Minimal elements of stopping time $\sigma$-algebras

Tom Fischer*
University of Wuerzburg

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Abstract

We show how minimal elements of a stopping time $\sigma$-algebra can be expressed in terms of the minimal elements of the $\sigma$-algebra of the underlying filtration. This facilitates an intuitive interpretation of stopping time $\sigma$-algebras. An example is provided.

Key words: Stopped $\sigma$-algebra, stopping time, stopping time $\sigma$-algebra.

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We denote the time axis by $T$, where $T \subset \mathbb{R}$.

**DEFINITION 1.** Let $\tau$ be a stopping time on a filtered probability space $(\Omega, F_\infty, (F_t)_{t \in T}, \mathbb{P})$. We define the stopping time $\sigma$-algebra w.r.t. $\tau$ as

\[
F_\tau = \{ F \in F_\infty : F \cap \{ \tau \leq t \} \in F_t \text{ for all } t \in T \}.
\]

It is well known and straightforward to show that $F_\tau$ indeed is a $\sigma$-algebra. Note again that $F_t \subset F_\infty$ is assumed for $t \in T$.

**DEFINITION 2.** For a measurable space $(\Omega, F)$, we define the set of the minimal elements in the $\sigma$-algebra $F$ by

\[
A(F) = \{ A \in F : A \neq \emptyset, \text{ and if } F \in F \text{ and } F \subset A, \text{ then } F = A \}.
\]

Eq. (2) means that $A \in A(F)$ can not be ‘split’ in $F$, which is why elements of $A(F)$ are also referred to as ‘atoms’ of $F$. Obviously, $A(F) \subset F$. For $|F| < +\infty$, it is therefore easy to see that $A(F)$ is a partition of $\Omega$ and that

\[
F = \sigma(A(F))
\]

since any non-empty $F \in F$ can be written as a finite union of elements in $A(F)$.

*Institute of Mathematics, University of Wuerzburg, Campus Hubland Nord, Emil-Fischer-Strasse 30, 97074 Wuerzburg, Germany. Tel.: +49 931 3188911. E-mail: tom.fischer@uni-wuerzburg.de.
DEFINITION 3. For \( t \in \mathbb{T} \cup \{+\infty\} \), we denote the set of minimal elements in \( \mathcal{F}_t \) by

\[
\mathcal{A}_t = \mathcal{A}(\mathcal{F}_t) = \{ A \in \mathcal{F}_t : A \neq \emptyset, \text{ and if } F \in \mathcal{F}_t \text{ and } F \subset A, \text{ then } F = A \}.
\]

Further, we define

\[
\mathcal{A}_t^\tau = \{ A \in \mathcal{A}_T : A \subset \{ \tau = t \} \} \quad (t \in \mathbb{T} \cup \{+\infty\}),
\]

\[
\mathcal{A}_\tau = \bigcup_{t \in \mathbb{T} \cup \{+\infty\}} \mathcal{A}_t^\tau.
\]

Note that the \( \mathcal{A}_t^\tau \) are disjoint for \( t \in \mathbb{T} \cup \{+\infty\} \).

THEOREM 1. The elements of \( \mathcal{A}_\tau \) are minimal elements of \( \mathcal{F}_\tau \), i.e. \( \mathcal{A}_\tau \subset \mathcal{A}(\mathcal{F}_\tau) \). If \( |\mathcal{F}_\infty| < +\infty \), then \( \mathcal{A}_\tau \) is the set of all minimal elements of \( \mathcal{F}_\tau \), i.e. \( \mathcal{A}_\tau = \mathcal{A}(\mathcal{F}_\tau) \), and \( \mathcal{F}_\tau = \sigma(\mathcal{A}_\tau) = \sigma(\mathcal{A}(\mathcal{F}_\tau)) \).

Proof. (i) \( \mathcal{A}_\tau \subset \mathcal{F}_\tau \): Let \( A \in \mathcal{A}_\tau \), so, for some \( s \in \mathbb{T} \cup \{+\infty\} \), \( A \in \mathcal{A}_s \) and \( A \subset \{ \tau = s \} \) and therefore \( A \cap \{ \tau = s \} = A \). Assume now \( t < s \) for some \( t \in \mathbb{T} \). Then

\[
A \cap \{ \tau \leq t \} = A \cap \{ \tau = s \} \cap \{ \tau \leq t \} = \emptyset \in \mathcal{F}_t.
\]

For \( t \in \mathbb{T} \), assume now \( t \geq s \). Then

\[
A \cap \{ \tau \leq t \} = A \cap \{ \tau = s \} \cap \{ \tau \leq t \} = A \cap \{ \tau = s \} = A \in \mathcal{F}_s \subset \mathcal{F}_t.
\]

Hence, \( A \cap \{ \tau \leq t \} \in \mathcal{F}_t \) for all \( t \in \mathbb{T} \), and therefore \( \mathcal{A}_\tau \subset \mathcal{F}_\tau \). (ii) \( \mathcal{A}_\tau \subset \mathcal{F}_\tau \) and \( F \subset A \) implies \( F = A \): (a) Assume that \( A \in \mathcal{A}_\infty \) and \( F \in \mathcal{F}_\tau \) with \( F \subset A \). Clearly, \( A \in \mathcal{A}_\infty \), implying \( F = A \) since \( F \in \mathcal{F}_\infty \). (b) Assume \( A \in \mathcal{A}_t^\tau \) for some \( t \in \mathbb{T} \) and \( F \in \mathcal{F}_\tau \) with \( F \subset A \). Therefore, \( A \in \mathcal{A}_t \) and \( F \subset A \subset \{ \tau = t \} \), and hence \( F \cap \{ \tau = t \} = F \). As \( F \in \mathcal{F}_\tau \), one has \( F \cap \{ \tau \leq t \} \in \mathcal{F}_t \). Since \( \{ \tau = t \} \in \mathcal{F}_t \), \( F \cap \{ \tau \leq t \} \cap \{ \tau = t \} = F \cap \{ \tau = t \} = F \cap \{ \tau = t \} = F \in \mathcal{F}_t \), but \( A \in \mathcal{A}_t \), and therefore \( F = A \). (i) and (ii) prove the first statement of the theorem. Assume now \( |\mathcal{F}_\infty| < +\infty \). \( \mathcal{A}_\tau \) is then a partition of \( \Omega \), because any two distinct sets in \( \mathcal{A}_\tau \) are disjoint, and, since \( \{ \tau = t \} \in \mathcal{F}_t \) for \( t \in \mathbb{T} \cup \{+\infty\} \), one has \( \bigcup \mathcal{A}_t^\tau = \{ \tau = t \} \), so \( \bigcup \mathcal{A}_\tau = \Omega \). This proves the second statement. The third statement follows by Eq. \( \text{[3]} \). \( \square \)

The following result is well known.

PROPOSITION 1. For \( |\mathcal{F}_\infty| < +\infty \), \( \sigma(\tau) \subset \mathcal{F}_\tau \) and, in general, \( \sigma(\tau) \neq \mathcal{F}_\tau \).

Proof. \( \{ \tau = t \} \) \( (t \in \mathbb{T}) \) and \( \{ \tau = +\infty \} \) are the minimal elements of \( \sigma(\tau) \) if they are non-empty, but it is well known and straightforward to see that these sets are elements of \( \mathcal{F}_\tau \), too. Therefore, \( \sigma(\tau) \subset \mathcal{F}_\tau \). It is an easy exercise to find examples where \( \sigma(\tau) \neq \mathcal{F}_\tau \) (see example below). \( \square \)

We can interpret the filtered probability space \( (\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P}) \) as a random ‘experiment’ or ‘experience’ that develops over time.

The common interpretation of the filtration \( (\mathcal{F}_t)_{t \in \mathbb{T}} \) is that if it is possible to repeat the experiment arbitrarily often until time \( t \in \mathbb{T} \), the maximum information that can be obtained about the experiment is \( \mathcal{F}_t \). Hence, \( \mathcal{F}_t \) represents (potentially) available information up to time \( t \).

Using Theorem \( \square \) for \( |\mathcal{F}_\infty| < +\infty \), we can interpret the stopping time \( \sigma \)-algebra \( \mathcal{F}_\tau \) as the maximum information that can be obtained from repeatedly carrying out the experiment up to the random time \( \tau \). This is straightforward from the definition of the
\(A_t^\tau\) in (5). While this interpretation is, of course, generally known, minimal sets are usually not used to derive it. However, the example below will illustrate that this is a very natural way of interpreting stopping time \(\sigma\)-algebras.

For an intuitive interpretation and representation of stopping times see Fischer (2011).

Example. In Figure 1, we see the usual interpretation of a discrete time finite space filtration as a stochastic tree. In such a setting, the set of paths of maximal length represents \(\Omega\). In this case, \(\Omega = \{\omega_1, \ldots, \omega_8\}\). A path up to some node at time \(t\) (here \(T = \{0, 1, 2, 3\}\) and \(\mathcal{F}_3 = \mathcal{F}_\infty = \mathcal{P}(\Omega)\)) represents the set of those paths of full length that have this path up to time \(t\) in common. The paths up to time \(t\) represent the minimal elements (atoms) \(A_t\) of \(\mathcal{F}_t\). For instance, in the example of Fig. 1,

\[
A_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \ldots, \omega_8\}\}.
\]

A stopping time \(\tau\) (see Fig. 1) and the corresponding stopping process \(X^\tau\) (see Fischer (2011)) are given (\(X^\tau\) is given by the values of the nodes of the tree). The intuitive interpretation of the stopping time as the random time at which \(X^\tau\) jumps to (‘hits’) 0 becomes clear (see boxed zeros). Furthermore, the paths up to those boxed zeros represent the minimal elements \(A_\tau\) of the stopping time \(\sigma\)-algebra \(\mathcal{F}_\tau\). This follows of course from the fact that for any \(A \in A_\tau^\tau\) one has \(A \subset \{\tau = t\}\) by (5) and therefore \(X^\tau_s(A) = 0\), but \(X^\tau_s(A) = 1\) for \(s < t\). So, in a tree example such as the given one, the ‘frontier of first zeros’ (or, precise, the paths leading up to it) describes the stopping time \(\sigma\)-algebra (as well as the stopping time itself). Therefore, it becomes very obvious in what sense \(\mathcal{F}_\tau\) contains the information in the system that can be explored up to time \(\tau\). In the case of the example,

\[
A_\tau = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7\}, \{\omega_8\}\}.
\]

It is also easy to see that in this case the minimal elements of \(\sigma(\tau)\), denoted by \(A(\sigma(\tau))\), are given by

\[
A(\sigma(\tau)) = \{\{\omega_1, \omega_2\}, \{\omega_5, \omega_6\}, \{\omega_3, \omega_4, \omega_7, \omega_8\}\}.
\]

Therefore, \(A_\tau \neq A(\sigma(\tau))\) and, hence, \(\sigma(\tau) \neq \mathcal{F}_\tau\), as stated in Prop. 1.

References

[1] Fischer, T. (2011): Stopping times are hitting times: a natural representation. arXiv:1112.1603v3 [math.PR].
Figure 1: Example.