DECOMPOSITIONS OF WEIGHTED CONDITIONAL EXPECTATION TYPE OPERATORS

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Abstract. In this paper we investigate boundedness, polar decomposition and spectral decomposition of weighted conditional expectation type operators on $L^2(\Sigma)$.

1. Introduction and Preliminaries

Let $(X, \Sigma, \mu)$ be a complete $\sigma$-finite measure space. For any sub-$\sigma$-finite algebra $A \subseteq \Sigma$ with $1 \leq p \leq \infty$, the $L^p$-space $L^p(X, A, \mu|_A)$ is abbreviated by $L^p(A)$, and its norm is denoted by $\|\cdot\|_p$. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set. The support of a measurable function $f$ is defined as $S(f) = \{x \in X; f(x) \neq 0\}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable functions on $X$ by $L^0(\Sigma)$.

For a sub-$\sigma$-finite algebra $A \subseteq \Sigma$, the conditional expectation operator associated with $A$ is the mapping $f \mapsto E_A f$, defined for all non-negative, measurable function $f$ as well as for all $f \in L^p(\Sigma)$, $1 \leq p \leq \infty$, where $E_A f$, by the Radon-Nikodym theorem, is the unique $A$-measurable function satisfying

$$\int_A f \, d\mu = \int_A E_A f \, d\mu, \quad \forall A \in A.$$

As an operator on $L^p(\Sigma)$, $E_A$ is idempotent and $E_A(L^p(\Sigma)) = L^p(A)$. If there is no possibility of confusion, we write $E(f)$ in place of $E_A(f)$. This operator will play a major role in our work and we list here some of its useful properties:

- If $g$ is $A$-measurable, then $E(fg) = E(f)g$.
- $|E(f)|^p \leq E(|f|^p)$.
- If $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.
- $|E(fg)| \leq E(|f|^p)E(|g|^q)^{\frac{1}{q}}$, where $\frac{1}{p} + \frac{1}{q} = 1$ (Hölder inequality).
- For each $f \geq 0$, $S(f) \subseteq S(E(f))$.

A detailed discussion and verification of most of these properties may be found in [11]. We recall that an $A$-atom of the measure $\mu$ is an element $A \in A$ with $\mu(A) > 0$ such that for each $F \in A$, if $F \subseteq A$, then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space $(X, \Sigma, \mu)$ with no atoms is called a non-atomic measure space. It is well-known fact that every $\sigma$-finite measure space $(X, A, \mu|_A)$ can be partitioned

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Throughout this paper we take $u = \mathcal{L}_{\text{Moy}}$ and characterized all operators on weighted composition operators and integral operators. Specifically, in [9], S.-T. C. A. disjoint $X_2$.

In [4, 5] we investigated some classic properties of multiplication conditional expectations. More recently, P.G. Dodds, C.B. Huijsmans and B. De Pagter, [3], extended these characterizations to the setting of function ideals and vector lattices. J. Herron presented some assertions about the operator $D_{\text{Pagter}}$, [3], extended these characterizations to the setting of function ideals and conditional expectations. More recently, P.G. Dodds, C.B. Huijsmans and B. De Pagter, [3], extended these characterizations to the setting of function ideals and vector lattices. J. Herron presented some assertions about the operator $D_{\text{Pagter}}$.

Throughout this paper we take $u$ and $w$ in $\mathcal{D}(E)$. In this paper we present some results on the boundedness, polar decomposition and spectral decomposition of this operators in $L^1(\Sigma)$, using different methods than those employed in [5].

2. Polar decomposition

**Theorem 2.1.** The operator $T = M_w E_{\text{Moy}} : L^2(\Sigma) \to L^2(\Sigma)$ is bounded if and only if $(E(|w|^2)^\frac{1}{2})(E(|u|^2)^\frac{1}{2}) \in L^\infty(\mathcal{A})$ and in this case $\|T\| = \|(E(|w|^2)^\frac{1}{2})(E(|u|^2)^\frac{1}{2})\|_{\infty}$.

**Proof** Suppose that $(E(|w|^2)^\frac{1}{2})(E(|u|^2)^\frac{1}{2}) \in L^\infty(\mathcal{A})$. Let $f \in L^2(\Sigma)$. Then

$$\|T(f)\|_2^2 = \int_X |wE(u/f)|^2d\mu = \int_X E(|w|^2)|E(u/f)|^2d\mu \leq \int_X E(|w|^2)E(|u|^2)E(|f|^2)d\mu.$$ 

Since $|E(u/f)| \leq (E(|u|^2)^\frac{1}{2})(E(|f|^2)^\frac{1}{2})$. Thus

$$\|T\| \leq \|(E(|w|^2)^\frac{1}{2})(E(|u|^2)^\frac{1}{2})\|_{\infty}.$$ 

To prove the converse, let $T$ be bounded on $L^2(\Sigma)$ and consider the case that $\mu(X) < \infty$. Then for all $f \in L^2(\Sigma)$ we have

$$\|T(f)\|_2^2 = \int_X |wE(u/f)|^2d\mu = \int_X E(|w|^2)|E(u/f)|^2d\mu \leq \|T\|^2 \int_X |f|^2d\mu.$$ 

For each $n \in \mathbb{N}$, define

$$E_n = \{x \in X : |u(x)|(E(|w|^2)^\frac{1}{2})(x) \leq n\}.$$ 

Each $E_n$ is $\Sigma$-measurable and $E_n \uparrow X$. Define $G_n = E_n \cap S$ for each $n \in \mathbb{N}$, where $S = S(|u|(E(|w|^2)^\frac{1}{2}))$. Let $A \in \mathcal{A}$ and define

$$f_n = u(E(|w|^2)^\frac{1}{2})\chi_{G_n \cap A}.$$
for each \( n \in \mathbb{N} \). It is clear that \( f_n \in L^\infty(\Sigma) \) for all \( n \) (which in our case implies \( f_n \in L^2(\Sigma) \)). For each \( n \),

\[
\|T(f_n)\|_2^2 = \int_X E(|w|^2)E(u f_n)^2 d\mu \\
= \int_X (E(|w|^2))^2(E(|u|^2 \chi_{G_n} \chi_A))^2 d\mu \\
= \int_A [E(|w|^2)] E(|u|^2 \chi_{G_n})^2 d\mu \\
\leq \|T\|^2 \int_X |f_n|^2 d\mu \\
= \|T\|^2 \int_A E(|u|^2 \chi_{G_n}) E(|w|^2) d\mu.
\]

Since \( A \) is an arbitrary \( A \)-measurable set and the integrands are both \( A \)-measurable functions, we have

\[
[E(|w|^2)] E(|u|^2 \chi_{G_n})^2 \leq \|T\|^2 E(|u|^2 \chi_{G_n}) E(|w|^2)
\]

almost everywhere. That is

\[
E((E(|w|^2))^\frac{1}{2}|u| \chi_{E_n})^2) = \|T\|^2 E((u| \chi_{E_n})^2)_{\chi_S}).
\]

Since

\[
S = \sigma(|u| (E(|w|^2))) = \sigma|u|^2 E(|w|^2)
\]

and

\[
|u|^2 E(|w|^2)_{\chi_S} = |u|^2 E(|w|^2),
\]

we have

\[
E((E(|w|^2))^\frac{1}{2}|u| \chi_{E_n})^2) \leq \|T\|^2.
\]

Thus

\[
(E|w|^2)^\frac{1}{2}(E|u|^2 \chi_{E_n})^\frac{1}{2} \leq \|T\|.
\]

This implies that \( (E|w|^2)^\frac{1}{2}(E|u|^2 \chi_{E_n})^\frac{1}{2} \in L^\infty(A) \) and

\[
\|(E|w|^2)^\frac{1}{2}(E|u|^2)\|_\infty \leq \|T\|.
\]

Moreover, since \( E_m \uparrow X \), the conditional expectation version of the monotone convergence theorem implies \( \|(E|w|^2)^\frac{1}{2}(E|u|^2)\|_\infty \leq \|T\| \). \( \square \)

**Proposition 2.2.** Let \( g \in L^\infty(A) \) and let \( T = M_{E} E M : L^2(\Sigma) \rightarrow L^2(\Sigma) \) be bounded. If \( M_{E} T = 0 \), then \( g = 0 \) on \( \sigma(E(|w|^2) E(|u|^2)) \).

**Proof.** Let \( f \in L^2(\Sigma) \). Then \( gw E(u f) = M_{E} T(f) = 0 \). Now, by Theorem 2.1

\[
0 = \|M_{E} T\|^2 = \|g|^2 E(|w|^2) E(|u|^2)\|_\infty,
\]

which implies that \( |g|^2 E(|w|^2) E(|u|^2) = 0 \), and so \( g = 0 \) on \( \sigma(E(|w|^2) E(|u|^2)) \). \( \square \)

**Theorem 2.3.** The bounded operator \( T = M_{E} E M \) is a partial isometry if and only if \( E(|w|^2) E(|u|^2) = \chi_A \) for some \( A \in A \).

**Proof.** Suppose \( T \) is partial isometry. Then \( TT^*T = T \), that is \( T f = E(|w|^2) E(|u|^2) T f \), and hence \( (E(|w|^2) E(|u|^2) - 1)T f = 0 \) for all \( f \in L^2(\Sigma) \). Put \( S = S(E(|u|^2)) \) and
G = S(E(|w|^2)). By Proposition 2.2, we get that \( E(|w|^2)E(|u|^2) = 1 \) on \( S \cap G \), which implies that \( E(|w|^2)E(|u|^2) = \chi_A \), where \( A = S \cap G \).

Conversely, suppose that \( E(|w|^2)E(|u|^2) = \chi_A \) for some \( A \in \mathcal{A} \). It follows that \( A = S \cap G \), and we have

\[
TT^*(f) = E(|w|^2)E(|u|^2)|f| = E(S^wE(u)|f|) = wE uf,
\]

where we have used the fact that \( S(Tf) = S(|Tf|^2) \subseteq S \cap G \), which this is a consequence of Hölder’s inequality for conditional expectation \( E \).

The spectrum of an operator \( A \) is the set

\[
\sigma(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \mbox{ is not invertible} \}.
\]

It is well known that any bounded operator \( A \) on a Hilbert space \( \mathcal{H} \) can be expressed in terms of its polar decomposition: \( A = UP \), where \( U \) is a partial isometry and \( P \) is a positive operator. (An operator is positive if \( P(f, f) \geq 0 \), for all \( f \in \mathcal{H} \).) This representation is unique under the condition that \( \ker U \cap \ker P = \ker A \). Moreover, \( P = |A| = (A^*A)^{1/2} \).

Let \( q(z) \) be a polynomial with complex coefficients: \( q(z) = \sum_{n=0}^{N} \alpha_n z^n \). If \( T \) is a bounded operator on \( L^2(\Sigma) \), then the operator \( q(T) \) is defined by \( q(z) = \alpha_0 I + \sum_{n=1}^{N} \alpha_n T^n \). Let \( M_\varphi \) be a bounded multiplication operator on \( L^2(\Sigma) \), then \( q(M_\varphi) \) is also bounded and \( q(M_\varphi) = M_{q\varphi} \). By the continuous functional calculus, for any \( f \in C(\sigma(M_\varphi)) \), we have \( g(M_\varphi) = M_{g\varphi} \).

**Proposition 2.4.** Let \( S = S(E(|u|^2)) \) and \( G = S(E(|w|^2)) \). If \( f \in C(\sigma(M_{E(|u|^2)})) \) and \( g \in C(\sigma(M_{E(|w|^2)})) \), then

\[
f(T^*T) = f(0)I + M_{E(|u|^2)}^{-1} \chi_S \left( M_{f\varphi(E(|u|^2)E(|w|^2))} - f(0)I \right) M_u EM_u
\]

and

\[
g(TT^*) = g(0)I + M_{E(|w|^2)}^{-1} \chi_G \left( M_{g\varphi(E(|u|^2)E(|w|^2))} - g(0)I \right) M_w EM_w.
\]

**Proof.** For all \( f \in L^2(\Sigma) \), \( T^*T(f) = \bar{u}E(|w|^2)E(u|f|) \) and \( TT^*(f) = wE(|u|^2)E(\bar{u}|f|) \). By induction, for each \( n \in \mathbb{N} \),

\[
(T^*T)^n(f) = \bar{u}(E(|w|^2))^n(E(|u|^2))^{-n-1}E(u|f|), \quad (TT^*)^n(f) = w(E(|u|^2))^n(E(|w|^2))^{-n-1}E(\bar{u}|f|).
\]

So

\[
q(T^*T) = q(0)I + M_{E(|u|^2)}^{-1} \chi_S \left( M_{q\varphi(E(|u|^2)E(|w|^2))} - q(0)I \right) M_u EM_u
\]

and

\[
g(TT^*) = g(0)I + M_{E(|w|^2)}^{-1} \chi_G \left( M_{g\varphi(E(|u|^2)E(|w|^2))} - g(0)I \right) M_w EM_w.
\]

By the Weierstrass approximation theorem we conclude that, for every \( f \in C(\sigma(M_{E(|u|^2)})) \) and \( g \in C(\sigma(M_{E(|u|^2)})) \),

\[
f(T^*T) = f(0)I + M_{E(|u|^2)}^{-1} \chi_S \left( M_{f\varphi(E(|u|^2)E(|w|^2))} - f(0)I \right) M_u EM_u
\]

and

\[
g(TT^*) = g(0)I + M_{E(|w|^2)}^{-1} \chi_G \left( M_{g\varphi(E(|u|^2)E(|w|^2))} - g(0)I \right) M_w EM_w.
\]

**Theorem 2.5.** The unique polar decomposition of \( T = M_u EM_u \) is \( U|T| \), where
\[ |T|(f) = \left( \frac{E(|w|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S \bar{u} E(uf), \quad U(f) = \left( \frac{\chi_{SG}}{E(|w|^2)E(|u|^2)} \right)^{\frac{1}{2}} w E(uf), \]

for all \( f \in L^2(\Sigma) \).

**Proof.** By Proposition 2.4 we have

\[ |T|(f) = (T^* T)^{\frac{1}{2}}(f) = \left( \frac{E(|w|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S \bar{u} E(uf). \]

Define a linear operator \( U \) whose action is given by

\[ U(f) = \left( \frac{\chi_{SG}}{E(|w|^2)E(|u|^2)} \right)^{\frac{1}{2}} w E(uf), \quad f \in L^2(\Sigma). \]

Then \( T = U|T| \) and by Theorem 2.3, \( U \) is a partial isometry. Also, it is easy to see that \( \mathcal{N}(T) = \mathcal{N}(U) \). Since for all \( f \in L^2(\Sigma) \), \( \|Tf\|_2 = \|T\|_2 \), hence \( \mathcal{N}(|T|) = \mathcal{N}(U) \) and so this decomposition is unique. \( \square \)

**Theorem 2.6.** The Aluthge transformation of \( T = M_w E_M \) is

\[ \hat{T}(f) = \frac{\chi_S E(uw)}{E(|w|^2)} \bar{u} E(uf), \quad f \in L^2(\Sigma). \]

**Proof.** Define operator \( V \) on \( L^2(\Sigma) \) as

\[ V(f) = \left( \frac{E(|w|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S \bar{u} E(uf), \quad f \in L^2(\Sigma). \]

Then we have \( V^2 = |T| \) and so by direct computation we obtain

\[ \hat{T}(f) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}(f) = \frac{\chi_S E(uw)}{E(|w|^2)} \bar{u} E(uf). \]

\( \square \)

3. **Spectral decomposition**

The normal operators form one of the best understood and most tractable of classes of operators. The principal reason for this is the spectral theorem, a powerful structure theorem that answers many (not all) questions about these operators. In this section we explore spectral measure and spectral decomposition corresponding to a normal weighted conditional expectation operator \( E_M \) on \( L^2(\Sigma) \).

**Definition 3.1.** If \( X \) is a set, \( \Sigma \) a \( \sigma \)-algebra of subsets of \( X \) and \( H \) a Hilbert space, a spectral measure for \( (X, \Sigma, H) \) is a function \( \mathcal{E} : \Sigma \to B(H) \) having the following properties.

(a) \( \mathcal{E}(S) \) is a projection.
(b) \( \mathcal{E}(\emptyset) = 0 \) and \( \mathcal{E}(X) = I \).
(c) If \( S_1, S_2 \in \Sigma \), \( \mathcal{E}(S_1 \cap S_2) = \mathcal{E}(S_1) \mathcal{E}(S_2) \).
(d) If \( \{S_n\}_{n=0}^\infty \) is a sequence of pairwise disjoint sets in \( \Sigma \), then

\[ \mathcal{E}(\cup_{n=0}^\infty S_n) = \Sigma_{n=0}^\infty \mathcal{E}(S_n). \]
The spectral theorem says that: For every normal operator $T$ on a Hilbert space $H$, there is a unique spectral measure $E$ relative to $(\sigma(T), H)$ such that $T = \int_{\sigma(T)} z dE$, where $z$ is the inclusion map of $\sigma(T)$ in $\mathbb{C}$.

J. Herron showed that $\sigma(EM_u) = \text{ess range}(Eu) \cup \{0\}$. Also, He has proved that: $EM_u$ is normal if and only if $u \in L^\infty(A)$. If $T = EM_u$ is normal, then $T^n = M_{u^n}E$ and $(T^*)^n = M_{\bar{u}^n}E$. So $(T^*)^nT^n = M_{(\bar{u}u)^n}E$ and

$$P(T,T^*) = \sum_{n,m=0}^{N,M} \alpha_{n,m}T^n(T^*)^m = \sum_{n,m=0}^{N,M} \alpha_{n,m}\bar{u}^nu^m E = P(u,\bar{u})E = EP(u,\bar{u}).$$

Where $p(z,t) = \sum_{n,m=0}^{N,M} \alpha_{n,m}z^n t^m$. If $q(z) = \sum_{n=0}^{N} \alpha_n z^n$, then $q(T) = \sum_{n=0}^{N} \alpha_n u^n E$.

Hence by the Weierstrass approximation theorem we have $f(T) = M_{f(u)}E$, for all $f \in C(\sigma(EM_u))$. Thus $\phi : C(\sigma(EM_u)) \rightarrow C^*(EM_u,I)$, by $\phi(f) = M_{f(u)}E$, is a unital $\ast$-homomorphism. Moreover, by Theorem 2.1.13 of [10], $\phi$ is also a unique $\ast$-isomorphism such that $\phi(z) = EM_u$, where $z : \sigma(EM_u) \rightarrow \mathbb{C}$ is the inclusion map.

If $EM_u$ is normal and compact, then $\sigma(EM_u) = \{0\} \cup \{\lambda_n\}_{n \in \mathbb{N}}$ where $\lambda_n \neq 0$ for all $n \in \mathbb{N}$.

So, for each $n \in \mathbb{N}$

$$E_n = \{0 \neq f \in L^2(\Sigma) : E uf = \lambda_n f\}$$

$$= \{0 \neq f \in L^2(A) : uf = \lambda_n f\}$$

$$= L^2(A_n,A_n,\mu_n),$$

where $A_n = \{x \in X : u(x) = \lambda_n\}$, $E_0 = \{f \in L^2(\Sigma) : E uf = 0\}$, $A_n = \{A_n \cap B : B \in A\}$ and $\mu_n \equiv \mu |_{A_n}$. It is clear that for all $n,m \in \mathbb{N} \cup \{0\}$, $E_n \cap E_m = \emptyset$. This implies that the spectral decomposition of $EM_u$ is as follows:

$$EM_u = \sum_{n=0}^{\infty} \lambda_n P_{E_n},$$

where $P_{E_n}$ is the orthogonal projection onto $E_n$. Since $EM_u$ is normal, then $\sigma(EM_u) = \text{ess range}(u) \cup \{0\}$. So $\{\lambda_n\}_{n=0}^{\infty}$ is a resolution of the identity on $X$.

Suppose that $W = \{u \in L^0(\Sigma) : E(|u|^2) \in L^\infty(A)\}$. If we set $\|u\| = \|(E(|u|^2))^{\frac{1}{2}}\|_{\infty}$, then $W$ is a complete $\ast$-subalgebra of $L^\infty(\Sigma)$.

In the sequel we assume that, $\varphi : X \rightarrow X$ is nonsingular transformation i.e, the measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to the measure $\mu$, and $\varphi^{-1}(\Sigma)$ is a sub-$\sigma$-finite algebra of $\Sigma$. Put $h = d\mu \circ \varphi^{-1}/d\mu$ and $E^\varphi = E^{\varphi^{-1}}(\Sigma)$.

For $S \in \Sigma$, let $E(S) : L^2(\Sigma) \rightarrow L^2(\Sigma)$ be defined by

$$E(S)(f) = E^\varphi M_{\chi_{\Delta^{-1}(S)}}(f),$$

i.e,

$$E(S) = E^\varphi M_{\chi_{\Delta^{-1}(S)}}.$$

$E$ defines a spectral measure for $(X,\Sigma,L^2(\mu))$. If $E^\varphi M_u$ is normal on $L^2(\Sigma)$, then by Theorem 2.5.5 of [10], $E$ is the unique spectral measure corresponding to $\ast$-homomorphism $\phi$ that is defined as follows:

$$\phi : C(\sigma(E^\varphi M_u)) \rightarrow C^*(EM_u,I), \phi(f) = E^\varphi M_{f(u)}.$$
So, for all \( f \in C(\sigma(E^\varphi M_u)) \) we have
\[
\phi(f) = \int_X f d\mathcal{E}.
\]
In [1] it is explored that which sub-\(\sigma\)-algebras of \( \Sigma \) are of the form \( \varphi^{-1}(\Sigma) \) for some nonsingular transformation \( \varphi : X \to X \). These observations establish the following theorem.

**Theorem 3.2.** Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite measure space, \( \varphi : X \to X \) be a nonsingular transformation and let \( u \) be in \( L^\infty(\varphi^{-1}(\Sigma)) \). Consider the operator \( E^\varphi M_u \) on \( L^2(\Sigma) \). Then the set function \( \mathcal{E} \) that is defined as: \( \mathcal{E}(S) = E^\varphi M_{\chi_{\varphi^{-1}(S)}} \) for \( S \in \Sigma \), is a spectral measure. Also, \( \mathcal{E} \) has compact support and
\[
E^\varphi M_u = \int z d\mathcal{E}.
\]

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