Canonical enhancement as a result of Poisson distribution

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Abstract

We point out that a certain finite size effect in heavy ion physics, the canonical enhancement, is based on the difference of conservation constrained pair statistics for Poisson and Gauss distributions, respectively. Consequently it should occur in a wide range of phenomena, whenever comparing rare and frequent events.

It is often so, in particular in relativistic heavy ion collisions observing thousands of newly produced particles in an event, that only statistical information is available for tracing back characteristica of an earlier stage of interacting matter. Aiming at the discovery of quark gluon plasma it is especially important to be aware of an unavoidable loss of information due to the very nature of statistics. Selecting out, however, relevant control parameters even statistical information can be used to make qualitative distinctions with respect to the kind of matter existed for a while in such experiments, in particular drawing conclusions about a possible equation of state.

A truly interesting control parameter is the size of the reacting system, which is often regarded as infinite in theory (as the so called thermodynamical limit). In reality it is, however, finite and to a certain degree controllable in experiments by varying the target nucleus and triggering measurements by centrality (multiplicity, transverse energy) of a collision. But even accepting the infinite size assumption (theoretically in any high energy collision even infinitely many particle pairs can be created for a while), there is a further distinction related to the mean particle number in an event.

Two famous, very characteristic limits can be considered: i) the limiting case of frequent events producing many particles on the mean, approximately described by the Gauss distribution, and ii) very rare events with few (less than
one) particles per event as a mean value, generally covered by the Poisson distribution. In this letter we point out that a characteristic finite size effect in relativistic heavy ion collisions, named canonical enhancement, can be understood as originating in the difference between rare and frequent events, between sparse and cupious particle production. Mathematically this reveals itself in the difference between a large and a small mean particle number. The large number case - as it is well known - gives results equivalent to those stemming from a Gauss distribution (central limit theorem of statistics).

The latter equivalence is the basis of the equivalence of the canonical and grand canonical approaches in the thermodynamical limit, while the Poisson statistics with a small mean number plots the difference between these approaches. The ratio of the canonical and grand canonical result in small systems is less than one, the same result can be obtained simply relying on the Poisson distribution alone. The effect is in principle more general than the canonical – grand canonical distinction; processes and phenomena featuring Poisson statistics occur also in physical situations far from equilibrium. Conversely, observation of this deviation signals truly a mesoscopic nature of the physical system (a fireball formed in heavy ion collisions), but does not prove in itself equilibrium, nor measures temperature or volume of the system. The mesoscopic nature reflected by the 'canonical' suppression factor can be extracted from mean particle numbers alone.

The importance of this finite size effect for relativistic heavy ion collisions, in particular its explanation power for finding more strange particles per nucleon produced in ion - ion than in proton - proton collisions (the so called strangeness enhancement), was first realized by Redlich et. al. Originally presented as a constraint stemming from the proper statistical treatment of a U(1) symmetry in the canonical approach (whence the name canonical enhancement), and referring to volume and temperature during the derivation of the result, soon was it rederived on the basis of rate equations, and has been shown that there can be an underlying master equation with the Poisson distribution as stationary solution. Rafelski and Letessier emphasized that this phenomenon is strongly related to pair statistics (to associated production of conserved charge and anticharge) and it holds also out of chemical equilibrium with general fugacity factors. Also a debate has been started about the relevance of this effect on the strangeness enhancement in heavy ion collisions, especially at CERN SPS using 40, 60 and 160 GeV/nucleon beam energies.

Our aim in this letter is to rederive the canonical enhancement factor relying on the Poisson distribution alone, not assuming even thermal equilibrium in the background. One does not need to refer either to temperature nor to volume in this derivation, - even if also thermal systems in a given volume may show Poisson statistics, this is not their only possible origin. Then we apply this finite number suppression factor to an analysis of particle production in the frame-
work of the sudden quark coalescence picture of hadron formation, ALCOR\[6\].

We recover qualitatively the linear coalescence assumption for the low particle number case, i.e. that the number of composite objects (mesons and baryons) are proportional to the product of the numbers of its constituents (the quarks and antiquarks).

From the probability \( P(n) \) of getting \( n \) particles of a certain kind the expectation value \( \langle n \rangle \) and higher moments like quadratic spread can easily be obtained by the use of the following generating function:

\[
Z(\gamma) = \sum_n P(n)e^{\gamma n}. \quad (1)
\]

The expected (mean) number becomes

\[
\langle n \rangle = \frac{\partial}{\partial \gamma} \log Z \bigg|_{\gamma=0}. \quad (2)
\]

Under certain circumstances we are particularly interested in pairs of particles carrying a sum or a difference of a given physical quantity, than in the one-particle distribution. On the basis of independent one-particle distributions the pair statistics and the distribution of the sum can be calculated. Without any constraint on the difference, the distribution of the sum is described by the convolution of the corresponding distributions:

\[
P_U(s) = \sum_{n,m} P_1(n)P_2(m)\delta_{m+n,s} = \sum_n P_1(n)P_2(s-n). \quad (3)
\]

This is, however, correct only if there is no information on the difference. In particular the production of particles and antiparticles happens always in pairs due to charge (and further) conservation laws, and as a consequence, even if the production process is statistical, their difference is bounded to be zero. In this case we obtain another distribution of the sum:

\[
P_C(s|0) = \frac{1}{P_0} \sum_{m,n} P_1(n)P_2(m)\delta_{m+n,s}\delta_{m-n,0} = \frac{P_1(s/2)P_2(s/2)}{\sum_n P_1(n)P_2(n)}. \quad (4)
\]

This is a conditional probability with the factor \( P_0 \) being the probability of getting zero difference in the number statistically, which ensures the normalization of the result to one. Fig.1 shows a geometrical interpretation of the distinction between these two pair statistics: the unconstrained distribution can be obtained by adding all points in \( (n,m) \) space over a given bin of the diagonal representing a given sum \( s = n + m \), while in case of the fixed zero difference only hits in the diagonal stripe counts (and the ratio of the stripe to the total area reflects the \( 1/P_0 \) normalization factor).
Fig. 1  Demonstrating the constrained (main diagonal stripe) and unconstrained (the whole square) distribution of the sum of two Poisson deviates with mean value 10 for each.

Applying this idea to restless coalescence of quarks into hadrons, we still constrain the difference to zero, but this time due to a confinement principle. The number of coalesced hadrons is the half sum in this case. Caring for pions only in a simplified world of light quarks and antiquarks, we arrive at having \( n \) from both, with zero difference and a sum of \( 2n \) forming exactly \( n \) pions. The generating function of the constrained distribution \( P_C(s|0) \) in this case reads as

\[
Z_C(\gamma) = \frac{1}{P_0} \sum_n P_q(n)P_{ar{q}}(n)e^{2\gamma n}.
\] (5)

Assuming Poisson distribution for the quarks this generating function becomes

\[
Z_C(\gamma) = \frac{1}{P_0} \sum_n \frac{\langle q \rangle^n}{n!}e^{-\langle q \rangle} \frac{\langle \bar{q} \rangle^n}{n!}e^{-\langle \bar{q} \rangle} e^{\gamma n} = \frac{1}{P_0} e^{-\langle q \rangle - \langle \bar{q} \rangle} I_0(2e^{\gamma} \sqrt{\langle q \rangle \langle \bar{q} \rangle}).
\] (6)

with \( I_0(x) \) being the Bessel function of the first kind with imaginary argument. The normalization factor \( 1/P_0 \) can be obtained from \( Z(0) = 1 \), giving

\[
Z_C(\gamma) = \frac{I_0(2e^{\gamma} \sqrt{\langle q \rangle \langle \bar{q} \rangle})}{I_0(2\sqrt{\langle q \rangle \langle \bar{q} \rangle})}.
\] (7)

The expectation value of the pions is

\[
\langle \pi \rangle_P = \frac{1}{2} \langle s \rangle = \sqrt{\langle q \rangle \langle \bar{q} \rangle} \frac{I_1(2\sqrt{\langle q \rangle \langle \bar{q} \rangle})}{I_0(2\sqrt{\langle q \rangle \langle \bar{q} \rangle})}.
\] (8)

with the label \( P \) reminding us to the Poisson distribution. For small mean number of quarks this leads to the product assumption of the simple linear coalescence model:

\[
\langle \pi \rangle_P \approx 2\langle q \rangle \langle \bar{q} \rangle.
\] (9)
For large mean numbers on the other hand the ratio of the Bessel functions, \( I_1/I_0 \) approaches one. This latter factor is the canonical suppression factor.

As opposed to the above analysis of the Poisson distribution, for the Gauss distribution both the unconstrained and the zero difference constrained pair statistics leads again to a Gauss distribution (but with a larger width). Assuming Gauss distributed quark and antiquark numbers with respective square widths equal to the mean values, as it is typical for equilibrium ideal gases, the expectation value of the half sum, the pion number becomes the harmonic mean of the expectations (because by convolution of Gauss distributions the inverse square widths are additive):

\[
\langle \pi \rangle_G = \frac{2\langle q \rangle\langle \bar{q} \rangle}{\langle q \rangle + \langle \bar{q} \rangle}.
\]  
(10)

In case of zero baryon charge \( \langle q \rangle = \langle \bar{q} \rangle \) and these formul\( i \) reduce to a very simple result:

\[
\langle \pi \rangle_G = \langle q \rangle; \quad \langle \pi \rangle_P = \langle q \rangle \frac{I_1(2\langle q \rangle)}{I_0(2\langle q \rangle)}.
\]  
(11)

As a consequence one can express canonical enhancement in this case as a relation between Poisson and Gauss expectation values.

It is particularly interesting when we compare rare particles, for example \( K^+ \) mesons coalesced from Poisson distributed \( u \) and \( \bar{s} \) quarks with copiously produced \( \pi^+ \) pions glued from \( u \) and \( d \) quarks. The expected meson numbers, \( \langle K^+ \rangle_P = \sqrt{u\bar{s}} \frac{I_1(2\sqrt{u\bar{s}})}{I_0(2\sqrt{u\bar{s}})} \),

for kaons and

\[
\langle \pi^+ \rangle_G = \frac{2ud}{u+d}
\]  
(13)

for pions (with \( u, \bar{s} \) and \( d \) denoting here the expectation values of the respective quark numbers) for a baryon free and strangeness free, and isotopically symmetric fireball due to \( d = d = \bar{u} = u \) and \( \bar{s} = s = f \bar{d} \) with a fixed strangeness ratio \( f = \bar{d}/\bar{s} \) reduce to

\[
\langle K^+ \rangle_P = u\sqrt{f} \frac{I_1(2u\sqrt{f})}{I_0(2u\sqrt{f})}
\]  
(14)

and

\[
\langle \pi^+ \rangle_G = u.
\]  
(15)

In this simplified scenario the kaon to pion ratio can be expressed as a function of the pion number and so can be compared with experimental results. Of course the result can only be qualitative on this level, since by using the above
assumptions for the quark and antiquark numbers, one is bound to predict the same ratio for \( K^+/\pi^+ \) and for \( K^−/\pi^− \). It is not quite fulfilled in experiments: comparing the heavy ion results with those of proton - proton collisions, these ratios increase by a factor of roughly 2 and 1.6 respectively.[7]

The result of this simple idea can be inspected in Fig.2, where the normalized ratio \( ⟨K^+⟩_P/⟨\pi^+⟩_G\sqrt{f} \) is plotted versus the scaled pion number, \( 2⟨\pi^+⟩_G\sqrt{f} \). This is exactly the canonical suppression factor. Using the value \( f \approx 0.5 \) for estimating the Wroblinski factor \( f \) one concludes that the kaon / pion ratio falls to its half at an expected pion number of about 0.5 as compared to large pion numbers. This value corresponds to a situation before resonance decays, so it is not directly observable in experiments, but correcting for resonance decays is always possible in the framework of a theoretical model. We used ALCOR for this purpose and concluded that the kaon-pion ratios normalized to the pp case do not change much due to resonance decay, as the unnormalized ratios do.

![Canonical suppression factor](image)

**Fig.2**  Canonical suppression factor obtained from the expectation value of the sum of two (numerically simulated) Poisson distributions. It coincides with the analytically derived ratio of two Bessel functions \( I_1/I_0 \).

Assuming that both pions and kaons are constructed from Poisson distributed quarks, their ratio changes between two limiting values corresponding to few and many quarks respectively:

\[
\frac{⟨K^+⟩_P}{⟨\pi^+⟩_P} = \sqrt{f} \frac{I_1(2u\sqrt{f})}{I_0(2u\sqrt{f})} \frac{I_0(2u)}{I_1(2u)} \quad (16)
\]
gives $f$ for $u \ll 1$ and $\sqrt{f}$ for $u \gg 1$. Considering the light quark (pion) number before resonance decay the former case is realized in $pp$ collisions (according to ALCOR $\langle \pi \rangle \sim 0.1$) and the latter case in $PbPb$ collisions ($\langle \pi \rangle \sim 10$, before resonance decay). The double ratio,

$$\frac{(K/\pi)_{PbPb}}{(K/\pi)_{pp}} \approx \frac{1}{\sqrt{f}} \quad (17)$$

would be about 1.4, which is to be compared with the experimental values 1.6 for negative and 2 for positive kaon to pion ratios. The transition is around $\langle \pi \rangle \approx 0.7 \ldots 1.4$, probably occurring in a collision of relatively light ions.

In conclusion we pointed out that the canonical suppression factor of a ratio of Bessel functions can be derived solely relying on properties of the Poisson distribution, and hence is only as much related to thermal equilibrium properties as the latter realizes a Poisson distribution for rare particles. This distribution, however, may stem also from a series of dynamical events and therefore the corresponding factor should show in a wide class of phenomena. We gave a rough estimate of reflecting this size (expected number) dependence in heavy ion collisions by the strangeness enhancement in the framework of a quark coalescence hadronization model, ALCOR.

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