NODAL COUNT FOR DIRICHLET-TO-NEUMANN OPERATORS WITH POTENTIAL

ASMA HASSANNEZHAD AND DAVID SHER

Abstract. We consider Dirichlet-to-Neumann operators associated to $\Delta + q$ on a Lipschitz domain in a smooth manifold, where $q$ is an $L^\infty$ potential. We prove a Courant-type bound for the nodal count of the extensions $u_k$ of the $k$th Dirichlet-to-Neumann eigenfunctions $\phi_k$ to the interior satisfying $(\Delta + q)u_k = 0$. The classical Courant nodal domain theorem is known to hold for Steklov eigenfunctions, which are the harmonic extension of the Dirichlet-to-Neumann eigenfunctions associated to $\Delta$. Our result extends it to a larger family of Dirichlet-to-Neumann operators. Our proof makes use of the duality between the Steklov and Robin problems.

Keywords. Dirichlet-to-Neumann operator, nodal count, Courant-type bound, Steklov problem.

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1. Introduction

We consider Dirichlet-to-Neumann operators associated to the Laplace operator with a potential. Let $M$ be a smooth Riemannian manifold, $\Omega \subseteq M$ a connected Lipschitz domain, and $q \in L^\infty(\Omega)$ a potential function. Consider the operator $\Delta_q := \Delta + q$ on $\Omega$, where $\Delta = -\text{div} \nabla$ is the positive Laplacian. Denote by $\Delta^D_q$ the operator $\Delta_q$ with Dirichlet boundary condition on $\partial \Omega$. The operator $\Delta^D_q$ has discrete spectrum whose only accumulation point is $+\infty$.

Now let $\lambda \in \mathbb{R}$. We consider the Dirichlet-to-Neumann operator $D_{q,\lambda}$ associated to $\Delta_q - \lambda$. We first define this in the case where $\lambda$ is not an eigenvalue of $\Delta^D_q$. In that case, for any $g \in L^2(\partial \Omega)$, the equation

$$\begin{cases}
\Delta_{q,\lambda}u = 0 & \text{in } \Omega \\
u = f & \text{on } \partial \Omega
\end{cases}$$

has a unique solution $u$, and we set

$$D_{q,\lambda}f := \partial_n u,$$

where $\partial_n u$ is the outward pointing normal derivative of $u$ along $\partial \Omega$. If $\lambda$ is an eigenvalue of $\Delta^D_q$, the solution is no longer unique, but we may still define $D_{q,\lambda}$ by projecting off the subspace consisting of normal derivatives of Dirichlet eigenfunctions. As we will see, in either event, $D_{q,\lambda}$ is a semi-bounded self-adjoint operator and has discrete, real spectrum whose only accumulation point is $+\infty$. We denote its eigenvalues, with multiplicity, by $\{\sigma_k\}_{k=1}^\infty$, and fix
a corresponding basis of eigenfunctions for $L^2(\partial \Omega)$ by $\{\phi_k\}_{k=1}^\infty$. Finally, define $\{u_k\}_{k=1}^\infty$ to be the interior extensions of $\phi_k$, that is, the functions for which

$$\begin{cases} (\Delta_q - \lambda)u_k = 0 & \text{in } \Omega \\ u_k = \phi_k & \text{on } \partial \Omega, \end{cases}$$

again with the appropriate modifications when $\lambda$ is an eigenvalue of $\Delta_q^D$. As in the case $q = 0$, we call $\{u_k\}$ the corresponding Steklov eigenfunctions.

In this paper, we discuss the nodal counts of both the Steklov eigenfunctions $u_k$ and the Dirichlet-to-Neumann eigenfunctions $\phi_k$. Throughout, we let $N_k$ be the number of nodal domains of $u_k$ on $\Omega$ and let $M_k$ be the number of nodal domains of $\phi_k$ on $\partial \Omega$.

In the case $q = 0$, it is well-known that we have an analogue of the Courant nodal domain theorem for Steklov eigenfunctions (see [11, 10, 8]). Specifically,

$$N_k \leq k.$$  

In this case, the proof essentially uses three ingredients: the variational principle for eigenvalues, the unique continuation theorem for the solutions of a second order elliptic PDE, and the fact that harmonic functions are the unique minimizers of the Dirichlet energy for given boundary data. The statement does not hold when $q$ is an arbitrary nonzero potential. However, as we show, there is a replacement:

**Theorem 1.1.** With terminology as above, let $d$ be the number of non-positive Dirichlet eigenvalues of $\Delta_{q, \lambda}$, or equivalently the number of eigenvalues of $\Delta_q^D$ which are less than or equal to $\lambda$. Then for all $k \in \mathbb{N},$

$$N_k \leq k + d.$$  

**Remark 1.2.** This theorem is sharp in the sense that for any $d \in \mathbb{N}$, there exists a domain $\Omega$, a potential function $q$, and an integer $k$ for which $N_k = k + d$.

**Remark 1.3.** If $\Omega$ is a fixed subdomain of $\mathbb{R}^n$ and $q$ is sufficiently small, then perturbation theory (see e.g. [14, Page 76]) implies that $\Delta_q$ has only positive Dirichlet eigenvalues. The same is true when $q \geq 0$. Thus, by Theorem 1.1 $N_k \leq k$ for the operator $\mathcal{D}_{q, 0}$ in these cases.

Very little is known about the nodal count of the Dirichlet-to-Neumann eigenfunctions $\phi_k$. See Open Problem 9 in [8]. The statement that $M_k \leq k$ is certainly not true in general, for the same reasons as for $N_k$. In fact, the situation is worse, as $\partial \Omega$ may be disconnected, in which case, even if $q = 0$, the Courant nodal domain theorem cannot hold for the ground state $k = 1$. When $q = 0$ and the dimension of $\Omega$ is two, the fact that no nodal line is a closed curve implies an estimate on $M_k$ in terms of $k$ and the topology of the domain. For example, for a simply connected domain, the bound is $2k$ [1, Lemma 3.4]. However, for $q \neq 0$ no such bound exists. See Example 1 below. In higher dimension, nothing is known regarding bounds for $M_k$ even when
The main difficulty is that \( \mathcal{D}_{q,\lambda} \) is nonlocal and the method of the proof we employ to study the nodal count of \( u_k \) cannot be generalised to study the nodal count of the Dirichlet-to-Neumann eigenfunctions \( \phi_k \).

We conjecture the following asymptotic version of the Courant nodal domain theorem:

**Conjecture 1.4.** With terminology as above,

\[
\limsup_{k \to \infty} \frac{M_k}{k} \leq 1.
\]

**Remark 1.5.** Note that the corresponding result for \( N_k \),

\[
\limsup_{k \to \infty} \frac{N_k}{k} \leq 1,
\]

follows immediately from Theorem 1.1.

**Remark 1.6.** If Conjecture 1.4 is true, it would immediately imply

\[
M_k \leq k + o(k).
\] (1)

This would yield a partial answer to Open Question 9 in [8].

We also conjecture the following sharpened version in dimension at least three. This is motivated by the Pleijel theorem for the nodal count of the Laplace operator [5, 13].

**Conjecture 1.7.** When the dimension of \( \Omega \) is at least three,

\[
\limsup_{k \to \infty} \frac{M_k}{k} < 1
\]

and

\[
\limsup_{k \to \infty} \frac{N_k}{k} < 1.
\]

In fact, this sharpened version is true in a number of special cases. For example, suppose that \( \Omega \) is a cylinder \([0,1] \times \Sigma\), where \( \Sigma \) is a compact manifold of dimension at least two. One can use separation of variables and Pleijel’s theorem [5, 13] to show that

\[
\limsup_{k \to \infty} \frac{M_k}{k} \leq c < 1.
\]

The same result is true if \( M_k \) is replaced by \( N_k \). A similar result holds if \( \Omega \) is a ball in \( \mathbb{R}^n \), with \( n \geq 3 \).

The key example to keep in mind is the following, motivated by [7, Figure 1]. In particular, it shows that \( \frac{N_k}{k} \) and \( \frac{M_k}{k} \) are only asymptotically bounded by one.
Example 1. Let $\Omega$ be the unit disk, set $\lambda = 0$, and let $q$ be the constant function $-\mu$ for some $\mu \geq 0$. Then the spectrum of $D_{q,\lambda}$ is of the form
\[
\left\{ \frac{\sqrt{\mu} J_n'(\mu)}{J_n(\mu)}, \ n \in \mathbb{N}_0 \right\},
\]
with a corresponding basis of eigenfunctions $J_n(\sigma r) e^{\pm i n \theta}$.

Note that $J_n(\mu)$ is zero if and only if $\mu$ is a Dirichlet eigenvalue of $\Delta + q = \Delta - \mu$. So fix a particular $n$ and consider what happens as $\mu$ approaches the first zero $j_{n,1}$ of $J_n(x)$ from below. The eigenvalue of $D_{-\mu,\lambda}$ corresponding to that particular $n$ will go to $-\infty$. (It is simple if $n = 0$ and double if $n > 0$.) Since the Dirichlet eigenvalues of a disk all have multiplicity at most 2, all other eigenvalues stay bounded below. If we choose $\mu = j_{n,1} - \epsilon$ for a sufficiently small $\epsilon > 0$, then the smallest eigenvalue of $D_{-\mu,\lambda}$ will be $\sigma = \frac{\sqrt{\mu} J_n'(\mu)}{J_n(\mu)}$, with eigenfunction(s) $J_n(\sigma r) e^{\pm i n \theta}$. So these eigenfunction(s) are the ground state eigenfunction(s) for $D_{-\mu,\lambda}$, i.e. they have $k = 1$. However, each of them has $n$ boundary nodal domains and $n$ interior nodal domains as well, so we have $N_k = M_k = n$. Since $n$ is arbitrary, not only can we have $N_k > k$, but we can have as large a discrepancy as we like, illustrating the sharpness in Remark 1.2.

The key method for the proof of Theorem 1.1 is to make use of Steklov-Robin duality. This is the observation that the two-parameter problem
\[
\begin{align*}
\Delta_q u &= \lambda u & \text{in } \Omega \\
\partial_n u &= \sigma u & \text{on } \partial \Omega
\end{align*}
\]
may be viewed either as a Steklov problem for fixed $\lambda$, with eigenvalue parameter $\sigma$, or as a Robin problem for fixed $\sigma$, with eigenvalue parameter $\lambda$. This idea has a long history, at least in the case $q = 0$. It was first written down in [9] but seems to have been known to others, including Caseau and Yau (see the discussion in [3]). In 1991, L. Friedlander rediscovered it and used it to give a proof of the interlacing of Dirichlet and Neumann eigenvalues for domains in $\mathbb{R}^n$ [6, 12]. In [2, 3], Arendt and Mazzeo generalized the Steklov-Robin duality to manifolds; though Friedlander’s inequalities fail in that setting, the duality results themselves still hold. Some duality results with nonzero potential, though nominally in the Euclidean setting only, are given in [4]. Finally, we should note that Steklov-Robin duality has been used to compare Steklov eigenvalues and eigenvalues of the boundary Laplacian, see for example [7], which gave us the idea for Example 1.

2. Modified Courant nodal domain theorem for Steklov eigenfunctions with potential

Let $\Omega$ be a Lipschitz domain in a smooth Riemannian manifold $M$. Let $q \in L^\infty(\Omega)$ be a potential. It is enough to prove Theorem 1.1 for $\lambda = 0$, as $\lambda$...
may be absorbed into the potential \( q \). Therefore, Theorem 1.1 is an immediate consequence of the following result:

**Theorem 2.1.** Suppose that \( \Omega \) and \( q \) are as above. Suppose that \( \{u_k\}_{k=1}^\infty \) is a complete set of Steklov eigenfunctions for \( \Delta_q \), and \( N_k \) is the number of nodal domains of \( u_k \) on \( \Omega \). Then

\[
N_k \leq k + d,
\]

where \( d \) is the number of non-positive eigenvalues of the following Dirichlet eigenvalue problem:

\[
\begin{align*}
\Delta_q u &= \lambda u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

The proof of Theorem 2.1 uses Steklov-Robin duality. Let introduce two parameters, \( \lambda \) and \( \sigma \), and consider the problem

\[
\begin{align*}
\Delta_q u &= \lambda u, \quad \text{in } \Omega \\
\partial_n u &= \sigma u, \quad \text{on } \partial \Omega.
\end{align*}
\]

One may consider \( \lambda \) as the spectral parameter, in which case we have a Robin problem with fixed \( \sigma \), or consider \( \sigma \) as the spectral parameter, in which case we have a Steklov-type problem with fixed \( \lambda \). We let \( \lambda_{q,k}(\sigma) \) be the \( k \)th eigenvalue of \( \Delta_{q,\sigma} \). Observe that the \( k \)th Steklov eigenfunction \( u_k \) is an eigenfunction with eigenvalue \( \lambda = 0 \) for the Robin problem:

\[
\begin{align*}
\Delta_q u &= \lambda u, \quad \text{in } \Omega \\
\partial_n u &= \sigma_k u, \quad \text{on } \partial \Omega.
\end{align*}
\]

The question is for which \( m \lambda_{q,m}(\sigma_k) \) is equal to 0.

The duality results we need essentially follow from [2], [3], and [4]. However, they are not stated in quite this much generality, and so we give a proof here. Our approach is modeled primarily on [3].

First, we define the Robin Laplacian \( \Delta_{q,\sigma} \) by using the weak formulation. Consider the form, for \( u, v \in H^1(\Omega) \),

\[
b_{q,\sigma}(u, v) = \int_{\Omega} (\nabla u \cdot \nabla v + quv) \, dV - \int_{\partial \Omega} u\sigma dV_{\partial \Omega}.
\]

Since \( q \in L^\infty \), this form is coercive, and so it determines an operator \( \Delta_{q,\sigma} \), which is the Robin Laplacian. The domain of \( \Delta_{q,\sigma} \) is the same as the domain of \( \Delta_{0,\sigma} \), namely

\[
\{ u : u \in L^2(\Omega), \Delta u \in L^2(\Omega), \partial_n u = \sigma u \text{ on } \partial \Omega \}.
\]

A Dirichlet Laplacian with potential, \( \Delta_q^D \), may also be defined as usual.

For each \( \lambda \) which is not in the spectrum of \( \Delta_q^D \), we define the Dirichlet-to-Neumann operator \( D_{q,\lambda} \). If \( g \in L^2(\partial \Omega) \) and \( u \in H^1(\Omega) \) is the unique solution of

\[
\begin{align*}
(\Delta_q - \lambda) u &= 0 \quad \text{in } \Omega \\
u &= f \quad \text{on } \partial \Omega
\end{align*}
\]
then we set $D_{q,\lambda}f = \partial_n u$. This is enough for many purposes. However, we need to consider $\lambda$ which are in the Dirichlet spectrum of $\Delta_q$. There are several ways to do this, the simplest of which is to restrict to the orthogonal complement of the kernel. Following [3], we define

$$K(\lambda) = \{ \partial_n w : \Delta_q w = \lambda w \text{ weakly}, w|_{\partial\Omega} = 0, \partial_n w \in L^2(\partial\Omega) \}.$$  

Then $D_{q,\lambda}$ may in all cases be defined as an operator on $L^2(\partial\Omega) \cap (K(\lambda))^\perp$.

Proposition 2.2. For any $\lambda \in \mathbb{R}$, $D_{q,\lambda}$ is self-adjoint, has compact resolvent, and is bounded below.

Proof. A proof is given in [3] when $q = 0$. However, it depends on a result of Grégoire, Nédélec, and Planchard [9], which is only stated in the setting $q = 0$. We instead use the machinery of [4], which instead views the Dirichlet-to-Neumann operator as a graph, that is, as a multi-valued operator. From [4, Proposition 3.3], it suffices to prove that this graph is self-adjoint, has compact resolvent, and is bounded below. Yet this is essentially the content of [4, Example 4.9]. Although stated in the setting $M = \mathbb{R}^n$ and $\lambda = 0$, every assertion there holds when $M$ is an arbitrary Riemannian manifold, and a nonzero $\lambda$ may be treated as part of the potential. The three parts of our Proposition then follow from Theorem 4.5, Proposition 4.8, and Theorem 4.15 of [4]. □

As a consequence, the spectrum of $D_{q,\lambda}$ is contained in the real axis, discrete, and has only the accumulation point at infinity.

In what follows we use the notational conventions:

$$D_{q,0} := D_q, \quad D_{0,0} := D.$$

Obviously, we have $D_{q,\lambda} = D_{q-\lambda}$. However, it will be convenient to separate the role of $\lambda$ from the potential $q$ to highlight the connection between $D_{q,\lambda}$ and the Robin problem.

The following proposition, encapsulating the Steklov-Robin duality, is the analogue of [3, Theorem 3.1] and is proved in identical fashion.

Proposition 2.3. For any $\lambda, \sigma \in \mathbb{R}$, the trace map is an isomorphism from $\ker(\Delta_{q,\sigma} - \lambda)$ to $\ker(D_{q,\lambda} - \sigma)$.

Remark 2.4. Since $D_{q,\lambda} = D_{q-\lambda}$, Proposition 2.3 is equivalent to show that the trace map is an isomorphism from $\ker(\Delta_{q,\sigma})$ to $\ker(D_q - \sigma)$ for any $\sigma \in \mathbb{R}$ and $q \in L^\infty(\Omega)$.

Proof. First we show that the trace map indeed maps into the indicated space. Suppose that $u \in \ker(\Delta_{q,\sigma} - \lambda)$. Then $u \in H^1(\Omega)$, so certainly $\text{Tr}(u) \in L^2(\partial\Omega)$. Since $u$ is in the domain of $\Delta_{q,\lambda}$, $\partial_n u$ exists and equals $\sigma \text{Tr}(u)$. And as long as $\text{Tr}(u) \in (K(\lambda))^\perp$, it is in the domain of $D_{q,\lambda}$. In that event, we can say that $D_{q,\lambda}(\text{Tr}(u)) = \sigma \text{Tr}(u)$, hence $\text{Tr}(u) \in \ker(D_{q,\lambda} - \sigma)$.
To show that $\text{Tr}(u) \in (K(\lambda))^\perp$, suppose that $\partial_n w \in K(\lambda)$, with $w \in \ker(\Delta^D - \lambda)$. Then by Green’s identity,
\[
\langle \partial_n w, \text{Tr}(u) \rangle_{L^2(\partial\Omega)} = \langle \nabla w, \nabla u \rangle_{L^2(\Omega)} - \langle \Delta_0^D w, u \rangle_{L^2(\Omega)}.
\]
Using Green’s identity again, combined with the facts that $w$ is in the domain of the Dirichlet Laplacian and $w \in \ker(\Delta^D q - \lambda)$, we have
\[
\langle \partial_n w, \text{Tr}(u) \rangle_{L^2(\partial\Omega)} = \langle w, \Delta u \rangle_{L^2(\Omega)} - \langle (\lambda - q) w, u \rangle_{L^2(\Omega)}.
\]
(3)

However, since $u \in \ker(\Delta^D q, \sigma - \lambda)$, we know that $\Delta u = (\lambda - q)u$. Since $\lambda \in \mathbb{R}$ and $q$ is real-valued, the right-hand side of (3) is zero. Thus $\text{Tr}(u) \in (K(\lambda))^\perp$ and therefore the trace map does indeed map into $\ker(D^q_\sigma - \lambda)$.

To show that the trace map is injective, suppose that $u \in \ker(\Delta^D q, \sigma - \lambda)$ with $\text{Tr}(u) = 0$. From our definition of $D^q_\sigma$, we know that for any $g \in (K(\lambda))^\perp$, the problem
\[
\begin{cases}
(\Delta q - \lambda)u = 0 & \text{in } \Omega \\
u = g & \text{on } \partial\Omega
\end{cases}
\]
has a unique solution whose trace is in $(K(\lambda))^\perp$. Since both $u$ and $0$ are solutions to this problem with $g = 0$, we must have $u = 0$.

Finally, surjectivity is straightforward: suppose that $g \in \ker(D^q_\sigma - \lambda)$. By definition there is a function $u \in H^1(\Omega)$ such that $(\Delta q - \lambda)u = 0$, $\text{Tr}(u) = g$, and $\partial_n u = \sigma g$. This $u$ is an element of $\ker(\Delta^D q, \sigma - \lambda)$ whose trace is $g$. □

An immediate consequence is

**Corollary 2.5.** For any $\lambda, \sigma \in \mathbb{R}$, $\sigma$ is an element of the (Steklov) spectrum of $D^q_\lambda$ if and only if $\lambda$ is an element of the (Robin) spectrum of $\Delta^D q, \sigma$. Moreover their geometric multiplicities are the same.

The following statement describes its behaviour as $\sigma$ varies.

**Proposition 2.6.** For every $k \geq 1$ the following hold:

(a) $\lambda_{q,k}(\sigma)$ is strictly decreasing.

(b) $\lambda_{q,k}$ as a function of $\sigma$ is continuous on $[-\infty, \infty)$. In particular,
\[
\lim_{\sigma \to -\infty} \lambda_{q,k}(\sigma) = \lambda^D_k,
\]
where $\lambda^D_k$ is the $k$-th Dirichlet eigenvalue of $\Delta^D q$.

(c) $\lim_{\sigma \to \infty} \lambda_{q,k}(\sigma) = -\infty$.

The proof of this proposition follows, nearly verbatim, the proof presented in [3, Proposition 3] and [2, Section 2]. For the sake of completeness and the reader’s convenience, we give the proof.

**Proof.** To prove a), note that by the max-min principle for eigenvalues, we have
\[
\lambda_{q,k}(\sigma) = \sup_{V_{n-1}} \inf \{ b_{q,\sigma}(u) : u \in V_{n-1}, \|u\|_{L^2(\Omega)} = 1 \}
\]
where the supremum is taken over all subspaces $V_{n-1} \subset H^1(\Omega)$ of codimension $n - 1$. Since $b_{q,\sigma}(u)$ is strictly decreasing in $\sigma$, it follows that $\lambda_{q,k}(\sigma)$ is decreasing. To show that it is strictly decreasing, assume to the contrary that for some $\sigma < \tilde{\sigma}$, $\lambda_{q,k}(\sigma) = \lambda_{q,k}(\tilde{\sigma})$. It implies that $\lambda := \lambda_{q,k}(\sigma)$ is constant on $[\sigma, \tilde{\sigma}]$. By Corollary 2.5, $[\sigma, \tilde{\sigma}]$ must be a subset of the spectrum of $D_{q,\lambda}$. This contradicts the fact that the spectrum of $D_{q,\lambda}$ is discrete.

To prove b), we first show the continuity of the resolvents $(\mu + \Delta_{q,\sigma})^{-1}$, $\sigma \in [-\infty, \infty)$. It was shown for sufficiently large $\mu$, in the $q = 0$ case, in [2, Proposition 2.6]. The proof remains the same and the statement remains true. Thus, for $\mu$ large enough,

$$\lim_{s \to \sigma}(\mu + \Delta_{q,s})^{-1} = (\mu + \Delta_{q,\sigma})^{-1}$$

and in particular when $\sigma = -\infty$, $\Delta_{q,-\infty} = \Delta_q^D$. Hence

$$\lim_{s \to -\infty} (\mu + \Delta_{q,s})^{-1} = (\mu + \Delta_q^D)^{-1}.$$

We can now use [2, Proposition 2.8] to conclude that for every $k \geq 1$ and $s \in [-\infty, \infty)$,

$$\lim_{\sigma \to s} \lambda_{q,k}(\sigma) = \lambda_{q,k}(s).$$

In particular, for $s = -\infty$

$$\lim_{\sigma \to -\infty} \lambda_{q,k}(\sigma) = \lambda_{q,k}^D.$$

For c), assume that $\lambda_{q,k}(\sigma)$ is bounded below by some $\lambda \in \mathbb{R}$ for all $\sigma \in \mathbb{R}$, i.e.

$$\lambda_{q,k}(\sigma) > \lambda, \quad \sigma \in \mathbb{R}.$$  

Note that $\lambda < \lambda_{q,k}(\sigma) \leq \lambda_{q,k+1}(\sigma)$ for all $\sigma$. By Corollary 2.5, we have that the spectrum of $D_{q,\lambda}$ is the set

$$\{ \sigma \in \mathbb{R} : \lambda = \lambda_{q,j}(\sigma) \text{ for some } j = 1, \ldots, k - 1 \}.$$  

However, this set is finite by part a). This is impossible. \hfill \Box

**Proposition 2.7.** For any $\lambda \in \mathbb{R}$, consider $d \in \mathbb{N} \cup \{0\}$ such that $\lambda_{q,d}^D \leq \lambda < \lambda_{q,d+1}^D$. By convention $\lambda_{q,0}^D = -\infty$. Then for every $k \geq 1$, there exists a unique $s_k \in \mathbb{R}$ such that $\lambda_{q,k+d}(s_k) = \lambda$. Moreover, $s_k = \sigma_k(D_{q,\lambda})$ for every $k \geq 1$.

The proof follows the same line of argument as in the proof of Proposition 4.5 in [2]; see also [3, Proposition 4].

**Proof.** By Proposition 2.6, we have

$$\lim_{\sigma \to -\infty} \lambda_{q,k+d}(\sigma) = \lambda_{q,k+d}^D > \lambda \geq \lambda_{q,d}^D, \quad \lim_{\sigma \to -\infty} \lambda_{q,k}(\sigma) = -\infty,$$

for every $k \geq 1$. Thus, the existence and uniqueness of $s \in \mathbb{R}$ follows from the fact that $\lambda_{q,k+d}$ is a strictly decreasing continuous function. If $s \in \{ \sigma_j(D_{q,\lambda}) \}$, then there exists $m \in \mathbb{N}$ such that $\lambda_{q,m}(s) = \lambda$. Hence, $m \geq d + 1$ and $s = s_k$, where $k = m - d$. Indeed, thanks to Proposition 2.6, for every $m \leq d$ and $s \in \mathbb{R}$, $\lambda_{q,m}(s) < \lambda_{q,d}^D \leq \lambda$. This shows that $\{ \sigma_j(D_{q,\lambda}) \}$ and $\{ s_j \}$ are
equal as sets. It remains to show that they are equal as multisets, i.e. their multiplicities are equal.

It is easy to observe that
\[ s_k \leq s_{k+1}. \]
Indeed, if \( s_k > s_{k+1} \), then
\[ \lambda_{q,d+k+1}(s_{k+1}) = \lambda = \lambda_{q,d+k}(s_k) < \lambda_{q,d+k+1}(s_{k+1}) \]
gives a contradiction. Assume that \( s_k \) has multiplicity \( p \) and \( s := s_k < s_{k+p} \). Hence, \( \lambda_{q,k+d+j}(s) = \lambda, \ j = 0, \ldots, p-1 \). But \( \lambda_{q,k+d+j}(s) < \lambda_{q,k+d+1}(s_{k+p}) \). Therefore, in both cases, the multiplicity of \( \lambda_{q,k+d}(s) \) is equal to \( p \) and so, by Proposition 2.3, is the multiplicity of \( \sigma_k(D_{q,\lambda}) \).

\[ \square \]

Theorem 2.1 is now an immediate consequence of Propositions 2.7 and 2.3.

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University of Bristol, School of Mathematics, Fry Building, Woodland Road, Bristol, BS8 1UG, U.K.

*Email address: asma.hassannezhad@bristol.ac.uk*

DePaul University, Department of Mathematical Sciences, 2320 N Kenmore Ave., Chicago, IL, 60614, U.S.A.

*Email address: dsher@depaul.edu*