Non-Perturbative Superpotentials from Membrane Instantons in Heterotic M-Theory

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ABSTRACT

A formalism for calculating the open supermembrane contribution to the non-perturbative superpotential of moduli in heterotic M-theory is presented. This is explicitly applied to the Calabi-Yau (1,1)-moduli and the separation modulus of the end-of-the-world BPS three-branes, whose non-perturbative superpotential is computed. The role of gauge bundles on the boundaries of the open supermembranes is discussed in detail, and a topological criterion presented for the associated superpotential to be non-vanishing.


1 Introduction

In a series of papers [1, 2, 3, 4], it was shown that Hořava-Witten theory [5, 6], when compactified on a Calabi-Yau threefold $CY_3$ with non-vanishing $G$-flux, leads to a natural brane universe theory of particle physics, called heterotic M-theory. Heterotic M-theory has a five-dimensional bulk space consisting of a specific gauging of $N = 2$ supergravity coupled to both hyper- and vector supermultiplets. This space has finite length in the fifth-direction, terminating in two five-dimensional BPS three-branes (eleven-dimensional nine-branes wrapped on $CY_3$), one, the observable brane, at one end of the universe and one, the hidden brane, at the other. By constructing holomorphic vector bundles on Calabi-Yau threefolds with both trivial and non-trivial homotopy, it was demonstrated that both three-family grand unified theories [7] and the standard model [8, 9, 10, 11] can appear on the observable brane, while an appropriate hidden sector arises on the other brane. An important feature of heterotic M-theory is that the non-trivial vector bundles required to give realistic particle physics gauge groups and spectra generically predict, via anomaly cancellation, the existence of additional five-dimensional BPS three-branes (eleven-dimensional five-branes wrapped on a holomorphic curve $C \subset CY_3$). These branes live in the bulk space and represent new non-perturbative vacua. They may play an important phenomenological role, both in particle physics [12, 13, 14, 15, 16, 17] and cosmology [18, 19, 20, 21, 22]. One interesting aspect of these bulk branes, their absorption and expulsion from the boundary three-branes via “small instanton” phase transitions, was discussed in [23].

In this paper, we begin an investigation of the stability and dynamics of non-perturbative heterotic M-theory vacua. All work referenced above assumed that the potential energy for the various moduli fields in the low energy four-dimensional effective theory vanished. This is certainly true for moduli at the perturbative level. However, it is well known [24, 25, 26, 27, 28] that non-vanishing contributions to the moduli potential energy can arise from various aspects of non-perturbative physics. Specifically, in the four-dimensional $N = 1$ supersymmetric effective theory of heterotic M-theory, a non-vanishing superpotential for moduli, $W$, can arise from non-perturbative effects. In this paper, we will initiate our study of this topic by explicitly computing the contribution to $W$ of open supermembranes stretched between the observable and hidden BPS boundary three-branes. This superpotential will depend on the $(1,1)$-moduli $T_I$ which carry information about the Kähler metric deformations of $CY_3$ and the separation modulus $R$ of the two boundary three-branes. Specifically, we will do the following.

In Section 2, we discuss BPS membranes in heterotic M-theory and show that they must
stretch between the boundary three-branes in order to preserve $N = 1$ supersymmetry. That is, we must consider “open” supermembranes only. In the next section, we present the $\kappa$-invariant action for an open supermembrane coupled to the gauge bundles of the boundary three-branes. Section 4 studies the BPS conditions for the open membrane when wrapped on a complex curve $C$ in $CY_3$. It is shown that curve $C$ must be holomorphic. In Section 5, we perform the dimensional reduction of the membrane theory on the fifth-dimensional interval $S^1/Z_2$ and show that this theory becomes the heterotic superstring coupled to $E_8 \times E_8$ and wrapped on holomorphic curve $C$. The following section is devoted to a careful discussion of the moduli of heterotic M-theory and their four-dimensional low energy effective theory. The formalism for computing a non-perturbative contribution to the superpotential via instanton contributions to the fermion two-point functions is presented. In Sections 7 and 8, the fermion two-point functions generated by open supermembranes stretched between the two boundary three-branes are explicitly computed; Section 7 discussing the contributions of the membrane worldvolume while Section 8 computes the contributions of the membrane boundaries. Finally, in Section 9, using the formalism presented in Section 6 and the results of Sections 7 and 8, we extract the expression for the non-perturbative superpotential $W$ generated by open supermembranes. We refer the reader to equation (9.3) for the final result. Various necessary technical remarks and formalism are presented in two Appendices A and B. We want to emphasize that our goal in this paper is to compute the superpotential for moduli associated with open supermembranes, a quantity holomorphic on chiral superfields and independent of the dilaton. Since it is independent of the dilaton, the result, to lowest order, can be computed taking the low energy limit where the wrapped supermembrane is replaced by the heterotic string. We take this approach here. Non-holomorphic quantities, such as the Kähler potential, are more subtle, but we do not compute them in this paper.

Within this context, it is clear that similar calculations can be carried out for the open supermembrane contributions to the 1) boundary three-brane—bulk three-brane (eleven-dimensional nine-brane—five-brane) superpotential and to the 2) bulk three-brane—bulk three-brane (eleven-dimensional five-brane—five-brane) superpotential. Perhaps not surprisingly, these calculations and results are similar to those given in this paper, although there are some fundamental differences. Due to the length of the present paper, we will present the results involving at least one bulk brane in another publication. The formalism both in this paper and in relies heavily on the ground breaking work of Strominger, Becker and Becker and Harvey and Moore. Recently, a paper
due to Moore, Peradze and Saulina [30] appeared whose results overlap substantially with the results in this paper and in [29]. We acknowledge their work and appreciate their pre-announcement of our independent study of this subject.

2 Membranes in Heterotic M-Theory:

Eleven-Dimensional Supergravity and BPS Membranes:

As is well-known, $D = 11, N = 1$ supersymmetry consists of a single supergravity multiplet [31]. This multiplet contains as its component fields a graviton $\hat{g}_{\hat{M}\hat{N}}$, a three-form $\hat{C}_{\hat{M}\hat{N}\hat{P}}$ and a Majorana gravitino $\hat{\Psi}_{\hat{M}}$. The dynamics of this supermultiplet is specified by a unique action, whose bosonic terms are

$$S_{SG} = -\frac{1}{2\kappa^2} \int_{M_{11}} d^{11}\hat{x} \sqrt{-\hat{g}} [\hat{R} + \frac{1}{24} \hat{G}_{\hat{M}\hat{N}\hat{O}\hat{P}} \hat{G}^{\hat{M}\hat{N}\hat{O}\hat{P}}$$

$$+ \frac{\sqrt{2}}{1728} \epsilon_{\hat{M}_{1}\ldots\hat{M}_{11}} \hat{C}_{\hat{M}_{1}\hat{M}_{2}\hat{M}_{3}} \hat{G}_{\hat{M}_{4}\ldots\hat{M}_{7}} \hat{G}_{\hat{M}_{8}\ldots\hat{M}_{11}}],$$

where $(\hat{x}^{0}, \ldots, \hat{x}^{10})$ are the eleven-dimensional coordinates, $\hat{g} = \text{det} \hat{g}_{\hat{M}\hat{N}}$ and $\hat{G}_{\hat{M}\hat{N}\hat{P}\hat{Q}} = 24 \partial_{[\hat{M}} \hat{C}_{\hat{N}\hat{P}\hat{Q}]}$ is the field strength of the three-form $\hat{C}$ defined by $\hat{G} = d\hat{C}$. This action is invariant under the supersymmetry transformations of the component fields. For our purposes, we need only specify the supersymmetry variation of the gravitino field $\hat{\Psi}_{\hat{M}}$, which is given by

$$\delta_{\hat{\epsilon}} \hat{\Psi}_{\hat{M}} = \hat{D}_{\hat{M}} \hat{\epsilon} + \frac{\sqrt{2}}{288} (\hat{\Gamma}_{\hat{M}}^{\hat{N}\hat{P}\hat{Q}\hat{R}} - 8 \delta_{\hat{M}}^{\hat{N}} \hat{\Gamma}^{\hat{P}\hat{Q}\hat{R}}) \hat{\epsilon} \hat{G}_{\hat{N}\hat{P}\hat{Q}\hat{R}} + \cdots,$$

where $\hat{\epsilon}$ is the Majorana supersymmetry parameter and the dots denote terms that involve the fermion fields of the theory. The 11-dimensional spacetime Dirac matrices $\hat{\Gamma}_{\hat{M}}$ satisfy $\{\hat{\Gamma}_{\hat{M}}, \hat{\Gamma}_{\hat{N}}\} = 2\hat{g}_{\hat{M}\hat{N}}$. Throughout this paper, we will follow convention and refer to the theory of $D = 11, N = 1$ supergravity as $M$-theory, although the complete $M$-theory must really include, presently unknown, higher order physics.

It is also well-known that there is a 2+1-dimensional membrane solution of the $M$-theory equations of motion that preserves one-half of the supersymmetries [32]. This “electrically charged” solution is equivalent, upon double dimensional reduction on $S^1$, to the elementary string solution of the ten-dimensional type $IIA$ supergravity equations of motion. This BPS membrane solution is of the form

$$ds^2 = e^{2A} dz^i dz^j \eta_{ij} + e^{2B} dy^m dy^n \delta_{mn},$$

$$\hat{C}_{ijk} = -\frac{1}{6\sqrt{2}} \hat{\epsilon}_{ijk} e^C,$$
where $\hat{z}^i$ are the three coordinates oriented in the direction of the membrane (with $i = \hat{0}, \hat{1}, \hat{2}$) and $\hat{y}^m (\hat{m} = \hat{3}, \ldots, \hat{10})$ are the eight coordinates transverse to the membrane. Note that these coordinates do not necessarily coincide with $\hat{x}^M$. Generically, there is a sign ambiguity on the right-hand side of the second equation in (2.3). The choice of this sign defines an orientation of the preserved supersymmetry on the membrane. In this paper, we have arbitrarily chosen the $-$ sign. That corresponds to positive chiral supersymmetry from the normal spin bundle point of view. This choice of sign is for specificity only, our conclusions being independent of it. The functions $A, B, C$ depend on the transverse coordinates only. Equations (2.3) represent a solution of eleven-dimensional supergravity when

$$A = -2B = C/3, \text{ and } e^{-C} = 1 + \frac{1}{\hat{r}^6},$$

(2.4)

where

$$\hat{\epsilon} \equiv \sqrt{(\hat{y}^\hat{m} - \hat{g}_0^\hat{m})(\hat{y}^\hat{n} - \hat{g}_0^\hat{n})\delta_{\hat{m}\hat{n}}},$$

(2.5)

and $\hat{g}_0^\hat{m}$ are constants. This solution describes a membrane located at $(\hat{y}_0^\hat{3}, \ldots, \hat{y}_0^{\hat{10}})$.

That this solution preserves one-half of the supersymmetry can be seen as follows. The supersymmetry transformation of the gravitino is given in (2.2). Now make the three-eight split

$$\tilde{\Gamma}_A = (\hat{\tau}_{a'} \otimes \hat{\gamma}, 1 \otimes \hat{\gamma}_a),$$

(2.6)

where $\hat{\tau}_{a'}$ and $\hat{\gamma}_a$ are the three- and eight-dimensional Dirac matrices, respectively, with flat indices $\hat{A} = \hat{0}, \ldots, \hat{10}, \hat{a}' = \hat{0}, \hat{1}, \hat{2}$ and $\hat{a} = \hat{3}, \ldots, \hat{10}$, and

$$\hat{\gamma} = \prod_{a = 3}^{\hat{10}} \hat{\gamma}_a.$$  

(2.7)

Then the supersymmetry variation (2.2) vanishes for spinor parameters $\hat{\epsilon}$ of the form

$$\hat{\epsilon} = \hat{\lambda}_0 \otimes \hat{\nu}_0 e^{C/6},$$

(2.8)

where $\hat{\lambda}_0$ and $\hat{\nu}_0$ are constant three- and eight-dimensional spinors, and $\hat{\nu}_0$ satisfies the chirality condition

$$\frac{1}{2}(1 - \hat{\gamma})\hat{\nu}_0 = 0.$$  

(2.9)

The minus sign in (2.9), which determines the chirality of the preserved supersymmetry, is correlated to the sign arbitrarily chosen in the membrane configuration (2.3).

This solution solves the eleven-dimensional supergravity equations of motion everywhere except at the singularity $\hat{r} = 0$. This implies that it is necessary to include delta function
source terms in the supergravity equations of motion \[32\]. The source terms that must be added to the pure supergravity action are precisely the supermembrane action \([33]\)

\[
S_{SM} = -T_M \int_{\Sigma} d^3\hat{\sigma} \left( \sqrt{-\det \hat{g}_{ij}} - \frac{1}{6} \hat{\varepsilon}^{ijk} \hat{\Pi}_i \hat{\Pi}_j \hat{\Pi}_k \hat{C}_{\hat{C}B\hat{A}} \right),
\]

where

\[
T_M = \left( \frac{2\pi^2/\kappa^2}{3} \right)^{1/3}.
\]

is the membrane tension of mass dimension three,

\[
\hat{g}_{ij} = \hat{\Pi}_i \hat{\Pi}_j \eta_{\hat{A}\hat{B}}, \quad \hat{\Pi}_i = \partial_i \hat{Z}^{\hat{B}} \hat{E}_\hat{M}^{\hat{A}},
\]

and \(\hat{\sigma}^0, \hat{\sigma}^1, \hat{\sigma}^2\) are the worldvolume coordinates of the membrane which are, generically, independent of any target space coordinates. This action represents the superembedding \(\hat{Z} : \Sigma^{3|0} \to M^{11|32}\), whose bosonic and fermionic component fields are the background coordinates, separated as

\[
\hat{Z}^{\hat{B}}(\hat{\sigma}) = (\hat{X}^{\hat{M}}(\hat{\sigma}), \hat{\Theta}^{\hat{\theta}}(\hat{\sigma})),
\]

respectively. The action is a sigma-model since the super-elfbeins \(\hat{E}_\hat{M}^{\hat{A}}\) and the super-three-form \(\hat{C}_{\hat{C}B\hat{A}}\) both depend on the superfields \(\hat{Z}^{\hat{M}}\). The super-elfbeins have, as their first bosonic and fermionic component in the \(\hat{\Theta}\) expansion, the bosonic elfbeins \(\hat{E}_\hat{M}^{\hat{A}}\) and the gravitino \(\hat{\Psi}_\hat{M}^{\hat{A}}\) respectively, while the super-three-form has the bosonic three-form from eleven-dimensional supergravity as its leading field component. The superfields \(\hat{E}_\hat{M}^{\hat{A}}\) and \(\hat{C}_{\hat{C}B\hat{A}}\) represent the background into which the supermembrane is embedded and, therefore, must satisfy the eleven-dimensional supergravity equations of motion \[34\]. Generically, there is a sign ambiguity in the second term of (2.10). Here we choose the sign that is consistent with our choice of sign in Eq.(2.3).

The fact that membrane configuration \([2.3]\) is a solution of \(M\)-theory which preserves one-half of the supersymmetries translates, when speaking in supermembrane worldvolume language, into the fact that the action (2.10) exhibits a local fermionic invariance, \(\kappa\)-invariance, that is used to gauge away half of the fermionic degrees of freedom. Specifically, the supermembrane action is invariant under the local fermionic symmetries

\[
\delta_\dot{\kappa} \hat{\Theta} = 2 \hat{P}_+ \dot{\kappa} + \cdots, \quad \delta_\dot{\kappa} \hat{X}^{\hat{M}} = 2 \hat{\Theta} \hat{\Gamma}^{\hat{M}} \hat{P}_+ \dot{\kappa} + \cdots,
\]

where \(\dot{\kappa}(\hat{\sigma})\) is an eleven-dimensional local spinor parameter and \(\hat{P}_\pm\) are the projection operators

\[
\hat{P}_\pm \equiv \frac{1}{2} (1 \pm \frac{1}{6 \sqrt{-\det \hat{g}_{ij}}} \hat{\varepsilon}^{ijk} \hat{\Pi}_i \hat{\Pi}_j \hat{\Pi}_k \hat{C}_{\hat{A}B\hat{C}}) \cdot
\]

where

\[
\hat{g}_{ij} = \hat{\Pi}_i \hat{\Pi}_j \eta_{\hat{A}\hat{B}}, \quad \hat{\Pi}_i = \partial_i \hat{Z}^{\hat{B}} \hat{E}_\hat{M}^{\hat{A}},
\]

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\]

where \(\dot{\kappa}(\hat{\sigma})\) is an eleven-dimensional local spinor parameter and \(\hat{P}_\pm\) are the projection operators

\[
\hat{P}_\pm \equiv \frac{1}{2} (1 \pm \frac{1}{6 \sqrt{-\det \hat{g}_{ij}}} \hat{\varepsilon}^{ijk} \hat{\Pi}_i \hat{\Pi}_j \hat{\Pi}_k \hat{C}_{\hat{A}B\hat{C}}) \cdot
\]
It follows from the first equation in (2.14) that the $\hat{P}_+$ component of spinor $\hat{\Theta}$ can be transformed away by a $\kappa$-transformation. Note that (2.14) includes only the leading order terms in $\hat{\Theta}$, which is all that is required to discuss the supersymmetry properties of the membrane. The theory is not, in general, invariant under global supersymmetry transformations

$$\delta_{\hat{\epsilon}} \hat{\Theta} = \hat{\epsilon},$$

$$\delta_{\hat{\epsilon}} \hat{X}^M = \bar{\hat{\epsilon}} \hat{\Gamma}^M \hat{\Theta},$$

(2.17)

where $\hat{\epsilon}$ is an eleven-dimensional spinor independent of $\hat{\sigma}$. For example, a general bosonic configuration $\hat{X}(\hat{\sigma})$ breaks all global supersymmetries generated by $\hat{\epsilon}$. However, one-half of the supersymmetries will remain unbroken if and only if (2.17) can be compensated for by a $\kappa$-transformation with a suitable parameter $\hat{\kappa}(\hat{\sigma})$. That is

$$\delta \hat{\Theta} = \delta_{\hat{\epsilon}} \hat{\Theta} + \delta_{\hat{\kappa}} \hat{\Theta}$$

$$= \hat{\epsilon} + 2 \hat{P}_+ \hat{\kappa}(\hat{\sigma}) = 0.$$  

(2.18)

In order for this to be satisfied, a necessary condition is that

$$\hat{P}_- \hat{\epsilon} = 0.$$  

(2.19)

Of course, the sign choice of $\hat{P}_+$ in (2.14) is correlated to the sign of the second term in the supermembrane action (2.10), the sign which fixes the membrane supersymmetry chirality [33]. Had we chosen the opposite chirality, the symbols $\hat{P}_+$ and $\hat{P}_-$ would be interchanged in the present discussion.

**Membranes in Hořava-Witten Theory:**

When M-theory is compactified on $S^1/\mathbb{Z}_2$, it describes the low energy limit of the strongly coupled heterotic string theory [3,3]. We choose $\hat{x}^{11}$ as the orbifold direction and parametrize $S^1$ by $\hat{x}^{11} \in [-\pi \rho, \pi \rho]$ with the endpoints identified. The $\mathbb{Z}_2$ symmetry acts by further identifying any point $\hat{x}^{11}$ with $-\hat{x}^{11}$ and, therefore, gives rise to two ten-dimensional fixed hyperplanes at $\hat{x}^{11} = 0$ and $\hat{x}^{11} = \pi \rho$. Since, at each $\mathbb{Z}_2$ hyperplane, only the field components that are even under the $\mathbb{Z}_2$ action can survive, the eleven-dimensional supergravity in the bulk space is projected into $N = 1$ ten-dimensional chiral supergravity on each boundary. Furthermore, cancellation of the chiral anomaly in this theory requires the existence of an $N = 1$, $E_8$ super-Yang-Mills multiplet on each fixed hyperplane [3,3]. Therefore, the effective action for M-theory on $S^1/\mathbb{Z}_2$ describes the coupling of two ten-dimensional $E_8$
super-Yang-Mills theories, one on each hyperplane, to eleven-dimensional supergravity in the bulk space. The bosonic part of the Hořava-Witten action is given by

\[ S_{HW} = S_{
abla G} + S_{YM}, \quad (2.20) \]

where \( S_{
abla G} \) is the eleven-dimensional supergravity bulk action (2.1) and \( S_{YM} \) describes the two \( E_8 \) Yang-Mills theories on the orbifold planes

\[ S_{YM} = -\frac{1}{8\pi^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \sum_{n=1,2} \int_{M_{10}} d^{10} x \sqrt{-g} \left[ \text{tr}(F^{(n)})^2 - \frac{1}{2} \text{tr}R^2 \right] \quad (2.21) \]

Here \( F^{(n)}_{MN} \), with \( M, N = 0, 1, \ldots, 9 \) and \( n = 1, 2 \), are the two \( E_8 \) gauge field-strengths where \( F^{(n)}_{MN} = F^{(n)\alpha} T^\alpha \) (\( T^\alpha \) being \( E_8 \) generators, \( a = 1, \ldots, 248 \)). \( R \) is the eleven-dimensional curvature two-form restricted to the orbifold planes \( M_{10}^{(n)} \). The supersymmetry transformations of the fermionic fields are given by (2.2) for the gravitino and

\[ \delta \hat{\psi}^{(n)a} = F^{(n)\alpha} A^\alpha \hat{\psi} + \ldots \quad (2.22) \]

for the fixed hyperplane gauginos \( \hat{\psi}^{(n)a} \), where \( A, B = 0, 1, \ldots, 9 \) are flat ten-dimensional indices.

In order to cancel all chiral anomalies on the hyperplanes, the action \( S_{HW} \) has to be supplemented by the modified Bianchi identity\footnote{The normalization of \( \hat{G} \) adopted here differs from \cite{5} by a factor of \( \sqrt{2} \) but it agrees with \cite{35}, in which one considers, as we will do in this paper, the superfield version of the Bianchi identities.}

\[ (d\hat{G})_{11MNPQ} = -\frac{1}{4\pi} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left\{ J^{(1)} \delta(\hat{x}^{11}) + J^{(2)} \delta(\hat{x}^{\hat{1}1} - \pi \rho) \right\}_{MNPQ}, \quad (2.23) \]

where

\[ J^{(n)} = \text{tr} F^{(n)} \wedge F^{(n)} - \frac{1}{2} \text{tr} R \wedge R, \quad (2.24) \]

for \( n = 1, 2 \). The solutions to the equations of motion resulting from action \( S_{HW} \) must respect the \( \mathbb{Z}_2 \) orbifold symmetry. Under \( \mathbb{Z}_2 \), the bosonic fields in the bulk behave as

\[ \hat{g}_{MN}(\hat{x}^{11}) = \hat{g}_{MN}(-\hat{x}^{11}), \quad \hat{C}_{MNP}(\hat{x}^{11}) = -\hat{C}_{MNP}(-\hat{x}^{11}), \]
\[ \hat{g}_{M11}(\hat{x}^{11}) = -\hat{g}_{M11}(-\hat{x}^{11}), \quad \hat{C}_{11MN}(\hat{x}^{11}) = \hat{C}_{11MN}(-\hat{x}^{11}), \quad (2.25) \]

while the gravitino transforms as

\[ \hat{\Psi}_M(\hat{x}^{11}) = \hat{\Gamma}_{11} \hat{\Psi}_M(-\hat{x}^{11}), \quad \hat{\Psi}_1(\hat{x}^{11}) = -\hat{\Gamma}_{11} \hat{\Psi}_1(-\hat{x}^{11}). \quad (2.26) \]
where $\hat{\Gamma}_{11} = \hat{\Gamma}_0 \hat{\Gamma}_1 \cdots \hat{\Gamma}_9$. In order for the supersymmetry transformations of the gravitino to be consistent with the $\mathbb{Z}_2$ symmetry, the eleven-dimensional Majorana spinor in (2.2) $\hat{\epsilon}$ must satisfy

$$\hat{\epsilon}(\hat{x}^{11}) = \hat{\Gamma}_{11} \hat{\epsilon}(-\hat{x}^{11}).$$

(2.27)

This equation does not restrict the number of independent components of the spinor fields $\hat{\epsilon}$ at any point in the bulk space. However, at each of the $\mathbb{Z}_2$ hyperplanes, constraint (2.27) becomes the ten-dimensional chirality condition

$$\frac{1}{2}(1 - \hat{\Gamma}_{11}) \hat{\epsilon} = 0, \quad \text{at} \quad \hat{x}^{11} = 0, \pi \rho.$$  

(2.28)

This leads to the correct amount of supersymmetry, $N = 1$, on each of the ten-dimensional orbifold fixed planes. Generically, there is a sign ambiguity in equations (2.26) and (2.27). The choice made here coincides with $\mathbb{F}$. This choice is consistent with the previous choice of supersymmetry orientation of the membrane.

Membrane solutions were explicitly constructed for Hořava-Witten theory in [36]. There, the membrane solution of eleven-dimensional supergravity was shown to satisfy, when appropriately modified and the boundary gauge fields are turned off, the equations of motion of the theory subject to the $\mathbb{Z}_2$ constraints.

There are two different ways to orient the membrane with respect to the orbifold direction, that is, $\hat{x}^{11}$ can either be a transverse coordinate or a coordinate oriented in the direction of the membrane. In the first case, the membrane is parallel to the hyperplanes. In the second case, it extends between the two hyperplanes and intersects each of them along a $1+1$-dimensional string. This latter configuration is sometimes referred to as an open supermembrane.

Let us consider the question of which type of supersymmetry each of these two membrane configurations preserve. Note that each such membrane is, by construction, a supersymmetry-preserving solution of supergravity. The associated singular worldvolume theory is, according to the previous section, a supersymmetric theory in the sense that it has $\kappa$-invariance that can gauge away only half of the fermions. Therefore, each of the two membrane orientations discussed above preserves one-half of the supersymmetries. More interesting is the question as to whether the particular supersymmetries that each of the two configurations preserves can be made consistent with the supersymmetry, defined in Eq. (2.28), that is imposed on the orbifold fixed hyperplanes.

We have seen in (2.19) that, in order for supersymmetry to be preserved, the global supersymmetry parameter $\hat{\epsilon}$ of the membrane worldvolume theory must satisfy $\hat{P} \cdot \hat{\epsilon} = 0$.

\[\text{We will discuss the } \kappa\text{-invariance of the open supermembrane worldvolume theory in the next section.}\]
where $\hat{P}_-$ is given in (2.15). Consider first a configuration in which the membrane is oriented parallel to the ten-dimensional hyperplanes. Choose the fields $\hat{X}$ and $\hat{\Theta}$ such that

$$\hat{X}^\hat{0} = \hat{\sigma}^\hat{0}, \quad \hat{X}^\hat{i} = \hat{\sigma}^\hat{i}, \quad \hat{X}^{\hat{2}} = \hat{\sigma}^{\hat{2}},$$

$$\hat{X}^{\hat{m}} = 0, \quad (\hat{m} = 3, \ldots, 9, 11) \quad \hat{\Theta} = 0. \quad (2.29)$$

Then $\hat{P}_- \hat{\varepsilon} = 0$ simplifies to

$$\hat{P}_- \hat{\varepsilon} = \frac{1}{2} (1 - \hat{\Gamma}_0 \hat{\Gamma}_1 \hat{\Gamma}_2) \hat{\varepsilon} = 0. \quad (2.30)$$

Because the membrane is embedded in a ten-dimensional space perpendicular to the orbifold direction, we need only consider eleven-dimensional spinors on the membrane worldvolume that can be decomposed into linearly independent chiral spinors

$$\hat{\varepsilon} = \hat{\varepsilon}_+ + \hat{\varepsilon}_-, \quad (2.31)$$

where

$$\hat{\varepsilon}_\pm = \frac{1}{2} (1 \pm \hat{\Gamma}_{\hat{1}1}) \hat{\varepsilon}. \quad (2.32)$$

Then (2.30) becomes

$$\frac{1}{2} (1 - \hat{\Gamma}_0 \hat{\Gamma}_1 \hat{\Gamma}_2) \hat{\varepsilon}_+ = 0, \quad \frac{1}{2} (1 - \hat{\Gamma}_0 \hat{\Gamma}_1 \hat{\Gamma}_2) \hat{\varepsilon}_- = 0. \quad (2.33)$$

Now multiply these expressions on the left by $\hat{\Gamma}_{\hat{1}1}$ and use the chirality properties of $\hat{\varepsilon}_\pm$ defined by (2.32). It follows that

$$\frac{1}{2} (1 + \hat{\Gamma}_0 \hat{\Gamma}_1 \hat{\Gamma}_2) \hat{\varepsilon}_+ = 0, \quad \frac{1}{2} (1 + \hat{\Gamma}_0 \hat{\Gamma}_1 \hat{\Gamma}_2) \hat{\varepsilon}_- = 0. \quad (2.34)$$

Combining the two sets of equations gives $\hat{\varepsilon}_\pm = 0$. Therefore, even though the membrane preserves one-half of the supersymmetries, they do not coincide with the supersymmetries preserved on the boundaries.

Next, consider a configuration in which the membrane is oriented perpendicular to the ten-dimensional hyperplanes. We choose the fields such that

$$\hat{X}^\hat{0} = \hat{\sigma}^\hat{0}, \quad \hat{X}^\hat{i} = \hat{\sigma}^\hat{i}, \quad \hat{X}^{\hat{1}1} = \hat{\sigma}^{\hat{2}},$$

$$\hat{X}^{\hat{m}} = 0, \quad (\hat{m} = 2, 3, \ldots, 9) \quad \hat{\Theta} = 0. \quad (2.35)$$

so that $\hat{P}_- \hat{\varepsilon} = 0$ now becomes

$$\hat{P}_- \hat{\varepsilon} = \frac{1}{2} (1 - \hat{\Gamma}_0 \hat{\Gamma}_1 \hat{\Gamma}_{\hat{1}1}) \hat{\varepsilon} = 0. \quad (2.36)$$
This is as far as one can go in the bulk space. However, on the orbifold boundary planes, (2.28) can be substituted in (2.37) to give

$$\frac{1}{2}(1 - \hat{\Gamma}_{01})\hat{\epsilon} = 0, \quad \text{at} \quad \hat{x}^{\hat{\Pi}} = 0, \pi \rho.$$  \hspace{1cm} (2.37)

This expression implies that the eleven-dimensional Majorana spinor $\hat{\epsilon}$, when restricted to the $1 + 1$-dimensional boundary strings (thereafter denoted by $\epsilon$), is a non-vanishing Majorana-Weyl spinor, as it should be. We thus see that this configuration preserves one-half of the supersymmetries on the $\mathbb{Z}_2$ hyperplanes.

Therefore, we conclude that a configuration in which the supermembrane is oriented parallel to the orbifold hyperplanes breaks all supersymmetries. On the other hand, the configuration for the open supermembrane is such that the hyperplane and membrane supersymmetries are compatible.

3 \kappa-Invariant Action for Open Membranes:

We have shown that for a supermembrane to preserve supersymmetries consistent with the boundary fixed planes, the membrane must be open, that is, stretched between the two $\mathbb{Z}_2$ hyperplanes. In this section, we want to find the action associated with such a membrane. Action (2.10) is a good starting point. However, it is not obvious that it will correspond to the desired configuration, even in the bulk space. For this to be the case, one needs to ask whether this action respects the $\mathbb{Z}_2$ symmetry of Hořava-Witten theory. The answer was provided in [36], where it was concluded that, for an appropriate extension of the $\mathbb{Z}_2$ symmetry to the worldvolume coordinates and similar constraints for the worldvolume metric, the open supermembrane equations of motion are indeed $\mathbb{Z}_2$ covariant. Therefore, we can retain action (2.10). Does it suffice, however, to completely describe the open membrane configuration? Note that the intersection of an open membrane with each orbifold fixed plane is a $1 + 1$-dimensional string embedded in the ten-dimensional boundary. We denote by $\sigma^i, \ i = 1, 2$, the worldsheet coordinates of these strings. Intuitively, one expects extra fields, which we generically denote by $\phi(\sigma)$, to appear on each boundary string in addition to the bulk fields $\hat{Z}_{\hat{M}}(\hat{\sigma})$. These would naturally couple to the pullback onto each boundary string of the background $E_8$ super-gauge fields $A_{\hat{M}}$. As we will see in this section, new supermembrane fields are indeed required and form a chiral Wess-Zumino-Witten multiplet

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3 When we switch to Euclidean space later in this paper, we must regard $\hat{\epsilon}$ as an eleven-dimensional Dirac spinor and $\epsilon$ as a ten-dimensional Weyl spinor, since in these dimensions one cannot impose the Majorana condition.
for each $E_8$ gauge group.\footnote{This section follows closely the original proof in [35].}

As discussed previously, the supergravity theory of the background fields exhibits both
gauge and gravitational anomalies that can only be cancelled by modifying the Bianchi
identity as in (2.23). Integrating (2.23) along the $\hat{x}^{11}$ direction, and promoting the result
to superspace, we find for the $n$-th boundary plane that

$$
\hat{G}_{MNPQ} \mid_{M^{(n)}} = -\frac{1}{8\pi T_M} (\text{tr} F^{(n)} \wedge F^{(n)})_{MNPQ},
$$

(3.1)

where $F^{(n)}$ is the super-field-strength of the fields $A^{(n)}$. Note that we have dropped the
curvature term in the modified Bianchi identity, since it is a higher-order derivative term
and it does not contribute to the one-loop calculation of the superpotential in this paper.
As a consequence, we are allowed to use the eleven-dimensional supergravity in the bulk
space. The reason for expressing the integrated Bianchi identity in superspace is to make
it compatible with the bulk part of supermembrane action (2.10), which is written in terms
of the pullbacks of superfields $\hat{A}_M^A$ and $\hat{C}_{\hat{C}\hat{B}\hat{A}}$ onto the worldvolume. Noting that, locally,$\hat{G} = d\hat{C}$, it follows from (3.1) that on the $n$-th boundary plane

$$
\hat{C}_{MNP} \mid_{M^{(n)}} = -\frac{1}{8\pi T_M} \Omega_{MNP}(A^{(n)}),
$$

(3.2)

where

$$
\Omega_{MNP}(A^{(n)}) = 3! \left( \text{tr}(A^{(n)} \wedge dA^{(n)}) + \frac{2}{3} \text{tr}(A^{(n)} \wedge A^{(n)} \wedge A^{(n)}) \right)_{MNP}
$$

(3.3)

is the Chern-Simons three-form of the super-one-form $A^{(n)}$.

Note that each $A^{(n)}$ is a super-gauge-potential and, as such, transforms under super-
gauge transformations as

$$
\delta_L A^a_M = \partial_M L^a + f^{abc} A^b_M L^c,
$$

(3.4)

with $a, b, c = 1, \ldots, 248$. Note that for simplicity, here and elsewhere where it is inessential,
we drop the superscript $(n)$ indicating the boundaries. If we define the pullback of $A$ as

$$
A_i \equiv \partial_i \hat{Z}^M A_M,
$$

(3.5)

the gauge transformation in superspace (3.4) induces a gauge transformation on the string
worldsheet, which acts on the pullback of $A$ as

$$
\delta_L A^a_i = (D_i L)^a = \partial_i L^a + f^{abc} A^b_i L^c,
$$

(3.6)
where $L = L(Z^M(\sigma))$. It follows from (3.2), (3.3) and (3.4) that, on each boundary fixed plane,
\[
\delta_L \hat{\mathcal{C}}_{MNP} = -\frac{3}{4\pi T_M} \left[ \delta_L \left( \text{tr}(A \wedge dA) + \frac{2}{3} \text{tr}(A \wedge A \wedge A) \right) \right]_{MNP} = -\frac{3}{4\pi T_M} \text{tr}(\partial_M \partial_N A_P).
\]
(3.7)

Now consider the variation of the supermembrane action (2.10) under a super-gauge transformation. Clearly, a non-zero variation arises from the second term in (2.10)
\[
\delta_L S_{SM} = \frac{T_M}{6} \int d^8 \sigma \, \hat{\epsilon}^{ij} \partial_i Z^M \partial_j Z^N \delta \hat{C}_{PMN} = \frac{1}{8\pi} \int_{\partial \Sigma} d^2 \sigma \, \hat{\epsilon}^{ij} \partial_i Z^M \partial_j Z^N \text{tr}(\partial_N A_M),
\]
(3.8)

where $\partial \Sigma$ is the sum over the two boundary strings $\sum_{n=1,2} \partial \Sigma^{(n)}$, and we have integrated by parts. Therefore, action (2.10) is not invariant under gauge transformations. This symmetry is violated precisely at the boundary planes. It follows that to restore gauge invariance, one must add appropriate boundary terms to the supermembrane action.

Before doing that, however, let us consider the transformation of the action $S_{SM}$ under a $\kappa$-transformation, taking into account the boundary expression (3.2). Note that the $\kappa$-transformation acts on the super-three-form $\hat{C}$ as
\[
\delta \kappa \hat{C} = L_\kappa \hat{C} = i_\kappa d\hat{C} + (d i_\kappa) \hat{C},
\]
(3.9)

where $L_\kappa$ is the Lie derivative in the $\kappa$-direction and the operator $i_\kappa$ is defined, for any super-$l$-form $\hat{H}$, as
\[
i_\kappa \hat{H} = \frac{1}{l!} \hat{H}_{\tilde{M}_1 \cdots \tilde{M}_l} i_\kappa (d\hat{Z}^{\tilde{M}_1} \wedge \cdots \wedge d\hat{Z}^{\tilde{M}_l}) = \frac{1}{(l-1)!} \hat{H}_{\tilde{M}_1 \cdots \tilde{M}_{l-1} \tilde{H}} (\hat{P}_+ \hat{\kappa}^\beta)(d\hat{Z}^{\tilde{M}_{l-1}} \wedge \cdots \wedge d\hat{Z}^{\tilde{M}_1}).
\]
(3.10)

Importantly, we use the positive projection $\hat{P}_+$ of $\hat{\kappa}$, as defined in (2.14), in order to remain consistent with the previous choices of supersymmetry orientation. Varying action (2.10) under (3.9) and under the full $\kappa$-variations of $\hat{Z}$, we observe that $\kappa$-symmetry is also violated at the boundaries
\[
\delta \kappa S_{SM} = -\frac{1}{6} T_M \int_{\partial \Sigma} d^2 \sigma \, \hat{\epsilon}^{ij} \partial_i Z^M \partial_j Z^N \hat{C}_{NM\tilde{\mu}} \hat{P}_+ \hat{\kappa}^\tilde{\mu} = \frac{1}{48\pi} \int_{\partial \Sigma} d^2 \sigma \, \hat{\epsilon}^{ij} \partial_i Z^M \partial_j Z^N \hat{\Omega}_{NM\tilde{\mu}} (\hat{A}) \hat{P}_+ \hat{\kappa}^\tilde{\mu}.
\]
(3.11)

In deriving this expression we have used the eleven-dimensional supergravity constraints. It proves convenient to consider, instead of this $\kappa$-transformation, the modified $\kappa$-transformation
\[
\Delta \kappa = \delta \kappa - \delta L_\kappa,
\]
(3.12)
where $\delta_{L\hat{\kappa}}$ is a super-gauge transformation with the special gauge parameter

$$L_{\hat{\kappa}} = i_{\hat{\kappa}} A = 2 \hat{\kappa}_\mu \hat{P}_+ \hat{k}^{\mu}.$$  \hspace{1cm} (3.13)

Under this transformation, the supermembrane action behaves as

$$\Delta_{\hat{\kappa}} S_{SM} = \frac{1}{8\pi} \int_{\partial \Sigma} \varepsilon^{ij} \partial_i Z^M \partial_j Z^N \partial_{\hat{\mu}} \hat{P}_+ \hat{k}^{\mu} \hat{A}_M.$$  \hspace{1cm} (3.14)

It is also useful to note that the pullback of the boundary background field $A$ transforms as

$$\Delta_{\hat{\kappa}} A_i = 2 \partial_i Z^M \partial_{\hat{\mu}} \hat{P}_+ \hat{k}^{\mu}$$  \hspace{1cm} (3.15)

under this modified $\kappa$-transformation, where we have used the fact that

$$\delta_{\hat{\kappa}} A = L_{\hat{\kappa}} A$$  \hspace{1cm} (3.16)

is the $\kappa$-transformation of $A$, just as in (3.9).

It was shown in [35] that the gauge and modified $\kappa$ anomalies can be cancelled if the supermembrane action is augmented to include a chiral level one Wess-Zumino-Witten model on each boundary string of the membrane. The fields thus introduced will couple to the pullback of the background fields $A$ at each boundary.

On each boundary string, the new fields can be written as

$$g(\sigma) = e^{i\phi^a(\sigma) T^a},$$  \hspace{1cm} (3.17)

where $T^a$ are the generators of $E_8$ (with $a = 1, \ldots, 248$) and $\phi^a(\sigma)$ are scalar fields that transform in the adjoint representation, and parametrize the group manifold, of $E_8$. Note that $g$ is a field living on the worldsheet of the boundary string. The left-invariant Maurer-Cartan one-forms $\omega_i(\sigma)$ are defined by

$$\omega_i = g^{-1} \partial_i g.$$  \hspace{1cm} (3.18)

The variation of $g(\sigma)$ under gauge and modified $\kappa$-transformations can be chosen to be

$$\delta_{L} g = g L, \quad \Delta_{\hat{\kappa}} g = 0,$$  \hspace{1cm} (3.19)

where $L = L(Z(\sigma))$. The coupling of this model to the external gauge fields is accomplished by replacing the left-invariant Maurer-Cartan one-form $\omega_i = g^{-1} \partial_i g$ by the “gauged” version

$$g^{-1} D_i g = \omega_i - \partial_i Z^M \hat{A}_M,$$  \hspace{1cm} (3.20)

where $D_i$ is the covariant derivative for the right-action of the gauge group.
The gauge- and $\kappa$-invariant action for the open supermembrane is then given by [35]

$$S_{OM} = S_{SM} + S_{WZW},$$

(3.21)

where $S_{SM}$ is the bulk action given in (2.10) and

$$S_{WZW} = \frac{1}{8\pi} \int_{\partial \Sigma} d^2 \sigma \, \text{tr} \left[ \frac{1}{2} \sqrt{-g} g^{ij} (\omega_i - \partial_i Z^M A_M) \cdot (\omega_j - \partial_j Z^N A_N) + \varepsilon^{ijk} \partial_j Z^M \omega_i A_M \right]$$

$$- \frac{1}{24\pi} \int_{\Sigma} d^3 \hat{\sigma} \, \varepsilon^{ijk} \Omega_{kji}(\hat{\omega}),$$

(3.22)

where

$$\Omega_{kji}(\hat{\omega}) = \text{tr}(\hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega})_{kji}.$$  

(3.23)

The first term in (3.22) describes the kinetic energy for the scalar fields $\phi^a(\sigma)$ and their interactions with the pullback of the super-gauge potential $\hat{A}$ on each of the boundary strings. The second term is the integral over the membrane of the Wess-Zumino-Witten three-form, constructed in (3.23) from a worldvolume one-form $\hat{\omega} = \hat{g}^{-1} d\hat{g}$, where $\hat{g} : \Sigma \rightarrow E_8$. The map $\hat{g}$ must satisfy

$$\hat{g} \mid_{\partial \Sigma^{(1)}} = g^{(1)}, \quad \hat{g} \mid_{\partial \Sigma^{(2)}} = g^{(2)},$$

(3.24)

but is otherwise unspecified. That such a $\hat{g}$ exists will be shown below. It is straightforward to demonstrate that the variation of $S_{WZW}$ under both gauge and local modified $\kappa$-transformations, $\delta_L$ and $\Delta_\kappa$ respectively, exactly cancels the variations of the bulk action $S_{SM}$ given in (3.8) and (3.14) provided we choose the parameter $\kappa$ on each boundary to obey

$$P_- \kappa = \frac{1}{2} (1 - \frac{1}{2\sqrt{-\det g_{ij}}} \varepsilon^{ijk} \Pi_i^A \Pi_j^B \Gamma_{AB} ) \kappa = 0.$$  

(3.25)

Note that this is consistent with (2.37). On the boundary strings we can denote $\kappa$ by $\kappa$. In proving this cancellation, it is necessary to use the super Yang-Mills constraints on each boundary plane.

We now prove that maps $\hat{g} : \Sigma \rightarrow E_8$ with property (3.24) indeed exist. Restoring the boundary index, the two sets of scalar fields $\phi^{(n)a}$, one at each boundary, correspond to two maps $g^{(n)} : \partial \Sigma^{(n)} \rightarrow E_8$ with $n = 1, 2$. From the Hořava-Witten point of view, these maps are completely independent of each other, as are the $E_8$ gauge groups that they map to. Now let us assume, as we do later in this paper when computing the superpotential, that

$$\partial \Sigma^{(n)} = \mathbb{C} P^1 = S^2$$

(3.26)

for $n = 1, 2$. Since

$$\pi_2(E_8) = 1,$$

(3.27)
it follows that maps $g^{(1)}$ and $g^{(2)}$ must be homotopically equivalent. Formally, this means that there exists a continuous map

$$\hat{g} : S^2 \times I \to E_8,$$  

where $I$ can be taken to be the closed interval $[0, \pi \rho]$, with the property that

$$\hat{g}(\sigma, 0) = g^{(1)}(\sigma), \quad \hat{g}(\sigma, \pi \rho) = g^{(2)}(\sigma),$$

for $\sigma \in S^2$. Clearly, however, the membrane manifold is

$$\Sigma = S^2 \times I.$$  

It follows that one can extend the boundary maps to membrane worldvolume maps

$$\hat{g} : \Sigma \to E_8,$$

where

$$\hat{g} \big|_{\partial \Sigma^{(n)}} = g^{(n)},$$

for $n = 1, 2$, as required. From this point of view, there appears to be a single $E_8$ gauge group. Note, however, that one can still vary the boundary maps independently. This is consistent with the Hořava-Witten two $E_8$ group interpretation. This result can be generalized to $\partial \Sigma^{(n)}$ being any compact Riemann surface. We conclude that the $\int_\Sigma \Omega(\hat{\omega})$ term in (3.22) can be constructed in a well-defined way.

It will be useful in this paper to discuss some specific properties of this term. Since $\Omega(\hat{\omega})$ is a closed three-form, it can be represented locally as $\Omega = dB$, where $B(\hat{\omega})$ is an $E_8$ Lie algebra-valued two-form. If we allow for Dirac-like singularities in $B$, this representation can, with care, be used globally. We can then write

$$\int_\Sigma \Omega(\hat{\omega}) = \int_{\partial \Sigma} B(\hat{\omega}) = \sum_{n=1,2} \int_{\partial \Sigma^{(n)}} B(\omega^{(n)}),$$

where $B(\omega^{(n)})$ is the restriction of $B$ to the boundary $\partial \Sigma^{(n)}$ of the membrane and $\omega^{(n)} = g^{(n)-1} dg^{(n)}$. Therefore, $\int_\Sigma \Omega$, although formally expressed as an integral over the entire membrane worldvolume, is completely determined by the values of $\phi^{(n)}(\sigma)$ on each of the boundary strings. This can also be seen by noticing that small variations $\phi^{(n)} \to \phi^{(n)} + \delta \phi^{(n)}$ which are zero on the boundaries leave the action invariant. Furthermore, notice that the equations of motion derived from the action by using either the left- or the right-hand side of (3.33) will agree.
4 BPS Conditions on a Calabi-Yau Threefold:

We have shown that, in order to preserve the same supersymmetry as the orbifold fixed planes, a supermembrane must be oriented along the orbifold direction and terminate on each of the fixed planes. Such an open supermembrane must contain chiral Wess-Zumino-Witten fields $\omega = g^{-1}dg$ as dynamical degrees of freedom. These couple to the boundary background gauge fields $A$ in such a way that action (3.21) exhibits both $\kappa$-invariance and gauge invariance. Recall that the supermembrane has three unphysical bosonic degrees of freedom. Hence, one can choose a gauge by specifying three of the bosonic fields $\hat{X}^M$ to be explicit functions of the supermembrane coordinates $\hat{\sigma}^i$ (with $i = \hat{0}, \hat{1}, \hat{2}$). In the eleven-dimensional Hořava-Witten background, the canonical gauge choice was specified in the first line of (2.35). In the curved backgrounds of heterotic M-theory, which we will shortly study, this gauge can also be chosen. Here, since the membrane must be oriented in the orbifold direction, we will take

$$\hat{X}^{11} = \hat{\sigma}^2,$$

leaving the rest of the gauge unspecified for the moment (we will specify the rest of the gauge in Section 7).

In this paper, we are interested in obtaining an effective four-dimensional theory with $N = 1$ supersymmetry. In particular, we want to compute and study non-perturbative corrections to the superpotential of such a theory. These corrections arise from the non-perturbative interaction between the background and the supermembrane embedded in it. The total action of this theory is

$$S_{\text{Total}} = S_{\text{HW}} + S_{\text{OM}}$$

$$= (S_{\text{SG}} + S_{\text{YM}}) + (S_{\text{SM}} + S_{\text{WZW}}),$$

(4.2)

where $S_{\text{SG}}$, $S_{\text{YM}}$, $S_{\text{SM}}$ and $S_{\text{WZW}}$ are given in (2.1), (2.21), (2.10) and (3.22) respectively.

In addition to compactifying on $S^1/\mathbb{Z}_2$, which takes eleven-dimensional supergravity to Hořava-Witten theory, there must also be a second dimensional reduction on a real six-dimensional manifold. This space, which reduces the theory from ten- to four-dimensions on each orbifold boundary plane, and from eleven- to five-dimensions in the bulk space, is taken to be a Calabi-Yau threefold, denoted $CY_3$. A Calabi-Yau space is chosen since such a configuration will preserve $N = 1$ supersymmetry in four-dimensions. That is, we now consider M-theory and open supermembranes on the geometrical background

$$M_{11} = R_4 \times CY_3 \times S^1/\mathbb{Z}_2,$$

(4.3)
where $R_4$ is four-dimensional flat space.

It is essential that the vacuum state of this theory be Lorentz invariant in four-dimensions. Any open supermembrane has an embedding geometry given by

$$\Sigma = C \times S^1 / \mathbb{Z}_2,$$

where $C$ is a real, two-dimensional surface. Clearly, the requirement of four-dimensional Lorentz invariance implies that

$$C \subset CY_3.$$  \hfill (4.5)

Since $CY_3$ is purely space-like, it follows that we must, henceforth, use the Euclidean version of the supermembrane theory.

In this section, we consider the question of which conditions, if any, are necessary for the supermembrane theory in such a background to preserve an $N = 1$ supersymmetry. The $\kappa$-transformations (2.14) of the superspace coordinates $\hat{Z}^M = (\hat{X}^M, \hat{\Theta}^\mu)$ do not receive boundary corrections. Therefore, equation (2.19) continues to be necessary for preservation of supersymmetry. For a purely bosonic configuration, this expression becomes

$$\frac{1}{2}(1 - \frac{i}{\sqrt{\det g_{ij}}} \varepsilon^{ijk} \partial_i \hat{X}^M \partial_j \hat{X}^N \partial_k \hat{X}^P \hat{\Gamma}^\rho_{\mu\nu}) \hat{\epsilon} = 0,$$  \hfill (4.6)

where $\hat{\epsilon}$ is a covariantly constant spinor and we have used (2.12), (2.13) and (2.15). Note that an $i$ appears since we are now in Euclidean space. The gauge fixing condition $\hat{X}^{11} = \hat{\sigma}^2$ reduces this expression to

$$\frac{1}{2}(1 - \frac{i}{\sqrt{\det g_{ij}}} \varepsilon^{ijk} \partial_i \hat{X}^M \partial_j \hat{X}^N \hat{\Gamma}_{MN}) \hat{\epsilon} = 0.$$  \hfill (4.7)

Recall from (2.28) that, on the boundary planes, the spinor $\hat{\epsilon}$ satisfies $\hat{\Gamma}^{11} \hat{\epsilon} = \hat{\epsilon}$, that is, it has positive ten-dimensional chirality. Therefore, on the boundary strings, we can denote $\hat{\epsilon}$ by $\hat{\eta}$ and write

$$\frac{1}{2}(1 - \frac{i}{\sqrt{\det g_{ij}}} \varepsilon^{ijk} \partial_i \hat{X}^M \partial_j \hat{X}^N \hat{\Gamma}_{MN}) \hat{\eta} = 0.$$  \hfill (4.8)

Next, we use the assumption that if $X^M$ describes a coordinate in $R^4$, denoted by $y^n$, with $n = 6, 7, 8, 9$, then $\partial_i y^n = 0$. Denote a coordinate in $CY_3$ by $\tilde{y}^U$, with $\tilde{U} = 0, 1, 2, 3, 4, 5$. Now choose the spinor $\epsilon$ to be of the form $\epsilon = \theta \otimes \eta$, where $\eta$ and $\theta$ are covariantly constant spinors of $CY_3$ and $R_4$, respectively. In this case, the above condition becomes

$$\frac{1}{2}(1 - \frac{i}{\sqrt{\det g_{ij}}} \varepsilon^{ijk} \partial_i \tilde{y}^\tilde{U} \partial_j \tilde{y}^\tilde{V} \hat{\Gamma}_{UV}) \eta = 0.$$  \hfill (4.9)

Another reason to Euclideanize the theory is that, in this paper, we will perform the calculation of quantum corrections using the path-integral formalism.
where $\tilde{\gamma}_U = \tilde{\gamma}_U^K \tilde{\gamma}_K$, $\tilde{\gamma}_K$ are CY$_3$ sechsbeins and $\tilde{\gamma}_K$ are the six-dimensional Dirac matrices, with flat indices $\tilde{K} = 0, 1, 2, 3, 4, 5$.

We now switch notation to exploit the complex structure of the Calabi-Yau space. The complex coordinates of CY$_3$ are denoted by $\tilde{y}^m$ and $\tilde{y}^\bar{m}$ with $m, \bar{m} = 1, 2, 3$. For the Kähler metric $g_{m\bar{n}}$, the Clifford relations for the Dirac matrices take the form

$$\{\tilde{\gamma}_m, \tilde{\gamma}_n\} = 0, \quad \{\tilde{\gamma}_m, \tilde{\gamma}_n\} = 0, \quad \{\tilde{\gamma}_m, \tilde{\gamma}_n\} = 2g_{m\bar{n}}, \quad (4.10)$$

It is known [37, 27] that there are two covariant constant spinors $\eta_-$ and $\eta_+$ that can exist on CY$_3$. They can be chosen such that

$$\tilde{\gamma}_m \eta_+ = 0, \quad \tilde{\gamma}_m \eta_- = 0. \quad (4.11)$$

It follows that

$$\tilde{\gamma}_{m\bar{n}} \eta_+ = \frac{1}{2}(\tilde{\gamma}_m \tilde{\gamma}_n - \tilde{\gamma}_n \tilde{\gamma}_m)\eta_+ = -\frac{1}{2}(\tilde{\gamma}_m \tilde{\gamma}_n + \tilde{\gamma}_n \tilde{\gamma}_m)\eta_+ = -g_{m\bar{n}}\eta_+, \quad (4.12)$$

and, similarly, that

$$\tilde{\gamma}_{m\bar{n}} \eta_- = g_{m\bar{n}}\eta_- \quad (4.13)$$

Here we have chosen a normalization of $\eta_{\pm}$ such that $\eta_{\pm} = \eta_{\pm}^*$. In this basis, for an arbitrary covariantly constant spinor expressed as

$$\eta = \epsilon^\alpha \eta_+ + \epsilon^{-\alpha} \eta_- \quad (4.14)$$

the condition for unbroken supersymmetry (4.9) can be written as

$$\eta_\pm = \frac{i}{2\sqrt{\det g_{\bar{z}z}}} \tilde{z}^{ij} (\partial_i \tilde{y}^m \partial_j \tilde{y}^\bar{n} \tilde{\gamma}_{m\bar{n}} + \partial_i \tilde{y}^m \partial_j \tilde{y}^{\bar{n}} \tilde{\gamma}_{mn})\eta_\pm. \quad (4.15)$$

Note that there are no terms that mix spinors of different type.

We now write the curve $C$ in complex coordinates, defining $z = \sigma^0 + i\sigma^1$. The derivatives are

$$\partial_0 = \partial_z + \partial_{\bar{z}}, \quad \partial_1 = i(\partial_z - \partial_{\bar{z}}). \quad (4.16)$$

Using this and (4.11), conditions (4.15) can be rewritten as

$$\eta_+ = \frac{i}{\sqrt{\det g_{\bar{z}z}}} (2\partial_z \tilde{y}^m \partial_z \tilde{y}^n \tilde{\gamma}_{m\bar{n}} + 2\partial_z \tilde{y}^m \partial_{\bar{z}} \tilde{y}^n \tilde{\gamma}_{m\bar{n}} + \partial_z \tilde{y}^m \partial_{\bar{z}} \tilde{y}^n \tilde{\gamma}_{m\bar{n}})\eta_+, \quad (4.17)$$

$$\eta_- = \frac{i}{\sqrt{\det g_{\bar{z}z}}} (2\partial_z \tilde{y}^m \partial_z \tilde{y}^{\bar{n}} \tilde{\gamma}_{m\bar{n}} + 2\partial_z \tilde{y}^m \partial_{\bar{z}} \tilde{y}^{\bar{n}} \tilde{\gamma}_{m\bar{n}} + \partial_z \tilde{y}^m \partial_{\bar{z}} \tilde{y}^{\bar{n}} \tilde{\gamma}_{m\bar{n}})\eta_-.$$
Since $\bar{\gamma}_{mn} \eta_+$ and $\eta_+$ transform differently under the holonomy group (and similarly for $\bar{\gamma}_{m\bar{n}} \eta_-$ and $\eta_-$), it follows from (4.12) and (4.13) that the coefficient of $\bar{\gamma}_{mn} \eta_+$ and $\bar{\gamma}_{m\bar{n}} \eta_-$ in each of the above equations has to vanish. That is,

$$\partial \bar{z} \partial \bar{y}^m \partial z \partial \bar{y}^n = 0,$$

(4.18)

These can be satisfied by either a holomorphic curve (for which $\partial \bar{z} \partial \bar{y}^m = \partial z \partial \bar{y}^n = 0$) or an anti-holomorphic curve (for which $\partial \bar{z} \partial \bar{y}^n = \partial z \partial \bar{y}^m = 0$). Suppose the curve is holomorphic. Then we obtain $\eta_- = 0$ and no further conditions on $\eta_+$. Therefore, the holomorphic curve leaves the supersymmetry generated by $\eta_+$ unbroken. Of course, if the curve is anti-holomorphic, only $\eta_-$ survives. Therefore, the necessary condition for a supermembrane in the background $M_{11} = R_4 \times CY_3 \times S^1/\mathbb{Z}_2$ to preserve an $N = 1$ supersymmetry when the membrane $\Sigma = C \times S^1/\mathbb{Z}_2$ is embedded as $C \subset CY_3$ is that $C$ must be either a holomorphic or anti-holomorphic curve. It is conventional to assume that $C$ is holomorphic, thus specifying the surviving four-dimensional $N = 1$ supersymmetry in terms of the covariantly constant spinor $\eta_+$. We adopt this convention and, in the remainder of this paper, take $C$ to be a holomorphic curve.

5 Low-Energy Limit and the Heterotic Superstring:

Thus far, we have shown that for a membrane to be supersymmetric in the background (4.3), it has to span the interval $S^1/\mathbb{Z}_2$ and wrap around a holomorphic curve $C \subset CY_3$. In this section, we take the limit as the radius $\rho$ of $S^1$ becomes small and explicitly compute the small $\rho$ limit of the open supermembrane theory. The result will be the heterotic superstring embedded in ten-dimensional space

$$M_{10} = R_4 \times CY_3,$$

(5.1)

and wrapped around a holomorphic curve $C \subset CY_3$.

We begin by rewriting the action (3.21) for a supermembrane with boundary strings as

$$S_{OM} = T_M \int_\Sigma d^3 \sigma \left( \sqrt{\det \hat{\Pi}_i \hat{\Pi}_j \eta_{\hat{A}} \hat{B}} - \frac{i}{6} \hat{\varepsilon}^{ij} \hat{\Pi}_i \hat{\Pi}_j \hat{\Pi}_k \hat{\varepsilon} \hat{\varepsilon} \hat{\varepsilon} \right)$$

$$- \frac{1}{8\pi} \sum_{n=1,2} \int_{\partial \Sigma^{(n)}} d^2 \sigma \left\{ \text{tr} \left[ \frac{1}{2} \sqrt{g} \hat{g}^{ij} (\omega_i^{(n)} - \hat{A}_i^{(n)}) \cdot (\omega_j^{(n)} - \hat{A}_j^{(n)}) + i\hat{\varepsilon}^{ij} \omega_i^{(n)} \hat{A}_j^{(n)} \right] ight\} - i\hat{\varepsilon}^{ij} B_{ij}(\omega^{(n)}) \right\},$$

(5.2)

where, again, an $i$ appears multiplying $\hat{\varepsilon}^{ij}$ as well as $\hat{\varepsilon} \hat{\varepsilon}$ because we are in Euclidean space, and we have used (3.33). Furthermore, it is important to note that the requirement
that we work in Euclidean space changes the sign of each term in (5.2) relative to the
Minkowski signature action (3.21). The boundary terms describe the gauged chiral Wess-
Zumino-Witten model. Since they are defined only on the boundary, they are not affected
by the compactification on \( S^1/\mathbb{Z}_2 \). As for the bulk action, we identify \( \dot{X}^{11} = \dot{\sigma}^2 \) and for all
remaining fields keep only the dependence on \( \dot{\sigma}^0, \dot{\sigma}^1 \). The Ansatz for compactification on
\( S^1/\mathbb{Z}_2 \) of the \( N = 1 \) eleven-dimensional super-elfbeins is given by

\[
\hat{E}_A^M = \begin{pmatrix} \hat{E}_A^M & \hat{E}_A^{11} & \hat{E}_A^\dot{\alpha} \\ \hat{E}_A^{11} & \hat{E}_A^{11} & \hat{E}_A^\dot{\alpha} \\ \hat{E}_A^\dot{\alpha} & \hat{E}_A^\dot{\alpha} & \hat{E}_A^\dot{\alpha} \end{pmatrix} = \begin{pmatrix} E_A^M & \phi V_M & E_M^\alpha + \chi^\dot{\alpha} V_M \\ 0 & \phi & \chi^\dot{\alpha} \end{pmatrix}. \tag{5.3} \]

Here, \( E_A^M = (E_A^M, E_\alpha^M) \) describes the super-zehnbeins of \( N = II A \) ten-dimensional super-space, \( V_M \) is a ten-dimensional vector superfield describing a \( U(1) \) super-gauge field and \( \phi \) and \( \chi^\dot{\alpha} \) are superfields whose leading components are the dilaton and the dilatino, respectively. Note that we have made a partial local Lorentz gauge choice by setting \( \hat{E}_A^{11} = 0 \).

The Ansatz for the super-three-form potential is

\[
\hat{C}_{MN11} = B_{MN}, \quad \hat{C}_{MNP} = B_{MNP}. \tag{5.4} \]

We must also find which conditions are imposed on these quantities by the \( \mathbb{Z}_2 \)-properties of the eleven-dimensional fields, given in (2.25) and (2.26). We will demand that a superfield have the same \( \mathbb{Z}_2 \) transformation properties as its bosonic component. Note that \( \hat{g}_{M11} \) in (2.25) has odd parity under \( \mathbb{Z}_2 \). Since we assume that all fields are independent of \( \dot{X}^{11} = \dot{\sigma}^2 \), these components must vanish. Therefore, using (5.3), we obtain

\[
\hat{g}_{M11} = \hat{E}_A^M \hat{E}_B^N \eta_{AB} + \hat{E}_A^{11} \hat{E}_B^{11} = \phi^2 V_M = 0, \tag{5.5} \]

where we use the same symbol, \( \hat{g}_{M11} \), for the superfield and its bosonic metric component.

That is, we must set either \( V_M = 0 \) or \( \phi = 0 \). Since \( \phi \) is a diagonal element in the super-elfbeins, \( \hat{E}_A^{11} = \phi \), its vanishing would imply that the determinant of the induced metric given in (5.9) below is zero. This is unacceptable. Therefore, we must set \( V_M = 0 \). The ten-dimensional three-form \( \hat{C}_{MNP} \) components of \( \hat{C}_{MN11} \) must also vanish for the same reasons. That is,

\[
B_{ABC} = 0. \tag{5.6} \]

We are left with the super-zehnbeins \( E_A^M = (E_A^M, E_\alpha^M) \), the super-two-form \( B_{AB} \), the dilaton superfield \( \phi \) and the dilatino superfield \( \chi^\dot{\alpha} \). We note in passing that

\[
\hat{g}_{1111} = \hat{E}_A^{11} \hat{E}_B^{11} \eta_{AB} + \hat{E}_A^{11} \hat{E}_B^{11} = \phi^2. \tag{5.7} \]

This relation will be useful in the next section when discussing low energy moduli fields.
The pullback of the super-elfbeins that enter the first two terms in (5.2) become

\[ \hat{\Pi}^A_i = \hat{E}_M^A M^i = (\partial_i Z^M) \hat{E}_M^A, \]
\[ \hat{\Pi}^{A\dagger} = (\partial_i Z^{\dagger M}) \hat{E}_M^{A\dagger} = (\partial_i Z^{\dagger M}) \hat{E}_M^{A\dagger} = 0, \]
\[ \hat{\Pi}^{A\dagger} = (\partial_{A\dagger} Z^{\dagger M}) \hat{E}_M^{A\dagger} = 0, \]
\[ \hat{\Pi}^{A\dagger} = \hat{E}_M^{A\dagger} = \phi. \]  

(5.8)

It is now straightforward to calculate the determinant of the induced metric. We obtain

\[ \det(\hat{\Pi}^A_i \hat{\Pi}^{A\dagger}_j \eta_{AB}) = \det(\Pi^A_i \Pi^B_j \eta_{AB} + \hat{E}_{A\dagger} \hat{E}_{B\dagger}) = \phi^2 \det(\Pi^A_i \Pi^B_j \eta_{AB}), \]  

(5.9)

Finally, we calculate the dimensional reduction of the closed three-form and find

\[ -\frac{1}{6} \varepsilon^{ijk} \partial_i Z^M \partial_j Z^N \partial_k Z_{PM} = -\frac{1}{2} \varepsilon^{ijkl} \partial_l Z^M \partial_k Z^N \eta_{BM}, \]

(5.10)

where we have used (5.4) and (5.6). Therefore, the first part of the action (5.2) reduces in the small \(\rho\) limit to the string action

\[ S_S = T_S \int C d^2 \sigma (\phi \sqrt{\det \Pi^A_i \Pi^B_j \eta_{AB} - \frac{i}{2} \varepsilon^{ijkl} \Pi^A_i \Pi^B_j \eta_{BM}}), \]  

(5.11)

where

\[ T_S = T_M \pi \rho \equiv (2\pi \alpha')^{-1} \]

(5.12)

is the string tension of mass dimension two.

Before we can write the total action for the open supermembrane compactified on \(S^1/\mathbb{Z}_2\), we must discuss the boundary terms in (5.2). In the limit that the radius \(\rho\) of \(S^1\) shrinks to zero, the two orbifold fixed planes coincide. Generically, the two different boundaries of the supermembrane need not be identified. However, since our supersymmetric embedding Ansatz assumes all quantities to be independent of the orbifold coordinate, the two boundary strings coincide as the zero radius limit is taken. Putting everything together, we find that the resulting action is

\[ S_C = T_S \int_C d^2 \sigma (\phi \sqrt{\det \Pi^A_i \Pi^B_j \eta_{AB} - \frac{i}{2} \varepsilon^{ijkl} \Pi^A_i \Pi^B_j \eta_{BM}}) \]

\[ -\frac{1}{8\pi} \sum_{n=1,2} \int_C d^2 \sigma \left\{ \frac{1}{2} \sqrt{g} g^{ij} (\omega^{(n)}_i - \Lambda^{(n)}_i) \cdot (\omega^{(n)}_j - \Lambda^{(n)}_j) + i \varepsilon^{ij} \omega^{(n)}_i \Lambda^{(n)}_j \right\}, \]

(5.13)

where

\[ \Pi^A_i = \partial_i Z^M \hat{E}_M^A. \]  

(5.14)
Note that the gauge supermultiplets \( \mathbb{A}^{(1)} \) and \( \mathbb{A}^{(2)} \) now collectively represent the connection for the single gauge group \( E_8 \times E_8 \) pulled back onto \( C \). We will, henceforth, denote this connection by \( \bar{A} \). Furthermore, \( \omega^{(1)} \) and \( \omega^{(2)} \) are now both one-forms on \( C \). Define

\[
\bar{g}(\sigma) = e^{\phi^{(1)}(\sigma)T_a^{(1)} + \phi^{(2)}(\sigma)T_a^{(2)}},
\]

where, collectively, \( T_a^{(1)} \) and \( T_a^{(2)} \) are the generators of \( E_8 \times E_8 \) and let \( \bar{\omega} = \bar{g}^{-1}d\bar{g} \). Then, using the fact that \( T_a^{(1)} \) and \( T_a^{(2)} \) commute, one can show that (5.13) can be rewritten as

\[
S_C = T_S \int_C d^2\sigma \left( \phi \sqrt{\det \Pi_i^A \Pi_j^B \eta_{AB}} - \frac{i}{2} \varepsilon^{ij} \Pi_i^A \Pi_j^B B_{AB} \right)
\]

\[
- \frac{1}{8\pi} \int_C d^2\sigma \left\{ \text{tr} \left[ \frac{1}{2} \sqrt{\bar{g}} g^{ij} (\bar{\omega}_i - \bar{\omega}_i) \cdot (\bar{\omega}_j - \bar{\omega}_j) + i \varepsilon^{ij} \bar{\omega}_i \bar{\omega}_j \right] - i \varepsilon^{ij} B_{ji}(\bar{\omega}) \right\}.
\]

Finally, following [38], we note that if \( C \) is taken to be the boundary of some three-ball \( B \), then one can rewrite

\[
\frac{1}{4\pi} \int_C d^2\sigma i \varepsilon^{ij} B_{ji}(\bar{\omega}) = \frac{1}{12\pi} \int_B d^3\hat{\sigma} i \varepsilon^{ijk} \Omega_{k\hat{j}\hat{i}}(\bar{\omega}'),
\]

where \( \bar{\omega}' \) is the homotopic extension of \( \bar{\omega} \) onto the ball \( B \). We recognize the action (5.16) and (5.17) as that of the heterotic \( E_8 \times E_8 \) superstring wrapped on a holomorphic curve \( C \subset CY_3 \).

6 Superpotential in 4D Effective Field Theory:

It is essential when constructing the superpotential to have a detailed understanding of all the moduli in five-dimensional heterotic M-theory. Furthermore, we must know explicitly how they combine to form the moduli of the four-dimensional low-energy theory. With this in mind, we now briefly review the compactification of Hořava-Witten theory to heterotic M-theory on a Calabi-Yau threefold with \( G \)-flux. We then further compactify this theory on \( S^1/\mathbb{Z}_2 \), arriving at the \( N = 1 \) supersymmetric action of the effective four-dimensional theory. We emphasize that, throughout this paper, we take the bosonic components of all superfields to be of dimension zero, both in five-dimensional heterotic M-theory and in the associated four-dimensional effective theory.

First consider the compactification from Hořava-Witten theory to heterotic M-theory. A peculiar feature of this compactification is that the Bianchi identity for the three-form field \( C \) is modified due to both gauge and gravitational anomalies on the boundary fixed planes. This implies that its field-strength \( G \) has nonzero components in the \( CY_3 \) direction.
As a consequence of this nontrivial $G$-flux, the five-dimensional effective theory of strongly coupled heterotic string theory is given by a specific gauged version of five-dimensional supergravity. This compactification is carried out as follows. Consider the metric

$$ds_{11}^2 = V^{-2/3} \hat{g}_{\hat{u}\hat{v}} d\hat{y}^\hat{u} d\hat{y}^\hat{v} + \hat{g}_{\hat{U}\hat{V}} d\hat{y}^\hat{U} d\hat{y}^\hat{V},$$

(6.1)

where $\hat{y}^\hat{u}, \hat{u} = 6, 7, 8, 9, 11$ are the coordinates of the five-dimensional bulk space of heterotic M-theory, $\hat{y}^\hat{U}, \hat{U} = 0, 1, 2, 3, 4, 5$ are the Calabi-Yau coordinates and $\hat{g}_{\hat{U}\hat{V}}$ is the metric on the Calabi-Yau space $CY_3$. The factor $V^{-2/3}$ in (6.1) has been chosen so that metric $\hat{g}_{\hat{u}\hat{v}}$ is the five-dimensional Einstein frame metric.

The non-metric five-dimensional zero-mode fields are obtained by expanding the fields of eleven-dimensional supergravity in terms of the cohomology classes of $CY_3$. We state in advance that the five-dimensional hypermultiplets associated with the $(2,1)$-forms of $CY_3$ cannot contribute to the superpotential generated by supermembranes and, hence, will not be discussed further. Using complex notation, the Kähler form in $CY_3$ is defined by

$$\omega_{m\bar{n}} = i g_{m\bar{n}},$$

(6.2)

and can be expanded in terms of the harmonic $(1,1)$-forms $\omega_{I m\bar{n}}, I = 1, \ldots, h^{1,1}$ as

$$\omega_{m\bar{n}} = \sum_{I=1}^{h^{1,1}} a^I \omega_{I m\bar{n}}.$$

(6.3)

The coefficients $a^I = a^I (y^\hat{u})$ are the $(1,1)$-moduli of the Calabi-Yau space. The Calabi-Yau volume modulus $V = V (y^\hat{u})$ is defined by

$$V = \frac{1}{v} \int_{CY_3} \sqrt{\hat{g}},$$

(6.4)

where $\hat{g}$ is the determinant of the Calabi-Yau metric $\hat{g}_{\hat{U}\hat{V}}$ and $v$ is a dimensionful parameter necessary to make $V$ dimensionless. The $h^{1,1}$ moduli $a^I$ and $V$ are not completely independent. It can be shown that

$$V = \frac{1}{6} \sum_{I,J,K=1}^{h^{1,1}} d_{IJK} a^I a^J a^K,$$

(6.5)

where coefficients $d_{IJK}$ are the Calabi-Yau intersection numbers defined by

$$d_{IJK} = \int_{CY_3} \omega_I \wedge \omega_J \wedge \omega_K.$$

(6.6)

Therefore, we can take $h^{1,1} - 1$ out of the $h^{1,1}$ moduli $a^I$, which we denote as $a^{\hat{I}}$ with $\hat{I} = 1, \ldots, h^{1,1} - 1$, and $V$ as the independent five-dimensional zero-modes. Now consider
the zero-modes of the antisymmetric tensor field $\hat{C}_{MNP}$. These are given by $\hat{C}_{\hat{u}\hat{v}\hat{w}}$ as well as

$$\hat{C}_{\hat{u}\hat{v}\hat{w}} = \sum_{I=1}^{h^{1,1}} \frac{1}{6} A^I_\hat{u}(y^\hat{v}) \omega_{I\hat{m}\hat{n}} \tag{6.7}$$

and

$$\hat{C}_{nm\hat{p}} = \frac{1}{6} \xi(y^\hat{v}) \Omega_{mnp}, \tag{6.8}$$

where $\Omega_{mnp}$ is the harmonic $(3,0)$-form on $CY_3$. Therefore, in addition to the graviton $g_{\hat{u}\hat{v}}$, the zero-mode fields of the five-dimensional effective theory are $h^{1,1} - 1$ real scalar fields $a^I$, a real scalar $V$, $h^{1,1}$ vector fields $A^I_\hat{u}$, a complex scalar $\xi$ and $\hat{C}_{\hat{u}\hat{v}\hat{w}}$.

Therefore, in addition to the graviton $g_{\hat{u}\hat{v}}$, the zero-mode fields of the five-dimensional effective theory are $h^{1,1} - 1$ real scalar fields $a^I$, a real scalar $V$, $h^{1,1}$ vector fields $A^I_\hat{u}$, a complex scalar $\xi$ and $\hat{C}_{\hat{u}\hat{v}\hat{w}}$.

These fields all must be the bosonic components of specific $N = 1$ supermultiplets in five-dimensions. These supermultiplets are easily identified as follows.

1. Supergravity: the bosonic part of this supermultiplet is

$$(g_{\hat{u}\hat{v}}, A_\hat{u}, \ldots). \tag{6.9}$$

This accounts for $g_{\hat{u}\hat{v}}$ and a linear combination of the vector moduli $A^I_\hat{u}$ which combine to form the graviphoton $A_\hat{u}$. We are left with $h^{1,1} - 1$ vector fields, denoted by $A^I_\hat{u}$.

2. Vector supermultiplets: the bosonic part of these supermultiplets is

$$(A^I_\hat{u}, b^I, \ldots). \tag{6.10}$$

Clearly, there are $h^{1,1} - 1$ such vector multiplets in the theory, accounting for the remaining $A^I_\hat{u}$ vector moduli. The $h^{1,1} - 1$ scalars $b^I$ can be identified as

$$b^I = V^{-1/3} a^I, \tag{6.11}$$

thus accounting for all $h^{1,1} - 1 (1,1)$-moduli.

3. Universal hypermultiplet: the bosonic part of this supermultiplet is

$$(V, C_{\hat{u}\hat{v}\hat{w}}, \xi, \ldots), \tag{6.12}$$

which accounts for the remaining zero-modes discussed above. Having identified the appropriate $N = 1$, five-dimensional superfields, one can read off the zero-mode fermion spectrum to be precisely those fermions that complete these supermultiplets.

We now move to the discussion of the compactification of heterotic M-theory in five-dimensions to the effective $N = 1$ supersymmetric theory in four-dimensions. This compactification was carried out in detail in [3]. Here we simply state the resultant four-dimensional
zero-modes and their exact relationship to the five-dimensional moduli of heterotic M-theory. The bulk space bosonic zero-modes coincide with the $\mathbb{Z}_2$-even fields. One finds that the metric is

$$ds_5^2 = R^{-1} g_{uv} dy^u dy^v + R^2 (dy^{11})^2,$$

(6.13)

where $g_{uv}$ is the four-dimensional metric and $R = R(y^u)$ is the volume modulus of $S^1/\mathbb{Z}_2$. The remaining four-dimensional zero-modes are

$$V = V(y^u), \quad b^I = b^I(y^u),$$

(6.14)

where $I = 1, \ldots, h^{1,1} - 1$. In addition, one finds $h^{1,1}$ scalar fields

$$A^I_{11} = p^I(y^u),$$

(6.15)

$h^{1,1} - 1$ of them arising as $A^I_{11}$ and one extra field descending from the eleven-component of the graviphoton $A_{11}$, which is $\mathbb{Z}_2$-even. Finally, there is a two-form field

$$C_{uv11} = \frac{1}{3} B_{uv}(y^u).$$

(6.16)

This two-form can be dualized to a scalar $\sigma$ as

$$H_{uvm} = V^{-2} \epsilon^{x}_{uwm} \partial_x \sigma,$$

(6.17)

where $H = dB$. It is conventional to redefine these fields into the dilaton $S$ and $h^{1,1}$ T-moduli $T^I$ as

$$S = V + i \sqrt{2} \sigma, \quad T^I = R b^I + i \frac{1}{6} p^I.$$

(6.18)

Note that in the definition of the $T^I$ moduli, we include all $h^{1,1}$ $(1, 1)$-moduli, even though they satisfy the constraint

$$6 = \sum_{I,J,K=1}^{h^{1,1}} d_{IJK} b^I b^J b^K.$$

(6.19)

This constraint reduces the number of $b^I$ moduli by one, but this is replaced by the $S^1/\mathbb{Z}_2$ volume modulus $R$. Hence, there remain $2h^{1,1}$ scalar degrees of freedom from which to form the $h^{1,1} T^I$ chiral supermultiplets. It is then easily seen that these modes form the following four-dimensional, $N = 1$ supermultiplets.

1. Supergravity: the full supermultiplet is

$$ (g_{uv}, \psi^a_i),$$

(6.20)

Note that the definition of the imaginary part of $T^I$ differs from that in [3] by a factor of $6 \sqrt{2}$. The factor chosen here has the same Kähler potential as in [3] and, as we will see, is more natural.
where $\psi_u^\alpha$ is the gravitino.

2. Dilaton and T-moduli chiral supermultiplets: the full multiplets are

$$(S, \lambda_S), \quad (T^I, \lambda^I_T), \quad (6.21)$$

where $I = 1, \ldots, h^{1,1}$ and $\lambda_S, \lambda^I_T$ are the dilatino and T-modulinos, respectively.

The fermions completing these supermultiplets arise as zero-modes of the fermions of five-dimensional heterotic M-theory. The action for the effective, four-dimensional, $N = 1$ theory has been derived in detail in [3]. Here we simply state the result. The relevant terms for our discussion of the superpotential are the kinetic terms for the $S$ and $T^I$ moduli and the bilinear terms of their superpartner fermions. If we collectively denote $S$ and $T^I$ as $Y^I$, where $I = 1, \ldots, h^{1,1} + 1$, and their fermionic superpartners as $\lambda^I$, then the component Lagrangian is given by

$$L_{4D} = K^I_{J'} \bar{\lambda}^{I'} \partial_{\lambda^J} + e^{\kappa_p^2 K} \left( K^{I'J'} D_I W \bar{D}_J W - 3\kappa_p^2 |W|^2 \right) + K^I_{J'} \bar{\lambda}^{I'} \partial_{\lambda^J} - e^{\kappa_p^2 K/2} (D_I D_J W) \lambda^I \lambda^J + \text{h.c.} \quad (6.22)$$

Here $\kappa_p^2$ is the four-dimensional Newton’s constant,

$$K^I_{J'} = \partial_I \partial_{J'} K \quad (6.23)$$

are the Kähler metric and Kähler potential respectively, and

$$D_I W = \partial_I W + \kappa_p^2 \frac{\partial K}{\partial Y^I} W \quad (6.24)$$

is the Kähler covariant derivative acting on the superpotential $W$. The Kähler potential was computed in [3]. In terms of the $S$ and $T^I$ moduli it is given by

$$\kappa_p^2 K = -\ln(S + \bar{S}) - \ln \left( \frac{1}{6} \sum_{I,J,K=1}^{h^{1,1}} d_{IJK} (T + \bar{T})^I (T + \bar{T})^J (T + \bar{T})^K \right) \quad (6.25)$$

It is useful at this point to relate the low energy fields of the heterotic superstring action derived in Section 5 to the four-dimensional moduli derived here from heterotic M-theory. Specifically, we note from (5.7) that

$$\hat{g}_{1111} |_{\Theta = 0} = \phi^2 |_{\Theta = 0}, \quad (6.26)$$

and from (6.1) and (6.13) that

$$ds_{11}^2 = \cdots + R^2 V^{-2/3} (dy^1)^2. \quad (6.27)$$
Identifying them implies that
\[ \phi |_{\Theta=0} = RV^{-1/3}. \] (6.28)

Similarly, it follows from (5.10), (6.7) and (6.15) that
\[ B_{m\bar{n}} |_{\Theta=0} = B_{m\bar{n}} = \sum_{I=1}^{n} \frac{1}{6} p^I \omega_{Im\bar{n}}. \] (6.29)

We will use these identifications in the next section.

Following the approach of [27] and [28], we will calculate the non-perturbative superpotential by computing instanton induced fermion bilinear interactions and then comparing these to the fermion bilinear terms in the low energy effective supergravity action. In this paper, the instanton contribution arises from open supermembranes wrapping on a product of the \( S^1 / \mathbb{Z}_2 \) interval and a holomorphic curve \( \mathcal{C} \subset CY_3 \). Specifically, we will calculate this instanton contribution to the two-point function of the fermions \( \lambda^I \) associated with the \( T^I \) moduli. The two-point function of four-dimensional space-time fermions \( \lambda^I, \lambda^J \) located at positions \( y_1^u, y_2^u \) is given by the following path integral expression
\[
\langle \lambda^I(y_1^u)\lambda^J(y_2^u) \rangle = \int \mathcal{D}\Phi e^{-S_{4D}} \lambda^I(y_1^u)\lambda^J(y_2^u) \cdot \int \mathcal{D}\hat{Z}\mathcal{D}\omega e^{-S_\Sigma(\hat{Z},\omega;\hat{A}^A,\hat{C}^{MN},A^{(n)})}, \] (6.30)

where \( S_\Sigma \) is the open supermembrane action given in (5.2). Here \( \Phi \) denotes all supergravity fields in the \( N = 1 \) supersymmetric four-dimensional Lagrangian (6.22) and \( \hat{Z}, \omega \) are all the worldvolume fields on the open supermembrane. In addition, the path-integral is performed over all supersymmetry preserving configurations of the membrane in the eleven-dimensional Horava-Witten background \( (\hat{B}^A, \hat{C}^{MN}, A^{(n)}_M) \) compactified down to four-dimensions on \( CY_3 \times S^1 / \mathbb{Z}_2 \). The integration will restore \( N = 1 \) four-dimensional supersymmetry. The result of this calculation is then compared to the terms in (6.22) proportional to \( (D_ID_J W)\lambda^I \lambda^J \) and the non-perturbative contribution to \( W \) extracted.

7 String Action Expansion:

In this paper, we are interested in the non-perturbative contributions of open supermembrane instantons to the two-point function (6.30) of chiral fermions in the four-dimensional effective field theory. In order to preserve \( N = 1 \) supersymmetry, the supermembrane must be of the form \( \Sigma = \mathcal{C} \times S^1 / \mathbb{Z}_2 \), where curve \( \mathcal{C} \subset CY_3 \) is holomorphic. As we have shown in previous sections, this is equivalent, in the low energy limit, to considering the non-perturbative contributions of heterotic superstring instantons to the same fermion two-point

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7 In the remained of this paper, we will drop the subscript \( T \) in \( \lambda_T \) given in (6.21).

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function in the effective four-dimensional theory. Of course, in this setting, the superstring must wrap completely around a holomorphic curve $C \subset CY_3$ in order for the theory to be $N = 1$ supersymmetric.

Since we are interested only in non-perturbative corrections to the two-point function $\langle \lambda^I(y_1^u)\lambda^J(y_2^u) \rangle$, the perturbative contributions to this function, which arise from the interaction terms in the effective four-dimensional action $S_{4D}$ in (6.30), will not be considered in this paper. Therefore, we keep only the kinetic terms of all four-dimensional dynamic fields in $S_{4D}$. Furthermore, we can perform the functional integrations over all these fields except $\lambda^I$, obtaining some constant determinant factors which we need not evaluate. Therefore, we can rewrite (6.30) as

$$\langle \lambda^I(y_1^u)\lambda^J(y_2^u) \rangle \propto \int D\lambda e^{-\int d^4y \sum_{K=1}^{1,1} \lambda^K \partial^K \lambda^I(y_1^u)\lambda^J(y_2^u)} \cdot \int DZ \omega e^{-S_C(Z,\omega; E_{MN}, \phi, \bar{A}_M)},$$

(7.1)

where $S_C$ is the heterotic superstring action given in (5.13). As we will see shortly, the functional dependence of $S_C$ on the fields $\lambda^I$ comes from the interaction between the superstring fermionic field $\Theta$ and the ten-dimensional gravitino (from which $\lambda^I$ is derived in the Kaluza-Klein compactification). Both of these fermions are Weyl spinors in ten-dimensions.

Clearly, to perform the computation of the two-point function (7.1), we must write the action $S_C$ in terms of its dynamical fields and their interactions with the dimensionally reduced background fields. This means that we must first expand all superfield expressions in terms of component fields. We will then expand the action in small fluctuations around its extrema (solutions to the superstring equations of motion), corresponding to a saddle-point approximation. We will see that because there exists two fermionic zero-modes arising from $\Theta$, their interaction with the gravitino will produce a non-vanishing contribution to (7.1). Therefore, when performing the path-integrals over the superstring fields, we must discuss the zero-modes with care. The next step will be to consider the expression for the superstring action and to write it in terms of the moduli of the compactification space $CY_3 \times S^1/\mathbb{Z}_2$. Finally, we will perform all remaining path integrals in the saddle-point approximation, obtaining the appropriate determinants.

This will entail a lengthy calculation. Let us then start by expanding the ten-dimensional superfields in the action $S_C$ in terms of the component fields.

*Note that in Euclidean space one does not have Majorana-Weyl spinors in ten-dimensions.*
Expanding in Powers of $\Theta$:

We begin by rewriting action $S_C$ in (5.13) as

$$S_C = S_S + S_{WZW},$$

where

$$S_S(Z; E_{\mathcal{A}M}(Z), B_{MN}(Z), \phi(Z)) = T_S \int_C d^2\sigma \left( \phi \sqrt{\det \partial_i Z^M \partial_j Z^N} E_{\mathcal{A}M}B_{\mathcal{B}A} \right) - \frac{i}{2} \epsilon^{ij} \partial_i Z^M E_{\mathcal{A}M} \partial_j Z^N E_{\mathcal{B}A}$$

is the supermembrane bulk action dimensionally reduced on $S^1/\mathbb{Z}_2$ and

$$S_{WZW}(Z, \omega; A^{(n)}_{\mathcal{A}M}(Z)) = -\frac{1}{8\pi} \int_C d^2\sigma \text{tr} \left[ \frac{1}{2} \sqrt{g} g^{ij} (\bar{\omega}_i - \bar{A}_i) \cdot (\omega_j - A_j) + i \epsilon^{ij} \omega_i \bar{A}_j \right] + \frac{1}{24\pi} \int_B d^3\hat{\sigma} \hat{\epsilon}^{ijk} \Omega_{k\bar{j}j}(\omega')$$

is the gauged Wess-Zumino-Witten action, where

$$\bar{A}_i = \partial_i Z^M \bar{A}_M(Z).$$

Note that this action is a functional of $Z(\sigma) = (X(\sigma), \Theta(\sigma))$. We now want to expand the superfields in (5.2) in powers of the fermionic coordinate $\Theta(\sigma)$. For the purposes of this paper, we need only keep terms up to second order in $\Theta$. We begin with $S_S$ given in (7.3).

Using an approach similar to [39] and using the results in [40], we find that, to the order in $\Theta$ required, the super-zehnbeins are given by

$$E_{\mathcal{A}M} = \left( \begin{array}{c} E^A_M - i \bar{\Psi}_M \Gamma^A \Theta \frac{1}{2} \Psi^\alpha_M + \frac{1}{4} \omega^C_{\mathcal{A}M} (\Gamma_{CD})^\alpha \Theta^\nu \\ -i \Gamma^\alpha_{\mu \nu} \Theta^\nu \delta^\alpha_{\mu} \end{array} \right),$$

where $E^A_M(X(\sigma))$ are the bosonic zehnbeins, $\Psi(X(\sigma))$ is the ten-dimensional gravitino, and $\omega^C_{\mathcal{A}M}(X(\sigma))$ is the ten-dimensional spin connection, defined in terms of derivatives of $E^A_M(X)$. The super-two-form fields are, up to the required order for the action to be at most quadratic in $\Theta$,

$$B_{MN} = B_{MN} + i \phi (\Theta \Gamma_{[M} \Psi_{N]} + \frac{i}{4} \Theta \Gamma_{[M} \Gamma^{CD} \Theta \omega_{N]CD}),$$

$$B_{MN} = -i \phi (\Gamma_M \Theta)_{\mu},$$

$$B_{\mu \nu} = 0,$$

where $B_{MN}$ is the ten-dimensional bosonic two-form field. Finally, we can write

$$\phi = RV^{-1/3}.$$
where we have used (6.28). Substituting these expressions into action (7.3), it can be written as

$$S_s = S_0 + S_\Theta + S_{\Theta^2},$$  

(7.9)

where $S_0$ is purely bosonic

$$S_0(X; E^A_M(X), B_{MN}(X)) = T_S \int \frac{d^2\sigma}{C} (RV^{-1/3} \sqrt{\det \partial_i X^M \partial_j X^N E^A_M E^B_N \eta_{AB}}$$

$$- \frac{i}{2} \varepsilon^{ij} \partial_i X^M \partial_j X^N B_{MN}),$$  

(7.10)

and $S_\Theta$ and $S_{\Theta^2}$ are the first two terms (linear and quadratic) in the $\Theta$ expansion. Straightforward calculation gives

$$S_\Theta(X, \Theta; E^A_M(X), \Psi_M(X)) = T_S \int \frac{d^2\sigma}{C} (RV^{-1/3} \sqrt{\det \partial_i X^M \partial_j X^N E^A_M E^B_N \eta_{AB}}$$

$$\times \frac{1}{2} (\bar{\Psi}_M \gamma^M - \bar{\nu}^M \Psi_M)$$  

(7.11)

and

$$S_{\Theta^2}(X, \Theta; E^A_M(X)) = T_S \int \frac{d^2\sigma}{C} (RV^{-1/3} \sqrt{\det \partial_i X^M \partial_j X^N E^A_M E^B_N \eta_{AB}}$$

$$\times (\bar{\Theta} \epsilon^{ij} \Gamma_i D_j \Theta - i \epsilon^{ij} \bar{\Theta} \Gamma_i D_j \Theta),$$  

(7.12)

where $D_i \Theta$ is the covariant derivative

$$D_i \Theta = \partial_i \Theta + \partial_i X^N \omega^A_N \Gamma_{AB} \Theta,$$  

(7.13)

$\Gamma_i$ is the pullback of the eleven-dimensional Dirac matrices

$$\Gamma_i = \partial_i X^M \Gamma_M,$$  

(7.14)

and $\gamma^M$ is the vertex operator for the gravitino $\Psi_M$, given by

$$\gamma^M = g^{ij} \partial_i X^M \partial_j X^N \Gamma_N \Theta - i \epsilon^{ij} \partial_i X^M \partial_j X^N \Gamma_N \Theta,$$  

(7.15)

where $\epsilon^{ij} = \epsilon^{ij} / \sqrt{g}$. Now consider the expansion of the superfields in $S_{WZW}$ given in (7.4). Here, we need only consider the bosonic part of the expansion

$$S_{0_{WZW}}(X, \omega; A_M(X), E^A_M(X)) = -\frac{1}{8\pi} \int_C \frac{d^2\sigma}{C} \mathrm{tr} \left[ \frac{1}{2} g^{ij} \bar{\omega}_i - A_i \right] \cdot \left[ \bar{\omega}_j - A_j \right] + \frac{1}{24\pi} \int_B \frac{d^3 \xi \epsilon^{ijk} \Omega_{kji}(\omega')}$$  

(7.16)

In a space with Minkowski signature, where the spinors are Majorana-Weyl, the fermion product would be $\bar{\Psi}_M \gamma^M$. However, in Euclidean space, the fermions are Weyl spinors only and this product becomes the hermitian sum $\frac{1}{2} (\bar{\Psi}_M \gamma^M - \bar{\nu}^M \Psi_M)$. 

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where $\tilde{A}_i(\sigma) = \partial_iX^M \tilde{A}_M(X(\sigma))$ is the bosonic pullback of $\tilde{A}_M$. For example, the expansion of $\tilde{A}_M$ to linear order in $\Theta$ contains fermions that are not associated with the moduli of interest in this paper. Hence, they can be ignored. Similarly, we can show that all other terms in the $\Theta$ expansion of $S_{WZW}$ are irrelevant to the problem at hand.

Note that, in terms of the coordinate fields $X$ and $\Theta$, the path integral measure in (7.1) becomes\[10]

$$\mathcal{D}Z\mathcal{D}\omega = \mathcal{D}X\mathcal{D}\Theta\mathcal{D}\omega. \quad (7.17)$$

We can now rewrite the two-point function as

$$\langle \lambda^I(y_1^u)\lambda^J(y_2^u) \rangle \propto \int \mathcal{D}\lambda e^{-\int d^4y \sum_{K=1}^{8} \lambda^K \partial^{\lambda^K} \lambda^I(y_1^u)\lambda^J(y_2^u)} \cdot \int \mathcal{D}X\mathcal{D}\Theta e^{-(S_0 + S_\Theta + S_{WZW})} \cdot \int \mathcal{D}\omega e^{-S_{WZW}}. \quad (7.18)$$

The last factor

$$\int \mathcal{D}\omega e^{-S_{WZW}} \quad (7.19)$$

behaves somewhat differently and will be evaluated in the next section. Here, we simply note that it does not contain the fermion $\lambda^I$ and, hence, only contributes an overall determinant to the superpotential. This determinant, although physically important, does not affect the rest of the calculation, to which we now turn. To perform the $X, \Theta$ path integral, it is essential that we fix any residual gauge freedom in the $X$ and $\Theta$ fields.

**Fixing the $X$ and $\Theta$ Gauge:**

First, let us fix the gauge of the bosonic coordinate fields $X$ by identifying

$$X^{m'}(\sigma) = \delta^{m'}_i \sigma^i, \quad (7.20)$$

where $m' = 0, 1$. This choice, which corresponds to orienting the $X^0$ and $X^1$ coordinates along the string worldvolume, can always be imposed. This leaves eight real bosonic degrees of freedom, which we denote as

$$X^{m''}(\sigma) \equiv y^{m''}(\sigma), \quad (7.21)$$

where $m'' = 2, \ldots, 9$. Next, let us fix the gauge of the fermionic coordinate fields $\Theta$. Recall that $\Theta$ is a Weyl spinor in ten-dimensional Euclidean space. Note that there are 16 complex

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\[10\]Since we are working in Euclidean space, the spinor fields $\Theta$ are complex. To be consistent one must use the integration measure $\mathcal{D}\mathcal{D}\Theta$. In this paper we write the integration measure $\mathcal{D}\Theta$ as a shorthand for $\mathcal{D}\mathcal{D}\Theta$.
(or 32 real) independent components in this Weyl spinor. Now make an two-eight split in the Dirac matrices

\[ \Gamma_A = (\tau_{a'} \otimes \tilde{\gamma}, 1 \otimes \gamma_{a''}), \tag{7.22} \]

where \( a' = 0, 1 \) and \( a'' = 2, \ldots, 9 \) are flat indices, and \( \tau_{a'} \) and \( \gamma_{a''} \) are the two- and eight-dimensional Dirac matrices, respectively. Then \( \Gamma_{11} \equiv -i\Gamma_0\Gamma_1 \cdots \Gamma_9 \) can be decomposed as

\[ \Gamma_{11} = \tilde{\tau} \otimes \tilde{\gamma} \tag{7.23} \]

where \( \tilde{\gamma} = \gamma_2 \gamma_3 \cdots \gamma_9 \) and

\[ \tilde{\tau} = -i\tau_0 \tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.24} \]

More explicitly,

\[ \Gamma_{11} = \begin{pmatrix} \tilde{\gamma} & 0 \\ 0 & -\tilde{\gamma} \end{pmatrix}. \tag{7.25} \]

In general, the Weyl spinor \( \Theta \) can be written in a generic basis as

\[ \Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}. \tag{7.26} \]

Note that \( SO(10) \) contains \( SO(2) \times SO(8) \) as a maximal subgroup. Under \( SO(8) \), \( \Theta_1 \) and \( \Theta_2 \) transform independently as spinors. The Weyl condition is chosen to be

\[ \frac{1}{2}(1 - \Gamma_{11})\Theta = 0. \tag{7.27} \]

Using (7.22), this condition implies

\[ \tilde{\gamma}\Theta_1 = \Theta_1, \quad \tilde{\gamma}\Theta_2 = -\Theta_2. \tag{7.28} \]

That is, \( \Theta_1 \) (\( \Theta_2 \)) has positive (negative) eight-dimensional chirality. It follows from the relation \( \Gamma_{11} = \tilde{\tau} \otimes \tilde{\gamma} \) that the two- and eight-dimensional chiralities of \( \Theta \) are correlated. Since \( \Theta \) has positive ten-dimensional chirality (7.27), this implies that the two- and eight-dimensional chiralities are either both positive or both negative. That is, \( \Theta \) is in the representation

\[ 16^+ = 1^+ \otimes 8^+ \oplus 1^- \otimes 8^- \tag{7.29} \]

From (7.28), we see that \( \Theta_1 \) is in \( 1^+ \otimes 8^+ \) and \( \Theta_2 \) is in \( 1^- \otimes 8^- \).

Recall from our discussion of \( \kappa \)-symmetry in Section 2 that, because we can use the \( \kappa \)-invariance of the worldvolume theory to gauge away half of the 16 independent components of \( \Theta \), only half of these components represent physical degrees of freedom. For the
superstring, we can define the projection operators

\[ P_\pm = \frac{1}{2} \left( 1 \pm \frac{i}{\sqrt{g}} \epsilon^{ij} \Pi_i^A \Pi_j^B \Gamma_{AB} \right) \] (7.30)

and write

\[ \Theta = P_+ \Theta + P_- \Theta. \] (7.31)

Now note from (2.14) that \( P_+ \Theta \) can be gauged away, while the physical degrees of freedom are given by \( P_- \Theta \). Using (7.26), it follows that \( \Theta_2 \) in (7.26) can be gauged to zero, leaving only \( \Theta_1 \) as the physical degrees of freedom. We thus can fix the fermion gauge so that

\[ \Theta = \begin{pmatrix} \theta \\ 0 \end{pmatrix}, \] (7.32)

where \( \theta \) is an \( SO(8) \) spinor with positive chirality,

\[ \tilde{\gamma} \theta = \theta. \] (7.33)

That is, the physical fermions in the worldsheet theory belong to the representation \( 1^+ \otimes 8^+ \) of \( SO(2) \times SO(8) \).

We conclude that the physical degrees of freedom contained in \( \mathcal{Z} = (X, \Theta) \) are

\[ y^{m''}(\sigma), \quad \theta^{\dot{q}}(\sigma), \] (7.34)

where \( m'' = 2, \ldots, 9 \), and \( \dot{q} = 1, \ldots, 8 \). The spinor index \( \dot{q} \) corresponds to the positive chirality \( SO(8) \)-Weyl representation. Therefore, the \( X, \Theta \) path-integral measure in (7.18) must be rewritten as

\[ \mathcal{D}X\mathcal{D}\Theta \propto \mathcal{D}y\mathcal{D}\theta, \] (7.35)

where there is an unimportant constant of proportionality representing the original gauge redundancy.\(^1\)

**Equations of Motion:**

We can now rewrite the two-point function (7.18) as

\[ \langle \lambda^I(\nu_1)\lambda^J(\nu_2) \rangle \propto \int \mathcal{D}\lambda e^{-\int d^4 y \sum_{K=1}^{4} \lambda^K \partial \lambda^K} \lambda^I(\nu_1)\lambda^J(\nu_2) \cdot \int \mathcal{D}y\mathcal{D}\theta e^{-(S_0 + S_\Theta + S_{\Theta 2})} \cdot \int \mathcal{D}\omega e^{-S_{\omega 2}}. \] (7.36)

\(^1\)Here, again, we write \( \mathcal{D}\theta \) as a shorthand for \( \mathcal{D}\bar{\theta}\mathcal{D}\theta \).
In this paper, we want to use a saddle-point approximation to evaluate these path-integrals. We will consider small fluctuations $\delta y$ and $\delta \theta$ of the superstring degrees of freedom around a solution $y_0$ and $\theta_0$ to the equations of motion

$$y = y_0 + \delta y, \quad \theta = \theta_0 + \delta \theta.$$ (7.37)

However, before expanding the action using (7.37), we need to discuss the equations of motion for the fields $y$ and $\theta$, as well as their zero-modes.

Consider first the equations of motion for the bosonic fields $y(\sigma)$. The bosonic action (7.10) can be written as

$$S_0 = T_S \int_C d^2 \sigma (RV^{-1/3} \sqrt{\det g_{ij}} + \frac{i}{2} \varepsilon^{ij} b_{ij}),$$ (7.38)

where

$$g_{ij} = \partial_i X^M \partial_j X^N g_{MN}, \quad b_{ij} = \partial_i X^M \partial_j X^N B_{MN}. \quad (7.39)$$

We now assume that the background two-form field $B_{MN}(X)$ satisfies $dB = 0$. This can be done if we neglect corrections of order $\alpha'$. Then, locally, $B = d\Lambda$, where $\Lambda$ is a one-form. Thus the second term in (7.38) can be written as a total derivative and so does not contribute to the equations of motion. Varying the action, we obtain the bosonic equations of motion

$$\frac{1}{2} \sqrt{\det g_{kl} g^{ij} \partial_i X^M \partial_j X^N} \frac{\partial g_{MN}}{\partial X^L} - \partial_i (\sqrt{\det g_{kl} g^{ij} \partial_j X^M g_{LM}}) = 0.$$ (7.40)

where $M, N, L = 0, \ldots, 9$. Since we are considering the product metric on $R_4 \times CY_3$, the ten-dimensional metric can be written as

$$g_{MN} = \begin{pmatrix} g_{U\bar{V}} & 0 \\ 0 & \eta_{uv} \end{pmatrix}. \quad (7.41)$$

Equation (7.40) then breaks into two parts

$$\frac{1}{2} \sqrt{\det g_{kl} g^{ij} \partial_i X^U \partial_j X^\bar{V}} \frac{\partial g_{U\bar{V}}}{\partial X^W} - \partial_i (\sqrt{\det g_{kl} g^{ij} \partial_j X^U g_{U\bar{V}}}) = 0.$$ (7.42)

and

$$\partial_i (\sqrt{\det g_{kl} g^{ij} \partial_j X^n \eta_{uv}}) = 0. \quad (7.43)$$

It is straightforward to show that the first equation of motion (7.42) is consistent with the BPS conditions (4.18) obtained in Section 4, as they should be. We will consider the second equation shortly.
Next consider the equations of motion for the fermionic degrees of freedom. In action (7.3) the terms that contain $\Theta$ are (7.11) and (7.12), whose sum can be written, taking into account the gauge fixing condition (7.32), as
\[
2T S \int_C d^2\sigma RV^{-1/3} \sqrt{\det g_{ij}} \left( \frac{1}{2} (\bar{\Psi}_M V^M - \bar{V}^M \Psi_M) + \bar{\Theta} \Gamma^i D_i \Theta \right),
\]
(7.44)
where
\[
V^M = g^{ij} \partial_i X^M \partial_j X^N \Gamma_N \Theta.
\]
(7.45)
It follows from the gauge fixing condition
\[
\Theta = \begin{pmatrix} \theta \\ 0 \end{pmatrix}
\]
(7.46)
that only half of the components of $\Psi$ couple to the physical degrees of freedom in $\Theta$, namely
\[
\Psi^+ = \frac{1}{2} (1 + 1 \otimes \tilde{\tau}) \Psi.
\]
(7.47)
The equations of motion for $\Theta$ are then found to be
\[
D_{0i} \Theta_0 = \frac{1}{2} \partial_i \bar{y}_0^U \Psi^+_U,
\]
(7.48)
where
\[
D_{0i} \Theta_0 = \partial_i \Theta_0 + \partial_i \bar{y}_0^U \omega_U^i \Gamma_{KL} \Theta_0,
\]
(7.49)
and we consider only the physical degrees of freedom $\theta_0$ in $\Theta_0$.

**Zero-Modes:**

The saddle-point calculation of the path-integrals $\mathcal{D}y$ and $\mathcal{D}\theta$ around a solution to the equations of motion can be complicated by the occurrence of zero-modes. First consider bosonic solutions of the equations of motion (7.40). By construction, all such solutions are maps from a holomorphic curve $\mathcal{C} \subset CY_3$ to the target space normal to the curve. Since $\mathcal{C} \subset CY_3$, the four functions which map to $R_4$, which we denote by $y_0^u$ with $u = 6, 7, 8, 9$, are constants independent of $\sigma$. Clearly, these can take any value in $R_4$, so we can write
\[
y_0^u \equiv x^u,
\]
(7.50)
where $x^u$ are coordinates of $R_4$. Therefore, any solution of the equations of motion will always have these four translational zero-modes. These modes are the solution of the second equation of motion (7.43). Are additional zero-modes possible? Generically, the
remaining functions \( y_0^U(\sigma) \), \( U = 2,3,4,5 \) can have other zero-modes. However, to avoid further technical complications we will, in this paper, consider only curves \( \mathcal{C} \) such that

\[
\mathcal{C} = \mathbb{C} \mathbb{P}^1 = S^2,
\]  

(7.51)

where the \( S^2 \) are rigid spheres isolated in \( CY_3 \). In this case, there are clearly no additional zero-modes. It follows that for a saddle-point calculation of the path-integrals around a rigid, isolated sphere the bosonic measure can be written as

\[
\mathcal{D} y^{m''} = d^4 x \mathcal{D} \delta y^{m''},
\]  

(7.52)

where we have expanded

\[
y^{m''} = y_0^{m''} + \delta y^{m''}
\]  

(7.53)

for small fluctuations \( \delta y^{m''} \).

Now consider fermionic solutions \( \theta_0 \) of the equation of motion (7.48). To any \( \Theta_0 \) can always be added a solution of the homogeneous ten-dimensional Dirac equation

\[
D_0 \iota \Theta' = 0.
\]  

(7.54)

This equation has the general solution

\[
\Theta' = \vartheta \otimes \eta_{-},
\]  

(7.55)

where \( \eta_{-} \) is the covariantly constant spinor on \( CY_3 \), discussed in Section 4, which is broken by the membrane embedding and \( \vartheta \) is an arbitrary Weyl spinor satisfying the Weyl equation in \( R_4 \). Therefore, any solution \( \theta_0 \) of the equations of motion will always have two complex component fermion zero-modes \( \vartheta^\alpha \), \( \alpha = 1,2 \). The rigid, isolated sphere has no additional fermion zero-modes. Hence, for a saddle-point calculation of the path integrals around a rigid, isolated sphere the fermionic measure can be written as

\[
\mathcal{D} \theta = d\vartheta^1 d\vartheta^2 \mathcal{D} \delta \theta,
\]  

(7.56)

where we have expanded

\[
\theta = \theta_0 + \delta \theta
\]  

(7.57)

for small fluctuations \( \delta \theta \). To conclude, in the saddle-point approximation the \( y, \theta \) part of the path integral measure can be written as

\[
\mathcal{D} y^{m''} \mathcal{D} \theta = d^4 x d\vartheta^1 d\vartheta^2 \mathcal{D} \delta y^{m''} \mathcal{D} \delta \theta.
\]  

(7.58)
Saddle-Point Calculation:

We are now ready to calculate the two-point function (7.36), which can be rewritten as

\[ \langle \lambda^I (y_1^u) \lambda^J (y_2^u) \rangle \propto \int \mathcal{D} \lambda e^{-\int d^4y \sum_{K=1}^{h_{1,1}} \lambda^K \lambda^K \lambda^I (y_1^u) \lambda^J (y_2^u)} \cdot \int d^4x d\phi^1 d\phi^2 d\delta y^{m_\nu} D\delta \theta e^{-(S_0 + S_\Theta + S_{\Theta^2})} \cdot \int \mathcal{D} \omega e^{-S_{WZW}}. \]

(7.59)

Substituting the fluctuations (7.37) around the solutions \( y_0 \) and \( \theta_0 \) into

\[ S = S_0 + S_\Theta + S_{\Theta^2}, \]

(7.60)

we obtain the expansion

\[ S = S_0 + S_2, \]

(7.61)

where, schematically

\[ S_0 = S \big|_{y_0, \theta_0} \]

(7.62)

and

\[ S_2 = \frac{\delta^2 S}{\delta y \delta y} |_{y_0, \theta_0} (\delta y)^2 + 2 \frac{\delta^2 S}{\delta y \delta \theta} |_{y_0, \theta_0} (\delta y \delta \theta) + \frac{\delta^2 S}{\delta \theta \delta \theta} |_{y_0, \theta_0} (\delta \theta)^2. \]

(7.63)

The terms in the expansion linear in \( \delta y \) and \( \delta \theta \) each vanish by the equations of motion. To avoid further complicating our notation, we state in advance the following simplifying facts. First, note that all terms in \( S_2 \) contribute to the two-point function to order \( \alpha' \) on the superstring worldsheet. Therefore, we should evaluate these terms only to classical order in \( y_0^{m_\nu} \) and \( \theta_0 \). To classical order, one can take \( \theta_0 = 0 \) since, to this order, the background gravitino on the right-hand side of (7.48) vanishes. Therefore, \( S_2 \) simplifies to

\[ S_2 = \frac{\delta^2 S}{\delta y \delta y} |_{y_0, \theta_0=0} (\delta y)^2 + \frac{\delta^2 S}{\delta \theta \delta \theta} |_{y_0, \theta_0=0} (\delta \theta)^2. \]

(7.64)

It is useful to further denote

\[ S_0 = S_0^y + S_0^\theta, \]

(7.65)

where

\[ S_0^y = (S_0) |_{y_0}, \quad S_0^\theta = (S_\Theta + S_{\Theta^2}) |_{y_0, \theta_0}, \]

(7.66)

and to write

\[ S_2 = S_2^y + S_2^\theta, \]

(7.67)

with

\[ S_2^y = \frac{\delta^2 S}{\delta y \delta y} |_{y_0, \theta_0=0} (\delta y)^2, \quad S_2^\theta = \frac{\delta^2 S}{\delta \theta \delta \theta} |_{y_0, \theta_0=0} (\delta \theta)^2. \]

(7.68)
We can then rewrite two-point function (7.59) as

\[
\langle \lambda^I(y_1^u) \lambda^J(y_2^u) \rangle \propto \int D\lambda e^{-\int d^4y \sum_{i=1}^{1,1} \lambda^{1,1}_i(y_1^u) \lambda^{1,1}_j(y_2^u) + \int d^4x e^{-S_0^y} \cdot \int d\theta e^{-S_0^\theta} \cdot \int D\delta \gamma e^{-S_0^\gamma} \cdot \int D\omega e^{-S_{\text{WZW}}}}.
\]  
(7.69)

We will now evaluate each of the path-integral factors in this expression one by one. We begin with \( \int d^4x e^{-S_0^y} \).

The \( S_0^y \) Term:

It follows from (7.66) that \( S_0^y \) is simply \( S_0 \), given in (7.38) and (7.39), evaluated at a solution of the equations of motion \( y_0^{m''} \). Using (7.50), which implies that \( \partial_i y_0^a = 0 \) for coordinates \( y_0^a \) of \( R_4 \), and the form of the ten-dimensional metric

\[
d s_{10}^2 = g_{uv} dy^u dy^v + g_{m\bar{n}} d\bar{y}^m d\bar{y}^\bar{n},
\]  
(7.70)

with \( \bar{y}^m, \bar{y}^\bar{n} \) complex coordinates of \( CY_3 \), we see that

\[
S_0^y = T_S \int_C d^2\sigma (RV^{-1/3} \sqrt{\det g_{ij}} + \frac{i}{2} e^{ij} b_{ij}),
\]  
(7.71)

where

\[
g_{ij} = \partial_i \bar{y}_0^m \partial_j \bar{y}_0^n g_{m\bar{n}}, \quad b_{ij} = \partial_i \bar{y}_0^m \partial_j \bar{y}_0^n B_{m\bar{n}}.
\]  
(7.72)

Let us evaluate the term involving \( g_{ij} \). To begin, we note that

\[
\int_C d^2\sigma \sqrt{\det g_{ij}} = \frac{1}{2} \int_C d^2\sigma \sqrt{g} \partial_i \bar{y}_0^m \partial_j \bar{y}_0^n g_{m\bar{n}},
\]  
(7.73)

where the first term is obtained from the second using the worldvolume metric equation of motion. Noting that \( g_{ij} \) is conformally flat, and going to complex coordinates \( z = \sigma^0 + i\sigma^1 \), \( \bar{z} = \sigma^0 - i\sigma^1 \), it follows from (7.73) that

\[
\int_C d^2\sigma \sqrt{\det g_{ij}} = \frac{1}{2} \int_C d^2z \partial_z \bar{y}_0^m \partial_{\bar{z}} \bar{y}_0^n \omega_{m\bar{n}},
\]  
(7.74)

where \( \omega_{m\bar{n}} = i g_{m\bar{n}} \) is the Kähler form on \( CY_3 \). In deriving (7.74) we used the fact, discussed in Section 4, that the functions \( \bar{y}^m \) must be holomorphic. Recall from (6.3) that

\[
\omega_{m\bar{n}} = \sum_{l=1}^{1,1} a^l \omega_{1m\bar{n}}.
\]  
(7.75)
Therefore, we can write

\[ RV^{-1/3} \int_C d^2 \sigma \sqrt{\det g_{ij}} = \frac{v_C}{2} \sum_{I=1}^{h^{1,1}} R b^I \omega_I, \quad (7.76) \]

where \( b^I = V^{-1/3} a^I \),

\[ \omega_I = \frac{1}{v_C} \int_C d^2 \zeta \omega_{I \zeta \bar{\zeta}} \quad (7.77) \]

and

\[ \omega_{I \zeta \bar{\zeta}} = \partial_\zeta \bar{y}^m \partial_{\bar{\zeta}} \bar{y}^n \omega_{I m\bar{n}} \quad (7.78) \]

is the pullback onto the holomorphic curve \( C \) of the \( I \)-th harmonic \((1,1)\)-form. Note that we have introduced the parameter \( v_C \) of mass dimension minus two to make \( \omega_I \) dimensionless. Parameter \( v_C \) can naturally be taken to be the volume of curve \( C \). Now consider the second term in (7.71) involving \( b_{ij} \). First, we note that

\[ \int_C d^2 \sigma \frac{i}{2} \varepsilon^{ij} b_{ij} = \frac{i}{2} \int_C d^2 z \partial_\zeta \bar{y}^m \partial_{\bar{\zeta}} \bar{y}^n B_{m\bar{n}}. \quad (7.79) \]

Remembering from (6.29) that

\[ B_{m\bar{n}} = \sum_{I=1}^{h^{1,1}} \frac{1}{6} \hat{p}^I \omega_{I m\bar{n}}, \quad (7.80) \]

it follows that

\[ \int_C d^2 \sigma \frac{i}{2} \varepsilon^{ij} b_{ij} = \frac{v_C}{2} \sum_{I=1}^{h^{1,1}} \frac{i}{6} \hat{p}^I \omega_I. \quad (7.81) \]

Putting everything together, we see that

\[ S_0^y = \frac{T}{2} \sum_{I=1}^{h^{1,1}} \omega_I (R b^I + \frac{i}{6} \hat{p}^I), \quad (7.82) \]

where

\[ T = T_{S v_C} = T_M \pi \rho v_C \quad (7.83) \]

is a dimensionless parameter. Recalling from (5.18) the \( T^I \) moduli are defined by

\[ T^I = R b^I + \frac{i}{6} \hat{p}^I, \quad (7.84) \]

it follows that we can write \( S_0^y \) as

\[ S_0^y = \frac{T}{2} \sum_{I=1}^{h^{1,1}} \omega_I T^I. \quad (7.85) \]

We conclude that the \( \int d^4 x e^{-S_0^y} \) factor in the path-integral is given by

\[ \int d^4 x e^{-S_0^y} = \int d^4 x e^{-\frac{T}{2} \sum_{I=1}^{h^{1,1}} \omega_I T^I}. \quad (7.86) \]

We next evaluate the path integral factor \( \int d^4 x d^2 \sigma e^{-S_0^y} \).
The $S^\theta_0$ Term and the Fermionic Zero-Mode Integral:

It follows from (7.66) that $S^\theta_0$ is the sum of $S_\Theta$ and $S_{\Theta^2}$, given in (7.44), evaluated at a solution of the equations of motion $y_0^\mu$, $\theta_0$. Varying (7.44) with respect to $\bar{\Theta}$ leads to the equation of motion $D_i \Theta_0 = \frac{1}{2} \partial_i X_0^M \Psi_M$, where we have used (7.15) and the fact that the Dirac matrices can be taken to be hermitean. Inserting the equation of motion into (7.44) we find

$$S^\theta_0 = TS \int_C d^2 \sigma RV^{-1/3} \sqrt{\det g_{ij}} \bar{\Psi}_M V_0^M, \quad (7.87)$$

where

$$V_0^M = g^{ij} \partial_i X_0^M \partial_j X_0^N \Gamma_N \Theta_0. \quad (7.88)$$

Recalling that $\partial_i y_0^\mu = 0$ for all coordinates $y_0^\mu$ of $R_4$, it follows that the only non-vanishing components of $V_0^M$ are

$$V_0^U = g^{ij} \partial_i \tilde{y}_0^U \partial_j \tilde{y}_0^W \Gamma_W \Theta_0, \quad (7.89)$$

where $g_{ij}$ is given by (7.72). We conclude that

$$S^\theta_0 = TS \int_C d^2 \sigma RV^{-1/3} \sqrt{\det g_{ij}} \bar{\Psi}_U V_0^U. \quad (7.90)$$

Now $V_0^U \propto \Theta_0$, where $\Theta_0$ satisfies the equation of motion (7.48). As discussed above, any such solution can be written as the sum

$$\Theta_0 = \hat{\Theta}_0 + \Theta', \quad (7.91)$$

where $\Theta'$ is a solution of the purely homogeneous Dirac equation (7.54) and has the form (7.55). Since in the path-integral we must integrate over the two zero-modes $\vartheta^\alpha$, $\alpha = 1, 2$ in $\Theta'$, it follows that terms involving $\hat{\Theta}_0$ can never contribute to the fermion two-point function. Therefore, when computing the superpotential, one can simply drop $\hat{\Theta}_0$. Hence, $S^\theta_0$ is given by (7.90) where

$$V_0^U = g^{ij} \partial_i \tilde{y}_0^U \partial_j \tilde{y}_0^W \Gamma_W \Theta'. \quad (7.92)$$

Inserting expression (7.55), and using the decomposition $\Gamma_W = 1 \otimes \bar{\gamma}_W$, implies that

$$V_0^U = g^{ij} \partial_i \tilde{y}_0^U \partial_j \tilde{y}_0^W \vartheta \otimes (\bar{\gamma}_W \eta_-). \quad (7.93)$$

Next, we note that the Kaluza-Klein Ansatz for the ten-dimensional gravitino in the Calabi-Yau directions is given by

$$RV^{-1/3} \Psi_U = - \sum_{L=1}^{h^{1,1}} i \omega_L \bar{\psi}_U \lambda^L \otimes (\bar{\gamma}_V \eta_+), \quad (7.94)$$
where $\omega_{LUV}(\dot{y}^U)$, $L = 1, \ldots, h^{1,1}$ are the harmonic $(1,1)$-forms on $CY_3$, $\lambda^L(y^u)$ are the fermionic superpartners of the moduli $T^L$ and $\eta_{+}(\dot{y}^U)$ is the Calabi-Yau covariantly constant spinor. Note that the left-hand side of (7.94) includes a factor of $RV^{-1/3}$, while the right-hand side has a factor of $-i$ which makes the Ansatz consistent with the compactification moduli defined in (5.2), (5.11) and (5.18). Using (7.93) and (7.94), one can evaluate the product $\tilde{\Psi}_U V_0^U$, which is found to be

$$RV^{-1/3} \tilde{\Psi}_U V_0^U = -ig^{ij} \partial_i \dot{y}_0^U \partial_j \dot{y}_0^W \sum_{L=1}^{h^{1,1}} \omega_{LUV}(\eta_{+}^\dagger \gamma^Y \bar{\gamma}^\dagger \eta_{-}) \cdot (\lambda^L \partial). \quad (7.95)$$

where $\lambda^L \partial = \lambda^L_\alpha \partial^\alpha$. Substituting this expression into (7.90) then gives

$$S_0^\theta = T \sum_{L=1}^{h^{1,1}} \omega_L \lambda^L \partial \quad (7.96)$$

where we have used complex coordinates $z = \sigma^0 + i \sigma^1$, $\bar{z} = \sigma^0 - i \sigma^1$ for the holomorphic curve, as well as $\dot{y}^m$, $\bar{y}^m$ for the Calabi-Yau coordinates $\dot{y}^U$. We have also used the property

$$\eta_{+}^\dagger \gamma^m \bar{\gamma}^n \eta_{-} = 2\delta_n^m \quad (7.97)$$

of the covariantly constant spinors on $CY_3$, derived from (4.11)–(4.12). The coefficients $\omega_L$ are given in (7.77). It follows that the $\int d\partial^1 d\partial^2 e^{-S_0^\theta}$ factor in the path-integral is

$$\int d\partial^1 d\partial^2 e^{-S_0^\theta} = \int d\partial^1 d\partial^2 e^{-T \sum_{L=1}^{h^{1,1}} \omega_L \lambda^L \partial}. \quad (7.98)$$

Expanding the exponential, and using the properties of the Berezin integrals, we find that

$$\int d\partial^1 d\partial^2 e^{-S_0^\theta} = \frac{T^2}{2} \sum_{L,M=1}^{h^{1,1}} \omega_L \omega_M \lambda^L \lambda^M, \quad (7.99)$$

where we have suppressed the spinor indices on $\lambda^L \lambda^M$.

Collecting the results we have obtained thus far, two-point function (7.69) can now be written as

$$\langle \lambda^I(y^u_1) \lambda^J(y^u_2) \rangle \propto \int \mathcal{D} \lambda e^{-\int d^4 y \sum_{K=1}^{h^{1,1}} \lambda^K \bar{\lambda}^K \lambda^I(y^u_1) \lambda^J(y^u_2)} \cdot \int d^4 x e^{-\frac{T}{2} \sum_{I=1}^{h^{1,1}} \omega_I T^I(x) \sum_{L,M=1}^{h^{1,1}} \omega_L \omega_M \lambda^L(x) \lambda^M(x)} \cdot \int \mathcal{D} \delta y^{\nu^u} e^{-S_2^u} \cdot \int \mathcal{D} \delta \theta e^{-S_2^\theta} \cdot \int \mathcal{D} \omega e^{-S_{WZW}}. \quad (7.100)$$

Next, we evaluate the bosonic path-integral factor $\int \mathcal{D} \delta y^{\nu^u} e^{-S_2^u}$. 

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The $S^y_2$ Quadratic Term:

It follows from (7.68) that $S^y_2$ is simply the quadratic term in the $y = y_0 + \delta y$ expansion of $S_0$, given in (7.38) and (7.39). Note that $S_\Theta + S_\Theta^2$ does not contribute since the second derivative is to be evaluated for $\theta_0 = 0$. Furthermore, since this contribution to the path-integral is already at order $\alpha'$, $S_0$ should be evaluated to lowest order in $\alpha'$. As discussed above, to lowest order $dB = 0$ and, hence, the $b_{ij}$ term in (7.38) is a total divergence which can be ignored. Within this bosonic gauge (7.20), choose a system of coordinates such that the metric tensor restricted to the holomorphic curve $\mathcal{C}$ can be written locally as

$$g_{MN}|_C = \begin{pmatrix} h_{m'n'}(\sigma) & 0 \\ 0 & h_{m''n''}(\sigma) \end{pmatrix}. \quad (7.101)$$

Performing the expansion, we find that

$$S^y_2 = T_S \int_C d^2\sigma RV^{-1/3} \sqrt{\det g_{ij}} \left( \frac{1}{2} g^{ij} (D_i \delta y^{m''})(D_j \delta y^{n''}) h_{m''n''} - \delta y^{m''} U_{m''n''} \delta y^{n''} \right), \quad (7.102)$$

where the induced covariant derivative of $\delta y$ is given by

$$D_i \delta y^{m''} = \partial_i \delta y^{m''} + \omega^{m''}_{n''} \delta y^{n''} \quad (7.103)$$

and the mass matrix is

$$U_{m''n''} = \frac{1}{2} R^{m'n'}_{m''n''} + \frac{1}{8} Q^{m'n'}_{m''n''}, \quad (7.104)$$

where $R^{m'n'}_{m''n''}$ is the ambient curvature tensor restricted to the string and $Q^{m'n'}_{m''n''}$ is the second fundamental form.\[12\] Note that the eight quantities $\delta y^{n''}$ are scalars from the point of view of the curve $\mathcal{C}$. Four of these scalars form a vector in the tangent bundle of $CY_3$ and the remaining four form a vector in $R_4$. By definition, metric $h_{m''n''}$ can be decomposed as

$$h_{m''n''} = \begin{pmatrix} \eta_{uv} & 0 \\ 0 & h_{UV}(\sigma) \end{pmatrix}, \quad (7.105)$$

where $\eta_{uv}$ is the flat metric of $R_4$ and $h_{UV}$ is the metric of the normal space $CY_\perp$ defined by the four directions of $CY_3$ that are perpendicular to the curve $\mathcal{C} \subset CY_3$. It follows that all connection components $\omega^{m''}_{n''}$ with either $m''$ and/or $n''$ in $R_4$ will vanish. Then (7.102) becomes

$$S^y_2 = T_S \int_C d^2\sigma RV^{-1/3} \sqrt{\det g_{ij}} \left\{ \frac{1}{2} g^{ij} (\partial_i \delta y^v)(\partial_j \delta y^v) \eta_{uv} + \frac{1}{2} g^{ij} (D_i \delta y^U)(D_j \delta y^V) h_{UV} - \delta y^U U_{UV} \delta y^V \right\}. \quad (7.106)$$

\[12\]These are standard results. For a proof, see for example [11].
Integrating the derivatives by parts then gives

\[
S_2^y = T_S \int_C d^2 \sigma RV^{-1/3} \{ -\frac{1}{2} \delta y^u [\eta_{uv} \sqrt{g} g^{ij} D_i \partial_j] \delta y^v \\
- \frac{1}{2} \delta y^v [\sqrt{g} (g^{ij} D_i h_{UV} D_j + 2 U_{UV})] \delta y^V \},
\]

(7.107)

where the symbol $D_i$ indicates the covariant derivative with respect to both the induced connection in (7.103) and the worldvolume connection of $C$. Generically, the fields $R, V$ are functions of $x^u$. However, as discussed above, at the level of the quadratic contributions to the path-integrals all terms should be evaluated at the classical values of the background fields. Since $R$ and $V$ are moduli, these classical values can be taken to be constants, rendering $RV^{-1/3}$ independent of $x^u$. Hence, the factor $T_S RV^{-1/3}$ can simply be absorbed by a redefinition of the $\delta y$'s. Using the relation

\[
\int D \delta y e^{-\frac{1}{2} \int d^2 \sigma \delta y \bar{O} \delta y} \propto \frac{1}{\sqrt{\det O}},
\]

(7.108)

we conclude that

\[
\int D dy^{m''} e^{-S_2^y} \propto \frac{1}{\sqrt{\det O_1}} \frac{1}{\sqrt{\det O_2}}
\]

(7.109)

where

\[
O_1 = \eta_{uv} \sqrt{g} g^{ij} D_i \partial_j, \\
O_2 = \sqrt{g} (g^{ij} D_i h_{UV} D_j + 2 U_{UV}).
\]

(7.110)

We next turn to the evaluation of the $\int D \delta \theta e^{-S_2^\theta}$ factor in the path-integral.

**The $S_2^\theta$ Quadratic Term:**

It follows from (7.68) that $S_2^\theta$ is the quadratic term in the $\theta = \theta_0 + \delta \theta$ expansion of $S_{\Theta^2}$, given in (7.12). Note that $S_0 + S_{\Theta}$ does not contribute. Performing the expansion and taking into account the gauge fixing condition (7.32), we find that

\[
S_2^\theta = 2T_S \int_C d^2 \sigma RV^{-1/3} \sqrt{\det g_{ij}} \delta \theta \Gamma^i D_i \delta \Theta.
\]

(7.111)

One must now evaluate the product $\delta \theta \Gamma^i D_i \delta \Theta$ in terms of the gauged-fixed quantities $\delta \theta$. We start by rewriting

\[
\delta \theta \Gamma^i D_i \delta \Theta = g^{ij} \partial_j X^M \delta \theta \Gamma_M \partial_i \delta \Theta \\
+ g^{ij} \partial_j X^M \partial_i X^N \omega_N^{AB} \delta \theta \Gamma_M \Gamma_{AB} \delta \Theta,
\]

(7.112)
where we have used (7.13) and (7.14). After fixing the gauge freedom of the bosonic fields $X^M(\sigma)$ as in (7.20), expression (7.111) becomes

\[
\delta \Theta A_i \partial_i \delta \Theta = g^{ij} \epsilon_{m'} e_{m''} \partial_j y^{m''} \delta \Theta \Gamma_{a'} \partial_i \delta \Theta + g^{ij} \epsilon_{m''} \partial_j y^{m''} \delta \Theta \Gamma_{a''} \partial_i \delta \Theta
\]

\[
+ g^{ij} \epsilon_{m''} \partial_j y^{m''} (\delta_{m''} \omega_{AB} + \partial_j y^{m''} \omega_{AB}) \delta \Theta \Gamma_{a''} \Gamma_{AB} \delta \Theta
\]

\[
+ g^{ij} \epsilon_{a''} \partial_j y^{m''} (\delta_{m''} \omega_{AB} + \partial_j y^{m''} \omega_{AB}) \delta \Theta \Gamma_{a''} \Gamma_{AB} \delta \Theta,
\]

(7.113)

where $A = (a', a'')$. We see that we must evaluate the fermionic products

\[
\delta \Theta \Gamma_{a'} \partial_i \delta \Theta, \quad \delta \Theta \Gamma_{a''} \partial_i \delta \Theta, \quad \delta \Theta \Gamma_{a''} \Gamma_{AB} \delta \Theta, \quad \delta \Theta \Gamma_{a''} \Gamma_{AB} \delta \Theta
\]

(7.114)

in terms of $\delta \theta$. After fixing the fermionic gauge according to (7.32), we can compute the relevant terms in the expression (7.113). Consider a product of the type $\delta \Theta M \delta \Theta$, where $M$ is a $32 \times 32$ matrix-operator,

\[
M = \left( \begin{array}{cc} M_1 & M_2 \\ M_3 & M_4 \end{array} \right).
\]

(7.115)

Using (7.22) and (7.32), we have

\[
\delta \Theta M \delta \Theta = \delta \Theta^\dagger M \delta \Theta = \delta \theta^\dagger M_1 \delta \theta
\]

(7.116)

Therefore, using (7.22), we have the following results

\[
\delta \Theta \Gamma_{a'} \partial_i \delta \Theta = 0, \quad \delta \Theta \Gamma_{a''} \partial_i \delta \Theta = \delta \theta^\dagger \gamma_{a'c} \partial_i \delta \theta,
\]

\[
\delta \Theta \Gamma_{a'} \Gamma_{bc} \delta \Theta = 0, \quad \delta \Theta \Gamma_{a''} \Gamma_{bc} \delta \Theta = (\delta_{a'c} - i \varepsilon_{a'u'}) \delta \theta^\dagger \gamma_{a''b} \delta \theta,
\]

\[
\delta \Theta \Gamma_{a''} \Gamma_{ab} \delta \Theta = 0, \quad \delta \Theta \Gamma_{a''} \Gamma_{ab} \delta \Theta = -i \varepsilon_{a''b'} \delta \theta^\dagger \gamma_{ab} \delta \theta,
\]

(7.117)

Substituting these expressions into (7.113) yields

\[
\delta \Theta \Gamma_i D_i \delta \Theta = \frac{1}{2} \epsilon_{a''} \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta - i \omega_{a''} \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta + \omega_{a''} \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta
\]

\[
+ \frac{1}{2} \epsilon_{a''} \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta + (\delta_{a''} - i \varepsilon_{a'u'}) \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta
\]

(7.118)

where

\[
\omega_{a''} = \frac{1}{2} \epsilon_{a''} \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta
\]

\[
\omega_{b''} = \frac{1}{2} \epsilon_{a''} \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta
\]

\[
\omega_{c''} = \frac{1}{2} \epsilon_{a''} \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta
\]

(7.119)

Then (7.111) becomes

\[
S_2^0 = 2T_S \int_C d^2 \sigma R V^{-1/3} \delta \theta^\dagger \{ \sqrt{g} \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta - i \omega_{a''} \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta
\]

\[
+ \omega_{a''} \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta + \sqrt{g} \epsilon_{a''} \gamma_{a''} \gamma_{b''} \gamma_{c''} \partial_i \delta \Theta \}\delta \theta.
\]

(7.120)
As discussed in the previous section, at the level of the quadratic contributions to the path-integrals all terms should be evaluated at the classical values of the background fields. Therefore, the factor $2T S RV^{-1/3}$ can be absorbed by a redefinition of the $\delta \theta$'s. Next we use the relation

$$\int D\delta \theta e^{-S_2} \propto \sqrt{\det O},$$

(7.121)

Note, however, that when going to Euclidean space, we have doubled the number of fermion degrees of freedom. Therefore, one must actually integrate over only one half of these degrees of freedom. This amounts to taking the square-root of the determinant on the right-hand side of (7.121). Hence, we conclude that

$$\int D\delta \theta e^{-S_2} \propto \sqrt{\det O/3},$$

(7.122)

where

$$O/3 = \sqrt{g\gamma^a g^{a'b'}} \{ e_{m''}^a \partial_j g^{m''} [\partial_i - i\omega_i^a b' \varepsilon_{a'b'} + \omega_i^{b''} c' \gamma_{b'c'}] 
+ \delta_j^{m''} e_{m'}^a \omega_i^{b''} c' (\delta_{a'b'} - i\varepsilon_{a'b'}) \}. $$

(7.123)

Collecting the results we have obtained thus far, two-point function (7.69) can now be written as

$$\langle \lambda^I(y_1) \lambda^J(y_2) \rangle \propto \frac{\sqrt{\det O_3}}{\sqrt{\det O_1 \sqrt{\det O_2}}} \cdot \int D\lambda e^{-\int d^4 y \sum_{K=1}^{h^{1,1}} \lambda^K \phi K \lambda^I(y_1) \lambda^J(y_2)} 
\cdot \int d^4 x e^{-\frac{1}{2} \sum_{l=1}^{h^{1,1}} \omega_l T^l(x) \sum_{L,M=1}^{h^{1,1}} \omega_L \omega_M \lambda^L(x) \lambda^M(x)} 
\cdot \int D\omega e^{-S_{0WZW}}.$$  

(7.124)

It, therefore, remains to evaluate the $\int D\omega e^{-S_{0WZW}}$ factor in the path-integral, which we now turn to.

**8 The Wess-Zumino-Witten Determinant:**

In this section, we will discuss the $E_8 \times E_8$ Wess-Zumino-Witten part of the action, its quadratic expansion and one loop determinant. Recall from (7.16) that the relevant action is

$$S_{0WZW} = -\frac{1}{8\pi} \int_C d^2 \sigma \text{ tr} \left[ \frac{1}{2} \sqrt{g} g^{ij} (\tilde{\omega}_i - \tilde{A}_i) \cdot (\tilde{\omega}_j - \tilde{A}_j) + i \varepsilon^{ij} \tilde{\omega}_i \tilde{A}_j \right] 
+ \frac{1}{24\pi} \int_B d^3 \sigma i \varepsilon^{ijk} \Omega_{kji}(\tilde{\omega})$$

(8.1)
where \( \bar{\omega} = \bar{g}^{-1}d\bar{g} \) is an \( E_8 \times E_8 \) Lie algebra valued one-form and \( \bar{g} \) is given in (5.15). In order to discuss the equation of motion and the chirality of this action, it is convenient to use the complex coordinates 
\[ z = \sigma^0 + i\sigma^1, \quad \bar{z} = \sigma^0 - i\sigma^1 \]
on \( C \) and to define the complex components of \( \bar{A} \) by 
\[ \bar{A} = \bar{A}_z dz + \bar{A}_{\bar{z}} d\bar{z}. \]
Then action (8.1) can be written as 
\[ S_{0WZW} = -\frac{1}{8\pi} \int_C d^2z \text{tr} \left( \bar{g}^{-1} \partial_z \bar{g} \bar{g}^{-1} \partial_{\bar{z}} \bar{g} - 2\bar{A}_z \bar{g}^{-1} \partial_{\bar{z}} \bar{g} + \bar{A}_{\bar{z}} \bar{A}_z \right) \]
\[ + \frac{1}{24\pi} \int_B d^3\hat{\sigma} i\hat{\epsilon}^{ijk} \Omega^{jk}(\hat{\omega}'). \] (8.2)
It is useful to define the two \( E_8 \times E_8 \) currents 
\[ J_z = (D_z \bar{g})\bar{g}^{-1}, \quad J_{\bar{z}} = \bar{g}^{-1}D_{\bar{z}}\bar{g}, \] (8.3)
where \( D_z \) and \( D_{\bar{z}} \) are the \( E_8 \times E_8 \) covariant derivatives. It follows from (8.2) that the \( \bar{g} \) equations of motion are 
\[ \partial_z J_z = 0, \] (8.4)
or, equivalently,
\[ D_z J_{\bar{z}} + F_{z\bar{z}} = 0, \] (8.5)
where \( F_{z\bar{z}} \) is the \( E_8 \times E_8 \) field strength which, generically, is non-vanishing.

In order to perform the path-integral over \( \omega \), it is necessary to fix any residual gauge freedom in the \( \omega \) fields. Recall from the discussion in Section 3 that the entire action is invariant under both local gauge and modified \( \kappa \)-transformations \( \delta_L \) and \( \Delta_{\hat{k}} \), respectively. It follows from (3.19) and (3.12) that 
\[ \delta_{\hat{k}} \bar{g} = \bar{g} i_{\hat{k}} \bar{\kappa}. \] (8.6)
It is not difficult to show that using this transformation, one can choose a gauge where 
\[ J_z = 0. \] (8.7)
Henceforth, we work in this chiral gauge. Note that this is consistent with the equations of motion (8.4) and (8.5). Indeed, (8.4) is now vacuous, being replaced by (8.7) itself.

Thus, the on-shell theory we obtain from the gauged Wess-Zumino-Witten action is an \( E_8 \times E_8 \) chiral current algebra at level one. The level can be read off from the coefficient of the Chern-Simons term in (8.2). We would now like to evaluate the Wess-Zumino-Witten contribution to the path-integral using a saddle-point approximation. To do this, we should expand \( \bar{g} \) as small fluctuations 
\[ \bar{g} = \bar{g}_0 + \delta \bar{g}. \] (8.8)
around a classical solution $\bar{g}_0$ of (8.7). However, it is clearly rather difficult to carry out the quadratic expansion and evaluate the determinant in this formalism. Luckily, there is an equivalent theory which is more tractable in this regard, which we now describe.

To start the discussion, set all $E_8 \times E_8$ background gauge fields to zero. It is well known that this is equivalent, on an arbitrary Riemann surface, to an $SO(16) \times SO(16)$ theory of free fermions. The partition function of the $E_8 \times E_8$ theory can be obtained by computing the partition function of the free fermion theory, where one sums over all spin structures of each $SO(16)$ factor on the Riemann surface \[42\]. We now want to turn on an arbitrary non-vanishing $E_8 \times E_8$ gauge field background. Here one meets some difficulty. Since only the $SO(16) \times SO(16)$ ($\subset E_8 \times E_8$) symmetry is manifest in the free fermion theory, only background gauge fields associated with this subgroup can be coupled to the fermions in the usual way. The coupling of $E_8 \times E_8$ gauge backgrounds with a structure group larger than $SO(16) \times SO(16)$ is far from manifest and cannot be given a Lagrangian description. Happily, for our purposes, it is sufficient to restrict the gauge field backgrounds to lie within the $SO(16) \times SO(16)$ subgroup of $E_8 \times E_8$. The reason is that, as explained in a number of papers \[7, 8, 9, 10, 11\], realistic heterotic M-theory requires gauge bundles with structure group of rank 4 or smaller within the observable $E_8$ factor. In addition, the structure group in the hidden $E_8$ factor can always be chosen to be of rank 4 or less. Hence, realistic heterotic M-theory models can generically be chosen to have gauge field backgrounds within the $SO(16) \times SO(16)$ subgroup of $E_8 \times E_8$. Henceforth, we consider only such restricted gauge field backgrounds. With this in mind, we can now write the action for $SO(16) \times SO(16)$ fermions coupled to background gauge fields. It is given by \[45, 46, 47, 48, 49\]

$$S_\psi = \int_C d^2 \sigma (\bar{\psi}^a D^a_A \psi^b + \bar{\psi}^{a'} D^{a'}_{A'} \psi^{b'}), \tag{8.9}$$

where $\psi^a, \psi^{a'}$ denote the two sets of $SO(16)$ fermions with $a, a' = 1, \ldots, 16$ and

$$D^a_A = \sqrt{g} \tau^i (D_i \delta^{ab} - A_i^{ab}), \quad D^{a'}_{A'} = \sqrt{g} \tau^i (D_i \delta^{a'b'} - A_i^{a'b'}) \tag{8.10}$$

are the covariant derivatives on $C$ with $A_i^{ab}, A_i^{a'b'}$ the two sets of $SO(16)$ background gauge fields. Recall that $\tau^i$ are the Dirac matrices in two-dimensions. It follows from the above discussion that we can write

$$\int \mathcal{D} \omega e^{-S_{\text{WZW}}} \propto \int \mathcal{D} \psi^a \mathcal{D} \psi^{a'} e^{-S_\psi}, \tag{8.11}$$

where the gauge fixing of variable $\omega$ described by (8.7) is inherent in the $\psi^a, \psi^{a'}$ formalism, as we will discuss below. The equations of motion are given by

$$\mathcal{D}^a_A \psi^b = 0, \quad \mathcal{D}^{a'}_{A'} \psi^{b'} = 0. \tag{8.12}$$
We now expand
\[ \psi^a = \psi_0^a + \delta \psi^a, \quad \psi'^a = \psi_0'^a + \delta \psi'^a \] (8.13)
around a solution \( \psi_0^a, \psi_0'^a \) of (8.12) and consider terms in \( S_{\psi} \) up to quadratic order in the fluctuations \( \delta \psi^a, \delta \psi'^a \). We find that
\[ S_{\psi} = S_{0\psi} + S_{2\psi}, \] (8.14)
where
\[ S_{0\psi} = \int C d^2 \sigma (\bar{\psi}_0^a D_A^b \psi_0^b + \bar{\psi}_0'^a D_A'^{b'} \psi_0'^{b'}) \] (8.15)
and
\[ S_{2\psi} = \int C d^2 \sigma (\delta \bar{\psi}^a D_A^b \delta \psi^b + \delta \bar{\psi}'^a D_A'^{b'} \delta \psi'^{b'}). \] (8.16)
The terms linear in \( \delta \psi \) vanish by the equations of motion. It follows immediately from (8.12) that \( S_{0\psi} = 0 \). Then, using (7.121), one finds from (8.16) that
\[ \int D \psi^a D \psi'^a e^{-S_}\propto \sqrt{\det D_A} \sqrt{\det D_A'}. \] (8.17)
Note, again, that by going to Euclidean space we have doubled the number of fermionic degrees of freedom. Therefore, one must actually integrate over only one half of these degrees of freedom. This requires the square-root of the determinants to appear in (8.17).

It is important to discuss how the chiral gauge fixing condition (8.7) is manifested in the \( \psi^a, \psi'^a \) formalism. Condition (8.7) imposes the constraint that \( \tilde{g} \) couples only to the \( \bar{A}_z \) component of the gauge fields and not to \( \bar{A}_{\bar{z}} \). It follows that in evaluating \( \det D_A \det D_{A'} \), we should keep only the \( \bar{A}_z \) components of the gauge fields. That is, we should consider the Dirac determinants of \( SO(16) \times SO(16) \) holomorphic vector bundles on Riemann surface \( \mathcal{C} \). Gauge fixing condition (8.7) also imposes a constraint on the definition of determinants \( \det D_A \det D_{A'} \) as follows. Consider one of the \( SO(16) \) Dirac operators, say \( D_A \). Recall that on the Euclidean space \( \mathcal{C} \), each spinor \( \psi \) is a complex two-component Weyl spinor
\[ \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \] (8.18)
Rescaling this basis to
\[ \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} (g_{zz})^{-1/4} \tilde{\psi}_+ \\ (g_{zz})^{1/4} \tilde{\psi}_- \end{pmatrix} \] (8.19)
and using the standard representation for \( \tau^0, \tau^1 \) then, locally, one can write
\[ D_A = \begin{pmatrix} 0 & D_{-A} \\ D_{+A} & 0 \end{pmatrix}, \] (8.20)
where
\[ D_{-A} = (g_{zz})^{3/4} \left( (g_{zz})^{-1/2} \frac{\partial}{\partial z} (g_{zz})^{1/2} - A_z \right), \quad D_{+A} = (g_{zz})^{1/4} \frac{\partial}{\partial \bar{z}}. \] (8.21)

Since the operator \( \mathcal{P}_A \) must be Hermitean, it follows that \( D_{+A} = D_{-A}^\dagger \). Now, in addition to disallowing any coupling to \( A_z \), gauge condition (8.7) imposes the constraint that
\[ \psi_a^0 = 0 \] (8.22)
for all \( a = 1, \ldots, 16 \). Then, using the fact that
\[ \det \mathcal{P}_A = \sqrt{\det(\mathcal{P}_A)^2} \] (8.23)
and gauge condition (8.22), we see that the proper definition of the determinant is
\[ \det \mathcal{P}_A = \sqrt{\det D_{-A}^\dagger D_{-A}}. \] (8.24)

In this paper, it is not necessary to determine the exact value of \( \det \mathcal{P}_A \). We need only compute whether it vanishes or is non-zero, and the conditions under which these two possibilities occur. To do this, we must examine the global properties of the holomorphic vector bundle. As we did in Section 7, we will, henceforth, restrict
\[ \mathcal{C} = \mathbb{C}P^1 = S^2. \] (8.25)

With this restriction, the condition for the vanishing of \( \det \mathcal{P}_A \) can be given explicitly, as we now show.

It follows from (8.22) that the chiral fermions realizing the \( SO(16) \) current algebra are elements of the negative chiral spinor line bundle \( S_- \) of the sphere. Note from (8.20) that \( D_{-A} \) is the part of the Dirac operator which acts on \( S_- \). With respect to a non-trivial \( SO(16) \) gauge bundle background \( A \), the complete operator we should consider is
\[ D_{-A} : S_- \otimes A \to S_+ \otimes A, \] (8.26)
where \( S_\pm \) denotes the positive chiral spinor bundle on the sphere. This is the global description of the local \( D_{-A} \) operator defined in (8.20) and (8.21).\(^\text{13}\) In order to have nonzero determinant \( \det \mathcal{P}_A \), it is necessary and sufficient that \( D_{-A} \) should not have any zero-modes. This follows from the fact that, for \( SO(16) \times SO(16) \), the index theorem guarantees that
\[ \text{coker} D_{-A}^\dagger = \ker D_{-A}. \] (8.27)

\( ^\text{13} \)To be even more precise, globally \( D_{-A} \) is a holomorphic section of the determinant line bundle \( \det(S_- \otimes A) \otimes \det(S_k \otimes A)^* \) over the gauge fixed configuration space. Furthermore, definition (8.24) corresponds to the Quillen norm of this holomorphic section. For a careful definition of the determinant of infinite dimensional spaces \( S_- \otimes A \) and \( S_k \otimes A \) and the Quillen norm, see [43, 44].
In Appendix B, we discuss how to calculate the number of zero-modes of the $D - A$ operator globally defined in (8.26). Since the number of zero-modes is invariant under a smooth deformation of the bundle, we can choose a convenient bundle for the purpose of the calculation. It is known that by a judicious choice of the connection, every $E_8$ bundle can be reduced to a $U(1)^8$ bundle. Since the maximal torus of $E_8$ coincides with that of $SO(16)$, every $U(1)^8$ bundle should have the form

$$\oplus_{i=1}^{8} O(m_i) \oplus O(-m_i),$$

(8.28)

where $O(m_i)$ stands for the $U(1)$ bundle on the sphere with degree $m_i$. Let us now suppose that the bundle $A$ is written in the form of (8.28). It follows that operator $D - A$ decomposes as

$$D - A = \oplus_{i=1}^{8} D_{-O(m_i)} \oplus D_{-O(-m_i)}.$$  

(8.29)

In Appendix B, we show that the number of zero-modes of the factor $D_{-O(m)} : S^{-} \otimes O(m) \rightarrow S^{+} \otimes O(m)$

(8.30)

is $m$ if $m$ is non-negative and zero otherwise. From this fact, we can see that if any of the $m_i$ in (8.28) is nonzero, then either $D_{-O(m_i)}$ or $D_{-O(-m_i)}$ has zero-modes since either $m_i$ or $-m_i$ must be positive. We conclude from this and expression (8.29) that $D - A$ will have at least one zero-mode, and hence $\det D_A$ will vanish, if and only if at least one $U(1)$ sub-bundle $O(m_i)$ has degree $m_i \neq 0$. That is, $\det D_A = 0$ if and only if holomorphic bundle $A$, restricted to the sphere $C = S^2$ and denoted by $A|_C$, is non-trivial. Clearly, the exact same conclusions apply to the other $SO(16)$ determinant $\det D_{A'}$ as well.

It is obviously very important to know, within the context of realistic heterotic M-theory vacua, when the restriction of the $SO(16) \times SO(16)$ gauge bundle $A \oplus A'$ to $C = S^2$, denoted $A|_C \oplus A'|_C$, is non-trivial, in which case this curve produces no instanton contributions to the superpotential $W$, and when $A|_C \oplus A'|_C$ is trivial, in which case it gives a non-zero contribution to $W$. We have shown that both situations are possible, the conclusion depending on the explicit choice of the gauge bundle, the specific Calabi-Yau threefold and the choice of the isolated sphere within a given Calabi-Yau threefold. These results will be presented elsewhere [29].

9 Final Expression for the Superpotential:

We are now, finally, in a position to extract the final form of the non-perturbative superpotential from the fermion two-point function. Combining the results of the previous section
with expression (7.124), we find that

\[
\langle \lambda^I(y_1^u)\lambda^J(y_2^u) \rangle \propto \frac{\sqrt{\det O_3}}{\sqrt{\det O_1}\sqrt{\det O_2}} \cdot \sqrt{\det P_A} \sqrt{\det P_{A'}} 
\cdot \int \mathcal{D}\lambda e^{-\int d^4y \sum_{k=1}^{h^{1,1}} \lambda^K \phi^K \lambda^I(y_1^u)\lambda^J(y_2^u)} 
\cdot \int d^4x e^{-\frac{T}{2} \sum_{I=1}^{h^{1,1}} \omega_I^T I} \sum_{L,M=1} \omega_L \omega_M \lambda^L(x)\lambda^M(x). 
\] (9.1)

Comparing this with the purely holomorphic part of the quadratic fermion term in the four-dimensional effective Lagrangian (6.22)

\[
(\partial_I \partial_J W)\lambda^I\lambda^J, 
\] (9.2)

we obtain

\[
W \propto \frac{\sqrt{\det O_3}}{\sqrt{\det O_1}\sqrt{\det O_2}} \cdot \sqrt{\det P_A} \sqrt{\det P_{A'}} \cdot e^{-\frac{T}{2} \sum_{I=1}^{h^{1,1}} \omega_I^T I}. 
\] (9.3)

In this expression, the dimensionless fields \( T^I \) correspond to the \((1,1)\)-moduli and the volume modulus of \( S^1/\mathbb{Z}_2 \). The \( \omega_I \) are dimensionless coefficients defined by

\[
\omega_I = \frac{1}{v_C} \int_C d^2z \omega_{Izz}, 
\] (9.4)

where \( v_C \) is the volume of curve \( C \) and \( \omega_{Izz} \) is the pullback (7.78) to the holomorphic curve \( C \) of the \( I \)-th harmonic \((1,1)\)-form on \( CY_3 \). \( T \) is a dimensionless parameter given by

\[
T = T_M \pi \rho \ v_C, 
\] (9.5)

with \( T_M \) the membrane tension and \( \pi \rho \) the \( S^1/\mathbb{Z}_2 \) interval length. The operators \( O_1, O_2 \) and \( \Phi_3 \) are presented in (7.110) and (7.123), respectively. The operator \( P_A \) and its determinant \( \det P_A \) are defined in (8.10), (8.20), (8.21) and (8.24). This determinant and, hence, the superpotential \( W \) will be non-vanishing if and only if the pullback of the associated \( SO(16) \) bundle \( A \) to the curve \( C \) is trivial. The same results apply to \( P_{A'} \), \( \det P_{A'} \) and the pullback of the other \( SO(16) \) bundle \( A' \) to \( C \). All the determinants contributing to \( W \) are non-negative real numbers. We emphasize that \( W \) given in (9.3) is the contribution of open supermembranes wrapped once around \( C \times S^1/\mathbb{Z}_2 \), where \( C = S^2 \) is a sphere isolated in the Calabi-Yau threefold \( CY_3 \). The existence of supermembranes multiply wrapped around \( C \), and the computation of their superpotential, is not straightforward. However, we expect that the appropriate contribution to the superpotential of a supermembrane wrapped once around \( S^1/\mathbb{Z}_2 \) and \( n \) times around \( C \) is

\[
e^{-\frac{nT}{2} \sum_{I=1}^{h^{1,1}} \omega_I^T I}. 
\] (9.6)
Further generalizations and discussions of the complete open supermembrane contributions to the non-perturbative superpotential in heterotic M-theory will be presented elsewhere [29].

A Notation and Conventions:

We use a notation such that symbols and indices without hats represent fields in the ten-dimensional fixed hyperplanes of Hořava-Witten theory (as well as the two-dimensional heterotic string theory), while hatted indices relate to quantities of eleven-dimensional bulk space (and the three-dimensional open membrane theory).

Bosons:

For example,

\[ X^M, \ M = 0, 1, \ldots, 9, \quad \text{and} \quad \hat{X}^\hat{M}, \ \hat{M} = \hat{0}, \hat{1}, \ldots, \hat{9}, \hat{11}, \]  

are, respectively, the coordinates of ten- and eleven-dimensional spacetimes. We do not change notation when switching from Minkowskian signature to Euclidean signature.

Eleven-dimensional space is, by assumption, given by

\[ M_{11} = R^4 \times CY_3 \times S^1 / \mathbb{Z}_2, \]  

while the ten-dimensional space obtained by compactifying it on \( S^1 / \mathbb{Z}_2 \) is, clearly,

\[ M_{10} = R^4 \times CY_3. \]

The membrane world volume \( \Sigma \) is decomposed as

\[ \Sigma = C \times S^1 / \mathbb{Z}_2, \]

where the (two-dimensional) curve \( C \) lies within \( CY_3 \).

The two dimensional heterotic string theory is represented by fields of the worldsheet coordinates \( \sigma^i \), with \( i = 0, 1 \). Bosonic indices of ten-dimensional spacetime are split into indices parallel to the worldsheet \( (m' = 0, 1) \) and indices perpendicular to it \( (m'' = 2, \ldots, 9) \). The space normal to the worldsheet is an eight-dimensional space. Since it is assumed that the worldsheet \( C \) is contained in the Calabi-Yau three-fold \( CY_3 \), these eight directions \( y^{m''} \) can be split in two sets of four, the first being directions of \( CY_3 \) but perpendicular to \( C \), denoted by \( y^{U} \), with \( U = 2, 3, 4, 5 \), and the second being directions normal to \( CY_3 \), that is, directions of \( R_4 \), denoted by \( y^{u} \), with \( u = 6, 7, 8, 9 \).
Coordinates of $CY_3$ are denoted by

$$\tilde{y}^{\tilde{U}} = (X^{m'}, y^{U}), \quad \text{with} \quad \tilde{U} = 0, 1, 2, 3, 4, 5, \quad m' = 0, 1,$$

(A.5)
or, using the complex structure notation,

$$\tilde{y}^{m}, \quad \tilde{y}^{\bar{m}}, \quad m = 1, 2, 3, \quad \bar{m} = \bar{1}, \bar{2}, \bar{3}.$$

(A.6)

The bosonic indices in (A.1)-(A.6) are coordinate (or “curved”) indices. The corresponding tangent space (or “flat”) indices are given in the following table,

| $M_{10}$ | $M_{11}$ | $C$ | $M_\perp$ | $R^4$ | $CY_\perp$ |
|---------|---------|-----|-----------|------|---------|
| $M, N$  | $M, \bar{N}$ | $m', n'$ | $m'', n''$ | $u, v$ | $U, V$   |
| $A, B$  | $\hat{A}, \hat{B}$ | $a', b'$ | $a'', b''$ | $k, l$ | $K, L$   |

where $M_\perp$ and $CY_\perp$ are subspaces of $M_{10}$ and $CY_3$ perpendicular to $C$, respectively.

**Spinors:**

In ten-dimensional spacetime with Euclidean signature, the $32 \times 32$ Dirac matrices $\Gamma_A$ satisfy

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$$

(A.7)
or, with curved indices, (since $\Gamma_A = e_\hat{A}^M \Gamma_M$)

$$\{\Gamma_M, \Gamma_N\} = 2g_{MN}. \quad \text{(A.8)}$$

One defines ten-dimensional chirality projection operators $\frac{1}{2}(1 \pm \hat{\Gamma})$, where

$$\Gamma_{11} = -i\Gamma_0 \Gamma_1 \cdots \Gamma_9. \quad \text{(A.9)}$$

A useful representation for $\Gamma_A$ is given by the two-eight split

$$\Gamma_A = (\tau_{a'} \otimes \gamma, 1 \otimes \gamma_{a''}), \quad \text{(A.10)}$$

where the two-dimensional Dirac matrices $\tau_0, \tau_1$ and their product defined by $\hat{\tau} = -i\tau_0 \tau_1$ are explicitly given by

$$\tau_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\tau} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \text{(A.11)}$$
These ten-dimensional Dirac matrices are more explicitly written as

\[
\begin{align*}
\Gamma_0 &= \begin{pmatrix} 0 & \tilde{\gamma} \\ \tilde{\gamma} & 0 \end{pmatrix}, & \Gamma_1 &= \begin{pmatrix} 0 & -i\tilde{\gamma} \\ i\tilde{\gamma} & 0 \end{pmatrix}, \\
\Gamma_a^{\prime\prime} &= \begin{pmatrix} \gamma_a^{\prime\prime} & 0 \\ 0 & \gamma_a^{\prime\prime} \end{pmatrix}, & \Gamma_{11} &= \begin{pmatrix} \tilde{\gamma} & 0 \\ 0 & -\tilde{\gamma} \end{pmatrix}
\end{align*}
\]

(A.12)

where \(\gamma_a^{\prime\prime}\) are 16 \times 16 Dirac matrices, and the product

\[
\tilde{\gamma} = \gamma_2\gamma_3 \cdots \gamma_9
\]

(A.13)

is used in the definition of eight-dimensional chirality projection operators \(\frac{1}{2}(1 \pm \tilde{\gamma})\).

Note that \(\Gamma_{11}^2 = 1\), \(\tilde{\gamma}^2 = 1\), and \(\tilde{\tau}^2 = 1\). In eleven-dimensions, the 32 \times 32 Dirac matrices are given by

\[
\hat{\Gamma}_{\hat{A}} = \Gamma_{\hat{A}}, \quad (\hat{A} = \hat{0}, \hat{1}, \ldots, \hat{9}), \quad \text{and} \quad \hat{\Gamma}_{\hat{11}} = \Gamma_{11}.
\]

(A.14)

\section*{B Dirac Operator on the Sphere:}

In this Appendix, we discuss the spinors, the chiral Dirac operator on the sphere and the number of zero modes of the Dirac operator. One nice thing on the sphere and generally on the Riemann surfaces is that we can identify a chiral spinor with a holomorphic bundle and the associated Dirac operator with a suitable differential operator on the corresponding holomorphic bundle. We follow [43] closely in the exposition.

When we consider a sphere \(C\), locally we can always write the metric as

\[
ds^2 = e^{2\phi}((d\sigma^0)^2 + (d\sigma^1)^2) = 2dzd\bar{z}
\]

(B.1)

with \(z = \sigma^0 + i\sigma^1\) and \(\bar{z} = \sigma^0 - i\sigma^1\). When we cover \(C\) with such complex coordinate patches, the transition functions on the overlaps are holomorphic and thus define a complex structure. Using this complex structure, we can divide one forms

\[
T^*C = T^{*(1,0)}C \oplus T^{*(0,1)}C
\]

(B.2)

according to whether they are locally of the form \(f(z, \bar{z})dz\) or \(f(z, \bar{z})d\bar{z}\).

In order to introduce spinors we choose a local frame \(e^a\)

\[
ds^2 = \delta_{ab}e^a \otimes e^b.
\]

(B.3)

If \(U_\alpha, U_\beta\) are two overlapping coordinate patches, the corresponding frames \(e^a_\alpha, e^a_\beta\) are related by a local \(SO(2)\) rotation

\[
e^a_\alpha = R^a_b(\theta)e^b_\beta.
\]

(B.4)
Hence, the spinor bundles have transition functions $\tilde{R}(\theta/2)$ such that $\tilde{R}(\theta/2)^2 = R(\theta)$.

One of the pleasant features of the sphere or, more generally, of the Riemann surfaces is that we can describe spinors in terms of half-order differential. This is most easily done by choosing frames

$$
e^w = e^0 + ie^1 = e^\phi dz \quad \text{for} \quad T^*(1,0),$$
$$
e^\bar{w} = e^0 - ie^1 = e^\bar{\phi} d\bar{z} \quad \text{for} \quad T^*(0,1),$$

where $\phi$ is the factor in (B.1). Then across patches $U_\alpha, U_\beta$ with coordinates $z^\alpha, z^\beta$, we have

$$e^{2\phi_\alpha}|dz_\alpha|^2 = e^{2\phi_\beta}|dz_\beta|^2,$$

which implies that

$$2\phi_\alpha = 2\phi_\beta + \ln |\frac{dz_\beta}{dz_\alpha}|^2.$$  \hspace{1cm} (B.6)

Thus we have

$$e^w_{(\alpha)} = e^{i\theta} e^w_{(\beta)},$$

where

$$e^{i\theta} = \frac{dz_\alpha}{dz_\beta} |\frac{dz_\beta}{dz_\alpha}|.$$  \hspace{1cm} (B.9)

The left- and right-spinors $\psi_{\pm} \in S^{\pm}$ transform as

$$\psi_{\pm \alpha} = e^{\pm i\theta} \psi_{\pm \beta}$$

on the sphere. When we refer the spinors $\psi_{\pm}$ to the frame $(e^w)^{\frac{1}{2}}, (e^{\bar{w}})^{\frac{1}{2}}$, the transformation functions are one-by-one unitary matrices.

It is sometimes more convenient to consider the bundle $S^{\pm}$ as holomorphic bundles. These will have transition functions $(\frac{dz_\alpha}{dz_\beta})^{\frac{1}{2}}$ for $S^+$ and $(\frac{dz_\alpha}{dz_\beta})^{-\frac{1}{2}}$ for $S^-$. In the holomorphic category more appropriate local sections are the holomorphic half order differential $(dz_\alpha)^{\frac{1}{2}}$. The relation between the standard and the holomorphic description of spinors is given by

$$\psi_+(e^w)^{\frac{1}{2}} = \tilde{\psi}_+(dw_\alpha)^{\frac{1}{2}}.$$  \hspace{1cm} (B.11)

The holomorphic line bundle defined by $S_+$ will be denoted by $L$. As suggested by the notation, this bundle can be interpreted as a holomorphic square root of the bundle of $(1,0)$ form,

$$T^*(1,0) \mathcal{C} = K = L^2,$$  \hspace{1cm} (B.12)

where $K$ denote the canonical bundle. Once we have introduced the spinor bundle $L$, we can define tensor powers $L^n$ corresponding to the differentials $\psi(dz_\alpha)^{\frac{n}{2}}$. Note that by
raising and lowering indices with the metric (B.1), any tensor can be decomposed into such differentials.

In the local coordinate (B.1), the covariant derivative for fields in $L^n$ is

$$\nabla_z^n : L^n \rightarrow L^{n+2},$$

such that

$$\nabla_z^n \tilde{\psi} = (g_{zz})^\frac{n}{2} \frac{\partial}{\partial z} (g_{zz})^\frac{n}{2} \tilde{\psi}.$$  (B.14)

We can introduce a scalar product in $L^n$

$$< \tilde{\phi} | \tilde{\psi} > = \int d^2 \sigma \sqrt{g} (g^{zz})^\frac{n}{2} \tilde{\phi}^* \tilde{\psi}.$$  (B.15)

The operator $\nabla_z^n$ is the unique holomorphic connection on $L^n$ compatible with the inner product (B.15). With respect to (B.15), the adjoint is

$$\nabla_{n+2}^z = (\nabla_z^n)^\dagger : L^{n+2} \rightarrow L^n,$$  (B.16)

with

$$\nabla_{n+2}^z \tilde{\psi} = g^{zz} \frac{\partial}{\partial z} \tilde{\psi}.$$  (B.17)

One can check that $\nabla_z^{-1} : L^{-1} \rightarrow L^1$ coincides with the Dirac operator $D_- : S_- \rightarrow S_+$. Together with $\nabla_1^z : L^1 \rightarrow L^{-1}$, they form a Dirac operator on the sphere.

Let $O(m)$ be a holomorphic bundle on the sphere with degree $m$. On the sphere $L \simeq O(-1)$ since the holomorphic cotangent bundle is $O(-2)$. Under this notation, $S_- \simeq O(1)$. Note that

$$S_+ \simeq \Lambda^{even} T^{*(0,1)} C \otimes L,$$

$$S_- \simeq \Lambda^{odd} T^{*(0,1)} C \otimes L$$  (B.18)

and, using the metric (B.1), $T^{*(0,1)} C$ is identified with the holomorphic tangent bundle $O(2)$.

If we consider

$$D_{-(m)} : S_- \otimes O(-m) \rightarrow S_+ \otimes O(-m),$$  (B.19)

where the connection on $O(-m)$ is induced from the metric, then one can see that this coincides with the operator $\nabla_z^{m-1} : L^{m-1} \rightarrow L^{m+1}$.

According to the Riemann-Roch theorem $D_{-(m)}$ or $\nabla_z^{m-1}$ has $m$ zero modes if $m$ is non-negative and zero otherwise [43, 26].

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