D-branes in an asymmetric orbifold

Matthias R. Gaberdiel\textsuperscript{\tiny †,\tiny a} and Sakura Schäfer-Nameki\textsuperscript{\tiny *,\tiny b}

\textsuperscript{a}School of Natural Sciences, Institute for Advanced Study
Princeton, NJ 08540, USA

\textsuperscript{b}Department of Applied Mathematics and Theoretical Physics
Wilberforce Road, Cambridge CB3 OWA, U.K.

Abstract

We consider the asymmetric orbifold that is obtained by acting with T-duality on a 4-torus, together with a shift along an extra circle. The chiral algebra of the resulting theory has non-trivial outer automorphisms that act as permutations on its simple factors. These automorphisms play a crucial role for constructing D-branes that couple to the twisted sector of the orbifold.

10/2002

\textsuperscript{†} On leave of absence from: Department of Mathematics, King’s College London, Strand, London WC2R 2LS, UK, e-mail: mrg@sns.ias.edu, mrg@mth.kcl.ac.uk
\textsuperscript{*} e-mail: S.Schafer-Nameki@damtp.cam.ac.uk
1. Introduction

Asymmetric orbifolds [1] have played a prominent rô le in constructing string back-
grounds with phenomenologically interesting features [2,3,5,7,8,10,11]. In particular,
they provide interesting non-supersymmetric string theories as well as dS-like backgrounds.
From a conformal field theoretic point of view, asymmetric orbifolds are of interest since
they provide a way to construct new non-diagonal, so-called heterotic modular invariants,
see e.g. [12] and references therein.

Despite the large literature on asymmetric orbifolds, little is known about D-branes in
these backgrounds. Some results for specific orbifolds have been obtained in [13,14,15,16],
but a complete understanding still seems to be lacking. One natural candidate for a
symmetry that acts differently on left and right moving degrees of freedom is T-duality.
Orbifolds including an element of the T-duality group have been considered in [2,3,7,8] in
the context of non-supersymmetric string compactifications. In this paper we will construct
the D-brane boundary states in a bosonic relative of these models. While this model is
only a toy model, some of the techniques we shall discuss will generalise to orbifolds of the
superstring.

Our construction also possesses a few novel conformal field theoretic features. Al-
though it is rather straightforward to construct a class of boundary states that are invari-
ant under the orbifold action, it is typically difficult to construct D-branes that couple to
states in the twisted sector of an asymmetric orbifold. In order to obtain such boundary
states in our example, we will have to consider D-branes that preserve the maximal chiral
symmetry only up to an automorphism of the symmetry algebra. Symmetry breaking
boundary conditions, in particular for simple chiral algebras, have been studied e.g. in
[17,18,19,20,21,22,23,24]. The orbifold chiral algebra that we shall encounter here consists
of powers of a simple chiral algebra, and thus allows for extra outer automorphisms, which
act as permutations on the factors (compare [25]). In fact we will be able to construct all
boundary states that preserve the orbifold chiral algebra up to an arbitrary automorphism.
Taken together, these boundary states couple to all different sectors of the theory.

The plan of the paper is as follows. In section 2 we describe the model we are discussing
and fix the notation. In section 3 we explain that the asymmetric orbifold can alternatively
be described as a simple current extension of the $\widehat{su}(2)_1^5$ diagonal theory. The permutation

† Orbifolds of a closed string theory by a permutation group have been discussed e.g. in
[26,27,28,29].
branes of this diagonal theory are constructed, and the NIM-rep property of the twisted fusion algebra is checked. In section 4 we then construct permutation branes for the asymmetric orbifold and verify Cardy’s condition. Section 5 explains how to generalise the construction to ‘conformal’ boundary states, and how superpositions of conventional Dirichlet- and Neumann-branes fit into our picture. Our conclusions are contained in section 6. There are three appendices where some more technical details are spelled out.

2. The setup

In this paper we shall be interested in the following asymmetric orbifold. Consider the toroidal compactification of the (bosonic) string on a 4-torus at the SO(8) point. At this point in moduli space, the theory is simply the (diagonal) level \( k = 1 \) \( \hat{so}(8)_1 \) Kac-Moody theory. Let us choose the Cartan-Weyl basis for the generators of \( \hat{so}(8)_1 \), and let us denote the ‘Cartan generators’ by \( H_n^i \), while the ‘root generators’ are \( E_{\pm \alpha}^n \). We are interested in the orbifold action that acts on the left-moving currents as

\[
g_L : E_{\pm \alpha}^n \rightarrow -E_{\mp \alpha}^n, \quad H_n^i \rightarrow -H_n^i,
\]

while the action on the right-movers is trivial, \( g_R = I \). The invariant algebra for the left-movers is then

\[
\langle \{ E^\alpha - E^{-\alpha} | \alpha \in \Delta^+ \} \rangle,
\]

which is isomorphic to \( \mathcal{A}_L^q = \hat{so}(4)_1 \oplus \hat{so}(4)_1 \). The invariant algebra for the right-movers is obviously \( \mathcal{A}_R^q = \hat{so}(8)_1 \). The action (2.1) can be thought of as some form of T-duality along the four torus directions†.

In order to describe the action of \( g_L \) on the four different representations of \( \hat{so}(8)_1 \) it is useful to decompose them with respect to \( \mathcal{A}_L^q \). Let us label the representations of the \( \hat{so}(4n)_1 \) chiral algebra by \( o, v, s, c \), and let us distinguish the two \( so(4) \) copies by \( I \) and \( II \). Then the relevant decomposition is

\[
\begin{align*}
\tilde{\mathcal{H}}_{o}^{so(8)} &= (\mathcal{H}_o^I \otimes \mathcal{H}_o^{II}) \oplus (\mathcal{H}_o^I \otimes \mathcal{H}_v^{II}) \\
\tilde{\mathcal{H}}_{v}^{so(8)} &= (\mathcal{H}_v^I \otimes \mathcal{H}_o^{II}) \oplus (\mathcal{H}_v^I \otimes \mathcal{H}_o^{II}) \\
\tilde{\mathcal{H}}_{s}^{so(8)} &= (\mathcal{H}_s^I \otimes \mathcal{H}_s^{II}) \oplus (\mathcal{H}_c^I \otimes \mathcal{H}_c^{II}) \\
\tilde{\mathcal{H}}_{c}^{so(8)} &= (\mathcal{H}_c^I \otimes \mathcal{H}_s^{II}) \oplus (\mathcal{H}_c^I \otimes \mathcal{H}_s^{II}).
\end{align*}
\]

† As is explained in [16], this transformation differs from the usual T-duality transformation by a rotation on the right-movers.
On these representations $g_L$ then acts as $\psi = \pm 1$.

The orbifold defined by $g$ does not satisfy the level matching condition. Indeed, since we invert four left-moving modes without modifying the right-movers, the energy shifts of the left- and right-moving ground state in the twisted sector are

$$
\Delta_L = \frac{4}{48} + \frac{4}{24} = \frac{1}{4}, \quad \Delta_R = 0,
$$

and thus the action by $g$ is not consistent by itself. The apparent level-matching problem (2.4) of the orbifold by $g$ can be cured by introducing additional shifts in the Narain lattice. Let us denote the inequivalent choices as in [30] by

$$
A_1 = \left( \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} \right), \quad A_2 = \left( \frac{1}{\sqrt{2}}, 0 \right), \quad A_3 = \left( \frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}} \right).
$$

On states with momentum and winding $(p_L, p_R) = \left( \frac{n+w}{\sqrt{2}}, \frac{n-w}{\sqrt{2}} \right)$ these shifts act via the phase $\exp(2\pi ip \cdot A)$, i.e. as

$$
(-1)^w, \quad (-1)^{w+n}, \quad (-1)^n,
$$

respectively.

For the supersymmetric version of this theory, a resolution of the level-mismatch has been discussed earlier in the literature [2,3,7,8]. To this end, one combines the orbifold action with a shift by $A_1$ along all four torus directions. This resolves the level-matching problem since the shift changes the right-moving ground state energy in the twisted sector by $\Delta_R = 4\frac{A_1^2}{2} = \frac{1}{4}$, while it does not contribute to the ground state energy of the left-movers, since the left-movers are reflected (and the shift can therefore be undone). If this resolution is applied to the bosonic theory in question, the resulting theory becomes actually left-right symmetric, and is therefore not of immediate interest.

In order to obtain a genuinely asymmetric orbifold, we therefore choose another way of satisfying the level matching condition which is similar to what was proposed in [4]. Let us compactify the theory on an additional circle for which we choose the radius to take the self-dual value, and combine $g$ with an $A_2$ shift along this additional circle,

$$
T_{so(8)}^4 \times T_{su(2)}^1 / f = \left( (-1)_L^4, A_2 \right).
$$

The additional shift changes the left-moving ground state energy by

$$
\Delta_L = \frac{(A_2)^2}{2} = \frac{1}{4}, \quad \Delta_R = 0,
$$

respectively.
and therefore resolves (2.4). The shift $A_2$ of (2.3) has the action

$$
\psi' = 1 \quad \psi' = -1
$$

$$
H_{T_{su(2)}} = H_{0,1} \otimes \bar{H}_{0,1} \oplus H_{1,1} \otimes \bar{H}_{1,1},
$$

where $H_{j,1}$ for $j = 0, 1$ denote the two irreducible highest weight representations of $\hat{su}(2)_1$. The spectrum of the resulting theory is then genuinely asymmetric, and is explicitly given as

$$
H_U = \left( (H_{0} \otimes H_{0}^{II}) \otimes \bar{H}_{o}^{so(8)} \oplus (H_{o} \otimes H_{v}^{II}) \otimes \bar{H}_{v}^{so(8)} \right. \\
\oplus \left. (H_{s} \otimes H_{s}^{II}) \otimes \bar{H}_{s}^{so(8)} \oplus (H_{v} \otimes H_{c}^{II}) \otimes \bar{H}_{c}^{so(8)} \right) \otimes H_{0,1} \otimes \bar{H}_{0,1}
$$

$$
H_T = \left( (H_{c} \otimes H_{c}^{II}) \otimes \bar{H}_{c}^{so(8)} \oplus (H_{c} \otimes H_{c}^{II}) \otimes \bar{H}_{c}^{so(8)} \right. \\
\oplus \left. (H_{v} \otimes H_{v}^{II}) \otimes \bar{H}_{v}^{so(8)} \oplus (H_{s} \otimes H_{s}^{II}) \otimes \bar{H}_{s}^{so(8)} \right) \otimes (H_{0,1} \otimes \bar{H}_{1,1}) \\
\oplus \left. (H_{0} \otimes H_{0}^{II}) \otimes \bar{H}_{o}^{so(8)} \oplus (H_{o} \otimes H_{o}^{II}) \otimes \bar{H}_{o}^{so(8)} \right) \otimes (H_{1,1} \otimes \bar{H}_{0,1}),
$$

where $H_U$ and $H_T$ denote the untwisted and twisted sectors, respectively. Here the representations have been written in terms of the chiral algebras

$$
A_L = \hat{so}(4)_1 \oplus \hat{so}(4)_1 \oplus \hat{su}(2)_1, \quad A_R = \hat{so}(8)_1 \oplus \hat{su}(2)_1.
$$

The maximal diagonal chiral subalgebra is therefore

$$
A_{diag} = \hat{so}(4)_1 \oplus \hat{so}(4)_1 \oplus \hat{su}(2)_1 \cong \hat{su}(2)_1^{10}.
$$

For definiteness, we identify the first two $su(2)$s with the first $so(4)$ in (2.12), and the second two $su(2)$s with the second $so(4)$. Furthermore, we identify the representations of $\hat{so}(4)_1$ with those of $\hat{su}(2)_1 \oplus \hat{su}(2)_1$ as

$$
O \simeq (0, 0), \quad V \simeq (1, 1), \quad S \simeq (1, 0), \quad C \simeq (0, 1).
$$
3. Permutation-twisted boundary states

The main aim of this paper is to construct all the D-branes of the asymmetric orbifold that preserve (2.12) up to automorphisms. This is to say, we want to construct the boundary states that preserve

\[( J_n^\alpha - \sigma(\bar{J}_{-n}^\alpha) ) \parallel \alpha \parallel = 0 , \] (3.1)

where \( J_n^\alpha \) is a current in (2.12), and \( \sigma \) is an arbitrary automorphism of \( \mathcal{A}_{\text{diag}} \). In particular, we are interested in D-branes that couple to the twisted sector states of the asymmetric orbifold. It is obvious from (2.10) that the boundary states that involve Ishibashi states from the twisted sector components can only arise for automorphisms \( \sigma \) that are outer. The group of outer automorphisms of \( \mathcal{A}_{\text{diag}} \) is isomorphic to the permutation group in five objects, \( S_5 \), where \( \sigma \in S_5 \) acts on the \( \hat{su}(2)_1 \) factors as

\[ \sigma : \hat{su}(2)_1^{(i)} \rightarrow \hat{su}(2)_1^{(\sigma i)} . \] (3.2)

For fixed \( \sigma \in S_5 \), the boundary states that satisfy (3.1) can be written in terms of the \( \sigma \)-twisted Ishibashi states,

\[ \parallel \alpha \parallel^\sigma = \sum_l B_{\alpha,1} \parallel l \parallel^\sigma , \] (3.3)

where \( \alpha \) labels the different \( \sigma \)-twisted boundary states, while \( \parallel l \parallel^\sigma \) denotes the \( \sigma \)-twisted Ishibashi state that originates from the representation of \( \mathcal{A}_{\text{diag}} \otimes \mathcal{A}_{\text{diag}} \)

\[ (l_1, l_2, l_3, l_4, l_5) \otimes \sigma^{-1}(l_1, l_2, l_3, l_4, l_5) , \quad l_i = 0, 1 . \] (3.4)

In principle one could now go ahead and determine the set of \( \sigma \)-twisted Ishibashi states for each element \( \sigma \in S_5 \), and then make an ansatz for the boundary states that can be written as a linear superposition of these Ishibashi states. However, the discussion simplifies drastically if we relabel our theory as follows. Let us redefine the right-moving chiral algebra \( \mathcal{A}_{\text{diag}}^R \) by exchanging the second and fifth \( \hat{su}(2)_1 \) factors. (In terms of the boundary states in (3.1) this amounts to replacing \( \sigma \) by \( \sigma(25) \).) After a little calculation one then finds that, with respect to this redefined chiral algebra, the spectrum of the theory simply becomes

\[ \mathcal{H} = \bigoplus_{J_1=+1} 1 \otimes 1 \bigoplus_{J_1=+1} 1 \otimes J_1 . \] (3.5)
Here $l = (l_1, l_2, l_3, l_4, l_5)$ labels representations of $\widehat{su}(2)_1^5$, and the sum runs over all such representations for which the eigenvalue $J_1$, defined by

$$J_1 = (-1)^{l_1 + l_3 + l_4 + l_5},$$

equals +1. Furthermore, the ‘simple current’ $J$ acts on $l$ as

$$J(l_1, l_2, l_3, l_4, l_5) = (l_1 + 1, l_2, l_3 + 1, l_4 + 1, l_5 + 1),$$

where addition is understood modulo 2. The first sum in (3.5) is the subsector of the diagonal $\widehat{su}(2)_1^5$ theory that has eigenvalue +1 under the action of $J$, while the second sum describes the $J$-twisted sector. We can therefore think of this theory as a simple current extension of the original diagonal $\widehat{su}(2)_1^5$ theory (see for example [31,32,33,34]). As an aside it may be worth pointing out that the first and second sum of (3.5) (i.e. the untwisted and twisted sector of the simple current extension) do not correspond to the untwisted and twisted sector of the original orbifold in (2.10), respectively. In order to construct D-branes for the theory in question, it is now convenient to begin with constructing permutation-twisted branes for the diagonal $\widehat{su}(2)_1^5$ theory.

3.1. Permutation-twisted boundary states for $\text{su}(2)_1^n$

The construction of the permutation-twisted boundary states can actually be formulated more generally for the diagonal $\widehat{su}(2)_1^n$ theory, and we shall therefore be more general in the following (compare also [25]). In particular, we shall show that the boundary states we construct satisfy the various Cardy conditions [35].

Suppose $\sigma$ is a permutation in $S_n$, and $\hat{m}$ is a $\sigma$-twisted representation of $\widehat{su}(2)_1^n$. For each such $\hat{m}$ we define the $\sigma$-twisted boundary state as

$$\langle \hat{m} \rangle_{\sigma} = \sum_{l} \frac{\hat{S}_{\hat{m}, l}^\sigma}{\sqrt{S_{0, l}}} \langle l \rangle_{\sigma},$$

where the sum runs over the sectors $l = (l_1, \ldots, l_n)$ of the diagonal $\widehat{su}(2)_1^n$ theory for which $\sigma(l) = l$, $S_{0, l}$ is a product of $n$ $\widehat{su}(2)_1$ S-matrices, and $\hat{S}_{\hat{m}, l}^\sigma$ is the $\sigma$-twisted S-matrix. In order to define the latter, consider the ‘twining character’

$$\chi_1^{(\sigma)}(\tau) = \text{Tr}_{\mathcal{H}_1}(\sigma q^{L_0 - c/24}), \quad q = e^{2\pi i \tau},$$

\[3.9\]
which is non-zero if and only if \( l \) is invariant under \( \sigma \), \( \sigma(l) = l \). Under the \( S \)-modular transformation these twining characters transform into characters of \( \sigma \)-twisted representations:

\[
\chi_1^{(\sigma)}(-1/\tau) = \sum_{\hat{m}} \hat{S}_{\hat{m},1} \chi_{\hat{m}}(\tau),
\]

where

\[
\chi_{\hat{m}}(\tau) = \text{Tr}_{H_{\hat{m}}}(q^{L_0-c/24}), \quad q = e^{2\pi i \tau}.
\]

(3.11)

The number of \( \sigma \)-twisted representations always equals the number of \( \sigma \)-invariant representations, and, in fact, \( \hat{S}_{\hat{m},1} \) is actually a unitary matrix.

The formula (3.8) is a natural generalisation of the formula proposed in [22,18]. In the following we want to show that it satisfies the Cardy condition. The Cardy condition states that the overlap of two boundary states must give rise, after an \( S \)-modular transformation, to a positive integer linear combination of (twisted) characters of the chiral algebra. The calculation of the overlap between two boundary states can be reduced to the overlap of two Ishibashi states. The overlap between the latter can be easily calculated, and it is given as

\[
\langle \langle 1 | q^{L_0+L_0-c/12} | k \rangle \rangle_{\tau} = \delta_{l,k} \chi_1^{(\sigma^{-1})}(q),
\]

(3.12)

where \( \chi_1^{(\sigma^{-1})}(q) \) is the twining character (3.9). Since only the \( \sigma \)-invariant states contribute to the twining character, it is obvious that

\[
\chi_1^{(\sigma)}(q) = \chi_1^{(\sigma^{-1})}(q).
\]

(3.13)

Furthermore, since each permutation commutes with \( L_0 \), the identity

\[
\chi_1^{(\sigma\tau)}(q) = \chi_1^{(\tau\sigma)}(q)
\]

holds. In particular, it therefore follows that (3.12) also equals

\[
\langle \langle \hat{n} | q^{L_0+L_0-c/12} | \hat{m} \rangle \rangle_{\tau} = \delta_{\hat{n},\hat{m}} \chi_1^{(\sigma\tau^{-1})}(q) = \delta_{\hat{n},\hat{m}} \chi_1^{(\tau\sigma^{-1})}(q) = \delta_{\hat{n},\hat{m}} \chi_1^{(\sigma^{-1}\tau)}(q).
\]

(3.15)

With these preparations it is now immediate to write down the overlap between two boundary states,

\[
\langle \langle \hat{n} | q^{L_0+L_0-c/12} | \hat{m} \rangle \rangle_{\tau} = \sum_{\hat{p}} \sum_{l} \left( \hat{S}_{\hat{n},l}^{\sigma} \right)^* \hat{S}_{\hat{p},l}^{\sigma\tau^{-1}} \hat{S}_{\hat{m},l}^{\tau} \chi_{\hat{p}}(\tilde{q}) = \sum_{\hat{p}} (\sigma,\tau) N_{\hat{p},\hat{m}}^{\hat{n}} \chi_{\hat{p}}(\tilde{q}),
\]

(3.16)

* A twisted representation of a chiral algebra is, by definition, the same as an untwisted representation of the corresponding twisted algebra, see [36] for an introduction.
where \( \hat{p} \) labels the set of \( \sigma \tau^{-1} \)-twisted representations of \( \widehat{su}(2)^{n} \), and \( \tilde{q} = e^{-2\pi i/\tau} \). The Cardy condition is thus satisfied provided that

\[
(\sigma, \tau) N_{\hat{p}, \hat{m}}^{\hat{n}} = \sum_{l} \left[ \frac{\hat{S}_{\sigma, 1}^{l} \hat{S}_{\sigma \tau^{-1} 1}^{l} \hat{S}_{\tau, 1}^{l} \hat{S}_{0, 1}^{l}}{S_{0, 1}} \right]
\]

are non-negative integers; this can easily be confirmed explicitly case by case. It is clear from the results of appendix B that our formula for the boundary states agrees with the more explicit formula given in [25]; the fact that our boundary states satisfy the Cardy condition follows then also from the analysis given there. Finally, we have given a more abstract proof of this property in appendix B.

It is very tempting to identify (3.17) with the fusion rules describing the fusion of the \( \tau \)-twisted representation \( \hat{m} \) with the \( \sigma \tau^{-1} \)-twisted representation \( \hat{p} \) to give the \( \sigma \)-twisted representation \( \hat{n} \). Formula (3.17) generalises then the Verlinde formula to twisted fusion rules; it is a natural further generalisation of the formula proposed in [22] (see also [21,37]). For a specific example we have checked that (3.17) does indeed describe the twisted fusion rules; this is described in appendix C.

3.2. The NIM-rep property

If the integers (3.17) describe the twisted fusion rules, they must define a non-negative integer matrix representation of the fusion rules (or NIM-rep for short). This is to say, the matrices (3.17) must satisfy

\[
\sum_{\hat{m}} (\sigma, \tau) N_{\hat{p}, \hat{m}}^{\hat{n}} (\tau, \rho) N_{\hat{m}, \hat{k}}^{\hat{q}} = \sum_{\hat{f}} (\sigma, \rho) N_{\hat{f}, \hat{k}}^{\hat{n}} (\sigma \rho^{-1}, \tau \rho^{-1}) N_{\hat{p}, \hat{q}}^{\hat{r}}. \quad (3.18)
\]

Here \( \hat{k} \) and \( \hat{n} \) are \( \rho \)-twisted and \( \sigma \)-twisted representations, respectively, and the sum on the left hand side runs over all \( \tau \)-twisted representations \( \hat{m} \), while the sum on the right hand side runs over all \( \sigma \rho^{-1} \)-twisted representations \( \hat{f} \). Furthermore, \( \hat{p} \) and \( \hat{q} \) are \( \sigma \tau^{-1} \)-twisted and \( \tau \rho^{-1} \)-twisted representations, respectively.

To prove (3.18) we write the left hand side as

\[
\sum_{\hat{m}} (\sigma, \tau) N_{\hat{p}, \hat{m}}^{\hat{n}} (\tau, \rho) N_{\hat{m}, \hat{k}}^{\hat{q}} = \sum_{\hat{m}} \sum_{l, l'} \left[ \frac{\hat{S}_{\sigma, 1}^{l} \hat{S}_{\sigma \tau^{-1} 1}^{l} \hat{S}_{\tau, 1}^{l} \hat{S}_{0, 1}^{l}}{S_{0, 1}} \right] \frac{\hat{S}_{\rho, 1}^{l'} \hat{S}_{\rho \tau^{-1} 1}^{l'} \hat{S}_{\tau, 1}^{l'} \hat{S}_{0, 1}^{l'}}{S_{0, 1}} \quad (3.19)
\]

\[= \sum_{l} \left[ \frac{\hat{S}_{\sigma, 1}^{l} \hat{S}_{\rho, 1}^{l'}}{S_{0, 1}} \right] \frac{\hat{S}_{\sigma \tau^{-1} 1}^{l} \hat{S}_{\rho \tau^{-1} 1}^{l'} \hat{S}_{\tau, 1}^{l} \hat{S}_{0, 1}^{l'}}{S_{0, 1}} \right]. \]
where the sum over $l$ in the first line extends over all $\sigma$- and $\tau$-invariant representations, while the sum over $l'$ runs over all $\tau$ and $\rho$-invariant representations. In going to the second line we have used that $\hat{S}^\tau$ is unitary; the sum over $l$ in the second line extends over all representations that are simultaneously $\sigma$, $\tau$- and $\rho$-invariant. (3.19) has to equal the right hand side of (3.18).

\[
\sum_{\tilde{f}} \mathcal{N}_{\tilde{f},k} (\sigma,\rho^{-1},\tau^{-1}) \mathcal{N}_{\tilde{p},\tilde{q}} = \sum_{\tilde{f}} \sum_{1,1'} \left( \frac{\hat{S}_\sigma^{\tilde{f},1}}{S_{0,1}} \right)^* \left( \frac{\hat{S}_\sigma^{\tilde{f},1'}}{S_{0,1'}} \right) \left( \frac{\hat{S}_\rho^{\tilde{p},1}}{S_{0,1}} \right)^* \left( \frac{\hat{S}_\rho^{\tilde{p},1'}}{S_{0,1'}} \right)^{-1} (3.20)
\]

where the sum in the last line is over $\sigma$, $\tau$- and $\sigma\rho^{-1}$-invariant, and hence also $\rho$-invariant, representations $l$. This then agrees with (3.19).

4. Permutation-twisted D-branes in the asymmetric orbifold

Let us now return to the description of the D-branes in the asymmetric orbifold. Recall that the asymmetric orbifold could be described as a simple current extension of a tensor product of $\hat{su}(2)_1$ theories (3.3). In the previous section we have shown how to construct the boundary states for this tensor product theory. Now we need to implement the simple current extension. Let us begin by defining an action of $J$ on the set of $\sigma$-twisted representations by

\[
\hat{S}_\sigma^{\tilde{m},1} = J_1 \hat{S}_\sigma^{\tilde{m},1}. (4.1)
\]

Given the structure of $\hat{S}_\sigma$ described in appendix B, this prescription defines the action of $J$ uniquely.

The boundary states can now be constructed as for usual simple current extensions [19,20]. There are two cases to distinguish. If a given twisted weight $\tilde{m}$ is not a fixed point under the action of $J$, the boundary state is defined by

\[
\| [\tilde{m}] \rangle \rangle^\sigma = \frac{1}{\sqrt{2}} (\| \tilde{m} \rangle \rangle^\sigma + \| J \tilde{m} \rangle \rangle^\sigma). (4.2)
\]

Here $\| \tilde{m} \rangle \rangle^\sigma$ and $\| J \tilde{m} \rangle \rangle^\sigma$ are boundary states of the diagonal $\hat{su}(2)_1^3$ theory defined by (3.8). The sum in (4.2) guarantees that only $J$-invariant Ishibashi states contribute. These
boundary states therefore only involve Ishibashi states that come from the first sum in (3.3). They are labelled by \( J \)-orbits, where \( \hat{m} \) denotes the orbit with representative \( \hat{m} \).

On the other hand, if \( \hat{m} \) is invariant under the action of \( J \) the above construction has to be modified. A \( \sigma \)-twisted representation \( \hat{m} \) is invariant under \( J \) if and only if \( \sigma \) has the property that \( J_1 = +1 \) for all \( l \) in the \( \hat{su}(2)_1 \) theory for which \( \sigma(l) = l \). (The simplest example for such a permutation is \( \sigma = (1345) \).) If this is the case, then there exist non-trivial Ishibashi states from the \( J \)-twisted sector, i.e. from the second sum in (3.5), and conversely, whenever such Ishibashi states from the \( J \)-twisted sector exist, the corresponding permutation has fixed points. In fact, for each such permutation there is an equal number of Ishibashi states from the \( J \)-untwisted and the \( J \)-twisted sector of (3.3). In order to see this, observe that there is a \( \sigma \)-twisted Ishibashi state from the first sum in (3.5) for each representation \( l \) with \( J_1 = +1 \) that satisfies

\[
\sigma(1) = 1. \tag{4.3}
\]

On the other hand, there is a \( \sigma \)-twisted Ishibashi state from the second sum in (3.5) for each representation \( l \) with \( J_1 = +1 \) that satisfies

\[
\sigma(Jl) = 1. \tag{4.4}
\]

Every solution to (4.4) can be obtained by adding all solutions of (4.3) to one fixed solution, \( l_0 \), of (4.4). (As always, addition is understood mod 2 here.) In particular, their number is therefore the same.

We have now assembled all the notation necessary to describe the boundary states for those permutations that have \( J \)-fixed points. These boundary states are labelled by a \( \sigma \)-twisted representation \( \hat{m} \) together with one additional sign. Explicitly they are given as

\[
\| \hat{m}, \pm \rangle \rangle_0^\sigma = \frac{1}{\sqrt{2}} \left( \sum_{l} \frac{\hat{S}_{\hat{m},l}^\sigma}{\sqrt{S_{0,1}}} |l\rangle\rangle^\sigma \pm \sum_{l} \frac{\hat{S}_{\hat{m},l}^\sigma}{\sqrt{S_{0,1}}} |l+l_0\rangle\rangle^\sigma \right). \tag{4.5}
\]

The Ishibashi state \( |l\rangle\rangle^\sigma \) lies in the \( l \otimes l \) sector of (3.5), while the Ishibashi state \( |l+l_0\rangle\rangle^\sigma \) lies in \( (l+l_0) \otimes J(l+l_0) \). Both sums run over the \( \sigma \)-invariant representations of \( \hat{su}(2)_1^{5} \) with \( J_1 = +1 \). The set of boundary states is independent of the special solution \( l_0 \) to (4.4); in fact, we have

\[
\| \hat{m}, \pm \rangle \rangle_0^\sigma = \| \hat{m}, \pm(-1)^{\hat{m} \cdot (l_0 - l'_0)} \rangle \rangle_{l'_0}^\sigma, \tag{4.6}
\]

where the inner product of \( \sigma \)-invariant and \( \sigma \)-twisted representations is defined by

\[
\hat{S}_{\hat{m},l+k}^\sigma = (-1)^{\hat{m} \cdot k} \hat{S}_{\hat{m},l}^\sigma, \tag{4.7}
\]

where \( k \) is a \( \sigma \)-invariant representation of \( \hat{su}(2)_1^{5} \).
4.1. The analysis of the overlaps

In the previous subsection we have described the \( \sigma \)-twisted boundary states of the asymmetric orbifold. It should be clear from the description of the twisted \( S \)-matrix in appendix B how to write down these states explicitly. Given these explicit descriptions, it is easy to check by hand that the boundary states do indeed satisfy the Cardy condition.

Now we want to give a more structural argument to this effect. Since there are two types of boundary states (namely those defined by (4.2) and those defined by (4.5)), there are three cases to consider. First we analyse the overlaps between two boundary states of the form (4.2). Using (3.16) it follows that

\[
\sigma \langle \langle \hat{n} \rangle \| q^{\frac{1}{2}\left(L_0 + L_0 - c/12\right)} \| \hat{m} \rangle \rangle^\tau = \frac{1}{2} \sum_{\hat{p}} (\mathcal{N}_{\hat{p},\hat{m}}^{\hat{n}} + \mathcal{N}_{\hat{p},\hat{m}}^{J\hat{n}} + \mathcal{N}_{\hat{p},J\hat{m}}^{\hat{n}} + \mathcal{N}_{\hat{p},J\hat{m}}^{J\hat{n}}) \chi_{\hat{p}}(\tilde{q})
\]

where we have dropped the superscripts \((\sigma, \tau)\) and have used that

\[
\mathcal{N}_{\hat{p},J\hat{m}}^{J\hat{n}} = \sum_l \left( \hat{S}_{J\hat{n},l}^{\sigma} \right)^* \hat{S}_{J\hat{m},l}^{\sigma} \hat{S}_{\hat{p},l}^{\sigma \tau - 1} S_{0,1} = \sum_l J_1 (J_1)^* \left( \hat{S}_{\hat{n},l}^{\sigma} \right)^* \hat{S}_{\hat{m},l}^{\sigma} \hat{S}_{\hat{p},l}^{\sigma \tau - 1} S_{0,1} = \mathcal{N}_{\hat{p},\hat{m}}^{\hat{n}}. \quad (4.9)
\]

The case when one of the two boundary states is of the form (4.5), is very similar. In this case one finds that

\[
\sigma \langle \langle \hat{n}, \pm \rangle \| q^{\frac{1}{2}\left(L_0 + L_0 - c/12\right)} \| \hat{m} \rangle \rangle^\tau = \frac{1}{2} \sum_{\hat{p}} (\mathcal{N}_{\hat{p},\hat{m}}^{\hat{n}} + \mathcal{N}_{\hat{p},\hat{m}}^{J\hat{n}}) \chi_{\hat{p}}(\tilde{q})
\]

where we have used (4.9) in the second line, and the invariance of \( \hat{n} \) in the last.

Finally, suppose that both boundary states are of the form (4.5), with ‘in’-state \( \langle \hat{n}, \pm \rangle_{\tilde{t}_0}^\sigma \) and ‘out’-state \( \| \hat{m}, \pm \rangle_{\tilde{t}_0}^\tau \). By the same argument as in (4.10), the contribution from the first sum in (4.5) gives rise to

\[
\frac{1}{2} \sum_{\hat{p}} \mathcal{N}_{\hat{p},\hat{m}}^{\hat{n}} \chi_{\hat{p}}(\tilde{q}). \quad (4.11)
\]
On the other hand, the contribution from the second sum in (4.5) depends on whether there are weights $l$ in $\hat{su}(2)_{\chi}$ that satisfy simultaneously (4.4) with $\sigma$ and $\tau$. If such a weight $l_0'$ exists, then the contribution from the second sum in (4.5) gives

$$(-1)^{\hat{m}-(l_0-l_0')}\frac{1}{2}\sum_{\hat{p}}\left(\hat{S}_{\hat{n},1}^{\sigma}\right)^* \frac{\hat{S}_{l_0'}^{\sigma \tau \nu} \hat{S}_{l_0',l_0'}^{-1}}{S_{0,1}} \chi_{\hat{p}}(\tilde{q})$$

$$= \frac{1}{2}(-1)^{\hat{m}-(l_0-k_0')} \sum_{\hat{p}}(-1)^{\hat{p}} l_0' \sum_{l_0} \left(\hat{S}_{\hat{n},1}^{\sigma}\right)^* \frac{\hat{S}_{\hat{n},l_0}^{\sigma \tau} \hat{S}_{\hat{n},l_0'}^{-1}}{S_{0,1}} \chi_{\hat{p}}(\tilde{q})$$

$$= \frac{1}{2}(-1)^{\hat{m}-(l_0-k_0')} \sum_{\hat{p}}(-1)^{\hat{p}} l_0' \sum_{l_0} \left(\hat{S}_{\hat{n},1}^{\sigma}\right)^* \frac{\hat{S}_{\hat{n},l_0}^{\sigma \tau} \hat{S}_{\hat{n},l_0'}^{-1}}{S_{0,1}} \chi_{\hat{p}}(\tilde{q}),$$

where we have used (4.6). Depending on whether the signs of the two boundary states are the same or opposite, (4.12) has to be added or subtracted from (4.11). In either case, taking both terms together we obtain a non-negative integer linear combination of characters in the open string.

Finally, if none of the $\hat{su}(2)_{\chi}$ weights satisfies (4.4) simultaneously for $\sigma$ and $\tau$, the contribution from the second sum in (4.3) vanishes. If this is the case, then one can show that for one of the sets $\mathcal{F}_i$ defined in appendix B, $m$ in (B.6) is non-zero. This implies that the coefficients (4.11) are actually even integers. The coefficients in (4.11) are then again integers, thus proving the Cardy condition.

5. Some generalisations

Up to now we have only discussed the D-branes that preserve the $\hat{su}(2)_{\chi}$ symmetry up to outer automorphisms, i.e. that satisfy (1.1) with $\sigma \in S_5$. It is relatively straightforward to include also inner automorphisms. The group of inner automorphisms is isomorphic to $SU(2)^5$, and it acts on the algebra $\hat{su}(2)_{\chi}$ by conjugation. The induced action on the boundary states is simply given by the global action of $SU(2)^5$ on the right-moving states, say. Since this action corresponds to a marginal deformation by a local field, the resulting boundary states satisfy the relevant consistency conditions [38].

It was shown in [39] that the D-branes that preserve the conformal symmetry Vir for $\hat{su}(2)_{1}$, necessarily preserve the full chiral algebra $\hat{su}(2)_{1}$ up to an inner automorphism. It therefore follows that the D-branes that preserve $\hat{su}(2)_{\chi}$ up to an inner automorphism in $SU(2)^5$ account already for all D-branes that preserve Vir$^5$. We have therefore managed
to construct all D-branes that preserve Vir$^5$ up to the outer automorphisms that are isomorphic to $S_5$.

The orbifold we have been considering acts as (some version of) T-duality on the four-torus part. One would therefore expect that the theory has D-branes that are simply superpositions of Dp-D(4-p) branes. In addition to this T-duality, there is the shift action along the additional circle; the pairs of branes will therefore be localised at opposite points along the fifth circle.

The combinations of Dp-D(4-p) branes preserve the full chiral algebra (2.12) up to certain inner automorphisms. In terms of the original description of the orbifold theory (2.10), the corresponding boundary states therefore only involve Ishibashi states from the untwisted sector. Following the discussion leading to (3.3), they correspond to (25)-twisted boundary states in the second formulation of the theory. Since $\sigma = (25)$ does not have any fixed points (in the sense discussed above in section 4), the corresponding boundary states then also only involve Ishibashi states from the first sum in (3.3). Of the $32 = 2^5$ representations of $\hat{su}(2)_1^5$, 16 = $2^4$ representations are invariant under the action of (25). There are therefore sixteen (25)-twisted representations of $\hat{su}(2)_1^5$, each of which lies in an orbit of length two under the action of $J$. Thus there are eight different boundary states that satisfy (3.1) with $\sigma = id$. We want to show that these eight boundary states can be identified with combinations of D0-D4 branes, where the position of the D4-brane is shifted along the fifth circle relative to the position of the D0-brane.

Recall from appendix B that the (25)-twisted $S$-matrix is simply a product of four $S$-matrices of $\hat{su}(2)_1$. One of the (25)-twisted representations, that we shall denote by $\hat{0}$ in the following, has thus the property that

$$\hat{S}_{0,1}^{(25)} = \frac{1}{\sqrt{2}} = \frac{1}{4},$$

for all (25)-invariant representations $\mathbf{l}$ of $\hat{su}(2)_1^5$. The boundary state associated to $\hat{0}$ via (4.2), $|\mathbf{0}\rangle^{(25)}$, is therefore simply the sum (with overall normalisation $2^{-1/4}$ but without signs) of the eight (25)-twisted Ishibashi states coming from the first sum of (3.3). Alternatively, it is the same sum over the eight (untwisted) Ishibashi states coming from the untwisted sector of (2.10).

The same boundary state can now be obtained starting from the original $\hat{so}(8)_1 \oplus \hat{su}(2)_1$ theory as follows. Consider the Cardy boundary state associated to the representation $(o, 0)$ of $\hat{so}(8)_1 \oplus \hat{su}(2)_1$. From a geometrical point of view, this boundary
state describes a single D0-brane. By the usual Cardy formula the boundary state is the sum (with overall normalisation $2^{-3/4}$ but without signs) over all eight Ishibashi states of $\widehat{\mathfrak{so}}(8) \oplus \widehat{\mathfrak{su}}(2)$. Because of the decomposition (2.3), each such Ishibashi state is the sum of two $\widehat{\mathfrak{su}}(2)$ Ishibashi states, and therefore the Cardy state corresponding to $(o,0)$ is the sum of sixteen Ishibashi states of $\widehat{\mathfrak{so}}(8) \oplus \widehat{\mathfrak{su}}(2)$ (with overall normalisation $2^{-3/4}$ but without signs). Finally, when we impose the orbifold projection, only half of these Ishibashi states survive, and the overall normalisation becomes $\sqrt{2} 2^{-3/4} = 2^{-1/4}$. The resulting boundary state therefore agrees with the boundary state $\langle \hat{0} \rangle^{(25)}$ described above.

On the other hand, the action of the orbifold acts geometrically on the $SO(8)$ lattice and the circle, and simply maps the D0-brane to a D4-brane located at the opposite point of the extra circle.

We have therefore shown that the boundary state $\langle \hat{0} \rangle^{(25)}$ describes the boundary state

$$\langle \hat{0} \rangle^{(25)} = \frac{1}{\sqrt{2}} (\langle D0, x \rangle + \langle D4, x + \pi R_5 \rangle).$$

(5.2)

Similarly, the other seven (25)-twisted boundary states can be obtained by the same construction starting with the Cardy state corresponding to some other representation of $\widehat{\mathfrak{so}}(8) \oplus \widehat{\mathfrak{su}}(2)$; they therefore describe combinations of D0-D4 branes where the position of the D0-brane is at a different point in the $SO(8)$ torus. It should also be clear that the combinations of Dp-D(4-p) branes can be obtained from the above by the action of the inner automorphism of $SU(2)^5$.

6. Conclusions

In this paper we have determined the boundary states for an asymmetric orbifold of the bosonic string. More precisely, we have constructed all the boundary states that preserve five copies of the Virasoro algebra at $c = 1$ up to permutations. The corresponding D-branes include the usual superpositions of Dp-D(4-p) branes that only couple to the untwisted sector of the asymmetric orbifold. However, we have also constructed branes that couple to twisted sector states.

The boundary states that only couple to the untwisted sector of the orbifold involve at most eight Ishibashi states; their overall normalisation (which is proportional to the tension) is therefore at least $2^{-1/4}$. These boundary states therefore do not describe the ‘lightest’ D-branes. Indeed, some of the boundary states (for example those that correspond
to $\sigma = (25)$ in the first formulation of the theory) involve sixteen Ishibashi states, and the overall normalisation is then $2^{-3/4}$. It would be interesting to understand the geometrical interpretation of these boundary states.

The techniques we have employed in our construction should generalise to other asymmetric orbifolds. In particular, it would be interesting to apply these ideas to superstring orbifolds, for example the one considered in [4].

Acknowledgments
We thank Ilka Brunner, Peter Goddard, Axel Kleinschmidt, Christian Stahn and in particular Andreas Recknagel for useful discussions.

MRG is grateful to the Royal Society for a University Research Fellowship. He thanks the Institute for Advanced Study for hospitality while this paper was being completed; his research there was supported by a grant in aid from the Funds for Natural Sciences. S.S.-N. is grateful to St. John’s College, Cambridge, for a Jenkins Scholarship. We also acknowledge partial support from the PPARC Special Programme Grant ‘String Theory and Realistic Field Theory’, PPA/G/S/1998/0061 and the EEC contract HPRN-2000-00122.

Appendix A. Character and Theta-function conventions

In this paper we are using the theta functions

$$\Theta_{m,k}(q) = \sum_{n \in \mathbb{Z} + \frac{m}{2k}} q^{n^2 k}, \quad (A.1)$$

which have the modular transformation properties

$$S \Theta_{m,k} = \left( \frac{i\tau}{2k} \right)^{\frac{1}{2}} \sum_{n \in \mathbb{Z}/2k} e^{-inm\pi/k} \Theta_{n,k} \quad \text{and} \quad T \Theta_{m,k} = e^{i\pi m^2/2k} \Theta_{m,k}. \quad (A.2)$$

The two characters of $\widehat{su}(2)_1$ corresponding to $j = 0$ and $j = 1/2$ are denoted by $\chi_0$ and $\chi_1$, respectively. They are related to the theta-functions (A.1) as

$$\chi_0(q) = \frac{\Theta_{0,1}(q)}{\eta(q)}, \quad \chi_1(q) = \frac{\Theta_{1,1}(q)}{\eta(q)}, \quad (A.3)$$
where $\eta(q)$ is the Dedekind eta-function

$$\eta(q) = q^{-1/24} \prod_{n=1}^{\infty} (1-q^n). \tag{A.4}$$

The modular $S$-matrix of $\widehat{su}(2)_1$ is then given by

$$S_{\widehat{su}(2)_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & e^{-i\pi/2} \\ 1 & -1 & e^{i\pi/2} & e^{-i\pi/2} \\ 1 & -1 & e^{-i\pi/2} & e^{i\pi/2} \end{pmatrix}. \tag{A.5}$$

The functions (A.1) are related to the usual $\theta_i$ Jacobi theta-functions as

$$\begin{align*}
\theta_2(\tau) &= \Theta_{1,1}(\tau/2) = \sqrt{\Theta_{0,1}(\tau)\Theta_{1,1}(\tau)} = \Theta_{1,2}(\tau) + \Theta_{3,2}(\tau) \\
\theta_3(\tau) &= \Theta_{0,1}(\tau/2) = \Theta_{0,4}(\tau/2) + \Theta_{4,4}(\tau/2) = \sqrt{\Theta_{0,1}^2(\tau) + \Theta_{1,1}^2(\tau)} \\
&= \Theta_{0,2}(\tau) + \Theta_{2,2}(\tau) \tag{A.6} \\
\theta_4(\tau) &= \Theta_{0,1}(\tau/2, z = 1/2) = \Theta_{0,4}(\tau/2) - \Theta_{4,4}(\tau/2) = \sqrt{\Theta_{0,1}^2(\tau) - \Theta_{1,1}^2(\tau)} \\
&= \Theta_{0,2}(\tau) - \Theta_{2,2}(\tau).
\end{align*}$$

In terms of these, the (specialised) characters for $\widehat{so}(2p)_1$, $p \in \mathbb{N}$, are

$$\begin{align*}
O_{2p} &= \frac{\theta_3^p + \theta_4^p}{2\eta^p}, \quad V_{2p} = \frac{\theta_3^p - \theta_4^p}{2\eta^p}, \quad S_{2p} = C_{2p} = \frac{\theta_2^p}{2\eta^p}, \tag{A.7}
\end{align*}$$

and their modular transformation matrices are

$$\begin{align*}
S_{\widehat{so}(2p)} &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & e^{-i\pi/2} \\ 1 & -1 & e^{i\pi/2} & e^{-i\pi/2} \\ 1 & -1 & -e^{-i\pi/2} & e^{i\pi/2} \end{pmatrix}, \quad T_{\widehat{so}(2p)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 0 & e^{i\pi/4} \end{pmatrix}. \tag{A.8}
\end{align*}$$

Appendix B. The Cardy condition for the permutation-twisted boundary states

Let us first compute the $\sigma$-twisted $S$-matrix for $\widehat{su}(2)_1^n$, where $\sigma \in S_n$. Suppose first that $\sigma$ consists of a single non-trivial cycle of length $k > 1$. Then there are $2^{n-k+1}$ $\sigma$-invariant representations of $\widehat{su}(2)_1^n$; they can be labelled by $(l_1, \ldots, l_{n-k}, l)$, where $l$ is the representation label for the $k$ representations that are permuted among each other by $\sigma$. The twining character is then

$$\chi_{(l_1, \ldots, l_{n-k}, l)}^{(\sigma)}(q) = \chi_l(q^k) \prod_{i=1}^{n-k} \chi_{l_i}(q). \tag{B.1}$$
Upon a modular transformation this becomes

\[ \chi_{(l_1, \ldots, l_{n-k}, l)}(\sigma) (q) = \sum_{(m_1, \ldots, m_{n-k}, \tilde{m})} S_{l, \tilde{m}} \chi_{\tilde{m}}(\tilde{q}^{1/k}) \prod_{i=1}^{n-k} S_{l_i, m_i} \chi_{m_i}(\tilde{q}), \]

(B.2)

where all S-matrices here are the S-matrix of \( \tilde{su}(2)_1 \). The characters \( \chi_{\tilde{m}}(\tilde{q}^{1/k}) \) can be identified with the \( \sigma \)-twisted characters of \( \tilde{su}(2)_{1} \), and thus the twisted S-matrix is simply a suitable product of S-matrices of \( \tilde{su}(2)_1 \). It is easy to see that this argument generalises directly to the case of an arbitrary permutation: if \( \sigma \) has \( c_\sigma \) cycles (counting trivial cycles) then the \( \sigma \)-invariant representations are labelled by \( c_\sigma \) labels (each of which can take the values 0, 1), and the \( \sigma \)-twisted S-matrix is the product of \( c_\sigma \) S-matrices of \( \tilde{su}(2)_1 \).

With these preparations we can now show that the coefficients (3.17)

\[ \mathcal{N}_{\tilde{p}, \tilde{m}}^{\tilde{n}} = \sum_{l=0}^{T} \frac{\left( \hat{S}_{\tilde{n}, l}^{\sigma} \right)^* \hat{S}_{\tilde{m}, l}^{\tau} \hat{S}_{\tilde{p}, l}^{\sigma \tau^{-1}}}{S_{0, l}} \]

(B.3)

define indeed non-negative integers. (The following argument is similar to the argument given in appendix A of [25].) The idea of the argument is to reduce this expression to a product of conventional fusion rules of \( \tilde{su}(2)_1 \), using the Verlinde formula. The main problem in doing so is that the sum over \( l \) only runs over those indices \( l = (l_1, \ldots, l_n) \) that are simultaneously invariant under \( \sigma \) and \( \tau \). Furthermore, if we write out the twisted S-matrices in the numerator in terms of the S-matrices of \( \tilde{su}(2)_1 \), there are only \( c_\sigma + c_\tau + c_{\sigma \tau^{-1}} \) S-matrices, whereas we would need \( 3n \) S-matrices in order to group them into Verlinde-formula expressions.

For each \( i \in \{1, \ldots, n\} \) let us denote by \( \mathcal{F}_i \) the subset of labels in \( \{1, \ldots, n\} \) that have to take the same value as \( l_i \) in the sum over \( l \) above, i.e. \( l_j = l_i \) for \( j \in \mathcal{F}_i \). In order to prove the above formula we can consider each such set at a time (since the total formula will just be the product of the expressions corresponding to each such set). Without loss of generality, let \( i = 1 \), and let \( \mathcal{F}_1 = \{1, \ldots, r\} \). Let us restrict \( \sigma, \tau \) and \( \sigma \tau^{-1} \) to the set \( \mathcal{F}_1 = \{1, \ldots, r\} \), and let \( s \) be the number of cycles of \( \sigma \) among \( \mathcal{F}_1 \) (including trivial cycles). Similarly, define \( t \) to be the number of cycles of \( \tau \), and \( u \) the number of cycles of \( \sigma \tau^{-1} \). The contribution to (B.3) coming from \( \mathcal{F}_1 \) is then

\[ \mathcal{N}' = \sum_{l=0,1} \frac{S_{n_1, l} \cdots S_{n_s, l} S_{m_1, l} \cdots S_{m_t, l} S_{p_1, l} \cdots S_{p_u, l}}{S_{0, l}} \]

(B.4)

17
where the product in the denominator contains \( r \) powers of \( S_{0,l} \), and we have used that \( S \) is real. Next we want to turn this into \( r \) sums by inserting the identity

\[
\sum_a S_{a,l} S_{a,l'} = \delta_{l,l'} \tag{B.5}
\]

\( r - 1 + m \) times, where \( m \) is the non-negative integer defined by

\[
s + t + u + 2m = r + 2. \tag{B.6}
\]

(As we shall see momentarily, (B.6) defines indeed a non-negative integer \( m \).) We can then distribute the remaining factors of \( S \) as

\[
N' = \sum_{l_1, \ldots, l_r} \sum_{a_1, \ldots, a_{r-1} + m} \frac{S_{n_1,l_1} S_{m_1,l_1} S_{a_1,l_1} S_{a_1,l_2} S_{n_2,l_2} S_{a_2,l_2}}{S_{0,l_1} S_{0,l_2} \ldots S_{0,l_r}} S_{a_{r-1} + m,l_r} S_{m_1,l_r} S_{p_a,l_r}. \tag{B.7}
\]

Each sum over \( l_i \) gives now a fusion rule coefficient via the Verlinde formula, and it is therefore manifest that \( N' \) is a non-negative integer.

It therefore only remains to prove (B.6). This can be done by induction on \( r \). The case \( r = 1 \) is trivial. Assume therefore that the statement holds for \( r - 1 \), and let \( \sigma \) and \( \tau \) be as above. Then we can find transpositions \( (jr) \) and \( (kr) \) (where either \( j \) or \( k \) but not both may be equal to \( r \)) so that \( \sigma = (jr)\sigma' \) and \( \tau = (kr)\tau' \), where \( \sigma' \) and \( \tau' \) leave \( r \) invariant. Thus \( \sigma' \) and \( \tau' \) satisfy the assumptions of the statement with \( r - 1 \), and we have that \( s' + t' + u' + 2m' = r + 1 \), where \( s', t' \), and \( u' \) are defined in the obvious manner. By construction

\[
s' = \begin{cases} s & \text{if } j \neq r \\ s - 1 & \text{if } j = r \end{cases} \quad t' = \begin{cases} t & \text{if } k \neq r \\ t - 1 & \text{if } k = r \end{cases}. \tag{B.8}
\]

The number of cycles of a permutation is the same in each conjugacy class, and therefore \( u \) is equal to the number of cycles of the permutation

\[
(kr)\sigma\tau^{-1}(kr) = (kr)(jr)\sigma'\tau'^{-1}. \tag{B.9}
\]

Now the product \((kr)(jr)\) is equal to

\[
(kr)(jr) = \begin{cases} (kr) & \text{if } j = r \\ (jr) & \text{if } k = r \\ \text{id} & \text{if } j = k \neq r \\ (jkr) & \text{otherwise.} \end{cases} \tag{B.10}
\]

18
Thus it follows from \((\text{B.9})\) that
\[
u' = \begin{cases} 
u & \text{if } j = r \text{ or } k = r \\ u - 1 & \text{if } j = k \neq r. \end{cases} \tag{\text{B.11}}
\]

So if \(j = r\) or \(k = r\) or \(j = k \neq r\), then \(s' + t' + u' = s + t + u - 1\), and the induction step follows (with \(m = m'\)). This leaves us with analysing the case when \(j \neq k\) with neither \(j\) nor \(k\) equal to \(r\). The answer then depends on whether \(j\) and \(k\) lie in the same cycle of \(\sigma'\tau' - 1\) or whether they do not. In the former case, the permutation in \((\text{B.9})\) is
\[
(jkr)(jv_1v_2\cdots v_Lkw_1w_2\cdots w_M)(\text{other cycles})
\]
\[
= (jv_1v_2\cdots v_Lr)(kw_1w_2\cdots w_M)(\text{other cycles})
\]
and thus \(u' = u - 1\), and hence \(s' + t' + u' = s + t + u - 1\). As before the induction step then follows with \(m = m'\). In the other case we have instead of \((\text{B.12})\)
\[
(jkr)(jv_1v_2\cdots v_L)(kw_1w_2\cdots w_M)(\text{other cycles})
\]
\[
= (jv_1v_2\cdots v_Lkw_1w_2\cdots w_Mr)(\text{other cycles})
\]
and thus \(u' = u + 1\), and hence \(s' + t' + u' = s + t + u + 1\). In this case the induction step follows with \(m = m' + 1\). This proves the statement \((\text{B.6})\).

**Appendix C. Twisted fusion rules**

In this appendix we want to demonstrate that for a simple example \((3.17)\) does indeed describe the twisted fusion rules. The example we want to consider is \(\widehat{su}(2)_1^2\), where \(\sigma\) is the transposition that exchanges the two factors of \(\widehat{su}(2)_1\). As was explained in the previous appendix, there are two \(\sigma\)-twisted representations. Every \(\sigma\)-twisted representation of \(\widehat{su}(2)_1^2\) defines an untwisted representation of \(\widehat{su}(2)_2 \oplus \text{Vir}_{1/2}\), where \(\text{Vir}_{1/2}\) is the Virasoro algebra at \(c = 1/2\). In general, an irreducible \(\sigma\)-twisted representation of \(\widehat{su}(2)_1^2\) will contain a number of irreducible representations of \(\widehat{su}(2)_2 \oplus \text{Vir}_{1/2}\).

\[\uparrow\] In the general case where \(\sigma\) is a permutation acting on \(\widehat{su}(2)_1^n\) with \(c\) cycles of length \(l_i \geq 1\), \(i = 1, \ldots, c\), the relevant chiral algebra is
\[
\bigoplus_{i=1}^c \widehat{su}(2)_{l_i} \oplus M_{rem},
\]
where \(M_{rem}\) is a chiral algebra of suitable central charge.
In order to determine the characters of the twisted representations let us consider the twining characters

\[
\text{Tr}_{\mathcal{H}_0 \otimes \mathcal{H}_0} (\sigma q^{L_0 - \frac{c}{24}}) = \chi_0(q^2) = \frac{\Theta_{0,2}(q)\eta(q^2)}{\eta(q^2)} = \chi_0^{(2)}(q) \chi_0(q) - \chi_2^{(2)}(q) \chi_{1/2}(q), \\
\text{Tr}_{\mathcal{H}_1 \otimes \mathcal{H}_1} (\sigma q^{L_0 - \frac{c}{24}}) = \chi_1(q^2) = \frac{\Theta_{2,2}(q)\eta(q^2)}{\eta(q^2)} = -\chi_0^{(2)}(q) \chi_{1/2}(q) + \chi_2^{(2)}(q) \chi_0(q),
\]

which we have written in terms of (conventional) characters of \(\tilde{su}(2)_2 \oplus \text{Vir}_{1/2}\). (The characters of \(\tilde{su}(2)_2\) are denoted by \(\chi_j^{(2)}(q)\), while the characters of \(\text{Vir}_{1/2}\) are \(\chi_h(q)\).) Upon an S-modular transformation we then find

\[
S \chi_{\mathcal{H}_0 \otimes \mathcal{H}_0}^{(\sigma)} = \frac{(\theta_3^{3/2} \theta_2^{1/2} + \theta_3^{1/2} \theta_2^{3/2})}{2\eta^2} \\
= \frac{1}{\sqrt{2}} \left( (\chi_0^{(2)} + \chi_2^{(2)}) \chi_{1/16} + \chi_1^{(2)}(\chi_0 + \chi_{1/2}) \right), \\
S \chi_{\mathcal{H}_1 \otimes \mathcal{H}_1}^{(\sigma)} = \frac{(\theta_3^{3/2} \theta_2^{1/2} - \theta_3^{1/2} \theta_2^{3/2})}{2\eta^2} \\
= \frac{1}{\sqrt{2}} \left( (\chi_0^{(2)} + \chi_2^{(2)}) \chi_{1/16} - \chi_1^{(2)}(\chi_0 + \chi_{1/2}) \right).
\]

The characters of the twisted representations are therefore \(\chi_U = \chi_1^{(2)}(\chi_0 + \chi_{1/2})\) and \(\chi_V = (\chi_0^{(2)} + \chi_2^{(2)}) \chi_{1/16}\), and the \(\hat{S}\)-matrix is given by

\[
\hat{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]

in agreement with the discussion of the previous appendix. Furthermore, the two twisted representations \(U\) and \(V\) decompose with respect to \(\tilde{su}(2)_2 \oplus \text{Vir}_{1/2}\) as

\[
U = \mathcal{H}_1^{(2)} \otimes (\mathcal{H}_0 \oplus \mathcal{H}_{1/2}) , \quad V = \left( \mathcal{H}_0^{(2)} \oplus \mathcal{H}_2^{(2)} \right) \otimes \mathcal{H}_{1/16}.
\]

The conjectured formula for the twisted fusion rules (B.17) now predicts that the fusion rules are

\[
\mathcal{N}_{(0,0)} = \mathcal{N}_{(1,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{N}_{(1,0)} = \mathcal{N}_{(0,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

where \((l, m)\) labels the untwisted representation of \(\tilde{su}(2)_2^2\), and the matrix acts on the space of twisted representations with basis \(U\) and \(V\). These fusion rules are in agreement (and could have been derived) from the description of these representations in terms of \(\tilde{su}(2)_2 \oplus \text{Vir}_{1/2}\). For example, the representation \((1, 1)\) corresponds to \((j = 0, h = 1/2) \oplus (j = 2, h = 0)\) of \(\tilde{su}(2)_2 \oplus \text{Vir}_{1/2}\), and therefore the fusion of \((1, 1)\) with \(U(V)\) can only contain \(U(V)\). Similarly, \((1, 0)\) corresponds to \((j = 1, h = \frac{1}{16})\) of \(\tilde{su}(2)_2 \oplus \text{Vir}_{1/2}\), and the fusion of \((1, 0)\) with \(U(V)\) can therefore only contain \(V(U)\). The other two cases are identical.
References

[1] K.S. Narain, M.H. Sarmadi, C. Vafa, *Asymmetric orbifolds*, Nucl. Phys. **B288**, 551 (1987).
[2] S. Kachru, J. Kumar, E. Silverstein, *Vacuum energy cancellation in a non-supersymmetric string*, Phys. Rev. **D59**, 106004 (1999); [hep-th/9807076](http://arxiv.org/abs/hep-th/9807076).
[3] S. Kachru, E. Silverstein, *4d conformal theories and strings on orbifolds*, Phys. Rev. Lett. **80**, 4855 (1998); [hep-th/9802183](http://arxiv.org/abs/hep-th/9802183).
[4] J.A. Harvey, *String duality and non-supersymmetric strings*, Phys. Rev. **D59**, 026002 (1999); [hep-th/9807213](http://arxiv.org/abs/hep-th/9807213).
[5] E. Silverstein, *(A)dS backgrounds from asymmetric orientifolds*, [hep-th/0106209](http://arxiv.org/abs/hep-th/0106209).
[6] A. Maloney, E. Silverstein, A. Strominger, *De Sitter space in noncritical string theory*, hep-th/0205316.
[7] R. Blumenhagen, L. Görlich, *Orientifolds of non-supersymmetric asymmetric orbifolds*, Nucl. Phys. **B551**, 601 (1999); [hep-th/9812158](http://arxiv.org/abs/hep-th/9812158).
[8] C. Angelantonj, I. Antoniadis, K. Foerger, *Non-supersymmetric type I strings with zero vacuum energy*, Nucl. Phys. **B555**, 116 (1999); [hep-th/9904092](http://arxiv.org/abs/hep-th/9904092).
[9] C. Angelantonj, R. Blumenhagen, M.R. Gaberdiel, *Asymmetric orientifolds, brane supersymmetry breaking and non-BPS branes*, Nucl. Phys. **B589**, 545 (2000); [hep-th/0006033](http://arxiv.org/abs/hep-th/0006033).
[10] B. Körs, *D-brane spectra of nonsupersymmetric, asymmetric orbifolds and nonperturbative contributions to the cosmological constant*, Journ. High Energy Phys. **9911**, 028 (1999); [hep-th/9907007](http://arxiv.org/abs/hep-th/9907007).
[11] R. Blumenhagen, L. Görlich, B. Körs, D. Lüst, *Asymmetric orbifolds, noncommutative geometry and type I string vacua*, Nucl. Phys. **B582**, 44 (2000); [hep-th/0003024](http://arxiv.org/abs/hep-th/0003024).
[12] T. Gannon, M.A. Walton, *Heterotic modular invariants and level-rank duality*, Nucl. Phys. **B536**, 553 (1998); [hep-th/9804040](http://arxiv.org/abs/hep-th/9804040).
[13] I. Brunner, A. Rajaraman, M. Rozali, *D-branes on asymmetric orbifolds*, Nucl. Phys. **B558**, 205 (1999); [hep-th/9905024](http://arxiv.org/abs/hep-th/9905024).
[14] M. Gutperle, *Non-BPS D-branes and enhanced symmetry in an asymmetric orbifold*, Journ. High Energy Phys. **0008**, 036 (2000); [hep-th/0007126](http://arxiv.org/abs/hep-th/0007126).
[15] L.-S. Tseng, *A note on c=1 Virasoro boundary states and asymmetric shift orbifolds*, Journ. High Energy Phys. **0204**, 051 (2002); [hep-th/0201254](http://arxiv.org/abs/hep-th/0201254).
[16] B. Craps, M.R. Gaberdiel, J.A. Harvey, *Monstrous branes*, to appear in Commun. Math. Phys., [hep-th/0202074](http://arxiv.org/abs/hep-th/0202074).
[17] H. Ooguri, Y. Oz, Z. Yin, *D-Branes on Calabi-Yau spaces and their mirrors*, Nucl. Phys. **B477**, 407 (1996); [hep-th/9606112](http://arxiv.org/abs/hep-th/9606112).
[18] L. Birke, J. Fuchs, C. Schweigert, *Symmetry breaking boundary conditions and WZW orbifolds*, Adv. Theor. Math. Phys. **3**, 671 (1999); [hep-th/9905033](http://arxiv.org/abs/hep-th/9905033).
[19] J. Fuchs, C. Schweigert, *Symmetry breaking boundaries. I: General theory*, Nucl. Phys. B558, 419 (1999); [hep-th/9902132](http://arxiv.org/abs/hep-th/9902132).

[20] J. Fuchs, C. Schweigert, *Symmetry breaking boundaries. II: More structures, examples*, Nucl. Phys. B568, 543 (2000); [hep-th/9908025](http://arxiv.org/abs/hep-th/9908025).

[21] J. Fuchs, C. Schweigert, *Solitonic sectors, alpha-induction and symmetry breaking boundaries*, Phys. Lett. B490, 163 (2000); [hep-th/0006181](http://arxiv.org/abs/hep-th/0006181).

[22] M.R. Gaberdiel, T. Gannon, *Boundary states for WZW models*, Nucl. Phys. B639, 471 (2002); [hep-th/0202067](http://arxiv.org/abs/hep-th/0202067).

[23] J. M. Maldacena, G. W. Moore, N. Seiberg, *Geometrical interpretation of D-branes in gauged WZW models*, Journ. High Energy Phys. 0107, 046 (2001); [hep-th/0105033](http://arxiv.org/abs/hep-th/0105033).

[24] T. Quella, V. Schomerus, *Symmetry breaking boundary states and defect lines*, Journ. High Energy Phys. 0206, 028 (2002); [hep-th/0203161](http://arxiv.org/abs/hep-th/0203161).

[25] A. Recknagel, *Permutation branes*, [hep-th/0208119](http://arxiv.org/abs/hep-th/0208119).

[26] A. Klemm, M.G. Schmidt, *Orbifolds by cyclic permutations of tensor product conformal field theories*, Phys. Lett. B245, 53 (1990).

[27] P. Bantay, *Characters and modular properties of permutation orbifolds*, Phys. Lett. B419, 175 (1998); [hep-th/9708120](http://arxiv.org/abs/hep-th/9708120).

[28] P. Bantay, *Permutation orbifolds*, Nucl. Phys. B633, 365 (2002); [hep-th/9910079](http://arxiv.org/abs/hep-th/9910079).

[29] K. Barron, C. Dong, G. Mason, *Twisted sectors for tensor product vertex operator algebras associated to permutation groups*, Commun. Math. Phys. 227, 349 (2002); [math.QA/9803118](http://arxiv.org/abs/math.QA/9803118).

[30] C. Vafa, E. Witten, *Dual string pairs with N=1 and N=2 supersymmetry in four dimensions*, Nucl. Phys. Proc. Suppl. 46, 225 (1996); [hep-th/9507050](http://arxiv.org/abs/hep-th/9507050).

[31] A.N. Schellekens, S. Yankielowicz, *Extended chiral algebras and modular invariant partition functions*, Nucl. Phys. B327, 673 (1989).

[32] K. Intriligator, *Bonus symmetry in conformal field theory*, Nucl. Phys. B332, 541 (1990).

[33] A.N. Schellekens, S. Yankielowicz, *Simple currents, modular invariants and fixed points*, Int. J. Mod. Phys. A5, 2903 (1990).

[34] M. Kreuzer, A.N. Schellekens, *Simple currents versus orbifolds with discrete torsion: a complete classification*, Nucl. Phys. B411, 97 (1994); [hep-th/9306143](http://arxiv.org/abs/hep-th/9306143).

[35] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, Nucl. Phys. B324, 581 (1989).

[36] P. Goddard, D.I. Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, Int. J. Mod. Phys. A1, 303 (1986).

[37] T. Quella, I. Runkel, C. Schweigert, *An algorithm for twisted fusion rules*, [math.qa/0203133](http://arxiv.org/abs/math.qa/0203133).

[38] A. Recknagel, V. Schomerus, *Boundary deformation theory and moduli spaces of D-branes*, Nucl. Phys. B545, 233 (1999); [hep-th/9811237](http://arxiv.org/abs/hep-th/9811237).

[39] M.R. Gaberdiel, A. Recknagel, G.M.T. Watts, *The conformal boundary states for SU(2) at level 1*, Nucl. Phys. B626, 344 (2002); [hep-th/0108102](http://arxiv.org/abs/hep-th/0108102).