Non-chiral fusion rules and structure constants of $D_m$
minimal models.

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Abstract

We present a technique to construct, for $D_m$ unitary minimal models, the non-
chiral fusion rules which determines the operator content of the operator product
algebra. Using these rules we solve the bootstrap equations and therefore determine
the structure constants of these models. Through this approach we emphasize the
role played by some discrete symmetries in the classification of minimal models.

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1 Introduction

The first and the most known process to deal with the classification problem of two dimensional conformal field theories (CFT) is the bootstrap approach. This approach initially developed in the seminal work of Belavin, Polyakov and Zamolodchikov (PBZ) \cite{1} is based, among other things, on the associativity property of the operator product algebra (OPA) of the four point correlation functions on the plane. Formally speaking, this property known as crossing symmetry is expressed by the so-called bootstrap equations. These equations are the master equations since their resolution gives in principle a complete classification of a conformal theory. (PBZ) mentioned the existence of a particular class of (CFT) associated to degenerate representation of the Virasoro algebra for which the bootstrap equations can be solved. This class of CFT’s designed as minimal models include the finite discrete unitary models with central charge $c < 1$ \cite{2}.

In this framework, Dotsenko and Fattev (DF) \cite{3} proposed to solve the bootstrap equations and therefore they determined the structure constants of the operator algebra for correlations with spinless fields. They use in their construction the monodromy property of conformal blocs defining the coordinate dependencies of the correlation functions in the halomorphic (antihalomorphic) sector $z (\bar{z})$ in the Coulomb gas formalism. Indeed for spineless fields the conformal blocs in the two sectors are complex conjugate so that only diagonal combinations of this blocs survive to the monodromy constraint. After this work Capelli, Itzykson and Zuber \cite{4} made a complete classification of modular invariant partition functions: the $ADE$ classification of minimal conformal models. Their results gave the operator content of all the minimal models. The $(\Lambda)$ present only spinless primary fields and the models of this series are those studied by (DF).

Nevertheless, in the case of the $(D)$ series solving the bootstrap equations seems to be more difficult. Two difficulties arise; namely the presence of spin fields and the existence of two copies of certain fields. Because of the presence of the non-vanishing spin fields, it is difficult to formulate the monodromy constraint in the (DF) coulomb gas approach spirit.
Moreover, with the monodromy constraint alone we cannot differentiate the behavior of the different copies of doubled fields in the (OPA).

Many approaches have been developed to solve these problems. Apparently the most complete and general approach is the one developed by Petkova [5]. This consists globally in an adaptation of the principle of monodromy in the (DF) coulomb gas formalism for the (D) series models. The importance of this adaptation resides in that it permits to understand the technical aspects of the (DF) formulation. However, despite its importance, which will be of a great utility for us, this approach present many ambiguities regarding its principle [6]. It is important to mention the existence of another approach [7] which call on other formalisms namely the lattice representation (generalized RSOS models) of the ADE models [8]. In this approach the ratios of the structure constants of the D or E theories over the corresponding structure constants of the A theory of the same Coxeter number are determined. It was also mentioned in Ref.[7] that these results do not seem easy derived from crossing (bootstrap) equations and only phenomenological observations are given.

What we propose in the present work is a simple and general approach to solve the bootstrap equations without using the monodromy constraint. Our approach is a generalization of the ideas initially developed in Ref.[9] for the particular case of the $D_5$ ($c = \frac{3}{5}$) model. The principle of the idea lays on the construction of the so called non-chiral fusion rules. The non-chiral fusion rules determine the operator content of the operator algebra. Initially the term fusion rules was used to express how two representations of

\[ I (|\alpha\rangle , |\beta\rangle ) = \lim_{z \to \infty, w \to 0} z^{2h_\alpha - 2h_\beta} \langle 0 | \Phi^\alpha (z, \bar{z}) \Phi^\beta (w, \bar{w}) | 0 \rangle = (-1)^s \delta_{\alpha \beta} \]

We see that the factor $(-1)^s$ would give negative norm states.

---

\(^1\)One of these adaptations is the normalization of the two point correlations by a factor equal to $(-1)^s$ rather than 1, where $s$ is the spin two point field. This redefinition of the normalization factor is in complete contradiction with unitarity. In fact, the inner product $I$ of two of the highest weight states is determined by [10]:
the Virasoro algebra combine in the (OPA). The non-chiral fusion rules permit to avoid
the monodromy problem and hence to make a new step in the resolution of the bootstrap
equations. For the determination of the non-chiral fusion rules we use some considerations
imposing strict constraints. The first constraint lays on the consistency of the non-chiral
fusion rules and the fusion rules (chiral). The second consideration, rather obvious, con-
sist on imposing the consistency of the non-chiral fusion rules with the operator content
determined by the modular constraint. The third important consideration, which is given
to construct the non-chiral fusion rules lays on symmetry considerations. This last consid-
eration is manifested by a discrete symmetry $Z_2$ which is defined by its action on the two
components of a doubled field $(\Phi^{\pm})$: $Z_2(\Phi^{\pm}) = \pm \Phi^{\pm}$. Thus, this symmetry ($Z_2$) permit
to separate the doubled field contribution by imposing the consistency of the non-chiral
fusion rules with its action. Under the action of the $Z_2$ symmetry the scalar fields are
found to have a positive parity contrary to spin fields which have negative parity. As a
consequence; the consistency of the non-chiral fusion rules with the $Z_2$ action gives them
a $Z_2 - grading$ structure.

Once the non-chiral fusion rules are determined, it is possible to solve the bootstrap
equations by considering that at short distance these equations should be consistent with
the (OPA) which we have expressed through these rules. This is precisely what was done
in Ref. [9] for the $D_5$ model. Here, we propose a generalization of these calculations to
the all set of $D_m$ models. For this end we use general analytic and duality proprieties of
conformal blocs in the coulomb gas formalism developed in Ref. [5]. As a result, we have
found that the structure constants of the $D_m$ series models factories out in those of the
chiral algebra expressed by the $A_m$ series. The signs of the structure constants are also
determined.

In our construction a particular role is played by $Z_2$ symmetry which reflects the
$Z_2 - grading$ structure of the non-chiral fusion rules. A physical interpretation of the
$Z_2$ symmetry as a scaling limit symmetry of some lattice models namely the generalized
RSOS [8] models is given. Furthermore, we show the importance of discrete symmetries
and particularly the symmetry $Z_2$ in the classification of minimal models. These remarks make our approach applicable in general for the large set of rational conformal field theories [11].

This article is organized as follows. In sec.2 we briefly review some important results of minimal models classification. We give in particular a precise analysis of the operator content spectrum of $(D_m)$ unitary series. In sec.3 we present our approach by constructing the non-chiral fusion rules. In sec.4 we solve the bootstrap equations and determine the structure constants. In sec.5 we give a general analysis of our approach and extract some important consequences and we conclude in sec.6.

2 General features of minimal models

One of the fundamental requirements in conformal field theories is the existence of a closed operator algebra. At two dimensions this requirement is expressed by the operator product algebra of two primary fields which can be written in the form:

$$\Phi_I(z, \bar{z}) \cdot \Phi_J(w, \bar{w}) = \sum_K C_{IJK} (z-w)^{h_k-h_i-h_j} (\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j} [\Phi_K (w, \bar{w}) + \ldots].$$

where $I = (i, \bar{i})$ indicates different fields with $i (\bar{i})$ representing the chiral contributions of the holomorphic (anti holomorphic) sectors. $h_i$ and $\bar{h}_i$ are the conformal dimensions of $\Phi_I$ and the ellipsis stands for terms involving descendant fields. $C_{IJK}$ are complex numbers known as structure constants of the operator algebra.

The associativity of the operator product algebra in four point correlation functions on the plane yields to symmetry relations termed as bootstrap equations:

$$G_{IJKP} (z, \bar{z}) = \langle \Phi_I (0) \cdot \Phi_J (z, \bar{z}) \cdot \Phi_L (1) \cdot \Phi_K (\infty) \rangle$$

$$= \sum_{(p, \bar{p})} C_{IJP} C_{LMP} F_{ij}^{lm} (p \mid z) \bar{F}_{\bar{i}j}^{\bar{l}m} (\bar{p} \mid \bar{z})$$

$$= \sum_{(q, \bar{q})} C_{ILQ} C_{JMQ} F_{ij}^{lm} (q \mid 1-z) \bar{F}_{\bar{i}j}^{\bar{l}m} (\bar{q} \mid 1-\bar{z}).$$
where $F_{ij}^m (p \mid z)$ denotes the conformal blocs. These two expressions (3) and (4) of the correlation $G$ are noted as t-channel and s-channel development respectively.

In the case of degenerate representations of the Virasoro algebra projecting out null states imposes a strict constraint on conformal models. This yields a particular class of conformal theories known as minimal models. For unitary minimal models the central charge and the conformal dimensions of fields are confined to discrete values:

$$c(m) = 1 - \frac{6}{m(m+1)},$$

$$h_{rs}(m) = h_{m-r,m+1-s}(m) = \frac{(m+1)r - ms)^2 - 1}{4(m+1)m},$$

$$1 \leq r \leq m - 1, \ 1 \leq s \leq m.$$  

In these cases the conformal blocs are solutions of differential equations. This later important fact, proved to give hard constraints on the operator algebra. These constraints are expressed through the fusion rules, which determine how the chiral parts of physical fields combine in the (OPA). If we associate to each primary field $\Phi_I, I = (r, s \mid \tau, \bar{\tau})$, its chiral part $(r, s)$; the fusion rules for minimal models will be given by:

$$(r_1, s_1) \times (r_2, s_2) = \sum_{k=|r_1-r_2|+1}^{\min(r_1+r_2,2m-r_1-r_2+1)} \sum_{l=|s_1-s_2|+1}^{\min(s_1+s_2,2m-s_1-s_2+1)} (k, l). \quad (5)$$

On another hand, the modular constraint on the partition function, give a complete classification of the operator content of minimal models: $ADE$ patterns [4]. First the $(A)$ series is diagonal so that the operator content is composed by spinless scalar fields. For the $D$ series the partition functions are non-diagonal. They can be organized in the unitary case as follows:

- $m = 4\rho + 2$

$$Z = \frac{1}{4} \sum_{s=1}^{m} \sum_{r(odd)=1}^{m-1} |\chi_{rs} + \chi_{m-r,s}|^2. \quad (6)$$
\[ m = 4\rho + 1 \]
\[ Z = \frac{1}{4} \sum_{r=1}^{m-1} \sum_{s(\text{odd})=1}^{m} |\chi_{rs} + \chi_{r,m+1-s}|^2. \]  
(7)

\[ m = 4\rho \]
\[ Z = \frac{1}{2} \sum_{s=1}^{m} \left\{ \sum_{r(\text{odd})=1}^{m-1} |\chi_{rs}|^2 + \sum_{r(\text{even})=2}^{m-2} \chi_{m-r,s}\chi_{rs}^* \right\}. \]  
(8)

\[ m = 4\rho + 3 \]
\[ Z = \frac{1}{2} \sum_{r=1}^{m-1} \left\{ \sum_{s(\text{odd})=1}^{m} |\chi_{rs}|^2 + \sum_{s(\text{even})=2}^{m-1} \chi_{r,m+1-s}\chi_{rs}^* \right\}. \]  
(9)

2.1 Spectrum analysis

From the partition functions of the \( D \) series unitary minimal models presented above, we can summarize the operator content spectrum with respect to \( m \) values modulo 4 as follows:

| \( m \) value | Scalar fields | Spin fields |
|--------------|---------------|-------------|
| \( m = 2 \mod 4 \) | \( r, s \mid r, s \) | \( r, s \mid m - r, s \) | \( r = \text{odd} \) |
| \( m = 1 \mod 4 \) | \( r, s \mid r, s \) | \( r, s \mid r, m + 1 - s \) | \( s = \text{odd} \) |
| \( m = 0 \mod 4 \) | \( r, s \mid r, s \) (\( r = \text{odd} \)) | \( r, s \mid m - r, s \) (\( r = \text{even} \)) |
| \( m = 3 \mod 4 \) | \( r, s \mid r, s \) (\( s = \text{odd} \)) | \( r, s \mid r, m + 1 - s \) (\( s = \text{paire} \)) |

According to the operator spectrum one can distinguish two categories:

- \( m = 0 \mod 4 \) and \( m = 3 \mod 4 \) : these cases are noted as automorphism or permutation invariant solutions. This notation comes from the fact that the modular invariant partition functions can be written as: \( Z = \sum_{(rs)} \chi_{rs}\chi_{\mu(rs)} \), where \( \mu \) is some automorphism of fusion rules [12]. We remark in this category the existence of a difference in parity with respect to the indices \( r \) or \( s \) between the scalar and spin fields. Also, for the values \( r = m/2 \) (\( m = 0 \mod 4 \)) and \( s = (m+1)/2 \) (\( m = 3 \mod 4 \))
(\(m = 3 \text{ mod } 4\)) the corresponding spin fields have a null spin; they are thus scalar fields. For technical reasons, that we will see later, they are called null spin field.

- \(m = 2 \text{ mod } 4\) and \(m = 1 \text{ mod } 4\): these cases are noted as integer invariant solutions.

The partition functions of this category \([1-7]\) can be regarded as diagonal invariant solutions of a larger chiral algebra than originally considered (Virasoro algebra) \([11]\). The field extending this algebra is represented by a character appearing in the same term as the identity character. This field is thus: \((1,1 | 1, \frac{m+1}{2})\) and \((1,1 | 1, \frac{m}{2})\) for \(m = 2 \text{ mod } 4\) and \(m = 1 \text{ mod } 4\) respectively. What characterizes the operator content of these cases is not the difference in parity between the scalar and spin fields with respect to the indices \((r, s)\), but the appearance of two copies of certain scalar fields. In fact, for \(m = 2 \text{ mod } 4\) \((m = 1 \text{ mod } 4)\) with \(r = m/2\) \((s = (m + 1)/2)\), the corresponding spin fields have the form:

\[
(m/2, s | m/2, s), \quad m = 2 \text{ mod } 4. \tag{10}
\]
\[
(r, (m + 1)/2 | r, (m + 1)/2), \quad m = 1 \text{ mod } 4. \tag{11}
\]

and have, thus, a form of a null spin fields. Furthermore we note that the same fields are present in the scalar fields set so that we have two copies of these fields \([10-11]\) in the operator content spectrum. We note these two copies by \(\Phi^+\) and \(\Phi^-\).

The operator content analysis in the case of the \(D\) unitary minimal models we have developed will permit us later to give a simple and general description of the operator algebra through the non-chiral fusion rules. At this point it is interesting to note that the two cases \(m = 2 \text{ mod } 4\) and \(m = 1 \text{ mod } 4\) are symmetric via the permutations \(s \leftrightarrow r, m \leftrightarrow m+1\). This same remark is valid for \(m = 0 \text{ mod } 4\) and \(m = 3 \text{ mod } 4\). Thanks to this, the treatment of the \(D\) series unitary models can be reduced to the cases \(m = 3 \text{ mod } 4\) for the automorphism invariant cases and \(m = 1 \text{ mod } 4\) for the integer invariant cases.
3 Non-chiral fusion rules

By definition the non-chiral fusion rules determine the operator content of the operator algebra. Therefore, the fusion of the two fields $\Phi_I$ and $\Phi_J$ produce the field $\Phi_K$ if and only if the structure constant $C_{IJK}$ is non-vanishing. To construct the non-chiral fusion rules in the case of minimal models we will use two important facts which are the consistency of these rules with the fusion rules (chiral) and the operator content structure together.

In fact, from the bootstrap equations we can get the following condition:

$$C_{IJK} \neq 0 \Rightarrow \left\{ \begin{array}{c}
(i) \times (j) \rightarrow (k) \\
(\bar{i}) \times (\bar{j}) \rightarrow (\bar{k})
\end{array} \right..$$

(12)

with $(i) \times (j) \rightarrow (k)$ translating the fusion condition of $(i)$ and $(j)$ which give the field $(k)$. This condition indicates the consistency of the non-chiral fusion rules with the fusion rules. The second condition that must be considered is the compatibility of the operator algebra with the operator content spectrum. If we designate this spectrum by the set $\mathcal{A}$ this condition becomes:

$$C_{IJK} \neq 0 \Rightarrow I, J, K \in \mathcal{A}.$$

(13)

For the series $(A)$ models the application of these conditions lead to the known result obtained by (DF) by the monodromy invariance $[3]$:

$$(s_1, r_1 \mid s_1, r_1) \times (s_2, r_2 \mid s_2, r_2) = \sum_{k=|r_1-r_2|+1}^{\min(r_1+r_2,2m-r_1-r_2+1)} \sum_{l=|s_1-s_2|+1}^{\min(s_1+s_2,2m-s_1-s_2+1)} (k, l \mid k, l).$$

(14)

3.1 $D$ series non-chiral fusion rules

3.1.1 Automorphism invariant cases

Due to the different parities of indices indicating the scalar and spin fields, the non-chiral fusion rules can be easily determined.
**Example of** $D_7$ **model:** For this model the partition function \( Z \) produce the following operator content:

\[
Z = \chi_{11}^* \chi_{11} + \chi_{13}^* \chi_{13} + \chi_{15}^* \chi_{15} + \chi_{17}^* \chi_{17} + \chi_{16}^* \chi_{12} + \chi_{14}^* \chi_{14} + \chi_{12}^* \chi_{16} + \\
\chi_{51}^* \chi_{51} + \chi_{53}^* \chi_{53} + \chi_{55}^* \chi_{55} + \chi_{57}^* \chi_{57} + \chi_{56}^* \chi_{52} + \chi_{54}^* \chi_{54} + \chi_{52}^* \chi_{56} + \\
\chi_{31}^* \chi_{31} + \chi_{33}^* \chi_{33} + \chi_{35}^* \chi_{35} + \chi_{37}^* \chi_{37} + \chi_{36}^* \chi_{32} + \chi_{34}^* \chi_{34} + \chi_{32}^* \chi_{36}.
\]

We begin by determining the fusion rules \( \mathcal{F} \) of the model. Among other things we find:

\[
(15) \times (15) = (11) + (13) + (15),
\]

\[
(17) \times (53) = (55),
\]

\[
(16) \times (14) = (13) + (15), \quad (12) \times (16) = (15) + (17)
\]

\[
(16) \times (36) = (31) + (33), \quad (12) \times (32) = (31) + (33)
\]

\[
(15) \times (12) = (14) + (16), \quad (15) \times (15) = (12) + (14).
\]

Finally to deduce the non-chiral fusion rules of the model $D_7$, we combine the fusion rules in the two sectors and use the consistency with the operator content. As a final result we obtain:

\[
(1,5 | 1,5) \times (1,5 | 1,5) = (1,1 | 1,1) + (1,3 | 1,3) + (1,5 | 1,5) \quad (a)
\]

\[
(1,7 | 1,7) \times (5,3 | 5,3) = (5,5 | 5,5) \quad (b)
\]

\[
(1,6 | 1,2) \times (1,4 | 1,4) = (1,5 | 1,5) \quad (c)
\]

\[
(1,5 | 1,5) \times (1,6 | 1,2) = (1,4 | 1,4) + (1,2 | 1,6) \quad (d)
\]

\[
(1,6 | 1,2) \times (3,6 | 3,2) = (3,5 | 3,5) + (3,1 | 3,1) \quad (e)
\]

We remark that the fusion of two scalar fields or two spin fields produce only scalar fields (rule \( a \), \( b \) and \( e \)). Whereas, the fusion of scalar fields and spin fields produce only spin fields (rule \( d \)). It is to be noted that the field \( 1,4 | 1,4 \) which is a null spin field behaves like a spin field in the non-chiral fusion rules (rule \( c \)). This justifies the name null spin field.
The conclusions of the preceding example are valid in the most general case. This is due, indeed, to the difference in parity of the indices between the scalar fields and spin fields. In fact, for \( m = 3 \mod 4 \) the index \( s \) is odd for scalar fields and even for spin fields. As a consequence, the fusion products have an index \( s \) with the parity of \(|s_1 - s_2| + 1\), that is: odd if two fields have the same parity and even if they have different parity. By symmetry \( s \leftrightarrow r \), the same argument is valid for the case \( m = 0 \mod 4 \).

In conclusion, if we designate by \( A_0 \) the set of scalar fields and \( A_1 \) the set of spin and null spin fields, the non-chiral fusion rules will have the form:

\[
A_i * A_j = A_k, \quad k = (i + j) \mod 2
\]

The so constructed non-chiral fusion rules have thus a \( \mathbb{Z}_2 \)–grading structure.

These fusion rules structure we have just determined translates an important fact (especially for the remaining) which is the conservation of a parity in these rules. Indeed, if we affect a positive parity charge to the set of scalar fields \( (A_0) \) and a negative parity charge to the set of spin and null spin fields \( (A_1) \) then the \( \mathbb{Z}_2 \)–grading structure translates the conservation of this charge in the non-chiral fusion rules.

### 3.2 Integer invariant cases

For these \( D_m \) series models there is no difference in parity between the scalar and spin fields and it is not possible to deduce simple rules as above. In addition and as we have already remarked there is a doubling of certain fields. The two components of these fields which we have noted \( \Phi^+ \) and \( \Phi^- \) will have the same contribution in the (OPA) and it is not possible, thus, to distinguish between the behavior of each one in a correlation function.

To overcome these difficulties we use an important physical fact namely that by the state-field correspondence principle the presence of many copies of primary fields translates the degeneracy of the ground state. To lift this degeneracy we introduce a discrete
parity symmetry $Z_2$:

\[ Z_2 (\Phi^\pm) = \pm \Phi^\pm \quad (16) \]

The characterization of the $Z_2$ symmetry is complete if one arrives at defining its action on the other fields of the model \{\(\Phi_\alpha\}\}. To this end we use an important consideration which is the consistency of the operator product algebra with the action of this symmetry. This last consideration is expressed by adding to the two construction bases (12-13) a third constraint which is the consistency of the non-chiral fusion rules with the $Z_2$ symmetry action, i.e. the different members in a fusion rule must have the same parity (the conservation of $Z_2$ parity charge in the non-chiral fusion rules).

For the $Z_2$ symmetry construction we use the $D_m$ series with $m = 1 \mod 4$. The same argument is valid by symmetry for $m = 2 \mod 4$.

### 3.2.1 Construction of $Z_2$

The technique of a $Z_2$ parity symmetry in the construction of the non-chiral fusion rules was initiated in Ref.[9] for the particular case of the $D_5$ model. In this work, the authors were concerned only with thermic sub-algebra:

\[ \{(15,15) , (15,11), (11,15), \Phi^+_{13}, \Phi^-_{13}\} \]

The construction of this symmetry is based on the consistence of its action with the operator algebra. Thus, from the fact that:

\[ (15,15) \times (15,11) = (11,15) \]

one can see that the $Z_2$ action will be limited to the following cases [9]:
It worth noting that for the construction of the non-chiral fusion rules the first two construction bases along with the definition itself of these rules was not taken explicitly in Ref.[9]. This has as a consequence a great problem as we will see hereafter.

The consistency with the \((Z_2)\) action (A)–(D) leads each one of its own to different structure of non-chiral fusion rules. To select the physical structure the consistency of each one structure with the bootstrap equations was taken into consideration. Therefore, by a counter example the fusion rules constructed on the basis of the case (C) and (D) are found inconsistent. The case (B) is presumed inconsistent and only the case (A) is retained.

The most striking result of the calculus of the structure constants in the case (A) is the vanishing of one of the constants [9]:

\[ C_{+++} = 0 \]  

(with the notation \( + = (1, 3 | 13)^+ \)) although the coupling \((+++\)\) is permitted by the fusion rules. This result was considered as specific by noting that “\( \cdots \) the vanishing of \( C_{+++} \) follows from our calculations and not from the fusion rules”. If one returns to the definition of the non-chiral fusion rules as describing the operator content of the operator algebra the result (17) found in Ref.[9] has nothing specific. In other words, this result is simply not consistent with the bootstrap equations. The action of the \( Z_2 \) symmetry that produce non-chiral fusion rules consistent with the bootstrap equations is limited only to the case (B) which was not considered in Ref.[9].
In this case as an example of the non-chiral fusion rules for the $D_5$ model we find:

\[
\begin{align*}
\Phi^+ \cdot \Phi^+ &= (1,1 \mid 1,1) + \Phi^+ + (1,5 \mid 1,5) \\
\Phi^- \cdot \Phi^- &= (1,1 \mid 1,1) + \Phi^+ + (1,5 \mid 1,5) \\
\Phi^- \cdot \Phi^+ &= \Phi^- \\
\end{align*}
\]

$\Phi^\pm$ are the two copies of the degenerate field $(1,3 \mid 1,3)$.

The construction of $Z_2$ symmetry for the thermic subalgebra of the $D_5$ model can be straightforwardly extended to the remaining fields of the $D_5$ model. In fact, the consistency of the $Z_2$ symmetry action with the (OPA):

\[
(1,1 \mid 1,5) \times (2,1 \mid 2,1) = (2,1 \mid 2,5), \quad (1,1 \mid 1,5) \times (2,1 \mid 2,1) = (2,5 \mid 2,1) \\
(1,5 \mid 1,5) \times (2,1 \mid 2,1) = (2,5 \mid 2,5)
\]

leads to the following tree possibilities for the $Z_2$ action:

B1) $(21;21) \rightarrow + (21;21), \quad (25;25) \rightarrow + (25;25), \quad (25;2,1) \rightarrow + (25;21), \quad (21;25) \rightarrow + (21;25)$.

B2) $(21;21) \rightarrow - (21;21), \quad (25;25) \rightarrow - (25;25), \quad (25;2,1) \rightarrow + (25;21), \quad (21;25) \rightarrow + (21;25)$.

B3) $(21;21) \rightarrow + (21;21), \quad (25;25) \rightarrow + (25;25), \quad (25;2,1) \rightarrow - (25;21), \quad (21;25) \rightarrow - (21;25)$.

By a straightforward manipulation of the bootstrap equations in these three cases, one can see that only the case (B3) is consistent with bootstrap constraint.

A first looking on the $Z_2$ symmetry for the $D_5$ model which we have just determined, one remark that scalar fields are singlet under $Z_2$ contrary to spin fields which has a negative parity under $Z_2$. The negative parity component ($\Phi^-$) of a degenerate field behave like a spin field in the (OPA) (null spin structure field).

This construction of the $Z_2$ symmetry is done for the $D_5$ model thermic subalgebra and is worth to be generalized for the remaining $D_m$ models, $m = 1 \text{ mod } 4$. Applying the
same method as for the $D_5$ for the whole set of $D_m$ ($m = 1 \mod 4$) is somewhat a delicate thing. The set of fields to take in consideration increases with ($m$) and the number of the possibilities of the $Z_2$ becomes, therefore, important. To overcome this problem we shall consider a simple intuitive analysis that lays on the following consideration: seeing that the symmetry $Z_2$ is introduced by an *ad-hoc* manner in order to separate the contribution of fields that double in the (OPA) and having in mind the structure of the operator content that is the same one is tempted to find that the $Z_2$ action follows a general law independently from the model (i.e. from $(m)$).

Thus the $Z_2$ symmetry action for the general $D_m$ models ($m = 1 \mod 4$) have the same structure action as the $D_5$ model.

$$Z_2 (\Phi^\pm) = \pm \Phi^\pm, \quad Z_2 (\Phi^s_\alpha) = -\Phi^s_\alpha, \quad Z_2 (\Phi^c_\alpha) = +\Phi^c_\alpha. \quad (19)$$

where $s =$ spin fields and $c =$ scalar fields.

Now as the action of the $Z_2$ symmetry is found the non-chiral fusion rules are constructed by imposing the consistency of these rules with the action of this symmetry. In other words this turn out to consider the conservation of a parity charge in these rules. In consequence, the non-chiral fusion rules in the $D_m$ case $m = 1, 2 \mod 4$, will have a $Z_2 - grading$ structure at the same title as the cases $m = 0, 3 \mod 4$.

**Conclusion for the non-chiral fusion rules** If we designate by $A_0$, the set of scalar fields and by $A_1$ the set of spin and null spin fields the fusion rules will be of the forme

$$A_i * A_j = A_k, \quad k = (i + j) \mod 2. \quad (20)$$

---

2This general symmetry structure is perceptible from the simple currents construction of the $D$ series \(1\).
4 Structure constants

Now as the fusion rules are constructed it is possible then to solve the bootstrap equations in order to obtain the structure constants. We present thereafter the complete demonstration of this calculation. Since the $D_m$ models have the same non-chiral fusion rules structure (20) we restrict ourselves to the cases $m = 1 \mod 4$.

4.1 Notations

Here we sketch the most intriguing properties of conformal blocs in coulomb gas construction (for more details see [3] and [5]). For simplicity we limit ourselves to thermic subalgebra. First let us consider the four point correlation functions:

$$G(z, \bar{z}) = \langle \Phi_N(z_1, \bar{z}_1) \cdot \Phi_K(z_2, \bar{z}_2) \cdot \Phi_K(z_3, \bar{z}_3) \cdot \Phi_N(z_4, \bar{z}_4) \rangle.$$  \hspace{1cm} (21)

with $N = (1, n \mid 1, \overline{n})$ denotes a field in unitary minimal model spectrum and $z = \left(\begin{smallmatrix} \frac{z_1 \bar{z}_1}{z_3 \bar{z}_3} \\ \frac{z_2 \bar{z}_2}{z_4 \bar{z}_4} \end{smallmatrix}\right)$. These correlations can be written in Coulomb gas formalism as follows:

$$G(z, \bar{z}) = f(z_i) f(\bar{z}_i) \sum_{i,j} \gamma_{ij} (a, b, c) I_i^n (a, b, c; z) I_j^n' (\overline{a}, \overline{b}, \overline{c}; \overline{z}) \cdot$$  \hspace{1cm} (22)

where $I_i(a, b, c; z)$ are conformal blocs in coulomb gas formulation and $\gamma_{ij} (a, b, c)$ are coupling constants with the notations: $a = 2\alpha_+ \alpha_n$, $b = c = 2\alpha_+ \alpha_k$, $d = 2\alpha_+ (2\alpha_0 - \alpha_n)$, $\alpha_+ = \left[\frac{m}{m+1}\right]^{1/2}$, $\alpha_0 = -\left[\frac{1}{m(m+1)}\right]^{1/2}$, $\alpha_i = \frac{1}{2} (1 - i) \alpha_+$ and $f(z_i) = \frac{(z_{14})^{2(h_k-h_n)}}{(z_{13} z_{24})^{2h_k}} z^{2\alpha_n \alpha_m} (1 - z)^{2\alpha_m \alpha_n}$.

The conformal blocs have the short distance development:

$$\lim_{z \to 0} I_i^n (a, b, c; z) = \left[z^{-2\alpha_n \alpha_m - (h_n + h_k - h_p)} \cdot \mathcal{N}_i^n (a, b, c) \cdot (1 + O(z))\right]_{p = n - k + 2i - 1}.$$  \hspace{1cm} (23)

$\mathcal{N}_i^n (a, b, c)$ is a normalization constant.
The correlations (22) are in the t-channel development. To express the s-channel development we use the conformal blocks transformation under duality:

$$I^n_i (a, b, c; z) = \sum_j \alpha_{ij} (a, b, c) I^n_j (b, a, c; 1 - z).$$  (24)

where $\alpha_{ij} (a, b, c)$ are elements of the monodromy matrices. Therefore, the s-channel correlations can be written as follows:

$$G (z, \bar{z}) = f (z_i) \bar{f} (\bar{z}_i) \sum_{k, i, j, l} \gamma_{ij} (a, b, c) \alpha_{ik} \alpha_{jl} I^n_k (b, a, c; 1 - z) T^n_l (\bar{b}, \bar{a}, \bar{c}; 1 - \bar{z}),$$  (25)

4.2 Bootstrap equations resolution

Before solving the bootstrap equations for the $D_m$ series models let us consider first this resolution for the simplest cases of $A_m$ series models. For these models and from the (OPA) we can write the correlation functions (21) at short distance as:

$$G (z) \sim \sum_{p} \frac{\left( C_{N K}^P \right)^2}{|z|^{2(h_a + h_b - h_p)}} (1 + O (z)).$$  (26)

where $(P)$ denotes a field permitted by the non-chiral fusion rules law (14) of $(N)$ and $(K)$. The consistency of the correlations in the s-channel with the non-chiral fusion rules imposes that only the diagonal terms are present in (22) and that from (23) and (26) the structure constants are given by:

$$\left( C_{N K}^P \right)^2 \propto \left( \gamma_i (a, b, c) N_i^2 (a, b, c) \right)_{p = n - k + 2i - 1}. $$  (27)

The $\gamma_i (a, b, c)$ are obtained by imposing the consistency of the correlations in the t-channel with non-chiral fusion rules and which leads to the known result of (DF) namely:

$$\sum_k \gamma_k (a, b, c) \alpha_{ki} (a, b, c) \alpha_{kj} (a, b, c) = \gamma_i (b, a, c) \delta_{ij}. $$  (28)
By imposing that \( C_{NN}^1 = 1 \) one can deduce the proportionality factor in (27) and thus obtain the final forms of the structure constants of the \((A)\) series models:

\[
C_{N}^{P} K = \sqrt{\frac{\gamma_i (a, b, c)}{\gamma_1 (b, a, c)}} \frac{N_i (a, b, c)}{N_1 (b, a, c)}.
\]  

(29)

Now we propose to solve in the same manner the bootstrap equations for the \(D_m\) series models. The correlations to deal with are of the form (21) where \(N\) (with bold character) is a spin field \((\bar{\eta} = m + 1 - n)\) or a negative parity copies \(\Phi^-\) of a degenerate field and \(K\) is a scalar field \((\bar{\pi} = m)\).

**t-channel:**

In the channel-t the correlations at short distance written on the basis of fusion rules (20) are of the form:

\[
G_1 = \sum_{p, \bar{\pi} \in \mathbb{N}} \frac{\left( C_{N}^{P} K \right)^2}{\delta_{m, m+1-p}} \frac{\delta_{m, m+1-p}}{z_{12}^{h_n+h_k-h_p} z_{34}^{h_n+h_k-h_p}} \frac{I_i (\bar{\pi}, b, c; z)}{I_j (\bar{\pi}, b, c; \bar{\pi})},
\]  

(30)

In terms of the conformal blocks these correlations are written as:

\[
G_1 = f (z_i) \bar{f} (\bar{z}_i) \sum_{i,j} \gamma_i^{(D)} (a, b, c | \bar{\eta}, b, c) I_i^n (a, b, c; z) I_j^n (\bar{\eta}, b, c; \bar{z}).
\]  

(31)

with:

\[
\begin{align*}
\alpha &= 2\alpha_+ \alpha_n, \\
\bar{\alpha} &= 2\alpha_+ \alpha_{m+1-n}, \\
b &= c = 2\alpha_+ \alpha_k.
\end{align*}
\]

At short distance one has:

\[
I_i (a, b, c; z) \to z^{-2\alpha_+ \alpha_n - (h_n + h_k - h_p)},
\]

\[
I_j (\bar{\eta}, b, c; z) \to z^{-2\alpha_+ \alpha_k - (h_\bar{\eta} + h_k - h_\bar{\eta})}.
\]  

(32)

with:

\[
\begin{align*}
p &= n - k + 2i - 1, \\
\bar{p} &= \bar{\eta} - k + 2j - 1.
\end{align*}
\]  

(33)
For the combination of conformal blocs \[^{[31]}\] to be consistent with the non-chiral fusion rules expressed by the development \(^{[30]}\) it is necessary that:

\[ p = m + 1 - p. \]

If one takes this result in the system \[^{[33]}\], along with the fact that \( \overline{p} = m + 1 - n \), one finds that:

\[ j = n + 1 - i. \]

In consequence the consistency of the non-chiral fusion rules and the combinations of conformal blocs at short distance at t-channel imposes that only the coefficients \( \gamma_{i,n+1-i} \) are non zero. Thus \[^{[3]}\]

\[
G_1 = f(z_i) \overline{f}(\overline{z}_i) \sum_i \gamma_{i,n+1-i}^{(D)} (a, b, c \mid \overline{a}, b, c) I^n_i (a, b, c; z) I^n_{n+1-i} (\overline{a}, b, c; \overline{z}). \tag{34}
\]

For more convenience, we adopt the following notation for the coupling constants:

\[
\gamma_{i,n+1-i}^{(D)} (a, b, c \mid \overline{a}, b, c) = \gamma_{i}^{(D)} (a, b, c) = \gamma_{n+1-i}^{(D)} (\overline{a}, b, c). \tag{35}
\]

The structure constants are obtained as limits at short distance of \(^{[34]}\)

\[
\left( C_{N,K}^P \right)^2 \propto \gamma_{i}^{(D)} (a, b, c) N_i (a, b, c) N_{n+1-i} (\overline{a}, b, c). \tag{36}
\]

**s-channel:**

In order to impose the bootstrap constraint we will develop \(^{[34]}\) in the s-channel. This is done by considering the duality transformation of the conformal blocs \(^{[24]}\). Thus, the s-channel correlation functions are of the form:

\[
G_1 = f(z_i) \overline{f}(\overline{z}_i) \sum_{i,l,l'} \gamma_{i}^{(D)} (a, b, c) \alpha_{il} (a, b, c) \alpha_{n+1-i,l'} (\overline{a}, b, c)
\]

\[
I^n_i (b, a, c; 1 - z) I^n_{l'} (b, \overline{a}, c; 1 - \overline{z}). \tag{37}
\]

This form of the correlations must be consistent at short distance with the non-chiral fusion rules which state that only scalar fields are present in the fusion of two fields of the

\[^{3}\]Here we note the net difference between our form of correlation function in the t-channel \(^{[34]}\) and the analogue (A.6) in the work \[^{[3]}\]. One of these differences is the absence of signs factor in our form. It is in order to lift this signs factor that the normalization of the two point correlation functions was redefined in \[^{[3]}\].
same nature. In other words only the diagonal terms are present in the correlations (37):

\[ G_1 = f(z_i) \bar{f}(\bar{z}_i) \sum_l \gamma_l^{(D)}(b, a, c) I^n_l(b, a, c; 1 - z) \bar{I}^n_l(b, \bar{a}, c; 1 - \bar{z}). \]  (38)

By comparing these two latter forms of s-channel correlations one can deduce that the coupling constants \( \gamma_i^{(D)}(a, b, c) \) are solutions of the algebraic equation:

\[ \sum_i \gamma_i^{(D)}(a, b, c) \alpha_{il}(a, b, c) \alpha_{n+1-i,l'}(\bar{a}, b, c) = \gamma_i^{(D)}(b, a, c) \delta_{ll'}. \]  (39)

The problem of determining the structure constants of the \((D_m)\) series models is reduced then to the resolution of the algebraic equation (39). For this goal, we consider the following analytic property of conformal blocs [5]:

\[ I^n_i(a, b, c; z) = z^{-2 \alpha_n \alpha_m -(h_n+h_m-h_p)} I^n_{n+1-i}(d, c, b; z). \]  (40)

This will permit in fact to deduce that:

\[ \alpha_{n+1-i,l'}(\bar{a}, b, c) = \alpha_{i,l'}(d, c, b). \]  (41)

with

\[ \bar{d} = 2 \alpha_+ (2 \alpha_0 - \alpha_{m+1-k}). \]  (42)

At this level we were inspired by Petkova’s work [5]; i.e. by using the fact that:

\[ \bar{d} - a = (2 - m) \in \mathbb{N}. \]  (42)

and that in these conditions:

\[ \alpha_{il}(\bar{d}, c, b) = \alpha_{il}(\bar{a}, b, c) = (-1)^{(\bar{d}-a)(l-1)} \alpha_{il}(a, b, c). \]  (43)

Now if we report these relations in (39) we arrive at the equation:

\[ \sum_i \gamma_i^{(D)}(a, b, c) \alpha_{il}(a, b, c) \alpha_{il'}(a, c, b) = (-1)^{(m-2)(l-1)} \gamma_i^{(D)}(b, a, c) \delta_{ll'}. \]  (44)

To find the solutions of this equation we consider its analogue of the \((A_m)\) series (28). By comparison we can derive the solutions of (44) under the form:

\[ \gamma_i^{D}(a, b, c; z) = \gamma_i^{A}(a, b, c; z), \]  (45)

\[ \gamma_i^{D}(b, a, c; z) = (-1)^{(m-2)(l-1)} \gamma_i^{A}(b, a, c; z). \]  (46)
Once the coupling constants are determined the structure constants of the \(D_m\) series models are given by:

\[
(C^P_{\mathbf{N}_K})^2 \propto \gamma_i^{(A)}(a, b, c)\mathcal{N}_i(a, b, c)\mathcal{N}_{n+1-i}(\overline{a}, b, c).
\]

To write this last result in more convenient form we use the fact that if \(|a'-a|\) is an integer then\[\int\]

\[
\gamma_i^{(A)}(a, b, c) = \gamma_i^{(A)}(a', b, c).
\]

and immediately we can show that:

\[
\gamma_i^{(A)}(a, b, c) = \gamma_{n+1-i}^{(A)}(d, c, b) = \gamma_{n+1-i}^{(A)}(\overline{a}, c, b) = \gamma_{n+1-i}^{(A)}(\overline{a}, b, c).
\]

Using this last result in (47) we find finally:

\[
(C^P_{\mathbf{N}_K})^2 \propto \left(\sqrt{\gamma_i^{(A)}(a, b, c)\mathcal{N}_i(a, b, c)}\right) \left(\sqrt{\gamma_{n+1-i}^{(A)}(\overline{a}, b, c)\mathcal{N}_{n+1-i}(\overline{a}, b, c)}\right),
\]

\[
= C^P_{\mathbf{N}_K} \cdot C^P_{\overline{\mathbf{N}_K}}.
\]

We see thus that the structure constants of the \(D_m\) series factorizes out in those of the chiral algebra expressed by the \(A_m\) series. Another important result can be deduced from (46) concerning the signs of the product of the structure constants namely:

\[
S \left(C^F_{\mathbf{N}_K}D^F_{KK}\right) = (-1)^{(m-2)(\frac{F-1}{2})}.
\]

Since the scalar fields constitute a subalgebra in the (OPA) we can chose the signs of the structure constants of this subalgebra arbitrarily. By opting for positive signs we can deduce that:

\[
S \left(C^F_{\mathbf{N}_N}\right) = (-1)^{(m-2)(\frac{F-1}{2})}
\]

This last result is obtained by the resolution of the bootstrap equations realized from duality symmetry of correlations of the form (21). To determine the signs of the structure constants between general couplings \(C^F_{\mathbf{N}_K}\) we must consider more general correlation forms.

\[\text{This can be readily deduced from}\]

\[\text{by using the properties}\]
(which generalize (43)) of the breading (monodromy) matrices determined by the connection established in Refs. [5, 7]. We find that the result (49) is general.

\[ S(C_{N \mathbf{K}}^F) = (-1)^{(m-2)(F-2)} \]

\[ (50) \]

5 Discrete symmetries of minimal models

We make appear a \( Z_2 \) discrete symmetry in the construction of the non-chiral fusion rules. This symmetry has appeared automatically in the automorphism invariant cases through the \( Z_2 - \text{grading} \) structure of the non-chiral fusion rules. In the integer invariant cases, the \( Z_2 \) symmetry was put into evidence in a different manner and this as a consequence of the existence of two copies of some scalar fields. The fusion rules obtained have also a \( Z_2 - \text{grading} \) structure.

Essentially, the \( Z_2 \) symmetry appear for the whole set of the \( D_m \) series as a consequence of the \( Z_2 - \text{grading} \) structure of the fusion rules. These structure expresses the conservation of a parity charge in these rules. In fact, the \( Z_2 \) permit to associate a positive parity to the scalar fields and a negative one to spin and null spin fields. This important fact suggests us to find a physical interpretation of the \( (Z_2) \) symmetry.

5.1 The \( ADE \) classification as lattice models

The fact that the universal critical properties are controlled by the long range fluctuations enable to treat them by a continuum field theory; conformal invariant at the critical point. The richness of the conformal symmetry in two dimensions makes it possible the classification of the universality classes. The \( ADE \) classification of the minimal models present a typical example of such classification.

In addition the universality principle of critical phenomena makes it conceivable to construct a statistical model of spin on lattice for all universality classes (conformal model).
Thus, one finds that the critical properties of the critical and the tricritical three states Potts models are given by the $D$ series with $m = 5$ and $m = 4$ respectively; the Ising model at the other hand is described by the $A$ series with $m = 3$.

Nowadays it is established that the whole set of unitary minimal models of the $ADE$ classification expresses the critical proprieties of the models said (RSOS) [8]. The formulation of these models (RSOS) is realized on the basis of the simple Lie algebra of type $ADE$ where at each site of the lattice is attributed a *weight* variable. These weights correspond to those of Coxeter-Dynkin diagram for a simple Lie $ADE$ algebra with the condition that two closest neighbors have neighboring weights in the Coxeter-Dynkin diagram.

None of these models has a continuum symmetry but on the contrary they have discrete symmetries. In this respect, we find as an example that the three state Potts models have a discrete symmetry $S_3$ which is a sum of cyclic discrete symmetries $Z_3$ and $Z_2$ and the Ising model has a $Z_2$ cyclic symmetry. For the whole set of RSOS models the discrete symmetries are nothing but the automorphism group of the Coxeter-Dynkin diagrams. In consequence the whole set of these models have a $Z_2$ symmetry except the $D_4, D_5, E_7$ and $E_8$ models. The diagrams of $D_5$ and $D_4$ have an $S_3$ symmetry and those of $E_7$ and $E_8$ have no symmetry.

The minimal models and their $ADE$ classification involve in their construction only the conformal symmetry and modular invariance (i.e. periodic) of partition functions. So it is of importance to know if this classification is consistent with the presence of other discrete symmetries. This is very important, indeed, because the critical properties have a strong dependence on symmetries and if the $ADE$ classification describes the critical behavior ( universality classes) of the RSOS models then it must be consistent with the presence of discrete symmetries of these lattice models. This is exactly what was done in a recent work [14] where the consistency of the $ADE$ classification with the presence of discrete cyclic ($Z_n$) symmetries is was investigated. The result found therein confirm that only symmetries which are present in the RSOS models are consistent with the $ADE$ classification. Another important result determined in Ref.[14] is the action of
these discrete symmetries on primary fields. For the particular cases of the $D$ series and $Z_2$ symmetry this action is exactly identical to that of our $Z_2$ symmetry, found in the construction of the non-chiral fusion rules. Instead the discrete symmetry $Z_2$ of the $D$ series RSOS models appear through the consistency of the non-chiral fusion rules with the action of this symmetry.

This important constatation leads us to think to construct for the three state Potts models $D_5$ ($D_4$) the non-chiral fusion rules consistent with the other symmetry of these models namely ($Z_3$) symmetry. This construction will be done for the $D_5$ critical three state Potts model and it will be available by symmetry to the $D_4$ tricritical three state Potts model.

5.2 $Z_3$ construction for $D_5$ model

The critical three states Potts model is a spin lattice model with discrete spin complex variable $\sigma = \exp (i\phi); \phi = 0, \pm \frac{2\pi}{3}$. The lattice Hamiltonian of this model is given by:

$$H = J \sum_{x,i} \frac{1}{2} (\sigma_x \sigma_{x+i}^* + \sigma_{x+i} \sigma_x^*) \eqno (51)$$

where $(x)$ denotes lattice spin position and $(i)$ the neighboring position.

At the scaling limit the discrete spin variables become continues operators $\sigma (x)$. It is natural also to identify the density energy operator $\varepsilon (x)$ from (51) as the scaling limit of the interaction term $\sigma_x \sigma_{x+i}^* + \sigma_{x+i} \sigma_x^*$. These two identified operator were known to have a scaling dimension equal to $\Delta_\sigma = \frac{2}{15}$ for the spin complex operator $\sigma (x) (\sigma^* (x))$ and $\Delta_\varepsilon = \frac{4}{5}$ for the energy density operator. The complex nature of the spin variable $\sigma (x)$ is one of the reasons that the critical three states Potts model is identified with the $D_5$ model rather than the diagonal $A_5$ model [13]. In fact, from the spin variable we can define two real spin variables: $\sigma + \sigma^*$ and $\sigma - \sigma^*$ which reflect the presence of two copies of the same scalar real primary field in the model. As a consequence and from the operator content of the $D_5$ we can identify $\sigma + \sigma^*$ and $\sigma - \sigma^*$ with the two copies $\Phi^\pm$ of
the doubled field \((2, 3 \mid 2, 3)\) and hence write the complex spin variable as:

\[
\sigma = \frac{1}{\sqrt{2}} (\Phi^+ + i \Phi^-)
\]

(52)

In addition the energy density field can be identified with the scalar field \((2, 1 \mid 2, 1)\).

The Hamiltonian (51) is invariant under the discrete cyclic symmetry \((Z_3)\) defined with its action on the spin variable as:

\[
Z_3 (\sigma (x)) = \exp \left( \frac{2 \pi i}{3} \right) \sigma (x)
\]

(53)

From the operator product expansion, one can show that the second doubled field \((2, 3 \mid 2, 3)\) which can be represented as a complex field \((\Omega)\) like \((52)\) transforms under \((Z_3)\) in the same way as \(\sigma (x)\) \((53)\). The other non doubled fields were invariant under \((Z_3)\) because each of them are conjugate to itself in the operator product expansion.

### 5.2.1 The \(Z_3\) non-chiral fusion rules

From the action of the \(Z_3\) symmetry we can easily deduce, as was done for the \(Z_2\) symmetry, the non-chiral fusion rules consistent with its action. This is what was considered in a second work \([13]\) by the author of Ref.\([9]\). For example one finds that:

\[
\begin{align*}
\Omega \times \Omega &= \Omega^* \\
\Omega \times \Omega^* &= 1 + (1, 5 \mid 1, 5) + (1, 5 \mid 11) + (1, 1 \mid 1, 5) \\
\Omega \times (1, 1 \mid 1, 5) &= \Omega \\
\Omega \times (2, 1 \mid 2, 1) &= \sigma + \Omega
\end{align*}
\]

(54)

In \([13]\) it was noted that the solutions obtained from the two constructions namely \((Z_2)\) and \((Z_3)\) are “..., of course, inequivalent.”. If the two constructions are really inequivalent one
can deduce that if (54) produce the (OPA) content of the critical three states Potts model then the \((Z_2)\) (OPA) structure describes another critical model, which is ambiguous. The \((Z_2)\) as \((Z_3)\) are both discrete symmetries of the three state Potts model so that the (OPA) structure obtained from these two symmetries must be equivalent. The error committed in [9] is that the good action (the \((A)\) case rather than the \((B)\) case) of the \((Z_2)\) symmetry was not considered. We propose now to establish the equivalence between the \((Z_3)\) and \((Z_2)\) structure of the (OPA). This is done easily by calculating the fusion for example of \(\Omega\) and \(\Omega^*\). From the rules (18) one finds that:

\[
\Omega \times \Omega^* = \frac{1}{2} (\Phi^+ + i\Phi^-) \times (\Phi^+ - i\Phi^-)
\]

\[
= \frac{1}{2} \left[ \Phi^+ \times \Phi^+ + \Phi^- \times \Phi^- + i \left( \Phi^+ \times \Phi^- - \Phi^- \times \Phi^+ \right) \right]
\]

\[
\Leftrightarrow \begin{bmatrix}
1 + C_{++} + C_{++(15|15)} + 1 + C_{--} + C_{--(15|15)} \\
+i \left( C_{+-} + C_{-(15|11)} + C_{+-(11|15)} - C_{--} \right) \\
-C_{-+} + C_{-(15|11)} - C_{-+ (11|15)}
\end{bmatrix}
\]

(55)

Using the fact that: \(C_{--} = -C_{++}\) and \(C_{++(15,15)} = C_{--(15,15)}\) deduced from the signs of the structure constants (50) and the fact that:

\[
C_{abc} = (-1)^{s(a)+s(b)+s(c)} C_{bac} = (-1)^{s(a)+s(b)+s(c)} C_{cba}
\]

we deduce that:

\[
\Omega \times \Omega^* = 1 + (1,5 | 1,5) + (1,5 | 1,1) + (1,1 | 1,5)
\]

In the same way we can proof that:

\[
\Omega \times \Omega = \Omega^*
\]  

(56)

What we have just proved through equation (56) is the equivalence of the two constructions of the non-chiral fusion rules based on the \(Z_2\) and \(Z_3\) symmetries. What we have exactly done is the following: If we consider the non-chiral fusion rules as a commutative and associative ring (in the same way as the fusion rules [15]) with as a basis the set of primary fields of the \(D_m\) model then by transformation (52) we have achieved a change
of basis. The $Z_2$ symmetry structure of the fusion rules in the real basis is manifested by
the complex $Z_3$ symmetry in the new complex basis (52).

The question that is of interest to answer at this stage is the very prediction of the
existence of the complex cyclic symmetry ($Z_N$) for the other models of the ($D_m$) series
as we have done for the $D_5$ ($D_4$) models of the ($Z_3$) symmetry. Indeed if such symmetry
exists it cannot be but a cyclic symmetry of order 3 ($Z_3$). This turns out to prove the
equivalent of the equation (56) with:

$$\Omega = \frac{1}{\sqrt{2}} (\Phi^+ + i\Phi^-)$$

$$\Phi^\pm = \left(1, \frac{m+1}{2} \mid 1, \frac{m+1}{2}\right)^\pm$$

In fact, the change to the complex basis cannot be consistent with the (OPA) i.e. $\Phi \times \Phi^* \sim$
1; but for the values of $m = 5 \text{ mod } 8$ ($m = 6 \text{ mod } 8$) and for this cases precisely and by
following the same approach as for ($D_5$) model we can show that it is not possible to have
a form consistent with (56). This meets the results found in Ref.[14].

6 General discussions

In this work we have proposed an approach to solve the bootstrap equations in the case
of the minimal and unitary models of $D_m$ series. This approach consists in the very
construction of the non-chiral fusion rules which determines the operator content of the
operator algebra. Once these fusion rules are determined it will be possible to solve the
bootstrap equations by considering the consistency of these equations at short distance
with these rules.

For the $D_m$ series models the non-chiral fusion rules found have a $Z_2 - grading$ struc-
ture. This later reflects, directly in the automorphism invariant cases, the existence of a
$Z_2$ symmetry. In the integer invariant cases the $Z_2 - grading$ structure was instead de-
duced as a consequence of the doubling of certain scalar fields and therefore the existence
of discrete parity symmetry $Z_2$. This symmetry has been interpreted as scaling limit of $Z_2$ symmetry of the $D$-like (RSOS) spin lattice models. In addition, beginning from the non-chiral fusion rules consistent with the $Z_2$ action and the signs of the structure constants, we succeeded in finding the non-chiral fusion rules consistent with $Z_3$ symmetry for the three state Potts models $D_5$ ($D_4$). Also we have proved that the existence of other discrete cyclic symmetries was not possible for the remaining of the $D_m$ models. Regarding the RSOS lattice construction this symmetries structure of the $D_m$ model is nothing but the automorphism group of the $D$ Coxeter-Dynkin diagrams [8].

A Further important interpretation of the manifestation of these discrete symmetries for minimal models is given in [18]. In fact, Goddard, Kent and Olive (GKO) [17] gave a “coset” construction to generate the unitary minimal models from unitary representation of $SU_2(K) \times SU_2(1)/SU_2(K+1)$. The $SU_2$ models are $Z_2$ invariant and so is the (GKO) coset construction of minimal unitary models. At another hand, it turns out that the three state Potts models can also be realized as coset construction of $SU_3(K) \times SU_3(1)/SU_3(K+1)$ which is $Z_3$ invariant. Thus these last models carry a $Z_3$ as well as $Z_2$ symmetry.

These remarks give to our approach a possibility to be generalized to other conformal models namely rational models. For these models and particularly in the case of Kac-Moody chiral algebras the non-chiral fusion rules in the $D$–like series may be structured following a discrete symmetry which is a center or a subalgebra of the center of the chiral algebra. For a coset construction $g/h$ the discrete symmetry is a subalgebra of the center of $g$ that preserves $h$. A convenient manner to formulate the problem is presented in another work of the present auctors [11]. This later is based, in the same way as in this work, on the consistency of the non-chiral fusion rules with the chiral fusion rules and with the operator content derived from the modular constraint. The discrete symmetry structure is introduced simply by the simple currents construction of $D$–like series. In this framework this symmetry is nothing but the effective center of the simple currents utilized in modular invariant construction [19].
Finally it is important to mention a possible connection of the discrete symmetry structure of the non-chiral fusion rules with a reflection group of what is known as graphs construction [7, 16]. These graphs are a generalization of the ADE Coxeter-Dynkin diagrams; so a possible integrable lattice interpretation and construction of the (OPA) and structure constants may be envisaged.

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