A purely Kerr nonlinear model admitting flat-top solitons

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The objective of this Letter is to demonstrate another natural setting, which gives rise to families of 1D and 2D flat-top solitons. This is a known variety of self-trapped modes, with a potential for applications [24], which are usually supported by systems with competing focusing and defocusing nonlinearities [25], such as cubic-quintic [26–29], cubic-quartic [30, 31], and quadratic-cubic [32–34] combinations, as well as by cubic terms with an additional logarithmic factor [33, 35] (the two former types of the nonlinearity were recently realized experimentally in optics [36] and as “quantum droplets” BEC [37, 38], respectively). However, stable flat-top solitons were not previously found in physical models with the cubic-only nonlinearity, while here we demonstrate that this is possible, in 1D and 2D geometries alike, for various soliton species (fundamental, multipole, vortical), if the coefficient of the cubic defocusing is subjected to an appropriate spatial modulation.

In addition to the systematically produced numerical results, we also obtain particular exact stable solutions for the 1D fundamental flat-top solitons, and develop the Thomas-Fermi (TF) approximation for generic soliton states and 2D vortices. Stability of all the solutions is investigated via the linear-stability analysis and numerical simulations.

The basic model is introduced as the scaled Schrödinger equation governing the evolution of the dimensionless complex amplitude of a light beam propagating in a cubic nonlinear medium, or the mean-field wave function in a Bose-Einstein condensate (BEC), \( \psi(r,t) \):

\[
    i\psi_t = -(1/2)\nabla^2\psi + \sigma (r) |\psi|^2 \psi, \tag{1}
\]

which is written in the 2D form, with \( r = (x, y) \). Here \( t \) is the evolution variable, representing time in the BEC version of the model, or the propagation distance in optics. The axially symmetric nonlinearity-modulation profile, \( \sigma (r) > 0 \), is chosen in the form which readily helps to create flat-top modes of radius \( r_0 \),

\[
    \sigma (r) = \begin{cases} 
    \sigma_{\text{int}}, r \leq r_0, \\
    \sigma_{\text{ext}} \exp [\alpha (r + \alpha^{-1/2} - r_0)^2 - 1], r > r_0,
    \end{cases} \tag{2}
\]

with constants \( \sigma_{\text{int}}, \sigma_{\text{ext}} > 0 \) and \( \alpha > 0 \). Note that Eq. (2) implies that \( \sigma (r = r_0) = \sigma_{\text{ext}} \). The 1D version of the model...
The modulation profile (here applied to Eq. (2)) provide D modes were found by means of the Newton’s method applied to Eq. (2). This profile can be created by means of the above-mentioned methods – for instance, by the application of an FR-controlling magnetic-field profile to the BEC layer, with a constant detuned value at \( r < r_0 \), and one approaching the exact resonant value at \( r > r_0 \). It is relevant to mention that similar cylindrical optical-box potentials are used in current BEC experiments [39].

Wave functions of stationary states with real chemical potential \( \mu \) (alias propagation constant \(-\mu\) in terms of optics) and integer vorticity \( m \) are sought for, in polar coordinates \((r, \theta)\), as \( \psi(r, \theta) = w(r) \exp(i m \theta - i \mu t) \), with real \( w \) determined by the equation

\[
\mu w = -\frac{1}{2} (w'' + r^{-1} w' - m^2 r^{-2} w) + \sigma(r) w^3
\]

in 2D, or its counterpart in 1D. First, the 1D version of the model admits an exact solution, with \( \mu = 3\alpha/2 \),

\[
w(x \leq |x_0|) = \frac{\sqrt{3} \alpha/2}{\sigma_{\text{ext}}} \\
\times \exp\left(-\frac{1}{2}\right) \alpha (|x|^2/4 - 1).
\]

Note that Eq. (4) gives \( dw/dx(x = x_0) = 0 \), which is necessary for the continuity at \( x = x_0 \), while the remaining continuity condition for \( w(x) \) imposes a relation on constants of modulation profile (2): \( \sigma_{\text{int}} = 3\sigma_{\text{ext}} \). This profile is continuous in the limit case of \( x_0 = 0 \), for which exact solution (4) remains valid. Numerical solutions are displayed below for the continuous modulation profile with \( \sigma_{\text{int}} = \sigma_{\text{ext}} \).

An analytical approximation for generic soliton shapes can be obtained in the Thomas-Fermi (TF) approximation, which neglects derivatives in Eq. (2): \( \sigma_{\text{int}} = 3\sigma_{\text{ext}} \). This profile is continuous in the limit case of \( x_0 = 0 \), for which exact solution (4) remains valid. Numerical solutions are displayed below for the continuous modulation profile with \( \sigma_{\text{int}} = \sigma_{\text{ext}} \).

An analytical approximation for generic soliton shapes can be obtained in the Thomas-Fermi (TF) approximation, which neglects derivatives in Eq. (2): \( \sigma_{\text{int}} = 3\sigma_{\text{ext}} \).

\[
\sigma_{\text{TF}} = \sigma(r)^{-1} \left[ \mu - m^2/(2r^2) \right]
\]

at \( r^2 > m^2/2\mu \), and \( \sigma_{\text{TF}} = 0 \) at \( r^2 < m^2/2\mu \). This approximation makes it possible to predict the dependence of the norm of the soliton family on the chemical potential,

\[
N_{\text{2D}} = 2\pi \int_0^\infty w^2(r)dr \approx \pi \sigma_{\text{int}}^{-1} \times \\
[\mu r_0 + \alpha^{-1/2} - (m^2/2) \ln(2\mu r_0^2/m^2)]
\]

(here \( e \) is the base of the natural logarithm), which is valid at \( \mu > m^2/(2r_0^2) \).

In the numerical form, stationary profiles of both 1D and 2D modes were found by means of the Newton’s method applied to Eq. (2). The subsequent stability analysis was based on the usual ansatz, \( \psi = |w(r)| p_+(r) \exp(i m \theta + \lambda t) + p_-(r) \exp(-i m \theta + \lambda^* t) |\exp(i m \theta - i \mu t) \), where \( p_\pm(r) \) represent perturbation eigenmodes with eigenvalue \( \lambda \), \( \ast \) stands for the complex conjugate, and \( n \) is an integer azimuthal index. Then, the eigenvalue problem amounts to the solution of linear equations,

\[
i\lambda p_\pm = \mp (1/2) [p_\pm' + r^{-1} p_\pm' - (m \pm n)^2 r^{-2} p_\pm] p \\
\mp \mu p_\pm + \sigma w^2 (2p_\pm + p_\pm),
\]

or their 1D counterparts. In particular, the exact solution given by Eq. (4) is found to be always stable.

Typical profiles of 1D flat-top solitons with the number of nodes \( k = 0 \), 1, and 2 (fundamental, dipole, and tripole solitons, respectively) at different values of \( \mu \) are displayed in Fig. 1, which clearly shows that the solitons’ shape gets flatter with the increase of \( \mu \). The functional form of solitons changes considerably as \( \mu \) increases. For instance, solitons with large
shows that flat-top states are generally a dipole \((k = 1)\) with \(\mu = 10\); (b) a tripole \((k = 2)\) with \(\mu = 10\); (c) a multipole \((k = 6)\) with \(\mu = 20\). The evolution of unstable multipoles, also for \(x_0 = 3\): (d) \(a k = 4\) with \(\mu = 4.5\); (e) \(k = 5\) with \(\mu = 7.5\); (f) \(k = 6\) with \(\mu = 11\). The evolution range is \(0 < t < 1000\).

Typical examples of the evolution of stable 1D flat-top solitons are displayed in Figs. 3(a)-(c), while evolution of their unstable counterparts is shown in Figs. 3(d-f). Unstable multipoles spontaneously develop oscillations, keeping the number of nodes.

The profiles of 2D solitons with vorticities \(m = 0\) (fundamental solitons), \(m = 1\), and \(m = 2\) for different values of \(r_0\) are displayed in Fig. 4(a), which shows that the width of the flat-top solitons increases with \(r_0\), similar to their 1D counterparts. Further, Fig. 4(b) displays profiles of the vortices with \(m = 2\) and different values of \(\mu\), demonstrating that 2D solitons also get flatter with the increase of \(\mu\).

Typical dependencies \(N(\mu)\) for the vortex families with \(m = 2\) are displayed in Fig. 5(a), featuring a nearly linear form for all values of \(r_0\), and considerable growth of \(N\) with the increase of \(r_0\). These features are well predicted by the TF approximation, as seen in the figure. The 2D modes with \(m = 0\) and 1 are completely stable, at least up to \(\mu = 40\), while the vortices with \(m \geq 2\) demonstrate alternation of stability and instability domains in the \((r_0, \mu)\) plane in Fig. 5(b). Finally, the evolution of the flat-top 2D vortices is displayed in Fig. 6, demonstrating that those with \(m = 2\) or 3, which are unstable, split into persistently rotating pairs or triplets of unitary vortices.

In conclusion, we have demonstrated that families of stable flat-top solitons, including 1D multipoles and 2D vortices, can be created in media with cubic self-repulsive nonlinearity whose local strength is subject to an appropriate spatial modulation. To our knowledge, this model predicts the first stable flat-top solitons realized with the cubic-only nonlinearity, in contrast to previous results which demonstrated such modes solely in systems with competing attractive and repulsive nonlinearities, thus providing an alternative way to create and stabilize the flat-top modes with free intrinsic parameters.
FIG. 6. The evolution of perturbed 2D vortex solitons at $r_0 = 4$: (a) a stable soliton with $m = 2$, $\mu = 30$; (b) an unstable soliton with $m = 2$, $\mu = 40$; (c) an unstable one with $m = 3$, $\mu = 23$. $k = 0, 1$, and 2 nodes, as well as 2D solitons with vorticities $m = 0$ and $1$ are completely stable, while higher-order modes, with $k \geq 3$ and $m \geq 2$, respectively, feature alternating stability and instability domains. Such self-trapped modes can be created in BEC and optics by means of available experimental techniques.

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