Particles with negative and zero energies in black holes and cosmological models

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Abstract Particles with negative energies are considered for three different cases: inside of horizon of non-rotating Schwarzschild black hole, Milne’s coordinates in flat Minkowski space-time (Milne’s universe using nonsynchronous coordinates) and in cosmological Gödel model of the rotating universe. It is shown that differently from the Gödel model with nondiagonal term where it occurs that negative energies are impossible they are present in all other cases considered in the paper. Particles with zero energy are also possible in first two cases.

Keywords Negative energy · Zero energy · Black hole · Milne universe · Gödel universe

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1 Introduction

The possibility of relativistic particles with negative energy is important because it makes possible to get large energy in interactions or decays of bodies. The simple example of this situation was proposed by R. Penrose

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in the case of the decays of particles in ergosphere of rotating black holes [1,2].

It looks that in order to have negative energy of the relativistic particle with nonzero mass one must have very strong external field leading to large potential energy as it is in the case for rotating black holes. However it is well known that the value of the energy depends on the choice of the reference frame and the time coordinate or Killing vector in case of conserved energy. It leads to the situation that states with negative energies in relativistic case occur in case of rotating coordinate system out of the static limit [3], where the effect analogous to Penrose effect turns out to be observable [4], and in nonsynchronous coordinate system in cosmology [5,6].

It seems from these examples that negative energies arise in case of the existence of nondiagonal terms in metrical tensor (gravymagnetism) but in this paper we show that in Gödel universe in spite of the presence of such terms negative energies are absent. The negative energies are present in cases of the movement inside of the horizon of the Schwarzschild black hole and in Milne’s universe where nondiagonal terms are present in nonsynchronous coordinate frame.

2 Negative energy in nonrotating black hole

Nonrotating black hole of mass $M$ in Schwarzschild coordinates is described by metric

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2), (1)$$

where $r_g = 2GM/c^2$ is the gravitational radius of the black hole, $G$ is gravitational constant, $c$ is the light velocity. Geodesic complete space-time of the nonrotating black hole one can be described in Kruskal–Szekeres
coordinates, \( \{u, v\} \in (-\infty, +\infty) \), which in region \( u > 0 \) are connected with Schwarzschild coordinate in \( r > r_g \) in the following way:

\[
\begin{align*}
u &= \sqrt{\frac{1-\frac{r}{r_g}}{1-\frac{r}{2r_g}}} \exp \frac{r}{2r_g} \sinh \frac{ct}{2r_g}, \\
u &= \sqrt{\frac{1-\frac{r}{r_g}}{1-\frac{r}{2r_g}}} \exp \frac{r}{2r_g} \cosh \frac{ct}{2r_g}.
\end{align*}
\]

For \( r < r_g \) and \( v > |u| \) the transformation from Schwarzschild coordinate in the Kruskal–Szekeres coordinates has the form

\[
\begin{align*}
u &= \sqrt{\frac{1-\frac{r}{r_g}}{1-\frac{r}{2r_g}}} \exp \frac{r}{2r_g} \sinh \frac{ct}{2r_g}, \\
u &= \sqrt{\frac{1-\frac{r}{r_g}}{1-\frac{r}{2r_g}}} \exp \frac{r}{2r_g} \cosh \frac{ct}{2r_g}.
\end{align*}
\]

Schwarzschild coordinates are singular at \( r = r_g \). Their connection with Kruskal–Szekeres coordinates for other \( u, v \), see in [7], Sec. 31.5.

Any possible movement of physical bodies and particles must satisfy the condition \( ds^2 > 0 \) leading to

\[
\begin{align*}
&\text{for } r > r_g \Rightarrow \left| \frac{dr}{d\lambda} \right| \leq c \left( 1 - \frac{r_g}{r} \right), \\
&\text{for } r < r_g \Rightarrow \left| \frac{dr}{d\lambda} \right| \geq c \left( \frac{r_g}{r} - 1 \right),
\end{align*}
\]

For \( r = r_g \) the coordinate \( c t \) is space and \( r/c \) — time coordinate.

Geodesic equations in Schwarzschild coordinates in the plane \( \theta = 0 \) are [8]

\[
\begin{align*}
\frac{dt}{d\lambda} &= \frac{r}{r-r_g} \frac{E}{c^2}, \\
\frac{d\phi}{d\lambda} &= \frac{J}{r^2}, \\
\left( \frac{dr}{d\lambda} \right)^2 &= \frac{E^2}{c^2} + \frac{r_g-r}{r} J^2 + \frac{r_g-r}{r} m^2 c^2,
\end{align*}
\]

where \( E \) is the energy of the moving particle, \( J \) is the conserved projection of the particle angular momentum on the axis orthogonal to the plane of motion, \( m \) is the particle mass, \( \lambda \) is affine parameter on geodesic. For massive particle \( \lambda = \tau/m \), where \( \tau \) is the proper time.

In external region of the black hole \( (r > r_g) \) for any physical object the time coordinate \( t \) is always increasing and so the energy \( E \) of the particle is positive (see [9]). Inside the horizon of the black \( (r < r_g) \), where \( t \) is space like \( (g_{tt} < 0) \) one has movement as in increasing as in decreasing \( t \). As it is seen from the first formula in [8] for a particle moving inside the horizon in the direction of decreasing of the coordinate \( t \) the energy \( E \) of the particle will be positive while for increasing coordinate \( t \) the energy \( E \) is negative. For constant \( t \) inside the black hole \( E = 0 \) due to formula [8].

Surely \( t \) inside of the black hole is space like and \( E \) is proportional to the \( t \)-component of the momentum.

Inside black hole one can use other reference frames and corresponding energies [9]. But for the observer outside of the black hole the conserved \( E \) along all trajectories of the free fall is equal to [10]

\[
E = mc^2 \sqrt{\frac{(1-r_g/r)}{1-v^2/c^2}},
\]

where \( v \) is the velocity measured by the observer at rest in the Schwarzschild coordinates. So we can call \( E \) inside the black hole following [7], as “energy at infinity”.

On Fig. 1 the trajectories for radial movement with positive, zero and negative energies in Kruskal–Szekeres coordinates are represented by red, green and blue lines. As one can see from [2], [3] the coordinate lines of constant \( t \) in Kruskal–Szekeres coordinates are straight lines through the origin of coordinates. In region II coordinates \( t \) decreases for moving from \( H^+ \) to \( F^+ \) (positive \( E \)) and increases for moving from \( H^- \) to \( F^- \) (negative value of \( E \)). Direct lines \((M^0O^k)\) correspond to constant \( t \) \( = \pm r_g/c \) and therefore \( E = 0 \).

Let us consider the problem of back influence of falling particles on metric of the black hole space-time. For macroscopic bodies with 4-velocity \( (u^i) \), with the energy density \( \varepsilon \) and pressure \( p \) in space-time with metric \( g_{ik} \) the energy-momentum density tensor is [10]

\[
T_{ik} = (\varepsilon + p) u_i u_k - pg_{ik},
\]

\( i, k = 0, 1, 2, 3 \). The trace of the energy-momentum tensor

\[
T^0_i = \varepsilon - 3p \tag{9}
\]

is invariant and it will be negative for \( \varepsilon - 3p < 0 \), in particular, for dust like matter \( (p = 0) \) with negative energy \( \varepsilon < 0 \). The back influence of falling particles with negative energy will be determined by the such energy-momentum tensor in the right hand side of Einstein equations. Notion of the existence of particles with negative energies as it is known was used by S. Hawking to predict Hawking effect for black holes [11].

3 Negative and zero energies in flat space-time

The geodesic lines equations can be obtained for space-time with metric \( g_{ik} \) from the Lagrangian

\[
L = \frac{g_{ik} dx^i dx^k}{2 \frac{d\lambda}{d\lambda}},
\]

where \( \lambda \) is the affine parameter on the geodesic [8]. The energy of the particle \( E \) is equal to the zero covariant component of the momentum \( (p_i) \) multiplied on the light velocity

\[
p_i = \frac{\partial L}{\partial \left( \frac{dx_i}{d\lambda} \right)} = g_{ik} \frac{dx^k}{d\lambda},
\]

\( i = 0 \).
Defining the affine parameter for the massive particle one obtains

\[ E = cp_0 = cg_\alpha \frac{dx^\alpha}{d\tau}. \quad (12) \]

Here we suppose generally that the mass can be negative.

Using notation \((\zeta') = (1,0,0,0)\) for the translation in time coordinate generator one can write \((12)\) for the energy of the particle as

\[ E = c(p, \zeta). \quad (15) \]

If the metric components don’t depend on the time coordinate \(x^0\), then \(\zeta\) is the time like Killing vector and the energy \(E\) is conserved on the geodesics. For time like vector \(\zeta\) and massive particle one has \((12)\)

\[ \sqrt{\langle \zeta, \zeta \rangle} \leq \frac{E}{|m|c^2} < +\infty \quad (16) \]

and the energy \((15)\) is positive. For space like vector \(\zeta\), as it take place in the ergosphere of rotating black hole, the arbitrary positive and negative values are possible (see \((12)\), p. 325).

Note that in spite of the invariance of the scalar product \((12)\) the value \((14)\) of the energy depends on the choice of the reference frame. This occurs due to the fact that by changing the reference frame in which the physical measurements are made the observer is changing vector \(\zeta\). The analysis of the situation in rotating coordinate system in flat space-time is made in \([3]\).

In Minkowski space-time in Galilean coordinate system or any other coordinate system with \(g_{\alpha \alpha} = 0\), \((\alpha = 1, 2, 3)\) the energy \((12)\) is

\[ E = c^2 \frac{dt}{d\lambda} \quad (17) \]

and it is always positive in movement “forward” in time because in the future light cone one has \(dt/d\lambda > 0\). For massive particle as with positive as negative mass \(m\) the energy is

\[ E = |m|c^2 \frac{dt}{d\tau} > 0. \quad (18) \]

Now let us give an example showing that in flat space-time the energy of the relativistic particle can be negative and zero in case of the special choice of the coordinate frame.

Consider the coordinate system in which metric of flat space-time has the form of the metric of the expanding homogeneous isotropic Universe — Milne universe \([13]\)

\[ ds^2 = c^2 dt^2 - c^2 t^2 \left( d\lambda^2 + \sinh^2 \chi d\Omega^2 \right), \quad (19) \]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2\), coordinate \(\chi\) is changing from 0 to +\(\infty\). In new coordinates

\[ T = t \cosh \chi, \quad r = ct \sinh \chi, \quad cT > r > 0 \quad (20) \]

the interval \((19)\) coincides with Minkowski interval

\[ ds^2 = c^2 dT^2 - dr^2 - r^2 d\Omega^2. \quad (21) \]
This space-time with coordinate $t \geq 0$, $\chi \geq 0$ is corresponds to the future cone in coordinates $cT, r$.

The radial distance from points $\chi = 0$ and $\chi$ in metric (14) is $D = \frac{cT}{\chi}$. Taking $D$ as the radial coordinate [15] one obtains the interval as

$$ds^2 = \left(1 - \frac{D^2}{c^2t^2}\right)c^2dt^2 + 2\frac{D}{t}dtdD - dD^2$$

$$- c^2t^2 \sinh^2\left(\frac{D}{ct}\right)d\Omega^2. \quad (22)$$

From the condition $ds^2 \geq 0$ one obtains that if $D$ is larger than $D_\chi = \frac{cT}{\chi}$, then no physical object can be at rest in coordinates $t, D, \theta, \phi$. The value $D_\chi$ corresponds to $\chi = 1$ and it plays the role of the static limit for the rotating black hole in Boyer-Lindquist [15] coordinates.

The energy of the particle with mass $m$ in coordinates $t, D, \theta, \phi$ is

$$E = mg_{\tau\lambda} \frac{dx^k}{d\lambda} = mc^2 \frac{dt}{d\tau} \left(1 - \frac{D^2}{c^2t^2} + \frac{D}{ct} \frac{dD}{dt}\right)$$

$$= mc^2 \frac{dt}{d\tau} \left(1 + \chi^2 \frac{d\chi}{dt}\right). \quad (23)$$

From (19) one obtains for any physical object the inequality

$$1 \leq \left| \frac{d\chi}{dt} \right|. \quad (24)$$

So particle can have negative energy only for $\chi > 1$, i.e. out of the static limit, if

$$\frac{d\chi}{dt} < -\frac{1}{\chi}. \quad (25)$$

Note that the components of metric (22) depend on time and the energy (23) in general is not conserved on the geodesics. If the energy is zero then particle is moving noninertial according to the law

$$\frac{d\chi}{dt} = -\frac{1}{\chi t} \Leftrightarrow \chi = \sqrt{\chi_0^2 - 2 \log(t/t_0)},$$

$$t \in [t_0, t_0 \exp((\chi_0^2 - 1)/2)]. \quad (26)$$

The trajectory of such movement for case $\chi_0 = 2$, $t_0 = 0.11$ is represented by the curve on Fig. 2 in coordinates $T, r$ (see [19]). In case of the inertial movement trajectory in these coordinates is the direct line. Possible region of movement of particles with negative and zero energies in the reference frame $T, D$ is defined in the coordinates $T, r$ by conditions $1 \leq cT/r \leq \coth t = 1.313.$

Velocities of movement in coordinates $T, r$ and $t, \chi$ satisfy condition [16]

$$t \frac{d\chi}{dt} = \frac{dr}{dt} - c \tanh \chi \frac{d\chi}{dt} \tanh \chi. \quad (27)$$

![Fig. 2 Possible region of movement of particle with negative and zero energies in the reference frame $t, D$ in flat coordinate $T, r$.](image)

So for

$$\chi \tanh \chi \geq 1 \quad (28)$$

particles at rest in inertial frame $T, r$ will have negative energies in the frame $t, D$. This region can be seen on Fig. 2 as the region above the blue line in red district. Zero energy of the particle at rest in $T, r$ coordinates is possible only the blue line defined by the root of equation $\chi \tanh \chi = 1$, i.e. $\chi \approx 1.1997$.

So one can see that for specific choice of coordinates one can obtain negative and zero energies for particles at rest in inertial frame.

4 Negative energy in Gödel universe

Metric of the Gödel cosmological model of the rotating Universe proposed in 1949 (see [17] or [18]) can be written as

$$ds^2 = c^2dt^2 - dx_1^2 + \frac{\exp\left(2\sqrt{2}\omega x_1/c\right)}{2}dx_2^2$$

$$+ 2\exp\left(\sqrt{2}\omega x_1/c\right)c\lambda dt dx_2 - dx_3^2, \quad (29)$$

where $\omega$ is constant. Such metric is the exact solution of Einstein’s equation with background matter as ideal liquid without pressure and negative cosmological constant $\Lambda$

$$R_{ik} - \frac{1}{2}Rg_{ik} + Ag_{ik} = -8\pi \frac{G}{c^2}T_{ik}, \quad (30)$$

were

$$-\Lambda = \left(\frac{\omega}{c}\right)^2 = 4\pi \frac{G}{c^2} \rho, \quad T_{ik} = \rho c^2 u_i u_k, \quad (31)$$
$u^i = \delta^i_0$. Here \( \omega \) has the sense of the angular velocity of rotation of the vector of fluid of the background matter \( u^i \).

Taking instead of \( t, x_1, x_2 \) new coordinates \( t', r, \phi \):

\[
\exp \left( \sqrt{2} \omega x_1 / c \right) = \cosh 2r + \cos \phi \sinh 2r,  
\]

\[
\omega x_2 \exp \left( \sqrt{2} \omega x_1 / c \right) = \sin \phi \sinh 2r,  
\]

\[
\tan \frac{1}{2} \left( \phi + \omega t - \sqrt{2} t' \right) = \exp(-2r) \tan \frac{\phi}{2},  
\]

one writes the interval (29) in the form \([17,19]::\)

\[
ds^2 = 2 \left( \frac{c}{\omega} \right)^2 \left( dt'^2 - dr^2 + (\sinh^4 r - \sin^2 r d\phi^2 + 2\sqrt{2} \sinh^2 r d\phi dt') - dx^2 \right),
\]

where \(-\infty < t' < \infty, 0 \leq r < \infty, 0 \leq \phi < 2\pi\) with identifying \( \phi = 0 \) and \( \phi = 2\pi \).

Now consider the general case of space-time \( t', r, \phi, z \) with the interval

\[
ds^2 = a^2 \left[ (dt' + \Phi(r) d\phi)^2 - dr^2 - dz^2 - R^2(r) d\phi^2 \right],
\]

where \( a \) is constant, \(-\infty < t < \infty, 0 \leq r < \infty, -\infty < z < \infty, 0 \leq \phi < 2\pi\) with identifying \( \phi = 0 \) and \( \phi = 2\pi \).

Let’s \( \Phi(r) > 0 \) and \( R(r) > 0 \) for \( r > 0 \).

For Gödel universe \( a = \sqrt{2}c/\omega, z = x_3/a \) and

\[
\Phi(r) = \sqrt{2} \sinh^2 r, \quad R(r) = \sinh r \cosh r.
\]

The metrical tensor is

\[
(g_{ik}) = a^2 \begin{pmatrix} 1 & \Phi & 0 & 0 \\ \Phi & \Phi^2 - R^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (g^{ik}) = \frac{1}{a^2 R^2(r)} \begin{pmatrix} R^2 - \Phi^2 & \Phi & 0 & 0 \\ \Phi & -1 & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2 \end{pmatrix},
\]

where indexes \( i, k = 0, 1, 2, 3 \) correspond to \( t', \phi, r, z \). Note that for any \( r > 0 \) the metrical tensor is not degenerate \( \det (g_{ik}) = -a^4 R^2(r) < 0 \). The degeneration for \( r = 0 \) in Gödel universe is coordinate degeneracy. The eigenvalues of \( g_{ik} \) tensor are

\[
\lambda_{1,2} = \frac{a^2}{2} \left( \Phi^2 - R^2 + 1 \pm \sqrt{(\Phi^2 - R^2 + 1)^2 + 4R^2} \right),
\]

\[
\lambda_{3,4} = -a^2.
\]

For \( r > 0 \) one has

\[
\lambda_1 \geq a^2, \quad 0 > \lambda_2 \geq -a^2 R^2.
\]

Note that inspite \( g_{\phi\phi} \) is positive for \( \Phi(r) > R(r) \) the signature of \( g_{ik} \) for all \( r > 0 \) is the standard \((+, - , - , -)\).

Possible movement of particles is defined by \( ds^2 \geq 0 \) so for the interval (30) one has

\[
dt'^2 + (\Phi^2(r) - R^2(r)) d\phi^2 + 2\Phi(r) d\phi dt' - dr^2 - dz^2 \geq 0.
\]

It is important that for any coordinate system with interval (30) the physical body for any values of \( r, \phi, z \) can be at rest, i.e. there is no static limit! However in (30) there is non diagonal term \( dt' d\phi \) like in Kerr’s metric. But differently from the case of rotating coordinate system (31) there is the possibility of the change of the sign before \( d\phi^2 \).

From (12) one obtains

\[
\left( \frac{dt'}{d\phi} \right)^2 + 2\Phi(r) \frac{dt'}{d\phi} + \Phi^2(r) - R^2(r) \geq 0.
\]

The solution of this inequality is the union of two intervals

\[
\frac{dt'}{d\phi} \in (-\infty, -(\Phi(r) + R(r))] \cup [R(r) - \Phi(r), +\infty). \quad (44)
\]

Considering cases of different signs of \( d\phi \), one obtains the following sets of solutions of (43):

\[
d\phi \geq 0 \Rightarrow dt' \geq (R - \Phi) d\phi \vee dt' \leq -(R + \Phi) d\phi, \quad (45)
\]

\[
d\phi \leq 0 \Rightarrow dt' \geq -(R + \Phi) d\phi \vee dt' \leq (R - \Phi) d\phi. \quad (46)
\]

These sets define light “cones” of future and past for the metric (30). The form of these cones in cases \( \Phi \ll R, \Phi = R \) and \( \Phi > R \) is shown in Fig. 6 for the Gödel universe with

\[
\Phi(r) > R(r) \iff r > r_0 = \log(1 + \sqrt{2}).
\]

Let us find limitations on possible values of the energy of particles moving in such universe. The coordinate \( t' \) is dimensionless, so the “physical energy” of the particle is expressed through the time component of the momentum as

\[
E = p_0 = g_{00} \frac{c}{a} \frac{dt'}{d\lambda}.
\]

For the frame with coordinates (36) the covariant \( t', \phi, z \) components are conserved, because the component of metric depend only on \( r \). So the conserved energy on the geodesic for the interval (30) is

\[
E = ca \left( \frac{dt'}{d\lambda} + \Phi(r) \frac{d\phi}{d\lambda} \right). \quad (49)
\]

From (14), (40) for the case of movement “forward” in time, i.e. in the future light cone one obtains

\[
dt' + \Phi d\phi \geq R |d\phi|,
\]

\[
\lambda_1 \geq a^2, \quad 0 > \lambda_2 \geq -a^2 R^2.
\]
Fig. 3 Light “cones” of future (blue color) and past (yellow color) for Gödel universe in coordinates $t', \phi$ for cases $r = 10^{-3}$ (left), $r = r_0$ (center) and $r = 2r_0$ (right).

so

$$E \geq caR \frac{|d\phi|}{d\lambda} \quad (51)$$

It means that for particle moving in the future cone in Gödel universe the energy is not negative.

For movement “back in time” the energy is limited from above by

$$E \leq -caR \frac{|d\phi|}{d\lambda} \quad (52)$$

and so it can be less or equal zero. However such movement physically is inconsistent. The “time machine” effect in Gödel universe corresponds to continuous movement in the future cone. So for $r > r_0$, where $\Phi(r) > R(r)$ closed loops (they are not geodesic lines) $r = \text{const}, \ z = \text{const}$, called Gödel cycles, from $\phi = 0$ to $\phi = 2\pi$ are closed time-like curves [19]. Particle moving along such cycle is moving “forward” in time but due to identification of values $\phi$ different on $2\pi$ it occurs in the past after the whole cycle. It’s energy is positive due to (51). Such “time machine” is different from that moving in the past by the sign of particle energy.

5 Conclusion

Three different cases are investigated concerning the possibility of existence of particles with negative and zero energies.

1. Schwarzschild black hole of the mass $M$. Trajectories of particles with negative and zero energies exist inside of the horizon of black hole which can be shown in Kruskal–Székeres coordinates.

2. Flat space-time in Milne’s coordinates. Here one also has the possibility of existence of particles with negative and zero energies if nonsynchronous system of coordinates is used.

3. Gödel cosmological model with rotation. Here we proved that in this model in Gödel’s coordinates particles with negative and zero energies don’t exist.

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