NONLOCAL p-KIRCHHOFF EQUATIONS WITH SINGULAR AND CRITICAL NONLINEARITY TERMS

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Abstract. The objective of this work is to investigate a nonlocal problem involving singular and critical nonlinearities:

\[
\begin{cases}
   (|u|^{p}_{s,p})^{\sigma-1}(-\Delta)_s^p u = \frac{u^\gamma}{|x|^\gamma} + u^{p^*-1} \\
   u > 0, \quad & \text{in } \Omega, \\
   u = 0, \quad & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with the smooth boundary \(\partial \Omega\), 0 < s < 1 < p < \infty, N > sp, 1 < \sigma < \frac{p_s^*}{p}, \text{ with } p_s^* = \frac{Np}{N-ps}, (-\Delta)_s^p \text{ is the nonlocal } p\text{-Laplace operator and } |u|_{s,p} \text{ is the Gagliardo } p\text{-seminorm.}

We combine some variational techniques with a truncation argument in order to show the existence and the multiplicity of positive solutions to the above problem.

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1. Introduction

In this paper, we shall consider the following singular critical nonlocal problem:

\[
\begin{cases}
\left( [u]_{s,p}^p \right)^{p-1} (-\Delta)^s_p u = \frac{\lambda}{u^\gamma} + u^{p^*_s - 1} & \text{in } \Omega, \\
u > 0, & \text{in } \Omega, \\
u = 0, & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with a smooth boundary \( \partial \Omega \), \( 0 < s < 1 < p < \infty \), \( N > sp \), \( 1 < \sigma < p^*_s/p \), with \( p^*_s = \frac{Np}{N - ps} \), \((-\Delta)^s_p\) is a nonlocal operator defined by

\[
(-\Delta)^s_p u(x) := 2 \lim_{\epsilon \to 0} \int_{\Omega \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy, \quad x \in \Omega,
\]

where \( B_\epsilon(x) := \{ y \in \Omega : |x - y| < \epsilon \} \), and \([u]_{s,p}\) is the Gagliardo \( p \)-seminorm given by

\[
[u]_{s,p}^p := \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy.
\]

Problems of this type describe diffusion processes in heterogeneous or complex medium (anomalous diffusion) due to random displacements executed by jumpers that are able to walk to neighbouring nearby sites. These problems are also due to excursions to remote sites by way of Lévy flights, they can be used in modelling turbulence, chaotic dynamics, plasma physics and financial dynamics. For more details, see [1, 6] and references therein.

For \( p = 2 \), problem (1.1) has been investigated by many authors in order to show the existence and the multiplicity of solutions. For further details, one can refer the reader to [8, 9, 11, 17, 18, 19, 20, 21, 22, 31] and the references therein.

For \( s = 1 \), the local setting case has been extensively investigated in the recent past. The existence, the uniqueness, the multiplicity of weak solutions and regularity of solutions have been studied in [5, 7, 10, 13, 14, 16, 26, 28, 30, 32] and the references therein.

Motivated by the previous results, and the work of Fiscella [11], who established the existence and the multiplicity of positive solutions using some variational methods combined with an appropriate truncation. The aim of this work is to extend the multiplicity results to a more general non-local problem. More precisely, we shall establish the following result.

**Theorem 1.1.** Suppose that the parameters in problem (1.1) satisfy the following two conditions

\[0 < 1 - \gamma < 1 < ps < p^*_s \quad \text{and} \quad 1 < \sigma < p^*_s/p,\]

Then there exists a parameter \( \lambda_0 > 0 \) such that for every \( \lambda \in (0, \lambda_0) \), problem (1.1) has at least two positive solutions.
2. Preliminaries

This section is devoted to basic definitions, notations, and function spaces that will be used in the forthcoming sections. For for the other background material we refer the reader to [24, 27]. We begin by defining the fractional Sobolev space

\[ W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : u \text{ measurable}, \left| u \right|_{s,p} < \infty \right\}, \]

with the Gagliardo norm

\[ \left|\left| u \right|\right|_{s,p} = \left( \left|\left| u \right|\right|_p^p + \left|\left| u \right|\right|_{s,p}^p \right)^{\frac{1}{p}}. \]

Denote

\[ Q := \mathbb{R}^{2N} \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)) \]

and define the space

\[ X := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ Lebesgue measurable} : u \mid_{\Omega} \in L^p(\Omega) \text{ and } \frac{\left| u(x) - u(y) \right|^p}{|x - y|^{N+sp}} \in L^p(Q) \right\} \]

with the norm

\[ \left\| u \right\|_X = \left\| u \right\|_{L^p(\Omega)} + \left( \int_Q \frac{\left| u(x) - u(y) \right|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{1/p}. \]

Throughout this paper, we shall consider the space

\[ X_0 := \{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}, \]

with the norm

\[ \left\| u \right\| := \left( \int_Q \frac{\left| u(x) - u(y) \right|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{1/p} \]

and the scalar product

\[ \langle u, \varphi \rangle_{X_0} := \int_{\mathbb{R}^N} \frac{\left| u(x) - u(y) \right|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \, dx \, dy. \]

We define a weak solution to problem (1.1) as follows:

**Definition 2.1.** We say that \( u \in X_0 \) is a weak solution of problem (1.1) if for all \( \varphi \in X_0 \), one has

\[ \begin{align*}
\left( [u]_{s,p} \right)^{\sigma-1} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \, dx \, dy \\
= \lambda \int_{\Omega} (u^+)^{1-\gamma} \varphi \, dx + \int_{\Omega} (u^+)^{p^*_s-1} \, dx.
\end{align*} \]

In order to find solutions of problem (1.1), we shall use the variational approach. More precisely, we shall find two distinct critical points of the energy functional \( J_\lambda : X_0 \to (-\infty, \infty] \) defined by

\[ J_\lambda(u) := \frac{1}{p^*} \left\| u \right\|^{p^*} - \frac{\lambda}{1-\gamma} \int_{\Omega} (u^+)^{1-\gamma} \, dx - \frac{1}{p^*_s} \int_{\Omega} (u^+)^{p^*_s} \, dx. \]

Now, we prove the following result.
Lemma 2.1. There exist $\rho \in (0, 1]$, $\lambda_1$ and $\alpha > 0$ such that for every $\lambda \in (0, \lambda_1]$, we have

$$J(\lambda u) \geq \alpha \quad \text{for all } u \in X_0 \text{ with } \|u\| = \rho.$$ 

Moreover, the following holds

$$m\lambda := \inf \{J(\lambda u) : u \in B_\rho\} < 0,$$

where $B_\rho := \{u \in X_0 : \|u\| \leq \rho\}$.

Proof. Let $\lambda > 0$. Then by virtue of the Hölder inequality and the Sobolev embedding theorem, we get for any $u \in X_0$

$$\int_\Omega u^{1-\gamma} dx \leq |\Omega| \frac{p^*-1+\gamma}{p^*} \|u\|_{p^*}^{1-\gamma} \leq C \|u\|^{1-\gamma}.$$ 

So from the Sobolev embedding, we obtain

$$J(\lambda u) = \frac{1}{p^*} \|u\|^{p^*} - \frac{\lambda}{1-\gamma} \int_\Omega u^{1-\gamma} dx - \frac{1}{p^*} \int_\Omega \varphi u^{p^*} dx \geq \|u\|^{1-\gamma} \left( \varphi(\|u\|) - \frac{CA}{1-\gamma} \right)$$

where $\varphi(t) = \frac{1}{p^*} t^{p^*-1+\gamma} - \frac{c_1}{p^*} t^{p^*-1+\gamma}$. Since $1-\gamma < 1 < p^* < p^*_s$, we find $\rho \in (0, 1)$ sufficiently small and satisfying

(2.3) $\max_{0 < t < 1} \varphi(t) = \varphi(\rho)$.

Put

(2.4) $\lambda_1 := \frac{(1-\gamma) \varphi(\rho)}{2C}.$

Thus, for all $u \in X_0$ with $\|u\| = \rho$ and all $\lambda \leq \lambda_1$, one has

$$J(\lambda u) \geq C \rho^{1-\gamma} (2\lambda_1 - \lambda) > \frac{C\rho^{1-\gamma}}{1-\gamma} \lambda_1 = \alpha > 0.$$ 

Moreover, since $1-\gamma < 1 < p^* < p^*_s$, it follows that for $u \in X_0$ with $u^+ \neq 0$ and for $t \in (0, 1)$ sufficiently small, one has

$$J(\lambda tu) = \frac{tp^*}{p^*} \|u\|^{p^*} - \frac{\lambda t^{1-\gamma}}{1-\gamma} \int_\Omega (u^+)^{1-\gamma} dx - \frac{tp^*_s}{p^*_s} \int_\Omega (u^+) t u^{p^*} dx < 0.$$ 

Lemma 2.2. For every $\lambda \in (0, \lambda_1]$, problem (1.1) has a positive solution $u_\lambda \in X_0$ with $J(\lambda u_\lambda) \leq 0.$
Proof. Let $\rho$ and $\lambda_1$ be the constants given respectively by (2.3) and (2.4). Let $\{u_k\} \subset B_\rho$ be a minimizing sequence for $m_\lambda$, i.e.

$$\lim_{k \to \infty} J_\lambda(u_k) = m_\lambda.$$ 

As $\{u_k\}$ is bounded, for any $1 \leq r < p_s^*$, one has

$$\begin{cases}
u_k \rightharpoonup u_\lambda \text{ weakly in } X_0, \\
u_k \to u_\lambda \text{ weakly in } L^{p_s^*}(\Omega), \\
u_k \to u_\lambda \text{ strongly in } L^r(\Omega), \\
u_k \to u_\lambda \text{ a.e. in } \Omega.
\end{cases} \quad (2.5)$$

By the Hölder inequality, we get for all integers $k$,

$$\left| \int \Omega (u_k^+)^{1-\gamma} dx - \int \Omega (u_\lambda^+)^{1-\gamma} dx \right| \leq \int \Omega |u_k^+ - u_\lambda^+|^{1-\gamma} dx \leq |\Omega| \frac{p-1-\gamma}{p} \|u_k^+\|_p^{\gamma} - \|u_\lambda^+\|_p^{\gamma}.$$ 

(2.6)

Combining (2.5) and (2.6), we obtain

$$\lim_{k \to \infty} \int \Omega (u_k^+)^{1-\gamma} dx = \int \Omega (u_\lambda^+)^{1-\gamma} dx.$$ 

(2.7)

Put $\widetilde{u}_k := u_k - u_\lambda$. Then, by invoking the Brezis-Lieb Lemma [4], we obtain

$$\lim_{k \to \infty} \|u_k\|_p^p - \|\widetilde{u}_k\|_p^p = \|u_\lambda\|_p^p$$

and

$$\lim_{k \to \infty} \|u_k\|_{p_s^*}^p - \|\widetilde{u}_k\|_{p_s^*}^p = \|u_\lambda\|_{p_s^*}^p.$$ 

Since $\{u_k\} \subset B_\rho$, it follows that (2.8) implies that for $k$ large enough, $\widetilde{u}_k \in B_\rho$. So, from Lemma 2.1 we deduce that for all $u \in X_0$ with $\|u\| = \rho$,

$$\frac{1}{p\sigma}\|u\|_p^{p\sigma} - \frac{1}{p_s^*}\int \Omega u_{p_s^*}^s dx \geq \alpha > 0,$$

that is, if $\rho \leq 1$ and $k$ large enough,

$$\frac{1}{p\sigma}\|\widetilde{u}_k\|_p^{p\sigma} - \frac{1}{p_s^*}\int \Omega \widetilde{u}_k_{p_s^*}^s dx > 0,$$

(2.9)

since $\{u_k\}$ is a minimizing sequence. Hence, by combining (2.7)-(2.9), we obtain for $k$ large enough,

$$m_\lambda = J_\lambda(u_k) + o(1)$$

$$= \frac{1}{p\sigma}\|\widetilde{u}_k + u_\lambda\|_p^{p\sigma} - \frac{1}{1-\gamma}\int \Omega ((\widetilde{u}_k + u_\lambda^+)^{1-\gamma} dx - \frac{1}{p_s^*}\int \Omega ((\widetilde{u}_k + u_\lambda^+)^{p_s^*} dx + o(1)$$

$$\geq \frac{1}{p\sigma}\|\widetilde{u}_k\|_p^{p\sigma} + \frac{1}{p\sigma}\|u_\lambda\|_p^{p\sigma} - \frac{1}{1-\gamma}\int \Omega (u_\lambda^+)^{1-\gamma} dx - \frac{1}{p_s^*}\int \Omega (\widetilde{u}_k^+)^{p_s^*} dx - \frac{1}{p_s^*}\int \Omega (u_\lambda^+)^{p_s^*} dx + o(1)$$

$$\geq J_\lambda(u_\lambda) + \frac{1}{p\sigma}\|\widetilde{u}_k\|_p^{p\sigma} - \frac{1}{p_s^*}\int \Omega (\widetilde{u}_k^+)^{p_s^*} dx + o(1)$$

$$\geq J_\lambda(u_\lambda) + o(1)$$

$$\geq m_\lambda,$$

Hence, $J_\lambda(u_\lambda) = m_\lambda < 0$. 

5 Critical Singular Kirchhoff type equations
Now, let us prove that $u_\lambda$ is a positive solution to problem (1.1). Our proof uses similar techniques as [12]. Consider $\phi \in X_0$ and $0 < \epsilon < 1$. Let $\Psi \in X_0$ be defined by $\Psi := (u_\lambda + \epsilon \phi)^+$ with $(u_\lambda + \epsilon \phi)^+ := \max\{u_\lambda + \epsilon \phi, 0\}$. Let $\Omega_\epsilon := \{u_\lambda + \epsilon \phi \leq 0\}$ and $\Omega^\epsilon := \{u_\lambda + \epsilon \phi < 0\}$. Put $\Theta_\epsilon := \Omega_\epsilon \times \Omega^\epsilon$. Since $u_\lambda$ is a local minimizer for $J_\lambda$, replacing $\varphi$ with $\Psi$ in (2.1), one gets

\[
0 \geq ([u_{\lambda,s,p}]^p)^{-1} \int_{\Omega} \frac{|u_{\lambda}(x) - u_{\lambda}(y)|^{p-2}(u_{\lambda}(x) - u_{\lambda}(y))(\psi(x) - \psi(y))}{|x-y|^{N+ps}}dx dy
- \lambda \int_{\Omega} \left(\int_{\Omega} (u_\lambda^+)^{-\gamma} \Psi dx - \int_{\Omega} (u_\lambda^+)^{p^*} \Psi \right) dx
= ([u_{\lambda,s,p}]^p)^{-1} \int_{\Omega} \frac{|u_{\lambda}(x) - u_{\lambda}(y)|^{p-2}(u_{\lambda}(x) - u_{\lambda}(y))(u_{\lambda} + \epsilon \phi)(x) - (u_{\lambda} + \epsilon \phi)(y)}{|x-y|^{N+ps}}dx dy
- \int_{\{x,y\in \Omega^\epsilon\}} \left(\lambda(u_\lambda^+)^{-\gamma}(u_{\lambda} + \epsilon \phi) + (u_\lambda^+)^{p^*} (u_{\lambda} + \epsilon \phi)\right) dx
= ([u_{\lambda,s,p}]^p)^{-1} \int_{\Omega} \frac{|u_{\lambda}(x) - u_{\lambda}(y)|^{p-2}(u_{\lambda}(x) - u_{\lambda}(y))(u_{\lambda} + \epsilon \phi)(x) - (u_{\lambda} + \epsilon \phi)(y)}{|x-y|^{N+ps}}dx dy
- \int_{\{x,y\in \Theta_\epsilon\}} \left(\lambda(u_\lambda^+)^{-\gamma}(u_{\lambda} + \epsilon \phi) + (u_\lambda^+)^{p^*} (u_{\lambda} + \epsilon \phi)\right) dx
= ([u_{\lambda,s,p}]^p)^{-1} \|u_{\lambda}\|_p - \lambda \int_{\Omega} (u_\lambda^+)^p dx - \lambda \int_{\Omega} (u_\lambda^+)^{p^*} dx - \int_{\Omega} \left(\lambda(u_\lambda^+)^{-\gamma}\phi + (u_\lambda^+)^{p^*}\phi\right) dx
+ \epsilon([u_{\lambda,s,p}]^p)^{-1} \int_{\Omega} \frac{|u_{\lambda}(x) - u_{\lambda}(y)|^{p-2}(u_{\lambda}(x) - u_{\lambda}(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}}dx dy
- ([u_{\lambda,s,p}]^p)^{-1} \int_{\{x,y\in \Theta_\epsilon\}} \frac{|u_{\lambda}(x) - u_{\lambda}(y)|^{p-2}(u_{\lambda}(x) - u_{\lambda}(y))(u_{\lambda} + \epsilon \phi)(x) - (u_{\lambda} + \epsilon \phi)(y)}{|x-y|^{N+ps}}dx dy
- \int_{\{x,y\in \Theta_\epsilon\}} \left(\lambda(u_\lambda^+)^{-\gamma}(u_{\lambda} + \epsilon \phi) + (u_\lambda^+)^{p^*} (u_{\lambda} + \epsilon \phi)\right) dx
= \epsilon([u_{\lambda,s,p}]^p)^{-1} \int_{\Omega} \frac{|u_{\lambda}(x) - u_{\lambda}(y)|^{p-2}(u_{\lambda}(x) - u_{\lambda}(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}}dx dy
- \epsilon \int_{\Omega} \left(\lambda(u_\lambda^+)^{-\gamma}\phi + (u_\lambda^+)^{p^*}\phi\right) dx
- ([u_{\lambda,s,p}]^p)^{-1} \int_{\{x,y\in \Theta_\epsilon\}} \frac{|u_{\lambda}(x) - u_{\lambda}(y)|^{p-2}(u_{\lambda}(x) - u_{\lambda}(y))(u_{\lambda} + \epsilon \phi)(x) - (u_{\lambda} + \epsilon \phi)(y)}{|x-y|^{N+ps}}dx dy
- \int_{\{x,y\in \Theta_\epsilon\}} \left(\lambda(u_\lambda^+)^{-\gamma}(u_{\lambda} + \epsilon \phi) + (u_\lambda^+)^{p^*}(u_{\lambda} + \epsilon \phi)\right) dx
The equality holds if we change \( \phi \).

Associated to problem (3.1), we consider the functional defined by

\[
\text{Lemma 3.1.}
\]

Moreover, there exists \( e \) such that

\[
\text{since the measure } \Omega_\varepsilon \text{ goes to zero as } \varepsilon \to 0^+. \text{ We deduce that,}
\]

\[
\left(\begin{array}{c}
[(u_\lambda^p)_{s,p}]^\sigma - 1 \\
(x,y) \in \Theta_\varepsilon
\end{array}\right)
\int |u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))
\]

\[
\frac{|x - y|^{N + ps}}{dxdy} 
\]

\[
\text{as } \varepsilon \to 0^+. \text{ We divide by } \varepsilon \text{ and passing to the limit as } \varepsilon \to 0^+, \text{ one has}
\]

\[
\left(\begin{array}{c}
[(u_\lambda^p)_{s,p}]^\sigma - 1 \\
(x,y) \in \Theta_\varepsilon
\end{array}\right)
\int |u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))
\]

\[
\frac{|x - y|^{N + ps}}{dxdy} 
\]

\[- \int \left( \frac{\lambda(u_\lambda^+)^{-\gamma}}{(u_\lambda^+)^{p^* - 1}} \right) \frac{dxdy}{x - y} \geq 0.
\]

The equality holds if we change \( \phi \) by \( -\phi \). So we deduce that \( u_\lambda \) is a non-negative solution of problem (1.1). \( \square \)

3. A perturbed problem

Since \( J_\lambda \) is not Fréchet differentiable due to the singular term, we cannot apply the usual variational theory to the functional energy. Therefore, in order to establish the existence of a second solution, we introduce the following perturbed problem

\[
\left\{ \begin{aligned}
[(u_\lambda^p)_{s,p}]^\sigma - 1 & - \Delta_p u = \frac{\lambda}{(u_\lambda^+)^{\gamma + 1}} + (u_\lambda^+)^{p^* - 1} \quad \text{in } \Omega, \\ u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{aligned} \right.
\]

Associated to problem (3.1), we consider the functional \( J_{n,\lambda} \colon X_0 \to \mathbb{R} \) defined by

\[
J_{n,\lambda}(u) := \frac{1}{p^*} \|u\|^{p^*} - \frac{1}{1 - \gamma} \int_\Omega \left( (u_\lambda^+)^{1-\gamma} - \left( \frac{1}{n} \right)^{1-\gamma} \right) dx - \frac{1}{p^*} \int_\Omega (u_\lambda^+)^{p^*} dx.
\]

It is clear that \( J_{n,\lambda} \) is Fréchet differentiable, and for all \( \varphi \in X_0 \), we have

\[
< J^\prime_{n,\lambda}(u), \varphi > = \|u\|^{p^* - 2} < u, \varphi > - \lambda \int_\Omega \frac{\varphi}{(u_\lambda^+)^{1-\gamma}} dx - \int_\Omega (u_\lambda^+)^{p^* - 1} \varphi dx.
\]

(3.2)

**Lemma 3.1.** Let \( \rho \in (0, 1] \), \( \lambda_1 \) and \( \alpha \) be the constants given by Lemma 2.1.

Then for any \( \lambda \in (0, \lambda_1) \), one has

\[
J_{n,\lambda}(u) \geq \alpha, \quad \text{for all } u \in X_0 \text{ with } \|u\| \leq \rho.
\]

Moreover, there exists \( e \in X_0 \), with \( \|e\| > \rho \) and \( J_{n,\lambda}(e) < 0 \).
Therefore, Lemma 2.1 implies that the first part of Lemma 3.1 has been proved.

Now, let $c$ and $\delta$ be such that $c < C_\lambda$ for any $\lambda > 0$.

Proof. Let $\{u_k\} \subset X_0$ be a (PS) minimizing sequence for the functional $J_{n,\lambda}$ at level $c \in \mathbb{R}$, that is

\begin{equation}
J_{n,\lambda}(u_k) \to c \quad \text{and} \quad J'_{n,\lambda}(u_k) \to 0 \quad \text{as} \quad k \to \infty.
\end{equation}

Then by the Sobolev embedding and the Hölder inequality, there exist $\epsilon > 0$ and $C > 0$ satisfying

\begin{equation}
\|u_k\| + o(1) \geq J_{n,\lambda} - \frac{1}{p_s^*} - J'_{n,\lambda}(u_k), u_k >
\end{equation}

Since $1 - \gamma < 1 < p\sigma < p_s^*$, it follows that $\{u_k\}$ is bounded. Moreover, $\{u_k^\pm\}$ is bounded in $X_0$. So from (3.4), we have

\begin{equation}
\lim_{k \to \infty} < J'_{n,\lambda}(u_k), u_k > = \lim_{k \to \infty} \|u_k\|^{p\sigma-1} < u_k, -u_k^\pm >.
\end{equation}

On the other hand, by an elementary inequality

\begin{equation}
(a - b)(a^- - b^-) \leq -(a^- - b^-)^2
\end{equation}
Moreover, for a fixed
\[ (3.7) \]
and 
\[ \mu \]
It is easy to see that if \( \sigma \cdot \)
Now, using (3.8) and (3.9), we can deduce that:
\[ \text{From (3.5), we have} \]
\[ (3.5) \]
\[ \text{It follows from the above assertion that} \]
\[ (3.6) \]
\[ \text{we have} \]
\[ (3.8) \]
\[ \lim_{k \to \infty} \int_\Omega \frac{u_k - u}{(u_k + \frac{1}{n})^\gamma} \] 
\[ = 0 \]
Hence, the Bresis-Lieb Lemma \cite{1} yields
\[ (3.9) \]
\[ \|u_k\|^p = \|u_k - u\|^p + \|u\|^p + o(1) \text{ and } \|u_k\|^{p_2^*} = \|u_k - u\|^{p_2^*} + \|u\|^{p_2^*} + o(1) \]
Now, using (3.8) and (3.9), we can deduce that:
\[ o(1) = \lambda J_{n,\lambda}'(u_k), u_k - u \]
\[ = \|u_k\|^{p(\sigma - 1)} - u_k, u_k - u - \lambda \int_\Omega \frac{u_k - u}{(u_k + \frac{1}{n})^\gamma} dx - \int_\Omega u_k^{p_2^* - 1}(u_k - u) dx \]
\[ = \mu^{p(\sigma - 1)}(\|u_k\|^p - \|u\|^p) - \|u_k\|^{p_2^*} + \|u\|^{p_2^*} + o(1) \]
\[ = \mu^{p(\sigma - 1)}\|u_k - u\|^p - \|u_k - u\|^{p_2^*} + o(1). \]
Therefore,
\[
\mu^{p(\sigma-1)} \lim_{k \to \infty} \|u_k - u\|^p = \lim_{k \to \infty} \|u_k - u\|^{p^*_s} = l.
\]
Since \( \mu > 0 \), if \( l = 0 \), we obtain that \( u_k \to u \) in \( X_0 \) and the proof is complete.

Now, let us prove that \( l = 0 \). Proceeding by contradiction, suppose that \( l > 0 \). Then from (3.10) and the Sobolev embedding, we get
\[
S \mu^{p(\sigma-1)} p \leq p^*_s,
\]
that is
\[
\mu^{p(\sigma-1)} p - p \geq S \mu^{p(\sigma-1)}.
\]
On the other hand, by combining (3.9) and (3.10), we obtain
\[
\mu^{p(\sigma-1)} (\mu^p - \|u\|^p) = p^*_s
\]
that is,
\[
l = \mu \frac{p(\sigma-1)}{p^*_s} (\mu^p - \|u\|^p) \frac{N-p\sigma}{Np}.
\]
So using (3.12), we get
\[
p^*_s - p = \mu \frac{p(\sigma-1)(\sigma-1)}{p_s^*} (\mu^p - \|u\|^p) \frac{(N-p\sigma)(p^*_s - p)}{Np} \geq S \mu^{p(\sigma-1)} p.
\]
We deduce that
\[
\mu^\frac{2}{N} \geq (\mu^p - \|u\|^p) \frac{(N-p\sigma)(p^*_s - p)}{Np} \geq S \left( \mu^{p(\sigma-1)} \right) \frac{N-p\sigma}{Np}.
\]
Since \( 1 < \sigma < \frac{p^*_s}{p} \), it follows that \( ps\sigma - N(\sigma - 1) > 0 \). So
\[
(3.13)
\]
\[
\mu^p \geq S^{\frac{N}{ps\sigma - N(\sigma - 1)}}.
\]
Now, the fact that \((u^+ + \frac{1}{n})^{1-\gamma} - (\frac{1}{n})^{1-\gamma} \leq (u^+)^{1-\gamma}\) implies that for all integers \( k \) and \( n \) we have
\[
J_{n,\lambda}(u_k) - \frac{1}{p^*_s} < J_{n,\lambda}(u_k), u_k \geq (\frac{1}{p^*\sigma} - \frac{1}{p^*_s}) \|u_k\|^{p^*\sigma} - \lambda(\frac{1}{1-\gamma} + \frac{1}{p^*_s}) \int_\Omega u_k^{1-\gamma} \, dx.
\]
So from (3.9), (3.13), the Hölder inequality and the Young inequality, if \( k \) tends to infinity, we get
\[
c \geq (\frac{1}{p^*\sigma} - \frac{1}{p^*_s}) (\mu^{p^*\sigma} + \|u\|^{p^*\sigma}) - \lambda(\frac{1}{1-\gamma} + \frac{1}{p^*_s}) \|\Omega\|^{p^*_s-1+\gamma} \frac{p^*_s-1+\gamma}{p^*_s} S^{-1-\gamma} \|u\|^{1-\gamma}
\]
\[
\geq (\frac{1}{p^*\sigma} - \frac{1}{p^*_s}) (\mu^{p^*\sigma} + \|u\|^{p^*\sigma}) - (\frac{1}{p^*\sigma} - \frac{1}{p^*_s}) \|u\|^{p^*\sigma}
\]
\[
\geq (\frac{1}{p^*\sigma} - \frac{1}{p^*_s}) \frac{N_p}{p^*\sigma - N(\sigma - 1)} - (\frac{1}{p^*\sigma} - \frac{1}{p^*_s})^{1-\gamma} \|\Omega\|^{p^*_s-1+\gamma} \frac{p^*_s-1+\gamma}{p^*_s} S^{-1-\gamma} \|u\|^{1-\gamma}
\]
\[
= C_\lambda,
\]
which is a contradiction. \( \square \)
4. Existence of an upper bound

Under some suitable condition, we shall prove that \( J_{n,\lambda} \) is bounded from above. To this end, we can assume without loss of generality, that \( 0 \in \Omega \) and we fix \( r > 0 \) such that \( B_{4r} \subset \Omega \) where \( B_{4r} := \{ x \in \mathbb{R}^N : |x| < 4r \} \). Let \( \varepsilon > 0 \) and \( \psi \) be the function defined by

\[
\psi_\varepsilon := \frac{\phi U_\varepsilon}{\|\phi U_\varepsilon\|_{p^*_s}},
\]

where \( U_\varepsilon \) is the family of functions (for more details see [25]) and \( \phi \in C^\infty(\mathbb{R}^N, [0, 1]) \) is satisfying

\[
\phi = \begin{cases} 
1 & \text{in } B_r, \\
0 & \text{in } \mathbb{R}^N \setminus B_{2r}
\end{cases}
\]

**Lemma 4.1.** There exist \( \lambda_2 > 0 \) and \( \psi \in E \) satisfying

\[
\sup_{t > 0} J_{n,\lambda}(t\psi) < C \lambda,
\]

for all \( \lambda \in (0, \lambda_1) \).

*Proof.* Let \( \varepsilon > 0 \) and let \( u_\varepsilon \) and \( \psi_\varepsilon \) be as above. Since \( 0 < 1 - \gamma < p\sigma < p^*_s \), it is easy to see that

\[
J_{n,\lambda}(t\psi_\varepsilon) \to -\infty \quad \text{as} \quad t \to \infty.
\]

Thus, there exists \( t_\varepsilon > 0 \) satisfying

\[
J_{n,\lambda}(t_\varepsilon \psi_\varepsilon) = \max_{t \geq 0} J_{n,\lambda}(t\psi_\varepsilon).
\]

From Lemma 2.1 we get \( J_{n,\lambda} \geq \alpha > 0 \). So since the functional \( J_{n,\lambda} \) is continuous, we deduce the existence of two values \( t_0, t_1 > 0 \) satisfying

\[
t_0 < t_\varepsilon < t_1, \quad \text{and} \quad J_{n,\lambda}(t_0 \psi_\varepsilon) = J_{n,\lambda}(t_1 \psi_\varepsilon) = 0.
\]

On the other hand, since \( \|u_\varepsilon\|_{p^*_s} \) is independent from \( \varepsilon \), it follows from [23] that

\[
\|\psi_\varepsilon\|_p \leq \frac{\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N + ps}} \, dx \, dy}{\|\phi u_\varepsilon\|_{p^*_s}} = S + O(\varepsilon^{N_p - 1}).
\]

In fact, for any \( a > 0, b \in [0, 1], p \geq 1, \)

\[
(a + b)^p \leq a^p + (a + 1)^{p-1}b.
\]

We obtain for \( \varepsilon \) small enough,

\[
\|\psi_\varepsilon\|_{ps} \leq (S + O(\varepsilon^{N_p - 1}))^\sigma \leq S^\sigma + O(\varepsilon^{N_p - 1}).
\]

Hence, for any \( \varepsilon > 0 \) sufficiently small, and using the fact that \( t_0 < t_\varepsilon < t_1 \) and \( \|\psi_\varepsilon\|_{ps} = 1 \), we obtain

\[
J_{n,\lambda}(t_\varepsilon \psi_\varepsilon) \leq \left( \frac{t_\varepsilon^{ps}}{\sigma p} S^\sigma - \frac{t_\varepsilon^{p^*_s}}{p^*_s} \right) - \frac{\lambda}{1 - \gamma} \int_\Omega \left( t_0 \psi_\varepsilon + \frac{1}{n} \right)^{1-\gamma} \left( \frac{1}{n} \right)^{1-\gamma} \, dx
\]
Combining this with (4.4), we get
\[ c \]
for some positive constant \( c \).

Since
\[ \max_{t>0} \left( \frac{t^p}{\sigma p} S^\sigma - \frac{t^{p_s}}{p_s} \right) = \frac{1}{\sigma p} - \frac{1}{p_s} \]
\[ \frac{p_s^p}{p^s}, \]

it follows by (4.2) and (4.3) that
\[ J_{n,\lambda}(t_\epsilon \psi_\epsilon) \leq \left( \frac{1}{\sigma p} - \frac{1}{p_s} \right) S^\sigma \frac{p_s^p}{p^s} - \frac{\lambda}{1-\gamma} \int_\Omega \left( (t_\epsilon \psi_\epsilon + \frac{1}{n})^{1-\gamma} - \left( \frac{1}{n} \right)^{1-\gamma} \right) dx \]
\[ \geq c_1 (1-\gamma) \epsilon \]
\[ \int_{x \in \Omega : |x| \leq \epsilon} \left( \frac{1}{|x|^{p'-\epsilon \rho}} \right)^{\frac{N-\rho s}{\rho_s}} dx \]
\[ \geq c_1 (1-\gamma) \epsilon \]
\[ \int_{x \in \Omega : |x| \leq \epsilon} \left( \frac{1}{|x|^{p'-\epsilon \rho}} \right)^{\frac{N-\rho s}{\rho_s}} dx \]
\[ \geq c_1 (1-\gamma) \epsilon \]

We can now deduce that for all \( q > 0 \) small enough, we can establish the existence of \( c_1 > 0 \) satisfying
\[ \int_\Omega \left( (t_\epsilon \psi_\epsilon + \frac{1}{n})^{1-\gamma} - \left( \frac{1}{n} \right)^{1-\gamma} \right) dx \]
\[ \geq c_1 (1-\gamma) \epsilon \]
\[ \int_{x \in \Omega : |x| \leq \epsilon} \left( \frac{1}{|x|^{p'-\epsilon \rho}} \right)^{\frac{N-\rho s}{\rho_s}} dx \]
\[ \geq c_1 (1-\gamma) \epsilon \]

Combining this with (4.4), we get
\[ J_{n,\lambda}(t_\epsilon \psi_\epsilon) \leq \frac{\lambda c_1 \epsilon}{\rho_s} \]
\[ + O(\epsilon^{\frac{N-\rho_s}{\rho_s}}) \]
\[ \geq c_1 (1-\gamma) \epsilon \]
\[ \int_{x \in \Omega : |x| \leq \epsilon} \left( \frac{1}{|x|^{p'-\epsilon \rho}} \right)^{\frac{N-\rho s}{\rho_s}} dx \]
\[ \geq c_1 (1-\gamma) \epsilon \]

for some positive constant \( c_2 \).

Now, let \( \lambda > 0 \) be such that \( C_\lambda > 0 \) for all \( \lambda \in (0, \bar{\lambda}) \), where \( C_\lambda \) is given by (3.3) and let us set
\[ \beta := 1 + \frac{p(p-1)\sigma ((N - ps)(1-\gamma) - p(p-1)q(N - ps)(1-\gamma) + p_s^s q N)}{p_s^s (p\sigma - 1 + \gamma)(N - ps)} - \frac{p\sigma}{p\sigma - 1 + \gamma}, \]
\[ \theta := \left( \frac{1}{p\sigma} - \frac{1}{p_s^s} \right)^{\frac{1-\gamma}{ps - \epsilon + \gamma - \frac{1}{\epsilon}}} \left[ \frac{1}{1-\gamma} + \frac{1}{p_s^s} \right]^{\frac{p_s^s - 1 + \gamma}{ps - \epsilon + \gamma}} S^{-\frac{1-\gamma}{ps - \epsilon + \gamma}}, \]
and
\[ \lambda_2 := \min \left\{ \bar{\lambda}, r \left( \frac{p\sigma - 1 + \gamma(N - ps)}{p\sigma} \right)^{\frac{1}{\epsilon}}, \left( \frac{c_2 + \theta}{c_1} \right)^{\frac{1}{\epsilon}} \right\}. \]
where $r > 0$ is such that $B_{4r} \subset \Omega$ and $q > 0$ is such that $\beta < 0$.

Now, for $\lambda \in (0, \lambda_1)$, if we choose

$$
\varepsilon := \lambda \frac{p(i(s-1) \sigma)}{(p - 1 + \gamma)(N - ps)}
$$

in (4.5). Then using the fact that $c_1 \lambda^\beta > c_1 \lambda_1^\beta \geq c_2 + \theta$, we obtain

$$
J_n,\lambda(t \psi_\varepsilon) \leq \left( \frac{1}{\sigma p} - \frac{1}{p_k^s} \right) S \frac{p \gamma \sigma}{p - ps} - \lambda c_1 \lambda^{\beta} + \frac{p \sigma}{p - 1 + \gamma} + c_2 \lambda \frac{p \sigma}{p - 1 + \gamma}
$$

$$
= \left( \frac{1}{\sigma p} - \frac{1}{p_k^s} \right) S \frac{p \gamma \sigma}{p - ps} + \lambda \frac{p \sigma}{p - 1 + \gamma} (c_2 - c_1 \lambda^\beta)
$$

$$
< \left( \frac{1}{\sigma p} - \frac{1}{p_k^s} \right) S \frac{p \gamma \sigma}{p - ps} - \theta \lambda \frac{p \sigma}{p - 1 + \gamma} = C_\lambda.
$$

Set

$$
\lambda_0 := \min(\lambda_1, \lambda_2).
$$

Then we have the following important result.

**Lemma 4.2.** Problem (3.1) has a nonnegative solution $v_n \in X_0$ satisfying

$$
\alpha < J_n,\lambda(v_n) < C_\lambda,
$$

for all $\lambda \in (0, \lambda_0)$, where $\alpha$ is from Lemma 2.1.

**Proof.** Let $\lambda \in (0, \lambda_0)$. By Lemma 2.1, $J_n,\lambda$ satisfies the Mountain Pass geometry. So we can define the Mountain Pass level

$$
c_{n,\lambda} := \inf \max_{g \in \Gamma, t \in [0,1]} J_n,\lambda(g(t)),
$$

where

$$
\Gamma := \{ g \in C([0,1], E) : g(0) = 0, J_n,\lambda(g(1)) < 0 \}.
$$

Moreover,

$$
0 < \alpha < c_{n,\lambda} \leq \sup_{t \geq 0} J_n,\lambda(t \psi) < C_{n,\lambda}.
$$

Hence, by Lemma 3.2, $J_{n,\lambda}$ satisfies the (PS) condition at the level $c_{n,\lambda}$, i.e., there exists a non-regular point $v_n$ for $J_{n,\lambda}$ at level $c_{n,\lambda}$. Moreover, $J_{n,\lambda}(v_n) = c_{n,\lambda} > \alpha > 0$. We can therefore deduce that $v_n$ is a nontrivial critical point of the functional energy $J_{n,\lambda}$ and also a solution to problem (3.1). If we now replace $\varphi$ by $v_n^\gamma$ in (3.2) and use (3.5), we get $\|v_n\| = 0$, that is, $v_n$ is nonnegative. This leads to the positivity of $v_n$ by the maximum principle. \qed
In order to complete the proof of our main result it now remains to obtain a second positive solution to problem (1.1) as a limit of the some subsequence of \(|v_n|\). To this end, let \(\lambda \in (0, \lambda_0)\) and \(|v_n|\) be a family of the positive function given by Lemma 4.2. By Lemma 4.2, the Hölder inequality and since \((v_n + \frac{1}{n})^{1-\gamma} - (\frac{1}{n})^{1-\gamma} \leq v_n^{1-\gamma}\), we see that \(C_\lambda > J_{n, \lambda} - \frac{1}{p_s} < J'_{n, \lambda}(v_n), v_n >\)

\[
C_\lambda > J_{n, \lambda} - \frac{1}{p_s} < J'_{n, \lambda}(v_n), v_n > \quad \text{with} \quad p_s = p - \frac{1}{\gamma} > \frac{n}{n-1} \quad \text{and} \quad \gamma = \min\left(1, p - \frac{1}{p_s}\right).
\]

Since \(0 < 1 - \gamma < 1 < p\sigma\), \(v_n\) is bounded in \(X_0\). So, there is \(v_\lambda \in X_0\) satisfying

\[
\begin{align*}
  v_n &\rightharpoonup v_\lambda \quad \text{weakly in } X_0, \\
  v_n &\rightharpoonup v_\lambda \quad \text{weakly in } L^{p_s}(\Omega), \\
  v_n &\rightarrow v_\lambda \quad \text{strongly in } L^r(\Omega), \quad \text{for any } r \in [1, p_s) \\
  v_n &\rightarrow v_\lambda \quad \text{a.e. in } \Omega.
\end{align*}
\]

We shall now prove that \(v_n \rightarrow v_\lambda\) strongly in \(X_0\), i.e. \(||v_n - v_\lambda|| \rightarrow 0\) as \(n \rightarrow \infty\).

First, we observe that if \(||v_n|| \rightarrow 0\), then \(v_n \rightarrow v_\lambda\) strongly in \(X_0\), so we assume that \(||v_n|| \rightarrow \eta > 0\). Since

\[
0 \leq \frac{v_n}{(v_n + \frac{1}{n})^{\gamma}} \leq v_n^{1-\gamma} \quad \text{a.e. in } \Omega,
\]

it follows by the Vitali theorem that

\[
\lim_{n \to \infty} \int_{\Omega} \frac{v_n}{(v_n + \frac{1}{n})^{\gamma}} \, dx = \int_{\Omega} v_\lambda^{1-\gamma} \, dx.
\]

Now, replace both \(u\) and \(\varphi\) by \(v_n\) in (3.2) to get

\[
\eta^{p_s} - \lambda \int_{\Omega} v^{1-\gamma}_\lambda \, dx + ||v_n||^{p_s} \rightarrow 0.
\]

On the other hand, by a simple calculation in (3.1) we get

\[
||v_n||^{p_s}(-\Delta)^s v_n \geq \min(1, \frac{\lambda}{p_s}) \quad \text{in } \Omega,
\]

since \(v_n\) is bounded in \(X_0\). Now, by the strong maximum principle \([3]\), there exist \(\bar{v} \subset \Omega\) such that

\[
\eta^{p_s} - \lambda \int_{\bar{v}} v^{1-\gamma}_\lambda \, dx + ||v_n||^{p_s} \rightarrow 0.
\]
for any integer $n$. Let $\varphi \in C_0^\infty(\Omega)$ satisfy $\operatorname{supp}(\varphi) = \Omega \subset \Omega$. Then by (5.2),

$$0 \leq \left| \frac{\varphi}{(v_n + \frac{1}{n})^{\gamma}} \right| \leq \frac{|\varphi|}{c}, \text{ a.e. in } \Omega.$$ 

Then the dominated convergence theorem implies that

$$\lim_{n \to \infty} \int_\Omega \frac{\varphi}{(v_n + \frac{1}{n})^{\gamma}} \, dx = \int_\Omega v_\lambda^{-\gamma} \varphi \, dx.$$ 

Thus, by replacing $u$ with $v_\lambda$ in (3.2) and by letting $n$ to infinity, we obtain

$$\eta^p(\sigma - 1) < v_\lambda, \varphi > -\lambda \int_\Omega v_\lambda^{-\gamma} \varphi \, dx + \int_\Omega v_\lambda^{p_\ast - 1} \varphi \, dx = 0.$$ 

Now, if we replace $\varphi$ by $v_\lambda$ in (5.3) and invoke (3.2), we obtain

$$= \eta^p(\sigma - 1) \left( \eta^p - \|v_\lambda\|^p \right) \lim_{n \to \infty} \left( \|v_n\|_{p_\ast}^p - \|v_\lambda\|_{p_\ast}^p \right).$$ 

Therefore, by the Brezis-Lieb Lemma [4], we obtain

$$\eta^p(\sigma - 1) \lim_{n \to \infty} \left( \|v_n - v_\lambda\|^p \right) = l^{p_\ast}.$$ 

Now, let us prove that $l = 0$, by contradiction, i.e. we assume that $l > 0$. As in Lemma 3.2 we can prove that

$$l^{p_\ast - p} \geq S \mu^p(\sigma - 1).$$ 

Therefore, by Lemma 4.2 combined with Young inequality and Hölder inequality, we deduce

$$C_\lambda > J_{n,\lambda}(v_n) - \frac{1}{p_\ast^s} < J'_{n,\lambda}(v_n), v_n > \geq \left( \frac{1}{p_\ast} - \frac{1}{p_s^\ast} \right) \left( \eta^{p_\sigma} + \|v_\lambda\|^{p_\sigma} \right) - \lambda \left( \frac{1}{1 - \gamma} + \frac{1}{p_s^\ast} \right) \Omega^{\frac{p_\ast - 1 + \gamma}{p_\ast}} S^{-\frac{1 - \gamma}{p - 1}} \|v_\lambda\|^{1 - \gamma} \geq C_\lambda.$$ 

Clearly, this is a contradiction, so $l = 0$ and $v_n \to v_\lambda$ strongly in $X_0$. In addition, one can easily see that $v_\lambda$ is a solution of problem (1.1). Therefore by Lemma 4.2, $J_\lambda(v_\lambda) \geq \alpha > 0$ so $v_\lambda$ is nontrivial. We can now proceed as in the proof of Lemma 4.2 and deduce that $v_\lambda$ is a positive solution of problem (1.1). In conclusion, since $J_\lambda(u_\lambda) < 0 < J_\lambda(v_\lambda)$, this completes the proof.

\[\square\]

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