The C-compact-open topology on function spaces

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1. Introduction

The set-open topology (the $\lambda$-open topology) is a generalization of the compact-open topology and of the topology of pointwise convergence. This topology was first introduced by Arens and Dugundji [1]. Let $\lambda$ be a collection of some subsets of a Tychonoff space $X$, then the set-open topology on $C(X)$ generated by $\lambda$ is as follows. All sets of the form $\{F, U\} = \{f \in C(X) : f(F) \subseteq U\}$, where $F \in \lambda$ and $U$ is an open subset of real line $\mathbb{R}$, form a subbase of the $\lambda$-open topology on $C(X)$.

The topology of uniform convergence is given by a base at each point $f \in C(X)$. This base consists of all sets $\{g \in C(X) : \sup_{x \in X} |g(x) - f(x)| < \varepsilon\}$. The topology of uniform convergence on elements of a family $\lambda$ (the $\lambda$-topology), where $\lambda$ is a fixed family of non-empty subsets of the set $X$, is a natural generalization of this topology. All sets of the form $\{g \in C(X) : \sup_{x \in F} |g(x) - f(x)| < \varepsilon\}$, where $F \in \lambda$ and $\varepsilon > 0$, form a base of the $\lambda$-topology at a point $f \in C(X)$.

Note that a $\lambda$-open topology coincides with a $\lambda$-topology, when the family $\lambda$ consists of all finite (compact, countable compact, pseudocompact, sequentially compact, C-compact) subsets of $X$. Therefore $C(X)$ with the topology of pointwise convergence (compact-open, countably compact-open, sequentially compact-open, pseudocompact-open, C-compact-open topology) is a topological vector space (TVS).

Moreover, if a $\lambda$-open topology coincides with a $\lambda$-topology, then $\lambda$ consists of C-compact subsets of space $X$ and the space $C_\lambda(X)$ is a topological algebra under the usual operations of addition and multiplication (and multiplication by scalars).

2. Main definitions and notation

In this paper, we consider the space $C(X)$ of all real-valued continuous functions defined on a Tychonoff space $X$. We denote by $\lambda$ a family of non-empty subsets of the set $X$. We use the following notation for various topological spaces with
the underlying set $C(X)$:

$$C_\lambda(X)$$ for the $\lambda$-open topology,

$$C_{\lambda,u}(X)$$ for the $\lambda$-topology.

The elements of the standard subbases of the $\lambda$-open topology and $\lambda$-topology will be denoted as follows:

$$[F, U] = \{ f \in C(X): f(F) \subseteq U \},$$

$$(f, F, \varepsilon) = \{ g \in C(X): \sup_{x \in F} |f(x) - g(x)| < \varepsilon \},$$ where $F \in \lambda$, $U$ is an open subset of $\mathbb{R}$ and $\varepsilon > 0$.

If $X$ and $Y$ are any two topological spaces with the same underlying set, then we use the notation $X = Y$, $X \leq Y$, and $X < Y$ to indicate, respectively, that $X$ and $Y$ have the same topology, that the topology on $Y$ is finer than or equal to the topology on $X$, and that the topology on $Y$ is strictly finer than the topology on $X$.

The closure of a set $A$ will be denoted by $\overline{A}$; the symbol $\emptyset$ stands for the empty set. As usual, $f(A)$ and $f^{-1}(A)$ are the image and the complete preimage of the set $A$ under the mapping $f$, respectively. The constant zero function defined on $X$ is denoted by $f_0$. We call it the constant zero function in $C(X)$.

We denote by $\mathbb{R}$ the real line with the natural topology.

We recall that a subset of $X$ is compact in $C(X)$ is called a zero-set. A subset $O$ of a space $X$ is called functionally open (or a cozero-set) if $X \setminus O$ is a zero-set. A family $\lambda$ of non-empty subsets of a topological space $(X, \tau)$ is called a $\pi$-network for $X$ if for any non-empty open set $U \in \varpi$ there exists $A \in \lambda$ such that $A \subseteq U$.

Throughout this paper, a family $\lambda$ of non-empty subsets of the set $X$ is a $\pi$-network. This condition is equivalent to the space $C_\lambda(X)$ being a Hausdorff space.

Recall that a subset $A$ of a space $X$ is a $C$-compact subset of $X$ if, for any real-valued function $f$ continuous on $X$, the set $f(A)$ is compact in $\mathbb{R}$. Note that if a subset $A$ of $X$ is such that every image $f(A)$ under a continuous real function $f$ on $X$ is closed in $\mathbb{R}$, then every continuous real image of $A$ is in fact closed and bounded in $\mathbb{R}$; hence compact. Indeed, let $f(A)$ be closed and unbounded in $\mathbb{R}$. We take $h(t) = \arctg(t)$. Then, $h(f(A))$ is not closed. So the notion of $C$-compactness ("every image compact") in fact reduces to "every image closed in $\mathbb{R}$".

Note (see Theorem 3.9 in [8]) that the set $A$ is a $C$-compact subset of $X$ if and only if every countable functionally open (in $X$) cover of $A$ has a finite subcover.

Let $\lambda$ be a family of non-empty $C$-compact subsets of the set $X$ and $\overline{\lambda} = \{ A: A \in \lambda \}$, then note that the same set-open topology is obtained if $\lambda$ is replaced by $\overline{\lambda}$. This is because for each $f \in C(X)$ we have $f(\overline{A}) \subseteq \overline{f(A)} = f(A)$. Consequently, $C_\overline{\lambda}(X) = C_\lambda(X)$. From now on, $\lambda$ denotes a family of non-empty closed $C$-compact subsets of the set $X$.

The set-open topology does not change when $\lambda$ is replaced with the finite unions of its elements. Therefore we assume that $\lambda$ is closed under finite unions of its elements.

The remaining notation can be found in [2].

3. Topological-algebraic properties of function spaces

Interest in studying the $C$-compact-open topology was generated by a Theorem 3.3 in [7] which characterizes some topological-algebraic properties of the set-open topology. It turns out that if $C_\lambda(X)$ is a paratopological group (TVS, locally convex TVS) then the family $\lambda$ consists of $C$-compact subsets of $X$.

Given a family $\lambda$ of non-empty subsets of $X$, let $\lambda(C) = \{ A \in \lambda: \text{for every } C \text{-compact subset } B \text{ of the space } X \text{ with } B \subseteq A, \text{the set } [B, U] \text{ is open in } C_\lambda(X) \text{ for any open set } U \text{ of the space } R \}$.

Let $\lambda_m$ be a maximal $\pi$-network (with respect to inclusion) of closed sets such that $C_{\lambda_m}(X) = C_\lambda(X)$. Note that a family $\lambda_m$ is simply the union of all such families $\mu$ that $C_\mu(X) = C_\lambda(X)$.

A family $\lambda$ of $C$-compact subsets of $X$ is said to be hereditary with respect to $C$-compact subsets if it satisfies the following condition: whenever $A \in \lambda$ and $B$ is a $C$-compact (in $X$) subset of $A$, then $B \in \lambda$ also.

We look at the properties of the family $\lambda$ which imply that the space $C_\lambda(X)$ is a topological algebra under the usual operations of addition and multiplication (and multiplication by scalars).

The following theorem is a generalization of Theorem 3.3 in [7].

**Theorem 3.1.** For a space $X$, the following statements are equivalent.

1. $C_\lambda(X) = C_{\lambda,u}(X)$.
2. $C_\lambda(X)$ is a paratopological group.
3. $C_\lambda(X)$ is a topological group.
4. $C_\lambda(X)$ is a topological vector space.
5. $C_\lambda(X)$ is a locally convex topological vector space.
6. $C_\lambda(X)$ is a topological ring.
7. $C_\lambda(X)$ is a topological algebra.
8. $\lambda$ is a family of $C$-compact sets and $\lambda = \lambda(C)$.
9. $\lambda_m$ is a family of $C$-compact sets and it is hereditary with respect to $C$-compact subsets.

**Proof.** Equivalence of the statements (1), (3), (4), (5) and (8) proved in [7, Theorem 3.3].

Note that the proof of Lemmas 3.1 and 3.2 in [7] used only the condition that the space $X$ is a paratopological space. Thus (2) $\Rightarrow$ (8).

(8) $\Rightarrow$ (7). As (8) $\Leftrightarrow$ (4), we only need to show that the operation of multiplication is continuous. Indeed, let $\beta$ be the neighborhood filter of the zero function in $C(X)$. Let $W = [A, V] \in \beta$, where $A \in \lambda$ and $V$ is an open set of the space $\mathbb{R}$. Then there is an open set $V_1$ such that $V_1 \cup V_1 \subseteq V$. Show that $W_1 = [A, V_1] \subseteq W$. Indeed $W_1 = [V_1 \cup V_1, V] = [f * g : f \in V_1, g \in V_1] = [f * g : f(A) \subseteq V_1 \text{ and } g(A) \subseteq V_1]$. Clearly $f(x) \ast g(x) \subseteq V_1 \ast V_1$ for each $x \in A$. Therefore $(f \ast g)(A) \subseteq V$ and $W_1 \ast W_1 \subseteq V$.

It remains to prove that if $W = [A, V] \in \beta$ and $f \in C(X)$ then there is an open set $V_1 \ni 0$ such that $f(A) \ast V_1 \subseteq V$ and $V_1 \ast f(A) \subseteq V$. Indeed let $g = f * h$ and $g_1 = h_1 * f$ where $h, h_1 \in W_1$. Then $g(x) = f(x) \ast h(x) \in f(A) \ast V_1$ and $g_1(x) = h_1(x) \ast f(x) \subseteq V_1 \ast f(A)$ for each $x \in A$. Note that $g(A) \subseteq V$ and $g_1(A) \subseteq V$.

(8) $\Rightarrow$ (9). Since $C_{\lambda_m}(X) = C_{\lambda}(X)$, the space $C_{\lambda_m}(X)$ is a topological group and $\lambda_m$ is a family of $C$-compact sets and consequently, $\lambda_m = \lambda_m(C)$. But if the set $[B, U]$ is open in $C_{\lambda_m}(X)$ for any open set $U$ of the space $\mathbb{R}$ then $B \in \lambda_m$.

The remaining implications are obvious and follow from Theorem 3.3 in [7] and the definitions. $\square$

### 4. Comparison of topologies

In this section, we compare the $C$-compact-open topology with several well-known and lesser-known topologies.

We use the following notations to denote the particular families of $C$-compact subsets of $X$.

- $F(X)$ — the collection of all finite subsets of $X$.
- $MC(X)$ — the collection of all metrizable compact subsets of $X$.
- $K(X)$ — the collection of all compact subsets of $X$.
- $SC(X)$ — the collection of all sequentially compact subsets of $X$.
- $CC(X)$ — the collection of all countably compact subsets of $X$.
- $PS(X)$ — the collection of all pseudocompact subsets of $X$.
- $RC(X)$ — the collection of all countable-compact subsets of $X$.

Note that $F(X) \subseteq MC(X) \subseteq K(X) \subseteq SC(X) \subseteq PS(X) \subseteq RC(X)$ and $MC(X) \subseteq SC(X) \subseteq CC(X)$. When $\lambda = F(X)$, $MC(X)$, $K(X)$, $SC(X)$, $CC(X)$, $PS(X)$ or $RC(X)$, we call the corresponding $\lambda$-open topologies on $C(X)$ point-open, metrizable compact-open, compact-open, sequentially compact-open, countable-compact-open, pseudocompact-open and $C$-compact-open, respectively. The corresponding spaces are denoted by $C_{\lambda}(X)$, $C_{mc}(X)$, $C_{k}(X)$, $C_{cc}(X)$, $C_{cc}(X)$, $C_{ps}(X)$ and $C_{rc}(X)$, respectively.

We obtain from Theorem 3.1 the following result.

**Theorem 4.1.** For any space $X$ and $\lambda \in \{F(X), MC(X), K(X), SC(X), CC(X), PS(X), RC(X)\}$, the $\lambda$-open topology on $C(X)$ is the same as the topology of uniform convergence on elements of a family $\lambda$, that is, $C_{\lambda}(X) = C_{\lambda, u}(X)$. Moreover, $C_{\lambda}(X)$ is a Hausdorff locally convex topological vector space (TVS).

When $X$ is equipped with the topology of uniform convergence on $X$, we denote the corresponding space by $C_u(X)$.

**Theorem 4.2.** For any space $X$,

$$
C_p(X) \subseteq C_{mc}(X) \subseteq C_k(X) \subseteq C_{cc}(X) \subseteq C_{ps}(X) \subseteq C_{rc}(X) \subseteq C_u(X)
$$

and

$$
C_{mc}(X) \subseteq C_{sc}(X) \subseteq C_{cc}(X).
$$

Now we determine when these inequalities are equalities and give examples to illustrate the differences.

**Example 4.3.** Let $X$ be the set of all countable ordinals $[\alpha : \alpha < \omega_1]$ equipped with the order topology. The space $X$ is sequentially compact and collectionwise normal, but not compact. For this space $X$, we have $C_{cc}(X) > C_k(X)$.

Indeed, let $f = f_\alpha$ and $U = (\alpha, 1)$. Consider the neighborhood $[X, U]$ of $f$. Assume that there are a family of neighborhoods $([A_i, U_i])_{i=1}^n$, where $A_i$ is compact, and $f \in \bigcap_{i=1}^n[A_i, U_i] \subseteq [X, U]$. Then $\exists \alpha < \omega_1$ such that $\forall \beta < \alpha$. Define function $g : g(\beta) = 0$ for $\beta \leq \alpha$ and $g(\beta) = 1$ for $\beta > \alpha$. Then $g \in \bigcap_{i=1}^n[A_i, U_i]$, but $g \notin [X, U]$, a contradiction.

Note that for this space $X$, we have:

$$
C_p(X) < C_{mc}(X) < C_k(X) < C_{cc}(X) = C_{cc}(X) = C_{ps}(X) = C_{rc}(X) = C_u(X).
$$
Example 4.4. Let $Y = \beta\mathbb{N}$ be Stone–Čech compactification of natural numbers $\mathbb{N}$. Note that every sequentially compact subset of $\beta\mathbb{N}$ is finite. For this space $Y$, we have:

$$C_p(Y) = C_{mc}(Y) = C_{sc}(Y) < C_k(Y) = C_{cc}(Y) = C_{ps}(Y) = C_{te}(Y) = C_u(Y).$$

Example 4.5. Let $Z = X \oplus Y$ where $X$ is the space of Example 4.3 and $Y$ is the space of Example 4.4. Then the sequentially compact-open topology is incomparable with the compact-open topology on the space $C(Z)$.

Example 4.6. Let $X = I^c$ be the Tychonoff cube of weight $c$. The space $X$ is compact and contains a dense sequentially compact subset. Thus, we have:

$$C_p(X) < C_{mc}(X) < C_k(X) = C_{sc}(X) = C_{cc}(X) = C_{ps}(X) = C_{te}(X) = C_u(X).$$

Example 4.7. Let $X = \omega_1 + 1$ be the set of all ordinals $\leq \omega_1$ equipped with the order topology. The space $X$ is compact and sequentially compact but not metrizable. Then, for space $X$ we have:

$$C_p(X) < C_{mc}(X) < C_k(X) = C_{sc}(X) = C_{cc}(X) = C_{ps}(X) = C_{te}(X) = C_u(X).$$

The following example is an example of the space in which every sequentially compact and every compact subset is finite.

Example 4.8. Let $K_0 = \mathbb{N}$. By using transfinite induction, we construct a subspace of $\beta\mathbb{N}$. Suppose that $K_\beta \subset \beta\mathbb{N}$ is defined for each $\beta < \alpha$ and $|K_\beta| \leq c$. Then for each $A \in [\bigcup_{\beta<\alpha}K_\beta]^{\omega_1}$ choose $x_A$ such that $\{x_A: A \in \bigcup_{\beta<\alpha}K_\beta]^{\omega_1}\}$. The space $M = \bigcup_{\beta<\alpha}K_\beta$ is a countably compact space in which every sequentially compact and every compact subset is finite. Thus, we have:

$$C_p(M) = C_{mc}(M) = C_{sc}(M) = C_k(M) < C_{cc}(M) = C_{ps}(M) = C_{te}(M) = C_u(M).$$

Example 4.9. Let $\mathcal{M}$ be a maximal infinite family of infinite subsets of $\mathbb{N}$ such that the intersection of any two members of $\mathcal{M}$ is finite, and let $\Psi = \mathbb{N} \cup \mathcal{M}$, where a subset $U$ of $\Psi$ is defined to be open provided that for any set $M \in \mathcal{M}$, if $M \in U$ then there is a finite set $F$ of $M$ such that $\{M\} \cup M \setminus F \subset U$. The space $\Psi$ is then a first-countable pseudocompact Tychonoff space that is not countably compact. The space $\Psi$ is due independently to J. Isbell and S. Mrówka.

Every compact, sequentially compact, countable compact subset of $\Psi$ has the form $\bigcup_{i=1}^m([x_i] \cup (x_i \setminus S_i)) \cup S$, where $x_i \in E$, $|S_i| < \omega$, $|S| < \omega$. We thus obtain the following relations:

$$C_p(\Psi) < C_{mc}(\Psi) = C_{sc}(\Psi) = C_k(\Psi) = C_{cc}(\Psi) < C_{ps}(\Psi) = C_{te}(\Psi) = C_u(\Psi).$$

Example 4.10. Let $X = \beta\mathbb{N} \oplus (\omega_1 + 1) \oplus M \oplus \Psi$, where $M$ is the space of Example 4.8 and $\Psi$ is the space of Example 4.9. We have the following relations:

$$C_p(X) < C_{mc}(X) < C_{sc}(X) < C_k(X) < C_{cc}(X) < C_{ps}(X) = C_{te}(X) = C_u(X).$$

Example 4.11. Let $G = \omega_1 \oplus M \oplus \Psi \oplus I^c$, where $M$ is the space of Example 4.8 and $\Psi$ is the space of Example 4.9. We have the following relations:

$$C_p(G) < C_{mc}(G) < C_k(G) < C_{sc}(G) < C_{cc}(G) < C_{ps}(G) = C_{te}(G) = C_u(G).$$

Example 4.12. Let $Y = [0, \omega_2] \times [0, \omega_1] \setminus \{(\omega_2, \omega_1)\}$, with the topology $\tau$ generated by declaring open each point of $[0, \omega_2] \times [0, \omega_1]$, together with the sets $U_p(\beta) = \{\beta, s\}: s \in ([0, \omega_1]) \setminus P$, where $P$ is finite and $\beta, \omega_1 \notin P$ and $V_\alpha(s) = \{\gamma, s\}: \alpha < \gamma < \omega_2$. Let $A = \{(\omega_2, s): 0 \leq s < \omega_1\}$ and $f \in C(Y)$.

Suppose that $f(A)$ is not a closed set, then there are $c \in f(\bar{A}) \setminus f(A)$ and sequence $\{a_n\} \subset A$ such that $\{f(a_n)\} \to c$. Since $a_n = (\omega_2, s_n)$, there is $\alpha_n$ such that $f(\alpha, s_n) = f(a_n)$ for each $\alpha > \alpha_n$. Moreover, there exists $\beta \in \omega_2$, such that $f(\alpha, s) = f(\omega_2, s)$ for each $s \in [0, \omega_1]$ and $\alpha > \beta$. Clearly $f(\beta, \omega_1) = c$. Then there exists $\delta \in \omega_1$, such that $f(\beta, s) = c$ for each $s \geq \delta$. It follows that $f(\omega_2, s) = c$, but $(\omega_2, s) \in A$ and $c \notin f(A)$, a contradiction. Thus, the set $A$ is a $C$-compact subset of the space $Y$. 
Let $B$ be a non-empty pseudocompact subset of $Y$. Since $\alpha \times [0, \omega_1]$ is a clopen set (functionally open) for each $\alpha < \omega_2$, $((0, \omega_2) \times \{s\}) \cap B$ has at most a finite number of points for each $s \leq \omega_1$. It follows that $B$ is a compact subset of $Y$.

As $A$ is infinite set and closed and the pseudocompact subsets of $Y$ are compact and have at most a finite intersection with $A$, $A$ provides an example of a $C$-compact subset which is not contained in any closed pseudocompact subset of $Y$. Since $Y$ has infinite compact subsets, for this space we have

$$C_k(Y) = C_{ps}(Y) < C_{rc}(Y).$$

**Example 4.13.** Let $Z = Y \oplus G$, where $Y$ is the space of Example 4.12 and $G$ is the space of Example 4.11. We have the following relations:

$$C_p(Z) < C_{mc}(Z) < C_k(Z) < C_{sc}(Z) < C_{cc}(Z) < C_{ps}(Z) < C_{rc}(Z).$$

Recall that a space $X$ is called submetrizable if $X$ admits a weaker metrizable topology.

Note that for a subset $A$ in a submetrizable space $X$, the following are equivalent:

1. $A$ is metrizable compact,
2. $A$ is compact,
3. $A$ is sequentially compact,
4. $A$ is countably compact,
5. $A$ is pseudocompact,
6. $A$ is C-compact subset of $X$.

**Theorem 4.14.** Let $X$ be a submetrizable space, then

$$C_{mc}(X) = C_k(X) = C_{sc}(X) = C_{cc}(X) = C_{ps}(X) = C_{rc}(X).$$

Similarly to Corollary 3.7 in [3] on the bounded-open topology we have

**Theorem 4.15.** For every space $X$,

1. $C_k(X) = C_{rc}(X)$ iff every closed $C$-compact subset of $X$ is compact.
2. $C_{rc}(X) = C_{u}(X)$ iff $X$ is pseudocompact.

**Proof.** (1) Note that for a subset $A$ of $X$, $(f, A, \varepsilon) \subseteq (f, A, \varepsilon)$. So if every closed $C$-compact subset of $X$ is compact, then $C_{rc}(X) \subseteq C_k(X)$. Consequently, in this case, $C_{rc}(X) = C_k(X)$.

Conversely, suppose that $C_k(X) = C_{rc}(X)$ and let $A$ be any closed $C$-compact subset of $X$. So $(0, A, 1)$ is open in $C_k(X)$ and consequently, there exist a compact subset $K$ of $X$ and $\varepsilon > 0$ such that $(0, K, \varepsilon) \subseteq (0, A, 1)$. If possible, let $x \in A \setminus K$. Then there exists a continuous function $g : X \mapsto [0, 1]$ such that $g(x) = 1$ and $g(y) = 0 \ \forall y \in K$. Note that $g \in (0, K, \varepsilon) \setminus (0, A, 1)$ and we arrive at a contradiction. Hence, $A \subseteq K$ and consequently, $A$ is compact.

(2) First, suppose that $X$ is pseudocompact. So for each $f \in C(X)$ and each $\varepsilon > 0$, $(f, X, \varepsilon)$ is a basic open set in $C_{rc}(X)$ and consequently, $C_{u}(X) = C_{rc}(X)$.

Now let $C_{rc}(X) = C_{u}(X)$. Since $(0, X, 1)$ is a basic neighborhood of the constant zero function $0$ in $C_{u}(X)$, there exist a $C$-compact subset $A$ of $X$ and $\varepsilon > 0$ such that $(0, A, \varepsilon) \subseteq (0, X, 1)$. As before, by using the complete regularity of $X$, it can be shown that we must have $X = A$. But the closure of a $C$-compact set is also a $C$-compact set. Hence, $X$ is pseudocompact. \)

Note that for a closed subset $A$ in a normal Hausdorff space $X$, the following are equivalent:

1. $A$ is countably compact,
2. $A$ is pseudocompact,
3. $A$ is $C$-compact subset of $X$.

**Corollary 4.16.** For any normal Hausdorff space $X$, $C_k(X) = C_{rc}(X)$ iff every closed countably compact subset of $X$ is compact.

5. Submetrizable and metrizable

One of the most useful tools in function spaces is the following concept of induced map. If $f : X \mapsto Y$ is a continuous map, then the induced map of $f$, denoted by $f^* : C(Y) \mapsto C(X)$ is defined by $f^*(g) = g \circ f$ for all $g \in C(Y)$. 


Recall that a map \( f : X \mapsto Y \), where \( X \) is any non-empty set and \( Y \) is a topological space, is called almost onto if \( f(X) \) is dense in \( Y \).

**Theorem 5.1.** Let \( f : X \mapsto Y \) be a continuous map between two spaces \( X \) and \( Y \). Then

1. \( f^* : C_{rc}(Y) \mapsto C_{rc}(X) \) is continuous;
2. \( f^* : \mathcal{C}(Y) \mapsto \mathcal{C}(X) \) is one-to-one if and only if \( f \) is almost onto;
3. if \( f^* : \mathcal{C}(Y) \mapsto C_{rc}(X) \) is almost onto, then \( f \) is one-to-one.

**Proof.** (1) Suppose \( g \in C_{rc}(Y) \). Let \( (f^*(g), A, \varepsilon) \) be a basic neighborhood of \( f^*(g) \) in \( C_{rc}(X) \). Then \( f^*(\langle g, f(A), \varepsilon \rangle) \subseteq \langle f^*(g), A, \varepsilon \rangle \) and consequently, \( f^* \) is continuous. (2) and (3) See Theorem 2.2.6 in [6].

**Remark 5.2.** (1) If a space \( X \) has a \( G_\delta \)-diagonal, that is, if the set \( \{(x, x) : x \in X\} \) is a \( G_\delta \)-set in the product space \( X \times X \), then every point in \( X \) is a \( G_\delta \)-set. Note that every metrizable space has a zero-set diagonal. Consequently, every submetrizable space has also a zero-set-diagonal.

(2) Every compact set in a submetrizable space is a \( G_\delta \)-set. A space \( X \) is called an \( E_\delta \)-space if every point in the space is a \( G_\delta \)-set. So the submetrizable spaces are \( E_\delta \)-spaces.

For our next result, we need the following definitions.

**Definition 5.3.** A completely regular Hausdorff space \( X \) is called \( \sigma \)-\( C \)-compact if there exists a sequence \( \{A_n\} \) of \( C \)-compact sets in \( X \) such that \( X = \bigcup_{n=1}^{\infty} A_n \). A space \( X \) is said to be almost \( \sigma \)-\( C \)-compact if it has a dense \( \sigma \)-\( C \)-compact subset.

**Theorem 5.4.** For any space \( X \), the following are equivalent.

1. \( C_{rc}(X) \) is submetrizable.
2. Every \( C \)-compact subset of \( C_{rc}(X) \) is a \( G_\delta \)-set in \( C_{rc}(X) \).
3. Every compact subset of \( C_{rc}(X) \) is a \( G_\delta \)-set in \( C_{rc}(X) \).
4. \( C_{rc}(X) \) is an \( E_\delta \)-space.
5. \( X \) is almost \( \sigma \)-\( C \)-compact.
6. \( C_{rc}(X) \) has a zero-set-diagonal.
7. \( C_{rc}(X) \) has a \( G_\delta \)-diagonal.

**Proof.** (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) are all immediate.

(4) \( \Rightarrow \) (5). If \( C_{rc}(X) \) is an \( E_\delta \)-space, then the constant zero function 0 defined on \( X \) is a \( G_\delta \)-set. Let \( \{0\} = \bigcap_{n=1}^{\infty} (0, A_n, \varepsilon) \) where each \( A_n \) is a \( C \)-compact subset in \( X \) and \( \varepsilon > 0 \). We claim that \( X = \bigcup_{n=1}^{\infty} A_n \).

Suppose that \( x_0 \in X \setminus \bigcup_{n=1}^{\infty} A_n \). So there exists a continuous function \( f : X \mapsto [0, 1] \) such that \( f(x) = 0 \) for all \( x \in \bigcup_{n=1}^{\infty} A_n \) and \( f(x_0) = 1 \). Since \( f(x) = 0 \) for all \( x \in A_n, f \in (0, A_n, \varepsilon) \) for all \( n \) and hence, \( f \in \bigcap_{n=1}^{\infty} (0, A_n, \varepsilon) = \{0\} \). This means \( f(x) = 0 \) for all \( x \in X \). But \( f(x_0) = 1 \). By this contradiction, we conclude that \( X \) is almost \( \sigma \)-\( C \)-compact.

(5) \( \Rightarrow \) (1). By Theorem 4.10 in [4] and Theorem 5.1.

By Remark (1) \( \Rightarrow \) (6) \( \Rightarrow \) (7) \( \Rightarrow \) (4). □

**Corollary 5.5.** Suppose that \( X \) is almost \( \sigma \)-\( C \)-compact. If \( K \) is a subset of \( C_{rc}(X) \), then the following are equivalent.

1. \( K \) is metrizable compact.
2. \( K \) is compact.
3. \( K \) is sequentially compact.
4. \( K \) is countably compact.
5. \( K \) is pseudocompact.
6. \( K \) is \( C \)-compact subset of \( C_{rc}(X) \).

A space \( X \) is said to be of (pointwise) countable type if each (point) compact set is contained in a compact set having countable character.

A space \( X \) is a \( q \)-space if for each point \( x \in X \), there exists a sequence \( \{U_n : n \in \mathbb{N}\} \) of neighborhoods of \( x \) such that if \( x_0 \in U_n \) for each \( n \), then \( \{x_0 : n \in \mathbb{N}\} \) has a cluster point. Another property stronger than being a \( q \)-space is that of being an \( M \)-space, which can be characterized as a space that can be mapped onto a metric space by a quasi-perfect map (a continuous closed map in which inverse images of points are countably compact). Both a space of pointwise countable type and an \( M \)-space are \( q \)-spaces.
Theorem 5.6. For any space $X$, the following are equivalent.

1. $C_{rc}(X)$ is metrizable.
2. $C_{rc}(X)$ is of first countable.
3. $C_{rc}(X)$ is of countable type.
4. $C_{rc}(X)$ is of pointwise countable type.
5. $C_{rc}(X)$ has a dense subspace of pointwise countable type.
6. $C_{rc}(X)$ is an $M$-space.
7. $C_{rc}(X)$ is a $q$-space.
8. $X$ is hemi-$C$-compact; that is, there exists a sequence of $C$-compact sets $\{A_n\}$ in $X$ such that for any $C$-compact subset $A$ of $X$, $A \subseteq A_n$ holds for some $n$.

Proof. From the earlier discussions, we have $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (7)$, $(1) \Rightarrow (6) \Rightarrow (7)$, and $(1) \Rightarrow (2) \Rightarrow (7)$.

$(4) \Leftrightarrow (5)$. It can be easily verified that if $D$ is a dense subset of a space $X$ and $A$ is a compact subset of $D$, then $A$ has countable character in $D$ if and only if $A$ is of countable character in $X$. Now since $C_{rc}(X)$ is a locally convex space, it is homogeneous. If we combine this fact with the previous observation, we have $(4) \Leftrightarrow (5)$.

$(7) \Rightarrow (8)$. Suppose that $C_{rc}(X)$ is a $q$-space. Hence, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of the zero-function $0$ in $C_{rc}(X)$ such that if $f_n \in U_n$ for each $n$, then $\{f_n : n \in \mathbb{N}\}$ has a cluster point in $C_{rc}(X)$. Now for each $n$, there exists a closed $C$-compact subset $A_n$ of $X$ and $\varepsilon_n > 0$ such that $0 \in (0, A_n, \varepsilon_n) \subseteq U_n$.

Let $A$ be a $C$-compact subset of $X$. If possible, suppose that $A$ is not a subset of $A_n$ for any $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there exists $a_n \in A \setminus A_n$. So for each $n \in \mathbb{N}$, there exists a continuous function $f_n : X \mapsto [0, 1]$ such that $f_n(a_n) = n$ and $f_n(x) = 0$ for all $x \in A_n$. It is clear that $f_n \in (0, A_n, \varepsilon_n)$. But the sequence $\{f_n\}_{n \in \mathbb{N}}$ does not have a cluster point in $C_{rc}(X)$. If possible, suppose that this sequence has a cluster point $f$ in $C_{rc}(X)$. Then for each $k \in \mathbb{N}$, there exists a positive integer $n_k > k$ such that $f_{n_k} \in (f, A, 1)$. So for all $k \in \mathbb{N}$, $f(a_{n_k}) > f_{n_k}(a_{n_k}) - 1 = n_k - 1 \geq k$. But this means that $f$ is unbounded on the $C$-compact set $A$. So the sequence $\{f_n\}_{n \in \mathbb{N}}$ cannot have a cluster point in $C_{rc}(X)$ and consequently, $C_{rc}(X)$ fails to be a $q$-space. Hence, $X$ must be hemi-$C$-compact.

$(8) \Rightarrow (1)$. Here we need the well-known result which says that if the topology of a locally convex Hausdorff space is generated by a countable family of seminorms, then it is metrizable. Now the locally convex topology on $C(X)$ generated by the countable family of seminorms $\{p_{A_n} : n \in \mathbb{N}\}$ is metrizable and weaker than the $C$-compact-open topology. However, since for each $C$-compact set $A$ in $X$, there exists $A_n$ such that $A \subseteq A_n$, the locally convex topology generated by the family of seminorms $\{p_{A_n} : A \in SC(X)\}$, that is, the $C$-compact-open topology, is weaker than the topology generated by the family of seminorms $\{p_{A_n} : n \in \mathbb{N}\}$. Hence, $C_{rc}(X)$ is metrizable.

6. Separable and second countability

Theorem 6.1. For any space $X$ and $\lambda \in \{MC(X), SC(X), CC(X), PS(X), RC(X)\}$, the following are equivalent.

1. $C_\lambda(X)$ is separable.
2. $C_k(X)$ is separable.
3. $X$ has a weaker separable metrizable topology.
4. $C_\lambda(X)$ is separable.

Proof. First, note by Corollary 4.2.2 in [6] that $(1)$, $(2)$, and $(3)$ are equivalent. Also, since $C_p(X) \subseteq C_\lambda(X)$, for $\lambda \in \{MC(X), SC(X), CC(X), PS(X), RC(X)\}$, $(4) \Rightarrow (1)$.

$(3) \Rightarrow (4)$. If $X$ has a weaker separable metrizable topology, then $X$ is submetrizable. By Theorem 4.14, $C_{mc}(X) = C_k(X) = C_{cc}(X) = C_{cc}(X) = C_{ps}(X) = C_{rc}(X)$. Since $(3) \Rightarrow (2)$, $C_\lambda(X)$ is separable for each $\lambda \in \{MC(X), SC(X), CC(X), PS(X), RC(X)\}$.

Corollary 6.2. If $X$ is pseudocompact and $\lambda \in \{MC(X), K(X), SC(X), CC(X), PS(X), RC(X)\}$, then the following statements are equivalent.

1. $C_\lambda(X)$ is separable.
2. $C_\lambda(X)$ has ccc.
3. $X$ is metrizable.

Proof. $(1) \Rightarrow (2)$. This is immediate.

$(2) \Rightarrow (3)$. By Corollary 4.8 in [7], $X$ is metrizable.

$(3) \Rightarrow (1)$. If $X$ is metrizable, then $X$, being pseudocompact, is also compact. Hence $X$ is separable and consequently by Theorem 6.1, $C_\lambda(X)$ is separable.

Recall that a family of non-empty open sets in a space $X$ is called a $\pi$-base for $X$ if every non-empty open set in $X$ contains a member of this family.
The following theorems are analogues of Theorem 4.6 and Theorem 4.8 in [5].

**Theorem 6.3.** For a space $X$ and $\lambda \in \{MC(X), K(X), SC(X), CC(X), PS(X), RC(X)\}$, the following statements are equivalent.

1. $C_\lambda(X)$ contains a dense subspace which has a countable $\pi$-base.
2. $C_\lambda(X)$ has a countable $\pi$-base.
3. $C_\lambda(X)$ is second countable.
4. $X$ is hemicompact and $\aleph_0$-space.

**Theorem 6.4.** For a locally compact space $X$ and $\lambda \in \{MC(X), K(X), SC(X), CC(X), PS(X), RC(X)\}$, the following statements are equivalent.

1. $C_\lambda(X)$ is second countable.
2. $X$ is hemicompact and submetrizable.
3. $X$ is Lindelöf and submetrizable.
4. $X$ is the union of a countable family of compact metrizable subsets of $X$.
5. $X$ is second countable.

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