A Delay Model of Multiple-Valued Logic Circuits Consisting of Min, Max, and Literal Operations

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SUMMARY
Delay models for binary logic circuits have been proposed and clarified their mathematical properties. Kleene’s ternary logic is one of the simplest delay models to express transient behavior of binary logic circuits. Goto first applied Kleene’s ternary logic to hazard detection of binary logic circuits in 1948. Besides Kleene’s ternary logic, there are many delay models of binary logic circuits, Lewis’s 5-valued logic etc. On the other hand, multiple-valued logic circuits recently play an important role for realizing digital circuits. This is because, for example, they can reduce the size of a chip dramatically. Though multiple-valued logic circuits become more important, there are few discussions on delay models of multiple-valued logic circuits. Then, in this paper, we introduce a delay model of multiple-valued logic circuits, which are constructed by Min, Max, and Literal operations. We then show some of the mathematical properties of our delay model.

key words: multiple-valued logic, multiple-valued logic circuits, hazard detection, delay model

1. Introduction

The customary model of a logic gate is its Boolean function. It should be clear that this model does not take into account all of the properties of a physical gate. For example, physical gates have delays associated with their operations. Thus, if an input of a gate changes at some time, its output will respond to this change only at some later time, whereas the Boolean function model treats the response as instantaneous. Therefore, researchers paid their attention to logical models using more than two values.

One of the non-binary delay models is Kleene’s ternary logic [4], which was first applied to hazard detection of binary logic circuits by Goto [1] in 1948. In 1972 a quinary logic was described by Lewis [3]. Besides these two studies, many researchers have proposed multiple-valued logic models as delay models to express transient behavior of binary logic circuits [2], [11], [12]. In several papers mentioned above, little attention has been paid to the mathematical properties of delay models. However, for example, Mukaidono has been clarified the mathematical properties of Kleene’s ternary logic [5], [7], and Brzozowski has recently introduced a 5-valued logic model and studied its mathematical properties [9].

Multiple-valued logic circuits (MVL circuits) play an important role for realizing digital circuits [13], [14]. This is because, for example, they can reduce the chip size dramatically. An MVL circuit is often realized using the current mode CMOS technology. However, some researchers have introduced the different idea to realize MVL circuits [15], [16]. Since an MVL gate is realized using physical device such as CMOS devices, a signal propagation delay exists in the gate. But, though many researchers paid their attention to hazard detection of binary logic circuits, there exist few studies of introducing a delay model of MVL circuits.

Based on this research background, we have introduced a delay model of MVL circuits that are realized using the current mode CMOS technology [8], [10]. In this model, it is assumed that when a switching between truth values a and b was occurred at a signal line, it is possible to observe all truth values between a and b at the signal line. On the other hand, for an MVL circuit based on set-valued logic devices [16], a switching between a and b completes without passing any other truth values.

Let us explain the difference of transient behavior between MVL circuits realized by the current mode CMOS technology and the device technology discussed in [15] and [16]. Consider the 4-valued logic circuit given in Fig. 1. In this example, suppose that the signal at x2 is switching from 0 to 3, while the signals at x1 and x3 are stably 0 and 3, respectively. First, let us consider the case where the logic gates were realized by the current mode CMOS technology. Since the current changes continuously from 0 to 3, we can assume that all values between 0 and 3 are observed at x2. Therefore, though the stable values at the line a before and after this switching 0 → 3 are 0 and 2, it is possible to observe the undesirable transient value 1 at the line a. In the delay model of [8] and [10], these switchings at x2 and the line a are expressed as the set values {0, 1, 2, 3} and {0, 1, 2}, respectively. Next, consider the case where the logic gates were realized by the device technology discussed in [15] and [16]. In this case, the switching at x2 completes without passing any other truth values between 0 and 3. Therefore, we can observe only the values 0 and 2 at the line a during
Let \( P_r \) be a function on \( S_r \) that this necessary condition is also sufficient. Then, Sect. 5 is the conclusion.

This paper is organized as follows. Our delay model is defined in Sect. 2. A truth value of our model is a non-empty subset of the conventional multiple-valued set \( E_r = \{0,1,\ldots,r-1\} \). The set of all non-empty subsets of \( E_r \) will be denoted as \( P_r \). Then, Sect. 2 also clarified a necessary condition for a function on \( P_r \) to be expressed by a logic formula. Section 3 provides the properties of functions on \( P_r \) satisfying this necessary condition. Then, we prove in Sect. 4 that this necessary condition is also sufficient condition for a function on \( P_r \) to be expressed by a logic formula. Lastly, Sect. 5 is the conclusion.

2. Functions on \( P_r \), Represented by Logic Formulas

Let \( E_r = \{0,1,\ldots,r-1\} \) for a fixed integer \( r > 1 \), and let \( P_r \) denote the set of all non-empty subsets of \( E_r \), i.e., \( P_r = 2^{E_r} - \{0\} \), where \( 2^{E_r} \) is the power set of \( E_r \). The set of all singletons of \( P_r \) is denoted as \( S_r \), i.e., \( S_r = \{\{0\},\{1\},\ldots,\{(r-1)\}\} \). It is clear that the set \( P_r \) is a partial order set with respect to the set inclusion \( \subseteq \). Figure 2 shows a Hasse diagram of the partial order set \( (P_r, \subseteq) \) when \( r = 3 \).

For simplicity, an element of \( P_r \) is sometimes expressed by underlining its elements. For example, \( P_3 \) is expressed as \( \{0,1,2\}, \{0,1\}, \{0\} \). According to Definition 2, \( f(x) \) is a 3-valued function \( f \) of Table 2. One of its sum-of-products expressions is as follows.

\[
F(A_1,\ldots,A_n) = \bigcup_{a_i \in A_i,\ldots,a_n \in A_n} \{f(a_1,\ldots,a_n)\},
\]

where \( A_i \in P_r \) (\( i = 1,\ldots,n \)).

The following min (\(-\)), max (\(+\)), and a literal (\(x^a\)) are basic operations on \( E_r \), because multiple-valued logic circuits are often realized using the operations [6].

![Fig. 2 Partial order set \((P_3, \subseteq)\).](image)

\[
a \cdot b = \min(a,b),
\]
\[
a + b = \max(a,b),
\]
\[
x^a = \begin{cases} r - 1 & \text{if } x = a, \\ 0 & \text{otherwise.} \end{cases}
\]

where \( a, b \in E_r \).

By Definition 1, min, max, and a literal can be expanded into operations on \( P_r \). We denote these three operations as \( \land, \lor, \text{ and } X^a \), where \( A \in S_r \). Table 1 shows the truth tables of \( \land, \lor, \text{ and } X^a \), when \( r = 3 \).

**Definition 2:** Logic formulas are defined inductively as follows.

1. Constants \( \{0,\ldots,r-1\} \) and literals \( X^a_1,\ldots,X^a_n \) (\( A_1,\ldots,A_n \in S_r \)) are logic formulas.
2. If \( G \) and \( H \) are logic formulas, then \( (G \land H) \) and \( (G \lor H) \) are also logic formulas.
3. It is a logic formula if and only if we get it from (1) and (2) in a finite number of steps.

Note that Definition 2 corresponds to sum-of-products expressions of MVL circuits consisting of min, max, and literals [6].

Consider the 2-variable 3-valued function \( f \) of Table 2. One of its sum-of-products expressions is as follows.

\[
f(x_1, x_2) = 1x_1^0x_2^1 + 1x_2^0 + x_2^1
\]

Figure 3 shows a realization of this sum-of-products expression. Then, consider a switching between \( (0,0) \) and \( (0,1) \). The outputs for these two inputs are the value 1. But, in this switching, it is possible to observe the undesirable signal 0 at the output line. This is due to the signal propagation delay of the path A and the path B.

**Table 1** Truth tables of \( \land, \lor, \) and \( X^a \).

(a) Truth Table of \( \land \)

| \( X \) \( \land Y \) | 0   | 1   | 2   | 01  | 02  | 03  |
|----------------------|-----|-----|-----|-----|-----|-----|
| 0                    | 0   | 0   | 0   | 01  | 02  | 03  |
| 01                   | 0   | 0   | 1   | 01  | 02  | 03  |
| 02                   | 0   | 0   | 02  | 01  | 02  | 03  |
| 03                   | 0   | 0   | 03  | 01  | 02  | 03  |
| 1                    | 1   | 1   | 1   | 1   | 1   | 1   |
| 2                    | 2   | 2   | 2   | 2   | 2   | 2   |
| 12                   | 12  | 12  | 12  | 12  | 12  | 12  |
| 012                  | 012 | 012 | 012 | 012 | 012 | 012 |

(b) Truth Table of \( \lor \)

| \( X \) \( \lor Y \) | 0   | 1   | 2   | 01  | 02  | 03  |
|----------------------|-----|-----|-----|-----|-----|-----|
| 0                    | 0   | 0   | 0   | 01  | 02  | 03  |
| 01                   | 0   | 0   | 01  | 01  | 02  | 03  |
| 02                   | 0   | 0   | 02  | 02  | 02  | 03  |
| 03                   | 0   | 0   | 03  | 03  | 03  | 03  |
| 1                    | 1   | 1   | 1   | 1   | 1   | 1   |
| 2                    | 2   | 2   | 2   | 2   | 2   | 2   |
| 12                   | 12  | 12  | 12  | 12  | 12  | 12  |
| 012                  | 012 | 012 | 012 | 012 | 012 | 012 |

(c) Truth Table of \( X^a \)

| \( X \) \( X^a \) | 0   | 1   | 2   | 01  | 02  | 03  |
|-------------------|-----|-----|-----|-----|-----|-----|
| \( X^a_1 \)       | 0   | 0   | 0   | 02  | 02  | 02  |
| \( X^a_2 \)       | 0   | 0   | 0   | 02  | 02  | 02  |
| \( X^a_3 \)       | 0   | 0   | 0   | 02  | 02  | 02  |
Table 2  
| $x_1 \times x_2$ | 0 | 1 | 2 |
|-----------------|---|---|---|
| 0               | 1 | 1 | 2 |
| 1               | 1 | 0 | 2 |
| 2               | 1 | 0 | 2 |

Fig. 3  
Realization of $f$.

Now, consider the following logic formula $F$, which corresponds to the sum-of-products expression of $f$.

$$ F(X_1, X_2) = 1 X_1^0 X_2^1 \lor 1 X_2^0 \lor X_2^1 $$

$F$ takes the value 01 when its input is (0, 01). Then, since the element (0, 01) is interpreted as a switching between (0, 0) and (0, 1), the output 01 of $F$ can be interpreted as the set of all possible outputs. That is, this implies that in this switching it is possible to observe the undesirable signal 0 at the output line when this switching has been occurred. Further, $F$ takes the value 012 when the input (0, 02). Then, though $f$ takes the values 1 and 2 when inputs are (0, 0) and (0, 2), we may observe the undesirable signal 0 at the output line.

A logic formula is a function on $P_r$, if each variable $X_i$ takes a truth value of $P_r$. It is clear that not all functions on $P_r$ can be expressed by logic formulas. This means that operations $\land$, $\lor$, and $X^A$ are not functionally complete over $P_r$. Thus, the subject of this paper is to show a necessary and sufficient condition for a function on $P_r$ to be expressed by a logic formula.

**Theorem 1**: Let $F$ be a function on $P_r$, expressed by a logic formula. Then, for any element $A \in S_r^n$, $F(A)$ is also an element of $S_r$.

(Proof) This theorem is proved directly from Definitions 1 and 2. $

Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be elements of $P_r^n$. Then, in the following discussion, $A \subseteq B$ means $A_i \subseteq B_i$ for every $i = 1, 2, \ldots, n$.

**Theorem 2**: Let $F$ be a function on $P_r$, expressed by a logic formula. Then, $F$ is monotonic in $\subseteq$, that is, $F(A) \subseteq F(B)$ for any elements $A$ and $B$ of $P_r^n$ such that $A \subseteq B$.

(Proof) This theorem is proved by the induction of the number of operations. It is evident that constants $0, \ldots, r - 1$ and any literal $X^A_i$ ($i = 1, \ldots, n$) satisfy the theorem. Let $G$ and $H$ be functions on $P_r$, expressed by logic formulas, and let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be elements of $P_r^n$ such that $A \subseteq B$. Then, suppose $G$ and $H$ satisfy the theorem. By Definition 1, we have the following two equations.

$$ (G \land H)(A) = \bigcup_{a_i \in A_i, \ldots, a_n \in A_n} \{g(a) \cdot h(a)\} $$

$$ (G \land H)(B) = \bigcup_{b_i \in B_i, \ldots, b_n \in B_n} \{g(b) \cdot h(b)\}, $$

where $a = (a_1, \ldots, a_n) \in E_r$ and $b = (b_1, \ldots, b_n) \in E_r$. Since $A$ is a subset of $B$, the above two equations implies $(G \land H)(A) \subseteq (G \land H)(B)$. Therefore, $(G \land H)$ satisfies the theorem. Similarly, we can prove $(G \lor H)(A) \subseteq (G \lor H)(B)$. This completes the proof of the theorem.

Let $A = (A_1, \ldots, A_n)$ be an element of $P_r^n$, and let $B$ be an element of $P_r$. Then, in the following, $A(B)$ stands for the element given by exchanging $A_i$ with $B_i$. Furthermore, define $\min B$ as the minimum number of the set $B$, where the order is the conventional linear order \( \leq \) of $E_r$. For example, $\min\{1, 2, 3\} = 1$, $\min\{0, 3\} = 0$, and so on.

**Theorem 3**: Let $F$ be a function on $P_r$, expressed by a logic formula, and let $A$ be an element of $P_r^n$. Then, $\min F(A(A)) = \min F(A(B))$ holds for any elements $A$ and $B$ of $P_r - S_r$, where $\min F(X)$ means the minimum element of $F(X)$.

(Proof) It is evident that constants $0, \ldots, r - 1$, and any literal $X^A$ (where $A \in S_r$) satisfy this theorem. Let $S$ and $T$ be elements of $P_r$. Then, it follows by Definition 1 that the minimum elements of $S \land T$ and $S \lor T$ are determined by the minimum elements of $S$ and $T$. Therefore, we can prove this theorem by the induction of the number of operations $\land$ and $\lor$.

It has been proved that the following Condition I is a necessary condition for a function $F$ on $P_r$ to be expressed by a logic formula.

**Condition I**:  
1. If $A \in S_r^n$, then $F(A) \in S_r$.  
2. For any $A$ and $B$ of $P_r^n$, $A \subseteq B$ implies $F(A) \subseteq F(B)$.  
3. Let $A$ be an element of $P_r^n$. Then, there exists a fixed element $s \in E_r$ such that $\min F(A(A)) = s$ holds for every $A \in P_r - S_r$.

3. Properties on Functions Satisfying Condition I

Condition I is also a sufficient condition for a function on $P_r$ to be expressed by a logic formula. Before showing this, we provide some properties between functions satisfying Condition I and logic formulas.

**Lemma 1**: Let $A = (A_1, \ldots, A_n)$ be an element of $S_r^n$, and let $a = X_1^{A_1} \cdot \cdots \cdot X_n^{A_n}$. Then, for any $B = (B_1, \ldots, B_n) \in P_r^n$, 

$$ \alpha(B) = \begin{cases} 
\{r - 1\} & \Leftrightarrow A_i \subseteq B_i \text{ for all } i, \\
\{0\} & \Leftrightarrow A_i \cap B_i = \emptyset \text{ for some } i, \\
\{0, r - 1\} & \Leftrightarrow A_i \subseteq B_i \text{ for all } i, \text{ and } \\
& A_i \neq B_i \text{ for some } i 
\end{cases} $$


(Proof) \( \alpha(B) = \{ r - 1 \} \) if and only if \( X_{Ai} = \{ r - 1 \} \) for all \( i \), which is equivalent to \( A_i = B_i \) for all \( i \). \( \alpha(B) = \{ 0 \} \) if and only if \( X_{Ai} = \{ 0 \} \) for some \( i \), which is equivalent to \( A_i \subseteq B_i \) for such \( i \). From the above, \( \alpha(B) = \{ 0, r - 1 \} \) if and only if \( A_i \neq B_i \) for some \( i \) and \( A_i \cap B_i \neq \emptyset \) for some \( i \). Since \( A_i \) is a singleton, that \( \alpha(B) = \{ 0, r - 1 \} \) if and only if \( A_i \neq B_i \) for some \( i \) and \( A_i \subseteq B_i \) for all \( i \). ■

**Lemma 2**: Let \( A \) be an element of \( P_r - S_r \), and let \( \alpha = \bigwedge_{a \in A} X_{Ai} \). Then, for any \( B \in P_r \),

\[
\alpha(B) = \begin{cases} 
0, & A \not\subseteq B, \\
0, & A \not\subseteq B 
\end{cases}
\]

(Proof) \( \alpha(B) = \{ 0 \} \) if and only if \( B_{Ai} = \{ 0 \} \) for some \( a \in A \). Since \( B_{Ai} = \{ 0 \} \) implies \( \{ a \} \cap B = \emptyset \), \( \alpha(B) = \{ 0 \} \) if and only if \( A \not\subseteq B \). It is evident from the above that \( \alpha(B) = \{ 0, r - 1 \} \) if and only if \( A \subseteq B \), because \( \alpha(B) = \{ 0 \} \) or \( \{ 0, r - 1 \} \) for any \( B \in P_r \).

**Definition 3**: Let \( F \) be a function on \( P_r \), and let \( A \) be an element of \( S^n_r \).

First, for every \( s \in E_r \), define \( P_A^*(s) \) as a subset of \( P_r \) as follows.

\[
P_A^*(s) = \{ B \in P_r \mid \min F(A(B)) = s \}.
\]

Note that \( P_A^*(s) \) is a partial order set with respect to the set inclusion \( \subseteq \). So, there exist maximal elements of \( P_A^*(s) \). The set of all maximal elements of \( P_A^*(s) \) is denoted as \( P^*_A(s) \). Then, \( F^*_A(X) \) is defined as the following logic formula.

\[
F^*_A(X) = \bigvee_{s \in E_r} \alpha_s(X)
\]

where \( \alpha_s(X) = \{ s \} \land \bigvee_{B \in P^*_A(s)} X^B \).

Next, for every \( S \in P_r - S_r \), define \( Q^s_A(S) \) as a subset of \( P_r \) as follows.

\[
Q^s_A(S) = \{ B \in P_r \mid F(A(B)) = S \}.
\]

Note that \( Q^s_A(S) \) is a partial order set with respect to the set inclusion \( \subseteq \). So, there exist minimal elements of \( Q^s_A(S) \). The set of all minimal elements of \( Q^s_A(S) \) is denoted as \( Q^*_A(S) \). Then, we define \( F^*_A(X) \) as the following logic formula.

\[
F^*_A(X) = \bigwedge_{S \in P_r - S_r} \bigvee_{s \in S} (\{ s \} \land \beta_s(X)),
\]

where \( \beta_s(X) = \bigwedge_{B \in Q^*_A(S)} X^B \).

**Lemma 3**: Let \( F \) be a function satisfying Condition I-3, and let \( A \) and \( s \) be elements of \( S^n_r \) and \( E_r \), respectively. Then, one of the following two conditions holds.

1. \( P_A^*(s) = P_r - S_r \)
2. \( P_A^*(s) \subseteq S_r \)

**Table 3**: Truth table of \( F \) in Example 1.

| \( X_1 \setminus X_2 \) | 0 | 1 | 2 | 01 | 02 | 12 | 012 |
|-------------------------|---|---|---|----|----|----|-----|
| 0                       | 0 | 0 | 0 | 0  | 0  | 0  | 0   |
| 1                       | 1 | 1 | 2 | 1  | 12 | 12 | 12  |
| 2                       | 2 | 2 | 2 | 0  | 0  | 0  | 0   |
| 01                      | 01| 01| 02| 02 | 02 | 02 | 02  |
| 02                      | 02| 02| 02| 02 | 02 | 02 | 02  |
| 12                      | 12| 12| 12| 12 | 12 | 12 | 12  |
| 012                     | 012| 012| 012| 012| 012| 012| 012 |

(Proof) If an element of \( P_r - S_r \) belongs to \( P_A^*(s) \), then (1) holds by Condition I-3, (2) otherwise. ■

Lemma 3 implies that \( P^*_A(s) \) is either \( \{ E_r \} \) or \( P^*_A(s) \), that is, \( P^*_A(s) = \{ E_r \} \) if (1) holds, \( P^*_A(s) = P_A^*(s) \) otherwise.

**Example 1**: Consider the 2-variable function \( F \) of Table 3. It is not difficult to check that this truth table satisfies Condition I. Let \( A \) be any element of \( S_r \). Then, since

\[
P_A^{1+}(0) = \{ 012 \}, P_A^{1+}(1) = \{ 1 \}, \text{ and } P_A^{1+}(2) = \emptyset,
\]

\( F_A^{1+}(X_1) \) can be expressed as

\[
F_A^{1+}(X_1) = X_1^2.
\]

Similarly,

\[
F_A^{1+}(X_1) = X_1^2, \quad F_A^{1+}(X_1) = X_1^2, \quad F_A^{1+}(X_2) = X_2^2.
\]

\( F_A^{1+}(X_2) \) can be expressed as

\[
F_A^{1+}(X_2) = X_1^2 X_2^2 \vee X_1^2 X_2^2 \vee X_1^2 X_2^2.
\]

Similarly,

\[
F_A^{1+}(X_1) = X_1^2 X_2^2, \quad F_A^{1+}(X_1) = X_1^2 X_2^2, \quad F_A^{1+}(X_2) = X_1^2 X_2^2, \quad F_A^{1+}(X_2) = X_1^2 X_2^2.
\]

\( F_A^{1+}(X_2) \) can be expressed as

\[
F_A^{1+}(X_2) = X_1^2 X_2^2 \vee X_1^2 X_2^2 \vee X_1^2 X_2^2.
\]

**Lemma 4**: Let \( F \) be a function satisfying Condition I, and let \( A \) be an element of \( S^n_r \). Then, the following equality holds for every \( A \in P_r \).
\[ F^*_A(A) = \begin{cases} F(A(A)) & \text{if } F(A(A)) \in S_r \\ K & \text{otherwise} \end{cases} \]

where \( K \) is a subset of \( E_r \), such that \( \{\min F(A(A))\} \subseteq K \subseteq F(A(A)) \).

(Proof) Let \( B \) be an element of \( S_r \). Then, since \( X^B = \{0\}, \{0, r - 1\}, \) or \( \{r - 1\} \) holds for any \( X \in P_r \),

\[ \alpha_r(X) = [0], [0, s], \text{ or } [s] \text{ for every } s \in E_r. \quad (1) \]

(1) First, consider the case where \( F(A(A)) \in S_r \). Then, let \( F(A(A)) = \{s_0\} \), where \( s_0 \in E_r \). Since \( \min F(A(A)) = s_0 \), this element \( A \) is a member of \( P^*_A(s_0) \). Therefore, there exists an element \( B \in P^*_A(s_0) \) such that \( A \subseteq B \), and it follows by Lemma 3 that \( B \) is either \( E_r \) or the element \( A \). This implies

\[ \alpha_{s_0}(A) = \{s_0\}. \quad (2) \]

Next, let \( s \) be an element of \( E_r \), such that \( s > s_0 \). Then, \( A \cap B = \emptyset \) holds for every \( B \) of \( P^*_A(s) \), because if not, we can show a contradiction in the following way: If \( A \cap B \neq \emptyset \) was true for some \( B \) of \( P^*_A(s) \), then there exists an \( C \in P_r \) such that \( C \subseteq A \) and \( C \subseteq B \). Then, by Condition I-2, these two relations imply \( F(A(C)) \subseteq F(A(A)) \) and \( F(A(C)) \subseteq F(A(B)) \). Therefore, \( s_0 \) is a member of \( F(A(B)) \), which contradicts to \( \min F(A(B)) = s \). Thus, \( A \cap B = \emptyset \) holds for every \( B \) of \( P^*_A(s) \) when \( s > s_0 \). It follows by Lemma 1 that

\[ \alpha_{s_0}(A) = \{0\} \text{ for any } s \in E_r \text{ such that } s > s_0. \quad (3) \]

It has been proved by (1), (2) and (3) that \( F^*_A(A) = F(A(A)) \) holds if \( F(A(A)) \in S_r \).

(2) Next, consider the case where \( F(A(A)) \notin S_r \). Then, let \( F(A(A)) = \{s_0, s_1, \ldots, s_m\} \), where \( s_0 \leq s_1 \leq \ldots \leq s_m \) and \( m > 1 \). As we discussed above, since \( A \) is a member of \( P^*_A(s_0) \),

\[ \alpha_{s_0}(A) = \{s_0\}. \quad (4) \]

Next, let \( s \) be an element of \( E_r \), such that \( s \notin \{s_0, \ldots, s_m\} \). Then, \( A \cap B = \emptyset \) holds for every \( B \) of \( P^*_A(s) \) because if not, we can show a contradiction in the following way: Suppose \( A \cap B \neq \emptyset \) was true for some \( B \) of \( P^*_A(s) \). Then, there exists a \( C \in P_r \) such that \( C \subseteq A \) and \( C \subseteq B \). Then, by Condition I-2, these relations imply \( F(A(C)) \subseteq F(A(A)) \) and \( F(A(C)) \subseteq F(A(B)) \). Here, \( F(A(B)) \notin S_r \) holds, because if not, then \( F(A(C)) = \{s\} \), and therefore, \( s \) is an element of \( F(A(A)) \), which contradicts to \( s \notin F(A(A)) \).

Then, since \( F(A(B)) \notin S_r \), it follows by Condition I-1 that \( A(B) \) is not an element of \( S_r^p \). So, by Condition I-3, \( \min F(A(B')) = s \) holds for every \( B' \) of \( P_r - S_r \). Therefore, since \( \min F(A(A)) = s_0 \neq s \) holds, it follows by the result of \( \min F(A(B')) = s \) for every \( B' \in P_r - S_r \) that \( A \) is an element of \( S_r \). Thus, \( A_i(A) \) is an element of \( S_r^p \), and by Condition I-1 this implies \( F(A_i(A)) \in S_r \), which contradicts to \( F(A_i(A)) \notin S_r \). Thus, \( A \cap B = \emptyset \) holds for every \( B \) of \( P^*_A(s) \). By Lemma 1, this implies

\[ \alpha_r(A) = \{0\} \text{ for any } s \notin \{s_0, \ldots, s_m\}. \quad (5) \]

It has been proved by (1), (4) and (5) that \( \{s_0\} \subseteq F^*_A(A) \subseteq \{s_0, \ldots, s_m\} \) if \( F(A(A)) \notin S_r \). This completes the proof of the lemma.

Lemma 5: Let \( F \) be a function satisfying Condition I, and let \( A \) be an element of \( S_r^p \). Then, the following equality holds for every \( A \in P_r \)

\[ F^*_A(A) = \begin{cases} [0] & \text{if } F(A(A)) \in S_r \\ \{0\} \cup F(A(A)) & \text{otherwise} \end{cases} \]

(Proof) Let \( B \) be an element of \( P_r - S_r \). Then, since \( \bigwedge_{b \in B} X^b \) is either \( \{0\} \) or \( \{0, r - 1\}, \)

\[ \beta_{s_0}(X) = \{0\} \text{ or } \{0, r - 1\} \text{ for any } S \in P_r - S_r. \]

Therefore, the following equality holds for any \( S \in P_r - S_r \)

\[ \bigwedge_{a \in S} (\{s\} \land \beta_{s_0}(X)) = \begin{cases} \{0\} & \text{if } \beta_{s_0}(X) = \{0\} \\ \{0\} \cup S & \text{if } \beta_{s_0}(X) = \{0, r - 1\} \end{cases} \quad (6) \]

(1) First, consider the case where \( F(A(A)) \in S_r \). Let \( S \) be an element of \( P_r - S_r \) and then, let \( B \) be an element of \( Q^*_A(S) \). Then, \( B \in Q^*_A(S) \) implies \( F(A(B)) = S \), and so, since \( F(A(B)) \notin F(A(A)) \), \( B \notin A \) holds by Condition I-2. Therefore, it follows by Lemma 2 that \( \beta_{s_0}(A) = \{0\} \) holds for every \( S \in P_r - S_r \). It has been proved that \( F^*_A(A) = \{0\} \) holds if \( F(A(A)) \in S_r \).

(2) Next, consider the case where \( F(A(A)) \notin S_r \). Let \( F(A(A)) = \{s_0\} \), and then, there exists an element \( B \) of \( Q^*_A(S_0) \) such that \( B \subseteq A \). So, the following equality holds by Lemma 2.

\[ \bigwedge_{a \in S} (\{s\} \land \beta_{s_0}(A)) = \begin{cases} \{0\} & \text{if } \beta_{s_0}(X) = \{0, r - 1\} \\ \{0\} \cup S & \text{if } \beta_{s_0}(X) = \{0\} \end{cases} \quad (7) \]

Let \( S \) be an element of \( P_r - S_r \) such that \( S \not

\[ \bigwedge_{a \in S} (\{s\} \land \beta_{s_0}(A)) = \begin{cases} \{0\} & \text{if } \beta_{s_0}(X) = \{0, r - 1\} \\ \{0\} \cup S & \text{if } \beta_{s_0}(X) = \{0\} \end{cases} \quad (8) \]

holds for any \( S \in P_r - S_r \) such that \( S \not

\[ \bigwedge_{a \in S} (\{s\} \land \beta_{s_0}(A)) \subseteq \{0\} \cup S_0 \quad (9) \]

holds for any \( S \in P_r - S_r \) such that \( S \subseteq S_0 \). It has been proved by (7), (8) and (9) that \( F^*_A(A) = \{0\} \cup F(A(A)) \) holds if \( F(A(A)) \notin S_r \).

Lemma 6: Let \( F \) be a function satisfying Condition I, and let \( A \) be an element of \( S_r^p \). Then, the following equality holds for every \( A \in P_r \)

\[ F^*_A(A) \lor F^*_A(A) = F(A_i(A)) \]
(Proof) This lemma can be proved directly from Lemmas 4 and 5.

**Example 2:** Consider the function $F$ of Example 1. Then, for any $(A_1, A_2) \in S^2_2$, each logic formula $F^i_{(A_1, A_2)}(X_i)$ is given as follows.

$$F^i_{(A_1, A_2)}(X_i) = F^{i+1}_{(A_1, A_2)}(X_i) \lor F^{i+1}_{(A_1, A_2)}(X_i).$$

For example, when $(A_1, A_2) = (A, 0)$ and $i = 1$,

$$F^1_{(A, 0)}(X_1) = F^{1+1}_{(A, 0)}(X_1) \lor F^{1+1}_{(A, 0)}(X_1) = X_1^1 \lor X_1^2 X_1^1 \lor X_1^2 X_1^1 \lor X_1^1 X_1^2$$

4. Realization of Logic Formulas for Functions Satisfying Condition I

This section proves that a function $F$ satisfying Condition I can be expressed by a logic formula, and also shows a method of constructing a logic formula from $F$.

**Definition 4:** Let $F$ be a function on $P_r$. Then, $F_1$ is defined as an $n$-variable function on $P_r$ expressed by the following logic formula.

$$F_1(X) = \bigvee_{i=1}^n F^i(X),$$

where

$$F^i(X) = \bigvee_{A=(A_1, \ldots, A_n) \in S^n_r} \left( \gamma^i_A(X) \land F^i_A(X_i) \right),$$

and

$$\gamma^i_A(X) = \bigwedge_{j=1, j \neq i}^n X_j^{A_j},$$

$$F^i_A(X_i) = F^i_{A_i}(X_i) \lor F^i_{A_i}(X_i).$$

**Example 3:** Consider the function $F$ of Example 1. Then, $F_1(X_1, X_2)$ is obtained as the following logic formula.

$$F_1(X_1, X_2) = X_1^0 F_{(A, 0)}^i(X_1) \lor X_1^1 F_{(A, 1)}^i(X_1) \lor X_2^0 F_{(A, 2)}^i(X_2) \lor X_2^1 F_{(A, 2)}^i(X_2) \lor X_1^1 X_2^2 F_{(A, 2)}^i(X_2),$$

where $F^i_{(A_1, A_2)}(X_i)$'s are the logic formulas given in Example 2. Table 4 is the truth table of $F_1$.

**Definition 5:** Let $I_i$ be a collection of elements $(A_1, \ldots, A_n) \in P^n_r$ satisfying the following condition.

$$A_i \in P_r - S_r, \text{ and } A_j \in S_r, \text{ for } j \neq i$$

Then, define a subset $I$ of $P^n_r$ as $I = \bigcup_{i=1}^n I_i$.

| Table 4 | Truth table of $F_1$ in Example 3. |
|---------|----------------------------------|
| $X_1 \setminus X_2$ | 0 | 1 | 2 | 01 | 02 | 12 | 012 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 2 | 1 | 12 | 12 | 12 |
| 2 | 0 | 2 | 0 | 0 | 0 | 0 | 0 |
| 01 | 01 | 01 | 02 | 012 | 012 | 012 |
| 02 | 02 | 02 | 02 | 02 | 02 | 02 |
| 12 | 012 | 012 | 012 | 012 | 012 | 012 |
| 012 | 012 | 012 | 012 | 012 | 012 | 012 |

**Lemma 7:** Let $F$ be a function satisfying Condition I. Then,

$$F_1(A) = \begin{cases} F(A) & \text{if } A \in S^n_r \cup I \\ K & \text{otherwise} \end{cases}$$

holds for every $A$ of $P^n_r$, where $K$ is a subset of $E_r$ such that $[0] \subseteq K \subseteq [0] \cup F(A)$.

(Proof) (1) First, consider the case where $A = (A_1, \ldots, A_n) \in S^n_r$. Then, since $A$ is an element of $S^n_r$, it follows by Lemma 1 that

$$\gamma^i_B(A) = \begin{cases} |r - 1| & \text{if } A_j = B_j \text{ for every } j \neq i \\ 0 & \text{otherwise} \end{cases}$$

where $B = (B_1, \ldots, B_n) \in S^n_r$. Therefore, by Lemma 6, the following equality holds for every $i = 1, 2, \ldots, n$.

$$F_i(A) = \bigvee_{B \in S^n_r} (\gamma^i_B(A) \land F^i_B(A_i))$$

$$= \{0, r - 1\} \land F^i_A(A_i) = F(A).$$

(2) Next, consider the case where $A = (A_1, \ldots, A_n) \in I$. We can assume without loss of generality that $A_1 \in P_r - S_r$ and $A_i \in S_r$ for any $i \neq 1$. Then, the following equalities hold by (10) and Lemma 6.

$$F_1(A) = \bigvee_{B \in S^n_r} (\gamma^i_B(A) \land F^i_B(A_i))$$

$$= \{0, r - 1\} \land F^i_A(A_1) = F(A).$$

Further, consider the value of $F^2(A)$. Since $A_1$ is a member of $P_r - S_r$, it follows by Lemma 1 that

$$\gamma^2_B(A) = \begin{cases} \{0, r - 1\} & \text{if } B_i \subseteq A_i \text{ for every } i \neq 2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$F^2(A) = \bigvee_{B \in S^n_r} (\gamma^2_B(A) \land F^2_B(A_2))$$

$$= \bigvee_{B \subseteq A(2)} (\{0, r - 1\} \cup F^2_B(A_2))$$

$$= \bigvee_{B \subseteq A(2)} (\{0\} \cup F^2_B(A_2))$$

$$= \bigcup_{B \subseteq A(2)} (\{0\} \cup F^2_B(A_2)).$$
where
\[ A(2) = \{ B \in S_i^n | B_i \subseteq A_i \text{ for every } i \neq 2 \}. \]

Since \( B_2(A_2) \subseteq A \) holds for every \( B \in A(2) \), it follows by Condition I-2 that \( F(B_2(A_2)) \subseteq F(A) \) holds for every \( B \in A(2) \). Then, \( F(B_2(A_2)) = F_B^*(A_2) \) by Lemma 6, and so,
\[ F_B^*(A_2) \subseteq F(A) \text{ for every } B \in A(2). \] (13)

Therefore, it has been proved by (12) and (13) that \( \{0\} \subseteq F^2(A) \subseteq \{0\} \cup F(A) \). Similarly,
\[ \{0\} \subseteq F_i(A) \subseteq \{0\} \cup F(A) \] (14)

holds for every \( i = 3, \ldots, n \). Therefore, by (11), (12), and (14), \( F_1(A) = \bigvee_{i=1}^n F_i(A) = F(A) \) holds for every \( A \in I \).

Similarly, it follows by the discussion on (12) and (13) that for every \( i, \{0\} \subseteq F_i(A) \subseteq \{0\} \cup F(A) \) holds for any \( A \notin S_i^n \cup I \).

Definition 6: Let \( F \) be a function on \( P_r \). Then, \( F_2 \) is defined as an \( n \)-variable function on \( P_r \) expressed by the following logic formula.
\[ F_2(X) = \{ s_0 \} \lor \bigvee \bigcup \bigcup \{ (s) \land F_S(X) \}, \]

where
\[ F_S(X) = \bigwedge_{A \in S_i^n \cup I} \varphi_A(X), \]
\[ \varphi_A(X) = \bigvee_{a \in A} X^{[a]}, \]
\[ T^-(S) \text{ is the set of all minimal elements of the set } T(S) = \{ A \in P_r^n | F(A) = S \text{ and } A \notin S_i^n \cup I \}, \]
and \( s_0 \) is the minimum element of \( \bigcup_{A \in P_r^n \setminus I} F(A) \).

Example 4: Consider the function \( F \) of Example 1. Then, in this example, we show the logic formula of \( F_2 \).

For each \( S \in P_r - S_r, T^-(S) \) is obtained as follows.
\[ T^-(\{0\}) = \emptyset, \]
\[ T^-(\{1\}) = \emptyset, \]
\[ T^-(\{0, 1\}) = \{0, 1\}, \]
\[ T^-(\{0, 2\}) = \{0, 2\}, \]
\[ T^-(\{0, 1, 2\}) = \{0, 1, 2\}, \]
\[ T^-(\{0, 1, 2, 12\}) = \{0, 1, 2, 12\}. \]

Then, for each \( S \in P_r - S_r, F_2(X_1, X_2) \) is given as follows.
\[ F_{01}(X_1, X_2) = X_1^0 X_2^0 X_1^2 X_2^2 \lor X_1^1 X_2^1 X_2^1 X_2^1, \]
\[ F_{02}(X_1, X_2) = X_1^0 X_2^1 X_2^1 X_2^1 \lor X_1^2 X_2^2 X_2^2, \]
\[ F_{012}(X_1, X_2) = X_1^0 X_2^1 X_2^1 X_2^1 \lor X_1^2 X_2^2 X_2^2 \lor X_1^1 X_2^1 X_2^1 X_2^1, \]
\[ \lor X_1^1 X_2^1 X_2^1 X_2^1 \lor X_1^1 X_2^1 X_2^1 X_2^1 \]

Table 5: Truth table of \( F_2 \) in Example 4.

| \( X_1 \setminus X_2 \) | 0 | 1 | 2 | 01 | 02 | 12 | 012 |
|------------------------|---|---|---|----|----|----|-----|
| 0                      | 0 | 0 | 0 | 0  | 0  | 0  | 0   |
| 1                      | 0 | 0 | 0 | 0  | 0  | 0  | 0   |
| 2                      | 0 | 0 | 0 | 0  | 0  | 0  | 0   |
| 01                     | 0 | 0 | 0 | 0  | 0  | 0  | 0   |
| 02                     | 0 | 0 | 0 | 0  | 0  | 0  | 0   |
| 12                     | 0 | 0 | 0 | 0  | 0  | 0  | 0   |
| 012                    | 0 | 0 | 0 | 0  | 0  | 0  | 0   |

Note that since \( T^-(\{01\}) = T^-(\{12\}) = \emptyset, F_{01} \) and \( F_{12} \) do not appear in the logic formula of \( F_2 \). Lastly, \( F_2(X_1, X_2) \) is obtained as the following logic formula.
\[ F_2(X_1, X_2) = \lor F_{01}(X_1, X_2) \lor F_{02}(X_1, X_2) \lor F_{012}(X_1, X_2) \]

Table 5 shows the truth table of \( F_2 \).

Lemma 8: Let \( F \) be a function satisfying Condition I. Then,
\[ F_2(A) = \begin{cases} \{s_0\} & \text{if } A \in S_i^n \cup I \\ F(A) & \text{otherwise} \end{cases} \]

where \( s_0 \) is the minimum element of \( \bigcup_{A \in P_r^n} F(A) \).

Proof: (1) First, consider the case where \( A = (A_1, \ldots, A_n) \in S_i^n \cup I \). Let \( S \) be an element of \( P_r - S_r \), and then, let \( B = (B_1, \ldots, B_n) \) be an element of \( T(S) \). There exists at least one \( i \) such that \( B_i \notin A_i \) because \( A \in S_i^n \cup I \) and \( B \notin S_i^n \cup I \). So, it follows by Lemma 2 that \( \varphi_{B_i}(A_i) = \{0\} \) holds for the element \( B_i \). This implies \( F_S(A) = \{0\} \). Then,
\[ F_2(A) = \{s_0\} \lor \{0\} = \{s_0\} \]
holds for every \( A \in S_i^n \cup I \).

(2) Next, consider the case where \( A \notin S_i^n \cup I \). In the following, let \( U = F(A) \).

(2-1) Since \( A \in T(U) \), there exists an element \( B \in T^-(U) \) such that \( B \subseteq A \). Then, \( F_U(A) = \{0, r-1\} \) holds by Lemma 2. Therefore,
\[ \bigcup \{ (u) \land F_U(A) \} = \bigcup \{0, u\} = \{0\} \cup U. \] (15)

(2-2) Let \( U' \) be an element of \( P_r - S_r \), and consider the case where \( U' \cup U \). Since \( F_{U'}(A) = \{0\} \) or \( \{0, r-1\} \) holds by Lemma 2, the following relations hold.
\[ \bigcup \{ (u) \land F_{U'}(A) \} = \{0\} \text{ or } \{0, U' \} \] (16)

(2-3) Let \( U' \) be an element of \( P_r - S_r \), and consider the case where \( U' \subseteq U \). In this case, suppose \( B \subseteq A \) for some \( B \in T^-(U') \). Then, by Condition I-2, \( F(B) \subseteq F(A) \), which contradicts to \( U' \subseteq U \). Therefore, \( B \notin A \) holds for every \( B \in T^-(U') \). This implies that \( F_{U'}(A) = \{0\} \) holds by Lemma 2. Thus,
\[ \bigcup \{ (u) \land F_{U'}(A) \} = \{0\}. \] (17)

From (15), (16), and (17), it has been proved that \( F_2(A) = F(A) \) holds if \( A \notin S_i^n \cup I \).
Theorem 4: Let \( F \) be a function satisfying Condition I. Then, for every \( A \) of \( P^n_r \),
\[
F(A) = F_1(A) \lor F_2(A).
\]
(Proof) The theorem can be proved directly from Lemmas 7 and 8.

Example 5: The function \( F \) of Example 1 is identical with the function that is obtained by ORing of the two functions \( F_1 \) and \( F_2 \) of Examples 3 and 4.

5. Conclusions

In this paper, we proposed a delay model of MVL circuits consisting of min, max, and literal operations. Then, we proved that Condition I is a necessary and sufficient condition for a function on \( P_r \) to be expressed by a logic formula. This result implies something about the limitation of transient behavior of MVL circuits. That is, under the assumption where a switching between two values completes without passing any other values, Condition I expresses the characteristic of transient behavior of MVL circuits.

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