LETTER

New reflection matrices for the $U_q(gl(m|n))$ case

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Abstract. We examine supersymmetric representations of the B-type Hecke algebra. We exploit such representations to obtain new non-diagonal solutions of the reflection equation associated with the super algebra $U_q(gl(m|n))$. The boundary super algebra is briefly discussed and it is shown to be central to the supersymmetric realization of the B-type Hecke algebra.

Keywords: algebraic structures of integrable models, integrable spin chains (vertex models), symmetries of integrable models
1. Introduction

There have been various investigations regarding the solution of the reflection equation \([1,2]\), and although these studies in the case of the usual Lie algebras and their \(q\)-deformed counterparts \([3]–[13]\) are rather exhaustive there seem to exist gaps when one looks at the corresponding solutions associated with the supersymmetric algebras. An exhaustive classification of solutions of the reflection equation associated with a specific algebra is a fundamental physical and mathematical issue, given that such investigations provide new boundary conditions that may for instance drastically alter the physical behavior as well as the symmetry governing the corresponding system.

One finds numerous works presenting diagonal and non-diagonal solutions of the reflection equation associated with various super algebras. More precisely, in \([14]\) diagonal and non-diagonal solutions are derived for the super Yangians \(osp(m|n)\) and \(gl(m|n)\), while in \([15]\) non-diagonal solutions are obtained for the \(sl(2|1)\) case (supersymmetric t-J model). In \([16,17]\) purely diagonal solutions for \(U_q(gl(m|n))\) are presented. Non-diagonal solutions of the reflection equation associated with the deformed algebras \(U_q(sl(r|2m)(2))\) and \(U_q(osp(r|2m))\) are presented in \([18]\), whereas in \([19]\) a generic solution for the \(U_q(sl(1|1))\) case is derived. Here, we shall focus on the general \(U_q(gl(m|n))\) algebra and we shall find non-diagonal solutions of the reflection equation. Note that this is the first time that non-diagonal solutions for the generic \(U_q(gl(m|n))\) case have been derived.

The main aim of the present investigation is the study of non-diagonal supersymmetric representations of the B-type Hecke algebra \([20]–[22]\). Using these representations we shall be able to identify new non-diagonal solutions of the reflection equation associated with \(U_q(gl(m|n))\). Our main result is the derivation of non-diagonal solutions of purely bosonic or fermionic type as well as mixed ones. The mixed solutions occur only in the \(U_q(gl(m|2k))\) case and are associated with the symmetric Dynkin diagram. Moreover, based on these non-diagonal solutions we extract the associated boundary non-local charges, which form the boundary super algebra and in the fundamental representation are central to the supersymmetric realization of the B-type Hecke algebra.

The outline of this paper is as follows: in the next section we introduce some basic notation for the super algebras and we also recall the \(U_q(gl(m|n))\) \(R\)-matrix and the corresponding supersymmetric representations of the A-type Hecke algebra. We then derive novel supersymmetric representations of the B-type Hecke algebra and we identify
the relevant non-diagonal reflection matrices. In section 3 we discuss the boundary super algebras associated with the aforementioned reflection matrix. We extract the boundary non-local charges, which form the boundary super algebra and are central to the B-type Hecke algebra.

2. Deriving reflection matrices

Our primary objective in this section is to derive new non-diagonal solutions of the reflection equation. This will be achieved by using supersymmetric representations of the B-type Hecke algebra, which will be defined subsequently. Both the representations as well as the non-diagonal $K$-matrices are novel.

2.1. Preliminaries

Before we proceed to present our main results it is necessary to introduce some useful notation associated with super algebras. Consider the $m+n = N$ dimensional column vectors $\hat{e}_i$, with 1 at position $i$ and 0 everywhere else, and the $N \times N \hat{e}_{ij}$ matrices defined as $(\hat{e}_{ij})_{kl} = \delta_{ik} \delta_{jl}$. Then define the gradings:

$$[\hat{e}_i] = [i], \quad [\hat{e}_{ij}] = [i] + [j]. \quad (2.1)$$

The tensor product is graded as:

$$(A_{ij} \otimes A_{kl})(A_{mn} \otimes A_{pq}) = (-1)^{([k]+[l])([m]+[n])} A_{ij} A_{mn} \otimes A_{kl} A_{pq}. \quad (2.2)$$

It is also convenient for what follows to introduce the distinguished and symmetric grading, corresponding apparently to the distinguished and symmetric Dynkin diagrams. In the distinguished grading we define:

$$[i] = \begin{cases} 0, & 1 \leq i \leq m, \\ 1, & m+1 \leq i \leq m+n. \end{cases} \quad (2.3)$$

In the $gl(m|2k)$ case we also define the symmetric grading as

$$[i] = \begin{cases} 0, & 1 \leq i \leq k, \quad m + k + 1 \leq i \leq m + 2k, \\ 1, & k + 1 \leq i \leq m + k. \end{cases} \quad (2.4)$$

We shall focus here, as already mentioned, on the $U_q(gl(m|n))$ algebra. The $R$-matrix associated with this algebra satisfies the graded Yang–Baxter equation [23, 24]

$$R_{12}(\lambda_1 - \lambda_2)R_{13}(\lambda_1)R_{23}(\lambda_2) = R_{23}(\lambda_2)R_{13}(\lambda_1)R_{12}(\lambda_1 - \lambda_2), \quad (2.5)$$

and is given by the following expressions [25]:

$$R(\lambda) = \sum_{i=1}^{N} a_i(\lambda) e_{ii} \otimes e_{ii} + b(\lambda) \sum_{i \neq j=1}^{N} e_{ii} \otimes e_{jj} + \sum_{i \neq j=1}^{N} c_{ij}(\lambda) e_{ij} \otimes e_{ji}, \quad (2.6)$$

where we define

$$a_j(\lambda) = \sinh(\lambda + i\mu - 2i\mu[j]), \quad b(\lambda) = \sinh(\lambda),$$

$$c_{ij}(\lambda) = \sinh(i\mu)e^{\text{sgn}(j-i)\lambda}(-1)^{|j|}. \quad (2.7)$$

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It is clear that the $R$-matrix may be written in the following form:

$$R(\lambda) = e^{\lambda} R^+ - e^{-\lambda} R^-.$$  \hfill (2.8)

Also the matrix $\tilde{R} = PR$ may be written in terms of supersymmetric representations of the A-type Hecke algebra; $P$ is the permutation operator and it has the following form:

$$P = \sum_{i,j} (-1)^{|j|} e_{ij} \otimes e_{ji}. \hfill (2.9)$$

Indeed we have (see also [26])

$$\tilde{R}_{ii+1}(\lambda) = \sinh(\lambda) U_i + \sinh(\lambda + i\mu), \hfill (2.10)$$

where $U_i$ is a supersymmetric representation of the A-type Hecke algebra [20,21]. We recall the A-type Hecke algebra $H_N(q)$ is defined by generators $U_i$, $i = 1, \ldots, N - 1$, and exchange relations:

$$U_i U_{i+1} U_i - U_i = U_{i+1} U_i U_{i+1} - U_{i+1}, \hfill (2.11)$$

$$U_i^2 = \delta U_i, \hfill (2.12)$$

$$[U_i, U_j] = 0, \quad |i - j| > 1 \hfill (2.13)$$

where $\delta = -(q + q^{-1})$ and $q = e^{i\mu}$.

Consider the following supersymmetric representation [21] associated with the $U_q(gl(m|n))$ $R$-matrix. Let

$$U = \sum_{a,b=1}^N f_{ab} e_{ab} \otimes e_{ba} + \sum_{a,b=1}^N t_{ab} e_{aa} \otimes e_{bb}, \quad \text{where} \hfill (2.14)$$

$$f_{aa} = 0, \quad f_{ab} = (-1)^{|b|}, \quad a \neq b,$$

$$t_{aa} = (-1)^{|a|} q^{1-2|a|} - q, \quad t_{ab} = -q^{-\text{sgn}(a-b)}, \quad a \neq b,$$

then we obtain the supersymmetric representation $\pi: H_N(q) \hookrightarrow \text{End}((\mathbb{C}^N)^{\otimes N})$ such that

$$\pi(U_i) = U_i = \mathbb{I} \otimes \cdots \mathbb{I} \otimes U_{\sum_{i,i+1}} \otimes \cdots \otimes \mathbb{I}. \hfill (2.15)$$

2.2. Reflection matrices from the B-type Hecke algebra

Our aim now is to find supersymmetric representations of the B-type Hecke algebra as candidate solutions of the super reflection equation [1,2]

$$R_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 + \lambda_2) K_2(\lambda_2) = K_2(\lambda_2) R_{12}(\lambda_1 + \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2), \hfill (2.16)$$

where $R$ here is given by (2.6), (2.7).

Recall that the B-type Hecke algebra [22] $B_N(q,Q)$ is defined by generators $U_i$ that satisfy (2.11)–(2.13) and $U_0$ with

$$U_1 U_0 U_1 U_0 - \kappa U_1 U_0 = U_0 U_1 U_0 U_1 - \kappa U_0 U_1, \hfill (2.17)$$

$$U_0^2 = \delta_0 U_0, \hfill (2.18)$$

$$[U_0, U_i] = 0, \quad i > 1, \hfill (2.19)$$

where it is convenient to parametrize as $\delta_0 = -(Q + Q^{-1})$ and $\kappa = qQ^{-1} + q^{-1}Q$.

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Inspired essentially by the structure of the representations in the non-supersymmetric case [9, 11, 10] we consider the following form for the non-diagonal supersymmetric representation. Let

$$e = \sum_{a=1}^N h_a e_{aa} + \sum_{a=1}^N c_a e_{a\bar{a}}, \quad (2.20)$$

where the parameters $h_a, c_a$ are a priori free. We define generally the conjugate index $\bar{a}$ such that $[a] = [\bar{a}]$, and more specifically

$$\bar{a} = 2k + m + 1 - a; \quad \text{symmetric diagram}$$

$$\bar{a} = m + 1 - a, \quad \text{a bosonic; \ distinguished diagram.} \quad (2.21)$$

$$\bar{a} = 2m + n + 1 - a, \quad \text{a fermionic; \ distinguished diagram.}$$

Consider tensor type supersymmetric representations of the B-type Hecke algebra; $\pi : B_N(q, Q) \hookrightarrow \text{End}(\mathbb{C}^N \otimes \bar{N})$ such that

$$\pi(U_i) = U_i = \mathbb{I} \otimes \cdots \otimes \underbrace{U_i \otimes \cdots \otimes \mathbb{I}}_{i, i+1} \quad \pi(U_0) = U_0 = e \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}. \quad (2.22)$$

To prove that (2.20), (2.22) is a supersymmetric representation, although tedious, is quite straightforward. We want $U_1, U_0$ to satisfy the B-type Hecke algebraic relations. First from the constraint (2.18)

$$U_0^2 = \sum_{a=1}^N h_a(h_a + c_a c_{\bar{a}} h_a^{-1}) e_{aa} + \sum_{a=1}^N c_a(h_a + h_{\bar{a}}) e_{a\bar{a}}. \quad (2.23)$$

From the equation above and (2.18) we conclude

$$h_a = h_{\bar{a}} = 1, \quad h_a + h_{\bar{a}} = -(Q + Q^{-1}), \quad c_a c_{\bar{a}} = 1, \quad \text{or} \quad h_a = h_{\bar{a}} = c_a = c_{\bar{a}} = 0. \quad (2.24)$$

Moreover, after some quite tedious algebra we conclude

$$U_0 U_1 U_0 U_1 - \kappa U_0 U_1 = \cdots$$

$$= \sum_{a,b} (t_{ab} f_{ab} h_{\bar{a}}^2 + t_{\bar{a}b} f_{\bar{a}b} c_a c_{\bar{a}} + f_{ab} f_{ba} h_a h_b - \kappa f_{ab} h_a) e_{ab} \otimes e_{ba}$$

$$+ \sum_{a,b} (t_{ab} t_{\bar{a}b} c_a h_a + f_{ab} f_{ba} c_a h_b (1)^{|a|+|b|} + t_{ab} t_{\bar{a}b} h_a c_{\bar{a}} - \kappa c_a t_{ab} e_{a\bar{a}} \otimes e_{bb})$$

$$+ \sum_{a,b} f_{ab} h_{ba} h_a c_b e_{ab} \otimes e_{ba}$$

$$+ \sum_{a,b} (f_{ab} t_{ba} c_{\bar{a}} h_{\bar{a}} + f_{ab} t_{\bar{a}b} c_{\bar{a}} h_{\bar{a}} + f_{ab} f_{ba} c_a h_a - \kappa c_{\bar{a}} f_{ab} e_{a\bar{a}} \otimes e_{ba})$$

$$+ \sum_{a,b} f_{ab} h_{ba} c_a e_{\bar{a}b} \otimes e_{ba} + \sum_{a,b} f_{ab} f_{ba} c_a c_b (1)^{|a|+|b|} e_{a\bar{a}} \otimes e_{b\bar{b}}. \quad (2.25)$$

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and

\[ U_1 U_0 U_0 U_0 - \kappa U_1 U_0 = \cdots \]

\[ = \sum_{a,b} (f_{ab} f_{ba} c_a c_b + t_{ab} f_{ab} h_b + f_{ab} t_{ba} h_b^2 - \kappa f_{ab} h_b) e_{ab} \otimes e_{ba} \]

\[ + \sum_{a,b} (f_{ab} f_{ba} c_a h_b (-1)^{|a|+|b|} + t_{ab} t_{ba} c_b h_a + t_{ab} t_{ba} h_b c_a - \kappa t_{ab} c_a) e_{ba} \otimes e_{ab} \]

\[ + \sum_{a,b} (f_{ab} f_{ba} h_b c_b + f_{ab} t_{ba} c_b h_b + f_{ab} t_{ba} h_b c_b - \kappa f_{ab} h_b) e_{ab} \otimes e_{ba} \]

\[ + \sum_{a,b} f_{ab} t_{ab} c_a h_b e_{ab} \otimes e_{ba} + \sum_{a,b} f_{ab} f_{ba} c_a h_b (-1)^{|a|+|b|} e_{ba} \otimes e_{ab}. \] (2.26)

Finally, after appropriately combining the terms of the equations above we show that the quadratic constraint (2.17) is indeed satisfied provided that \( \kappa = qQ^{-1} + q^{-1}Q \), and we can now derive the explicit form of the non-diagonal solution.

More specifically, the generic solution we find reads as follows:

A. Symmetric Dynkin diagram

\[ h_a = h_{\bar{a}}^{-1} = -Q^{-1}, \quad c_a c_{\bar{a}} = 1, \quad 1 \leq a \leq L, \]

\[ h_a = h_{\bar{a}} = c_a = c_{\bar{a}} = 0, \quad L < a \leq m + 2k, \quad 1 \leq L \leq \frac{m + 2k}{2}; \] (2.27)

\[ h_{(m+2k+1)/2} = 0 \quad \text{if } m \text{ odd.} \]

Notice that in this case the solution may be purely bosonic purely fermionic and mixed.

B. Distinguished Dynkin diagram

We obtain only purely bosonic solutions

\[ h_a = h_{\bar{a}}^{-1} = -Q^{-1}, \quad c_a c_{\bar{a}} = 1, \quad 1 \leq a \leq L, \]

\[ h_a = h_{\bar{a}} = c_a = c_{\bar{a}} = 0, \quad L < a \leq \frac{m}{2}, \quad 1 \leq L \leq \frac{m}{2}; \] (2.28)

or purely fermionic ones

\[ h_a = h_{\bar{a}}^{-1} = -Q^{-1}, \quad c_a c_{\bar{a}} = 1, \quad m + 1 \leq a \leq L, \]

\[ h_a = h_{\bar{a}} = c_a = c_{\bar{a}} = 0, \quad L < a \leq m + \frac{n}{2}, \quad m + 1 \leq L \leq m + \frac{n}{2}; \] (2.29)

\[ h_{m+(n+1)/2} = 0 \quad \text{if } n \text{ odd,} \]

and \( h_a = h_{\bar{a}} = c_a = c_{\bar{a}} = 0, \quad a \leq m. \)

Note that the main technical reason why we obtain only purely bosonic or fermionic solutions in the distinguished Dynkin diagram is that in this case \( t_{ab} \neq t_{ba} \) if the two indices are for instance \( a \) fermionic and \( b \) bosonic, or vice versa, as opposed to the symmetric Dynkin diagram or the isotropic case \((q = 1)\). This inequality makes the annihilation
of certain terms appearing in (2.25) and (2.26) impossible, and thus the two expressions cannot coincide.

The supersymmetric representation is then expressed as:

A. Symmetric Dynkin diagram

\[ e = \sum_{a=1}^{L} (-Q^{-1}e_{aa} - Qe_{\bar{a}a}) + \sum_{a=1}^{L} (c_{a}e_{aa} + c_{\bar{a}}e_{\bar{a}a}), \quad c_{a}c_{\bar{a}} = 1, \]
\[ 1 \leq L \leq \frac{m + 2k}{2}. \]  

(2.30)

B. Distinguished Dynkin diagram

We have either bosonic solutions

\[ e = \sum_{a=1}^{L} (-Q^{-1}e_{aa} - Qe_{\bar{a}a}) + \sum_{a=1}^{L} (c_{a}e_{aa} + c_{\bar{a}}e_{\bar{a}a}), \quad c_{a}c_{\bar{a}} = 1, \]
\[ 1 \leq L \leq \frac{m}{2}. \]  

(2.31)

or fermionic ones

\[ e = \sum_{a=m+1}^{L} (-Q^{-1}e_{aa} - Qe_{\bar{a}a}) + \sum_{a=m+1}^{L} (c_{a}e_{aa} + c_{\bar{a}}e_{\bar{a}a}), \quad c_{a}c_{\bar{a}} = 1, \]
\[ m + 1 \leq L \leq m + \frac{n}{2}. \]  

(2.32)

Using the theorem proved in [7, 8, 11] we may express the \( K \)-matrix solution of the reflection equation as

\[ K(\lambda) = x(\lambda)I + y(\lambda)e, \]
\[ x(\lambda) = -\frac{\delta_{0}}{2i \sinh(i\mu)} \cosh(2\lambda + i\mu) - \frac{\kappa}{2i \sinh(i\mu)} \cosh(2\lambda) - \cosh 2i\mu\zeta; \]
\[ y(\lambda) = i \sinh(2\lambda). \]

Let also \( Q = ie^{i\mu m} \) then the non-zero entries of the reflection matrix are written below.

A. Symmetric Dynkin diagram

\( K_{a\bar{a}}(\lambda) = e^{2\lambda} \cosh i\mu m - \cosh 2i\mu\zeta, \quad K_{\bar{a}a}(\lambda) = e^{-2\lambda} \cosh i\mu m - \cosh 2i\mu\zeta, \)
\( K_{a\bar{a}}(\lambda) = ic_{a} \sinh 2\lambda, \quad K_{\bar{a}a}(\lambda) = ic_{\bar{a}} \sinh 2\lambda, \quad 1 \leq a \leq L \)
\( K_{aa}(\lambda) = K_{\bar{a}\bar{a}}(\lambda) = \cosh(2\lambda + im\mu) - \cosh 2i\mu\zeta, \)
\[ K_{a\bar{a}}(\lambda) = K_{\bar{a}a}(\lambda) = 0, \quad L < a \leq \frac{m + 2k}{2}; \quad 1 \leq L \leq \frac{m + 2k}{2} \]  

(2.34)

\( K_{AA} = \cosh(2\lambda + im\mu) - \cosh 2i\mu\zeta, \quad A = \frac{m + 2k + 1}{2} \) if \( m \) odd.
B. Distinguished Dynkin diagram

bosonic:

\[ K_{aa}(\lambda) = e^{2\lambda} \cosh i\mu m - \cosh 2i\mu \zeta, \quad K_{\bar{a}\bar{a}}(\lambda) = e^{-2\lambda} \cosh i\mu m - \cosh 2i\mu \zeta, \]
\[ K_{a\bar{a}}(\lambda) = i c_a \sinh 2\lambda, \quad K_{\bar{a}a}(\lambda) = i c_{\bar{a}} \sinh 2\lambda, \quad 1 \leq a \leq L \]
\[ K_{aa}(\lambda) = K_{\bar{a}\bar{a}}(\lambda) = \cosh(2\lambda + im\mu) - \cosh 2i\mu \zeta, \]
\[ K_{a\bar{a}}(\lambda) = K_{\bar{a}a}(\lambda) = 0, \quad L < a \leq \frac{m}{2}, \quad 1 \leq L \leq \frac{m}{2} \quad (2.35) \]
\[ K_{AA} = \cosh(2\lambda + im\mu) - \cosh 2i\mu \zeta, \quad A = \frac{m+1}{2} \quad \text{if } m \text{ odd}, \]

and

\[ K_{aa}(\lambda) = K_{\bar{a}\bar{a}}(\lambda) = \cosh(2\lambda + im\mu) - \cosh 2i\mu \zeta, \]
\[ K_{a\bar{a}}(\lambda) = K_{\bar{a}a}(\lambda) = 0, \quad a > m, \]

fermionic:

\[ K_{aa}(\lambda) = e^{2\lambda} \cosh i\mu m - \cosh 2i\mu \zeta, \quad K_{\bar{a}\bar{a}}(\lambda) = e^{-2\lambda} \cosh i\mu m - \cosh 2i\mu \zeta, \]
\[ K_{a\bar{a}}(\lambda) = i c_a \sinh 2\lambda, \quad K_{\bar{a}a}(\lambda) = i c_{\bar{a}} \sinh 2\lambda, \quad m+1 \leq a \leq L \]
\[ K_{aa}(\lambda) = K_{\bar{a}\bar{a}}(\lambda) = \cosh(2\lambda + im\mu) - \cosh 2i\mu \zeta, \]
\[ K_{a\bar{a}}(\lambda) = K_{\bar{a}a}(\lambda) = 0, \quad L < a \leq m + \frac{n}{2}, \quad m+1 \leq L \leq m + \frac{n}{2} \quad (2.36) \]
\[ K_{AA} = \cosh(2\lambda + im\mu) - \cosh 2i\mu \zeta, \quad A = m + \frac{n+1}{2} \quad \text{if } n \text{ odd}, \]

and

\[ K_{aa}(\lambda) = K_{\bar{a}\bar{a}}(\lambda) = \cosh(2\lambda + im\mu) - \cosh 2i\mu \zeta, \]
\[ K_{a\bar{a}}(\lambda) = K_{\bar{a}a}(\lambda) = 0, \quad a \leq m. \]

In the \( q = 1 \) case we get non-diagonal solutions of the isotropic \( gl(m|n) \) case and the hyperbolic functions become rational. Also, in the case that \( n = 0 \) we recover non-supersymmetric solutions found in [9,10].

An interesting problem to pursue is the derivation of non-diagonal solutions associated with cyclotomic Hecke algebras, that is affine Hecke type algebras with more generic constraints instead of the quadratic one (2.18), i.e.

\[ \sum_{k=1}^{p} \alpha_k U_0^k = 0. \quad (2.37) \]

The latter is a significant question even in the context of \( U_q(gl(n)) \) algebras (see e.g. [9,10]), especially when one analyzes the boundary behavior of \( A_{n-1}^{(1)} \) affine Toda field theories [27]. In this case, in particular as discussed in [27], the most general solution possible is needed in order to eliminate certain discrepancies arising in the classical Hamiltonian formalism. An exhaustive classification of the solutions of the (super) reflection equation based on the cyclotomic Hecke algebra will be pursued elsewhere.

3. The boundary super algebra; central elements

We shall now briefly discuss the so-called boundary super algebra, which turns out to be central to the supersymmetric realization of the B-type Hecke algebra. Let us first recall

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the associated double row transfer matrix [2]

\[ t(\lambda) = \text{str}_0\{M_0 K_0^{(L)}(\lambda)\mathbb{T}_0(\lambda)\}. \] (3.1)

\( \mathbb{T} \) is a tensor representation of the reflection algebra

\[ \mathbb{T}_0(\lambda) = T_0(\lambda) K_0(\lambda) \tilde{T}_0(\lambda), \quad \tilde{T}(\lambda) = T^{-1}(-\lambda), \] (3.2)

where \( T \) is the monodromy matrix defined as [28]–[30]

\[ T_0(\lambda) = R_0N(\lambda) R_0N-1(\lambda) \ldots R_{02}(\lambda) R_{01}(\lambda). \] (3.3)

Also the matrix \( M \) is defined in the \( U_q(\mathfrak{gl}(m|n)) \) case as

\[ M = \sum_{k=1}^{N} q^{N-2k+1} q^{-2[k]+4} \sum_{i=1}^{k} e_{kk}. \] (3.4)

It is also shown, using the fact that \( T \) and \( K^{(L)} \) satisfy the reflection equation, that [2]

\[ [t(\lambda), t(\mu)] = 0, \] (3.5)

ensuring the integrability of the associated system. We consider here for simplicity the left boundary \( K^{(L)} \propto \mathbb{I} \).

Due to the fact that the \( R \)-matrix reduces to the permutation operator for \( \lambda = 0 \), a local Hamiltonian may be deduced from \( (\mathcal{H} \propto (dt(\lambda)/d\lambda)|_{\lambda=0}) \), which can be expressed in terms of supersymmetric representations of the B-type Hecke algebra (for more details see e.g. [8,11]). Also, from the asymptotic behavior of \( \mathbb{T} \), by keeping the leading contribution, i.e. \( \mathbb{T}(\lambda \to \infty) \sim \mathbb{T}^\pm + \mathcal{O}(e^{\pm 2\lambda}) \), we obtain the so-called boundary super algebra. More precisely, the elements \( T^\pm_{ab} \) are the boundary non-local charges, which form the boundary super algebra—a non-Abelian algebra in general—with exchange relations dictated by

\[ R^\pm_{12} T^\pm_{1} R^\pm_{21} T^\pm_{2} = T^\pm_{2} R^\pm_{21} T^\pm_{1} R^\pm_{12}. \] (3.6)

In this context, it may be shown that the boundary super algebra is central to the supersymmetric realization of the B-type Hecke algebra, i.e. (the proof goes along the same lines as in [31])

\[ [U_i, T^\pm_{ab}] = 0, \quad [U_0, T^\pm_{ab}] = 0, \quad i \in \{1, \ldots, N-1\}, \quad a, b \in \{1, \ldots, N\}, \] (3.7)

where apparently \( T^\pm_{ab} \in \text{End}((\mathbb{C}^N)^{\otimes N}) \). The boundary super algebra also turns out to be an exact symmetry of the double row transfer matrix; however, the explicit proof as well as a detailed discussion will be presented elsewhere (see [32]).

It is worth mentioning that, as in the case of the non-supersymmetric quantum algebras, appropriate choice of boundary conditions leads to known quantum algebras as boundary symmetries. For instance, if we choose \( K \propto \mathbb{I} \) then one may show that the boundary algebra is the \( U_q(\mathfrak{gl}(m|n)) \) (see [32] for details of the proof), which apparently is central to the supersymmetric representation of the A-type Hecke algebra (\( U_0 \) in this case is trivial).

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