Inhomogeneous chiral phases away from the chiral limit

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The effect of explicit chiral-symmetry breaking on inhomogeneous chiral phases is studied within a Nambu–Jona-Lasinio model with nonzero current quark mass. Generalizing an earlier result obtained in the chiral limit, we show within a Ginzburg-Landau analysis that the critical endpoint of the first-order chiral phase boundary between two homogeneous phases gets replaced by a “pseudo-Lifshitz point” when the possibility of inhomogeneous order parameters is considered. Performing a stability analysis we also show that the unstable mode along the phase boundary is in the scalar but not in the pseudoscalar channel, suggesting that modulations which contain pseudoscalar condensates, like a generalized dual chiral density wave, are disfavored against purely scalar ones. Numerically we find that the inhomogeneous phase shrinks as one moves away from the chiral limit, but survives even at significantly large values of the current quark mass.

I. INTRODUCTION

The conjectured existence of a chiral critical point in the QCD phase diagram at nonvanishing temperature $T$ and chemical potential $\mu$ \cite{1} has triggered tremendous experimental and theoretical activities, e.g., \cite{2–5}. While lattice gauge simulations with physical quark masses have revealed that at $\mu = 0$ the transition from the low-temperature phase with spontaneously broken chiral symmetry to the approximately restored phase at high temperature is not a true phase transition but only a smooth crossover \cite{6}, the situation is still unsettled at nonvanishing chemical potential where the application of standard lattice techniques is hampered by the sign problem. In this regime, calculations within effective models, like the Nambu–Jona-Lasinio (NJL) model or the quark-meson (QM) model, typically predict the existence of a first-order phase boundary, where the chiral order parameter changes discontinuously. When increasing the temperature, the discontinuity decreases until the first-order phase boundary vanishes at a critical endpoint (CEP) \cite{7, 8}. Beyond the CEP one finds again a crossover, consistent with the lattice results at $\mu = 0$. A special case is the chiral limit, where an exactly chirally restored phase exists. For two quark flavors, instead of a crossover, one then finds a second-order phase transition at high temperature and instead of a CEP there is a tricritical point (TCP) where the second-order phase boundary joins with the first-order one.

A tacit assumption in these model studies is that the chiral order parameter is constant in space. Allowing for spatially varying order parameters, it turns out that there can be a region in the phase diagram where an inhomogeneous phase is favored over homogeneous ones (see Ref. \cite{9} for a review). In particular, in the NJL model it was found that the first-order phase boundary is entirely covered by an inhomogeneous phase \cite{10}. In the chiral limit there is then a so-called Lifshitz point (LP) where the three different phases, i.e., the homogeneous and inhomogeneous chirally broken ones, and the restored phase meet. Moreover, it was shown within a Ginzburg-Landau (GL) analysis that the LP exactly coincides with the TCP obtained when only considering homogeneous phases \cite{11}. The same behavior, which was already known from the 1+1 dimensional Gross-Neveu model \cite{12} was also found in the QM model for a special choice of the sigma-meson mass \cite{13}.

Away from the chiral limit, the situation is less clear. In Ref. \cite{14} it was reported for the QM model that, when the symmetry breaking parameter is increased, the inhomogeneous phase quickly shrinks and disappears already at a pion mass of 37 MeV, way below the physical value. For the NJL model, on the other hand, it was found in Ref. \cite{10} that, although the inhomogeneous phase also shrinks with increasing explicit symmetry breaking, the effect is less dramatic, and the inhomogeneous phase is still present for realistic values of the current quark mass $m$. In particular it was stated that for $m \neq 0$ the inhomogeneous phase still reaches out to the CEP. This statement, however, was only based on numerical evidence and not shown with the same rigor as the coincidence of the LP with the TCP in the chiral limit.

We therefore want to revisit this question and investigate the relation between the “tip” of the inhomogeneous phase and the CEP within a GL analysis. We restrict ourselves to the NJL model and postpone the investigation of the QM model, which is analogous but technically more involved, to a later publication.
FIG. 1. Qualitative behavior of the thermodynamic potential $\Omega$ as a function of a spatially homogeneous order parameter $M$ at the left spinodal (a), at the right spinodal (b), at a first-order phase boundary (c), and at the CEP (d).

II. GINZBURG-LANDAU ANALYSIS OF CRITICAL AND LIFSHITZ POINTS

Consider the free-energy density of a physical system at temperature $T$ and chemical potential $\mu$, given by the value of the thermodynamic potential $\Omega[M]$. The latter is a functional of some order-parameter field $M(x)$, which generally depends on the spatial position $x$. Specifically we may think of $M$ being a space-dependent “constituent quark mass”1 proportional to the chiral condensate $\langle \bar{\psi} \psi \rangle$ at $x$. In the case of an exact chiral symmetry we can then expand $\Omega[M]$ about the chirally restored solution $M \equiv 0$ as [11]

$$\Omega[M] = \Omega[0] + \frac{1}{V} \int d^3x \left( \alpha_2 M^2(x) + \alpha_{4,a} M^4(x) + \alpha_{4,b} (\nabla M(x))^2 + \ldots \right),$$

where $V$ is the quantization volume and the GL coefficients $\alpha_i$ are $T$ and $\mu$ dependent functions. Odd powers of $M$ are prohibited by chiral symmetry. The ellipsis indicates higher-order terms, which we assume to be positive for the stability of the ground state.

If $\alpha_{4,b}$ is positive, gradients are suppressed and we have a homogeneous ground state, characterized by $M = \text{const}$. If $\alpha_{4,a}$ is positive as well, the system is in the restored phase ($M = 0$) for $\alpha_2 > 0$ and in the broken phase for $\alpha_2 < 0$ with a second-order phase transition at $\alpha_2 = 0$. In contrast, there is a first-order phase transition for $\alpha_{4,a} < 0$. Hence, the TCP, where the first-order phase boundary goes over into a second-order one, is given by the condition

$$\text{TCP: } \alpha_2 = \alpha_{4,a} = 0.$$  

For $\alpha_{4,b} < 0$, on the other hand, gradients are favored, so that inhomogeneous order parameters become possible. Typically the phase transition between the inhomogeneous phase and the restored phase is of second order. At the LP it meets the second-order phase boundary between the homogeneous broken and restored phases, so that this point is determined by the condition

$$\text{LP: } \alpha_2 = \alpha_{4,b} = 0.$$  

We note that away from the LP, $\alpha_{4,b}$ is negative along the second-order phase boundary between inhomogeneous and restored phase, balanced by positive contributions from $\alpha_2 > 0$ and higher-order gradient terms. As a consequence, while the amplitude of $M$ vanishes at the phase boundary, the wave number of the modulation stays nonzero and vanishes only at the LP.

The analysis becomes more complicated when the chiral symmetry is explicitly broken, e.g., by the presence of a nonvanishing current quark mass. In this case there is no chirally restored solution and we therefore expand the thermodynamic potential about an – a priori unknown – constituent mass $M_0$, which we assume to be spatially constant but which may depend on $T$ and $\mu$. Writing $M(x) = M_0 + \delta M(x)$ and assuming that $\delta M(x)$ and its gradients are small, the expansion then takes the form

$$\Omega[M] = \Omega[M_0] + \frac{1}{V} \int d^3x \left( \alpha_1 \delta M(x) + \alpha_2 \delta M^2(x) + \alpha_3 \delta M^3(x) + \alpha_{4,a} \delta M^4(x) + \alpha_{4,b} (\nabla \delta M(x))^2 + \ldots \right).$$

Note that here, in contrast to Eq. (1), odd powers of the expansion parameter are allowed. In the following we require that $M_0$ corresponds to a stationary point at given $T$ and $\mu$, meaning that the linear term of the expansion vanishes, $\alpha_1(T,\mu; M_0) = 0$. It is crucial however to keep the cubic term.

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1 For this discussion we will focus on real order parameters, an assumption which will be justified in the following section.
Again, we will first consider the case of homogeneous order parameters, $\delta M = \text{const}$. Since there is no restored phase, we cannot have a second-order transition to it, but it is possible to have a first-order boundary between solutions with different values of $M$, ending at a CEP (where the phase transition is second order). If at temperature $T$ there is a first-order phase transition at a critical chemical potential $\mu_c = \mu_c(T)$, the thermodynamic potential at $T$ and $\mu_c$ as a function of $M$ has two degenerate minima $M_0^{(\text{min1})}$ and $M_0^{(\text{min2})}$ with $\alpha_2(T, \mu_c; M_0^{(\text{min1, min2})}) > 0$, separated by a maximum $M_0^{(\text{max})}$ with $\alpha_2(T, \mu_c; M_0^{(\text{max})}) < 0$ (see Fig. 1 (c)). Then, if we move towards the CEP, the maximum gets more and more shallow until it vanishes at $(T_{\text{CEP}}, \mu_{\text{CEP}})$, corresponding to $\alpha_2(T_{\text{CEP}}, \mu_{\text{CEP}}; M_0^{(\text{CEP})}) = 0$ (Fig. 1 (d)). Moreover, since there is a minimum at this point and not a saddle point, $\alpha_3(T_{\text{CEP}}, \mu_{\text{CEP}}; M_0^{(\text{CEP})})$ vanishes as well. The CEP is therefore given by the equations

$$\text{CEP}: \quad \alpha_1 = \alpha_2 = \alpha_3 = 0,$$

which determine the values of $T_{\text{CEP}}$ and $\mu_{\text{CEP}}$ together with the corresponding constituent mass $M_0^{(\text{CEP})}$.

An alternative way to derive this condition is to approach the CEP along the spinodal lines. Assuming again a first-order phase transition at temperature $T$ and $\mu = \mu_c$, the spinodals correspond to the chemical potentials $\mu_{\text{sp1}} < \mu_c$ and $\mu_{\text{sp2}} > \mu_c$ at which the left or the right minimum in Fig. 1 (c) merges with the maximum to a saddle point (cf. Fig. 1 (a) and (b)). If these saddle points are located at $M = M_0^{(\text{sp1})}$ and $M = M_0^{(\text{sp2})}$, respectively, it follows that $\alpha_2(T, \mu_{\text{sp1}}; M_0^{(\text{sp1})}) = 0$ and $\alpha_3(T, \mu_{\text{sp1}}; M_0^{(\text{sp1})}) < 0$, while at the right spinodal $\alpha_2(T, \mu_{\text{sp2}}; M_0^{(\text{sp2})}) = 0$ but $\alpha_3(T, \mu_{\text{sp2}}; M_0^{(\text{sp2})}) > 0$. We thus find again that at the CEP, where the two spinodals meet, both $\alpha_2$ and $\alpha_3$ vanish.

Without a chirally restored phase and a second-order boundary which separates it from the homogeneous broken phase, we also cannot have a Lifshitz point, where this phase boundary is supposed to meet with the boundaries of an inhomogeneous phase. It is still possible, however, to have a second-order boundary between an inhomogeneous and a homogeneous phase along which the amplitude of the inhomogeneous phase. It is still possible, however, to have a second-order boundary between solutions with coupling constant $G$. Allowing for (possibly space dependent but time independent) scalar and pseudoscalar condensates

$$\phi_S(x) = \langle \bar{\psi}(x) \psi(x) \rangle, \quad \phi_P(x) = \langle \bar{\psi}(x)i\gamma_5\tau^3\psi(x) \rangle,$$

the mean-field thermodynamic potential at temperature $T$ and quark chemical potential $\mu$ takes the form

$$\Omega_{\text{MF}} = -\frac{T}{V} \text{Tr} \log \left( \frac{S^{-1}}{T} \right) + G \frac{1}{V} \int d^3 x \left( \phi_S^2(x) + \phi_P^2(x) \right),$$

where $V$ is again a quantization volume, and the functional trace runs over the Euclidean space $V_4 = [0, \frac{1}{T}] \times V$, Dirac, color, and flavor degrees of freedom. The inverse dressed quark propagator is given by

$$S^{-1}(x) = i\partial + \mu^0 - m + 2G \left( \phi_S(x) + i\gamma_5\tau^3\phi_P(x) \right).$$

We first neglect the possibility of inhomogeneous condensates. At nonvanishing $m$ the ground state at given $T$ and $\mu$ is then given by a constant scalar condensate, $\phi_S(x) = \phi_{S,0}$, and a vanishing pseudoscalar condensate, $\phi_P = 0$. Accordingly the inverse quark propagator takes the form

$$S_0^{-1}(x) = i\partial + \mu^0 - M_0, \quad M_0 = m - 2G\phi_{S,0},$$

III. STABILITY ANALYSIS

To be specific we consider the standard NJL-model Lagrangian

$$\mathcal{L} = \bar{\psi} \left( i\partial - m \right) \psi + G \left\{ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau^3\psi)^2 \right\},$$

for a two-flavor quark field $\psi$ with $N_c = 3$ colors and bare mass $m$, and a chirally symmetric four-point interaction in the scalar-isoscalar and pseudoscalar-isovector channels with coupling constant $G$. Allowing for (possibly space dependent but time independent) scalar and pseudoscalar condensates

$$\phi_S(x) = \langle \bar{\psi}(x) \psi(x) \rangle, \quad \phi_P(x) = \langle \bar{\psi}(x)i\gamma_5\tau^3\psi(x) \rangle,$$

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$$S_0^{-1}(x) = i\partial + \mu^0 - M_0, \quad M_0 = m - 2G\phi_{S,0},$$

(11)
corresponding to a free particle with constituent quark mass \( M_0 \).

Next we consider inhomogeneous fluctuations around the homogeneous ground state,

\[
\phi_S(\mathbf{x}) = \phi_{S,0} + \delta \phi_S(\mathbf{x}) , \quad \phi_P(\mathbf{x}) \equiv \delta \phi_P(\mathbf{x}).
\]  

(12)

Searching for a possible second-order phase boundary between the homogeneous and an inhomogeneous phase, we assume that the amplitudes (but not necessarily the gradients), of the fluctuations are small and expand \( \Omega_{MF} \) in powers of \( \delta \phi_S \) and \( \delta \phi_P \). The thermodynamic potential can then be written as

\[
\Omega_{MF} = \sum_{n=0}^{\infty} \Omega^{(n)} ,
\]

(13)

where \( \Omega^{(n)} \) is of the \( n \)th order in the fluctuating fields. Specifically we obtain

\[
\Omega^{(1)} = \frac{T}{V} \text{Tr} \left( S_0 \delta \hat{\Sigma} \right) + 2G \phi_{S,0} \frac{1}{V} \int d^3x \delta \phi_S(\mathbf{x})
\]

(14)

for the linear contribution,

\[
\Omega^{(2)} = \frac{1}{2} \frac{T}{V} \text{Tr} \left( S_0 \delta \hat{\Sigma} \right)^2 + G \frac{1}{V} \int d^3x \left( \delta \phi_S^2(\mathbf{x}) + \delta \phi_P^2(\mathbf{x}) \right)
\]

(15)

for the quadratic contribution, and

\[
\Omega^{(n \geq 3)} = \frac{1}{n} \frac{T}{V} \text{Tr} \left( S_0 \delta \hat{\Sigma} \right)^n
\]

(16)

for all higher-order contributions, where \( \delta \hat{\Sigma}(\mathbf{x}) = -2G \left( \delta \phi_S(\mathbf{x}) + i\gamma_5 \tau_3 \delta \phi_P(\mathbf{x}) \right) \) is the quark self-energy related to the fluctuating fields.

Assuming spatially periodic condensates we can perform the Fourier decompositions

\[
\delta \phi_S(\mathbf{x}) = \sum_{\mathbf{q}_k} \delta \phi_{S,\mathbf{q}_k} e^{i\mathbf{q}_k \cdot \mathbf{x}} , \quad \delta \phi_P(\mathbf{x}) = \sum_{\mathbf{q}_k} \delta \phi_{P,\mathbf{q}_k} e^{i\mathbf{q}_k \cdot \mathbf{x}} ,
\]

(17)

with \( \mathbf{q}_k \) being the elements of the corresponding reciprocal lattice. We require that the condensates and, thus, their fluctuations are real functions in coordinate space. The Fourier coefficients then obey the relations \( \delta \phi_{S,-\mathbf{q}_k} = \delta \phi_{S,\mathbf{q}_k}^* \) and \( \delta \phi_{P,-\mathbf{q}_k} = \delta \phi_{P,\mathbf{q}_k}^* \). Recalling that \( S_0 \) is the standard propagator of a free fermion with constant mass \( M_0 \) it is then straightforward to evaluate the above expressions for \( \Omega^{(n)} \). For the linear contribution we find

\[
\Omega^{(1)} = -\delta \phi_{S,0} (M_0 - m + 2GM_0 F_1)
\]

(18)

where for later convenience we define

\[
F_n = 8N_c \int \frac{d^3p}{(2\pi)^3} s_n(p)
\]

(19)

with

\[
s_n(p) = T \sum_j \frac{1}{[(i\omega_j + \mu)^2 - \mathbf{p}^2 - M_0^2]^n}
\]

(20)

and fermionic Matsubara frequencies \( \omega_j = (2j + 1)\pi T \).

Eq. (18) shows that only the homogeneous (\( \mathbf{q}_k = 0 \)) part of the scalar fluctuations contribute to this order. However, since we assumed \( \phi_{S,0} \) to be the homogeneous ground-state solution, it follows that \( \Omega^{(1)} \) should vanish. In turn this means that the term in parentheses must be equal to zero, which is nothing but the gap equation for \( M_0 \).

For the quadratic contribution one finds

\[
\Omega^{(2)} = 2G^2 \sum_{\mathbf{q}_k} \left\{ |\delta \phi_{S,\mathbf{q}_k}|^2 \Gamma_{S}^{-1}(\mathbf{q}_k) + |\delta \phi_{P,\mathbf{q}_k}|^2 \Gamma_{P}^{-1}(\mathbf{q}_k) \right\}
\]

(21)
FIG. 2. Inverse correlation functions \( \Gamma^{-1}_S \) (red solid line) and \( \Gamma^{-1}_P \) (blue dashed line) for \( m = 10 \text{ MeV}, T = 10 \text{ MeV} \) and \( \mu = 344 \text{ MeV} \) as functions of the momentum \( q^2 \).

with the inverse scalar and pseudoscalar correlation functions

\[
\Gamma^{-1}_S(q^2) = \frac{m}{M_0 2G} - \frac{1}{2} \left( q^2 + 4M_0^2 \right) L_2(q^2),
\]

(22)

\[
\Gamma^{-1}_P(q^2) = \frac{m}{M_0 2G} - \frac{1}{2} q^2 L_2(q^2),
\]

(23)

which can be interpreted as inverse meson propagators at vanishing energy and nonzero 3-momentum \( q \). Here the gap equation has been exploited to eliminate a term proportional to \( F_1 \), while

\[
L_2(q^2) = -8N_c \int \frac{d^3p}{(2\pi)^3} T \sum_n \frac{1}{[i\omega_n + \mu)^2 - (p + q)^2 - M_0^2][i\omega_n + \mu)^2 - p^2 - M_0^2]}. \]

(24)

As obvious from Eq. (21) the homogeneous ground state is unstable against the formation of inhomogeneous modes if \( \Gamma^{-1}_S \) or \( \Gamma^{-1}_P \) is negative in some region of \( q \). According to Eqs. (22) and (23) a necessary condition for this to happen is that the function \( L_2(q^2) \) is positive in this regime (since \( m, M_0 \) and \( G \) are positive). In this case we always have \( \Gamma^{-1}_S < \Gamma^{-1}_P \), meaning that the instability occurs first in the scalar channel. This is illustrated in Fig. 2 where \( \Gamma^{-1}_S \) and \( \Gamma^{-1}_P \) at \( T = 10 \text{ MeV} \) and \( \mu = 344 \text{ MeV} \) are displayed as functions of the momentum \( q^2 \) for a current quark mass \( m = 10 \text{ MeV} \), roughly corresponding to a vacuum pion mass \( m_\pi = 135 \text{ MeV} \). While the function \( \Gamma^{-1}_S \) just touches the zero-axis, indicating that the chosen values of \( T \) and \( \mu \) correspond to a point on the phase boundary, \( \Gamma^{-1}_P \) is strictly positive, i.e., the system is stable against pseudoscalar fluctuations. This is qualitatively different from the situation in the restored phase of the chiral limit, which was discussed in Ref. [15]. There we have \( M_0 = 0 \) and thus the instabilities in the scalar and pseudoscalar channels occur simultaneously.

In particular we find that near the second-order phase boundary the inhomogeneous mode is purely scalar for \( m \neq 0 \), so that modulations like the dual chiral density wave [15] or modifications thereof which contain pseudoscalar condensates [16, 17] are excluded in this regime. This can also be seen in Fig. 3. The shaded area corresponds to the inhomogeneous region of Ref. [10] where a so-called real kink crystal (“solitonic”) modulation was considered, which is purely scalar and allows for a smooth transition to the homogeneous phase, consistent with our ansatz Eq. (12). Its phase boundary on the right coincides with the border of the instability region with respect to scalar fluctuations (pink dotted line). If instead only pseudoscalar fluctuations were allowed, the inhomogeneous phase would already end at the red dash-dotted line, i.e., at considerably lower chemical potentials. In the following we will therefore drop the pseudoscalar condensate and concentrate on the scalar channel. In analogy to the definition of \( M_0 \) in Eq. (11) we then define \( \delta M(x) = -2G\delta \phi_S(x) \), which can be identified with the \( \delta M(x) \) introduced in the previous section. Plugging this, together with the Fourier decomposition Eq. (17), into Eq. (4) and comparing the terms proportional to \( |\delta \phi_{S,q^2}|^2 \) with Eq. (21) we can identify

\[
\alpha_2 = \frac{1}{2} \Gamma^{-1}_S(0), \quad \alpha_{4,b} = \frac{1}{2} \frac{d\Gamma^{-1}_S}{dq^2}\bigg|_{q^2=0}.
\]

(25)

\(^2\) In all our numerical calculations we use Pauli-Villars regularization with three regulator terms and adopt the coupling constant \( G \) and the Pauli-Villars parameter \( \Lambda \) from Ref. [10] where they have been fitted in the chiral limit to a vacuum constituent quark mass of 300 MeV and a pion decay constant of 88 MeV.

\(^3\) The picture might change if magnetic fields [18, 19] or vector interactions [20] are included.

\(^4\) Note that this type of analysis cannot say anything about the left boundary of the inhomogeneous phase where the amplitudes of the oscillating fields are large.

\(^5\) Generally this yields the coefficients associated with terms of the form \( (\nabla_i \nabla_j \ldots M)^2 \), which are used in the improved GL analysis developed in [21].
FIG. 3. Relevant lines in the phase diagram of the NJL model for $m = 10 \text{ MeV}$: GL coefficients $\alpha_2 = 0$ (black solid) and $\alpha_3 = 0$ (blue dashed), meeting at the CEP (black dot). As discussed in Sec. IV, the $\alpha_3 = 0$ line is identical to the $\alpha_{4,b} = 0$ line and therefore the CEP coincides with the PLP. At each point $M_0$ was determined by simultaneously solving the equation $\alpha_1 = 0$.

The shaded area indicates the region where the inhomogeneous solution of Ref. [10] is favored over homogeneous phases. Its phase boundary to the right coincides with the instability line against scalar fluctuations (pink dotted). The red dash-dotted line indicates the line where the instability against pseudoscalar fluctuations would occur if scalar fluctuations were suppressed.

As illustrated in Fig. 2, the second-order phase boundary between homogeneous and inhomogeneous phase is given by the condition that $\Gamma_s^{-1}(q^2)$ just touches the $\Gamma_s^{-1} = 0$-axis at a single momentum. This momentum is then the wave number of the inhomogeneous modulation at the phase boundary. Hence, at the PLP, which we defined to be the point of the phase boundary where the wave number vanishes, we have $\Gamma_s^{-1}(0) = 0$ and $\frac{d\Gamma_s^{-1}}{dq^2}|_{q^2=0} = 0$. Comparing this with Eq. (25), we conclude that $\alpha_2 = \alpha_{4,b} = 0$ at the PLP, in agreement with Eq. (6).

Of course, what we are really interested in is the tip of the inhomogeneous phase in the phase diagram, which, from a logical point of view, is not necessarily the same as our definition of the PLP. However, if one continues to move in the direction of the phase boundary beyond the PLP, the value of $q^2$ where $\Gamma_s^{-1}$ touches the zero-axis becomes negative, and therefore there is no instability in this region. Here we assumed that $\Gamma_s^{-1}$ is a smooth function of $T$ and $\mu$ at the PLP. But even if this is not the case, e.g., if there is a discontinuous jump of $M_0$, this should reflect itself in the behavior of the phase boundary at this point, most likely producing a kink. So in any case the PLP, as we defined it, should be a significant point of the phase boundary. Indeed, numerically we confirm that it corresponds to the tip of the inhomogeneous phase.

IV. EVALUATION OF THE GINZBURG-LANDAU COEFFICIENTS

According to Eq. (25), we can calculate $\alpha_2$ and $\alpha_{4,b}$ directly from $\Gamma_s^{-1}$. Analogously, $\alpha_1$ can be derived from $\Omega^{(1)}$, while $\alpha_3$ and $\alpha_{4,a}$ can be obtained from $\Omega^{(3)}$ and $\Omega^{(4)}$, respectively. Instead of using the Fourier decomposition of the condensates, as done in the previous section, an equivalent but perhaps more straightforward procedure to calculate the Ginzburg-Landau coefficients is to keep the coordinate-space representation and perform a gradient expansion [11]. One finds

$$\alpha_1 = \frac{M_0 - m}{2G} + M_0 F_1,$$

$$\alpha_2 = \frac{1}{4G} + \frac{1}{2} F_1 + M_0^2 F_2,$$

$$\alpha_3 = M_0 \left( F_2 + \frac{4}{3} M_0^2 F_3 \right),$$

$$\alpha_{4,a} = \frac{1}{4} F_2 + 2M_0^2 F_3 + 2M_0^4 F_4,$$

$$\alpha_{4,b} = \frac{1}{4} F_2 + \frac{1}{3} M_0^2 F_3.$$
with the integrals $F_n$ defined in Eq. (19). We immediately see that the stationarity condition $\alpha_1 = 0$ is equivalent to $\Omega^{(1)} = 0$, Eq. (18), as expected. In addition, the above results have several interesting consequences:

(i) In the chiral limit, where we can expand about the restored solution $M_0 = 0$, we reproduce that $\alpha_{4,a} = \alpha_{4,b}$ [11], meaning that TCP and LP coincide.

(ii) Assuming $M_0 \neq 0$ and then taking the limit $M_0 \to 0$, the $\alpha_3 = 0$ line converges to the $F_2 = 0$ line, which also determines the $\alpha_{4,a} = 0$ line in this limit. Hence the CEP converges to the TCP in the chiral limit, as one would expect.

(iii) For arbitrary values of $M_0 \neq 0$ we find $\alpha_3 = 4M_0 \alpha_{4,b}$ and, thus, the PLP coincides with the CEP.

The lines $\alpha_2 = 0$ and $\alpha_3 = 0$ (coinciding with $\alpha_{4,b} = 0$) are also shown in Fig. 3. As one can see, both spinodals (i.e., the left and the right part of the $\alpha_2 = 0$ line) and the $\alpha_3 = 0$ line become parallel at the CEP. Therefore from a practical point of view the $\alpha_3 = 0$ line does not really help to localize the CEP, which is found more easily by determining the point where the two spinodals meet. However, as we have seen above, the condition $\alpha_3 = 0$ at the CEP is very important conceptually in order to show its coincidence with the PLP.

In practical calculations we have to deal with the fact that some of the integrals are divergent and have to be regularized. One must then be careful not to spoil the above relations by an improper choice of the regularization procedure. In fact, the derivation of the Ginzburg-Landau coefficients relies at several places on integrations by parts and the assumption that surface terms can be dropped. This was already pointed out in Ref. [11] for the chiral limit and gets additional importance in our case for showing the proportionality of $\alpha_{4,b}$ to $\alpha_3$. Here a straightforward calculation along the lines of Ref. [11] yields

$$\alpha_{4,b} = \frac{1}{4} F_2 - M_0^2 \frac{16}{3} N_c \int \frac{d^3 p}{(2\pi)^3} p^2 s_4(p),$$

and the result given in Eq. (30) is obtained by noting that $s_4 = \frac{1}{6|p|} \frac{\partial s_3}{\partial |p|}$ and integrating by parts, assuming that the surface terms can be dropped. As already found out earlier when dealing with inhomogeneous phases, we should therefore not regularize the integrals by a momentum cutoff, while for instance Pauli-Villars (which was employed in all numerical calculations presented here) or proper-time regularization are permitted.

V. DISCUSSION

The coincidence of the PLP (which we have argued to be the tip of the inhomogeneous phase) with the CEP in the NJL model with a nonzero current quark mass is our main result. It means that there is an inhomogeneous phase in the model if the analysis of homogeneous phases predicts the existence of a first-order chiral phase transition with a CEP. Of course this result does not tell us whether there is a CEP in the first place, and how it behaves as a function of the quark mass.
In Fig. 4 we therefore show the positions of the CEP (and hence of the PLP) in the phase diagram for different values of \( m \), keeping the other model parameters as in the numerical examples shown before. For small \( m \) we find that the temperature of the PLP strongly decreases with increasing quark mass, suggesting that the inhomogeneous phase will quickly disappear at higher \( m \). It turns out, however, that this is not the case, as the movement of the PLP as a function of \( m \) slows down, behaving more like the increase of \( m_* = \sqrt{m} \). Moreover, the temperature levels off, although we cannot exclude that this is a regularization artifact.\(^6\) It is therefore not obvious from these calculations whether or not inhomogeneous condensates could also play a role in the strange-quark sector.

Independent of the numerical details, the coincidence of CEP and PLP means that the possibility of having an inhomogeneous phase should be taken as seriously as the possibility of the existence of a CEP. We note that the coincidence of the two points could have been anticipated not only from the same feature in the \( 1+1 \) dimensional Gross Neveu model \(^2\) or from the numerical evidence of Ref. \(^10\) but already from the earlier observation that near the CEP the scalar meson propagator has a space-like peak which at the CEP diverges for \(|q| \to 0\) \(^{23}\). As discussed in Sec. III, this indicates an instability towards an “inhomogeneous” mode with vanishing wave number at this point, in agreement with our definition of the PLP.

We should mention that already in the chiral limit the coincidence of TCP and LP does not necessarily hold if additional interaction terms are considered. However, as shown in Ref. \(^24\) for the case of vector interaction, these terms typically affect the TCP stronger than the LP, so that the inhomogeneous phase is rather robust under model extensions. We therefore expect a similar behavior at nonzero \( m \).

Finally, we recall that fluctuations beyond the mean-field approximation discussed here are expected to play an important role on the phase structure of the model, as they are known to prevent the formation of any true one-dimensional long-range ordering at finite temperature \(^{25}\). One expects instead a phase characterized by a quasi-long-range ordering, similar to that of smectic liquid crystals. Moreover, it was recently argued that, at least in the chiral limit, the presence of fluctuations prevents the coefficients \( \alpha_2 \) and \( \alpha_4, b \) from vanishing simultaneously, so that a LP cannot exist and gets instead replaced by a Lifshitz regime in the neighboring region of the phase diagram \(^{26}\). The properties of low-energy modes and their effects on crystalline chiral condensates in the chiral limit have been explicitly investigated in \(^{27}\) and \(^{28}\) by employing GL functionals, and more recently in \(^{29}\). It would be very interesting to extend these analyses to finite quark masses, as the explicit breaking of chiral symmetry might affect the properties of these fluctuations and the possible disappearance of the LP from the phase diagram \(^{28}\). The GL functional employed in this work could provide an effective starting point for this kind of analysis.

It is rather straightforward to extend our analysis to the QM model with an explicitly symmetry-breaking term. A calculation analogous to Sec. III then reveals that instabilities towards inhomogeneous modes are again signalled by poles in the dressed scalar and pseudoscalar propagators at vanishing energy and finite momentum. Unlike in the NJL model, however, these poles are not only determined by the quark-antiquark polarization functions (which again favor instabilities in the scalar modes) but also by the tree-level meson masses. The situation is therefore more involved and will be discussed in a future publication.\(^7\) In particular it will be interesting to see whether the early disappearance of the inhomogeneous phase at nonzero pion mass, which has been found in Ref. \(^14\) for a dual chiral density wave modulation, will still persist if a purely scalar modulation, like the real kink crystal, is considered.

Ultimately, we are of course interested in the phase diagram of QCD. In Ref. \(^31\) inhomogeneous phases have been studied within a Dyson-Schwinger approach applied to two-flavor QCD in the chiral limit. Using a relatively simple truncation scheme, an inhomogeneous selfconsistent solution of the Dyson-Schwinger equations was found to exist in a region of the phase diagram quite similar to that in the NJL model. In particular, within numerical precision, the LP agrees with the TCP in that approach. Our present finding of the coincidence of PLP and CEP might therefore also have relevance for QCD away from the chiral limit.

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\[\begin{align*}
[1] & \text{M. A. Stephanov, K. Rajagopal, and E. V. Shuryak, Phys. Rev. Lett. 81, 4816 (1998), arXiv:hep-ph/9806219.} \\
\text{\textsuperscript{6} For even higher quark masses we find that the temperature of the PLP rises again.} \\
\text{\textsuperscript{7} A stability analysis in the pseudoscalar channel has been performed in Ref. [30] within a functional renormalization-group framework.}
\end{align*}\]
