GEOMETRIC REALIZATIONS OF LUSZTIG’S SYMMETRIES

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Abstract. In this paper, we give geometric realizations of Lusztig’s symmetries. We also give projective resolutions of a kind of standard modules. By using the geometric realizations and the projective resolutions, we obtain the categorification of the formulas of Lusztig’s symmetries.

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1. Introduction

Let $U$ be the quantum group and $\mathfrak{f}$ be the Lusztig’s algebra associated with a Cartan datum. Denote by $U^+$ and $U^-$ the positive part and the negative part of $U$ respectively. There are two well-defined $\mathbb{Q}(v)$-algebra homomorphisms $+: \mathfrak{f} \to U$ and $-: \mathfrak{f} \to U$ with images $U^+$ and $U^-$ respectively.

Lusztig introduced the canonical basis $B$ of $\mathfrak{f}$ in [11, 13, 16]. Let $Q = (I, H)$ be a quiver corresponding to $\mathfrak{f}$ and $V$ be an $I$-graded vector space such that $\dim V = \nu \in \mathbb{N}$. He studied the variety $E_V$ consisting of representations of $Q$ with dimension vector $\nu$, and a category $Q_V$ of some semisimple perverse sheaves on $E_V$. Let $K(Q_V)$ be the Grothendieck group of $Q_V$. Considering all dimension vectors, he proved that $\bigoplus_{\nu \in \mathbb{N}} K(Q_V)$ realizes $\mathfrak{f}$ and the set of isomorphism classes of simple objects realizes the canonical basis $B$.

Lusztig also introduced some symmetries $T_i$ on $U$ for all $i \in I$ in [10, 12]. Note that $T_i(U^+)$ is not contained in $U^+$. Hence, Lusztig introduced two subalgebras $i\mathfrak{f}$ and $i\mathfrak{f}$ of $\mathfrak{f}$ for any $i \in I$, where $i\mathfrak{f} = \{ x \in \mathfrak{f} | T_i(x^+) \in U^+ \}$ and $i\mathfrak{f} = \{ x \in \mathfrak{f} | T_i^{-1}(x^+) \in U^+ \}$. Let $T_i: i\mathfrak{f} \to i\mathfrak{f}$ be the unique map satisfying $T_i(x^+) = T_i(x)^+$. The algebra $\mathfrak{f}$ has the following direct sum decompositions $\mathfrak{f} = i\mathfrak{f} \bigoplus \theta i\mathfrak{f} = i\mathfrak{f} \bigoplus i\mathfrak{f}$. Denote by $i\pi: \mathfrak{f} \to i\mathfrak{f}$ and $\pi: \mathfrak{f} \to i\mathfrak{f}$ the natural projections.

Associated to a finite dimensional hereditary algebra $\Lambda$, Ringel introduced the Hall algebra and the composition subalgebra $F$ in [18], which gives a realization of the positive part of the quantum group $U$. If we use the notations of Lusztig in [13], we have the canonical isomorphism between the composition subalgebra $F$ and the Lusztig’s algebra $\mathfrak{f}$. Via the Hall algebra approach, one can apply BGP-reflection functors to quantum groups to give precise constructions of Lusztig’s symmetries ([19, 15, 22, 24, 3, 25]).

To a Lusztig’s algebra $\mathfrak{f}$, Khovanov, Lauda ([6]) and Rouquier ([20]) introduced a series of algebras $R_{\nu}$ respectively. The category of finitely generated projective modules of $R_{\nu}$ gives a categorification of $\mathfrak{f}$ and $R_{\nu}$ are called Khovanov-Lauda-Rouquier (KLR) algebras. Varagnolo, Vasserot ([23]) and Rouquier ([21]) realized the KLR algebra $R_{\nu}$ as the extension algebra of semisimple perverse sheaves in $Q_V$ and proved that the set of indecomposable projective modules of $R_{\nu}$ can categorify the canonical basis $B$.

In [4, 5], Kato gave the categorification of the PBW-type bases of quantum groups of finite type. He constructed some modules (which are called standard modules) of the KLR algebras $R_{\nu}$ and proved that there standard modules can categorify the PBW-type basis of $\mathfrak{f}$ by using the geometric realizations of $R_{\nu}$ given by Varagnolo, Vasserot and Rouquier. He proved that the length of the projective resolution of any standard module is finite, which is the categorification of the following fact: the transition matrix between the PBW-type basis of $\mathfrak{f}$ and the canonical basis $B$ is triangular with diagonal entries equal to 1. This result implies that the global dimensions of the KLR algebras $R_{\nu}$ are also finite. In [17, 2], Brundan, Kleshchev and McNamara proved the same result by using an algebraic method.
Let \( i \in I \) be a sink (resp. source) of \( Q \). Similarly to the geometric realization of \( f \), consider a subvariety \( i_! E_V \) (resp. \( i^! E_V \)) of \( E_V \) and a category \( i^! Q_V \) (resp. \( i_! Q_V \)) of some semisimple perverse sheaves on \( i_! E_V \) (resp. \( i^! E_V \)). In Section 3.2 we verify that \( \bigoplus_{r \in \mathbb{N}} K(\mathcal{Q}_V) \) (resp. \( \bigoplus_{r \in \mathbb{N}} K(\mathcal{Q}_V) \)) realizes \( f \) (resp. \( f' \)).

Let \( i \in I \) be a sink of \( Q \). Let \( Q' = \sigma_i Q \) be the quiver by reversing the directions of all arrows in \( Q \) containing \( i \). Hence, \( i \) is a source of \( Q' \). Consider two \( I \)-graded vector spaces \( V \) and \( V' \) such that \( \overline{\dim V'} = s_i \overline{\dim V} \). In the case of finite type, Kato introduced an equivalence \( \tilde{\omega}_i : \mathcal{Q}_{V,Q} \to \mathcal{Q}_{V',Q'} \) and studied the properties of this equivalence in [4] [5]. In this paper, we generalize his construction to all cases and prove that the map induced by \( \tilde{\omega}_i \) realizes the Lusztig's symmetry \( T_i : \mathfrak{f} \to \mathfrak{f}' \).

For the proof of the result, we shall study the relations between the map induced by \( \tilde{\omega}_i \) and the Hall algebra approach to \( T_i \) in [1].

In [1], Lusztig showed that Lusztig's symmetries and canonical bases are compatible. Let \( \mathfrak{B} = \mathfrak{f}(B) \), which is a \( \mathbb{Q}(v) \)-basis of \( \mathfrak{f} \). Similarly, \( \mathfrak{B}' = \mathfrak{f}'(B) \) is a \( \mathbb{Q}(v) \)-basis of \( \mathfrak{f}' \). Lusztig proved that \( T_i : \mathfrak{f} \to \mathfrak{f}' \) maps any element of \( \mathfrak{B} \) to an element of \( \mathfrak{B}' \).

For any simple perverse sheaf \( \mathcal{L} \) in \( \mathcal{Q}_{V,Q} \), the restriction \( i_! \mathcal{L} = j^*_V(\mathcal{L}) \) on \( i_! E_{V,Q} \) is also a simple perverse sheaf and belongs to \( i^! Q_{V,Q} \), where \( j_V : i^! E_{V,Q} \to E_{V,Q} \) is the canonical embedding. Let \( i^! \mathcal{L} = \tilde{\omega}_i (i_! \mathcal{L}) \in i^! Q_{V',Q'} \). The simple perverse sheaf \( i^! \mathcal{L} \) can be written as \( i^! \mathcal{L} = j^*_V(\mathcal{L}') \), where \( \mathcal{L}' \) is a simple perverse sheaf in \( Q_{V',Q'} \) and \( j'_V : i^! E_{V',Q'} \to E_{V',Q'} \) is the canonical embedding. Since the map induced by \( \tilde{\omega}_i \) realizes \( T_i : \mathfrak{f} \to \mathfrak{f}' \), this result gives a geometric interpretation of Lusztig's result in [1].

For any \( m \leq -a_{ij} \), let

\[
f(i, j; m) = \sum_{r + s = m} (-1)^r v^{-r(-a_{ij} - m + 1)} \theta_i^{(r)} \theta_j^{(s)} \in \mathfrak{f},
\]

and

\[
f'(i, j; m) = \sum_{r + s = m} (-1)^r v^{-r(-a_{ij} - m + 1)} \theta_i^{(s)} \theta_j^{(r)} \in \mathfrak{f}.
\]

In [16], Lusztig proved that \( T_i(f(i, j; m)) = f'(i, j; m') \), where \( m' = -a_{ij} - m \). The following formula

\[T_i(E_j) = \sum_{r + s = -a_{ij}} (-1)^r v^{-r} E_i^{(s)} E_j E_i^{(r)}\]

is a special case of \( T_i(f(i, j; m)) = f'(i, j; m') \). In this paper, our main result is the categorification of these formulas. Consider an \( I \)-graded vector space \( V \) such that \( \overline{\dim V} = \nu = m_i + j \). Let \( D_{G_V}(E_V) \) be the bounded \( G_V \)-equivariant derived category of complexes of \( I \)-adic sheaves on \( E_V \). We construct a series of distinguished triangles in \( D_{G'_V}(E_V) \), which represent the constant sheaf \( 1_{i,E_V} \) in terms of some semisimple perverse sheaves \( I_p \in D_{G_V}(E_V) \) geometrically. Note that, \( 1_{i,E_V} \) corresponds to a standard module \( K^p \) of the KLR algebra \( R_v \) and \( I_p \) correspond to projective modules of \( R_v \). This result means that we find projective resolutions of the standard modules.
Consider two $I$-graded vector spaces $V$ and $V'$ such that $\dim V = ni + j$ and $\dim V' = n'i + j$. Applying to the Grothendieck group, $1_{E_{V},Q}$ (resp. $1_{E_{V'},Q'}$) corresponds to $f(i, j; m)$ (resp. $f'(i, j; m')$). The property of BGP-reflection functors implies $\tilde{\omega}_i(v^{-mN}1_{E_{V},Q}) = v^{-m'N}1_{E_{V'},Q'}$, therefore $\tilde{T}_i(f(i, j; m)) = f'(i, j; m')$.

In Example D of [5], Kato constructed a short exact sequence

$$0 \longrightarrow P_1 \ast P_2[2] \longrightarrow P_2 \ast P_1 \longrightarrow Q_{12} \longrightarrow 0$$

which coincides with the projection resolution in our main result in the case of finite type. In Theorem 4.10 of [2], Brundan, Kleshchev and McNamara constructed a shout exact sequence of standard modules

$$0 \longrightarrow v^{-\beta} \Delta(\beta) \circ \Delta(\gamma) \longrightarrow \Delta(\gamma) \circ \Delta(\beta) \longrightarrow [p_{\beta, \gamma} + 1]\Delta(\alpha) \longrightarrow 0.$$ 

In the case of finite type, the projection resolution in our main result is a special case of the shout exact sequence above where $\alpha = \alpha_i + \alpha_j$.

2. Quantum groups and Lusztig’s symmetries

2.1. Quantum groups. Let $I$ be a finite index set with $|I| = n$ and $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix. Let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum associated with $A$, where

(1) $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots;
(2) $\Pi^\vee = \{\beta_i \mid i \in I\}$ is the set of simple coroots;
(3) $P$ is the weight lattice;
(4) $P^\vee$ is the dual weight lattice.

In this paper, we always assume that the generalized Cartan matrix $A$ is symmetric. Fix an indeterminate $v$. For any $n \in \mathbb{Z}$, set $[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}} \in \mathbb{Q}(v)$. Let $[0]_v! = 1$ and $[n]_v! = [n]_v[n - 1]_v \cdots [1]_v$ for any $n \in \mathbb{Z}_{>0}$.

The quantum group $U$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is an associative algebra over $\mathbb{Q}(v)$ with unit element 1, generated by the elements $E_i$, $F_i(i \in I)$ and $K_\mu(\mu \in P^\vee)$ subject to the following relations

$$K_0 = 1, \quad K_\mu K_{\mu'} = K_{\mu + \mu'} \text{ for all } \mu, \mu' \in P^\vee;$$

$$K_\mu E_i K_{-\mu} = v^{\alpha_i(\mu)} E_i \text{ for all } i \in I, \mu \in P^\vee;$$

$$K_\mu F_i K_{-\mu} = v^{-\alpha_i(\mu)} F_i \text{ for all } i \in I, \mu \in P^\vee;$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_{-j}}{v - v^{-1}} \text{ for all } i, j \in I;$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k E_i^{(k)} E_j E_i^{(1-a_{ij}-k)} = 0 \text{ for all } i \neq j \in I;$$
Let \( f \) be the associative algebra defined by Lusztig in [16]. The algebra \( f \) is generated by \( \theta_i (i \in I) \) subject to the following relations

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \theta_i^{(k)} \theta_j \theta_i^{(1-a_{ij}-k)} = 0 \text{ for all } i \neq j \in I,
\]

where \( \theta_i^{(n)} = \theta_i^n/[n]_v! \).

There are two well-defined \( \mathbb{Q}(v) \)-algebra homomorphisms \( + : f \to U \) and \( - : f \to U \) satisfying \( E_i = \theta_i^+ \) and \( F_i = \theta_i^- \) for all \( i \in I \). The images of \( + \) and \( - \) are \( U^+ \) and \( U^- \) respectively.

### 2.2. Lusztig’s symmetries.

Corresponding to \( i \in I \), Lusztig introduced the Lusztig’s symmetry \( T_i : U \to U \) ([10] [12] [16]). The formulas of \( T_i \) on the generators are:

\[
\begin{align*}
T_i(E_j) &= -F_i K_i, \quad T_i(F_i) = -K_i E_i; \\
T_i(E_j) &= \sum_{r+s=-a_{ij}} (-1)^r v^{-r} E_i^{(s)} E_j^{(r)} \quad \text{for } i \neq j \in I; \\
T_i(F_j) &= \sum_{r+s=-a_{ij}} (-1)^r v^r F_i^{(r)} F_j^{(s)} \quad \text{for } i \neq j \in I; \\
T_i(K_\mu) &= K_{\mu-\alpha_i(\mu) h_i}.
\end{align*}
\]

Lusztig introduced two subalgebras \( i f \) and \( i' f \) of \( f \). For any \( j \in I, i \neq j, m \in \mathbb{N} \), define

\[
f(i, j; m) = \sum_{r+s=m} (-1)^r v^{-r(-a_{ij}-m+1)} \theta_i^{(r)} \theta_j \theta_i^{(s)} \in f,
\]

and

\[
f'(i, j; m) = \sum_{r+s=m} (-1)^r v^{-r(-a_{ij}-m+1)} \theta_i^{(s)} \theta_j \theta_i^{(r)} \in f.
\]

The subalgebras \( i f \) and \( i' f \) are generated by \( f(i, j; m) \) and \( f'(i, j; m) \) respectively.

Note that \( i f = \{ x \in f \mid T_i(x^+) \in U^+ \} \) and \( i' f = \{ x \in f \mid T_i^{-1}(x^+) \in U^+ \} \) ([16]). Hence there exists a unique \( T_i : i f \to i' f \) such that \( T_i(x^+) = T_i(x)^+ \). Lusztig also showed that \( f \) has the following direct sum decompositions \( f = i f \oplus \partial_i f = i' f \oplus f \theta_i \).

Denote by \( i : f \to i f \) and \( i' : f \to i' f \) the natural projections.

Lusztig also proved the following formulas.
Proposition 2.1 ([16]). For any $-a_{ij} \geq m \in \mathbb{N}$, $T_i(f(i, j; m)) = f'(i, j; -a_{ij} - m)$.

The formulas (1) and (2) are two special cases of Proposition 2.1.

3. Geometric realizations

3.1. Geometric realization and canonical basis of \( f \). In this subsection, we shall review the geometric realization of \( f \) introduced by Lusztig ([11, 13, 16]).

A quiver \( Q = (I, H, s, t) \) consists of a vertex set \( I \), an arrow set \( H \), and two maps \( s, t : H \to I \) such that an arrow \( \rho \in H \) starts at \( s(\rho) \) and terminates at \( t(\rho) \). Let \( h_{ij} = \#\{i \to j\} \), \( a_{ij} = h_{ij} + h_{ji} \) and \( f \) be the Lusztig’s algebra corresponding to \( A = (a_{ij}) \). Let \( p \) be a prime and \( q \) be a power of \( p \). Denote by \( \overline{\mathbb{F}_q} \) the finite field with \( q \) elements and \( K = \overline{\mathbb{F}_q} \).

For a finite dimensional \( I \)-graded \( K \)-vector space \( V = \bigoplus_{i \in I} V_i \), define
\[
E_V = \bigoplus_{\rho \in H} \text{Hom}_K(V_{s(\rho)}, V_{t(\rho)}).
\]
The dimension vector of \( V \) is defined as \( \dim V = \sum_{i \in I} (\dim_K V_i) i \in NI \). The algebraic group \( G_V = \prod_{i \in I} GL_K(V_i) \) acts on \( E_V \) naturally.

Fix a nonzero element \( \nu \in NI \). Let
\[
Y_\nu = \{ y = (i, a) \mid \sum_{l=1}^k a_l i_l = \nu \},
\]
where \( i = (i_1, i_2, \ldots, i_k), \ i_l \in I, \ a = (a_1, a_2, \ldots, a_k), \ a_l \in \mathbb{N}, \) and
\[
I^\nu = \{ i = (i_1, i_2, \ldots, i_k) \mid \sum_{l=1}^k i_l = \nu \}.
\]
Fix a finite dimensional \( I \)-graded \( K \)-vector space \( V \) such that \( \dim V = \nu \). For any element \( y = (i, a) \), a flag of type \( y \) in \( V \) is a sequence
\[
\phi = (V = V^k \supset V^{k-1} \supset \cdots \supset V^0 = 0)
\]
of \( I \)-graded \( K \)-vector spaces such that \( \dim V^l/V^{l-1} = a_l i_l \). Let \( F_y \) be the variety of all flags of type \( y \) in \( V \). For any \( x \in E_V \), a flag \( \phi \) is called \( x \)-stable if \( x_{\rho}(V_{s(\rho)}^l) \subset V_{t(\rho)}^l \) for all \( l \) and all \( \rho \in H \). Let
\[
\tilde{F}_y = \{(x, \phi) \in E_V \times F_y \mid \phi \text{ is } x \text{-stable}\}
\]
and \( \pi_y : \tilde{F}_y \to E_V \) be the projection to \( E_V \).

Let \( \mathbb{Q} \) be the \( l \)-adic field and \( \mathcal{D}_{G_V}(E_V) \) be the bounded \( G_V \)-equivariant derived category of complexes of \( l \)-adic sheaves on \( E_V \). For each \( y \in Y_\nu \), \( \mathcal{L}_y = (\pi_y)_!(1_{\tilde{F}_y}) \mathcal{D}_{G_V}(E_V) \) is a semisimple perverse sheaf, where \( d_y = \dim \tilde{F}_y \). Let \( \mathcal{P}_V \) be the set of isomorphism classes of simple perverse sheaves \( \mathcal{L} \) on \( E_V \) such that \( \mathcal{L}[r] \) appears as a direct summand of \( \mathcal{L}_i \) for some \( i \in I^\nu \) and \( r \in \mathbb{Z} \). Let \( \mathcal{Q}_V \) be the full subcategory of \( \mathcal{D}_{G_V}(E_V) \) consisting of all complexes which are isomorphic to finite direct sums of complexes in the set \( \{ \mathcal{L}[r] \mid \mathcal{L} \in \mathcal{P}_V, r \in \mathbb{Z} \} \).
Let $K(\mathcal{Q}_V)$ be the Grothendieck group of $\mathcal{Q}_V$. Define
\[ v^+|\mathcal{L}| = |\mathcal{L}|[\pm 1](\pm \frac{1}{2}), \]
where $\mathcal{L}(d)$ is the Tate twist of $\mathcal{L}$. Then, $K(\mathcal{Q}_V)$ is a free $\mathcal{A}$-module, where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Define
\[ K(\mathcal{Q}) = \bigoplus_{\nu \in \mathbb{N}I} K(\mathcal{Q}_V). \]

For $\nu, \nu', \nu'' \in NI$ such that $\nu = \nu' + \nu''$ and three $I$-graded $\mathbb{K}$-vector spaces $V, V', V''$ such that $\dim V = \nu, \dim V' = \nu'$, $\dim V'' = \nu''$, Lusztig constructed a functor
\[ * : \mathcal{Q}_V \times \mathcal{Q}_{V'} \to \mathcal{Q}_V. \]
This functor induces an associative $\mathcal{A}$-bilinear multiplication
\[ \otimes : K(\mathcal{Q}_V) \times K(\mathcal{Q}_{V'}) \to K(\mathcal{Q}_V) \]
\[ ([\mathcal{L}'], [\mathcal{L}'']) \mapsto [\mathcal{L}' \otimes [\mathcal{L}''] = [\mathcal{L}' \otimes [\mathcal{L}'']] \]
where $\mathcal{L}' \otimes [\mathcal{L}'' = (\mathcal{L}' \otimes [\mathcal{L}'])(m_{\nu_1' \nu_2' \nu_3'})$ and $m_{\nu_1' \nu_2' \nu_3'} = \sum_{\rho \in H} \nu_1'(s(\rho)\nu_2'(\rho) - \sum_{i \in I} \nu_i'' \nu_i'$. Then $K(\mathcal{Q})$ becomes an associative $\mathcal{A}$-algebra and the set $\{[[\mathcal{L}] \mid \mathcal{L} \in \mathcal{P}_V\}$ is a basis of $K(\mathcal{Q}_V)$.

**Theorem 3.1** ([13]). There is a unique $\mathcal{A}$-algebra isomorphism
\[ \lambda_\mathcal{A} : K(\mathcal{Q}) \to \mathfrak{f}_\mathcal{A} \]
such that $\lambda_\mathcal{A}(\mathcal{L}_y) = \theta_y$ for all $y \in Y_\nu$, where $\theta_y = \theta^{(a_1)}(\theta^{(a_2)} ... \theta^{(a_k)})$ and $\mathfrak{f}_\mathcal{A}$ is the integral form of $\mathfrak{f}$.

Let $B_\nu = \{b_\mathcal{L} = \lambda_\mathcal{A}(\mathcal{L}) \mid \mathcal{L} \in \mathcal{P}_V\}$ and $B = \sqcup_{\nu \in NI} B_\nu$. Then $B$ is the canonical basis of $\mathfrak{f}$ introduced by Lusztig in [11, 13].

3.2. **Geometric realizations of $\mathfrak{f}$ and $^i\mathfrak{f}$**. Assume that $i \in I$ is a sink. Let $V$ be a finite dimensional $I$-graded $\mathbb{K}$-vector space such that $\dim V = \nu$. Consider a subvariety $^iE_V$ of $E_V$
\[ ^iE_V = \{x \in E_V \mid \bigoplus_{h \in H, t(h) = i} x_h : \bigoplus_{h \in H, t(h) = i} V_{s(h)} \to V_i \text{ is surjective}\}. \]
Let $j_V : ^iE_V \to E_V$ be the canonical embedding. For any $y = (i, a) \in Y_\nu$, let
\[ ^i\tilde{F}_y = \{(x, \phi) \in ^iE_V \times F_y \mid \phi \text{ is $x$-stable}\} \]
and $^i\pi_y : ^i\tilde{F}_y \to ^iE_V$ be the projection to $^iE_V$.

For any $\mathcal{L} \in \mathcal{P}_V$, $\mathcal{L} = (\lambda_\mathcal{A})(1, \tilde{F}_y)[d_y] \in \mathcal{D}_\mathcal{G}_V(^iE_V)$ is a semisimple perverse sheaf.

Let $^i\mathcal{P}_V$ be the set of isomorphism classes of simple perverse sheaves $\mathcal{L}$ on $^iE_V$ such that $\mathcal{L}[r]$ appears as a direct summand of $^i\mathcal{L}$ for some $i \in I'$ and $r \in \mathbb{Z}$. Let $^i\mathcal{Q}_V$ be the full subcategory of $\mathcal{D}_\mathcal{G}_V(^iE_V)$ consisting of all complexes which are isomorphic to finite direct sums of complexes in the set $\{\mathcal{L}[r] \mid \mathcal{L} \in ^i\mathcal{P}_V, r \in \mathbb{Z}\}$. 
Let $K(i\mathcal{Q}_V)$ be the Grothendieck group of $i\mathcal{Q}_V$ and

$$K(i\mathcal{Q}) = \bigoplus_{[V]} K(i\mathcal{Q}_V).$$

Naturally, we have two functors $j_V^! : \mathcal{D}_{GV}(iE_V) \to \mathcal{D}_{GV}(E_V)$ and $j_V^* : \mathcal{D}_{GV}(E_V) \to \mathcal{D}_{GV}(iE_V)$.

For any $y \in Y_V$, we have the following fiber product

$$
\begin{array}{ccc}
\tilde{F}_y & \xrightarrow{j_V} & F_y \\
\downarrow i\pi_y & & \downarrow \pi_y \\
\tilde{E}_V & \xrightarrow{j_V} & E_V
\end{array}
$$

So

$$j_V^! L_y = j_V^!(\pi_y)_!(1_{\tilde{F}_y})[d_y] = (i\pi_y)_! j_V^!(1_{\tilde{F}_y})[d_y] = (i\pi_y)_!(1_{\tilde{F}_y})[d_y] = iL_y.
$$

That is $j_V^!(\mathcal{Q}_V) = i\mathcal{Q}_V$. Hence $j_V^! : \mathcal{Q}_V \to i\mathcal{Q}_V$ and $j^* : K(\mathcal{Q}) \to K(i\mathcal{Q})$ can be defined.

Consider the following diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{\theta_i f_A} & iE_V \\
& & \downarrow \lambda_A \\
& & K(\mathcal{Q}) \\
& & \xrightarrow{j^*} K(i\mathcal{Q}) & \xrightarrow{\lambda'_A} 0
\end{array}
$$

where $\lambda'_A$ is the inverse of $\lambda_A$. Since $j^* \circ \lambda'_A \circ i = 0$, there exists a map $i\lambda'_A : i\mathcal{Q}_V \to K(i\mathcal{Q})$ such that the above diagram commutes.

**Proposition 3.2.** The map $i\lambda'_A : i\mathcal{Q}_V \to K(i\mathcal{Q})$ is an isomorphism of $\mathcal{A}$-algebras.

The proof of Proposition 3.2 will be given in Section 4.2.

Assume that $i \in \mathcal{I}$ is a source. We can give a geometric realization of $i\mathcal{Q}$ similarly. Consider a subvariety $iE_V$ of $E_V$

$$iE_V = \{ x \in E_V \mid \bigoplus_{h \in H, s(h) = i} x_h : V_i \to \bigoplus_{h \in H, s(h) = i} V_{t(h)} \text{ is injective} \}.$$

Let $j_V : iE_V \to E_V$ be the canonical embedding. The definitions of $i\mathcal{Q}_V$, $K(i\mathcal{Q}_V)$ and $K(i\mathcal{Q})$ are similar to those of $i\mathcal{Q}_V$, $K(i\mathcal{Q}_V)$ and $K(i\mathcal{Q})$ respectively. We can also define $j_V^* : \mathcal{Q}_V \to i\mathcal{Q}_V$, $j^* : K(\mathcal{Q}) \to K(i\mathcal{Q})$ and $i\lambda'_A : i\mathcal{Q}_V \to K(i\mathcal{Q})$.

Similarly to Proposition 3.2, we have the following proposition.

**Proposition 3.3.** The map $i\lambda'_A : i\mathcal{Q}_V \to K(i\mathcal{Q})$ is an isomorphism of $\mathcal{A}$-algebras.

□
3.3. **Geometric realization of** $T_i : f \to ^if$. Assume that $i$ is a sink of $Q = (I, H, s, t)$. So $i$ is a source of $Q' = \sigma_i Q = (I, H', s, t)$, where $\sigma_i Q$ is the quiver by reversing the directions of all arrows in $Q$ containing $i$. For any $\nu, \nu' \in NI$ such that $\nu' = s_i \nu$ and $I$-graded $\mathbb{K}$-vector spaces $V$, $V'$ such that $\dim V = \nu$, $\dim V' = \nu'$, consider the following correspondence ([15],[5])

$$i_{E,d} : Z_{VV'} \xrightarrow{\beta} _{E,d} \cdot$$

where

(1) $Z_{VV'}$ is the subset in $E_{V,Q} \times E_{V',Q'}$ consisting of all $(x, y)$ satisfying the following conditions

(a) for any $h \in H$ such that $t(h) \neq i$ and $h \in H'$, $x_h = y_h$;

(b) the following sequence is exact

$$0 \longrightarrow V_i \xrightarrow{\bigoplus_{h \in H', s(h) = i} y_h} \bigoplus_{h \in H, t(h) = i} \bigoplus_{h \in (H', s(h)) = i} x_h V_i \longrightarrow 0$$

(2) $\alpha(x, y) = x$ and $\beta(x, y) = y$.

From now on, $i_{E,d} Q$ is denoted by $i_{E,d}$ and $i_{E,d} Q'$ is denoted by $i_{E,d}$. Let

$$G_{VV'} = GL(V_i) \times GL(V_i') \times \prod_{j \neq i} GL(V_j) \cong GL(V_i) \times GL(V_i') \times \prod_{j \neq i} GL(V_j'),$$

which acts on $Z_{VV'}$ naturally.

By (5), we have

$$\mathcal{D}_{G_{d}}(i_{E,d}) \xrightarrow{\alpha^*} \mathcal{D}_{G_{d}} Z_{VV'} \xrightarrow{\beta^*} \mathcal{D}_{G_{d}} (i_{E,d}').$$

Since $\alpha$ and $\beta$ are principal bundles with fibers $Aut(V_i')$ and $Aut(V_i)$ respectively, $\alpha^*$ and $\beta^*$ are equivalences of categories by Section 2.2.5 in [1]. Hence, for any $L \in \mathcal{D}_{G_{d}}(i_{E,d})$ there exists a unique $L' \in \mathcal{D}_{G_{d}}(i_{E,d}')$ such that $\alpha^*(L) = \beta^*(L')$. Define

$$\tilde{\omega}_d : \mathcal{D}_{G_{d}}(i_{E,d}) \rightarrow \mathcal{D}_{G_{d}}(i_{E,d}')$$

$$L \mapsto L'[-s(V)](-\frac{s(V)}{2})$$

where $s(V) = \dim GL(V_i) - \dim GL(V_i')$. Since $\alpha^*$ and $\beta^*$ are equivalences of categories, $\tilde{\omega}_d$ is also an equivalence of categories.

**Proposition 3.4.** It holds that $\tilde{\omega}_d (i_{Q,V}) = i_{Q,V'}$. 

The proof of Proposition 3.4 will be given in Section 4.3.

Hence, we can define $\tilde{\omega}_d : i_{Q,V} \rightarrow i_{Q,V'}$ and $\tilde{\omega}_d : K(i_{Q}) \rightarrow K(i_{Q'})$. We have the following theorem.
Theorem 3.5. We have the following commutative diagram

\[
\begin{array}{ccc}
  i^*f_A & \xrightarrow{T_i} & i^!f_A \\
  \downarrow i^*\lambda' & & \downarrow i^!\lambda' \\
  K(iQ) & \xrightarrow{\tilde{\omega}_i} & K(i'^!Q)
\end{array}
\]

The proof of Theorem 3.5 will be given in Section 4.3.

3.4. \(T_i : f \rightarrow i^!f\) and canonical bases. In [14], Lusztig showed that Lusztig’s symmetries and canonical bases are compatible. In this section, we shall give a geometric interpretation of this result by using the geometric realization of \(T_i\).

Let \(B\) be the canonical basis of \(f\). Since \(\theta_i f\) is the kernel of \(i^!\pi : f \rightarrow i^!f\) and \(B \cap \theta_i f\) is a \(Q(v)\)-basis of \(\theta_i f\), \(i^!B = i^!\pi(B)\) is a \(Q(v)\)-basis of \(i^!f\). Similarly, \(i^!B = i^!\pi(B)\) is a \(Q(v)\)-basis of \(i^!f\).

Lusztig proved the following theorem.

Theorem 3.6 ([14]). Lusztig’s symmetry \(T_i : f \rightarrow i^!f\) maps any element of \(i^!B\) to an element of \(i^!B\). Thus, there exists a unique bijection \(\kappa_i : B - B \cap \theta_i f \rightarrow B - B \cap f\theta_i\) such that \(T_i(i^!\pi(b)) = i^!\pi(\kappa_i(b))\).

Let \(i\) be a sink of a quiver \(Q\). So \(i\) is a source of \(Q' = \sigma_i Q\). By Theorem 3.1, Proposition 3.3, the formula (3) and the commutative diagram (4), we have

\[
i^!B = \bigcup_{\nu \in NI} \{ b_L = i^!\lambda_A([L]) \mid L \in i^!\mathcal{P}_V, \dim V = \nu \}.
\]

Similarly, we have

\[
i'^!B = \bigcup_{\nu' \in NI} \{ b_L = i'^!\lambda_A([L]) \mid L \in i'^!\mathcal{P}_{V'}, \dim V' = \nu' \}.
\]

Fix any \(\nu, \nu' \in NI\) such that \(\nu' = s_i \nu\) and \(I\)-graded \(K\)-vector spaces \(V, V'\) such that \(\dim V = \nu, \dim V' = \nu'\).

In (6), the functors \(\alpha^*\) and \(\beta^*\) are equivalences of categories. Hence the functor

\[
\tilde{\omega}_i : i^!Q_V \rightarrow i'^!Q_{V'}
\]

maps any simple perverse sheaf in \(i^!Q_V\) to a simple perverse sheaf in \(i'^!Q_{V'}\). That is, \(\tilde{\omega}_i(i^!P_V) = i'^!P_{V'}\). So the map

\[
\tilde{\omega}_i : K(i^!Q) \rightarrow K(i'^!Q)
\]

satisfies

\[
\tilde{\omega}_i([L] \mid L \in i^!\mathcal{P}_V) = [L] \mid L \in i'^!\mathcal{P}_{V'}\}
\]

By Theorem 3.5, (7) and (8), it holds that \(T_i(i^!B) = i'^!B\) and we get a geometric interpretation of Theorem 3.6.
4. Hall algebra approaches

4.1. Hall algebra approach to $f$. In this subsection, we shall review the Hall algebra approach to $f$ ([18] [15] [8] [9]).

Let $Q = (I, H, s, t)$ be a quiver. In Section 3.1 $E_V$ and $G_V$ are defined for any $I$-graded $\mathbb{K}$-vector space $V$. Let $F^n$ be the Frobenius morphism. The sets $E^F_V$ and $G^F_V$ consist of the $F^n$-fixed points in $E_V$ and $G_V$ respectively.

Lusztig defined $\mathcal{F}_V$ as the set of all $G^F_V$-invariant $\mathbb{Q}_f$-functions on $E^F_V$ and we can give a multiplication on $\mathcal{F}_V = \bigoplus_{v \in \mathbb{N}} \mathcal{E}_V^n$ to obtain the Hall algebra. For any $i \in I$, let $V_i$ be the $I$-graded $\mathbb{K}$-vector space with dimension vector $i$ and $f_i$ be the constant function on $E^F_V$ with value 1. Denote by $F^n$ the composition subalgebra of $\mathcal{F}_V$ generated by $f_i$ and $\mathcal{F}_V^n = \mathcal{F}_V \cap F^n$. Let $\mathcal{F} = \bigoplus_{v \in \mathbb{N}} \mathcal{F}_V$ be the generic form of $\mathcal{F}_V$ and $\mathcal{F}_A$ be the integral form of $\mathcal{F}$ ([15]).

Theorem 4.1 ([18] [15]). There exists an isomorphism of $\mathcal{A}$-algebras

$$ \varpi_A : \mathcal{F}_A \to \mathcal{F}_A $$

such that $\varpi_A(\theta_i) = f_i$.

For any $\mathcal{L} \in D_G(E_V)$, there is a function $\chi^n_\mathcal{L} : E^F_V \to \overline{\mathbb{Q}}_f$ (Section I.2.12 in [7]). Hence, we have the following map

$$ \chi^n : D_G(E_V) \to \mathcal{F}_V^n $$

$$ \mathcal{L} \mapsto \chi^n_\mathcal{L} $$

The restriction of this map on the subcategory $Q_V$ is also denoted by

$$ \chi^n : Q_V \to \mathcal{F}_V^n. $$

Lusztig proved that $\chi^n(Q_V) \subset \mathcal{F}_V^n$ in [15]. Hence, we can define $\chi^n : Q_V \to \mathcal{F}_V^n$, which induces $\chi : Q_V \to \mathcal{F}_V$ naturally. Hence, we get a map $\chi_A : K(Q) \to \mathcal{F}_A$.

Lusztig proved the following proposition.

Proposition 4.2 ([15]). $\chi_A : K(Q) \to \mathcal{F}_A$ is an isomorphism of $\mathcal{A}$-algebras such that $\chi_A([\mathcal{L}_i]) = f_i$ and the following diagram is commutative

$$ \xymatrix{ K(Q) \ar[r]^{\lambda_A} \ar[d]_{\chi_A} & \mathcal{F}_A \ar[dl]_{\varpi_A} \ar[d]_{\varpi_A} \\ \mathcal{F}_A & } $$

4.2. Hall algebra approaches to $i^f$ and $i^f$. Let $i$ be a sink of $Q$. In Section 3.2 $i^fE_V$ and $j_V : i^fE_V \to E_V$ are defined for any $I$-graded $\mathbb{K}$-vector space $V$. Similarly, $i^fE^n_V$ is defined as the $F^n$-fixed points set in $i^fE_V$ and we have $j_V : i^fE^n_V \to E^n_V$.

Lusztig also defined $i^f\mathcal{F}_V$ as the set of all $G^F_V$-invariant $\mathbb{Q}_f$-functions on $i^fE^n_V$. Similarly to the case in Section 4.1, the Hall algebra is denoted by $i^f\mathcal{F}_V = \bigoplus_{v \in \mathbb{N}} i^f\mathcal{F}_V^n$, the composition subalgebra is denoted by $i^f\mathcal{F} = \bigoplus_{v \in \mathbb{N}} i^f\mathcal{F}_V^n$ and the generic form is denoted by $i^f\mathcal{F} = \bigoplus_{v \in \mathbb{N}} i^f\mathcal{F}_V^n$. 

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Naturally, we have two maps \( j^*_V : \mathcal{F}_V \to i\mathcal{F}_V \) and \( j^*_V : i\mathcal{F}_V \to \mathcal{F}_V \). Considering all dimension vectors, we have \( j_1 : i\mathcal{F} \to \mathcal{F} \) and \( j^* : \mathcal{F} \to i\mathcal{F} \).

**Proposition 4.3** ([15]). We have the following commutative diagram
\[
\begin{array}{ccc}
  j^* & \cong & j^* \\
  \downarrow \cong & \cong & \downarrow \cong \\
  i\mathcal{F} & \xrightarrow{j_1} & i\mathcal{F} \\
\end{array}
\]

where \( i\varpi \) is the isomorphism induced by \( \varpi \).

Next, we shall prove Proposition 3.2.

For any \( L \in \mathcal{D}_G(iE_V) \), there is also a function \( \chi^n_L : iE^n_V \to \bar{Q}_l \). Hence, we have the following map
\[
i\chi^n : \mathcal{D}_G(iE_V) \to iE^n_V \\
\mathcal{L} \mapsto \chi^n_L
\]
The restriction of this map on the subcategory \( i\mathcal{Q}_V \) is also denoted by
\[
i\chi^n : i\mathcal{Q}_V \to iE^n_V.
\]

**Proposition 4.4.** It holds that \( i\chi^n(i\mathcal{Q}_V) \subset i\mathcal{F}_V^n \).

**Proof.** By the properties of \( \chi \) and \( i\chi \) (Theorem III.12.1(5) in [7]), we have the following commutative diagram
\[
\begin{array}{ccc}
  \mathcal{Q}_V & \xrightarrow{j_\mathcal{Q}_V} & i\mathcal{Q}_V \\
  \downarrow \chi^n & & \downarrow i\chi^n \\
  \mathcal{F}_V & \xrightarrow{j^*_V} & i\bar{Q}_V
\end{array}
\]

By the commutative diagram (9), \( j^*_V(\mathcal{Q}_V) \subset i\mathcal{F}_V^n \) and \( j^*_V(\mathcal{Q}_V) = i\mathcal{Q}_V \), we have \( i\chi^n(i\mathcal{Q}_V) \subset i\mathcal{F}_V^n \).

Hence, we can define \( i\chi^n : i\mathcal{Q}_V \to i\mathcal{F}_V^n \), which induces \( i\chi : i\mathcal{Q}_V \to i\mathcal{F}_V \) and \( i\chi_A : K(iQ) \to i\mathcal{F}_A \).

The commutative diagram (9) implies the following proposition.

**Proposition 4.5.** We have the following commutative diagram
\[
\begin{array}{ccc}
  K(Q) & \xrightarrow{j_*} & K(iQ) \\
  \downarrow \chi^A & & \downarrow i\chi_A \\
  \mathcal{F}_A & \xrightarrow{j^*} & i\mathcal{F}_A
\end{array}
\]
Proof of Proposition 3.2. First, we shall prove the following commutative diagram

Consider the following diagram

Since three squares and the triangle in the left are commutative, the triangle in the right is also commutative.

Proposition 4.3 implies that $i\varphi_A : iF_A \to i\lambda_A' iF_A$ is isomorphic. Hence $i\lambda_A' : iF_A \to K(iQ)$ is injective. The commutative diagram (4) in the definition of $i\lambda_A'$ implies $i\lambda_A' : iF_A \to K(iQ)$ is surjection. Hence, $i\lambda_A' : iF_A \to K(iQ)$ is isomorphic.

In the proof, we get the following proposition.

Proposition 4.6. We have the following commutative diagram

where all maps are isomorphisms of $A$-algebras and $i\lambda_A$ is the inverse of $i\lambda_A'$.

Assume that $i$ is a source of $Q$. The notations and results in this case are completely similar to the case that $i$ is a sink. We can define $iF^n = \bigoplus_{\nu \in \mathbb{N}} iF^n_{\nu}$ and $iF = \bigoplus_{\nu \in \mathbb{N}} iF_{\nu}$. We also have two maps $j^*_\nu : iF_{\nu} \to iF$ and $j^*_\nu : iF_{\nu} \to iF$. Considering all dimension vectors, we have $j_\nu : iF \to F$ and $j_\nu^* : F \to iF$.

Proposition 4.7 ([15]). We have the following commutative diagram
where $i\varphi$ is the isomorphism induced by $\varphi$.

We can also define $i\chi : iQ \rightarrow iF_V$ and $i\chi_A : K(iQ) \rightarrow iF_A$.

**Proposition 4.8.** We have the following commutative diagram

$$
\begin{array}{ccc}
K(Q) & \xrightarrow{j^*} & K(iQ) \\
\downarrow{\chi_A} & & \downarrow{i\chi_A} \\
F_A & \xrightarrow{j^*} & iF_A
\end{array}
$$

□

**Proposition 4.9.** We have the following commutative diagram

$$
\begin{array}{ccc}
K(iQ) & \xrightarrow{i\lambda_A} & iF_A \\
\downarrow{i\chi_A} & \nearrow{i\varphi_A} & \downarrow{\lambda} \\
iF_A & & iF_A
\end{array}
$$

where all maps are isomorphisms of $A$-algebras and $i\lambda_A$ is the inverse of $i\lambda'_A$.

4.3. Hall algebra approach to $T_i : iF \rightarrow iF$ and the proof of Theorem 3.5.

Let $i$ be a sink of a quiver $Q = (I, H, s, t)$. So $i$ is a source of $Q' = \sigma_i Q = (I, H', s, t)$. For any $\nu$ and $\nu' \in NI$ such that $\nu' = s_i \nu$, and two $I$-graded $K$-vector spaces $V$ and $V'$ such that $\dim V = \nu$ and $\dim V' = \nu'$, the following correspondence is considered in Section 3.3

$$iE_{V, Q} \xrightarrow{\alpha} Z_{VV'} \xrightarrow{\beta} iE_{V', Q'} .$$

Similarly, $Z_{VV'}^F$ is defined as the $F_n$-fixed points set in $Z_{VV'}$ and we have

$$iE_{V, Q}^F \xrightarrow{\alpha} Z_{VV'}^F \xrightarrow{\beta} iE_{V', Q'}^F .$$

Note that $\alpha$ and $\beta$ are principal bundles with fibers $Aut(V'_i)$ and $Aut(V'_i)$ respectively. Hence, for any $f \in iF^n_V$, there exists a unique $g \in iF^n_V$, such that $\alpha^*(f) = \beta^*(g)$. Define

$$\omega_i : iF^n_V \rightarrow iF^n_{V'},
\quad f \mapsto (p^n)^{-\frac{\dot{\lambda}(\nu)}{2}} g$$

Lusztig proved that $\omega_i(iF^n_V) \subset iF^n_{V'}$. Hence, we have $\omega_i : iF^n_V \rightarrow iF^n_{V'}$ and $\omega_i : iF_V \rightarrow iF_{V'}$. Considering all dimension vectors, we have $\omega_i : iF \rightarrow iF$.

Lusztig proved the following theorem.
Theorem 4.10 ([15]). We have the following commutative diagram

\[
\begin{array}{c}
\mathcal{F} \\
\downarrow \omega_i \\
\mathcal{F}
\end{array}
\quad \xrightarrow{egin{array}{c}iF \\ \downarrow i\omega \\
iF
\end{array}}
\begin{array}{c}
iF \\
\downarrow i\omega \\
iF
\end{array}
\quad \xrightarrow{egin{array}{c}T_i \\
\downarrow i\omega \\
iF
\end{array}}
\begin{array}{c}
iF \\
\downarrow i\omega \\
iF
\end{array}
\]

Proof of Proposition 3.4. By the properties of \(i\chi^n\) and \(i\chi^n\) (Theorem III.12.1(4,5) in [7]), we have the following commutative diagram

\[
\begin{array}{c}
\mathcal{D}_{G_V}(iE_V) \\
\downarrow i\chi^n \\
\mathcal{F}_V
\end{array}
\quad \xrightarrow{egin{array}{c}\tilde{\omega}_i \\
\downarrow \omega_i^n \\
\mathcal{F}_V
\end{array}}
\begin{array}{c}
i\mathcal{F}_V^n \\
\downarrow i\chi^n \\
i\mathcal{F}_V^n
\end{array}
\quad \xrightarrow{egin{array}{c}\omega_i^n \\
\downarrow \omega_i^n \\
i\mathcal{F}_V^n
\end{array}}
\begin{array}{c}
i\mathcal{F}_V^n \\
\downarrow i\chi^n \\
i\mathcal{F}_V^n
\end{array}
\]

Hence, we have

\[
\begin{array}{c}
\mathcal{D}_{G_V}(iE_V) \\
\downarrow \Pi_{n\in\mathbb{Z}_{\geq 1}} i\chi^n \\
\mathcal{F}_V^n
\end{array}
\quad \xrightarrow{egin{array}{c}\tilde{\omega}_i \\
\downarrow \Pi_{n\in\mathbb{Z}_{\geq 1}} \omega_i^n \\
\mathcal{F}_V^n
\end{array}}
\begin{array}{c}
i\mathcal{F}_V^n \\
\downarrow \Pi_{n\in\mathbb{Z}_{\geq 1}} i\chi^n \\
i\mathcal{F}_V^n
\end{array}
\quad \xrightarrow{egin{array}{c}\omega_i^n \\
\downarrow \Pi_{n\in\mathbb{Z}_{\geq 1}} \omega_i^n \\
i\mathcal{F}_V^n
\end{array}}
\begin{array}{c}
i\mathcal{F}_V^n \\
\downarrow \Pi_{n\in\mathbb{Z}_{\geq 1}} i\chi^n \\
i\mathcal{F}_V^n
\end{array}
\]

By Proposition 4.4, \(i\chi^n(iQ_V) \subset i\mathcal{F}_V^n\). Hence, we have

\[
\begin{array}{c}
iQ_V \\
\downarrow i\chi \\
iF_V
\end{array}
\quad \xrightarrow{egin{array}{c}\tilde{\omega}_i \\
\downarrow \omega_i \\
iF_V
\end{array}}
\begin{array}{c}
i\mathcal{F}_V^n \\
\downarrow \Pi_{n\in\mathbb{Z}_{\geq 1}} i\chi^n \\
i\mathcal{F}_V^n
\end{array}
\quad \xrightarrow{egin{array}{c}\omega_i \\
\downarrow \omega_i \\
iF_V
\end{array}}
\begin{array}{c}
i\mathcal{F}_V^n \\
\downarrow \Pi_{n\in\mathbb{Z}_{\geq 1}} i\chi^n \\
i\mathcal{F}_V^n
\end{array}
\]

Hence,

\[
(\prod_{n\in\mathbb{Z}_{\geq 1}} i\chi^n) \circ \tilde{\omega}_i(iQ_V) \subset \omega_i \circ i\chi(iQ_V).
\]

Since \(\omega_i(iF_V) \subset iF_V^n\),

\[
(\prod_{n\in\mathbb{Z}_{\geq 1}} i\chi^n) \circ \tilde{\omega}_i(iQ_V) \subset iF_V^n.
\]

For any two semisimple perverse sheaves \(\mathcal{L}\) and \(\mathcal{L}'\) in \(\mathcal{D}_{G_V}(iE_V)\) such that

\[
(\prod_{n\in\mathbb{Z}_{\geq 1}} i\chi^n)(\mathcal{L}) = (\prod_{n\in\mathbb{Z}_{\geq 1}} i\chi^n)(\mathcal{L}'),
\]

\(\mathcal{L}\) is isomorphic to \(\mathcal{L}'\) by Theorem III.12.1(3) in [7]. Since \((\prod_{n\in\mathbb{Z}_{\geq 1}} i\chi^n)(iQ_V) = iF_V^n\) and the objects in \(\tilde{\omega}_i(iQ_V)\) are semisimple, \(\tilde{\omega}_i(iQ_V) \subset iQ_V\). \(\square\)
Proposition 4.11. We have the following commutative diagram

\[
\begin{array}{ccc}
K(iQ) & \xrightarrow{\tilde{\omega}_i} & K(iQ) \\
\downarrow^{i\mathcal{X}} & & \downarrow^{i\mathcal{X}} \\
i\mathcal{F}_A & \xrightarrow{\omega_i} & i\mathcal{F}_A
\end{array}
\]

Proof. By the properties of \(i\chi_n\) and \(i\chi_n\), we have the following commutative diagram

\[
\begin{array}{ccc}
iQ & \xrightarrow{\tilde{\omega}_n} & iQ' \\
\downarrow^{i\chi_n} & & \downarrow^{i\chi_n} \\
i\mathcal{F}_V & \xrightarrow{\omega_n} & i\mathcal{F}_V'
\end{array}
\]

Hence, we get the commutative diagram in this proposition. \(\square\)

At last, Theorem 4.10 and Proposition 4.11 imply Theorem 3.5.

5. Projective resolutions of a kind of standard modules

5.1. KLR algebras. First let us review the definitions of KLR algebras (\cite{6, 23}).

Let \(Q = (I, H, s, t)\) be a quiver corresponding to the Lusztig’s algebra \(\mathfrak{f}\). Let \(\mathbb{K}\) be an algebraic closed field. Fix an \(I\)-graded \(\mathbb{K}\)-vector space \(V\) such that \(\text{dim} V = \nu \in NI\). In Section 3.1 the semisimple perverse sheaves \(L_i \in \mathcal{D}_{G\mu}(E_V)\) are defined for all \(i \in I^\nu\). Let \(\mathcal{L}_\nu = \bigoplus_{i \in I^\nu} L_i\).

The KLR algebra \(R_\nu\) is defined as

\[R_\nu = \bigoplus_{k \in \mathbb{Z}} \text{Ext}^k_{G\mu}(\mathcal{L}_\nu, \mathcal{L}_\nu).\]

\(R_\nu\) is a graded algebra and the degree of any element in \(\text{Ext}^k_{G\mu}(\mathcal{L}_\nu, \mathcal{L}_\nu)\) is \(k\).

Let \(R_\nu\)-gmod be the category of graded \(R_\nu\)-modules and \(R_\nu\)-proj be the category of finitely generated graded projective \(R_\nu\)-modules. Let \(K(R_\nu\)-proj) be the Grothendieck group of \(R_\nu\)-proj.

Define \(v^\pm [P] = [P[\pm 1]]\). So \(K(R_\nu\)-proj) is a free \(A\)-module. Define

\[K(R\text{-proj}) = \bigoplus_{\nu \in NI} K(R_\nu\text{-proj}).\]

For \(\nu, \nu', \nu'' \in NI\) such that \(\nu = \nu' + \nu''\) and three \(I\)-graded \(\mathbb{K}\)-vector spaces \(V, V', V''\) such that \(\dim V = \nu, \dim V' = \nu', \dim V'' = \nu''\), Khovanov and Lauda (\cite{6}) defined a functor

\[\text{Ind}_{\nu', \nu''} : R_{\nu'}\text{-proj} \times R_{\nu''}\text{-proj} \to R_{\nu}\text{-proj}.\]
which induces an $\mathcal{A}$-bilinear multiplication
\[
[\text{Ind}_{\nu'}^{\nu}]: K(R_{\nu'}\text{-proj}) \otimes_{\mathcal{A}} K(R_{\nu}\text{-proj}) \to K(R_{\nu}\text{-proj}).
\]
Khovanov and Lauda (6) proved that $K(R\text{-proj})$ becomes an associative $\mathcal{A}$-algebra.

For any $y \in Y_\nu$, let
\[
P_y = \bigoplus_{k \in \mathbb{Z}} \text{Ext}^k_{G\nu}(\mathcal{L}_y, \mathcal{L}_\nu).
\]

**Theorem 5.1** (6, 20). There is a unique isomorphism of $\mathcal{A}$-algebras
\[
\gamma_\mathcal{A}: f\mathcal{A} \to K(R\text{-proj})
\]
such that $\gamma_\mathcal{A}(\theta_y) = P_y$ for all $y \in Y_\nu$.

Let $B_\mathbb{Z} = \{v^sb \mid b \in B, s \in \mathbb{Z}\}$, which is a $\mathbb{Z}$-basis of $f\mathcal{A}$. Varagnolo, Vasserot and Rouquier proved the following theorem.

**Theorem 5.2** (23, 21). The map $\gamma_\mathcal{A}$ takes $B_\mathbb{Z}$ to the $\mathbb{Z}$-basis of $K(R\text{-proj})$ consisting of all indecomposable projective modules.

5.2. **Projective resolutions.** Let $i$ and $j$ be two vertices of the quiver $Q$ such that there are no arrows from $i$ to $j$. Let $N = \#\{j \to i\}$ and $m$ be a non-negative integer such that $m \leq N$. Let $\nu^{(m)} = mi + j \in N\mathbb{N}$. Fix an $I$-graded $\mathbb{K}$-vector space $V^{(m)}$ such that $\dim V^{(m)} = \nu^{(m)}$.

Denote by $1_{iE(V^{(m)})} \in D_{G(V^{(m)})}(iE(V^{(m)}))$ the constant sheaf on $iE(V^{(m)})$. The following functor is defined in Section 3.2
\[
j_{V^{(m)}}: D_{G(V^{(m)})}(iE(V^{(m)})) \to D_{G(V^{(m)})}(E(V^{(m)})).
\]
Define
\[
\mathcal{E}^{(m)} = j_{V^{(m)}}(v^{-mN}1_{iE(V^{(m)})}) \in D_{G(V^{(m)})}(E(V^{(m)}))
\]
and
\[
K_m = \bigoplus_{k \in \mathbb{Z}} \text{Ext}^k_{G(V^{(m)})}(\mathcal{E}^{(m)}, \mathcal{L}_{\nu^{(m)}}).
\]

$K_m$ is an object in $R_{\nu^{(m)}}\text{-gmod}$ for any $m$. Note that $K_m$ is a standard module in the sense of Kato (5). We shall give projective resolutions of these standard modules. For convenience, the complex $j_{V^{(m)}}(1_{iE(V^{(m)})}) \in D_{G(V^{(m)})}(E(V^{(m)}))$ is also denoted by $1_{iE(V^{(m)})}$.

For each $m \geq p \in \mathbb{N}$, consider the following variety
\[
\check{S}_p^{(m)} = \{(x, W) \mid x \in E(V^{(m)}), W \subset V_i, \dim(W) = p, \text{Im} \bigoplus_{h \in H, t(h) = i} x_h \subset W\}.
\]

Let $\pi_p: \check{S}_p^{(m)} \to E(V^{(m)})$ be the projection taking $(x, W)$ to $x$ and $S_p^{(m)} = \text{Im}\pi_p$.

By the definitions of $S_p^{(m)}$, we have
\[
E(V^{(m)}) = S_p^{(m)} \supset S_{m-1}^{(m)} \supset S_{m-2}^{(m)} \supset \cdots \supset S_0^{(m)}.
\]
For each $1 \leq p \leq m$, let

$$N_p^{(m)} = S_p^{(m)} \setminus S_{p-1}^{(m)}.$$  

Denote by $i_p^{(m)} : S_{p-1}^{(m)} \to S_p^{(m)}$ the close embedding and $j_p^{(m)} : N_p^{(m)} \to S_p^{(m)}$ the open embedding.

Define

$$I_p^{(m)} = (\pi_p)_!(1_{\tilde{S}_p^{(m)}})[\dim \tilde{S}_p^{(m)}].$$

In [13], Lusztig proved that $I_p^{(m)}$ are semisimple perverse sheaves in $\mathcal{D}_{G^{(m)}}(E_{V^{(m)}})$. Hence $I_p^{(m)}$ correspond to projective modules in $R_{\nu^{(m)}}$-proj.

The following theorem is the main result in this section.

**Theorem 5.3.** For $E^{(m)}$, there exists $s_m \in \mathbb{N}$. For each $s_m \geq p \in \mathbb{N}$, there exists $E_p^{(m)} \in D_{G^{(m)}}(E_{V^{(m)}})$ such that

1. $E_{s_m}^{(m)} = E^{(m)}$ and $E_{0}^{(m)}$ is the direct sum of some semisimple perverse sheaves of the form $I_p^{(m)}[l]$;
2. For each $p \geq 1$, there exists a distinguished triangle

$$E_p^{(m)} \to G_p^{(m)} \to E_{p-1}^{(m)},$$

where $G_p^{(m)}$ is the direct sum of some semisimple perverse sheaves of the form $I_p^{(m)}[l]$.

The proof of Theorem 5.3 will be given in Section 5.3.

Let

$$P_0^{(m)} = \bigoplus_{k \in \mathbb{Z}} \text{Ext}^k_{G^{(m)}}(E_0^{(m)}, \mathcal{L}_{\nu^{(m)}})$$

and

$$P_s^{(m)} = \bigoplus_{k \in \mathbb{Z}} \text{Ext}^k_{G^{(m)}}(G_s^{(m)}, \mathcal{L}_{\nu^{(m)}}) \quad (1 \leq s \leq m),$$

which are projective modules in $R_{\nu^{(m)}}$-proj.

As a corollary of Theorem 5.3, we have the following theorem.

**Theorem 5.4.** For any $N \geq m \in \mathbb{N}$, there exists a finite length projective resolution of $K_m$:

$$0 \longrightarrow P_0^{(m)} \longrightarrow P_1^{(m)} \longrightarrow \cdots \longrightarrow P_{s_m-1}^{(m)} \longrightarrow P_{s_m}^{(m)} \longrightarrow K_m \longrightarrow 0.$$

In the case of finite type, Kato proved that the projective dimension of any standard module is finite ([4, 5]). Theorem 5.3 show that the projective dimensions of a kind of standard modules are also finite in the general case.
5.3. The proof of Theorem 5.3. For convenience, a sheaf $A \in \mathcal{D}_G \mathcal{V}(m)(E \mathcal{V}(m))$ is called with Property $A(m)$, if $A$ satisfies the following conditions. There exists $s_A \in \mathbb{N}$. For each $s_A \geq p \in \mathbb{N}$, there exists $A_p \in \mathcal{D}_G \mathcal{V}(m)(E \mathcal{V}(m))$ such that

1. $A_{s_A} = A$ and $A_0$ is the direct sum of some semisimple perverse sheaves of the form $I_p^{(m)}[l]$;
2. for each $p \geq 1$, there exists a distinguished triangle

$$A_p \to G_p^A \to A_{p-1}$$

where $G_p^A$ is the direct sum of some semisimple perverse sheaves of the form $I_p^{(m)}[l]$.

Theorem 5.3 means that $E^{(m)}$ is with Property $A(m)$.

For the proof of Theorem 5.3, we need the following lemma.

Lemma 5.5. Fix any distinguished triangle

$$A \longrightarrow A' \longrightarrow A'' \longrightarrow,$$

where $A, A', A'' \in \mathcal{D}_G \mathcal{V}(m)(E \mathcal{V}(m))$. If $A$ and $A''$ are with Property $A(m)$, $A'$ is with Property $A(m)$ and $s_A = s_A + s_A'' + 1$.

Proof. We shall prove this lemma by induction on $s_A''$.

1. For $s_A'' = 0$, $A''$ is the direct sum of some semisimple perverse sheaves of the form $I_p^{(m)}[l]$. Let $A'_{s_A'} = A'$ and $A'_p = A_p[1]$ for any $0 \leq p \leq s_A = s_A' - 1$. Let $G_{s_A'} = A''$ and $G_p^{A'} = G_p^{A'}[1]$ for any $1 \leq p \leq s_A = s_A' - 1$. The distinguished triangle

$$A \longrightarrow A' \longrightarrow A'' \longrightarrow$$

implies

$$A'_{s_A'} \longrightarrow G_{s_A'}^{A'} \longrightarrow A'_{s_A' - 1}$$

and the distinguished triangles

$$A_p \longrightarrow G_p^A \longrightarrow A_{p-1}$$

imply

$$A'_p \longrightarrow G_p^{A'} \longrightarrow A'_{p-1}$$

for $1 \leq p \leq s_A' - 1$. Hence, $A'$ is with Property $A(m)$.

2. Assume that the lemma is true for $s_A'' < k$, we shall prove the lemma for $s_A'' = k$.

Now we have the following two distinguished triangles

$$A' \longrightarrow A'' \longrightarrow A[1]$$

and

$$A'' \longrightarrow G_k^{A''} \longrightarrow A''_{k-1}.$$
Then we can construct the following distinguished triangle

\[
\mathcal{A}' \xrightarrow{vu} G_k^{\mathcal{A}''} \longrightarrow \mathcal{B} \longrightarrow \mathcal{A}'' \xrightarrow{\text{id}} \mathcal{A}''_{k-1} \xrightarrow{\text{id}} \mathcal{B} \\
\]

By the octahedral axiom, there exist two maps \( f : \mathcal{A}[1] \to \mathcal{B} \) and \( g : \mathcal{B} \to \mathcal{A}_{k-1}'' \) such that the following diagram commutes and the third row is a distinguished triangle

\[
\begin{array}{c}
\mathcal{A}' \\
\mathcal{A}'' \\
\mathcal{A}[1] \\
\end{array} \xrightarrow{\text{id}} \begin{array}{c}
\mathcal{A}' \\
\mathcal{A}'' \\
\mathcal{A}[1] \\
\end{array} \xrightarrow{\text{id}} \begin{array}{c}
\mathcal{B} \\
\mathcal{A}_{k-1}'' \\
\mathcal{A}_{k-1}'' \\
\end{array} \xrightarrow{\text{id}} \begin{array}{c}
\mathcal{B} \\
\mathcal{A}_{k-1}'' \\
\mathcal{A}_{k-1}'' \\
\end{array} \\
\]

Consider the following distinguished triangle

\[
\mathcal{A}[1] \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{A}''_{k-1} \\
\]

Since \( \mathcal{A}' = \mathcal{A}'' \) is with Property \( \mathcal{A}(m) \), \( \mathcal{A}_{k-1}'' \) is also with Property \( \mathcal{A}(m) \) and \( s_{\mathcal{A}_{k-1}''} = k - 1 \). By the induction hypothesis, \( \mathcal{B} \) is with Property \( \mathcal{A}(m) \) and \( s_{\mathcal{B}} = s_{\mathcal{A}} + k \). Hence, for each \( s_{\mathcal{B}} \geq p \in \mathbb{N} \), there exists \( \mathcal{B}_p \in \mathcal{D}_{\mathcal{G}_V(m)}(E_{V(m)}) \) such that

1) \( \mathcal{B}_p = \mathcal{B} \) and \( \mathcal{B}_0 \) is the direct sum of some semisimple perverse sheaves of the form \( I_{p'}^m[l] \);
2) for each \( p \geq 1 \), there exists a distinguished triangle

\[
\begin{array}{c}
\mathcal{B}_p \\
\mathcal{G}^\mathcal{B}_p \\
\mathcal{B}_{p-1} \\
\end{array} \xrightarrow{\text{id}} \begin{array}{c}
\mathcal{B}_p \\
\mathcal{G}^\mathcal{B}_p \\
\mathcal{B}_{p-1} \\
\end{array} \xrightarrow{\text{id}} \begin{array}{c}
\mathcal{B}_p \\
\mathcal{G}^\mathcal{B}_p \\
\mathcal{B}_{p-1} \\
\end{array} \\
\]

where \( \mathcal{G}^\mathcal{B}_p \) is the direct sum of some semisimple perverse sheaves of the form \( I_{p'}^m[l] \).

Note that \( s_{\mathcal{A}} = s_{\mathcal{B}} + 1 \). Let \( \mathcal{A}'_{s_{\mathcal{A}}} = \mathcal{A}' \) and \( \mathcal{A}_p = \mathcal{B}_p \) for any \( 0 \leq p \leq s_{\mathcal{B}} = s_{\mathcal{A}} - 1 \). Let \( \mathcal{G}^\mathcal{A}_{s_{\mathcal{A}}} = \mathcal{G}^\mathcal{A} \) and \( \mathcal{G}^\mathcal{A}_p = \mathcal{G}^\mathcal{B}_p \) for any \( 1 \leq p \leq s_{\mathcal{B}} = s_{\mathcal{A}} - 1 \). The distinguished triangle

\[
\mathcal{A}' \xrightarrow{vu} G_k^{\mathcal{A}''} \longrightarrow \mathcal{B} \\
\]

implies

\[
\mathcal{A}'_{s_{\mathcal{A}}} \xrightarrow{vu} G_k^{\mathcal{A}''} \longrightarrow \mathcal{A}'_{s_{\mathcal{A}}-1} \\
\]

and the distinguished triangles

\[
\mathcal{B}_p \xrightarrow{\text{id}} \mathcal{G}^\mathcal{B}_p \longrightarrow \mathcal{B}_{p-1} \\
\]
For each $i$, there exists a distinguished triangle

$$v_{m}^{(m)}(\mathcal{L}_{(m-p)i} \otimes 1_{E_{(p)})}) \longrightarrow C_{p}^{(m)} \longrightarrow C_{p-1}^{(m)} \longrightarrow,$$

where $a_{p}^{(m)} = p(m - p) - mN$.

Proof. We shall construct $C_{p}^{(m)}$ for each $p$ by induction.

1. For $p = m$, let $C_{m}^{(m)} = I_{m}^{(m)}$. It is clear that $I_{m}^{(m)} \simeq v^{-mN} 1_{E_{(m)}}$, that is $C_{m}^{(m)} \simeq v^{a_{m}^{(m)}} 1_{E_{(m)}}$.

2. For each $p < m$, we shall construct $C_{p}^{(m)}$ and show that it satisfies the following conditions:

   1. there exists a distinguished triangle

   $$v^{a_{p+1}^{(m)}}(\mathcal{L}_{(m-p-1)i} \otimes 1_{E_{(p+1)})}) \longrightarrow C_{p+1}^{(m)} \longrightarrow C_{p}^{(m)} \longrightarrow;$$

   2. $C_{p}^{(m)} = \mathcal{L}_{(m-p)i} \otimes \hat{C}_{p}^{(m)}$, where $\hat{C}_{p}^{(m)} \in \mathcal{D}_{G_{(m)}}(E_{(m)})$.

First, we construct $C_{p}^{(m)}$ for $p = m - 1$. There is a distinguished triangle

$$(j_{m}^{(m)})_{!}(j_{m}^{(m)})_{!}(C_{m}^{(m)}) \longrightarrow C_{m}^{(m)} \longrightarrow (i_{m}^{(m)})_{*}(i_{m}^{(m)})_{*}(C_{m}^{(m)}) \longrightarrow.$$

Since $C_{m}^{(m)} \simeq v^{a_{m}^{(m)}} 1_{E_{(m)}}$,

$$(j_{m}^{(m)})_{!}(j_{m}^{(m)})_{!}(C_{m}^{(m)}) \simeq v^{a_{m}^{(m)}} 1_{E_{(m)}}$$

and

$$(i_{m}^{(m)})_{*}(i_{m}^{(m)})_{*}(C_{m}^{(m)}) \simeq v^{a_{m}^{(m)}} 1_{S_{m-1}^{(m)}}.$$

Let $C_{m-1}^{(m)} = (i_{m}^{(m)})_{*}(i_{m}^{(m)})_{*}(C_{m}^{(m)})$. By (10), there exists a distinguished triangle

$$v^{a_{m}^{(m)}} 1_{E_{(m)}} \longrightarrow C_{m}^{(m)} \longrightarrow C_{m-1}^{(m)} \longrightarrow.$$

Since the support of $C_{m-1}^{(m)}$ is in $S_{m-1}^{(m)}$, it can be wrote as

$$C_{m-1}^{(m)} = \mathcal{L}_{i} \otimes \hat{C}_{m-1}^{(m)}.$$
where \( C_{m-1}^{(m)} \in \mathcal{D}_{G_{V(m-1)}}(E_{V(m-1)}) \). We have \( C_{m-1}^{(m)} \simeq v^{a_{m}^{(m)}} S_{m-1}^{(m)} = v^{-mN} 1_{S_{m-1}^{(m)}} \). Hence
\[
v^{-(m-1)} C_{m-1}^{(m)} \simeq v^{-mN} 1_{E_{V(m-1)}},
\]
that is,
\[
\hat{C}_{m-1}^{(m)} \simeq v^{-mN} v^{m-1} 1_{E_{V(m-1)}} \simeq v^{a_{m-1}^{(m)}} 1_{E_{V(m-1)}}.
\]

Now, we have constructed \( C_{m-1}^{(m)} \) satisfying the following conditions:

1) there exists a distinguished triangle
\[
v^{a_{m}^{(m)}} 1_{E_{V(p)}} \longrightarrow C_{m}^{(m)} \longrightarrow C_{m-1}^{(m)} \longrightarrow ;
\]

2) \( C_{m-1}^{(m)} = L_{(m-p)} \hat{C}_{m-1}^{(m)}, \) where \( \hat{C}_{m-1}^{(m)} \in \mathcal{D}_{G_{V(m-1)}}(E_{V(m-1)}) \) and \( \hat{C}_{m-1}^{(m)} \simeq v^{a_{m-1}^{(m)}} 1_{E_{V(m-1)}} \).

(3) Assume that we have constructed \( C_{p}^{(m)} \) satisfying the following conditions:

1) there exists a distinguished triangle
\[
v^{a_{p}^{(m)}} (L_{(m-p)} 1_{E_{V(p+1)}}) \longrightarrow C_{p+1}^{(m)} \longrightarrow C_{p}^{(m)} \longrightarrow ;
\]

2) \( C_{p}^{(m)} = L_{(m-p)} \circ \hat{C}_{p}^{(m)}, \) where \( \hat{C}_{p}^{(m)} \in \mathcal{D}_{G_{V(p)}}(E_{V(p)}) \) and \( \hat{C}_{p}^{(m)} \simeq v^{a_{p}^{(m)}} 1_{E_{V(p)}} \).

We shall construct \( C_{p-1}^{(m)} \). First, there is a distinguished triangle
\[
(j_{p}^{(p)})^{t}(j_{p}^{(p)})^{t}(\hat{C}_{p}^{(m)}) \longrightarrow \hat{C}_{p}^{(m)} \longrightarrow (i_{p}^{(p)})^{t}(i_{p}^{(p)})^{t}(\hat{C}_{p}^{(m)}) \longrightarrow .
\]
Hence, we have
\[
L_{(m-p)} \circ (j_{p}^{(p)})^{t}(j_{p}^{(p)})^{t}(\hat{C}_{p}^{(m)}) \longrightarrow L_{(m-p)} \circ \hat{C}_{p}^{(m)}
\]
(11)
\[
\longrightarrow L_{(m-p)} \circ (i_{p}^{(p)})^{t}(i_{p}^{(p)})^{t}(\hat{C}_{p}^{(m)}) \longrightarrow .
\]
Since \( \hat{C}_{p}^{(m)} \simeq v^{a_{p}^{(m)}} 1_{E_{V(p)}} \),
\[
(j_{p}^{(p)})^{t}(j_{p}^{(p)})^{t}(\hat{C}_{p}^{(m)}) \simeq v^{a_{p}^{(m)}} 1_{E_{V(p)}},
\]
and
\[
(i_{p}^{(p)})^{t}(i_{p}^{(p)})^{t}(\hat{C}_{p}^{(m)}) \simeq v^{a_{p}^{(m)}} 1_{S_{p-1}^{(p)}} .
\]
Let \( C_{p-1}^{(m)} = L_{(m-p)} \circ (i_{p}^{(p)})^{t}(i_{p}^{(p)})^{t}(\hat{C}_{p}^{(m)}) \). By (11), there exists a distinguished triangle
\[
v^{a_{p}^{(m)}} (L_{(m-p)} 1_{E_{V(p)}}) \longrightarrow C_{p}^{(m)} \longrightarrow C_{p-1}^{(m)} \longrightarrow .
\]
Since the support of \( C_{p-1}^{(m)} \) is in \( S_{p-1}^{(m)} \), it can be wrote as
\[
C_{p-1}^{(m)} = L_{(m-p+1)} \circ \hat{C}_{p-1}^{(m)},
\]
where $C_{p-1}^{(m)} \in \mathcal{D}_{G_{\mathbf{V}(p-1)}}(E_{\mathbf{V}(p-1)})$. Since
\[
(\tilde{\psi}_p^{(p)})^* (\tilde{\psi}_p^{(p)})^* (\hat{C}_p^{(m)}) \simeq v^{a_p^{(m)}} 1_{S_{p-1}^{(p)}},
\]
we have
\[
C_{p-1}^{(m)} = \mathcal{L}_{(m-p)i} \otimes (\tilde{\psi}_p^{(p)})^*(\hat{C}_p^{(m)}) \simeq v^{-(m-p)p} v^{a_p^{(m)}} 1_{S_{p-1}^{(m)}} \simeq v^{-mN} 1_{S_{p-1}^{(m)}}.
\]
Hence
\[
v^{-(m-p+1)(p-1)} \hat{C}_{p-1}^{(m)} = v^{-mN} 1_{E_{\mathbf{V}(p-1)}},
\]
that is
\[
\hat{C}_{p-1}^{(m)} \simeq v^{-mN} v^{(m-p+1)(p-1)} 1_{E_{\mathbf{V}(p-1)}} \simeq v^{a_p^{(m)}} 1_{E_{\mathbf{V}(p-1)}}.
\]
Now, we have constructed $C_{p-1}^{(m)}$ satisfying the following conditions:

1) there exists a distinguished triangle
\[
v^{a_p^{(m)}} (\mathcal{L}_{(m-p)i} \otimes 1_{E_{\mathbf{V}(p)}}) \rightarrow C_{p-1}^{(m)} \rightarrow C_{p-1}^{(m)} \rightarrow \]

2) $C_{p-1}^{(m)} = \mathcal{L}_{(m-p+1)i} \otimes \hat{C}_{p-1}^{(m)}$, where $\hat{C}_{p-1}^{(m)} \in \mathcal{D}_{G_{\mathbf{V}(p-1)}}(E_{\mathbf{V}(p-1)})$ and $\hat{C}_{p-1}^{(m)} \simeq v^{a_p^{(m)}} 1_{E_{\mathbf{V}(p-1)}}$.

By induction, the proof is finished.

In Section 4.1 we have $\chi : K(Q) \rightarrow \mathcal{F}$. In this section, we identify the Lusztig’s algebra $\mathfrak{f}$ with the corresponding composition subalgebra $\mathcal{F}$.

Lusztig proved the following theorem.

**Theorem 5.7** ([13]). $\chi(I_{\mathbf{V}(m)}^{(m)}) = \theta_i^{(m-p)} \theta_j^{(p)}$ for each $m \geq p \in \mathbb{N}$.

By Proposition 5.6 and Theorem 5.7, we have the following corollary.

**Corollary 5.8.** We have the following formula in $\mathfrak{f}$

\[
\theta_j \theta_i^{(m)} = \sum_{p=0}^{m} v^{b_p^{(m)}} \theta_i^{(m-p)} \chi(\mathcal{E}(p)),
\]

where $b_p^{(m)} = (p - N)(m - p)$.

**Proof.** By Proposition 5.6 and Theorem 5.7, we have

\[
\theta_j \theta_i^{(m)} = \sum_{p=0}^{m} v^{a_p^{(m)}} \theta_i^{(m-p)} \chi(1_{E_{\mathbf{V}(p)}}) = \sum_{p=0}^{m} v^{a_p^{(m)}} v^{pN} \theta_i^{(m-p)} \chi(\mathcal{E}(p)).
\]

Since $a_p^{(m)} + pN = b_p^{(m)}$, we have

\[
\theta_j \theta_i^{(m)} = \sum_{p=0}^{m} v^{b_p^{(m)}} \theta_i^{(m-p)} \chi(\mathcal{E}(p)).
\]

□
We shall use Lemma 5.5 and Proposition 5.6 to prove Theorem 5.3 by induction.

**Proof of Theorem 5.3.** We shall prove this result by induction on $m$.

(1) For $m = 0$, $\mathcal{E}^{(0)} = I^{(0)}_0$. It is clear that $\mathcal{E}^{(0)}$ is with Property $A(0)$.

(2) For $m = 1$, by Proposition 5.6 there exists a distinguished triangle

$$v^{-N}1_{E\circ(1)} \rightarrow C^{(1)}_1 \rightarrow C^{(1)}_0 \rightarrow,$$

where $C^{(1)}_1 = I^{(1)}_1$ and $C^{(1)}_0 = v^{-N}(L \otimes 1_{E\circ(0)})$. Since $\mathcal{E}^{(0)} = I^{(0)}_0$,

$$C^{(1)}_0 = v^{-N}(L \otimes 1_{E\circ(0)}) = L \otimes \mathcal{E}^{(0)}$$

is the direct sum of some semisimple perverse sheaves of the form $I^{(1)}_{p'}[l]$. Hence, $\mathcal{E}^{(1)} = v^{-N}1_{E\circ(1)}$ is with Property $A(1)$.

(3) Assume the $\mathcal{E}^{(k)}$ is with Property $A(k)$ for all $k < m$. Let us prove $\mathcal{E}^{(m)}$ is with Property $A(m)$.

For any $k < m$, there exists $s_k \in \mathbb{N}$. For each $s_k \geq p \in \mathbb{N}$, there exists $\mathcal{E}_p^{(k)} \in \mathcal{D}_{G\circ(1)}(E\circ(k))$ such that

1) $\mathcal{E}_0^{(k)} = \mathcal{E}^{(k)}$ and $\mathcal{E}_0^{(k)}$ is the direct sum of some semisimple perverse sheaves of the form $I^{(k)}_{p'}[l]$;
2) for each $p \geq 1$, there exists a distinguished triangle

$$\mathcal{E}_p^{(k)} \rightarrow \mathcal{G}_p^{(k)} \rightarrow \mathcal{E}_{p-1}^{(k)} \rightarrow,$$

where $\mathcal{G}_p^{(k)}$ is the direct sum of some semisimple perverse sheaves of the form $I^{(k)}_{p'}[l]$.

Hence, we have the following distinguished triangle for each $p \geq 1$

$$L_{(m-k)i} \otimes \mathcal{E}_p^{(k)} \rightarrow L_{(m-k)i} \otimes \mathcal{G}_p^{(k)} \rightarrow L_{(m-k)i} \otimes \mathcal{E}_{p-1}^{(k)} \rightarrow.$$

Denote $\tilde{\mathcal{E}}_p^{(k)} = L_{(m-k)i} \otimes \mathcal{E}_p^{(k)}$ and $\tilde{\mathcal{G}}_p^{(k)} = L_{(m-k)i} \otimes \mathcal{G}_p^{(k)}$. Then, we have

$$\tilde{\mathcal{E}}_p^{(k)} \rightarrow \tilde{\mathcal{G}}_p^{(k)} \rightarrow \tilde{\mathcal{E}}_{p-1}^{(k)} \rightarrow.$$

Because $\tilde{\mathcal{E}}_0^{(k)}$ and $\tilde{\mathcal{G}}_0^{(k)}$ are the direct sums of some semisimple perverse sheaves of the form $I^{(k)}_{p'}[l]$, $\tilde{\mathcal{E}}_k^{(k)}$ is with Property $A(m)$. Since

$$\tilde{\mathcal{E}}_k^{(k)} = L_{(m-k)i} \otimes \mathcal{E}_k^{(k)} = v^{-kN}(L_{(m-k)i} \otimes 1_{E\circ(k)})$$

$L_{(m-k)i} \otimes 1_{E\circ(k)}$ is with Property $A(m)$.

By Proposition 5.6, for each $m \geq k \in \mathbb{N}$, there exists $\mathcal{C}_k^{(m)} \in \mathcal{D}_{G\circ(m)}(E\circ(m))$ such that

1) $\mathcal{C}_m^{(m)} = I^{(m)}_m$ and $\mathcal{C}_0^{(m)} = v^{-mN}(L_{mi} \otimes 1_{E\circ(0)})$;
2) for each $k \geq 1$, there exists a distinguished triangle

$$v^k_*(\mathcal{L}(m-k)L, \mathbb{1}_E) \to \mathcal{C}_k \to \mathcal{C}_{k-1}.$$ 

We have proved that $\mathcal{C}_0$ and $\mathcal{L}(m-k)L$ are with Property $A(m)$. Hence, by Lemma 5.5, $\mathcal{C}_m$ is with Property $A(m)$. At last, by Lemma 5.5 and the distinguished triangle

$$\mathcal{C}_{m-1} \rightarrow v^{-mN} \mathbb{1}_E \rightarrow \mathcal{I}_m,$$

$\mathcal{E}(m) = v^{-mN} \mathbb{1}_E$ is with Property $A(m)$.

By induction, the proof is finished. 

□

As a corollary of Theorem 5.3, we have

**Corollary 5.9.** For each $N \geq m \in \mathbb{N}$, we have the following formula

$$\chi(\mathcal{E}(m)) = \sum_{p=0}^{m} (-1)^p v^{-p(1+N-m)} \theta_j^{(p)} \theta_i^{(m-p)} = f(i, j; m).$$

**Proof.** By Theorem 5.3 we have

$$\chi(\mathcal{E}(m)) = \sum_{p=0}^{m} c_p^{(m)} \theta_j^{(p)} \theta_i^{(m-p)}.$$

We shall prove that $c_p^{(m)} = (-1)^p v^{-p(1+N-m)}$ ($0 \leq p \leq m$) by induction on $m$.

(1) For $m = 0$, by Corollary 5.8

$$\theta_j = \chi(\mathcal{E}(0)).$$

That is $c_0^{(0)} = 1$. Hence, the corollary is true in this case.

(2) Assume that $c_p^{(k)} = (-1)^p v^{-p(1+N-k)}$ ($0 \leq p \leq k$) for any $k < m$. We shall prove that $c_q^{(m)} = (-1)^q v^{-q(1+N-m)}$ ($0 \leq q \leq m$).

By Corollary 5.8

$$\theta_j \theta_i^{(m)} = \sum_{k=0}^{m} v^{k}_{*}\theta_i^{(m-k)} \chi(\mathcal{E}(k))$$

$$= \sum_{k=0}^{m-1} v^{k}_{*}\theta_i^{(m-k)} \chi(\mathcal{E}(k)) + v^m \theta_i \chi(\mathcal{E}(m))$$

$$= \sum_{k=0}^{m-1} v^{k}_{*}\theta_i^{(m-k)} \chi(\mathcal{E}(k)) + \chi(\mathcal{E}(m)).$$
Hence,

\[
\chi(\mathcal{E}^{(m)}) = \theta_j \theta_i^{(m)} - \sum_{k=0}^{m-1} v_k^{\ell(m)} \theta_i^{(m-k)} \chi(\mathcal{E}^{(k)})
\]

\[
= \theta_j \theta_i^{(m)} - \sum_{k=0}^{m-1} v_k^{\ell(m)} \theta_i^{(m-k)} \sum_{p=0}^{k} c_p^{(k)} \theta_i^{(p)} \theta_j^{(k-p)}
\]

\[
= \theta_j \theta_i^{(m)} - \sum_{k=0}^{m-1} \sum_{p=0}^{k} v_k^{\ell(m)} c_p^{(k)} \frac{(m-k+p)!}{(m-k)! (p)!} \theta_i^{(m-k+p)} \theta_j^{(k-p)}.
\]

For any \(q \geq 1\),

\[
c_q^{(m)} = - \sum_{k=m-q}^{m-1} v_k^{\ell(m)} c_{q+k-m}^{(k)} \frac{[q]_v!}{[m-k]_v! [q+k-m]_v!}.
\]

By the induction hypothesis,

\[
c_q^{(m)} = - \sum_{k=0}^{q-1} v_k^{\ell(m)} c_{k+m-q}^{(k+m-q)} \frac{[q]_v!}{[q-k]_v! [k]_v!}.
\]

\[
= - \sum_{k=0}^{q-1} (-1)^k v^{(k+m-q-N)(q-k)} v^{-k(1+N-k-m+q)} \frac{[q]_v!}{[q-k]_v! [k]_v!}.
\]

\[
= - v^{q(m-q-N)} \sum_{k=0}^{q-1} (-1)^k v^{(q-k-1)} \frac{[q]_v!}{[q-k]_v! [k]_v!}.
\]

\[
= - v^{q(m-q-N)} \sum_{k=0}^{q} (-1)^k v^{(q-k-1)} \frac{[q]_v!}{[q-k]_v! [k]_v!} + v^{q(m-q-N)} (-1) v^{q(q-1)}
\]

\[
= v^{q(m-q-N)} (-1) v^{q(q-1)} = (-1)^q v^{-q(1+N-m)}
\]

Note that

\[
c_0^{(m)} = 1 = (-1)^0 v^{-0(1+N-m)}.
\]

Hence, \(c_q^{(m)} = (-1)^q v^{-q(1+N-m)}\) for any \(0 \leq q \leq m\).

By induction, for each \(N \geq m \in \mathbb{N}\), \(c_p^{(m)} = (-1)^p v^{-p(1+N-m)}\) \((0 \leq p \leq m)\) and

\[
\chi(\mathcal{E}^{(m)}) = \sum_{p=0}^{m} (-1)^p v^{-p(1+N-m)} \theta_i^{(p)} \theta_j^{(m-p)}.
\]

\(\square\)
5.4. The formulas of Lusztig’s symmetries. In this section, we shall give a new proof of Proposition 2.1.

Consider the following quiver

\[ Q : i \xleftarrow{\beta} j \]

with vertex set \( I = \{i, j\} \) and \( N \) arrows from \( j \) to \( i \). Let \( Q' = \sigma_i Q \) be the quiver by reversing the directions of all arrows

\[ Q' : i \xrightarrow{\alpha} j \]

Let \( m \) be a non-negative integer such that \( m \leq N \) and \( m' = N - m \). Let \( \nu = mi + j \in NI \) and \( \nu' = si \nu = m'i + j \in NI \). Fix two \( I \)-graded \( \mathbb{K} \)-vector spaces \( V \) and \( V' \) such that \( \dim V = \nu \) and \( \dim V' = \nu' \).

Denote by \( 1_{i, E_V, Q} \in D_{G_V}(iE_V, Q) \) the constant sheaf on \( iE_V, Q \) and \( 1_{i, E_{V'}, Q'} \in D_{G_{V'}}(iE_{V'}, Q') \) the constant sheaf on \( iE_{V'}, Q' \). For convenience, denote \( iE_V \) (resp. \( iE_{V'} \)) by \( iE \) (resp. \( iE_{V'} \)) and \( 1_{i, E_V, Q} \) (resp. \( 1_{i, E_{V'}, Q'} \)) by \( 1_{iE} \) (resp. \( 1_{iE_{V'}} \)).

Denote

\[ \mathcal{E}^{(m)} = j_{VV!(v^{-mN}1_{i, E_V})} \in D_{G_V}(E_V) \]

and

\[ \mathcal{E}'^{(m')} = j_{VV!(v^{-m'N}1_{i, E_{V'}})} \in D_{G_{V'}}(E_{V'}). \]

In Section 5.3, we give the following geometric realization of the Lusztig’s symmetry \( T_i \):

\[ \tilde{\omega}_i : D_{G_V}(iE_V) \to D_{G_{V'}}(iE_{V'}). \]

**Proposition 5.10.** For any \( N \geq m \in \mathbb{N} \), \( \tilde{\omega}_i(v^{-mN}1_{i, E_V}) = v^{-m'N}1_{i, E_{V'}} \).

**Proof.** By the definitions of \( \alpha \) and \( \beta \) in the diagram (5) of Section 5.3

\[ \alpha^*(1_{i, E_V}) = 1_{Z_{VV'}} = \beta^*(1_{i, E_{V'}}). \]

Hence

\[ \tilde{\omega}_i(1_{i, E_V}) = v^{(m-m')N}1_{i, E_{V'}}. \]

That is

\[ \tilde{\omega}_i(v^{-mN}1_{i, E_V}) = v^{-m'N}1_{i, E_{V'}}. \]

\[ \square \]

Corollary 5.9 implies \( \chi(\mathcal{E}^{(m)}) = f(i, j; m) \). Similarly, we have \( \chi(\mathcal{E}'^{(m')}) = f'(i, j; m') \). Hence, Proposition 5.10 implies Proposition 2.1.
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