Deterministic Entanglement of Assistance and Monogamy Constraints

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Certain quantum information tasks require entanglement of assistance, namely a reduction of a tripartite entangled state to a bipartite entangled state via local measurements. We establish that ‘concurrence of assistance’ (CoA) identifies capabilities and limitations to producing pure bipartite entangled states from pure tripartite entangled states and prove that CoA is an entanglement monotone for \((2 \times 2 \times n)\)-dimensional pure states. Moreover, if the CoA for the pure tripartite state is at least as large as the concurrence of the desired pure bipartite state, then the former may be transformed to the latter via local operations and classical communication, and we calculate the maximum probability for this transformation when this condition is not met.

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Entanglement is crucial for many quantum information processing tasks. More specifically bipartite entanglement underpins ubiquitous tasks such as quantum teleportation, entanglement swapping, and remote state preparation. Three alternatives to producing bipartite entanglement include processing a product state of two qubits through a two-qubit unitary transformation such as a CNOT gate, creating a source of two-qubit entanglement (such as parametric down conversion for producing polarization-entangled two-photon states), and the reduction of a multipartite entangled state to an entangled state over fewer parties (e.g. bipartite) via measurements. Each process is important and the choice of which process should be used depends on the physics of the implementation. The latter case, which we call “assisted entanglement”, quantified by the entanglement of assistance (EoA), is especially important for quantum communication, where quantum repeaters are needed to establish bipartite entanglement over a long length scale, and for spin systems that are all coupled via Ising or similar interactions resulting in a multipartite entangled state (which is the resource for one-way quantum computation).

An important application of assisted entanglement concerns the deterministic creation of a bipartite entangled state from a tripartite entangled state, where the tripartite state consists of three parties: the two qubits to be prepared in a bipartite entangled state and a party in an \(n\)-dimensional Hilbert space that corresponds to all other particles in the system. Here we fully assess deterministic creation of a bipartite entangled state from a pure tripartite entangled state, including (i) proving that concurrence of assistance (CoA) is an entanglement monotone, (ii) presenting a condition for CoA which, if met, guarantees that deterministic distillation of a bipartite pure state from a single copy of a tripartite state can be achieved, (iii) calculating the maximum probability for obtaining the desired bipartite state if this condition is not satisfied, and (iv) showing that CoA satisfies monogamy constraints that are dual to the Coffman-Kundu-Wootters (CKW) monogamy constraints for concurrence and recently proven in the general case by Osborne.

Thus we have provided a strong foundation to analyzing assisted capabilities and limitations of assisted entanglement, which is an important tool for creating bipartite entanglement in certain important physical systems.

We now consider a tripartite pure state shared between three parties referred to as Alice, Bob, and Sapna: the entanglement supplier, Sapna, performs a measurement on her share of the tripartite state, which yields a known bipartite entangled state for Alice+Bob. Specifically we study a pure \((2 \times 2 \times n)\)-dimensional tripartite entangled state, \(|\psi\rangle_{\text{ABS}} \in \mathcal{H}_2 \times \mathcal{H}_2 \times \mathcal{H}_n\), with \(n \geq 2\) the dimension of Sapna’s system. Tracing over Sapna’s system yields the bipartite \((2 \times 2)\)-dimensional mixed state \(\rho_{\text{AB}} = \text{Tr}_S (|\psi\rangle_{\text{ABS}} \langle \psi|)\) shared by Alice+Bob, and any decomposition of \(\rho_{\text{AB}}\) can be realized by a generalized measurement performed by Sapna.

Sapna’s aim is to maximize entanglement for Alice+Bob, and the maximum average entanglement she can create is the EoA, which was originally defined in terms of entropy of entanglement (dual to the entanglement of formation) but extended here to include any bipartite entanglement measure: the EoA \(E_a\) for a tripartite pure state \(|\psi\rangle_{\text{ABS}}\) is dual to any entanglement measure \(E\) according to

\[ E_a(|\psi\rangle_{\text{ABS}}) = E_a(\rho_{\text{AB}}) = \max \sum_k p_k E(|\phi_k\rangle_{\text{AB}}), \]  

which is maximized over all possible decompositions of \(\rho_{\text{AB}} = \sum_k p_k |\phi_k\rangle_{\text{AB}} \langle \phi_k|\).

**Remark.** In general, a distribution of states that maximizes Eq. (1) for a given entanglement measure \(E\) will not necessarily be the optimal distribution for a different measure. Therefore, the choice of measure is important and depends on the planned quantum information task by Alice and Bob subsequent to Sapna’s assistance.
For any choice of entanglement monotone $E$, EoA is bounded, namely

$$E_a(|\psi\rangle_{\text{ABS}}) \leq \min\{E_{\text{A(BS)}}, E_{\text{B(A(S)}}\} ,$$

(2)

for $E_{\text{A(BS)}}$ ($E_{\text{B(A(S)}}$) the bipartite entanglement shared by Alice with Bob+Sapna (Bob with Alice+Sapna). Eq. (2) holds because both bipartite entanglements $E_{\text{A(BS)}}$ and $E_{\text{B(A(S)}}$ cannot increase by any general local operations and classical communications (LOCC) by Alice, Bob, and Sapna. This bound is not tight in general, and tighter bounds have been obtained (e.g. [10, 11]). Recently, it has been shown [12] (see also [13] for the generalization of this result) that in the asymptotic limit the upper bound in Eq. (2) is saturated [13].

In general the EoA [11] is difficult to calculate, in contrast to the CoA for the $2 \times 2 \times n$ pure state $|\psi\rangle_{\text{ABS}}$ [5]:

$$C_a(|\psi\rangle_{\text{ABS}}) = F(\hat{\rho}_{AB}, \hat{\rho}_{AB}) ,$$

(3)

with $\hat{\rho}_{AB}$ defined by the “spin flip” transformation [10]

$$\hat{\rho}_{AB} = \sigma_y \otimes \sigma_y \hat{\rho}_{AB} \sigma_y \otimes \sigma_y$$

and $F(\rho, \sigma) = \text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}$ the fidelity. Although EoA (including CoA) is not a bipartite entanglement monotone [15], we consider whether EoA can be an entanglement monotone for pure tripartite states. In general, it is not known if the EoA [11] provides the maximum average of entanglement under general three-party LOCC (i.e., not only by Sapna’s measurement) and hence may not be an entanglement monotone, but, in the following theorem, we prove that CoA is indeed an entanglement monotone.

**Theorem 1.** CoA is an entanglement monotone for $2 \times 2 \times n$ pure states.

**Proof.** We need to prove that general LOCC by Alice, Bob and Sapna cannot increase the CoA for a tripartite $2 \times 2 \times n$ pure state $|\psi\rangle_{\text{ABS}}$. Thus, let us consider the following three-way LOCC. First, Sapna performs a measurement represented by Kraus operators $M(j)$ and sends result $j$ to Alice+Bob. Based on this result, Alice performs a measurement represented by Kraus operators $\hat{A}_j^{(k)}$ and transmits her result $k$ to Bob+Sapna. Based on outcomes $j, k$ from Sapna+Alice, Bob performs a measurement represented by Kraus operator $\hat{B}_j^{(n)}$ and sends his result $n$ to Sapna. Finally Sapna performs a second measurement with Kraus operator denoted by $\hat{F}_{jn}^{(i)}$ and sends the result $i$ to Alice and Bob.

The final distribution of entangled states shared between Alice and Bob is denoted by $\{N_{jkn}, |\phi_{jkn}\rangle_{\text{ABS}}\}$, where $N_{jkn}$ is the probability for outcome $j, k, n, i$. The state $|\phi_{jkn}\rangle_{\text{ABS}}$ is given by

$$|\phi_{jkn}\rangle_{\text{ABS}} = \frac{\hat{A}_j^{(k)} \otimes \hat{B}_j^{(n)} \otimes \hat{F}_{jn}^{(i)} M(j)}{\sqrt{N_{jkn}}} |\psi\rangle_{\text{ABS}} ,$$

(4)

and the final average of concurrence is given by $C_{AB} = \sum_{jkn} N_{jkn} C(\sigma^{jkn}_{AB})$ for $\sigma^{jkn}_{AB} = \text{Tr}_S |\phi_{jkn}\rangle_{\text{ABS}} \langle \phi_{jkn}|$. As the concurrence of any bipartite state $|\phi\rangle_{AB}$ satisfies $C(\hat{A}_j^{(k)} \otimes \hat{B}_j^{(n)} |\phi\rangle) = |\text{Det}(\hat{A}_j^{(k)})||\text{Det}(\hat{B}_j^{(n)})| C(|\phi\rangle)$, we obtain

$$C_{AB} = \sum_{jkn} |\text{Det}(\hat{A}_j^{(k)})||\text{Det}(\hat{B}_j^{(n)})| \leq \sum_{j} C_a (M(j) |\psi\rangle_{\text{ABS}}) \leq C_a (|\psi\rangle_{\text{ABS}}) .$$

(5)

The first inequality follows from the fact that the second measurement performed by Sapna (represented by the Kraus operators $\hat{F}_{jn}^{(i)}$) yields a probability distribution of states with average concurrence smaller than the concurrence of assistance of $M(j) |\psi\rangle_{\text{ABS}}$. The second inequality follows from the geometric-arithmetic inequality:

$$\sum_{j} |\text{Det}(\hat{A}_j^{(k)})||\text{Det}(\hat{B}_j^{(n)})| \leq \frac{1}{2} \sum_{j} |\text{Det}(\hat{A}_j^{(k)})|^2 = 1$$

and similarly for $\hat{A}_j^{(k)}$. Evidently all operations that are performed by Alice, Bob and Sapna cannot yield a probability distribution with average concurrence (between Alice+Bob) that exceeds the CoA.

CoA is a readily computed measure of entanglement for tripartite systems that can serve as a convenient mathematical tool to determine when a transformation between two tripartite states cannot be realized by LOCC.

**Example 1.** Consider two tripartite states $|\psi\rangle = \sqrt{\sigma}(110) + \sqrt{\sigma}(010) + \sqrt{\sigma}(001)$ and $|\phi\rangle = \sqrt{\sigma}(010) + \sqrt{\sigma}(001) + \sqrt{\sigma}(101)$ with $a + b + c = 1$ and $b > c$ (both states belong to the W-class [17]). Let $C_{3}^{k}(k = 1, 2, 3)$ be the CoA with respect to Sapna, who holds the $k$th qubit. From Eq. (3), we obtain $C_{3}^{k}(|\psi\rangle) \geq C_{3}^{k}(|\phi\rangle)$, but $C_{3}^{1}(|\psi\rangle) \leq C_{3}^{1}(|\phi\rangle)$.

As both $C_{3}^{1}$ and $C_{3}^{3}$ are entanglement monotones, deterministic LOCC transformations between $|\psi\rangle$ and $|\phi\rangle$ cannot be realized.

In the following theorem we establish a sufficient condition, based on CoA, which indicates if a transformation from tripartite state to bipartite state can be realized deterministically by LOCC.

**Theorem 2.** A deterministic map $T : \mathcal{H}_2 \times \mathcal{H}_2 \times \mathcal{H}_n \rightarrow \mathcal{H}_2 \times \mathcal{H}_2 : |\psi\rangle_{\text{ABS}} \mapsto |\phi\rangle_{\text{ABS}}$ can be realized by LOCC iff

$$C_a (|\psi\rangle_{\text{ABS}}) \geq C(|\phi\rangle_{\text{ABS}}) .$$

(6)

**Proof.** Inequality (6) is a necessary condition because CoA is an entanglement monotone; thus the sum is now to prove that inequality (6) is a sufficient condition.
Beginning with the decomposition $\rho_{AB} = \sum_{k=1}^n |\phi_k\rangle \langle \phi_k|$ for $n \leq 4$ the rank of $\rho_{AB}$ and $|\phi_k\rangle$ subnormalized such that $\langle \phi_k | \phi_k\rangle = \lambda_k |\psi_{ik}\rangle$ (see (12)), we observe that the average concurrence of this decomposition is $\sum_{k=1}^n \lambda_k$, which is optimal; c.f. Eq. (9). Any other decomposition of $\rho_{AB} = \sum_{i=1}^m |\chi_i\rangle \langle \chi_i|$ is given by $|\chi_i\rangle = \sum_{k=1}^n U_{ik} |\phi_k\rangle$ with $m \geq n$ and $U$ an $m \times m$ unitary matrix. Thus, the average concurrence of the decomposition, $\chi$, is given by $\langle C \rangle = \sum_{i=1}^m \sum_{k=1}^n (U_{ik})^2 \lambda_k$. Thus, any other decomposition, $\chi$, has the same average concurrence as $\phi$ as long as the matrix elements $U_{ik}$ are all real (i.e. $U$ is an $m \times m$ orthogonal matrix). Hence, as discussed in (10), one can always find an optimal decomposition such that all states in the decomposition have the same concurrence. According to Eq. (10) this concurrence exceeds $C(|\phi\rangle_{AB})$ so, since all bipartite states in this decomposition have dimension $2 \times 2$, they are all majorized by $|\phi\rangle_{AB}$, and it follows from Nielsen’s theorem (25) that Alice, Bob and Sapna can transform $|\psi_{ABS}\rangle$ to $|\phi_{AB}\rangle$ by LOCC.

In addition to CoA being valuable for testing whether deterministic LOCC transformations map from a pure tripartite state to tripartite or bipartite states, the CoA for the $2 \times 2 \times 2$ pure state $|\psi_{ABS}\rangle$ also exhibits monogamy constraints (7) (entanglement tradeoffs) analogous to those for the usual concurrence. Here we derive another monogamy constraint which is in some sense the dual to the CKW constraint (8).

**Theorem 3.** For a pure tripartite state in $\mathcal{H}_2 \times \mathcal{H}_2 \times \mathcal{H}_2$ and $\tau_{ABS} = C^2_A(\rho_{AB}) - C^2(\rho_{AS}) - C^2(\rho_{PS})$ the 3-tangle (7) with $\rho_{AB}$ and $\rho_{AS}$ the reduced density matrices after tracing over Sapna’s and Bob’s systems, respectively,

$$\tau_{ABS} = C^2_a(\rho_{AB}) + C^2_a(\rho_{AS}) - C^2_A(\rho_{PS}) \geq 0.$$  

**Proof.** We employ CKW’s notation, wherein $\lambda_1^{AB}, \lambda_2^{AB}$ and $\lambda_1^{AS}, \lambda_2^{AS}$ denote the eigenvalues of $R_{AB} \equiv \sqrt{\rho_{AB}\rho_{AB}}$, and $R_{AS} \equiv \sqrt{\rho_{AS}\rho_{AS}}$, respectively. CKW have shown that $Tr(\rho_{AB}^2) + Tr(\rho_{AS}) = C^2_A$. Therefore,

$$C^2(\rho_{AB}) + C^2(\rho_{AS}) - C^2_A(\rho_{PS}) = (TrR_{AB})^2 + (TrR_{AS})^2 - TrR_{AB}^2 - TrR_{AS}^2 = 2(\lambda_1^{AB}\lambda_2^{AB} + \lambda_1^{AS}\lambda_2^{AS}).$$  

CKW have shown that $\tau_{ABS} = 2(\lambda_1^{AB}\lambda_2^{AB} + \lambda_1^{AS}\lambda_2^{AS})$, which proves the theorem.

CKW conjectured (8) (recently proven by Osborne (9)) that, for $n$ qubits (labeled by 1, 2, ..., $n$),

$$C^2(\rho_{12}) + C^2(\rho_{13}) + \cdots + C^2(\rho_{1n}) \leq C^2(\rho_{1(23\cdots n)}).$$  

Similarly, we are willing to conjecture that the dual to this conjecture also holds.

**Conjecture 4.**

$$C^2(\rho_{12}) + C^2(\rho_{13}) + \cdots + C^2(\rho_{1n}) \geq C^2(\rho_{1(23\cdots n)}).$$  

It is interesting to note that for states of the form

$$|\phi\rangle = \alpha_1|1000\cdots0\rangle + \alpha_2|0100\cdots0\rangle + \cdots + \alpha_n|000\cdots0\rangle,$$

both inequalities (10) become equalities.

In Theorem 2 we established a necessary and sufficient condition for the existence of a deterministic transformation from a pure $2 \times 2 \times n$ tripartite state to a bipartite state. When this condition is violated, such a transformation may be possible but only probabilistically. Here we consider a $d \times d \times n$ pure tripartite state $|\psi\rangle_{ABS}$ shared by Alice, Bob and Sapna and investigate the maximum probability, $P_m$, to ‘distill’ a bipartite state $|\phi\rangle_{AB}$ according to the protocol that Sapna first performs a generalized measurement, whose outcome is communicated to Alice and Bob, and Alice and Bob subsequently perform pairwise LOCC.

The maximum probability, $P_m$, to transform locally one $d \times d$ bipartite state to another is given by (10):

$$P_m(|\psi\rangle \rightarrow |\phi\rangle) = \min \left\{ \frac{E_1(|\psi\rangle)}{E_1(|\phi\rangle)}, \frac{E_2(|\psi\rangle)}{E_2(|\phi\rangle)}, \ldots, \frac{E_d(|\psi\rangle)}{E_d(|\phi\rangle)} \right\}$$

for $E_i(|\psi\rangle) = \sum_{k=1}^d \lambda_k$, where $\lambda_k$ are the Schmidt numbers of $|\psi\rangle$ in a decreasing order. NB: for a given fixed state $|\phi\rangle$, $E_0(|\phi\rangle) = P_m(|\psi\rangle \rightarrow |\phi\rangle)$ is an entanglement monotone. The probability to ‘distill’ the bipartite state $|\phi\rangle_{AB}$ from $|\psi\rangle_{ABS}$ is, therefore, given by

$$P_m(|\psi\rangle_{ABS} \rightarrow |\phi\rangle_{AB}) = \max \sum_i^d p_i E_0(|\psi_i\rangle_{AB}),$$

where the maximum is taken over all the decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}$. That is, the EoA when measured in terms of the monotone, $E_0$, can be interpreted as the maximum probability to distill the bipartite state $|\phi\rangle$. In particular, the maximum probability to distill a $d \times d$ maximally entangled state is

$$P_m = \max \sum_i^d p_i E_0(|\psi_i\rangle_{AB}),$$

where the normalized entanglement monotone, $E_0(|\psi\rangle) \equiv d_{\min}(|\psi\rangle)$, and $\lambda_{\min}(|\psi\rangle)$ is the minimum Schmidt number (including zero) of $\psi$. In general, it is quite difficult to calculate $P_m$, however, in the following we calculate it for a large class of $(2 \times 2 \times n)$-dimensional pure states.

**Lemma 5.** For $a_{i,j}^l = A_i^l |l\rangle_A \langle j|_B \langle l|_S$ for Alice, Bob and Sapna, respectively, such that the matrices $A_i^l$ are diagonal for all $l$ with $A_i \equiv (a_{i,j}^l)^l$ a $2 \times 2$ matrix of components; similarly, there exists orthonormal bases $|i\rangle_A$, $|j\rangle_B$, and $|l\rangle_S$ for Alice, Bob and Sapna, respectively, that the matrices $A_i^l$ are diagonal for all $l$. 


Theorem 6. If $|\psi\rangle_{\text{ABS}} \in \mathcal{A}$ then the maximum probability $P_m$ is given by

$$P_m = \min\{E_{A(\text{BS})}, E_{B(\text{AS})}\}.$$  

Proof. The average of $E_2$ (see Eq. (14) with $d = 2$) between Alice and Bob after Sapna performs the projective basis measurement in the basis $|l\rangle_S$ (as defined in Lemma 6) gives the desired result of Eq. (19). As $P_m$ cannot be greater than $\min\{E_{A(\text{BS})}, E_{B(\text{AS})}\}$ this is also the optimal result.

As $E_2 \leq E$ for all entanglement measures $E$ (normalized such that $E = 1$ for a Bell state), the EoA in terms of $E_2$ provides a lower bound for the EoA when measured with respect to any measure $E$.

In the following example the states belong to the class $\mathcal{A}$ so Eq. (19) is correct for these cases.

Example 2. The state (a) $\sqrt{a}|00\rangle + \sqrt{b}|01\rangle + \sqrt{c}|10\rangle$ from the $W$ class, with $a + b + c = 1$, the state (b) $\sqrt{a}|000\rangle + \sqrt{b}|111\rangle$ from the GHZ class, with $a + b = 1$, and the state (c) $|\phi_1\rangle_{A_1}, |\phi_2\rangle_{A_2} \in \mathcal{H}_2 \times \mathcal{H}_2 \times \mathcal{H}_2$ (with Sapna holding the two qubits $S_1$ and $S_2$, which is the case for entanglement swapping) are in $\mathcal{A}$.

In conclusion we have proved that CoA is an entanglement monotone that provides a condition to assess whether a deterministic transformation exists for creating a bipartite entangled state from a $2 \times 2 \times n$ pure tripartite state, and we have calculated the maximum probability for obtaining such states when this condition is not satisfied. The CoA provides an elegant approach to studying assisted entanglement and obeys monogamy constraints. Our analysis provides a foundation for studying the capabilities and limitations of assisted entanglement for producing bipartite entangled states from multipartite entangled states.

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