Two-fluid evolving Lorentzian wormholes

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We investigate the evolution of a family of wormholes sustained by two matter components: one with homogeneous and isotropic properties \( \rho(t) \) and another inhomogeneous and anisotropic \( \rho_{\text{in}}(t, r) \). The rate of expansion of these evolving wormholes is only determined by the isotropic and homogeneous matter component \( \rho(t) \). Particularly, we consider a family of exact two-fluid evolving wormholes expanding with constant velocity and satisfying the dominant and the strong energy conditions in the whole spacetime. In general, for the case of vanishing isotropic fluid \( \rho(t) \) and cosmological constant \( \Lambda \) the space expands with constant velocity, and for \( \rho(t) = 0 \) and \( \Lambda \neq 0 \) the rate of expansion is determined by the cosmological constant. The considered here two-fluid evolving wormholes are a generalization of single fluid models discussed in previous works of the present authors [Phys. Rev. D 78, 104006 (2008); Phys. Rev. D 79, 024005 (2009)].

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I. INTRODUCTION

Wormhole spacetimes have become one of the most popular and intensively studied topics in general relativity. Throughout the last decades there has been an accumulating volume of works on the analytic wormhole geometries. The various approaches include both static [1] and evolving relativistic versions [2]. They principally consider static wormhole spacetimes sustained by a single fluid component which requires the violation of the null energy condition (NEC), and the interest has been focused on traversable wormholes, which have no horizons, allowing two-way passage through them. These hypothetical tunnels in spacetime allow effective superluminal travels, although the speed of light is not locally surpassed [3].

It is interesting to note that for constructing wormhole geometries in general is adopted the reverse approach for solving the Einstein field equations. This means that one first fixes the form of the spacetime metric (such as the redshift and shape functions) and then, by computing the field equations, one finds the energy-momentum tensor components needed to support such a spacetime geometry. The obtained in such a way stress components automatically satisfy local conservation equations, by virtue of the Bianchi identities. This reverse method helps us to find that a static traversable wormhole violates the NEC [4, 5], thus in general relativity an exotic type of matter is required for sustaining a static traversable wormhole. It is interesting to note that there are explicit static wormholes solutions respecting the energy conditions in the whole spacetime in Einstein-Gauss-Bonnet gravity [6]. Notice also that in higher dimensions, the presence of terms with higher powers in the curvature provided by certain class of Lovelock theories, allows to remove the possibility of violating energy conditions even in vacuum, since the whole spacetime is devoid of any kind of stress-energy tensor [7].

However, it is well known that in Einstein gravity there are nonstatic Lorentzian wormholes which do not require NEC violating matter to sustain them. Such wormholes may exist for arbitrarily small or large intervals of time [8].

On the purely gravitational side, most of the efforts are directed to study Lorentzian wormholes sustained by a single exotic fluid in classical general relativity. However, one can consider also gravitational configurations filled with two or more fluids [9]. For example, in cosmology such two-fluid models are widely considered today in order to explain the observed accelerated expansion of the Universe [10].

In this paper we shall study evolving wormholes sustained by two fluids: one with homogeneous and isotropic properties \( \rho(t) \) and another inhomogeneous and anisotropic \( \rho_{\text{in}}(t, r) \). The theoretical construction of these wormholes will be performed by imposing conditions on the stress-energy tensor threading the evolving wormhole geometry. Specifically, we shall consider the radial and tangential pressures of the inhomogeneous and anisotropic matter to obey barotropic equations of state with constant state parameters. On the other hand, the homogeneous and isotropic fluid is taken to be that of a
perfect fluid described by the energy-momentum tensor
\[ T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta - pg_{\alpha\beta}, \] (1)
where \( u_\alpha \) is the four-velocity of the fluid, \( \rho \) and \( p \) are the energy density and the pressure of the cosmic fluid respectively.

We shall suppose that the dynamics of the gravitational fields is governed by Einstein field equations
\[ R_{\alpha\beta} - \frac{R}{2}g_{\alpha\beta} = -\kappa T_{\alpha\beta} - \Lambda g_{\alpha\beta}, \]
where \( \kappa = 8\pi G \) and \( \Lambda \) is the cosmological constant, and the evolving wormhole metric will be given by
\[ ds^2 = -e^{2\Phi(t,r)}dt^2 + a(t)^2 \left( \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2d\Omega^2 \right), \] (2)
where \( \Phi(t,r) \) is the redshift function, \( a(t) \) is the scale factor of the wormhole universe, \( b(r) \) is the shape function and \( d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \). Note that the essential characteristics of a wormhole geometry are encoded in the spacelike section of the metric (7).

The organization of the paper is as follows: In Sec. II we present the dynamical field equations for wormhole models with a matter source composed of an ideal isotropic cosmic fluid and an anisotropic and inhomogeneous one. In Sec. III some aspects of the geometry of the general solution are discussed. In Sec. IV expanding wormholes are discussed, and in Sec. V we conclude with some remarks.

II. FIELD EQUATIONS

Let us now consider the dynamical field equations describing evolving wormhole models (7). We shall be interested in studying wormhole scenarios filled with two fluids \( \rho = \rho(t) + \rho_\alpha \) and \( p_\alpha = p_\alpha(t, r) \), where the first cosmic fluid always remains homogeneous and isotropic and the other component is in general an inhomogeneous and anisotropic fluid. Since we have a spherically symmetric space-time, the cosmic fluid \( \rho_\alpha(t, r) \) in general may have anisotropic pressures, which we shall define as \( p_r(t, r) \) and \( p_\alpha(t, r) \) for the radial and lateral components respectively. If \( p_r(t, r) = p_\alpha(t, r) \) we have an isotropic inhomogeneous pressure.

Thus, for spherically symmetric spacetimes written in comoving coordinates (7) and filled with these two kinds of cosmic fluids, the Einstein field equations may be written in the following form:

\[ 3e^{-2\phi(t,r)}H^2 + \frac{b'}{a^2r^2} = \kappa \rho_\alpha(t, r) + \kappa \rho(t) + \Lambda, \] (3)
\[ -e^{-2\phi(t,r)} \left( \frac{\dot{a}}{a} + H^2 \right) - \frac{b}{a^2r^2} + 2e^{-2\phi(t,r)}H \frac{\partial \phi}{\partial t} + 4 \]

where it was assumed that the 4-velocity of both fluids is the timelike vector \( u^a = (e^{-\phi}, 0, 0, 0) \), \( \kappa = 8\pi G \), \( H = \dot{a}/a \), and an overdot and a prime denote differentiation \( d/dt \) and \( d/dr \) respectively.

In attempting to find solutions to the field equations we first note that Eq. (6) gives some constraints on relevant metric functions which separate the wormhole solutions into two branches: one static branch given by the condition \( \dot{a} = 0 \) and another non-static branch for \( \partial \phi/\partial r = 0 \).

In what follows we shall restrict our discussion to non-static branch. The condition \( \partial \phi/\partial r = 0 \) implies that the redshift function can only be a function of \( t \), i.e. \( \phi(t, r) = f(t) \) so, without any loss of generality, by rescaling the time coordinate we can set \( \phi(t, r) = 0 \). Thus we shall look for the solutions to Einstein equations described by the metric form

\[ ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2d\Omega^2 \right), \] (7)
so we have to put \( \Phi = 0 \) into the Eqs. (3)-(6). As we shall see below this will imply that the anisotropic and inhomogeneous matter component \( \rho_\alpha(t, r) \) cannot be isotropic.

Notice that the field equations (3)-(6) generalize the Einstein equations considered in Refs. (11) and (12). Thus, in order to quickly solve the field equations we shall use the conservation equations \( T^\mu{}_{\nu\mu} = 0 \). By supposing that each fluid satisfies the standard conservation equation separately we obtain

\[ \frac{\partial \rho}{\partial t} + 3H(\rho + p) = 0, \] (8)
\[ \frac{\partial \rho_\alpha}{\partial t} + H(3\rho_\alpha + p_r + 2p_\alpha) = 0, \] (9)
\[ \frac{2(p_r - p_\alpha)}{r} = \frac{\partial p_r}{\partial r}, \] (10)
where Eq. (8) states the conservation of the isotropic and homogeneous component and Eqs. (9) and (11) are valid for the anisotropic and inhomogeneous cosmic fluid and may be interpreted as the conservation equation and the relativistic Euler equation (or the hydrostatic equation for equilibrium for the anisotropic matter supporting the gravitational configuration) respectively.
Notice that from equations (9) and (10) we see that for an isotropic but still inhomogeneous matter component \( \rho_{in} \), i.e. \( p_r = p_t = p_{in} \), we have to require \( \partial p_r / \partial r = 0 \), so the pressure will depend only on time \( t \), obtaining the standard cosmological conservation equation \( \dot{\rho}_{in} (t) + 3H [\rho_{in} (t) + p_{in} (t)] = 0 \). Thus in this case we have a noninteracting superposition of two homogeneous and isotropic fluids \( \rho \) and \( \rho_{in} \). This leads us to conclude that if we want to study evolving wormholes filled with a mixture of homogeneous, isotropic fluid \( \rho (t) \) and an inhomogeneous fluid \( \rho_{in} (t, r) \), we must consider only anisotropic matter component \( \rho_{in} (t, r) \) with \( p_r \neq p_t \).

For solving the field equations of the considered gravitational configuration we shall consider the following anzats: we shall require that the radial and the lateral pressures have barotropic equations of state. Thus we shall write for them

\[
\begin{align*}
p_r (t, r) &= \omega_r \rho_{in} (t, r), \\
p_t (t, r) &= \omega_t \rho_{in} (t, r),
\end{align*}
\]  
(11)

where \( \omega_r \) and \( \omega_t \) are constant state parameters (note that in this case we have that \( p_r = p_o = p_t \) due to the spherical symmetry).

In the following, for solving the field equations, we shall require that \( p_r (t, r) \neq p_t (t, r) \). By taking into account Eq. (11), from Eq. (10) we get

\[
\rho_{in} (t, r) = F (t) r^{2 (\omega_t - \omega_r) / \omega_r},
\]  
(12)

where \( F (t) \) is an integration function, and introducing this expression into Eq. (3) we have for the energy density of the anisotropic matter

\[
\rho_{in} (t, r) = \frac{C r^{2 (\omega_t - \omega_r) / \omega_r}}{a^{3 + \omega_r + 2 \omega_t}},
\]  
(13)

where \( C \) is an integration constant.

Now, by subtracting Eqs. (4) and (5), and using the full energy density (13), we obtain the differential equation

\[
\frac{\kappa (\omega_t - \omega_r)}{a^{3 + \omega_r + 2 \omega_t}} C r^{2 (\omega_t - \omega_r) / \omega_r} = 3 \dot{b} - \frac{rb'}{2a^2 r^3}.
\]  
(14)

It is straightforward to see that in order to have a solution for the shape function \( b = b (r) \) we must impose the constraint

\[
\omega_r + 2 \omega_t + 1 = 0
\]  
(15)
on the state parameters \( \omega_r \) and \( \omega_t \), thus obtaining for the shape function

\[
b (r) = C_3 r^3 - \kappa C \omega_r r^{-1 / \omega_r},
\]  
(16)

where \( C_3 \) is a new integration constant. Notice that the constraint (15) implies that the radial and tangential pressures are given by

\[
p_r = \omega_r \rho_{in}, \quad p_t = -\frac{1}{2} (1 + \omega_r) \rho_{in},
\]  
(17)

so the energy density and pressures satisfy the following relation:

\[
\rho_{in} + p_r + 2p_t = 0.
\]  
(18)

This equation implies that the inhomogeneous and anisotropic component satisfies the strong energy condition. In this case for \( \omega_r \leq -1, -1 \leq \omega_t \leq 0 \) and \( \omega_t \geq 1 \) we have that \( \omega_t \geq 0, -1 \leq \omega_t \leq 0 \) and \( \omega_t \leq -1 \) respectively.

Now, from Eqs. (4), (13), (16) and taking into account the constraint (15) we obtain the following master equation for the scale factor:

\[
3H^2 = \frac{3C_3}{a^2} + \kappa \rho + \Lambda.
\]  
(19)

Note that by taking into account the metric (7) we conclude that \( C_3 \) may be absorbed by rescaling the \( r \)-coordinate as follows: \( C_3 = 1 \) for \( C_3 > 0 \) and \( C_3 = -1 \) for \( C_3 < 0 \), so without any loss of generality we can identify it with the spatial curvature parameter \( k \) by putting \( C_3 = k \), with \( k = -1, 0, 1 \). Thus the master equation (19) may be written in the form

\[
3H^2 + \frac{3k}{a^2} = \kappa \rho (t) + \Lambda.
\]  
(20)

Summarizing, we have shown that for the gravitational configuration

\[
ds^2 = dt^2 - a(t)^2 \left( \frac{dr^2}{1 - kr^2 + \kappa C \omega_r r^{-1 - 1 / \omega_r}} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right),
\]  
(21)

filled with the inhomogeneous cosmic fluid

\[
\rho_{in} (t, r) = \frac{C r^{-3 - 1 / \omega_r}}{a^2 (t)},
\]  
(22)

(with anisotropic pressures \( p_r = \omega_r \rho_{in} \) and \( p_t = -\frac{1}{2} (1 + \omega_r) \rho_{in} \)) and another noninteracting arbitrary homogeneous and isotropic \( \rho (t) \), the evolution of the scale factor \( a(t) \) is governed by the standard Friedmann equation (20) and the conservation equation (15).

In conclusion, the rate of expansion of these evolving wormholes is only determined by the matter component \( \rho (t) \) which may be in principle an ideal barotropic fluid, a scalar field or any other isotropic and homogeneous cosmic fluid considered in literature. Notice that if \( \rho (t) = \Lambda = 0 \) the space expands with constant velocity (11), and for \( \rho (t) = 0 \) and \( \Lambda \neq 0 \) the rate of expansion is determined by the cosmological constant (12).

It is worth noticing that for \( \omega_r = -1/3 \) we have a limiting case since the 3-space becomes isotropic and homogeneous and the anisotropic matter behaves as an ideal string gas (i.e. \( p_r = p_t = -\rho_{in} (t)/3 \)). From Eq. (22) we conclude that in this case the matter component behaves as \( \rho_{in} = C / a^2 (t) \), implying that we have a FRW cosmology filled with a mixture of a curvature fluid with a cosmic fluid \( \rho (t) \).
III. SOME REMARKS ON THE GEOMETRY OF THE SPACE-TIME

It is clear that the gravitational configuration (21) is sustained via a matter source made of the inhomogeneous and anisotropic cosmic fluid (22). Let us point out some properties of the discussed geometry. In general the metric (21) is not conformally flat since the Weyl tensor does not vanish for this metric, except for \( C = 0 \) or \( C \neq 0 \) and \( \omega_r = -1/3 \). On the other hand, this acceleration-free space-time \((g_{\alpha t} = 1)\) is characterized by zero anisotropic stress \( \sigma_{\alpha \beta}(t, r) \) and zero heat-flux vector \( q_{\alpha} \).

Note that the metric (21) is conformal to the following static metric:

\[
ds^2 = dt^2 - \left( \frac{dr^2}{1 - kr^2 + \kappa C \omega_r r^{-1-1/\omega_r}} + r^2 d\Omega^2 \right), \tag{23}
\]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \) and \( \tau = \int dt/a(t) \) is the conformal time. In general the component \( g_{\tau \tau} \) of the metric (23) may be valid for all \( r > 0 \) or vanish for some value of the radial coordinate \( r_0 > 0 \); however this does not mean that this space-time contains an event horizon at \( r_0 \) since \( g_{\tau \tau} = 1 \). So in principle, for some sets of the model parameters, this space-time may contain a naked singularity at \( r = 0 \) which may be observable from the outside.

In general this geometry admits three-dimensional slices \( t = t_0 = \text{const} \) with a variable curvature. In this case the 3-curvature may be written as

\[3R = -a_0^{-2} \left( 6k + 2\kappa Cr^{-3-1/\omega_r} \right), \tag{24}\]

where \( a_0 = a(t_0) \). If \( \omega_r < -1/3 \) or \( \omega_r > 0 \) these slices are asymptotically flat for \( k = 0 \) asymptotically de-Sitter for \( k = 1 \) or asymptotically anti de-Sitter for \( k = -1 \). Note that for these ranges of \( \omega_r \) there may arise a naked singularity at \( r = 0 \), on the other hand the energy density of the inhomogeneous matter component vanishes as \( r \to \infty \) since from Eq. (22) we have that \( \rho_{\omega}(t_0, r) \to 0 \).

It is remarkable that the 3-dimensional slices \( t = t_0 \) of metric (21) include as a particular case the 3-dimensional slices \( t = t_0 \) of the Kottler metric

\[
ds^2 = \left( 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2} + r^2 d\Omega^2, \tag{25}\]

which includes the de-Sitter (\( \Lambda > 0 \)) and anti de-Sitter (\( \Lambda < 0 \)) space-times. The 3-curvature of its slices \( t = t_0 \) is given by \( 3R = -2\Lambda \), so they are also spaces of constant curvature. By comparing metrics (21) and (25) we conclude that Kottler slices \( t = t_0 \) are obtained from slices of metric (21) by putting \( a_0^{-2} k = \Lambda/3, \kappa C \omega_r = -2M \) and \( \omega_r \to \pm \infty \). From Eq. (21) we see that

\[
3Ra_0^{-2} = -6k - 2\kappa C r^{-3-1/\omega_r} \equiv -6k - 2\kappa C \omega_r r^{-3-1/\omega_r} \to -6k, \tag{26}
\]

for \( \omega_r \to \pm \infty \), thus effectively we have in this case a space of constant curvature. It can be shown that for this limit the inhomogeneous energy density (22) vanishes since

\[
\rho_{\omega}(t, r) = \frac{C \omega^{-3-1/\omega_r}}{a^2(t)} = \frac{C \omega_r r^{-3-1/\omega_r}}{a^2(t)} \to 0 \tag{26}
\]

for \( \omega_r \to \pm \infty \), while the anisotropic pressures take the forms \( p_r(t, r) \to \frac{\omega_r C}{2 \omega^2(t)r} \) and \( p_t(t, r) \to \frac{\omega_r C}{2 \omega^2(t)r} \). Unfortunately this model with a vanishing inhomogeneous and anisotropic energy density \( \rho_{\omega}(t, r) \) and non-vanishing pressures \( p_r(t, r) \) and \( p_t(t, r) \) is non-physical, so we rule it out from consideration.

If \(-1/3 < \omega_r < 0 \) the curvature increases with radius and as \( r \to \infty \) we have the asymptotic forms \( 3R \sim r^\alpha \), \( \rho_{\omega} \sim r^\alpha \) and \( f^2(r) \sim r^\beta \) with \( \alpha > 0 \) and \( \beta > 0 \) respectively. These gravitational configurations do not contain any singularity at \( r = 0 \) for slices \( t = t_0 \).

IV. EXPANDING WORMHOLE UNIVERSES

Now we shall study gravitational configurations where the considered solution represents an expanding wormhole geometry (1, 5). Before treating Lorentzian wormhole geometries in more detail, let us note again that the metric ansatz (7) provides an explicit class of dynamic wormhole geometries in more detail, let us note again that the considered solution represents an expanding wormhole geometry (4, 5). Before treating Lorentzian wormhole geometries in more detail, let us note again that the considered solution represents an expanding wormhole geometry (4, 5).
\(e^{\Phi(r)} = 1\), thus the first constraint is automatically fulfilled. It must be remarked that, for a general dynamical wormhole, the definition of the location of a wormhole throat is not straightforward. In this case, the position of the wormhole throat depends on the time-slicing, and for a time-dependent wormhole it may not be possible to locate the entire throat within one time slice, as the dynamic throat is an extended object in spacetime \([14]\). In our case, the evolving metric \((21)\) is conformal to the static spacetime \((23)\), which represents a static wormhole for \(\omega_r < -1\) or \(\omega_r > 0\), and then we always may determine the location of the wormhole throat on the hypersurface with \(t = t_0 = \text{const}\). In general, for time-dependent spherically symmetric wormhole spacetimes, alternative definitions of a wormhole throat are required, equally valid for static as well as for dynamical wormholes. Several different definitions have been given in Refs. \([3, 14, 16]\). For example, Hochberg and Visser \([14]\) and Hayward \([16]\) have introduced two independent quasilocal definitions of a throat for dynamical wormholes. These authors do not consider global properties of the wormholes, i.e. they make no assumptions about symmetries, asymptotic flatness, topology, etc., and the wormhole throat is a two-dimensional surface of nonvanishing minimal area on a null hypersurface. On the other hand, in the Ref. \([17]\) the authors have defined a wormhole throat quasilocally in terms of a surface of nonvanishing minimal area on a spacelike hypersurface. Some properties of these three definitions are compared in \([13]\).

From the second constraint, \(g_{rr}^{-1}(r_0) = 0\), we can find the minimum value \(r_0\) of the radial coordinate where the wormhole throat must be located. From this throat condition, and by taking into account the metric \((21)\), we obtain for the integration constant

\[
C = \left(\frac{kr_0^2 - 1}{\kappa \omega_r} \right)^{\frac{1}{1 + \omega_r}} r_0^{\frac{1}{1 + \omega_r}},
\]

yielding for the shape function \(b(r)\) and the metric component \(g_{rr}\)

\[
b(r) = r_0 \left( \frac{r}{r_0} \right)^{-1/\omega_r} \left( 1 - \left( \frac{r}{r_0} \right)^{-1/\omega_r} \right)^{\frac{1}{1 + \omega_r}},
\]

\[
a^2(t)g_{rr}^{-1} = 1 - \left( \frac{r}{r_0} \right)^{-1/\omega_r} \left( 1 - \left( \frac{r}{r_0} \right)^{-1/\omega_r} \right)^{\frac{1}{1 + \omega_r}},
\]

respectively.

It is easy to verify that the wormhole throat is located at \(r_0\) since \(b(r_0) = r_0\). In this case the energy density of the matter threading the wormhole takes the following form:

\[
\kappa \rho_{an}(t, r) = \frac{kr_0^2 - 1}{r_0^2 \omega_r a(t)^2} \left( \frac{r}{r_0} \right)^{-\left(1 + 3 \omega_r\right) / \omega_r}.
\]

Clearly, in order to have an evolving wormhole we must require \(\omega_r < -1\) or \(\omega_r > 0\) (in both of these cases, in the \(g_{rr}\) metric component, \((1 + \omega_r) / \omega_r > 0\) and \((1 + 3 \omega_r) / \omega_r > 0\), implying that the inhomogeneous and anisotropic cosmic fluid \((30)\) can support the existence of evolving wormholes. It can be shown that the form of the wormhole is preserved during all evolution. Notice that for an anisotropic matter with \(\omega_r < -1\) we have that \(\rho_{an}(t, r) > 0\), while for an anisotropic matter with \(\omega_r > 0\) we have that \(\rho_{an}(t, r) < 0\). For \(\rho(t) = \Lambda = 0\) the shape of the wormhole expands with constant velocity \([11]\). In the presence of a cosmological constant with \(\rho_{an}(t, r) = 0\) the wormhole configurations have an accelerated expansion (contraction) \([12]\). Other properties of such evolving wormholes are discussed by authors of the Ref. \([11, 12]\) and references cited therein.

As explicit examples of evolving wormholes, let us first consider the case where the isotropic and homogeneous component is given by a perfect fluid with the barotropic state equation \(p(t) = \omega \rho(t)\) and \(k = \Lambda = 0\). Thus the scale factor is given by

\[
a(t) = a_0 t^{2/3(1 + \omega)}
\]

and the energy density by

\[
\rho(t) = \frac{4}{3 \kappa (1 + \omega)^2 t^2},
\]

while the metric takes the form

\[
ds^2 = dt^2 - a_0^2 t^{4/3(1 + \omega)} \times \left( \frac{dv^2}{1 - \left( \frac{v}{r_0} \right)^{(1 + 3 \omega_r) / \omega_r}} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right).
\]

In this case, the energy density of the anisotropic matter and its pressure components are given by

\[
\kappa \rho_{an}(t, r) = -\frac{1}{\omega_r r_0^2 a_0^2 t^{4/3(1 + \omega)}} \left( \frac{r}{r_0} \right)^{-\left(1 + 3 \omega_r\right) / \omega_r}
\]

and \([17]\) respectively. Note that if \(\omega_r < -1\) or \(\omega_r > 0\) we have asymptotically flat FRW regions for \(r \rightarrow \infty\) and \(\omega > -1\). Let us suppose that the isotropic fluid satisfies the dominant energy condition, i.e. \(|p| \leq \rho, \rho \geq 0\). Thus if \(-1 \leq \omega < -1/3\) the expansion of the evolving wormhole is accelerated, i.e. both universes and the throat of the wormhole are simultaneously expanding with acceleration, while for \(-1/3 < \omega \leq 1\) the expansion is decelerated.

On the other hand, the total matter content is given by \(\rho_{\text{total}} = \rho(t) + \rho_{an}(t, r)\) and for any \(\omega_r < 0\) we have...
that $\rho_{\text{total}} \geq 0$. For $\omega_r > 0$ we can have in general time intervals where the total energy is positive or negative. For $\omega > -1/3$ the wormhole model starts with a positive total energy density (since for a fixed $r = \text{const}$ the isotropic component dominates over the another one), then decreases till zero at certain $t_{eq}$, and becomes negative for $t > t_{eq}$. For $\omega < -1/3$ the total energy density starts negative, then increases till zero at certain $t_{eq}$, and becomes positive for $t > t_{eq}$.

The case $\omega = -1/3$ is more interesting, since it allows us to consider evolving wormhole models satisfying the dominant energy condition (DEC) in the whole spacetime. In this case the total energy density and the corresponding total pressure components are given by

$$
\rho_r = \rho + \rho_{\text{in}} = \left(3 - \frac{(r/r_0)^{-1+\omega_r}}{\omega t_{eq}^2 a_0^2}\right) t^{-2},
$$

$$
p_{r,r} = -\frac{1}{3} \rho + \omega_r \rho_{\text{in}},
$$

$$
p_{r,l} = -\frac{1}{3} \rho - \frac{1}{2} (1 + \omega_r) \rho_{\text{in}}.
$$

From this expressions we conclude that the total matter content satisfies the strong energy condition since $\rho_r + p_{r,r} = 0$.

In order to fulfill the DEC we need to satisfy the following conditions: $\rho_T \geq 0$, $p_T + p_{r,r} \geq 0$, $\rho_{T} - p_{r,r} \geq 0$, $\rho_r + p_{r,l} \geq 0$ and $p_T - p_{r,l} \geq 0$. These conditions imply that the following constraints must be satisfied:

$$
\frac{1}{3 \omega_r} \leq a_0^2 r_0^2,
$$

$$
\frac{1 + \omega_r}{2 \omega_r} \leq a_0^2 r_0^2,
$$

$$
\frac{1 - \omega_r}{4 \omega_r} \leq a_0^2 r_0^2,
$$

$$
\frac{3 + \omega_r}{8 \omega_r} \leq a_0^2 r_0^2.
$$

Hence, for any given $\omega_r > 0$, the DEC is fulfilled by choosing the parameters $a_0^2$ and $r_0^2$ satisfying the conditions (36)-(39). Note that in this case the isotropic fluid gives a scale factor given by $a(t) = t$. Thus, the class of analytic two-fluid evolving wormholes with $\omega = -1/3$ satisfies the dominant and strong energy conditions in the whole spacetime and expands with constant velocity. It is interesting to note that the violation of not of the NEC, at and near the throat, of a dynamical wormhole, may be properly connected to the generalizations of the flare-out condition for an arbitrary wormhole discussed by above cited authors of the Refs. [14,16]. In the definitions of [14,16] dynamical wormhole throats are trapping horizons, i.e. hypersurfaces foliated by “marginally trapped surfaces”, and the violation of the NEC is a generic property of such wormhole throats. While in the Ref. [15] a dynamical spherically symmetric wormhole throat is defined as a “trapped sphere”, and the NEC can still be satisfied for some wormhole configurations.

As a second example, we shall consider an evolving wormhole with $k = 0$ and filled with a minimally coupled scalar field with the exponential potential $V(\phi) = V_0 e^{-\lambda \phi}$, where $\lambda$ and $V_0$ are constant parameters. In this case the energy density and the pressure of the homogeneous and isotropic component are given by $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$ and $p_\phi = \dot{\phi}^2/2 - V(\phi)$ respectively. Thus Eqs. (36) and (40) imply that

$$
\dot{\phi}(t) + 3H \dot{\phi} + \lambda V_0 e^{-\lambda \phi} = 0,
$$

$$
3H^2 = \kappa \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right).
$$

A particular exact solution, describing the power-law expansion

$$
a(t) = a_0 t^p
$$

of the evolving wormhole, is given by

$$
\phi(t) = \frac{2}{\lambda} \ln t,
$$

where

$$
\chi^2 = \frac{2 \kappa}{p}, V_0 = \frac{p(3p - 1)}{\kappa}.
$$

The first relation implies that $p > 0$, and from the second one we have that $V_0 < 0$ if $0 < p < 1/3$ and $V_0 > 0$ if $p > 1/3$.

Finally, as a last example we shall consider evolving wormholes with $k = 0$ and filled with a tachyon field giving the power-law expansion (42), where $p$ is a constant parameter. In this case the energy density and the pressure of the homogeneous and isotropic component are given by $\rho_T = \frac{V(\phi)}{\sqrt{1 - \dot{\phi}^2}}$ and $p_T = -V(\phi) \sqrt{1 - \dot{\phi}^2}$ respectively. Thus Eqs. (37) and (40) imply that

$$
\frac{\dot{V}(\phi)}{V(\phi)} + \frac{3}{V(\phi)} \frac{dV(\phi)}{d\phi} = 0,
$$

$$
3H^2 = \frac{\kappa V(\phi)}{\sqrt{1 - \dot{\phi}^2}}.
$$

It can be shown that in order to have the power-law expansion (42) the tachyon potential takes the form $V(\phi) = \alpha \phi^{-2}$, where $\phi(t) = \phi_0 t$, $\phi_0 = \sqrt{2/(3p)}$ and $\kappa = 2p \sqrt{1 - 2/(3p)}$.

Notice that for discussed expanding wormholes, filled with a scalar and tachyon fields, the wormhole geometry is given by

$$
\frac{ds^2}{dt^2} = -a_0^2 t^{2p} \times
$$

$$
\left( \frac{dt^2}{1 - \left( \frac{r}{r_0} \right)^{-1+\omega_r}/\omega_r} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right).
$$

(47)
In this case, the energy density of the anisotropic component and its pressure are given by
\[ \kappa \rho_{in}(t, r) = -\frac{1}{\omega_r r^3 d^2 t^3} \left( \frac{r}{r_0} \right)^{-(1+3\omega_r)/\omega_r} \]
and (17) respectively.

V. CONCLUSIONS

We have developed models for evolving wormholes sustained by two cosmic fluids: one with homogeneous and isotropic properties and another inhomogeneous and anisotropic. It is remarkable that the energy density of the matter threading and sustaining such a wormhole is the inhomogeneous and anisotropic component, while the rate of expansion of the evolving wormhole is determined by the isotropic and homogeneous component. This matter component may be in principle an ideal barotropic fluid, a scalar field or any other cosmic fluid satisfying the homogeneity and isotropy requirements. For the case where the cosmological constant and the isotropic and homogeneous component are absent the inhomogeneous space expands with constant velocity, and when only the isotropic and homogeneous component is absent the rate of expansion is determined by the cosmological constant.

In general, we have wormhole universes for \( \omega_r < -1 \) or \( \omega_r > 0 \). The present results generalize our previous works [11] and [12]. The case when the studied inhomogeneous geometry represents wormhole configurations expanding with constant velocity, i.e. \( \omega_r < -1 \) or \( \omega_r > 0 \) and \( \rho(t) = \Lambda = 0 \), was discussed in Ref. [11], while the scenarios where the expansion rate is determined by the cosmological constant, i.e. \( \rho(t) = 0 \), were discussed in Ref. [12]. If now \( \rho(t) \neq 0 \) and \( \Lambda = 0 \), the expansion rate of the wormhole is determined by this isotropic and homogeneous fluid. It is interesting to note that the results of Ref. [12] were generalized to the case of evolving wormholes sustained by a single inhomogeneous and anisotropic fluid \( g(t, r) \), by imposing the generalized equation of state \( \gamma + \alpha P_r + 2\beta P_t = 0 \), where \( \alpha \) and \( \beta \) are constant parameters, and \( P_r \) and \( P_t \) are the radial and transverse pressures, respectively [19]. All particular wormhole solutions discussed in this Ref. are related to solutions reported in Ref. [12].

Note that the wormhole geometries [31]-[48] far from the throat look like a flat FRW Universe. At first glance, if the wormhole throat is located outside of the cosmological horizon of any observer, then he is not in causal contact with the throat. Thus, for late times, an observer in this wormhole Universe located too far from the wormhole throat will see the Universe isotropic and homogeneous and it will be in principle unable for him to make a decision about whether he lives in a space of constant curvature or in a space of a wormhole spacetime.

An interesting feature of the discussed here solutions is that one can consider evolving wormholes with positive total energy density, i.e. \( \rho_{total} = \rho(t) + \rho_{in}(t, r) > 0 \). We always are free to consider positive homogeneous and isotropic energy density \( \rho(t) \). In order to have positive \( \rho_{in}(t, r) \) we must choose \( \omega_r \leq -1 \). In this case the inhomogeneous and anisotropic matter component sustaining the evolving wormhole may be considered a generalization of the hypothetical phantom energy used in cosmology in order to explain accelerated expansion of the Universe. Effectively, this cosmic phantom source is characterized by a positive homogeneous energy density, i.e. \( \rho_{DE}(t) > 0 \), and by an isotropic pressure satisfying \( p_{DE}(t) < -\rho_{DE}(t) \). Clearly for \( p_{DE} = \gamma \rho_{DE} \) we have that \( \gamma < -1 \). In our case the phantom energy is realized by the inhomogeneous matter component \( \rho_{in}(t, r) > 0 \) with linear but highly anisotropic equation of state \( p_r < -\rho_r < 0, p_t > 0 \). Spherically symmetric distribution of phantom energy, depending only on the radial coordinate \( r \), with such linear equation of state for the radial pressure were considered in Refs. [17], where the authors constructed static wormholes sustained by a positive energy density \( \rho(r) > 0 \).

Lastly, let us note that, to our knowledge, the results described in this article, for evolving Lorentzian wormholes, with energy-momentum tensor associated with a mixture of one isotropic and homogeneous fluid with an inhomogeneous and anisotropic component, are firstly reported here. There are in the literature many works reporting on evolving wormhole solutions to the Einstein field equations, however, a large number of these papers discuss dynamic wormhole solutions with a single fluid source [2, 8, 18, 19]. It is worth to mention here that a general class of higher evolving dimensional wormholes sustained by a single fluid was studied in Ref. [20]. Most specifically, it was considered a quasi-static spherically symmetric evolving wormholes, with static four non-compact dimensions and an arbitrary number of extra time-dependent compact dimensions. The results of the study show that the WEC cannot be satisfied at the throat. This is mainly due to that the matter content is not distributed in the whole space and is matched to the vacuum. The presence of this matter-vacuum boundary places restrictions on the time dependence of extra compact dimensions, consequently implying the violation of the WEC. On the contrary, as shown in the previous Sec., we can have evolving wormhole configurations satisfying the WEC.

On the other hand, with respect to wormholes involving a mixture of two fluids, a dynamical wormhole, filled with a perfect fluid and a ghost scalar field, is provided in Ref. [15]. The considered in this Ref. dynamical wormhole metric (4.53),
\[ ds^2 = dt^2 - a^2(t) \left( dx^2 - (x^2 + \tilde{b}^2)d\Omega^2 \right) , \]
may be rewritten, by using the transformation \( x^2 = r^2 - \tilde{b}^2 \), as
\[ ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1 - \tilde{b}^2/r^2} - r^2d\Omega^2 \right) . \]
where $\hat{b}$ is a constant parameter and $a(t) = t/t_0$. Thus the solution (4.58)-(4.60) of the Ref. [13] is a particular case of the discussed in this paper wormhole geometries (31)-(33), with $\omega = -1/3$, $\omega_r = 1$, $r_0 = \hat{b}$ and $a_0 = 1/t_0$. In this case the anisotropic and inhomogeneous component with $\omega_r = 1$ may be identified as a massless ghost scalar field (note that the energy density (33) becomes negative). The conditions (36)-(39) imply the inequalities $a_0^2 r_0^2 / \Lambda^2 \geq 1/3$, $a_0^2 r_0^2 \geq 1$, $a_0^2 r_0^2 \geq 0$ and $a_0^2 r_0^2 \geq 1/2$ respectively. Hence, for this particular solution the DEC is satisfied in the whole space for $a_0 r_0 \geq 1$. Notice that the wormhole throat definitions of Hochberg-Visser [14] or Hayward [16] do not apply to this evolving wormhole since the whole spacetime is foliated by trapped surfaces and there is no trapping horizon [15].

As far as we know, there is only one more paper where two-fluid evolving lorentzian wormholes are considered. In Ref. [21] FRW models with a traversable wormhole are considered. In this case the matter content is divided into two parts: the cosmic part $\rho$ that depends on cosmological time only, and the wormhole part $\rho_w$ that depends on the radial coordinate only. This wormholes can be finally connected with the particular wormholes solutions (31)-(33).

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