Polynomiality of off-forward distribution functions
in the chiral quark soliton model

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Mellin moments of off-forward distribution functions are, at $t=0$, even polynomials of
the skewedness parameter $\xi$. It is proven that the unpolarized off-forward distribution
functions in the chiral quark soliton model satisfy this so called polynomiality property.
The proof is an important contribution to the demonstration that the description of off-
forward distribution functions in the model is consistent.

Introduction

Off-forward parton distribution functions (OFDFs) – see \cite{1-3} for recent reviews – are a
promising source of new information on the internal nucleon structure. The understanding
of non-perturbative properties of OFDFs – which at present relies on models – is essential
in order to interpret first data on deeply virtual Compton scattering \cite{4-5}, or to make
predictions for future experiments \cite{6-8}. Important contributions to our intuition on
non-perturbative aspects of OFDFs are based on calculations in the chiral quark soliton
model (\chiQSM) \cite{11,12}. The model has been derived from the instanton model of the QCD
vacuum \cite{13}. It describes – without free adjustable parameters – nucleon properties, like
form factors \cite{14} and forward quark and antiquark distribution functions \cite{15,16}, typically
within (10-30)%.

The reliability of the \chiQSM, however, is not only based on its phenomenological success.
More important – from a theoretical point of view – is the fact that it is possible to prove
analytically that the model description of the nucleon is consistent. E.g., in ref.\cite{15} it has
been proven that forward quark and antiquark distribution functions in the model satisfy
all general requirements such as sum rules, positivity and inequalities. With the same
rigour it has been shown in refs.\cite{11,12}, that the model expressions for OFDFs reduce to
usual parton distributions in the forward limit, and that their first moments yield form
factors. In this note a further contribution is made which demonstrates the consistency
of the \chiQSM. It is proven – or rather the proof sketched – that the model expression for
$(H^u + H^d)(x, \xi, t)$ satisfies polynomiality, i.e. the property that the $m^{th}$ moment in $x$ of
an OFDF at $t=0$ is an even polynomial in $\xi$ of degree less than or equal to $m$. In QCD,
this property follows from Lorentz invariance.

\textsuperscript{*}Presented by P. Schweitzer at the “European Workshop on the QCD Structure of the Nucleon” (QCD-
N’02), Ferrara, Italy, 3-6 Apr 2002.

\textsuperscript{†}Supported by the contract HPRN-CT-2000-00130 of the European Commission.
OFDFs in the chiral quark soliton model

The $\chi$QSM is based on an effective chiral low-energy field theory with explicit quark, antiquark ($\bar{\psi}, \psi$) and Goldstone boson (for flavour SU(2) pion $\pi^0$) degrees of freedom. The effective theory is valid for energies below 600 MeV and given by the partition function

$$Z_{\text{eff}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\pi \exp \left( i \int d^4x \, \bar{\psi} \left( i \gamma^\mu \partial_\mu - M U^{\gamma_5} \right) \psi \right), \quad U^{\gamma_5} = e^{i\gamma^5 \sigma^a \tau^a}.$$

(1)

$M$ is the effective quark mass due to spontaneous breakdown of chiral symmetry. In the limit of a large number of colours $N_c$ the nucleon emerges as a classical soliton of the pion field in the effective theory eq.(1). In leading order of the large $N_c$ limit the soliton field is static and one can determine the spectrum of the one-particle Hamiltonian

$$\hat{H}_{\text{eff}} \Phi_n(x) = E_n \Phi_n(x), \quad \hat{H}_{\text{eff}} = -i\gamma^0 \gamma^k \partial_k + M \gamma^0 U^{\gamma_5}$$

(2)

of the effective theory eq.(1). The spectrum consists of a discrete bound state level and upper and lower Dirac continua. The nucleon is obtained by occupying the bound state and the states of the lower continuum each by $N_c$ quarks in an anti-symmetric colour state, and considering the zero modes of the soliton solution.

The model allows to express nucleon matrix elements of QCD quark bilinear operators in terms of the single quark wave-functions $\Phi_n(x)$. In leading order of the large $N_c$ limit

$$\langle \mathcal{P}' | \bar{\psi}(z_1) \Gamma \psi(z_2) | \mathcal{P} \rangle = c \gamma \int d^3x \, e^{i(\mathcal{P}' - \mathcal{P}) x} \Phi_n(z_1 - x) \bar{\Phi}_n(z_2 - x) e^{i E_n (z_1 - z_2)}$$

(3)

where $\Gamma$ denotes some Dirac- and flavour-matrix (which determines the constant $c_{\gamma}$) and the sum goes over occupied states. $1/N_c$ corrections to eq.(3) can be taken systematically into account. Eq.(3) enables one to evaluate in the model, e.g.

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} (\mathcal{P}', S_3) \bar{\psi}_f(-\lambda n/2) \not\! \psi_f(\lambda n/2) | \mathcal{P}, S_3 \rangle$$

$$= H_f(x, \xi, t) \bar{U}(\mathcal{P}', S_3) \not\! n U(\mathcal{P}, S_3) + E_f(x, \xi, t) \bar{U}(\mathcal{P}', S_3) \frac{i \sigma^{\mu \nu} n_\mu \Delta_\nu}{2 M_N} U(\mathcal{P}, S_3),$$

(4)

which defines the twist-2 unpolarized OFDFs $\mathcal{P}_f$. In eq.(4) $n^\mu$ satisfies $n^2 = 0$ and $n(\mathcal{P}' + P) = 2$. The four-momentum transfer is defined as $\Delta^\mu = (\mathcal{P}' - P)^\mu$, the skewedness parameter as $\xi = - \Delta n/2$, and the Mandelstam variable as $t = \Delta^2$.

In the large $N_c$ limit the nucleon mass $M_n$ is $O(N_c)$ but $|\mathcal{P}'|, |\mathcal{P}'| = O(N_c^0)$, thus $t = - \Delta^2 = O(N_c^{-1})$. The variables $x, \xi$ are both $O(N_c^{-1})$. With the convenient choice $n^\mu = (1, -\mathbf{e}^3)/M_N$ one has $\xi = - \Delta^3/2M_N$. The spin-non-flip OFDF is given by

$$(H^u + H^d)(x, \xi, t) = M_N N_c \int d^3x \, e^{i \Delta x} \sum_{n, \text{occ}} \int \frac{dz^0}{2\pi} e^{iz^0(xM_N - E_n)}$$

$$\times \Phi_n^*(x + \frac{z^0}{2} \mathbf{e}^3) \left( 1 + \gamma^0 \gamma^3 \right) \Phi_n(x - \frac{z^0}{2} \mathbf{e}^3)$$

(5)

in LO of the large $N_c$ limit, while the flavour-nonsinglet combination $(H^u - H^d)(x, \xi, t)$ vanishes at this order. Eq.(5) has been derived and numerically evaluated in ref.[1]. The spin-flip OFDF $E_f(x, \xi, t)$ will be discussed elsewhere.
Sketch of the proof of polynomiality

In this section the proof is sketched that the model expression for \((H^u + H^d)(x, \xi, t)\), eq.(3), satisfies polynomiality. For the detailed proof see ref.[17]. From eq.(3) one obtains the model expression for the \(m\)th moment \(M_H^{(m)}(\xi, t) = \int dx x^{m-1}(H^u + H^d)(x, \xi, t)\) for physical values of \(t\). In eq.(3) the dependence of \(M_H^{(m)}(\xi, t)\) on \(\xi\) and \(t\) is entirely determined by \(\Delta\) which appears in the exponential \(\exp(i\Delta X)\) in eq.(4). The latter can be continued analytically to \(t = -\Delta^2 \to 0\) as

\[
\lim_{\text{analytical continuation}} \exp(i\Delta X) = \sum_{l_e=0}^{\infty} \frac{(-2i\xi M_N|X|)^{l_e}}{l_e!} P_{l_e}
\]

with \(P_{l_e}\) denoting Legendre polynomials \(P_{l_e}(\cos \theta)\). This yields for the moments at \(t = 0\)

\[
M_H^{(m)}(\xi, 0) = \frac{N_e}{M_N} \sum_{n, \text{occ}} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{E_n^{m-1-k}}{2^k} \sum_{j=0}^{k} \frac{(k)}{(j)} \sum_{l_e=0}^{\infty} \frac{(-2i\xi M_N)^{l_e}}{l_e!} \langle n| \gamma^0 \gamma^3 | \hat{\mathbf{p}}^3 |^j \rangle \hat{\mathbf{X}}_{l_e} P_{l_e}(\cos \hat{\theta}) \langle \hat{\mathbf{p}}^3 |^{k-j} | n \rangle .
\]

The “ket” states \(|n\rangle\) are related to coordinate space wave functions by \(\Phi_n(x) = \langle x|n\rangle\), and \((\gamma^0 \gamma^3)^k\) equal to unity for even \(k\) and to \(\gamma^0 \gamma^3\) for odd \(k\) is introduced to simplify notation. Finally, \(\hat{\mathbf{p}}^3\) is the free-momentum operator, which was “implicitly present” in eq.(2) due to \(\Phi_n(x - \frac{v^0}{2} e^3) = \langle x|e^{-ix \hat{p}^3}|n\rangle\). In \(M_H^{(m)}(\xi, 0)\), eq.(3), only even powers of \(\xi\) appear: Odd powers of \(\xi\) vanish due to symmetries of the model. What remains to be shown is that the series in \(\xi^2\) terminates at an \(l_{\text{max}}\leq m\) for the \(m\)th moment \(M_H^{(m)}(\xi, 0)\).

To see this note that the Hamiltonian \(\hat{H}_{\text{eff}}\) eq.(2) commutes with the grand-spin operator \(\hat{K}\) defined as sum of quark orbital angular momentum, spin and isospin. In other words, simultaneous rotations in 3-space and isospin-space – around some axis \(n\) and an angle \(\alpha\) given by \(\hat{R} = \exp(i\alpha \hat{K})\) – leave \(\hat{H}_{\text{eff}}\) invariant. Under these rotations \(\hat{H}_{\text{eff}}\) and \(|\hat{X}\rangle\) transform as rank 0, \(\gamma^0 \gamma^3\) and \(\hat{p}^3\) as rank 1, and \(\hat{P}_{l_e}(\cos \hat{\theta}) \propto Y_{l_e}^0(\hat{\Omega})\) as rank \(l_e\) irreducible tensor operators. An operator \(\hat{T}_L^L\) is said to be an irreducible tensor operator of rank \(L\), if it transforms as \(\hat{R} \hat{T}_L^L \hat{R}^\dagger = \sum_{M} D_{M,L}(R) \hat{T}_{L,M} \hat{T}_L^L \), where \(D_{M,L}(R)\) are finite rotations Wigner matrices. The single quark states – which are simultaneously eigenfunctions of \(\hat{H}_{\text{eff}}\), \(\hat{K}^2\) and \(\hat{K}^3\), i.e. \(|n\rangle \equiv |E_n, K, M\rangle\) – transform as \(\hat{R}|E_n, K, M\rangle = \sum_{M'} D_{M,M'}^{(K)}(R) |E_n, K, M'\rangle\).

The product of two irreducible tensor operators \(\hat{T}_{L,M}^L\) and \(\hat{T}_{L',M'}^{L'}\) is a sum of some other irreducible tensor operators \(\hat{T}_{L,M}^{L'}\) with ranks \(|L - L'| \leq L \leq L' + L''\). So the product of irreducible tensor operators sandwiched between \(|n\rangle...|n\rangle\) in eq.(4) is a sum of irreducible tensor operators with ranks \(L\) ranging between 0 \(\leq L \leq k + l_e + 1\) for odd \(k\), and 0 \(\leq L \leq k + l_e\) for even \(k\). Thus, in eq.(3) one deals with traces (sums over matrix elements diagonal in \(K\) and \(M\)) of irreducible tensor operators. The trace of an irreducible tensor operator, however, vanishes unless the operator has rank zero [18].

So \(l_e\) in eq.(3) cannot take arbitrary values, but is bound as

\[
l_e \leq l_{\text{max}}(k) = \begin{cases} k + 1 & \text{for odd } k, \\ k & \text{for even } k. \end{cases}
\]

If \(l_e\) were larger than this \(l_{\text{max}}(k)\), it would be impossible to compensate the rank \(l_e\) of \(P_{l_e}\) to obtain a rank zero operator, even if the ranks of the other operators in eq.(3)
\((\gamma^0\gamma^3)^k, (\not{p}^3)^j, (\not{p}^3)^{m-j-1}\) would all add up. Inserting the result eq.(8) into eq.(7) yields the desired result: The \(m\)th moment of \((H^u + H^d)(x, \xi, t)\) at \(t = 0\), \(M_H^{(m)}(\xi, 0)\), is an even polynomial in \(\xi\) of degree less than or equal to \(m\), i.e.

\[
\int_{-1}^{1} dx x^{m-1} (H^u + H^d)(x, \xi, 0) = h_0^{(m)} + h_2^{(m)} \xi^2 + \ldots + \begin{cases} h_m^{(m)} \xi^m & \text{for even } m, \\ h_{m-1}^{(m)} \xi^{m-1} & \text{for odd } m \end{cases} \tag{9}
\]

with coefficients \(h_i^{(m)}, i = 0, 2, 4, \ldots \leq m\), explicitly given in eq.(7).

**Conclusions**

The proof has been sketched, that the chiral quark soliton model expression for the OFDF \((H^u + H^d)(x, \xi, t)\) satisfies polynomiality. The method can be generalized to prove polynomiality for other OFDFs in the model. As a byproduct analytical expressions for moments at \(t = 0\) have been derived. This opens the possibility, e.g., to evaluate the phenomenologically particularly interesting coefficients in the Gegenbauer expansion of the \(D\)-term \([9, 21]\). These coefficients have been extracted in \([9]\) from model results at physical values of \(t\) by numerical extrapolation to \(t = 0\), but with the results reported here they can be evaluated directly at the unphysical point \(t = 0\).

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