On recurrence in zero-dimensional locally compact flow with compactly generated phase group

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Abstract

We define recurrence for a compactly generated para-topological group $G$ acting continuously on a locally compact Hausdorff space $X$ with $\dim X = 0$, and then, show that if $G_x$ is compact for all $x \in X$, the conditions (i) this dynamics is pointwise recurrent, (ii) $X$ is a union of $G$-minimal sets, (iii) the $G$-orbit closure relation is closed in $X \times X$, and (iv) $X \ni x \mapsto G_x \in 2^X$ is continuous, are pairwise equivalent. Consequently, if this dynamics is pointwise product recurrent, then it is pointwise regularly almost periodic and equicontinuous; moreover, a distal, compact, and non-connected $G$-flow has a non-trivial equicontinuous pointwise regularly almost periodic factor.

Keywords: Recurrence · Distality · Zero-dimensional flow · Compactly generated group

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0. Introduction

Let $(G, X)$ be a flow with phase group $G$ and with phase space $X$, which is in the following context, unless stated otherwise:

(i) $G$ is a “Hausdorff para-topological group”, namely: $G$ is a multiplicative group with a Hausdorff topology under which $G \times G \xrightarrow{(t,s)\mapsto ts} G$ is jointly continuous, but $G \xrightarrow{t\mapsto t^{-1}} G$ need not be continuous;

(ii) $X$ is a locally compact, Hausdorff, uniform space; and moreover, there is a left-action of $G$ on $X$, denoted $G \times X \xrightarrow{(t,x)\mapsto tx} X$ such that

(i) Continuity: $(t, x) \mapsto tx$ is jointly continuous;

(ii) Transformation group: $ex = x$ and $(st)x = s(tx)$ for all $x \in X$ and $s, t \in G$, where $e$ is the identity of $G$;

(iii) Lagrange stability: $Gx$ is relatively compact (i.e. $Gx$ is a compact set) in $X$ for all $x \in X$.

If the phase space $X$ is compact itself, then $(G, X)$ is Lagrange stable. Here a Hausdorff para-topological group need not be a topological group; see, e.g., [19, (4.20a)]. However, if $t_n \to t$ and $t_n^{-1} \to \tau$ in $G$, then $\tau = t^{-1}$; moreover, if $G$ is a locally compact Hausdorff para-topological group, then it is a topological group by Ellis’ Joint Continuity Theorem.
If Homeo \((X)\) stands for the set of self homeomorphisms of \(X\) endowed with the topology of uniform convergence on compacta, then Homeo \((X)\) is a Hausdorff para-topological group such that Homeo \((X) \times X \xrightarrow{f \times f} X\) is a flow having no Lagrange stability in general.

**0.1.** It is well known that every recurrent point is almost periodic (in fact periodic) for any continuous-time flow on the plane [24]. In [16, p. 764] W. H. Gottschalk suggested an interesting question that is to determine conditions under which pointwise recurrence of a flow implies pointwise almost periodicity.

The main aim of this paper is to consider the relationships of the following important dynamics of \((G, X)\):

1. The pointwise “recurrence” in certain sense (cf. Def. 2.2).
2. The pointwise almost periodicity (cf. Def. 1.5a).
3. \(X\) is a union of minimal sets.
4. The orbit closure relation is closed (cf. Def. 1.3).
5. The orbit closure mapping is “continuous” on \(X\) (cf. Def. 1.2a).
6. The orbit closure mapping is continuous on \(X\) (cf. Def. 1.2b).
7. The local weak almost periodicity (cf. Def. 1.5b).

It is evident that (6) \(\Rightarrow\) (5) \(\Leftrightarrow\) (4) \(\Rightarrow\) (2) \(\Rightarrow\) (1). However, even for \(G = \mathbb{Z}\) or \(\mathbb{R}\) and \(X\) a compact metric space, that “(1) \(\Rightarrow\) (5)” is not generally true is shown by very simple examples. On the other hand, we consider the following problem, which is motivated by Furstenberg’s work [11].

**8.** If \((G, X)\) is a distal (cf. Def. 1.6) compact non-minimal flow, does \((G, X)\) have an equicontinuous factor?

**0.2.** Note that if \((G, X)\) is distal and the orbit closure mapping of \((G, X \times X)\) is continuous, then it is not difficult to prove that the regionally proximal relation \(RP\) of \((G, X)\) is equal to \(\Delta_X\) so that \((G, X)\) is equicontinuous [6] (cf. Theorem 3.1.1). In McMahon and Wu [21, (3.4)], there exists an example of a countable group \(G\) acting on a 0-dimensional compact \(X\) such that \((G, X)\) is minimal distal non-equicontinuous (also cf. Example 3.1.5). So in this example, \((G, X \times X)\), with 0-dimensional compact phase space \(X \times X\) and with countable (so \(\sigma\)-compact) phase group \(G\), satisfies that (1) \(\Rightarrow\) (5) and (3) \(\Rightarrow\) (5). Thus we need to impose some restrictive condition on the phase group \(G\) for (1) or (3) \(\Rightarrow\) (5).

On the other hand, there are \(\mathbb{R}\)-flows and \(\mathbb{Z}\)-flows that are distal non-equicontinuous with compactly generated phase groups [4, 11] so that (1) or (2) \(\not\Rightarrow\) (6) in 0.1. Thus we also need impose some restrictive condition on the phase space \(X\).

Let \(X\) be a 0-dimensional compact space. In [16, Theorems 5 and 6], Gottschalk shows that (1) \(\Rightarrow\) (5) \(\Leftrightarrow\) (7) for \(\mathbb{Z}_+\)-flow on \(X\), and see also [2, Theorem 7.12] for \(\mathbb{Z}\)-flow on \(X\).

Recall that \(G\) is generative provided that \(G\) is abelian and generated by some compact neighborhood of \(e\) (cf. [18, Def. 6.01]). Using “replete semigroup”, in [18] Gottschalk and Hedlund formulate a definition of recurrence. Then, with this notion, they prove that (1) \(\Rightarrow\) (5) \(\Leftrightarrow\) (7) for each flow with generative phase group and with 0-dimensional phase space (cf. [18, Theorems 7.07 and 7.08]).

In [3], Auslander et al. introduce a definition of recurrence in terms of “cone”. With their notion of recurrence, they show that (1) \(\Rightarrow\) (5) for every flow \((G, X)\) with finitely generated phase group \(G\) not necessarily abelian and with 0-dimensional compact metric phase space \(X\) (cf. [3, Theorem 1.8] and Proposition 4.5 below).
We will formulate a recurrence and extend the theorem of Auslander et al. [3, Theorem 1.8] to any flow \((G, X)\) with compactly generated phase group \(G\) not necessarily countable or abelian, and, with 0-dimensional phase space \(X\) not necessarily compact metrizable, as follows:

**Theorem A** (see Thm. 2.6a for the full statement). Let \((G, X)\) be a flow, where \(G\) is compactly generated and \(X\) is 0-dimensional (cf. Def. 1.7). Then properties (1) – (7) in 0.1 are pairwise equivalent.

**Warning.** The orbit closure relation being closed or orbit closure mapping \(\text{continuous} \Leftrightarrow \text{orbit closure mapping being upper semi-continuous in general non-compact flows (see Theorem 2.6c).}

Comparing with the case \(X\) compact, we have to face the essential point that a net in \(X\) need not have a convergent subnet. In addition, comparing with the case \(G = \mathbb{Z}_+\), for any point \(x\) and any closed subset \(U\) of \(X\) we have no the “first visit time” \(t \in G\) with \(tx \in U\).

In addition, as an application of Theorem A, we can conclude the following using the component relation:

(9) Every distal, compact, and non-connected flow has a non-trivial equicontinuous pointwise regularly almost periodic factor if the phase group is compactly generated like \(\mathbb{Z}\) and \(\mathbb{R}\) (see Thm. 3.1.2 and Corollary to Thm. 3.1.3). See 1.5c for the definition of regular almost periodicity.

If a flow \((G, X)\) is pointwise product almost periodic (i.e. \((G, X \times X)\) is pointwise almost periodic) then \((G, X)\) is distal (cf. [6, Theorem 1]}; and moreover, if in addition \(G\) is generative and \(X\) is 0-dimensional, then \((G, X)\) is equicontinuous (cf. [6, Theorem 2]). Now we shall improve Ellis’ theorems as follows:

**Theorem B** (see Thm. 3.1.1 and Thm. 3.1.3). Let \((G, X)\) be any flow with \(G\) compactly generated and with dim \(X = 0\). If \((G, X)\) is pointwise product recurrent, then \((G, X)\) is pointwise regularly almost periodic and equicontinuous.

**0.3 Remark.** It should be mentioned that a part of Theorem A has been proved by Reid [26] under the framework that \(G\) is “equicontinuously generated” by a subset \(S\) of \(\text{Homeo}(X)\); that is, \(G = \bigcup_{k \in S} S^k\), where \(id_X \in S = S^{-1}\), and \(S\) is equicontinuous. However, our approaches are different completely with and much simpler than Reid [26, Theorems 1.2, 1.3] of using very technical arguments of compact-open invariant sets. It turns out that Theorem C concisely proved here implies the “equicontinuously generated” case of Reid [26] (see Propositions 5.5 and 5.6 for alternative simple proofs of Reid’s results).

**0.4 Standing notation.** By \(\mathcal{U}\) there is a uniformity structure, denoted \(\mathcal{U}_X\) or \(\mathcal{U}_x\), on \(X\). In the sequel, \(\mathcal{R}_x\) stands for the neighborhood filter at \(x\) of \(X\) and \(N_G(x, U) = \{t \in G : tx \in U\}\) for all \(x \in X\) and all \(U \in \mathcal{R}_x\).

1. Preliminaries

**1.1.** If \(Y\) is a Hausdorff space then \(2^Y\) will stand for the collection of closed subsets of \(Y\). For a net \(\{Y_i\}\) in \(2^Y\) and \(K \subseteq Y\), we say \(Y_i \to K\) in \(2^Y\), denoted \(\limsup Y_i = K\), if

\[K = \{y \in Y \mid \exists \text{ a subnet } (Y_{i_k}) \text{ from } (Y_i) \text{ and } y_{i_k} \in Y_{i_k} \text{ s.t. } y_{i_k} \to y\}.
\]

It turns out that in the case of \(\to\), every net \(\{K_i\}\) in \(2^Y\) converges and \(\limsup K_i \in 2^Y\). However, it is possible that \(\limsup K_i = \emptyset\).
Lemma. If $K_i \rightarrow K$ in $2^X$ and $(K_i)$ is any subnet of $(K_i)$ with $K_i \rightarrow K'$, then $K' \subseteq K$. Moreover, if $K_i$ is invariant for all $i$, then $\limsup_i K_i$ is also invariant.

If $Y$ is a discrete space and $(Y_i)_{i=1}^\infty$ with $Y_i \rightarrow K$, then $K = \bigcap_{i=1}^\infty \bigcup_{j\geq i} Y_j = \limsup_{i\rightarrow \infty} Y_i$; in other words, $K = \{Y_i \text{ i.o.}\}$. We will concern with $Y = G$ or $X$ for a flow $(G, X)$.

1.2. Associated to $(G, X)$ we define the ‘orbit closure mapping’ $\bar{O}_G : X \rightarrow 2^X$ by $x \mapsto \bar{O}_G x$. Then we say that:

a. $\bar{O}_G$ is 'continuous', provided that if $x_i \rightarrow x$ in $X$, then $\bar{O}_G x_i \rightarrow \bar{O}_G x$ in $2^X$.

b. $\bar{O}_G$ is upper semi-continuous, provided that for every $x \in X$ and every neighborhood $U$ of $\bar{O}_G x$ there is a $V \in \mathcal{O}_x$ such that $\bar{O}_G x \subseteq U$ for all $y \in V$. Further, $\bar{O}_G$ is called continuous if $x_i \rightarrow x$ in $X$ implies that $\bar{O}_G x_i \rightarrow \bar{O}_G x$ in $2^X$ with the Hausdorff topology.

Lemma. If $\bar{O}_G$ is upper semi-continuous, then $\bar{O}_G$ is 'continuous', and moreover, $\bar{O}_G$ is continuous.

Proof. If $x_i \rightarrow x$ in $X$ and $\bar{O}_G x_i \rightarrow L$ in $2^X$, then $\bar{O}_G x_i \subseteq L$ for $L$ is closed $G$-invariant and $x$ is in $L$. On the other hand, if we additionally assume $\bar{O}_G$ is upper semi-continuous, then for every neighborhood $V$ of $\bar{O}_G x$ we have $\bar{O}_G x_i \subseteq V$ eventually. Since $\bar{O}_G x$ is compact and $X$ is Hausdorff, $\bar{O}_G x$ is eventually separated from any point outside $\bar{O}_G x$, so $L \subseteq \bar{O}_G x$. Thus $\bar{O}_G$ is 'continuous'.

On the other hand, since the phase mapping $(t, x) \mapsto tx$ of $G \times X$ onto $X$ is jointly continuous, $\bar{O}_G$ is automatically lower semi-continuous (i.e. $\forall x' \in \bar{O}_G x \exists x'_i \in \bar{O}_G x_i$ s.t. $x'_i \rightarrow x'$ for $x_i \rightarrow x$), $\bar{O}_G$ is continuous. The proof is completed.

Given $U \subset X$, define invariant sets

c. $U^\circ_G = \{x \in X | \bar{O}_G x \subseteq U\}$ and $U^{\circ\circ}_G = \bigcap_{g \in G} g U$.

If $U$ is closed, then $U^\circ_G = U^{\circ\circ}_G$ is closed. However, $U^\circ_G \not\subseteq U^{\circ\circ}_G$ in general.

d. $(G, X)$ is equicontinuous at a point $x \in X$ if given $\varepsilon \in \mathbb{N}$ there is $U \in \mathcal{O}_x$ such that $gU \subseteq \varepsilon[gx] \forall g \in G$. $(G, X)$ is equicontinuous iff $(G, X)$ is pointwise equicontinuous.

Clearly, if $(G, X)$ is equicontinuous at $x_0 \in X$ then $\bar{O}_G : X \rightarrow 2^X$ is continuous at the point $x_0$. So if $(G, X)$ is equicontinuous, then $\bar{O}_G$ is continuous on $X$.

Note that equicontinuity is independent of the topology of $G$, but it depends on the specific uniform structure of $\mathcal{O}$ and not merely on the topology of $X$.

1.3. The ‘orbit-closure relation $R_o(X)$ of $(G, X)$’ is defined by $(x, y) \in R_o(X)$ iff $y \in \bar{O}_G x$.

a. If $R_o(X)$ is closed, then $R_o(X)$ is symmetric, i.e., $R_o = R_o^{-1}$.

Proof. This is because for $(x, y) \in R_o(X)$ there is a net $(t_n, x) \rightarrow y$ and $(t_n, x) \in R_o(X)$ so that $(t_n, x) \rightarrow (y, x) \in R_o(X)$.

b. $\bar{O}_G$ is 'continuous' on $X$ if and only if $R_o(X)$ is closed in $X \times X$. (Note: Lagrange stability is surplus here.)

Proof. Indeed, suppose $\bar{O}_G$ is 'continuous' and, let $(x_i, y_i) \in R_o(X) \rightarrow (x, y)$. Then $\bar{O}_G x_i \rightarrow \bar{O}_G x$ by 1.2a and $y_i \in \bar{O}_G x_i$ with $y_i \rightarrow y$. So $y \in \bar{O}_G x$ by 1.1 and $(x, y) \in R_o(X)$. Thus $R_o(X)$ is closed.

Conversely, assume $R_o(X)$ is closed, and, let $x_i \rightarrow x$ in $X$ and $\bar{O}_G x_i \rightarrow \bar{O}_G x$ in $2^X$. If $\bar{O}_G x \not\subseteq \mathcal{O}$, then $\bar{O}_G x \not\subseteq \mathcal{O}$. For $y \in \mathcal{O} \setminus \bar{O}_G x$ there is a net $y_i \in \bar{O}_G x_i$ with $y_i \rightarrow y$. Since $(x_i, y_i) \in R_o(X)$, so $(x, y) \in R_o(X)$ and $y \in \bar{O}_G x$ contrary to $y \in \mathcal{O} \setminus \bar{O}_G x$. Thus $\bar{O}_G x = \mathcal{O}$ and $\bar{O}_G$ is 'continuous'.


1.4. We say that a set $K$ in $X$ is ‘minimal’ under $(G, X)$ if $\overline{Gx} = K$ for all $x \in K$. In this case $K$ is a closed invariant subset of $X$, and moreover, $K$ is compact by (iii) in $\mathbb{C}$. So every minimal flow is compact in our setting.

1.4a Lemma. Let $R_o[y] = \{x \mid x \in X, (x, y) \in R_o(X)\}$ for $(G, X)$ and $y \in X$. If $R_o[y]$ is closed, then $\overline{Gy}$ is minimal. Here $(G, X)$ need not be Lagrange stable.

Note. Here $\{x \in X \mid (y, x) \in R_o(X)\}$ is always closed but it gives no minimality.

Proof. Let $x \in \overline{Gy}$. There is a net $t_n \in G$ with $t_n y \rightarrow x$. Since $(t_n y, y) \in R_o(X)$, so $(x, y) \in R_o(X)$ and $y \in \overline{Gx}$. Thus $\overline{Gy} = \overline{Gx}$ is minimal for $x \in \overline{Gy}$ is arbitrary.

Clearly $R_o(X)$ is symmetric iff $\overline{Gx}$ is minimal for all $x \in X$. Moreover, if $R_o(X)$ is closed, then $R_o(X)$ is an invariant closed equivalence relation on $X$, and, $X/G := X/R_o(X)$ with the quotient topology is locally compact Hausdorff by Lemma 1.4b below.

1.4b Lemma. If $R_o(X)$ is symmetric for $(G, X)$, then the quotient mapping $\rho: X \rightarrow X/G$ is open.

Note. $X/G$ need not be Hausdorff if $R_o(X)$ is not closed.

Proof. Let $U$ be an open subset of $X$, $U \neq \emptyset$. Then $\rho^{-1} \rho U = GU$. Indeed, for each $x \in U$, choose a compact $V \in \mathbb{N}_x$ with $V \subseteq U$. Since $x$ is almost periodic and $N_U(x, V)$ is discretely syndetic (cf. 1.5a below), there is a finite set $K \subset G$ such that $Gx \subseteqKV$ and $\overline{Gx} \subseteq K \subseteq GU$. By $\rho^{-1} \rho U = \bigcup_{t \in U} \overline{Gx}$, it follows that $\rho^{-1} \rho U$ is open in $X$. Thus $\rho U$ is open in $X/G$. The proof is complete.

1.4c Lemma (cf. [17, Thm. 1] for $X$ compact). The orbit-closure relation $R_o(X \times X)$ of $(G, X \times X)$ is closed if and only if $(G, X)$ is equicontinuous.

Proof. Sufficiency is obvious from 1.2d and that if $(G, X)$ is equicontinuous, then so is $(G, X \times X)$. Conversely, suppose $R_o(X \times X)$ is closed. Let $x_0 \in X$. To show that $(G, X)$ is equicontinuous at $x_0$, suppose the contrary that $(G, X)$ were not equicontinuous at $x_0$. Then there would be nets $(x_n, x'_n) \in X \times X$, $t_n \in G$ and an open index $\varepsilon \in \mathcal{V}$ such that $(x_n, x'_n) \rightarrow (x_0, x_0)$ and $t_n(x_n, x'_n) \notin \varepsilon$.

By $G(x_n, x'_n) = G(t_n(x_n, x'_n))$, it follows that

$$
\varepsilon \supseteq \Delta_X \supseteq \overline{G(x_0, x_0)} = \lim_n \overline{G(x_n, x'_n)} = \lim_n \overline{G(t_n(x_n, x'_n))} \subseteq \varepsilon
$$

which is a contradiction, where $\Delta_X$ is the diagonal of $X \times X$. This contradiction shows that $(G, X)$ is equicontinuous.

In particular, if $X$ is compact and the relation $R_o(X \times X)$ of $(G, X \times X)$ is closed, then $(G, X)$ is uniformly equicontinuous.

1.5. A subset $A$ of $G$ is ‘syndetic’ in $G$ if there exists a compact subset $K$ of $G$ with $G = K^{-1}A$. A set $B$ is said to be ‘thick’ in $G$ if for all compact set $K$ in $G$ there corresponds an element $t \in B$ such that $Kt \subseteq B$. It is a well-known fact that

- $A$ is syndetic in $G$ iff it intersects non-voidly with every thick subset of $G$.

Although $G$ need not be discrete here, yet if the syndetic/thick is defined in the sense of discrete topology of $G$ then it will be called ‘discretely syndetic/thick’. For instance, in $\mathbb{R}$ with the usual Euclidean topology, $\mathbb{Z}$ is a discrete subgroup and $\mathbb{Z}$ is syndetic; however, $\mathbb{Z}$ is not discretely syndetic in $\mathbb{R}$. 

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a. A point \( x \in X \) is ‘almost periodic’ (a.p) under \((G, X)\) if \( N_G(x, U) \) is a syndetic subset of \( G \) for all \( U \in \mathfrak{U} \). Since \( Gx \) is compact by (iii) in @ here, \( x \in X \) is a.p if and only if \( Gx \) is minimal under \((G, X)\) (cf., e.g. [2, Note 2.5.1]). Thus

\[ N_G(x, U) \text{ is discreetly syndetic in } G \text{ for every a.p point } x \text{ of } (G, X). \]

In addition, \( x \) is a.p iff given \( U \in \mathfrak{U} \) there is a finite set \( K \) in \( G \) such that \( Gx \subseteq KU \).

b. Following [18, 17], we say that:

1) \((G, X)\) is ‘weakly a.p’ if given \( \alpha \in \mathcal{Y} \) there is a compact subset \( F \) of \( G \) such that \( Fix \cap \alpha[x] \neq \emptyset \) for all \( x \in X \) and \( t \in G \).

2) \((G, X)\) is ‘locally weakly a.p’ if given \( \alpha \in \mathcal{Y} \) and \( x \in X \) there is a compact subset \( F \) of \( G \) and a \( V_0 \in \mathcal{U} \) such that \( Fty \cap \alpha[y] \neq \emptyset \) for all \( y \in V \) and all \( t \in G \). Equivalently, \((G, X)\) is locally weakly a.p iff for each \( x \in X \) and all \( U \subseteq \mathcal{U} \), there is a compact set \( F \) in \( G \) and a set \( V \in \mathcal{U} \) such that \( GV \subseteq U^{-1}U \).

Clearly, if \( X \) is compact, then \((G, X)\) is weakly a.p iff \((G, X)\) is locally weakly a.p; if \((G, X)\) is locally weakly a.p, it is pointwise a.p under \((G, X)\). Moreover, if \((G, X)\) is locally weakly a.p, then \( \partial_G: X \to 2^X \) is upper semi-continuous (see Lemma 2.4).

c. (see [18, 6, 22] for \( G \) an abelian group).

1) A point \( x \) is said to be regular a.p under \((G, X)\), denoted \( x \in P_{r.a.p}(X) \), if \( N_G(x, U) \) contains a syndetic normal subgroup of \( G \) for every \( U \in \mathfrak{U} \); if every point of \( X \) is regularly a.p under \((G, X)\), i.e., \( \forall x \in X \), then \((G, X)\) is called pointwise regularly a.p.

2) We say \((G, X)\) is (uniformly) regularly a.p if every \( \alpha \in \mathcal{Y} \) there is a syndetic normal subgroup \( A \) of \( G \) such that \( Ax \subseteq \alpha[x] \) for all \( x \in X \).

3) We call \((G, X)\) a point-regularly a.p flow if there is a regularly a.p point that has a dense orbit in \( X \).

See [18, Theorem 12.55] for an example of point-regularly a.p \( \mathbb{Z} \)-flows that is not equicontinuous. Note that even for \( X \) a compact metric space, a pointwise regularly a.p (in fact, pointwise periodic) \( \mathbb{Z} \)-flow need not be regularly a.p (see Example 3.3.1).

1.5d Lemma (cf. [18, Lem. 5.04]). Let \( T \) be a discrete group and let \( A, B_1, \ldots, B_n \) be syndetic subgroups of \( T \). Then:

1) \( \bigcap_{i=1}^n B_i \) is a syndetic subgroup of \( T \).

2) There exists a syndetic normal subgroup \( H \) of \( T \) such that \( H \leq A \). In fact, we can take \( H = N(A) = \bigcap_{t \in T} tA^{-1}t \).

Proof. (1). As \( T/B_1, \ldots, T/B_n \) are finite, it follows from \( tB_1 \cap \cdots \cap B_n = tB_1 \cap \cdots \cap tB_n \forall t \in T \) that \( T/ \left( \bigcap_{i=1}^n B_i \right) \) is finite, whence \( \bigcap_{i=1}^n B_i \) is a syndetic subgroup of \( T \).

(2). Choose a finite set \( K \) in \( T \) such that \( KA = T \). Then \( N(A) = \bigcap_{k \in K} tA^{-1}t \), which is normal and is syndetic by (1). The proof is complete.

Comparing with [18] where Gottschalk and Hedlund is by using the permutation group of \( T/A \) for proving (2), the point of our alternative proof is to choose \( H = N(A) \) here.

1.5e Lemma (cf. [7, Prop. 2.8] and [1, Thm. 1.13] for \( G \) a para-topological group). Let \((G, X)\) be a flow and \( A \) a syndetic normal subgroup of \( G \). Then \((G, X)\) is pointwise a.p iff \((A, X)\) is pointwise a.p.
Proof. Sufficiency is obvious (without using the assumption that $A$ is normal). Conversely, suppose $(G, X)$ is pointwise a.p and let $x \in X$. Let $y \in \overline{A x}$ be an a.p point under $(A, X)$. Take $t_n \in G$ with $t_n y \to x$. Let $K$ be a compact subset of $G$ with $G = K^{-1}A$. Then there are $k_n \in K$ with $k_n t_n \in A$ and $k_n \to k \in K$. So $k_n t_n y \to k x \in \overline{A x}$. Thus $k x$ is a.p under $(A, X)$. Since $A$ is normal, it is not difficult to prove that $x$ is also a.p for $(A, X)$. The proof is complete.

Corollary. Let $(G, X)$ be a flow with $G$ a discrete group and $A$ a syndetic subgroup of $G$. Then $(G, X)$ is pointwise a.p iff $(A, X)$ is pointwise a.p.

Proof. By Lemma 1.5d, there is a syndetic normal subgroup $H$ of $G$ such that $H \leq A$. Then by Lemma d) above and $H \triangleleft A$, $(G, X)$ is pointwise a.p iff $(H, X)$ is pointwise a.p iff $(A, X)$ is pointwise a.p. The proof is complete.

Note. If $(G, X)$ is minimal in addition, then $\theta_G : X \to 2^X$ is continuous; see Theorem 2.7. However, there is no an analogous inheritance in general for the regular almost periodicity of non-discrete group actions.

1.6. We say that $(G, X)$ is distal iff we have for all $x, y \in X$ with $x \neq y$ that $\Delta x \cap G(x, y) = \emptyset$. Clearly, distality is independent of the topology of $G$. Moreover, a distal flow is pointwise a.p.

1.7. A Hausdorff space is said to be 0-dimensional if it has a base consisting of clopen sets. For example: 1. A locally compact, Hausdorff, and totally disconnected topological space is 0-dimensional ([19, Theorem 3.5]); 2. Ellis’ “two-circle minimal set” is compact Hausdorff 0-dimensional ([7, Example 5.29]).

$G$ is called compactly generated if there exists a compact subset $\Gamma$ of $G$ with $e \in \Gamma = \Gamma^{-1}$, called a generating set, such that $G = \bigcup_{n=0}^{\infty} \Gamma^n$. In this case, $G$ is $\sigma$-compact, not necessarily locally compact. Moreover, the generating set $\Gamma$ is not unique; e.g., $t^{-1} \Gamma t$, for $t \in G$, is so. In addition, $\Gamma$ need not be a neighborhood of $e$. For instance, $\Gamma = [-1, 1] \times \{0\} \cup \{0\} \times [-1, 1]$ is a compact generating set of $(\mathbb{R}^2, +)$ but it is not a neighborhood of $o = (0, 0) \in \mathbb{R}^2$.

1.8 (inheritance). If $G$ is compactly generated and $S$ a closed syndetic subgroup of $G$, is $S$ compactly generated too? Towards a positive solution we need the following lemma, which is a variation of [18, Theorem 6.10].

1.8a Lemma. Let $T$ be a group, let $S$ be a subgroup of $T$, $\Gamma \subseteq T$ a set with $e \in \Gamma = \Gamma^{-1}$ such that $T = \Gamma S$. Define $\Psi = \Gamma^3 \cap S$. Then $\Gamma^n \cap S \subseteq \Psi^n$ for all $n \in \mathbb{Z}$.

Proof. Let $n$ be any positive integer, and, let $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $s_n := \gamma_n \cdots \gamma_1 \in S$; that is, $s_n \in \Gamma^n \cap S$. By $T = \Gamma S = \Sigma \Gamma$, it follows that, for $1 \leq i < n$, there exists $s_i \in S$ with $s_i \in \Gamma \gamma_i \cdots \gamma_1$. Clearly, $s_n = \gamma_n \cdots \gamma_1$. Now $s_{i+1} s_i^{-1} \in \Gamma \gamma_{i+1} \Gamma \subseteq \Gamma^3$ and $s_{i+1} s_i^{-1} \in S$ for all $1 \leq i < n$. So $s_{i+1} s_i^{-1} \in \Psi$ and $s_{i+1} \in \Psi s_i$ for $1 \leq i < n$. Also $s_j \in \Gamma \gamma_j \subseteq \Gamma^3$ and $s_j \in S$, thus $s_1 \in \Psi$. Then $s_n \in \Psi s_1 \subseteq \Psi^2 s_1 \subseteq \cdots \subseteq \Psi^{n-1} s_1 \subseteq \Psi^n$. The proof is completed.

1.8b Corollary. Let $G$ be a compactly generated topological group and $S$ a syndetic closed subgroup of $G$. Then $S$ is also compactly generated.

Proof. Let $G$ be compactly generated by $\Gamma$ with $e \in \Gamma = \Gamma^{-1}$. Since $S$ is syndetic, we can take a compact symmetric set $K \subseteq G$ with $G = K S$. Define $\Psi = (\Gamma \cup K)^3 \cap S$, which is compact in $S$ such that $e \in \Psi = \Psi^{-1}$. By Lemma 1.8a, $S = \bigcup_{i \geq 2} \Psi^i$. Thus $S$ is compactly generated by $\Psi$. The proof is completed.
1.9 (a note to para-topological group). Let \((G, X)\) be a flow. Let \(K \subset G\) be a compact set. Then \(K^{-1}\) need to be compact subset of \(G\). However, we can assert that \(K^{-1}W\) is compact for every compact set \(W\) in \(X\) for \((G, X)\):

**Lemma.** If \(K \subset G \text{ and } W \subset X\) are compact sets. Then \(K^{-1}W\) is a compact subset of \(X\).

**Proof.** We equip \(\text{Homeo}(X)\) with the topology of uniform convergence on compacta. Then the mapping \(G \to \text{Homeo}(X)\) defined by \(t \mapsto t_x\), where \(t_x \colon x \in X \mapsto tx \in X\), is continuous so \(\mathcal{K} = \{tx \mid t \in K\}\) is a compact subset of \(\text{Homeo}(X)\). Let \(t_0 \to t \in K\) and \(x \in X\). Since \(G\) is compact, we may assume a (subset of) \((t_{aX})^{-1}x \to x^t\) so that \(x = t_{aX}(t_{aX})^{-1}x \to t_{X}x^t\). Thus \(x = t_{X}x^t\) and \((t_{X})^{-1}x = x^t\). This implies that \((t_{X})^{-1} \to (t_{X})^{-1}\). Thus \(K \to \text{Homeo}(X)\) defined by \(t \mapsto (t_{X})^{-1}\) is continuous. Since \(t_{Y}(t_{X})^{-1} = t_{X}t_{Y}^{-1}x = x\) for all \(t \in G\) and \(x \in X\), so \((t_{X})^{-1} = (t_{Y})x\) for all \(t \in G\). Then \(\mathcal{K}^{-1}\) is compact in \(\text{Homeo}(X)\). Therefore, \(K^{-1}W = \mathcal{K}^{-1}W\) is compact in \(X\). The proof is completed.

\[ \square \]

2. Pointwise recurrence in 0-dimensional flows

This section will be devoted to proving the full statement of Theorem A stated in §0, which implies that for some class of flows, the pointwise recurrence is equivalent to the pointwise almost periodicity.

2.1. Let \(G\) be compactly generated by \(\Gamma\) such that \(e \in \Gamma = \Gamma^{-1}\) as in Def. 1.7. For example, if \(G\) is finitely generated or if \(G\) is generative, then \(G\) is compactly generated.

2.1a (word length). For any \(g \in G\) with \(g \neq e\), the \(\Gamma\)-length of \(g\), denoted \([g]\), is the smallest integer \(r\) such that \(g = \gamma_1 \cdots \gamma_r\) with \(\gamma_i \in \Gamma\) for \(1 \leq i \leq r\), and, write \(K(g) = \Gamma^{[g]} \cdot g\), where \(\Gamma^{[g]} = [e] and \(K(g)\) is a compact subset of \(G\). Clearly, \(e \notin K(g)\) for \(g \in G \setminus [e]\) with \([g] \geq 1\).

2.1b (cone). A subset \(C\) of \(G\) is called a \(\Gamma\)-cone in \(G\) if there exists a net \(\{g_i \mid i \in \Lambda\}\) in \(G\) such that \([g_i] \not\to \infty\) and \(K(g_i) \to C\) in \(2^G\). Here \([g_i] \not\to \infty\) means that if \(i \leq i'\) in the ordering of \(\Lambda\) then \([g_i] \leq [g_{i'}]\). Clearly, every \(\Gamma\)-cone in \(G\) is a closed set in \(G\).

Moreover, if \(G\) is discrete (i.e. finitely generated), then \(e \notin C\). However, this is not the case in non-discrete \(G\). For instance, let \(G = \langle \mathbb{R}, + \rangle\) with the usual euclidean topology and let \(\Gamma = [-1, 1]\); then for \(g_i = i + 1/i, i = 1, 2, \ldots\), we have that \(0 \notin K(g_i) \to C = [0, \infty)\) with \(0 \in C\).

2.1c Lemma (cf. [3, Prop. 1.2 and Prop. 1.5] for \(G\) finitely generated). Let \(G\) be compactly generated by \(\Gamma\). Then:

1. If \([g_i] \not\to \infty\) in \(G\) with \([g_i] \not\to \infty\), then \(K(g_i) \to C \neq \emptyset\).
2. If \(C\) is a \(\Gamma\)-cone in \(G\), then \(C\) is a discretely thick subset of \(G\).

**Proof.** Let \(\{g_i \mid i \in \Lambda\}\) be a net in \(G\) with \([g_i] \not\to \infty\) and \(K(g_i) \to C\). Let \(K\) be a finite subset of \(G\) with \(e \in K\). We shall find an element \(t \in G\) with \(Kt \subset C\). Since \(G\) is generated by \(\Gamma\), there exists an integer \(r \geq 1\) such that \(K \subset \Gamma^r\). For each \(i \in \Lambda\), let \(t_i = [g_i] \not\to \infty\) and write

\[ g_i = \gamma_1 \cdots \gamma_{r-1} \gamma_{r-1} \cdots \gamma_{1} \in \Gamma^r, \quad a_i = \gamma_{r-1} \cdots \gamma_{1} \in \Gamma^{r+1}. \]

Then \(K \subset C\) for \(i \in \Lambda\) with \(t_i > r\). Since \(\Gamma^{r+1}\) is compact, we can assume (a subnet of) \(a_i \to t\) in \(G\). Thus \(t \in C\) and \(Kt \subset C \neq \emptyset\) by Lemma 1.1. This also proves that \(C\) is a discretely thick subset of \(G\). The proof is completed.

\[ \square \]
This simple observation will be very useful. From the choice of \( \alpha_i \) in its proof above we can easily conclude the following lemma:

**2.1d Lemma** (cf. [26, Lem. 3.6]). Let \( F \) be any finite subset of \( G \). Then there is an integer \( n > 1 \) such that for all \( g \in G \) with \( |g| \geq n \), there is an element \( t \in G \) such that \(|t| = n \) and \( Ft \subseteq K(g) \).

Note that if \( r_1 \to \infty \) and \( |g| \to \infty \) with no any restriction on \( |g| - r_1 \), then it is possible that \( \Gamma^+ g_i \to \emptyset \) even though \( G = \mathbb{Z} \) or \( \mathbb{R} \). Moreover, the compactness of the generator set \( \Gamma \) plays an important role in the above proof of Lemma 2.1c and also in Theorem 2.6a below.

2.2 (recurrrence). Let \( (G, X) \) be a flow with a compactly generated phase group \( G \) that has a compact generating set \( \Gamma \) and let \( x \in X \). We will define recurrence at \( x \) in two ways.

2.2a. We say that \( x \) is \( \Gamma \)-\textit{recurrent of type I} under \( (G, X) \) if for all \( Cx \cap U \neq \emptyset \) for all \( \Gamma \)-cone \( C \) in \( G \) and every \( U \in \mathcal{U}_x \). We say that \( x \) is \( \Gamma \)-\textit{recurrence of type I} under \( (G, X) \) if \( x \) is \( \Gamma \)-recurrent of type I for all compact generating set \( \Gamma \) of \( G \). Clearly, if \( x \) is recurrent of type I, then \( tx \) is recurrent of type I for every \( t \in G \).

2.2b. We say that \( x \) is \( \Gamma \)-\textit{recurrent of type II} under \( (G, X) \) if for each \( U \in \mathcal{U}_x \) and every net \( \{g_i\} \) in \( G \) with \( |g_i| \to \infty \) there exists an integer \( n \geq 1 \) such that there is a subnet \( \{g_{i_j}\} \) from \( \{g_i\} \) such that for each \( j \), there is an element \( t_j \in G \) with \( F t_j \subseteq K(g_{i_j}) \) and \(|t_j| = n \). Take \( f_j \in F \) with \( c_j = f_j t_j \in A \), then \(|c_j| \leq 2n \) and \( c_jx \in U \) for all \( j \). Thus \( x \) is \( \Gamma \)-recurrent of type II.

(2). Suppose \( x \) is \( \Gamma \)-recurrent of type II. Let \( C \) be a \( \Gamma \)-cone in \( G \). That is, \( K(g_i) \to C \), where \( \{g_i\} \) is a net in \( G \) with \( |g_i| \nrightarrow \infty \). Let \( U, V \in \mathcal{U}_x \) with \( V \subseteq U \). By Def. 2.2b, there is an integer \( n \geq 1 \) and a subnet \( \{g_{i_j}\} \) from \( \{g_i\} \) such that \( c_jx \in V \) and \(|c_j| \leq n \). Since \( \Gamma^+ \) is compact and \( c_j \in \Gamma^+ \), we can assume (a subnet of) \( c_j \to c \in G \) and \( c_jx \to cx \in V \). Since \( c \in C \) by definition and \( Cx \cap U \neq \emptyset \), \( x \) is \( \Gamma \)-recurrent of type I. The proof is complete.

**2.3 Lemma** ([18, Thm. 4.16, 4.17, 4.24]). A flow \( (G, X) \) is locally weakly a.p with \( G \) under the discrete topology if and only if \( \mathcal{O}_G : X \to 2^X \) is upper semi-continuous.

**Proof.** Sufficiency. Assume \( \mathcal{O}_G \) is upper semi-continuous. By 1.3b and Lemma 1.4a, every orbit-closure is minimal so \( (G, X) \) is pointwise a.p. Let \( \alpha \in \mathcal{U} \) and \( x \in X \). Let \( U \in \mathcal{U}_x \) be so small that \( \alpha[y] \supseteq U \) for all \( y \in U \). Since \( \overline{Gx} \) is compact minimal, there exists a finite set \( \{z_1, \ldots, z_k\} \subseteq Gx \) with \( W_i \in \mathcal{U}_y, t_i \in G \) such that \( t_i W_i \subseteq U \) and \( \overline{Gx} \subseteq \bigcup_{i=1}^k W_i \). There is a \( V \in \mathcal{U}_x \) with \( V \subseteq U \) such that \( \overline{Gx} \subseteq \bigcup_{i=1}^k W_i \) for all \( y \in V \). Then for \( F = \{t_1, \ldots, t_k\}, Ftx \cap \alpha[y] \neq \emptyset \) for all \( y \in V \) and \( t \in G \). Thus \( (G, X) \) is locally weakly a.p with \( G \) under the discrete topology.

Necessity. Suppose \( (G, X) \) is locally weakly a.p with \( G \) under the discrete topology and let \( x \in X \) and \( N \) an open neighborhood of \( \overline{Gx} \) such that \( N \) is compact. It is not difficult to show that \( \overline{Gx} \) is compact.
Indeed, for each \( y \in \overline{N} \) and each compact \( U_y \in \mathfrak{U} \), there is a finite set \( K_y \) in \( G \) and an open set \( V_y \in \mathfrak{U} \) such that \( G V_y \subseteq K_y^{-1} U_y \). Since \( \overline{N} \) is compact, there is a finite set \( \{y_1, \ldots, y_n\} \subset \overline{N} \) with \( \overline{N} \subseteq V_{y_1} \cup \cdots \cup V_{y_n} \) so that \( GN \subseteq \bigcup_{i=1}^n K_{y_i}^{-1} U_{y_i} \). Thus \( G \overline{N} \) is compact.

Now replacing \( X \) by \( G \overline{N} \) if necessary, we may assume \( X \) is compact. It is enough to find a \( V \in \mathfrak{U} \) such that \( GV \subseteq N \). For this, choose a closed neighborhood \( W \) of \( Gx \) with \( W \subset N \). For \( X - W \) is open and \( X - N \subset X - W \), there exists a finite set \( K \subset G \) such that \( G(X - N) \subseteq K(X - W) \). Take \( V \in \mathfrak{U} \) so small that \( K^{-1} V \subseteq W \). Then \( K^{-1} V \cap (X - W) = \emptyset \), \( V \cap K(X - W) = \emptyset \), and \( V \cap G(X - N) = \emptyset \). So \( GV \cap (X - N) = \emptyset \) and \( GV \subseteq N \). The proof is complete.

It should be mentioned that the Lagrange stability plays an important role for the sufficiency of Lemma 2.4. The main ideas of the proof of Theorem A are contained in Lemma 2.4 above and Lemma 2.5 below.

2.5 Lemma. Let \((G, X)\) be any flow with \( G \) compactly generated by \( \Gamma \), where \( G \overline{V} \) need not be compact for \( y \in X \). Let \( x \in X \) and suppose there is an open neighborhood \( U \) of \( G \overline{x} \) with \( \overline{U} \) compact, such that \( x \notin \operatorname{int} U^G \). Then there are \( g_i \in G \) with \( |g_i| \to \infty \) and \( x_j \in X \) with \( x_i \to x \), such that \( g_i x_i \to y \in X \setminus U \) and \( K(g_i^{-1}) \to C \) with \( C_y \subseteq \overline{U} \) and \( \{C_y\}^\infty_{i=1} \neq \emptyset \).

Proof. Since \( x \notin \operatorname{int} U^G \) there is a net \( x_j \in U \setminus U^G \) with \( x_j \to x \). We can obviously take \( g_i \in G \) such that \( g_i x_j \notin U \) and \( |g_i| = \min \{|t| : t \in G, tx_j \notin U\} \). Obviously (a subnet of) \( |g_i| \to \infty \). For all \( i \) we write \( g_i = \gamma_{i,1} \gamma_{i,2} \cdots \gamma_{i,|g_i|} \) and \( g_i' = \gamma_{i,2} \cdots \gamma_{i,|g_i|} \). Then \( g_i' x_i \in U \) and \( \gamma_{i,1} \in \Gamma \) for all \( i \). Since \( \overline{U} \) and \( \Gamma \) are compact, we can assume (a subnet of) \( \gamma_{i,1} \to \gamma \) and \( g_i' x_i \to y' \in U \). Let \( y = y y' \); then \( g_i x_i \to y \notin U \).

Let \( K(g_i^{-1}) \to C \), a \( \Gamma \)-cone in \( G \). Given \( c \in C \), we can take a subnet \( \{g_{i_j}\} \) from \( \{g_i\} \) and \( c_j = k_j g_{i_j}^{-1} \) with \( k_j \in \Gamma^{\{g_{i_j}\}} \) and \( c_j \to c \). Thus by \( |k_j| < |g_{i_j}| \), it follows that \( cy = \lim_j c_j y = \lim_j (k_j g_{i_j}^{-1}) (g_{i_j} x_i) \in U \).

So \( \overline{C_y} \subseteq \overline{U} \). Finally, to show \( \overline{C_y} \) contains a \( G \)-orbit, let \( \mathcal{F} \) be the collection of finite subsets of \( G \) with a partial order by inclusion \( F \subseteq F' \) if \( F \subsetneq F' \). Then \( \mathcal{F} \) is a directed system. Since \( C \) is discretely thick in \( G \) by Lemma 2.1c, we have for \( F \in \mathcal{F} \) that there exists an element \( \gamma_F \in C \) with \( F t_F \subseteq C \). Since \( \overline{U} \) is compact and \( t_F y \in \overline{U} \), we can assume (a subnet of) \( t_F y \to y' \in \overline{U} \). Then for any fixed \( t \in G \) there is some \( F_0 \in \mathcal{F} \) with \( t t_F \in F_0 \). Then \( t \in F \) for all \( F \geq F_0 \) so that \( t \in \Gamma \) eventually and \( t y' \in \overline{C_y} \) for all \( t \in G \). Then \( G \overline{C_y} \subseteq \overline{C_y} \). The proof is completed.

2.6. We are ready to give concise and self-contained proofs of our main theorems. Recall that \( \partial_G : X \to 2^X \) is defined by \( x \mapsto G \overline{x} \), and, \( U^G_C = \{x \in X \mid G \overline{x} \subseteq U\} \) is closed invariant for all subset \( U \) of \( X \).

2.6a Theorem. Let \((G, X)\) be any flow, where \( G \) is compactly generated by \( \Gamma \) and \( \dim X = 0 \). Then the following conditions are pairwise equivalent:

1. \((G, X)\) is pointwise \( \Gamma \)-recurrent of type I.
2. \((G, X)\) is pointwise \( \Gamma \)-recurrent of type II.
3. \((G, X)\) is pointwise a.p.
4. \( X \) is a union of \( G \)-minimal sets.
(5) $R_\gamma(X)$ is closed.
(6) $\mathcal{O}_G$ is \(^\text{\`continuous}\) on $X$.
(7) $\mathcal{O}_G$ is upper semi-continuous on $X$.
(8) Given any compact-open subset $U$ of $X$, $U^*_G$ is compact open.
(9) $(G, X)$ is locally weakly a.p with $G$ under the discrete topology.

**Notes.**

1. The above (3) and (5) implies that the quotient mapping $\rho : X \twoheadrightarrow X/G$ is open and closed by Lemma 1.4b.

2. If in addition $(G, X)$ is topologically transitive (i.e. $GU = X$ for every non-empty open set $U$ in $X$), then each of conditions (1) \~ (9) implies that $(G, X)$ is minimal. For (3) implies that $X/G$ is a singleton.

**Proof.**

(2)\~(1): By (2) of Lemma 2.3.

(3)\~(2): By (1) of Lemma 2.3.

(4)\~(3): By 1.5a.

(5)\~(4): By Lemma 1.4a.

(6)\~(5): By 1.3b.

(7)\~(6): By Def. 1.2.

(8)\~(7): Assume (8). Let $x \in X$ and $V$ an open neighborhood of $\overline{Gx}$. Because $X$ is locally compact 0-dimensional and $\overline{Gx}$ is compact, there is a compact-open set $U$ with $\overline{Gx} \subseteq U \subseteq V$. Then $x \in \overline{Gx} \subseteq U^*_G \subseteq V$. Since $U^*_G$ is open, $\mathcal{O}_G$ is upper semi-continuous. Thus (8)\~(7).

(1)\~(8): Assume (1). Let $U$ be a compact-open subset of $X$. Then $U^*_G = U^\omega_G$ is a $G$-invariant compact set in $X$. To prove that $U^*_G$ is open, suppose the contrary that $U^*_G$ is not open in $X$. Then $U^*_G \neq \emptyset$ and there is a point $x \in U^\omega_G \setminus \operatorname{int} U^*_G$. Then by Lemma 2.5, it follows that there exists a point $y \in X \setminus U$ and a $\Gamma$-cone $C$ such that $Cy \cap (X \setminus U) = \emptyset$, contrary to that $X \setminus U \in \mathfrak{N}_y$ and that $y$ is $\Gamma$-recurrent of type I by condition (1). Thus (1) implies (8).

(7)\~(9): By Lemma 2.4.

The proof of Theorem 2.6a is thus completed.

The idea of (1)\~(8) of using the $\Gamma$-lengths of elements in $G$ could date back to [27]. It is also useful for the following theorem, where $V^\omega_G = \bigcap_{\gamma \in G} gV$ as in 1.2c and in particular there is no condition dim $X = 0$.

**2.6b Theorem.** Let $(G, X)$ be any flow with $G$ compactly generated by $\Gamma$, where $\overline{Gx}$ need not be compact for $x \in X$. Let $V$ be an open subset of $X$ such that $V^\omega_G$ is compact. Then $V^\omega_G$ is open if and only if $\overline{Gx} \cap V^\omega_G = \emptyset$ for every $x \in X \setminus V^\omega_G$.

**Proof.** Necessity is obvious, for $X \setminus V^\omega_G$ is a closed $G$-invariant subset of $X$. To show sufficiency, suppose the contrary that $V^\omega_G$ is not open (so $V^\omega_G \neq \emptyset$). Since $X$ is a locally compact Hausdorff space and $V^\omega_G$ is compact, we can find an open neighborhood $U$ of $V^\omega_G$ with $\overline{U} \subseteq V$ such that $\overline{U}$ is compact. Clearly, $V^\omega_G = \bigcap_{\gamma \in G} gU = U^\omega_G$. Then $U^\omega_G \setminus \operatorname{int} U^\omega_G \neq \emptyset$. Further by Lemma 2.5, it follows that there exists some point $y \in X \setminus V^\omega_G$ and some $\Gamma$-cone $C$ in $G$ such that $Cy \subseteq \overline{U} \subseteq V$ so that $\overline{Cy} \cap V^\omega_G = \emptyset$, contrary to the sufficiency condition. The proof is completed.

**2.6c Theorem.** Let $(G, X)$ be a flow with $G$ compactly generated. Then $\mathcal{O}_G$ is \(^\text{\`continuous}\) if and only if $\mathcal{O}_G$ is upper semi-continuous.

**Proof.** Sufficiency is obvious by Def. 1.2a and 1.2b. Necessity follows easily from Lemma 2.5. We omit the details here.
2.7 Theorem (cf. [18, Thm. 2.32, Thm. 4.29] for G a topological group). Let \((G, X)\) be a minimal flow and \(H\) a syndetic normal subgroup of \(G\). Then \(\partial_H^G: X \rightarrow 2^X\) is continuous; that is, \((H, X)\) is a weakly a.p flow with \(H\) under the discrete topology.

Proof. 1 Let \(K\) be a compact subset of \(G\) with \(G = K^{-1}H\) (cf. 1.5). Let \(x, y \in X\). By \(\overline{tx} = X\) there is a net \(t_n \in G\) such that \(y = \lim_n t_nx\). Take \(s_n \in K\) with \(s_n \rightarrow s \in K\) such that \(s_n t_n x \in H\). Since \(G \times X \rightarrow X\) is continuous, it follows that \(sy = \lim_n s_n t_n x \in \overline{tx}\). Hence \(y \in s^{-1}\overline{tx} = Hx^{-1}x\).

So by Lemma 1.5e, \(\{k^{-1}Hx | k \in K\}\) is a partition of \(X\) into \(H\)-minimal sets. Let \(k_n^{-1} x \rightarrow k^{-1} x\) with \(k_n, k \in K\) and let \(k_n^{-1}\overline{tx} \rightarrow L\). We need prove \(L = k^{-1}\overline{tx}\). We may assume \(k_n \rightarrow \ell \in K\). Then \(x = \ell k^{-1} x\) (equivalently \(\ell^{-1}x = k^{-1}x\)), and, \(\overline{tx} = \ell L\) for \(G \times 2^X \rightarrow 2^X\) is continuous. Thus, \(L = \ell^{-1}\overline{tx} = H\ell^{-1}x = Hk^{-1}x = k^{-1}\overline{tx}\). Then \(\partial_H^G\) is continuous. The proof is complete. 2

2.8 Remarks. (1) It should be noticed that since we do not know whether the \(\Gamma\)-recurrence of type I at a point of \(X\) is equivalent to that of type II for non-discrete \(G\), so (1) \(\Rightarrow\) (2) in Theorem 2.6a is not trivial. In addition, Theorem 2.6b can be utilized for proving (1) \(\Rightarrow\) (8) in Theorem 2.6a.

(2) If \(X\) has a neighborhood base at a point \(x_0\) of clopen sets instead of \(\dim X = 0\), is \(x_0\) a.p whenever \((G, X)\) is pointwise \(\Gamma\)-recurrent of type I?

3. Distality, equicontinuity and regular almost periodicity

The main goal of §3.1 is to prove Theorem B stated in §0 using Theorem 2.6a. According to Furstenberg’s structure theorem [11] every minimal distal compact flow has an equicontinuous factor. In §3.1 we also consider the problem: When does a distal compact flow have an equicontinuous or regularly a.p factor? Note here that without minimality of \(X\) the classical \(F\)-topology and \(\tau\)-topology arguments (cf. [11, 28] and [8, 13, 5]) are invalid. Moreover, we consider weakly rigid \(\mathbb{Z}\)-flows in §3.2 and pointwise periodic \(\mathbb{Z}\)-flows in §3.3.

3.1. Distal 0-dimensional flows

It is easy to prove that if \((G, X)\) is equicontinuous, then \((G, X)\) is distal. No converse is valid in general. However, using Theorem 2.6a we can then conclude the following generalization of an important theorem of R. Ellis:

3.1.1 Theorem. Let \((G, X)\) be a flow such that \(\dim X = 0\) and \(G\) is compactly generated by \(\Gamma\). Then the following conditions are pairwise equivalent:

1. \((G, X)\) is pointwise product \(\Gamma\)-recurrent of type I (i.e. \((G, X \times X)\) is pointwise \(\Gamma\)-recurrent of type I).
2. \((G, X)\) is a distal flow.
3. \((G, X)\) is an equicontinuous flow.

Notes. 1 As to (2)\(\Leftrightarrow\)(3), see [6, Theorem 2] for \(G\) generative, [21, Corollary (3.11)] for \((G, X)\) a minimal flow with \(G\) a direct product of a compactly generated separable group with a compact group, and [3, Corollary 1.9] for \(G\) finitely generated.

\(^1\)Note here that \(G\) is only a para-topological group. So if \(K\) is a compact set in \(G, K^{-1}\) need not be compact. If \(G\) is a topological group, the proof of this theorem might be simplified, and moreover, the locally weakly a.p is independent of the topology of \(G\).
2. Let $T$ be a Hausdorff topological group compactly generated by $\Gamma$ and $\mathbb{K}$ a compact Hausdorff topological group. Let $G = T \times \mathbb{K}$. Then $G = \bigcup_{n=1}^{\infty} (\Gamma^n) \times \mathbb{K} = \bigcup_{n=1}^{\infty} (\Gamma \times \mathbb{K})^n$. Since $\Gamma = \mathbb{K}$ is compact in $G$, $G$ is compactly generated by $\Gamma$. So Theorem 3.1.1 generalizes the result of McMahon-Wu [21, Corollary (3.11)].

3. Equicontinuity implies the pointwise regional distality under our situation. However, since $X$ need not be compact here, it need not imply equicontinuity that $(G, X)$ is pointwise regionally distal (RP$[x] = [x] \forall x \in X$) in general. In view of this, the approach of McMahon-Wu [21] (using the Furstenberg tower of minimal distal flow) does not work for the non-minimal case.

4. To what extent can the condition of zero-dimensionality be relaxed for “(1) $\Rightarrow$ (2)” in Theorem 3.1.1? By a slight modification of the proof of [14, Proposition 6.5], we can show that there exists a homeomorphism $f: X \to X$ on a compact metric space $X$ with $\dim X \geq 1$ such that $(f, X)$ is minimal, weakly mixing, and $f^n \to \text{id}_X$ and $f^{-m} \to \text{id}_X$ uniformly w.r.t. some sequences $n_k \nearrow \infty$ and $m_k \searrow -\infty$. Thus $(f, X)$, as a Z-flow, is pointwise product $\{\pm 1, 0\}$-recurrent of type I (see Proposition 4.6) but it is not distal (for it is weakly mixing).

Proof. “(3) $\Rightarrow$ (2) $\Rightarrow$ (1)” is obvious. The remainder is to prove that “(1) $\Rightarrow$ (3)”. Now assume (1); then $(G, X \times X)$ is pointwise $\Gamma$-recurrent of type I with $\dim X = 0$. By Theorem 2.6a, the orbit-closure relation $R_\Gamma$ of $(G, X \times X)$ is closed and so by Lemma 1.4c, it follows that $(G, X)$ is equicontinuous. The proof is completed.

3.1.2 Theorem. Let $(G, X)$ be a compact flow, where $X$ is not connected and $G$ is compactly generated. If $(G, X)$ is distal, then $(G, X)$ has an equicontinuous non-trivial factor $(G, Y)$.

Proof. Define the “component relation” $R_\mathcal{C}$ of $X$ as follows: $(x, x') \in R_\mathcal{C}$ if and only if $x$ and $x'$ are in the same connected component of $X$. Then it is not difficult to verify that $R_\mathcal{C}$ is an $\mathcal{G}$-invariant closed equivalence relation on $X$ (cf. [21, Def. 2.2 and Proposition 2.3]). Set $Y = X/R_\mathcal{C}$ equipped with the quotient topology and let $\rho: X \to Y$ be the canonical quotient mapping. Then $(G, Y)$ is a distal flow with $Y$ to be compact 0-dimensional, and, $\rho : (G, X) \to (G, Y)$ is an extension. Since $X$ is not connected, so $Y$ is not a singleton. By Theorem 3.1.1, $(G, Y)$ is equicontinuous. The proof is completed.

3.1.3 Theorem (cf. [6, Thm. 3] for $G$ generative; [6, Problem (1)]). Let $(G, X)$ be a distal flow with $G$ compactly generated, such that $\dim Gx = 0$ for all $x \in X$, where $X$ need not be locally compact. Then, under the discrete topology of $G$, $(G, X)$ is pointwise regularly a.p. and, $(G, Gx)$ is a regularly a.p. subflow for every $x \in X$.

Note. A normal discretely syndetic closed subgroup of $G$ is clopen and has the finite index in $G$. However, the subgroup itself need not be discrete.

Proof. Let $x \in X$. Using $G^x$ instead of $X$, we can assume $G^x = X$ with $\dim X = 0$. By Theorem 3.1.1, $(G, X)$ is an equicontinuous minimal compact 0-dimensional flow. Let $E(x)$, the Ellis enveloping semigroup of $(G, X)$, be the closure of $G$ in $X^X$ with the topology of pointwise/uniform convergence. Then $E(x)$ is a compact topological group [7, 4.4]. Since $\dim X = 0$, we can take $U \in \mathcal{O}_c$ to be clopen. Since $E(x) \to X$, defined by $p \mapsto px$ with $e = \text{id}_X \mapsto x$, is continuous, thus $\bigcup \{ p \mid p \in E(x), px \in U \}$ is a clopen neighborhood of $e$ in $E(x)$. By [19, Theorem 7.6],

\footnote{\textit{Theorem} (Hewitt and Ross; cf. [19, Thm. 7.6]). Let $G$ be a compact topological group and let $U$ be a clopen neighborhood of the identity. Then $U$ contains a clopen normal subgroup $N$ of $G$ and $G/N$ is finite.}
follows that there exists a clopen normal subgroup $N$ of $E(X)$ with $N \subseteq U$ and $E(X)/N$ is finite.
Since $G$ is dense in $E(X)$, there is a finite subset $K$ of $G$ such that $KN = E(X)$. Set $H = N \cap G$. Then $KH = G$ and $H$ is a closed normal discretely syndetic subgroup of $G$. By $H \subseteq N \subseteq U$, $Hx \subseteq U$ and $x$ is a regularly $a.p$ point under $(G, X)$ with discrete phase group.

**Corollary.** Let $(G, X)$ be a minimal distal flow with $G$ compactly generated. If $X$ is not connected, then $(G, X)$ is has a regularly $a.p$ factor.

**3.1.4 Theorem.** Let $(G, X)$ be a metric flow, which is pointwise regularly $a.p$ under the discrete topology of $G$. Suppose $G$ contains only countably many clopen normal subgroups of finite index. Then the set of points $R$ at which $G$ is complete.

**Proof.** Let $\{H_n | n = 1, 2, \ldots \}$ be the set of clopen normal subgroups with finite index of $G$ and let $\rho$ be a metric on $X$. For all $m, n \in \mathbb{N}$ set $E(n, m) = \{x \in X | H_n x \subseteq B(x, 1/m)\}$, where $B(x, r) = \{y | \rho(x, y) \leq r\}$ for $r > 0$. Then $E(n, m)$ is a closed subset of $X$ and $\bigcup\{E(n, m) | n = 1, 2, \ldots \} = X$. Hence $E(m) = \bigcup_n \text{int} E(n, m)$ is an everywhere dense open subset of $X$. Let $E = \bigcap_mE(m)$. Then $E$ is a residual subset of $X$. We will prove that $E \subseteq R$.

First, from the definition of $E$, it follows that given any $U \subseteq \mathcal{R}_x$, $x \in E$, there exists a $V \subseteq \mathcal{R}_x$ and a discretely syndetic normal subgroup $A$ of $G$ such that $AV \subseteq U$.

Let $x \in E$. Let $U \subseteq \mathcal{R}_x$ be compact; then there are $V \subseteq \mathcal{R}_x$ and $A$ a discretely syndetic subset of $G$ such that $AV \subseteq U$. Let $K$ be a finite subset of $G$ such that $K^{-1}A = G$. Since $K$ is finite, it follows that $K^{-1}(U \times U)$ is compact in $X \times X$. Then

$$G(V \times V) = K^{-1}A(V \times V) \subseteq K^{-1}(U \times U) \subseteq G(U \times U)$$

shows that $G(V \times V)$ is compact with $G(V \times V) \subseteq G(U \times U)$ and that

$$\bigcap\{G(V \times V) | V \in \mathcal{R}_x \text{ and } V \subseteq U\} = \bigcap\{G(V \times V) | V \in \mathcal{R}_x \text{ and } V \subseteq U\}.$$

Then the proof that $(G, X)$ is equicontinuous at $x$ may be completed as in Proof of [17, Lemma 1].

Indeed, suppose the contrary that $G$ is not equicontinuous at $x$; then there exists an open index $\epsilon \in \mathscr{U}$ such that $G(N \times N) \not\subseteq \epsilon$ for every $N \in \mathcal{R}_x$. We set $\epsilon' = X \times X - \epsilon$ and define $\overline{\mathcal{R}} = \{G(N \times N) \cap \epsilon'| N \in \mathcal{R}_x\}$. Since $\overline{\mathcal{R}} = \{F | F \in \mathcal{F}\}$ has the finite intersection property, hence $0 \not\in \bigcap \overline{\mathcal{R}} = \bigcap \mathcal{R}$. This implies that $G$ is not distal on $X$. However, as $(G, X)$ is pointwise regularly $a.p$ under the discrete topology of $G$, it follows by Lemma 1.5d that $(G, X)$ is distal. The proof is complete.

By a slight modification of the above proof with ‘$K$ compact’ instead of ‘$K$ finite’ and using Lemma 1.9, we can present another formulation of Theorem 3.1.4 as follows:

**3.1.4’ Theorem.** Let $(G, X)$ be a distal pointwise regularly $a.p$ flow with $X$ a metric space. Suppose $G$ contains only countably many normal subgroups. Then the set of points $R$ at which $(G, X)$ is equicontinuous is residual in $X$.

**Corollary.** (cf. [6, Thm. 4] for $G$ generative; [6, Problem (1)]). Let $(G, X)$ be a distal metric flow such that $\dim GX = 0$ for all $x \in X$. Suppose $G$ is compactly generated and contains only countably many normal clopen subgroups of finite index. Then the set of points $R$ at which $(G, X)$ is equicontinuous is a residual subset of $X$.

**Proof.** This follows easily from Theorem 3.1.3 and Theorem 3.1.4. □
3.1.5 Example (An explicit realization of [21, Exa. (3.4.1)]). Let \( \mathbb{Z}_2 = \{0, 1\} \) be the cyclic group of order 2 \((1 + 1 = 0)\). Let \( Y = \prod_{n=1}^{\infty} \mathbb{Z}_2 \). Then \( Y \) is a compact totally disconnected topological group under the product topology. Let \( T = \{ t \in Y \mid t(n) = 0 \text{ for all but a finite set of } n's \} \). Clearly, \( T \) is a dense subgroup of \( Y \). Now we can define an equicontinuous minimal flow \((T, Y)\) by

\[
T \times Y \to Y, \quad (t, y) \mapsto ty = (t(n) + y(n))_{n \in \mathbb{Z}}.
\]

Let \( X = Y_{-1} \cup Y_{-2} \), where \( Y_{-1} \) and \( Y_{-2} \) are two copies of \( Y \). Let \( o \in Y \) with \( o(n) = 0 \) for all \( n \in \mathbb{Z} \).

For \( i \geq 1 \) take \( \xi_i, \eta_i \in Y \) such that

\[
\xi_i(n) = \begin{cases} 
0 & \text{if } n \leq i, \\
1 & \text{if } n > i,
\end{cases}
\]

\[
\eta_i(n) = \begin{cases} 
1 & \text{if } n < -i, \\
0 & \text{if } n \geq -i.
\end{cases}
\]

Then \( \xi_i \to o \) and \( \eta_i \to o \) as \( i \to \infty \). Moreover, the cylinders

\[
[0, j, \ldots, 0, 1, j_{i+1}] = \{ y \in Y \mid y(n) = 0 \text{ for } |n| \leq j \text{ and } y(j + 1) = 1 \}, \quad j = 1, 2, \ldots,
\]

are disjoint clopen subsets of \( Y \) such that \( \xi_i \in [0, j_i, \ldots, 0, 1, j_{i+1}] \) and \( \eta_i \notin \bigcup_{j=1}^{\infty} [0, j_i, \ldots, 0, 1, j_{i+1}] \) for all \( i \geq 1 \). For each \( i \geq 1 \), define \( b_i: X \to X \) by:

for \( y_x \in Y_x \), \( \epsilon = \pm 1 \), put

\[
b_i y_x = \begin{cases} 
y_x & \text{if } y_x \notin [0, j_i, \ldots, 0, 1, j_{i+1}]; \\
y_{-x} & \text{if } y_x \in [0, j_i, \ldots, 0, 1, j_{i+1}];
\end{cases}
\]

and for \( t \in T \), define \( t: X \to X \) by \( t y_x = (ty)_x \). Let \( G \) be the countable discrete group generated by \( T \) and \( \{ b_i \}_{i=1}^{\infty} \). Then \( (G, X) \xrightarrow{\pi, y \mapsto \pi y} (G, Y) \) is a 2-to-1 distal/equicontinuous extension of the minimal equicontinuous \((G, Y)\); moreover, \((o_{n+1}, o_n)\) is a regionally proximal pair under \((G, X)\), hence \((G, X)\) is a minimal non-equicontinuous (non-a.a) flow, where \( X \) is a 0-dimensional compact metric space and \( G \) is not compactly generated. In fact, the regionally proximal cell \( \text{RP} \{ y_x \} \) for all \( y_x \in X \) (noting \( \text{RP} \{ \} = \Delta_X \)). Moreover, although \( X \) is a homogeneous space, \((G, X)\) is not dynamically homogeneous (i.e. \( \text{Aut} (G, X) \neq \{ X \} \) for \( X \) in \( X \); otherwise, \((G, X)\) is equicontinuous by Auslander [1, Theorem 2.13]).

3.1.6 Example (a modification of D. McMahon’s example; cf. [21, Exa. (3.4.2)]). As in Example 3.1.5, let \( Y = \prod_{n=1}^{\infty} \mathbb{Z}_2 \) be the direct product such that \( y_1 y_2 = (y_1(n) + y_2(n))_{n \in \mathbb{Z}} \) for all \( y_1, y_2 \in Y \). Let \( o \in Y \) such that \( o(n) \equiv 0 \) for all \( n \in \mathbb{Z} \), and, set \( 0' = 1 \) and \( 1' = 0 \). Let \( X = Y \times \mathbb{Z}_2 \) and \( \delta = 0 \) or 1. Then \( X \) and \( Y \) both are totally disconnected compact metric spaces such that \( \pi: (y, \delta) \mapsto y \) from \( X \) onto \( Y \) is open 2-to-1 continuous. Given \( i \in \mathbb{Z} \) we define the dual homeomorphisms on \( Y \) and \( X \), respectively, as follows:

\[
\theta_i: Y \to Y, \quad y \mapsto y'_i, \quad \text{where } y'_i(n) = y(n) \text{ if } n \neq i \text{ and } = y(n)' \text{ if } n = i;
\]

and

\[
\theta_i: X \to X, \quad (y, \delta) \mapsto (y', \delta + y(i)).
\]

Clearly, for all \( i, j \in \mathbb{Z} \), it holds that \( \theta_j \circ \theta_i = \theta_i \circ \theta_j \circ \theta_i^j \mid Y = \text{id} \) and \( \theta_i^j \mid X = \text{id} \).

\[\text{If } \theta_i: (y, \delta) \mapsto (y', \delta + y(i) - 1 + y(i)) \text{ for each } i \in \mathbb{Z}, \text{ then we return to McMahon’s case. Here our definition simplifies the proof of } \theta_i \circ \theta_j = \theta_j \circ \theta_i.\]
Let $G = \bigoplus_{n \in \mathbb{Z}} H_i$, where $H_i = \mathbb{Z}_4$. Now we may define actions of $G$ on $Y$ and $X$ as follows: for all
\[ t = (t(i))_{i \in \mathbb{Z}} = (\ldots, 0, 0, t(i_1), \ldots, t(i_n), 0, 0, \ldots) \in G, \]
set
\[ t|Y = \theta_{t(i_1)}^{(i_1)} \circ \cdots \circ \theta_{t(i_n)}^{(i_n)} : Y \to Y \quad \text{and} \quad t|X = \theta_{t(i_1)}^{(i_1)} \circ \cdots \circ \theta_{t(i_n)}^{(i_n)} : X \to X. \]
Then $(G, Y)$ is a minimal equicontinuous flow and $\pi : (G, X) \to (G, Y)$ is a 2-to-1 equicontinuous extension. Next we shall prove that $(G, X)$ is a minimal distal non-equicontinuous flow. Since $\pi$ is distal and $Y$ is minimal distal, $(G, X)$ is distal and pointwise a.p having at most two minimal subsets. By $\theta_j^o(o, 0) = (o, 1)$, it follows that $(G, X)$ is minimal distal. For each integer $j \geq 1$ take a point $y_j \in Y$ such that
\[ y_j(n) = 0 \text{ for } |n| \leq j \quad \text{and} \quad y_j(n) = 1 \text{ for } |n| > j. \]
Then $(y'_j, 1) \to (o, 1), (y_j, 0) \to (o, 0)$ in $X$ and $t_j((y'_j, 1), (y_j, 0)) \to ((o, 0), (o, 0))$ in $X \times X$ as $j \to \infty$, where $t_j = (\ldots, 0, 0, 1, 0, 0, \ldots) \in G$ with 1 is at the $j$-coordinate. Hence $(o, 1)$ is regionally proximal to $(o, 0)$ under $(G, X)$. This shows that $(G, X)$ is not equicontinuous (so not locally a.p and not almost automorphic).

It is well known that distality alone does not imply equicontinuity. Therefore, for equicontinuity, none of the conditions — $\dim X = 0$, $G$ compactly generated and $(G, X)$ distal, of Theorem 3.1.1 is surplus.

3.1.7 Remark. $(G, X)$ in Example 3.1.6 has no regularly a.p points. In addition, $(G, Y)$ in Example 3.1.6 is regularly a.p (see Proof of Theorem 3.1.3). So the regularly a.p cannot be lifted by finite-to-one covering maps if no restriction to the phase group $G$.

3.2. Weakly rigid compact $\mathbb{Z}$-flows

Let $X$ be a compact Hausdorff space, not necessarily metric, and $f : X \to X$ a self homeomorphism of $X$ in this subsection.

3.2.1. Following [14] we say that
\begin{enumerate}
  \item [(i)] $(f, X)$ is weakly rigid if for every $\varepsilon \in \mathcal{U}$ and points $x_1, \ldots, x_n \in X$ there exists $n \in \mathbb{Z}, n \neq 0$, such that $(f^nx, x) \in \varepsilon$ for $i = 1, \ldots, n$.
  \item [(ii)] $(f, X)$ is rigid (w.r.t. a net $n_k \in \mathbb{Z}$ with $n_k \nrightarrow \infty$) if $f^{n_k}x \to x$ for all $x \in X$.
  \item [(iii)] $(f, X)$ is uniformly rigid (w.r.t. a net $n_k \nrightarrow \infty$) if $f^{n_k}x \to x$ uniformly for all $x \in X$.
\end{enumerate}

Clearly, (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) but (i) $\not\Rightarrow$ (ii) $\not\Rightarrow$ (iii) in general (cf. [14]).

3.2.2 Theorem (cf. [14, Prop. 6.7] for (2) $\Leftrightarrow$ (5) and [22, Cor. 8.4] for (2) $\Leftrightarrow$ (7), in the case $(f, X)$ minimal metric). Let $(f, X)$ be such that $\dim X = 0$. Then the following are pairwise equivalent:
\begin{enumerate}
  \item [(1)] $(f, X)$ is pointwise product $\Gamma$-recurrent of type I, where $\Gamma = [0, \pm1]$.
  \item [(2)] $(f, X)$ is weakly rigid.
  \item [(3)] $(f, X)$ is rigid.
  \item [(4)] $(f, X)$ is uniformly rigid.
  \item [(5)] $(f, X)$ is equicontinuous.
\end{enumerate}
(6) \((f, X)\) is positively/negatively pointwise product recurrent (i.e. for every \((x_1, x_2)\) \(\in X \times X\) there is a net \(n_k \rightarrow +\infty\) \(n_k \rightarrow -\infty\) such that \((f^{n_k}x_1, f^{n_k}x_2) \rightarrow (x_1, x_2)\)).

(7) \((f, X)\) is regularly a.p.

**Proof.** (1) \(\Rightarrow\) (6): By the fact that the only \(\Gamma\)-cones are \(\mathbb{N}\) and \(-\mathbb{N}\) in \((\mathbb{Z}, +)\).

(6) \(\Rightarrow\) (5): Assume (6). Let \(V\) be a clopen non-empty subset of \(X\). Let \(\pi_V : X \rightarrow [0, 1]^{\mathbb{Z}}\) be defined by \(x \mapsto (1/\pi f^x)_t\in\mathbb{Z}\). Since \(V\) is clopen, \(\pi_V\) is continuous. Let \(Y_v = \pi_1 X\) and \(\sigma_V : Y_v \rightarrow Y_v\) the canonical shift map. Then \(\pi_V \circ f = \sigma_V \circ \pi_V\). So \((\sigma_V, Y_v)\) is positively/negatively pointwise product recurrent. This implies that \((\sigma_V, Y_v)\) is distal. For if otherwise, then by [2, Proposition 5.10] there were a pair \((y_1, y_2)\) \(\in Y_v \times Y_v\) such that \(y_1\) is positively/negatively proximal to \(y_2\) and \(y_1 \neq y_2\); further, we can find a positively/negatively asymptotic pair in \(Y\) (or by [14, Lemma 6.6]). Thus by Theorem 3.1.1, \((\sigma_V, Y_v)\) is equicontinuous. Let \(Y = \{Y_v\}_{v\in\mathcal{Y}}\) be the collection of clopen subsets of \(X\). Then \((\sigma, \prod_{v\in\mathcal{Y}} Y_v)\) is equicontinuous. Define \(\pi : X \rightarrow \prod_{v\in\mathcal{Y}} Y_v\) by \(x \mapsto (\pi_V x)_{v\in\mathcal{Y}}\). Since \(X\) is 0-dimensional, \(\pi\) is obviously continuous 1-1 with \(\pi \circ f = \sigma \circ \pi\). As \((\sigma, \pi X)\) is equicontinuous, it follows that \((f, X)\) is equicontinuous.

(5) \(\Rightarrow\) (1): By Theorem 3.1.1.

(5) \(\Rightarrow\) (4) \(\Rightarrow\) (3) \(\Rightarrow\) (2): By definitions.

(2) \(\Rightarrow\) (5): By a slight modification of the above proof of (6) \(\Rightarrow\) (5).

(5) \(\Leftarrow\) (7): Assume (5). By Theorem 3.1.3, \((f, X)\) is pointwise regularly a.p. Further by equicontinuity, \((f, X)\) is regularly a.p. Finally, (7) \(\Rightarrow\) (5) is obvious.

The proof is thus completed.

It should be noticed that in the above proof of (6) \(\Rightarrow\) (5), \(\{(\sigma, Y_v) | V \in \mathcal{Y}\}\) need not be a directed system, since we did not define the connecting homeomorphism \(\phi^y_U : Y_U \rightarrow Y_V\) for \(U, V \in \mathcal{Y}\) with \(U \subset V\) or \(V \subset U\).

3.3. Pointwise periodic homeomorphisms

Let \(f : X \rightarrow X\) be a homeomorphism. If each point of \(X\) is periodic under \(f\), then \((f, X)\) is said to be pointwise periodic. If there is some positive integer \(k\) such that \(f^k = \text{id}_X\), then \((f, X)\) is said to be periodic. The least integer \(p\) greater than 0 such that \(f^p(x) = x\) for all \(x\) \(\in X\) is called the period of \(f\).

The following example shows that the pointwise periodic does not imply that \((f, X)\) is periodic in general.

3.3.1 Example. Let \(r_n, n = 1, 2, \ldots\) be a sequence of rational numbers with \(0 < r_n < 1\) such that \(r_n \downarrow 0\) as \(n \rightarrow \infty\). Let \(S = \{z|z \in \mathbb{C}, |z| = 1\}\) and \(S_n = \{z|z \in \mathbb{C}, |z| = 1 - r_n\}\) for \(n = 1, 2, \ldots\) are concentric circles in the complex plane. Define \(f_n : S_n \rightarrow S_n\) by \(z \mapsto e^{2\pi i r_n z}\), for \(n = 1, 2, \ldots\). Let \(X = S \cup (\bigcup_{n=1}^{\infty} S_n)\) and define \(f : X \rightarrow X\) by \(f|S_n = f_n\) and \(f|S = \text{id}_S\). Clearly, the \(Z\)-flow induced by the pointwise periodic cascade \((f, X)\) is pointwise (regularly almost) periodic but it is not equicontinuous at every point of \(S\). In fact, in view of Lemma 2.4, \((f, X)\) is not a locally weakly a.p. \(Z\)-flow.

However, if the phase space \(X\) is a locally euclidean space, then a pointwise periodic uniform transformation is periodic:

3.3.2 Theorem (Montgomery [23]). If \(f\) is a pointwise periodic uniform homeomorphism of a boundaryless \(n\)-manifold \(M\) into itself, then \((f, M)\) is periodic.
A sharp case beyond Montgomery’s Theorem is a compact metric $\mathbb{Z}$-flow with zero-dimensional phase space as follows:

**3.3.3 Theorem.** Let $X$ be a compact space with dim $X = 0$. If $f$ is a pointwise periodic homeomorphism of $X$ into itself then $(f, X)$ is regularly a.p but it is not necessarily periodic.

**Proof.** 1). As a $\mathbb{Z}$-flow $(f, X)$ is distal so $(f, X)$ is (uniformly) equicontinuous by Theorem 3.1.1. Further by compactness of $X$ and Lemma 1.5d, it follows easily that $(f, X)$ is a regularly a.p cascade.

2). For the second part, let’s consider a counterexample (due to the referee) as follows: Let $X = \prod_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z}$, and, let $f : X \to X$ be defined in the ways: for every $x \in X$, if $x = (0, 0, 0, \ldots)$ then $fx = x$, otherwise $fx$ is given by adding 1 to the digit after the first nonzero entry of $x$. Then $(f, X)$ is pointwise periodic but has infinite order. The proof is completed.

Therefore, if $(f, X)$ is a pointwise periodic cascade with $X$ a compact boundaryless $n$-manifold or $X$ a 0-dimensional compact space, then $f$ has the topological entropy zero without using the variational principle of entropy.

Notice that as shown by Example 3.3.1 and Theorem 3.3.3, if $X$ is not a manifold, then a pointwise periodic homeomorphism of $X$ need not be periodic.

4. The case when $G$ is a finitely generated group

Let $G$ be discrete and finitely generated by $e \in \Gamma = \Gamma^{-1}$, and let $(G, X)$ be a flow. We shall show that the $\Gamma$-recurrence of type I (Def. 2.2a) is consistent with [3] in this case. See Lemma 4.4 below.

4.1. Let $B_r = \{g \in G | g \neq e, |g| \leq r\}$, for all $r \geq 1$. For $g \in G$ with $g \neq e$, write $K(g) = B_{|g|^{-1}} \cdot g$.

4.2. A subset $C$ of $G$ is an AGW-cone, namely, a cone in the sense of Auslander, Glasner and Weiss [3], if there is a sequence $g_n \in G$ with $|g_n| \to \infty$ such that for each $r \geq 1$ there exists $n_r$ such that $B_r \cap K(g_{n_r})$ is independent of $n$ for all $n \geq n_r$, and, $C = \lim_{n \to \infty} K(g_{n})$. Since $G$ is discrete here, so $c \in C$ iff $c \in K(g_{n})$ as $n$ sufficiently large. Moreover, $e \notin C$ and by the proof of Lemma 2.1c, it is easy to see that $C$ is thick in $G$.

4.3 (cf. [3, Def. 1.6]). We say that $x \in X$ is AGW-recurrent, if $C_x \cap U \neq \emptyset$ for every $U \in \mathcal{U}$, and every AGW-cone $C$ in $G$.

4.4 Lemma. A point $x$ is AGW-recurrent under $(G, X)$ if and only if $x$ is $\Gamma$-recurrent of type I in the sense of Def. 2.2a.

**Proof.** 1). Let $C$ be a $\Gamma$-cone in $G$ with $K(g_i) = \Gamma^{|g_i|^{-1}} \cdot g_i \to C$, where $\{g_i | i \in \Lambda\}$ is a net in $G$ such that $|g_i| \not\to \infty$. For all integer $k \geq 1$, there is an $i_k \in \Lambda$ such that $|g_i| \geq k$ for all $i \geq i_k$ and $|g_i| < |g_{i_{k+1}}|$. Clearly $|g_{i_k}| \to \infty$ as $k \to \infty$. Moreover, the sequence $\{g_{i_k}\}_{k=1}^{\infty}$ is a subnet of $\{g_i\}$. Indeed, for $i' \in \Lambda$, there is an integer $k \geq 1$ with $|g_{i'}| < k \leq |g_{i_k}|$ for all $k' > k$ so $i' \leq i_k$ for all $k' > k$. Therefore, using $\{i_k\}$ in place of $\Lambda$ if necessary, we can assume $\{g_{i_k}\}_{k=1}^{\infty}$ is a sequence in $G$ with $|g_{i_k}| \not\to \infty$ as $k \to \infty$.

Each $B_r$ is finite, so we may choose a subsequence $\{i_k\}$ from $\Lambda$ so that each $K(g_{i_k}) \cap B_{r_k}$ is eventually constant. Then by a diagonal process we can choose a subsequence (and relabel) $\{i_{k_{n+1}}\}_{n=1}^{\infty}$ from $\Lambda$ such that $K(g_{i_k}) \to C$ as $k \to \infty$. Then $C$ is an AGW-cone in $G$ such that $C \subseteq C$ by Lemma 1.1.
2). Let $C$ be a cone in $G$ with $C = \lim_{i \to \infty} K(g_i)$ in the sense of Def. 4.2, where $\{g_i\}_{i=1}^\infty$ is a sequence in $G$ such that $|g_i| \to \infty$ and for each $r \geq 1$ there exists an $n_r \geq 1$ such that $B_r \cap K(g_i)$ is independent of $i$ for all $i \geq n_r$. By induction, we may require $n_r < n_{r+1}$ and $|g_{n_r}| < |g_{n_{r+1}}|$ for $r \geq 1$. Then

$$C = \bigcup_{r=1}^\infty (B_r \cap K(g_{n_r})).$$

Let $K(g_{n_r}) \to C$ in the sense of Def. 2.1b. Let $c \in C$. Then there exists a subnet $\{n_j\}$ from $\{n_r\}$ and $k_{n_j} \in K(g_{n_j})$ such that $k_{n_j} \to c$. Since $G$ is discrete and $[c]$ is an open neighborhood of $c$ in $G$, we can assume $k_{n_j} = c$ for all $n_j \in [c]$. Since we have for all $n_j$ that $n_j > n_r$, $c$ is sufficiently large, so $c \in C$ and $C \subseteq C$.

The above discussion of 1) and 2) implies that $x$ is AGW-recurrent iff $x$ is also $\Gamma$-recurrent of type I. The proof is complete.

4.5 Proposition (cf. [3, Thm. 1.8]). Let $(G, X)$ be a flow, where $G$ is finitely generated and $X$ is a 0-dimensional space. Then the following conditions are equivalent:

1. $(G, X)$ is pointwise AGW-recurrent.
2. $(G, X)$ is pointwise a.p.
3. The $G$-orbit closure relation $R(G)(X)$ is closed.

Proof. In view of Lemma 4.4, this is a special case of Theorem 2.6a.

4.6 Proposition (cf. [15, Thm. 1], [10, Thm. 1(a)] and [12, Thm. 1.4] for (2) by different approaches). Suppose $f : X \to X$ is a homeomorphism of $X$, which is thought of as a $\mathbb{Z}$-flow. Let $x \in X$.

1. $x$ is recurrent of type I (cf. Def. 2.2a) iff $x$ is “stable in the sense of Poisson”, i.e., there exists a net $\{i_n\}$ in $\mathbb{Z}$ with $i_n \to +\infty$ and a net $\{j_n\}$ in $\mathbb{Z}$ with $j_n \to -\infty$, such that $f^{i_n}x \to x$ and $f^{j_n}x \to x$ simultaneously.

2. If $k \in \mathbb{N}$ and $i_n \in \mathbb{N}$ is a net with $i_n \to \infty$ such that $f^{i_n}x \to x$, then there is a net $k_n \in \mathbb{N}$ with $\tau \to k_n \to \infty$ such that $f^{k_n}x \to x$.

Proof. (1). Note that a $\Gamma$-cone $C$ in $\mathbb{Z}$ with $\Gamma = \{-1, 0, 1\}$ is either $C = \mathbb{N}$ or $C = -\mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. Then the statement follows easily from this fact.

2. Let $i_n = k_n - \tau_n$, where $k_n \in k\mathbb{N}$ and $\tau_n \in [0, \kappa) \cap \mathbb{Z}$. We may assume $\tau_n \to \tau \in \mathbb{Z}$. Then $f^{k_n-\tau_n}x \to x$ so that $f^{k_n}x \to f^{\tau}x$. Further $f^{k_n+\tau}x \to f^{2\tau}x$. We shall show that $f^{2\tau}x$ is a limit of the set $f^{i_n}x$ for $i \in k\mathbb{N}$. For this, let $\varepsilon, \alpha \in \mathbb{W}$ with $\alpha^2 \leq \varepsilon$. Let $n_0$ be an index then there is an index $n > n_0$ such that $(f^{\tau}x, f^{\tau+\tau}x) \in \alpha$. There is an index $\delta \in \mathbb{W}$ such that if $(q, f^{\tau}x) \in \delta$ then $(f^{\delta}q, f^{\delta+\tau}x) \in \alpha$. Then by $f^{\delta}x \to f^{\tau}x$ and $(f^{\delta+\tau}x, f^{\delta+\tau}x) \in \alpha$, it follows that as $n$ sufficiently big, $(f^{\delta}q, f^{\tau+n}x, f^{2\tau}x) \in \varepsilon$. Thus $f^{2\tau}x$ is a limit of $f^{i_n}x$.

The same reasoning would show that $f^{i_n}x, f^{i_{n+1}}x, \ldots$ are also limits of $f^{i_n}x$. Then $f^{i_n}x$ and so $x$ is a limit of $f^{i_n}x$. The proof is complete.

Condition (3) in the following corollary is exactly the definition of the recurrence in [18] for generative groups.

Corollary. Let $f : X \to X$ be a homeomorphism of $X$ and $x \in X$. Then: (1) $x$ is Poisson stable iff (2) $f^{i_n}x \cap U \neq \emptyset$ for all $U \in \mathcal{R}_x$ and all subsemigroup $S$ of $\mathbb{Z}$ iff (3) $f^{i_n}x \cap U \neq \emptyset$ for all $U \in \mathcal{R}_x$ and all thick subsemigroup $S$ of $\mathbb{Z}$.
Proof. (1)⇒(2) follows obviously from Proposition 4.6. (2)⇒(3) is trivial. Finally (3)⇒(1) follows from that \( \mathbb{N} \) and \(-\mathbb{N}\) both are thick semigroups in \( \mathbb{Z} \).

There exists an example [3] of a \( \mathbb{Z} \)-flow where all points are positively recurrent, but there are points which are not negatively recurrent. Thus the recurrence of type I is stronger than the positive recurrence.

4.7 Proposition (cf. [18, Thm. 7.10] for \( G \) abelian finitely generated). Let \( (G, X) \) be a flow with \( G \) finitely generated and with \( X \) 0-dimensional compact. If \( (G, X) \) is pointwise regularly a.p, then \( (G, X) \) is regularly a.p.

Proof. By Lemma 1.5d, \( (G, X) \) is distal. Indeed, let \( x_1, x_2 \in X, U \in \mathfrak{N}_{x_1}, \) and \( V \in \mathfrak{N}_{x_2}. \) There are normal syndetic subgroups \( H_1 \) and \( H_2 \) of \( G \) with \( H_1 x_1 \subseteq U \) and \( H_2 x_2 \subseteq V. \) By Lemma 1.5d, there is a normal syndetic subgroup \( A \) of \( G \) with \( A \subseteq H_1 \cap H_2. \) Thus \( A(x_1, x_2) \subseteq U \times V. \) Then \( (G, X \times X) \) is pointwise a.p and this shows that \( (G, X) \) is distal.

Further by Theorem 3.1.1, it follows that \( (G, X) \) is equicontinuous. Finally by [18, Remark 5.02, Theorem 5.17], we see that \( (G, X) \) is regularly a.p. The proof is complete.

4.8 Proposition (cf. [18, Thm. 7.11] for \( G \) abelian finitely generated). Let \( (G, X) \) be a flow with \( G \) finitely generated and with \( X \) a compact metric space. Then \( (G, X) \) is regularly a.p if and only if \( (G, X) \) is a pointwise regularly a.p and weakly a.p flow.

Proof. The proof of [18, Thm. 7.11] is still valid for this case using Theorem 2.6a in place of [18, Theorem 7.08]. In fact, the necessity is trivial. To prove the sufficiency, let \( S \) be any normal syndetic subgroup of \( G. \) By [18, Theorems 5.17 and 4.24], it is enough to show that \( \mathcal{O}_S : x \in X \mapsto \overline{Sx} \subseteq 2^X \) is continuous. Note that by Inheritance Theorem (cf. Lemma 1.5c), \( (S, X) \) is pointwise a.p so \( \overline{Sx} \) is a partition of \( X. \) Let there be a sequence \( x_n \to x_0 \) in \( X. \) Then by Lemma 2.4, \( \overline{Gx_0} \to \overline{Gx_0} \) in \( 2^X \) and further \( Y = \bigcup_{n=0}^{\infty} \overline{Gx_0} \) is closed in \( X. \) By [18, Theorem 5.08], \( \dim \overline{Gx} = 0 \) for all \( x \in X. \) Since a separable metric space that is countable union of closed zero-dimensional subsets is itself zero-dimensional (cf. [9, Theorem 1.3.1]), \( Y \) is \( G \)-invariant such that \( \dim Y = 0. \) Then \( (S, Y) \) is locally weakly a.p by Theorem 2.6a and Corollary 1.8b. Thus \( \overline{Sx_n} \to \overline{Sx_0}. \) The proof is completed.

4.9 Proposition. Let \( (G, X) \) be a flow with \( G \) finitely generated. Let \( U \) be a compact open subset of \( X, \) which consists of a.p points. Then there is a finite set \( K \) in \( G \) such that \( GU = KU. \)

Proof. Let \( G \) be finitely generated by \( \Gamma \) with \( \Gamma = \Gamma^{-1}. \) For \( \ell \in \mathbb{N} \) set \( W_\ell = \Gamma^\ell U. \) If there exists \( \ell \in \mathbb{N} \) such that \( \Gamma W_\ell \subseteq W_\ell, \) then by induction \( GW_\ell \subseteq W_\ell. \) Hence in particular, \( GU \subseteq \Gamma U, \) and this completes the proof by letting \( K = \Gamma. \) Now assume that for all \( \ell \in \mathbb{N} \) we have \( \Gamma W_\ell \not\subseteq W_\ell; \) i.e., \( \Gamma^{\ell+1} U \not\subseteq \Gamma^\ell U. \) Then for every \( \ell \in \mathbb{N} \) there are \( t_\ell \in \Gamma^{\ell+1} \) with \( |t_\ell| = \ell + 1 \) and \( t_\ell \in U \) such that \( t_\ell y \not\in \Gamma^\ell U. \) We may assume (a subnet of) \( y_\ell \to y \in U. \) Let \( s \in N_G(y, U); \) then \( sy_\ell \to sy \in U, \) so as \( \ell \) sufficiently large, \( sy \in U, \) hence \( t_\ell y \in t_\ell s^{-1} U, \) and therefore \( t_\ell s^{-1} \not\in \Gamma^\ell \) and \( s^{-1} \not\in t_\ell^{-1} \Gamma^\ell, \) or equivalently, \( s \not\in \Gamma t_\ell \). Let \( K(t_\ell) \to C, \) a \( \Gamma \)'-cone in \( G \) as in 2.1b. Then \( s \not\in C \) and \( C \cap N_G(y, U) = \emptyset, \) contrary to that \( C \) is thick and \( N_G(y, U) \) is syndetic in \( G. \) The proof is complete.

4.10 Remarks. (1) The \( \Gamma \)-recurrence of type I (Def. 2.2a) is conceptually dependent of the generating set \( \Gamma \) of \( G. \) So it should be interested to generalize Proposition 4.6 from \( G = \mathbb{Z} \) to a more general non-abelian case.

(2) Can we remove the “metric” condition in Proposition 4.8?

(3) Can the assumption that \( G \) is finitely generated be replaced by that \( G \) is compactly generated in any of Propositions 4.8 and 4.9?
5. The case when $G$ is equicontinuously generated

Let $C_d(X, X)$ be the space of continuous maps from $X$ to itself with the topology of uniform convergence on compacta. Clearly, $(\text{Homeo}(X), X)$ defined by $\text{Homeo}(X) \times X \xrightarrow{(f, x) \mapsto f x} X$ is a flow, where $(f, x) \mapsto f x$ is jointly continuous but $\text{Homeo}(X)x$ need not be compact for $x \in X$. In the sequel let $S \subset \text{Homeo}(X)$ such that:

1) $e = \text{id}_X \in S = S^{-1}$ and
2) $S$ acting equicontinuously on $X$ (i.e., given $e \in \Upsilon$ and $x_0 \in X$, there is a $U \in \mathcal{U}_{x_0}$ such that $(sx, sx_0) \in e$ for all $x \in U$ and all $s \in S$; cf. Def. 1.2d).

5.1. Set $(S) = \bigcup_{n=0}^{\infty} S^n$. Clearly $(S)$ is a subgroup of $\text{Homeo}(X)$, which is said to be equicontinuously generated by $S$.

5.2. Write $\Gamma = \text{cls}_a S$, where $\text{cls}_a$ denotes the closure relative to $C_d(X, X)$. Then:

**Lemma.** If $Sx$ is relatively compact in $X$ for all $x \in X$, then $\Gamma$ is a compact subset of $C_d(X, X)$ and $\Gamma \subset \text{Homeo}(X)$ such that $e \in \Gamma = \Gamma^{-1}$.

**Proof.** Since $S$ is equicontinuous, so is $\Gamma$. Then $\Gamma$ is compact in $C_d(X, X)$ by Ascoli’s Theorem (cf. [20, Theorem 7.17]). Moreover, $\Gamma x$ is compact in $X$. We need prove that $\Gamma \subset \text{Homeo}(X)$. For this, let $S \ni f_n \to f \in \Gamma$. Let $x \neq y$ and $e \in \Upsilon$ with $(x, y) \notin e$. If $f x = f y = z$, then for $\delta \in \Upsilon$ we have $x, y \in f_n^{-1}\delta[z]$ as $n$ sufficiently large, contrary to equicontinuity of $S$. Thus $f$ is injective.

Let $x \in X$. Since $f_0x = X$, there is a point $x_0 \in \Gamma x$ with $f_n x_n = x$ or $x_n = f_n^{-1}x$. We can assume (a subnet of) $x_n \to x'$ for $\Gamma x$ is compact. Then $f x' = x$ so $f X = X$. Thus $f$ is a continuous bijection. Now let (a subnet of) $f_n S \to g \in \Gamma$. Since $e = f_n f_n^{-1} \to fg$, $fg = e$, $g = f^{-1} \in \Gamma$, and $f \in \text{Homeo}(X)$. The proof is complete. $\Box$

5.3. Let $Sx$ be relatively compact in $X$ for all $x \in X$. Set $\langle \Gamma \rangle = \bigcup_{r=1}^{\infty} \Gamma^r$. Then $\langle \Gamma \rangle$ is a compactly generated subgroup of $\text{Homeo}(X)$ with a generating set $\Gamma$. Moreover, $\langle \Gamma \rangle x$ is a flow; however, $\langle \Gamma \rangle x$ need not be compact for $x \in X$. It is evident that $\langle \Gamma \rangle$ is dense in $\Gamma$ for all $r \geq 1$ and $\langle S \rangle$ is dense in $\langle \Gamma \rangle$ under the topology of uniform convergence.

Following Def. 2.2b, a point $x \in X$ is $S$-recurrent of type II under $(\langle S \rangle, X)$ if for every $U \in \mathcal{U}_x$ and every net $\{g_i\}$ in $(\langle S \rangle)$ with $|g_i| \to \infty$, there is an integer $n$, a subnet $\{g_i\}$ from $\{g_i\}$ and $c_j \in S^{[|g_i|]+1}$ such that $c_j x \in U$ and $|c_j| \leq n$.

We do not know if a point $S$-recurrent of type II is $\Gamma$-recurrent of type II, yet it is $\Gamma$-recurrent of type I by Lemma 5.4 below.

5.4 Lemma. Let $Sx$ be relatively compact in $X$ for all $x \in X$. If $x_0$ is $S$-recurrent of type II under $(\langle S \rangle, X)$, then $x_0$ is $\Gamma$-recurrent of type I under $(\langle \Gamma \rangle, X)$.

**Proof.** Let $\{g_i | i \in \Lambda\}$ be a net in $(\langle \Gamma \rangle)$ with $|g_i| \to \infty$ and $K(g_i) \to C$. Let $U, V \in \mathcal{U}_{x_0}$ with $\overline{V} \subseteq U$. We need prove that $C x_0 \cap U \neq \emptyset$. Since $(\langle S \rangle)$ is dense in $(\Gamma)$, we can assume $g_i | i \in \Lambda$ for all $i \in \Lambda$. Further by definition, there is an integer $n \geq 1$, a subnet $\{g_i\}$ from $\{g_i\}$ and $c_j \in S^{[|g_i|]+1}$ such that $c_j x \in U$ and $|c_j| \leq n$. By $c_j \in \Gamma^n$ and $\Gamma^n$ is compact, it follows that (a subnet of) $c_j \to c \in C$ and $c x_0 \in U$. The proof is complete. $\Box$

5.5 Proposition (cf. [25, Thm. 2.2] and [26, Thm. 1.2] for $X$ s.t. dim $X = 0$). Let $Sx$ be relatively compact in $X$ for all $x \in X$. Let $G$ be an equicontinuously generated group by $S$. Let $V$ be an open subset of $X$. Suppose $V_G^\infty$ is compact. Then $V_G^\infty$ is open if and only if $\overline{Gx} \cap V_G^\infty = \emptyset$ for every $x \in X \setminus V_G^\infty$. 21
Let \( G = \langle S \rangle \), where \( S \) is the equicontinuously generating set of \( G \). Then \( \overline{Gx} = \langle \Gamma \rangle x \) for all \( x \in X \). Since \( V_G^0 \) is closed, so \( V_G^0 = \bigcap_{g \in \langle \Gamma \rangle} gV = V_{\langle \Gamma \rangle}^0 \). Then Proposition 5.5 follows from Theorem 2.6b. The proof is complete.

5.6 Proposition (cf. [26, Thm. 1.3] using Prop. 5.5). Let \( \dim X = 0 \) and \( G = \langle S \rangle \) such that \( Gx \) is compact for all \( x \in X \). Then the following are pairwise equivalent:

i) \((G, X)\) is pointwise \( S \)-recurrent of type II.
ii) \((G, X)\) is pointwise a.p.
iii) Given any compact open subset \( V \) of \( X \), \( V^*_{G} \) is open.
iv) The orbit closure relation \( R_o(X) \) of \((G, X)\) is closed.
v) There exists an extension \( \rho : (G, X) \to (G, Y) \) such that:
   1) \( Y \) is locally compact 0-dimensional,
   2) \( gy = y \) for all \( y \in Y \) and \( g \in G \), and
   3) \( \rho^{-1} \) is a minimal subset of \((G, X)\) for each \( y \in Y \).

Proof. By Lemma 5.4, \((G, X)\) is pointwise \( \Gamma \)-recurrent of type I. Since \( \overline{Gx} = \langle \Gamma \rangle x \) for all \( x \in X \), \( x \) is a.p under \((G, X)\) iff \( x \) is a.p under \((\langle \Gamma \rangle, X)\). Then i) \( \iff \) ii) \( \iff \) iii) \( \iff \) iv) \( \iff \) iv') by Theorem 2.6a, where

iv') \( \overline{G} : X \to 2^X \) is upper semi-continuous.

v) \( \iff \): By 3) of v).

v') \( \iff \): Assume iv'). Define \( Y = X/G = X/R_o(X) \) and let \( \rho : X \to Y \) be the canonical quotient mapping. Then \( \rho \) is closed (cf. [20, Theorem 3.12]), and moreover, by ii) and Lemma 1.4b it is easy to see that \( \rho \) is open so \( Y \) is Hausdorff. Since \( \rho \) is clopen and \( X \) is locally compact 0-dimensional, \( Y \) is locally compact 0-dimensional. Thus \((G, Y)\) is a flow having the properties 1), 2) and 3).

The proof of Proposition 5.6 is thus completed.

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