Harmonic Oscillator in Snyder Space, the Classic and the Quantum

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The harmonic oscillator in Snyder space is investigated in its classical and quantum versions. The classical trajectory is obtained and the semiclassical quantization from the phase space trajectories is discussed. In the meanwhile, an effective cutoff to high frequencies is found. The quantum version is developed and an equivalent usual harmonic oscillator is obtained through an effective mass and an effective frequency introduced in the model. This modified parameters give us an also modified energy spectra.

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I. INTRODUCTION

Today, the possibility that the space could be noncommutative is an idea more and more accepted. The non-commutativity is usually set through a constant parameter. But exists another more general formulation, that is the Snyder space. R. Snyder investigated these ideas long time ago and built a noncommutative Lorentz invariant space-time where the non-commutativity of space operators is proportional to non-linear combinations of phase space operators through a free parameter that is usually identified with the Planck longitude . Kontsevich worked out these kind of space and since then, Snyder-like spaces in the sense of non-commutativity are of ever-increasing interest. Snyder space is also interesting because it can be mapped to the k-Minkowski space-time. This space can be canonically and elegantly obtained in its classical version through a lagrangian

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and hamiltonian approach \[7\], and a dimensional reduction from a \((D+1,2)\) space with two time dimensions, to a \((D,1)\) space with just one time dimension \[8\].

Nowadays Snyder space is also of increasing interest because it could be seen as an environment where it could be possible being successfully in quantizing gravity. In fact, in that direction it is possible to find a plausible explanation to the Bekenstein conjecture for the area spectrum of a black hole horizon through the area quantum in this kind of space \[9\].

In this paper the harmonic oscillator is analyzed in its classical and quantum versions. The quantum version of this simple but fundamental system was studied in \[10\], but the treatment here is simpler and we don’t need to bet about the right operators and despite that, we can shed some light in probably applications to problems like infinities in Quantum Field Theory (QFT). That is the importance of building a well defined harmonic oscillator in this kind of space, it could be possible to develop a QFT in it with very desirable properties. Furthermore, in the paper it is shown that we can build an Harmonic Oscillator with an effective mass related to the \(l\) parameter.

The paper is organized as follows. In section 2, the classical version is investigated and some possible quantum consequences postulated, in section 3 the quantum version is developed and the energy spectra is obtained. Finally the results are discussed in section 4.

## II. THE CLASSIC

Classical \(n\) dimensional Snyder Space is characterized by its non linear commutation relation between the variables of the phase space:

\[
\{ q_i, q_j \} = -l^2 L_{ij}, \quad (1)
\]

\[
\{ q_i, p_j \} = \delta_{ij} - l^2 p_i p_j, \quad (2)
\]

\[
\{ p_i, p_j \} = 0, \quad (3)
\]

where \(l\) is a tiny constant parameter (usually identified with Planck longitude), that measures the deformation introduced in the canonical Poisson brackets, and \(L_{ij}\), the angular momentum.

Let’s consider the usual Hamiltonian of an Harmonic Oscillator:
\[ H = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2, \]  
(4)

where \( m = 1 \), so \( \omega^2 = k \). The canonical equations are:

\[ \dot{q} = \{q, H\} = p - l^2p^3, \]  
(5)

and for \( p \):

\[ \dot{p} = \{p, H\} = -\omega^2q + \omega^2l^2qp^2. \]  
(6)

If we solve \( p(q) \), we find the usual relation \( p^2 + \omega^2q^2 = \omega^2 \). So the orbits in the phase space are untouched after the deformation of the Poisson brackets.

Solving the simultaneous equations (5) and (6) we obtain for \( q \):

\[ q = \pm \frac{\tan\{\omega t + d\sqrt{1-l^2\omega^2}\}}{\sqrt{1-l^2\omega^2} + \tan^2\{\omega t + d\sqrt{1-l^2\omega^2}\}}, \]  
(7)

where \( d \) is a suitable constant in order to achieve the initial condition \( q(t = 0) = 1 \), and \( p \) can be expressed in terms of \( q \):

\[ p = \pm \omega \sqrt{1 - q^2}. \]  
(8)

FIG. 1: Plot for the two branches of Snyder \( q(t) \) with \( l = 10^{-5} \) and \( \omega = 8.5 \cdot 10^{-4} \) (continue line), and for normal \( q(t) \) (dashed).

Figures 1 and 2 show the behavior of the positions \( q \) and momentum \( p \). We can see from \( q \) graph, that Snyder oscillator is periodic but contains harmonics that deform the trajectory.
FIG. 2: Plot for the two branches of Snyder $p(t)$ with $l = 10^{-5}$ and $\omega = 8.5 \cdot 10^{-4}$ (continue line), and for normal $p(t)$ (dashed).

Furthermore, it is possible to see that the Snyder oscillator has a different equivalent period than the normal one.

Another conspicuous feature is that $\omega = 1/l$ is effectively a cutoff to high frequencies. Indeed this is a good new in the search of possible QFT theory in Snyder space, because there is some hope of avoiding infinities.

Due to the orbits are not affected by the non linear version of Poisson brackets, we expect that the energy spectra from the Sommerfeld-Wilson quantization method $\int pdq = nh$, should be formally the same that the one of the linear oscillator: $E_n = nh\omega$, but as we have seen, the real equivalent period is not the same as the one of the normal oscillator, so some variations in the values should be expected.

Of course, Snyder Oscillator is not longer an expression in single $\sin$ or $\cos$ functions i.e it is not a pure oscillator, but we still can express it as a Fourier transform and formulate it as a linear infinite combination of harmonics of the frequency $\omega$. That is, the single Snyder oscillator looks like a set of normal coupled oscillators.

In fact, using the fact that $l^2$ must be considered as a tiny parameter in relation with the all other quantities, we can use a perturbation method, among others, to solve these equations.

Let’s start with the usual solution to $p$:

$$p_0 = -\omega \sin(\omega t),$$

(9)

where we normalized the initial $q$ perturbation, that is: $q(t = 0) = 1$. With $p_0$, we can
integrate $\dot{q}$ in order to obtain the first order $q_1$:

$$q_1(t) = (1 - \frac{3}{4}l^2\omega^2)\cos(\omega t) + \frac{1}{12}l^2\omega^2\cos(3\omega t) + q_1^0,$$

where $q_1^0$ is the constant evaluated in order of having the initial value of $q$. Now, we can introduce $q_1$ in (6) and integrate to obtain $p_1$

$$p_1(t) = (-1 + l^2\omega^2 - \frac{5}{24}l^4\omega^5)\sin(\omega t) + (-\frac{1}{9}l^2\omega^3 + \frac{11}{144}l^4\omega^5)\sin(3\omega t) - \frac{1}{240}l^4\omega^5\sin(5\omega t),$$

and so on. The method gives us, as we expected, the expansion in terms of harmonic functions in harmonic frequencies of $\omega$.

**III. THE QUANTUM**

After the Dirac quantization recipe, we can postulate the commutation relations of the Snyder Space:

$$[\hat{Q}_i, \hat{Q}_j] = -il^2\hat{L}_{ij},$$  \hspace{1cm} (12)

$$[\hat{Q}_i, \hat{P}_j] = i\delta_{ij} - il^2\hat{P}_i\hat{P}_j,$$  \hspace{1cm} (13)

$$[\hat{P}_i, \hat{P}_j] = 0.$$  \hspace{1cm} (14)

The (12) relation is a nonlinear version of the usual non commutativity one, where the commutator of the position operators is proportional to a constant $[2, 3]$. Here it is proportional to the angular momentum operator: $\hat{L}_{ij}$

The (13) equation is related to the different models with generalized commutation relations $[11, 12, 13]$.

To study the one dimensional Harmonic Oscillator we start considering an standard Hamiltonian:

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2,$$  \hspace{1cm} (15)

We define the usual creation and annihilation operators, where $\hbar = 1$:

$$a = \sqrt{\frac{m\omega}{2}}\hat{Q} + i\sqrt{\frac{1}{2m\omega}}\hat{P},$$

$$a^\dagger = \sqrt{\frac{m\omega}{2}}\hat{Q} - i\sqrt{\frac{1}{2m\omega}}\hat{P}.$$  \hspace{1cm} (16)
Using the commutation rules between operators of position and momentum, the commutation rules of the operators $a$ and $a^\dagger$ are $[a, a] = [a^\dagger, a^\dagger] = 0$, and:

$$[a, a^\dagger] = 1 - l^2 \hat{P}^2 = 1 + \frac{l^2}{2} (a^\dagger - a)^2.$$  \hspace{1cm} (18)

Writing the Hamiltonian in terms of the creation and annihilation operators and using the commutation relation between $a$ and $a^\dagger$, we obtain:

$$H = \omega \{a^\dagger a + \frac{1}{2}\} + \frac{\omega l^2}{2} (a^\dagger a^\dagger - a^\dagger a - aa^\dagger + a^2).$$  \hspace{1cm} (19)

Due the structure of the Hamiltonian, $|n\rangle$ is not longer an eigenvalue of the Hamiltonian, in fact:

$$H|n\rangle = \omega \{n[1 - \frac{l^2}{(1 + l^2)^2}] + \frac{1}{2}[1 + \frac{l^2}{(1 + l^2)^2}]\}|n\rangle$$

$$+ \omega \left\{\frac{l^2}{2(1 + l^2)} \sqrt{n + 1} \sqrt{n + 2}\right\}|n + 2\rangle$$

$$+ \omega \left\{\frac{l^2}{2(1 + l^2)} \sqrt{n} \sqrt{n - 1}\right\}|n - 2\rangle.$$  \hspace{1cm} (20)

So, the Snyder oscillator mixes states as we expected from the classical version.

But, encouraged by the semiclassical quantization that says us that we could find an standard spectra of the energy, we will use a QFT trick: because the extra term in the Hamiltonian induced by the non linearity of the commutators of $a$ and $a^\dagger$ is proportional to the dynamic term, we can add an counter term to the original Hamiltonian:

$$\tilde{H} = \frac{1}{2m} \hat{P}^2 + \frac{1}{2} m \omega^2 \hat{X}^2 + \frac{l^2}{2} \hat{P}^2.$$  \hspace{1cm} (21)

Now, it is possible to define a new mass parameter, $\tilde{m} = m/(1 + ml^2)$ and modify the frequency, $\tilde{\omega} = \omega \sqrt{(1 + ml^2)}$, then $\tilde{H}$ becomes:

$$\tilde{H} = \frac{1}{2\tilde{m}} \hat{P}^2 + \frac{1}{2} \tilde{m} \tilde{\omega}^2 \hat{X}^2.$$  \hspace{1cm} (22)

This Hamiltonian has as eigenvector $|n - 2\rangle$ and as eigenvalue $n$, and its spectra is:

$$E = \tilde{\omega}(n + \frac{1}{2}).$$  \hspace{1cm} (23)
This mass renormalization like procedure allows us to see the Snyder oscillator as the usual one, at least in the energy spectra, but with an effective mass. The energy spectra have been modified due the $l$ parameter. So, the zero energy is $\tilde{E}_0 = \tilde{\omega}/2$ and the $\Delta E = \tilde{\omega}$ (remember that $\hbar = 1$).

IV. DISCUSSION AND OUTLOOK

In the paper we have found the classical trajectory of an oscillator in Snyder space and found that we can see it as a set of coupled oscillators that can be described by an expansion in harmonic functions in the harmonic frequencies of $\omega$. We found also that there is a high frequency cutoff, because beyond it the oscillator has no response. Due this, we can hope that infinities in QFT theories could be avoided in Snyder space. Furthermore, we could see that the Sommerfeld-Wilson quantization method indicates us that the spectra should be formally like the usual harmonic oscillator, and consequentially in the quantum formulation, we saw that the oscillator in this space effectively mixes states, but through a QFT of mass renormalization, we could build an standard harmonic oscillator with an energy spectra modified due the presence of the non-commutative parameter $l$. We could expect that in the following we will can couple infinite oscillators in order to build an effective QFT in Snyder space. On the other hand, it will be worth to investigate the movement integrals of this kind of systems in higher dimensions.

Acknowledgments

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