Abstract. We establish triviality of some holomorphic Banach vector bundles on the maximal ideal space $M(H^\infty)$ of the Banach algebra $H^\infty$ of bounded holomorphic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$ with pointwise multiplication and supremum norm. We apply the result to the study of the Sz.-Nagy operator corona problem.

1. Formulation of Main Results

1.1. We continue our study started in [Br2] of analytic objects on the maximal ideal space $M(H^\infty)$ of the Banach algebra $H^\infty$ of bounded holomorphic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$ with pointwise multiplication and supremum norm. The present paper deals with holomorphic Banach vector bundles defined on $M(H^\infty)$ and the operator corona problem posed by Sz.-Nagy. Recall that for a commutative unital complex Banach algebra $A$ with dual space $A^*$ the maximal ideal space $M(A)$ of $A$ is the set of nonzero homomorphisms $A \to \mathbb{C}$ equipped with the Gelfand topology, the weak* topology induced by $A^*$. It is a compact Hausdorff space contained in the unit ball of $A^*$. In the case of $H^\infty$ evaluation at a point of $\mathbb{D}$ is an element of $M(H^\infty)$, so $\mathbb{D}$ is naturally embedded into $M(H^\infty)$ as an open subset. The famous Carleson corona theorem [C] asserts that $\mathbb{D}$ is dense in $M(H^\infty)$.

Let $U \subset M(H^\infty)$ be an open subset and $X$ be a complex Banach space. A continuous function $f \in C(U; X)$ is said to be $X$-valued holomorphic if its restriction to $U \cap \mathbb{D}$ is $X$-valued holomorphic in the usual sense.

By $O(U; X)$ we denote the vector space of $X$-valued holomorphic functions on $U$.

Let $E$ be a continuous Banach vector bundle on $M(H^\infty)$ with fibre $X$ defined on an open cover $U = (U_i)_{i \in I}$ of $M(H^\infty)$ by a cocycle $\{g_{ij} \in C(U_i \cap U_j; GL(X))\}$; here $GL(X)$ is the group of invertible elements of the Banach algebra $L(X)$ of bounded linear operators on $X$ equipped with the operator norm. We say that $E$ is holomorphic if all $g_{ij} \in O(U_i \cap U_j; GL(X))$. In this case $E|_\mathbb{D}$ is a holomorphic Banach vector bundle on $\mathbb{D}$ in the usual sense. Recall that $E$ is defined as the quotient space of the disjoint union $\sqcup_{i \in I} U_i \times X$ by the equivalence relation:

$$U_j \times X \ni u \times x \sim u \times g_{ij}(u)x \in U_i \times X.$$  

The projection $p : E \to X$ is induced by natural projections $U_i \times X \to U_i$, $i \in I$. 

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A morphism \( \varphi : (E_1, X_1, p_1) \to (E_2, X_2, p_2) \) of holomorphic Banach vector bundles on \( M(H^\infty) \) is a continuous map which sends each vector space \( p_1^{-1}(w) \cong X_1 \) linearly to vector space \( p_2^{-1}(w) \cong X_2 \) for each \( w \in M(H^\infty) \), and such that \( \varphi|_\partial : E_1|_\partial \to E_2|_\partial \) is a holomorphic map of complex Banach manifolds. If, in addition, \( \varphi \) is bijective, then \( \varphi \) is called an isomorphism.

We say that a holomorphic Banach vector bundle \((E, X, p)\) on \( M(H^\infty) \) is holomorphically trivial if it is isomorphic to the trivial bundle \( M(H^\infty) \times X \). (For the basic facts of the theory of bundles, see, e.g., [Hus].)

Let \( GL_0(X) \) be the connected component of \( GL(X) \) containing the identity map \( I_X := \text{id}_X : X \to X \). Then \( GL_0(X) \) is a clopen normal subgroup of \( GL(X) \). By \( q : GL(X) \to GL(X)/GL_0(X) := C(GL(X)) \) we denote the continuous quotient homomorphism onto the discrete group of connected components of \( GL(X) \). Let \( E \to M(H^\infty) \) be a holomorphic Banach vector bundle with fibre \( X \) defined on a finite open cover \( U = (U_i)_{i \in I} \) of \( M(H^\infty) \) by a cocycle \( g = \{g_{ij} \in O(U_i \cap U_j; GL(X))\} \). By \( E_{C(GL(X))} \) we denote the principal bundle on \( M(H^\infty) \) with fibre \( C(GL(X)) \) defined on \( U \) by the locally constant cocycle \( q(g) = \{q(g_{ij}) \in C(U_i \cap U_j; C(GL(X)))\} \).

**Theorem 1.1.** \( E \) is holomorphically trivial if and only if the associated bundle \( E_{C(GL(X))} \) is trivial in the category of principal bundles with discrete fibres.

**Corollary 1.2.** \( E \) is holomorphically trivial in one of the following cases:

1. The image of each function \( g_{ij} \) in the definition of \( E \) belongs to \( GL_0(X) \) (e.g., this is true if \( GL(X) \) is connected);

2. \( E \) is trivial in the category of continuous Banach vector bundles.

In particular, the result is valid for spaces \( X \) with contractible group \( GL(X) \). The class of such spaces include infinite-dimensional Hilbert spaces, spaces \( l^p \) and \( L^p[0, 1] \), \( 1 \leq p \leq \infty \), \( c_0 \) and \( C[0, 1] \), spaces \( L_p(\Omega, \mu) \), \( 1 < p < \infty \), of \( p \)-integrable measurable functions on an arbitrary measure space \( \Omega \), and some classes of reflexive symmetric function spaces; the class of spaces \( X \) with connected but not simply connected group \( GL(X) \) include finite dimensional Banach spaces, finite direct products of James spaces etc., see, e.g., [M] and references therein. There are also Banach spaces \( X \) whose linear groups \( GL(X) \) are not connected. E.g., the groups of connected components of spaces \( l^p \times l^q \), \( 1 \leq p < q < \infty \), are isomorphic to \( \mathbb{Z} \), see [D0].

We deduce Theorem 1.1 from more general results on triviality of holomorphic principal bundles on \( M(H^\infty) \), see Theorems 6.4–7.1.

1.2. We apply Theorem 1.1 to the Sz.-Nagy operator corona problem [SN] posed in 1978. In its formulation \( H^\infty(L(X, Y)) \) stands for the Banach space of holomorphic functions \( F \) on \( \mathbb{D} \) with values in the space of bounded linear operators \( X \to Y \) of complex Banach spaces \( X, Y \) with norm \( \|F\| := \sup_{z \in \mathbb{D}} \|F(z)\|_{L(X,Y)} \).

**Problem (Sz.-Nagy).** Let \( F \in H^\infty(L(H_1, H_2)) \), where \( H_i \), \( i = 1, 2 \), are separable Hilbert spaces, satisfy \( \|F(z)x\| \geq \delta \|x\| \) for every \( x \in H_1 \) and every \( z \in \mathbb{D} \), where \( \delta > 0 \) is a constant. Does there exist \( G \in H^\infty(L(H_2, H_1)) \) such that \( G(z)F(z) = I_{H_1} \) for every \( z \in \mathbb{D} \)?

This problem is of great interest in operator theory (angles between invariant subspaces, unconditionally convergent spectral decompositions), as well as in control theory. It is also related to the study of submodules of \( H^\infty \) and to many other
Theorem 1.3. Let \(L \rightarrow F \rightarrow G \rightarrow H \rightarrow \) be such that for every \(z \in \mathbb{D} \) there exists a left inverse \(G_z \) of \(F(z) \) satisfying \(\sup_{z \in \mathbb{D}} \|G_z\| < \infty \). Does there exist \(G \in H^\infty(L(X_2, X_1)) \) such that \(G(z)F(z) = I_{X_1} \) for every \(z \in \mathbb{D} \)?

Problem 1. Let \(X_1, X_2 \) be complex Banach spaces and \(F \in H^\infty(L(X_1, X_2)) \) be such that for each \(z \in \mathbb{D} \) there exists a left inverse \(G_z \) of \(F(z) \) satisfying \(\sup_{z \in \mathbb{D}} \|G_z\| < \infty \). Does there exist \(G \in H^\infty(L(X_2, X_1)) \) such that \(G(z)F(z) = I_{X_1} \) for every \(z \in \mathbb{D} \)?

Since in this general setting the answer is negative, as in [V] we restrict ourselves to the case of \(F \in H^\infty_{\text{comp}}(L(X_1, X_2)) \), the space of holomorphic functions on \(\mathbb{D} \) with relatively compact images in \(L(X_1, X_2) \). Then the answer is positive for \(F \) that can be uniformly approximated by finite sums \(\sum f_k(z)L_k \), where \(f_k \in H^\infty \) and \(L_k \in L(X_1, X_2) \), see [V] Th. 2.1. The question of whether each \(F \in H^\infty_{\text{comp}}(L(X_1, X_2)) \) can be obtained in that form is closely related to the still open problem about the Grothendieck approximation property for \(H^\infty \).

Now, under the above restriction we obtain a much stronger result.

Theorem 1.3. Let \(F \in H^\infty_{\text{comp}}(L(X_1, X_2)) \), where \(X_i, i = 1, 2 \), are complex Banach spaces, be such that for every \(z \in \mathbb{D} \) there exists a left inverse \(G_z \) of \(F(z) \) satisfying \(\sup_{z \in \mathbb{D}} \|G_z\| < \infty \). Let \(Y := \text{Ker} G_0 \). Assume that \(GL(Y) \) is connected. Then there exist functions \(H \in H^\infty_{\text{comp}}(L(X_1 \oplus Y, X_2)) \), \(G \in H^\infty_{\text{comp}}(L(X_2, X_1 \oplus Y)) \) such that \(H(z)G(z) = I_{X_2}, G(z)H(z) = I_{X_1 \oplus Y} \) and \(H(z)|_{X_1} = F(z) \) for all \(z \in \mathbb{D} \).

Corollary 1.4. Let \(F \in H^\infty_{\text{comp}}(L(H_1, H_2)) \), where \(H_i, i = 1, 2 \), are Hilbert spaces, satisfy \(\|F(z)x\| \geq \delta \|x\| \) for every \(x \in H_1 \) and every \(z \in \mathbb{D} \), where \(\delta > 0 \) is a constant. Let \(Y := (F(0)(H_1))^\perp \). Then there exist functions \(H \in H^\infty_{\text{comp}}(L(H_1 \oplus Y, H_2)) \), \(G \in H^\infty_{\text{comp}}(L(H_2, H_1 \oplus Y)) \) such that \(H(z)G(z) = I_{H_2}, G(z)H(z) = I_{H_1 \oplus Y} \) and \(H(z)|_{H_1} = F(z) \) for all \(z \in \mathbb{D} \).

This follows immediately from Theorem 1.3 because in the Hilbert case the condition of the corollary implies existence of a uniformly bounded family of left inverses of \(F(z) \), \(z \in \mathbb{D} \).

Finally, we obtain the positive answer in Problem 1 for spaces \(H^\infty_{\text{comp}} \).

Theorem 1.5. Let \(X_1, X_2 \) be complex Banach spaces and \(F \in H^\infty_{\text{comp}}(L(X_1, X_2)) \) be such that for every \(z \in \mathbb{D} \) there exists a left inverse \(G_z \) of \(F(z) \) satisfying \(\sup_{z \in \mathbb{D}} \|G_z\| < \infty \). Then there exist \(G \in H^\infty_{\text{comp}}(L(X_2, X_1)) \) such that \(G(z)F(z) = I_{X_1} \) for every \(z \in \mathbb{D} \).

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2. Preliminary Results

In this part we collect some preliminary results used in the proof of Theorem 1.1.

2.1. Runge-type approximation theorem. A compact subset $K \subset M(H^\infty)$ is called holomorphically convex if for any $x \notin K$ there is $f \in H^\infty$ such that

$$\max_K |f| < |f(x)|.$$ 

Proposition 2.1 ([Br2], Lemma 5.1). Let $N \subset M(H^\infty)$ be an open neighbourhood of a holomorphically convex set $K$. Then there exists an open set $U \Subset N$ containing $K$ such that the closure $\overline{U}$ is holomorphically convex.

Theorem 2.2 ([Br2], Theorem 1.7). Let $B$ be a complex Banach space. Any $B$-valued holomorphic function defined on a neighbourhood of a holomorphically compact set $K \subset M(H^\infty)$ can be uniformly approximated on $K$ by functions from $\mathcal{O}(M(H^\infty); B)$.

2.2. Maximal Ideal Space of $H^\infty$.

2.2.1. Recall that the pseudohyperbolic metric on $\mathbb{D}$ is defined by

$$\rho(z, w) := \left| \frac{z - w}{1 - \bar{w}z} \right|, \quad z, w \in \mathbb{D}.$$ 

For $x, y \in M(H^\infty)$ the formula

$$\rho(x, y) := \sup\{ |\hat{f}(y)| : f \in H^\infty, \hat{f}(x) = 0, \|f\| \leq 1 \}$$

gives an extension of $\rho$ to $M(H^\infty)$. The Gleason part of $x \in M$ is then defined by $\pi(x) := \{ y \in M(H^\infty) : \rho(x, y) < 1 \}$. For $x, y \in M(H^\infty)$ we have $\pi(x) = \pi(y)$ or $\pi(x) \cap \pi(y) = \emptyset$. Hoffman’s classification of Gleason parts [H] shows that there are only two cases: either $\pi(x) = \{ x \}$ or $\pi(x)$ is an analytic disk. The former case means that there is a continuous one-to-one and onto map $L_x : \mathbb{D} \to \pi(x)$ such that $\hat{f} \circ L_x \in H^\infty$ for every $f \in H^\infty$. Moreover, any analytic disk is contained in a Gleason part and any maximal (i.e., not contained in any other) analytic disk is a Gleason part. By $M_a$ and $M_s$ we denote the sets of all non-trivial (analytic disks) and trivial (one-pointed) Gleason parts, respectively. It is known that $M_a \subset M(H^\infty)$ is open. Hoffman proved that $\pi(x) \subset M_a$ if and only if $x$ belongs to the closure of some interpolating sequence in $\mathbb{D}$.

2.2.2. Structure of $M_a$. In [Br1] $M_a$ is described as a fibre bundle over a compact Riemann surface. Specifically, let $G$ be the fundamental group of a compact Riemann surface $S$ of genus $\geq 2$. Let $\ell_\infty(G)$ be the Banach algebra of bounded complex-valued functions on $G$ with pointwise multiplication and supremum norm. By $\beta G$ we denote the Stone-Čech compactification of $G$, i.e., the maximal ideal space of $\ell_\infty(G)$ equipped with the Gelfand topology.

The universal covering $r : \mathbb{D} \to S$ is a principal fibre bundle with fibre $G$. Namely, there exists a finite open cover $U = (U_i)_{i \in I}$ of $S$ by sets biholomorphic to $\mathbb{D}$ and a locally constant cocycle $\tilde{g} = \{ g_{ij} \} \in Z^1(U; G)$ such that $\mathbb{D}$ is biholomorphic to the quotient space of the disjoint union $V = \bigsqcup_{i \in I} U_i \times G$ by the equivalence relation $U_i \times G \ni (x, g) \sim (x, g g_{ij}) \in U_j \times G$. The identification space is a fibre bundle with projection $r : \mathbb{D} \to S$ induced by projections $U_i \times G \to U_i$, see, e.g., [Hi] Ch. 1.
Next, the right action of $G$ on itself by multiplications is extended to the right continuous action of $G$ on $\beta G$. Let $\tilde{r} : E(S, \beta G) \to S$ be the associated with this action bundle on $S$ with fibre $\beta G$ constructed by cocycle $\tilde{g}$. Then $E(S, \beta G)$ is a compact Hausdorff space homeomorphic to the quotient space of the disjoint union $\tilde{V} = \sqcup_{i\in I} U_i \times \beta G$ by the equivalence relation $U_i \times \beta G \ni (x, \xi) \sim (x, \xi g_i) \in U_j \times \beta G$. The projection $\tilde{r} : E(S, \beta G) \to S$ is induced by projections $U_i \times \beta G \to U_i$. Note that there is a natural embedding $V \hookrightarrow \tilde{V}$ induced by the embedding $G \hookrightarrow \beta G$. This embedding commutes with the corresponding equivalence relations and so determines an embedding of $\mathbb{D}$ into $E(S, \beta G)$ as an open dense subset. Similarly, for each $\xi \in \beta G$ there exists a continuous injection $V \to \tilde{V}$ induced by the injection $G \to \beta G$, $g \mapsto \xi g$, commuting with the corresponding equivalence relations. Thus it determines a continuous injective map $i_\xi : \mathbb{D} \to E(S, \beta G)$. Let $X_G := \beta G / G$ be the set of co-sets with respect to the right action of $G$ on $\beta G$. Then $i_\xi(\mathbb{D}) = i_{\xi_2}(\mathbb{D})$ if and only if $\xi_1$ and $\xi_2$ determine the same element of $X_G$. If $\xi$ represents an element $x \in X_G$, then we write $i_\xi(\mathbb{D})$ instead of $i_\xi(\mathbb{D})$. In particular, $E(S, \beta G) = \sqcup_{x \in X_G} i_x(\mathbb{D})$.

Let $U \subset E(S, \beta G)$ be open. We say that a function $f \in C(U)$ is holomorphic if $f|_{U \cap \mathbb{D}}$ is holomorphic in the usual sense. The set of holomorphic on $U$ functions is denoted by $O(U)$. It was shown in [Br1] Th. 2.1 that each $h \in H^\infty(U \cap \mathbb{D})$ is extended to a unique holomorphic function $\hat{h}$ on $O(U)$. In particular, the restriction map $O(E(S, \beta G)) \to H^\infty(\mathbb{D})$ is an isometry of Banach algebras. Thus the quotient space of $E(S, \beta G)$ (equipped with the factor topology) by the equivalence relation $x \sim y \iff f(x) = f(y)$ for all $f \in O(E(S, \beta G))$ is homeomorphic to $M(H^\infty)$. By $q$ we denote the quotient map $E(S, \beta G) \to M(H^\infty)$.

A sequence $\{g_n\} \subset G$ is said to be interpolating if $\{g_n(0)\} \subset \mathbb{D}$ is interpolating for $H^\infty$ (here $G$ acts on $\mathbb{D}$ by Möbius transformations). Let $G_m \subset \beta G$ be the union of closures of all interpolating sequences in $G$. It was shown that $G_m$ is an open dense subset of $\beta G$ invariant with respect to the right action of $G$. The associated with this action bundle $E(S, G_m)$ on $S$ with fibre $G_m$ constructed by the cocycle $\tilde{g} \in Z^1(U, G)$ is an open dense subbundle of $E(S, \beta G)$ containing $\mathbb{D}$. It was established in [Br1] that $q$ maps $E(S, G_m)$ homeomorphically onto $M_a$ so that for each $\xi \in G_m$ the set $q(i_\xi(\mathbb{D}))$ coincides with the Gleason part $\pi(q(i_\xi(\mathbb{D})))$. Also, for distinct $x, y \in E(S, \beta G)$ with $x \in E(S, G_m)$ there exists $f \in O(E(S, \beta G))$ such that $f(x) \neq f(y)$. Thus $q(x) = x$ for all $x \in E(S, G_m)$, i.e., $E(S, G_m) = M_a$.

From the definition of $E(S, \beta G)$ follows that for a simply connected open subset $U \subset S$ restriction $E(S, \beta G)|_U := \tilde{r}^{-1}(U)$ is a trivial bundle, i.e., there exists an isomorphism of bundles (with fibre $\beta G$) $\varphi : E(S, \beta G)|_U \to U \times \beta G$, $\varphi(x) := (\tilde{r}(x), \tilde{g}(x))$, $x \in E(S, \beta G)|_U$, mapping $\tilde{r}^{-1}(U) \cap \mathbb{D}$ biholomorphically onto $U \times \beta G$. A subset $W \subset \tilde{r}^{-1}(U)$ of the form $R_{U,H} := \varphi^{-1}(U \times H)$, $H \subset \beta G$, is called rectangular. The base of topology on $E(S, G_m)(:= M_a)$ consists of rectangular sets $R_{U,H}$ with $U \subset S$ biholomorphic to $\mathbb{D}$ and $H \subset G_m$ being the closure of an interpolating sequence in $G$ (so $H$ is a clopen subset of $\beta G$). Another base of topology on $M_a$ is given by sets of the form $\{x \in M_a : |\tilde{B}(x)| < \varepsilon\}$, where $B$ is an interpolating Blaschke product (here $\tilde{B}$ is the extension of $B$ to $M(H^\infty)$ by means of the Gelfand transform). This follows from the fact that for a sufficiently small $\varepsilon$ the set $B^{-1}(\mathbb{D}_\varepsilon) \subset \mathbb{D}$, $\mathbb{D}_\varepsilon := \{z : |z| < \varepsilon\}$, is biholomorphic to $\mathbb{D}_\varepsilon \times B^{-1}(0)$, see [Ga] Ch. X, Lm. 1.4. Hence, $\{x \in M_a : |\tilde{B}(x)| < \varepsilon\}$ is biholomorphic to $\mathbb{D}_\varepsilon \times \tilde{B}^{-1}(0)$. 

2.2.3. Structure of $M_a$. It was proved in [S2], that the set $M_s$ of trivial Gleason parts is totally disconnected, i.e., $\dim M_s = 0$ (because $M_s$ is compact).

Also, it was proved in [S1] that the covering dimension of $M(H^\infty)$ is 2 and the Čech cohomology group $H^2(M(H^\infty), \mathbb{Z}) = 0$.

2.3. $\overline{\partial}$-equations with Support in $M_a$. Let $X$ be a complex Banach space and $U \subset S$ be open simply connected. Let $\varphi : E(S, \beta G)|U \to U \times \beta G$ be a trivialization as in subsection 2.2.2. We say that a function $f \in C(E(S, \beta G)|U; X)$ belongs to the space $C^k(E(S, \beta G)|U; X)$, $k \in \mathbb{N} \cup \{\infty\}$, if its pullback to $U \times \beta G$ by $\varphi^{-1}$ is of class $C^k$. In turn, a continuous $X$-valued function on $U \times \beta G$ is of class $C^k$ if regarded as a Banach-valued map $U \to C(\beta G; X)$ it has continuous derivatives of order $\leq k$ (in local coordinates on $U$).

For a rectangular set $R_{U,H} \subset E(S, \beta G)|U$ with clopen $H$ a function $f$ on $R_{U,H}$ is said to belong to the space $C^k(R_{U,H}; X)$ if its extension to $E(S, \beta G)|U$ by 0 belongs to $C^k(E(S, \beta G)|U; X)$.

For an open $V \subset E(S, \beta G)$ a continuous function $f$ on $V$ belongs to the space $C^k(V; X)$ if its restriction to each $R_{U,H} \subset V$ with $H$ clopen belongs to $C^k(R_{U,H}; X)$.

In the proofs we use the following results.

**Proposition 2.3** ([Br2], Proposition 3.2). For a finite open cover of $E(S, \beta G)$ there exists a $C^\infty$ partition of unity subordinate to it.

**Corollary 2.4** ([Br2], Corollary 3.4). Let $U \subset V \subset E(S, \beta G)$ be open. Then there exists a nonnegative $C^\infty$ function $\rho$ on $E(S, \beta G)$ such that $\rho = 1$ in an open neighbourhood of $\bar{U}$ and $\text{supp} \, \rho \subset V$.

An $X$-valued $(0,1)$-form $\omega$ of class $C^k$ on an open $U \subset E(S, \beta G)$ is defined in each coordinate chart $R_{V,H} \subset U$ with local coordinates $(z, \xi)$ (pulled back from $V \times \beta G$ by $\varphi$) by the formula $\omega|_{R_{V,H}} := f(z, \xi)d\bar{z}$, $f \in C^k(R_{V,H}; X)$, so that the restriction of the family $\{\omega|_{R_{V,H}} : R_{V,H} \subset U\}$ to $\mathbb{D}$ determines a global $X$-valued $(0,1)$-form of class $C^k$ on the open set $U \cap \overline{\mathbb{D}} \subset \mathbb{D}$.

By $\mathcal{E}^{0,1}(U; X)$ we denote the space of $X$-valued $(0,1)$-forms on $U \subset E(S, \beta G)$. The operator $\bar{\partial} : C^\infty(U; X) \to \mathcal{E}^{0,1}(U; X)$ is defined in each $R_{V,H} \subset U$ equipped with the local coordinates $(z, \xi)$ as $\bar{\partial}f(z, \xi) := \frac{\partial f}{\partial \bar{z}}(z, \xi)d\bar{z}$. Then the composite of the restriction map to $U \cap \mathbb{D}$ with this operator coincides with the standard $\bar{\partial}$ operator defined on $C^\infty(U \cap \mathbb{D}; X)$.

It is easy to check, using Cauchy estimates for derivatives of families of uniformly bounded holomorphic functions on $\mathbb{D}$, that if $f \in \mathcal{O}(U; X)$, $U \subset E(S, \beta G)$, then $f \in C^\infty(U; X)$ and in each $R_{V,H} \subset U$ with local coordinates $(z, \xi)$ the function $f(z, \xi)$ is holomorphic in $z$. Thus $\bar{\partial}f = 0$.

By $\mathcal{E}^{0,1}_{\text{comp}}(M_a; X)$ we denote the class of $X$-valued $C^\infty$ $(0,1)$-forms on $E(S, \beta G)$ with compact supports in $M_a := E(S, G_m)$, i.e., $\omega \in \mathcal{E}^{0,1}_{\text{comp}}(M_a; X)$ if there is a compact subset of $M_a$ such that in each local coordinate representation $\omega = fd\bar{z}$ support of $f$ belongs to it. By $\text{supp} \, \omega$ we denote the minimal set satisfying this property. Let $\mathcal{E}^{0,1}_{K}(X) \subset \mathcal{E}^{0,1}_{\text{comp}}(M_a; X)$ be the subspace of forms with supports in the compact set $K \subset M_a$.

**Theorem 2.5** ([Br2], Theorem 3.5). There exist a norm $\| \cdot \|_{K; X}$ on $\mathcal{E}^{0,1}_{K}(X)$ and a continuous linear operator $L_{K; X} : (\mathcal{E}^{0,1}_{K}(X), \| \cdot \|_{K; X}) \to (C(M(H^\infty); X), \text{sup} \, M(H^\infty)) \|$. 
\[ \|x\| \] with norm bounded by a constant depending only on \( K \) such that for each \( \omega \in \mathcal{E}_{K}^{0,1}(X) \)

(a) \( L_{K;X}(\omega)|_{M_{\alpha}} \in C^\infty(M_{\alpha};X) \) and \( \partial(L_{K;X}(\omega)|_{M_{\alpha}}) = \omega \);

(b) \( L_{K;X}(\omega)|_{M(H^\infty) \setminus K} \in \mathcal{O}(M(H^\infty) \setminus K;X) \).

Moreover, if \( \omega|_{\bar{D}} = F(z)d\bar{z} \) for \( \omega \in \mathcal{E}_{K}^{0,1}(X) \), then

\[
\|\omega\|_{K;X} \leq C_{K} \sup_{z \in \partial\text{supp}\omega} \{ \|F(z)\|_{X} \cdot (1 - |z|) \}
\]

for a constant \( C_{K} \geq 1 \) depending only on \( K \), and if \( H \) is a \( C^\infty \) function on a neighbourhood of \( \text{supp}\ \omega \) with values in the space \( L(X;Y) \) of bounded linear operators between complex Banach spaces \( X \) and \( Y \), then

\[
\|H(\omega)\|_{K;Y} \leq \left\{ \sup_{z \in \text{supp}\omega} \|H(z)\|_{L(X,Y)} \right\} \cdot \|\omega\|_{K};
\]

here \( H(\omega)|_{\bar{D}} := H(z)(F(z))d\bar{z} \).

3. Cousin-type Lemma

Let \( X \) be a complex Banach space. A continuous \( X \)-valued function on a compact subset \( K \subset M(H^\infty) \) is called holomorphic if it is the restriction of a holomorphic function defined in an open neighbourhood of \( K \). The space of such functions is denoted by \( \mathcal{O}(K;X) \). By \( A(K;X) \) we denote the closure of \( \mathcal{O}(K;X) \) in the Banach space \( C(K;X) \) equipped with norm \( \|f\|_{K;X} := \sup_{z \in K} \|f(z)\|_{X} \).

**Theorem 3.1.** Suppose that \( U_{1}, U_{2} \subset M(H^\infty) \) are open such that \( \bar{U}_{1} \cap \bar{U}_{2} \subset M_{\alpha} \) and \( W_{i} \subset U_{i}, \ i = 1, 2 \), are compact. We set \( W := W_{1} \cup W_{2} \). There exists a constant \( C \) such that for every function \( f \in A(\bar{U}_{1} \cap \bar{U}_{2} \cap W;X) \) there are functions \( f_{i} \in A(U_{i} \cap W;X) \) such that

\[
(1) \quad f_{1} + f_{2} = f \quad \text{on} \quad U_{1} \cap U_{2} \cap W;
\]

\[
(2) \quad \|f_{i}\|_{\bar{U}_{i} \cap W;X} \leq C \|f\|_{\bar{U}_{i} \cap \bar{U}_{2} \cap W;X}, \quad i = 1, 2.
\]

**Proof.** We consider \( M_{\alpha} \) as an open subset of \( E(S,\beta G) \). Let \( \rho_{i}, \ i = 1, 2 \), be nonnegative \( C^\infty \) functions on \( E(S,\beta G) \) such that \( \rho_{i}|_{W_{i}} > 0 \) and \( \text{supp}\ \rho_{i} \subset U_{i} \), cf. Corollary 2.4. Let \( Z := \{ x \in E(S,\beta G) : \rho_{1}(x) = \rho_{2}(x) = 0 \} \). By definition, \( Z \cap W = \emptyset \).

Therefore there is an open neighbourhood \( U \) of \( W \) such that \( U \cap Z = \emptyset \). Then we define nonnegative \( C^\infty \) functions \( \psi_{i} \) on \( U \) by the formulas

\[
\psi_{i} := \frac{\rho_{i}}{\rho_{1} + \rho_{2}}, \quad i = 1, 2;
\]

here \( \text{supp}\ \psi_{i} \subset U_{i} \cap U \).

Without loss of generality we may assume that \( \emptyset \neq \bar{U}_{1} \cap \bar{U}_{2} \cap W \neq W \). (For otherwise, the statement of the theorem is trivial.) We fix an open neighbourhood \( N \subset M_{\alpha} \) of \( \bar{U}_{1} \cap \bar{U}_{2} \cap W \).

By definition, a function \( f \in \mathcal{O}(\bar{U}_{1} \cap \bar{U}_{2} \cap W;X) \) admits a holomorphic extension (denoted also by \( f \)) to an open set \( O_{f} \subset N \) such that \( O_{f} \supset \bar{U}_{1} \cap \bar{U}_{2} \cap W \) and \( \|f\|_{O_{f};X} \leq 2\|f\|_{\bar{U}_{1} \cap \bar{U}_{2} \cap W;X} \). Without loss of generality we may assume that \( O_{f} \subset \)
By our construction, the functions are well defined. We check it, say, for $f'_i$. Indeed, by our construction $f'_1$ is defined in a neighbourhood of the closure of the union of $U'_1 \cap U'_2 \cap W'$. Since

$$(U'_1 \cap W') \setminus (U'_1 \cap U'_2 \cap W') = (U'_1 \setminus U'_2) \cap W' \subset (U'_1 \setminus U) \setminus (U_2 \cap U),$$

we obtain

$$\psi = 0 \text{ on } (U'_1 \cap W') \setminus (U'_1 \cap U'_2 \cap W').$$

Hence, $f'_1 = 0$ on $(U'_1 \cap W') \setminus (U'_1 \cap U'_2 \cap W')$ as well. Similarly, one proves that $f'_i$ is of class $C^\infty$ on $U'_i \cap W', i = 1, 2$. Thus,

$$\omega_f := \partial f'_i \text{ on } U'_i \cap W', \ i = 1, 2,$$

is an $X$-valued $C^\infty (0, 1)$ form on $W'$ (because $f'_1 - f'_2 = f$ on $U'_1 \cap U'_2 \cap W'$). By the definition $\supp \omega_f \subset O_f$. Let $\rho_f$ be a $C^\infty$ function on $E(S, \beta G)$ with values in $[0, 1]$ equals 1 in an open neighbourhood of $W$ with support in $W'$, cf. Corollary 2.4. Then $\eta_f := \rho_f \omega_f$ is an $X$-valued $C^\infty (0, 1)$ form on $E(S, \beta G)$ with support in $O_f \subset N$.

Let us apply this construction to $X := C$ and $f(z) := I, z \in \bar{U}_1 \cap \bar{U}_2 \cap W$, where $I := 1$ is the unit of $C$. In this case $\rho_I$ and $\omega_I$ are some specific function and form depending only on the choice of the above sets for $I$. Therefore for our space $X$ and function $\bar{f}$ we may choose $\rho_f$ so that $\rho_f = 1$ on $\supp \rho_f$. Since $\supp \eta_f$ is a compact subset of $O_f$, there is a $C^\infty$ function $h_f$ on $E(S, \beta G)$ with values in $[0, 1]$ equals 1 in an open neighbourhood of $\supp \eta_f$ with support in $O_f$. In particular, we obtain

$$\eta_f = h_f \cdot \chi_{O_f} \cdot f|_{O_f} \cdot \rho_f \cdot \eta_I,$$

where $\chi_{O_f}$ is the characteristic function of $O_f$. The $X$-valued function $h_f \cdot \chi_{O_f} \cdot f|_{O_f} \cdot \rho_f$ is of class $C^\infty$ on $E(S, \beta G)$. Therefore applying estimate (2.2) we get

$$\|\eta_f\|_{N; X} \leq \left\{ \sup_{z \in O_f} \|h_f(z) \cdot \chi_{O_f}(z) \cdot f(z) \cdot \rho_f(z)\|_X \right\} \cdot \|\eta_I\|_{N; \mathbb{C}}$$

$$\leq (2\|\eta_I\|_{N; \mathbb{C}})\|f\|_{\bar{U}_1 \cap \bar{U}_2 \cap W; X}.$$
We set \( h_n := g_n - g_{n-1}, n \in \mathbb{N} \). By the above result, there exist functions \( h_{in} \in A(\bar{U}_i \cap W; X) \) such that \( h_{1n} + h_{2n} = h_n \) on \( \bar{U}_1 \cap \bar{U}_2 \cap W \) and \( \|h_{in}\|_{\bar{U}_1 \cap \bar{U}_2 \cap W; X} \leq C\|h_n\|_{\bar{U}_1 \cap \bar{U}_2 \cap W; X} \), \( i = 1, 2 \). We set

\[
f_i := \sum_{n=1}^{\infty} h_{in}, \quad \text{on} \quad \bar{U}_i \cap W, \quad i = 1, 2.
\]

The above estimates imply that the series converge to elements of \( A(\bar{U}_i \cap W; X), \ i = 1, 2 \). Moreover,

\[
\|f_i\|_{\bar{U}_i \cap W; X} \leq 3C\|f\|_{\bar{U}_1 \cap \bar{U}_2 \cap W; X}, \quad i = 1, 2,
\]

and

\[
f_1 + f_2 = \sum_{n=1}^{\infty} (h_{1n} + h_{2n}) = \sum_{n=1}^{\infty} (g_n - g_{n-1}) = f \quad \text{on} \quad \bar{U}_1 \cap \bar{U}_2 \cap W.
\]

The proof of the theorem is complete. \( \square \)

4. Cartan-type Lemma

For the basic facts of the theory of Banach Lie groups see, e.g., [Mai].

Let \( B^{-1} \) be the group of invertible elements of a complex unital Banach algebra \( B \). Let \( \psi : G \to B_0^{-1} \) be a regular covering of the connected component \( B_0^{-1} \) of \( B^{-1} \) containing the unit \( 1_{B^{-1}} \) of \( B^{-1} \). Then \( G \) is complex Banach Lie group and \( \psi \) is a surjective morphism of complex Lie groups whose kernel is a discrete central subgroup of \( G \), see, e.g., [Po]. By \( \exp_G : \mathcal{L}_G \to G, \exp_{B^{-1}} : \mathcal{L}_{B^{-1}} \to B^{-1} \) we denote the corresponding exponential maps of Lie algebras of \( G \) and \( B^{-1} \). Then differential \( d\psi(1_G) : \mathcal{L}_G \to \mathcal{L}_{B^{-1}} \) at the unit \( 1_G \) of \( G \) is an isomorphism of Lie algebras and

\[
(4.1) \quad \psi \circ \exp_G = \exp_{B^{-1}} \circ d\psi(1_G).
\]

A continuous map \( f : W \to G, \) where \( W \subset M(H^\infty) \) is open, is called holomorphic if \( f|_{\bar{W} \cap W} : \mathbb{D} \cap W \to G \) is a holomorphic map of complex Banach manifolds. For a compact set \( K \subset M(H^\infty) \) a map \( K \to G \) is called holomorphic if it is a restriction of a holomorphic map into \( G \) defined on an open neighbourhood of \( K \). The space of such maps is denoted by \( \mathcal{O}(G; X) \). We say that a continuous map \( f : K \to G \) belongs to the space \( A(K; G) \) if \( \psi(f|_S) \in A(S; B) \) for every compact subset \( S \subset K \). The space \( A(K; G) \) has a natural group structure induced by the product on \( G \). Clearly, \( \mathcal{O}(K; G) \subset A(K; G) \) (because \( \psi \) is a holomorphic map). The unit of \( A(K; G) \), i.e., the map \( z \mapsto 1_G, z \in K \), will be denoted by \( I \).

A map \( \gamma : [0, 1] \to A(K; G) \) is called a path if the induced map \( [0, 1] \times K \to G, (t, x) \mapsto \gamma(t)(x) \), is continuous. The set of all maps in \( A(K; G) \) that can be joined by paths (in \( A(K; G) \)) with \( I \) will be called the connected component of \( I \).

We retain notation of Theorem 3.1.

**Theorem 4.1.** Assume that \( W_1 \) is holomorphically convex and \( W_1 \cap W_2 \neq \emptyset \). Let \( F \in A(\bar{U}_1 \cap \bar{U}_2; G) \) belong to the connected component of \( I \). Then there exist \( F_i \in A(W_i; G) \) such that \( F_1 \cdot F_2 = F \) on \( W_1 \cap W_2 \).

**Proof.** Since \( W_1 \) is holomorphically convex, according to Proposition 2.1, there exists a sequence of open sets \( U_{1, i} \subset U_1 \) containing \( W_1 \) such that \( U_{1, i+1} \subset U_{1, i} \) and \( U_{1, i} \) is holomorphically convex for all \( i \in \mathbb{N} \). Let us choose a sequence of open sets \( U_{2, i} \subset U_2 \)
containing $W_2$ such that $U_{2,i+1} \subset U_{2,i}$ for all $i \in \mathbb{N}$. Then Theorem 3.1 is also valid with $U_j$ replaced by $U_{j,i}$; $j = 1, 2$, $W := W_1 \cup W_2$ replaced by $W^j := \bar{U}_{1,i+1} \cup U_{2,i+1}$ and $C$ replaced by some $C_i$.

**Proposition 4.2.** There exists a positive number $\varepsilon_1$ such that for every function $F \in A(U_{1,i} \cap U_{2,i} \cap W^i; G)$ satisfying $\|\psi \circ F - \psi \circ I\|_{\bar{U}_{1,i} \cap \bar{U}_{2,i} \cap W^i; B} < \varepsilon_1$ there exist $F_j \in A(\bar{U}_{j,i} \cap W^j; G)$, $j = 1, 2$, such that

1. $F_1 \cdot F_2 = F$ on $\bar{U}_{1,i} \cap \bar{U}_{2,i} \cap W^i$ and
2. $\|\psi \circ F_j - \psi \circ I\|_{\bar{U}_{j,i} \cap \bar{U}_{2,i} \cap W^i; B} \leq 4C_i \|\psi \circ F - \psi \circ I\|_{\bar{U}_{1,i} \cap \bar{U}_{2,i} \cap W^i; B}$, $j = 1, 2$.

Moreover, $\psi$ has an inverse $\psi^{-1}$ on the ball $\{v \in B; \|v - 1_{B-1}\|_B < 4C_i \varepsilon_1\} \subset B$ such that $\psi^{-1}(1_{B-1}) = 1_G$.

**Proof.** Since $G$ is locally isomorphic to $B^{-1}$, cf. (1.1), it suffices to prove the result for $G := B^{-1}$. In this case the proof repeats literally the classical proof of Cartan’s lemma with the classical Cousin lemma replaced by Theorem 3.1 and with matrix norm replaced by norm on $B$, see, e.g., [GR Ch. III.1].

Further, since $F$ in the statement of the theorem belongs to the connected component $C_0$ containing $I$ of the group $A(U_1 \cap U_2; G)$, there exists a path $\gamma : [0, 1] \to C_0$ such that $\gamma(0) = I$ and $\gamma(1) = F$. From the continuity of $\gamma$ follows that there is a partition $0 - t_0 < t_1 < \cdots < t_k = 1$ of $[0, 1]$ such that

\[
\|\psi \circ (\gamma(t_i)^{-1} \cdot \gamma(t_{i+1})) - \psi \circ I\|_{\bar{U}_{1,i} \cap \bar{U}_{2,i} \cap W^i; B} < \varepsilon_1 \quad \text{for all} \quad i.
\]

(4.2) Applying Proposition 4.2 to each $\gamma(t_i)^{-1} \cdot \gamma(t_{i+1})$ we obtain

\[
\gamma(t_i)^{-1} \cdot \gamma(t_{i+1}) = F^i_1 \cdot F^i_2 \quad \text{on} \quad \bar{U}_{1,i} \cap \bar{U}_{2,i} \cap W^1
\]

for some $F^i_1 \in A(U_\ell,1 \cap W^1; G)$, $\ell = 1, 2$.

Next, we have

\[
F = \prod_{i=0}^{k-1} (\gamma(t_i)^{-1} \cdot \gamma(t_{i+1})).
\]

We will use induction on $j$ for $1 \leq j \leq k$. The induction hypothesis is if

\[
F^j := \prod_{i=0}^{j-1} (\gamma(t_i)^{-1} \cdot \gamma(t_{i+1})),
\]

then

\[
F^j = F^j_1 \cdot F^j_2 \quad \text{on} \quad \bar{U}_{1,j} \cap \bar{U}_{2,j} \cap W^j
\]

for some $F_{\ell,j} \in A(U_{\ell,j} \cap W^j; G)$, $\ell = 1, 2$.

By (4.3) the statement is true for $j = 1$. Assuming that it is true for $j - 1$ let us prove it for $j$.

By the induction hypothesis we have

\[
F^j = F^{j-1} \cdot \gamma(t_j)^{-1} \cdot \gamma(t_{j+1}) = F^j_{1,j-1} \cdot F_{2,j-1} \cdot F^{j-1}_{1,j} \cdot F^{j-1}_{2,j} \quad \text{on} \quad \bar{U}_{1,j-1} \cap \bar{U}_{2,j-1} \cap W^{j-1}.
\]

Since $\bar{U}_{1,j} \subset \bar{U}_{1,j-1} \cap W^{j-1}$ is holomorphically convex, by Theorem 2.2, assuming that $\varepsilon_1$ is sufficiently small such that $\exp^{-1}_{B^{-1}}((\psi \circ F^{j-1}_1))$ is well-defined on $\bar{U}_{1,j}$ (see Proposition 4.2(2)), we approximate $\exp^{-1}_{B^{-1}}((\psi \circ F^{j-1}_1))$ uniformly on $\bar{U}_{1,j}$ by a sequence of functions from $O(M(H^\infty); L_{B^{-1}})$. Applying to the functions of this
sequence map $\exp_G \circ (d\psi(1_G))^{-1}$, see (4.1), we obtain that for every $\varepsilon > 0$ there exists $F_\varepsilon \in \mathcal{O}(M(H^\infty); G)$ such that
\begin{equation}
\|\psi(F_1^{j-1} \cdot F_\varepsilon^{-1}) - \psi(I)\| \leq \varepsilon.
\end{equation}

We write
\begin{align*}
F_2,j^{-1} \cdot F_1^{j-1} \cdot F_2^{-1} &= [F_2,j^{-1}, F_1^{j-1}, F_\varepsilon^{-1}] \cdot (F_1^{j-1} \cdot F_\varepsilon^{-1}) \cdot F_2,j^{-1} \cdot F_\varepsilon \cdot F_2^{-1},
\end{align*}
where $[A_1, A_2] := A_1 \cdot A_2 \cdot A_1^{-1} \cdot A_2^{-1}$.

According to (4.4) for a sufficiently small $\varepsilon$ we have
\begin{equation}
\|\psi([F_2,j^{-1}, F_1^{j-1}, F_\varepsilon^{-1}] \cdot (F_1^{j-1} \cdot F_\varepsilon^{-1})) - \psi(I)\| \leq \varepsilon_j.
\end{equation}
Thus by Proposition 4.2, there exist $H_\ell \in A(\bar{U}_{\ell,j} \cap W^j; G)$, $\ell = 1, 2$, such that
\begin{align*}
[F_2,j^{-1}, F_1^{j-1}, F_\varepsilon^{-1}] \cdot (F_1^{j-1} \cdot F_\varepsilon^{-1}) &= H_1 \cdot H_2 \quad \text{on } \bar{U}_{1,j} \cap \bar{U}_{2,j} \cap W^j.
\end{align*}
In particular, we obtain
\begin{align*}
F_j &= F_{1,j}^{-1} \cdot F_{2,j}^{-1} \cdot F_1^{j-1} \cdot F_2^{-1} = F_{1,j}^{-1} \cdot H_1 \cdot H_2 \cdot F_{2,j}^{-1} \cdot F_\varepsilon \cdot F_2^{-1} \quad \text{on } \bar{U}_{1,j} \cap \bar{U}_{2,j} \cap W^j.
\end{align*}
We set
\begin{align*}
F_{1,j} := F_{1,j}^{-1} \cdot H_1, \quad \text{on } \bar{U}_{1,j} \cap W^j, \quad F_{2,j} := H_2 \cdot F_{2,j}^{-1} \cdot F_\varepsilon \cdot F_2^{-1} \quad \text{on } \bar{U}_{2,j} \cap W^j.
\end{align*}
This completes the proof of the induction step.

From this result for $j := k - 1$ and the fact that $W_\ell \subset \bar{U}_{\ell,k+1}$, $\ell = 1, 2$, we obtain that there exist $F_i \in A(W_i; G)$ such that $F_1 \cdot F_2 = F$ on $W_1 \cap W_2$. \hfill \Box

## 5. Maximal Ideal Space of Algebra $A(K)$

We set $A(K) := A(K; \mathbb{C})$, where $K \subset M_a$ is compact. By definition, $A(K)$ is a complex commutative unital Banach algebra with norm $\|f\|_\infty := \sup_{z \in K}|f(z)|$.

**Theorem 5.1.** The maximal ideal space of $A(K)$ is $K$.

**Proof.** Since $A(K)$ is the closure of $\cup_{U \supset K} (\mathcal{O}(U)|_K)$, where $U$ runs over all possible open neighbourhoods of $K$, to prove the result it suffices to establish the following (cf. [Ga, Ch. V, Th. 1.8]):

**Lemma 5.2.** Suppose that $f_1, \ldots, f_n \in \mathcal{O}(U)$ for an open $U$, $K \subset U \subset M_a$, satisfy
\begin{equation}
\sum_{j=1}^n |f_j(z)| > \delta > 0 \quad \text{for all } z \in U.
\end{equation}

Then there exist $g_1, \ldots, g_n \in A(K)$ such that
\begin{equation}
\sum_{j=1}^n g_j f_j = 1 \quad \text{on } K.
\end{equation}

**Proof.** Using (5.1) we find open sets $W_k \subset U$, $1 \leq k \leq \ell$, and bounded functions $g_{ik} \in \mathcal{O}(W_k)$ such that $K \subset \bigcup_{k=1}^\ell W_k$ and for all $k$
\begin{equation}
\sum_{i=1}^n g_{ik} f_i = 1 \quad \text{on } W_k.
\end{equation}
Let \( \{ \varphi_k \}_{1 \leq k \leq \ell} \) be nonnegative \( C^\infty \) functions on \( E(S, \beta G) \) such that \( \text{supp } \varphi_k \subseteq W_k \) and \( \sum_{k=1}^{\ell} \varphi_k = 1 \) in an open neighbourhood \( O \subseteq M_0 \) of \( K \) (see subsection 2.3). We set
\[
c_{i,rs} := g_{ir} - g_{is} \quad \text{on} \quad W_r \cap W_s \neq \emptyset,
\]
and
\[
h_{ir} := \sum_s \varphi_s c_{i,rs},
\]
where the sum is taken over all \( s \) for which \( W_s \cap W_r \neq \emptyset \). Then \( h_{ir} \in C^\infty(W_r) \) and
\[
h_{ir} - h_{is} = c_{i,rs} \quad \text{on} \quad O \cap W_r \cap W_s \neq \emptyset.
\]
Next, we define
\[
h_{i} := g_{ir} - h_{ir} \quad \text{on} \quad O \cap W_r.
\]
Clearly, each \( h_i \in C^\infty(O) \). In particular, \( \bar{\partial}h_i \) are \( C^\infty(0,1) \) forms on \( O \).

Consider equations
\[
\bar{\partial}b_{is} = h_i \bar{\partial}h_s \quad \text{on} \quad O.
\]
According to [Br2, Lm. 4.2], for an open set \( O' \) such that \( K \subset O' \subseteq O \) there are \( b_{is} \in C^\infty(O') \) solving these equations on \( O' \). We set
\[
g_i := h_i + \sum_{s=1}^{n} (b_{is} - b_{si}) f_s \quad \text{on} \quad O'.
\]
Since by (5.3) \( \sum_{i=1}^{n} h_i f_i = 1 \) on \( O \),
\[
\sum_{i=1}^{n} g_i f_i = 1 \quad \text{on} \quad O' \quad \text{and}
\]
\[
\bar{\partial}g_i = \bar{\partial}h_i + \sum_{s=1}^{n} f_s \cdot (h_i \bar{\partial}h_s - h_s \bar{\partial}h_i) = \bar{\partial}h_i + h_i \bar{\partial} \left( \sum_{s=1}^{n} f_s h_s \right) - \bar{\partial}h_i \sum_{s=1}^{n} f_s h_s = 0.
\]
Hence, \( g_i, \bar{\partial}h_i, \bar{\partial}h_s \) are \( C^\infty(0,1) \) forms on \( O \). The proof of the theorem is complete.

**Remark 5.3.** Theorem 5.1 is valid for any compact \( K \subset M(H^\infty) \). This fact is not used in the proof.

### 6. Triviality of Holomorphic Principal Bundles

#### 6.1. Bundles with Simply Connected Fibres

Let \( B \) be a complex unital Banach algebra and \( \psi : \tilde{B}_0^{-1} \rightarrow B_0^{-1} \) be the universal covering of the connected component \( B_0^{-1} \) of \( B^{-1} \) containing \( 1_{B^{-1}} \). Let \( p : E \rightarrow M(H^\infty) \) be a holomorphic principal bundle with fibre \( \tilde{B}_0^{-1} \). It is defined on an open cover \( \mathcal{U} = (U_i)_{i \in I} \) of \( M(H^\infty) \) by a cocycle \( g = \{ g_{ij} \in \mathcal{O}(U_i \cap U_j; \tilde{B}_0^{-1}) \} \) similarly to the construction of subsection 1.1 (with \( X \) replaced by \( \tilde{B}_0^{-1} \)). For a compact set \( K \subset M(H^\infty) \), we say that \( E|_K \) is holomorphically trivial if it is holomorphically trivial in an open neighbourhood of \( K \) (the notion is defined analogously to that of subsection 1.1).

Any open set of the form
\[
O_{f, \varepsilon} := \{ x \in M(H^\infty) ; |\hat{f}(x)| < \varepsilon \},
\]
where \( f \) is an interpolating Blaschke product (\( \hat{f} \) is the extension of \( f \) to \( M(H^\infty) \) by means of the Gelfand transform) and \( \epsilon \) is so small that \( O_{f,\epsilon} \subseteq M_a \) and is homeomorphic to \( \mathbb{D}_\epsilon \times \hat{f}^{-1}(0) \), see subsection 2.2.2, will be called an open Blaschke set.

**Lemma 6.1.** Let \( N \) be a compact subset of the Blaschke set \( O_{f,\epsilon} \). Then every continuous map \( N \to \tilde{B}_0^{-1} \) is homotopic to a constant map.

**Proof.** For the basic facts of algebraic topology see, e.g., [ES].

Since \( f^{-1}(0) \) is an interpolating sequence for \( H^\infty \), the set \( f^{-1}(0) \) is homeomorphic to the Stone-Čech compactification \( \beta N \). By the result of [Ma], \( \beta N \) can be obtained as the (iterated) inverse limit of a family of finite sets. Hence, \( O_{f,\epsilon} \) can be obtained as the (iterated) inverse limit of a family of sets of the form \( \mathbb{D}_\epsilon \times \mathcal{F} \), where \( \mathcal{F} \) is a finite set. In turn, \( N \) can be obtained as the (iterated) inverse limit of a family of compact subsets \( K \subset \mathbb{D}_\epsilon \times \mathcal{F} \). Since \( \tilde{B}_0^{-1} \) is an absolute neighbourhood retract, see [P], any continuous map \( N \to \tilde{B}_0^{-1} \) is homotopic to a continuous map into \( \tilde{B}_0^{-1} \) pulled back from some \( K \subset \mathbb{D}_\epsilon \times \mathcal{F} \) in the inverse limit construction, see, e.g., an argument in the proof of Theorem 1.2 of [BRS]. Thus it suffices to prove that every continuous map \( K \to \tilde{B}_0^{-1} \) with such a \( K \) is homotopic to a constant map. Further, since \( \tilde{B}_0^{-1} \) is an absolute neighbourhood retract, every continuous map \( K \to \tilde{B}_0^{-1} \), can be extended to an open neighbourhood \( W \subset \mathbb{D}_\epsilon \times \mathcal{F} \) with a smooth boundary. In particular, \( W \) has finitely many connected components homeomorphic to multiply connected domains in \( \mathbb{C} \). This implies that it suffices to establish the result for continuous maps into \( \tilde{B}_0^{-1} \) defined on a multiply connected domain \( D \). Finally, there exists a finite connected one-dimensional CW complex \( S \subset D \) such that \( S \) is a strong deformation retract of \( D \). Thus every continuous map \( D \to \tilde{B}_0^{-1} \) is homotopic to the pullback under the retraction of a continuous map \( S \to \tilde{B}_0^{-1} \). Since \( \tilde{B}_0^{-1} \) is simply connected, every continuous map \( S \to \tilde{B}_0^{-1} \) is homotopic to a constant map. \( \square \)

**Proposition 6.2.** Suppose \( W_i \subseteq U_i \subset M(H^\infty) \), \( i = 1, 2 \), are open such that \( \tilde{W}_1 \) is holomorphically convex and \( \tilde{U}_1 \cap \tilde{U}_2 \) is a subset of an open Blaschke set \( O_{B,\epsilon} \). We set \( W := W_1 \cup W_2 \). If \( E|_{U_i}, i = 1, 2 \), are holomorphically trivial, then \( E|_W \) is holomorphically trivial.

**Proof.** Without loss of generality we may assume that \( \tilde{W}_1 \cap \tilde{W}_2 \) is nonempty and \( \tilde{U}_1 \cap \tilde{U}_2 \) is proper in each \( \tilde{U}_i \), \( i = 1, 2 \) (for otherwise the statement is trivial). Then, by the hypothesis, \( E|_{\tilde{U}_1 \cap \tilde{U}_2} \) is determined by a cocycle \( c \in \mathcal{O}(\tilde{U}_1 \cap \tilde{U}_2; \tilde{B}_0^{-1}) \). Let \( \mathcal{C} \) be a connected component of the Banach group \( A(\tilde{U}_1 \cap \tilde{U}_2; B^{-1}) \) containing \( \psi \circ c \). According to [Br2] Th. 1.19 \( A(\tilde{U}_1 \cap \tilde{U}_2; B) \) is the completion of the algebraic tensor product \( A(\tilde{U}_1 \cap \tilde{U}_2) \otimes B \) with respect to norm

\[
\left\| \sum_{k=1}^m a_k \otimes b_k \right\| := \sup_{x \in \tilde{U}_1 \cap \tilde{U}_2} \left\| \sum_{k=1}^m a_k(x) b_k \right\|_B \quad \text{with} \quad a_k \in A(\tilde{U}_1 \cap \tilde{U}_2), b_k \in B.
\]

Therefore (because \( \mathcal{C} \) is open) there exists \( \tilde{c} \in \mathcal{C} \cap (A(\tilde{U}_1 \cap \tilde{U}_2) \otimes B) \) sufficiently close to \( \psi \circ c \) such that \( t(\psi \circ c) + (1-t)\tilde{c} \in \mathcal{C} \) for all \( t \in [0, 1] \). In turn, \( A(\tilde{U}_1 \cap \tilde{U}_2) \otimes B \) belongs to the projective tensor product \( A(\tilde{U}_1 \cap \tilde{U}_2) \hat{\otimes} B \). By the result of [D] the set of connected components of the group of invertible elements of the Banach algebra \( A(\tilde{U}_1 \cap \tilde{U}_2) \hat{\otimes} B \) is in a one-to-one correspondence (defined by the Gelfand transform on \( A(\tilde{U}_1 \cap \tilde{U}_2) \))
with the set of homotopy classes of continuous maps $M(A(\hat{U}_1 \cap \hat{U}_2)) \to B^{-1}$. By Theorem 5.1, $M(A(\hat{U}_1 \cap \hat{U}_2)) = \hat{U}_1 \cap \hat{U}_2 \subset O_{B', \varepsilon}$. Then by Lemma 6.1 the map $\psi \circ c$ is homotopic inside $B^{-1}$ to $I$ (here $I(z) := 1_{B^{-1}}$ for all $z$). In turn, the map $\tilde{c} : \hat{U}_1 \cap \hat{U}_2 \to B^{-1}$ is homotopic to $I$. Thus $\tilde{c} \in (A(\hat{U}_1 \cap \hat{U}_2) \otimes B)^{-1}$ belongs to the connected component containing $I$. Observing that $(A(\hat{U}_1 \cap \hat{U}_2) \otimes B)^{-1} \subset A(\hat{U}_1 \cap \hat{U}_2; B^{-1})$ we get that $\psi \circ c$ can be joined by a path $\gamma : [0, 1] \to A(\hat{U}_1 \cap \hat{U}_2; B^{-1})$ with $I$. Further, since $\psi : \tilde{B}_0^{-1} \to B_0^{-1}$ is a principal bundle with discrete fibre and $\hat{U}_1 \cap \hat{U}_2$ is compact, by the covering homotopy theorem, see, e.g., [Hu], there exists a unique continuous map $R : [0, 1] \times \hat{U}_1 \cap \hat{U}_2 \to \tilde{B}_0^{-1}$ such that

$$
\psi(R(t, x)) = \gamma(t)(x), \quad t \in [0, 1], \quad \text{and} \quad R(1, \cdot) = c.
$$

Since $\psi$ is locally biholomorphic, $R(t, \cdot) \in A(\hat{U}_1 \cap \hat{U}_2; \tilde{B}_0^{-1})$ for each $t \in [0, 1]$. We define a path $\overline{\gamma} : [0, 1] \to A(\hat{U}_1 \cap \hat{U}_2; \tilde{B}_0^{-1})$ by the formula $\overline{\gamma}(t) := R(t, \cdot), \ t \in [0, 1]$. By the definition $\overline{\gamma}(0) : \hat{U}_1 \cap \hat{U}_2 \to \tilde{B}_0^{-1}$ is a continuous map into the discrete group $\psi^{-1}(1_{B^{-1}})$. Since $\hat{U}_1 \cap \hat{U}_2$ is compact, the image of $\overline{\gamma}(0)$ consists of finitely many points. In particular, there exists a path $\overline{\gamma}' : [0, 1] \to A(\hat{U}_1 \cap \hat{U}_2; \tilde{B}_0^{-1})$ which joins $\overline{\gamma}(0)$ with $I$ (here $I(z) := 1_{\tilde{B}_0^{-1}}$ for all $z$). The product of $\overline{\gamma}$ and $\overline{\gamma}'$ gives a path in $A(\hat{U}_1 \cap \hat{U}_2; \tilde{B}_0^{-1})$ joining $c$ with $I$. Hence, we can apply Theorem 4.1 to $c$. Then we find $c_1, c_2 \in A(\hat{W}_i; G)$, $i = 1, 2$, such that $c_1 \circ c_2 = c$ on $\hat{W}_1 \cap \hat{W}_2$. But this shows that $E|_W$ is holomorphically trivial, as required.

**Theorem 6.3.** Let $p : E \to M(H^{\infty})$ be a holomorphic principal bundle with fibre $\tilde{B}_0^{-1}$. Then $E$ is holomorphically trivial.

**Proof.** Since $M_s$ is totally disconnected, there exists a finite open cover $(V_j)$ of $M_s$ such that $V_i \cap V_j = \emptyset$ for all $i \neq j$ and $E|_{V_j}$ is trivial for all $j$. We set $U_j := \cup V_j$. Then $E|_{U_j}$ is trivial. Since $M(H^{\infty}) \setminus U_1$ is a compact subset of $M_a$, we can cover it by open Blaschke sets $U_2, \ldots, U_N$ such that $U_j \subset M_a$ and $E|_{U_j}$ is trivial for all $2 \leq j \leq N$. Next, we can find a refinement $W_j$ of the cover $(U_j)$ such that $W_j \subset U_j$ and all $W_j$, $j \neq 1$, are holomorphically convex compact subsets of $M_a$, see subsection 2.2.2. Choosing an open neighbourhood of each $\hat{W}_j$ containing in $\hat{U}_j$, $\tau$ is the refinement map, without loss of generality we may assume that the sets of indices of these covers coincide, i.e., $\hat{W}_j \subset U_j$ for all $1 \leq j \leq N$.

Further, according to Proposition 2.3 we can find a sequence of open sets $W_{ij}$, $i \in \mathbb{N}, 1 \leq j \leq N$, such that $W_j \subset W_{ij}$ for all $i$, each $W_{ij} \subset U_j$ and $W_{ij}$, $2 \leq j \leq N$, are holomorphically convex. We set $W_i := \cup_{1 \leq j \leq N}. Then $W_i$ are finite open covers of $M(H^{\infty})$ such that all nonempty $W_{ij}$ with $j_1 \neq j_2$ are relatively compact in $M_a$ (and belong to open Blaschke sets). For $1 \leq k \leq N$ we set

$$
Z_{ik} := \bigcup_{j=1}^k W_{ij}.
$$

Using induction on $k$ we prove that if $E|_{Z_{k-1k-1}}$ is holomorphically trivial in the category of $\tilde{B}_0^{-1}$-bundles, then $E|_{Z_{kk}}$ is holomorphically trivial in this category as well.

By our definition $E|_{Z_{11}}$ is holomorphically trivial in the category of $\tilde{B}_0^{-1}$-bundles. Provided that the statement is valid for $k - 1$ let us prove it for $k$. 




To this end in Proposition 6.2 consider $U_1 := W_{k-1,k}$, $U_2 := Z_{k-1,k-1}$, $W_1 := W_{kk}$, $W_2 := Z_{kk}$. Then the required statement follows from the proposition.

For $k := N$ we obtain $Z_{NN} = M(H^\infty)$. Thus $E$ is holomorphically trivial as required.

6.2. Bundles with Connected Fibres.

**Theorem 6.4.** Let $p : E \to M(H^\infty)$ be a holomorphic principal bundle with fibre $B_0^{-1}$. Then $E$ is holomorphically trivial.

**Proof.** Suppose that $E$ is defined on a finite open cover $U = (U_i)_{i \in I}$ of $M(H^\infty)$ by a cocycle $g = \{g_{ij} \in O(U_i \cap U_j; B_0^{-1})\}$. Passing to a suitable refinement, if necessary, without loss of generality we may assume that for each $i, j \in I$ there exists an open simply connected set $W_{ij} \subset B_0^{-1}$ such that $g_{ij}(U_i \cap U_j) \subset W_{ij}$ and each $g_{ij}$ admits a continuous extension to the compact set $U_i \cap U_j$. Since $W_{ij}$ is simply connected, an inverse $\psi_{ij} : W_{ij} \to \tilde{B}_0^{-1}$ of $\psi : \tilde{B}_0^{-1} \to B_0$ is defined. Since $g_{ii} \equiv 1_{B_0^{-1}}$ and $g_{ij} = g_{ji}^{-1}$, we can choose $\psi_{ii}$ so that $\psi_{ii}(1_{B_0^{-1}}) = 1_{\tilde{B}_0^{-1}}$ and $W_{ji} := \{z \in B_0^{-1} ; z^{-1} \in W_{ij}\}$, $\psi_{ji}(z) := (\psi_{ij}(z^{-1}))^{-1}$, $z \in W_{ji}$.

We set $h_{ij} := \psi_{ij} \circ g_{ij}$. Then $h_{ii} \equiv 1_{\tilde{B}_0^{-1}}$, $h_{ji} = h_{ij}^{-1}$ and for pairwise distinct $i, j, k \in I$ \begin{equation}
\psi(h_{ij} \cdot h_{jk} \cdot h_{ki}) \equiv 1_{B_0^{-1}} \quad \text{on} \quad U_i \cap U_j \cap U_k \neq \emptyset.
\end{equation}

Let $Z \subset \tilde{B}_0^{-1}$ be the kernel of $\psi$. Then $Z$ is a discrete abelian subgroup of $\tilde{B}_0^{-1}$. Due to (6.1) the family $h := \{h_{ijk}\}$, $h_{ijk} := h_{ij} \cdot h_{jk} \cdot h_{ki}$ on $U_i \cap U_j \cap U_k \neq \emptyset$, is a (continuous) 2-cocycle on $U$ with values in $Z$. Since by our assumption $h_{ijk}$ admits a continuous extension to the compact set $U_i \cap U_j \cap U_k$, the image of each $h_{ijk}$ is a finite subset of $Z$. Thus, because the family $\{h_{ijk}\}$ is finite, images of all functions of this family generate a finitely-generated subgroup $Z'$ of $Z$. Hence, $h$ determines an element of the Čech cohomology group $H^2(M(H^\infty), Z')$.

**Lemma 6.5.** $H^2(M(H^\infty), Z') = 0$.

**Proof.** Since $Z'$ is a direct product of finitely many copies of groups $Z$ and $Z_{p^m}$, where $p \in \mathbb{N}$ is prime, it suffices to show that $H^2(M(H^\infty), Z) = 0$ and $H^2(M(H^\infty), Z_{p^m}) = 0$. The former group is trivial by [SL] Cor. 3.9. Next, let $c = \{c_{ijk}\} \in Z^2(V, Z_{p^m})$ be a continuous cocycle defined on a finite open cover $V = (V_i)_{i \in I}$ of $M(H^\infty)$. According to [SL] the covering dimension of $M(H^\infty)$ is 2, hence, passing to a refinement of $V$, if necessary, without loss of generality we may assume that the order of the cover $V$ is 3. Since each $c_{ijk} : V_i \cap V_j \cap V_k \to Z_{p^m}$ is a continuous locally constant function, there exists a continuous locally constant function $\tilde{c}_{ijk} : V_i \cap V_j \cap V_k \to Z$ such that $q(\tilde{c}_{ijk}) = c_{ijk}$, where $q : Z \to Z_{p^m}$ is the quotient homomorphism, and $\tilde{c}_{ik} = 0$, $\tilde{c}_{ij} = -\tilde{c}_{jk} = \tilde{c}_{ki}$ for all $i, j, k$. As the order of $V$ is 3, $\tilde{c} = \{\tilde{c}_{ijk}\}$ is a continuous integer-valued 2-cocycle on $V$. Due to [SL], $\tilde{c}$ determines zero element of $H^2(M(H^\infty), Z)(= 0)$. Thus the restriction of $\tilde{c}$ to a finite open refinement of $V$ is a coboundary. To avoid abuse of notation we will assume that $\tilde{c}$ is yet a coboundary on $V$. Thus there exist a continuous cochain $\{\tilde{c}_{ij} \in Z_i \cap V_j \to Z\}_{i, j \in I}$ such that $\tilde{c}_{ij} = \tilde{c}_{jk} + \tilde{c}_{ki} = \tilde{c}_{ijk}$ on $V_i \cap V_j \cap V_k \neq \emptyset$. This implies that $c_{ij} - c_{jk} + c_{ki} = c_{ijk}$ on $V_i \cap V_j \cap V_k \neq \emptyset$, where $c_{st} := q(\tilde{c}_{st})$. Thus $c$ is a coboundary on $V$. □
According to the lemma the restriction of the cocycle $h$ to a suitable refinement of $\mathcal{U}$ is a coboundary. As before, without loss of generality, we will assume that $h$ is yet a coboundary on $\mathcal{U}$. Hence, there exist a continuous cochain $\{\tilde{h}_{ij} : U_i \cap U_j \rightarrow Z'\}_{i,j \in I}$ such that

$$\tilde{h}_{ij} \cdot \tilde{h}_{jk} \cdot \tilde{h}_{ki} = h_{ijk} \quad \text{on} \quad U_i \cap U_j \cap U_k \neq \emptyset.$$  

We set

$$\tilde{g}_{ij} := h_{ij} \cdot \tilde{h}_{ij}^{-1} \quad \text{on} \quad U_i \cap U_j \neq \emptyset.$$  

Then, as $Z'$ is a central subgroup of $\tilde{B}_0^{-1}$,

$$\tilde{g}_{ij} \cdot \tilde{g}_{jk} \cdot \tilde{g}_{ki} = 1_{\tilde{B}_0^{-1}} \quad \text{on} \quad U_i \cap U_j \cap U_k \neq \emptyset.$$  

In particular, $\tilde{g} = \{\tilde{g}_{ij}\}$, $\tilde{g}_{ij} \in \mathcal{O}(U_i \cap U_j; \tilde{B}_0^{-1})$, is a 1-cocycle determining a holomorphic principal bundle on $M(H^\infty)$ with fibre $\tilde{B}_0^{-1}$. By Theorem 6.3 this bundle is holomorphically trivial. Hence, there exist $\tilde{g}_i \in \mathcal{O}(U_i; \tilde{B}_0^{-1})$, $i \in I$, such that

$$\tilde{g}_i^{-1} \cdot \tilde{g}_j = \tilde{g}_{ij} \quad \text{on} \quad U_i \cap U_j.$$  

Since $\psi(\tilde{g}_{ij}) = g_{ij}$ for $g_i := \psi(\tilde{g}_i) \in \mathcal{O}(U_i; B_0^{-1})$, $i \in I$, we obtain

$$g_i^{-1} \cdot g_j = g_{ij} \quad \text{on} \quad U_i \cap U_j.$$  

This shows that the bundle $E$ is holomorphically trivial.

The proof of Theorem 6.4 is complete.  

\begin{center}
7. Proofs of Theorem 1.1 and Corollary 1.2
\end{center}

We will prove a more general result. Let $q : B^{-1} \rightarrow B^{-1}/B_0^{-1} =: C(B^{-1})$ be the (continuous) quotient homomorphism onto the discrete group of connected components of $B^{-1}$. Let $p : E \rightarrow M(H^\infty)$ be a holomorphic principal bundle with fibre $B^{-1}$ defined on a finite open cover $\mathcal{U} = (U_i)_{i \in I}$ of $M(H^\infty)$ by a cocycle $g = \{g_{ij} \in \mathcal{O}(U_i \cap U_j; B^{-1})\}$. By $E_{C(B^{-1})} \rightarrow M(H^\infty)$ we denote the principal bundle with fibre $C(B^{-1})$ defined on $\mathcal{U}$ by the locally constant cocycle $q(g) = \{q(g_{ij}) \in \mathcal{O}(U_i \cap U_j; C(B^{-1}))\}$.

**Theorem 7.1.** $E$ is holomorphically trivial if and only if the associated bundle $E_{C(B^{-1})}$ is trivial (in the category of principal bundles with discrete fibres).

**Proof.** If $E$ is holomorphically trivial, then there exist $g_i \in \mathcal{O}(U_i; B^{-1})$, $i \in I$, such that

$$g_i^{-1} \cdot g_j = g_{ij} \quad \text{on} \quad U_i \cap U_j.$$  

This implies that

$$q(g_i)^{-1} \cdot q(g_j) = q(g_{ij}) \quad \text{on} \quad U_i \cap U_j,$$

i.e., $E_{C(B^{-1})}$ is isomorphic to $M(H^\infty) \times C(B^{-1})$ in the category of principal bundles with discrete fibres.

Conversely, suppose $E_{C(B^{-1})}$ is isomorphic to $M(H^\infty) \times C(B^{-1})$ in the category of principal bundles with discrete fibres. Then there exist $h_i \in C(U_i; C(B^{-1}))$, $i \in I$, such that

$$h_i^{-1} \cdot h_j = q(g_{ij}) \quad \text{on} \quad U_i \cap U_j.$$  

Taking a refinement of $\mathcal{U}$, if necessary, without loss of generality we may assume that each $h_i$ admits a continuous extension to the compact set $\bar{U}_i$, $i \in I$. Since
A \ Fix \ x = g_0 \ there \ exists \ a \ finite \ open \ cover \ U \ \text{such that} \ \tilde{h}_i \in \mathcal{O}(U; B^{-1}). \ Let \ us \ define \ cocycle \ \tilde{g} = \{\tilde{g}_{ij} \in \mathcal{O}(U_i \cap U_j; B^{-1})\} \ by \ the \ formulas

\( \tilde{g}_{ij} := \tilde{h}_i \cdot g_{ij} \cdot \tilde{h}_j^{-1} \) on \ \( U_i \cap U_j \).

Then \( \tilde{g} \) determines a holomorphic principal bundle \( \tilde{E} \) on \( M(H^\infty) \) with fibre \( B^{-1} \) isomorphic to the bundle \( E \). Also, from (7.1) we obtain for all \( i, j \in I \)

\[ q(\tilde{g}_{ij}) = 1_{C(B^{-1})}. \]

Thus each \( \tilde{g}_{ij} \) maps \( U_i \cap U_j \) into \( B^{-1}_0 \). In particular, \( \tilde{g} \) determines also a subbundle of \( \tilde{E} \) with fibre \( B^{-1}_0 \). According to Theorem 6.4 this subbundle is holomorphically trivial. Thus there exist \( \tilde{g}_i \in \mathcal{O}(U_i; B^{-1}_0), \ i \in I, \) such that

\[ \tilde{g}_i^{-1} \cdot \tilde{g}_j = \tilde{g}_{ij} \] on \( U_i \cap U_j \).

From here and (7.2) we obtain for all \( i, j \in I \)

\[(\tilde{g}_i \cdot h_i)^{-1} \cdot (\tilde{g}_j \cdot h_j) = g_{ij} \] on \( U_i \cap U_j \).

This shows that \( E \) is holomorphically trivial. \( \square \)

**Proof of Theorem 7.1** Since a holomorphic Banach vector bundle on \( M(H^\infty) \) with fibre \( X \) is associated with a holomorphic principal bundle on \( M(H^\infty) \) with fibre \( GL(X) \), the required result follows from Theorem 7.1 with \( B := L(X) \). \( \square \)

**Proof of Corollary 7.3** (1) The result follows directly from Theorem 6.4.

(2) Let \( E \) is defined on a finite open cover \( U = \{U_i\}_{i \in I} \) of \( M(H^\infty) \) by a cocycle \( g = \{g_{ij} \in \mathcal{O}(U_i \cap U_j; GL(X))\} \). Since \( E \) is topologically trivial, there exist \( g_i \in C(U_i; GL(X)), \ i \in I, \) such that

\[ g_i^{-1} \cdot g_j = g_{ij} \] on \( U_i \cap U_j \).

This implies that

\[ q(g_i)^{-1} \cdot q(g_j) = q(g_{ij}) \] on \( U_i \cap U_j, \)

i.e., \( E_{C(GL(X))} \) is isomorphic to \( M(H^\infty) \times C(GL(X)) \) in the category of principal bundles with discrete fibres. Thus the required result follows from Theorem 1.1. \( \square \)

8. **Proof of Theorem 1.3**

**Proof.** Since \( F \in H^{\text{comp}}_{\text{cont}}(L(X_1, X_2)) \), by [Br2, Prop. 1.3] it is extended to a holomorphic function \( M(H^\infty) \to L(X_1, X_2) \) (denoted by the same symbol). We set \( M := \max_{z \in M(H^\infty)} \|F(z)\| \).

**Lemma 8.1.** There exist a finite open cover \( (U_i)_{1 \leq i \leq k} \) of \( M(H^\infty) \) and operators \( G_i \in \mathcal{O}(U_i; L(X_2, X_1)) \) such that \( G_i(z)F(z) = I_{X_1} \) for all \( z \in U_i \).

**Proof.** Let \( C := \sup_{z \in \mathbb{D}} \|G_z\| \). Since \( F \) is uniformly continuous on \( M(H^\infty) \), for every \( \varepsilon > 0 \) there exists a finite open cover \( (U_i)_{1 \leq i \leq k} \) of \( M(H^\infty) \) such that

\[ \max_{1 \leq i \leq k} \sup_{x, y \in U_i} \|F(x) - F(y)\| < \varepsilon. \]

Fix \( x_i \in U_i \cap \mathbb{D}, 1 \leq i \leq k \), and define \( A_i(z) := G_{x_i}F(z) : X_1 \to X_1, z \in U_i \). We have \( A_i(x_i) = I_{X_1} \) and \( \|A_i(z) - A_i(x_i)\| < C\varepsilon \). Choose \( \varepsilon \) so small that \( C\varepsilon < \frac{1}{2} \).
Then $A_i^{-1} := \sum_{j=0}^{\infty} (I_{X_1} - A_i)^j \in \mathcal{O}(U_i; L(X_1))$ is well defined, and $\|A_i^{-1}\| < 2$. We set

$$G_i(z) := A_i(z)^{-1}G_{x_i}.$$ Clearly, $G_i(z)F(z) = A_i(z)^{-1}G_{x_i}F(z) = I_{X_1}$ and $\|G_i(z)\| < 2C$ for all $z \in U_i$. □

Choosing $\varepsilon < \frac{1}{8C(1+2MC)}$ sufficiently small we may obtain for all $z_1, z_2 \in U_i$, $1 \leq i \leq k$,

$$\|G_i(z_1) - G_i(z_2)\| \leq C\|A_i(z_1)^{-1} - A_i(z_2)^{-1}\| \leq C\|A_i(z_1) - A_i(z_2)\| \cdot \left(1 + \sum_{j=2}^{\infty} j \cdot (C\varepsilon)^{j-1}\right) \leq \frac{1}{4M(1+2MC)}.$$

(8.1)

Let $Y_{iz} := \text{Ker}(G_i(z))$, $P_i(z) := I_{X_2} - F(z)G_i(z)$, $z \in U_i$. Then $P_i(z)$ is a projector onto $Y_{iz}$. For $z, w \in U_i$ consider the diagram

$$Y_{iz} \xrightarrow{P_i(w)} Y_{iw} \xrightarrow{P_i(z)} Y_{iz}.$$ If $H_i(z, w) := P_i(z)P_i(w)$, then $H_i(z, z) = I_{Y_{iz}}$ and by (8.1)

$$\|H_i(z, w) - H_i(z, z)\| \leq \|P_i(z)\| \cdot \|F(w)G_i(w) - F(z)G_i(z)\| \leq (1 + 2MC) \cdot \frac{2C\varepsilon + \frac{M}{4M(1+2MC)}}{2} < \frac{1}{2}.$$ Thus $H_i(z, w)^{-1}$ exists for all $1 \leq i \leq k$ and $z, w \in U_i$. This implies that $P_i(z)$ is surjective and $P_i(w)$ is injective for all $z, w \in U_i$, i.e., all $P_i(z)$ are isomorphisms. Choosing as one of $x_i$ point 0, we, hence, obtain (since $M(H^\infty)$ is connected) that all $Y_{iz}$ are isomorphic to $Y := \text{Ker}(G_0)$. By $S_i : Y \to Y_{ix_i}$, $1 \leq i \leq k$, we denote the corresponding isomorphisms.

Next, we define $B_i \in \mathcal{O}(U_i; L(X_1 \oplus Y, X_2))$ as

$$B_i(z)(v_1, v_2) := F(z)(v_1) + P_i(z)S_i(v_2), \quad z \in U_i.$$ Clearly, all $B_i(z)$ are isomorphisms. We set for all $z \in U_i \cap U_j \neq \emptyset$

$$B_{ij}(z) := B_i(z)^{-1}B_j(z) : X_1 \oplus Y \to X_1 \oplus Y.$$ Since $B_{ij}(z)(v) = v$ for all $v \in X_1$,

(8.2)

$$B_{ij} = \begin{pmatrix} I_{X_1} & E_{ij} \\ 0 & D_{ij} \end{pmatrix}$$

for some $D_{ij} \in \mathcal{O}(U_i \cap U_j; GL(Y))$, $E_{ij} \in \mathcal{O}(U_i \cap U_j; L(Y, X_1))$.

The holomorphic 1-cocycle $\{D_{ij}\}$ determines a holomorphic Banach vector bundle on $M(H^\infty)$ with fibre $Y$. Since $Y$ satisfies conditions of Theorem 1.1, this bundle is holomorphically trivial. In particular, there exist $\tilde{D}_i \in \mathcal{O}(U_i; GL(Y))$ such that $\tilde{D}_i\tilde{D}_j^{-1} = D_{ij}$ on $U_i \cap U_j \neq \emptyset$. We set $D_i := \text{diag}(I_{X_1}, \tilde{D}_i)$ on $U_i$ and define

$$C_{ij}(z) := D_i(z)^{-1}B_{ij}(z)D_j(z), \quad z \in U_i \cap U_j \neq \emptyset.$$ Then

$$C_{ij} = \begin{pmatrix} I_{X_1} & F_{ij} \\ 0 & I_Y \end{pmatrix}$$
for some $F_{ij} \in \mathcal{O}(U_i \cap U_j; L(Y, X_1))$.

Clearly, $\{F_{ij}\}$ is an additive holomorphic 1 cocycle on cover $(U_i)$ with values in $L(Y, X_1)$. According to [Bz2, Th. 1.4], $\{F_{ij}\}$ represents zero element in the corresponding Čech cohomology group. This implies that there exist $\tilde{F}_i \in \mathcal{O}(U_i; L(Y, X_1))$ such that $\tilde{F}_i - \tilde{F}_j = F_{ij}$ on $U_i \cap U_j \neq \emptyset$.

We set

$$F_i = \begin{pmatrix} I_{X_1} & \tilde{F}_i \\ 0 & I_Y \end{pmatrix}$$

and define

$$H(z) := B_i(z)D_i(z)F_i(z), \quad z \in U_i.$$ 

Then $H \in H^\infty_{\text{comp}}(L(H_1 \oplus Y, H))$ has inverse $G \in H^\infty_{\text{comp}}(L(H_2, H_1 \oplus Y))$ and $H(z)|_{X_1} = F(z)$ for all $z$.

9. Proof of Theorem 1.5

Proof. We retain notation of the proof of Theorem 1.3. According to (8.2) cocycle $\{B_{ij}\}$ determines the trivial holomorphic Banach vector bundle $E$ on $M(H^\infty)$ with fibre $X_1 \oplus Y$ which has the trivial holomorphic Banach vector subbundle $E_1$ with fibre $X_1$ (determined by trivial cocycle $\{I_{X_1}\}$) and the holomorphic quotient Banach vector bundle $E_2$ with fibre $Y$ (determined by cocycle $\{D_{ij}\}$). Thus we have an exact sequence of bundles

$$0 \to E_1 \to E \to E_2 \to 0$$

which induces an exact sequence of Banach holomorphic bundles

$$0 \to \text{Hom}(E_2, E_1) \to \text{Hom}(E_2, E) \to \text{Hom}(E_2, E_2) \to 0.$$ 

In turn, the latter produces the long cohomology sequences of sheaves of germs of holomorphic sections of these bundles:

$$0 \to H^0(M(H^\infty), \text{Hom}(E_2, E_1)) \to H^0(M(H^\infty), \text{Hom}(E_2, E))$$

$$\to H^0(M(H^\infty), \text{Hom}(E_2, E_2)) \to H^1(M(H^\infty), \text{Hom}(E_2, E_1)) \to \cdots.$$ 

The identity map $E_2 \to E_2$ determines an element $I \in H^0(M(H^\infty), \text{Hom}(E_2, E_2))$. Then sequence (9.1) splits if and only if $\delta(I) = 0 \in H^1(M(H^\infty), \text{Hom}(E_2, E_1))$, for the basic results related to extensions of bundles, see, e.g., [A]. In our notation $\delta(I)$ is represented by the holomorphic 1-cocycle $\{E_{ij}\}$, see (8.2). Since $M(H^\infty)$ is paracompact, $\delta(I) = 0$ in the corresponding cohomology group of sheaves of germs of continuous sections of $\text{Hom}(E_2, E_1)$. Therefore $E$ is isomorphic in the category of continuous bundles on $M(H^\infty)$ to the bundle $E_1 \oplus E_2$ (the Whitney sum). Further, since $E$ is holomorphically trivial, $E_1 \oplus E_2$ is the topologically trivial bundle. Thus according to Corollary 1.2(2), the holomorphic Banach vector bundle $E_1 \oplus E_2$ is holomorphically trivial. By definition we have

$$\text{Hom}(E_1 \oplus E_2, E_1) = \text{Hom}(E_1, E_1) \oplus \text{Hom}(E_2, E_1).$$

Thus there exist homomorphisms

$$i : H^1(M(H^\infty), \text{Hom}(E_2, E_1)) \to H^1(M(H^\infty), \text{Hom}(E_1 \oplus E_2, E_1))$$
and
\[ j : H^1(M(H^\infty), \text{Hom}(E_1 \oplus E_2, E_1)) \to H^1(M(H^\infty), \text{Hom}(E_2, E_1)) \]
such that \( j \circ i = \text{id} \). Consider \( i(\delta(I)) \in H^1(M(H^\infty), \text{Hom}(E_1 \oplus E_2, E_1)) \). Since \( E_1 \oplus E_2 \) and \( E_1 \) are holomorphically trivial, the holomorphic Banach vector bundle \( \text{Hom}(E_1 \oplus E_2, E_1) \) is holomorphically trivial as well, i.e., it is holomorphically isomorphic to the bundle \( M(H^\infty) \times L(X_1 \oplus Y, X_1) \). Then \( i(\delta(I)) \) is naturally identified with an element of the cohomology group \( H^1(M(H^\infty); L(X_1 \oplus Y, X_1)) \) of the sheaf of germs of holomorphic functions on \( M(H^\infty) \) with values in the Banach space \( L(X_1 \oplus Y, X_1) \). According to [Br² Th. 1.4], \( H^1(M(H^\infty); L(X_1 \oplus Y, X_1)) = 0 \). Hence, \( \delta(I) = j(i(\delta(I))) = 0 \in H^1(M(H^\infty), \text{Hom}(E_1 \oplus E_2, E_1)) \).

Thus we have proved that sequence (9.1) splits. This is equivalent to the fact that there exist \( \tilde{E}_i \in \mathcal{O}(U_i, L(Y, X_i)), i \in I, \) such that

\[
\begin{pmatrix}
I_{X_i} & 0 \\
0 & D_{ij}
\end{pmatrix} = \begin{pmatrix}
I_{X_i} & \tilde{E}_i \\
0 & I_Y
\end{pmatrix}^{-1} \cdot \begin{pmatrix}
I_{X_i} & E_{ij} \\
0 & D_{ij}
\end{pmatrix} \cdot \begin{pmatrix}
I_{X_i} & \tilde{E}_j \\
0 & I_Y
\end{pmatrix}
\text{ on } U_i \cap U_j.
\]

We set
\[
E_i := \begin{pmatrix}
I_{X_i} & \tilde{E}_i \\
0 & I_Y
\end{pmatrix} \in \mathcal{O}(U_i; GL(X_1 \oplus Y)).
\]

Then
\[
(B_i \cdot E_i)\cdot^{-1} (B_j \cdot E_j) = \text{diag} (I_{X_i}, D_{ij}) \text{ on } U_i \cap U_j,
\]

and \( B_i \cdot E_i \in \mathcal{O}(U_i; L(X_1 \oplus Y, X_2)) \) are invertible and such that \( (B_i(z) \cdot E_i(z))|_{X_1} = F(z), z \in U_i, i \in I \). Let \( P_{X_i} := \text{diag} (I_{X_i}, 0) : X_1 \oplus Y \to X_1 \) be the natural projection. We set
\[
G := P_{X_i} \cdot (B_i \cdot E_i)^{-1} \text{ on } U_i.
\]

Then on \( U_i \cap U_j \neq \emptyset \) we have
\[
P_{X_1} \cdot ((B_i \cdot E_i)^{-1}) - P_{X_1} \cdot ((B_j \cdot E_j)^{-1})
= P_{X_1} \cdot (((B_i \cdot E_i)^{-1} (B_j \cdot E_j) - I_{X_1 \oplus Y}) \cdot (B_j \cdot E_j)^{-1})
= \text{diag} (I_{X_1}, 0) \cdot \text{diag} (0, D_{ij} - I_Y) \cdot (B_j \cdot E_j)^{-1} = 0.
\]

Thus \( G \in \mathcal{O}(M(H^\infty); L(X_2, X_1)) \). Moreover, by our construction \( G(z)F(z) = I_{X_1} \) for all \( z \in M(H^\infty) \).

This completes the proof of the theorem. \( \square \)

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