When can a formality quasi-isomorphism over $\mathbb{Q}$ be constructed recursively?

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Abstract

Let $\mathcal{O}$ be a differential graded (possibly colored) operad defined over rationals. Let us assume that there exists a zig-zag of quasi-isomorphisms connecting $\mathcal{O} \otimes K$ to its cohomology, where $K$ is any field extension of $\mathbb{Q}$. We show that for a large class of such dg operads, a formality quasi-isomorphism for $\mathcal{O}$ exists and can be constructed recursively. Every step of our recursive procedure involves a solution of a finite dimensional linear system and it requires no explicit knowledge about the zig-zag of quasi-isomorphisms connecting $\mathcal{O} \otimes K$ to its cohomology.

1 Introduction

A differential graded (dg) operad $\mathcal{O}$ is formal if there exists a sequence of quasi-isomorphisms (of dg operads)

$$\mathcal{O} \sim \bullet \sim \bullet \sim \ldots \sim H^\bullet(\mathcal{O})$$

connecting $\mathcal{O}$ to its cohomology $H^\bullet(\mathcal{O})$. Formality for dg operads (and other algebraic structures) is a subtle phenomenon. Currently, there are no effective tools for determining whether a given dg operad is formal or not. Moreover, in various interesting examples (including the braces operad $Br$ [9], [18], [22], its “framed” version $CBr$ [2], [25] and the Kontsevich-Soibelman operad $KS$ [19], [26]) all known proofs of formality require transcendental tools [17], [20], [24], [26].

In this paper we consider a dg operad $\mathcal{O}$ defined over the field $\mathbb{Q}$ of rationals and assume that $\mathcal{O} \otimes \mathbb{Q} K$ is formal for some field extension $\mathbb{Q} K$ of $\mathbb{Q}$. We consider a cobar resolution $\text{Cobar}(\mathcal{C}) \sim \to H^\bullet(\mathcal{O})$ of $H^\bullet(\mathcal{O})$ and show that, under some mild conditions on $\mathcal{O}$ and on the resolution $\text{Cobar}(\mathcal{C})$, there is an explicit algorithm which allows us to produce a formality quasi-isomorphism

$$\text{Cobar}(\mathcal{C}) \sim \to \mathcal{O} \quad (1.1)$$

over $\mathbb{Q}$ recursively. The proof that this algorithm works is based on the existence of a sequence of quasi-isomorphisms connecting $\mathcal{O} \otimes \mathbb{Q} K$ to its cohomology. However, no explicit knowledge about this sequence of quasi-isomorphisms is required at any step of this algorithm.

We would like to mention that the existence of a formality quasi-isomorphism $(1.1)$ over $\mathbb{Q}$ (from the existence of a formality quasi-isomorphism over an extension of $\mathbb{Q}$) was proved in paper [15] by F. Guillén Santos, V. Navarro, P. Pascual, and A. Roig. More precisely, see Theorem 6.2.1 in loc. cit.

Our paper is organized as follows. In Section 1.1 we recall some basic concepts and fix the notational conventions. In Section 2 we introduce the concept of an MC-sprout,
which can be viewed as an approximation to a formality quasi-isomorphism \([1.1]\). Using this concept, we formulate the main theorem of this paper (see Theorem \([2.14]\)) and deduce it from a technical lemma (see Lemma \([2.17]\)). Section \([3]\) is devoted to the proof of this lemma and Appendix \([A]\) contains the proof of a useful lifting property for cobar resolutions. Finally, Appendix \([B]\) displays a third MC sprout in Conv\((\text{Ger}^\vee, \text{Br})\) which can be extended to a genuine MC element in Conv\((\text{Ger}^\vee, \text{Br})\). This MC sprout was found using the software \([6]\) developed by the authors.

We should mention that our construction is inspired by Proposition 5.8 from classical paper \([11]\] by V. Drinfeld.

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### 1.1 Preliminaries

In this paper, \(\mathbb{K}\) is any field extension of the field \(\mathbb{Q}\) of rational numbers and \(\otimes := \otimes_{\mathbb{Q}}\). For a cochain complex \(V\), the notation \(\mathcal{Z}(V)\) is reserved for the subspace of cocycles. The degree of a vector \(v\) in a graded vector space (or a cochain complex) \(V\) is denoted by \(|v|\). The notation \(s\) (resp. \(s^{-1}\)) is reserved for the operator which shifts the degree up by 1 (resp. down by 1), i.e.

\[
(s V)^\bullet = V^{\bullet - 1}, \quad (s^{-1} V)^\bullet = V^{\bullet + 1}.
\]

The notation \(S_n\) is reserved for the symmetric group on \(n\) letters.

The abbreviation “dg” always means “differential graded”.

For a dg Lie algebra \(L\), Curv is the map \(\text{Curv} : L^1 \to L^2\) defined by the formula

\[
\text{Curv}(\alpha) := \partial \alpha + \frac{1}{2}[\alpha, \alpha].
\]

(1.2)

For example, Maurer-Cartan (MC) elements of \(L\) are precisely elements of the zero locus of Curv.

Let us recall \([4]\), \([14]\), \([16]\) that for every filtered dg Lie algebra \(L\) (in the sense of \([4]\), Section 1)), the set of MC elements of \(L\) can be upgraded to a groupoid\(^4\) with MC elements being objects. Recall that two MC elements \(\alpha, \tilde{\alpha}\) of a filtered dg Lie algebra \(L\) are isomorphic (in this groupoid) if there exists a degree 0 element \(\xi \in L\) such that

\[
\tilde{\alpha} = \exp([\xi, \;]) \alpha - \frac{\exp([\xi, \;]) - 1}{[\xi, \;]} \partial \xi,
\]

where the expressions \(\exp([\xi, \;])\) and

\[
\frac{\exp([\xi, \;]) - 1}{[\xi, \;]}
\]

are defined via the corresponding Taylor series\(^5\).

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\(^3\)See also Theorem 4 and Corollary 4.1 in D. Bar-Natan’s beautiful paper \([1]\).

\(^4\)This groupoid is actually a truncation of an \(\infty\)-groupoid (i.e. a fibrant simplicial set). However, for our purposes, we will not need cells of dimension \(\geq 2\).

\(^5\)These series are well defined because \(L = F_1 L\) and \(L\) is complete with respect to the filtration.
In this paper, we will freely use the language of (colored) operads [3, 12, 21]. For a coaugmented cooperad $C$, the notation $C^{\circ}$ is reserved for the cokernel of the coaugmentation. For a dg pseudo-cooperad $P$, we denote by $P^{\diamond}$ the dg cooperad which is obtained from $P$ by formally adjoining the counit. Clearly, for every coaugmented cooperad $C$, the cooperad $C^{\diamond}$ is canonically identified with $C$. The notation $\Xi$ is reserved for the ordinal of colors. A ($\Xi$-colored) collection $V$ is a family of cochain complexes $\{V(q)\}_q$ indexed by all $\Xi$-colored corollas $q$ (with the standard labeling). For every $\Xi$-colored corolla $q$, $V(q)$ is equipped with the left action of the group

$$S_{k_1(q)} \times S_{k_2(q)} \times \cdots \times S_{k_m(q)},$$

where $m$ is the total number of colors of the incoming edges and $k_i(q)$ is the number of incoming edges of the $i$-th color. For example, if the ordinal of colors $\Xi$ is the singleton, the collection is simply a family of cochain complexes $\{V(n)\}_{n \geq 0}$, where each $V(n)$ is equipped with a left action of $S_n$.

The notation $\text{Coll}$ is reserved for the category of $\Xi$-colored collections of graded vector spaces. For objects $Q_1, Q_2$ of $\text{Coll}$ the notation $\text{Hom}_{\text{Coll}}(Q_1, Q_2)$ is reserved for the vector space of homomorphisms (of all degrees) from the collection $Q_1$ to the collection $Q_2$. For example, if the ordinal of colors is the singleton, then

$$\text{Hom}_{\text{Coll}}(Q_1, Q_2) := \prod_{n \geq 0} \text{Hom}_{S_n}(Q_1(n), Q_2(n)), \quad (1.4)$$

where

$$\text{Hom}_{S_n}(Q_1(n), Q_2(n)) = \left(\text{Hom}(Q_1(n), Q_2(n))\right)^{S_n}$$

and $\text{Hom}(Q_1(n), Q_2(n))$ is the inner hom in the category of graded vector spaces.

For a dg pseudo-cooperad $P$ and a dg operad $O$, the notation $\text{Conv}(P, O)$ is reserved for the convolution Lie algebra [3, Section 2.3], [5, Section 4]. The underlying graded vector space of $\text{Conv}(P, O)$ is $\text{Hom}_{\text{Coll}}(P, O)$ and the Lie bracket is given by the formula

$$[f, g] := f \bullet g - (-1)^{|f||g|} g \bullet f,$$

where $f \bullet g$ is the pre-Lie multiplication of $f$ and $g$ defined in terms of comultiplication on $P$ and multiplications on $O$.

Let us recall [5, Proposition 5.2] that MC elements of $\text{Conv}(P, O)$ are in bijection with operad morphisms $F : \text{Cobar}(P^{\diamond}) \to O$. In particular, the operad morphism corresponding to a MC element $\alpha \in \text{Conv}(P, O)$ will be denoted by $F_{\alpha}$.

In this paper, we assume that

**Condition 1.1** Every dg pseudo-cooperad $P$ carries an ascending filtration

$$0 = F^0 P \subset F^1 P \subset F^2 P \subset F^3 P \subset \ldots \quad (1.5)$$

which is compatible with the differential and the comultiplications in the following sense:

$$\partial_P(F^m P) \subset F^{m-1} P, \quad (1.6)$$

See eq. (2.41) in [3].
\[ \Delta_t(\mathcal{F}^m P) \subset \bigoplus_{m_1 + \cdots + m_k = m} \mathcal{F}^{m_1} P \otimes \mathcal{F}^{m_2} P \otimes \cdots \otimes \mathcal{F}^{m_k} P, \]  

where \( t \) is a (\( \Xi \)-colored) planar tree with the set of leaves \{1, 2, \ldots, n\} and \( k \) nodal vertices. Moreover, \( P \) is cocomplete with respect to filtration \((1.5)\), i.e.

\[ P = \bigcup_m \mathcal{F}^m P. \]  

**Remark 1.2** Cobar resolutions \( \text{Cobar}(P^\diamond) \) for which \( P \) satisfies Condition \((1.1)\) may be thought of as analogs of Sullivan algebras from rational homotopy theory. Let us also mention that, due to [23, Proposition 38], such dg operads \( \text{Cobar}(P^\diamond) \) are cofibrant.

For example, if the ordinal of colors \( \Xi \) is the singleton, and \( P(0) = P(1) = 0 \), then the filtration “by arity”

\[ \mathcal{F}^m P(n) := \begin{cases} P(n) & \text{if } n \leq m + 1 \\ 0 & \text{otherwise.} \end{cases} \]

satisfies Condition \((1.1)\).

Condition \((1.1)\) guarantees that, for every dg operad \( \mathcal{O} \), the dg Lie algebra

\[ \text{Conv}(P, \mathcal{O}) \]  

is equipped with the complete descending filtration:

\[ \text{Conv}(P, \mathcal{O}) = \mathcal{F}_1 \text{Conv}(P, \mathcal{O}) \supset \mathcal{F}_2 \text{Conv}(P, \mathcal{O}) \supset \ldots \]

\[ \mathcal{F}_m \text{Conv}(P, \mathcal{O}) := \{ f \in \text{Conv}(P, \mathcal{O}) \mid f \big|_{\mathcal{F}_{m-1} P} = 0 \}. \]

In other words, \( \text{Conv}(P, \mathcal{O}) \) is a filtered dg Lie algebra in the sense of [4, Section 1].

### 2 The recursive construction of formality quasi-isomorphisms

Let \( \mathcal{O} \) be a dg operad and \( \mathcal{H} \) be the cohomology operad for \( \mathcal{O} \):

\[ \mathcal{H} := H^\bullet(\mathcal{O}). \]

We assume that \( \mathcal{H} \) admits a cobar resolution \( \text{Cobar}(P^\diamond) \) where \( P^\diamond \) is a dg pseudo-cooperad satisfying Condition \((1.1)\).

Due to Corollary A.3 from Appendix A, the problem of constructing a zig-zag of quasi-isomorphisms (of dg operads) connecting \( \mathcal{O} \) to \( \mathcal{H} \) is equivalent to the problem of constructing a single quasi-isomorphism (of dg operads)

\[ F : \text{Cobar}(P^\diamond) \to \mathcal{O}. \]

The latter problem is, in turn, equivalent to the problem of constructing a MC element

\[ \alpha \in \text{Conv}(P, \mathcal{O}) \]

whose corresponding morphism \( F_\alpha : \text{Cobar}(P^\diamond) \to \mathcal{O} \) is a quasi-isomorphism of dg collections.

In this paper, we consider a dg operad \( \mathcal{O} \) and a cobar resolution

\[ \rho : \text{Cobar}(P^\diamond) \xrightarrow{\sim} \mathcal{H} := H^\bullet(\mathcal{O}). \]  

We assume that the pair \((P, \mathcal{O})\) satisfies the following conditions:
The dg pseudo-operad $P$ is equipped with an additional grading

$$P = \bigoplus_{k \geq 1} G^k P, \quad G^0 P = 0 \quad (2.2)$$

which is compatible with the differential $\partial_P$ and the comultiplications $\Delta_t$ in the following sense:

$$\partial_P(G^k P) \subset G^{k-1} P, \quad (2.3)$$

$$\Delta_t(G^m P) \subset \bigoplus_{r_1 + \ldots + r_q = m} G^{r_1} P \otimes G^{r_2} P \otimes \ldots \otimes G^{r_q} P, \quad (2.4)$$

where $t$ is $(\Xi$-colored) tree with $q$ nodal vertices.

$G^k P$ is finite dimensional for every $k$ and the graded components of $O(q)$ are finite dimensional for every $\Xi$-colored corolla $q$.

The operad $H$ is generated by $\rho(s G^1 P)$ and $\rho|_{s G^k P} = 0 \quad \forall \ k \geq 2. \quad (2.5)$

Example 2.1 Suppose that the ordinal of colors $\Xi$ is the singleton, $P(0) = P(1) = 0$ and the differential $\partial_P = 0$. Then the grading by arity

$$G^k P(n) := \begin{cases} P(n) & \text{if } n = k + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.6)$$

satisfies Condition C1. Moreover, if $P(n)$ is finite dimensional for all $n$ and each graded component of $O(n)$ is finite dimensional, then Condition C2 is also satisfied. In particular, for $P = \text{Ger}_\Xi^\vee$ the Koszul dual of the Gerstenhaber operad, and $O = \text{Br}$ the braces operad [9], [10], all these assumptions are met.

Remark 2.2 Conditions C1, C2, and C3 are also satisfied for the pairs $(P_{BV}, \text{CBr})$ and $(\text{calc}^\vee, \text{KS})$, where $P_{BV}$ is the dg pseudo-cooperad used for the cobar resolution [13] of the operad $BV$ governing $BV$-algebras and $\text{calc}^\vee$ shows up in the cobar resolution for the operad governing calculus algebras [7], [8, Definition 3].

Remark 2.3 Clearly, every dg pseudo-operad $P$ with a grading $\mathcal{G}^* P$ satisfying the above conditions has the ascending filtration

$$\mathcal{F}^m P := \bigoplus_{k \leq m} G^k P \quad (2.7)$$

and this filtration satisfies Condition 1.1.

Remark 2.4 If we forget about the differential $\partial_P$ on $P$, every $G^k P$ can be viewed as a collection of graded vector spaces. So we will tacitly identify elements in $\text{Hom}_{\text{Coll}}(G^k P, O)$ with elements $f \in \text{Conv}(P, O)$ which satisfy the condition $f|_{G^m P} \equiv 0$ for all $m \neq k$. It is clear that $\text{Hom}_{\text{Coll}}(G^k P, O)$ is closed with respect to the differential $\partial_O$ on $O$ for every $k$. However, for the map $f \mapsto f \circ \partial_P$, we have

$$f \in \text{Hom}_{\text{Coll}}(G^k P, O) \mapsto f \circ \partial_P \in \text{Hom}_{\text{Coll}}(G^{k+1} P, O).$$
Remark 2.5 In many examples, the gradation on the (dg) pseudo-operad $P$ from Condition C1 is precisely the syzygy gradation [L3 Appendix A], [21 Sections 3.3, 7.3].

Let $F$ be an arbitrary morphism of dg operads

$$F : \text{Cobar}(P^\otimes) \otimes K \to \mathcal{O} \otimes K$$

and $\pi_\mathcal{H}$ be the canonical projection

$$\pi_\mathcal{H} : \mathcal{Z}(\mathcal{O}) \to \mathcal{H}$$

from the sub-operad $\mathcal{Z}(\mathcal{O}) := \mathcal{O} \cap \ker(\partial)$ to $\mathcal{H}$.

Since every vector in $s \mathcal{G}^1P$ is a cocycle in Cobar($P^\otimes$) the restriction $F|_{s \mathcal{G}^1P}$ gives us a map of dg collections

$$F|_{s \mathcal{G}^1P} : s \mathcal{G}^1P \to \mathcal{Z}(\mathcal{O}).$$

We claim that

**Proposition 2.6** If the image of

$$\pi_\mathcal{H} \circ F|_{s \mathcal{G}^1P} : s \mathcal{G}^1P \to \mathcal{H}$$

generates the operad $\mathcal{H}$ then $F$ is a quasi-isomorphism of dg operads. The same statement holds if the base field $\mathbb{Q}$ is replaced by its extension $\mathbb{K}$.

**Proof.** Since all vectors in $s \mathcal{G}^1P$ are cocycles in Cobar($P^\otimes$) and the sub-collection $\pi_\mathcal{H} \circ F(s \mathcal{G}^1P)$ generates the operad $\mathcal{H}$, the map

$$H^\bullet(F) : H^\bullet(\text{Cobar}(P^\otimes)) \to \mathcal{H}$$

is surjective.

Since each graded component of $\mathcal{O}(q)$ is finite dimensional for every corolla $q$ (see Condition C2), we know that each graded component of $\mathcal{H}(q)$ is finite dimensional for every $q$.

On the other hand, $H^\bullet(\text{Cobar}(P^\otimes))$ is isomorphic to $\mathcal{H}$.

Thus the proposition follows from the fact a surjective map between isomorphic finite dimensional vector spaces is an isomorphism.

Since this proof works for any base field (of characteristic zero), the last assertion in the proposition is obvious. \qed

2.1 MC-sprouts in Conv($P, \mathcal{O}$)

**Definition 2.7** Let $\mathcal{F}_n \text{Conv}(P, \mathcal{O})$ be the descending filtration on Conv($P, \mathcal{O}$) coming from the ascending filtration (2.7) on $P$ and $n$ be an integer $\geq 1$. An $n$-th MC-sprout in Conv($P, \mathcal{O}$) is a degree 1 element $\alpha \in \text{Conv}(P, \mathcal{O})$ such that

$$\text{Curv}(\alpha) \in \mathcal{F}_{n+1} \text{Conv}(P, \mathcal{O})$$

or equivalently

$$\text{Curv}(\alpha)(X) = 0, \quad \forall \ X \in \mathcal{G}^{\leq n}P. \quad (2.8)$$
Remark 2.8 Since $P$ is graded, every element $\alpha \in \text{Conv}(P, \mathcal{O})$ can be uniquely written as

$$\alpha = \sum_{k=1}^{\infty} \alpha_k, \quad \alpha_k \in \text{Hom}_{\text{Coll}}(G^kP, \mathcal{O}).$$

Moreover, since $\alpha_k$ for $k > n$ do not contribute to the left hand side of (2.8), we may only consider $n$-th MC-sprouts of the form

$$\alpha = \sum_{k=1}^{n} \alpha_k, \quad \alpha_k \in \text{Hom}_{\text{Coll}}(G^kP, \mathcal{O}).$$

Due to our conditions on $\mathcal{O}$ and $P$, any such MC-sprout is determined by a finite number of coefficients.

Example 2.9 Let $\alpha$ be a genuine MC element of $\text{Conv}(P, \mathcal{O})$. The $n$-th truncation of $\alpha$ is the degree 1 element $\alpha[n]$ of $\text{Conv}(P, \mathcal{O})$ defined by the formula

$$\alpha[n](X) = \begin{cases} \alpha(X) & \text{if } X \in G^{\leq n}P, \\ 0 & \text{otherwise.} \end{cases} \quad (2.9)$$

Clearly, the $n$-th truncation of any MC element $\alpha$ of $\text{Conv}(P, \mathcal{O})$ is an $n$-th MC-sprout in $\text{Conv}(P, \mathcal{O})$. It is also easy to see that the same formula (2.9) defines an $n$-th MC-sprout in $\text{Conv}(P, \mathcal{O})$ provided $\alpha$ is an $m$-th MC-sprout and $m \geq n$. We also call $\alpha[n]$ the $n$-th truncation of $\alpha$ even if $\alpha$ is not a genuine MC element of $\text{Conv}(P, \mathcal{O})$ but merely an $m$-th MC-sprout for $m \geq n$.

Example 2.10 Let $\mathcal{B}r$ be the braces operad and $T_{12}, T_{\cup}, T_{1,23}, T^\odot_{123}$ and $T_{1,\bullet,23}$ be the brace trees shown in figures 2.4 and 2.2. Let $\alpha'$ be the following vector in $\mathcal{B}r(2) \otimes \Lambda^{-2}\text{Ger}(2) \oplus \mathcal{B}r(3) \otimes \Lambda^{-2}\text{Ger}(3)$:

$$\alpha' := T_{12} \otimes b_1 b_2 + \frac{1}{2} T_{\cup} \otimes \{b_1, b_2\} + \frac{1}{2} T_{1,23} \otimes b_1 \{b_2, b_3\} - \frac{1}{3} T^\odot_{123} \otimes \{b_1, \{b_2, b_3\}\}$$

$$- \frac{1}{6} T^\odot_{123} \otimes \{b_2, \{b_1, b_3\}\} - \frac{1}{6} T_{1,\bullet,23} \otimes \{b_2, \{b_1, b_3\}\} - \frac{1}{12} T_{1,\bullet,23} \otimes \{b_1, \{b_2, b_3\}\}. \quad (2.10)$$

A direct computation shows that $\text{Av}(\alpha')$ is a second MC-sprout in $\text{Conv}(\text{Ger}^\vee, \mathcal{B}r)$. Here $\text{Av}$ is the operator \(\bigoplus_{n \geq 2} \mathcal{B}r(n) \otimes \Lambda^{-2}\text{Ger}(n) \to \bigoplus_{n \geq 2} \text{Hom}_{S_n}(\text{Ger}^\vee(n), \mathcal{B}r(n))\) defined in eq. (C.3) in [10, Appendix C] and, for $\alpha'$, we use the notation for vectors in $\Lambda^{-2}\text{Ger}(n)$ from [10, Section 4.3].

Since all vectors in $sG^1P$ are cocycles in $\text{Cobar}(P^\odot)$,

$$\alpha(G^1P) \subset \mathcal{Z}(\mathcal{O})$$

for every MC-sprout $\alpha \in \text{Conv}(P, \mathcal{O})$. Let us observe that
Fig. 2.1: The brace trees $T_{12}, T_U$, and $T_{1,23}$, respectively

Fig. 2.2: The brace trees $T_{123}^U$, and $T_{1,•,23}$, respectively

**Proposition 2.11** If $H^\ast(O) \cong \mathcal{H}$ and $\alpha_H$ is the MC element in Conv$(P,\mathcal{H})$ corresponding to (2.1), then there exists a second MC-sprout $\alpha \in$ Conv$(P,O)$ such that the diagram

$$Z(O) \xrightarrow{\alpha} \mathcal{H}$$
$$\downarrow \pi_H$$
$$G^1 P \xrightarrow{\alpha_H} \mathcal{H}$$

commutes.

**Proof.** Since we work with vector spaces, there exist splittings

$$\eta_H : \mathcal{H}(q) \to Z(O(q))$$

(2.12)

of the projections $\pi_H : Z(O(q)) \to \mathcal{H}(q)$ for every $\Xi$-colored corolla $q$.

Moreover, since our base field has characteristic zero, we can use the standard averaging operators (for products of symmetric groups) and turn the splittings (2.12) into a map of collections

$$s : \mathcal{H} \to Z(O)$$

(2.13)

for which

$$\pi \circ s = \text{id}_\mathcal{H}.$$ 

(2.14)

The similar argument, implies that there exists a map of collections

$$\tilde{s} : \ker(Z(O) \to \mathcal{H}) \to O$$

(2.15)

which splits $\partial_O : O \to \ker(Z(O) \to \mathcal{H})$.

By setting

$$\alpha^{(1)} := s \circ \alpha_H$$

(2.16)

we get a first MC-sprout in Conv$(P,O)$ for which

$$\pi_H \circ \alpha^{(1)} = \alpha_H.$$ 

(2.17)

Note that, due to (2.5), $\alpha^{(1)}(X) = 0$ for every $X \in \mathcal{G}^{2,2}P$. 

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Let us observe that, since $\alpha^{(1)}$ lands in $Z(O)$, the assignment

$$X \in G^2 P \mapsto \alpha^{(1)} \partial_P(X) + \alpha^{(1)} \bullet \alpha^{(1)}(X)$$

gives us a map of collections:

$$G^2 P \to Z(O).$$

(2.18)

Since $\pi_{H}$ is compatible with the operadic multiplications, the composition of (2.18) with $\pi_{H}$ sends $X \in G^2 P$ to

$$\alpha_{H}(\partial_P X) + \alpha_{H} \bullet \alpha_{H}(X) \in H$$

(2.19)

On the other hand, the vector (2.19) is zero since $\alpha_{H}$ satisfies the MC equation and $H$ has the zero differential.

Since the composition of (2.18) with $\pi_{H}$ is zero, the map (2.18) lands in $\ker (Z(O) \to H)$ and hence (2.18) can be composed with the splitting (2.15).

Setting

$$\alpha(X) = \begin{cases} 
\alpha^{(1)}(X) & \text{if } X \in G^1 P \\
-\tilde{s}(\alpha^{(1)} \partial_P(X) + \alpha^{(1)} \bullet \alpha^{(1)}(X)) & \text{if } X \in G^2 P \\
0 & \text{otherwise}
\end{cases}$$

(2.20)

we get a degree 1 element $\alpha$ which satisfies

$$\partial_O \alpha(X) + \alpha(\partial_P X) + \alpha \bullet \alpha(X) = 0 \quad \forall \ X \in G^1 P \oplus G^2 P.$$ 

In other words, $\alpha$ is a second MC-sprout in $\text{Conv}(P, O)$.

Equations (2.17) and (2.20) imply that the diagram (2.11) commutes. \hfill \Box

**Remark 2.12** Let $n$ be an integer $\geq 2$ and

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \quad \alpha_k \in \text{Hom}_{\text{Coll}}(G^k P, O)$$

be an $n$-th MC-sprout in $\text{Conv}(P, O)$. Proposition 2.6 implies that, if $\alpha$ is a truncation of a genuine MC element $\beta \in \text{Conv}(P, O)$ and the diagram (2.11) commutes then the corresponding map of dg operads

$$F_{\beta} : \text{Cobar}(P^\vee) \to O$$

is a quasi-isomorphism. Thus, for our purposes, it makes sense to consider only MC-sprouts in $\text{Conv}(P, O)$ for which the diagram (2.11) commutes.

**Remark 2.13** Due to Proposition 2.11, a second MC-sprout $\alpha$ exists even if $O$ is non-formal. Of course, if $O$ is non-formal, such $\alpha$ is not a truncation of any MC element in $\text{Conv}(P, O)$.

### 2.2 The main theorem

Let, as above, $O$ be a dg operad defined over $Q$, $H := H^\bullet(O)$, and

$$\rho : \text{Cobar}(P^\vee) \sim \to H$$

be a cobar resolution for $H$, where $P$ is a dg pseudo-cooperad.

The main result of this paper is the following theorem.
Theorem 2.14 Let us assume that the pair $(\mathcal{O}, P)$ satisfies Conditions C1, C2, C3, and \( \mathcal{O} \otimes K \) is formal for some field extension \( K \) of \( \mathbb{Q} \). Let, furthermore, \( n \) be an integer \( \geq 2 \) and \( \alpha = \alpha_1 + \cdots + \alpha_n \in \text{Conv}(P, \mathcal{O}), \quad \alpha_k \in \text{Hom}_{\text{Coll}}(\mathcal{G}^k P, \mathcal{O}) \) \hspace{1cm} (2.21)

be an \( n \)-th MC-sprout in \( \text{Conv}(P, \mathcal{O}) \) for which the diagram \((2.11)\) commutes. Then there exists an \((n+1)\)-th MC-sprout \( \tilde{\alpha} \) such that

\[ \tilde{\alpha}_k = \alpha_k \quad \forall \quad k < n. \]

Moreover, the unknown vectors \( \tilde{\alpha}_n \) and \( \tilde{\alpha}_{n+1} \) can be found by solving a finite dimensional linear system.

Theorem 2.14 has the following immediate corollaries:

**Corollary 2.15** Under the above conditions on the pair \((\mathcal{O}, P)\), a quasi-isomorphism of operads

\[ \text{Cobar}(P^\wedge) \xrightarrow{\sim} \mathcal{O} \] \hspace{1cm} (2.22)

can be constructed recursively. Moreover the algorithm for constructing \((2.22)\) requires no explicit knowledge about a sequence of quasi-isomorphisms (of operads) connecting \( \mathcal{O} \otimes K \) to \( \mathcal{H} \otimes K \).

**Corollary 2.16** If the assumptions of Theorem 2.14 hold and

\[ \alpha = \alpha_1 + \cdots + \alpha_n \in \text{Conv}(P, \mathcal{O}), \quad \alpha_k \in \text{Hom}_{\text{Coll}}(\mathcal{G}^k P, \mathcal{O}) \]

is an \( n \)-th MC-sprout in \( \text{Conv}(P, \mathcal{O}) \) for which the diagram \((2.11)\) commutes, then there exists a genuine MC element \( \alpha_{MC} \in \text{Conv}(P, \mathcal{O}) \) whose \((n-1)\)-th truncation \( \alpha_{MC}^{[n-1]} \) coincides with

\[ \alpha_1 + \cdots + \alpha_{n-1}. \]

The proof of Theorem 2.14 is based on the following technical statement:

**Lemma 2.17** Let \( n \) be an integer \( \geq 2 \) and

\[ \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \quad \alpha_k \in \text{Hom}_{\text{Coll}}(\mathcal{G}^k P, \mathcal{O}) \]

be an \( n \)-th MC-sprout in \( \text{Conv}(P, \mathcal{O}) \) for which the diagram \((2.11)\) commutes. Then there exists a genuine MC element \( \beta \in \text{Conv}(P, \mathcal{O} \otimes K) \) such that

\[ \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = \beta^{[n-1]}, \]

where \( \beta^{[n-1]} \) is the \((n-1)\)-th truncation of \( \beta \).

**2.3 Theorem 2.14 follows from Lemma 2.17**

Lemma 2.17 is proved in Section 3 below. Here we show that Theorem 2.14 is a consequence of Lemma 2.17.

Our goal is to find

\[ \tilde{\alpha} := \tilde{\alpha}_1 + \tilde{\alpha}_2 + \cdots + \tilde{\alpha}_{n+1}, \quad \tilde{\alpha}_k \in \text{Hom}_{\text{Coll}}(\mathcal{G}^k P, \mathcal{O}) \]
satisfying
\[ \tilde{\alpha}_k = \alpha_k, \quad \forall \ k \leq n - 1 \]
and
\[ \text{Curv}(\tilde{\alpha})(X) = 0 \quad \forall \ X \in G^{\leq n+1}P. \]  
(2.23)

So we set
\[ \tilde{\alpha}_k := \alpha_k, \quad \forall \ k \leq n - 1 \]  
(2.24)
and observe that the unknown terms \( \tilde{\alpha}_n \) and \( \tilde{\alpha}_{n+1} \) show up only in the equations
\[ \partial_O \tilde{\alpha}_n(X) + \alpha_{n-1}(\partial_P X) + \frac{1}{2} \sum_{i+j=n, \ i,j \geq 1} [\alpha_i, \alpha_j](X) = 0 \quad X \in G^nP, \]  
(2.25)
\[ \partial_O \tilde{\alpha}_{n+1}(Y) + \tilde{\alpha}_n(\partial_P Y) + [\alpha_1, \tilde{\alpha}_n](Y) + \frac{1}{2} \sum_{i+j=n+1, \ i,j < n} [\alpha_i, \alpha_j](Y) = 0 \quad Y \in G^{n+1}P. \]  
(2.26)

Moreover, the unknown terms enter these equations linearly.

Due to the finite dimensionality condition (see \textbf{C2}), equations (2.25) and (2.26) can be viewed as a finite dimensional inhomogeneous linear system for the unknown vectors \( \tilde{\alpha}_n \) and \( \tilde{\alpha}_{n+1} \).

Thanks to Lemma 2.17, there exists a genuine MC element in \( \text{Conv}(P, \mathcal{O} \otimes \mathbb{K}) \)
\[ \beta = \sum_{k=1}^{\infty} \beta_k, \quad \beta_k \in \text{Hom}_{\text{Coll}}(G^kP, \mathcal{O} \otimes \mathbb{K}) \]
such that
\[ \beta \big|_{G^1P} = \alpha \big|_{G^1P}. \]  
\textbf{3 The proof of Lemma 2.17}

Let us first prove the following statement.

\textbf{Proposition 3.1} Let \( n \) be an integer \( \geq 2 \) and \( \alpha \) be an \( n \)-th MC sprout in \( \text{Conv}(P, \mathcal{O}) \) for which the diagram (2.11) commutes. Then there exists a MC element
\[ \beta = \sum_{k=1}^{\infty} \beta_k, \quad \beta_k \in \text{Hom}_{\text{Coll}}(G^kP, \mathcal{O} \otimes \mathbb{K}) \]
in \( \text{Conv}(P, \mathcal{O} \otimes \mathbb{K}) \) such that
\[ \beta \big|_{G^1P} = \alpha \big|_{G^1P}. \]  
(3.1)
Remark 3.2 Proposition 2.6 and Condition C3 imply that the operad morphism

\[ F_\beta : \text{Cobar}(P^\diamondsuit) \otimes K \to O \otimes K \]

corresponding to the above MC element \( \beta \) is a quasi-isomorphism.

Proof of Proposition 3.1 Since \( O \otimes K \) is formal, there exists a quasi-isomorphism of dg operads

\[ F : \text{Cobar}(P^\diamondsuit) \otimes K \to O \otimes K \quad (3.2) \]

Both \( F \) and \( \rho \) (2.1) induce the isomorphisms of operads

\[ H^\bullet (F) : H^\bullet (\text{Cobar}(P^\diamondsuit) \otimes K) \to \mathcal{H} \otimes K \quad (3.3) \]

and

\[ H^\bullet (\rho) : H^\bullet (\text{Cobar}(P^\diamondsuit) \otimes K) \to \mathcal{H} \otimes K. \quad (3.4) \]

Hence there exists (a unique) operad automorphism

\[ T : \mathcal{H} \otimes K \to \mathcal{H} \otimes K \]

such that

\[ T \circ H^\bullet (F) = H^\bullet (\rho). \quad (3.5) \]

Due to Corollary A.2 from Appendix A there exists a map of operads

\[ \tilde{T} : \text{Cobar}(P^\diamondsuit) \otimes K \to \text{Cobar}(P^\diamondsuit) \otimes K \]

such that the diagram

\[ \begin{align*}
\text{Cobar}(P^\diamondsuit) \otimes K & \xrightarrow{\tilde{T}} \text{Cobar}(P^\diamondsuit) \otimes K \\
\rho \downarrow & \quad \quad \quad \quad \quad \downarrow \rho \\
\mathcal{H} \otimes K & \xrightarrow{T} \mathcal{H} \otimes K
\end{align*} \quad (3.6) \]

commutes up to homotopy.

Since \( \rho \circ \tilde{T} \) is homotopic to \( T \circ \rho \), \( \rho \) is a quasi-isomorphism, and \( T \) is an automorphism of operads, \( \tilde{T} \) is a quasi-isomorphism of dg operads. Hence so is the composition

\[ \tilde{F} := F \circ \tilde{T} : \text{Cobar}(P^\diamondsuit) \otimes K \to O \otimes K. \quad (3.7) \]

Again, since diagram (3.6) commutes up to homotopy, we have

\[ H^\bullet (\tilde{T}) = H^\bullet (\rho)^{-1} \circ T \circ H^\bullet (\rho). \quad (3.8) \]

Combining (3.5) with (3.8), we deduce that

\[ H^\bullet (\tilde{F}) = H^\bullet (F) \circ H^\bullet (\tilde{T}) = T^{-1} \circ H^\bullet (\rho) \circ H^\bullet (\rho)^{-1} \circ T \circ H^\bullet (\rho) = H^\bullet (\rho). \]

In other words, both \( \tilde{F} \) and \( \rho \) induce the same map at the level of cohomology.

Let us denote by \( \tilde{\beta} \) the MC element in \( \text{Conv}(P, O \otimes K) \) corresponding to the map \( \tilde{F} \).

Since the diagram (2.11) for \( \alpha \) commutes and the maps \( \tilde{F}, \rho \) induce the same map at the level of cohomology, we have

\[ \pi_H \circ (\tilde{\beta} - \alpha)|_{\mathcal{G}_1 P} \equiv 0, \]
where $\pi_H$ is the canonical projection from $\mathcal{Z}(\mathcal{O}) \to \mathcal{H}$.

Hence, composing $(\tilde{\beta} - \alpha)|_{G^1P}$ with a splitting (2.15), we get a degree 0 map of collections

$$h := \tilde{s} \circ (\tilde{\beta} - \alpha) : G^1P \otimes \mathbb{K} \to \mathcal{O} \otimes \mathbb{K}$$

such that

$$\tilde{\beta}(X) - \alpha(X) = \partial \circ h(X) \quad \forall \ X \in G^1P$$

or equivalently\(^8\)

$$\tilde{\beta}(X) - \alpha(X) = \partial \circ h(X) + h \circ \partial_P(X) \quad \forall \ X \in G^1P. \quad (3.10)$$

Let us extend $h$ to the degree zero element in $\text{Conv}(P, \mathcal{O} \otimes \mathbb{K})$ by setting

$$h|_{G^{>1}P} = 0,$$

and form the new MC element of $\text{Conv}(P, \mathcal{O} \otimes \mathbb{K})$

$$\beta := \exp([h, \ ]\tilde{\beta} - \frac{\exp([h, \ ] - 1)}{[h, \ ]} \partial h, \quad (3.11)$$

where $\partial$ is the differential on $\text{Conv}(P, \mathcal{O} \otimes \mathbb{K})$.

Equation (3.10) and Condition C1 imply that equation (3.1) holds.

So the desired statement is proved. \(\square\)

Note that Proposition 3.1 already implies the statement of Lemma 2.17 for $n = 2$. So we can now assume that $n \geq 3$. For this case, Lemma 2.17 is a consequence of the following statement.

**Proposition 3.3** Let $n > m \geq 2$ be integers and

$$\alpha = \sum_{k=1}^{n} \alpha_k, \quad \alpha_k \in \text{Hom}_{\text{Coll}}(G^kP, \mathcal{O})$$

be an $n$-th MC-sprout in $\text{Conv}(P, \mathcal{O})$ for which the diagram (2.11) commutes. Furthermore, let

$$\beta = \sum_{k=1}^{\infty} \beta_k \quad \beta_k \in \text{Hom}_{\text{Coll}}(G^kP, \mathcal{O} \otimes \mathbb{K})$$

be a genuine MC element in $\text{Conv}(P, \mathcal{O} \otimes \mathbb{K})$ such that

$$\beta_k = \alpha_k \quad \forall \ 1 \leq k \leq m - 1. \quad (3.12)$$

Then there exists a MC element

$$\tilde{\beta} = \sum_{k=1}^{\infty} \tilde{\beta}_k \quad \tilde{\beta}_k \in \text{Hom}_{\text{Coll}}(G^kP, \mathcal{O} \otimes \mathbb{K})$$

of $\text{Conv}(P, \mathcal{O} \otimes \mathbb{K})$ such that $\tilde{\beta}_k = \alpha_k$ for every $k \leq m$.

\(^8\)Recall that $\partial_P|_{G^1P} = 0$.  

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3.1 The sub-spaces $\text{Der}^{(t)} \subset \text{Der}(\text{Cobar}(P^\wedge))$

Let us recall that, as the operad in the category of graded vector spaces\footnote{In this subsection, we assume that the base field is any field of characteristic zero.}, $\text{Cobar}(P^\wedge)$ is the free operad generated by the collection $sP$. So, using the grading on the dg pseudo-operad $P$, we introduce the following grading on $\text{Cobar}(P^\wedge)$:

$$\text{Cobar}(P^\wedge) = \bigoplus_{q \geq 0} \text{Cobar}(P^\wedge)^{(q)},$$

(3.13)

where $\text{Cobar}(P^\wedge)^{(q)}$ is spanned by operadic monomials in $sX_1 \in sG^{k_1}P$, $sX_2 \in sG^{k_2}P$, \ldots such that

$$\sum_{i \geq 1} (k_i - 1) = q.$$

For example, $\text{Cobar}(P^\wedge)^{(0)}$ is precisely $\mathbb{OP}(sG^1P)$ and $\text{Cobar}(P^\wedge)^{(1)}$ is spanned by operadic monomials in $\mathbb{OP}(sG^1P \oplus sG^2P)$ for which a vector in $sG^2P$ appears exactly once.

This grading is clearly compatible with the operadic multiplications on $\text{Cobar}(P^\wedge)$. In addition, Conditions C1 and C3 imply that

$$\partial(Cobar(P^\wedge)^{(q)}) \subset Cobar(P^\wedge)^{(q-1)} \quad \forall \quad q \geq 0,$$

(3.14)

$$\rho |_{Cobar(P^\wedge)^{(0)}} = 0 \quad \forall \quad q \geq 1,$$

(3.15)

and the map

$$\rho |_{Cobar(P^\wedge)^{(0)}} : Cobar(P^\wedge)^{(0)} \to \mathcal{H}$$

(3.16)

is onto.

We claim that

**Claim 3.4** There exist maps of collections (for $q \geq 1$) of degree $-1$

$$h_q : \mathcal{Z}(Cobar(P^\wedge)^{(q)}) \to Cobar(P^\wedge)^{(q+1)}$$

(3.17)

and a degree $-1$ map of collections

$$h_0 : \ker(Cobar(P^\wedge)^{(0}) \xrightarrow{\rho} \mathcal{H}) \to Cobar(P^\wedge)^{(1)}$$

(3.18)

such that

$$\partial \circ h_q(Y) = Y \quad \forall \quad Y \in \mathcal{Z}(Cobar(P^\wedge)^{(q)}), \quad q \geq 1,$$

$$\partial \circ h_0(Y) = Y \quad \forall \quad Y \in \ker(Cobar(P^\wedge)^{(0}) \xrightarrow{\rho} \mathcal{H}).$$

Proof. Since $\rho$ is a quasi-isomorphism, the existence of the desired maps follows from (3.14), (3.15), (3.16) and the fact that we work with collections of vector spaces over a field of characteristic zero. \qed

Let us denote by $\alpha_{\text{id}}$ the MC element of $\text{Conv}(P, Cobar(P^\wedge))$ corresponding to

$$\text{id} : Cobar(P^\wedge) \to Cobar(P^\wedge)$$

and consider

$$\text{Conv}(P, Cobar(P^\wedge))$$

(3.19)
as the cochain complex with the differential $\partial + \partial_P + [\alpha_{id}, \cdot]$, where $\partial$ (resp. $\partial_P$) is the differential coming from the one on Cobar$(P^{\hat{\diamond}})$ (resp. $P$).

Since Cobar$(P^{\hat{\diamond}})$ is freely generated by $sP$, the assignment

$$D \mapsto D|_{sP} \circ s$$

gives us an isomorphism of graded vector spaces

$$\text{Der}(\text{Cobar}(P^{\hat{\diamond}})) \cong \text{Conv}(P, \text{Cobar}(P^{\hat{\diamond}}))$$

with the obvious shift: every degree $d$ derivation $D$ corresponds to a degree $d+1$ vector in \text{Conv}(P, \text{Cobar}(P^{\hat{\diamond}})).$

Using the grading (3.13) on Cobar$(P^{\hat{\diamond}})$, we introduce the following subspaces of (3.19) for $t \in \mathbb{Z}$

$$L^{(t)} := \{ f \in \text{Conv}(P, \text{Cobar}(P^{\hat{\diamond}})) \mid f(G^qP) \subset \text{Cobar}(P^{\hat{\diamond}})^{(q-1)+t}, \ \forall \ q \geq 1 \}. \tag{3.21}$$

Let us denote by $\{\text{Der}^{(t)}\}_{t \in \mathbb{Z}}$ the corresponding subspaces in $\text{Der}(\text{Cobar}(P^{\hat{\diamond}}))$, i.e.

$$\text{Der}^{(t)} := \{ D \in \text{Der}(\text{Cobar}(P^{\hat{\diamond}})) \mid D|_{sP} \circ s \in L^{(t)} \}. \tag{3.22}$$

It is clear that the commutator $[\cdot, \cdot]$ on $\text{Der}(\text{Cobar}(P^{\hat{\diamond}}))$ satisfies

$$[\cdot, \cdot] : \text{Der}^{(t_1)} \otimes \text{Der}^{(t_2)} \subset \text{Der}^{(t_1+t_2)} \ \forall \ t_1, t_2 \in \mathbb{Z}. \tag{3.23}$$

Moreover, due to (3.14)

$$[\partial, \cdot] : \text{Der}^{(t)} \to \text{Der}^{(t-1)} \ \forall \ t \in \mathbb{Z}, \tag{3.24}$$

where $\partial$ is the full differential on Cobar$(P^{\hat{\diamond}})$.

Let us prove the following statement

**Claim 3.5** Let $t$ be a negative integer and $D$ be a degree 0 derivation in $\text{Der}^{(t)}$. Then $D$ acts locally nilpotently on Cobar$(P^{\hat{\diamond}})$, i.e. for every $X \in \text{Cobar}(P^{\hat{\diamond}})$, there exists an integer $m$ such that

$$D^m(X) = 0.$$ 

**Proof.** Since every vector in $\text{Cobar}(P^{\hat{\diamond}})$ is a finite linear combination of operadic monomials in $sP$, it suffices to prove that for every $X \in sP$, there exists $m$ such that

$$D^m(X) = 0.$$ 

Again, since every $X \in sP$ is a linear combination of vectors in $sG^kP$ for various $k$'s, we may assume without loss of generality, that $X \in sG^kP$ for some $k \geq 1$.

By definition of $\text{Der}^{(t)}$, we have

$$D^m(X) \subset \text{Cobar}(P^{\hat{\diamond}})^{(k-1)+mt)}.$$ 

So the desired statement follows from the fact that

$$\text{Cobar}(P^{\hat{\diamond}})^{(r)} = 0 \ \forall \ r < 0. \tag{3.25}$$

Claim 3.5 implies that
Claim 3.6 For every negative integer \( t \), every \( \partial \)-closed degree degree 0 derivation
\[
\mathcal{D} \in \text{Der}^{(t)} \subset \text{Der}(\text{Cobar}(P^{\diamond}))
gives us the automorphism of the dg operad Cobar\(P^{\diamond}\)
\[
\exp(\mathcal{D}) : \text{Cobar}(P^{\diamond}) \xrightarrow{\cong} \text{Cobar}(P^{\diamond}).
\]
Proof. Claim 3.5 implies that the formal Taylor series
\[
\exp(\mathcal{D}) := \text{id} + \sum_{k \geq 1} \frac{1}{k!} \mathcal{D}^k
\]
is a well defined automorphism of the graded operad \( \mathcal{O}_\mathcal{P}(sP) \).
Since \( \mathcal{D} \) is \( \partial \)-closed, this automorphism is also compatible with the differential on Cobar\(P^{\diamond}\).

Let us now prove the following technical statement:

Proposition 3.7 Let \( m \) be an integer \( \geq 2 \) and
\[
\psi \in \text{Hom}_{\text{Coll}}(G^m P, \mathcal{H}) \in \text{Conv}(P, \mathcal{H})
\]
be a degree 1 vector satisfying
\[
\psi \circ \partial + [\alpha_\mathcal{H}, \psi] = 0. \tag{3.25}
\]
Then there exists a degree 0 \( \partial \)-closed derivation \( \mathcal{D} \in \text{Der}^{(1-m)} \subset \text{Der}(\text{Cobar}(P^{\diamond})) \) such that
\[
\rho \circ \mathcal{D} \circ s \mid P = \psi \tag{3.26}
\]
and
\[
\mathcal{D}(sX) = 0 \quad \forall \ X \in \mathcal{G}^{<m} P. \tag{3.27}
\]
Proof. Since
\[
\rho\big|_{\mathcal{O}_\mathcal{P}(sG^1 P)} : \mathcal{O}_\mathcal{P}(sG^1 P) \to \mathcal{H}
\]
is onto (and we work with fields of characteristic zero), there exists a degree 1 vector
\[
\Psi_m \in \text{Hom}_{\text{Coll}}(G^m P, \mathcal{O}_\mathcal{P}(sG^1 P)) \subset \text{Conv}(P, \text{Cobar}(P^{\diamond}))
\]
such that \( \rho \circ \Psi_m(X) = \psi(X) \) for all \( X \in G^m P \). Clearly, \( \Psi_m \in \mathcal{L}^{(1-m)} \) and \( \Psi_m \) satisfies the equation
\[
\partial \Psi_m(X) + \Psi_m(\partial P X) + [\alpha_{\text{id}}, \Psi_m](X) = 0 \quad \forall \ X \in \mathcal{G}^{<m} P.
\]
Due to (3.25), the map
\[
\left( \Psi_m \circ \partial + [\alpha_{\text{id}}, \Psi_m] \right)\big|_{G^{m+1}P} : G^{m+1} P \to \text{Cobar}(P^{\diamond})^{(0)}
\]
lands in the kernel of \( \rho \).
Hence, by Claim 3.4, the map
\[
\Psi_{m+1}(Y) := -h_0\left( \Psi_m(\partial P Y) + [\alpha_{\text{id}}, \Psi_m](Y) \right) : \mathcal{G}^{m+1} P \to \text{Cobar}(P^{\diamond})^{(1)}
\]
satisfies
\[
\partial \Psi_{m+1}(Y) + \Psi_m(\partial P Y) + [\alpha_{\text{id}}, \Psi_m](Y) = 0 \quad \forall \ Y \in \mathcal{G}^{(m+1)} P.
\]
Therefore, the sum \( \Psi^{(m+1)} = \Psi_m + \Psi_{m+1} \) satisfies the equation
\[
\partial \Psi^{(m+1)}(Y) + \Psi^{(m+1)}(\partial P Y) + [\alpha_{id}, \Psi^{(m+1)}](Y) = 0 \quad \forall \ Y \in G^{\leq (m+1)} P.
\]

Moreover, \( \Psi^{(m+1)} \) belongs to \( \mathcal{L}^{(1-m)} \) by construction.

Let us assume that we extended \( \Psi^{(m+1)} \) to a vector (for some \( k \geq 1 \))
\[
\Psi^{(m+k)} = \Psi_m + \Psi_{m+1} + \cdots + \Psi_{m+k}, \quad \Psi_j \in \text{Hom}_{\text{Coll}}(G^j P, \text{Cobar}(P^\wedge)(j-m))
\]
such that
\[
\partial \Psi^{(m+k)}(Y) + \Psi^{(m+k)}(\partial P Y) + [\alpha_{id}, \Psi^{(m+k)}](Y) = 0 \quad \forall \ Y \in G^{\leq (m+k)} P. \tag{3.28}
\]

Let \( X \in G^{m+k+1} P \). Using (3.28) and the MC equation
\[
\partial \circ \alpha_{id} + \alpha_{id} \circ \partial P + \frac{1}{2} [\alpha_{id}, \alpha_{id}] = 0
\]
for \( \alpha_{id} \), we deduce that
\[
\partial \left( \Psi^{(m+k)}(\partial P X) + [\alpha_{id}, \Psi^{(m+k)}](X) \right) = -[\alpha_{id}, \Psi^{(m+k)}](\partial P X) + \partial \left( [\alpha_{id}, \Psi^{(m+k)}](X) \right)
\]
\[
= [\partial \circ \alpha_{id} + \alpha_{id} \circ \partial P, \Psi^{(m+k)}](X) - [\alpha_{id}, \partial \circ \Psi^{(m+k)} + \Psi^{(m+k)} \circ \partial P](X)
\]
\[
= \left( [\alpha_{id}, [\alpha_{id}, \Psi^{(m+k)}]] - \frac{1}{2} [[\alpha_{id}, \alpha_{id}], \Psi^{(m+k)}] \right)(X) = 0.
\]

In other words, the map
\[
\left. \left( \Psi^{(m+k)} \circ \partial P + [\alpha_{id}, \Psi^{(m+k)}] \right) \right|_{G^{m+k+1} P} : G^{m+k+1} P \to \text{Cobar}(P^\wedge)^{(k)}
\]
lands in \( Z(\text{Cobar}(P^\wedge)^{(k)}) \).

Hence, by Claim 3.4, the map
\[
\Psi_{m+k+1}(X) := -h_k \left( \Psi^{(m+k)}(\partial P X) + [\alpha_{id}, \Psi^{(m+k)}](X) \right) : G^{m+k+1} P \to \text{Cobar}(P^\wedge)^{(k+1)}
\]
satisfies the equation
\[
\partial \Psi_{m+k+1}(X) + \Psi^{(m+k)}(\partial P X) + [\alpha_{id}, \Psi^{(m+k)}](X) = 0 \quad \forall \ X \in G^{m+k+1} P. \tag{3.29}
\]

Therefore the vector
\[
\Psi^{(m+k+1)} := \Psi^{(m+k)} + \Psi_{m+k+1} = \Psi_m + \Psi_{m+1} + \cdots + \Psi_{m+k+1}
\]
satisfies the equation
\[
\partial \Psi^{(m+k+1)}(X) + \Psi^{(m+k+1)}(\partial P X) + [\alpha_{id}, \Psi^{(m+k+1)}](X) = 0 \quad \forall \ X \in G^{\leq (m+k+1)} P. \tag{3.30}
\]

Moreover, since \( \Psi_{m+k+1} \in \mathcal{L}^{(1-m)} \), the vector \( \Psi^{(m+k+1)} \) also belongs to \( \mathcal{L}^{(1-m)} \).

This inductive argument shows that there exists a degree 1 vector
\[
\Psi = \sum_{j=m}^{\infty} \Psi_j, \quad \Psi_j \in \text{Hom}_{\text{Coll}}(G^j P, \text{Cobar}(P^\wedge)^{(j-m)})
\]
such that
\[ \partial \circ \Psi + \Psi \circ \partial_P + [\alpha_{id}, \Psi] = 0 \] (3.31)
and
\[ \rho \circ \Psi_m = \psi. \] (3.32)
Since \( \rho(Z) = 0 \) for every \( Z \in \text{Cobar}(P^\wedge)(t) \) if \( t \geq 1 \), equation (3.32) implies that
\[ \rho \circ \Psi = \psi. \] (3.33)
Equation (3.31) implies that the (degree 0) derivation
\[ D \in \text{Der}^{(1-m)} \subset \text{Der}(\text{Cobar}(P^\wedge)) \]
corresponding to \( \Psi \) is \( \partial \)-closed. Furthermore, equation (3.33) implies (3.26). Finally, equation (3.27) is a consequence of
\[ \Psi \big|_{G^mP} = 0. \]
\[ \square \]

3.2 The proof of Proposition 3.3

We will now use Proposition 3.7 to prove Proposition 3.3.

Since \( \alpha \) is an \( n \)-th MC-sprout and \( \beta \) is a genuine MC element of \( \text{Conv}(P, \mathcal{O} \otimes \mathbb{K}) \), we have
\[ \partial_{\mathcal{O}} \circ \beta_m + \beta_{m-1} \circ \partial_P + \sum_{k=1}^{m-1} \beta_k \cdot \beta_{m-k} = 0, \] (3.34)
\[ \partial_{\mathcal{O}} \circ \alpha_m + \alpha_{m-1} \circ \partial_P + \sum_{k=1}^{m-1} \alpha_k \cdot \alpha_{m-k} = 0, \] (3.35)
\[ \partial_{\mathcal{O}} \circ \beta_{m+1} + \beta_m \circ \partial_P + [\beta_1, \beta_m] + \sum_{k=2}^{m-1} \beta_k \cdot \beta_{m+1-k} = 0, \] (3.36)
and
\[ \partial_{\mathcal{O}} \circ \alpha_{m+1} + \alpha_m \circ \partial_P + [\alpha_1, \alpha_m] + \sum_{k=2}^{m-1} \alpha_k \cdot \alpha_{m+1-k} = 0. \] (3.37)
Subtracting (3.35) from (3.34) and using (3.12), we get
\[ \partial_{\mathcal{O}} \circ (\beta_m - \alpha_m) = 0. \]
In other words \( \beta_m - \alpha_m \) is a map from \( G^mP \) to \( Z(\mathcal{O} \otimes \mathbb{K}) \).
Let
\[ \psi_m := \pi_{\mathcal{H}} \circ (\beta_m - \alpha_m) \in \text{Conv}(P, \mathcal{H} \otimes \mathbb{K}). \] (3.38)
Subtracting (3.37) from (3.36) and using (3.12) again, we get
\[ (\beta_m - \alpha_m) \circ \partial_P + [\alpha_1, \beta_m - \alpha_m] = -\partial_{\mathcal{O}} \circ (\beta_{m+1} - \alpha_{m+1}). \] (3.39)
\[ 18 \]
Next, we observe that both sides of (3.39) are maps which land in $\mathcal{Z}(\mathcal{O} \otimes \mathbb{K})$. So applying $\pi_H$ to both sides of (3.39) and using $\pi_H \circ \alpha_1 = \alpha_H$, we deduce that

$$\psi_m \circ \partial_P + [\alpha_H, \psi_m] = 0.$$ 

In other words, $\psi_m$ is a cocycle in the cochain complex

$$\text{Conv}(P, \mathcal{H} \otimes \mathbb{K})$$

with the differential $\partial_P + [\alpha_H, ]$.

Due to Proposition 3.7, there exists a $\partial$-closed degree zero derivation $D \in \text{Der}^{1-m} \subset \text{Der}(\text{Cobar}(P^\Diamond) \otimes \mathbb{K})$ such that

$$\rho \circ D \circ s |_P = \psi_m$$

(3.40)

and

$$D(sX) = 0 \quad \forall \ X \in \mathcal{G}^{<m}P.$$ (3.41)

Thanks to Claim 3.6, $-D$ can be exponentiated to the automorphism $\exp(-D)$ of the dg operad $\text{Cobar}(P^\Diamond) \otimes \mathbb{K}$.

Let $F_{\beta}$ be the quasi-isomorphism of dg operads $\text{Cobar}(P^\Diamond) \otimes \mathbb{K} \to \mathcal{O} \otimes \mathbb{K}$ corresponding to the MC element $\beta$. Due to (3.41), the quasi-isomorphism

$$F := F_{\beta} \circ \exp(-D) : \text{Cobar}(P^\Diamond) \otimes \mathbb{K} \to \mathcal{O} \otimes \mathbb{K}$$

satisfies

$$F_{\beta} \circ \exp(-D)(sX) = F_{\beta}(sX) \quad \forall \ X \in \mathcal{G}^{<m}P.$$ 

Furthermore,

$$F_{\beta} \circ \exp(-D)(sX) - F_{\beta}(sX) \in \mathcal{Z}(\mathcal{O} \otimes \mathbb{K}) \quad \forall \ X \in \mathcal{G}^{m}P.$$ 

Using equations $\pi_H \circ \beta_1 = \alpha_H$ and (3.40), we deduce that

$$\pi_H(F_{\beta} \circ \exp(-D)(sX) - F_{\beta}(sX)) = -\psi_m(X).$$

Thus the MC element

$$\beta^\circ = \sum_{k=1}^{\infty} \beta_k^\circ \in \text{Hom}_{\mathcal{G}P}(\mathcal{G}^kP, \mathcal{O} \otimes \mathbb{K})$$

corresponding to $F$ has these properties:

$$\beta_k^\circ = \beta_k(= \alpha_k) \quad \forall \ k < m,$$

$$(\beta_m^\circ - \beta_m)(X) \in \mathcal{Z}(\mathcal{O}) \quad \forall \ X \in \mathcal{G}^mP$$

and

$$\pi_H \circ (\beta_m^\circ - \beta_m)(X) = -\psi_m \quad \forall \ X \in \mathcal{G}^mP$$

or equivalently

$$\pi_H \circ (\beta_m^\circ - \alpha_m)(X) = 0 \quad \forall \ X \in \mathcal{G}^mP.$$ (3.42)
Hence, using the splitting (2.15), we define the following degree 0 vector
\[ \xi \in \text{Hom}_{\text{Coll}}(G^m P, O \otimes K) \]
\[ \xi(X) := \tilde{s} \circ (\beta_m^\circ - \alpha_m)(X) \quad X \in G^m P, \] (3.43)
which satisfies
\[ \beta_m^\circ(X) = \alpha_m(X) + \partial O \circ \xi(X). \] (3.44)

The desired MC element \( \tilde{\beta} \) is defined by the formula
\[ \tilde{\beta} = \exp([\xi,])\beta^\circ - \frac{\exp([\xi,]) - 1}{[\xi,]} \partial \xi. \]

Indeed, since \( \xi(X) = 0 \) for all \( X \in G^m P \),
\[ \tilde{\beta}_k = \beta_k = \alpha_k \quad \forall \ k < m. \]

Moreover, equation (3.44) implies that
\[ \tilde{\beta}_m = \alpha_m. \]

Thus Proposition 3.3 is proved. □

A The lifting property for cobar resolutions

Let us recall that the functor Conv(\( P, ? \)) preserves quasi-isomorphisms:

**Proposition A.1** If \( P \) is a dg pseudo-operad satisfying Condition 1.1 and \( f : A \to B \) is a quasi-isomorphism of dg operads, then the restriction of
\[ f_* : \text{Conv}(P, A) \to \text{Conv}(P, B) \]

\[ f_*|_{F_m\text{Conv}(P, A)} : F_m\text{Conv}(P, A) \to F_m\text{Conv}(P, B) \] (A.1)

for every \( m \geq 1 \).

**Proof.** This statement was proved in [5, Section 4.4] for non-colored operads under the assumption that \( P \) has the zero differential. Here we will give the proof of the more general statement.

Since \( f_* \) is compatible with the Lie brackets, we may forget about the Lie brackets on \( F_m\text{Conv}(P, A) \) and \( F_m\text{Conv}(P, B) \) and treat both the source and the target of (A.1) as cochain complexes with the differentials coming from those on \( P, A, \) and \( B \).

Since we deal with cochain complexes of vector spaces, there exists degree zero maps
\[ \tilde{g}_q : B(q) \to A(q) \] (A.2)
and degree \(-1\) maps
\[ \tilde{\chi}_{q, A} : A(q) \to A(q), \] (A.3)
\[ \tilde{\chi}_{q, B} : B(q) \to B(q), \] (A.4)
such that
\[ f \circ \tilde{g}_q(v) = v + \partial_B \circ \tilde{\chi}_{q,B}(v) + \tilde{\chi}_{q,B} \circ \partial_B(v), \quad \forall \ v \in \mathcal{B}(q) \] (A.5)
and
\[ \tilde{g}_q \circ f(w) = w + \partial_A \circ \tilde{\chi}_{q,A}(w) + \tilde{\chi}_{q,A} \circ \partial_A(w), \quad \forall \ w \in \mathcal{A}(q), \] (A.6)
where \( q \) is any \( \Xi \)-colored corolla and \( \partial_A \) (resp. \( \partial_B \)) is the differential on \( A \) (resp. on \( B \)).

Moreover, since our base field has characteristic zero, we can use the standard averaging operators (for products of symmetric groups) and turn the maps (A.2), (A.3), and (A.4) into maps of collections
\[ g : \mathcal{B} \to \mathcal{A}, \quad \chi_A : \mathcal{A} \to \mathcal{A}, \quad \chi_B : \mathcal{B} \to \mathcal{B}, \] (A.7)
satisfying
\[ g \circ f = \text{id}_A + \partial_A \circ \chi_A + \chi_A \circ \partial_A, \quad f \circ g = \text{id}_B + \partial_B \circ \chi_B + \chi_B \circ \partial_B. \] (A.8)

Inclusion (1.6) guarantees that if \( f \in \mathcal{F}_m \text{Conv}(P, \mathcal{O}) \) (for any dg operad \( \mathcal{O} \)) then
\[ f \circ \partial_P \in \mathcal{F}_{m+1} \text{Conv}(P, \mathcal{O}). \]

Hence the differential on the associated graded complex
\[ \bigoplus_{k \geq m} \mathcal{F}_k \text{Conv}(P, \mathcal{O}) / \mathcal{F}_{k+1} \text{Conv}(P, \mathcal{O}) \] (A.9)
comes solely from the differential \( \partial_O \) on \( \mathcal{O} \).

Therefore, equations in (A.8) imply that the map (A.1) induces a quasi-isomorphism for the associated graded complexes:
\[ \bigoplus_{k \geq m} \mathcal{F}_k \text{Conv}(P, \mathcal{A}) / \mathcal{F}_{k+1} \text{Conv}(P, \mathcal{A}) \overset{\sim}{\longrightarrow} \bigoplus_{k \geq m} \mathcal{F}_k \text{Conv}(P, \mathcal{B}) / \mathcal{F}_{k+1} \text{Conv}(P, \mathcal{B}). \]

Thus, since \( \mathcal{F}_m \text{Conv}(P, \mathcal{A}) \) (resp. \( \mathcal{F}_m \text{Conv}(P, \mathcal{B}) \)) is complete with respect to the filtration \( \mathcal{F}_m \text{Conv}(P, \mathcal{A}) \supset \mathcal{F}_{m+1} \text{Conv}(P, \mathcal{A}) \supset \ldots \) (resp. \( \mathcal{F}_m \text{Conv}(P, \mathcal{B}) \supset \mathcal{F}_{m+1} \text{Conv}(P, \mathcal{B}) \supset \ldots \)), the map (A.1) is indeed a quasi-isomorphism. (See, for example, Lemma D.1 from [10]). □

Proposition A.1 has the following corollaries:

**Corollary A.2** Let \( \Psi : \mathcal{A} \to \mathcal{B} \) be a quasi-isomorphism of dg operads and \( P \) be a dg pseudo-operad satisfying Condition 1.1. Then for every operad map \( R_B : \text{Cobar}(P^\circ) \to \mathcal{B} \) there exists an operad map \( R_A : \text{Cobar}(P^\circ) \to \mathcal{A} \) such that the diagram

\[ \begin{array}{ccc}
\text{Cobar}(P^\circ) & \xrightarrow{R_A} & \mathcal{A} \\
& \searrow{\Psi} & \downarrow{R_B} \\
& \mathcal{B} &
\end{array} \]
(A.10)

commutes up to homotopy. Moreover, if \( \tilde{R}_A \) is another operad map \( \text{Cobar}(P^\circ) \to \mathcal{A} \) for which \( \Psi \circ \tilde{R}_A \) is homotopy equivalent to \( R_B \) then \( \tilde{R}_A \) is homotopy equivalent to \( R_A \).
Corollary A.3 Let $\mathcal{O}$ be a dg operad defined over a field $\mathbb{K}$ of characteristic zero and $\text{Cobar}(P^{\circ})$ be a cobar resolution of another dg operad $\tilde{\mathcal{O}}$, where $P$ is a dg pseudo-cooperad satisfying Condition 1.1. Then the existence of a zig-zag of quasi-isomorphisms (of dg operads)

$$\mathcal{O} \cong \cdots \cong \mathcal{O}$$

is equivalent to the existence of a quasi-isomorphism of dg operads

$$F : \text{Cobar}(P^{\circ}) \cong \mathcal{O}. \tag{A.12}$$

Proof of Corollary A.2 Due to Proposition A.1, the map

$$\Psi^* : \text{Conv}(P, A) \cong \text{Conv}(P, B)$$

induced by the quasi-isomorphism $\Psi : A \to B$ is a quasi-isomorphism of filtered dg Lie algebras satisfying the necessary conditions of [4, Theorem 1.1].

Therefore, there exists a MC element

$$\alpha_A \in \text{Conv}(P, A)$$

for which $\Psi^*(\alpha_A)$ is equivalent to the MC element $\alpha_B \in \text{Conv}(P, B)$ corresponding to the operad map $R_B$. Hence [5, Theorem 5.6] implies that $R_B$ is homotopy equivalent to $\Psi \circ R_A$, where $R_A$ is the operad map $\text{Cobar}(P^{\circ}) \to A$ corresponding to the MC element $\alpha_A$.

Let $\tilde{R}_A$ be another operad map $\text{Cobar}(P^{\circ}) \to A$ for which $\Psi \circ \tilde{R}_A$ is homotopy equivalent to $R_B$ and $\tilde{\alpha}_A$ be the MC element of $\text{Conv}(P, A)$ corresponding to $\tilde{R}_A$.

Since $\Psi \circ \tilde{R}_A$ is homotopy equivalent to $R_B$, the MC elements $\Psi^*(\tilde{\alpha}_A)$ and $\alpha_B$ are isomorphic. Hence, applying Theorem [4, Theorem 1.1] once again, we conclude that $\tilde{\alpha}_A$ is isomorphic to $\alpha_A$.

Thus $\tilde{R}_A$ is homotopy equivalent to $R_A$. □

Proof of Corollary A.3 Let us denote by $\rho$ the quasi-isomorphism

$$\rho : \text{Cobar}(P^{\circ}) \cong \tilde{\mathcal{O}}. \tag{A.13}$$

Given a quasi-isomorphism $F$ in (A.12), we produce the zig-zag of quasi-isomorphisms of dg operads

$$\mathcal{O} \leftarrow F \to \text{Cobar}(\mathcal{C}) \xrightarrow{\rho} \tilde{\mathcal{O}}.$$ 

So the implication $\Leftarrow$ is obvious.

To proof of the implication $\Rightarrow$ is based on the obvious application of the lifting property from Corollary A.2 and the 2-out-of-3 property for quasi-isomorphisms. □

B Tamarkin’s Ger$_\infty$-structure up to arity 4

In [6], we developed a software which implements the recursive construction of a quasi-isomorphism $\text{Ger}_\infty \to \mathcal{B}r$ over rationals.

---

10This theorem is proved in [5] only for non-colored operads. However, the proof can be easily generalized to the case of colored operads.
Let us recall that an \( n \)-th sprout in Conv\((\mathsf{Ger}^\vee, \mathsf{Br})\) is identified with a degree 1 vector in

\[
\alpha \in \bigoplus_{m=2}^{n+1} (\mathsf{Br}(m) \otimes \Lambda^{-2}\mathsf{Ger}(m))_{S_m}
\]

for which

\[
\partial \alpha + \frac{1}{2}[\alpha, \alpha] \in (\mathsf{Br}(n+2) \otimes \Lambda^{-2}\mathsf{Ger}(n+2))_{S_{n+2}} \oplus (\mathsf{Br}(n+3) \otimes \Lambda^{-2}\mathsf{Ger}(n+3))_{S_{n+3}} \oplus \ldots
\]

In other words, \( n \)-th MC-sprout \( \alpha \) involves terms in arities \( 2, 3, \ldots, n+1 \) and all terms of Curv\((\alpha)\) have arities \( \geq n+2 \).

Using this software, we found a 4-th MC-sprout \( \alpha \). This sprout has 1265 terms and the truncation \( \alpha^{[3]} \) of \( \alpha \) is shown in figures B.1, B.2, and B.3. Due to Corollary 2.16, there exists a genuine MC element \( \alpha_{MC} \in \text{Conv}(\mathsf{Ger}^\vee, \mathsf{Br}) \) such that

\[ \alpha^{[3]} = \alpha_{MC}^{[3]} \, . \]

In other words, the truncation \( \alpha^{[3]} \) shown in figures B.1, B.2, and B.3 can be extended to a genuine MC element in Conv\((\mathsf{Ger}^\vee, \mathsf{Br})\).

We would like to remark that, if we subtract the blue terms (in figure B.1) from \( \alpha^{[3]} \), then the resulting degree 1 element \( \bar{\alpha} \) will still be a third MC-sprout. However, we proved\(^\text{11}\) that \( \bar{\alpha} \) is not a truncation of any 4-th MC-sprout in Conv\((\mathsf{Ger}^\vee, \mathsf{Br})\). So \( \bar{\alpha} \) cannot be extended to a genuine MC element in Conv\((\mathsf{Ger}^\vee, \mathsf{Br})\).

\(^{11}\)The verification of this fact on a modern MacAir using [6] took approximately 5 hours.
\[
\frac{1}{2} \otimes \{b_1, b_2\} + \frac{1}{2} \otimes b_1 b_2 + \frac{1}{3} \otimes b_1 \{b_2, b_3\} - \frac{1}{9} \otimes \{b_1, \{b_2, b_3, b_4\}\} \\
- \frac{1}{5} \otimes b_2 \{b_1, b_3\} - \frac{1}{12} \otimes \{b_1, \{b_2, b_3\}\} - \frac{1}{12} \otimes \{b_2, \{b_1, b_3\}\}
\]

\[
- \frac{1}{3} \otimes b_1 \{b_2, \{b_3, b_4\}\} - \frac{1}{12} \otimes b_1 \{b_2, \{b_3, b_4\}\} - \frac{1}{6} \otimes b_1 \{b_3, \{b_2, b_4\}\}
\]

\[
- \frac{1}{12} \otimes b_1 \{b_3, \{b_2, b_4\}\} + \frac{1}{24} \otimes \{b_1, b_2\}\{b_3, b_4\} + \frac{1}{24} \otimes \{b_1, b_3\}\{b_2, b_4\}
\]

\[
+ \frac{1}{24} \otimes \{b_1, b_3\}\{b_2, b_4\} + \frac{1}{24} \otimes \{b_1, b_3\}\{b_2, b_4\} + \frac{1}{24} \otimes \{b_1, b_4\}\{b_2, b_3\}
\]

\[
+ \frac{1}{24} \otimes \{b_1, b_4\}\{b_2, b_3\} - \frac{1}{6} \otimes \{b_1, \{b_2, \{b_3, b_4\}\}\} - \frac{1}{12} \otimes \{b_1, \{b_2, \{b_3, b_4\}\}\}
\]

\[
- \frac{1}{23} \otimes \{b_1, \{b_2, \{b_3, b_4\}\}\} - \frac{137}{1140} \otimes \{b_1, \{b_2, \{b_3, b_4\}\}\} - \frac{1}{36} \otimes \{b_1, \{b_2, \{b_3, b_4\}\}\}
\]

\[
- \frac{7}{360} \otimes \{b_1, \{b_2, \{b_3, b_4\}\}\} - \frac{1}{36} \otimes \{b_1, \{b_2, \{b_3, b_4\}\}\} - \frac{1}{36} \otimes \{b_1, \{b_2, \{b_3, b_4\}\}\}
\]

\[
- \frac{1}{6} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\} - \frac{7}{60} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\} - \frac{11}{216} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\}
\]

\[
- \frac{157}{1440} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\} - \frac{11}{180} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\} - \frac{11}{45} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\}
\]

\[
- \frac{1}{18} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\} - \frac{1}{18} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\} - \frac{1}{18} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\}
\]

\[
- \frac{13}{720} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\} - \frac{1}{24} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\} - \frac{1}{144} \otimes \{b_1, \{b_3, \{b_2, b_4\}\}\}
\]

Fig. B.1: The first part of $\alpha_4$
Fig. B.2: The second part of $\alpha_4$
Fig. B.3: The last part of $\alpha_4$

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