \varepsilon] \forall s \leq t \), otherwise \( \theta_j^{s} > \varepsilon \) which implies persistence), i.e

\[
\theta_j^{t+1} \geq (1 - D)\eta \theta_j^{t} \geq [(1 - D)\eta]^{t} \theta_j^{0} > \theta_j^{0} > 0.
\]

To study persistence in partial differential equations, we use the following results [3] with slight modifications:

**Theorem 1:** (i) Suppose that \( f(\varphi, u) \) is Lipschitz in \( \varphi \) and continuously differentiable in \( u \) with

\[
\frac{\partial F}{\partial u} \leq 0 \text{ for } u \geq 0, \ f(\varphi, u) \leq 0 \text{ if } u \geq l,
\]

for some constant \( l \), and \( f(\varphi, 0) > 0 \) at some point in the domain \( \Omega \). Then the following problem

\[
\frac{\partial u}{\partial \tau} = D\nabla^2 u + uf(\varphi, u) \quad \text{in } \Omega \times (0, \infty), \quad (21)
\]

with Dirichlet or Neumann boundary conditions has a unique positive steady state \( u^{**} \) which is a global attractor for nontrivial non-negative solutions (hence the system (21) is persistent), if the following problem has a positive eigenvalue \( \sigma \)

\[
\sigma u = D\nabla^2 u + uf(\varphi, 0) \quad \text{in } \Omega, \quad (22)
\]

with the same boundary conditions as (21).

(ii) Suppose that \( f_1(\varphi, u_1, u_2) \) and \( f_2(\varphi, u_1, u_2) \) are \( C^2 \), bounded that \( f_1(\varphi, u_1, 0) \) satisfies the conditions of part (i), with positive \( \sigma \), and \( f_2(\varphi, 0, u_2) \leq 0 \) for \( u_2 \geq 0 \). Let \( \sigma_0 \) be the largest eigenvalue of the system, then

\[
\sigma_0 u_1 = D_1\nabla^2 u_1 + u_1 f_1(\varphi, 0, 0) \quad \text{in } \Omega, \quad (23)
\]

with Dirichlet or Neumann boundary conditions. Also let \( \sigma_2 \) be the largest eigenvalue of the system

\[
\sigma_2 u_2 = D_2\nabla^2 u_2 + u_2 f_2(\varphi, u_1^{**}, 0) \quad \text{in } \Omega, \quad (24)
\]

where \( u_1^{**} \) is given by \( f_1(\varphi, u_1^{**}, 0) = 0 \), then the following system is persistent if both \( \sigma_0 \) and \( \sigma_2 \) are positive

\[
\frac{\partial u}{\partial \tau} = D_1\nabla^2 u_1 + u_1 f_1(\varphi, u_1, u_2), \quad \text{in } \Omega \times (0, \infty), \quad (25)
\]

\[
\frac{\partial u}{\partial \tau} = D_2\nabla^2 u_2 + u_2 f_2(\varphi, u_1, u_2), \quad \text{in } \Omega \times (0, \infty).
\]
Now theorem (1) is applied to some 2-dimensional systems. The domain \( \Omega \) is chosen to be the rectangle \( x \in [0, L_1], \ y \in [0, L_2] \). An example for two species problems is host-parasitoid, where \( u_1 \ (u_2) \) is host (parasitoid), respectively and \( f_1 \) and \( f_2 \) in Eq. (25) are given by

\[
f_1 = b \frac{1 - au_1}{1 + u_2} - 1, \quad f_2 = \frac{u_1}{1 + u_2} - 1,
\]

and persistence conditions become

\[
\begin{align*}
 b - 1 & > D_2 \left( \frac{\pi^2}{L_1^2} + \frac{\pi^2}{L_2^2} \right), \\
 b - 1 & > D_1 \left( \frac{\pi^2}{L_1^2} + \frac{\pi^2}{L_2^2} \right).
\end{align*}
\]

(27)

Another example is the predator-prey or epidemic model where \( u_1 \ (u_2) \) is prey (predator) or susceptible (infected and infectious) and \( f_1 \) and \( f_2 \) in Eq. (25) are

\[
f_1 = 1 - u_1 - u_2, \quad f_2 = u_1 - \gamma,
\]

(28)

and persistence conditions are

\[
1 > D_1 \left( \frac{\pi^2}{L_1^2} + \frac{\pi^2}{L_2^2} \right) \quad \text{and} \quad 1 > \gamma + D_2 \left( \frac{\pi^2}{L_1^2} + \frac{\pi^2}{L_2^2} \right).
\]

(29)

For the case of rabies one sets \( D_1 = 0 \), since only infected foxes spread, while susceptible ones do not. This indicates that a method to fight infection spread is to put susceptibles in small groups (to decrease the domain).

Now the Krill-Whale model \[3\] will be studied including diffusion. The functions \( f_1 \) and \( f_2 \) in Eq. (25) are

\[
f_1 = ru_1(1 - u_1) - u_1u_2, \quad f_2 = su_2 \frac{1 - u_2}{u_1},
\]

where \( u_1 \ (u_2) \) represents krill (whale), respectively and \( r, s \) are positive constants. The uniform coexistence steady states are \( u_1 = 1 - 1/r, \ u_2 = 1, \ r > 1 \). Its stability requires \( s < 1 - 1/r, \ 1 < r < 2 \). Sufficient conditions for persistence can be derived (using theorem (1)) in the following form

\[
r > D_1 \left( \frac{\pi^2}{L_1^2} + \frac{\pi^2}{L_2^2} \right) + 1, \quad s > D_2 \left( \frac{\pi^2}{L_1^2} + \frac{\pi^2}{L_2^2} \right).
\]

(30)

Furthermore using Dulac-Bendixon lemma, one can see that this system has no cycles.

There is a weakness in the definition of persistence in that it does not guarantee that the population is larger than a prescribed value. To find the minimum of the population is part of the practical persistence \[7,6\]. Studying this concept for the krill-whale model, one can derive the following limits for the populations

\[
1 \geq u_1 \geq 1 - \frac{1}{r}, \quad u_2 \simeq 1.
\]

These values agreed with the numerical simulations.

One may argue that advection should be included in the above diffusion model. So an interesting question arises as to what is the effect of advection on the persistence conditions of theorem (1)? Consider the following diffusion-advection system

\[
\frac{\partial u}{\partial t} = \nabla^2 u + uf(\nabla u) - v \frac{\partial u}{\partial x} \quad \text{in} \quad \Omega x(0, \infty).
\]

(31)

If the system (31) admits a solution of the following form

\[
u(x, t) = \sum_{n=1}^{\infty} e^{\lambda_n t} (a_n \sin \frac{n\pi x}{L} + b_n \cos \frac{n\pi x}{L}),
\]

(32)

and linearizing around \( u = 0 \), one gets that the advection term in Eq. (31) affects only the imaginary part of the eigenvalue \( \sigma \) in Eq. (22). Hence we conclude the following proposition:

**Proposition 6:** If the system (31) admits a solution of the form (32), then the persistence conditions of theorem (1) are valid.
V. A METHOD TO STUDY DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

We begin by proposing the following correspondence between differential equations with periodic coefficients and some difference equations. Consider the following system

\[ \frac{dx_i}{dt} = \sum_j p_{ij}(t)x_j, \quad p_{ij}(t) = p_{ij}(t + T). \] (33)

Approximating the left hand side using

\[ \frac{dx_i}{dt} \approx \frac{x_i(t + T) - x_i(t)}{T}, \]

thus one gets

\[ x_i(t + T) = x_i(t) + T \sum_j p_{ij}(t)x_j, \quad t = 0, T, 2T, \ldots. \] (34)

Since the discrete system (34) has time step \( T \), then \( p_{ij}(t) \) will be constant in its evolution, thus the system (34) is autonomous.

Recalling that the stability conditions for discrete systems are more stringent than the corresponding continuous one, then stability conditions for the system (34) will be sufficient conditions for those of (33). Thus we have

**Proposition 7**: If all the eigenvalues \( \lambda \) of the matrix \([\delta_{ij} + Tp_{ij}(t)]\) satisfy \(|\lambda| \leq 1 \ \forall t \in [0, T]\), then the periodic solution of Eq. (33) is stable.

As an example consider the following equation

\[ \frac{d^2x}{dt^2} + p(t)x = 0, \quad p(t + T) = p(t). \] (35)

Approximating it by

\[ x(t + T) + x(t - T) + x(t)(-2 + T^2p(t)) = 0, \quad t = T, 2T, 3T, \ldots. \] (36)

Notice that in Eq. (36), \( p(t) \) is constant since \( t \) increases by \( T \), and \( p(t) = p(t + T) \). Thus one gets that Eq. (36) is stable if

\[ 0 < T^2p(t) < 4 \ \forall t \in [0, T], \] (37)

which yields the known Lyapunov theorem for the sufficient conditions for the stability of the periodic motion of Eq. (36)

\[ p(t) > 0, \quad 0 < T \int_0^T p(s)ds < 4. \] (38)

Applying these results to Lame’s equation

\[ \frac{d^2x}{dt^2} - (h + 2k^2\sin^2 t)x = 0, \]

one gets the following sufficient conditions for the stability of its periodic motion

\[ 0 \leq -4K^2(h + 2k^2\sin^2 t) \leq 4, \forall t \in [0, 2K], \]

\[ K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2 \phi}}. \] (39)

Applying them to Hill’s equation

\[ \frac{d^2x}{dt^2} + (a - 2b \cos 2\pi t)x = 0, \]

one gets that the sufficient conditions for the stability of its periodic motion are
The previous method is also useful for the case of periodically forced Lotka-Volterra predator-prey models of the following form

\[
\frac{dp}{dt} = -\alpha p + \beta pv, \quad \frac{dv}{dt} = (r + \epsilon \sin 2\pi t)v(1 - \frac{\nu}{K}) - \gamma pv.
\]

Following the above procedure, one gets the following stability conditions:

\[
0 < \alpha \frac{r + \epsilon \sin 2\pi t}{\beta K} < 4 \quad \forall t \in [0, 1].
\]

This method is expected to simplify the study of seasonality.

VI. CONCLUSIONS

In conclusion, ecologists have stated that it is important to include several factors in some biological and epidemic models e.g. spatial heterogeneity, synchronization and seasonality. Since a crucial question is whether the epidemic (or the population) will persist, we studied the persistence of some spatially heterogenous systems (using CML or partial differential equations). Synchrony in CML and CA is studied. Since chaos implies asynchrony, we studied bounded chaos control on CA and CML. A simple method to study seasonally in forced systems (systems with periodic coefficients) is introduced.

ACKNOWLEDGEMENTS

We are grateful to R. Cantrell and C. Cosner for sending some copies of their work to us and for discussions.

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