Chaotic Signal Denoising Based on Hierarchical Threshold Synchrosqueezed Wavelet Transform

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Abstract: In order to overcoming the shortcoming of single threshold synchrosqueezed wavelet transform(SWT) denoising method, an adaptive hierarchical threshold SWT chaotic signal denoising method is proposed. Firstly, a new SWT threshold function is constructed based on Stein unbiased risk estimation, which is two order continuous derivable. Then, by using of the new threshold function, a threshold process based on the minimum mean square error was implemented, and the optimal estimation value of each layer threshold in SWT chaotic denoising is obtained. The experimental results of the simulating chaotic signal and measured sunspot signals show that, the proposed method can filter the noise of chaotic signal well, and the intrinsic chaotic characteristic of the original signal can be recovered very well. Compared with the EEMD denoising method and the single threshold SWT denoising method, the proposed method can obtain better denoising result for the chaotic signal.

1. Introduction

Chaos is a seemingly random irregular motion, which refers to the stochastic behavior in deterministic systems without additional stochastic factors[1,2]. Chaos exists many fields, such as electronics, meteorology, hydrology and communication[3]. At present, chaos theory has been widely used in secure communications[4,5], weak signal detection[6], image encryption[7,8] and so on. But the actual observation chaotic signals usually are polluted by varying degrees noise, the noise makes the calculation of invariant system parameters of chaotic signal, such as the Lyapunov index, correlation dimension and Kolmogorov entropy, become very difficult or even impossible[9]. In order to analyze and deal with the chaotic signals effectively, it is the precondition of the chaotic signal processing to effectively suppress the noise. The chaotic signals have wide frequency spectrum characteristic, which is to the frequency spectrum of noise. So, the traditional linear filtering and spectral analysis methods are not suitable for chaotic signal denoising[10]. So, it is very meaningful to research the denoising methods which are suitable for chaotic signal.

The synchrosqueezed wavelet transform(SWT) is a new time-frequency analysis method based on the continuous wavelet transform[11]. SWT can obtain the higher accuracy time-frequency curve by squeezing the diagram of continuous wavelet transform in frequency direction. SWT is also robust to noise, when the signal is contaminated by strong noise, the SWT can still obtain clear time-frequency curves and basically invariant decomposing results[12]. SWT has been widely used in nonlinear signal...
denoising\cite{13}. For simultaneously taking advantage of the frequency and amplitude information of the signal, so SWT achieves the better denoising effect than wavelet and EMD. The existing SWT denoising methods all use a single threshold algorithm, that is, each layer coefficients of SWT decomposition uses the same threshold for denoising. However, after the noisy chaotic signals are decomposed by SWT, the noise intensity of different layer SWT coefficients is not the same. Therefore, it is not suitable and reduces the denoising effect to use single threshold to denoise the chaotic signal.

In order to overcome the shortcoming of single threshold SWT method in chaotic signal denoising, an adaptive hierarchical threshold SWT denoising method is proposed. Firstly, we construct a new threshold function which has two order continuous derivatives. Then, the optimal denoising threshold of each layer SWT coefficients is calculated iteratively based on the Stein’s unbiased risk estimation(SURE). Lastly, the effectiveness of the proposed method for chaotic signal denoising are analyzed by simulation experiments.

2. denoising theory of Synchrosqueezed wavelet transform

Supposing the multi-component signal \( f(t) \) is
\[
f(t) = \sum_{k=1}^{N} f_k(t) + n(t) = \sum_{k=1}^{N} A_k(t) \cos[2\pi\phi_k(t)] + n(t)
\]

**Theorem 1:** the synchrosqueezing value of the wavelet coefficient \( W_f(a,b) \) with threshold \( \gamma \) and accuracy \( \delta \) is
\[
S_{f,\gamma}^\delta(b,\omega) = \int_{A_f(b)} W_f(a,b) \frac{1}{\delta} \left( \frac{\omega - \omega_f(a,b)}{\delta} \right) a^{-3/2} da
\]

Where \( W_f(a,b) = \int f(t) a^{-1/2} \psi(t) dt \) denotes the wavelet transform coefficient of \( f(t) \).

**Theorem 2:** when the \( \varepsilon \) is enough small, the harmonic component \( f_k(t) \) can be reconstructed completely by SWT, that is to let
\[
\hat{f}_k(b) = \lim_{\delta \to 0} \left( R_\gamma^{-1} \int_{[\omega(a,b),\gamma]} S_{f,\gamma}^\delta(b,\omega) d\omega \right), \quad k \in \{1,\ldots,N\}
\]

There exists a constant \( C, \gamma \) such that for any \( b \),
\[
|\hat{f}_k(b) - A_k(b) \cos[2\pi\phi_k(b)]| \leq C \varepsilon
\]

For the noisy multi-component signal \( f(t) \), the formula (2) shows that the component signal \( f_k(t) \) can be reconstructed completely, so the chaotic signal can be denoised by SWT as following
\[
\hat{f}(t) = \sum_{k=1}^{K} \hat{f}_k(t)
\]

3. Adaptive threshold SST denoising of chaotic signal based on SURE

3.1 SST threshold denoising of chaotic signal

Supposing the noisy chaotic signal to be
\[
Y = X + n
\]

If the scale series of SWT transform are \( a = \{a(j), j = 1,2,\cdots,J\} \), then the \( j \)-th SWT coefficients of noisy chaotic signal can be written
\[
S_j^c = S_j^a + S_j^n
\]
According to the denoising formula (3), we know that the retained coefficients during SWT denoising is

\[
\hat{S}_{j,i} = \begin{cases} 
S_{j,i}, & |S_{j,i}| > \tau_j \cap |\omega_{j,i} - \phi_{j,i}| < \epsilon \\
0, & \text{else} 
\end{cases}
\]

(5)

Where the \( \tau_j \) indicates the amplitude threshold. In the existing SWT denoising algorithm, the amplitude threshold \( \tau_j \) is the global threshold. However, after the noisy chaotic signal is decomposed by SWT, the intensity of noise contained in each layer SWT coefficients \( S_j^y \) not the same. In order to overcome the deficiency of the existing single threshold SWT denoising algorithm, in this paper, we propose an adaptive hierarchical threshold SWT denoising method.

3.2 Iterating calculation of optimal hierarchical threshold

For the noisy chaotic signal \( Y = X + n \), the purpose of noise elimination is to find the estimated value \( \hat{X} \) of the ideal \( X \) according to the observed value \( Y \), and makes the mean square error between \( \hat{X} \) and \( X \) to be minimum. Replacing expectation with mean, the mean square error can be obtained

\[
\xi(\hat{X}, X) = E(\| \hat{X} - X \|^2) = \frac{1}{N} \left[ \| \hat{X} - X \|^2 \right] = \frac{1}{N} \sum_{i=0}^{N-1} (\hat{x}_i - x_i)^2
\]

(6)

The following formula can be obtained by Parseval formula

\[
\xi(\hat{X}, X) = E(\| \hat{S}^y - S^y \|^2) = \frac{1}{N} \sum_{j,i} (\hat{S}_{j,i}^y - S_{j,i}^y)^2
\]

(6)

The gradient of the \( R(\tau) \) is

\[
\frac{\partial R(\tau)}{\partial \tau} = 2 \sum_{i=0}^{N-1} g_i \frac{\partial g_i}{\partial \tau} + 2 \sum_{i=0}^{N-1} \frac{\partial^2 g_i}{\partial S_i^y \partial \tau}
\]

(7)

The minimum value of \( R(\tau) \) is calculated iteratively by the steepest descent method, that is

\[
\tau(k+1) = \tau(k) - \mu \cdot \Delta \tau(k)
\]

(8)

Where \( \mu \) is step length and the \( \Delta \tau(k) = \frac{\partial R[\tau(k)]}{\partial \tau(k)} \). So, in the threshold calculation based on MMSE, the key is to calculate the gradient \( \Delta \tau(k) \). So

\[
\Delta \tau(k) = \frac{\partial R[\tau(k)]}{\partial \tau(k)} = 2 \sum_{i=0}^{N-1} g_i \cdot \frac{\partial g_i}{\partial \tau(k)} + 2 \sum_{i=0}^{N-1} \frac{\partial^2 g_i}{\partial S_i^y \partial \tau(k)} \cdot g_i = T[S_i^y, \tau(k)] - S_i^y
\]

(8)

So,

\[
\Delta \tau(k) = 2 \sum_{i=0}^{N-1} g_i \cdot \frac{\partial T[S_i^y, \tau(k)]}{\partial \tau(k)} + 2 \sum_{i=0}^{N-1} \frac{\partial^2 T[S_i^y, \tau(k)]}{\partial S_i^y \partial \tau(k)}
\]

(9)

In order to overcome the deficiency of soft threshold function, an improved threshold function is proposed, which has higher order derivatives and can be used in iterative calculation based on steepest descent.

3.3 Improved threshold function and iterative operation

In this paper, an improved Sigmoid function is selected as the threshold function of SWT chaos denoising. The improved Sigmoid threshold function is
Where, \( S \) denotes the coefficients of SWT. As can be obtained by formula (15)

\[
T_s(S, \tau) = \begin{cases} 
S + \tau - \tau/(2k + 1), & S < -\tau \\
S^{2k+1} / [(2k + 1) \tau^{2k}], & |S| \leq \tau \\
S - \tau + \tau/(2k + 1), & S > \tau 
\end{cases}
\] (10)

\[
\frac{\partial T_s(S^\gamma, \tau(k))}{\partial \tau(k)} = \begin{cases} 
1 - \tau/(2k + 1), & S^\gamma < -\tau(k) \\
[(2 - \tau/k(k + 1)](S^\gamma / \tau(k))^{2k+1}, & S^\gamma \leq \tau(k) \\
1 + \tau/(2k + 1), & S^\gamma > \tau(k) 
\end{cases}
\] (11)

\[
\frac{\partial^2 T_s(S^\gamma, \tau(k))}{\partial S^\gamma \partial \tau(k)} = \begin{cases} 
0, & |S| \geq \tau(k) \\
[-(2k / \tau(k))^{2k+1}] \cdot (S^\gamma)^{2k+1}, & |S^\gamma| \leq \tau(k) 
\end{cases}
\] (12)

the denoising threshold \( \tau_j \) can be iterative calculated as following formula

\[
\tau_j(k + 1) = \tau_j(k) - \mu \Delta \tau_j(k), \quad j = 1, 2, \ldots, J
\]

4. Experiments and analysis

In order to compare the performance of the proposed SWT hierarchical threshold denoising method, the simulated chaotic signal and the measured sunspot signal are analyzed respectively in experiments. The chaotic signals are denoised by the Ensemble empirical mode decomposition (EEMD) method\(^9\), single threshold SWT method (ST-SWT)\(^14\) and the proposed hierarchical threshold SWT method (HT-SWT) respectively.

4.1 Simulated chaotic signal denoising

The Duffing chaotic signal can be generated by the following system equation:

\[
x'' + cx' - f_0^2 x + dx^3 = P \cos(ft)
\]

The four order Runge Kutta method is used to solve the equation, and the parameter values of the equation are \( c = 0.05 \), \( f_0^2 = 0.2 \), \( d = 1 \), \( f = 2 \), \( P = 10 \). The experimental chaotic data are added the Gauss white noise with SNR=-5dB, 0dB, 5dB, 10dB, 15dB, 20dB, EEMD method, single threshold SWT method and the proposed method are used to denoise the noisy chaotic signal respectively. For different noise intensity, the SNR and MSE of denoised signals by three methods are shown in Tab1.

| Tab.1 SNR and RMSE of denoised chaotic signal by three methods |
|---------------------------------|----------------|----------------|----------------|
| Input SNR | EEMD method | Single threshold SWT method | The proposed method |
| SNR(dB)/RMSE | SNR(dB)/RMSE | SNR(dB)/RMSE |
| -5 | 7.9846/0.8126 | 8.86997/0.7552 | 13.8229/0.4913 |
| 0 | 9.6105/0.6883 | 11.3827/0.5976 | 16.2362/0.3176 |
| 5 | 13.348/0.5335 | 15.1612/0.4825 | 20.1809/0.2125 |
| 10 | 17.055/0.4870 | 19.6562/0.3638 | 25.9117/0.1838 |
| 15 | 20.911/0.3296 | 23.8719/0.3162 | 27.7002/0.1266 |
| 20 | 25.742/0.2146 | 28.7002/0.1728 | 31.4432/0.0406 |
| Sunspot data | 13.588/2.1135 | 13.9734/2.0051 | 14.6288/1.5098 |

The figure 1 shows the phase diagrams of chaotic attractors of denoised signal by three methods when SNR=5dB, figure 1(a) is the phase diagram of noisy chaotic signal, and figure 1(b)-(d)
respectively is the denoised phase diagram of the EEMD method, the single threshold SWT method and the proposed method. As can be seen from figure 1, The chaotic attractor phase diagram of denoised signal by the proposed is more regular and smoother, and has higher similarity with the standard phase diagram.

![Fig. 1 phase graphs of noisy and denoised Duffing signal](image)

(a) phase graph of noisy Duffing signal  
(b) denoised phase graph by EEMD  
(c) denoised phase graph by single threshold SWT  
(d) denoised phase graph by the proposed method

5. Conclusion
In this paper, an improved SWT denoising method is proposed and used to denoise the chaotic signal. Firstly, a new threshold function is constructed. The new threshold function has a continuous two order derivative, and overcomes the discontinuity of the discontinuity of the derivative of the classic soft threshold. The experimental results show that the proposed method can effectively filter the noise of chaotic signal, the phase diagram of denoised signal can clearly restore the topology of chaotic attractor. The denoising effect of the proposed method is better than the EEMD method and single threshold SWT method.

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