Approximation by sequence of operators including Dunkl Appell polynomials

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Abstract In this article, we give a sequence of operators for producing an approximation result. The relation between the rate of approximation of sequence operators including Dunkl variant of exponential function with first and second-order modulus of continuity are shown. A specific application of sequence of operators which include Gould-Hopper type polynomials is constructed.

Keywords Dunkl Appell polynomials · Rate of approximation · Gould-Hopper type polynomials

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1 Introduction

In applied science and engineering, the approximation theory plays a significant role. The main purpose of the theory of approximation is related to representation of functions which are difficult to comprehend of its characteristic properties. In this area, the first result that a continuous function which is defined on a closed and bounded interval can be represented with the help of polynomials was expressed and proved by Weierstrass in 1885. With this theorem proved by Weierstrass, there have been important developments in other fields of theoretical mathematics and applied mathematics.

The different proofs of Weierstrass approximation theorem were given by Weierstrass, Picard, Lebesgue, Fejér, Mittag-Leffler, Landau, de la Vallée-Poussin, Bernstein and Montel using dissimilar methods. By means of probabilistic considerations, Bernstein gave an elementary proof of this theorem
by giving an explicit construction of the polynomials. For each function $\varphi$ which is continuous on $[0, 1]$, the corresponding polynomials having degree $n$ are defined

$$B_n(\varphi; x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1 - x)^{n-i} \varphi \left( \frac{i}{n} \right), \quad x \in [0, 1]$$

by Bernstein in 1912 [1]. In 1950, the following operators

$$M_n(\varphi; x) = e^{-nx} \sum_{i=0}^{\infty} \frac{(nx)^i}{i!} \varphi \left( \frac{i}{n} \right)$$

(1)

was carefully examined by Szasz [2] who proved that the sequence $\{M_n(\varphi; x)\}_{n \geq 1}$ converges to $\varphi(x)$ for all $x \in [0, \infty)$ provided the function $\varphi$ is continuous.

Let $r$ be holomorphic in $\{\omega : |\omega| < R, R > 1\}$ such that $r(1) \neq 0$ and suppose that it has a Taylor expansion as the following form

$$r(\omega) e^{\omega x} = \sum_{i=0}^{\infty} q_i(x) \omega^i.$$  

(2)

Following operators $J_n(\varphi; x)$ with $q_i(x) \geq 0$ for $x \in [0, \infty)$,

$$J_n(\varphi; x) = \frac{e^{-nx}}{r(1)} \sum_{i=0}^{\infty} q_i(nx) \varphi \left( \frac{i}{n} \right)$$

(3)

considered in [3].

Recently, there have been significant developments in theoretical mathematics and applied mathematics in quantum calculus. In some applications, $q$ -Bernstein polynomials are more useful than Bernstein polynomials. Because of the importance of $q$ -Bernstein polynomials in many fields of application such as pattern recognition and computer aided geometry design, these kinds of polynomials have been extensively studied in the literature. There are many papers and several books published on $q$ -Bernstein polynomials ([4],[5],[6],[7],[8]).

For $\mu + 1/2 > 0$ extended exponential function [9] defined by

$$e_\mu(x) = \sum_{i=0}^{\infty} \frac{x^i}{\gamma_\mu(i)}.$$  

(4)

where

$$\gamma_\mu(2i) = \frac{2^{2i+1} \Gamma(i + \mu + 1/2)}{\Gamma(\mu + 1/2)} \quad \text{and} \quad \gamma_\mu(2i + 1) = \frac{2^{2i+1} \Gamma(i + \mu + 3/2)}{\Gamma(\mu + 1/2)}$$

(5)
has been used to solve the approximation problem of the sequence of operators defined as [10]

\[
S_n (\varphi; x) = \frac{1}{\gamma_n (x)} \sum_{i=0}^{\infty} \frac{(nx)^i}{\gamma_i^n (i)} \varphi \left( \frac{i + 2\mu \theta_i}{n} \right).
\]  

(6)

If \( \theta_i \) is defined as

\[
\theta_i = \begin{cases} 
0 & i \in \{0, 2, 4, \ldots, 2n, \ldots\} \\
1 & i \in \{1, 3, \ldots, 2n + 1, \ldots\}
\end{cases}
\]

then following recursion relation

\[
\gamma_i (i + 1) = (i + 1 + 2\mu \theta_{i+1}) \gamma_i (i), \quad i \in \mathbb{N}_0
\]  

(7)

is satisfied.

There are many authors considering the exponential function based on gamma function. For the approximation results related to (4), we refer the reader to studies by authors ([11], [12], [13], [14]) who proved the quantitative convergence theorems for the different kinds of operators sequence generated by the exponential function based on gamma function.

The Dunkl operator \( A_\mu \) has the form

\[
(A_\mu \vartheta) (x) = A_\mu \vartheta (x) = \frac{d \vartheta (x)}{dx} + \mu \frac{\vartheta (x) - \vartheta (-x)}{x},
\]  

where \( \mu \) is a real number satisfying \( \mu > -1/2 \) and \( \vartheta \) is an entire function [9]. Repeating this process we can define

\[
(A_\mu^2 \vartheta) (x) = A_\mu^2 \vartheta (x) = \frac{d^2 \vartheta (x)}{dx^2} + \frac{2\mu d \vartheta (x) - \vartheta (-x)}{x^2}.
\]  

(9)

The starting point of our work here is the paper of Ben Cheikh and Gaied [15]. A polynomial set \((q_i)\) is called Dunkl-Appell polynomial set if and only if for \( i \in \mathbb{N}_0 \)

\[
A_\mu q_{i+1} (x) = \gamma_i (i + 1) q_i (x).
\]

The authors have proved that the following propositions are equivalent:

(i) \((q_i)\) is a Dunkl-Appell polynomial set.

(ii) The polynomials \(q_i\) can be written as

\[
q_i (x) = \sum_{j=0}^{i} \binom{i}{j} a_{i-j} x^j, \quad (a_0 \neq 0),
\]

where the sequence \((a_j)\) is not depend on \( i \) and

\[
\binom{i}{j} = \frac{\gamma_i (i)}{\gamma_i (i-j) \gamma_j (i-j)}.
\]

(iii) \((q_i)\) is generated by

\[
Q (t) e_\mu (xt) = \sum_{i=0}^{\infty} \frac{q_i (x)}{\gamma_i (i)} t^i.
\]
where
\[ Q(t) = \sum_{i=0}^{\infty} \frac{a_i}{\gamma_{\mu}(i)} t^i, \quad (a_0 \neq 0). \]

The main purpose of this article is to explain the approximation problem described below in detail:

\((P)\) Find the sequence of operators generated by Dunkl-Appell polynomial set \((q_i)\) defined by \([15]\)

\[ Q(t) e^\mu(xt) = \sum_{i=0}^{\infty} \frac{q_i(x)}{\gamma_{\mu}(i)} t^i, \quad (10) \]

where the coefficients \(\gamma_{\mu}\) are defined as in \((5)\).

For \(\mu \geq 0\), the sequence of operators providing a solution of the above-mentioned problem \((P)\) is expressed as follows:

\[ K_{\mu}^n (f; x) = \frac{1}{Q(1) e_\mu(nx)} \sum_{i=0}^{\infty} \frac{q_i(nx)}{\gamma_{\mu}(i)} f \left( i + \frac{2\mu \theta_i}{n} \right). \quad (11) \]

The operators \((11)\) are the generalized form of operators \((3)\). When we take \(\mu = 0\) in \((11)\), mentioned expression is obtained explicitly.

Following section gives a sequence of operators for producing an approximation. The relation between the approximation order of operators defined by \((11)\) with first and second-order modulus of continuity are shown. A specific application of sequence of operators given in \((11)\) which include Gould-Hopper type polynomials is constructed.

2 Main Results

The starting point for this construction is the some crucial preliminary results that are used in this article.

\((11)\) yields the following assertion:

**Lemma 1** \(K_{\mu}^n\) operators are linear and positive operators.
Lemma 2 Let \( \{K_n^\mu\}_{n \geq 1} \) be the sequence of operators defined by (11). Then there are following assertions:

\[
\begin{align*}
K_n^\mu (1; x) &= 1, \\
K_n^\mu (\xi; x) &= x + \frac{(e_\mu(nx) - e_\mu(-nx)) Q' (1) + e_\mu (-nx) (A_\mu Q) (1)}{Q (1) ne_\mu(nx)}, \\
K_n^\mu (\xi^2; x) &= x^2 + \frac{e_\mu(nx) \left( 2Q' (1) + Q (1) \right) + 2\mu Q (1) e_\mu (-nx)}{Q (1) ne_\mu(nx)} x \\
&\quad + \frac{(A_\mu Q) (1) e_\mu (-nx)}{Q (1) n^2 e_\mu(nx)} \\
&\quad + \frac{\left( 2Q'' (1) - (A_\mu Q)' (1) - (A_\mu Q') (1) + Q' (1) - 2\mu Q' (-1) \right) (e_\mu(nx) - e_\mu(-nx))}{Q (1) n^2 e_\mu(nx)} \\
&\quad + \frac{(A_\mu^2 Q) (1) + 2\mu (A_\mu Q) (-1)}{Q (1) n^2}. \tag{12}
\end{align*}
\]

Proof The proof of first relation in (12) is quite simple, because it is enough to replace \( x \) with \( nx \) and \( t \) with 1 in the following statement

\[
\sum_{i=0}^{\infty} q_i (x) t^i = Q (t) e_\mu (xt).
\]

To obtain the required second statement in (12) it suffices to apply the Dunkl operator \( A_\mu \) to the both side of above equality. By reason of the fact that

\[
A_\mu \left( t^i \right) := A_{\mu,t} \left( t^i \right) = \frac{\gamma_\mu (i)}{\gamma_\mu (i - 1)} t^{i-1} \quad \text{and} \quad A_\mu \left( e_\mu (xt) \right) := A_{\mu,t} \left( e_\mu (xt) \right) = xe_\mu (xt) \tag{13}
\]

and the following product rule

\[
(A_\mu (\partial \zeta)) (t) = \partial (t) A_\mu \zeta (t) + \zeta (-t) A_\mu \partial (t) + \partial' (t) (\zeta (t) - \zeta (-t)) \tag{14}
\]

it is clear that

\[
\sum_{i=1}^{\infty} q_i (x) \frac{(i + 2\mu \theta (i)) t^{i-1}}{\gamma_\mu (i)} = xQ (t) e_\mu (xt) + e_\mu (-xt) (A_\mu Q) (t) + Q' (t) \left[ e_\mu (xt) - e_\mu (-xt) \right].
\]

To achieve the third statement in (12) it is enough again to apply the Dunkl operator \( A_\mu \) to the both side of above equality. Similarly, by virtue of (13) and
we conclude that

$$\sum_{i=2}^{\infty} g_i(x) (i + 2\mu \theta_i)(i - 1 + 2\mu \theta_{i-1}) t^{i-2} = x^2 Q(t) e_\mu(x) + x e_\mu(-x) (A_\mu Q)(t)$$

$$+ x Q'(t) [e_\mu(x) - e_\mu(-x)]$$

$$- x e_\mu(-x) (A_\mu Q)(t) + e_\mu(x) (A_\mu^2 Q)(t)$$

$$+ [e_\mu(-x) - e_\mu(x)] \left( A_\mu Q' \right)(t)$$

$$+ x e_\mu(x) + x e_\mu(-x) [A_\mu Q](t)$$

$$+ 2 [e_\mu(x) - e_\mu(-x)] Q''(t).$$

If we replace $x$ with $nx$ and $t$ with 1 in the above expressions, these equalities enable us to obtain the proof of lemma.

**Theorem 1** Let $f$ be a real valued continuous function on $[0, \infty)$ with

$$f(x) = \frac{A}{1 + x^2}, \quad A \in \mathbb{R}, \quad (x \to \infty).$$

Then the sequence $\{K_n(f)\}_{n \geq 1}$ does converge uniformly to the $f$ function on $[a, b]$ for every $0 \leq a < b < \infty$.

**Proof** The proof of this fact is the substance of Korovkin-type property which is given by Altomare [16]. The formula obtained in Lemma 2 allows us to get

$$K_n^\mu(t^i; x) \equiv x^i, \quad i = 0, 1, 2,$$

on every interval $[a, b] \subset [0, \infty)$. Returning now to the universal Korovkin-type property, the theorem is herewith proved.

We shall now give a lemma which will be useful later. By combining relations in [12], we arrive at the following assertion:

For every natural number $n$, we define two notations for convenience as follows:

$$\Omega_1 := K_n^\mu(\xi - x; x) \quad \text{and} \quad \Omega_2 := K_n^\mu((\xi - x)^2; x).$$
Lemma 3 For $K_n^\mu$ operators, the following relations
\[
\Omega_1 = \frac{(e_\mu (nx) - e_\mu (-nx)) Q'(1) + e_\mu (-nx) (A_\mu Q)(1)}{Q(1) n e_\mu (nx)},
\]
\[
\Omega_2 = \left( 1 + \frac{2e_\mu (-nx)}{Q(1) e_\mu (nx)} \left[ \frac{\mu Q(-1) + Q'(1) - (A_\mu Q)(1)}{Q(1) e_\mu (nx)} \right] \right) \frac{x}{n}
\]
\[
+ \frac{e_\mu (-nx) (A_\mu Q)(1)}{Q(1) n^2 e_\mu (nx)} + \frac{2Q''(1) - (A_\mu Q)'(1) - (A_\mu Q)'(1) + Q'(1) - 2\mu Q'(-1)}{Q(1) n^2 e_\mu (nx)}
\]
\[
+ \frac{(A_\mu^2 Q)(1) + 2\mu (A_\mu Q)(-1)}{Q(1) n^2}
\]
can be expressed.

Let $\delta \geq 0$ and $UC[0, \infty)$ be the family of uniformly continuous functions on positive real axis. Assume that $\varphi \in UC[0, \infty)$. For $\zeta_1, \zeta_2 \in [0, \infty)$ such that $|\zeta_1 - \zeta_2| \leq \delta$, the least upper bound of $|\varphi(\zeta_1) - \varphi(\zeta_2)|$ is represented by $w(\varphi; \delta)$. We shall give the modulus of continuity of $\varphi$.

We continue this section with the following useful relation between sequence of operators $\{K_n^\mu(f;\cdot)\}_{n \geq 1}$ with modulus of continuity.

Theorem 2 Let $f$ be a uniformly continuous real valued function on $[0, \infty)$. Then the following holds
\[
|K_n^\mu(f; x) - f(x)| \leq (1 + \lambda_n(x)) w\left(f; \frac{1}{\sqrt{n}}\right),
\]
where $\lambda_n(x) = \sqrt{\Omega_2}$.

Proof According to Lemma 2 and well known Schwarz inequality, it follows that
\[
|K_n^\mu(f; x) - f(x)| \leq \frac{1}{Q(1) e_\mu (nx)} \sum_{i=0}^{\infty} q_i(nx) \left| f\left( i + \frac{2\mu \theta_i}{n} \right) - f(x) \right|
\]
\[
\leq \left\{ 1 + \frac{1}{\delta} \sqrt{\Omega_2} \right\} w(f; \delta)
\]
If we put $\delta = \frac{1}{\sqrt{n}}$, we arrive at desired result.
Functions \( \varphi \) satisfying the following inequality are referred to as H"older functions with exponent \( \beta > 0 \) on \([0, \infty)\) if there is a positive constant \( M \) such that

\[
|\varphi(\zeta_1) - \varphi(\zeta_2)| \leq M |\zeta_1 - \zeta_2|^{\beta}
\]

for all \( \zeta_1, \zeta_2 \in [0, \infty) \).

**Theorem 3** If \( \varphi \) satisfies a H"older condition with exponent \( \beta > 0 \), then the following estimate

\[
|K_n^\mu (\varphi; x) - \varphi(x)| \leq M [\Omega_2]^{\beta}
\]

holds.

**Proof** Monotonicity property of \( K_n^\mu \), combined with the Lemma 2, shows that

\[
|K_n^\mu (\varphi; x) - \varphi(x)| = |K_n^\mu (\varphi(\xi) - \varphi(x); x)| 
\leq |K_n^\mu (|\varphi(\xi) - \varphi(x)|; x) 
\leq MK_n^\mu \left(|\xi - x|^{\beta}; x\right).
\]

Finally, from H"older inequality we deduce the following expression

\[
\frac{1}{Q(1) e_\mu (nx)} \sum_{i=0}^{\infty} \frac{q_i (nx)}{\gamma_\mu (i)} \left| \frac{i + 2\mu \theta_i}{n} - x \right|^{\beta} \leq \frac{1}{Q(1) e_\mu (nx)} 
\times \sum_{i=0}^{\infty} \left( \frac{q_i (nx)}{\gamma_\mu (i)} \right)^{2+\alpha} \left( \frac{q_i (nx)}{\gamma_\mu (i)} \right)^{\frac{\alpha}{2}} \left| \frac{i + 2\mu \theta_i}{n} - x \right|^{\beta} 
\leq \left[ \sum_{i=0}^{\infty} \frac{q_i (nx)}{\gamma_\mu (i)} \right]^{2+\alpha} \left[ \frac{i + 2\mu \theta_i}{n} - x \right]^{\beta} 
\leq \left[ K_n^\mu (1; x) \right]^{2+\alpha} \left[ \Omega_2 \right]^{\beta}.
\]

So, the theorem is established.

**Theorem 4** Under the condition \( \psi \in C [0, \infty) \), the relation

\[
|K_n^\mu (\psi; x) - \psi(x)| \leq \frac{3}{4} (2 + a + s^2) w_2 (\psi; s) + \frac{2s^2}{a} \| \psi \| , \ x \in [0, a]
\]

holds, where \( s = \sqrt{\Omega_2} \) and \( \| \psi \| = \sup_{x \in [0, \infty)} |\psi(x)| \) and \( w_2 \) is second-order modulus of continuity.
Proof Suppose that \( a \) is a positive real number. By virtue of Lemma 2 and on the basis of linearity property of \( K^\mu_n \),
\[
K^\mu_n (\psi; x) - \psi (x) = K^\mu_n (\psi - \psi_s; x) + K^\mu_n (\psi_s; x) - \psi_s (x) - \psi (x)
\]
is verified. Taking into consideration the above equality we have, by virtue of Lemma 2,
\[
|K^\mu_n (\psi; x) - \psi (x)| \leq |K^\mu_n (\psi - \psi_s; x)| + |K^\mu_n (\psi_s; x) - \psi_s (x)| + |\psi_s (x) - \psi (x)| 
\]
In accordance with the property of second-order Steklov function of the \( \psi \) function [17], \( \psi_s \in C^2 [0, a] \) is satisfied. Therefore, from the expansion of Taylor and inequality of Cauchy-Schwarz, the right member of the above inequality
\[
|K^\mu_n (\psi_s; x) - \psi_s (x)| \leq \left\| \psi_s' \right\| \sqrt{\Omega^2 + \frac{1}{2} \left\| \psi_s'' \right\|} \Omega^2 
\]
can be written [18]. As a consequence of Landau inequality we have the following fact
\[
\left\| \psi_s' \right\| \leq \frac{2}{a} \left\| \psi_s \right\| + \frac{a}{2} \left\| \psi_s'' \right\| 
\]
Combining this formula with (15) gives us
\[
|K^\mu_n (\psi; x) - \psi (x)| \leq 2 \left\| \psi - \psi_s \right\| + |K^\mu_n (\psi_s; x) - \psi_s (x)| 
\]
\[
\leq 2 \left\| \psi - \psi_s \right\| + |K^\mu_n (\psi_s; x) - \psi_s (x)| 
\]
If we select \( s = \sqrt{\Omega^2} \), above inequality implies that
\[
|K^\mu_n (\psi; x) - \psi (x)| \leq 3 \left( \frac{2}{a} \left\| \psi \right\| + \frac{3a}{4s^2} w_2 (\psi; s) \right) \sqrt{\Omega^2} + \frac{3}{4s^2} w_2 (\psi; s) \Omega^2. 
\]
This completes the proof.

The following significant example is an explicit form of the \( K^\mu_n \) operators.

Example 1 Polynomials having the following generating functions
\[
e^{a\omega^{d+1}} e^{i\mu} (x\omega) = \sum_{i=0}^{\infty} g_i^{(d+1)} (x, a, \mu) \frac{\omega^i}{\gamma_\mu (i)} 
\]
are called Gould-Hopper type polynomials [15]. Gould-Hopper type polynomials set \( \{ g_i^{(d+1)} (x, a, \mu) \}_{i=0}^{\infty} \) is a \( d \)-orthogonal polynomial set [15]. From [16],
it is clear that Gould-Hopper type polynomials are the $A_\mu$ Appell polynomial set with
\[ Q(\omega) = e^{ax^{d+1}}. \]

Under the assumption $a \geq 0$, $K^\mu_n$ operators which include Gould-Hopper type polynomials are
\[ K^\mu_n (f; x) = \frac{e^{-a}}{e^\mu (nx)} \sum_{i=0}^{\infty} \frac{g_i(d+1)(nx,a,\mu)}{\gamma \mu (i)} f\left( i + \frac{2\mu \theta n}{n} \right). \]

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