Non-Abelian Kubo Formula and the Multiple Time-Scale Method

Zhang Xiaofei      Li Jiarong
Institute of Particle Physics, Hua-Zhong Normal University, Wuhan 430070, China

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Abstract

The non-Abelian Kubo formula is derived from the kinetic theory. That expression is compared with the one obtained using the eikonal for a Chern-Simons theory. The multiple time-scale method is used to study the non-Abelian Kubo formula, and the damping rate for longitudinal color waves is computed.
1. Introduction

Substantive progress is made in the study of non-equilibrium quark-gluon plasma in recent years\[1\]. The Kubo formula is a fundamental tool for describing a non-equilibrium system. Hence it is important to get the Kubo formula of a non-Abelian system. In Ref.[2], meaningful results are given. In this paper, we derive a form of the non-Abelian generation of the Kubo formula from kinetic theory directly, and the multiple time-scale method is used to investigate it. To show the intention of our work concretely, we first give the result in Ref.[2]. Using the relation between the current and the generating functional, i.e., \( j^\mu(x) = -\delta\Gamma[A]/\delta A_\mu(x) \), and the Yang-Mills equation, a form of non-Abelian Kubo formula is obtained with the color current being expressed in terms of the eikonal for a chern-Simons theory\[3\]:

\[
[D_\nu F^{\nu\mu}(x)]_a = j^\mu_a(x),
\]

where \( D_\nu = \partial_\nu - ig[A_\nu(x), \cdot \cdot \cdot] \),

\[
j^\mu_a(x) = \frac{1}{(2\pi)^4} \int d^4k e^{-ik\cdot x} j^\mu_a(k) = \frac{1}{(2\pi)^4} \int d^4k e^{-ik\cdot x} \sum_{n=1}^\infty j^\mu_a(k),
\]

and \( j^\mu_a(k) \) is expressed in terms of powers of \( A_\mu \),

\[
j^\mu_a(k) = -(N_c + \frac{N_f}{2}) \frac{g^2}{12\pi} \sum_{k_1+\cdots+k_n=k} \int d\Omega \left[ \text{tr}\left\{ (I_a v^\mu)(A(k_1) \cdot v) \right\} \delta_{n1} ight. \\
- g^n \left. \text{tr}\left\{ (I_a v^\mu)(A(k_1) \cdot v) \cdots (A(k_n) \cdot v) \right\} F(k_1, \cdots, k_n) \right],
\]

\[
F(k_1, \cdots, k_n) = \sum_{i=0}^n \frac{\delta_{q_i}}{(q_0 - q_i)(q_1 - q_i) \cdots (q_{i-1} - q_i)(q_{i+1} - q_i) \cdots (q_n - q_i)},
\]

where \( v = (1, v) \) is a four light-like vector, and \( v' = (1, -v) \), \( q_0 = 0 \), \( q_i = \sum_{j=0}^i (k_j \cdot v + i0^+) \), \( q_i = \sum_{j=0}^i (k_j \cdot v') \). \( N_f \) being the number of flavors for massless quarks, \( I_a \) is the SU(\( N_c \)) group generator in the fundamental representation. \( d\Omega \) denotes integration over angular direction of the unit vector \( v \). In the above expression, the relation \( \int d\Omega f[A \cdot v] = \int d\Omega f[A \cdot v'] \) has been taken consider of, and we have defined,

\[
\sum_{k_1+\cdots+k_n=k} = \int \frac{d^4k_1}{(2\pi)^4} \cdots \int \frac{d^4k_n}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k - k_1 - \cdots - k_n).
\]
The non-Abelian Kubo formula describes Landau damping of fields in a quark-gluon plasma as well as Debye screening and propagation of plasma waves in a gauge covariant way. Finding solutions to the non-Abelian Kubo formula is appealing, since such solutions give collective effects of the quark-gluon plasma at high temperature. R. Jackiw, Q.Liu and C. Lucchesi investigate the static non-Abelian Kubo formula and show that there are no hard thermal solitons. However, the above expression of the Kubo formula is quite complicated, this makes it much more difficult to use it to analysis relativistic physical problems. Recently, P. F. Kelly, Q.Liu C. Lucchesi and C. Manuel have shown that the generating functional of hard thermal loops can be derived from classical transport theory, and it has been shown that the solution of the kinetic equations taking into account the non-Abelian and nonlinear effects can be obtained using the multiple time-scale method. The multiple time-scale method is one of the most powerful asymptotic methods in solving nonlinear equations, and it is proved successful in studying nonlinear properties of electromagnetic plasmas. This gives us a chance of investigating the non-Abelian Kubo formula from kinetic theory. In this paper, the non-Abelian Kubo formula will be derived from the kinetic theory, then the multiple time-scale method will be used to study it and damping rate will be calculated.

This paper is organized as follows: In section 2, the non-Abelian Kubo formula is derived from the kinetic theory and the identity of it with the form obtained using the eiknoal for a Chern-Simons theory is shown. In section 3, the multiple time-scale method is introduced to study the non-Abelian generation of the Kubo formula. In section 4, using the multiple time-scale method, the damping rate due to non-Abelian and nonlinear nature of QGP eigenwaves is calculated in the temporal gauge for longitudinal color waves, and the result is compared with the hard thermal result in thermal QCD.
2. Non-Abelian Kubo formula derived from the kinetic approach

In this section, we will derive the non-Abelian Kubo formula from the kinetic approach. The phase space of the distribution function of colored particles is \((x, p, Q)\), where \(Q^a, a = 1, 2 \cdots N^2 - 1\) are color charges, and the use of the color charge as phase-space coordinates is justified recently[5]. Taking cosider of physical constraints on the phase space volume element \(dxdPdQ\), the momentum and the color charges measure

\[
dP = \frac{d^4P}{(2\pi)^3}2\theta(p^0)\delta(p^2 - m^2),
\]

\[
dQ = d^8Q\delta(Q_\alpha Q^\alpha - q_2)\delta(d_{abc}Q^aQ^bQ^c - q_3),
\]

where the constants \(q_2\) and \(q_3\) fix the values of the Casimir invariants and \(d_{abc}\) are the totally symmetric group constants.

When the classical transport equations for colored particles is expanded consistently in the coupling constant \(g\), at the leading order, the transport equations for the distribution function \(f(x, p, Q)\) is reduced to(neglecting spin effects)[5]

\[
p\mu \left[\partial_\mu - g f^{abc} A^\mu_a Q^c\right] f(x, p, Q) = -gQ_\alpha F^{\alpha a}_{\mu\nu} \frac{\partial}{\partial p^\nu} f^{(0)}(p),
\]

where distribution functions \(f^{(0)}(p)\) in the above equation is the equilibrium distribution function in the absence of a net color color field,

\[
f^{(0)}(p) = C n_{B,F},
\]

where C is a normalization constant, \(n_{B,F}\) are Fermi-Dirac and Bose-Einstein equilibrium distribution function respectively

\[
n_F = (e^{\beta p \cdot U} + 1)^{-1}, \quad n_B = (e^{\beta p \cdot U} - 1)^{-1},
\]

where \(\beta\) is the inverse of the system temperature \(T\), \(U^\mu\) is the local four velocity.
The Yang-Mills mean field equation coupled with the transport equation is

\[ [D_\nu F^{\nu\mu}(x)]_a = j_\mu^a(x), \]  

where color current \( j^\nu \) generated by quarks and gluons is expressed as,

\[ j^\nu(x) = \sum_{\text{species}} \sum_{\text{helicites}} \int dPdQ Q^a f(x, p, Q) \]  

Using the above equation, the constraint on the first order current demanding by the leading order transport equation can be obtained

\[ [p \cdot D J^\mu(x, p)]_a = 2g^2 p^\rho p^\nu F_{\rho\nu}^a(x) \frac{d}{dp_0} [N_f f^{(0)}(p) + N_c G^{(0)}(p)], \]  

where the field tensor is only related to \( F_{\rho\nu}^a \).

The constraint equation (13) can lead the generating functional of hard thermal loops. This means that Eq.(13) determines the color current generated by hard thermal loops and can be used to study the response of quark-gluon plasma to fields consistently. In the following discussion, we will use the constraint equation (13) to get the expression for the color current.

In the momentum space the constraint equation (13) yields,

\[ (p \cdot k) J^\mu_{1a}(k, p) + ig f^{abc} \sum_{k' + k'' = k} (p \cdot A(k'))_b J^\mu_{k'' a}(k', p) \]

\[ = 2g^2 p^\rho [(p \cdot k) g_\rho^0 - \omega p^\rho] A^\rho_{\nu} A^\mu_{\nu} \frac{d}{dp_0} [N_c G^{(0)}(p) + N_f f^{(0)}(p)] \]

\[ + 2ig^3 p^\rho p^\nu \sum_{k_1 + k_2 = k} f^{abc} A^b_{\nu}(k_1) A^c_{\mu}(k_2) \frac{d}{dp_0} [N_c G^{(0)}(p) + N_f f^{(0)}(p)]. \]  

The color current can be expressed as a series of powers in \( A_\mu \), and in the above equation there is no constraint on the field potential \( A_\mu \), thus Eq.(14) can be reduced to the equations of every power in \( A_\mu \). After integrating over the momentum space one can get

\[ j_{1, a}^\mu(k) = -m^2 g_0^a g_\mu^0 - \int \frac{d\Omega}{4\pi} \frac{\omega^{\mu\nu} v^\nu}{v \cdot k + i0^+} A^\nu_a(k), \]  

\[ j_{2, a}^\mu(k) = -igm^2 f^{abc} \int \frac{d\Omega}{4\pi} \sum_{k_1 + k_2 = k} \frac{\omega_1}{(v \cdot k + i0^+)(v \cdot k_1 + i0^+)} \]

\[ v^\mu v^\lambda v^\nu A^b_{\nu}(k_2) A^c_{\mu}(k_1), \]
\[ j^\mu_{n,a}(k) = -igf^{abc} \int d\Omega \frac{1}{v \cdot k + i0^+} \sum_{k' + k'' = k} (v \cdot A(k')) j^\mu_{n-1,c}(k'') \quad n > 2, \quad (17) \]

where \( m_D^2 = \frac{1}{3}(N_c + \frac{1}{2}N_f)g^2T^2 \), \( j^\mu_{n-1,a}(k) = \int d\Omega j^\mu_{n-1,c}(k) \) and to avoid the poles in the above integrand, the retard condition, i.e., replacing \( \omega \) by \( \omega + i0^+ \), has been imposed.

The expression for the current to any power of \( A_\mu \) can be obtained easily using the above equation. Further, \( j^\mu_{n,a} \) can be expressed as

\[
 j^\mu_{n,a}(k) = -\delta_{n1}m_D^2 \left[ g_0^\mu g_0^\nu - \int \frac{d\Omega}{4\pi} \frac{\omega v^\mu v^\nu}{v \cdot k + i0^+} \right] A_\nu(k) \\
 + (-ig)^{n-1}m_D^2 \int \frac{d\Omega}{4\pi} \sum_{k_1 + \cdots + k_n = k} f^{a_1b_{n-1}} f^{b_{n-1}a_{n-2}} \cdots f^{b_{2}a_{2}b_{1}} \\
 \left( v \cdot (k + i0^+) \right) (v \cdot (k - k_n) + i0^+) \cdots (v \cdot (k - k_2 - \cdots - k_2 + i0^+)) \\
 (A(k_n) \cdot v)_{a_n} \cdots (A(k_2) \cdot v)_{a_2} (A(k_1) \cdot v)_{b_1}. \quad (18) \]

Using Eq.(2) and the above equation to express the color current in the Yang-Mills mean field equation (11), the non-Abelian Kubo formula is derived from the kinetic theory. The above expression for the color current is relatively simple, plain, and the non-Abelian Kubo formula expressed by it is easier for being studied. In the case when the mean field is weak, the non-Abelian Kubo formula can be solved by the multiple time-scale method.

Now we turn to show that the expression for the color current obtained from the kinetic theory is equivalent to the one obtained using the eiknoal for a Chern-Simons theory.

In the kinetic theory, the expression for the color current is determined by Eqs.(15-17). One can check that \( j^\mu_{1,a}(k) \) and \( j^\mu_{2,a}(k) \) in Eq.(15) and Eq.(16) are equivalent to those got from Eqs.(3-4). Thus to show the identity of these two expressions, one only need to show \( j^\mu_{n,a} \) in Eqs.(3-4) satisfy Eq.(17). Now we turn to show this.

Using the relation

\[
 \frac{1}{\bar{q}_i(\bar{q}_n - \bar{q}_i)} = \frac{1}{\bar{q}_i\bar{q}_n} + \frac{1}{\bar{q}_n(\bar{q}_n - \bar{q}_i)}, \quad (19) \]

And taking note of the delta function in the definition (5), one can get from Eq.(4),

\[
 F(k_1, \cdots, k_n) = \frac{1}{v \cdot k + i0^+} [F(k_1, \cdots, k_{n-1}) - F(k_2, \cdots, k_n)]. \quad (20) \]
Then using the invariance of $j_{\mu_n}^{\mu}$ when $k_1, \cdots, k_n$ are exchanged, we obtained from Eq.(3)

$$j_{\mu_n}^{\mu}(k) = \frac{1}{v \cdot k + i0^+} (N_c + \frac{N_f}{2}) \frac{T^2}{12\pi} \sum_{k_1 + \cdots + k_n = k} \int d\Omega \text{Tr}\left\{ [I_b, I_a]^\mu \right\} \left( A(k_1) \cdot v \right) \left( A(k_{n-1}) \cdot v \right) F(k_1, \cdots, k_{n-1}, k_n) (A(k_n) \cdot v)^b \left( A(k_1) \cdot v \right) \left( A(k_{n-1}) \cdot v \right) F(k_1, \cdots, k_{n-1}, k_n).$$

This concludes our proof of the identity of the expression of the non-Abelian Kubo formula from kinetic theory and the one obtained using the eiknoal for a Chern-Simons theory.

In the following section, we will investigate the non-Abelian Kubo formula obtained from the kinetic theory, i.e., Eq.(11) coupled with Eq.(2) and Eq.(18), using the multiple time-scale method.

### 3. The multiple time-scale method

The multiple time-scale method is a kind of asymptotic methods of solving nonlinear equations. In plasma theory, it is used to analysis the nonlinear effects of a electromagnetic plasma\[7\]. Generally speaking, when the field strength is not very strong, the mean field equation may be solved by perturbation methods. However, As is known, the nonlinear wave equation in a plasma can not be solved by naive perturbation analysis, which may yield singularity in the solutions by virtue of the resonance in the relevant equations. This difficulty may be overcome if it is solved by means of the multiple time-scale perturbation approach\[7,8\]. In this method, first of all, the independent time variable $t$ is extended to many time variables, each with different scale, by introducing the new time variables \[7\]

$$t^{(n)} \equiv \kappa^n t, \quad n = 0, 1, 2, \cdots, N,$$

where $\kappa$ is a dimensionless parameter introduced to denote the order of a small quantity. Then, every function dependent on $t$, such as $Y[t]$, is expanded into an asymptotic series of
the form
\[ Y[t, \kappa] = \sum_{n=0}^{N} \kappa^n Y^{(n)}[t^{(0)}, t^{(1)}, \ldots, t^{(N)}] + O(\kappa^{N+1}). \] (23)

Every function dependent on \( t \) is assumed to be a function of the multiple time, so the time derivative should be replaced by
\[ \frac{\partial}{\partial t} = \sum_{n=0}^{N} \kappa^n \frac{\partial}{\partial t^{(n)}}. \] (24)

Usually, one prefers to discuss the nonlinear wave equations in momentum space. The multiple time-scale procedure can also be used there by introducing the multiple scale Fourier transformation
\[ Y[t^{(0)}, t^{(1)}, \ldots, t^{(N)}] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} Y(\omega^{(0)}, \omega^{(1)}, \ldots, \omega^{(n)}) \exp\left(-i \sum_{n=0}^{N} \omega^{(n)} t^{(n)}\right) \prod_{n=0}^{N} d\omega^{(n)}. \] (25)

Using this transformation, the multiple time-scale method can be used in momentum space simply by taking the following expansions instead of Eqs.(22-24)[8],
\[ \omega = \sum_{n=0}^{N} \kappa^n \omega^{(n)}, \] (26)
\[ Y[\omega, \kappa] = \sum_{n=0}^{N} \kappa^n Y^{(n)}[\omega^{(0)}, \omega^{(1)}, \ldots, \omega^{(N)}] + O(\kappa^{N+1}). \] (27)

In this paper, we will use the multiple time-scale method to study the non-Abelian Kubo formula in momentum space. The corresponding equation of Eq.(11) in momentum space is obtained by using
\[ A^\mu(x) = \int \frac{d^3k}{(2\pi)^3} e^{-ik \cdot x} A^\mu(k). \] (28)

The non-Abelian Kubo formula in the momentum space is
\[ - (k^2 g^{\nu\mu} - k^{\nu} k^{\mu}) A_\nu(k) - g \sum_{k_1+k_2=k} (k^{\nu} + k^{\mu}_2) [A_\nu(k_1), A^{\mu}(k_2)] \]
\[ + g \sum_{k_1+k_2=k} (k^{\mu}_2) [A^{\nu}(k_1), A^{\nu}(k_2)] + g^2 \sum_{k_1+k_2=k} \sum_{k_3+k_4=k_2} [A^{\nu}(k_1), [A_\nu(k_3), A^{\mu}(k_4)]] \]
\[ = j^{\mu}(k), \] (29)
where $j^\mu(k)$ is expressed as Eq.(18).

As we have discussed in the beginning of this section, when the mean field is not very strong, the non-Abelian Kubo formula can be solved by the multiple time-scale method. If we work in the momentum space, following Eq.(26) and Eq.(27), the frequency and the color field potential are expanded into asymptotic series of forms as

$$\omega = \omega^{(0)} + \kappa \omega^{(1)} + \kappa^2 \omega^{(2)} + \kappa^3 \omega^{(3)} + \cdots,$$

(30)

$$A_\mu = \kappa A^{(1)}_\mu (\omega^{(0)}, \omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \cdots) + \kappa^2 A^{(2)}_\mu (\omega^{(0)}, \omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \cdots) + \kappa^3 A^{(3)}_\mu (\omega^{(0)}, \omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \cdots) + \cdots.$$

(31)

Inserting the expansions of all the variables into the nonlinear equation, and equating the coefficients of corresponding powers of $\kappa$ in the two sides of this equation, the perturbation expansions of the nonlinear equation at every order are given. The nonlinear equation then can be discussed iteratively.

Now, we first discuss the physical meaning of the above expansions given by the multiple time-scale method. In the linear response approximation, any perturbation in a plasma may be taken to be a superposition of independent eigenwaves whose dispersion relation is determined by the solution to the non-Abelian Kubo formula in the linear approximation. When the nonlinear terms in the non-Abelian Kubo formula are retained, the nonlinear interactions of the eigenwaves will be taken into account. The nonlinear interactions of the eigenwaves will change the dispersion relation obtained in the linear approximation, i.e., frequency corrections $\omega^{(1)}, \omega^{(2)} \cdots$ expressed by Eq.(30) will be given to the eigenfrequency $\omega^{(0)}$ correspondingly. If the imaginary parts of the frequency corrections for eigenmodes are not vanishing, there will be instability or damping for the eigenmodes. In next section, we will calculate the damping rate due to interactions of QGP eigenwaves in the temporal gauge.
4. Nonlinear damping rate in the temporal gauge

The non-Abelian Kubo formula determines the dispersion relation of the eigenwaves in QGP. The damping rate of plasma eigenwaves is determined by the imaginary part of the dispersion relation. In this section we will calculate the damping rate for longitudinal color eigenwaves in the long-wavelength limit from the non-Abelian Kubo formula.

In this section, for convenience we will work in the temporal gauge, $A_0 = 0$. The relation between the color electric field $E$ and the vector potential in that gauge is simple,

$$E^i_a(x) = -\partial A^i_a(x)/\partial t. \quad (32)$$

The non-Abelian Kubo formula in the momentum space and in the temporal gauge is

$$-(\omega^2 - K^2) A^h(k) - k^i k^h A^i(k) + g \sum_{k_1+k_2=k} k^i [A^i(k_1), A^h(k_2)]$$

$$+ g \sum_{k_1+k_2=k} k_2^i [A^i(k_1), A^h(k_2)] - g \sum_{k_1+k_2=k} k_2^h [A^i(k_1), A^i(k_2)]$$

$$+ g^2 \sum_{k_1+k_2=k} \sum_{k_3+k_4=k_2} [A^i(k_1), [A^i(k_3), A^h(k_4)]] = j^h(k), \quad (33)$$

where $K = |k|$ and $j^h(k)$ is expressed as Eq.(18).

Though the method used in this paper is general, for simplicity of calculation, we suppose that there is only longitudinal vector potential, i.e., the vector potential is parallel to the wave vector,

$$A^i(k) = k^i A(k)/K. \quad (34)$$

Inserting the expansions (32-34) into Eq.(33) and Eq.(18), for longitudinal field, the first order non-Abelian Kubo formula is,

$$- (\omega^{(0)})^2 - K^2) A^{(1)h}_a(k) - k^i k^h A^{(1)i}_a(k) = -m_D^2 \int d\Omega \frac{\omega^{(0)} v^h v^j}{4\pi v \cdot k + i0^+} A^{(1)}_a(k). \quad (35)$$

The above non-Abelian Kubo formula can be reduced to

$$\epsilon(\omega^{(0)}, k) A^{(1)}(k) = 0, \quad (36)$$
where \( \epsilon \) is the color electric permeability,
\[
\epsilon(\omega(0), k) = 1 + \frac{3\omega^2}{K^2} \left[ 1 - \frac{\omega(0)}{2K} \left( \ln \frac{|K + \omega(0)|}{|K - \omega(0)|} - i\pi \Theta(K - \omega(0)) \right) \right].
\]

where \( \omega_p^2 = (2N_c + N_f)g^2T^2/18 \).

From the above equation, the solution to the first order non-Abelian Kubo formula can be got
\[
A_k^{(1)} = -\frac{i\pi}{\omega(0)} E_k^\sigma \left[ e^{-i\phi_k^\sigma} \delta(\omega(0) - \omega_k^\sigma) + e^{i\phi_k^\sigma} \delta(\omega(0) + \omega_k^\sigma) \right],
\]

where \( E_k^\sigma \) and \( \phi_k^\sigma \) are the initial amplitude and phase of the oscillation respectively (\( \sigma \) is introduced to label different eigenwaves). The frequencies and the wave vectors must satisfy the dispersion relation,
\[
\epsilon(\omega(0), k) = 0.
\]

Eq.(39), which is quite similar to the dispersion relation for the abelian-like plasma eigenwaves, is the first order dispersion relation for the color eigenwaves in a QGP. It agrees with the leading order result using the finite temperature QCD[9]. It has been shown by U.Heinz that as the eigenwaves in QGP are always time-like, \( \omega(0)^2 > K^2 \), the color eigenwaves in a QGP do not undergo Landau damping in the linear approximation[10].

Using the results of the first order equation, the second order equation now can be considered. The second order non-Abelian Kubo formula for the longitudinal field is
\[
-\omega(0)^2 \epsilon(\omega(0), k) A_a^{(2)}(k) \frac{k^h}{K} - 2\omega(1) \omega(0) A_a^{(2)}(k) \frac{k^h}{K}
\]
\[
-2g \sum_{k_1 + k_2 = k} \frac{k \cdot k_1}{K_1} \frac{k_2}{K_2} \text{tr} \{ I_a[A^{(1)}(k_1), A^{(1)}(k_2)] \}
\]
\[
-4\pi \int \frac{d\Omega}{K} \frac{v^h A_a^{(1)}(k)}{v \cdot k + i0^+} \frac{v^h \omega_1^0}{(v \cdot k_1 + i0^+)(v \cdot k_2 + i0^+)}
\]
\[
-2gm^2 \sum_{k_1 + k_2 = k} \int \frac{d\Omega}{K_1K_2} \frac{v \cdot k_1 v \cdot k_2}{4\pi} \frac{u^h A_a^{(1)}(k_1)}{(v \cdot k + i0^+)(v \cdot k_1 + i0^+)} \text{tr} \{ I_a[A^{(1)}(k_2), A^{(1)}(k_1)] \},
\]

where the right side of the above equation comes from the second order contribution of \( j_{a,1}^\mu \) and \( j_{a,2}^\mu \) in Eqs.(15-16).
Eq. (40) describes the nonlinear effects owing to the three-wave processes which are the merging of two eigenwaves with eigenfrequencies $\omega_{k_1}^{(0)\sigma}$, $\omega_{k_2}^{(0)\lambda}$ into a third wave, or the corresponding inverse processes. The appearance of the term $[A^{(1)}(k_1), A^{(1)}(k_2)]$ in Eq. (40) indicates that the non-Abelian nature of the mean color field has been taken into account and the interaction of the QGP eigenwaves.

Now, we consider the nonlinear effects due to the three-wave interactions. We will see that the nonlinear effects owing to the three-wave interactions on the eigenfrequency are vanishing. In the three-wave processes, when the third wave is also an eigenwave, this kind of processes is called three-wave resonance. Taking note of Eq. (5), the following matching condition representing the conservation of energy should be satisfied for the three-wave resonance,

$$|\omega_{k_1}^{(0)\sigma} \pm \omega_{k_2}^{(0)\lambda}| = \omega_{k}^{(0)\tau},$$

(41)

where $\sigma$, $\lambda$, and $\tau$ in the above equation indicate that the corresponding frequency is the eigenfrequency of a certain eigenmode.

The three-wave resonance processes require that the matching condition Eq. (41) must be satisfied strictly; even a small mismatch of Eq. (41) may significantly decrease the efficiency of the three-wave resonance[11]. This condition limits the permissible region of the three-wave resonance interaction in $k$ space. Indeed for some forms of dispersion relation, the three-wave resonance is completely forbidden. It can be proved that for dispersion Eq. (37) and Eq. (39), where the phase velocity decreases with $K$ [1,10], the three-wave resonance is forbidden [12]. The absence of the three-wave resonance here leads the eigenfrequency correction $\omega^{(1)}$ to vanish, which is similar to the corresponding situation in an electromagnetic plasma[8]. Using this result and taking note of Eq. (38), the second order field potential is expressed as

$$\omega^{(0)2} \epsilon(\omega^{(0)}, k) A^{(2)}(k)$$
\[-2g m_D^2 \sum_{k_1 + k_2 = k} \int \frac{d\Omega}{4\pi} \frac{v \cdot k_1 v \cdot k_2}{K_1 K_2} \omega_1^{(0)} v \omega_1^{(0)} \operatorname{tr}\{I_a[A^{(1)}(k_2), A^{(1)}(k_1)]\}.\]

From the above discussion, we can see that there is no effect of three-wave resonance on the eigenfrequencies. Hence the next order perturbation equation should be considered in order to obtain non-vanishing effects of the non-Abelian and nonlinear wave interactions on the eigenfrequencies.

Using the above result Eq. (12) and \(\omega^{(1)} = 0\), the third order non-Abelian Kubo formula for the longitudinal field is expressed as

\[-\omega^{(0)}_2 \epsilon(\omega^{(0)}, k) A_a^{(3)}(k) - 2\omega^{(2)} \omega^{(0)} A_a^{(1)}(k)\]

\[+2g^2 \sum_{k_1 + k_2 = k} \sum_{k_3 + k_4 = k} \frac{k_1 \cdot k_3 k_4 \cdot k}{K_1 K_3 K_4} \operatorname{tr}\{I_a[A^{(1)}(k_1), [A^{(1)}(k_3), A^{(1)}(k_4)]]\}\]

\[= -2g^2 m_D^2 \sum_{k_1 + k_2 = k} \sum_{k_3 + k_4 = k} g^6 \int d\Omega \frac{1}{v \cdot k_2^{(0)} + i0^+} \frac{v \cdot k_1 v \cdot k_2}{K_1 K_2} \frac{1}{\omega_2^{(0)} - \omega_1^{(0)}} \frac{1}{\omega_2^{(0)} - \epsilon(k_3^{(0)})} \int d\Omega' \frac{1}{v' \cdot k_2^{(0)} + i0^+} \frac{v' \cdot k_2 v' \cdot k_3 v' \cdot k_4}{K_2 K_3 K_4} \operatorname{tr}\{I_a[A^{(1)}(k_1), [A^{(1)}(k_3), A^{(1)}(k_4)]]\}\]

\[= -2g^2 m_D^2 \sum_{k_1 + k_2 = k} \sum_{k_3 + k_4 = k} g^4 \int d\Omega \frac{1}{v \cdot k_2^{(0)} + i0^+} \frac{v \cdot k_1 v \cdot k_2}{K_1 K_2} \frac{1}{\omega_2^{(0)} - \omega_1^{(0)}} \operatorname{tr}\{I_a[A^{(1)}(k_1), [A^{(1)}(k_3), A^{(1)}(k_4)]]\}.\]

We only need to discuss the four-wave resonance processes, as we are interested in the effects of wave interactions on the eigenfrequencies. Therefore, \(k^{(0)}\) in Eq. (13) satisfies the dispersion relation Eq. (37) and Eq. (39) in the following discussion. In this section, we only calculate damping rate being the imaginary part of the eigenfrequency, i.e., \(\gamma = -\text{Im}\omega^{(2)}\).

Since only the imaginary part of Eq. (43) is related to the damping rate, it is also sufficient to consider the imaginary part of Eq. (43)

As the oscillations are developed from random thermal motions, the eigenmodes can be considered completely incoherent to each other. For a color field being in random phase, we
have
\[ \langle A^{(1)\alpha}(k) \rangle = 0. \] (44)

where \( \langle \cdot \rangle \) means the average with respect to the random phase of the oscillations. Then, using Eq.(38), one can obtain
\[ \langle A^{(1)\alpha}(k)A^{(1)\beta}(k') \rangle = (2\pi)^4 \delta^{(4)}(k - k') \delta^{\alpha\beta} \langle A^{(1)} \rangle^2_{k\omega}, \] (45)
\[ \langle A^{(1)} \rangle_{k\omega} = \frac{\pi}{\omega^{(0)}D} [\delta(\omega^{(0)} - \omega_k) + \delta(\omega^{(0)} + \omega_k)]I_k, \] (46)
\[ I_k = \frac{|E_k|^2}{2V}. \] (47)

where \( I_k \) characterizes the total intensity of the fluctuation oscillation with frequency \( \omega_k \) and \( -\omega_k \), \( V \) is the volume of the plasma.

As the oscillations are incoherent to each other, the average of product of the field potential can be expanded as
\[ \langle A^{(1)}(k_1)A^{(1)}(k_2)A^{(1)}(k_3)A^{(1)}(k_4) \rangle \]
\[ = \langle A^{(1)}(k_1)A^{(1)}(k_2) \rangle \langle A^{(1)}(k_3)A^{(1)}(k_4) \rangle + \langle A^{(1)}(k_1)A^{(1)}(k_3) \rangle \langle A^{(1)}(k_2)A^{(1)}(k_4) \rangle \]
\[ + \langle A^{(1)}(k_1)A^{(1)}(k_4) \rangle \langle A^{(1)}(k_2)A^{(1)}(k_3) \rangle. \] (48)

Multiplying both sides of Eq.(33) by \( A^{(1)\beta}(k') \), then performing the average of the result with respect to the random phase using Eq.(44) and Eq.(48), one can obtain the damping rate of the longitudinal eigenwaves in QGP
\[ \gamma(K) = -\text{Im}\omega^{(2)} \]
\[ = \frac{3m^2_{\rho}g^2}{\omega_p^2} \text{Im} \left\{ \int \frac{d\Omega}{4\pi} \int \frac{d^4k_1}{(2\pi)^4} \langle A^{(1)} \rangle_{k_1\omega} \frac{1}{v \cdot k^{(0)} + i0^+} \right. \]
\[ \times \left[ \frac{(v \cdot k)^2}{K^2} \frac{(v \cdot k_1)^2}{K_1^2} \frac{1}{v \cdot (k^{(0)} - k_1^{(0)}) + i0^+} \left( \frac{\omega_1^{(0)}}{v \cdot k_1^{(0)} - i0^+} - \frac{\omega_1^{(0)}}{v \cdot k^{(0)} + i0^+} \right) \right. \]
\[ + m^2_{\rho} \frac{(v \cdot k)^2}{K^2} \frac{(v \cdot k_1)^2}{K_1^2} \frac{1}{v \cdot (k^{(0)} - k_1^{(0)}) + i0^+} \left( \frac{\omega_1^{(0)}}{v \cdot k_1^{(0)} - i0^+} - \frac{\omega_1^{(0)}}{v \cdot k^{(0)} + i0^+} \right) \]
\[ \left. + \int \frac{d\Omega'}{4\pi} \frac{v' \cdot (k - k_1)}{|k - k_1|} \right]_1 \langle A^{(1)} \rangle_{k \omega} \frac{1}{v \cdot k^{(0)} + i0^+} \]
\[ \left. \times \frac{1}{\epsilon(\omega^{(0)} - \omega_1^{(0)}, k - k_1) (\omega^{(0)} - \omega_1^{(0)})^2} \right\} \]
\[ \frac{1}{v' \cdot (k(0) - k_1(0)) + i0^+} \left( \frac{\omega_1(0)}{v' \cdot k_1(0) - i0^+} - \frac{\omega(0)}{v' \cdot k(0) + i0^+} \right) \} \right). \]  

(49)

Now we perform the integrals in Eq. (49) in cylindrical coordinates and in plasma particle local rest frame. For simplicity of calculation the direction of polar axis is selected to be the same as the direction of \( k_1 \); then the coordinates for \( k, p, p' \) are \( k(K, \alpha, \beta) \), \( p(E_p, \theta, \varphi) \), \( p'(E_p', \theta', \varphi') \) correspondingly. As only the damping rate in the long-wavelength limit is calculated numerically later, we will use the approximate relation in the long-wavelength region in the following calculation. In the long-wavelength region, the dispersion relation Eq.(39) is reduced to the form of\[ \omega(0)^2 = \omega_p^2 + \frac{3}{5} K^2. \]  

(50)

For the long-wavelength limit case\( (K = 0) \), using the equation \( p^0/(p \cdot k' + ip^00^+) = P[1/(\omega' - \mathbf{v} \cdot \mathbf{k}')] \) \(-i\pi\delta(\omega' - \mathbf{v} \cdot \mathbf{k}') \), where \( \mathbf{v} = p/p^0 \) and \( P \) stands for principal value of the function, one can check that the condition for nonvanishing of Eq.(49) is the satisfaction of the delta function \( \delta[\omega(0) - \omega_1(0) - \mathbf{v} \cdot (\mathbf{k} - \mathbf{k}_1)] \). This delta function restricts the upper bound of the integrals over \( K_1 \) in Eq.(49) to equal \( gT \) approximately. We obtain from Eq.(49) the expression of the damping rate for \( k = 0 \) modes,

\[ \gamma = \frac{3g'^2m_D^2}{32\omega_p^2\pi^2} \int d\omega_1(0) \int_0^{gT} dK_1 I_{k_1} \frac{K_1}{\omega_1(0) \omega_p} (\frac{\omega_1(0)}{\omega_p} - 1) \delta[\omega(0) - \sqrt{\omega_p^2 + \frac{3}{5} K_1^2}] \]

\[ \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\varphi \delta(\cos\theta - \omega_p K_1) \int_0^{2\pi} \frac{\cos^2\theta \cos^2\varphi}{K_1} \cos^2\theta \cos^2\varphi' \]

\[ \left\{ \frac{\cos^2\phi \cos^2\alpha + \frac{m_D^2}{4} \int_{-1}^1 d(\cos\theta') \int_0^{2\pi} d\varphi' \cos^2\theta \cos^2\theta'}{\omega_p - \omega_1(0) + K_1 \cos \theta' - (\omega_p - \omega_1(0)) \omega_p - \omega_1(0) + K_1 \cos \theta'} \right\} \]

\[ \cos \phi \cos \phi' \left[ \left( \frac{\omega_1(0)}{\omega_p} - 1 \right) + \left( \frac{K_1 \cos \theta'}{\omega_1(0) - K_1 \cos \theta'} \right) \right] \]

\[ + \frac{\omega_1(0)}{\omega_1(0) - K_1 \cos \theta'} \left( \frac{\omega_1(0)}{\omega_p} - 1 \right) \right\}, \]  

(51)

where \( \phi, \phi' \) are the corresponding angles between \( \mathbf{p} \) and \( \mathbf{k} \), \( \mathbf{p}' \) and \( \mathbf{k} \). They can be expressed
as
\[
\cos \phi = \sin \theta \sin \alpha \cos (\varphi - \beta) + \cos \theta \cos \alpha.
\] (52)

And the expression for \(\cos \phi ' \) is similar to the above equation.

In Eqs. (51), the damping rate is proportional to the wave intensity \(I_k\). In an equilibrium plasma, the wave intensity is governed by the plasma temperature[7][13],
\[
I_k = 8\pi T \frac{T}{\epsilon_k \omega_k},
\] (53)
\[
\epsilon_k' = \frac{\partial \text{Re} (\omega, k)}{\partial \omega_k}.
\] (54)

For the long-wavelength case, using Eq.(50), we can obtain
\[
I_k = 4\pi T.
\] (55)

Then, after finishing all the integrals in Eq.(51), the numerical result of the damping rate for pure gluon gas is obtained
\[
\gamma \sim 0.21 g^2 T.
\] (56)

Now, compare the above result with the damping rate for long-wavelength gluons calculated by the effective perturbation theory based on the resummation of the hard thermal loops. The hard thermal result is [14]
\[
\gamma \sim 0.26 g^2 T.
\] (57)

Note that both of them are obtained approximately, we can conclude that they are coincide. This indicates the consistency between the high temperature limit of thermal QCD and the non-Abelian Kubo formula derived from the kinetic theory for quarks and gluons.

In this paper, we start from the non-Abelian Kubo formula which is gauge invariant, and we chose a physical gauge to express the gauge invariant equation, then the multiple time-scale method is used to expand the equation according the field strength. Our result is coincide with the gauge invariant result obtained using the effective perturbation theory. As
to the question if the multiple method guarantees the gauge symmetry generally, this need further study.

In conclusion: in this paper we derive the non-Abelian generation of the Kubo formula from the kinetic theory and show it is coincide with the form obtained using the eiknoal for a Chern-Simons theory. The multiple time-scale method is used to solve the non-Abelian Kubo formula obtained in this paper to calculate the damping rate for the eigenwaves in the long-wavelength limit numerically,

the result is coincide with that obtained using the effective perturbation theory in the finite temperature field theory. This show that the multiple time-scale method is effective on solving the non-Abelian generation of the Kubo formula.

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