DISPERSE MIXED-ORDER SYSTEMS IN \(L^p\)-SOBOLEV SPACES AND APPLICATION TO THE THERMOELASTIC PLATE EQUATION

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Abstract. We study dispersive mixed-order systems of pseudodifferential operators in the setting of \(L^p\)-Sobolev spaces. Under the weak condition of quasi-hyperbolicity, these operators generate a semigroup in the space of tempered distributions. However, if the basic space is a tuple of \(L^p\)-Sobolev spaces, a strongly continuous semigroup is in many cases only generated if \(p = 2\) or \(n = 1\). The results are applied to the linear thermoelastic plate equation inertial term and with Fourier’s or Maxwell-Cattaneo’s law of heat conduction.

1. Introduction

Our investigation is motivated by the analysis of the linear thermoelastic plate equation in the whole space which is given by

\[
\begin{align*}
    u_{tt} + \Delta^2 u - \mu \Delta u_{tt} + \Delta \theta &= 0 \quad \text{in} \ (0, \infty) \times \mathbb{R}^n, \\
    \theta_t + \text{div} \ q - \Delta u_t &= 0 \quad \text{in} \ (0, \infty) \times \mathbb{R}^n, \\
    \tau q_t + q + \nabla \theta &= 0 \quad \text{in} \ (0, \infty) \times \mathbb{R}^n,
\end{align*}
\]

supplemented by initial conditions. In (1-1), the unknown functions are \(u, \theta, \) and \(q, \) where \(u\) describes the elongation of a plate and \(\theta\) and \(q\) model the temperature (relative to a fixed reference temperature) and the heat flux, respectively. The parameters \(\tau, \mu \geq 0\) are chosen depending on the underlying model. For \(\mu > 0,\) an inertial term is included, for \(\tau = 0\) the classical Fourier law of heat conduction is assumed, while for \(\tau > 0\) we take the Cattaneo-Maxwell law. System (1-1) was investigated in many papers, in particular in the setting of \(L^2\)-Sobolev spaces. We refer, e.g., to the papers by Lasiecka and Triggiani ([13], [14]), Racke and Ueda [21], Said-Houari [24], and Ueda, Duan, and Kawashima [25] and the references therein.

The aim of the present paper is to study system (1-1) and general mixed-order systems of pseudodifferential operators in the setting of \(L^p\)-Sobolev spaces for \(p \neq 2.\) It is well known that the wave equation is well-posed in \(L^p\) if and only if \(n = 1\) (see Littman [15], Peral [19]). Well-posedness in the \(L^p\)-setting for symmetric hyperbolic systems was investigated by Brenner [4]. For such systems, the symbol has the form \(a(\xi) = i \sum_{j=1}^n \xi_j a_j\) with symmetric matrices \(a_j \in \mathbb{R}^{N \times N},\) and it was shown that such a system gives rise to a well-posed Cauchy problem in \(L^p\) if and only if the matrices \(a_1, \ldots, a_n\) commute ([4], Theorem 1). In the present paper, we study more general mixed-order systems with symbol \(a(\xi) = (a_{ij}(\xi))_{i,j=1,\ldots,N}\) where each entry belongs to the Hörmander symbol class \(S^{\mu_{ij}}_{\alpha_j}(\mathbb{R}^n)\) of classical pseudodifferential operators of order \(\mu_{ij}.\) In order to solve the Cauchy problem

\[
(\partial_t - a(D))u(t) = 0 \ (t > 0), \quad u(0) = u_0,
\]

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in $\mathbb{R}^n$, one has to study the symbol $e^{\iota \alpha(\xi)}$. If the equation is quasi-hyperbolic (or correct in the sense of Petrovskii), the operator generates a locally uniformly bounded semigroup in the space $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N)$ of tempered distributions (see Theorem 2.2 below). For a survey on distributional Cauchy problems, we refer to the monograph by Ortner and Wagner [18]. The generation of a strongly continuous semigroup (or, equivalently, the well-posedness of the Cauchy problem) in $L^p$-Sobolev spaces can be described by a condition on the multiplier norm of the symbol $e^{\iota \alpha(\xi)}$, see Theorem 2.8. This result is a slight generalization of classical results by Brenner [4] and Hörmander [10]. We remark that the symbol $e^{\iota \alpha(\xi)}$ can formally also be seen as the symbol of a Fourier integral operator with matrix-valued and complex phase function. For the scalar (and homogeneous) case of phase functions, many results are known on dispersive estimates in $L^p$, see, e.g., Ruzhansky [23], and Coriasco and Ruzhansky [5]. The main problem in our case and for (1-1) is the mixed-order structure of the system.

The basic space for a general mixed-order system will be of the form $X_p = \prod_{j=1}^N H^s_j(\mathbb{R}^n)$, where $H^s_p(\mathbb{R}^n)$ stands for the standard (Bessel potential) Sobolev space. If $s$ is an integer, this space coincides with the classical Sobolev space $W^s_p(\mathbb{R}^n)$. By real interpolation, also Sobolev-Slobodetskii spaces $W^s_p(\mathbb{R}^n)$ for non-integer $s$ and Besov spaces $B^s_p(\mathbb{R}^n)$ can be considered. One of the main results of this paper, Theorem 3.11 below, states that dispersive mixed-order systems generate a $C_0$-semigroup in $X_p$ only in special cases. In particular, after order reduction due to the definition of the space $X_p$, the operator has to be of order one. Even if this holds, there are restrictions on the eigenvalues if $n > 1$. Roughly speaking, the general picture which is known for symmetric hyperbolic systems carries over to more general mixed-order systems.

In Section 4, we apply the above results to the thermoelastic plate equation (1-1). In the case $\tau = \mu = 0$, it is known that the related operator even generates an analytic semigroup in $L^p$ for every $p \in (1, \infty)$ (see Denk and Racke [6]). For the Cattaneo-Maxwell setting $\tau > 0$, a $C_0$-semigroup is generated in $L^p$, $p \neq 2$ only in the case $n = 1$ and $\mu > 0$ (Theorem 4.3 and Theorem 4.4). We remark here that for $\tau > 0$ and $\mu = 0$ the Cauchy problem is not well-posed even for $n = 1$.

For the Fourier law $\tau = 0$ (and $\mu > 0$), the generation of $C_0$-semigroups again holds if and only if $n = 1$ (Theorem 4.5). This result cannot be obtained by a straightforward application of the general results, as the relevant part of the symbol is still a combination of first- and second-order. The only nontrivial eigenvalue of the principal symbol (which is of second order) has negative real part which does not lead to a contradiction with the generation of a semigroup. Therefore, to prove Theorem 4.5, we explicitly apply an approximate diagonalization procedure (up to operators of order 0) which is motivated by the method in Kozhevnikov [12] (see also Denk, Saal, and Seiler [7]). This procedure gives a separation of the first-order and the second-order part of the symbol which yields the results on well-posedness in $L^p$.

2. Well-posedness of the Cauchy problem

In the following, let $\text{op}[a]$ be a mixed-order $N \times N$-system of pseudo-differential operators in $\mathbb{R}^n$ with $x$-independent symbols, i.e., $a = (a_{ij})_{i,j=1,\ldots,N}$, where $a_{ij} \in S^{\mu_{ij}}(\mathbb{R}^n)$, $\mu_{ij} \in \mathbb{R}$. Here, $S^0(\mathbb{R}^n) = S^0_{0,0}(\mathbb{R}^n)$ stands for the standard Hörmander class of $x$-independent symbols of order $\mu \in \mathbb{R}$, i.e., $S^\mu(\mathbb{R}^n)$ is the set of all smooth complex-valued functions $b \in C^\infty(\mathbb{R}^n)$ such that for each $\alpha \in \mathbb{N}_0^n$ there exists a $C_\alpha > 0$ satisfying
\[
|\partial_\xi^\alpha b(\xi)| \leq C_\alpha (\xi)^{\mu - |\alpha|} \quad (\xi \in \mathbb{R}^n).
\]
Here we have used the standard multi-index notation $\partial_\xi^\mu = \partial_{\xi_1}^{\mu_1} \ldots \partial_{\xi_n}^{\mu_n}$ and have set $(\xi) := (1+|\xi|^2)^{1/2}$. Note that in this situation we have $a \in S^\mu(\mathbb{R}^n; \mathbb{C}^{N \times N})$ with $\mu := \max_{i,j=1,\ldots,n} \mu_{ij}$. As usual, the pseudo-differential operator related to the symbol $a$ is defined by $\text{op}[a] \varphi = F^{-1} a F \varphi$ for all $\varphi$ belonging to the $\mathbb{C}^N$-valued Schwartz space $\mathscr{S} (\mathbb{R}^n; \mathbb{C}^N)$. In the above formula, $F$ stands for the Fourier transform which is defined by

$$(F \varphi)(\xi) := \hat{\varphi}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i \xi \cdot x} \varphi(x) \, dx \quad (\xi \in \mathbb{R}^n)$$

for $\varphi \in \mathscr{S} (\mathbb{R}^n; \mathbb{C}^N)$ and by duality extended to the space of tempered $\mathbb{C}^N$-valued distributions $\mathscr{S}'(\mathbb{R}^n; \mathbb{C}^N) := L(\mathscr{S}(\mathbb{R}^n); \mathbb{C}^N)$.

Let $\mathcal{O}_M(\mathbb{R}^n; \mathbb{C}^{N \times N})$ denote the space of all slowly increasing smooth functions, i.e., the space of all $a \in C^\infty (\mathbb{R}^n; \mathbb{C}^{N \times N})$ for which for each $\alpha \in \mathbb{N}_0^N$ there exist $C_\alpha, m_\alpha > 0$ such that

$$|\partial^\alpha a(\xi)|_{\mathbb{C}^{N \times N}} \leq C_\alpha |\xi|^{m_\alpha} \quad (\xi \in \mathbb{R}^n).$$

By definition of the Hörmander class, we have $S^\mu(\mathbb{R}^n; \mathbb{C}^{N \times N}) \subseteq \mathcal{O}_M(\mathbb{R}^n; \mathbb{C}^{N \times N})$. It was shown in [1], Thm. 1.6.4, that for $a \in \mathcal{O}_M(\mathbb{R}^n; \mathbb{C}^{N \times N})$ the multiplication operator $\varphi \mapsto a \varphi$ is a continuous linear operator belonging to $L(\mathscr{S}(\mathbb{R}^n; \mathbb{C}^N))$. Moreover, there exists a unique hypocontinuous and bilinear map

$$\mathcal{O}_M(\mathbb{R}^n; \mathbb{C}^{N \times N}) \times \mathscr{S}'(\mathbb{R}^n; \mathbb{C}^N) \to \mathscr{S}'(\mathbb{R}^n; \mathbb{C}_N), \quad (a, u) \mapsto au,$$

induced by the dual pairing

$$(au, \varphi)_{\mathscr{S}'(\mathbb{R}^n; \mathbb{C}^N)} = (u, a^T \varphi)_{\mathscr{S}'(\mathbb{R}^n; \mathbb{C}^N)} = \sum_{j=1}^N u_j \left( \sum_{k=1}^N a_{kj} \varphi_k \right)$$

([1], Thm. 1.6.4). Therefore, for $a \in \mathcal{O}_M(\mathbb{R}^n; \mathbb{C}^{N \times N})$, we obtain by this duality an operator $\text{op}[a] \in L(\mathscr{S}(\mathbb{R}^n; \mathbb{C}^N))$ (cf. also [1], Remark 1.9.11). For $a \in S^0(\mathbb{R}^n; \mathbb{C}^{N \times N})$, we consider the Cauchy problem

$$\begin{align*}
\partial_t u - \text{op}[a]u &= 0 \quad (t > 0), \\
u(0) &= u_0.
\end{align*}$$

(2-1)

The following definition of quasi-hyperbolicity is classical and can be found, e.g., in [17]. This condition is also called correct in the sense of Petrovskii or Petrovskiǐ condition, see [8], Definition 2 on p. 168, and [11], p. 143.

**Definition 2.1.** Let $a \in \mathcal{O}_M(\mathbb{R}^n; \mathbb{C}^{N \times N})$. Then the Cauchy problem (2-1) is called quasi-hyperbolic if there exists a constant $M_a \in \mathbb{R}$ such that

$$\det (\lambda - a(\xi)) \neq 0 \quad (\text{Re} \lambda > M_a, \xi \in \mathbb{R}^n).$$

(2-2)

For a survey on distributional Cauchy problems and fundamental solutions, we mention the monograph [18]. For differential operators, the following result can be found in [3], Proposition 3.

**Theorem 2.2.** Let $a \in S^0(\mathbb{R}^n; \mathbb{C}^{N \times N})$, and assume that equation (2-1) is quasi-hyperbolic. Then for every $u_0 \in \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)$ there exists a unique solution $u \in C^1([0,\infty); \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N))$ of (2-1). This solution is given by $u(t) = \text{op}[e^{at(\xi)}]u_0$. Moreover, the family $(T(t))_{t \geq 0}$ with $T(t) := \text{op}[e^{at(\xi)}]$ is a locally uniformly bounded semigroup on $\mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)$. The analog results hold with $\mathscr{S}'(\mathbb{R}^n; \mathbb{C}_N)$ being replaced by $\mathscr{S}'(\mathbb{R}^n; \mathbb{C}^N)$.

**Proof.** By (2-2), we see that for all $\lambda \in \mathbb{C}$ with $\text{Re} \lambda > M_a + 1$, all eigenvalues of the matrix $\lambda - a(\xi)$ have real part not less than 1. Therefore, $|\det(\lambda - a(\xi))| \geq 1$
if Re\(\lambda \geq M_a + 1\). By Cramer’s rule, every entry of the matrix \((\lambda - a(\xi))^{-1}\) is a quotient of the form \(\frac{c'_{ij}(\xi, \lambda)}{\det(\lambda - a(\xi))}\), where \(c_{ij}(\xi, \lambda)\) stands for the cofactor.

It is well-known from the theory of pseudo-differential operators that sums and products of scalar symbols in \(S^\ast := \bigcup_{\nu \in \mathbb{R}} S^n(\mathbb{R}^n)\) belong to \(S^\ast\) again. We also remark that derivatives with respect to \(\xi\) of \(\frac{c'_{ij}(\xi, \lambda)}{\det(\lambda - a(\xi))}\) are again of the form \(\frac{c_{ij}(\xi, \lambda)}{\det(\lambda - a(\xi))}\) with some \(m \in \mathbb{N}\), where \(c_{ij}\) depend polynomially on the entries of the matrix \(\lambda - a(\xi)\).

From this and the above estimate on the determinant we obtain

\[(\lambda - a(\xi))^{-1} \in S^\ast(\mathbb{R}^n; \mathbb{C}^{N \times N}) \quad (\Re \lambda > M_a + 1)\]

for some \(\tilde{\mu} \in \mathbb{R}\). In particular, \((\lambda - a(\xi))^{-1} \in \mathcal{O}_M(\mathbb{R}^n; \mathbb{C}^{N \times N})\) if \(\Re \lambda > M_a + 1\). Therefore, Theorem 3.1.3 in [1] can be applied which states that there exists a unique fundamental solution for the Cauchy problem (2-1). This implies that (2-1) has a unique distributional solution (cf. [1], Theorem 3.1.1). On this other hand, by [1], Remark 3.2.3(c), \(T(t) := \text{op}[e^{\mu t}]\) defines a locally uniformly bounded semigroup \((T(t))_{t \geq 0}\) on \(\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N)\) and on \(\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N)\), and the unique solution \(u\) of (2-1) is given by \(u(t) = T(t)u_0\) for \(t > 0\).

Whereas well-posedness of the Cauchy problem (2-1) holds in \(\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N)\) under very weak assumptions, the situation is different if we consider \(\text{op}[a(t)]\) as an unbounded operator in some Banach space \(X \subset \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N)\). In particular, we are interested in the case \(X = L^p(\mathbb{R}^n; \mathbb{C}^N)\).

**Definition 2.3.** Let \(X\) be a Banach space with norm \(\| \cdot \|\), and let \(A: X \supset D(A) \to X\) be a closed and densely defined linear operator. Then the Cauchy problem

\[
\begin{align*}
\partial_t u - Au &= 0 \quad (t > 0) \\
u(0) &= u_0
\end{align*}
\]

is called well-posed if for every \(u_0 \in D(A)\) there exists a unique (classical) solution \(u \in C^1([0, \infty), X)\) of (2-3) with \(u(t) \in D(A)\) \((t > 0)\), and if for all \(T > 0\) there exists a constant \(C_T > 0\) such that for all \(u_0 \in D(A)\) we have that

\[\|u(t)\| \leq C_T \|u_0\| \quad (t \in [0, T]).\]

(2-4)

It is well known that well-posedness is equivalent to the generation of a \(C_0\)-semigroup in \(X\). On the other hand, this is also equivalent to the existence of a mild solution for all initial values \(u_0 \in X\). First, we give the definition (see, e.g., [2], Def. 3.1.1).

**Definition 2.4.** A function \(u \in C([0, \infty), X)\) is called a mild solution of the Cauchy problem (2-3) if for all \(t \in [0, \infty)\) we have

\[
\int_0^t u(s)ds \in D(A) \quad \text{and} \quad A \int_0^t u(s)ds = u(t) - u_0.
\]

**Theorem 2.5.** Let \(A: X \supset D(A) \to X\) be a closed densely defined linear operator. Then the following statements are equivalent:

(i) The operator \(A\) generates a \(C_0\)-semigroup on \(X\).
(ii) For all \(u_0 \in X\) there exists a unique mild solution of (2-3).
(iii) There exists a subspace \(D \subset D(A)\) which is dense in \(X\) such that for all \(u_0 \in D\) the Cauchy problem (2-3) has a unique classical solution \(u\), and for every \(T > 0\) there exists \(C_T > 0\) such that (2-4) holds for all \(u_0 \in D\).

**Proof.** The equivalence of (i) and (ii) is shown in [2], Theorem 3.1.12. In the same theorem, it is also shown that (i) implies well-posedness of the Cauchy problem...
Let $u_0 \in X$. Since $D$ is dense in $X$, there is a sequence $(u_{k,0})_{k \in \mathbb{N}} \subset D$ such that $u_{k,0} \to u_0$ in $X$ as $k \to \infty$. For all $k \in \mathbb{N}$ let $u_k \in C^1([0, \infty), X)$ be the classical solution to (2.3) with initial value $u_{k,0}$. Since $A$ is closed, the $u_k$ are mild solutions (cf. [2], Prop. 3.1.2). By assumption, for any $T > 0$ there is a constant $C_T > 0$ such that

$$
\| u_k(t) - u_\ell(t) \| \leq C_T \| u_{k,0} - u_{\ell,0} \| \to 0 \quad (\ell, k \to \infty)
$$

Therefore, $(u_{k,0})_{k \in \mathbb{N}}$ converges uniformly on compact subsets of $[0, \infty)$. We define the limit $u(t) := \lim_{n \to \infty} u_k(t)$ $(t > 0)$. Due to the estimate (2.4) for $u_0 \in D$, the definition of $u(t)$ does not depend on the chosen sequence $(u_{k,0})_{k \in \mathbb{N}} \subset D$. Moreover, we get that

$$
\int_0^t u(s) \ ds = \lim_{k \to \infty} \int_0^t u_k(s) \ ds
$$

and

$$
\lim_{k \to \infty} A \int_0^t u_k(s) \ ds = \lim_{k \to \infty} (u_k(t) - u_{k,0}) = u(t) - u_0
$$

for all $t \geq 0$. By the closedness of $A$, we get that $\int_0^t u(s) \ ds \in D(A)$ as well as $A \int_0^t u(s) \ ds = u(t) - u_0$. It remains to show that this mild solution is unique. Let $v \in C([0, \infty), X)$ be another mild solution. Then, for all $t \geq 0$ we get

$$
u(t)(v(t) - A \int_0^t (u(s) - v(s)) \ ds = 0.
$$

Defining $w(t) := \int_0^t (u(s) - v(s)) \ ds$ we obtain a classical solution for the Cauchy problem to the initial value $w(0) = 0$. By (2.4) with $u_0 = 0 \in D$ we obtain $w(t) = 0$ for all $t \geq 0$ and therefore $u = v$. □

For mixed-order systems of pseudo-differential operators, the closedness of $\text{op}[a]$ holds if we consider the maximal domain. More precisely, for $p \in (1, \infty)$ and $s \in \mathbb{R}$ let $H^s_p(\mathbb{R}^n)$ denote the Bessel potential space $H^s_p(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) : \text{op}[\langle \cdot \rangle^s] u \in L^p(\mathbb{R}^n) \}$ with its canonical norm. Then we obtain the following result.

**Lemma 2.6.** Let $s_1, \ldots, s_n \in \mathbb{R}$, and let $X := \prod_{j=1}^n H^s_j(\mathbb{R}^n)$. For the symbol $a \in S^0(\mathbb{R}^n; \mathcal{C}^N)$, define the unbounded operator $A := \text{op}[a]$ in $X$ with (maximal) domain $D(A) := \{ u \in X : A u \in X \}$. Then $A$ is densely defined and closed.

**Proof.** Because of $\mathcal{S}'(\mathbb{R}^n; \mathcal{C}^N) \subset D(A)$, $A$ is densely defined. Let $(u_k)_{k \in \mathbb{N}} \subset D(A)$ with $u_k \to u$ in $X$ and $Au_k \to v$ in $X$ for $k \to \infty$. By the continuity of the embedding $X \subset \mathcal{S}'(\mathbb{R}^n; \mathcal{C}^N)$, we get that $Au_k \to v$ in $\mathcal{S}'(\mathbb{R}^n; \mathcal{C}^N)$. On the other hand, we also have $u_k \to u$ in $\mathcal{S}'(\mathbb{R}^n; \mathcal{C}^N)$, and $\text{op}[a]$ is continuous in $\mathcal{S}'(\mathbb{R}^n; \mathcal{C}^N)$ which gives $Au_k \to Au$ in $\mathcal{S}'(\mathbb{R}^n; \mathcal{C}^N)$.

Since $\mathcal{S}'(\mathbb{R}^n, \mathcal{C}^N)$ is a Hausdorff space, it follows that $Au = v$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{C}^N)$ and by the injectivity of the embedding $X \hookrightarrow \mathcal{S}'(\mathbb{R}^n; \mathcal{C}^N)$, we get $u \in D(A)$ and $Au = v$ in $X$. □

For the investigation of $C_0$-semigroups in the space $X$ from the previous lemma, the notion of an $L^p$-Fourier multiplier is useful (cf. [1], Def. 1.3, and [4], Section 2). In the following, we always assume $p \in (1, \infty)$.

**Definition 2.7.** A function $m \in L^\infty(\mathbb{R}^n; \mathcal{C}^{N \times N})$ is called an $L^p$-Fourier multiplier if there exists a constant $c_p > 0$ such that for all $u \in \mathcal{S}'(\mathbb{R}^n; \mathcal{C}^N)$ we have $\text{op}[m]u := \mathcal{F}^{-1} m \mathcal{F}u \in L^p(\mathbb{R}^n; \mathcal{C}^N)$ and

$$
\| \text{op}[m]u \|_{L^p(\mathbb{R}^n; \mathcal{C}^N)} \leq c_p \| u \|_{L^p(\mathbb{R}^n; \mathcal{C}^N)} \quad (u \in \mathcal{S}'(\mathbb{R}^n; \mathcal{C}^N)).
$$
In this case, \( \text{op}[m] \) extends by continuity to a bounded linear operator \( \text{op}[m] \in L(L^p(\mathbb{R}^n; C^N)) \). We denote by \( M^N_p \), the space of all \( L^p \)-Fourier multipliers and endow \( M^N_p \) with the norm \( \|m\|_{M^N_p} := \|\text{op}[m]\|_{L(L^p(\mathbb{R}^n; C^N))}. \)

**Theorem 2.8.** Let \( a \in S^0(\mathbb{R}^n; C^{N \times N}) \) be quasi-hyperbolic, and define the unbounded operator \( A \) in \( X := L^p(\mathbb{R}^n; C^N) \) by \( A = \text{op}[a] \) and \( D(A) := \{u \in X : Au \in X\} \). Then the Cauchy problem (2.3) is well-posed if and only if for all \( T > 0 \) there is a \( C_T > 0 \) such that

\[
\|e^{ta(t)}\|_{M^N_p} \leq C_T \quad (t \in [0, T]). \tag{2.5}
\]

In this case, the semigroup generated by \( A \) is given by \( \text{op}[e^{ta(t)}] \) \( t \geq 0 \).

**Proof.** (i) Let (2.3) be well-posed, and let \( u_0 \in D(A) \). From Theorem 2.2, we know that \( u(t) := \text{op}[e^{ta(t)}]u_0 \) is the unique solution in \( \mathcal{S}(\mathbb{R}^n; C^N) \). By the definition of well-posedness, for \( T > 0 \) there exists a \( C_T > 0 \) such that

\[
\|u(t)\|_{L^p(\mathbb{R}^n; C^N)} = \|\text{op}[e^{ta(t)}]u_0\|_{L^p(\mathbb{R}^n; C^N)} \leq C_T \|u_0\|_{L^p(\mathbb{R}^n; C^N)} \quad (t \in [0, T]).
\]

As \( \mathcal{S}(\mathbb{R}^n; C^N) \subset D(A) \), the function \( e^{ta(t)} \) is an \( L^p \)-Fourier multiplier, and its multiplier norm satisfies (2.5).

(ii) Assume now that (2.5) holds. We define \( m(t, \xi) := e^{ta(\xi)} \) and fix the initial value \( u_0 \in \mathcal{S}(\mathbb{R}^n; C^N) \). By Theorem 2.2, for \( u(t) := \text{op}[e^{ta(t)}]u_0 \) we obtain \( u \in C^1([0, \infty), \mathcal{S}(\mathbb{R}^n; C^N)) \) with

\[
\partial_t u(t) = \text{op}[a]u(t) = \text{op}[a] \text{op}[e^{ta(t)}]u_0 = \text{op}[e^{ta(t)}] \text{op}[a]u_0 = \text{op}[e^{ta(t)}]Au_0 \tag{2.6}
\]

in \( \mathcal{S}(\mathbb{R}^n; C^N) \) for every \( t \geq 0 \). An iteration gives \( \partial^2 u(t) = \text{op}[e^{ta(t)}]A^2 u_0 \). In particular, we obtain \( u(t) \in L^p(\mathbb{R}^n; C^N) \) and \( Au(t) \in L^p(\mathbb{R}^n; C^N) \) and therefore \( u(t) \in D(A) \) for every \( t \geq 0 \).

Applying twice the fundamental theorem of calculus, we get as equality in \( \mathcal{S}(\mathbb{R}^n; C^N) \) for \( t, h \geq 0 \):

\[
\frac{1}{t} (u(t + h) - u(t)) - \text{op}[a]u(t) = \int_0^1 \int_0^1 sh \text{op}[e^{t+sh}a(t)]A^2 u_0 dr ds. \tag{2.7}
\]

By assumption, \( \|e^{ta(t)}\|_{M^N_p} \) is uniformly bounded on bounded intervals. Therefore, we can estimate the \( L^p \)-norm of the right-hand side of (2.7) by

\[
\int_0^1 \int_0^{t+h} s|h| \left( \sup_{\tau \in [t, t+h]} \|e^{ta(t)}\|_{M^N_p} \right) \|A^2 u_0\|_{L^p(\mathbb{R}^n; C^N)} dr ds \leq C|a| \|A^2 u_0\|_{L^p(\mathbb{R}^n; C^N)}.
\]

The same argument holds for \( t \geq 0 \) and \( h < 0 \) with \( t+h \geq 0 \). Therefore, we see that the left-hand side of (2.7) tends to zero in \( L^p(\mathbb{R}^n; C^N) \) for \( h \to 0 \). Consequently, we have \( \partial_t u(t) = Au(t) \) in \( L^p(\mathbb{R}^n; C^N) \) for every \( t \geq 0 \).

In particular, the above differentiability yields \( u \in C([0, \infty), L^p(\mathbb{R}^n; C^N)) \). Due to the identity (2.6), \( \partial_t u \) is a solution of the Cauchy problem (2.1) with \( u_0 \) being replaced by \( Au_0 \). Therefore, \( \partial_t u \) is continuous, too, and we have that \( u \in C^1([0, \infty), L^p(\mathbb{R}^n; C^N)) \) is a classical solution.

By the assumption (2.5),

\[
\|u(t)\|_{L^p(\mathbb{R}^n; C^N)} = \|\text{op}[e^{ta(t)}]u_0\|_{L^p(\mathbb{R}^n; C^N)} \leq C_T \|u_0\|_{L^p(\mathbb{R}^n; C^N)} \quad (t \in [0, T]).
\]

Therefore all assumptions of Theorem 2.5 (iii) are satisfied with \( D = \mathcal{S}(\mathbb{R}^n; C^N) \), and by Theorem 2.5 (i) we see that \( A \) generates a \( C_0 \)-semigroup which implies well-posedness of (2.3). \( \square \)

**Corollary 2.9.** If in the situation of Theorem 2.8 the Cauchy problem (2.3) is well-posed in \( L^p(\mathbb{R}^n; C^N) \) then it is well-posed in every \( L^r(\mathbb{R}^n; C^N) \) with \( r \in [\min\{p, q\}, \max\{p, q\}] \). Here \( q \) is the conjugate exponent to \( p \), i.e. \( \frac{1}{p} + \frac{1}{q} = 1 \).
Proof. This follows from the equivalence in Theorem 2.8 and the fact that $M_p^N = M_q^N \subset M^N$, see [10], Theorem 1.3. □

**Corollary 2.10.** Let $a \in S^p(\mathbb{R}^n; C^{N \times N})$ be quasi-hyperbolic, and define $A$ in $X := \prod_{j=1}^N H^s_p(\mathbb{R}^n)$ as in Lemma 2.6. Then the following statements are equivalent:

(i) The Cauchy problem (2-3) is well-posed in $X = \prod_{j=1}^N H^s_p(\mathbb{R}^n)$.

(ii) Let $\Lambda(\xi) := \text{diag}((\xi)^{s_1}, \ldots, (\xi)^{s_n})$. Then for all $T > 0$ there exists a $C_T > 0$ such that

$$\|\Lambda a(t)\Lambda^{-1}\|_{M_p^N} \leq C_T \quad (t \in [0, T]).$$

(iii) For every $s \in \mathbb{R}$, the Cauchy problem is well-posed in $X = \prod_{j=1}^N H^{s_j+s_p}(\mathbb{R}^n)$.

Proof. By definition, $\text{op}[A] \colon X \rightarrow L^p(\mathbb{R}^n; C^N)$ is an isometric isomorphism. Therefore, a function $u \in C^1([0, \infty), X)$ is a classical solution to (2-3) with the initial value $u_0 \in D(A)$ if and only if $\tilde{u} := \text{op}[A]u \in C^1([0, \infty), L^p(\mathbb{R}^n; C^N))$ is a classical solution of

$$\text{op}[\Lambda^{-1}]\partial_t \tilde{u} - \text{op}[a] \text{op}[\Lambda^{-1}] \tilde{u} = 0 \quad (t > 0),$$

$$\tilde{u}(0) = \tilde{u}_0 \quad (2-9)$$

with $\tilde{u}_0 := \text{op}[A]u_0$. Also the continuous dependence on the initial value (inequality (2-4)) is maintained. Applying $\text{op}[A]$ to the first line in (2-9), we see that (2-3) is well-posed in $X$ if and only if

$$\partial_t \tilde{u} - \text{op}[a] \tilde{u} = 0 \quad (t > 0),$$

$$\tilde{u}(0) = \tilde{u}_0 \quad (2-10)$$

is well-posed in $L^p(\mathbb{R}^n; C^N)$, with $\tilde{a} := \Lambda a \Lambda^{-1}$. Now Theorem 2.8 yields the equivalence of (i) and (ii) if we take into account that $\Lambda(\xi) \exp(ta(\xi))\Lambda^{-1}(\xi) = \exp\left(t\Lambda(\xi)a(\xi)\Lambda^{-1}(\xi)\right)$.

As we have

$$\text{diag}((\xi)^{s_1+s}, \ldots, (\xi)^{s_n+s})a(\xi)\text{diag}((\xi)^{-s_1-s}, \ldots, (\xi)^{-s_n-s}) = \text{diag}((\xi)^{s_1}, \ldots, (\xi)^{s_n})a(\xi)\text{diag}((\xi)^{-s_1}, \ldots, (\xi)^{-s_n}),$$

condition in (ii) holds for $s_1, \ldots, s_n$ if and only if it holds for $s_1+s, \ldots, s_n+s$ for any $s \in \mathbb{R}$. This gives the equivalence of condition (iii) to (i) and (ii). □

**Remark 2.11.** a) For $s \in \mathbb{R}$, $p \in (1, \infty)$, and $q \in [1, \infty]$, let $B^s_{pq}(\mathbb{R}^n)$ denote the standard Besov space. Then, if the conditions of Corollary 2.10 are satisfied, the Cauchy problem is well-posed in the space $\prod_{j=1}^N B^s_{pq}(\mathbb{R}^n)$. This follows by real interpolation, as, e.g., $B^s_{pq}(\mathbb{R}^n) = (H^{s-1}_p(\mathbb{R}^n), H^{s+1}_p(\mathbb{R}^n))_{1/2, q}$. In particular, we get well-posedness in the Sobolev-Slobodeckii spaces $W^s_p(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^n)$, $s \notin \mathbb{Z}$.

b) In the situation of Corollary 2.10, consider the perturbed Cauchy problem

$$\partial_t u - \text{op}[a]u - \text{op}[b]u = 0 \quad (t > 0),$$

$$u(0) = u_0 \quad (2-11)$$

If $a$ satisfies (2-8) and if $b : \mathbb{R}^n \rightarrow C^{N \times N}$ is a function satisfying $\tilde{b} := \Lambda b \Lambda^{-1} \in M_p^N$, then (2-11) is well-posed in $X = \prod_{j=1}^N H^s_p(\mathbb{R}^n)$. This follows from the fact that $\text{op}[a] + \text{op}[\tilde{b}]$ is a bounded perturbation of $\text{op}[a]$ in (2-10), and the set of generators of $C_0$-semigroups is stable under bounded perturbations (see [2], Corollary 3.5.6).
3. Multipliers and mixed-order systems in $L^p$-spaces

In this section, we want to investigate in which cases condition (2.5) from Theorem 2.8 can hold. We will consider systems of classical (polyhomogeneous) pseudo-differential operators with constant ($x$-independent) coefficients. Therefore, we start with the definition of homogeneity.

**Definition 3.1.** Let $d \in \mathbb{R}$. A function $a \in C(\mathbb{R}^n \setminus \{0\}, \mathbb{C}^{N \times N})$ is called homogeneous of degree $d$ if there exists an $R > 0$ such that

$$f(t\xi) = t^d f(\xi)$$

holds for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq R$ and all $t > 1$. If this equality holds for all $\xi \neq 0$ and all $t > 1$, then $a$ is called strictly homogeneous.

Let $R > 0$, and let $V$ be an open subset of the unit sphere $S^{n-1} := \{ \eta \in \mathbb{R}^n : |\eta| = 1 \}$. If (3-1) holds for all $t > 1$ and all $\xi$ in a truncated cone of the form

$$S_{R,V} := \{ r\eta : r > R, \eta \in V \},$$

then $a$ is called homogeneous in $S_{R,V}$.

**Remark 3.2.** If $a \in C^{[n/2]+1}(\mathbb{R}^n \setminus \{0\}, \mathbb{C}^{N \times N})$ is strictly homogeneous of degree 0, then every derivative of order $k$ is strictly homogeneous of degree $-k$. Therefore, $a \in M^N_N$ by the theorem of Mikhlin (see, e.g., [2], Theorem E.3).

We want to compare the multiplier properties of a matrix-valued function and the eigenvalues of the matrix. For this, we start with two remarks on the smoothness of eigenvalues and eigenvectors which should be well known but which we could not find in literature.

**Lemma 3.3.** Let $U \subset \mathbb{R}^n$ be open and non-empty, and let $a \in C^\infty(U, \mathbb{C}^{N \times N})$. Then there exists an open non-empty set $\tilde{U} \subset U$ such that (with appropriate numbering) the eigenvalues $\lambda_1(\xi), \ldots, \lambda_N(\xi)$ of $a(\xi)$ satisfy $\lambda_1, \ldots, \lambda_N \in C^\infty(\tilde{U})$.

**Proof.** We prove the statement by induction on $N \in \mathbb{N}$, the case $N = 1$ being trivial. For $\xi \in U$, let $p(\lambda, \xi) := \det(\lambda - a(\xi)) = \prod_{j=1}^N (\lambda - \lambda_j(\xi))$. We may assume that the numbering of the eigenvalues is chosen such that all $\lambda_j$ are continuous (see [22], Chapter 1.3). By induction, there exists an open non-empty set $U_0 \subset U$ and $\tau_1, \ldots, \tau_{N-1} \in C^\infty(U_0)$ such that $\partial_{p(\lambda, \xi)} = N \prod_{j=1}^{N-1} (\lambda - \tau_j(\xi))$. (Note that for the induction we formally have to write the zeros of the polynomial $\partial_{p(\lambda, \cdot)}$ as the eigenvalues of its companion matrix.) By continuity of $\lambda_1, \tau_1, \ldots, \tau_{N-1}$, the set

$$W := \bigcap_{j=1}^{N-1} \{ \xi \in U_0 : \lambda_1(\xi) \neq \tau_j(\xi) \}$$

is open. Moreover, the implicit function theorem yields $\lambda_1|_W \in C^\infty(W)$. Therefore, if $W \neq \emptyset$ we define $U_1 := W$. If $W = \emptyset$, then we choose $A \subset U_0$ as the closure of a non-void open ball contained in $U_0$. Again by continuity it follows that the sets

$$A_j := \{ \xi \in A : \lambda_1(\xi) = \tau_j(\xi) \}$$

for $j = 1, \ldots, N-1$ are closed. Moreover, since $W = \emptyset$ we have that that

$$A = U_0 \cap A = \bigcup_{j=1}^{N-1} A_j.$$ 

Therefore, there exists a $j \in \{1, \ldots, N-1\}$ such that $A_j$ contains a non-void open ball $U_1$ (this can be seen, e.g., by an application of the Baire category theorem). As $\lambda_1|_{U_1} = \tau_j|_{U_1}$, we have $\lambda_1 \in C^\infty(U_1)$.
Replacing now $U$ by $U_1$ and repeating the same procedure for $\lambda_2, \ldots, \lambda_N$, we obtain open sets $U_0 \supset U_1 \supset \ldots \supset U_N \neq \emptyset$ with $\lambda \in C^\infty(U)$. In particular, we have $\lambda_1, \ldots, \lambda_N \in C^\infty(U_N)$. Setting $\tilde{U} := U_N$, we obtain the statement.

\textbf{Lemma 3.4.} Let $U \subset \mathbb{R}^n$ be open and non-empty, and let $a \in C^\infty(U; \mathbb{C}^{n \times N})$ and $\lambda \in C^\infty(U)$ such that $\lambda(\xi)$ is an eigenvalue of $a(\xi)$ for each $\xi \in U$. Then there exists an open non-empty set $\tilde{U} \subset U$ and a function $v \in C^\infty(\tilde{U}; \mathbb{C}^N)$ such that $v(\xi)$ is an eigenvector to the eigenvalue $\lambda(\xi)$.

\textbf{Proof.} We define $b \in C^\infty(U; \mathbb{C}^{N \times N})$ by $b(\xi) := a(\xi) - \lambda(\xi)I_N$, so we have to consider the kernel of $b$. We set $k := \max_{\xi \in U} \text{rank } b(\xi) < N$ and choose $\xi_0 \in U$ with $\text{rank } b(\xi_0) = k$. Without loss of generality, we may assume that the upper $k \times k$ corner of $b(\xi_0)$ is invertible. Accordingly, we write

$$b(\xi) = \begin{pmatrix} b(1,1)(\xi) & b(1,2)(\xi) \\ b(2,1)(\xi) & b(2,2)(\xi) \end{pmatrix}$$

with $b(1,1) \in C^\infty(U; \mathbb{C}^{k \times k})$, $b(1,2) \in C^\infty(U; \mathbb{C}^{k \times (N-k)})$, $b(2,1) \in C^\infty(U; \mathbb{C}^{(N-k) \times k})$ and $b(2,2) \in C^\infty(U; \mathbb{C}^{(N-k) \times (N-k)})$. By continuity, there exists an open ball $\tilde{U} \subset U$ such that $b(1,1)(\xi)$ is invertible for all $\xi \in \tilde{U}$. By the definition of $k$, for all $\xi \in \tilde{U}$ the last $N-k$ columns are linear combinations of the first $k$ columns. Therefore, we obtain

$$b(\xi) = \begin{pmatrix} b(1,1)(\xi) & b(1,2)(\xi) \\ b(2,1)(\xi) & b(2,2)(\xi) \end{pmatrix}$$

where $c(\xi) := (b(1,1)(\xi))^{-1}b(1,2)(\xi)$. Note that $c \in C^\infty(\tilde{U}; \mathbb{C}^{k \times (N-k)})$. Let $e_1$ be the first unit vector in $\mathbb{C}^{N-k}$, and set $v(\xi) := c(\xi)e_1 \in \mathbb{C}^N$. Then $v(\xi)$ is an eigenvector of $a(\xi)$ to the eigenvalue $\lambda(\xi)$ for all $\xi \in \tilde{U}$ and depends smoothly on $\xi$.

The following definition is essentially taken from [4], p. 30.

\textbf{Definition 3.5.} Let $U \subset \mathbb{R}^n$ be open, and let $m \in L^\infty(U; \mathbb{C}^{N \times N})$. Then $m$ is called a local $L^p$-Fourier multiplier in $U$ if there exists $\tilde{m} \in M_N^p$ such that $\tilde{m}|_U = m$. The space of all such functions will be denoted by $M_N^p(U)$. For $m \in M_N^p(U)$, we define $\|m\|_{M_N^p(U)}$ as the supremum over all $\|\hat{f}\|_{L_p(\mathbb{R}^n; \mathbb{C}^N)}$ where $f \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ with $\text{supp } \hat{f} \subset U$ and $\|f\|_{L_p(\mathbb{R}^n; \mathbb{C}^N)} \leq 1$ and where $\tilde{m} \in M^p_N$ with $\tilde{m}|_U = m$.

\textbf{Remark 3.6.} a) Let $a \in C^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{N \times N})$. If $a$ is homogeneous of degree $d \leq 0$, then $a$ is a local $L^p$-Fourier multiplier in $U := \{ \xi \in \mathbb{R}^n : |\xi| > \varepsilon \}$ for all $\varepsilon > 0$ by the theorem of Mikhlin, applied to a smooth extension of $a|_U$. Similarly, if $a$ is strictly homogeneous of degree $d > 0$, then $a$ is a local $L^p$-Fourier multiplier in $\{ \xi \in \mathbb{R}^n \setminus \{0\} : |\xi| < R \}$ for all $R > 0$.

b) Let $R > 0$ and let $V \subset S^{n-1}$ be open. If $a \in C^\infty(S_{R,V}; \mathbb{C}^{N \times N})$ is homogeneous in $S_{R,V}$ of degree $d \in \mathbb{R}$, then there exists an open subset $\tilde{V} \subset V$ such that the eigenvalues and eigenvectors of $a$ are smooth and homogeneous in $S_{R,V}$ of degree $d$. In fact, by Lemma 3.3 and 3.4, there exists an open set $U_0 \subset S_{R,V}$ where the eigenvalues and eigenvectors are smooth. We choose $r_0 > R$ such that $V_0 := r_0S^{n-1} \cap U_0 \neq \emptyset$ and set $\tilde{V} := r_0^{-1}V_0 \subset S^{n-1}$. Then we can extend the eigenvalues and eigenvectors by homogeneity from $V_0$ to $S_{R,V}$.

\textbf{Theorem 3.7.} In the situation of Theorem 2.8, assume that $a \in S^\infty(\mathbb{R}^n; \mathbb{C}^{N \times N})$ is homogeneous of degree $\mu \in \mathbb{R}$.

a) If $\mu \leq 0$, then the Cauchy problem \eqref{eq:2} is well-posed in $L^p(\mathbb{R}^n; \mathbb{C}^N)$ for all $p \in (1, \infty)$.
b) Let $\mu > 0$, and assume that for sufficiently large $\xi \in \mathbb{R}^n$ all eigenvalues of $a(\xi)$ have negative real part. Then the Cauchy problem (2-3) is well-posed for all $p \in (1, \infty)$.

c) Let $\mu > 0$, and assume that there exists an eigenvalue $\lambda(\xi)$ of $a(\xi)$ with $\lambda(\xi) \in i\mathbb{R} \setminus \{0\}$ for all $\xi \in S_{R,V}$ for some $R > 0$ and some open set $\emptyset \neq V \subset S^{n-1}$. If (2-3) is well-posed for some $p \neq 2$, then $\mu = 1$.

Proof. a) It is well known that $A = \text{op}[a]$ is a bounded operator in $L^p(\mathbb{R}^n, \mathbb{C}^N)$ for all $p \in (1, \infty)$ if $a \in S^0(\mathbb{R}^n, \mathbb{C}^{N \times N})$, see, e.g., [26], Theorem 10.7. Therefore, $A$ generates a $C_0$-semigroup in $L^p(\mathbb{R}^n, \mathbb{C}^N)$.

b) Under the assumption b), the symbol $a$ satisfies the classical condition of parameter-ellipticity, i.e. we have

$$\det(\lambda - a(\xi)) \neq 0 \quad (\text{Re} \lambda \geq 0, |\xi| \geq R)$$

for sufficiently large $R > 0$. Therefore, the operator $A$ even generates a holomorphic semigroup and (2-3) is well-posed in $L^p(\mathbb{R}^n, \mathbb{C}^N)$ (see, e.g., [9], Theorem 1.7 and Theorem 2.3).

c) We may assume that $a$ is homogeneous in $S_{R,V}$. Moreover, by Remark 3.6 b), we may also assume that $\lambda \in C^\infty(S_{R,V})$, that we have an eigenvector $v \in C^\infty(S_{R,V}, \mathbb{C}^N)$ and that both $\lambda$ and $v$ are homogeneous in $S_{R,V}$ of degree $\mu$.

Let $\emptyset \neq U \subset S_{R,V}$ be an open ball. As $v(\xi) \neq 0$ for all $\xi \in U$, we can apply [4], Lemma 4, which yields that there exists an open $\emptyset \neq U_0 \subset U$ such that for all $f \in \mathcal{H}(\mathbb{R}^n)$ with supp $f \subset U_0$ we have

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C \|\text{op}[v]f\|_{L^p(\mathbb{R}^n, \mathbb{C}^N)}. \quad (3-2)$$

Here, $\text{op}[v]f := (\text{op}[v_1]f, \ldots, \text{op}[v_n]f)^\top$ for $v = (v_1, \ldots, v_n)^\top$. Let $k \in \mathbb{N}$. We choose $f \in \mathcal{H}(\mathbb{R}^n)$ with supp $f \subset U_0$. We apply (3-2) to $\text{op}[e^{\lambda(\xi)}]f$ instead of $f$ and obtain

$$\|\text{op}[e^{\lambda(\xi)}]f\|_{L^p(\mathbb{R}^n)} \leq C \|\text{op}[e^{\lambda(\xi)}v(\xi)]f\|_{L^p(\mathbb{R}^n, \mathbb{C}^N)}$$

$$= C \|\text{op}[e^{\lambda(\xi)}v(\xi)]f\|_{L^p(\mathbb{R}^n, \mathbb{C}^N)}$$

$$= C \|\text{op}[e^{a(\xi)}v(\xi)]f\|_{L^p(\mathbb{R}^n, \mathbb{C}^N)}.$$ 

For the last equality, we used the homogeneity of $a$ in $S_{R,V}$ and supp $\hat{f} \subset S_{R,V}$ (here we also used $\mu > 0$). Let $v_0 \in \mathcal{H}(\mathbb{R}^n)$ be an extension of $v|_{U_0}$. From the elementary fact that

$$\|e^{a(\xi)}\|_M^p = \|e^{a(\xi)}\|_M^p$$

(see [2], Proposition E.2 e)), we obtain

$$\|\text{op}[e^{\lambda(\xi)}f]\|_{L^p(\mathbb{R}^n)}$$

$$\leq C \|\text{op}[e^{a(\xi)}]f\|_{L(\mathbb{R}^n, \mathbb{C}^N)} \|\text{op}[v_0]\|_{L(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n, \mathbb{C}^N))} \|f\|_{L^p(\mathbb{R}^n)}$$

$$\leq C \|e^{a(\xi)}\|_M^p \|\text{op}[v_0]\|_{L(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n, \mathbb{C}^N))} \|f\|_{L^p(\mathbb{R}^n)}$$

$$\leq C' \|f\|_{L^p(\mathbb{R}^n)}$$

with a constant $C'$ depending on $a$ and $v_0$ but not on $f$ or $k$. Note that we have $\lambda(\xi) \in i\mathbb{R}$ and therefore $|e^{\lambda(\xi)}| = 1$ for $\xi \in U_0$. Thus, we may apply [4], Lemma 5, to get

$$\lambda(\xi) = i\xi^\top \xi + i\lambda_0 \quad (\xi \in U_0) \quad (3-3)$$

for some $\lambda_0 \in \mathbb{R}$ and $\xi_0 \in \mathbb{R}^n$. However, as $\lambda$ is homogeneous of degree $\mu > 0$, we obtain $\lambda_0 = 0$ and $\xi_0 \neq 0$ as well as $\mu = 1$. □
Remark 3.8. a) The statement in c) also holds if \( a \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{C}^{N \times N}) \) is strictly homogeneous of degree \( \mu > 0 \), as this is a bounded perturbation of some homogeneous symbol \( \tilde{a} \in C^\infty(\mathbb{R}^n, \mathbb{C}^{N \times N}) \) (cf. Remark 3.6 a)).

b) Well-posedness is invariant under similarity transformations: Assume that \( S \in L(L^p(\mathbb{R}^n; \mathbb{C}^N)) \) is an isomorphism. Then the Cauchy problem is well-posed in \( L^p(\mathbb{R}^n; \mathbb{C}^N) \) for \( \text{op}[a] \) if and only if it is well-posed for \( S^{-1} \text{op}[a]S \). This can be seen as in the proof of Corollary 2.10. In particular, this holds if \( S = \text{op}[s] \) with \( s, s^{-1} \in S^0(\mathbb{R}^n; \mathbb{C}^{N \times N}) \).

c) In the proof of Theorem 3.7 c), we have seen that well-posedness in \( L^p \) implies a strong condition on the eigenvalues of \( a(\xi) \): Assume in the situation of Theorem 3.7 that \( \mu = 1 \). If there exists an eigenvalue \( \lambda(\xi) = \lambda(0) + i \xi_0 + \lambda_0 \) for all \( \xi \) in a nonempty open set which is only possible for \( n = 1 \). In fact, we have seen above that the eigenvalues of \( a(\xi) \) have the form (3-3). Therefore, \( |\xi|\lambda(\xi) = \xi_j^2 + i + \lambda_0 \) for all \( \xi \) in a nonempty open set which is only possible for \( n = 1 \). This situation occurs, in particular, if \( a(\xi) = |\xi|a(0) \) with a constant matrix \( a(0) \in \mathbb{C}^{N \times N} \) with at least one purely imaginary eigenvalue.

Remark 3.9. In the situation of Theorem 3.7, let us consider the particular case that \( a \) is homogeneous of degree 1 and linear in \( \xi \), i.e., \( a(\xi) = \sum_{j=1}^n \xi_j a_j \) with \( a_j \in \mathbb{C}^{N \times N} \). To our knowledge, there is no characterization of all matrices which lead to a well-posed problem, and we only state some remarks on this.

a) If \( a_j = i\tilde{a}_j \) with symmetric real matrices \( \tilde{a}_j \in \mathbb{R}^{N \times N} \), then (2-3) is well-posed in \( L^p(\mathbb{R}^n; \mathbb{C}^N) \) if and only if all matrices \( \tilde{a}_1, \ldots, \tilde{a}_n \) commute. This is a classical result by Brenner ([4], Theorem 1).

b) If all eigenvalues of \( a(\xi) \) have negative real part for large \( \xi \in \mathbb{R}^n \), then (2-3) is well-posed for all \( p \in (1, \infty) \) by Theorem 3.7 b).

c) If there exists a \( j \in \{1, \ldots, n\} \) and an eigenvalue \( \lambda_0 \in i\mathbb{R} \) of \( a_j \) with nontrivial Jordan structure, then (2-3) is not well-posed even for \( p = 2 \). In fact, setting \( \xi = (0, \ldots, \xi_j, \ldots, 0)^\top \), we see that the symbol \( e^{\lambda_0(\xi)} = e^{i\xi_j a_j} \) is unbounded.

d) Let \( n = 1 \), and let \( a(\xi) = \xi a_1 \) with a diagonalizable matrix \( a_1 \in \mathbb{C}^{N \times N} \). Then (2-3) is well-posed in \( L^p \), \( p \neq 2 \), if and only if all eigenvalues of \( a_1 \) have nonpositive real part.

The result of Theorem 3.7 extends to a system of classical pseudo-differential operators. Here a symbol \( \tilde{a} \in S^\mu(\mathbb{R}^n, \mathbb{C}^{N \times N}) \) belongs to the space \( S^\mu_{cl}(\mathbb{R}^n, \mathbb{C}^{N \times N}) \) of classical (polyhomogeneous) symbols if there exists an asymptotic expansion \( a \sim \sum_{j=0}^\infty a_j \) with \( a_j \in S^{\mu-j}(\mathbb{R}^n, \mathbb{C}^{N \times N}) \) is homogeneous of degree \( \mu - j \). In this case, \( a_0 \) is called the principal symbol of \( a \).

Lemma 3.10. Let \( a \in S^\mu_{cl}(\mathbb{R}^n, \mathbb{C}^{N \times N}) \) be quasi-hyperbolic. Then the statement of Theorem 3.7 c) hold analogously, where now the eigenvalues of the principal symbol \( a_0 \) have to be considered.

Proof. Assume that \( \mu > 0 \) and that the Cauchy problem (2-3) for \( \text{op}[a] \) is well-posed in \( L^p(\mathbb{R}^n; \mathbb{C}^N) \) for some \( p \neq 2 \). We choose \( m \in \mathbb{N}_0 \) with \( \mu - m > 0 \) and \( \mu - m - 1 \leq 0 \). Then \( a - \sum_{j=0}^m a_j \in S^0(\mathbb{R}^n, \mathbb{C}^{N \times N}) \), and by bounded perturbation (see Remark 2.11 b)), we may assume that \( a = \sum_{j=0}^\infty a_j \).

We choose \( R > 0 \) such that \( a_0, \ldots, a_m \) are homogeneous for \( |\xi| \geq R \) and fix \( \chi \in C^\infty(\mathbb{R}^n) \) with \( \chi = 0 \) for \( |\xi| \leq R \) and \( \chi = 1 \) for \( |\xi| \geq R + 1 \). As \( \chi \in M^N_p \) by Mikhlin’s theorem, we have by Theorem 2.8

\[
\|e^{a_0(\xi)}(\cdot)\|_{M^N_p} \leq C_T \quad (t \in [0, T]).
\]
In particular, the same estimate holds if we replace $t$ by $tk^{-\mu} \leq t$ for $k \in \mathbb{N}$. By [2], Proposition E.2 e) again, we see that

$$\|e^{tk^{-\mu}a(k\cdot)}\chi(k\cdot)\|_{M^s_p} \leq C_T \quad (t \in [0,T]).$$

For every $\xi \in \mathbb{R}^n \setminus \{0\}$, the homogeneity of $a_j$ yields

$$k^{-\mu}a(k\xi) = \sum_{j=0}^m k^{-\mu}a_j(k\xi) = \sum_{j=0}^m k^{-3}a_j^{(h)}(\xi) \to a_0^{(h)}(\xi) \quad (k \to \infty),$$

where $a_j^{(h)}$ denotes the strictly homogeneous version of $a_j$, i.e. the strictly homogeneous function which coincides for large $\xi$ with $a_j$. We also have $\chi(k\xi) \to 1$ for every $\xi \neq 0$. Therefore, the sequence $(\exp(tk^{-\mu}a(k\cdot))\chi(k\cdot))_{k \in \mathbb{N}}$ is a bounded sequence in $M^s_p$ converging pointwise almost everywhere to $\exp(ta_0^{(h)})$. Consequently, $\exp(ta_0^{(h)}) \in M^s_p$ and

$$\|\exp(ta_0^{(h)})\|_{M^s_p} \leq C_T \quad (t \in [0,T]) \quad (3-4)$$

(see [2], Proposition E.2 f)).

Since $m(t,\xi) := e^{ta_0(\xi)} - e^{ta_0^{(h)}(\xi)}$ is smooth in $\mathbb{R}^n \setminus \{0\}$ and has compact support, and since $a_0^{(h)}$ is homogenous of positive degree, it is easy to see that for every $\alpha \in \mathbb{N}^n$, the expression $\xi^\alpha \partial^\alpha m(t,\xi)$ is bounded by a constant independent of $\xi$ and of $t \in [0,T]$. By Mikhlin’s theorem, $m(t,\cdot) \in M^s_p$, and from (3-4) we get

$$\|e^{ta_0(\cdot)}\|_{M^s_p} \leq C_T \quad (t \in [0,T]).$$

Therefore, we can apply Theorem 3.7 c) to $a_0$ and obtain $\mu = 1$ if $a_0$ satisfies the assumptions of Theorem 3.7.

We summarize Corollary 2.10, Theorem 3.7, and the above remarks in the following theorem which is one of the main results of the present paper.

**Theorem 3.11.** Let $a = (a_{ij})_{i,j=1,\ldots,N} : \mathbb{R}^n \to \mathbb{C}^{N \times N}$ be a quasi-hyperbolic mixed-order system of classical pseudodifferential operators with constant coefficients, $a_{ij} \in S^m_{cl}(\mathbb{R}^n)$. For $p \in (1,\infty)$, let $A_p$ be the realization of $\text{op}[a]$ in the basic space $X_p = \prod_{j=1}^n H^m_p(\mathbb{R}^n)$ with maximal domain.

Define $\Lambda(\xi) := \text{diag}((\xi)^{s_1}, \ldots, (\xi)^{s_n})$ and $\tilde{a} := \Lambda a \Lambda^{-1} \in S^m_{cl}(\mathbb{R}^n; \mathbb{C}^{N \times N})$ where $\mu$ is the maximal order of the entries of $\tilde{a}$. Let $\tilde{a}_0$ be the principal symbol of $\tilde{a}$.

a) If $\mu \leq 0$, then the Cauchy problem (2-3) is well-posed for all $p \in (1,\infty)$.

b) If $\mu > 0$ and for sufficiently large $\xi \in \mathbb{R}^n$ all eigenvalues of $\tilde{a}_0(\xi)$ have negative real part, then (2-3) is well-posed for all $p \in (1,\infty)$.

c) Let $\mu > 0$ and assume that there exists an eigenvalue $\lambda(\xi) \in i\mathbb{R} \setminus \{0\}$ of $\tilde{a}_0(\xi)$ for all $\xi \in S_{R,V} := \{ v \in V \}$. Moreover, if $\lambda(\xi)$ only depends on $|\xi|$ for all $\xi \in S_{R,V}$, then well-posedness is only possible if $p = 2$ or $n = 1$.

**Proof.** a) and b) follow in exactly the same way as in the proof of Theorem 3.7. The necessity of $\mu = 1$ in c) is stated in Lemma 3.10, and the case of eigenvalues depending only on $|\xi|$ is discussed in Remark 3.8 c), applied to $\tilde{a}_0$.

4. Application to the thermoelastic plate equation

In this section, we apply the previous results to the thermoelastic plate equation with Fourier and Maxwell-Cattaneo type heat conduction model, respectively. The
dissipative structure of this equation in $L^2$-spaces has been studied, e.g., in [21].

Omitting physical constants, the linear thermoelastic plate equation is given by
\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} + \Delta^2 u - \mu \Delta \frac{\partial u}{\partial t} + \Delta \theta &= 0, \\
\theta_t + \text{div} q - \Delta u_t &= 0, \\
\tau q_t + q + \nabla \theta &= 0.
\end{aligned}
\end{equation}

In (4-1), the unknowns are the elongation $u = u(t, x)$ of the plate at time $t \geq 0$ and position $x \in \mathbb{R}^n$, the temperature (difference) $\theta = \theta(t, x)$, and the heat flux $q = q(t, x)$. The two parameters $\mu, \tau \geq 0$ describe whether an inertial term is present ($\mu > 0$) and which type of heat conduction model is used ($\tau = 0$ for Fourier’s law and $\tau > 0$ for Cattaneo-Maxwell’s law). In the $L^2$-setting, many results are known, for instance on (non-)exponential stability and regularity loss phenomena. For this, we refer to [13], [16], [20], [21] and the references therein.

4.1. Cattaneo-Maxwell’s law. We first consider the case $\tau > 0$, i.e., Cattaneo-Maxwell’s law of heat conduction. We start with the additional assumption $\mu > 0$.

In this case, we apply the operator $(1 - \mu \Delta)^{-1}$ to the first equation in (4-1) and set $U := (u, v, \theta, q)^T$ with $v := u_t$. We obtain the Cauchy problem
\begin{equation}
\begin{aligned}
(\partial_t - A(D))U(t) &= 0 \ (t > 0), \\
U(0) &= U_0
\end{aligned}
\end{equation}

with
\begin{equation}
\begin{aligned}
A(D) := \begin{pmatrix}
0 & 1 & 0 & 0 \\
-(1 - \mu \Delta)^{-1} \Delta^2 & 0 & -(1 - \mu \Delta)^{-1} \Delta & 0 \\
0 & \Delta & 0 & -\text{div} \\
0 & 0 & -\frac{1}{\tau} \nabla & -\frac{1}{\tau}
\end{pmatrix}
\end{aligned}
\end{equation}

and $U_0 := (u_0, u_1, \theta_0, q_0)^T$. The symbol of $A(D)$ is given by
\begin{equation}
\begin{aligned}
a(\xi) := \begin{pmatrix}
0 & |\xi|^2 & 0 & \cdots & 0 \\
\frac{|\xi|^2}{1 + |\xi|^2} & 0 & |\xi|^2 & \cdots & 0 \\
0 & \frac{|\xi|^2}{1 + |\xi|^2} & 0 & \cdots & \frac{1}{\tau} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \frac{|\xi|^2}{1 + |\xi|^2} & \cdots & -\frac{1}{\tau}
\end{pmatrix}
\end{aligned}
\end{equation}

As the basic space for the Cauchy problem, we natural choice is $X_p := W_p^2(\mathbb{R}^n) \times W_p^1(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n; \mathbb{C}^n)$.

The realization of $A(D)$ in $X_p$ is given by the operator $A_p: X_p \supset D(A_p) \to X_p$ with maximal domain
\begin{equation}
D(A_p) := \{U \in X_p : A(D)U \in X_p\}, \quad A_p U := A(D)U.
\end{equation}

By the structure of the matrix $A(D)$, we immediately obtain
\begin{equation}
\begin{aligned}
D(A_p) &= \{(u, v, \theta, q)^T \in W_p^2(\mathbb{R}^n) \times W_p^1(\mathbb{R}^n) \times W_p^1(\mathbb{R}^n) \times L^p(\mathbb{R}^n; \mathbb{C}^n) : \\
\text{div} q &\in L^p(\mathbb{R}^n)\}\}.
\end{aligned}
\end{equation}

We start with some remarks on the $L^2$-case. Part b) of the following lemma shows that the choice of the space $X_p$ essentially is the only possible one even for $p = 2$.

Lemma 4.1. Let $\tau > 0$, $\mu > 0$, and $p = 2$.

a) The operator $A_2$ generates a $C_0$-semigroup in $X_2$.

b) For $s = (s_1, \ldots, s_4)$, let $A_2^{(s)}$ be the realization of $A(D)$ in the basic space $X_2^{(s)} := H^{s_1}(\mathbb{R}^n) \times H^{s_2}(\mathbb{R}^n) \times H^{s_3}(\mathbb{R}^n) \times H^{s_4}(\mathbb{R}^n; \mathbb{C}^n)$
with maximal domain. If \( A^s_2 \) generates a \( C_0 \)-semigroup in \( X^s_2 \), then \( s = (c + 2, c + 1, c, c) \) for some \( c \in \mathbb{R} \).

**Proof.** a) This can be seen by a standard application of the Lumer-Phillips theorem where it is convenient to endow \( X_2 \) with the equivalent norm

\[
\|u\|^2_{X_2} = \|u\|^2_{L^2(\mathbb{R}^n)} + \|(1 - \mu \Delta)^{-1}u\|^2_{L^2(\mathbb{R}^n)} + \|\theta\|^2_{L^2(\mathbb{R}^n)} + \|q\|^2_{L^2(\mathbb{R}^n; \mathbb{C}^n)}.
\]

With this norm, it is straightforward to see that the operator

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-(1 - \mu \Delta)^{-1}(\Delta^2 - 1) & 0 & -(1 - \mu \Delta)^{-1} \Delta & 0 \\
0 & \Delta & -1 & -\text{div} \\
0 & 0 & -\nabla & -\nabla
\end{pmatrix}
\]

is a bounded perturbation of \( A_2 \) and dissipative in \( X_2 \).

b) For \( s \in \mathbb{R}^4 \) and \( \xi \in \mathbb{R}^n \), we define

\[
A^s(\xi) := \text{diag}\left(\langle \xi \rangle^{s_1}, \langle \xi \rangle^{s_2}, \langle \xi \rangle^{s_3}, \langle \xi \rangle^{s_4}\right) \in \mathbb{R}^{(n+3) \times (n+3)}.
\]

Assume that \( A^s \) generates a \( C_0 \)-semigroup in \( X^s_2 \). By the Hille-Yosida theorem, there exists a \( C > 0 \) such that for large \( \lambda > 0 \) we have \( \|\lambda(\lambda - A^s_2)^{-1}\|_{L(X^s_2)} \leq C \).

For the symbol, this implies

\[
\|A^s(\xi)\lambda(\lambda - a(\xi))^{-1}A^{-s}(\xi)\|_{L^\infty(\mathbb{R}^n; \mathbb{C}^{(n+3) \times (n+3)})} \leq C.
\]

Let \( s^{(0)} := (2, 1, 0, 0) \). Setting \( \xi = (\rho, 0, \ldots, 0)^T \) and \( \lambda = \lambda_0 \rho \) with large \( \rho > 0 \) and fixed \( \lambda_0 > 0 \), we obtain

\[
A^s(\xi)\lambda(\lambda - a(\xi))^{-1}A^{-s}(\xi) = A^{s-s^{(0)}}(\xi)\begin{pmatrix} b_0 - \lambda_0 I_4 & 0 \\ 0 & -\lambda_0 I_4 \end{pmatrix}^{-1}A^{-s+s^{(0)}}(\xi)
\]

modulo lower-order terms with respect to \( \rho \to \infty \). Here,

\[
b_0 := \begin{pmatrix}
0 & 1 & 0 & 0 \\
-\frac{1}{\rho} & 0 & \frac{1}{\rho} & 0 \\
0 & -1 & 0 & i \\
0 & 0 & -\frac{1}{\rho} & 0
\end{pmatrix}.
\]

Direct calculations show that every entry of the matrix \((b^{ij}(\lambda_0))_{i,j=1,\ldots,4} := (b_0 - \lambda_0 I_4)^{-1}\) is a nontrivial rational function of \( \lambda_0 \) with coefficients depending polynomially on \( \frac{1}{\rho} \) and \( i \). Therefore, for every fixed \( \mu > 0 \) and \( \tau > 0 \), we can choose a \( \lambda_0 > 0 \) such that every entry \( b_{ij}(\lambda_0) \) is non-zero. For \( i, j = 1, \ldots, 4 \), the entry of the matrix \((4-3)\) at position \((i, j)\) is given by

\[
\langle \xi \rangle^{s_i-s_i^{(0)}-s_j+s_j^{(0)}}b^{ij}(\lambda_0).
\]

Due to \( b^{ij}(\lambda_0) \neq 0 \), we obtain from the boundedness of \((4-3)\)

\[
s_i - s_i^{(0)} - s_j + s_j^{(0)} = 0 \quad \text{for all } i, j = 1, \ldots, 4.
\]

This implies \( s_i - s_i^{(0)} = c \) for some \( c \in \mathbb{R} \). \( \square 

**Lemma 4.2.** Let \( \tau > 0, \mu > 0, \) and \( p \in (1, \infty) \). Then \( A_p \) does not generate an analytic semigroup in \( X_p \).

**Proof.** It was shown in [21], (4.43), that there exists an eigenvalue \( \lambda_1(\xi) \) of \( a(\xi) \) with \( |\text{Im} \lambda_1(\xi)| \to \infty \) and \( |\text{Re} \lambda_1(\xi)| \leq C \) for \( |\xi| \to \infty \). Therefore, the resolvent set of \( A_p \) does not contain any sector of the complex plane with angle larger than \( \frac{\pi}{\tau} \). By this, \( A_p - \lambda_0 \) is not sectorial for any \( \lambda_0 > 0 \) which implies that \( A_p \) does not generate an analytic semigroup. \( \square \)
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Theorem 4.3. Let $\tau > 0$, $\mu > 0$, and $p \in (1, \infty) \setminus \{2\}$. Then the operator $A_p$ generates a $C_0$-semigroup in $X_p$ (and the Cauchy problem (4.2) is well-posed) if and only if $n = 1$.

Proof. By Corollary 2.10, we have to study the symbol $\tilde{a}(\xi) := \Lambda^{(s)}(\xi)a(\xi)\Lambda^{(-s)}(\xi)$ with $s := (2, 1, 0, 0)$. We have $\tilde{a}(\xi) \in S^1(R^n; C^{(n+3)\times(n+3)})$ with principal symbol

$$\tilde{a}_0(\xi) = \begin{pmatrix}
0 & |\xi| & 0 & 0 & 0 & \cdots & 0 \\
-\frac{i\xi}{\mu} & 0 & |\xi| & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & |\xi| & 0 & i\xi_1 & i\xi_2 & \cdots & i\xi_n \\
0 & 0 & \frac{i\xi}{\mu} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \frac{i\xi}{\mu} & 0 & 0 & \cdots & 0
\end{pmatrix}$$

(for large $|\xi|$). Its characteristic polynomial is given by

$$\det(\lambda - \tilde{a}_0(\xi)) = \lambda^{n-1}\left(\lambda^4 + (\frac{1}{\tau} + \frac{1}{\mu})|\xi|^2\lambda^2 + \frac{1}{\tau \mu}|\xi|^4\right).$$

Therefore, all eigenvalues of $\tilde{a}_0(\xi)$ are functions of $|\xi|$ and lie on the imaginary axis. If $A_p$ generates a $C_0$-semigroup in $X_p$, then $n = 1$ by Remark 3.8 c).

Now let $n = 1$. We write $\tilde{a}_0(\xi)$ in the form $\tilde{a}_0(\xi) = \xi a_+ \chi_{[0, \infty)}(\xi) + \xi a_- \chi_{(-\infty, 0)}(\xi)$ with the constant matrices

$$a_{\pm} := \begin{pmatrix}
0 & \pm 1 & 0 & 0 \\
\mp \frac{1}{\mu} & 0 & \pm \frac{1}{\mu} & 0 \\
0 & \mp 1 & 0 & i \\
0 & 0 & \pm \frac{1}{\mu} & 0
\end{pmatrix}.$$

We apply Theorem 2.8 to $e^{i\tilde{a}_0(\xi)} = e^{i\xi a_+ \chi_{[0, \infty)}(\xi)} + e^{i\xi a_- \chi_{(-\infty, 0)}(\xi)}$. As both matrices $a_+, a_-$ have four different purely imaginary eigenvalues, they are diagonalizable, and Remark 3.9 d) yields that $e^{it\tilde{a}_0}$ satisfies (2.5). On the other hand, the characteristic functions $\chi_{[0, \infty)}$ and $\chi_{(-\infty, 0)}$ are $L^p$-Fourier multipliers by Mikhlin’s theorem. Therefore, the realization of $\tilde{a}_0$ generates a $C_0$-semigroup in $X_p$ for $n = 1$. As $\tilde{a}$ is a bounded perturbation of $\tilde{a}_0$, we see that $A_p$ generates a $C_0$-semigroup in $X_p$ for $n = 1$.

Now we consider the case $\mu = 0$. Now the natural setting is $X_p := W^{2}_p(R^{n}) \times L^{p}(R^{n}) \times L^{p}(R^{n}; C^{n})$. The maximal domain is given by

$$D(A_p) = \{ U = (u, v, \theta, q)^T \in W^{2}_p(R^{n}) \times W^{2}_p(R^{n}) \times W^{1}_p(R^{n}) \times L^{p}(R^{n}; C^{n}) :$$

$$\text{div } u \in L^{p}(R^{n}), \Delta u + \theta \in W^{2}_p(R^{n}) \}. $$

In the case $\mu = 0$, the results in the $L^2$-case are analog to the case $\mu > 0$. However, even for $n = 1$, the operator $A_p$ does not generate a $C_0$-semigroup:

Theorem 4.4. Let $\tau > 0$ and $\mu = 0$.

a) Let $p = 2$. Then the operator $A_2$ generates a $C_0$-semigroup in $X_2$ but no analytic semigroup.

b) Let $p \in (1, \infty) \setminus \{2\}$. Then $A_p$ does not generate a $C_0$-semigroup in $X_p$.

Proof. a) This can be shown in an analogous way as for the case $\mu > 0$.

b) Again we have to consider $\tilde{a}(\xi) = \Lambda^{(s)}(\xi)a(\xi)\Lambda^{(-s)}(\xi)$ where now $s = (2, 0, 0, 0)$. Now we have $\tilde{a} \in S^{2}(R^{n}; C^{(n+3)\times(n+3)})$, and the principal symbol is given by

$$\tilde{a}_0(\xi) = \begin{pmatrix}
0 & |\xi|^2 & 0 & 0 \\
-|\xi|^2 & 0 & |\xi|^2 & 0 \\
0 & -|\xi|^2 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \in C^{(n+3)\times(n+3)}.$$
As this matrix has the purely imaginary eigenvalues $\pm \sqrt{2}|\xi|^2 i$, Theorem 3.11 c) shows that $A_p$ is no generator of a $C_0$-semigroup. \hfill \Box

4.2. Fourier’s law. Now let us consider the case $\tau = 0$, i.e., the thermoelastic plate equation with Fourier’s law of heat conduction. We first remark that for $\tau = 0$ and $\mu = 0$, the operator generates an analytic semigroup in the $L^p$-setting, see [6], Theorem 3.5. Therefore, we only have to investigate the case $\mu > 0$.

So we consider for $\mu > 0$ the equation

$$u_{tt} + \Delta^2 u - \mu \Delta u_{tt} + \Delta \theta = 0,$$

$$\theta_t - \Delta \theta - \Delta u_t = 0$$

in $\mathbb{R}^n$ with initial conditions $u|_{t=0} = u_0$, $u_t|_{t=0} = u_1$, $\theta|_{t=0} = \theta_0$. Setting $U := (u, v, \theta)^\top$ with $v := u_t$, we obtain the Cauchy problem

$$(\partial_t - A(D))U(t) = 0 \ (t > 0), \quad U(0) = U_0$$

with $U_0 := (u_0, u_1, \theta_0)^\top$ and

$$A(D) := \begin{pmatrix}
-1 - \mu \Delta & 1 & 0 \\
0 & -1 & -\Delta \\
0 & \Delta & \Delta
\end{pmatrix}.$$}

The natural basic space for the operator related to \eqref{eq:4-4} is given by

$$X_p := W^2_p(\mathbb{R}^n) \times W^1_p(\mathbb{R}^n) \times L^p(\mathbb{R}^n),$$

and the operator is defined as the realization of $A(D)$ in $X_p$ with maximal domain

$$D(A_p) = W^3_p(\mathbb{R}^n) \times W^2_p(\mathbb{R}^n) \times W^2_p(\mathbb{R}^n).$$

The symbol of $A(D)$ equals

$$a(\xi) := \begin{pmatrix}
0 & 1 & 0 \\
-\frac{2|\xi|^2}{1 + |\xi|^2} & 0 & -|\xi|^2 \\
0 & |\xi|^2 & -|\xi|^2
\end{pmatrix}.$$}

Setting $\Lambda(\xi) := \text{diag}(|\xi|^2, |\xi|^2, 1)$, we have to study the mixed-order symbol $\tilde{a}(\xi) := \Lambda(\xi)a(\xi)\Lambda(\xi)^{-1}$. We have $\tilde{a} \in S^{2,0}_c(\mathbb{R}^n; \mathbb{C}^{3 \times 3})$ and, for large $|\xi|$, $\tilde{a}(\xi) = |\xi|^2 a_0 + |\xi| a_1 + a_2(\xi)$ with $a_2 \in S^{0,0}_c(\mathbb{R}^n; \mathbb{C}^{3 \times 3})$ and

$$a_0 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -1 & 0
\end{pmatrix}, \quad a_1 = \begin{pmatrix}
0 & 1 & 0 \\
-\frac{1}{n} & 0 & \frac{1}{n} \\
0 & -1 & 0
\end{pmatrix}.$$}

Theorem 4.5. Let $\tau = 0$ and $\mu > 0$.

a) Let $p = 2$. Then the operator $A_2$ generates a $C_0$-semigroup in $X_2$ but no analytic semigroup.

b) Let $p \in (1, \infty) \setminus \{2\}$. Then $A_p$ generates a $C_0$-semigroup if and only if $n = 1$.

Proof. a) Again, the first statement is a straightforward application of the Lumer-Phillips theorem. For the second statement, we use the fact that $a(\xi)$ has eigenvalues with bounded real part and unbounded imaginary part, see [21], (4.40).

b) In contrast to the proof of Theorem 4.4 b), we cannot apply Theorem 3.11 c) as the nontrivial eigenvalue of the principal symbol $a_0$ has negative imaginary part. Therefore, we use the idea of an approximate diagonalization procedure which was introduced in [12], Section 2.4, on closed manifolds and in [7], Section 3.3, in $\mathbb{R}^n$. 
By bounded perturbation, the operator $A_p$ generates a $C_0$-semigroup in $X_p$ if and only if the operator $B_p$ corresponding to the mixed-order symbol
\[
b(\xi) := |\xi|^2 a_0 + |\xi| a_1 = \begin{pmatrix} 0 & |\xi| & 0 \\ -\frac{|\xi|}{\mu} & 0 & \frac{|\xi|}{\mu} \\ 0 & -|\xi| & -|\xi|^2 \end{pmatrix}
\]
(for large $|\xi|$) generates a $C_0$-semigroup in $L^p(\mathbb{R}^n; \mathbb{R})$. We define the transformation matrix
\[
S := \frac{1}{\sqrt{\mu}} \begin{pmatrix} 0 & \sqrt{\mu} i & \sqrt{\mu} \\ -\sqrt{\mu} i & -\frac{\mu}{|\xi|} & -\frac{\mu}{|\xi|} \\ \frac{\mu}{|\xi|} & -\frac{\mu}{|\xi|} & \frac{\mu}{|\xi|} \end{pmatrix}.
\]
As in the proof of Corollary 2.10, we see that well-posedness is invariant under similarity transforms $b(\xi) \mapsto S^{-1}(\xi)b(\xi)S(\xi)$. An explicit calculation shows
\[
S^{-1}(\xi) = \frac{1}{2(\mu|\xi|^2 - 1)} \begin{pmatrix} \mu |\xi|^2 - 1 & -2 \mu |\xi|^2 & -2 \mu |\xi|^2 \\ \mu |\xi|^2 - 1 & -\mu^{3/2} |\xi|^2 i & -\sqrt{\mu} |\xi| i \\ |\xi|^2 - 1 & -\mu^{3/2} |\xi|^2 i & -\sqrt{\mu} |\xi| i \end{pmatrix}
\]
and
\[
\tilde{b}(\xi) := S^{-1}(\xi)b(\xi)S(\xi) = \begin{pmatrix} -|\xi|^2 & 0 & 0 \\ 0 & \frac{|\xi|}{\sqrt{\mu}} i & 0 \\ 0 & 0 & -\frac{|\xi|}{\sqrt{\mu}} i \end{pmatrix} + R(\xi)
\]
with
\[
R(\xi) = \frac{1}{2(\mu|\xi|^2 - 1)} \begin{pmatrix} 2|\xi|^2 & 2|\xi|^2 & 2|\xi|^2 \\ |\xi|^2 - \frac{1}{\mu} & |\xi|^2 - \frac{1}{\mu} & |\xi|^2 - \frac{1}{\mu} \\ |\xi|^2 - \frac{1}{\mu} & |\xi|^2 - \frac{1}{\mu} & |\xi|^2 - \frac{1}{\mu} \end{pmatrix}.
\]
We see that $S,S^{-1} \in S^0_{\lambda}(\mathbb{R}^n; \mathbb{C}^{3 \times 3})$. Therefore, the symbol $S$ induces an isomorphism $S(D) : L^p(\mathbb{R}^n; \mathbb{C}^3) \to L^p(\mathbb{R}^n; \mathbb{C}^3)$, and $B_p$ is well-posed in $L^p(\mathbb{R}^n; \mathbb{C}^3)$ if and only if $\tilde{B}_p := S(D)^{-1}B_pS(D)$ is well-posed in $L^p(\mathbb{R}^n; \mathbb{C}^3)$. Moreover, we have $R \in S^0_{\lambda}(\mathbb{R}^n; \mathbb{C}^{3 \times 3})$. Therefore, $\tilde{B}_p$ is a bounded perturbation of the operator related to the diagonal matrix in (4-5).

Altogether we have seen that $A_p$ generates a $C_0$-semigroup in $X_p$ if and only if the operator related to the symbol
\[
\begin{pmatrix} -|\xi|^2 & 0 & 0 \\ 0 & \frac{|\xi|}{\sqrt{\mu}} i & 0 \\ 0 & 0 & -\frac{|\xi|}{\sqrt{\mu}} i \end{pmatrix}
\]
generates a $C_0$-semigroup in $L^p(\mathbb{R}^n; \mathbb{R})$. Now we can consider each component separately. If $A_p$ generates a $C_0$-semigroup, then the eigenvalue $\mu^{-1/2}|\xi|$ has to be a linear function of $\xi$ at the points of differentiability amann03 which implies $n = 1$. On the other hand, in the case $n = 1$ we can write $|\xi| = -\xi(\chi_{(-\infty,0)}(\xi) + \xi(0,\infty)(\xi)$ (cf. the proof of Theorem 4.3) and obtain that $A_p$ generates a $C_0$-semigroup. □

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