A SIMPLE PROOF OF REGULARITY FOR $C^{1,\alpha}$ INTERFACE TRANSMISSION PROBLEMS

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ABSTRACT. We give a simple proof of a recent result in [1] by Caffarelli, Soria-Carro, and Stinga about the $C^{1,\alpha}$ regularity of weak solutions to transmission problems with $C^{1,\alpha}$ interfaces. Our proof does not use the mean value property or the maximum principle, and also works for more general elliptic systems. Some extensions to $C^{1,\text{Dini}}$ interfaces and to domains with multiple sub-domains are also discussed.

1. INTRODUCTION AND MAIN RESULTS

In a recent paper [1], Caffarelli, Soria-Carro, and Stinga studied the following transmission problem. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain with $d \geq 2$, and $\Omega_1$ be a sub-domain of $\Omega$ such that $\Omega_1 \subset \subset \Omega$ and $\Omega_2 = \Omega \setminus \overline{\Omega_1}$. Assume that the interfacial boundary $\Gamma (= \partial \Omega_1)$ between $\Omega_1$ and $\Omega_2$ is $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Consider the elliptic problem with the transmission conditions

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega_1 \cup \Omega_2, \\
u = 0 & \text{on } \partial \Omega, \\
|u|^+_{\Gamma} - |\nu u|^+ - |\nu u|^- = g,
\end{cases}
\]

where $g$ is a given function on $\Gamma$, $\nu$ is the unit normal vector on $\Gamma$ which is pointing inside $\Omega_1$, and $|u|^+$ and $|\nu u|^+$ (and $\nu u|_{\Gamma}^+$) are the left and right limit of $u$ (and its normal derivative, respectively) on $\Gamma$ in $\Omega_1$ and $\Omega_2$. The main result of [1] can be formulated as the following theorem.

Theorem 1.1 (Caffarelli, Soria-Carro, and Stinga [1]). Under the assumptions above, for any $g \in C^{\alpha}(\Gamma)$, there is a unique weak solution $u \in H^1(\Omega)$ to (1.1), which is piecewise $C^{1,\alpha}$ up to the boundary in $\Omega_1$ and $\Omega_2$ and satisfies

\[\|u\|_{C^{1,\alpha}(\Omega_1)} + \|u\|_{C^{1,\alpha}(\Omega_2)} \leq N\|g\|_{C^{\alpha}(\Gamma)},\]

where $N = N(d, \alpha, \Omega, \Gamma) > 0$ is a constant.

The proof in [1] uses the mean value property for harmonic functions and the maximum principle together with an approximation argument. We refer the reader to [1] for earlier results about the transmission problem with smooth interfacial boundaries. The main feature of Theorem 1.1 is that $\Gamma$ is only assumed to be in $C^{1,\alpha}$, which is weaker than those in the literature.

In this short paper, we give a simple proof of Theorem 1.1 which does not invoke the mean value property or the maximum principle, and also works for more general
elliptic systems in the form

\[ \begin{cases} \mathcal{L} u := D_k(A^{kl} D_l u) = \text{div } F + f \quad \text{in } \Omega_1 \cup \Omega_2, \\ u = 0 \quad \text{on } \partial \Omega, \\ u|_i^+ = u|_i^-, \quad A^{kl} D_l w_{k}|_i^+ - A^{kl} D_l w_{k}|_i^- = g, \end{cases} \tag{1.2} \]

where the Einstein summation convention in repeated indices is used,

\[ u = (u^1, \ldots, u^n)^T, \quad F_k = (F^1_k, \ldots, F^n_k)^T, \quad f = (f^1, \ldots, f^n)^T, \quad g = (g^1, \ldots, g^n)^T \]

are (column) vector-valued functions, for \( k, l = 1, \ldots, d \), \( A^{kl} = A^{kl}(x) \) are \( n \times n \) matrices, which are bounded and satisfy the strong ellipticity with ellipticity constant \( \kappa > 0 \):

\[ \kappa|\xi|^2 \leq A^{kl}_i \xi_i \xi_l, \quad |A^{kl}| \leq \kappa^{-1} \]

for any \( \xi = (\xi_i^j) \in \mathbb{R}^{n \times d} \).

Theorem 1.2. Assume that \( \Omega_1, \Omega_2, \text{ and } \Gamma \) satisfy the conditions in Theorem 1.1. \( A^{kl} \) and \( F \) are piecewise \( C^\alpha \) in \( \Omega_1 \) and \( \Omega_2 \), \( g \in C^\alpha(\Gamma) \), and \( f \in L_\infty(\Omega) \). Then there is a unique weak solution \( u \in H^1(\Omega) \) to (1.2), which is piecewise \( C^{1,\alpha} \) up to the boundary in \( \Omega_1 \) and \( \Omega_2 \) and satisfies

\[ \sum_{j=1}^{2} \|u\|_{C^{1,\alpha}(\partial \Omega_j)} \leq N \|g\|_{C^\alpha(\Gamma)} + N \sum_{j=1}^{2} \|F\|_{C^\alpha(\Omega_j)} + N \|f\|_{L_\infty(\Omega)}, \]

where \( N = N(d, n, \kappa, \alpha, \Omega, \Gamma, [A]_{C^\alpha(\Omega_j)}) > 0 \) is a constant.

We also consider the transmission problem with multiple disjoint sub-domains \( \Omega_1, \ldots, \Omega_M \) with \( C^{1,\alpha} \) interfacial boundaries in the setting of [6, 5]. As in these papers, we assume that any point \( x \in \Omega \) belongs to the boundaries of at most two of the \( \Omega_j \)’s, so that if the boundaries of two \( \Omega_j \) touch, then they touch on a whole component of such a boundary. Without loss of generality assume that \( \Omega_j \subset \subset \Omega, j = 1, \ldots, M - 1 \) and \( \partial \Omega \subset \partial \Omega_M \). The transmission problem in this case is then given by

\[ \begin{cases} \mathcal{L} u = \text{div } F + f \quad \text{in } \bigcup_{j=1}^{M} \Omega_j, \\ u = 0 \quad \text{on } \partial \Omega, \\ u|_{\partial \Omega_j}^+ = u|_{\partial \Omega_j}^-, \quad A^{kl} D_l w_{k}|_{\partial \Omega_j}^+ - A^{kl} D_l w_{k}|_{\partial \Omega_j}^- = g_j, \quad j = 1, \ldots, M - 1. \end{cases} \tag{1.3} \]

In the following theorem, we obtain an estimate which is independent of the distance of interfacial boundaries, but may depend on the number of sub-domains \( M \).

Theorem 1.3. Assume that \( \Omega_j \) satisfy the conditions above, \( A^{kl} \) and \( F \) are piecewise \( C^{\alpha'} \) for some \( \alpha' \in (0, \alpha/(1 + \alpha)] \), \( g_j \in C^{\alpha'}(\partial \Omega_j), j = 1, \ldots, M - 1 \), and \( f \in L_\infty(\Omega) \). Then there is a unique weak solution \( u \in H^1(\Omega) \) to (1.3), which is piecewise \( C^{1,\alpha'} \) up to the boundary in \( \Omega_j, j = 1, \ldots, M \), and satisfies

\[ \sum_{j=1}^{j} \|u\|_{C^{1,\alpha'}(\Omega_j)} \leq N \sum_{j=1}^{M-1} \|g\|_{C^{\alpha'}(\partial \Omega_j)} + N \sum_{j=1}^{M} \|F\|_{C^{\alpha'}(\Omega_j)} + N \|f\|_{L_\infty(\Omega)}, \]

where \( N = N(d, n, M, \kappa, \alpha, \Omega_j, [A]_{C^{\alpha'}(\Omega_j)}) > 0 \) is a constant.
It is worth noting that in the special case when $A^{\alpha \beta}$ and $F$ are Hölder continuous in the whole domain, by the linearity the result of Theorem 1.3 still holds with $\alpha' = \alpha$.

Our last result concerns the case when the interfaces are $C^1$ and $Dini$, and $A^{kl}$ satisfy the piecewise $L_2$-Dini mean oscillation in $\Omega$, i.e.,

$$\omega_A(r) := \sup_{x_0 \in \Omega} \inf_{\bar{A} \in \mathcal{A}} \left( \int_{\Omega_r(x_0)} |A(x) - \bar{A}|^2 \, dx \right)^{1/2}$$

satisfies the Dini condition, where $\Omega_r(x_0) = B_r(x_0) \cap \Omega$ and $\mathcal{A}$ is the set of piecewise constant functions in $\Omega_j, j = 1, \ldots, M$.

**Theorem 1.4.** Assume that $\Omega_j$ satisfy the $C^1, Dini$ condition, $A^{kl}$ and $F$ are of piecewise $L_2$-Dini mean oscillation in $\Omega$, $g_j$ is Dini continuous on $\partial \Omega_j, j = 1, \ldots, M-1$, and $f \in L_\infty(\Omega)$. Then there is a unique weak solution $u \in H^1(\Omega)$ to (1.3), which is piecewise $C^1$ up to the boundary in $\Omega_j, j = 1, \ldots, M$.

We note that the piecewise $L_2$-Dini mean oscillation condition is weaker than the usual piecewise Dini continuity condition in the $L_\infty$ sense.

## 2. Proofs

The idea of the proof is to reduce the transmission problem to an elliptic equation (system) with piecewise Hölder (or Dini) nonhomogeneous terms, by solving a conormal boundary value problem. These equations arose from composite material and have been extensively studied in the literature. See, for instance, [6, 5], and also recent papers [2, 4]. We will apply the results in the latter two papers, the proofs of which in turn are based on Campanato’s approach.

**Proof of Theorem 1.2.** Let $w \in H^1(\Omega)$ be the weak solution to the conormal boundary value problem

$$\begin{cases}
\Delta w = c & \text{in } \Omega_1, \\
w_{\nu} = g & \text{on } \partial \Omega_1, \\
\int_{\Omega_1} w \, dx = 0,
\end{cases}$$

where $c = -|\Gamma|^{-1} \int_{\Gamma} g$ is a constant. The existence and uniqueness of such solution $w$ follows from the trace theorem and the Lax–Milgram theorem, and

$$\|w\|_{H^1(\Omega_1)} \leq N \|g\|_{L_2(\Gamma)},$$

where $N = N(d, \Omega_1)$. Since $g \in C^\alpha(\Omega_1)$, by the classical elliptic theory (see, for instance, [7] Theorem 5.1), we have

$$\|w\|_{C^{1, \alpha}(\Omega_1)} \leq N \|g\|_{C^\alpha(\Gamma)},$$

where $N = N(d, \alpha, \Omega_1)$. By using the weak formulation of solutions, from (2.1) it is easily seen that (2.2) is equivalent to

$$\begin{cases}
Lu = \text{div} \tilde{F} + \tilde{f} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where

$$\tilde{F} = 1_{\Omega_1 \cup \Omega_2} F - 1_{\Omega_1} \nabla w, \quad \tilde{f} = f + 1_{\Omega_1} c.$$
By the Lax–Milgram theorem, there is a unique solution $u \in H^1(\Omega)$ to (2.4) and
\[ \|u\|_{H^1(\Omega)} \leq N\|F\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_1)}, \] where we used (2.2) in the second inequality. Since $\tilde{F}$ and $A^{\alpha\beta}$ are piecewise $C^\alpha$, it follows from [2, Corollary 2 and Remark 3 (ii)], (2.3), and (2.5) that
\[ \sum_{j=1}^2 \|u\|_{C^{1,\alpha}(\Omega_j)} \leq N\|g\|_{C^{\alpha}(\partial\Omega_j)} + N\sum_{j=1}^2 \|F\|_{C^{\alpha}(\Omega_j)} + N\|f\|_{L^\infty(\Omega_1)} \] The theorem is proved. □

Proof of Theorem 1.3 The proof is similar to that of Theorem 1.2. In each $\Omega_j, j = 1, \ldots, M-1$, we find a weak solution to
\[ \begin{cases}
\Delta w_j = c_j & \text{in } \Omega_j, \\
\partial_{\nu} w_j |_{\partial\Omega_j} = g_j & \text{on } \partial\Omega_j, \\
\int_{\Omega_j} w_j \, dx = 0,
\end{cases} \] (2.6) where $c_j = -|\partial\Omega_j|^{-1} \int_{\partial\Omega_j} g_j$, and $w_j$ satisfies
\[ \|w_j\|_{C^{1,\alpha}(\Omega_j)} \leq N\|g_j\|_{C^{\alpha}(\partial\Omega_j)}. \] (2.7) By using the weak formulation of solutions, it is easily seen that (1.3) is equivalent to
\[ \begin{cases}
Lu = \text{div} \tilde{F} + \tilde{f} & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases} \] (2.8) where
\[ \tilde{F} = 1_{j=1}^{M-1} \Omega_j F - \sum_{j=1}^{M-1} 1_{\Omega_j} \nabla w_j, \quad \tilde{f} = f + \sum_{j=1}^{M-1} 1_{\Omega_j} c_j. \] As before, by the Lax–Milgram theorem, there is a unique solution $u \in H^1(\Omega)$ to (2.8). Since $\tilde{F}$ and $A^{\alpha\beta}$ are piecewise $C^\alpha$, and $\partial\Omega_j$ is piecewise $C^{1,\alpha}$, by using (2.7) and appealing to [4, Corollary 1.2 and Remark 1.4], we conclude the proof of the theorem. □

Finally, we give

Proof of Theorem 1.4. We claim that under the conditions of the theorem, if $w_j$ is the solution to (2.6), then $ Dw_j$ satisfies the $L_2$–Dini mean oscillation condition in $\Omega_j$. Assuming this is true, then the conclusion of the theorem follows from the proof of Theorem 1.3 and [4, Theorem 1.1]. We remark that the $C^1$ continuity of $ Dw_j$ was proved in [7, Theorem 5.1] for more general quasilinear equations, but in general $ Dw_j$ may not be Dini continuous in the $L_\infty$ sense.

To prove the claim, we follow the argument in the proof of Theorem 1.7 of [3]. We only give the boundary estimate since the corresponding interior estimate is
simpler. By using the $C^{1, \text{Dini}}$ regularity of $\Omega_j$ and locally flattening the boundary, it then suffices to verify Lemma 2.1 below. □

In the sequel, we denote $x = (x^1, x^2, \ldots, x^d)$, where $x^i = (x_1, x_2, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$, and $\Gamma_r(x) := B_r(x) \cap \{x_d = 0\}$ for $x \in \mathbb{R}^d$ and $r > 0$.

**Lemma 2.1.** Let $u \in H^1(B_1^+) \cap L_2^2$ be a weak solution to

$$D_k(a^{kl}D_{kl}u) = 0 \quad \text{in} \quad B_1^+,$$

with the conormal boundary condition $a^{dl}D_{dl}u = g(x)$ on $\Gamma_1 = B_1 \cap \{x_d = 0\}$, where $a^{kl} = a^{kl}(x)$ satisfy the uniform ellipticity condition and are of $L_2$-Dini mean oscillation, and $g$ is a Dini continuous function on $\Gamma_1$. Then $Du$ is of $L_2$-Dini mean oscillation in $B_1^+$.

**Proof.** We set

$$g^d(x) = g^d(x^1, x^2, \ldots, x^d) := g(x^1),$$

which satisfies $D_d g^d = 0$. Therefore, the above problem is reduced to the standard conormal boundary problem

$$\begin{align*}
D_k(a^{kl}D_{kl}w) &= D_d g^d \quad \text{in} \quad B_1^+ \\
-a^{dl}D_{dl}w &= g^d \quad \text{on} \quad \Gamma_1.
\end{align*}$$

Similar to [3] Section 3, for $x \in \overline{B_1^+}$ and $r \in (0, 1)$, we define

$$\phi(x, r) := \left( \int_{B_r(x) \cap B_1^+} |Du - (Du)_{B_r(x) \cap B_1^+}|^2 \right)^{\frac{1}{2}},$$

where

$$(Du)_{B_r(x) \cap B_1^+} = \int_{B_r(x) \cap B_1^+} Du.$$

Fix a smooth domain $\mathcal{D}$ satisfying

$$B_{1/2}^+ \subset \mathcal{D} \subset B_1^+$$

and for $\bar{x} \in \partial B_1^+$, we set $\mathcal{D}_r(\bar{x}) = r \mathcal{D} + \bar{x}$. We decompose $u = w + v$, where $w \in H^1(\mathcal{D}_r(\bar{x}))$ is a weak solution of the problem

$$\begin{align*}
D_k(\bar{a}^{kl}D_{kl}w) &= -D_k((a^{kl} - \bar{a}^{kl})D_{kl}u) + D_d(g^d - \bar{g}^d) \quad \text{in} \quad \mathcal{D}_r(\bar{x}), \\
\bar{a}^{kl}D_{kl}w &= -(a^{kl} - \bar{a}^{kl})D_{kl}u + (g^d - \bar{g}^d)v_d \quad \text{on} \quad \partial \mathcal{D}_r(\bar{x}),
\end{align*}$$

where $\bar{a}^{kl}$ and $\bar{g}^d$ are the average of $a^{kl}$ and $g^d$ in $\mathcal{D}_r(\bar{x})$, respectively. By the $H^1$-estimate, we have

$$\left( \int_{B_1^+(\bar{x})} |Dw|^2 \right)^{1/2} \leq N\omega_A(2r) \|Du\|_{L^2(\mathcal{D}_r(\bar{x}))} + N\omega_g(2r). \quad (2.9)$$

Note that $v := u - w$ satisfies

$$D_k(\bar{a}^{kl}D_{kl}v) = D_d(\bar{g}^d) \quad \text{in} \quad B_1^+(\bar{x}), \quad \bar{a}^{dl}D_{dl}v = \bar{g}^d \quad \text{on} \quad \Gamma_r(\bar{x}).$$

Then for any $c \in \mathbb{R}$ and $k = 1, 2, \ldots, d - 1$, $\tilde{v} := D_k v - c$ satisfies

$$D_k(\tilde{a}^{kl}D_{kl}\tilde{v}) = 0 \quad \text{in} \quad B_1^+(\bar{x}), \quad \tilde{a}^{dl}D_{dl}\tilde{v} = 0 \quad \text{on} \quad \Gamma_r(\bar{x}).$$
By the standard elliptic estimates for equations with constant coefficients and zero conormal boundary data, we have for any \( c \in \mathbb{R} \),

\[
\|DD_k v\|_{L^\infty(B_r^+(\bar{x}))} \leq N r^{-1} \left( \int_{B_r^+(\bar{x})} |D_k v - c|^2 \right)^{1/2}, \quad k = 1, \ldots, d - 1.
\]

Then by using \( D_{dd} v = -\frac{1}{\delta_{dd}} \sum_{(i,j) \neq (d,d)} \tilde{a}^{ij} D_{ij} v \), we obtain

\[
\|D^2 v\|_{L^\infty(B_r^+(\bar{x}))} \leq N \|D D_{x} v\|_{L^\infty(B_r^+(\bar{x}))} \leq N r^{-1} \left( \int_{B_r^+(\bar{x})} |D_{x} v - c|^2 \right)^{1/2},
\]

where we used the notation \( D_{x} v = (D_1 v, \ldots, D_{d-1} v) \). Therefore, we have

\[
\|D^2 v\|_{L^\infty(B_r^+(\bar{x}))} \leq N r^{-1} \left( \int_{B_r^+(\bar{x})} |D v - q|^2 \right)^{1/2}, \quad \forall q \in \mathbb{R}^d.
\]

Let \( \mu \in (0, 1/2) \) be a small number. Since

\[
\left( \int_{B_r^+(\bar{x})} |D v - (D v)_{B_r^+(\bar{x})}|^2 \right)^{1/2} \leq 2 \mu r \|D^2 v\|_{L^\infty(B_r^+(\bar{x}))},
\]

we see that there is a constant \( N_0 = N_0(d, \kappa) > 0 \) such that

\[
\left( \int_{B_r^+(\bar{x})} |D v - (D v)_{B_r^+(\bar{x})}|^2 \right)^{1/2} \leq N_0 r \left( \int_{B_r^+(\bar{x})} |D v - q|^2 \right)^{1/2}, \quad \forall q \in \mathbb{R}^d.
\]

By using the decomposition \( u = v + w \), we obtain from the above and the triangle inequality that

\[
\left( \int_{B_r^+(\bar{x})} |D u - (D v)_{B_r^+(\bar{x})}|^2 \right)^{1/2} \leq \left( \int_{B_r^+(\bar{x})} |D u - D v|^2 \right)^{1/2} + \left( \int_{B_r^+(\bar{x})} |D w|^2 \right)^{1/2} \leq N_0 r \left( \int_{B_r^+(\bar{x})} |D u - q|^2 \right)^{1/2} + N \mu^{-d/2} \left( \int_{B_r^+(\bar{x})} |D w|^2 \right)^{1/2}.
\]

By setting \( q = (D u)_{B_r^+(\bar{x})} \) and using (2.9), we obtain

\[
\phi(\bar{x}, \mu r) \leq N_0 \mu \phi(\bar{x}, r) + N \mu^{-d/2} \left( \omega_A(2r) \|D u\|_{L^\infty(B_r^+(\bar{x}))} + \omega_g(2r) \right). \tag{2.10}
\]

By using an iteration argument as in the proof of [3 Theorem 1.7], from (2.10) and the corresponding interior estimate, it is easily seen that \( D u \) is of \( L^2_{Dini} \) mean oscillation in \( B_r^+ \) with a modulus of continuity depending on \( d, \kappa, \|D u\|_{L^2(B_r^+)} \), \( \omega_g \), and \( \omega_A \). The lemma is proved. \( \square \)
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