High Spins Beyond Rarita-Schwinger Framework

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Abstract

We study the eigenvalue problem of the squared Pauli-Lubanski vector, $W^2$, in the Rarita-Schwinger representation space and derive from it that the $(-s(s+1)m^2)$ subspace with $s = 3/2$, i.e. spin $3/2$ in the rest frame, is pinned down by the one sole Klein-Gordon like equation,

\[
\left[ (p^2 - m^2)g_{\alpha\beta} - \frac{2}{3}p_{\beta}p_{\alpha} - \frac{1}{3}(p_{\alpha}\gamma_{\beta} + p_{\beta}\gamma_{\alpha}) \not{p} + \frac{1}{3}\gamma_{\alpha} \not{p}\gamma_{\beta} \not{p} \right] \psi^\beta = 0.
\]

Upon gauging this $W^2$ invariant subspace of $\psi^\mu$ is shown to couple to the electromagnetic field in a fully covariant fashion already at zeroth order of $1/m$ and with the correct gyromagnetic factor of $g_3^2 = \frac{2}{3}$. The gauged equation is hyperbolic and hence free from the Velo-Zwanziger problem of acausal propagation within an electromagnetic field at least to that order.

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I. INTRODUCTION

High spin particles occupy an important place in theoretical physics. For the first time they were observed as resonant excitations in pion-nucleon scattering. The Particle Data Group [1] lists more than thirty non-strange baryon resonances with spins ranging from 3/2 to 15/2, and more than twenty strange ones with spins from 3/2 to 9/2. Baryon resonances have been extensively investigated in the past at the former Los Alamos Meson Physics Facility (LAMPF), and at present their study continues at the Thomas Jefferson National Accelerator Facility (TJNAF) [2]. Such particles are of high relevance in the description of photo- and electro-pion production off proton, where they appear as intermediate states, studies to which the Mainz Microtron (MAMI) devotes itself since many years [3]. Search for high spin solutions to the QCD lagrangian has been recently reported by the Lattice collaboration in Ref. [4]. Moreover, also the twistor formalism has been employed in the construction of high spin fields [5]. Integer high spins meson resonances with spins ranging from 0 to 6 can have importance in various processes revealing the fundamental features of QED at high energies such like pair production [6]. However, the most attractive high spin fields appear in proposals for physics beyond the standard model which invoke supersymmetry [7] and contain gauge fields of fractional spins such as the gravitino- the supersymmetric partner of the ordinary spin-2 graviton. Supersymmetric theories open the venue to the production of fundamental spin $\frac{3}{2}$ particles at early stages of the universe, whose understanding can play an important role in its evolution [8].

The description of high spins takes its origin from Refs. [9], [10], [11] who suggest to consider any fractional spin-$s$ as the highest spin in the totally symmetric rank-$(s - 1/2)$ Lorentz tensor with Dirac spinor components, $\psi_{\mu_1...\mu_{s-1}}$. For spin 3/2 one has to consider the four-vector–spinor, $\psi_{\mu}$,

$$\psi_{\mu} = A_{\mu} \otimes \psi \simeq \left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left[\left(\frac{1}{2}, 0\right) \oplus \left(\frac{1}{2}, 0\right)\right],$$

the direct product between the four vector, $A_{\mu}$, and the Dirac spinor, $\psi$. As shown in [12], in the latter case it is possible to derive the free particle equations from a family of lagrangians depending on a free parameter $A$,

$$\mathcal{L}(A) = \overline{\psi} \left[ p_{\alpha} \Gamma_{\mu} \, ^\alpha_{\nu}(A) - m g_{\mu\nu} \right] \psi,$$

(2)
where
\[ p_{\alpha} \Gamma^\alpha_{\nu}(A) \psi^\nu = \not{p} \psi_{\mu} + B(A) \gamma^\mu \not{p} \gamma \cdot \psi + A(\gamma^\mu p \cdot \psi + p^\mu \gamma \cdot \psi) + C(A) m \gamma^\mu \gamma \cdot \psi, \]  
(3)

\[ A \neq \frac{1}{2}, B(A) \equiv \frac{3}{2} A^2 + A + \frac{1}{2}, \quad C(A) = 3 A^2 + 3 A + 1. \]  
(4)

The wave equation following from this lagrangian is obtained as
\[ (\not{p} - m) \psi_{\mu} + A (\gamma^\mu p \cdot \psi + p^\mu \gamma \cdot \psi) + B(A) (\gamma^\mu \not{p} \gamma \cdot \psi) + C(A) m \gamma^\mu \gamma \cdot \psi = 0, \]  
(5)

which for \( A = -1 \) can be written in a compact form as
\[ (i \varepsilon_{\mu \nu \beta \alpha} \gamma^5 \gamma^\beta p^\alpha - mg_{\mu \nu} + m \gamma^\mu \gamma^\nu) \psi^\nu = 0. \]  
(6)

Equations of this type are equivalent to
\[ \frac{1}{2m} (\not{p} + m) \psi_{\mu} = \psi_{\mu}, \]  
(7)
\[ (-g_{\mu \nu} + \frac{1}{m^2} p^\mu p^\nu) \psi^\nu = -\psi_{\mu}, \]  
(8)
\[ \gamma^\mu \psi_{\mu} = 0, \]  
(9)

known as the Rarita-Schwinger framework. Notice that for the sake of convenience of the point we are going to make in the next section, we here wrote the respective Dirac and Proca equations in terms of covariant projectors picking up spin-1/2\( ^+ \) and spin-1\( ^- \) states, respectively. Spin 3/2\( ^+ \) needs an axial four vector. The freedom represented by \( A \) reflects invariance under the so called point transformations which mix the two spin 1/2 sectors residing in the RS representation space besides spin 3/2. Unfortunately for interacting fields this freedom gives rise to undetermined "off-shell" parameters yielding an ambiguous theory. As we will see below, this is not to remain the only disadvantage of the RS framework. Over the years, Eqs. (8)-(10) have been widely applied in hadron physics to the description of mainly the \( \Delta(1232) \) and occasionally the \( D_{13}(1520) \) resonances and their contributions to various processes. A recent account of effective theories for the calculation of properties of light hadrons using the Rarita-Schwinger formalism along the line of Chiral Perturbation Theory can be found in [16].

Yet, a detailed study of Eqs. (8), (9), and (10) reveals that the Rarita-Schwinger framework suffers two more fundamental weaknesses. These are:
1. The interacting quantum spin-3/2 field is plagued by several problems ranging from loss of constraints to a failure to propagate, observations reported by Johnson and Sudarshan in Ref. [17].

2. The wave fronts of the classical solutions of the Rarita-Schwinger spin-3/2 equations suffer acausal propagation within the electromagnetic environment, an observation due to Velo and Zwanziger [18], [19].

Although several remedies to Eq. (2) have been suggested over the years (Ref. [20] being the most recent), none of them resulted in a covariant, unique, parameter– and pathology free high spin wave equation. The reason for that, as we see it, has been to not have worried sufficiently about the fundamental principles behind the high-spin description.

It is the goal of the present study to

1. unveil importance of Poincaré invariance and its Casimir operators, the squared four momentum, $P^2$, and the squared Pauli-Lubanski vector, $W^2$, for the dynamics and unique identification of the spin-degrees of freedom in $\psi_{\mu_1...\mu_s-1/2}$ with the special emphasis on spin-3/2 in $\psi_\mu$,

2. derive a unique, parameter free, wave equation for spin-3/2 that preserves the quality of the Rarita-Schwinger field of having correct electromagnetic couplings such as a correct gyromagnetic factor and which is furthermore free from the Velo-Zwanziger problem at least to leading (zeroth) order with respect to $\frac{1}{m}$.

The literature devoted to the description of high spins by means of the Rarita-Schwinger formalism is overwhelming and contains creative treatments at many occasions. We here focus on suggesting a solution to the Velo-Zwanziger problem and aim to keep the presentation as concise as possible. In view of this goal, we restrict ourselves to quote only few works which are indispensable for the point we have to make, hence we leave aside a vast amount of otherwise interesting and important articles.

The paper is organized as follows. In the next Section we reveal the relationship between the Rarita-Schwinger lagrangian and the eigenvalue problem of the squared Pauli-Lubanski vector in the four-vector–spinor. In Section III we briefly review the Velo-Zwanziger problem for completeness of the presentation. In Section IV we present the dynamics of a spin-3/2 particle in terms of the Casimir operators of the Poincaré group, analyze the minimal
coupling to an external electromagnetic field and the Velo-Zwanziger problem. The paper closes with a brief summary and has one Appendix.

II. THE $W^2$ BACKGROUND TO THE RARITA–SCHWINGER LAGRANGIAN.

We envisage our goal in taking a radically new look on the old problem. We remark that although the Rarita-Schwinger formalism is supposed to describe spin $3/2$, unlike the Dirac (8) and Proca (9) cases, none of the equations is a manifest covariant projector onto the subspace in $\psi_\mu$ of the desired spin and parity.

The Rarita-Schwinger formalism does not do anything more but successively track first the Dirac spin $1/2$ piece, then the Proca spin $1$ piece, and finally their supposed coupling to spin $3/2$ by means of the respective Eqs. (8), Eq. (9), and Eq. (10). In other words, the search for spin $3/2$ in $\psi_\mu$ is realized along the line of ordinary $SU(2)$ spin–angular-momentum coupling.

In the present work we aim to proceed differently and identify spin $3/2$ directly and in a covariant fashion. To do so we first retreat from the $SU(2)$ labeling and switch to Poincaré group labels. Recall that according to their definition as irreducible representations of the Poincaré group particles have to be labeled by numbers that relate to the eigenvalues of the two Casimir invariants of the Poincaré group, the squared four-momentum $P^2$, and the squared Pauli-Lubanski vector, $W^2$, according to (11),

\[
P^2 \Psi^{(m,s)} = m^2 \Psi^{(m,s)},
\]
\[
W^2 \Psi^{(m,s)} = -m^2 s(s + 1) \Psi^{(m,s)}.
\]

Here $m$ stands for the mass, while $\Psi^{(m,s)}$ denotes a generic Poincaré group representation of mass $m$ and rest-frame spin $s$. The reason for which the Pauli-Lubanski (PL) vector plays a pivotal role in particle classification is that the eigenvalue problem of its square encodes in any inertial frame the rest-frame $SU(2)$ spin (see Appendix). We will apply Eqs. (11) to the vector-spinor and search for the $s = 3/2$ value. The Pauli-Lubanski vector in $\psi_\mu$ is defined as

\[
W_\mu = W_\mu \otimes 1_4 + 1_4 \otimes w_\mu,
\]

where $W_\mu$ and $w_\mu$ in turn stand for the Pauli-Lubanski vectors in the four-vector and Dirac spinor spaces, respectively, while $1_4$ denotes the $4 \times 4$ unit matrix. Applied to the spin $3/2$
sector of the Rarita-Schwinger field, \( \psi^{(m,3/2)} \), Poincaré invariance requires it to satisfy

\[
\left[ \mathcal{W}^2 + m^2 \frac{15}{4} \gamma_+ \otimes \gamma_- \right] \psi^{(m,3/2)} = 0,
\]

\[
\psi^{(m,3/2)} = [A \otimes \psi]^{(m,3/2)}.
\]  \( \tag{13} \)

The action of the Casimir invariant \( \mathcal{W}^2 \) onto the RS field now results in

\[
[\mathcal{W}^2]_\alpha^\beta \psi^{(m,3/2)}_\beta = [w^2 + W^2] \psi^{(m,3/2)} + 2 (W^\mu)_\alpha^\beta [A \otimes w_\mu \psi]^{(m,3/2)}.
\]  \( \tag{14} \)

In using Eqs. (59) from the Appendix we find

\[
[\mathcal{W}^2]_\alpha^\beta \psi^{(m,3/2)}_\beta = \left[ -\frac{3}{4} p^2 - 2(p^2 g_\alpha^\beta - p^\beta p_\alpha) \right] \psi^{(m,3/2)} + 2 (W^\mu)_\alpha^\beta [A_\beta \otimes w_\mu \psi]^{(m,3/2)}
\]

\[
= -m^2 \frac{15}{4} \psi^{(m,3/2)}.
\]  \( \tag{15} \)

In making use of the mass shell condition \( p^2 = m^2 \) in Eq. (15) allows to reduce it to

\[
2 (W^\mu)_\alpha^\beta [A \otimes w_\mu \psi]^{(m,3/2)} + m^2 \psi^{(m,3/2)} = -2p_\alpha \cdot \psi^{(m,3/2)}.
\]  \( \tag{16} \)

Notice that spin 3/2 or spin 1/2 are identified by the eigenvalues of the first term on the left hand side of Eq. (16), \(-m^2\) versus \(2m^2\). In order to construct explicitly this term which will be referred to as the Dirac–Proca “intertwining term” we exploit, \( \gamma^\nu \gamma^\rho = g^{\nu\rho} - i\sigma^{\nu\rho} \), to cast \( w_\mu \) from Eq. (58) into the more convenient form

\[
w_\mu = \frac{1}{2} \gamma_5 (p_\mu - \gamma_\mu \slashed{\dot{p}}).
\]  \( \tag{17} \)

Substitution of Eq. (17) into the intertwining term yields

\[
2 (W^\mu)_\alpha^\beta w_\mu = i\epsilon^\mu_\alpha^\beta \sigma^\rho \gamma_5 (p_\mu - \gamma_\mu \slashed{\dot{p}})
\]

\[
= -i\epsilon^\mu_\alpha^\beta \sigma^\rho \gamma_5 \gamma_\mu \slashed{\dot{p}}.
\]  \( \tag{18} \)

Substitution of Eq. (18) in Eq. (16) results in the following equation:

\[
(i\epsilon_{\alpha\beta\sigma} \gamma_5 \gamma^\mu \sigma^\rho \slashed{\dot{p}} - m^2 g_{\alpha\beta} + 2p_\beta p_\alpha) \psi^{(m,3/2)} = 0.
\]  \( \tag{19} \)

The resemblance between Eqs. (19) and (17) is hardly to be overlooked. The kinetic and mass terms are identical (modulo the Dirac condition \( \slashed{\dot{p}} \psi_\mu = m \psi_\mu \)), while the remaining terms only represent different auxiliary conditions, a difference that should be irrelevant at
that stage as both wave equations provide equivalent free particle descriptions. Through
above observation the Rarita-Schwinger lagrangian unexpectedly acquires the status of a
substitute to the complete $\mathcal{W}^2$ eigenvalue problem. Within the context of this new reading
of the Rarita-Schwinger lagrangian the question arises as to what extent the problems of
the traditional high-spin description may be resolved in recovering the complete $\mathcal{W}^2$ action
on $\psi_\mu$ given in Eq. (15).

Below we will make the case that

the complete $\mathcal{W}^2$ eigenvalue problem gives rise to a simple, transparent and
easy to handle equation for the spin-$3/2$ degrees of freedom in $\psi_\mu$ that has the
appealing advantage to be free from the chronic deficit of the Rarita-Schwinger
framework, the acausal propagation in the presence of an electromagnetic field,
at least to leading (zeroth) order in $\frac{1}{m}$.

Before presenting the mentioned equation in Section IV below, we briefly review the
Velo-Zwanziger problem in the next Section.

III. THE VELO-ZWANZIGER PROBLEM OF LINEAR HIGH-SPIN LA-
GRANGIANS

The work of Giorgio Velo and Daniel Zwanziger on the defects of the linear RS lagrangian
for interacting particles with high spins \cite{18} occupies the central place in the present study.
For the sake of self sufficiency of the present paper we here highlight it in brief. Also from
here onwards we will suppress upper labels $^{(m,s)}$ for the sake of simplicity of the notations.

The main point of Ref. \cite{18} is that Eq. (7) provided by the lagrangian device (3) is not
a genuine equation of motion because it is of first order but the time derivative of $\psi_0$
never occurs. This defect reveals itself in multiple ways through

1. the complete cancellation of all $\partial_0 \psi_0$ terms in Eq. (7) for any $\mu$,

2. the complete cancellation of $\partial_0 \psi_\alpha$ terms for $\mu = 0$, in which case one finds instead of
   a wave equation the constraint
   \begin{equation}
   [\vec{p} + (\vec{p} \cdot \vec{\gamma} - \gamma^0 m) \vec{\gamma}] \cdot \vec{\psi} = 0, \tag{20}
   \end{equation}
3. the absence of $\psi_0$ in Eq. (20) that leaves the time-component of the Rarita-Schwinger field undetermined,

In fact above deficits are due to the constraints incorporated in the wave equations and could be tolerated if remediable upon gauging. Velo and Zwanziger elaborate in Ref. [18] the gauging procedure in replacing $p_\mu$ by $\pi_\mu = p_\mu + eA_\mu$, and succeed in shaping Eq. (7) to a genuine equation [31]. The remedy starts with first contracting the gauged equation successively by $\gamma^\mu$ and $\pi^\mu$ and generating the covariant gauged constraints as [32]

$$\gamma \cdot \psi = \frac{2}{3m^2} e^5 \gamma \cdot \widetilde{F} \cdot \psi,$$

$$\pi \cdot \psi = (\gamma \cdot \pi + \frac{3}{2}m) \frac{2}{3m^2} e^5 \gamma \cdot \widetilde{F} \cdot \psi,$$

and ends with substituting Eqs. (21,22) back into the gauged Eq. (7). The resulting new wave equation,

$$(\not{\pi} - m)\psi_\mu - (\pi_\mu + \frac{1}{2}\gamma_\mu) \frac{2}{3m^2} e^5 \gamma \cdot \widetilde{F} \cdot \psi = 0,$$

is a true one because it specifies the time derivatives of $\psi_\mu$ for any given $\mu$. Also $\psi_0$ is now defined by means of Eq. (21). The ultimate step is testing hyperbolicity and causality of the solutions to Eq. (23) by means of the Courant-Hilbert criterion [33]. The result is that in general the equation is not hyperbolic and the wave front velocity of its solutions can exceed the speed of light. Strong fields destroy the hyperbolic character of Eq. (23). It was found that solely sufficiently weak fields guarantee hyperbolicity of Eq. (23).

To the best of our knowledge no solution to the Velo-Zwanziger problem has been suggested so far. It solely has been pointed out by Hurley [24] that the generalized Feynman–Gell-Mann equations for $(s,0) \oplus (0,s)$ representations,

$$(\pi^2 - m^2)\Psi^{(s,0)\oplus(0,s)} = \frac{c}{2s} S^\mu F_{\mu\nu} \Psi^{(s,0)\oplus(0,s)},$$

are obviously manifestly hyperbolic. However, $(s,0) \oplus (0,s)$ states are difficult to couple to the pion-nucleon or photon-nucleon system due to dimensionality mismatch, a reason that represents a serious obstacle to their application in phenomenology.

In the next Section we show that Poincaré invariance of space time suggests a different equation in which (i) the wave fronts propagate causally, (ii) the coupling to the electromagnetic field appears in a fully covariant fashion already at leading order with respect to $1/m$ and wears the correct gyromagnetic factor of $g_s = 1/s$. 

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IV. BEYOND RARITA-SCHWINGER FRAMEWORK–CONSISTENT DESCRIPTION OF HIGH SPINS.

The observed kinship between the lagrangian equation (7) and the $W^2$ eigenvalue problem discussed in Section II above is suggestive of the idea to test predictive power of Eq. (15) for the description of high spins.

A. The free case

In taking this venue we substitute Eq. (18) in Eqs. (15) to obtain the full $W^2$ operator as

$$(W^2)_{\alpha\beta} = -\frac{3}{4} p^2 g_{\alpha\beta} - 2(p^2 g_{\alpha\beta} - p_\beta p_\alpha) - p^2 g_{\alpha\beta} + (p_\alpha \gamma_\beta + p_\beta \gamma_\alpha - \gamma_\alpha \gamma_\beta) \hat{p}$$

$$= -\frac{15}{4} p^2 g_{\alpha\beta} + 2p_\beta p_\alpha + (p_\alpha \gamma_\beta + p_\beta \gamma_\alpha - \gamma_\alpha \gamma_\beta) \hat{p}.$$  \hfill (25)

Hence the associated spin 3/2 equation reads

$$\left[\frac{15}{4}(p^2 - m^2)g_{\alpha\beta} - 2p_\beta p_\alpha - (p_\alpha \gamma_\beta + p_\beta \gamma_\alpha - \gamma_\alpha \gamma_\beta) \hat{p}\right] \psi^\beta = 0.$$  \hfill (26)

If this equation is to pin down correctly the eight spin-3/2 degrees of freedom in $\psi_\mu$ it has to incorporate the supplementary condition which should be found by contracting successively with $\gamma^\alpha$, and $p^\alpha$. In so doing we find

$$\left[p^2 - 5m^2\right] \gamma \cdot \psi = 0,$$  \hfill (27)

$$\left[p^2 - 5m^2\right] p \cdot \psi = 0.$$  \hfill (28)

This result reflects certain occasional confusion in the $s$ value. In the concrete case we are facing the twofold ambiguity in the decomposition of the $W^2$ eigenvalues according to

$$-\frac{15}{4} m^2 = \left\{ -\frac{3}{2}(\frac{3}{2} + 1)m^2, -\frac{1}{2}(\frac{1}{2} + 1)5m^2 \right\},$$  \hfill (29)

that attributes a mass of $\sqrt{5}m$ to the spin-1/2 fields $\gamma \cdot \psi$ and $p \cdot \psi$, respectively \[25\]. In other words, at times the $W^2$ eigenvalue by itself ceases to fix unambiguously the spin unless the eigenvalue of the first Casimir invariant, $p^2$, has been specified. The problem is resolved upon setting the Dirac sector on mass shell, i.e. in assuming

$$w^2 = -\frac{3}{4} p^2 = -\frac{3}{4} m^2.$$  \hfill (30)
In substituting the latter equation in Eq. (26) amounts to the following free Klein-Gordon like spin 3/2 equation

\[(W^2)_{\alpha\beta}\psi^\beta + 3m^2\psi_\alpha + 2(W^\mu)_{\alpha\beta}w_\mu\psi^\beta = 0,\]  

(31)

the explicit form of which reads

\[
\left( p^2 - m^2 \right) g_{\alpha\beta} - \frac{2}{3} p_\beta p_\alpha - \frac{1}{3} \left( p_\alpha \gamma_\beta + p_\beta \gamma_\alpha - \gamma_\alpha \hat{p}_\beta \right) \hat{p} \right) \psi^\beta = 0. 
\]

(32)

In now contracting this equation successively by \( p^\alpha \) and \( \gamma^\alpha \) one recovers the standard auxiliary conditions

\[ p \cdot \psi = 0, \]  

(33)

\[ \gamma \cdot \psi = 0. \]  

(34)

Introducing these constraints into Eq. (32) amounts to the Klein-Gordon condition for all the field \( \psi_\mu \) components

\[ (p^2 - m^2)\psi_\mu = 0, \]  

(35)

as it should be. Our free Eq. (32) has one problem in common with the RS framework—it is also not a genuine equation because the highest time derivative of \( \psi_0 \) never occurs. On the other side, it can be shown that its \( \mu = 0 \) component does not amount to a constraint as in Eq. (20) but fixes \( \psi_0 \) and also contains first order time derivatives. In the next subsection we will couple Eq. (32) to electromagnetism.

**B. The interacting case**

In now replacing \( p^\mu \) by \( \pi^\mu = p^\mu - eA_\mu \) in Eq. (31), the gauged equation is rewritten as

\[
\left( \hat{W}^2 \right)_{\alpha\beta} + 3m^2g_{\alpha\beta} + 2(\hat{W}^\mu)_{\alpha\beta}\hat{w}_\mu \right) \psi^\beta = 0, 
\]

(36)

where hat-quantities are obtained from the bare ones by the replacement \( p^\mu \rightarrow \pi^\mu \). The gauged Dirac-Proca intertwining term is now calculated as

\[ 2(\hat{W}^\mu)_{\alpha\beta}\hat{w}_\mu = (ie^{\mu}_{\alpha\beta\sigma}\pi^\sigma)\left( \frac{i}{2}\gamma_5\sigma_{\mu\nu}\pi^\nu \right) = -\hat{T}_{\alpha\beta} \not{\pi} - e\gamma_5\tilde{F}_{\alpha\beta}, \]  

(37)

\[ \hat{T}_{\alpha\beta} = i\epsilon_{\alpha\beta\mu\sigma}\gamma_5\gamma^\mu\pi^\sigma, \]  

(38)
where \( \tilde{F}_{\alpha\beta} \) is the dual to the electromagnetic field strength tensor, \( \tilde{F}_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\mu\sigma} F^{\mu\sigma} \), while \( e \) stands for the electric charge of the field. Finally, \( [\pi_\alpha, \pi_\beta] = ie F_{\alpha\beta} \), with \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \).

Putting everything together we obtain the following equation:

\[
\left[ \tilde{T}_{\alpha\beta} \not{\!}{\!}_\alpha + (2\pi^2 - 3m^2) g_{\alpha\beta} - 2\pi_\alpha \pi_\beta + e\gamma_5 \tilde{F}_{\alpha\beta} + 2ie F_{\alpha\beta} \right] \psi_\beta = 0 ,
\]

(39)

Contracting Eq. (39) by \( \pi^\alpha \) and using \( \tilde{T}_{\alpha\beta} \) from Eq. (38) results in the first gauged constraint,

\[
\pi \cdot \psi = \frac{e}{3m^2} \left[ -\gamma_5 (\gamma^\mu \tilde{F}_{\mu\beta} \not{\!}{\!}_\alpha + \pi^\mu \tilde{F}_{\mu\beta}) + 2i F_{\beta\mu} \pi^\mu \right] \psi_\beta ,
\]

(40)

whereas contraction by \( \gamma^\alpha \) leads to

\[
\left( 1 - \frac{e}{3m^2} \sigma^{\mu\nu} F_{\mu\nu} \right) \gamma \cdot \psi = \frac{e}{3m^2} \gamma^\mu \left( \gamma_5 \tilde{F}_{\mu\beta} + 4i F_{\mu\beta} \right) \psi_\beta .
\]

(41)

To obtain the latter equation we used the equivalent representation for \( T_{\alpha\beta} \)

\[
T_{\alpha\beta} = g_{\alpha\beta} \not{\!}{\!}_\alpha - (\gamma_\alpha \pi_\beta + \gamma_\beta \pi_\alpha - \gamma_\alpha \not{\!}{\!}_\beta \gamma_\beta ) .
\]

(42)

In the \( 1/m \) expansion the gauged constraints read

\[
\pi \cdot \psi = 0 + \mathcal{O}(1/m^2) ,
\]

\[
\gamma \cdot \psi = 0 + \mathcal{O}(1/m^2) .
\]

(43)

Substitution of Eqs. (43) into Eq. (39) amounts to our prime result – the explicit form of the gauged equation (36):

\[
(\pi^2 - m^2) \psi_\alpha + \frac{2i}{3} e F_{\alpha\beta} \psi_\beta + \frac{e}{6} F^{\mu\nu} \sigma_{\mu\nu} \psi_\alpha + \frac{ie}{3} \gamma_\alpha \gamma^\eta F_{\eta\beta} \psi_\beta + \frac{e}{3} \gamma_5 \tilde{F}_{\alpha\beta} \psi_\beta = 0 .
\]

(44)

In recalling that the magnetic interaction is described by the covariant term

\[
M_{\mu\nu} F^{\mu\nu} ,
\]

(45)

where \( M_{\mu\nu} \) stand for the Lorentz group generators (see Appendix), one finds that for Dirac theory the magnetic interaction is given by

\[
\mathcal{L}^{(s=1/2)}_{mag} = \frac{ge}{2} \bar{\psi}_a \left( \frac{\sigma_{\mu\nu}}{2} \right)_{ab} \psi_b F^{\mu\nu} .
\]

(46)

Similarly, for Proca particles one encounters

\[
\mathcal{L}^{(s=1)}_{mag} = \frac{ge}{2} W_\alpha \left( L^{\mu\nu} \right)^{\alpha\beta} W_\beta F^{\mu\nu} = i ge W_\mu W_\nu F^{\mu\nu} ,
\]

(47)
where we used \((L^\mu_\nu)^{\alpha\beta} = i(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})\). In a similar way we expect for any representation
\[
L_{\text{mag}} = \frac{g e}{2} \bar{\psi}_{A}(M_{\mu\nu})_{AB}\psi_{B}F^{\mu\nu},
\] (48)
where capital Latin indices stand for the complete set of quantum numbers that label \(M_{\mu\nu}\) in the internal space of \(\psi\) and its adjoint, respectively. In particular for the Rarita-Schwinger space under consideration
\[
L^{(s=3/2)}_{\text{mag}} = \frac{g e}{2} \bar{\psi}_{a}(M_{\mu\nu})_{\alpha\beta;ab}\psi_{b}F^{\mu\nu}.
\] (49)
As long as
\[
(M_{\mu\nu})_{\alpha\beta;ab} = (L_{\mu\nu})_{\alpha\beta}\delta_{ab} + g_{\alpha\beta}(\frac{1}{2}\sigma^{\mu\nu})_{ab},
\] (50)
and in suppressing Dirac indices leads to the following interacting lagrangian
\[
L^{(s=3/2)}_{\text{mag}} = \frac{g e}{2} \bar{\psi}_{a}[(L_{\mu\nu})_{\alpha\beta} + g_{\alpha\beta}(\frac{1}{2}\sigma^{\mu\nu})_{ab}^{\beta}] \psi_{b}F^{\mu\nu} = i g e \bar{\psi}_{\mu}\psi_{\nu}F^{\mu\nu} + \frac{g e}{2} \bar{\psi}_{a}[g_{\alpha\beta}(\frac{1}{2}\sigma^{\mu\nu})_{ab}]^{\beta} \psi_{b}F^{\mu\nu}.
\] (51)
In comparing the latter equation to Eq. (44) allows to conclude the gyromagnetic factor as
\[
g = \frac{2}{3} = \frac{1}{s}, \quad s = \frac{3}{2}.
\] (52)
In conclusion, Eq. (39) predicts the correct value for the gyromagnetic factor in accordance with Belinfante’s conjecture \([26]\) on the inverse proportionality between \(g_s\) and spin. Moreover, Eq. (44) is manifestly hyperbolic and causal as the second order time derivative enter the equation only via its Klein-Gordon building block. A further advantage of Eq. (44) is that the electromagnetic interaction of the \(s = 3/2\) sector is already turned on at leading order in the \(\frac{1}{m}\) expansion. Notice that to same order the Rarita-Schwinger equation (23) gauged by Velo and Zwanziger reduces to the mere gauged Dirac equation and leaves the coupling of the spin 3/2 degrees of freedom unspecified. Within the Rarita-Schwinger framework the gyromagnetic factor is extracted at the non-relativistic level \([12],[23],[27]\) or from calculating pion-nucleon bremsstrahlung and a subsequent comparison to low energy theorems \([28]\). Our equation (44) has the advantage to allow identification of the magnetic coupling in a fully covariant fashion.

Incorporation of the auxiliary conditions beyond leading order is in work. In Ref. \([29]\) we studied the propagation of the wave fronts of the complete \(W^2\) eigenvalue problem and delivered the proof that at least in the basis where \(W^2\) diagonalizes, the associated spin 3/2
equation of motion is free from the Velo-Zwanziger problem. However, in this basis, one no longer has separation between Lorentz and Dirac indices. In case the Velo-Zwanziger problem turns out to be related to the particular representation choice in Eq. (1), then we at least succeeded in pushing away the inconsistencies to order $O(1/m^2)$ in the $1/m$ expansion.

V. SUMMARY.

Our suggested solution to the problem of the covariant and consistent description of spin $3/2$ within an electromagnetic environment is the Klein-Gordon-like equation (52) (gauged (44)) following from the $W^2$ eigenvalue problem in $\psi_\mu$ and which is (i) fully covariant, (ii) parameter free, (iii) hyperbolic and causal, (iv) covariantly coupled to the electromagnetic field by a gyromagnetic factor of $g_{3/2} = 2/3$. Though we argued at the level of the vector-spinor, our approach in being based upon the complete $W^2$ eigenvalue problem—a quantity that is well defined in any representation space—obviously can be generalized to any totally symmetric rank-$(s-\frac{1}{2})$ Lorentz tensor with Dirac spinor components, $\psi_{\mu_1...\mu_{s-\frac{1}{2}}}$, and thereby to any fractional spin $s$. Compared to the Dirac-like spin $3/2$ equations, the advantage of the Klein-Gordon like one is to have achieved a parameter free description and a covariant coupling to the electromagnetic field already to leading order in the $1/m$ expansion. We expect the proposed scheme to help improving description of various processes containing spin $3/2$ fermions.

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VI. APPENDIX 1

In this appendix we collect conventions and some results on the symmetry of spacetime under rotations, boosts and translations—transformations that constitute the Poincaré group.
for which the squared Pauli-Lubanski vector is a Casimir invariant. In terms of the Poincaré group generators, $M_{\mu\nu}$ and $P_\eta$ and their algebra

\begin{equation}
[M_{\mu\nu}, M_{\alpha\beta}] = -i(g_{\mu\alpha}M_{\nu\beta} - g_{\mu\beta}M_{\nu\alpha} + g_{\nu\beta}M_{\mu\alpha} - g_{\nu\alpha}M_{\mu\beta}), \tag{53}
\end{equation}

\begin{equation}
[M_{\alpha\beta}, P_\mu] = -i(g_{\mu\alpha}P_\beta - g_{\mu\beta}P_\alpha), \quad [P_\mu, P_\nu] = 0, \tag{54}
\end{equation}

where $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the metric tensor, the Pauli–Lubanski (PL) vector is defined as

\begin{equation}
W_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}M^{\nu\alpha}p^\beta, \tag{55}
\end{equation}

with $\epsilon_{0123} = 1$. This operator can be shown to satisfy the following commutation relations

\begin{equation}
[M_{\mu\nu}, W_\alpha] = -i(g_{\alpha\mu}W_\nu - g_{\alpha\nu}W_\mu), \quad [W_\alpha, P_\mu] = 0, \quad [W_\alpha, W_\beta] = -i\epsilon_{\alpha\beta\mu\nu}W^{\mu\nu}, \tag{56}
\end{equation}

i.e. it transforms as a four-vector under Lorentz transformations. Moreover, its square commutes with all the generators and is a group invariant. For this reason elementary particles are required to transform invariantly under the action of $W^2$ and to be labeled by the $W^2$ eigenvalues next to those of $p^2$. The representation spaces of interest for the present work are the four-vector and the Dirac-spinor. Their respective Pauli-Lubanski vectors, $W_\mu$, and $w_\mu$ are found as

\begin{equation}
[W_\mu]_\alpha^\beta = i\epsilon_{\mu\alpha}^\beta\sigma p^\sigma, \tag{57}
\end{equation}

\begin{equation}
w_\mu = i2\gamma_5\sigma_{\mu\rho}p^\rho. \tag{58}
\end{equation}

To obtain above expressions we substituted for the generators in the four-vector and the Dirac-spinor in Eq. \( \text{[M}^{\mu\nu}\text{]}_\alpha^\beta = i(g_\alpha^{\mu\nu}g^{\rho\beta} - g_\rho^{\mu\beta}g_\alpha^{\nu\rho}), \) \text{and} \( M_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}, \) \text{respectively.}

The Casimir invariants in the four-vector and Dirac-spinor spaces are now calculated as

\begin{equation}
[W^2]_\alpha^\beta = -2(g_\alpha^\beta p^2 - p^\beta p_\alpha), \quad w^2 = -\frac{3}{4}p^2. \tag{59}
\end{equation}

In particular Proca’s equation is just the corresponding eigenvalue equation for the spin 1 subspace

\begin{equation}
[W^2]_\alpha^\beta A_\beta = -2m^2A_\alpha. \tag{60}
\end{equation}

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[31] Velo and Zwanziger consider the case of $A = -1$, the $A = -\frac{1}{3}$ has been worked out in Refs. [12], [23].

[32] Notice that we are using different conventions for $\gamma_5$ and the Levi-Civita tensor.

[33] The Courant-Hilbert criterion for hyperbolicity requires the determinant of the coefficient matrix obtained by replacing the highest derivatives by $n_\mu$, the normals to the characteristic surfaces, to vanish only for real $n_0$. 