Comparison and verification of enthalpy schemes for polythermal glaciers and ice sheets with a one-dimensional model

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Abstract

The enthalpy method for the thermodynamics of polythermal glaciers and ice sheets is tested and verified by a one-dimensional problem (parallel-sided slab). The enthalpy method alone does not include explicitly the Stefan-type energy- and mass-flux matching conditions at the cold-temperate transition surface (CTS) that separates the upper cold from the lower temperate layer. However, these transition conditions are important for correctly determining the position of the CTS. For the numerical solution of the polythermal slab problem, we consider a two-layer front-tracking scheme as well as three different one-layer schemes that feature a single grid for both layers. Computed steady-state temperature and moisture profiles are verified with exact solutions, and transient solutions computed by the one-layer schemes are compared with those of the two-layer scheme, considered to be a reliable reference. While the conventional one-layer enthalpy scheme (that does not include the transition conditions at the CTS) can produce correct solutions for melting conditions at the CTS, it is more reliable to enforce the transition conditions explicitly. For freezing conditions, it is imperative to enforce them because the conventional one-layer scheme cannot handle the associated discontinuities. The suggested numerical schemes are suitable for implementation in three-dimensional glacier and ice-sheet models.

Keywords:
Glacier, Ice sheet, Polythermal ice, Modeling, Enthalpy method
1. Introduction

The decrease of the ice viscosity with increasing content of liquid water (moisture) in temperate ice was first confirmed and measured by Duval (1977). It is therefore desirable to simulate the moisture content in glaciers and ice sheets realistically, especially if the temperate ice occurs in a basal layer where shear deformation is largest. Mathematical models of polythermal ice masses were introduced and further developed by Fowler and Larson (1978), Hutter (1982), Fowler (1984) and Hutter (1993). We distinguish essentially two types of polythermal glaciers, Canadian-type polythermal glaciers, which are cold in most of the ice mass except for a temperate basal layer in the ablation zone, and Scandinavian-type glaciers, which are temperate in most parts except for a cold surface layer in the ablation zone (Blatter and Hutter, 1991).

This work attempts to verify thermodynamic schemes used in shallow ice sheet models. Therefore, we do not investigate processes which are not usually included in ice sheet models, such as possible diffusion of water in temperate ice (Hutter, 1993) and pre-melting in ice at sub-freezing temperatures (Dash et al., 2006). For verification of numerical solutions with exact solutions, we neglect the pressure dependence of the melting point and the temperature dependence of the heat conductivity and specific heat capacity. Following Aschwanden and Blatter (2005), “ice is treated as temperate if a change in heat content leads to a change in liquid water content alone, and is considered cold if a change in heat content leads to a temperature change alone.” This implies that temperate ice is at the melting point and the temperatures in cold ice are below the melting point.

Polythermal schemes that solve the field equations for the cold and temperate layers separately were implemented for both types of polythermal glaciers, in one dimension for the Scandinavian-type Storglaciären, Sweden (Pettersson et al., 2007), in two dimensions for the Canadian-type Laika Glacier, Canada (Blatter and Hutter, 1991) and for three-dimensional ice sheets, which are Canadian-type polythermal (Greve, 1997). With the assumption that moisture mostly accumulates along the trajectories of ice particles in the temperate layer, Aschwanden and Blatter (2005) used a trajectory model to determine the position of the cold-temperate transition surface (CTS) and the moisture content in the temperate part of Storglaciären.
Aschwanden et al. (2012) suggested an enthalpy scheme with the idea that, with enthalpy, only one thermodynamic field variable must be computed, and the temperature and moisture content result from the enthalpy as diagnostic fields. The domains of cold and temperate ice are discriminated by the contour of the enthalpy with no liquid water content at the melting point.

A crucial point in polythermal enthalpy schemes is their treatment of the Stefan-type energy- and mass-flux matching conditions at the CTS, which are important for determining its position (Greve, 1997). These transition conditions are not included explicitly in the formulation of the enthalpy scheme according to Aschwanden et al. (2012).

Two different cases must be distinguished. Melting conditions occur if cold ice flows across the CTS into the temperate region. At the CTS, the particles consist of ice at melting temperature without liquid water, and, after the transition, start to accumulate moisture due to strain heating. Thus, the boundary condition on the temperate side of the CTS is zero moisture content. To match the vanishing latent heat flux, the diffusive heat flux and corresponding enthalpy gradient on the cold side must also vanish.

The situation is different for freezing conditions at the CTS, where the ice flows from the temperate region into the cold region and the liquid water content of the temperate ice freezes at the CTS. The advective latent heat flux on the temperate side then changes into a diffusive heat flux on the cold side. Thus, a drop of a non-vanishing moisture content to zero results in a non-vanishing temperature (enthalpy) gradient in the cold layer at the CTS.

This work attempts to verify and test modified enthalpy methods, and in particular to test how the modified schemes handle the internal boundary between cold and temperate ice. For the verification, we use an exact solution which is available for steady states in a parallel-sided slab, which reduces the problem to one dimension (Greve, 1997; Greve and Blatter, 2009). In Section 2, we review the main concepts of the enthalpy method, and in Section 3, we formulate the enthalpy method for the special case of the parallel-sided slab. Section 4 deals with different one- and two-layer methods to solve this problem, the two-layer front-tracking scheme being used to provide reference solutions against which the performance of the simpler one-layer methods can be tested. Concrete numerical experiments are defined in Section 5, and results are presented and discussed in Sections 6 and 7.
2. Enthalpy formulation

In this paper we follow the formulation of Aschwanden et al. (2012) and use the notation of Greve and Blatter (2009). All physical parameters, such as the stress exponent \(n = 3\) and the rate factor \(A = 5.3 \cdot 10^{-24} \text{ s}^{-1}\text{Pa}^{-3}\) of Glen’s flow law, the thermal conductivity of ice, \(\kappa = 2.1 \text{ W m}^{-1}\text{K}^{-1}\), the melting point of ice, \(T_m = 0^\circ\text{C}\), the density of ice (cold and temperate), \(\rho = 910 \text{ kg m}^{-3}\), the specific heat content of ice, \(c = 2009 \text{ J kg}^{-1}\text{K}^{-1}\), and the specific heat of fusion, \(L = 3.35 \cdot 10^5 \text{ J kg}^{-1}\), are assumed to be constant for simplicity.

Let \(h_m = cT_m\) be the enthalpy of ice at the melting temperature with vanishing moisture content, which is defined by

\[
W = \frac{\rho_w}{\rho},
\]

where \(\rho_w\) is the partial density of liquid water in the mixture. For cold ice with a temperature \(T\) and temperate ice with a moisture content \(W\), the enthalpy is given by

\[
h = \begin{cases} 
  cT, & T < T_m, \\
  h_m + LW, & T = T_m \text{ and } 0 \leq W < 1,
\end{cases}
\]

and the corresponding balance equation reads (Greve and Blatter, 2009)

\[
\rho \frac{dh}{dt} = -\nabla \cdot \mathbf{q} + \text{tr}(\mathbf{t} \cdot \mathbf{D}),
\]

where \(t\) is time, \(\mathbf{q}\) is the diffusive heat flux, \(\mathbf{t}\) is the Cauchy stress tensor, \(\mathbf{D}\) is the strain-rate tensor and \(\text{tr}\) denotes the trace of a tensor. The diffusive heat flux in cold ice is given by Fourier’s law,

\[
\mathbf{q} = -\kappa \nabla T, \quad T < T_m.
\]

Combining Eqns (2) and (4) allows us to write the heat flux in terms of the enthalpy gradient in cold ice,

\[
\mathbf{q} = -\frac{\kappa}{c} \nabla h, \quad T < T_m.
\]

The diffusive enthalpy flux in the temperate layer is omitted because of the vanishing temperature gradient (\(\nabla T_m = 0\)) and the negligibly small (at least for small moisture content, \(W \ll 1\)) moisture diffusion,

\[
\mathbf{q} = 0, \quad T = T_m.
\]
3. Polythermal slab

To reduce the problem of a polythermal ice mass to a one-dimensional problem, we apply the plane strain approximation for a two-dimensional flow in the vertical $x$-$z$ plane of a parallel-sided slab with constant and steady thickness $H$ and constant inclination angle $\gamma$, and without dependencies on the transverse $y$ coordinates (Fig. 1). Furthermore, we impose uniformity in the down-slope ($x$) direction, $\partial(\cdot)/\partial x = 0$, for all field quantities. Thermomechanical coupling is omitted, thus strain heating due to horizontal shearing is prescribed (Greve, 1997; Greve and Blatter, 2009),

$$\text{tr}(t \cdot D) = 2A (\rho g \sin \gamma)^4 (H - z)^4. \quad (7)$$

Otherwise, the downslope velocity profile $v_x(z)$ and basal sliding are not relevant for the problem in consideration. Owing to the assumed uniformity in $x$-direction and the plane strain approximation, the continuity equation (mass balance) takes the form $\partial v_z / \partial z = 0$ for the slab problem, so that the velocity component in $z$-direction, $v_z$, is constant over depth, $v_z = \text{const.}$

![Figure 1: Polythermal parallel-sided ice slab geometry and coordinate system, modified after Greve and Blatter (2009).](image)

With all these conditions, and neglect of water diffusion, the balance equation for enthalpy, Eqn (3), is reduced to

$$\frac{\partial h}{\partial t} = -v_z \frac{\partial h}{\partial z} + \frac{1}{\rho c} \frac{\partial}{\partial z} \left( \kappa_{c,t} \frac{\partial h}{\partial z} \right) + \frac{2A}{\rho} (\rho g \sin \gamma)^4 (H - z)^4, \quad (8)$$
where \( v_z \) is the velocity component in \( z \)-direction, and the conductivity \( \kappa_{c,t} \) is

\[
\kappa_{c,t} = \begin{cases} 
\kappa, & h < h_m, \\
0, & h \geq h_m.
\end{cases}
\]  

(9)

We restrict this study to the Canadian-type polythermal slab. The imposed enthalpy boundary condition at the surface of the cold layer is

\[
h = h_s(t) = c T_s(t),
\]  

(10)

and at the CTS, we impose continuity of velocity and temperature,

\[
v^+_x = v^-_x, \quad v^+_z = v^-_z, \quad T^+ = T^- = T_m,
\]  

(11)

where the cold layer is defined as the positive, and the temperate layer as the negative side of the CTS. Depending on the direction of the ice flow through the CTS, we have to distinguish between melting and freezing conditions.

For melting conditions, the volume flux \( a^\perp_m \) through the possibly moving CTS is

\[
a^\perp_m = w_z - v_z > 0,
\]  

(12)

where \( w_z \) is the kinematic (migration) velocity of the CTS in \( z \)-direction \citep{GreveBlatter2009}. Ice at the melting temperature with no liquid water flows into the temperate layer, where moisture is produced along a trajectory of an ice particle by strain heating, thus

\[
h^+ = h^- = h_m, \quad (W^- = 0) \quad \text{and} \quad \frac{d h^+}{dz} = 0.
\]  

(13)

Together with the imposed surface enthalpy \( h_s \), the vanishing enthalpy gradient defines the thickness of the cold layer and thus the position of the CTS. Therefore, the required enthalpy \( h_m \) at the CTS alone defines the enthalpy profile in the temperate layer. No additional basal boundary condition is required unless a regularising small diffusion is applied to the otherwise pure advection equation in the temperate layer.

For freezing conditions, \( a^\perp_m < 0 \), the enthalpy released by freezing of moisture flows into the cold ice along the enthalpy gradient,

\[
\frac{\kappa}{\rho c} \frac{d h^+}{dz} = (h^- - h_m) a^\perp_m \quad \text{with} \quad \frac{dT^+}{dz} \leq 0.
\]  

(14)

With \( a^\perp_m < 0 \), \( d h^+/dz \) can be strictly negative and \( h^- - h_m \) strictly positive (equivalent to \( W^- > 0 \), discontinuous moisture content at the CTS). The
advection equation for enthalpy in the temperate layer requires one boundary
condition, e.g. the moisture content and thus the basal enthalpy $h_b$ in the
ice entering the slab at the base. Together with the resulting enthalpy $h^-$ at
the CTS, this defines the position of the CTS.

Equation (8) with $\kappa_t = 0$ for the temperate layer is equivalent to the accu-
mulation of moisture produced by strain heating along trajectories. Melting
and freezing conditions correspond to downwards and upwards ice motion,
respectively. Therefore, the moisture content increases downwards from the
CTS for melting conditions and upwards from the bed for freezing conditions.

Equations (7)–(14) constitute the problem of the Canadian-type poly-
thermal parallel sided slab. The steady state equations for enthalpy profiles,
with the assumption of $v_z = \text{const}$, can be solved exactly, with the exception
of the position of the CTS that must be obtained with a numerical root finder
(for details see Greve (1997) or Greve and Blatter (2009)).

4. Numerical Methods

One method to compute polythermal ice masses numerically splits the
computational domain into two distinct layers of cold and temperate ice and
computes the respective temperature and moisture content on two different
grids (Blatter and Hutter, 1991; Greve, 1997; Pettersson et al., 2007). In this
method, the CTS is fixed with the lower and upper boundaries of the cold and
temperate domains, respectively, and thus can be tracked with the transition
condition at the CTS. In this paper, we consider this solution to be reliable
and accurate. However, for the sake of simplicity, it is desirable to produce a
numerical solution of the problem with a one-layer scheme, i.e., on one grid
that spans the entire polythermal domain. This requires tracking the CTS
on a discrete grid. Here, we describe possibilities for one-layer schemes for
both melting and freezing CTS, and test them with respect to the two-layer
reference solutions.

In all numerical schemes we use explicit and implicit schemes with 2nd-
order centred finite differences for the first and second derivatives with respect
to $z$ for the diffusion-advection equation for the cold layer, and upstream first-
order differences for the advection equation in the temperate layer. In the
two-layer tracking scheme, both the cold and temperate layers are resolved
with 100 to 400 grid points, counted by the grid indices $k_c$ and $k_t$, respectively.
For the one-layer schemes, grid resolutions of 0.5, 1 and 2 m (corresponding to
400, 200 and 100 grid points, counted by the grid index $k$; see Section 5) were
compared, although not always shown. In the explicit schemes, time steps of 0.02 to 0.1 years are required for stability reasons. Implicit schemes allow for time steps up to 100 years, although then the accuracy of the transient solutions is affected. As long as small time steps are used, the results of the explicit and implicit schemes are very small and virtually indistinguishable in normal plots.

4.1. Two-layer front-tracking scheme

The surface of the cold layer is at $z = H$, the bottom of the cold and top of the temperate layer (thus the CTS) at $z = M(t)$, and the bottom of the temperate layer at $z = 0$ (Fig. 1). To solve the equations, we map both layers separately to layers of thickness unity,

$$
\zeta_c = \frac{z - M(t)}{H - M(t)}, \quad \zeta_t = \frac{z}{M(t)}, \quad \tau = t,
$$

(15)

where $\zeta_c$ and $\zeta_t$ are the transformed coordinates in the cold and temperate layer, respectively, and $\tau$ is the transformed time. The transformed Eqns (8) are for the cold layer

$$
\frac{\partial h}{\partial \tau} = \frac{w_z (1 - \zeta_c) - v_z}{H - M} \frac{\partial h}{\partial \zeta_c} + \frac{\kappa}{\rho c} \frac{1}{(H - M)^2} \frac{\partial^2 h}{\partial \zeta_c^2} + \frac{2A}{\rho} (\rho g \sin \gamma)^4 (H - M)^4 (1 - \zeta_c)^4,
$$

(16)

where $w_z = dM/dt$ is the kinematic velocity of the CTS, and for the temperate layer

$$
\frac{\partial h}{\partial \tau} = \frac{w_z \zeta_t - v_z}{M} \frac{\partial h}{\partial \zeta_t} + \frac{2A}{\rho} (\rho g \sin \gamma)^4 (H - M \zeta_t)^4.
$$

(17)

For melting conditions at the CTS, the enthalpy $h_s$ at the surface, the enthalpy $h_m$ and the enthalpy gradient on the cold side of the CTS are defined, thus the position of the CTS is determined by the enthalpy profile in the cold layer alone. With the two boundary conditions for the cold layer, given surface enthalpy $h_s$ and given enthalpy $h_m$ at the given CTS, $M_1$, obtained for a given time $t_1$, a time step in the integration of Eqn (16) does not generally result in a vanishing enthalpy gradient at $M_1$. By approximating the enthalpy profile around $M_1$ by a quadratic parabola, the position of the vertex of the parabola is a first approximation $M_i$ of the CTS at time
\( t_2 = t_1 + \Delta t \), where \( \Delta t \) is the time step. Finally, the CTS at time \( t_2 \) can be iterated during this time step to the desired accuracy,

\[
M_{i+1} = M_i - \frac{\text{d}h^+}{\text{d}z} \bigg|_i \frac{\text{d}^2h^+}{\text{d}z^2} \bigg|_i ,
\]

with the first and second derivative of the enthalpy profile at \( M_i \). Furthermore, from the displacement of the CTS, we obtain the kinematic velocity \( w_z \) of the CTS.

For freezing conditions, the transition condition at the CTS (Eqn (14)) in the transformed cold layer yields an equation for the volume flux \( a_m^\perp \) of ice through the CTS,

\[
a_m^{\perp} = \kappa (h - h_m) \rho c \frac{\partial h^+}{\partial \zeta} ,
\]

and the kinematic velocity of the CTS is given by Eqn (12),

\[
w_z = a_m^{\perp} + v_z .
\]

### 4.2. Conventional one-layer scheme

In contrast to the two-layer scheme discussed above, in the conventional one-layer scheme, which corresponds to the enthalpy scheme by Aschwanden et al. (2012), Eqn (8) is solved for the entire polythermal slab on one grid, and the cold and temperate layers are discriminated by the contour \( h = h_m \). Transition conditions at the CTS are not accounted for explicitly. In our one-dimensional implementation, for a melting CTS, where the ice flows downwards, the surface boundary condition (given enthalpy) and the assumed continuity of the enthalpy field at the CTS define the entire profile. Boundary conditions at the base are not necessary because of the advection equation in the temperate layer. Therefore, only one boundary condition at the surface of the cold layer is required to obtain a unique solution.

If required for numerical stability (in our case not), the conductivity \( \kappa_t \) in the temperate layer can be set to a small, non-zero value rather than the value zero used here. The diffusion term then serves as a regularization term. However, this requires an additional basal boundary condition, which then should be chosen carefully in order not to influence the numerical solution significantly.

Although the conductivity is piece-wise constant (\( \kappa_c \) and \( \kappa_t \) in the cold and temperate layers, respectively), the discretization of the diffusion term
in Eqn (8) must take into account the variation of the conductivity at least
at the CTS, i.e., at the uppermost grid point in the temperate part (personal
communication, T. Kleiner, February 2014). Omission of this results in a
faulty enthalpy profile that violates the melting-CTS transition condition
\textit{\textsuperscript{(13)}}, (zero enthalpy gradient at the cold side of the CTS).

For freezing conditions with upward-moving ice, the method must fail
because it is not consistent with the discontinuity of the enthalpy field at
the CTS that results from the discontinuity of the moisture content. Fur-
thermore, the basal boundary condition is defined by the moisture content
of the ice entering the slab and thus is a Dirichlet condition, or, if there is
an additional moisture flux relative to the ice, a Robin condition.

4.3. One-layer melting CTS scheme

Here, we propose a second scheme that enforces explicitly the zero en-
thalpy gradient at the cold side of the CTS and retains the enthalpy corre-
sponding to the accumulated moisture content in the temperate zone. For
this scheme, the discretization of the diffusion term in Eqn (8) need not
necessarily consider a conductivity that depends on the position $z$.

From the previous time step (time $t_1$), the position of the CTS is given
by the grid point $k_{cts}$, which is the uppermost point in the temperate layer.
Additionally, the conductivities $\kappa_c$ and $\kappa_t$ for the cold and temperate parts,
respectively, are defined for each grid point $k$ according to Eqn (9).

Let $h_{1,k}$ be the enthalpy at the given time $t_1$ and the grid point $k$ along
the vertical profile, and $h_{2,k}$ the enthalpy at the new time $t_2$. Each time step
is now divided into two iteration steps. The predictor step for the enthalpy,
Eqn (8), is carried out for the entire slab from the temperate glacier bed to
the cold ice surface, giving a preliminary enthalpy at the new time, $\tilde{h}_{2,k}$.
For this profile, the position of the CTS and the diffusivies for each grid point
are determined.

Using the predictor enthalpy, we determine the conductivities $\kappa_c$ and $\kappa_t$
for the cold and temperate parts according to Eqn (9). The corrector step
affects only the cold layer, from the grid point $k_{cts}$ to the surface. The explicit
forward step for the enthalpy equation is repeated for the cold layer alone,
enforcing a zero enthalpy gradient on the cold side of the CTS by setting
$h_{2,k_{cts}} = h_{2,k_{cts}+1}$. The updated enthalpy profile $h_{2,k}$ is assembled by the
predictor step for the temperate layer and the corrector step for the cold
layer.
4.4. One-layer freezing CTS scheme

Because of the non-vanishing temperature gradient on the cold side of the CTS, a one-layer scheme for a freezing CTS is more difficult to implement. Since the non-vanishing enthalpy gradient on the cold side of the CTS and the non-vanishing moisture content on the temperate side of the CTS define the migration of the CTS (Eqn (14)), this can be used for a sub-grid tracking of the CTS at every time step.

Since the enthalpy is discontinuous at a freezing CTS, the enthalpy profile in the temperate part must and can be computed independently of the cold side. The advection equation only requires one boundary condition, which in this case is a Dirichlet condition with an imposed value of the basal enthalpy. Unless ice with a given moisture content accumulates at the ice bed, we can impose \( h_b = h_m \). The temperate enthalpy profile in the temperate layer is independent of the conditions in the cold layer, thus it is advisable to compute it for the entire (temperate and cold) layer or at least for the range of possible positions of the CTS to have the enthalpy at the CTS available for the transition condition at all time steps. A minor disadvantage of this method is the fact that for cold and temperate parts two different variables, both meaning the enthalpy, must be applied. This method only works if the equation for enthalpy in the temperate part is purely advective. For regularization and stabilization of the integration, a small diffusion can be introduced, but it must be small enough that the additionally required boundary condition does not affect the solution.

For a time step from time \( t_1 \) to time \( t_2 \) we preferably first carry out the step for the enthalpy in the temperate domain, which can be the entire polythermal slab. For the time step in the cold layer we first choose the grid point just below the position \( M \) of the CTS, which is denoted as \( k_{cts} \). The value of the corresponding enthalpy is obtained by carrying out the time step for the cold part and extrapolate the obtained enthalpy to this grid point. Since the value of the enthalpy on this grid point is unknown, the extrapolation may be implemented as a boundary condition calculated with the values of the obtained enthalpy from the next two or three grid points, depending on the order (linear or quadratic) of the chosen extrapolation. With the gradient of the enthalpy at the cold side of the CTS and the interpolated enthalpy at \( M \) on the enthalpy profile in the temperate part, we obtain the volume flux \( a_m^\perp \) (Eqn (14)), the velocity of the CTS, \( w_z \) (Eqn (20)) and thus, the displacement of the CTS and the new CTS at the time \( t_2 \).
5. Set-up of the numerical experiments

For all numerical solutions presented in this work, the thickness of the slab is $H = 200$ m and the inclination angle $\gamma = 4^\circ$ (Fig. 1). Under the additional assumption of steady-state conditions ($\partial(\cdot)/\partial t = 0$ for all fields), the field equations of temperature and moisture content become ordinary linear differential equations with constant coefficients, for which exact solutions are available (Greve, 1997; Greve and Blatter, 2009). For melting conditions at the CTS, we set $v_z = -0.2$ m a$^{-1}$, and for freezing conditions $v_z = 0.2$ m a$^{-1}$.

Steady states for enthalpy profiles, or, equivalently, for temperature and moisture profiles are computed for a melting CTS with surface temperatures of $-1^\circ$C and $-3^\circ$C, and for a freezing CTS with $-6^\circ$C and $-10^\circ$C. Transient experiments from one of the steady states to the other, with a step change in the surface temperature at the starting time of the transition, are computed for both melting and freezing CTS. Furthermore, we perform experiments with sinusoidal variations of the surface temperature with a mean value of $-2^\circ$C and an amplitude of 1 K for a melting CTS, and a mean value of $-8^\circ$C and an amplitude of 2 K for a freezing CTS. Two different periods of 100 and 500 years are employed for both cases.

6. Results

6.1. Two-layer front-tracking scheme

The steady-state results of the two-layer CTS tracking scheme can be verified with the exact solutions (Greve, 1997; Greve and Blatter, 2009), and the transient results serve as references to test the performance of the other schemes. Figure 2 shows the steady states for both a melting and freezing CTS and prescribed constant surface enthalpies $h_s = cT_s$ corresponding to the surface temperatures listed in Section 5. These steady states correspond to high accuracy to the exact steady states (not shown explicitly). The positions of the CTS coincide within about 0.3 m, better than grid resolution, slightly depending on the chosen grid size.

We also calculated various transient situations, both from one steady state to another with a step change in the surface boundary conditions and periodic variations with the surface conditions oscillating sinusoidally (see Section 5). The results of the transitions are shown in the corresponding
Figure 2: Steady-state profiles of temperature and moisture content in the cold and temperate layers, respectively, of the parallel-sided slab with a melting CTS, computed with the two-layer front-tracking scheme with a time step of 0.1 years and 100 grid points in each layer, (a) surface temperature $T_s = -1^\circ$C, (b) $T_s = -3^\circ$C. Same for a freezing CTS with (c) surface temperature $T_s = -6^\circ$C, (d) $T_s = -10^\circ$C.

figures of the one-layer schemes for comparison. All runs with the two-layer scheme were performed with both the explicit and the implicit version. Steady-state solutions are almost independent on the time step. For time steps short enough compared with the rate of changes in the conditions, the results of both types of schemes coincide within grid resolution. Long time steps act like a low-pass filter, so that the amplitudes of the oscillating solutions are reduced with a slight phase shift.

6.2. Conventional one-layer scheme

Figure 3 shows the steady-state solution for a melting CTS and a surface enthalpy corresponding to $T_s = -3^\circ$C computed with the conventional one-layer scheme, which corresponds to the enthalpy scheme by Aschwanden et al. (2012). The solution is almost indistinguishable from the corresponding exact solution and the one computed with the two-layer front-tracking scheme (Fig. 2b). However, this result is only obtained if the jump of the conductivity at the CTS is properly accounted for in the discretization of the diffusion term in the enthalpy equation (Eqn (8)). Otherwise (discontinuity of the
conductivity at the CTS disregarded), a faulty enthalpy profile results that
does not meet the required transition condition (upper curve in Fig. 3).

Figure 3: Lower line: Steady-state profiles of temperature and moisture content corre-
sponding to the steady-state solution shown in Fig. 2b for a melting CTS, computed with
the conventional one-layer scheme with a time step of 0.1 years and a grid resolution of
1 m. Upper line: Same, but the jump of the conductivity at the CTS was disregarded in
the discretization of the diffusion term in Eqn (8).

6.3. One-layer melting CTS scheme

Figure 4 shows a comparison of the one-layer melting CTS scheme with
the two-layer front-tracking scheme that is considered to provide reference
solutions. It shows the position of the CTS during a transition from one
steady state to the other, with a step change of the surface enthalpy at
the starting time \( t = 0 \) of the transition. The steady states correspond to
surface enthalpies of \( h_s = cT_s \), with surface temperatures of \( T_s = -1^\circ C \) and
\( T_s = -3^\circ C \) (see Section 5). The transitions between the two steady states
show some asymmetric behaviour depending on whether the CTS moves in
the direction of the cold or temperate layer.

The differences between the one-layer melting CTS scheme and the ref-
ence solutions from the two-layer scheme in the shown examples, although
relatively small, must partly be attributed to the relatively rapid changes
forced by the abrupt changes in the surface conditions. The possible accu-
metry depends on the grid resolution in the profile. This is especially true
Figure 4: Comparison between the one-layer melting CTS scheme (solid lines) and the two-layer front-tracking scheme (dashed lines) with a time step of 0.1 years and a grid resolution of 1 m. (a) Evolution of the position of the CTS and (b) evolution of the basal moisture content for the transition from the steady state with surface temperature $T_s = -3^\circ C$ to the steady state with $T_s = -1^\circ C$ (rising curves) and vice versa (falling curves) after a step change of the corresponding surface enthalpies at time $t = 0$. 
Figure 5: Comparison between the one-layer melting CTS scheme (solid lines) and the two-layer front-tracking scheme (dashed lines). Evolution of the position of the CTS for a sinusoidal oscillation of the surface temperature centred at $T_s = -2^\circ C$ with an amplitude of 1 K, (a) with a period of 100 years, computed with a time step of 0.02 years and a grid resolution of 0.5 m, (b) for a period of 500 years, computed with a time step of 0.1 years and grid resolutions of 2 m (upper solid line) and 0.5 m (lower solid line).
for the sinusoidal variations with a short period (100 years), where only grid resolutions of 0.5 m and better give reasonable results (Fig. 5). For substantially longer periods (500 years), the results are less susceptible to the grid resolution, and for a grid resolution of 0.5 m, the one-layer melting CTS and two-layer front-tracking solutions match closely (within grid resolution).

Figure 6: Comparison between the one-layer freezing CTS scheme (solid lines) and the two-layer front-tracking scheme (dashed lines). Evolution of the position of the CTS from the steady state with surface temperature $T_s = -10^\circ$C to the steady state with $T_s = -6^\circ$C (rising curves) and vice versa (falling curves) after a step change of the corresponding surface enthalpies at time $t = 0$. The upper solid lines are computed with a grid resolution $\Delta z = 1$ m and the lower solid lines with $\Delta z = 2$ m, both with a time step of 0.1 years. The solutions computed with $\Delta z = 0.5$ m are not shown because they are almost identical to the reference solutions (dashed lines; two-layer front-tracking scheme).

6.4. One-layer freezing CTS scheme

Figure 6 shows the evolution of the height of the CTS above the bed between two steady states for a freezing CTS with surface enthalpies $h_s = cT_s$, corresponding to surface temperatures of $T_s = -6^\circ$C and $T_s = -10^\circ$C (see Section 5), and the position of the CTS during the transition between the steady states with a step change of the surface condition at time $t = 0$. The agreement between the one-layer freezing CTS scheme and the two-layer front-tracking scheme lies within about the grid resolution of the one-layer scheme. The freezing CTS requires a longer time for adjustment compared to the melting CTS (compare with Fig. 4). One reason seems to be the upward
Figure 7: Comparison between the one-layer freezing CTS scheme (solid lines) and the two-layer front-tracking scheme (dashed lines). (a) Evolution of the position of the CTS for a sinusoidal oscillation of the surface temperature centred at $T_s = -8^\circ$C with an amplitude of 2 K and a period of 100 years. The three different solid lines correspond to grid resolutions of 0.5, 1 and 2 m. (b) Same as (a), but with a period of 500 years. Due to the long time scale, after 1000 years, the system is still in a transient state influenced by the initial conditions.
motion of the ice, which reduces the response to surface perturbations at a given depth.

Figure 7 shows the evolutions of the positions of the freezing CTS for periodic surface conditions with periods of 100 and 500 years, a mean temperature of $-8^\circ C$ and an amplitude of 2 K (see Section 5), computed with the one-layer freezing CTS and two-layer schemes. The agreement between the different methods is within about the grid resolution of the one-layer scheme. The amplitude of the variations of the CTS is substantially smaller than that of the melting CTS although the amplitude of the surface perturbation is larger, again because of the upward direction of the ice motion.

7. Discussion and conclusion

The conventional one-layer scheme, which corresponds to the implementation of the enthalpy method by Aschwanden et al. (2012), does not explicitly take into account the Stefan-type energy- and mass-flux matching conditions at the CTS that are crucial for determining the position of the CTS. Nevertheless, we have demonstrated that this scheme can determine the CTS position for the case of melting conditions correctly. This depends, however, critically on details of the numerical implementation and is therefore prone to failure if the implementation is not done with great care. For the case of freezing conditions with its associated jump of the moisture content at the CTS, the conventional one-layer scheme fails inevitably.

Two-layer front-tracking schemes, using a time-dependent terrain-following coordinate transformation for the cold and temperate layers separately (Blatter and Hutter, 1991; Greve, 1997; Pettersson et al., 2007), can also be used in conjunction with the enthalpy method. We have constructed such a scheme, and have shown that it produces very good results for both melting and freezing conditions. However, schemes using only one grid for the entire polythermal slab are simpler to implement in existing ice sheet models and therefore more desirable.

Therefore, we have proposed one-layer methods that modify the original enthalpy scheme by Aschwanden et al. (2012) in order to treat explicitly the transition conditions at the CTS for both cases of a melting and freezing CTS. The proposed methods work well in our one-dimensional model, provided that the time steps and grid resolutions are sufficiently small. We expect them to work as well in shallow ice sheet models, where the thermodynamics neglects horizontal diffusive heat fluxes and thus treats vertical enthalpy or
temperature profiles essentially in a one-dimensional way. Horizontal advective heat fluxes can be treated as source terms of the vertical profiles. In fact, we have already implemented the one-layer melting CTS scheme in the ice sheet model SICOPOLIS (e.g., Sato and Greve, 2012; Greve and Herzfeld, 2013, URL http://www.sicopolis.net/), which, with the appropriate adjustments for the additional physics (pressure dependence of the melting point as well as temperature-dependent heat conductivity and specific heat capacity accounted for), could be done in a fairly straightforward way (paper in preparation). With some additional effort due to the complicating horizontal diffusive heat fluxes, implementations in non-shallow (higher-order or full Stokes) ice sheet and glacier models should also be feasible.

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References

Aschwanden, A., Blatter, H., 2005. Meltwater production due to strain heating in Storglaciären, Sweden. J. Geophys. Res. 110 (F4), F04024.

Aschwanden, A., Bueler, E., Khroulev, C., Blatter, H., 2012. An enthalpy formulation for glaciers and ice sheets. J. Glaciol 58 (209), 441–457.

Blatter, H., Hutter, K., 1991. Polythermal conditions in Artic glaciers. J. Glaciol. 37 (126), 261–269.

Dash, J. G., Rempel, A. W., Wettlaufer, J. S., 2006. The physics of premelted ice and its geophysical consequences. Rev. Mod. Phys. 78, 261–269.

Duval, P., 1977. The role of water content on the creep of polycrystalline ice. In: Isotopes and Impurities in Snow and Ice – Proceedings of the Grenoble Symposium, August-September 1975. IAHS Publication No. 118. International Association of Hydrological Sciences, Wallingford, UK, pp. 29–33.
Fowler, A. C., 1984. On the transport of moisture in polythermal glaciers. Geophys. Astrophys. Fluid Dyn. 28 (2), 99–140.

Fowler, A. C., Larson, D. A., 1978. Flow of polythermal glaciers: 1. Model and preliminary analysis. Proc. R. Soc. Lond. A 363 (1713), 217–242.

Greve, R., May 1997. A continuum-mechanical formulation for shallow polythermal ice sheets. Phil. Trans. R. Soc. Lond. A 355 (1726), 921–974.

Greve, R., Blatter, H., 2009. Dynamics of Ice Sheets and Glaciers. Springer, Berlin, Germany, etc.

Greve, R., Herzfeld, U. C., 2013. Resolution of ice streams and outlet glaciers in large-scale simulations of the Greenland ice sheet. Ann. Glaciol. 54 (63), 209–220.

Hutter, K., 1982. A mathematical model of polythermal glaciers and ice sheets. Geophys. Astrophys. Fluid Dyn. 21 (3-4), 201–224.

Hutter, K., 1993. Thermo-mechanically coupled ice-sheet response – cold, polythermal, temperate. J. Glaciol. 39 (131), 65–86.

Pettersson, R., Jansson, P., Huwald, H., Blatter, H., 2007. Spatial pattern and stability of the cold surface layer of Storglaciären, Sweden. J. Glaciol. 53 (180), 99–109.

Sato, T., Greve, R., 2012. Sensitivity experiments for the Antarctic ice sheet with varied sub-ice-shelf melting rates. Ann. Glaciol. 53 (60), 221–228.