Hydraulic Fracturing of Poorly Consolidated Reservoir During Waterflooding

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This paper describes a KGD-type model of a hydraulic fracture created by injecting fluid in weak, poorly consolidated rocks. The model is based on the key assumption that the hydraulic fracture is propagating within a domain where the hydraulic fields are quasi-stationary. By further assuming a negligible toughness, the fracture is shown to grow as a square root of time. The asymptotic fracture propagation regimes at small and large time are constructed and the transient solution is computed by solving a nonlinear system of algebraic equations formulated in terms of the fracture aperture. At early time the fracture is hydraulically invisible and the injection pressure increases with time \( t \) as \( \log t \), while at late time leak-off from the borehole is negligible and the injection pressure decreases as \( t^{-1/4} \). According to this model, the peak injection pressure observed when injecting fluid in weak, poorly consolidated rocks should not be interpreted as indicating a breakdown of the formation, but rather as marking the transition between two asymptotic flow regimes. The timescale that legislates the transition between the small and large time asymptotic regimes is shown to be a strongly nonlinear function of a dimensionless injection rate.

1. Introduction

The efficiency of waterflooding carried out to increase oil recovery from hydrocarbon-bearing rocks is predicated in part on the initiation and propagation of hydraulic fractures at injection wells to ensure a more efficient sweep of the reservoir [Sharma et al., 2000, van den Hoek and Mclennan, 2000, Noirot et al., 2003]. These fracture allow the injected water to leak through the crack surfaces, which eventually leads to the establishment of a quasi-linear flow pattern around the borehole-fracture system.

In the waterflooding of poorly consolidated reservoirs, unusually large breakdown pressures are observed. This is a priori unexpected in view of the small fracture toughness.

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of these rocks. It has been proposed that the observed anomalous breakdown pressure is a consequence of the large apparent toughness associated with the development of plastic zones in the crack tip region [Cleary et al., 1991, Papanastasiou and Thiercelin, 1993, Papanastasiou, 1997, van Dam et al., 2002]. However, laboratory fluid injection experiments in weak sandstone show evidence of injection-induced hairline cracks [Ispas et al., 2012], which contradicts the blunt crack tip predicted by plasticity-based models [Germanovich et al., 2012]. In view of this inconsistency, this paper explores instead an alternative explanation, which is rooted in the hydraulic nature of the problem rather than in the apparent strength of the formation.

Hydraulic fractures created during waterflooding of poorly consolidated oil reservoirs cannot be analyzed, however, using models developed for predicting the growth of hydraulic fractures in conventional reservoirs. The reason is twofold. First, the volume balance equation of hydraulic fracturing— the injected fluid volume equals the crack volume plus the leak-off volume — is essentially meaningless. Indeed, the volume of fluid stored in the crack is negligibly small compared to the volume of injected fluid, in view of the large permeability of these reservoirs. In other words the treatment efficiency — the ratio of the stored fluid volume to the injected volume — is virtually zero. Second, the conventional models do not account for the large scale perturbations of the pore pressure due to the prolonged duration, of order of months or years, of water injection.

The main objective of this paper is thus to develop a model for analyzing the propagation of a KGD-type hydraulic fracture in the context of a waterflooding operation. The paper is structured as follows. First the problem is formulated within the framework of porous media flow, lubrication theory, and linear elastic fracture mechanics. The model is then expressed as a nonlinear system of integro-differential equations in terms of the fracture aperture, by taking advantage of the linearity of the equations governing the hydraulic and mechanical fields in the porous medium. A scaling analysis indicates that the scaled fields depend only on the dimensionless space and time variables, with all the parameters of the problem absorbed in the scales. Small and large-time asymptotic solutions are then constructed and the transient solution is obtained by solving numerically a nonlinear system of algebraic equations deduced by discretizing the integro-differential system of equations governing the fracture aperture.

2. Problem Definition

We consider the propagation of a KGD-type (plane strain) hydraulic fracture driven by injection of a Newtonian fluid in a permeable elastic rock, as sketched in Fig. 1. The far-field boundary conditions correspond to minimum compressive stress $\sigma_o$ (normal to the hydraulic fracture) and pore pressure $p_o$. A fluid of viscosity $\mu$ is injected at a constant rate $Q_o$ per unit length of the hole. Prior to injection, the stress and pore pressure fields are uniform. The rock is characterized by intrinsic permeability $k$, or equivalently by mobility $\kappa = k/\mu$, diffusivity $c$, elastic modulus $E$, and Poisson ratio $\nu$. The rock toughness is assumed to be negligible.

In view of the plane strain nature of the problem, the two elastic constants $E$ and $\nu$ can
be combined into the plane strain modulus $E'$. Furthermore, to avoid carrying numerical factors in the equations, we introduce the alternate constant $\mu'$. The constants $E'$ and $\mu'$ are defined as

$$E' = \frac{E}{1-\nu^2} \quad \mu' = 12\mu$$

The final set of model parameters consists therefore of 7 constants: $\mathcal{P} = \{\mu', E', \kappa, c, \sigma_o, p_o, Q_o\}$. Using Buckingham $\Pi$-theorem, this set can in principle be reduce to 4 dimensionless parameters. However, further analysis of the problem indicates that a much greater simplification of the problem is possible.

The hydraulic fracturing problem can be formulated as follows. Given the fluid and solid properties, the far-field stress and pore pressure, and the injection rate, determine how the crack half-length (one wing) $\ell(t)$ and injection pressure $p_w(t)$ evolve with time $t$.

Figure 1: Problem definition. The model is constructed assuming that the borehole can be reduced to a point source.

3. Mathematical Model

Here we formulate a mathematical model that combines (i) the diffusion equation controlling the pore pressure evolution in the permeable medium; (ii) the equations of Linear Elastic Fracture Mechanics (LEFM), one linking crack aperture and fluid pressure, the other defining the propagation criterion; (iii) the lubrication equation governing fluid flow in the fracture; and finally (iv) the boundary conditions and interface conditions between the fracture and the porous medium.

The model is based on the following assumptions: (i) the fracture length is large compared to the borehole radius so that the fluid is injected via a point source; (iii) the viscosity of the injected fluid is identical to that of the pore fluid; (iv) the crack propagates inside the region where the flow is quasi-stationary (i.e., where the pore pressure is governed by Laplace equation); (v) the crack volume is negligible compared
to the volume of fluid injected, or equivalently, the rate of change of the fluid volume stored in the crack is negligible compared to the injection rate.

Consider the system of coordinates $(x, y)$ with the origin at the point source and the $x$-axis oriented in the direction normal to the minimum in-situ stress $\sigma_o$. Hence, as illustrated in Fig. 1, the fracture propagates along the $x$-axis. Pore pressure $p_r(x, y, t)$ is defined in the plane $(x, y)$, while fluid pressure $p_f(x, t)$ or equivalently net pressure $p(x, t) = p_f(x, t) - \sigma_o$, crack aperture $w(x, t)$, and leak-off $g(x, t)$ are defined along the crack $-\ell(t) \leq x \leq \ell(t), y = 0$.

### 3.1. Diffusion

The pore pressure field $p_r(x, y, t)$ in the permeable medium is governed by the diffusion equation

$$c \nabla^2 p_r - \frac{\partial p_r}{\partial t} = \frac{1}{S} [g(x, t) \delta(y) + \Omega_o (1 - \Phi(t)) \delta(x, y)], \quad (2)$$

where $\delta(y)$ and $\delta(x, y)$ denote the Dirac $\delta$-function in one- and two-dimensional space, respectively. The storage coefficient is defined as $S = \kappa/c$. The leak-off $g(x, t)$ represents the normal flux discontinuity across the crack and corresponds therefore to a source density ($g > 0$) from the point of view of the porous medium.

The presence of the crack causes the injection of fluid into the rock to be partly distributed along the crack rather than to take place solely at the source. The unknown function $\Phi(t)$ characterizes the partitioning of the injected fluid between the crack and the porous medium at the injection point. Thus, $0 \leq \Phi(t) \leq 1$ with the lower limit corresponding to a fracture of relatively zero conductivity and the upper limit to a fracture of relatively infinite conductivity. Also, $\Phi(t) = 0$ for $\tau \leq 0$ as injection starts at $t = 0$. Moreover, because storage of fluid in the fracture is negligible on the ground of the high permeability of the porous medium, the leak-off $g(x, t)$ is constrained by

$$\int_{-\ell(t)}^{\ell(t)} g(x, t) dx = \Phi(t) \Omega_o. \quad (3)$$

The pore pressure field $p_r(x, y, t)$ is subject to the conditions at infinity

$$\lim_{r \to \infty} p_r = p_o, \quad (4)$$

where $r = \sqrt{x^2 + y^2}$ and to the initial conditions, here assumed to correspond to a uniform pressure field $p_o$

$$p_r(x, y, 0) = p_o. \quad (5)$$

### 3.2. Lubrication

The flow of fluid in the crack is governed by Reynolds lubrication equation [Batchelor, 1967]

$$\frac{1}{\mu'} \frac{\partial}{\partial x} \left( w^3 \frac{\partial p_f}{\partial x} \right) = g(x, t) - \Omega_o \Phi(t) \delta(x), \quad (6)$$
which is obtained by combining Poiseuille's law

\[ q = -\frac{w^3}{\mu'} \frac{\partial p_f}{\partial x} \] \hspace{2cm} (7)

with the continuity equation

\[ \frac{\partial q}{\partial x} + g - Q_o \Phi(t) \delta(x) = 0. \] \hspace{2cm} (8)

The source term in the continuity equation implies a flux discontinuity at the origin, which can be translated as

\[ q(0^\pm, t) = \pm \frac{1}{2} Q_o \Phi(t) \] \hspace{2cm} (9)

in view of the problem symmetry. Finally, the flux \( q(x, t) \) in the crack vanishes at the crack tip [Detournay and Peirce, 2014]

\[ q(\pm \ell, t) = 0. \] \hspace{2cm} (10)

The fluid pressure \( p_f \) in the crack is continuous with the pore pressure field

\[ p_f(x, t) = p_r(x, 0, t) \quad |x| \leq \ell(t). \] \hspace{2cm} (11)

This equation together with the continuity of the normal flux at the crack walls, which is implicitly satisfied by equating source density along the crack to the leak-off rate, as done in (2), represent the two hydraulic conditions at the interface between the crack and the porous medium.

### 3.3. Linear Elastic Fracture Mechanics

The net pressure in the crack and the crack aperture are also related by a singular integral equation obtained by superposition of dislocation dipoles

\[ p_f(x, t) - \sigma_o = -\frac{E'}{4\pi} \int_{-\ell(t)}^{\ell(t)} \frac{w(s, t)}{(x - s)^2} ds. \] \hspace{2cm} (12)

According to the theory of singular integral equations [Muskhelishvili, 1946], the above equation necessarily implies that the crack closes at its tip, i.e., \( w(\pm \ell(t), t) = 0 \). Finally, the propagation criterion \( K_I = 0 \) can be expressed as a weighted integral of the net pressure (12) [Bueckner, 1970, Rice, 1972]

\[ \int_0^{\ell(t)} \frac{(p_f(x, t) - \sigma_o) \, dx}{\sqrt{\ell^2(t) - x^2}} = 0. \] \hspace{2cm} (13)

To simplify the writing of equations, we introduce the net pressure \( p(x, t) \) defined as

\[ p = p_f - \sigma_o. \] \hspace{2cm} (14)
4. Asymptotic Mathematical Model

The problem can thus be formulated only in terms of variables that are defined on the 1D crack. The system of equations (2)-(6), (10)-(9), and (12)-(13), is closed and can be used to evolve the solution, in particular crack-wing length \( \ell(t) \), and fields \( p(x, t) \), \( w(x, t) \), and \( g(x, t) \). The formulation can be simplified, however, by further adopting the assumption that the crack propagates in a region where the pore pressure is quasi-stationary. As a result, the problem loses its evolutionary nature: time is no longer an independent variable, but becomes a parameter of the problem. This change in the nature of the problem will be reflected in the notation of the fields defined along the crack by using a semi-colon rather than a coma in front of \( t \), i.e., \( p(x; t) \), \( w(x; t) \), and \( g(x; t) \).

4.1. Decomposition of the pore pressure field

First, we express the pore pressure field \( \rho r(x, y, t) \) as the superposition of three fields

\[
\rho r(x, y, t) = \rho_0 + \Delta \rho_w(r, t) + \Delta \rho_c(x, y, t)
\]

where \( \Delta \rho_w(r, t) \) is the pore pressure field induced by injection of fluid in the absence of a crack, \( \Delta \rho_c(x, y, t) \) is the pore pressure perturbation caused by the exchange of fluid between the crack and the medium, and \( \rho_0 \) is the initial homogeneous field at the onset of injection. As a consequence of the linearity of the diffusion equation, the two fields \( \Delta \rho_w \) and \( \Delta \rho_c \) are respectively governed by

\[
c \nabla^2 \Delta \rho_w - \frac{\partial \Delta \rho_w}{\partial t} = \frac{Q_o}{S} H(t) \delta(x, y)
\]

and

\[
c \nabla^2 \Delta \rho_c - \frac{\partial \Delta \rho_c}{\partial t} = \frac{1}{S} \left[ g(x, t) \delta(y) - \rho_0 \Phi(t) \delta(x, y) \right]
\]

noting also the initial conditions

\[
\Delta \rho_w = 0 \quad \Delta \rho_c = 0, \quad t = 0
\]

and the boundary conditions at infinity

\[
\Delta \rho_w = 0 \quad \Delta \rho_c = 0, \quad r \to \infty
\]

The magnitude of the singular source terms in (16) and (17) is consistent with our definition of \( \Delta \rho_w \) and \( \Delta \rho_c \); i.e., \( \Delta \rho_w \) representing the injection-induced pore pressure in the absence of a crack, and \( \Delta \rho_c \) representing the additional pore pressure perturbation associated with the existence of a fracture. It will be shown that the far-field signature of the source terms in (17) —the right-hand member of (17)— is a singular double dipole at the origin.

The pore pressure field in the absence of a crack is given by the well-known source solution, often referred to as Thys solution in geo-hydrology [Cheng and Detournay, 1998]

\[
\Delta \rho_w = \frac{Q_o}{4\pi \kappa} E_1 \left( \frac{r^2}{4ct} \right)
\]
where $E_1$ is the exponential integral. This solution has the interesting property, deduced from the asymptotics of $E_1$, that inside a radius $r_s = \chi_s \sqrt{ct}$ with $\chi_s$ of order $O(10^{-1})$, $\Delta p_w$ can be approximated as

$$\Delta p_w \simeq -\frac{Q_o}{2\pi\kappa} \left( \ln \frac{r}{2\sqrt{ct}} + \frac{\gamma}{2} \right), \quad r < r_s$$

In the above, $\gamma = 0.5772...$ is Euler’s constant. With an accuracy of about 1%, $\Delta p_w$ can be approximated by the asymptotic solution inside $r = r_s$ if $\chi_s \approx 0.35$.

Thus inside the quasi-stationary region $r < r_s(t)$, the pore pressure evolves only because of the movement of the diffusion front. Provided that the crack is well inside the quasi-stationary region, we can assume that there are no transients associated with the exchange of fluid between the crack and the medium. Thus, we assume that

$$\nabla^2 \Delta p_c \simeq \frac{1}{\kappa} \left[ g(x,t)\delta(y) - Q_o \Phi(t)\delta(x,y) \right], \quad r < r_s(t)$$

noting that the leak-off function $g(x,t)$ is only defined along the crack. Hence the pore pressure field $\Delta p_c$ can be written as an integral expression using a superposition of source solution of the Laplace equation, i.e.,

$$\Delta p_c(x,y; t) = \int_{-\ell(t)}^{\ell(t)} P_s(x-s,y)g(s; t)ds - Q_o \Phi(t)P_s(x,y)$$

where the source influence function $P_s$ is given by

$$P_s(x,y) = -\frac{1}{2\pi\kappa} \ln r, \quad r = \sqrt{x^2 + y^2}.$$  

It is important to stress that time only enters the expression for $\Delta p_c$ via the bounds of the integral and through the dependence of the leak-off on time. Pore pressure component $\Delta p_c$ decays as $1/r^2$ away from the crack, as the far-field hydraulic effect of the crack is equivalent to a double dipole.

### 4.2. Reformulation

It is now possible to reformulate the problem only in terms of the net pressure $p(x;t)$, crack opening $w(x;t)$, leak-off $g(x;t)$, fluid fraction $\Phi(t)$, and crack length $\ell(t)$. Thanks to the integral representation of $p(x;t)$ in terms of leak-off $g(x;t)$ and crack aperture $w(x;t)$, we can reduce the formulation of the problem to a set of integro-differential equations on the crack and to unknowns only associated with the crack. With this formulation, the elastic fields and pore pressure field in the domain containing the crack are calculated by solving singular integrals equations on the crack. These integral equations make use of free-field singular solutions: dislocation dipoles for elasticity and source solutions for porous media flow.

The set of equations is given by
\[
\frac{1}{\mu'} \frac{\partial}{\partial x} \left( w' \frac{\partial p}{\partial x} \right) = g(x; t) - \mathcal{Q}_0 \Phi(t) \delta(x), \quad (25)
\]

\[
\mathcal{Q}_0 \Phi(t) = \int_{-\ell(t)}^{\ell(t)} g(x; t) dx
\]  

(26)

\[
p(x; t) + \sigma_o - p_o - \Delta \bar{p}_w(x; t) = -\frac{1}{2\pi\kappa} \int_{-\ell(t)}^{\ell(t)} \ln |x - s| g(s; t) ds + \frac{\mathcal{Q}_0}{2\pi\kappa} \Phi(t) \ln |x|, \quad (27)
\]

\[
p(x; t) = -\frac{E'}{4\pi} \int_{-\ell(t)}^{\ell(t)} \frac{w(s; t)}{(x - s)^2} ds,
\]

(28)

where \( \Delta \bar{p}_w(x, t) = \Delta p_w(r, t)|_{y=0} \) reads

\[
\Delta \bar{p}_w(x, t) = -\frac{\mathcal{Q}_0}{2\pi\kappa} \left( \ln \frac{|x|}{2\sqrt{ct}} + \frac{\gamma}{2} \right). \quad (29)
\]

The propagation criterion is expressed in terms of a weighted integral of the net pressure

\[
\int_{0}^{\ell(t)} p(x; t) dx \sqrt{\ell^2(t) - x^2} = 0, \quad (30)
\]

Note that (26) implies that the flux vanishes at the crack tip

\[
w' \frac{\partial p}{\partial x} = 0, \quad x = \pm \ell(t), \quad (31)
\]

as can readily be verified by integrating (25) using (26). The condition \( w(\pm \ell, t) = 0 \) is also automatically met by the integral equation (28).

The above system of equations (25)-(31) is closed. Hence, given \( t \) and the set of parameters \( \mathcal{P} = \{ \mu', \kappa, E', \sigma_o, p_o, \mathcal{Q}_0 \} \), the solution \( S(x; t) = \{ w(x; t), p(x; t), g(x; t), \Phi(t), \ell(t) \} \) can be determined from the system of equations (25)-(31). In other words, \( S(x; t) \) can be calculated without the need to know the solution at prior times.

5. Scaling

Scaling indicates that this problem is characterized by one time scale, \( t_K \), which is associated with the transition between the rock-flow to the fracture-flow regime.

5.1. Scales

First we introduce scales for the aperture, pressure, leak-off, length, and time. These scales, respectively denoted as \( w_K, p_K, g_K, \ell_K, t_K \), are undetermined for the time being.
Dimensionless crack aperture $\Omega(\xi; \tau)$, net pressure $\Pi(\xi; \tau)$, flux $\Psi(\xi; \tau)$, leak-off rate $\Gamma(\xi; \tau)$, and crack length $\Lambda(\tau)$ are then naturally defined as

$$
\Omega = \frac{w}{w_K}, \quad \Pi = \frac{p}{p_K}, \quad \Psi = \frac{q}{Q_0}, \quad \Gamma = \frac{g}{g_K}, \quad \Lambda = \frac{\ell}{\ell_K \tau^{1/2}} \tag{32}
$$

with scaled coordinate $\xi$ along the crack and dimensionless time $\tau$ given by

$$
\xi = \frac{x}{\ell(t)}, \quad \tau = \frac{t}{t_k} \tag{33}
$$

The motivation to define $\Lambda$ according to (32) stems from the search of similarity solutions at small and large time. Indeed, the existence of similarity solutions requires that $\ell \sim t^{1/2}$. We will show that $\Lambda = 1$ if the toughness is zero.

### 5.2. Dimensionless Groups

The system of equations (25)-(29) is rewritten in terms of the newly introduced quantities

$$
\frac{G_m}{\tau \Lambda^2} \frac{\partial}{\partial \xi} \left( \Omega^3 \frac{\partial \Pi}{\partial \xi} \right) = \Gamma - \frac{G_i}{G_c} \Phi \delta(\xi) \tau^{1/2} \Lambda \tag{34}
$$

$$
\frac{G_i}{G_c} \Phi(\tau) = \tau^{1/2} \Lambda \int_{-1}^{1} \Gamma(\xi) d\xi \tag{35}
$$

$$
\Pi - \Pi_o = \frac{G_c \tau^{1/2} \Lambda}{2\pi} \int_{-1}^{1} \ln |\xi - \zeta| \Gamma(\zeta) d\zeta + \frac{G_i}{2\pi} \Phi \ln |\xi| \tag{36}
$$

$$
\Pi = \frac{G_e}{4\pi \tau^{1/2} \Lambda} \int_{-1}^{1} \frac{\Omega d\xi}{(\xi - \zeta)^2} \tag{37}
$$

$$
\int \frac{\Pi d\xi}{\sqrt{1 - \xi^2}} = 0 \tag{38}
$$

where

$$
\Pi_o = -G_i \left[ 1 + \frac{1}{2\pi} \left( \ln \left( \frac{\Lambda|\xi|}{2G_i} \right) + \frac{\gamma}{2} \right) \right] \tag{39}
$$

noting that $\delta(\xi) = \tau^{1/2} \Lambda \ell_K \delta(x)$. The dimensionless injection rate $J$ appearing in (34)-(39) is defined as

$$
J = \frac{Q_o}{\kappa(\sigma_o - p_o)} \tag{40}
$$

Thus, five dimensionless groups emerge from the above equations:

$$
G_m = \frac{w^3 p_K}{\mu g_K \ell_K^3}, \quad G_i = \frac{\sigma_o - p_o}{p_K}, \quad G_c = \frac{g_K \ell_K}{K p_K}, \quad G_e = \frac{E' w_K}{\ell_K p_K}, \quad G_l = \frac{\sqrt{ct_K}}{\ell_K} \tag{41}
$$
5.3. Characteristic quantities

The five scales \( w_k, p_k, g_k, \ell_k, \) and \( t_k \) become explicit by imposing a value to the five dimensionless groups defined in (41). While there is arbitrariness in the chosen values for these groups, they should in principle be selected in such a way that the norm of the fields \( \Omega(\xi; \tau), \Pi(\xi; \tau), \Gamma(\xi; \tau), \) as well as \( \Lambda(\tau) \) are of order \( O(1) \) when \( \tau = 1 \).

Here we choose
\[
G_m = 1, \quad G_e = 1, \quad G_c = 1, \quad G_i = 1, \quad G_l = 1 \quad \text{exp} \left( \frac{2\pi I}{J} + \frac{\gamma}{2} \right)
\]  

(42)

As a result of enforcing (42), explicit expressions for the scales are obtained:
\[
w_k = \frac{\ell_k Q_o}{K}, \quad p_k = \frac{Q_o}{K}, \quad g_k = \frac{Q_o}{\ell K}, \quad \ell_k = \left( \frac{\mu' \kappa^4 E^3 \beta}{Q_0^3} \right)^{1/2}, \quad t_k = \frac{\mu' \kappa^4 E^3 \beta}{16 c Q_0^3} \exp \left( \frac{4\pi I}{J} + \gamma \right)
\]  

(43)

Note that with this scaling, expression (39) for \( \Pi_o(\xi) \) reduced to
\[
\Pi_o(\xi) = -\frac{1}{2\pi} \ln(2\Lambda|\xi|)
\]  

(44)

5.4. Scaled Mathematical Model

It is noteworthy to point out that the scaled solution \( S(\xi; \tau) = \{\Omega, \Pi, \Gamma, \Phi, \Lambda\} \) depends only on the spatial variable \( \xi \) and the time parameter \( \tau \), as all the physical parameters have been absorbed in the scales. The solution is obtained by solving the closed system of integro-differential equations
\[
\frac{1}{\tau^{1/2} \Lambda} \frac{\partial}{\partial \xi} \left( \Omega^3 \frac{\partial \Pi}{\partial \xi} \right) = \tau^{1/2} \Lambda \Gamma - \Phi(\tau) \delta(\xi)
\]  

(45)

\[
\Phi(\tau) = \tau^{1/2} \Lambda \int_{-1}^{1} \Gamma(\zeta) d\zeta
\]  

(46)

\[
\Pi(\xi) = -\frac{1}{2\pi} \left[ \ln(2\Lambda) + (1 - \Phi(\tau)) \ln|\xi| + \tau^{1/2} \Lambda \int_{-1}^{1} \ln|\xi - \zeta| \Gamma(\zeta) d\zeta \right]
\]  

(47)

\[
\Pi(\xi) = -\frac{1}{4\pi \tau^{1/2} \Lambda} \int_{-1}^{1} \Omega(\zeta) d\zeta
\]  

(48)

\[
\int_{0}^{1} \frac{\Pi d\xi}{\sqrt{1 - \xi^2}} = 0
\]  

(49)

Interestingly, the solution \( \Pi(\xi) \) of the reduced system of equations (45)-(48) automatically satisfies the propagation criterion (49) if \( \Lambda = 1 \). This result can readily be confirmed at small time, when \( \Pi(\xi) \approx \Pi_o(\xi) \) as shown in Section (6.2). Indeed, replacing \( \Pi(\xi) \) in the propagation criterion (49) by expression (44) implies that \( \Lambda = 1 \). Although this result has not be formally proven to hold at any time, it has been confirmed numerically. In the following, we will impose that \( \ell = \tau^{1/2} \) and thus remove the propagation criterion (49) from the set of governing equations.
5.5. Alternative Scaled Mathematical Model

The system of equations (45)-(49) can be rewritten so as to remove $\Phi(\tau)$ from the formulation. To that effect, express $\Gamma(\xi; \tau)$ as the superposition of two functions

$$
\Gamma(\xi; \tau) = \tilde{\Gamma}(\xi; \tau) + \frac{1}{\tau^{1/2}}\Phi(\tau)\delta(\xi) 
$$

(50)

Substituting (50) into (46) shows that

$$
\hat{\beta}_1 - \frac{1}{\tau^{1/2}}\tilde{\Gamma}(\zeta)\,d\zeta = 0 \quad (51)
$$

Hence, the system of equations to be solved for $\tilde{S}(\xi; \tau) = \{\Omega, \Pi, \tilde{\Gamma}\}$ is

$$
\frac{1}{\tau}\frac{\partial}{\partial \xi} \left( \Omega^3 \frac{\partial \Pi}{\partial \xi} \right) = \tilde{\Gamma} 
$$

(52)

$$
\Pi(\xi) = \Pi_o(\xi) - \frac{1}{2\pi}\tau^{1/2}\int_{-1}^{1} \ln |\xi - \zeta| \tilde{\Gamma}(\zeta)\,d\zeta \quad (53)
$$

$$
\Pi(\xi) = -\frac{1}{4\pi\tau^{1/2}}\int_{-1}^{1} \frac{\Omega(\zeta)d\zeta}{(\xi - \zeta)^2} \quad (54)
$$

$$
\Omega^3 \frac{\partial \Pi}{\partial \xi} = 0, \quad \xi = \pm 1 \quad (55)
$$

where

$$
\Pi_o(\xi) = -\frac{1}{2\pi} \ln(2|\xi|) \quad (56)
$$

The constraint (51) is automatically satisfied by the zero-flux boundary condition (55) at the crack tip and Reynolds equation (52).

5.6. Intermediate-Range Pore Pressure Field

The existence of a hydraulic fracture causes a perturbation of the pore pressure field, $\Delta \Pi(\xi, \eta) = \Pi - \Pi_o$ given by

$$
\Delta \Pi = -\frac{1}{2\pi}\tau^{1/2}\int_{-1}^{1} \ln \sqrt{(\xi - \zeta)^2 + \eta^2} \tilde{\Gamma}(\zeta)\,d\zeta \quad (57)
$$

At distance large enough from the crack but still within the quasi-stationary region, the pore pressure perturbation $\Delta \Pi$ is equivalent the pore pressure induced by a steady-state quadripole centered at the origin and aligned along the $\xi$-axis, i.e.,

$$
\Delta \Pi \simeq \frac{1}{4\pi} \tau^{1/2} M^{(2)} \frac{\xi^2 - \eta^2}{(\xi^2 + \eta^2)^2} \quad (58)
$$
where $M^{(2)}$ is the second moment of leak-off $\tilde{\Gamma}(\xi)$ over the crack length

$$M^{(2)} = \int_{-1}^{1} \xi^2 \tilde{\Gamma}(\xi) \, d\xi$$

(59)

Expression (58) is an intermediate asymptotic solution. It is worth noting that $\tilde{\Gamma}(\xi)$ and $\Gamma(\xi)$ have identical moments, except for the zeroth moment since the two functions differ by a Dirac $\delta$-function. Indeed, $M^{(0)} = 0$ while $\tilde{M}^{(0)} = \Phi(\tau)$.

5.7. Discussion

As shown above, the model can be formulated either in terms of $\Gamma(\xi)$ or $\tilde{\Gamma}(\xi)$. Consider first the model formulated in terms of $\Gamma(\xi; \tau)$ and $\Phi(\tau)$. The porous medium equation (47) indicates that the pore pressure field induced by injection of fluid is the superposition of a discrete continuous source of strength $(1 - \Phi(\tau))$ and of a distributed source density $\Gamma(\xi; \tau)$ with a combined strength $\Phi(\tau)$

$$\tau^{1/2} \int_{-1}^{1} \Gamma(\xi; \tau) d\xi = \Phi(\tau), \quad \Psi(0^\pm, \tau) = \pm \frac{1}{2} \Phi(\tau)$$

(60)

The above equation expresses the property that all the fluid entering the crack at the inlet leaves the fracture via leak-off.

Formulating the model in terms of $\tilde{\Gamma}(\xi; \tau)$ implies instead that fluid enters the fracture via the leak-off function — as a Dirac, and is then expelled along the crack into the porous medium. Hence,

$$\int_{-1}^{1} \tilde{\Gamma}(\xi; \tau) d\xi = 0, \quad \tilde{\Psi}(0^\pm, \tau) = 0$$

(61)

The two approaches are essentially equivalent in regard to the determination of the crack opening $\Omega(\xi; \tau)$ and net pressure $\Pi(\xi; \tau)$. However, the injection pressure is directly affected by the strength of the logarithmic singularity of $\Pi$, which depends directly on $\Phi$, as can be surmised from (47). A correct evaluation of the fluid fraction $\Phi$ requires, therefore, consideration of the borehole radius $a$.

6. Asymptotic Regimes

6.1. Similarity Solutions with Power Law Dependence on Time

For small- and large-time, one searches for solution of the type

$$\Omega = \Omega_s(\xi) \tau^{\beta_s}, \quad \Pi = \Pi_s(\xi) \tau^{\alpha_s}, \quad \Gamma = \Gamma_s(\xi) \tau^{\gamma_s}, \quad \Phi = \Phi_s \tau^{\delta_s}$$

(62)

Note that the existence of a similarity solution hinges necessarily on the requirement that the fracture length grows at the square root or time, which is indeed met at all time for $K_{Ic} = 0$. Using (62), the system of equations (45)-(49) becomes
\[ \begin{align*}
\tau^{\alpha_s + 3\beta_s - 1/2} \frac{d}{d\xi} \left( \Omega_s^{3/2} \frac{d\Pi_s}{d\xi} \right) &= \tau^{\gamma_s + 1/2} \Gamma_s - \tau^{\delta_s} \Phi_s \delta(\xi) \quad \text{(63)} \\
\tau^{\delta_s} \Phi_s &= \tau^{\gamma_s + 1/2} \int_{-1}^{1} \Gamma_s(\zeta) d\zeta \\
\tau^{\alpha_s} \Pi_s &= -\frac{1}{2\pi} \left[ \ln 2 + (1 - \tau^{\delta_s} \Phi_s) \ln |\xi| + \tau^{\gamma_s + 1/2} \int_{-1}^{1} \ln |\xi - \zeta| \Gamma_s(\zeta) d\zeta \right] \\
\tau^{\alpha_s} \Pi_s &= -\tau^{\beta_s - 1/2} \frac{1}{4\pi} \int_{-1}^{1} \Omega_s(\zeta) d\zeta \\
\end{align*} \]

We examine now how time balances in those equations. An examination of Reynolds equation (63), balance equation (64), and elasticity equation (66) reveals that these equations are homogeneous of degree zero in time — a requirement for the existence of a similarity solution of the form (62), provided that

\[ \begin{align*}
\delta_s &= \gamma_s + 1/2 \quad \text{(67)} \\
\alpha_s + 3\beta_s - \gamma_s - 1 &= 0 \quad \text{(68)} \\
\alpha_s - \beta_s + \frac{1}{2} &= 0 \quad \text{(69)} \\
\end{align*} \]

In contrast, the porous media flow (65) contains three terms, each with a different power law dependence on time. In view of the constraints (67)-(69), it is not possible to strictly enforce homogeneity of degree zero in time in (65). However, it is conceivable that one term vanishes in the limit of \( \tau \to 0 \) and \( \tau \to \infty \), with the two other terms balancing each other and the resulting balance being then homogeneous of degree zero in time.

- In the rock flow regime, \( \gamma_s > 0 \) since \( \lim_{\tau \to 0} \Gamma = 0 \). Thus \( \tau^{\alpha_s} \Pi_s \) balances \( \ln |\xi| \) at small time. We conclude therefore that

\[ \begin{align*}
\alpha_r &= 0, \quad \beta_r = \frac{1}{2}, \quad \gamma_r = \frac{1}{2}, \quad \delta_r = 1. \\
\end{align*} \] (70)

- In the fracture flow regime, \( \alpha_s < 0 \) since \( \lim_{\tau \to \infty} \Pi = 0 \). The right-hand side of (65) must therefore balance in time. It then follows that

\[ \begin{align*}
\alpha_f &= -\frac{1}{4}, \quad \beta_f = \frac{1}{4}, \quad \gamma_f = -\frac{1}{2}, \quad \delta_r = 0. \\
\end{align*} \] (71)

In the above, the subscript \( r \) denotes the small time solution (rock-flow regime) and the subscript \( f \) the large time solution (fracture-flow regime).
6.2. $R$-Regime (Small-Time Solution)

The above analysis shows therefore the existence of a similarity solution at small time. This solution, denoted as the $R$-vertex solution, is of the form

$$
\Omega = \Omega_r(\xi)\tau^{1/2}, \quad \Pi = \Pi_r(\xi), \quad \Gamma = \Gamma_r(\xi)\tau^{1/2}, \quad \Phi = \Phi_r\tau
$$

(72)

Replacing $\Omega$, $\Pi$, and $\Gamma$ by their asymptotic expressions (72) in the system of equations (45)-(49), and furthermore noting that the source term in the Reynolds and in the porous medium flow equations vanishes at small time —as it proportional to $\tau$— leads to the following time-independent equations

$$
\frac{\partial}{\partial \xi} \left( \Omega^3 \frac{\partial \Pi_r}{\partial \xi} \right) = \Gamma_r - \Phi_r \delta(\xi)
$$

(73)

$$
\Phi_r = \int_{-1}^{1} \Gamma_r(\zeta) d\zeta
$$

(74)

$$
\Pi_r(\xi) = -\frac{1}{2\pi} \ln(2|\xi|)
$$

(75)

$$
\Pi_r(\xi) = -\frac{1}{4\pi} \int_{-1}^{1} \Omega_r(\zeta) d\zeta
$$

(76)

With the additional condition of zero flux that at the crack tip

$$
\Omega^3 \frac{\partial \Pi_r}{\partial \xi} = 0, \quad \xi = \pm 1
$$

(77)

the system of equations (73)-(77) is closed and can be solved for $S_r(\xi) = \{\Omega_r, \Pi_r, \Gamma_r, \Phi_r\}$. However, the crack aperture is determined by integrating the inverse integral equation [Sneddon and Lowengrub, 1969]

$$
\Omega_r(\xi) = \frac{4}{\pi} \int_{0}^{1} \Pi_r(\zeta) \ln \left| \frac{\sqrt{1 - \xi^2} + \sqrt{1 - \zeta^2}}{\sqrt{1 - \xi^2} - \sqrt{1 - \zeta^2}} \right| d\zeta
$$

(78)

rather than by solving the hypersingular integral equation (76). Reynolds lubrication equation (45) with $\Gamma = 0$ can only be strictly satisfied at $\tau = 0$ and asymptotically at small time $\tau$.

In summary, this similarity solution is given by

$$
\Pi_r = -\frac{1}{2\pi} \ln 2|\xi|, \quad \Omega_r = \sqrt{\frac{\pi}{8}} \left[ \sqrt{1 - \xi^2} - \frac{\pi}{2} |\xi| + |\xi| \arctan \left( \frac{|\xi|}{\sqrt{1 - \xi^2}} \right) \right]
$$

(79)

The small-time asymptotic solution $\Omega_r(\xi)$ and $\Pi_r(\xi)$ is illustrated in Figs 2 and 3.

The asymptotic strength of the double dipole that reflect the pore pressure perturbation caused by leak-off from the fracture is given by $M^{(2)} = \tau^{1/2}M^{(2)}_r$ with

$$
M^{(2)}_r = \int_{-1}^{1} \xi^2 \Gamma_r(\xi) d\xi
$$

(80)
Figure 2: Crack opening profile at the $\mathcal{R}$- and $\mathcal{F}$-vertex (small- and large-time asymptotes).

Figure 3: Net pressure profile at the $\mathcal{R}$- and $\mathcal{F}$-vertex (small- and large-time asymptotes).
6.3. \(\mathcal{F}\)-Regime (Large-Time Solution)

The large-time solution is a similarity solution of the form

\[
\Omega = \Omega_f(\xi)\tau^{1/4}, \quad \Pi = \Pi_f(\xi)\tau^{-1/4}, \quad \Gamma = \Gamma_f\tau^{-1/2}, \quad \Phi = \Phi_f.
\]  

(81)

However, necessarily \(\Phi_f = 1\), since all the injected fluid enters the crack directly. The equations governing the solution \(\delta_f(\xi, \mathcal{X}) = \{\Omega_f, \Pi_f, \Gamma_f, \Phi_f\}\) are

\[
\frac{d}{d\xi} \left( \frac{\Omega_f^2 \Pi_f}{d\xi} \right) = \Gamma_f - \delta(\xi)
\]  

(82)

\[
\int \ln|\xi - \zeta| \Gamma_f(\zeta)d\zeta = -\ln 2
\]  

(83)

\[
\Pi_f(\xi) = -\frac{1}{4\pi} \int_{-1}^{1} \Omega_f(\zeta)\zeta d\zeta
\]  

(84)

With the additional conditions that at the crack tip

\[
\Omega_f^2 \frac{d\Pi_f}{d\xi} = 0, \quad \xi = \pm 1
\]  

(85)

The system of equations (82)-(85) is closed and can be solved for \(\delta_f(\xi)\).

The porous media flow equation (83) can be solved in closed-form for \(\Gamma_f\)

\[
\Gamma_f = \frac{1}{\pi \sqrt{1 - \xi^2}}
\]  

(86)

Then after defining the flux \(\Psi_f\) (and noting also that \(\Psi = \Psi_f\)) as

\[
\Psi_f = -\Omega_f^2 \frac{d\Pi_f}{d\xi}
\]  

(87)

integration of (82) yields

\[
\Psi_f = \text{sgn}(\xi) \left( \frac{1}{2} - \frac{1}{\pi} \arcsin |\xi| \right)
\]  

(88)

The two fields \(\Omega_f(\xi)\) and \(\Pi_f(\xi)\) are then computed by solving numerically the elasticity equation (84) and the integrated Reynolds equation

\[
\Omega_f^2 \frac{d\Pi_f}{d\xi} = -\text{sgn}(\xi) \left( \frac{1}{2} - \frac{1}{\pi} \arcsin |\xi| \right)
\]  

(89)

Appendix A outlines the numerical scheme. Figures 2-3 illustrate the large-time asymptotic profile for the opening \(\Omega_f(\xi)\) and the net pressure \(\Pi_f(\xi)\), respectively.

Finally, \(M^{(2)} = \tau^{-1/2}M_f^{(2)}\) with

\[
M_f^{(2)} = \int_{-1}^{1} \xi^2 \Gamma_f(\xi) d\xi
\]  

(90)
7. Numerical Algorithm

7.1. Preamble

The numerical scheme is based on the mathematical model without $\Phi$ as a primary variable, as formulated in Section (5.5) Discretization of the set of integro-differential equations (52)-(54) and boundary conditions (55) leads to a nonlinear algebraic system of equations in terms of the crack opening $\Omega$ at discrete points on the crack. The starting point is to segment the crack on $[-1, 1]$ into $2n$ elements, each of equal length with the mid-point of element $i$ located at $\xi_i = (2i - 1)h - 1$. The discretization of the system of equations rests on assuming that both $\tilde{\Gamma}$ —the discontinuity of the normal flux, and $\Omega$ —the discontinuity of the displacement normal to the crack, are uniform along each element. Thus three quantities are assigned to element $i$: $\tilde{\Gamma}_i$, $\Pi_i$, and $\Omega_i$, with $\Pi_i$ representing the net pressure at $\xi_i$.

Introducing the piecewise constant approximation of the discontinuity fields $\tilde{\Gamma}(\xi)$ and $\Omega(\xi)$ into the singular integral equations (53) and (54) leads to two linear systems of equations

$$\Pi - \Pi_o = \tau^{1/2}B \cdot \tilde{\Gamma}, \quad \tau^{1/2}\Pi = A \cdot \Omega$$

(91)

where $\Pi = \{\Pi_1, \cdots, \Pi_{2n}\}^T$, $\tilde{\Gamma} = \{\tilde{\Gamma}_1, \cdots, \tilde{\Gamma}_{2n}\}^T$, and $\Omega = \{\Omega_1, \cdots, \Omega_{2n}\}^T$, with vector $\Pi_o$ consisting of values of function $\Pi_o(\xi)$ evaluated at the element mid-points located at $\xi_i$, $i = 1, \cdots, 2n$. Also, $A$ denotes the elasticity matrix linking the net pressure to the crack opening and $B$ is the porous flow matrix linking the fluid pressure to the leak-off. These two equations are then combined with a discretized Reynolds equation (52) to yield a nonlinear system of equations to be solved for $\Omega$.

7.2. Elasticity

As a consequence of assuming that $\Omega(\xi)$ is piecewise constant, integral equation (54) simplifies to

$$\Pi(\xi) = -\frac{1}{4\pi\tau^{1/2}} \sum_{j=1}^{2n} \Omega_j \int_{\xi_j - h}^{\xi_j + h} \frac{d\zeta}{(\xi - \zeta)^2}$$

(92)

or after integration to

$$\Pi(\xi) = -\frac{1}{4\pi\tau^{1/2}} \sum_{j=1}^{2n} \Omega_j \left( \frac{1}{\xi - \xi_j - h} - \frac{1}{\xi - \xi_j + h} \right)$$

(93)

Hence, evaluating $\Pi_i = \Pi(\xi_i)$ from (93) yields

$$\tau^{1/2}\Pi_i = \sum_{j=1}^{2n} A_{ij} \Omega_j, \quad i = 1, \cdots, 2n$$

(94)

where

$$A_{ij} = -\frac{1}{2\pi} \frac{h}{(\xi_i - \xi_j)^2 - h^2}$$

(95)
To improve accuracy of the solution, the self-coefficient of the two edge elements \((i = 1 \text{ and } i = 2n)\) is increased by \(1/24 h\) [Gordeliy and Detournay, 2011], following the theoretical argument put forward by Ryder and Napier [1985].

### 7.3. Porous Medium Flow

With \( \Gamma(\xi) \) also assumed to be piecewise constant, the integral equation (53) simplifies to

\[
\Pi(\xi) - \Pi_o(\xi) = -\frac{\tau^{1/2}}{2\pi} \sum_{i=1}^{2n} \tilde{r}_j \int_{\xi_j - h}^{\xi_j + h} \ln(|\xi - \zeta|) d\zeta \tag{96}
\]

Hence, evaluating \( \Pi_i = \Pi(\xi_i) \) from (96) and \( \Pi_{oi} = \Pi_o(\xi_i) \) from (55) yields

\[
\Pi_i - \Pi_{oi} = \tau^{1/2} \sum_{i=1}^{2n} B_{ij} \tilde{r}_j, \quad i = 1, \ldots, 2n \tag{97}
\]

where

\[
B_{ij} = \frac{1}{2\pi} \left[ 2h + (\xi_i - \xi_j - h) \ln |\xi_i - \xi_j - h| - (\xi_i - \xi_j + h) \ln |\xi_i - \xi_j + h| \right]. \tag{98}
\]

### 7.4. Lubrication

Discretizing Reynolds equation (52) using a finite volume approach leads to

\[
C(\Omega) \cdot \Pi = \tau \tilde{\Gamma}, \tag{99}
\]

where the matrix \( C(\Omega) \) is given by

\[
C = \frac{1}{4h^2} \begin{bmatrix}
-K_{3/2} & K_{3/2} & & & \\
K_{3/2} & -(K_{3/2} + K_{5/2}) & K_{5/2} & & \\
& \ddots & \ddots & \ddots & \\
& & K_{2n-3/2} & -(K_{2n-3/2} + K_{2n-1/2}) & K_{2n-1/2} \\
& & & K_{2n-1/2} & -K_{2n-1/2}
\end{bmatrix} \tag{100}
\]

with

\[
K_{i-1/2} = (\Omega_{i-1}^3 + \Omega_{i}^3)/2. \tag{101}
\]

### 7.5. Discrete System of Equations

In summary, the discrete solution at time \( \tau \) is governed by the following system of algebraic equations

\[
C(\Omega) \cdot \Pi = \tau \tilde{\Gamma} \tag{102}
\]

\[
\Pi - \Pi_o = \tau^{1/2} B \cdot \tilde{\Gamma} \tag{103}
\]

\[
\tau^{1/2} \Pi = A \cdot \Omega \tag{104}
\]
Combining (102)-(104) to eliminate \( \Pi \) and \( \tilde{\Gamma} \) results in a system of equations in terms of \( \Omega \) only
\[
\left( I - \tau^{-1/2} B \cdot C(\Omega) \right) \cdot A \cdot \Omega = \tau^{1/2} \Pi_0,
\]
(105)
where \( I \) denotes the identity matrix of size \( 2n \). The system of \( 2n \) nonlinear algebraic equations (105) can be solved for \( \Omega \), using a fixed-point algorithm. Arrays \( \Pi \) and \( \tilde{\Gamma} \) are then readily deduced from \( \Omega \) using (102) and (104).

8. Results

Figures 4 and 5 show the profiles of aperture and net pressure in the \( R \)-regime and compare the analytical solutions for \( \Omega_r(\xi) \) and \( \Pi_r(\xi) \) given by (79) with the transient solution computed numerically at \( \tau = 10^{-2} \). The numerical solution was calculated using 200 source and displacement discontinuity elements. Since \( \Omega \sim \tau^{-1/2} \) at small time, \( \Omega \) is multiplied by \( 10^{-1} \) to enable comparison between the numerical and analytical solutions. In contrast, \( \Pi \) does not depend on time at small \( \tau \). These figures show the high quality of the numerical solutions and confirm that the solution is still in the \( R \)-regime at \( \tau = 10^{-2} \).

Figures 6 and 7 provide a similar comparison between the \( F \)-vertex solution (computed numerically using a dedicated algorithm) and the transient solution computed at time \( \tau = 10^7 \). The transient solution was calculated using 200 source and displacement discontinuity elements while the \( F \)-vertex solution was obtained using 40 elements. Now, \( \Omega \) is multiplied by \( \tau^{-1/4} \) and \( \Pi \) by \( \tau^{1/4} \) to enable comparison between the two solutions. These figures show again the high quality of the numerical solution and confirm that the solution is in the \( F \)-regime at \( \tau = 10^7 \).

The changing nature of the solution with time is illustrated in Figs. 8 and 9, which show profiles of \( \Omega(\xi) \) and \( \Pi(\xi) \) at time \( \tau = 10^{-2}, 1, 10^2 \) in a time-independent scaling. We can observe a large spatial variation of the net pressure at \( \tau = 10^{-2} \) compared to \( \Pi(\xi) \) at \( \tau = 10^2 \), but a small aperture at \( \tau = 10^2 \) compared to \( \Omega(\xi) \) at \( \tau = 10^2 \).

9. Conclusions

This paper has described a KGD-type model to simulate for the propagation of a hydraulic fracture in very permeable rocks. The model is based on the key assumptions that the volume of fluid stored in the hydraulic fracture is negligibly small compared to the injected volume and that the crack is propagating within a domain where the hydraulic fields are quasi-stationary. Scaling of the equations indicates that the solution depends only on a dimensionless time \( \tau \), as all the physical parameters defining the problem are absorbed in the scales. A peculiarity of this problem is the extreme sensitivity of the timescale on a dimensionless injection rate.

Two asymptotic regimes bound the solution: the rock-flow dominated regime or \( R \)-regime at small time (approximately \( \tau \lesssim 10^{-2} \)), and the fracture-flow dominated regime or \( F \)-regime at large time (approximately \( \tau \gtrsim 10^2 \)). The fracture has a zero conductivity and is thus hydraulically invisible in the \( R \)-regime, while it has a large conductivity in the
Figure 4: Comparison between $\mathcal{R}$-vertex opening $\Omega_r(\xi)$ computed either directly or deduced from the transient calculation at time $\tau = 10^{-2}$.

Figure 5: Comparison between $\mathcal{R}$-vertex net pressure $\Pi_r(\xi)$ computed either directly or deduced from the transient calculation at time $\tau = 10^{-2}$. 
Figure 6: Comparison between $I$-vertex opening $\Omega_f(\xi)$ computed either directly or deduced from the transient calculation at time $\tau = 10^7$.

Figure 7: Comparison between $I$-vertex net pressure $\Pi_f(\xi)$ computed either directly or deduced from the transient calculation at time $\tau = 10^7$. 

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Figure 8: Opening profile in time scaling at $\tau = 10^{-2}, 1, 10^2$.

Figure 9: Net pressure profile in time scaling at $\tau = 10^{-2}, 1, 10^2$. 
Thus all the injected fluid enters the reservoir via the borehole at small time, but via the crack at large time. Furthermore, the injection pressure increases as $\ln \tau$ in the $R$-regime, but decreases as $\tau^{-1/4}$ in the $F$-regime. The peak injection pressure, which should not be interpreted as the breakdown pressure, takes place in the transition between the two regimes.

This study suggests that the hydraulic aspects are important, likely critical, in this class of problems, which have so far been viewed only through the prism of strength.

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P J van den Hoek and J D Mclennan. Hydraulic Fracturing in Produced Water Reinjection. Proc. Workshop on Three-Dimensional and Advanced Hydraulic Fracture Modeling, pages 88–109, 2000.
A. Numerical Scheme for $F$-Vertex Solution

The two fields $\Omega_f(\xi)$ and $\Pi_f(\xi)$ are computed by solving numerically the elasticity equation (84), the integrated Reynolds equation (89), and the propagation criterion (49) expressed in terms $\Pi_f(\xi)$. Discretization of (84), (89) and (49) leads to system of algebraic equations consists of $2n$ linear equations of elasticity

$$\Pi = A \cdot \Omega$$

(106)

$2n - 1$ non-linear equations formulated by discretizing the integrated Reynolds equation at the mid-nodes, and 1 linear equation expressing the propagation criterion. The combined discretized non-linear equations and propagation criterion can be written as

$$D(\Omega) \cdot \Pi = -\Psi,$$

(107)

where the matrix $D(\Omega)$ is given by

$$D = \frac{1}{2h} \begin{bmatrix}
-K_{3/2} & K_{3/2} \\
-K_{5/2} & K_{5/2} \\
& \ddots & \ddots \\
W_1 & W_2 & \cdots & W_{2n-1} & K_{2n-1/2} & K_{2n-1/2} & W_{2n}
\end{bmatrix}, \quad \Psi = \begin{bmatrix}
\Psi_{f,1/2} \\
\Psi_{f,3/2} \\
\vdots \\
\Psi_{f,2n-1/2} \\
0
\end{bmatrix}$$

(108)

with

$$K_{i-1/2} = (\Omega_{i,i-1}^2 + \Omega_{i,i}^2)/2$$

(109)

and

$$W_i = \arcsin(\xi_i + a) - \arcsin(\xi_i - a)$$

(110)

Note that the flux at the common boundary of elements $n$ and $n + 1$, $\Psi_{f,n+1/2} = 0$, as it is the average of a discontinuous flux of opposite sign on both sides of the origin.

Combining (106) and (107) leads to a system of $2n$ non-linear algebraic equations

$$D(\Omega) \cdot A \cdot \Omega = -\Psi$$

to be solved for $\Omega$. 

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