We study the edge-mode excitations of a fractional quantum Hall droplet by expressing the edge state wavefunctions as linear combinations of Jack polynomials with a negative parameter. We show that the exact diagonalization within subspace of Jack polynomials can be used to generate the chiral edge-mode excitation spectrum in the Laughlin phase and the Moore-Read phase with realistic Coulomb interaction. The truncation technique for the edge excitations simplifies the procedure to extract reliably the edge-mode velocities, which avoids the otherwise complicated analysis of the full spectrum that contains both edge and bulk excitations. Generalization to the Read-Rezayi state is also discussed.

I. INTRODUCTION

The fractional quantum Hall (FQH) effect offers us a unique arena to study strongly correlated electron systems with a collection of effective tools, including many-body model wavefunction exact diagonalization (successful in often surprisingly small systems), the composite-fermion theory conformal field theories (CFT), and Jack polynomials (or Jacks).

Of particular interest of these bulk-gapped topological phases of matter are gapless edge excitations, which lead to experimentally measurable charge and neutral currents including highly nontrivial signals in quasiparticle interference measurements.

Not only are edge excitations key to transport experiments, they also manifest along artificially cut internal boundary in quantum entanglement studies, clearly demonstrated in the entanglement spectrum of FQH systems. The multiplicities in the low-lying part of the entanglement spectrum matches identically with those of the edge excitations with the corresponding boundary conditions. In fact the entanglement spectrum in the case of a real-space cut can be generated by a local field theory along the cut.

Explicitly in the disk geometry the model wavefunctions of edge excitations can be obtained by multiplying the corresponding ground-state wavefunction by symmetric polynomials. The Hilbert space of the edge excitations is robust even in microscopic systems in the presence of long-range interaction, when their excitation energies are comparable to those of bulk excitations. In the CFT construction edge states can be expressed as the correlators of bulk CFT primary fields and additional edge fields in the generated chiral algebra. In the Laughlin case, the edge-state Hilbert space can also be generated by a composite-fermion approach.

Alternatively, Jack polynomials emerged as one of the effective tools for ground state wavefunction construction. Jacks are homogeneous symmetric polynomials and Jacks with negative parameter are shown to be the correlator of the $\mathcal{W}A_{k-1}(k+1,k+r)$ conformal field theory, which includes the $Z_k$ parafermion theory. One advantage of using Jack polynomials is that a recursive construction algorithm exists, which renders the exact diagonalization of model Hamiltonians obsolete for a large class of FQH model wavefunctions. The Jack polynomials with a negative rational Jack parameter $\alpha$ generate ground-state wavefunctions as well as quasihole wavefunctions whose constructions differ only in the corresponding root configurations. By the bulk and edge correspondance in the FQH systems, this implies that one can use Jack polynomials to span the space of FQH states on a disk geometry with single or multiple edge excitations, although they are generically not orthogonal.

The importance of the inner products of edge states in the CFT construction has been emphasized in the context of real-space entanglement spectrum of model FQH states. In particular, these inner products take universal values in the thermodynamic limit, reflecting a correspondance of the edge CFT and the bulk CFT under generalized screening.

In this paper we present a framework to study edge excitations using Jack polynomials, whose coefficients are integers and can be conveniently generated in a computer recursively. The main purpose of this paper is to show that an edge state wavefunction, expressed as the corresponding ground state wavefunction (a Jack polynomial itself) multiplied by a symmetric polynomial, can be written alternatively as a linear combination of several Jacks in the corresponding momentum subspace. There is a linear map between the edge-state Hilbert space and the set of admissible root configurations for Jacks with the proper Jack parameter. In fact, the coefficients of the linear combination are universal for all system sizes. By using Jack polynomial to span the edge-state space, we can facilitate numerical calculations involving edge states in the presence of realistic interaction and confinement.

Our paper is organized as follows. In section III we review general properties of FQH wavefunctions and basic ingredients of the Jack polynomial approach of the FQH wavefunctions. We discuss the framework for constructing the edge Hilbert space by Jack polynomials in Sec. III. In section IV we apply the approach to generate the edge spectrum for FQH systems in both the Laughlin and the Moore-Read phases with realistic long-range Coulomb interaction in the edge-excitation space. Finally, in section V we summarize the paper and discuss potential applications of the framework. In Appendices, we provide the details on several crucial statements in the main text on the Jack polynomial construction of
II. FQH WAVEFUNCTIONS AND JACK POLYNOMIALS

A. Basic notations

The wavefunction of a free particle in the lowest Landau level (LLL) with the symmetric gauge in a plane is given by

$$\phi_m(z) = \frac{1}{\sqrt{2\pi 2^m m!}} z^m e^{-z^2/4}, \quad z = x + iy$$

where \(m\) is a non-negative integer representing the angular momentum. Since the Gaussian factor is the same for all \(m\), we neglect it in later discussions and only pay attention to polynomials of \(z\). It is obvious from the single-particle wavefunction that a many-particle wavefunction must be a multivariate complex polynomial. Depending on the statistics of the particles, the polynomial for the corresponding interacting system is either antisymmetric (for fermions) or symmetric (for bosons). Since a bosonic wavefunction can be mapped to a fermionic wavefunction by the multiplication of a Vandermonde determinant, the discussion hereafter on bosonic wavefunctions (unless we specify otherwise) is sufficient for our purposes.

Since symmetric polynomials form a vector space, one needs to choose a basis. For the description of bosonic wavefunctions by symmetric polynomials the basis of symmetrized monomial (from now on monomial implicitly means symmetrized monomial) is the natural choice; a monomial is not ambiguous once the total number of particles is fixed but an unnormalized wavefunction for a free particle in the LLL. We will denote a monomial by a sequence of integers, which label the occupied orbitals. For instance, \(\{4, 2, 2, 1\}\) is a partition of 9; in other words, it represents a monomial with total angular momentum 9: one particle in the \(z^4\) orbital, two in the \(z^2\) orbital, and the other one in the \(z^1\) orbital. Normally, the number of particles in the \(z^0\) orbital is not specified, but its occupation is not ambiguous once the total number of particles is fixed so we can assume that the unspecified particles are in the \(z^0\) orbital. We denote the weight (the sum of the elements) of a partition \(\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) as \(|\lambda| = \sum_{i=1}^{n} \lambda_i\), or the total degree of its representative monomial.

Sometimes, it is more convenient to use the occupation representation (or configuration) to represent a basis. The representations of partition and configuration are of course equivalent, but, just for the sake of convenience and clarity in discussion, we use both representations interchangeably. For instance, the following three notations are considered to be equivalent: \(|N - 4, 1, 2, 0, 1\rangle\), \({4, 2, 2, 1}\), and \(m_{\{4,2,2,1\}}\), where \(N \geq 5\) is the total number of particles. Note that we use the curly bracket for the partition representation and the ket notation for the configuration representation, respectively. Note that the sequence of integers in a configuration specify the corresponding numbers of particles in the orbitals \(z^0, z^1, z^2, z^3, z^4\), respectively.

One can order distinct partitions of an integer by the lexicographical order, but in the context of quantum Hall wavefunctions the dominance order is more convenient. The dominance order is determined by comparing the partial sums of two partitions. Consider two partition \(\lambda\) and \(\mu\). If \(\sum_{i=1}^{n} \lambda_i \geq \sum_{j=1}^{n} \mu_j\) for every \(j\), \(\mu\) is dominated by \(\lambda\); the relation is denoted by \(\mu \preceq \lambda\). In particular, when \(\mu\) and \(\lambda\) are different, we say \(\mu\) is strictly dominated by \(\lambda\), or \(\mu \prec \lambda\). Dominance is transitive, i.e., if \(\nu \preceq \mu\) and \(\mu \preceq \lambda\), then \(\nu \preceq \lambda\). The dominance ordering only renders a partial order to the whole set of partitions of an integer and admits the lexicographical ordering (i.e., if \(\mu \prec \lambda\), \(\lambda\) is larger than \(\mu\) in lexicographical order).

One can define an instructive operation called squeezing on a partition, which is closely related to the dominance ordering. A squeezing on a partition moves one pair of particles in the angular-momentum space closer \(\{\cdots, p, \cdots, q, \cdots\} \to \{\cdots, p - \delta m, \cdots, q + \delta m, \cdots\}\) (reordering can be done afterward to make it a partition). Obviously, the total angular momentum does not change after squeezing. We call the original partition a parent and the result a descendant; the parent partition strictly dominates the descendant partition.

Applying all the possible sequence of squeezing to a partition (which one refers to as a root partition) generates partitions dominated by the root partition. All these descendant partitions, together with the root partition, span a Hilbert space (or a symmetric polynomial space) with a fixed total angular momentum. Studies show that the Hilbert spaces of a certain series of root partitions are closely relate to quantum Hall wavefunctions, which can be expressed as a single Jack-polynomial or the linear combination of a finite set of Jack polynomials. This latter will be elaborated when we introduce Jack polynomial and, in particular, its application to the edge states of FQH systems. For now, let us denote the Hilbert space spanned by all the partitions dominated by a root configuration \(\lambda\) (i.e., \(\lambda\) and all the partitions constructed from it by repeated squeezing) as \(H_{\text{sq}}(\lambda)\).

B. A primer on Jack polynomial

Jack polynomials are homogeneous symmetric polynomials specified by a root configuration and a rational parameter. Jacks satisfy a number of differential equations and exhibit clustering properties. Explicitly, a Jack is one of the polynomial solutions of the following Calogero-Sutherland Hamiltonian

$$H_{\text{CS}}(\{z_i\}) = \sum_i (z_i \partial_i)^2 + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} (z_i \partial_i - z_j \partial_j)$$

\(2\)
where $\alpha$ is a rational parameter and $\partial_i \equiv \partial / \partial z_i$. The definition of a Jack also requires a root configuration (or partition), such that the Jack polynomial of a given root configuration $\lambda$ is defined in the Hilbert space $H_{\text{eq}}(\lambda)$, which is spanned by all the partitions that can be squeezed from $\lambda$. This fact is due to the structure of the Calogero-Sutherland Hamiltonian, which only couples two partitions if one can be squeezed from another. For a given $\alpha$ we can have a number of Jacks by choosing different root configuration. If $\alpha$ is positive we can choose any kind of partition as root partition and these Jacks are all linearly independent. Therefore, for a positive $\alpha$ Jacks form a basis for symmetric polynomial.

A recursive construction algorithm exists for Jacks of a root configuration $\lambda$ and a Jack parameter $\alpha$. Hence one can compute the coefficients of the following monomial expansion of a Jack

$$J^{\alpha}_{\lambda}(\{z\}) = m_{\lambda} + \sum_{\mu < \lambda} c_{\mu} m_{\mu}$$  \hspace{1cm} (3)

The convention is to fix the coefficient of the root configuration as unity and other coefficients are scaled accordingly. The use of this set of coefficients in subsequent numerical calculations for a given geometry (e.g., disk or sphere) often requires a proper normalization for each monomial.

Some Jacks of a negative $\alpha$ can be directly related to FQH wavefunctions. It was conjectured that Jacks with a negative parameter with proper root configuration are correlators of certain conformal theories. Bernevig and Halvane and co-workers further explored the idea extensively and showed that the Jack polynomial approach is an efficient and insightful development to obtain and to exploit FQH wavefunction. It was proven for general $(k, r)$ that Jacks are the correlators of $\mathcal{W}_{\alpha}^{k-1}(k+1, k+r)$ conformal theories. In other words, the correlators are eigenfunctions of the Calogero-Sutherland Hamiltonian [Eq. (2)]. Further more, the application of Jack polynomial on a certain type of quasihole wavefunctions was also proposed and proven to be correct with the discovery of interesting duality structure between electron and quasihole wavefunction.

In contrast to a positive $\alpha$, we have, for a negative $\alpha$, restrictions on the choice of root configuration. One allowed choice is to use a $(k, r)$ admissible root configuration, which leads to legitimate FQH trial wavefunctions. The $(k, r)$-admissibility means that there can be at most $k$ particles in $r$ consecutive orbitals. More precisely, a partition $\lambda$ is said to be $(k, r)$ admissible if $\lambda_i - \lambda_{i+k} \geq r$. The densest $(k, r)$ admissible root configuration and a corresponding Jack parameter $\alpha = -(k+1)/(r-1)$ (with the condition that $k+1$ and $r-1$ are coprime) generate the FQH ground state with a filling fraction $\nu = k/r$ (corresponding to $\nu = k/(k+r)$ in the fermionic case). Note that $\alpha$ is negative.

The applicability of the Jacks to the fermionic case is highly nontrivial since the multiplication of a Jack (with a monomial expansion) with a Vandermonde determinant cannot be straightforwardly mapped to a sum of Slater determinants. Nevertheless, a recursive procedure was found to generate the Slater-determinant expansion of $\prod_{i<j} (z_i - z_j) J^{\alpha}_{\lambda}(\{z\})$. Hence, all the discussion on bosonic wavefunctions, including numerical calculation, is applicable to fermionic wavefunctions in practice.

### III. JACK POLYNOMIALS FOR FQH EDGE STATES

#### A. Laughlin edge states: an example

In the simplest case of an Abelian Laughlin state, edge excitations are deformation of edge density of the incompressible liquid. One can write down the trial wavefunction for an edge state as a symmetric polynomial multiplied by the ground state wavefunction$^{16}$

$$\Psi(\{z\}) = S(\{z\}) \Psi_L(\{z\})$$  \hspace{1cm} (4)

where $\Psi_L = \prod_{i<j} (z_i - z_j)^r$ is the Laughlin state at a filling fraction $\nu = 1/r$ (with total angular momentum of $M_0 = rN(N-1)/2$ and $S(\{z\})$) a homogeneous symmetric polynomial. In this expression we encounter the multiplication of a monomial by another, which is then to be expanded as a sum of monomials. The expansion is not straightforward and often computationally expensive. Similarly, a symmetrized monomial multiplied by a antisymmetric Vandermonde determinant is in general a complex superposition of antisymmetrized monomials (or Slator determinants). The same difficulty also arises in the wavefunctions of quasihole states. Therefore, the nontrivial multiplication of physically natural basis states hinders us to efficiently use ansatz edge-state or quasihole-state wavefunctions in computation. The difficulty, however, can be resolved by the introduction of Jack polynomial$^{23}$. In this paper, we focus on the applications for the edge-state wavefunctions.

The edge state, as well as the ground state, is a zero-energy eigenstate of an ideal Hamiltonian with short-range interaction. One remarkable fact about Eq. (4) is that the number of linearly independent edge states for a given momentum $(M = M_0 + \Delta M)$ predicted by the chiral Luttinger liquid theory is the same as the number of the symmetric polynomials of degree $\Delta M$. To see this, it is sufficient to check the dimension of the legitimate symmetric polynomials for the given momentum $M$. By counting the number of possible monomials of degree $\Delta M$ or possible partitions of $\Delta M$, we obtain the dimension of the edge-state subspace with a total angular momentum $M = M_0 + \Delta M$. Hence the subspace dimension for the given momentum $\Delta M$ is exactly the number of partition of the integer $\Delta M$ and the counting is consistent with the edge theory of a single branch of chiral bosons.

| $(k, r)$ | bosonic | $(k, k+r)$ | fermionic |
|---------|---------|------------|----------|
| $(1, 2)$ | $[1, 0, 1, 0, 1, \cdots, 1]$ | $(1, 3)$ | $[1, 0, 0, 1, 0, 1, \cdots, 1]$ |
| $(1, 4)$ | $[1, 0, 0, 1, \cdots, 1]$ | $(1, 5)$ | $[1, 0, 0, 1, 0, 1, \cdots, 1]$ |
| $(2, 2)$ | $[2, 0, 0, 2, \cdots, 2]$ | $(2, 4)$ | $[1, 1, 0, 0, 1, 1, \cdots, 1]$ |

**TABLE I:** The densest $(k, r)$ admissible configurations. Fermionic configuration is $(k, k+r)$ admissible configurations which is the counter part state of $(k, r)$ admissible bosonic configuration.
Given the fact that the Laughlin state can be expressed as a single Jack with a root configuration being a \((k, r)\) admissible partition of \(M_0\), one naturally wants to explore the connection of the edge states to Jacks with root configurations being \((k, r)\) admissible partitions of \(M > M_0\). The clustering property is enforced such that the Jacks are also zero-energy states of the ideal Hamiltonian. In other words, these Jacks are edge states. One immediate question is whether the distinct Jacks span the same Hilbert space as the edge states described by dominance ordering. It means that the dominance relation between the root configurations is sufficient to describe the inclusion relation of the Hilbert space for the Jacks generated by the respective root configurations.

Now we consider the concrete construction of the edge-state space for the Laughlin case, i.e. \(k = 1\). The dominance relation between \(\lambda\) is same as the corresponding \(\eta\). Therefore, in lieu of Eq. \((4)\) one may expect a set of equations that relate the Jacks with the edge partition \(\eta\) as their roots and the Jacks with the full partition \(\lambda\). Indeed, we find

\[ J^\alpha_{\Omega + \eta} = J^\beta_{\eta} J^\alpha_{\Omega} \]  

where \(\alpha = -\frac{2}{r-1}\) and \(\beta = -\frac{2}{r}\). We leave the derivation in Appendix \(A\) Note that there is no restriction on the root configuration for the positive rational parameter \(\beta\). The result means that \(J^\beta_{\eta}\) is a representation of \(J^\beta_{\Omega + \eta}\) that can be readily used to decompose any edge state in the form of Eq. \((4)\). For an arbitrary symmetric polynomial \(S(z)\), one can expand it in terms of Jacks \(J^\beta_{s}\), since the Jacks form a basis for the homogeneous symmetric polynomial space. Therefore, it is possible to obtain the monomial expansion of \(S(z)J^\alpha_{\Omega}\) efficiently.

As an important example, we consider the monomial expansion of an arbitrary monomial multiplied by the Laughlin ground state \(J^\alpha_{\Omega}\) for \(\alpha = -2/(r-1)\). In other words, we want to calculate the coefficient \(c_\mu\) in the expression

\[ m_{\eta} J^\alpha_{\Omega} = \sum_{\mu \leq \Omega, \eta} c_\mu m_\mu. \]

This can be easily achieved once we know the monomial expansion of \(J^\beta_{\eta}\). By inverting the relation between \(m_{\eta}\) and \(J^\beta_{\eta}\)s, one can express any \(m_{\eta}\) as a linear combination of \(J^\beta_{\eta}\)s. In other words, one then obtains a Jack-polynomial expansion of the monomial multiplied by the ground state wavefunction. We can, equivalently, write down the array of equations in a compact form for any \(\Delta M\) as

\[ J m J^\alpha_{\Omega} = j_E, \]

where \(j_E\) is a column vector whose components are the edge Jacks \(J^\alpha_{\Omega + \eta}\) and \(m\) a vector of \(m_{\eta}\)s; both are ordered in the lexicographical order of \(\eta\). Explicitly, we have

\[ j_E = \left( J^\alpha_{\Omega + \{1, \ldots, 1\}}, \ldots, J^\alpha_{\Omega + \{\Delta M\}} \right)^T, \]

\[ m = \left( m_{\{1, \ldots, 1\}}, \ldots, m_{\{\Delta M\}} \right)^T. \]

They have the same dimension, which is the number of all possible partitions of \(\Delta M\). \(J\) is a matrix, which consists of the corresponding coefficients of \(J^\beta_{\eta}\) in terms of \(m_\mu\), i.e.,

\[ J^\beta_{\eta} = c_\mu(\eta), \]
if we assume
\[ J^2_{\eta} = \sum_{\nu \leq \eta} c^{(\eta)}_{\nu} m_{\nu}. \tag{11} \]

Again, we recall the convention \( c^{(\eta)}_{\eta} = 1 \). By inverting \( J \), we obtain
\[ m J^2_{\Omega} = J^{-1} J_E. \tag{12} \]

Since the rank of \( J \) is only the number of distinct partitions of \( \Delta M \), the computation of \( J \) and its inverse is computationally cheap.

We point out two important properties of \( J \) in the following. Firstly, when we arrange the basis (for both the Jacks by their root partitions and the monomials) in lexicographical order or any other order that respects the dominance ordering, we obtain a lower triangular matrix with all identities at its diagonal (i.e., a unitriangular matrix) for \( J \). In other words, if \( \mu > \lambda \) by dominance ordering or has no relation between \( \mu, J_{\Omega} \) and \( \lambda \), then the inner product of \( m_\mu \) and \( J_\lambda \) is zero, as the two share no common monomials in their respective expansions. We note that \( H_{sq}(\Omega) H_{sq}(\eta) \subset H_{sq}(\Omega + \eta) \) for our understanding, which can be proved as in Appendix C and which indicates that \( m J^2_{\Omega} \subset H_{sq}(\Omega + \eta) \). Secondly, as long as the number of particles is larger than \( \Delta M \), \( J \) is independent of system size. As an example, let us define \( \eta_1 = \{1, 1, 1, 1\} \), \( \eta_2 = \{2, 1, 1\} \), \( \eta_3 = \{2, 2\} \), \( \eta_4 = \{3, 1\} \), and \( \eta_5 = \{4\} \) for \( k = 1, r = 2, \) and \( \Delta M = 4 \). We have
\[
\begin{pmatrix}
  m_{\eta_1} \\
  m_{\eta_2} \\
  m_{\eta_3} \\
  m_{\eta_4} \\
  m_{\eta_5}
\end{pmatrix}
J^2_{\Omega} =
\begin{pmatrix}
  1 & 0 & 0 & 0 & 0 \\
  -\frac{36}{11} & 1 & 0 & 0 & 0 \\
  \frac{27}{11} & -\frac{6}{5} & 1 & 0 & 0 \\
  \frac{27}{35} & \frac{6}{5} & -\frac{6}{5} & 1 & 0 \\
  \frac{27}{35} & \frac{6}{5} & \frac{6}{5} & -\frac{6}{5} & 1
\end{pmatrix}
\begin{pmatrix}
  J_{\alpha}^{\Omega + \nu_1} \\
  J_{\alpha}^{\Omega + \nu_2} \\
  J_{\alpha}^{\Omega + \nu_3} \\
  J_{\alpha}^{\Omega + \nu_4} \\
  J_{\alpha}^{\Omega + \nu_5}
\end{pmatrix}
\tag{13}
\]

where the matrix on right hand side is \( J^{-1} \), the inverse of
\[
J =
\begin{pmatrix}
  1 & 0 & 0 & 0 & 0 \\
  \frac{36}{11} & 1 & 0 & 0 & 0 \\
  \frac{27}{11} & -\frac{6}{5} & 1 & 0 & 0 \\
  \frac{27}{35} & \frac{6}{5} & -\frac{6}{5} & 1 & 0 \\
  \frac{27}{35} & \frac{6}{5} & \frac{6}{5} & -\frac{6}{5} & 1
\end{pmatrix}
\tag{14}
\]

The electron number independence of \( J \) is guaranteed by product rule 23,29 of Jacks. If the electron number \( N \) is larger than \( \Delta M \), then the admissible root partitions are in one-to-one correspondence with those of the system with \( N = \Delta M \). Hence the product rule, the coefficients of the admissible root partitions remain unchanged with increasing system size.

Before we move on to the more general cases (i.e., \( k > 1 \)), we emphasize that the form of \( J \) for \( k = 1 \) is the direct consequence of Eq. (5). We do not know any such simple relation for \( k > 1 \), hence the form of \( J \) is not as simple but still can be computed. The reason for the complication is because an edge sequence is no longer a simple partition, but consists of \( k \) subpartitions. However, the Jacks with admissible root configurations still exhaust the Hilbert space of the edge states, therefore, we can still expand any edge-state wavefunction as a linear combination of the edge Jack polynomials. That is to say, we look for a generalization of Eq. (7) or Eq. (12). As the edge space is no longer spanned by simple monomials multiplied by the ground state. Nevertheless, we can still write down a complete set of edge states for each \( \Delta M \) (which we label as \( m_E \) that replaces \( m J^{2}_{\Omega} \)), as Milovanović and Read did for the \( k = 2 \) Moore-Read case. Now the question converts to how to create a dictionary \( J \) to translate \( m_E \), which is normally easier for analytical discussions, to the set of edge Jack polynomials \( j_E \), which is more convenient for numerical calculations. Note that, in the basis of monomials of degree \( M_0 + \Delta M \) [i.e. in the Hilbert space \( H_{sq}(\Omega + \{\Delta M\}) \)], \( m_E \) and \( j_E \) can be viewed, alternatively, as matrices. A key observation is that the dimension of edge state (number of rows) is usually much smaller than the dimension of \( H_{sq}(\Omega + \{\Delta M\}) \) (number of columns), such that we do not need to know all elements of \( m_E \) to obtain \( J \); in other words, we have a set of overdetermined linear equations. The most obvious solution to the problem which subspace in \( H_{sq}(\Omega + \{\Delta M\}) \), whose dimension is the same as that of the edge space, should be chosen to determine \( J \) turns out to be the space spanned by admissible root partitions.

This can be further extended to express, in terms of Jacks, quasiholes states (including multiple quasiholes), which can be written as the superpositions of edge states with powers of the quasihole coordinates as coefficient and which we will not elaborate.

IV. NUMERICAL APPLICATIONS

A. Diagonalization in the truncated space of edge states

One of the applications of the edge Jacks is the exact diagonalization of the Hamiltonian with realistic Coulomb interaction within the edge-state subspace spanned by the Jacks. The exact diagonalization in the full Hilbert space is difficult because of its exponentially growing size as particle number increases. Fortunately, the low-energy eigenfunctions of the Hamiltonian are well represented by CFT correlators or Jacks so we can overcome this difficulty by projecting the Hamiltonian to the edge-state subspace. For electrons we use the fermionic Jacks, which are symmetric Jacks multiplied by a Vandermonde determinant. The recursive construction procedure is well documented in Ref [23]. Here we consider a realistic Hamiltonian
\[
H = \frac{1}{2} \sum_{mnl} V_{mn}^{l} c_{m+n+l}^{\dagger} c_{m} c_{n+l} + \sum_{m} U_{m} c_{m}^{\dagger} c_{m}, \tag{15}
\]
where the Coulomb matrix elements \( V_{mn}^{l} \) are
\[
V_{mn}^{l} = \int d^2 r_1 \int d^2 r_2 \phi_{m+n+l}(\vec{r}_1) \phi_{m}^{*}(\vec{r}_2) \frac{e^2}{|\vec{r}_1 - \vec{r}_2|^2} \phi_{n+l}(\vec{r}_1) \phi_{m}(\vec{r}_2), \tag{16}
\]
and \( U_{m} \) is the matrix element of the rotationally invariant confining potential due to the disk-shaped positive background.
charge at a setback distance \( d \)

\[
U_m = \frac{N_e e^2}{\pi R^2 \varepsilon} \int d^2 r \int_{\rho<R} d^2 \rho \frac{|\phi_m(\vec{r})|^2}{\sqrt{f^2 - \rho^2} + d^2}
\]  

(17)

Here \( \varepsilon \) is the dielectric constant and \( R \) the radius of the background charge disc.

The diagonalization can be performed in the subspace of Jacks with a fixed momentum due to the rotational invariance. For instance, for \( \Delta M = 3 \), available \( \eta_{3\lambda} \) of Laughlin sequence are \{3\}, \{2, 1\} and \{1, 1, 1\}. Hence the subspace we deal with are only 3 dimension consists in \( J_{\Omega_3(\{3\})}, J_{\Omega_3(\{2, 1\})} \) and \( J_{\Omega_3(\{1, 1, 1\})} \). Since Jacks are not orthogonal to each other, orthogonalization is needed. We project the Hamiltonian to the orthogonalized edge basis and then perform the exact diagonalization. The result for 9 electrons at \( d = 0.6l_B \) is shown in Fig.1(a). The low-lying spectrum is in good agreement with that from the exact diagonalization in the full Hilbert space. A similar comparison for the Moore-Read phase with 12 electrons are shown in Fig.1(b) and (c).

We illustrate here, as in Ref. 19, the diagonalization results for a mixed Hamiltonian with both Coulomb and three-body interaction, i.e., \( H = (1-\lambda) H_{\text{Coulomb}} + \lambda H_{3B} \). Explicitly, the three-body interaction \( H_{3B} \), which generates the Moore-Read wavefunction as its exact ground state, has the form

\[
H_{3B} = -\sum_{i<j<k} S_{ijk} [\nabla_i^2 \nabla_j^2 (\nabla_i \nabla_j^2) \delta(r_i - r_j) \delta(r_i - r_k)],
\]

(18)

where \( S \) is a symmetrizer: \( S_{123}[f_{123}] = f_{123} + f_{231} + f_{312} \), where \( f \) is symmetric in its first two indices. When \( \lambda = 0.5 \), the edge states, which are zero-energy states for \( H_{3B} \), pick up finite energies but are still well separated from the bulk states in small excitation-momentum sectors. Fig.1(b) shows that the edge-state spectrum agrees well with that from the full diagonalization. In the case of the pure Coulomb interaction in Fig.1(c), the agreement is not as good due to the mixing of the edge states with the bulk states.

From the comparison of the excitation spectra in Fig.1 of the full-space diagonalization and the truncated-space diagonalization we have the following observation. For small excitation-momentum sectors, the energies are almost identical in two cases. Even for \( \nu = 5/2 \) state with the Coulomb interaction, their difference decreases as the system size increases and vanishes in the thermodynamic limit. This is particularly useful to extract the edge velocities, to be discussed in the next subsection, from the truncated-space diagonalization for system sizes significantly larger than those can be handled by the full diagonalization. For regular computing systems, the main bottleneck of this method for large systems is the large storage space (or memory) for the edge Jacks.

### B. Edge-Mode Velocities

Edge-mode velocities in the Moore-Read phase and the Read-Reaży phase are important non-universal quantities, as they are closely related to the decoherence length of the non-Abelian quasiparticles propagating along the edge of, say, a quasiparticle interferometer. Their calculation can be made more efficient with the help of the diagonalization in the space spanned by the edge Jacks. Charge mode velocity can be defined as \( v_{\text{ch}} = L \frac{(E_0(\Delta M = 1) - E_0(\Delta M = 0))}{[\epsilon^2/c^2 ]^{1/4}} \) where the perimeter of the quantum droplet is \( L = 2\pi R = \sqrt{2N(k + \alpha)/k} \). Neutral mode velocity can be defined as \( v_{\text{fe}} = L \frac{(E_0(\Delta M = 2) - E_0(\Delta M = 0))}{[\epsilon^2/c^2 ]^{1/4}} \) \( E_0(\Delta M) \) is the lowest eigenenergy for the given momentum \( \Delta M \). Through finite-size scaling we can extrapolate the edge-mode velocities in the thermodynamic limit. We have demonstrated the applicability of the finite-size scaling with data from the diagonalization in the full Hilbert space in Ref. 34.

We first apply the velocity calculation to the 1/2-filling...
Moore-Read case, i.e., \( k = 2 \) and \( r = 2 \), in the first excited Landau level. With the diagonalization in the edge-state space we can handle systems with up to 20 electrons. Fig. 2 shows that a charge-mode velocity of \( 0.44e^2/\hbar \) can be extrapolated in the thermodynamic limit. This value agrees with the previous finite-size scaling result based on the exact diagonalization of systems of 8-14 electrons. The neutral-mode velocity is fitted by a linear dependence on \( 1/N \). In the thermodynamic limit, the velocity reads \( 0.033e^2/\hbar \).

FIG. 2: Finite-size scaling of the edge-mode velocities for \( \nu = 5/2 \) with Coulomb interaction and confinement from neutralizing background charge. The charge-mode velocity is fitted by a quadratic function in \( 1/N \) and, in the thermodynamic limit, the velocity reads \( 0.44e^2/\hbar \) (based on data from 8-20 particles). The neutral-mode velocity is fitted by a linear dependence on \( 1/N \). In the thermodynamic limit, the velocity reads \( 0.033e^2/\hbar \).

V. CONCLUSION

In summary, we discuss the identification of the Hilbert space of edge excitations in the Laughlin–Moore–Read–Read–Rezayi series as spanned by the appropriate Jack polynomials with admissible root configurations. In particular, we elaborate on the Laughlin case \( (k = 1) \), which contains a single charged mode. We explain how to establish a linear map between the polynomial edge wavefunctions and edge Jack polynomials. The map is of particular use in the presence of realistic interaction and confinement, in which case the edge spectrum has a nontrivial dispersion with a nonuniversal edge-mode velocity. The mapping formalism and its numerical application can be generalized to larger \( k \)s. The applicability has been checked by the comparison of the edge spectra obtained in the truncated edge Jack polynomial space and in the full Hilbert space with exact diagonalization. The truncation approach has the advantage of being able to treat larger systems than the conventional exact diagonalization approach. As an example, we are able to calculate the edge-mode velocities in the Moore–Read phase for \( \nu = 5/2 \) up to 20 electrons and confirm with greater confidence that the neutral-mode velocity is an order of magnitude smaller than the charge-mode velocity.

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Appendix A: Edge partitions for \( k > 1 \)

To discuss \((k, r)\) admissible states with \( k > 1 \) is more complicated, as \( \eta \) cannot be a simple partition. To systematically classify these admissible edge states, it is necessary to separate a partition into \( k \) sequences. By doing so we get a set of \( \eta^{(1)}, \ldots, \eta^{(k)} \), where \( \eta^{(i)} = \{ \eta_{1}, \eta_{1+k}, \eta_{1+2k}, \ldots, \eta_{n-k+i} \} \). The \((k, r)\) admissible condition of \( \lambda \) implies that

\[
\eta_{[i/k]}^{(i \ mod \ k)} - \eta_{[i/k]+1}^{(i \ mod \ k)} = \eta_{i+k} = (\lambda_{i} - \lambda_{i+k}) - r \geq 0
\]

where \([x]\) is the smallest integer greater than or equal to \( x \). The first condition is that \( \eta_{j}^{(i)} \geq \eta_{j}^{(i+1)} \). This condition implies that \( \eta^{(i)} \) is a partition of \( \eta^{(i)} \). Unlike the \( k = 1 \) case, not every partition of a given angular momentum is qualified as \( \eta^{(i)} \). If \( k \neq 1 \) we have two conditions that exist among \( \eta_{j}^{(i)} \)s. The first condition is that \( \eta_{j}^{(i)} \geq \eta_{j}^{(i+1)} \). This condition implies that \( \eta^{(i)} \geq \eta^{(i+1)} \), which is weaker. So, \( \{\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(k)}\} \) is itself a partition of \( \Delta M \). The second condition is that \( \eta_{j}^{(i)} < \eta_{j}^{(i+1)} + r \) for every \( 1 \leq i \leq k \) and \( j \geq 1 \). Without such conditions, two different \( \eta \) can be assign to the same state. For example, the additions of two sets of partitions \( \eta^{(1)} = \{2\}, \eta^{(2)} = \{1\} \) and \( \xi^{(1)} = \{1\}, \xi^{(2)} = \{2\} \) generate the same state for the Moore-Read ground state with a densest root configuration \( \{2, 0, 2, \ldots\} \). Careful examination leads to the conclusion that the above
two conditions are sufficient to generate a unique representation. These conditions are easily depicted by a Ferres diagram in Fig. 3. The counting of \( \eta \) is not as straightforward as in the \( k = 1 \) case, but can be done by partitioning the angular momentum of the edge excitation into \( k \) ordered partitions as discussed here. The counting of \( (k, 2) \) and \( (k, 3) \) admissible partitions are also considered in Ref. 22 to compute the specific heat of FQH states.

**Appendix B: Derivation of Eq. (5)**

To verify the relation \( J_{\Omega+\eta}^{\alpha} = J_{\eta}^{\beta} J_{\Omega}^{\alpha} \) where \( \alpha = -\frac{2}{r-1} \) and \( \beta = \frac{2}{r-1} \) (i.e. \( k = 1 \)), we start with the Calogero-Sutherland Hamiltonian \( H_{CS} = \sum_{i} z_{i} D_{i}^{L-1} z_{i} D_{i}^{L} r \), where \( D_{i}^{L} r = \partial_{i} - r \sum_{j \neq i} \frac{1}{z_{i} - z_{j}} \) and, importantly, \( D_{i}^{L} J_{\Omega}^{\alpha} = 0 \), and apply it to \( J_{\eta}^{\beta} J_{\Omega}^{\alpha} \).

\[
\begin{align*}
\sum_{i} z_{i} D_{i}^{L-1} z_{i} D_{i}^{L} r J_{\Omega}^{\alpha} &= \sum_{i} z_{i} D_{i}^{L-1} z_{i} J_{\Omega}^{\alpha} \partial_{i} J_{\eta}^{\beta} \\
&= \sum_{i} \left[ z_{i} J_{\Omega}^{\alpha} \partial_{i} J_{\eta}^{\beta} + z_{i}^{2} J_{\Omega}^{\alpha} \partial_{i}^{2} J_{\eta}^{\beta} + z_{i}^{2} J_{\Omega}^{\alpha} \partial_{i} J_{\eta}^{\beta} \right] \\
&\quad + z_{i}^{2} \partial_{i} J_{\Omega}^{\alpha} \partial_{j} J_{\eta}^{\beta} + z_{i}^{2} \partial_{i} J_{\Omega}^{\alpha} \left( \sum_{j \neq i} \partial_{j} J_{\eta}^{\beta} \right)
\end{align*}
\]

The first and the third equalities come from the relation \( D_{i}^{L} r J_{\Omega} = 0 \). But

\[
2 \sum_{i \neq j} \frac{z_{i}^{2} \partial_{i}}{z_{i} - z_{j}} - \sum_{i \neq j} \partial_{i} z_{i} = \sum_{i \neq j} \left( \frac{z_{i}^{2} + z_{i} z_{j}}{z_{i} - z_{j}} \right) \partial_{i}
\]

Hence,

\[
\sum_{i \neq j} \frac{z_{i}^{2} \partial_{i}}{z_{i} - z_{j}} = \frac{1}{2} \sum_{i<j} \left( z_{i} \partial_{i} - z_{j} \partial_{j} \right)
\]

In order to be the solution of the equation, \( J_{\eta}^{\beta} \) is the solution of the

\[
\sum_{i} \left( z_{i} \partial_{i} \right)^{2} + \frac{1}{2} \sum_{i<j} \left( z_{i} \partial_{i} - z_{j} \partial_{j} \right) + \frac{1}{2} \sum_{i \neq j} z_{i} \partial_{i}
\]

It is the Calogero-Sutherland Hamiltonian with a constant offset (note that \( \sum_{i} z_{i} \partial_{i} \) is nothing but the total angular momentum) and a positive parameter \( \frac{1}{2} \). By choosing \( \beta = \frac{1}{r-1} \) and following the conventional normalization of the Jack, we complete the proof of Eq. (5).

**Appendix C: Proof of** \( H_{sq}(\nu)H_{sq}(\omega) \subset H_{sq}(\nu + \omega) \)

In this appendix we examine the multiplication of two monomials labeled by \( \lambda \in H_{sq}(\nu) \) and \( \mu \in H_{sq}(\omega) \). We
will show that \( m_\lambda m_\mu \) is in \( H_{sq}(\nu + \omega) \), i.e., the Hilbert space of monomials labeled by partitions that can be squeezed from \( \nu + \omega \). The addition of two partitions is understood as element-wise addition. For example, \( \{9, 5, 2\} + \{4, 2\} = \{13, 7, 2\} \). Obviously, the multiplication of two monomials is not necessarily a single symmetric monomial but a sum of them, or a generic symmetric polynomial.

First, we need to examine what kind of symmetric monomial is included in this generic symmetric polynomial. Consider the multiplication of two monomials \( m_\lambda \) and \( m_\mu \).

\[
m_\lambda m_\mu = \text{Sym} z_1^{\lambda_1 + \mu_1} \cdots z_n^{\lambda_n + \mu_n},
\]

which can then be rewritten as an expansion of symmetric monomials. Consider an arbitrary symmetric monomial in the expansion. The key step is to show that the corresponding partition is dominated by \( \lambda + \mu \). To show this, let us fix \( \lambda \) and permute \( \mu \). The permutation will generate terms belonging to other symmetric monomials, but they are related to \( m_{\lambda + \mu} \) by squeezing. After a permutation of \( \lambda_i \) and \( \lambda_j \) for example, we obtain a term included in a symmetric monomial \( \{\cdots, \lambda_i + \mu_j, \cdots, \lambda_j + \mu_i\} \). If we assume that \( i > j \), the permutation brings the momenta of the corresponding pair of particles closer and the new partition is dominated by the original one; this is what we call squeezing. Starting from the monomial \( m_{\lambda + \mu} \), we can show that all terms in \( m_\lambda m_\mu \) can be obtained by squeezing and, therefore, their corresponding partitions are dominated by \( \lambda + \mu \). Similarly, one can show that \( \lambda + \mu \) is dominated by \( \nu + \mu \), which is then dominated by \( \nu + \omega \). If we define the multiplication of two Hilbert spaces \( H_{sq}(\nu) \) and \( H_{sq}(\omega) \) as

\[
\{m : \langle m | m_\lambda m_\mu \rangle \neq 0, \lambda \in H_{sq}(\nu), \mu \in H_{sq}(\omega)\},
\]

we can say that the multiplication of two Hilbert spaces squeezed from two separate partitions \( \nu \) and \( \omega \) is included in the Hilbert space squeezed from the sum of two partitions, i.e.,

\[
H_{sq}(\nu) H_{sq}(\omega) \subset H_{sq}(\nu + \omega).
\]

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1. R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
2. J. K. Jain, Composite Fermions (Cambridge University Press, New York, 2007), ISBN 978-0-521-86232-5.
3. B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. 100, 246802 (2008).
4. A. M. Chang, Rev. Mod. Phys. 75, 1449 (2003).
5. I. P. Radu, J. B. Miller, C. M. Marcus, M. A. Kastner, L. N. Pfeiffer, and K. W. West, Science 320, 899 (2008).
6. A. Bid, N. Ofek, H. Inoue, M. Heiblum, C. Kane, et al., Nature 466, 585 (2010), 1005.5724.
7. R. L. Willett, L. N. Pfeiffer, and K. W. West, Proceedings of the National Academy of Sciences 106, 8853 (2009).
8. R. L. Willett, L. N. Pfeiffer, and K. W. West, Phys. Rev. B 82, 205301 (2010).
9. R. Willett, L. Pfeiffer, and K. West, arXiv:1204.1993 (unpublished).
10. H. Li and F. D. M. Haldane, Phys. Rev. Lett. 101, 010504 (2008).
11. J. Dubail, N. Read, and E. H. Rezayi, Phys. Rev. B 85, 115321 (2012).
12. A. Sterdyniak, A. Chandran, N. Regnault, B. A. Bernevig, and P. Bonderson, Phys. Rev. B 85, 125308 (2012).
13. I. D. Rodríguez, S. H. Simon, and J. K. Slingerland, Phys. Rev. Lett. 108, 256806 (2012).
14. J. Dubail, N. Read, and E. H. Rezayi, Phys. Rev. B 86, 245310 (2012).
15. X.-G. Wen, Int. J. Mod. Phys. B 6, 1711 (1992).
16. M. Milovanović and N. Read, Phys. Rev. B 53, 13559 (1996).
17. X. Wan, E. H. Rezayi, and K. Yang, Phys. Rev. B 68, 125307 (2003).
18. X. Wan, Z.-X. Hu, E. H. Rezayi, and K. Yang, Phys. Rev. B 77, 165316 (2008).
19. S. Jolad, D. Sen, and J. K. Jain, Phys. Rev. B 82, 075315 (2010).
20. B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. 100, 246802 (2008).
21. B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. 101, 246806 (2008).
22. B. A. Bernevig and N. Regnault, Phys. Rev. Lett. 103, 206801 (2009).
23. B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, Int. Math. Res. Not. 2002, 1223 (2002).
24. B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, Int. Math. Res. Not. 2003, 1015 (2003).
25. B. Estienne and R. Santachiara, J. Phys. A: Math. Theor. 42, 445209 (2009).
26. B. Yang, Z.-X. Hu, Z. Papić, and F. D. M. Haldane, Phys. Rev. Lett. 108, 256807 (2012).
27. B. A. Bernevig and F. D. M. Haldane, Phys. Rev. B 77, 184502 (2008).
28. B. A. Bernevig, R. Thomale, B. Estienne, and R. Santachiara, J. Phys. A: Math. Theor. 42, 587 (2007).
29. I. Dumer, A. Edelman, and G. Shuman, J. Sym. Comp. 42, 587 (2007).
30. R. Thomale, B. Estienne, N. Regnault, and B. A. Bernevig, Phys. Rev. B 84, 045127 (2011).
31. B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. 102, 66802 (2009).
32. B. A. Bernevig, V. Guralie, and S. H. Simon, J. Phys. A: Math. Theor. 42, 245206 (2009).
33. B. Estienne, B. A. Bernevig, and R. Santachiara, Phys. Rev. B 82, 205307 (2010).
34. Z.-X. Hu, E. H. Rezayi, X. Wan, and K. Yang, Phys. Rev. B 80, 235330 (2009).
35. R. Willett, L. Pfeiffer, K. West, and M. Manfra, arXiv:1301.2594 (unpublished).
36. J. S. Xia, W. Pan, C. L. Vicente, E. D. Adams, N. S. Sullivan, H. L. Stormer, D. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett 93, 176809 (2004).
37. A. Kumar, G. A. Csatky, M. J. Manfra, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett 105, 246808 (2010).
38. C. Zhang, C. Huang, J. S. Xia, N. S. Sullivan, W. Pan, K. W. Baldwin, K. W. West, L. N. Pfeiffer, and D. C. Tsui, Phys. Rev. B 85, 241302 (2012).