NORMAL DEVIATION OF SYNCHRONIZATION OF STOCHASTIC COUPLED SYSTEMS

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Abstract. This paper will prove the normal deviation of the synchronization of stochastic coupled system. According to the relationship between the stationary solution and the general solution, the martingale method is used to prove the normal deviation of the fixed initial value of the multi-scale system, thereby obtaining the normal deviation of the stationary solution. At the same time, with the relationship between the synchronized system and the multi-scale system, the normal deviation of the synchronization is obtained.

1. Introduction. Synchronization has a wide range of applications in chemical kinetics, materials science, fluid dynamics and finance, see e.g. [1, 2]. In mathematics, there are also a lot of works on synchronization, see e.g. [3, 6, 5, 19]. In particular, Al-Azzawi et al. [3] and Caraballo et al. [6] investigated the effect of additive noise on the synchronization of coupled dissipative system driven by Itô stochastic differential equations (SDEs) with asymptotically stationary solution. In addition, when the coupled system meets certain conditions, the convergence rate of synchronization is also obtained.

Caraballo and Kloeden in [6] considered the following two SDEs in $\mathbb{R}^{2d}$

$$
\begin{align*}
    dX_t &= f(X_t)dt + \alpha dW^1_t, \\
    dY_t &= g(Y_t)dt + \beta dW^2_t,
\end{align*}
$$

where $\alpha, \beta \in \mathbb{R}^d$ are constant vectors with no components equal to zero, $W^1_t, W^2_t$ are independent two-sided scalar Wiener processes and the continuously differentiable functions $f, g$ satisfy the one-sided dissipative Lipschitz conditions in Assumption 7. The coupled synchronized system is

$$
\begin{align*}
    dX'_t &= f(X'_t)dt + \nu(Y'_t - X'_t)dt + \alpha dW^1_t, \\
    dY'_t &= g(Y'_t)dt + \nu(X'_t - Y'_t)dt + \beta dW^2_t
\end{align*}
$$

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with a coupling coefficient $\nu > 0$. They proved that the coupled synchronized system has a unique stationary solution $(X^\nu_t, Y^\nu_t)$, which is pathwise globally asymptotically stable [6, 7]. Moreover,

$$(X^\nu_t, Y^\nu_t) \to (Z_t, Z_t) \text{ as } \nu \to \infty,$$

where $Z_t$ is the unique stationary solution of the averaged SDE

$$dZ_t = \frac{1}{2} [f(Z_t) + g(Z_t)] dt + \frac{1}{2} \alpha dW^1_t + \frac{1}{2} \beta dW^2_t.$$

This phenomenon is called synchronization, that is, the unique asymptotically stationary solution of the coupled system converges to the unique asymptotically stationary solution of the averaged system.

To be more precise, in this paper we consider the following system in [3] with additive noise in $\mathbb{R}^{2d}$

$$
\begin{align*}
\{ & \quad dX_t = f(X_t)dt + \alpha dW_t, \\
& \quad dY_t = g(Y_t)dt + \beta dW_t,
\end{align*}
$$

where $\alpha, \beta \in \mathbb{R}^{d \times n}$ are constant matrices, $W_t$ is a two-sided $\mathbb{R}^n$ valued Wiener process and the continuously differentiable functions $f, g$ satisfy the one-sided dissipative Lipschitz conditions in Assumption 7. The coupled system is

$$
\begin{align*}
\{ & \quad dX^\nu_t = f(X^\nu_t)dt + \nu (Y_t^\nu - X^\nu_t)dt + \alpha dW_t, \\
& \quad dY^\nu_t = g(Y^\nu_t)dt + \nu (X_t^\nu - Y^\nu_t)dt + \beta dW_t
\end{align*}
$$

with a coupling coefficient $\nu > 0$. They proved that the coupled synchronized system (1) has a unique stationary solution $(X^\nu_t, Y^\nu_t)$, which is pathwise globally asymptotically stable. Moreover,

$$(X^\nu_t, Y^\nu_t) \to (Z_t, Z_t) \text{ as } \nu \to \infty,$$

where $Z_t$ is the unique stationary solution of the averaged SDE

$$dZ_t = \frac{1}{2} [f(Z_t) + g(Z_t)] dt + \frac{1}{2} (\alpha + \beta) dW_t. \quad (2)$$

Furthermore, the convergence rate of synchronization is obtained, that is

$$E|X^\nu_t - Y^\nu_t|^2 + E|X^\nu_t - Z_t|^2 \leq \frac{C}{\nu}.$$

This result can be viewed as a version of the law of large numbers. The central limit theorem corresponds to the law of large numbers, so that the following problem is to prove the central limit theorem for the synchronized system.

From the above, to the best knowledge of the authors, the existing literature about synchronization only shows the results of synchronization and the corresponding convergence rate, leaving the central limit theorem of synchronization unsolved. Therefore, this paper mainly introduces the central limit theorem of synchronized system. We show the normalized difference $\frac{1}{\sqrt{\nu}}(X^\nu_t - Y^\nu_t)$ converges weakly to $Z_1(t)$ as $\nu$ tends to infinity, where $Z_1(t)$ is the unique stationary solution of the SDE

$$dZ_1(t) = -2Z_1(t)dt + \frac{1}{2} (\alpha - \beta) dW_t,$$

the normalized difference $\sqrt{\nu} \left( \frac{1}{2}(X^\nu_t + Y^\nu_t) - Z_t \right)$ converges weakly to $Z_2(t)$ as $\nu$ tends to infinity, where $Z_2(t)$ is the unique stationary solution of the SDE

$$dZ_2(t) = \frac{1}{2} [D_x f(Z_t) + D_x g(Z_t)] Z_2(t) dt.$$
Comparing with the synchronization conclusions in previous articles, these results provide a better approximation of the limit behavior of the synchronized system. Here is a simple example to illustrate the significance to the problem we are studying.

**Example 1.1.** Let us consider the linear equation in \( \mathbb{R}^2 \)

\[
\begin{align*}
\left\{ \begin{array}{l}
 dX'_t = -X'_t \, dt + \nu(Y'_t - X'_t) \, dt + \alpha dW_t, \\
 dY'_t = -Y'_t \, dt + \nu(X'_t - Y'_t) \, dt + \beta dW_t,
\end{array} \right.
\]

where \( \alpha, \beta \in \mathbb{R} \) are constants, \( W_t \) is a two-sided scalar Wiener process. The averaged SDE is

\[
dZ_t = -Z_t \, dt + \frac{1}{2}(\alpha + \beta) dW_t.
\]

Then

\[
\begin{align*}
\left\{ \begin{array}{l}
 d(X'_t + Y'_t) = -(X'_t + Y'_t) \, dt + (\alpha + \beta) dW_t, \\
 d(X'_t - Y'_t) = -(2\nu + 1)(X'_t - Y'_t) \, dt + (\alpha - \beta) dW_t.
\end{array} \right.
\]

We have

\[
X'_t - Y'_t = (\alpha - \beta) \int_{-\infty}^{t} e^{-2(\nu+1)(t-s)} \, dW_s
\]

and

\[
X'_t + Y'_t = 2Z_t = (\alpha + \beta) \int_{-\infty}^{t} e^{-(t-s)} \, dW_s,
\]

where \( X'_t, Y'_t \) and \( Z_t \) are the stationary solutions of the coupled SDEs (3) and the averaged SDE (4). Therefore,

\[
\begin{align*}
2X'_t &= (\alpha + \beta) \int_{-\infty}^{t} e^{-(t-s)} \, dW_t + (\alpha - \beta) \int_{-\infty}^{t} e^{-2(\nu+1)(t-s)} \, dW_t, \\
2Y'_t &= (\alpha + \beta) \int_{-\infty}^{t} e^{-(t-s)} \, dW_t - (\alpha - \beta) \int_{-\infty}^{t} e^{-2(\nu+1)(t-s)} \, dW_t
\end{align*}
\]

and

\[
2(X'_t - Z_t) = -2(Y'_t - Z_t) = X'_t - Y'_t = (\alpha - \beta) \int_{-\infty}^{t} e^{-2(\nu+1)(t-s)} \, dW_s.
\]

Thus, we obtain the synchronization result with convergence rate

\[
E \left| \frac{1}{2} X'_t - \frac{1}{2} Y'_t \right|^2 = \frac{(\alpha - \beta)^2}{4} E \left| \int_{-\infty}^{t} e^{-2(\nu+1)(t-s)} \, dW_s \right|^2 \leq \frac{(\alpha - \beta)^2}{8(2\nu + 1)},
\]

Meanwhile, we get

\[
\frac{1}{2} \sqrt{\nu}(X'_t - Y'_t) = \frac{1}{2}(\alpha - \beta) \sqrt{\nu} \int_{-\infty}^{t} e^{-2(\nu+1)(t-s)} \, dW_s.
\]

Let \( \nu \) tend to infinity,

\[
\lim_{\nu \to \infty} E \frac{1}{2} \sqrt{\nu}(X'_t - Y'_t) = 0
\]

and

\[
\lim_{\nu \to \infty} Var \left[ \frac{1}{2} \sqrt{\nu}(X'_t - Y'_t) \right] = \lim_{\nu \to \infty} \frac{1}{4}(\alpha - \beta)^2 \nu \int_{-\infty}^{t} e^{-2(\nu+1)(t-s)} \, ds = \frac{1}{16}(\alpha - \beta)^2.
\]
Using Lemma 3.8, one can then derive that the distribution of \( \frac{1}{\sqrt{\nu}}(X^n_t - Y^n_t) \) is asymptotically Gaussian with expectation zero and variance \( \frac{1}{16}(\alpha - \beta)^2 \).

From the example, the previous articles only introduced the convergence rate of synchronization, like (5). However, in this paper we give a better estimate that \( \frac{1}{2}\sqrt{\nu}(X^n_t - Y^n_t) \) converges weakly to a Gaussian process as \( \nu \) tends to infinity. At the same time, according to Theorem 4.4, we get the Gaussian process is the stationary solution of

\[
dZ_1(t) = -2Z_1(t)dt + \frac{1}{2}(\alpha - \beta)dW_t.
\]

This is the main innovation of our research.

In order to solve these problems, we apply the method in [14], which mainly transforms the coupled system (1) to a multi-scale system and then discusses the normal deviation of synchronization under the framework of the normal deviation of stationary solution of the multi-scale system. We can construct some equivalence relations and convert the synchronized system (1) into the multi-scale system, as shown below.

Substituting \( \frac{1}{\nu} = \epsilon \), \( 2\tilde{X}^\epsilon_t = X^n_t + Y^n_t \) and \( 2\sqrt{\epsilon}\tilde{Y}^\epsilon_t = X^n_t - Y^n_t \) into the SDEs (1) gives that

\[
\begin{align*}
d\tilde{X}^\epsilon_t &= \frac{1}{2}[f(\tilde{X}^\epsilon_t + \sqrt{\epsilon}\tilde{Y}^\epsilon_t) + g(\tilde{X}^\epsilon_t - \sqrt{\epsilon}\tilde{Y}^\epsilon_t)]dt + \frac{1}{2}(\alpha + \beta)dW_t, \\
d\tilde{Y}^\epsilon_t &= \frac{1}{2\sqrt{\epsilon}}[f(\tilde{X}^\epsilon_t + \sqrt{\epsilon}\tilde{Y}^\epsilon_t) - g(\tilde{X}^\epsilon_t - \sqrt{\epsilon}\tilde{Y}^\epsilon_t)]dt - \frac{\beta}{2\sqrt{\epsilon}}\tilde{Y}^\epsilon_t dt + \frac{1}{2\sqrt{\epsilon}}(\alpha - \beta)dW_t. 
\end{align*}
\]

Thus, to achieve the normal deviation of synchronization results of the coupled stochastic system (1), we need to verify there exists a unique stationary solution \((\tilde{X}^\epsilon_t, \tilde{Y}^\epsilon_t)\), such that \( \frac{1}{\sqrt{\epsilon}}(\tilde{X}^\epsilon_t - Z_t) \) and \( \tilde{Y}^\epsilon_t \) converge weakly to a Gaussian process and a stochastic process as \( \epsilon \) decreases to zero, respectively. For this reason, we must consider the normal deviation of multi-scale system (6).

More generally, we consider the following fast-slow system in \( \mathbb{R}^{2d} \)

\[
\begin{align*}
dX^\epsilon_t &= f(X^\epsilon_t, Y^\epsilon_t, \epsilon)dt + \sigma_1dW_t, \\
dY^\epsilon_t &= \frac{1}{\sqrt{\epsilon}}g(X^\epsilon_t, Y^\epsilon_t, \epsilon)dt + \frac{1}{\sqrt{\epsilon}}\sigma_2dW_t 
\end{align*}
\]

with initial value \( X^\epsilon_0 = x_0, \ Y^\epsilon_0 = y_0, \sigma_1, \sigma_2 \in \mathbb{R}^{d \times n} \) are constant matrices and \( W_t \) is a two-sided \( \mathbb{R}^{n} \) valued Wiener process.

Under some dissipative conditions, the dynamics for \( Y^\epsilon_t \) with \( X^\epsilon_t = x \) fixed is ergodic with a unique invariant measure \( \mu^\epsilon_x(dy) \). When \( \epsilon \) tends to zero, it follows from Theorem 4.1 in [15] that \( X^\epsilon_t \) converges in mean square sense to a SDE

\[
dZ_t = \tilde{f}(Z_t)dt + \sigma_1dW_t
\]

with initial value \( Z_0 = x_0 \), where

\[
\tilde{f}(x) = \lim_{\epsilon \to 0} \int f(x, y, \epsilon)\mu^\epsilon_x(dy).
\]

We consider the limiting behavior of the normal deviation of the stationary solution that is different from the normal deviation of the general solution which is the solution with the fixed initial value in the previous articles. The history of similar central limit theorem in multi-scale SDEs with fixed initial value is long. For \( f(X^\epsilon_t, Y^\epsilon_t, \epsilon) = f(t) \), which is a stochastic process that satisfies a strong mixing condition and periodicity, it follows from [11, 17, 22] that the normalized difference \( \frac{1}{\sqrt{\epsilon}}(X^\epsilon_t - Z_t) \) has a Gaussian limit distribution. When \( f(X^\epsilon_t, Y^\epsilon_t, \epsilon) \) only depends
on the slow system $X^\epsilon_t$ and the random process $\xi^\epsilon_t$, the normalized difference converges weakly to a Gaussian Markov process in [10, 12]. Actually, here they do not introduce the normal deviation of a system with fully coupled system i.e. the fast system depends on the slow variable as well as the fast ones.

Recently, Cerrai [8] considered the normalized difference from a finite dimension to an infinite dimension. The techniques used in [8] to prove its results mainly come from paper [10], only the case of $f(X^\epsilon_t, Y^\epsilon_t, \epsilon) = f(X^\epsilon_t, \xi^\epsilon_t)$ is considered, but the case of fully coupled is also not considered. However, at present such techniques do not allow us to treat the more general case of fully coupled stochastic systems for which normal deviation phenomenon occurs. Wang et al. [23] proposed a new technique of a martingale approach to prove that the normal deviation is described by a Gaussian process with the fully coupled stochastic system in an infinite dimension. To the best knowledge of the authors, the existing literature does not address the normal deviation of stationary solutions for the multi-scale system.

Note that we can not directly apply the arguments about the normal deviation that have been presented in the previous literature. There are two reasons for this. The first reason has to do with modeling considerations. We do not make periodic assumptions, but impose conditions on the fast system to guarantee ergodicity. We are also interested in the study of the normal deviations of the $f, g$ dependent on $X^\epsilon_t, Y^\epsilon_t$ and $\epsilon$, whereas the $f, g$ in [23] only dependent on $X^\epsilon_t, Y^\epsilon_t$. The second reason is that the normal deviation of the stationary solution needs to be proved in the synchronized system. This is the main innovation of this paper.

Let us now explain the organization of the rest of this paper. In Section 2, we introduce the sufficient condition for the stationary solution, which is used for proving the existence and uniqueness of stationary solution and invariant measure. And we construct the relationship between the stationary solution and the general solution of an equation. Moreover, we consider the asymptotic behavior of $Y^\epsilon_t$ and the corresponding invariant measure $\mu^\epsilon_x(dy)$. Section 3 contains proof of tightness and weak convergence of the normalized difference $\frac{1}{\sqrt{\epsilon}}(X^\epsilon_t - Z_t)$ as $\epsilon$ tends to zero with fixed initial value. Section 4 extends our results from the general solution to the stationary solution and gives normal deviation in the sense of synchronization.

We will make some assumptions.

Assumption 1. The function $f(x, y, \epsilon)$ has bounded continuous first and second partial derivatives with respect to $x$.

Assumption 2. (One-sided dissipative Lipschitz condition) For all $x, y, x_1, x_2, y_1, y_2$, there exist constants $k_1, k_2 > 0$ such that

$$\langle x_1 - x_2, f(x_1, y, \epsilon) - f(x_2, y, \epsilon) \rangle \leq -k_1|x_1 - x_2|^2,$$

$$\langle y_1 - y_2, g(x, y_1, \epsilon) - g(x, y_2, \epsilon) \rangle \leq -k_2|y_1 - y_2|^2.$$ 

Assumption 3. (Lipschitz condition) The function $g(x, y, \epsilon)$ is Lipschitz continuous with respect to $x, y$, specially for all $x, y, x_1, x_2, y_1, y_2$, there exists a constant $L > 0$ such that

$$|g(x_1, y, \epsilon) - g(x_2, y, \epsilon)|^2 \leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2).$$
Assumption 4. (Linear growth condition) For all $x, y$, there exists a constant $K > 0$ such that

$$|f(x, y, \epsilon)|^2 + |g(x, y, \epsilon)|^2 \leq K(|x|^2 + |y|^2 + 1).$$

Assumption 5. For any fixed $x, y$,

$$\lim_{\epsilon \to 0} f(x, y, \epsilon) = \hat{f}(x, y),$$

$$\lim_{\epsilon \to 0} g(x, y, \epsilon) = \hat{g}(x, y).$$

Moreover there exists a constant $C > 0$ independent of $\epsilon$ such that

$$|f(x, y, \epsilon) - \hat{f}(x, y)|^2 \leq C\epsilon(|x|^2 + |y|^2 + 1).$$

We will introduce some notations. Let us denote $\langle \cdot, \cdot \rangle$ and $|\cdot|$ for the usual scalar product and the Euclidean norm in $\mathbb{R}^d$, respectively. $\text{Var}(X)$ denotes the variance of the random variable $X$ defined on a probability space $(\Omega, \mathcal{F}, P)$. Throughout this paper, the capital letter $C$ denotes a constant whose value may change from line to line. And $C$ is independent of $\epsilon$.

2. Asymptotic behavior of the fast system. This section is dedicated to considering the asymptotic behavior of the fast system $Y_t^\epsilon$ and the corresponding invariant measure $\mu_{t, x}(dy)$. Meanwhile, we introduce the concept of stationary solution and give a theorem to guarantee the existence and uniqueness of a stationary solution and invariant measure. Moreover, we construct the relationship between the stationary solution and the general solution of a SDE.

We consider the asymptotic behavior of the fast system $Y_t^\epsilon$ and the corresponding invariant measure $\mu_{t, x}(dy)$. The exponentially mixing of an invariant measure is defined as follows.

Definition 2.1. Assume that $P_t, t > 0$, is a stochastically continuous Markov semigroup on a polish space $U$ with invariant measure $\mu$. There exist $r > 0$ and a positive function $c(\cdot)$, such that, for any bounded Lipschitz continuous function $\varphi$, for all $x \in U$ and $t > 0$,

$$\left| P_t \varphi(x) - \int_U \varphi(y) \mu(dy) \right| = \left| \int_U P_t(x, dy) \varphi(y) - \int_U \varphi(y) \mu(dy) \right| \leq c(x) e^{-rt} ||\varphi||_{Lip}$$

holds, then the measure $\mu$ will be called exponentially mixing (see [9, pages 39-40]), where $||\varphi||_{Lip}$ is the Lipschitz constant of $\varphi$.

Theorem 2.2. If Assumptions 1-4 are satisfied, for any fixed $x$, the system

$$dY_t^{x, \epsilon} = \frac{1}{\epsilon} g(x, Y_t^{x, \epsilon}, \epsilon) dt + \frac{1}{\sqrt{\epsilon}} \sigma_d dW_t$$

(9)

with initial value $Y_0^{x, \epsilon} = y_0$, has a unique invariant measure denoting $\mu_{t, x}(dy)$. Moreover, the exponentially mixing property of $\mu_{t, x}(dy)$ is also established.

Before we give the proof of Theorem 2.2, we introduce the concept of random dynamical system generated by SDEs and give a theorem to guarantee the existence and uniqueness of stationary solution.
Let \((\Omega, \mathcal{F}, P)\) be a probability space. According to Arnold [4], a random dynamical system (RDS) \((\theta, \phi)\) on \(\Omega \times \mathbb{R}^d\) consists of a metric dynamical system \(\theta\) on \(\Omega\) and a cocycle mapping \(\phi: \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d\), with the following cocycle property
\[
\phi(t + s, \omega, x) = \phi(t, \phi(s, \omega, x)).
\]

In the next, we recall a definition of the stationary solution in [14, 18].

**Definition 2.3.** Let \(\varphi(t, \omega, x)\) be a RDS generated by a SDE. If there is a random variable \(X(\omega)\) which satisfies
\[
\varphi(t, \omega, X(\omega)) = X(\theta_t \omega), \text{ a.s.}
\]
we call \(X_t(\omega) := X(\theta_t \omega)\) a stationary solution of the SDE.

The main purpose of this section is to study stationary solutions of the SDE,
\[
\begin{cases}
    dX_t = f_1(X_t, Y_t)dt + \sigma_1 dW_t, \\
    dY_t = f_2(X_t, Y_t)dt + \sigma_2 dW_t
\end{cases}
\] (10)
with initial value \(X_{t_0} = x_0, Y_{t_0} = y_0\), \(\sigma_1, \sigma_2 \in \mathbb{R}^{d \times n}\) are constant matrices, \(W_t\) is a two-sided \(\mathbb{R}^n\) valued Wiener process. Taking \(\zeta_t = (X_t, Y_t)^T, G_1(\zeta_t) = (f_1(\zeta_t), f_2(\zeta_t))^T\) and \(G_2 = (\sigma_1, \sigma_2)^T\), (10) can be transformed into
\[
d\zeta_t = G_1(\zeta_t)dt + G_2 dW_t.
\]

Denote the unique solution \(\zeta_{\tau,t}\) starting at time \(\tau\), with initial value \(\zeta_{\tau,\tau} = (X_{t_0}, Y_{t_0})^T = (x_0, y_0)^T : = z_0 \in \mathbb{R}^{2d}, t \in (\tau, \infty)\) for any \(\tau \leq 0\). The solution \(\zeta_{\tau,t}\) generates an RDS \(\phi\) satisfying cocycle property
\[
\phi(t - \tau, \theta_\tau \omega, z_0) = \phi(t, \omega, \phi(-\tau, \theta_\tau \omega, z_0)) = \zeta_{\tau,t}
\]
with \(\phi(0, \theta_\tau \omega, z_0) = z_0\).

The following theorem guarantees the existence and uniqueness of stationary solution, which is taken from Theorem 2.2 in [14].

**Theorem 2.4.** If the one-sided dissipative Lipschitz condition
\[
\langle z_1 - z_2, G_1(z_1) - G_1(z_2) \rangle \leq -\lambda |z_1 - z_2|^2
\] (11)
is satisfied for \(\lambda > 0\), then we have the following conclusions

(I)
\[
E |z_\tau^0 - z_t^\omega|^2 \leq E |z_0 - z_0^t|^2 e^{-2\lambda(t-t_0)}
\] (12)
and
\[
E |z_t^\omega|^2 \leq C E(1 + |z_0|^2),
\]
where \(z_t^\omega\) is a solution of (10) with initial value \(z_0\).

(II) We can define \(\xi_t = \lim_{\tau \to -\infty} \zeta_{\tau,t}^0\). There exists a unique stationary solution \(\xi_t\) with a stationary distribution \(\mathcal{L}\). Moreover, stationary probability distribution \(\mathcal{L}\) agrees with the invariant measure \(\mu\), that is \(\lim_{t \to \infty} \mathcal{L}(\xi_t) = \mu\). In addition,
\[
E |\xi_t - z_t^\omega|^2 \leq E(1 + |z_0|^2) e^{-2\lambda(t-t_0)}.
\]

We use Theorem 2.4 to prove Theorem 2.2.
Proof of Theorem 2.2. For any solutions \( Y_{t}^{x,y_{0},\epsilon} \) and \( \bar{Y}_{t}^{x,y_{0},\epsilon} \) of the SDE (9) with the different initial values \( y_{0}, y'_{0} \), the Itô formula yields
\[
\frac{dE| Y_{t}^{x,y_{0},\epsilon} - Y_{t}^{x,y'_{0},\epsilon} |^{2}}{dt} = 2E \left( Y_{t}^{x,y_{0},\epsilon} - Y_{t}^{x,y'_{0},\epsilon}, \frac{1}{\epsilon} g(x, Y_{t}^{x,y_{0},\epsilon}, \epsilon) - \frac{1}{\epsilon} g(x, Y_{t}^{x,y'_{0},\epsilon}, \epsilon) \right) 
\leq -2k_{2}E \left( Y_{t}^{x,y_{0},\epsilon} - Y_{t}^{x,y'_{0},\epsilon} \right)^{2}.
\]
By the Gronwall inequality,
\[
E \left| Y_{t}^{x,y_{0},\epsilon} - Y_{t}^{x,y'_{0},\epsilon} \right|^{2} \leq e^{-2k_{2}(t-t_{0})} E|y_{0} - y'_{0}|^{2}.
\]
In view of Theorem 2.4, we then obtain that there exists a unique stationary solution \( \bar{Y}_{t}^{x,\epsilon} \) of the SDE (9). Hence,
\[
E| Y_{t}^{x,y_{0},\epsilon} - \bar{Y}_{t}^{x,\epsilon} |^{2} \leq e^{-2k_{2}(t-t_{0})} E(1 + |x|^{2} + |y_{0}|^{2}).
\]
Furthermore, by Theorem 2.4, the \( Y_{t}^{x,\epsilon} \) has a unique invariant probability measure \( \mu_{t}^{x}(dy) \). Then we get the exponentially mixing property of the invariant measure \( \mu_{t}^{x}(dy) \),
\[
\left| Eg(x, Y_{t}^{x,y_{0},\epsilon}, \epsilon) - \int_{-\infty}^{\infty} g(x, y, \epsilon) \mu_{t}^{x}(dy) \right| = \left| Eg(x, Y_{t}^{x,y_{0},\epsilon}, \epsilon) - Eg(x, \bar{Y}_{t}^{x,\epsilon}, \epsilon) \right| \leq \| g \|_{Lip} E| Y_{t}^{x,y_{0},\epsilon} - \bar{Y}_{t}^{x,\epsilon} | \leq C \sqrt{1 + |x|^{2} + |y_{0}|^{2}} e^{-\frac{t}{t_{0}}}(t-t_{0}).
\]
We now show what the invariant probability measure \( \mu_{t}^{x}(dy) \) converges to as \( \epsilon \to 0 \). By the time scale \( t = se \), the SDE (9) is transformed into
\[
\bar{Y}_{t}^{x,\epsilon} = g(x, \bar{Y}_{s}^{x,\epsilon}, \epsilon)ds + \sigma_{2}d\bar{W}_{s},
\]
where \( W_{s} = \frac{1}{\sqrt{\epsilon}} W_{se} \) is the scaled version of \( W_{s} \) and with the same distribution. Then the SDE (9) has a unique solution \( \bar{Y}_{t}^{x,\epsilon} \) with the same distribution as that of \( \bar{Y}_{t}^{x,\epsilon} \). We have the following lemma, see Lemma 3.3 in [14].

Lemma 2.5. If Assumptions 2-5 are satisfied, there exists a constant \( C > 0 \), which is independent of \( \epsilon \) and \( t \), such that
\[
E| \bar{Y}_{t}^{x,\epsilon} - \bar{Y}_{t}^{x,\epsilon} |^{2} \leq C \epsilon,
\]
where \( \bar{Y}_{t}^{x,\epsilon} \) is the unique stationary solution of (13) and \( \bar{Y}_{t}^{x} \) is the unique stationary solution of
\[
dY_{t}^{x} = \hat{g}(x, Y_{t}^{x})ds + \sigma_{2}d\bar{W}_{t}.
\]

3. Normal deviation of general solution. In this section, we prove the normal deviation of the general solution, which establishes weak convergence of the normalized difference \( \frac{1}{\sqrt{\epsilon}}(X_{t} - Z_{t}) \). We adopt the martingale method in [23] to prove the weak convergence, but the system in this paper depends on \( x, y, \epsilon \), i.e. \( f = f(x, y, \epsilon) \), \( g = g(x, y, \epsilon) \). Wang and Roberts in [23] dealt with the infinite dimensional case, a lot of restrictions of abstract operators are considered. However, these abstract operators are not needed in our paper.

Now we formulate the fundamental result of this section.
Theorem 3.1. If Assumptions 1 - 5 are satisfied, then the normalized difference
\[ Z_t^\epsilon := \frac{X_t^\epsilon - Z_t}{\sqrt{\epsilon}} \] in the space \( C([0,T];\mathbb{R}^d) \), where \( X_t^\epsilon \) and \( Z_t \) are the solutions of (7) and (8) with the same initial value \( x_0 \), converges weakly to \( Z_0^0 \), which \( Z_0^0 \) solves the SDE
\[ dZ_t^0 = F(Z_t)Z_t^0 dt + \sqrt{B(Z_t)}dW_t \]
with initial value \( Z_0^0 = 0 \), where \( F(x) = \frac{\partial f(x)}{\partial x} \) and
\[ B(x) = 2 \int_0^\infty E[\tilde{f}(x,\tilde{Y}^x_s) - \tilde{f}(x)](\tilde{f}(x,\tilde{Y}^x_0) - \tilde{f}(x))ds. \]

Proof. We have the following decomposition \( Z_t^\epsilon = \frac{1}{\sqrt{\epsilon}}(X_t^\epsilon - Z_t) = \lambda_t^\epsilon + \bar{\lambda}_t^\epsilon \), for \( 0 \leq t \leq T \),
\[ \frac{d\lambda_t^\epsilon}{dt} := \frac{1}{\sqrt{\epsilon}}[f(Z_t, \tilde{Y}^Z_t, \epsilon) - \tilde{f}(Z_t)], \lambda_0^\epsilon = 0, \]
\[ \frac{d\bar{\lambda}_t^\epsilon}{dt} := \frac{1}{\sqrt{\epsilon}}[f(X_t^\epsilon, Y_t^\epsilon, \epsilon) - f(Z_t, \tilde{Y}^Z_t, \epsilon)], \bar{\lambda}_0^\epsilon = 0, \]
where \( \tilde{Y}^x_t \) is the stationary solution of the SDE (9) for fixed \( x \in \mathbb{R}^d \).

Before discussing the normal deviation of the general solutions in detail, we give some auxiliary processes and conclusions which are used in the next proof. We also define the following auxiliary process in [10], which is a modification of the process \( (X_t^\epsilon, Y_t^\epsilon) \). Denote \( [x] \) to be the largest integer less than or equal to \( x \). Partitioning \([0,T]\) into subintervals of the same length \( \Delta \), we construct for \( t \in [k\Delta, (k+1)\Delta), k \geq 0 \), the process \( (\hat{X}^\epsilon_t, \hat{Y}^\epsilon_t) \) such that
\[ d\hat{Y}^\epsilon_t = \frac{1}{\epsilon}g(X_{k\Delta}^\epsilon, \hat{Y}^\epsilon_t, \epsilon)dt + \frac{1}{\epsilon}\sigma_2 dW_t, \hat{Y}_t^\epsilon = Y_0^\epsilon, t \in [k\Delta, (k+1)\Delta], \]
\[ \hat{X}^\epsilon_t = x + \int_0^t f(X_{s\Delta}^\epsilon, \hat{Y}_s^\epsilon, \epsilon)ds + \int_0^t \sigma_1 dW_s. \]

To derive normal estimate, we further need the following lemmas. The proofs are the same as the proofs of Lemma 3.7-Lemma 3.9 in [13], Lemma 3.5 and Lemma 3.6 in [15].

Lemma 3.2. For any \( T > 0 \), there exists a constant \( C > 0 \) independent of \( (\epsilon, \Delta) \) such that
\[ E|Y_t^\epsilon - \hat{Y}_t^\epsilon|^2 \leq C\Delta \]
for any \( t \in [0,T] \).

Lemma 3.3. The family of process \( \{Z_t^\epsilon, 0 \leq t \leq T, 0 < \epsilon \leq 1\} \), \( \{\lambda_t^\epsilon, 0 \leq t \leq T, 0 < \epsilon \leq 1\} \), \( \{\bar{\lambda}_t^\epsilon, 0 \leq t \leq T, 0 < \epsilon \leq 1\} \) are weakly compact in \( C([0,T];\mathbb{R}^d) \).
Proof. There exists a convenient criterion for tightness: Kolmogorov’s criterion of Remark A.5 in [21]. What we only need to verify is that there exist \( \epsilon \), and \( \Delta = \epsilon \), for all \( t \in [0, T] \). Assumption 1 yields that \( f \) satisfies the Lipschitz condition. Since \( f \) satisfies the Lipschitz condition, according to the definition of \( \bar{f} \), \( \bar{f} \) also satisfies the Lipschitz condition. Using Hölder’s inequality, Jensen’s inequality, some elementary inequalities and the Lipschitz conditions of \( f \) and \( \bar{f} \), we get

\[
E|Z_{t+h} - Z_t|^2 = \frac{1}{\epsilon^4} E \left| \int_t^{t+h} f(X_s^\epsilon, Y_s^\epsilon, \epsilon) - \bar{f}(Z_s) \, ds \right|^2 \\
\leq \frac{Ch^2}{\epsilon^4} E \left( \int_0^T |f(X_s^\epsilon, Y_s^\epsilon, \epsilon) - \bar{f}(Z_s)|^2 \, ds \right)^{\frac{1}{2}} \\
\leq \frac{Ch^2}{\epsilon^4} \left( \int_0^T E |f(X_s^\epsilon, Y_s^\epsilon, \epsilon) - \bar{f}(Z_s)|^2 \, ds \right)^{\frac{1}{2}} \\
\leq \frac{Ch^2}{\epsilon^4} \left( \int_0^T E \left| f \left(X_{s/\Delta}^\epsilon, Y_{s/\Delta}^\epsilon, \epsilon\right) - \bar{f} \left(X_{s/\Delta}^\epsilon\right) \right|^2 \, ds \right)^{\frac{1}{2}} \\
+ \frac{Ch^2}{\epsilon^4} \left( \int_0^T E \left| f \left(X_{s/\Delta}^\epsilon, Y_{s/\Delta}^\epsilon, \epsilon\right) - \bar{f} \left(X_{s/\Delta}^\epsilon\right) \right|^2 \, ds \right)^{\frac{1}{2}} \\
+ \frac{Ch^2}{\epsilon^4} \left( \int_0^T E \left| f \left(X_{s/\Delta}^\epsilon, Y_{s/\Delta}^\epsilon, \epsilon\right) - \bar{f} \left(X_{s/\Delta}^\epsilon\right) \right|^2 \, ds \right)^{\frac{1}{2}} \\
\leq \frac{Ch^2}{\epsilon^4} \left( \int_0^T E \left| X_s^\epsilon - X_{s/\Delta}^\epsilon \right|^2 + E|Y_s^\epsilon - \bar{y}_{s/\Delta}^\epsilon|^2 \, ds \right)^{\frac{1}{2}} \\
+ \frac{Ch^2}{\epsilon^4} \left( \int_0^T E \left| f \left(X_{s/\Delta}^\epsilon, Y_{s/\Delta}^\epsilon, \epsilon\right) - \bar{f} \left(X_{s/\Delta}^\epsilon\right) \right|^2 \, ds \right)^{\frac{1}{2}} \\
+ \frac{Ch^2}{\epsilon^4} \left( \int_0^T E \left| f \left(X_{s/\Delta}^\epsilon, Y_{s/\Delta}^\epsilon, \epsilon\right) - \bar{f} \left(X_{s/\Delta}^\epsilon\right) \right|^2 \, ds \right)^{\frac{1}{2}} \\
+ \frac{Ch^2}{\epsilon^4} \left( \int_0^T E\left| X_s^\epsilon - Z_s \right|^2 \, ds \right)^{\frac{1}{2}}.
\]

Taking \( \Delta = \epsilon \), we have, in light of Lemma 3.2,

\[
\frac{1}{\epsilon} \int_0^T E \left| X_s^\epsilon - X_{s/\Delta}^\epsilon \right|^2 \, ds \leq \frac{C_T \Delta}{\epsilon} = C_T,
\]

\[
\frac{1}{\epsilon} \int_0^T E|Y_s^\epsilon - \bar{y}_{s/\Delta}^\epsilon|^2 \, ds \leq \frac{C_T \Delta}{\epsilon} = C_T,
\]

\[
\frac{1}{\epsilon} \int_0^T E\left| X_s^\epsilon - Z_s \right|^2 \, ds \leq C_T.
\]
An argument similar to the one used in the proof of Theorem 3.10 in [13] shows that
\[ \frac{1}{\epsilon^2} \int_0^T E \left( f \left( X^\epsilon_s, Y^\epsilon_s, \epsilon \right) - f \left( Z_s, \bar{Y}^\epsilon_s, \epsilon \right) \right)^2 ds \leq C. \]

We get \( E \left| Z^\epsilon_t - Z \right|^2 \leq C h \frac{\epsilon}{\sqrt{r}}. \) This implies the weak compactness of the family of the processes \( Z^\epsilon_t \) in \( C([0,T]; \mathbb{R}^d). \)

Upon using Hölder’s inequality, Jensen’s inequality, some elementary inequalities and the Lipschitz condition of \( f, \) we get
\[
E |\bar{\lambda}^{t+h} - \bar{\lambda}^t|^2 = \frac{1}{\epsilon^2} E \left( \int_t^{t+h} f(X^\epsilon_s, Y^\epsilon_s, \epsilon) - f(Z_s, \bar{Y}^\epsilon_s, \epsilon) ds \right)^2
\leq \frac{Ch}{\epsilon^2} E \left( \int_0^T |f(X^\epsilon_s, Y^\epsilon_s, \epsilon) - f(Z_s, \bar{Y}^\epsilon_s, \epsilon)|^2 ds \right)^{\frac{3}{2}}
\leq \frac{Ch}{\epsilon^2} \left( \int_0^T E \left| X^\epsilon_s - Z_s \right|^2 + E \left| Y^\epsilon_s - \bar{Y}^\epsilon_s \right|^2 ds \right)^{\frac{3}{2}}
\leq \frac{Ch}{\epsilon^2} \left( \int_0^T E \left| X^\epsilon_s - Z_s \right|^2 + E \left| Y^\epsilon_s - \bar{Y}^\epsilon_s \right|^2 ds \right)^{\frac{3}{4}}
+ \frac{Ch}{\epsilon^2} \left( \int_0^T E \left| Y^\epsilon_s - \bar{Y}^\epsilon_s \right|^2 ds \right)^{\frac{1}{4}}.
\]

Firstly, it follows from \( X^\epsilon_s \) converges in the mean square sense to \( Z_s \) as \( \epsilon \) tends to zero that
\[
E \left| X^\epsilon_s - Z_s \right|^2 \leq C \epsilon.
\]

Secondly, taking account of Corollary 4.1, we obtain
\[
\int_0^T \frac{1}{\epsilon} E \left| Y^\epsilon_s - \bar{Y}^\epsilon_s \right|^2 ds \leq \int_0^T \frac{1}{\epsilon} (1 + |g_0|^2 + E|Z_s|^2) e^{-\frac{\epsilon}{2} s} ds \leq C.
\]

Thirdly, combining (7) and (9), we see that
\[
\frac{dE \left| Y^\epsilon_s - Y^\epsilon_{s+\epsilon} \right|^2}{ds} = \frac{2}{\epsilon} E \left( Y^\epsilon_s - Y^\epsilon_{s+\epsilon}, g(X^\epsilon_s, Y^\epsilon_s, \epsilon) - g(Z_s, Y^\epsilon_s, \epsilon) \right)
\leq \frac{2}{\epsilon} E \left( Y^\epsilon_s - Y^\epsilon_{s+\epsilon}, g(X^\epsilon_s, Y^\epsilon_s, \epsilon) - g(Z_s, Y^\epsilon_{s+\epsilon}, \epsilon) \right)
+ \frac{2k_2}{\epsilon} E \left| Y^\epsilon_s - Y^\epsilon_{s+\epsilon} \right|^2
+ \frac{k_2}{\epsilon^2} E \left| X^\epsilon_s - Z_s \right|^2
\leq \frac{k_2}{\epsilon} E \left| Y^\epsilon_s - Y^\epsilon_{s+\epsilon} \right|^2 + C.
\]
The Gronwall inequality then yields that
\[ E|Y_s^\varepsilon - Y_s^{Z_s,\varepsilon}|^2 \leq \frac{C}{k^2} \left( e^{-\frac{k}{2}s} - 1 \right). \]

Above all, the weak compactness of the family of the processes \( \tilde{\lambda}_t \) in \( C([0, T]; \mathbb{R}^d) \) is obtained.

In order to prove the weak compactness of measure corresponding to the \( \lambda_t \), we note that the relation
\[ E|\lambda_{t+h} - \lambda_t|^2 \leq CE|Z_{t+h}^\varepsilon - Z_t^\varepsilon|^2 + CE|\tilde{\lambda}_{t+h} - \tilde{\lambda}_t|^2 \leq Ch^2. \]

This estimate guarantees the weak compactness of the family of the processes \( \lambda_t, t \in [0, T] \). \( \square \)

From the weak compactness of the family of measures corresponding to the processes in the space \( C([0, T]; \mathbb{R}^d) \), in view of the Prohorov’s theorem, we can extract every sequence of such process contains a subsequence converging to a process. The weak convergence will be proved, if we can show that the distribution of the limit process does not depend on the choice of the subsequence. Our task now is to characterize the limit process.

Denote \( F(x, y, \varepsilon) = \frac{\partial f(x, y, \varepsilon)}{\partial x} \). According to the definition of \( \lambda_t \), it follows that
\[ Z_t^\varepsilon = \lambda_t + \frac{1}{\sqrt{\varepsilon}} \int_0^t \left[ f(X_s^\varepsilon, Y_s^\varepsilon, \varepsilon) - f(Z_s, Y_s^{Z_s,\varepsilon}, \varepsilon) - F(Z_s, Y_s^{Z_s,\varepsilon}, \varepsilon) \sqrt{\varepsilon} Z_s^\varepsilon ds \right. \\
+ \int_0^t (F(Z_s, Y_s^{Z_s,\varepsilon}, \varepsilon) - F(Z_s)) Z_s^\varepsilon ds + \int_0^t F(Z_s) Z_s^\varepsilon ds \\
+ \frac{1}{\sqrt{\varepsilon}} \int_0^t f(Z_s, Y_s^{Z_s,\varepsilon}, \varepsilon) - f(Z_s, Y_s^{Z_s,\varepsilon}, \varepsilon) ds \\
= \lambda_t + \int_0^t F(Z_s) Z_s^\varepsilon ds + \int_0^t I_1(s, \varepsilon, \omega) ds + \int_0^t I_2(s, \varepsilon, \omega) Z_s^\varepsilon ds \\
+ \int_0^t I_3(s, \varepsilon, \omega) ds, \tag{17} \]

where
\[ I_1(s, \varepsilon, \omega) = \frac{1}{\sqrt{\varepsilon}} \left[ f(X_s^\varepsilon, Y_s^\varepsilon, \varepsilon) - f(Z_s, Y_s^{Z_s,\varepsilon}, \varepsilon) - F(Z_s, Y_s^{Z_s,\varepsilon}, \varepsilon) \sqrt{\varepsilon} Z_s^\varepsilon \right], \]
\[ I_2(s, \varepsilon, \omega) = F(Z_s, Y_s^{Z_s,\varepsilon}, \varepsilon) - F(Z_s), \]
\[ I_3(s, \varepsilon, \omega) = \frac{1}{\sqrt{\varepsilon}} \left[ f(Z_s, Y_s^{Z_s,\varepsilon}, \varepsilon) - f(Z_s, Y_s^{Z_s,\varepsilon}, \varepsilon) \right]. \]

We will show that the third term, the fourth term and the fifth term on the right side of (17) have vanishing effect as \( \varepsilon \) tends to zero.

Hence we consider the simplified linearized equation
\[ \xi_t^\varepsilon = \lambda_t + \int_0^t F(Z_s) \xi_s^\varepsilon ds. \tag{18} \]

In accordance with our proof, we have to prove that \( \xi_t^\varepsilon \) converges weakly to a process \( Z_t^0 \). We divide our proof in two steps.

On the one hand, we need to verify the limit measure of \( \lambda_t \) on the space \( C([0, T]; \mathbb{R}^d) \) solves the martingale problem. Let \( Q^\varepsilon \) denote the probability measure
of $\lambda_\varepsilon^t$ on the space $C([0, T]; \mathbb{R}^d)$. According to Lemma 3.3, we can suppose $\lambda_t$ is one weak limit point of $\lambda_\varepsilon^t$. Then we have the following lemma in [16, pages 138-139].

**Lemma 3.4.** Given any $T > 0$ and a process $X_t$ in the space $C([0, T]; \mathbb{R}^d)$, then for any $f \in C^2_0(\mathbb{R})$, the process

$$f(X_t) - f(X_0) - \int_0^t f'(X_s)b(X_s)ds - \frac{1}{2} \int_0^t f''(X_s)B(X_s)ds$$

is a martingale in $C([0, T]; \mathbb{R}^d)$ if and only if the following SDE

$$dX_t = b(X_t)dt + \sqrt{B(X_t)}dW_t$$

with initial value $X_0$, has a weak solution $X_t$.

According to the above lemma, we will prove the following lemma.

**Lemma 3.5.** Any limiting measure of $Q^\varepsilon$, denoted by $Q^0$, solves the following martingale problem on $C([0, T]; \mathbb{R}^d)$, $Q^0(\lambda_0 = 0) = 1$. For any $h \in C^2_0$,

$$h(\lambda_\varepsilon^t) - h(\lambda_0^t) - \frac{1}{2} \int_0^t h''(\lambda_\varepsilon^s)B(Z_s)ds$$

is a martingale. Here, for a fixed $x$,

$$B(x) = 2\int_0^\infty E\left[\hat{f}(x, \bar{Y}_s^\varepsilon) - \hat{f}(x)\right] \left[\hat{f}(x, \bar{Y}_s^\varepsilon) - \hat{f}(x)\right] ds,$$

where the $\bar{Y}_t^\varepsilon$ is the unique stationary solution of (15).

**Proof.** This is a slight extension of Lemma 10 in [23], but the proof is similar. We omit the details here.

According to Lemma 3.4 and the uniqueness of the solution, we show that $\lambda_\varepsilon^t$ converges weakly to $\lambda_t$, where $\lambda_t$ is the solution of

$$d\lambda_t = \sqrt{B(Z_t)}dW_t$$

with initial value $\lambda_0 = 0$.

Now, (18) defines a continuous mapping $\Phi : \lambda_\varepsilon^t \to \xi_\varepsilon^t$ of $C([0, T]; \mathbb{R}^d)$ into itself, for any $\varepsilon > 0$, $\Phi(\lambda_\varepsilon^t) = \xi_\varepsilon^t$. By Lemma 3.5 one can further show that $\lambda_\varepsilon^t$ is weakly convergent in $C([0, T]; \mathbb{R}^d)$ as $\varepsilon \to 0$ to the Gaussian process $\lambda^t$. Then the continuous mapping theorem shows that $\xi_\varepsilon^t$ converges weakly to the measure corresponding to $\Phi(\lambda_t) = Z^0_t$.

On the other hand, we estimate the difference $Z_t^\varepsilon - \xi^t = \eta^t_\varepsilon$. Combining (17) and (18), we see that

$$\eta^t_\varepsilon - \int_0^t F(Z_s, Y_s^{Z_s}, \varepsilon, \eta^t_\varepsilon)ds = \int_0^t I_1(s, \varepsilon, \omega)ds + \int_0^t I_2(s, \varepsilon, \omega)\xi^t_\varepsilon ds + \int_0^t I_3(s, \varepsilon, \omega)ds.$$

Then by Assumption 1,

$$E|\eta^t_\varepsilon| \leq e^{Ct}\left(E\left[\int_0^t I_1(s, \varepsilon, \omega)ds\right] + E\left[\int_0^t I_2(s, \varepsilon, \omega)\xi^t_\varepsilon ds\right] + E\left[\int_0^t I_3(s, \varepsilon, \omega)ds\right]\right).$$

(20)

According to Lemma 3.3, we have proved the weak compactness of the family of measure induced by process $Z_t^\varepsilon$ in $C([0, T]; \mathbb{R}^d)$. The theorem will be proved if we can show the right side of (20) converges to zero in probability.
Let’s consider each of these cases separately. For \( E \left| \int_0^t I_1(s, \epsilon, \omega) ds \right| \), we can derive that
\[
\int_0^t I_1(s, \epsilon, \omega) ds = \frac{1}{\sqrt{\epsilon}} \int_0^t \left| f(Z_s + \sqrt{\epsilon} Z(\epsilon), Y_s, \epsilon) - f(Z_s, Y_s^{Z, \epsilon}, \epsilon) - F(Z_s, Y_s^{Z, \epsilon}, \epsilon) \sqrt{\epsilon} Z_s \right| ds.
\]
It follows from Taylor’s formula that for some \( C \), we have \( |I_1(s, \epsilon, \omega)| \leq C \epsilon |Z_s|^2 \).

Using the fact of \( E|Z_s|^2 \leq C < \infty \), \( s \in [0, T] \), one can derive that
\[
E \left| \int_0^t I_1(s, \epsilon, \omega) ds \right| \leq \int_0^t E|I_1(s, \epsilon, \omega)| ds \leq C t \epsilon. \tag{21}
\]

For \( E \left| \int_0^t I_2(s, \epsilon, \omega) ds \right| \), Assumptions 3 yields that
\[
E \left| \int_0^t I_2(s, \epsilon, \omega) ds \right| \leq \frac{1}{\sqrt{\epsilon}} \int_0^t E|f(Z_s, Y_s^{Z, \epsilon}, \epsilon) - f(Z_s, \bar{Y}_s^{Z, \epsilon}, \epsilon)| ds \\
\leq \frac{C}{\sqrt{\epsilon}} \int_0^t \left( E|Y_s^{Z, \epsilon} - \bar{Y}_s^{Z, \epsilon}|^2 \right)^{\frac{1}{2}} ds.
\]

With the relationship between the stationary solution \( \bar{Y}_s^{Z, \epsilon} \) and the general solution \( Y_s^{Z, \epsilon} \), we get
\[
E|Y_s^{Z, \epsilon} - \bar{Y}_s^{Z, \epsilon}|^2 \leq (1 + y_0 + E|Z_s|^2) e^{-\frac{2k^2}{\epsilon} s}.
\]
We have
\[
E \left| \int_0^t I_3(s, \epsilon, \omega) ds \right| \leq \frac{C}{\sqrt{\epsilon}} \int_0^t e^{-\frac{2k^2}{\epsilon} s} ds \leq C_1 \sqrt{\epsilon}. \tag{22}
\]

For \( E \left| \int_0^t I_2(s, \epsilon, \omega) \xi_s^\epsilon ds \right| \),
\[
E \left| \int_0^t I_2(s, \epsilon, \omega) \xi_s^\epsilon ds \right| \leq \left[ E \left( \int_0^t \left| f(Z_s, Y_s^{Z, \epsilon}, \epsilon) - f(Z_s) \right| ds \right)^2 \right]^{\frac{1}{2}} \\
* \sup_{0 < t < T} \left| \xi_t^\epsilon \right|^2 \frac{1}{2}.
\]
To derive \( E \left| \int_0^t I_2(s, \epsilon, \omega) \xi_s^\epsilon ds \right| \to 0 \) as \( \epsilon \to 0 \), we further need the following lemma.

**Lemma 3.6.** If Assumptions 1 - 4 are satisfied and suppose \( Y_t^{Z, \epsilon} \) is the unique solution of the SDE (7) with any fixed \( x = Z_t \), there exists a constant \( C \) such that
\[
E \left( \int_0^t \left| f(Z_s, Y_s^{Z, \epsilon}, \epsilon) - f(Z_s) \right| ds \right)^2 \leq C \epsilon.
\]

The above lemma yields that
\[
E \left| \int_0^t I_2(s, \epsilon, \omega) \xi_s^\epsilon ds \right| \leq C \epsilon. \tag{23}
\]

Combining (21), (23) and (22), we have \( E|\eta_t^\epsilon| \to 0 \) as \( \epsilon \to 0 \). This completes the proof of the theorem.

Now, we give the proof of Lemma 3.6.
Proof of Lemma 3.6. Assumption 1 yields that $F(x, y, \epsilon)$ is the bounded Lipschitz function with respect to $y$. Hence

\[ |E[F(Z_t, Y_t^{Z_t, \epsilon}, \epsilon) - F(Z_t)]| \leq CE[Y_t^{Z_t, \epsilon} - \bar{Y}_t^{Z_t, \epsilon}]^2 \leq Ce^{-\frac{2\epsilon}{t}}. \quad (24) \]

Let $U_t^\epsilon = \int_0^1 F(Z_s, Y_s^{Z_s, \epsilon}, \epsilon) - F(Z_s)|ds$. For $0 < \alpha < 1$, taking

\[ C_\alpha = \int_s^t (r-s)^{-\alpha}(t-r)^{\alpha-1} dr = B(\alpha, 1-\alpha) \]

into account with the Beta function $B(x, y)$. Using Fubini’s theorem, we have

\[ U_t^\epsilon = \frac{1}{C_\alpha} \int_0^t F(Z_s, Y_s^{Z_s, \epsilon}, \epsilon) - F(Z_s) \int_s^t (r-s)^{-\alpha}(t-r)^{\alpha-1} dr ds \]

\[ \quad = \frac{1}{C_\alpha} \int_0^t (r-s)^{\alpha-1} V_r^\epsilon dr, \]

where

\[ V_r^\epsilon = \int_s^r (r-s)^{-\alpha}[F(Z_s, Y_s^{Z_s, \epsilon}, \epsilon) - F(Z_s)]|ds. \]

Moreover, for some positive $C$,

\[ E[U_t^\epsilon]^2 \leq C \int_0^t (t-r)^{2(\alpha-1)} E[V_r^\epsilon]^2 dr. \]

Hence

\[ E[V_r^\epsilon]^2 = \int_0^r \int_0^r (r-u)^{-\alpha}(r-v)^{-\alpha} E[F(Z_u, Y_u^{Z_u, \epsilon}, \epsilon) - F(Z_u)][F(Z_v, Y_v^{Z_v, \epsilon}, \epsilon) - F(Z_v)]|duds \]

\[ \quad = \frac{2}{C_\alpha} \int_0^r \int_v^r (r-u)^{-\alpha}(r-v)^{-\alpha} E[F(Z_u, Y_u^{Z_u, \epsilon}, \epsilon) - F(Z_u)][F(Z_v, Y_v^{Z_v, \epsilon}, \epsilon) - F(Z_v)]|duds. \]

For any fixed $x$, denote by $P_t, t > 0$, as the transition semigroup associated with the SDE (9) for any bounded Lipschitz function $G : \mathbb{R}^d \to \mathbb{R}^d$, such that $P_tG(x) = EG(Y_t^{x, \epsilon})$. Using the property of conditional expectation, the Markovian property of process $Y_t^{Z_t, \epsilon}$ and Assumption 1, one can derive that

\[ E[F(Z_u, Y_u^{Z_u, \epsilon}, \epsilon) - F(Z_u)]E[F(Z_v, Y_v^{Z_v, \epsilon}, \epsilon) - F(Z_v)] |P_{u-v}[F(Z_v, Y_v^{Z_v, \epsilon}, \epsilon) - F(Z_v)]|^\frac{1}{2} \]

\[ \leq Ce^{-\frac{2\epsilon}{t(u-v)}}, \]

the last inequality is deduced by (24). We have

\[ E[U_t^\epsilon]^2 \leq C \int_0^t \int_0^r \int_v^r (r-u)^{-\alpha}(r-v)^{-\alpha}(t-r)^{2(\alpha-1)} e^{-\frac{2\epsilon}{t(u-v)}} dudsdr \]

\[ \leq C \int_0^t dv \int_v^t e^{-\frac{2\epsilon}{t(u-v)}} du \int_u^t (r-u)^{-\alpha}(r-v)^{-\alpha}(t-r)^{2(\alpha-1)} dr \]

\[ \leq C \int_0^t dv \int_v^t e^{-\frac{2\epsilon}{t(u-v)}} du \int_u^t (r-u)^{-2\alpha}(t-r)^{2(\alpha-1)} dr \]
Through a simple example, we will explicitly illustrate that normal deviation is effective for SDEs in the form of (7).

**Example 3.7.** Let us consider the following equation in $\mathbb{R}^2$

\[
\begin{align*}
\frac{dX^\epsilon}{t} &= Y_t^\epsilon dt + dW_t, \\
\frac{dY_t^\epsilon}{t} &= -\frac{1}{\epsilon} Y_t^\epsilon dt + \frac{1}{\sqrt{\epsilon}} dW_t
\end{align*}
\]

with initial value $X_0 = x$, $Y_0^\epsilon = y$, where $W_t$ is a two-sided scalar Wiener process. The corresponding averaged SDE is

\[dZ_t = dW_t\]

with initial value $Z_0 = x$. Hence

\[Y^\epsilon_t = e^{-\frac{1}{2}t}y + \frac{1}{\sqrt{\epsilon}} \int_0^t e^{-\frac{1}{2}(t-s)} dW_s.\]

Denote $Z^\epsilon_t = \frac{1}{\sqrt{\epsilon}} (X^\epsilon_t - Z_t)$. The definition of $Z^\epsilon_t$ yields that

\[dZ^\epsilon_t = \frac{1}{\sqrt{\epsilon}} Y^\epsilon_t dt.\]

Then, $Z^\epsilon_t$ satisfies the following SDE

\[Z^\epsilon_t = \frac{1}{\sqrt{\epsilon}} \int_0^t Y^\epsilon_s ds.\]

We get

\[
\begin{align*}
\lim_{\epsilon \to 0} \mathbb{E}Z^\epsilon_t &= \lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \int_0^t Y^\epsilon_s ds = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \int_0^t e^{-\frac{1}{2}s} y ds = \lim_{\epsilon \to 0} [-\sqrt{\epsilon} e^{-\frac{1}{2}t} y + \sqrt{\epsilon} y] \\
&= 0
\end{align*}
\]

and

\[
\begin{align*}
\lim_{\epsilon \to 0} \text{Var} Z^\epsilon_t &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E} \left( \int_0^t Y^\epsilon_s ds \right)^2 = \lim_{\epsilon \to 0} \frac{2}{\epsilon} \int_0^t \int_s^t Y^\epsilon_r Y^\epsilon_s dr ds \\
&= \lim_{\epsilon \to 0} \frac{2}{\epsilon} \int_0^t \int_s^t e^{-\frac{1}{2}(s+r)} y^2 + \frac{1}{2} \left( e^{-\frac{1}{2}(r-s)} - e^{-\frac{1}{2}(s+r)} \right) dr ds \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( (2y^2 - 1) \left[ \frac{1}{2} \epsilon^2 (e^{-\frac{1}{2}2t} + 1) - e^2 e^{-\frac{1}{2}t} \right] + ct - \epsilon^2 + c e^{-\frac{1}{2}t} \right) \\
&= t.
\end{align*}
\]

We have the following result in [20, page 322], which provides the method to prove the weak convergence of the Gaussian distribution.

**Lemma 3.8.** Let $F_n$ be a sequence of distribution function $F_n = F_n(x)$ and let $\varphi_n$ be the corresponding sequence of characteristic functions,

\[\varphi_n(t) = \int_{\mathbb{R}} e^{itx} dF_n(x).\]
If \( \lim_{n} \varphi_n(t) \) exists for each \( t \) and \( \varphi(t) = \lim_{n} \varphi_n(t) \) is continuous at \( t = 0 \), then \( \varphi(t) \) is the characteristic function of a probability distribution \( F(t) \) and \( F_n(t) \) converges weakly to \( F(t) \).

Writing

\[
\sigma^2(\epsilon, t) = \frac{1}{\epsilon} \left\{ (2y^2 - 1) \left[ \frac{1}{2} \epsilon^2 (e^{-\frac{1}{2}t} + 1) - \epsilon^2 e^{-\frac{1}{2}t} \right] + \epsilon t - \epsilon^2 + \epsilon^2 e^{-\frac{1}{2}t} \right\}
\]

and

\[
\phi(\epsilon, t) = -\sqrt{\epsilon} e^{-\frac{1}{2}t} y + \sqrt{\epsilon} y.
\]

The corresponding characteristic function \( F^*(t) \) is \( e^{i\phi(\epsilon, t)t - \frac{1}{2} \sigma^2(\epsilon, t)t^2} \). And \( \lim_{\epsilon \to 0} F^*(t) = e^{-\frac{1}{2}t^3} \). In conclusion, we show that \( Z^*_t \) converges weakly to \( Z^0_t \), where \( Z^0_t \) is the solution of

\[
dZ^0_t = dW_t
\]

with initial value \( Z^0_0 = 0 \).

Next we show that this conclusion is consistent with the conclusion of Theorem 3.1. Let \( f(x, y, \epsilon) = y, \sigma_1 = 1, g(x, y, \epsilon) = y \) and \( \sigma_2 = 1 \). We have \( f(x, y) = \lim_{\epsilon \to 0} f(x, y, \epsilon) = y \). It is easily checked

\[
dY_t = -Y_t dt + dW_t.
\]

Hence the stationary solution is

\[
\bar{Y}_t = \int_{-\infty}^{t} e^{-(t-s)} dW_s.
\]

Then

\[
B(x) = 2 \int_{-\infty}^{\infty} E[\bar{Y}_t \bar{Y}_0] dt = 2 \int_{0}^{\infty} E \left[ \int_{-\infty}^{t} e^{-(t-s)} dW_s \int_{-\infty}^{0} e^{s} dW_s \right] dt
\]

\[
= 2 \int_{0}^{\infty} e^{-t} \int_{-\infty}^{0} e^{2s} ds dt = \int_{0}^{\infty} e^{-t} dt = 1.
\]

Consequently, \( Z^*_t \) converges weakly to \( Z^0_t \), where \( Z^0_t \) is the solution of

\[
 dZ^0_t = dW_t \quad (25)
\]

with initial value \( Z^0_0 = 0 \).

From the above, for one thing, the normal deviation of SDE with the general solution is valid. For another thing, \( (25) \) does not exist a unique stationary solution. Through the demonstration of the normal deviation of the general solution, the relation between the general solution and the stationary solution is given, which provides a new perspective to investigate the normal deviation of the stationary solution. Next, the normal deviation of the stationary solution will be considered.

4. Deviation estimate for stationary solution and synchronization. In this section, based on the arguments of the normal deviation of the general solution in Section 3 and the relationship between the general solution and the stationary solution in Section 2, the normal deviation of the stationary solution can be investigated in this section. We extend our result from the general solution in Theorem 3.1 to the stationary solution in Theorem 4.2. No matter how complex of the slow and fast systems of the SDEs (6) are in Section 1, which contain \( \bar{X}^*_t \) and \( \sqrt{\epsilon} \bar{Y}^*_t \), the method of the normal deviation of the general solution with the multi-scale system
can still be applied to the synchronized system. We concentrate on studying the normal deviation of synchronized system. These are the main innovation of this paper.

We will use the same proof of Corollary 2.3 in [14] to prove that coupled synchronized system has a unique stationary solution.

**Lemma 4.1.** If Assumptions 1-4 are satisfied and the corresponding coefficients satisfy the constraint $-2 \rho + 1 + L < 0$, where $-\rho = \max\{-k_1, -k_2\}$, then there exists a unique stationary solution $(X^\epsilon_t, Y^\epsilon_t)$, such that they satisfy the SDE (7) with the stationary probability distribution.

In the next, we will consider our synchronized system as the following SDE

\[
\begin{align*}
\frac{dX^\epsilon_t}{\sqrt{\epsilon}} &= f(X^\epsilon_t, Y^\epsilon_t)dt + \sigma_1 dW_t, \\
\frac{dY^\epsilon_t}{\sqrt{\epsilon}} &= \frac{1}{\epsilon} g(X^\epsilon_t, Y^\epsilon_t, \epsilon)dt + \frac{1}{\sqrt{\epsilon}} \sigma_2 dW_t.
\end{align*}
\]

The corresponding averaged SDE is

\[
dZ_t = f(Z_t, 0)dt + \sigma_1 dW_t.
\]

It follows from Lemma 4.1 that the synchronized system (26) and the averaged system (27) have the unique stationary solution. Denote $(X^\epsilon_t, Y^\epsilon_t)$ as the stationary solution of the synchronized system (26) and $\bar{Z}_t$ as the stationary solution of the averaged system (27). Besides, set the random dynamical system $\varphi^\epsilon$ generated by the first SDE of (26), that is $\varphi^\epsilon(t, \omega, x) = \bar{X}^\epsilon(t, x)$ and the random dynamical system $\psi^\epsilon$ generated by the averaged SDE (27), that is $\psi^\epsilon(t, \omega, z) = \bar{Z}^\epsilon(t, z)$. Then we get

\[
\bar{X}^\epsilon_t(\omega) = \varphi^\epsilon(t, \omega, \bar{X}^\epsilon_t(\omega)) = \bar{X}^\epsilon_t(\theta_t \omega)
\]

and

\[
\bar{Z}_t(\omega) = \psi^\epsilon(t, \omega, \bar{Z}_t(\omega)) = \bar{Z}_t(\theta_t \omega).
\]

By Lemma 3.3 in [14], we can get the boundedness of the stationary solutions, that is,

\[
E|\bar{X}_t^\epsilon|^2 \leq M, E|\bar{Y}_t^\epsilon|^2 \leq M, E|\bar{Z}_t|^2 \leq M.
\]

**Theorem 4.2.** If Assumptions 1-5 are satisfied, then the normal deviation of the stationary solution $\bar{X}^\epsilon_t$ and $\bar{Z}_t$, $Z^0_t := \frac{\bar{X}^\epsilon_t - \bar{Z}_t}{\sqrt{\epsilon}}$, converges in probability to $Z^0_t$, where $Z^0_t$ is the unique stationary solution of the SDE

\[
dZ^0_t = F(\bar{Z}_t)Z^0_t dt.
\]

**Proof.** By the definition of the stationary solution, we get

\[
Z^\epsilon_t = \frac{\bar{X}^\epsilon_t - \bar{Z}_t}{\sqrt{\epsilon}} = \frac{1}{\sqrt{\epsilon}} [\varphi^\epsilon(t, \omega, \bar{X}^\epsilon_t(\omega)) - \psi^\epsilon(t, \omega, \bar{Z}_t(\omega))] \\
= \frac{1}{\sqrt{\epsilon}} [\varphi^\epsilon(t, \omega, \bar{X}^\epsilon_t(\omega)) - \varphi^\epsilon(t, \omega, \bar{Z}_t(\omega))] \\
+ \frac{1}{\sqrt{\epsilon}} [\psi^\epsilon(t, \omega, \bar{Z}_t(\omega)) - \psi^\epsilon(t, \omega, \bar{Z}_t(\omega))] \\
= u^\epsilon_t + \tilde{u}^\epsilon_t,
\]

where

\[
u^\epsilon_t = \frac{1}{\sqrt{\epsilon}} [\varphi^\epsilon(t, \omega, \bar{X}^\epsilon_t(\omega)) - \varphi^\epsilon(t, \omega, \bar{Z}_t(\omega))]
\]

and

\[
\tilde{u}^\epsilon_t = \frac{1}{\sqrt{\epsilon}} [\psi^\epsilon(t, \omega, \bar{Z}_t(\omega)) - \psi^\epsilon(t, \omega, \bar{Z}_t(\omega))].
\]
and
\[ \tilde{u}_t^\epsilon = \frac{1}{\sqrt{\epsilon}} [\varphi^\epsilon(t, \omega, \tilde{Z}_t(\omega)) - \psi^\epsilon(t, \omega, \tilde{Z}_t(\omega))]. \]

Observing that \( \varphi^\epsilon(t, \omega, \tilde{Z}_t(\omega)) \) is corresponding to the general solution of slow system in (7) and \( \psi^\epsilon(t, \omega, \tilde{Z}_t(\omega)) \) is corresponding to that of averaged system (8), both with the same initial random variable \( \tilde{Z}_t(\omega) \). In fact, it is just the conclusion of the normal deviation of the general solution that we have demonstrated in the first part of Theorem 3.1. Then \( \tilde{u}_t^\epsilon \) converges in probability to \( \tilde{u}_t \), where \( \tilde{u}_t \) is the unique stationary solution of the SDE
\[ d\tilde{u}_t = F(Z_t)\tilde{u}_t dt. \] (28)

For \( u_t^\epsilon \), we show the normal deviation of the slow system with different initial values \( X_t^0, \tilde{Z}_t \). Writing \( Y_t^{y_0, \epsilon} \) is the solution of \( Y_t^\epsilon \) of (26) with initial value \( y_0 \). We can write the \( N_t^\epsilon = \frac{1}{\sqrt{\epsilon}} [X_t^{X_t^0, \epsilon} - X_t^{Z_t^0, \epsilon}] \). We have
\[
\frac{d}{dt} N_t^\epsilon = \frac{1}{\sqrt{\epsilon}} [f(X_t^{X_t^0, \epsilon}, \sqrt{\epsilon} Y_t^{y_0, \epsilon}) - f(X_t^{Z_t^0, \epsilon}, \sqrt{\epsilon} Y_t^{y_0, \epsilon})]
= \frac{1}{\sqrt{\epsilon}} [f(X_t^{X_t^0, \epsilon}, \sqrt{\epsilon} Y_t^{y_0, \epsilon}) - f(X_t^{Z_t^0, \epsilon}, \sqrt{\epsilon} Y_t^{y_0, \epsilon})]
+ \frac{1}{\sqrt{\epsilon}} [f(X_t^{X_t^0, \epsilon}, \sqrt{\epsilon} Y_t^{y_0, \epsilon}) - f(X_t^{Z_t^0, \epsilon}, \sqrt{\epsilon} Y_t^{y_0, \epsilon})]
= \frac{d}{dt} v_t^\epsilon + \frac{d}{dt} \tilde{v}_t^\epsilon,
\]
where
\[
\frac{d}{dt} v_t^\epsilon = \frac{1}{\sqrt{\epsilon}} [f(X_t^{X_t^0, \epsilon}, \sqrt{\epsilon} Y_t^{y_0, \epsilon}) - f(X_t^{Z_t^0, \epsilon}, \sqrt{\epsilon} Y_t^{y_0, \epsilon})]
\]
and
\[
\frac{d}{dt} \tilde{v}_t^\epsilon = \frac{1}{\sqrt{\epsilon}} [f(X_t^{X_t^0, \epsilon}, \sqrt{\epsilon} Y_t^{y_0, \epsilon}) - f(X_t^{Z_t^0, \epsilon}, \sqrt{\epsilon} Y_t^{y_0, \epsilon})].
\]
For any fixed \( X_t^{X_t^0, \epsilon} \),
\[
E[v_t^\epsilon] \leq \frac{1}{\sqrt{\epsilon}} C \int_0^t (E[Y_s^{y_0, \epsilon} - Y_s^{\tilde{y}_0, \epsilon}]^2)^{\frac{1}{2}} ds.
\]
Due to
\[
\frac{d}{dt} E[Y_t^{y_0, \epsilon} - Y_t^{\tilde{y}_0, \epsilon}]^2 \leq \frac{2}{\epsilon} E(Y_t^{y_0, \epsilon} - Y_t^{\tilde{y}_0, \epsilon}, g(X_t^{X_t^0, \epsilon}, Y_t^{y_0, \epsilon}, \epsilon) - g(X_t^{X_t^0, \epsilon}, Y_t^{\tilde{y}_0, \epsilon}, \epsilon)) - \frac{2k_2}{\epsilon} E[Y_t^{y_0, \epsilon} - Y_t^{\tilde{y}_0, \epsilon}]^2,
\]
the Gronwall inequality yields that
\[
E[Y_t^{y_0, \epsilon} - Y_t^{\tilde{y}_0, \epsilon}]^2 \leq e^{-\frac{2k_2 t}{\epsilon}} |y_0 - y_0'|^2.
\]
We have $E|u'_t| \to 0$ as $\epsilon \to 0$. Suppose that $N_t$ is one weak point of $N'_t$. Now
\[
\int_0^t f(X_s^{\epsilon,\epsilon}, \sqrt{\epsilon} Y_s^{\epsilon,\epsilon}) - f(X_s^{\epsilon,\epsilon}, \sqrt{\epsilon} Y_s^{0,\epsilon}) \, ds
= \int_0^t f(X_s^{\epsilon,\epsilon}, \sqrt{\epsilon} Y_s^{\epsilon,\epsilon}) - f(X_s^{\epsilon,\epsilon}, \sqrt{\epsilon} Y_s^{0,\epsilon}) + F(\bar{Z}_s, \sqrt{\epsilon} Y_s^{0,\epsilon}) N'_s \, ds
+ \int_0^t (F(\bar{Z}_s, \sqrt{\epsilon} Y_s^{0,\epsilon}) - F(\bar{Z}_s)) N'_s \, ds + \int_0^t F(\bar{Z}_s) N'_s \, ds.
\]

Proceeding as in the proof of Theorem 3.1, we observe the fact that $N_t$ converges in probability to $N_t$, where $N_t$ solves
\[dN_t = F(\bar{Z}_t) N_t \, dt.
\]
Therefore, we can get $u'_t$ converges in probability to $u_t$, where $u_t$ is the unique stationary solution of the SDE
\[du_t = F(\bar{Z}_t) u_t \, dt.\] (29)
Combining (28) and (29) shows that $Z'_t$ converges in probability to $\bar{Z}_t$, where $\bar{Z}_t$ is the unique stationary solution of the SDE
\[dZ'_t = F(\bar{Z}_t) Z'_t \, dt.
\]

We present a simple example.

**Example 4.3.** Let us consider the following equation in $\mathbb{R}^2$
\[
\begin{cases}
    dX_t^\epsilon = -X_t^\epsilon \, dt - \sqrt{\epsilon} Y_t^\epsilon \, dt + \sigma_1 \, dW_t, \\
    dY_t^\epsilon = -\frac{1}{\epsilon} Y_t^\epsilon \, dt + \frac{1}{\sqrt{\epsilon}} \sigma_2 \, dW_t
\end{cases}
\]
with initial value $X_{t_0}^\epsilon = x$, $Y_{t_0}^\epsilon = y$, where $\sigma_1, \sigma_2 \in \mathbb{R}$ are constants, $W_t$ is a two-sided scalar Wiener process. The corresponding averaged SDE is
\[dZ_t = -Z_t \, dt + \sigma_1 \, dW_t\]
with initial value $Z_0 = x$. Denote
\[A = \begin{pmatrix}
    -1 & -\sqrt{\epsilon} \\
    0 & -\frac{1}{\epsilon}
\end{pmatrix}.
\]
We have the general solution
\[
\begin{pmatrix}
    X_t^\epsilon \\
    Y_t^\epsilon
\end{pmatrix} = e^{A(t-t_0)} \begin{pmatrix}
    x \\
    y
\end{pmatrix} + \int_{t_0}^t e^{A(t-s)} \begin{pmatrix}
    \sigma_1 \\
    \frac{1}{\sqrt{\epsilon}} \sigma_2
\end{pmatrix} dW_s
+ \int_{t_0}^t \begin{pmatrix}
    e^{-(t-s)} & e^{\sqrt{\epsilon} (e^\frac{1}{\epsilon} (t-s) - e^{-(t-s)})} \\
    0 & e^{\frac{1}{\epsilon} (t-s)}
\end{pmatrix} \begin{pmatrix}
    X_{t_0}^\epsilon \\
    Y_{t_0}^\epsilon
\end{pmatrix}
+ \int_{t_0}^t \begin{pmatrix}
    e^{-(t-s)} & e^{\sqrt{\epsilon} (e^\frac{1}{\epsilon} (t-s) - e^{-(t-s)})} \\
    0 & e^{\frac{1}{\epsilon} (t-s)}
\end{pmatrix} \begin{pmatrix}
    \sigma_1 \\
    \frac{1}{\sqrt{\epsilon}} \sigma_2
\end{pmatrix} dW_s
\]
\[
\begin{align*}
&= \left( e^{-(t-t_0)}X_{t_0} + \epsilon \sqrt{\epsilon} \left( e^{-\frac{1}{\epsilon}(t-t_0)} - e^{-(t-t_0)} \right) Y_{t_0} \right) \\
&+ \int_{t_0}^{t} \left( \sigma_1 e^{-(t-s)} + \sigma_2 \left( e^{-\frac{1}{\epsilon}(t-s)} - e^{-(t-s)} \right) \right) dW_s.
\end{align*}
\]

Accordingly, the stationary solutions are
\[
\bar{X}_t = \int_{-\infty}^{t} \sigma_1 e^{-(t-s)} + \sigma_2 \left( e^{-\frac{1}{\epsilon}(t-s)} - e^{-(t-s)} \right) ds
\]
and
\[
\bar{Z}_t = \int_{-\infty}^{t} \sigma_1 e^{-(t-s)} ds.
\]

Denote \( Z_t^\epsilon = \frac{1}{\sqrt{\epsilon}} (\bar{X}_t^\epsilon - \bar{Z}_t) \). Hence
\[
Z_t^\epsilon = \int_{-\infty}^{t} \sigma_2 \sqrt{\epsilon} e^{-\frac{1}{\epsilon}(t-s)} - \sqrt{\epsilon} e^{-(t-s)} ds.
\]

We get
\[
\lim_{\epsilon \to 0} EZ_t^\epsilon = \lim_{\epsilon \to 0} E \int_{-\infty}^{t} \sigma_2 \frac{\sqrt{\epsilon} e^{-\frac{1}{\epsilon}(t-s)} - \sqrt{\epsilon} e^{-(t-s)}}{\epsilon - 1} ds = 0
\]
and
\[
\lim_{\epsilon \to 0} \text{Var}Z_t^\epsilon = \lim_{\epsilon \to 0} \sigma_2^2 \int_{-\infty}^{t} \frac{\epsilon (e^{-\frac{1}{\epsilon}(t-s)} - e^{-(t-s)})^2}{(\epsilon - 1)^2} ds = 0.
\]

Next we show that this conclusion is consistent with the conclusion of Theorem 4.2.

Let \( f(x, y, \epsilon) = -x - \sqrt{\epsilon} y, \hat{f}(x, y) = \lim_{\epsilon \to 0} f(x, y, \epsilon) = -x \) and \( g(x, y, \epsilon) = -y \).

For any fixed \( x \), it is easily checked
\[
dY_s = -Y_s ds + \sigma_2 dW_s.
\]

Hence the stationary solution is
\[
\bar{Y}_t = \sigma_2 \int_{-\infty}^{t} e^{-(t-s)} ds.
\]

Hence \( B(x) = 0 \). Consequently, \( Z_t^\epsilon \) converges weakly to \( \bar{Z}_t^0 \), where \( \bar{Z}_t^0 \) is the stationary solution of the SDE
\[
dZ_t^0 = -Z_t^0 dt
\]
with initial value \( Z_0^0 = 0 \). The corresponding stationary solution is \( \bar{Z}_t^0 = 0 \). We have
\[
E[\bar{Z}_t^0] = 0, \text{Var}[\bar{Z}_t^0] = 0.
\]

We concentrate on studying the normal deviation of synchronized system. In this situation, we will make some assumptions.

**Assumption 6.** The functions \( f(x), g(x) \) have bounded continuous first and second derivatives.
Assumption 7. (One-sided dissipative Lipschitz condition) For all $x_1, x_2$, there exist constants $\tilde{k}_1, \tilde{k}_2 > 0$, such that

\[
\langle x_1 - x_2, f(x_1) - f(x_2) \rangle \leq -\tilde{k}_1 |x_1 - x_2|^2, \\
\langle x_1 - x_2, g(x_1) - g(x_2) \rangle \leq -\tilde{k}_2 |x_1 - x_2|^2.
\]

Theorem 4.4. If Assumptions 6-7 are satisfied, then $\frac{1}{2} \sqrt{\nu}(X^\nu_1 - Y^\nu_1)$ converges weakly to $Z_1(t)$ as $\nu$ tends to infinity, where $Z_1(t)$ is the stationary solution of the SDE

\[
dZ_1(t) = -2Z_1(t)dt + \frac{1}{2}(\alpha - \beta)dW_t
\]

and $\sqrt{\nu}(\frac{1}{2}(X^\nu_1 + Y^\nu_1) - Z_1)$ converges in probability to $Z_2(t)$ as $\nu$ tends to infinity, where $Z_2(t)$ is the stationary solution of the SDE

\[
dZ_2(t) = \frac{1}{2}[D_x f(Z_t) + D_y g(Z_t)]Z_2(t)dt,
\]

where $(X^\nu_1, Y^\nu_1)$ is the unique stationary solution of the coupled system (1) and $Z_1$ is the unique stationary solution of the averaged system (2).

Proof. In Section 1, we show the synchronized system (1) can be transformed into multi-scale system (6). If Assumptions 1 - 5 are satisfied, Theorem 4.2 holds. Note that

\[
f(x, \sqrt{\epsilon}y) = \frac{1}{2} [f(x + \sqrt{\epsilon}y) + g(x - \sqrt{\epsilon}y)], \sigma_1 = \frac{1}{2}(\alpha + \beta),
\]

\[
g(x, y, \epsilon) = \frac{\sqrt{\epsilon}}{2} [f(x + \sqrt{\epsilon}y) - g(x - \sqrt{\epsilon}y)] - 2y, \sigma_2 = \frac{1}{2}(\alpha - \beta).
\]

Firstly, we prove that Assumption 2 is satisfied. For any $x_1, x_2, y_1, y_2$ and $x, y$, by Assumption 7, we get

\[
\langle x_1 - x_2, f(x_1, \sqrt{\epsilon}y) - f(x_2, \sqrt{\epsilon}y) \rangle = \frac{1}{2} \langle x_1 - x_2, f(x_1 + \sqrt{\epsilon}y) - f(x_2 + \sqrt{\epsilon}y) \rangle \\
+ \frac{1}{2} \langle x_1 - x_2, g(x_1 - \sqrt{\epsilon}y) - g(x_2 - \sqrt{\epsilon}y) \rangle \leq -\frac{1}{2} \left( \tilde{k}_1 + \tilde{k}_2 \right) |x_1 - x_2|^2,
\]

\[
\langle y_1 - y_2, g(x, y_1, \epsilon) - g(x, y_2, \epsilon) \rangle = \frac{\sqrt{\epsilon}}{2} \langle y_1 - y_2, f(x + \sqrt{\epsilon}y_1) - f(x + \sqrt{\epsilon}y_2) \rangle \\
+ \frac{\sqrt{\epsilon}}{2} \langle y_1 - y_2, g(x - \sqrt{\epsilon}y_2) - g(x - \sqrt{\epsilon}y_1) \rangle -2 |y_1 - y_2|^2 \leq - \left( \frac{1}{2} \tilde{k}_1 \epsilon + \frac{1}{2} \tilde{k}_2 \epsilon + 2 \right) |y_1 - y_2|^2.
\]

We can choose $-k_1 = -\frac{1}{2} \left( \tilde{k}_1 + \tilde{k}_2 \right) < 0$ and $-k_2 = - \left( \frac{1}{2} \tilde{k}_1 \epsilon + \frac{1}{2} \tilde{k}_2 \epsilon + 2 \right) < 0$, then the one-sided dissipative Lipschitz conditions of $f(x, y, \epsilon)$, $g(x, y, \epsilon)$ are satisfied.

Secondly, we prove that Assumption 3 is satisfied. For any $x_1, x_2, y_1, y_2$ and $x$, $y$, Assumption 6 yields that there exists $\tilde{L} > 0$, such that

\[
|f(x_1) - f(x_2)|^2 + |g(x_1) - g(x_2)|^2 \leq \tilde{L} |x_1 - x_2|^2.
\]

We get

\[
|g(x_1, y, \epsilon) - g(x_2, y, \epsilon)|^2 = \frac{\epsilon}{2} |f(x_1 + \sqrt{\epsilon}y) - f(x_2 + \sqrt{\epsilon}y)|^2
\]
We can choose \( L = \frac{1}{2} \tilde{L}e > 0 \), then the Lipschitz condition of \( g(x, y, \epsilon) \) is satisfied.

Thirdly, we prove that Assumption 4 is satisfied.

\[
|f(x, \sqrt{\epsilon}y)|^2 = \frac{1}{4} |f(x + \sqrt{\epsilon}y) + g(x - \sqrt{\epsilon}y)|^2
\]

\[
\leq \frac{\epsilon}{4} |f(x + \sqrt{\epsilon}y) - g(x - \sqrt{\epsilon}y)|^2 + 4|y|^2
\]

\[
\leq \frac{\epsilon}{4} |f(x + \sqrt{\epsilon}y) - f(0) - g(x - \sqrt{\epsilon}y) + g(0) + f(0) - g(0)|^2
\]

\[
+ 4|y|^2
\]

\[
\leq \epsilon |f(x + \sqrt{\epsilon}y) - f(0)|^2 + \epsilon |g(x - \sqrt{\epsilon}y) - g(0)|^2
\]

\[
+ \epsilon |f(0)|^2 + \epsilon |g(0)|^2 + 4|y|^2
\]

\[
\leq 2\tilde{L}e|x|^2 + 2\tilde{L}e|y|^2 + \epsilon |f(0)|^2 + \epsilon |g(0)|^2 \leq K(|x|^2 + |y|^2 + 1),
\]

where \( K = \max\{2\tilde{L}, 2\tilde{L}e^2 + 4, \epsilon |f(0)|^2, \epsilon |g(0)|^2\} \). Then linear growth conditions of \( f(x, y, \epsilon), g(x, y, \epsilon) \) are satisfied.

Fourthly, we prove that Assumption 5 is satisfied. On the one hand, we have

\[
\hat{f}(x, y) = \lim_{\epsilon \to 0} f(x, \sqrt{\epsilon}y) = \frac{1}{2} [f(x) + g(x)], \hat{g}(x, y) = \lim_{\epsilon \to 0} f(x, y, \epsilon) = -2y.
\]

On the other hand, by Assumption 7, we have

\[
|f(x, \sqrt{\epsilon}y) - \hat{f}(x, y)|^2 = \left| \frac{1}{2} [f(x + \sqrt{\epsilon}y) + g(x - \sqrt{\epsilon}y)] - \frac{1}{2} [f(x) + g(x)] \right|^2
\]

\[
\leq \frac{1}{2} |f(x + \sqrt{\epsilon}y) - g(x - \sqrt{\epsilon}y)|^2 + \frac{1}{2} |g(x - \sqrt{\epsilon}y) - g(x)|^2
\]

\[
\leq \frac{1}{2} \tilde{L}e|y|^2
\]

and

\[
|g(x, y, \epsilon) - \hat{g}(x, y)|^2 = \left| \frac{\sqrt{\epsilon}}{2} [f(x + \sqrt{\epsilon}y) - g(x - \sqrt{\epsilon}y)] - 2y + 2y \right|^2
\]

\[
\leq \epsilon |(f(x + \sqrt{\epsilon}y) - f(0))^2 + |g(x - \sqrt{\epsilon}y) - g(0)|^2
\]

\[
+ \epsilon |f(0)|^2 + |g(0)|^2\right|^2
\]

\[
\leq 2\tilde{L}e|x|^2 + 2\tilde{L}e^2|y|^2 + \epsilon |f(0)|^2 + \epsilon |g(0)|^2.
\]

Above all, Assumptions 2 - 4 hold. As a result, Theorem 4.2 holds.
With $\frac{1}{\nu} = \epsilon$, $2\tilde{X}_t^\epsilon = X_t^\nu + Y_t^\nu$ and $2\sqrt{\epsilon}\tilde{Y}_t^\epsilon = X_t^\nu - Y_t^\nu$, we get

$$Z_2^\nu(t) := \sqrt{\nu} \left( \frac{1}{2}(X_t^\nu + Y_t^\nu) - Z_t \right) = \frac{1}{\sqrt{\epsilon}}(\tilde{X}_t^\epsilon - Z_t).$$

Therefore, $Z_2^\nu(t) = \frac{1}{\sqrt{\epsilon}}(\tilde{X}_t^\epsilon - Z_t)$ converges in probability to $Z_2(t)$, where $Z_2(t)$ is stationary solution of

$$dZ_2(t) = \frac{1}{2}[D_x f(Z_t) + D_y g(Z_t)]Z_2(t)dt.$$

With $\frac{1}{\nu} = \epsilon$, $2\tilde{X}_t^\epsilon = X_t^\nu + Y_t^\nu$ and $2\sqrt{\epsilon}\tilde{Y}_t^\epsilon = X_t^\nu - Y_t^\nu$, we get

$$Z_2^\epsilon(t) := \frac{1}{\epsilon}\sqrt{\nu}(X_t^\nu - Y_t^\nu) = \tilde{Y}_t^\epsilon.$$

In the next, we want to present that $\tilde{Y}_t^\epsilon$ converges weakly to the stationary solution of the SDE (30). For each fixed $(Z_t, \epsilon)$,

$$d\tilde{Y}_t^{Z_t,\epsilon} = \frac{1}{2\sqrt{\epsilon}} \left[ f \left( Z_t + \sqrt{\nu} \tilde{Y}_t^{Z_t,\epsilon} \right) - g \left( Z_t - \sqrt{\nu} \tilde{Y}_t^{Z_t,\epsilon} \right) \right] dt - \frac{2}{\epsilon} \tilde{Y}_t^{Z_t,\epsilon} dt$$

where $Z_t$ is the unique stationary solution of the averaged SDE (2). By the definitions of $\tilde{Y}_t^\epsilon$ and $\tilde{Y}_t^{Z_t,\epsilon}$, we get

$$\frac{dE[|\tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon}|^2]}{dt} = 2E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} , \frac{1}{2\sqrt{\epsilon}} \left[ f \left( \tilde{X}_t^\epsilon + \sqrt{\nu} \tilde{Y}_t^\epsilon \right) - g \left( \tilde{X}_t^\epsilon - \sqrt{\nu} \tilde{Y}_t^\epsilon \right) \right] \right)$$

$$- \frac{1}{\sqrt{\epsilon}} E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} , f \left( Z_t + \sqrt{\nu} \tilde{Y}_t^{Z_t,\epsilon} \right) - g \left( Z_t - \sqrt{\nu} \tilde{Y}_t^{Z_t,\epsilon} \right) \right)$$

$$- \frac{4}{\epsilon} E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} , \frac{1}{2\sqrt{\epsilon}} \right)^2$$

$$= \frac{1}{\sqrt{\epsilon}} E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} , f \left( \tilde{X}_t^\epsilon + \sqrt{\nu} \tilde{Y}_t^\epsilon \right) - f \left( \tilde{X}_t^\epsilon + \sqrt{\nu} \tilde{Y}_t^{Z_t,\epsilon} \right) \right)$$

$$+ \frac{1}{\sqrt{\epsilon}} E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} , f \left( \tilde{X}_t^\epsilon + \sqrt{\nu} \tilde{Y}_t^{Z_t,\epsilon} \right) - f \left( Z_t + \sqrt{\nu} \tilde{Y}_t^{Z_t,\epsilon} \right) \right)$$

$$+ \frac{1}{\sqrt{\epsilon}} E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} , g \left( Z_t - \sqrt{\nu} \tilde{Y}_t^{Z_t,\epsilon} \right) - g \left( Z_t - \sqrt{\nu} \tilde{Y}_t^\epsilon \right) \right)$$

$$+ \frac{1}{\sqrt{\epsilon}} E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} , g \left( Z_t - \sqrt{\nu} \tilde{Y}_t^\epsilon \right) - g \left( \tilde{X}_t^\epsilon - \sqrt{\nu} \tilde{Y}_t^\epsilon \right) \right)$$

$$- \frac{4}{\epsilon} E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} , \frac{1}{2\sqrt{\epsilon}} \right)^2$$

$$\leq \frac{2}{\epsilon} E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} , \frac{1}{2\sqrt{\epsilon}} \right)^2 - k_1 E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} \right)^2 + k_2 E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} \right)^2$$

$$+ \frac{L_1}{4\epsilon} E \left( \tilde{X}_t^\epsilon - Z_t \right)^2 - k_2 E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} \right)^2 + k_2 E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} \right)^2$$

$$+ \frac{L_2}{4\epsilon} E \left( \tilde{X}_t^\epsilon - Z_t \right)^2$$

$$\leq \frac{2}{\epsilon} E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} , \frac{1}{2\sqrt{\epsilon}} \right)^2$$

Now the Gronwall inequality yields that

$$E \left( \tilde{Y}_t^\epsilon - \tilde{Y}_t^{Z_t,\epsilon} \right)^2 \leq E \left( \tilde{Y}_{t_0}^\epsilon - \tilde{Y}_{t_0}^{Z_t,\epsilon} \right)^2 e^{-\frac{\nu}{2}(t-t_0)} + 4C\epsilon \left( 1 - e^{-\frac{\nu}{2}(t-t_0)} \right).$$
For fixed $\epsilon$, recall the facts that the stationary solution of (6) and (32) are bounded. Letting $t_0$ tend to negative infinity, we get that

$$\mathbb{E} \left| \tilde{Y}_t^{\epsilon} - \tilde{Z}_t^{\epsilon} \right|^2 \leq 4C\epsilon.$$  

By the time scale $t = s\epsilon$, the SDE (32) is transformed to

$$dY_{se}^{Z_t^{\epsilon},\epsilon} = \sqrt{\epsilon} \left[ f \left( Z_t + \sqrt{\epsilon} \tilde{Y}_{se}^{Z_t^{\epsilon},\epsilon} \right) - g \left( Z_t - \sqrt{\epsilon} \tilde{Y}_{se}^{Z_t^{\epsilon},\epsilon} \right) \right] ds - 2\tilde{Y}_{se}^{Z_t^{\epsilon},\epsilon} ds$$

$$+ \frac{1}{2\sqrt{\epsilon}} (\alpha - \beta) dW_{se}.$$  

Denote $\tilde{Y}_{s}^{Z_t^{\epsilon},\epsilon} = \tilde{Y}_{se}^{Z_t^{\epsilon},\epsilon}$,

$$d\tilde{Y}_{s}^{Z_t^{\epsilon},\epsilon} = \sqrt{\epsilon} \left[ f \left( Z_t + \sqrt{\epsilon} \tilde{Y}_{s}^{Z_t^{\epsilon},\epsilon} \right) - g \left( Z_t - \sqrt{\epsilon} \tilde{Y}_{s}^{Z_t^{\epsilon},\epsilon} \right) \right] ds - 2\tilde{Y}_{s}^{Z_t^{\epsilon},\epsilon} ds$$

$$+ \frac{1}{2\sqrt{\epsilon}} (\alpha - \beta) d\tilde{W}_s,$$

where $\tilde{W}_s = \frac{1}{\sqrt{\epsilon}} W_{se}$ is the scaled version of $W_s$ and with the same distribution.

Lemma 3.2 in [18] yields that the stationary solution $\tilde{Y}_{t}^{Z_t^{\epsilon},\epsilon}$ has the same distribution of the stationary solution $\tilde{Y}_{s}^{Z_t^{\epsilon},\epsilon}$. $Z_1(t) = \frac{1}{2}(\alpha - \beta) \int_{-\infty}^{t} e^{-2(t-s)} d\tilde{W}_s$ is a unique stationary solution of the following SDE

$$dZ_1(t) = -2Z_1(t) dt + \frac{1}{2}(\alpha - \beta) d\tilde{W}_t.$$  

Moreover, it follows from Lemma 2.5 that

$$\mathbb{E}|\tilde{Y}_{t}^{Z_t^{\epsilon},\epsilon} - Z_1(t)|^2 \leq C\epsilon.$$  

Above all, the stationary solution $\tilde{Y}_t^{\epsilon}$ converges weakly to the stationary solution of the SDE

$$dZ_1(t) = -2Z_1(t) dt + \frac{1}{2}(\alpha - \beta) d\tilde{W}_t.\qedhere$$

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