Near-horizon dynamics of a particle in extreme Reissner–Nordström and Clément–Gal’tsov black hole backgrounds: action-angle variables

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Abstract

We analyze the periodic motion in the conformal mechanics describing particles moving near the horizon of extreme Reissner–Nordström and axion–dilaton (Clément–Gal’tsov) black holes. For this purpose, we extract the (two-dimensional) compact (angular) parts of these systems and construct their action-angle variables. In the first case, we obtain the well-known spherical Landau problem, which possesses hidden $so(3)$ symmetry, while in the latter case the system does not have a hidden constant of motion. In both the cases, we indicate the existence of ‘critical points’, separating the regions of periodic motions with qualitatively different properties.

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1. Introduction and summary

The black hole solutions allowed in supersymmetric field theories have an extremality property, i.e. the inner and outer horizons of the black hole coalesce. In this case, one can pass to the near-horizon limit, which brings us to new solutions of Einstein equations. In this limit (near-horizon extreme black hole), the solutions become conformal invariant. The conformal invariance was one of the main reasons why the extreme black holes have been payed so much attention for the last 15 years. Indeed, due to conformal invariance, black hole solutions are a good research area for studying conformal field theories and AdS/CFT correspondence (for the recent review see [1]). The simplest way to research this type of configurations is to study the motion of a (super)particle in this background. The first paper that considered such a problem is [2], where the motion of a particle near the horizon of an extreme Reissner–Nordström black hole has been considered. Later, similar problems in various extreme black hole backgrounds were studied by several authors (see [3, 4] and references therein). It is obvious that the particle moving on a conformal-invariant background inherits the property of (dynamical) conformal symmetry, i.e. one can present additional generators $K$ and $D$, which form, together with the Hamiltonian $H$, the conformal algebra $so(2, 1)$:

$$\{H, D\} = H, \quad \{H, K\} = 2D, \quad \{D, K\} = K. \tag{1}$$
On the other hand from general reasoning (in the spirit of Darboux’s theorem), one can state that a conformal mechanics can be presented in a non-relativistic ‘canonical’ form

$$H = \frac{p_R^2}{2} + \frac{2I(u)}{R^2}, \quad \Omega = dp_R \wedge dR + \frac{1}{2} \omega_{\alpha\beta}(u) du^\alpha \wedge du^\beta,$$

(2)

where $R$ and $p_R$ are the effective radial coordinate and the momentum, respectively, and $I = HK - D^2$ is the Casimir element of $so(2,1)$ algebra. This means that all the characteristics of conformal mechanics are encoded in such an (angular) system $\omega_{\alpha\beta}(u) du^\alpha \wedge du^\beta, I(u))$ (see, e.g., [5–7]). One can also reverse the arguments and construct new integrable conformal mechanical systems starting from the known ones [8]. However, there is no general canonical transformation known which transforms arbitrary conformal mechanics to the form (2). For the particular case of the near-horizon motion of a particle in the extreme Reissner–Nordström background, such transformation has been suggested in [9], while recently it was extended to the case of a general four-dimensional near-horizon extreme black hole in [10]. In the latter, the authors, taking into account the integrability of this system, suggested the generic canonical transformation, assuming that the angular system $\omega_{\alpha\beta}(u) du^\alpha \wedge du^\beta, I(u))$ is formulated in action-angle variables. They exemplified their scheme, constructing the action-angle variables for a neutral particle near the horizon of an extreme Reissner–Nordström black hole, as well as discussed the case of the charged particle near the horizon of an extreme axion–dilaton (Clément–Gal’tsov) black hole [11], without actually constructing the action-angle variables for the second system. Then, the action-angle formulation for the angular part of a near-horizon particle dynamics in the extreme Kerr black hole background has also been presented [12].

The purpose of current paper is to construct the action-angle variables for the angular parts of the following two exactly solvable systems:

- charged particle moving near the horizon of an extreme Reissner–Nordström black hole with magnetic charge (unlike the neutral particle in [10]),
- particle moving near the horizon of extreme Clément–Gal’tsov black hole (explicit expressions).

The study of this problem not only fills the gap in [10], but also presents its own interest. Namely,

- The construction of action-angle variables is the most sequential method of the study of periodic motion. Particularly, the formulation of the system in action-angle variables explicitly indicates the existence of hidden symmetries in the system.
- Action-angle variables give a precise indication of the (non)equivalence of different integrable systems
- Finally, action-angle variables form a bridge between classical and quantum-mechanical systems and allows us to perform the semiclassical quantization of the system.

Recently, in a series of papers, an attempt was made to show the effectiveness of action-angle variables in problems of modern theoretical physics [8, 12–14].

Let us reinforce the above-mentioned observations on the near-horizon dynamics of a particle in the background of extreme Kerr black hole [12]: the use of action-angle variables allowed us to find there a critical point, where the trajectories become closed. We will show that there are similar singular points in the dynamics of a charged particle moving near the horizons of extreme Reissner–Nordström and Clément–Gal’tsov black holes. They are defined by the relation $p_\phi = \pm s$, where $s = ep$ for the Reissner–Nordström case (with $e$ being the electric charge of the probe particle, and $p$ being the magnetic charge of the extreme Reissner–Nordström black hole), and $s = e$ for the case of the extreme Clément–Gal’tsov black hole (with $e$ being an effective electric charge of the probe particle).
The remainder of the paper is organized as follows. In section 2, we describe, following [10], the canonical transformation that leads the Hamiltonian of the particle moving near the horizon of a four-dimensional extreme black hole to the canonical form (2). Then, we present the procedure of constructing the action-angle variables for this particular case. In section 3, we construct the action-angle variables for the angular part of a charged particle moving near the horizon of an extreme Clément–Gal’tsov black hole. We construct the action-angle variables for the angular part of this system and use them to analyze the properties of periodic motion. In contrast with the Reissner–Nordström case, the system does not possess a hidden constant of motion. We find a ‘critical point’ that divides the different phases of effective periodic motion. In section 4, we discuss a charged particle moving near the horizon of an extreme black hole in four dimensions is described by the triple 

\[ H = r \sqrt{(rp_r)^2 + L(\theta, p_\theta, p_\varphi) - q(p_\varphi))}, \]

\[ K = \frac{1}{r} \sqrt{(rp_r)^2 + L(\theta, p_\theta, p_\varphi) + q(p_\varphi),} \quad D = rp_r, \tag{3} \]

which involves the Hamiltonian \( H \), the generator of dilatations \( D \) and the generator of special conformal transformations \( K \). Under the Poisson bracket, they form conformal algebra \( so(2, 1) \) \( (1) \).

The functions \( L(\theta, p_\theta, p_\varphi) \) and \( q(p_\varphi) \) that appear in equation (3) depend on the details of a particular black hole under consideration: see [9, 4] for the near-horizon extremal Reissner–Nordström black hole, [15] for the rotating extremal dilaton–axion black hole, [16] for the extremal Kerr solution and [17] for the extremal Kerr–Newman and Kerr–Newman–AdS–dS black holes. Note that the angular sector of the system under consideration defined by the Hamiltonian system

\[ (I = L(\theta, p_\theta, p_\varphi) - q^2(p_\varphi), \quad \omega_0 = d\theta \wedge d\varphi + dp_\varphi \wedge d\varphi) \tag{4} \]

is an integrable system. Thus, it can be formulated in terms of action-angle variables \( (I_a, \Phi^a) \):

\[ (I = L(I_1, I_2) - q^2(I_2), \quad \omega_0 = dI_a \wedge d\Phi^a), \quad \Phi^a \in [0, 2\pi), \quad a = 1, 2. \tag{5} \]

The Hamiltonian can be put into the conventional conformal mechanics form

\[ H = \frac{1}{2} R^2 + \frac{2\Omega}{R^2}, \quad \Omega = dP_k \wedge dR + dI_a \wedge d\Phi^a \tag{6} \]

by the following canonical transformation:

\[ R = \sqrt{2K}, \quad P_R = -\frac{2D}{\sqrt{2K}}, \quad \Phi^a = \Phi^a + \frac{1}{2} \int_{x=0 \text{ to } x=\text{const.}} dx \frac{\partial \log(\sqrt{x^2/4 + L(I_1, I_2) + q(I_2))}}{\partial I_a}. \tag{7} \]

However, this transformation assumes the action-angle variables of the angular sector of the system. Let us recall that for the construction of action-angle variables for the system (4), we should introduce the generating function [18]

\[ S(I, p_\varphi, \theta, \varphi) = p_\varphi \varphi + \int_{I=\text{const.}} \int_{P_\varphi=\text{const.}} p_\varphi(I, p_\varphi, \theta) d\theta = p_\varphi \varphi + S_0(I, p_\varphi, \theta). \tag{8} \]
Then, we define, by its use, the action-angle variables

$$I_1(\mathcal{I}, p_\psi) = \frac{1}{2\pi} \oint p_\psi (\mathcal{I}, p_\psi, \theta) \, d\theta, \quad I_2 = p_\psi, \quad \Phi_{1,2} = \frac{\partial S(\mathcal{I}(I_1, I_2), I_2, \theta, \varphi)}{\partial I_{1,2}},$$

where $\mathcal{I}(I_1, I_2)$ is obtained from the first expression above.

### 3. Reissner–Nordström black hole

In this section, we construct the action-angle variables for the angular part of the conformal mechanics describing the motion of a charged particle near the horizon of an extreme Reissner–Nordström black hole (which defines the electrically and magnetically charged static black hole configuration).

The conformal generators which characterize the charged probe particle read [4]

$$H = \frac{r}{M^2} \left( \sqrt{(mM)^2 + (rp_\psi)^2 + p_\psi^2 + \sin^2 \theta (p_\psi + ep \cos \theta)^2 + eq} \right), \quad D = rp_r,$$

$$K = \frac{M^2}{r} \left( \sqrt{(mM)^2 + (rp_\psi)^2 + p_\psi^2 + \sin^2 \theta (p_\psi + ep \cos \theta)^2 - eq} \right),$$

where $m$ and $e$ are the mass and the electric charge of a particle, while $M$, $q$ and $p$ are, respectively, the mass, the electric charge and the magnetic charge of the black hole. From these expressions, we immediately obtain the angular part of our system

$$\mathcal{I} = p_\psi^2 + \frac{(p_\psi + s \cos \theta)^2}{\sin^2 \theta} - (mM)^2 - (eq)^2, \quad \omega = dp_\psi \wedge d\theta + dp_\psi \wedge d\varphi.$$

It is precisely the spherical Landau problem (Hamiltonian system, describing the motion of the particle on the sphere in the presence of a constant magnetic field generated by a Dirac monopole) shifted on the constant $\mathcal{I}_0 = (mM)^2 - (eq)^2$. Here and throughout this section, we will use the notation

$$s = ep,$$

which is precisely the Dirac’s ‘monopole number’.

For the construction of action-angle variables of the obtained system, let us introduce the generating function (8), where the second term is as follows:

$$S_0(\mathcal{I}, p_\psi, \theta) = \int_{\mathcal{I} = \text{const}} d\theta \sqrt{\mathcal{I} - (mM)^2 - (eq)^2 - \frac{(p_\psi + s \cos \theta)^2}{\sin^2 \theta}}$$

$$= \begin{cases} 2|s| \arcsin \frac{|l|}{\sqrt{\mathcal{I}}} \cot \frac{\theta}{2} - 2\sqrt{\mathcal{I} + s^2} \arctan \frac{\sqrt{\mathcal{I} + s^2} \cot \frac{\theta}{2}}{\sqrt{\mathcal{I} - s^2} \sin \frac{\theta}{2}}, & \text{for } p_\psi = s \\ -2|s| \arcsin \frac{|l|}{\sqrt{\mathcal{I}}} \tan \frac{\theta}{2} + 2\sqrt{\mathcal{I} + s^2} \arctan \frac{\sqrt{\mathcal{I} + s^2} \tan \frac{\theta}{2}}{\sqrt{\mathcal{I} - s^2} \tan \frac{\theta}{2}}, & \text{for } p_\psi = -s \end{cases}$$

$$= \sqrt{\mathcal{I} + s^2} \left[ 2 \arctan t - \sum_{\pm} \sqrt{((1 \pm b)^2 - a^2)} \arctan \frac{(1 \pm b)^2 \pm \pm \pm}{((1 \pm b)^2 - a^2)} \right], \quad \text{for } |p_\psi| \neq |s|. \quad (14)$$

Here, we introduce the notation

$$\tilde{\mathcal{I}} = \mathcal{I} - (mM)^2 - (eq)^2, \quad a^2 \equiv \frac{\tilde{\mathcal{I}}^2 - (p_\psi^2 - s^2)\tilde{\mathcal{I}}}{(\tilde{\mathcal{I}} + s^2)^2},$$

$$b \equiv -sp_\psi \frac{\tilde{\mathcal{I}}}{\tilde{\mathcal{I}} + s^2}, \quad t = \frac{a - \sqrt{a^2 - (\cos \theta - b)^2}}{\cos \theta - b}. \quad (15)$$

Hence, the equation $p_\psi = \pm s$ defines critical points, where the system changes its behavior.
For the non-critical values $p_\psi \neq \pm s$, we obtain, by the use of standard methods [18], the following expressions for the action-angle variables:

$$I_1 = \sqrt{\frac{\tilde{\mathcal{I}} + s^2}{2}} - \frac{|s + p_\psi| + |s - p_\psi|}{2}, \quad \Phi_1 = - \arcsin \left( \frac{\tilde{\mathcal{I}} + s^2}{\sqrt{2} - (p_\psi^2 - s^2)} \right),$$

$$I_2 = p_\psi, \quad \Phi_2 = \varphi + \gamma_1 \Phi_1 + \gamma_2 \arctan \left( \frac{a - (1 - b)t}{\sqrt{(1 - b)^2 - a^2}} \right) + \gamma_3 \arctan \left( \frac{a + (1 + b)t}{\sqrt{(1 + b)^2 - a^2}} \right),$$

where

$$(\gamma_1, \gamma_2, \gamma_3) = \begin{cases} 
\text{sgn}(I_2)(1, -1, 1) & \text{for } |I_2| > |s| \\
\text{sgn}(s)(0, -1, -1) & \text{for } |I_2| < |s|.
\end{cases}$$

Thus, the Hamiltonian reads

$$\mathcal{I} = \left( I_1 + \frac{|s + I_2| + |s - I_2|}{2} \right)^2 + (mM)^2 - (\Omega_1)^2 - s^2$$

$$\mathcal{I} = \begin{cases} 
(I_1 + |I_2|)^2 + (mM)^2 - (\Omega_1)^2 - s^2 & \text{for } |I_2| > |s| \\
(I_1 + |s|)^2 + (mM)^2 - (\Omega_1)^2 - s^2 & \text{for } |I_2| < |s|.
\end{cases}$$

The effective frequencies $\Omega_{1,2} = \partial \mathcal{I} / \partial I_{1,2}$ are as follows:

$$\Omega_1 = \begin{cases} 
\frac{2(I_1 + |I_2|)}{2(I_1 + |s|)} & \text{for } |I_2| > |s| \\
\frac{2(I_1 + |s|)}{2(I_1 + |I_2|)} & \text{for } |I_2| < |s|.
\end{cases} \quad \Omega_2 = \begin{cases} 
2(I_1 + |I_2|)\text{sgn}I_2 & \text{for } |I_2| > |s| \\
0 & \text{for } |I_2| < |s|.
\end{cases}$$

It is shown that in the subcritical regime, when $|I_2| < s$, the frequency $\Omega_2$ becomes zero, while the frequency $\Omega_1$ depends on $I_1$ only. In the overcritical regime, when $|I_2| > s$, the frequencies $\Omega_1$ and $\Omega_2$ coincide modulo to sign: the frequency $\Omega_2$ is positive for the positive values of $I_2$ (which is precisely the angular momentum $p_\psi$), and vice versa. This is essentially different from the periodic motion in the spherical part of the ‘Kerr particle’ observed in [12], where the critical point separated two phases, both of which corresponded to the two-dimensional motion, but with the opposite sign of $\Omega_2$.

So, in both regimes, the trajectories are closed, and the motion is effectively one-dimensional. It reflects the existence of the additional constant of motion in the system (11), reflecting the $so(3)$ invariance of the spherical Landau problem. In other words, it is a superintegrable one. In action-angle variables, the additional constant of motion reads $I_{add} = \sin(\Phi_1 - \Phi_2)$ (cf [19, 8]).

Now, let us write down the expressions for action-angle variables at the critical point $p_\psi = \pm s$:

$$I_1 = 2(\sqrt{\tilde{\mathcal{I}} + s^2} - |s|), \quad \Phi_1 = \begin{cases} 
-2 \arctan \sqrt{\frac{\tilde{\mathcal{I}} + s^2 \cot^2 \frac{\theta}{2}}{2}} & \text{for } p_\psi = s \\
2 \arctan \sqrt{\frac{\tilde{\mathcal{I}} + s^2 \tan^2 \frac{\theta}{2}}{2}} & \text{for } p_\psi = -s.
\end{cases}$$

Similarly, the Hamiltonian reads

$$\mathcal{I} = \left( \frac{I_1}{2} + |s| \right)^2 + (mM)^2 - (\Omega_1)^2 - s^2.$$  

Note that the obtained action variable is not the corresponding limit of (16).
4. Clément–Gal’tsov black hole

Now, let us consider the motion of a particle near the horizon of an extremal rotating Clément–Gal’tsov (dilaton–axion) black hole [15]. The conformal generators of this particle system (with mass \( m \) and ‘effective monopole number’ \( s \), which was refereed in [11, 15] as ‘effective electric charge’ \( e \)) read [11]

\[
H = r \left( \sqrt{m^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2} \theta (p_\phi - s \cos \theta)^2} - p_\phi \right), \quad D = rp_r,
\]

\[
K = \frac{1}{r} \left( \sqrt{m^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2} \theta (p_\phi - s \cos \theta)^2} + p_\phi \right).
\]

The Casimir of conformal algebra is given by the expression

\[
I = p_\phi^2 + \frac{(p_\phi \cos \theta - s)^2}{\sin^2 \theta} + m^2.
\]

The second term in the generating function for the action-angle variables (8) can be explicitly integrated in the elementary functions (its explicit expression could be found, e.g., in the appendix of [14])

\[
S_0 = \int_{I = \text{const}} \frac{d\theta}{\sin^2 \theta} \sqrt{I - m^2 - \frac{(p_\phi \cos \theta - s)^2}{\sin^2 \theta}}
\]

\[
= \begin{cases} 
-2|s| \arcsin \frac{|s| \tan \frac{\pi}{4}}{\sqrt{I - m^2 + s^2}}, & \text{for } p_\phi = s \\
2|s| \arcsin \frac{|s| \cot \frac{\pi}{4}}{\sqrt{I - m^2 + s^2}}, & \text{for } p_\phi = -s
\end{cases}
\]

\[
= \sqrt{I - m^2 + p_\phi^2} \left[ 2 \arctan t - \sum_{\pm} \sqrt{(1 \pm b)^2 - a^2} \arctan \frac{(1 \pm b)^2 - a^2}{(1 \pm b)^2 - a^2} \right],
\]

for \(|p_\phi| \neq |s|\).

where we introduce the notation

\[
a^2 = \frac{(I - m^2)^2 + (I - m^2)(p_\phi^2 - s^2)}{(I - m^2 + p_\phi^2)^2}, \quad b = \frac{sp_\phi}{I - m^2 + p_\phi^2}, \quad t = \frac{a - \sqrt{a^2 - (\cos \theta - b)^2}}{\cos \theta - b}.
\]

Hence, the equation \( p_\phi = \pm s \) defines the critical points, where the system changes its behavior.

For the non-critical values \( p_\phi \neq \pm s \), we obtain, by the use of standard methods [18], the following expressions for the action-angle variables:

\[
I_1 = \sqrt{I - m^2 + p_\phi^2} \frac{|s + p_\phi| + |s - p_\phi|}{2},
\]

\[
\Phi_1 = - \arcsin \frac{(I - m^2 + p_\phi^2) \cos \theta - sp_\phi}{\sqrt{(I - m^2)^2 + (I - m^2)(p_\phi^2 - s^2)}},
\]

\[
I_2 = p_\phi, \quad \Phi_2 = \varphi + \gamma_1 \Phi_1 + \gamma_2 \arctan \left( \frac{a - (1 - b)t}{\sqrt{(1 - b)^2 - a^2}} \right) + \gamma_3 \arctan \left( \frac{a + (1 + b)t}{\sqrt{(1 + b)^2 - a^2}} \right),
\]

where

\[
(\gamma_1, \gamma_2, \gamma_3) = \begin{cases} 
\text{sgn}(I_2)(1, -1, 1) & \text{for } |I_2| > |s| \\
\text{sgn}(s)(0, 1, 1) & \text{for } |I_2| < |s|.
\end{cases}
\]
Similarly, the Hamiltonian reads

\[ I = \left( I_1 + \frac{|s + I_2| + |I_2 - s|}{2} \right)^2 - I_2^2 + m^2 = \begin{cases} \frac{(I_1 + |I_2|)^2 - I_2^2 + m^2}{(I_1 + |s|)^2 - I_2^2 + m^2} & \text{for } |I_2| > |s| \\ \frac{(I_1 + |s|)^2 - I_2^2 + m^2}{(I_1 + |s|)^2 - I_2^2 + m^2} & \text{for } |I_2| < |s|. \end{cases} \]  

(31)

In the critical points \( p_\phi = \pm s \), the action-angle variables read

\[ I_1 = 2(\sqrt{I - m^2 + s^2} - |s|), \quad \Phi_1 = \begin{cases} 2 \arctan \frac{\sqrt{I - m^2 + s^2} \tan \frac{\theta}{2}}{\sqrt{I - m^2 - s^2} \tan \frac{\theta}{2}} & \text{for } p_\phi = s \\ -2 \arctan \frac{\sqrt{I - m^2 + s^2} \cot \frac{\theta}{2}}{\sqrt{I - m^2 - s^2} \cot \frac{\theta}{2}} & \text{for } p_\phi = -s. \end{cases} \]  

(32)

Inverting the first expression, we shall obtain the expression for the Hamiltonian

\[ I = \left( \frac{I_1}{2} + |s| \right)^2 + m^2 - s^2. \]  

(33)

Let us note that the action variable at the critical point is different from the corresponding limit of (28).

To clarify the meaning of critical point, let us calculate the effective frequencies of the system, i.e. \( \Omega_{1,2} = \frac{\partial^2 I}{\partial I_{1,2}} \):

\[ \Omega_1 = \begin{cases} 2(I_1 + |I_2|) & \text{for } |I_2| > |s| \\ 2(I_1 + |s|) & \text{for } |I_2| < |s| \end{cases}, \quad \Omega_2 = \begin{cases} 2I \text{ sgn } I_2 & \text{for } |I_2| > |s| \\ -2I_2 & \text{for } |I_2| < |s|. \end{cases} \]  

(34)

It can be seen from these expressions that in contrast with the previous case the system does not possess the hidden symmetries. The motion is nondegenerated in noncritical regimes. So, in contrast with the Reissner–Nordström case, the system is essentially a two-dimensional one, and its trajectories are unclosed. The frequencies \( \Omega_1 \) are the same in both the cases, while \( \Omega_2 \) are essentially different. Moreover, the frequency \( \Omega_2 \) behaves in essentially different ways in subcritical and overcritical regimes. In the first case, it is proportional to \( I_1 \), and in the second case to \( I_2 \).

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References

[1] Bredberg I, Keeler C, Lysov V and Strominger A 2011 Cargese lectures on the Kerr/CFT correspondence arXiv:1103.2355
[2] Claus P, Derix M, Kallosh R, Kumar J, Townsend P K and Van Proeyen A 1998 Phys. Rev. Lett. 81 4553 (arXiv:hep-th/9804177)
[3] Ivanov E, Krivonos S and Niederle J 2004 Nucl. Phys. B 677 485 (arXiv:hep-th/0210196)
[4] Galajinsky A 2008 Phys. Rev. D 78 044014 (arXiv:0806.1629 [hep-th])
[5] Hakobyan T, Nersessian A and Yeghikyan V 2009 J. Phys. A: Math. Theor. 42 205206 (arXiv:0808.0430)
[6] Hakobyan T, Krivonos S, Lechtenfeld O and Nersessian A 2010 Phys. Lett. A 374 801 (arXiv:0908.3290)
[7] Hakobyan T, Lechtenfeld O, Nersessian A and Saghatelian A 2011 *J. Phys. A: Math. Theor.* **44** 055205 (arXiv:1008.2912)

[8] Hakobyan T, Lechtenfeld O, Nersessian A, Saghatelian A and Yeghikyan V 2012 *Phys. Lett. A* **376** 679 (arXiv:1108.5189 [hep-th])

[9] Bellucci S, Galajinsky A, Ivanov E and Krivonos S 2003 *Phys. Lett. B* **555** 99 (arXiv:hep-th/0212204)

[10] Galajinsky A and Nersessian A 2011 *J. High Energy Phys.* JHEP11(2011)135 (arXiv:1108.3394 [hep-th])

[11] Clément G and Gal’tsov D 2001 *Nucl. Phys. B* **619** 741 (arXiv:hep-th/0105237)

[12] Bellucci S, Nersessian A and Yeghikyan V 2012 *Mod. Phys. Lett. A* **27** 1250191 (arXiv:1112.4713 [hep-th])

[13] Bellucci S, Nersessian A, Saghatelian A and Yeghikyan V 2011 *J. Comput. Theor. Nanosci.* **8** 769 (arXiv:1008.3865)

[14] Lechtenfeld O, Nersessian A and Yeghikyan V 2010 *Phys. Lett. A* **374** 4647 (arXiv:1005.0464)

[15] Clément G and Gal’tsov D 2001 *Phys. Rev. D* **63** 124011 (arXiv:gr-qc/0102025)

[16] Galajinsky A 2010 *J. High Energy Phys.* JHEP11(2010)126 (arXiv:1009.2341)

[17] Galajinsky A and Orekhov K 2011 *Nucl. Phys. B* **850** 339 (arXiv:1103.1047)

[18] Arnold V I 1973 *Mathematical Methods in Classical Mechanics* (Moscow: Nauka)

[19] Gonera C 2012 *Phys. Lett. A* **376** 2341 (arXiv:1010.2915 [math-ph])