ON THE EXISTENCE OF ORDINARY AND ALMOST ORDINARY PRYM VARIETIES

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Abstract. We study the relationship between the $p$-rank of a curve and the $p$-ranks of the Prym varieties of its cyclic covers in characteristic $p > 0$. For arbitrary $p$, $g \geq 3$ and $0 \leq f \leq g$, we generalize a result of Nakajima by proving that the Prym varieties of all unramified cyclic degree $\ell \neq p$ covers of a generic curve $X$ of genus $g$ and $p$-rank $f$ are ordinary. Furthermore, when $p \geq 5$, we prove that there exists a curve of genus $g$ and $p$-rank $f$ having an unramified degree $\ell = 2$ cover whose Prym is almost ordinary. Using work of Raynaud, we use these two theorems to prove results about the (non)-intersection of the $\ell$-torsion group scheme with the theta divisor of the Jacobian of a generic curve $X$ of genus $g$ and $p$-rank $f$. The proofs involve geometric results about the $p$-rank stratification of the moduli space of Prym varieties.

Keywords: Prym, curve, Jacobian, $p$-rank, theta divisor, torsion point, moduli space.

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1. Introduction

Suppose $X$ is a smooth projective connected curve of genus $g \geq 2$ defined over an algebraically closed field $k$ of characteristic $p > 0$. Suppose $\pi : Y \to X$ is an unramified cyclic cover of degree $\ell$ for some prime $\ell \neq p$. Then $Y$ has genus $g_Y = \ell(g-1) + 1$ by the Riemann-Hurwitz formula. For each of the $\ell^2g-1$ unramified $\mathbb{Z}/\ell$-covers $\pi : Y \to X$, the Jacobian $J_Y$ is isogenous to $J_X \oplus P_\pi$ for an abelian variety $P_\pi$ of dimension $(\ell-1)(g-1)$, called the Prym variety of $\pi$. In particular, when $\ell = 2$ and $\pi : Y \to X$ is an unramified double cover, then $Y$ has genus $2g-1$ and $P_\pi$ is a principally polarized abelian variety of dimension $g-1$.

In this paper, we study the relationship between the $p$-ranks of $J_X$ and $P_\pi$. The $p$-rank $f_A$ of an abelian variety $A/k$ of dimension $g_A$ is the integer $0 \leq f_A \leq g_A$ such that the number of $p$-torsion points in $A(k)$ is $p^{f_A}$. One says that $A$ is ordinary if its $p$-rank is as large as possible ($f_A = g_A$) and is almost ordinary if its $p$-rank equals $g_A - 1$.

By a result of Nakajima, the Prym varieties of the unramified $\mathbb{Z}/\ell$-covers of the generic curve $X/k$ of genus $g \geq 2$ are ordinary [15, Theorem 2]. As the first main result of the paper (Theorem 1.3, Theorem 4.5), we generalize Nakajima’s result by adding a condition on the $p$-rank of $X$, which yields the following application.

Application 1.1. Let $\ell$ be a prime distinct from $p$. Let $g \geq 2$ and $0 \leq f \leq g$ with $f \neq 0$ if $g = 2$. Let $X$ be a generic curve of genus $g$ and $p$-rank $f$. Then the Prym varieties of all of the unramified $\mathbb{Z}/\ell$-covers of $X$ are ordinary.

The second result (Theorem 1.4, Theorem 7.1) demonstrates the existence of degree 2 unramified covers $\pi : Y \to X$ such that $Y$ is not ordinary. As an application, we prove:

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Application 1.2. Let $\ell = 2$. Let $p \geq 5$, $g \geq 2$ and $0 \leq f \leq g$. Then there exists a curve of genus $g$ and $p$-rank $f$ having an unramified double cover whose Prym is almost ordinary ($f_{P_{\pi}} = g - 2$).

Raynaud considered similar questions using theta divisors \cite{16, 17, 18}. In Section 2.6 we define the theta divisor $\Theta_X$ of the Jacobian and compare our results with those of Raynaud. Briefly, Raynaud’s results are stronger in that they apply to an arbitrary base curve $X$ but are weaker in other ways: his result about ordinary Pryms applies only when $\ell > (p - 1)g$, in which case he shows that at least one of the Pryms is ordinary; and, in his result for non-ordinary covers, the Galois group of the cover is solvable but not cyclic. We use Raynaud’s work to phrase our results in terms of the (non)-existence of points of order $\ell$ contained in the theta divisor $\Theta_X$.

We now state the results of the paper more precisely. Consider the moduli space $M_g$ whose points represent smooth curves of genus $g$ and the moduli space $R_{g, \ell}$ whose points represent unramified $\mathbb{Z}/\ell$-covers $\pi : Y \to X$, where $X$ is a smooth curve of genus $g$. There is a finite morphism of degree $\ell^{2g} - 1$, denoted

$$\Pi_\ell : R_{g, \ell} \to M_g,$$

which takes the point representing a cover $\pi : Y \to X$ to the point representing the curve $X$ \cite[Page 6]{7}. For $0 \leq f \leq g$, let $M^f_g$ denote the $p$-rank $f$ stratum of the moduli space $M_g$ of smooth curves of genus $g$. For most $g$ and $f$, it is not known whether $M^f_g$ is irreducible, however, every component of $M^f_g$ has dimension $2g - 3 + f$ \cite[Theorem 2.3]{9].

Theorem 1.3. Let $\ell \neq p$ be prime. Let $g \geq 2$ and $0 \leq f \leq g$ with $f \neq 0$ if $g = 2$. Let $S$ be an irreducible component of $M^f_g$.

1. Then $\Pi^{-1}_\ell(S)$ is irreducible (of dimension $2g - 3 + f$) and the Prym of the generic point of $\Pi^{-1}_\ell(S)$ is ordinary.
2. If $X$ is the curve represented by the generic point of $S$, then the theta divisor $\Theta_X$ of the Jacobian of $X$ does not contain any point of order $\ell$.

Theorem 1.4. Let $\ell = 2$. Let $p \geq 5$, $g \geq 2$ and $0 \leq f \leq g$. Let $S$ be an irreducible component of $M^f_g$.

1. Then the locus of points of $\Pi^{-1}_2(S)$ for which the Prym $P_\pi$ is almost ordinary is non-empty with dimension $2g - 4 + f$ (codimension 1 in $\Pi^{-1}_2(S)$).
2. The locus of points of $S$ representing curves $X$ for which $\Theta_X$ contains a point of order 2 is non-empty with codimension 1 in $S$.

Here is an outline of the paper. Section 2 contains background about Prym varieties, moduli spaces of Prym varieties, the $p$-rank, and Raynaud’s work on theta divisors. Proposition 2.4 in Section 2.6 demonstrates that items (1) and (2) in Theorem 1.3 are equivalent and that item (1) implies item (2) in Theorem 1.4. In Section 3 we analyze the $p$-rank stratification of the boundary of $\overline{R}_{g, \ell}$. This uses a description of the components of the boundary of $\overline{R}_{g, \ell}$ from \[5\].

Section 4 contains the proof of Theorem 1.3. Like Nakajima, we use the technique of degeneration to the boundary of $\overline{R}_{g, \ell}$. The argument is more complicated, however, because $M^f_g$ may not irreducible. A key point is that the $\mathbb{Z}/\ell$-monodromy of the tautological curve $X \to S$ is as large as possible, namely $\text{Sp}_{2g}(\mathbb{Z}/\ell)$ \cite[Theorem 4.5]{11}. This monodromy group is the image of the fundamental group $\pi_1(S, s)$ in $\text{Aut}(\text{Pic}^0(C)[\ell]_s)$. We use this to prove that
\( \Pi_\ell^{-1}(S) \) is irreducible and that it degenerates to a particular boundary component \( \Delta_{i;g-i} \) which is convenient for proving the result inductively.

In the second part of the paper, we restrict to the case \( \ell = 2 \). In Section 5, we stratify \( \mathcal{R}_{g,2} \) by the pair \((f, f')\) where \( f \) is the \( p \)-rank of \( X \) and \( f' \) is the \( p \)-rank of \( P_\pi \). Using purity, we prove that the dimension of the \((f, f')\) stratum of \( \mathcal{R}_{g,2} \) is bounded below by \( g - 2 + f + f' \) Proposition 5.4.

Section 6 contains some results about non-ordinary Pryms of unramified double covers of curves of low genus \( g = 2 \) and \( g = 3 \). These generalize work from [9]. We illustrate the difficulty in proving these results computationally, even for one fixed small prime \( p \), in Section 6.2.

Section 7 contains the proof of Theorem 1.4. The statement about the dimension follows from the purity lower bound and Theorem 1.3, so most of the proof is focused on proving the non-emptiness statement. This is also an inductive argument which uses the boundary component \( \Delta_{i;j} \), but it relies on more refined information from Sections 5 and 6. Finally, the paper ends with some open questions in Section 7.2.

2. Pryms, moduli, \( p \)-rank, and theta divisors

Suppose \( X \) is a smooth curve of genus \( g \) defined over a field \( K \) of characteristic \( p > 0 \). The Jacobian \( J_X \) of \( X \) is a principally polarized abelian variety of dimension \( g \). For a prime \( \ell \neq p \), recall that there is a bijection between points of order \( \ell \) on \( J_X \) and unramified connected \( \mathbb{Z}/\ell \)-covers \( \pi : Y \to X \).

2.1. Prym varieties. Suppose \( \pi : Y \to X \) is an unramified \( \mathbb{Z}/\ell \)-cover. The Prym variety \( P_\pi \) is the connected component containing 0 of the norm map on Jacobians. More precisely, if \( \sigma \) is the endomorphism of \( J_X \) induced by a generator of \( \text{Gal}(Y/X) \), then

\[
P_\pi = \text{Im}(1 - \sigma) = \ker(1 + \sigma + \ldots + \sigma^{\ell-1})^0.
\]

The canonical principal polarization of \( \text{Jac}(Y) \) induces a polarization on \( P_\pi \) [14, Page 6]. This polarization is principal when \( \ell = 2 \).

2.2. Moduli spaces of Prym varieties. Let \( \mathcal{R}_{g,\ell} \) denote the moduli space of Prym varieties of unramified \( \mathbb{Z}/\ell \)-covers of smooth projective curves of genus \( g \); it is a smooth Deligne-Mumford stack [6, Page 5].

The points of \( \mathcal{R}_{g,\ell} \) can also represent triples \((X, \eta, \phi)\) where \( X \) is a smooth genus \( g \) curve equipped with a line bundle \( \eta \in \text{Pic}(X) \) and an isomorphism \( \phi : \eta^{\otimes \ell} \cong \mathcal{O}_X \). This is because the data of the \( \mathbb{Z}/\ell \)-cover \( \pi : Y \to X \) is equivalent to the data \((X, \eta, \phi)\).

Recall that the morphism \( \Pi_\ell : \mathcal{R}_{g,\ell} \to \mathcal{M}_g \) sends the point representing \( \pi : Y \to X \) to the point representing \( X \). Then \( \Pi_\ell \) is surjective, étale, and finite of degree \( \ell^{2g} - 1 \). Thus \( \dim(\mathcal{R}_{g,\ell}) = 3g - 3 \).

For \( \ell = 2 \), consider the Prym map \( Pr_g : \mathcal{R}_{g,2} \to \mathcal{A}_{g-1} \) which sends the point representing \( \pi : Y \to X \) to the point representing the principally polarized abelian variety \( P_\pi \). The image and fibers of \( Pr_g \) are well understood for \( 2 \leq g \leq 6 \), but not for \( g \geq 7 \).

2.3. Marked curves. A marking of a curve \( X \) consists of the choice of a point \( x \in X \). A marking of an unramified \( \mathbb{Z}/\ell \)-cover \( \pi : Y \to X \) consists of a marking \( x \in X \), together with the choice of a point \( x' \in \pi^{-1}(x) \). This marking will be denoted \( x' \mapsto x \). It is equivalent to the choice of a labeling of the \( \ell \) points of the fiber \( \pi^{-1}(x) \) because of the \( \mathbb{Z}/\ell \)-action.
A point of $\mathcal{M}_{g,1}$ represents a curve $X$ of genus $g$ together with a marking $x \in X$. A point of $\mathcal{R}_{g,\ell,1}$ represents an unramified $\mathbb{Z}/\ell$-cover $\pi : Y \to X$ such that $X$ has genus $g$, together with a marking $x' \mapsto x$. There are forgetful maps $\psi_{\mathcal{M}} : \mathcal{M}_{g,1} \to \mathcal{M}_g$ and $\psi_{\mathcal{R}} : \mathcal{R}_{g,1} \to \mathcal{R}_g$.

**Lemma 2.1.** If $S \subset \mathcal{M}_g$ is irreducible, then $\psi_{\mathcal{M}}^{-1}(S)$ is irreducible in $\mathcal{M}_{g,1}$. If $T \subset \mathcal{R}_g$ is irreducible, then $\psi_{\mathcal{R}}^{-1}(T)$ is irreducible in $\mathcal{R}_{g,1}$.

**Proof.** The fiber of $\psi_{\mathcal{M}}$ above a point of $\mathcal{M}_g$ is isomorphic to the curve it represents and is thus irreducible. Similarly, the fiber of $\psi_{\mathcal{R}}$ above the point of $\mathcal{R}_g$ representing a cover $\pi : Y \to X$ is isomorphic to $Y$ and is thus irreducible. The result then follows from Zariski’s main theorem. $\square$

2.4. The $p$-rank. The $p$-rank $f_A$ of an abelian variety $A$ of dimension $g_A$ is the integer $0 \leq f_A \leq g_A$ such that the number of $p$-torsion points in $A(\overline{K})$ is $p^{f_A}$. The $p$-rank of a curve is that of its Jacobian. There are three $p$-ranks associated with the cover $\pi : Y \to X$, namely the $p$-rank $f$ of $X$, the $p$-rank $f'$ of $P_\pi$, and the $p$-rank of $Y$ which equals $f + f'$. The $p$-rank is invariant under isogeny $\sim$ of abelian varieties.

The $p$-rank of $P_\pi$ can also be computed using theory from [18 Page 343]. Then $\pi_*(\mathcal{O}_Y) = \oplus_{0 \leq i \leq \ell-1} L^i$ where $L$ is the invertible sheaf of order $\ell$ associated with $\eta$. Since $\ell$ is prime, the new part of $\pi_*(\mathcal{O}_Y)$ is $\mathcal{L} := \oplus_{1 \leq i \leq \ell-1} L^i$. The Frobenius $F$ acts semi-linearly on $\pi_*(\mathcal{O}_Y)$ and on $\mathcal{L}$ taking $L^i$ to $L^{pi}$. Thus $F$ acts on $H^1(Y, \mathcal{O}_Y)$ and on $H^1(X, \mathcal{L})$.

Let $H^1_{et}(Y, \mathbb{Z}/p)$ be the kernel of $\text{Id} - F$ on $H^1(Y, \mathcal{O}_Y)$. Let $H^1_{et, new}(Y, \mathbb{Z}/p)$ be the kernel of $\text{Id} - F$ on $H^1(X, \mathcal{L})$. Note that the dimension of $H^1_{et, new}(Y, \mathbb{Z}/p)$ over $\mathbb{F}_p$ is at most $(\ell - 1)(g - 1)$ [18 page 343].

As in [18 Definition 2.1.1], the new part of $\pi : Y \to X$ is *ordinary* if the action of $F$ on $H^1(X, \mathcal{L})$ is semi-simple. This is equivalent to the condition that the dimension of $H^1_{et, new}(Y, \mathbb{Z}/p)$ over $\mathbb{F}_p$ equals $(\ell - 1)(g - 1)$. More generally, the $p$-rank of $P_\pi$ is the stable rank of $F$ on $H^1(X, \mathcal{L})$.

2.5. The $p$-rank stratification. If $X/S$ is a semiabelian scheme over a Deligne-Mumford stack, then there is a stratification $S = \cup S^f$ by locally closed substacks such that $s \in S^f(k)$ if and only if $f(X_s) = f$ (this follows from [13 Theorem 2.3.1], see, e.g., [11 Lemma 2.1]). For example, $\mathcal{M}^f_g$ is the locus in $\mathcal{M}_g$ whose points represent curves of genus $g$ having $p$-rank $f$.

**Definition 2.2.** For $0 \leq f \leq g$, define $W^f_g = \Pi_\ell^{-1}(\mathcal{M}^f_g)$. Hence, the points of $W^f_g$ represent unramified $\mathbb{Z}/\ell$-covers $\pi : Y \to X$ such that $X$ has genus $g$ and $p$-rank $f$.

**Lemma 2.3.** If $0 \leq f \leq g$, then $W^f_g$ is non-empty and every component of $W^f_g$ has dimension $2g - 3 + f$.

**Proof.** The lemma follows from the facts that $\Pi_\ell : \mathcal{R}_{g,\ell} \to \mathcal{M}_g$ is finite and surjective and $\mathcal{M}^f_g$ is pure of dimension $2g - 3 + f$ [9 Theorem 2.3]. $\square$

2.6. Theta divisors. We recall the definition of the theta divisor from [18 Section 1.1]. Let $X^1$ be the curve induced from $X/S$ by base change by the absolute Frobenius of $S$. Let $F : X \to X^1$ be the relative Frobenius morphism. There is an exact sequence of sheaves:

$$0 \to \mathcal{O}_{X^1} \to F_*(\mathcal{O}_X) \to B \to 0,$$
where $B$ is the sheaf of locally exact differential forms on $X^1$. In other words, if $C$ denotes the Cartier operator, then there is an exact sequence

$$0 \rightarrow B \rightarrow F_\ast(\Omega^1_X) \rightarrow \Omega^1_{X^1} \rightarrow 0.$$ 

Then $B$ is a vector bundle of rank $p - 1$ and slope $g - 1$, where the slope is the quotient of the degree by the rank. More precisely, if $X$ is not smooth, then $B$ is a torsion-free sheaf, which is locally free of rank $p - 1$ outside the singularities of $X$.

By [16, Theorem 4.1.1], $B$ admits a theta divisor $\Theta_X$. This is a positive divisor on the Jacobian $J^1$ of $X^1$ (the determinant of the universal cohomology). If $L$ is an invertible sheaf of degree 0 over $X^1$, then $h^0(B \otimes L)$ is nonzero if and only if $L$ defines a point on $J^1$ which belongs to the support of $\Theta_X$.

This construction is compatible with the classical construction of the theta divisor. Consider $J^{[g-1]}$, the torsor for $J$ representing line bundles of degree $g - 1$. After choosing a base point of $X$, there is a canonical map

$$\text{Sym}^{g-1}(X) \rightarrow J^{[g-1]}, \quad (x_1, x_2, \ldots, x_{g-1}) \mapsto \mathcal{O}_X(x_1 + \cdots + x_{g-1}).$$

The image $\Delta$ is a translation of the classical theta divisor $\Theta_{\text{class}}$ on $J$.

### 2.7. Theta divisors and ordinarity.

By work of Raynaud, the theta divisor gives information about the $p$-rank of unramified covers of the curve $X$. By [17, Proposition 1], $X$ is ordinary if and only if the theta divisor of $B$ does not contain the origin of $J^1$. To generalize this, for an abelian variety $A/k$ and a point $x \in A[\ell]$, define $\text{Sat}(x) = \{ix \mid \gcd(i, \ell) = 1\}$ to be the orbit of $x$ under $(\mathbb{Z}/\ell)^\ast$.

**Proposition 2.4.** [18, Proposition 2.1.4] Let $L$ be an invertible sheaf of order $\ell$ with $p \nmid \ell$ and let $\pi : Y \rightarrow X$ be the corresponding unramified $\mathbb{Z}/\ell$-cover. Then the new part of $\pi : Y \rightarrow X$ is ordinary if and only if $\text{Sat}(L)$ does not intersect the theta divisor $\Theta_X$.

The Prym is the new part of $\pi$. Thus Proposition 2.4 shows that items (1) and (2) in Theorem 1.3 are equivalent. and that item (1) implies item (2) in Theorem 1.4.

Using theta divisors, Raynaud proves:

**Theorem 2.5.** Let $X$ be a smooth projective $k$-curve of genus $g \geq 2$.

1. [16, Theorem 4.3.1] Under the condition $\ell \geq (p - 1)g - 1$, there is an unramified $\mathbb{Z}/\ell$-cover $\pi : Y \rightarrow X$ such that $P_\pi$ is ordinary.
2. [17, Theorem 2] There is an unramified Galois cover $Z \rightarrow X$, with solvable Galois group, such that $Z$ is not ordinary.

The results in this paper strengthen Raynaud’s results for a generic curve $X$ of genus $g$ and $p$-rank $f$. Specifically, Theorem 1.3 shows that the condition on $\ell$ in Theorem 2.5(1) can be removed, and that all (not just one) of the Pryms of the $\mathbb{Z}/\ell$-covers of $X$ are ordinary, for a generic curve $X$ of genus $g$ and $p$-rank $f$. Theorem 1.4 shows the existence of unramified degree two covers $\pi : Y \rightarrow X$ such that $Y$ is not ordinary under a codimension 1 condition on a generic curve $X$ of genus $g$ and $p$-rank $f$.

**Remark 2.6.** If $C$ is a curve of genus $g$ and $d = (g - 1)/2$, then the Jacobian $J_C$ contains the top difference variety $V_d = C_d - C_d$ which consists of divisors of the form $\sum_{i=1}^{d} P_i - \sum_{i=1}^{d} Q_i$. One can check that $\dim(V_d) = g - 1$. In [5, Corollary 0.4], for a generic curve of odd genus, the authors prove that $V_d$ contains no points of order $\ell$. 


3. The \( p \)-rank stratification of the boundary of the Prym variety

In this section, we study the Pryms of singular curves and then analyze the \( p \)-rank stratification of components of the boundary of \( \mathcal{R}_{g,\ell} \). The goal is to prove that every component of the boundary of \( \Pi_{\ell}^{-1}(\mathcal{M}_{g}) \) has dimension \( 2g - 4 + f \), Proposition \[3.14\]. The strategy is to study the clutching maps, whose images are the boundary strata of \( \mathcal{R}_{g,\ell} \), in terms of the \( p \)-rank. The following lemma will be useful.

**Lemma 3.1.** [20, p. 614]. A smooth proper stack has the same intersection-theoretic properties as a smooth proper scheme.

3.1. Compactification of \( \mathcal{M}_g \). Suppose \( X \) is a stable curve with irreducible components \( C_i \), for \( 1 \leq i \leq s \). Let \( \tilde{C}_i \) denote the normalization of \( C_i \). By [3] Example 8, Page 246, \( J_X \) is canonically an extension by a torus \( T \) and there is a short exact sequence:

\[
1 \to T \to J_X \to \bigoplus_{i=1}^{s} J_{\tilde{C}_i} \to 1.
\]

The rank \( r_T \) of \( T \) is the rank of the cohomology group \( H^1(\Gamma_X, \mathcal{Z}) \), where \( \Gamma_X \) denotes the dual graph of \( X \). One says that \( X \) has compact type if \( T \) is trivial.

Let \( \mathcal{M}_g \) be the Deligne-Mumford compactification of \( \mathcal{M}_g \). If \( S \) is a substack of \( \mathcal{M}_g \), let \( \bar{S} \) denote its closure in \( \mathcal{M}_g \). The boundary \( \delta \mathcal{M}_g = \mathcal{M}_g - \mathcal{M}_g \) is the union of the components \( \Delta_i[\mathcal{M}_g] \) for \( 1 \leq i \leq [g/2] \).

3.1.1. Compact type. For \( 1 \leq i \leq [g/2] \), the boundary component \( \Delta_i[\mathcal{M}_g] \) is the image of the clutching morphism:

\[
\kappa_i : \mathcal{M}_{i;1} \times \mathcal{M}_{g-i;1} \to \mathcal{M}_g,
\]

defined as follows. For \( i = 1, 2 \), let \( (C_i, x_i) \) be a marked curve with \( C_1 \) of genus \( i \) and \( C_2 \) of genus \( g - i \). Consider the genus \( g \) curve \( X \) which has components \( C_1 \) and \( C_2 \) and is formed by identifying the points \( x_1 \in C_1 \) and \( x_2 \in C_2 \) in an ordinary double point. If \( \xi_1 \) is the point of \( \mathcal{M}_{i;1} \) representing \( (C_1, x_1) \) and \( \xi_2 \) is the point of \( \mathcal{M}_{g-i;1} \) representing \( (C_2, x_2) \), then \( \kappa_i(\xi_1, \xi_2) \) is the point representing \( X \).

In this situation,

\[
J_X \cong J_{C_1} \oplus J_{C_2}.
\]

If \( f_i \) is the \( p \)-rank of \( C_i \) for \( i = 1, 2 \), then \( X \) has \( p \)-rank \( f_1 + f_2 \). Thus \( \kappa_i \) restricts to a map

\[
\kappa_i : \mathcal{M}^{f_1}_{i;1} \times \mathcal{M}^{f_2}_{g-i;1} \to \Delta_i[\mathcal{M}_g^{f_1+f_2}].
\]

3.1.2. Non-compact type. The boundary component \( \Delta_0[\mathcal{M}_g] \) is the image of the clutching morphism:

\[
\kappa_0 : \mathcal{M}_{g-1;2} \to \mathcal{M}_g,
\]

defined as follows. Let \( (C', x, y) \) be a curve of genus \( g - 1 \) with 2 marked points. Consider the curve \( C \) of genus \( g \) which is formed by identifying \( x \) and \( y \) in an ordinary double point. If \( \xi' \) is the point of \( \mathcal{M}_{g-1;2} \) representing \( (C', x, y) \), then \( \kappa_0(\xi') \) is the point representing \( C \).

In this situation, there is an exact sequence

\[
1 \to T \to J_{C} \to J_{C'} \to 1.
\]

If \( C' \) is smooth, then the toric rank of \( J_{C} \) is \( r_T = 1 \). If \( f \) is the \( p \)-rank of \( C' \), then \( C \) has \( p \)-rank \( f + 1 \). Thus \( \kappa_0 \) restricts to a map

\[
\kappa_0 : \mathcal{M}^f_{g-1;2} \to \Delta_0[\mathcal{M}_g^{f+1}].
\]
3.2. **Compactification of** $\mathcal{R}_{g,\ell}$. A twisted curve $C$ is a one dimensional stack such that the corresponding coarse moduli space $C$ is a stable curve whose smooth locus is represented by a scheme and whose singularities are nodes with local picture $\{(xy = 0)/\mu_\ell\}$ with $\zeta \in \mu_\ell$ acting as $\zeta(x, y) = (\zeta x, \zeta^{-1} y)$.

As stated on [6, page 6], the space $\mathcal{R}_{g,\ell}$ admits a compactification $\overline{\mathcal{R}}_{g,\ell}$, which is the smooth proper Deligne-Mumford stack of level-$\ell$ twisted curves. The points of $\overline{\mathcal{R}}_{g,\ell}$ represent triples $[C, \eta, \phi]$, where $C$ is a genus $g$ twisted curve, where $\eta \in \text{Pic}(C)$ is a faithful line bundle (see [6, Definition 1.5]), and where $\phi : \eta^{\otimes \ell} \rightarrow \mathcal{O}_C$ is an isomorphism. The morphism $\Pi_\ell$ extends to a morphism $\Pi_\ell : \overline{\mathcal{R}}_{g,\ell} \rightarrow \mathcal{M}_g$.

Some points of $\overline{\mathcal{R}}_{g,\ell}$ cannot be interpreted in terms of $\ell$-torsion line bundles or $\mathbb{Z}/\ell$-covers of a scheme-theoretic curve. For the sake of intuition, however, in the following sections we will most often describe the generic point of the boundary components of $\overline{\mathcal{R}}_{g,\ell}$ in terms of the cover $\pi : Y \rightarrow X$ it represents.

3.3. Boundary components of $\mathcal{R}_{g,\ell}$. Consider the boundary $\delta \mathcal{R}_g = \overline{\mathcal{R}}_{g,\ell} - \mathcal{R}_{g,\ell}$, whose points represent Pryms of singular curves. Let $\delta \mathcal{R}_g,\ell = \delta \mathcal{R}_{g,\ell} - \Pi_\ell^{-1}(\mathcal{R}_{0,\ell})$. For Pryms of singular curves of compact type, the boundary components lie above $\Delta_i[\mathcal{M}_g]$ for some $1 \leq i \leq \lfloor g/2 \rfloor$ and are denoted $\Delta_{v-g-i}, \Delta_i$, and $\Delta_{g-i}$. For Pryms of singular curves of non-compact type, the boundary components lie above $\Delta_0[\mathcal{M}_g]$ and are denoted $\Delta_{0,0}, \Delta_{0,II}$ and $\Delta_{0,III}$.

In Sections 3.4 and 3.5, we recall the definition of these boundary components and investigate them in terms of the $p$-rank. Before doing this, recall the following results.

**Proposition 3.2.** [6, Equation (16)] For $1 \leq i < \lfloor g/2 \rfloor$, there is an equality of divisors

$$\Pi_\ell^*(\Delta_i[\mathcal{M}_g]) = \Delta_i[\overline{\mathcal{R}}_{g,\ell}] + \Delta_{g-i}[\overline{\mathcal{R}}_{g,\ell}] + \Delta_{v-g-i}[\overline{\mathcal{R}}_{g,\ell}].$$

If $g$ is even, there is an equality of divisors

$$\Pi_\ell^*(\Delta_{g/2}[\mathcal{M}_g]) = \Delta_{g/2}[\overline{\mathcal{R}}_{g,\ell}] + \Delta_{g/2-g/2}[\overline{\mathcal{R}}_{g,\ell}].$$

**Proposition 3.3.** [5, page 14] or [6, Equation (17)] There is an equality of divisors

$$\Pi_\ell^*(\Delta_0[\mathcal{M}_g]) = \Delta_{0,0}[\overline{\mathcal{R}}_{g,\ell}] + \Delta_{0,II}[\overline{\mathcal{R}}_{g,\ell}] + \ell \sum_{a=0}^{\lfloor \ell/2 \rfloor} \Delta_{0,III}^{(a)}[\overline{\mathcal{R}}_{g,\ell}].$$

It follows from Propositions 3.2 and 3.3 that every component of $\delta \mathcal{R}_{g,\ell}$ has dimension $3g - 4$. We generalize this statement in Proposition 3.14, replacing the boundary of $\mathcal{R}_{g,\ell}$ by that of the pullback of the $p$-rank strata of $\mathcal{M}_g$. For $0 \leq f \leq g$, recall that $W_{g,f}^\ell = \Pi_\ell^{-1}(\mathcal{M}_g^f)$. Let $W_g^\ell = \Pi_\ell^{-1}(\mathcal{M}_g^f)$.

3.4. Pryms of covers of singular curves of compact type. Let $\xi = \kappa_i(\xi_1, \xi_2)$ be the point of $\Delta_i[\mathcal{M}_g]$ representing the singular curve $X$ of compact type defined in Section 3.1.1. Then an unramified cyclic degree $\ell$ cover $\pi : Y \rightarrow X$ is determined by two line bundles $\eta_{C_1} \in \text{Pic}^0(C_1)[\ell]$ and $\eta_{C_2} \in \text{Pic}^0(C_2)[\ell]$, which are not both trivial. The points of $\Delta_i[\overline{\mathcal{R}}_{g,\ell}]$ (resp. $\Delta_{g-i}[\overline{\mathcal{R}}_{g,\ell}]$) above $\xi$ represent covers $\pi$ for which $\eta_{C_1}$ (resp. $\eta_{C_2}$) is trivial; the points of $\Delta_{v-g-i}[\overline{\mathcal{R}}_{g,\ell}]$ above $\xi$ represent covers $\pi$ for which both $\eta_{C_1}$ and $\eta_{C_2}$ are nontrivial.

3.4.1. The boundary component $\Delta_{v-g-i}$. The boundary divisor $\Delta_{v-g-i}[\overline{\mathcal{R}}_g]$ is the image of the clutching map

$$\kappa_i : \mathcal{R}_{g-\ell\cdot 1} \times \mathcal{R}_{g-\ell\cdot 1} \rightarrow \mathcal{R}_{g,\ell},$$

defined as follows. Let $\tau_1$ be a point of $\mathcal{R}_{g-\ell\cdot 1}$ representing $(\pi_1 : C'_1 \rightarrow C_1, x'_1 \mapsto x_1)$ and let $\tau_2$ be a point of $\mathcal{R}_{g-\ell\cdot 1}$ representing $(\pi_2 : C'_2 \rightarrow C_2, x'_2 \mapsto x_2)$. Let $X$ be the curve of compact
type defined in Section 3.1.1. Let \( Y \) be the curve with components \( C'_1 \) and \( C'_2 \), formed by identifying \( x'_1 \) and \( x'_2 \) in an ordinary double point and identifying the other points in the fiber according to the \( \mathbb{Z}/\ell \)-action. More precisely, if \( \sigma \) generates \( \mathbb{Z}/\ell \), then \( \sigma^k(x'_1) \) and \( \sigma^k(x'_2) \) are identified in an ordinary double point for \( 1 \leq k \leq \ell \). Then \( \kappa_{i: g-i} : g-\iota \) is the point representing the unramified \( \mathbb{Z}/\ell \)-cover \( Y \to X \). This is illustrated in Figure 1 for \( \ell = 2 \).

**Figure 1.** \( \Delta_{i: g-i} : \eta_{C_1} \not\cong \mathcal{O}_{C_1}, \ \eta_{C_2} \not\cong \mathcal{O}_{C_2} \)

**Lemma 3.4.** The clutching map \( \kappa_{i: g-i} \) restricts to a map

\[
\kappa_{i: g-i} : \bar{W}_{f_1}^{g-f_2} \to \Delta_{i: g-i}[\bar{W}_{g}^{f_1+f_2}].
\]

*Proof.* This follows from Equation (3). \( \square \)

**Lemma 3.5.** Suppose \( \pi : Y \to X \) is an unramified \( \mathbb{Z}/\ell \)-cover represented by a point of \( \Delta_{i: g-i}[^2\mathbb{R}_g, \ell] \). Then \( P_\pi \) is an extension by a torus \( T \) of rank \( r_T = \ell - 1 \) and, up to isogeny, there is an exact sequence

\[
1 \to T \to P_\pi \to P_{\pi_1} \oplus P_{\pi_2} \to 1.
\]

If \( f'_i \) is the \( p \)-rank of \( P_{\pi_i} \), then the \( p \)-rank of \( P_\pi \) is \( f'_1 + f'_2 + (\ell - 1) \).

*Proof.* By (1), \( J_Y \) is an extension by a torus \( T \) with rank \( r_T \) equal to the rank of \( H^1(\Gamma_Y, \mathbb{Z}) \). Since \( \Gamma_Y \) consists of two vertices, corresponding to the two irreducible components \( C'_1, C'_2 \), which are connected with \( \ell \) edges, corresponding to the \( \ell \) intersection points \( \sigma^k(x'_1) = \sigma^k(x'_2) \), the rank is \( r_T = \ell - 1 \).

The statement about the \( p \)-rank follows from the exact sequence and the fact that \( T \) contributes \( \ell - 1 \) to the \( p \)-rank of \( P_\pi \).

Consider the exact sequence

\[
1 \to T \to J_Y \xrightarrow{f} J_{C'_1} \oplus J_{C'_2} \to 1.
\]

There is an exact sequence of abelian varieties up to isogeny

\[
1 \to J_X \xrightarrow{\pi^*} J_Y \xrightarrow{\phi} P_\pi \to 1.
\]

For \( i = 1, 2 \), let \( \phi_i \) be the restriction of \( \phi \) to \( J_{C'_i} \). Then there is an exact sequence

\[
1 \to J_{C'_1} \oplus J_{C'_2} \xrightarrow{\pi^*_1 \oplus \pi^*_2} J_{C'_1} \oplus J_{C'_2} \xrightarrow{\phi_1 \oplus \phi_2} P_{\pi_1} \oplus P_{\pi_2} \to 1.
\]
Since $J_X$ is an abelian variety, the intersection of $\pi^*(J_X)$ with the image of $T$ in $J_Y$ is trivial. Thus $f(\pi^*(J_X)) \simeq J_X$, which is isomorphic to $(\pi_1^* \oplus \pi_2^*)(J_{C_1} \oplus J_{C_2})$ by (2). Thus $f$ descends to a well-defined surjective map $\bar{f} : P_{\pi} \to P_{\pi_1} \oplus P_{\pi_2}$.

Consider the restriction $\bar{\phi}$ of $\phi$ to the image of $T$ in $J_Y$. Since $\pi^*(J_X)$ and $T$ intersect trivially, it follows that $\bar{\phi}$ is injective. If $\bar{r} \in \text{Ker}(\bar{f})$, then there exists $r \in J_Y$ such that $f(r) \in (\pi_1^* \oplus \pi_2^*)(J_{C_1} \oplus J_{C_2})$. Without loss of generality, one can suppose that $f(r) = 0$, or $r \in T$, since $\text{Ker}(\phi) = \pi^*(J_X)$. Thus $\bar{\phi}$ surjects onto $\text{Ker}(\bar{f})$. It follows that $\text{Ker}(\bar{f}) \simeq T$, proving there is an exact sequence as claimed. $\square$

3.4.2. The boundary component $\Delta_{i, i > 0}$: The boundary divisor $\Delta_{i}[\bar{R}_g]$ is the image of the clutching map

$$\kappa_i : \bar{R}_{i, \ell, 1} \times \bar{\mathcal{M}}_{g-i, 1} \to \bar{R}_{g, \ell},$$

defined as follows. Let $\tau$ be a point of $\bar{R}_{i, \ell, 1}$ representing $(\pi'_1 : C'_1 \to C_1, x' \mapsto x)$ and let $\omega$ be a point of $\bar{\mathcal{M}}_{g-i, 1}$ representing $(C_2, x_2)$. Let $X$ be the curve of compact type defined in Section 3.1.1. Let $Y$ be the curve with components $C'_1$ and $\ell$ copies of $C_2$, formed by identifying the point $x_2$ on each copy of $C_2$ with a point of $\pi_1^{-1}(x_1)$. Then $\kappa_i(\tau, \omega)$ represents the unramified $\mathbb{Z}/\ell$-cover $\pi : Y \to X$. This is illustrated in Figure 2 for $\ell = 2$.

![Figure 2](image)

**Lemma 3.6.** The clutching map $\kappa_i$ restricts to a map:

$$\kappa_i : \bar{W} f_{i, 1}^{f_1} \times \bar{\mathcal{M}}_{g-i, 1}^{f_2} \to \Delta_{i}[\bar{W} f_{1}^{f_1} + f_2].$$

**Proof.** This follows from Equation (3). $\square$

**Lemma 3.7.** Suppose $\pi : Y \to X$ is an unramified $\mathbb{Z}/\ell$-cover represented by a point of $\Delta[\bar{R}_g]$. Then $\bar{P}_\pi \simeq P_{\pi_1} \oplus J_{C_2}^{f_2-1}$. If $f'_1$ is the $p$-rank of $P_{\pi_1}$, then the $p$-rank of $P_{\pi}$ is $f'_1 + (\ell - 1)f_2$.

**Proof.** By definition, $J_Y \simeq J_{C'_1} \oplus J_{C_2}^{f_2}$. Since $J_{C'_1}$ is isogenous to $J_{C_1} \oplus P_{\pi_1}$, it follows that $J_Y$ is isogenous to $P_{\pi_1} \oplus J_{C'_1} \oplus J_{C_2}^{f_2-1}$. Applying (2) completes the proof. $\square$

3.4.3. The boundary component $\Delta_{g-i}$: Analogously, one can study the boundary divisor $\Delta_{g-i}[\bar{R}_g]$, which is the image of the clutching map

$$\kappa_{g-i} : \bar{\mathcal{M}}_{i, 1} \times \bar{R}_{g-i, \ell, 1} \to \bar{R}_{g, \ell}.$$
3.5. Pryms of covers of singular curves of non-compact type. The main reference for this section is [10, Example 6.5] when \( \ell = 2 \) and [5, Section 1.4] and [6, Section 1.5.2] for general \( \ell \). Let \( C \) be the nodal curve of genus \( g \) with normalization \( C' \) defined in Section 3.1.2. Recall that if \( C' \) has \( p \)-rank \( f_1 \), then \( C \) has \( p \)-rank \( f = f_1 + 1 \).

The statements below about the \( p \)-rank of the Prym are similar to Lemma 3.5 and rely on Equation (1). We omit the details of the proofs.

3.5.1. The Boundary Component \( \Delta_{0, I} \). The boundary divisor \( \Delta_{0, I}[\bar{\mathcal{R}}_{g, \ell}] \) is the image of the clutching map

\[
\kappa_{0, I} : \bar{\mathcal{R}}_{g-1, \ell; 2} \to \bar{\mathcal{R}}_{g, \ell},
\]
defined as follows. Let \( \tau \) be a point of \( \bar{\mathcal{R}}_{g-1, \ell; 2} \) representing \( (\pi' : Y' \to C', x' \mapsto x, y' \mapsto y) \) (two markings). Let \( Y \) be the nodal curve of non-compact type with normalization \( Y' \), formed by identifying \( x' \) and \( y' \) in an ordinary double point and identifying the other points in the fiber according to the \( \mathbb{Z}/\ell \)-action. Then \( \kappa_{0, I}(\tau) \) is the point representing the unramified \( \mathbb{Z}/\ell \)-cover \( \pi : Y \to C \). This is illustrated in Figure 3 for \( \ell = 2 \).

![Figure 3. \( \Delta_{0, I}, \ell = 2 \)](image)

**Lemma 3.8.** The clutching map \( \kappa_{0, I} \) restricts to a map: \( \kappa_{0, I} : \bar{W}_{g-1, \ell; 2}^f \to \Delta_{0, I}[\bar{\mathcal{W}}_{g}^{f_1+1}] \).

*Proof.* This follows from Equation (5). \( \square \)

**Lemma 3.9.** Suppose \( \pi : Y \to X \) is an unramified \( \mathbb{Z}/\ell \)-cover represented by a point of \( \Delta_{0, I}[\bar{\mathcal{R}}_{g, \ell}] \). Then \( P_\pi \) is an extension by a torus \( T \) with \( r_T = \ell - 1 \) and there is an exact sequence

\[
1 \to T \to P_\pi \to P_\pi' \to 1.
\]

If \( f_1 \) is the \( p \)-rank of \( P_\pi' \), then the \( p \)-rank of \( P_\pi \) is \( f = f_1 + (\ell - 1) \).

3.5.2. The Boundary Component \( \Delta_{0, II} \). The boundary divisor \( \Delta_{0, II}[\bar{\mathcal{R}}_{g, \ell}] \) is the image of the clutching map

\[
\kappa_{0, II} : \bar{\mathcal{M}}_{g-1, 2} \to \bar{\mathcal{R}}_{g, \ell},
\]
defined as follows. Let \( \omega \) be a point of \( \bar{\mathcal{M}}_{g-1, 2} \) representing \( (C', x, y) \) (with 2 markings). Consider a disconnected curve with components \( (C'_i, x_i, y_i) \) indexed by \( i \in \mathbb{Z}/\ell \) such that each component is isomorphic to \( (C', x, y) \).
Lemma 3.10. The clutching map $\kappa_{0,II}$ restricts to a map $\kappa_{0,II} : \mathcal{M}_{g-1;2}^f \to \Delta_{0,II}[^{f_1+1}_g]$.

Proof. This follows from Equation (5). \hfill $\Box$

Lemma 3.11. Suppose $\pi : Y \to X$ is an unramified $\mathbb{Z}/\ell$-cover represented by a point of $\Delta_{0,II}[\mathcal{R}_{g,\ell}]$. Then $P_\pi \simeq J^{f_1-1}_C$. If $f_1$ is the p-rank of $C'$, then the p-rank of $P_\pi$ is $(\ell - 1)f_1$.

3.5.3. The Boundary Component $\Delta_{0,III}$. The points of the last boundary component(s) represent level-$\ell$ twisted curves $[X, \tilde{\eta}, \tilde{\phi}]$ which have the following structure. Let $(C', x, y)$ be a point of $\mathcal{M}_{g-1;2}$ and let $E$ be a projective line. The genus $g$ curve $X = C' \cup_{x,y} E$ has components $C'$ and $E$, with two ordinary double points formed by identifying $x$ with 0 and $y$ with $\infty$. Then $\eta_E = \mathcal{O}_E(1)$ and, for some $1 \leq a \leq \ell - 1$,

$$\eta_{C'}^{-\ell} = \mathcal{O}_{C'}(ax + (\ell - a)y).$$

Let $\tilde{\eta}$ be the line bundle on $X$ which restricts to $\eta$ on $C'$ and to $\eta_E$ on $E$.

The boundary divisor $\Delta_{0,III}^{(a)}$ is the closure in $\mathcal{R}_{g-1,\ell}$ of points representing level-$\ell$ twisted curves $[X, \tilde{\eta}, \tilde{\phi}]$ of this form. For more details and proofs, see [5] Section 1.4]. One can define a clutching map

$$\kappa_{0,III}^{(a)} : \mathcal{M}_{g-1;2}^f \to \Delta_{0,III}^{(a)}[\mathcal{R}_{g,\ell}].$$

Lemma 3.12. The clutching map $\kappa_{0,III}^{(a)}$ restricts to a map $\kappa_{0,III}^{(a)} : \mathcal{W}_{g-1;2}^f \to \Delta_{0,III}^{(a)}[\mathcal{W}_{g}^{f_1+1}]$.

Proof. This follows from Equation (5). \hfill $\Box$

Lemma 3.13. Suppose $[X, \tilde{\eta}, \tilde{\phi}]$ is a level-$\ell$ twisted curve represented by a point of $\Delta_{0,III}^{(a)}[\mathcal{R}_{g,\ell}]$. Then its Prym $P_\pi$ is an extension by a torus $T$ of rank $r_T = \ell - 1$ and there is an exact sequence

$$1 \to T \to P_\pi \to P_{\pi|_{C'}} \to 1.$$ 

If $f'_1$ is the p-rank of $P_{\pi|_{C'}}$, then the p-rank of $P_\pi$ is $f' = f'_1 + (\ell - 1)$. 

Figure 4. $\Delta_{0,II,\ell=2}$
3.6. Dimension of the boundary of $W^f_g$. For $0 \leq f \leq g$, recall that $W^f_g = \Pi_{\ell}^{-1}(\mathcal{M}^f_g)$. Let $\hat{W}^f_g = \Pi_{\ell}^{-1}(\mathcal{M}^f_g)$. Since the map $\Pi_{\ell}$ is finite, $\dim(\hat{W}^f_g) = \dim(\mathcal{M}^f_g) = 2g - 3 + f$. Let $\delta \mathcal{R}_{g,\ell} = \mathcal{R}_{g,\ell} - \Pi_{\ell}^{-1}(\mathcal{R}_{g,\ell})$.

**Proposition 3.14.** Let $g \geq 2$ and $0 \leq f \leq g$. If $Q$ is an irreducible component of $\hat{W}^f_g$, then the dimension of $Q \cap \delta \mathcal{R}_{g,\ell}$ is $2g - 4 + f$ and the generic point of $Q$ represents a smooth curve.

**Proof.** By Lemma 2.3, $\dim(Q) = 2g - 3 + f$. In particular, this means that the first statement implies the second. One sees that $\dim(Q \cap \delta \mathcal{R}_{g,\ell}) \geq 2g - 3 + f$ by Lemma 3.1, since $\delta \mathcal{R}_{g,\ell}$ has codimension 1 in $\mathcal{R}_{g,\ell}$.

A component of $\hat{W}^f_g \cap \delta \mathcal{R}_g$ is in the image of one of the clutching morphisms. Applying Lemma 2.3, the following calculations yield the result.

- **Lemma 3.4.** The dimension of $\Delta_{i,g-i}[Q]$ is at most
  \[\dim(\kappa_{i,g-i}(\hat{W}^f_{g-1,i} \times \hat{W}^f_{g-2,i-1})) = (2i - 3 + f_1) + 1 + (2(g - i) - 3 + f_2) + 1 = 2g - 4 + f.\]
- **Lemma 3.6.** The dimension of $\Delta_i[Q]$ is at most
  \[\dim(\kappa_i(\hat{W}^f_{g-1,i} \times \hat{M}^f_{g-2,i-1})) = (2i - 3 + f_1) + 1 + (2(g - i) - 3 + f_2) + 1 = 2g - 4 + f.\]
- **Lemma 3.8.** The dimension of $\Delta_{0,\ell}[Q]$ is at most
  \[\dim(\kappa_{0,\ell}(\hat{W}^f_{g-2,1})) = (2g - 1) - 3 + f_1) + 2 = 2g - 4 + f.\]
- **Lemma 3.10.** The dimension of $\Delta_{0,II}[Q]$ is at most
  \[\dim(\kappa_{0,II}(\hat{M}^f_{g-2,1})) = (2(g - 1) - 3 + f_1) + 2 = 2g - 4 + f.\]
- **Lemma 3.12.** The dimension of $\Delta_{0,III}[Q]$ is at most
  \[\dim(\kappa_{0,III}(\hat{W}^f_{g-2,2})) = (2g - 1) - 3 + f_1) + 2 = 2g - 4 + f.\]

\[\square\]

4. The Prym of a generic curve of given $p$-rank is ordinary

The main result of this section is that the Prym of an unramified $\mathbb{Z}/\ell$-cover of a generic curve of genus $g$ and $p$-rank $f$ is ordinary, Theorem 4.5.

4.1. Earlier work on the $p$-rank stratification of $\mathcal{M}_g$.

**Proposition 4.1.** [11 Proposition 3.4] Suppose $g \geq 2$ and $0 \leq f \leq g$. Suppose $1 \leq i \leq g - 1$ and $(f_1, f_2)$ is a pair such that $f_1 + f_2 = f$ and $0 \leq f_1 \leq i$ and $0 \leq f_2 \leq g - i$. Let $S$ be an irreducible component of $\mathcal{M}^f_g$.

1. Then $\hat{S}$ intersects $\kappa_{i,g-i}(\hat{M}^f_{i-1} \times \hat{M}^f_{g-i-1})$.
2. Each irreducible component of the intersection contains the image of a component of $\hat{M}^f_{i-1} \times \hat{M}^f_{g-i-1}$.

Let $\mathcal{C} \rightarrow S$ be a relative proper semi-stable curve of compact type of genus $g$ over $S$. Then $\text{Pic}^0(\mathcal{C})[\ell]$ is an étale cover of $S$ with geometric fiber isomorphic to $(\mathbb{Z}/\ell)^{2g}$. The fundamental group $\pi_1(S, s)$ acts linearly on the fiber $\text{Pic}^0(\mathcal{C})[\ell]_s$, and the monodromy group $\text{M}_\ell(\mathcal{C} \rightarrow S, s)$ is the image of $\pi_1(S, s)$ in $\text{Aut}(\text{Pic}^0(\mathcal{C})[\ell]_s)$. The next result states that $\text{M}_\ell(S) := \text{M}_\ell(\mathcal{C} \rightarrow S, s)$ is as large as possible, when $S$ is an irreducible component of $\mathcal{M}^f_g$ and $\mathcal{C} \rightarrow S$ is the tautological curve.
Theorem 4.2. [1] Theorem 4.5] Let \( \ell \) be a prime distinct from \( p \); let \( g \geq 2 \) and \( 0 \leq f \leq g \) with \( f \neq 0 \) if \( g = 2 \). Let \( S \) be an irreducible component of \( \mathcal{M}_g^f \), the \( p \)-rank \( f \) stratum in \( \mathcal{M}_g \). Then \( M_\ell(S) \simeq \text{Sp}_{2g}(\mathbb{Z}/\ell) \) and \( M_{2\ell}(S) \simeq \text{Sp}_{2g}(\mathbb{Z}/\ell) \).

4.2. Irreducibility of fibers of \( \Pi_\ell \) over \( \mathcal{M}_g^f \). Recall that the morphism \( \Pi_\ell : \mathcal{R}_{g,\ell} \to \mathcal{M}_g \), which sends the point representing the cover \( \pi : Y \to X \) to the point representing the curve \( X \), is finite and flat with degree \( \ell^{2g} - 1 \).

Proposition 4.3. Under the hypotheses of Theorem 4.2, if \( S \) is an irreducible component of \( \mathcal{M}_g^f \), then \( \Pi_\ell^{-1}(S) \) is irreducible.

Proof. Equip \((\mathbb{Z}/\ell)^{2g}\) with the standard symplectic pairing \( \langle \cdot , \cdot \rangle_{\text{std}} \), and let
\[
S[\ell] := \text{Isom}((\text{Pic}^0(\mathcal{C}/S)[\ell], \langle \cdot , \cdot \rangle_{\lambda}), ((\mathbb{Z}/\ell)^{2g}, \langle \cdot , \cdot \rangle_{\text{std}})).
\]
There is an \( \ell \)th root of unity on \( S \), so \( S[\ell] \to S \) is an étale Galois cover, possibly disconnected, with covering group \( \text{Sp}_{2g}(\mathbb{Z}/\ell) \). By Theorem 4.2, \( M_\ell(S) \simeq \text{Sp}_{2g}(\mathbb{Z}/\ell) \). The geometric interpretation of the monodromy group being \( \text{Sp}_{2g}(\mathbb{Z}/\ell) \) (big) is that \( S[\ell] \) is irreducible.

Suppose \( \xi \) is a point of \( S[\ell] \). Then \( \xi \) represents a curve \( X \), together with an isomorphism between \((\mathbb{Z}/\ell)^{2g}\) and \( \text{Pic}^0(X)[\ell] \). The isomorphism identifies \((1,0)\) with a point of order \( \ell \) on \( \text{Jac}(X) \). It follows that \( \xi \) determines an unramified \( \mathbb{Z}/\ell \)-cover \( \pi : Y \to X \). Thus there is a forgetful morphism \( F : S[\ell] \to \Pi_\ell^{-1}(S) \).

Since \( S[\ell] \) is irreducible, it follows that \( \Pi_\ell^{-1}(S) \) is irreducible. \( \square \)

4.3. Key degeneration result.

Theorem 4.4. Let \( g \geq 3 \) and \( 0 \leq f \leq g \). Let \( Q \) be an irreducible component of \( \tilde{W}_g^f \).

1. Then \( Q \) intersects \( \Delta_{i;g-i} \) for each \( 1 \leq i \leq \lfloor g/2 \rfloor \).
2. More generally, if \((f_1, f_2)\) is a pair such that \( f_1 + f_2 = f \) and \( 0 \leq f_1 \leq i \) and \( 0 \leq f_2 \leq g-i \), then \( Q \) contains the image of a component of \( \kappa_{i;g-i}(\tilde{W}_{i;1}^{f_1} \times \tilde{W}_{g-i;1}^{f_2}) \).

Proof. By Proposition 4.3, \( Q = \Pi_\ell^{-1}(S) \) for some irreducible component \( S \) of \( \mathcal{M}_g^f \). By Proposition 4.1, \( S \) contains the image of a component of \( \tilde{M}_{i;1}^{f_1} \times \tilde{M}_{g-i;1}^{f_2} \). Consider a point \( \xi \) of \( Q \) lying above this image. Then \( \xi \) represents an unramified \( \mathbb{Z}/\ell \)-cover \( \pi : Y \to X \) as in Section 3.4.1. By definition, \( Y \) is a stable curve having components \( C_1 \) and \( C_2 \) of genera \( i \) and \( g-i \) and \( p \)-ranks \( f_1 \) and \( f_2 \).

The \( \mathbb{Z}/\ell \)-cover \( \pi \) is determined by a point of order \( \ell \) on \( J_X \simeq J_{C_1} \oplus J_{C_2} \). Note that \( J_X[\ell] \simeq J_{C_1}[\ell] \oplus J_{C_2}[\ell] \). The point \( \xi \) is in \( \Delta_i \) or \( \Delta_{g-i} \) if and only if the point of order \( \ell \) is in either \( J_{C_1}[\ell] \oplus \{0\} \) or \( \{0\} \oplus J_{C_2}[\ell] \). There are \( \ell^{2g} - \ell^{2i} - \ell^{2(g-i)} + 1 \) points of order \( \ell \) which do not have this property. Since \( Q = \Pi_\ell^{-1}(S) \), without loss of generality, one can suppose that the point of order \( \ell \) is one of these or, equivalently, that \( \xi \) is in \( \Delta_{i;g-i} \). This completes the proof of part (1).

Every component of \( Q \cap \Delta_{i;g-i} \) has dimension \( 2g - 4 + f \). By Proposition 3.14, this equals the dimension of the components of \( \kappa_{i;g-i}(\tilde{W}_{i;1}^{f_1} \times \tilde{W}_{g-i;1}^{f_2}) \), finishing the proof of part (2). \( \square \)

4.4. Ordinary Pryms. The first main result is that the Prym of an unramified \( \mathbb{Z}/\ell \)-cover of a generic curve of genus \( g \) and \( p \)-rank \( f \) is ordinary, for any \( 0 \leq f \leq g \).
Theorem 4.5. Let $\ell$ be a prime distinct from $p$; let $g \geq 2$ and $0 \leq f \leq g$ with $f \neq 0$ if $g = 2$. If $Q$ is an irreducible component of $W^f_g = \Pi^{-1}_\ell(M^f_g)$, then the Prym of the generic point of $Q$ is ordinary ($f'_{Q} = (\ell - 1)(g - 1)$).

Proof. The proof is by induction on $g$, with the base case $g = 1$ being vacuous. Suppose the result is true for all $1 \leq g' < g$. Let $Q$ be an irreducible component of $W^f_g$. Let $Q$ be its closure in $\mathcal{R}_{g,\ell}$. Choose $i$ such that $1 \leq i \leq g - 1$ and a pair $(f_1, f_2)$ such that $f_1 + f_2 = f$ and $0 \leq f_1 < i$ and $0 \leq f_2 \leq g - i$. Note that one can avoid the choice $i = 2$ and $f_1 = 0$. By Theorem 4.4, $Q$ contains a component of $\kappa_{i,g-i}(\bar{W}^{f_1}_{g,1} \times \bar{W}^{f_2}_{g-i,1})$.

The $p$-rank $f'_{Q}$ of the Prym of the generic point of $Q$ is at least as big as the $p$-rank $f'_Q$ of the Prym of a generic point of $\bar{Q} \cap \Delta_{i,g-i}$. By Lemma 3.5, $f'_{\bar{Q}} = f'_1 + f'_2 + (\ell - 1)$, where $f'_1$ (resp. $f'_2$) equals the $p$-rank of the Prym of the generic point of a component of $\bar{W}^{f'_1}_{g,1}$ (resp. $\bar{W}^{f'_2}_{g-i,1}$). By the inductive hypothesis, $f'_1 = (\ell - 1)(i - 1)$ and $f'_2 = (\ell - 1)(g - i - 1)$. Thus $f'_{Q} = (\ell - 1)(g - 1)$ which equals dim $P_{\pi}$. □

Remark 4.6. The proof of Theorem 1.3 is now complete by Proposition 4.3, Theorem 4.5 and Proposition 2.4.

4.5. The hyperelliptic case. We expect that one can prove an analogue of Theorem 4.5 for the hyperelliptic locus $\mathcal{H}_g$. Specifically, one could ask whether the Prym of the generic point of each irreducible component of $\Pi^{-1}_\ell(\mathcal{H}^f_g)$ is ordinary. We expect that the answer to this question is yes for $1 \leq f \leq g$. The reason is that the analogue to Proposition 4.1 (and thus Theorem 4.4) for the hyperelliptic locus are true under the restriction that $i = 1$; Corollary 3.13]. Furthermore, the analogue of Theorem 4.2 for the hyperelliptic locus is true under the restriction that $f > 0$ (or for $f = 0$ and $\ell$ sufficiently large) Theorems 5.2, 5.7]. However, there may be complications with Propositions 3.2, 3.3 over the hyperelliptic locus, especially when $\ell = 2$. For this reason, we leave the hyperelliptic analogue as an open question.

5. A purity result

When $\ell = 2$, we consider the stratification of the moduli space of Prym varieties by $p$-rank. We use purity to give a lower bound for the dimension of the $p$-rank strata, Proposition 5.4. Since we only consider double covers in this section, the subscript $\ell = 2$ is dropped from the notation for simplicity.

If $\pi : Y \to X$ is an unramified double cover, let $f'$ denote the $p$-rank of $P_\pi$. Recall that $\mathcal{A}^f_{g-1}$ denotes the $p$-rank $f'$ stratum of the moduli space $\mathcal{A}_{g-1}$ of principally polarized abelian varieties of dimension $g - 1$. Consider the Prym map $P_{\pi} : (\mathcal{R}_g - \Delta_0) \to \mathcal{A}_{g-1}$.

Definition 5.1. For $0 \leq f' \leq g - 1$, define $V^f_g = P^{-1}_{\pi}(\mathcal{A}^f_{g-1})$. Hence, the points of $V^f_g$ represent unramified double covers $\pi : Y \to X$ of a curve $X$ of compact type such that $X$ has genus $g$ and $P_{\pi}$ has $p$-rank $f'$.

Definition 5.2. For $0 \leq f \leq g$ and $0 \leq f' \leq g - 1$, define $R^{(f,f')}_{g} = W^f_g \cap V^f_g$. Hence, the points of $R^{(f,f')}_{g}$ represent unramified double covers $\pi : Y \to X$ of a curve $X$ of compact type such that $X$ has genus $g$ and $p$-rank $f$ and the Prym $P_{\pi}$ has $p$-rank $f'$.

Remark 5.3. It is not easy to study $V^f_g$ since neither the image nor the fibers of $P_{\pi}$ are understood in general.
Applying purity yields the following result about the dimension of components of $\mathcal{R}^{(f,f')}\!$.

**Proposition 5.4.** Let $\ell = 2$ and $g \geq 2$. If $\mathcal{R}^{(f,f')}\!$ is non-empty then its components have dimension at least $g - 2 + (f + f')$. The same conclusion is true for $\bar{\mathcal{R}}^{(f,f')}\!$.

**Proof.** Consider the forgetful morphism $\psi_g : \mathcal{R}_g \to \mathcal{M}_{2g-1}$ which sends the point representing the cover $\pi : Y \to X$ to the point representing the curve $Y$. If $g \geq 2$, then the genus of $Y$ is at least 3 and $Y$ has only finitely many automorphisms. So $\psi_g$ is finite-to-1 and its image has dimension $3g - 3$.

If $\pi : Y \to X$ is represented by a point of $\mathcal{R}_g^{(f,f')}$, then the $p$-rank of $Y$ is $f + f'$. Thus the image of the restriction of $\psi_g$ to $\mathcal{R}_g^{(f,f')}\!$ is contained in $\mathcal{M}_{2g-1}^{f+f'}$.

The dimension of a component of $\mathcal{R}_g^{(f,f')}$ equals the dimension of its image $Z$ under $\psi_g$. Note that $Z$ is a component of $\text{Im}(\psi_g) \cap \mathcal{M}_{2g-1}^{f+f'}$. By Lemma 3.1

$$\text{codim}(\text{Im}(\psi_g) \cap \mathcal{M}_{2g-1}^{f+f'}, \text{Im}(\psi_g)) \leq \text{codim}(\mathcal{M}_{2g-1}^{f+f'}, \mathcal{M}_{2g-1}) .$$

By [9] Theorem 2.3, $\text{codim}(\mathcal{M}_{2g-1}^{f+f'}, \mathcal{M}_{2g-1}) = 2g - 1 - (f + f')$. Thus

$$\text{dim}(Z) \geq 3g - 3 - (2g - 1 - (f + f')) = g + 2 + f + f'.$$

The proof for $\bar{\mathcal{R}}_g^{(f,f')}$ is almost identical. \hfill $\Box$

**Remark 5.5.** The hypothesis in Proposition 5.4 that $\mathcal{R}_g^{(f,f')} \neq \emptyset$ is not superfluous. Note that $\mathcal{R}^{(0,0)}_2$ is empty when $p = 3$ [9] Theorem 6.1.

**Remark 5.6.** The strategy of the proof of Proposition 5.4 does not give much information when $\ell \geq 3$ because $g_Y$ is too big relative to $3g - 3$.

6. Results for low genus when $\ell = 2$

This section contains results about non-ordinary Pryms of unramified double covers of curves of low genus $g = 2$ and $g = 3$. Since we only consider double covers in this section, the subscript $\ell = 2$ is dropped from the notation for simplicity.

It is useful to keep in mind that $\mathcal{A}_g^f$ is irreducible for all $g \geq 2$ and $0 \leq f \leq g$ except $(g,f) = (2,0)$ [4] Theorem A. When either $g = 2$, $f = 1, 2$ or $g = 3$, $0 \leq f \leq 3$, then the image of $\mathcal{M}_g^f$ under the Torelli map is open and dense in $\mathcal{A}_g^f$ and thus $\mathcal{M}_g^f$ is irreducible as well.

6.1. Base Case: Genus 2. The Pryms of covers of genus two curves arise from an explicit fiber product construction. Even so, the problem is surprisingly subtle from a computational perspective. This section contains a proof that $\mathcal{R}_2^{(f,f')}$ is non-empty with the expected dimension for all six choices of $(f,f')$ when $p \geq 5$.

Recall the well-known fiber product construction. Suppose $X$ is a genus 2 curve and $f_1 : X \to \mathbb{P}^1$ is a hyperelliptic cover branched above a set $B_X$ of cardinality 6. For a set $B_E \subset B_X$ of cardinality 4, let $f_2 : E \to \mathbb{P}^1$ be the hyperelliptic cover branched above $B_E$. The fiber product $f : Y \to \mathbb{P}^1$ of $f_1$ and $f_2$ is a Klein four cover of $\mathbb{P}^1$. By Abhyankar’s Lemma, the degree two subcover $\pi : Y \to X$ is unramified since $B_E \subset B_X$. Then $\text{Jac}(Y)$ is isogenous to $\text{Jac}(X) \oplus E$ by [11] Theorem B. It follows that the Prym of $\pi : Y \to X$ is isogenous to $E$. 
Furthermore, each of the 15 unramified degree two connected covers $\pi : Y \to X$ arises via the fiber product construction (from one of the 15 choices of $B_E \subset B_X$). This is because the hyperelliptic involution $\iota$ on $X$ fixes each point of order 2 on $\text{Jac}(X)$ and thus extends to $Y$.

For a fixed (small) prime $p$, it is thus computationally feasible to find equations for curves represented by points of $R_2^{(f,f')}$. However, if $f + f'$ is small, then trying to prove that $R_2^{(f,f')}$ is non-empty for all primes $p$ using computational algebra is not easy, as illustrated in Section 6.2.

**Proposition 6.1.** Let $g = 2$. Suppose $0 \leq f \leq 2$ and $0 \leq f' \leq 1$. Then $R_2^{(f,f')}$ is non-empty and has dimension $f + f'$ (except when $p = 3$ and $(f, f') = (0, 0)$ or $(1, 0)$).

\[
R_2^{(0,1)} \quad R_2^{(1,1)} \quad R_2^{(2,1)} \\
R_2^{(0,0)} \quad R_2^{(1,0)} \quad R_2^{(2,0)}
\]

**Figure 5.** $R_2$

**Proof.** By Lemma 2.3, $W_2^f$ is non-empty with dimension $1 + f$ for $0 \leq f \leq 2$. If $f = 1, 2$, then $M^f_2$ is irreducible and so $W_2^f$ is irreducible by Proposition 4.3.

When $p = 3$, if $X$ is a smooth curve of genus 2 having a supersingular Prym, then $X$ is ordinary by [9, Section 7.1]. In other words, $R_2^{(1,0)}$ and $R_2^{(0,0)}$ are empty when $p = 3$.

- $(0,0)$. By [9, Theorem 6.1], if $p \geq 5$, then $R_2^{(0,0)}$ is nonempty with dimension 0.
- $(0,1)$. When $f = 0$, then $W_2^0 = R_2^{(0,1)} \cup R_2^{(0,0)}$. For all $p$, it follows that $R_2^{(0,1)}$ is open and dense in $W_2^0$ and so $\dim(R_2^{(0,1)}) = 1$.
- $(2,1)$. Since $W_2^2$ is open and dense in $R_2$, which contains $R_2^{(0,1)}$, the generic point of $W_2^2$ has $f' = 1$. This implies that $R_2^{(2,1)}$ is open and dense in $W_2^2$ and $\dim(R_2^{(2,1)}) = 3$.
- $(1,1)$. By purity, applied to $R_2^{(0,1)} \subset R_2$, one sees that $R_2^{(1,1)}$ is non-empty with dimension 2. Thus $R_2^{(1,1)}$ is open and dense in $W_2^1$.
- $(2,0)$. For $f' = 0$, note that $\dim(V_2^0) = 2$ by the fiber product construction. If $p = 3$, then $R_2^{(2,0)} = V_2^0$ since $R_2^{(0,0)}$ and $R_2^{(1,0)}$ are empty. If $p \geq 5$, then no component of $V_2^0$ can be contained in $W_2^2$ since the latter is irreducible with dimension 2 and has generic $f' = 1$. Thus $R_2^{(2,0)}$ is non-empty and open and dense in $V_2^0$.
- $(1,0)$. If $p \geq 5$, applying purity to $R_2^{(0,0)} \subset V_2^0$ shows that $R_2^{(1,0)}$ is non-empty. Also $R_2^{(1,0)} \subset W_2^1$ which is irreducible of dimension 2 and generic $f' = 1$. Thus every component of $R_2^{(1,0)}$ has dimension at most 1. By purity applied to $R_2^{(1,0)} \subset V_2^0$, finally $\dim(R_2^{(1,0)}) = 1$.

\[\square\]

**Remark 6.2.** For all odd primes $p$, there is a one-dimensional family of singular curves of genus 2 and $p$-rank $f = 1$ having a supersingular Prym. To see this, consider $M_1^{0,2}$,
whose points represent supersingular elliptic curves $E$ with 2 marked points. Note that $\dim(\mathcal{M}_{1;2}^0) = 1$. Then $\kappa_{0,1}(\mathcal{M}_{1;2}^0)$ is a one-dimensional subspace fully contained in the boundary of $\mathcal{R}_{2}^{(1,0)}$.

6.2. Examples when $g = 2$.

6.2.1. Computational approach to Proposition 6.1. There is a computational approach to Proposition 6.1, but it is not easy to implement for all $p \geq 5$.

Let $\lambda \in k - \{0, 1\}$. Consider the elliptic curve $E_\lambda : y_2^2 = x(x - 1)(x - \lambda)$. One says that $\lambda$ is supersingular when $E_\lambda$ is supersingular. For $t_1, t_2 \in k - \{0, 1, \lambda\}$ with $t_1 \neq t_2$, consider the genus two curve

$$X : y_1^2 = f_\lambda(t_1, t_2) := x(x - 1)(x - \lambda)(x - t_1)(x - t_2).$$

As above, $E_\lambda$ is the Prym of a degree 2 unramified cover $\pi : Y \to X$.

Let $M_\lambda(t_1, t_2)$ be the matrix of the Cartier operator on $H^0(X, \Omega^1)$ with respect to the basis \{$dx/y, xdx/y$\}. Recall, from [21, page 381], that the entries of $M_\lambda(t_1, t_2)$ depend on the coefficients $c_i$ of $x^i$ in $f_\lambda(t_1, t_2)^{(p-1)/2}$. Specifically,

$$M_\lambda(t_1, t_2) = \begin{pmatrix} c_{p-1} & c_{p-2} \\ c_{2p-1} & c_{2p-2} \end{pmatrix}.$$

Let $D_\lambda = \det(M_\lambda(t_1, t_2))$. By [21, Theorem 2.2], $X$ is ordinary if and only if $D_\lambda \neq 0$; and, more generally, the $p$-rank of $X$ is the stable rank of $V$ on $H^0(X, \Omega^1)$, which is the rank of $N_\lambda(t_1, t_2) = M_\lambda(t_1, t_2)M_\lambda(t_1, t_2)^{(p)}$ (where the exponent $(p)$ means to raise each coefficient of the matrix to the $p$th power). Let $S_\lambda \subset \mathbb{A}^2$ be the vanishing locus of $D_\lambda$. Here is a computational perspective on Proposition 6.1 when $f' = 0$.

1. The case $(f, f') = (2, 0)$. For each value of $\lambda$ such that $E_\lambda$ is supersingular, to see that $X$ is generically ordinary, one would need to check that $D_\lambda \in k[t_1, t_2]$ is non-zero.

2. The case $(f, f') = (\leq 1, 0)$. To see that $\mathcal{R}_{2}^{(1,0)} \cup \mathcal{R}_{2}^{(0,0)}$ is non-empty, one would need to find some supersingular $\lambda$ such that the determinant $D_\lambda \in k[t_1, t_2]$ is non-constant and $S_\lambda$ is not contained in the union $L$ of the lines $t_i = 0$, $t_i = 1$, $t_i = \lambda$, and $t_1 = t_2$.

3. The case $(f, f') = (0, 0)$ for $p \geq 5$. To see that $\mathcal{R}_{2}^{(0,0)}$ has dimension 0, one would need to show that the stable rank of the Cartier operator is 1 (not zero) for every supersingular $\lambda$ and for each generic point of $S_\lambda$ not in $L$. To see that $\mathcal{R}_{2}^{(0,0)}$ is non-empty, one would need to find some supersingular $\lambda$ and distinct $t_1, t_2 \in k - \{0, 1, \lambda\}$ such that the stable rank of the Cartier operator is 0.

6.2.2. Examples when $g = 2$ and $p = 5$. For example, when $p = 5$, then

$$D_\lambda = c_4c_8 - c_3c_9$$

where

\begin{align*}
    c_3 &= 3\lambda^2t_1t_2 + 3\lambda^2t_1^2t_2 + 3\lambda^2t_1t_2^2 + 3\lambda^2t_1^2t_2^2 \\
    c_4 &= \lambda^2t_1^2t_2^2 + 4\lambda^2t_1t_2^2 + 4\lambda^2t_1^2t_2 + 4\lambda^2t_1t_2^2 + 4\lambda t_1t_2^2 + 4\lambda t_1^2t_2^2 + 4\lambda t_1^2t_2 + 4\lambda t_1t_2^2 + t_1^2t_2^2 \\
    c_8 &= \lambda^2 + 4\lambda t_1 + 4\lambda t_2 + 4\lambda + t_1^2 + 4t_1t_2 + 4t_1 + t_2^2 + 4t_2 + 1 \\
    c_9 &= 3\lambda + 3t_1 + 3t_2 + 3.
\end{align*}

Let $\lambda = a^4$ where $a \in \mathbb{F}_{25}$ is a root of $x^2 + 4x + 2$. Then $E_\lambda$ is supersingular and

$$D_\lambda = (t_1 + 4t_2)^2(t_1^2t_2 + t_1t_2^2 + a^{17}t_1 + a^{17}t_2 + a^5t_1t_2 + a^4t_1 + a^4t_2).$$

Here are some examples of parts of Proposition 6.1 when $f' = 0$ and $p = 5$. 
(1) The case \((f, f') = (2, 0)\). One sees that \(X\) generically ordinary because \(D_\lambda \neq 0\).
(2) The case \((f, f') = (1, 0)\). One can check that \(X\) has \(p\)-rank 1 when \(\lambda = a^4\) and 
\((t_1, t_2) = (a^{16}, a)\).
(3) The case \((f, f') = (0, 0)\). By [9, Section 7.2], up to isomorphism, there is exactly one unramified double cover \(\pi : Y \to X\) such that \(X\) has genus 2 and \(Y\) has \(p\)-rank 0. An equation for \(X\) is \(y^2 = x(x^4 + x^3 + 2x + 3)\).

6.3. Base case: \(g = 3\).

**Lemma 6.3.** Let \(g = 3\). Let \(p \geq 3\) and \(0 \leq f \leq 2\) with \(f \neq 0\) when \(p = 3\). Then \(\mathcal{R}_3^{(f,1)}\) is non-empty and its components have dimension \(2 + f\).

**Proof.** Recall that \(\mathcal{R}_2^{(f,0)}\) is non-empty and has dimension \(f\) by Proposition 6.1 when \(f = 1, 2\) and by [9, Theorem 6.1] when \(f = 0\) and \(p \geq 5\). Consider

\[ I = \kappa_{2,1}(\mathcal{R}_2^{(f,0)} \times \mathcal{R}_1^{(0,0)}) \subset \mathcal{R}_3. \]

Then \(I\) is non-empty and \(\dim(I) = \dim(\mathcal{R}_2^{(f,0)}) + 1 = f + 1\).

By Lemmas 3.4 and 3.5, \(I\) is contained in a component \(T\) of \(\mathcal{R}_3^{(f,1)}\). Now \(\dim(T) \geq f + 2\) by Proposition 5.4. The generic point of \(T\) is not contained in \(\Delta_{2,1}[\mathcal{R}_3^{(f,1)}]\) since the dimension of the latter is bounded by \(f + 1\). Moreover, from the construction of \(I\), one can see that the generic point of \(T\) cannot be contained in any other boundary component. Thus the generic point of \(T\) is in \(\mathcal{R}_3^{(f,1)}\).

The fact that \(\dim(T) = f + 2\) follows from Lemma 3.1 and Proposition 3.14 because

\[ \text{codim}(\Delta_{2,1}[\mathcal{R}_3^{(f,1)}], \mathcal{R}_3^{(f,1)}) \leq \text{codim}(\Delta_{2,1}[\mathcal{R}_3], \mathcal{R}_3) = 1. \]

\[ \square \]

**Remark 6.4.** If \(\mathcal{R}_3^{(0,0)}\) is non-empty, then its components have dimension 1 by [9, Proposition 4.2](i). If \(p = 3\), then \(\mathcal{R}_3^{(0,0)}\) is non-empty by [9, Example 5.5]. In [9, Proposition 4.2], the authors show that there are components of the boundary of \(\mathcal{R}_3^{(0,0)}\) which have dimension 1 or 2. For \(p \geq 5\), it is not known whether \(\mathcal{R}_3^{(0,0)}\) is non-empty.

**Proposition 6.5.** Let \(g = 3\). Let \(p \geq 3\) and \(0 \leq f \leq 3\) with \(f \neq 0\) when \(p = 3\). Then \(\Pi^{-1}(\mathcal{M}_3^f)\) is irreducible and \(\mathcal{R}_3^{(f,1)} = \Pi^{-1}(\mathcal{M}_3^f) \cap V_3^1\) is non-empty with dimension \(2 + f\).

**Proof.** When \(g = 3\), then \(\mathcal{M}_3^f\) is irreducible for \(0 \leq f \leq 3\). By Proposition 4.3, \(\Pi^{-1}(\mathcal{M}_3^f)\) is irreducible. Also \(\Pi^{-1}(\mathcal{M}_3^f)\) has dimension \(3 + f\). By Theorem 4.5, the Prym of the generic point of \(\Pi^{-1}(\mathcal{M}_3^f)\) has \(p\)-rank \(f' = 2\). So, if \(\mathcal{R}_3^{(f,1)} = \Pi^{-1}(\mathcal{M}_3^f) \cap V_3^1\) is non-empty, then its components have dimension \(2 + f\) by Proposition 5.4. If \(0 \leq f \leq 2\), then \(\mathcal{R}_3^{(f,1)}\) is non-empty by Lemma 6.3.

The case \(f = 3\): To show that \(\mathcal{R}_3^{(3,1)}\) is non-empty, consider the restriction \(Pr_3 : V_3^1 \to \mathcal{A}_2^1\), which is surjective. Note that \(\mathcal{A}_2^1\) is irreducible of dimension 2 by [8, Example 11.6]. The generic point \(\eta\) of \(\mathcal{A}_2^1\) represents an abelian variety \(A\) which is indecomposable. Let \(\Theta\) be the theta divisor of the abelian variety represented by \(\eta\). By [19, Remark 4.1], \(\text{Aut}(\Theta_A) \simeq \mathbb{Z}/2\) for all but finitely many abelian surfaces \(A\) which are not a product. Thus \(\text{Aut}(\Theta) \simeq \mathbb{Z}/2\).

By [19, Theorem 4.2], the fiber of \(Pr_3 : \mathcal{R}_3 \to \mathcal{A}_2\) over \(\eta\) is a blow-up of a variety \(F\) of (relative) dimension 3. In fact, \(F\) is isomorphic to a Siegel modular threefold or to \(\mathcal{A}_2^1\) with
added level structure on the 2-torsion points. The exceptional components are described in equations 3.14, 3.15, 3.16 of [19]. They each have (relative) dimension \( \leq 2 \).

Thus the fiber of \( P_{\mathbb{R}} : \mathcal{R}_3 \to \mathcal{A}_2 \) over \( \eta \) has a unique irreducible component \( F \) of dimension 5. By definition, \( F \subset V^3_f \). Let \( f \) be the \( p \)-rank of the generic point of \( \Pi(F) \). If \( f \leq 2 \), then \( F \) would be contained in the closure of \( W^g_3 \), which is irreducible of dimension 5. But this would contradict the fact that the generic point of \( W^g_3 \) has \( f' = 2 \). Thus \( f = 3 \). \( \square \)

7. Almost ordinary Pryms

In this section, we prove Theorem 1.4 which demonstrates the existence of unramified double covers \( \pi : Y \to X \) whose Prym is not ordinary. More precisely, we show there is a codimension one condition on a generic curve \( X \) of genus \( g \) and \( p \)-rank \( f \) for which the Prym of an unramified double cover \( \pi : Y \to X \) is almost ordinary. Recall that the almost ordinary condition means that the \( p \)-rank of \( P_{\pi} \) is \( g - 2 = \dim(P_{\pi}) - 1 \). Section 7.2 contains questions about several possible generalizations of this result.

7.1. Almost ordinary. Consider the stratum \( \mathcal{P}_{g}^{(f,g-2)} = W^g_f \cap V^g_{g-2} \) whose points represent unramified double covers \( \pi : Y \to X \) such that \( X \) is a smooth curve of genus \( g \) and \( p \)-rank \( f \) and such that \( P_{\pi} \) has \( p \)-rank \( f' = g - 2 \) (or, equivalently, such that \( P_{\pi} \) is almost ordinary).

**Theorem 7.1.** Let \( \ell = 2 \). Suppose \( p \geq 5 \), \( g \geq 2 \), and \( 0 \leq f \leq g \). Let \( S \) be a component of \( \mathcal{M}^f_g \). Then the locus of points of \( \Pi_{2}^{-1}(S) \) for which the Prym \( P_{\pi} \) is almost ordinary is non-empty with dimension \( 2g - 4 + f \) (codimension 1 in \( \Pi_{2}^{-1}(S) \)).

**Proof.** Let \( \Pi = \Pi_{2} \). The claim is that \( \Pi_{2}^{-1}(S) \cap V^g_{g-2} \) is non-empty with dimension \( 2g - 4 + f \).

**Dimension:** If \( T \) is a component of \( \Pi_{1}^{-1}(S) \cap V^g_{g-2} \), then \( \dim(T) \geq 2g - 4 + f \) by purity, Proposition 5.4. Also, \( \dim(\Pi_{1}^{-1}(S)) = 2g - 3 + f \) and the generic point of \( \Pi_{1}^{-1}(S) \) has Prym of \( p \)-rank \( f' = g - 1 \) by Theorem 4.5 (or Proposition 6.1 if \( g = 2 \) and \( f = 0 \)). Thus \( \dim(T) = 2g - 4 + f \). It thus suffices to show that \( \Pi_{1}^{-1}(S) \cap V^g_{g-2} \) is non-empty.

**Base cases:** When \( g = 2 \) and \( 0 \leq f \leq 2 \), then \( \Pi_{1}^{-1}(S) \cap V^0_2 \) is non-empty with dimension \( f \) by Proposition 6.1. When \( g = 3 \) and \( 0 \leq f \leq 3 \), then \( \Pi_{1}^{-1}(\mathcal{M}_3^f) \) is irreducible and \( \Pi_{1}^{-1}(\mathcal{M}_3^f) \cap V^1_3 \) is non-empty with dimension \( 2 + f \) by Proposition 6.5.

**Strategy:** Suppose \( g \geq 4 \). Let \( \overline{S} \) be the closure of \( S \) in \( \mathcal{M}_g^f \). The plan is to show that \( \Pi_{1}^{-1}(\overline{S}) \cap V^g_{g-2} \) is non-empty and that one of its components is not contained in \( \delta \mathcal{R}_g \).

**Non-empty:** Let \( i = 3 \). Choose \( f_1, f_2 \) such that \( f_1 + f_2 = f \) with \( 0 \leq f_1 \leq i \) and \( 0 \leq f_2 \leq g - i \). By Proposition 4.1, there is a component \( S_{1,1} \) of \( \overline{\mathcal{M}}_i^{f_1} \) and a component \( S_{2,1} \) of \( \overline{\mathcal{M}}_{g-i,1}^{f_2} \) such that

\[
\kappa_{i,g-i}(S_{1,1} \times S_{2,1}) \subset \overline{S}.
\]

Let \( S_1 = \psi_M(S_{1,1}) \) and \( S_2 = \psi_M(S_{2,1}) \) which are irreducible components of \( \overline{\mathcal{M}}_i^{f_1} \) and \( \overline{\mathcal{M}}_{g-i}^{f_2} \), respectively.

The Prym of the generic point of \( \Pi_{1}^{-1}(S_2) \) has \( p \)-rank \( f'_1 = g - i - 1 \) by Theorem 4.5. By Proposition 4.3, \( \Pi_{1}^{-1}(S_1) \) is irreducible. By Proposition 6.5, there exists a point of \( \Pi_{1}^{-1}(S_1) \) whose Prym has \( p \)-rank \( f'_1 = i - 2 = 1 \). Recall that \( \overline{\mathcal{M}}_3^{f_1} \) is irreducible, so \( \Pi_{1}^{-1}(S_1) = W_3^{f_1} \).
Continuing the proof of non-emptiness, consider
\[ N := \kappa_{i;g-1}(\Pi^{-1}(S_{1;1}) \times \Pi^{-1}(S_{2;1})) \subset \Pi^{-1}(\overline{S}). \]
By Lemma 3.5, \( N \) contains a point whose Prym has \( p \)-rank \( f' = f'_1 + f'_2 + 1 = g - 2 \). In other words, \( N \) contains a point whose Prym is almost ordinary, thus finishing the proof that \( \Pi^{-1}(\overline{S}) \cap V_g^{g-2} \) is non-empty.

**Generically smooth:** Let \( T \) be a component of \( \Pi^{-1}(\overline{S}) \cap V_g^{g-2} \) containing \( N \). By the remarks above, \( T \) intersects the image of
\[ \kappa_{i;g-1} : \mathcal{R}_{i;1}^{(f,i-2)} \times W_{g-i;1}^{f_2} \to \Delta_{i;g-1}[\mathcal{R}_g^{(f,g-2)}]. \]
This image has dimension
\[ (2 + f_1) + 1 + (2(g - 3) - 3 + f_2) + 1 = 2g - 5 + f, \]
which is strictly smaller than \( g - 2 + f + (g - 2) = \dim(\Pi^{-1}(S) \cap V_g^{g-2}) \), Proposition 5.4. Thus the generic point of \( T \) is not contained in \( \Delta_{i;g-1}[\mathcal{R}_g] \).

Furthermore, the generic point of \( T \) is not contained in any other boundary component of \( \delta\mathcal{R}_g \). This is because the generic point of \( S_1 \) and \( S_2 \) represent a smooth curve. It follows that the generic point of \( T \) is contained in \( \Pi^{-1}(S) \cap V_g^{g-2} \). \( \Box \)

**Remark 7.2.** The proof of Theorem 1.4 is now complete by Theorem 7.1 and Proposition 2.4.

7.2. Open questions.

7.2.1. A question about purity. Suppose \( \ell = 2 \). Recall that the points of \( \mathcal{R}_g^{(f,f')} = W_g^f \cap V_g^{f'} \) represent unramified double covers \( \pi : Y \to X \) such that \( X \) is a smooth curve of genus \( g \) and \( p \)-rank \( f \) and such that the Prym of \( \pi \) has \( p \)-rank \( f' \). By Proposition 5.4, if \( \mathcal{R}_g^{(f,f')} \) is non-empty, then its components have dimension at least \( g - 2 + (f + f') \).

**Question 7.3.** Let \( p \) be an odd prime. Let \( g \geq 2 \) and \( 0 \leq f \leq g \) and \( 0 \leq f' \leq g - 1 \). If \( \mathcal{R}_g^{(f,f')} \) is non-empty, then do all its components have dimension exactly \( g - 2 + (f + f') \)?

The answer to Question 7.3 is yes for any \( 0 \leq f \leq g \) in the following cases:

1. when \( f' = g - 1 \) by Theorem 4.5 (and Proposition 6.1 for the case \( g = 2 \) and \( f = 0 \));
2. and when \( f' = g - 2 \) and \( p \geq 5 \) by Theorem 7.1.

One complication in answering Question 7.3 for \( f' < g - 2 \) is that there are families of singular curves in the compactification \( \overline{\mathcal{R}}_{g,2}^{(f,f')} \) whose dimension exceeds \( g - 2 + (f + f') \) as in Remark 6.2.

7.2.2. A question about non-ordinary cyclic covers. The second question is about whether Theorem 7.1 can be generalized to the case \( \ell > 2 \).

**Question 7.4.** Suppose \( \ell \) is an odd prime. For which curves \( X \) of genus \( g \) and \( p \)-rank \( f \), do there exist unramified \( \mathbb{Z}/\ell \)-covers \( \pi : Y \to X \) such that the Prym of \( \pi \) is non-ordinary?

The following example shows that the answer to Question 7.4 may be complicated for small primes.

**Example 7.5.** [15, Section 6] Let \( p = 2 \) and \( g = 2 \) and \( \ell = 3 \). If \( X \) is a curve of genus 2 which is not ordinary \((f < 2)\), then the Prym of every unramified \( \mathbb{Z}/3 \)-cover of \( X \) is ordinary.
7.2.3. Pryms of low $p$-rank. The next question arises when trying to generalize Theorem 7.1 to the case $f' = g - 3$.

Let $\ell = 2$ and $p \geq 5$ and $g = 3$. Consider $V_3^1 = Pr_{3-1}(A_3^1)$ whose points represent unramified double covers $\pi : Y \to X$ such that $X$ has genus 3 and $P_\pi$ has $p$-rank 1. Applying [19, Theorem 4.2, equations 3.14-3.16], one sees that $V_3^1$ has one irreducible component of dimension 5, and three exceptional components of lower dimension.

Consider the closure of $\overline{V}_3^1$ of $V_3^1$ inside $R_{g;2}$. In $\overline{V}_3^1$ is the locus $V_3^0$, whose points represent unramified double covers $\pi : Y \to X$ such that $X$ has genus 3 and $P_\pi$ has $p$-rank 0. By [19, Theorem 4.2], $V_3^0$ has codimension 1 in $\overline{V}_3^1$. However, $A_3^0$ is not irreducible for $p > 11$ by [12]. Thus $V_3^0$ contains many irreducible components of dimension 4.

Furthermore, inside $\overline{V}_3^1$ is the locus $T = W_3^2 \cap \overline{V}_3^1$, whose points represent unramified double covers $\pi : Y \to X$ such that $X$ has genus 3 and $p$-rank 2 and $P_\pi$ has $p$-rank 0. By Proposition 6.5, $T$ is non-empty with dimension 4. It is not known whether or not $T$ is irreducible.

**Question 7.6.** With notation as above, is any component of $T$ or $V_3^0$ contained in a component of the other?

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