NON-LINEAR MORSE-BOTT FUNCTIONS
ON QUATERNIONIC STIEFEL MANIFOLDS

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Abstract. In the Stiefel manifold $X_{n,k}$, we replace Frankel linear height function by a quadratic one. We prove this is still a Morse-Bott function, whose structure of critical levels presents a dichotomy according to the sign of $n - 2k$. The critical submanifolds are no longer Grassmannians but total spaces of fibrations of basis a product of two Grassmannians. We explicitly integrate the gradient flow.

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Date: April 14, 2020.
2010 Mathematics Subject Classification. Primary 58E05; Secondary 22E20; 22F30.
Key words and phrases. Quaternionic Stiefel manifold; Morse-Bott function.
The three authors are partially supported by the MINECO and FEDER research project MTM2016-78647-P. The first author was partially supported by Xunta de Galicia ED431C 2019/10 with FEDER funds.
Introduction

Morse-Bott functions are a very effective tool for studying compact manifolds. The pioneer work of T. Frankel ([3]) on Lie groups and Stiefel manifolds uses height functions defined, on the unitary group for instance, by the real part of the trace, $A \mapsto \Re \text{Tr} A$. The associated critical submanifolds are Grassmannians and from this fact H. Miller ([12]) proves that Stiefel manifolds admit a stable decomposition as Thom spaces of bundles over these Grassmannians, see also [7]. These critical submanifolds have also been the origin of the work of H. Kadzisa and M. Mimura ([6]) for the construction of cone-decompositions of Stiefel manifolds and their application to Lusternik-Schnirelmann category. Some recent papers took up again the work of Frankel, by using linear and quadratic Morse-Bott functions on orthogonal groups and Stiefel manifolds, as in [14, 2].

In [10], height functions with respect to the hyperplane orthogonal to a given matrix $\omega$ are considered on quaternionic Stiefel manifolds. All those functions are Morse-Bott with Grassmannians as critical submanifolds and, among them, the Morse functions are characterized according to $\omega$. Some consequences on the Lusternik-Schnirelmann category are given. In this work, we replace height functions by a non-linear function and prove that the situation is different: for instance, instead of Grassmannians, the critical submanifolds are associated fibrations to the principal bundles of a product of two Stiefel manifolds, with basis a product of two Grassmannians, see Proposition 4.3.

Finally, we are able to explicitly integrate the gradient flow, for arbitrary initial data.

Let us specify the notations used. Let $\mathbb{H}^n$ be the quaternionic $n$-space endowed with the structure of a right $\mathbb{H}$-vector space and the hermitian product $\langle u, v \rangle = u^* v$. For $0 \leq k \leq n$, let $X_{n,k}$ be the Stiefel manifold of linear maps $\mathbb{H}^k \to \mathbb{H}^n$ preserving the Hermitian product. Such map is identified with a matrix $x \in \mathbb{H}^{n \times k}$ represented by two blocks,

$$
\begin{bmatrix}
T \\
P
\end{bmatrix},
$$

where $T, P$ are quaternionic matrices of order $(n-k) \times k$ and $k \times k$, respectively. The condition $x^* x = I_k$ becomes $T^* T + P^* P = I_k$. Let $\text{Sp}(n)$ be the Lie group of $n \times n$ matrices $A$ such that $A^* A = I_n$. The linear left action of $\text{Sp}(n)$ on $X_{n,k}$ is transitive and the isotropy group of $x_0 = \begin{bmatrix} 0 & I_k \end{bmatrix}$ is isomorphic to $\text{Sp}(n-k)$, so $X_{n,k}$ is diffeomorphic to $\text{Sp}(n)/\text{Sp}(n-k)$.

If $x = \begin{bmatrix} T \\
P
\end{bmatrix} \in X_{n,k}$, we define the real number $h(x)$ as the trace of $P^* P$,

$$
(1) \quad h(x) = \text{Tr}(P^* P).
$$

The purpose of this work is the study of the function $h: X_{n,k} \to \mathbb{R}$. In what follows we always assume $k < n$ since we have $X_{n,n} = \text{Sp}(n)$ where the function $h$ is constant.

Main Theorem. Let $0 \leq k < n$ and $X_{n,k} = \text{Sp}(n)/\text{Sp}(n-k)$ be the quaternionic Stiefel manifold. The following properties are satisfied.

1) The function $h: x \mapsto \text{Tr}(P^* P)$ is Morse-Bott.

2) The critical levels of $h$ are
3. The critical submanifold $\Sigma_q$ is the total space of a principal fibration with fibre $\text{Sp}(k)$ and base space the product of Grassmannians $\text{Gr}_{n-k,p} \times \text{Gr}_{k,k-p}$. The index of $\Sigma_q$ is $4(n - 2k + q)q$, with $q = k - p$.

Let us observe that the dichotomy between $n \geq 2k$ and $n < 2k$, appearing in the previous statement, is also reflected in Nishimoto’s work ([13]) on the Lusternik-Schnirelmann category of quaternionic Stiefel manifolds $X_{n,k}$.

The proof of the Main Theorem occupies most of the rest of this work. In Section 1, we compute the gradient of $h$ and prove Assertion 2) of the Main Theorem. In Section 2, we give explicit expressions of the Hessian whereas the determination of its eigenvalues and eigenvectors is done in Section 3. In Section 4, we show that the group $K_{n,k} = \text{Sp}(n-k) \times \text{Sp}(k) \times \text{Sp}(k)$ acts transitively on the critical submanifold $\Sigma_q$. For this action, we also determine the isotropy subgroup as $L_{n,k,q} = \text{Sp}(k) \times \text{Sp}(n + q - 2k) \times \text{Sp}(k) \times \text{Sp}(q)$.

The inclusion $L_{n,k,q} \hookrightarrow K_{n,q}$, described in Corollary 4.1, brings up diagonal terms on certain factors and thus the writing of $\Sigma_q$ as a homogeneous space is not direct. Still, we express it as announced in the Assertion 3) and detail some particular cases. Assertion 1) is also a consequence of this determination, see Corollary 4.2.

Finally, in Section 5, we give an explicit description of the gradient flow of $h$.

1. Critical points

Let $h : X_{n,k} \rightarrow \mathbb{R}$ be the function defined by $h(x) = \text{Tr}(P^* P)$ for $x = \begin{bmatrix} T \\ P \end{bmatrix}$.

1.1. Gradient. In this paragraph, we explicit the value of the gradient of $h$.

**Proposition 1.1.** The gradient of $h$ at the point $x = \begin{bmatrix} T \\ P \end{bmatrix}$ is given by

$$\text{grad} \ h_x = -2 \begin{bmatrix} TP^* P \\ (PP^* - I_k)P \end{bmatrix}.$$  

We proceed as usual by immersing the manifold $X_{n,k}$ in $\mathbb{H}^{n \times k} = \mathbb{R}^{4n \times 4k}$, where the Euclidean metric is given by $|x|^2 = \text{Tr}(x^* x)$. The gradient of $h$ is obtained from the gradient of an extension $\phi : \mathbb{R}^{4n \times 4k} \rightarrow \mathbb{R}$ of $h$ that we project on the tangent space $T_x X_{n,k}$ of $X_{n,k}$:

$$\text{grad} \ h_x = \text{proj}_x (\text{grad} \  \phi_x),$$  

where we denote by $\text{proj}_x$ the projection onto the tangent space. Let us begin by some lemmas in this direction.

**Lemma 1.2.** The function $\phi : \mathbb{R}^{4n \times 4k} \rightarrow \mathbb{R}$, defined by $\phi(x) = \text{Tr}(P^* P)$ for $x = \begin{bmatrix} T \\ P \end{bmatrix}$, extends $h$ to the whole Euclidean space. If $x = \begin{bmatrix} T \\ P \end{bmatrix}$ then $\text{grad} \  \phi_x = 2 \begin{bmatrix} 0 \\ P \end{bmatrix}$.  

---

• $q = 0, \ldots, k$, if $n \geq 2k$;
• $q = 2k - n, \ldots, k$, if $n \leq 2k$.  

---
Lemma 1.3. If tangent space is given by \( \nu \) and the result follows. □

\( x \) other point of the Stiefel manifold can be written as \( x = x + tu \) and that the first one lies on the tangent space described above. Now, in general, since \( t \) for all \( u \in T_x \mathbb{R}^{4n \times 4k} \), that is,

\[ 2\Re \text{Tr}(P^* \pi) = \Re \text{Tr}(C^* \tau + D^* \pi), \]

for all \( \tau \) and \( \pi \). This implies \( C = 0 \) and \( D = 2P \). □

For the next step, we need to determine the tangent space \( T_x X_{n,k} \). Let us begin with \( x_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \). The tangent space in \( x_0 \) is formed by the vectors \( v = \begin{bmatrix} X \\ Y \end{bmatrix}, X \in \mathbb{H}^{n-k \times k}, Y \in \mathbb{H}^{k \times k}, \) with \( Y + Y^* = 0 \). (This corresponds to the equation \( x_0^* v + v^* x_0 = 0 \).) Any other point of the Stiefel manifold can be written as \( x = Ax_0 \), with \( A \in \text{Sp}(n) \), so the tangent space is given by \( T_x X_{n,k} = A T_{x_0} X_{n,k} \).

**Lemma 1.3.** If \( x \in X_{n,k} \), the projection of a vector \( u \in \mathbb{R}^{4n \times 4k} \) onto the normal vector space \( \nu_x \) is

\[ \text{proj}_x^\perp u = \frac{1}{2} x(x^* u + u^* x). \]

**Proof.** We begin with \( x_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \).

If \( u = \begin{bmatrix} A \\ B \end{bmatrix} \), then its tangent part is \( u^\top = \begin{bmatrix} A \\ (1/2)(B - B^*) \end{bmatrix} \)

and its normal part is

\[ u^\perp = \begin{bmatrix} 0 \\ (1/2)(B + B^*) \end{bmatrix} = \frac{1}{2} x_0(x_0^* u + u^* x_0). \]

This can be checked by proving that they are orthogonal for the Hermitian product \( v^* w \)

and that the first one lies on the tangent space described above. Now, in general, since \( x = Ax_0 \) and the product with \( A \) is an isometry, we have:

\[ \text{proj}_x^\perp u = A \text{proj}_{x_0}^\perp(A^* u) = A \left( \frac{1}{2} x_0(x_0^* A^* u + u^* A x_0) \right) = \frac{1}{2} x(x^* u + u^* x). \]

**Proof of Proposition 1.1** From Lemmas 1.2 and 1.3 the normal projection of \( u = \text{grad} \phi_x \) is

\[ \text{proj}_x^\perp u = \frac{1}{2} \begin{bmatrix} x^* \\ 0 \\ 2P \end{bmatrix} + \begin{bmatrix} 0 \\ 2P \end{bmatrix}^* x = \begin{bmatrix} T \\ P \end{bmatrix} (2P^* P), \]

so

\[ \text{grad} h_x = \text{proj}_x u = u - \text{proj}_x^\perp u = \begin{bmatrix} 0 \\ 2P \end{bmatrix} - \begin{bmatrix} 2TP^* P \\ 2PP^* P \end{bmatrix}, \]

and the result follows. □
1.2. Critical points and levels. The previous determination of the gradient gives a characterization of the critical points and values.

**Proposition 1.4.**

1) The point \( x = \left[ \begin{array}{c} T \\ P \end{array} \right] \) is a critical point of \( h \) if, and only if, \( TP^* = 0 \).

2) If \( n \geq 2k \), the function \( h \) has \( k+1 \) critical values \( q_0, \ldots, q_k \).

3) If \( n \leq 2k \), then \( h \) has \( n-k+1 \) critical values, \( q = 2k-n, \ldots, k \).

**Proof.** 1) The condition \( \text{grad } h_x = 0 \) implies that \( P = PP^* \), so \( P^*P = (P^*P)^2 \). But since \( P \in \mathbb{H}^{k \times k} \) is a square matrix, it follows that \( P^*P \) is a Hermitian matrix \( S \), which is semidefinite positive since for any \( u \in \mathbb{H}^k \), \( u^*Su = \langle u, P^*Pu \rangle = \langle P^*Pu, Pu \rangle = |Pu|^2 \geq 0 \).

The eigenvalues of \( S \) are thus non-negative real numbers \( s_1, \ldots, s_k \), verifying \( s_i^2 = s_i \), so \( s_i = 0, 1 \). We write a singular value decomposition (henceforth SVD) of \( P \) as

\[
P = a \begin{bmatrix} 0_p & 0_q \end{bmatrix} b^*, \quad a, b \in \text{Sp}(k), \quad p + q = k,
\]

and the corresponding diagonalization

\[
S = P^*P = b \begin{bmatrix} 0_p & 0_q \end{bmatrix} b^*.
\]

The condition \( \text{grad } h_x = 0 \) in Proposition 1.1 also implies \( TP^*P = 0 \), so

\[
Tb \begin{bmatrix} 0_p & 0_q \end{bmatrix} b^* = 0
\]

and, using \( b^*b = I_k \),

\[
TP^* = Tb \begin{bmatrix} 0_p & 0_q \end{bmatrix} b^*a^* = 0.
\]

For the reciprocal, it is clear that the two conditions \( TP^* = 0 \) and \( PP^*P = P \) imply \( \text{grad } h_x = 0 \). But the second condition is implied by the first one, since \( TP^* = 0 \) implies \( (I_k - P^*P)P^*T^*TP^* = 0 \) and \( P^*PP^* = P^* \).

2) Let \( x \) be a critical point and suppose \( n-k \geq k \). From the relative SVD developed in [11], there exists a SVD of \( T \) as

\[
m \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} b^*, \quad D = \text{diag}[t_1, \ldots, t_k].
\]

So \( T^* = b \begin{bmatrix} D & 0 \end{bmatrix} m^*, T^*T = bD^2b^* \) and the condition \( T^*T + P^*P = I_k \) implies

\[
D^2 + \begin{bmatrix} 0_p & 0_q \\ 0_p & 0_q \end{bmatrix} = I_k
\]

then \( D = \begin{bmatrix} I_p & 0_q \\ 0_p & 0_q \end{bmatrix} \). From \( q = k-p \) and \( k \leq n-k \), we deduce \( 0 \leq q \leq k \).

3) Let \( x \) be a critical point and suppose \( n-k < k \). We write

\[
T = m \begin{bmatrix} D & 0 \end{bmatrix} b^*, \quad D = \text{diag}[t_1, \ldots, t_{n-k}].
\]
The same argument gives \( D = \begin{bmatrix} I_p & 0 \\ 0 & 0_{n-k-p} \end{bmatrix} \) and \( 0 \leq p \leq n - k \). With \( q = k - p \), this gives \( 2k - n \leq q \leq k \).

Let us detail some easy examples of critical submanifolds.

**Example 1.5.** If \( k = 1 \), then \( n > 1 \) hence \( n - k \geq 1 = k \) and the function \( h \) has two critical levels \( q = 0, 1 \). In fact, \( X_{n,1} = \text{Sp}(n)/\text{Sp}(n-1) \) is the sphere \( S^{4n-1} \) and \( P \in \mathbb{H}^{1\times 1} \) is a quaternion verifying \( P^*P = |P|^2 \leq 1 \). The function \( h \), defined by \( h(x) = |P|^2 \), has 1 as maximum and 0 as minimum values.

**Example 1.6 (Maximum level).** The maximum level \( q = k \) implies \( P^*P = I_k \) and \( P = ab^* \in \text{Sp}(k) \), hence \( T^*T = 0 \) and \( T = 0 \). So, the critical level \( \Sigma_k \) is the space of matrices \( \begin{bmatrix} 0 \\ P \end{bmatrix} \) diffeomorphic to the symplectic group \( \text{Sp}(k) \) which can also be viewed as the Stiefel manifold \( X_{k,k} \).

**Example 1.7 (Minimum level).** Suppose \( n \geq 2k \). The minimum level corresponds to \( q = 0 \), thus \( p = k \) and \( P = 0 \). We deduce \( T^*T = I_k - P^*P = I_k \), which means that \( x = \begin{bmatrix} T \\ 0 \end{bmatrix} \in X_{n,n-k} \). Thus \( \Sigma_0 = X_{n-k,k} \).

The second case, \( n - k < k \), is more involved and is considered in Example 4.4.

2. Hessian

The tangent vector field \( \text{grad} \ h \) when differentiated at a critical point \( x \in X_{n,k} \) gives rise to a linear map

\[
Hh_x : v \in T_x X_{n,k} \mapsto \nabla_v \text{grad} \ h \in T_x X_{n,k},
\]

called the Hessian of \( H \). We determine it in this section but, before, let us recall some notations.

**Notations.** Let \( x_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \). For any \( x = \begin{bmatrix} T \\ P \end{bmatrix} \in X_{n,k} \), there exists an element \( A = \begin{bmatrix} \alpha & T \\ \beta & P \end{bmatrix} \in \text{Sp}(n) \) such that \( x = Ax_0 \). Moreover, as \( T_x X_{n,k} = A T_{x_0} X_{n,k} \), any vector \( v \in T_x X_{n,k} \) can be written as \( v = A \begin{bmatrix} X \\ Y \end{bmatrix} \) with \( Y + Y^* = 0 \). We keep these notations all along this section.

Now comes the main result of this section.

**Theorem 2.1.** The Hessian of the map \( h(x) = \text{Tr}(P^*P) \), at the critical point \( x \) and the vector \( v \in T_x X_{n,k} \), is given by

\[
Hh_x(v) = -2(vx^* + xv^*)x_0P - 2 \begin{bmatrix} 0 \\ PP^* - I \end{bmatrix} x_0^*v = -2(vx^* + xv^*)x_0P + 2 \begin{bmatrix} 0 \\ \beta X \end{bmatrix}.
\]
Remark 2.2. The first expression shows that the Hessian only depends on $x$ and $v$. In this form, it is not obvious at all that this vector belongs to the tangent space $T_x X_{n,k}$. Replacing $x$, $x_0$ and $v$ by their value in (3) shows that

$$Hh_x(v) = -2A \left[ X^* \beta^* P - P^* \beta X \right]$$

Since the matrix $X^* \beta^* P - P^* \beta X$ is skew-Hermitian, we can now notice that $Hh_x(v) \in A \cdot T_{x_0} X_{n,k} = T_x X_{n,k}$.

To prove this theorem, let us go back to the beginning. In (2), $\nabla$ is the connection on the Riemannian submanifold $X_{n,k} \subset \mathbb{R}^N$, with $N = 4n \times 4k$, so if $\tilde{\text{grad}} h$:

$$\nabla_v \tilde{\text{grad}} h = \text{proj}_x D_x (\tilde{\text{grad}} h)(v).$$

We can denote it

$$\text{proj}_x D_x (\tilde{\text{grad}} h)$$

since the connection in $\mathbb{R}^N$ is the usual derivative. Hence, we have

$$Hh_x(v) = \text{proj}_x \frac{d}{dt} |_{t=0} \tilde{\text{grad}} h_{x+tv}. \quad (5)$$

This situation is covered by [1] which contains an explicit formula for the Hessian of any map $h: M \to \mathbb{R}$, which is the restriction to a Riemannian submanifold $M \subset \mathbb{R}^N$ of a function $\phi: \mathbb{R}^N \to \mathbb{R}$ defined in the Euclidean space. We follow this pattern for $M = X_{n,k}$ and $h$, $\phi$ the functions defined in Section 1. Recall that the authors of [1] introduce

- the orthogonal projectors on the tangent and the normal spaces, $P_x: \mathbb{R}^N \to T_x X_{n,k}$ and $P_x^\perp = \text{id} - P_x: \mathbb{R}^N \to \nu_x$,
- the Weingarten map, $\mathfrak{A}_x: T_x X_{n,k} \times \nu_x \to T_x X_{n,k}$, of $X_{n,k}$ at $x$, which is given by

$$\mathfrak{A}_x(v, w) = -P_x D_v W,$$

where $W$ is any normal vector field extending locally the vector $w$, and they prove the two following results.

**Proposition 2.3.** [1 Theorem 1] For any $x \in X_{n,k}$, $u \in \mathbb{R}^N$ and $v \in T_x X_{n,k}$, the Weingarten map verifies

$$\mathfrak{A}_x(v, P_x^\perp u) = P_v D_v P_x^\perp u. \quad (6)$$

**Proposition 2.4.** [1 Formula (10)] The Hessian of the map $h$ whose extension is $\phi$ is given by

$$Hh_x(v) = \nabla_v \text{grad} h = P_x H\phi_x(v) + \mathfrak{A}(v, P_x^\perp \text{grad} \phi_x). \quad (7)$$

We develop some lemmas for the computation of the different expressions in the formulae (5) and (7). Let us begin with the orthogonal projection. We already know (Lemma 1.3) that the normal projection, $\text{proj}_x^\perp: \mathbb{R}^N \to \nu_x$, is defined, for each point $x \in X_{n,k}$ as

$$\text{proj}_x^\perp u = \frac{1}{2} x(x^* u + u^* x).$$
Thus we extend the operator proj\(_x\) to the whole space by
\[
\mathcal{P}_x(u) = u - \frac{1}{2}x(x^*u + u^*x), \quad \text{for any } x, u \in \mathbb{R}^N.
\]

**Lemma 2.5.** For each \(x \in \mathbb{R}^N\), the \(\mathbb{R}\)-linear map \(\mathcal{P}_x : \mathbb{R}^N \to \mathbb{R}^N\) is a projector, that is, it verifies \(\mathcal{P}_x \circ \mathcal{P}_x = \mathcal{P}_x\).

**Proof.** From
\[
x^*\mathcal{P}_x(u) = x^*u - \frac{1}{2}x^*x(x^*u + u^*x) = \frac{1}{2}(x^*u - u^*x),
\]
we deduce
\[
(x^*\mathcal{P}_x(u))^* = -x^*\mathcal{P}_x(u)
\]
and
\[
\mathcal{P}_x\mathcal{P}_x(u) = \mathcal{P}_x(u) - \frac{1}{2}(x^*\mathcal{P}_x(u) + \mathcal{P}_x(u)^*x) = \mathcal{P}_x(u). \quad \square
\]

Then we consider the Hessian of the extension \(\phi\).

**Lemma 2.6.** The Hessian of the map \(\phi : \mathbb{R}^{4n \times 4k} \to \mathbb{R}\) is
\[
H\phi_x(v) = 2\begin{bmatrix} 0 \\ \beta X + PY \end{bmatrix}.
\]

Notice that \(\beta X + PY = x_0^*v\) only depends on \(v\).

**Proof.** Recall from Lemma 1.2 that \(\text{grad} \phi_x = \begin{bmatrix} 0 \\ 2P \end{bmatrix}\). Thus, we have
\[
H\phi_x(v) = \frac{d}{dt}|_{t=0} \text{grad} \phi_{x+tv} = 2\frac{d}{dt}|_{t=0} \begin{bmatrix} 0 \\ P + t(\beta X + PY) \end{bmatrix} = 2\begin{bmatrix} 0 \\ \beta X + PY \end{bmatrix}. \quad \square
\]

**Lemma 2.7.** The normal projection of the Hessian of \(\phi\) is
\[
\text{proj}_x^\perp H\phi_x(v) = x \begin{bmatrix} T^*P \\ \beta X + PY \end{bmatrix}^* + \begin{bmatrix} T^*P \\ \beta X + PY \end{bmatrix}^* \begin{bmatrix} 0 \\ \beta X + PY \end{bmatrix}.
\]

**Proof.** This follows from a direct computation
\[
\text{proj}_x^\perp H\phi_x(v) = \frac{1}{2}x(x^*H\phi_x(v) + H\phi_x(v)^*x)
\]
\[
= x \begin{bmatrix} T^*P \\ \beta X + PY \end{bmatrix}^* + \begin{bmatrix} T^*P \\ \beta X + PY \end{bmatrix}^* \begin{bmatrix} 0 \\ \beta X + PY \end{bmatrix}. \quad \square
\]

Finally, we specify the Weingarten map.

**Lemma 2.8.**
1) The Weingarten map of the Stiefel manifold \(X_{n,k}\) is given by
\[
\mathfrak{A}_x(v, w) = -vx^*w - \frac{1}{2}x(v^*w + w^*v), \quad v \in T_x X_{n,k}, \ w \in \nu_x.
\]

2) In particular, if \(x\) is a critical point and \(v = A \begin{bmatrix} X \\ Y \end{bmatrix} \in T_x X_{n,k}\), we have
\[
\mathfrak{A}_x(v, \text{proj}_x^\perp \text{grad} \phi_x) = -2v^*P - x(P^*(\beta X + PY) + (\beta X + PY)^*P).
\]
Proof. 1) This formula is established in [1, Section 4.1] for the real Stiefel manifolds. The same proof works, word for word, in the quaternionic case.

2) Notice first that, in a critical point \( x \) for \( h \), we have \( \text{proj}_x \text{grad} \phi_x = \text{grad} h_x = 0 \), thus \( \text{proj}_x^\perp \text{grad} \phi_x = \text{grad} \phi_x \). Then, from formula (8), we deduce

\[
\mathfrak{A}_x(v, \text{proj}_x^\perp \text{grad} \phi_x) = \mathfrak{A}_x(v, \left[ \begin{array}{c} 0 \\ 2P \end{array} \right]) = -vx^* \left[ \begin{array}{c} 0 \\ 2P \end{array} \right] - \frac{1}{2} x \left( v^* \left[ \begin{array}{c} 0 \\ 2P \end{array} \right] + \left[ \begin{array}{c} 0 \\ 2P \end{array} \right]^* v \right)
\]

\[
= -2vP^* P - \frac{1}{2} x \left( \left[ \begin{array}{c} X \\ Y \end{array} \right]^* \left[ \begin{array}{c} 2\beta^* P \\ 2P^* P \end{array} \right] + \left[ \begin{array}{c} 2P^* \beta P^* P \\ \beta P^* \end{array} \right] \left[ \begin{array}{c} X \\ Y \end{array} \right] \right)
\]

\[
= -2vP^* P - x \left( \left[ \begin{array}{c} X \\ Y \end{array} \right] \left[ \begin{array}{c} \beta^* P \\ P^* \beta \end{array} \right] P + P^* \left[ \begin{array}{c} \beta P^* P \\ \beta \end{array} \right] \left[ \begin{array}{c} X \\ Y \end{array} \right] \right)
\]

\[
= -2vP^* P - x ((\beta X + PY)^* P + P^*(\beta X + PY)). \quad \square
\]

From these lemmas, we can now prove the formula given for the Hessian.

Proof of Theorem 2.1. Recall \( x^*_0 v = \beta X + PY \). With Lemma 2.7, we have

\[
\text{proj}_x H \phi_x(v) = 2 \left[ \begin{array}{c} 0 \\ x^* \end{array} \right] - x(P^* x^*_0 v + v^* x_0 P).
\]

From Lemma 2.8 we compute the Weingarten map:

\[
\mathfrak{A}_x(v, \text{proj}_x^\perp \text{grad} \phi_x) = -2vP^* P - x(P^* x^*_0 v + v^* x_0 P).
\]

From Proposition 2.4 we get

\[
Hh_x(v) = 2 \left[ \begin{array}{c} 0 \\ x^*_0 \end{array} \right] - 2vP^* P - 2x(P^* x^*_0 v + v^* x_0 P)
\]

\[
= 2 \left[ \begin{array}{c} 0 \\ x^*_0 \end{array} \right] - 2vx^* x_0 P - 2xP^* x^*_0 v - 2xv^* x_0 P
\]

\[
= 2 \left[ \begin{array}{c} 0 \\ x^*_0 \end{array} \right] - 2vx^* x_0 P - 2 \left[ \begin{array}{c} T \\ P \end{array} \right] P^* x^*_0 v - 2xv^* x_0 P
\]

\[
= -2 \left[ \begin{array}{c} 0 \\ PP^* - I \end{array} \right] x^*_0 v - 2(vx^* + xv^*)x_0 P,
\]

since \( TP^* = 0 \) for a critical point. \quad \square

3. Eigenvalues and Eigenvectors

In this section, we compute the eigenvalues of the Hessian, by solving the equation \( Hh_x(v) = v\lambda \), where \( x \in \Sigma_q \), \( v \in T_x X_{n,k} \) and \( \lambda \in \mathbb{H} \). Recall that the critical submanifold is nondegenerate if the kernel of the Hessian coincides with the tangent space at each point. Also, the index of \( \Sigma_q \) is the dimension of the largest subspace on which the Hessian is negative definite.
Example 3.1. Let us begin with the simplest case: $x_0 = \begin{bmatrix} 0 \\ I \end{bmatrix}$. Since it verifies the condition $TP^* = 0$, this is a critical point. In fact, we have $h(x_0) = k$, so it is a maximum. For the value of the Hessian, we apply (4), with $A = I_n$, $P = I_k$ and $\beta = 0$ and get the equation giving the eigenvalues:

$$Hh_{x_0}(v) = -2 \begin{bmatrix} X \\ 0 \end{bmatrix} = \begin{bmatrix} X\lambda \\ Y\lambda \end{bmatrix}.$$ 

Therefore, the eigenvalues are $\lambda = 0$ with multiplicity $k$, and $\lambda = -2$ with multiplicity $n - k$ equal to the index, showing that the transverse directions are all going down. Below, we develop this structure in the general case.

As it will soon appear, it is sufficient to determine the eigenvalues in some particular points, as $x_0$ for the case $q = k$. We construct now such particular point in each singular submanifold. Let $p, q \geq 0$, $p + q = k$ and $p + q' = n - k$.

Definition 3.2. The notable point $x_0^q$ of $\Sigma_q$ is defined by

$$x_0^q = \begin{bmatrix} T_0 \\ P_0 \end{bmatrix} \text{ with } P_0 = \begin{bmatrix} 0_p & 0 \\ 0 & I_q \end{bmatrix} \text{ and } T_0 = \begin{bmatrix} I_p & 0 \\ 0 & 0_{q' \times q} \end{bmatrix}.$$ 

Indeed, an easy computation gives the equalities $T_0^*T_0 + P_0^*P_0 = I_k$ and $T_0P_0^* = 0$ which justify the assertion $x_0^q \in \Sigma_q$. (The previous point $x_0$ of Example 3.1 is $x_0^k$.)

Let us return to the general case. Using (4), the eigenvalues are the solutions of the system

$$\begin{aligned}
-2(XP^*P - \beta^*\beta X) &= X\lambda \\
-2(X^*\beta^*P - P^*\beta X) &= Y\lambda.
\end{aligned} \quad (9)$$

Instead of solving it in general, we show its invariance by a transitive action, which reduces the resolution to the case of the notable points.

Theorem 3.3.

1) The group $K_{n,k} = \text{Sp}(n-k) \times \text{Sp}(k) \times \text{Sp}(k)$ acts transitively on the left on each critical level $\Sigma_q$ as

$$(m,a,b) \cdot x = (m,a,b) \cdot \begin{bmatrix} T \\ P \end{bmatrix} = \begin{bmatrix} mTb^* \\ aPb^* \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} T \\ P \end{bmatrix} b^*.$$ 

2) The Hessian is invariant by the action, that is if $x = g \cdot x_0^q \in \Sigma_q$ and $v = g \cdot v_0 \in T_xX_{n,k}$, with $g \in K_{n,k}$ and $v_0 \in T_{x_0}X_{n,k}$, then

$$Hh_x(v) = g \cdot Hh_{x_0^q}(v_0).$$

Proof. 1) Let $x \in \Sigma_q$. Set $T' = mTb^*$ and $P' = aPb^*$. We verify $T'(P')^* = mTb^*bP^*a = m(TP^*)a = 0$, thus, from Proposition 1.4, $(m,a,b) \cdot x$ is a critical point. We compute now its image by $h$ and get directly from the definition,

$$h((m,a,b) \cdot x) = \text{Tr}((P')^*P') = \text{Tr}(bP^*a^*aPb^*) = \text{Tr}(bP^*bP^*) = \text{Tr}(P^*P) = q.$$ 

Thus $(m,a,b) \cdot x \in \Sigma_q$. Let $x_0^q$ be the notable point of $\Sigma_q$. From [11] Theorem 3.1, there exist $a, b \in \text{Sp}(k)$ and $m \in \text{Sp}(n-k)$ such that $P = aP_0b^*$ and $T = mT_0b^*$. This implies the transitivity of the action.
2) If \( x = Ax_0, \) \( x = g \cdot x_0^q, \) and \( x_0^q = Bx_0, \) where \( A, B \in \text{Sp}(n), \) \( x_0 = x_0^k = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \) and \( T_0 \) and \( P_0 \) as in Definition 3.2. We first determine \( A \) and \( B. \) Let us observe,

\[
Ax_0 = x = g \cdot x_0^q = \begin{bmatrix} m \\ 0 \\ a \end{bmatrix} \begin{bmatrix} T_0 \\ P_0 \end{bmatrix} b^* = \begin{bmatrix} m \\ 0 \\ a \end{bmatrix} Bx_0 b^* = \begin{bmatrix} m \\ 0 \\ a \end{bmatrix} B \begin{bmatrix} I_p \\ 0 \\ b^* \end{bmatrix} x_0.
\]

Thus, we can take

\[
A = \begin{bmatrix} m \\ 0 \\ a \end{bmatrix} B \begin{bmatrix} I_p \\ 0 \\ b^* \end{bmatrix}
\]

so

\[
B = \begin{bmatrix} m^* \\ 0 \\ a^* \end{bmatrix} A \begin{bmatrix} I_p \\ 0 \\ b \end{bmatrix}. \]

If we denote \( A = \begin{bmatrix} \alpha & \beta \\ \beta & P \end{bmatrix} \) and \( B = \begin{bmatrix} \alpha_0 & T_0 \\ \beta_0 & P_0 \end{bmatrix}, \) the equality (10) implies \( \beta_0 = a^* \beta. \)

Now, let \( v = A \begin{bmatrix} X \\ Y \end{bmatrix} = g \cdot v_0 = g \cdot B \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}, \) we have

\[
A \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} m \\ 0 \\ a \end{bmatrix} B \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} b^* = A \begin{bmatrix} I_p \\ 0 \\ b \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} b^*
\]

which gives \( X_0 = Xb \) and \( Y_0 = b^T Y b. \) By replacing \( X = X_0 b^* = P = aP \) \( b^* \) and \( \beta = a \beta_0 \) by their values, we get

\[
\begin{align*}
XP^* \beta - P^* \beta X & = (X_0 P_0 \beta - \beta_0 \beta_0 X_0) b^*, \\
x^* \beta P - P^{*} \beta X & = b(X_0 \beta_0 P_0 - P_0 \beta_0 X_0) b^*,
\end{align*}
\]

Then, we deduce from formula (4):

\[
\text{H}h_x(v) = -2 \begin{bmatrix} m \\ 0 \\ a \end{bmatrix} B \begin{bmatrix} X_0 P_0 \beta_0 P_0 - \beta_0 \beta_0 X_0 \\ X_0 \beta_0 P_0 - P_0 \beta_0 X_0 \end{bmatrix} b^* = g \cdot \text{H}h_x^2(v_0). \]

From the properties of a left action, we reduce the search of the eigenvalues and eigenvectors of the Hessian to the case of the notable points.

**Corollary 3.4.** The Hessian has the same eigenvalues in all points of the critical level \( \Sigma_q. \) Moreover, the vector \( v_0 \) is a \( \lambda \)-eigenvector of \( \text{H}h_x^2 \) if, and only if, \( g \cdot v_0 \) is a \( \lambda \)-eigenvector of \( g \cdot \text{H}h_x^2 \).

### 3.1. Eigenvalues, kernel and index at the notable point \( x_0^q. \)

We first have to solve the system

\[
\begin{align*}
-2(X_0 P_0 \beta_0 P_0 - \beta_0 \beta_0 X_0) & = X_0 \lambda, \\
-2(X_0 \beta_0 P_0 - P_0 \beta_0 X_0) & = Y_0 \lambda,
\end{align*}
\]

with \( T_0 \) and \( P_0 \) as in Definition 3.2. We complete \( \begin{bmatrix} T_0 \\ P_0 \end{bmatrix} \) to a symplectic matrix \( B = \begin{bmatrix} \alpha_0 & T_0 \\ \beta_0 & P_0 \end{bmatrix} \) with \( \beta_0 = \begin{bmatrix} I_p \\ 0 \\ 0 \end{bmatrix}. \) Then \( \beta_0 \beta_0 = \begin{bmatrix} I_p \\ 0 \\ 0 \end{bmatrix} \) and \( P_0 \beta_0 \beta_0 = 0. \) With these values, the second equation of (11) becomes \( Y_0 \lambda = 0. \) For the first one, after a writing of \( X_0 \) in block of the adequate size, \( X_0 = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \) this equation is equivalent to the system:

\[
2a_1 = a_1 \lambda, \quad a_2 \lambda = 0, \quad a_3 \lambda = 0, \quad -2a_4 = a_4 \lambda. \]

Therefore, the eigenvalues and eigenvectors are:
• \( \lambda = 0 \) with \( a_1 = 0, a_4 = 0 \). So \( X_0 = \begin{bmatrix} 0 & a_2 \\ a_3 & 0 \end{bmatrix} \) and \( Y_0 \) is an arbitrary skew-Hermitian matrix.
• \( \lambda = 2 \), with \( Y_0 = 0 \) and \( a_2 = 0, a_3 = 0, a_4 = 0 \).
• \( \lambda = -2 \), with \( Y_0 = 0 \) and \( a_1 = 0, a_2 = 0, a_3 = 0 \).

From that, we deduce immediately the dimension of the kernel and the index. For the kernel, we observe that \( a_2 \in \mathbb{H}^{p \times (k-p)} \) and \( a_3 \in \mathbb{H}^{(n-k-p) \times p} \).

**Corollary 3.5.** The (real) dimension of the kernel of \( H_{x_0^q}^q \), is

\[
4p(k-p) + 4(n-k-p)p + 3k + 4 \frac{k^2 - k}{2} = 4np - 8p^2 + 2k^2 + k.
\]

**Corollary 3.6.** The index of the critical point is the dimension of the eigenspace for \( \lambda = -2 \), that is, the dimension of the vector space of matrices

\[
X_0 = \begin{bmatrix} 0 & 0 \\ 0 & a_4 \end{bmatrix},
\]

which equals \( 4(n-k-p)(k-p) = 4(n-2k+q)q \), with \( q = k-p \)

4. Critical submanifolds

We can describe the critical level \( \Sigma_q \) as a homogeneous space of the group \( K_{n,k} = \text{Sp}(n-k) \times \text{Sp}(k) \times \text{Sp}(k) \), which acts transitively. For that, we compute the isotropy subgroup of the notable point \( x_0^q = \begin{bmatrix} T_0 \\ P_0 \end{bmatrix} \) by solving the equation \( g \cdot x_0^q = x_0^q \). With \( g = (m, a, b) \in K_{n,q} \), this equation becomes the system,

\[
mT_0 = T_0b \quad \text{and} \quad aP_0 = P_0b. \tag{12}
\]

We decompose again the matrices in boxes of the adequate sizes:

\[
m = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.
\]

Replacing these expressions for \( m, a \) and \( b \) in (12), we obtain

\[
m_{11} = b_{11}, \quad m_{21} = 0, \quad b_{12} = 0, \quad a_{12} = 0, \quad a_{22} = b_{22} \quad \text{and} \quad b_{21} = 0.
\]

In short, we have

\[
m = \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \in \text{Sp}(p) \times \text{Sp}(n-k-p), \quad a = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \in \text{Sp}(p) \times \text{Sp}(k-p),
\]

together with

\[
b = \begin{bmatrix} m_{11} & 0 \\ 0 & a_{22} \end{bmatrix}.
\]

**Corollary 4.1.** The critical submanifold \( \Sigma_q \), with \( q = k-p \), is diffeomorphic to the quotient \( K_{n,k}/L_{n,k,q} \) with

\[
K_{n,k} = \text{Sp}(n-k) \times \text{Sp}(k) \times \text{Sp}(k)
\]

and

\[
L_{n,k,q} = \text{Sp}(p) \times \text{Sp}(n-k-p) \times \text{Sp}(p) \times \text{Sp}(k-p).
\]
The injection $L_{n,k,q} \hookrightarrow K_{n,k}$ is given by
\[
(m_1, m_2, a_1, a_2) \mapsto \left( \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} m_1 & 0 \\ 0 & a_2 \end{bmatrix} \right).
\]

Corollary 4.2. The dimension of $\Sigma_q$ coincides with the dimension of the kernel of the Hessian, thus its is a nondegenerate critical submanifold.

Proof. Using $\dim \text{Sp}(m) = 2m^2 + m$, we find $\dim \Sigma_q = \dim K_{n,k} - \dim L_{n,k,q} = 4np - 8p^2 - 2k^2 + k$. The equality $\dim \Sigma_q = \dim \ker H_{h,q}$, follows from Corollary 3.3. □

Proposition 4.3. The critical level $\Sigma_q$ is the total space of a fibre bundle of fibre $\text{Sp}(k)$, of structural group $\text{Sp}(k-q) \times \text{Sp}(q)$ and of basis the product of two Grassmannians $\text{Gr}_{n-k,q} \times \text{Gr}_{k,q}$.

Proof. With the notations of Corollary 4.1 let $(m, a, b)$ be an element of $K_{n,k}$ and $(m_1, m_2, a_1, a_2)$ an element of $L_{n,k,q}$. As proved in the computation of the isotropy group, the action on the right of $L_{n,k,q}$ on $K_{n,k}$ is given by
\[
(m, a, b) \cdot (m_1, m_2, a_1, a_2) = \left( m \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, a \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, b \begin{bmatrix} m_1 & 0 \\ 0 & a_2 \end{bmatrix} \right).
\]

We first consider the group of elements $(m_2, a_1) \in \text{Sp}(n-k-p) \times \text{Sp}(p)$. The quotient of $K_{n,k}$ by this subgroup is
\[
\mathcal{E} = X_{n-k,p} \times X_{k,k-p} \times \text{Sp}(k).
\]
Moreover, the action of $(m_1, a_2) \in \text{Sp}(p) \times \text{Sp}(k-p)$ on $(\bar{m}, \bar{a}, b) \in \mathcal{E}$ is given by
\[
(\bar{m}, \bar{a}, b) \cdot (m_1, a_2) = \left( \bar{m} m_1, \bar{a} a_2, b \begin{bmatrix} m_1 & 0 \\ 0 & a_2 \end{bmatrix} \right).
\]
The quotient is known to be the critical submanifold $\Sigma_q$. By definition, this is also the fiber bundle associated to the principal bundle
\[
\text{Sp}(p) \times \text{Sp}(k-p) \rightarrow X_{n-k,p} \times X_{k,k-p} \rightarrow \text{Gr}_{n-k,p} \times \text{Gr}_{k,k-p}
\]
and the left $(\text{Sp}(p) \times \text{Sp}(k-p))$-space $\text{Sp}(k)$. Its fibre is $\text{Sp}(k)$ and the structural group is $\text{Sp}(p) \times \text{Sp}(k-p)$, see [5, §5 of Chapter 4] for more details on this construction. □

Example 4.4. Let $n < 2k$, and $q = 2k - n$ corresponding to the minimal level. The critical manifold $\Sigma_{2k-n}$ is the total space of a fiber bundle, with fiber $\text{Sp}(k)$, and basis
\[
\text{Gr}_{n-k,n-k} \times \text{Gr}_{k,2k-n} = \text{Gr}_{k,n-k}.
\]
For instance, if $n = 2k - 1$, the minimal level corresponds to $q = 1$ and $\Sigma_1$ fibers over the projective space $\text{Gr}_{k,k-1} = \text{Gr}_{k,1} = \mathbb{H}P^{k-1}$ with fiber $\text{Sp}(k)$.

5. Integration of the gradient flow

In this section we show how to give an explicit description of the gradient flow for the function $h$. There are already examples in the literature of gradient flows which are naturally integrable: for instance linear functions on compact symmetric spaces [9].
According to Proposition 1.1, the gradient of \( h(x) = \text{Tr}(P^*P) \) at the point \( x = \begin{bmatrix} T \\ P \end{bmatrix} \) is

\[
\text{grad} \ h_x = -2 \begin{bmatrix} TP^*P \\ (PP^* - I)P \end{bmatrix},
\]

so if \( \alpha(t) = \begin{bmatrix} T(t) \\ P(t) \end{bmatrix} \) the equation \( \alpha'(t) = \text{grad} \ h_{\alpha(t)} \) reduces to

\[
T' = -2TP^*P, \quad P' = -2(PP^* - I)P.
\]

(14)

Let us fix an initial condition \( \alpha(0) = \begin{bmatrix} T(0) \\ P(0) \end{bmatrix} \). According to [11, Theorem 3.1], the \( k \times k \) matrix \( P(0) \) admits a singular value decomposition

\[
P(0) = a \begin{bmatrix} I_{p \times p} & 0 & 0 \\ 0 & \text{diag}[c_i]_{q \times q} & 0 \\ 0 & 0 & 0_{r \times r} \end{bmatrix} b^*,
\]

(15)

with \( p + q + r = k \), \( a, b \in \text{Sp}(k) \), \( p, q, r \geq 0 \) and \( 0 < c_i < 1 \).

Analogously,

\[
T(0) = m \begin{bmatrix} 0_{p' \times p} & 0 & 0 \\ 0 & \text{diag}[-s_i]_{q \times q} & 0 \\ 0 & 0 & -I_{r \times r} \end{bmatrix} b^*,
\]

(16)

with \( 0 < s_i = \sqrt{1 - c_i^2} < 1 \), \( m \in \text{Sp}(n - k) \) and \( p' + q + r = n - k \).

Moreover, if \( \alpha(0) \) is not a critical point, then \( q > 0 \), according to point 1) of Proposition 1.4.

For each \( i = 1, \ldots, q \) choose constants \( C_i > 0 \) such that

\[
c_i^2 = \frac{C_i}{1 + C_i},
\]

(17)

Lemma 5.1. With the notations of (15), the equation \( P' = -2(PP^* - I)P \) in (14) has the solution

\[
P(t) = a \begin{bmatrix} I_{p \times p} & 0 & 0 \\ 0 & \text{diag}[c_i(t)]_{q \times q} & 0 \\ 0 & 0 & 0_{r \times r} \end{bmatrix} b^*,
\]

where

\[
c_i(t)^2 = \frac{C_i e^{4t}}{1 + C_i e^{4t}}.
\]

(18)

Proof. We have

\[
P' = a D' b^*
\]

while

\[
-2(PP^* - I)P = -2(aa^* Db - I)ab^* = -2a(D^2 - I)a^* Db^* = -2a(D^2 - I)Db^*.
\]
So we have to prove that $D' = -2(D^2 - I)D$. This identity is trivially true for the entries 1 or 0 in $D$. For the other entries $c_i(t)$, it follows easily from (15).

Notice that the initial condition is $P(0)$, because $c_i(0)^2 = c_i^2$, as it follows from (17).

**Lemma 5.2.** With the notations of (16), the solution of the equation $T' = -2TP^*P$ in (14) is

$$T(t) = m \begin{bmatrix} 0_{p' \times p} & 0 & 0 \\ 0 & \text{diag}[-s_i(t)]_{q \times q} & 0 \\ 0 & 0 & -I_{r \times r} \end{bmatrix} b^*,$$

where

$$s_i(t)^2 = 1 - c_i(t)^2 = \frac{1}{1 + Ce^{4t}}. \tag{19}$$

**Proof.** We have

$$-2TP^*P = 2m \begin{bmatrix} 0_{p' \times p} & 0 & 0 \\ 0 & \text{diag}[s_i(t)c_i(t)]_{q \times q} & 0 \\ 0 & 0 & 0_{r \times r} \end{bmatrix} b^*,$$

so we only have to show that

$$-s_i' = 2s_i(1 - s_i^2),$$

which follows easily from (19).

Notice that the initial condition is $T(0)$ because $s_i(0) = s_i$.

Finally, notice that the solutions in Lemmas 5.1 and 5.2 verify $T^*T + P^*P = I_k$ for all $t$, so we have completely integrated the gradient equation.

**Example 5.3.** On the sphere $S^7 = X_{2,1}$, the two coordinates of $x = \begin{bmatrix} T \\ P \end{bmatrix}$ are quaternions. The function $h(x) = |P|^2$ has two critical levels $h = 0, 1$, each one diffeomorphic to $S^3$ (they correspond to $P = 0$ or $T = 0$, respectively). Take a regular point $\alpha(0) = \begin{bmatrix} T_0 \\ P_0 \end{bmatrix}$. The flow line passing through it is

$$T(t) = -s(t)\frac{T_0}{|T_0|}, \quad P(t) = c(t)\frac{P_0}{|P_0|},$$

where

$$s(t)^2 = \frac{1}{1 + Ce^{4t}}, \quad c(t)^2 = \frac{Ce^{4t}}{1 + Ce^{4t}}$$

and

$$C = \frac{|P_0|}{1 - |P_0|}.$$

Taking limits to $\pm \infty$ we observe that the flow line goes from $\begin{bmatrix} T_0/|T_0| \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ P_0/|P_0| \end{bmatrix}$. 

\[\Box\]
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