SYzygies AND Tensor Product of Modules

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Abstract. We give an application of the New Intersection Theorem and prove the following: let $R$ be a local complete intersection ring of codimension $c$ and let $M$ and $N$ be nonzero finitely generated $R$-modules. Assume $n$ is a nonnegative integer and that the tensor product $M \otimes_R N$ is an $(n+c)$th syzygy of some finitely generated $R$-module. If $\text{Tor}^R_{>0}(M, N) = 0$, then both $M$ and $N$ are $n$th syzygies of some finitely generated $R$-modules.

1. Introduction

Auslander’s rigidity theorem [2, 2.2] states that finitely generated modules over unramified (or equi-characteristic) regular local rings are Tor-rigid, that is, if $R$ is such a ring, and $M$ and $N$ are such modules, then the vanishing of $\text{Tor}^R_n(M, N)$ for some nonnegative integer $n$ forces the vanishing of all subsequent Tor modules. This result yielded several consequences and received considerable attention; Lichtenbaum [28] extended it to all regular rings by proving the ramified case.

Auslander [2] made use of his rigidity result, analyzing torsion submodules of the tensor products of finitely generated modules over unramified regular local rings. Recall that, if $R$ is a commutative Noetherian ring, then the torsion submodule $\text{T}_R M$ of an $R$-module $M$ is defined as the set $\{ x \in M : r x = 0 \text{ for some nonzero-divisor } r \in R \}$. $M$ is called torsion-free if $\text{T}_R M = 0$, and torsion if $\text{T}_R M = M$; see [25]. An important consequence of Auslander’s rigidity theorem we are concerned with in this paper is:

Theorem 1.1. (Auslander [2, 3.1]) Let $R$ be an unramified (or equi-characteristic) regular local ring and let $M$ and $N$ be finitely generated $R$-modules. Assume that the tensor product $M \otimes_R N$ is torsion-free. Then (i) $\text{Tor}^R_{>0}(M, N) = 0$, and (ii) $M$ and $N$ are both torsion-free.

Let us remark that regular local rings are complete intersections of codimension zero. Furthermore, in this setting, torsion-freeness is equivalent to Serre’s condition $(S_1)$; see [24]. Therefore it is quite natural to ask whether a similar conclusion of Theorem 1.1 holds, for tensor products of modules satisfying higher $(S_n)$ conditions, over complete intersection rings of higher codimension. Huneke and R. Wiegand [21] pioneered this question and extended many of Auslander’s results to hypersurfaces, that is, complete intersections of codimension one. They proved, with an unavoidable hypothesis, that if $R$ is a hypersurface ring and the tensor product $M \otimes_R N$ is reflexive, then $\text{Tor}^R_{>0}(M, N) = 0$. Their result is known as the Second Rigidity Theorem:

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Theorem 1.2. (Second Rigidity Theorem, [22] Theorem 1]) Let $R$ be a local hypersurface and let $M$ and $N$ be finitely generated $R$-modules. Assume $M$ or $N$ has constant rank. Assume further that the tensor product $M \otimes_R N$ is reflexive. Then $\text{Tor}^R_{>0}(M, N) = 0$.

Theorem 1.2 is a substantial generalization of Auslander’s result, stated as Theorem 1.1 to hypersurface rings; over such rings, Serre’s condition $(S_2)$ is equivalent to reflexivity, or being a second syzygy module; see (2.3). Recall that $M$ is said to have constant rank if there exists an integer $r$ such that $M_p \cong R_p^{(r)}$ for all associated primes $p$ of $R$. For example, finitely generated modules of finite projective dimension have constant rank [7, 1.4.6]. The constant rank hypothesis in Theorem 1.2 is necessary; see [21, 1.2(1)].

A nice consequence of Theorem 1.2 is that one of the modules considered must be reflexive. This fact is not explicitly stated in [21], so we discuss this next; see the proof of [21, 2.7] for more details. Note that $\text{depth}(0) = \infty$; this is indeed the correct convention which is needed for several fundamental results in commutative algebra such as the depth lemma [7, 1.2.9]; see also [22].

Remark 1.3. Let $R$, $M$, and $N$ be as in Theorem 1.2 with $M \neq 0 \neq N$. Assume that $M$ is the module of constant rank, say of rank $r$. Then $r \neq 0$; otherwise $\text{Tor}_R M = M$ and this would imply that $\text{Tor}_R (M \otimes_R N) = M \otimes_R N$, which is not possible as $M \otimes_R N$ is torsion-free. It follows, as each prime ideal contains a minimal prime, that $\text{Supp}(M) = \text{Spec}(R)$. Furthermore the vanishing of $\text{Tor}^R_{>0}(M, N)$ yields an equality of depths, known as the depth formula, $\text{depth}(M) + \text{depth}(N) = \text{depth}(R) + \text{depth}(M \otimes_R N)$; see (2.8). Now, if $p \in \text{Supp}(N)$, then one can localize the depth formula at $p$ and obtain an equality all of whose terms are finite. As $\text{depth}(R_p) - \text{depth}_{R_p}(M_p) \geq 0$, we have:

$$\text{depth}_{R_p}(N_p) \geq \text{depth}_{R_p}(N_p \otimes_{R_p} M_p) \geq \min\{2, \text{height}(p)\}.$$  

Therefore $N$ satisfies $(S_2)$, that is, $N$ is reflexive; see (2.4) and (2.5).

Our main aim in this paper is to generalize the observation we discussed in 1.1 to complete intersections of arbitrary codimension for modules that may not have constant rank. We obtain such a generalization in Theorem 3.1; a special case of our argument can be stated as follows:

Theorem 3.1. Let $R$ be a local complete intersection ring of codimension $c$ and let $M$ and $N$ be nonzero finitely generated $R$-modules. Assume that $M \otimes_R N$ satisfies $(S_{n+c})$ for some nonnegative integer $n$, equivalently $M \otimes_R N$ is an $(n+c)$th syzygy of some finitely generated $R$-module. If $\text{Tor}^R_{>0}(M, N) = 0$, then both $M$ and $N$ satisfy $(S_n)$.

The main ingredient that makes the argument in Remark 1.3 work is the vanishing of $\text{Tor}^R_{>0}(M, N)$, which follows from the fact that $M$ or $N$ has constant rank over a hypersurface. We should note that we do not know whether the reflexivity of the tensor product of two finitely generated modules, either of which has constant rank, implies the vanishing of $\text{Tor}^R_{>0}(M, N)$ over complete intersections that are not hypersurfaces. Therefore, in order to generalize the observation of Remark 1.3 to complete intersection rings, it is reasonable to assume the vanishing of such Tor modules, as we do in Theorem 3.1. Since modules over domains have constant rank, it seems worth posing the following question; see also [3, 3.14].
Question 1.5. Let \( R \) be a complete intersection domain and let \( M \) and \( N \) be nonzero finitely generated \( R \)-modules. If \( M \otimes_R N \) is reflexive, then must \( \text{Tor}^R_{\geq 0}(M, N) = 0 \)? What if \( M \), \( N \) and \( M \otimes_R N \) are all maximal Cohen-Macaulay?

Theorem 3.1 is stated and proved for homologically bounded complexes. We use this generality for two reasons: one of the main ingredients of our proof, a generalization of the New Intersection Theorem, was first developed for complexes \([36, 3.3]\); see also \((2.12.1)\). Secondly, a reader interested merely in modules may follow our arguments word for word by simply replacing each inf and sup by zero, so there is no penalty for this extra generality.

We are also interested in analyzing the torsion of both of the modules \( M \) and \( N \) considered in the Second Rigidity Theorem; see Theorem 1.2. We observed in 1.3 that the support of \( M \) or \( N \) is the whole spectrum of the ring so that one can use the depth formula and easily deduce the reflexivity of either of those modules; see \((2.8)\). One can also see, following the arguments given in the proof of the Second Rigidity Theorem, that both of the modules considered do satisfy (S1), that is, both are torsion-free modules; see the discussion following Corollary 4.4. Indeed our main argument shows that this fact carries over to complete intersection rings of arbitrary codimension; see Corollary 4.5.

We should note that, if \( M \) and \( N \) are arbitrary nonzero finitely generated modules over a complete intersection ring \( R \) that is not a hypersurface, then the vanishing of \( \text{Tor}^R_{\geq 0}(M, N) \) does not necessarily imply that \( M \) or \( N \) has finite projective dimension; for an example see \([25, 4.2]\). Therefore, when \( M \otimes_R N \) is reflexive and \( \text{Tor}^R_{\geq 0}(M, N) = 0 \), the support of both \( M \) and \( N \) may be a proper subset of \( \text{Spec}(R) \), and hence one cannot repeat the arguments of 1.3 to examine the reflexivity of \( M \) or \( N \): the depth formula does not immediately yield any information because localizing the depth formula at a prime ideal that is not in the support of both \( M \) and \( N \) yields the void equality \( \infty = \infty \); see also \([22]\). We are able to overcome this difficulty by using the New Intersection Theorem in the proof of Theorem 3.1 see also Corollaries 4.2 and 4.3.

One can also ask whether there exists an analogous observation to that of 1.3 for the torsion-free tensor products of modules:

**Question 1.6.** Let \( R \) be a hypersurface domain and let \( M \) and \( N \) be nonzero finitely generated \( R \)-modules. If \( M \otimes_R N \) is torsion-free, then must \( M \) or \( N \) be torsion-free?

One can find several interesting results related to Question 1.6 in \([9]\) where it was studied in detail via different techniques.

### 2. Definitions and Preliminary Results

Our proofs use computations in the derived category over commutative Noetherian rings. Again, our arguments may be read just for modules by simply replacing each inf and each sup by zero. However, for readers interested in more details for the hyperhomology techniques, we suggest \([12\) Appendix] or \([13\).

A local ring \((R, \mathfrak{m}, k)\) is a complete intersection if the defining ideal of some (equivalently every) Cohen presentation of the \(\mathfrak{m}\)-adic completion \(\hat{R}\) of \(R\) can be generated by a regular sequence. If \(R\) is such a ring, then \(\hat{R}\) has the form \(Q/(x)\), where \(\{x\}\) is a \(Q\)-regular sequence and \(Q\) is a ring of formal power series over the residue field \(k\), or
over a complete discrete valuation ring with residue field \( k \). The codimension of \( R \) is the nonnegative integer \( \dim_k(m/m^2) - \dim(R) \). A complete intersection of codimension one is called a hypersurface. We should note that Huneke and R. Wiegand define a hypersurface ring in [21] as a quotient of a regular local ring by a nonzero element. Although this differs from our definition, the difference does not affect our results; constant rank and the \((S_n)\) conditions are preserved under completion; see the discussion following Theorem 1.2 and (2.5).

Let \( R \) be a commutative Noetherian ring. We grade complexes \((M, \partial^M)\) of \( R \)-modules homologically. The \( n \)th homology module of \( M \) is denoted by \( H_n(M) \). The supremum and infimum of \( M \) are given by

\[
\sup(M) = \sup\{i \in \mathbb{Z} : H_i(M) \neq 0\} \text{ and } \inf(M) = \inf\{i \in \mathbb{Z} : H_i(M) \neq 0\}.
\]

\( M \) is homologically bounded provided that both \( \sup(M) \) and \( \inf(M) \) are finite.

2.1. Conventions. Through the paper \( R \) denotes a local ring, that is, a commutative Noetherian ring with unique maximal ideal \( m \) and residue class field \( k = R/m \). All modules considered over \( R \) are finitely generated. By an \( R \)-complex, we mean a complex of finitely generated \( R \)-modules which is always assumed to be homologically bounded.

2.2. Support and Dimension. The annihilator and support of an \( R \)-complex \( M \) are defined to be the following:

\[
\text{Ann}(M) = \bigcap_{i \in \mathbb{Z}} \text{Ann}(H_i(M)) \text{ and } \text{Supp}(M) = \bigcup_{i \in \mathbb{Z}} \text{Supp}(H_i(M)).
\]

The (Krull) dimension of an \( R \)-complex \( M \) is defined as

\[
\dim(M) = \sup\{\dim(R/p) - \inf(M_p) : p \in \text{Supp}(M)\} = \sup\{\dim_R H_i(M) - i : i \in \mathbb{Z}\}.
\]

These definitions extend the usual concepts for modules, see [12, A.8.4].

2.3. Depth. The depth of an \( R \)-complex \( M \) is defined by

\[
\text{depth}(M) = n - \sup(K \otimes_R \Omega^1 M).
\]

Here \( K \) is the Koszul complex on a set of generators \( x_1, \ldots, x_n \) of \( m \); see [24] §2 and [12] (A.6.1). It is known that this definition is independent of the choice of generators of \( m \) [24, 1.3]. Notice, by our convention for \( R \)-complexes (2.1), \( \text{depth}(0) = \infty \) and \( \text{depth}(M) \) is finite if \( H(M) \neq 0 \).

The Koszul complex \( K \) is a finite free complex; hence \( K \otimes_R M \simeq K \otimes_R \Omega^1 M \). For any fixed choice of generators of \( m \), the quantity \( n \) and the collection \( \{H_i(K \otimes_R M) : i \in \mathbb{Z}\} \) of homology modules of \( K \otimes_R M \) are invariant under suspension of \( M \), but the depth is not: the suspension \( \Sigma M \) of \( M \) satisfies \( \text{depth}(\Sigma M) = \text{depth}(M) - 1 \).

2.4. Serre’s Condition. We say an \( R \)-complex \( M \) satisfies Serre’s condition \((S_n)\) provided that the following inequality holds for all \( p \in \text{Supp}(M)\):

\[
\text{depth}_{R_p}(M_p) + \inf(M_p) \geq \min\{n, \text{height}(p)\}.
\]

When \( M \) is a finitely generated \( R \)-module, this definition agrees with the classical one defined by Evans and Griffith [14], that is, \( \text{depth}(M_p) \geq \min\{n, \text{height}(p)\} \) for all \( p \in \text{Supp}(M) \). Note that their definition for Serre’s condition for finitely generated modules differs from the one given in [10, 5.7.2.I] and [7, page 62].
We record the following facts about Serre’s condition \((S_n)\).

**2.5. Facts.** [14 3.8], [18 1.7], [27 1.3] and [34 Proposition 6]) Assume \(R\) is a Gorenstein local ring and \(M\) is a finitely generated \(R\)-module.

(i) The following conditions are equivalent:

(a) \(M\) satisfies \((S_n)\).

(b) \(M\) is an \(n\)th syzygy module.

(c) All \(R\)-regular sequences of length at most \(n\) are also \(M\)-regular.

(ii) \(M\) is torsion-free, that is, the natural map \(M \to \text{Hom}(\text{Hom}(M, R), R)\) is injective, if and only if \(M\) satisfies \((S_1)\).

(iii) \(M\) is reflexive, that is, the natural map \(M \to \text{Hom}(\text{Hom}(M, R), R)\) is bijective, if and only if \(M\) satisfies \((S_2)\).

(iv) If \(R \to S\) is a flat ring map and \(M\) satisfies \((S_n)\) over \(R\), then \(M \otimes_R S\) satisfies \((S_n)\) over \(S\).

**2.6. Projective Dimension.** Recall that the \(n\)th Betti number of an \(R\)-complex \(M\) is defined as

\[
\beta^R_n(M) := \dim_k(\text{Tor}^R_n(M, k)) = \dim_k(\text{H}_n(M \otimes_R L^R k)).
\]

The projective dimension of \(M\) can be given as follows [12, (A.5.3)]:

\[
\text{pd}(M) = \sup\{n \in \mathbb{Z} : \beta^R_n(M) \neq 0\}.
\]

**2.7. Quasi-projective and Complete Intersection Dimension.** A diagram of local ring maps \(R \to R' \leftarrow Q\) is called a quasi-deformation provided that \(R \to R'\) is flat and the kernel of the surjection \(R' \leftarrow Q\) is generated by a \(Q\)-regular sequence, see [5, 1.1].

The quasi-projective dimension of an \(R\)-complex \(M\) [3 3.3], see also [36 3.1], is:

\[
\text{qpd}^R_R(M) = \inf\{\text{pd}_Q(M \otimes_R^L R') \mid R \to R' \leftarrow Q\text{ is a quasi-deformation}\}.
\]

The complete intersection dimension of \(M\), see [5 1.2] and [35 3.1], is quite similar:

\[
\text{CI-dim}_R(M) = \inf\{\text{pd}_Q(M \otimes_R^L R') - \text{pd}_Q(R') \mid R \to R' \leftarrow Q\text{ is a quasi-deformation}\}.
\]

Notice \(\text{CI-dim}(M) < \infty\) if and only if \(\text{qpd}^R_R(M) < \infty\). If \(R\) is a complete intersection, then \(\text{CI-dim}(M) < \infty\) [5 1.3]. Furthermore, by [35 3.3], we have:

\[
(2.7.1) \sup(M) \leq \text{CI-dim}(M) \leq \text{pd}(M).
\]

**2.8. Derived Depth Formula.** Let \(M\) and \(N\) be \(R\)-complexes. We say that the pair \((M, N)\) satisfies the derived depth formula provided the following equality holds:

\[
\text{depth}(M) + \text{depth}(N) = \text{depth}(R) + \text{depth}(M \otimes_R^L N).
\]

Christensen and Jorgensen proved in [11 5.4(a)] that if \(\text{Tor}_R^{\geq 0}(M, N) = 0\) and \(M\) has finite complete intersection dimension, see [2.7], then the pair of complexes \((M, N)\) satisfies the derived depth formula.

The depth formula, for the tensor product of finitely generated modules, is initially due to Auslander [2]; see also Foxby [15] and Iyengar [24].

**2.9. Complexity.** The complexity of an \(R\)-complex \(M\) is a measure for the polynomial growth rate of its Betti numbers and is defined as follows, see [4 4.2] and [5 5.1].

\[
\text{cx}_R(M) = \inf\{r \in \mathbb{N} \mid \exists \beta \in \mathbb{R} \text{ so that } \beta_n^R(M) \leq \beta n^{r-1} \forall n \gg 0\}
\]
Note that \(cx(M) = 0\) if and only if \(\text{pd}(M) < \infty\). Moreover, the finiteness of complete intersection dimension implies the finiteness of complexity; this is essentially due to Gulliksen [17, 3.3]. More precisely, when \(\text{CI-dim}_R(M) < \infty\), the complexity \(cx(M)\) cannot exceed \(\text{embdim}(R) - \text{depth}(R)\) [35, 3.10(v)].

2.10. Avramov-Gasharov and Peeva [5, 5.11] obtained a relationship between quasi-projective and complete intersection dimension using complexity: if \(M\) is an \(R\)-complex, then the following equality holds:

\[
\text{qpd}_R(M) = \text{CI-dim}(M) + cx_R(M).
\]

This equality was originally obtained in [5] for finitely generated modules. Its proof mainly depends on a factorization result of quasi-deformations [5, 5.9] which can be restated for complexes. A proof of this equality for \(R\)-complexes follows the arguments in [5, §5] using the analogous results in [6, §5] and [35].

2.11. Auslander-Buchsbaum Formula. Complexes of finite projective dimension also have finite complete intersection dimension. The finiteness of either of these homological dimensions implies an equality, known as the Auslander-Buchsbaum formula: if \(M\) is an \(R\)-complex and if \(\text{pd}(M) < \infty\) (respectively, if \(\text{CI-dim}(M) < \infty\)), then \(\text{pd}_R(M) = \text{depth}(R) - \text{depth}(M)\) (respectively, \(\text{CI-dim}(M) = \text{depth}(R) - \text{depth}(M)\)).

If \(\text{qpd}_R(M) < \infty\), then the Auslander-Buchsbaum formula and (2.10) give that:

(2.11) \[
\text{qpd}_R(M) = \text{depth}(R) - \text{depth}(M) + cx_R(M).
\]

2.12. New Intersection Theorem. The New Intersection Theorem of Hochster [19], Peskine and Szpiro [30] and Roberts [31], [32] states that, if \(M\) and \(N\) are finitely generated \(R\)-modules, then:

\[
\dim_R(N) \leq \text{pd}_R(M) + \dim_R(M \otimes_R N).
\]

This inequality can also be given for \(R\)-complexes, see for example [13, §18].

If \(M\) and \(N\) are \(R\)-complexes with \(H(M) \neq 0\) and \(\text{qpd}(M) < \infty\), then Sharif and Yassemi [36, 3.3], see also (2.7), generalized the New Intersection Theorem as follows:

(2.12) \[
\dim_R(N) \leq \text{qpd}(M) + \dim_R(M \otimes_R N).
\]

We finish this section by recording some inequalities that will be used later.

2.13. Let \(M\) and \(N\) be \(R\)-complexes such that \(\text{CI-dim}_R(M) < \infty\) and \((M, N)\) satisfies the derived depth formula. Then it follows from (2.11) and (2.8) that:

\[
\text{qpd}_R(M) = \text{depth}(R) - \text{depth}(M) + cx_R(M)
= \text{depth}(N) - \text{depth}(M \otimes_R N) + cx_R(M).
\]

2.14. Let \(M\) and \(N\) be \(R\)-complexes. Then, by [12] (A.6.2), we have

\[
\text{depth}_R(N) \leq \text{depth}_{R_p}(N_p) + \dim(R/p)\text{ for all } p \in \text{Spec}(R).
\]
3. An Application of the New Intersection Theorem

One can use the depth formula, see (2.8), to make local conclusions about either $M$ or $N$ only on the intersection of their supports [22]. Localizing the depth formula at a prime ideal that is outside of the support of $M$ or $N$ yields a void equality; depth$(0) = \infty$, see (2.1), (2.3) and also [21, 3.3].

In Theorem 3.1 we make use of the depth formula (2.8) and analyze the depth of the derived tensor product $M \otimes^L_R N$ of $R$-complexes. Our aim is to determine certain conditions so that, if $M \otimes^L_R N$ satisfies $(S_{n+r})$, where $r = cx_R(M)$, then $N$ satisfies $(S_n)$, see also (2.4) and (2.9). We combine an argument, analogous to that of Yoshida, see his proof [37, 4.1, Case 2], with the inequality (2.12.1), due to Sharif and Yassemi [36], that generalizes the New Intersection Theorem. The result we obtain is, to our knowledge, new, even for finitely generated modules.

**Theorem 3.1.** Let $R$ be a local ring, $M$ and $N$ be $R$-complexes with $H(M) \neq 0$ and let $n$ be a nonnegative integer. Set $r = cx_R(M)$ and assume the following conditions:

(i) $\text{CI-dim}_R(M) < \infty$,
(ii) $M \otimes^L_R N$ satisfies $(S_{n+r})$, and
(iii) $\text{Tor}_{n+r}^R(M, N) = 0$.

Then $N$ satisfies $(S_n)$.

**Proof.** We shall prove that, for all $p \in \text{Supp}(N)$, the following inequality holds:

\[
\text{depth}_{R_p}(N_p) + \text{inf}(N_p) \geq \min\{n, \text{height}(p)\}. \tag{3.1.1}
\]

We start by recording some observations. Note that, by (iii), $M \otimes^L_R N$ satisfies our convention to be an $R$-complex, see (2.4). Notice also that the complexity, Serre’s condition, finiteness of complete intersection dimension, the vanishing of $\text{Tor}_{n+r}^R(M, N)$ as well as the conclusion of the theorem are invariant under suspension of $M$, see (2.4), (2.7) and (2.9). Therefore we may assume that $\text{inf}(M) = 0$. Furthermore, for all $q \in \text{Supp}(M)$, the following inequality holds, see (2.7) and (2.11):

\[
0 = \text{inf}(M) \leq \text{inf}(M_q) \leq \text{CI-dim}_{R_q}(M_q) = \text{depth}(R_q) - \text{depth}(M_q)
\]

In particular, setting $q = m$, we see that $\text{depth}(M) \leq \text{depth}(R)$.

The hypotheses imply that $(M, N)$ satisfies the derived depth formula (2.8); hence

\[
\text{depth}(M) + \text{depth}(N) = \text{depth}(R) + \text{depth}(M \otimes^L_R N).
\]

Depth formula together with the fact that $\text{depth}(M) \leq \text{depth}(R)$ gives

\[
\text{depth}(N) \geq \text{depth}(M \otimes^L_R N) \tag{3.1.2}
\]

We know, by [12, (A.4.16)], that $\text{inf}(M \otimes^L_R N) = \text{inf}(M) + \text{inf}(N) = \text{inf}(N)$. Hence (ii) and (3.1.2) show that

\[
\text{depth}(N) + \text{inf}(N) \geq \text{depth}(M \otimes^L_R N) + \text{inf}(M \otimes^L_R N) \geq \min\{n + r, \text{dim}(R)\}.
\]
This yields, for any \( p \in \text{Supp}(N) \), the following inequalities:

\[
\text{depth}_{R_p}(N_p) + \inf(N_p) \geq \text{depth}_{R_p}(N_p) + \inf(N) \\
\geq \text{depth}(N) + \inf(N) - \dim(R/p) \\
\geq \min\{n + r, \dim(R)\} - \dim(R/p).
\]

Here the second inequality follows from (3.1.1).

Let \( p \in \text{Supp}(N) \). We now divide our proof into two main cases:

1. \( \dim(R) \leq n + r \) or \( p \in \text{Supp}(M) \).
2. \( \dim(R) > n + r \) and \( p \not\in \text{Supp}(M) \).

**Case 1.** Assume \( p \in \text{Supp}(M) \). Then \( p \in \text{Supp}(M \otimes^L_R N) = \text{Supp}(M) \cap \text{Supp}(N) \) and hence \( H(M_p) \neq 0 \neq H(N_p) \). The hypotheses and the conclusion are preserved under localization at \( p \). Hence we may consider the pair \((M_p, N_p)\) over \( R_p \). Therefore (3.1.3) yields

\[
\text{depth}(N) + \inf(N) \geq \text{depth}(M \otimes^L_R N) + \inf(M \otimes^L_R N) \geq \min\{n + r, \dim(R)\},
\]

which justifies (3.1.1). Now assume \( \dim(R) \leq n + r \). Then it follows from (3.1.3) that

\[
\begin{align*}
\text{depth}_{R_p}(N_p) + \inf(N_p) & \geq \min\{n + r, \dim(R)\} - \dim(R/p) \\
& \geq \dim(R) - \dim(R/p) \\
& \geq \text{height}(p).
\end{align*}
\]

Therefore we have established (3.1.1) in this case, too.

**Case 2.** Assume \( p \not\in \text{Supp}(M) \) and that \( \dim(R) > n + r \). Recall that \( p \in \text{Supp}(N) \). Let \( q \) be a minimal prime ideal of \( p + \text{Ann}_R(M) \). Then \( q \in \text{Supp}(M \otimes^L_R N) \); see (2.2).

Suppose first that \( \text{height}(q) \leq n + r \). Then, since \( H(M_q) \neq 0 \neq H(N_q) \), we can consider the pair \((M_q, N_q)\) over \( R_q \) and deduce from Case 1 that (3.1.1) holds. Therefore we proceed under the assumption that \( \text{height}(q) > n + r \).

The New Intersection Theorem (2.12.1), applied to the pair \((M_q, (R/p)_q)\) over the ring \( R_q \), gives

\[
\text{qpd}_{R_q}(M_q) + \dim_{R_q}(M_q \otimes^L_{R_q} R_q/pR_q) \geq \dim_{R_q}(R_q/pR_q).
\]

Furthermore the inequality (2.14) applied to the \( R_q \) complex \( N_q \) yields that

\[
\text{depth}_{R_q}(N_q) \geq \text{depth}_{R_q}(N_q) - \text{dim}_{R_q}(R_q/pq).
\]

These last two observations imply

\[
\text{depth}_{R_q}(N_p) \geq \text{depth}_{R_q}(N_q) - (\text{qpd}_{R_q}(M_q) + \dim_{R_q}(M_q \otimes^L_{R_q} R_q/pR_q)).
\]

Note that \( \text{Supp}_{R_q}(M_q \otimes^L_{R_q} R_q/pR_q) = \{qR_q\} \). Hence, by (2.2), we deduce

\[
\dim(M_q \otimes^L_{R_q} R_q/pR_q) = \dim(R_q/qR_q) - \inf((M_q \otimes^L_{R_q} R_q/pR_q)_{qR_q}) = -\inf(M_q).
\]

Therefore, substituting this into (3.1.4), we have

\[
\text{depth}_{R_q}(N_p) \geq \text{depth}_{R_q}(N_q) - \text{qpd}_{R_q}(M_q) + \inf(M_q).
\]

Note that (2.13) gives

\[
\text{qpd}_{R_q}(M_q) = \text{depth}_{R_q}(N_q) - \text{depth}_{R_q}(M_q \otimes^L_{R_q} N_q) + \text{cx}(M_q).
\]

Thus

\[
\text{depth}_{R_q}(N_p) \geq \text{depth}_{R_q}(N_q) - \text{depth}_{R_q}(M_q \otimes^L_{R_q} N_q) + \text{cx}(M_q).
\]
Combining (3.1.5) with (3.1.6), we obtain
\[ \text{depth}_{R_p}(N_p) \geq \text{depth}_{R_q}(N_q) + \inf(M_q) - (\text{depth}_{R_q}(N_q) - \text{depth}_{R_q}(M_q \otimes_{R_q} N_q) + \text{cx}(M_q)), \]
which simplifies to
\[ \text{depth}_{R_p}(N_p) \geq \text{depth}_{R_q}(M_q \otimes_{R_q} N_q) - \text{cx}(M_q) + \inf(M_q). \]
It is clear that \( \text{cx}_{R_q}(M_q) \leq \text{cx}_R(M) = r \) and \( \inf(M_q) + \inf(N_q) = \inf(M_q \otimes_{R_q} N_q) \).
Moreover, \( \inf(N_q) \geq \inf(M_q) \). Hence,
\[ \text{depth}_{R_p}(N_p) + \inf(N_p) \geq \text{depth}_{R_p}(N_p) + \inf(N_q) \]
\[ \geq \text{depth}_{R_q}(M_q \otimes_{R_q} N_q) - \text{cx}(M_q) + \inf(M_q) + \inf(N_q) \]
\[ = \text{depth}_{R_p}(M_q \otimes_{R_q} N_q) + \inf(M_q \otimes_{R_q} N_q) - \text{cx}(M_q) \]
\[ \geq \min\{n + r, \text{height}(q)\} - r = n, \]
where the last equality follows from the fact that \( \text{height}(q) > n + r \). Thus the inequality (3.1.17) holds and hence \( N \) satisfies \( (S_n) \).

4. Applications of Theorem 3.1

**Corollary 4.1.** Let \( R \) be a local ring, \( M \) and \( N \) nonzero finitely generated \( R \)-modules and let \( n \) be a nonnegative integer. Set \( r = \text{cx}_R(M) \) and assume the following conditions hold:

(i) CI-dim \( R(M) \) < \( \infty \),

(ii) \( M \otimes_R N \) satisfies \( (S_{n+r}) \), and

(iii) \( \text{Tor}_{>0}^R(M, N) = 0 \).

Then \( N \) satisfies \( (S_n) \).

**Proof.** By (iii), \( M \otimes^L_R N \cong M \otimes_R N \), so the conclusion follows from Theorem 3.1. Therefore we have:

**Corollary 4.2.** Let \( R \) be a local complete intersection ring and let \( M \) and \( N \) be nonzero finitely generated \( R \)-modules. Set \( r = \text{cx}_R(M) \) and assume \( n \) is a nonnegative integer. If \( M \otimes_R N \) satisfies \( (S_{n+r}) \) and \( \text{Tor}_{>0}^R(M, N) = 0 \), then \( N \) satisfies \( (S_n) \).

The complexity of a module over a complete intersection cannot exceed the codimension of the ring; see [2.9]. This yields:

**Corollary 4.3.** Let \( R \) be a local complete intersection ring of codimension \( c \) and let \( M \) and \( N \) be nonzero finitely generated \( R \)-modules. Assume that \( M \otimes_R N \) satisfies \( (S_{n+c}) \) for some nonnegative integer \( n \). If \( \text{Tor}_{>0}^R(M, N) = 0 \), then \( M \) and \( N \) satisfy \( (S_n) \).

The assertion that \( M \) or \( N \) is reflexive in the next result is implicitly contained in the proof of the Second Rigidity Theorem; see Theorem 1.2 and also Remark 1.3.

**Corollary 4.4.** (Huneke–Wiegand [21]) Let \( R \) be a local hypersurface ring and let \( M \) and \( N \) be nonzero finitely generated \( R \)-modules, either of which has constant rank. If \( M \otimes_R N \) is reflexive, then (i) \( M \) and \( N \) are both torsion-free, and (ii) \( M \) or \( N \) is reflexive.
Proof. It follows from Theorem 1.2 that \( \text{Tor}^R_{>0}(M, N) = 0 \). This implies that either \( M \) or \( N \) has finite projective dimension [23, 1.9], that is, the complexity of \( M \) or \( N \) is zero. As the tensor product \( M \otimes_R N \) satisfies \((S_2)\), we conclude, setting \( n = 2 \), that the claim of (ii) follows from Corollary 1.2. Furthermore, as the complexity of an \( R \)-module cannot exceed one, setting \( n = 1 \) in Corollary 1.2 we obtain the required conclusion of (i). □

One can also obtain the conclusion of Corollary 1.4(i) from the proof of the Second Rigidity Theorem, see Theorem 1.2: assume \( R \), \( M \) and \( N \) are as in Corollary 4.4. Then \( M \otimes_R N \cong (M/\sqrt{R}M) \otimes_R N \) and hence it follows from the Second Rigidity Theorem that \( \text{Tor}^R_{>0}(M/\sqrt{R}M, N) = 0 = \text{Tor}^R_{>0}(M, N) \). Therefore, tensoring the short exact sequence \( 0 \to \sqrt{R}M \to M \to M/\sqrt{R}M \to 0 \) with \( N \), we deduce that there is an injection \( \sqrt{R}M \otimes_R N \to M \otimes_R N \). This yields, as \( N \neq 0 \), that \( \sqrt{R}M = 0 \), that is, \( M \) is torsion-free. Similarly one can see that \( N \) is torsion-free, too.

We finish this section by pointing out, under the hypotheses of Corollary 1.2, the relationship between the depth and the complexity of \( M \).

**Corollary 4.5.** Let \( R \) be a local complete intersection ring of codimension \( c \) and let \( M \) and \( N \) be nonzero finitely generated \( R \)-modules. Set \( r = cx_R(M) \) and assume \( M \otimes_R N \) satisfies \((S_c)\). Assume further that \( \text{Tor}^R_{>0}(M, N) = 0 \).

(i) If \( r < c \), then \( N \) satisfies \((S_{c-r})\).

(ii) If \( r = c \), then \( M \) satisfies \((S_c)\).

Proof. There is nothing to prove if \( c = 0 \). Hence we may assume \( c > 0 \). The claim of (i) immediately follows from Corollary 1.2. If \( r = c \), then the vanishing of \( \text{Tor}^R_{>0}(M, N) \) implies that \( \text{pd}(N) < \infty \) [29, 2.1]. Therefore, since \( M \otimes_R N \) is torsion-free, we have, as in Remark 1.3, that \( \text{Supp}(N) = \text{Spec}(R) \) and hence \( M \) satisfies \((S_c)\). □

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