First Order Corrections to the Unruh Effect

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Abstract

First order corrections to the Unruh effect are calculated from a model of an accelerated particle detector of finite mass. We show that quantum smearing of the trajectory and large recoil essentially do not modify the Unruh effect. Nevertheless, we find corrections to the thermal distribution and to the Unruh temperature. In a certain limit, when the distribution at equilibrium remains exactly thermal, the corrected temperature is found to be $T = T_U (1 - T_U / M)$, where $T_U$ is the Unruh temperature. We estimate the consequent corrections to the Hawking temperature and the black hole entropy, and comment on the relationship to the problem of trans-planckian frequencies.

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1 Introduction

In many respects the Unruh effect and the Hawking effect are manifestations of the same phenomenon [1-3]. In both cases the accelerated detector (or asymptotic observer) observes a thermal spectrum of particles. The thermal radiation is associated with information (or entropy) which is hidden beyond an event horizon. In both cases the thermal radiation originates due to an exponentially increasing red shift between the rest frame (or asymptotic infinity) and the initial Minkowski vacuum (ingoing fields). In fact, the similarity between the two effect is much more than formal. This can be seen by examining the relationship of the two effects near a black hole. The existence of a Hawking flux of emitted radiation at infinity depends on a suitable boundary condition at the horizon. However a sufficient condition that Hawking radiation will be observed at infinity is that a stationary particle detector located at a constant \( r \) near the horizon will detect a thermal bath of radiation with the Unruh temperature \( T_U = \frac{a}{2\pi} \), where \( a \) is the detector’s proper acceleration. The converse is also true; if Hawking radiation is seen at infinity, this implies that the stationary detector must see the Unruh temperature near the horizon. Therefore, to some extent, the red shifted Hawking radiation near the horizon is a “hot” Unruh radiation.

The close relationship between the two effects suggests that a better understanding of the role of quantum effects in the case of an accelerated Unruh detector might shed light on the case of the black-hole. When the mass of the detector is taken to be finite, one can no longer ignore the quantum mechanical smearing of the trajectory and the recoil back-reaction when the detector is excited. In the case of the black hole, the first effect of quantum smearing might be analogous to the quantum smearing of the black-hole horizon, i.e. of the causal structure. The second effect, that of the detector’s recoil, might be related to the back-reaction of the black hole when a Hawking quanta is emitted. In this paper
we shall study these effects for the case of the Unruh detector and attempt to estimate the implications for the case of the black hole.

In his original paper [1], Unruh suggested a two-field model for a finite mass accelerated detector. Two scalar fields $\Phi_M$ and $\chi_{M'}$, of masses $M$ and $M' = M + \Omega$, respectively, were taken to represent two states of a detector of mass $M$ with two internal energy levels with an energy gap $\Omega$. By introducing a coupling of the form $\epsilon \phi \chi_{M'} \Phi_M$ with a field $\phi$, this detector can detect quanta associated with the field $\phi$. Recently, Parentani studied a similar two-field detector model [4]. Using the WKB approximation to describe the fields and a stationary phase approximation to calculate the transition amplitudes, Parentani showed that when the quantum smearing is smaller than the typical length $1/a$, the Unruh effect is unmodified.

In this work, the problem is approached by using another model. In Section 2 we present a model for a first quantized relativistic particle detector that is accelerated by a constant external electric field. The geometry of the detector’s trajectory is described by introducing future and past Rindler horizon operators [5]. We then compute in Section 3 the first order transition amplitude. What we find is that a large quantum smearing in detector’s trajectory and the (possibly) exponentially large recoil of the detector do not modify the Unruh effect. Nevertheless, the recoil back-reaction does induce corrections in the probability distribution at equilibrium and in the Unruh temperature. The origin of these corrections is that different energy levels of the detector experience different acceleration and hence “see” different temperatures. We calculate the first order correction to the thermal spectrum. Only in two limits – that of $\Omega/T_U << 1$ and $\Omega/T_U >> 1$, where $\Omega$ is the excitation energy – does the probability distribution remain exactly thermal. In the first limit, we once again obtain back the Unruh temperature. However in the second limit, of $\Omega >> T_U$, we find a correction to the Unruh temperature given by $T = T_U (1 - T_U/m)$,
where $m$ is the detector’s rest mass.

In Section 4 we study more qualitatively the nature of the final state of the detector + field system. Using the geometrical event horizon operators, the final state is represented as an entangled state of field and detector or horizon states. The back reaction can then be expressed as a shift in the location of the Rindler horizons. The location of the horizons with respect to the initial state of the detector is shifted with respect to the location of the horizons with respect to the final state. This shift can be exponentially large.

In the final section we attempt to apply our results to the case of the black hole. Since the acceleration in this case is determined by the black hole’s mass (and not by the detector’s mass as is in the case of the electric field), the correction is a genuine property of the black hole. The connection between the Unruh temperature near the horizon, and the Hawking temperature at infinity, is used to extrapolated from the Unruh temperature to a corrected Hawking temperature. The modification of the latter leads to a logarithmic correction to the black hole entropy. We also comment on the relation of our results to the problem of trans-planckian frequencies.

In the following we adopt the units in which $\hbar = k_B = c = G = 1$.

2 Accelerated Detector with Finite Mass

In this section we present a model for a particle detector of finite mass which takes into account also the quantum nature of the detector’s trajectory.

Consider a particle detector of rest mass $m_0$ and charge $q$ in a constant external electric field $E_x$ in $1 + 1$ dimensions. Let us describe the internal structure by a harmonic oscillator with a coordinate $\eta$ and frequency $\Omega$. The internal oscillator is coupled to a free scalar
field \( \phi \). The total effective action is

\[
S = -m_0 \int d\tau - qE_x \int X dt + \frac{1}{2} \int \left( \left( \frac{d\eta}{d\tau} \right)^2 - \Omega^2 \eta^2 \right) d\tau + \int g_0 \eta \phi(X(t(\tau)), t(\tau)) d\tau + S_F. \tag{1}
\]

Here, \( \tau \) is the proper time in the detector’s rest frame, \( X \) is the position of the detector, \( g_0 \) is the coupling strength with a scalar field \( \phi \) and \( S_F \) is the action of the field. Since we would like to describe the back reaction on the trajectory let us rewrite this action in terms of the inertial frame time \( t \). The action of the accelerated detector is then given by

\[
\int \left[ \left( -m_0 - g_0 \eta \phi(X, t) \right) \sqrt{1 - X^2 - qE_x X} \right] dt + \frac{1}{2} \int \left[ \frac{1}{\sqrt{1 - \dot{X}^2}} \left( \frac{d\eta}{dt} \right)^2 - \sqrt{1 - \dot{X}^2} \Omega^2 \eta^2 \right] dt. \tag{2}
\]

This yields a simple expression for the Hamiltonian of the total system with respect to the inertial frame:

\[
H = \sqrt{P^2 + M^2} - qE_x X + H_F, \tag{3}
\]

where the effective mass \( M \) is given by

\[
M = m_0 + \frac{1}{2} \left( \pi_\eta^2 + \Omega \eta^2 \right) + g_0 \eta \phi(X), \tag{4}
\]

and \( \pi_\eta = \frac{\partial L}{\partial \dot{\eta}} = \dot{\eta}/\sqrt{1 - \dot{X}^2} \). The validity of our model rest upon a the assumption that the Schwinger pair creation effect can be neglected for our detector. Since the the Schwinger pair creation process is damped by the factor \( \exp(-\pi M^2/qE_x) \) this implies the limitation \( M^2 > qE_x \). Notice that since the acceleration is \( a = qE_x/M \), this implies that \( M > a = 2\pi T_U \). In the following we set \( E_x = 1 \) for convenience.

To obtain a quantum mechanical model we simply need to impose quantization conditions on the conjugate pairs \( X, P \) and \( q, \pi_q \) and use the standard quantization procedure for the scalar field. It is convenient to introduce internal energy level raising and lowering operators \( A^\dagger \) and \( A \). The harmonic oscillator Hamiltonian can then be replaced by \( \Omega A^\dagger A \equiv \Omega N \) and the internal coordinate by \( \eta = i(A^\dagger - A)/\sqrt{2\Omega} \). This form can also be
used in other, more general cases, however the simple commutation relation $[A, A^\dagger] = 1$ in the case of a harmonic oscillator, needs to be modified accordingly.

So far we have not imposed a limitation on the coupling strength $g_0$. In the case of small coupling $g_0(t) = \epsilon(t)$ the Hamiltonian can be written to first order in $\epsilon(t)$ as

$$H = H_D - qX + H_F + H_I.$$  \hspace{1cm} (5)

Here

$$H_D = H_D(P, N) = \sqrt{P^2 + (m_0 + \Omega A^\dagger A)^2}$$  \hspace{1cm} (6)

is the free detector Hamiltonian, $H_F$ is the free field Hamiltonian

$$H_F = \frac{1}{2} \int dx' [\Pi_\phi^2 + (\nabla \phi)^2 + m^2_\phi \phi^2],$$  \hspace{1cm} (7)

and

$$H_I = i\epsilon(t)\left\{ \frac{m_N}{H_D}, (A^\dagger - A)\phi(X, t) \right\},$$  \hspace{1cm} (8)

where $m_N \equiv m_0 + N\Omega = m_0 + \Omega A^\dagger A$ and the anti-commutator, $\{A, B\} = \frac{1}{2}(AB + BA)$, maintains hermiticity. We have also absorbed a factor of $1/\sqrt{2\Omega}$ in the definition of $\epsilon(t)$. Comparing this interaction term with that used in the absence of a back-reaction we note that apart from the appearance of an anti-commutator there is also a new factor $\frac{m_N}{H_D}$. As we shall see, it corresponds to an operator boost factor from the inertial rest frame to the detector’s rest frame.

In the Hiesenberg representation the eqs. of motion for the detector’s coordinates $X$ and $P$ are given by:

$$\dot{X} = \frac{P}{H_D} - i\epsilon(t)\left\{ \frac{m_N P}{H_D^2}, (A^\dagger - A)\phi(X) \right\},$$  \hspace{1cm} (9)

$$\dot{P} = q - i\epsilon(t)\left\{ \frac{m_N}{H_D}, (A^\dagger - A)\phi'(X, t) \right\},$$  \hspace{1cm} (10)
where \( \phi' = \frac{\partial \phi}{\partial x} \). We also have

\[
\dot{A} = -i(H_{D,N+1} - H_{D,N})A - i[A, H_I],
\]

(11)

and

\[
(\Box - m_f^2)\phi(x,t) = i\epsilon(t)\left\{\frac{m_N}{H_D}, (A^\dagger - A)\delta(x - X)\right\}.
\]

(12)

In the zeroth order approximation \((\epsilon = 0)\) the solution of eqs. (9-11) is

\[
X^{(0)}(t) = X_0 + \frac{1}{q}[H_D(t) - H_D(t_0)], \quad P^{(0)}(t) = P_0 + q(t - t_0),
\]

(13)

\[
H_D^{(0)}(t) = \sqrt{(P_0 + q(t - t_0))^2 + (m_0 + \Omega A_0^\dagger A_0)^2},
\]

(14)

and

\[
A^{(0)}(t) = \exp\left[-i\int_{t_0}^{t}(H_{D,N_0+1} - H_{D,N_0})dt'\right]A_0.
\]

(15)

Here the subscript was used to denote the operator at time \( t = t_0 \) and the superscript to denote the zeroth order solution. To simplify notation we shall drop the superscript. Notice that \( N_0 = A_0^\dagger A_0 \) is a constant of motion in the zeroth order approximation.

It is now useful to introduce a proper time operator \( \tau(t) \):

\[
\tau = \int_{t_0}^{t} \frac{m_0 + \Omega A_0^\dagger A}{H_D} dt = \frac{m_0 + \Omega A_0^\dagger A}{q} \sinh^{-1}\left[\frac{q(t - t_0) + P_0}{m_0 + \Omega A_0^\dagger A}\right].
\]

(16)

We see that the factor \((m_0 + \Omega A_0^\dagger A)/H_D = m_N/H_D\) appearing in eq. (16) is the operator boost factor \(\frac{d\tau(t)}{dt} \), from the inertial frame to the detector’s rest frame. Notice that \( \tau \) depends only on \( P_0 \) and \( N \).

In terms of the proper time operator, the detector’s trajectory can be simplified to:

\[
t - t_0 - \tilde{T}_0 = \frac{1}{a} \sinh a\tau,
\]

(17)

\[
X - \tilde{X}_0 = \frac{1}{a} \cosh a\tau,
\]

(18)

6
where
\[
\tilde{T}_0 = -\frac{P_0}{q}, \quad \tilde{X}_0 = -\frac{H_D}{q},
\] (19)
and the acceleration \(a\) is given by the operator
\[
a = a_N = \frac{q}{m_0 + \Omega A^\dagger A} = \frac{q}{m_N}.
\] (20)

The operators \(\tilde{T}_0\) and \(\tilde{X}_0\) determine the location of the Rindler coordinate system of the detector with respect to the Minkowski coordinates \((t,x)\). The space-time location of the intersection point of the future and past Rindler horizons is given by \((-t_0 - \tilde{T}_0, -\tilde{X}_0)\).

Since
\[
[\tilde{X}_0, \tilde{T}_0] = \frac{i\hbar}{q},
\] (21)
the location of this space-time point becomes quantum mechanically smeared.

Another set of useful operators we shall introduce is that of the location of the future and past Rindler horizons \(\mathcal{H}_+\) and \(\mathcal{H}_-\), respectively. They can be found from the relations
\[
\mathcal{H}_+(t) = \lim_{t \to \infty} X(t), \quad \mathcal{H}_-(t) = \lim_{t \to -\infty} X(t).
\] (22)

We find
\[
\mathcal{H}_+(t) = -\tilde{T}_0 + \tilde{X}_0 + t - t_0 = \frac{P(t)}{q} - \frac{H_D}{q},
\] (23)
and
\[
\mathcal{H}_-(t) = \tilde{T}_0 + \tilde{X}_0 - (t - t_0) = -\frac{P(t)}{q} - \frac{H_D}{q}.
\] (24)

Therefore we can express \(X(t)\) as
\[
X(t) = \mathcal{H}_+(t) + \frac{1}{a} e^{-a\tau} \lim_{t \to \infty} \mathcal{H}_+(t),
\] (25)
and
\[
X(t) = \mathcal{H}_-(t) + \frac{1}{a} e^{a\tau} \lim_{t \to -\infty} \mathcal{H}_-(t).
\] (26)
In terms of $H_\pm$, the Hamiltonian of the detector in an external electric field has the simple form:

$$H_{acc} = H_D - qX = -\frac{q}{2}(H_+ + H_-).$$  \hspace{1cm} (27)

Finally, $H_\pm$ satisfy the commutation relation:

$$[H_-, H_+] = \frac{2i\hbar}{q} \hspace{1cm} (28)$$

Examining eqs. (21) and (28), we notice that since $q = am$, in the limit of constant acceleration but large mass, the commutators vanish as $m^{-1}$ and the classical trajectory limit is restored.

## 3 The Transition Amplitude

We shall now proceed to calculate the first order transition amplitude between the internal energy levels $n$ and $n + 1$ of the detector. To this end it will be most convenient to use the interaction representation. The operators in this representation are the solutions of the free equations of motion given by (15,16,17,18), and the wave function satisfies the Schrödinger equation

$$i\partial |\Psi\rangle = H_I |\Psi\rangle. \hspace{1cm} (29)$$

Given at $t = t_0$ by the initial wave function $|\Psi_0\rangle$, to first order in $\epsilon$ the final state at time $t$ is given by

$$|\Psi(t)\rangle = \left[ 1 - i \int_{t_0}^{t} \epsilon(t') \left\{ \frac{m_0 + \Omega A^\dagger A}{H_D}, i(A^\dagger - A)\phi(X,t') \right\} dt' \right] |\Psi(t_0)\rangle. \hspace{1cm} (30)$$

Let us set initial conditions for the internal oscillator to be in the $n$’th exited state $|n\rangle$, and for the scalar field to be in a Minkowski vacuum state $|0_M\rangle$. The initial state of the total system is therefore given by $|\Psi(t_0)\rangle = |0_M\rangle \otimes |n\rangle \otimes |\psi_D\rangle$, where $|\psi_D\rangle$ denotes the
space component of the detector’s wave function. Using the solution (15) for $A$ and $A^\dagger$, the transition amplitude can be expressed as:

$$|\Psi(t)\rangle = |\Psi(t_0)\rangle - \frac{\epsilon}{2} \int_{t_0}^{t} dt' \left[ \sqrt{n+1} |n+1\rangle \left( \frac{m_{n+1}}{H_{D,n+1}} e^{i \int_{t_0}^{t'} \Delta H_{n+1} dt''} \phi(X_n, t') + e^{i \int_{t_0}^{t'} \Delta H_{n+1} dt''} \phi(X_n, t') \frac{m_n}{H_{D,n}} \right) \
- \sqrt{n} |n-1\rangle \left( \frac{m_{n-1}}{H_{D,n-1}} e^{-i \int_{t_0}^{t'} \Delta H_n dt''} \phi(X_n, t') + e^{-i \int_{t_0}^{t'} \Delta H_n dt''} \phi(X_n, t') \frac{m_n}{H_{D,n}} \right) \right] \otimes |0_M\rangle \otimes |\psi_D\rangle. \tag{31}$$

Here we used the notation $\Delta H_n = H_{D,n} - H_{D,n-1}$. The subscript $n$ (e.g. in $X_n$), means that we need to substitute the free solutions with $N = n$. In two dimensions the solutions for a free massless scalar field can always be separated into right and left moving waves, i.e. $\phi = \phi_L(V) + \phi_R(U)$ where $U = t - x$, $V = t + X$. For simplicity we will limit the discussion to massless scalar fields and examine the solution only for right moving waves. Therefore, we substitute for $\phi$:

$$\phi_R(U) = \int \frac{d\omega}{\sqrt{4\pi\omega}} \left( e^{-i\omega U} a_\omega + e^{i\omega U} a_\omega^\dagger \right). \tag{32}$$

Using eqs. (17,18,23) we find that on the trajectory of the detector the light cone coordinate $U$ is given by

$$U|_D = t - X = -\mathcal{H}_{+0} - \frac{1}{a} e^{-a \tau}. \tag{33}$$

The final state can be written as:

$$|\Psi(t)\rangle = |\Psi(t_0)\rangle - \frac{\epsilon}{2} \int \frac{d\omega}{\sqrt{4\pi\omega}} \int_{t_0}^{t} dt' \left[ \sqrt{n+1} |n+1\rangle \left( \frac{m_{n+1}}{H_{D,n+1}} e^{i \int_{t_0}^{t'} \Delta H_{n+1} dt''} e^{i \omega(-\mathcal{H}_{+0} - \frac{1}{a} e^{-a \tau}) + e^{i \int_{t_0}^{t'} \Delta H_{n+1} dt''} e^{i \omega(-\mathcal{H}_{+0} - \frac{1}{a} e^{-a \tau})} \frac{m_n}{H_{D,n}} \right) \
- \sqrt{n} |n-1\rangle \left( \frac{m_{n-1}}{H_{D,n-1}} e^{-i \int_{t_0}^{t'} \Delta H_n dt''} e^{i \omega(-\mathcal{H}_{+0} - \frac{1}{a} e^{-a \tau})} + e^{-i \int_{t_0}^{t'} \Delta H_n dt''} e^{i \omega(-\mathcal{H}_{+0} - \frac{1}{a} e^{-a \tau})} \frac{m_n}{H_{D,n}} \right) \right]. \tag{34}$$
This is an exact result in the first order approximation in $\epsilon$. So far we have not introduced additional assumptions on $m_0$, $\Omega$ or $a_n = q/m_n$. We shall now apply a large mass limit. We shall assume that

$$m_0 >> a_0 = \frac{q}{m_0}.$$  \hspace{1cm} (35)

This restriction is indeed equivalent to a suppression of the Schwinger pair production process. Since the Unruh radiation has temperature $T_U = a/2\pi$ we shall need only energy gaps with $\Omega \sim a$. therefore we can also set

$$m_0 > n\Omega,$$  \hspace{1cm} (36)

where $n = O(1)$. Under these assumptions we can simplify the terms in (34). First consider the term $\exp(\int \Delta H_{n+1} dt)$. Using (36) we expand:

$$i \int_{t_0}^{t} \Delta H_{n+1} dt' = i\Omega \int_{t_0}^{t} \frac{m_n}{H_{D,n}} \left[ 1 + \frac{1}{2} \Omega \frac{P^2}{m_n H_{D,n}} \right] + O(\Omega^3/m^3) \hspace{1cm} (37)$$

$$\approx i\Omega \tau_n \left( 1 + \frac{1}{2} \frac{\Omega}{m_n} \frac{a_n}{q} \exp(-a_n \tau_n) \right) + c(P_0) + O(\Omega^3/m^3),$$

where $c(P_0)$ is a constant, and in the last line we have used the large $\tau$ approximation. This approximation is justified since the transition amplitude is dominated by contributions arising from integration over large $\tau$. In the following we shall hence neglect the exponential correction and the constant $c(P_0)$ which gives rise only to an overall phase, and use the approximation:

$$i \int_{t_0}^{t} \Delta H_{n+1} dt' = i\Omega \tau_n \left( 1 + \frac{1}{2} \frac{\Omega}{m_n} \right).$$  \hspace{1cm} (38)
Next consider the exponential terms in (34) which contain the horizon operator $H_+$. Only these terms maintain a dependence on the operator $X$ as $H_{+0} = X + G(P_0)$, where $G$ is a function of $P_0$. Using the Baker-Hansdorff identity we obtain:

$$
\exp[-i\omega(H_{+0} + \frac{1}{a}e^{-ar})] = \exp\left[\frac{i}{2q}\omega^2 e^{-ar} \frac{m_n}{H_{D,n}} + O\left(\frac{1}{q^2}\right)\right] \exp(-i\omega H_{+0} + i\frac{\Omega}{m_n} H_{D,n} + O(\Omega^2/m_n^2))
$$

(39)

The $O(q^{-2})$ corrections will be neglected in the following. Notice that since $[H_+, P_0] = i\hbar/q$, the unitary operator $e^{-i\omega H_{+0}}$ generates the translation: $p_0 \rightarrow p_0 + \omega$. In other words, this unitary operator generates the recoil which is required to conserve the total momentum when the detector is exited and a scalar Minkowski photon is emitted.

Finally, we consider the boost operator:

$$
\frac{m_{n+1}}{H_{D,n+1}} = \frac{m_n}{H_{D,n}} \left[1 + \frac{\Omega}{m_n} \left(1 - \frac{m_n^2}{H_{D,n}^2}\right) + O(\Omega^2/m_n^2)\right].
$$

(40)

Since for large $\tau$

$$
\frac{m_n}{H_{D,n}} = \frac{1}{\cosh(a_n\tau_n)} = 2e^{-a\tau_n} - O(2e^{-3a\tau_n}),
$$

(41)

we shall approximate this boost factor by

$$
\frac{m_{n+1}}{H_{D,n+1}} = \frac{m_n}{H_{D,n}} \left[1 + \frac{\Omega}{m_n}\right].
$$

(42)

We can now return to the transition amplitude (34) and for simplicity focus only on the amplitude $A(\omega, n + 1, p_0) = \langle 1_\omega, n + 1, p_0 | \Psi(t) \rangle$ using eqs. (38,39,42) we find

$$
A(\omega, n + 1, p_0) = -\frac{i\epsilon}{2 \sqrt{4\pi\omega}} \int_0^t dt' \left(\frac{m_n}{H_{D,n}(p_0 + \omega)} + \frac{m_n}{H_{D,n}(p_0)} \left(1 + \frac{\Omega}{m_n}\right)\right) \times
$$

$$
\exp\left[i\Omega(1 + \frac{1}{2m_n})\tau_n - i\omega \frac{1}{a}e^{-a\tau_n} + i\frac{\omega^2}{2q}e^{-a\tau_n} \frac{m_n}{H_{D,n}}\right] \phi_D(p_0 + \omega).
$$

(43)

Here, $\phi_D(p) = \langle p | \psi_D \rangle$. To obtain (38) we used a representation with $H_{+0}$ and $P_0$ as conjugate operators, and acted with the unitary operator $\exp(-i\omega H_{+0})$ to generate translations in the momentum. At this point the transition amplitude is expressed as a c-number integral.
Let us proceed to investigate this integral. For large \( t \) the phase \( \theta \) of the integrand can be approximated by
\[
\theta = \Omega \left( 1 + \frac{1}{2} \frac{\Omega}{m_n} \right) \tau_n - \omega \frac{1}{a} e^{-a_n \tau_n} + \frac{1}{q} \omega^2 e^{-2a_n \tau_n}.
\] (44)
The stationary phase condition yield
\[
\omega \simeq -\Omega \left( 1 - \frac{\Omega}{2m_n} \right) e^{a_n \tau_n}.
\] (45)
This can be compared with the case of a classical trajectory obtained by sending \( m \to \infty \).
In the present case, the frequency at the stationary point is shifted. However, with the assumption \( \frac{\Omega}{m_n} < 1 \), the correction is small and this frequency remains exponentially high.

Next notice that the recoil affects only one of the boost factor \( \frac{m_n}{H_{D, n}(p_0 + \omega)} \) in eq. (43), by a shift of the momentum. This has a simple physical interpretation. The transition amplitude is a superposition of two terms which correspond to two different “histories”. In one history, the detector is first boosted by \( \omega \) and only then it “absorbs” a scalar photon. In the second term, the detector first absorbs a photon and only afterwards it is boosted. Therefore in this term the boost factor is not affected by the recoil.

The shift of \( p_0 \to p_0 + \omega \) in the boost factor, is equivalent to a shift in time given by \( t \to t' = t + \frac{\omega}{e} \). In terms of the proper time (which is now a \( c \)-number) this correspond to the transformation
\[
\tau \to \tau' = \tau + \frac{\omega}{q} e^{-a \tau}.
\] (46)
For transitions with \( \tau(t) - \tau(t_0) \gg 1/a \), this transformation does not modify the integral. Hence in terms of \( \tau' \):
\[
\frac{m_n}{H_{D, n}(p_0 + \omega)} = \frac{d\tau'_n}{dt}.
\] (47)
The second, unshifted, boost factor can be expressed in terms of \( \tau' \) as
\[
\frac{m_n}{H_{D, n}} \left( 1 + \frac{\Omega}{m_n} \right) = \frac{d\tau'}{dt} \left( 1 + \frac{1}{m} (\Omega + \omega e^{-a \tau'}) + O(\Omega^2/m^2) \right).
\] (48)
Hence by expressing the integral \((43)\) in terms of \(\tau'\) we find that the two terms are equal up to order \(O(\Omega^2/m^2)\) and an additional piece that (up to this order) vanishes at the stationary point \((45)\).

Expressing the phase in terms of \(\tau'\) we find

\[
\theta = \Omega \left( 1 + \frac{1}{2} \frac{\Omega}{m_n} \right) \tau_n' - \frac{\omega}{a_n} \left( 1 + \frac{\Omega}{m_n} \right) e^{-a_n \tau_n'} + O(\Omega^2/m^2). \tag{49}
\]

where the term involving \(\frac{\omega^2}{q} e^{-2\alpha \tau}\) in eq. \((44)\) has dropped out and we are left only with the higher order corrections \(O(\Omega^2/m^2)\), which we will neglect.

In terms of \(\tau'\) the amplitude \(A(\omega, n + 1, p_0)\) can be written as:

\[
-i \epsilon \sqrt{\frac{n+1}{4\pi \omega}} \phi_D(p_0 + \omega) \left[ \int d\tau_n' \exp \left( i\Omega \left( 1 + \frac{1}{2} \frac{\Omega}{m_n} \right) \tau_n' - i\omega \frac{1}{a_n} \left( 1 + \frac{\Omega}{m_n} \right) e^{-a_n \tau_n'} \right) + \frac{\xi}{m_n} \right] \tag{50}
\]

where

\[
\xi = \frac{1}{2} \int d\tau (\Omega + \omega e^{-\alpha \tau}) \exp \left( i\Omega \tau - i\omega \left( 1 + \frac{\Omega}{m} \right) e^{-\alpha \tau} \right) \tag{51}
\]

For large \(\tau\), \(\xi \sim O(\Omega/m)\), and the term \(\xi/m\) can be neglected.

Finally we obtain

\[
A(\omega, n + 1, p_0) = i \epsilon \sqrt{\frac{n+1}{4\pi \omega}} \phi_D(p_0 + \omega) a_n^{-1} \left( \frac{\omega}{a_n} \right) \left( 1 + \frac{\Omega}{m_n} \right) \frac{\Omega'}{a_n} \Gamma(-i\frac{\Omega'}{a_n} e^{-\alpha \tau}) + O(\Omega^2/m^2), \tag{52}
\]

where \(\Gamma\) is the Gamma function, and

\[
\Omega' = \Omega \left( 1 + \frac{1}{2} \frac{\Omega}{m_n} \right). \tag{53}
\]

Comparing this amplitude to that obtained in the case of a fixed classical trajectory, we notice that it appears to be modified only by a pure phase factor and by the shift \(\Omega \rightarrow \Omega'\).

To first order, the transition probability is therefore given by

\[
P(\omega, n \rightarrow n + 1, p_0) = \frac{\epsilon^2 (n+1)}{4\pi \omega} |\phi_D(p_0 + \omega)|^2 \left( \frac{2\pi}{a_n \Omega'} \right) e^{-\pi \Omega'/a_n} e^{-\pi \Omega'/a_n} \tag{54}
\]
This expression has the same form as of a thermal transition probability with a shifted Unruh temperature \( T'_U \):

\[
T'_U(n) = \frac{q}{m_0 + n\Omega'} \simeq T_U \left( 1 - \frac{n\Omega^2}{m_n^2} \right). \tag{55}
\]

Nevertheless, this transition probability does not imply a thermal distribution at equilibrium. Notice that the “transition” temperature \( (55) \) depends on the energy level \( n \). Indeed, since the acceleration depends on \( n \), each energy state of the detector “sees” a slightly different temperature. The temperature gradient between two neighboring levels is given by \( \Delta T_n/T_n = \Omega/m_n \).

In order to find the distribution at equilibrium we need to compare the probability of excitation, \( P(\omega, n+1 \to n, p_0) \), to the transition probability for de-excitation, \( P(\omega, n+1 \to n, p_0) \). By examining eq. \( (34) \) we find that up to corrections of \( O(\Omega^2/m^2) \), the de-excitation probability \( P(\omega, n + 1 \to n, p_0) \) is obtained by the substitution \( a_n \to a_{n+1} \) and \( \Omega' \to -\Omega' \) in eq. \( (34) \). Using the approximation \( a_{n+1} = a_n(1 - \Omega/m_n) \) we find that the probability distribution at equilibrium satisfies, up to a correction of \( O(\Omega^2/m^2) \), the relation

\[
P(n \to n+1) = \left( 1 + \frac{\pi\Omega\Omega'}{a_n m_n} \coth(\pi\Omega'/a_n) \right) \exp \left( -\frac{2\pi\Omega'}{a_n} \left( 1 + \frac{\Omega}{2m_n} + \frac{a_n}{2\pi m_n} \right) \right) P(n+1 \to n), \tag{56}
\]

which is satisfied for every \( \omega \) and \( p_0 \).

This probability distribution can be simplified in two limiting cases. For \( \Omega << T_U \) we get back the ordinary thermal relation

\[
P(n \to n+1) = \exp \left( -\frac{\Omega}{T_U} \right) P(n+1 \to n), \tag{57}
\]

where \( T_U = a_n/2\pi \). This should have been anticipated. In this limit, the temperature gradient \( \Delta T/T \) between nearby energy levels, vanishes.
The more interesting limit is obtained for \( \Omega > T_U \). In this limit we obtain back an exact thermal distribution:

\[
P(n \rightarrow n + 1) = \exp\left(\frac{-\Omega}{T_{acc}}\right)P(n + 1 \rightarrow n).
\]

However the Unruh temperature receives a correction:

\[
T_{acc} = \frac{a_n}{2\pi} \left(1 - \frac{a_n}{2\pi m_n}\right) = T_U \left(1 - \frac{T_U}{m}\right)
\]

By repeating the stages of this calculation it can be verified that the same correction to the probability distribution and to the temperature is also obtained from the transition amplitude involving left moving photons, i.e. from the interaction with the part \( \phi_L(V) \) of the scalar field. Therefore it seems that in this limit eq. (59) constitutes a genuine first order correction to the Unruh temperature. Since in higher levels the effective acceleration is smaller, this correction indeed acts to reduce the Unruh temperature.

It is interesting to notice that even in the limit of \( \Omega/m << 1 \), when the Unruh temperature is restored, the recoil back-reaction can be still large. The recoil shifts the momentum by \( \omega \), which by eq. (45) can be exponentially large even for small \( \Omega \). A further discussion of this recoil back-reaction and of the quantum smearing effect is given in the next section.

4 Recoil and Quantum Smearing

In this section we examine the back-reaction effect on the trajectory of the detector. Let us re-state the results of the last section in a more qualitative way. For the case of a classical trajectory, it was shown by Unruh and Wald \[8\] that if the detector is initially in the ground state then the final state can be written as

\[
|\Psi(t)\rangle = |\Psi_0\rangle - i |n = 1\rangle \otimes a_R |0_M\rangle.
\]
Here, $a_{R\Omega}$ is the annihilation operator of a quantum with frequency $\Omega$ with respect to the Rindler coordinate system that is defined by the detector’s trajectory. Using the well known relation (1) of $a_{R\Omega}$ to Minkowski creation and annihilation operators $a_M$ and $a_M^\dagger$, they get

$$|\Psi(t)\rangle = |\Psi(0)\rangle - iC(\Omega, a)|n = 1\rangle \frac{e^{-\pi\Omega/a}}{(e^{\pi\Omega/a} - e^{-\pi\Omega/a})^{1/2}} a_M^\dagger|0_M\rangle,$$

(61)

where $C$ is a normalization factor. Note that $a_M^\dagger$ creates a positive frequency Minkowskian photon, which is not in a state of definite frequency $\omega$. Qualitatively we can use the stationary phase approximation eq. (44) to relate the typical frequency of this photon to the time of emission $\tau$.

We can now use the result obtained in the last section to replace eq. (61) with

$$|\Psi(t)\rangle = |n = 0, \psi_D, 0_M\rangle$$

(62)

$$-iC(\Omega, a_n)|n = 1\rangle \frac{e^{-2\pi\Omega/a}}{(e^{\pi\Omega/a} - e^{-\pi\Omega/a})^{1/2}} \left(e^{-iH_F\mathcal{H}_+} a_{MR}^\dagger + e^{+iH_F\mathcal{H}_-} a_{ML}^\dagger\right)|0_M, \psi_D\rangle$$

Here we ignored the effect of temperature gradient between different energy levels, which gave rise to the correction found in the previous section. We have also restored the full coupling with the left and right moving waves. The operators $a_{MR}^\dagger$ and $a_{ML}^\dagger$, correspond to creation operators of right and left moving waves respectively. This equation can be easily generalized to the case of transitions between any two levels $n$ to $n + 1$, as well as to the case of de-excitations. We have assumed that the scalar field is massless. However, for a massive field we simply need to replace $e^{-iH_F\mathcal{H}_+}$ by $e^{iH_F\tilde{T}_0 - iP_J\tilde{X}_0}$ etc.

The new feature of eq. (62) is the insertion of the horizon shift operators $\exp(\pm iH_F\mathcal{H}_{\pm})$ which act on the wave function of the detector and of the scalar field. These shift operators generate correlations between the “emitted” Minkowski scalar photon and the trajectory of the detector.
To illustrate these correlations, let us concentrate only on the left moving waves and express $a^\dagger_{MR}$ in terms of creation operators of definite Minkowski frequency:

$$a^\dagger_{MR} = \int f(\omega)a^\dagger_\omega d\omega$$

Eq. (63) can now be written as

$$|\delta \Psi\rangle = -iC'|n = 1\rangle \int d\omega dh_+ e^{-i\omega h_+} f(\omega)\psi(h_+)|1_\omega\rangle \otimes |h_+\rangle.$$  

(64)

Here we used a basis of $\mathcal{H}_{0+}$: $\mathcal{H}_+|h_+\rangle = h_+ |h_+\rangle$. We see that the recoil interaction generates correlation between the shift $h_+$ in the $u$-time of the right moving “emitted” Minkowski photons with the “horizons states” $|h_+\rangle$ of the detector. Therefore, the effect of “smearing the horizon” yields after emission the final entangled state (64). In each component of this state, the Unruh effect is manifested, with the correction discussed in the previous section. Since the corrections do not depend on the uncertainty or the smearing $\Delta h_+$ of the future event horizon, the overall wave function still manifests the Unruh effect.

In order to examine the effect of the emission on the detector we can re-write eq. (64) by using as a basis the past horizon operator $\mathcal{H}_-$. We obtain:

$$|\delta \Psi\rangle = -iC'|n + 1\rangle \int d\omega dh_- f(\omega)\psi(h_-)|1_\omega\rangle \otimes |h_- - \omega\rangle,$$

(65)

where $\psi(h_-) = \langle h_- | \Psi_D \rangle$. Since $\mathcal{H}_\pm$ are conjugate operators, the operator $\exp -i\omega \mathcal{H}_+$ has shifted the past horizon operator by $\omega$. It is interesting to notice that the shift by $\omega$ of the past horizon can be exponentially large. In fact, from the stationary phase approximation we get that it is related to the time of emission $\tau$ as: $\Omega \simeq \Omega \exp(a\tau)$. Therefore a detection of a particle of energy $\Omega$ generates an exponential shift in the location of the past horizon of the detector:

$$\delta h_- = h_{-out} - h_{-in} \simeq \Omega \exp(a\tau)$$
The meaning of this shift is as follows. We can use the initial state $\psi_{in}$ to define the location $h_{-in}$ of the past horizon. We can also use the final state $\psi_f$ of the detector and by propagating it to the past (with the free Hamiltonian) determine the location $h_{-out}$. These two locations differ by an exponential shift.

The use of propagation of a wave function to the past might seem strange. However the same phenomenon occurs if the detector is excited in the past at $\tau < 0$. In this case it emits a left moving Minkowski particle. We find that this induces an exponentially large shift in the location of the future event horizon operator $H_+$:

$$\delta h_+ = h_{+out} - h_{-in} \simeq \Omega \exp(-a\tau)$$

The manifestation of the back reaction as an exponentially large shift is related to the method of 't Hooft [6] and of Schoutens, Verlinde and Verlinde [7]. In their case, infalling matter into the black hole, induces an exponential shift of the time of emission of the Hawking photon in the future. The reason is that the Hawking photons stick so close to the horizon that even a small shift of the horizon still modifies the time of emission. In our case this exponential shift is related to the exponential energy of the emitted Minkowski photon. In both cases, the back reaction requires the existence of exponentially high frequencies in the vacuum. As in the case of Hawking radiation, a naive cutoff eliminates the thermal spectrum seen by the Unruh detector.

5 Correction to Black Hole Radiation

As noted in the introduction, the Unruh effect and the Hawking effect are very closely related. It is therefore conceivable that the same type of corrections are relevant in both cases. Let us recall how the Unruh effect is manifested in the case of the black hole. A
fiducial particle detector (i.e. stationary at constant angles and Schwarzschild radius $r$), in
the gravitational field of a black hole will in general observe radiation. Only in two limiting
cases, that of $r$ going to infinity and of $r$ close to the horizon, does this radiation take a
simple form. In the first case, the detector observes at spatial infinity Hawking radiation
with temperature $T_H = 1/(8\pi M)$, where $M$ is the mass of the black hole. In the second
case, the detector will see the Unruh radiation with temperature

$$T_U(r) = \frac{a(r)}{2\pi} \approx \frac{1}{8\pi M} \frac{1}{\sqrt{g_{00}(r)}} = \frac{T_H}{\sqrt{g_{00}(r)}}, \quad (68)$$

where $a(r)$ is the proper acceleration at a constant radius $r$. This equation relates the Unruh
temperature seen very near to the horizon and the Hawking temperature at $r \gg 2M$.

The origin of the Unruh radiation, i.e. the proportionality of the temperature to the
acceleration near the horizon, can be seen as follows. The Schwarzschild metric

$$ds^2 = (1 - \frac{2M}{r}) dt^2 - (1 - \frac{2M}{r})^{-1} dr^2 - r^2 d\Omega^2 \quad (69)$$

can be approximated near the horizon by the metric

$$ds^2 = \left(\frac{\rho}{4M}\right)^2 dt^2 - d\rho^2 - r^2(\rho)d\Omega^2, \quad (70)$$

where

$$\rho \equiv 4M\left(1 - \frac{2M}{r}\right)^{1/2} \approx \int_{2M}^{r} \frac{dr}{\sqrt{1 - \frac{2M}{r}}} \quad (71)$$

is approximately the proper distance of the point $r$ to the horizon. Therefore near the
horizon, the reduced 2-dimensional Schwarzschild metric with coordinates $t - \rho$ can be ap-
proximated by a Rindler metric. World lines with constant $\rho(r)$ correspond to trajectories
of constant proper acceleration $a = 1/\rho$.

Consider now a particle detector of mass $m$ which is held by an external force at a fixed
radius $r$. What is the nature of the corrections to the Unruh effect in this case? First we
notice that contrary to the case studied in the previous sections, the acceleration (68) in
this case is a function of the black hole mass, and not of the mass of the detector. Since
we have seen that the corrections originate from modifications to the acceleration, we can
expect that in the present case they are determined by the back reaction on the black hole.

When an Unruh particle is detected, the rest frame mass of the detector is modified
from \( m \) to \( m + \delta m \). If the total mass at infinity is unchanged by this process, the excitation
must be accompanied by a decrease of the back hole’s mass by

\[
\delta M = -\delta m \sqrt{g_{00}(r_D)}.
\]  

(72)

In other words, an excitation of the detector corresponds to an emission by the black hole.
The emitted photon is absorbed by the detector and therefore the total black-hole and
detector mass, remains unchanged as seen from infinity. Nevertheless, due to the back
reaction effect (72), the proper distance \( \rho \) between the horizon and the detector, or the
acceleration \( 1/\rho \) of the detector have been changed.

We have seen in Section 3. that the correction to the Unruh temperature arises from
the decrease of the acceleration due to an absorption. The correction (59) was in fact due
to the ratio \( a_{n+1}/a_n = (1 + \delta a_n/a_n) \). In the present case the decrease by \( \delta M \) in the mass
of the black hole causes to a decrease in the proper acceleration. Using eqs. (68) and (72)
we obtain

\[
\delta a(r) = -\frac{\delta M}{4M \sqrt{g_{00}(r)}} \left( \frac{1}{M} - \frac{1}{r g_{00}(r)} \right),
\]  

(73)

\[
\approx \frac{\delta M}{8M^2 g_{00}^{3/2}} = -\frac{\delta m}{8M^2 g_{00}}.
\]

Therefore,

\[
\frac{a_{M+\delta M}}{a_M} \simeq 1 - 4\pi \delta m T_U(r) \simeq \exp \left(-4\pi \delta m T_U(r) \right).
\]  

(74)
The Boltzmann factor is hence shifted to
\[ \exp \left( - \frac{\delta m}{T_U(r)} \left( 1 + 4\pi T_U^2(r) \right) \right). \tag{75} \]
This corresponds to a correction of the Unruh temperature:
\[ T_{acc}(r) = T_U(r) \left( 1 - 4\pi T_U^2(r) \right). \tag{76} \]
Indeed, since the back reaction effect is larger when the detector is closer to the horizon the correction increases accordingly. We note that the correction (76) tends to decrease the temperature. This effect becomes large only as the detector is lowered to a distance of \( \rho \sim l_{pl} \). However, at this point we can no longer trust our model since in this limit the detector’s mass needs to be of order of \( T_U(r) \).

What does eq. (76) imply for an observer at infinity? When the back reaction is ignored, the Unruh temperature near the horizon, is seen at infinity as red shifted to the Hawking temperature:
\[ \sqrt{g_{00}} T_U(r) = T_H. \tag{77} \]
In our case however, we can not simply multiply eq. (76) by the red shift factor in order to obtain the correction to the Hawking temperature. The reason is that the source of this correction is the modification in \( g_{00}(r_D) \) which is cause by the back reaction effect. Therefore, by using the classical metric we will obtain the answer that the corrected Hawking temperature depends on the location of the detector, which is of course incorrect.

Since we still expect only a small correction to the transformation (77), it seems suggestive to extrapolate (76) by the substitution \( T_U(r) \rightarrow T_H \). Therefore,
\[ T_{BH} = T_H \left( 1 - 4\pi T_H^2 \right). \tag{78} \]
The correction to the Hawking temperature is very small for \( M >> 1 \). Although (78) was obtained only for Hawking particles of energy \( E > T_H \), it can still be a good estimate for
the modification for the complete spectrum. This allows us to obtain, using the first law of thermodynamics, the corresponding modification in the black hole entropy. Up to an additive constant we obtain:

\[ S_{BH} = 4\pi M^2 + \frac{1}{2} \ln M. \]  

(79)

We shall conclude with a few remarks. One of the main motivations for studying this problem of a quantum Unruh detector was the hope that the quantum smearing or the back reaction effects would render the problem of exponentially high trans-planckian frequency manageable. For small accelerations, it seems that nevertheless these new effects this problem is still unavoidable in our model. Only for a large accelerations or temperature of the order of the detectors mass does the back reaction have a significant effect. At this limit however, the validity of our model breaks down. It is interesting however, that if we naively extrapolate the corrections to the regime of large acceleration, i.e., lower the detector nearer to the horizon, the Unruh temperature decreases toward zero instead of diverging to infinity. Such an effect could indicates an effective cutoff of high frequencies near the horizon which does not eliminate the Hawking effect.

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