ON THE QUANTITATIVE DYNAMICAL MORDELL-LANG CONJECTURE

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Abstract. The dynamical Mordell-Lang conjecture concerns the structure of the intersection of an orbit in an algebraic dynamical system and an algebraic variety. In this paper, we bound the size of this intersection for various cases when it is finite.

1. Introduction

1.1. Motivation. Let $X$ be an algebraic variety defined over the complex numbers $\mathbb{C}$, and let $\Phi : X \to X$ be a morphism. For any integer $n \geq 0$, we denote by $\Phi^{(n)}$ the $n$-th iteration of $\Phi$ with $\Phi^{(0)}$ denoting the identity map.

Throughout the paper, a single integer is viewed as an arithmetic progression with common difference 0.

The following is the so-called Mordell-Lang conjecture for self-morphisms of algebraic varieties in the dynamical setting; see [11, 16, 17].

Conjecture 1.1 (Dynamical Mordell-Lang Conjecture). Let $X$ and $\Phi$ be given as the above, let $V \subseteq X$ be a closed subvariety, and let $P \in X(\mathbb{C})$. Then, the following subset of integers

$$\{n \geq 0 : \Phi^{(n)}(P) \in V(\mathbb{C})\}$$

is a finite union of arithmetic progressions.

Conjecture 1.1 has been studied extensively in recent years. However, so far there are only a few related results. These include results on maps of various special types [4, 5, 7, 14, 16, 17, 23, 24] (especially diagonal maps), and analogues for Noetherian spaces [6] and Drinfeld modules [15].

Recently, Silverman and Viray [23, Corollary 1.4] have given results regarding the uniform boundedness (only in terms of $m$) of intersections of orbits of the power map (with the same exponent) at a point of the
projective $m$-space $\mathbb{P}^m(\mathbb{C})$ with non-zero multiplicatively independent coordinates, with any linear subspace of $\mathbb{P}^m(\mathbb{C})$. However, they have not provided quantitative results. In fact, such a result follows, even in a more general case, directly from the uniform bound on the number of zeros of simple and non-degenerate linear recurrence sequences.

We also note that the uniform boundedness condition has recently been considered in [10] to obtain, using an explicit version of Hilbert’s Nullstellensatz, several results for the problem of bounding the frequency of the points in an orbit of an algebraic dynamical system that belong to a given algebraic variety under the reduction modulo primes $p$.

In this paper, we study the quantitative version of Conjecture 1.1 for polynomial morphisms of several special types when $\mathcal{X}$ is the affine $m$-space $\mathbb{A}^m(\mathbb{C})$ and $V$ is a hypersurface; see Section 3. Our main objective is to find as many classes of polynomial morphisms as possible having uniform bounds (or as close as possible to uniformity), and not to investigate detailedly the quality of the bounds. To the best of our knowledge, this is the first work on the quantitative dynamical Mordell-Lang conjecture.

Here, we extend the results of [23] in two directions. First, we consider monomial systems with different exponents. Second, we estimate the size of the intersection of an orbit with a hypersurface rather than with a hyperplane.

1.2. Convention and notation. Let

$$\Phi = (F_1, \ldots, F_m) : \mathbb{A}^m(\mathbb{C}) \to \mathbb{A}^m(\mathbb{C}), \quad F_1, \ldots, F_m \in \mathbb{C}[X_1, \ldots, X_m],$$

be a morphism defined by a system of $m$ polynomials in $m$ variables over $\mathbb{C}$. For each $i = 1, \ldots, m$ we define the $n$-th iteration of the polynomials $F_i$ by the recurrence relation

$$F_i^{(0)} = X_i, \quad F_i^{(n)} = F_i \left( F_1^{(n-1)}, \ldots, F_m^{(n-1)} \right), \quad n = 1, 2, \ldots,$$

so that

$$\Phi^{(n)} = (F_1^{(n)}, \ldots, F_m^{(n)}).$$

See [3, 21, 22] for a background on dynamical systems associated with such iterations.

For a vector $w = (w_1, \ldots, w_m) \in \mathbb{C}^m$, we denote by

$$\text{Orb}_w(\Phi) = \{\Phi^{(n)}(w) : n = 0, 1, 2, \ldots\}$$

the orbit of $\Phi$ at the point $w$. For an algebraic variety $V = Z(G_1, \ldots, G_s)$ defined by the equations $G_1 = \cdots = G_s = 0$, $G_i \in \mathbb{C}[X_1, \ldots, X_m]$,
We consider the elements of the orbit $\text{Orb}_w(\Phi)$ which fall into $V$ and denote

\begin{equation}
S_w(\Phi, V) = \{ n \geq 0 : \Phi^{(n)}(w) \in V \}.
\end{equation}

We say that the complex numbers $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent if all of them are non-zero and there is no non-zero integer vector $(i_1, \ldots, i_n)$ such that $\alpha_1^{i_1} \cdots \alpha_n^{i_n} = 1$.

In the sequel, we denote by $|S|$ the cardinality of a finite set $S$. Our objective in this paper is to bound the size of $|S_w(\Phi, V)|$ for various cases when it is finite.

Throughout the paper, let $\overline{\mathbb{Q}}$ be the algebraic closure of the rational numbers $\mathbb{Q}$. For any field $K$, we write $K^*$ for the multiplicative group of all the non-zero elements of $K$. For any multiplicative group $\Lambda$ and any integer $k \geq 1$, let $\Lambda^k$ be the direct product consisting of $k$-tuples $x = (x_1, \ldots, x_k)$ with $x_i \in \Lambda, 1 \leq i \leq k$. As usual, the multiplication of the group $\Lambda^k$ is defined by $xy = (x_1y_1, \ldots, x_ky_k)$ for any $x, y \in \Lambda^k$.

2. Preliminaries

In this section, we gather some results which are used afterwards.

Recall that a linear recurrence sequence (LRS) of order $m \geq 1$ is a sequence $\{u_0, u_1, u_2, \ldots\}$ with elements in $\mathbb{C}$ satisfying a linear relation

\begin{equation}
\sum_{i=0}^m a_i u_{n+i} = 0 \quad (n = 0, 1, 2, \ldots),
\end{equation}

where $a_i \in \mathbb{C}, a_m \neq 0$ and $u_j \neq 0$ for at least one $j$ in the range $0 \leq j \leq m - 1$. We assume that relation (2.1) is minimal, that is the sequence $\{u_n\}$ does not satisfy a relation of type (2.1) of smaller length.

The characteristic polynomial of this LRS $\{u_n\}$ is

$$f(X) = X^m - a_1X^{m-1} - \cdots - a_m = \prod_{i=1}^k (X - \alpha_i)^{e_i} \in \mathbb{C}[X]$$

with distinct $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $e_i > 0$ for $1 \leq i \leq k$. Then, $u_n$ can be expressed as

$$u_n = \sum_{i=1}^k f_i(n)\alpha_i^n,$$

where $f_i$ is some polynomial of degree $e_i - 1 \ (i = 1, 2, \ldots, k)$. We call the sequence $\{u_n\}$ simple if $k = m$ (that is $e_1 = \cdots = e_m = 1$) and non-degenerate if $\alpha_i/\alpha_j$ is not a root of unity for any $i \neq j$ with $1 \leq i, j \leq k$. 
One fundamental problem of the LRS (2.1) is to describe the structure or bound the size of the following set
\[ \{ n \geq 0 : u_n = 0 \}, \]
which is called the zero set of the sequence (2.1). Equivalently, we want to study the integer roots of the exponential polynomial \( \sum_{i=1}^{k} f_i(z) \alpha_i^z \).

The well-known Skolem-Mahler-Lech theorem says that the zero set of any LRS is a finite union of arithmetic progressions, and furthermore it is a finite set if the sequence is non-degenerate; for example see [9, Theorem 2.1]. There are rich results on the quantitative version of this theorem. Here we restate some results in the setting of exponential polynomials, which are used later on.

In the rest of this section, we fix an exponential polynomial over \( \mathbb{C} \)
\[
F(z) = \sum_{i=1}^{k} f_i(z) \alpha_i^z
\]
with distinct \( \alpha_1, \ldots, \alpha_k \in \mathbb{C}^* \), and non-zero \( f_i \in \mathbb{C}[z] \) for \( 1 \leq i \leq k \). We also define
\[ m = \deg f_1 + \cdots + \deg f_k + k \]
and denote
\[ \mathcal{Z}(F) = \{ n \in \mathbb{Z} : F(n) = 0, n \geq 0 \}. \]

Note that \( F(z) \) corresponds to a LRS of order \( m \), and the set \( \mathcal{Z}(F) \) is exactly the zero set of the corresponding sequence. Especially, when \( f_1, \ldots, f_k \) are constants, \( F(z) \) corresponds to a simple LRS.

The following result comes from [1, Corollary 6.3] and [2, Theorem 1.1].

**Lemma 2.1.** Let \( F(z) \) be given by (2.2). Then the set \( \mathcal{Z}(F) \) is the union of at most \( \exp(\exp(70m)) \) arithmetic progressions. Moreover, if \( f_1, \ldots, f_k \) are non-zero constants, then the set \( \mathcal{Z}(F) \) is the union of at most \( (8m)^{4m^5} \) arithmetic progressions.

Lemma 2.1 can yield some quantitative results concerning Conjecture 1.1. However, here we are more interested with the case when the subset of integers in Conjecture 1.1 is a finite set.

As mentioned above, if \( F(z) \) corresponds to a non-degenerate LRS, the set \( \mathcal{Z}(F) \) is in fact a finite set, and furthermore we can bound the cardinality \( |\mathcal{Z}(F)| \). The following result follows from [1, Corollary 6.3] and [2, Theorem 1.2].

**Lemma 2.2.** Let \( F(z) \) be given by (2.2). Suppose that \( F(z) \) corresponds to a non-degenerate LRS, and \( \deg f_i + 1 \leq a \) for \( 1 \leq i \leq k \).
Then we have

$$|Z(F)| \leq (8k^a)^{8k^6a};$$

furthermore if $f_1, \ldots, f_k$ are non-zero constants, then we have

$$|Z(F)| \leq (8m)^{4m^5}.$$ 

In fact, if there exists some index $i$ such that the ratio $\alpha_i/\alpha_j$ is not a root of unity for any $j \neq i$, then the set $Z(F)$ is still a finite set; see [9, Theorem 2.1 (iii)]. Here we want to bound $|Z(F)|$ in this case by using Lemma 2.2 and following the arguments in [9].

**Lemma 2.3.** Let $F(z)$ be given by (2.2). Let $D$ be the order of the group of roots of unity generated by all those roots of unity which are of the form $\alpha_i/\alpha_j$ for some $1 \leq i, j \leq k$. Suppose that there exists some index $i_0$ such that the ratio $\alpha_{i_0}/\alpha_j$ is not a root of unity for any $j \neq i_0$, and $\deg f_i + 1 \leq a$ for $1 \leq i \leq k$. Then we have

$$|Z(F)| \leq D(8k^a)^{8k^6a}.$$ 

**Proof.** We partition $\alpha_1, \ldots, \alpha_k$ into equivalence classes according to the equivalence relation where $b \sim c$ if and only if the ratio $b/c$ is a root of unity. By renumbering, we can assume that $\alpha_1, \ldots, \alpha_s$ are representatives of these equivalence classes. Then, fixing an integer $b$ with $0 \leq b < D$, we consider the equation

$$F(b + nD) = 0$$

with integer unknown $n \geq 0$. By the choice of $D$, we can express $F(b + nD)$ as

$$F(b + nD) = \sum_{i=1}^{s} g_i(n)(\alpha_i^D)^n$$

for some polynomials $g_i \in \mathbb{C}[z]$ with $\deg g_i + 1 \leq a$. Under the assumption of $\alpha_{i_0}$, there indeed exists some index $j$ such that $g_j \not= 0$. So, using Lemma 2.2, we deduce that the cardinality of the set $\{n \geq 0 : F(b + nD) = 0\}$ is at most $(8k^a)^{8k^6a}$. The final result follows from the fact that there are $D$ choices of the integer $b$. □

Note that if $F(z)$ corresponds to a non-degenerate sequence, then $D = 1$ and Lemma 2.3 is exactly the first upper bound in Lemma 2.2.

Moreover, if $\alpha_1, \ldots, \alpha_k$ are roots of a polynomial $f(X)$ over a number field $K$, then the quantity $D$ can be bounded by

$$D < \exp \left( (1.05314 + \sqrt{6d}) \sqrt{m \log(dm)} \right),$$

where $d = [K : \mathbb{Q}]$ and $m = \deg f \geq 2$; see [12, Theorem 1].
Except for studying the set $\mathcal{Z}(F)$, we also need to estimate the number of integers $n \geq 0$ such that $F(n)$ is equal to a fixed non-zero complex number.

**Corollary 2.4.** Let $F(z)$ be given by (2.2). Define $\alpha_{k+1} = 1$. Suppose that there exists some index $i_0$ such that the ratio $\alpha_{i_0}/\alpha_j$ is not a root of unity for any $j \neq i_0$ with $1 \leq j \leq k + 1$. Let $D$ be the order of the group of roots of unity generated by all those roots of unity which are of the form $\alpha_i/\alpha_j$ for some $1 \leq i, j \leq k + 1$. Assume that $\deg f_i + 1 \leq a$ for $1 \leq i \leq k$. Then for any $\mu \in \mathbb{C}$ with $\mu \neq 0$, we have

$$|\{n \in \mathbb{Z} : F(n) = \mu, n \geq 0\}| \leq D(8(k + 1)^a)^{8(k+1)a};$$

furthermore if $F(z)$ corresponds to a non-degenerate LRS, no $\alpha_i$ ($1 \leq i \leq k$) is a root of unity, and $f_1, \ldots, f_r$ are non-zero constants, we have

$$|\{n \in \mathbb{Z} : F(n) = \mu, n \geq 0\}| \leq (8(m + 1))^{4(m+1)^5}.$$

**Proof.** Under the assumptions, we can get the desired results by applying Lemmas 2.2 and 2.3 to the following equation

$$\sum_{i=1}^{k} f_i(n)\alpha_i^n + (-\mu) \cdot 1 = 0, \quad n \geq 0,$$

with coefficients $f_1(n), \ldots, f_k(n), -\mu$. \hfill \Box

When $\alpha_1, \ldots, \alpha_k$ are algebraic numbers, the results in Lemma 2.3 and Corollary 2.4 can be improved in some sense. The following lemma is derived from [19, Theorem 1] with a slight refinement. Although the arguments in [19] were presented only for non-degenerate sequences, they are also valid for more general cases under minor changes.

**Lemma 2.5.** Let $F(z)$ be given by (2.2). Suppose that $\alpha_1, \ldots, \alpha_k$ are algebraic numbers, and let $D$ be the order of the group of roots of unity generated by all those roots of unity which are of the form $\alpha_i/\alpha_j$ for some $1 \leq i, j \leq k$. Denote $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_k)$, and assume that $f_1, \ldots, f_k \in K[z]$. Let $d = [K : \mathbb{Q}]$, and let $\omega$ be the number of prime ideals occurring in the decomposition of the fractional ideals $\langle \alpha_i \rangle$ in $K$. Assume that there exists some index $i_0$ such that the ratio $\alpha_{i_0}/\alpha_j$ is not a root of unity for any $j \neq i_0$. Then, we have

$$|\mathcal{Z}(F)| < D(4(d + \omega))^{2(d+1)}(m - 1);$$

furthermore if $K/\mathbb{Q}$ is a Galois extension but not a cyclic extension, we have

$$|\mathcal{Z}(F)| < D(4(d + \omega))^{d+2}(m - 1).$$
Proof. Here, we sketch the proof for the convenience of the reader.

We first choose a rational prime \( p \) such that none of the prime ideals \( p_1, \ldots, p_\omega \) from the decomposition in \( K \) of the ideals \((\alpha_i) (1 \leq i \leq k)\) divides the ideal \( (p) \). Let \( p \) be a prime ideal of \( K \) lying above \( p \), and let \( f \) denote the residue class degree of \( K_p \) over \( \mathbb{Q}_p \), where \( \mathbb{Q}_p \) is the \( p \)-adic completion of \( \mathbb{Q} \) and \( K_p \) is the completion of \( K \) with respect to \( p \). Let \( C_p \) be the completion of the algebraic closure of \( \mathbb{Q}_p \). We denote the valuation of \( C_p \) by \( |\cdot|_p \), normalised such that \( |p|_p = p^{-1} \). Note that \( \mathbb{Q}_p \subseteq K_p \subseteq C_p \).

By the choice of \( p \), we have

\[
|\alpha_i|_p = 1, \quad i = 1, \ldots, k.
\]

Furthermore, by [19, Equation (3.4)] we know

\[
|\alpha_i^{p^{f-1}} - 1|_p < p^{-1/(p(p-1)) - 1/(p-1)}, \quad i = 1, \ldots, k.
\]

Then, we have

\[
|\alpha_i^{D(p^{f-1}) - 1}|_p \leq |\alpha_i^{p^{f-1} - 1}|_p < p^{-1/(p(p-1)) - 1/(p-1)}, \quad i = 1, \ldots, k.
\]

Fix an integer \( a \) with \( 0 \leq a < D(p^f - 1) \), we consider the equation

\[
F(a + nD(p^f - 1)) = 0
\]

with integer unknown \( n \geq 0 \).

As in the proof of Lemma 2.3, we partition \( \alpha_1, \ldots, \alpha_k \) into equivalence classes, and we assume that \( \alpha_1, \ldots, \alpha_s \) are representatives of these equivalence classes. Then, by the choice of \( D \), we can express \( F(a + nD(p^f - 1)) \) as

\[
F(a + nD(p^f - 1)) = \sum_{i=1}^{s} g_i(n)(\alpha_i^{D(p^{f-1})})^n
\]

for some polynomials \( g_i \in K[z] \). Under the assumption of \( \alpha_{i_0} \), there indeed exists some index \( j \) such that \( g_j \neq 0 \).

As solving the equation (3.6) of [19], we immediately see that the cardinality of the set \( \{n \in \mathbb{Z} : F(a + nD(p^f - 1)) = 0, n \geq 0\} \) is at most \((m - 1)(p + 1)\). Thus, we obtain

\[
|Z(F)| \leq D(p^f - 1)(m - 1)(p + 1).
\]

From [19, Section 4], the prime \( p \) can be chosen such that

\[
p < (4(d + \omega))^2.
\]

Then, the first desired upper bound follows from the fact that \( f \leq d \).

Now, we assume that \( K/\mathbb{Q} \) is a Galois extension but not a cyclic extension. In order to prove the second claimed upper bound, it suffices
to show that \( p \) does not remain inert in \( K \). Because if this is true, then \( f \leq d/2 \), which can conclude the proof.

Let \( D_p \) denote the decomposition group of \( p \) in \( K/Q \). Suppose that \( p \) remains inert in \( K \). Then, \( f = d \), and \( D_p \) is a cyclic group of order \( d \). Since \( [K : Q] = d \), \( D_p \) is exactly the Galois group \( \text{Gal}(K/Q) \). So, \( K/Q \) is a cyclic extension, this leads to a contradiction. \( \square \)

Applying the same arguments as in the proof of Corollary 2.4, we can obtain similar results as Lemma 2.5 for the cardinality \( \{ n \in \mathbb{Z} : F(n) = \mu, n \geq 0 \} \), where \( \mu \) is a non-zero algebraic number.

We also need a result on solutions of linear equations in several variables. The following result is given in [2, Lemma 2.1] and is derived from [1, Theorem 6.2].

**Lemma 2.6.** Let \( \Gamma \) be finitely generated subgroup of \((\mathbb{C}^*)^k\) of rank \( r \), and let \( a_1, \ldots, a_k \in \mathbb{C}^* \). Then, up to a factor of proportionality, the equation

\[
(2.4) \quad a_1x_1 + \cdots + a_kx_k = 0
\]

has less than \((8k)^{4(k-1)^4(k+r)}\) non-degenerate solutions in \( \Gamma \).

Here, we call a solution of (2.4) non-degenerate if no subsum of the left hand side of the equation vanishes. Let \( K \) be a number field, let \( S \) be a finite set of places of \( K \) containing all the archimedean ones and write \( \mathcal{O}_S^* \) for the group of \( S \)-units of \( K \). If the coefficients \( a_1, \ldots, a_k \in K \setminus \{0\} \), then the number of such solutions of (2.4) in \( \Gamma \subseteq (\mathcal{O}_S^*)^k \) can be bounded better than Lemma 2.6; for example see [13, Theorem 3]. So, some results in this paper can be improved in this case.

Now, we want to count all the solutions of (2.4) up to a factor of proportionality.

**Corollary 2.7.** Let \( \Gamma \) be finitely generated subgroup of \((\mathbb{C}^*)^k\) of rank \( r \), and let \( a_1, \ldots, a_k \in \mathbb{C}^* \). Then, up to a factor of proportionality, the equation

\[
(2.5) \quad a_1x_1 + \cdots + a_kx_k = 0
\]

has less than

\[
(0.792k)^k(8k)^{4(k-1)^4(k+r)}
\]

solutions in \( \Gamma \).

**Proof.** Let \( \mathcal{P} \) be a partition of the set \( I = \{1, \ldots, k\} \). The subsets \( \lambda \subseteq I \) occurring in the partition \( \mathcal{P} \) are considered as elements of \( \mathcal{P} \).
Then, the system of equations

\[(2.4 \ P) \quad \sum_{i \in \lambda} a_i x_i = 0 \quad (\lambda \in \mathcal{P})\]

is a refinement of \( (2.5) \). Note that in order to ensure that the system \( (2.4 \ P) \) has solutions in \( \Gamma \) we must have that \( |\lambda| \geq 2 \) for any \( \lambda \in \mathcal{P} \). So, we can assume that \( |\mathcal{P}| \leq k/2 \).

If \( \mathcal{Q} \) is another partition of \( I \) such that \( \mathcal{Q} \) is a refinement of \( \mathcal{P} \), then the equation \( (2.4 \ Q) \) implies \( (2.4 \ P) \). Let \( \mathcal{T}(\mathcal{P}) \) consist of solutions of \( (2.4 \ P) \) in \( \Gamma \) up to a factor of proportionality, which are not solutions of any \( (2.4 \ Q) \) where \( \mathcal{Q} \) is a proper refinement of \( \mathcal{P} \).

According to the partition \( \mathcal{P} \), we can treat \( \Gamma \) as a direct product

\[\Gamma = \prod_{\lambda \in \mathcal{P}} \Gamma(\lambda),\]

where \( \Gamma(\lambda) \) is the projection of \( \Gamma \) corresponding to \( \lambda \). For each \( \lambda \in \Gamma \), \( \Gamma(\lambda) \) is also a finitely generated group and let \( r(\lambda) \) be its rank. Obviously, we have

\[\sum_{\lambda \in \mathcal{P}} r(\lambda) = r.\]

For each equation in \( (2.4 \ P) \)

\[\sum_{i \in \lambda} a_i x_i = 0,\]

by Lemma 2.6 it has less than \((8|\lambda|)^{4(|\lambda|-1)^4(|\lambda|+r(\lambda))}\) non-degenerate solutions in \( \Gamma(\lambda) \) up to a factor of proportionality. Thus, we have

\[|\mathcal{T}(\mathcal{P})| < \prod_{\lambda \in \mathcal{P}} (8|\lambda|)^{4(|\lambda|-1)^4(|\lambda|+r(\lambda))}
\leq (8k)^{4(k-1)^4(k+r)}.

Recall that the Bell numbers count the number of partitions of a set. By [8, Theorem 2.1], the number of partitions of \( I \) is less than

\[(0.792k/ \log(k + 1))^k.\]

Note that every solution of the equation \( (2.5) \) lies in \( \mathcal{T}(\mathcal{P}) \) for some \( \mathcal{P} \), and we can assume that \( k \geq 2 \). So, up to a factor of proportionality, the number of solutions of \( (2.5) \) in \( \Gamma \) is at most

\[\sum_{\mathcal{P}} |\mathcal{T}(\mathcal{P})| < (0.792k/ \log(k + 1))^k \cdot (8k)^{4(k-1)^4(k+r)}
\leq (0.792k)^k (8k)^{4(k-1)^4(k+r)},\]

which completes the proof. \( \square \)
Finally, we state a result due to Schlickewei and Schmidt [20, Theorem 1]. For a vector \( \mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{Z}^m \) and \( \mathbf{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \overline{\mathbb{Q}}^m \), we denote
\[
\mathbf{\alpha}^{\mathbf{x}} = \alpha_1^{x_1} \cdots \alpha_m^{x_m}.
\]

Lemma 2.8. Fix \( \alpha_i \in (\overline{\mathbb{Q}}^*)^m, i = 1, \ldots, k \), such that for any \( 1 \leq i \neq j \leq k \) the set of \( z \in \mathbb{Z}^m \) with \( \mathbf{\alpha}^z_i = \mathbf{\alpha}^z_j \) contains only the zero vector. Then, the number of solutions to the equation
\[
\sum_{i=1}^{k} a_i \mathbf{\alpha}^x_i = 0
\]
with non-zero algebraic numbers \( a_i \) is less than
\[
(0.792k)^{k \cdot 3^B \cdot d^{6B^2}},
\]
where \( B = \max(m, k) \) and \( d \) is the degree of the number field generated by the coefficients \( a_i \) and the vectors \( \mathbf{\alpha}_i \).

Proof. The desired result can be easily proved by using [20, Theorem 1] and counting the solutions through all the partitions of the set \( \{1, 2, \ldots, k\} \).

3. Main results

We first introduce a notation. For any polynomial \( G \in \mathbb{C}[X_1, \ldots, X_m] \), we let
\[
n(G) = \text{the number of monomials of } G.
\]
As usual, here we treat a non-zero constant as a monomial.

Recall that for any point \( \mathbf{w} \in \mathbb{C}^m \), we denote its coordinates by \( (w_1, \ldots, w_m) \). We start the discussions by dealing with the simplest case.

Theorem 3.1. Let \( \Phi = (X_1^d, \ldots, X_m^d) \) with integer \( d \geq 2 \). Fix a hypersurface \( V = Z(G) \), where
\[
G = \sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \ldots, X_m]
\]
with \( G \neq 0 \) and \( i_j \geq 0 \) for \( 1 \leq j \leq m \). Then, for any \( \mathbf{w} \in (\mathbb{C}^*)^m \) with multiplicatively independent coordinates, we have
\[
|S_{\mathbf{w}}(\Phi, V)| \leq (8n(G))^{4n(G)}.
\]
Proof. For the given point \( w \), we want to bound the number of integers \( n \geq 0 \) such that \( \Phi^{(n)}(w) \in V \), that is
\[
\sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} \left( w_1^{i_1} \cdots w_m^{i_m} \right)^n = 0.
\]
This is upper bounded by the number of integers \( n \geq 0 \) such that
\[
\sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} \left( w_1^{i_1} \cdots w_m^{i_m} \right)^n = 0.
\]
Since the coordinates of \( w \) are multiplicatively independent, the upper bound follows from Lemma 2.2 and Corollary 2.4 by noticing whether \( G \) has a non-zero constant term or not.
\[ \square \]

One can relieve the multiplicative independence condition on the point \( w \) in some special cases.

**Theorem 3.2.** Let \( \Phi = (X_1^d, \ldots, X_m^d) \) with integer \( d \geq 2 \). Fix a hypersurface \( V = Z(G) \), where
\[
G = \sum_{j=1}^{m} a_j X_j^{e_j} \in \mathbb{C}[X_1, \ldots, X_m]
\]
with \( G \neq 0 \) and \( e_j \geq 0 \) for \( 1 \leq j \leq m \). For any point \( w \in \mathbb{C}^m \), let \( D \) be the order of the group of roots of unity generated by all those roots of unity which are of the form \( w_i/w_j \) for some \( 1 \leq i, j \leq m \). Suppose that there exists some index \( j_0 \) such that \( w_{j_0} \) is not a root of unity, \( w_{j_0} \neq 0, a_{j_0} \neq 0, e_{j_0} \neq 0 \) and the ratio \( w_j/w_{j_0} \) is not a root of unity for any \( j \neq j_0 \). Then, we have
\[
|S_w(\Phi, V)| \leq D(8n(G))^8n(G)^6.
\]

**Proof.** Applying the same arguments as in the proof of Theorem 3.1, the desired result follows directly from Lemma 2.3 and Corollary 2.4.
\[ \square \]

The upper bound in Theorem 3.2 is not uniform because of the quantity \( D \). However, we can make it uniform in some sense. In fact, if we choose the point \( w \in K^m \), where \( K \) is a number field, then we have \( D \leq [K : \mathbb{Q}] \), thus we can replace \( D \) by \([K : \mathbb{Q}]\) in the upper bound. Alternatively, one can also use (2.3).

We also want to indicate that in Theorem 3.2 if we further assume that \( w \in \overline{K}^m \) and \( G \in \overline{K}[X_1, \ldots, X_m] \), then under the same assumptions as in Lemma 2.5, we can get another upper bound for \( |S_w(\Phi, V)| \).

Obtaining results on the size of \( S_w(\Phi, V) \) even for the slightly more general case when \( F_i = X_i^{d_i}, \ i = 1, \ldots, m \), where the degrees \( d_i \) are not necessarily the same, seems not to be quite straightforward.
Theorem 3.3. Let $Φ = (X_1^{d_1}, \ldots, X_m^{d_m})$ with integers $d_i \geq 0$ ($1 \leq i \leq m$) and some index $ℓ$ such that $d_ℓ \geq 2$. Fix a hypersurface $V = Z(G)$, where

$$G = \sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \ldots, X_m]$$

with $i_j \geq 0$ for $1 \leq j \leq m$, such that $G$ has a non-zero constant term $c$ and $X_ℓ$ is a factor of some monomial of $G$. Then, for any $w = (w_1, \ldots, w_m) \in (\mathbb{C}^*)^m$ with multiplicatively independent coordinates, we have

$$|S_w(Φ, V)| \leq (0.792n(G))^n(G)(8n(G))^{4n(G)(n(G)−1)^4(m+1)}.$$

Proof. Since $G$ has $n(G)$ monomials, we renumber the indices $(i_1, \ldots, i_m)$ as $1, 2, \ldots, n(G)$. If the index $(i_1, \ldots, i_m)$ corresponds to $j$ ($1 \leq j \leq n(G)$), then accordingly we write $a_{i_1, \ldots, i_m}$ and $X_1^{i_1} \cdots X_m^{i_m}$ as $b_j$ and $Y_j$ respectively. So, we have

$$G = \sum_{j=1}^{n(G)} b_j Y_j.$$

Moreover, we assume that $b_1 = c, Y_1 = 1$ and $Y_2$ has a factor $X_ℓ$.

For the given point $w$, bounding $|S_w(Φ, V)|$ is exactly to bound the number of integers $n \geq 0$ such that $Φ^{(n)}(w) \in V$, that is

$$b_1 + \sum_{j=2}^{n(G)} b_j Y_j(Φ^{(n)}(w)) = 0. \tag{3.1}$$

Let $Λ$ be the group generated by the coordinates of $w$, and let $Γ = Λ^{n(G)}$. Since the rank of $Λ$ is $m$, the rank of $Γ$ is at most $mn(G)$. In view of (3.1), we consider the solutions $(x_1, \ldots, x_{n(G)})$ of the equation

$$b_1 x_1 + \cdots + b_{n(G)} x_{n(G)} = 0 \tag{3.2}$$

in $Γ$. By Corollary 2.7, the equation (3.2) has less than

$$(0.792n(G))^{n(G)}(8n(G))^{4(n(G)−1)^4(n(G)+mn(G))}$$

solutions in $Γ$ up to a factor of proportionality.

Thus, the number of those solutions of (3.2) with the form

$$(1, Y_2(Φ^{(n)}(w)), \ldots, Y_{n(G)}(Φ^{(n)}(w))) \quad (n = 0, 1, 2, \ldots) \tag{3.3}$$

is less than

$$(0.792n(G))^{n(G)}(8n(G))^{4n(G)(n(G)−1)^4(m+1)}$$

because they have the same first component 1. Now, we claim that if $n \neq k$, then $Y_2(Φ^{(n)}(w)) \neq Y_2(Φ^{(k)}(w))$. Then, there is a one-to-one correspondence between the solutions (3.3) and the integers $n \geq 0$, and thus we can complete the proof.
Indeed, for any integers \( n, k \) with \( 0 \leq n < k \), since \( \Phi(n) = (X_1^{d_1}, \ldots, X_m^{d_m}) \) with \( d_\ell \geq 2 \) and there is a factor \( X_\ell \) in \( Y_2 \), we have \( \deg_{X_\ell} Y_2(\Phi(n)) < \deg_{X_\ell} Y_2(\Phi(k)) \). So, by the multiplicative independence of the coordinates of \( \mathbf{w} \), \( Y_2(\Phi(n)(\mathbf{w})) \neq Y_2(\Phi(k)(\mathbf{w})) \) for any \( n \neq k \). \( \square \)

In Theorem 3.3 we assume that \( G \) has a non-zero constant term, the situation is slightly different if it has zero constant term.

**Theorem 3.4.** Let \( \Phi = (X_1^{d_1}, \ldots, X_m^{d_m}) \) with integers \( d_i \geq 2 \) \((1 \leq i \leq m)\). Fix a hypersurface \( V = Z(G) \), where

\[
G = \sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \ldots, X_m]
\]

with \( G \neq 0 \) and \( i_j \geq 1 \) for \( 1 \leq j \leq m \), such that \( G \) has zero constant term. Then, for any \( \mathbf{w} \in (\mathbb{C}^*)^m \) with multiplicatively independent coordinates, we have

\[
|S_{\mathbf{w}}(\Phi, V)| \leq (0.792n(G))^n(G)(8n(G))^{4n(G)(n(G)-1)^4(m+1)}.
\]

**Proof.** Here we can assume that \( G \) has at least two different monomials. As in the proof of Theorem 3.3, we can write

\[
G = \sum_{j=1}^{n(G)} b_j Y_j.
\]

Now, we need to bound the number of integers \( n \geq 0 \) such that

\[
\sum_{j=1}^{n(G)} b_j Y_j(\Phi(n)(\mathbf{w})) = 0.
\]

Write \( Y_1 = X_1^{j_1} \cdots X_m^{j_m} \) and \( Y_2 = X_1^{l_1} \cdots X_m^{l_m} \). Suppose that there exist integers \( n, k \) with \( 0 \leq n < k \) such that

\[
Y_1(\Phi(k)(\mathbf{w}))Y_2(\Phi(k)(\mathbf{w})) = Y_1(\Phi(n)(\mathbf{w}))Y_2(\Phi(n)(\mathbf{w})).
\]

Then due to \( \Phi(n) = (X_1^{d_1}, \ldots, X_m^{d_m}) \), we obtain

\[
w_1^{j_1(d_1-1)} \cdots w_m^{j_m(d_m-1)} = w_1^{l_1(d_1-1)} \cdots w_m^{l_m(d_m-1)}.
\]

Noticing that \( d_\ell \geq 2 \) \((1 \leq i \leq m)\) and the coordinates \( w_1, \ldots, w_m \) are multiplicatively independent, we must have \( j_1 = l_1, \ldots, j_m = l_m \), which implies that \( Y_1 = Y_2 \). This contradicts with \( Y_1 \neq Y_2 \).

Thus as before, the equation \( b_1 x_1 + \ldots + b_{n(G)} x_{n(G)} = 0 \) has less than

\[
(0.792n(G))^n(G)(8n(G))^{4n(G)(n(G)-1)^4(m+1)}
\]
solutions with the form
\[(Y_1(\Phi^{(n)}(w)), Y_2(\Phi^{(n)}(w)), \ldots, Y_n(G)(\Phi^{(n)}(w))) \quad (n = 0, 1, 2, \ldots).\]

In addition, it is easy to see that \(\deg Y_j(\Phi^{(n)}) < \deg Y_j(\Phi^{(k)})\) for any integers \(n < k\) and \(1 \leq j \leq m\). So, if \(n \neq k\), then \(Y_j(\Phi^{(n)}(w)) \neq Y_j(\Phi^{(k)}(w))\) for any \(1 \leq j \leq n(G)\) by using the multiplicative independence condition. Thus, the desired upper bound follows. \(\square\)

Now, we want to use Lemma 2.8 to give another method on handling a similar case as in Theorem 3.4.

Theorem 3.5. Let \(K\) be a number field of degree \(d\). Let \(\Phi = (X_1^{d_1}, \ldots, X_m^{d_m})\) with integers \(d_i \geq 0\) \((1 \leq i \leq m)\) and some index \(\ell\) such that \(d_\ell \geq 2\).

Fix a hypersurface \(V = Z(G)\), where
\[G = \sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in K[X_1, \ldots, X_m]\]
with \(G \neq 0\) and \(i_j \geq 0\) for \(1 \leq j \leq m\), such that \(G\) has zero constant term. Suppose that for any two monomials \(a_{i_1, \ldots, i_m} X_1^{i_1} \cdots X_m^{i_m}\) and \(a_{j_1, \ldots, j_m} X_1^{j_1} \cdots X_m^{j_m}\) of \(G\), we have \(i_1 \neq j_1, \ldots, i_m \neq j_m\). Then, for any \(w \in (K^*)^m\) with multiplicatively independent coordinates, we have
\[|S_w(\Phi, V)| \leq (0.792n(G))^{n(G)} 2^{35B^3} d^{6B^2},\]
where \(B = \max(m, n(G))\).

Proof. Since \(\Phi^{(n)} = (X_1^{d_1}, \ldots, X_m^{d_m})\), we need to bound the number of integers \(n \geq 0\) such that
\[\sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} w_1^{i_1 d_1} \cdots w_m^{i_m d_m} = 0.\]
For each index \((i_1, \ldots, i_m)\), we write \(\alpha_i = (w_1^{i_1}, \ldots, w_m^{i_m})\) and \(x = (d_1^{i_1}, \ldots, d_m^{i_m})\), then the above equation becomes
\[(3.4) \sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} \alpha_i^x = 0.\]
Moreover, for any two indices \((i_1, \ldots, i_m)\) and \((j_1, \ldots, j_m)\), if \(\alpha_i^z = \alpha_j^z\) for \(z = (z_1, \ldots, z_m) \in \mathbb{Z}^m\), then, we must have \(i_1 z_1 = j_1 z_1, \ldots, i_m z_m = j_m z_m\) by using the multiplicative independence condition. Under the assumption that \(i_1 \neq j_1, \ldots, i_m \neq j_m\), we get \(z_1 = \cdots = z_m = 0\). That is, \(z\) is the zero vector.

Now, applying Lemma 2.8 to the equation (3.4) we know that the equation (3.4) has less than
\[(0.792n(G))^{n(G)} 2^{35B^3} d^{6B^2}\]
solutions with the form \((d_1^n, \ldots, d_m^n), n = 0, 1, 2, \ldots\). Besides, since \(d_\ell \geq 2\), we have \((d_1^n, \ldots, d_m^n) \neq (d_1^k, \ldots, d_m^k)\) for any integers \(n \neq k\). So, the desired result follows. □

It seems natural to expect that the classes of dynamical systems and hypersurfaces that satisfy the uniform boundedness condition are quite wide. We confirm this by the following three theorems.

**Theorem 3.6.** Let \(\Phi = (X_1^{d_1}, X_2^{d_2}, F_3, \ldots, F_m)\) with \(m \geq 3\), integers \(d_1, d_2 \geq 1\) and

\[
F_i = X_1^{s_{i1}} \cdots X_m^{s_{im}}
\]

for \(s_{ij} \geq 0, 3 \leq i \leq m, 1 \leq j \leq m\). Fix a hypersurface \(V = Z(G)\), where

\[
G = aX_1^{e_1} + bX_2^{e_2} + \sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m}X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \ldots, X_m]
\]

with \(a \neq 0, b \neq 0, e_1 \geq 1, e_2 \geq 1\) and \(i_j \geq 0\) for \(1 \leq j \leq m\). Suppose that \(d_1d_2 \geq 2\). Then, for any \(w \in (\mathbb{C}^*)^m\) such that \(w_1\) and \(w_2\) are multiplicatively independent, we have

\[
|S_w(\Phi, V)| \leq (0.792n(G))^{n(G)}(8n(G))^{4n(G)(n(G)-1)^4(m+1)}.
\]

**Proof.** As in the proof of Theorem 3.3 and under the assumptions, we can write

\[
G = \sum_{j=1}^{n(G)} b_j Y_j
\]

such that \(b_1 = a, Y_1 = X_1^{e_1}, b_2 = b\) and \(Y_2 = X_2^{e_2}\). We need to bound the number of integers \(n \geq 0\) such that

\[
\sum_{j=1}^{n(G)} b_j Y_j(\Phi^{(n)}(w)) = 0.
\]

Suppose that there exist integers \(n, k\) with \(0 \leq n < k\) such that

\[
\frac{Y_1(\Phi^{(k)}(w))}{Y_1(\Phi^{(n)}(w))} = \frac{Y_2(\Phi^{(k)}(w))}{Y_2(\Phi^{(n)}(w))}.
\]

Then, we get

\[
w_1^{e_1(d_1^n-d_1^k)} = w_2^{e_2(d_2^n-d_2^k)}.
\]

So, noticing \(d_1d_2 \geq 2, w_1\) and \(w_2\) are multiplicatively dependent. This contradicts with the assumption that \(w_1\) and \(w_2\) are multiplicatively independent.

Thus as before, the equation \(b_1x_1 + \ldots + b_{n(G)}x_{n(G)} = 0\) has less than

\[
(0.792n(G))^{n(G)}(8n(G))^{4n(G)(n(G)-1)^4(m+1)}.
\]
solutions with the form
\[ (Y_1((\Phi^{(n)}(w))), Y_2((\Phi^{(n)}(w))), \ldots, Y_{n(G)}((\Phi^{(n)}(w)))) \quad (n = 0, 1, 2, \ldots). \]

On the other hand, due to \(d_1d_2 \geq 2\), for any integers \(n \neq k\) we have \(Y_1((\Phi^{(n)}(w))) \neq Y_1((\Phi^{(k)}(w)))\) or \(Y_2((\Phi^{(n)}(w))) \neq Y_2((\Phi^{(k)}(w)))\). Thus as before, the desired upper bound follows. \(\square\)

**Theorem 3.7.** Let \(\Phi = (F_1, \ldots, F_m)\) with \(m \geq 2\) be defined by
\[ F_i = X_i^{s_i}X_{i+1}^{s_{i+1}} \cdots X_m^{s_m}, \]
with
\[ s_i > 1, s_{ij} \geq 0, \quad j = i + 1, \ldots, m, \]
or
\[ s_i \geq 1, s_{ij} \geq 1 \quad \text{for at least one} \quad j = i + 1, \ldots, m, \]
\[ i = 1, \ldots, m - 1, \quad \text{and} \]
\[ F_m = X_m. \]

Fix a hypersurface \(V = Z(G)\), where
\[ G = \sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m}X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \ldots, X_m] \]
with one monomial \(cX_m^{e_m} \in \mathbb{C}^*, e_m \geq 0\), such that \(G\) has a non-constant monomial different from \(cX_m^{e_m}\) containing at least one factor \(X_j\) for certain \(1 \leq j \leq m - 1\). Then, for any \(w \in (\mathbb{C}^*)^m\) with multiplicatively independent coordinates, we have
\[ |S_w(\Phi, V)| < (0.792n(G))^n(G)(8n(G))^{4n(G)(n(G)-1)^4(m+1)}. \]

**Proof.** We first note that if \(e_m = 0\), then \(G\) has a non-zero constant term \(c\).

As in the proof of Theorem 3.3 and under the assumptions, we can write
\[ G = \sum_{j=1}^{n(G)} b_jY_j \]
such that \(b_1 = c, Y_1 = X_m^{e_m}\) and \(Y_2\) is not a constant.

For the given point \(w\), noticing \(F^{(n)}_m(w) = w_m\) for any \(n \geq 0\), we need to bound the number of integers \(n \geq 0\) such that
\[ b_1w_m^{e_m} + \sum_{j=2}^{n(G)} b_jY_j((\Phi^{(n)}(w))) = 0 \]

Similarly, the equation \(b_1x_1 + \ldots + b_{n(G)}x_{n(G)} = 0\) has less than
\[ (0.792n(G))^n(G)(8n(G))^{4n(G)(n(G)-1)^4(m+1)}. \]
solutions with the form
\[(w^m, Y_2(\Phi^{(n)}(w)), \ldots, Y_n(G)(\Phi^{(n)}(w))) \quad (n = 0, 1, 2, \ldots).\]

Now, we claim that if \(n \neq k\), then \(Y_2(\Phi^{(n)}(w)) \neq Y_2(\Phi^{(k)}(w))\). Then, the desired upper bound follows directly from this claim.

Finally, we want to prove the above claim. Assume that \(n < k\). We note that, by the conditions on \(s_i\) and \(s_{ij}\) and the fact that the degree \(d_{i,n} = \deg \Phi^{(n)}(w)\) satisfies
\[d_{i,n} = s_i d_{i,n-1} + s_{i,i+1} d_{i+1,n-1} + \ldots + s_{i,m-1} d_{m-1,n-1} + s_m,\]
by induction one easily proves that \(\deg \Phi^{(n)}(w) < \deg \Phi^{(k)}(w)\) for any \(1 \leq i \leq m\) (the case \(s_i = 1\) for all \(i = 1, \ldots, m - 1\) is also proved in [18, Lemma 1]). Thus, by the multiplicative independence of the coordinates of \(w\), we deduce that \(Y_2(\Phi^{(n)}(w)) \neq Y_2(\Phi^{(k)}(w))\).

**Theorem 3.8.** Let \(\Phi = (F_1, \ldots, F_m)\) be defined by
\[F_i = \prod_{j=1}^m X_j^{s_{ij}}, \quad s_{ij} \geq 0\]
for \(i = 1, \ldots, m\) and \(j = 1, \ldots, m\) such that for any \(1 \leq i \leq m\) the degree \(\deg F_i \geq 2\). Fix a hypersurface \(V = Z(G)\), where
\[G = \sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \ldots, X_m]\]
with a non-zero constant term \(c\). Then, for any \(w \in (\mathbb{C}^*)^m\) with multiplicatively independent coordinates, we have
\[|S_w(\Phi, V)| < (0.792n(G))^n(G)(8n(G))^{4n(G)(n(G)-1)^4(m+1)}\]

**Proof.** First we can assume that \(G\) has at least two monomials. As in the proof of Theorem 3.3, we can write
\[G = \sum_{j=1}^{n(G)} b_j Y_j\]
with \(b_1 = c\) and \(Y_1 = 1\). What we need is to bound the number of integers \(n \geq 0\) such that
\[b_1 + \sum_{j=2}^{n(G)} b_j Y_j(\Phi^{(n)}(w)) = 0.\]
Similarly, the equation \(b_1 x_1 + \ldots + b_n(G) x_n(G) = 0\) has less than
\[(0.792n(G))^n(G)(8n(G))^{4n(G)(n(G)-1)^4(m+1)}\]
solutions with the form
\[(1, Y_2(\Phi^{(n)}(w)), \ldots, Y_{n(G)}(\Phi^{(n)}(w))) \quad (n = 0, 1, 2, \ldots).\]

Now, we claim that for any integers \(n \neq k\), we have \(Y_j(\Phi^{(n)}(w)) \neq Y_j(\Phi^{(k)}(w))\) for any \(2 \leq j \leq n(G)\), which implies the desired upper bound.

Indeed, assume that \(n < k\), and fix arbitrary \(j\) with \(2 \leq j \leq n(G)\). Since the coordinates of \(w\) are multiplicatively independent, if \(Y_j(\Phi^{(n)}(w)) = Y_j(\Phi^{(k)}(w))\), then we must have \(\deg Y_j(\Phi^{(n)}) = \deg Y_j(\Phi^{(k)})\).

However, in view of the form of \(\Phi\), by induction we in fact have \(\deg F_i^{(n)} < \deg F_i^{(k)}\) for any \(1 \leq i \leq m\), so \(\deg Y_j(\Phi^{(n)}) < \deg Y_j(\Phi^{(k)})\).

We remark that the assumption on \(\Phi\) put in Theorem 3.8 is reasonable. For example, let \(m = 2\) and fix such a point \(w\), choose \(\Phi = (X_1, X_2^2)\) and \(G = X_1 - w_1\), then for any \(n \geq 0\) we have \(\Phi^{(n)}(w) \in Z(G)\).

In the above, we can see that most of our results depend on the multiplicative independence of the coordinates of the point \(w\). In fact, this condition can be relaxed in many cases. The following theorem is an example, which can be proved similarly as before.

**Theorem 3.9.** Let \(\Phi = (F_1, \ldots, F_m)\) be defined by

\[F_i = \prod_{j=1}^{m} X_j^{s_{ij}}, \quad s_{ij} \geq 0\]

for \(i = 1, \ldots, m - 1\) and \(j = 1, \ldots, m\), and

\[F_m = X_m^k, \quad k \geq 2.\]

Fix a hypersurface \(V = Z(G)\), where

\[G = \sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \ldots, X_m]\]

with one constant monomial \(a \neq 0\) and one non-constant monomial \(bX_m^{e_m}, b \in \mathbb{C}^*, e_m \geq 1\). Then, for any \(w \in (\mathbb{C}^*)^m\) with \(w_m\) not a root of unity, we have

\[|S_w(\Phi, V)| < (0.792n(G))^{n(G)}(8n(G))^{4n(G)n(G)-1}^{4(m+1)}.\]

4. **Comments**

It is quite sure that one can get more partial results concerning the quantitative dynamical Mordell-Lang conjecture by using the methods presented in Section 3. It is also likely that several upper bounds in Section 3 can be improved in some special cases.
It is worth remarking that our methods can also be employed to study the synchronized intersection of two orbits. Indeed, let \( \mathcal{F} \) and \( \mathcal{H} \) be two polynomial systems from \( \mathbb{C}^m \) to itself, then one can ask to bound the size of the subset of integers

\[
\{ n \geq 0 : \mathcal{F}(n)(w_1) = \mathcal{H}(n)(w_2) \},
\]

where \( w_1, w_2 \) are two vectors in \( \mathbb{C}^m \). Now, we take \( \Phi = (\mathcal{F}, \mathcal{H}) \) as in our results (but we see \( \mathcal{F} \) in variables \( X_1, \ldots, X_m \) and \( \mathcal{H} \) in variables \( Y_1, \ldots, Y_m \), so we have \( 2m \) variables), and \( V = Z(G) \), where \( G = \sum_{i=1}^{m} (X_i - Y_i) \). Clearly, we have

\[
\{ n \geq 0 : \mathcal{F}(n)(w_1) = \mathcal{H}(n)(w_2) \} \subseteq S_{(w_1, w_2)}(\Phi, V),
\]

where \( S_{(w_1, w_2)}(\Phi, V) \) is defined as in (1.1). So, the problem turns out to bound the size \( |S_{(w_1, w_2)}(\Phi, V)| \), which can be done in many cases by applying the methods in Section 3.

In Section 3, we consider uniform boundedness of the size of the intersection \( \text{Orb}_w(\Phi) \cap V \) for various cases. One can also consider obtaining upper bounds under which there is indeed an integer \( n \) such that \( \Phi(n)(w) \in V \), or which are greater than any integer \( n \) with \( \Phi(n)(w) \in V \). For example, in Theorem 3.2, if we further assume that \( |w_{j_0}| > |w_j| \) and \( e_{j_0} \geq e_j \) for any \( j \neq j_0 \), and define

\[
a = \max_{j \neq j_0} |a_j| \quad \text{and} \quad \text{the index } i \text{ such that } |w_{e_i}^i| = \max_{j \neq j_0} |w_{e_j}^j|,
\]

then we can easily get a lower bound of \( n \) from the following inequality

\[
|a_{j_0} w_{e_{j_0}}^{e_{j_0} d^n} | > (m - 1)a |w_{e_i}^{e_i d^n}|
\]

such that this bound is larger than any integer \( n \) with \( \Phi(n)(w) \in V \).

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