Partially Optimal Edge Fault-Tolerant Spanners

Greg Bodwin$^1$  Michael Dinitz$^2$  Caleb Robelle$^3$

$^1$University of Michigan

$^2$Johns Hopkins University

$^3$MIT

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Graph Spanners: Basics

Definition

Given graph $G = (V, E)$, subgraph $H$ of $G$ is a $t$-spanner of $G$ if

$$d_H(u, v) \leq t \cdot d_G(u, v) \quad \text{for all } u, v \in V$$

- $t$ is the stretch of the spanner.
- In this paper: $G$ undirected, connected
- Sufficient for stretch condition to hold for all edges $\{u, v\} \in E$
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Main Theorem

Theorem (Althöfer et al ’93)

- For any positive integer $k$, all graphs have a $(2k - 1)$-spanner with $O(n^{1+1/k})$ edges, and
- There exist graphs in which all $(2k - 1)$-spanners have $\Omega(n^{1+1/k})$ edges (assuming Erdős Girth Conjecture).
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Upper bound statement existential, but actually algorithmic: greedy algorithm

\[
\begin{align*}
H & \leftarrow (V, \emptyset) \\
\text{for all } \{u, v\} \in E \text{ in nondecreasing weight order do} \\
& \quad \text{if } d_H(u, v) > (2k - 1) \cdot w(u, v) \text{ then} \\
& \quad \quad \text{add } \{u, v\} \text{ to } H \\
& \quad \text{end if} \\
\text{end for} \\
\text{return } H
\end{align*}
\]
Spanners For Distributed Systems

Go back to # edges.

**In Theory**, we’re done. We have a simple, optimal, textbook algorithm.

**In Practice**, spanners useful in many applications, but commonly used in distributed systems.

Imagine we build a 3-spanner of this network of computers, which need to talk to each other...
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**In Practice**, spanners useful in many applications, but commonly used in distributed systems.

Imagine we build a **3-spanner** of this network of computers, which need to talk to each other . . . but then one breaks.
A subgraph $H \subseteq G$ is an $f$-Edge Fault Tolerant (EFT) $(2k - 1)$-spanner of $G$ if, for every possible set $F$ of $|F| = f$ edges, we have

\[ H \setminus F \text{ is a } (2k - 1)\text{-spanner of } G \setminus F. \]

Equivalently: for all $u, v \in V$ and $F \subseteq E$ with $|F| \leq f$,

\[ d_{H \setminus F}(u, v) \leq (2k - 1) \cdot d_{G \setminus F}(u, v) \]

$f$-Vertex Fault Tolerant ($f$-VFT): $F \subseteq V$. 
Fault-Tolerant Spanners

Definition (Chechik, Langberg, Peleg, Roditty ’09)

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$\mathbf{f}$-Vertex Fault Tolerant ($\mathbf{f}$-VFT): $F \subseteq V$.

**Subtle definition:** $H$ only has to be “fault-tolerant” if $G$ is “fault-tolerant”

- Relative fault-tolerance
Fault-Tolerant Spanners

**Question:** How much “extra” above \( n^{1+1/k} \) do we need to pay for \( f \)-fault tolerance?
Fault-Tolerant Spanners

**Question:** How much “extra” above $n^{1+1/k}$ do we need to pay for $f$-fault tolerance?

Reasonable intuition:

- Natural approach: redundancy. Build a bunch of different spanners so that for all $F$, at least one spanner is unaffected.
- Needs at least $f + 1$ redundancy, pay extra factor of $f$. 

Theorem (Bodwin, Dinitz, Robelle '18)

Existential lower bounds on $f$-fault tolerance:

- $f$-VFT $(2k - 1)$-spanner: \(\Omega(1 - \frac{1}{k})n + 1/k)\) edges.
- For $k = 2$:
  \[
  f - EFT(2k - 1) - \text{spanner: } \Omega(f \left(1 - \frac{1}{k}\right)n^3) \text{ edges.}
  \]
- For $k \geq 3$:
  \[
  \Omega(f^2(1 - \frac{1}{k})n + fn) \text{ edges.}
  \]
Fault-Tolerant Spanners

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**Theorem (Bodwin, D, Parter, Vassilevska Williams ’18)**

*Existential lower bounds on $f$-fault tolerance:*

- **$f$-VFT $(2k - 1)$-spanner:** $\Omega \left(f^{1-1/k}n^{1+1/k}\right)$ edges.
- **$f$-EFT $(2k - 1)$-spanner:**
  - $k = 2$: $\Omega \left(f^{1-1/k}n^{1+1/k}\right) = \Omega \left(f^{1/2}n^{3/2}\right)$ edges.
  - $k \geq 3$: $\Omega \left(f^{\frac{1}{k}(1-1/k)}n^{1+1/k} + fn\right)$ edges.
### Vertex Fault-Tolerant Spanner Bounds

| Spanner Size                  | Runtime               | Greedy? | Citation       |
|------------------------------|-----------------------|---------|----------------|
| $\tilde{O} (k^{O(f)} \cdot n^{1+1/k})$ | $\tilde{O} (k^{O(f)} \cdot n^{3+1/k})$ |         | [CLPR '10]     |
| $\tilde{O} (f^{2-1/k} \cdot n^{1+1/k})$ | $\tilde{O} (f^{2-2/k} \cdot mn^{1+1/k})$ |         | [DK '11]       |
| $O (\exp(k)f^{1-1/k} \cdot n^{1+1/k})$ | $O (\exp(k) \cdot mn^{O(f)})$ | ✓       | [BDPV '18]     |
| $O (f^{1-1/k} \cdot n^{1+1/k})$ | $O (mn^{O(f)})$ | ✓       | [BP '19]       |
| $O (kf^{1-1/k} \cdot n^{1+1/k})$ | $\tilde{O} (f^{2-1/k} \cdot mn^{1+1/k})$ | ✓*      | [DR '20]       |
| $O (f^{1-1/k} \cdot n^{1+1/k})$ | $\tilde{O} (f^{1-1/k}n^{2+1/k} + mf^2)$ | ✓*      | [BDR '21]      |
Vertex Fault-Tolerant Spanner Bounds

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| $O\left(\exp(k)f^{1-1/k} \cdot n^{1+1/k}\right)$ | $O\left(\exp(k) \cdot mn^{O(f)}\right)$ | ✓ | [BDPV '18] |
| $O\left(f^{1-1/k} \cdot n^{1+1/k}\right)$ | $O\left(mn^{O(f)}\right)$ | ✓ | [BP '19] |
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| $O\left(f^{1-1/k} \cdot n^{1+1/k}\right)$ | $\tilde{O}\left(f^{1-1/k}n^{2+1/k} + mf^2\right)$ | ✓* | [BDR '21] |

So VFT essentially resolved!
Edge Fault-Tolerant Spanner Bounds

What about edge fault-tolerance?

- Lower bound: $\Omega \left( f^{1/k} \left( (1-1/k) n^{1+1/k} + fn \right) \right) \ [BDPV \ '18]$
- Upper bound: $O \left( f^{1-1/k} \cdot n^{1+1/k} \right) \ [BDR \ '21]$
Edge Fault-Tolerant Spanner Bounds

What about edge fault-tolerance?

- Lower bound: $\Omega \left( f^{1/(1-k)} n^{1+1/k} + fn \right)$ [BDPV '18]
- Upper bound: $O \left( f^{1-1/k} \cdot n^{1+1/k} \right)$ [BDR '21]

**Theorem**

*Every* $n$-node graph has an $f$-EFT $(2k - 1)$-spanner $H$ with

$$|E(H)| = \begin{cases} 
O \left( k^2 f^{1/2-1/(2k)} n^{1+1/k} + kfn \right) & \text{k is odd} \\
O \left( k^2 f^{1/2} n^{1+1/k} + kfn \right) & \text{k is even.}
\end{cases}$$
Edge Fault-Tolerant Spanner Bounds

dependence on $f$ (log$_f$ scale)
Greedy Fault-Tolerant Spanner Algorithm

Originally proposed by [Bodwin, D, Parter, Vassilevska Williams '18]

\[ H \leftarrow (V, \emptyset) \]

for all \( \{u, v\} \in E \) in nondecreasing weight order

do
  if there exists \( F \subseteq V \setminus \{u, v\} \) (for VFT) or \( F \subseteq E \) (for EFT) with \( |H| \leq f \) such that \( d_{H \setminus F}(u, v) > (2k - 1) \cdot w(u, v) \) then
    add \( \{u, v\} \) to \( H \)
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return \( H \)
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\[ k = 2, f = 1 \]
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**Intuition:** \( H \) should be “almost” high-girth

- How do we define “almost”?
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- How do we define “almost”?

**Structural:** There is a large high-girth subgraph in \( H \)

**Moore-like:** Suitable adaptations of the arguments for high-girth graphs also apply
Main Difficulty for Edge Fault-Tolerance

VFT: structural approach (BP ’19, BDR ’21)
  ▶ Greedy spanner “almost” high-girth because it has large high-girth subgraph
    ▶ Blocking sets (BP’19), or direct from algorithm (BDR ’21)

Problem: Can’t use this idea to get improved bounds for edge fault-tolerance!
  ▶ Bodwin-Patel showed can’t use blocking sets to get below $f^{1-1/k}n^{1+1/k}$
  ▶ (this paper, informal): if there was a structural argument, then it would imply the Erdős girth conjecture for $k = 7$ (currently unknown)
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**Approach:**
- *Strong* blocking sets
- More sophisticated version of Moore bounds on graphs with small strong blocking sets
Strong Blocking Sets

Definition (strong $t$-blocking set)

A *strong $t$-blocking set* of a graph $G = (V, E)$ is a set $B \subseteq E \times E$ where for every cycle $C$ in $G$ with $|C| \leq t$, there exists $(e, e') \in B$ such that:

- $e, e' \in C$ and $e \neq e'$, and
- Either $e$ or $e'$ is the *highest-weight* edge in $C$

If $G$ unweighted, “highest-weight” determined by ordering used by greedy algorithm.
**Strong Blocking Sets**

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If $G$ unweighted, “highest-weight” determined by ordering used by greedy algorithm.

**Lemma**

*The subgraph $H$ output by the greedy algorithm has a strong $2k$-blocking set of size at most $f|E(H)|$.*

**Proof sketch:** Same as non-strong lemma from Bodwin-Patel ’19
Moore Bounds

Can’t use structural approach – use strong blocking set for modified Moore bounds.

Theorem (Moore Bounds)

Any graph $G$ with girth at least $2k + 1$ has at most $O(n^{1+1/k})$ edges
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Theorem (Moore Bounds)

Any graph \( G \) with girth at least \( 2k + 1 \) has at most \( O(n^{1+1/k}) \) edges.

Counting Lemma: Let \( d \) be average degree of \( G \). Then \( G \) has at least \( \Omega(n \cdot d^k) \) simple \( k \)-paths.
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*Any graph* $G$ *with girth at least* $2k + 1$ *has at most* $O(n^{1+1/k})$ *edges*

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**Dispersion Lemma**: No two simple $k$-paths can share the same endpoints $\implies \leq n^2$ simple $k$-paths
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\[
n \cdot d^k \leq n^2 \implies d \leq n^{1/k} \implies |E| = nd/2 = O(n^{1+1/k})
\]
Generalized Dispersion Lemma

Greedy spanner $H$ has small strong blocking set.

Idea: For each $(x, y) \in S_{uv}$, inductively bound # of $u \rightarrow x$ paths and # of $y \rightarrow v$ paths.

To make induction work, helpful if heaviest edge not first or last, and for this to be true throughout induction.

Only count simple alternating $k$-paths: each even hop heavier than adjacent (odd) hops.
Generalized Dispersion Lemma

Greedy spanner $H$ has small strong blocking set.

$$\Rightarrow$$ for all $u, v \in V$, there is some set $S_{uv}$ of $O(kf)$ edges such that all simple $u - v$ $k$-paths use some edge of $S_{uv}$ as heaviest edge.

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Lemma (Generalized Dispersion)

For any nodes $u, v$, the number of simple alternating $u - v$ $k$-paths is

- $O(k^2 f)^{(k-1)/2}$ if $k$ is odd
- $O(k^2 f)^{k/2}$ if $k$ is even
Generalized Counting Lemma

Need to show there are many simple alternating $k$-paths.

Lemma:
Any graph with at least $k^n$ edges has at least one simple alternating $k$-path.

Proof:
Induction on $k$.

Lemma (Generalized Counting):
$H$ has $\Omega(n \cdot (d/\text{slash}k)^k)$ simple alternating $k$-paths.

Proof Sketch.
Sample nodes to get subgraph, use previous lemma to argue many simple alternating $k$-paths, scale back up.
Generalized Counting Lemma

Need to show there are many simple alternating $k$-paths.

**Lemma:** Any graph with at least $kn$ edges has at least one simple alternating $k$-path

**Proof:** Induction on $k$
Generalized Counting Lemma

Need to show there are many simple alternating $k$-paths.

**Lemma:** Any graph with at least $kn$ edges has at least one simple alternating $k$-path

**Proof:** Induction on $k$

**Lemma:** Any graph with at least $2kn$ edges has at least $kn$ simple alternating $k$-paths

**Proof:** Find a distinct simple alternating $k$-path for each “extra” edge
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**Lemma (Generalized Counting)**

$H$ has $\Omega(n \cdot (d/k)^k)$ simple alternating $k$-paths.
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**Proof Sketch.**

Sample nodes to get subgraph, use previous lemma to argue many simple alternating $k$-paths, scale back up.
Putting It Together

Count simple alternating $k$-paths.

**Odd $k$:**

$$\Omega(n \cdot (d/k)^k) = O\left(n^2 (k^2 f)^{(k-1)/2}\right)$$

$$\implies d/k = O\left(n^{1/k} (k^2 f)^{1/2 (1-1/k)}\right)$$

$$\implies |E(H)| = \frac{nd}{2} = O\left(k^2 f^{1/2 (1-1/k)} n^{1+1/k}\right)$$

**Even $k$:**

$$\Omega(n \cdot (d/k)^k) = O\left(n^2 (k^2 f)^{k/2}\right)$$

$$\implies d/k = O\left(n^{1/k} (k^2 f)^{1/2}\right)$$

$$\implies |E(H)| = \frac{nd}{2} = O\left(k^2 f^{1/2} n^{1+1/k}\right)$$
Final Notes

Optimal for odd constant $k$, off by $f^{1/(2k)}$ for even constant $k$.

Main open question: close gap for even $k$!
  - Essentially always see difference between even/odd stretch due to bipartiteness (hence why stretch is always $2k - 1$)
  - Rare (but not unheard of) to see difference between even/odd $k$.
  - What is the correct bound???

Also off by $k^2$, but WLOG $k \leq O(\log n)$. Still would like to get rid of $k$ factors!

Algorithm as stated takes exponential time!
  - Can turn into polytime using same idea as [D, Robelle PODC ’20]. Extra loss of $O(k^{1/2})$
Thanks!