Abstract—This letter presents a unified analytical framework for relay-assisted mixed FSO/RF transmission. In addition to accounting for different FSO detection techniques, the mathematical model offers a twofold unification of mixed FSO/RF systems by considering mixed Málaga-$\mathcal{M}$/$\kappa$-$\mu$ shadowed fading, which includes as special cases nearly all linear turbulence/fading models adopted in the open literature.

I. INTRODUCTION

RECENTLY, free-space optical (FSO) communications have gained a significant attention due to their advantages of higher bandwidth in unlicensed spectrum and higher throughput compared to their RF counterparts [1]. Hence, the gathering of both FSO and RF technologies arises as a promising solution for securing connectivity between the RF access network and the fiber-optic-based backbone network. As such, there has been prominent interest in mixed FSO/RF systems where RF transmission is used at one hop and FSO transmission at the other. Most contributions within this research line consider restrictive irradiance and channel gain probability density function (PDF) models for the FSO and RF links, respectively. The most commonly utilized models for the irradiance in FSO links are the lognormal and the Gamma-Gamma [2], [3]. Recently, a new generalized statistical model, the Málaga-$\mathcal{M}$ distribution, was proposed in [5] to model the irradiance fluctuation of an unbounded optical wavefront (plane or spherical waves) propagating through a turbulent medium under all irradiance conditions in homogeneous and isotropic turbulence. The Málaga-$\mathcal{M}$ distribution is a generalized distribution that unifies most statistical models proposed so far with its ability to better reflect a wider range of turbulence conditions [6]. On the RF side, previous works typically assume either Rayleigh or Nakagami-$m$ fading [3], [4], thereby lacking the flexibility to account for disparate signal propagation mechanisms as those characterized in 5G communications which will accommodate a wide range of usage scenarios with diverse link requirements. To bridge this gap in the literature, the $\kappa$-$\mu$ shadowed fading model, recently derived in [7], is an attractive proposition. In addition to presenting an excellent fit to the fading observed in a range of real-world applications (e.g., device-to-device, and body-centric fading channels [8]), the $\kappa$-$\mu$ shadowed fading encompasses several RF channel models such as Nakagami-$m$, Rayleigh, Ricean, $\kappa$-$\mu$ and shadowed Rician fading distributions. This new channel fading model offers far better and much more flexible representations of practical fading LOS (line of sight), NLOS (non-LOS), and shadowed channels than the Rayleigh and Nakagami-$m$ distributions.

This letter presents a unified mathematical framework to study the ergodic capacity of mixed FSO/RF amplify and forward (AF) relay systems operating over Málaga-$\mathcal{M}$ and $\kappa$-$\mu$ shadowed fading channels. The paper further accounts for pointing errors on the FSO link, while considering two detection techniques: heterodyne and intensity modulation/direct detection (IM/DD) [6].

II. CHANNEL AND SYSTEM MODELS

We consider a relay-assisted mixed FSO/RF transmission composed of both Málaga-$\mathcal{M}$ with pointing errors and $\kappa$-$\mu$ shadowed fading environments. The source communicates with the destination through an intermediate relay, able to activate both heterodyne and IM/DD detection techniques at the reception of the optical beam.

The FSO $(S-R)$ link irradiance is assumed to follow a Málaga-$\mathcal{M}$ distribution with pointing errors impairments for which the PDF of the irradiance, $I$, is given by [6, Eq.(5)]

$$f_I(x) = \frac{\xi^2 A}{x\Gamma(\alpha)} \sum_{k=1}^{\beta} b_k \Gamma(k) x^{k-1.3} \left[ \frac{\alpha\beta}{g\beta + \Omega} A_0 \right] \xi^2 + 1,$$

(1)

where $\xi$ is the ratio between the equivalent beam radius and the pointing error displacement standard deviation (i.e., jitter) at the relay (for negligible pointing errors $\xi \rightarrow \infty$), $A_0$ defines the pointing loss [2], $A = \alpha^\beta (g\beta/(g\beta + \Omega))^{\beta + \frac{\mu}{\kappa}} \gamma_{1-\frac{\mu}{\kappa}}$ and $b_k = (\frac{g\beta}{g\beta + \Omega})^{1-\frac{\mu}{\kappa}} (g\beta + \Omega)^{-\frac{\mu}{\kappa}} \Gamma(\alpha) / \Gamma(g)$, where $\alpha$, $\beta$, $g$ and $\Omega$ are the fading parameters related to the atmospheric turbulence conditions [5]. Moreover in [1], $G_{\text{pdf}}(\cdot)$ and $\Gamma(\cdot)$ stand for the Meijer-G function [9, Eq.(9.301)] and the incomplete gamma function [9, Eq.(8.310.1)], respectively.

It is worth highlighting that the $\mathcal{M}$ distribution unifies most of the proposed statistical models characterizing the optical irradiance in homogeneous and isotropic turbulence [5]. Hence both Gamma-Gamma and $\mathcal{K}$ models are special cases of the Málaga-$\mathcal{M}$ distribution, where they mathematically derives from [1] by setting $(\rho = 1, \Omega = 1)$ and $(\rho = 0, \Omega = 0$ or $\beta = 1)$, respectively [5 Table 1].

The RF $(R-D)$ link, experiences the $\kappa$-$\mu$ shadowed fading with non-negative real shape parameters $\kappa$, $\mu$ and $m$, for which the PDF of instantaneous SNR, $\gamma_2$, is given by [2, Eq.(4)]

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where $F_1(\cdot)$ is the confluent hypergeometric function and $\gamma_2 = E[\gamma_2]$. This fading model jointly includes large-scale and small-scale propagation effects, by considering that only the dominant components (DSCs) are affected by Nakagami-$m$ distributed shadowing. The shadowed $\kappa$-$\mu$ distribution is an extremely versatile fading model that includes as special cases nearly all linear fading models pertaining to LOS and NLOS scenarios, such as $\kappa$-$\mu$ ($m \to \infty$), $\eta$-$\mu$ ($\mu = 2\mu$, $\kappa = (1-\eta)/2\eta$, and $m = \mu$), Nakagami-$m$ ($\mu = m$ and $\kappa \to 0$), Rayleigh ($\mu = m = 1$ and $\kappa \to 0$), and Rice ($\mu = 1$, $\kappa = K$ and $m \to \infty$), to name a few [10 Table I].

Assuming AF relaying with channel state information (CSI), then the end-to-end SNR can be expressed as [4 Eq.(7)]

$$\gamma = \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2 + 1},$$

where $\gamma_1 = (A_0\xi^2(g + \Omega)/\xi^2 + 1) \frac{\mu_\text{heterodyne}}{\kappa \mu}$ and $\gamma_2$ is the instantaneous SNR of the FSO link $(S-R)$ with $r$ being the parameter that describes the detection technique at the relay (i.e., $r = 1$ is associated with heterodyne detection and $r = 2$ is associated with IM/DD) and $\mu_r$ refers to the electrical SNR of the FSO hop [6]. In particular, for $r = 1$, $\mu_1 = \mu_\text{heterodyne} = E[\gamma_1] = \gamma_1$ and for $r = 2$, $\mu_2 = \mu_\text{IM/DD} = \mu_1 A_0\xi^2(g + \Omega)/\xi^2 + 1$.

**III. ANALYSIS OF THE ERGODIC CAPACITY**

In this section, a new mathematical framework investigating the average capacity of the mixed FSO/RF transmission composed of both Málaga-$M$ with pointing errors and shadowed $\kappa$-$\mu$ fading environments and accounting for both detection techniques is presented. To the best of the author’s knowledge, there are few works that consider the average capacity of mixed FSO/RF systems, yet mostly considering the mixed Gamma-Gamma/Nakagami-$m$ fading [3, 4] and references therein. This paper complements and extends the efforts of [3-4] by unifying the ergodic capacity analysis for any turbulence/fading model under both types of detection techniques. Hereafter, we provide capacity formulas for the considered system by using the complementary moment generation function CMGF-based approach [13].

Form [13], the ergodic capacity can be computed as

$$C = \frac{\Delta E[\ln(1 + \gamma)]}{2\ln(2)} = \frac{1}{2\ln(2)} \int_0^\infty se^{-s} M_{\gamma_1}^{(c)}(s)M_{\gamma_2}^{(c)}(s) ds,$$

where $M_{\gamma_1}^{(c)}(s) = \int_0^\infty e^{-x} F_{\gamma_1}^{(c)}(x) dx$ stands for the CMGF with $F_{\gamma_1}^{(c)}(x)$ denoting the complementary cumulative distribution function (CCDF) of $\gamma_1$.

**Lemma 1:** The CMGF of the $S$-$R$ link SNR $\gamma_1$ under Málaga-$M$ distribution with pointing errors impairments for both detection techniques is obtained as

$$M_{\gamma_1}^{(c)}(s) = \frac{\xi^2 A \mu_r}{\Gamma(\alpha)B^s} \sum_{k=1}^\beta \frac{b_k}{\Gamma(k)} H_{1,3}^{4,1} \left[ \frac{\mu_r}{B_r^s} \left( \delta, \sigma \right) \right],$$

where $B = \alpha \beta \xi^2(g + \Omega)/[(gb + \Omega)(\xi^2 + 1)]$, $H_{m,n}^{p,q}[\cdot]$ is the Fox-H function [12 Eq.(1.2)] whereby $\delta = (1-r, r)$, $\sigma = (0, 1, \xi^2 - r, -r)$.

**Proof:** The CCDF of the irradiance of the FSO link is derived from (1) as

$$F_1(x) = \int_x^\infty f_1(y) dy = \frac{\xi^2 A \mu_r}{\Gamma(\alpha)} \sum_{k=1}^\beta \frac{b_k}{\Gamma(k)} g_{\gamma_2}^t \left[ \left( \frac{x}{\mu_r} \right)^{\xi^2 + 1}, \alpha, \delta, \sigma \right]$$

where $\alpha$ follows from applying [9]. In its turn, the CCDF of $\gamma_1$ is obtained after resoring to a simple variable transformation $\gamma_1 = (A_0\xi^2(g + \Omega)/\xi^2 + 1)\frac{\mu_\text{heterodyne}}{\kappa \mu}$ as

$$F_{\gamma_1}^{(c)}(x) = \frac{\xi^2 A \mu_r}{\Gamma(\alpha)} \sum_{k=1}^\beta \frac{b_k}{\Gamma(k)} g_{\gamma_2}^t \left[ \left( \frac{x}{\mu_r} \right)^{\xi^2 + 1}, \alpha, \delta, \sigma \right].$$

Applying the Laplace transform to the FSO link’s CCDF yields its CMGF as

$$M_{\gamma_2}^{(c)}(s) = \frac{\xi^2 A \mu_r}{\Gamma(\alpha)B^s} \sum_{k=1}^\beta \frac{b_k}{\Gamma(k)} H_{1,3}^{4,1} \left[ \frac{\mu_r}{B_r^s} \left( \delta, \sigma \right) \right]$$

where (a) follows after expressing the Meijer-G function in terms of Fox-H function by means of [12 Eq.(1.111)] and resorting to [12 Eq.(2.19)]; $(\Lambda_i, \nu_i) = (0, 1/r)$, $(\xi^2 + 1, 1)$; $(\Delta j, \tau_j) = (0, 1), (\xi^2 + 1, 1), (\beta, 1)$; and (b) is obtained after applying the identities in [12 Eqs.(1.58),(1.59)] and (1.60) where $\delta = (1 - \Delta j - r\tau_j, r\tau_j)$ and $\sigma = (1 - \Lambda_i - r\nu_i, r\nu_i)$.

On the RF side, the CMGF of $\gamma_2$ under shadowed $\kappa$-$\mu$ fading is given by

$$M_{\gamma_2}^{(c)}(s) = \frac{1 - M_{\gamma_2}^{(c)}(s)}{s} = \frac{1}{s} \left( 1 - \left( \frac{\theta_1 s + 1}{\theta_2 s + 1} \right)^{\mu - \mu} \right),$$

where (a) follows from the recent result in [7 Eq.(6)], where $\theta_1 = \frac{\gamma_2}{\mu(1+\kappa)}$ and $\theta_2 = \frac{\gamma_2 (\mu + m)}{\mu(1+\kappa)}$. Plugging the CMGF expressions obtained in (5) and (9) into (4) yields

$$C = \frac{\xi^2 A \mu_r}{2 ln(2) \Gamma(\alpha)B^s} \sum_{k=1}^\beta \frac{b_k}{\Gamma(k)} \int_0^\infty e^{-s} \left( 1 - \left( \frac{\theta_1 s + 1}{\theta_2 s + 1} \right)^{\mu - \mu} \right) H_{1,3}^{4,1} \left[ \frac{\mu_r}{B_r^s} \left( \delta, \sigma \right) \right] ds.$$

The integral in (10) reveals that the obtainment of $C$ in closed-form requires separate treatment in the case of integer-valued shape parameters of the $\kappa$-$\mu$ shadowed fading $m$ and $\mu$. However, for arbitrary $m$, $\mu$, we are unaware of any solution to (10) but should be able to provide a series based-approximation of the ergodic capacity as will be shown later.

**Proposition 1:** Assuming positive integers $m$ and $m$, with $m - \mu \geq 0$, then a closed-form expression for the ergodic capacity is obtained as
where (a) follows form applying the binomial expansion and all terms vanish except for the term when

\[\chi_l = \begin{cases} \frac{m}{l}, & \text{for } 0 \leq l \leq m - \mu, \\
\frac{m-\mu}{l}, & \text{for } l > m - \mu, \end{cases}\]

and \(H[\cdot, \cdot]\) denotes the Fox-H function (FHF) of two variables \[\text{(11)}\] also known as the bivariate FHF.

**Proof:** For integer-valued \(m, \mu\) with \(m - \mu \geq 0\), the CMGF of the RF link can be rewritten as

\[M^{(c)}_2(s) = \sum_{m=0}^{\infty} \frac{\chi_{l+1}^{l+1}}{(\theta s + 1)^m} \Gamma(\mu + 1) H_{1,1}^{1,1} \left[\begin{array}{c} 1-m+1 \\ 0 \end{array} \right], \] (12)

where (a) follows form applying the binomial expansion with \(\chi_l\) given after \[\text{(11)}\] and (b) follows after applying the transformation \(\Gamma(z)(1+z)^{-\alpha} = H_{1,1}^{1,1}[z]^{(1-\alpha,1)}\) in \[\text{(11)}\].

Plugging \[\text{(12)}\] and \[\text{(5)}\] into \[\text{(14)}\] yields

\[C = \sum_{m=0}^{\infty} \frac{\chi_{l+1}^{l+1}}{(\theta s + 1)^m} \Gamma(\mu + 1) H_{1,1}^{1,1} \left[\begin{array}{c} 1-m+1 \\ 0 \end{array} \right] ds. \] (13)

By resorting to \[\text{(11)}\] Eq.(2.2), \(C\) is obtained in \[\text{(11)}\].

**Corollary 1:** The ergodic capacity of mixed FSO/RF AF relay systems over Gamma-Gamma/Nakagami-m fading is obtained as

\[C = \sum_{m=0}^{\infty} \frac{\chi_{l+1}^{l+1}}{(\theta s + 1)^m} \Gamma(\mu + 1) H_{1,1}^{1,1} \left[\begin{array}{c} 1-m+1 \\ 0 \end{array} \right] ds. \] (13)

By resorting to \[\text{(11)}\] Eq.(2.2), \(C\) is obtained in \[\text{(11)}\].

**Proposition 2:** For positive integers \(m, \mu\), with \(m - \mu < 0\), the ergodic capacity is obtained as in \[\text{(15)}\] shown at the top of the next page.

**Proof:** For integer-valued \(m, \mu\) with \(m - \mu < 0\), the CMGF of \(\gamma_2\) in \[\text{(9)}\] can be rewritten in a more convenient form as

\[M^{(c)}_2(s) = \sum_{m=0}^{\infty} \frac{\chi_{l+1}^{l+1}}{(\theta s + 1)^m} \Gamma(\mu + 1) H_{1,1}^{1,1} \left[\begin{array}{c} 1-m+1 \\ 0 \end{array} \right], \] (12)

where (a) follows form applying a partial fraction expansion \[\text{(9)}\] Eq.(2.102) with \(\Psi_i = \frac{(m-i+1)(\theta s + 1)^m}{\theta s + 1}\) and \(\Phi_j = (-1)^{m-j}(\mu-m-j-2)!\theta s + 1\). Plugging \[\text{(16)}\] and \[\text{(5)}\] into \[\text{(14)}\] and following the same steps as in \[\text{(11)}\] yield the result after some manipulations.

**Corollary 2:** The ergodic capacity of mixed FSO/RF AF relay systems over Gamma-Gamma/\(\kappa\)-\(\mu\) fading is obtained as

\[C = \sum_{m=0}^{\infty} \frac{\chi_{l+1}^{l+1}}{(\theta s + 1)^m} \Gamma(\mu + 1) H_{1,1}^{1,1} \left[\begin{array}{c} 1-m+1 \\ 0 \end{array} \right] ds. \] (13)

By resorting to \[\text{(11)}\] Eq.(2.2), \(C\) is obtained in \[\text{(11)}\].

**Proposition 3:** The ergodic capacity of mixed FSO/RF AF relay system over Gamma-Gamma/\(\kappa\)-\(\mu\) fading with arbitrary \(m, \mu\) is obtained as in \[\text{(18)}\] at the top of the next page.

**Proof:** Under shadowed \(\kappa\)-\(\mu\) fading with arbitrary real positive valued shape parameters \(m, \mu\) and \(\kappa, \mu\), the CCDF of \(\gamma_2\) can be expressed as a series expansion as follows \[\text{(14)}\] Eq.(64)

\[F^{(c)}(x) = \frac{\Gamma(\mu, x)}{\Gamma(\mu)} - \sum_{n=1}^{\infty} \frac{C_n}{(n + \mu)} x^n e^{-x} L_{n-1}(x), \] (19)

where \(C_n = \sum_{j=0}^{n} \binom{\mu}{j} (\mu - j) \frac{E_{\gamma_2}^j}{\Gamma(\mu)\theta_j}\) with \(E_{\gamma_2} = (\mu - j) \frac{E_{\gamma_2}^j}{\Gamma(\mu)\theta_j}\) and \(\theta_j = (\mu - j)/(\Gamma(\mu)\theta_j)\).
where $2F_1(a; b; c; x)$ denotes the Gauss hypergeometric function [9] Eq. (9.100.1), and $L^0_n(\cdot)$ denotes the generalized Laguerre polynomials [9] Eq.(8.970.1). Applying the Laplace transform to [19] and resorting to [9] Eq.(6.451.2) and [9] Eq.(7.414.8]) yield the CMGF of $\gamma_2$ as

$$M_{\gamma_2}(s) = \frac{1 - (s + 1)^{-\mu}}{s} - \sum_{n=0}^{\infty} C_{n+1} \frac{(a + 1)n^n}{(n+1)!} \cdot (20)$$

Finally, plugging (5) and (20) into (4) and using the identity $\Gamma(\alpha)(1 - (1 + z)^{-\alpha}) = H_{2,2}^1 \left[ \left( \begin{array}{c} \alpha \\ 1,1,1,0,1 \end{array} \right) \right]$ while resorting to [12] Eq.(2.19) and [11] Eq.(2.2) yield the desired result after following the same manipulations as in [13].

IV. NUMERICAL RESULTS

Fig.1(a) depicts the ergodic capacity of mixed FSO/RF relay systems in Málaga-$\mathcal{M}$ and shadowed $\kappa$-$\mu$ fading channels for both heterogeneous and IM/DD detection at the relay. In the legend, please note that we have identified some particular fading distribution cases that simply stem from the general $\kappa$-$\mu$ shadowed fading scenario. We observe that the performance trends obtained by Monte-Carlo simulations are in perfect agreement with those calculated by the analysis of Section III.

Fig.1(b) plots the ergodic capacity of mixed FSO/RF relay systems over Málaga-$\mathcal{M}$/shadowed $\kappa$-$\mu$ fading channels versus $\kappa$. We observe that larger $\kappa$ improves the ergodic capacity when $m > \mu$. However, when $m < \mu$, increasing the parameter $\kappa$ is detrimental for capacity. When $m = \mu$ the shadowed $\kappa$-$\mu$ distribution boils down to the Nakagami-$m$ distribution, [10], whence the ergodic capacity’s independency of $\kappa$. Fig.1(b) also shows that increasing $\mu$, which denotes the number of multipath clusters [7], [10], reduces the fading severity of the small-scale propagation effects thereby improving the ergodic capacity.

Fig.1(c) investigates the impacts of pointing errors on the system performance when the RF link is subject to Rician shadowed fading distribution for $\kappa = 5$, $\mu = 1$, and $m = 2$. Once again we highlight the good match between analytical and simulation results. As expected, the ergodic capacity deteriorates by decreasing the pointing error displacement standard deviation, i.e., for smaller $\xi$.

V. CONCLUSION

We have presented a unified analytical framework for relay-assisted mixed FSO/RF systems that remarkably accommodates generic turbulence/fading models including Málaga-$\mathcal{M}$ with pointing errors and shadowed $\kappa$-$\mu$ distribution that account for shadowed LOS and NLOS scenarios. The results demonstrate the unification of various FSO turbulent/RF fading scenarios into a single closed-form expression for the ergodic capacity while accounting for both IM/DD and heterogeneous detection techniques at the relay.

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