AN IMEX FINITE ELEMENT METHOD
FOR A LINEARIZED CAHN-HILLIARD-COOK EQUATION
DRIVEN BY THE SPACE DERIVATIVE
OF A SPACE-TIME WHITE NOISE

GEORGIOS E. ZOURARIS†

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ABSTRACT. We consider a model initial- and Dirichlet boundary- value problem for a linearized Cahn-Hilliard-Cook equation, in one space dimension, forced by the space derivative of a space-time white noise. First, we introduce a canvas problem the solution to which is a regular approximation of the mild solution to the problem and depends on a finite number of random variables. Then, fully-discrete approximations of the solution to the canvas problem are constructed using, for discretization in space, a Galerkin finite element method based on $H^2$ piecewise polynomials, and, for time-stepping, an implicit/explicit method. Finally, we derive a strong a priori estimate of the error approximating the mild solution to the problem by the canvas problem solution, and of the numerical approximation error of the solution to the canvas problem.

1. Introduction

Let $T > 0$, $D := (0, 1)$ and $(\Omega, \mathcal{F}, P)$ be a complete probability space. Then, we consider the model initial- and Dirichlet boundary- value problem for a linearized Cahn-Hilliard-Cook equation formulated in [7], which is as follows: find a stochastic function $u : [0, T] \times \overline{D} \rightarrow \mathbb{R}$ such that
\begin{align}
    u_t + u_{xxxx} + \mu u_{xx} &= \partial_x \tilde{W}(t, x) \quad \forall (t, x) \in (0, T] \times D, \\
    u(t, \cdot)|_{\partial D} &= u_{xx}(t, \cdot)|_{\partial D} = 0 \quad \forall t \in (0, T], \\
    u(0, x) &= 0 \quad \forall x \in D,
\end{align}
a.s. in $\Omega$, where $\tilde{W}$ denotes a space-time white noise on $[0, T] \times D$ (see, e.g., [12], [5]) and $\mu$ is a real constant. We recall that the mild solution of the problem above (cf. [3]) is given by
\begin{align}
    u(t, x) &= \int_0^t \int_D \Psi_{t-s}(x, y) dW(s, y),
\end{align}
where
\begin{align}
    \Psi_t(x, y) := -\sum_{k=1}^{\infty} \lambda_k e^{-\lambda_k^2 (\lambda_k^2 - \mu) t} \varepsilon_k(x) \varphi_k(y) \\
    &= -\partial_y G_t(x, y) \quad \forall (t, x, y) \in (0, T] \times \mathbb{D} \times \overline{D},
\end{align}
$\lambda_k := k \pi$ for $k \in \mathbb{N}$, $\varepsilon_k(z) := \sqrt{2} \sin(\lambda_k z)$ and $\varphi_k(z) := \sqrt{2} \cos(\lambda_k z)$ for $z \in \mathbb{D}$ and $k \in \mathbb{N}$, and $G_t(x, y)$ is the space-time Green kernel of the solution to the deterministic parabolic problem: find

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†Department of Mathematics and Applied Mathematics, University of Crete, P.O. Box 2208, GR–710 03 Heraklion, Crete, Greece.
$w : [0, T] \times \overline{D} \to \mathbb{R}$ such that
\begin{equation}
\begin{aligned}
w_t + w_{xxxx} + \mu w_{xx} &= 0 \quad \forall (t, x) \in (0, T] \times D, \\
w(t, \cdot)|_{\partial D} &= w_{xx}(t, \cdot)|_{\partial D} = 0 \quad \forall t \in (0, T], \\
w(0, x) &= w_0(x) \quad \forall x \in D.
\end{aligned}
\end{equation}

In the paper at hand, our goal is to propose and analyze a numerical method for the approximation of $u$ that has less stability requirements and lower complexity than the method proposed in [7].

1.1. A canvas problem. A canvas problem is an initial- and boundary-value problem the solution to which: i) depends on a finite number of random variables and ii) is a regular approximation of the mild solution $u$ to (1.1). Then, we can derive computable approximations of $u$ by constructing numerical approximations of the canvas problem solution via the application of a discretization technique for stochastic partial differential equations with random coefficients. The formulation of the canvas problem depends on the way we replace the infinite stochastic dimensionality of the problem (1.1) by a finite one.

In our case the canvas problem is formulated as follows (cf. [1], [6], [7]): Let $M, N \in \mathbb{N}$, $\Delta t := \frac{T}{N}$, $t_n := n \Delta t$ for $n = 0, \ldots, N$, $T_n := (t_{n-1}, t_n)$ for $n = 1, \ldots, N$, and $u : [0, T] \times \overline{D} \to \mathbb{R}$ such that
\begin{equation}
\begin{aligned}
&u_t + u_{xxxx} + \mu u_{xx} = \partial_x W \quad \text{in} \quad (0, T] \times D, \\
&u(t, \cdot)|_{\partial D} = u_{xx}(t, \cdot)|_{\partial D} = 0 \quad \forall t \in (0, T], \\
&u(0, x) = 0 \quad \forall x \in D,
\end{aligned}
\end{equation}

where
\begin{equation}
W(\cdot, x)|_{t_n} := \frac{1}{\Delta t} \sum_{i=1}^{M} R^n_i \varphi_i(x) \quad \forall x \in D, \quad n = 1, \ldots, N,
\end{equation}

\begin{equation}
R^n_i := \int_{t_n}^{t_{n+1}} \int_D \varphi_i(x) \, dW(t, x) = B^i(t_{n+1}) - B^i(t_n), \quad i = 1, \ldots, M, \quad n = 1, \ldots, N,
\end{equation}

and $B^i(t) := \int_{0}^{t} \int_D \varphi_i(x) \, dW(s, x)$ for $t \geq 0$ and $i \in \mathbb{N}$. According to [12], $(B^i)_{i=1}^{\infty}$ is a family of independent Brownian motions, and thus, the random variables $((R^n_i)_{n=1}^{M})_{i=1}^{M}$ are independent and satisfy
\begin{equation}
R^n_i \sim \mathcal{N}(0, \Delta t), \quad i = 1, \ldots, M, \quad n = 1, \ldots, N.
\end{equation}

Thus, the solution $u$ to (1.5) depends on $NM$ random variables and the well-known theory for parabolic problems (see, e.g., [8]) yields its regularity along with the following representation formula:
\begin{equation}
\begin{aligned}
u(t, x) &= \int_{0}^{t} \int_D G_{t-s}(x, y) \partial_y W(s, y) \, ds \, dy \\
&= \int_{0}^{t} \int_D \Psi_{t-s}(x, y) W(s, y) \, ds \, dy \quad \forall (t, x) \in [0, T] \times \overline{D}.
\end{aligned}
\end{equation}

Remark 1.1. In [2] the definition of $W$ is based on a uniform partition of $[0, T]$ in $N$ subintervals and on a uniform partition of $D$ in $J$ subintervals. At every time slab, $W$ has a constant value with respect to the time variable, but, with respect to the space variable, is defined as the $L^2(D)$-projection of a random, piecewise constant function onto the space of linear splines, the computation of which leads to the numerical solution of a $(J+1) \times (J+1)$ tridiagonal linear system of algebraic equations. Finally, $W$ depends on $N(J+1)$ random variables and its construction has $O(N(J+1))$ complexity, that must to be added to the complexity of the numerical method used for the approximation of $u$. On the contrary, the stochastic load $W$ of the canvas problem (1.5) we propose here, is given explicitly by the formula (1.6), and thus no extra computational cost is required for its formation.
1.2. An IMEX finite element method. Let \( M \in \mathbb{N}, \Delta \tau := \frac{T}{M}, \tau_m := m \Delta \tau \) for \( m = 0, \ldots, M \), and \( \Delta_m := (\tau_{m-1}, \tau_m) \) for \( m = 1, \ldots, M \). Also, for \( r = 2 \) or \( 3 \), let \( M_r^h \subset H^2(D) \cap H_0^1(D) \) be a finite element space consisting of functions which are piecewise polynomials of degree at most \( r \) over a partition of \( D \) in intervals with maximum mesh-length \( h \).

The fully-discrete method we propose for the numerical approximation of \( u \) uses an implicit/explicit (IMEX) time-discretization treatment of the space differential operator along with a finite element variational formulation for space discretization. Its algorithm is as follows: first sets

\[
U_h^0 := 0
\]

and then, for \( m = 1, \ldots, M \), finds \( U_h^m \in M_h^r \) such that

\[
(1.11) \quad (U_h^m - U_h^{m-1}, \chi)_{0,D} + \Delta \tau \left[ (\partial_t^2 U_h^m, \partial_x^2 \chi)_{0,D} + \mu (\partial_t^2 U_h^{m-1}, \chi)_{0,D} \right] = \int_{\Delta_m} (\partial_x W, \chi)_{0,D} \, d\tau
\]

for all \( \chi \in M_h^r \), where \((\cdot, \cdot)_{0,D}\) is the usual \( L^2(D)\)-inner product.

**Remark 1.2.** It is easily seen that the numerical method above is unconditionally stable, while the Backward Euler finite element method is stable under the time-step restriction \( \Delta \tau \mu^2 \leq 4 \) (see [7]).

1.3. An overview of the paper. In Section 2 we introduce notation and we recall several results that are often used in the rest of the paper. In Section 3, we focus on the estimation of the error we made by approximating the solution \( u \) to (1.1) by the solution \( u \) to (1.5), arriving at the bound

\[
\max_{[0,T]} \mathbb{E} \left[ \left\| u - u \right\|_{L^2(D)}^2 \right] \leq C \left( M - \frac{\Delta \tau}{2} + \Delta \tau \right)
\]

(see Theorem 3.1). Section 4 is dedicated to the definition and the convergence analysis of modified IMEX time-discrete and fully-discrete approximations of the solution \( u \) to the deterministic problem (1.4). The results obtained are used later in Section 5 where we analyze the numerical method for the approximation of \( u \), given in Section 1.2. Its convergence is established by proving the following strong error estimate

\[
\max_{0 \leq m \leq M} \mathbb{E} \left[ \left\| U_h^m - u(\tau_m, \cdot) \right\|_{0,D}^2 \right] \leq C \left( \epsilon_1^{-\frac{1}{2}} \Delta \tau^\frac{\frac{1}{2} - \epsilon_1}{2} + \epsilon_2^{-\frac{1}{2}} h^{\frac{3}{2} - \epsilon_2} \right)
\]

for all \( \epsilon_1 \in (0, \frac{1}{2}] \) and \( \epsilon_2 \in (0, \frac{1}{2}] \) (see Theorem 5.3). We obtain the latter error bound, by applying a discrete Duhamel principle technique to estimate separately the *time discretization error* and the *space discretization error*, which are defined using as an intermediate the corresponding IMEX time-discrete approximations of \( u \), specified by (5.1) and (5.2) (cf., e.g., [0], [7], [11]).

Since we have no assumptions on the sign, or, the size of \( \mu \), the elliptic operator in (1.5) is, in general, not invertible. This is the reason that the Backward Euler/finite element method is stable and convergent after adopting a restriction on the time-step size (see [7], Remark 1.2). On the contrary, the IMEX/finite element method we propose here is unconditionally stable and convergent, because the principal part of the elliptic operator is treated implicitly and its lower order part explicitly. Another characteristic in our method is the choice to build up the canvas problem using spectral functions, which allow us to avoid the numerical solution of an extra linear system of algebraic equation at every time step that is required in the approach of [7] (see Remark 1.1).

The error analysis of the IMEX finite element method is more technical than that in [7] for the Backward Euler finite element method. The main difference is due to the fact that the representation of the time-discrete and fully discrete approximations of \( u \) is related to a modified version of the IMEX time-stepping method for the approximation of the solution to the deterministic problem (1.4), the error analysis of which is necessary in obtaining the desired error estimate and is of independent interest (see Section 4).
2. Preliminaries

We denote by $L^2(D)$ the space of the Lebesgue measurable functions which are square integrable on $D$ with respect to the Lebesgue measure $dx$. The space $L^2(D)$ is provided with the standard norm $\|g\|_{0,D} := \left(\int_D |g(x)|^2 \, dx\right)^{1/2}$ for $g \in L^2(D)$, which is derived by the usual inner product $(g_1, g_2)_{0,D} := \int_D g_1(x) g_2(x) \, dx$ for $g_1, g_2 \in L^2(D)$. Also, we employ the symbol $\mathbb{N}_0$ for the set of all nonnegative integers.

For $s \in \mathbb{N}_0$, we denote by $H^s(D)$ the Sobolev space of functions having generalized derivatives up to order $s$ in $L^2(D)$, and by $\| \cdot \rVert_{s,D}$ its usual norm, i.e., $\|g\rVert_{s,D} := \left(\sum_{i=0}^{s} \|\partial^i g\rVert_{0,D}^2\right)^{1/2}$ for $g \in H^s(D)$. Also, by $H^s_0(D)$ we denote the subspace of $H^1(D)$ consisting of functions which vanish at the endpoints of $D$ in the sense of trace.

The sequence of pairs $\{(\lambda^2_i, \varepsilon_i)\}_{i=1}^{\infty}$ is a solution to the eigenvalue/eigenfunction problem: find nonzero $\varphi \in H^2(D) \cap H^1_0(D)$ and $\lambda \in \mathbb{R}$ such that $-\varphi'' = \lambda \varphi$ in $D$. Since $(\varepsilon_i)_{i=1}^{\infty}$ is a complete $(\cdot, \cdot)_{0,D}$-orthonormal system in $L^2(D)$, for $s \in \mathbb{N}_0$, we define by

$$V^s(D) := \left\{v \in L^2(D) : \sum_{i=1}^{\infty} \lambda_i^2 (v, \varepsilon_i)_0,D^2 < \infty \right\}$$

a subspace of $L^2(D)$ provided with the natural norm $\|v\|_{V^s} := \left(\sum_{i=1}^{\infty} \lambda_i^2 (v, \varepsilon_i)_0,D^2\right)^{1/2}$ for $v \in V^s(D)$. For $s \geq 0$, the space $(V^s(D), \| \cdot \Vert_{V^s})$ is a complete subspace of $L^2(D)$ and we define $(\tilde{H}^s(D), \| \cdot \Vert_{\tilde{H}^s}) := (V^s(D), \| \cdot \Vert_{V^s})$. For $s < 0$, the space $(\tilde{H}^s(D), \| \cdot \Vert_{\tilde{H}^s})$ is defined as the completion of $(V^s(D), \| \cdot \Vert_{V^s})$, or, equivalently, as the dual of $(\tilde{H}^{-s}(D), \| \cdot \Vert_{H^{-s}})$.

Let $m \in \mathbb{N}_0$. It is well-known (see [10]) that

$$\tilde{H}^m(D) = \left\{v \in H^m(D) : \partial^2 v \rvert_{\partial D} = 0 \text{ if } 0 \leq 2\ell < m \right\}$$

and that there exist constants $C_{m,a}$ and $C_{m,b}$ such that

$$C_{m,a} \|v\rVert_{m,D} \leq \|v\rVert_{H^m} \leq C_{m,b} \|v\rVert_{m,D} \quad \forall v \in \tilde{H}^m(D).$$

Also, we define on $L^2(D)$ the negative norm $\| \cdot \rVert_{-m,D}$ by

$$\|v\rVert_{-m,D} := \sup \left\{\frac{(v, \varphi)_{0,D}}{\varphi \rVert_{-m,D}}: \varphi \in \tilde{H}^m(D) \text{ and } \varphi \neq 0\right\} \quad \forall v \in L^2(D),$$

for which, using (2.1), follows that there exists a constant $C_{-m} > 0$ such that

$$\|v\rVert_{-m,D} \leq C_{-m} \|v\rVert_{\tilde{H}^{-m}} \quad \forall v \in L^2(D).$$

Let $L_2 = (L^2(D), (\cdot, \cdot)_{0,D})$ and $L(L_2)$ be the space of linear, bounded operators from $L_2$ to $L_2$. An operator $\Gamma \in L(L_2)$ is Hilbert-Schmidt, when $\|\Gamma\rVert_{HS} := \left(\sum_{i=1}^{\infty} \|\Gamma \varepsilon_i\rVert_{0,D}^2\right)^{1/2} < +\infty$, where $\|\Gamma\rVert_{HS}$ is the so called Hilbert-Schmidt norm of $\Gamma$. We note that the quantity $\|\Gamma\rVert_{HS}$ does not change when we replace $(\varepsilon_i)_{i=1}^{\infty}$ by another complete orthonormal system of $L_2$. It is well known (see, e.g., [4, 9]) that an operator $\Gamma \in L(L_2)$ is Hilbert-Schmidt iff there exists a measurable function $\gamma : D \times D \rightarrow \mathbb{R}$ such that $\Gamma[v](\cdot) = \int_D \gamma(\cdot, y) v(y) \, dy$ for $v \in L^2(D)$, and then, it holds that

$$\|\Gamma\rVert_{HS} = \left(\int_{D \times D} \gamma^2(x, y) \, dx \, dy\right)^{1/2}.$$

Let $L_{HS}(L_2)$ be the set of Hilbert Schmidt operators of $L(L^2)$ and $\Phi : [0, T] \rightarrow L_{HS}(L_2)$. Also, for a random variable $X$, let $E[X]$ be its expected value, i.e., $E[X] := \int_{\Omega} X \, dP$. Then, the Itô isometry property for stochastic integrals reads

$$E \left[\left\|\int_{0}^{T} \Phi \, dW\right\|_{0,D}^2\right] = \int_{0}^{T} \|\Phi(t)\rVert_{HS}^2 \, dt.$$

For later use, we recall that if $(\mathcal{H}, (\cdot, \cdot)_n)$ is a real inner product space with induced norm $| \cdot |_n$, then

$$2(g - v, g)_n = |g|^2_n - |v|^2_n + |g - v|^2_n \quad \forall g, v \in \mathcal{H}.$$
Finally, for any nonempty set \( A \), we denote by \( \chi_A \) the indicator function of \( A \).

2.1. A projection operator. Let \( \mathcal{O} := (0, T) \times D \), \( \mathcal{S}_m := \text{span}(\varphi_i)_{i=1}^n \), \( \mathcal{S}_n := \text{span}(X_{r_{\text{rn}}})_{n=1}^n \) and \( \Pi : L^2(\mathcal{O}) \to \mathcal{S}_n \otimes \mathcal{S}_m \) the usual \( L^2(\mathcal{O}) \)-projection operator which is given by the formula

\[
\Pi g := \frac{1}{\Delta t} \sum_{i=1}^M \left( \sum_{n=1}^N X_{r_{\text{rn}}} \int_{T_n} (g, \varphi_i)_{0, D} \, dt \right) \varphi_i \quad \forall g \in L^2(\mathcal{O}).
\]

Then, the following representation of the stochastic integral of \( \Pi \) holds (cf. Lemma 2.1 in [6]).

**Lemma 2.1.** For \( g \in L^2(\mathcal{O}) \), it holds that

\[
\int_0^T \int_D \Pi g(t, x) \, dW(t, x) = \iint_{\mathcal{O}} W(s, y) g(s, y) \, dsdy.
\]

**Proof.** Using (2.6) and (1.7), we have

\[
\int_0^T \int_D \Pi g(t, x) \, dW(t, x) = \frac{1}{\Delta t} \sum_{i=1}^M \left( \int_{T_n} \int_D g(s, y) \varphi_i(y) \, ds \, dy \right) R_i^n
\]

\[
= \frac{1}{\Delta t} \sum_{i=1}^M \left( \int_{T_n} \int_D X_{r_{\text{rn}}}(s) R_i^n g(s, y) \varphi_i(y) \, ds \, dy \right)
\]

\[
= \int_{\mathcal{O}} g(s, y) \left( \frac{1}{\Delta t} \sum_{n=1}^N \sum_{i=1}^M X_{r_{\text{rn}}}(s) R_i^n \varphi_i(y) \right) \, ds \, dy
\]

which along (1.6) yields (2.7). \( \Box \)

2.2. Linear elliptic and parabolic operators. Let \( T_E : L^2(D) \to \dot{H}^2(D) \) be the solution operator of the Dirichlet two-point boundary value problem: for given \( f \in L^2(D) \) find \( v_E \in \dot{H}^2(D) \) such that \( v_E'' = f \) in \( D \), i.e. \( T_E f := v_E \). It is well-known that

\[
(T_E f, g)_{0, D} = (f, T_E g)_{0, D} \quad \forall f, g \in L^2(D)
\]

and, for \( m \in \mathbb{N}_0 \), there exists a constant \( C_E^m > 0 \) such that

\[
\|T_E f\|_{m, D} \leq C_E^m \|f\|_{m-2, D} \quad \forall f \in H^{\max\{0, m-2\}}(D).
\]

Let, also, \( T_B : L^2(D) \to \dot{H}^4(D) \) be the solution operator of the following Dirichlet biharmonic two-point boundary value problem: for given \( f \in L^2(D) \) find \( v_B \in \dot{H}^4(D) \) such that

\[
v_B'''' = f \quad \text{in} \quad D,
\]

i.e. \( T_B f := v_B \). It is well-known that, for \( m \in \mathbb{N}_0 \), there exists a constant \( C_B^m > 0 \) such that

\[
\|T_B f\|_{m, D} \leq C_B^m \|f\|_{m-4, D} \quad \forall f \in H^{\max\{0, m-4\}}(D).
\]

Due to the type of boundary conditions of (2.10), we have

\[
T_B f = T_E^2 f \quad \forall f \in L^2(D),
\]

which, after using (2.8), yields

\[
(T_B v_1, v_2)_{0, D} = (T_E v_1, T_E v_2)_{0, D} = (v_1, T_B v_2)_{0, D} \quad \forall v_1, v_2 \in L^2(D).
\]

Let \( (S(t)w_0)_{t \in [0,T]} \) be the standard semigroup notation for the solution \( w \) to (1.4). Then (see Appendix A in [7]) for \( \ell \in \mathbb{N}_0 \), \( \beta \geq 0 \) and \( p \geq 0 \), there exists a constant \( C_{\beta, \ell, \mu, \mu^2T} > 0 \) such that

\[
\int_{t_a}^{t_b} \tau^\beta \|d^{2\ell}_t S(\tau)w_0\|_{H_p}^2 \, d\tau \leq C_{\beta, \ell, \mu, \mu^2T} \|w_0\|_{H^{p+4\ell-2\beta-2}}^2
\]

for all \( w_0 \in \dot{H}^{p+4\ell-2\beta-2}(D) \) and \( t_a, t_b \in [0, T] \) with \( t_b > t_a \).
2.3. Discrete operators. Let $r = 2$ or $3$, and $M^r_h \subset H^1_0(D) \cap H^2(D)$ be a finite element space consisting of functions which are piecewise polynomials of degree at most $r$ over a partition of $D$ in intervals with maximum length $h$. It is well-known (cf., e.g., \cite{[6]}) that

\begin{equation}
\inf_{\chi \in M^r_h} \|v - \chi\|_{s,D} \leq C_r h^{s-2} \|v\|_{s,D} \quad \forall v \in H^{s+1}(D) \cap H^1_0(D), \quad s = 3, \ldots, r+1,
\end{equation}

where $C_r$ is a positive constant that depends on $r$ and $D$, and is independent of $h$ and $v$. Then, we define the discrete biharmonic operator $B_h : M^r_h \to M^r_h$ by $(B_h \varphi, \chi)_{0,D} = (\partial_x^2 \varphi, \partial_x^2 \chi)_{0,D}$ for $\varphi, \chi \in M^r_h$, the $L^2(D)$—projection operator $P_h : L^2(D) \to M^r_h$ by $(P_h f, \chi)_{0,D} = (f, \chi)_{0,D}$ for $\chi \in M^r_h$ and $f \in L^2(D)$, and the standard Galerkin finite element approximation $v_{b,h} \in M^r_h$ of the solution $v_{b}$ to (2.10) by requiring

\begin{equation}
B_h(v_{b,h}) = P_h f.
\end{equation}

Let $T_{b,h} : L^2(D) \to M^r_h$ be the solution operator of the finite element method (2.10), i.e. $T_{b,h} f := v_{b,h} = B_h^{-1} P_h f$ for all $f \in L^2(D)$. Then, we can easily conclude that

\begin{equation}
\langle T_{b,h} f, g \rangle_{0,D} = (\partial_x^2 (T_{b,h} f), \partial_x^2 (T_{b,h} g))_{0,D} = (f, T_{b,h} g)_{0,D} \quad \forall f, g \in L^2(D)
\end{equation}

and

\begin{equation}
\|\partial_x^2 (T_{b,h} f)\|_{0,D} \leq C \|f\|_{-2,D} \quad \forall f \in L^2(D).
\end{equation}

Finally, the approximation property (2.15) of the finite element space $M^r_h$ yields (see, e.g., Proposition 2.2 in \cite{[6]}) the following error estimate

\begin{equation}
\|T_{b} f - T_{b,h} f\|_{0,D} \leq C h^r \|f\|_{-1,D} \quad \forall f \in L^2(D), \quad r = 2, 3.
\end{equation}

3. An approximation estimate for the canvas problem solution

Here, we establish the convergence of $u$ towards $u$ with respect to the $L^\infty(L^2_0(L^2_0))$ norm, when $\Delta t \to 0$ and $M \to \infty$ (cf. \cite{[6], [7]}).

**Theorem 3.1.** Let $u$ be the solution to (1.1), $u$ be the solution to (1.3) and $\kappa \in \mathbb{N}$ such that $\kappa^2 \pi^2 > \mu$. Then, there exists a constant $C_{\text{CRF}} > 0$, independent of $\Delta t$ and $M$, such that

\begin{equation}
\max_{[0,T]} \Theta(t) := \mathbb{E} \left[ \|\langle u(t,\cdot) - u(t,\cdot)\|_{0,D}^2 \right]^{1/2} \quad \forall M \geq \kappa,
\end{equation}

where $\Theta(t) := (\mathbb{E} \left[ \|u(t,\cdot) - u(t,\cdot)\|_{0,D}^2 \right])^{1/2}$ for $t \in [0, T]$.

**Proof.** In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\Delta t$ and $M$ and may change value from one line to the other.

Using (1.2), (1.9) and Lemma 2.1, we conclude that

\begin{equation}
u(t, x) - u(t, x) = \int_0^L \int_D X_{(0,l)}(s) \Psi_{t-s}(x,y) - \tilde{\Psi}(t,x,s,y) \, dW(s,y)
\end{equation}

for $(t, x) \in [0, T] \times \bar{D}$, where $\tilde{\Psi} : (0, T) \times D \to L^2(D)$ is given by

\begin{equation}\tilde{\Psi}(t; x, s, y) := \frac{1}{\lambda_n} \sum_{i=1}^M \left[ \int_{T_n} X_{(0,i)}(s') \left( \int_D \Psi_{t-s'}(x, y') \varphi_i(y') \, dy' \right) ds' \right] \varphi_i(y)
\end{equation}

for $(s, y) \in T_n \times D$, $n = 1, \ldots, N$, and for $(t, x) \in (0, T) \times D$. Now, we use (1.3) and the $L^2(D)$—orthogonality of $(\varphi_k)_{k=1}^\infty$ to obtain

\begin{equation}\Psi(t, x; s, y) = \frac{1}{\lambda_n} \int_{T_n} X_{(0,i)}(s') \left( \sum_{i=1}^M \lambda_i e^{-\lambda_i^2 (\mu - \nu) (t-s)} \varepsilon_i(x) \varphi_i(y) \right) ds'.
\end{equation}
for \((s, y) \in T_n \times D, n = 1, \ldots, N\), and for \((t, x) \in (0, T] \times D\). Also, we use \((3.2)\), \((2.4)\) and \((2.3)\), to get

\[
\Theta(t) = \left( \int_0^T \int_D \left( \sum_{n=1}^N \int_D \int_{T_n} \left[ \lambda_i e^{-\lambda_i^2 (t-s^2)} \phi_i(x) \varphi_i(y) \right] ds' \right)^2 dxdy ight)^{\frac{1}{2}}
\]

(3.4)

\[
\leq \sqrt{\Theta_A(t)} + \sqrt{\Theta_B(t)} \quad \forall t \in (0, T],
\]

where

\[
\Theta_A(t) := \sum_{n=1}^N \int_D \int_D \int_{T_n} \left[ \lambda_i e^{-\lambda_i^2 (t-s^2)} \phi_i(x) \varphi_i(y) \right] ds' \right]^{2} dxdy
\]

and

\[
\Theta_B(t) := \sum_{n=1}^N \int_D \int_D \int_{T_n} \left[ \lambda_i e^{-\lambda_i^2 (t-s^2)} \phi_i(x) \varphi_i(y) \right] ds' \right]^{2} dxdy.
\]

Proceeding as in the proof of Theorem 3.1 in [7] we arrive at

\[
\sqrt{\Theta_A(t)} \leq C \Delta t^{\frac{1}{2}} \quad \forall t \in (0, T].
\]

(3.5)

Combining \((4.3)\) and \((4.3)\) and using the \(L^2(D)\)–orthogonality of \((\varepsilon_k)_{k=1}^\infty\) and \((\varphi_k)_{k=1}^\infty\) we have

\[
\Theta_B(t) = \frac{1}{\Delta t} \sum_{n=1}^N \int_D \int_D \left[ \sum_{i=1}^M \lambda_i e^{-\lambda_i^2 (t-s^2)} \varepsilon_i(x) \varphi_i(y) \right] ds' \right]^{2} dxdy
\]

\[
= \frac{1}{\Delta t} \sum_{n=1}^N \int_D \int_D \left[ \sum_{i=M+1}^\infty \lambda_i e^{-\lambda_i^2 (t-s^2)} \varepsilon_i(x) \varphi_i(y) \right] ds' \right]^{2} dxdy
\]

\[
= \frac{1}{\Delta t} \sum_{n=1}^N \int_D \int_D \left[ \sum_{i=M+1}^\infty \left( \int_{T_n} \lambda_i e^{-\lambda_i^2 (t-s^2)} ds' \right) \right] \varepsilon_i(x) \varphi_i(y) \right]^{2} dxdy
\]

\[
= \frac{1}{\Delta t} \sum_{n=1}^N \sum_{i=M+1}^\infty \left( \int_{T_n} \lambda_i e^{-\lambda_i^2 (t-s^2)} ds' \right)^2 \quad \forall t \in (0, T].
\]

For \(M \geq \kappa\), using the Cauchy-Schwarz inequality, we obtain

\[
\sqrt{\Theta_B(t)} \leq \left[ \sum_{i=M+1}^\infty \lambda_i^2 \left( \int_0^t e^{-2 \lambda_i^2 (t-s^2)} ds' \right) \right]^{\frac{1}{2}}
\]

(3.6)

\[
\leq \frac{1}{\sqrt{2 \pi}} \left( \sum_{i=M+1}^\infty \frac{1}{\lambda_i^2} \right)^{\frac{1}{2}}
\]

\[
\leq \frac{\kappa+1}{\sqrt{2 \pi} \kappa} \left( \sum_{i=M+1}^\infty \frac{1}{\lambda_i^2} \right)^{\frac{1}{2}}
\]

\[
\leq \frac{\kappa+1}{\sqrt{2 \pi} \kappa} \left( \int_M^\infty \frac{1}{x^2} dx \right)^{\frac{1}{2}}
\]

\[
\leq \frac{\kappa+1}{\sqrt{2 \pi} \kappa} M^{-\frac{1}{2}} \quad \forall t \in (0, T].
\]

The error bound \((3.1)\) follows by observing that \(\Theta(0) = 0\) and by combining the bounds \((3.4)\), \((3.5)\) and \((3.6)\). \(\square\)
4. Deterministic Time-Discrete and Fully-Discrete Approximations

In this section we define and analyze auxiliary time-discrete and fully-discrete approximations of the solution to the deterministic problem (4.1). The results of the convergence analysis will be used in Section 5 for the derivation of an error estimate for numerical approximations of $u$ introduced in Section 1.

4.1. Time-Discrete Approaches. We define an auxiliary modified-IMEX time-discrete method to approximate the solution $w$ to (1.4), which has the following structure: First sets

$$W^0 := w_0$$

and determines $W^1 \in \hat{H}^1(D)$ by

$$W^1 - W^0 + \Delta \tau \partial_x^2 W^1 = 0.$$ (4.1)

Then, for $m = 2, \ldots, M$, finds $W^m \in \hat{H}^1(D)$ such that

$$W^m - W^{m-1} + \Delta \tau \left( \partial_x^2 W^m + \mu \partial_x^2 W^{m-1} \right) = 0.$$ (4.2)

In the proposition below, we derive a low regularity priori error estimate in a discrete in time $L^2(L^2)$-norm.

**Proposition 4.1.** Let $(W^m)_{m=0}^M$ be the time-discrete approximations defined in (4.1) and (4.2), and $w$ be the solution to the problem (1.4). Then, there exists a constant $C > 0$, independent of $\Delta \tau$, such that

$$\left( \Delta \tau \sum_{m=1}^{M} \|W^m - w^m\|_{L^2(D)}^2 \right)^{\frac{1}{2}} \leq C \Delta \tau \theta \|w_0\|_{H^4(D)} \quad \forall \theta \in [0, 1], \quad \forall w_0 \in \hat{H}^2(D),$$ (4.3)

where $w^\ell(\cdot) := w(\tau_\ell, \cdot)$ for $\ell = 0, \ldots, M$.

**Proof.** In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\Delta \tau$ and may changes value from one line to the other.

Let $E^m := w^m - W^m$ for $m = 0, \ldots, M$, and

$$\sigma_m(\cdot) := \int_{\Delta_m} (w(\tau_m, \cdot) - w(\tau, \cdot)) \, d\tau + \mu \int_{\Delta_m} T_k (w(\tau_m - 1, \cdot) - w(\tau, \cdot)) \, d\tau$$

for $m = 1, \ldots, M$. Thus, combining (1.4), (1.2) and (4.3), we conclude that

$$T_k (E^1 - E^0) + \Delta \tau E^1 = \sigma_1 - \Delta \tau \mu T_k w_0,$$ (4.4)

$$T_k (E^m - E^{m-1}) + \Delta \tau (E^m + \mu T_k E^{m-1}) = \sigma_m, \quad m = 2, \ldots, M.$$ (4.5)

First take the $L^2(D)$-inner product of both sides of (4.5) with $E^1$ and of (4.6) with $E^m$, and then use (2.13) to obtain

$$\langle T_k E^1 - T_k E^0, T_k E^1 \rangle_{0,D} + \Delta \tau \|E^1\|_{L^2(D)}^2 = \langle \sigma_1, E^1 \rangle_{0,D} - \Delta \tau \mu (T_k w_0, E^1)_{0,D},$$

$$\langle T_k E^m - T_k E^{m-1}, T_k E^m \rangle_{0,D} + \Delta \tau \|E^m\|_{L^2(D)}^2 = -\mu \Delta \tau (T_k E^{m-1}, E^m)_{0,D} + \langle \sigma_m, E^m \rangle_{0,D}$$

for $m = 2, \ldots, M$. Then, using that $E^0 = 0$ and applying (2.5) along with the arithmetic mean inequality, we get

$$\|T_k E^1\|_{0,D}^2 + \Delta \tau \|E^1\|_{L^2(D)}^2 \leq \Delta \tau^{-1} \|\sigma_1\|_{L^2(D)}^2 - 2 \Delta \tau \mu (T_k w_0, E^1)_{0,D},$$ (4.7)

$$\|T_k E^m\|_{0,D}^2 + \frac{1}{2} \Delta \tau \|E^m\|_{L^2(D)}^2 \leq (1 + 2 \mu^2 \Delta \tau) \|T_k E^{m-1}\|_{0,D}^2 + \Delta \tau^{-1} \|\sigma_m\|_{L^2(D)}^2, \quad m = 2, \ldots, M.$$ (4.8)

Observing that (4.8) yields

$$\|T_k E^m\|_{0,D}^2 \leq (1 + 2 \mu^2 \Delta \tau) \|T_k E^{m-1}\|_{0,D}^2 + \Delta \tau^{-1} \|\sigma_m\|_{L^2(D)}^2, \quad m = 2, \ldots, M,$$
we use a standard discrete Gronwall argument to arrive at

\[ \max_{1 \leq m \leq M} \| T_E E^m \|_{W^3}^2 \leq C \left( \| T_E E^1 \|_{W^3}^2 + \Delta \tau^{-1} \sum_{m=2}^{M} \| \sigma_m \|_{W^3}^2 \right). \]

Summing both sides of (4.8) with respect to \( m \), from 2 up to \( M \), we obtain

\[ \| T_E E^M \|_{W^3}^2 + \Delta \tau \sum_{m=2}^{M} \| \sigma_m \|_{W^3}^2 \leq \| T_E E^1 \|_{W^3}^2 + 2 \mu^2 \Delta \tau \sum_{m=1}^{M-1} \| T_E E^m \|_{W^3}^2 + \Delta \tau^{-1} \sum_{m=2}^{M} \| \sigma_m \|_{W^3}^2, \]

which, along with (4.9), yields

\[ \Delta \tau \sum_{m=1}^{M} \| \sigma_m \|_{W^3}^2 \leq C \left( \| T_E E^1 \|_{W^3}^2 + \Delta \tau \| E^1 \|_{W^3}^2 + \Delta \tau^{-1} \sum_{m=2}^{M} \| \sigma_m \|_{W^3}^2 \right). \]

Using (4.1), (2.8), the Cauchy-Schwarz inequality and the arithmetic mean inequality, we have

\[
\| T_E E^1 \|_{W^3}^2 + \Delta \tau \| E^1 \|_{W^3}^2 \leq \Delta \tau^{-1} \| \sigma_1 \|_{W^3}^2 + 2 \Delta \tau \| \mu \| \| w_0 \|_{W^3} \| T_E E^1 \|_{W^3}^2 \\
\leq \Delta \tau^{-1} \| \sigma_1 \|_{W^3}^2 + 2 \Delta \tau \| \mu \| \| w_0 \|_{W^3} \| T_E E^1 \|_{W^3}^2 \\
\leq \Delta \tau^{-1} \| \sigma_1 \|_{W^3}^2 + \frac{1}{2} \| T_E E \|_{W^3}^2 + 2 \Delta \tau^2 \| w_0 \|_{W^3}^2.
\]

which, finally, yields

\[ \| T_E E^1 \|_{W^3}^2 + \Delta \tau \| E^1 \|_{W^3}^2 \leq C \left( \Delta \tau^2 \| w_0 \|_{W^3}^2 + \Delta \tau^{-1} \| \sigma_1 \|_{W^3}^2 \right). \]

Next, we use the Cauchy-Schwarz inequality and (2.9) to get

\[ \| \sigma_m \|_{W^3}^2 \leq 2 \Delta \tau^3 \int_{\Delta m} \| \partial \tau w(s, \cdot) \|_{W^3}^2 ds + 2 \mu^2 \Delta \tau^3 \int_{\Delta m} \| T_E (\partial \tau w(s, \cdot)) \|_{W^3}^2 ds \\
\leq C (\Delta \tau)^3 \int_{\Delta m} \| \partial \tau w(s, \cdot) \|_{W^3}^2 ds, \quad m = 1, \ldots, M.
\]

Finally, we use (4.10), (4.11), (4.12) and (2.14) (with \( \beta = 0, \ell = 1, p = 0 \)) to obtain

\[
\Delta \tau \sum_{m=1}^{M} \| E^m \|_{W^3}^2 \leq C \left( \Delta \tau^2 \| w_0 \|_{W^3}^2 + \Delta \tau^{-1} \sum_{m=1}^{M} \| \sigma_m \|_{W^3}^2 \right) \\
\leq C \left( \Delta \tau^2 \| w_0 \|_{W^3}^2 + \Delta \tau^2 \int_{0}^{T} \| \partial \tau w(s, \cdot) \|_{W^3}^2 ds \right) \\
\leq C \Delta \tau^2 \| w_0 \|_{W^3}^2,
\]

which establishes (4.3) for \( \theta = 1 \).

From (4.2), (4.3) and (2.12) follows that

\[
T_b (W^1 - W^0) + \Delta \tau W^1 = 0, \quad T_b (W^{m} - W^{m-1}) + \Delta \tau (W^m + \mu T_E W^{m-1}) = 0, \quad m = 2, \ldots, M.
\]

Taking the \( L^2(D) \)-inner product of both sides of the first equation above with \( W^1 \) and of the second one with \( W^m \), and then applying (2.13), (2.5) and the arithmetic mean inequality, we obtain

\[ \| T_E W^1 \|_{W^3}^2 - \| T_E W^0 \|_{W^3}^2 + 2 \Delta \tau \| W^1 \|_{W^3}^2 \leq 0, \]

\[ \| T_E W^m \|_{W^3}^2 - \| T_E W^{m-1} \|_{W^3}^2 + \Delta \tau \| W^m \|_{W^3}^2 \leq \mu^2 \Delta \tau \| T_E W^{m-1} \|_{W^3}^2, \quad m = 2, \ldots, M.
\]

The inequalities (4.13) and (4.14), easily, yield that

\[ \| T_E W^m \|_{W^3}^2 \leq (1 + \mu^2 \Delta \tau) \| T_E W^{m-1} \|_{W^3}^2, \quad m = 1, \ldots, M,
\]

from which, after the use of a standard discrete Gronwall argument, we arrive at

\[ \max_{0 \leq m \leq M} \| T_E W^m \|_{W^3}^2 \leq C \| T_E W^0 \|_{W^3}^2.
\]
We sum both sides of (4.14) with respect to \( m \), from 2 up to \( M \), and then use (4.15), to have

\[
\Delta \tau \sum_{m=2}^{M} \| W^m \|^2_{0,D} \leq \| T_k W^1 \|^2_{0,D} + \mu^2 \Delta \tau \sum_{m=1}^{M-1} \| T_k W^m \|^2_{0,D} \\
\leq C \left( \| T_k W^1 \|^2_{0,D} + \| T_k W^0 \|^2_{0,D} \right).
\]

Thus, using (4.16), (4.13), (4.1), (2.9) and (2.2) we obtain

\[
\Delta \tau \sum_{m=1}^{M} \| W^m \|^2_{0,D} \leq C \left( \| T_k W^1 \|^2_{0,D} + \Delta \tau \| W^1 \|^2_{0,D} + \| T_k w_0 \|^2_{0,D} \right)
\]

(4.17)

In addition we have

\[
\Delta \tau \sum_{m=1}^{M} \| w^m \|^2_{0,D} = \sum_{m=1}^{M} \int_{D} \left( \int_{\Delta m} \partial_{\tau} \left[ (\tau - \tau_{m-1}) w^2(\tau, x) \right] d\tau \right) dx
\]

\[
= \sum_{m=1}^{M} \int_{D} \left( \int_{\Delta m} \left[ w^2(\tau, x) + 2(\tau - \tau_{m-1}) w(\tau, x)\right] d\tau \right) dx
\]

\[
\leq \sum_{m=1}^{M} \int_{\Delta m} \left( 2 \| w(\cdot, \cdot) \|^2_{0,D} + (\tau - \tau_{m-1})^2 \| w(\tau, \cdot) \|^2_{0,D} \right) d\tau
\]

\[
\leq 2 \int_{0}^{T} \| w(\tau, \cdot) \|^2_{0,D} d\tau + \int_{0}^{T} \tau^2 \| w(\tau, \cdot) \|^2_{0,D} d\tau,
\]

which, along with (2.14) (with (\( \beta, \ell, p \)) = (0, 0, 0) and (\( \beta, \ell, p \)) = (2, 1, 0)), yields

\[
\Delta \tau \sum_{m=1}^{M} \| w^m \|^2_{0,D} \leq C \| w_0 \|^2_{H^{-2}}.
\]

(4.18)

Thus, (4.17) and (4.18) establish (4.4) for \( \theta = 0 \).

Finally, the estimate (4.3) follows by interpolation, since it is valid for \( \theta = 1 \) and \( \theta = 0 \). \( \square \)

We close this section by deriving, for later use, the following a priori bound.

**Lemma 4.1.** Let \((W^m)_{m=0}^{M}\) be the time-discrete approximations defined by (4.1) -(4.3). Then, there exist a constant \( C > 0 \), independent of \( \Delta \tau \), such that

\[
\left( \Delta \tau \sum_{m=1}^{M} \| \partial_x^3 W^m \|^2_{0,D} \right)^{\frac{1}{2}} \leq C \| w_0 \|_{H^{1}} \quad \forall \; w_0 \in \mathcal{H}^1(D).
\]

(4.19)

**Proof.** In the sequel, we will use the symbol \( C \) to denote a generic constant that is independent of \( \Delta \tau \) and may changes value from one line to the other.

Taking the \( (\cdot, \cdot)_{0,D} \)-inner product of (4.8) with \( \partial_x^2 W^m \) and of (4.9) with \( \partial_x^2 W^1 \), and then integrating by parts, we obtain

\[
(\partial_x W^1 - \partial_x W^0, \partial_x W^1)_{0,D} + \Delta \tau \| \partial_x^3 W^1 \|^2_{0,D} = 0,
\]

(4.20)

\[
(\partial_x W^m - \partial_x W^{m-1}, \partial_x W^m)_{0,D} + \Delta \tau \left[ \| \partial_x^3 W^m \|^2_{0,D} + \mu (\| \partial_x^3 W^m \|^2_{0,D} + \partial_x^3 W^{m-1} \right)_{0,D} = 0,
\]

(4.21)

for \( m = 2, \ldots, M \). Using (2.5) and the arithmetic mean inequality, from (4.20) and (4.21) follows that

\[
\| \partial_x W^1 \|^2_{0,D} - \| \partial_x W^0 \|^2_{0,D} + 2 \Delta \tau \| \partial_x^3 W^1 \|^2_{0,D} \leq 0,
\]

(4.22)

\[
\| \partial_x W^m \|^2_{0,D} - \| \partial_x W^{m-1} \|^2_{0,D} + \Delta \tau \| \partial_x^3 W^m \|^2_{0,D} \leq \Delta \tau \mu^2 \| \partial_x^3 W^{m-1} \|^2_{0,D}, \quad m = 2, \ldots, M.
\]

(4.23)
Now, (4.23) and (4.22), easily, yield that
\[ \| \partial_x W^m \|_{0,D}^2 \leq (1 + \mu^2 \Delta \tau) \| \partial_x W^{m-1} \|_{0,D}^2, \quad m = 2, \ldots, M, \]
which, after a standard induction argument, leads to
\[ (4.24) \quad \max_{1 \leq m \leq M} \| \partial_x W^m \|_{0,D}^2 \leq C \| \partial_x W^1 \|_{1,D}^2. \]
After summing both sides of (4.23) with respect to \(m\), from 2 up to \(M\), we obtain
\[(4.25) \quad \Delta \tau \sum_{m=2}^M \| \partial_x^3 W^m \|_{0,D}^2 \leq \| \partial_x W^1 \|_{0,D}^2 + \mu^2 \Delta \tau \sum_{m=1}^{M-1} \| \partial_x W^m \|_{0,D}^2,\]
which, after using (4.24), yields
\[ (4.26) \quad \Delta \tau \sum_{m=1}^M \| \partial_x^3 W^m \|_{0,D}^2 \leq C \| \partial_x W^0 \|_{0,D}^2, \]
Finally, we combine (4.25), (4.22) and (2.1) to get
\[ (4.27) \quad \Delta \tau \sum_{m=1}^M \| \partial_x^3 W^m \|_{0,D}^2 \leq C \| \partial_x W^0 \|_{0,D}^2, \]
which, easily, yields (4.19).

### 4.2. Fully-Discrete Approximations

The modified-IMEX time-stepping method along with a finite element space discretization yields a fully discrete method for the approximation of the solution to the deterministic problem (1.4). The method begins by setting
\[ (4.28) \quad W^0 := P_h w_0 \]
and specifying \(W^1_h \in M_h^r\) such that
\[ (4.29) \quad W^1_h - W^0_h + \Delta \tau B_h W^1_h = 0. \]
Then, for \(m = 2, \ldots, M\), it finds \(W^m_h \in M_h^r\) such that
\[ (4.30) \quad W^m_h - W^{m-1}_h + \Delta \tau \left[ B_h W^m_h + \mu P_h \left( \partial_x W^{m-1}_h \right) \right] = 0. \]

Adopting the viewpoint that the fully-discrete approximations defined above are approximations of the time-discrete ones defined in the previous section, we estimate below the corresponding approximation error in a discrete in time \(L_2^1(L_2^2)\)-norm.

**Proposition 4.2.** Let \(r = 2\) or \(3\), \((W^m)_M^m = 0\) be the time discrete approximations defined by (4.11)–(4.13), and \((W^m)_M^M = 0 \subset M_h^r\) be the fully discrete approximations specified in (4.26)–(4.28). Then, there exist a constant \(C > 0\), independent of \(\Delta \tau\) and \(h\), such that
\[ (4.31) \quad \left( \sum_{m=1}^M \| W^m - W^m_h \|_{0,D}^2 \right)^{\frac{1}{2}} \leq C h^{\theta} \| w_0 \|_{\dot{H}^{4-\theta}} \quad \forall w_0 \in \dot{H}^4(D), \quad \forall \theta \in [0, 1]. \]

**Proof.** In the sequel, we will use the symbol \(C\) to denote a generic constant which is independent of \(\Delta \tau\) and \(h\), and may changes value from one line to the other.

Let \(Z^m := W^m - W^m_h\) for \(m = 0, \ldots, M\). Then, from (4.22), (4.27), (4.29) and (4.28), we obtain the following error equations:
\[ (4.32) \quad T_{b,h}(Z^1 - Z^0) + \Delta \tau Z^1 = \Delta \tau \xi^1, \]
\[ (4.33) \quad T_{b,h}(Z^m - Z^{m-1}) + \Delta \tau \left[ Z^m + \mu T_{b,h}(\partial_x^2 Z^{m-1}) \right] = \Delta \tau \xi^m, \quad m = 2, \ldots, M, \]

where

$$\xi^m := (T_B - T_{B,h}) \partial_x^2 W^m, \quad m = 1, \ldots, M.$$  

Taking the $L^2(D)$–inner product of both sides of (4.31) with $Z^m$, we obtain

$$\langle T_{B,h}(Z^m - Z^{m-1}), Z^m \rangle_{0,D} + \Delta \tau \|Z^m\|_{0,D}^2 = -\mu \Delta \tau (T_{B,h}(\partial_x^2 Z^{m-1}), Z^m)_{0,D}$$

$$+ \Delta \tau \langle \xi^m, Z^m \rangle_{0,D}, \quad m = 2, \ldots, M,$$

which, along with (2.17) and (2.3), yields

$$\langle \partial_x^2(T_{B,h} Z^m) \rangle_{0,D}^2 - \langle \partial_x^2(T_{B,h} Z^{m-1}) \rangle_{0,D}^2 + \langle \partial_x^2(T_{B,h} (Z^m - Z^{m-1})) \rangle_{0,D}^2$$

$$+ 2 \Delta \tau \|Z^m\|_{0,D}^2 = A_1^m + A_2^m$$

for $m = 2, \ldots, M$, where

$$A_1^m := 2 \Delta \tau \langle \xi^m, Z^m \rangle_{0,D},$$

$$A_2^m := -2 \mu \Delta \tau (T_{B,h}(\partial_x^2 Z^{m-1}), Z^m)_{0,D} .$$

Using (2.17), integration by parts, the Cauchy-Schwarz inequality, the arithmetic mean inequality, we have

$$A_1^m \leq \Delta \tau \left( \|Z^m\|_{0,D}^2 + \|\xi^m\|_{0,D}^2 \right)$$

and

$$A_2^m = -2 \mu \Delta \tau \left( \partial_x^2(Z^{m-1}, T_{B,h} Z^m)_{0,D} \right.$$

$$- 2 \mu \Delta \tau \left( Z^{m-1}, \partial_x^2(T_{B,h} Z^{m-1}) \right)_{0,D}$$

$$\leq 2 |\mu| \Delta \tau \|Z^{m-1}\|_{0,D} \|\partial_x^2(T_{B,h} (Z^m - Z^{m-1}))\|_{0,D}$$

$$+ 2 |\mu| \Delta \tau \|Z^{m-1}\|_{0,D} \|\partial_x^2(T_{B,h} Z^{m-1})\|_{0,D}$$

$$\leq \Delta \tau \mu^2 \|Z^{m-1}\|_{0,D}^2 + \|\partial_x^2(T_{B,h} (Z^m - Z^{m-1}))\|_{0,D}^2$$

$$+ \Delta \tau \mu^2 \|Z^{m-1}\|_{0,D}^2 + 2 \Delta \tau \mu^2 \|\partial_x^2(T_{B,h} Z^{m-1})\|_{0,D}^2, \quad m = 2, \ldots, M.$$ 

Now, we combine (4.33), (4.34) and (4.35) to get

$$\|\partial_x^2(T_{B,h} Z^m)\|_{0,D}^2 + \Delta \tau \|Z^m\|_{0,D}^2 \leq \|\partial_x^2(T_{B,h} Z^{m-1})\|_{0,D}^2 + \Delta \tau \|\xi^m\|_{0,D}^2$$

$$+ 2 \Delta \tau \mu^2 \left( \|\partial_x^2(T_{B,h} Z^{m-1})\|_{0,D}^2 + \Delta \tau \|Z^{m-1}\|_{0,D}^2 \right)$$

for $m = 2, \ldots, M$. Let $Y^\ell := \|\partial_x^2(T_{B,h} Z^\ell)\|_{0,D}^2 + \Delta \tau \|Z^\ell\|_{0,D}^2$ for $\ell = 1, \ldots, M$. Then (4.36) yields

$$Y^m \leq (1 + 2 \mu^2 \Delta \tau) Y^{m-1} + \Delta \tau \|\xi^m\|_{0,D}^2, \quad m = 2, \ldots, M,$$

from which, after applying a standard discrete Gronwall argument, we conclude that

$$\max_{1 \leq m \leq M} Y^m \leq C \left( Y^1 + \Delta \tau \sum_{m=2}^M \|\xi^m\|_{0,D}^2 \right).$$

Since $T_{B,h} Z^0 = 0$, after taking the $L^2(D)$–inner product of both sides of (4.30) with $Z^1$, and then using (2.17) and the arithmetic mean inequality, we obtain

$$\|\partial_x^2(T_{B,h} Z^1)\|_{0,D}^2 + \Delta \tau \|Z^1\|_{0,D}^2 \leq \Delta \tau \|\xi^1\|_{0,D}^2,$$

which, along with (4.37), yields

$$\max_{1 \leq m \leq M} Y^m \leq C \Delta \tau \sum_{m=1}^M \|\xi^m\|_{0,D}^2.$$
Now, summing both sides of (4.36) with respect to $m$, from 2 up to $M$, we obtain
\[
\Delta \tau \sum_{m=2}^{M} \|Z^m\|_{0,D}^2 \leq \|\partial_z^2(T_{B,h}Z^1)\|_{0,D}^2 + \frac{\Delta \tau}{2} \sum_{m=1}^{M-1} \|Z^m\|_{0,D}^2 \\
+ \Delta \tau \sum_{m=2}^{M} \|\xi^m\|_{0,D}^2 + 2 \mu^2 \Delta \tau \sum_{m=1}^{M-1} \gamma^m,
\]
which, along with (4.39), yields
\[
\Delta \tau \sum_{m=1}^{M} \|Z^m\|_{0,D}^2 \leq \|\partial_z^2(T_{B,h}Z^1)\|_{0,D}^2 + \Delta \tau \|Z^1\|_{0,D}^2 \\
+ \Delta \tau \sum_{m=2}^{M} \|\xi^m\|_{0,D}^2 + 2 \mu^2 \Delta \tau \sum_{m=1}^{M-1} \gamma^m \\
\leq C \left( \max_{1 \leq m \leq M-1} \gamma^m + \Delta \tau \sum_{m=2}^{M} \|\xi^m\|_{0,D}^2 \right) \\
\leq C \Delta \tau \sum_{m=1}^{M} \|\xi^m\|_{0,D}^2.
\]
Therefore, (4.40) yields (4.41) for $\theta = 1$.

From (4.27) and (4.28) we conclude that
\[
T_{B,h}(W^1_h - W^{0}) + \Delta \tau W^1_h = 0,
\]
\[
T_{B,h}(W^m_h - W^{m-1}_h) + \Delta \tau W^m_h = -\mu \Delta \tau T_{B,h}(\partial_z^2 W^{m-1}_h), \quad m = 2, \ldots, M.
\]
Taking the $L^2(D)$-inner product of both sides of the first equation above with $W^1_h$ and of the second one with $W^m_h$, and then applying (2.17) and (2.5), we obtain
\[
\|\partial_z^2(T_{B,h}W^1_h)\|_{0,D}^2 + \|\partial_z^2(T_{B,h}W^m_h)\|_{0,D}^2 + 2 \mu^2 \Delta \tau \|W^m_h\|_{0,D}^2 \leq 0,
\]
\[
\|\partial_z^2(T_{B,h}W^m_h)\|_{0,D}^2 + 2 \Delta \tau \|W^m_h\|_{0,D}^2 = \|\partial_z^2(T_{B,h}W^{m-1}_h)\|_{0,D}^2 + A^m_3, \quad m = 2, \ldots, M,
\]
where
\[
A^m_3 := -2 \mu \Delta \tau (T_{B,h}(\partial_z^2 W^{m-1}_h), W^m_h)_{0,D}.
\]
Using (2.17), integration by parts, the Cauchy-Schwarz inequality, and the arithmetic mean inequality, we have
\[
A^m_3 = -2 \mu \Delta \tau (W^{m-1}_h, \partial_z^2 (T_{B,h}W^m_h))_{0,D} \\
= -2 \mu \Delta \tau (W^{m-1}_h, \partial_z^2 (T_{B,h} (W^m_h - W^{m-1}_h)))_{0,D} \\
- 2 \mu \Delta \tau (W^{m-1}_h, \partial_z^2 (T_{B,h} W^{m-1}_h))_{0,D} \\
\leq \Delta \tau \mu^2 \|W^{m-1}_h\|_{0,D}^2 + \|\partial_z^2 (T_{B,h} (W^m_h - W^{m-1}_h))\|_{0,D}^2 \\
+ \Delta \tau \mu^2 \|W^{m-1}_h\|_{0,D}^2 + 2 \Delta \mu \|\partial_z^2 (T_{B,h} W^{m-1}_h)\|_{0,D}^2, \quad m = 2, \ldots, M.
\]
Combining (4.43) and (4.44), we arrive at
\[
\|\partial_z^2(T_{B,h}W^m_h)\|_{0,D}^2 + 2 \Delta \tau \|W^m_h\|_{0,D}^2 \leq \|\partial_z^2(T_{B,h}W^{m-1}_h)\|_{0,D}^2 + \Delta \tau \mu^2 \|W^{m-1}_h\|_{0,D}^2 \\
+ 2 \Delta \mu \|\partial_z^2 (T_{B,h} W^{m-1}_h)\|_{0,D}^2 + \Delta \tau \|W^{m-1}_h\|_{0,D}^2, \quad m = 2, \ldots, M.
\]
Let \( \Omega^\ell := \| \partial_x^2 (T_{B,h} W^\ell_h) \|_{0,D}^2 + \Delta \tau \| W^\ell_h \|_{0,D}^2 \) for \( \ell = 1, \ldots, M \). Then, we use (4.12), (4.26), (2.18), (2.2) and (4.45) to obtain

\[
\Omega^1_h \leq \| \partial_x^2 (T_{B,h} P_h w_0) \|_{0,D}^2 \\
\leq \| \partial_x^2 (T_{B,h} w_0) \|_{0,D}^2 \\
\leq \| w_0 \|_{2,D}^2 \\
\leq \| w_0 \|_{H^{-2}}^2
\]

(4.46)

and

\[
\Omega^m_h \leq (1 + 2 \mu^2 \Delta \tau) \Omega^{m-1}_h, \quad m = 2, \ldots, M.
\]

(4.47)

From (4.47), after the application of a standard discrete Gronwall argument and the use of (4.46), we conclude that

\[
\max_{1 \leq m \leq M} \Omega^m_h \leq C \Omega^1_h \\
\leq C \| w_0 \|_{H^{-2}}^2.
\]

(4.48)

Summing both sides of (4.45) with respect to \( m \), from 2 up to \( M \), we have

\[
\Delta \tau \sum_{m=2}^M \| W^m_h \|_{0,D}^2 \leq \| \partial_x^2 (T_{B,h} W^1_h) \|_{0,D}^2 + \Delta \tau \sum_{m=1}^{M-1} \| W^m_h \|_{0,D}^2 + 2 \mu^2 \Delta \tau \sum_{m=1}^{M-1} \Omega^m_h,
\]

which, along with (4.48), yields

\[
\Delta \tau \sum_{m=1}^M \| W^m_h \|_{0,D}^2 \leq \Omega^1_h + 2 \mu^2 \Delta \tau \sum_{m=1}^{M-1} \Omega^m_h \\
\leq C \| w_0 \|_{H^{-2}}^2.
\]

(4.49)

Thus, (4.49) and (4.17) yield (4.29) for \( \theta = 0 \).

Thus, the error estimate (4.29) follows by interpolation, since it holds for \( \theta = 1 \) and \( \theta = 0 \).

5. Convergence analysis of the IMEX finite element method

In order to estimate the approximation error of the IMEX finite element method given in Section 1.2, we use, as a tool, the corresponding IMEX time-discrete approximations of \( u \), which are defined first by setting

\[
U^0 := 0
\]

and then, for \( m = 1, \ldots, M \), by seeking \( U^m \in \text{H}^4(D) \) such that

\[
U^m - U^{m-1} + \Delta \tau \left( \partial_x^4 U^m + \mu \partial_x^2 U^{m-1} \right) = \int_{\Delta_m} \partial_x W d\tau \quad \text{a.s.}
\]

(5.2)

Thus, we split the total error of the IMEX finite element method as follows

\[
\max_{0 \leq m \leq M} \left( \mathbb{E} \left[ \| u^m - U^m_h \|_{0,D}^2 \right] \right)^{1/2} \leq \max_{0 \leq m \leq M} E^m_{\text{TDR}} + \max_{0 \leq m \leq M} E^m_{\text{SDR}},
\]

(5.3)

where \( u^m := u(\tau_m, \cdot) \), \( E^{m}_{\text{TDR}} := (\mathbb{E} \left[ \| u^m - U^m_h \|_{0,D}^2 \right])^{1/2} \) is the time discretization error at \( \tau_m \), and \( E^{m}_{\text{SDR}} := (\mathbb{E} \left[ \| U^m - U^m_h \|_{0,D}^2 \right])^{1/2} \) is the space discretization error at \( \tau_m \).
5.1. **Estimating the time discretization error.** The convergence estimate of Proposition 4.1 is the main tool in providing a discrete in time $L^\infty_t(L^2_x)$ error estimate of the time-discretization error (cf. [11], [6], [7]).

**Proposition 5.1.** Let $u$ be the solution to (1.3) and $(U^m)_{m=0}^M$ be the time-discrete approximations of $u$ defined by (5.1)–(5.2). Then, there exists a constant $\hat{c}_{\text{TDR}}$, independent of $\Delta t$, $M$ and $\Delta \tau$, such that

$$\max_{0 \leq m \leq M} c_{\text{TDR}}^m \leq \hat{c}_{\text{TDR}} \epsilon^{-\frac{1}{2}} \Delta \tau^{\frac{1}{2}} \epsilon \quad \forall \epsilon \in \left(0, \frac{1}{2}\right].$$

**Proof.** In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\Delta t$, $M$ and $\Delta \tau$, and may changes value from one line to the other. First, we introduce some notation by letting $I : L^2(D) \to L^2(D)$ be the identity operator, $Y : H^2(D) \to L^2(D)$ be the differential operator $Y := 1 - \Delta \tau \mu \mathcal{D}_{\mathcal{D}}$, and $\Lambda : L^2(D) \to \mathcal{H}^1(D)$ be the inverse elliptic operator $\Lambda := (1 + \Delta \tau \partial_x^2)^{-1}$. Then, for $m = 1, \ldots, M$, we define the operator $Q^m : L^2(D) \to \mathcal{H}^1(D)$ by $Q^m := (\Lambda \circ Y)^{-1} \circ \Lambda$. Also, for given $w_0 \in \mathcal{H}^2(D)$, let $(S^m_{\Delta \tau}(w_0))_{m=0}^M$ be time-discrete approximations of the solution to the deterministic problem (1.3), defined by (4.1)–(4.3). Then, using a simple induction argument, we conclude that

$$S^m_{\Delta \tau}(w_0) = Q^m(w_0), \quad m = 1, \ldots, M.$$

Let $m \in \{1, \ldots, M\}$. Applying a simple induction argument on (5.2) we conclude that

$$U^m = \sum_{\ell=1}^m \int_{\Delta \ell} Q^{m-\ell+1}(\partial_t X(\tau, \cdot)) \, d\tau,$$

which, along with (1.4), yields

$$U^m = \frac{1}{\Delta t} \sum_{i=1}^n \sum_{n=1}^N R_{ni} \lambda_i \left( \sum_{\ell=1}^m \int_{\Delta \ell} X_{\Delta \tau}(\tau) S^{m-\ell+1}_{\Delta \tau}(\epsilon_i) \, d\tau \right),$$

(5.6)

$$= \frac{1}{\Delta t} \sum_{i=1}^n \sum_{n=1}^N R_{ni} \lambda_i \left( \int_0^\tau X_{\Delta \tau}(\tau) \left( \sum_{\ell=1}^m X_{\Delta \tau}(\tau) S^{m-\ell+1}_{\Delta \tau}(\epsilon_i) \right) \, d\tau \right),$$

(5.7)

Also, using (1.5) and (1.6), and proceeding in similar manner, we arrive at

$$u^m = \int_0^{\tau_m} S(\tau_m - \tau) (\partial_x X(\tau, \cdot)) \, d\tau$$

$$= \frac{1}{\Delta t} \sum_{i=1}^n \sum_{n=1}^N R_{ni} \lambda_i \left( \int_{\tau_n}^{\tau_m} \left( \sum_{\ell=1}^m X_{\Delta \tau}(\tau) S(\tau_m - \tau)(\epsilon_i) \right) \, d\tau \right).$$

Thus, using (5.6) and (5.7) along with Remark 4.3, we obtain

$$E_{\text{TDR}}^m = \frac{1}{\Delta t} \sum_{i=1}^n \sum_{n=1}^N \lambda_i^2 \int_D \left( \int_{\tau_n}^{\tau_m} \left( \sum_{\ell=1}^m X_{\Delta \tau}(\tau) \left[ S^{m-\ell+1}_{\Delta \tau}(\epsilon_i) - S(\tau_m - \tau)(\epsilon_i) \right] \right) \, d\tau \right)^2 \, dx$$

$$\leq \sum_{i=1}^n \sum_{n=1}^N \lambda_i^2 \int_D \int_{\tau_n}^{\tau_m} \left( \sum_{\ell=1}^m X_{\Delta \tau}(\tau) \left[ S^{m-\ell+1}_{\Delta \tau}(\epsilon_i) - S(\tau_m - \tau)(\epsilon_i) \right] \right)^2 \, d\tau \, dx$$

$$\leq \sum_{i=1}^n \lambda_i^2 \int_0^{\tau_m} \int_D \left( \sum_{\ell=1}^m X_{\Delta \tau}(\tau) \left[ S^{m-\ell+1}_{\Delta \tau}(\epsilon_i) - S(\tau_m - \tau)(\epsilon_i) \right] \right)^2 \, dx \, d\tau$$

$$\leq \sum_{i=1}^n \lambda_i^2 \left( \sum_{\ell=1}^m \int_{\Delta \ell} \left\| S^{m-\ell+1}_{\Delta \tau}(\epsilon_i) - S(\tau_m - \tau)(\epsilon_i) \right\|_{0,D}^2 \, d\tau \right),$$

where $\| \cdot \|_{0,D}$ denotes the $L^2(D)$ norm.
which, easily, yields
\begin{equation}
\mathcal{E}_{TDR}^m \leq \sqrt{B_1^m} + \sqrt{B_2^m},
\end{equation}
with
\begin{align*}
B_1^m &:= \sum_{i=1}^{M} \lambda_i^2 \left( \sum_{\ell=1}^{M} \Delta \tau \| S_{m-\ell+1}^m (\varepsilon_i) - S(\tau_{m-\ell+1})(\varepsilon_i) \|_{W_{\text{D},\varepsilon}}^2 \right), \\
B_2^m &:= \sum_{i=1}^{M} \lambda_i^2 \left( \sum_{\ell=1}^{M} \int_{\Delta \tau} \| S(\tau_{m-\ell+1})(\varepsilon_i) - S(\tau_{m-\tau})(\varepsilon_i) \|_{W_{\text{D},\varepsilon}}^2 \, d\tau \right).
\end{align*}
Proceeding as in the proof of Theorem 4.1 in [7] we get
\begin{equation}
\sqrt{B_2^m} \leq C \Delta \tau^{-\frac{3}{2}}.
\end{equation}
Also, using the error estimate (4.3) it follows that
\begin{align*}
\sqrt{B_1^m} &\leq C \Delta \tau^\theta \left( \sum_{i=1}^{M} \lambda_i^2 \| \varepsilon_i \|_{W_{\text{D},\varepsilon}}^2 \right)^{\frac{1}{2}} \\
&\leq C \Delta \tau^\theta \left( \sum_{i=1}^{M} \frac{1}{\lambda_i^{2\theta}} \right)^{\frac{1}{2}} \forall \theta \in [0,1].
\end{align*}
Setting \( \theta = \frac{1}{8} - \epsilon \) with \( \epsilon \in \left(0, \frac{1}{8}\right] \), we have
\begin{equation}
\sqrt{B_1^m} \leq C \Delta \tau^{\frac{1}{4} - \epsilon} \left( \sum_{i=1}^{M} \frac{1}{\lambda_i^{4\epsilon}} \right)^{\frac{1}{2}} \leq C \Delta \tau^{\frac{1}{4} - \epsilon} \left( 1 + \int_1^M x^{-1-8\epsilon} \, dx \right)^{\frac{1}{2}} \leq C \Delta \tau^{\frac{1}{4} - \epsilon} \left( 1 - \frac{1}{M^{8\epsilon}} \right)^{\frac{1}{2}}.
\end{equation}
Thus, the estimate (5.3) follows, easily, as a simple consequence of (5.8), (5.9) and (5.10).

5.2. Estimating the space discretization error. The outcome of Proposition 5.2 will be used below in the derivation of a discrete in time \( L_1^\infty(L_2^\infty(L_2^2)) \) error estimate of the space discretization error (cf. [11], [6], [21]).

Proposition 5.2. Let \( r = 2 \) or \( 3 \), \((U_h^m)_{m=0}^M\) be the fully discrete approximations defined by (1.10)–(1.11) and \((U_m^m)_{m=0}^M\) be the time discrete approximations defined by (5.1)–(5.2). Then, there exists a constant \( \hat{c}_{\text{DR}} > 0 \), independent of \( M, \Delta t, \Delta \tau \) and \( h \), such that
\begin{equation}
\max_{0 \leq m \leq M} \mathcal{E}_{\text{DR}}^m \leq \hat{c}_{\text{DR}} \epsilon^{-\frac{1}{2}} h^{\frac{1}{2} - \epsilon} \forall \epsilon \in \left(0, \frac{1}{6}\right].
\end{equation}

Proof. In the sequel, we will use the symbol \( C \) to denote a generic constant that is independent of \( \Delta t, M, \Delta \tau \) and \( h \), and may changes value from one line to the other.

Let us denote by \( I : L_2^2(D) \to L_2^2(D) \) the identity operator, by \( Y_h : M_h^r \to M_h^r \) the discrete differential operator \( Y_h := I - \mu \Delta \tau (P_h \circ \partial_2^2) \), \( \Lambda_h : L_2^2(D) \to M_h^r \) be the inverse discrete elliptic operator \( \Lambda_h := (I + \Delta \tau B_h)^{-1} \circ P_h \). Then, for \( m = 1, \ldots, M \), we define the auxiliary operator \( Q_h^m : L_2^2(D) \to M_h^r \) by \( Q_h^m := (\Lambda_h \circ Y_h)^{m-1} \circ \Lambda_h \). Also, for given \( w_0 \in \tilde{H}^2(D) \), let \( (S_h^m(w_0))_{m=0}^M \) be fully discrete discrete approximations of the solution to the deterministic problem (1.4), defined by (4.20)–(4.28). Then, using a simple induction argument, we conclude that
\begin{equation}
S_h^m(w_0) = Q_h^m(w_0), \quad m = 1, \ldots, M.
\end{equation}
Let \( m \in \{1, \ldots, M\} \). Using a simple induction argument on (1.11), (1.6) and (5.12), we conclude that
\[
U_m^h = \sum_{\ell=1}^{m} \int_{\Delta_t} Q_{\ell}^{m-\ell+1} (\partial_x W(\tau, \cdot)) \, d\tau
\]
(5.13)
\[
= -\frac{1}{\Delta t} \sum_{i=1}^{m} \sum_{n=1}^{N} R_i^n \lambda_i \left[ \int_{\tau_n}^{\tau_{n+1}} \left( \sum_{\ell=1}^{m} X_{\Delta t} (\tau) S_{\ell}^{m-\ell+1} (\varepsilon_i) \right) \, d\tau \right].
\]
After, using (5.13), (5.6) and Remark 1.8, and proceeding as in the proof of Proposition 5.1, we arrive at
\[
E_m^{SDR} \leq \sum_{i=1}^{m} \lambda_i^2 \left( \sum_{\ell=1}^{m} \Delta \tau \| S_{\ell}^{m-\ell+1} (\varepsilon) - S_{\ell}^{m-\ell+1} (\varepsilon_i) \|_{0.D}^2 \, d\tau \right)^{\frac{1}{2}},
\]
which, along (4.29), yields
\[
E_m^{SDR} \leq C h^{r_0} \frac{1}{\theta} \left( \sum_{i=1}^{m} \lambda_i^2 \| \varepsilon_i \|_{0.D}^2 \right)^{\frac{1}{2}} \quad \forall \theta \in [0, 1].
\]
Setting \( \theta = \frac{1}{\theta} - \delta \) with \( \delta \in (0, \frac{1}{\theta}] \), we have
\[
E_m^{SDR} \leq C h^{r_0} \left( \sum_{i=1}^{m} \lambda_i^2 \right)^{\frac{1}{2}} \leq C h^{r_0} \left( \sum_{i=1}^{m} \frac{1}{\lambda_i^2} \right)^{\frac{1}{2}} \quad \forall \theta \in [0, 1].
\]

5.3. Estimating the total error.

**Theorem 5.3.** Let \( r = 2 \) or \( 3 \), \( u \) be the solution to the problem (1.5), and \( (U_m^h)^M_{m=0} \) be the finite element approximations of \( u \) constructed by \( (1.10)-(1.11) \). Then, there exists a constant \( \varepsilon_{TTL} > 0 \), independent of \( h, \Delta t \), and \( M \), such that
\[
(5.15) \quad \max_{0 \leq m \leq M} \left( E \left[ \| U_m^h - u_m^m \|_{0.D}^2 \right] \right)^{\frac{1}{2}} \leq \varepsilon_{TTL} \left( \epsilon_1^{\frac{r_0}{2}} \Delta t^{\frac{r_0}{2}} + \epsilon_2^{\frac{r_0}{2}} h^{r_0} \right)
\]
for all \( \epsilon_1 \in (0, \frac{1}{8}] \) and \( \epsilon_2 \in (0, \frac{1}{8}] \).

**Proof.** The error bound (5.15) follows easily from (5.4), (5.11) and (5.3). \( \square \)

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E-mail address: georgios.zouraris@uoc.gr