INTEGRABLE CONFORMAL FIELD THEORY IN FOUR DIMENSIONS
AND FOURTH-RANK GEOMETRY

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Abstract. We consider the conformal properties of geometries described by higher-rank line elements. A crucial role is played by the conformal Killing equation (CKE). We introduce the concept of null-flat spaces in which the line element can be written as $ds^r = r!d\zeta_1 \cdots d\zeta_r$. We then show that, for null-flat spaces, the critical dimension, for which the CKE has infinitely many solutions, is equal to the rank of the metric. Therefore, in order to construct an integrable conformal field theory in 4 dimensions we need to rely on fourth-rank geometry. We consider the simple model $\mathcal{L} = \frac{1}{4} G^{\mu \nu \lambda \rho} \partial_\mu \phi \partial_\nu \phi \partial_\lambda \phi \partial_\rho \phi$ and show that it is an integrable conformal model in 4 dimensions. Furthermore, the associated symmetry group is $Vir^4$.

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1. Introduction

It is expected that at very high energies physical processes become scale invariant. In fact, in such regime all masses involved in any physical process are small in comparison with the energies, and they can be put equal to zero. There is therefore, no fundamental mass setting the scale of energies, and therefore, all physical processes must be scale invariant.

The above speculation is confirmed by several experiments, as deep inelastic scattering, which reveal that, at very high energies, physical phenomena become scale invariant. Therefore, any field theory attempting to provide a unified description of interactions must, at high energies, exhibit this behaviour.

It can furthermore be shown, from a mathematical point of view, that scale invariance implies conformal invariance.

Conformal field theories can be constructed in any dimension but only for \( d = 2 \) they exhibit a radically different behaviour. Chief among them is the fact that there exists an infinite number of conserved quantities making the theory an exactly solvable, or integrable, theory. The associated symmetry group becomes infinite-dimensional and after convenient parametrisation of its generators in terms of Fourier components is the familiar Virasoro algebra. To be more precise the group is \( \text{Vir} \oplus \text{Vir} \), with one Virasoro algebra associated to each null space-time direction. Furthermore, two-dimensional conformal integrable field theories hold several other properties which are missing when formulated into higher-dimensional space-times. Chief among them are those properties, such as renormalisability, which make of its quantum field theoretical version a mathematically consistent model. All the previous facts gave rise to the success of string theories in the recent years; cf. ref. 1 for further details.

Let us closer analyse the situation. In a conformal field theory the symmetry generators are the conformal Killing vectors. They are solutions of the conformal Killing equation

\[
\partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu} - \frac{2}{d} g_{\mu\nu} \partial_{\lambda} \xi^\lambda = 0. \tag{1.1}
\]

Only for two-dimensional spaces the solutions are infinitely many giving rise to an infinite-dimensional symmetry group. A closer analysis of the conformal Killing equation shows that this critical dimension is closely related to the rank of the metric. In fact, since the metric is a second-rank tensor, in the conformal Killing equation will appear two terms containing derivatives of the Killing vectors. After contraction with the metric a 2 is contributed which leads to the critical dimension \( d = 2 \). Therefore, the critical dimension for which the theory exhibits the critical behaviour is equal to the rank of the metric.

However, 2 is quite different from 4, the accepted dimension of space-time. Therefore, the ideal situation would be to have a field theory with the previous properties holding in four dimensions: scale invariance, the appearance of an infinite-dimensional symmetry group and the hope for a mathematically consistent quantum field theory. Many attempts have been done in order to reach this purpose. However, all existing proposals are not exempt from criticism and no one can claim success. All of them have so far met with considerable difficulties and in spite of the tremendous amount of work done on the subject there is still not a generally accepted integrable conformal field theory in four dimensions.

In order to obtain an integrable conformal field theory two ingredients are necessary.
The first one is the scale invariance and the second is the existence of an infinite number of conserved quantities.

As mentioned above, scale invariance implies conformal invariance. However, the concept of conformality strongly depends on the metric properties of the base space. In order to clarify this point let us consider some general properties of metric spaces.

Our approach is based on the use of line elements of the form
\[ ds^r = G_{\mu_1 \cdots \mu_r} \, dx^{\mu_1} \cdots dx^{\mu_r}. \] (1.2)

Here the natural object is the rth-rank metric \( G_{\mu_1 \cdots \mu_r} \). With it we can define generalised inner products, norms and angles.

The inner product of two generic vectors \( A^\mu \) and \( B^\mu \) is given by
\[ N_{p,q}(A, B) = G_{\mu_1 \cdots \mu_r} \, A^{\mu_1} \cdots A^{\mu_p} \, B^{\mu_{p+1}} \cdots B^{\mu_{p+q}}, \] (1.3)

where, obviously, \( p + q = r \). Next we can define the norm of the vector \( A^\mu \) by
\[ A = |N_{p,q}(A, A)|^{1/r} = |N_{r,0}(A, 0)|^{1/r}, \] (1.4)

and the same definition for \( B^\mu \). Next we define the *generalised angles*
\[ \alpha_{p,q}(A, B) = A^{-p} B^{-q} N_{p,q}(A, B). \] (1.5)

In the second-rank case the above formulae coincide with the usual definitions. These formulae seem to be the natural generalisations to higher-rank geometries of the concepts of inner product, norm and angle, in Riemannian geometry.

It can then be shown that the generalised angles are invariant under the scale transformation
\[ G_{\mu_1 \cdots \mu_r} \rightarrow \Omega \, G_{\mu_1 \cdots \mu_r}. \] (1.6)

Therefore, it is reasonable to call conformal this kind of transformations. This is the concept of conformality we use in our approach.

The next step is to determine how conformal transformations of the metric \( G_{\mu_1 \cdots \mu_r} \) can be obtained. For this let us consider the transformations
\[ x^\mu \rightarrow x^\mu + \xi^\mu, \] (1.7)

with \( \xi^\mu \) an infinitesimal function. At first order in \( \xi^\mu \) the metric is changed by
\[ \delta G_{\mu_1 \cdots \mu_r} = G_{\mu_2 \cdots \mu_r \mu} \, \partial_{\mu_1} \xi^\mu + \cdots + G_{\mu_1 \cdots \mu_{r-1} \mu} \, \partial_{\mu_r} \xi^\mu + \xi^\mu \, \partial_{\mu} G_{\mu_1 \cdots \mu_r}. \] (1.8)

A conformal transformation will be induced on the metric if the previous variation is proportional to the metric
\[ \delta G_{\mu_1 \cdots \mu_r} = \alpha \, G_{\mu_1 \cdots \mu_r}. \] (1.9)
We obtain then a generalised conformal Killing equation

\[ G_{\mu_2 \cdots \mu_r \mu} \partial_{\mu_1} \xi^\mu + \cdots + G_{\mu_1 \cdots \mu_{r-1} \mu} \partial_{\mu_r} \xi^\mu + \xi^\mu \partial_\mu G_{\mu_1 \cdots \mu_r} \]

\[- \frac{r}{d} G_{\mu_1 \cdots \mu_r} G^{-1/r} \partial_\mu (G^{1/r} \xi^\mu) = 0, \] (1.10)

(the value of \( \alpha \) has been fixed by taking the trace of this equation) where \( G \) is the determinant of \( G_{\mu_1 \cdots \mu_r} \). Before continuing the analysis of this equation let us turn our attention to the curvature properties of differentiable manifolds.

Curvature properties are described by the curvature tensor

\[ R^\lambda_{\rho \mu \nu} = \partial_\mu \Gamma^\lambda_{\nu \rho} - \partial_\nu \Gamma^\lambda_{\mu \rho} + \Gamma^\lambda_{\mu \sigma} \Gamma^\sigma_{\nu \rho} - \Gamma^\lambda_{\nu \sigma} \Gamma^\sigma_{\mu \rho}, \] (1.11)

constructed in terms of a connection \( \Gamma^\lambda_{\mu \nu} \).

The metric and the connection are, in general, independent objects. They can be related through a metricity condition. The natural metricity condition is

\[ \nabla_\mu G_{\mu_1 \cdots \mu_r} = \partial_\mu G_{\mu_1 \cdots \mu_r} - \Gamma^\lambda_{\mu \mu_1} G_{\mu_2 \cdots \mu_r} = 0. \] (1.12)

The number of unknowns for a symmetric connection \( \Gamma^\lambda_{\mu \nu} \) is \( \frac{1}{2} d^2 (d+1) \), while the number of equations is

\[ \frac{d (d+r-1)!}{(d-1)! r!}. \] (1.13)

This number is always greater than \( \frac{1}{2} d^2 (d+1) \). Therefore the system is overdetermined and some differentio-algebraic conditions must be satisfied by the metric. Since, in general, such restrictions will not be satisfied by a generic metric, one must deal with the connection and the metric as independent objects. Therefore, for physical applications, the connection and the metric must be considered as independent fields.

The exception is \( r = 2 \), Riemannian geometry, in which the number of unknowns and the number of equations are the same. Since (1.12) is an algebraic linear system, the solution is unique and is given by the familiar Christoffel symbols of the second kind.

A metricity condition can be imposed consistently only if the number of independent components of the metric is less than that naively implied by (1.13). The maximum acceptable number of independent components is \( \frac{1}{2} d^2 (d+1) \). This can be achieved, for instance, for null-flat spaces, for which the line element is given by

\[ ds^r = r! d\zeta^1 \cdots d\zeta^r. \] (1.14)

Spaces described by this line element have \( r \) null directions. The only non-null component of the metric is

\[ G_{1 \cdots r} = 1. \] (1.15)

In this case eq. (1.12) has the unique solution \( \Gamma^\lambda_{\mu \nu} = 0 \), therefore these spaces are flat. That is the reason to call them null-flat spaces.
A simple counting of equations and unknowns shows that the situation for eq. (1.10) is similar to that for eq. (1.12). Therefore a consistent solution will exist only for certain kinds of metrics. A particular case is for null-flat metrics. In this case one can prove the following result:

**Theorem.** The critical dimension, for which the conformal Killing equation has infinitely many solutions, is equal to the rank of the metric.

In this case one can furthermore prove that the symmetry group is $\text{Vir}^r$.

Therefore, if we want to construct an integrable conformal field theory in four dimensions we must rely on fourth-rank geometry. In this case the conformal Killing equation must appear as the condition for the existence of conserved quantities.

Let us now mimic the introductory remarks for a fourth-rank metric. Conformal field theories can be constructed in any dimension but only for $d = 4$ they exhibit a radically different behaviour. Chief among them is the fact that the symmetry group becomes infinite-dimensional. The group, after convenient parametrisation of its generators in terms of Fourier components, is nothing more than the familiar Virasoro algebra. To be more precise the group is \( \text{Vir} \oplus \text{Vir} \oplus \text{Vir} \oplus \text{Vir} \), with one Virasoro algebra for each null space-time direction. The fact that the symmetry group is infinite-dimensional implies that there is an infinite number of conserved quantities making the theory an exactly solvable, or integrable, theory. Furthermore, four-dimensional field theories hold several other properties which are missing when formulated in other dimensions. Chief among them are those properties which make of its quantum field theoretical version a mathematically consistent model.

Let us closer analise the situation. In a conformal field theory the symmetry generators are the conformal Killing vectors. They are solutions of the conformal Killing equation

\[
G_{\alpha \mu \nu \lambda} \partial_\rho \xi^\alpha + G_{\alpha \nu \lambda \rho} \partial_\mu \xi^\alpha + G_{\alpha \rho \mu \nu} \partial_\nu \xi^\alpha + G_{\alpha \rho \mu \nu} \partial_\lambda \xi^\alpha - \frac{4}{d} G_{\mu \nu \lambda \rho} G^{-1/4} \partial_\alpha (G^{1/4} \nabla^\alpha) = 0.
\]  

(1.16)

Only for four-dimensional spaces the solutions are infinitely many giving rise to an infinite-dimensional symmetry group. A closer analysis of the conformal Killing equation shows that this critical dimension is closely related to the rank of the metric. In fact, since the metric is a fourth-rank tensor, in the conformal Killing equation will appear four terms containing derivatives of the Killing vectors. After contraction with the metric a 4 is contributed which leads to the critical dimension $d = 4$. Therefore, the critical dimension for which the theory exhibits the critical behaviour is equal to the rank of the metric.

Comparison of eq. (1.16) with eq. (1.1) illustrates the comments at the introduction concerning the appearance of a number of terms equal to the rank of the metric.

The simple Lagrangian

\[
\mathcal{L} = G^{\mu \nu \lambda \rho} \phi_\mu \phi_\nu \phi_\lambda \phi_\rho G^{1/4},
\]  

(1.17)

where \( \phi_\mu = \partial_\mu \phi \), and \( \phi \) is a scalar field, exhibits all the properties we are looking for: it is conformally invariant and integrable. Furthermore the Lagrangian (1.17) is renormalisable, by power counting, in four dimensions.
The work is organised as follows: In Section 2 we start by considering the metric properties of differentiable manifolds. In Section 3 we consider the curvature properties of differentiable manifolds and introduce the concept of null-flat spaces. Section 4 is dedicated to the conformal Killing equation in null-flat spaces. In Section 5 we introduce the fundamentals of conformal field theory. Section 6 reviews the results on conformal field theory for second-rank, Riemannian, geometry. Section 7 presents the results on conformal field theory for fourth-rank geometry. Section 8 is dedicated to the conclusions.

To our regret, due to the nature of this approach, we must bore the reader by exhibiting some standard and well known results in order to illustrate where higher-rank geometry departs from the standard one.

### 2. Metric Properties of Differentiable Manifolds

The metric properties of a differentiable manifold are related to the way in which one measures distances. Let us remember the fundamental definitions concerning the metric properties of a manifold. Here we take recourse to the classical argumentation by Riemann.\(^2\)

Let \( M \) be a \( d \)-dimensional differentiable manifold, and let \( x^\mu, \mu = 1, \ldots, d \), be local coordinates. The infinitesimal element of distance \( ds \) is a function of the coordinates \( x \) and their differentials \( dx \)’s

\[
 ds = f(x, dx), \quad (2.1)
\]

which is homogeneous of the first-order in \( dx \)’s

\[
 f(x, \lambda dx) = \lambda f(x, dx), \quad (2.2a)
\]

\( \lambda > 0 \), and is positive definite

\[
 f \geq 0. \quad (2.2b1)
\]

Condition (2.2b1) was written, so to say, in a time in which line elements were thought to be positive definite. With the arrival of General Relativity one got used to work with line elements with undefined signature. Condition (2.2b1) was there to assure that the distance measured in one direction is the same one measures in the opposite direction. Therefore, condition (2.2b1) can be replaced by the weaker condition

\[
 f(x, -dx) = f(x, dx). \quad (2.2b2)
\]

However, the above conditions can be summarised into the single condition

\[
 f(x, \lambda dx) = |\lambda| f(x, dx). \quad (2.2)
\]

Of course the possibilities are infinitely many. Let us restrict our considerations to monomial functions. Then we will have

\[
 ds = (G_{\mu_1 \cdots \mu_r}(x) dx^{\mu_1} \cdots dx^{\mu_r})^{1/r}. \quad (2.3)
\]
Condition (2.2a) is satisfied by construction. In order to satisfy condition (2.2b2) \( r \) must be an even number.

The simplest choice is \( r = 2 \)

\[
ds^2 = g_{\mu \nu} \, dx^\mu \, dx^\nu ,
\]

which corresponds to Riemannian geometry. The coefficients \( g_{\mu \nu} \) are the components of the covariant metric tensor. The determinant of the metric is given by

\[
g = \frac{1}{d!} \epsilon^{\mu_1 \ldots \mu_d} \epsilon^{\nu_1 \ldots \nu_d} g_{\mu_1 \nu_1} \cdots g_{\mu_d \nu_d} .
\]

If \( g \neq 0 \), the inverse metric is given by

\[
g^{\mu \nu} = \frac{1}{(d - 1)!} \frac{1}{g} \epsilon^{\mu \mu_1 \ldots \mu_{d-1}} \epsilon^{\nu \nu_1 \ldots \nu_{d-1}} g_{\mu_1 \nu_1} \cdots g_{\mu_{d-1} \nu_{d-1}} .
\]

and satisfies

\[
g^{\mu \lambda} g_{\lambda \nu} = \delta^\mu_\nu .
\]

The next possibility is \( r = 4 \). In this case the line element is given by

\[
ds^4 = G_{\mu \nu \lambda \rho} \, dx^\mu \, dx^\nu \, dx^\lambda \, dx^\rho .
\]

The determinant of the metric \( G_{\mu \nu \lambda \rho} \) is given by

\[
G = \frac{1}{d!} \epsilon^{\mu_1 \ldots \mu_d} \ldots \epsilon^{\rho_1 \ldots \rho_d} G_{\mu_1 \nu_1 \lambda_1 \rho_1} \cdots G_{\mu_d \nu_d \lambda_d \rho_d} ,
\]

where the \( \epsilon \)'s can be chosen as the usual completely antisymmetric Levi-Civita symbols. If \( G \neq 0 \), the inverse metric is given by

\[
G^{\mu \nu \lambda \rho} = \frac{1}{(d - 1)! \, G} \epsilon^{\mu_1 \ldots \mu_{d-1}} \ldots \epsilon^{\rho_1 \ldots \rho_{d-1}} G_{\mu_1 \nu_1 \lambda_1 \rho_1} \cdots G_{\mu_{d-1} \nu_{d-1} \lambda_{d-1} \rho_{d-1}} ,
\]

and satisfies the relations

\[
G^{\mu \alpha \beta \gamma} G_{\nu \alpha \beta \gamma} = \delta^\mu_\nu .
\]

That this is true can be verified by hand in the two-dimensional case and with computer algebraic manipulation for 3 and 4 dimensions.\(^3\)

All the previous results can be generalised to an arbitrary even \( r \). In the generic case the line element is given by

\[
ds^r = G_{\mu_1 \ldots \mu_r} \, dx^{\mu_1} \cdots dx^{\mu_r} .
\]

The determinant of the metric \( G_{\mu_1 \ldots \mu_r} \) is given by

\[
G = \frac{1}{d!} \epsilon^{\alpha_1 \ldots \alpha_d} \ldots \epsilon^{\rho_1 \ldots \rho_d} G_{\alpha_1 \ldots \rho_1} \cdots G_{\alpha_d \ldots \rho_d} ,
\]
where again the $\epsilon$'s can be chosen as the usual completely antisymmetric Levi-Civita symbols. If $G \neq 0$, the inverse metric is given by

$$G^{\alpha\cdots\rho} = \frac{1}{(d-1)!} \frac{1}{G} \epsilon^{\alpha_{d-1}\cdots\alpha_{1}} \cdots \epsilon^{\rho_{d-1}\cdots\rho_{1}} G_{\alpha_{1}\cdots\rho_{1}} \cdots G_{\alpha_{d-1}\cdots\rho_{d-1}}, \quad (2.14)$$

and satisfies the relations

$$G_{\mu_1\nu_1\cdots\mu_s\nu_s} = g(\mu_1\nu_1 \cdots g_{\mu_s\nu_s}). \quad (2.16)$$

In this case the line element reduces to a quadratic form and therefore all the results obtained for a generic $G_{\mu_1\cdots\mu_r}$ reduce to those for Riemannian geometry.

2.1. Inner Products and Angles

Let us consider the inner products of two generic vectors $A^\mu$ and $B^\mu$

$$N_{p,q}(A, B) = G_{\mu_1\cdots\mu_r} A^{\mu_1} \cdots A^{\mu_p} B^{\mu_{p+1}} \cdots B^{\mu_{p+q}}, \quad (2.17)$$

where, obviously, $p + q = r$. Next we can define the norm of the vector $A^\mu$ by

$$A = | N_{p,q}(A, A) |^{1/r} = | N_{r,0}(A, 0) |^{1/r}, \quad (2.18)$$

and the same definition for $B^\mu$. Next we define the generalised angles

$$\alpha_{p,q}(A, B) = A^{-p} B^{-q} N_{p,q}(A, B). \quad (2.19)$$

In the second-rank case the above formulae coincide with the usual definitions. These formulae seem to be the natural generalisations to higher-rank geometries of the concepts of inner product, norm and angle, in Riemannian geometry. Furthermore, it must be observed that for higher-rank geometries we can consider inner products of more than two vectors. For our purposes it is enough to restrict our considerations to two vectors.

Let us now observe that under the transformation

$$G_{\mu_1\cdots\mu_r} \to \Omega G_{\mu_1\cdots\mu_r}, \quad (2.20)$$

the generalised angles remain unchanged, they are scale invariant. This is a good reason to call the previous transformations conformal, since they preserve the (generalised) angles.
2.2. The Conformal Killing Equation

Let us next analyse how one can obtain conformal symmetries of the metric. Let us consider the transformation

\[ x^\mu \rightarrow x^\mu + \xi^\mu(x), \tag{2.21} \]

with \( \xi \) an infinitesimal function. Under this transformation the metric is changed, at first-order in \( \xi \), by

\[ \delta G_{\mu_1\cdots \mu_r} = G_{\mu_2\cdots \mu_r\mu} \partial_\mu \xi^\mu + \cdots + G_{\mu_1\cdots \mu_{r-1}\mu} \partial_\mu \xi^\mu + \xi^\mu \partial_\mu G_{\mu_1\cdots \mu_r}. \tag{2.22} \]

In order for this variation to induce a conformal transformation over the metric, it must be

\[ \delta G_{\mu_1\cdots \mu_r} = \alpha G_{\mu_1\cdots \mu_r}. \tag{2.23} \]

One arrives then to the conformal Killing equation

\[ G_{\mu_2\cdots \mu_r\mu} \partial_\mu \xi^\mu + \cdots + G_{\mu_1\cdots \mu_{r-1}\mu} \partial_\mu \xi^\mu + \xi^\mu \partial_\mu G_{\mu_1\cdots \mu_r} \]

\[ - \frac{r}{d} G_{\mu_1\cdots \mu_r} G_{\mu_1/r} \partial_\mu (G_{\mu_1/r} \xi^\mu) = 0. \tag{2.24} \]

(\( \text{The value of } \alpha \text{ has been fixed by taking the trace of this equation.} \) This equation is completely written in terms of the metric since there is no Christoffel symbol associated to it. This equation is furthermore overdetermined. In fact the number of derivatives \( \partial_\nu \xi^\mu \) is much lesser than the number of equations (2.24). Therefore, solutions will exist only for certain classes of metrics. A solution can be obtained for \textit{null flat} spaces; cf. Section 3.1.

3. Curvature Properties of Differentiable Manifolds

Curvature properties are based on affine properties which in turn are related to how one moves from one point to a close one. These properties are mathematically described by the connection \( \Gamma^\lambda_{\mu \nu} \). In terms of the connection one can define the Riemann tensor

\[ R^\lambda_{\rho \mu \nu} = \partial_\mu \Gamma^\lambda_{\nu \rho} - \partial_\nu \Gamma^\lambda_{\mu \rho} + \Gamma^\lambda_{\mu \sigma} \Gamma^\sigma_{\nu \rho} - \Gamma^\lambda_{\nu \sigma} \Gamma^\sigma_{\mu \rho}. \tag{3.1} \]

Up to now the connection \( \Gamma^\lambda_{\mu \nu} \) and the metric \( G_{\mu \nu \lambda \rho} \) are unrelated. They can be related through a metricity condition. In Riemannian geometry this metricity condition reads

\[ \nabla_\lambda g_{\mu \nu} = \partial_\lambda g_{\mu \nu} - \Gamma^\rho_{\lambda \nu} g_{\mu \rho} - \Gamma^\rho_{\lambda \mu} g_{\nu \rho} = 0. \tag{3.2} \]

The number of unknowns for a symmetric \( \Gamma \) and the number of equations (3.2) are the same, \( \frac{1}{2} d^2 (d + 1) \). Therefore, since this is an algebraic linear system, the solution is unique and is given by the familiar Christoffel symbols of the second kind.
$$\Gamma^\lambda_{\mu\nu} = \{^\lambda_{\mu\nu}\}(g) = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}).$$  \hfill (3.3)

In the case of a higher-rank metric a condition analogous to (3.2) would read

$$\nabla_\mu G_{\mu_1\cdots\mu_r} = \partial_\mu G_{\mu_1\cdots\mu_r} - \Gamma^\lambda_{\mu(\mu_1} G_{\mu_2\cdots\mu_r)\lambda} = 0.$$  \hfill (3.4)

However, in this case, the number of unknowns $\Gamma^\lambda_{\mu\nu}$ is, as before, $\frac{1}{2} d^2 (d+1)$, while the number of equations is

$$d \frac{(d+r-1)!}{r!(d-1)!} > d \frac{d(d+1)}{2}, \text{ for } r > 2.$$  \hfill (3.5)

Therefore the system is overdetermined and some differential-algebraic conditions must be satisfied by the metric. Since, in general, such restrictions will not be satisfied by a generic metric, one must deal with $\Gamma^\lambda_{\mu\nu}$ and $G_{\mu_1\cdots\mu_r}$ as independent objects. A metricity condition can be imposed consistently only if the number of independent components of the metric is lesser than that naively implied by (3.5). The maximum acceptable number of independent components is $\frac{1}{2} d (d+1)$. This can be achieved, for instance, if the metric is that corresponding to a null-flat one.

### 3.1 Null-Flat Spaces

As we will see in the next section, there is a close connection between the dimension of the manifold and the rank of the metric. For a second-rank geometry in a two-dimensional manifold we have that the metric of a flat space can always be brought to the forms

$$ds^2 = dt^2 - dx^2,$$  \hfill (3.6a)

$$ds^2 = du^2 + dv^2,$$  \hfill (3.6b)

for Minkowskian and Euclidean signatures, respectively. However, both of them can be brought to the simple form

$$ds^2 = 2 dz^+ dz^-.$$  \hfill (3.7)

where

$$z^\pm = \frac{1}{\sqrt{2}} (t \pm x),$$  \hfill (3.8a)

$$z^\pm = \frac{1}{\sqrt{2}} (u \pm iv),$$  \hfill (3.8b)

for the Minkowskian and Euclidean cases, respectively. Therefore the canonical form (3.7) is independent of the signature of the underlying space. Since for higher-rank geometries there is no concept of flatness, eq. (3.7) seems to be a good definition to be generalised.
The concept of null-flat space is defined only for spaces in which the dimension and the rank of the metric coincide. Then, the line element is given by

$$ds^r = r! d\zeta^1 \cdots d\zeta^r.$$  \hfill (3.9)

(In higher-rank geometry not all flat spaces are null.) It is clear that each coordinate $\zeta_\mu$ is associated to a null direction of the manifold. This is the reason to call these spaces null. The only non-null component of the metric is

$$G_{1 \cdots r} = 1.$$  \hfill (3.10)

One can now easily verify that in this case the metricity condition for higher-rank metrics, eq. (3.4), has the only solution $\Gamma^\lambda_{\mu\nu} = 0$. Therefore these spaces are flat. This is the reason to call these spaces flat.

### 4. The Conformal Killing Equation in Null-Flat Spaces

Now we come to what we consider to be our most important result. Let us consider the conformal Killing equation in a null-flat space

$$G_{\mu_2 \cdots \mu_r \mu} \partial_{\mu_1} \xi^\mu + \cdots + G_{\mu_1 \cdots \mu_{r-1} \mu} \partial_{\mu_r} \xi^\mu - G_{\mu_1 \cdots \mu_r} \partial_\mu \xi^\mu = 0.$$  \hfill (4.1)

(The $r!$ factor has disappeared since in a null-flat space $r = d$.) Then we can establish the following:

**Theorem.** The critical dimension, for which the conformal Killing equation has infinitely many solutions, is equal to the rank of the metric.

The proof is quite simple. Let us observe that at most two indices can be equal. The number of equations in which two indices are equal is $d(d - 1)$. They are equivalent to

$$\partial_\nu \xi^\mu = 0, \nu \neq \mu.$$  \hfill (4.2)

The solution is (no summation)

$$\xi^\mu = f^\mu(\zeta^\mu).$$  \hfill (4.3)

The equation in which all indices are different is identically zero. Therefore, the components of the conformal Killing vectors are arbitrary functions of the single coordinate along the associated direction and therefore the solutions are infinitely many.

Let us now define the operators (no summation)

$$U^\mu(f) = \xi^\mu(\zeta^\mu) \partial_\mu = f^\mu(\zeta^\mu) \partial_\mu.$$  \hfill (4.4)

One can then easily verify that

$$\{U^\mu(f_1), U^\nu(f_2)\} = U^\mu(f_1) U^\nu(f_2) - U^\nu(f_2) U^\mu(f_1) = \delta^{\mu\nu} U(f_1 f_2' - f_2 f_1').$$  \hfill (4.5)
Therefore the symmetry group is the direct product of $r$ times the group
\[
\{U(f_1), U(f_2)\} = U(f_1)U(f_2) - U(f_2)U(f_1).
\] (4.6)
which, after Fourier parametrisation, we recognise as the Virasoro group. The symmetry group is therefore $Vir^r$.

5. Integrable Conformal Field Theories

Now we state the fundamentals for the construction of an integrable conformal field theory.

Let us start by making some elementary considerations about field theory. We will restrict our considerations to generic fields $\phi^A, A = 1, \ldots, n$, described by a Lagrangian
\[
\mathcal{L} = \mathcal{L}(\phi^A, \phi^A_{\mu}),
\] (5.1)
where $\phi^A_{\mu} = \partial_{\mu}\phi^A$. The field equations are
\[
\frac{\delta\mathcal{L}}{\delta\phi^A} = \frac{\partial\mathcal{L}}{\partial\phi^A} - d_{\mu}\pi^A_{\mu} = 0,
\] (5.2)
where we have introduced the generalised canonical momentum
\[
\pi^A_{\mu} = \frac{\partial\mathcal{L}}{\partial\phi^A_{\mu}}.
\] (5.3)
The energy-momentum tensor is given by
\[
\mathcal{H}^\mu_\nu = \phi^A_{\nu} \pi^A_{\mu} - \delta^\mu_\nu \mathcal{L},
\] (5.4)
and satisfies the continuity equation
\[
d_{\mu}\mathcal{H}^\mu_\nu = -\phi^A_{\nu} \frac{\delta\mathcal{L}}{\delta\phi^A} = 0.
\] (5.5)
The first comment relevant to our work is in order here. The definition (5.4), of the energy-momentum tensor, guarantees, through (5.5), its conservation on-shell. This definition is independent of the existence of a metric or other background field. This is what we need in the next stages where we are going to independise from the usual second-rank metric.

Let us now make some considerations about conformal field theory. The main properties that a conformal theory must have are:

C1. Translational invariance, which implies that the energy-momentum tensor $\mathcal{H}^\mu_\nu$ is conserved, i.e., eq. (5.5).
C2. Invariance under scale transformations which implies the existence of the dilaton current. This current can be constructed to be
\[
D^\mu = \mathcal{H}^\mu_\nu x^\nu.
\] (5.6)
The conservation of $D^\mu$ implies that $\mathcal{H}^{\mu\nu}$ is traceless

$$d_\mu D^\mu = \mathcal{H}^{\mu\mu} = 0, \quad (5.7)$$

where the conservation of $\mathcal{H}^{\mu\nu}$, eq. (5.5), has been used.

Now we look for the possibility of constructing further conserved quantities. We concentrate on quantities of the form

$$J^\mu = \mathcal{H}^{\mu\nu} \xi^\nu. \quad (5.8)$$

Then it must be

$$d_\mu J^\mu = \mathcal{H}^{\mu\nu} d_\mu \xi^\nu = 0. \quad (5.9)$$

In order to obtain more information from this equation we need to introduce a further geometrical object allowing us to raise and low indices.

In the next sections we apply the above condition to second- and fourth-rank geometries.

6. Scale Invariant Field Theory in Second-Rank Geometry

Now we consider the properties of scale invariant field theories in second-rank geometry.

In order to obtain the consequences of scale invariance in second-rank geometry we consider a constant flat metric $g_{\mu\nu}$. Then we define

$$\mathcal{H}^{\mu\nu} = \mathcal{H}^{\mu\lambda} g_{\lambda\nu}, \quad (6.1)$$

$$\xi_\mu = g_{\mu\nu} \xi^\nu. \quad (6.2)$$

If (6.1) happens to be symmetric then eq. (5.9) can be written as

$$d_\mu J^\mu = \frac{1}{2} \mathcal{H}^{\mu\nu} (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) = 0. \quad (6.3)$$

Furthermore, if the energy-momentum tensor is traceless, as required by scale invariance, the most general solution to (6.4) is

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{2}{d} g_{\mu\nu} \partial_\lambda \xi^\lambda = 0. \quad (6.4)$$

i.e., the $\xi$’s are conformal Killing vectors for the metric $g_{\mu\nu}$.

As shown in section 4, the critical dimension for this equation is $d = 2$, i.e., the solutions to (6.4) are infinitely many. For the metric (3.7) the solutions to eq. (6.4) are

$$\xi^+ = f(z^+), \quad (6.5a)$$
\[ \xi^- = g(z^-), \]  
where \( f \) and \( g \) are arbitrary functions. Now we define

\[ U_+(f) = \xi^+ \partial_+ = f(z^+) \partial_+, \]  
(6.6a)

\[ U_-(g) = \xi^- \partial_- = g(z^-) \partial_. \]  
(6.6b)

These quantities satisfy the commutation relations

\[ \{U_+(f_1), U_+(f_2)\} = U_+(f_1 f_2' - f_2 f_1'), \]  
(6.7a)

\[ \{U_+(f), U_-(g)\} = 0, \]  
(6.7b)

\[ \{U_-(g_1), U_-(g_2)\} = U_-(g_1 g_2' - g_2 g_1'). \]  
(6.7c)

Relations (6.7) are essentially the algebra of two-dimensional diffeomorphisms. After conveniently parametrise them in terms of Fourier components one gets the familiar Virasoro algebra. To be more precise there is one Virasoro algebra for each null direction.

The next step is to find a conformal field theory for which the conserved quantities be determined by eq. (6.4). The simplest example is

\[ \mathcal{L} = \frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu g^{1/2}, \]  
(6.8)

where \( \phi_\mu = \partial_\mu \phi \). This simple Lagrangian has many important properties. It is conformally invariant, it possesses infinitely many conserved quantities and is renormalisable, by power counting, for \( d = 2 \). The energy-momentum tensor is

\[ \mathcal{H}^{\mu \nu} = \phi_\nu g^{\mu \lambda} \phi_\lambda g^{1/2} - \delta_\nu^{\mu} \mathcal{L}. \]  
(6.9)

The contravariant form is

\[ \mathcal{H}^{\mu \nu} = (g^{\nu \rho} \phi_\rho) (g^{\mu \lambda} \phi_\lambda) g^{1/2} - g^{\mu \nu} \mathcal{L}. \]  
(6.10)

In this case the contravariant \( \partial \phi \) coincides with the momentum such that this expression becomes symmetric. It is furthermore traceless. Therefore we will have an infinite set of conserved quantities, for \( d = 2 \).

The above properties of the CKE in \( d = 2 \) is the origin of the great success of string theories. In fact, string theories have all the good properties one would like for a consistent quantum field theory of the fundamental interactions. All these reasons gave hope for strings to be the theory of everything. However, string theories are interesting only when formulated in 2 dimensions, which is quite different from 4, the accepted dimension of space-time. Therefore, the ideal situation would be to have a field theory formulated in 4 dimensions and exhibiting the good properties of string theories. This is the problem to which we turn our attention now.
7. Integrable Conformal Field Theory in Four Dimensions

As mentioned previously, in order to obtain an integrable scale invariant model, two ingredients are necessary: scale invariance and an infinite number of conserved quantities. In order to obtain an infinite number of conserved quantities we need that they be generated by an equation admitting infinitely many solutions.

In 4 dimensions an infinite number of solutions can be obtained with the conformal Killing equation for fourth-rank geometry in null-flat spaces, as shown in section 4. Now we need to establish the equivalence between the condition (5.9) and the fourth-rank CKE

\[ G_{\alpha\mu\lambda} \partial_{\rho} \xi^\alpha + G_{\alpha\nu\lambda\rho} \partial_{\mu} \xi^\alpha + G_{\alpha\lambda\rho\mu} \partial_{\nu} \xi^\alpha + G_{\alpha\rho\mu\nu} \partial_{\lambda} \xi^\alpha - \frac{4}{d} G_{\mu\nu\lambda\rho} G^{-1/4} \partial_{\alpha} (G^{1/4} \xi^\alpha) = 0. \]  

(7.1)

In the fourth-rank case, however, the operation of raising and lowering indices is not well defined, therefore the operations involved in (6.1)-(6.2) do not exist. In fact this procedure works properly for second-rank metrics due to the fact that only for them the operations of raising and lowering indices are well defined (the tangent and cotangent bundles are diffeomorphic). This is not a real problem since the only thing we must require is that (7.1) gives rise to the conformal Killing equation. This can be done for a simple Lagrangian which is the natural generalisation of (6.8) to fourth-rank geometry.

Let us consider the Lagrangian

\[ \mathcal{L} = \frac{1}{4} G^{\alpha\beta\gamma\delta} \phi_{\alpha} \phi_{\beta} \phi_{\gamma} \phi_{\delta} G^{1/4}. \]  

(7.2)

This simple Lagrangian exhibits the properties we are interested in. It is conformally invariant, it possesses infinitely many conserved quantities and it is renormalisable, by power counting, for \( d = 4 \).

The generalised momenta are given by

\[ \pi^\mu = G^{\mu\beta\gamma\delta} \phi_{\beta} \phi_{\gamma} \phi_{\delta} G^{1/4}. \]  

(7.3)

The energy-momentum tensor is defined as in (5.4). Condition (5.9) reads

\[ \phi_{\nu} G^{\mu\beta\gamma\delta} \phi_{\beta} \phi_{\gamma} \phi_{\delta} \partial_{\mu} \xi^\nu - 1/4 G^{\alpha\beta\gamma\delta} \phi_{\alpha} \phi_{\beta} \phi_{\gamma} \phi_{\delta} \partial_{\mu} \xi^\mu = 0. \]  

(7.4)

Since the \( \xi \)'s do not depend on \( \partial\phi \)'s, what must be zero is the completely symmetric coefficient with respect to \( \partial\phi \)'s. The result is the conformal Killing equation (7.1). Therefore, we will have an infinite-dimensional symmetry group for \( d = 4 \).

Therefore, we have succeeded in implementing conformal invariance for \( d = 4 \). We have seen furthermore that the rank of the metric is essential to implement conformal invariance in higher dimensions. It must furthermore be observed that the appearance of the conformal behaviour for some critical dimension is a geometrical property of the base space and therefore it is model independent. Therefore any attempt at the implementation of conformal invariance in four dimensions by relying only on the second-rank metric is condemned to fail.
The next step is of course to construct a more realistic model on lines, for example, similar to the Polyakov string.

7.1. Comments

That the symmetry group for an integrable conformal model in 4 dimensions should be $Vir^4$ was advanced by Fradkin and Linetsky. $^5$ While $Vir$ and $Vir^2$ are clearly related to Riemannian geometry, a similar geometrical concept was lacking for $Vir^4$. This missing geometrical link is provided by fourth-rank geometry. According to our previous results the conformal Killing equation for a fourth-rank metric exhibits the desired behaviour. Our problem is therefore reduced to construct a field theory in which the conformal Killing equation plays this central role.

8. Conclusions

We have seen that integrable conformal field theories can be constructed in 4 dimensions if one relies on fourth-rank geometry. Furthermore, all desirable properties of a quantum field theory are present when using fourth-rank geometry, viz., renormalisability (by power counting), integrability, etc. There seems to be a close connection between the dimension and the rank of the geometry. When they coincide, as happens for null-flat spaces, all good properties show up. The why this is so is a question still waiting for an answer.

Our future plan of work is to develop further models, even realistic, with the previous properties which perhaps will provide clues to answer the question asked in the above paragraph.

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