On the possibility of finite quantum Regge calculus

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**Abstract**

The arguments were given in a number of our papers that the discrete quantum gravity based on the Regge calculus possesses nonzero vacuum expectation values of the triangulation lengths of the order of Plank scale $10^{-33} \text{cm}$. These results are considered paying attention to the form of the path integral measure showing that probability distribution for these linklengths is concentrated at certain nonzero finite values of the order of Plank scale. That is, the theory resembles an ordinary lattice field theory with fixed spacings for which correlators (Green functions) are finite, UV cut off being defined by lattice spacings. The difference with an ordinary lattice theory is that now lattice spacings (linklengths) are themselves dynamical variables, and are concentrated around certain Plank scale values due to *dynamical* reasons.

PACS numbers: 04.60.-m Quantum gravity
The formal nonrenormalisability of quantum version of general relativity (GR) may cause us to try to find alternatives to the continuum description of underlying spacetime structure. An example of such the alternative description may be given by Regge calculus (RC) suggested in 1961 [1]. In RC the exact GR is developed in the piecewise flat spacetime which is a particular case of general Riemannian spacetime [2]. In turn, the general Riemannian spacetime can be considered as limiting case of the piecewise flat spacetime [3]. Any piecewise flat spacetime can be represented as collection of a (countable) number of the flat 4-dimensional simplices (tetrahedrons), and its geometry is completely specified by the countable number of the freely chosen lengths of all edges (or 1-simplices). Thus, RC implies a discrete description alternative to the usual continuum one. For a review of RC and alternative discrete gravity approaches see, e.g., [4].

The discrete nature of the Regge’s description presents a difficulty in the (canonical) quantization of such the theory due to the absence of a regular continuous coordinate playing the role of time. Therefore one cannot immediately develop Hamiltonian formalism and canonical (Dirac) quantization. To do this we need to return to the partially continuum description, namely, with respect to only one direction shrinking sizes of all the simplices along this direction to those infinitely close to zero. The linklengths and other geometrical quantities become functions of the continuous coordinate taken along this direction. We can call this coordinate time $t$ and develop quantization procedure with respect to this time. The result of this procedure can be formulated as some path integral measure. It is quite natural to consider this measure as a (appropriately defined) limiting continuous time form of a measure on the set of the original completely discrete Regge spacetimes. This last completely discrete measure is just the object of interest to be found. The requirement for this measure to have the known limiting continuous time form can be considered as a starting postulate in our construction. The issuing principles are of course not unique, and another approaches to defining quantum measure in RC based on another physical principles do exist [5, 6].

The above condition for the completely discrete measure to possess required continuous time limit does not defines it uniquely as long as only one fixed direction which defines $t$ is considered. However, different coordinate directions should be equivalent and we have a right to require for the measure to result in the canonical quantization measure in the continuous time limit whatever coordinate direction is chosen to define
a time. These requirements are on the contrary a priori too stringent, and it is important that on some configuration superspace (extended in comparison with superspace of the genuine Regge geometries) such the measure turns out to exist.

Briefly speaking, we should, first, find continuous time limit for Regge action, recast it in the canonical Hamiltonian form and write out the Hamiltonian path integral, the measure in the latter being called for a moment the continuous time measure; second, we should check for existence and (if exists) find the measure obeying the property to tend in the continuous time limit (with concept ”to tend” being properly defined) to the found continuous time measure irrespectively of the choice of the time coordinate direction. When passing to the continuous time RC we are faced with the difficulty that the description of the infinitely flattened in some direction simplex purely in terms of the lengths is singular.

The way to avoid singularities in the continuous time limit is to extend the set of variables via adding the new ones having the sense of angles and considered as independent variables. Such the variables are the finite rotation matrices which are the discrete analogs of the connections in the continuum GR. The situation considered is analogous to that one occurred when recasting the Einstein action in the Hilbert-Palatini form,

$$\frac{1}{2} \int R \sqrt{|g|} d^4x \equiv \frac{1}{8} \int \epsilon^{abcd} \epsilon^{\lambda \mu \nu \rho} e^a_\lambda e^b_\nu [\partial_\nu + \omega_\nu, \partial_\rho + \omega_\rho]^{cd} d^4x,$$

(1)

where the tetrad $e^a_\lambda$ and connection $\omega^{ab}_\lambda = -\omega^{ba}_\lambda$ are independent variables, the RHS being reduced to LHS in terms of $g_{\lambda \mu} = e^a_\lambda e_{a\mu}$ if we substitute for $\omega^{ab}_\lambda$ solution of the equations of motion for these variables in terms of $e^a_\lambda$. The Latin indices $a, b, c, ...$ are the vector ones with respect to the local Euclidean frames which are introduced at each point $x$.

Now in RC the Einstein action in the LHS of (1) becomes the Regge action,

$$\sum_{\sigma^2} \alpha_{\sigma^2} |\sigma^2|,$$

(2)

where $|\sigma^2|$ is the area of a triangle (the 2-simplex) $\sigma^2$, $\alpha_{\sigma^2}$ is the angle defect on this triangle, and summation run over all the 2-simplices $\sigma^2$. The discrete analogs of the tetrad and connection, edge vectors and finite rotation matrices, were first considered in [7]. The local Euclidean frames live in the 4-simplices now, and the analogs of the connection are defined on the 3-simplices $\sigma^3$ and are the matrices $\Omega_{\sigma^3}$ connecting the
frames of the pairs of the 4-simplices $\sigma^4$ sharing the 3-faces $\sigma^3$. These matrices are the finite SO(4) rotations in the Euclidean case (or SO(3,1) rotations in the Lorentzian case) in contrast with the continuum connections $\omega^{ab}_\lambda$ which are the elements of the Lee algebra so(4)(so(3,1)) of this group. This definition includes pointing out the direction in which the connection $\Omega_{\sigma^3}$ acts (and, correspondingly, the opposite direction, in which the $\Omega_{\sigma^3}^{-1} = \bar{\Omega}_{\sigma^3}$ acts), that is, the connections $\Omega$ are defined on the oriented 3-simplices $\sigma^3$. Instead of RHS of (1) we use exact representation which we suggest in our work [8],

$$S(v, \Omega) = \sum_{\sigma^2} |v_{\sigma^2}| \arcsin \frac{v_{\sigma^2} \circ R_{\sigma^2}(\Omega)}{|v_{\sigma^2}|}$$  \hspace{1cm} (3)

where we have defined $A \circ B = \frac{1}{2} A^{ab} B_{ab}$, $|A| = (A \circ A)^{1/2}$ for the two tensors $A$, $B$; $v_{\sigma^2}$ is the dual bivector of the triangle $\sigma^2$ in terms of the vectors of its edges $l^a_1$, $l^a_2$,

$$v_{\sigma^2} = \frac{1}{2} \epsilon_{abcd} l^b_1 l^d_2$$  \hspace{1cm} (4)

(in some 4-simplex frame containing $\sigma^2$). The curvature matrix $R_{\sigma^2}$ on the 2-simplex $\sigma^2$ is the path ordered product of the connections $\Omega_{\sigma^3}^{\pm 1}$ on the 3-simplices $\sigma^3$ sharing $\sigma^2$ along the contour enclosing $\sigma^2$ once and contained in the 4-simplices sharing $\sigma^2$,

$$R_{\sigma^2} = \prod_{\sigma^3 \supset \sigma^2} \Omega_{\sigma^3}^{\pm 1}$$.  \hspace{1cm} (5)

As we can show, when substituting as $\Omega_{\sigma^3}$ the genuine rotations connecting the neighbouring local frames as functions of the genuine Regge lengths into the equations of motion for $\Omega_{\sigma^3}$ with the action (3) we get exactly the closure condition for the surface of the 3-simplex $\sigma^3$ (vanishing the sum of the bivectors of its 2-faces) written in the frame of one of the 4-simplices containing $\sigma^3$, that is, the identity. This means that (3) is the exact representation for (2).

We can pass to the continuous time limit in (3) in a nonsingular manner and recast it to the canonical (Hamiltonian) form [9]. This allows us to write out Hamiltonian path integral. The above problem of finding the measure which results in the Hamiltonian path integral measure in the continuous time limit whatever coordinate is chosen as time has solution in 3 dimensions [10]. A specific feature of the 3D case important for that is commutativity of the dynamical constraints leading to a simple form of the functional integral. The 3D action looks like (3) with area tensors $v_{\sigma^2}$ substituted by the edge vectors $l_{\sigma^3}$ independent of each other. In 4 dimensions, the variables $v_{\sigma^2}$
are not independent but obey a set of (bilinear) intersection relations. For example, tensors of the two triangles \( \sigma_1^2, \sigma_2^2 \) sharing an edge satisfy the relation

\[
\epsilon_{abcd} v_{\sigma_1^2}^{ab} v_{\sigma_2^2}^{cd} = 0.
\]

These purely geometrical relations can be called kinematical constraints. The idea is to construct quantum measure first for the system with formally independent area tensors. That is, originally we concentrate on quantization of the dynamics while kinematical relations of the type (6) are taken into account at the second stage. Note that the RC with formally independent (scalar) areas have been considered in the literature [4, 11].

The theory with formally independent area tensors can be called area tensor RC. Consider the Euclidean case. The Einstein action is not bounded from below, therefore the Euclidean path integral itself requires careful definition. Our result for the constructed in the above way completely discrete quantum measure [12] can be written as a result for vacuum expectations of the functions of the field variables \( v, \Omega \). Upon passing to integration over imaginary areas with the help of the formal replacement of the tensors of a certain subset of areas \( \pi \) over which integration in the path integral is to be performed,

\[
\pi \to -i\pi,
\]

the result reads

\[
< \Psi(\{\pi\}, \{\Omega\}) > = \int \Psi(-i\{\pi\}, \{\Omega\}) \exp \left( - \sum_{t\text{-like} \sigma^2} \tau_{\sigma^2} \circ R_{\sigma^2}(\Omega) \right) \cdot \exp \left( i \sum_{not \ t\text{-like} \sigma^2} \pi_{\sigma^2} \circ R_{\sigma^2}(\Omega) \right) \prod_{not \ t\text{-like} \sigma^2} d^6 \pi_{\sigma^2} \prod_{\sigma^3} D\Omega_{\sigma^3} \\
= \int \Psi(-i\{\pi\}, \{\Omega\}) d\mu_{\text{area}}(-i\{\pi\}, \{\Omega\}),
\]

where \( D\Omega_{\sigma^3} \) is the Haar measure on the group SO(4) of connection matrices \( \Omega_{\sigma^3} \). Appearance of some set \( \mathcal{F} \) of triangles \( \sigma^2 \) integration over area tensors of which is omitted (denoted as "t-like" in (7)) is connected with that integration over all area tensors is generally infinite, in particular, when normalizing measure (finding \( < 1 > \)). Indeed, different \( R_{\sigma^2} \) for \( \sigma^2 \) meeting at a given link \( \sigma^1 \) are connected by Bianchi identities [1]. Therefore for the spacetime of Minkowsky signature (when exponent is oscillating over all the area tensors) the product of \( \delta^6(R_{\sigma^2} - \bar{R}_{\sigma^2}) \) for all these \( \sigma^2 \) which follow upon
integration over area tensors for these $\sigma^2$ contains singularity of the type of $\delta$-function squared. To avoid this singularity we should confine ourselves by only integration over area tensors on those $\sigma^2$ on which $R_{\sigma^2}$ are independent, and complement $\mathcal{F}$ to this set of $\sigma^2$ are those $\sigma^2$ on which $R_{\sigma^2}$ are by means of the Bianchi identities functions of these independent $R_{\sigma^2}$. Let us adopt regular way of constructing 4D Regge structure of the 3D Regge geometries (leaves) of the same structure. The $t$-like edges connect corresponding vertices in the neighboring leaves (do not mix with the term ”timelike” which is reserved for the local frame components). The diagonal edges connect a vertex with the neighbors of the corresponding vertex in the neighboring leaf. The $t$-like simplices (in particular, $t$-like triangles) are then defined as those containing $t$-like edges; the leaf simplices are those completely contained in the leaf; the diagonal simplices are all others. It can be seen that the set of the $t$-like triangles is fit for the role of the above set $\mathcal{F}$. In the case of general 4D Regge structure we can deduce that the set $\mathcal{F}$ of the triangles with the Bianchi-dependent curvatures pick out some one-dimensional field of links, and we can simply take it as definition of the coordinate $t$ direction so that $\mathcal{F}$ be just the set of the $t$-like triangles. Also existence of the set $\mathcal{F}$ naturally fits our initial requirement that limiting form of the full discrete measure when any one of the coordinates (not necessarily $t$!) is made continuous by flattening the 4-simplices in the corresponding direction should coincide with Hamiltonian path integral (with that coordinate playing the role of time). Namely, in the Hamiltonian formalism absence of integration over area tensors of triangles which pick out some coordinate $t$ ($t$-like ones) corresponds to some gauge fixing.

Given the above (spontaneously arisen) asymmetry between the different area tensors we nevertheless can ask about maximally symmetrical form of the measure extended by inserting possible integrations [13]. We can integrate over $d^6\tau_{\sigma^2}$ in a non-singular way if $\delta$-functions are inserted which fix the scale of these tensors, say, at the level $\varepsilon \ll 1$. If the number of these $\delta$-functions is 4 per vertex, on physical hypersurface of ordinary RC this corresponds to fixing 4 degrees of freedom connected with lapse-shift vectors. The latter define location of the next in $t$ 3D leaf relative to the current leaf. Their continuum version in the Arnowitt-Deser-Misner Hamiltonian approach in the continuum GR are nondynamical variables and can be chosen by hand. Further, area tensor of $\sigma^2$ could be defined in any one of the 4-simplices $\sigma^4 \supset \sigma^2$, and the more detailed notation is $v_{\sigma^2|\sigma^4}$. Above the $v_{\sigma^2}$ means $v_{\sigma^2|\sigma^4}$ at some $\sigma^4$ = $\sigma^4(\sigma^2)$
⊃ 𝜎^2 \text{ (function of } 𝜎^2). \text{ Insert } d^6v_{\sigma^2|\sigma^4} \text{ for all } \sigma^4 \supset \sigma^2. \text{ As applied to functions of the above } v_{\sigma^2} \text{ in the frames of certain } \sigma^4(\sigma^2) \text{ only, the new integrations over } d^6v_{\sigma^2|\sigma^4}, \sigma^4 \neq \sigma^4(\sigma^2), \text{ simply contribute into a normalization factor (some intermediate regularization is implied which sets large but finite values for the integration limits over area tensors). Such the extended form of the measure is just used in the following when passing to physical hypersurface (of the ordinary RC).}

There is the invariant (Haar) measure $D\Omega$ in (7) which looks natural from symmetry considerations. From the formal point of view, in the Hamiltonian formalism (when one of the coordinates is made continuous) this arises when we write out standard Hamiltonian path integral for the Lagrangian with the kinetic term $\pi_{\sigma^2} \circ \bar{\Omega}_{\sigma^2} \dot{\Omega}_{\sigma^2}$ [10, 12]. To this end, one might pass to the variables $\Omega_{\sigma^2} \pi_{\sigma^2} = P_{\sigma^2}$ and $\Omega_{\sigma^2}$ (in 3D case used in [14, 10]). The kinetic term $P\dot{\Omega}$ with arbitrary matrices $P$, $\Omega$ leads to the standard measure $d^{16}P d^{16}\Omega$, but there are also $\delta$-functions taking into account II class constraints to which $P$, $\Omega$ are subject, $\delta^{10}(\bar{\Omega}\Omega - 1)\delta^{10}(\bar{\Omega}P + \bar{P}\Omega)$. Integrating out these just gives $d^6\pi D^6\Omega$. Following our strategy of recovering full discrete measure from requirement that it reduces to the Hamiltonian path integral whatever coordinate is made continuous, the same Haar measure should be present also in the full discrete measure.

One else specific feature of the quantum measure is the absence of the inverse trigonometric function 'arcsin' in the exponential, whereas the Regge action (3) contains such functions. This is connected with using the canonical quantization at the intermediate stage of derivation: in gravity this quantization is completely defined by the constraints, the latter being equivalent to those ones without arcsin (in some sense on-shell).

The theory with independent area tensors is locally trivial (just as 3D RC). In the considered formalism this explicitly exhibits at the negligibly small values of $\tau_{\sigma^2}$ when we get factorisation of the quantum measure obtained into the "elementary" measures on separate areas of the type

$$\exp (i\pi \circ R)d^6\pi D R.$$ \hspace{1cm} (8)

Upon splitting antisymmetric matrices ($\pi$ and generator of $R$) into self- and antiselfdual parts like

$$\pi_{ab} \equiv \frac{1}{2}^{+\pi_k} \Sigma^k_{ab} + \frac{1}{2}^{-\pi_k} \Sigma^k_{ab}$$ \hspace{1cm} (9)
\[ \pm R = \exp(\pm \hat{\phi} \hat{\Sigma}) = \cos \pm \hat{\phi} + \pm \hat{\Sigma} \mp n \sin \pm \hat{\phi} \]

(\(+n = \pm \hat{\phi} / \pm \hat{\phi}\) is unit vector and the basis of self- and antiselfdual matrices \(i \pm \Sigma_{ab}\) obeys the Pauli matrix algebra) the measure (8) splits as

\[ \exp(i \pi \circ \mp R) d^3 \mp \pi \mathcal{D} \mp R \cdot \exp(i \mp \pi \circ \pm R) d^3 - \mp \pi \mathcal{D} - \mp R \]

(10)

where \(\mathcal{D} \pm R = (4 \pi^2 \pm \phi^2)^{-1} \sin^2 \pm \phi d^3 \pm \phi\). When calculating expectations of powers of area vectors \(\pm \mp\) integrals over \(d^3 \pm \mp\) give (derivatives of) \(\delta\)-functions which are then easily integrated out giving

\[ \langle (\pm \mp)^{2k} \rangle = \text{const} \cdot \int (-i \mp \pi)^{2k} d^3 \pm \pi \int e^{i \pm \pi \circ \pm \mp R} \mathcal{D} \pm \mp R \]

\[ = \text{const} \cdot \int \left[ \partial^2_{\pm r} \delta^{(\pm r)} \right] \frac{d^3 \pm r}{\sqrt{1 - \pm r^2}} = \frac{4^{-k}(2k + 1)!(2k)!}{k!^2} \]  

(11)

where \(\pm r = \pm n \sin \pm \phi\), and ”const” is a normalization factor. Knowing how monomials are averaged we can select the measure needed for that; thereby the result extends to averaging arbitrary polynomial or, by continuity, practically arbitrary function,

\[ \langle f(\mp) \rangle = \int f(-i \mp) d^6 \pi \int e^{i \mp \circ R} \mathcal{D} \mathcal{R} \]

\[ = \int f(\mp) \nu(\mp \mp) \nu(\pm \pm) d^3 \mp \pi d^3 - \mp \pi \]

\[ = \frac{s}{\pi} \int \exp (-s \cosh \eta) \cosh \eta \, d\eta = \frac{2s}{\pi} K_1(s). \]

(12)

\(K_1\) is the modified Bessel function. (A shorter way to get the same is to proceed by moving integration contours over curvatures to complex plane [15].)

Next, as considered below the equation (6), we are aiming at implementing kinematical relations of that type in order to get quantum measure in the genuine ordinary RC from the obtained measure in area tensor RC. For that we find the measure of interest as the result of reducing the measure obtained in the superspace of independent area tensors onto the hypersurface \(\Gamma_{\text{Regge}}\) corresponding to the ordinary RC in this superspace. The quantum measure can be considered as a linear functional \(\mu_{\text{area}}(\Psi)\) on the space of functionals \(\Psi(\{v\})\) on the configuration space (for our purposes here it is sufficient to restrict ourselves to the functional dependence on the area tensors \(\{v\}\); the dependence on the connections is unimportant). The physical assumption is that we can consider ordinary RC as a kind of the state of the more general system with independent area tensors. This state is described by the following functional,

\[ \Psi(\{v\}) = \psi(\{v\}) \delta_{\text{Regge}}(\{v\}), \]

(13)
where $\delta_{\text{Regge}}(\{v\})$ is the (many-dimensional) $\delta$-function with support on $\Gamma_{\text{Regge}}$. The derivatives of $\delta_{\text{Regge}}$ have the same support, but these violate positivity in our subsequent construction. To be more precise, delta-function is distribution, not function, but can be treated as function if regularised. If the measure on such the functionals exists in the limit when regularisation is removed, this allows to define the quantum measure on $\Gamma_{\text{Regge}},$

$$\mu_{\text{Regge}}(\cdot) = \mu_{\text{area}}(\delta_{\text{Regge}}(\{v\}) \cdot).$$  \hfill (14)

Uniqueness of the construction of $\delta_{\text{Regge}}$ follows under quite natural assumption of the minimum of lattice artefacts. Let the system be described by the metric $g_{\lambda\mu}$ constant in each of the two 4-simplices $\sigma_4^1$, $\sigma_4^2$ separated by the 3-face $\sigma^3 = \sigma_4^1 \cap \sigma_4^2$ formed by three 4-vectors $\iota^\lambda_a$. These vectors also define the metric induced on the 3-face, $g^\parallel_{ab} = \iota^\lambda_a g_{\lambda\mu}$. The continuity condition for the induced metric is taken into account by the $\delta$-function of the induced metric variation,

$$\Delta_{\sigma^3} g^\parallel_{ab} \overset{\text{def}}{=} g^\parallel_{ab}(\sigma_4^1) - g^\parallel_{ab}(\sigma_4^2).$$  \hfill (15)

As for the $\delta_{\text{Regge}}$, it is of course defined up to a factor which is arbitrary function nonvanishing at nondegenerate field configurations. In the spirit of just mentioned principle of minimizing the lattice artefacts it is natural to choose this factor in such the way that the resulting $\delta$-function factor would depend only on hyperplane defined by the 3-face but not on the form of this face, that is, would be invariant with respect to arbitrary nondegenerate transformations $\iota_a^\lambda \mapsto m_b^a \iota_b^\lambda$. To ensure this, the $\delta$-function should be multiplied by the determinant of $g^\parallel_{ab}$ squared. This gives

$$[\det(\iota_a^\lambda \iota_b^\mu g_{\lambda\mu})]^2 \delta^6(\iota_a^\lambda \iota_b^\mu \Delta_{\sigma^3} g_{\lambda\mu}) = V_{\sigma^3}^4 \delta^6(\Delta_{\sigma^3} S_{\sigma^3}).$$  \hfill (16)

Here $S_{\sigma^3}$ is the set of the 6 edge lengths squared of the 3-face $\sigma^3$, $V_{\sigma^3}$ is the volume of the face.

Further, the product of the factors (16) over all the 3-faces should be taken. As a result, we have for each edge the products of the $\delta$-functions of the discontinuity of its length between the 4-simplices taken along closed contours, $\delta(s_1-s_2)\delta(s_2-s_3)\ldots\delta(s_N-s_1)$ containing singularity of the type of the $\delta$-function squared. In other words, the conditions equating (15) to zero on the different 3-faces are not independent. The more detailed consideration allows us to cancel this singularity in a way symmetrical with respect to the different 4-simplices (thus extracting irreducible conditions), the
resulting $\delta$-function factor remaining invariant with respect to arbitrary deformations of the faces of different dimensions keeping each face in the fixed plane spanned by it [16].

Besides factors (16), we need to impose the conditions ensuring that tensors of the 2-faces in the given 4-simplex define a metric in this simplex [13]. These conditions of the type of (6) can be easily written in general form. Let a vertex of the given 4-simplex be the coordinate origin and the edges emitted from it be the coordinate lines $\lambda, \mu, \nu, \rho, \ldots = 1, 2, 3, 4$. Then the (ordered) pair $\lambda\mu$ means the (oriented) triangle formed by the edges $\lambda, \mu$. The conditions of interest take the form

$$
\epsilon_{abcd} v^a_{\lambda\mu} v^d_{\nu\rho} - \frac{1}{4!} \left( \epsilon^{\sigma\varphi\chi} \epsilon_{abcd} v_{\xi\sigma} v^c_{\nu\rho} \right) \epsilon_{\lambda\mu\nu\rho} = 0.
$$

(17)

This expresses proportionality of $\epsilon_{abcd} v^a_{\lambda\mu} v^d_{\nu\rho}$ to $\epsilon_{\lambda\mu\nu\rho}$. The LHS is symmetric $6 \times 6$ matrix w.r.t. the antisymmetric pairs $\lambda\mu, \nu\rho$. It has 21 different nontrivial elements of which 20 are independent ones (contraction with $\epsilon^{\lambda\mu\nu\rho}$ gives identical zero). These 20 equations define the 16-dimensional surface $\gamma(\sigma^4)$ in the 36-dimensional configuration space of the six antisymmetric tensors $v^a_{\lambda\mu}$. The factor of interest in quantum measure is the product of the $\delta$-functions with support on $\gamma(\sigma^4)$ over all the 4-simplices $\sigma^4$. The covariant form of the constraints (17) with respect to the world index means that the product of these $\delta$-functions in each the 4-simplex is the scalar density of a certain weight with respect to the world index, that is, the scalar up to power of the 4-volume $V_{\sigma^4}$. Therefore introducing the factors of the type $V_{\sigma^4}^\eta$ we get the scalar at some parameter $\eta$. Namely, the product of the factors

$$
\prod_{\sigma^4} \int_{V_{\sigma^4}^\eta} \delta^{21}(\epsilon_{abcd} v^a_{\lambda\mu} v^d_{\nu\rho} - V_{\sigma^4} \epsilon_{\lambda\mu\nu\rho}) \, dV_{\sigma^4}
$$

(18)

at $\eta = 20$ is the world index scalar, i.e. invariant w.r.t. arbitrary deformations of the 4-simplex $v^a_{\lambda\mu} \mapsto \xi^a_{\lambda\mu} v^a_{\nu\rho}$ as is desirable from the viewpoint of minimization of the lattice artefacts. The (18) is a short symmetrical way to write our 20 irreducible conditions. Here $V_{\sigma^4}$ serves as dummy variable, integrating over it simply excludes $V_{\sigma^4}$ from 21 conditions and yields 20 delta functions of 20 independent conditions per $\sigma^4$.

Qualitatively, it is important that $\delta$-factors (16), (18) automatically turn out to be invariant w.r.t. the overall rescaling area tensors. Therefore introducing these into the

1There are also the linear constraints of the type $\sum \pm v = 0$ providing closing surfaces of the 3-faces of our 4-simplex. These are assumed to be already resolved.
measure functional (14) turns out to keep convergence properties of the corresponding integrals. The integrals convergent in area tensor RC remains convergent on physical hypersurface of ordinary RC (both at infinite or at infinitely small area tensors). To be exact, invariance of the additional factors w.r.t. the scaling only $\pi_{\sigma^2}$ is needed for that. As these stand, these factors possess this property. For example, it is seen for (16) upon rewriting it in terms of the triples of area vectors of the 3-face $\sigma^3 = \sigma_1^4 \cap \sigma_4^2$, namely, $v^{(1)}_1, v^{(1)}_2, v^{(1)}_3$ defined in $\sigma_1^4$ and $v^{(2)}_1, v^{(2)}_2, v^{(2)}_3$ defined in $\sigma_2^4$, as $[v^{(1)}_1 \times v^{(1)}_2 \cdot v^{(1)}_3]_4 \delta^6 (v^{(1)}_\alpha \cdot v^{(1)}_\beta - v^{(2)}_\alpha \cdot v^{(2)}_\beta)$ (modulo (18) there is no matter whether $v$ means $+v$ or $-v$ here). Violation of this property of invariance w.r.t. rescaling $\pi_{\sigma^2}$ might arise when some $\pi_{\sigma^2}$ in these factors are expressed (see footnote on page 10) as an algebraic sum of some another $\pi_{\sigma^2}$ chosen as independent variables plus some $\tau_{\sigma^2}$. However, the role of this circumstance is that different $\pi_{\sigma^2}$ on physical hypersurface cannot achieve 0 simultaneously, and convergence properties of the measure cannot become worse when passing from area tensor to ordinary RC.

Now consider to what extent our system can be similar to the ordinary lattice field theory in which correlators (Green functions) are well defined due to the lattice regularization, UV cut off being determined by the (fixed) lattice spacing. Knowing the lengths expectation values is not sufficient to make conclusion on possible finiteness of the theory, we need a more detailed study of the probability distribution for the linklengths, that is, of the quantum measure.

Indeed, let the linklengths are allowed to be arbitrarily close to zero with some probability. (This does not contradict to the statement on their expectation values being finite and nonzero.) We may speak of the *dynamical* lattice with spacings being dynamical variables (linklengths) themselves. In fact, we have an ensemble of the lattices with different spacings. If linklengths can be found with noticeable probability in the arbitrarily small neighborhood of zero, this means that the ensemble includes the lattices in the limit of zero spacings, that is, in the limit of the regularization removed, and finiteness of the theory is not evident.

On the contrary, let the quantum measure has the support strictly separated from zero lengths. In this case the theory is thought to be finite like a lattice theory with the difference that now the lattice is the *dynamical* one. (At large areas/lengths suitable properties are provided by the exponential cut off in the measure.)

Qualitative considerations lead to namely this last possibility. Let us make a simple
scaling estimate. Consider estimation model used in [13]. The half of 36 components of the 6 antisymmetric tensors $\tau_{\lambda \mu}^{ab}$ in a given 4-simplex are dynamical $\pi_{\lambda \mu}^{ab} (\lambda \mu = 12, 23, 31)$, another half are $\tau_{\lambda \mu}^{ab} (\lambda \mu = 14, 24, 34$, and a scale $\varepsilon \ll 1$ for $\tau_{\sigma 2}$ is chosen). Denote by $x$ a scale of tensors $\pi_{\sigma 2}$ in the given 4-simplex. Then $d^{18} \pi_{\lambda \mu}^{ab}$ behaves like $x^{17} dx$ (also together with the factor (18) due to its invariance w.r.t. the rescaling $\pi_{\sigma 2}$). Besides that, there is the factor in the measure (12) which gives $x^{-4} e^{-x}$ for each leaf/diagonal triangle (one might write $e^{-\lambda x}$ with $\lambda = O(1)$ but $x$ is itself defined up to a value of the order of unity). Finally, the factors like (16) serve to equate the scales $x_1$ and $x_2$ in the 4-simplices sharing a 3-face and in our scaling estimate effect of such factor is equivalent to the effect of $x_1 \delta(x_1 - x_2)$ on the two measures $f_1(x_1) dx_1$ and $f_2(x_2) dx_2$: $f_1(x_1) dx_1 f_2(x_2) dx_2 x_1 \delta(x_1 - x_2) \Rightarrow x f_1(x) f_2(x) dx$. Collecting together just considered factors in the measure according to this rule we find

\begin{equation}
(e^{-x} x^{-4})^{L_2} x^{18 N_4} dx/x = (e^{-x} x^{8})^{3N_4/2} dx/x \tag{19}
\end{equation}

where $L_2$ is the number of the leaf/diagonal triangles and $N_4$ is the number of the 4-simplices. We have used simple combinatorial relation $N_4/L_2 = 2/3$ [13] for the above considered (after eq. (7)) regular way of constructing 4D Regge structure from the 3D ones. As $N_4$ grows, the factor in (19) approaches $\delta(x - 8)$.

Thus, the presumable picture looks as if the main contribution to the path integral were from linklengths being the sum of some constant (uniform) part $l_0$ of the order of Plank scale plus small fluctuations around it. The finite nonzero $l_0$ provides finite calculational framework for a correlator (say, of two or more defect angles or bivectors located at a certain distances), as if it were considered on the lattice with fixed spacing $l_0$.

Of course, the more detailed analytical and, probably, numerical investigations are in order to confirm this picture.

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