Modelling rogue waves through exact dynamical lump soliton controlled by ocean currents

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Rogue waves are extraordinarily high and steep isolated waves, which appear suddenly in a calm sea and disappear equally fast. However, though the rogue waves are localized surface waves, their theoretical models and experimental observations are available mostly in one dimension, with the majority of them admitting only limited and fixed amplitude and modular inclination of the wave. We propose two dimensions, exactly solvable nonlinear Schrödinger (NLS) equation derivable from the basic hydrodynamic equations and endowed with integrable structures. The proposed two-dimensional equation exhibits modulation instability and frequency correction induced by the nonlinear effect, with a directional preference, all of which can be determined through precise analytic result. The two-dimensional NLS equation allows also an exact lump soliton which can model a full-grown surface rogue wave with adjustable height and modular inclination. The lump soliton under the influence of an ocean current appears and disappears preceded by a hole state, with its dynamics controlled by the current term. These desirable properties make our exact model promising for describing ocean rogue waves.

1. Introduction

The mysterious ocean rogue waves (RWs) are reported to be observed in a relatively calm sea, where they, as a localized and isolated surface waves, apparently appear from nowhere, make a sudden hole in the sea just before attaining surprisingly high amplitude and disappear again without a trace ([1–7], http://news.bbc.co.uk/2/hi/8548547.stm). This
elusive freak wave caught the imagination of the broad scientific community quite recently
[8–15], triggering off an upsurge in theoretical [7,16–18] and experimental [5,8–15] studies
of this unique phenomenon. For identifying such extreme waves, the suggested signature of
these rare events is a deviation of the probability distribution function (PDF) of the wave
amplitude from its usual random Gaussian distribution (GD), by having a long-tail, indicating
that the appearance of high-intensity pulses more often has much higher probability than that
predicted by the GD [19]. In conformity with this definition, RWs were detected in a photonic
crystal fibre [12], in a multi-stable state of an erbium-doped fibre laser [15], in chaotic but
deterministic regime of optical injected semiconductor lasers [5,8], in nonlinear optical cavity
[11], in acoustic turbulence in He II [9] and other set-ups [13]. On the formation of the ocean
RWs, a number of supporting theories have been developed [1–4]. Among various possible
factors contributing to the creation of the RW, the modulation instability (MI) supported by
the nonlinear effect is believed to play a crucial role, by inducing preliminary amplification of
water wave height, which may trigger self-attractive nonlinear interaction, initiating the RW
formation [20]. The MI can also cause wave–wave interaction leading to the four-wave mixing
at matching frequencies and wave numbers, inducing resonance effect which might also develop
into an RW [7,13,21]. Like the four-wave nonlinear interaction, a leading order nonlinear effect
in deep-sea waves is found also to be a dominant interaction in the nonlinear Schrödinger
(NLS) equation
\[ \imath \frac{\partial q}{\partial t} = q_{xx} + 2|q|^2 q, \tag{1.1} \]
with the subscripts denoting partial differentiation, the NLS-based nonlinear models are the
most accepted ones for the RW, though often with certain modifications to include higher order
dispersion or ocean currents, which are suspected to have a deciding role in the formation of
the RW [6]. In extended space dimensional systems, the nonlinear effect owing to the MI in
combination with a space-asymmetry directional spectra and broken symmetry owing to non-
local coupling is suspected to be the major cause of such extreme waves [10,11,18]. The NLS
equation (1.1) is a well-known evolution equation with integrability properties, such as having
a Lax pair and exact soliton solutions [22]. Some models of RW generalize the NLS equation
with the addition of extra terms on physical grounds, such as ocean current [6], nonlinear
dispersion [16,23], etc. However such modifications of the NLS equation (1.1) make the system
non-integrable, allowing only numerical solutions. The most popular one-dimensional RW model
is a unique analytic rational solution of the original NLS equation (1.1) [24], given by the
Peregrine breather (PB) [12,14,17] or its higher order versions [25–27] and the trigonometric
variants [28,29]. However, as the RW is an aperiodic event with a single appearance, the
trigonometric breather solutions, owing to their periodic nature, are not very suitable for a
direct description of the RW. Nevertheless, interestingly these breather solutions, periodic in
time [28] or space [29], degenerate to the rational PB solution (1.2) at their periods going to
infinity [30].

Note that the conventional soliton solution of the NLS equation (1.1) representing a localized
translational wave behaves like a stable particle and unlike a RW propagates with unchanged
shape and amplitude. Tsunami waves, though highly devastating, also exhibit a different nature
from ocean RWs. The ocean RWs are deep-sea waves with two-dimensional character, localized
in both space dimensions and appearing as a single-peak event for a short interval of time.
Tsunami waves, on the other hand, manifest only in shallow water near the sea shore, though
generated in the deep sea and propagated across a long distance. In the deep sea, tsunami
waves behave like one-dimensional translatory waves moving very fast with insignificant
amplitude [31]. Therefore, tsunamis and RWs exhibit different features and dynamics and
need different types of modelling, which for the RW is still an open problem. More details
on the progress in the study of ocean RWs can be found in some excellent reviews on the
subject [1–4].
Figure 1. Amplitude variation of the full-grown one-dimensional rogue wave, modelled by the modulus of the static PB $|q_P(x, 0)|$. The maximum amplitude 3 is attained at $x = 0$, while it goes to its asymptotic value 1 at $x \to \pm \infty$. The maximum inclination attainable is $3\sqrt{3}$ at $x = \sqrt{3}/6$, and becomes 0 both at $x = 0$ and $x \to \pm \infty$.

(a) Rogue wave model on a one-dimensional line

In contrast to the soliton or the trigonometric breather solutions of the NLS equation (1.1), its exact rational PB solution

$$q_P(x, t) = e^{-2it}(u + iv), \quad u = G - 1, \quad v = -4tG,$$

where $G = \frac{1}{F(x, t)}$, $F(x, t) = x^2 + 4t^2 + \frac{1}{4}$, (1.2)

represents a breather mode with unit amplitude at both distant past and future. The amplitude of the wave rises suddenly at $t = 0$, attaining its maximum at $x = 0$, though subsiding with time again to the same breathing state. This intriguing behaviour makes the PB a popular candidate for the RW [12,14,17].

As the characteristics of the envelop wave is the most significant in the description for the RW, the modulus of the PB solution (1.2)

$$|q_P(x, t)| = (u^2 + v^2)^{1/2} = [(G - 1)^2 + (4tG)^2]^{1/2},$$

with $G$ as in (1.2), is used in describing the RW profile. The full-grown one-dimensional RW at $t = 0$ therefore may be represented by

$$|q_P(x, 0)| = (G - 1)|_{t=0} = \left[ \frac{1}{x^2 + 1/4} - 1 \right],$$

as shown in figure 1. The maximum amplitude as seen from (1.4) is attained at $x = 0$, as $|q_P(0, 0)| = 3$. The modular inclination defined as

$$S^x_{P}(x) = \frac{d}{dx}|q_P(x, 0)| = -\frac{2x}{(x^2 + 1/4)^2}$$

attains its maximum $S^x_{P\text{max}}(x_m) = 3\sqrt{3}$ at $x_m = \pm 1/2\sqrt{3}$.

Note however that NLS equation (1.1) together with its different generalizations are equations in $(1 + 1)$-dimensions and therefore all of their solutions, including the PB and its higher order generalizations, can describe the time evolution of a wave only along a one-dimensional line (as in figure 1). Looking more closely into the PB, we also realize that the maximum amplitude of this solution describing a one-dimensional RW is fixed and just three times that of the background waves (figure 1). The modular inclination of this wave as well as the fastness of
its appearance is also fixed, as solution (1.2) admits no free parameters. This situation can be improved to obtain higher amplitude and modular inclination of the PB model by using higher order rational solutions [25]. For example, the next higher order PB known also as Akhmediev-PB can enhance the maximum wave elevation by a factor of five, while the next one by a factor of seven and so on, with an intriguing enhancement of factors by increasing odd numbers. Such increments in amplitude, however, are discrete and could be achieved at the cost of going to new solutions with increasingly complicated structures involving higher and higher order polynomials [25]. The maximum amplitude and modular inclination reachable by this class of solutions are fixed owing to the absence of relevant free tunable parameters, making it difficult to adjust to the continuously varied range of shape and sizes of the observed oceanic RWs. However, recently, higher order rational solutions to the NLS equation allowing free parameters have been discovered [26,27,32], though they seem to represent multi-peak waves in the \( x-t \) plane for the non-trivial choice of parameters [27]. The single-peak solution, which is suitable for describing RWs having a single appearance in time, is obtained unfortunately for a trivial choice of the free parameters. The trigonometric breathers [28,29] also contain free parameters [30], though such periodic solutions, as mentioned already, are different in nature from the single crest RW event. The crucial fact however is that the one-dimensional spatial nature remains the same for the whole class of the PB solutions, including its higher order rational and trigonometric generalizations. Therefore, modelling an ocean RW, which is a two-dimensional surface wave, by this class of one-dimensional PB solutions remains problematic.

(b) Need for a rogue wave model on a two-dimensional plane

Therefore, though the well-accepted class of PB or other solutions of the generalized NLS equation could fit into the working definition of the ocean RW, saying any wave with height more than twice the nearby significant height (average height among one-third of the highest waves) could be treated as the RW [19], they perhaps, with their restricted characteristics, can explain successfully only fixed and moderately intense RW-like events on a one-dimensional line, as observed in water channels [14], optical fibres [12,15] or optical lasers [5,8], but seem to be not satisfactory for modelling the ocean surface RWs.

Oceanic RWs are said to be consist of an almost vertical wall of water preceded by a trough so deep that it was referred to as a hole in the sea (http://wikipedia.org/wiki/Rogue_Wave). In march 2001, two ships named Bremen and Calendonian Star carrying hundreds of tourists across the South Atlantic had a devastating encounter with RW-like events. It was reported by the witnesses that a giant isolated wave of around 30 m high fell upon the ships like a wall of water; it appeared out of nowhere and disappeared again without a trace (http://news.bbc.co.uk/2/hi/3917539.stm). At the initiative of 11 organizations from several countries in the EU, the Earth-scanning satellites, named ERS-1 and ERS-2, were given the task to send images from a localized area of \( 10 \times 5 \) km\(^2 \) on the sea surface at certain locations to spot the possible occurrence of RWs (http://news.bbc.co.uk/2/hi/3917539.stm).

All these available facts and information suggest that unlike the tsunami and internal waves, pictures of which can be seen through satellite images (http://earthobservatory.nasa.gov/IOTT/View.php?id=44567), ocean RWs with hole states must have a two-dimensional character, localized in both the space dimensions. In two-dimensional water basin experiments as well as in the related simulations the amplitude and the modular inclination of the RWs were found to be higher [10,13,17,18] than those predicted and observed in one dimension [14,17].

The above arguments should be convincing enough to go beyond the one-dimensional equations and search for a suitable \((2+1)\)-dimensional equations, to find a two-dimensional alternative to the PB and other solutions of the one-dimensional NLS equation, for constructing a more realistic model for the ocean RWs.

There are many nonlinear equations known in \((2+1)\) dimensions having fruitful applications in various fields. Some of them allow exact analytic solutions, while others permit only approximate numerical simulations.
The well-known Kadomtsev–Petviashvili (KP) equation is an integrable extension of the Korteweg–de Vries (KdV) equation to two-dimensional space [33] describing the dynamics of a real field. However, the KP equation like the KdV is a shallow-water model, whereas the RW is naturally a deep-water phenomenon.

There are also several equations extending the one-dimensional NLS equation to \((2 + 1)\) dimensions. From the basic hydrodynamic equations, by taking the perturbation analysis to a higher order, Dysthe has derived for the deep water waves a two-dimensional evolution equation [34]. The Dysthe equation in general is non-integrable.

The Davey–Stewartson equation [35] is an integrable generalization where the existence of RWs has been analysed [36]. However, such RW solutions are reducible to the PB solution by a simple rotation in the plane. The Boiti–Leon–Pempinelli (BLP) equation [37] is another \((2 + 1)\)-dimensional integrable equation, defined through two real coupled equations. Recently, an RW-type solution has been found in this equation allowing a free parameter [38]. However as the BPL equation describes wave propagation along a channel, its applicability in modelling the ocean RW is questionable.

Zakharov has proposed several two-dimensional equations, some of them are integrable [39,40] while others are not [21]. Although these equations are applicable in other fields, the model proposed in Dyachenko & Zakharov [21] seems to be a successful model for the RW.

A straightforward two-dimensional extension of the NLS equation

\[
iq_t = d_1 q_{xx} - d_2 q_{yy} + 2|q|^2 q, \tag{1.6}
\]

where \(q(x, y, t)\) is a slowly varying envelop and \(d_1\) and \(d_2\) represent linear dispersion coefficients [18] of the deep-water gravity wave, was proposed in connection with RWs [17,18].

Note however that the two-dimensional NLS equation (1.6) is not an integrable system and gives only approximate numerical solutions with no stable soliton. Nevertheless, this unlikely candidate is found to exhibit RW-like structures numerically, with higher amplitude and modular inclination and with an intriguing directional preference [10,18] with broken spatial symmetry [11,17]. However, though the experimental and theoretical studies on nonlinear systems in two-dimensional space have shown promise in describing more realistic situations in the formation of two-dimensional ocean RWs, unfortunately, all of them can give only approximate numerical results and most of these models could not consider the effect of ocean current, which is supposed to play a crucial role in the formation of ocean RWs [17,18].

2. Proposed integrable two-dimensional nonlinear Schrödinger equation

In the light of the not so satisfactory present state in modelling deep-sea RWs, we propose an integrable extension of the two-dimensional NLS equation

\[
iq_t = d_1 q_{xx} - d_2 q_{yy} + 2iq(\sqrt{d_1} \frac{x}{|x|} - \sqrt{d_2} \frac{y}{|y|}), \quad j^a \equiv q_{a}^a - q^a q_a, \tag{2.1}
\]

allowing an exact lump soliton as a suitable RW model. In (2.1), the linear dispersion relation is exactly the same as the conventional water wave dispersion as described in (1.6), with the only difference from this well-known two-dimensional NLS equation being in the nonlinear term. Note that when the conventional amplitude-like nonlinear term in the non-integrable equation (1.6) is replaced by a nonlinear current-like term (expressed through \(j_x, j_y\)), the resulting equation (2.1) miraculously becomes a completely integrable system with all its characteristic properties, which is much rarer in two dimensions than in one dimension. Before proceeding further, observe that through scaling and a \(\pi/4\) rotation on the plane: \((x, y) \rightarrow (\tilde{x}, \tilde{y})\) with \(\tilde{x} = \frac{1}{2}(-x/\sqrt{d_1} + 1/\sqrt{d_2})y\), \(\tilde{y} = \frac{1}{2}(x/\sqrt{d_1} + 1/\sqrt{d_2})y\) and \(\tilde{t} = 2t\), our two-dimensional NLS equation (2.1) can be simplified to

\[
iq_t + q_{xy} + 2iq(q_{a}^a - q^a q_a) = 0, \tag{2.2}
\]

where the bar over the coordinates is omitted. Encouragingly, our two-dimensional NLS equation (2.2), at par with the well-known one-dimensional NLS equation is derivable from the more fundamental hydrodynamic equations and exhibits MI together with a nonlinear frequency...
correction, as we show below. Equation (2.2) also admits exact soliton and breather solutions through the standard formalism of Hirota’s bilinearization and an associated Lax pair as well as an infinite set of conserved charges [41,42], thus proving the integrability of this nonlinear equation. More satisfactorily, equation (2.2), as we see below, admits an exact two-dimensional generalization of the PB with the desirable properties of a realistic surface RW. It is promising that many characteristic properties, such as directional preference, MI, appearance of higher amplitude, etc., observed theoretically and experimentally in connection with the formation of RWs in two-dimensional models [10,11,17,18], which remained as numerical approximations, are confirmed through analytic result in our model based on integrable equation (2.2).

(a) Nonlinear frequency correction and modulation instability

Instability of a planar wave, appearing owing to the interplay between dispersion and nonlinear effect called Benjamin Feir or MI [43], which has been in continuous focus for many years [44,45] has gained more importance recently in the context of the RW. The nonlinearity and the MI are supposed to be the basic reasons behind the formation of RWs. Therefore, before progressing further with our two-dimensional NLS equation (2.2), we focus on the correction of its linear frequency induced by the nonlinear effect and the appearance of the MI mediated by such nonlinearity in the system. For investigating the contributions to the frequency owing to the linear dispersive and the nonlinear term in (2.2), we insert the plane wave solution with \( A_0 \) as the real constant amplitude, \( \omega \) as the frequency and \((k^x, k^y)\) as the wave vector. For the plane wave to be an exact solution of (2.2), the frequency should be \( \omega = \omega_L + \omega_{NL} \), \( \omega_L = -k^x k^y \), \( \omega_{NL} = 4A_0^2 k^x \), where \( \omega_L \) is the frequency owing to linear dispersion and \( \omega_{NL} \) is its nonlinear correction, which depends on the amplitude of the wave as well as on the \( x \) component of the wave vector.

Now to explore the onset of MI in the system affecting this plane wave solution, we perturb it by a small parameter function \( \epsilon(x, y, t) \). Note that the perturbation is considered in both the space directions because its importance in the instability in two dimensions is emphasized in the context of RW formation [17]. The solution

\[ q_c = (A_0 + \epsilon) e^{i(\omega t + k^x x + k^y y)}, \]

(2.3)

neglecting the higher order terms in \( \epsilon \) yields from (2.2) a linear equation for \( \epsilon \) as

\[ i\epsilon_t + \epsilon_{xy} + i(k^y \epsilon_x + k^x \epsilon_y) + 2iA_0^2 \epsilon_x^* \epsilon_x + 4A_0^2 k^x(\epsilon^* \epsilon + \epsilon) = 0. \]

(2.4)

The appearance of the last two terms in equation (2.4) is due to the nonlinearity. For detecting the instability of the perturbation, we represent \( \epsilon = c_1 e^{i(\omega_m t + k^x m x + k^y m y)} + c_2 e^{-i(\omega_m t + k^x m x + k^y m y)} \). Inserting this form of perturbation in equation (2.4) and arranging the independent terms we get a set of two homogeneous equations for the arbitrary coefficients \( c_1, c_2 \), non-trivial solutions of which can exist only when the determinant of the matrix vanishes leading to the necessary relation \( \omega_m^2 = K^2 - \Omega_c \), where \( \omega_m = \omega_m - \omega_0 \), and \( \omega_0 = 2A_0^2 k^x k^y - k^x k^y - k^x k^y \), \( K = k^x k^y + 4A_0^2 k^x \), \( \Omega_c = 4A_0^4(4k^x^2 - k^x m^2) \), which gives finally

\[ \omega_m = \omega_0 \pm i\omega_1 \quad \text{and} \quad \omega_1 = (\Omega_c - K^2)^{1/2}. \]

(2.5)

Therefore, under the condition \( K^2 < \Omega_c \) with \( \Omega_c > 0 \), i.e. when \(|k^x_m| < 2|k^x|\) the modulation frequency \( \omega_m \) can acquire an imaginary part \( \omega_1 \), initiating an exponential growth of perturbation with time \( t \), and hence onsetting the MI. \( \omega_1 \) is the growth rate of the instability given by (2.5), a graphical form of which is shown in figure 2, showing its dependence on the longitudinal and transverse directions through \( k^x_m \) and \( k^y_m \), respectively.

The stability plot is drawn in figure 3 in the \((k^x_m, k^y_m)\) plane, with the shaded region showing the domain of MI.

Both these figures show clearly that the behaviour of MI as well as the growth rate has a strong directional preference and range as observed also earlier in two-dimensional models [10,17,18].
Figure 2. The growth rate $\omega_I$ of the MI given by (2.5), arising in our two-dimensional NLS equation, exhibiting how it changes (for $A_0 = 1.0, k^x = 1.0$) along the longitudinal ($k^x_m$) and transverse ($k^y_m$) directions, showing a strong directional preference. (Online version in colour.)

Figure 3. Graphical representation of the MI region, where the instability can occur only within the shaded area (for fixed values of $A_0 = 1.0, k^x = 1.0$). The instability region, showing dependence on the wave vector ($k^x_m, k^y_m$), varies asymmetrically along the longitudinal and transverse direction, as seen clearly from the figure. (Online version in colour.)

We have confirmed such properties through exact analytic results showing explicitly that in the MI as well as in the growth rate the components ($k^x_m, k^y_m$) of the wave vector do not enter symmetrically, in addition to a directional range $|k^x_m| < 2|k^y_m|$.

A comparison here with the analysis of MI in the case of the known one-dimensional NLS equation [46] may be illuminating. The condition for the onset of instability in the one-dimensional case involves only the nonlinear amplitude $A_0$ expressed in the form $|k^y_m| < 2A_0$, while in the present situation the condition is more complicated involving all components $k^x_m, k^y_m$. 
Apart from $A_0$, together with an allowed range on the wave vector component, as found above analytically and shown graphically in figure 3. A similar situation is also true for the growth rates, where the one-dimensional case is given by $\omega_I = |k_x m| [(2A_0)^2 - (k_x m)^2]^{1/2}$ [46], while in the present case the form of $\omega_I$ is more complicated and depends on both longitudinal and transverse directions, as shown above.

Thus, the overall picture for the onset of the MI is similar to that occurring in the one-dimensional NLS equation [46], though in the case of the two-dimensional NLS equation (2.2) the details are different and more intricate with a directional preference and range, as seen also for the MI, initiating RW formation in some other systems in higher space dimensions [10,11,17,18]. We emphasize, however, that in place of the approximate numerical result obtained earlier, we found here similar properties in exact analytic form in our model. This is a strong point of our exact model. As in the case of the well-studied one-dimensional NLS model, we may expect the MI to play a key role in the creation of RWs based on our two-dimensional NLS model (2.2).

3. Modelling of two-dimensional rogue waves

Apart from finding a novel two-dimensional integrable equation (2.2), our aim, relevant to the present problem, is to construct a two-dimensional RW model as an exact solution of this equation.

(a) Static lump soliton

Before presenting the dynamical lump soliton related to (2.2), we consider first its static two-dimensional lump-like structure

$$q_{P(2D)}(x, y) = e^{i4y (u + iv)}, \quad u = G - 1, \quad v = -4yG,$$

where $G \equiv \frac{1}{F(x, y)}$, $F(x, y) = \alpha x^2 + 4y^2 + c$,

\begin{equation}
(3.1)
\end{equation}

localized in both space directions and describing a fully developed RW. One can check by direct insertion that (3.1), having two arbitrary parameters $\alpha$ and $c$, is an exact static solution of the two-dimensional nonlinear equation (2.2). Solution (3.1), in spite of its close resemblance to the well-known PB solution (1.2), marks some important differences. The static wave profile $|q_{P(x, 0)}|$ (1.4) obtained from PB solution (1.2) (figure 1) at time $t = 0$ is a curve, representing full-blown one-dimensional RWs admitting no free parameters of relevance. On the other hand,

$$|q_{P(2D)}(x, y)| = (u^2 + v^2)^{1/2} = [(G - 1)^2 + (4yG)^2]^{1/2}$$

obtained from static solution (3.1) represents a two-dimensional lump (figure 4) with two independent free parameters, the significance of which will be is explained below and shown in figure 4a–d.

(b) Rogue wave with adjustable amplitude, inclination and hole waves

Note that the static lump soliton (3.8) can be obtained from the dynamical RW solution (1.4) at the static point $t = 0$ (figure 5c) similar to static PB profile obtained from (1.3), and hence it physically represents a full-grown RW solution, as shown in figure 4d. Looking more closely into solution (3.1) for understanding the physical relevance of its free parameters $c$ and $\alpha$, we notice that the wave attains its maximum amplitude, $|q_{P(2D)}(0, 0)| = A_{\text{rog}}(c) = (1/c - 1)$, at the centre ($x = 0, y = 0$), while at large distances ($|x| \to \infty, |y| \to \infty$) the wave goes to the background plane wave, with its amplitude decreasing to $A_\infty = 1$. Therefore, the maximum amplitude reachable by our RW solution relative to that of the background wave is $A_{\text{rog}}(c)/A_\infty = (1/c - 1)$. Consequently, the amplitude of the full grown RW described by the lump soliton can be changed continuously by changing parameter $c$ (with $A_{\text{rog}}(c)$ increasing with decreasing $c$) and could therefore be adjusted to fit the heights of any observed RW. Consequently, the maximum RW amplitude in our...
Figure 4. Full-grown two-dimensional rogue wave modelled by the modulus (|q_{P(2D)}|) of the static lump soliton (3.1) with different shapes and sizes, generated from the same single-peak solution. The maximum amplitude and modular inclination are tunable through two free parameters $c$ and $\alpha$ showing (a) high amplitude: 12 and high modular inclination, for $c = \frac{1}{13}$, $\alpha = 4.0$. (b) High amplitude: 12 but low modular inclination, for $c = \frac{1}{13}$, $\alpha = 0.4$. (c) Low amplitude: 2 and low modular inclination, for $c = \frac{1}{3}$, $\alpha = 0.4$. (d) Moderate amplitude: 5 and moderate modular inclination, for $c = \frac{1}{5}$, $\alpha = 1.2$. The last situation is the same as figure 5c, obtained at $t = 0$. (Online version in colour.)

Extending the modular inclination in case of one dimension, $S_x^P(x)$, as defined in (1.5), we get for the full grown two-dimensional RW solution (3.1) the modular inclination as

$$S_{x}^{P(2D)}(x,y) = \frac{\partial}{\partial x} |q_{P(2D)}(x,y)| \quad \text{and} \quad S_{y}^{P(2D)}(x,y) = \frac{\partial}{\partial y} |q_{P(2D)}(x,y)|.$$  

(3.3)

Focusing on the inclination $S_{x}^{P(2D)}(x,0)$, as observed at the middle of the wave front, we note that it is linked also to another free parameter $\alpha$ and attains its maximum

$$S_{x}^{P(2D)\max}(x_m,0) = -2\alpha x_m G^2(x_m,0)$$  

(3.4)

at $x_m = \sqrt{c}/\sqrt{3\alpha}$ with function $G(x,y)$ as defined in (3.1). We see that the maximum modular inclination of a full-grown two-dimensional RW in our model depends on both the parameters $c$
and $\alpha$ in an intricate way and can be changed continuously by varying two arbitrary parameters to fit varied situations (figure 4e–d). Note that this inclination will be influenced by the physical steepness of the wave contributing from the wave vector of the career wave. We can identify another intriguing feature of our solution, by noting that the amplitude of the wave (3.2) falls to its minimum $A_0 = 0$, at $y = 0$, $x = \pm x_0$, where $x_0 = \sqrt{(1/\alpha)(1 - c)}$, which depends again on two free parameters. This significant feature emerging from our RW model, as will be demonstrated in figure 5a,b, is related to the hole-wave formation observed during ocean RWs (http://wikipedia.org/wiki/Rogue_Wave [21]).

(c) Topological consideration

Although the static lump soliton (3.1) can describe the profile of a full-grown two-dimensional RW, for modelling an evolving realistic RW we need to find a time-dependent solution, which would smoothly go to its static form (3.1) at the moment $t = 0$. Our next aim therefore is to construct a dynamical lump soliton out of the static lump soliton, to create a true picture of an RW which can appear and disappear fast with time. However, for constructing such a solution we have to clarify first, whether it is possible in principle for our lump soliton to disappear without a trace, i.e. whether the soliton is free from all topological restrictions, which otherwise would prevent such a vanishing. The reason for such suspicion is owing to an interesting lesson from topology, stating that when a complex field $q(x, y)$ is defined on a two-dimensional space with non-vanishing boundary condition $|q| \to 1$ at large distances, but having vanishing values $q \to 0$ close to the centre, we can define a unit vector $\hat{\phi} = q/|q|$ on a 1-sphere $S^1$. However, this vector $\hat{\phi} = (\phi^1, \phi^2)$ is well defined only at the space boundaries: $\partial \mathbb{R}^2 \sim S^1$ (as $q = 0$ at inner points); realizing a smooth map: $S^1 \to S^1$ with possible non-trivial topological charge $Q = n$. This charge

Figure 5. Snap shots of a two-dimensional rogue wave with two-dimensional holes during its formation at different times, described by the modulus $|q_{P(2)}(x, y, t)|$ of the dynamical lump soliton (3.8) with parameter values $c = \frac{1}{8}$, $\alpha = 1.2$ and $\mu = 1.2$, at three crucial moments of time: (a) At $t = t_h = -0.83$: creation of a two-dimensional hole at the centre. (b) At $t = -0.40$: the hole splits into two, which are drifting away from the centre. (c) At $t = 0.0$: The full-grown RW corresponds also to the static lump soliton (3.1), as shown in figure 4d. (Online version in colour.)
with integer values \( n = 0, 1, 2, \ldots \) labels the distinct homotopy classes and is defined as the degree of the map, which unlike a Nöther charge is conserved irrespective of the dynamics of the system. Such a situation occurs, for example, in type II superconductors with the charge linked to the quantized flux of vortices for the magnetic field \( B(x, y) \) [47]

\[
2\pi Q = \int dS \cdot B = \int_C dl \cdot A,
\]

where \( B = \text{curl} \, A = \dot{z}(\partial_x \phi_1 \partial_y \phi_2 - \partial_x \phi_2 \partial_y \phi_1) \).

Note that our complex field solution \( q_{p(2D)}(x, y) \) possesses clearly the features of \( \phi \) discussed above, as (3.1) goes to a constant modulation \(-e^{4i\psi}\) at large distances and vanishes at points \((0, \pm x_0)\). Note that such a solution related to a sphere to sphere map cannot go to a trivial configuration, if it belongs to a homotopy class with non-trivial topological charge, \( Q = n, n = 1, 2, 3, \ldots \), owing to conservation of the charge, with the only exception for the class with zero charge \( Q = 0 \). Therefore, for confirming the possible appearance/disappearance property of an RW for solution (3.1), we have to establish first that in spite of defining a non-trivial topological map, it belongs nevertheless to the sector with topological charge \( Q = 0 \), i.e. our lump soliton is indeed shrinkable to the vacuum solution. For this, we can calculate explicitly the topological charge (3.5) associated with (3.1) as

\[
2\pi Q = \int_C dl \cdot A = \int (dx A_x + dy A_y),
\]

where \( A_q = \phi_1 \partial_x \phi_2, \phi_1 = \text{Re} \, q/|q|, \phi_2 = \text{Im} \, q/|q| \), where the contour integral along \( x \) and \( y \) are taken along a closed square at the boundaries of the plane. Substituting explicit form of solution \( q(x, y) \) from (3.1) and arguing about the oddness and evenness of the integrand with respect to \( x, y \) or checking directly by any analytic computational package, one can show that the related charge is indeed \( Q = 0 \), and therefore the solution belongs to the trivial topological sector, as we wanted.

The intriguing reason behind this fact is that the two holes appearing here have opposite charges resulting in their combined charge being zero.

(d) Construction of dynamical lump soliton

For constructing a dynamical extension of the two-dimensional static lump soliton (3.1), we realize that a sudden change of amplitude with time, as necessary to mimic the two-dimensional RW behaviour, might result in non-conservation of energy. This however cannot be described by an integrable equation alone, as the integrability demands strict conservation of all charges and therefore our integrable equation (2.2) needs certain modification for allowing the appearing/disappearing nature of its lump soliton. On the other hand, the importance of ocean currents in the formation of RWs is documented and repeatedly emphasized [6,20,21], which however is absent in equation (2.2). This motivates us to solve both these problems in one go, by modifying equation (2.2) with the inclusion of the effect of an ocean current, as in [6], by adding a term in the form \( I = -iU_c q_s \). For obtaining an exact dynamical RW solution to the modified two-dimensional NLS equation, we choose the current flowing along longitudinal directions and changing with time and location as \( U_c(x, t) = \mu t/\alpha x \). Looking closely into the structure of this current term for RW solution (3.8)

\[
I(x, y, t) = i \left( \frac{\mu t}{\alpha x} \right) \frac{\partial}{\partial x} [q_{p(2D)}(x, y, t)] = -2\mu t(4y - i)G^2 e^{4iy},
\]

with \( G \) as defined in (3.8), it becomes apparent that the currents would flow to the centre of formation of the RW \((x = y = 0)\) from both of the longitudinal and the transverse sides, though with a directional preference, with their magnitude \( |I(x, y, t)| \) increasing as they approach the centre, but stopping completely at the moment of the full surge at \( t = 0 \). The picture gets reversed after the RW event with currents flowing back quickly, away from the centre with the intensity of the current \( |I| \) diminishing as the distance increases. Such an in- and outflow of energy seems to be physically consistent with the formation of a two-dimensional ocean RW. Note that, though the current factor \( U_c \) looks ill defined, the multiplicative factor \( q_s \) makes the term \( I(x, y, t) \) well
behaved on the RW solution (3.8), with the ocean current term becoming a smooth and bounded function in all space and time variables, as evident from (3.7). It has been suspected in earlier studies that spatially non-uniform current should be responsible in the development of ocean rogue waves [20]. Such a non-uniform dependence on space variables can be seen in our current term \( I(x, y, t) \). Interestingly, the modified two-dimensional NLS equation ((2.2) with the inclusion of the current term \( I \)) admits now an exact dynamical two-dimensional extension of the Peregrine soliton in the analytic form, though the modified equation loses its integrability in the sense, discussed in §4b. The dynamical RW solution has a similar form as (3.1), only with the function \( G \) becoming dynamical by the inclusion of time variable

\[
q_{P2D}(x, y, t) = e^{4iy}[-1 + (1 - i4y)G] \\
and \quad G = \frac{1}{F(x, y, t)} F(x, y, t) = \alpha x^2 + 4y^2 + \mu t^2 + c.
\]

(3.8)

The arbitrary parameter \( \mu \) appearing in solution (3.8) is related to the ocean current and can control how fast the RW would appear and how long it would stay. Note again that (3.8) at \( t = 0 \), representing a full-grown RW (figure 5c) coinciding with the exact static lump soliton (3.1) of the two-dimensional NLS equation (2.2) (as in figure 4d), justifying the physical relevance of the static lump soliton. At this stage, a comparison between one-dimensional PB soliton

\[
q_P(x, t) = \left[1 - \frac{(1 - 4it)}{x^2 + 4t^2 + 1/4}\right] e^{-2it}
\]

and our two-dimensional lump soliton

\[
q_{P2D}(x, 0, t) = \left[1 + \frac{(1)}{\alpha x^2 + \mu t^2 + c}\right], \quad \text{at } y = 0
\]

(3.10)

might be interesting. This shows that though there is some similarity between these two solutions there are many differences as well at \( y = 0 \). In the absence of the transverse coordinate, the two-dimensional solution (3.10) of our modified equation becomes real, though still having three independent free parameters. The one-dimensional PB soliton, on the other hand, is complex with a breathing mode but without any free parameter.

We should mention here that the two-dimensional extension of PB solution (3.8), unlike the standard one-dimensional PB, unfortunately could not be derived as a limiting case from the breather solution of two-dimensional NLS equation (2.2) owing to the two-dimensional nature of the solution and has to be constructed by direct insertion through an ansatz.

(e) Proposed two-dimensional rogue wave model and its dynamics

It is convincingly demonstrated in figure 5 (fixing the free parameters to certain values), how the envelop wave \( |q_{P2D}(x, y, t)| \) corresponding to the exact dynamical two-dimensional lump soliton (3.8), dependent on time \( t \) and two space variables \( x \) and \( y \) on a plane, evolves from a background plane wave existing in the distance past and how it could acquire a sudden two-dimensional hole at the centre \( (x = 0, y = 0) \) at the moment \( t_b = -\sqrt{(1/\mu)}(1 - \bar{c}) \), \( (t_b = -0.83 \text{ for } \bar{c} = \frac{1}{6}, \mu = 1.2 \text{ in figure 5a}) \), as told in marine lore ([7,17] http://wikipedia.org/wiki/Rogue_Wave). The hole subsequently splits into two and shifts apart from the centre (figure 5b), to make space for a high-steep upsurge of the lump forming the full-grown RW (figure 5c) at time \( t = 0 \). Note that we have derived analytically the exact positions of these holes in §3b. With the passage of time, the picture gets reversed and the two-dimensional RW disappears fast into the background waves with the two-dimensional holes merging at the centre and vanishing again. Thus, our model describes vividly well the reported picture of the ocean surface RWs [6,15,21] as well as those found in large-scale two-dimensional experiments [10]. As our model is an exact one, we could work out these details analytically. The surface RWs modelled by our solution (3.8), and as visible from figure 5 (similarly from solution (3.1) and figure 4), show a distinct directional preference and an asymmetry between the two space variables \( x, y \) (similar to the report of [6,15]).
maximum amplitude attained by the full grown RW (as shown in figure 5c) is five times that of the background waves, owing to our choice of $c = 1/6$. Examples of other amplitudes and modular inclination of full-grown RWs for some other choices of the free parameters $c$ and $\alpha$, as modelled by the static solution (3.1), are presented already in figure 4a–d.

4. Physical origin of the proposed two-dimensional nonlinear Schrödinger equation and its integrability

We have shown that the two-dimensional NLS equation (2.2) which is equivalent to a nonlinear equation (2.1) can give an exact model, considerably successful in describing realistic two-dimensional RWs. In this section, we show the direct link of the two-dimensional NLS equation with basic hydrodynamic equations. Moreover, we show the underlying integrable structures of the proposed equation.

(a) Derivation of the integrable two-dimensional nonlinear Schrödinger equation from basic hydrodynamic equations

For emphasizing the physical significance of our main nonlinear integrable equation (2.2), on which the ocean rogue wave model is based, we show its direct link with basic hydrodynamic equations. The procedure is based on a multi-scale expansion, on par with the celebrated equations, such as KdV, NLS, etc. [48,49], though one should include here an extra space dimension with an asymmetric scaling in space variables, considering the perturbative expansion to the next higher order. This is consistent, however, with the modelling of an ocean RW, which is a surface phenomenon with a likely broken space symmetry and directional preference [11].

Before entering into the detailed calculation, three-dimensionless entities $\epsilon = a/h_0$, $\delta = h_0/\lambda_x$ and $\mu = \lambda_x/\lambda_y$ are defined, where $a$ is the maximum amplitude, $h_0$ is the constant water depth, $\lambda_x$ and $\lambda_y$ are the wavelengths of the surface wave along longitudinal and transverse directions, respectively. The nonlinear parameter $\epsilon$ is responsible for the slow evolution of a harmonic wave of wavenumber $k_x, k_y$. The wave is thus slowly modulated as $\epsilon$ tends to 0, and therefore this small parameter can be used for perturbative expansion. Smallness of $\epsilon$ is consistent with the deep-water limit with $a \ll h_0$, and hence with the formation of oceanic RWs. Note that parameters $\epsilon$ and $\delta$ are similar to those appearing in the derivation of the well-known one-dimensional NLS equation, with $\epsilon$ small and $\delta$ without any restriction, as also true in our case. However, an additional parameter $\mu$, also without any restriction on its value appears in our two-dimensional case, owing to the presence of an additional transverse direction.

The first step in the derivation is to write the basic hydrodynamic equations for inviscid, irrotational and incompressible fluid in dimensionless variables for the velocity potential field $\phi(t, x, y)$ and the gravity wave $\eta(t, x, y)$ as the free surface displacement above the mean water depth $h_0$ in the form

$$\phi_{zz} + \delta^2(\phi_{xx} + \mu^2 \phi_{yy}) = 0,$$  \hspace{1cm} (4.1)

at $0 < z < 1 + \epsilon \eta$, which comes from the continuity equation. The equation

$$\phi_z = \delta^2[\eta_t + \epsilon(\phi_x \eta_x + \mu^2 \phi_y \eta_y)],$$  \hspace{1cm} (4.2)

called kinematic condition, is valid on $z = 1 + \epsilon \eta$. The equation

$$\phi_t + \eta + \frac{1}{2} \epsilon \left[ \frac{\phi_x^2}{\delta^2} + \phi_x^2 + \mu^2 \phi_y^2 \right] = 0$$  \hspace{1cm} (4.3)

is Bernoulli’s equation and also valid at $z = 1 + \epsilon \eta$, while

$$\phi_z = 0$$  \hspace{1cm} (4.4)

is the fixed boundary condition valid at $z = 0$, i.e. at the bottom.
where $\omega$, $M_x$ are frequency and velocity parameters to be determined later. Note that the two space variables are treated with a non-symmetric scaling and using this set of variables, equations (4.1)–(4.4) become

\begin{equation}
\phi_{zz} + \delta^2(\delta^2\phi_{\xi\xi} + \epsilon^2\phi_{\zeta\zeta}) + \mu^2\delta^2(k_x^2\phi_{\xi\xi} + 4\epsilon k_x\phi_{\xi\zeta} + 2\epsilon k_y\phi_{\zeta\zeta} + 2\epsilon^2 k_y\phi_{\zeta\zeta}) = 0, \quad \phi_{z\zeta} = 0, \quad \text{at } z = 0.
\end{equation}

Using this expansion of dependent variables along with the scaled independent variables, different sets of equations are obtained from the basic set (4.6)–(4.9) at different powers of $\epsilon$. In each $\epsilon$, different order equations are obtained for various powers of $E$ and $E^*$. We would consider these equations sequentially at each order of parameter $\epsilon$.

(1) $\epsilon^0$ order. The solution of interest in this case takes the form

\begin{equation}
\phi_0 = f_0 + F_0^*E + F_0^{*}E^* \quad \text{and} \quad \eta = A_0^*E + A_0 E^*,
\end{equation}

where $F_0^*(\xi, \zeta, \tau, z)$, $A_0^*(\xi, \zeta, \tau)$ are complex functions with $F_0^*$, $A_0^*$ as complex conjugates, while $f_0(\xi, \zeta, \tau)$ is a real function and $E = \exp(i\xi)$. Using (4.6) and (4.9), $F_0$, can be determined as

\begin{equation}
F_0 = G_0 \cosh(\delta K_1 z), \quad \text{where} \quad G_0 = \frac{-iA_00\omega}{K_1 \sinh(\delta K_1)}, \quad K_1 = \sqrt{k_x^2 + \mu^2k_y^2}.
\end{equation}

Using other two nonlinear boundary conditions (4.7) and (4.8), we obtain the dispersion relation $\omega^2 = (K_1/\delta) \tanh(\delta K_1)$.

(2) $\epsilon$ order. Expanding $\phi_n, \eta_n$ as

\begin{equation}
\phi_n = \sum_{m=0}^{n+1} F_{nm}E^m + c.c. \quad \text{and} \quad \eta_n = \sum_{m=0}^{n+1} A_{nm}E^m + c.c.,
\end{equation}

where $F_{nm}(\xi, \zeta, \tau, z)$ and $A_{nm}(\xi, \zeta, \tau)$ are to be determined for various powers of $E$, at each power of $\epsilon$.

At $\epsilon$ order, the components $F_{10}, F_{11}, F_{12}, A_{10}, A_{11}, A_{12}$ and the velocity parameter $M_x$ are determined from the equations corresponding to $E$, $E^2$ and $E^0$, explicit forms of which are appended in A1.

(3) $\epsilon^2$ order. At this order, an NLS-type equation (space coordinate $Y$ replacing the time coordinate) is obtained, collecting the coefficients of $E$ from (4.7) and (4.8) and by using the quantities already determined. Before calculating the final form of this equation, some other components namely $F_{21}, F_{20}, f_{0\xi}$ at this order need to be evaluated, which are given in appendix A2.
The final form of the NLS-like equation is obtained eliminating the unknown terms and expressing other terms through the single function $A_0$ as

$$i\alpha_1 A_{0t} + \alpha_2 A_{0\xi} + \beta_2 |A_0|^2 A_0 = 0,$$

(4.14)

where the constant coefficients $\alpha_1$, $\alpha_2$ and $\beta_2$ are also given in appendix A2.

Following the same procedure, the components $F_{22}, A_{22}, A_{20}, f_{0Y}$ are determined, which we are not furnishing here owing to their cumbersome expressions.

(4) $e^3$ order. In this order, an evolution equation is obtained, for which some relevant components i.e. $F_{31}, F_{30}$, etc., are also determined by continuing with the same procedure. The explicit forms of these coefficients presented in A3.

The evolution equation obtained by using equations (4.7) and (4.8) and collecting coefficients of $E$ takes the form

$$i A_{0t} + \alpha_3 A_{0\xi Y} + i\beta_3 |A_0|^2 A_{0\xi} + i\beta_{31} |A_0|^2 A_{0\xi} + i\beta_{32} |A_0|^2 A_{0\xi} + i\beta_{31} |A_0|^2 A_{0\xi} + i\beta_{32} |A_0|^2 A_{0\xi} = 0,$$

(4.15)

where $\alpha, \alpha_3, \beta_3, e, f, \alpha_3$ are real constants dependent on parameters $k_x, k_y, \mu, \delta$.

If it is assumed that the term $G_{11}$ depends also on $A_0$ like the other terms as $F_0 \sim A_0$ and $G_{12}, G_{12} \sim A_0^2$, etc. (see appendix), then the only consistent relation would be $G_{11} = P_1 A_{0t}$, where $P_1$ is a real constant, dependent only on $k_x, k_y, \mu, \delta$. Using this relation in (4.15), one simplifies it in the form

$$i A_{0t} + \alpha_3 A_{0\xi Y} + i\alpha_3 |A_0|^2 A_{0\xi} + i(\beta_{31} |A_0|^2 A_{0\xi} + \beta_{32} |A_0|^2 A_{0\xi}) = 0,$$

(4.16)

where $\beta_{31}, \beta_{32}$, are another set of constant coefficients expressed through earlier coefficients. Note that equation (4.16) is similar to but not the same as our integrable two-dimensional NLS equation owing to the appearance of the term $i\alpha_3 A_{0\xi Y}$. However, fortunately we have another equation (4.14) at our disposal, obtained at a lower order. Taking derivative of (4.14) with respect to $\zeta$, we derive the relation

$$i \alpha_2 A_{0\xi Y} = \alpha_1 A_{0\xi Y} - i\beta_2 (|A_0|^2 A_0)_{\xi},$$

(4.17)

using which we can eliminate this unwanted term from (4.16) to obtain an equation in the form

$$i C_0 A_{0t} + C_1 A_{0\xi Y} + i C_2 A_0 (A_0 A_{0\xi}^* - A_{0\xi}^* A_0) = 0,$$

(4.18)

under the condition on the coefficients of the original equation as

$$\frac{\beta_2}{\alpha_2} = \frac{(\beta_{32} + \beta_{31})}{3\alpha_3}.$$

Rescaling $\zeta, Y$ and $\tau$ and renaming $A_0$ equation (4.18) goes directly to the two-dimensional NLS equation (2.2), which is equivalent to (2.1) proposed by us. Note that for constraint (4.19), we have to impose for deriving our integrable two-dimensional NLS equation from the basic hydrodynamic equations, though it does not hold for general water wave problems, this loss of generality is compensated for by the gain of our important exact results. This in general is true for all integrable models.

(b) Integrable structures of the proposed equation

We present here the associated integrability properties of equation (2.3). The one-soliton solution of this equation is given in the form $q_{nl}(x, y, t) = \text{sech}^{2}(y) \times v(t)$, while also allowing higher soliton solutions and an infinite set of conserved quantities [41,42]. One can also
find the associated linear system

$$\Phi_y = U(\lambda)\Phi \quad \text{and} \quad \Phi_t = V(\lambda)\Phi,$$

with a Lax pair given by

$$U(\lambda) \equiv V_2(\lambda) = 2\lambda V_1(\lambda) + V_2^{(0)} \quad \text{and} \quad V(\lambda) \equiv V_3(\lambda) = 2\lambda V_2(\lambda) + V_3^{(0)},$$

(4.20)

where

$$V_1(\lambda) = i(\lambda\sigma^3 + U^{(0)}), \quad V_2^{(0)} = \sigma^3(U_x^{(0)} - iU_t^{(0)})$$

and

$$V_3^{(0)} = -\sigma^3 U_y^{(0)} - [U^{(0)}, U_x^{(0)}], \quad U^{(0)} = q\sigma^+ + q^*\sigma^-$$

(4.21)

with $\sigma^a, a = \pm, 3$, Pauli matrices, the flatness condition $U_t - V_y + [U, V] = 0$, of which generates our two-dimensional NLS equation (2.2). Note that unlike the known Lax pair of the one-dimensional NLS, the pair $U(\lambda), V(\lambda)$ associated to our system have higher order dependence on the spectral parameter $\lambda$. It is not difficult to show that the flatness condition yields from (4.21) different relations at different powers of $\lambda$. The equation linked to the $\lambda$ corresponds to our (2 + 1)-dimensional NLS equation (2.2), while the relation with $\lambda^0$ gives another intriguing nonlinear equation

$$i\phi_{xt} + q_{yy} + 2i|q|^2q_y + 2qq_x(q^*q_x - q^*q_x) = 0.$$ 

(4.22)

Our main concern here however is the two-dimensional NLS equation (2.2), which we intend to use for constructing a two-dimensional RW model. Note, however, the modification of (2.2) by the addition of the current term as considered in the §3d yields exact analytic RW solution no longer remains integrable in the sense described here.

5. Concluding remarks

We conclude by listing a few distinguishing features of our proposed dynamical lump soliton (3.8), which are important for a realistic ocean RW model.

(1) This is the first two-dimensional dynamical RW model given in an analytic form.

(2) It is a two-dimensional extension of a Peregrine-like soliton, representing an exact lump soliton linked to a novel (2 + 1)-dimensional integrable NLS equation, derivable from the basic hydrodynamic equations.

(3) The dynamics of the RW solution is induced by an ocean current term and controlled by it. Importance of the current in the formation of RW is strongly emphasized [6,10], though perhaps for the first time this effect is attempted to be analysed analytically in two dimensions in our model.

(4) Both the height and the inclination of the single peak RW are adjustable by two independent free parameters present of our model.

(5) The fastness of appearance of the RW and the duration of its stay can be regulated by yet another parameter linked to the ocean current.

(6) The proposed solution and MI exhibit broken spatial symmetry as well as a directional preference, which are suspected to be the crucial features in the formation of a two-dimensional RW [10,11,17,18]. Note again that these features obtained earlier through observation or numerical simulation found and confirmed in our model through exact analytic result.

(7) Strange appearance (and disappearance) of a two-dimensional hole just before (and after) the formation of the rogue wave ([17,21], http://wikipedia.org/wiki/Rogue_Wave) is also confirmed in our model, graphically as well as by analytic findings.

In comparison, the original Peregrine soliton (1.2) (together with its higher order solutions), by far the most popular model of the RW, does not exhibit most of these essential properties, owing to its inherently one-dimensional nature and the absence of free parameters. Therefore,
while the class of Peregrine solitons are successful in modelling one-dimensional RW-like structures observed in many experiments, the two-dimensional RW model reported here should complement it, to stand close to a realistic model for ocean surface RWs.

We hope that this breakthrough in describing large ocean RWs by an analytic dynamical lump soliton with adjustable height, inclination and duration would also be valuable for experimental findings of two-dimensional RWs in other systems, such as capillary fluid waves [13] optical cavity waves [11] and basin water waves [10]. Derivation of our exact lump soliton from the breather solution of the integrable two-dimensional NLS equation presented here in a systematic way as well as to find higher order rational lump solitons would be challenging theoretical problems.

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Appendix A

A1: coefficients appearing in order \( \epsilon \):

\[
F_{10} = G_{10}(\xi, \eta, \tau), \quad A_{10} = M_2 f_0 \xi - \frac{2\delta K_1}{\sinh(2\delta K_1)}|A_0|^2,
\]

\[
F_{12} = G_{12} \cosh(2\delta K_1 \xi), \quad \text{where } G_{12} = -\frac{3\omega \delta^2}{4 \sinh^4(\delta K_1)} A_0^2,
\]

\[
A_{12} = \frac{\delta K_1 \cosh(\delta K_1)}{2[\sinh(\delta K_1)]^3} [1 + 2 \cosh^2(\delta K_1)] A_0^2,
\]

\[
F_{11} = G_{11} \cosh(\delta K_1 \xi - \frac{i\delta K_x}{K_1} G_{0\xi} \xi \sinh(\delta K_1 \xi),
\]

\[
A_{11} = i\omega \left[ G_{11} \cosh(\delta K_1 \xi - \frac{i\delta K_x}{K_1} G_{0\xi} \xi \sinh(\delta K_1 \xi) + M_2 |G_{0\xi} \cosh(\delta K_1)| \right].
\]

The velocity parameter:

\[
M_2 = \frac{\omega K_1}{2K_1^2} \left[ 1 + \frac{2\delta K_1}{\sinh(2\delta K_1)} \right].
\]

A2: coefficients appearing in order \( \epsilon^2 \):

\[
F_{21} = G_{21} \cosh(\delta K_1 \xi - \frac{i\delta K_x}{K_1} G_{11\xi} \xi \sinh(\delta K_1 \xi) - \frac{i\delta K_y}{K_1} \mu^2 G_{0\eta} \eta \sinh(\delta K_1 \xi)
\]

\[
+ G_{0\xi} \left[ \left(-\frac{\delta}{2K_1}\right) \xi \sinh(\delta K_1 \xi) + \left(\frac{\delta K_x^2}{2K_1^2}\right) \xi \sinh(\delta K_1 \xi) - \left(\frac{\delta K_y^2}{2K_1^2}\right) \xi^2 \cosh(\delta K_1 \xi) \right],
\]

\[
F_{20} = -\delta^2 f_{0\xi} \frac{\xi^2}{2} + G_{10}(\xi, \eta, \tau),
\]

\[
\alpha_1 = -k_y \mu^2 \tanh(K_1 \delta) \left[ \frac{2K_1 \delta + \sinh(2K_1 \delta)}{2\omega \cosh^2(K_1 \delta)} \right],
\]

\[
\alpha_2 = \frac{\delta}{2\omega K_1^3} \left[ K_1^3 \delta \left\{ 2M_2^2 - \frac{1}{\cosh^2(K_1 \delta)} \right\} - K_1^2 \tanh(K_1 \delta) + k_x \tanh(K_1 \delta)
\]

\[
+ 4K_1 k_x M_2 \delta^3 \omega^3 - K_1 k_x^2 \delta[1 + \tanh^2(K_1 \delta)] \right],
\]
A3: coefficients appearing in order $\epsilon^3$:

$$ F_{30} = -\delta^2 F_{10}\zeta \frac{z^2}{2} + G_{30}(\zeta, Y, \tau), \quad (A\, 1) $$

$$ F_{31} = G_{31} \cosh(\delta K_1 z) + G_{21\zeta} \left\{ \left( \frac{-i\delta k_x}{K_1} \right) z \sinh(\delta K_1 z) \right\} $$

$$ + G_{11\zeta} \left\{ \left( \frac{i\delta k_x}{K_1} \right)^2 \frac{z^2}{2} \cosh(\delta K_1 z) - \left( \frac{i\delta k_x}{K_1} \right)^2 \left( \frac{z}{2\delta K_1} \right) \sinh(\delta K_1 z) \right\} $$

$$ - \left( \frac{\delta}{2K_1} \right) z \sinh(\delta K_1 z) + \left( \frac{i\delta k_x}{K_1} \right) G_{0\zeta\zeta} \left\{ \left( \frac{\delta}{2K_1} \right) \left( \frac{z^2}{2} \right) \cosh(\delta K_1 z) \right\} $$

$$ - \left( \frac{\delta}{2K_1} \right) \left( \frac{z}{2\delta K_1} \right) \sinh(\delta K_1 z) - \left( \frac{\delta}{2K_1} \right) \left( \frac{\delta}{2K_1} \right) \left( \frac{z}{2\delta K_1} \right) \sinh(\delta K_1 z) $$

$$ + \left( \frac{\delta}{K_1} \right)^2 \left( \frac{z}{2\delta K_1} \right)^2 \sinh(\delta K_1 z) - \frac{\delta^2 k_x^2}{2K_1^2} \left( \frac{z}{2\delta K_1} \right) \sinh(\delta K_1 z) $$

$$ + \frac{\delta^2 k_y^2}{3} \sinh(\delta K_1 z) - \frac{\delta^2 k_x^2}{2K_1^2} \left( \frac{z}{2\delta K_1} \right) \cosh(\delta K_1 z) + \frac{\delta^2 k_y^2}{2K_1^2} \left( \frac{z}{2\delta^2 K_1^2} \right) \sinh(\delta K_1 z) \right\} $$

$$ + G_{0\zeta Y} \left\{ \left( \frac{i\delta k_y}{K_1} \right) \left( \frac{i\delta k_y}{K_1} \right) \left[ \mu^2 z^2 \cosh(\delta K_1 z) - \mu^2 \sinh(\delta K_1 z) \left( \frac{z}{\delta K_1} \right) \right] \right\} $$

$$ + G_{11Y} \left\{ -\left( \frac{i\delta k_y}{K_1} \right) \mu^2 z \sinh(\delta K_1 z) \right\}. $$

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