Algorithms for computing the sets of stochastic matrices with predefined properties, based on automaton models

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Abstract. The formed sets of ergodic stochastic matrices are focused on solving the problem of classifying probabilistic automaton models by certain criteria/parameters of similarities or differences in the structures of ergodic stochastic matrices using the methods of applied multivariate mathematical statistics. The algorithms developed allow forming a variety of the sets of stochastic matrices through changing the transition and random entry function in a probabilistic automaton. Transition function of the autonomous probabilistic automaton allows using the probabilistic automaton operation algorithm proposed to form the sets of ergodic stochastic matrices that differ in their potencies and structures, based on implementing state set permutations with repetitions and changing the random input variable probability distributions. Defining various autonomous deterministic automaton output functions, we can use the algorithm developed to form the sets of ergodic stochastic matrices that differ in their potencies, with a specified limiting vector, based on implementing the permutations of a set of output letters with repetitions. We are also presenting the evaluations of the potencies of the sets of ergodic stochastic matrices with rational elements, represented by autonomous probabilistic automata under the given constraints.

1. Introduction
Automaton probabilistic models (APMs) [1–4] defined by stochastic matrices are widely used to describe complex systems.

Properties of stochastic matrices depend on their structure defined as a relationship of the values of its elements to each other. Therefore, a naturally arising APM analysis problem is that of classifying stochastic matrices by certain criteria/parameters of similarities or differences in their structures [5–7]. The approach to solving this problem, considered in [5, 7], is based on the methods of applied multivariate mathematical statistics, using software tools. To implement this approach, we should form various predefined sets of stochastic matrices, i. e., matrices with a predefined structure, such as triangular, block, or circulant matrices with the required entropy, etc. [5–7], and matrices having predefined asymptotic properties, such as ergodicity, and a specified limiting vector [8]. In [5, 7], this problem is being solved based on partitioning a unit segment into a predefined number of randomintervals using a pseudo-random sequence generator and the normalization of the elements of the stochastic matrix. However, such approach does not allow obtaining the required variety of the sets of stochastic matrices having the predefined structures and asymptotic properties.

The goal of the presented work is to develop the algorithms of synthesizing the sets of ergodic stochastic matrices with predefined structures and specified limiting vectors, based on probabilistic...
and deterministic models represented by autonomous automata, and evaluate the potencies of the sets of stochastic matrices obtained, depending on the dimensionality of the automaton models specified.

2. Probabilistic Automaton Model

Let us consider the following autonomous probabilistic automaton (APA) [2].

\[
ABA = \left( S, \hat{X}, \Delta(x, s) = s, \pi_0 \right),
\]

where \( S = \{ s_i \}, \ i = 0, n-1 \) is the set of states; \( \hat{X} \) is a discrete random value \( \hat{X} = \left( x_0, x_1, \ldots, x_{i-1} \right) \), \( 0 < p_i < 1, \ \sum_{i=0}^{i-1} p_i = 1; \ \Delta(x, s) \) is the APA transition function (1) defined by the automaton table sized \( n \times l \), both arbitrarily and by algorithms [9, 10], based on partitioning a given SM (denoted as \( P_3 \)) sized \( n \times n \). Parameters \( n \) and \( l \) define the dimensionality of the automaton model. Let us represent \( P_3 \) as a partition [1, 9], in turn, represented as:

\[
P_3 = \sum_{k=0}^{l-1} p_k M(k),
\]

where \( p_k, k = 0, l-1 \) are elements of \( \bar{P} \), \( M(k), k = 0, l-1 \) is a simple matrix sized \( n \times n \). and corresponding with symbol \( x_k \); \( l \) satisfies the relationship [1,9]

\[
l \leq n^2 - n + 1.
\]

Note. Evaluation of the potency of the automaton table set sized \( n \times l \) to define functions \( \Delta(x, s) \) is defined based on the function defining the number of the permutations of set \( S \) with repetitions [11]:

\[
L_4 = \sum_{i_1, i_2, \ldots, i_n} \frac{k!}{i_1! i_2! \ldots i_n!},
\]

where \( i_1, i_2, \ldots, i_n \) are natural numbers, such that \( \sum_{j=1}^{n} i_j = n \cdot l = k \).

Law (SM \( P_3 \) sized \( n \times n \)) of Markov chain obtained in automaton (1) can be uniquely computed as (2) in accordance with [12] by the predefined elements (1): \( \hat{X} \) and \( \Delta(x, s) \).

Algorithm of computing the set of SMs represented as \( P_3 \) (hereinafter, Algorithm 1) based on relationships (2) and (3) can be represented as the following three stages:

1) Defining by the given \( \Delta(x, s) \) simple matrices \( M(x_k), k = 0, l-1 \).

2) Multiplying the \( M(x_k) \) obtained by the relevant elements \( p_k, k = 0, l-1 \), of the specified vector \( \bar{P} \).

3) Computing \( P_3 \) in accordance with (2).

Specifying for (1) various \( \Delta(x, s) \) represented as the relevant automaton tables at a fixed stochastic vector \( \bar{P} \), we can use Algorithm 1 to obtain various sets (subclasses) of SMs \( P_3 \) having different structures, such as ergodic quasi-triangular ones, ergodic block ones, etc. [5,7].

Changing in (1) stochastic vector \( \bar{P} \) at the fixed \( \Delta(x, s) \), we will obtain a set of SMs \( P_3 \) from a fixed subclass defined by function \( \Delta(x, s) \). Let us evaluate the total number of the possible variety of SMs based on model (1) by Algorithm 1. Let us introduce some limitations. Suppose for vector represented as \( \bar{P} \), the elements are computed by formula:
\[ p_i = a_i / N, \quad \sum_{i=0}^{l-1} a_i = N, \]  

where \( a_i \) are non-negative integers, and \( N \) is power of \( l \). Then \( \bar{P} \) belongs to the probability distribution class, the potency of which is evaluated by the value of [2].

\[ L_2 = C_{i,N-1}^{N-1}. \]  

Value \( L_2 \) defines the potency of the set of SMs obtained by changing \( \bar{P} \). The problem of constructing a set of these vectors with the potency of (6) can be solved in accordance with [2].

The validity of the next assumption follows from relationships (4)-(6)

\textbf{Statement.} Where the potency of the set of automaton tables for defining \( \Delta(x, s) \) in automaton (1) is defined by the value of (4), then the potency of the set of stochastic matrices \( P_s \) obtained on automaton model (1) and satisfying conditions (5) and (6) is evaluated by the product of \( L_1 \cdot L_2 \).

\section{3. Deterministic Automaton Model}

Let us consider a deterministic autonomous (DA) automaton with output [13].

\[ DA = (S, Y, \delta, \lambda), \]  

where \( S = \{ s_0, s_1, \ldots, s_{N-1} \} \) is the set of states; \( Y = \{ y_0, y_1, \ldots, y_{m-1} \} \) is the output alphabet; \( \delta : S \to S \) is the transition function; and \( \lambda : S \to Y \) is the output function displaying \( S \) on \( Y \). The automaton transition trajectory forms an \( N \)-long sequence covering all the states. We will define function \( \lambda(s) \) by partitioning set \( S \) into \( m \) non-overlapping subsets \( \{ A_0, A_1, \ldots, A_{m-1} \} \), the potencies of which are equal to \( a_i \geq 1, \sum_{i=0}^{m-1} a_i = N \), respectively. Let us denote these subsets, respectively, with the symbols of set \( Y = \{ y_0, y_1, \ldots, y_{m-1} \} \). Parameters \( N, m \) defined the dimensionality of DA.

Over the execution of the cycle, an \( N \)-long sequence of output letters is formed at the automaton output. Let us assign to this sequence the matrix of relative frequencies, \( P' = p'_{ij} = a_{ij} / a_i \) sized \( m \times m \), where \( a_{ij} \) is the number of occurrences of letter \( y_j \) in the \( N \)-long sequence, \( a_i \geq 1 \), \( a_{ij} \) is the number of occurrences of the pair of adjacent letters \( y_i y_j \), \( i, j = 0, m - 1 \) (we will consider that \( y_{N} \) is followed by \( y_0 \)). \( P' \) is an ergodic SM having the properties of [13]:

1) Elements \( p'_{ij} = a_{ij} / \sum_{j=0}^{m-1} a_{ij} \),

\[ \sum_{j=0}^{m-1} a_{ij} = \sum_{j=0}^{m-1} a_{ji} = a_i \geq 0 \quad \text{and} \quad \sum_{i=0}^{m-1} a_{ij} = N, \ i, j = 0, m - 1; \]  

2) The limiting stochastic vector of matrix \( P'_N \) is equal to

\[ \bar{P}_N = (a_0 / N \ a_1 / N \ \ldots \ a_{m-1} / N). \]  

We have defined automaton (7) and stochastic vector (9) corresponding with the specified output function, \( \lambda : S \to Y \). We have also developed an algorithm of using automaton (7) to obtain matrix \( P' \) that has the limiting stochastic vector equaling to the predefined vector \( \bar{P}_N \) represented as (9) (hereinafter, Algorithm 2).

At each step of the \( N \)-long cycle, Algorithm 2 executes the following three procedures:

1) Transition function implements the transition to a new state, \( s \in S \).
2) By the obtained value of \( s \), the output function returns the relevant value of \( y \in Y \).
3) The value of the relevant element of \( p'_{ij} = a_j / a_i \) is computed according to (8).

The limiting stochastic vector of this matrix is equal to the predefined vector, \( \overline{P_N} \), represented as (9), which follows from property (8).

Based on defining various vectors \( \overline{P_N} = (a_0/N \ a_1/N \ ... \ a_{m-1}/N) \), satisfying limitation (5) Algorithm 2 considered above allows obtaining a set of the relevant \( P' \) matrices with a potency that can be evaluated with the potency of the probability distribution class represented as \( \overline{P_N} \), determined by the value of [2]

\[
L_3 = c_{m^rN-1}^{N-1}.
\]  

(10)

Defining in automaton (7) various transition functions implementing cyclic mappings, with the fixed output function (and, accordingly, with the fixed stochastic vector \( \overline{P_N} \)), we can obtain a set of metrics \( P' = p'_{ij} = a_j / a_i \) having, in accordance with properties (8) and (9), the same limiting stochastic vector that is equal to the predefined vector \( \overline{P_N} \). Particularly, defining in automaton (7) various transition functions implemented based on different LSRs, i.e., \( M \)-sequence generators with period \( N = 2^n - 1 \), with different characteristic primitive \( n \)-power polynomials \( F(x) \) at the fixed output function (at the fixed stochastic vector \( \overline{P_N} \)), we can obtain the set of metrics \( P' \) with the potency of \( Q = \phi(2^n - 1)/n \) [14], where \( \phi( ) \) is the Euler function, having the same limiting stochastic vector \( \overline{P_N} \). True is

**Statement 2.** Evaluation of a set of ergodic stochastic matrices satisfying limitations (8) and (9) and obtained on automaton model (7) is determined by the product of \( L_3 \cdot Q \).

4. **Examples of Using the Algorithms**

**Example 1.** Suppose there is a discrete random value, \( \hat{X} = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ p_0 & p_1 & p_2 & p_3 \end{pmatrix} \), defined as to satisfy relationship (3); the set of states, \( S = \{s_1, s_2, s_3, s_4\} \), and transition function \( \Delta(x, s) \) are represented as Automaton Table 1.

Matrices \( M (x_k) \), \( k = 0, 1, \ldots, 4 \), and \( P_3 \) obtained using **Algorithm 1** are shown in figure 1.

| \( s_0 \) | \( s_1 \) | \( s_2 \) | \( s_3 \) | \( s_0 \) |
|-----------|-----------|-----------|-----------|-----------|
| \( s_1 \) | \( s_3 \) | \( s_1 \) | \( s_2 \) | \( s_3 \) |
| \( s_2 \) | \( s_0 \) | \( s_3 \) | \( s_1 \) | \( s_2 \) |
| \( s_3 \) | \( s_3 \) | \( s_2 \) | \( s_2 \) | \( s_3 \) |

**Table 1.** Transition function \( \Delta(x, s) \).
\[ P_0 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + P_1 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + P_2 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + P_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \]

\[ = \begin{pmatrix} P_3 & P_0 & P_1 & P_2 \\ 0 & P_1 & P_2 & P_0 + P_3 \\ P_0 & P_2 & P_3 & P_1 \\ 0 & 0 & P_1 + P_2 & P_0 + P_3 \end{pmatrix} = P_S. \]

**Figure 1.** Using Algorithm 1 to compute an ergodic stochastic matrix represented as \( P_S \).

**Example 2.** Suppose for the data from Example 1, the automaton table is represented as shown in Table 2:

| \( \Delta(x,s) \) | \( x_0 \) | \( x_1 \) | \( x_2 \) | \( x_3 \) |
|------------------|--------|--------|--------|--------|
| \( s_0 \)       | \( s_0 \) | \( s_0 \) | \( s_0 \) | \( s_0 \) |
| \( s_1 \)       | \( s_0 \) | \( s_0 \) | \( s_2 \) | \( s_1 \) |
| \( s_2 \)       | \( s_2 \) | \( s_1 \) | \( s_3 \) | \( s_2 \) |
| \( s_3 \)       | \( s_3 \) | \( s_2 \) | \( s_2 \) | \( s_3 \) |

The relevant stochastic matrix is shown in figure 2.

\[ P_0 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + P_1 \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + P_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + P_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \]

\[ = \begin{pmatrix} p_1 + p_2 + p_3 & p_0 & 0 & 0 \\ p_0 + p_1 & p_3 & p_2 & 0 \\ 0 & p_1 & p_0 + p_3 & p_2 \\ 0 & 0 & p_1 + p_2 & p_0 + p_3 \end{pmatrix} = P_S. \]

**Figure 2.** Computing the ergodic block matrix represented as \( P_S \).

**Example 3.** Constructing a set of ergodic stochastic matrices with a specified limiting vector by defining various output functions in (7).

Suppose in automaton (7), \( S = \{s_1, s_2, ..., s_{15}\} \), \( N = 15 \). Set \( Y = \{y_1, y_2, y_3\} \). Vector \( \overline{P_N} = (6/15, 5/15, 4/15) \). According to Algorithm 2, matrix \( P' = \begin{pmatrix} 2/6 & 4/6 & 0 \\ 3/5 & 0 & 2/5 \\ 1/4 & 1/4 & 2/4 \end{pmatrix} \) is computed, which has the properties of (8) and (9) and the specified limiting vector \( \overline{P_N} \).
Using the data from Example 3 and having defined another vector, \( \bar{P}_N = (2/15, 9/15, 4/15) \), we will obtain the following matrix: 
\[
P' = \begin{pmatrix}
1/2 & 1/2 & 0 \\
2/9 & 5/9 & 2/9 \\
0 & 2/4 & 2/4
\end{pmatrix}.
\]

5. Conclusion

Algorithms have been developed aimed at constructing stochastic matrices with a predefined structure, based on the autonomous probabilistic automaton represented as (1) and having the predefined asymptotic properties based on an autonomous deterministic automaton with output (7). Variety of the sets of stochastic matrices can be obtained due to changeability in automata (1) and (7) of the transition functions and the output function in automaton (7) and random entry function in automaton (1). We have presented the potency evaluations of the sets of stochastic matrices being obtained, depending on the dimensionality of the predefined automaton models. For the presented implementations of automaton (1), the evaluation is defined by the value of \( L_1 \cdot L_2 \), while it is defined by the value of \( L_5 \cdot Q \) for automaton (7).

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