COMPLEX HADAMARD MATRICES OF ORDER 9, AND MUBS

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Abstract. A new type of complex Hadamard matrices of order 9 are constructed. The studied matrices are symmetric, block circulant with circulant blocks (BCCB) and form an until now unknown non-reducible and non-affine two-parameter orbit. Several suborbits are identified, including a one-parameter intersection with the Fourier orbit $F_N^{(4)}$. The defect of this new type of Hadamard matrices is observed to vary, from a generic value 2 to the anomalous values 4 and 10 for some sub-orbits, and to 12 and 16 for some single matrices. The latter matrices are shown to be related to complete sets of MUBs in dimension 9.

1. Introduction

Complex Hadamard matrices have turned out hard to describe in a uniform manner and a comprehensive understanding of such matrices has only been achieved in orders $N \leq 5$. At higher orders, the number of known complex Hadamard matrices, or orbits of such matrices, is growing (see the catalogue in [1]), but no general construction or classification principle has emerged. For example, almost all known complex Hadamard matrices are either isolated, or elements in affine orbits stemming from a seed matrix, typically but not exclusively the Fourier matrix $F_N$. However, non-affine orbits have also been discovered, and in order 6 such orbits play a major role. As another example, prime order Hadamard matrices might be thought of as more elementary than those of composite order. Indeed, almost all known complex Hadamard matrices of orders 4, 6, 8, 9, 10 and 12 can be seen as composed from $F_2$, $F_3$ and $F_5$. However, the most general, non-affine orbit in order 6 is not composed of $F_2$ and/or $F_3$. For an overview of complex Hadamard matrices, see [1, 2, 3, 4].

In view of this situation it is of some interest to identify and categorize as many different (orbits of) complex Hadamard matrices as possible. As a contribution to these efforts, in this paper we report a new orbit in order 9, of a kind not encountered before and therefore of relevance for the ongoing efforts to better understand the full set of complex Hadamard matrices.

2. Notation and definitions.

A complex Hadamard matrix $H_N$ is an $N \times N$ matrix with complex elements of modulus 1, and such that $H_N^*H_N = NI$ (the unitarity constraint). Here, $I$ is the identity matrix. In this paper, all matrices referred to as Hadamard will be of this kind. Hadamard matrices exist for any $N$, as exemplified by the Fourier matrix with elements $(F_N)_{ij} = \omega_N^{(i-1)(j-1)}$, with $\omega_N = \exp(2\pi i/N)$.

Two Hadamard matrices $H$ and $\tilde{H}$ are said to be equivalent, $H \sim \tilde{H}$, if there exist diagonal, unitary matrices $D_1$, $D_2$, and permutation matrices $P_1$, $P_2$ such that

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For each Hadamard matrix there is an equivalent dephased matrix with all elements in the first row and first column equal to 1.

In higher orders, several-parameter orbits of Hadamard matrices are prevalent. Two such orbits will be considered equivalent if for each matrix in one there is an equivalent matrix in the other. A matrix which is not part of an orbit is termed isolated.

An orbit of dephased complex Hadamard matrices is affine if the phases of the elements are linear functions of the orbit parameters.

A Hadamard matrix of even order is \( H_2 \)-reducible [5] if it is equivalent to a matrix where all the \( 2 \times 2 \) submatrices are also (in general enphased) Hadamard matrices. More generally, a composite order Hadamard matrix is reducible if it can be seen as built from Hadamard submatrices of order 2 or more.

In a circulant matrix [6], each row is a copy of the previous row shifted one step to the right, with wrap around. The columns of \( F_n \) are eigenvectors of any such \( N \times N \) circulant matrix. If in a circulant matrix \( C_{n_1} = \text{circ}(a^{(1)}, a^{(2)}, \ldots, a^{(n_1)}) \) of order \( n_1 \) the elements \( a^{(i)} \) are replaced by order \( n_2 \) circulant submatrices \( A^{(i)}_{n_2} \), the result is an order \( N = n_1 n_2 \) block circulant with circulant blocks (BCCB) matrix, which has the columns of \( F_{n_1} \otimes F_{n_2} \) as eigenvectors.

The defect [1, 2] of a unitary \( N \times N \) matrix with elements \( H_{ij} = \exp(iR_{ij}) \) equals \( d(H) = r - (2N - 1) \), where \( r \) is the dimension of the solution space for \( \frac{d}{dt}(H^\dagger H) = 0 \). In a \( p \)-parameter orbit of Hadamard matrices, \( d(H) \geq p \), and its generic defect is the smallest defect encountered along the orbit. There may exist suborbits with larger generic defect, and in particular individual matrices with a larger defect.

3. Low order Hadamard matrices

In order 2, all complex Hadamard matrices are equivalent to the Fourier matrix on standard or circulant form,

\[
F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{or} \quad C_2 = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \sim F_2.
\]

This matrix is isolated.

Also in order 3, all Hadamard matrices are equivalent to the Fourier matrix on standard or circulant form,

\[
F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad \text{or} \quad C_3 = \begin{pmatrix} 1 & \omega & \omega \\ \omega & 1 & \omega \\ \omega & \omega & 1 \end{pmatrix} \sim F_3
\]

where \( \omega = \omega_3 = \exp(2\pi i/3) \). Again, this matrix is isolated.

In order 4, all Hadamard matrices are equivalent to an element in a one-parameter affine orbit, the Fourier orbit \( F_4^{(1)}(a) \) in the notation of [1]. If written on manifestly reducible form, or circulant and BCCB form, this orbit can be represented by

\[
\begin{pmatrix} F_2 & \Delta F_2 \\ \Delta F_2 & -F_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & t & -1 & t \\ t & 1 & t & -1 \\ -1 & t & 1 & t \\ t & -1 & t & 1 \end{pmatrix}
\]

where \( \Delta = \text{diag}(1, x) \) is a diagonal, enphasing matrix with \( x = \exp(ia) \), and where \( t^2 x = 1, x, t \in T \). This orbit is \( H_2 \)-reducible and it is the lowest order example of an orbit of reducible Hadamard matrices. While the generic defect of the orbit is 1, at the point \( x = 1 \), i.e. for the matrix \( F_2 \otimes F_2 \), the defect takes the value 3 [1].

In order 5, all complex Hadamard matrices are equivalent to \( F_5 \), an isolated Hadamard matrix.
In order 6, not only affine but also non-affine orbits have been found. The three-parameter, non-affine orbit \( K_6^{(3)} \) [7] contains the affine suborbit \( F_6^{(2)} \) but also all previously described, smaller non-affine orbits (\( B_6^{(1)} \) [8], \( M_6^{(1)} \) [9], \( K_6^{(2)} \) [10], \( X_6^{(2)} \) and \( X_6^{(2)} \) [11]). It is itself a suborbit of a larger but not yet fully understood non-affine orbit \( G_6^{(4)} \) [12]. Furthermore, \( K_6^{(3)} \) exhausts the set of \( H_2 \)-reducible matrices of order 6, making the general elements of \( G_6^{(4)} \) together with the isolated matrix \( S_9^{(0)} \) stand out as the lowest, composite order Hadamard matrices that are not reducible. In this order, a few circulant [13, 14] or \( BCCB \) Hadamard matrices exist, e.g. \( C_2 \otimes C_3 \), but there are no orbits of such matrices [15]. The generic defect of \( K_6^{(3)} \) and \( F_6^{(2)} \) is 4.

In order 9, which is the case of main interest here, the Fourier orbit \( F_9^{(4)}(a, b, c, d) \) [1] is the only orbit found until now. It is an affine orbit with generic defect 4, and it can be written on the equivalent and manifestly reducible form

\[
F_9^{(4)}(a, b, c, d) = \begin{pmatrix}
F_3 & \Delta_1 F_3 & \Delta_2 F_3 \\
F_3 & \omega \Delta_1 F_3 & \omega^2 \Delta_2 F_3 \\
F_3 & \omega^2 \Delta_1 F_3 & \omega \Delta_2 F_3
\end{pmatrix}
\]

where \( \Delta_1 = \text{diag}(1, 1, 1, 1, 1, 1, 1, 1, 1) \), \( \Delta_2 = \text{diag}(1, 1, 1, 1, 1, 1, 1, 1, 1) \), \( x_1 = \omega_9 e^{ia} \), \( x_2 = \omega_9^2 e^{ic} \), \( x_3 = \omega_9^2 e^{ib} \) and \( x_4 = \omega_9^{1+3n} \) with \( \omega_9 = e^{2\pi i/9} \). For later reference, note that the Kronecker product \( F_3 \otimes F_3 \) is equivalent to an element of \( F_9^{(4)} (\Delta_1 = \Delta_2 = I \text{ in } (3.4)) \), but also to the symmetric block circulant with circulant blocks matrix

\[
C_3 \otimes C_3 = \begin{pmatrix}
C_3 & \omega C_3 & \omega C_3 \\
\omega C_3 & C_3 & \omega C_3 \\
\omega C_3 & \omega C_3 & C_3
\end{pmatrix} \sim F_3 \otimes F_3
\]

and these matrices have defect 16.

The orbit \( F_9^{(4)} \) has a two-parameter suborbit of circulant Hadamard matrices \( FB_{9}^{(2)} = \text{circ}(1, u, v, 1, \omega_9 u, \omega_9^2 v, 1, \omega_9^2 u, \omega_9^3 v) \) [16, 17] with \( u, v \in \mathbb{T} \) related to the parameters of \( F_9^{(4)} \) in (3.4) through \( x_1 = uw^2 \), \( x_2 = uv \) with \( x_3 = \omega_9^2 x_2 \) and \( x_4 = \omega_9 x_2 / x_1 \). Like \( F_9^{(4)} \), this Backelin orbit has generic defect 4. It has 9 circulant one-parameter suborbits with generic defect 6, for \( v = \omega_9^{1+3n} u^2 \), \( u = \omega_9^{1+3n} v \), \( uv = \omega_9^{1+3n} \), \( n = 0, 1, 2 \). At the 27 points where these orbits intersect, the matrices have defect 10.

The orbit \( F_9^{(4)} \) has also a one-parameter suborbit of symmetric \( BCCB \) Hadamard matrices (see \( BC_{9B}^{(1)} \) below), equivalent to the \( F_9^{(4)} \) of (3.4) with \( x_1 = x_2 = x_3 = x_4 \), and with generic defect equal to 10.

Among other suborbits, there is a one-parameter orbit of self-adjoint Hadamard matrices ([4], Prop. 3.4.12), with generic defect equal to 12.

Two isolated matrices of order 9 are also known: the matrix \( N_9^{(0)}[8, 1] \) is equivalent to a symmetric, not reducible matrix, while the recently described matrix \( S_9^{(0)} \) [18] is reducible. Two other matrices, \( B_9^{(1)} \) [8, 1] and \( W_9A[19] \), will be identified below as equivalent to elements in a new non-affine and not reducible \( BCCB \) orbit \( BC_9^{(2)} \).

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1As pointed out by a referee, a matrix \( Q_9 \) and a one-parameter orbit \( Z_9(x) \) are reported in [28]. \( Q_9 \) is an isolated matrix neither equivalent to \( N_9^{(0)} \) nor to \( S_9^{(0)} \), while \( Z_9(x) \), which has the generic defect 2, is equivalent to a non-affine suborbit of the full \( BC_9^{(2)} \) orbit of the present paper.

2The finding that \( B_9^{(0)} \) is not isolated but an element in a two-parameter orbit was anticipated in [8] based on the observation that its defect is 2.
4. Numerical experiments

Numerical experiments have indicated that there exists an orbit of Hadamard matrices of order 9 on the form (all elements of absolute value 1, $x, y, u, w \in T$)

\[
H = \begin{pmatrix}
1 & x & x & y & u & w & y & w & u \\
x & 1 & x & w & y & u & u & y & w \\
x & x & 1 & u & w & y & w & u & y \\
y & w & u & 1 & x & y & u & w & u \\
u & y & w & x & 1 & x & w & y & u \\
w & u & y & x & x & 1 & u & w & y \\
y & u & w & y & w & u & 1 & x & x \\
w & y & u & u & y & w & x & 1 & x \\
w & w & y & w & u & y & x & x & 1 \\
\end{pmatrix}
\]

(4.1)

A matrix of this form has several outstanding properties.

1. Any permutation of $(xyuw)$ results in a matrix that, after permutations of rows and columns, coincides with the original one.
2. It is symmetric.
3. It is block circulant, with circulant blocks ($BCCB$),

\[
H = \begin{pmatrix}
A & B & B^T \\
B^T & A & B \\
B & B^T & A \\
\end{pmatrix}
\]

(4.2)

with

\[
A = \begin{pmatrix}
1 & x & x \\
x & 1 & x \\
x & x & 1 \\
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
y & u & w \\
w & y & u \\
u & w & y \\
\end{pmatrix}
\]

(4.3)

Let $\sigma = x + y + u + w$. It follows from the $BCCB$ property that

\[
A = \frac{1}{9} (F_3 \otimes F_3) H (F_3 \otimes F_3)^\dagger = \begin{pmatrix}
1 + 2\sigma & 0 \\
1 - \sigma + 3y & 1 - \sigma + 3y \\
1 - \sigma + 3x & 1 - \sigma + 3x \\
1 - \sigma + 3w & 1 - \sigma + 3w \\
0 & 1 - \sigma + 3w \\
\end{pmatrix}
\]

(4.4)

is a diagonal matrix with entries equal to the eigenvalues of $H$, and that any two matrices of the general form (4.1) commute.

For $H$ in (4.1) to be Hadamard, the elements $x, y, u, w \in T$ cannot all be chosen independently. By enforcing the unitarity constraint, the missing relations between $x, y, u$ and $w$ can be found, and the most general orbit of complex Hadamard matrices compatible with (4.1) can be constructed. As will be shown below, the result is either a non-affine, two-parameter $BCCB$ orbit, to be denoted $BC^{(2)}_9$, or a set of matrices all equivalent to the $BCCB$ matrix $C_3 \otimes C_3$ of (3.5).
5. The unitarity constraint

The unitarity constraint $H^*H = 9I$ on the matrix $H$ in (4.1) is most easily imposed by requiring that the eigenvalues in (4.4) have modulus 3,

\begin{align}
4\sigma \bar{\sigma} + 2(\sigma + \bar{\sigma}) + 1 &= 9 \\
3x(1-\sigma) + 3\bar{x}(1-\sigma) + (1-\bar{\sigma})(1-\sigma) &= 0
\end{align}

(5.1)

where in the last equation the condition $x\bar{x} = 1$ has been used. There are three additional equations where $x$ is replaced by $y$, $u$, or $w$.

If $\sigma = x + y + u + w = 1$, all the unitarity conditions (5.1) are satisfied, with no additional constraints on $x, y, u, w \in T$, and the eigenvalue matrix is simply $\Lambda = 3 \times \text{diag}(1, y, x, w, u, x, u, w)$.

Otherwise, if $\sigma \neq 1$, the second equation in (5.1) reads

\[ \text{Re}(3x/(1-\sigma)) = -1/2 \]

or, again using that $x\bar{x} = 1$,

\[ \frac{3x}{1-\sigma} = -\frac{1}{2} \pm i \sqrt{\frac{9}{|1-\sigma|^2} - \frac{1}{4}} \]

(5.2)

with similar expressions for $y$, $u$ and $w$. Adding these relations, and taking into account all sign combinations for the square root terms, one finds five possible conditions on $\sigma$,

\[ \frac{3\sigma}{1-\sigma} = -2 + \left\{ \begin{array}{ccc} 4 & 2 \\ 2 & 0 \\ 0 & -2 \\ -2 & -4 \end{array} \right\} \sqrt{\frac{9}{|1-\sigma|^2} - \frac{1}{4}} \]

In the first and last cases,

\[ \frac{3|\sigma|}{|1-\sigma|} = 4 \frac{3}{|1-\sigma|} \]

i.e. $|\sigma| = 4$. Such sigmas are not compatible with the first of the equations (5.1).

Similarly, in the second and fourth cases, $|2\sigma + 1| = 3\sqrt{3}$, again not allowed by (5.1). In the third case, finally, $\sigma$ equals $-2$. This is a value compatible with (5.1) and implies (see (5.2)) that $x$ equals $\omega$ or $\omega^2$, and correspondingly for $y$, $u$ and $w$.

Taking into account that the sum of $x$, $y$, $u$ and $w$ should equal $-2$, the final result is that two of these parameters must have the value $\omega$, and the other two $\omega^2$. For the corresponding eigenvalue matrix one finds $\Lambda = 3 \times \text{diag}(-1, 1 + y, 1 + y, 1 + x, 1 + w, 1 + u, 1 + x, 1 + u, 1 + w)$. Recall that $x = \omega$ implies $1 + x = -\omega^2$ etc, i.e. in addition to $-3$ there are four eigenvalues equal to $-3\omega$ and four equal to $-3\omega^2$.

The above findings are collected in

**Theorem 1.** For the elements $x, y, u, w \in T$ of $H$ in (4.1), the unitarity condition $H^*H = 9I$ implies that either

\[ x + y + u + w = 1 \]

or that one pair of $x, y, u, w$ equals $\omega$ and the other pair equals $\omega^2$, where $\omega = \exp(2\pi i/3)$. In this last case

\[ x + y + u + w = -2. \]

In the $x + y + u + w = 1$ case, the interdependence between $x, y, u$ and $w$ can be illustrated with the help of a complex parameter $\zeta$:

**Lemma 2.** Let $x + y + u + w = 1$ with $x, y, u, w \in T$, and let $\zeta = 2(x + y) - 1 = -2(u + w) + 1$. Then, for $\zeta \neq \pm 1$, 

1. \( \zeta \) is a complex number in the intersection of the circular discs \(|1 + \zeta| \leq 4\) and \(|1 - \zeta| \leq 4\).

2. \( x = \frac{1}{4}(1 + \zeta)(1 + i\sqrt{\frac{16}{|1 + \zeta|^2} - 1}) \) and \( y = \frac{1}{4}(1 + \zeta)(1 - i\sqrt{\frac{16}{|1 + \zeta|^2} - 1}) \) or vice versa.

3. \( u = \frac{1}{4}(1 - \zeta)(1 + i\sqrt{\frac{16}{|1 - \zeta|^2} - 1}) \) and \( w = \frac{1}{4}(1 - \zeta)(1 - i\sqrt{\frac{16}{|1 - \zeta|^2} - 1}) \) or vice versa.

Proof. If \( z_1 \) and \( z_2 \) are of modulus 1, and \( z_1 + z_2 = c \), then, for any \( 0 < |c| \leq 2 \),
\[ z_1 = \frac{x}{2} \left(1 \pm i\sqrt{\frac{1}{|c|^2} - 1}\right) \text{ and } z_2 = \frac{x}{2} \left(1 \mp i\sqrt{\frac{1}{|c|^2} - 1}\right). \]

For each \( \zeta \) in the allowed region there are therefore 4 different, but equivalent Hadamard matrices. In the exceptional point where \( \zeta = 1 \), either \( x = -\omega \) and \( y = -\omega^2 \), or vice versa, with \( u = -w \) arbitrary in \( \mathbb{T} \). This single point in the \( \zeta \) parameter space therefore corresponds to two instances of a one-parameter set. This phenomenon is an artifact of the \( \zeta \) parametrization that also manifests itself in the above expressions for \( u \) and \( v \): with \( \zeta = 1 + \epsilon e^{i\phi} \), the limit \( \epsilon \to 0^+ \) depends on the parameter \( \phi \). The situation where \( \zeta \to -1 \) is analogous.

6. The new non-affine orbit \( BC_9^{(2)} \) and the set \( BC_9(-2) \)

In terms of orbits of Hadamard matrices, the results of the previous section can be summarized as follows. The full set of symmetric \( BCCB \) Hadamard matrices \( BC_9(x, y, u, w) \) on the form (4.1) can be seen as 4 instances \((x \leftrightarrow y \text{ and/or } u \leftrightarrow w)\) of a two parameter orbit \( BC_9^{(2)}(\zeta) \) where

\[
\begin{align*}
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{4}(1 + \zeta)(1 \pm i\sqrt{\frac{16}{|1 + \zeta|^2} - 1}) \\
\begin{bmatrix} u \\ w \end{bmatrix} &= \frac{1}{4}(1 - \zeta)(1 \pm i\sqrt{\frac{16}{|1 - \zeta|^2} - 1})
\end{align*}
\]

(6.1)

In addition there are six different but equivalent matrices where one pair of \( x, y \), \( u \) and \( w \) is equal to \( \omega \), and the other pair equals \( \omega^2 \), so that \( x + y + u + w = -2 \). These matrices will collectively be denoted \( BC_9(-2) \).

No other complex Hadamard matrices are compatible with the form (4.1).

The orbit \( BC_9^{(2)} \) is a non-affine orbit of symmetric \( BCCB \) complex Hadamard matrices, and the defect of a generic element is 2. Since the value of the defect coincides with the number of continuous parameters, \( BC_9^{(2)} \) is not contained in any orbit with additional continuous parameters, and it is in particular not a suborbit of the Fourier orbit \( F_9^{(4)} \) (for which the defect of a generic element is 4).

The six \( BC_9(-2) \) matrices are all equivalent to the matrix \( C_3 \otimes C_3 \) (or \( F_3 \otimes F_3 \)), a matrix which is also in the Fourier orbit \( F_9^{(4)} \), and their defect is 16.

7. Non-reducibility of \( BC_9^{(2)} \)

For the new orbit \( BC_9^{(2)} \) constructed in the previous sections, the question of reducibility has not yet been addressed. A Hadamard matrix of order 9 can at most be \( H_3 \)-reducible, and if this is the case, it must be equivalent to a dephased Hadamard matrix where the upper left \( 3 \times 3 \) submatrix is \( F_3 \). If the new orbit were \( H_3 \)-reducible, in the dephased form there would be at least two elements \( \omega \), and two elements \( \omega^2 \), in such positions that an \( F_3 \) submatrix can be generated through equivalence transformations. For arbitrary parameters \( x, y \in \mathbb{T} \), this cannot happen. Indeed, there are \( 9 \times 9 = 81 \) instances of dephased matrices, and they all have elements taken from the set \( \{1, a, a^2, \frac{1}{a}, \frac{1}{a^2}, \frac{2}{a}, \frac{2}{a^2}, \frac{a}{c}, \frac{a^2}{c}, \frac{b}{c}, \frac{b^2}{c}\} \) where \( a, b \) and...
c are (all different and) equal to any combination of \(x, y, u\) and \(w\). No member in the set equals to \(\omega\) or \(\omega^2\) over the entire parameter space, implying that the orbit \(BC_9^{(2)}\) is not reducible. For special values of the parameters, on the other hand, \(H_3\)-reducibility may occur, as exemplified by the appearance of the matrix \(C_3^\dagger \otimes C_3\) for \(x = \omega, y = \omega^2, u = w = 1\).

8. Affine suborbits of \(BC_9^{(2)}\)

One-parameter suborbits of \(BC_9^{(2)}\) will in general be non-affine. However, two cases of affine suborbits can be identified.

8.1. The affine suborbit \(BC_{9A}^{(1)}\).

If \((xyuw)\) is a permutation of \((\mu, -\mu, -\omega, -\omega^2)\), with \(\mu \in \mathbb{T}\), then \(\zeta = \pm 1, \pm \sqrt{3} \pm 2\mu\) or \(-1 \pm \sqrt{3} \pm 2\mu\). The last two possibilities correspond to two cases of two overlapping circles in the complex \(\zeta\) plane, of radius 2 and with midpoints at \(\pm \sqrt{3}\) (Figure 8.1). The situation at the points \(\zeta = \pm 1\) is more subtle since the \(\mu\) degree of freedom is hidden in how these points are approached in the complex plane. Indeed, let for instance \(\zeta = -1 - \epsilon \mu\), \(\mu \in \mathbb{T}\), and take \(\epsilon \to 0^+\) in (6.1). As a result, \(x, y \to \pm \mu\) and \(u, w \to -\omega^2\) or \(-\omega\), as advertised. As already mentioned, this behavior is an artifact of the parametrization (6.1), and has been noted before in a similar case [10].

Taking \((xyuw) = (\mu, -\mu, -\omega, -\omega^2)\), the resulting suborbit \(BC_{9A}^{(1)}\) has the representation (4.2) with

\[
A = \begin{pmatrix}
1 & \mu & \mu \\
\mu & 1 & \mu \\
\mu & \mu & 1
\end{pmatrix}
\quad \text{and} \quad
B = -\begin{pmatrix}
\mu & \omega & \omega^2 \\
\omega^2 & \mu & \omega \\
\omega & \omega^2 & \mu
\end{pmatrix}.
\]

The defect of a generic element of \(BC_{9A}^{(1)}\) is 2. Interestingly, although affine, \(BC_{9A}^{(1)}\) is not a suborbit of the affine orbit \(F_9^{(4)}\).
8.2. The affine suborbit $BC^{(1)}_{9B}$.

If $(x y u w)$ is a permutation of $(1, \xi, \omega \xi, \omega^2 \xi)$, with $\xi \in \mathbb{T}$, then $\zeta = \pm (1 + 2\xi)$, $\pm (1 + 2\omega \xi)$ or $\pm (1 + 2\omega^2 \xi)$ and there are two cases of three overlapping circles in the complex $\zeta$ plane, again of radius 2 but with midpoints at $\pm 1$ (Figure 8.1). Taking $(x y u w) = (1, \xi, \omega \xi, \omega^2 \xi)$, the resulting affine suborbit $BC^{(1)}_{9B}$ has the representation (4.2) with

\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \xi \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix},
\]

(8.2)

It is straightforward to show that the matrices of $BC^{(1)}_{9B}$ are equivalent to Fourier matrices on the form (3.4) with $\Delta_1 = \text{diag}(1, \xi, \xi^2)$ and $\Delta_2 = \text{diag}(1, \xi^2, \bar{\xi})$, i.e. that the $BC^{(1)}_{9B}$ orbits can be seen as one-parameter intersections of $BC^{(2)}_9$ with the Fourier orbit $F_9^{(4)}$. The defect of a generic element of $BC^{(1)}_{9B}$ turns out to be 10.

**Conjecture 3.** $BC^{(1)}_{9A}$ and $BC^{(1)}_{9B}$ are the only affine suborbits of $BC^{(2)}_9$.

As support for this conjecture, consider the graphs in the complex plane representing the relation $x + y + u + w = 1$ characteristic for $BC^{(2)}_9$. In the $BC^{(1)}_{9A}$ case two links add up to zero and can for that reason have any orientation (the $\mu$ parameter). In the $BC^{(1)}_{9B}$ case three links add up to zero and the resulting triangle can have any orientation (the $\xi$ parameter). We see no other way for a subgraph to exhibit a similar rotational invariance, and hence no room for an associated affine parameter.

9. One-parameter suborbits with defect 4

The occurrence of a full suborbit with the anomalous defect 10, the $BC^{(1)}_{9B}$ of the previous section, has motivated the search for additional suborbits with a defect different from the generic value 2. A numerical search has revealed 24 one-variable suborbit pieces in $BC^{(2)}_9$ with the anomalous defect 4 (Figure 9.1). Lacking an analytic description, we have chosen not to try to put these pieces together into full suborbits. However, if the two generic pieces shown in Figure 9.2 are complemented by other instances of themselves, obtained by permuting the $(x y u w)$ parameters, one half of the full suborbit structure in Figure 9.1 is obtained. The remaining half follows by taking the mirror image in the real $\zeta$-axis.

10. Special matrices

Of some interest are the points where the $BC^{(1)}_{9A}$ and $BC^{(1)}_{9B}$ orbits intersect each other, other instances of themselves, or the boundary of the parameter domain, see Figure 8.1.

The $BC^{(1)}_{9B}$ orbit intersects another such orbit at the points $\zeta = \pm i \sqrt{3}$, and the boundary at $\zeta = \pm 3$. The corresponding matrices are equivalent to $C_3 \oplus C_3 \sim F_3 \odot F_3$, see equation (3.5). This matrix is in the Butson class $BH(9, 3)$, and its defect is known to be 16 [1].

A $BC^{(1)}_{9B}$ orbit intersects a $BC^{(1)}_{9A}$ orbit at the six points $\zeta = \pm 1, \pm (1 - 2\omega)$ and $\pm (1 - 2\omega^2)$. For instance, with $\mu = 1$ or $\xi = -1$, $(x y u w)$ equals $(1, -1, -\omega, -\omega^2)$ and $\zeta = -1$. The resulting matrix $BC_{9A \supset B}$ is a symmetric, block circulant with circulant blocks matrix in the Butson class $BH(9, 6)$.

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3 based on \( \sim 700 \) points.
(10.1)  \[ BC_{9A\cap B} = \begin{pmatrix} A & B & B^T \\ B^T & A & B \\ B & B^T & A \end{pmatrix} \]

with

(10.2)  \[ A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = -\begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}, \]

and it is equivalent to the (Fourier orbit) matrix (3.4) with \( \Delta_1 = \text{diag}(1, -1, 1) \) and \( \Delta_2 = \text{diag}(1, 1, -1) \). In whichever form, this matrix is a natural, common seed matrix for all three affine orbits \( F^{(4)}_9, BC^{(1)}_9 \) and \( BC^{(1)}_{9B} \), and its defect is 12.

Instances of the \( BC^{(1)}_{9A} \) orbit intersect the domain boundary at the four points \( \zeta = \pm(1+4\omega) \) and \( \zeta = \pm(1+4\omega^2) \). Taking \( (xyuw) = (\omega, -\omega^2,-\omega,-\omega) \) at \( \zeta = 1+4\omega \)

**Figure 9.1.** Defect 4 suborbits of \( BC^{(2)}_9 \)

**Figure 9.2.** Defect 4 generic suborbit(s) of \( BC^{(2)}_9 \)
results in the BCCB matrix,

\[
(10.3) \quad BC_{9,Ab} = \begin{pmatrix}
    C_3 & -\omega^2 C_3^* & -\omega C_3^*
    -\omega^2 C_3^* & C_3 & -\omega^2 C_3^*
    -\omega C_3^* & -\omega^2 C_3^* & C_3
\end{pmatrix}
\]

Similarly, taking \((xyuw) = (\omega^2, -\omega, -\omega^2, -\omega^2)\) at \(\zeta = 1 + 4\omega^2\) results in \(BC_{9,Ab}^\dagger\). These two matrices are both in the Butson class \(BH(9,6)\) and have defect 10. A direct calculation has shown that they are not equivalent.

**Proposition 4.** The Butson matrices \(BC_{9,Ab}\) and \(BC_{9,Ab}^\dagger\) are not elements in the Fourier orbit \(F_9^{(4)}\).

**Proof.** The two matrices \(BC_{9,Ab}\) and \(BC_{9,Ab}^\dagger\) are in \(BH(9,6)\) and have defect 10. Taking \(x_1, x_2, x_3\) and \(x_4\) in (3.4) as any combination of \(\pm 1, \pm \omega\) and \(\pm \omega^2\), and \(\omega^2\), a direct evaluation shows that for the resulting \(6^4\) \(BH(9,6)\) matrices in \(F_9^{(4)}\) the defect is 4 in 864 cases, 8 in 243 cases, 12 in 162 cases and 16 in 27 cases, but never 10. \(\square\)

Finally, two complex Hadamard matrices of order 9 reported earlier can now be identified as elements in \(BC_{9}^{(2)}\). For \((xyuw)\) some permutation of \((\tau \tau \bar{\tau} \bar{\tau})\), with \(\tau = \frac{1}{2}(1 + i\sqrt{15})\), either \(\zeta = 0\) or \(\zeta = \pm i\sqrt{15}\). The corresponding element of \(BC_{9}^{(2)}\) is equivalent to the matrix \(W_{9A}\) of [19]. For \((xyuw)\) some permutation of \((e^{-i} i^{-i} e^{-i} e^{-i})\), with \(\epsilon = \exp(2\tau i/10)\), the corresponding elements of \(BC_{9}^{(2)}\) are equivalent to the \(B_{9}^{(0)}\) of [8, 1]. This matrix is found at the six points where \(\zeta = \pm \sqrt{5}\), \(\zeta = \pm i(\sqrt{5} + \sqrt{5})/2 + \sqrt{(5 - \sqrt{5})/2}\) or \(\zeta = \pm i(\sqrt{5} + \sqrt{5})/2 - \sqrt{(5 - \sqrt{5})/2}\). The defect for the matrices \(B_{9}^{(0)}\) and \(W_{9A}\) is 2.

**11. MUBS for \(N = 3^2\)**

Mutually unbiased bases (MUBs) in \(\mathbb{C}^N\) are closely related to complex Hadamard matrices: taking the basis vectors as columns of a matrix, the standard basis can be represented by the unit matrix, and all other bases will then appear as (mutually unbiased and) in general emphaed complex Hadamard matrices. This section serves to illustrate how the results of the present paper are related to such bases in dimension \(N = 9\).

Complete sets of \(N+1\) MUBs exist for all prime and powers of a prime dimensions \(N\) (for recent reviews, see [20, 21]). For example, for \(N = 9 = 3^2\), the complete set \(\{B_i\}_{i=0..9}\) of MUBs given in [22, 23] have the form \(B_0 = I\) and \(B_i = \frac{1}{3}D_i(F_3 \otimes F_3)\) where the unitary, diagonal matrices \(D_i\) are specified in Table 1. The \(D_i\) matrices form a closed set under the product \(D_i^\dagger D_j = D_k\) according to the pattern displayed for the \(M_i\) matrices in Table 2.

Several other complete sets of MUBs have been shown to be equivalent to this set in the sense that one set can be obtained from another by means of an overall unitary transformation, c.f. Ref [19].

Of particular interest in the context of the present paper is the complete set \(\{M_i\}_{i=0..9}\) which is obtained from the \(\{B_i\}\) set through left multiplication by (the unitary matrix) \(\frac{1}{3}(F_3 \otimes F_3)^\dagger\), i.e. \(^4\)

\[
\begin{align*}
M_0 &= \frac{1}{3}(F_3 \otimes F_3)^\dagger, \text{ or equivalently, } \frac{1}{3}F_3 \otimes F_3 \\
M_1 &= I \\
M_i &= \frac{1}{9}(F_3 \otimes F_3)^\dagger D_i(F_3 \otimes F_3), \ i = 2, \ldots, 9
\end{align*}
\]

\(^4\)note that \((F_3 \otimes F_3)^\dagger\) and \(F_3 \otimes F_3\) are column permutation equivalent.
all properties inherited from the matrices of the general form (4.1) investigated in the previous sections. The out-

lant blocks

Except for

σ

identified through inspection of the eigenvalues. A comparison of Table 1 with

Table 8 (the intersection of the

matrices are therefore equal to elements in the suborbit

3. These matrices are therefore equal to elements in the suborbit

BC

COMPLEX HADAMARD MATRICES OF ORDER 9, AND MUBS

Table 1. Diagonals of the MUB unitaries

D

as obtained from

Table 2.

Multiplication table for

M

j

. Note the block circulant with circulant blocks structure.

Table 2. Multiplication table for

M

i

j

. Note the block circulant with circulant blocks structure.

ExCEPT for

M

6

and

M

1

, the

M

i

matrices are symmetric, block circulant with circulant blocks mutually unbiased Hadamard matrices, i.e. (apart from normalization) matrices of the general form (4.1) investigated in the previous sections. The outstanding features [24] of this set are that, for

i, j, k in

\{1, 2, ..., 9\},

\[
M_i^3 = I \\
M_i M_j = M_j M_i \\
M_i^k M_j = M_k, \text{ see Table 2,}
\]

all properties inherited from the \(\{D_i\}_{i=1,...,9}\) matrices. Furthermore, \(M_3 = M_2^\dagger\), \(M_7 = M_4^\dagger\), \(M_6 = M_5^\dagger\) and \(M_9 = M_1^\dagger\), i.e. for each \(M_i\) there is an \(M_j\) such that \(M_i^k = M_j\). This set of MUBs is therefore invariant under Hermitian conjugation.

The parameters in (4.1) corresponding to the \(\{M_i\}_{i=2,...,9}\) matrices are easily identified through inspection of the eigenvalues. A comparison of Table 1 with the eigenvalues (4.4) for \(\sigma = 1\) and \(\sigma = -2\) shows that for \(M_2, M_3, M_5\) and \(M_9\) the parameters \((xyuw)\) in (4.1) are permutations of \((1, 1, \omega, \omega^2)\), see Table 3. These matrices are therefore equal to elements in the suborbit \(BC_{9B}^{(1)}\) in Section 8 (the intersection of the \(BC_{9B}^{(2)}\) of Theorem 1 with \(E_{9B}^{(3)}\) corresponding to \(\xi = 1\). For \(-M_4, -M_6, -M_7\) and \(-M_8\) the parameters \((xyuw)\) are permutations of
Table 3. Parameters in the $BC_9(x,y,u,w)$ orbit (see (4.1)) corresponding to the $M_i$ matrices. Note that $u = \bar{x}$ and $w = \bar{y}$.

| $M_2$ | $M_3$ | $-M_4$ | $M_5$ | $-M_6$ | $-M_7$ | $-M_8$ | $M_9$ |
|-------|-------|--------|-------|--------|--------|--------|-------|
| $x$   | 1     | 1      | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ |
| $y$   | $\omega^2$ | $\omega$ | $\omega^2$ | 1 | $\omega$ | $\omega^2$ | 1 |
| $u$   | 1     | 1      | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega^2$ | $\omega$ |
| $w$   | $\omega$ | $\omega^2$ | 1 | $\omega^2$ | $\omega^2$ | $\omega$ | 1 |

$(\omega, \omega, \omega^2, \omega^2)$, see Table 3, characteristic for matrices in the exceptional set $BC_9(-2)$ also identified in Theorem 1.

It should be pointed out that as Hadamard matrices, the $M_i$ matrices for $i \geq 2$ are all equivalent to each other, and to the $\frac{1}{4}C_3 \otimes C_3$ matrix of (3.5) (for instance, $-M_7 = \frac{1}{4}C_3 \otimes C_3$). Their common defect is therefore 16.

As a final remark, recall from Section 4 that a permutation of the $(x,y,u,w)$ parameters in (4.1) can be undone by permutations of rows and columns. Specifically, a permutation of the parameters $(x,y,u,w)$ in Table 3 either leaves the set $\{M_i\}_{i=1,\ldots,9}$ invariant, or gives rise to one of two additional, but equivalent, sets $\{P_1^iM_iP_1\}_{i=1,\ldots,9}$ or $\{P_2^iM_iP_2\}_{i=1,\ldots,9}$ where $P_1$ and $P_2$ are (unitary) permutation matrices that interchange $y$ and $u$, or $u$ and $w$, respectively.

Explicit expressions for complete sets of MUBs in 9 dimensions have been published several times. It has been verified that the versions given in references [25, 26, 27] indeed are equivalent to the $\{M_i\}$ set of the present section (and therefore also to the set of references [22, 23]).

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5There is a misprint in Table 5 of [25]. The corrected entries read $|3_1\rangle = 1 \omega 1 \omega 1 \omega 1 \omega$ and $|3_2\rangle = 1 \omega \omega \omega \omega \omega \omega$. There is also a misprint in the $B_3$ of Appendix G in [27]. The corrected matrix elements read $B_3(2,5) = \alpha^2_3$, $B_3(3,5) = 1$ and $B_3(4,5) = \alpha_3$. 
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