Approximation Algorithms for a Balanced Capacity and Distance Constrained Vehicle Routing Problem

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Abstract
In the current paper we have investigated the capacitated distance constraint vehicle routing problem for the standard approximation algorithm. This problem consists of a number of different types of vehicles at the depot which differ at their capacity, cost and maximum distance bound. We have designed an approximation algorithm for this problem in the case that the tours are balanced.

Keywords
Vehicle Routing Problem, Approximation Algorithms, Distance Constraint VRP, Capacity constraint VRP

1 Introduction
Vehicle routing problem is one of the most important and extensively studied combinatorial optimization problems [1-12]. It calls for the determination of the optimal set of routes to be performed by a fleet of vehicles to serve a given set of customers. The basic version of the VRP is the Capacitated VRP (CVRP) in which all the customers correspond to the demands are deterministic. The vehicles are identical and based at a single central depot, and only the capacity restrictions for the vehicles are imposed. The objective is to minimize the total cost (i.e., a weighted function of the number of routes and their length) to serve all the customers. The problem with distance constraints on each route where the objective is to minimize the number of vehicles, called the Distance Constrained Vehicle Routing Problem (DVRP) [13-17]. The distance bound translates to a quality of service guarantee for all customers to be served on the day they are scheduled. Nagarajan and Ravi investigated this problem for an approximation algorithm [17].

The DVRP with capacity constraint on the vehicles called Capacity and Distance constrained Vehicle Routing Problem (CDVRP) arises in many logistics and distribution operations [15-17]. The distance bound may translate a constraint on the overall cost of a vehicle during its service. The vehicles at the depot may differ with their capacity and distance constraint. There exist a number of customers with known demand levels that must be visited from a single depot. The problem is defined under the following constraints:

- There exists different type of vehicles at the depot with different capacity and distance constraint,
• Each customer must be visited by one of the vehicles only once,
• Each vehicle starts and ends its route at the depot,
• Total demand serviced by each vehicle is limited and,
• The total distance traveled by a vehicle at its route should be restricted with corresponding distance bound.

The aim of the CDVRP is to minimize the number tours with mentioned constraints. Minimizing the number of vehicles over a period of typical demands also allows for saving costs, better fleet driver planning and management. These problems generalize the classical TSP and are NP-complete.

In this paper, we investigate CDVRP for an approximation algorithm to measure the performance of heuristics. An approximation algorithm for a minimization problem is said to achieve an approximation ratio \( \alpha \) (which may be a function of the input instance), if on every instance, the cost of the solution obtained by the algorithm is at most \( \alpha \) times the cost of an optimal solution. Such an algorithm is also referred to as an \( \alpha \) -approximation algorithm [18-28].

2 Problem Formulation

We use the mathematically classical description of VRP. The demand locations are assumed as vertices of the graph \( G = (V,E) \) in a finite metric space \( (V,d) \) with \( |V| = n \). The edge set \( E = \{(i,j) : i,j \in V : i \neq j\} \) represents the distance between two locations. The distance function \( d : V \times V \rightarrow N \) is symmetric and satisfies the triangle inequality. The input of the capacity and distance constrained vehicle routing problem (CDVRP) consists of a graph \( G = (V,E) \), a depot \( r \in V \), a set of vehicles \( \{1,2,...,N\} \) stationed at the depot with capacities \( \{Q_1,Q_2,...,Q_N\} \) and distance bounds \( \{T_1,T_2,...,T_N\} \), a metric space \( (V,d) \) and a request function \( q : V \longrightarrow R \). The objective is to find a minimum cardinality set of tours which starts and ends at the depot that covers all the vertices in \( V \). Each tour corresponding to a vehicle \( i \in \{1,2,...,N\} \) is required to have length at most \( T_i \in \{T_1,T_2,...,T_N\} \) (the distance constraint) and capacity at most \( Q_i \in \{Q_1,Q_2,...,Q_N\} \) (the capacity constraint). Tours originating from \( r \) are referred to as \( r \)-tours. The minimum distance constraint is denoted by \( T_{\min} = \min_{i=1,N} T_i \). In order to avoid the feasible solutions, we assume that the maximum distance of the vertices from the depot, \( \Delta \), satisfies \( \Delta \leq T_{\min} / 2 \).

The variable Bin-Packing problem is defined as follows: given a number of bin types and a list of items the goal is to pack the list of items by choosing appropriate bins such that the total size of bins used is minimized.

3 Capacity and Distance Constrained Vehicle Routing Problem
In this section, we present an approximation algorithm for CDVRP on general metrics. Our algorithm uses as a subroutine, the variable bin-packing problem and the \( O(\log \frac{D}{D-2\Delta+2}) \)-approximation algorithm for DVRP from Nagarajan et al. [17]. We partition the vertices of the graph into bins, according to their requests, and solve the DVRP for all vertices in the same bins (with corresponding distance bounds). Algorithm min NT for CDVRP on general metrics is described below.

1. Run the variable bin-packing with bins equal to the set of vehicles and items as the set of requests. Let \( K_1, K_2, \ldots, K_N \) be the number of bins obtained by the above algorithm.
2. For the sub-graphs \( G_{K_1}, G_{K_2}, \ldots, G_{K_N} \) associated to the vertices in the corresponding bins, run the DVRP algorithm with distance bounds \( \{T_1, T_2, \ldots, T_N\} \) and capacities \( \{Q_1, Q_2, \ldots, Q_N\} \). Let \( \Pi_i, i \in \{1, 2, \ldots, N\} \) be the number of tours obtained.
3. Return: \( \Pi = \sum_{i=1}^{N} \Pi_i \).

**Theorem 1.** Algorithm min NT obtains an approximation algorithm to CDVRP.

The obtained tours in the previous algorithm aren’t balanced in the sense that their lengths are not necessarily near each other. Although, this point has been partly regarded in the min VR sub-routine algorithm [17] used in the min NT algorithm. We call this problem as balanced distance and capacity constraint vehicle routing problem (BDCVRP). In the next section we will improve the min NT algorithm from this point of view. First, we prove that this problem is NP-complete and then will give an approximation algorithm for it.

### 4 Balanced Distance and Capacity Constraint Vehicle Routing Problem (BDCVRP)

In the HDCVRP, we have assumed an additional constraint on the length of tours. For a given parameter \( \alpha < 1 \) the length of each tour should satisfy in the following condition:

\[
\alpha \frac{L(i)}{L(j)} < 1, \text{ for each } i, j \in \Pi,
\]

in which the length of each tour, \( L \), is computed by the sum of its edge lengths.

#### 4.1 NP-Completeness of BDCVRP

In this section, we investigate the NP-completeness of BDCVRP.

**Theorem 2.** The BDCVRP is an NP-complete problem.

**Proof.**

We show that \( DCVRP \prec BDCVRP \). So, this problem is NP-complete. DCVRP is NP-complete, since it is a generalization of the famous traveling salesman problem. We model a
problem using a given DCVRP, such that there exists a solution to the DCVRP only if exists for the BDCVRP.

Let \( T_1, T_2, \ldots, T_\pi \) be the solution tours for a given DCVRP and \( T_m \) be the tour with maximum length: \( L(T_m) = \max_{i=1,\ldots,\pi} L(T_i) \). Assume that \( \frac{L(T_i)}{L(T_m)} < \alpha < 1 \) and \( L = L(T_m) - L(T_i) \). Let \( d(r, a_n) = b \) and 
\[
 s = \left\lfloor \frac{L - b}{\Delta} \right\rfloor, \quad a = \frac{L - b}{\Delta} \left\lfloor \frac{L - b}{\Delta} \right\rfloor
\]
for each tour \( T_i := r, a_1, a_2, \ldots, a_n, r \). We remove the vertex \( a_n \) from the tour \( T_i \) and add the following edges by the vertices \( a_{n+1}, a_{n+2}, \ldots, a_{n+s}, a_{n+s+1} \):
\[
 d(a_n, a_{n+1}) = d(a_{n+1}, a_{n+2}) = \ldots = d(a_{n+s-1}, a_{n+s}) = \Delta, \quad d(a_{n+s}, a_{n+s+1}) = a, \quad d(a_{n+s+1}, r) = b.
\]
We assume that \( Q(a_{n+i}) = 0 \) for each \( i = 1 : s + 1 \). Therefore, we have an optimal solution to BDCVRP only if have an optimal solution for DCVRP. \( \square \)

4.2 Approximation Algorithm for BDVRP

In this sub section, we give an approximation algorithm for HDVRP on general metrics and call min NHT algorithm. Our algorithm is as follows:

Input: a complete graph \( (V, E) \), a metric \( l: E \to R \) and a real number \( \lambda > 0 \).
Output: paths \( P_1, P_2, \ldots, P_k \) such that \( \lambda / 2 \leq L(P_i) \leq \lambda, \quad i = 1, 2, \ldots, k \).
Measure: \( k \)

1. begin
2. \( F: \) a minimum weight tree on \( G \) with \( n-1 \) edges,
3. \( \text{Max}=0 \)
4. \( \text{Min}=0 \)
5. \( \text{Sol}= \emptyset \)
6. \( \text{N}=1 \)
7. While \( L(F) > \lambda :\)

   \( b: = \) The deepest node such that the sub tree is of length \( > \lambda \),
   \( a: = \) The deepest node such that the sub tree is of length \( > 3 / 2 \lambda \),
   \( T= \) sub tree
   \( \text{TOURT}= \) a tour on the vertices of \( T \) obtained by doubling \( T \) and short cutting.
   \( P= \) a path resulting from TOURT by deleting one edge.

If \( L(P) > \text{Max} \), \( \text{Max}= L(P) \) and If \( L(P) < \text{Min} \) \( \text{Min}= L(P) \).
Sol = Sol ∪ P,
N = N + 1
F = F \ T
end while
8. Choose h such that \( \lambda < \Delta_{\min} \times 2^h < 3/2 \lambda \).
9. \( P_t := \) a path resulting from F obtained by \( 2^{h(t)} \) times F and deleting one edge.
10. Return Sol, N, \( \alpha = \frac{\text{Min}}{\text{Max}} \).

Fig. 1. The min NHT algorithm.

**Theorem 3.** The min NHT is an approximation algorithm for BDVRP.

In the sub section we improve min NHT algorithm for BCDVRP on general metrics.

### 4.3 Approximation Algorithm for BDCVRP

In this sub section we consider balanced distance and capacity constraint vehicle routing problem.

Input: a complete graph \( G = (V, E) \), a metric \( l : E \rightarrow R \), real numbers \( D > 0, \ C > 0 \) and a request function \( q : V \rightarrow R \geq 0 \).

Output: Tours \( T_1, T_2, \ldots, T_k \) such that \( \sum q(T_i) < C, \ \alpha < \frac{L(T_i)}{L(T_j)} < 1 \).

Measure: k

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