Finite $N$ Analysis of Matrix Models for $n$-Ising Spin on a Random Surface

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Abstract

The saddle point equation described by the eigenvalues of $N \times N$ Hermitian matrices is analyzed for a finite $N$ case and the scaling relation for the large $N$ is considered. The critical point and the critical exponents of matrix model are obtained by the finite $N$ scaling. One matrix model and two-matrix model are studied in detail. Small $N$ behavior for $n$-Ising model on a random surface is investigated.

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1. Introduction

The large N limit of matrix models has attracted theoretical interests in various fields including random matrix model, gauge theory and phase transition. Recently, it becomes clearer that the matrix model is closely related to the string field theory. Ising model on a random surface is described by a two-matrix model, in which the planar surface is generated in the large N limit and the up spin and down spin assigned by two different matrices. In this case, an external magnetic field on Ising spins can be introduced and the model is exactly solvable [1]. The critical exponents of the specific heat, the susceptibility and the spontaneous magnetization are obtained exactly, and they agree with the conformal field theoretical result [2]. Their values are different from the well known Onsager solution of 2D Ising model on the regular lattice. The difference is caused by the random surface, which is interpreted as 2D quantum gravity. The Ising spin has $c=1/2$ central charge due to fermionic character, and consequently Ising model on a random surface describes $c=1/2$ matter field coupled to 2D quantum gravity.

The large N limit of matrix model gives planar Feynman diagrams in the perturbation of the coupling constant [3]. This planar diagrams gives us a random surface of the fixed genus. On each vertex of the descretized random surface, one can put different n-species Ising spins, which are independent on different species and only interact through random statistical average [4,5]. This n-Ising model on a random surface is particularly interesting since the matter field has central charge $c=n/2$ which can be larger than one. Unfortunately, this model has not been solved yet for $n > 2$ although it is described by $2^n$ matrix model. Only numerical analyses exist in this case [4,5]. To investigate the large N limit of the matrix model, several techniques have been introduced. The saddle point equation [3], Schwinger Dyson equation, the orthogonal polynomial method [6], the double scaling method [7] and the renormalization group method [8] are considered. In this paper, we pursue the saddle point equation method for matrix models, since it is simple and it gives clear interpretation for the transition. Although the large N limit should be taken for applying the saddle point method, it may be interesting to investigate the finite N case, and to understand the mechanism of the phase transition together with a finite N scaling behavior. We hope that finite N saddle point analysis becomes complementary one to the series expansion of [4]. It is expected that our method also work for n-Ising case, and a small N calculation is presented for this as a preliminary study.
2. One Matrix Model

The one matrix model has a following Hamiltonian,

\[ H = \frac{1}{2} \text{Tr} M^2 + \frac{g}{N} \text{Tr} M^4 \] (1)

where \( M \) is a \( N \times N \) Hermitian matrix. Denoting the eigenvalues of this matrix by \( \lambda \) which are all real, the partition function is given by

\[ Z = \int d\lambda \prod_{i<j}(\lambda_i - \lambda_j)^2 \exp\left[-\frac{1}{2} \sum \lambda_i^2 - \frac{g}{N} \sum \lambda_i^4\right]. \] (2)

The large \( N \) limit of this partition function is evaluated by the saddle point equation of \( \lambda \). Exponentiating the measure, and differentiating the exponent by \( \lambda_i \) we obtain the saddle point equation.

\[-\lambda_i - \frac{4g}{N} \lambda_i^3 + \sum \frac{2}{\lambda_i - \lambda_j} = 0.\] (3)

There appears a critical value of \( g \), beyond which there is no real eigenvalue of \( \lambda \). The free energy is expanded in the power series of \( g \),

\[ F = \sum c_k g^k \] (4)

where \( c_k \) behaves for large \( k \)

\[ c_k \sim A^k k^{-3+\gamma_{st}}. \] (5)

The series of \( F \) is a convergent series for the large \( N \) limit, and \( A \) is \(-1/g_c\). For the one matrix model of (1), \( g_c \) is \(-1/48\) \cite{3}. The exponent \( \gamma_{st} \) is called as a string susceptibility and becomes \(-1/2\) for one matrix model.

Although the saddle point equation is justified for the large \( N \) limit we apply it for a finite \( N \) matrix model. We approximate the large \( N \) Riemann-Hilbert integral equation by a finite discrete eigenvalue equation. It is easy to find the critical value of \( g_c \) for such finite \( N \) saddle point equation and increasing number of \( N \), \( g_c \) is expected to become \( g_c \) of the large \( N \) limit. Our aim is to develop the method for obtaining the critical coupling \( g_c \) or cosmological constant \( A \) of (5), instead of calculating the partition function which requires the integration of the eigenvalue \( \lambda \). Our method is similar to Lipatov large order calculation, but for finite \( N \) the series of (4) becomes asymptotic and there appear \( k \)-factorial coefficient instead of (5). Therefore, Lipatov method \cite{9} is not appropriate for our problem. We will show that this finite \( N \) saddle point method is practically useful for obtaining the critical value \( g_c \) and the string susceptibility \( \gamma_{st} \).
For example in the case of \( N=2 \), we have only two eigenvalue \( \lambda_1 \) and \( \lambda_2 \). The effective Hamiltonian after exponentiating the measure part,

\[
H_{\text{eff}} = \frac{1}{2}(\lambda_1^2 + \lambda_2^2) + \frac{g}{2}(\lambda_1^4 + \lambda_2^4) - \ln(\lambda_1 - \lambda_2)^2
\]

The saddle point equation is obtained by the differentiating this Hamiltonian by \( \lambda_1 \) and \( \lambda_2 \). There is a solution that two eigenvalues are symmetric around zero, \( \lambda_1 = -\lambda_2 \). Thus the equations reduce to one equation,

\[
\lambda + 2g\lambda^3 - \frac{1}{\lambda} = 0
\]

This quadratic equation gives \( \lambda = (-1 \pm (1 + 8g)^{1/2})/4g \) and \( g_c = -1/8 \). The physical solution appears in + sign, which becomes finite for \( g \to 0 \). Two solutions degenerates at \( g_c \) and beyond this, there is no real eigenvalue \( \lambda \). The eigenvalue should be real since a matrix \( M \) is Hermitian.

The obtained critical value \( g_c = -1/8 \) is far from -1/48, but increasing \( N \), we see that the result approaches to the correct one smoothly. We see later that the root singularity near \( g_c \) found in \( N=2 \) case is a correct answer for \( N = \infty \). For \( N=3 \), considering \( \lambda_1 = -\lambda_3 \), \( \lambda_2 = 0 \) solution for three eigenvalues, we obtain \( g_c = -1/16 \). When \( N \) is even number, we take a solution \( \lambda_1 = -\lambda_N \), \( \lambda_2 = -\lambda_{N-1},..., \) and for \( N \) odd, we have \( \lambda_1 = -\lambda_N,...,\lambda_{N+1/2} = 0 \). The critical value of \( g_c \) is easily obtained numerically by the investigation of the largest eigenvalue \( \lambda_1 \). The obtained critical value \( g_c \) is shown in table 1. They are indeed approaching to -1/48.

Since the finite \( N \) scaling is expected, we discuss the critical exponent \( \nu \) defined by

\[
A_N = A_\infty + \frac{c}{N^\nu}
\]

where \( A_N = -1/g_c \) for a finite \( N \), and \( A_\infty = 48 \). The exponent \( \nu \) is scaling exponent defined by

\[
F \sim (g - g_c)^{2-\gamma_{\text{st}}}(N^\nu(g - g_c))
\]

Denoting the difference between \( A_\infty \) and \( A_N \) by \( D_N \)

\[
D_N = A_\infty - A_N \sim N^{-\nu}
\]

we have a ratio \( R_N = D_N/D_{N-1} \sim ((N - 1)/N)^\nu \sim 1 - \nu/N \) which is plotted in Fig. 1. The asymptotic coefficient of 1/N is estimated as \( \nu \) is 0.8 which is precisely same as the exact value of the pure gravity case with \( c = 0 \).

It is somehow remarkable that without calculating the free energy, we obtain the correct values of \( A \) and the string susceptibility through a finite \( N \) saddle point equation up to order \( N=8 \). The string susceptibility \( \gamma_{\text{st}} \) is related to \( \nu \) by

\[
\gamma_{\text{st}} + \frac{2}{\nu} = 2
\]
and we have $\gamma_{st} = -1/2$ since $\nu = 4/5$.

It is also interesting to study how the largest eigenvalue $\lambda_1$ behaves near $g_c$ for finite $N$, and to see the eigenvalue density behaves at $g_c$. It is known that the density of the eigenvalue, which obeys usually the semi-circle law, becomes singular at $g_c$. The density $\rho$ vanishes as

$$\rho \sim (\lambda - \lambda_c)^{\gamma_{st} - 1/2}.$$  \hspace{1cm} (12)

We find that the histogram of the density of eigenvalues for $N = 8$ agrees with this behavior. The density of eigenvalue at the critical value $g_c$ vanishes in general as $\rho \sim (\lambda - \lambda_c)^{-1/\gamma_{st} - 1/2}$.

The scaling between $(\lambda - \lambda_c)$ and $(g_c - g)$ exists [10], and it is given by

$$\lambda - \lambda_c \sim (g_c - g)^{-\gamma_{st}}.$$  \hspace{1cm} (13)

Since $\gamma_{st} = -1/2$ for one matrix model, this scaling relation is satified apparently in $N=2$ case (7) as a solution of a quadratic equation. For the large value of $N$, also this root singularity is preserved and we checked it up to $N=8$.

3. One Matrix Model with $(\text{Tr}M^2)^2$ Interaction

It is straightforward to apply our method to one matrix model with $(\text{Tr}M^2)^2$ interaction,

$$H = \frac{1}{2} \text{Tr}M^2 + \frac{g}{N} \text{Tr}M^4 + \frac{g'}{N^2} (\text{Tr}M^2)^2.$$  \hspace{1cm} (14)

where $g'$ is an additional coupling constant. When $g = 0$, this model becomes equivalent to $O(N)$ vector model with $\gamma_{st} = 1/2$ and $-1/g_c' = 16$ [11]. Recently, it was found that there is a critical point at $g = -3/256$ and $g' = -9/256$, where the string susceptibility has a positive value $\gamma_{st} = 1/3$ [12]. It is interesting to investigate this model by our saddle point equation, since $\gamma_{st} = 1/3$ is positive and clearly (13) can not be applied. The calculation is very easy and we have evaluated the finite $N$ saddle point equation and obtained the critical values up to order $N=7$. The results of the evaluation for the critical values of $g_c$ and $g_c'$ are given in Fig. 2. The lines of finite $N$ solutions smoothly converge to $n = \infty$ solution as expected. Therefore, our finite $N$ analysis works well for this model.

The scaling relation of (10) is investigated near the critical point $-1/g_c = 3/256$ and $-1/g_c' = 9/256$. For fixed $-1/g_c' = 9/256$, we evaluated $g_c$ from the finite $N$ saddle point equation. The critical value of $g_c$ is determined such that there is
no real solution of the eigenvalue beyond $g_c$. In table 2, the obtained value of $g_c$ is represented. Using the same ratio method $R_N = D_N/D_{N-1}$, we estimated the scaling exponent $\nu$ in (10). We obtained for this critical point $\nu = 6/5$. Assuming (11) is still valid in this case, we obtain precisely $\gamma_{st} = 1/3$.

We have also verified that at a fixed $g' = -1/16$, near $g_c = 0$, the ratio method gives $\nu = 4/3$ which leads to $\gamma_{st} = 1/2$ by (11). This point at $g = 0$ and $g' = -1/16$ corresponds to the vector O(N) model and it is known that this model has $\gamma_{st} = 1/2$ [12].

It is unexpected result that we have a correct scaling exponent $\nu$ by the finite N saddle point equation. Usually the scaling relation of (9) and the exponent $\nu$ are derived by the double scaling limit [7], based upon the orthogonal polynomial analysis for the free energy. We have restricted our investigation only on the saddle point equation.

4. Two Matrix Model

Two matrix model represents Ising model on a random surface and it is given by

$$H = \frac{1}{2} \text{Tr}(M_1^2 + M_2^2) - a \text{Tr}M_1M_2 + \frac{g}{N} \text{Tr}(M_1^4 + M_2^4)$$ (15)

where $a$ is a coupling constant related to the nearest neighbour spin interaction $J$ divided by the temperature $kT$, $\beta = J/kT$,.

$$a = \exp(-2\beta)$$ (16)

The first and the third term of the Hamiltonian are written by the eigenvalues $\lambda$ and $\xi$ of the matrix $M_1$ and $M_2$. For the two matrix, the Hermitian matrix is diagonalized by the unitary matrix and it is possible to integrate this unitary matrix. Then, partition function is written only by the eigenvalues with Haar measure [8],

$$\Xi = C \int \Pi_i d\lambda_i a^{-\frac{N(N-1)}{2}} \Pi_{i<j}(\lambda_i - \lambda_j)(\xi_i - \xi_j) \exp(-V_0 + a \sum \lambda_i \xi_i))$$ (17)

where $V_0$ is the first and the third term in (15),

$$V_0 = \frac{1}{2} \sum (\lambda_i^2 + \xi_i^2) + \frac{g}{N} \sum (\lambda_i^4 + \xi_i^4)$$ (18)

For $N=2$, after exponentiating the measure part, we have an effective Hamiltonian,

$$H_{eff} = (\lambda_1^2 + \xi_1^2) + g(\lambda_1^4 + \xi_1^4) - \ln(\frac{\lambda_1 \xi_1}{2} \sinh(2a\lambda_1 \xi_1))$$ (19)
where we use $\lambda_1 = -\lambda_2$, $\xi_1 = -\xi_2$. We dropped the irrelevant term which is vanishing in the integration of eigenvalues before the exponentiation. When we take the solution that these eigenvalues are symmetric, $\lambda_1 = \xi_1$, we have

$$\lambda_1 + 2g\lambda_1^3 - \frac{1}{2\lambda_1} - \frac{a\lambda_1}{\tanh(2a\lambda_1^2)} = 0$$

(20)

The critical value of $g_c$ is obtained from this equation. There is no real solution beyond $g_c$. We obtain for $a=0$, $g_c = -1/8$ and $-(1-a^2)^2/g_c = 16$ for $a = 1$. The result that the value of $(1-a^2)^2/g_c$ at $a=1$ becomes twice of the value at $a=0$ is consistent with the observation in the perturbation of $g$ [4]. In Fig. 3, the value of $g_c$ is plotted as a function of $a$. Also the large $N$ exact value of $g_c$ is given; they consist of two solutions [1]: the low temperature phase and the high temperature phase respectively given by

$$-(1-a^2)^2/g_c = \frac{48(1-a^2)^2}{1-\frac{8}{3}a^2}$$

(21)

$$-(1-a^2)^2/g_c = \frac{18(1+\sqrt{a})^2(1+a)^2}{\sqrt{a}(2+\sqrt{a})}$$

(22)

Taking the useful analogy that the partition function of (17) is a grand canonical partition function with identification of $g$ as the exponential of chemical potential $e^\mu$, and that the canonical partition function is $A = -1/g_c$ itself, we have the free energy of the Ising spin on a random surface as

$$F(a) = -\ln A(a)$$

(23)

For the two-matrix model, there is a critical value of $a$, where the spontaneous magnetization vanishes. The phase described for $a > 1/4$ corresponds to the disorder phase (22) and the phase for $a < 1/4$ is a low temperature ordered phase (21).

The magnetic field $B$ for Ising spin can be introduced by the change of the coupling $g$ in $V_0$ as

$$V_0(B) = \frac{1}{2} \text{Tr}(M_1^2 + M_2^2) + \frac{ge^B}{N} \text{Tr}M_1^4 + \frac{ge^{-B}}{N} \text{Tr}M_2^4$$

(24)

At zero temperature $a=0$, the free energy $F(a, B)$ and the magnetization $M$ are

$$F(a, B) = \ln g - B$$

(25)

$$M = -dF/dB = 1$$

(26)

For a finite magnetic field $B$, the saddle point equation gives the asymmetric solution, $\lambda_i \neq \xi_i$. The low temperature phase, i.e. the symmetry breaking phase
also is described by the asymmetric solution. The asymmetric $\lambda \neq \xi$ solution is obtained easily, and $g_c$ is plotted by the dotted line in Fig. 3. Expanding (17) for small $a$, we have up to order $a^2$,

$$\frac{1}{g_c} = \frac{8}{1 - \frac{8}{3}a^2}$$

(27)

Although it diverges at $a = 1/2$ when the higher order is included, instead of $a^2 = 3/8$, up to order $a^2$ the behavior is similar to the exact solution of (21) which shows the divergence at $a^2 = 3/8$.

The asymmetric solution has a larger free energy than the symmetric one, and thus there is no symmetry breaking for $N=2$. The difference appears at order $a^4$. However for $a < 1/4$, the difference is very small as shown in Fig. 3, and the magnetization evaluated for a finite B shows effectively that there is a phase transition. We notice that it is necessary to subtract $N(N-1)/N$ terms in the expansion of $a$ of $\exp(a \sum \lambda_i \xi_i)$. Otherwise, the saddle point solution gives a wrong answer specially for small $a$. It is related to the factor of $a^{-N(N-1)/2}$ which is divergent for $a \to 0$. Applying the saddle point equation for the two matrix model is subtle and already discussed in [13]. Since the first few terms in the expansion of $a$ have no contribution in the integral up to order $a^{N(N-1)/2}$, we safely replace $\exp(a \sum \lambda_i \xi_i)$ term by

$$\sum_{k=N(N-1)/2+1} a^k (\sum \lambda_i \xi_i)^k / \Gamma(k+1)$$

(28)

For $N=5$ and $N=7$ with the expression of (28), the symmetric solution $\lambda_i = \xi_i$ gives the similar curves as the exact solution of the high temperature phase of (22). For small $a$, they indicate the divergence in the large $N$ limit same as the correct expression of (22). Our (28) may be not sufficient for deriving the correct saddle point equation, since it also has terms which do not contribute to the integral. Indeed for even number of $N$, the curve of $-1/g_c$ of the symmetric solution becomes flat for small $a$.

Since the critical value $-1/g_c$ should be same as one matrix model for $a = 0$, and the low temperature phase has a continuous curve starting at $a = 0$, if we obtain the high temperature curve which gives divergence at $a = 0$, it concludes that there is a phase transition.

For $N=7$ case, we observe that the largest eigenvalue $\lambda_1$ has a maximum value at certain value of $a = 0.4$, and below this value $\lambda_1$ starts to decrease. This point $a$ seems to be a critical point. This point is greater than the point where the value of $-1/g_c$ becomes minimum. The situation is similar to the exact solution shown in Fig. 3.
More precise small $a$ expansion using (17) is possible. Noting that the measure $J$ is written by Vandermonde determinant in (17), we have invariance under cyclic exchange of eigenvalues and we have antisymmetric exchange between two different eigenvalue. The nonvanishing term in integration of (28) at order $a^k$ takes a following form due to these symmetries

$$\lambda_1^{l_1} \lambda_2^{l_2} ... \lambda_N^{l_N}$$

(29)

where $l_1 + ... + l_N = k$, and $l_i > l_j$ for $i < j$. Using these nonvanishing terms in a small $a$ expansion, we obtain for $N=2$ as

$$- \frac{1}{g_c} = \frac{8}{1 - \frac{8}{3}a^2} + O(a^4)$$

(30)

This result coincides with the previous one (27). For $N=3$, we find

$$- \frac{1}{g_c} = \frac{16}{1 - 3a^2} + O(a^4)$$

(31)

Since the value $-(1-a^2)^2/g_c$ at $a = 1$ is twice of the value at $a = 0$ [4], we apply Padé approximation of this quantity based upon the small $a$ expansion. Before making Padé analysis, we check the validity of this approximation using the exact expression of (21). Up to order $a^2$, we have

$$- \frac{(1-a^2)^2}{g_c} = 48(1 + \frac{2}{3}a^2)$$

$$= 48 \frac{1 + b_1a^2}{1 + c_1a^2},$$

(32)

where $b_1$ and $c_1$ are determined with the condition that at $a=1$ we have $(1+b_1a^2)/(1+c_1a^2) = 2$. We get $b_1 = 1/3$ and $c_1 = -1/3$. This [1,1] Padé is crude and it gives 50.04 for $a = 1/4$ while the exact value at $a = 1/4$ is 50.63.

Next order, up to order $a^4$, [2,1] Padé approximation becomes

$$- \frac{(1-a^2)^2}{g_c} = 48 \frac{(1 + 8a^2 + \frac{23}{3}a^4)}{(1 + \frac{23}{3}a^2)}$$

(33)

This [2,1] Padé gives 50.36 for $a = 1/4$, and the result is improved. The difference from the exact value is 0.5 percent. Thus we see that Padé approximation is effective. Moreover, the second derivative of the logarithm of (33) by $a$ shows the maximum at $a = 0.24$, which agrees with the exact critical value at $a = 1/4$. This means that we have a method of estimation of $a_c$ as a specific heat peak.
Using the results of (30) and (31), we have Padé approximation for $N = 2$ and $N = 3$ as

\[
- \frac{(1 - a^2)^2}{g_c} = 8 \frac{1 + \frac{1}{3}a^2}{1 - \frac{1}{3}a^2} \quad (N = 2) \tag{34}
\]

\[
= 16(1 + a^2) \quad (N = 3) \tag{35}
\]

they become 8.34 and 17.0 at $a = 1/4$ for $N=2$ and $N=3$, respectively.

The next order of $O(a^4)$ is calculated for $N = 2$ and approximated by $[1,2]$ Padé,

\[
- \frac{(1 - a^2)^2}{g_c} = 8(1 + \frac{2}{3}a^2 - \frac{1}{15}a^4) \simeq 8 \frac{1 + \frac{34}{15}a^2}{1 + \frac{7}{5}a^2 - \frac{13}{15}a^4} \tag{36}
\]

which becomes 8.332 at $a = 1/4$. If we estimate the exponent $\nu$ from these small matrix result by the ratio method (10), we obtain $R_3 = D_3/D_2 = 0.795$ at $a=1/4$, which leads to $\nu = 0.82$ according to the same estimation of Fig.1. This value is slightly larger than the pur gravity result $\nu = 0.8$. The exact value of two matrix case is known as $\nu = 6/7 = 0.853$.

The small $a$ expansion without use of the formula of (17) is also possible. It is indeed necessary to develop such method for general $n$-Ising case, $2^n$ matrix models, since (17) is only applied to two matrix model. For example, in the $N=2$ case, the $2 \times 2$ matrix is represented by

\[
M_2 = \begin{pmatrix} c & b^* \\ b & d \end{pmatrix}, \tag{37}
\]

where we take $M_1 = diag(\lambda_1, \lambda_2)$. Then Jacobian becomes

\[
J = \frac{\xi_1 - \xi_2}{\sqrt{(\xi_1 - \xi_2)^2 - 4|b|^2}} \tag{38}
\]

with two eigenvalues of $M_2 \xi_1$ and $\xi_2$. Expanding $a$-dependent term in the exponent, and integrating by $|b|$, we have an effective Hamiltonian,

\[
\int d|b|^2 \frac{(\lambda_1 - \lambda_2)^2(\xi_1 - \xi_2)^2}{\sqrt{(\xi_1 - \xi_2)^2 - 4|b|^2}} \exp \left[\frac{a}{2}(\lambda_1 + \lambda_2)(\xi_1 + \xi_2)\right] \\
+ \frac{a}{2}(\lambda_1 - \lambda_2) \times \sqrt{(\xi_1 - \xi_2)^2 - 4|b|^2}] \\
= \frac{(\lambda_1 - \lambda_2)^2(\xi_1 - \xi_2)^2}{4} \left[1 + \frac{a^2}{24}(\lambda_1 - \lambda_2)^2(\xi_1 - \xi_2)^2\right] \tag{39}
\]

where we neglected the term of order $a$ which vanishes after integration of $\lambda$, and also we dropped terms which vanishes for $\lambda_1 = -\lambda_2$. We have checked that the result of (30) is obtained by this method up to order $a^2$. 

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It is possible to expand the off-diagonal elements $b$ and truncate at the sufficient order both for the measure and for the exponent. The diagonal elements $c$ and $d$ are given by solving the characteristic equation in a perturbation of $|b|^2$,

$$c = \xi_1 - \frac{|b|^2}{\xi_1 - \xi_2} + O(|b|^4)$$ \hspace{1cm} (40)

$$d = \xi_2 + \frac{|b|^2}{\xi_1 - \xi_2} + O(|b|^4)$$ \hspace{1cm} (41)

Jacobian is $J = \partial(c, d)/\partial(\xi_1, \xi_2)$,

$$J = 1 + 2 \frac{|b|^2}{(\xi_1 - \xi_2)^2} + O(|b|^4)$$ \hspace{1cm} (42)

After integration by these off-diagonal elements $|b|^2$, which is bounded as $|b| \leq (\xi_1 - \xi_2)/2$, we have an effective Hamiltonian written only by the eigenvalues of two matrices. Taking order $|b|^2$ term and up to order $a^2$, we obtain very close result of (27). Including higher expansion of $|b|^2$, the result becomes improved. This method can be applied for general $N \times N$ matrix.

5. n-Ising Model

We have considered Ising model on a random surface as two-matrix model in the large $N$ limit. The extension of this model corresponds to the increasing the number of species or colors of Ising spin. Only same color Ising spin can interact each other. Denoting the number of colors by $n$, this n-Ising model is represented by $2^n$ matrix model [4,5]. $2^n$ configurations of up and down $n$-spins on a vertex are represented by $2^n$ matrices.

The Hamiltonian of this matrix model is given by multi matrix model similar as two matrix model, in which $g$ is a common coupling constant of $\text{Tr}M^4$ term and the coefficients of $\text{Tr}M_iM_j$ is obtained by the inverse matrix of Boltzmann weight of spin interaction.

In the previous works [4], we have calculated numerically the cosmological constant $A = -1/g_c$, the critical value $a_c$, and various critical exponents including the string susceptibility. It is desirable, however, to develop the method to calculate directly the cosmological constant $-1/g_c$. The main difficulty may be that there is no available formula for the integration of matrix, and no systematic method to rewrite the Hamiltonian only by the eigenvalues of matrices.

We apply the method which has been explained in the previous section, (38) or (42). Since the number of matrices increases rapidly as $2^n$, we represent only
n=2 and n=3 Hamiltonian here. They are described by four matrices and eight matrices, respectively.

\[
H(n = 2) = \frac{1}{2} \text{Tr}(M_1^2 + M_2^2 + M_3^2 + M_4^2) - a \text{Tr}(M_1M_2 + M_2M_3 + M_3M_4 + M_1M_4) \\
- a^2 \text{Tr}(M_1M_3 + M_2M_4) + \frac{g}{4} \text{Tr}(M_1^4 + M_2^4 + M_3^4 + M_4^4)
\]

(43)

\[
H(n = 3) = \frac{1}{2} \sum_{i=1}^{8} \text{Tr}M_i^2 \\
- a \text{Tr}(M_1M_2 + M_1M_3 + M_1M_5 + M_2M_4 + M_2M_6 + ...) \\
- a^2 \text{Tr}(M_1M_4 + M_1M_6 + M_1M_7 + M_2M_3 + M_2M_5 + ...) \\
- a^3 \text{Tr}(M_1M_8 + M_2M_7 + M_3M_6 + M_4M_5) \\
+ \frac{g}{N} \text{Tr}(\sum_{i=1}^{8} M_i^4)
\]

(44)

For N=2 case, using the representation of (37), we have for small \(a\),

\[-\frac{1}{g_c} = \frac{8}{1 - \frac{8n}{3}a^2} + O(a^4)\]  \hspace{1cm} (45)

For n-Ising case, the value of \(- (1 - a^2)^{2n}/g_c\) at \(a=1\) becomes \(2^n\) times of \(a=0\) [4]. Therefore, we take [1,1] Padé approximation using this condition as

\[-\frac{(1 - a^2)^{2n}}{g_c} = 8 \frac{[1 + \left(\frac{2n-3}{3(2^n-1)}\right)a^2]}{[1 + \left(-1 + \frac{2n}{3(2^n-1)}\right)a^2]}
\]

(46)

This Padé approximation gives 8.34 for \(n=1\) and 8.69 for \(n=2\) at \(a = 1/4\). In the previous series analysis [4], the critical value of \(- (1 - a^2)^{2n}/g_c\) is estimated as 54.0 at \(a_c = 1/4\) for \(n=2\), and 59.2 at \(a_c = 0.23\) for \(n=4\), and 63.8 at \(a_c = 0.21\) for \(n=6\).

It is important to find \(n\) dependence of \(g_c\) for finite \(N\) matrix as (45). We have only discussed \(N=2\) case, and therefore we can not make any analysis of the exponent \(\nu\) at this stage.

6. Discussion

In this paper, we have discussed the finite \(N\) saddle point equation and and its solution. We have obtained the correct scaling exponent for one matrix model and a modified one matrix model through finite \(N\) scaling. The exponent \(\nu\) has been determined by how finite \(N\) critical value of \(g_c\) approaches to the \(n = \infty\) critical
value $g_c$ (10). Our result shows that even small $N$, the finite size $N$ scaling works very well. This is rather remarkable and it may be related to the fact that the small $g$ series expansion [4] gives also correct estimation of $\gamma_{st}$ by small orders. Although we did not study the multicritical behavior described by $\text{Tr}M^2l$ terms, we believe that our method works for such cases.

For two matrix model, we find the saddle point method is also effective although we restricted our analysis on small matrices. The more detailed and higher order analysis are necessary for this case. We have shown that small $a$ expansion is useful and Padé analysis will give quite accurate numerical value for the critical value of $g_c$ for a finite $N$.

For n-Ising model, we have only studied very small matrices. Using small $a$ expansion and Padé approximation, we have discussed $n$ dependence of the critical value $g_c$. In this case also, the approach to the $N = \infty$ critical value $g_c$ seems smooth, and analysis is possible. This is consistent with the series expansion by planar diagrams [4]. There is no tachyonic instability in this model.

The most interesting problem of n-Ising model is to find the value of $\gamma_{st}$ for large $n$. In the previous analysis of the perturbation of $g$, $\gamma_{st}$ seems increasing for $c = n/2 > 1$. For the large value of $n$, we need higher order calculation in the perturbation method [4]. Our present finite $N$ saddle point method is complementary one in this respect. We are planning to study more detail for n-Ising model by making combined analysis with a series expansion.

We have considered only $d = 0$ matrix models. It is also interesting to extend our finite $N$ analysis for $d = 1$ matrix models which becomes quantum mechanical problems. This problem will be discussed in other place.

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Table 1. The critical value of $g$ obtained for one matrix model with size $N$.

| $N$ | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|----|----|----|----|----|----|----|
| $-1/g_c$ | 8  | 16 | 21.2 | 24.8 | 27.4 | 29.4 | 31.1 |

Table 2. The critical value of $g$ at a fixed $g' = -9/256$ for $(\text{Tr}M^2)^2$ model.

| $N$ | 2  | 3  | 4  | 5  | 6  | 7  |
|-----|----|----|----|----|----|----|
| $-1/g_c$ | 11.13 | 25.60 | 35.92 | 43.29 | 48.78 | 52.99 |
**Figure caption**

**Fig. 1.** The ratio method for the scaling exponent $\nu$ of one matrix model. The ratio $R_N = D_N / D_{N-1}$ is plotted against $1/(N + 1)$ where $D_N = 48 + 1/g_c$. The slope gives $\nu = 4/5$.

**Fig. 2.** The critical lines obtained from finite N saddle point solutions for $g'(\text{Tr}M^2)^2/N^2$ interaction. The lines are N=2,3,...,7 from the left respectively. The dotted line is the critical line for $N = \infty$ in $g - g'$ plane. The string susceptibility $\gamma_{st}$ becomes $1/3$ at $g = -3/256$ and $g' = -9/256$.

**Fig. 3.** The critical value of $-(1 - a^2)^2/g_c$ is shown for N=2,3,5,7. The line of $N = \infty$ is exact value and it has a critical point $a = 1/4$. Two lines of N=\infty are expressed by (21) and (22). The dotted lines are the low temperature solutions.