THE MEASURE TRANSFER FOR SUBSHIFTS INDUCED BY A MORPHISM OF FREE MONOIDS

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Abstract. Every non-erasing monoid morphism \( \sigma : A^* \rightarrow B^* \) induces a measure transfer map \( \sigma_X^M : \mathcal{M}(X) \rightarrow \mathcal{M}(\sigma(X)) \) between the measure cones \( \mathcal{M}(X) \) and \( \mathcal{M}(\sigma(X)) \), associated to any subshift \( X \subseteq A^\mathbb{Z} \) and its image subshift \( \sigma(X) \subseteq B^\mathbb{Z} \) respectively. We define and study this map in detail and show that it is continuous, linear and functorial. It also turns out to be surjective [2]. Furthermore, an efficient technique to compute the value of the transferred measure \( \sigma_X^M(\mu) \) on any cylinder \([w]\) (for \( w \in B^* \)) is presented.

Theorem: If a non-erasing morphism \( \sigma : A^* \rightarrow B^* \) is injective on the shift-orbits of some subshift \( X \subseteq A^\mathbb{Z} \), then \( \sigma_X^M \) is injective.

The assumption on \( \sigma \) that it is “injective on the shift-orbits of \( X \)” is strictly weaker than “recognizable in \( X \)”, and strictly stronger than “recognizable for aperiodic points in \( X \)”. The last assumption does in general not suffice to obtain the injectivity of the measure transfer map \( \sigma_X^M \).

1. Introduction

The prime object of this paper are morphism \( \sigma : A^* \rightarrow B^* \) of free monoids \( A^* \) and \( B^* \) over finite sets \( A \) and \( B \) respectively. These sets are called alphabets, and their elements are letters, denoted here by \( a_k \in A \) or \( b_j \in B \). To \( A \) and \( B \) there are canonically associated the spaces \( A^\mathbb{Z} \) and \( B^\mathbb{Z} \), equipped both with the product topology and a shift operator, here always denoted by \( T \). All morphisms \( \sigma \) in this paper are non-erasing (i.e. none of the \( a_k \in A \) are mapped to the empty word), and such \( \sigma \) induces canonically a map \( \sigma^Z : A^\mathbb{Z} \rightarrow B^\mathbb{Z} \). But while \( \sigma^Z \) is continuous, in general it fails to be a “morphism of dynamical systems” from \( (A^\mathbb{Z}, T) \) to \( (B^\mathbb{Z}, T) \) in the classical topological dynamics meaning, in that most of the time we have:

\[
T \circ \sigma^Z \neq \sigma^Z \circ T
\]

This creates a number of well known technical problems; for instance any non-empty, closed, shift-invariant subset (called a subshift) \( X \subseteq A^\mathbb{Z} \) has \( \sigma^Z \)-image that is in general not shift-invariant, and thus not a subshift by itself. Still, there is a well defined image subshift (see Definition-Remark 2.2 below), denoted here by \( \sigma(X) \), which has been studied previously in many occasions.

The focus of this paper is on the set \( \mathcal{M}(A^\mathbb{Z}) \) of measures \( \mu \) on \( A^\mathbb{Z} \) that are invariant, which means that \( \mu \) is a finite Borel measure, and that the shift operator preserves \( \mu \). The map \( \sigma^Z \) gives us directly the classical “push forward” measure \( \mu_\sigma \) on \( B^\mathbb{Z} \), but because of (1.1) this measure will almost never be invariant, as it will not be preserved by the shift operator on \( B^\mathbb{Z} \). Nevertheless, any invariant measure \( \mu \) on \( A^\mathbb{Z} \) can be canonically be “transferred” by \( \sigma \) to define an invariant measure on \( B^\mathbb{Z} \). Moreover, if the subshift \( X \) is the support of \( \mu \), then \( \sigma(X) \) will be the support of this transferred measure. Setting up properly and studying carefully this measure transfer map

\[
\sigma_X^M : \mathcal{M}(A^\mathbb{Z}) \rightarrow \mathcal{M}(B^\mathbb{Z})
\]

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induced by any non-erasing free monoid morphism \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \) is the main purpose of this paper. There are several occasions where the measure transfer has already been employed sporadically (see for instance Example 1.5 below); the present paper, however, seems to be the first where a systematic treatment is pursued.

We show (see section 3):

**Proposition 1.1.** Let \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \) be any non-erasing morphism, and let \( X \subseteq \mathcal{A}^\mathbb{Z} \) be any subshift over \( \mathcal{A} \), with image subshift \( Y := \sigma(X) \subseteq \mathcal{B}^\mathbb{Z} \). Then the induced measure transfer map \( \sigma^M \) restricts/co-restricts to a well defined map

\[
\sigma^M_X : \mathcal{M}(X) \to \mathcal{M}(Y), \quad \mu \mapsto \mu^\sigma
\]

which has the following properties:

1. \( \sigma^M_X \) is an \( \mathbb{R}_{\geq 0} \)-linear map of cones.
2. \( \sigma^M_X \) is continuous.
3. \( \sigma^M_X \) is surjective.
4. \( \sigma^M_X \) is functorial: \((\sigma' \circ \sigma)^M_X = \sigma'^M_X \circ \sigma^M_X \)
5. If the subshift \( X' \subseteq X \) is the support of a measure \( \mu \in \mathcal{M}(X) \), then \( \sigma(X') \) is the support of \( \sigma^M_X(\mu) \).

We would like to stress that the transferred measure is in general much different from the well known push-forward measure. In particular, the transferred measure of a probability measure will in general not be probability. Indeed, we have (see Remark 4.4 and Proposition 4.5):

**Proposition 1.2.** For any invariant measure \( \mu \) on the full shift \( \mathcal{A}^\mathbb{Z} \) and any non-erasing morphism \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \) the transferred measure \( \mu^\sigma = \sigma^M(\mu) \) satisfies the following formula:

\[
\mu^\sigma(\mathcal{B}^\mathbb{Z}) = \sum_{a_k \in \mathcal{A}} \sum_{b_j \in \mathcal{B}} |\sigma(a_k)|_{b_j} \cdot \mu([a_k])
\]

More specifically, for the “vectors” of letter frequencies we have the following matrix equality:

\[
([\mu^\sigma(b_j)])_{b_j \in \mathcal{B}} = \mathcal{M}(\sigma) \cdot ([\mu([a_k])])_{a_k \in \mathcal{A}}
\]

Here \(|w|_u\) denotes the number of (possibly overlapping) occurrences of the word \( u \) as factor in the word \( w \), and \([w]\) is the “cylinder” that consists of all bi-infinite words \( \ldots x_n x_{n+1} \ldots \) for which the positive half-word \( x_1 x_2 \ldots \) starts with \( w \) as prefix. By \( \mathcal{M}(\sigma) \) we denote the incidence matrix of \( \sigma \), which has coefficient \(|\sigma(a_k)|_{b_j}\) at the position \((j, k)\).

In order to compute the transferred measure \( \mu^\sigma([w']) \) of an arbitrary cylinder \([w'] \subseteq \mathcal{B}^\mathbb{Z} \), as done in Proposition 1.2 for the special case \( w' = b_j \in \mathcal{B} \), it turns out that it is useful to introduce the number \(|\sigma(w)|_u\) of essential occurrences of \( u \) as a factor of \( \sigma(w) \), by which we mean that the first letter of \( u \) occurs in the \( \sigma \)-image of first letter of \( w \), and the last letter of \( u \) occurs in the \( \sigma \)-image of last letter of \( w \). The following is proved in Proposition 4.2 below, and several examples of concrete computations of cylinder values \( \mu^\sigma([w']) \) are given in sections 3 and 4.

**Proposition 1.3.** Let \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \) be any non-erasing monoid morphism, and let \( \mu \) be any invariant measure on \( \mathcal{A}^\mathbb{Z} \). Then for any \( w' \in \mathcal{B}^* \) with \(|w'| \geq 2\) the transferred measure \( \mu^\sigma := \sigma^M(\mu) \), evaluated on the cylinder \([w']\), gives

\[
\mu^\sigma([w']) = \sum_{w \in W(w')} |\sigma(w)|_{w'} \cdot \mu([w]),
\]

for \( W(w') = \{ w \in \mathcal{A}^* : |w| \leq \frac{|w'| - 2}{\min_{a_k \in \mathcal{A}} |\sigma(a_k)|} + 2 \}. \)
As stated above in property (3) of Proposition 1.1, any invariant measure \( \mu' \) on the image subshift \( Y = \sigma(X) \) is equal to the transfer \( \sigma_X^M(\mu) \) of some measure \( \mu \) on the given preimage subshift \( X \). However, for a general non-erasing morphism \( \sigma \), this measure \( \mu \) will be far from uniquely determined by \( \mu' \). A sufficient condition for the injectivity of the measure transfer map \( \sigma_X^M \) is the assumption that \( \sigma \) is recognizable in \( X \) (see Corollary 3.9 of [2]). We show here the following stronger result (see Theorem 5.5 below):

**Theorem 1.4.** Let \( \sigma : A^* \to B^* \) be a non-erasing morphism of free monoids on finite alphabets, and let \( X \subseteq A^\mathbb{Z} \) be any subshift.

If the map induced by \( \sigma \) on the shift-orbits of \( X \) is injective, then the measure transfer map \( \sigma_X^M : M(X) \to M(\sigma(X)) \) is injective.

To be specific, we’d like to note that the hypothesis “injectivity on the set of shift-orbits of \( X \)” is strictly weaker than “recognizable in \( X \)”, and strictly stronger than “recognizable for aperiodic points in \( X \)” (see Proposition 5.7 and the discussion around it). The last condition doesn’t suffice to deduce the conclusion of the above Theorem 1.4 (see Remark 6.8).

To terminate this introduction, we’d like to list some incidents where our readers may already have encountered the measure transfer map, perhaps “disguised” in a different setting or language:

**Example 1.5.** (1) Let \( S \) be a compact surface with non-empty boundary, and \( \tau \subseteq S \) a train track that fills \( S \) and satisfies the usual conditions on its complementary components (see for instance [7] for details). Then a proper choice of arcs \( \alpha_1, \ldots, \alpha_d \) transverse to \( \tau \) gives rise to intervals with the following property: Any oriented geodesic lamination \( \Lambda \subseteq S \), for which we assume that it can be isotoped into an interval-fibered neighborhood \( \mathcal{N}(\tau) \) of \( \tau \) in such a way that \( \Lambda \) becomes transverse to all interval fibers, defines an interval exchange transformation system (IET) on the intervals \( \alpha_k \), and thus a subshift \( X_\Lambda \subseteq A^\mathbb{Z} \) for \( A = \{\alpha_1, \ldots, \alpha_d\} \). Any transverse measure \( \mu_\Lambda \) on \( \Lambda \) defines an invariant measure \( \mu \) on \( X_\Lambda \).

Assume now that (as shown by Thurston for any “pseudo-Anosov” homeomorphism) that some homeomorphism \( h : S \to S \), after being properly isotoped, maps \( \mathcal{N}(\tau) \) in an interval-fiber preserving fashion into \( \mathcal{N}(\tau) \). Then any transverse measure \( \mu_\Lambda \) gives rise to an “h-image transverse measure” \( \mu'_\Lambda \), which, when translated back to the corresponding invariant measure \( \mu' \) on \( X_h(\Lambda) \), turns out to be precisely the transferred measure \( \sigma_X^M(\mu) \), where \( \sigma \) is the morphism on \( A^* \) induced by \( h \).

In the frequently considered pseudo-Anosov case the \( h \)-invariant lamination \( \Lambda \) as well as the corresponding subshift \( X_\Lambda \) turn out to be both, minimal and uniquely ergodic, so that we have \( \sigma_X^M(\mu) = \lambda \mu \), where \( \lambda > 1 \) is the celebrated “stretching factor” (= the Perron-Frobenious eigenvalue of \( M(\sigma) \)) for \( h \).

(2) Any word \( w \in A^* \) (or rather, its conjugacy class in the free group \( F(A) \)) defines a finite subshift \( X_w \) that consists of the sequence \( w^{\pm \infty} = \ldots w w w \ldots \) and its finitely many shift translates. To \( w \) there is canonically associated a characteristic measure \( \mu_w \in \mathcal{M}(A^\mathbb{Z}) \) with support equal to \( X_w \) and total measure \( \mu_w(X_w) = \mu_w(A^\mathbb{Z}) = |w| \) (see (2.5) for more details).

For any non-erasing morphism \( \sigma : A^* \to B^* \) the transferred measure of any characteristic measure is again a characteristic measure, given by:

\[
\sigma_X^M(\mu_w) = \mu_{\sigma(w)}
\]

(3) For the special case that \( \sigma : A^* \to B^* \) extends to a free group automorphism \( \varphi : F(A) \to F(B) \), one can use the well established 1-1 relationship between invariant measures on \( A^\mathbb{Z} \) on one hand and currents on \( F(A) \) on the other, as well as the similarly well established action of \( \text{Aut}(F(A)) \) on the current space \( \text{Curr}(F(A)) \cong \mathcal{M}(A^\mathbb{Z}) \) (see [8] for details). Denoting by \( \mu \) also the current on \( F(A) \) defined by the invariant measure \( \mu \), one has:

\[
\varphi(\mu) = \sigma_X^M(\mu)
\]
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2. Notation and conventions

2.1. Standard terminology and well known facts.

Throughout this paper we denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ finite non-empty sets, called alphabets. For any such alphabet, say $\mathcal{A}$, we denote by $\mathcal{A}^*$ the free monoid over the set $\mathcal{A}$, given by all finite words $w = x_1 x_2 \ldots x_n$ with $x_i \in \mathcal{A}$, and equipped with the multiplication defined by concatenation. We denote by $|w| := n$ the length of any such word. The empty word $\varepsilon \in \mathcal{A}^*$ is defined by $|\varepsilon| = 0$; it is the unit element with respect to the multiplication in $\mathcal{A}^*$.

The elements $a_i \in \mathcal{A}$ are called the letters; they constitute a set of generators of $\mathcal{A}^*$ and moreover a basis of the associated free group $\mathcal{F}(\mathcal{A})$. Any subset $\mathcal{L} \subseteq \mathcal{A}^*$ (often assumed to be infinite) is called a language over $\mathcal{A}$.

A monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is well defined by knowing the image word $\sigma(a_i) \in \mathcal{B}^*$ for any of the letters $a_i \in \mathcal{A}$. Conversely, each choice of such image words defines a monoid morphism $\sigma$ as above. The monoid morphism $\sigma$ is non-erasing if $|\sigma(a_i)| \geq 1$ for each of the letters $a_i \in \mathcal{A}$. In this paper we will only consider non-erasing morphisms. Note that any non-erasing morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is “finite-to-one”, i.e. for any element $w \in \mathcal{B}^*$ the preimage set $\sigma^{-1}(w)$ is finite.

Every monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ defines an incidence matrix
\[
M(\sigma) = (|\sigma(a_j)|_{b_i})_{b_i \in \mathcal{B}, a_j \in \mathcal{A}}
\]
where $|\sigma(a_j)|_{b_i}$ denotes the number of occurrences of the letter $b_i \in \mathcal{B}$ in the $\sigma$-image of any $a_j \in \mathcal{A}$. One easily verifies the formula $M(\sigma) = M(\sigma_2) \cdot M(\sigma_1)$ for any composition of monoid morphisms $\sigma = \sigma_2 \circ \sigma_1$.

To any alphabet $\mathcal{A}$ there is canonically associated the shift space $\mathcal{A}^\mathbb{Z}$. Its elements are written as bi-infinite words $x = \ldots x_{i-1}x_ix_{i+1}\ldots$ with $x_i \in \mathcal{A}$. The set $\mathcal{L}(x) \subseteq \mathcal{A}^*$ of all finite subwords (called factors) $x[k \ell] := x_kx_{k+1}\ldots x_{\ell}$ is the language associated to $x$. The one-sided infinite positive half-word $x_1x_2\ldots$ of $x$ is denoted by $x_{[1,\infty)}$.

The shift space $\mathcal{A}^\mathbb{Z}$ is canonical equipped with the shift operator $T : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ which maps the word $x = \ldots x_{i-1}x_ix_{i+1}\ldots$ to the word $y = \ldots y_{j-1}y_jy_{j+1}\ldots$ given by $y_k = x_{k+1}$ for all indices $k \in \mathbb{Z}$. Similarly, $\mathcal{A}^\mathbb{Z}$ is naturally equipped with the product topology (with respect to the discrete topology on $\mathcal{A}$), which makes $\mathcal{A}^\mathbb{Z}$ into a compact space, indeed a Cantor set (unless $\text{card}(\mathcal{A}) = 1$). For any $w \in \mathcal{A}^*$ the cylinder $[w] \subseteq \mathcal{A}^\mathbb{Z}$ is open and closed (and thus compact); it consists of all bi-infinite words $x = \ldots x_{i-1}x_ix_{i+1}\ldots$ with $x_{[1,|w|]} = w$. The set of all cylinders and their shift-translates constitutes a basis for the topology on $\mathcal{A}^\mathbb{Z}$. The shift operator $T$ acts as homeomorphism on $\mathcal{A}^\mathbb{Z}$.

A subset $X \subseteq \mathcal{A}^\mathbb{Z}$ is called a subshift if $X$ is non-empty, closed, and if it satisfies $T(X) = X$. To any infinite language $\mathcal{L} \subseteq \mathcal{A}^*$ there is canonically associated the subshift $X(\mathcal{L})$ generated by $\mathcal{L}$: It is defined through
\[
x \in X(\mathcal{L}) \iff \mathcal{L}(x) \subseteq \mathcal{L}^f,
\]
where $\mathcal{L}^f$ denotes the factorial closure of $\mathcal{L}$, which is the language obtained from $\mathcal{L}$ by adding in all factors (= subwords) of any word $w \in \mathcal{L}$. Conversely, every subshift $X \subseteq \mathcal{A}^\mathbb{Z}$ determines an associated subshift language $\mathcal{L}(X) := \bigcup \{\mathcal{L}(x) \mid x \in X\}$, which is infinite and equal to its factorial closure, so that one has $X = X(\mathcal{L}(X))$.

A subshift $X$ is minimal if for any element $x \in X$ the set $X$ is equal to the closure of the shift-orbit $O(x) := \{T^n(x) \mid n \in \mathbb{Z}\}$.
Important particular examples are minimal subshifts that are finite: They consist of a single orbit 
\(O(w)\) that is given by the finitely many shift-translates of a periodic word \(w = \ldots w w \ldots\). To
be specific, we fix the indexing of such a periodic word by the requirement that \(w\) is a prefix of
the positive half-word \(w_{[1,\infty)}\); in this case \(w\) is denoted by \(w^{\pm \infty}\), and we write \(w_{[1,\infty)} = : w^{+ \infty}\) and 
\(w_{(-\infty, 0]} = : w^{- \infty}\).

The space \(\Sigma(A)\) of all subshifts \(X \subseteq A^Z\) is naturally equipped with the partial order given
by the inclusion; the minimal elements with respect to this partial order are precisely the minimal
subshifts. The shift space \(A^Z\) itself is the only maximal element with respect to this partial order;
it is often also called the full shift over \(A\). Similarly, a subshift \(X \subseteq A^Z\) is sometimes called a
subshift over \(A\).

The space \(\Sigma(A)\) is also equipped with a natural topology, inherited from the canonical embedding
\(\Sigma(A) \subseteq \mathcal{P}(A^*): \) which is defined by the above bijection between subshifts and their associated
languages. Since the topology of the shift space doesn’t play a role in this paper, we will not give
details here and refer the reader instead to [11]. Just for “general interest” we note that the subset
of \(\Sigma(A)\), which consists of all subshifts that are a union of finitely many shift-orbits, is dense in
\(\Sigma(A)\). More information about the topology on \(\Sigma(A)\) can be found in [11], where in particular it
is shown that \(\Sigma(A)\) is a Pelczyński space.

An invariant measure \(\mu\) on a subshift \(X \subseteq A^Z\) is a finite Borel measure on \(A^Z\) with support in \(X\)
that is invariant under \(T\), i.e. \(\mu(T^{-1}(B)) = \mu(B)\) for every measurable set \(B \subseteq X\). Any invariant
measure \(\mu\) defines a weight function
\[
\omega_\mu : A^* \to \mathbb{R}_{\geq 0}, \ w \mapsto \mu([w]) ,
\]
by which we mean any function \(\omega : A^* \to \mathbb{R}_{\geq 0}\) that satisfies the Kirchhoff equalities
\[
\omega(w) = \sum_{a_i \in A} \omega(a_i w) = \sum_{a_i \in A} \omega(wa_i)
\]
for all \(w \in A^*\). Conversely, it is well known (see [1], [12]) that every weight function \(\omega : A^* \to \mathbb{R}_{\geq 0}\)
defines an invariant measure \(\mu_\omega\) via \(\mu_\omega([w]) := \omega(w)\) for all \(w \in A^*\). This gives \(\mu_\omega \mu_\omega = \mu\) and 
\(\omega_\mu_\omega = \omega\), and hence a bijective relation between invariant measures and weight functions. This
bijection respects the addition and the multiplication with scalars \(\lambda \in \mathbb{R}_{\geq 0}\), both of which are
naturally defined for invariant measures as well as for weight functions. We thus simplify our
notation by writing
\[
\mu(w) := \mu([w]) = \omega_\mu(w).
\]

An invariant measure \(\mu\) is ergodic if \(\mu\) can not be written in any non-trivial way as sum \(\mu_1 + \mu_2\)
of two invariant measures \(\mu_1\) and \(\mu_2\) (i.e. \(\mu_1 \neq 0 \neq \mu_2\) and \(\mu_1 \neq \lambda \mu_2\) for any \(\lambda \in \mathbb{R}_{>0}\)). An invariant
measure is called a probability measure if \(\mu(X) = 1\), which is equivalent to \(\sum_{a_i \in A} \mu([a_i]) = 1\).

We denote by \(\mathcal{M}(X)\) the set of invariant measures on \(X\), and by \(\mathcal{M}_1(X) \subseteq \mathcal{M}(X)\) the subset
of probability measures. The set \(\mathcal{M}(X)\) is naturally equipped with the weak*-topology (see [1],
[8], [12]). Using the above bijection to the set of weight functions this topology turns out to be
equivalent to the topology inherited from the canonical embedding of \(\mathcal{M}(X)\) into the product space
\(\mathbb{R}_{\geq 0} A^*\) given by \(\mu \mapsto (\mu(w))_{w \in A^*}\).

It follows that the set \(\mathcal{M}(X)\) is a convex linear cone which is naturally embedded into the
non-negative cone of the infinite dimensional vector space \(\mathbb{R} A^*\). The cone \(\mathcal{M}(X)\) is closed, and
the extremal vectors of \(\mathcal{M}(X)\) are in 1-1 relation with the ergodic measures on \(X\). Furthermore,
\(\mathcal{M}_1(X)\) is compact, and it is the closed convex hull of its extremal points.
It is well known (see [1], [12]) that for any subshift $X \subseteq A^\mathbb{Z}$ the set $\mathcal{M}(X)$ of invariant measures is not empty. If $\mathcal{M}_1(X)$ consists of a single point (which then must be ergodic), then $X$ is called uniquely ergodic.

An important class of ergodic invariant measures on the full shift $A^\mathbb{Z}$ is given by the characteristic measures $\mu_w$, for any non-empty $w \in A^* \setminus \{\varepsilon\}$, which are defined as follows: If $w$ is not a proper power, i.e.

$$\mu_w := \frac{1}{|w|} \mu_w$$

then $\mu_w$ simply counts for any measurable set $B \subseteq A^\mathbb{Z}$ the number of intersections of $B$ with the minimal finite subshift $\mathcal{O}(w^{\pm \infty})$ associated to $w$:

$$\mu_w(B) := \text{card}(B \cap \mathcal{O}(w^{\pm \infty}))$$

If on the other hand $w = w_0^r$ for some $w_0 \in A^*$ and some integer $r \geq 2$, where $w_0$ is assumed not to be a proper power, one sets:

$$\mu_w := r \cdot \mu_{w_0}$$

In either case, it follows that $\frac{1}{|w|} \mu_w$ is a probability measure. The set of weighted characteristic measures $\lambda \mu_w$ (for any $\lambda > 0$ and any $w \in A^* \setminus \{\varepsilon\}$) is known to be dense in $\mathcal{M}(A^\mathbb{Z})$ (see [1], [8], [12]).

It is well known (see [1], [12]) and easy to verify that the support $X_\mu := \text{Supp}(\mu)$ of any non-zero invariant measure $\mu \in \mathcal{M}(A^\mathbb{Z})$ is a subshift. Its language $\mathcal{L}(X_\mu)$ is given by

$$\mu(w) > 0$$

for any $w \in A^*$. If $\mu$ is uniquely ergodic, then $X_\mu$ is minimal, but the converse is famously wrong (see [9]). For any characteristic measure $\mu_w$ one has $\text{Supp}(\mu_w) = \mathcal{O}(w^{\pm \infty})$. The support map

$$\text{Supp} : \mathcal{M}(A^\mathbb{Z}) \setminus \{0\} \rightarrow \Sigma(A), \mu \mapsto X_\mu$$

has some nice natural properties: for example, if $\mu_1, \mu_2 \in \mathcal{M}(A^\mathbb{Z})$ and $\lambda_1 > 0, \lambda_2 > 0$ are given, then for $\mu = \lambda_1 \mu_1 + \lambda_2 \mu_2$ one has $X_\mu = X_{\mu_1} \cup X_{\mu_2}$. Also, every minimal subshift is the support of some measure $\mu$, but (by a variety of reasons) there are non-minimal subshifts that are not in the image of the map Supp. An example is given by the subshift that consists of the three orbits $\mathcal{O}(a^{\pm \infty}), \mathcal{O}(b^{\pm \infty})$ and $\mathcal{O}(\ldots, \text{aaabb}, \ldots)$.

Unfortunately, however, there is no topology on $\Sigma(A)$ which is at the same time Hausdorff and for which the support map is continuous: For any $a_1, a_2 \in A$ and $\mu_n := \frac{1}{n} \mu_{a_1} + \mu_{a_2}$ for any $n \in \mathbb{N}$ we have $\text{Supp}(\mu_n) = \mathcal{O}(a_1^{\pm \infty}) \cup \mathcal{O}(a_2^{\pm \infty})$ but $\lim \mu_n = \mu_{a_2}$ and thus $\text{Supp}(\lim \mu_n) = \mathcal{O}(a_2^{\pm \infty})$.

An invariant measure $\mu$ on some subshift $X \subseteq A^\mathbb{Z}$ extends canonically to an invariant measure on all of $A^\mathbb{Z}$. For the corresponding weight functions this extension is obtained by simply declaring $\omega_\mu(w) = 0$ for any $w \notin \mathcal{L}(X)$. We will notationally not distinguish between $\mu$ and its canonical extension, or conversely, between $\mu$ and the restriction of $\mu$ to its support. In particular, for any subshift $X \subseteq A^\mathbb{Z}$ we understand $\mathcal{M}(X)$ as canonical subset of $\mathcal{M}(A^\mathbb{Z})$.

2.2. “Not so standard” basic facts and terminology.

Let $\sigma : A^* \rightarrow B^*$ a non-erasing monoid morphism. Then there is a canonically induced map

$$\sigma^Z : A^\mathbb{Z} \rightarrow B^\mathbb{Z}$$

that maps a biinfinite word $x = \ldots x_{i-1}x_ix_{i+1}\ldots \in A^\mathbb{Z}$ to the biinfinite word $y = \ldots y_{j-1}y_jy_{j+1}\ldots \in B^\mathbb{Z}$ obtained from concatenating the $\sigma(x_i)$ in the obvious way, starting with the convention

$$\sigma(x_1) = y_1 \ldots y_{|\sigma(x_1)|} \ldots.$$
This map \( \sigma^Z \) will in general not inherit any of the properties of \( \sigma \) (for instance, \( \sigma^Z \) is almost never surjective and almost never finite-to-one). However, it satisfies \( \sigma^Z(T(x)) = T^k(y) \) for suitable \( k \geq 0 \) depending on \( x \), so that the shift-orbit \( \mathcal{O}(x) \) has a well defined image shift-orbit \( \mathcal{O}(y) \). Hence \( \sigma^Z \) induces a map

\[
\sigma^T : \mathcal{A}^Z/\langle T \rangle \to \mathcal{B}^Z/\langle T \rangle, \quad \mathcal{O}(x) \mapsto \mathcal{O}(\sigma^Z(x))
\]
on the associated “leaf spaces” (i.e. the set of shift-orbits), which turns out to be a lot more meaningful than the map \( \sigma^T \) itself.

**Remark 2.1.** It follows straight from the definitions that both induced maps \( \sigma^Z \) and \( \sigma^T \) behave functorially: in particular, for any two non-erasing monoid morphisms \( \sigma_1 : \mathcal{A}^* \to \mathcal{B}^* \) and \( \sigma_2 : \mathcal{B}^* \to \mathcal{C}^* \) we have:

\[
(\sigma_2 \circ \sigma_1)^Z = \sigma_2^Z \circ \sigma_1^Z \quad \text{and} \quad (\sigma_2 \circ \sigma_1)^T = \sigma_2^T \circ \sigma_1^T
\]

We now come to a third map induced by any non-erasing monoid morphism \( \sigma : \mathcal{A}^* \to \mathcal{A}^* \), namely the map

\[
\sigma^\Sigma : \Sigma(\mathcal{A}) \to \Sigma(\mathcal{B}), \quad X \mapsto \sigma^\Sigma(X)
\]

from the space \( \Sigma(\mathcal{A}) \) of subshifts \( X \subseteq \mathcal{A}^\mathbb{Z} \) to the space \( \Sigma(\mathcal{B}) \) of subshifts \( Y \subseteq \mathcal{B}^\mathbb{Z} \). There are three natural ways to define the image \( \sigma^\Sigma(X) \subseteq \mathcal{B}^\mathbb{Z} \) under this map, for any given subshift \( X \subseteq \mathcal{A}^\mathbb{Z} \), listed below as follows. Notice that here the assumption on \( \sigma \) to be non-erasing is necessary.

**Definition-Remark 2.2.** If \( \sigma \) is non-erasing, then the following three definitions of the *image subshift* \( Y := \sigma^\Sigma(X) \), for simplicity usually denoted by \( Y = \sigma(X) \), are equivalent:

1. \( Y \) is the intersection of all subshifts that contain the set \( \sigma^Z(X) \) (which in general is not shift-invariant and hence not a subshift itself).
2. \( Y \) is the closure of the union of all image orbits \( \sigma^T(\mathcal{O}(x)) \), for any \( x \in X \). In fact (see Lemma 2.4 below), taking the closure in the previous sentence can be omitted.
3. \( Y \) is the subshift generated by the language \( \sigma(\mathcal{L}(X)) \). Thus \( Y \) consists of all biinfinite words \( y \in \mathcal{B}^\mathbb{Z} \) with the property that every finite factor of \( y \) is also a factor of some word in \( \sigma(\mathcal{L}(X)) \).

In particular, the map \( \sigma^Z \) restricts/co-restricts to a map

\[
\sigma_X^Z : X \to \sigma(X).
\]

The proof of the equivalence of the statements (1) - (3) above is straight forward and hence left here to the reader (except for the part proved below in Lemma 2.4). We do however illustrate the terms used above by making them explicit in the following special case:

**Example 2.3.** Let \( \mathcal{A} = \{a,b\} \) and \( \mathcal{B} = \{c,d\} \), and define \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \) via

\[
a \mapsto (cd)^2 = cdcd, \quad b \mapsto (cd)^3 = cdcdc
\]

We set \( X := \{a^{\pm \infty}, b^{\pm \infty}\} \) and obtain:

- \( \sigma^\Sigma(X) = \{(cd)^{\pm \infty}\} \)
- \( \sigma^\Sigma(X) = \{(cd)^{\pm \infty}, (dc)^{\pm \infty}\} \)
- \( \sigma^T(\mathcal{O}(a^{\pm \infty})) = \sigma^T(\mathcal{O}(b^{\pm \infty})) = \mathcal{O}((cd)^{\pm \infty}) = \mathcal{O}((dc)^{\pm \infty}) \)
- \( \mathcal{L}(X) = \{a^k, b^k \mid k \geq 0\} \)
- \( \sigma(\mathcal{L}(X)) = \{(cd)^k \mid k \geq 2\} \)
- \( \mathcal{L}(\sigma(X)) = \{(cd)^k, (dc)^k, (cd)^k c, d(cd)^k \mid k \geq 0\} \)

The following basic fact is used below in the proofs of Lemma 5.2 and of Lemma 6.1; since we don’t know a reference for it, we include here a proof.
Lemma 2.4. (1) For any non-erasing monoid morphism \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \) and any subshift \( X \subseteq \mathcal{A}^\infty \) the union of all image orbits \( \sigma^T(\mathcal{O}(x)) \), for any \( x \in X \), is a closed subset of \( \mathcal{B}^\infty \).

(2) In particular, the map \( \sigma^T_X : X/\langle T \rangle \to \sigma(X)/\langle T \rangle \) induced by the map \( \sigma^T \) is surjective.

Proof. We need to show the following fact:

(\#) For any integer \( n \geq 0 \) let \( x(n) \in X \) be a biinfinite word with image \( y(n) := \sigma^\infty(x(n)) \in \sigma(X) =: Y \), and assume that for suitable shift exponents \( k(n) \in \mathbb{Z} \) the sequence of the biinfinite words \( T^{k(n)}(y(n)) \) converges to some \( y \in Y \). Then there exists a biinfinite word \( x \in X \) and a subsequence \( (x(n_m))_{m \in \mathbb{N}} \) such that for a suitable set of integer exponents \( \ell_m \in \mathbb{Z} \) one has

\[
x = \lim_{m \to \infty} T^{\ell_m}(x(n_m))
\]

as well as \( \sigma^\infty(x) = T^k(y) \) for some \( k \in \mathbb{Z} \).

In order to prove (\#) we first observe that without loss of generality we can replace any \( x(n) \) by some shift-translate and thus achieve that the set of exponents \( k(n) \) is bounded, and indeed, that \( 0 \leq k(n) < |\sigma(x_0(n))| \), where \( x_0(n) \in \mathcal{A} \) is the 0-th letter of the biinfinite word \( x(n) = \ldots x_{-1}(n)x_0(n)x_1(n) \ldots \) for any \( n \in \mathbb{N} \).

Hence, by extracting a subsequence of the \( x(n) \), we can achieve that the letter \( x_0 = x_0(n) \) as well as the exponent \( k = k(n) \) is independent of \( n \). We now use the assumption that the \( T^k(y(n)) \) converge, in order to extract again a subsequence of the previous subsequence of the \( x(n) \) in order to ensure that the letters \( x_{-1}(n) \) and \( x_1(n) \) adjacent to \( x_0(n) \) on \( x(n) \) are also independent of \( n \). We iteratively proceed in this manner and extract finally a diagonal subsequence that defines a biinfinite word \( x \) which is the limit of some subsequence \( (x(n_m))_{m \in \mathbb{N}} \) of the \( x(n) \). From our construction and the definition of the map \( \sigma^\infty \) we see directly that \( \sigma^\infty(x) = T^{-k}(y) \).

\[\square\]

Remark 2.5. The concept of an image subshift given in Definition-Remark 2.2 is a very natural one; it is used frequently, in particular in the \( S \)-adic context. A systematic treatment, however, doesn’t seem to be available anywhere. One derives easily from the above definitions the following properties of the map (see (2.9))

\[
\sigma^\Sigma : \Sigma(\mathcal{A}) \to \Sigma(\mathcal{B})
\]
on the subshift spaces over \( \mathcal{A} \) and \( \mathcal{B} \) respectively. We recall that for simplicity we allow ourselves to denote the image of a subshift \( X \subseteq \mathcal{A}^\infty \) by \( \sigma(X) \) rather than by \( \sigma^\Sigma(X) \).

(1) The map \( \sigma^\Sigma \) is functorial. In particular, for any second morphism \( \sigma' : \mathcal{B}^* \to \mathcal{C}^* \) one has:

\[
\sigma'(\sigma(X)) = \sigma' \circ \sigma(X)
\]

(2) The map \( \sigma^\Sigma \) respects the inclusion: For any two subshifts \( X' \subseteq X \) in \( \Sigma(\mathcal{A}) \) one has \( \sigma(X') \subseteq \sigma(X) \). For any third subshift \( X'' \) in \( \Sigma(\mathcal{A}) \) one has \( \sigma(X \cap X'') \subseteq \sigma(X) \cap \sigma(X'') \) and \( \sigma(X \cup X'') = \sigma(X) \cup \sigma(X'') \); the analogous statements hold for infinite intersections and for the closure of infinite unions.

(3) The map \( \sigma^\Sigma \) respects the subshift closure: If \( L \subseteq \mathcal{A}^* \) is an infinite set, then the subshift \( X(L) \in \Sigma(\mathcal{A}) \) generated by \( L \) has as image the subshift generated by \( \sigma(L) \):

\[
\sigma(X(L)) = X(\sigma(L))
\]

(4) For any subshift \( X \in \Sigma(\mathcal{A}) \) the map \( \sigma^\Sigma \) induces on the subset \( \Sigma(X) \subseteq \Sigma(\mathcal{A}) \) of subshifts \( X' \subseteq X \) a map

\[
\sigma^\Sigma_X : \Sigma(X) \to \Sigma(\sigma(X)).
\]

(5) For any subshift \( Y \subseteq \mathcal{B}^\infty \) the preimage \( \sigma^{-1}(Y) \subseteq \mathcal{A}^\infty \) is either the empty set, or else it is a subshift over \( \mathcal{A} \). Alternatively this subshift is obtained as union of all subshifts \( X_i \) with \( \sigma^\Sigma(X_i) \subseteq Y \). For simplicity we denote this subshift by \( \sigma^{-1}(Y) \subseteq \mathcal{A}^\infty \).
(6) By considering for any subshifts \( X \subseteq A^\mathbb{Z} \) and \( Y \subseteq \sigma(X) \) the subshift \( X \cap \sigma^{-1}(Y) \) we observe that the above map \( \sigma_X^Y : \Sigma(X) \to \Sigma(\sigma(X)) \) is surjective. It also preserves the partial order given by the inclusion of subshifts.

(7) In particular, if \( X \) is minimal, then so is \( \sigma(X) \).

(8) The map \( \sigma^X \) is continuous with respect to the canonical topology on the subshift spaces.

**Remark 2.6.** Although not central to the topics evoked in this paper, for completeness we would like to state here how a non-erasing morphism \( \sigma : A^* \to B^* \) acts on the complexity and the topological entropy of a subshift. Recall that for any subshift \( X \subseteq A^\mathbb{Z} \) the complexity (also called word complexity) is given by the function \( p_X : \mathbb{N} \to \mathbb{N} \), defined via \( p_X(n) = \text{card}\{w \in \mathcal{L}(X) \mid |w| = n\} \). The topological entropy of \( X \) is defined as \( h_X = \lim_{n \to \infty} \frac{\log p_X(n)}{n} \). A fairly standard exercise (see for instance [10], Lemma 2.1, or [6], Lemma 2.9 and Proposition 2.11) shows, for \( Y := \sigma(X) \):

1. Setting \( ||\sigma|| := \max\{|\sigma(a_k)| \mid a_k \in A\} \) one has:
   \[
   p_Y(n) \leq ||\sigma|| \cdot p_X(n)
   \]

2. As direct consequence one derives:
   \[
   h_Y \leq h_X
   \]

2.3. About injectivity.

Injectivity of monoid morphisms \( \sigma \) and of their induced maps \( \sigma^X, \sigma^T \), and \( \sigma^\Sigma \) is an important and often tricky issue. We start out in sub-subsection 2.3.1 to list several problems and give examples where these often undesired phenomena do occur. In sub-subsection 2.3.2, which can be read independently, we will then define the subtle notion of “shift-period preservation”, which will be used in sections 5 and 6 below as well as in [2].

2.3.1. Typical injectivity problems.

For starters, we have the following well known phenomenon:

**Remark 2.7.** There exist non-erasing morphisms \( \sigma : A^* \to B^* \) which are injective, while the associated free group homomorphism \( F(A) \to F(B) \) is not injective. An example is given by

\[
\sigma : \{a, b, c\}^* \to \{d, e\}^*, \quad a \mapsto dde, \quad b \mapsto dee, \quad c \mapsto dde
\]

which maps \( c(ac^{-1}b)^{-2} \) to the neutral element \( 1 \in F(\{d, e\}) \).

A second, also well known disturbance comes from the fact that injective free group morphisms need not induce injective maps on the set of conjugacy classes. This happens for example for the morphism induced by the quotient map, if two boundary curves of a surface (with at least one more boundary component to ensure free fundamental groups) are glued together. In a classical monoid morphism situation we observe that for the “Thue-Morse morphism”

\[
\sigma_{TM} : \{a, b\}^* \to \{c, d\}^*, \quad a \mapsto cd, \quad b \mapsto dc
\]

the images \( \sigma_{TM}(a) \) and \( \sigma_{TM}(b) \) are conjugate in \( F(\{c, d\}) \), while \( \sigma_{TM} \) is injective on all of \( F(\{a, b\}) \).

This yields:

**Remark 2.8.** There exist non-erasing monoid morphisms \( \sigma \) which are injective, but for which the induced map \( \sigma^T \) is not injective. For example, for the Thue-Morse morphism one has \( \mathcal{O}(\sigma_{TM}(a^{\pm \infty})) = \mathcal{O}(\sigma_{TM}(b^{\pm \infty})) \).

Two further, more subtle “injectivity disturbances” can be observed by the following two examples, despite the fact that in both cases the induced map on the shift-orbits is injective. Indeed, in both cases we consider a subshift which consists of a single periodic shift-orbit: in the first case
Lemma 3.2. Let $\sigma: \mathcal{A}^* \rightarrow \mathcal{A}_\ell^*$ be a subdivision morphism as in (3.1). Then the following holds:

Remark 2.9. (1) The morphism $\sigma_1: \{a, b\}^* \rightarrow \{c\}^*$, $\sigma_1(a) = \sigma_1(b) = c$ maps the orbit $\mathcal{O}((ab)^{\pm\infty})$ to $\mathcal{O}(c^{\pm\infty})$, but the restriction of $\sigma_1^2$ to $\mathcal{O}((ab)^{\pm\infty})$ is not injective.

(2) The morphism $\sigma_2: \{a\}^* \rightarrow \{b\}^*$, $\sigma_2(a) = b^2$ maps the orbit $\mathcal{O}(a^{\pm\infty})$ injectively to $\mathcal{O}(b^{\pm\infty})$, but the “shift-period” is not preserved: We have $\sigma_2(T(a^{\pm\infty})) = T^2(\sigma_2(a^{\pm\infty}))$ [so that the image of the shift-period has length 2], while $\sigma_2(a^{\pm\infty}) = b^{\pm\infty} = T(b^{\pm\infty})$ [so that the image orbit has shift-period of length 1].

2.3.2. Shift-period preservation.

For any periodic word $x \in \mathcal{A}^\mathbb{Z}$ we define the shift-period exponent of $x$ to be the smallest integer $k \geq 1$ such that $T^k(x) = x$. If $x = w^{\pm\infty}$ for some $w \in \mathcal{A}^*$, then $k$ divides $|w|$. If $w$ cannot be written as proper power (see (2.4)), then the shift-period exponent of $x$ is given by $k = |w|$. In this case any cyclic permutation of $w$ will be called a shift-period of the periodic word $x$.

Definition 2.10. A morphism $\sigma: \mathcal{A}^* \rightarrow \mathcal{B}^*$ is said to preserve the shift-period of some biinfinite periodic word $x \in \mathcal{A}^\mathbb{Z}$ if any shift-period $w \in \mathcal{A}^*$ of $x$ is mapped by $\sigma$ to a shift-period of the image word $\sigma^\mathbb{Z}(x) \in \mathcal{B}^\mathbb{Z}$.

In other words, if $x = w^{\pm\infty}$ and $\sigma(w)$ is a proper power, then so is $w$.

Remark 2.11. In the special case where $|\sigma(a_i)| = 1$ for all $a_i \in \mathcal{A}$ we observe that $\sigma$ preserves the shift-period of some periodic biinfinite word $x$ if and only if the shift-period exponents of $x$ and of its image $\sigma^\mathbb{Z}(x)$ agree.

3. THE MEASURE TRANSFER

In this section we will carefully define for any non-erasing monoid morphism $\sigma: \mathcal{A}^* \rightarrow \mathcal{B}^*$ and any invariant measure $\mu$ on $\mathcal{A}^\mathbb{Z}$ a shift-invariant “image measure” $\mu^\sigma$ on $\mathcal{B}^\mathbb{Z}$. The simplest and most natural way to understand this measure transfer is achieved by decomposing the given morphism $\sigma$ in a canonical way into two morphisms of very elementary type. We start our detailed presentation by considering first each of these two elementary morphism types separately.

3.1. Subdivision morphisms.

Let $\mathcal{A}$ be a finite alphabet, and let $\ell: \mathcal{A} \rightarrow \mathbb{Z}_{\geq 1}$ be any map, called subdivision length function.

We now define a new subdivision alphabet $\mathcal{A}_\ell$ which consists of letters $a_i(k)$ for any $a_i \in \mathcal{A}$ and any $k \in \{1, \ldots, \ell(a_i)\}$. We then define the associated subdivision morphism as follows:

(3.1) $\pi_\ell: \mathcal{A}^* \rightarrow \mathcal{A}_\ell^*$, $a_i \mapsto a_i(1) \ldots a_i(\ell(a_i))$

Remark 3.1. (1) The name and the intuition here comes from picturing $\mathcal{A}$ as edge labels of an oriented “rose” $R_\mathcal{A}$, i.e. a 1-vertex graph with card($\mathcal{A}$) oriented edges. Then any edge with label $a_i$ is subdivided by introducing $\ell(a_i) - 1$ new vertices in its interior, and by labeling the obtained new edges (in the order given by the orientation) by $a_i(1), a_i(2), \ldots, a_i(\ell(a_i))$. Then any edge path $\gamma(w)$ in $R_\mathcal{A}$, which reads off a word $w \in \mathcal{A}^*$, will after the subdivision read off the word $\pi_\ell(w)$.

(2) The natural “geometrization” of the subdivision monoid $\mathcal{A}_\ell^*$ as rose $R_\mathcal{A}_\ell$ is however not quite the above subdivision of the rose $R_\mathcal{A}$, but is obtained from the latter by identifying all subdivision vertices into a single vertex. This is reflected by the fact that the subdivision morphism $\pi_\ell$ defined above is not surjective (except in the trivial case where all $\ell(a_i) = 1$ so that $\pi_\ell$ is a bijection). Its image generates a subshift of finite type in $\mathcal{A}_\ell^\mathbb{Z}$.

From the above definitions we deduce directly the following:

Lemma 3.2. Let $\pi_\ell: \mathcal{A}^* \rightarrow \mathcal{A}_\ell^*$ be a subdivision morphism as in (3.1). Then the following holds:
(1) The monoid morphism $\pi_\ell$ is injective.
(2) The induced map $\pi_\ell^T : \mathcal{A}_\ell^Z \to \mathcal{A}_\ell^Z$ is injective.
(3) The map $\pi_\ell^T$ induced on shift-orbits is injective.
(4) The morphism $\pi_\ell$ preserves the shift-period of any biinfinite periodic word $x \in \mathcal{A}_\ell^Z$.
(5) The map $\pi_\ell^T$ induced on subshifts over $\mathcal{A}$ is injective.

Proof. All the listed properties follow directly from the definition of the map $\pi_\ell$, since for any element $w' \in \pi_\ell(\mathcal{A}^*)$ the (uniquely determined) preimage $w \in \mathcal{A}^*$ is directly visible through replacing every factor $a_i(1) \ldots a_i(\ell(a_i))$ by the letter $a_i$.

Definition 3.3. Let $\pi_\ell : \mathcal{A}^* \to \mathcal{A}_\ell^*$ be a subdivision morphism as in (3.1), and let $\mu$ be an invariant measure on $\mathcal{A}^*_\ell$. Consider the weight function $\mathcal{A}^* \to \mathbb{R}_{\geq 0}$, $w \mapsto \mu([w])$ associated to $\mu$, which for simplicity is also denoted by $\mu$ (see (2.3)).

Define a function $\mu_\ell : \mathcal{A}_\ell^* \to \mathbb{R}_{\geq 0}$ by

$$
\mu_\ell(w) = \mu(\hat{w}),
$$

where $\hat{w} \in \mathcal{A}^*$ is the shortest word in $\mathcal{A}^*$ such that $\pi_\ell(\hat{w})$ contains $w$ as factor. If such $\hat{w}$ exists, then it is uniquely defined by $w$. If there is no such word $\hat{w}$, we set formally $\mu(\hat{w}) = 0$ and thus $\mu_\ell(w) = 0$.

It is shown in Lemma 3.4 just below that the function $\mu_\ell$ is a weight function, so that (see section 2.1) it defines an invariant measure on $\mathcal{A}_{\ell}^Z$ which will also be denoted by $\mu_\ell$. We call $\mu_\ell$ the subdivision measure defined by $\mu$.

Lemma 3.4. The function $\mu_\ell$ inherits from $\mu$ the Kirchhoff equalities (2.2), so that $\mu_\ell$ is itself a weight function.

Proof. By symmetry it suffices to prove the first equality of (2.2) for the function $\mu_\ell$. We have to distinguish three cases:

If $w \in \mathcal{A}_\ell^*$ is not a factor of any element from $\pi_\ell(\mathcal{A}^*)$, then the same is true for $xw$ for any $x \in \mathcal{A}_\ell$. In this case both, the left and the right hand side of the desired equality, are equal to 0.

If $w$ is a factor of some element in $\pi_\ell(\mathcal{A}^*)$, then we consider the first letter $a_i(k)$ of $w$. If we have $k \geq 2$, then there is only one letter $x \in \mathcal{A}_\ell$ such that $xw$ is a factor of some element from $\pi_\ell(\mathcal{A}^*)$, namely $x = a_i(k - 1)$. In this case we have $\hat{xw} = \hat{w}$, so that again the desired equality holds.

Finally, if the first letter of $w$ is equal to some $a_i(1)$, then $xw$ is a factor of some element from $\pi_\ell(\mathcal{A}^*)$ precisely if $x = a_j(\ell(j))$ for any of the $a_j \in \mathcal{A}$. In this case we have $\hat{xw} = a_j \hat{w}$, so that the first Kirchhoff equality for $\mu_\ell(\hat{w})$ gives directly the desired equality for $\mu_\ell(w)$.

Remark 3.5. The subdivision measure defined by a probability measure will in general not be probability: Unless $\ell$ is the constant function with value 1, for the total measure we will have

$$
\mu_\ell(\mathcal{A}^Z_\ell) > 1.
$$

In fact, one easily derives from (3.2) that $\mu_\ell(a_i(k)) = \mu(a_i)$, which yields the following formula:

$$
\mu_\ell(\mathcal{A}^Z_\ell) = \sum_{a_i(k) \in \mathcal{A}_\ell} \mu_\ell(a_i(k)) = \sum_{a_i \in \mathcal{A}} \ell(a_i) \cdot \mu(a_i)
$$

3.2. Letter-to-letter morphisms.

Recall that a monoid morphism $\alpha : \mathcal{A}^* \to \mathcal{B}^*$ is called letter-to-letter if for any letter $a_i \in \mathcal{A}$ the length of its image is equal to $|\alpha(a_i)| = 1$. In other words: $\alpha$ is induced by a map $\alpha_A : \mathcal{A} \to \mathcal{B}$ on the alphabets\(^1\).

\(^1\) Some authors (see for instance [5]) require in addition that a letter-to-letter morphism must be surjective. All letter-to-letter morphisms occurring in this paper are indeed surjective, but formally we do not need this condition anywhere.
It follows directly that both, the letter-to-letter morphism \( \alpha : \mathcal{A}^* \to \mathcal{B}^* \) and the induced map \( \alpha_Z : \mathcal{A}^Z \to \mathcal{B}^Z \), are injective and/or surjective if and only if \( \alpha_A \) is injective and/or surjective. Furthermore, \( \alpha_Z \) commutes with the shift maps (both denoted by \( T \)) on \( \mathcal{A}^Z \) and \( \mathcal{B}^Z \):

\[
T \circ \alpha = \alpha \circ T
\]

As a consequence, the image \( \alpha_Z(X) \) of any subshift \( X \subseteq \mathcal{A}^Z \) is equal to the image subshift \( \alpha_Z^X(X) \) over \( \mathcal{B} \). Furthermore, for any invariant measure \( \mu \) on \( \mathcal{A}^Z \) the classical push-forward measure \( \alpha_*(\mu) \) is an invariant measure on \( \mathcal{B}^Z \), with support equal to \( \alpha_Z^X(\text{Supp}(\mu)) \). In particular, if \( \mu \) is a probability measure, then so is \( \alpha_*(\mu) \).

According to the defining equation \( \mu_*(S) := \mu(f^{-1}(S)) \) for any measurable set \( S \) in the range of a map \( f \), we obtain for the weight function associated to \( \alpha_*(\mu) \) the following finite sum decomposition, for any \( w \in \mathcal{B}^* \):

\[
(3.3) \quad \alpha_*(\mu)(w) = \sum_{u \in \alpha^{-1}(w)} \mu(u)
\]

Note that any \( u \in \alpha^{-1}(w) \) has length \(|u| = |w|\).

3.3. The induced measure morphisms.

We now consider an arbitrary non-erasing monoid morphism \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \) (as usual for finite alphabets \( \mathcal{A} \) and \( \mathcal{B} \)). Then \( \sigma \) defines a subdivision length function \( \ell_\sigma : \mathcal{A} \to \mathbb{Z}_{\geq 1} \), given by

\[
\ell_\sigma(a_i) := |\sigma(a_i)|
\]

for any \( a_i \in \mathcal{A} \). This gives a subdivision alphabet \( \mathcal{A}_\sigma := \mathcal{A}_{\ell_\sigma} \) as well as a subdivision morphism

\[
\pi_\sigma := \pi_{\ell_\sigma} : \mathcal{A}^* \to \mathcal{A}_\sigma^*.
\]

Furthermore, \( \sigma \) defines a letter-to-letter morphism given by

\[
\alpha_\sigma : \mathcal{A}^*_\sigma \to \mathcal{B}^* \quad , \quad a_i(k) \mapsto \sigma(a_i)_k,
\]

for any \( a_i \in \mathcal{A} \) and \( 1 \leq k \leq |\sigma(a_i)| \), where \( \sigma(a_i)_k \in \mathcal{B} \) denotes the \( k \)-th letter of the word \( \sigma(a_i) \in \mathcal{B}^* \).

**Definition-Remark 3.6.** (1) From the above definitions we observe that any non-erasing monoid morphism \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \) admits a canonical decomposition

\[
(3.4) \quad \sigma = \alpha_\sigma \circ \pi_\sigma
\]

as product of a subdivision morphism \( \pi_\sigma \) with a subsequent letter-to-letter morphism \( \alpha_\sigma \).

(2) As a consequence, the morphism \( \sigma \) induces a canonical measure transfer \( \sigma^M : \mathcal{M}(\mathcal{A}^Z) \to \mathcal{M}(\mathcal{B}^Z) \) which maps any invariant measure \( \mu \) on \( \mathcal{A}^Z \) to the measure \( (\alpha_\sigma)_*(\mu_\ell) \), which for simplicity will sometimes be denoted by \( \mu^\sigma \). For the associated weight function \( \sigma^M(\mu) \) on \( \mathcal{B}^* \) we obtain, for any \( w \in \mathcal{B}^* \), the formula

\[
(3.5) \quad \sigma^M(\mu)(w) \ [= : \mu^\sigma(w)] = \sum_{u \in \alpha^{-1}_\sigma(w)} \mu_\ell (u) = \sum_{u \in \alpha^{-1}_\sigma(w)} \mu(\widehat{u}),
\]

where \( \widehat{u} \in \mathcal{A}^* \) is defined above in Definition 3.3. Recall from Definition 3.3 that if \( \widehat{u} \) doesn’t exist for some \( u \in \alpha^{-1}_\sigma(w) \), then one has formally set \( \mu(\widehat{u}) = 0 \).

The above canonical decomposition of an arbitrary decomposition has been used previously; it can be found for instance in Lemma 3.4 of [4].
3.4. Basic properties of the measure transfer map.

In this subsection we want to show some first properties of the measure transfer map $\sigma^M$ defined in the previous subsection. We start out with some basic facts: their proof is an elementary (and not very illuminating) exercise, based on the canonical decomposition (3.4) of any morphism $\sigma$; it is hence not carried through here.

Lemma 3.7. Let $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a non-erasing monoid morphism. Then the induced measure transfer map $\sigma^M$ has the following properties:

(a) The map $\sigma^M : \mathcal{M}(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}(\mathcal{B}^\mathbb{Z})$ is $R_{\geq 0}$-linear.
(b) The map $\sigma^M$ is functorial: For any second non-erasing monoid morphism $\sigma' : \mathcal{B}^* \to \mathcal{C}^*$ one has $(\sigma' \sigma)^M = \sigma'^M \sigma^M$.
(c) The image measure $\sigma^M(\mu)$ of a probability measure $\mu$ on $\mathcal{A}^\mathbb{Z}$ is in general not a probability measure on $\mathcal{B}^\mathbb{Z}$.
(d) For any word $w \in \mathcal{A}^*$ the characteristic measure $\mu_w$ is mapped by $\sigma^M$ to the characteristic measure $\mu_{\sigma(w)}$.
(e) For any word $w \in \mathcal{A}^*$ the measures of corresponding cylinders $[w] \subseteq \mathcal{A}^\mathbb{Z}$ and $[\sigma(w)] \subseteq \mathcal{B}^\mathbb{Z}$ satisfy the following inequality:

\[
\mu([w]) \leq \sigma^M(\mu)([\sigma(w)])
\]

If $\sigma$ is a subdivision morphism, then (3.6) becomes an equality, but for a letter-to-letter morphism the inequality (3.6) will in general be strict. □

For the next observation we recall from section 2.1 that the classical weak*-topology on the space $\mathcal{M}(\mathcal{A}^\mathbb{Z})$ is equivalent to the topology induced by the product topology on the space $\mathbb{R}_{\geq 0}^{\mathcal{A}^*}$, via the embedding $\mathcal{M}(\mathcal{A}^\mathbb{Z}) \subseteq \mathbb{R}_{\geq 0}^{\mathcal{A}^*}$ given by $\mu \mapsto (\mu([w]))_{w \in \mathcal{A}^*}$.

Lemma 3.8. For any non-erasing monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ the induced map $\sigma^M : \mathcal{M}(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}(\mathcal{B}^\mathbb{Z})$ is continuous.

Proof. Following (3.4) we decompose $\sigma$ as product $\sigma = \alpha_\sigma \circ \pi_\sigma$. In order to show that the first factor of this decomposition, the morphism $\pi_\sigma : \mathcal{A}^* \to \mathcal{A}_\sigma^*$, induces a continuous map on $\mathcal{M}(\mathcal{A}^\mathbb{Z})$, we recall that by definition this map is defined via $\mu \mapsto \mu_{\ell_\sigma}$, with $\mu_{\ell_\sigma}(w) = \mu(\tilde{w})$ for any $w \in \mathcal{A}_\sigma^*$, where $\tilde{w}$ is the shortest word in $\mathcal{A}^*$ such that $w$ is a factor of $\pi_\sigma(\tilde{w})$. It follows directly that small variations of $\mu$ imply small variations of $\mu_{\ell_\sigma}$, so that $\pi_\sigma$ induces a continuous map $\mathcal{M}(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}(\mathcal{A}_\sigma^\mathbb{Z})$.

The second factor $\alpha_\sigma : \mathcal{A}_\sigma^* \to \mathcal{B}^*$ of the above decomposition induces a map $\mathcal{M}(\mathcal{A}_\sigma^\mathbb{Z}) \to \mathcal{M}(\mathcal{B}^\mathbb{Z})$ that is given by the classical push-forward definition $\mu' \mapsto (\alpha_\sigma)_* (\mu')$ for any $\mu' \in \mathcal{M}(\mathcal{A}_\sigma^\mathbb{Z})$, for which the continuity is well known (and easy to prove, by a similar approach as used in the previous paragraph).

It follows that the composition $\mathcal{M}(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}(\mathcal{A}_\sigma^\mathbb{Z}) \to \mathcal{M}(\mathcal{B}^\mathbb{Z})$, $\mu \mapsto \mu_{\ell_\sigma} \mapsto (\alpha_\sigma)_* (\mu_{\ell_\sigma})$ is continuous. But this is precisely how the map $\sigma^M : \mathcal{M}(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}(\mathcal{B}^\mathbb{Z})$ is defined (see Definition-Remark 3.6 (2)). □

Remark 3.9. (1) Using the density of the set of weighted characteristic measures $\lambda \mu_w$ (see section 2.1) within $\mathcal{M}(\mathcal{A}^\mathbb{Z})$ on one hand and the continuity of the map $\sigma^M$ on the other, one can use statement (d) of Lemma 3.7 in order to alternatively determine the image $\sigma^M(\mu)$ for any invariant measure $\mu$ on $\mathcal{A}^\mathbb{Z}$ as limit of weighted characteristic measures. This methods has been used for instance in [8] to define the map induced by an automorphism of a free group $F_N$ on the space of currents on $F_N$. The approach presented here, however, has many practical advantages; for instance it is more efficient for most computations.

(2) A third alternative to determine the image $\sigma^M(\mu)$ for any invariant measure $\mu$ on $\mathcal{A}^\mathbb{Z}$ is given in [2] by means of everywhere growing $S$-adic developments and vector towers over them.
To conclude this section we want to note:

**Lemma 3.10.** Let \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \) be a non-erasing monoid morphism, and let \( \mu \in \mathcal{M}(\mathcal{A}^\mathbb{Z}) \setminus \{0\} \) be an invariant measure with support contained in some subshift \( X \subseteq \mathcal{A}^\mathbb{Z} \) that has image subshift \( Y := \sigma(X) \subseteq \mathcal{B}^\mathbb{Z} \). Then the transferred measure \( \sigma^\mathcal{M}(\mu) \left[= \mu^\sigma \right] \) has support in \( Y \). More precisely, using the terminology from (2.7), we have:

\[
(3.7) \quad \text{Supp}(\sigma^\mathcal{M}(\mu)) = \sigma^\Sigma(\text{Supp}(\mu))
\]

In particular, the morphism \( \sigma \) induces for any subshift \( X \subseteq \mathcal{A}^\mathbb{Z} \) a well defined map

\[
\sigma^\mathcal{M}_X : \mathcal{M}(X) \to \mathcal{M}(\sigma(X))
\]

which satisfies all the properties analogous to statements (a) - (e) of Lemma 3.7, as well as to Lemma 3.8.

**Proof.** Recall first from equivalence (2.6) that the language of the support of any shift-invariant measure is given by all words with positive measure of the associated cylinder.

In particular, any \( w \in \mathcal{A}^* \) belongs to \( \mathcal{L}(\text{Supp}(\mu)) \) if and only if \( \mu(w) > 0 \). In this case \( \sigma(w) \) belongs to \( \mathcal{L}(\sigma^\Sigma(\text{Supp}(\mu))) \), and any \( w' \in \mathcal{L}(\sigma^\Sigma(\text{Supp}(\mu))) \) is a factor of some such \( \sigma(w) \). From (3.6) we know \( \mu^\sigma(\sigma(w)) \geq \mu(w) \), which implies that \( \sigma(w) \) belongs to \( \mathcal{L}(\text{Supp}(\mu^\sigma)) \). Since \( \mu^\sigma(w') \geq \mu^\sigma(\sigma(w)) \), the same applies to any factor \( w' \) of \( \sigma(w) \). We thus obtain:

\[
\mathcal{L}(\sigma^\Sigma(\text{Supp}(\mu))) \subseteq \mathcal{L}(\text{Supp}(\mu^\sigma))
\]

Conversely (again using (2.6)), any \( w' \in \mathcal{B}^* \) belongs to \( \mathcal{L}(\text{Supp}(\mu^\sigma)) \) if and only if \( \mu^\sigma(w') > 0 \). In this case there exists a word \( \hat{w} \in \mathcal{A}^* \) with \( \alpha(\hat{w}) = w' \), and with \( \mu(\hat{w}) > 0 \). Hence (see formula (3.5)) there is a word \( \hat{w} \in \mathcal{A}^* \) with \( \mu(\hat{w}) > 0 \) such that \( \pi_\sigma(\hat{w}) \) contains \( w \) as factor. It follows that \( w' \) is a factor of \( \sigma(\hat{w}) \) and that \( \hat{w} \in \mathcal{L}(\text{Supp}(\mu)). \) Since \( \sigma(\mathcal{L}(\text{Supp}(\mu))) \subseteq \mathcal{L}(\sigma^\Sigma(\text{Supp}(\mu))) \), this implies:

\[
\mathcal{L}(\text{Supp}(\mu^\sigma)) \subseteq \mathcal{L}(\sigma^\Sigma(\text{Supp}(\mu)))
\]

Hence we have \( \mathcal{L}(\text{Supp}(\mu^\sigma)) = \mathcal{L}(\sigma^\Sigma(\text{Supp}(\mu))) \), which implies the equality (3.7).

\[\square\]

### 4. Evaluation of the transferred measure \( \sigma^\mathcal{M}(\mu) \)

**4.1. A first example for the measure transfer.**

We will illustrate in this subsection the induced measure transfer map \( \sigma^\mathcal{M} \) from Definition-Remark 3.6 for an explicitly given morphism \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \). For this example we will carry through in all detail, for any invariant measure \( \mu \) on \( \mathcal{A}^\mathbb{Z} \), the computation of the values of \( \sigma^\mathcal{M}(w) \) for any word \( w \in \mathcal{A}^* \) of length \(|w| \leq 2 \).

**Convention 4.1.** We simplify here (as well as in some other concrete computations below) the notation used before by writing \( a, b, c, \ldots \) instead of \( a_1, a_2, a_3, \ldots \) for the elements of the given alphabet \( \mathcal{A} \), and we write \( a_k \) or \( b_k \) instead of \( a(k) \) or \( b(k) \) for the letters of a corresponding subdivision alphabet.

For the alphabets \( \mathcal{A} = \{a, b\} \) and \( \mathcal{B} = \{c, d\} \) consider the morphism given by:

\[
\sigma : \mathcal{A}^* \to \mathcal{B}^*, \quad a \mapsto cdc, \; b \mapsto dcc
\]

We derive \( \ell_\sigma(a) = \ell_\sigma(b) = 3 \) and \( \mathcal{A}_\sigma = \{a_1, a_2, a_3, b_1, b_2, b_3\} \) as well as the corresponding subdivision morphisms \( \pi_\sigma : \mathcal{A}^* \to \mathcal{A}_\sigma^* \) given by

\[
\pi_\sigma(a) = a_1 a_2 a_3, \quad \pi_\sigma(b) = b_1 b_2 b_3 .
\]

Similarly, the corresponding letter-to-letter morphism \( \alpha_\sigma : \mathcal{A}_\sigma^* \to \mathcal{B}_\sigma^* \) is given by:

\[
\alpha_\sigma(a_1) = \alpha_\sigma(a_3) = \alpha_\sigma(b_2) = \alpha_\sigma(b_3) = c, \; \alpha_\sigma(a_2) = \alpha_\sigma(b_1) = d
\]
We obtain (for $\mu_\sigma := \mu_\ell$):

$$\mu_\sigma(a_1) = \mu_\sigma(a_2) = \mu_\sigma(a_3) = \mu(a), \quad \mu_\sigma(b_1) = \mu_\sigma(b_2) = \mu_\sigma(b_3) = \mu(b)$$

and thus (following (3.3)):

$$\begin{align*}
(\alpha_\sigma)_* (\mu_\sigma)(c) &= \mu_\sigma(a_1) + \mu_\sigma(a_3) + \mu_\sigma(b_2) + \mu_\sigma(b_3) = 2\mu(a) + 2\mu(b) \\
(\alpha_\sigma)_* (\mu_\sigma)(d) &= \mu_\sigma(a_2) + \mu_\sigma(b_1) = \mu(a) + \mu(b)
\end{align*}$$

Similarly, one computes:

$$\begin{align*}
\mu_\sigma(a_1a_2) &= \mu_\sigma(a_2a_3) = \mu(a), \quad \mu_\sigma(b_1b_2) = \mu_\sigma(b_2b_3) = \mu(b) \\
\mu_\sigma(a_2a_1) &= \mu(aa), \quad \mu_\sigma(a_3b_1) = \mu(ab), \quad \mu_\sigma(b_3a_1) = \mu(ba), \quad \mu_\sigma(b_3b_1) = \mu(bb)
\end{align*}$$

and $\mu_\sigma(w) = 0$ for any other $w \in \mathcal{A}^*$ of length $|w| = 2$.

Correspondingly, one obtains:

$$\begin{align*}
(\alpha_\sigma)_* (\mu_\sigma)(cc) &= [\mu_\sigma(a_1a_1) + \mu_\sigma(a_1a_3) + \mu_\sigma(a_1b_2) + \mu_\sigma(a_1b_3)] \\
&\quad + [\mu_\sigma(a_3a_1) + \mu_\sigma(a_3a_3) + \mu_\sigma(a_3b_2) + \mu_\sigma(a_3b_3)] \\
&\quad + [\mu_\sigma(b_2a_1) + \mu_\sigma(b_2a_3) + \mu_\sigma(b_2b_2) + \mu_\sigma(b_2b_3)] \\
&\quad + [\mu_\sigma(b_3a_1) + \mu_\sigma(b_3a_3) + \mu_\sigma(b_3b_2) + \mu_\sigma(b_3b_3)] \\
&= \mu(aa) + \mu(b) + \mu(ba),
\end{align*}$$

further

$$\begin{align*}
(\alpha_\sigma)_* (\mu_\sigma)(cd) &= [\mu_\sigma(a_1a_2) + \mu_\sigma(a_1b_1)] + [\mu_\sigma(a_3a_2) + \mu_\sigma(a_3b_1)] \\
&\quad + [\mu_\sigma(b_2a_2) + \mu_\sigma(b_2b_1)] + [\mu_\sigma(b_3a_2) + \mu_\sigma(b_3b_1)] \\
&= \mu(a) + \mu(ab) + \mu(bb),
\end{align*}$$

and

$$\begin{align*}
(\alpha_\sigma)_* (\mu_\sigma)(cd) &= [\mu_\sigma(a_2a_1) + \mu_\sigma(a_2a_3) + \mu_\sigma(a_2b_2) + \mu_\sigma(a_2b_3)] \\
&\quad + [\mu_\sigma(b_1a_1) + \mu_\sigma(b_1a_3) + \mu_\sigma(b_1b_2) + \mu_\sigma(b_1b_3)] \\
&= \mu(a) + \mu(b),
\end{align*}$$

and finally

$$\begin{align*}
(\alpha_\sigma)_* (\mu_\sigma)(dd) &= [\mu_\sigma(a_2a_2) + \mu_\sigma(a_2b_1)] + [\mu_\sigma(b_1a_2) + \mu_\sigma(b_1b_1)] = 0.
\end{align*}$$

Since by definition we have $\sigma^M(\mu) = (\alpha_\sigma)_* (\mu_\sigma)$, we have computed:

$$\begin{align*}
\sigma^M(\mu)(c) &= 2(\mu(a) + \mu(b)) \\
\sigma^M(\mu)(d) &= \mu(a) + \mu(b) \\
\sigma^M(\mu)(cc) &= \mu(b) + \mu(aa) + \mu(ba) \\
\sigma^M(\mu)(cd) &= \mu(a) + \mu(ab) + \mu(bb) \\
\sigma^M(\mu)(dc) &= \mu(a) + \mu(b) \\
\sigma^M(\mu)(dd) &= 0.
\end{align*}$$

4.2. An alternative evaluation method.

As illustrated by the example considered in the previous subsection, already for fairly simple morphisms $\sigma$ the preimage set $\alpha_\sigma^{-1}(w)$ may become rather large, even for small $|w|$. In this subsection we explain how a more efficient evaluation technique is obtained (compare formulas (4.3) and (3.5)) and we give an example of a typical computation.

Given a morphism $\sigma : \mathcal{A}^* \rightarrow \mathcal{B}^*$, consider any word $w = x_1 \ldots x_n \in \mathcal{A}^*$ and its image $\sigma(w) = y_1 \ldots y_m \in \mathcal{B}^*$, with letters $x_i \in \mathcal{A}$ and $y_j \in \mathcal{B}$. An occurrence of $w' \in \mathcal{B}^*$ in $\sigma(w)$ is a factor $y_{k'} \ldots y_{r'}$ of $y_1 \ldots y_m$ which satisfies $w' = y_{k'} \ldots y_{r'}$.

An occurrence of $w'$ in $\sigma(w)$ is called essential if its first letter occurs in $\sigma(x_1)$ and its last letter in $\sigma(x_n)$. In particular, for $|w| = 1$ any occurrence of $w'$ in $\sigma(w)$ is essential. If $w$ has length $|w| = 2$, then an occurrence of $w'$ in $\sigma(w)$ is essential if the factor $w'$ overlaps from $\sigma(x_1)$ into $\sigma(x_2)$. For $|w| \geq 3$ a factor $w'$ of $\sigma(w)$ is an essential occurrence if $w'$ contains the image $\sigma(x_2 \ldots x_{n-1})$ of the “maximal inner factor” $x_2 \ldots x_{n-1}$ of $w$ as factor, but not as prefix or suffix.
The number of all essential occurrences of \( w' \) in \( \sigma(w) \) will be denoted by \( |\sigma(w)|_{w'} \). It follows directly from these definitions that for any \( w' \in \mathcal{B}^* \) the set of all words \( w \) in \( \mathcal{A}^* \), for which \( \sigma(w) \) contains at least one essential occurrence of \( w' \), is finite. Indeed, for \( |w'| \geq 2 \) one easily verifies that any such \( w \) must satisfy

\[
|w'| \leq |\sigma| \leq |w'| - 2 + 2,
\]

where \( ||\sigma|| \) and \( \langle \sigma \rangle \) denote the biggest and smallest length respectively of any of the letter images \( \sigma(a_i) \).

**Proposition 4.2.** Let \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \) be any non-erasing morphism of free monoids, and let \( \mu \) be any invariant measure on \( \mathcal{A}^\mathbb{Z} \). Then for any \( w' \in \mathcal{B}^* \) the transferred measure \( \sigma^\mathcal{M}(\mu) \), evaluated on the cylinder \( \{w'\} \), has the value

\[
\sigma^\mathcal{M}(\mu)(w') = \sum_{u \in \mathcal{A}^*} |\sigma(u)|_{w'} \cdot \mu(u)
\]

In particular, for \( |w'| \geq 2 \), we have:

\[
\sigma^\mathcal{M}(\mu)(w') = \sum_{\{u \in \mathcal{A}^* \mid |u| \leq |w'| - 2 + 2 \}} |\sigma(u)|_{w'} \cdot \mu(u)
\]

**Proof.** Recall that in formula (3.5), for any \( u \in \alpha_{\sigma}^{-1}(w') \), either \( \hat{u} \) is defined as shortest word in \( \mathcal{A}^* \) with the property that \( \pi_{\sigma}(\hat{u}) \) contains \( u \) as factor, or else (if such \( \hat{u} \) doesn’t exist) we have formally set \( \mu(\hat{u}) = \mu_{\sigma}(u) = 0 \). In the first case the factor \( u \) of \( \pi_{\sigma}(\hat{u}) \) defines an essential occurrence of \( w' = \alpha_{\sigma}(u) \) in \( \sigma(\hat{u}) \).

Conversely, every essential occurrence of \( w' \) in \( \sigma(v) \), for any \( v \in \mathcal{A}^* \), defines a factor \( u \) of \( \pi_{\sigma}(v) \) with \( u \in \alpha_{\sigma}^{-1}(w') \) for which we have \( \hat{u} = v \). Indeed, the word \( \hat{u} \) can not be a proper factor of \( v \), or else the given occurrence of \( w' \) as factor of \( \sigma(v) \) would not have been essential.

It follows that the sums in the formulas (3.5) and (4.3) differ only in their organization of the indexing, so that the results of the right hand sides of (3.5) and of (4.3) must be equal. \( \square \)

We will illustrate now that the new formula (4.3) is a lot more convenient in practice (in particular in view of the fact that in the example below the set \( \alpha_{\sigma}^{-1}(w) \) consists of 648 elements): 

**Example 4.3.** Let us consider \( \mathcal{A} = \{a, b, c\} \), \( \mathcal{B} = \{d, e\} \) and \( \sigma \) given by:

\[
a \mapsto ded, \ b \mapsto de, \ c \mapsto dedd
\]

Let us compute \( \sigma^\mathcal{M}(\mu)(w) \) for any invariant measure \( \mu \) on \( \mathcal{A}^\mathbb{Z} \), for the randomly picked word

\[
w = dedd.
\]

By (4.1) it suffices to consider any \( u \in \mathcal{A}^* \) of length \( |u| \leq 3 \). We quickly check that that

\[
|\sigma(u)|_w = 0 \quad \text{for} \quad u \in \{a, b, c, ab, ba, bb, bc, cb\}
\]

and

\[
|\sigma(u)|_w = 1 \quad \text{for} \quad u \in \{aa, ac, ca, cc\}.
\]

For any word \( u = x_1x_2x_3 \in \mathcal{A}^* \) with \( x_1, x_2, x_3 \in \mathcal{A} \) and \( x_2 \neq b \) the definition of \( |\sigma(u)|_w \) gives directly \( |\sigma(u)|_w = 0 \). It remains to check that

\[
|\sigma(u)|_w = 0 \quad \text{for} \quad u \in \{bba, bbb, bbc\},
\]

and

\[
|\sigma(u)|_w = 1 \quad \text{for} \quad u \in \{aba, abb, abc, cba, cbb, cbc\}.
\]
in order to obtained the desired formula:
\[ \sigma^M(\mu)(w) = \mu(aa) + \mu(ac) + \mu(ca) + \mu(cc) + \mu(aba) + \mu(abb) + \mu(abc) + \mu(cba) + \mu(cbb) + \mu(cbc) \]
\[ = \mu(aa) + \mu(ac) + \mu(ca) + \mu(cc) + \mu(ab) + \mu(cb) = \mu(a) + \mu(c) \]

**Remark 4.4.** Since a factor \( w' \in B^* \) of length \(|w'| = 1\) in any \( \sigma(w) = \sigma(x_1 \ldots x_n) \) cannot “overlap” from some \( \sigma(x_i) \) into the adjacent \( \sigma(x_{i+1}) \), the only essential occurrences of \( w' \) in any \( \sigma(w) \) can take place if one has \(|w| = 1\). Hence we deduce from (4.2) for any \( b_j \in B \) the formula
\[ (4.4) \quad \sigma^M(\mu)(b_j) = \sum_{a_k \in A} |\sigma(a_k)| |b_j| \cdot \mu(a_k). \]

Every shift-invariant measure \( \mu \) on \( A^\mathbb{Z} \) defines canonically a **letter frequency vector** \( \vec{v}(\mu) = (\mu([a_k]))_{a_k \in A} \in \mathbb{R}_\geq 0^A \), which plays an important role in many contexts (see for instance [2] or [3]).

We observe directly from (4.4):

**Proposition 4.5.** Let \( \sigma : A^* \to B^* \) be a non-erasing monoid morphism, and let \( \mu \in M(A^\mathbb{Z}) \) be an invariant measure. Then the letter frequency vectors \( \vec{v}(\mu) \) and \( \vec{v}(\sigma^M(\mu)) \), associated to \( \mu \) and to its image measure \( \sigma^M(\mu) \in M(B^\mathbb{Z}) \) respectively, satisfy
\[ (4.5) \quad \vec{v}(\sigma^M(\mu)) = M(\sigma) \cdot \vec{v}(\mu), \]
where \( M(\sigma) \) denotes the incidence matrix of \( \sigma \) - see (2.1).

\[ \square \]

5. **Shift-orbit injectivity and related notions**

In this section we will establish a natural criterion which garantees that the measure transfer map \( \sigma^M \) is \( 1 - 1 \), when restricted to measures which are supported by suitable subshifts. We first need to recall and specify the notation from section 2.3:

**Definition 5.1.** Let \( \sigma : A^* \to B^* \) be a non-erasing monoid morphism, and let \( X \subseteq A^\mathbb{Z} \) be any subshift.

(1) We say that \( \sigma \) is **shift-orbit injective in \( X \)** if the map \( \sigma^T \) restricted to the shift-orbits of \( X \) is injective.

(2) We say that \( \sigma \) is **shift-period preserving in \( X \)** if \( \sigma \) preserves the shift-period for every periodic orbit in \( X \) (see Definition 2.10).

**Lemma 5.2.** Let \( \sigma' : A^* \to B^* \) and \( \sigma'' : B^* \to C^* \) be two non-erasing morphisms (so that the composition \( \sigma := \sigma'' \circ \sigma' \) is also non-erasing.) For any subshift \( X \subseteq A^\mathbb{Z} \) consider the image subshift \( Y = \sigma(X) \). Then we have:

(1) The map \( \sigma \) is shift-orbit injective in \( X \) if and only if \( \sigma' \) is shift-orbit injective in \( X \) and \( \sigma'' \) is shift-orbit injective in \( Y \).

(2) The map \( \sigma \) is shift-period preserving in \( X \) if and only if \( \sigma' \) is shift-period preserving in \( X \) and \( \sigma'' \) is shift-period preserving in \( Y \).

**Proof.** Using the surjectivity from Lemma 2.4 (2), we obtain the first of these statements as direct application of the fact that for any two composable surjective maps \( f \) and \( g \) the composition \( g \circ f \) is injective if and only if both maps \( f \) and \( g \) are injective. Statement (2) follows directly from the trivial observation that \( \sigma(w) \) is a proper power (see (2.4)) if and only if one of the three, \( w, \sigma'(w) \) or \( \sigma''(\sigma'(w)) \) is a proper power. \[ \square \]

Next we consider a subdivision length function \( \ell \) and the associated subdivision morphism \( \pi_\ell : A^* \to A^*_\ell \) as in subsection 3.1.

**Lemma 5.3.** Any subdivision morphism \( \pi_\ell : A^* \to A^*_\ell \) is both, shift-orbit injective and shift-period preserving in the full shift \( A^\mathbb{Z} \).
Proof. In order to see that $\pi_\ell$ is shift-orbit injective in the full shift we only need to observe that $\pi_\ell^*> : A^* \to A_\ell^*$ is injective, and that two biinfinite words from $\pi_\ell^*(A^*)$ belong to the same shift-orbit if and only if their preimages in $A^*$ belong to the same orbit.

Similarly, from the definition of $\pi_\ell$ it follows directly that the image of any $w \in A^*$ is a proper power if and only $w$ itself is a proper power. This implies directly (see Definition 2.10) that $\pi_\ell$ is shift-period preserving in the full shift.

□

Lemma 5.4. For any subdivision morphism $\pi_\ell : A^* \to A_\ell^*$ the induced transfer map $\pi_\ell^M : M(A^*) \to M(A_\ell^*)$, $\mu \mapsto \mu_\ell$ on the measure cones is injective.

Proof. Two measures $\mu, \mu' \in M(A^*)$ are distinct if and only there exists a word $w \in A^*$ where the associated weight functions satisfy $\mu(w) \neq \mu'(w)$. From the definition of the subdivision measure in Definition 3.3 we see directly that $\mu_\ell(\pi_\ell(w)) = \mu(w)$ and $\mu'_\ell(\pi_\ell(w)) = \mu'(w)$. It follows that $\mu_\ell \neq \mu'_\ell$.

□

We can now start the proof of main result of this paper; its core (Theorem 6.7) will however be delayed until the next section.

Theorem 5.5. Let $\sigma : A^* \to B^*$ be a non-erasing morphism of free monoids on finite alphabets, and let $X \subseteq A^*$ be a subshift.

If $\sigma$ is shift-orbit injective in $X$, then the measure transfer map $\sigma^M_X : M(X) \to M(\sigma(X))$ is injective.

Proof. We consider the canonical decomposition $\sigma = \alpha_\sigma \circ \pi_\sigma$ from equality (3.4) and obtain from the functoriality of the measure transfer (see Lemma 3.7 (b)) the decomposition $\sigma^M_X = (\alpha_\sigma)^M_M \circ (\pi_\sigma)^M_M(X)$.

From Lemma 5.2 (1) and Lemma 5.3 we obtain directly that the morphism $\alpha_\sigma$ is shift-orbit injective. We can thus apply Lemma 5.4 to $(\pi_\sigma)^M_M$ and Theorem 6.7 to $(\alpha_\sigma)^M_M$ to deduce that $\sigma^M_X$ is injective.

□

We will terminate this section with a discussion that compares the above introduced notions to the more frequently used notions of morphisms that are “recognizable” or a “recognizable for aperiodic points”. For the convenience of the reader we briefly recall the definitions:

Definition 5.6. Let $\sigma : A^* \to B^*$ be a non-erasing morphism, and let $X \subseteq A^*$ be a subshift over $A$.

(1) Then $\sigma$ is said to be recognizable in $X$ if the following conclusion is true: Consider biinfinite words $x, x' \in X \subseteq A^*$ and $y \in B^*$ which satisfy

(*) $y = T^k(\sigma^{-}(x))$ and $y = T^\ell(\sigma^{-}(x'))$ for some integers $k, \ell$ which satisfy $0 \leq k \leq |\sigma(x_1)| - 1$ and $0 \leq \ell \leq |\sigma(x'_1)| - 1$, where $x_1$ and $x'_1$ are the first letters of the positive half-words $x_{[1,\infty)} = x_1 x_2 \ldots$ of $x$ and $x'_{[1,\infty)} = x'_1 x'_2 \ldots$ of $x'$ respectively.

Then one has $x = x'$ and $k = \ell$.

(2) The morphism $\sigma$ is called recognizable for aperiodic points in $X$ if the same conclusion as in (1) is true, but under the strengthened hypothesis that in addition $y$ is assumed not to be a periodic word.

It turns out (see Proposition 3.8 of [2]) that for any non-erasing morphism $\sigma$ as above the condition “recognizable in a subshift $X$” is equivalent to the condition “shift-orbit injective and shift-period preserving in $X$”. Indeed, a quick proof for the injectivity of the measure transfer map $\sigma^M_X$ as in Theorem 5.5, under the stronger hypothesis of “recognizability in $X$” is given in section 3.3 of [2].

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Some authors say “recognizable on $X$”. 18
However, since recognizability can only be achieved for an everywhere growing $S$-adic development of a given subshift $X$ (for the terminology see for instance section 2 of [2]) if $X$ is aperiodic, the much more popular hypothesis used by the $S$-adic community is not “recognizable” but “recognizable for aperiodic points”, or “eventually recognizable for aperiodic points”. The relation of this notion to the concepts introduced in this section (see Fig. 1) is given by the following; a formal proof is given in Remark 3.10 (3) of [2].

**Proposition 5.7.** If a non-erasing morphism $\sigma : A^* \to B^*$ is shift-orbit injective in a subshift $X \subseteq A^\mathbb{Z}$, then $\sigma$ is recognizable for aperiodic points in $X$.

The converse implication of Proposition 5.7 turns however out to be wrong: An obvious counterexample is given by the morphism $\sigma_1$ from Remark 2.9 and the subshift $X = \{a^\pm, b^\pm\}$, which only consists of periodic words, so that $\sigma_1$ is automatically recognizable for aperiodic points on $X$. However, since $\sigma_1^T(O(a^\pm)) = O(c^\pm)$ and $\sigma_1^T(O(b^\pm)) = O(c^\pm)$, the map $(\sigma_1)_X$ is not shift-orbit injective.

This example shows also that “$\sigma$ recognizable for aperiodic points in $X$” is in general too weak to be able to deduce that the measure transfer map $\sigma_\mathcal{M} : \mathcal{M}(X) \to \mathcal{M}(\sigma(X))$ is injective: For $\sigma_1$ as above one observes directly from Lemma 3.7 (d) that $\sigma_1^\mathcal{M}(\mu_a) = \sigma_1^\mathcal{M}(\mu_b) = \mu_c$.

6. **The injectivity of the measure transfer for letter-to-letter morphisms**

Throughout this section we assume that $\sigma : A^* \to B^*$ is a letter-to-letter morphism of free monoids, and that $X \subseteq A^\mathbb{Z}$ is a given subshift. From Definition-Remark 3.6 we observe directly that in this special case the induced measure transfer map $\sigma_\mathcal{M} : \mathcal{M}(X) \to \mathcal{M}(\sigma(X))$ is injective: For $\sigma_1$ as above one observes directly from Lemma 3.7 (d) that $\sigma_1^\mathcal{M}(\mu_a) = \sigma_1^\mathcal{M}(\mu_b) = \mu_c$.

**Lemma 6.1.** If $\sigma$ is shift-orbit injective in $X$, and if for some $x \in X$ the image word $\sigma(x)$ is periodic, then $x$ is periodic too.

**Proof.** From the definition of the image subshift (see Definition-Remark 2.2 and Lemma 2.4) it follows directly that for any biinfinite word $x \in X$ the closure $\overline{O(x)}$ of its orbit $O(x)$ is a subshift which satisfies:

$$\sigma(\overline{O(x)}) = \overline{O(\sigma(x))}$$

The claim is hence a direct consequence of the well known and easy to prove fact that the shift-orbit of any biinfinite word is closed if and only if the word is periodic.

For any word $w \in \mathcal{L}(X)$ and any integer $n \geq 0$ let us consider the set $W_n(w)$ of both-sided prolongations of $w$ in $\mathcal{L}(X)$ by $n$ letters on each side, i.e.

$$W_n(w) := \{uwv \in \mathcal{L}(X) \mid |u| = |v| = n\}.$$
For any \( w' = uvw \in W_n(w) \) we now consider the image \( \sigma(w') \) and its preimage set \( \sigma^{-1}(\sigma(w')) \cap \mathcal{L}(X) \). We split the set \( W_n(w) \) into a disjoint union,

\[
W_n(w) = U_n(w) \sqcup A_n(w),
\]

by setting

\[
w' \in U_n(w) \iff \sigma^{-1}(\sigma(w')) \cap \mathcal{L}(X) \subseteq W_n(w)
\]

and

\[
w' \in A_n(w) \iff \sigma^{-1}(\sigma(w')) \cap \mathcal{L}(X) \not\subseteq W_n(w)
\]

In other words, for any \( w' = uvw \in A_n(w) \) there exists a word \( u'w''v' \in \mathcal{L}(X) \) with \( |u'| = |v'| = n \) and \( |w''| = |w| \) such that \( w'' \neq w \) and \( \sigma(u'w''v') = \sigma(uvw) \), while for any \( w' = uvw \in U_n(w) \) any word \( u'w''v' \in \mathcal{L}(X) \) with \( |u'| = |v'| = n \) and \( \sigma(u'w''v') = \sigma(uvw) \) one has \( w'' = w \) (and thus \( u'w''v' \in W_n(w) \)).

We now fix some \( w \in \mathcal{L}(X) \) and observe directly from the definition of the sets \( A_n(w) \) that their elements define cylinders for which their union \( \{ A_n(w) \} := \bigcup \{ [uvw] \mid uvw \in A_n(w) \} \subseteq X \) satisfies

\[
T^{-n-1}([A_{n+1}(w)]) \subseteq T^{-n}([A_n(w)]) .
\]

**Remark 6.2.** (1) We denote by \( A_\infty(w) \) the intersection of the nested family of the \( T^{-n}([A_n(w)]) \). Since by definition we have \( A_n(w) \subseteq \mathcal{L}(X) \) for all \( n \geq 0 \) we know that \( A_\infty(w) \subseteq X \).

(2) For any invariant measure \( \mu \) on \( X \) and any \( n \geq 0 \) one has thus \( \mu([A_n+1(w)]) < \mu([A_n(w)]) \) and

\[
\lim \mu([A_n(w)]) = \mu(A_\infty(w)) .
\]

**Lemma 6.3.** If \( \sigma \) is shift-orbit injective in \( X \), then for any \( w \in \mathcal{L}(X) \) the intersection \( A_\infty(w) \) consists only of periodic words.

**Proof.** For any biinfinite word \( x \in A_\infty(w) \) there exists, by definition of \( A_\infty(w) \), a sequence of words \( u_n, v_n, u'_n, v'_n, w'_n \in \mathcal{L}(X) \) which have the following properties:

1. \( |u_n| = |v_n| = |u'_n| = |v'_n| = n \) and \( |w'_n| = |w| \)
2. \( u'_n v'_n v_n \in \mathcal{L}(X) \) and \( u_n w v_n \in \mathcal{L}(X) \)
3. \( \sigma(u_n w v_n) = \sigma(u'_n v'_n v_n) \)
4. \( w'_n \neq w \)
5. \( \lim w v_n = x_{[1, +\infty)} \) and \( \lim u_n = x_{(-\infty, 0]} \)

The statement (5) needs a bit of interpretation: We think of the words \( w v_n \) as being indexed from 1 to \( |w v_n| \) and of \( u_n \) as being indexed from \(-|u_n| + 1 \) to 0, and we pass to the limit while keeping the indices fixed. In other words, statement (5) is equivalent to

\[
(5') \lim y(n) = x \text{ in } \mathcal{A}^\mathbb{Z}, \text{ for any } n \geq 0 \text{ the biinfinite word } y(n) \text{ is defined through } y_{[1, +\infty]}(n) = w v_n^{+\infty} \text{ and } y_{(-\infty, 0]}(n) = u_n^{-\infty} .
\]

We now proceed analogously with the sequence of words \( u'_n v'_n \) (indexed from 1 to \( |u'_n v'_n| \)) and the words \( u'_n \) (indexed from \(-|u_n| + 1 \) to 0), and pass to a subsequence of the integers \( n \) such that there exists a biinfinite “limit word” \( z \in \mathcal{A}^\mathbb{Z} \) which satisfies:

\[
(6) \lim u'_n v'_n = z_{[1, +\infty)} \text{ and } \lim u'_n = z_{(-\infty, 0]}
\]

Again, these limits are meant to be equivalent to the statement

\[
(6') \lim z(n) = z \text{ in } \mathcal{A}^\mathbb{Z}, \text{ where the } z(n) \in \mathcal{A}^\mathbb{Z} \text{ are defined via } z_{[1, +\infty]}(n) = u'_n v'_n^{+\infty} \text{ and } z_{(-\infty, 0]}(n) = u'_n^{-\infty} .
\]
Since by property (2) we have \( u_n'w_n'v_n' \in \mathcal{L}(X) \), for all indices \( n \) in the limits (6) or (6'), we obtain \( z \in X \). Furthermore, from the above limit set-up we know that the factor \( z_{[1,|w|]} \) of \( z \) is equal to some \( w_n' \), and thus (by property (4)) distinct from \( w \). We thus deduce:

\[ z \neq x \]

On the other hand, we obtain from property (3) directly:

\[ \sigma(z) = \lim \sigma(z_n) = \lim \sigma(y_n) = \sigma(x) \]

We now use the assumption that \( \sigma \) is shift-orbit injective in \( X \) to deduce that \( x \) and \( z \) are shift-translates of each other: There exists some integer \( k \neq 0 \) such that \( T^k(x) = x \). We thus have \( T^k(\sigma(x)) = T^k(\sigma(z)) = \sigma(T^k(z)) = \sigma(x) \). But any biinfinite word which is equal to a non-trivial shift-translate of itself must be periodic. Hence \( \sigma(x) \) is periodic, and now we can employ Lemma 6.1 in order to deduce that the biinfinite word \( x \) is periodic. \( \square \)

We denote by \( \text{Per}(X) \) the subset of all biinfinite periodic words in \( X \). We observe that both, the countable set \( \text{Per}(X) \) and its complement \( X \setminus \text{Per}(X) \) are measurable subsets of \( \mathcal{X}^2 \). Any measure \( \mu \in \mathcal{M}(X) \) satisfies \( \mu(\text{Per}(X)) = 0 \) if and only if \( \mu \) is non-atomic. This gives rise to a decomposition of \( \mu \) as sum of a non-atomic measure \( \mu^{\text{na}} \) and a periodic measure \( \mu^{\text{per}} \), by which we mean that \( \mu^{\text{per}} \) is a countable or finite sum of scalar multiples of the characteristic measures \( \mu_w \) from (2.5). Note that this decomposition is canonical, in that for any measurable set \( B \subseteq X \) one has

\[
\mu^{\text{per}}(B) = \mu(B \cap \text{Per}(X)) \quad \text{and} \quad \mu^{\text{na}}(B) = \mu(B \cap (X \setminus \text{Per}(X)))
\]

**Proposition 6.4.** Let \( \mu \) and \( \mu' \) be two invariant measures on \( X \) which are both non-atomic. Assume furthermore that \( \sigma \) is shift-orbit injective. Then one has:

\[ \sigma_*(\mu) = \sigma_*(\mu') \implies \mu = \mu' \]

**Proof.** We fix an arbitrary word \( w \in \mathcal{L}(X) \) and consider for any \( n \geq 0 \) the subsets \( W_n(w), U_n(w) \) and \( A_n(w) \) as defined in the equalities (6.1), (6.3) and (6.4), abbreviated here to \( W_n, U_n \) and \( A_n \) respectively.

Let us also fix some integer \( n \geq 0 \). From the definition of a weight function (see Section 2.1) we see directly that for any measure \( \mu \) on \( X \) the corresponding weight function satisfies:

\[
\mu(w) = \sum_{w' \in W_n(w)} \mu(w')
\]

From (6.2) we know that the set \( W_n \) decomposes as disjoint union of \( U_n \) and \( A_n \), so that (6.7) gives us

\[
\mu(w) = \mu(U_n) + \mu(A_n),
\]

where we set \( \mu(U_n) := \sum_{w' \in U_n} \mu(w') \) and \( \mu(A_n) := \sum_{w' \in A_n} \mu(w') \).

We now define \( V_n := \sigma(U_n) \) and observe from the definition of \( U_n \) that one has \( \sigma^{-1}(V_n) \cap \mathcal{L}(X) = U_n \). Hence we deduce from the definition of the push-forward measure \( \sigma_*(\mu) \), together with the fact that by definition of \( \mu \in \mathcal{M}(X) \) one has \( \mu(w_0) = 0 \) for any \( w_0 \notin \mathcal{L}(X) \), that

\[
\mu(U_n) = \sum_{v \in V_n} \sigma_*(\mu)(v) =: \sigma_*(\mu)(V_n).
\]

We thus deduce from the equalities (6.8) and (6.9) that for any integer \( n \geq 0 \) one has:

\[
\mu(w) = \mu(U_n) + \mu(A_n) = \sigma_*(\mu)(V_n) + \mu(A_n).
\]
We now pass to the limit for \( n \to \infty \) and recall from Remark 6.2 (2) that \( \lim \mu(A_n) = \mu(A_x(w)) \).

From Lemma 6.3 we know \( A_x(w) \subseteq \text{Per}(X) \), so that our assumption that \( \mu \) is non-atomic and hence zero on \( \text{Per}(X) \) implies \( \lim \mu(A_n) = 0 \). Hence \( \lim \sigma_*(\mu)(V_n) \) does exist and satisfies:

\[
\lim_{n \to \infty} \sigma_*(\mu)(V_n) = \mu(w)
\]

Since the sets \( V_n \) depend only on \( w \) and not on \( \mu \), equality (6.10) shows that \( \mu(w) \) is entirely determined by the values of the induced measure \( \sigma_*(\mu) \), which proves our claim.

**Lemma 6.5.** Let \( \mu \) be any shift-invariant measure on the subshift \( X \). Then we have:

1. If \( \mu \) is periodic (i.e. \( \mu(X \setminus \text{Per}(X) = 0) \), then \( \sigma_*(\mu) \) is also periodic (i.e. \( \sigma_*(\mu)(\sigma(X) \setminus \text{Per}(\sigma(X)) = 0) \).

2. Under the additional hypothesis that \( \sigma \) is shift-orbit injective, we also have: If \( \mu \) is non-atomic (i.e. \( \mu(\text{Per}(X)) = 0 \), then \( \sigma_*(\mu) \) is also non-atomic (i.e. \( \sigma_*(\mu)(\text{Per}(\sigma(X))) = 0) \).

**Proof.** (1) If \( \mu(X \setminus \text{Per}(X)) = 0 \), then \( \mu \) is carried entirely by the (countable) set of periodic words in \( X \): There exist a coefficient \( \lambda_w \geq 0 \) for any \( w \in L(X) \) such that, using the characteristic measures \( \mu_w \) from (2.5), the measure \( \mu \) can be expressed as countable sum

\[
\mu = \sum_{w \in L(X)} \lambda_w \mu_w
\]

But then Lemma 3.7 (d) gives directly \( \sigma_*(\mu) = \sum_{w \in L(X)} \lambda_w \mu_{\sigma(w)} \), which is clearly the zero-measure outside of \( \text{Per}(\sigma(X)) \).

(2) For any \( x \in \text{Per}(\sigma(X)) \) we know from Lemma 6.1 that \( \sigma^{-1}(x) \) consist of periodic words only. Hence the assumption \( \mu(\text{Per}(X)) = 0 \) implies that \( \sigma_*(\mu)(x) = \mu(\sigma^{-1}(x)) = 0 \). Since \( \text{Per}(X) \) is countable, this implies \( \sigma_*(\mu)(\text{Per}(X)) = 0 \).

**Proposition 6.6.** For the subshift \( X \subseteq A^\mathbb{Z} \) let \( M_{\text{per}}(X) \subseteq M(X) \) and \( M_{\text{na}}(X) \subseteq M(X) \) be the subsets of periodic and of non-atomic invariant measures on \( X \) respectively.

For any shift-orbit injective morphism \( \sigma : A^* \to B^* \) the injectivity of the push-forward map \( \sigma_*^X : M(X) \to M(\sigma(X)) \), \( \mu \mapsto \sigma_*(\mu) \) follows if one proves the injectivity for the restrictions of \( \sigma_*^X \) to both, \( M_{\text{per}}(X) \) and \( M_{\text{na}}(X) \).

**Proof.** Let \( \mu_1, \mu_2 \in M(X) \) be invariant measures which satisfy \( \sigma_*(\mu_1) = \sigma_*(\mu_2) =: \mu_0 \). Consider the canonical decompositions into a periodic and an non-atomic measure from (6.5) given by \( \mu_1 = \mu_1^\text{per} + \mu_1^\text{na} \), \( \mu_2 = \mu_2^\text{per} + \mu_2^\text{na} \) and \( \mu_0 = \mu_0^\text{per} + \mu_0^\text{na} \). From Lemma 6.5 and the uniqueness of the canonical decomposition for \( \mu_0 \) (see the equalities (6.6)) we derive that \( \sigma_*(\mu_1^\text{per}) = \sigma_*(\mu_2^\text{per}) = \mu_0^\text{per} \) and \( \sigma_*(\mu_1^\text{na}) = \sigma_*(\mu_2^\text{na}) = \mu_0^\text{na} \). Injectivity of \( \sigma_*^X \) on \( M_{\text{per}}(X) \) and \( M_{\text{na}}(X) \) implies \( \mu_1^\text{per} = \mu_2^\text{per} \) and \( \mu_1^\text{na} = \mu_2^\text{na} \), which shows \( \mu_1 = \mu_2 \).

**Theorem 6.7.** For any letter-to-letter morphism \( \sigma : A^* \to B^* \) and any subshift \( X \subseteq A^\mathbb{Z} \) one has: If \( \sigma \) is shift-orbit injective in \( X \), then the push-forward map \( M(X) \to M(\sigma(X)) \), \( \mu \mapsto \sigma_*(\mu) \) is injective.

**Proof.** From Proposition 6.6 we know that it suffices to prove the injectivity of the measure map \( \mu \mapsto \sigma_*(\mu) \) for the two cases, where \( \mu \) is periodic or where \( \mu \) is non-atomic.

The latter case is already dealt with in Proposition 6.4, so that we can from now on assume that \( \mu \) is carried by \( \text{Per}(X) \). But in this case the injectivity of the map \( \mu \mapsto \sigma_*(\mu) \) is trivial, since by the injectivity assumption on the shift-orbits together with Lemma 6.1 we know that for every periodic word \( x \in \sigma(X) \) the set \( \sigma^{-1}(x) \cap X \) consists of words \( y_i \) which are periodic as well and belong all to a single shift-orbit \( O(y_i) \). Hence we have \( \mu(O(y_i)) = \sigma_*(\mu)(O(x)) \). Since any invariant measure on \( X \) which is carried by \( \text{Per}(X) \) is determined by knowing its evaluation on every periodic orbit, it follows that \( \mu \) is entirely determined by \( \sigma_*(\mu) \).
The following remark serves as continuation of the discussion started at the end of last section:

**Remark 6.8.** (1) We would like to point out a subtlety in the last proof: Although the preimage set $\sigma^{-1}(O(x))$ of any periodic orbit $O(x)$ is a single periodic orbit, it is in general not true the the preimage set $\sigma^{-1}(x)$ consists of a single biinfinite word $y_1 \in \sigma^{-1}(O(x))$. It is here that the missing assumption “shift-period preserving” materializes, compared to the case where we assume “$\sigma$ recognizable in $X$” instead of our weaker assumption “$\sigma$ is shift-orbit injective”.

(2) On the other hand, as pointed out at the end of last section, “shift-orbit injective” is a slightly stronger property than “recognizable for aperiodic points”, and as we have observed in the last section, the latter doesn’t suffice to deduce the injectivity of the push-forward map on $M(x)$. In fact, this injectivity fails in general already for non-atomic measures on the given subshift $X$, as can be seen easily from the morphism which sends all letters of an alphabet of seize $\geq 2$ to a single letter of the image alphabet; any such morphism is a fortiori recognizable for aperiodic points.

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