Taking a Detour to Zero: An Alternative Formalization of Functions Beyond PR

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Abstract

There are two well known systems formalizing total recursion beyond primitive recursion (PR), system T by Gödel and system F by Girard and Reynolds. System T defines recursion on typed objects and can construct every function of Heyting arithmetic (HA). System F introduces type variables which can define the recursion of system T. The result is a system as expressive as second-order Heyting arithmetic (HA_2). Though, both are able to express unimaginably fast growing functions, in some applications a more flexible formalism is needed. One such application is CERES cut-elimination for schematic LK-proofs (CERES_s) where the shape of the recursion is important. In this paper we introduce a formalism for fast growing functions without a type theory foundation. The recursion is indexed by ordered sets of natural numbers. We highlight the relationship between our recursion and the Wainer hierarchy to provide an comparison to existing systems. We can show that our formalism expresses the functions expressible using system T. We leave comparison to system F and beyond to future work.

1 Introduction

Primitive recursion dates back to [16, 22] when Dedekind introduced the concept of recursive. After Ackermann developed the function [1] carrying his name, it became obvious that the Dedekind’s definition needed to be distinguished from the general concept of recursion. Dedekind’s definition only handles recursion up to what is now known as primitive recursion. This distinction was solidified by Skolem [20]. Initially, Dedekind’s primitive recursion represented what we know as terminating total recursive functions which are to be distinguished from functions requiring a μ-operator [15] to define. Though, such functions can still be total and terminating, they, in general, represent the class of partial functions, or more specifically computable functions.

To make a finer categorization of the concept of terminating total recursive functions without reliance on the class of partial functions, one needs to make a distinction concerning what objects the recursive operator is defined over. In defining system T [12], Gödel typed the object over which the recursive operator is defined using simple type theory [8]. Thus, instead of producing a number the recursive operator can produce functions. One can think of this as producing f^n(f_0) for some function f and base function f_0 rather than s^n(0) where s(·) is the successor function. This formulation can easily construct the Ackermann function using recursion of type Nat → Nat, where Nat is the type of the natural numbers.

One can go even further by typing objects using a stronger type theory, that is polymorphic lambda calculus. Girard introduced system F [11], which includes quantifiers over types. The
expressive power of system $F$ is beyond that of system $T$. Using system $F$, one can construct inductive data types. It has been shown equivalent to $\text{HA}_2$.

The work presented here does not attempt to raise the bar of expressivity beyond system $F$, but rather attempts to find an elementary solution to the problem of expressing functions beyond primitive recursion, one that does not require iteration of complex type structures. Rather than using typed objects, we develop recursive operators whose iteration is defined over a more complex ordering than the simple total ordering of the natural numbers. We use lexicographically ordered sequences of natural numbers of length $n$. This alone does not provide expressive power, rather it is our interpretation of a recursive step which provides a much more expressive formalization than $\text{PR}$ (See Fig. 1). The key to our definition of a recursive step can be found in the definition of the Ackermann function, that is $A(m+1,0) \Rightarrow A(m,1)$. Rather than decreasing the sum of $m$ and $n$ we decrease the most significant position in the order by 1 and add 1 to a less significant position. Our recursive operators generalize this concept.

What we haven’t mentioned yet is how to get to the zero sequence. We did promise a detour to zero. In Fig. 1 the least significant position loops. That is, the only way to produce an element lower in a lexicographic ordering from the least significant position is to reduce its value. Though, our choice can be generalized, we follow the following procedure, every position can move a 1 to the next less significant position or to the least significant position as a recursive step. Adding more possibilities results in more recursive calls, more complexity, but not necessarily more expressive power.

An unexpected result of this choice is that over sequences of length 2 the set of expressible functions is the same as primitive recursion. Though for sequences of length 3 we can construct the Ackermann function and thus construct the base function of the Wainer hierarchy $f_\omega(n)$.

A direct application of these recursive operators is to CERES type cut-elimination for Proof schemata (CERES$_s$) [5, 10, 21]. In [5, 10] the focus was to get around the problem of cut-elimination in the presence of induction. When an induction rule is present in the sequent calculus, cut elimination on the object level is not possible, but after a metalevel proof transformation the cuts can be eliminated, though, only when the end sequent is strong quantifier free. This results in the loss of the subformula property and the mid-sequent theorem [21]. The CERES$_s$ method [2, 3, 4] allows cut-elimination for certain inductive proofs without the loss of the subformula property [10]. Though, other proof systems have been designed to deal with induction in an elegant way, again, at the cost of the subformula property [17]. Even though keeping the subformula property is beneficial, using the CERES$_s$ method is difficult and the existing system does not promise to work in every case [10]. The main issue is formalizing the resolution refutation of the cut structure’s clause set. The difficulty was highlighted in [7] where a simple mathematical statement required non-trivial analysis in order to perform cut-elimination, though the subformula property was retained and a Herbrand sequent was extracted. More complex statements than the one analysed in [7] seem to require a much more general recursion for defining the refutation of the characteristic clause set [5, 6]. A set of clauses representing the cut structure within a proof. The goal of future work is to use the recursion we define here to develop a more general recursive resolution calculus. If the recursive resolution calculus is expressive enough, we hope it can lead to a completeness proof concerning the refutation of characteristic clause sets of certain inductive proofs, and in doing so, remove one of the major stumbling blocks when using the method. There is already some progress towards this goal [6].

The rest of the paper is organized as follows: In Section 2 we discuss the necessary
literature and concepts needed for understanding this work. In Section 5 we build an intuitive introduction to our recursive operators. In Section 4 we provide a termination argument for the generalized primitive recursion (GPR) hierarchy and show that the functions definable in PR are equivalent to those definable in PR1 which is PR. Also we show that PR3 is more expressive than PR2. In Section 5 Also we provide a method to describe the operator for each PRi and show that for i ≥ 3 the Ackermann function is part of the expressible functions. Finally, we construct a function still definable in PA, but not easily expressible by other formalism. In Section 6 We show that our construction can express the Wainer hierarchy. Finally, in Section 7 we discuss future work and open problems.

2 Preliminaries

For the rest of this paper we will use the symbol \( \mathbb{N} \) to represent the natural numbers including zero, that is \( \{0, 1, 2, 3, 4, \cdots \} \). We assume the standard ordering of the natural numbers. A sequence of natural numbers \( s \) of length \( n \geq 1 \) is an object \( s \in \mathbb{N}^n = \mathbb{N} \times \cdots \times \mathbb{N} \). We write sequences as \( (x_0, \cdots, x_{n-1}) \) and order them lexicographically where the most significant position is on the left side and least significant position is on the right. That is \( (x_0, \cdots, x_{n-1}) < (y_0, \cdots, y_{n-1}) \) iff there is a position \( x_i < y_i \) and for every \( 0 \leq j < i \), \( x_j = y_j \). When possible, without causing confusion, we will abbreviate a list of arguments \( m_1, \cdots, m_k \) as \( \overline{m}_k \).

2.1 Primitive Recursion

Though ubiquitously known, we provide a short introduction to primitive recursion to give a precise definition of our usage of the phrase. We use non-standard notation for the sets of primitive recursively expressible total functions \( f : \mathbb{N} \to \mathbb{N} \) that correspond to our extended notation. Given \( i \geq 0 \) primitive recursion level \( i \) PRi is the set of expressible functions using the first \( i \) operators and the basic function. The more general operators will be defined in Section 5.

► Definition 1. We define primitive recursion level 0 PR0 as all total functions \( f : \mathbb{N} \to \mathbb{N} \) definable using the following basic functions:

1. (Zero function): for all \( n \), \( 0(n) = 0 \).
2. (Successor function): for all \( n \), \( s(n) = n + 1 \).
3. (Projection functions): \( p^k_i(m_0, \cdots, m_{k-1}) \) for \( k \geq 1 \) and \( 0 \leq i < k \).
4. (Composition level 0, \( C_0 \)): Let \( g \in \text{PR}_0 \) be of arity \( i \geq 1 \) and \( h_0, \cdots, h_{i-1} \in \text{PR}_0 \) of arity \( k \geq 1 \), then we can construct \( f \in \text{PR}_0 \), \( f = g(h_0(\overline{m}_k), \cdots, h_{i-1}(\overline{m}_k)) \)

Note that composition can be defined for any level of recursion and we will assume that PRi allows \( C_i \) composition for the rest of this paper.

► Definition 2. We define primitive recursion level 1 PR1 as all total functions \( f : \mathbb{N} \to \mathbb{N} \) definable using the following operations:

1. (Basic functions): Any \( \text{PR}_0 \) function
2. (Primitive recursion): Let \( g \in \text{PR}_1 \) be arity \( k \) and \( h \in \text{PR}_1 \) be arity \( k + 2 \) then we can define \( f \) of arity \( k + 1 \) using primitive recursion as follows: \( f(0, \overline{m}_k) = g(\overline{m}_k) \) and \( f(s(x), \overline{m}_k) = h(x, \overline{m}_k, f(x, \overline{m}_k)) \)

For more details concerning what is expressible, see [9], the source of the above definitions.
2.2 Ordinals and Cantor Normal Form

Both the Grzegorczyk hierarchy [14] and Wainer hierarchy [23] use an ordinal notation to denote growth speed of functions. The Grzegorczyk hierarchy is concerned with finite ordinals, i.e., $\alpha < \omega$ where $\omega$ corresponds to the first countable limit ordinal. It is related to the loop hierarchy [18, 19]. The Wainer hierarchy is concerned with ordinals $\alpha \leq \varepsilon_0$, the proof-theoretic strength of $\text{PA}$ [21]. This seems like an arbitrary stopping point being that $\varepsilon_0$ is, with respect to the set of countable ordinals, quite a small ordinal, but beyond $\varepsilon_0$ the relationship between large countable ordinals and the first limit is more complex. That is, the concept of a fundamental sequence is hard to uniquely define. Before defining a fundamental sequences of a limit ordinal we need to introduce Cantor normal form base-$\omega$.

**Definition 3** (Cantor Normal Form Base-$\omega$). Let $k \in \mathbb{N}$, $B_0, \ldots, B_k$ be ordinals such that $B_0 \geq B_1 \geq \cdots \geq B_k \geq 0$. Then for an ordinal $\alpha < \varepsilon_0$ the Cantor normal form base-$\omega$ is as follows:

$$\alpha = \omega^{B_0} + \omega^{B_1} + \cdots + \omega^{B_k}.$$  

Note that $\omega^0 = 1$ and $\omega^1 = \omega$, thus the above definition defines all finite ordinals as well. Also, we abbreviate $\omega$ stacks using the following function: $\omega(n+1) = \omega^n(n)$ and $\omega(0) = 1$. Now we can give a recursive definition of the fundamental sequence for ordinals $\alpha < \varepsilon_0$:

**Definition 4** (Fundamental Sequence). Let $n \in \mathbb{N}$, and $\omega < \alpha < \varepsilon_0$ an ordinal in Cantor normal form base-$\omega$. Then the $n^{th}$ member of the fundamental sequence $\alpha[n]$ of $\alpha$ is defined recursively as follows:

$$\alpha[n] = \begin{cases} 
\beta + m & \alpha = \beta + m \land \omega \leq \beta \\
(m-1) \cdot \omega + n & \alpha = m \cdot \omega \\
n \cdot \omega^{(m-1)} & \alpha = \omega^m \\
\omega^\beta + \omega^\beta_1 + \cdots + \omega^\beta_k[n] & \alpha = \omega^{\beta_0} + \omega^{\beta_1} + \cdots + \omega^{\beta_k} 
\end{cases}$$

Now we can define the Wainer hierarchy which includes the Grzegorczyk hierarchy and is sometimes referred to the extended Grzegorczyk hierarchy.

**Definition 5** (Grzegorczyk hierarchy). Let $n, m \in \mathbb{N}$. We define a function $f_m(n)$ of the Grzegorczyk hierarchy as follows:

$$f_m(n) = \begin{cases} 
n + 1 & m = 0 \\
 f_{m-1}(n) = f_{m-1}(f_{m-1}(\cdots f_{m-1}(n)\cdots)) & m > 0 
\end{cases}$$

**Definition 6** (Wainer hierarchy). Let $n, m \in \mathbb{N}$, and $\omega \leq \alpha, \beta < \varepsilon_0$ be ordinals in Cantor normal form base-$\omega$. We define a function $f_\alpha(n)$ of the Wainer hierarchy as follows:

$$f_\alpha(n) = \begin{cases} 
f_{\beta + m}(n) = f_{\beta + m}(f_{\beta + m}(\cdots f_{\beta + m}(n)\cdots)) & \alpha = \beta + (m + 1) \\
f_{\alpha[n]}(n) & \text{otherwise} 
\end{cases}$$

By definition, $f_\alpha(n)$ is equivalent to $A(n, n)$, where $A(\cdot, \cdot)$ is the Ackermann function. Thus our construction of the Wainer hierarchy in GPR starts with formalizing $A(\cdot, \cdot)$.
2.3 System T

Concerning simple type theory and system T, we refrain from going into the details of their formalism due to space constraints. For a good introduction to the subjects we refer the reader to “Proofs and Types” by Girard [11]. Nonetheless, we discuss the relationship between system T and the Wainer hierarchy being that it is an integral part of this work. System T can construct any function definable by Heyting arithmetic (HA); also, by negative translation, any function definable in \( PA \) [13]. For every function definable by \( PA \) there is a function of the Wainer hierarchy which grows faster than it. Thus, the Wainer hierarchy is a way to deconstruct the expressible functions of system T into recursive classes. It turns out that \( f_{co}(n) \) grows faster than the Goodstein sequence which cannot be defined in \( PA \), and thus, cannot be defined in system T. This fact is important for analysis of the mutual Ackermann function. Essentially, the Wainer hierarchy is categorization of expressible functions of system T.

3 Generalized Primitive Recursion

Unlike other methods of generalizing recursion beyond \( PR \), we change the ordering over which the recursion is defined rather than the objects over which the recursion is defined. To the best of our knowledge this approach has not be attempted. Our generalization defines recursion over sequences of natural numbers of length \( n \). Rather than treating each position in the sequence as an individual primitive recursion, we allow a position of higher significance to shift a 1 to the next position or the position of lowest significance (position \( n-1 \)). This can be interpreted as a kind of recursion over the length of the sequence. A graphical representation of the recursion for \( PR_2 \), that is recursion over sequences of length 2, can be found in Fig. 1 in the middle of the top row. The graph in Fig. 1 on the left side of the top row is for \( PR_1 \). The precise meaning of this graph concerning the sequence \((x, y)\) of \( PR_2 \) is as follows (remember significance is left to right):

\[
(x + 1, y) \Rightarrow (x, y + 1) \\
(x, y + 1) \Rightarrow (x, y)
\]

Notice that removing \((x + 1, y) \Rightarrow (x, y + 1)\) results in a system with no rule defined on position 0. Such a system would be equivalent to the system for \( PR_1 \). Essentially, \( PR_2 \) is \( PR_1 \) with an additional rule for sequences of length 2. For \( PR_i \), \( i \geq 3 \), two rules are added to the previous system to get the next system. \( PR_2 \) is an exception because the next position and the last position are the same for position 0. This informally implies the result that \( PR_2 \) and \( PR_1 \) express the same functions.

Putting this into context, let us consider \( PR_3 \), the next position of position zero is not the last position. Thus, we add two rules to \( PR_2 \) for position zero to get \( PR_3 \):

\[
(x + 1, y, z) \Rightarrow (x, y + 1, z) \\
(x, y + 1, z) \Rightarrow (x, y, z + 1) \\
(x + 1, y, z + 1) \Rightarrow (x, y, z + 1) \\
(x, y, z + 1) \Rightarrow (x, y, z)
\]

A graphical representation can be found in Fig. 1 on the right side of the top row. For every \( i \geq 3 \) we can extend the system of transitions for \( PR_i \), to \( PR_{i+1} \) by adding two rules to position zero and shifting the other rules up by one position. It follows that there are \( 2 \cdot (i - 1) \) transition rules for \( PR_i \). A graphical representation can be found in Fig. 1 bottom. We will refer to the transition rules of \( PR_i \) as \( T_i \). An application of some rule in \( T_i \) to a sequence \( s \) of length \( n \) will be written as \( s \Rightarrow_{T_i} s' \).

Using sequences of numbers rather than a single number for recursion in ambiguity concerning the meaning of stepcase and basecase. Of course we have a basecase, the sequence
of all zeros, but can all of the other cases be called the stepcase? Notice that the transition rules require the value of the significant position to be non-zero. Once a position is zero it cannot be the significant position of a transition rule. Thus, the number of applicable rules drops as positions are reduced to zero, i.e. less recursive calls. Essentially, this implies that we need to differentiate between the various configurations of a sequence, more specifically, which positions are zero and which are not. We will refer to a sequence as the stepcase if no position is zero, a soft-basecase if at least one position is zero and at least one position is nonzero, and a hard-basecase if all positions are zero. Notice that this implies that PR\(_i\) will have \(2^i\) cases. So, PR\(_1\) has two cases (like PR), PR\(_2\) has four cases, and PR\(_3\) has 8 cases. This can make defining PR\(_i\) a bit cumbersome, but it can be elegantly organized.

### 4 Termination, PR Equivalence, and the Ackermann Function

Unlike system T, termination of our formalization is almost free because our transition rules always step down the lexicographical ordering of length \(n\) sequences of natural numbers. Like primitive recursion we only allow the stepcase or soft-basecases to use the predecessors of the numbers in a given number sequence and the hard-basecase can only use the non-recursive arguments provided to the function.

\[\text{Theorem 7. For all } i \geq 1 \text{ and for all } s \in \mathbb{N}^i \text{ there exists a } j \geq 0 \text{ such that} \]

\[s = s_0 \rightarrow_{T_i} s_1 \rightarrow_{T_i} \cdots \rightarrow_{T_i} s_j\]

and \(s_j\) is the zero sequence of length \(i\).

\[\text{Proof.} \quad \text{We prove this statement by induction on } i \text{ and induction on the lexicographical ordering of } \mathbb{N}^i. \text{ When } i \text{ is 1 there is one transition rule, the rule of PR. The theorem obviously holds in this case. Now let us assume the theorem holds for all } j \leq i \text{ and show it holds for } i + 1. \text{ Let } s \in \mathbb{N}^{i+1} \text{ such that the most significant position is zero. Then there is a bijective mapping } f \text{ of the transition rules applicable to } s, T_{i+1}, \text{ and } T_i \text{ such that if } t \in T_{i+1} \text{ applies to a position in } s \text{ and adds to the next position, } f(t) \text{ applies to the previous}\]

\[\]
position of a sequence \( s' \in \mathbb{N}^i \) and adds to the next position. Same can be stated for adding to the last position. Also there is a sequence \( s' \in \mathbb{N}^i \) such that the \( i^{\text{th}} \) position of \( s \) is equal to the \( (i - 1)^{\text{th}} \) position of \( s' \). Thus, by the induction hypothesis we know that \( s' \) goes to the zero sequence and the above shows that \( s \) must go to the zero sequence as well. Now let us assume that the theorem holds when the most significant position of \( s \in \mathbb{N}^{i+1} \) is \( l \leq k \) and show it holds for \( k + 1 \). This simply requires application of one of the rules of \( T_{i+1} \) applicable to the most significant position. Thus, we have shown the theorem holds by induction.

\[ \text{Corollary 8.} \quad \text{For all } i \geq 1, \text{ for all } s \in \mathbb{N}^i, \text{ and } \]

\[ s \rightarrow_{T_i} s_1 \rightarrow_{T_i} \cdots \rightarrow_{T_i} s_j \]

for all valid applications of \( j \geq 0 \) transition rules from \( T_{i+1} \) there exists a further \( k \geq 0 \) valid applications of transition rules from \( T_{i+1} \) such that the sequence \( s_{j+k} \) is the zero sequence of length \( i \).

**Proof.** Follows from Thm. \[ \Box \]

\[ \text{Corollary 9.} \quad \text{For all } i \geq 1, \text{ for all } s \in \mathbb{N}^i \text{ such that the maximum value in } s \text{ is } n \geq 0, \text{ then there exist } j(n) \geq 0 \text{ such that any valid sequence of applications of transition rules from } T_i \text{ will reach the zero sequence of length } i \text{ in less than } j(n) \geq 0 \text{ steps.} \]

Corollary 9 implies that no matter the order of the recursive calls we will always reach the zero sequence, essentially termination. Before moving on, we want to point of that termination is based on the lexicographical ordering. If a function \( f \) with recursive variables \( x_1, \ldots, x_n \) calls itself in a soft-basecase non-recursively using values \( x'_1, \ldots, x'_n \) it must be the case that \( (x_1, \ldots, x_n) > (x'_1, \ldots, x'_n) \), otherwised an infinite loop might occur. Such issues do not occur in \( \text{PR} \). We use this fact in Section 6. Now we move on to defining \( \text{PR}_2 \) and \( \text{PR}_3 \), proving the expressive equivalence of \( \text{PR}_1 \) and \( \text{PR}_2 \). Also, we formalize the Ackermann function in \( \text{PR}_3 \).

### 4.1 Equivalence to Primitive Recursion

We number the cases in our definition of \( \text{PR}_2 \) so that the definition will be consistent with the general definition of \( \text{PR}_i \).

\[ \text{Definition 10.} \quad \text{Let } x, y, k \geq 0, \text{ } g^3, g^2, g^1, g^0, \overline{n}_k = n_1, \ldots, n_k \in \text{PR}_2, \text{ where the arity of the } g^i \text{'s is } k + 4, k + 2, k + 2, \text{ and } k \text{ respectively. Then we define the function } f \in \text{PR}_2 \left[ g^3, g^2, g^1, g^0 \right] (x, y) \text{ of arity } k + 2 \text{ as follows:} \]

\[
\begin{align*}
f(\overline{n}_k, x + 1, y + 1) & \Rightarrow g^3(\overline{n}_k, x, y, f(\overline{n}_k, x, y + 2), f(\overline{n}_k, x + 1, y)) \\
f(\overline{n}_k, x + 1, 0) & \Rightarrow g^2(\overline{n}_k, x, f(\overline{n}_k, x, 1)) \\
f(\overline{n}_k, 0, y + 1) & \Rightarrow g^1(\overline{n}_k, y, f(\overline{n}_k, 0, y)) \\
f(\overline{n}_k, 0, 0) & \Rightarrow g^0(\overline{n}_k)
\end{align*}
\]

By proving the following two theorems we show expressive equivalence to \( \text{PR}_1 \).

\[ \text{Theorem 11.} \quad \text{For every function } f \in \text{PR}_1 \text{ there exists a function } f' \in \text{PR}_2 \text{ such that for all } x \geq 0 \text{ there exists } y, z \geq 0, \text{ where } x = y + z, \text{ such that } f(\overline{n}_k, x) = f'(\overline{n}_k, y, z). \]
Proof. If \( f \) is of the form \( f(\overline{n}_k, x + 1) \Rightarrow g(\overline{n}_k, x, f(\overline{n}_k, x)) \) \( f(\overline{n}_k, 0) \Rightarrow h(\overline{n}_k) \) Then we define the function \( f' \) as follows:

\[
\begin{align*}
  f(\overline{n}_k, x + 1, y + 1) & \Rightarrow f(\overline{n}_k, x, y + 2) \\
  f(\overline{n}_k, 0, y + 1) & \Rightarrow g(\overline{n}_k, y, f(\overline{n}_k, 0, y)) \\
  f(\overline{n}_k, x + 1, 0) & \Rightarrow f(\overline{n}_k, x, 1) \\
  f(\overline{n}_k, 0, 0) & \Rightarrow h(\overline{n}_k)
\end{align*}
\]

\[\blacktriangleright\]

**Theorem 12.** For every function \( f \in \text{PR}_2 \) there exists functions \( A, B \in \text{PR}_1 \) such that for all \( x, y \geq 0 \), \( f(\overline{n}_k, x, y) = A(\overline{n}_k, x, y) \).

Proof. The following two \( \text{PR}_1 \) functions are sufficient:

\[
\begin{align*}
  A(\overline{n}_k, x + 1, y) & \Rightarrow \text{If}(y > 0) \text{ Then } g^3(\overline{n}_k, x, y, A(\overline{n}_k, x, y + 1), B(\overline{n}_k, y - 1, x + 1)) \\
  & \quad \text{Else } g^4(\overline{n}_k, x, A(\overline{n}_k, x, 1)) \\
  A(\overline{n}_k, 0, y) & \Rightarrow \text{If}(y > 0) \text{ Then } g^4(\overline{n}_k, y, B(\overline{n}_k, y - 1, 0)) \\
  & \quad \text{Else } g^4(\overline{n}_k) \\
  B(\overline{n}_k, y + 1, x) & \Rightarrow \text{If}(x > 0) \text{ Then } g^3(\overline{n}_k, x, y, A(\overline{n}_k, x - 1, y + 2), B(\overline{n}_k, y, x)) \\
  & \quad \text{Else } g^4(\overline{n}_k, x, B(\overline{n}_k, y, 0)) \\
  B(\overline{n}_k, 0, x) & \Rightarrow \text{If}(x > 0) \text{ Then } g^4(\overline{n}_k, x, A(\overline{n}_k, x - 1, 1)) \\
  & \quad \text{Else } g^4(\overline{n}_k)
\end{align*}
\]

\[\blacktriangleright\]

**Corollary 13.** \( \text{PR}_1 \) and \( \text{PR}_2 \) express the same functions.

### 4.2 Constructing the Ackermann Function using GPR

Now we define \( \text{PR}_3 \) and provide a construction of the Ackermann function.

**Definition 14.** Let \( x, y, z, k \geq 0 \), \( g^7, g^8, g^5, g^3, g^2, g, g^0, \overline{n}_k = n_1, \cdots, n_k \in \text{PR}_3 \). Then we define the function \( f \in \text{PR}_3 \) \( [g^7, g^8, g^5, g^3, g^2, g, g^0] \) \( (x, y, z) \) as follows:

\[
\begin{align*}
  f(\overline{n}_k, x + 1, y + 1, z + 1) & \Rightarrow g^7(\overline{n}_k, x, y, z, f(\overline{n}_k, x, y + 1, z + 2), f(\overline{n}_k, x, y + 2, z + 1), \\
  f(\overline{n}_k, x + 1, y + 1, 0) & \Rightarrow g^8(\overline{n}_k, x, y, f(\overline{n}_k, x, y + 1, 1), f(\overline{n}_k, x, y + 2, 0), \\
  f(\overline{n}_k, x + 1, 0, z + 1) & \Rightarrow g^5(\overline{n}_k, x, z, f(\overline{n}_k, x, 0, z + 2), f(\overline{n}_k, x, 1, z + 1), \\
  f(\overline{n}_k, x + 1, 0, 0) & \Rightarrow g^4(\overline{n}_k, x, f(\overline{n}_k, x, 0, 1), f(\overline{n}_k, x, 1, 0)) \\
  f(\overline{n}_k, 0, y + 1, z + 1) & \Rightarrow g^3(\overline{n}_k, y, z, f(\overline{n}_k, 0, y, z + 2), f(\overline{n}_k, 0, y + 1, z)) \\
  f(\overline{n}_k, 0, y + 1, 0) & \Rightarrow g^2(\overline{n}_k, y, f(\overline{n}_k, 0, y, 1)) \\
  f(\overline{n}_k, 0, 0, z + 1) & \Rightarrow g^1(\overline{n}_k, z, f(\overline{n}_k, 0, 0, z)) \\
  f(\overline{n}_k, 0, 0, 0) & \Rightarrow g^0(\overline{n}_k)
\end{align*}
\]

Not every soft-basecase is needed to define the Ackermann. We distinguish between variables which must be greater than zero and those which do not have to be by adding a 1 in the latter case.
Theorem 15. There exists \( f \in \text{PR}_3 \) such that \( f \notin \text{PR} \).

Proof. We prove this statement by construction the Ackermann function using the \( \text{PR}_3 \) recursive operator.

1. \( A(x + 1, y + 1, z) \Rightarrow A(x, A(x + 1, y, z + 1), z) \)
2. \( A(x + 1, 0, z) \Rightarrow A(x, 1, z) \)
3. \( A(0, y, z) \Rightarrow y + 1 \)

Notice that 1. is both \( g^7 \) and \( g^8 \), 2. is \( g^4 \) and \( g^5 \), and 3. is the rest.

5 General Formulation and the Mutual Ackermann Function

The higher in the \( \text{GPR} \) hierarchy the more recursive variables are needed. Thus, the hierarchy gets increasingly more complex to present. One way of presenting \( \text{PR}_i \) for \( i > 3 \) is to use functions instead of sequences. The value of the function at 0 is the most significant. Let us consider the set \( \mathbb{F} \) of all total recursive functions \( f: \mathbb{N} \rightarrow \mathbb{N} \). We can define the \( \text{PR}_i \) transition rules as a set operators \( \mathcal{O}_k^j : \mathbb{F} \rightarrow \mathbb{F} \) for \( 1 \leq j < i \) and \( k \in \{ j + 1, i - 1 \} \), where \( j \) is the position which we are subtracting from and \( k \) is the position we are adding to. We will refer to the function \( f_0 \in \mathbb{F} \) as the zero function, that is \( \forall j \in \mathbb{N} (f_0(j) = 0) \), and \( F_0 \subseteq \mathbb{F} \) as the zero functions with respect to \( i \), that is, \( \forall f_0^i \in F_0^i \forall j < i (f_0^i(j)) = 0 \). The operators \( \mathcal{O}_k^j \) are defined as follows:

\[
f^{\mathcal{O}_k^j} = \begin{cases} f' & f(j) \neq 0 \land j \neq i - 1 \\ f' & f'(k) = f(k + 1), \\ & \text{and } \forall l (l \neq j \land l \neq k \rightarrow f'(l) = f(l)) \\ f' & f'(i - 1) = f(i - 1) - 1, \\ & \text{and } \forall l (l \neq i - 1 \rightarrow f'(l) = f(l)) \\ f^{\mathcal{O}_{i-1}^j} & \exists r \in \{1, \cdots, i - 1\} (f(r) > 0) \\ f & \text{otherwise} \\
\end{cases}
\]

The third component of the definition allows every soft-basecase to have the same arity by repeating arguments where there would not normally be one. The following operator constructs a function which provides the recursive variable values to be used by the various cases.

\[
f \downarrow_i (j) = \begin{cases} f(j) - 1 & (f(j) > 0 \land 0 \leq j < i) \\ f(j) & \text{otherwise} \end{cases}
\]

The last operator we need to define is one allowing us to pick which case we are in given the current function:

\[
\Sigma_i(f, j + 1) = \sigma(f(j + 1)) \cdot 2^{(i - j)} + \Sigma_j(f, j) \quad \Sigma_i(f, 0) = \sigma(f(i)) \cdot 2^{(i - 1)}
\]

Where \( \sigma(x) = 1 \) when \( x > 0 \) and 0 otherwise. Now we can define the entire \( \text{GPR} \) hierarchy.

Definition 16. Let \( i \geq 1 \), \( f \in \mathbb{F}, f_0^i \in F_0^i \), such that for \( j \geq i\), \( f(j) = f_0^i(j) \), and \( g^{a-1}, \cdots, g^1, g^0, \pi_k = n_1, \cdots, n_k \in \text{PR}_i \). Then \( h \in \text{PR}_i \left[ g^{a-1}, \cdots, g^1, g^0 \right] (f) \) is

\[
h(\pi_k, f) \Rightarrow g^{\Sigma_i(f, i + 1)}(\pi_k, f \downarrow_i, h(\pi_k, f^{\mathcal{O}_k^i}), h(\pi_k, f^{\mathcal{O}_k^1}), \cdots, h(\pi_k, f^{\mathcal{O}_{i-1}^1}))
\]

\[
h(\pi_k, f_0^i) \Rightarrow g^0(\pi_k)
\]
Using Definition [16] we can prove that every PR, for i ≥ 3 is more expressive than PR.

**Theorem 17.** For all i ≥ 3, there exists f ∈ PR, such that f /∈ PR.

**Proof.** We can prove this theorem by showing that the Ackermann function is expressible in PR, for i ≥ 3. An additional function is needed to define the Ackermann function using the above definition.

\[
f'_i(x) = \begin{cases} 
0 & 0 ≤ x < n - 3 \\
\downarrow (i - 3) & x = i - 3 \\
A(f_{i-1}) & x = i - 2 \\
\downarrow (i - 1) & x = i - 1 
\end{cases}
\]

The Ackermann function is as follows:

\[
A(f) = \begin{cases} 
A(f_{i-1}) & 7 < Σ_i(f, i - 1) \\
A(f_i) & 6 ≤ Σ_i(f, i - 1) ≤ 7 \\
A(f_{i-2}) & 4 ≤ Σ_i(f, i - 1) ≤ 5 \\
f(i - 2) + 1 & 2 ≤ Σ_i(f, i - 1) ≤ 3 \\
1 & 0 ≤ Σ_i(f, i - 1) ≤ 1 
\end{cases}
\]

higher levels of the GPR hierarchy allow for interesting function definitions not easily expressible in other formalisms. For example we define the mutual Ackermann function in PR5.

\[
A(x + 1, y + 1, z + 1, w + 1, r) ⇒ A(x, A(x + 1, y + 1, z + 1, w + 1, r + 1), z, A(x + 1, y + 1, z + 1, w, r + 1), r) \\
A(x + 1, y + 1, z + 1, w + 1, r) ⇒ A(0, A(x + 1, y + 1, z + 1, w + 1, r), z, A(x + 1, 0, z + 1, w, r + 1), r) \\
A(x + 1, 0, y + 1, z + 1, w + 1, r) ⇒ A(0, y, z, A(0, y + 1, z + 1, w, r + 1), r) \\
A(x + 1, y, z + 1, w + 1, r) ⇒ A(x, A(x + 1, y + 1, z + 1, w + 1, r + 1), z, A(x + 1, y + 1, z, r), r) \\
A(x + 1, y + 1, 0, w + 1, r) ⇒ A(x + 1, y, 0, z + 1, w + 1, r, 1), r) \\
A(x + 1, 0, y + 1, z + 1, w + 1, r) ⇒ A(x, A(x + 1, y, 0, z + 1, w + 1, r + 1), 0, w, r) \\
A(x + 1, 0, z + 1, 0, r) ⇒ A(x, A(x + 1, y + 1, 0, w + 1, r), z, A(x + 1, 0, z, r, 1) \\
A(x + 1, 0, 0, w + 1, r) ⇒ A(x + 1, 0, 0, w, r) + w \\
A(x + 1, 0, 0, 0, r) ⇒ A(0, x, 1, r) + y \\
A(0, y, 0, w, r) ⇒ y + w + 1
\]

There should be 32 cases but the above abbreviation suffices as it did with the Ackermann function. System T can construct every function of HA [12] and through translation every function of PA [13]. It is not clear if our mechanism goes beyond the expressive power of System T. In the case of the mutual Ackermann function a similar construction as found in Thm. [12] is able to formalize it. We do show in Section [6] that the Wainer hierarchy is constructible and thus all functions expressible by system T, but this is only one direction of the proof of equivalence. A note on the growth rate, A(2, 1, 0, 0, 0) = A(2, 1, 0) = 5 and A(0, 0, 1, 1, 0) = A(1, 1, 0) = 3, but A(2, 1, 1, 1, 0) = 25556.

**6 Relationship to the Wainer Hierarchy**

Now we discuss the relationship to the Wainer hierarchy. As we can see, PR3 can construct the base of the hierarchy, the Ackermann function. This leaves the question, how much of the hierarchy can be constructed using PR3.

**Theorem 18.** The Wainer hierarchy up to \(f_{n^2}(n)\) is constructable in PR4.
Proof. We can start with a construction using primitive recursion, \( \text{PR}_1 \), and then add addition recursive variables. Consider the function \( A'(0, x, 1, y) \) (note \( A \) is the Ackermann function):

- \( A'(0, x, 1, y) = A(A'(0, x, y), A'(0, x, y), 0) \)
- \( A'(0, 0, y) = A(y, y, 0) \)

This \( \text{PR}_1 \) function represents the function \( f_{\omega + 1}(n) \) in the Wainer hierarchy, i.e. \( A'(0, n, n) = f_{\omega + 1}(n) \). To construct \( f_{\omega + 2}(n) \) we just need to call the above function \( n \) times:

- \( A'(1, x + 1, y) = A(0, A'(1, x, y), A'(1, x, y)) \)
- \( A'(1, 0, y) = A'(0, y, y) \)

What we illustrate above are special cases of the \( \text{PR}_2 \) function defined below:

- \( A'(x + 1, y + 1, z) = A'(x, A'(x + 1, y, z), A'(x + 1, y, z)) \)
- \( A'(x + 1, 0, z) = A'(x, x, z) \)
- \( A'(0, y + 1, z) = A'(A'(0, y, z), A'(0, y, z), 0) \)
- \( A'(0, 0, z) = A(z, z, z) \)

This function is exactly the function \( f_{2\omega}(n) = f_{\omega + n}(n) \). Rather than constructing the various limit ordinals \( f_{3\omega}(n), f_{4\omega}(n), \) and so on, we can construct \( f_{\omega^2}(n) \) directly in \( \text{PR}_3 \) as follows:

- \( A''(w + 1, x + 1, y + 1, z) = A''(w, A''(w + 1, x + 1, y, z), A''(w + 1, x + 1, y, z), A''(w + 1, x + 1, y, z)) \)
- \( A''(w + 1, 0, 0, z) = A''(w + 1, x, z, z) \)
- \( A''(0, y + 1, z) = A''(A''(0, y, z), A''(0, y, z), A''(0, y, z)) \)
- \( A''(0, 0, z) = A''(A''(0, 0, z), A''(0, 0, z), A''(0, 0, z)) \)
- \( A''(0, 0, 0, z) = A''(z, z, z) \)

Again, we can skip the limit ordinals \( f_{3\omega}(n), f_{4\omega}(n), \cdots \) by replacing \( A'' \) with a \( \text{PR}_4 \) function of similar structure \( A'' \), and replacing \( A' \) in \( A'' \) with \( A'' \). The resulting function is equivalent to \( f_{\omega^2}(n) \).

This construction procedure can be carried out for higher and higher functions of the Wainer hierarchy as long as the function has a corresponding ordinal \( \alpha \), where \( \alpha < \varepsilon_0 \). The problem with \( \varepsilon_0 \) is that the fundamental sequence of \( \varepsilon_0 \) is constructed using variable number of recursive variables rather than using a variable value of the recursive variables. This would mean that the definition of \( f_{\varepsilon_0} \) would require infinitely many functions. Thus, the construction we use in Theorem [18] would not be enough for formalization of \( \varepsilon_0 \). But it can be used to construct the Wainer hierarchy below \( f_{\varepsilon_0} \). Our construction is not sufficient for the defining of \( f_{\varepsilon_0} \), but another method might allow for the construction of this function.

To construct the functions corresponding to the ordinals up \( \varepsilon_0 \) we need to find the number of parameters corresponding to the destruction of the ordinal into a sequence of \( \omega \). For example, it takes for parameters to properly deconstruct \( \omega^2 \). Notice that the number of parameters needed for the deconstruction is not one to one with the number of times \( \omega \) shows up in an exponent. Though it is a function of ordinal complex. Though, for our work, we do not really care how many parameters are needed by rather that there exists a number of parameters which can be used to construct the function. This should be clearly possible from Theorem [18] for all ordinals \( \alpha < \varepsilon_0 \). The following partial construction illustrates the
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idea where $\alpha > \beta$.

\[ A^\alpha(x_1 + 1, x_2 + 1, \ldots, x_i + 1, y) \Rightarrow A^\alpha(x_1, A^\alpha(x_1 + 1, x_1, y), \ldots, A^\alpha(x_1 + 1, \ldots, x_i, y)) \]
\[ A^\alpha(x_1 + 1, \ldots, x_{i-1} + 1, 0, y) \Rightarrow A^\alpha(x_1 + 1, \ldots, x_{i-1}, y, y) \]
\[ \vdots \]
\[ A^\alpha(0, \ldots, 0, x_1 + 1, y) \Rightarrow A^\beta(A^\alpha(0, \ldots, 0, x_1, y), \ldots, A^\alpha(0, \ldots, 0, x_i, y)) \]
\[ A^\alpha(0, 0, 0, 0, y) \Rightarrow A^\beta(y, \ldots, y) \]

> Corollary 19. There exists $3 \leq i \in \mathbb{N}$ such that the Wainer hierarchy up to $f_{\omega^\omega}(n)$ is constructable in $\text{PR}_i$.

Proof (sketch). Repeat the procedure of Theorem 18 defining the transition from $\omega^2$ to $\omega^\omega$ until $\omega^\omega$ is constructed.

> Theorem 20. There exists $3 \leq i \in \mathbb{N}$ such that the Wainer hierarchy up to $f_{\omega^{(j)}}(n)$ is constructable in $\text{PR}_i$ for $j \geq 0$.

Proof (sketch). By the above construction.

> Theorem 21. The Wainer hierarchy up to $f_{\varepsilon_0}(n)$ is constructable using GPR.

Proof. By Theorem 20.

> Corollary 22. The GPR hierarchy can express the total recursive functions of PA.

Proof. By Theorem 21.

7 Future Work and Conclusions

In this paper we present an alternative formalism for total recursive functions beyond primitive recursion. This formalism is based on a lexicographical ordering of sequences of ordered natural numbers. We define a specific method of iterating the lexicographical ordering allowing the formalizing of the Ackermann function, a well known function beyond primitive recursion. The formalization of the Ackermann function is done in a weak fragment of the GPR hierarchy. To investigate the hierarchy’s expressive power in greater detail we compare it with the Wainer hierarchy up to $\varepsilon_0$. All total recursive functions which can be formalized in Peano arithmetic fall into a fragment of the Wainer hierarchy. Thus, proving that all classes of the Wainer hierarchy can be expressed using generalized primitive recursion shows that the concept is as expressive as Peano arithmetic and thus, as expressive as system $\text{T}$. Though, it is an open question whether there are functions which can be expressed using generalized primitive recursion which cannot be expressed in system $\text{T}$. The mutual Ackermann function is an example of an interesting function whose complexity is not easily derivable. Thus, it is still open whether generalized primitive recursion goes beyond system $\text{T}$ and Peano arithmetic. The mutual Ackermann function can be generalized to any $\text{PR}_{2+i+1}$ for $i \geq 2$, thus we know it is not a special case. Comparison to more expressive systems like system $\text{F}$ is left to future work. Also, we plan to construct operators similar to those of system $\text{T}$ using the concepts defined in this paper. That is a sequence recursive operators over simple type theory similar to those of system $\text{T}$. In terms of application, we plan to use a type of recursion definable in $\text{PR}_3$ to develop a recursive resolution calculus for use in schematic cut-elimination. This has been done to some extent, but without taking advantage of the properties of the GPR hierarchy [6].
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