Variational problems on product spaces
Different obstacle constraints

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Abstract

We study two principle minimizing problems, subject of different constraints. Our open sets are assumed bounded, except mentioning otherwise; precisely $\Omega = [0,1]^n \in \mathbb{R}^n$, $n = 1$ or $n = 2$.

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1 Introduction

Theorem 1.1 (Rellich-Kondrachov theorem). Suppose $\Omega$ is bounded and of class $C^1$ then, $W^{1,p} \subset L^p$ with compact injection for all $p$ (and all $n$).

Let $p \geq 2$ and $W^{1,p}(]0,1[; \mathbb{R}^2) = \{ u = (u_1, u_2); u_1 \in W^{1,p}(]0,1[; \mathbb{R}), u_2 \in W^{1,p}(]0,1[; \mathbb{R}) \}$

Define the functionals

1. $W^{1,p}(]0,1[; \mathbb{R}^2) \to \mathbb{R}_+$
   
   $u \to F(u) = \int_0^1 |u'(x)|_2^p = \int_0^1 \left( |u_1'(x)|^2 + |u_2'(x)|^2 \right)^{\frac{p}{2}}$

2. $W_0^{1,p}(\Omega; \mathbb{R}^2) \to \mathbb{R}_+$
   
   $u \to K(u) = \int_\Omega |\nabla u(x)|_2^p = \int_\Omega \left( |\nabla u_1(x)|_2^2 + |\nabla u_2(x)|_2^2 \right)^{\frac{p}{2}}$

Mainly, our goals are:

- show if that there exists $u_0 \in A_i$ unique such that, $G(u_0) = \inf \{ G(u); u \in A_i \}$
write the Euler-Lagrange equation satisfied by a 'smooth' $u_0$

Let us define the constraint sets:

1. $A_1 = \{ u \in W^{1,p}(]0,1[; \mathbb{R}^2) :$ 
   \[ |u|^2_2 = (|u_1|^2 + |u_2|^2) = 1 \text{ a.e. so } |u_1| \leq 1, u_1^2 = 1 - u_2^2; u(0) = (0,1), u(1) = (1,0) \}\]

2. $A_2 = \{ u \in W^{1,p}(\Omega; \mathbb{R}^2); u_1 = 0 \text{ & } u_2 = 1 \text{ on } \partial \Omega; u_1 \in W_0^{1,p}(\Omega), |u|^2_2 = (|u_1|^2 + |u_2|^2) = 1 \text{ a.e. so } |u_i| \leq 1, u_1^2 = 1 - u_2^2 \}$

Note that the condition a.e. is implicitly important. One can notice that it could be written directly into equation $u_2 = \sqrt{1-(u_1)^2}$; without loss of generality we didn’t do so. Clearly, boundary condition does not define a vector space, if $u_1(0) = 0, u_1(1) = 1$, we write $u_1 = g$ and $u_2 = 1 - g$ on $\partial \Omega$ and $g$ may be a function defined on the open set $\Omega$ as well.

## 2 Solutions

**Lemma 2.1.** $A_i \neq \phi$ for all $i$.

*Proof.** For $i = 1$ consider the bounded smooth functionals 

$x \to u_1 = \begin{cases} 
\exp \left( \frac{x}{xp - 1} \right) & \text{for any } p > 0 \text{ if } x \in [0,1[ \\
0 & \text{if not}
\end{cases}$

For $i = 2$, similarly but more explicitly we use the following proposition about partition of unity which lead to the result after a regularization process.

**Proposition 2.1.** Let $\Omega$ be an open set of $\mathbb{R}^d$ and $K$ a compact $\subset \Omega$.

Then $\exists \Phi \in C_c(\mathbb{R}^d)$, such that 

$$0 \leq \Phi \leq 1, \quad \text{supp}(\Phi) \subset \Omega.$$
Definition 2.1. The $p$-norm on $\mathbb{R}^n$ is defined as:

$$x = (x_1, \cdots, x_n) \in \mathbb{R}^n, p \in ]0; +\infty[: x \to |x|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^\frac{1}{p}$$

and it is denoted by $\| \cdot \|_p$.

Lemma 2.2. If $u_1 \in L^p(\Omega)$, and $u_2 \in L^p(\Omega)$ then $\left( |u_1(x)|^2 + |u_2(x)|^2 \right)^\frac{2}{p} \in L^1(\Omega)$.

Proof. Write $|u|_2 \leq C|u|_p$ for some $C > 0$. \qed

2.1 Existence and uniqueness

Note that product spaces such $V \times V$ are equipped with the sum norm that is $\|u\| + \|v\|$.

Usually we will study $K(u)^\frac{1}{p}$, and $F(u)^\frac{1}{p}$ as the norm $L^p$ will appear explicitly. Before we state the main theorem, we have:

Proposition 2.2. $|v(\partial \Omega)| \leq C\|v\|_{W^{1,p}} \quad \forall v \in W^{1,p}[0,1[$

where $\partial \Omega := \{0,1\}$

Remark 2.1. A minimizer of a positive valued function $f$ is also a minimizer of $f^p$ and conversely , $\forall p > 0$.

Theorem 2.1.

1. There exists at least one function $u = (u_1, u_2) \in A_1$ solving $F(u) = \min_{w \in A_1} F(w)$.

2. There exists at least one function $u = (u_1, u_2) \in A_2$ solving $K(u) = \min_{w \in A_2} K(w)$.

Proof. First $F(u)^\frac{1}{p}$ and $K(u)^\frac{1}{p}$ are both continuous convex functions thus weakly lower semi continuous. Also the constraints sets are weakly closed, in the sense that, if $u_n \rightharpoonup u$, and $u_n$ satisfies any of the constraint, $u$ will be
as well. For the boundary condition that is \( u = g \) on the boundary, choose any \( h \) satisfying same constraints, \( u_n - h \) is a sequence \( \in W_{0}^{1,p} \times W_{0}^{1,p} \), a convex closed subspace of \( W^{1,p} \times W^{1,p} \), hence weakly closed.

For the condition of \( |u_i|_{\infty} \leq 1 \text{ a.e.} \) it suffices to show that \( |u_1|_{\infty} \leq 1 \text{ a.e.} \) Take a sequence weakly convergent to \( u \) in \( W^{1,p} \) by Rellich-K. Theorem we have a strong convergence at least in one of the \( L^p \)'s. Thus we can extract a subsequence that converges a.e. to \( u \). Giving that \( |\Omega| < \infty \), by Egoroff theorem the a.e convergence is equivalent to uniform convergence, up to arbitrarily negligible sets. Since the set is closed for the uniform convergence, we conclude that \( |u_i|_{\infty} \leq 1, i = 1, 2 \text{ a.e.} \)

It could be said directly after the extraction of a subsequence a.e. convergent, that we have

\[
|u_{k_1}|^2 + |u_{k_2}|^2 \rightarrow |u_1|^2 + |u_2|^2 = 1 \text{ a.e.}
\]

Remaining to show that the functionals verify a coercivity condition over the product space.

1. Set \( f := \inf_{u \in A_1} F(u) \). If \( f = +\infty \) we are done, suppose \( f \) is finite. Select a minimizing sequence \( \{u_k\} \), then \( F(u_k) \rightarrow f \) because we are in \( \mathbb{R} \)

\[
F(u) \geq C \left( |u_1|^p_{W_0^{1,p}} + |u_2|^p_{W_0^{1,p}} \right) \geq C'C \left( |u_1|_{W_0^{1,p}} + |u_2|_{W_0^{1,p}} \right)^p
\]

One can verify because of boundary conditions (on \( A_1 \)) that we have equivalence between the two norms \( \|\cdot\|_{W_0^{1,p}} \) and \( \|\cdot\|_{W^{1,p}} \) i.e.

\[
F(u_k) \geq \alpha \|u_k\| := \alpha \left( |u_{k_1}|_{W^{1,p}} + |u_{k_2}|_{W^{1,p}} \right)^p.
\]

This estimate implies that \( \{u_k\} \) is bounded in \( W^{1,p} \times W^{1,p} \). Consequently there exist a subsequence \( \{u_{k_j}\} \) and a function \( u \in W^{1,p} \times W^{1,p} \) such that; \( u_{k_j} \rightharpoonup u \) weakly in \( W^{1,p} \times W^{1,p} \), thus \( F(u) \) is weakly lower
semi continuous. \( F(u) \leq \lim \inf_{j \to \infty} F(u_k) = f \), since \( u \in A_1 \) it follows that

\[
F(u) = f = \min_{u \in A_1} F(u).
\]

2. Similarly, set \( m := \inf_{u \in A_2} K(u) \). If \( m = +\infty \) we are done, suppose \( m \) is finite, select a minimizing sequence \( \{u_k\} \), then \( K(u_k) \to m \) because we are in \( \mathbb{R} \).

\[
K(u) = \int_{\Omega} |\nabla u(x)|^p_2 = \int_{\Omega} \left( |\nabla u_1(x)|^2 + |\nabla u_2(x)|^2 \right)^{\frac{p}{2}}
\]

\[
K(u) = \int_{\Omega} \left( \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_1}{\partial x_2} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_2} \right)^2 \right)^{\frac{p}{2}}
\]

\[
\geq C \int_{\Omega} \left( \frac{\partial u_1}{\partial x_1} \right)^p + \left( \frac{\partial u_1}{\partial x_2} \right)^p + \left( \frac{\partial u_2}{\partial x_1} \right)^p + \left( \frac{\partial u_2}{\partial x_2} \right)^p
\]

\[
\geq CC'\left( \|u_1\|_{W^{1,p}}^p + C(\|\nabla \sqrt{1 - u_1^2}\|_{L^p}^p) \right)
\]

Consequently

\[
K(u_k) \geq \min(CC', C)(\|u_k_1\|_{W^{1,p}}^p + \|\nabla \sqrt{1 - u_1^2}\|_{L^p}^p),
\]

and \( u_1 \) is bounded. But if \( u_1 \) is bounded so is \( u_2 \) and conversely for:

\[
1 - \|u_1\|^2 \leq 1 - \|u_2\|^2 \leq \|1 - u_1^2\| = \|u_2^2\| \leq \|u_2\|^2
\]

Thus we conclude that the sequence \( \{u_k\} \) is bounded in \( W^{1,p} \times W^{1,p} \) and the proof is similar to that of \( F(u) \).

\[\square\]

**Theorem 2.2.** The minimizing problem: \( F(u) = \min_{w \in A_1} F(w) \) has a unique solution

**Proof.** Suppose not, if \( u \) is a minimizer and a distinct minimizer \( v \) exists,

\( v := (v_1, v_2) \) write \( w = (w_1, w_2) = \left( \frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2} \right) \) and recall that the Euclidean norm \( |.|_2 \) is strictly convex, which means that as long as

\[
v' \neq \alpha u' \text{ a.e.}
\]

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we have this strict inequality:

\[
G(w)^{\frac{1}{p}} = \left[ \int_0^1 |w'(x)|_2^p \right]^{\frac{1}{p}} = \left[ \int_0^1 \left( \left( \frac{u_1' + v_1'}{2} \right)^2 + \left( \frac{u_2' + v_2'}{2} \right)^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{p}} \leq \frac{1}{2} G(u)^{\frac{1}{p}} + \frac{1}{2} G(v)^{\frac{1}{p}}
\]

which contradicts the minimum property. This contradiction completes the proof if we showed that \( v' \neq \alpha u' \) a.e., suppose the converse and let \( u = \beta v + \text{cte} \), if \( u_1 = \beta_1 v_1 + \text{cte}_1 \), applying boundary constraints and using Proposition 2.2 we conclude that \( u_1 \neq \beta_1 v_1 + \text{cte}_1 \) a.e. for any \( \beta_1 \) and any constant \( \text{cte}_1 \)

\[ \square \]

3 Euler-Lagrange

**Lemma 3.1.** \( F(u) \) and \( K(u) \) are both differentiable \((C^1)\) on the product space except at \((0,0)\)

**Proof.** This follows by the regularity of the \(|.|_2\) norm and derivation under integral sign. \[ \square \]

From this, we can compute the Euler-Lagrange equations giving the existence of a minimizer \((u_0, u_0) \neq 0\). Bearing in mind that \( C^1 \) Gateaux differentiable is the same as \( C^1 \) Frechet -differentiable. We will give the 'equation' satisfied by the 'minimizer' of \( K(u) \) as it is the most general case. Fix \( v \in W_0^1, p(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2) \). Since \(|u|_2 = 1 \) a.e, we have

\[ |u + \tau v|_2 \neq 0 \text{ a.e.} \]
for each sufficiently small $\tau$ by continuity. Consequently

$$v(\tau) := \frac{u + \tau v}{|u + \tau v|_2} \in A_2$$

Thus

$$k(\tau) := K(v(\tau))$$

has a minimum at $\tau = 0$, and so

$$k'(0) = 0.$$  

Norms on product spaces are of course Euclidean norms, that is $|.|_2$. Matrices such the gradient matrix (here it’s a $2 \times 2$ matrix) can be identified to a vector $\in \mathbb{R}^4$, and let $(.)$ denotes the usual scalar product on $\mathbb{R}^n$, by a direct computation we have:

**Proposition 3.1.** $v'(\tau) = \frac{v}{|u + \tau v|} - \frac{[(u + \tau v).v(u + \tau v)]}{|u + \tau v|^3}$

**Theorem 3.1.** Let $u \in A_2$ satisfy

$$K(u) = \min_{w \in A_2} K(w).$$

Then

$$\int_{\Omega} p|Du|^{\frac{p}{2}-1}[((Du.Dv) - |Du|^2(u.v))]$$

for each $v \in W^{1,p}_0(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2)$.

**Proof.** In fact

$$K(u) = \int_{\Omega} |Du|^p$$

where $Du$ is the gradient matrix associated to $u$ and the norm as said is the one associated to the scalar product: $<A, B> = Tr(B^t.A)$.

$$k'(0) = 0 = \int_{\Omega} p|Du|^{\frac{p}{2}-1} Du.Dv'(0)$$

$$= \int_{\Omega} p|Du|^{\frac{p}{2}-1} Du.D(v - (u.v)u)$$

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Upon differentiating $|u|^2 = 1$ a.e., we have

$$(Du)^T u = 0$$

Using this fact, we then verify

$$Du.(D(u.v)u) = |Du|^2(u.v) \text{ a.e. in } \Omega$$

This identity employed in (7) gives (6). We leave details to the interested reader. \[2\]
References

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