NONLINEAR PHYSICS AND MECHANICS

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On a Method of Introducing Local Coordinates in the Problem of the Orbital Stability of Planar Periodic Motions of a Rigid Body

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A method is presented of constructing a nonlinear canonical change of variables which makes it possible to introduce local coordinates in a neighborhood of periodic motions of an autonomous Hamiltonian system with two degrees of freedom. The problem of the orbital stability of pendulum-like oscillations of a heavy rigid body with a fixed point in the Bobylev–Steklov case is discussed as an application. The nonlinear analysis of orbital stability is carried out including terms through degree six in the expansion of the Hamiltonian function in a neighborhood of the unperturbed periodic motion. This makes it possible to draw rigorous conclusions on orbital stability for the parameter values corresponding to degeneracy of terms of degree four in the normal form of the Hamiltonian function of equations of perturbed motion.

Keywords: rigid body, rotations, oscillations, orbital stability, Hamiltonian system, local coordinates, normal form

1. Introduction

In many problems of classical mechanics and satellite dynamics, the equations of motion admit partial solutions corresponding to plane motions. In a number of problems such motions are described by the equation of a mathematical pendulum. A well-known example of pendulum-like motions is the motion of a rigid body with one fixed point in a homogeneous gravitational...
field where one of its principal axes of inertia invariably occupies a horizontal position in space.

Pendulum-like motions also include body motions relative to the center of mass in the central Newtonian gravitational field in a circular orbit where one of its principal central axes of inertia is perpendicular to the plane of the orbit. The above-mentioned motions have been well explored. They are either periodic motions (oscillations or rotations of the body relative to the principal axis of inertia) or motions asymptotically approaching an unstable equilibrium point. Pendulum-like periodic motions are Lyapunov unstable with respect to perturbations of the coordinate — the angle of rotation of the body relative to the axis of rotation. However, the problem of their orbital stability is of great interest from a theoretical and an applied point of view. In the above-mentioned problems the equations of rigid body motion can be represented in the form of an autonomous Hamiltonian system. Therefore, to solve the problem of orbital stability, one can apply the methods and algorithms developed to date in the theory of stability of Hamiltonian systems. In accordance with the general methods for investigating the stability of Hamiltonian systems, in a neighborhood of periodic motions it is necessary to introduce the so-called local coordinates [1] and to write equations of perturbed motion. A rigorous stability analysis of the perturbed system thus obtained can be carried out using the well-developed methods of normal forms [2, 3] and KAM theory [4, 5].

In concrete dynamics problems, introducing local coordinates in a neighborhood of periodic motions and obtaining equations of perturbed motion in explicit form can turn out to be a difficult problem. One of the possible methods of introducing local coordinates is to make a canonical change of variables allowing a transformation to action-angle variables in the region of periodic motion on a two-dimensional invariant plane. If such a change of variables can be constructed in explicit form, then one can choose the perturbation of the action variable and perturbations orthogonal to the invariant plane as local coordinates describing the motion in a neighborhood of the periodic trajectory. This approach was applied, in particular, in problems of the orbital stability of pendulum-like periodic motions of a rigid body [6–12]. This approach is very efficient in cases where periodic solutions can be written in explicit analytic form. However, in using this approach one can generally encounter a number of technical difficulties, both in constructing the above-mentioned canonical change of variables and in obtaining explicit expressions for the coefficients of series expansion of a Hamiltonian in a neighborhood of unperturbed periodic motion.

Another method for introducing local coordinates was proposed in [13]. This method is used to introduce the local coordinates by making a linear canonical change of variables. It allows one to avoid the above-mentioned technical difficulties and to solve a wide range of problems of orbital stability, but in some cases its application can also be difficult due to the appearance of a singularity in the coefficients of a corresponding linear transformation. For example, this obstacle prevents this method from being applied for transformation to local coordinates in a neighborhood of pendulum-like oscillations.

In cases where equations of motion are integrated by quadratures, the orbital stability of periodic motions can be investigated using topological methods [14].

The method proposed in this paper for introducing local coordinates uses construction of a nonlinear canonical change of variables. A constructive algorithm is presented for making the above-mentioned change of variables in the form of series in powers of a new variable — a normal perturbation in the neighborhood of a periodic orbit on an invariant plane. This method makes it possible to avoid a singularity when introducing local coordinates and may be applied to investigate the orbital stability of periodic motions both in the presence of their analytic representation and in the case where periodic motions have been found numerically.
As an application of the above method, we will consider the problem of the orbital stability of periodic motions of a heavy rigid body with a fixed point in the Bobylev–Steklov case. The results of analysis of the orbital stability are in complete agreement with the results obtained previously by a different method in [10].

2. A method of transformation to local coordinates in a neighborhood of periodic motions

Consider a Hamiltonian system with two degrees of freedom

\[ \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2), \]  

where the Hamiltonian \( H \) does not explicitly depend on time.

We assume that the system (2.1) has a one-parameter family of periodic solutions of the form

\[ q_1 = f(t), \quad p_1 = g(t), \quad q_2 = p_2 = 0. \]  

(2.2)

In this case the Hamiltonian function \( H \) has the following form:

\[ H = H_0(q_1, p_1) + H^{(2)}(q_1, p_1, q_2, p_2), \]  

(2.3)

where the expansion of the function \( H^{(2)} \) as a series in powers of the canonical variables \( q_2, p_2 \) begins with terms of degree at least two:

\[ H^{(2)}(q_1, p_1, q_2, p_2) = \sum_{k=2}^{\infty} \sum_{i+j=k} h_{ij}(q_1, p_1)q_2^i p_2^j. \]  

(2.4)

We pose the problem of constructing a canonical change of variables

\[ q_1 = \varphi(\xi, \eta), \quad p_1 = \psi(\xi, \eta), \]  

(2.5)

such that in the new variables the family of periodic solutions (2.2) has the form

\[ \xi = f(t) + \xi(0), \quad \eta = q_2 = p_2 = 0. \]  

(2.6)

The problem of the orbital stability of the periodic solution (2.6) is equivalent to the problem of Lyapunov stability with respect to the variables \( \eta, q_2, p_2 \), which we will call local coordinates in what follows.

We will search for the canonical change of variables (2.5) in the form of series in powers of \( \eta \):

\[ \varphi(\xi, \eta) = f(\xi) + a_1(\xi)\eta + a_2(\xi)\eta^2 + a_3(\xi)\eta^3 + \ldots, \]  

\[ \psi(\xi, \eta) = g(\xi) + b_1(\xi)\eta + b_2(\xi)\eta^2 + b_3(\xi)\eta^3 + \ldots. \]  

(2.7)

From the condition that the change of variables (2.5) is canonical it follows that

\[ \frac{\partial \varphi}{\partial \xi} \frac{\partial \psi}{\partial \eta} - \frac{\partial \psi}{\partial \xi} \frac{\partial \varphi}{\partial \eta} = 1. \]  

(2.8)
Equation (2.8) can be satisfied by a suitable choice of the coefficients \(a_i(\xi), b_i(\xi)\) \((i = 1, 2, \ldots)\). For this purpose we substitute (2.7) into (2.8) and, by equating to zero the coefficients with powers of \(\eta\), we obtain equations from which we can sequentially define \(a_i(\xi), b_i(\xi)\):

\[
a_1 \frac{\partial g}{\partial \xi} - b_1 \frac{\partial f}{\partial \xi} = 1, \quad (2.9)
\]

\[
2b_2 \frac{\partial f}{\partial \xi} + b_1 \frac{\partial a_1}{\partial \xi} - 2a_2 \frac{\partial g}{\partial \xi} - a_1 \frac{\partial b_1}{\partial \xi} = 0, \quad (2.10)
\]

\[
3b_2 \frac{\partial f}{\partial \xi} + 2b_2 \frac{\partial a_1}{\partial \xi} + b_1 \frac{\partial a_2}{\partial \xi} - 3a_3 \frac{\partial g}{\partial \xi} - 2a_2 \frac{\partial b_1}{\partial \xi} - a_1 \frac{\partial b_2}{\partial \xi} = 0, \quad (2.11)
\]

\[
\ldots
\]

\[
nb_i \frac{\partial f}{\partial \xi} + \sum_{k=1}^{n-1} k b_k \frac{\partial a_{n-k}}{\partial \xi} - na_n \frac{\partial g}{\partial \xi} - \sum_{k=1}^{n-1} k a_k \frac{\partial b_{n-k}}{\partial \xi} = 0, \quad (2.12)
\]

\[
\ldots
\]

From the equations thus obtained the coefficients \(a_i(\xi), b_i(\xi)\) are defined nonuniquely. This allows us to choose \(a_i(\xi), b_i(\xi)\) in such a way as to avoid a singularity in the expansions (2.7). In particular, equation (2.9) can be satisfied by the following choice of \(a_1(\xi), b_1(\xi)\):

\[
a_1(\xi) = -\frac{\partial g}{\partial \xi}, \quad b_1(\xi) = \frac{\partial f}{\partial \xi}, \quad (2.13)
\]

where

\[
V^2 = \left(\frac{\partial f}{\partial \xi}\right)^2 + \left(\frac{\partial g}{\partial \xi}\right)^2. \quad (2.14)
\]

We note that the function \(V^2\) cannot vanish. Indeed, it follows from the equation \(V^2 = 0\) that \(\frac{\partial f}{\partial \xi} = \frac{\partial g}{\partial \xi} = 0\). But this is impossible since the functions \(f(\xi)\) and \(g(\xi)\) define the periodic solution to the system (2.1) and their derivatives cannot vanish simultaneously.

In a similar way one can choose a solution to Eq. (2.10). Indeed, substituting (2.13) into (2.10), we obtain

\[
2b_2(\xi) \frac{\partial f}{\partial \xi} - \frac{\partial f \partial^2 g}{\partial \xi^2 \partial \xi^2} - 2a_2 \frac{\partial g}{\partial \xi} + \frac{\partial g \partial^2 f}{\partial \xi^2 \partial \xi^2} = 0. \quad (2.15)
\]

The last equation can be satisfied by setting

\[
a_2(\xi) = \frac{\partial^2 f}{\partial \xi^2}, \quad b_2(\xi) = \frac{\partial^2 g}{\partial \xi^2}. \quad (2.16)
\]

We note that (2.13) is not a unique solution to Eq. (2.10), which allows us to avoid a singularity in the denominator.
In a similar way one can find a solution to Eq. (2.11) that contains no singularity in the denominator. This solution can be chosen, for example, in the following form:

\[
\begin{align*}
    a_3(\xi) &= \frac{\partial^4 g}{\partial \xi^4} \left( \frac{\partial^2 f}{\partial \xi^2} \right)^2 + \frac{f}{3V^8} \frac{\partial^2 g}{\partial \xi^2} \frac{\partial g}{\partial \xi} \\
    b_3(\xi) &= -\frac{\partial^4 f}{\partial \xi^4} \left( \frac{\partial^2 f}{\partial \xi^2} \right)^2 + \frac{f}{3V^8} \frac{\partial^2 g}{\partial \xi^2} \frac{\partial f(\xi)}{\partial \xi}.
\end{align*}
\]  

(2.17)

The process of constructing the expansion (2.7) can be continued up to any power of \( \eta \). Indeed, a suitable choice of the coefficients \( a_n(\xi), b_n(\xi) \) can be made, for example, as follows. Set

\[
a_n(\xi) = -C \frac{\partial g}{\partial \xi}, \quad b_n(\xi) = C \frac{\partial f}{\partial \xi}.
\]

(2.18)

Now, substituting (2.18) into (2.12), we obtain an equation for finding the function \( C(\xi) \):

\[
nCV^2(\xi) + \sum_{k=1}^{n-1} kb_k \frac{\partial a_{n-k}}{\partial \xi} - \sum_{k=1}^{n-1} ka_k \frac{\partial b_{n-k}}{\partial \xi} = 0,
\]

(2.19)

whose solution yields the following expressions for \( a_n(\xi), b_n(\xi) \):

\[
\begin{align*}
    a_n(\xi) &= \frac{\partial g}{nV^2} \left\{ \sum_{k=1}^{n-1} kb_k \frac{\partial a_{n-k}}{\partial \xi} - \sum_{k=1}^{n-1} ka_k \frac{\partial b_{n-k}}{\partial \xi} \right\}, \\
    b_n(\xi) &= \frac{\partial f}{nV^2} \left\{ \sum_{k=1}^{n-1} ka_k \frac{\partial b_{n-k}}{\partial \xi} - \sum_{k=1}^{n-1} kb_k \frac{\partial a_{n-k}}{\partial \xi} \right\}.
\end{align*}
\]

(2.20)

The right-hand sides of (2.20) are expressed in terms of \( a_k(\xi), b_k(\xi) \) \((k = 1, 2, \ldots, n-1)\) and their derivatives which were obtained at the previous steps and contain no singularities. Therefore, the expressions (2.20) contain no singularities either.

Making a change of variables by formulae (2.7), we obtain the following expansion of the Hamiltonian function in the neighborhood of \( \eta = q_2 = p_2 = 0 \):

\[
\Gamma = \Gamma_2 + \Gamma_3 + \Gamma_4,
\]

(2.21)

where

\[
\Gamma_2 = \eta + \varphi_2(q_2, p_2, \xi), \quad \Gamma_3 = \varphi_3(q_2, p_2, \xi), \quad \Gamma_4 = \chi(\xi)\eta^2 + \psi_2(q_2, p_2, \xi)\eta + \varphi_4(q_2, p_2, \xi).
\]

(2.22)

The functions \( \chi, \psi, \varphi_i \) \((i = 2, 3, 4)\) are given by the following explicit expressions:

\[
\begin{align*}
    \chi(\xi) &= \frac{1}{2V^2} \left\{ \left[ \frac{\partial^2 H_0}{\partial q_i^2} - \frac{\partial^2 H_0}{\partial p_i^2} \right] \left[ \left( \frac{\partial H_0}{\partial q_1} \right)^2 - \left( \frac{\partial H_0}{\partial p_1} \right)^2 \right] + 4 \frac{\partial^2 H_0}{\partial q_1 \partial p_1} \frac{\partial H_0}{\partial q_1} \frac{\partial H_0}{\partial p_1} \right\}, \\
    \psi_2(q_2, p_2, \xi) &= \frac{1}{V^2} \sum_{i+j=2} \left\{ \frac{\partial h_{ij}}{\partial p_1} \frac{\partial H_0}{\partial p_1} - \frac{\partial h_{ij}}{\partial q_1} \frac{\partial H_0}{\partial q_1} \right\} q_i^2 p_j^2, \\
    \varphi_k(q_2, p_2, \xi) &= \sum_{i+j=k} h_{ij} \left( f(\xi), g(\xi) \right) q_i^2 p_j^2, \quad k = 2, 3, 4.
\end{align*}
\]

(2.23)

(2.24)

(2.25)
The functions $h_{ij}$ in (2.25) are the coefficients in the expansion (2.4) of the initial Hamiltonian as a series in powers of $q_1, p_2$. The partial derivatives of the functions $h_{ij}$ and $H_0$ in the expressions (2.23) and (2.24) are calculated for $q_1 = f(\xi), p_1 = g(\xi)$.

3. Local coordinates in a neighborhood of pendulum-like motions of a heavy rigid body in the Bobylev – Steklov case

In this section we show how to apply the method described above to the problem of the orbital stability of pendulum-like motions of a heavy rigid body with one fixed point in the Bobylev – Steklov case.

Consider the motion of a rigid body about a fixed point $O$ in a homogeneous gravitational field. Let $mg$ be the weight of the body and let $l$ be the distance from the center of gravity to the fixed point $O$. Introduce a fixed coordinate system $OXYZ$ with the axis $Z$ directed vertically upwards, and a moving coordinate system $Oxyz$, attached to the rigid body, with the axes $x, y$ and $z$ directed along the principal axes of inertia of the body for point $O$. Denote by $A, B$ and $C$ the corresponding moments of inertia, and by $x_*, y_*, z_*$, the coordinates of the center of gravity in the moving coordinate system. Specify the position of the body (the moving axes $Oxyz$) relative to the fixed coordinate system $OXYZ$ by the Euler angles $\psi, \theta, \phi$.

Suppose that the Bobylev – Steklov case takes place, when $A = 2C$ and the center of mass of the body lies on the axis $x$ of the moving coordinate system, i.e., $x_* = l, y_* = z_* = 0$.

The Bobylev – Steklov case is remarkable for the fact that the equations of motion of a rigid body admit a family of partial solutions [15, 16] which can be obtained in terms of elliptic Jacobi functions (see, e.g., [17]).

In the Bobylev – Steklov case the equations of motion also admit a partial solution describing a planar rigid body motion such that the axis of inertia $z$ invariably occupies a horizontal position and the constant of the area integral is zero. For this motion, $\theta = \pi/2, \psi = \text{const}$, and the change in the angle $\phi$ is described by the following equation of a physical pendulum:

$$\frac{d^2\phi}{dt^2} + \mu^2 \cos \phi = 0, \quad \mu^2 = \frac{mg l}{C}.$$ (3.1)

Thus, in this motion the body either performs pendulum-like motions about the axis $z$ or asymptotically approaches an unstable equilibrium point.

Note that pendulum-like motions are also possible relative to the axis $y$. Since in the Bobylev – Steklov case the moment of inertia $C$ is the smallest moment of inertia, we will call the motions about the axis $z$ motions relative to the largest axis of the ellipsoid of inertia.

Choosing the Euler angles $\psi, \theta, \phi$ as generalized coordinates and introducing the corresponding generalized momenta $p_\psi, p_\theta, p_\phi$, one can write the equations of motion in Hamiltonian form. In this case the angle $\psi$ is a cyclic coordinate and the corresponding momentum $p_\psi$ is a first integral and hence takes zero value for the unperturbed motion. In what follows, we assume that $p_\psi = 0$ for the perturbed motion as well.

Let us introduce a dimensionless time variable $\tau = \mu t$ and dimensionless canonical variables $q_1, q_2, p_1, p_2$

$$q_1 = \phi - \frac{3\pi}{2}, \quad q_2 = \theta - \frac{\pi}{2}, \quad p_1 = \frac{p_\phi}{C\mu}, \quad p_2 = \frac{p_\theta}{C\mu}.$$ (3.2)
The Hamiltonian of the problem takes the form [10]

\[
H = \frac{1}{4} \left[ (2\alpha - 1) \sin^2 q_1 \tan^2 q_2 + \tan^2 q_2 + 2 \right] p_1^2 + \frac{1}{4} (2\alpha - 1) \sin 2q_1 \tan q_2 p_1 p_2 + \\
+ \frac{1}{4} [2\alpha - (2\alpha - 1) \sin^2 q_1] p_2^2 - \cos q_1 \cos q_2,
\]

(3.3)

where \(\alpha = C/B\), and in the Bobylev–Steklov case \(\frac{1}{3} \leq \alpha \leq 1\).

For solutions corresponding to planar pendulum-like motions of the rigid body relative to the fixed axis of inertia \(Oz\), the change of variables \(q_1, p_1\) is described by a system of canonical equations with the Hamiltonian \(H_0 = 1/2p_1^2 - \cos q_1\), and the variables \(q_2, p_2\) take zero values. Depending on the value of the constant \(h\) of the energy integral \(H_0 = h\), the planar motions are either asymptotic (\(h = 1\)) to the unstable equilibrium point \(\varphi = \pi/2\) of the rigid body or are periodic motions: pendulum-like oscillations (\(|h| < 1\)) in a neighborhood of the stable equilibrium point \(\varphi = 3\pi/2\) or rotations (\(h > 1\)) relative to the axis \(Oz\).

Let the solution \(q_1 = f(\tau - \tau_0, h), p_1 = g(\tau - \tau_0, h)\) describes the pendulum-like periodic motion of the rigid body. Then, by virtue of the system of equations with the Hamiltonian \(H_0\), the following equalities hold:

\[
\frac{\partial f}{\partial \tau} = g, \quad \frac{\partial g}{\partial \tau} = -\sin f.
\]

(3.4)

Differentiating both sides of equations (3.5), we have

\[
\frac{\partial^2 f}{\partial \tau^2} = -\sin f, \quad \frac{\partial^2 g}{\partial \tau^2} = -\cos f.
\]

(3.5)

In a similar way we obtain

\[
\frac{\partial^3 f}{\partial \tau^3} = -\cos f, \quad \frac{\partial^3 g}{\partial \tau^3} = \sin f (\cos f - g^2).
\]

(3.6)

Taking Eqs. (3.4)–(3.6) into account, we obtain the following explicit form of the canonical change of variables (2.7):

\[
q_1 = f + \frac{\sin f}{V^2} \eta - \frac{\sin f}{2V^4} \eta^2 + \left[ \frac{\sin f (\cos^2 f - g^2)}{6V^6} + \frac{g^2 \cos^2 f + \sin^2 f \sin f}{3V^8} \right] \eta^3 + O(\eta^4),
\]

\[
p_1 = g + \frac{g}{V^2} \eta - \frac{g \cos f}{2V^4} \eta^2 + \left[ \frac{g \cos f}{6V^6} + \frac{g^2 \cos^2 f + \sin^2 f g}{3V^8} \right] \eta^3 + O(\eta^4),
\]

(3.7)

where \(V^2 = g^2 + \sin^2 f\).

The argument of the functions \(f\) and \(g\) in (3.7) is the new canonical variable \(\xi\). The explicit expressions for \(f\) and \(g\) depend on the type of periodic motion. In the case of oscillations (when \(|h| < 1\)) they are given by the formulae [8]:

\[
f = 2 \arcsin[k_1 \sin(\xi, k_1)], \quad g = 2k_1 \cos(\xi, k_1), \quad k_1^2 = (h + 1)/2,
\]

(3.8)

and in the case of rotations (when \(h > 1\)) by the formulae

\[
f = 2m(\xi, k_2), \quad g = 2k_2^{-1} \cos(\xi, k_2), \quad k_2^2 = 2/(1 + h).
\]

(3.9)

In (3.8)–(3.9), standard notation is used for the elliptic functions [18].
The period of pendulum-like motions is equal to $2\pi/\omega$, where $\omega = \pi/(2K(k_1))$ in the case of oscillations and $\omega = \pi/(k_2K(k_2))$ in the case of rotations. $K$ denotes the complete elliptic integral of the first kind.

We make another canonical change of variables $\xi, \eta \rightarrow w, r$ by the formulae

$$\xi = \frac{1}{\omega}w, \quad \eta = \omega r.$$  

(3.10)

In the case of rotations the Hamiltonian of the problem is a $\pi$-periodic function $w$, and in the case of oscillations it is a $2\pi$-periodic function.

By substituting (3.7) and (3.10) into (3.3), we obtain the Hamiltonian of the system of equations of perturbed motion in a neighborhood of the periodic orbit

$$\Gamma = \Gamma_2 + \Gamma_4 + \ldots + \Gamma_{2m} + \ldots,$$  

(3.11)

where $\Gamma_{2m}$ is a form of degree $2m$ in $q_2, p_2, \eta^{1/2}$. The forms $\Gamma_2$ and $\Gamma_4$ of the expansion (3.11), which are required for further analysis, have the form

$$\Gamma_2 = \omega r + \Phi_2(q_2, p_2, w),$$

$$\Gamma_4 = \chi(w)r^2 + \Psi_2(q_2, p_2, w)r + \Phi_4(q_2, p_2, w),$$

(3.12)

(3.13)

where

$$\Psi_2(q_2, p_2, w) = \sum_{i+j=k} \psi_{ij}q_2^i p_2^j, \quad \Phi_k(q_2, p_2, w) = \sum_{i+j=k} \varphi_{ij}q_2^i p_2^j, \quad k = 2, 4.$$  

(3.14)

The coefficients of the forms $\Psi_2, \Phi_2, \Phi_4$ which periodically depend on $w$ are calculated by the formulae

$$\chi(w) = \frac{\omega^2}{2V^2}(\cos f_* - 1)(\sin^2 f_* - g_*^2),$$

$$\psi_{20}(w) = \frac{\omega^2}{2V^2}[(\sin^2 f_* (\cos f + 1)(2\alpha - 1) + 1)g_*^2 - \sin^2 f_*],$$

$$\psi_{11}(w) = -\frac{\omega}{2V^2}(2\alpha - 1)g_* \sin f [\sin^2 f_* - \cos f (\cos f_* + 1)],$$

$$\psi_{02}(w) = -\frac{\omega}{2V^2}(2\alpha - 1)\sin^2 f_* \cos f_*,$$

$$\varphi_{20}(w) = \frac{1}{4} [(2\alpha - (2\alpha - 1) \cos^2 f_*)g_*^2 + 2 \cos f_*],$$

$$\varphi_{11}(w) = \frac{1}{2}(2\alpha - 1)g_* \sin f_* \cos f_*,$$

$$\varphi_{40}(w) = \frac{1}{6} [(2\alpha \sin^2 f_* + \cos^2 f_*)g_*^2 - \frac{1}{4} \cos f_*],$$

$$\varphi_{31}(w) = \frac{1}{6}(2\alpha - 1)g_* \sin f_* \cos f_*,$$

(3.15)

where $f_*, g_*$ denotes the functions $f_*(w) = f(\omega^{-1}w), g_*(w) = g(\omega^{-1}w), T$-periodic in $w$, with $T = \pi$ in the case of rotations and $T = 2\pi$ in the case of oscillations.

By virtue of the equations of motion with the Hamiltonian (3.11), the coordinate $w$ is an increasing function of the variable $\tau$. Therefore, in the problem of the stability of motion this coordinate can play the role of time. To describe the motion on the zero isoenergetic level, we
take the coordinate $w$ to be a new independent variable. In addition, from the equation $\Gamma = 0$ with small $q_2, p_2, r$ we have $r = -K(q_2, p_2, w)$. The function $K(q_2, p_2, w)$ is the series

$$K = K_2 + K_4 + \ldots + K_k + \ldots,$$

where $K_k$ is a form of degree $k$ in $q_2, p_2$ with coefficients $T$-periodic in $w$. The forms $K_2$ and $K_4$ have the following explicit form:

$$K_2 = \frac{1}{\omega} \Phi_2(q_2, p_2, w),$$

$$K_4 = \frac{1}{\omega} \left[ \chi(w) \Phi_2^2(q_2, p_2, w) + \Psi_2(q_2, p_2, w) \Phi_2(q_2, p_2, w) + \Phi_4(q_2, p_2, w) \right].$$

The equations of motion on the isoenergetic level $\Gamma = 0$ can be written in the Hamiltonian form

$$\frac{dq_2}{dw} = \frac{\partial K}{\partial p_2}, \quad \frac{dp_2}{dw} = -\frac{\partial K}{\partial q_2}.$$

Thus, the problem of the stability of pendulum-like periodic motions of a rigid body reduces to investigating the stability of the equilibrium point $q_2 = p_2 = 0$ of the reduced system (3.19).

4. On the orbital stability of pendulum-like periodic motions of a heavy rigid body in the Bobylev – Steklov case

The problem of the stability of pendulum-like periodic motions of a heavy rigid body in the Bobylev – Steklov case has been investigated previously. In [9], the case of pendulum-like motions relative to the axis $O_y$ was discussed. The case considered here of pendulum-like motions relative to the axis $O_z$ was examined in [10].

In the above-mentioned papers, the local coordinates were introduced by first making a canonical change of variables which allows introduction of action-angle variables in the region of phase space corresponding to unperturbed periodic orbits. In this case the Hamiltonian of the perturbed motion is obtained in the form of a power series of the perturbation of the action variable and normal perturbations to the invariant manifold on which periodic orbits lie. In such an approach, it is necessary to have an explicit analytic representation of the unperturbed periodic motion. This is mainly due to the fact that construction of the expansion of a Hamiltonian in the neighborhood of a periodic orbit requires an explicit form of its partial derivatives with respect to the action variable. This leads to rather cumbersome and time-consuming calculations in degenerate cases where solving the problem of orbital stability requires an analysis including terms of order higher than four in the expansion of the Hamiltonian in a neighborhood of the periodic orbit. For that reason, in particular, the above-mentioned degenerate case was not considered in [9, 10].

In [10], a nonlinear analysis of the orbital stability of the pendulum-like motions considered here was carried out taking into account terms through order four in the expansion of the Hamiltonian in a neighborhood of the periodic orbit. We give a brief account of the main results of [10]. In the parameter plane $(\alpha, h)$ an orbital stability diagram, shown in Fig. 1, was obtained. In the hatched regions, the pendulum-like periodic motions are orbitally unstable. In the unhatched regions, orbital stability takes place. An exception can only be the curve $\Gamma$, on which a degeneracy takes place. For parameter values on this curve, the problem of orbital
stability is solved using terms of order six, or perhaps of higher orders, in the expansion of the Hamiltonian of the perturbed motion.

The boundaries separating the stability and instability regions were obtained in the following analytic form.

- The left and right boundaries of the region of orbital instability of pendulum rotations are given, respectively, by the equations
  \[ h = -\frac{1}{2\alpha - 1}, \quad h = \frac{1}{2\alpha - 1}. \]

- The regions of orbital stability and instability of pendulum-like oscillations are separated by the straight line \( h = 0 \).

Using the above explicit expressions for the Hamiltonian of the perturbed motion, one can investigate the orbital stability of pendulum-like oscillations for parameter values corresponding to the degeneracy curve \( \Gamma \). In this paper we consider the case of small oscillation amplitudes, when it is possible to introduce a small parameter and perform the analysis of orbital stability analytically.

As a small parameter of the problem we choose the quantity \( k_1 = \sin \beta / 2 \) (where \( \beta \) is the amplitude of oscillations). This quantity is the absolute value of the elliptic functions in the expressions (3.8) and is related to the constant energy \( h \) by \( k_1^2 = (h + 1)/2 \). In order to investigate the stability of the equilibrium point in the degenerate case, it is necessary to reduce the Hamiltonian (3.16) to normal form up to terms of degree six and to apply the criterion for stability of the Hamiltonian system with one degree of freedom [4, 5].

At the first stage of the normalization procedure we present the quadratic part of the Hamiltonian (3.16) to the form corresponding to a harmonic oscillator. Using the well-known
expansions of elliptic functions [18], we obtain from (3.17)

\[ K_2 = \frac{1}{2} q_2^2 + \frac{1}{2} \alpha p_2^2 + O(k_1^2). \]  

(4.1)

By making the change of variables

\[ q_2 = \alpha^{\frac{1}{4}} x, \quad p_2 = \alpha^{-\frac{1}{4}} y \]  

(4.2)

we reduce the autonomous part \( K_2 \) to the required form:

\[ K_2 = \frac{1}{2} \sqrt{\alpha} (x^2 + y^2) + O(k_1^2). \]  

(4.3)

By making a canonical near-identity linear \( \pi \)-periodic in \( w \) change of variables

\[ x = a_{11}(w) X + a_{12}(w) Y, \quad y = a_{21}(w) X + a_{22}(w) Y \]  

(4.4)

we can exclude the dependence of \( K_2 \) on \( w \) and bring the quadratic part of the Hamiltonian into the following form

\[ K_2 = \frac{1}{2} \Omega (X^2 + Y^2), \]  

(4.5)

The functions \( a_{11}(w), a_{12}(w), a_{21}(w), a_{22}(w) \) are \( \pi \)-periodic in \( w \) and analytic in \( k_1 \). They can be constructed in form of converging series in powers of the small parameter \( k_1 \). The coefficients of any finite power of \( k_1 \) in these series can be determined, for example, by the Birkhoff method [1, 3] or the Deprit–Hori method [2]. The calculation have shown

\[ a_{11}(w) = 1 + \frac{1 - 3 \alpha + 2 \alpha^2 + \alpha (3 - 4 \alpha) \cos 2t}{4 \alpha (\alpha - 1)} k_1^2 + O(k_1^4), \]

\[ a_{12}(w) = \frac{\sqrt{\alpha} (7 - 8 \alpha) \sin 2t}{4 (\alpha - 1)} k_1^4 + O(k_1^4), \]

\[ a_{21}(w) = \frac{(2 - 3 \alpha) \sin 2t}{4 \sqrt{\alpha} (\alpha - 1)} k_1^2 + O(k_1^4), \]

\[ a_{22}(w) = 1 - \frac{1 - 3 \alpha + 2 \alpha^2 + \alpha (3 - 4 \alpha) \cos 2t}{4 \alpha (\alpha - 1)} k_1^2 + O(k_1^4) \]  

(4.6)

and

\[ \Omega = \sqrt{\alpha} - \frac{(3 \alpha - 2)}{4 \sqrt{\alpha}} k_1^2 + O(k_1^4). \]  

(4.7)

We now transform to canonical polar coordinates \( \theta, \rho \) by the formulae

\[ X = \sqrt{2 \rho} \sin \varphi, \quad Y = \sqrt{2 \rho} \cos \varphi \]  

(4.8)

and, using the expansions of elliptic functions, obtain the following explicit form of the expansion of the Hamiltonian function through terms \( \rho^3 \):

\[ K = \Omega \rho + \rho^2 F_4(\theta, w) + \rho^3 F_6(\theta, w) + O(\rho^4), \]  

(4.9)
we have terms of order not manifest themselves in this approximation. The Hamiltonian function
\[ H = \frac{1}{6} \alpha \cos^4 \vartheta - \sin 2w \sqrt{\alpha} (2\alpha - 1) \cos \vartheta \sin \vartheta + (\sin^2 w - 4/3\alpha) \sin^2 \vartheta + \frac{23}{12} \alpha - \sin^2 w - 5/4 \alpha \cos^2 w - 1/2 \alpha \cos^4 w + O(k_1^2), \]
(4.10)

Since the Hamiltonian (4.9) depends \( \pi \)-periodically on \( w \), it follows that, if the equation
\[ m\Omega = 2n, \quad m, n \in \mathbb{N}, \]
(4.12)
is satisfied, resonance takes place in the corresponding canonical system. In the case of resonance, the normal form of the Hamiltonian contains additional (resonant) terms; therefore, resonance cases are considered separately.

In normalizing the Hamiltonian through terms of degree \( p^2 \) it is necessary to take into account only resonances of orders not higher than four (\( m \leq 4 \)). Higher-order resonances do not manifest themselves in this approximation. The Hamiltonian function (4.9) contains no terms of order \( p^{3/2} \), which implies that its normal form calculated through terms of order \( p^2 \) will also contain no terms corresponding to a resonance of order three, \( 3\Omega = 2n \). We finally note that, when \( k_1 \ll 1 \), first- and second-order resonances, corresponding to boundaries of the regions of parametric resonance, and a fourth-order resonance are impossible since the inequality
\[ 1/3 \leq \alpha < 1 \]
is satisfied. Thus, in the case of small-amplitude oscillations with any admissible values of the parameter \( \alpha \), the Hamiltonian (4.9) can be reduced to the following normal form by a suitable choice of canonical variables:
\[ \mathcal{H} = \Omega R + c_2 R^2 + \tilde{F}_6(\psi, w) R^3, \]
(4.13)

where
\[ c_2 = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^\pi F_4(\vartheta, w) \, dw \, d\vartheta. \]
(4.14)

The change of variables \( \rho, \vartheta \to R, \psi \), which reduces the Hamiltonian (4.9) to the form (4.13), is a near-identity transformation analytic in \( k_1 \) and \( \rho \), \( \pi \)-periodic in \( w \) and \( 2\pi \)-periodic in \( \psi \).

Calculations show that by taking into account in \( F_4(\vartheta, w) \) terms up to order \( k^2 \) inclusively we have
\[ c_2 = -\frac{1}{4} + \frac{3}{8} \alpha + \frac{(45 \alpha^3 - 29 \alpha^2 - 16 \alpha + 12)}{128 \alpha (\alpha - 1)} k_1^2 + O(k_1^4), \]
(4.15)
The corresponding canonical transformation \( \rho, \vartheta \to R, \psi \) is given by the following generating function:
\[ S(\vartheta, R) = \vartheta R + R^2 S_4(\vartheta, w), \]
(4.16)

where
\[ S_4(\vartheta, w) = \frac{\sqrt{\alpha}}{24} \sin \vartheta \cos^3 \vartheta + \frac{(24 \alpha (4\alpha - 3) \cos^2 w - 77\alpha^2 + 77\alpha - 12)}{48 \sqrt{\alpha} (\alpha - 1)} \sin \vartheta \cos \vartheta + \frac{\sin 2w (4\alpha^2 - 2\alpha - 1) \sin^2 \vartheta + \sin 2w (2\alpha (\alpha - 1) \cos^2 w - 3\alpha^2 - 9\alpha + 8)}{4(\alpha - 1)} + O(k_1^2). \]
(4.17)
The function $\tilde{F}_6$ in the Hamiltonian (4.13) is calculated from the formula

$$\tilde{F}_6(\psi, w, k_1) = F_6 + \left[ S_4 \frac{\partial F_4}{\partial \psi} - (F_4 + c_2) \frac{\partial S_4}{\partial \psi} \right]. \tag{4.18}$$

If $c_2 \neq 0$, then by the Arnold–Moser theorem [4, 5], the equilibrium point of the system (3.19) is Lyapunov stable. This implies that oscillations are orbitally stable. But if $c_2 = 0$, then the so-called case of degeneracy takes place and solving the problem of the stability requires an additional analysis including terms of degree $\rho^3$ in the Hamiltonian (4.13).

By solving the equation $c_2 = 0$ with respect to $\alpha$ we have the following equation for curve $\Gamma$ (see Fig. 1)

$$\alpha_\ast = \frac{2}{3} + \frac{k_1^2}{6} + O(k_1^4). \tag{4.19}$$

Let us now consider the case of degeneracy which takes place on curve $\Gamma$. To this end we put $\alpha = \alpha_\ast$. Then by substituting (4.10), (4.11), (4.17) into (4.18), one can obtain an explicit expression for $\tilde{F}_6$ as series in powers of $k_1$. With an explicit expression for $\tilde{F}_6$ in place, we can normalize the Hamiltonian up to terms of degree $\rho^3$ by a canonical near-identity change of variables $\psi, R \rightarrow \tilde{\psi}, \tilde{R}$, analytic in $k_1$ and $\tilde{R}$ and $\pi$–periodic in $w$.

The normalized Hamiltonian will take the form

$$\mathcal{H} = \Omega \tilde{R} + c_3 \tilde{R}^3 + O(\tilde{R}^3), \tag{4.20}$$

where

$$c_3 = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{\pi} \tilde{F}_6(\psi, w) \, dw \, d\psi. \tag{4.21}$$

Calculations show that

$$c_3 = \frac{\sqrt{6}}{216k_1^2} - \frac{13\sqrt{6}}{3456} + O(k_1^2). \tag{4.22}$$

Since the quantity $c_3$ is nonzero for sufficiently small $k_1$, it follows that, by the Arnold–Moser theorem, the equilibrium point of the system (3.19) is Lyapunov stable and hence the corresponding pendulum-like oscillations are orbitally stable as well.

Thus, in the Bobylev–Steklov case, pendulum-like periodic oscillations of a rigid body relative to the largest axis of an ellipsoid of inertia with sufficiently small amplitudes are orbitally stable.

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