Concurrence Vectors in Arbitrary Multipartite Quantum Systems

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Abstract

For a given pure state of multipartite system, the concurrence vector is defined by employing the defining representation of generators of the corresponding rotation groups. The norm of a concurrence vector is considered as a measure of entanglement. For multipartite pure state, the concurrence vector is regarded as the direct sum of concurrence subvectors in the sense that each subvector is associated with a pair of particles. It is proposed to use the norm of each subvector as the contribution of the corresponding pair in entanglement of the system.

Keywords: Quantum entanglement, Concurrence vector, Orthogonal group
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1 Introduction

Quantum entanglement, as the most intriguing features of quantum mechanics, has been investigated for decades in relation with quantum nonseparability and the violation of Bell’s inequality [1, 2, 3]. In the last decade it has been regarded as a valuable resource for quantum communications and information processing [4, 5, 6], so, as with other resources such as free energy and information, quantification of entanglement is necessary to understand and develop the theory.

From the various measures proposed to quantify entanglement, the entanglement of formation has been widely accepted which in fact intends to quantify the resources needed to create a given entangled state [6]. In the case of pure state, if the density matrix obtained from the partial trace over other subsystems is not pure the state is entangled. Consequently, for the pure state $|\psi\rangle$ of a bipartite system, entropy of the density matrix associated with either of the two subsystems is a good measure of entanglement

$$E(\psi) = -\text{Tr}(\rho_A \log_2 \rho_A) = -\text{Tr}(\rho_B \log_2 \rho_B),$$

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where $\rho_A = \text{Tr}_B (|\psi\rangle \langle \psi|)$ and $\rho_B$ is defined similarly. Due to classical correlations existing in the mixed state each subsystem can have non-zero entropy even if there is no entanglement, therefore the von Neumann entropy of a subsystem is no longer a good measure of entanglement. For a mixed state, the entanglement of formation (EoF) is defined as the minimum of an average entropy of the state over all pure state decompositions of the state [6],

$$E_f (\rho) = \min \sum_i p_i E (\psi_i). \quad (2)$$

Although the entanglement of formation has most widely been accepted as an entanglement measure, there is no known explicit formula for the EoF of a general state of bipartite systems except for $2 \otimes 2$ quantum systems [7] and special types of mixed states with definite symmetry such as isotropic states [8] and Werner states [9]. Remarkably, Wootters has shown that EoF of a two qubit mixed state $\rho$ is related to a quantity called concurrence as [7]

$$E_f (\rho) = H \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - C^2} \right), \quad (3)$$

where $H(x) = -x \ln x - (1 - x) \ln (1 - x)$ is the binary entropy and concurrence $C(\rho)$ is defined by

$$C(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \quad (4)$$

where the $\lambda_i$ are the non-negative eigenvalues, in decreasing order, of the Hermitian matrix $R \equiv \sqrt{\rho\rho^\dagger}$ and $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$, where $\sigma_y$ is the Pauli matrix in a local basis $\{|1\rangle, |2\rangle\}$. Furthermore, the EoF is monotonically increasing function of the concurrence $C(\rho)$, so one can use concurrence directly as a measure of entanglement. For pure state $|\psi\rangle = a_{11} |11\rangle + a_{12} |12\rangle + a_{21} |21\rangle + a_{22} |22\rangle$, the concurrence takes the form

$$C(\psi) = |\langle \psi | \tilde{\psi} \rangle| = 2 |a_{11} a_{22} - a_{12} a_{21}|. \quad (6)$$

Because of the relation between concurrence and entanglement of formation it is, therefore, interesting to ask whether concurrence can be generalized to larger quantum systems. Indeed attempts have been made to generalize the definition of concurrence to higher dimensional composite systems [10, 11, 12, 13, 14, 15, 16, 17]. Uhlmann generalized the concept of concurrence by considering arbitrary conjugations acting on arbitrary Hilbert spaces [10]. His motivation is based on the fact that the tilde operation on a pair of qubits is an example of conjugation, that is, an antiunitary operator whose square is the identity. Rungta et al defined the so-called I-concurrence in terms of a universal-inverter which is a generalization to higher dimensions of two qubit spin flip operation, therefore, the pure state concurrence in arbitrary dimensions takes the form [11]

$$C(\psi) = \sqrt{\langle \psi | S_{N_1} \otimes S_{N_2} (|\psi\rangle \langle \psi|) |\psi\rangle} = \sqrt{2(1 - \text{Tr}(\tilde{\rho}_A^2))}. \quad (7)$$

Another generalization is proposed by Audenaert et al [12] by defining a concurrence vector in terms of a specific set of antilinear operators. As pointed out by Wootters, it turns out that the length of the concurrence vector is equal to the definition given in Eq. (7) [18].
Albeverio and Fei also generalized the notion of concurrence by using invariants of local unitary transformations as [13]

\[ C(\psi) = \sqrt{\frac{N}{N-1}} (I_0^2 - I_1^2) = \sqrt{\frac{N}{N-1}} (1 - Tr(\rho_2^2)), \]

which turns out to be the same as that of Rungta et al up to a whole factor. In Eq. (8) \( I_0 \) and \( I_1 \) are two former invariants of the group of local unitary transformations. As a complete characterization of entanglement of a bipartite state in arbitrary dimensions may require a quantity which, even for pure states, does not reduce to single number [19, 20, 21, 22, 23], Fan et al defined the concept of a concurrence hierarchy as \( N - 1 \) invariants of a group of local unitary for \( N \)-level systems [14]. Badziag et al [15] also introduced multidimensional generalization of concurrence. Recently, Li et al used a fundamental representation of \( A_N \) Lie algebra and proposed concurrence vectors for a bipartite system of arbitrary dimension as [16]

\[ C = \{ \langle \psi | (E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta}) | \psi^* \rangle | \alpha, \beta \in \Delta^+ \} \],

where \( \Delta^+ \) denotes the set of positive roots of \( A_{N-1} \) Lie algebra. An extension of the notion of Wooters concurrence to multi-qubit systems is also proposed in Ref. [17].

In this contribution, I generalize the notion of concurrence vectors to arbitrary multipartite systems. The motivation is based on the fact that Wootters concurrence of a pair of qubits can be obtained by defining the tilde operation as \( |\tilde{\psi}\rangle = S \otimes S |\psi^*\rangle \) instead of \( |\tilde{\psi}\rangle = \sigma_y \otimes \sigma_y |\psi^*\rangle \), where here \( S \) is the only generator of rotation group \( SO(2) \) in such basis that \( (S)_{ij} = \epsilon_{ij} \) where \( \epsilon_{12} = -\epsilon_{21} = 1 \) and \( \epsilon_{11} = \epsilon_{22} = 0 \). Therefore, a natural generalization of spin flip operation for arbitrary bipartite systems leads to a vector whose components are obtained by employing tensor product of generators of the corresponding rotation groups. A suitable generalization of the definition for multipartite system is also proposed by defining concurrence vector as a direct sum of concurrence subvectors in the sense that each subvector corresponds to one pair of particles. Therefore, it is proposed to use the norm of each subvector as a measure of entanglement shared between the corresponding pair of particles. A criterion for the separability of bipartite states is then given as: a state is separable if and only if the norm of its concurrence vector vanish. For a multipartite systems the vanishing of the concurrence vectors is necessary but not sufficient condition for separability.

The paper is organized as follows: In section 2, the definition of concurrence vectors is given. In section 3, the generalization of the concurrence vector for multipartite system is proposed. Some examples are also considered in this section. The paper is concluded in section 4 with a brief conclusion.

## 2 Concurrence vectors for bipartite pure states

In this section, we give a generalization of the concurrence for an arbitrary bipartite pure state. For motivation, let us first consider a pure state \( |\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^3 \) with the following generic form,

\[ |\psi\rangle = \sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij} |e_i \otimes e_j\rangle, \]

where \( |e_i\rangle \ (i = 1, 2) \) and \( |e_j\rangle \ (j = 1, 2, 3) \) are orthonormal real basis of Hilbert space \( \mathbb{C}^2 \) and \( \mathbb{C}^3 \) respectively. Of course by means of the Schmidt decomposition one can consider \( |\psi\rangle \) as
a vector in a $\mathbb{C}^2 \otimes \mathbb{C}^2$ Hilbert space, but to see the main idea of the paper we do not use Schmidt decomposition. It can be easily seen that the entanglement of $|\psi\rangle$ can be written as $E_f(\psi) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - C^2}\right)$, where concurrence $C$ is defined by

$$C = 2\sqrt{|a_{12}a_{23} - a_{13}a_{22}|^2 + |a_{11}a_{23} - a_{13}a_{21}|^2 + |a_{11}a_{22} - a_{12}a_{21}|^2}.$$  (11)

On the other hand, Eq. (11) can also be written as

$$C = \sqrt{\sum_{\alpha=1}^{3} |\langle \psi|\tilde{\psi}_\alpha \rangle|^2},$$  (12)

where $|\tilde{\psi}_\alpha\rangle$ are defined by

$$|\tilde{\psi}_\alpha\rangle = (S \otimes L_\alpha)|\psi^*\rangle,$$  (13)

where $S$ is the only generator of two-dimensional rotation group $SO(2)$ with matrix elements $(S)_{ij} = \epsilon_{ij}$ and $L_\alpha$ with matrix elements $(L_\alpha)_{jk} = \epsilon_{\alpha jk}$ denote three generators of an $SO(3)$ group. Here $\epsilon_{ij}$ is defined by $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$ and $\epsilon_{\alpha jk}$ is antisymmetric under interchange of any two indices and $\epsilon_{123} = 1$.

Similarly, for pure state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^N$ with the generic form

$$|\psi\rangle = \sum_{i=1}^{2} \sum_{j=1}^{N} a_{ij} |e_i \otimes e_j\rangle,$$  (14)

entanglement $E(\psi)$ is obtained by Eq. (3) with the following:

$$C = 2\sqrt{\sum_{j<k}^{N} |a_{1j}a_{2k} - a_{2j}a_{1k}|^2}.$$  (15)

It is straightforward to see that Eq. (15) can be expressed as

$$C = \sqrt{\sum_{\alpha=1}^{N(N-1)/2} |\langle \psi|\tilde{\psi}_\alpha \rangle|^2}.$$  (16)

Here $|\tilde{\psi}_\alpha\rangle = (S \otimes L_\alpha)|\psi^*\rangle$, $\alpha = 1, ..., N(N-1)/2$, where $L_\alpha$ are generators of $SO(N)$ group with matrix elements $(L_\alpha)_{kl} = (L_{[j_1j_2...j_{N-2}])}_{kl} = \epsilon_{[j_1j_2...j_{N-2}]}_{kl}$ where $\alpha$ is used to denote the set of $N-2$ indices $[j_1j_2...j_{N-2}]$ with $1 \leq j_1 < j_2 < \cdots < j_{N-2} \leq N$ in order to label $N(N-1)/2$ generators of $SO(N)$, and $\epsilon_{[j_1j_2...j_{N-2}]}$ is antisymmetric under interchange of any two indices with $\epsilon_{12...N} = 1$. To achieve Eq. (15) from Eq. (16) we used the following equations:

$$\epsilon_{kl\ell k'\ell'} = \delta_{kk'}\delta_{ll'} - \delta_{kl}\delta_{k'l},$$  (17)

$$\sum_{1\leq j_1 < j_2 < \cdots < j_{N-2} \leq N} \epsilon_{[j_1j_2...j_{N-2}]}\epsilon_{[j_1j_2...j_{N-2}]} = \delta_{kk'}\delta_{ll'} - \delta_{kl}\delta_{k'l}.$$  (18)

Next, to generalize the above definition of concurrence for an arbitrary bipartite pure state let $|\psi\rangle$ be a pure state in Hilbert space $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ with following decomposition

$$|\psi\rangle = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} a_{ij} |e_i \otimes e_j\rangle,$$  (19)
Now we define concurrence vector $C$ with components $C_{\alpha\beta}$ as

$$C_{\alpha\beta} = \langle \psi | \tilde{\psi}_{\alpha\beta} \rangle, \quad \langle \tilde{\psi}_{\alpha\beta} | = (L_\alpha \otimes L_\beta) | \psi^* \rangle. \quad (20)$$

where $L_\alpha, \alpha = 1, \ldots, N_1(N_1 - 1)/2$ and $L_\beta, \beta = 1, \ldots, N_2(N_2 - 1)/2$ are generators of $SO(N_1)$ and $SO(N_2)$ respectively. Now the norm of the concurrence vector can be defined as a measure of entanglement, i.e.,

$$C = |C| = \sqrt{\sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} |C_{\alpha\beta}|^2}. \quad (21)$$

By using Eq. (18) we can evaluate concurrence in terms of parameters $a_{ij}$ where we get

$$C = 2 \sqrt{\sum_{i<j}^{N_1} \sum_{k<l}^{N_2} |a_{ik}a_{jl} - a_{il}a_{jk}|^2}. \quad (22)$$

It is clear that $C(\psi)$ is zero when $|\psi\rangle$ is factorizable, i.e., $a_{ij} = b_i c_j$ for some $b_i, c_j \in \mathbb{C}$. On the other hand, $C$ takes its maximum value $\sqrt{2(N-1)/N}$ with $N = \min(N_1, N_2)$, when $|\psi\rangle$ is a maximally entangled state. It should be noted that the result is the same as that obtained in [13], up to a whole factor, therefore it is also in accordance with the result obtained from the definition given in [11]. It should also be mention that the result is equal to the bipartite pure state concurrence that Badsiag et al have defined in terms of a trace norm of the concurrence matrix [15]. Interestingly, components of the concurrence vector given in Eq. (20) are same as elements of the concurrence matrix of Ref. [15]. As a matter of fact, the definition given in Eq. (20) for concurrence vectors is closely related to the definition proposed in Ref. [16]. Actually, all bipartite generalization of the concurrence lead to Eq. (22). However, our objective here is to generalize the definition for multipartite systems.

## 3 Concurrence vectors for multipartite systems

In order to further generalize the concept of concurrence vector to multipartite systems, let us first analyze the problem that arises in the definition of the pairwise entanglement between the particles. In Eq. (20) $|\psi^*\rangle$ can also be written as $|\psi^*\rangle = (K_1 \otimes K_2) |\psi\rangle$ where $K_1$ and $K_2$ are the complex conjugation operators acting in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. Although the action of the direct product of two anti-unitary transformation $K_1 \otimes K_2$ on a general ket $|\psi\rangle \in \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ can be properly defined, the combination of an anti-unitary and a unitary transformation such as $K_1 \otimes K_2 \otimes I_3$ cannot be properly defined on a general ket $|\psi\rangle \in \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \otimes \mathbb{C}^{N_3}$ except that $|\psi\rangle$ is factorized as $|\psi\rangle = |\psi_{12}\rangle \otimes |\psi_3\rangle$ where $|\psi_{12}\rangle \in \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ and $|\psi_3\rangle \in \mathbb{C}^{N_3}$. This ambiguity can be removed in the Hilbert Schmidt basis [24] of the corresponding system with

$$\rho^{T_{12}} = (K_1 \otimes K_2 \otimes I_3) \rho(K_1 \otimes K_2 \otimes I_3), \quad (23)$$

for any $\rho = |\psi\rangle \langle \psi|$, whether $|\psi\rangle$ is factorizable or not. In Eq. (23) $\rho^{T_{12}}$ is the partial transpose of $\rho$ with respect to particles 1 and 2.

Now to generalize the concept of a concurrence vector to multipartite systems, let us consider an $m$-partite pure state $|\psi\rangle \in \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \otimes \cdots \otimes \mathbb{C}^{N_m}$ which in the standard basis
has the following decomposition:

\[ |\psi\rangle = \sum_{i_1}^{N_1} \sum_{i_2}^{N_2} \cdots \sum_{i_m}^{N_m} a_{i_1i_2\cdots i_m} |e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m}\rangle. \]  

(24)

We first define the pairwise entanglement between the particles. Let \( \rho = |\psi\rangle \langle \psi| \) be the density matrix corresponding to pure state (24). With \( \rho^{T_{ij}} \), we denote the matrix obtained from \( \rho \) by partial transposition with respect to subsystems \( i \) and \( j \), i.e.,

\[ \rho^{T_{ij}} = (|\psi\rangle \langle \psi|)^{T_{ij}}. \]  

(25)

Next, we define \( \Pi_{i=1}^m N_i(N_i - 1)/2 \) dimensional concurrence vector \( C \) with components \( C_{\alpha_i\alpha_j}^{(ij)} \) as

\[ C_{\alpha_i\alpha_j}^{(ij)} = \sqrt{\langle \psi | \hat{\rho}_{\alpha_i\alpha_j}^{(ij)} | \psi \rangle}, \]

(26)

where

\[ \hat{\rho}_{\alpha_i\alpha_j}^{(ij)} = M_{\alpha_i\alpha_j}^{(ij)} \rho^{T_{ij}} M_{\alpha_i\alpha_j}^{(ij)} \]

(27)

with

\[ M_{\alpha_i\alpha_j}^{(ij)} = I_1 \otimes \cdots \otimes I_{i-1} \otimes L_{\alpha_i} \otimes I_{i+1} \otimes \cdots \otimes I_{j-1} \otimes L_{\alpha_j} \otimes I_{j+1} \otimes \cdots \otimes I_m, \]

(28)

for \( 1 \leq i < j \leq m \), \( \alpha_i = 1, \ldots, N_i(N_i - 1)/2 \) and \( \alpha_j = 1, \ldots, N_j(N_j - 1)/2 \). Here \( I_k \) denotes the identity matrix in the Hilbert space of particle \( k \), and \( L_{\alpha_i} \) represents the set of \( N_i(N_i - 1)/2 \) generators of an \( SO(N_i) \) group with the following matrix elements

\[ \langle k_i | L_{\alpha_i} | l_i \rangle = (L_{[i_1i_2\cdots i_{N_i-2}]})_{k_i l_i} = \epsilon_{[i_1i_2\cdots i_{N_i-2}]k_i l_i}, \]

\[ 1 \leq i_1 < i_2 < \cdots < i_{N_i-2} \leq N_i, \]

(29)

and \( L_{\alpha_j} \) are generators of an \( SO(N_j) \) group with similar a definition. The concurrence vector \( C \) is defined in such a way that it involves all two-level entanglement shared between all pairs of particles. Moreover, we can consider vector \( C \) as a direct sum of elementary subvectors \( C^{(ij)} \), i.e.

\[ C = \bigoplus_{ij} C^{(ij)}, \]

(30)

such that each subvector \( C^{(ij)} \) corresponds to pair \( i \) and \( j \) of particles. Accordingly the entanglement contribution of pair \( i \) and \( j \) in the entanglement of \( |\psi\rangle \) can be defined as the norm of the concurrence subvector \( C^{(ij)} \), that is

\[ C^{(ij)} = |C^{(ij)}| = \sqrt{\sum_{\alpha_i=1}^{N_i(N_i-1)/2} \sum_{\alpha_j=1}^{N_j(N_j-1)/2} \langle \psi | \hat{\rho}_{\alpha_i\alpha_j}^{(ij)} | \psi \rangle} \]

\[ = \left\{ \sum_{\{K\}} \sum_{\{L\}} \sum_{k_i \in K} \sum_{k_j \in L} |a_{k_i k_j K} a_{l_i l_j L} - a_{k_i k_j K} a_{l_i l_j K} - a_{k_i k_j L} a_{k_i k_j K} + a_{k_i k_j K} a_{k_i k_j L}|^2 \right\}^{1/2}, \]

(31)

where in the last line we used the following equations:

\[ \sum_{[\alpha_i]} \epsilon_{[\alpha_i]k_i l_i} \epsilon_{[\alpha_i]k'_i l'_i} = \delta_{k_i k'_i} \delta_{l_i l'_i} - \delta_{k_i l'_i} \delta_{k'_i l_i}, \]

\[ \sum_{[\alpha_j]} \epsilon_{[\alpha_j]k_i l_i} \epsilon_{[\alpha_j]k'_i l'_i} = \delta_{k_i k'_i} \delta_{l_i l'_i} - \delta_{k_i l'_i} \delta_{k'_i l_i}. \]

(32)
In Eq. (31), \( \{k_i, k_j, K\} \) stands for \( m \) indices such that \( k_i \) and \( k_j \) correspond to subsystems \( i \) and \( j \) respectively, and \( K \) denotes the set of \( m - 2 \) indices for other subsystems. Also \( \sum_{\{K\}} \) stands for summation over indices of all subsystems except subsystems \( i \) and \( j \).

It can be shown that \( C^{(ij)} \) is zero when \( \ket{\psi} \) is factorizable among index \( i \) and the rest of the system, i.e., when there exist some \( b_{k_i}, c_{k_j, K} \in \mathbb{C} \) such that \( a_{\{k_i, k_j, K\}} = b_{k_i} c_{k_j, K} \). A similar statement is also correct when \( \ket{\psi} \) is factorizable among index \( j \) and the rest of the system. Also when two subsystems \( i \) and \( j \) are disentangled from the rest of the system, i.e., \( a_{\{k_i, k_j, K\}} = b_{k_i, k_j} c_{k_j} \) for some \( b_{k_i, k_j}, c_{k_j} \in \mathbb{C} \), Eq. (31) takes the form of Eq. (22), as we expect. This feature of \( C^{(ij)} \) shows that it can be considered as the pairwise entanglement between the subsystems \( i \) and \( j \).

Finally, the total concurrence of \( \ket{\psi} \) may be defined as the norm of the concurrence vector \( C \), i.e.,

\[
C = |C| = \sqrt{\sum_{1 \leq i < j \leq m} |C^{(ij)}|^2} = \left\{ \sum_{1 \leq i < j \leq m} \sum_{\{K\}} \sum_{\{L\}} \sum_{k_i < k_L} \sum_{k_j < k_L} \left[ a_{\{k_i, k_j, K\}} a_{\{l, k_j, L\}} - a_{\{k_i, l, K\}} a_{\{k_j, k_j, L\}} - a_{\{k_i, k_j, K\}} a_{\{l, k_j, L\}} + a_{\{l, l, K\}} a_{\{l, l, L\}} \right]^2 \right\}^{1/2}.
\]

It is clear that if \( \ket{\psi} \) is completely separable, if \( a_{i_1 i_2 \cdots i_m} = a_{i_1} b_{i_2} \cdots \) for some \( a_{i_1}, b_{i_2}, \cdots \in \mathbb{C} \), then all \( C_{a_{i_1} a_{i_2}} \) are zero and entanglement of the state becomes zero. This is, of course, the first condition that any good measure of entanglement should satisfy. Furthermore, the entanglement measure should be invariant under local unitary transformation, and its expectation should not increase under local operation and classical communication (LOCC). First, it should be mentioned that for bipartite systems, where Eq. (22) gives concurrence of the state, the above conditions are satisfied. This follows from the fact that Eq. (22) is (up to a whole factor) the concurrence which is introduced by Albeverio et al in terms of local invariants [13]. On the other hand, it is also equal to two-level concurrence defined in Ref. [14]. In [14], Fan et al have defined a hierarchy of concurrence for bipartite pure states and have shown that they are entanglement monotones, i.e., cannot increase under LOCC. Now, what can we say about monotonicity of the concurrence for multipartite states? For any state we define a concurrence vector the norm of which quantifies entanglement of the state. The definition is such that for multipartite case, the concurrence vector can be regarded as a direct sum of concurrence subvectors (Eq. (30)) in such a way that the norm of each subvector denotes pairwise entanglement shared between particles. However, it should be stressed that this pairwise entanglement (Eq. 31) is not obtained through tracing out the remaining subsystems, but its definition comes directly from a generalization of the bipartite case. The definition is simple and computationally straightforward, and allows the concurrence to be read off from the state. Our conjecture is that such defined measure for multipartite systems is non-increasing under LOCC; however, its proof is still in progress.

Next, to demonstrate the nature of this measure, we give some multi-qubit examples in the following.

First, let us consider a three-qubit system with GHZ-state: \( \ket{GHZ_3} = (|000\rangle + |111\rangle) / \sqrt{2} \). For this state Eqs. (31) and (33) give \( C^{(ij)} = 1/\sqrt{2} \) (for all \( i, j \)), and \( C = \sqrt{3}/2 \) respectively.

As another example, let us consider the W-state and anti W-state defined with \( \ket{W_3} = (|110\rangle + |101\rangle + |011\rangle) / \sqrt{3} \) and \( \ket{\bar{W}_3} = (|001\rangle + |010\rangle + |100\rangle) / \sqrt{3} \), respectively. For these states we obtain \( C^{(ij)} = 2/3 \) (for all \( i, j \)), and \( C = 2/\sqrt{3} \).
For $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \otimes (|\alpha_3\rangle|0\rangle + |\beta_3\rangle|1\rangle)$ with $|\alpha_3|^2 + |\beta_3|^2 = 1$ we obtain $C^{(12)} = 1$, $C^{(13)} = C^{(23)} = 0$ and $C = 1$. The above examples show that $C(|GHZ_3\rangle) > C(|W_3\rangle) = C(|\tilde{W}_3\rangle) > C(|EPR_{12}\rangle \otimes |\psi_3\rangle)$. Actually, in an m-qubit system, the generalized GHZ-state $|GHZ_m\rangle = (|0\rangle^\otimes m + |1\rangle^\otimes m)/\sqrt{2}$ has the maximum total concurrence equal to $C(|GHZ_m\rangle) = \sqrt{m(m-1)/2}$.

Let us now consider two superposition states which are considered in Ref. [25]. First consider a superposition of the W-state and anti W-state, i.e., $|W\tilde{W}(s,\phi)\rangle \equiv \sqrt{s} |W\rangle + \sqrt{1-s} e^{i\phi} |\tilde{W}\rangle$. For this state, we obtain $C^{(ij)} = \frac{2}{3} \sqrt{\frac{3}{2} s(s-1)} + 1$ and $C = \sqrt{3} C^{(ij)}$. This result is in complete agreement with the entanglement that was obtained in [25] by using a geometric measure. Consider now a superposition of the W-state and the GHZ-state, i.e., $|GW(s,\phi)\rangle \equiv \sqrt{s} |GHZ\rangle + \sqrt{1-s} e^{i\phi} |W\rangle$. In this case we obtain $C^{(ij)} = \frac{1}{\sqrt{2}} \sqrt{s(5s-4) + 8}$ and $C = \sqrt{3} C^{(ij)}$. For this state, the geometric measure of Ref. [25] observes an entanglement which is dependent on phase $\phi$, but our measure shows an entanglement independent of phase $\phi$. However, in this example as far as $s$ is concerned, the behavior of our entanglement measure is closely related to the behavior of the geometric measure of Ref. [25].

4 Conclusion

In summary, we gave the definition of a concurrence vector and proposed to use its norm as a measure of entanglement. In the case of a bipartite pure state, it is shown that the norm of the concurrence vector leads to the other proposals of generalization of concurrence. In the multipartite case, the concurrence vector is regarded as the direct sum of concurrence subvectors, each one is associated with a pair of particles, therefore, the norm of each subvector is used as the entanglement contribution of the corresponding pair. We argue that the definition is not exhaustive in order to completely quantify entanglement, so the result of the paper is a small step towards quantifying the entanglement. Also the definition of concurrence vectors considered in this paper is just for pure states, and the problem of mixed states remains open.

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