Strong Solutions to Non–Stationary Channel Flows of Heat–Conducting Viscous Incompressible Fluids with Dissipative Heating

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Abstract

We study an initial-boundary-value problem for time-dependent flows of heat-conducting viscous incompressible fluids in channel-like domains on a time interval $(0, T)$. For the parabolic system with strong nonlinearities and including the artificial (the so called “do nothing”) boundary conditions, we prove the local in time existence, global uniqueness and smoothness of the solution on a time interval $(0, T^*)$, where $0 < T^* \leq T$.

1 Introduction

1.1 Preliminaries

Let $\Omega \in \mathcal{C}^{0,1}$ be a two-dimensional bounded domain with the boundary $\partial \Omega$. Let $\partial \Omega = \Gamma_D \cup \Gamma_N$ be such that $\Gamma_D$ and $\Gamma_N$ are open, not necessarily connected, the one-dimensional measure of $\Gamma_D \cap \Gamma_N$ is zero and $\Gamma_D \neq \emptyset$.

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\( \Gamma_N = \bigcup_i \Gamma_N^{(i)}, \bar{\Gamma}_N^{(i)} \cap \bar{\Gamma}_N^{(j)} = \emptyset \) for \( i \neq j \). In a physical sense, \( \Omega \) represents a “truncated” region of an unbounded channel system occupied by a moving heat–conducting viscous incompressible fluid. \( \Gamma_D \) will denote the “lateral” surface and \( \Gamma_N \) represents the open parts of the region \( \Omega \). We assume that in/outflow channel segments extend as straight pipes. All portions of \( \Gamma_N \) are taken to be flat and the boundary \( \Gamma_N \) and rigid boundary \( \Gamma_D \) form a right angle at each point where the boundary conditions change. Moreover, we assume that \( \Gamma_D \) is smooth (of class \( C^\infty \)).

The flow of a viscous incompressible heat–conducting constant–property fluid is governed by balance equations for linear momentum, mass and internal energy \[5\]

\[\begin{align*}
\rho (u_t + (u \cdot \nabla) u) - \nu \Delta u + \nabla \pi &= \rho (1 - \alpha \theta) f, \\
\text{div } u &= 0, \\
c_p \rho (\theta_t + (u \cdot \nabla) \theta) - \kappa \Delta \theta - \nu e(u) : e(u) &= \rho \alpha \theta f \cdot u + h.
\end{align*}\]

Here \( u = (u_1, u_2) \), \( \pi \) and \( \theta \) denote the unknown velocity, pressure and temperature, respectively. Tensor \( e(u) \) denotes the symmetric part of the velocity gradient. Data of the problem are as follows: \( f \) is a body force and \( h \) a heat source term. Positive constant material coefficients represent the kinematic viscosity \( \nu \), density \( \rho \), heat conductivity \( \kappa \), specific heat at constant pressure \( c_p \) and thermal expansion coefficient of the fluid \( \alpha \). The energy balance equation \[3\] takes into account the phenomena of the viscous energy dissipation and adiabatic heat effects. For rigorous derivation of the model like \((1)-(3)\) we refer the readers to \[11\].

Concerning the boundary conditions of the flow, it is a standard situation to prescribe the non-homogeneous Dirichlet boundary condition for temperature and homogeneous no-slip boundary condition for velocity of the fluid on the fixed walls of the channel. In the case of temperature outflow boundary condition on \( \Gamma_N \), we accept a frequently used assumption as “zero flux density”, which is equivalent to the condition \( \nabla \theta \cdot n = 0 \), sometimes referred to as a “do nothing” (or “natural”) boundary condition and rather used in numerical simulations (cf. \[24\]). However, it is really not clear which boundary condition for \( u \) should be prescribed on \( \Gamma_N \). The boundary condition

\[-\pi n_i + \nu \frac{\partial u}{\partial n_i} = F_i n_i\]

prescribed on \( \Gamma_N^{(i)} \) is again often called “do nothing” (or “free outflow”) boundary condition (cf. \[7\], \[8\], \[10\]). Here \( F_i \) are given functions of space and time and \( n_i \) is the outer unit normal to \( \Gamma_N^{(i)}, \; i = 1, \ldots, m \). The boundary condition \[4\] results from a variational principle and does not have a real
physical meaning but is rather used in truncating large physical domains. It has been proven to be convenient in numerical modeling of parallel flows. For more information about application of this boundary condition and the physical meaning of the quantities \( F_i \) we refer to [7].

**Remark 1.1** Assume that \( F_i \) are given smooth functions on \( \Gamma_{N(i)} \), \( i = 1, \ldots, m \), and consider the smooth extension \( F \) such that \( F(x,t)|_{\Gamma_{N(i)}} \equiv F_i(x,t) \). Introducing the new variable \( P = \pi + F \) this amounts to solving the homogeneous “do nothing” boundary condition transferring the data from the right-hand side of (4) to the right-hand side of the linear momentum balance equation. Hence we will assume throughout this paper, without loss of generality, that \( F_i \equiv 0 \) in (4), \( i = 1, \ldots, m \).

**Remark 1.2** To simplify the notation in the whole paper, we normalize material constants \( \rho, \nu, \kappa, \alpha \) and \( c_p \) to one.

The paper is organized as follows. In Subsection 1.2, we introduce basic notations and some appropriate function spaces in order to precisely formulate our problem. In Section 2, we present the strong form of the model for the non-stationary motion of viscous incompressible heat-conducting fluids in a channel considered in our work, impose compatibility conditions on initial data, specify our smoothness assumptions on data and formulate the problem in a variational setting. The main result of our work is established at the end of Section 2. In Section 3, we present basic results on the existence, uniqueness and energy estimates of the solution to an auxiliary linearized problems, the decoupled initial-boundary value problems for the Stokes and heat equations. The main result, stated in Section 2, is proved in Section 4 via the Banach contraction principle. In the proof of local in time existence, presented in Subsection 4.1, we rely on the energy estimates for linear problems, regularity of stationary solutions and interpolations-like inequalities. The global in time uniqueness of the strong solution is proved in Subsection 4.2 using the technique of Gronwall lemma.

**Remark 1.3** Let us note that our results can be extended to the problem if we consider the so called “free surface” boundary condition on \( \Gamma_N \) and replace (4) by

\[-\pi n + \nu (\nabla u + (\nabla u)^\top)n = 0.\]

However, to ensure the smoothness of the solution and exclude boundary singularities near the points where the boundary conditions change their type some additional requirements on the geometry of the domain need to be introduced. This means that \( \Gamma_N \) and \( \Gamma_D \) form an angle \( \omega < \pi/4 \) at each point where the boundary conditions change (see [22]).
1.2 Basic notation and some function spaces

Vector functions and operators acting on vector functions are denoted by bold-face letters. Unless specified otherwise, we use Einstein’s summation convention for indices running from 1 to 2. Throughout the paper, we will always use positive constants \(c, c_1, c_2, \ldots\), which are not specified and which may differ from line to line.

For an arbitrary \(r \in [1, +\infty]\), \(L^r(\Omega)\) denotes the usual Lebesgue space equipped with the norm \(\| \cdot \|_{L^r(\Omega)}\), and \(W^{k,p}(\Omega), k \geq 0\) (\(k\) need not to be an integer, see [20]), \(p \in [1, +\infty]\), denotes the usual Sobolev space with the norm \(\| \cdot \|_{W^{k,p}(\Omega)}\). For simplicity we denote shortly \(W^{k,p} \equiv W^{k,p}(\Omega)^2\) and \(L^r \equiv L^r(\Omega)^2\).

To simplify mathematical formulations we introduce the following notations:

\[
a_u(u, v) := \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, d\Omega, \quad (5)
\]

\[
b(u, v, w) := \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \, d\Omega, \quad (6)
\]

\[
a_\theta(\theta, \varphi) := \int_\Omega \nabla \theta \cdot \nabla \varphi \, d\Omega, \quad (7)
\]

\[
d(u, \theta, \varphi) := \int_\Omega u \cdot \nabla \theta \varphi \, d\Omega, \quad (8)
\]

\[
e(u, v, \varphi) := \int_\Omega e_{ij}(u)e_{ij}(v)\varphi \, d\Omega, \quad (9)
\]

\[
(u, v) := \int_\Omega u \cdot v \, d\Omega, \quad (10)
\]

\[
(\theta, \varphi)_\Omega := \int_\Omega \theta \varphi \, d\Omega. \quad (11)
\]

In (5)–(11) all functions \(u, v, w, \theta, \varphi\) are smooth enough, such that all integrals on the right-hand sides make sense. In (9) \(e_{ij}(u)\) denotes the components of the tensor \(e(u)\) defined by

\[
e_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2.
\]

Let

\[
\mathcal{E}_u := \{ u \in C^\infty(\overline{\Omega})^2; \ \text{div} \ u = 0, \ \text{supp} \ u \cap \Gamma_D = \emptyset \} \quad (12)
\]

and

\[
\mathcal{E}_\theta := \{ \theta \in C^\infty(\overline{\Omega}); \ \text{supp} \ \theta \cap \Gamma_D = \emptyset \} \quad (13)
\]
and $V^{k,p}_u$ be the closure of $\mathcal{E}_u$ in the norm of $W^{k,p}(\Omega)^2$, $k \geq 0$ and $1 \leq p \leq \infty$. Similarly, let $V^{k,p}_\theta$ be a closure of $\mathcal{E}_\theta$ in the norm of $W^{k,p}(\Omega)$. Then $V^{k,p}_u$ and $V^{k,p}_\theta$, respectively, are Banach spaces with the norms of the spaces $W^{k,p}(\Omega)^2$ and $W^{k,p}(\Omega)$, respectively. Note, that $V^{1,2}_u$, $V^{1,2}_\theta$, $V^{0,2}_u$ and $V^{0,2}_\theta$, respectively, are Hilbert spaces with scalar products (5), (7), (10) and (11), respectively.

Further, define the spaces

$$D_u := \{ u \mid f \in V^{0,2}_u, \ a_u(u, v) = (f, v) \text{ for all } v \in V^{1,2}_u \}$$ (14)

and

$$D_\theta := \{ \theta \mid h \in V^{0,2}_\theta, \ a_\theta(\theta, \varphi) = (h, \varphi)_\Omega \text{ for all } \varphi \in V^{1,2}_\theta \},$$ (15)

equipped with the norms

$$\| u \|_{D_u} := \| f \|_{V^{0,2}_u}$$ (16)

and

$$\| \theta \|_{D_\theta} := \| h \|_{V^{0,2}_\theta},$$ (17)

where $u$ and $f$ are corresponding functions via (14). Similarly, $\theta$ and $h$ are corresponding functions via (15).

The key embeddings $D_u \hookrightarrow W^{2,2}$ and $D_\theta \hookrightarrow W^{2,2}(\Omega)$ are consequences of assumptions setting on the domain $\Omega$ and the regularity results for the steady Stokes system in channel-like domains with “do-nothing” condition, see [2, Remark 2.2 and Corollary 2.3] and the “classical” regularity results for the stationary linear heat equation (the Poisson equation) with the mixed boundary conditions, see for instance [19].

2 Formulation of the problem and the main result

Let $T \in (0, \infty)$ be fixed throughout the paper and $Q_T = \Omega \times (0, T)$, $\Gamma_{DT} = \Gamma_D \times (0, T)$ and $\Gamma_{NT} = \Gamma_N \times (0, T)$. The strong formulation of our problem
is as follows:

\[ \begin{align*}
\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \Delta \mathbf{u} + \nabla P + \theta f &= f \quad \text{in } Q_T, \\
\text{div} \mathbf{u} &= 0 \quad \text{in } Q_T, \\
\theta_t + \mathbf{u} \cdot \nabla \theta - \Delta \theta - e(\mathbf{u}) : e(\mathbf{u}) &= \theta f \cdot \mathbf{u} + h \quad \text{in } Q_T, \\
\mathbf{u} &= 0 \quad \text{on } \Gamma_{DT}, \\
\theta &= g \quad \text{on } \Gamma_{DT}, \\
-P n + \frac{\partial \mathbf{u}}{\partial n} &= 0 \quad \text{on } \Gamma_{NT}, \\
\frac{\partial \theta}{\partial n} &= 0 \quad \text{on } \Gamma_{NT}, \\
\mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in } \Omega, \\
\theta(0) &= \theta_0 \quad \text{in } \Omega.
\end{align*} \]

Here \( g \) is a given function representing the distribution of the temperature \( \theta \) on \( \Gamma_D \). \( \mathbf{u}_0 \) and \( \theta_0 \) describe the initial velocity and temperature, respectively. Here we suppose that all functions in (18)–(26) are smooth enough and satisfy the compatibility conditions \( \mathbf{u}_0 = 0 \) and \( \theta_0 = g \) on \( \Gamma_D \).

**Remark 2.1** Throughout the paper, \( \mu \) denotes some fixed (arbitrarily small) positive real number (cf. (27)).

At this point we can formulate our problem. Suppose that

\[ \begin{align*}
f &\in L^{2+\mu}(0,T; V_u^{0.2}), \quad h \in L^2(0,T; V_\theta^{0.2}), \\
g &\in L^2(0,T; W^{2,2}(\Omega)), \quad g_t \in L^2(0,T; L^2(\Omega)), \\
\mathbf{u}_0 &\in D_u, \quad \theta_0 \in W^{2,2}(\Omega), \quad \theta_0 - g(0) \in D_\theta.
\end{align*} \]

Find a pair \([\mathbf{u}, \theta]\) such that

\[ \begin{align*}
\mathbf{u}_t &\in L^2(0,T; V_u^{0.2}), \quad \mathbf{u} \in L^2(0,T; D_u) \cap L^\infty(0,T; V_u^{1.2}), \\
\theta_t - g_t &\in L^2(0,T; V_\theta^{0.2}), \quad \theta - g \in L^2(0,T; D_\theta) \cap L^\infty(0,T; V_\theta^{1.2})
\end{align*} \]

and the following system

\[ \begin{align*}
(u_t, v) + a_u(u, v) + b(u, u, v) + (\theta f, v) &= (f, v), \\
(\theta_t, \varphi) + a_\theta(\theta, \varphi) + d(u, \theta, \varphi) - e(u, \varphi) - (\theta f \cdot u, \varphi) &= (h, \varphi),
\end{align*} \]

holds for every \([v, \varphi] \in V_u^{1.2} \times V_\theta^{1.2}\) and for almost every \( t \in (0,T) \) and

\[ \begin{align*}
\mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in } \Omega, \\
\theta(0) &= \theta_0 \quad \text{in } \Omega.
\end{align*} \]
The pair \([\mathbf{u}, \theta]\) is called the strong solution to the system (18)–(26).

Let us briefly describe some difficulties we have to solve in our work. The equations (18)–(20) represent the system with strong nonlinearities (quadratic growth of \(\nabla \mathbf{u}\) in dissipative term \(\mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{u})\)) without appropriate general existence and regularity theory. In [9], Frehse presented a simple example of discontinuous bounded weak solution \(\mathbf{U} \in L^\infty \cap H^1\) of nonlinear elliptic system of the type \(\Delta \mathbf{U} = B(\mathbf{U}, \nabla \mathbf{U})\), where \(B\) is analytic and has quadratic growth in \(\nabla \mathbf{U}\). However, for scalar problems, such existence and regularity theory is well developed (cf. [21, 22]).

Nevertheless, the main (open) problem of the system (18)–(26) consists in the fact that, because of the boundary condition (23), we cannot prove that \(b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0\). Consequently, we are not able to show that the kinetic energy of the fluid is controlled by the data of the problem and the solutions of (18)–(26) need not satisfy the energy inequality. This is due to the fact that some uncontrolled “backward flow” can take place at the open parts \(\Gamma_N\) of the domain \(\Omega\) and one is not able to prove global (in time) existence results. In [12–14], Kračmar and Neustupa prescribed an additional condition on the output (which bounds the kinetic energy of the backward flow) and formulated steady and evolutionary Navier–Stokes problems by means of appropriate variational inequalities. In [18], Kučera and Skalák proved the local–in–time existence and uniqueness of a “weak” solution of the evolution Navier–Stokes problem for iso-thermal fluids, such that

\[
\mathbf{u}_t \in L^2(0, T^*; V_{u}^{1,2}), \quad \mathbf{u}_{tt} \in L^2(0, T^*; (V_{u}^{1,2})^*), \quad 0 < T^* \leq T,
\]

under some smoothness restrictions on \(\mathbf{u}_0\) and \(\mathcal{P}\). In [25], the same authors established the similar results for the evolution Boussinesq approximations of the heat–conducting incompressible fluids. In [17], Kučera supposed that the “do nothing” problem for the Navier–Stokes system is solvable in suitable function class with some given data (the initial velocity and the right hand side). The author proved that there exists a unique solution for data which are small perturbations of the original ones.

In the present paper, we extend the results by Skalák and Kučera in [25]. However, our results are restricted to two dimensions. We shall prove local existence and global uniqueness of the strong solution to (18)–(26), i.e. for more general model than the Boussinesq approximations considered in cited references, and, moreover, such that i.a. \([\mathbf{u}, \theta] \in L^2(0, T^*; W^{2,2}) \times L^2(0, T^*; W^{2,2})\), which is strong in the sense that the solution possess second spatial derivatives. The main result of the paper is the following
Theorem 2.2 (Main result) Assume

\[ f \in L^{2+\mu}(0, T; V_u^{0,2}), \quad h \in L^2(0, T; V_\theta^{0,2}), \]
\[ g \in L^2(0, T; W^{2,2}(\Omega)), \quad g_t \in L^2(0, T; L^2(\Omega)), \]
\[ u_0 \in D_u, \quad \theta_0 \in W^{2,2}(\Omega), \quad \theta_0 - g(0) \in D_\theta. \]

Then there exists \( T^* \in (0, T] \) and the pair \([u, \theta]\),

\[ u_t \in L^2(0, T; V_u^{0,2}), \quad u \in L^2(0, T; D_u) \cap L^\infty(0, T; V_u^{1,2}), \]
\[ \theta_t - g_t \in L^2(0, T; V_\theta^{0,2}), \quad \theta - g \in L^2(0, T; D_\theta) \cap L^\infty(0, T; V_\theta^{1,2}), \]

such that \([u, \theta]\) is the strong solution of the system (18) - (26). This solution is also globally unique.

3 Auxiliary results

Before we proceed to prove the main result of our paper, let us establish the well-posedness property for appropriate linear problems, which can be found in literature.

Theorem 3.1 (The decoupled Stokes equations) Let \( f \in L^2(0, T; V_u^{0,2}). \)
Then there exists the unique function \( u \in L^2(0, T; D_u) \cap L^\infty(0, T; V_u^{1,2}), \)
\[ u_t \in L^2(0, T; V_u^{0,2}), \quad u \in L^2(0, T; D_u) \cap L^\infty(0, T; V_u^{1,2}), \]
\[ \theta_t - g_t \in L^2(0, T; V_\theta^{0,2}), \quad \theta - g \in L^2(0, T; D_\theta) \cap L^\infty(0, T; V_\theta^{1,2}), \]

such that \([u, \theta]\) is the strong solution of the system (18) - (26). This solution is also globally unique.

\[ (u_t, v) + a_u(u, v) = (f, v) \tag{38} \]

holds for every \( v \in V_u^{1,2} \) and for almost every \( t \in (0, T) \) and \( u(0) = 0 \). Moreover

\[ \|u\|_{L^2(0,T;V_u^{0,2})} + \|u\|_{L^2(0,T;D_\theta)} + \|u\|_{L^\infty(0,T;V_u^{1,2})} \leq c(\Omega)\|f\|_{L^2(0,T;V_u^{0,2})}. \tag{39} \]

Theorem 3.2 (The decoupled heat equation) Let \( h \in L^2(0, T; V_\theta^{0,2}). \)
Then there exists the unique function \( \theta \in L^2(0, T; D_\theta) \cap L^\infty(0, T; V_\theta^{1,2}), \)
\[ \theta_t \in L^2(0, T; V_\theta^{0,2}), \quad \theta \in L^2(0, T; D_\theta) \cap L^\infty(0, T; V_\theta^{1,2}), \]
\[ \theta - g \in L^2(0, T; D_\theta) \cap L^\infty(0, T; V_\theta^{1,2}), \]

such that \([\theta]\) is the strong solution of the system (18) - (26). This solution is also globally unique.

\[ (\theta_t, \varphi) + a_\theta(\theta, \varphi) = (h, \varphi) \tag{40} \]

holds for every \( \varphi \in V_\theta^{1,2} \) and for almost every \( t \in (0, T) \) and \( \theta(0) = 0 \). Moreover

\[ \|\theta_t\|_{L^2(0,T;V_\theta^{0,2})} + \|\theta\|_{L^2(0,T;D_\theta)} + \|\theta\|_{L^\infty(0,T;V_\theta^{1,2})} \leq c(\Omega)\|h\|_{L^2(0,T;V_\theta^{0,2})}. \tag{41} \]
Remark to proofs of Theorem 3.2 and 3.1: The assertion of Theorem 3.1 is proved in [3, Theorem 2.1] and [4, Theorem 3.4]. We omit the proof of Theorem 3.2 since it can be established in the same way, see also [9, Chapter 5]. Note that the well-known approach to the proof is based on the Galerkin approximation with spectral basis and the uniform boundedness of approximate solutions in suitable spaces, see also [26, Chapter 3].

4 Proof of the main result

4.1 Existence

Let us introduce the following reflexive Banach spaces

\[ X^u_T := \{ \phi \mid \phi \in L^2(0, T; D_u), \phi_t \in L^2(0, T; V_u^0), \phi(0) = 0 \}, \]
\[ X^p_T := \{ \psi \mid \psi \in L^2(0, T; D_p), \psi_t \in L^2(0, T; V_p^0), \psi(0) = 0 \}, \]
\[ X_T := \{ [\phi, \psi] \mid \phi \in X^u_T, \psi \in X^p_T \}, \]
respectively, with norms

\[ \| \phi \|_{X^u_T} := \| \phi \|_{L^2(0, T; D_u)} + \| \phi_t \|_{L^2(0, T; V_u^0)}, \]
\[ \| \psi \|_{X^p_T} := \| \psi \|_{L^2(0, T; D_p)} + \| \psi_t \|_{L^2(0, T; V_p^0)}, \]
\[ \| [\phi, \psi] \|_{X_T} := \| \phi \|_{X^u_T} + \| \psi \|_{X^p_T}. \]

Let us present some properties of \( X^u_T \), \( X^p_T \) and (consequently) \( X_T \). Note that [2] Remark 2.2 and Corollary 2.3] yields the embedding \( D_u \hookrightarrow W^{2, 2} \), which implies

\[ X^u_T \hookrightarrow L^\infty(0, T; W^{1, 2}) \hookrightarrow L^\infty(0, T; L^p), \quad 1 \leq p < \infty. \]

Let \( \phi \in X^u_T \). Using the interpolation inequality

\[ \| \phi \|_{W^{3/2, 2}} \leq c \| \phi \|_{W^{1, 2}}^{1/2} \| \phi \|_{W^{2, 2}}^{1/2} \]

we get

\[ \| \phi \|_{L^4(0, T; W^{3/2, 2})} \leq c \| \phi \|_{L^2(0, T; W^{2, 2})}^{1/2} \| \phi \|_{L^\infty(0, T; W^{1, 2})}^{1/2} \]

\[ \leq c \| \phi \|_{X^u_T}, \]

where \( c = c(\Omega) \). Hence we have

\[ X^u_T \hookrightarrow L^4(0, T; W^{3/2, 2}) \hookrightarrow L^4(0, T; W^{1, 4}) \hookrightarrow L^4(0, T; L^p) \]
for every $1 \leq p \leq \infty$. Similar properties hold for the space $X^\theta_T$. Since \cite{19} yields the embedding $\mathcal{D}_\theta \hookrightarrow W^{2,2}(\Omega)$, we get
\[
X^\theta_T \hookrightarrow L^\infty(0, T; W^{1,2}(\Omega)) \hookrightarrow L^\infty(0, T; L^p(\Omega)), \quad 1 \leq p < \infty,
\]
and
\[
X^\theta_T \hookrightarrow L^4(0, T; W^{3/2,2}(\Omega)) \hookrightarrow L^4(0, T; W^{1,4}(\Omega)) \hookrightarrow L^4(0, T; L^p(\Omega))
\]
for every $1 \leq p \leq \infty$.

Let $\phi \in L^2(0, T; W^{2,2}(\Omega)) \cap L^\infty(0, T; W^{1,2}(\Omega))$. Raising and integrating the interpolation inequality
\[
\|\phi(t)\|_{W^{1+j,2}(\Omega)} \leq c \|\phi(t)\|_{W^{2,2}(\Omega)}^{1-j} \|\phi(t)\|_{W^{1,2}(\Omega)}^{1+j}, \quad 0 < j \leq 1,
\]
from 0 to $T$ we get
\[
\left(\int_0^T \|\phi(t)\|_{W^{1+j,2}(\Omega)}^{2j} dt\right)^{j/2} \leq c \left(\int_0^T \|\phi(t)\|_{W^{2,2}(\Omega)}^{2j} \|\phi(t)\|_{W^{1,2}(\Omega)}^{2(1-j)/j} dt\right)^{j/2} \\
\leq c \|\phi\|_{L^2(0,T;W^{2,2}(\Omega))} \|\phi\|_{L^\infty(0,T;W^{1,2}(\Omega))}^{1-j},
\]
where $c = c(\Omega)$. Hence we have
\[
L^2(0, T; W^{2,2}(\Omega)) \cap L^\infty(0, T; W^{1,2}(\Omega)) \hookrightarrow L^{2/j}(0, T; W^{1+j,2}(\Omega)) \hookrightarrow L^{2/j}(0, T; L^\infty(\Omega))
\]
for arbitrarily small positive $j$ and consequently we deduce
\[
X^u_T \hookrightarrow L^q(0, T; L^\infty)
\]
and
\[
X^\theta_T \hookrightarrow L^q(0, T; L^\infty(\Omega))
\]
for $2 \leq q < \infty$.

**Remark 4.1** Setting $[w, \vartheta] = [u - u_0, \theta - \theta_0 - g]$ this amounts to solving the problem with homogeneous boundary and initial data
\[
(w_t, v) + a_w(w + u_0, v) + b(w + u_0, w + u_0, v) \\
- (f(\vartheta + \theta_0 + g), v) = (f, v),
\]
\[
(\vartheta_t + g_t, \varphi) + a_\theta(\vartheta + \theta_0 + g, \varphi) + d(w + u_0, \vartheta + \theta_0 + g, \varphi) \\
+ ((\vartheta + \theta_0 + g) f \cdot (w + u_0), \varphi) + e(w + u_0, w + u_0, \varphi) = (h, \varphi)_{\Omega},
\]
\[
w(0) = 0, \quad \vartheta(0) = 0
\]
for every \([v, \varphi] \in V^{1,2}_u \times V^{1,2}_\vartheta\) and for almost every \(t \in (0, T)\), where \(f \in L^2(0, T; V^{0,2}_u), h \in L^2(0, T; V^{0,2}_\vartheta)\), \(u_0 \in D_u\) and \(\theta_0 \in D_\vartheta\).

For arbitrary fixed \([\tilde{w}, \tilde{\varphi}] \in X_T\) we now consider the nonlinear problem

\[
(w_t, v) + a_u(w, v) = (f, v) - a_u(u_0, v) - b(u_0, u_0, v) - b(\tilde{u}, \tilde{w}, v) + (f(\theta_0 + g), v) + (f \tilde{\varphi}, v),
\]

\[
(\theta_t, \varphi)_{\Omega} + a_\theta(\vartheta, \varphi) + e(w, w, \varphi) = -2e(\tilde{w}, u_0, \varphi) - e(u_0, u_0, \varphi) + (h, \varphi)_{\Omega} - (g, \varphi)_{\Omega} - a_\theta(\theta_0 + g, \varphi) - d(u_0, \theta_0 + g, \varphi) - d(\tilde{w}, \tilde{\varphi}, \varphi) - d(u_0, \tilde{\varphi}, \varphi) + ((\theta_0 + g) f \cdot u_0, \varphi)_{\Omega} + (\tilde{\varphi} f \cdot u_0, \varphi)_{\Omega} + ((\tilde{\varphi} f \cdot \tilde{w}, \varphi)_{\Omega} + (\tilde{\varphi} f \cdot \tilde{w}), \varphi)_{\Omega},
\]

\[
w(0) = 0, \quad \vartheta(0) = 0
\]

for every \([v, \varphi] \in V^{1,2}_u \times V^{1,2}_\vartheta\) and for almost every \(t \in (0, T)\).

In the proof of the next lemma we verify that all terms on the right-hand sides of (62)–(63) and the dissipative term \(e(w, w, \cdot)\) are well defined. From Theorem 3.1 and Theorem 3.2 we deduce that for arbitrary fixed \([\tilde{w}, \tilde{\varphi}] \in X_T\) there exists \([w, \vartheta] = Z([\tilde{w}, \tilde{\varphi}]) \in X_T\), the solution of the problem (62)–(65).

Consequently, the mapping \(Z : X_T \to X_T\) is well defined.

Denote by \(B_R(T) \subset X_T\) the closed ball

\[
B_R(T) := \{[\phi, \psi] \in X_T; \|\phi, \psi\|_{X_T} \leq R\}.
\]

We are going to show that the formal map \(Z : [\tilde{w}, \tilde{\varphi}] \to [w, \vartheta]\), where \([w, \vartheta]\) is a solution of the problem (62)–(65), has a fixed point in \(B_R(T^*)\) for \(T^*\) sufficiently small.

Let us prepare the existence proof. We now claim the following

**Lemma 4.2** For all \(R > 0, T > 0\) and for every \([\tilde{w}, \tilde{\varphi}] \in B_R(T)\) we have

\[
\|Z([\tilde{w}, \tilde{\varphi}])\|_{X_T} \leq c_1\xi_1(T) + c_2\xi_2(T)\omega(R),
\]

where \(\omega\) is an increasing function depending solely on \(R\), functions \(\xi_1\) and \(\xi_2\) depend solely on \(T\) and \(\xi_1(T) \to 0_+, \xi_2(T) \to 0_+\) for \(T \to 0_+\) and the positive constants \(c_1\) and \(c_2\) are independent of \(T\) and \(R\).
The proof is rather technical. Obviously, there exists a positive function $K_1(T)$ such that
\[
\| (f, \cdot) - a_u(u_0, \cdot) - b(u_0, u_0, \cdot) + (f(\theta_0 + g), \cdot) \|_{L^2(0,T;\mathcal{V}_0^{1,2})} \\
+ \| (h, \cdot) - (g_t, \cdot) - a_\theta(\theta_0 + g, \cdot) - d(u_0, \theta_0 + g, \cdot) \|_{L^2(0,T;\mathcal{V}_0^{1,2})} \\
+ \| e(u_0, u_0, \cdot) + ((\theta_0 + g, f \cdot u_0, \cdot) - a_\theta(\theta_0 + g, \cdot)) \|_{L^2(0,T;\mathcal{V}_0^{1,2})} \leq cK_1(T)
\] (68)
and $K_1(T) \to 0_+$ for $T \to 0_+$, $c$ is independent of $T$.

Similarly and using the interpolation inequality we get
\[
\| b(\tilde{w}, u_0, \cdot) \|_{L^2(0,T;\mathcal{V}_0^{1,2})} \leq \left( \int_0^T \| \tilde{w} \|^2 \| \nabla u_0 \|^2 \|_{L^4} dt \right)^{1/2} \\
\leq \| u_0 \|_{W^{1,4}} \| \tilde{w} \|_{L^2(T;L^4)} \\
\leq T^{1/4} \| u_0 \|_{W^{1,4}} \| \tilde{w} \|_{L^4(T;L^4)} \\
\leq cT^{1/4} \| u_0 \|_{D_n} \| \tilde{w} \|_{X^2_T}.
\] (69)
The other terms in (62) can be handled in the same way
\[
\| b(u_0, \tilde{w}, \cdot) \|_{L^2(0,T;\mathcal{V}_0^{1,2})} \leq \left( \int_0^T \| u_0 \|_{L^\infty} \| \nabla \tilde{w} \|^2 \|_{L^4} dt \right)^{1/2} \\
\leq \| u_0 \|_{L^\infty} \| \tilde{w} \|_{L^2(T;W^{1,2})} \\
\leq T^{1/4} \| u_0 \|_{L^\infty} \| \tilde{w} \|_{L^4(T;W^{1,2})} \\
\leq cT^{1/4} \| u_0 \|_{D_n} \| \tilde{w} \|_{X^2_T},
\] (70)
\[
\| b(\tilde{w}, \tilde{w}, \cdot) \|_{L^2(0,T;\mathcal{V}_0^{1,2})} \leq \left( \int_0^T \| \tilde{w} \|^2 \| \nabla \tilde{w} \|^2 \|_{L^4} dt \right)^{1/2} \\
\leq T^{1/4} \| \tilde{w} \|_{L^\infty(T;L^4)} \| \tilde{w} \|_{L^4(T;W^{1,4})} \\
\leq cT^{1/4} \| \tilde{w} \|_{X^2_T}.
\] (71)
and finally the last term in (62) can be estimated using the interpolation inequality as
\[
\| (f, \tilde{\phi}, \cdot) \|_{L^2(0,T;\mathcal{V}_0^{1,2})} \leq \left( \int_0^T \| f \|^2 \| \tilde{\phi} \|^2 \|_{L^\infty(\Omega)} dt \right)^{1/2} \\
\leq T^{\mu/[2+\mu]}(4+\mu) \| f \|_{L^2(0,T;L^2)} \| \tilde{\phi} \|_{L^2(4+\mu)/\mu(0,T;L^\infty(\Omega))} \\
\leq cT^{\mu/[2+\mu]}(4+\mu) \| f \|_{L^2(0,T;L^2)} \| \tilde{\phi} \|_{X^2_T}.
\] (72)
The inequalities (68)–(72) and Theorem 3.1 yield the estimate
\[ \|w\|_{X_T^2} \leq K_1(T) + c(R^2 + R)T^{1/4} + cT^{\mu/[(2+\mu)(4+\mu)]}R, \] (73)
where \( K_1(T) \to 0_+ \) for \( T \to 0_+ \) and \( c \) is independent of \( T \). Successively, we use (50) and (73) to estimate the dissipative term \( e(w, w, \cdot) \) in (63).

\[ \|e(w, w, \cdot)\|_{L^2(0, T; V_0^{2,2})} \leq \left( \int_0^T \|e(w)\|_{L^4}^2 dt \right)^{1/2} \]
\[ \leq \|w\|_{L^4(0, T; W^{1,4})}^2 \]
\[ \leq c\|w\|_{X_T^2}^2 \]
\[ \leq [K_1(T) + c(R^2 + R)T^{1/4} + cT^{\mu/[(2+\mu)(4+\mu)]}R]^2. \] (74)

Now we are going to estimate all remaining terms on the right-hand side of (63). We use interpolation inequality and (50) to estimate the dissipative term \( e(\bar{w}, u_0, \varphi) \)
\[ \|e(\bar{w}, u_0, \cdot)\|_{L^2(0, T; V_0^{2,2})} \leq \left( \int_0^T \|\nabla u_0\|_{L^4}^2 \|\nabla \bar{w}\|_{L^4}^2 dt \right)^{1/2} \]
\[ \leq T^{1/4} \|\nabla u_0\|_{L^4} \|\nabla \bar{w}\|_{L^4(0, T; L^4)} \]
\[ \leq cT^{1/4} \|u_0\|_{D_\theta} \|\bar{w}\|_{X_T^2} \] (75)
and taking into account (50) and (52), we arrive at the estimate
\[ \|d(\bar{w}, \tilde{\vartheta}, \cdot)\|_{L^2(0, T; V_0^{2,2})} \leq \left( \int_0^T \|\bar{w}\|_{L^4}^2 \|\nabla \tilde{\vartheta}\|_{L^4(\Omega)}^2 dt \right)^{1/2} \]
\[ \leq T^{1/4} \|\bar{w}\|_{L^\infty(0, T; L^4)} \|\tilde{\vartheta}\|_{L^4(0, T; W^{4,4}(\Omega))} \]
\[ \leq cT^{1/4} \|\bar{w}\|_{X_T^2} \|\tilde{\vartheta}\|_{X_T^2} \]
\[ \leq cT^{1/4} \frac{1}{2} \left( \|\bar{w}\|_{X_T^2}^2 + \|\tilde{\vartheta}\|_{X_T^2}^2 \right). \] (76)

Let us proceed to estimate the remaining convective terms:
\[ \|d(\bar{w}, \theta_0 + g, \cdot)\|_{L^2(0, T; V_0^{2,2})} \leq \left( \int_0^T \|\bar{w}\|_{L^4}^2 \|\theta_0 + g\|_{W^{1,4}(\Omega)}^2 dt \right)^{1/2} \]
\[ \leq T^{1/4} \|\bar{w}\|_{L^\infty(0, T; L^4)} \|\theta_0 + g\|_{L^4(0, T; W^{4,4}(\Omega))} \]
\[ \leq cT^{1/4} \|\bar{w}\|_{X_T^2} \|\theta_0 + g\|_{X_T^2} \]
\[ \leq cT^{1/4} \frac{1}{2} \left( \|\bar{w}\|_{X_T^2}^2 + \|\theta_0 + g\|_{X_T^2}^2 \right) \]
\[ \leq cT^{1/4} \frac{1}{2} \|\bar{w}\|_{X_T^2}^2 + K_2(T), \] (77)
where the positive function $K_2(T) \to 0_+$ for $T \to 0_+$. Further

$$
\|d(u_0, \tilde{\omega}, \cdot)\|_{L^2(0,T;V^0_\theta)} \leq \left( \int_0^T \|u_0\|_{L^\infty}^2 \|\nabla \tilde{\omega}\|_{L^2}^2 \, dt \right)^{1/2} \\
\leq T^{1/4} \|u_0\|_{L^\infty} \|	ilde{\omega}\|_{L^4(0,T;W^{1,2}(\Omega))} \\
\leq cT^{1/4} \|u_0\|_{\mathcal{D}_\omega} \|	ilde{\omega}\|_{X^0_\theta}.
$$

(78)

In the similar way one can deduce for the adiabatic terms the estimates

$$
\|((\tilde{\omega} f, \tilde{\omega}), \cdot)_\Omega\|_{L^2(0,T;V^0_\theta)} \leq \left( \int_0^T \|\tilde{\omega}\|_{L^\infty(\Omega)}^2 \|f\|_{L^2(0,T;L^2)}^2 \|\tilde{\omega}\|_{L^4(0,T;L^\infty)} \right)^{1/2} \\
\leq c\|\tilde{\omega}\|_{X^0_\theta} \|f\|_{L^2(0,T;L^2)} \|\tilde{\omega}\|_{X^0_\theta},
$$

and, in the same way,

$$
\|((\theta_0 + g) f, \tilde{\omega} \cdot, \cdot)_\Omega\|_{L^2(0,T;V^0_\theta)} \leq \left( \int_0^T \|\theta_0 + g\|_{L^4(0,T;L^\infty)}^2 \|f\|_{L^2(0,T;L^2)}^2 \|\tilde{\omega}\|_{L^4(0,T;L^\infty)} \right)^{1/2} \\
\leq c\|\theta_0 + g\|_{X^0_\theta} \|f\|_{L^2(0,T;L^2)} \|\tilde{\omega}\|_{X^0_\theta}.
$$

(79)

Finally,

$$
\|((\tilde{\omega} f, u_0), \cdot)_\Omega\|_{L^2(0,T;V^0_\theta)} \leq \left( \int_0^T \|\tilde{\omega}\|_{L^\infty(\Omega)}^2 \|f\|_{L^2(0,T;L^2)}^2 \|u_0\|_{L^\infty}^2 \, dt \right)^{1/2} \\
\leq T^{\mu/[(2+\mu)(4+\mu)]} \|\tilde{\omega}\|_{L^{2+\mu}(0,T;L^\infty)} \|f\|_{L^{2+\mu}(0,T;L^2)} \|u_0\|_{L^\infty} \\
\leq cT^{\mu/[(2+\mu)(4+\mu)]} \|\tilde{\omega}\|_{X^0_\theta} \|f\|_{L^{2+\mu}(0,T;L^2)} \|u_0\|_{L^\infty}.
$$

(81)

Now the assertion is the straight consequence of inequalities (68)–(81), Theorem 3.2 and Theorem 3.1. The proof of Lemma 4.2 is complete.

**Lemma 4.3** For $T^* \in (0, T]$, sufficiently small, $Z$ maps $B_R(T^*)$ into $B_R(T^*)$ and realizes a contraction.

**Corollary 4.4** Lemma 4.2, Lemma 4.3 and the Banach fixed point theorem yield the existence of a fixed-point $[\omega, \tilde{\omega}] = Z([\omega, \tilde{\omega}])$ in the ball $B_R(T^*)$ for sufficiently small $T^* \in (0, T]$. By Remark 4.2 the function $[u, \theta] = [u_0, \tilde{\omega} + \theta + \theta_0 + g]$ is the strong solution to the system (18)–(26) on $(0, T^*)$.
Proof of Lemma 4.3 Fix $R > 0$ and choose $\tau \in (0, T]$ sufficiently small, such that (cf. (57))

$$c_1\xi_1(s) + c_2\xi_2(s)\omega(R) \leq R \quad \forall s \in (0, \tau].$$

Now we conclude that for every positive $T^*, 0 < T^* \leq \tau$, $Z$ maps $B_R(T^*)$ into $B_R(T^*)$.

Now we prove contraction. Let $[\tilde{\omega}_1, \tilde{\theta}_1], [\tilde{\omega}_2, \tilde{\theta}_2] \in X_T$, $z = \tilde{\omega}_2 - \tilde{\omega}_1$ and $\sigma = \tilde{\theta}_2 - \tilde{\theta}_1$. Theorem 3.2 and Theorem 3.1 imply the estimate

$$\|Z([\tilde{\omega}_2, \tilde{\theta}_2]) - Z([\tilde{\omega}_1, \tilde{\theta}_1])\|_{X_T} \leq c(\Omega) \left( \|b(z, u_0, \cdot)\|_{L^2(0, T; V^{0,2}_\theta)} + \|b(u_0, z, \cdot)\|_{L^2(0, T; V^{0,2}_\theta)} + \|\tilde{\omega}_2\|_{X_T} \right), \quad (82)$$

Following (68)–(78) we have (recall that $c = c(\Omega)$)

$$\|b(z, u_0, \cdot)\|_{L^2(0, T; V^{0,2}_\theta)} \leq c T^{1/4} \|u_0\|_{D_u} \|z\|_{X^p_T}, \quad (83)$$

$$\|b(u_0, z, \cdot)\|_{L^2(0, T; V^{0,2}_\theta)} \leq c T^{1/4} \|u_0\|_{D_u} \|z\|_{X^p_T}, \quad (84)$$

$$\|\tilde{\omega}_2\|_{X_T} \leq c T^{1/4} \|z\|_{X^p_T} \|\tilde{\omega}_2\|_{X^p_T}, \quad (85)$$

$$\|\tilde{\theta}_2\|_{X_T} \leq c T^{1/4} \|z\|_{X^p_T} \|\tilde{\theta}_2\|_{X^p_T}, \quad (86)$$

$$\|d(z, \tilde{\theta}_2, \cdot)\|_{L^2(0, T; V^{0,2}_\theta)} \leq c T^{1/4} \|\tilde{\theta}_2\|_{X^p_T} \|z\|_{X^p_T}, \quad (87)$$

$$\|d(u_0, \cdot, \cdot)\|_{L^2(0, T; V^{0,2}_\theta)} \leq c T^{1/4} \|u_0\|_{D_u} \|\sigma\|_{X^p_T}. \quad (88)$$

Estimating the dissipative terms we arrive at

$$\|e(w_2 - w_1, \cdot)\|_{L^2(0, T; V^{0,2}_\theta)} \leq c \|w_2 - w_1\|_{X^p_T} \|u_0\|_{D_u}, \quad (90)$$

$$\|e(w_2 - w_1, \cdot)\|_{L^2(0, T; V^{0,2}_\theta)} \leq c \|w_2 - w_1\|_{X^p_T} \|w_2\|_{X^p_T}, \quad (92)$$

$$\|e(w_1, w_2 - w_1, \cdot)\|_{L^2(0, T; V^{0,2}_\theta)} \leq c \|w_1\|_{X^p_T} \|w_2 - w_1\|_{X^p_T}. \quad (94)$$
Following Theorem 3.1 we can eliminate the term $\|w_2 - w_1\|_{X_T}$ in (92)–(94) using the energy-like estimate (cf. (39))

$$\|w_2 - w_1\|_{X_T} \leq c(\Omega) \left( \|b(z, u_0, \cdot)\|_{L^2(0,T;V_u^{0,2})} + \|b(u_0, z, \cdot)\|_{L^2(0,T;V_u^{0,2})} + \|b(\tilde{w}_2, z, \cdot)\|_{L^2(0,T;V_u^{0,2})} + \|b(z, \tilde{w}_1, \cdot)\|_{L^2(0,T;V_u^{0,2})} + \|(f \sigma, \cdot)\|_{L^2(0,T;V_u^{0,2})} \right)$$

(95)

and then apply the inequalities (83)–(87). Finally, the adiabatic terms can be estimated in the following way:

$$\|\sigma f \cdot u_0, \cdot\|_{L^2(0,T;V_u^{0,2})} \leq cT^{\mu/(2+\mu)(4+\mu)} \|\sigma\|_{X_T^2} \|f\|_{L^{2+\mu}(0,T;L^2)} \|u_0\|_{X_T^2}, \quad \text{(96)}$$

$$\|((\theta_0 + g) f \cdot z, \cdot)\|_{L^2(0,T;V_u^{0,2})} \leq cT^{\mu/(2+\mu)(4+\mu)} \|\theta_0 + g\|_{X_T^2} \|f\|_{L^{2+\mu}(0,T;L^2)} \|z\|_{X_T^2}, \quad \text{(97)}$$

$$\|\sigma f \cdot \tilde{w}_1, \cdot\|_{L^2(0,T;V_u^{0,2})} \leq cT^{\mu/(2+\mu)(4+\mu)} \|\sigma\|_{X_T^2} \|f\|_{L^{2+\mu}(0,T;L^2)} \|\tilde{w}_1\|_{X_T^2} \quad \text{(98)}$$

and

$$\|((\tilde{\theta}_2 f \cdot z, \cdot)\|_{L^2(0,T;V_u^{0,2})} \leq cT^{\mu/(2+\mu)(4+\mu)} \|\tilde{\theta}_2\|_{X_T^2} \|f\|_{L^{2+\mu}(0,T;L^2)} \|z\|_{X_T^2}. \quad \text{(99)}$$

Now let $[\tilde{w}_1, \tilde{\theta}_1], [\tilde{w}_2, \tilde{\theta}_2] \in B_R(T) \subset X_T$. Inequality (82) and the estimates (83)–(99) yield

$$\|Z([\tilde{w}_2, \tilde{\theta}_2]) - Z([\tilde{w}_1, \tilde{\theta}_1])\|_{X_T} \leq cK(T)\|\tilde{w}_2, \tilde{\theta}_2\| - [\tilde{w}_1, \tilde{\theta}_1]\|_{X_T}, \quad \text{(100)}$$

where the positive function $K(T) \to 0_+$ for $T \to 0_+$ and $c$ does not depend on $T$. Hence taking $T^* \in (0, T]$ sufficiently small, the contraction of $Z : B_R(T^*) \to B_R(T^*)$ can be easily established. The proof of Lemma 4.3 is complete.

### 4.2 Global uniqueness of the strong solutions

Here we prove the global uniqueness of the strong solution stated in the main result. Suppose that all assumptions of Theorem 2.2 are satisfied and...
there are two strong solutions $[u_1, \theta_1], [u_2, \theta_2]$ of (18)–(26) on $(0, T)$. Denote $z = u_1 - u_2 \in X^p_T$ and $\sigma = \theta_1 - \theta_2 \in X^q_T$. Then $z$ and $\sigma$ satisfy the equations

\[
(z, v) + a_u(z, v) + b(z, u_2, v) + b(u_1, z, v) - (f, v) = 0, \quad (101)
\]

\[
(\sigma, \varphi)_\Omega + a_\varphi(\sigma, \varphi) + d(z, \theta_1, \varphi) + d(u_2, \sigma, \varphi) - e(z, u_1, \varphi) - e(u_2, z, \varphi) - (\sigma f \cdot u_1, \varphi)_\Omega - (\theta_2 f \cdot z, \varphi)_\Omega = 0 \quad (102)
\]

for every $[v, \varphi] \in V_u^{1.2} \times V_\theta^{1.2}$ and for almost every $t \in (0, T)$ and $z(0) = 0$ and $\sigma(0) = 0$. Hence substituting $v = z$ and $\varphi = \sigma$ we obtain estimates

\[
\frac{1}{2} \frac{d}{dt} \|z(t)\|^2_{V_u^{0.2}} + \|z(t)\|^2_{V_\theta^{1.2}} \leq |b(u_1(t), z(t), z(t))| + |b(z(t), u_2(t), z(t))| + |(f(t)\sigma(t), z(t))| \quad (103)
\]

and

\[
\frac{1}{2} \frac{d}{dt} \|\sigma(t)\|^2_{V_\theta^{0.2}} + \|\sigma(t)\|^2_{V_\theta^{1.2}} \leq |d(z(t), \theta_1(t), \sigma(t))| + |d(u_2(t), \sigma(t), \sigma(t))| + |e(z(t), u_1(t), \sigma(t))| + |e(u_2(t), z(t), \sigma(t))| + |(\sigma(t)f(t) \cdot u_1(t), \sigma(t))_\Omega| + |(\theta_2(t)f(t) \cdot z(t), \sigma(t))_\Omega| \quad (104)
\]

for a.e. $t \in (0, T)$. To estimate term by term on the right-hand sides of (103) and (104) the Gagliardo–Nirenberg interpolation inequalities (cf. [1, Theorem 5.8])

\[
\|z(t)\|_{L^4} \leq c \|z(t)\|^{1/2}_{W^{1,2}} \|z(t)\|^{1/2}_{L^2},
\]

\[
\|\sigma(t)\|_{L^4(\Omega)} \leq c \|\sigma(t)\|^{1/2}_{W^{1,2}(\Omega)} \|\sigma(t)\|^{1/2}_{L^2(\Omega)}
\]

and the well-known Young’s inequality with parameter $\delta$

\[
ab \leq \delta a^p + C(\delta)b^q \quad (a, b > 0, \quad \delta > 0, \quad 1 < p, q < \infty, \quad 1/p + 1/q = 1)
\]

for $C(\delta) = (\delta p)^{-q/p}q^{-1}$, will be frequently used.

Estimating the right-hand side of the inequality (103) we arrive at

\[
|b(u_1(t), z(t), z(t))| \leq \|u_1(t)\|_{L^4} \|z(t)\|_{W^{1,2}} \|z(t)\|_{L^4}
\]

\[
\leq c \|u_1(t)\|_{L^4} \|z(t)\|^{1/2}_{W^{1,2}} \|z(t)\|^{1/2}_{L^2}
\]

\[
\leq \delta \|z(t)\|^2_{W^{1,2}} + C(\delta)\|u_1(t)\|^4_{L^4} \|z(t)\|^2_{L^2}, \quad (105)
\]
Similarly, let us estimate all terms on the right-hand side of (104). Successively

\[
|b(z(t), u_2(t), z(t))| \leq \|u_2(t)\|_{W^{1,2}} \|z(t)\|_{L^2}^2
\]

\[
\leq c \|u_2(t)\|_{W^{1,2}} \|z(t)\|_{W^{1,2}} \|z(t)\|_{L^2}
\]

\[
\leq \delta \|z(t)\|_{W^{1,2}}^2 + C(\delta) \|u_2(t)\|_{W^{1,2}}^2 \|z(t)\|_{L^2}^2
\]

(106)

and

\[
|f(t)\sigma(t), z(t)| \leq \|f(t)\|_{L^2} \|\sigma(t)\|_{L^4(\Omega)} \|z(t)\|_{L^4}
\]

\[
\leq c \|f(t)\|_{L^2} \|\sigma(t)\|_{L^2(\Omega)}^{1/2} \|\sigma(t)\|_{W^{1,2}(\Omega)}^{1/2} \|z(t)\|_{L^2}^{1/2} \|z(t)\|_{W^{1,2}}^{1/2}
\]

\[
\leq \delta \left( \|\sigma(t)\|_{W^{1,2}(\Omega)}^2 + \|z(t)\|_{W^{1,2}}^2 \right) + C(\delta) \|f(t)\|_{L^2}^2 \left( \|\sigma(t)\|_{L^2(\Omega)}^2 + \|z(t)\|_{L^2}^2 \right).
\]

(107)

Now the estimates (103) and (105)–(107) imply

\[
\frac{1}{2} \frac{d}{dt} \|z(t)\|_{W^{1,2}}^2 + \|z(t)\|_{W^{1,2}}^2 \leq \delta \left( \|\sigma(t)\|_{W^{1,2}(\Omega)}^2 + \|z(t)\|_{W^{1,2}}^2 \right)
\]

\[
+ C(\delta) \left( \|f(t)\|_{L^2}^2 + \|u_1(t)\|_{L^4}^4 + \|u_2(t)\|_{W^{1,2}}^4 \right) \|z(t)\|_{L^2}^2
\]

\[
+ C(\delta) \|f(t)\|_{L^2}^2 \|\sigma(t)\|_{L^2(\Omega)}^2.
\]

(108)

Similarly, let us estimate all terms on the right-hand side of (104). Successively

\[
|d(z(t), \theta_1(t), \sigma(t))| \leq \|z(t)\|_{L^4} \|\nabla \theta_1(t)\|_{L^2} \|\sigma(t)\|_{L^4(\Omega)}
\]

\[
\leq c \|\theta_1(t)\|_{W^{1,2}(\Omega)} \|\sigma(t)\|_{L^2(\Omega)}^{1/2} \|\sigma(t)\|_{W^{1,2}(\Omega)}^{1/2} \|z(t)\|_{L^2}^{1/2} \|z(t)\|_{W^{1,2}}^{1/2}
\]

\[
\leq \delta \left( \|\sigma(t)\|_{W^{1,2}(\Omega)}^2 + \|z(t)\|_{W^{1,2}}^2 \right)
\]

\[
+ C(\delta) \|\theta_1(t)\|_{W^{1,2}(\Omega)}^2 \left( \|\sigma(t)\|_{L^2(\Omega)}^2 + \|z(t)\|_{L^2}^2 \right).
\]

(109)

and

\[
|d(u_2(t), \sigma(t), \sigma(t))| \leq \|u_2(t)\|_{L^4} \|\nabla \sigma(t)\|_{L^2} \|\sigma(t)\|_{L^4(\Omega)}
\]

\[
\leq c \|u_2(t)\|_{L^4} \|\sigma(t)\|_{W^{1,2}(\Omega)}^{3/2} \|\sigma(t)\|_{L^2(\Omega)}^{1/2}
\]

\[
\leq \delta \|\sigma(t)\|_{W^{1,2}(\Omega)}^2 + C(\delta) \|u_2(t)\|_{L^4}^4 \|\sigma(t)\|_{L^2(\Omega)}^2.
\]

(110)
For the dissipative terms in (104) we have

\[ |e(z(t), u_1(t), \sigma(t))| \leq c \|z(t)\|_{W^{1,2}} \|u_1(t)\|_{W^{1,4}} \|\sigma(t)\|_{L^4(\Omega)} \]
\[ \leq c \|z(t)\|_{W^{1,2}} \|u_1(t)\|_{W^{1,4}} \|\sigma(t)\|_{L^{1/2}(\Omega)} \|\sigma(t)\|_{L^{1/2}(\Omega)} \]
\[ \leq \delta \|z(t)\|_{W^{1,2}}^2 + C(\delta) \|u_1(t)\|_{W^{1,4}} \|\sigma(t)\|_{W^{1,2}(\Omega)} \|\sigma(t)\|_{L^2(\Omega)} \]
\[ \leq \delta \left( \|z(t)\|_{W^{1,2}}^2 + \|\sigma(t)\|_{W^{1,2}(\Omega)}^2 \right) + C(\delta)^3 \|u_1(t)\|_{W^{1,4}} \|\sigma(t)\|_{L^2(\Omega)}^2 \]  
(111)

and in the same way we deduce the inequality

\[ |e(u_2(t), z(t), \sigma(t))| \leq c \|u_2(t)\|_{W^{1,4}} \|z(t)\|_{W^{1,2}} \|\sigma(t)\|_{L^4(\Omega)} \]
\[ \leq \delta \left( \|z(t)\|_{W^{1,2}}^2 + \|\sigma(t)\|_{W^{1,2}(\Omega)}^2 \right) + C(\delta)^3 \|u_2(t)\|_{W^{1,4}} \|\sigma(t)\|_{L^2(\Omega)}^2 \]  
(112)

The last two terms can be estimated as follows:

\[ |(\sigma(t)f(t) \cdot u_1(t), \sigma(t))_{\Omega}| \leq \|f(t)\|_{L^2} \|u_1(t)\|_{L^\infty} \|\sigma(t)\|_{L^2(\Omega)}^2 \]
\[ \leq c \|f(t)\|_{L^2} \|u_1(t)\|_{L^\infty} \|\sigma(t)\|_{W^{1,2}(\Omega)} \|\sigma(t)\|_{L^2(\Omega)} \]
\[ \leq \delta \|\sigma(t)\|_{W^{1,2}(\Omega)}^2 + C(\delta) \|f(t)\|_{L^2} \|u_1(t)\|_{L^\infty} \|\sigma(t)\|_{L^2(\Omega)}^2 \]  
(113)

and finally

\[ |(\theta_2(t)f(t) \cdot z(t), \sigma(t))_{\Omega}| \leq \|\theta_2(t)\|_{L^\infty(\Omega)} \|f(t)\|_{L^2} \|z(t)\|_{L^4(\Omega)} \|\sigma(t)\|_{L^1(\Omega)} \]
\[ \leq c \|	heta_2(t)\|_{L^\infty(\Omega)} \|f(t)\|_{L^2} \|z(t)\|_{L^4(\Omega)} \|\sigma(t)\|_{W^{1,2}(\Omega)} \|\sigma(t)\|_{L^2(\Omega)} \]
\[ \leq \delta \|z(t)\|_{W^{1,2}} \|\sigma(t)\|_{W^{1,2}(\Omega)} + C(\delta) \|	heta_2(t)\|_{L^\infty(\Omega)} \|f(t)\|_{L^2} \|z(t)\|_{L^2} \|\sigma(t)\|_{L^2(\Omega)} \]
\[ \leq \frac{\delta}{2} \left( \|\sigma(t)\|_{W^{1,2}(\Omega)}^2 + \|z(t)\|_{W^{1,2}}^2 \right) \]
\[ + C(\delta) \|	heta_2(t)\|_{L^\infty(\Omega)}^2 \|f(t)\|_{L^2}^2 \left( \|\sigma(t)\|_{L^2(\Omega)}^2 + \|z(t)\|_{L^2}^2 \right) \]  
(114)

Now the estimates (104) and (109)–(114) imply

\[ \frac{1}{2} \frac{d}{dt} \|\sigma(t)\|_{V^{0,2}}^2 + \|\sigma(t)\|_{V^{0,2}}^2 \leq \delta \left( \|z(t)\|_{W^{1,2}}^2 + \|\sigma(t)\|_{W^{1,2}(\Omega)}^2 \right) \]
\[ + C(\delta) \left( \|\theta_1(t)\|_{W^{1,2}(\Omega)}^2 + \|\theta_2(t)\|_{L^\infty(\Omega)}^2 \|f(t)\|_{L^2}^2 \right) \|z(t)\|_{L^2}^2 \]
\[ + C(\delta) \left( \|\theta_1(t)\|_{W^{1,2}(\Omega)}^2 + \|u_2(t)\|_{L^4}^2 + C(\delta)^2 \|u_1(t)\|_{W^{1,4}} \|\sigma(t)\|_{W^{1,4}} + C(\delta)^2 \|u_2(t)\|_{W^{1,4}} \right) \]
\[ + \|f(t)\|_{L^2}^2 \|u_1(t)\|_{L^\infty}^2 + \|\theta_2(t)\|_{L^\infty(\Omega)}^2 \|f(t)\|_{L^2}^2 \|\sigma(t)\|_{L^2(\Omega)}^2 \]  
(115)
Choosing $\delta$ sufficiently small and summing $\eqref{108}$ and $\eqref{115}$ we conclude that

$$
\frac{d}{dt} \left( \|z(t)\|_{V_{0,2}^u}^2 + \|\sigma(t)\|_{V_{0,2}^\theta}^2 \right) \leq \chi(t) \left( \|z(t)\|_{V_{0,2}^u}^2 + \|\sigma(t)\|_{V_{0,2}^\theta}^2 \right),
$$

\hfill \eqref{116}

where $\chi(t) \in L^1((0,T))$. Now the uniqueness follows from the fact that $z(0) = 0$ and $\sigma(0) = 0$ using the Gronwall lemma.

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