HOMOTOPY TYPE OF THE NILPOTENT ORBITS IN CLASSICAL LIE ALGEBRAS

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ABSTRACT. In [BCM] homotopy types of nilpotent orbits are explicitly described in the case of real simple classical Lie algebras for which any maximal compact subgroup in the associated adjoint group is not semisimple. In this paper we extend the above description of homotopy type of nilpotent orbits to the remaining cases of real simple classical Lie algebras for which any maximal compact subgroup in the associated adjoint group is semisimple.

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2010 Mathematics Subject Classification. 57T15, 17B08.
Key words and phrases. Nilpotent orbit, classical groups, homogeneous spaces.
1. Introduction

Let $\mathfrak{g}$ be a real simple Lie algebra, and let $G$ be the associated adjoint Lie group. An element $X \in \mathfrak{g}$ is called nilpotent if $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is a nilpotent operator. For any nilpotent $X$, let $O_X := \{ \text{Ad}(g)X \mid g \in G \}$ be the corresponding nilpotent orbit under the adjoint action of $G$ on $\mathfrak{g}$. Such nilpotent orbits form a rich class of homogeneous spaces. In fact they lie on the interface of several disciplines in mathematics such as Lie theory, symplectic geometry, representation theory, algebraic geometry; see [CoMc], [Mc]. Nevertheless, surprisingly for a very long period there seems to have been hardly any literature on the topological invariants of these orbits, other than the description of fundamental group in the case of simple Lie algebras. The computation of the fundamental groups of such orbits is attributed to T. Springer and R. Steinberg [SpSt] for the classical case and A. Alexeevski [Al] for the complex exceptional case; see [CoMc, Corollary 6.1.6, p. 91, pp. 128–134], [Mc, pp. 229–230]. We also refer the reader to the works of D. King [Ki] and E. Sommers [So] in this regard.

It is only recent that attentions have been drawn to topological invariants of such orbits other than the fundamental group; see [Ju] and [Cr]. The first two authors began their study on this topic in [BC] where the second cohomology groups of the nilpotent orbits in all the complex simple Lie algebras were computed; see [BC, Theorems 5.4, 5.5, 5.6, 5.11, 5.12]. Each adjoint orbit in a semisimple Lie algebra is equipped with the Kostant-Kirillov two form. The motivation for studying the second cohomology groups stemmed from, besides computing some invariant other than the fundamental group, the exactness criterion obtained in [BC, Proposition 1.2] for the Kostant-Kirillov form on adjoint orbits of arbitrary elements in the Lie algebra of a real semisimple Lie group with a semisimple maximal compact subgroup. It may be mentioned that [BC, Proposition 1.2] generalizes [ABB, Theorem 1.2] where the exactness criterion is obtained for the Kostant-Kirillov form on an adjoint orbit of a semisimple element in a complex semisimple Lie algebra. In [ChMa] the second cohomology groups of nilpotent orbits are computed for most of the nilpotent orbits in non-compact non-complex exceptional Lie algebras. For the rest of cases of nilpotent orbits, which are not covered in the above computations, an upper bound for the dimension of the second cohomology group is obtained; see [ChMa, Theorems 3.2–3.13].

To describe the results of this paper we need to recall our recent work in [BCM]. In [BCM], considering all the non-complex and non-compact real classical Lie algebras, we have given a complete description of the second and the first cohomology groups of all the nilpotent orbits in terms of their standard parametrizations involving the (signed) Young diagram. In doing so, first generalizing [BC, Theorem 3.3] we obtained a computable description of the second and first cohomologies of a general connected homogeneous space in terms of the ambient Lie group and the associated quotenting (closed) subgroup. In the setting of [BCM] it was convenient to assume that the simple real Lie group $G$ is the connected component of the $\mathbb{R}$-points of a $\mathbb{R}$-simple algebraic groups defined over $\mathbb{R}$.

Setting $\mathfrak{g} := \text{Lie} G$, let $X \in \mathfrak{g}$ be a non-zero nilpotent element, and let $O_X$ be its adjoint $G$-orbit. A Lie theoretic reformulation of the second and the first cohomology groups of $O_X$ was obtained in [BCM, Theorem 4.2], incorporating a $\mathfrak{sl}_2(\mathbb{R})$-triple containing $X$; the computations in [ChMa] also use this result crucially. Let $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{g}$, containing $X$, and let $Z_G(X, H, Y)$ be the centralizer of the triple $\{X, H, Y\}$.
in $G$. Let $K$ be a maximal compact subgroup in $Z_G(X, H, Y)$, and $M$ be a maximal compact subgroup in $G$ containing $K$. Let $\mathfrak{m}$ and $\mathfrak{z}(\mathfrak{k})$ be the Lie algebras of $M$ and the center of $K$, respectively. Then [BCM, Theorem 4.2] says that the computation of the second cohomology of the nilpotent orbits boils down to understanding the action of the component group $K/K^\circ$ on the subalgebra $\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m}, \mathfrak{m}]$. Thus, when $M$ is semisimple, this amounts to describing the action of $K/K^\circ$ on $\mathfrak{z}(\mathfrak{k})$, and hence knowing the isomorphism class of $K$ is enough to compute the second cohomology group in this case.

However, when $M$ is not semisimple, it does not suffice to know the isomorphism classes of $K$ and $M$, rather what is needed is an understand of the embedding of $K$ in $M$. The case of $\mathfrak{su}(p, q)$ dealt in [BCM, § 4.4] constitutes a typical example of such a situation. Although the isomorphism class of $M$ is a standard known object in the case of a non-compact classical simple real Lie group, and the isomorphism class of $K$ can be obtained immediately using either the work of Springer and Steinberg [SpSt] (see also [BCM, Lemma 4.4]), hardly anything can be concluded, from these isomorphism classes, on how $K$ is embedded in $M$. Consequently, one of the major objectives in [BCM] was to compute this embedding explicitly for all the nilpotent orbits in the classical real simple Lie algebras $\mathfrak{g}$ for which maximal compact subgroup of $G$ is not semisimple. This situation occurs in the cases when $\mathfrak{g}$ is either $\mathfrak{su}(p, q)$ or $\mathfrak{so}^*(2n)$ or $\mathfrak{sp}(n, \mathbb{R})$.

It follows from a minor variation of a general fact due to Mostow (see Theorem 2.2, [BC, Theorem 3.2]) that $M/K$ naturally embeds in $G/Z_G(X)$ as a deformation retract. In particular, the compact submanifold $M/K$ of $G/Z_G(X)$ is in the same homotopy class as that of $G/Z_G(X)$; see Theorem 2.3 for details. Thus computations in [BCM] in fact yield compact sub-homogeneous spaces as convenient and optimal homotopy types of nilpotent orbits in the case when maximal compact subgroups of $G$ are not semisimple (we refer to [BCM, Propositions 4.14, 4.30 and 4.36]). It should be mentioned here that, for a certain technical reason, the homotopy types of the nilpotent orbits in $\mathfrak{so}(p, q)$ were also described in [BCM, Proposition 4.22] under the following assumption on the partition associated to the parametrization: $N_d = O_d$; see (2.4) for the definitions of $N_d$ and $O_d$.

The objective of this paper is to extend the computations and complete the project, initiated in [BCM], of describing optimal homotopy types of nilpotent orbits by giving explicit embedding of maximal compact subgroups $K$ of $Z_G(X)$ in $M$ for the remaining cases of simple Lie algebras $\mathfrak{g}$ for which $M$ is semisimple; see Theorems 3.2, 3.4, 3.6 3.13, 3.19, 3.24 and 3.31.

The description of a suitable reductive part of the centralizer in $G$ of a nilpotent element in $\mathfrak{g}$, when $\mathfrak{g}$ is isomorphic to a complex simple Lie algebra or it is isomorphic to one of the Lie algebras $\mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{u}(p, q)$, $\mathfrak{o}(p, q)$ and $\mathfrak{sp}(n, \mathbb{R})$, is due to Springer and Steinberg [SpSt]. Since we are unable to find such descriptions in the literature when the classical simple Lie algebras are matrix subalgebras with entries from $\mathbb{H}$, we record them here in the final section as an appendix.

The paper is organized as follows. In Section 2 we fix some notation and terminology and we recall some necessary background. The explicit homotopy types of the nilpotent orbits are described in Section 3; they are spread across Theorems 3.2, 3.4, 3.6 3.13, 3.19, 3.24 and 3.31.
2. Notation and background

In this section we fix the notation, and recall some background material which will be used throughout. Subsequently, a few specialized notation are mentioned as and when they are needed. The notation and the background introduced in this section overlap with those in [BCM, § 2] to some extent. However, for the sake of completeness and clarity of the exposition we also include them here.

Once and for all fix a square root of $-1$ and call it $\sqrt{-1}$. The Lie groups will be denoted by the capital letters, while the Lie algebra of a Lie group will be denoted by the corresponding lower case German letter, unless a different notation is explicitly mentioned. Sometimes, for notational convenience, the Lie algebra of a Lie group $G$ is also denoted by $\text{Lie}(G)$. The connected component of $G$ containing the identity element is denoted by $G^0$, and the commutator subgroup of $G$ is denoted by $(G, G)$. For a subgroup $H$ of $G$, and a subset $S$ of $\mathfrak{g}$, the centralizer of $S$ in $H$ is

$$Z_H(S) := \{ h \in H \mid \text{Ad}(h)Y = Y \text{ for all } Y \in S \}.$$ 

Similarly, for a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, by $\mathfrak{z}_\mathfrak{h}(S)$ we denote the subalgebra

$$\{ X \in \mathfrak{h} \mid [X, Y] = 0 \text{ for all } Y \in S \}.$$ 

Let $G$ be a semisimple Lie group. An element $X \in \mathfrak{g}$ is called nilpotent if $\text{ad}(X) : \mathfrak{g} \rightarrow \mathfrak{g}$ is a nilpotent operator. The set of nilpotent elements in $\mathfrak{g}$ is denoted by $\mathcal{N}_{\mathfrak{g}}$. For any $X \in \mathcal{N}_\mathfrak{g}$, define

$$O_X := \{ \text{Ad}(g)X \mid g \in G \}$$

to be the nilpotent orbit of $X$ in $\mathfrak{g}$. The set of all nilpotent orbits in $\mathfrak{g}$ is denoted by $\mathcal{N}(\mathfrak{g})$.

2.1. Classical Lie groups and their Lie algebras. The notation $\mathbb{D}$ will stand for either $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, unless mentioned otherwise. Let $V$ be a right vector space over $\mathbb{D}$. Let $\text{End}_\mathbb{D}(V)$ be the right $\mathbb{D}$-algebra of $\mathbb{D}$-linear endomorphisms of $V$, and let $\text{GL}(V)$ be the group of invertible elements of $\text{End}_\mathbb{D}(V)$. For a $\mathbb{D}$-linear endomorphism $T \in \text{End}_\mathbb{D}(V)$, and an ordered $\mathbb{D}$-basis $\mathcal{B}$ of $V$, the matrix of $T$ with respect to $\mathcal{B}$ is denoted by $[T]_{\mathcal{B}}$. When $\mathbb{D}$ is either $\mathbb{R}$ or $\mathbb{C}$, let

$$\text{tr} : \text{End}_\mathbb{D}(V) \rightarrow \mathbb{D} \quad \text{and} \quad \det : \text{End}_\mathbb{D}(V) \rightarrow \mathbb{D}$$

respectively be the usual trace and determinant maps. When $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$, define

$$\text{SL}(V) := \{ z \in \text{GL}(V) \mid \det(z) = 1 \} \quad \text{and} \quad \mathfrak{sl}(V) := \{ y \in \text{End}_\mathbb{D}(V) \mid \text{tr}(y) = 0 \}.$$ 

If $\mathbb{D} = \mathbb{H}$, then define

$$\text{SL}(V) := \left( \text{GL}(V), \text{GL}(V) \right) \quad \text{and} \quad \mathfrak{sl}(V) := \left[ \text{End}_\mathbb{D}(V), \text{End}_\mathbb{D}(V) \right].$$

Let $\mathbb{D}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, as above. Let $\sigma$ be either the identity map $\text{Id}$ or an involution of $\mathbb{D}$; i.e., $\sigma$ is a $\mathbb{R}$-linear map with $\sigma^2 = \text{Id}$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in \mathbb{D}$. Let $\varepsilon = \pm 1$. Following [Bo, § 23.8, p. 264] we call a map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{D}$$

a $\varepsilon$-$\sigma$ Hermitian form if

- $\langle \cdot, \cdot \rangle$ is additive in each argument,
- $\langle v, u \rangle = \varepsilon \sigma(\langle u, v \rangle)$, and
The pair \((\langle \cdot, \cdot \rangle)\) is defined to be the $B$ by 2 is said to be $\not\in \langle \cdot, \cdot \rangle$ is denoted by $B$, a symmetric or Hermitian form $\langle \cdot, \cdot \rangle$.

Similarly, if $\langle \cdot, \cdot \rangle$ is a symmetric form on $V$ and $\sigma$ is a symmetric form on $V$. Therefore, from now on we will restrict ourselves to the involution $\sigma$ on $D$.

We next introduce some terminology associated to certain types of $\sigma$ and $\epsilon$. In this case an ordered basis $A := \{v_1, \cdots, v_n; v_{n+1}, \cdots, v_{2n}\}$ of $V$ is said to be symplectic if $\langle v_i, v_{n+i} \rangle = 1$ for all $1 \leq i \leq n$ and $\langle v_i, v_j \rangle = 0$ for all $j \neq n + i$. The ordered set $\{v_1, \cdots, v_n\}$ is called the positive part of $B$ and it is denoted by $B_+$. Similarly, the ordered set $\{v_{n+1}, \cdots, v_{2n}\}$ is called the negative part of $B$, and it is denoted by $B_-$. The complex structure on $V$ associated to the above symplectic basis $B$ is defined to be the $\mathbb{R}$-linear map

$$J_B : V \rightarrow V, \quad v_i \mapsto v_{n+i}, \quad v_{n+i} \mapsto -v_i \quad \forall \ 1 \leq i \leq n.$$
If \( \mathbb{D} = \mathbb{H} \), and \( \langle \cdot, \cdot \rangle \) is a skew-Hermitian form on \( V \), an orthogonal \( \mathbb{H} \)-basis

\[
\mathcal{B} := (v_1, \cdots, v_m)
\]
of \( V \) (\( m := \dim_\mathbb{H} V \)) is said to be standard orthogonal if \( \langle v_r, v_r \rangle = 1 \) for all \( 1 \leq r \leq m \) and \( \langle v_r, v_s \rangle = 0 \) for all \( r \neq s \).

For \( P = (p_{ij}) \in \mathbb{M}_{r \times s}(\mathbb{D}) \), let \( P^t \) denote the transpose of \( P \). If \( \mathbb{D} = \mathbb{C} \) or \( \mathbb{H} \), then define \( \overline{P} := (\sigma_c(p_{ij})) \). Let

\[
I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad J_n := \begin{pmatrix} I_n & -I_n \\ 0 & 0 \end{pmatrix}.
\] (2.1)

The classical groups and Lie algebras that we will be working with are:

\[
\begin{align*}
\text{SL}_n(\mathbb{C}) & := \{ g \in \text{GL}_n(\mathbb{C}) \mid \det(g) = 1 \}, & \text{sl}_n(\mathbb{C}) & := \{ Y \in M_n(\mathbb{C}) \mid \text{tr}(Y) = 0 \}; \\
\text{SL}_n(\mathbb{R}) & := \{ g \in \text{GL}_n(\mathbb{R}) \mid \det(g) = 1 \}, & \text{sl}_n(\mathbb{R}) & := \{ Y \in M_n(\mathbb{R}) \mid \text{tr}(Y) = 0 \}; \\
\text{SL}_n(\mathbb{H}) & := \{ g \in \text{GL}_n(\mathbb{H}) \mid \text{Nrd}_{M_n(\mathbb{H})}(g) = 1 \}, & \text{sl}_n(\mathbb{H}) & := \{ Y \in M_n(\mathbb{H}) \mid \text{Trd}_{M_n(\mathbb{H})}(Y) = 0 \}; \\
\text{SO}(n, \mathbb{C}) & := \{ g \in \text{SL}_n(\mathbb{C}) \mid g^t I_n g = I_n \}, & \text{so}(n, \mathbb{C}) & := \{ Y \in \text{sl}_n(\mathbb{C}) \mid Y^t I_n + I_n = 0 \}; \\
\text{SO}(p, q) & := \{ g \in \text{SL}_{p+q}(\mathbb{R}) \mid g^t I_{p,q} g = I_{p,q} \}, & \text{so}(p, q) & := \{ Y \in \text{sl}_{p+q}(\mathbb{R}) \mid Y^t I_{p,q} + I_{p,q} Y = 0 \}; \\
\text{Sp}(n, \mathbb{C}) & := \{ g \in \text{SL}_{2n}(\mathbb{C}) \mid g^t J_n g = J_n \}, & \text{sp}(n, \mathbb{C}) & := \{ Y \in \text{sl}_{2n}(\mathbb{C}) \mid Y^t J_n + J_n Y = 0 \}; \\
\text{Sp}(p, q) & := \{ g \in \text{SL}_{p+q}(\mathbb{H}) \mid g^t I_{p,q} g = I_{p,q} \}, & \text{sp}(p, q) & := \{ Y \in \text{sl}_{p+q}(\mathbb{H}) \mid Y^t I_{p,q} + I_{p,q} Y = 0 \}.
\end{align*}
\]

For any group \( H \), let \( H^n \) denote the diagonally embedded copy of \( H \) in the direct product \( H^n \). Let \( V \) be a vector space over \( \mathbb{D} \). Define \( \partial_V : \text{End}_V(V) \rightarrow \mathbb{D}^* \) to be \( \partial_V := \det \) if \( \mathbb{D} = \mathbb{C} \) or \( \mathbb{R} \), and \( \partial_V := \text{Nrd}_{\text{End}_V V} \) if \( \mathbb{D} = \mathbb{H} \). Let now \( V_i, 1 \leq i \leq m, \) be right vector spaces over \( \mathbb{D} \). As before, \( \mathbb{D} \) is either \( \mathbb{R} \) or \( \mathbb{C} \) or \( \mathbb{H} \). For every \( 1 \leq i \leq m \), let \( H_i \subset \text{GL}(V_i) \) be a matrix subgroup. Define the subgroup

\[
S(\prod_i H_i) := \{ (h_1, \cdots, h_m) \in \prod_i H_i \mid \prod_i \partial_{V_i}(h_i) = 1 \} \subset \prod_i H_i.
\]

The following notation will allow us to write block-diagonal square matrices with many blocks in a convenient way. For \( r\)-many square matrices \( A_i \in M_{m_i}(\mathbb{D}) \), \( 1 \leq i \leq r \), the block diagonal square matrix of size \( \sum m_i \times m_i \), with \( A_i \) as the \( i \)-th block in the diagonal, is denoted by \( A_1 \oplus \cdots \oplus A_r \). This is also abbreviated as \( \oplus_{i=1}^r A_i \). Furthermore, if \( B \in M_m(\mathbb{D}) \) and \( s \) is a positive integer, then denote \( B^s := \bigoplus_{s\text{-many}} B \).

2.2. Jacobson-Morozov Theorem. For a Lie algebra \( \mathfrak{g} \) over \( \mathbb{R} \), a subset \( \{X, H, Y\} \subset \mathfrak{g} \) is said to be a \( \mathfrak{sl}_2(\mathbb{R}) \)-triple if \( X \neq 0 \), \( [H, X] = 2X \), \( [H, Y] = -2Y \) and \( [X, Y] = H \). We now recall a well-known result due to Jacobson and Morozov.

**Theorem 2.1 (Jacobson-Morozov, cf. [CoMc, Theorem 9.2.1])**. Let \( X \in \mathfrak{g} \) be a non-zero nilpotent element in a real semisimple Lie algebra \( \mathfrak{g} \). Then there exist \( H, Y \in \mathfrak{g} \) such that \( \{X, H, Y\} \subset \mathfrak{g} \) is a \( \mathfrak{sl}_2(\mathbb{R}) \)-triple.
2.3. **Finite dimensional \( \mathfrak{sl}_2(\mathbb{R}) \)-modules.** Given an endomorphism \( T \in \text{End}_\mathbb{R}(W) \), where \( W \) is a \( \mathbb{R} \)-vector space, and any \( \lambda \in \mathbb{R} \), set

\[
W_{T,\lambda} := \{ w \in W \mid Tw = w\lambda \}.
\]

Let \( V \) be a right vector space of dimension \( n \) over \( D \), where \( D \) is, as before, \( \mathbb{R} \) or \( \mathbb{C} \) or \( \mathbb{H} \). Let \( \{X, H, Y\} \subset \mathfrak{sl}(V) \) be a \( \mathfrak{sl}_2(\mathbb{R}) \)-triple. Note that \( V \) is also a \( \mathbb{R} \)-vector space using the inclusion \( \mathbb{R} \hookrightarrow D \). Hence \( V \) is a module over \( \text{Span}_\mathbb{R}\{X, H, Y\} \simeq \mathfrak{sl}_2(\mathbb{R}) \). For any positive integer \( d \), let \( M(d - 1) \) denote the sum of all the \( \mathbb{R} \)-subspaces \( A \) of \( V \) such that

- \( \dim \mathbb{R} A = d \), and
- \( A \) is an irreducible \( \text{Span}_\mathbb{R}\{X, H, Y\} \)-submodule of \( V \).

Then \( M(d - 1) \) is the **isotypical component** of \( V \) containing all the irreducible submodules of \( V \) with highest weight \( d - 1 \). Let

\[
L(d - 1) := V_{Y,0} \cap M(d - 1).
\]

As the endomorphisms \( X, H, Y \) of \( V \) are \( \mathbb{D} \)-linear, the \( \mathbb{R} \)-subspaces \( M(d - 1), V_{Y,0} \) and \( L(d - 1) \) of \( V \) are also \( \mathbb{D} \)-subspaces. Let

\[
t_d := \dim \mathbb{D} L(d - 1).
\]

2.4. **Preliminary results on topology of homogeneous spaces.** In this section we will recall some well-known results. The following one gives an equivalence of homotopy types between a non-compact homogeneous space and certain compact homogeneous space.

**Theorem 2.2 ([Mo]).** Let \( G \) be a connected Lie group, and let \( H \subset G \) be a closed subgroup with finitely many connected components. Let \( M \) be a maximal compact subgroup of \( G \) such that \( M \cap H \) is a maximal compact subgroup of \( H \). Then the image of the natural embedding \( M/(M \cap H) \hookrightarrow G/H \) is a deformation retract of \( G/H \).

Theorem 2.2 is proved in [Mo, p. 260, Theorem 3.1] for connected \( H \). However, as mentioned in [BC], using [Ho, p. 180, Theorem 3.1], the proof as in [Mo] goes through when \( H \) has finitely many connected components.

The following result explains why one needs to identify a maximal compact subgroup of the centralizer of a \( \mathfrak{sl}_2(\mathbb{R}) \)-triple as a subgroup of a maximal compact subgroup of the ambient simple real Lie group.

**Theorem 2.3.** Let \( \mathfrak{g} \) be a real simple Lie algebra and \( G \) be the real (connected) adjoint group of \( \mathfrak{g} \). Let \( X \in \mathfrak{g} \) be a nilpotent element, and \( \{X, H, Y\} \) be a \( \mathfrak{sl}_2(\mathbb{R}) \)-triple in \( \mathfrak{g} \) containing \( X \). Let \( K \) be a maximal compact subgroup of \( Z_G(X, H, Y) \). Then \( K \) is a maximal compact subgroup of \( Z_G(X) \). Moreover, if \( M \) is a maximal compact subgroup of \( G \) containing \( K \), then the compact homogeneous space \( M/K \) embeds in \( G/Z_G(X) \) as a deformation retract.

**Proof.** It follows from the proof of [BCM, Theorem 4.2]. The last part follows from Theorem 2.2. \( \square \)
2.5. Partitions and (signed) Young diagrams. An ordered set of order \( n \) is a \( n \)-tuple \((v_1, \ldots, v_n)\), where \( v_1, \ldots, v_n \) are elements of some set, such that \( v_i \neq v_j \) if \( i \neq j \). If \( w \in \{v_1, \ldots, v_n\} \), then write \( w \in (v_1, \ldots, v_n) \). A pair of ordered sets \((v_1, \ldots, v_n)\) and \((w_1, \ldots, w_m)\) is said to be disjoint if \( \{v_1, \ldots, v_n\} \cap \{w_1, \ldots, w_m\} = \emptyset \). For a pair of ordered sets \((v_1, \ldots, v_n)\) and \((w_1, \ldots, w_m)\) which is disjoint, the ordered set \((v_1, \ldots, v_n, w_1, \ldots, w_m)\) will be denoted by \((v_1, \ldots, v_n) \lor (w_1, \ldots, w_m)\).

Furthermore, for \( k \)-many ordered sets \((v^i_1, \ldots, v^i_n)\), \( 1 \leq i \leq k \), which are pairwise disjoint, we define the ordered set \((v^1, \ldots, v^n, v^1_{n_1}, \ldots, v^k_{n_k})\) to be the following juxtaposition of ordered sets \((v^1_1, \ldots, v^1_n)\), \( \ldots \), \( (v^k_1, \ldots, v^k_{n_k})\) with increasing \( i \):

\[
(v^1_1, \ldots, v^1_n_1) \lor \cdots \lor (v^k_1, \ldots, v^k_{n_k}) := (v^1_1, \ldots, v^1_{n_1}, \ldots, v^k_1, \ldots, v^k_{n_k}).
\]

A partition of a positive integer \( n \) is an object of the form \([d^1_1, \ldots, d^s_s]\) where \( t_d, \, d_i \in \mathbb{N}, \, 1 \leq i \leq s \), such that \( \sum_{i=1}^{s} t_d d_i = n \), \( t_d \geq 1 \), and \( d_{i+1} \geq d_i \geq 0 \) for all \( i \); see [CoMc, § 3.1, p. 30]. Let \( P(n) \) denote the set of all partitions of \( n \). For a partition \( d = [d^1_1, \ldots, d^s_s] \) of \( n \), define

\[
N_d := \{d_i \mid 1 \leq i \leq s\}, \quad E_d := N_d \cap (2\mathbb{N}), \quad O_d := N_d \setminus E_d. \tag{2.4}
\]

Further define

\[
O^1_d := \{d \mid d \in O_d, \, d \equiv 1 \pmod{4}\} \quad \text{and} \quad O^3_d := \{d \mid d \in O_d, \, d \equiv 3 \pmod{4}\}. \tag{2.5}
\]

Following [CoMc, Theorem 9.3.3], a partition \( d \) of \( n \) will be called even if \( N_d = E_d \). Let \( P_{\text{even}}(n) \) be the subset of \( P(n) \) consisting of all even partitions of \( n \). We call a partition \( d \) of \( n \) to be very even if

- \( d \) is even,
- \( t_\eta \) is even for all \( \eta \in N_d \).

Let \( P_{\text{v.even}}(n) \) be the subset of \( P(n) \) consisting of all very even partitions of \( n \). Now define

\[
P_1(n) := \{d \in P(n) \mid t_\eta \text{ is even for all } \eta \in E_d\}
\]

and

\[
P_{-1}(n) := \{d \in P(n) \mid t_\theta \text{ is even for all } \theta \in O_d\}.
\]

Clearly, we have \( P_{\text{v.even}}(n) \subset P_1(n) \).

We next define certain sets using collections of matrices with entries comprising of signs \( \pm 1 \), which are easily seen to be in bijection with sets of equivalence classes of various types of signed Young diagrams. These sets will be used in parametrizing the nilpotent orbits in the classical Lie algebras.

For a partition \( d \in P(n) \) and \( d \in N_d \), we define the subset \( A_d \subset M_{t_d \times d}(\mathbb{C}) \) of matrices \((m^d_{ij})\) with entries in the set \( \{\pm 1\} \) as follows:

\[
A_d := \{(m^d_{ij}) \in M_{t_d \times d}(\mathbb{C}) \mid (m^d_{ij}) \text{ satisfies (Yd.1) and (Yd.2) below}\}. \tag{2.6}
\]

**Yd.1** There is an integer \( 0 \leq p_d \leq t_d \) such that

\[
m^d_{i1} := \begin{cases} +1 & \text{if } 1 \leq i \leq p_d \\ -1 & \text{if } p_d < i \leq t_d. \end{cases}
\]
Let $S$ and $S'$. We also define equivalence classes of signed Young diagrams of size $p$.

Further define $Y$ and $Y'$. For any $(m \in A_d)$ and $d \in \mathbb{N}$.

For any $(m_i \in A_d)$ set
\[
\text{sgn}_+(m_i) := \#\{(i, j) \mid 1 \leq i \leq t_d, 1 \leq j \leq d, m_i = +1\}
\]
and
\[
\text{sgn}_-(m_i) := \#\{(i, j) \mid 1 \leq i \leq t_d, 1 \leq j \leq d, m_i = -1\}.
\]

Let $S_d := A_d \times \cdots \times A_d$. For a pair of non-negative integers $(p, q)$ with $p + q = n$ we now define the set $S_d(p, q) \subset S_d$ by
\[
S_d(p, q) := \{(M_d, \ldots, M_d) \in S_d(n) \mid \sum_{i=1}^n sgn_+ M_{d_i} = p, \sum_{i=1}^n sgn_ M_{d_i} = q\}.
\]
We also define
\[
\mathcal{Y}(p, q) := \{(d, sgn) \mid d \in \mathcal{P}(n), \ sgn \in S_d(p, q)\}.
\]
It is easy to see that there is a natural bijection between the set $\mathcal{Y}(p, q)$ and the equivalence classes of signed Young diagrams of size $p + q$ with signature $(p, q)$. Hence, we will call $\mathcal{Y}(p, q)$ the set of equivalence classes of signed Young diagrams of size $p + q$ with signature $(p, q)$.

For any $d \in \mathcal{P}(n)$ and $d \in \mathbb{N}$, define the subset $A_{d,1}$ of $A_d$ by
\[
A_{d,1} := \{(m_i \in A_d) \mid m_i = +1 \ \forall \ 1 \leq i \leq t_d\}.
\]
Further define $S_d^{\text{even}}(p, q) \subset S_d(p, q)$ and $S_d^{\text{odd}}(n) \subset S_d(n)$ by
\[
S_d^{\text{even}}(p, q) := \{(M_d, \ldots, M_d) \in S_d(p, q) \mid M_{\eta} \in A_{\eta,1} \ \forall \ \eta \in \mathbb{E}_d\}
\]
and
\[
S_d^{\text{odd}}(n) := \{(M_d, \ldots, M_d) \in S_d(n) \mid M_{\theta} \in A_{\theta,1} \ \forall \ \theta \in \mathbb{O}_d\}.
\]

For a pair $(p, q)$ of non-negative integers we define the sets $\mathcal{Y}^{\text{even}}(p, q)$ and $\mathcal{Y}^{\text{even}}_i(p, q)$ by
\[
\mathcal{Y}^{\text{even}}(p, q) := \{(d, sgn) \mid d \in \mathcal{P}(n), \ sgn \in S_d^{\text{even}}(p, q)\},
\]
\[
\mathcal{Y}^{\text{even}}_i(p, q) := \{(d, sgn) \mid d \in \mathcal{P}_i(n), \ sgn \in S_d^{\text{even}}(p, q)\}.
\]
Similarly, for a non-negative integer $n$, set
\[
\mathcal{Y}^{\text{odd}}(n) := \{(d, sgn) \mid d \in \mathcal{P}(n), \ sgn \in S_d^{\text{odd}}(n)\},
\]
\[
\mathcal{Y}^{\text{odd}}_i(2n) := \{(d, sgn) \mid d \in \mathcal{P}_i(2n), \ sgn \in S_d^{\text{odd}}(2n)\}.
\]

Let $d \in \mathcal{P}(n)$. For $\theta \in \mathbb{O}_d$ and $M_{\theta} := (m_{rs}) \in A_{\theta}$, define
\[
l^{+}_{\theta,i}(M_{\theta}) := \#\{j \mid m_{ij} = +1\} \quad \text{and} \quad l^{-}_{\theta,i}(M_{\theta}) := \#\{j \mid m_{ij} = -1\}
\]
for all $1 \leq i \leq t_{\theta}$; set
\[
S_d'(p, q) := \{(M_d, \ldots, M_d) \in S_d^{\text{even}}(p, q) \mid \begin{array}{l}
l^{+}_{\theta,i}(M_{\theta}) \text{ is even} \forall \theta \in \mathbb{O}_d, \ 1 \leq i \leq t_{\theta} \quad \text{or} \quad l^{-}_{\theta,i}(M_{\theta}) \text{ is even} \forall \theta \in \mathbb{O}_d, \ 1 \leq i \leq t_{\theta}
\end{array}\}.
\]
3. Homotopy types of the nilpotent orbits

In this section we will explicitly write down homotopy types of the nilpotent orbits in classical Lie algebras as compact homogeneous spaces.

Let \( V \) be a right \( D \)-vector space of dimension \( n \), where \( D \) is, as before, \( \mathbb{R} \) or \( \mathbb{C} \) or \( \mathbb{H} \). Let \( \{X, H, Y\} \subset \mathfrak{sl}(V) \) be a \( \mathfrak{sl}_2(\mathbb{R}) \)-triple. Consider the non-zero irreducible \( \text{Span}_\mathbb{R}\{X, H, Y\}\)-submodules of \( V \). Let \( \{d_1, \ldots, d_s\} \), with \( d_1 < \cdots < d_s \), be the integers that occur as \( \mathbb{R} \)-dimension of such \( \text{Span}_\mathbb{R}\{X, H, Y\}\)-modules. Then using (2.3) and [BCM, Lemma A.1 (2)], it follows that

\[
\sum_{i=1}^{s} t_i d_i = \dim_D V = n.
\]

Thus

\[
d := [d_1^{t_1}, \ldots, d_s^{t_s}] \in \mathcal{P}(n).
\] (3.1)

Consider \( N_d, E_d \) and \( O_d \) as defined in (2.4). We have

\[
V = \bigoplus_{d \in N_d} M(d-1) \quad \text{and} \quad L(d-1) = V_{Y,0} \cap V_{H,1-d} \quad \text{for} \quad d \geq 1.
\] (3.2)

Let \( (v_1^d, \ldots, v_{d_1}^d) \) be the ordered \( D \)-basis of \( L(d-1) \) as in [BCM, Proposition A.2] for \( d \in N_d \). Then it follows that

\[
\mathcal{B}(d) := (X^l v_1^d, \ldots, X^l v_{d_1}^d) \quad (3.3)
\]
is an ordered \( D \)-basis of \( X^l L(d-1) \) for \( 0 \leq l \leq d-1 \) with \( d \in \mathbb{N}_d \); see [BCM, Proposition A.2(2)]. Define

\[
\mathcal{B} := \mathcal{B}(d_1) \vee \cdots \vee \mathcal{B}(d_s).
\] (3.4)

Let

\[
\mathbf{D}_{\mathfrak{sl}(V)} : M_{t_d_1}(\mathbb{D}) \times \cdots \times M_{t_d_s}(\mathbb{D}) \rightarrow M_n(\mathbb{D})
\]
\[
(A_{d_1}, \ldots, A_{d_s}) \mapsto \bigoplus_{j=1}^{s} (A_{d_j})_{d_j}^d
\] (3.5)
be the \( \mathbb{R} \)-algebra embedding. Let

\[
A_B : \text{End}(V) \rightarrow M_n(\mathbb{D})
\] (3.6)
be the isomorphism of \( \mathbb{R} \)-algebras with respect to the ordered basis \( \mathcal{B} \). Next define the character when \( D = \mathbb{R} \) or \( \mathbb{C} \)

\[
\chi_d : \prod_{d \in N_d} \text{GL}(L(d-1)) \rightarrow \mathbb{D}^*
\]
\[
(g_{t_{d_1}}, \ldots, g_{t_{d_s}}) \mapsto \prod_{i=1}^{s} (\det g_{t_{d_i}})^{d_i}.
\] (3.7)

Let \( \langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{D} \) be a \( \epsilon-\sigma \) Hermitian form. Assume that \( \{X, H, Y\} \) be a \( \mathfrak{sl}_2(\mathbb{R}) \)-triple in \( \mathfrak{su}(V, \langle \cdot, \cdot \rangle) \). Define the form

\[
\langle \cdot, \cdot \rangle_d : L(d-1) \times L(d-1) \rightarrow \mathbb{D}, \quad \langle v, u \rangle_d := \langle v, X^{d-1} u \rangle
\] (3.8)
as in [CoMc, p. 139].
3.1. Homotopy types of the nilpotent orbits in $\mathfrak{sl}_n(\mathbb{D})$. Let $n$ be a positive integer. In this subsection we write down the homotopy types of the nilpotent orbits in $\mathfrak{sl}_n(\mathbb{D})$ for $\mathbb{D} = \mathbb{C}, \mathbb{R}, \mathbb{H}$.

First recall a standard parametrization of $\mathcal{N}(\mathrm{SL}_n(\mathbb{D}))$, the set of all nilpotent orbits in $\mathfrak{sl}_n(\mathbb{D})$. Let $X \in \mathcal{N}_{\mathfrak{sl}_n(\mathbb{D})}$ be a nilpotent element. First assume that $X \neq 0$, and let $\{X, H, Y\} \subset \mathfrak{sl}_n(\mathbb{D})$ be an $\mathfrak{sl}_2(\mathbb{R})$-triple. Let $V := \mathbb{D}^n$ be the right $\mathbb{D}$-vector space of column vectors. Let $\{c_1, \ldots, c_l\}$, with $1 < \cdots < c_l$, be the finitely many ordered integers that occur as $\mathbb{R}$-dimension of non-zero irreducible $\mathrm{Span}_\mathbb{R}\{X, H, Y\}$-submodules of $V$. Recall that $M(c-1)$ is defined to be the isotypical component of $V$ containing all irreducible $\mathrm{Span}_\mathbb{R}\{X, H, Y\}$-submodules of $V$ with highest weight $(c-1)$, and as in (2.2), we set $L(c-1) := V_{Y,0} \cap M(c-1)$. Recall that the space $L(c_r-1)$ is a $\mathbb{D}$-subspace for $1 \leq r \leq l$. Let

$$t_{c_r} := \dim_\mathbb{D} L(c_r-1)$$

for $1 \leq r \leq l$. Then as $\sum_{r=1}^l t_{c_r} c_r = n$, we have $[c_1^{t_{c_1}}, \ldots, c_l^{t_{c_l}}] \in \mathcal{P}(n)$. Define $\Psi_{\mathrm{SL}_n(\mathbb{D})}(O_X) := [c_1^{t_{c_1}}, \ldots, c_l^{t_{c_l}}]$. It is easy to see that $\Psi_{\mathrm{SL}_n(\mathbb{D})}(O_X) \neq [1^n]$ as $X \neq 0$. By declaring $\Psi_{\mathrm{SL}_n(\mathbb{D})}(O_0) = [1^n]$ we obtain a map

$$\Psi_{\mathrm{SL}_n(\mathbb{D})} : \mathcal{N}(\mathrm{SL}_n(\mathbb{D})) \longrightarrow \mathcal{P}(n). \quad (3.9)$$

Now we will consider three cases $\mathbb{D} = \mathbb{C}, \mathbb{R}, \mathbb{H}$ separately.

3.1.1. The case of $\mathfrak{sl}_n(\mathbb{C})$. First we assume that $\mathbb{D} = \mathbb{C}$. Recall that the map as in (3.9) parametrizes the nilpotent orbits in $\mathfrak{sl}_n(\mathbb{C})$.

Theorem 3.1 ([CoMc, Theorem 5.1.1]). The map $\Psi_{\mathrm{SL}_n(\mathbb{C})}$ in (3.9) is a bijection.

Theorem 3.2. Let $X \in \mathcal{N}_{\mathfrak{sl}_n(\mathbb{C})}$, $X \neq 0$, and $\Psi_{\mathrm{SL}_n(\mathbb{C})}(O_X) = d$. Let $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sl}_n(\mathbb{C})$. Let $K$ be a maximal compact subgroup of $\mathbb{Z}_{\mathrm{SL}_n(\mathbb{C})}(X, H, Y)$. Let the maps $\Lambda_B, \mathbf{D}_{\mathrm{SL}(V)}$ and $\chi_d$ be defined as in (3.6), (3.5) and (3.7), respectively. Then $\Lambda_B(K)$ is given by

$$\Lambda_B(K) = \{ \mathbf{D}_{\mathrm{SL}(V)}(g) \mid g \in \prod_{i=1}^s U(t_{d_i}), \chi_d(g) = 1 \}.$$ 

Moreover, the nilpotent orbit $O_X$ in $\mathfrak{sl}_n(\mathbb{C})$ is homotopic to $\mathrm{SU}(n)/\Lambda_B(K)$.

Proof. Let $V$ be a right $\mathbb{C}$-vector space of column vectors such that $\dim_\mathbb{C} V = n$, and $\mathcal{B}$ be an ordered basis of $V$ as in (3.4). The proof follows from [BCM, Lemma 4.4(1)] and by writing the matrices of the elements of the maximal compact subgroup $K$ with respect to the ordered basis $\mathcal{B}$ as in (3.4).

The second part follows from Theorem 2.3 and the well-known fact that any maximal compact subgroup of $\mathrm{SL}_n(\mathbb{C})$ is isomorphic to $\mathrm{SU}(n)$. \qed

3.1.2. The case of $\mathfrak{sl}_n(\mathbb{R})$. Next we will consider $\mathbb{D} = \mathbb{R}$. The following known result says that the map $\Psi_{\mathrm{SL}_n(\mathbb{R})}$ as in (3.9) is “almost” a parametrization of the nilpotent orbits in $\mathfrak{sl}_n(\mathbb{R})$.\[\square\]
Theorem 3.3 ([CoMc, Theorem 9.3.3]). For the map \( \Psi_{\mathcal{SL}_n(\mathbb{R})} \) in (3.9),

\[
\#\Psi_{\mathcal{SL}_n(\mathbb{R})}^{-1}(d) = \begin{cases} 
1 & \text{for all } d \in \mathcal{P}(n) \setminus \mathcal{P}_{\text{even}}(n) \\
2 & \text{for all } d \in \mathcal{P}_{\text{even}}(n).
\end{cases}
\]

Theorem 3.4. Let \( X \in \mathcal{N}_{\mathfrak{sl}_n(\mathbb{R})}, \ X \neq 0 \) and \( \Psi_{\mathcal{SL}_n(\mathbb{R})}(O_X) = d \). Let \( \{X, H, Y\} \) be a \( \mathfrak{sl}_n(\mathbb{R}) \)-triple in \( \mathfrak{sl}_n(\mathbb{R}) \). Let \( K \) be a maximal compact subgroup of \( \mathcal{Z}_{\mathcal{SL}_n(\mathbb{R})}(X, H, Y) \). Let the maps \( \Lambda_B, \mathcal{D}_{\mathcal{SL}(V)} \) and \( \chi_d \) be defined as in (3.6), (3.5) and (3.7), respectively. Then \( \Lambda_B(K) \) is given by

\[
\Lambda_B(K) = \{ \mathcal{D}_{\mathcal{SL}(V)}(g) \mid g \in \prod_{i=1}^{s} O_{t_{d_i}}, \chi_d(g) = 1 \}.
\]

The nilpotent orbit \( O_X \) in \( \mathfrak{sl}_n(\mathbb{R}) \) is homotopic to \( \text{SO}_n/\Lambda_B(K) \).

Proof. Let \( V \) be a right \( \mathbb{R} \)-vector space of column vectors such that \( \dim_{\mathbb{R}} V = n \), and \( \mathcal{B} \) be an ordered basis of \( V \) as in (3.4). The proof follows from [BCM, Lemma 4.4(1)] and by writing the matrices of the elements of the maximal compact subgroup \( K \) with respect to the ordered basis \( \mathcal{B} \) as in (3.4).

The second part follows from Theorem 2.3 and the well-known fact that any maximal compact subgroup of \( \mathcal{SL}_n(\mathbb{R}) \) is isomorphic to \( \text{SO}_n \).\( \square \)

3.1.3. The case of \( \mathfrak{sl}_n(\mathbb{H}) \). Finally we will consider \( \mathcal{D} = \mathbb{H} \).

Theorem 3.5 ([CoMc, Theorem 9.3.3]). The map \( \Psi_{\mathcal{SL}_n(\mathbb{H})} \) in (3.9) is a bijection.

Theorem 3.6. Let \( X \in \mathcal{N}_{\mathfrak{sl}_n(\mathbb{H})}, \ X \neq 0 \) and \( \Psi_{\mathcal{SL}_n(\mathbb{H})}(O_X) = d \). Let \( \{X, H, Y\} \) be a \( \mathfrak{sl}_n(\mathbb{H}) \)-triple in \( \mathfrak{sl}_n(\mathbb{H}) \). Let \( K \) be a maximal compact subgroup of \( \mathcal{Z}_{\mathcal{SL}_n(\mathbb{H})}(X, H, Y) \). Let the maps \( \Lambda_B \) and \( \mathcal{D}_{\mathcal{SL}(V)} \) be defined as in (3.6) and (3.5), respectively. Then \( \Lambda_B(K) \) is given by

\[
\Lambda_B(K) = \{ \mathcal{D}_{\mathcal{SL}(V)}(g) \mid g \in \prod_{i=1}^{s} \text{Sp}(t_{d_i}) \}.
\]

Moreover, the nilpotent orbit \( O_X \) in \( \mathfrak{sl}_n(\mathbb{H}) \) is homotopic to \( \text{Sp}(n)/\Lambda_B(K) \).

Proof. Let \( V \) be a right \( \mathbb{H} \)-vector space of column vectors such that \( \dim_{\mathbb{H}} V = n \), and \( \mathcal{B} \) be an ordered basis of \( V \) as in (3.4). Now the proof follows from [BCM, Lemma 4.4(1)] and by writing the matrices of the elements of the maximal compact subgroup \( K \) with respect to the ordered basis \( \mathcal{B} \) as in (3.4).

The second part follows from Theorem 2.3 and the well-known fact that any maximal compact subgroup of \( \mathcal{SL}_n(\mathbb{H}) \) is isomorphic to \( \text{Sp}(n) \).\( \square \)

3.2. Homotopy types of the nilpotent orbits in \( \mathfrak{so}(n, \mathbb{C}) \). Let \( n \) be a positive integer such that \( n \geq 5 \). The aim in this subsection is to write down the homotopy types of the nilpotent orbits in the simple Lie algebra \( \mathfrak{so}(n, \mathbb{C}) \) as compact homogeneous spaces. Throughout this subsection \( \langle \cdot, \cdot \rangle \) denotes the symmetric form on \( \mathbb{C}^n \) defined by \( \langle x, y \rangle := x^t y \) for \( x, y \in \mathbb{C}^n \).

We first recall a suitable parametrization of \( \mathcal{N}((\text{SO}(n, \mathbb{C}))) \). Let

\[
\Psi_{\mathcal{SL}_n(\mathbb{C})} : \mathcal{N}((\mathcal{SL}_n(\mathbb{C}))) \rightarrow \mathcal{P}(n)
\]
be the parametrization of \( \mathcal{N}(\text{SL}_n(\mathbb{C})) \); see Section 3.1 for details. Since \( \text{SO}(n, \mathbb{C}) \subset \text{SL}_n(\mathbb{C}) \) (consequently as, the set of nilpotent elements \( \mathcal{N}_\text{so}(n, \mathbb{C}) \subset \mathcal{N}_\text{sl}_n(\mathbb{C}) \)) we have the inclusion map \( \Theta_{\text{SO}(n, \mathbb{C})} : \mathcal{N}(\text{SO}(n, \mathbb{C})) \rightarrow \mathcal{N}(\text{SL}_n(\mathbb{C})) \). Let

\[
\Psi_{\text{SO}(n, \mathbb{C})} := \Psi_{\text{SL}_n(\mathbb{C})} \circ \Theta_{\text{SO}(n, \mathbb{C})} : \mathcal{N}(\text{SO}(n, \mathbb{C})) \rightarrow \mathcal{P}(n)
\]

be the composition. Recall that \( \Psi_{\text{SO}(n, \mathbb{C})}(\mathcal{N}(\text{SO}(n, \mathbb{C}))) \subset \mathcal{P}_1(n) \), see [BCM, Proposition A.6]. Hence we have the following map:

\[
\Psi_{\text{SO}(n, \mathbb{C})} : \mathcal{N}(\text{SO}(n, \mathbb{C})) \rightarrow \mathcal{P}_1(n).
\]

The following known result says that the map \( \Psi_{\text{SO}(n, \mathbb{C})} \) is “almost” a parametrization of the nilpotent orbits in \( \mathfrak{so}(n, \mathbb{C}) \).

**Theorem 3.7** ([CoMc, Theorem 5.1.2, Theorem 5.1.4]). For the above map \( \Psi_{\text{SO}(n, \mathbb{C})} \),

\[
\#\Psi_{\text{SO}(n, \mathbb{C})}^{-1}(d) = \begin{cases} 
2 & \text{for all } d \in \mathcal{P}_{v.even}(n) \\
1 & \text{for all } d \in \mathcal{P}_1(n) \setminus \mathcal{P}_{v.even}(n).
\end{cases}
\]

Let \( 0 \neq X \in \mathcal{N}_{\text{so}(n, \mathbb{C})} \) and \( \{X, H, Y\} \) a \( \mathfrak{sl}_d(\mathbb{R}) \)-triple in \( \mathfrak{so}(n, \mathbb{C}) \). Let \( V \) be a right \( \mathbb{C} \)-vector space of column vectors such that \( \dim_{\mathbb{C}} V = n \). Recall that the form \( (\cdot, \cdot)_d \) on \( L(d-1) \) (see (3.8) for the definition) is symmetric when \( d \in \mathbb{O}_d \) and symplectic when \( d \in \mathbb{E}_d \). Let \( (v_1^d, \ldots, v_{t_\theta}^d) \) be a \( \mathbb{C} \)-basis of \( L(d-1) \) as in [BCM, Proposition A.6]. We may further assume that \( (v_1^d, \ldots, v_{t_\theta}^d) \) is an orthonormal basis of \( L(\theta - 1) \) for the form \( (\cdot, \cdot)_\theta \) for all \( \theta \in \mathbb{O}_d \), i.e.,

\[(v_j^\theta, v_j^\theta)_{\theta} = 1 \quad \text{for } 1 \leq j \leq t_\theta \quad \text{and} \quad (v_j^\theta, v_i^\theta)_{\theta} = 0 \quad \text{for } j \neq i. \tag{3.10}
\]

Similarly, for all \( \eta \in \mathbb{E}_d \) we may assume that \( (v_1^\eta, \ldots, v_{t_\eta}^\eta) \) is a symplectic basis of \( L(\eta - 1) \).

This is equivalent to say that

\[(v_j^\eta, v_{i/2+j}^\eta)_{\eta} = 1 \quad \text{for } 1 \leq j \leq t_\eta/2 \quad \text{and} \quad (v_j^\eta, v_i^\eta)_{\eta} = 0 \quad \text{for } i \neq j + t_\eta/2. \tag{3.11}
\]

Next we will construct an orthonormal basis of \( M(d-1) \). Following [BCM, Lemma A.9], for \( \theta \in \mathbb{O}_d \) and \( l \) even, \( 0 \leq l \leq \theta - 1 \), we define

\[
w_{j_l}^\theta := \begin{cases} 
\frac{(X^l v_j^\theta + X^{\theta-1-l} v_j^\theta)}{\sqrt{2}} & \text{if } 0 \leq l < (\theta - 1)/2 \\
X^l v_j^\theta & \text{if } l = (\theta - 1)/2 \\
\sqrt{-1}(X^{\theta-1-l} v_j^\theta - X^l v_j^\theta)/\sqrt{2} & \text{if } (\theta - 1)/2 < l \leq \theta - 1.
\end{cases}
\]

Similarly, for \( l \) odd, \( 0 \leq l \leq \theta - 1 \), let

\[
w_{j_l}^\theta := \begin{cases} 
\sqrt{-1}(X^l v_j^\theta + X^{\theta-1-l} v_j^\theta)/\sqrt{2} & \text{if } 0 \leq l < (\theta - 1)/2 \\
\sqrt{-1}X^l v_j^\theta & \text{if } l = (\theta - 1)/2 \\
(X^{\theta-1-l} v_j^\theta - X^l v_j^\theta)/\sqrt{2} & \text{if } (\theta - 1)/2 < l \leq \theta - 1.
\end{cases}
\]

Using [BCM, Lemma A.9(2)] and the relation in (3.10) we conclude that for all \( \theta \in \mathbb{O}_d \),

\[
\{w_{j_l}^\theta \mid 1 \leq j \leq t_\theta, \ 0 \leq l \leq \theta - 1\}
\]

is an orthonormal basis of \( M(\theta - 1) \) with respect to \( (\cdot, \cdot) \). For each \( 0 \leq l \leq \theta - 1 \), set

\[V^l(\theta) := \text{Span}_\mathbb{C}\{w_{l1}^\theta, \ldots, w_{lt_l}^\theta\}. \tag{3.14}\]
The orthonormal ordered basis \((w_{1l}^\theta, \ldots, w_{dl}^\theta)\) of \(V^l(\theta)\) with respect to \((\cdot, \cdot)\) is denoted by \(C^l(\theta)\). Recall that \(\mathcal{B}(d) = (X^1v_{1d}^d, \ldots, X^1v_{d}^d)\) is the ordered basis of \(X^1L(d-1)\) for \(0 \leq l \leq d-1, d \in \mathbb{N}_d\) as in (3.3).

**Lemma 3.8.** Let \(X\) be a nilpotent element in \(\mathfrak{so}(n, \mathbb{C})\) and \(\{X, H, Y\}\) be a \(\mathfrak{sl}_2(\mathbb{R})\)-triple in \(\mathfrak{so}(n, \mathbb{C})\) containing \(X\). The following holds:

\[
\mathcal{Z}_{\mathfrak{so}(n, \mathbb{C})}(X, H, Y) = \left\{ g \in \mathfrak{so}(n, \mathbb{C}) \right\mid g(V^l(\theta)) \subset V^l(\theta) \text{ and } g|_{V^l(\theta)} = g|_{V_0(\theta)} \right\}.
\]

**Proof.** The proof of this lemma follows from the observation that if \(\theta \in \mathbb{O}_d\) is fixed, then for \(0 \leq l \leq \theta - 1\) and \(g \in \mathcal{Z}_{\mathfrak{so}(n, \mathbb{C})}(X, H, Y)\) the following holds:

\[
g(X^1L(\theta - 1)) \subset X^1L(\theta - 1) \text{ if and only if } g(V^l(\theta)) \subset V^l(\theta),
\]

and moreover,

\[
[g|_{X^1L(\theta - 1)}]_{\mathcal{B}(\theta)} = [g|_{L(\theta - 1)}]_{\mathcal{B}(\theta)} \text{ if and only if } [g|_{V^l(\theta)}]_{C^l(\theta)} = [g|_{V_0(\theta)}]_{C^0(\theta)}.
\]

In fact, for any such \(g\) as above, \([g|_{L(\theta - 1)}]_{\mathcal{B}(\theta)} = [g|_{V_0(\theta)}]_{C^0(\theta)}\).

**Remark 3.9.** We follow the notation as in Lemma 3.8. Let \(g \in \mathcal{Z}_{\mathfrak{so}(n, \mathbb{C})}(X, H, Y)\). Let \(\theta \in \mathbb{O}_d\) and \(\eta \in \mathbb{E}_d\). Then it follows from Lemma 3.8 that \(g\) keeps the subspaces \(V^0(\theta)\) and \(L(\eta - 1)\) invariant. Since the restriction of \((\cdot, \cdot)\) is a symmetric form on \(V^0(\theta)\) we have \(g|_{V^0(\theta)} \in O(V^0(\theta), (\cdot, \cdot))\). Further recall that the form \((\cdot, \cdot)\), as defined in (3.8), is symplectic on \(L(\eta - 1)\), and \((gx, gy)_{\eta} = (x, y)_{\eta}\) for all \(x, y \in L(\eta - 1)\), see [BCM, Remark A.7]. Thus \(g|_{L(\eta - 1)} \in \text{Sp}(L(\eta - 1), (\cdot, \cdot)_{\eta})\).

For \(\eta \in \mathbb{E}_d, 0 \leq l \leq \eta/2 - 1\), set

\[
W^l(\eta) := X^1L(\eta - 1) + X^{\eta-l-1}L(\eta - 1).
\]

Next we will construct an orthonormal basis of \(W^l\). For \(l\) even, \(0 \leq l \leq \eta/2 - 1\), let

\[
w_{jl}^n := \begin{cases} 
(X^1v_j^n + X^{\eta-1-l}v_{j+n/2}^n) / \sqrt{2} & \text{if } 0 < j \leq t_{n/2} \\
(X^1v_j^n - X^{\eta-1-l}v_{j-n/2}^n) / \sqrt{2} & \text{if } t_{n/2} < j \leq t_n \\
-1(X^1v_j^n - X^{\eta-1-l}v_{j-n/2}^n) / \sqrt{2} & \text{if } t_n < j \leq 3t_n/2 \\
-1(X^1v_j^n + X^{\eta-1-l}v_{j+3n/2}^n) / \sqrt{2} & \text{if } 3t_n/2 < j \leq 2t_n.
\end{cases}
\]

For \(l\) odd, \(0 \leq l \leq \eta/2 - 1\), let

\[
w_{jl}^n := \begin{cases} 
(X^1v_j^n + X^{\eta-1-l}v_{j+n/2}^n) / \sqrt{2} & \text{if } 0 < j \leq t_{n/2} \\
(X^1v_j^n - X^{\eta-1-l}v_{j-n/2}^n) / \sqrt{2} & \text{if } t_{n/2} < j \leq t_n \\
-1(X^1v_j^n - X^{\eta-1-l}v_{j-n/2}^n) / \sqrt{2} & \text{if } t_n < j \leq 3t_n/2 \\
-1(X^1v_j^n + X^{\eta-1-l}v_{j+3n/2}^n) / \sqrt{2} & \text{if } 3t_n/2 < j \leq 2t_n.
\end{cases}
\]

Using (3.11) it follows that for all \(\eta \in \mathbb{E}_d\),

\[
\{w_{jl}^n \mid 1 \leq j \leq 2t_n, 0 \leq l \leq \eta/2 - 1\}
\]
is an orthonormal basis of \( M(\eta - 1) \) with respect to \( \langle \cdot, \cdot \rangle \). The orthonormal ordered basis
\((w_1^\eta, \cdots, w_{2m,l}^\eta)\) of \( W^l(\eta) \) with respect to \( \langle \cdot, \cdot \rangle \) is denoted by \( \mathcal{D}^l(\eta) \).

The next two lemmas are standard fact where we recall, without proofs, explicit descriptions of a maximal compact subgroup in an orthogonal group and a maximal compact subgroup in a symplectic group. Let \( V' \) be a \( \mathbb{C} \)-vector space, \( \langle \cdot, \cdot' \rangle \) be a non-degenerate symmetric form on \( V' \) and \( B' \) be an orthonormal basis of \( V' \). We set

\[
K_{B'} := \{ g \in SU(V', \langle \cdot, \cdot' \rangle) \mid [g]_{B'} = \overline{[g]}_{B'} \}.
\]

**Lemma 3.10.** Let \( V', \langle \cdot, \cdot' \rangle, B' \) be as above. Then \( K_{B'} \) is a maximal compact subgroup in \( SO(V, \langle \cdot, \cdot' \rangle) \).

Let \( \tilde{V} \) be a \( \mathbb{C} \)-vector space, \( \langle \cdot, \cdot \rangle \) be a non-degenerate symplectic form on \( \tilde{V} \) and \( \tilde{B} \) be a symplectic basis of \( \tilde{V} \). Let \( J_{\tilde{B}} \) be the complex structure on \( \tilde{V} \) associated to \( \tilde{B} \). Let

\[
2m := \dim_\mathbb{C} \tilde{V}.
\]

We set

\[
K_{\tilde{B}} := \{ g \in Sp(\tilde{V}, \langle \cdot, \cdot \rangle) \mid \tilde{J}_{\tilde{B}} J_{\tilde{B}} = J_{\tilde{B}} g \}.
\]

**Lemma 3.11.** Let \( \tilde{V}, \langle \cdot, \cdot \rangle, \tilde{B} \) be as above. Then

1. \( K_{\tilde{B}} \) is a maximal compact subgroup in \( Sp(\tilde{V}, \langle \cdot, \cdot \rangle) \).
2. \( K_{\tilde{B}} = \{ g \in Sp(\tilde{V}, \langle \cdot, \cdot \rangle) \mid [g]_{\tilde{B}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \) where \( A + jB \in Sp(m), A, B \in M_m(\mathbb{R}) \} \).

We next give the description of a suitable maximal compact subgroup of the group \( \mathbb{Z}_{SO(n,\mathbb{C})}(X, H, Y) \) in terms of the subspaces \( V^l(\theta), W^l(\eta) \) defined as in (3.14), (3.15), respectively.

**Lemma 3.12.** Let \( K \) be the subgroup of \( \mathbb{Z}_{SO(n,\mathbb{C})}(X, H, Y) \) consisting of elements \( g \) in \( \mathbb{Z}_{SO(n,\mathbb{C})}(X, H, Y) \) satisfying the following conditions:

1. For all \( \theta \in \mathbb{O}_d \) and \( 0 \leq l \leq \theta - 1 \) the inclusion \( g(V^l(\theta)) \subset V^l(\theta) \) holds.
2. For all \( \theta \in \mathbb{O}_d \), there exist \( A(\theta) \in O_{16} \) such that

\[
[g|_{V^l(\theta)}]_{C^l(\theta)} = A(\theta) \quad \text{for} \quad 0 \leq l \leq \theta - 1.
\]

3. For all \( \eta \in \mathbb{E}_d \) and \( 0 \leq l \leq \eta/2 - 1 \) the inclusion \( g(W^l(\eta)) \subset W^l(\eta) \) holds.
4. For all \( \eta \in \mathbb{E}_d \), \( 0 \leq l \leq \eta/2 - 1 \) there exist \( A_1(\eta), A_2(\eta), B_1(\eta), B_2(\eta) \in M_{\eta/2}(\mathbb{R}) \) with \( (A_1(\eta) + iA_2(\eta)) + j(B_1(\eta) + iB_2(\eta)) \in Sp(t_{\eta/2}) \) such that

\[
[g|_{W^l(\eta)}]_{D^l(\eta)} = \begin{pmatrix} A_1(\eta) & -B_1(\eta) & -A_2(\eta) & -B_2(\eta) \\ B_1(\eta) & A_1(\eta) & -B_2(\eta) & A_2(\eta) \\ A_2(\eta) & B_2(\eta) & A_1(\eta) & -B_1(\eta) \\ B_2(\eta) & -A_2(\eta) & B_1(\eta) & A_1(\eta) \end{pmatrix}.
\]

Then \( K \) is a maximal compact subgroup of \( \mathbb{Z}_{SO(n,\mathbb{C})}(X, H, Y) \).
Proof. Let \( K' \) be the subgroup consisting of all \( g \in \mathcal{Z}_{SO(n, \mathbb{C})}(X, H, Y) \) satisfying the conditions (3.18) and (3.19) below:
\[
\left[ g \right]_{\mathcal{C}(\theta)} \equiv \left[ g \right]_{\mathcal{C}(\theta)}^{\mathcal{O}(\theta)}, \quad \text{for all } \theta \in \mathcal{O}_d; \quad (3.18)
\]
\[
\left[ g \right]_{\mathcal{L}(\eta^{-1})} \lambda_{\mathcal{B}(\eta)} = \lambda_{\mathcal{B}(\eta)} \left[ g \right]_{\mathcal{L}(\eta^{-1})}, \quad \text{for all } \eta \in \mathcal{E}_d. \quad (3.19)
\]
In view of Lemma 3.8, Lemma 3.10 and Lemma 3.11 it is clear that \( K' \) is a maximal compact subgroup of \( \mathcal{Z}_{SO(n, \mathbb{C})}(X, H, Y) \). Thus to prove the lemma it is enough to show that \( K = K' \). Let \( g \in \mathcal{Z}_{SO(n, \mathbb{C})}(X, H, Y) \). In view of Lemma 3.10 and Remark 3.9 it is clear that \( g \) satisfies (1), (2) in the statement of the lemma if and only if \( g \) satisfies (3.18).

Clearly, \( g \) satisfies (3) of the Lemma 3.12. Let \( \left[ g \right]_{\mathcal{L}(\eta^{-1})} \lambda_{\mathcal{B}(\eta)} := \left( \begin{array}{cc} A(\eta) & C(\eta) \\ B(\eta) & D(\eta) \end{array} \right) \). Now suppose that \( g \) satisfies (3.19). Then it follows that \( C(\eta) = -B(\eta), D(\eta) = A(\eta) \) and
\[
A(\eta) + j B(\eta) \in \text{Sp}(t\eta/2).
\]
Set
\[
A(\eta) := A_1(\eta) + \sqrt{-1} A_2(\eta) \quad \text{and} \quad B(\eta) := B_1(\eta) + \sqrt{-1} B_2(\eta)
\]
for \( A_1(\eta), A_2(\eta), B_1(\eta), B_2(\eta) \in M_{\eta/2}(\mathbb{R}) \). Then \( g \) satisfies (4) of the Lemma 3.12. Next we assume that \( g \) satisfies (3), (4) of Lemma 3.12. We observe that
\[
\left[ g \right]_{\mathcal{L}(\eta^{-1})} \lambda_{\mathcal{B}(\eta)} = \left( \begin{array}{cc} A_1(\eta) + \sqrt{-1} A_2(\eta) & -B_1(\eta) + \sqrt{-1} B_2(\eta) \\ B_1(\eta) + \sqrt{-1} B_2(\eta) & A_1(\eta) - \sqrt{-1} A_2(\eta) \end{array} \right)
\]
which proves that (3.19) holds. This completes the proof. \( \square \)

Now we introduce some notation which will be required to state Theorem 3.13. For \( \eta \in \mathcal{E}_d \), set
\[
\mathcal{D}(\eta) := \mathcal{D}^0(\eta) \lor \cdots \lor \mathcal{D}^{\eta/2 - 1}(\eta),
\]
and for \( \theta \in \mathcal{O}_d \), set
\[
\mathcal{C}(\theta) := \mathcal{C}^0(\theta) \lor \cdots \lor \mathcal{C}^{\theta - 1}(\theta).
\]
Let \( \alpha := \# \mathcal{E}_d \) and \( \beta := \# \mathcal{O}_d \). We enumerate \( \mathcal{E}_d = \{ \eta_i \mid 1 \leq i \leq \alpha \} \) such that \( \eta_i < \eta_{i+1} \), and similarly \( \mathcal{O}_d = \{ \theta_j \mid 1 \leq j \leq \beta \} \) such that \( \theta_j < \theta_{j+1} \). Now define
\[
\mathcal{E} := \mathcal{D}(\eta_1) \lor \cdots \lor \mathcal{D}(\eta_\alpha); \quad \text{and} \quad \mathcal{O} := \mathcal{C}(\theta_1) \lor \cdots \lor \mathcal{C}(\theta_\beta).
\]
Finally, define
\[
\mathcal{H} := \mathcal{E} \lor \mathcal{O}. \quad (3.20)
\]
Define the \( \mathbb{R} \)-algebra embedding
\[
\varphi_{m, \mathbb{C}} : M_m(\mathbb{C}) \hookrightarrow M_{2m}(\mathbb{R}), \quad S + \sqrt{-1} T \longmapsto \begin{pmatrix} S & -T \\ T & S \end{pmatrix}, \quad (3.21)
\]
where \( S, T \in M_m(\mathbb{R}) \). Similarly, for \( P, Q \in M_m(\mathbb{C}) \) define the \( \mathbb{R} \)-algebra embedding
\[
\varphi_{m, \mathbb{H}} : M_m(\mathbb{H}) \hookrightarrow M_{2m}(\mathbb{C}), \quad P + j Q \longmapsto \begin{pmatrix} P & -Q \\ Q & P \end{pmatrix}. \quad (3.22)
\]
The $\mathbb{R}$-algebra $\prod_{i=1}^{\alpha} M \cdot t_{\alpha i/2}(\mathbb{H}) \times \prod_{k=1}^{\beta} M \cdot t_{\beta k}(\mathbb{R})$ is embedded into $M_n(\mathbb{R})$ in the following way:

$$D_{SO(n,\mathbb{C})} : \prod_{i=1}^{\alpha} M \cdot t_{\alpha i/2}(\mathbb{H}) \times \prod_{k=1}^{\beta} M \cdot t_{\beta k}(\mathbb{R}) \rightarrow M_n(\mathbb{R}) \quad (3.23)$$

$$(A_{\eta_1}, \ldots, A_{\eta_{\alpha}}; C_{\theta_1}, \ldots, C_{\theta_{\beta}})\mapsto \bigoplus_{i=1}^{\alpha} \left( \varphi_{t_{\alpha i}/2, \mathbb{H}}(A_{\eta_i}) \right)_{\eta_i/2}^\wedge \bigoplus_{k=1}^{\beta} \left( C_{\theta_k} \right)_{\theta_k}^\wedge.$$

Note that the basis $\mathcal{H}$ as in (3.20) is an orthonormal basis of $V$ with respect to $\langle \cdot, \cdot \rangle$. Let $\Lambda_{\mathcal{H}} : \text{End}_\mathbb{C} \mathbb{C}^n \rightarrow M_n(\mathbb{C})$ be the isomorphism of $\mathbb{C}$-algebras induced by the ordered basis $\mathcal{H}$.

**Theorem 3.13.** Let $X \in \mathcal{N}_{\mathfrak{so}(n,\mathbb{C})}(\mathcal{O}_X) \neq 0$, and $\Psi_{\mathfrak{so}(n,\mathbb{C})}(\mathcal{O}_X) = \mathfrak{d}$. Let $\alpha := \#\mathfrak{d}$ and $\beta := \#\mathcal{O}_{\mathfrak{d}}$. Let $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{so}(n,\mathbb{C})$. Let $K$ be the maximal compact subgroup of $\mathcal{Z}_{\mathfrak{so}(n,\mathbb{C})}(X, H, Y)$ as in Lemma 3.12. Let the map $D_{\mathfrak{so}(n,\mathbb{C})}$ be defined as in (3.23). Then $\Lambda_{\mathcal{H}}(K) \subset SO_n$ is given by

$$\Lambda_{\mathcal{H}}(K) = \left\{ D_{\mathfrak{so}(n,\mathbb{C})}(g) \mid g \in \prod_{i=1}^{\alpha} \text{Sp}(t_{\alpha i}/2) \times \prod_{j=1}^{\beta} \text{SO} \left( \prod_{k=1}^{\beta} \varphi_{t_{\alpha i}/2, \mathbb{H}}(A_{\eta_i}) \right)_{\eta_i/2}^\wedge \right\}.$$ 

The nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{so}(n,\mathbb{C})$ is homotopic to $SO_n/\Lambda_{\mathcal{H}}(K)$.

**Proof.** This follows by writing the matrices of the elements of the maximal compact subgroup $K$ in Lemma 3.12 with respect to the ordered basis $\mathcal{H}$ as in (3.20).

The second part follows from Theorem 2.3 and the fact that any maximal compact subgroup of $SO(n,\mathbb{C})$ is isomorphic to $SO_n$. \(\square\)

### 3.3. Homotopy types of the nilpotent orbits in $\mathfrak{so}(p,q)$. Let $n$ be a positive integer and $(p,q)$ be a pair of non-negative integers such that $p + q = n \geq 5$. In this subsection we write down the homotopy types of the nilpotent orbits in $\mathfrak{so}(p,q)$ under the adjoint action of $SO(p,q)^\circ$. Throughout this subsection $\langle \cdot, \cdot \rangle$ denotes the symmetric form on $\mathbb{R}^n$ defined by $\langle x, y \rangle := x^tI_{p,q}y$ where $x, y \in \mathbb{R}^n$ and $I_{p,q}$ is as in (2.1).

We first need to describe a suitable parametrization of $\mathcal{N}(SO(p,q)^\circ)$, the set of all nilpotent orbits in $\mathfrak{so}(p,q)$ under the adjoint action of $SO(p,q)^\circ$, see [BCM, §4.5]. Let

$$\Psi_{\text{SL}_n(\mathbb{R})} : \mathcal{N}(\text{SL}_n(\mathbb{R})) \rightarrow \mathcal{P}(n)$$

be the parametrization of $\mathcal{N}(\text{SL}_n(\mathbb{R}))$ as in Theorem 3.3. Since $SO(p,q) \subset \text{SL}_n(\mathbb{R})$ (consequently as, the set of nilpotent elements $\mathcal{N}_{\mathfrak{so}(p,q)} \subset \mathcal{N}_{\text{SL}_n(\mathbb{R})}$) we have the inclusion map

$$\Theta_{SO(p,q)^\circ} : \mathcal{N}(SO(p,q)^\circ) \rightarrow \mathcal{N}(\text{SL}_n(\mathbb{R})).$$

Let

$$\Psi'_{SO(p,q)^\circ} := \Psi_{\text{SL}_n(\mathbb{R})} \circ \Theta_{SO(p,q)^\circ} : \mathcal{N}(SO(p,q)^\circ) \rightarrow \mathcal{P}(n)$$

be the composition. Recall that $\Psi'_{SO(p,q)^\circ}(\mathcal{N}(SO(p,q)^\circ)) \subset \mathcal{P}_1(n)$. Let $X \in \mathfrak{so}(p,q)$ be a non-zero nilpotent element and $\mathcal{O}_X$ the corresponding nilpotent orbit in $\mathfrak{so}(p,q)$ under the adjoint action of $SO(p,q)^\circ$. Let $\{X, H, Y\} \subset \mathfrak{so}(p,q)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let $V := \mathbb{R}^n$ be the right $\mathbb{R}$-vector space of column vectors. Let $\{d_1, \ldots, d_s\}$, with $d_1 < \cdots < d_s,$
be ordered finite set of natural numbers that occur as dimension of non-zero irreducible \( \text{Span}_\mathbb{R}\{X, H, Y\}\)-submodules of \( V \). Recall that \( M(d-1) \) is defined to be the isotypical component of \( V \) containing all irreducible \( \text{Span}_\mathbb{R}\{X, H, Y\}\)-submodules of \( V \) with highest weight \( d-1 \) and as in (2.2) we set

\[
L(d-1) := V_{Y,0} \cap M(d-1).
\]

Let \( t_{d_r} := \dim_\mathbb{R} L(d_r - 1), 1 \leq r \leq s \). Then

\[
d := [d_{t_1}^1, \ldots, d_{t_s}^s] \in \mathcal{P}_1(n),
\]

and moreover, \( \Psi'_{\text{SO}(p,q)^o}(\mathcal{O}_X) = d \).

We next assign \( \text{sgn}_{\mathcal{O}_X} \in \mathcal{S}_d^{\text{even}}(p,q) \) to each \( \mathcal{O}_X \in \mathcal{N}(\text{SO}(p,q)^o) \); see (2.9) for the definition of \( \mathcal{S}_d^{\text{even}}(p,q) \). For each \( d \in \mathbb{N}_d \) (see (2.4) for the definition of \( \mathbb{N}_d \)) we will define a \( t_d \times d \) matrix \( (m_{ij}^d) \) in \( A_d \) that depends only on the orbit \( \mathcal{O}_X \); see (2.6) for the definition of \( A_d \). For this, recall that the form

\[
(\cdot, \cdot)_d : L(d-1) \times L(d-1) \to \mathbb{R},
\]

defined in (3.8), is symmetric or symplectic according as \( d \) is odd or even. Denoting the signature of \( (\cdot, \cdot)_d \) by \( (p_d, q_d) \) when \( d \in \mathbb{N}_d \), we now define

\[
m_{i_1}^\eta := +1 \quad \text{if} \quad 1 \leq i \leq t_\eta, \quad \eta \in \mathbb{E}_d;
\]

\[
m_{i_1}^\theta := \begin{cases} 
+1 & \text{if} \quad 1 \leq i \leq p_\theta \\
-1 & \text{if} \quad p_\theta < i \leq t_\theta, \quad \theta \in \mathbb{O}_d;
\end{cases}
\]

and for \( j > 1 \), define \( (m_{ij}^d) \) as follows:

\[
m_{ij}^d := (-1)_{j+1} m_{i_1}^d \quad \text{if} \quad 1 < j \leq d, \quad d \in \mathbb{E}_d \cup \mathbb{O}_d^1;
\]

\[
m_{ij}^\theta := \begin{cases} 
(-1)_{j+1} m_{i_1}^\theta & \text{if} \quad 1 < j \leq \theta - 1, \quad \theta \in \mathbb{O}_d^3 \\
-m_{i_1}^\theta & \text{if} \quad j = \theta
\end{cases}
\]

Then the matrices \( (m_{ij}^d) \) clearly verify (Yd.2). Set

\[
\text{sgn}_{\mathcal{O}_X} := ((m_{ij}^d), \ldots, (m_{ij}^d)).
\]

It now follows from the last paragraph of [BCM, Remark A.13] and the above definition of \( m_{ij}^\eta \) for \( \eta \in \mathbb{E}_d \) that \( \text{sgn}_{\mathcal{O}_X} \in \mathcal{S}_d^{\text{even}}(p,q) \). Thus we have the map

\[
\Psi_{\text{SO}(p,q)^o} : \mathcal{N}(\text{SO}(p,q)^o) \to \mathcal{Y}_1^{\text{even}}(p,q), \quad \mathcal{O}_X \mapsto (\Psi'_{\text{SO}(p,q)^o}(\mathcal{O}_X), \text{sgn}_{\mathcal{O}_X});
\]

where \( \mathcal{Y}_1^{\text{even}}(p,q) \) is as in (2.12). The map \( \Psi_{\text{SO}(p,q)^o} \) is surjective. The following theorem is standard, see [CoMc, Theorem 9.3.4], [BCM, Theorem 4.16].

**Theorem 3.14.** For the above map \( \Psi_{\text{SO}(p,q)^o} \),

\[
\#\Psi_{\text{SO}(p,q)^o}^{-1}(d, \text{sgn}) = \begin{cases} 
4 & \text{for all } d \in \mathcal{P}_{v,\text{even}}(n) \\
2 & \text{for all } d \in \mathcal{P}_1(n) \setminus \mathcal{P}_{v,\text{even}}(n), \text{sgn} \in \mathcal{S}_d'(p,q) \\
1 & \text{otherwise}.
\end{cases}
\]

Let \( 0 \neq X \in \mathcal{N}_{\text{so}(p,q)} \) and \( \{X, H, Y\} \subset \text{so}(p,q) \) a \( \mathfrak{s}\mathfrak{l}_2(\mathbb{R}) \)-triple. Let \( \Psi_{\text{SO}(p,q)^o}(\mathcal{O}_X) = (d, \text{sgn}_{\mathcal{O}_X}). \)
Recall that \( -d \dim v \in \theta \) that \( (\cdot, \cdot)_{\theta} \) on \( L(d - 1) \) (see (3.8) for the definition) is symmetric for \( d \in \mathcal{O}_d \) and symplectic for \( d \in \mathcal{E}_d \). Let \( (v^1_d, \ldots, v^d_t) \) be a \( \mathbb{R} \)-basis of \( L(d - 1) \) as in [BCM, Proposition A.6]. Recall that \( \text{sgn}_{\mathcal{O}_X} \) determines the signature of \( (\cdot, \cdot)_{\theta} \) on \( L(\theta - 1), \theta \in \mathcal{O}_d \); let \( (p_\theta, q_\theta) \) be the signature of \( (\cdot, \cdot)_{\theta} \). We may further assume that \( (v^1_\theta, \ldots, v^d_\theta) \) is a standard orthogonal basis of \( L(\theta - 1) \) for the form \( (\cdot, \cdot)_{\theta} \) for all \( \theta \in \mathcal{O}_d \), i.e.,

\[
(v^j_\theta, v^j_\theta)_{\theta} = \begin{cases} 
+1 & \text{if } 1 \leq j \leq p_\theta \\
-1 & \text{if } p_\theta < j \leq t_\theta.
\end{cases} \tag{3.26}
\]

Similarly, we may assume that \( (v^n_\eta, \ldots, v^n_{l_\eta}) \) is a symplectic basis of \( L(\eta - 1) \) for all \( \eta \in \mathcal{E}_d \). This is equivalent to say that

\[
(v^n_j, v^n_{n/2+j})_{\eta} = 1 \quad \text{for } 1 \leq j \leq \eta/2 \quad \text{and} \quad (v^n_j, v^n_i)_{\eta} = 0 \quad \text{for } i \neq j + \eta/2. \tag{3.27}
\]

For \( \theta \in \mathcal{O}_d \), let \( \{w_{l_\theta}^j \mid 1 \leq j \leq t_\theta, 0 \leq l \leq \theta - 1\} \) be the \( \mathbb{R} \)-basis of \( M(\theta - 1) \) as in [BCM, Lemma A.9(2)]. For each \( 0 \leq l \leq \theta - 1 \), define

\[
V^l(\theta) := \text{Span}_\mathbb{R}\{w^j_{l_\theta}, \ldots, w^j_{t_\theta}\}.
\]

The ordered basis \( (w^j_{l_\theta}, \ldots, w^j_{t_\theta}) \) of \( V^l(\theta) \) is denoted by \( C^l(\theta) \).

Next we will write down a general version of [BCM, Lemma 4.18] which will give a suitable description of reductive part of the centralizer of a nilpotent element in \( \mathfrak{so}(p, q) \). Recall that \( \mathcal{C}^l(d) = (X^l v^d_1, \ldots, X^l v^d_t) \) is the ordered basis of \( X^l L(d - 1) \) for \( 0 \leq l \leq d - 1, d \in \mathbb{N}_d \) as in (3.3).

**Lemma 3.15.** Let \( X \) be a nilpotent element in \( \mathfrak{so}(p, q) \) and \{\( X, H, Y \)\} be a \( \mathfrak{sl}_2(\mathbb{R}) \)-triple in \( \mathfrak{so}(p, q) \) containing \( X \). Then the following holds:

\[
\mathcal{Z}_{\mathfrak{so}(p,q)}(X, H, Y) = \left\{ g \in \text{SO}(p, q) \left| \begin{array}{c}
g(V^l(\theta)) \subset V^l(\theta) \text{ and} \\
g(V^l(\theta)) \subset V^l(\theta) \\text{ and} \\
g(X^l L(\eta - 1)) \subset X^l L(\eta - 1) \text{ and} \\
g(X^l L(\eta - 1)) \subset X^l L(\eta - 1) \forall \eta \in \mathcal{E}_d, 0 \leq l < \eta
\end{array} \right. \right\}.
\]

**Proof.** We omit the proof as it is very similar to that of Lemma 3.8. \( \Box \)

We next impose orderings on the sets \( \{v \in C^l(\theta) \mid \langle v, v \rangle > 0\}, \{v \in C^l(\theta) \mid \langle v, v \rangle < 0\} \). Define the ordered sets by \( C^l_+(\theta), C^l_-(\theta), C^l_+(\zeta) \) and \( C^l_-(\zeta) \) as in [BCM, (4.19), (4.20), (4.21), (4.22)], respectively according as \( \theta \in \mathcal{O}_d \) or \( \zeta \in \mathcal{O}_d^3 \). For all \( \theta \in \mathcal{O}_d \) and \( 0 \leq l \leq \theta - 1 \), set

\[
V^l_+(\theta) := \text{Span}_\mathbb{R}\{v \mid v \in C^l(\theta), \langle v, v \rangle > 0\},
\]

\[
V^l_-(\theta) := \text{Span}_\mathbb{R}\{v \mid v \in C^l(\theta), \langle v, v \rangle < 0\}.
\]

It is straightforward from (3.26), and the orthogonality relations in [BCM, Lemma A.9], that \( C^l_+(\theta) \) and \( C^l_-(\theta) \) are indeed ordered bases of \( V^l_+(\theta) \) and \( V^l_-(\theta) \), respectively. For \( \eta \in \mathcal{E}_d, 0 \leq l \leq \eta/2 - 1 \), set

\[
W^l(\eta) := X^l L(\eta - 1) + X^{\eta-1-l} L(\eta - 1).
\]
Now we will construct a standard orthogonal basis of $W^l$ as done in (3.16), (3.17). For $l$ even, $0 \leq l \leq \eta/2 - 1$, let

$$w^\eta_{jl} := \begin{cases} 
\left( X^l v^\eta_{j} + X^{\eta-1-l} v^\eta_{j+1} \right) / \sqrt{2} & \text{if } 1 < j \leq t_\eta/2 \\
\left( X^l v^\eta_{j} - X^{\eta-1-l} v^\eta_{j-1} \right) / \sqrt{2} & \text{if } t_\eta/2 < j \leq t_\eta \\
\left( X^l v^\eta_{j-t_\eta} + X^{\eta-1-l} v^\eta_{j-t_\eta} \right) / \sqrt{2} & \text{if } t_\eta < j \leq 3t_\eta/2 \\
\left( X^l v^\eta_{j-t_\eta} - X^{\eta-1-l} v^\eta_{j-3t_\eta/2} \right) / \sqrt{2} & \text{if } 3t_\eta/2 < j \leq 2t_\eta.
\end{cases}$$

For $l$ odd, $0 \leq l \leq \eta/2 - 1$, let

$$w^\eta_{jl} := \begin{cases} 
\left( X^l v^\eta_{j} - X^{\eta-1-l} v^\eta_{j+1} \right) / \sqrt{2} & \text{if } 1 < j \leq t_\eta/2 \\
\left( X^l v^\eta_{j} + X^{\eta-1-l} v^\eta_{j-1} \right) / \sqrt{2} & \text{if } t_\eta/2 < j \leq t_\eta \\
\left( X^l v^\eta_{j-t_\eta} + X^{\eta-1-l} v^\eta_{j-t_\eta} \right) / \sqrt{2} & \text{if } t_\eta < j \leq 3t_\eta/2 \\
\left( X^l v^\eta_{j-t_\eta} - X^{\eta-1-l} v^\eta_{j-3t_\eta/2} \right) / \sqrt{2} & \text{if } 3t_\eta/2 < j \leq 2t_\eta.
\end{cases}$$

Using (3.27) it follows that $\{w^\eta_{jl} \mid 1 \leq j \leq 2t_\eta, 0 \leq l \leq \eta/2 - 1\}$ is a standard orthogonal basis of $M(\eta - 1)$ with respect to $\langle \cdot, \cdot \rangle$ for all $\eta \in \mathbb{E}_d$. The ordered basis $(w^\eta_{1l}, \ldots, w^\eta_{2t_\eta l})$ of $W^l(\eta)$ is denoted by $D^l(\eta)$.

**Remark 3.16.** We follow the notation of Lemma 3.15. Let $g \in Z_{SO(p,q)}(X, H, Y)$. Let $\theta \in O_d$ and $\eta \in \mathbb{E}_d$. Then it follows from Lemma 3.15 that $g$ keeps the subspaces $V^0(\theta)$ and $L(\eta - 1)$ invariant. Since the restriction of $\langle \cdot, \cdot \rangle$ is a symmetric form on $V^0(\theta)$ we have $g|_{V^0(\theta)} \in O(V^0(\theta), \langle \cdot, \cdot \rangle)$. Further recall that the form $(\cdot, \cdot)_\eta$, as defined in (3.8), is symplectic on $L(\eta - 1)$, and $(g x, g y)_\eta = (x, y)_\eta$ for all $x, y \in L(\eta - 1)$; see [BCM, Remark A.7]. Thus $g|_{L(\eta - 1)} \in \text{Sp}(L(\eta - 1), (\cdot, \cdot)_\eta)$.

In the next lemma, which generalizes [BCM, Lemma 4.19], we specify a maximal compact subgroup of $Z_{SO(p,q)}(X, H, Y)$ which will be used in Theorem 3.18. The notation $(-1)^l$ stands for the sign ‘+’ or the sign ‘−’ according as $l$ is an even or odd integer. Recall that the $\mathbb{R}$-algebra embedding $\varphi_{m,c}$ is defined in (3.21).

**Lemma 3.17.** Let $K$ be the subgroup of $Z_{SO(p,q)}(X, H, Y)$ consisting of elements $g$ in $Z_{SO(p,q)}(X, H, Y)$ satisfying the following conditions:

1. $g(V^l_+ (\theta)) \subset V^l_+ (\theta)$ and $g(V^l_- (\theta)) \subset V^l_- (\theta)$, for all $\theta \in O_d$ and $0 \leq l \leq \theta - 1$.
2. When $\theta \in O^*_d$,

$$[g|_{V^l_+ (\theta)}]_{c^l_+ (\theta)} = \begin{cases} 
[g|_{V^l_- (\theta)}]_{c^l_- (\theta)} & \text{for all } 0 \leq l < (\theta - 1)/2 \\
[g|_{V^{(\theta-1)/2}_+ (\theta)}]_{c^{(\theta-1)/2}_+ (\theta)} & \text{for all } (\theta - 1)/2 < l \leq (\theta - 1)/2 \\
[g|_{V^{(\theta-1)/2}_- (\theta)}]_{c^{(\theta-1)/2}_- (\theta)} & \text{for all } (\theta - 1)/2 < l \leq \theta - 1.
\end{cases}$$
\[
\begin{aligned}
g\big|_{V^0(\theta)}\bigg|_{c_0(\theta)} &= \begin{cases} 
g\big|_{V^l(-1)^{l+1}(\theta)}\big|_{c_{(-1)^{l+1}(\theta)}} & \text{for all } 0 \leq l < (\theta - 1)/2 \\
g\big|_{V^l(-1)^{(\theta-1)/2}(\theta)}\big|_{c_{(-1)^{(\theta-1)/2}(\theta)}} & \\
g\big|_{V^l(-1)^l(\theta)}\big|_{c_{(-1)^l(\theta)}} & \text{for all } (\theta - 1)/2 < l \leq \theta - 1.
\end{cases}
\end{aligned}
\]

(3) When \( \zeta \in O_3^3 \),
\[
\begin{aligned}
g\big|_{V^\zeta(\zeta)}\bigg|_{c_0(\zeta)} &= \begin{cases} 
g\big|_{V^l(-1)^{l+1}(\zeta)}\big|_{c_{(-1)^{l+1}(\zeta)}} & \text{for all } 0 \leq l < (\zeta - 1)/2 \\
g\big|_{V^l(-1)^{(\zeta-1)/2}(\zeta)}\big|_{c_{(-1)^{(\zeta-1)/2}(\zeta)}} & \\
g\big|_{V^l(-1)^l(\zeta)}\big|_{c_{(-1)^l(\zeta)}} & \text{for all } (\zeta - 1)/2 < l \leq \zeta - 1,
\end{cases}
\end{aligned}
\]

(4) For all \( \eta \in E_d \) and \( 0 \leq l \leq \eta/2 - 1 \) the inclusion \( g(W^l(\eta)) \subseteq W^l(\eta) \) holds.

(5) For all \( \eta \in E_d, \) \( 0 \leq l \leq \eta/2 - 1, \) there exists \( A(\eta), B(\eta) \) such that \( A(\eta) + \sqrt{-1}B(\eta) \in U(t_{\eta/2}) \) such that
\[
\begin{aligned}
[g|_{W^l(\eta)}]_{D(\eta)} = (\psi_{t_{\eta/2},C}(A(\eta) + \sqrt{-1}B(\eta)))^2 = \begin{pmatrix} A(\eta) & -B(\eta) \\ B(\eta) & A(\eta) \end{pmatrix}.
\end{aligned}
\]

**Proof.** Let \( K' \) be the subgroup consisting of all \( g \in Z_{SO(p,q)}(X, H, Y) \) satisfying the conditions in (3.28), (3.29) and (3.30) below:
\[
\begin{aligned}
g(V^l_+(\theta)) \subseteq V^l_+(\theta), & \quad g(V^l_-(\theta)) \subseteq V^l_-(\theta), \quad \text{for all } \theta \in O_d, \quad 0 \leq l < \theta; \quad (3.28) \\
[g|_{V^l(\theta)}]_{c^l(\theta)} = [g|_{V^0(\theta)}]_{c^0(\theta)}, & \quad \text{for all } \theta \in O_d, \quad 0 \leq l < \theta; \quad (3.29) \\
g|_{L(\eta-1)} \text{ commutes with } J_{B^0(\eta)}, & \quad \text{for all } \eta \in E_d. \quad (3.30)
\end{aligned}
\]

In view of Lemma 3.15, Remark 3.16 and [BCM, Lemma 4.34] which is analogous to Lemma 3.11, it is straightforward that \( K' \) is actually a maximal compact subgroup of \( Z_{SO(p,q)}(X, H, Y) \). Thus to prove the lemma it is suffices to show that \( K = K' \). Let \( g \in Z_{SO(p,q)}(X, H, Y) \). We omit the proof of the fact that \( g \) satisfies (3.28) and (3.29) if and only if \( g \) satisfies (1), (2) and (3) in the statement of the lemma, as this follows from [BCM, Lemma 4.19] when \( \theta \in O_d^1 \) and \( \zeta \in O_3^3 \). Let
\[
[g|_{L(\eta-1)}]_{B^0(\eta)} := \begin{pmatrix} A(\eta) & C(\eta) \\ B(\eta) & D(\eta) \end{pmatrix}.
\]
Lastly, we assume that $g$ satisfies (3.30). Then it follows that $A(\eta) = D(\eta)$ and $B(\eta) = -C(\eta)$. Using [BCM, Lemma 4.34] and Remark 3.16, we have $A(\eta) + \sqrt{-1}B(\eta) \in U(t_{\eta}/2)$. Now statement (4) of the lemma follows from the definition of $D'(\eta)$. Also statement (5) of the lemma holds, as $[g|_{W'(\eta)}]_{D'(\eta)} = \begin{pmatrix} A(\eta) & -B(\eta) \\ B(\eta) & A(\eta) \end{pmatrix}$. Lastly, we assume that $g$ satisfies statements (4) and (5) of Lemma 3.17. Then it follows that $[g|_{L(\eta-1)}]_{B^g(\eta)} = \begin{pmatrix} A(\eta) & -B(\eta) \\ B(\eta) & A(\eta) \end{pmatrix}$. Now clearly (3.30) holds for $g$ and this completes the proof of the lemma. 

For $\eta \in \mathbb{E}_d$, define $D_+^0(\eta) := (w_{1\eta}, \ldots, w_{t\eta})$ and $D_-^0(\eta) := (w_{(t\eta)+1}, \ldots, w_{2t\eta})$. Set $D_+^0(\eta) := D_+^0(\eta) \lor \cdots \lor D_+^{\eta/2-1}(\eta)$ and $D_-^0(\eta) := D_-^0(\eta) \lor \cdots \lor D_-^{\eta/2-1}(\eta)$. When $\theta \in \mathbb{O}_d$, define $C_+^0(\theta)$ and $C_-^0(\theta)$ as in [BCM, (4.19), (4.20), (4.21), (4.22)]. Set $C_+^0(\theta) := C_+^0(\theta) \lor \cdots \lor C_+^{\eta/2-1}(\theta)$ and $C_-^0(\theta) := C_-^0(\theta) \lor \cdots \lor C_-^{\eta/2-1}(\theta)$.

Let $\alpha := \#\mathbb{E}_d$, $\beta := \#\mathbb{O}_d^1$ and $\gamma := \#\mathbb{O}_d^3$. We enumerate $\mathbb{E}_d = \{\eta_i \mid 1 \leq i \leq \alpha\}$ such that $\eta_i < \eta_{i+1}$, and $\mathbb{O}_d^1 = \{\theta_j \mid 1 \leq j \leq \beta\}$ such that $\theta_j < \theta_{j+1}$; similarly enumerate $\mathbb{O}_d^3 = \{\zeta_j \mid 1 \leq j \leq \gamma\}$ such that $\zeta_j < \zeta_{j+1}$. Now define $E_+ := D_+^0(\eta_i) \lor \cdots \lor D_+^0(\eta_{i})$; $O_+^1 := C_+^0(\theta_1) \lor \cdots \lor C_+^0(\theta_{\beta})$; $O_+^3 := C_+^0(\zeta_1) \lor \cdots \lor C_+^0(\zeta_{\gamma})$; $E_- := D_-^0(\eta_i) \lor \cdots \lor D_-^0(\eta_{i})$; $O_-^1 := C_-^0(\theta_1) \lor \cdots \lor C_-^0(\theta_{\beta})$ and $O_-^3 := C_-^0(\zeta_1) \lor \cdots \lor C_-^0(\zeta_{\gamma})$.

Also we define $H_+ := E_+ \lor O_+^1 \lor O_+^3$, $H_- := E_- \lor O_-^1 \lor O_-^3$ and $H := H_+ \lor H_-$. (3.31) It is clear that $H$ is a standard orthogonal basis of $V$ such that $H_+ = \{v \in H \mid \langle v, v \rangle = 1\}$ and $H_- = \{v \in H \mid \langle v, v \rangle = -1\}$. In particular, $\#H_+ = p$ and $\#H_- = q$. Also, we have the following relations:

$$\sum_{i=1}^{\alpha} \eta_i t_{\eta_i} + \sum_{j=1}^{\beta} \left( \frac{\theta_j + 1}{2} p_{\theta_j} + \frac{\theta_j - 1}{2} q_{\theta_j} \right) + \sum_{k=1}^{\gamma} \left( \frac{\zeta_k - 1}{2} p_{\zeta_k} + \frac{\zeta_k + 1}{2} q_{\zeta_k} \right) = p$$

and

$$\sum_{i=1}^{\alpha} \eta_i t_{\eta_i} + \sum_{j=1}^{\beta} \left( \frac{\theta_j - 1}{2} p_{\theta_j} + \frac{\theta_j + 1}{2} q_{\theta_j} \right) + \sum_{k=1}^{\gamma} \left( \frac{\zeta_k + 1}{2} p_{\zeta_k} + \frac{\zeta_k - 1}{2} q_{\zeta_k} \right) = q.$$
is embedded in $M_p(\mathbb{R})$ and in $M_q(\mathbb{R})$ as follows:

$$D_p : \prod_{i=1}^{\alpha} (M_{t_{n_i}/2}(\mathbb{R}) \times M_{t_{n_i}/2}(\mathbb{R})) \times \prod_{j=1}^{\beta} (M_{p_{t_{j}}}(\mathbb{R}) \times M_{q_{t_{j}}}(\mathbb{R})) \times \prod_{k=1}^{\gamma} (M_{p_{c_k}}(\mathbb{R}) \times M_{q_{c_k}}(\mathbb{R}))$$

$$\rightarrow M_p(\mathbb{R})$$

(3.32)

$$(A_{\eta_1}, B_{\eta_1}, \ldots, A_{\eta_2}, B_{\eta_2}; C_{\theta_1}, D_{\theta_1}, \ldots, C_{\theta_{\beta}}, D_{\theta_{\beta}}; E_{\zeta_1}, F_{\zeta_1}, \ldots, E_{\zeta_\gamma}, F_{\zeta_\gamma}) \rightarrow$$

$$\bigoplus_{i=1}^{\alpha} \varphi_{t_{n_i}/2, C} (A_i + \sqrt{-1}B_i)^{\eta_i/2} \bigoplus_{j=1}^{\beta} \left( (C_{\theta_j} \oplus D_{\theta_j})^{\theta_j^{-1}} \oplus C_{\theta_j} \oplus (C_{\theta_j} \oplus D_{\theta_j})^{\theta_j^{-1}} \right)$$

$$\bigoplus_{k=1}^{\gamma} \left( (E_{\zeta_k} \oplus F_{\zeta_k})^{\zeta_k+1/4} \oplus (F_{\zeta_k} \oplus E_{\zeta_k})^{\zeta_k-3/4} \oplus F_{\zeta_k} \right)$$

and

$$D_q : \prod_{i=1}^{\alpha} (M_{t_{n_i}/2}(\mathbb{R}) \times M_{t_{n_i}/2}(\mathbb{R})) \times \prod_{j=1}^{\beta} (M_{p_{t_{j}}}(\mathbb{R}) \times M_{q_{t_{j}}}(\mathbb{R})) \times \prod_{k=1}^{\gamma} (M_{p_{c_k}}(\mathbb{R}) \times M_{q_{c_k}}(\mathbb{R}))$$

$$\rightarrow M_q(\mathbb{R})$$

(3.33)

$$(A_{\eta_1}, B_{\eta_1}, \ldots, A_{\eta_2}, B_{\eta_2}; C_{\theta_1}, D_{\theta_1}, \ldots, C_{\theta_{\beta}}, D_{\theta_{\beta}}; E_{\zeta_1}, F_{\zeta_1}, \ldots, E_{\zeta_\gamma}, F_{\zeta_\gamma}) \rightarrow$$

$$\bigoplus_{i=1}^{\alpha} \varphi_{t_{n_i}/2, C} (A_i + \sqrt{-1}B_i)^{\eta_i/2} \bigoplus_{j=1}^{\beta} \left( (D_{\theta_j} \oplus C_{\theta_j})^{\theta_j^{-1}} \oplus D_{\theta_j} \oplus (D_{\theta_j} \oplus C_{\theta_j})^{\theta_j^{-1}} \right)$$

$$\bigoplus_{k=1}^{\gamma} \left( (F_{\zeta_k} \oplus E_{\zeta_k})^{\zeta_k+1/4} \oplus (E_{\zeta_k} \oplus F_{\zeta_k})^{\zeta_k-3/4} \oplus E_{\zeta_k} \right).$$

Define two characters

$$\chi_p : \prod_{i=1}^{\alpha} U(t_{n_i}/2) \times \prod_{j=1}^{\beta} (O_{p_{t_{j}}} \times O_{q_{t_{j}}}) \times \prod_{k=1}^{\gamma} (O_{p_{c_k}} \times O_{q_{c_k}}) \rightarrow \mathbb{R} \setminus \{0\}$$

(3.34)

$$(A_{\eta_1}, \ldots, A_{\eta_2}; C_{\theta_1}, D_{\theta_1}, \ldots, C_{\theta_{\beta}}, D_{\theta_{\beta}}; E_{\zeta_1}, F_{\zeta_1}, \ldots, E_{\zeta_\gamma}, F_{\zeta_\gamma})$$

$$\mapsto \prod_{i=1}^{\alpha} \det \varphi_{t_{n_i}/2, C} (A_{\eta_i})^{\eta_i/2} \prod_{j=1}^{\beta} \det C_{\theta_j} \prod_{k=1}^{\gamma} \det E_{\zeta_k}$$

and

$$\chi_q : \prod_{i=1}^{\alpha} U(t_{n_i}/2) \times \prod_{j=1}^{\beta} (O_{p_{t_{j}}} \times O_{q_{t_{j}}}) \times \prod_{k=1}^{\gamma} (O_{p_{c_k}} \times O_{q_{c_k}}) \rightarrow \mathbb{R} \setminus \{0\}$$

(3.35)

$$(A_{\eta_1}, \ldots, A_{\eta_2}; C_{\theta_1}, D_{\theta_1}, \ldots, C_{\theta_{\beta}}, D_{\theta_{\beta}}; E_{\zeta_1}, F_{\zeta_1}, \ldots, E_{\zeta_\gamma}, F_{\zeta_\gamma})$$

$$\mapsto \prod_{i=1}^{\alpha} \det \varphi_{t_{n_i}/2, C} (A_{\eta_i})^{\eta_i/2} \prod_{j=1}^{\beta} \det D_{\theta_j} \prod_{k=1}^{\gamma} \det F_{\zeta_k}. $$
Let $\Lambda_H : \text{End}_R \mathbb{R}^n \rightarrow M_n(R)$ be the isomorphism of $\mathbb{R}$-algebras induced by the ordered basis $H$ as in (3.31). Let $M$ be the maximal compact subgroup of $SO(p, q)$ which leaves invariant simultaneously the two subspaces spanned by $H_+$ and $H_-$. Clearly, $\Lambda_H(M) = S(O(p) \times O(q))$.

**Theorem 3.18.** Let $X \in \mathcal{N}_{so(p,q)}$ and $\Psi_{SO(p,q)^0}(O_X) = (d, \text{sgn}_{O_X})$. Let $\alpha := \#E_d$, $\beta := \#O_d^3$, and $\gamma := \#O_d^5$. Let $\{X, H, Y\} \subset so(p,q)$ be a $sl_3(\mathbb{R})$-triple, and let $(p_\theta, q_\theta)$ be the signature of the form $(\cdot, \cdot)_\theta$ for all $\theta \in O_d$. Let $K$ be the maximal compact subgroup of $Z_{SO(p,q)}(X, H, Y)$ as in Lemma 3.17. Let the maps $D_p, D_q, \chi_p$ and $\chi_q$ be defined as in (3.32), (3.33), (3.34) and (3.35), respectively. Then $\Lambda_H(K) \subset S(O(p) \times O(q))$ is given by

$$\Lambda_H(K) = \left\{D_p(g) \oplus D_q(g) \mid g \in \prod_{i=1}^\alpha U(t_{\eta_i}/2) \times \prod_{j=1}^\beta (O_{p_{\eta_j}} \times O_{q_{\eta_j}}) \times \prod_{k=1}^\gamma (O_{p_{\zeta_k}} \times O_{q_{\zeta_k}}) \right\}.$$

**Proof.** This follows by writing the matrices of the elements of the maximal compact subgroup $K$ in Lemma 3.17 with respect to the ordered basis $H$ as in (3.31). \(\square\)

Since $SO(p,q)^0$ is normal in $SO(p,q)$, so is $Z_{SO(p,q)^0}(X, H, Y)$ in $Z_{SO(p,q)}(X, H, Y)$. Recall that $K$ is a maximal compact subgroup in $Z_{SO(p,q)}(X, H, Y)$. Thus it follows that

$$\tilde{K} := K \cap Z_{SO(p,q)^0}(X, H, Y) = K \cap SO(p,q)^0$$

is a maximal compact subgroup of $Z_{SO(p,q)^0}(X, H, Y)$. In the next result we obtain an explicit description of $\Lambda_H(\tilde{K})$ in $SO(p) \times SO(q)$.

**Theorem 3.19.** Let $X \in \mathcal{N}_{so(p,q)}$ and $\Psi_{SO(p,q)^0}(O_X) = (d, \text{sgn}_{O_X})$. Let $\alpha := \#E_d$, $\beta := \#O_d^3$ and $\gamma := \#O_d^5$. Let $\{X, H, Y\} \subset so(p,q)$ be a $sl_3(\mathbb{R})$-triple, and let $(p_\theta, q_\theta)$ be the signature of the form $(\cdot, \cdot)_\theta$ for all $\theta \in O_d$. Let $\tilde{K}$ be the maximal compact subgroup of $Z_{SO(p,q)^0}(X, H, Y)$ as in the preceding paragraph. Let the maps $D_p, D_q, \chi_p$ and $\chi_q$ be defined as in (3.32), (3.33), (3.34) and (3.35), respectively. Then $\Lambda_H(\tilde{K}) \subset SO(p) \times SO(q)$ is given by

$$\left\{D_p(g) \oplus D_q(g) \mid g \in \prod_{i=1}^\alpha U(t_{\eta_i}/2) \times \prod_{j=1}^\beta (O_{p_{\eta_j}} \times O_{q_{\eta_j}}) \times \prod_{k=1}^\gamma (O_{p_{\zeta_k}} \times O_{q_{\zeta_k}}) \right\}.$$  

The nilpotent orbit $O_X$ in $so(p,q)$ is homotopic to $SO(p) \times SO(q)/\Lambda_H(\tilde{K})$.

**Proof.** Let $V_+$ and $V_-$ be the $\mathbb{R}$-span of $H_+$ and $H_-$, respectively. Let $M$ be the maximal compact subgroup in $SO(p,q)$ which simultaneously leaves the subspaces $V_+$ and $V_-$ invariant. It is clear that $M^o$ is a maximal compact subgroup of $SO(p,q)^0$. Hence,

$$M^o = SO(p,q)^0 \cap M = \{g \in SO(p,q) \mid \det g_{|V_+} = 1, \det g_{|V_-} = 1\}.$$

As $K \subset M$, we have that $K \cap SO(p,q)^0 = K \cap M^o$. The first part of the proposition now follows. For the second part we use Theorem 2.3. \(\square\)

### 3.4. Homotopy types of the nilpotent orbits in $sp(n, \mathbb{C})$.

Let $n$ be a positive integer. The aim in this subsection is to write down the homotopy types of the nilpotent orbits in the simple Lie algebra $sp(n, \mathbb{C})$ as compact homogeneous spaces. Throughout this subsection $\langle \cdot, \cdot \rangle$ denotes the symplectic form on $\mathbb{C}^{2n}$ defined by

$$\langle x, y \rangle := x^t J_n y, \quad x, y \in \mathbb{C}^{2n},$$
where $J_n$ is as in (2.1).

We first recall a suitable parametrization of the nilpotent orbits $\mathcal{N}(\text{Sp}(n, \mathbb{C}))$. Let

$$
\Psi_{\text{SL}_n(\mathbb{C})} : \mathcal{N}(\text{SL}_n(\mathbb{C})) \rightarrow \mathcal{P}(n)
$$

be the parametrization of $\mathcal{N}(\text{SL}_n(\mathbb{C}))$; see Section 3.1 for details. Since $\text{Sp}(n, \mathbb{C}) \subset \text{SL}_{2n}(\mathbb{C})$ (consequently as, the set of nilpotent elements $\mathcal{N}_{\text{sp}(n, \mathbb{C})} \subset \mathcal{N}_{\text{sl}_{2n}(\mathbb{C})}$) we have the inclusion map

$$
\Theta_{\text{Sp}(n, \mathbb{C})} : \mathcal{N}(\text{Sp}(n, \mathbb{C})) \rightarrow \mathcal{N}(\text{SL}_{2n}(\mathbb{C})).
$$

Let

$$
\Psi_{\text{Sp}(n, \mathbb{C})} := \Psi_{\text{SL}_n(\mathbb{C})} \circ \Theta_{\text{Sp}(n, \mathbb{C})} : \mathcal{N}(\text{Sp}(n, \mathbb{C})) \rightarrow \mathcal{P}(2n)
$$

be the composition. Recall that

$$
\Psi_{\text{Sp}(n, \mathbb{C})}(\mathcal{N}(\text{Sp}(n, \mathbb{C}))) \subset \mathcal{P}_1(2n)
$$

(this follows form [BCM, Proposition A.6]). Hence we have the following parametrizing map:

$$
\Psi_{\text{Sp}(n, \mathbb{C})} : \mathcal{N}(\text{Sp}(n, \mathbb{C})) \rightarrow \mathcal{P}_1(2n).
$$

**Theorem 3.20** ([CoMc, Theorem 5.1.3]). The above map $\Psi_{\text{Sp}(n, \mathbb{C})}$ is bijective.

Let $0 \neq X \in \mathcal{N}_{\text{sp}(n, \mathbb{C})}$ and $\{X, H, Y\}$ be a $\text{sl}_2(\mathbb{R})$-triple in $\mathfrak{sp}(n, \mathbb{C})$. Recall that the form $(\cdot, \cdot)_d$ on $L(d - 1)$ is symmetric when $d \in \mathbb{E}_d$ and symplectic when $d \in \mathbb{O}_d$. Let $(v_1^d, \ldots, v_d^d)$ be an $\mathbb{C}$-basis of $L(d - 1)$ as in [BCM, Proposition A.6]. It follows form [BCM, Proposition A.6] that $(v_1^n, \ldots, v_n^n)$ is an orthonormal basis of $L(\eta - 1)$ for the form $(\cdot, \cdot)_\eta$ for all $\eta \in \mathbb{E}_d$, i.e.,

$$(v_j^n, v_j^n)_\eta = 1 \quad \text{for } 1 \leq j \leq t_\eta \quad \text{and} \quad (v_j^n, v_i^n)_\eta = 0 \quad \text{for } j \neq i.$$

(3.36)

For all $\theta \in \mathbb{O}_d$, as $(\cdot, \cdot)_\theta$ is a symplectic form, we may assume that

$$(v_1^\theta, \ldots, v_{t_\theta/2}^\theta; v_{t_\theta/2+1}^\theta, \ldots, v_{t_\theta}^\theta)$$

is a symplectic basis of $L(\theta - 1)$. This is equivalent to saying that, for all $\theta \in \mathbb{O}_d$,

$$(v_j^\theta, v_{t_\theta/2+1}^\theta)_\theta = 1 \quad \text{for } 1 \leq j \leq t_\theta/2 \quad \text{and} \quad (v_j^\theta, v_i^\theta)_\theta = 0 \quad \text{for all } i \neq j + t_\theta/2.$$

(3.37)

Now fixing $\theta \in \mathbb{O}_d$, for all $1 \leq j \leq t_\theta$, define

$$
w_{ji}^\theta := \begin{cases} 
\frac{(X^l v_j^\theta + X^{\theta-1-l} v_j^\theta)}{\sqrt{2}} & \text{if } l \text{ is even and } 0 \leq l < (\theta - 1)/2 \\
\frac{(X^l v_j^\theta + X^{\theta-1-l} v_j^\theta)}{\sqrt{-2}} & \text{if } l \text{ is odd and } 0 \leq l < (\theta - 1)/2 \\
X^l v_j^\theta & \text{if } l = (\theta - 1)/2 \\
\frac{(X^{\theta-1-l} v_j^\theta - X^l v_j^\theta)}{\sqrt{-2}} & \text{if } l \text{ is even and } (\theta + 1)/2 \leq l \leq (\theta - 1) \\
\frac{(X^{\theta-1-l} v_j^\theta - X^l v_j^\theta)}{\sqrt{2}} & \text{if } l \text{ is odd and } (\theta + 1)/2 \leq l \leq (\theta - 1).
\end{cases}
$$

(3.38)
For $\zeta \in \mathbb{O}_d^3$, for all $1 \leq j \leq t_\zeta$, define

$$w^\zeta_{jl} := \begin{cases} 
(X^j v^\zeta_j + X^{j-1} v^\zeta_j)/\sqrt{2} & \text{if } l \text{ is even and } 0 \leq l < (\zeta - 1)/2 \\
(X^j v^\zeta_j + X^{j-1} v^\zeta_j)/\sqrt{-2} & \text{if } l \text{ is odd and } 0 \leq l < (\zeta - 1)/2 \\
X^j v^\zeta_j \sqrt{-1} & \text{if } l = (\zeta - 1)/2 \\
(X^{j-1} v^\zeta_j - X^j v^\zeta_j)/\sqrt{-2} & \text{if } l \text{ is even and } (\zeta + 1)/2 \leq l \leq (\zeta - 1) \\
(X^{j-1} v^\zeta_j - X^j v^\zeta_j)/\sqrt{2} & \text{if } l \text{ is odd and } (\zeta + 1)/2 \leq l \leq (\zeta - 1).
\end{cases}$$

(3.39)

For $\theta \in \mathbb{O}_d$, $0 \leq l \leq \theta - 1$, set

$$V^l(\theta) := \text{Span}_R\{w^\theta_{jl} \mid 1 \leq j \leq t_\theta\}.$$ 

(4.40)

Using (3.37) we observe that for each $\theta \in \mathbb{O}_d$ the space $M(\theta - 1)$ is a direct sum of the subspaces $V^l(\theta)$, $0 \leq l \leq \theta - 1$, which are mutually orthogonal with respect to $\langle \cdot, \cdot \rangle$. For $\theta \in \mathbb{O}_d$, define

$$C^l(\theta) := (w^\theta_{1l}, \ldots, w^\theta_{t_\theta/2}l^l) \vee (w^\theta_{(t_\theta/2+1)l}, \ldots, w^\theta_{t_\theta}l).$$

Then using (3.37), (3.38) and (3.39) it follows that $C^l(\theta)$ is a symplectic basis for $V^l(\theta)$. Recall that

$$B^l(d) = (X^l v^d_1, \ldots, X^l v^d_{t_d})$$

is the ordered basis of $X^l L(d-1)$ for $0 \leq l \leq d-1$, $d \in \mathbb{N}_d$ as in (3.3).

Lemma 3.21. Let $X$ be a nilpotent element in $\mathfrak{sp}(n, \mathbb{C})$ and $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sp}(n, \mathbb{C})$ containing $X$. Then the following holds:

$$Z_{\mathfrak{sp}(n, \mathbb{C})}(X, H, Y) = \left\{ g \in \text{Sp}(n, \mathbb{C}) \left| \begin{array}{c} g|_{V^l(\theta)_l} \subseteq V^l(\theta) \text{ and} \\
g(V^l(\theta)) \cap V^l(\theta) \text{ for all } \theta \in 0 \leq l < \theta, \\
g(X^l L(\eta - 1)) \subseteq X^l L(\eta - 1) \text{ and} \\
g|_{X^l L(\eta - 1)} \subseteq L(\eta - 1) \text{ for all } \eta \in \mathbb{N}_d, 0 \leq l < \eta \end{array} \right. \right\}.$$ 

Proof. The proof is similar to that of Lemma 3.8.

Remark 3.22. We follow the notation as in Lemma 3.21. Let $g \in Z_{\mathfrak{sp}(n, \mathbb{C})}(X, H, Y)$. Let $\theta \in \mathbb{O}_d$ and $\eta \in \mathbb{N}_d$. Then it follows from Lemma 3.21 that $g$ keeps the subspaces $V^0(\theta)$ and $L(\eta - 1)$ invariant. Since the restriction of $\langle \cdot, \cdot \rangle$ is a symplectic form on $V^0(\theta)$ we have $g|_{V^0(\theta)} \in \text{Sp}(V^0(\theta), \langle \cdot, \cdot \rangle)$. Further recall that the form $\langle \cdot, \cdot \rangle_\eta$, as defined in (3.8), is symmetric on $L(\eta - 1)$, and $(gx, gy)_\eta = (x, y)_\eta$ for all $x, y \in L(\eta - 1)$, see [BCM, Remark A.7]. Thus $g|_{L(\eta - 1)} \in O(L(\eta - 1), \langle \cdot, \cdot \rangle_\eta)$.

For $\eta \in \mathbb{N}_d$, $0 \leq l \leq \eta/2 - 1$, set

$$W^l(\eta) := X^l L(\eta - 1) + X^{\eta - 1} L(\eta - 1).$$

(4.41)

We re-arrange the ordered basis $B^l(\eta) \vee B^{\eta - l}(\eta)$ of $W^l(\eta)$ as follows:

$$D^l(\eta) := \begin{cases} 
(X^l v_1, \ldots, X^l v_\eta) \vee (X^{\eta - l} v_1, \ldots, X^{\eta - l} v_\eta) & \text{if } l \text{ is even} \\
(X^{\eta - l} v_1, \ldots, X^{\eta - l} v_\eta) \vee (X^l v_1, \ldots, X^l v_\eta) & \text{if } l \text{ is odd}.
\end{cases}$$

(4.42)
Using (3.36) it can be easily verified that $D^l(\eta)$ is a symplectic basis with respect to $\langle \cdot, \cdot \rangle$.

Let $J_{c(\theta)}$ be the complex structure on $V^l(\theta)$ associated to the basis $C^l(\theta)$ for $\theta \in \Omega_\mathfrak{d}$, $0 \leq l \leq \theta - 1$, and let $J_{D^l(\eta)}$ be the complex structure on $W^l(\eta)$ associated to the basis $D^l(\eta)$ for $\eta \in \mathbb{E}_\mathfrak{d}$, $0 \leq l \leq \eta - 1$.

In the next lemma we specify a maximal compact subgroup $Z_{Sp(n, \mathbb{C})}(X, H, Y)$ which will be used in Theorem 3.24. Recall that the $\mathbb{H}$-algebra embedding $\wp_m, \mathbb{H}$ is defined in (3.22).

**Lemma 3.23.** Let $K$ be the subgroup of $Z_{Sp(n, \mathbb{C})}(X, H, Y)$ consisting of elements $g$ in $Z_{Sp(n, \mathbb{C})}(X, H, Y)$ satisfying the following conditions:

1. For all $\theta \in \Omega_\mathfrak{d}$ and $0 \leq l \leq \theta - 1$ the inclusion $g(V^l(\theta)) \subset V^l(\theta)$ holds.
2. For all $\theta \in \Omega_\mathfrak{d}$, there exist $A_\theta, B_\theta \in M_{\theta/2}(\mathbb{C})$ with $A_\theta + jB_\theta \in \text{Sp}(t_{\theta/2})$ such that $[g|_{V^l(\theta)}]_{C^l(\theta)} = \wp_{t_{\theta/2}}(A_\theta + jB_\theta)$.
3. For all $\eta \in \mathbb{E}_\mathfrak{d}$ and $0 \leq l \leq \eta - 1$ the inclusion $g(X^lL(\eta - 1)) \subset X^lL(\eta - 1)$ holds.
4. For all $\eta \in \mathbb{E}_\mathfrak{d}$, there exist $C_\eta \in O_{t_{\eta}}$ such that $[g|_{X^lL(\eta - 1)}]_{B^0(\eta)} = C_\eta$.

Then $K$ is a maximal compact subgroup of $Z_{Sp(n, \mathbb{C})}(X, H, Y)$.

**Proof.** Let $K' \subset Z_{Sp(n, \mathbb{C})}(X, H, Y)$ be the subgroup consisting of all elements $g$ satisfying the conditions (3.43), (3.44) and (3.45) below:

\[
\overline{g}|_{V^0(\theta)} J_{C^0(\theta)} = J_{C^0(\theta)} g|_{V^0(\theta)}, \quad \text{for all } \theta \in \Omega_\mathfrak{d}; \tag{3.43}
\]

\[
g(W^l(\eta)) \subset W^l(\eta) \quad \text{and} \quad [g|_{W^l(\eta)}]_{D^l(\eta)} = [g|_{W^0(\eta)}]_{D^0(\eta)}, \quad \text{for all } \eta \in \mathbb{E}_\mathfrak{d}, 0 \leq l < \eta; \tag{3.44}
\]

\[
\overline{g}|_{W^0(\eta)} J_{D^0(\eta)} = J_{D^0(\eta)} g|_{W^0(\eta)}, \quad \text{for all } \eta \in \mathbb{E}_\mathfrak{d}. \tag{3.45}
\]

Using Lemma 3.21 and Lemma 3.11(1) it is clear that $K'$ is a maximal compact subgroup of $Z_{Sp(n, \mathbb{C})}(X, H, Y)$. Hence to prove the lemma it suffices to show that $K = K'$. Let $g \in Sp(n, \mathbb{C})$. Using Lemma 3.11(2) it is straightforward to check that $g$ satisfies (1), (2) in the statement of the lemma if and only if $g$ satisfies (3.43). Let $g \in Z_{Sp(n, \mathbb{C})}(X, H, Y)$ and

\[
C_\eta := [g|_{X^lL(\eta - 1)}]_{B^0(\eta)} = [g|_{X^lL(\eta - 1)}]_{B^0(\eta)}.
\]

Then we observe that

\[
[g|_{W^l(\eta)}]_{D^l(\eta)} = \begin{pmatrix} C_\eta & \cdot \end{pmatrix}.
\]

Now suppose that $g \in Sp(n, \mathbb{C})$ and $g$ satisfies (3) and (4) in the statement of the lemma. It is clear that (3.44) holds. Since $C_\eta \in O_{t_{\eta}}$, it follows that (3.45) holds.

Next assume that $g \in Z_{Sp(n, \mathbb{C})}(X, H, Y)$ and satisfies (3.44) and (3.45). From (3.45) it follows that $C_\eta = \overline{C_\eta}$. Using Lemma 3.11(2), we have $C_\eta + j0 \in \text{Sp}(t_{\eta})$. Thus $C_\eta \in O_{t_{\eta}}$. \qed

We next introduce some notation which will be needed in Theorem 3.24. Recall that the positive parts of the symplectic bases $D(\eta)$ and $C(\theta)$ are denoted by $D_+(\eta)$ and $C_+(\theta)$, respectively. Similarly the negative parts of $D(\eta)$ and $C(\theta)$ are denoted by $D_-(\eta)$ and $C_-(\theta)$, respectively. For $\eta \in \mathbb{E}_\mathfrak{d}$, set

\[
D_+(\eta) := D^0_+(\eta) \vee \cdots \vee D^{\eta/2-1}_+(\eta) \quad \text{and} \quad D_-(\eta) := D^0_-(\eta) \vee \cdots \vee D^{\eta/2-1}_-(\eta).
\]
For \( \theta \in \mathcal{O}_d \), set
\[
C_+(\theta) := C_0^0(\theta) \lor \cdots \lor C_{\theta-1}^0(\theta) \quad \text{and} \quad C_-(\theta) := C_0^0(\theta) \lor \cdots \lor C_{\theta-1}^0(\theta).
\]
Let \( \alpha := \#\mathcal{E}_d, \beta := \#\mathcal{O}_d \). We enumerate \( \mathcal{E}_d = \{ \eta_i \mid 1 \leq i \leq \alpha \} \) such that \( \eta_i < \eta_{i+1} \), and \( \mathcal{O}_d = \{ \theta_j \mid 1 \leq j \leq \beta \} \) such that \( \theta_j < \theta_{j+1} \). Now define
\[
\mathcal{E}_+ := \mathcal{D}_+(\eta_1) \lor \cdots \lor \mathcal{D}_+(\eta_\alpha); \quad \mathcal{O}_+ := C_+(\theta_1) \lor \cdots \lor C_+(\theta_\beta).
\]
\[
\mathcal{E}_- := \mathcal{D}_-(\eta_1) \lor \cdots \lor \mathcal{D}_-(\eta_\alpha); \quad \mathcal{O}_- := C_-(\theta_1) \lor \cdots \lor C_-(\theta_\beta).
\]
Also we define
\[
\mathcal{H}_+ := \mathcal{E}_+ \lor \mathcal{O}_+, \quad \mathcal{H}_- := \mathcal{E}_- \lor \mathcal{O}_- \quad \text{and} \quad \mathcal{H} := \mathcal{H}_+ \lor \mathcal{H}_-.
\] (3.46)

As before, for a matrix \( A = (a_{ij}) \in \mathcal{M}_r(\mathbb{C}) \), define \( \overline{A} := (\overline{a}_{ij}) \in \mathcal{M}_r(\mathbb{C}) \). Let
\[
D_{\text{Sp}(n,\mathbb{C})} := \prod_{i=1}^\alpha M_{t_{\eta_i}}(\mathbb{R}) \times \prod_{j=1}^\beta M_{t_{\theta_j}/2}(\mathbb{H}) \longrightarrow \mathcal{M}_n(\mathbb{H})
\] (3.47)
be the \( \mathbb{R} \)-algebra embedding defined by
\[
(C_{\eta_1}, \ldots, C_{\eta_\alpha}; A_{\theta_1}, \ldots, A_{\theta_\beta}) \longmapsto \bigoplus_{i=1}^\alpha (C_{\eta_i})^{\eta_i/2} \oplus \bigoplus_{j=1}^\beta (A_{\theta_j})^{\theta_j}.
\]

It is clear that the basis \( \mathcal{H} \) in (3.46) is a symplectic basis of \( V \) with respect to \( \langle \cdot, \cdot \rangle \). Let \( \overline{\mathcal{A}} : \{ x \in \text{End}_\mathbb{C}\mathcal{C}^{2n} \mid xJ = Jx \} \longrightarrow \mathcal{M}_n(\mathbb{H}) \) be the isomorphism of \( \mathbb{R} \)-algebras induced by the above symplectic basis \( \mathcal{H} \). Recall that \( \overline{\mathcal{A}} : K_H \longrightarrow \text{Sp}(n) \) is an isomorphism of Lie groups.

**Theorem 3.24.** Let \( X \in \mathcal{N}_{\text{Sp}(n,\mathbb{C})}, X \neq 0 \), and \( \Psi_{\text{Sp}(n,\mathbb{C})}(O_X) = d \). Let \( \alpha := \#\mathcal{E}_d \) and \( \beta := \#\mathcal{O}_d \). Let \( \{ X, H, Y \} \) be a \( \mathfrak{sl}_2(\mathbb{R}) \)-triple in \( \mathfrak{sp}(n, \mathbb{C}) \). Let \( K \) be the maximal compact subgroup of \( Z_{\text{Sp}(n,\mathbb{C})}(X, H, Y) \) as in Lemma 3.23, and the map \( \mathcal{D}_{\text{Sp}(n,\mathbb{C})} \) be defined as in (3.47). Then \( \overline{\mathcal{A}}_{\mathcal{H}}(K) \subset \text{Sp}(n) \) is given by
\[
\overline{\mathcal{A}}_{\mathcal{H}}(K) = \left\{ D_{\text{Sp}(n,\mathbb{C})}(g) \mid g \in \prod_{i=1}^\alpha O_{t_{\eta_i}} \times \prod_{j=1}^\beta \text{Sp}(t_{\theta_j}/2) \right\}.
\]
Moreover, the nilpotent orbit \( O_X \) in \( \mathfrak{sp}(n, \mathbb{C}) \) is homotopic to \( \text{Sp}(n)/\overline{\mathcal{A}}_{\mathcal{H}}(K) \).

**Proof.** This follows by writing the matrices of the elements of the maximal compact subgroup \( K \) in Lemma 3.23 with respect to the ordered basis \( \mathcal{H} \) as in (3.46).

The second part follows from Theorem 2.3 and the well-known fact that any maximal compact subgroup of \( \text{Sp}(n, \mathbb{C}) \) is isomorphic to \( \text{Sp}(n) \). \( \square \)

3.5. **Homotopy types of the nilpotent orbits in \( \mathfrak{sp}(p, q) \).** Let \( n \) be a positive integer, and let \( (p, q) \) be a pair of non-negative integers with \( p + q = n \). In this subsection we write down the homotopy types of the nilpotent orbits in \( \mathfrak{sp}(p, q) \) as compact homogeneous spaces. As we do not need to deal with compact groups, we will further assume that \( p > 0 \) and \( q > 0 \). Throughout this subsection \( \langle \cdot, \cdot \rangle \) denotes the Hermitian form on \( \mathbb{H}^n \) defined by \( \langle x, y \rangle := \overline{x}^T I_{p,q} y, x, y \in \mathbb{H}^n \), where \( I_{p,q} \) is as in (2.1). We will follow notation as defined in \( \S 2. \)
First we will recall a parametrization of nilpotent orbits in $\mathfrak{sp}(p,q)$, see [BCM, Section 4.8]. Let

$$\Psi_{\text{SL}_n(\mathbb{H})} : N(\text{SL}_n(\mathbb{H})) \rightarrow \mathcal{P}(n)$$

be the parametrization as in Theorem 3.5. As $\text{Sp}(p,q) \subset \text{SL}_n(\mathbb{H})$ (consequently, $N_{\text{sp}(p,q)} \subset N_{\text{sl}_n(\mathbb{H})}$) we have the inclusion map $\Theta_{\text{Sp}(p,q)} : N(\text{Sp}(p,q)) \rightarrow N(\text{SL}_n(\mathbb{H}))$. Let

$$\Psi'_{\text{Sp}(p,q)} := \Psi_{\text{sl}_n(\mathbb{H})} \circ \Theta_{\text{Sp}(p,q)} : N(\text{Sp}(p,q)) \rightarrow \mathcal{P}(n)$$

be the composition. Let $0 \neq X \in \mathfrak{sp}(p,q)$ be a nilpotent element and $\mathcal{O}_X$ be the corresponding nilpotent orbit in $\mathfrak{sp}(p,q)$. Let $\{X, H, Y\} \subset \mathfrak{sp}(p,q)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. We now use [BCM, Proposition A.6, Remark A.8(3)]. Let $V := \mathbb{H}^n$ be the right $\mathbb{H}$-vector space of column vectors. Let $\{d_1, \ldots, d_s\}$, with $d_1 < \cdots < d_s$, be a ordered finite subset of natural numbers that arise as $\mathbb{R}$-dimension of non-zero irreducible $\text{Span}_\mathbb{R}\{X, H, Y\}$-submodules of $V$. Recall that $M(d-1)$ is defined to be the isotypical component of $V$ containing all irreducible $\text{Span}_\mathbb{R}\{X, H, Y\}$-submodules of $V$ with highest weight $(d-1)$, and as in (2.2), we set $L(d-1) := V_{Y,0} \cap M(d-1)$. Recall that the space $L(d_r-1)$ is a $\mathbb{H}$-subspace for $1 \leq r \leq s$. Let $d_r := \dim_{\mathbb{H}} L(d_r-1)$ for $1 \leq r \leq s$. Then $d := [d_1^{d_1}, \ldots, d_s^{d_s}] \in \mathcal{P}(n)$, and moreover, $\Psi'_{\text{Sp}(p,q)}(\mathcal{O}_X) = d$.

We next assign $\text{sgn}_{\mathcal{O}_X} \in \mathcal{S}_d^{\text{even}}(p,q)$ to each $\mathcal{O}_X \in N(\text{Sp}(p,q))$; see (2.9) for the definition of $\mathcal{S}_d^{\text{even}}(p,q)$. For each $d \in \mathbb{N}_d$ (see (2.4) for the definition of $\mathbb{N}_d$) we will define a $t_d \times d$ matrix $(m_{ij}^d)$ in $\mathbb{A}_d$ which depends only on the orbit $\mathcal{O}_X$ containing $X$; see (2.6) for the definition of $\mathbb{A}_d$. For this, recall that the form $(\cdot, \cdot)_d : L(d-1) \times L(d-1) \rightarrow \mathbb{H}$ defined as in (3.8) is Hermitian or skew-Hermitian according as $d$ is odd or even. Denoting the signature of $(\cdot, \cdot)_d$ by $(p_\theta, q_\theta)$ when $\theta \in \mathbb{O}_d$, we now define

$$m_{i1}^\theta := +1 \quad \text{if} \quad 1 \leq i \leq t_\eta, \quad \eta \in \mathbb{E}_d;$$

$$m_{i1}^\theta := \begin{cases} +1 & \text{if} \quad 1 \leq i \leq p_\theta, \\ -1 & \text{if} \quad p_\theta < i \leq t_\theta, \end{cases} \quad \theta \in \mathbb{O}_d;$$

and for $j > 1$, define $(m_{ij}^d)$ as in (3.24) and (3.25). Then the matrices $(m_{ij}^d)$ clearly verify (Yd.2). Set $\text{sgn}_{\mathcal{O}_X} := [(m_{d1}^d), \ldots, (m_{ds}^d)]$. It now follows from the last paragraph of [BCM, Remark 5.21] that $\text{sgn}_{\mathcal{O}_X} \in \mathcal{S}_d^{\text{even}}(p,q)$. Thus we have the map

$$\Psi_{\text{Sp}(p,q)} : N(\text{Sp}(p,q)) \rightarrow \mathcal{Y}^{\text{even}}(p,q), \quad \mathcal{O}_X \mapsto (\Psi'_{\text{Sp}(p,q)}(\mathcal{O}_X), \text{sgn}_{\mathcal{O}_X});$$

where $\mathcal{Y}^{\text{even}}(p,q)$ is as in (2.11). The following theorem is standard, see [CoMc, Theorem 9.3.5], [BCM, Theorem 4.38].

**Theorem 3.25.** The above map $\Psi_{\text{Sp}(p,q)}$ is a bijection.

Let $0 \neq X \in N_{\text{sp}(p,q)}$ and $\{X, H, Y\}$ a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sp}(p,q)$. Let $\Psi_{\text{Sp}(p,q)}(\mathcal{O}_X) = (d, \text{sgn}_{\mathcal{O}_X})$. Then $\Psi'_{\text{Sp}(p,q)}(\mathcal{O}_X) = d$. Recall that $\text{sgn}_{\mathcal{O}_X}$ determines the signature of $(\cdot, \cdot)_\theta$ on $L(\theta-1)$ for all $\theta \in \mathbb{O}_d$; let $(p_\theta, q_\theta)$ be the signature of $(\cdot, \cdot)_\theta$ on $L(\theta-1)$. Let $(v_{1}^{d}, \ldots, v_{t_\theta}^{d})$ be an ordered $\mathbb{H}$-basis of $L(d-1)$ as in [BCM, Proposition A.6]. It now follows from Proposition [BCM, Proposition A.6 (3)(a)] that $(v_{1}^{d}, \ldots, v_{t_\theta}^{d})$ is an orthogonal basis of $L(d-1)$ for the form $(\cdot, \cdot)_d$ for all $d \in \mathbb{N}_d$. Since $(\cdot, \cdot)_\eta$ is skew-Hermitian for all $\eta \in \mathbb{E}_d$, we may assume that for all $\eta \in \mathbb{E}_d$ the orthogonal basis $(v_{1}^{\eta}, \ldots, v_{t_\eta}^{\eta})$ satisfies the following relations:

$$(v_{j}^{\eta}, v_{j}^{\eta}) = j \quad \text{for all} \quad 1 \leq j \leq t_\eta, \eta \in \mathbb{E}_d.$$
We next impose orderings on the sets \( \{ \eta \} \). Define the ordered sets by
\[
\eta^d_{jl} := \begin{cases} 
(X^j_0^d + X^{n-1-j_0^d})/\sqrt{2} & \text{if } 0 \leq l < \eta/2 \\
(X^{n-1-j_0^d} - X^j_0^d)/\sqrt{2} & \text{if } \eta/2 \leq l \leq \eta - 1.
\end{cases}
\]

Similarly, as in \([BCM, \text{Lemma A.11(2)}]\), for \( \theta \in \Omega_d \) we define,
\[
\theta^d_{jl} := \begin{cases} 
(X^j_0^d + X^{\theta-1-j_0^d})/\sqrt{2} & \text{if } 0 \leq l < (\theta - 1)/2 \\
X^j_0^d & \text{if } l = (\theta - 1)/2 \\
(X^{\theta-1-j_0^d} - X^j_0^d)/\sqrt{2} & \text{if } (\theta - 1)/2 < l \leq \theta - 1.
\end{cases}
\]

Let \( \{ w^d_{jl} \mid 1 \leq j \leq t_d, 0 \leq l \leq d - 1 \} \) be the \( \mathbb{H} \)-basis of \( M(d-1) \) constructed as above. For each \( d \in \mathbb{N}_d, 0 \leq l \leq d - 1 \) we set
\[
V^l(d) := \text{Span}_\mathbb{H}\{ w^d_{1l}, \ldots, w^d_{td} \}.
\]
The ordered basis \( (w^d_{1l}, \ldots, w^d_{td}) \) of \( V^l(d) \) is denoted by \( C^l(d) := (w^d_{1l}, \ldots, w^d_{td}) \). Set
\[
V^l_+(d) := \text{Span}_\mathbb{H}\{ v \mid v \in C^l(d), \langle v, v \rangle > 0 \}, \quad V^l_-(d) := \text{Span}_\mathbb{H}\{ v \mid v \in C^l(d), \langle v, v \rangle < 0 \}.
\]
Now it is clear that for \( \eta \in \mathbb{E}_d \),
\[
V^l_+(\eta) := \begin{cases} 
V^l(\eta) & \text{if } l \text{ odd and } l \text{ even} \\
\phi & \text{if } l \text{ even}
\end{cases}
\]
We next impose orderings on the sets \( \{ v \in C^l(d) \mid \langle v, v \rangle > 0 \}, \{ v \in C^l(d) \mid \langle v, v \rangle < 0 \} \). Define the ordered sets by \( C^l_+(\theta), C^l_-(\theta), C^l_+(\zeta) \) and \( C^l_-(\zeta) \) as in \([BCM, \text{(4.19), (4.20), (4.21), (4.22)}]\), respectively according as \( \theta \in \Omega^3_d \) or \( \zeta \in \Omega^3_d \). Set
\[
C^l_+(\eta) := \begin{cases} 
C^l(\eta) & \text{if } l \text{ odd and } l \text{ even} \\
\phi & \text{if } l \text{ even}
\end{cases}
\]
It is straightforward that \( C^l_+(d) \) and \( C^l_-(d) \) are indeed ordered bases of \( V^l_+(d) \) and \( V^l_-(d) \), respectively.

Next we will write down a suitable description of reductive part of the centralizer of a nilpotent element in \( \mathfrak{sp}(p, q) \).

**Lemma 3.26.** Let \( X \) be a nilpotent element in \( \mathfrak{sp}(p, q) \) and \( \{ X, H, Y \} \) be a \( \mathfrak{sl}_2(\mathbb{R}) \)-triple in \( \mathfrak{sp}(p, q) \) containing \( X \). Then the following holds:
\[
\mathcal{Z}_{\mathfrak{sp}(p, q)}(X, H, Y) = \left\{ g \in \text{Sp}(p, q) \ \bigg| \ g(V^l(d)) \subset V^l(d) \text{ and } \left[ g|_{V^l(d)} \right]_{C^l_+(d)} = \left[ g|_{V^l(d)} \right]_{C^l_-(d)} \text{ for all } d \in \mathbb{N}_d, 0 \leq l < d \right\}.
\]

**Proof.** We omit the proof as it is similar to that of Lemma 3.8. \( \square \)
Remark 3.27. We follow the notation as in Lemma 3.26. Let $g \in \mathcal{Z}_{\text{Sp}(p,q)}(X, H, Y)$. Let $\theta \in \mathcal{O}_d$ and $\eta \in \mathcal{E}_d$. Then it follows from Lemma 3.26 that $g$ keeps the subspace $V^0(\theta)$ invariant. Since the restriction of $\langle \cdot, \cdot \rangle$ is a hermitian form on $V^0(\theta)$ we have $g|_{V^0(\theta)} \in \text{SU}(V^0(\theta), \langle \cdot, \cdot \rangle)$. Further recall that the form $\langle \cdot, \cdot \rangle_\eta$, as defined in (3.8), is skew-hermitian on $L(\eta-1)$. Also $g$ keeps the subspace $L(\eta-1)$ invariant and $(gx, gy)_\eta = (x, y)_\eta$ for all $x, y \in L(\eta-1)$, see [BCM, Lemma 4.4(3) and Remark A.7]. Thus $g|_{L(\eta-1)} \in \text{SO}^*(L(\eta-1), \langle \cdot, \cdot \rangle_\eta)$.

Let $W$ be a right $\mathbb{H}$-vector space, $\langle \cdot, \cdot \rangle'$ be a non-degenerate skew-Hermitian form on $W$. Let $\dim\mathbb{H} W = m$ and $\mathcal{B}' := \langle v_1, \ldots, v_m \rangle$ be a standard orthogonal basis of $W$ such that $\langle v_r, v_r \rangle' = j$ for all $1 \leq r \leq m$. Let $J_{\mathcal{B}'} : W \rightarrow W$ be defined by $J_{\mathcal{B}'}(\sum_r v_rz_r) := \sum_r v_rjz_r$ for all column vectors $(z_1, \ldots, z_m)^t \in \mathbb{H}^m$. The next lemma is a standard fact where we recall an explicit description of maximal compact subgroups in the group $\text{SO}^*(W, \langle \cdot, \cdot \rangle')$.

We set

$$K_{\mathcal{B}'} := \{ g \in \text{SO}^*(W, \langle \cdot, \cdot \rangle') \mid gJ_{\mathcal{B}'} = J_{\mathcal{B}'}g \}.$$ 

Lemma 3.28 ([BCM, Lemma 4.28]). Let $W, \langle \cdot, \cdot \rangle'$ and $\mathcal{B}'$ be as above. Then

1. $K_{\mathcal{B}'}$ is a maximal compact subgroup in $\text{SO}^*(W, \langle \cdot, \cdot \rangle')$.
2. $K_{\mathcal{B}'} = \{ g \in \text{SL}(W) \mid [g]_{\mathcal{B}'} = A + jB \text{ where } A, B \in M_m(\mathbb{R}), A + \sqrt{-1}B \in U(m) \}.$

Recall that $\{ x \in \text{End}_{\mathbb{H}} W \mid xJ_{\mathcal{B}'} = J_{\mathcal{B}'}x \} = \{ x \in \text{End}_{\mathbb{H}} W \mid [x]_{\mathcal{B}'} \in M_m(\mathbb{R}) + jM_m(\mathbb{R}) \}.$

We now consider the $\mathbb{R}$-algebra isomorphism

$$\Lambda'_{\mathcal{B}'} : \{ x \in \text{End}_{\mathbb{H}} W \mid xJ_{\mathcal{B}'} = J_{\mathcal{B}'}x \} \rightarrow M_m(\mathbb{C})$$

by $\Lambda'_{\mathcal{B}'}(x) := A + \sqrt{-1}B$ where $A, B \in M_m(\mathbb{R})$ are the unique elements such that $[x]_{\mathcal{B}'} = A + jB$. In view of the above lemma it is clear that $\Lambda'_{\mathcal{B}'}(K_{\mathcal{B}'} \circ \text{U}(m))$ and hence $\Lambda'_{\mathcal{B}'} : K_{\mathcal{B}'} \rightarrow \text{U}(m)$ is an isomorphism of Lie groups.

Let $\widetilde{W}$ be a right $\mathbb{H}$-vector space, $\langle \widetilde{\cdot}, \cdot \rangle$ be a non-degenerate Hermitian form on $\widetilde{W}$ with signature $(\tilde{p}, \tilde{q})$. Let $\mathcal{B} := (v_1, \ldots, v_{\tilde{p}}, v_{\tilde{p}+1}, \ldots, v_{\tilde{p}+\tilde{q}})$ be a standard orthogonal basis of $W$ such that

$$\langle v_r, v_r \rangle = \begin{cases} +1 & \text{if } 1 \leq r \leq \tilde{p} \\ -1 & \text{if } \tilde{p} + 1 \leq r \leq \tilde{p} + \tilde{q} \end{cases}.$$ 

Let $\widetilde{W}_+ := \text{Span}_{\mathbb{H}}\{v_1, \ldots, v_{\tilde{p}}\}$ and $\widetilde{W}_- := \text{Span}_{\mathbb{H}}\{v_{\tilde{p}+1}, \ldots, v_{\tilde{p}+\tilde{q}}\}$. The next lemma is a standard fact where we recall, without a proof, an explicit description of maximal compact subgroups in the group $\text{SU}(\widetilde{W}, \langle \widetilde{\cdot}, \cdot \rangle)$.

$$\tilde{K} := \{ g \in \text{SU}(\widetilde{W}, \langle \widetilde{\cdot}, \cdot \rangle) \mid g(\widetilde{W}_+) \subset \widetilde{W}_+ \text{ and } g(\widetilde{W}_-) \subset \widetilde{W}_- \}$$

Lemma 3.29. Let $\tilde{W}$, $\langle \tilde{\cdot}, \cdot \rangle$, $\mathcal{B}$ be as above. Then $\tilde{K}$ is a maximal compact subgroup of $\text{SU}(\tilde{W}, \langle \tilde{\cdot}, \cdot \rangle)$.

In the next lemma we specify a maximal compact subgroup $\mathcal{Z}_{\text{Sp}(p,q)}(X, H, Y)$ which will be used in Theorem 3.31.

Lemma 3.30. Let $K$ be the subgroup of $\mathcal{Z}_{\text{Sp}(p,q)}(X, H, Y)$ consisting of all $g$ in $\mathcal{Z}_{\text{Sp}(p,q)}(X, H, Y)$ satisfying the following conditions:
(1) $g(V^l_+(d)) \subset V^l_+(d)$ and $g(V^l_-(d)) \subset V^l_-(d)$, for all $d \in \mathbb{N}_d$ and $0 \leq l \leq d - 1$.

(2) For all $\eta \in \mathbb{E}_d$, there exists $A_\eta, B_\eta \in M_{l_\eta}(\mathbb{R})$ with $A_\eta + \sqrt{-1}B_\eta \in U(t_\eta)$ such that

$$A_\eta + jB_\eta = \begin{bmatrix} g|V^0_+(\eta) \end{bmatrix}_{c^0_+(\eta)} = \begin{bmatrix} g|V^l_-(\eta) \end{bmatrix}_{c^l_-(\eta)}; \text{ for all } 0 \leq l \leq \eta - 1.$$ 

(3) For all $\theta \in \mathbb{O}_d^1$, there exist $C_\theta \in \text{Sp}(p_\theta), D_\theta \in \text{Sp}(q_\theta)$ such that

$$C_\theta = \begin{bmatrix} g|V^0_+(\theta) \end{bmatrix}_{c^0_+(\theta)} = \begin{bmatrix} g|V^l_-(\theta) \end{bmatrix}_{c^l_-(\theta)}; \text{ for all } 0 \leq l < (\theta - 1)/2$$

and

$$D_\theta = \begin{bmatrix} g|V^0_-(\theta) \end{bmatrix}_{c^0_-(\theta)} = \begin{bmatrix} g|V^l_-(\theta) \end{bmatrix}_{c^l_-(\theta)}; \text{ for all } 0 \leq l < (\theta - 1)/2$$

for all $0 \leq l \leq (\theta - 1)/2$.

(4) For all $\zeta \in \mathbb{O}_d^3$, there exist $C_\zeta \in \text{Sp}(p_\zeta), D_\zeta \in \text{Sp}(q_\zeta)$ such that

$$C_\zeta = \begin{bmatrix} g|V^0_+(\zeta) \end{bmatrix}_{c^0_+(\zeta)} = \begin{bmatrix} g|V^l_-(\zeta) \end{bmatrix}_{c^l_-(\zeta)}; \text{ for all } 0 \leq l < (\zeta - 1)/2$$

and

$$D_\zeta = \begin{bmatrix} g|V^0_-(\zeta) \end{bmatrix}_{c^0_-(\zeta)} = \begin{bmatrix} g|V^l_-(\zeta) \end{bmatrix}_{c^l_-(\zeta)}; \text{ for all } 0 \leq l < (\zeta - 1)/2$$

for all $0 \leq l \leq (\zeta - 1)/2$.

Then $K$ is a maximal compact subgroup of $\mathcal{Z}_{\text{Sp}(p,q)}(X, H, Y)$.

**Proof.** The proof of the lemma is similar to that of Lemma 3.17. \qed

We need some more notation to state Theorem 3.31. For $d \in \mathbb{N}_d$, define

$$\mathcal{C}_+(d) := c^0_+(d) \vee \cdots \vee c^{d-1}_+(d) \text{ and } \mathcal{C}_-(d) := c^0_-(d) \vee \cdots \vee c^{d-1}_-(d).$$

Let $\alpha := \#\mathbb{E}_d$, $\beta := \#\mathbb{O}_d^1$ and $\gamma := \#\mathbb{O}_d^3$. We enumerate $\mathbb{E}_d = \{\eta_i \mid 1 \leq i \leq \alpha\}$ such that $\eta_i < \eta_{i+1}$, $\mathbb{O}_d^1 = \{\theta_j \mid 1 \leq j \leq \beta\}$ such that $\theta_j < \theta_{j+1}$ and similarly $\mathbb{O}_d^3 = \{\zeta_j \mid 1 \leq j \leq \gamma\}$ such that $\zeta_j < \zeta_{j+1}$. Now define

$$\mathcal{E}_+ := \mathcal{C}_+(\eta_1) \vee \cdots \vee \mathcal{C}_+(\eta_\alpha); \quad \mathcal{O}_d^1 := \mathcal{C}_+(\theta_1) \vee \cdots \vee \mathcal{C}_+(\theta_\beta); \quad \mathcal{O}_d^3 := \mathcal{C}_+(\zeta_1) \vee \cdots \vee \mathcal{C}_+(\zeta_\gamma).$$
Let $H$ and $\Lambda \in \mathcal{H}$ be specific elements. The signature of the form $p$ associated with $H$ is clear that $R$ is a standard orthogonal basis with $\mathcal{H}_+ = \{ v \in \mathcal{H} \mid \langle v, v \rangle = 1 \}$ and $\mathcal{H}_- = \{ v \in \mathcal{H} \mid \langle v, v \rangle = -1 \}$. In particular, $\# \mathcal{H}_+ = p$ and $\# \mathcal{H}_- = q$.

For a complex matrix $A \in M_m(\mathbb{C})$, let $Re A \in M_m(\mathbb{R})$ denote the real part of $A$ and $Im A \in M_m(\mathbb{R})$ denote the imaginary part of $A$. Thus $A = Re A + \sqrt{-1} Im A$. The $\mathbb{R}$-algebra

$$
\prod_{i=1}^{\alpha} M_{t_{\eta_i}}(\mathbb{R}) \times \prod_{j=1}^{\beta} (M_{p_{\eta_j}}(\mathbb{R}) \times M_{q_{\eta_j}}(\mathbb{H})) \times \prod_{k=1}^{\gamma} (M_{p_{\zeta_k}}(\mathbb{H}) \times M_{q_{\zeta_k}}(\mathbb{H}))
$$

is embedded into $M_p(\mathbb{H})$ and $M_q(\mathbb{H})$ in the following two ways:

$$
D_p:\prod_{i=1}^{\alpha} M_{t_{\eta_i}}(\mathbb{R}) \times \prod_{j=1}^{\beta} (M_{p_{\eta_j}}(\mathbb{R}) \times M_{q_{\eta_j}}(\mathbb{H})) \times \prod_{k=1}^{\gamma} (M_{p_{\zeta_k}}(\mathbb{H}) \times M_{q_{\zeta_k}}(\mathbb{H})) \rightarrow M_p(\mathbb{H})
$$

and

$$
D_q:\prod_{i=1}^{\alpha} M_{t_{\eta_i}}(\mathbb{R}) \times \prod_{j=1}^{\beta} (M_{p_{\eta_j}}(\mathbb{R}) \times M_{q_{\eta_j}}(\mathbb{H})) \times \prod_{k=1}^{\gamma} (M_{p_{\zeta_k}}(\mathbb{H}) \times M_{q_{\zeta_k}}(\mathbb{H})) \rightarrow M_p(\mathbb{H})
$$

Let $\Lambda_\mathcal{H} : \text{End}_\mathbb{R} \mathbb{H}^n \rightarrow M_n(\mathbb{H})$ be the isomorphism of $\mathbb{R}$-algebras induced by the ordered basis $\mathcal{H}$ in (3.51). Let $M$ be the maximal compact subgroup of $\text{Sp}(p,q)$ which leaves invariant simultaneously the two subspace spanned by $\mathcal{H}_+$ and $\mathcal{H}_-$. Clearly, $\Lambda_\mathcal{H}(M) = \text{Sp}(p) \times \text{Sp}(q)$.

**Theorem 3.31.** Let $X \in \mathcal{N}_{\text{sp}(p,q)} \cdot \mathcal{V}_{\text{sp}(p,q)}(\mathcal{O}_X) = (d, \text{sgn}_\mathcal{O}_X)$. Let $\alpha := \# \mathcal{O}_d$, $\beta := \# \mathcal{O}_\mathcal{A}$ and $\gamma := \# \mathcal{O}_\mathcal{A}$. Let $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sp}(p,q)$ and $(p_\eta, q_\eta)$ be the signature of the form $(\cdot, \cdot)_\theta$, for all $\theta \in \mathcal{O}_\eta$. Let $K$ be the maximal compact subgroup of
$\mathcal{Z}_{\text{Sp}(p,q)}(X,H,Y)$ as in Lemma 3.30. Let the maps $D_p$ and $D_q$ be defined as in (3.52) and (3.53), respectively. Then $\Lambda_H(K) \subset \text{Sp}(p) \times \text{Sp}(q)$ is given by

$$\Lambda_H(K) = \left\{ D_p(g) \oplus D_q(g) \mid g \in \prod_{i=1}^{\alpha} U(t_{\alpha_i}) \times \prod_{j=1}^{\beta} \left( \text{Sp}(p_{\theta_j}) \times \text{Sp}(q_{\theta_j}) \right) \times \prod_{k=1}^{\gamma} \left( \text{Sp}(p_{\zeta_k}) \times \text{Sp}(q_{\zeta_k}) \right) \right\}.$$ 

Moreover, the nilpotent orbit $O_X$ in $\text{sp}(p,q)$ is homotopic to $(\text{Sp}(p) \times \text{Sp}(q))/\Lambda_H(K)$.

**Proof.** This follows by writing the matrices of the elements of the maximal compact subgroup $K$ in Lemma 3.30 with respect to the ordered basis $H$ as in (3.51).

The second part follows from Theorem 2.3 and the well-known fact that any maximal compact subgroup of $\text{Sp}(p,q)$ is isomorphic to $\text{Sp}(p) \times \text{Sp}(q)$.

**Appendix**

Let $G$ be a real Lie group with Lie algebra $\mathfrak{g}$. We further assume that the Lie algebra $\mathfrak{g}$ is simple and of classical type. Here we give the descriptions of the reductive part of the centralizers in $G$ of nilpotent elements in $\mathfrak{g}$ when $\mathfrak{g}$ is a subalgebra of a matrix algebra over $\mathbb{H}$. When $\mathfrak{g}$ is isomorphic to a complex simple Lie algebra or one of the Lie algebras $\mathfrak{gl}_n(\mathbb{R})$ or $\mathfrak{u}(p,q)$ or $\mathfrak{o}(p,q)$ or $\mathfrak{sp}(n,\mathbb{R})$ such descriptions are due to Springer and Steinberg; see [SpSt, 1.8,p.E-85; 2.25,p.E-95] and [CoMc, Theorem 6.1.3]. However, when the classical simple Lie algebras are matrix subalgebras with entries from $\mathbb{H}$, we are unable to locate such descriptions in the existing literature. Further, as mentioned in [BCM], the noncommutativity of $\mathbb{H}$ creates technical difficulties and does not allow direct extensions of the results of [SpSt] to the case of simple Lie algebras involving $\mathbb{H}$. In [BCM] the description of the reductive part, as above, was needed in the case when $\mathfrak{g}$ simple matrix Lie algebra involving $\mathbb{H}$. The reasons mentioned above together with the foregoing requirements in [BCM] led us to doing the computations. The first key point in our computations is the well-known fact (see [CoMc, Lemma 3.7.3, p. 50]) that the centralizer of a $\mathfrak{sl}_2(\mathbb{R})$-triple containing a nilpotent element constitutes a reductive part of the centralizer of the nilpotent element itself. In view of the Jacobson–Morozov theorem and the above result we use [BCM, Lemma 4.4] which was deduced applying the basics of $\mathfrak{sl}_2(\mathbb{R})$-representation theory. The above method was indicated in [Ma, Section 3.1–3.3] and in [CoMc, Section 9.3]. The advantage of this method lies in the uniform manner it deals with all the cases of simple classical Lie algebras involving $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. We record these computations below and these results should be viewed as complementary to those in [SpSt, 1.8,p.E-85; 2.25,p.E-95] and in [CoMc, Theorem 6.1.3].

In what follows we will use the notation as defined in §2, and in particular, we will employ the symbols $\mathbb{N}_d$, $\mathbb{E}_d$ and $\mathcal{O}_d$ as given in (2.4).

**Proposition 3.32.** Let $X \in \mathfrak{sl}_n(\mathbb{H})$ be a non-zero nilpotent element, and $\{X,H,Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sl}_n(\mathbb{H})$. Let the nilpotent orbit $O_X$ corresponds to the partition $\mathbf{d} \in \mathcal{P}(n)$. Then

$$\mathcal{Z}_{\text{Sl}_n(\mathbb{H})}(X,H,Y) \simeq S \left( \prod_{d \in \mathbb{N}_d} \text{GL}_d(\mathbb{H})^d \right) \Delta^d.$$

**Proof.** The proof follows from [BCM, Lemma 4.4(2)].
**Proposition 3.33.** Let $X \in \mathfrak{so}^\ast(2n)$ be a non-zero nilpotent element, and $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{so}^\ast(2n)$. Let the nilpotent orbit $\mathcal{O}_X$ corresponds to the signed Young diagram of size $n$. Let $p_\eta$ (respectively, $q_\eta$) be the number of $+1$ (respectively, $-1$) in the 1st column of the block of size $t_\eta \times \eta$ for $\eta \in \mathbb{E}_d$. Then

$$Z_{SO^\ast(2n)}(X, H, Y) \simeq \prod_{\theta \in \mathbb{O}_d} SO^\ast(2t_\theta) \times \prod_{\eta \in \mathbb{E}_d} \text{Sp}(p_\eta, q_\eta).$$

**Proof.** Recall that for $d \in \mathbb{N}_d$, the form $(\cdot, \cdot)_d$ as in (3.8) is Hermitian or skew-Hermitian according as $d$ is even or odd. For the nilpotent element $X$, the signature of $(\cdot, \cdot)_\eta$ on $L(\eta - 1)$ is $(p_\eta, q_\eta)$ for $\eta \in \mathbb{E}_d$. Now the proof follows from [BCM, Lemma 4.4(4)].

**Proposition 3.34.** Let $X \in \mathfrak{sp}(p, q)$ be a non-zero nilpotent element, and $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sp}(p, q)$. Let the nilpotent orbit $\mathcal{O}_X$ corresponds to the signed Young diagram of signature $(p, q)$, where $p_\theta$ (respectively, $q_\theta$) denotes the number of $+1$ (respectively, $-1$) in the 1st column of the block of size $t_\theta \times \theta$ for $\theta \in \mathbb{O}_d$. Then

$$Z_{Sp(p, q)}(X, H, Y) \simeq \prod_{\eta \in \mathbb{E}_d} SO^\ast(2t_\eta) \times \prod_{\theta \in \mathbb{O}_d} \text{Sp}(p_\theta, q_\theta).$$

**Proof.** Recall that for $d \in \mathbb{N}_d$, the form $(\cdot, \cdot)_d$ as in (3.8) is Hermitian or skew-Hermitian according as $d$ is odd or even. Now the proof follows from the fact that the signature of $(\cdot, \cdot)_\theta$ on $L(\theta - 1)$ is $(p_\theta, q_\theta)$ for $\theta \in \mathbb{O}_d$ and [BCM, Lemma 4.4(4)].

**Acknowledgements**

Indranil Biswas is supported by a J. C. Bose Fellowship. Pralay Chatterjee acknowledges support from the SERB-DST MATRICS project: MTR/2020/000007. Chandan Maity is supported by an NBHM PDF during the course of this work.

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