Modular categories, orbit method and character sheaves on unipotent groups

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Abstract

Let $G$ be a unipotent group over a field of characteristic $p > 0$. The theory of character sheaves on $G$ was initiated by V. Drinfeld and developed jointly with D. Boyarchenko. They also introduced the notion of L-packets of character sheaves. Each L-packet can be described in terms of a modular category. Now suppose that the nilpotence class of $G$ is less than $p$. Then the L-packets are in bijection with the set $g^*/G$ of coadjoint orbits, where $g$ is the Lie ring scheme obtained from $G$ using the Lazard correspondence and $g^*$ is the Serre dual of $G$. If $\Omega$ is a coadjoint orbit, then the corresponding modular category can be identified with the category of $G$-equivariant local systems on $\Omega$. This in turn is equivalent to the category of finite dimensional representations of a finite group. However, the associativity, braiding and ribbon constraints are nontrivial. Drinfeld gave a conjectural description of these constraints in 2006. In this article, we prove the formula describing the ribbon structure when $\dim(\Omega)$ is even.

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1 Introduction

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and let \( G \) be a unipotent group over \( k \). Fix a prime \( \ell \) different from \( p \) and let \( D(G) \) denote the bounded derived category of constructible complexes of \( \ell \)-adic sheaves on \( G \). Let \( D_G(G) \) be the category of \( G \)-equivariant objects of \( D(G) \), where \( G \) acts on itself by conjugation. The category \( D_G(G) \) is monoidal with respect to convolution with compact support. A (weak) idempotent in \( D_G(G) \) is an element \( e \in D_G(G) \) such that \( e * e \cong e \). A nonzero idempotent \( e \) is called minimal if for every idempotent \( e' \), one has \( e * e' \cong e \) or \( 0 \). The following fundamental result is proved in [BD2] (cf. Theorem 1.15): If \( e \in D_G(G) \) is a minimal idempotent, then \( eD_G(G) \) is the derived category of a modular category \( M_e \) (a precise definition of \( M_e \) is given in Section 2.1). Two fundamental problems in the theory of character sheaves over unipotent groups are:

1. Give a concrete description of the set \( \hat{G} \) of isomorphism classes of minimal idempotents in \( D_G(G) \).
2. For each \( e \in \hat{G} \), describe the modular category \( M_e \).

Partial solutions of the above problems are known in the general situation (with arbitrary \( G \)). Essential to this approach is a class of special minimal idempotents called Heisenberg idempotents. These idempotents as well as the corresponding modular categories can be described explicitly. Moreover, one can show that the study of arbitrary idempotents can be reduced such idempotents. For an exposition of this theory, we refer to [BD2] and [Des].

Now assume that \( G \) is connected and the nilpotence class of \( G \) is less than \( p \). In this case, a suitable modification of the orbit method applies and a complete answer to problem (1) is known. Let \( g \) be the Lie ring scheme associated to \( G \) via the Lazard correspondence and let \( g^* \) be the Serre dual of \( g \). Using Fourier-Deligne transform, one can identify \( \hat{G} \) with the set \( g^* / G \) of coadjoint orbits (cf. Section 2.5). Let \( e \in D_G(G) \) be a minimal idempotent and let \( \Omega \in g^* \) be the coadjoint orbit corresponding to \( e \). It can be shown that \( M_e \) is equivalent to the category of \( G \)-equivariant local systems on \( \Omega \). If \( \chi \in \Omega \) and \( G_\chi \) is the stabilizer of \( \chi \) in \( G \), then the later category is isomorphic to \( \text{Rep}(\Gamma) \), where \( \Gamma = \pi_0(G_\chi) \). Thus one has a equivalence \( M_e \simeq \text{Rep}(\Gamma) \). This equivalence transforms the monoidal functor on \( M_e \) to usual tensor product on \( \text{Rep}(\Gamma) \). However, the ribbon, braiding and associativity constraints in \( M_e \) induce nontrivial constraints in \( \text{Rep}(\Gamma) \). If \( A = \mathcal{O}_e[\Gamma] \), then these constraints are given by certain elements in \( A^\times \), \((A \otimes A)^\times\) and \((A \otimes A \otimes A)^\times\). A conjectural description of these elements were proposed by Drinfeld. The main result of this paper is a proof of his formula for the ribbon element in \( A^\times \). The precise statement appears in Theorem 3.3.

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2 The orbit method

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and let \( G \) be a connected unipotent group over \( k \). If the nilpotence class of \( G \) less than \( p \), then one can use the orbit method to study character sheaves on \( G \). In this section, we briefly recall the relevant ideas and constructions from [BD1].

In the remainder of this article we shall assume (unless otherwise stated) that:

Convention. The nilpotence class of \( G \) is less than \( p \) and \( G \) is a perfect unipotent group (cf. Section 1.9, [BD2]).

The first condition is necessary for the orbit method to work. The second condition is imposed to avoid technical difficulties involving Serre duality (of perfect commutative unipotent groups - cf. Section 2.3 below). However, the results of this paper are independent of this condition, as is explained in [BD2].
Fix a prime $\ell$ different from $p$. For a scheme $X$ over $k$, let $\mathcal{D}(X)$ denote the bounded derived category of constructible complexes of $\ell$-adic sheaves on $X$. If $X$ is equipped with a $G$ action, then let $\mathcal{D}_G(X)$ denote the category of $G$-equivariant objects of $\mathcal{D}(X)$ (cf. Appendix E, [BD2]).

### 2.1 The category $\mathcal{D}_G(G)$ and character sheaves on $G$

In this subsection, we give the definition of the category $\mathcal{M}_e$ mentioned in the introduction. The assumption on the nilpotent class of $G$ is not used.

Let $\mathcal{D}_G(G)$ be the category of $G$-equivariant objects of $\mathcal{D}(G)$, where $G$ acts on itself by conjugation. Let $M, N$ be objects on $\mathcal{D}_G(G)$. The convolution of $M$ and $N$ is the object $M \ast N$ of $\mathcal{D}_G(G)$ defined by

$$M \ast N = \mu_1(\text{pr}_1^*(M) \otimes \text{pr}_2^*(N))$$

where $\mu : G \times G \rightarrow G$ is the multiplication map and $\text{pr}_1, \text{pr}_2 : G \times G$ are projections onto the first and second factors. The category $\mathcal{D}_G(G)$ is a monoidal category with respect to the convolution bifunctor. Moreover, one can define natural isomorphisms

$$\beta_{M,N} : M \ast N \rightarrow N \ast M$$

which equips $\mathcal{D}_G(G)$ with the structure of a braided category. These isomorphisms are constructed using the $G$ equivariance of $N$ (cf. Section 5.5, [BD1]). Further, there is a canonical automorphism $\theta$ of the identity functor of $\mathcal{D}_G(G)$. If $M \in \mathcal{D}_G(G)$, then the automorphism $\theta_M : M \rightarrow M$ can be described as follows. For every $g, x \in G$, one has isomorphisms $\phi_{g,x} : M_g \rightarrow M_{g^x}$. Setting $g = x$ gives an automorphism of $M_x$, which is none other than $(\theta_M)_x$. More precisely, if $e : G \times G \rightarrow G$ is defined by $e(x,g) = xg^{-1}$, then the $G$ equivariant structure on $M$ yields an isomorphism $\text{pr}_2^* \rightarrow e^*M$. Pulling back this isomorphism by the diagonal map $\Delta : G \rightarrow G \times G$ gives $\theta_M$. One has the following relation for all $M, N \in \mathcal{D}_G(G)$:

$$\theta_{M \ast N} = \beta_{N,M} \circ \beta_{M,N} \circ (\theta_M \ast \theta_N)$$

This can be checked directly. An abstract proof is given in Proposition A.46, [BD2].

Let $e$ be a minimal idempotent of $\mathcal{D}_G(G)$. The modular category $\mathcal{M}_e$ mentioned in the introduction is defined as follows. Let $e\mathcal{D}_G(G)$ be the full subcategory of $\mathcal{D}_G(G)$ consisting of objects of the form $e \ast M$ where $M \in \mathcal{D}_G(G)$. Let $\mathcal{M}_e^{\text{perv}} \subset e\mathcal{D}_G(G)$ denote the full subcategory of perverse objects in $e\mathcal{D}_G(G)$. A character sheaf on $G$ is defined to be an indecomposable object of $\mathcal{M}_e^{\text{perv}}$ for some minimal idempotent $e$. It is shown in [BD2] that there exists a (unique) integer $n_e \geq 0$ such that $e[-n_e] \in \mathcal{M}_e^{\text{perv}}$. One now defines $\mathcal{M}_e$ to be the subcategory $\mathcal{M}_e^{\text{perv}}[n_e]$ of $\mathcal{D}_G(G)$. It is closed under convolution and is a modular category, with braiding and ribbon structure given by $\beta$ and $\theta$ (cf. Theorem 1.15, [BD2]).

### 2.2 Lazard correspondence

A Lie ring is an additive group $\mathfrak{p}$ equipped a biadditive map $[\cdot, \cdot] : \mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p}$ satisfying the Jacobi identity and the identity $[x,x] = 0$. Let $\mathfrak{nil}_L$ be the category of all nilpotent Lie rings (i.e., Lie algebras over $\mathbb{Z}$) $\mathfrak{p}$ satisfying the following condition:

If $n$ is the nilpotence class of $\mathfrak{p}$, then the map $\mathfrak{p} \times n! \rightarrow \mathfrak{p}$ of multiplication by $n!$ is invertible. (L)

One can define a multiplication operation $\mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p}$, $(x, y) \mapsto xy$ by using the Campbell-Hausdorff formula

$$xy = \log(e^x \cdot e^y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \cdots$$

If $\text{CH}_i$ is homogeneous component of degree $i$ is the above series, then one knows that $\text{CH}_i$ is a Lie polynomial over $\mathbb{Z}[1/i!]$ (cf. Section IV.8 of [Se]). The condition (L) ensures that the above formula is well defined. Let $\text{Exp}(\mathfrak{p})$ be the group whose underlying set is same as that of $\mathfrak{p}$ and whose product operation is given by the above formula. The inverse of $x$ is $-x$ and the identity element is $0 \in \mathfrak{p}$. If
φ : p₁ → p₂ is a homomorphism of Lie rings in nilpf, then it is immediate that φ, viewed as a map
Exp(p₁) → Exp(p₂), is a group homomorphism. In other words, Exp is a functor from nilpf to the
category of groups. Let Nilf be the category of all nilpotent groups G such that if n is the nilpotence
class of G, then the map G → G, g → gⁿ is invertible.

**Theorem (Lazard).** If p ∈ nilpf then Exp(p) ∈ Nilf. The functor Exp : nilpf → Nilf is an isomorphism
of categories.

The inverse of the functor Exp is denoted by Log. A proof of the theorem can be found in Laz or
Khui. The difficult part is to show that one can recover p from Exp(p), i.e., the addition and Lie bracket
operations can be expressed in terms of the product operation. For example, in nilpotence class 2, it can
be checked that [x, y] = (x, y), x + y = xy(x, y)^{-1/2} where (x, y) denotes xy⁻¹y⁻¹.

For x ∈ Exp(p), the conjugation action of x on Exp(p) gives an automorphism of Exp(p). By the
theorem of Lazard, it is also an isomorphism of the Lie ring p. This is easy to check directly using the
following formula:

**Lemma 1.** \(xyx^{-1} = e^{\text{ad}(x)}(y) = y + [x, y] + \frac{1}{2}[x, [x, y]] + \frac{1}{6}[x, [x, [x, y]]] + \cdots\)

**Proof.** We have \(e^{xyx^{-1}} = e^{x} \cdot e^{y} \cdot e^{-x} = 1 + e^{x} \cdot y \cdot e^{-x} + \frac{1}{2}e^{x} \cdot y^{2} \cdot e^{-x} + \cdots = e^{x} \cdot y \cdot e^{-x} \cdot e^{-x} \cdots \). An easy computation
shows that \(e^{x} \cdot y \cdot e^{-x} = x + [x, y] + \frac{1}{2}[x, [x, y]] + \cdots \). QED.

One can use the Lazard correspondence to produce a Lie ring scheme (i.e., a Lie ring object in the
category of schemes) from G. Indeed, for every integer n with 1 ≤ n ≤ p − 1, the multiplication by n
map from G to itself is invertible (this follows by using a filtration of G such that the successive
quotients are isomorphic to the additive group). Hence for every scheme S over k, the group G(S) belongs to Nilf.
Hence the set G(S) can be given the structure of a Lie ring. Therefore one obtains the structure of a Lie
ring scheme on G. This Lie ring scheme is denoted by Log(G). Note that G and Log(G) are equal as
schemes.

### 2.3 Serre duality and multiplicative local systems

We refer to Appendix F, [BDI] for a more detailed exposition. Let perf denote the category of perfect
schemes over k and let A be a commutative, unipotent, connected group scheme in perf. One considers
the following functor, from the category perf to abelian groups:

\[ S \mapsto \text{Ext}^1(A \times S, \mathbb{Q}/\mathbb{Z}_p) \]

where Ext¹ is computed in the category of commutative group schemes over S and \(\mathbb{Q}/\mathbb{Z}_p\) is viewed as
a discrete group scheme over S. One knows that the functor defined above is representable by a group
scheme A⁺ in perf (cf. [Beg], Prop. 1.2.1). The group A⁺ is called the Serre dual of A. One has the
following properties (loc. cit.):

i) A⁺ is a connected, commutative, unipotent group scheme isogenous to A.

ii) The canonical homomorphism of A onto A⁺⁺ is an isomorphism.

iii) If 0 → A' → A → A'' → 0 is an exact sequence of connected, commutative, unipotent group
schemes in perf, then so is 0 → A''⁺⁺ → A⁺ → A⁺⁺ → 0.

iv) If f : A → A' is an isogeny, then so is f⁺ : A⁺ → A⁺.

One also has the following alternative interpretation of the Serre dual. Let H be a group scheme
over k and let L be a \(\mathbb{Q}_ℓ\)-local system on H. If μ : H × H → H is the multiplication map and pr₁, pr₂ : H × H → H are the projection maps, then L is called a multiplicative local system if L is nonzero
and \(μ^* L \cong \text{pr}^*_1 L \otimes \text{pr}^*_2 L\). Fix an injective homomorphism (of groups):

\[ \psi : \mathbb{Q}/\mathbb{Z}_p \rightarrow \mathbb{Q}_ℓ^\times \]

An element \(χ ∈ A⁺\) is a (central) extension of A by \(\mathbb{Q}/\mathbb{Z}_p\). Using the embedding \(\psi : \mathbb{Q}/\mathbb{Z}_p \rightarrow \mathbb{Q}_ℓ^\times \), one
can view \(χ\) as a multiplicative \(\mathbb{Q}_ℓ\)-local system on A. Conversely, it can be proved that every multiplicative
\(\mathbb{Q}_ℓ\)-local system on A comes from an element of A⁺ via \(ψ\) (cf. Lemma 7.3 in [Boy1]).
2.4 Fourier-Deligne transform

Let \( \mathfrak{g} \) denote the Lie ring scheme \( \text{Log}(G) \) (cf. Section 2.2) obtained from \( G \) using the Lazard correspondence. In particular, \( \mathfrak{g} \) is a commutative group scheme and thus its Serre dual \( \mathfrak{g}^* \) is well defined. Let \( \mathcal{U} \) be the universal \( \mathbb{Q}_p / \mathbb{Z}_p \) torsor on \( \mathfrak{g} \times \mathfrak{g}^* \). If \( g \in \mathfrak{g} \) and \( \chi \in \mathfrak{g}^* \), then the stalk of \( \mathcal{U} \) at \( (g, \chi) \) is the stalk of \( \chi \) at \( g \). Let \( \mathcal{E} \) be the local system on \( \mathfrak{g} \times \mathfrak{g}^* \) obtained from \( \mathcal{U} \) using the homomorphism \( \psi \). The Fourier-Deligne transform is the functor

\[
\mathcal{F} : \mathcal{D}(\mathfrak{g}) \to \mathcal{D}(\mathfrak{g}^*)
\]

defined by

\[
\mathcal{F}(M) = \text{pr}'(\text{pr}^*(M) \otimes \mathcal{E})
\]

where \( \text{pr}, \text{pr}' \) are projections:

\[
\mathfrak{g} \xrightarrow{\text{pr}'} \mathfrak{g} \times \mathfrak{g}^* \xrightarrow{\text{pr}} \mathfrak{g}
\]

One also has the functor \( \mathcal{F}' : \mathcal{D}(\mathfrak{g}^*) \to \mathcal{D}(\mathfrak{g}) \) defined by \( \mathcal{F}'(N) = \text{pr}_!(\text{pr}^*(N) \otimes \mathcal{E}) \). Let \( d = \dim(\mathfrak{g}) \). For each \( M \in \mathcal{D}(\mathfrak{g}) \), there exist a natural isomorphism

\[
(\mathcal{F}' \circ \mathcal{F})(M) \cong (-1)^* M[-2d][-d]
\]

where \(-1 : \mathfrak{g} \to \mathfrak{g}\) denotes the inverse map. This is proved in [Sa] (cf. Théorème 2.2.4.1). It follows that \( \mathcal{F} \) is an equivalence of categories and a quasi-inverse functor is given by \( N \mapsto (-1)^* \mathcal{F}'(N)[2d]/(d) \). Further, \( \mathcal{F} \) transforms convolution in \( \mathcal{D}(\mathfrak{g}) \) (with respect to the additive group structure in \( \mathfrak{g} \)) to tensor product in \( \mathcal{D}(\mathfrak{g}^*) \). That is, for all \( M, N \in \mathcal{D}(\mathfrak{g}) \), there exist natural bifunctorial isomorphisms

\[
\mathcal{F}(M * N) \cong \mathcal{F}(M) \otimes \mathcal{F}(N)
\]

inducing an equivalence of the symmetric monoidal categories \( (\mathcal{D}(\mathfrak{g}), \ast) \) and \( (\mathcal{D}(\mathfrak{g}^*), \otimes) \). This is proved in Proposition H.11, [BD1].

2.5 The bijection between \( \hat{G} \) and \( \mathfrak{g}^*/G \)

Consider the action of \( G \) on itself by conjugation. Since \( G \) and \( \mathfrak{g} \) are equal as schemes, this induces an action of \( G \) on \( \mathfrak{g} \). One also has the corresponding contragradient action of \( G \) on \( \mathfrak{g}^* \). It is proved in [BD1] (cf. Appendix H) that \( \mathcal{F} \) can be lifted to an equivalence

\[
\mathcal{F} : \mathcal{D}_G(\mathfrak{g}) \to \mathcal{D}_G(\mathfrak{g}^*)
\]

of the monoidal categories \( (\mathcal{D}_G(\mathfrak{g}), \ast) \) and \( (\mathcal{D}_G(\mathfrak{g}^*), \otimes) \).

In the following, we shall identify \( \mathcal{D}_G(\mathfrak{g}) \) and \( \mathcal{D}_G(\mathfrak{g}^*) \). For \( M, N \in \mathcal{D}_G(\mathfrak{g}) \), let \( M \ast_m N \) denote that additive convolution with respect to \( \mathfrak{g} \) and let \( M \ast_m N \) denote the multiplicative convolution with respect to \( G \). The important observation is that there exist bifunctorial isomorphisms

\[
M \ast_m N \cong M \ast_m N
\]

for all \( M, N \in \mathcal{D}_G(\mathfrak{g}) \). These isomorphisms are not canonically defined and depend on the choice of certain Lie polynomials (cf. Proposition 5.11, loc. cit.). However, it follows that an element \( e \in (\mathcal{D}_G(\mathfrak{g}), \ast_m) \) is a minimal idempotent if and only if it is so in \( (\mathcal{D}_G(\mathfrak{g}), \ast_m) \). Via the Fourier-Deligne transform, such idempotents correspond to minimal idempotents in \( (\mathcal{D}_G(\mathfrak{g}^*), \otimes) \). Let \( \Omega \) be a \( G \)-orbit in \( \mathfrak{g}^* \). Since \( G \) is unipotent, one knows that \( \Omega \) is closed in \( \mathfrak{g}^* \). Let \( i_\Omega : \Omega \hookrightarrow \mathfrak{g}^* \) be the inclusion map and let \( (\underline{\mathcal{O}}_\Omega)_{\Omega} \) be the constant sheaf on \( \Omega \) with stalk \( \underline{\mathcal{O}}_\Omega \). Then \( e_\Omega = (i_\Omega)_!(\underline{\mathcal{O}}_\Omega)_{\Omega} \) is a minimal idempotent of \( \mathcal{D}_G(\mathfrak{g}^*) \). Moreover, all such idempotents are of the form \( e_\Omega \) for a suitable orbit \( \Omega \subset \mathfrak{g}^* \). Therefore one obtains a bijection between \( \hat{G} \) and \( \mathfrak{g}^*/G \).

Let \( e \) be the minimal idempotent of \( \mathcal{D}_G(G) \) corresponding to an orbit \( \Omega \). One has \( \mathcal{F}(e) \cong e_\Omega \). The Fourier-Deligne transform gives an equivalence between \( e \mathcal{D}_G(G) \) and \( e_\Omega \mathcal{D}_G(\mathfrak{g}^*) \), which can be identified
with \( D_G(\Omega) \). This equivalence transforms convolution in \( eD_G(G) \) to usual tensor product in \( D_G(\Omega) \). However, for general \( G \), the braiding and associativity constraints in \( eD_G(G) \) are transformed into nontrivial constraints in \( D_G(\Omega) \). This is because the isomorphisms in \( * \) above are not compatible (in general) with the braiding and associativity constraints in \( D_G(G) \) and \( D_G(g) \). Let \( \text{Loc}_G(\Omega) \) denote the full subcategory of \( D_G(\Omega) \) consisting of \( G \)-equivariant local systems on \( \Omega \).

**Lemma 2.** The Fourier-Deligne transform induces an equivalence between \( M_\epsilon \) and \( \text{Loc}_G(\Omega) \).

**Proof.** The category of the perverse objects in \( D_G(\Omega) \) is \( \text{Loc}_G(\Omega)[\dim\Omega] \). Let \( d = \dim G \). As \( F[d] \) takes perverse sheaves to perverse sheaves (cf Appendix H, [BDI]), it follows that \( M_\epsilon^{\text{per}} \) is equivalent to \( \text{Loc}_G(\Omega)[\dim\Omega - d] \). Therefore \( M_\epsilon^{\text{per}}[d - \dim\Omega] \) contains \( e \) and thus \( M_\epsilon = M_\epsilon^{\text{per}}[d - \dim\Omega] \), which is equivalent to \( \text{Loc}_G(\Omega) \), qed.

### 3 Statement of the main result

#### 3.1 Skew-symmetric biextensions

We recall several facts about biextensions from Appendix A of [Boy1]. For the definition of biextensions in general, the reader is referred to loc. cit. Let \( a, b \) be perfect, commutative, connected, unipotent groups and let \( E \) be a biextension of \( a \times b \) by \( \mathbb{Q}_p/\mathbb{Z}_p \). In particular, \( E \) is a \( \mathbb{Q}_p/\mathbb{Z}_p \) torsor over \( a \times b \). For each \( y \in b \), the restriction \( E_{|a \times y} \) is an extension of \( a \) by \( \mathbb{Q}_p/\mathbb{Z}_p \), and thus is an element of \( a^* \). The map \( y \mapsto E_{|a \times y} \) defines an additive homomorphism \( f : a \to b^* \). One can similarly define a map from \( b \) to \( a^* \), which is none other than the dual \( f^* \) of \( f \). The restriction of \( E \) to \( \ker f \times b \) and \( a \times \ker f^* \) are trivial and thus one obtains unique trivializations \( \lambda : \ker f \times b \to E \) and \( \rho : a \times \ker f^* \to E \) such that \( \lambda(0,0) = 0 = \rho(0,0) \). Define a map \( B : \ker f \times \ker f^* \to \mathbb{Q}_p/\mathbb{Z}_p \) by \( B(x,y) = \lambda(x,y) - \rho(x,y) \). Since \( \mathbb{Q}_p/\mathbb{Z}_p \) is discrete, this descends to a (biadditive) pairing

\[
B : \pi_0(\ker f) \times \pi_0(\ker f^*) \to \mathbb{Q}_p/\mathbb{Z}_p
\]

It is proved in Proposition A.19, loc. cit. that \( B \) is nondegenerate. Now suppose that \( E \) is a skew-symmetric biextension of \( a \times a \), i.e., the restriction of \( E \) to the diagonal \( \Delta(a) \subset a \times a \) is trivial. In this case one has \( f = -f^* \) and thus \( \ker f = \ker f^* \). Let \( \alpha : \Delta(a) \to E \) be the trivialization of \( E_{|\Delta(a)} \) such that \( \alpha(0) = 0 \). Define \( \tilde{q} : \ker f \to \mathbb{Q}_p/\mathbb{Z}_p \) by \( \tilde{q}(x) = \lambda(x, x) - \alpha(x, x) \). Then \( \tilde{q} \) descends to a map \( q : \pi_0(\ker f) \to \mathbb{Q}_p/\mathbb{Z}_p \). One has the following result (cf. Lemma A.26, loc. cit.):

1) \( q(nx) = n^2q(x) \) for all \( n \in \mathbb{Z} \).

2) \( B(x,y) = q(x+y) - q(x) - q(y) \).

In particular, \( q \) is a nondegenerate quadratic form on \( \pi_0(\ker f) \).

#### 3.2 The quadratic form \( q \)

Let \( \Omega \) be a \( G \)-orbit in \( g^* \). Fix an element \( \chi \in \Omega \). Let \( G_\chi \) be the stabilizer of \( \chi \) in \( G \). Let \( \Gamma = \pi_0(G_\chi) \) and let \( p = \text{Log}(\Gamma) \). Using the construction of the previous section, we shall now define a quadratic form \( q : p \to \mathbb{Q}_p/\mathbb{Z}_p \) on \( p \). This quadratic form is used in Drinfeld’s formula for the ribbon element in \( \mathbb{Q}_l[\Gamma] \) (cf. Section 3.3).

Let \( E \) be the biextension of \( g \times g \) obtained by pulling back \( \chi \) via the biadditive commutator map \( g \times g \to g \) given by \( (u,v) \mapsto [u,v] \). The restriction of \( E \) to the diagonal \( \Delta g \subset g \times g \) is trivial, whence \( E \) is a skew-symmetric biextension. If \( f : g \to g^* \) is the homomorphism corresponding to \( E \), then the kernel of \( E \) is defined to be \( \ker f \). Thus \( x \in \ker E \) if and only if \( E_{|x \times g} \) is trivial. Let \( g_\chi = \text{Log}(G_\chi) \).

**Lemma 3.** \( g_\chi \) is the kernel of \( E \).

If \( x \in G \), then we denote the element \( x(\chi) \in g^* \) by \( \chi_x \).
Proof. In the following, we identify $G$ and $g$ (resp. $G_\chi$ and $g_\chi$) as schemes. Let $c$ denote the kernel of $E$. It suffices to show that $c \subset g_\chi$ and $g_\chi \subset c$.

(i) $c \subset g_\chi$. Let $x \in c$. Then the restriction of $E$ to $x \times g$ is trivial. Equivalently, if $\phi_x : g \rightarrow g$ is the map defined by $\phi_x(y) = [x, y]$, then $\phi_x^* (\chi)$ is trivial. We want to show that $x \in g_\chi$, i.e., $\chi = x\chi$ in $g^*$. The extension $x\chi$ is obtained by pulling back $\chi$ by the automorphism $\varphi_x : g \rightarrow g$ defined by $\varphi_x(y) = x^{-1}yx$. Thus $\chi = x\chi = (\text{id} - \varphi_x)^* (\chi)$. Further, using Lemma 1 one obtains:

$$(\text{id} - \varphi_x)(y) = y - x^{-1}yx = y - e^{\text{ad}(-x)}(y) = [x, y] - \frac{1}{2}[x, [x, y]] + \frac{1}{6}[x, [x, [x, y]]] + \cdots = [x, \mu(y)]$$

where $\mu : g \rightarrow g$ is the additive map defined by $\mu(y) = y - \frac{1}{2}[x, y] + \frac{1}{6}[x, [x, y]] - \cdots$. It follows that $\mu$ can be written as the composition: $g \xrightarrow{\mu} g \xrightarrow{\varphi_x} g$. By hypothesis, $\varphi_x^* (\chi)$ is trivial, whence so is $(\text{id} - \varphi_x)^* (\chi)$. Therefore $x \in g_\chi$.

(ii) $g_\chi \subset c$. Let $x \in g_\chi$. Then $(\text{id} - \varphi_x)^* (\chi) = 0$. To show that $x$ is in the kernel of $E$, it suffices to find (additive) $\lambda : g \rightarrow g$ such that $\varphi_x = (\text{id} - \varphi_x) \circ \lambda$, i.e.,

$$[x, y] = [x, \lambda(y)] - \frac{1}{2}[x, [x, \lambda(y)]] + \frac{1}{6}[x, [x, [x, \lambda(y)]]] + \cdots$$

Set

$$\lambda(y) = y + \frac{1}{2}[x, y] + \frac{1}{6}[x, [x, y]] - \frac{1}{120}[x, [x, [x, y]]] + \cdots$$

where the coefficients $B_n$ of $(\text{ad} x)^n (y)$ satisfy the equation:

$$B_0 + B_1 t + B_2 t^2 + \cdots = \frac{t}{1 - e^{-t}}$$

It is easily checked that $\lambda(y)$ is well defined (i.e., the denominator of $B_n$ is coprime to the $p$ whenever $n + 1 < p$) and that it satisfies (3). This completes the proof.

Let $p = \pi_0(g_\chi)$. Note that $p = \text{Log}(\Gamma)$. Applying the construction of Section 3.1 to $E$, one obtains a quadratic form:

$$q : p \rightarrow \mathbb{Q}_p / \mathbb{Z}_p$$

Lemma 4. $q$ is invariant under the conjugation action of $\Gamma$ on $p$.

Proof. Let $\tilde{q} : g_\chi \rightarrow \mathbb{Q}_p / \mathbb{Z}_p$ be the function obtained by pulling back $q$ via the projection $g_\chi \rightarrow p$. It suffices to show that $\tilde{q}$ is invariant under the action of $G_\chi$. Let $x$ be an element of $G$ and let $\xi E$ be the biextension of $g \times g$ obtained from $x \chi \in g^*$ as above. If $\varphi_x$ is as in the proof of Lemma 3 then $\xi E = (\varphi_x \times \varphi_x)^* (E)$. Thus $\ker(\xi E) = \varphi_x^* (\ker E) = xg_\chi x^{-1}$. The corresponding function $\tilde{q}_x : \ker(\xi E) \rightarrow \mathbb{Q}_p / \mathbb{Z}_p$ is $\tilde{q} \circ \varphi_x$. Now assume that $x \in G_\chi$. Then $\xi \chi = \chi$, whence $\xi E \cong E$. Thus $\tilde{q}_x = \tilde{q}$, which proved the desired assertion.

3.3 Drinfeld’s formula

The category $\text{Loc}_{G}(\Omega)$ of $G$-equivariant local systems on $\Omega$ is equivalent to the category $\text{Rep}(\Gamma)$ of finite dimensional representation of $\Gamma$ over $\mathbb{Q}_\ell$ (the equivalence sends a local system $\mathcal{L}$ to its stalk $\mathcal{L}_\chi$ at $\chi$). It follows from Lemma 2 that $\mathcal{M}_c$ is equivalent to $\text{Rep}(\Gamma)$. The ribbon structure of $\mathcal{M}_c$ given by $\theta$ induces an automorphism of the identity functor of $\text{Rep}(\Gamma)$. This automorphism can be interpreted as an invertible element of the center of $\mathbb{Q}_\ell [\Gamma]$. A precise formula (due to Drinfeld) for this element is given in Theorem 4 below.

Let $\tilde{q}$ be the composition:

$$p \twoheadrightarrow \mathbb{Q}_p / \mathbb{Z}_p \xrightarrow{\psi} \mathbb{Q}_\ell^\times$$

Thus $\tilde{q}$ is a quadratic form on $p$ with values in $\mathbb{Q}_\ell^\times$. Let $p^* = \text{Hom}(p, \mathbb{Q}_\ell^\times)$ be the group of characters of $p$. Since $\tilde{q}$ is nondegenerate, the bimultiplicative pairing $p \times p \rightarrow \mathbb{Q}_\ell^\times$ defined by $\tilde{q}$ gives an isomorphism
of the abelian groups \( p \) and \( p^* \). This isomorphism transforms \( \widetilde{q} \) into a quadratic form on \( p^* \), which we denote by \( \widetilde{q'} \). It easily follows from Lemma 4 that \( \widetilde{q'} \) is invariant under the action of \( \Gamma \) on \( p^* \). Consider the Fourier transform
\[
\mathcal{F}_\ell[p] \to \text{Fun}(p^*, \mathcal{F}_\ell)
\]
which sends \( a \in p \) to the function \( \phi \mapsto \phi(a)^{-1} \). Let \( \widetilde{q} \) be the preimage of \( \widetilde{q'} \) under this map. Since this map is \( \Gamma \) invariant, it follows that \( \widetilde{q} \) is also \( \Gamma \) invariant. Using the identity map \( \Gamma \to p \), one can interpret \( \widetilde{q} \) as an element of the center of \( \mathcal{F}_\ell[\Gamma] \).

**Theorem 1.** Let \( \theta' \) be the automorphism of the identity functor of \( \text{Rep}(\Gamma) \) corresponding to the ribbon automorphism \( \theta \) under the equivalence \( M_e \simeq \text{Rep}(\Gamma) \). Then \( \theta' \) is multiplication by \( \widetilde{q} \).

Conjecturally, the braiding and associativity constraints in \( \text{Rep}(\Gamma) \) coming from \( M_e \) is also completely determined by \( \widetilde{q} \). We now give an explicit formula for \( \widetilde{q} \). Let \( |p| \) denote the cardinality of \( p \) and let \( G(p, \widetilde{q}) = \sum_{a \in p} \widetilde{q}(a) \). Note that \( G(p, \widetilde{q}) \) is the Gauss sum of the metric group \( (p, \widetilde{q}) \) (cf. Section 6 of \( [DGNO] \)).

**Lemma 5.** \( \widetilde{q} = \frac{G(p, \widetilde{q})}{|p|} \sum_{a \in p} \widetilde{q}(a)^{-1}a \)

**Proof.** For \( \phi \in p^* \), let \( 1_\phi \in \text{Fun}(p^*, \mathcal{F}_\ell) \) be the characteristic function of \( \phi \). The inverse image of \( 1_\phi \) under the Fourier transform is \( 1/|p| \sum_{a \in p} \phi(a) a \). Let \( \phi_a \in p^* \) be the character corresponding to \( a \in p \) (under the isomorphism \( p \to p^* \) defined using \( \widetilde{q} \)). Then \( \widetilde{q'} = \sum_{b \in p} \widetilde{q}(b) 1_{\phi_b} \). If \( \widetilde{B} : p \times p \to \mathcal{F}_\ell \) is the bimultiplicative pairing defined by \( \widetilde{q} \), then one has \( \phi_b(a) = \widetilde{B}(a, b) \). Thus
\[
\widetilde{q} = \frac{1}{|p|} \sum_{b \in p} \left( \sum_{a \in p} \widetilde{q}(b) \sum_{a \in p} \widetilde{B}(a, b) a \right)
\]
As \( \widetilde{q}(b) \widetilde{B}(a, b) = \widetilde{q}(a)^{-1} \widetilde{q}(a + b) \), it follows that the coefficient of \( a \) in the above sum is \( \widetilde{q}(a)^{-1} G(p, \widetilde{q})/|p| \), qed.

### 3.4 A reformulation of Theorem 1

Let \( L \) be the \( G \)-equivariant local system on \( \Omega \) corresponding to the regular representation of \( \Gamma \). To prove Theorem 1 it suffices to show that \( \theta' \) acts by multiplication by \( \widetilde{q} \) on the stalk \( L_{\chi} \). Let \( T \) be an element of \( M_e \) corresponding to \( L \). A formula for \( T \) is given in Lemma 7 using the inverse Fourier-Deligne transform. To understand \( \theta'_L : L_{\chi} \to L_{\chi} \), we can use the fact that it is obtained by applying \( \mathcal{F} \) to \( \theta_T : T \to T \). This approach yields an explicit interpretation of \( \theta'_L \), described in Proposition 1 below.

**Notation.** If \( \phi \) is an element of \( g^* \), then let \( \widetilde{\phi} \) denote the (multiplicative) local system on \( g \) obtained from \( \phi \) using \( \psi : Q_p/Z_p \to \mathcal{F}_\ell \).

Let
\[
w : g \times G \to g
\]
be the map \( (x, y) \mapsto x - y^{-1}xy \) and let \( W = w^*(\widetilde{\chi}) \). Then:

A) The map \( w \) is invariant under the automorphism \( (x, y) \mapsto (x, xy) \) of \( g \times G \).

B) Consider the action of \( G_\chi \) on \( g \times G \) such that \( g \in G_\chi \) takes \( (x, y) \) to \( (gxg^{-1}, gy) \). Then \( W \) has a natural \( G_\chi \)-equivariant structure. Indeed, let \( v : g \times G \to g \) be the map which takes \( (x, y) \) to \(-y^{-1}xy\). Since \( \widetilde{\chi} \) is multiplicative, one has \( W = pr_1^*(\widetilde{\chi}) \otimes v^*(\widetilde{\chi}) \). Let \( g \in G_\chi \) and let \( \sigma_g \) be the corresponding automorphism of \( g \times G \). Then
\[
\sigma_g^*(W) = pr_1^*(\overline{v^{-1}}\widetilde{\chi}) \otimes v^*(\widetilde{\chi})
\]
This is isomorphic to \( W \) since \( \overline{v^{-1}}\widetilde{\chi} = \chi \).
Let $V = H^i_W$, where $i = \dim G + \dim_{\mathcal{X}} G$ and $H^\cdot$ denotes $\ell$-adic cohomology with compact support. It follows from (B) that $V$ is a $\Gamma$ module. The automorphism in (A) induces an automorphism $\eta: V \to V$ of $V$.

**Proposition 1.** There exists an isomorphism $\mathcal{L}_\mathcal{X} \cong V$ of $\Gamma$ modules which maps $\theta'_{\mathcal{X}}$ to $\eta$.

The proof of the proposition is given in Section 3.6. It follows that Theorem 1 is equivalent to showing that $\eta$ acts by multiplication by $\tilde{q}$. This is proved by explicit computation in Sections 5, 6.

### 3.5 A formula for $T$ and the automorphism $\theta_T$

We use the following notation: if $\mu: X \to Y$ is a morphism, then $\mu_!(\underline{\mathcal{E}}) = \mu_!((\underline{\mathcal{E}})_X)$, where $(\underline{\mathcal{E}})_X$ is the constant sheaf on $X$ with stalk $\underline{\mathcal{E}}$. Let $d = \dim G$ and $n = \dim_{\mathcal{X}} G$.

**Lemma 6.** Let $\pi: G \to G/G = \Omega$ be the projection map. Then $\pi_!(\underline{\mathcal{E}}) = \mathcal{L}[-2n](-n)$.

**Proof.** The morphism $\pi$ is the composition of the projection maps: $G \xrightarrow{\pi_1} G/G \xrightarrow{\pi_2} G/G$. Note that $(\pi_1)_!(\underline{\mathcal{E}}) = \underline{\mathcal{E}}[-2n](-n)$. This follows since $G^\circ$ is isomorphic to the affine space of dimension $n$ and $G$ is a $G^\circ$-torsor over $G/G^\circ$. Further, $\pi_2$ is a Galois cover with Galois group $\Gamma$, whence $(\pi_2)_!(\underline{\mathcal{E}}) \cong \mathcal{L}$. This completes the proof. □

Let $\mathcal{E}$ be the sheaf on $\mathfrak{g}^*$ obtained by extending $\mathcal{L}$ by zero. Let $F^{-1}$ be a quasi-inverse of $F$ which sends $N$ to $(-1)^*F(N)[2d](d)$ (cf. Section 2.4). Then one has $T \cong F^{-1}(\mathcal{E})$.

**Lemma 7.** Let $u: \mathfrak{g} \times G \to \mathfrak{g}$ be the map $(x, y) \mapsto -y^*x$ and let $V = (\chi_1(x))$. If $\text{pr}_1: \mathfrak{g} \times G \to \mathfrak{g}$ is the first projection, then

$$T \cong (\text{pr}_1)_!(V)[2n + 2d](n + d)$$

**Proof.** Let $u: G \to \mathfrak{g}^*$ be the composition

$$G \xrightarrow{\pi} \Omega \xrightarrow{\tilde{q}} \mathfrak{g}^*$$

One has $u_!(\underline{\mathcal{E}}) = \mathcal{L}[-2n](-n)$. Consider the commutative diagram:

$$\begin{array}{ccc}
\mathfrak{g} \times G & \xrightarrow{\text{pr}_2} & G \\
\downarrow{\tilde{u}} & & \downarrow{\text{pr}} \\
\mathfrak{g} & \xrightarrow{u} & \mathfrak{g}^* \\
\mathfrak{g} & \xrightarrow{\text{pr}'} & \mathfrak{g}^*
\end{array}$$

where $\tilde{u} = \text{id}_{\mathfrak{g}} \times u$. The square is cartesian and using the projection formula, it follows that

$$\text{pr}'^*(\mathcal{E}) \otimes \mathcal{E} = (\text{pr}'^* u_!(\underline{\mathcal{E}}) \otimes \mathcal{E})[2n](n) \cong (\tilde{u}_!(\underline{\mathcal{E}}) \otimes \mathcal{E})[2n](n) \cong (\tilde{u}_! \tilde{u}^*(\mathcal{E}))[2n](n)$$

As $\text{pr}_1 = \text{pr} \circ \tilde{u}$, one has

$$F'(\mathcal{E}) = \text{pr}_!(\text{pr}'^*(\mathcal{E}) \times \mathcal{E}) \cong ((\text{pr}_1)_! \tilde{u}^*(\mathcal{E}))[2n](n)$$

Let $i: \mathfrak{g} \times G \to \mathfrak{g} \times G$ be defined by $(x, y) \mapsto (-x, y)$. Then

$$T \cong F^{-1}(\mathcal{E}) \cong (-1)^* (\text{pr}_1)_! \tilde{u}^*(\mathcal{E})[2n + 2d](n + d) \cong (\text{pr}_1)_!(i^* \tilde{u}^*(\mathcal{E}))[2n + 2d](n + d)$$

Now the stalk of $\tilde{u}^*(\mathcal{E})$ at $(x, y)$ is $\mathcal{E}_{x, u(y)}$. As $u(y) = y\chi_1$, the stalk is $\tilde{\chi}_y = \tilde{\chi}_{-x, y}$. Thus the stalk of $i^* \tilde{u}^*(\mathcal{E})$ at $(x, y)$ is $\tilde{\chi}_{-x, y} = \tilde{\chi}_{-y, xy}$. Therefore $i^* \tilde{u}^*(\mathcal{E}) = V$, qed. □
To understand the automorphism $\theta_T : T \to T$, note that
1) The diagram appearing in the proof above is $G$-equivariant where $G$ acts on itself by left multiplication. The map $i$ is obviously $G$-invariant.
2) The local system $E$ has a canonical $G$-invariant structure: if $x \in \mathfrak{g}$ and $\varphi \in \mathfrak{g}^*$, then $E_{(x, \varphi)} = \bar{\varphi}_x$ and thus for all $g \in G$, one has

$$E_{g(x, \varphi)} = E_{(gxg^{-1}, \varphi)} = (\bar{g}\varphi)_{gxg^{-1}} = \bar{\varphi}_x$$

It follows from (1) and (2) that all the sheaves appearing in the proof has a $G$-equivariant structure. It is easy to see using (2) that the $G$-equivariant structure on $V = \nu^*\nu^*(E)$ is the one that comes from the $G$-invariance of $v$ (i.e., $v(x, y) = v(gxg^{-1}, gy)$ for $g \in G$). Fix $g \in G$ and $x \in \mathfrak{g}$. Then $g$ maps $V|_{x \times G}$ to $V|_{gxg^{-1}, xG}$ and thus one gets an isomorphism $((\text{pr}_1)V)_x \to ((\text{pr}_1)V)_{gxg^{-1}}$. Let us put $g = x$. The resulting automorphism of $((\text{pr}_1)V)_x$ comes from the invariance of $v$ under $(x, y) \mapsto (x, xy)$ along the fiber $x \times G$ of $\text{pr}_1$. It follows that:

**Lemma 8.** The map $v$ is invariant under the automorphism $(x, y) \mapsto (x, xy)$ of $\mathfrak{g} \times G$ and the corresponding automorphism of $T$ (coming from the isomorphism of Lemma 7) equals $\theta_T$.

### 3.6 Proof of Proposition

Since $F^{-1}$ is a quasi-inverse of $F$, one has the isomorphisms $\mathcal{L} \cong F(T)$ and thus $L_\chi \cong F(T)_\chi$.

**Lemma 9.** If $M \in D(\mathfrak{g})$, then $F(M)_\chi \cong R\Gamma_c(M \otimes \bar{\chi})$.

**Proof.** Let $i$ be the inclusion of $\chi$ in $\mathfrak{g}^*$ and let $j : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}^*$ be the inclusion $g \mapsto (g, \chi)$. One has the cartesian square:

$$\begin{array}{ccc}
\mathfrak{g} & \to & \{\chi\} \\
i & \downarrow & \downarrow i \\
\mathfrak{g} \times \mathfrak{g}^* & \xrightarrow{j} & \mathfrak{g}^*
\end{array}$$

Note that $j^*(E) = \bar{\chi}$. Using the projection formula, one obtains:

$$F(M)_\chi = (\text{pr}_!(\text{pr}_*^*(M) \otimes E))_\chi$$
$$= i^* \text{pr}_!(\text{pr}_*^*(M) \otimes E)$$
$$= R\Gamma_c(j^*(\text{pr}_*^*(M) \otimes E))$$
$$= R\Gamma_c(M \otimes \bar{\chi}), \text{ qed}$$

Let $i = n + d$. One has

$$T \otimes \bar{\chi} \cong ((\text{pr}_1)(V \otimes \bar{\chi}))[2i](i)$$
$$= ((\text{pr}_1)(V \otimes \text{pr}_1^*(\bar{\chi}))[2i](i)$$
$$= ((\text{pr}_1)(W))[2i](i)$$

Thus it follows from the Lemma 9 that:

$$F(T)_\chi \cong R\Gamma_c((\text{pr}_1)(W))[2i](i) = R\Gamma_c(W)[2i](i)$$

As $L_\chi \cong F(T)_\chi$, we get an isomorphism

$$\sigma : L_\chi \to V = H^2_c(W)$$

(we can ignore the Tate twist since $k$ is algebraically closed). Further, under the identification of $L_\chi$ and $F(T)_\chi$, the automorphism $\theta'_c$ of $L_\chi$ is mapped to the restriction of $F(\theta_T) : F(T) \to F(T)$ to the stalk $F(T)_\chi$. This automorphism is obtained by applying $R\Gamma_c$ to $\theta_T \otimes \text{id}_{\bar{\chi}} : T \otimes \bar{\chi} \to T \otimes \bar{\chi}$. Using the description of $\theta_T$ from Lemma 8 it follows that $\theta_T \otimes \text{id}_{\bar{\chi}}$ is the one induced by the invariance of $W$ under the automorphism $(x, y) \mapsto (x, xy)$ of $\mathfrak{g} \times G$. Thus $\sigma$ maps $\theta'_c$ to $\eta$. Finally, it remains to prove that $\sigma$ is $T$-invariant. This follows easily from the $G_\chi$-equivariant structure of $V$ described before Lemma 8. The details are omitted.
4 Quasi-polarizations

In this section we introduce the notion of quasi-polarizations. This will be used in computing the $\Gamma$ module structure on $V$ as well as the automorphism $\eta \in \text{Aut}(V)$.

4.1 Isotropic subgroups

In this section the we shall only use the commutative group structure on $\mathfrak{g}$. The results below apply equally well whenever $E$ is any skew-symmetric biextension of $\mathfrak{g}$.

In the following, we fix an additive connected subgroup $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{h}^\perp$ denote the orthogonal complement of $\mathfrak{h}$ with respect to $E$. The group $\mathfrak{h}^\perp$ consists of all $x \in \mathfrak{g}$ such that the restriction of $E$ to $x \times \mathfrak{h}$ is trivial. If $f : \mathfrak{g} \rightarrow \mathfrak{g}^*$ is the homomorphism corresponding to $E$, then $\mathfrak{h}^\perp$ is the kernel of the composition:

$$\mathfrak{g} \xrightarrow{f} \mathfrak{g}^* \rightarrow \mathfrak{h}^*$$

where the second map is the dual of the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Note that $\ker E = \mathfrak{h}^\perp = \ker f$. For notational clarity, we shall denote $\ker E$ by $\mathfrak{c}$ (recall that $\ker E = \mathfrak{g}_\chi$ by Lemma 3). In the following, we assume that $\mathfrak{c}^\circ \subset \mathfrak{h}$ and $\mathfrak{h}$ is isotropic, i.e., $\mathfrak{h} \subset \mathfrak{h}^\perp$.

Lemma 10. $\dim \mathfrak{h}^\perp + \dim \mathfrak{h} = \dim \mathfrak{g} + \dim \ker E$

Proof. We claim that the image of $f$ is the subgroup $(\mathfrak{g}/\mathfrak{c})^\ast \subset \mathfrak{g}$ consisting of all elements of $\mathfrak{g}^*$ whose restriction to $\mathfrak{c}$ is trivial. Indeed, it is clear that $\text{im} f \subset (\mathfrak{g}/\mathfrak{c})^\ast$. Further, the kernel of $f$ is $\mathfrak{c}$, whence there exists an injection $f' : \mathfrak{g}/\mathfrak{c} \hookrightarrow (\mathfrak{g}/\mathfrak{c})^\ast$ such that $f$ factors through $f'$, i.e., $f$ is the composition:

$$\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{c} \xrightarrow{f'} (\mathfrak{g}/\mathfrak{c})^\ast \hookrightarrow \mathfrak{g}^*$$

The map $f'$ is an injective group homomorphism and thus a closed immersion. Further, one has:

$$\dim(\mathfrak{g}/\mathfrak{c})^\ast = \dim \mathfrak{g}/\mathfrak{c}^\circ = \dim \mathfrak{g}/\mathfrak{c}$$

whence $f'$ is an isomorphism. Thus $\text{im} f = (\mathfrak{g}/\mathfrak{c})^\ast$, as claimed. Let $\varphi$ denote the composition $\mathfrak{g} \xrightarrow{f} \mathfrak{g}^* \rightarrow \mathfrak{h}^*$. By definition, $\mathfrak{h}^\perp$ is the kernel of $\varphi$. Further, the image of $\varphi$ is the image of $(\mathfrak{g}/\mathfrak{c})^\ast$ under the projection $\mathfrak{g}^* \rightarrow \mathfrak{h}^*$. As $(\mathfrak{g}/\mathfrak{c})^\ast$ surjects onto $((\mathfrak{h}/\mathfrak{c})^\ast$ (since its dual is injective), it follows that $\text{im} \varphi = ((\mathfrak{h}/\mathfrak{c})^\ast$. Therefore $\mathfrak{g}/\mathfrak{h}^\perp$ is isomorphic to $((\mathfrak{h}/\mathfrak{c})^\ast$. The equality now follows as $\dim(\mathfrak{h}/\mathfrak{c})^\ast = \dim \mathfrak{h}/\mathfrak{c}^\circ = \dim \mathfrak{h} - \dim \mathfrak{c}^\circ$.

Let $\mathfrak{h}'$ denote the neutral connected component of $\mathfrak{h}^\perp$.

Lemma 11. The map $\pi_0(\mathfrak{c}) \rightarrow \pi_0(\mathfrak{h}^\perp)$ induced by the inclusion $\mathfrak{c} \subset \mathfrak{h}^\perp$ is surjective.

Proof. Let $x \in \mathfrak{h}^\perp$. It suffices to find $y \in \mathfrak{h}'$ such that $x - y \in \mathfrak{c}$, i.e., $f(x) = f(y)$. Note that $f(\mathfrak{h}^\perp)$, by definition of $\mathfrak{h}^\perp$, is a subset of $(\mathfrak{g}/\mathfrak{h})^\ast$. Further, we claim that $f(\mathfrak{h}') = (\mathfrak{g}/\mathfrak{h})^\ast$. Indeed, if $f' : \mathfrak{h}' \rightarrow (\mathfrak{g}/\mathfrak{h})^\ast$ is the map obtained by restricting $f$, then it suffices to show that:

$$\dim \mathfrak{h}' = \dim(\mathfrak{g}/\mathfrak{h})^\ast + \dim \ker f'$$

But $\ker f' = \mathfrak{h}' \cap \mathfrak{c}$, which has the same dimension as $\mathfrak{c}$ (because $\mathfrak{h}'$ contains $\mathfrak{c}$). Thus the above equality follows from the Lemma. As $f(\mathfrak{h}') = (\mathfrak{g}/\mathfrak{h})^\ast$, there exists $y \in \mathfrak{h}'$ such that $f(x) = f(y)$, qed.

As $\mathfrak{h}$ is connected, we have $\mathfrak{h} \subset \mathfrak{h}'$. One therefore has the following lattice of subgroups of $\mathfrak{g}$:
The inclusions along the vertical segments have finite index.

**Corollary 1.** The homomorphism $c/(c \cap h') \to h^\perp/h'$ induced by the inclusion $c \hookrightarrow h^\perp$ is an isomorphism.

**Proof.** Injectivity is obvious and surjectivity follows from Lemma 11. 

Recall that $\pi_0(c)$ was denoted by $p$. Define a filtration: $0 \subset p_h \subset p_{h'} \subset p$ of $p$ by setting $p_h = (c \cap h)/c^\circ$, $p_{h'} = (c \cap h')/c^\circ$.

**Lemma 12.** $p_h$ is isotropic with respect to the quadratic form $q$ on $p$. Further, one has $p_h^\perp = p_{h'}$.

**Proof.** Since $h$ is isotropic, the restriction of $E$ to $h \times h$ gives the trivial biextension of $h \times h$. This implies that $q|_{p_h} = 0$, whence the first assertion. Let $B : p \times p \to \mathbb{Q}_p/\mathbb{Z}_p$ be the biadditive map induced by $q$. Since the restriction of $E$ to $h \times h'$ is trivial, it follows that $B(p_h, p_{h'}) = 0$, i.e., $p_{h'} \subset p_h^\perp$. It suffices to show that $\text{Card}(p_{h'}) = \text{Card}(p/p_h)$, i.e., the cardinalities of $(c \cap h)/c^\circ$ and $c/(c \cap h')$ are equal. Consider the biextension of $h \times g$ obtained from $E$ by restriction. It induces a nondegenerate pairing:

$$\frac{c \cap h}{c^\circ} \times \frac{h^\perp}{h'} \to \mathbb{Q}_p/\mathbb{Z}_p$$

In particular, one has $\text{Card}((c \cap h)/c^\circ) = \text{Card}(h^\perp/h')$. The desired equality now follows from Corollary 11. 

**4.2 Quasi-polarizations: definition and existence results**

In this section, the hypothesis on the nilpotence class of $g$ is not used. We retain the notations introduced in the previous section.

**Definition 1.** A quasi-polarization of $E$ is a connected, Lie subring $h$ of $g$ such that

(i) $c^\circ \subset h$ and $h$ is isotropic;

(ii) $h^\perp$ is a Lie ring.

A quasi-polarization $h$ is called a **Heisenberg polarization** if $[h^\perp, h'] \subset h$.

**Proposition 2.** A Heisenberg polarization always exists.

To prove this, one uses the following:

**Proposition 3.** Let $h$ be a quasi-polarization. If $h$ is not Heisenberg, then there exists a quasi-polarization $r$ such that $r$ strictly contains $h$ and $[h^\perp, h^\perp] \subset r^\perp \subset h^\perp$. 

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The proof of this is given in Section 4.3. To deduce Proposition 2, one starts with the quasi-polarization \( h = c^\circ \) and applies Proposition 3 to get an increasing chain of quasi-polarizations terminating in a Heisenberg polarization. For Heisenberg polarizations, one has the following analogue of Proposition 3.

**Lemma 13.** Let \( h \) be a Heisenberg polarization and let \( r \) be any connected isotropic additive subgroup such that \( h \subset r \subset h' \). Then \( r \) is a Heisenberg polarization and \( [h^\perp, h] \subset r^\perp \subset h^\perp \).

**Proof.** Indeed, since \( [h', h'] \subset h \), any additive subgroup of \( h' \) containing \( h \) is a Lie ring. Thus \( r \) is a Lie ring. It remains to verify that \( [h^\perp, h] \subset r^\perp \subset h^\perp \) (this automatically implies that \( r^\perp \) is a Lie ring). The second inclusion is clear since \( h \subset r \). For the first inclusion, let us fix \( x, y \in h^\perp \) and let \( z = [x, y] \). We need to show that the restriction \( E|_{z \times r} \) is trivial. Let \( k \in r \). The Jacobi identity

\[
[z,k] = [x,[y,k]] + [y,[k,x]]
\]

implies that \( E|_{z \times r} \) is the sum of two \( \mathbb{Q}_p / \mathbb{Z}_p \) torsors \( E^1, E^2 \) where:

\[
E^1_{(z,k)} = \chi_{[x,[y,k]]}, \quad E^2_{(z,k)} = \chi_{[y,[k,x]]}
\]

It suffices to check that \( E^1, E^2 \) are trivial. Note that \( [y,k] \in h \) (since \( [h^\perp, h] \subset h \)). Thus \( E^1 \) is obtained by pulling back \( E|_{z \times h} \) to \( z \times r \) via the map \( (z,k) \mapsto (x, [y,k]) \). As \( x \in h^\perp \), it follows that \( E|_{z \times h} \) is trivial, whence so is \( E^1 \). One shows similarly that \( E^2 \) is trivial. \( \square \)

Let \( \text{rank} \ E \) denote the integer \( \dim g - \dim c \).

**Corollary 2.** If \( \text{rank} \ E \) is even, then there exists a quasi-polarization \( r \) such that \( \dim r = \dim r^\perp \) (equivalently, \( r = r' \)).

One can obtain an additive version of this corollary (disregarding the Lie ring structures) by replacing “polarization \( r \)” by “connected, isotropic (additive) subgroup \( r \) containing \( (\ker E)^{\perp} \).” In this case, the result is proved in Proposition A.28 of [Boy1].

**Proof of Corollary 2.** Let \( h \) be a Heisenberg polarization and let \( E' \) be the biextension of \( h' \times h' \) obtained from \( E \) by restriction. We claim that \( \text{rank} \ E' \) is even. Indeed, the homomorphism \( f' : h' \to h^* \) corresponding to \( E' \) is the composition

\[
h' \hookrightarrow g \xrightarrow{f'} g^* \to h'^*
\]

Thus \( \ker E' = \ker f' = h' \cap h'^\perp \). We claim that \( (\ker E')^{\perp} = h \). It suffices to show that \( h \) is neutral connected component of \( h'^\perp \). Since \( h' \subset h^\perp \), the restriction of \( E \) to \( h' \times h' \) is trivial. Thus \( h \subset h'^\perp \). Hence it remains to show that \( \dim h'^\perp = \dim h \). Indeed, replacing \( h \) by \( h' \) in Lemma 10 gives \( \dim h'^\perp = \dim g + \dim c - \dim h' \), which equals to \( \dim h \) (again by Lemma 10). Therefore \( \rank E' = \dim h' - \dim h \), whence it follows from Lemma 10 that \( \rank E = \rank E' = 2(\dim h - \dim c) \), qed. Applying Proposition A.28 of [Boy1] to \( E' \), one obtains a additive subgroup \( r \) of \( h' \) such that \( h \subset r \) and \( \dim r = \dim r^\perp \), where \( r^\perp \) is the orthogonal complement of \( r \) in \( E' \). But \( r^\perp = r^\perp \cap h' \), whence \( \dim r = \dim r^\perp \). It follows from Lemma 13 that \( r \) is a polarization. This completes the proof. \( \square \)

### 4.3 Proof of Proposition 3

We shall need the elementary result:

**Lemma 14.** Let \( a, b \) be Lie subrings of \( g \) such that \( a \) is connected. Then \( [a,b] \) is also connected.

**Proof.** Let \( t \) be any Lie ring. The image of the commutator map \( t^g \times t^g \to t \) is connected and contains \( 0 \), and thus is contained in \( t^0 \). This shows that \( t^0 \) is also a Lie ring. Now let \( t = [a,b] \) and let \( b_1 \) be an arbitrary connected component of \( b \). The image of the commutator map \( a \times b_1 \to t \), for reasons similar to above, is contained in \( t^0 \). Therefore whenever \( a \in a, b \in b \), one has \( [a,b] \in t^0 \). Since \( t^0 \) is a Lie ring, this proves \( t = t^0 \), qed. \( \square \)
It follows that $h'$ is an ideal in $h^\perp$. Define a sequence of Lie subrings $h^{(n)}$ of $h^\perp$ for integers $n \geq 0$ by:

$$
\begin{align*}
h^{(0)} &= h' \\
h^{(n)} &= [h^\perp, h^{(n-1)}] & \text{for } n \geq 1
\end{align*}
$$

One easily checks using Lemma 14 that the $h^{(n)}$ form a decreasing sequence of connected ideals of $h^\perp$. Let $k$ be the integer such that $h^{(k)} \not\subset h$ and $h^{(k+1)} \subset h$. Note that $k \geq 1$ since $h$ is not Heisenberg. Define $r$ by

$$
r = h + h^{(k)}
$$

It suffices to show that $r$ has the required properties. One proceeds in various steps.

i) $r$ is a connected Lie subring of $h^\perp$. Indeed, both $h$ and $h^{(k)}$ are connected, whence so is $r$. Further, $h^{(k)}$ is an ideal in $h^\perp$ and thus $r$ is a Lie subring of $h^\perp$.

ii) If $a, b$ are two connected subgroups of $g$, then $(a + b)^\perp$ contains the neutral connected component of $a^\perp \cap b^\perp$. Since $(a + b)^\perp$ is contained in $a^\perp \cap b^\perp$, it suffices to show that it has finite index in $a^\perp \cap b^\perp$. Note that $(a + b)^\perp$ is the kernel of the map $u : g \to (a + b)^*$ obtained as the composition

$$
g \xrightarrow{f} g^* \to (a + b)^*
$$

where the second map sends $\varphi \in g^*$ to $\varphi|_{a + b}$. Further, let $\pi : (a + b)^* \to a^* \otimes b^*$ be the map $\varphi \mapsto (\varphi_a, \varphi_b)$. Then $a^\perp \cap b^\perp$ is the kernel of $\pi \circ u : g \to a^* \otimes b^*$. Thus it suffices to show that $\pi$ has finite kernel, or equivalently, $\pi^*$ is surjective (the equivalence easily follows from properties (iii), (iv) of Section 2.3). This follows since $\pi^* : a \oplus b \to a + b$ is the addition map.

iii) $h^{(1)} = [h^\perp, h']$ is contained in $r^\perp$. As $h^{(1)}$ is connected, it suffices to show (using (ii)) that $h^{(1)} \subset h^\perp \cap h^{(k)} \perp$, i.e., $h^{(1)} \subset h^{(k)} \perp$. This can be shown using the Jacobi identity in the style of the proof of Lemma 13. One needs to use the fact that $[h^\perp, h^{(k)}] \subset h$.

iv) $[h^\perp, h^\perp] \subset r^\perp \subset h^\perp$. The second inclusion is obvious since $h \subset r$. For the first inclusion, note that $h^\perp = h' + c$ by Lemma 11. As $c \in r^\perp$, it suffices to show that $[r, h']$ and $[h', h']$ are contained in $r^\perp$. This follows from (iii).

v) $r^\perp$ is a Lie ring and $r$ is isotropic. The first statement is immediate from (iv). To show $r$ is isotropic, it suffices to prove that $h + h^{(1)} \subset r^\perp$. Using (iii), it remains to show that $h \subset r^\perp$. Indeed, since $r \subset h^\perp$, the restriction of $E$ to $h \times r$ is trivial and thus $h \subset r^\perp$.

This completes the proof of the proposition.

## 5 Proof of Theorem 1 when $\dim(\Omega)$ is even

In this section we prove Theorem 1 when $\dim(\Omega)$ is even by direct computation. Note that

$$
\dim(\Omega) = \dim(G) - \dim(G_\chi) = \dim(g) - \dim(c) = \rank(E)
$$

Thus we can apply Corollary 2 in this case. We shall use the notation of Section 3.3 as well as the following:

1) $\tilde{E}$ denotes the local system on $g \times g$ obtained by pulling back $\tilde{\chi}$ using the commutator map $g \times g \to g$ which sends $(u, v)$ to $[u, v]$. Note that $\tilde{E}$ is obtained from $E$ by means of $\psi : Q_\mathbb{F}_p \to \mathbb{Q}_\mathbb{F}_p$. 

2) Let $X$ be a scheme over $k$ and let $F \in D(X)$. We denote $R\Gamma_c(X, F)$ by $\int_X F$.

3) If $Y$ is a locally closed subscheme of $X$ and $i : Y \hookrightarrow X$ is the inclusion, then we denote $i^*(F)$ by $F|_Y$ and $R\Gamma_c(Y, F|_Y)$ by $\int_Y F$.

We note that if $Y$ is closed in $X$, then there is a natural morphism $\int_X F \to \int_Y F$, which is an isomorphism if $\int_X F$ is zero (this is standard and follows from the distinguished triangle described in Lemma 5.8 of [Boy1]).
5.1 A basis of $V$

For notational convenience, we shall identify $G$ with $g$ and consider $W$ as a local system on $g \times g$. Let $h$, $r$ be quasi-polarizations of $E$ such that $h \subset r$ (and thus $r^+ \subset h^+$).

**Proposition 4.** If $[h^+, h^+] \subset r^+$, then the natural map

$$\int_{h^+ \times h^+} W \to \int_{r^+ \times r^+} W$$

is an isomorphism.

The proof of this proposition is given in Section 5.4. By the results of Section 4.2, it follows that there exists a sequence

$$c^0 = h_0 \subset \cdots \subset h_n = h$$

of quasi-polarizations such that $\dim h = \dim h^+$ (i.e., $h = h'$) and $[h^+, h^+] \subset h^+_{i+1}$ for all $0 \leq i < n$. Thus Proposition 3 implies that the natural map $\int_{g \times g} W \to \int_{h^+ \times h^+} W$ is an isomorphism.

**Lemma 15.** The local system $W|_{h^+ \times h^+}$ is trivial.

Before proving the lemma, we introduce some notation. One has:

$$w(x, y) = x - y^{-1}xy = [y, x] - \frac{1}{2} [y, [y, x]] + \cdots = [y, \Phi(x, y)]$$

where

$$\Phi(x, y) = x - \frac{1}{2} [y, x] + \cdots + \frac{(-1)^n}{(n + 1)!} \text{ad}^n(y)(x) + \cdots$$

Let $\lambda$ be the automorphism of $g \times g$ which sends $(x, y)$ to $(y, \Phi(x, y))$. Note that $W = \lambda^*(\overline{E})$.

**Proof of Lemma 15.** As $\lambda$ maps $h^+ \times h^+$ to itself, it suffices to show that $\overline{E}|_{h^+ \times h^+}$ is trivial. Since $h$ is isotropic, the biextension $\overline{E}|_{h \times h}$ of $h \times h$ is trivial and thus $\overline{E}|_{h \times h}$ is a trivial local system. Let $\alpha$, $\beta \in \pi_0(h^+)$ and let $h_{\alpha}$, $h_{\beta}$ be the corresponding connected components of $h^+$. Choose $\tilde{\alpha} \in h_{\alpha}$, $\tilde{\beta} \in h_{\beta}$. Let $x, y \in h$. Then the stalk of $\overline{E}$ at $(x + \tilde{\alpha}, y + \tilde{\beta})$ is

$$\tilde{\chi}_{[x + \tilde{\alpha}, y + \tilde{\beta}]} = \tilde{\chi}_{[x, y]} \otimes \tilde{\chi}_{[\tilde{\alpha}, \tilde{\beta}]} \otimes \tilde{\chi}_{[\tilde{\alpha}, \tilde{\beta}]} \otimes \tilde{\chi}_{[\tilde{\alpha}, \tilde{\beta}]}$$

$$= \tilde{E}_{(x, \tilde{\alpha})} \otimes \tilde{E}_{(\tilde{\beta}, y)} \otimes \tilde{E}_{(\tilde{\alpha}, \tilde{\beta})} \otimes \tilde{E}_{(\tilde{\alpha}, \tilde{\beta})}$$

where the first equality uses the fact that $\tilde{\chi}$ is a multiplicative local system. It remains to note that both $\tilde{\alpha}$, $\tilde{\beta}$ belong to $h^+$, whence $\tilde{E}|_{h^+ \times h^+}$ and $\tilde{E}|_{h \times h}$ are trivial local systems, qed.

Recall that we denoted $\pi_0(c)$ by $p$ (cf. Section 3.2). Let $b = \pi_0(h^+) = h^+ / h$. By Lemma 11 the map $p \to b$ induced by the inclusion $c \hookrightarrow h^+$ is surjective. Let $a$ be the kernel of this map, so that $a$ has an exact sequence

$$0 \to a \to p \to b \to 0$$

Since $h = b'$, it follows from Lemma 12 that $a = a^\perp$. Thus $a$ is a Lagrangian ideal (it can be shown that it is abelian). In particular, one has $\text{Card}(a) = \text{Card}(b) = \text{Card}(p)^{1/2}$.

Fix a pair $(\alpha, \beta) \in b \times b$. Let $\tilde{\beta}$ be an element of $h_{\beta}$ such that $\beta \in c$ (this is possible since $p$ surjects onto $b$). For each such $\beta$, one can choose a trivialization of $h_{\alpha} \times h_{\beta}$, as follows. First, let $y_0$ be an element of $c$. Note that $W|_{g \times y_0}$ is a trivial local system. This follows since $\lambda$ sends $g \times y_0$ to $y_0 \times g$ and $\overline{E}|_{y_0 \times g}$ is trivial. Further, the stalk of $W$ at $(0, y_0)$ is $\tilde{\chi}_{0}$, which can be canonically identified with $\mathbb{Q}_x$ (since $\tilde{\chi}$ is
multiplicative, one has an isomorphism $\tilde{\chi}_0 \otimes \tilde{\chi}_0 \cong \tilde{\chi}_0$, which gives the identification). Thus $W|_{p \times \tilde{y}_0}$ has a natural trivialization. This holds in particular for $y_0 = \tilde{\beta}$. This trivialization induces trivializations

$$t^\alpha_\beta : \mathbb{Q}_\ell[A_\alpha \times B_\beta] \cong W|_{A_\alpha \times B_\beta}$$

for each $\alpha \in b$ (the difference between the various trivializations $t^\alpha_\beta$ when $\beta$ varies in $b \cap c$ is computed in Lemma [11] below). Let us fix a map $s : b \to c$ such that $s(\beta) \in b \cap c$ (in other words, $s$ is a section of the composite map $c \to p \to b$). We further assume that $s(0) = 0$. Using $t^\alpha_\beta$, one can identify $\int_{\mathbb{Q}_\ell}$ with $\int_{b \times b_\beta} W$, which can be canonically identified with $\mathbb{Q}_\ell[-2i](-i)$, where $i = \dim b_\alpha + \dim b_\beta$ (note that $i = \dim g + \dim c$ by Lemma [10]). Let $1_{a,\beta}$ denote the image of the element $1 \in \mathbb{Q}_\ell = \mathbb{Q}_\ell(-i)$ (we ignore the Tate twist) in $H^2(\mathbb{Q}_\ell \times \mathbb{Q}_\ell, W)$ by $t^\alpha_\beta$. Then $\{1_{a,\beta}\}$ form a $\mathbb{Q}_\ell$-basis of $V$.

### 5.2 The action of $\Gamma$ on $V$ and the automorphism $\eta$

Let $g$ be a point in $c = \text{Log}(G_X)$ and let $\sigma_g$ be the automorphism of $g \times g$ which sends $(x, y)$ to $(gxg^{-1}, gy)$. Since $W$ has a $G_X$-equivariant structure, one has isomorphisms $W(x, y) \to W(x, g)$ for all $x, y \in G$. By transport of structure, $\sigma_g$ sends a trivialization $t$ of $W|_{A_\alpha \times B_\beta}$ to a trivialization of $W|_{A_\beta \times B_\alpha}$, which we denote by $\sigma_g(t)$.

**Lemma 16.** Let $\tilde{\beta} \in b \cap c$. Then $\sigma_g(t^\alpha_\beta) = t^{\sigma_g g^{-1}}_\beta$.

**Proof.** Note that $\sigma_g$ sends $g \times \tilde{\beta}$ to $g \times \tilde{\beta}$. Both $W|_{g \times \tilde{\beta}}$ and $W|_{g \times \tilde{\beta}}$ are trivial with natural trivializations, coming from the equalities $W|_{g \times \tilde{\beta}} = 0 = \mathbb{Q}_\ell$ and $W|_{g \times \tilde{\beta}} = 0 = \mathbb{Q}_\ell$. It remains to show that the map $W(\tilde{\alpha}, \tilde{\beta}) \to W(\tilde{\alpha}, \tilde{\beta})$ induced by $\sigma_g$ is identity. This is immediate from the $G_X$-equivariant structure of $W$, defined in Section 3.4.

Let $B : p \times p \to \mathbb{Q}_p/\mathbb{Z}_p$ be the biadditive pairing induced by $q$ and let $\tilde{B} : b \times b \to \mathbb{Q}_\ell$ be defined by $\tilde{B} = \psi \circ B$. It is sometimes convenient to extend the domain of $B$ and $\tilde{B}$ to $c \times c$ by means of the projection $c \times c \to p \times p$.

**Lemma 17.** Let $\tilde{\beta}, \tilde{\beta}' \in b \cap c$. Then $t^\alpha_{\tilde{\beta}'} = \tilde{B}(\tilde{\beta}' - \tilde{\beta}, \Phi(\alpha, \beta))t^\alpha_\beta$.

The constant $\tilde{B}(\tilde{\beta}' - \tilde{\beta}, \Phi(\alpha, \beta))$ in this formula is to be understood in the following sense: since $a$ is isotrpic, the pairing $\tilde{B}$ induces a map from $a \times a$ to $\mathbb{Q}_\ell$, which we again denote by $\tilde{B}$. Note that $\Phi(\alpha, \beta) \in b$ and $\tilde{\beta}' - \tilde{\beta}$ belongs to $b \cap c$, whence its image in $p$ is contained in $a$. Thus $\tilde{B}(\tilde{\beta}' - \tilde{\beta}, \Phi(\alpha, \beta))$ makes sense.

**Proof of Lemma 17** Using the automorphism $\lambda$ of $g \times g$, we shall shift the computation from the local system $W$ to $\tilde{E}$. Note that:

1) $\lambda$ sends $\mathbb{Q}_\ell \times \mathbb{Q}_\ell$ to $\mathbb{Q}_\ell \times \mathbb{Q}_\ell$.

2) For each $y_0 \in \mathbb{Q}_\ell$, the trivial local system $\tilde{E}|_{\mathbb{Q}_\ell \times \mathbb{Q}_\ell}$ admits a natural trivialization coming from the equalities $\tilde{E}|_{\mathbb{Q}_\ell \times \mathbb{Q}_\ell} = \mathbb{Q}_\ell$. Further, the trivialization of $W|_{g \times \tilde{y}_0}$ described in Section 5.1 is obtained by pulling back that of $\tilde{E}|_{\mathbb{Q}_\ell \times \mathbb{Q}_\ell}$ by $\lambda$.

Thus it remains to show that the trivializations of $\tilde{E}|_{b \times b \Phi(\alpha, \beta)}$ induced by the trivializations of $\tilde{E}|_{b \times b}$ and $\tilde{E}|_{b \times b \Phi(\alpha, \beta)}$ differ by the desired constant. Choose $y_0 \in b \Phi(\alpha, \beta) \cap c$. Then these trivializations give isomorphisms $\mathbb{Q}_\ell \cong \tilde{E}(\tilde{\beta}, y_0)$ and $\mathbb{Q}_\ell \cong \tilde{E}(\tilde{\beta}, y_0)$, which induce trivializations $t_1$ and $t_2$ of $\tilde{E}|_{b \times \tilde{y}_0}$. Further, $\tilde{E}|_{b \times \tilde{y}_0}$ is equipped with a natural trivialization $t : \mathbb{Q}_\ell \times \tilde{y}_0 \to \tilde{E}|_{b \times \tilde{y}_0}$ (defined as in (2)). It follows from the definition of $B$ (cf. Section 3.1) that $t_1 = B(\tilde{\beta}', y_0)t$ and $t_2 = B(\tilde{\beta}, y_0)t$. Hence $t_1$ and $t_2$ differ by $B(\tilde{\beta}' - \tilde{\beta}, \tilde{\beta})$. This completes the proof. 

\[ \square \]
We can now describe the action of $\Gamma$ on the basis elements $1_{\alpha,\beta}$ of $V$.

**Corollary 3.** Let $g \in \Gamma$. Then $g(1_{\alpha,\beta}) = \tilde{B}(g\alpha g^{-1}, g\beta)1_{g\alpha g^{-1}, g\beta}$.

**Proof.** We may assume that $g \in \epsilon = \text{Log}(G_\chi)$. By Lemma 16 the map $g : \mathfrak{h}_\alpha \times \mathfrak{h}_\beta \to \mathfrak{h}_{g\alpha g^{-1}} \times \mathfrak{h}_{g\beta}$ transforms the trivialization $t^\alpha_{s(\beta)}$ to $t^{g\alpha g^{-1}}_{g\alpha g^{-1}}$. It remains to note that by Lemma 17 the trivializations $t^{g\alpha g^{-1}}_{g\alpha g^{-1}}$ and $t^{g\beta g^{-1}}_{g\beta g^{-1}}$ differ by $\tilde{B}(g\alpha g^{-1} - s(g\beta), \Phi(g\alpha g^{-1}, g\beta))$, qed.

It remains to compute the automorphism $\eta$. Let $x_0, y_0 \in \epsilon$. Then $W(x_0, y_0) = \tilde{\chi}_{x_0 - y_0^{-1} x_0 y_0} = W(x_0, x_0 y_0)$. Further, the trivializations of $W|_{\mathfrak{g} \times {x_0}}$ and $W|_{\mathfrak{g} \times {x_0 y_0}}$ give isomorphisms $t'_x : \mathfrak{g} \ell \to W(x, y_0)$ and $t''_x : \mathfrak{g} \ell \to W(x, x_0 y_0)$ for each $x \in \mathfrak{g}$.

**Lemma 18.** One has $t_{x_0} = \tilde{q}(x_0)^{-1} t''_{x_0}$.

**Proof.** For notational clarity, we shall denote the stalk $\tilde{\chi}_x$ by $\tilde{\chi}(x)$. One has

$W(x, x_0 y_0) = \tilde{\chi}(x - y_0^{-1} x_0^{-1} x x_0 y_0) = \tilde{\chi}(x - x_0 x_0^{-1}) \otimes \tilde{\chi}(x_0 x_0^{-1} - y_0^{-1} x_0^{-1} x x_0 y_0) = W(x, x_0) \otimes W(x_0^{-1} x x_0 y_0)$

Note that $W|_{\mathfrak{g} \times {x_0}}$ is also trivial and let $t''_x : \mathfrak{g} \ell \to W(x, x_0)$ be the trivialization. It follows from the above that

$t''_x = t''_x \otimes t_{x_0 x_0^{-1}}$.

Putting $x = x_0$, we get $t''_{x_0} = t''_{x_0} \otimes t_{x_0}$. Thus it remains to show that $t''_{x_0} : \mathfrak{g} \ell \to W(x_0, x_0) = \tilde{\chi}(0) = \mathfrak{g} \ell$ is multiplication by $\tilde{q}(x_0)$. Note that the automorphism $\lambda$ of $\mathfrak{g} \times \mathfrak{g}$ takes $\mathfrak{g} \times {x_0}$ to $x_0 \times \mathfrak{g}$ and the trivialization $t''$ to the (natural) trivialization of $E|_{x_0 \times \mathfrak{g}}$. Thus it remains to check that the map $\mathfrak{g} \ell \to E(x_0, x_0) = \mathfrak{g} \ell$ coming from the trivialization of $E|_{x_0 \times \mathfrak{g}}$ is multiplication by $\tilde{q}(x_0)$. But this follows from the definition of the quadratic form $q$ (cf. Section 3.1).

Let $\tau$ be the automorphism of $\mathfrak{g} \times \mathfrak{g}$ which sends $(x, y)$ to $(x, xy)$. Let $\tilde{\alpha} \in \mathfrak{h}_\alpha \cap \epsilon$ and $\tilde{\beta} \in \mathfrak{h}_\beta \cap \epsilon$. Putting $x_0 = \tilde{\alpha}$ and $y_0 = \tilde{\beta}$ in the lemma shows that

**Corollary 4.** $\tau$ takes the trivialization $t''_{\tilde{\alpha}}$ to the trivialization $\tilde{q}(\tilde{\alpha})^{-1} t''_{\tilde{\alpha}}$ of $\mathfrak{h}_\alpha \times \mathfrak{h}_\beta$.

Using the corollary, it is easy to give a formula for $\eta(1_{\alpha, \beta})$. However, we shall only need the following:

**Corollary 5.** $\eta(1_{\alpha, 0}) = \tilde{q}(s(\alpha))^{-1} 1_{\alpha, \alpha}$.

**Proof.** Putting $\tilde{\alpha} = s(\alpha)$ and $\tilde{\beta} = 0$ in Corollary 4.

### 5.3 Proof of Theorem [1]

We shall prove the theorem by means of explicit computation using Corollary 3 and Corollary 5. Put

$$u = \sum_{\alpha \in \mathfrak{b}} 1_{\alpha, 0}$$

**Lemma 19.** The elements $gu, g \in \Gamma$ form a $\mathfrak{g} \ell$-basis of $V$.

**Proof.** Let $\tilde{\beta}$ denote the image of $s(\beta) \in \epsilon$ in $\mathfrak{p}$. Fix $\beta_0 \in \mathfrak{b}$. Let $\pi : \mathfrak{p} \to \mathfrak{b}$ denote the projection. Note that $\pi^{-1}(\beta_0) = \alpha + \beta_0$. Let $g \in \pi^{-1}(\beta_0)$. It follows from Corollary 3 that

$$g(1_{\alpha, 0}) = \tilde{B}(g - s(g), \Phi(gg^{-1}, g))1_{gg^{-1}, g}$$

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The pair \((gog^{-1}, g) \in b \times b\) is none other than \((\beta_0\alpha\beta_0^{-1}, \beta_0)\). Thus
\[
\tilde{B}(g - s(g), \Phi(gog^{-1}, g)) = \tilde{B}(g - \tilde{\beta}_0, \Phi(\beta_0\alpha\beta_0^{-1}, \beta_0))
\]
Writing \(g = a + \tilde{\beta}_0\), it follows that \(g(1_{\alpha}, a) = \tilde{B}(a, \Phi(\beta_0\alpha\beta_0^{-1}, \beta_0))1_{\beta_0\alpha\beta_0^{-1}, \beta_0}\) and thus
\[
g(u) = \sum_{a \in b} \tilde{B}(a, \Phi(\beta_0\alpha\beta_0^{-1}, \beta_0))1_{\beta_0\alpha\beta_0^{-1}, \beta_0}
\]
\[
= \sum_{a \in b} \tilde{B}(a, \Phi(\alpha, \beta_0))1_{\alpha, \beta_0}
\]
The map \(\alpha \to \tilde{B}(a, \Phi(\alpha, \beta_0))\) from \(b\) to \(\mathbb{Q}_k^\times\) is a character of \(b\) (this follows since \(\Phi(x, y)\) is additive in \(x\)), which we denote by \(\zeta_a\). Note that \(\zeta_a\) is trivial if and only if \(a = 0\) (i.e., \(g = \tilde{\beta}_0\)). Indeed, as \(\alpha\) varies over \(b\), the elements \(\Phi(\alpha, \beta_0)\) exhaust \(b\). Thus if \(\zeta_a = 0\), then \(\tilde{B}(a, b) = 0\), which implies \(a = 0\). Also, \(\zeta_{a+b} = \zeta_a \zeta_b\), whence the map \(a \mapsto \zeta_a\) from \(a\) to \(b^\times\) is an isomorphism of groups. It follows that the elements \(g(u)\) as \(g\) varies over \(\pi^{-1}(\beta_0)\) form a basis of the subspace of \(V\) generated by \(\{1_{\alpha, \beta_0}\}, \alpha \in b\). This completes the proof. \(\square\)

In view of the lemma, it suffices to show that \(\eta(u) = \tilde{q}(u)\). Let \(h_{\beta_0}\) be the element of \(\mathbb{Q}_k[\Gamma]\) which sends \(u\) to \(1_{\beta_0, \beta_0}\). Now one has
\[
(a + \tilde{\beta}_0)(u) = \sum_{a \in b} \zeta_a(a)1_{\alpha, \beta_0}
\]
It follows from the Fourier inversion formula for finite abelian group that
\[
h_{\beta_0} = \frac{1}{\text{Card}(a)} \sum_{a \in a} \zeta_a(-\beta_0)(a + \tilde{\beta}_0)
\]
Note that \(\zeta_a(-\beta_0) = \tilde{B}(a, \Phi(-\beta_0, \beta_0)) = \tilde{B}(a, -\beta_0)\). Further:
\[
\tilde{B}(a, -\beta_0) = \frac{\tilde{q}(a)\tilde{q}(\tilde{\beta}_0)}{\tilde{q}(a + \beta_0)} = \frac{\tilde{q}(\tilde{\beta}_0)}{\tilde{q}(a + \beta_0)}
\]
where the second equality follows from the fact that \(\tilde{q}(a) = 1\) (indeed, the biextension \(E|_{b \times b}\) is trivial, whence the result). Thus
\[
h_{\beta_0} = \frac{\tilde{q}(\tilde{\beta}_0)}{\text{Card}(a)} \sum_{g \in \pi^{-1}(\beta_0)} \frac{1}{\tilde{q}(g)} g
\]
As \(\eta(u) = \sum_{\beta_0 \in b} \tilde{q}(\tilde{\beta}_0)^{-1}1_{\beta_0, \beta_0}\), it suffices to show that \(\tilde{q}\) is equal to
\[
\sum_{\beta_0 \in b} \frac{1}{\tilde{q}(\beta_0)} h_{\beta_0} = \frac{1}{\text{Card}(a)} \sum_{g \in \pi^{-1}(\beta_0)} \frac{1}{\tilde{q}(g)} g
\]
Comparing with Lemma\(^\text{[5]}\) it remains to show that \(G(p, \tilde{q}) = \text{Card}(a)\). This is a consequence of Corollary 6.2. \(\text{[DGNO]}\).

### 5.4 Proof of Proposition \(^\text{[4]}\)

Let \(A = \mathfrak{r}^+ \times \mathfrak{r}^+, A' = \mathfrak{h}^+ \times \mathfrak{h}^+\). It suffices to show that \(\int_{A'\setminus A} W = 0\). Let \(B = \mathfrak{h}^+ \times \mathfrak{r}^+\). One has \(A \subset B \subset A'\). Let
\[
U = A' \setminus B = \mathfrak{h}^+ \times (\mathfrak{h}^+ \setminus \mathfrak{r}^+)
\]
\[
V = B \setminus A = (\mathfrak{h}^+ \setminus \mathfrak{r}^+) \times \mathfrak{r}^+
\]
It remains to prove that: i) \(\int_U W = 0\) and ii) \(\int_V W = 0\).
Lemma 20. Let \( \mathcal{L} \) be a multiplicative local system on a group scheme \( H \). If \( \mathcal{L}|_{H^*} \) is nontrivial, then \( R\Gamma_c(H, \mathcal{L}) = 0 \).

This is well known and is proved in Lemma 9.4, [Boy1]. Using this, it remains to show that the local system \( i^*(\chi)|_{H^*} \) is nontrivial, i.e., \( \chi|_{H^*} \) is not isomorphic to \( \chi y_0|_{H^*} \).

Lemma 21. Let \( a \subset g \) be a Lie subalgebra and let \( b \in g \) be such that \( [b, a] \subset a \). Then \( \chi|_a = b \chi|_a \) if and only if \( b \in a^\perp \).

Proof. This is a generalization of Lemma 3 and the same proof applies (with \( x = b \) and \( y \in a \)).

As \( y_0 \in h^\perp \), one has \( [y_0, b'] \subset b' \). Thus it suffices to show that \( y_0 \notin (b')^\perp \). This follows since \( y_0 \notin r^\perp \) and \( (b')^\perp \subset r^\perp \) (as \( r \subset b' \)).

Proof of (ii). Using the first projection \( V \to h^\perp \setminus r^\perp \), it suffices to show that \( \int_{x_0 \times r^\perp} W = 0 \) for all \( x_0 \in h^\perp \setminus r^\perp \). Let \( i : r^\perp \to g \) be the map \( y \mapsto x_0 - y^{-1} x_0 y \). Then \( \int_{x_0 \times r^\perp} W \cong \int_{r^\perp} i^*(\chi) \). Unfortunately, \( i \) is not additive. However, it suffices to show that \( \int_{y_0 r} i^*(\chi) = 0 \) for all \( y_0 \in r^\perp \) (if \( \pi : r^\perp \to r^\perp/\pi \) is the projection map, then this would prove that \( \pi(i^*(\chi)) = 0 \)). Define \( j : r \to g \) by \( y \mapsto x_0 - y^{-1} x_0 y_0 y \). Then \( \int_{y_0 r} i^*(\chi) \cong \int_r j^*(\chi) \). Put \( t = y_0^{-1} x_0 y_0 \). One has

\[
j(y) = x_0 - y^{-1} t y = (x_0 - t) + [y, t] - \frac{1}{2} [y, [y, t]] + \cdots
\]

Let \( j_n \) denote that \( n \)-th term of this Lie series (i.e., \( j_0 = (x_0 - t), j_1 = [y, t] \) etc.). Since \( \chi \) is multiplicative, one has

\[
j^*(\chi) = j_0^*(\chi) \otimes j_1^*(\chi) \otimes \cdots
\]

Claim. \( j_n^*(\chi) \cong (Q_{\ell})^r \) if \( n \neq 1 \).

Proof. Since \( j_0 \) is a constant map, this is obvious for \( n = 0 \). Suppose that \( n = 2 \). Let \( s : r \to g \times g \) be the map \( y \mapsto (y, -\frac{1}{2} [y, t]) \). Then \( j_2^*(\chi) = s^*(\tilde{E}) \). As \( y \in r \subset h^\perp \) and \( t \in h^\perp \), one has \( [y, t] \subset [h^\perp, h^\perp] \), which is contained in \( r^\perp \) by assumption. Thus \( s \subset r \times r^\perp \). But \( \tilde{E}|_{r \times r^\perp} = (Q_{\ell})|_{r \times r^\perp} \) (cf. proof of Lemma 15), whence \( j_2^*(\chi) \cong (Q_{\ell})^r \). The proof for \( n > 2 \) is similar.

It follows that \( j^*(\chi) \cong j_1^*(\chi) \). Since \( j_1 \) is additive, the local system \( j_1^*(\chi) \) is multiplicative. If this is trivial, then \( t \in r^\perp \). As \( t = y_0 x_0 y_0^{-1} \), this implies \( x_0 \in r^\perp \), which is absurd. It now follows from Lemma 20 that \( \int_r j^*(\chi) = 0 \). This completes the proof (ii) as well as the proof of the proposition.

References

[Bég] Bégueri, L. Dualité sur un corps local à corps résiduel algébriquement clos, Mém. Soc. Math. France (N.S.) 1980/81, no. 4.

[Boy1] Boyarchenko, M. Characters of unipotent groups over finite fields, Selecta Math. (N.S.) 16 (2010), no. 4, 857–933.
[BD1] Boyarchenko, M., Drinfeld, V. A motivated introduction to character sheaves and the orbit method for unipotent groups in positive characteristic, Preprint, November 2010, arXiv: math.RT/0609769v2.

[BD2] Boyarchenko, M., Drinfeld, V. Character sheaves on unipotent groups in positive characteristic: foundations, Preprint, January 2013, arXiv: math.RT/0810.0794v3.

[Des] Deshpande, T. Heisenberg idempotents on unipotent groups, Math. Res. Lett. 17 (2010), no. 3, 415–434.

[DGNO] Drinfeld, V., Gelaki, S., Nikshych, D., Ostrik, V. On braided fusion categories I, Selecta Math. (N.S.) 16 (2010), no. 1, 1–119.

[Khu] Khukhro, E. I. p-Automorphisms of Finite p-groups, Lond. Math. Soc. Lect. Note Series 246, Cambridge University Press, 1998.

[Laz] Lazard, M Sur les groupes nilpotents et anneaux de Lie, Ann. Sci. Ecole Norm. Sup. (3) 71 (1954), 101–190.

[Sa] Saibi, M. Transformation de Fourier-Deligne sur les groupes unipotents, Ann. Inst. Fourier (Grenoble) 46 (1996), no. 5, 1205–1242.

[Ser] Serre, J.-P. Lie algebras and Lie groups, Lecture Notes in Mathematics, 1500, Springer-Verlag, Berlin-Heidelberg, 1992.