Equilibrium Refinement in Finite Evidence Games  

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Evidence games study situations where a sender persuades a receiver by selectively disclosing hard evidence about an unknown state of the world. Evidence games often have multiple equilibria. Hart et al. (2017) propose to focus on truth-leaning equilibria, i.e., perfect Bayesian equilibria where the sender prefers disclosing truthfully when indifferent, and the receiver takes off-path disclosure at face value. They show that a truth-leaning equilibrium is an equilibrium of a perturbed game where the sender has an infinitesimal reward for truth-telling. We show that, when the receiver’s action space is finite, truth-leaning equilibrium may fail to exist, and it is not equivalent to equilibrium of the perturbed game. To restore existence, we introduce a disturbed game with a small uncertainty about the receiver’s payoff. A purifiable equilibrium is a truth-leaning equilibrium in an infinitesimally disturbed game. It exists and features a simple characterization. A truth-leaning equilibrium that is also purifiable is an equilibrium of the perturbed game.

Keywords: Hard evidence, Verifiable disclosure, Equilibrium refinement

JEL Codes: C72, D82, D83

1. Introduction. In many real-life situations, communication relies on hard evidence. For example, a jury’s verdict should be based on evidence presented in the court, rather than exchanges of empty claims. Evidence games study such situations. There is a sender (e.g., a prosecutor) and a receiver (e.g., a jury). The sender has some hard evidence about an unknown state of the world (e.g., whether a defendant is guilty) and wants to persuade the receiver to take a certain action (e.g., to reach a guilty verdict) by selectively presenting evidence. Full revelation of evidence is often impossible in the presence of conflict of interest between the sender and the receiver—the receiver wants to learn the payoff relevant state and act accordingly, whereas the sender merely wants to induce her preferred receiver action. Therefore, the sender has an incentive to persuade the receiver that a certain state is more likely by partially revealing evidence.

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A distinguishing feature of evidence games (as opposed to signaling games and cheap-talk communication) is that the receiver’s private information (i.e., her evidence) is not payoff relevant. Instead, it affects her ability to persuade the receiver by restricting her set of feasible actions (i.e., the set of evidence she can present to the receiver). That is, the sender can selectively disclose evidence that she has but cannot fabricate evidence. For example, the prosecutor’s objective is to convict the defendant. This is not affected by what evidence she has. In equilibrium, her chance of convicting the defendant may depend on the evidence she has, because when she has more evidence, there are more ways to present evidence in the court, and thereby she may better persuade the jury.

Evidence games often have multiple (Nash) equilibria. For example, there is a trivial equilibrium where, regardless of her evidence, the prosecutor presents no evidence, and the jury always acquits the defendant (this must be optimal on the equilibrium path for the jury if the presumption of innocence is practiced). Obviously, it is not a sensible prediction of what happens in courtrooms. However, this equilibrium is perfect (Selten, 1975) and sequential (Kreps and Wilson, 1982). Consider a perturbation to the prosecutor’s strategy which puts higher probability on disclosing acquitting evidence than on disclosing convicting evidence, and a perturbation to the jury’s strategy such that the probability of convicting the defendant after seeing any evidence is smaller than that after seeing no evidence. As both perturbations converge to zero, this gives a sequence of $\varepsilon$-constrained equilibria in completely mixed strategies that converges to the trivial equilibrium. Hence, the trivial equilibrium is perfect. Moreover, given the perturbed sender’s strategy, it is consistent for the jury to believe that the actual evidence possessed by the prosecutor is more acquitting after seeing any disclosed evidence. Therefore, the trivial equilibrium is also a sequential equilibrium.

Hart et al. (2017) (henceforth HKP) propose the following refinement to perfect Bayesian equilibrium in evidence games. A truth-leaning equilibrium is a perfect Bayesian equilibrium such that

(Truth-leaning) Given the receiver’s strategy, the sender discloses her evidence truthfully if doing so is optimal;

(Off-path beliefs) The receiver takes any off-path disclosure at face value (i.e., he believes that the sender discloses truthfully).

These conditions (especially the first one) are intuitive for evidence games. As is argued in HKP, these conditions follow the simple intuition that there is a “slight inherent advantage” for the sender to tell the whole truth, and “there must be good reasons for not telling it.”

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1HKP defines truth-leaning equilibrium as a refinement to Nash equilibrium. However, we note that any truth-leaning equilibrium is a perfect Bayesian equilibrium and sequential equilibrium. We regard all solution concepts in the current paper as refinements to perfect Bayesian equilibrium.
H KP study evidence games where the receiver continuously chooses an action on the real line, the receiver’s expected payoff is a single-peaked function of his action given any distribution of the state, and the sender strictly prefers higher receiver action. They show that a truth-leaning equilibrium exists and is receiver optimal. That is, any truth-leaning equilibrium yields the same ex post payoffs as the optimal mechanism where the receiver can commit to an action plan. Jiang (2019) studies evidence games where the state of the world is binary, the receiver’s optimal action is monotone in his posterior belief, and the sender strictly prefers higher receiver action. He shows that the truth-leaning equilibrium is essentially unique in any evidence game and provides a method to solve the truth-leaning equilibrium.

However, in many applications of evidence games, the receiver takes a discrete action. For example, juries choose between conviction and acquittal, banks decide whether or not to grant a loan, and rating agencies rate financial assets into finitely many grades. In such situations, both assumptions of HKP and Jiang (2019) are violated. A part of this paper is to answer the following question: is truth-leaning equilibrium a “good” solution concept when the receiver’s action set is finite?

The short answer is “no,” and one reason is that a truth-leaning equilibrium may fail to exist. Loosely speaking, nonexistence arises because the sender lacks a strict incentive to persuade the receiver. To address this problem, we propose the following solution concept by introducing a small uncertainty (i.e., disturbance) to the receiver’s payoff à la Harsanyi (1973). Suppose that the receiver receives a random private payoff shock associated with each of his actions. In the disturbed game, the sender has a strict incentive to persuade the receiver, and a truth-leaning equilibrium exists in the disturbed game. We define a purifiable equilibrium as the limit of a sequence of truth-leaning equilibria in the disturbed games as the disturbances converge to zero. That is, a purifiable equilibrium is a truth-leaning equilibrium of an infinitesimally disturbed game. A purifiable equilibrium always exists.

Another problem of truth-leaning equilibrium in finite evidence games is that it may not follow the intuition that the sender is slightly more advantageous if she discloses truthfully.

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2In both HKP and Jiang (2019), if a piece of evidence $e'$ is inherently better than the sender’s evidence $e$ (i.e., the receiver’s optimal action knowing that the sender’s evidence is $e'$ is strictly higher than his optimal action knowing that the sender’s evidence is $e$) and the sender can feasibly disclose $e'$, then the sender’s payoff from any randomization between disclosing $e'$ and $e$ is strictly higher than her payoff from disclosing only $e$, given any Bayesian consistent system of beliefs of the receiver and any sequentially rational receiver strategy. This is not the case when the receiver’s action is finite.

3For example, the prosecutor does not know how lenient the jury is (i.e., how convinced the jury has to be in order to reach a conviction). However, she knows that after seeing more convicting evidence, the probability that the jury will choose to convict the defendant is higher. Therefore, the prosecutor strictly prefers presenting all convicting evidence.
To formalize this intuition, we revisit the perturbed game in HKP, where the sender receives a small reward if she discloses truthfully, and the sender must disclose truthfully with at least some small probability. We define a weakly truth-leaning equilibrium as the limit of a sequence of perfect Bayesian equilibria of the perturbed games as the perturbations converge to zero. HKP shows that truth-leaning equilibrium is equivalent to weakly truth-leaning equilibrium. When the receiver’s action space is finite, however, the equivalence is no longer true. It turns out that purifiability is the missing condition connecting truth-leaning and weakly truth-leaning equilibrium–a purifiable truth-leaning equilibrium is weakly truth-leaning, and a purifiable weakly truth-leaning equilibrium is truth-leaning in “almost all” evidence games.

Outline of the paper. Section 2 presents an example where truth-leaning equilibrium does not exist and illustrates the constructions of purifiable equilibrium and weakly truth-leaning equilibrium. Section 3 presents the model. Section 4 formally defines purifiable equilibrium and shows the relationship between purifiable equilibrium, truth-leaning equilibrium, and weakly truth-leaning equilibrium. The last section concludes. Proofs are in the Appendix.

2. A Simple Example and Discussion. Every new aircraft design has to be certified by the Federal Aviation Administration (FAA) before any aircraft built according to this design can enter service (in the U.S.). Like other innovations, altering the design of an aircraft often entails high level of risks. The FAA often has to rely on information and test results provided by airplane manufacturers, yet airplane manufacturers’ disclosure is far from complete.

Consider an airplane manufacturer (the sender) seeking to get a new aircraft design certified by the FAA (the receiver). The design can be good or bad with equal likelihood. If the design is bad, the aircraft manufacturer has some bad evidence (e.g., machanical failures during test flights) with probability \( \frac{2}{3} \). Otherwise, the aircraft manufacturer has no evidence. The FAA does not know the quality of the design and chooses to Approve or Reject the aircraft design based on evidence disclosed by the sender. The disclosure of bad evidence is voluntary and verifiable (i.e., the sender can disclose bad evidence or no evidence if it has bad evidence, and it can only disclose no evidence if it has no evidence).

4For example, a design deficiency of the battery system on board Boeing’s 787 Dreamliners had caused several aircraft fires in 2013, which led to the grounding of all 50 aircrafts at the time and a redesign of the battery system (see [https://www.cnn.com/travel/article/boeing-787-dreamliner-investigation-report](https://www.cnn.com/travel/article/boeing-787-dreamliner-investigation-report)). More recently, MCAS, a new fligh control software embedded into Boeing’s 737 MAX aircrafts, caused two deadly crashes within two years of the airliner’s first commercial operation. The entire fleet has since then been grounded (see [https://www.nytimes.com/2019/06/01/business/boeing-737-max-crash.html](https://www.nytimes.com/2019/06/01/business/boeing-737-max-crash.html)).

5In the 737 MAX incident, Boeing failed to inform the FAA of a design change of MCAS (see [https://www.nytimes.com/2019/10/11/business/boeing-737-max.html](https://www.nytimes.com/2019/10/11/business/boeing-737-max.html)).
The airplane manufacturer’s payoff depends only on the FAA’s action: it receives 1 if the design is approved and 0 if the design is rejected. The FAA, on the other hand, gains from approving a good design and loses from approving a bad design. Its payoff is 0 if it rejects the design, is 1 if it approves a good design, and is -2 if it approves a bad design. Hence, the FAA has a cutoff decision rule. If, after observing the disclosed evidence, its posterior belief that the design is good exceeds \( \frac{2}{3} \), its optimal action is Approve; if its posterior belief is less than \( \frac{2}{3} \), its optimal action is Reject; if its posterior belief is exactly \( \frac{2}{3} \), either action as well as any randomization between the two actions is optimal.

A strategy of the sender describes how it discloses bad evidence. Let \( p \) be the probability that the sender discloses no evidence if it has bad evidence. Since bad evidence fully reveals that the design is bad, the receiver always chooses Reject (thus the sender gets 0) after seeing bad evidence. Let \( q \) be the probability that the receiver chooses Approve after seeing no evidence. Let \( \mu \) be the receiver’s posterior belief that the design is good after seeing no evidence. Since no evidence is always disclosed with positive probability, Bayes’ rule requires that \( \mu = \frac{3}{4+2p} \).

2.1. Truth-leaning equilibrium. It is easy to verify that the game has a continuum of perfect Bayesian equilibria, where \( p \geq \frac{1}{4}, q = 0, \) and \( \mu = \frac{3}{4+2p} \leq \frac{2}{3} \). That is, the sender with bad evidence discloses no evidence with at least probability \( \frac{1}{4} \), and the receiver always rejects the design.

However, there is no truth-leaning equilibrium. Given the receiver’s strategy, the sender with bad evidence is indifferent between disclosing no evidence and disclosing truthfully (since both actions yield payoff 0). Truth-leaning therefore requires the sender to disclose bad evidence truthfully (i.e., \( p = 0 \)), which is not satisfied by any perfect Bayesian equilibrium.

2.2. Purifiable equilibrium. Suppose that the receiver receives a payoff shock \( \zeta \) for choosing Approve, where \( \zeta \) is normally distributed according to \( N(0, \varepsilon^2) \) and is private information of the receiver (hence the receiver’s type). That is, the receiver’s payoff from approving a good design is \( 1 + \zeta \), and that from approving a bad design is \( \zeta - 2 \). Hence, the receiver almost always has a unique optimal action after seeing no evidence, which is Approve if \( \mu > \frac{2-\zeta}{3} \) (equivalently, \( \zeta > 2 - 3\mu \)) and Reject if \( \mu < \frac{2-\zeta}{3} \) (equivalently, \( \zeta < 2 - 3\mu \)). This implies that the design is approved with probability \( \Phi \left( \frac{3\mu - 2}{\varepsilon} \right) > 0 \) if the sender discloses no evidence in any perfect Bayesian equilibrium of the disturbed game, where \( \Phi \) is the cdf of standard normal distribution. Hence, the sender’s expected payoff from disclosing no evidence is strictly higher than that from disclosing bad evidence truthfully, so the sender always discloses no evidence.

To summarize, let \( q(\zeta) \) be the probability that the type \( \zeta \) receiver approves the design
after observing no evidence. The disturbed game has a continuum of perfect Bayesian equilibria, where $p = 1$, $\mu = \frac{1}{2}$, and $q(\zeta) = 0$ if $\zeta < \frac{1}{2}$, $q(\zeta) \in [0, 1]$ if $\zeta = \frac{1}{2}$, $q(\zeta) = 1$ if $\zeta > \frac{1}{2}$. Since the sender strictly prefers disclosing no evidence, all perfect Bayesian equilibria of any disturbed game are truth-leaning. Moreover, in all equilibria, the receiver chooses $Approve$ with probability $\Phi(-\frac{1}{2\varepsilon})$ after observing no evidence. That is, the disturbed game has a unique truth-leaning equilibrium outcome, where the sender discloses no evidence, and after seeing no evidence, the receiver chooses $Approve$ with probability $\Phi(-\frac{1}{2\varepsilon})$ and believes that the design is good with $\frac{1}{2}$ probability.

As the disturbance diminishes (i.e., as $\varepsilon \downarrow 0$), the unique equilibrium outcome of the disturbed game converges to a perfect Bayesian equilibrium of the original evidence game, where $p = 1$, $q = 0$, $\mu = \frac{1}{2}$.

2.3. Weakly truth-leaning equilibrium. Let us consider the following perturbed game. Let $\varepsilon_1$ and $\varepsilon_2$ be small positive reals that are common knowledge to the sender and the receiver. The sender receives a reward $\varepsilon_1$ if it discloses (bad evidence) truthfully, and the sender must disclose truthfully with at least probability $\varepsilon_2$.

If its posterior belief $\mu > \frac{2}{3}$, then the receiver has a unique optimal action $Approve$ after observing no evidence. Then, for $\varepsilon_1 < 1$, the sender strictly prefers disclosing no evidence, so the Bayesian consistent belief is $\mu = \frac{1}{2} < \frac{2}{3}$. If $\mu < \frac{2}{3}$, the receiver’s unique optimal action is $Reject$ after observing no evidence. With the reward for truth-telling, the sender strictly prefers disclosing truthfully, so the Bayesian consistent belief is $\mu = \frac{3}{4} > \frac{2}{3}$. Hence, the receiver’s posterior belief $\mu = \frac{2}{3}$ in any perfect Bayesian equilibrium of the perturbed game. Indeed, for $\varepsilon_1 < 1$ and $\varepsilon_2 \leq \frac{3}{4}$, the perturbed game has a unique perfect Bayesian equilibrium, where $p = \frac{1}{4}$, $q = \varepsilon_1$, $\mu = \frac{2}{3}$.

As $\varepsilon_1, \varepsilon_2 \downarrow 0$, the perfect Bayesian equilibrium of the perturbed game converges to a perfect Bayesian equilibrium of the original game, where $p = \frac{1}{4}$, $q = 0$, $\mu = \frac{2}{3}$.

2.4. Discussions. Figure 1 summarizes the equilibria of the game. There is a continuum of perfect Bayesian equilibria which differ only on the sender’s strategy. Among them, the weakly truth-leaning equilibrium maximizes the probability that the sender discloses truthfully. The purifiable equilibrium maximizes the receiver’s posterior belief on the good design.

The fact that this simple game does not possess a truth-leaning equilibrium suggests that truth-leaning equilibrium may not be a proper solution concept for finite evidence games. A more fundamental problem of truth-leaning equilibrium is the discrepancy between the refinement and the intuition behind it. The requirement that the sender weakly prefers disclosing truthfully seemingly arises from the sender having infinitesimal reward for truth-
telling, but in the example, the weakly truth-leaning equilibrium constructed by adding an infinitesimal reward for truth-telling is not the same as imposing truth-leaning refinement on perfect Bayesian equilibria.

Weakly truth-leaning equilibrium also has a few shortcomings. First, it may not be robust to incomplete receiver payoff information. In our example, the sender strictly prefers disclosing no evidence once we introduce a small uncertainty to the receiver’s payoff. Therefore, the weakly truth-leaning equilibrium where the sender having bad evidence discloses no evidence with probability $\frac{1}{4}$ is not robust to incomplete receiver payoff information. In defense of weakly truth-leaning equilibrium, the perfect Bayesian equilibrium in every perturbed game where the sender receives a small reward for truth-telling (viz., $p = \frac{1}{4}$, $q = \varepsilon_1$, $\mu = \frac{2}{3}$) is robust to incomplete receiver payoff information. In the current example, suppose that the receiver chooses an action $a \in \mathbb{R}$, and the receiver’s payoff is quadratic, i.e., $-\frac{1}{2}(a-x)^2$, where $x$ is a random variable that equals $0$ if the design is bad and $1$ if the design is good. The unique truth-leaning equilibrium is as follows. The sender always discloses no evidence, the receiver’s belief and action are $\frac{1}{2}$ after seeing no evidence and $0$ after seeing bad evidence. This is also the unique weakly truth-leaning equilibrium.

Second, different sequences of perturbations may select different weakly truth-leaning equilibria, and not all sequences of perturbed games have a sequence of perfect Bayesian equilibria that converges as the perturbation goes to zero.

Consider a slight variant to our example, where the sender’s bad evidence is either type $1$ or type $2$ (think about software failures and hardware failures). If the design is bad, the

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6 Recall that HKP shows the equivalence of truth-leaning equilibrium and weakly truth-leaning equilibrium in evidence games where the receiver continuously chooses an action, and its payoff function is single-peaked given any belief. In the current example, suppose that the receiver chooses an action $a \in \mathbb{R}$, and the receiver’s payoff is quadratic, i.e., $-\frac{1}{2}(a-x)^2$, where $x$ is a random variable that equals $0$ if the design is bad and $1$ if the design is good. The unique truth-leaning equilibrium is as follows. The sender always discloses no evidence, the receiver’s belief and action are $\frac{1}{2}$ after seeing no evidence and $0$ after seeing bad evidence. This is also the unique weakly truth-leaning equilibrium.

7 To see this, consider a disturbed game where: (i) the sender receives $\varepsilon_1$ if it discloses truthfully; (ii) the sender must disclose truthfully with at least probability $\varepsilon_2$; (iii) the receiver receives a payoff shock $\zeta$ distributed according to $\mathcal{N}(0, \varepsilon^2)$ for choosing $\text{Approve}$, which is its private information. For $\varepsilon_1 < \frac{1}{2}$, $\varepsilon_2 < \frac{1}{2}$, and $\varepsilon < \frac{3 - 4\varepsilon}{2} - \frac{1}{\Phi^{-1}(\varepsilon_1)}$, the disturbed game has a continuum of perfect Bayesian equilibria, where $p = \frac{9}{4 + 2\varepsilon \Phi^{-1}(\varepsilon_1)} - 2$, $\mu = \frac{2 + \varepsilon \Phi^{-1}(\varepsilon_1)}{3}$, $q(\zeta) = 0$ if $\zeta < -\varepsilon \Phi^{-1}(\varepsilon_1)$, $q(\zeta) \in [0, 1]$ if $\zeta = -\varepsilon \Phi^{-1}(\varepsilon_1)$, and $q(\zeta) = 1$ if $\zeta > -\varepsilon \Phi^{-1}(\varepsilon_1)$. In any perfect Bayesian equilibrium, the design is approved with probability $\varepsilon_1$ after the receiver observes no evidence. As $\varepsilon \downarrow 0$, this equilibrium outcome converges to $p = \frac{1}{4}$, $q = \varepsilon_1$, $\mu = \frac{2}{3}$.

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sender has type 1 bad evidence, type 2 bad evidence, and no evidence each with \( \frac{1}{3} \) probability; if the design is good, the sender has no evidence. The sender with a certain type of bad evidence can disclose truthfully or no evidence but cannot disclose the other type of bad evidence. Let \( p_i \) be the probability that the sender with type \( i \) bad evidence discloses no evidence, \( q \) the probability that the receiver chooses \textit{Approve} after seeing no evidence, and \( \mu \) the receiver’s belief that the design is good after seeing no evidence. The game has a continuum of perfect Bayesian equilibria, where \( p_1 + p_2 \geq \frac{1}{2}, q = 0, \) and \( \mu = \frac{3}{4+p_1+p_2} \).

Now, let us consider the following perturbed game. Given small positive reals \( \varepsilon_1, \varepsilon_2 < 1 \) and \( \varepsilon_{1|1}, \varepsilon_{2|2} \leq \frac{1}{2} \), the sender receives a reward \( \varepsilon_i \) if it truthfully discloses type \( i \) bad evidence, and the sender with type \( i \) bad evidence must disclose truthfully with at least probability \( \varepsilon_{i|i} \). If \( \varepsilon_i < \varepsilon_j \), the unique perfect Bayesian equilibrium is \( p_i = \frac{1}{2}, p_j = 0, q = \varepsilon_i, \mu = \frac{2}{3} \). If \( \varepsilon_1 = \varepsilon_2 \), there is a continuum of perfect Bayesian equilibria, where \( p_1 + p_2 = \frac{1}{2}, q = \varepsilon_1 = \varepsilon_2, \mu = \frac{2}{3} \). Hence, as \( (\varepsilon_1, \varepsilon_{1|1}, \varepsilon_2, \varepsilon_{2|2}) \to 0 \), whether there exits a convergent sequence of perfect Bayesian equilibria depends on the rates of convergence of \( \varepsilon_1 \) and \( \varepsilon_2 \). If \( \varepsilon_1 = \varepsilon_2 \) almost always, then any perfect Bayesian equilibrium of the unperturbed game such that \( p_1 + p_2 = \frac{1}{2} \) is the limit point of a sequence of perfect Bayesian equilibria of the perturbed game. If \( \varepsilon_i \leq \varepsilon_j \) almost always, and \( \varepsilon_i < \varepsilon_j \) infinitely often, then the unique weakly truth-leaning equilibrium is such that \( p_i = \frac{1}{2}, p_j = 0, q = 0, \) and \( \mu = \frac{2}{3} \). If neither case happens, there is no convergent sequence of perfect Bayesian equilibria of the perturbed game. In conclusion, the unperturbed game has a continuum of weakly truth-leaning equilibria, where \( p_1 + p_2 = \frac{1}{2}, q = 0, \mu = \frac{2}{3} \), and different weakly truth-leaning equilibria may be selected by different sets of infinitesimal perturbations.

Purifiable equilibrium is spared from similar problems. For “almost all” evidence games, any purifiable equilibrium is infinitesimally close to a truth-leaning equilibrium of \textit{any} infinitesimally perturbed game. That is, purifiability does not depend on the selection of disturbances. The normality of the receiver’s payoff shock in our example is dispensable. Moreover, the set of purifiable equilibria has a simple structure, and we give a characterization of all purifiable equilibria in any evidence game.

3. The Evidence Game. There are two stages. Two players, a sender (she) and a receiver (he), move sequentially. At the outset of the game, a state of the world \( \omega \in \{G, B\} \) is realized with probability \( \pi_0 \in (0, 1) \) on \( \omega = G \). Neither player observes the realized state \( \omega \)\(^8\) and the prior \( \pi_0 \) is common knowledge. In the first stage, the sender observes a piece of hard evidence \( e \in E \) and discloses \( m \in E \) to the receiver, where \( E \) is a finite set of evidence. In the second stage, the receiver observes the disclosed evidence \( m \) and chooses an action

\(^8\)Since the sender’s payoff is independent of the realized state, so it does not affect our analysis if the realized state is known to the sender.
a ∈ A, where $A = \{a_1 < a_2 < \cdots < a_K\}$ is a finite subset of the real line.

3.1. Evidence and disclosure. Let $F_G$ and $F_B$ be two distributions over the set of evidence $E$. The sender’s evidence $e$ is a random draw from either $F_G$ or $F_B$, depending on the realized state. If $\omega = G$, $e$ is drawn from distribution $F_G$; if $\omega = B$, it is drawn from distribution $F_B$.

Disclosure is verifiable. That is, the set of evidence that the sender can feasibly disclose depends on the evidence she has (in contrast, in a signaling game, the sender, regardless of her type, chooses from the same set of signals). Throughout the paper, We maintain the following assumptions that are standard in the literature:

(Reflexivity) The sender can always truthfully disclose her evidence $e$;

(Transitivity) If the sender can disclose $e'$ when she has evidence $e$, and she can disclose $e''$ when she has evidence $e'$, then she can disclose $e''$ if she has evidence $e$.

Under these assumptions, we can represent the “disclosure rule” as a preorder $\preceq$ on $E$.\footnote{A preorder $\preceq$ is a binary relation satisfying reflexivity ($e \preceq e$ for all $e$) and transitivity ($e'' \preceq e' \preceq e \Rightarrow e'' \preceq e$).}

Disclosing $m$ is feasible given evidence $e$ if and only if $m \preceq e$, and the feasible set of disclosure given a piece of evidence $e$ is its lower contour set $LC(e) = \{m \in E : m \preceq e\}$.

3.2. Payoffs. The receiver’s payoff $u_R(a, \omega)$ depends on both his action and the realized state of the world (but not the true evidence or the disclosed evidence), and the receiver maximizes his expected payoff. We assume that the receiver’s payoff function satisfies the following assumption:

(Increasing differences) $u_R(a, G) - u_R(a, B)$ is strictly increasing in $a$.

Under this assumption, the receiver wants to match the state of the world. That is, his optimal action is weakly increasing in his posterior belief that the state is good. More precisely, given $\mu \in [0, 1]$, the solution to the receiver’s maximization problem

$$\phi(\mu) = \arg\max_{a \in A} \mu u_R(a, G) + (1 - \mu) u_R(a, B)$$

is upper hemicontinuous and weakly increasing in $\mu$.\footnote{Throughout the paper, we say a correspondence $\phi : [0, 1] \rightrightarrows A$ is weakly increasing if $a_i \leq a_j$ for all $\mu_i < \mu_j$, $a_i \in \phi(\mu_i)$, and $a_j \in \phi(\mu_j)$.}

The sender’s payoff equals the action of the receiver, i.e., $u_S(a, \omega) = a$. Given the assumption on the receiver’s payoff, the sender has a weak incentive to persuade the receiver that the state is good. Notice that the evidence $e$, the disclosed evidence $m$, and the realized state $\omega$ are payoff irrelevant to the sender.

An evidence game is summarized by a tuple $G = \langle \pi_0, (E, \preceq), F_G, F_B, A, u_R \rangle$.\footnote{An evidence game is a tuple $G = \langle \pi_0, (E, \preceq), F_G, F_B, A, u_R \rangle$. The set $E$ of possible observations is the set of possible realizations of the evidence; $\preceq$ is a preorder on $E$; $F_G$ and $F_B$ are the probability distributions that assign probabilities to pairs of observations and states; $A$ is the set of actions available to the receiver; and $u_R$ is the receiver’s payoff function.}
3.3. Strategies and perfect Bayesian equilibrium. A (behavioral) strategy of the sender is \( \sigma : E \to \Delta(E) \) such that \( \text{supp}(\sigma(\cdot|e)) \subset \text{LC}(e) \), a (behavioral) strategy of the receiver is \( \rho : E \to \Delta(A) \), and a system of beliefs of the receiver is \( \mu : E \to [0,1] \), where \( \mu(m) \) denotes the receiver’s posterior belief that the state is good after observing \( m \).

A \textit{perfect Bayesian equilibrium} of \( \mathcal{G} \) is a collection of the sender’s strategy, the receiver’s strategy, and the receiver’s system of belief \((\sigma, \rho, \mu)\) such that:

**Sender optimality** Given \( \rho \),

\[
\text{supp}(\sigma(\cdot|e)) \subset \arg\max\sum_{m \leq e} a \cdot \rho(a|m)
\]

for all \( e \in E \);

**Receiver optimality** Given \( \mu \),

\[
\text{supp}(\rho(\cdot|m)) \subset \phi(\mu(m))
\]

for all \( m \in E \);

**Bayesian consistency** For all on-path disclosure \( m \in \bigcup_{e \in E} \text{supp}(\sigma(\cdot|e)) \),

\[
\mu(m) = \frac{\sum_{e \in UC(m)} \sigma(m|e)FG(e)\pi_0}{\sum_{e \in UC(m)} \sigma(m|e)[FG(e)\pi_0 + FB(e)(1 - \pi_0)]}.
\]

4. Refinements of Perfect Bayesian Equilibrium. Sections 4.1 through 4.3 give formal definitions to truth-leaning equilibrium, purifiable equilibrium, and weakly truth-leaning equilibrium. Section 4.2 also characterizes purifiable equilibrium. Section 4.4 shows the relationship between the three refinements.

4.1. Truth-leaning equilibrium. A \textit{truth-leaning equilibrium} of \( \mathcal{G} \) is a perfect Bayesian equilibrium \((\sigma, \rho, \mu)\) such that:

**Truth-leaning** Given \( \rho \),

\[
e \in \arg\max\sum_{m \leq e} a \cdot \rho(a|m) \Rightarrow \sigma(e|e) = 1;
\]

**Off-path beliefs** For all off-path disclosure \( m \), \( \mu(m) = \nu(m) \), where

\[
\nu(m) = \frac{FG(m)\pi_0}{FG(m)\pi_0 + FB(m)(1 - \pi_0)}.
\]

As is shown in Section 2, a truth-leaning equilibrium of \( \mathcal{G} \) may not exist.
4.2. Disturbed games and purifiable equilibrium. A disturbed game is where the receiver has a private payoff shock (i.e., type) \( \zeta : A \to \mathbb{R} \). The receiver has type dependent payoff \( v_R(a, \omega | \zeta) = u_R(a, \omega) + \zeta(a) \). We identify the set of the receiver’s types with \( \mathbb{R}^K \), where \( K = |A| \) is the number of available receiver actions. Let \( \eta \) be a distribution over \( \mathbb{R}^K \) that has full support and is absolutely continuous with respect to the Lebesgue measure. Denote by \( \mathcal{G}_R(\eta) \) the disturbed game where the receiver’s type is distributed according to \( \eta \).

In the disturbed game, a strategy of the sender is \( \sigma : E \to \Delta(E) \) such that \( \text{supp}(\sigma(\cdot | e)) \subset LC(e) \), a strategy of the receiver in \( \mathcal{G}_R(\eta) \) is \( r : E \times \mathbb{R}^K \to \Delta(A) \), and a system of beliefs of the receiver is \( \mu : E \to [0, 1] \), where \( \mu(m) \) is the receiver’s posterior belief that the state is good after observing \( m \). Given any strategy of the receiver \( r \), let \( \rho : E \to \Delta(A) \) be the induced distributions over the receiver’s actions. That is, \( \rho(a|m) = \int r(a|m, \zeta) \eta(d\zeta) \) is the probability that the receiver takes action \( a \) after \( m \) is disclosed. We shall also use the shorthand notation and write this as \( \rho = \langle r, \eta \rangle \).

A truth-leaning equilibrium of \( \mathcal{G}_R(\eta) \) is a tuple \( (\sigma, r, \mu) \) such that:

(Receiver optimality in disturbed games) Given \( \mu \),

\[
\text{supp}(r(\cdot | m, \zeta)) \subset \tau(\mu(m), \zeta)
\]

for all \( m \in E \) and \( \zeta \in \mathbb{R}^K \), where \( \tau(\bar{\mu}, \zeta) \subset A \) is the solution to the type \( \zeta \) receiver’s problem given posterior belief \( \bar{\mu} \in [0, 1] \) on the good state, i.e.,

\[
\tau(\bar{\mu}, \zeta) = \argmax_{a \in A} \bar{\mu} u_R(a, G) + (1 - \bar{\mu}) u_R(a, B) + \zeta(a);
\]

(Sender optimality), (Bayesian consistency), (Truth-leaning), and (Off-path beliefs), as are defined above for the original game \( \mathcal{G} \).

If \( (\sigma, r, \mu) \) is a truth-leaning equilibrium, we say \( (\sigma, \rho, \mu) \) is a truth-leaning equilibrium outcome of \( \mathcal{G}_R(\eta) \).

In any disturbed game, the sender has a strict incentive to persuade the receiver—from the sender’s perspective, the expected value of the receiver’s optimal action is strictly increasing in his posterior belief. Therefore, a truth-leaning equilibrium exists in any disturbed game.

\footnote{The assumption that the receiver’s belief is independent of his type is without loss for finding truth-leaning equilibrium, as on-path beliefs are determined by Bayes’ rule, and off-path beliefs are determined by the refinement.}
Moreover, the truth-leaning equilibrium is essentially unique, and the receiver’s equilibrium system of beliefs depends only on the evidence structure, instead of the player’s utility functions. Hence, the receiver’s equilibrium system of beliefs is the same across all truth-leaning equilibria of all disturbed games. In fact, the set of truth-leaning equilibria is the same for all disturbed games.

**Lemma 1.** A truth-leaning equilibrium exists in all disturbed games. Moreover, there exist a closed set $\Sigma^* \subset \Delta(E)^E$ and a system of beliefs of the receiver $\mu^*$ such that for all disturbed games $G_R(\eta)$, $(\sigma, r, \mu)$ is a truth-leaning equilibrium of $G_R(\eta)$ if and only if $\sigma \in \Sigma^*$, $\mu = \mu^*$, and $\text{supp}(r(\cdot|m, \zeta)) \subset \tau(\mu(m), \zeta)$ for all $m \in E$ and $\zeta \in \mathbb{R}^K$.

A purifiable equilibrium is the limit point of a sequence of truth-leaning equilibria of disturbed games as the payoff uncertainty goes to zero. Formally, a purifiable equilibrium of $G$ is a tuple $(\sigma, \rho, \mu)$ such that there exists a sequence of disturbances $\{\eta^n\}_{n=1}^{\infty}$ and for each $\eta^n$, a truth-leaning equilibrium outcome $(\sigma^n, \rho^n, \mu^n)$ of $G_R(\eta^n)$ such that $\eta^n \overset{w}{\to} \delta_0$ and $(\sigma^n, \rho^n, \mu^n) \to (\sigma, \rho, \mu)$.

By Lemma 1, it is easy to see that a purifiable equilibrium exists, and in any purifiable equilibrium, $\sigma \in \Sigma^*$ and $\mu = \mu^*$. Since the receiver’s problem in any disturbed game depends only on his type and his posterior belief, the receiver’s action after seeing a disclosed evidence in any purifiable equilibrium should depend only on his posterior belief. That is, if two pieces of evidence $m$ and $m'$ are such that $\mu^*(m) = \mu^*(m')$, then $\rho(\cdot|m) = \rho(\cdot|m')$ in any purifiable equilibrium. It turns out that conversely, any perfect Bayesian equilibrium satisfying these conditions is a purifiable equilibrium.

**Proposition 2.** A purifiable equilibrium exists and is a perfect Bayesian equilibrium. Moreover, $(\sigma, \rho, \mu)$ is a purifiable equilibrium if and only if $\sigma \in \Sigma^*$, $\mu = \mu^*$, $\text{supp}(\rho(\cdot|m)) \subset \phi(\mu(m))$ for all $m \in E$, and $\mu(m) = \mu(m') \Rightarrow \rho(\cdot|m) = \rho(m', \cdot)$.

As is noted in Appendix A.1, the equilibrium system of beliefs $\mu^*$ and the set of the sender’s equilibrium strategies $\Sigma^*$ are determined by the evidence space $(E, \preceq)$ and the distributions $F_G$ and $F_B$. They are independent of the receiver’s payoff function $u_R$.

It is also worth noting that, since the sender’s action (i.e., the disclosed evidence) is not payoff relevant, evidence games are not generic in the sense of Haranyi (1973). However, all purifiable equilibria in “almost all” evidence games can be approached using an arbitrary

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12 That is, $\eta^n$ converges weakly to the point mass at 0, i.e., $\int f d\eta^n \to f(0)$ for all bounded continuous functions $f : \mathbb{R}^K \to \mathbb{R}$.

13 By contrast, a perfect Bayesian equilibrium only requires that $\text{supp}(\rho(\cdot|m)) = \text{supp}(\rho(\cdot|m'))$.

14 All Nash equilibria of an evidence game are irregular under the definition of van Damme (1996), since given the receiver’s strategy, the sender is indifferent between all of her strategies.
sequence of diminishing disturbances. By Proposition 2, if \( \phi(\mu^*(m)) \) is a singleton for all \( m \in E \), then the receiver’s purifiable equilibrium strategy is unique and is a pure strategy (viz. \( \rho(a|m) = 1_{a \in \phi(\mu^*(m))} \)). We show in Appendix A.2 that, in this case, any purifiable equilibrium can be approached using arbitrary disturbances. That is, for all purifiable equilibria \((\sigma, \rho, \mu)\) and all sequences of disturbances \( \eta^n \xrightarrow{w} \delta_0 \), there exists a sequence of truth-leaning equilibrium outcomes of the disturbed games \((\sigma^n, \rho^n, \mu^n)\) that converges to \((\sigma, \rho, \mu)\). In Appendix A.5 we show that \( \phi(\mu^*(m)) \) is a singleton for all \( m \in E \) in all but a measure zero set of evidence games.

If \( \phi(\mu^*(m)) \) is not a singleton for some \( m \in E \), there exists a continuum of the receiver’s purifiable equilibrium strategies. A given purifiable equilibrium may be the limit point of truth-leaning equilibrium outcomes only for some sequences of disturbed games, and not all sequences of disturbed games have a convergent sequence of truth-leaning equilibrium outcomes. For example, consider a slight variant of the example in Section 2 where the receiver’s payoff from approving a bad design is -1 (instead of -2). As a result, the receiver’s belief threshold is \( \frac{1}{2} \). There exists a continuum of purifiable equilibria, where \( \mu = \frac{1}{2} \). Specifically, there exists a purifiable equilibrium in which the receiver chooses Approve and Reject with equal probability after seeing no evidence (i.e., \( q = \frac{1}{2} \)). But in order to approach this equilibrium using truth-leaning equilibria of disturbed games, the sequence of disturbances \( \{\eta^n\}_{n=1}^\infty \) must be such that \( \eta^n(\{\zeta(\text{Approve}) > \zeta(\text{Reject})\}) \xrightarrow{w} \frac{1}{2} \). That is, along the sequence of disturbances, the probability that the receiver has a strict incentive to choose Approve at belief \( \frac{1}{2} \) must converge to \( \frac{1}{2} \), equating the probability that the receiver chooses Approve in the intended purifiable equilibrium.

4.3. Perturbed games and weakly truth-leaning equilibrium. Let \( \varepsilon = \{\varepsilon_e, \varepsilon_{e|e}\}_{e \in E} \) be a collection of positive real numbers. The perturbed game \( G_S(\varepsilon) \), as is defined in HKP, is an evidence game where the sender who has evidence \( e \) receives an extra payoff \( \varepsilon_e \) if she discloses truthfully, and she must disclose truthfully with at least probability \( \varepsilon_{e|e} \). That is, the sender’s payoff is \( v_S(a, e, m) = a + \varepsilon_e 1_{e=m} \), and a strategy of the sender is \( \sigma : E \rightarrow \Delta(E) \) such that \( \text{supp}(\rho(\cdot|e)) \subset \text{LC}(e) \), and \( \sigma(e|e) \geq \varepsilon_{e|e} \) for all \( e \).

A perfect Bayesian equilibrium of \( G_S(\varepsilon) \) is a collection of the sender’s strategy, the receiver’s strategy, and the receiver’s system of beliefs \((\sigma, \rho, \mu)\) such that:

(Sender optimality) Given \( \rho \),

\[
\sigma(m|e) > 0 \Rightarrow m \in \arg\max_{m \neq e} \sum_{a \in A} v_S(a, e, m) \cdot \rho(a|m)
\]

for all \( e \) and \( m \neq e \);
(Receiver optimality) and (Bayesian consistency), as are defined for \( \mathcal{G} \).

A weakly truth-leaning equilibrium of \( \mathcal{G} \) is a tuple \((\sigma, \rho, \mu)\) such that there exists a sequence of perturbations \(\{\varepsilon^n\}_{n=1}^{\infty}\) and for each \(\varepsilon^n\), a PBE \((\sigma^n, \rho^n, \mu^n)\) of \(\mathcal{G}_{S}(\varepsilon^n)\) such that \(\varepsilon^n \to 0\), and \((\sigma^n, \rho^n, \mu^n) \to (\sigma, \rho, \mu)\). A weakly truth-leaning equilibrium exists. As is shown in Section 2, different sequences of perturbations may select different weakly truth-leaning equilibria.

**Proposition 3** (Proposition 1 in HKP). A weakly truth-leaning equilibrium exists and is a perfect Bayesian equilibrium.

4.4. Relationship between truth-leaning, weakly truth-leaning, and purifiable equilibrium.

The example in Section 2 indicates that purifiable equilibrium and weakly truth-leaning equilibrium do not imply each other, and neither implies truth-leaning equilibrium.

In a slight variant of the example in Section 2, we shall see that an equilibrium that is both truth-leaning and weakly truth-leaning can fail to be purifiable. Suppose that we alter the distribution of the sender’s evidence when the design is good such that the sender has bad evidence and no evidence with equal probability. The distribution when the design is bad remains unchanged. The game has a unique truth-leaning equilibrium, where \(p = 0, q = 0, \mu = \frac{3}{5}\). That is, the sender discloses truthfully, the receiver always rejects the design, and the receiver’s belief on the good design is \(\frac{3}{5}\) after seeing no evidence. Notice that this is also the unique weakly truth-leaning equilibrium of the game. However, it is not a purifiable equilibrium. In the unique purifiable equilibrium of the game, the sender always discloses no evidence, the receiver always rejects the project, and his belief on the good design is \(\frac{1}{2}\) after seeing no evidence (i.e., \(p = 1, q = 0, \mu = \frac{1}{2}\)).

HKP shows that truth-leaning equilibrium and weakly truth-leaning equilibrium are equivalent in a setting where the receiver continuously chooses an action on the real line. This is not the case in finite evidence games. It turns out that purifiability is the missing condition. On the one hand, if a weakly truth-leaning equilibrium is also purifiable, then it is a truth-leaning equilibrium. On the other hand, for “almost all” evidence games, a truth-leaning equilibrium that is also purifiable is a weakly truth-leaning equilibrium.

**Proposition 4.** If a truth-leaning equilibrium is purifiable, then it is also a weakly truth-leaning equilibrium.

**Proposition 5.** Fix \(\pi_0, (E, \mathcal{G}), F_G, F_B, \) and \(A\). Let \(\mathcal{G}\) be the set of all evidence games

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15Although bad evidence is not fully revealing of the state, the receiver knows that the sender has bad evidence after seeing bad evidence. Therefore, the receiver’s posterior belief on the good design is \(\frac{3}{5}\), and the receiver chooses Reject after seeing bad evidence in any equilibrium. Hence, we can describe an equilibrium of the game using \(p, q, \mu\), as are defined in Section 2.
with prior \( \pi_0 \), evidence space \((E, \preceq)\), distributions of evidence \(F_G\) and \(F_B\), and receiver action space \(A\). Identify \(\mathcal{G}\) with a subset of \(\mathbb{R}^{2K}\) by the bijection

\[
\langle \pi_0, (E, \preceq), F_G, F_B, A, u_R \rangle \mapsto \{u_R(a, G), u_R(a, B)\}_{a \in A}.
\]

Let \(\mathcal{N} \subset \mathcal{G}\) be the set of evidence games that have a purifiable truth-leaning equilibrium that is not a weakly truth-leaning equilibrium. \(\mathcal{N}\) has Lebesgue measure zero.

For nongeneric games, a purifiable truth-leaning equilibrium need not be weakly truth-leaning. Consider again the example presented after Proposition 2 where the receiver’s belief threshold is \(\frac{1}{2}\). There exists a continuum of truth-leaning equilibria, where \(p = 1, q > 0\), and \(\mu = \frac{1}{2}\). As we have seen, all truth-leaning equilibria are purifiable in this game. However, there is a unique weakly truth-leaning equilibrium in which \(p = 0, q = 0\), and \(\mu = \frac{1}{2}\). All other truth-leaning equilibria are not weakly truth-leaning. This example is not generic, since the receiver is indifferent between Approve and Reject after seeing no evidence in truth-leaning equilibria.

5. Conclusion. HKP propose truth-leaning equilibrium as a solution concept in evidence games. The intuition is that the sender may find it slightly more advantageous to disclose evidence truthfully when indifferent. This paper points out two problems of applying this solution concept to finite evidence games. First, it may fail to exist. Second, it may not agree with the intuition that the sender receives an infinitesimal reward for truth-telling. That is, truth-leaning equilibrium is not equivalent to weakly truth-leaning equilibrium in finite evidence games.

We propose a simple solution to restore existence by adding a small payoff uncertainty to the receiver. In the disturbed game, the sender is as if she faces a single receiver whom she has strict incentive to persuade, and therefore, a truth-leaning equilibrium exists. A purifiable equilibrium is the limit point of a sequence of truth-leaning equilibria of disturbed games. That is, a purifiable equilibrium is a truth-leaning equilibrium in an infinitesimally disturbed game. We show that a purifiable equilibrium always exists and has a simple characterization.

Purifiability also solves the second problem. If a weakly truth-leaning equilibrium is also purifiable, then it is a truth-leaning equilibrium. Conversely, in almost all finite evidence games, a truth-leaning equilibrium that is also purifiable is a weakly truth-leaning equilibrium.

Appendix A. Proofs

A.1. Proof of Lemma [1]
Proof. Given any posterior belief $\mu \in [0, 1]$, two actions $a_i$ and $a_j$ are both optimal for type $\zeta$ receiver only if $\zeta(a_j) - \zeta(a_i) = \mu[u_R(a_i, G) - u_R(a_j, G)] + (1 - \mu)[u_R(a_i, B) - u_R(a_j, B)]$. By assumption, this is true only for an $\eta$-null set of $\zeta$. Hence, $\tau(\mu, \cdot)$ is $\eta$-a.e. a singleton set. This allows us to define

$$\varphi(\mu) = \int \sup \tau(\mu, \zeta) \eta(d\zeta) = \int \inf \tau(\mu, \zeta) \eta(d\zeta).$$

In any equilibrium $(\sigma, r, \mu)$ of the disturbed game, $\varphi(\mu(m))$ is the sender’s expected payoff if she discloses $m$.

Moreover, $\tau(\cdot, \zeta)$ is weakly increasing for all $\zeta \in \mathbb{R}^K$. Let $\mu_i < \mu_j$, $a_i \in \tau(\mu_i, \zeta)$, and $a_j \in \tau(\mu_j, \zeta)$. Then

$$\mu_i u_R(a_i, G) + (1 - \mu_i) u_R(a_i, B) + \zeta(a_i) \geq \mu_j u_R(a_j, G) + (1 - \mu_j) u_R(a_j, B) + \zeta(a_j),$$

$$\mu_j u_R(a_j, G) + (1 - \mu_j) u_R(a_j, B) + \zeta(a_j) \geq \mu_j u_R(a_i, G) + (1 - \mu_j) u_R(a_i, B) + \zeta(a_i).$$

Hence,

$$(A.1) \quad (\mu_j - \mu_i)[u_R(a_i, G) - u_R(a_j, B)] \geq (\mu_j - \mu_i)[u_R(a_i, G) - u_R(a_i, B)].$$

Since $u_R(a, G) - u_R(a, B)$ is strictly increasing in $a$, (A.1) implies that $a_j \geq a_i$.

Therefore, $\varphi : [0, 1] \to \mathbb{R}$ is strictly increasing. Suppose that, contrary to the claim, there exist $\mu_i < \mu_j$ such that $\varphi(\mu_i) = \varphi(\mu_j)$. Then, for an $\eta$-a.e. set of $\zeta$, $\tau(\mu_i, \zeta) = \tau(\mu_j, \zeta)$. This is true only if $u_R(a, G) - u_R(a, B)$ is constant across all $a \in A$, which contradicts the assumption of increasing differences.

Now consider an auxiliary evidence game $G(\varphi)$ without receiver type, where the receiver chooses an action in $\mathbb{R}$, and given any posterior belief $\mu \in [0, 1]$, he has a unique optimal action $\varphi(\mu)$. This is the standard setup in Jiang (2019). We are to establish a duality between truth-leaning equilibria of $G_R(\eta)$ and truth-leaning equilibria of $G(\varphi)$.

Let $(\hat{\sigma}, \hat{\mu}, \hat{\mu})$ be a truth-leaning equilibrium of $G(\varphi)$.[16] Let $r : E \times \mathbb{R}^K \to \Delta(A)$ be such that $\text{supp}(r(\cdot|m, \zeta)) \subset \tau(\hat{\mu}(m), \zeta)$ for all $m \in E$ and $\zeta \in \mathbb{R}^K$. We are to show that $(\hat{\sigma}, r, \hat{\mu})$ is a truth-leaning equilibrium of $G_R(\eta)$. By construction, it satisfies receiver optimality, Bayesian consistency, and the condition on off-path beliefs. We only need to verify sender optimality and truth-leaning. Since $\tau(\hat{\mu}(m), \cdot)$ is $\eta$-a.e. a singleton for all $m, r(a|m, \zeta) = 1_{a \in \tau(\hat{\mu}(m), \zeta)}$ for all $m, a$, and almost all $\zeta$. Hence, with slight abuse of notation, $\sum_{a \in A} a \cdot r(a|m, \zeta) = \tau(\hat{\mu}(m), \zeta)$ for all $m$ and almost all $\zeta$. Integrating over $\zeta$ on both sides, $\sum_{a \in A} a \cdot \rho(a|m) = \ldots$

[16]$\hat{a} : E \to \mathbb{R}$ is a pure strategy of the receiver. Since given any posterior belief $\mu$, the receiver has a unique optimal action $\varphi(\mu)$. Thus, he uses a pure strategy such that $\hat{a} = \varphi \circ \hat{\mu}$ in any equilibrium of $G(\varphi)$. 

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\(\varphi(\hat{\mu}(m)) = \hat{a}(m)\). That is, the sender’s problem given \(\hat{a}\) in \(G(\varphi)\) is the same as the sender’s problem given \(a\) in \(G_R(\eta)\). Since \((\hat{\sigma}, \hat{\rho}, \hat{\mu})\) is sender optimal and truth-leaning, \((\hat{\sigma}, a, \hat{\mu})\) is therefore also sender optimal and truth-leaning.

Conversely, let \((\hat{\sigma}, \hat{r}, \hat{\mu})\) be a truth-leaning equilibrium of \(G_R(\eta)\), and define \(a = \varphi \circ \hat{\mu}\). It is easy to see that \((\hat{\sigma}, a, \hat{\mu})\) is a truth-leaning equilibrium of \(G(\varphi)\).

Jiang (2019) characterizes the truth-leaning equilibria of the auxiliary disclosure game. Fixing a finite evidence space \((E, \mathcal{Z})\) and distributions \(F_G\) and \(F_B\), a truth-leaning equilibrium exists in \(G(\varphi)\) for all strictly increasing \(\varphi : [0, 1] \rightarrow \mathbb{R}\). Moreover, there exists a system of beliefs \(\mu^* : E \rightarrow [0, 1]\) such that for all strictly increasing \(\varphi\), \((\sigma, a, \mu)\) is a truth-leaning equilibrium of \(G(\varphi)\) if and only if \(\mu = \mu^*\), \(a = \varphi \circ \mu\), \(\sigma(e|e) = 1_{\mu(e) \leq \nu(e)}\), and

\[
\mu(m) = \min \left\{ \nu(m), \frac{\sum_{e \in E} \sigma(m|e) F_G(e) \pi_0}{\sum_{e \in E} \sigma(m|e) [F_G(e) \pi_0 + F_B(e)(1 - \pi_0)]} \right\}
\]

for all \(m \in E\), where \(\nu(\cdot) = \frac{F_G(\cdot|\pi_0)}{F_G(\cdot|\pi_0 + F_B(\cdot)(1 - \pi_0))}\). Notice that the right hand side of (A.2) is a continuous function of \(\sigma \in \Delta(E)^E\). Therefore, there exists a closed subset \(\Sigma^*\) of \(\Delta(E)^E\) such that \((\sigma, a, \mu)\) is a truth-leaning equilibrium of \(G(\varphi)\) if and only if \(\sigma \in \Sigma^*, \mu = \mu^*, \) and \(a = \varphi \circ \mu\). By the above duality, for all disturbances \(\eta\), \((\sigma, r, \mu)\) is a truth-leaning equilibrium of the disturbed game \(G_R(\eta)\) if and only if \(\sigma \in \Sigma^*, \mu = \mu^*, \) and \(\text{supp}(r(\cdot|m, \zeta)) \subset \tau(\mu(m), \zeta)\) for all \(m \in E\) and \(\zeta \in \mathbb{R}^K\).

**A.2. Proof of Proposition 2.**

**Proof.** The first statement is implied by the second statement, since \(\mu^*\) is Bayesian consistent with any sender’s strategy \(\sigma \in \Sigma^*\) by Lemma 1 and \(\phi\) is nonempty-valued.

For the “only if” part of the second statement, let \((\sigma, \rho, \mu)\) be a purifiable equilibrium. There exists a sequence of disturbances \(\eta^n \xrightarrow{w} \delta_0\) and for each \(\eta^n\), a truth-leaning equilibrium \((\sigma^n, r^n, \mu^n)\) of \(G_R(\eta^n)\) such that \((\sigma^n, r^n, \mu^n) \rightarrow (\sigma, \rho, \mu)\), where \(\rho^n = (r^n, \eta^n)\). By Lemma 1, \(\sigma^n \in \Sigma^*\) for all \(n\), and \(\Sigma^*\) is closed. Therefore, \(\sigma \in \Sigma^*\). Additionally, \(\mu^n = \mu^*\) for all \(n\), so \(\mu = \mu^*\). Fix any \(m \in E\) and \(a \in A\) such that \(a \notin \phi(\mu(m)) = \tau(\mu(m), 0)\). Since \(\tau\) is upper hemicontinuous in \(\zeta\), there exists a neighborhood \(U\) of \(0\) in \(\mathbb{R}^K\) such that \(a \notin \tau(\mu(m), \zeta)\) for all \(\zeta \in U\). By receiver optimality, \(r^n(a|m, \zeta) = 0\) for all \(n\) and \(\zeta \in U\). Hence, as \(\eta^n \xrightarrow{w} \delta_0\), \(\rho^n(a|m) = \int r^n(a|m, \zeta) \eta^n(d\zeta) \rightarrow 0\). That is, \(a \notin \text{supp}(\rho(\cdot|m))\). Lastly, let \(m, m' \in E\) be such that \(\mu(m) = \mu(m')\). Since \(\tau(\mu(m), \zeta) = \tau(\mu(m'), \zeta)\) for all \(\zeta\), \(r^n(a|m, \zeta) = r^n(a|m', \zeta)\) for all \(n, a, \) and almost all \(\zeta\). Therefore, \(\rho^n(\cdot|m) = \rho^n(\cdot|m')\) for all \(n\), so their limits also coincide, i.e., \(\rho(\cdot|m) = \rho(\cdot|m')\).

For the “if” part of the second statement, let \((\sigma, \rho, \mu)\) be such that \(\sigma \in \Sigma^*, \mu = \mu^*, \) \(\text{supp}(\rho(\cdot|m)) \subset \phi(\mu(m))\) for all \(m \in E\), and \(\mu(m) = \mu(m') \Rightarrow \rho(\cdot|m) = \rho(\cdot|m')\). We are
to show that it is a purifiable equilibrium. Let $\mu_1 < \mu_2 < \cdots < \mu_N$ be elements of $\mu(E)$, i.e., all possible posterior beliefs of the receiver. Since $\tau$ is upper hemicontinuous in $\zeta$, there exists $r > 0$ such that $\tau(\mu_i, \zeta) \subset \phi(\mu_i)$ for all $i$ and all $\zeta \in B_r(0)$, where $B_r(0)$ denotes the open ball of radius $r$ around $0$ in $\mathbb{R}^K$. For each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \times_{i=1}^N \phi(\mu_i)$, let $V_\alpha$ be the set of $\zeta \in \mathbb{R}^K$ such that $\tau(\mu_i, \zeta) = \{\alpha_i\}$. Notice that $V_\alpha$ are pairwise disjoint, $\bigcup_\alpha V_\alpha = B_r(0)$, and $\lambda \zeta \in V_\alpha$ for all $\zeta \in V_\alpha$ and $\lambda \in (0, 1)$. Moreover, each $V_\alpha$ has positive Lebesgue measure$^{17}$ Let $q^n \to \rho$ be a sequence such that $\text{supp}(q^n(\cdot|m)) = \phi(\mu(m))$ for all $m \in E$, and $\mu(m) = \mu(m') \Rightarrow q^n(\cdot|m) = q^n(\cdot|m')$. By abuse of notation, we write $q^n(a|m)$ as $q^n(a, \mu(m))$, and let $x^n_\alpha = \prod_{i=1}^N q^n(\alpha_i, \mu_i)$. $x^n_\alpha > 0$ for all $\alpha$ and all $n$. Therefore, for each $n$, we can define a distribution $\eta^n$ over $\mathbb{R}^K$ with full support and absolutely continuous with respect to the Lebesgue measure such that $\eta^n \left( \frac{1}{n}V_\alpha \right) = \frac{n-1}{n}x^n_\alpha$ for all $\alpha$, where $\frac{1}{n}V_\alpha = \{\zeta : n\zeta \in V_\alpha\}$ is a subset of $V_\alpha$. By construction $\eta^n \xrightarrow{w} \delta_0$. Let $r$ be any receiver strategy in the disturbed games such that $\text{supp}(r(\cdot|m, \zeta)) \subset \tau(\mu(m), \zeta)$ for all $m \in E$ and $\zeta \in \mathbb{R}^K$. By Lemma $\Pi$ $(\sigma, r, \mu)$ is a truth-leaning equilibrium of $G_R(\eta^n)$. Let $(\sigma, \rho^n, \mu)$ be the associated equilibrium outcome. Notice that $\rho^n(a|m) = \int r(a|m, \zeta)\eta^n(d\zeta)$ is bounded from below by $\frac{n-1}{n}q^n(a|m)$ and from above by $\frac{n-1}{n}q^n(a|m) + \frac{1}{n}$, and recall that $q^n \to \rho$. Hence, $\rho^n \to \rho$, and $(\sigma, \rho, \mu)$ is a purifiable equilibrium. \hfill $\square$

**Remarks.** The above proof implies that, if $\phi(\mu^*(m))$ is a singleton for all $m \in E$, there exists a sequence of truth-leaning equilibrium outcomes of the disturbed games $(\sigma^n, \rho^n, \mu^n)$ that converges to $(\sigma, \rho, \mu)$ for all purifiable equilibria $(\sigma, \rho, \mu)$ and all disturbances $\eta^n \xrightarrow{w} \delta_0$. Let $r$ be any receiver strategy in the disturbed games such that $\text{supp}(r(\cdot|m, \zeta)) \subset \tau(\mu^*(m), \zeta)$ for all $m \in E$ and $\zeta \in \mathbb{R}^K$. By Lemma $\Pi$ $(\sigma, r, \mu)$ is a truth-leaning equilibrium of all disturbed games $G_R(\eta^n)$. Since $\tau$ is upper hemicontinuous in $\zeta$, and $\phi(\mu^*(m))$ is a singleton for all $m \in E$, $r(a|m, \cdot)$ is constant on a small neighborhood of $0$ in $\mathbb{R}^K$ for all $a \in A$ and $m \in E$. Hence, $\rho^n(a|m) \to r(a|m, 0) = \rho(a|m)$ for all $a \in A$ and $m \in E$.

**A.3. Proof of Proposition $3$.**

**Proof.** The proof works similarly as in HKP despite different settings. First, observe that a perfect Bayesian equilibrium exists in every perturbed game $G_S(\varepsilon)$. The set of sender strategies in the perturbed game $\Sigma \subset \Delta(E)^E$ and the set of receiver strategies $\Delta(A)^E$ are convex and compact. Given a strategy of the receiver, the set of sender strategies that are sender optimal is closed and nonempty. This yields an upper hemicontinuous best response correspondence of the sender $\Gamma_S : \Delta(A)^E \rightrightarrows \Sigma$. Given a sender strategy $\sigma$, since all evidence

$^{17}$Since $\tau$ is upper hemicontinuous in $\zeta$, we only need to show that $V_\alpha$ is nonempty for all $\alpha$. Notice that $\tau(\mu_i, \zeta) = \{\alpha_i\}$ if and only if $\zeta(\alpha_i) > \zeta(a')$ for all $a' \in \phi(\mu_i)$, $a' \neq \alpha_i$. The assumption of increasing differences guarantees that all inequalities can be simultaneously satisfied for all $i$. Hence, $V_\alpha$ is nonempty.
is disclosed with positive probability, there is a unique Bayesian consistent system of beliefs \( \mu^* \), and the mapping \( \sigma \mapsto \mu^\sigma \) is continuous. Since the solution to the receiver’s optimality problem \( \phi \) is upper hemicontinuous, we have an upper hemicontinuous best response correspondence of the receiver \( \Gamma_R : \Sigma \rightrightarrows \Delta(A)^E \) such that \( \Gamma_R(\sigma) = \times_{m \in E} \Delta(\phi(\rho^\sigma(m))) \). Then by the Kakutani fixed point-theorem, there exists \( \sigma, \rho \) such that \( \sigma \in \Gamma_S(\rho) \) and \( \rho \in \Gamma_R(\sigma) \). That is, the perturbed game has a Nash equilibrium. The Nash equilibrium paired with the system of beliefs \( \mu^\sigma \) consists of a perfect Bayesian equilibrium of the perturbed game.

Since the set of sender’s strategies \( \{ \sigma : \text{supp}(\sigma(\cdot|e)) \subseteq LC(e) \} \subset \Delta(E)^E, \Delta(A)^E, \) and \([0,1]^E\) are compact, any sequence of perfect Bayesian equilibria of perturbed games \( \{ \langle \sigma^n, \rho^n, \mu^n \rangle \}_n \) has a convergent subsequence. Hence, a weakly truth-leaning equilibrium exists. It is easy to verify that any weakly truth-leaning equilibrium is a perfect Bayesian equilibrium. \( \square \)

### A.4. Proof of Proposition 4

**Proof.** Let \( \langle \sigma, \rho, \mu \rangle \) be a weakly truth-leaning equilibrium that is also purifiable. We show that (1) if \( \sigma(e|e) > 0 \), then \( \sigma(e|e) = 1 \), and (2) if \( \sigma(e|e) = 0 \), then \( e \notin \arg\max_{a \in A} \sum_{a \in A} a \cdot \rho(a|m) \), and \( \mu(e) = \nu(e) \).

The first claim is due to purifiability. Let \( \eta^n \rightharpoonup \delta_0 \), and \( \langle \sigma^n, \rho^n, \mu^n \rangle \to \langle \sigma, \rho, \mu \rangle \) be such that \( \langle \sigma^n, \rho^n, \mu^n \rangle \) is a truth-leaning outcome of \( G_R(\eta^n) \) for all \( n \). If \( \sigma(e|e) > 0 \), then there exists \( N \) such that \( \sigma^n(e|e) > 0 \) for all \( n \geq N \). However, \( \langle \sigma^n, \rho^n, \mu^n \rangle \) is truth-leaning, so \( \sigma^n(e|e) = 1 \) for all \( n \geq N \). Therefore, \( \sigma(e|e) = 1 \).

The second claim is due to weakly truth-leaning. Let \( \varepsilon^n \to 0 \), and \( \langle \sigma^n, \rho^n, \mu^n \rangle \to \langle \sigma, \rho, \mu \rangle \) be such that \( \langle \sigma^n, \rho^n, \mu^n \rangle \) is a perfect Bayesian equilibrium of \( G_S(\varepsilon^n) \) for all \( n \). If \( \sigma(e|e) = 0 \), \( e \notin \arg\max_{m \geq e} \sum_{a \in A} a \cdot \rho(a|m) \). Otherwise, for all \( n, e \) is the unique maximizer to the sender’s problem in \( G_S(\varepsilon^n) \), so \( \sigma^n(e|e) = 1 \), and \( \sigma^n \not\sim \sigma \). Hence, for all \( e' > e \) and all \( n \), \( \sigma^n(e'|e') = 0 \). By Bayes’ rule, \( \mu^n(e) = \nu(e) \) for all \( n \). Therefore, \( \mu(e) = \nu(e) \). \( \square \)

### A.5. Proof of Proposition 5

**Proof.** Notice that the receiver’s system of beliefs \( \mu^* \) is the same across all purifiable equilibria of all games in \( \mathcal{G} \). Moreover, given any two actions \( a_i, a_j \) and a belief \( \mu \), the receiver is indifferent between actions \( a_i \) and \( a_j \) at \( \mu \in [0,1] \) if and only if \( u(a_i, G), u(a_i, B), u(a_j, G), u(a_j, B) \) are on a hyperplane in \( \mathbb{R}^4 \). Therefore, the receiver is indifferent at some belief \( \mu^*(m) \) only on a Lebesgue null set of \( \mathcal{G} \). We are to show that, if \( \phi(\mu^*(m)) \) is a singleton for all \( m \in E \), then a truth-leaning equilibria that is also purifiable is weakly truth-leaning. This concludes that \( \mathcal{N} \) has Lebesgue measure zero.

Let \( \mathcal{G} \in \mathcal{G} \) be such that \( \phi(\mu^*(m)) \) is a singleton for all \( m \), and \( \langle \sigma, \rho, \mu^* \rangle \) a truth-leaning equilibrium of \( \mathcal{G} \) that is also purifiable. Given any perturbation \( \varepsilon = \{ \varepsilon_e, \varepsilon_{e|e} \}_{e \in E} \), we define
$(\sigma_\varepsilon, \rho_\varepsilon, \mu_\varepsilon)$ as follows:

1. $\sigma_\varepsilon(e|e) = 1$ if $\sigma(e|e) = 1$;
2. $\sigma_\varepsilon(e|e) = \varepsilon_{e|e}$, and $\sigma_\varepsilon(m|e) = (1 - \varepsilon_{e|e})\sigma(m|e)$ for all $m \neq e$ if $\sigma(e|e) = 0$;
3. $\rho_\varepsilon = \rho$;
4. $\mu_\varepsilon$ is by Bayes’ rule, i.e.,

$$\mu(m) = \frac{\sum_{e \in UC(m)} \sigma_\varepsilon(m|e) F_G(e) \pi_0}{\sum_{e \in UC(m)} \sigma_\varepsilon(m|e) [F_G(e) \pi_0 + F_B(e)(1 - \pi_0)]}.$$

For sufficiently small $\varepsilon$, $(\sigma_\varepsilon, \rho_\varepsilon, \mu_\varepsilon)$ is a perfect Bayesian equilibrium of $G(\varepsilon)$. Sender optimality is satisfied if

$$\varepsilon_e < \max_m \sum_{a \in A} a[\rho(a|m) - \rho(a|e)]$$

for all $e \in E$ such that $\sigma(e|e) = 0$. For all $m \in E$, since $\phi$ is upper hemi-continuous and $\mu^*(m)$ is a singleton for all $m$, there exists $\delta > 0$ such that $\phi(\mu) = \phi(\mu^*(m))$ for all $m$ and all $\mu \in [0,1]$ such that $|\mu - \mu^*(m)| < \delta$. Since $\mu_\varepsilon \rightarrow \mu^*$, when $\varepsilon$ is sufficiently small, $\rho_\varepsilon(a|m) = \rho(a|m) = 1_{a=\phi(\mu_\varepsilon(m))} = 1_{a=\phi(\mu^*(m))}$ for all $m \in E$. That is, receiver optimality is satisfied. By construction, it is also Bayesian consistent, and $(\sigma_\varepsilon, \rho_\varepsilon, \mu_\varepsilon) \rightarrow (\sigma, \rho, \mu)$ for any sequence $\varepsilon \rightarrow 0$. Therefore, $(\sigma, \rho, \mu)$ is a weakly truth-leaning equilibrium.

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