Spin projection operators in (A)dS and partial masslessness

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Abstract

We elaborate on the traceless and transverse spin projectors in four-dimensional de Sitter and anti-de Sitter spaces. The poles of these projectors are shown to correspond to partially massless fields. We also obtain a factorisation of the conformal operators associated with gauge fields of arbitrary Lorentz type \((m/2,n/2)\), with \(m\) and \(n\) positive integers.
1 Introduction

In four-dimensional Minkowski space $\mathbb{M}^4$, spin projection operators, also known as traceless and transverse (TT) spin-$s$ projectors, were constructed by Behrends and Fronsdal more than sixty years ago [1,2]. These TT projectors have found numerous applications. For instance, it is well-known that they determine the structure of massive spin-$s$ propagators in the quantum theory. They can also be used to construct gauge-invariant actions. An important example of the latter application is the formulation of conformal higher-spin actions proposed by Fradkin and Tseytlin [3], although the TT spin-$s$ projectors were given explicitly in [3] only for $s \leq 2$.

Refs. [1,2] made use of the four-vector notation in conjunction with the four-component spinor formalism, which resulted in rather complicated expressions for the TT spin-$s$ projectors. However, switching to the two-component spinor formalism leads to remarkably simple and compact expressions for these projectors [4,5].

For higher-dimensional Minkowski space $\mathbb{M}^d$, with $d > 4$, the TT integer-spin projectors were constructed by Segal [6] who used these operators to formulate bosonic conformal higher-spin actions. In the literature, there have appeared different forms of the bosonic TT projectors [7–10], which may be shown to be equivalent to the ones presented in the arXiv version of [6]. The TT half-integer-spin projectors for $d > 4$ were constructed for the first time in [10]. It should be pointed out that the three-dimensional case is somewhat special, and the corresponding spin projection operators were described in [11] (see also [12] for the superprojectors).

Unlike Minkowski space, both de Sitter (dS) and anti-de Sitter (AdS) spaces have non-vanishing curvature, which makes it more challenging to construct the TT spin-$s$ projectors. For this reason only the lower-spin cases corresponding to $s \leq 2$ have been considered in the literature [13]. In this paper we construct all the spin projection operators in (A)dS$_4$ and use them to derive various properties of higher-spin (i) partially massless fields [13–23]; (ii) massive fields; and (iii) conformal models.

Throughout the paper we make use of the two-component spinor formalism and follow the notation and conventions of [24]. In this notation the algebra of AdS covariant derivatives is

$$[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = -4S^2(\varepsilon_{\alpha\beta}\bar{M}_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}}M_{\alpha\beta}),$$

where the Lorentz generator $M_{\alpha\beta}$ is defined by $M_{\alpha\beta}\psi_\gamma = \varepsilon_{\gamma(\alpha}\psi_{\beta)}$. An analysis similar to that given below applies in the case of de Sitter space, one just needs to replace all occurrences of $S^2$ with $-S^2$.

\footnote{In vector notation this reads $[\mathcal{D}_a, \mathcal{D}_b] = -2S^2M_{ab}$.}
2 Spin projection operators

Of crucial importance to our subsequent analysis is the quadratic AdS Casimir operator:

$$Q := \Box - 2S^2(M^\gamma{}^\delta M_{\gamma\delta} + \tilde{M}^\gamma{}^\delta \tilde{M}_{\gamma\delta}) , \quad [Q, D_{\alpha\dot{\alpha}}] = 0 , \quad (2.1)$$

where $\Box = D^aD_a = -\frac{1}{2}D^{\alpha\dot{\alpha}}D_{\alpha\dot{\alpha}}$.

Denote by $V_{(m,n)}$ the space of fields $\phi_{\alpha(m)\dot{\alpha}(n)}$ which are totally symmetric in their dotted and, independently, in their undotted indices. To construct the projectors, we introduce two operators $P^{(m,n)}$, $\hat{P}^{(m,n)} : V_{(m,n)} \rightarrow V_{(m,n)}$ which are defined by their action on $\phi_{\alpha(m)\dot{\alpha}(n)}$ as

$$P_{\alpha(m)\dot{\alpha}(n)}(\phi) = D_{(\dot{\alpha}_1} \cdots D_{\dot{\alpha}_n)} \beta_1 \cdots D_{(\alpha_1} \cdots D_{\alpha_m)} \dot{\beta}_1 \cdots \beta_n \phi_{\alpha_1 \cdots \alpha_m}(\phi) ; \quad (2.2a)$$

$$\hat{P}_{\alpha(m)\dot{\alpha}(n)}(\phi) = D_{(\dot{\alpha}_1} \cdots D_{\dot{\alpha}_m)} \beta_1 \cdots D_{(\alpha_1} \cdots D_{\alpha_n)} \dot{\beta}_1 \cdots \beta_n \phi_{\alpha_1 \cdots \alpha_n}(\phi) . \quad (2.2b)$$

Both operators (2.2) project out the transverse component of the field $\phi_{\alpha(m)\dot{\alpha}(n)}$,

$$D^{\beta\dot{\beta}}P_{\beta\alpha(m-1)\dot{\beta}\dot{\alpha}(n-1)}(\phi) = 0 , \quad (2.3a)$$

$$D^{\beta\dot{\beta}}\hat{P}_{\beta\alpha(m-1)\dot{\beta}\dot{\alpha}(n-1)}(\phi) = 0 . \quad (2.3b)$$

However they are not projectors in the sense that they do not square to themselves. In fact, one may show that they instead satisfy

$$P^{(m,n)}P^{(m,n)} \phi_{\alpha(m)\dot{\alpha}(n)} = \prod_{t=1}^{n} (Q - \lambda_{(t,m,n)} S^2) P^{(m,n)} \phi_{\alpha(m)\dot{\alpha}(n)} , \quad (2.4a)$$

$$\hat{P}^{(m,n)}\hat{P}^{(m,n)} \phi_{\alpha(m)\dot{\alpha}(n)} = \prod_{t=1}^{m} (Q - \lambda_{(t,m,n)} S^2) \hat{P}^{(m,n)} \phi_{\alpha(m)\dot{\alpha}(n)} , \quad (2.4b)$$

where the parameters $\lambda_{(t,m,n)}$ are defined by

$$\lambda_{(t,m,n)} := (m + n - t + 3)(m + n - t - 1) + (t - 1)(t + 1) . \quad (2.5)$$

From Eq. (2.4) it follows that the two operators $\Pi^{(m,n)}$, $\hat{\Pi}^{(m,n)} : V_{(m,n)} \rightarrow V_{(m,n)}$ where

$$\Pi_{\alpha(m)\dot{\alpha}(n)}(\phi) = \left[ \prod_{t=1}^{n} (Q - \lambda_{(t,m,n)} S^2) \right]^{-1} P_{\alpha(m)\dot{\alpha}(n)}(\phi) , \quad (2.6a)$$

$$\hat{\Pi}_{\alpha(m)\dot{\alpha}(n)}(\phi) = \left[ \prod_{t=1}^{m} (Q - \lambda_{(t,m,n)} S^2) \right]^{-1} \hat{P}_{\alpha(m)\dot{\alpha}(n)}(\phi) . \quad (2.6b)$$

This operator may be compared with the quadratic Casimir operator of the $\mathcal{N} = 1$ AdS supergroup [25].

This means that from the beginning we work with traceless tensor fields.
square to themselves and project out the transverse subspace of \( V_{(m,n)} \),

\[
\Pi^{(m,n)} \Phi_{\alpha(m)\hat{\alpha}(n)} = \Pi^{(m,n)} \Phi_{\alpha(m)\hat{\alpha}(n)} , \quad D^{\beta\hat{\beta}} \Pi_{\beta\alpha(m-1)\hat{\beta}\hat{\alpha}(n-1)}(\phi) = 0 , \quad (2.7a)
\]

\[
\hat{\Pi}^{(m,n)} \hat{\Phi}_{\alpha(m)\hat{\alpha}(n)} = \hat{\Pi}^{(m,n)} \Phi_{\alpha(m)\hat{\alpha}(n)} , \quad D^{\beta\hat{\beta}} \hat{\Pi}_{\beta\alpha(m-1)\hat{\beta}\hat{\alpha}(n-1)}(\phi) = 0 . \quad (2.7b)
\]

Therefore the operators \( \Phi_{2.6} \) are the spin projection operators in AdS. Actually, the two types of projectors prove to coincide,

\[
\Pi_{\alpha(m)\hat{\alpha}(n)}(\phi) = \hat{\Pi}_{\alpha(m)\hat{\alpha}(n)}(\phi) , \quad (2.8)
\]

and so it suffices to consider only the first, \( \Phi_{2.6a} \).

In addition, it is possible to show that for any field \( \Phi_{\alpha(m)\hat{\alpha}(n)} \) such that the projector \( \Phi_{2.6a} \) is well defined, there exists some \( \zeta_{\alpha(m-1)\hat{\alpha}(n-1)} \) such that

\[
(1 - \Pi^{(m,n)}) \Phi_{\alpha(m)\hat{\alpha}(n)} = D(\alpha_1(\hat{\alpha}_1 \zeta_{\alpha_2...\alpha_m} \alpha_2...\alpha_n)) . \quad (2.9)
\]

This means that any field may be decomposed as

\[
\Phi_{\alpha(m)\hat{\alpha}(n)} = \sum_{l=0}^{m-1} D(\alpha_1(\hat{\alpha}_1 \cdots D(\alpha_l(\hat{\alpha}_{l+1}...\alpha_m)\hat{\alpha}_{l+1}...\alpha_n) + D(\alpha_1(\hat{\alpha}_1 \cdots D(\alpha_m)\hat{\alpha}_m \Phi_{\alpha_{m+1}...\hat{\alpha}_n}) , \quad (2.10)
\]

for some set of fields \( \{ \Phi_{\alpha(m)\hat{\alpha}(n)}^T, \Phi_{\alpha(m-1)\hat{\alpha}(n-1)}^T, \cdots, \Phi_{\alpha(n-m+1)}^T \} \) which are transverse and where we have assumed, without loss of generality, that \( n \geq m \).

3 Analysis of results and applications

It is of interest to understand the physical significance of the parameters \( \Phi_{2.5} \) which appear in the definition of the projectors \( \Phi_{2.6} \). With this in mind we now introduce on-shell fields, which are those satisfying the equations

\[
(\mathcal{Q} - \mu^2) \Phi_{\alpha(m)\hat{\alpha}(n)} = 0 , \quad (3.1a)
\]

\[
D^{\gamma\hat{\gamma}} \Phi_{\alpha(m-1)\gamma\hat{\gamma}(n-1)} = 0 . \quad (3.1b)
\]

We say that such a field describes a spin \( s = \frac{1}{2}(m + n) \) particle with pseudo-mass \( \mu \).

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4 A supersymmetric extension of \( \Phi_{2.10} \) is given in \( \Phi_{2.8} \).

5 This terminology is because with our conventions \( \mu \) is not the physical mass. For example, in the massless case \( \mu \neq 0 \).
3.1 Partially massless fields

It is typical to choose \( m = n = s \) for spin-\( s \) bosonic fields whereas the usual choice for spin-\( (s + \frac{1}{2}) \) fermionic fields is \( m = n - 1 = s \). By now it is well known that in these cases, the system of equations \(^{(3.1)}\) becomes invariant under gauge transformations of depth \( t \)

\[
\delta \zeta \phi_{\alpha(s)}^{(s)} = D_{(\alpha_1 (\tilde{\alpha}_1 \ldots D_{(\alpha_t \tilde{\alpha})} \zeta_{\alpha_{t+1} \ldots \alpha_s}) \tilde{\alpha}_{t+1} \ldots \tilde{\alpha}_s)}^{(s+1)},
\]

\[
\delta \zeta \phi_{\alpha(s)}^{(s+1)} = D_{(\alpha_1 (\tilde{\alpha}_1 \ldots D_{(\alpha_t \tilde{\alpha})} \zeta_{\alpha_{t+1} \ldots \alpha_s}) \tilde{\alpha}_{t+1} \ldots \tilde{\alpha}_{s+1})}^{(s+1)},
\]

for an on-shell gauge parameter when the mass squared takes the special values \(^{[21,23]}\)

\[
\mu^2_{(t,s)} = (2s - t + 3)(2s - t - 1) + (t - 1)(t + 1)|S^2|, \quad (3.3a)
\]

\[
\mu^2_{(t,s+\frac{1}{2})} = (2s - t + 4)(2s - t) + (t - 1)(t + 1)|S^2|, \quad (3.3b)
\]

where \( 1 \leq t \leq s \). Strictly massless fields correspond to \( t = 1 \) whilst all other values of \( t \) correspond to partially massless fields. Remarkably, we see that for these values of \( m \) and \( n \), the partially massless values coincide with the parameters in the projectors,

\[
\mu^2_{(t,m,n)} = \lambda_{(t,m,n)} S^2, \quad \mu^2_{(t,s+\frac{1}{2})} = \lambda_{(t,s,s+1)} S^2. \quad (3.4)
\]

Therefore, we can extend this notion and say that a field \( \phi_{\alpha(m)\tilde{\alpha}(n)} \) is partially massless when it satisfies \(^{(3.1)}\) with

\[
\mu^2_{(t,m,n)} = \lambda_{(t,m,n)} S^2, \quad 1 \leq t \leq \min(m, n). \quad (3.5)
\]

Indeed, as shown in the appendix, for these values a gauge invariance with depth \( t \) emerges in the system of equations \(^{(3.1)}\),

\[
\delta \zeta \phi_{\alpha(m)\tilde{\alpha}(n)} = D_{(\alpha_1 (\tilde{\alpha}_1 \ldots D_{(\alpha_t \tilde{\alpha})} \zeta_{\alpha_{t+1} \ldots \alpha_m}) \tilde{\alpha}_{t+1} \ldots \tilde{\alpha}_n)}^{(s+1)},
\]

with \( \zeta_{\alpha(m-t)\tilde{\alpha}(n-t)} \) being on-shell. This gauge symmetry\(^7\) is our main motivation for choosing the upper bound of \( \min(m, n) \) for \( t \) in the definition \(^{(3.5)}\).

For transverse fields \( \phi_{\alpha(m)\tilde{\alpha}(n)}^{T} \) satisfying \(^{(3.1b)}\), it follows from \(^{(2.4)}\) that upon application of \( \mathbb{P}^{(m,n)} \) and \( \mathbb{P}^{(m,n)} \), we obtain the following factorisations

\[
\mathbb{P}_{\alpha(m)\tilde{\alpha}(n)}(\phi^{T}) = \prod_{t=1}^{n}(Q - \lambda_{(t,m,n)} S^2) \phi_{\alpha(m)\tilde{\alpha}(n)}^{T}, \quad (3.7a)
\]

\[
\mathbb{P}_{\alpha(m)\tilde{\alpha}(n)}(\phi^{T}) = \prod_{t=1}^{m}(Q - \lambda_{(t,m,n)} S^2) \phi_{\alpha(m)\tilde{\alpha}(n)}^{T}. \quad (3.7b)
\]

So partially massless fields may be understood as those for which \(^{(3.7)}\) vanishes, or alternatively as those fields whose masses appear as poles in the projectors \(^{(2.6)}\).

\(^6\)Due to our definition \(^{(3.1a)}\), the mass values \(^{(3.3)}\) are shifted with respect to the usual ones.

\(^7\)Gauge-invariant Lagrangian formulations for partially massless fields in (A)dS were given in \(^{[22,23,26]}\).
3.2 Massive fields

Massive fields correspond to those values of $\mu^2$ which differ from (3.5). The tensor field $\phi_{\alpha(m)\dot{\alpha}(n)}$ is not the only type of field that can be used to describe a massive spin $s = \frac{1}{2}(m + n)$ particle in AdS. Such representations can be equivalently realised on $V_{(m+t,n-t)}$ where $1 \leq t \leq n$. To see this, consider the set of operators

$$\Delta^{(t,m,n)}_{\alpha\dot{\alpha}} = \frac{D_{\alpha\dot{\alpha}}}{\sqrt{Q - \lambda_{(t,m,n)}S^2}}.$$  

(3.8)

For a fixed $t$ the $\lambda_{(t,m,n)}$ with different values of $m$ and $n$ such that $m + n = \text{const}$ are all equal, therefore we may drop these two labels in (3.8), $\Delta^{(t)} \equiv \Delta^{(t,m,n)}$, if it is clear which family of fields it acts upon. If $\phi^T_{\alpha(m)\dot{\alpha}(n)}$ is transverse (3.1b), then we may use $\Delta^{(t)}$ to convert back and forth between tensor types,

$$\phi^T_{\alpha(m+t)\dot{\alpha}(n-t)} = \Delta^{(n-t+1)}_{\alpha_{1}} \cdots \Delta^{(m)}_{\alpha_{t}} \phi^T_{\alpha_{t+1} \cdots \alpha_{m+t} \dot{\alpha}(n-t) \delta(t)} ,$$  

(3.9a)

$$\phi^T_{\alpha(m)\dot{\alpha}(n)} = \Delta^{(m-t+1)}_{\alpha_{1}} \cdots \Delta^{(n)}_{\alpha_{t}} \phi^T_{\alpha_{t+1} \cdots \alpha_{n+t} \dot{\alpha}(n) \delta(t)} .$$  

(3.9b)

Alternatively, we could instead use (3.8) to trade dotted indices for undotted ones and convert to fields of the type belonging to $V_{(m-t,n+t)}$,

$$\phi^T_{\alpha(m-t)\dot{\alpha}(n+t)} = \Delta^{(m-t+1)}_{\alpha_{1}} \cdots \Delta^{(m)}_{\alpha_{t}} \phi^T_{\alpha_{t+1} \cdots \alpha_{n+t} \dot{\alpha}(n+t) \delta(t)} ,$$  

(3.10a)

$$\phi^T_{\alpha(m)\dot{\alpha}(n)} = \Delta^{(m-t+1)}_{\alpha_{1}} \cdots \Delta^{(n)}_{\alpha_{t}} \phi^T_{\alpha_{t+1} \cdots \alpha_{n+t} \dot{\alpha}(n) \delta(t)} .$$  

(3.10b)

where now $1 \leq t \leq m$. We note that in (3.9a) and (3.10a), each of the fields $\phi^T_{\alpha(m \pm t)\dot{\alpha}(n \mp t)}$ inherit their transversality from $\phi^T_{\alpha(m)\dot{\alpha}(n)}$ and so their right hand sides are totally symmetric.

The denominator of (3.8) is well defined for all on-shell fields except for those whose mass satisfies (3.5). This means that for partially massless fields, the different spaces $V_{(m,n)}$ with $m + n = \text{const}$ describe inequivalent representations, which is not surprising from the point of view of gauge symmetry.

The operators (3.8) may also be used to rewrite the projectors (2.6) as

$$\Pi_{\alpha(m)\dot{\alpha}(n)}(\phi) = \Delta^{(n)\dot{\beta}_{n}}_{(\dot{\alpha}_{n})} \cdots \Delta^{(1)\dot{\beta}_{1}}_{(\dot{\alpha}_{1})} \Delta^{(1)\dot{\beta}_{1}}_{\beta_{1}} \cdots \Delta^{(n)\dot{\beta}_{n}}_{\beta_{n}} \phi_{\alpha_{1} \cdots \alpha_{m}} \delta(n) ;$$  

(3.11a)

$$\hat{\Pi}_{\alpha(m)\dot{\alpha}(n)}(\phi) = \Delta^{(m)\dot{\beta}_{m}}_{(\dot{\alpha}_{m})} \cdots \Delta^{(1)\dot{\beta}_{1}}_{(\dot{\alpha}_{1})} \Delta^{(1)\dot{\beta}_{1}}_{\beta_{1}} \cdots \Delta^{(m)\dot{\beta}_{m}}_{\beta_{m}} \phi_{\beta(m)\dot{\alpha}_{1} \cdots \dot{\alpha}_{n}} .$$  

(3.11b)

This makes it clear that both $\Pi^{(m,n)}$ and $\hat{\Pi}^{(m,n)}$ act as the identity operator on the space of transverse fields of rank $(m,n)$.

8We would like to point out that the spin projection operators for conformal fields in arbitrary conformally-flat backgrounds appeared in [27]. Their structure is similar to (3.11). However, the projectors of Ref. [27] were formulated using the conformal covariant derivative and it is non-trivial to switch to a description in terms of the torsion-free Lorentz covariant derivative used here.
3.3 Conformal higher-spin models

As an application of our analysis, we would like to discuss the action for a conformal higher-spin field $\phi_{\alpha(m)}\dot{\alpha}(n)$ in AdS. Making use of the linearised higher-spin Bach operators $B(m,n), \hat{B}(m,n) : V(m,n) \to V(n,m)$ defined by

\begin{align}
B_{\alpha(n)}\dot{\alpha}(m)(\phi) &= D(\alpha_1^{\beta_1} \ldots D\dot{\alpha}_m^{\beta_m} D(\alpha_1^{\dot{\beta}_1} \ldots D\dot{\alpha}_n^{\dot{\beta}_n} \dot{\phi}_{\beta_1 \ldots \beta_m}) \dot{\beta}_n), \quad (3.12a) \\
\hat{B}_{\alpha(n)}\dot{\alpha}(m)(\phi) &= D(\alpha_1^{\dot{\beta}_1} \ldots D\dot{\alpha}_n^{\dot{\beta}_n} D(\alpha_1^{\beta_1} \ldots D\dot{\alpha}_m^{\beta_m} \dot{\phi}(m)\beta_1 \ldots \beta_n)), \quad (3.12b)
\end{align}

which were introduced in [27], the action takes the form

\begin{align}
S_{\text{CHS}}^{(m,n)}[\phi, \bar{\phi}] &= i^{m+n} \int d^4x \bar{\phi}_{\alpha}^{\alpha(n)}\dot{\alpha}(m) B_{\alpha(n)}\dot{\alpha}(m)(\phi) + c.c. \quad (3.13a) \\
&= i^{m+n} \int d^4x \phi_{\alpha}^{\alpha(n)}\dot{\alpha}(m) \hat{B}_{\alpha(n)}\dot{\alpha}(m)(\bar{\phi}) + c.c. \quad (3.13b)
\end{align}

This action is invariant under the gauge transformations

\[ \delta_\zeta \phi_{\alpha}^{\alpha(m)}\dot{\alpha}(n) = D(\alpha_1^{\zeta_1} \ldots \alpha_m^{\zeta_m})\dot{\alpha}(n) \]

which is equivalent to the fact that the descendants (3.12) are gauge invariant. The equation of motion for $\bar{\phi}_{\alpha}^{\alpha(n)}\dot{\alpha}(m)$ is the vanishing of the higher-spin Bach tensor

\[ B_{\alpha(n)}\dot{\alpha}(m)(\phi) = 0 \]

(3.15)

The gauge freedom (3.14) allows us to impose the transverse gauge

\[ \phi_{\alpha}^{\alpha(m)}\dot{\alpha}(n) \equiv \phi_{\alpha}^{\alpha(m)}\dot{\alpha}(n), \quad D^{\gamma\dot{\gamma}} \phi_{\alpha}^{\alpha(m-1)}\gamma\dot{\alpha}(n-1) = 0 \]

(3.16)

There are three separate scenarios that we should consider, the first of which occurs when $m = n = s$. In this case the bosonic higher-spin Bach operator $B^{(s,s)}$ coincides with $P^{(s,s)}$,

\[ B_{\alpha(s)}\dot{\alpha}(s)(\phi) = P_{\alpha(s)}\dot{\alpha}(s)(\phi) \]

(3.17)

As a consequence of (3.17a), this means that in the gauge (3.16) the Bach tensor factorises,

\[ B_{\alpha(s)}\dot{\alpha}(s)(\phi^T) = \prod_{t=1}^{s}(Q - \lambda_{(t,s,s)}S^2)\phi_{\alpha(s)}^{(s)}\dot{\alpha}(s) \]

(3.18)

\[ ^9\text{Gauge-invariant actions for conformal higher-spin fields on arbitrary conformally flat } d = 4 \text{ backgrounds were constructed in [27]. A few years earlier, Metsaev [28] developed the so-called ordinary derivative formulation for conformal higher-spin fields on AdS}_d. \]
and hence so too does the gauge fixed action.

Next, let us consider the case when \( n > m \). By taking appropriate derivatives of the Bach tensor (3.12a) one arrives at the following relation,

\[
D_{\dot{\alpha}m+1} \alpha_{n-1} \cdots D_{\dot{\alpha}n} \alpha_n \mathcal{B}(\alpha(n)\dot{\alpha}(m)) = \mathbb{P}_\alpha(m)\dot{\alpha}(n)(\phi),
\]

(3.19)

which may be inverted to give

\[
\mathcal{B}(\alpha(n)\dot{\alpha}(m))(\phi) = \left[ \prod_{t=m+1}^{n} (Q - \lambda(t,m,n)S^2) \right]^{-1} D_{\dot{\alpha}m+1} \beta_1 \cdots D_{\dot{\alpha}n} \beta_{n-m} \mathbb{P}_\alpha(m)\dot{\alpha}(m)\beta(n-m)(\phi).
\]

(3.20)

It follows from (3.19) that in the gauge (3.16), the Bach operator factorises as

\[
\mathcal{B}(\alpha(n)\dot{\alpha}(m))(\phi^T) = \prod_{t=1}^{m} (Q - \lambda(t,m,n)S^2) D_{\dot{\alpha}1} \beta_1 \cdots D_{\dot{\alpha}n-m} \beta_{n-m} \phi^T(\alpha_{n-m+1} \cdots \alpha_{n} \dot{\alpha}(m)\beta(n-m)).
\]

(3.21)

We see that due to the mismatch of \( m \) and \( n \), the conformal operator \( \mathcal{B}^{(m,n)} \) does not factorise wholly into products of second-order operators. However, using (3.19) it is easy to see that for transverse fields the following equation can be derived from (3.15),

\[
\prod_{t=1}^{n} (Q - \lambda(t,m,n)S^2) \phi^T(\alpha(n)\dot{\alpha}(m)) = 0.
\]

(3.22)

If on the other hand \( m > n \), then the Bach tensor may be written in terms of \( \mathbb{P}(m,n) \)

\[
\mathcal{B}(\alpha(n)\dot{\alpha}(m))(\phi) = \mathcal{D}_{\dot{\alpha}m+1} \beta_1 \cdots \mathcal{D}_{\dot{\alpha}n} \beta_{n-m} \mathbb{P}_\alpha(m)\dot{\alpha}(m)\beta(n-m)(\phi).
\]

(3.23)

Once again, from (3.19) it follows that in the transverse gauge \( \mathcal{B}^{(m,n)} \) factorises as

\[
\mathcal{B}(\alpha(n)\dot{\alpha}(m))(\phi^T) = \prod_{t=1}^{m} (Q - \lambda(t,m,n)S^2) \mathcal{D}_{\dot{\alpha}1} \beta_1 \cdots \mathcal{D}_{\dot{\alpha}n-m} \beta_{n-m} \phi^T(\alpha_{n-m+1} \cdots \alpha_{n} \dot{\alpha}(m)\beta(n-m)).
\]

(3.24)

This time the higher-derivative equation derivable from (3.15) is

\[
\prod_{t=1}^{m} (Q - \lambda(t,m,n)S^2) \phi^T(\alpha(n)\dot{\alpha}(m)) = 0.
\]

(3.25)

An interesting observation is that according to the definition (3.5), when \( m \neq n \) there is a discrete set of mass values corresponding to the range \( \min(m,n) < t \leq \max(m,n) \) which are not partially massless but which enter the spectrum of the operators (3.22) and (3.25).

\footnote{Due to (1.1), the left hand side of (3.19) is automatically totally symmetric in its dotted indices.}
To obtain the effective actions corresponding to the CHS models with $m \neq n$, it is convenient to make use of the method of squaring which is always applied in the spinor theory. In this method we have to deal with the operator \( \hat{B}^{(m,m)}(m,m) \phi_{\alpha(m)\dot{\alpha}(n)} = \prod_{t=1}^{m} (Q - \lambda_{(t,m,n)}S^2) \mathbb{P}^{(m,n)} \phi_{\alpha(m)\dot{\alpha}(n)} \) (3.26)

which for a transverse field becomes
\[
\hat{B}^{(m,m)}(m,m) \phi_{\alpha(m)\dot{\alpha}(n)} = \prod_{t=1}^{m} (Q - \lambda_{(t,m,n)}S^2) \prod_{k=1}^{n} (Q - \lambda_{(k,m,n)}S^2) \phi_{\alpha(m)\dot{\alpha}(n)}.
\]

Finally, we note that in the case of unconstrained fields, the action (3.13a) may be rewritten in terms of the projectors. In the ordinary bosonic $m = n = s$ case it takes the form
\[
S_{CHS}^{(s,s)} = (-1)^s \int d^4 x \phi_{(s)\dot{\alpha}(s)} \prod_{t=1}^{s} (Q - \lambda_{(t,s,s)}S^2) \Pi_{\alpha(s)\dot{\alpha}(s)}(\phi) + c.c.,
\]
while in the fermionic $m = n - 1 = s$ case it becomes
\[
S_{CHS}^{(s,s+1)} = (-1)^{s} \int d^4 x \phi_{(s+1)\dot{\alpha}(s)} \prod_{t=1}^{s} (Q - \lambda_{(t,s,s+1)}S^2) D_{\alpha(s+1)\beta(s)} \Pi_{\alpha(s)\dot{\alpha}(s)\beta}(\phi) + c.c.
\]

For lower-spin values $s = 3/2$ and $s = 2$, corresponding to $m = n - 1 = 1$ and $m = n = 2$ respectively, the factorisation of the conformal operators (3.12) was observed long ago in [14,29,30]. This factorisation was conjectured, based on lower-spin examples, by Tseytlin [31] for the bosonic $m = n$ and fermionic $m = n - 1$ cases, and also by Joung and Mkrtchyan [32,33] for certain bosonic CHS models. More recently, the factorisation was proved by several groups [28,34,35] for those bosonic CHS models on AdS$_d$, with even $d$, which are described by completely symmetric arbitrary spin conformal fields (the $m = n$ case in four dimensions).

Using the formalism of spin projector operators in AdS$_4$ developed in this work, the known factorisation properties for $m = n$ follow immediately, and are captured through the expression (3.18). We have also provided the first derivation of the factorisation for conformal operators of arbitrary Lorentz type ($m/2, n/2$), which is encapsulated by expressions (3.21) and (3.24). This encompasses the case of arbitrary fermionic spin, $m = n - 1$, which to our knowledge was not covered previously in the literature. As in the bosonic case, we find that the spectrum of (3.21) and (3.24) consists of all partial masses. In contrast however, we find that the spectrum of the wave equations (3.22) and (3.25) contain a discrete set of massive values.

It would be interesting to re-derive our results using the ambient space approach, see [36,38] and references therein.
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A Technical results

Here we prove that the system of equations (3.1) is invariant under depth $t$ gauge transforma-
tions (3.6) for the mass values (3.5).

To begin with, it is clear that (3.1a) is gauge invariant only if the gauge parameter is also
on-shell with the same (pseudo-)mass $\mu$

\[(Q - \mu^2)\zeta_{\alpha(m-t)\tilde{\alpha}(n-t)} = 0.\]  
(A.1)

Additionally, we need to ensure that the gauge variation of the transverse condition (3.1b)
vanishes,

\[0 = D^{\beta\tilde{\gamma}}\delta_\epsilon \phi_{\alpha(m-1)\tilde{\alpha}(n-1)\beta} \ . \]  
(A.2)

To compute the right hand side of (A.2) it is useful to introduce the auxiliary commuting
spinor variables $\Upsilon^\alpha$ and $\bar{\Upsilon}\dot{\alpha}$. Associated with a tensor field $\phi_{\alpha(m)\dot{\alpha}(n)}$ of Lorentz type $(\frac{m}{2}, \frac{n}{2})$ is
a homogeneous polynomial $\phi_{(m,n)}(\Upsilon, \bar{\Upsilon})$ of degree $(m, n)$ defined by

\[\phi_{(m,n)} := \Upsilon^{\dot{\alpha}1} \ldots \Upsilon^{\dot{\alpha}m} \bar{\Upsilon}\dot{\alpha}1 \ldots \bar{\Upsilon}\dot{\alpha}n \phi_{\alpha1\ldots\alpha m\dot{\alpha}1\ldots\dot{\alpha}n}. \]  
(A.3)

We denote the space of such homogeneous polynomials as $\mathcal{H}_{(m,n)}$.

Next we introduce two operators

\[D_{(1,1)} := \Upsilon^\alpha\bar{\Upsilon}\dot{\alpha}D_{\alpha\dot{\alpha}}, \quad D_{(-1,-1)} := D^{\alpha\dot{\alpha}} \frac{\partial}{\partial \Upsilon^\alpha} \frac{\partial}{\partial \bar{\Upsilon}\dot{\alpha}} \equiv D^{\alpha\dot{\alpha}} \partial_\alpha \bar{\partial}_\dot{\alpha}, \]  
(A.4)

which increase and decrease the degree of homogeneity by $(1, 1)$ and $(-1, -1)$ respectively. They may be shown to satisfy the algebra

\[\left[D_{(1,1)}, D_{(-1,-1)}\right] = (\Upsilon + 1)(\Box + 4S^2\bar{M}) + (\bar{\Upsilon} + 1)(\Box + 4S^2M), \]  
(A.5)
where we have defined

$$\Upsilon^\alpha = \Upsilon^\alpha \partial_\alpha , \quad \Upsilon_{\phi(m,n)} = m \phi(m,n), \tag{A.6a}$$

$$\bar{\Upsilon}^{\dot{\alpha}} = \bar{\Upsilon}^{\dot{\alpha}} \partial_{\dot{\alpha}}, \quad \bar{\Upsilon}_\phi(m,n) = n \phi(m,n), \tag{A.6b}$$

$$M^\alpha = \Upsilon^\alpha \partial^\beta M_{\alpha\beta}, \quad M_{\phi(m,n)} = -\frac{1}{2} m (m + 2) \phi(m,n), \tag{A.6c}$$

$$\bar{M}^{\dot{\alpha}} = \bar{\Upsilon}^{\dot{\alpha}} \partial^{\dot{\beta}} M_{\dot{\alpha}\dot{\beta}}, \quad \bar{M}_\phi(m,n) = -\frac{1}{2} n (n + 2) \phi(m,n), \tag{A.6d}$$

Then, via induction on $k$ it is possible to show that for any $\phi(m,n) \in \mathcal{H}(m,n)$, the following identity holds true

$$\left[ \mathcal{D}_{(-1, -1)} \mathcal{D}_{(1, 1)} \cdots \mathcal{D}_{(1, 1)} \right]^{-k\text{-times}} \phi(m,n) = -k(m + n + k + 1) \left( Q - \lambda_{(k; m + k, n + k)} S^2 \right) \times \left[ \mathcal{D}_{(1, 1)} \mathcal{D}_{(1, 1)} \cdots \mathcal{D}_{(1, 1)} \right]^{-\text{(k-1)-times}} \phi(m,n). \tag{A.7}$$

Finally, using the operators (A.4), one may show that the condition (A.2) is equivalent to

$$0 = \mathcal{D}_{(-1, -1)} \mathcal{D}_{(1, 1)} \cdots \mathcal{D}_{(1, 1)} \zeta(m-t, n-t),$$

$$= \mathcal{D}_{(1, 1)} \cdots \mathcal{D}_{(1, 1)} \mathcal{D}_{(-1, -1)} \zeta(m-t, n-t),$$

$$t(m + n - t + 1) \left( Q - \lambda_{(t, m, n)} S^2 \right) \mathcal{D}_{(1, 1)} \cdots \mathcal{D}_{(1, 1)} \zeta(m-t, n-t),$$

where $t$ is the depth and we have used (A.7) in the second line. The first term vanishes if $\zeta_{\alpha(m-t)\dot{\alpha}(n-t)}$ is transverse

$$0 = \mathcal{D}^{\beta\dot{\beta}} \zeta_{\alpha(m-t-1)\beta\dot{\alpha}(n-t-1)\dot{\beta}}, \tag{A.8}$$

whilst the second term vanishes if the mass in (A.1) satisfies $\mu^2 = \lambda_{(t, m, n)} S^2$.

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