Distribution of resources for equipping handling fronts in rail consolidation terminals

A Ş Stere*, E A Roman

1Transport, Traffic and Logistics Department, Faculty of Transport, University POLITEHNICA of Bucharest, Spl. Independenţei, 313, Bucharest, Romania

*Corresponding author’s e-mail: armand_serban.stere@upb.ro

Abstract. An indispensable factor of the socio-economic life, the transports realize the connection between the economic zones, which facilitates the attraction of all the companies to the economic circuit, to the capitalization of the available resources and to their overall development. The influence of transport volumes is clearly manifested on the capacity of the terminals, the number of means of transport necessary to satisfy the demand, the degree of their use, the economy of the goods handling installations, the capacity of the handling fronts and labour productivity. Therefore, it is of a great importance to design the constructions and equipments of the terminals according to a criteria based on the optimal structure of the technical endowment subject to variations in the volume of transport activities. The present paper aims to present a theoretical framework for the distribution of resources for equipping handling fronts of the rail terminals. A mathematical model and a algorithm based on the projected gradient method are developed to solve this problem.

1. Introduction

Freight transport is a vital component of the national economy. It is the basis of production, trade, consumer activities, ensuring the movement and availability of raw materials and products [1]. The analysis of freight transport sector represents a subject characterized by a high degree of uncertainty. The complexity of this sector is given by the variety of commodities transported and by the here are real constraints due to the location of goods production activities. Because it is a constantly evolving sector, the phenomena related to the transport of different type of commodities are very important in all fields of study that deal with territorial structures and their transformations. In recent years, there has been an increase in the demand for services in the transport of goods in the Member States of the European Union, in close correlation with the increase in the level of consumption of goods and services in society [4].

Variations in transport volumes affect all aspects of the operation of road, rail, river, sea and air transport. A correct assessment of the size of the capacity required, for example, traffic sections and technical and shunting yards, their processing capacity, the capacity of the cargo handling equipment at the terminals is the basis for all necessary construction and installations. Consequently, the volume of investments and the terms of their recovery depend on it.

Therefore, the problem of the appropriate distribution of the resources available for the endowment of the handling furnaces within the consolidation terminals arises as one of great importance. This is a sizing problem of the constructions and installations that are subject to variable operating demands. These requests are undesirable because they involve maintaining additional capacity during peak hours. If the installation is designed according to the maximum load then most of the time...
it will not be used in full [5]. From this we can deduce that it is not indicated the design of the installations corresponding to the demand during the rush hours because the investments and expenses corresponding to the additional technical endowment will not be recovered.

Also, it is not recommended to design the constructions and installations from the endowment of the consolidation terminals to the minimum volume because, in this case, it is practically impossible to accomplish the operation process. Therefore, the aim will be to identify that economic criterion that can allow the determination of the optimal structure of the technical endowment in the case of machinery and installations subject to variations in the volume of transport activity.

2. Mathematical formulation of the problem

This category of problems generated by variations in the volume of transport activity also includes the problem of allocating available resources in order to endow the handling fronts of the consolidation terminals. This is an example of a non-linear programming problem and involves developing a perspective plan that can distribute these resources over a m period of years to equip a n number of machine handling fronts. The plan must be in direct and close accordance with the requirement that the level of expenditure on the handling of goods must be minimized. Therefore, it can be stated that the problem consists of the distribution of the investment fund on each individual handling front for each year of the planning period.

The following matrix corresponding to the distribution of the volume of goods at each of the n handling fronts during the m years of the planning period shall be considered [5]:

\[
Q = \begin{pmatrix}
Q_{11} & Q_{12} & \cdots & Q_{1n} \\
Q_{21} & Q_{22} & \cdots & Q_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{m1} & Q_{m2} & \cdots & Q_{mn}
\end{pmatrix}
\]

while the distribution of the fleet of equipment used to handle goods on the n fronts during the m years is given by a matrix of the form:

\[
X = \begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix}
\]

The number of machines in year I of the period from front k can be calculated with a relation of the form \( \alpha_{ik} = \frac{I_{ik}}{V_k} \) in which \( I_{ik} \) represents the value of the existing resources in year I to the handling front k while \( V_k \) represents the value of a machine with which front k is equipped.

The main objective is to obtain an expression of the optimization criterion \( C_{ik} \), taking into account all the elements of expenditure that are dependent on the value of existing resources in year I at the handling front k. This category includes the depreciation rate and the expenses related to the repair of the handling equipment and those of stationary of the wagons in order to carry out the loading-unloading operations of the goods. Therefore, the value of the optimization criterion \( C_{ik} \) is given by the following formula [5]:

\[
C_{ik} (I_{ik}) = \alpha_k I_{ik} + \frac{e_k \cdot Q_{ik}^2 \cdot V_k}{I_{ik} \cdot r_{ik} \cdot \eta_{ik}}
\]

where:
• $a_k$ - the annual rate of return on investment for equipment of type $k$;
• $e_k$ - the value equivalent of a wagon – stationary hour which is related to the ton of goods loaded in the wagon;
• $r_{ik}$ - the daily number of rounds in which the wagons from year $i$ are introduced at front $k$;
• $\eta_k$ - the operating productivity achieved in year $i$ by the machines from front $k$.

Next we will consider a simplifying hypothesis according to which a number of two manipulation fronts will be endowed. Therefore, the Eq. (3) of the optimization criterion can be written as follows [5]:

$$C_{ik}(I_{ik}) = \sum_{i=1}^{m} \sum_{k=1}^{2} C_{ik}(I_{ik}) = \sum_{i=1}^{m} \sum_{k=1}^{2} a_k I_{ik} + \sum_{i=1}^{m} \sum_{k=1}^{2} e_k \cdot Q_i^2 \cdot V_k$$

(4)

It must be taken into account the condition that, for each of the years of the investment planning period, the following identity exists [5]: $\sum_{i=1}^{m} I_i = I_{i1} + I_{i2}$ (where $I_i$ represents the investment made in year $i$ while terms $I_{i1}$ and $I_{i2}$ define the existing resources on the two fronts in year $i$).

To obtain a mathematical expression of the optimization criterion, the sum given by Eq. (4) will be divided into two amounts. By developing the two sums, we obtain that:

$$S_1 = \sum_{i=1}^{m} \sum_{k=1}^{2} a_k I_{ik} = \sum_{i=1}^{m} (a_1 I_{i1} + a_2 I_{i2}) = a_1 \sum_{i=1}^{m} I_{i1} + a_2 \sum_{i=1}^{m} I_{i2}$$

(5)

If we consider $\sum_{i=1}^{m} I_i = I_{11} + I_{21} + ... + I_{m1} + I_{12} + I_{22} + ... + I_{m2}$ we will obtain the following result: $mI_1 + (m-1)I_2 + ... + I_m = I_{11} + I_{21} + ... + I_{m1} + I_{12} + I_{22} + ... + I_{m2}$

By numerical substitution in the expression of the first sum, we get that: $S_1 = (a_1 - a_2) (I_{11} + I_{21} + ... + I_{m1}) + a_2 [mI_1 + (m-1)I_2 + ... + I_m]$.

In the case of the second sum, we have: $S_2 = \sum_{i=1}^{m} \sum_{k=1}^{2} e_k \cdot Q_i^2 \cdot V_k = \sum_{i=1}^{m} \left( e_1 \cdot Q_i^2 \cdot V_1 \left( \frac{Q_{i1}^2}{I_{i1} \cdot r_{i1} \cdot \eta_{i1}} + \frac{Q_{i2}^2}{I_{i1} \cdot r_{i1} \cdot \eta_{i12}} + ... + \frac{Q_{i2}^2}{I_{i1} \cdot r_{i1} \cdot \eta_{i22}} \right) \right) + e_2 \cdot V_2 \left( \frac{Q_{i2}^2}{I_{i2} \cdot r_{i2} \cdot \eta_{i12}} + ... + \frac{Q_{i2}^2}{I_{i2} \cdot r_{i2} \cdot \eta_{i22}} \right)$.

By making the necessary substitutions on the terms that quantify the value of the existing resources at the two fronts of manipulation in each of the $m$ years of the planning period, we obtain that:
Therefore, the optimization criterion takes the following form:

\[
S_2 = e_1 \cdot V_1 \cdot \left( \frac{Q_{11}^2}{I_{11} \cdot r_{11} \cdot \eta_{11}} + \frac{Q_{21}^2}{I_{21} \cdot r_{21} \cdot \eta_{21}} + \ldots + \frac{Q_{m1}^2}{I_{m1} \cdot r_{m1} \cdot \eta_{m1}} \right) + e_2 \cdot V_2 \cdot \left[ \frac{Q_{12}^2}{(I_1 - I_{11}) \cdot r_{12} \cdot \eta_{12}} + \frac{Q_{22}^2}{(I_1 + I_2 - I_{21}) \cdot r_{22} \cdot \eta_{22}} + \ldots + \frac{Q_{m2}^2}{\sum_{j=1}^{m} (I_j - I_{m1}) \cdot r_{m2} \cdot \eta_{m2}} \right]
\]  

(6)

The conditions that must be imposed on the problem are the following:

\[
\sum_{i=1}^{m} I_i \leq I \quad (8a)
\]

\[
I_{011} \leq I_{ii} \leq \sum_{j=1}^{m} I_j - I_{012} \quad (8b)
\]

\[
\begin{align*}
&I_{011} + I_{012} \leq I_1 \leq I \\
&0 \leq I_2 \leq I - (I_{011} + I_{012}) \\
&\ldots \\
&0 \leq I_i \leq I - (I_{0(i-1)1} + I_{0(i-1)2})
\end{align*}
\]

(8c)

where:

- \( I \) – the total value of the investment allocated for the \( m \) years for the \( k \) fronts;
- \( I_{0ik} \) – the minimum amount of resources at its disposal in year \( I \) for handling the quantity of commodities \( Q_{ik} \).

Condition (8a) requires that the investments made in each of these years not exceed the total value \( I \) of the investment allocated to those \( k \) handling fronts. The restriction (8b) takes into account the need to ensure mechanized handling of the volumes of goods from each of the \( k \) handling fronts. The set of restrictions (8c) aims to limit the investments that are made each year. The value of parameter \( I_{0ik} \) can be determined by using the bellow formula [5]:

\[
I_{0ik} = \frac{Q_{ik} \cdot V_k}{365 \cdot n_s \cdot t_s \cdot \eta_{ik} \cdot \tau_0}
\]  

(9)
in which \( n_j \) represents the number of shifts needed to work at the front, \( t_j \) quantify the duration of a shift (in hours) and \( \tau_0 \) is the working time utilization coefficient. As a result, the problem will be to establish the minimum value of the objective function given by relationship (7) that satisfies the constraints (8a) - (8c).

To identify the minimum expression of the optimization criterion given by Eq. (7) (which can be seen to be nonlinear), the projected gradient method (or Rosen method) can be used for solving. This method is used to obtain the solution of minimizing a convex function that is subject to linear constraints. It is an iterative method that projects the opposite direction to the gradient on active constraints \[2\]. In essence, the projected gradient method involves following three steps \[3\] that suppose to determine an initial solution \( x^* \) of the problem either by transforming the set of inequations (8a) – (8c) into equations or by using Simplex algorithm, to calculate the gradient of the functions \( \nabla C(x^*) \) and check the optimality conditions \( P_r \nabla C(x^*) = 0 \) and \( P_r = I - A_r (A_r A_r')^{-1} A_r \geq 0 \). In case both optimality conditions are satisfied, the algorithm stops and the initial solution is the optimal one.

On the contrary, it is necessary to go to the next step by improving the initial solution \( x^* \) with another solution \( x^k \). If \( x^k \) does not represent a solution after following a number of \( k \) steps, we will try to find a new \( x^{k+1} \) as follows :

- If \( P_r \nabla f(x^*) \neq 0 \), we will consider \( s = P_r \nabla f(x^*) \)
- If \( P_r \nabla f(x^*) = 0 \) but \( A_r (A_r A_r')^{-1} A_r \nabla f(x^*) \neq 0 \), we will consider \( s = P_{r-1} \nabla f(x^*) \)

With the new \( s \) solution found, we will determine \[3\] :

\[
\beta = \min_{\beta \geq 0} \left\{ \beta_j \left| \beta_j = -\frac{g_j(x^*)}{\alpha_j s}, j = r + 1, r + 2, ..., m \right. \right\} \quad (10 \text{ a})
\]

and we will note \( z = x^k + \beta \). If \( s' \nabla f(x^*) \geq 0 \) we shall consider \( x^{k+1} = z \). In the contrary case \( (s' \nabla f(x^*) \leq 0) \) we will consider:

\[
x^{k+1} = \frac{s' \nabla f(x^*)}{s' \nabla f(x^*) - s' \nabla f(z)} z + \left(1 - \frac{s' \nabla f(x^*)}{s' \nabla f(x^*) - s' \nabla f(z)}\right)x^k \quad (10 \text{ b})
\]

With the new \( x^{k+1} \) we shall return to the second step. The logical scheme of the algorithm for the optimal distribution of resources for endowment of the handling fronts in the consolidation terminals applying the projected gradient method is represented in figure 1.

3. Numerical example

In this chapter we will provide a numerical example of the mathematical model of resource allocation for the endowment of handling fronts in the consolidation terminals. We consider the following complex machining process of two handling fronts that extends over a period of 4 years. The available funds are worth \( I = 4200000 \) €. The following variables are also known :

- annual depreciation rate for equipment type \( k \) \( a_1 = 0.21 \) and \( a_2 = 0.15 \);
- the equivalent value of a wagon - stationary hour in relation to the ton of goods loaded in the wagon \( e_1 = 0.06 \) € and \( e_2 = 0.04 \) €;
- the value of a machine of the type with which the handling front \( k \) is equipped : \( V_1 = 600000 \) € and \( V_2 = 900000 \) €;
- the operating productivity that is achieved in year \( i \) by the existing equipment at front \( k \) \( q_{1i} = q_{2i} = 55 \) t/h and \( q_{12} = q_{22} = q_{13} = q_{23} = 65 \) t/h;
- working time utilization coefficient : \( \lambda_0 = 0.55 \).
Figure 1. Projected gradient method to solve the distribution of resources for equipping handling fronts in consolidation terminals

The quantities to be handled on two fronts are centralized in the following table. The terms $Q_{r1}$ and $Q_{r2}$ indicate the quantities of goods handled during a recovery on each of the two handling fronts.

|          | $Q_{r1}$ | $Q_{r2}$ |
|----------|----------|----------|
| $Q_{r1}$ |          |          |
| $Q_{r2}$ |          |          |
Based on the values in Table 1, the value of $I_{0ik}$, which represents the minimum level of resources available to one of the two handling fronts in year $i$, must be determined in order to be able to handle the quantities of goods.

**Table 1.** Quantities of commodities handled and minimum values of resources available on the two handling fronts

| Year $i$ | Front I | Front II |
|---------|---------|---------|
|         | $Q_{i1}$ $\times 10^{-3}$ | $Q'_{i1} = \frac{Q_{i1}}{\eta}$ | $Q_{i2}$ $\times 10^{-3}$ | $Q'_{i2} = \frac{Q_{i2}}{\eta}$ |
| 1       | 610     | 428     | 590     | 401     |
| 2       | 690     | 479     | 620     | 414     |
| 3       | 740     | 510     | 700     | 483     |
| 4       | 780     | 534     | 720     | 532     |

By applying Eq. (9), the below results will be achieved:

\[
I_{011} = \frac{610000 \cdot 6 \cdot 10^5}{365 \cdot 3 \cdot 1 \cdot 8 \cdot 55 \cdot 0.55} = 13.81 \cdot 10^3 \, \text{€} \quad I_{033} = \frac{740000 \cdot 6 \cdot 10^5}{365 \cdot 3 \cdot 8 \cdot 55 \cdot 0.55} = 16.76 \cdot 10^3 \, \text{€}
\]

\[
I_{012} = \frac{590000 \cdot 9 \cdot 10^5}{365 \cdot 3 \cdot 8 \cdot 65 \cdot 0.55} = 16.96 \cdot 10^3 \, \text{€} \quad I_{032} = \frac{700000 \cdot 9 \cdot 10^5}{365 \cdot 3 \cdot 8 \cdot 65 \cdot 0.55} = 20.12 \cdot 10^3 \, \text{€}
\]

\[
I_{021} = \frac{690000 \cdot 6 \cdot 10^5}{365 \cdot 3 \cdot 8 \cdot 55 \cdot 0.55} = 15.62 \cdot 10^3 \, \text{€} \quad I_{041} = \frac{780000 \cdot 6 \cdot 10^5}{365 \cdot 3 \cdot 8 \cdot 55 \cdot 0.55} = 17.67 \cdot 10^3 \, \text{€}
\]

\[
I_{022} = \frac{620000 \cdot 9 \cdot 10^5}{365 \cdot 3 \cdot 8 \cdot 65 \cdot 0.55} = 17.82 \cdot 10^3 \, \text{€} \quad I_{042} = \frac{770000 \cdot 9 \cdot 10^5}{365 \cdot 3 \cdot 8 \cdot 65 \cdot 0.55} = 22.13 \cdot 10^3 \, \text{€}
\]

Therefore, the minimum resources required in each of the four years are:

\[
I_{01} = I_{011} + I_{012} = 30.78 \cdot 10^3 \, \text{€} \quad I_{03} = I_{031} + I_{032} = 36.88 \cdot 10^3 \, \text{€}
\]

\[
I_{02} = I_{021} + I_{022} = 33.44 \cdot 10^3 \, \text{€} \quad I_{04} = I_{041} + I_{042} = 39.80 \cdot 10^3 \, \text{€}
\]

It can be seen that the allocated investment fund ensures the mechanized handling of the entire quantity of goods ($I_{04} < I$). Therefore, the expression of the optimization criterion for the two handling fronts considered and for a four-year investment duration is:

\[
C = 0.06 (I_{11} + I_{21} + I_{31} + I_{41}) + 0.15 (4I_1 + 3I_2 + 2I_3 + I_4) + \frac{121}{I_{11}} + \frac{156}{I_{21}} + \frac{179}{I_{31}} + \frac{200}{I_{41}}
\]

\[
= \frac{96}{I_1 - I_{11}} + \frac{107}{I_1 + I_2 - I_{21}} + \frac{136}{I_1 + I_2 + I_3 - I_{31}} + \frac{165}{I_1 + I_2 + I_3 + I_4 - I_{41}}
\]

We are interested in obtaining a minimum value of the optimization criterion with the following restrictions:
\[ \begin{align*}
I_1 + I_2 + I_3 + I_4 & \leq 42 \\
30.78 \leq I_1 & \leq 42 \\
0 \leq I_2 & \leq 11.22 \\
0 \leq I_3 & \leq 8.56 \\
0 \leq I_4 & \leq 5.12 \\
\end{align*} \]

To find the minimum of the expression \( C \) (which is nonlinear with respect to the variables \( I_i \) and \( I_{ik} \), we can use for solving the projected gradient method:

\[
\frac{\partial C}{\partial I_1} = 0.6 - \frac{96}{(I_1 - I_{11})^2} - \frac{107}{(I_1 + I_2 - I_{21})^2} - \frac{136}{(I_1 + I_2 + I_3 - I_{31})^2} - \frac{165}{(I_1 + I_2 + I_3 + I_4 - I_{41})^2} \\
\frac{\partial C}{\partial I_2} = 0.45 - \frac{96}{(I_1 - I_{11})^2} - \frac{107}{(I_1 + I_2 - I_{21})^2} - \frac{136}{(I_1 + I_2 + I_3 - I_{31})^2} - \frac{165}{(I_1 + I_2 + I_3 + I_4 - I_{41})^2} \\
\frac{\partial C}{\partial I_3} = 0.3 - \frac{96}{(I_1 - I_{11})^2} - \frac{107}{(I_1 + I_2 - I_{21})^2} - \frac{136}{(I_1 + I_2 + I_3 - I_{31})^2} - \frac{165}{(I_1 + I_2 + I_3 + I_4 - I_{41})^2} \\
\frac{\partial C}{\partial I_4} = 0.155 - \frac{96}{(I_1 - I_{11})^2} - \frac{107}{(I_1 + I_2 - I_{21})^2} - \frac{136}{(I_1 + I_2 + I_3 - I_{31})^2} - \frac{165}{(I_1 + I_2 + I_3 + I_4 - I_{41})^2} 
\]

It can be noticed that:

\[
\frac{\partial^2 C}{\partial I_1^2} = \frac{\partial^2 C}{\partial I_2^2} = \frac{\partial^2 C}{\partial I_3^2} = \frac{\partial^2 C}{\partial I_4^2} = \frac{192}{(I_1 - I_{11})^2} + \frac{214}{(I_1 + I_2 - I_{21})^2} + \frac{272}{(I_1 + I_2 + I_3 - I_{31})^2} + \frac{320}{(I_1 + I_2 + I_3 + I_4 - I_{41})^2} 
\]

It is necessary to specify the restrictions of the problem (corresponding to the investments made in each of the four years):

\[ \begin{align*}
I_1 & \leq 11.22 \\
I_2 & \leq 8.56 \\
I_3 & \leq 5.12 \\
I_4 - 30.78 & \leq 11.22 \\
I_1 - 30.78 & \leq 11.22 \\
I_1 & \leq 0 \\
I_2 & \leq 0 \\
I_3 & \leq 0 \\
I_4 & \leq 0 \\
\end{align*} \]

The system of restrictions can be written as a matrix that has the rank \( r = 4 \):

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

The gradient of the objective function \( C \) is:

\[
\nabla C = \begin{bmatrix}
\frac{\partial C}{\partial I_1} \\
\frac{\partial C}{\partial I_2} \\
\frac{\partial C}{\partial I_3} \\
\frac{\partial C}{\partial I_4}
\end{bmatrix} = \begin{bmatrix}
0.6 - \frac{96}{(I_1 - I_{11})^2} - \frac{107}{(I_1 + I_2 - I_{21})^2} - \frac{136}{(I_1 + I_2 + I_3 - I_{31})^2} - \frac{165}{(I_1 + I_2 + I_3 + I_4 - I_{41})^2} \\
0.45 - \frac{96}{(I_1 - I_{11})^2} - \frac{107}{(I_1 + I_2 - I_{21})^2} - \frac{136}{(I_1 + I_2 + I_3 - I_{31})^2} - \frac{165}{(I_1 + I_2 + I_3 + I_4 - I_{41})^2} \\
0.3 - \frac{96}{(I_1 - I_{11})^2} - \frac{107}{(I_1 + I_2 - I_{21})^2} - \frac{136}{(I_1 + I_2 + I_3 - I_{31})^2} - \frac{165}{(I_1 + I_2 + I_3 + I_4 - I_{41})^2} \\
0.155 - \frac{96}{(I_1 - I_{11})^2} - \frac{107}{(I_1 + I_2 - I_{21})^2} - \frac{136}{(I_1 + I_2 + I_3 - I_{31})^2} - \frac{165}{(I_1 + I_2 + I_3 + I_4 - I_{41})^2}
\end{bmatrix} \]

\[(12)\]
A point \( M_1(1,2,3,4) \) must be chosen so that the conditions are met at the limit, namely: 
\[
M_1 = \begin{bmatrix} 30.78 \\ 3.74 \\ 3.74 \\ 3.74 \end{bmatrix}
\]. We must also keep in mind that compliance with the condition is also necessary: 
\[
P_4 \cdot \nabla C(M_1) = 0.
\]
Because \( A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \) and the matrix transpose is \( A_4^t = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \), we will obtain that
\[
(P_4 \cdot A_4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow (A_4 \cdot A_4)^{-1} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.
\]
From this we can deduce that: 
\[
(A_4 \cdot A_4)^{-1} = I_4 = \Rightarrow P_4 = I_4 - (A_4 \cdot A_4)^{-1}
\]
\[
(A_4 \cdot A_4)^{-1} = I_4 - I_4 = 0 = \Rightarrow P_4 \cdot \nabla C(M_1) = 0,
\]
which means that the first condition is met. The second optimality condition that must be met is that the following product has all the positive components:
\[
(A_4 \cdot A_4)^{-1} A_4 \cdot \nabla C(M_1) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} -0.743 \\ -0.893 \\ -1.043 \\ -1.188 \end{bmatrix} = \begin{bmatrix} 0.445 \\ 0.295 \\ 0.145 \\ 0 \end{bmatrix}
\]
Because all values \( u_j \geq 0 \) (where \( j=1,...,5 \)), the algorithm stops at this step, which means that our basic initial solution \( M_1 = \begin{bmatrix} 30.78 \\ 3.74 \\ 3.74 \\ 3.74 \end{bmatrix} \) is also the optimal one. By numerical substitution in equation (11), we obtain that the expression of the optimization criterion for the two handling fronts considered and for an investment duration of four years is worth \( C_{\text{optim}} = 93,130 * 10^5 \text{€} \).

4. Conclusions

Due to trends in today’s transport sector that requires an adequate planning of sustainable logistics processes in order to achieve successful business strategies, a correct evaluation of the size of the elements underlying the design of all constructions and installations in the endowment of the terminals is necessary. In this paper, a mathematical framework for solving the problem of the distribution of resources for equipping handling fronts in rail consolidation terminals was developed.

The main objective of the problem is to solve the optimization criterion that takes into account the depreciation rate and the expenses related to the repair of the handling equipment and those of
stationary of the wagons in order to carry out the loading-unloading operations of the commodities. To find the minimum of the optimization criterion, the projected gradient method was used starting from an initial point. The coordinates were chosen so as to correspond to a set of restrictions based on the values of level of resources available to one of the three handling fronts in year \( i \) for handling a certain volume of commodity and the total value of the investment allocated for the \( m \) year for each of the fronts. The number of the iterations of the algorithm depends on the scale coefficients adopted.

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