STABILITY AND REGULARITY OF SOLUTIONS OF THE MONGE-AMPÈRE EQUATION ON HERMITIAN MANIFOLDS

SLAWOMIR KOŁODZIEJ AND NGOC CUONG NGUYEN

Abstract. We prove stability of solutions of the complex Monge-Ampère equation on compact Hermitian manifolds, when the right hand side varies in a bounded set in $L^p$, $p > 1$ and it is bounded away from zero. Such solutions are shown to be Hölder continuous. As an application we extend a recent result of Székelyhidi and Tosatti on Kähler-Einstein equation from Kähler to Hermitian manifolds.

1. INTRODUCTION

Ever since the solution of the Calabi conjecture by S.T. Yau [Yau76] the complex Monge-Ampère equation played a prominent role in complex geometry. In the last decades the weak solutions of the equation on compact Kähler manifolds also found many applications in the study of degenerations of canonical metrics and the limits of the Kähler-Ricci flow. In those settings, often, when a family of Kähler metrics approaches the boundary of the Kähler cone their volume forms blow up, becoming unbounded, but still remain bounded in $L^p$ for some $p > 1$. Then the stability estimates [Ko03, Ko05] provide good control of the potentials of those metrics close to the singularity set. In particular the potentials are then Hölder continuous (see [Ko08], [DDGHKZ]). Those results found a number of applications in the works on the Kähler-Ricci flow of Tian-Zhang [TZh00], and Song-Tian [ST07, ST09, ST12]; in Tosatti’s description of the limits of families of Calabi-Yau metrics when the Kähler class degenerates [T09, T10]; and in the recent solution of the Donaldson-Tian-Yau conjecture. The proof of Chen-Donaldson-Sun [CDS15] uses stability and Hölder continuity results for approximation of cone Kähler-Einstein metrics by the smooth ones. The pluripotential approach was further developed in the papers of Eyssidieux-Guedj-Zeriahi [EGZ09], of those authors together with Boucksom [BEGZ], and in several other articles. An up-to-date account of those developments can be found in the survey of Phong, Song and Sturm [PSS12].

In this paper we are concerned with the complex Monge-Ampère equation on compact Hermitian (non-Kähler) manifolds. In the eighties Chérief [Ch87] made an attempt to prove the analogue of the Calabi-Yau theorem in this setting obtaining the result under a rather restrictive assumption. Further progress was made in [GL10] and [TW10a]. Finally Tosatti and Weinkove [TW10b] gave the complete proof. Those results came amid a considerable growth of research activity in Hermitian geometric analysis. In this context one should mention a paper by Fu and Yau [FY08] with a construction of some non-trivial solutions of the Strominger system. In relation to this problem Fu-Wang-Wu [FWW10a] introduced,

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form-type Calabi-Yau equation and solved it [FWW10b] for metrics of nonnegative orthogonal bisectional curvature. Tosatti and Weinkove [TW13] were able to solve the equation without the extra assumption. In the equation there are two reference metrics. If both are non-Kähler, the solution, which gives the confirmation of the Gauduchon conjecture (analogue of the one of Calabi for Gauduchon metrics), is still to be found. Popovici [Po13] considered a variant of this equation studying moduli spaces of Calabi-Yau ∂∂̄-manifolds. The Monge-Ampère equation is closely related to the Chern-Ricci flow, intensively studied recently in the papers of Gill [Gil11, Gil13], Tosatti-Weinkove [TW12a, TW12b], Tosatti-Weinkove-Yang [TWY13]. Chiose [Chi13] and the second named author [N15] used the solutions of the equation to prove a conjecture of Demailly and Paun [DP04] in some special cases.

The analysis of geometrically meaningful equations on Hermitian manifolds is harder than in the Kähler case due to the torsion terms which are difficult to handle in the estimates. This accounts for a somewhat technical character of the proofs.

Results. Throughout the paper (X, ω) will denote a compact manifold X with a Hermitian metric ω. S. Dinew and the first author obtained in [DK12] L∞ a priori estimates for the complex Monge-Ampère equation

\[(ω + dd^c u)^n = \text{const.} f ω^n,\]

with the nonnegative right hand side in L^p(ω^n), p > 1. The weak continuous solutions, under the same assumption, were obtained by the authors in [KN]. Up till now the uniqueness of solutions, normalized, for instance, by \(\sup_X u = 0\), has not been established. It follows, for strictly positive f, from the main result of this paper, which is the following stability statement.

**Theorem A.** Let 0 ≤ f, g ∈ L^p(ω^n), p > 1, be such that \(\int_X f ω^n > 0\), \(\int_X g ω^n > 0\). Consider two continuous ω-psh solutions of the complex Monge-Ampère equation

\[(ω + dd^c u)^n = f ω^n, \quad (ω + dd^c v)^n = g ω^n,\]

with \(\sup_X u = \sup_X v = 0\). Assume that

\[f ≥ c_0 > 0 \quad (c_0 \text{ a constant}).\]

Fix 0 < α < \(\frac{1}{n+1}\). Then, there exists C = C(c_0, α, \|f\|_p, \|g\|_p) such that

\[\|u - v\|_∞ ≤ C\|f - g\|_p^α.\]

As compared to the corresponding theorem for Kähler manifolds we have here an additional hypothesis that f is nondegenerate (f > c_0 > 0). It would be very desirable to remove or weaken it. Note, however, that on non-Kähler manifolds the notion of "stability" itself has an extra dimension, since \(\int_X f ω^n\) is no longer fixed. It means that if we consider a small perturbation \(\tilde{f}\) of f on the right hand side then the Monge-Ampère equation has a solution for \(c\tilde{f}\) and this constant is not determined by ω. On the bright side, the Hölder exponent in Theorem A is independent of (X, ω) and it is almost as good as in the Kähler case (see [DiZh10]), except that we consider L^1 norm of (f - g) instead of L^1 norm. We next prove the result with L^1 norm in Theorem 3.11 but then the exponent is worse. The proof of Theorem A required a completely new method. It may be obscured by many technicalities, so we described the main idea briefly in short guide to the proof at the beginning of the proof of Theorem 3.1.
With the stability at our disposal we could prove two other theorems. First, the Hölder continuity of solutions of the Monge-Ampère equation on compact Hermitian manifolds.

**Theorem B.** Consider the solution \( u \in PSH(\omega) \cap L^\infty(X) \) of the complex Monge-Ampère equation

\[
\omega_n^u = f \omega^n,
\]
on \((X, \omega)\) a compact Hermitian manifold, with \( f > c_0 > 0, \| f \|_p < \infty \). Then for any \( \alpha < \frac{2}{\rho(n(n+1)+1)} \) the function \( u \) is Hölder continuous, with Hölder exponent \( \alpha \).

Again we have the hypothesis that \( f \) is nondegenerate, which is not needed on Kähler manifolds. The Hölder exponent is worse than in the Kähler counterpart, (roughly) by factor \( 1/n \).

The second application of the stability result is an extension of a theorem of Székelyhidi and Tosatti [SzTo11] to the case of compact Hermitian manifolds.

**Theorem C.** Let \((X, \omega)\) be a compact \( n \)-dimensional Hermitian manifold. Suppose that \( u \in PSH(\omega) \cap L^\infty(X) \) is a solution of the equation

\[
(\omega + dd^c u)^n = e^{-F(u, z)} \omega^n
\]
in the weak sense of currents, where \( F : \mathbb{R} \times X \to \mathbb{R} \) is smooth. Then \( u \) is smooth.

In [SzTo11] the authors obtained this for Kähler manifolds to derive that if \( X \) is a Fano manifold and \( \omega \) represents the first Chern class, then any Kähler-Einstein current with bounded potentials is smooth. Nie [Nie13] recently generalised this result to special compact Hermitian manifolds, and observed that her higher order estimates and a stability result would give the theorem above. We have just provided this result.

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2. **Preliminaries**

In this section we give some auxiliary results used later for the proof of the stability estimates and uniqueness in Section 3. Some of them are interesting in themselves.

Let \((X, \omega)\) be a compact \( n \)-dimensional Hermitian manifold. \( PSH(\omega) \) stands for the set of all \( \omega \)-plurisubharmonic (\( \omega \)-psh) functions on \( X \). If \( u \in PSH(\omega) \), then

\[
\omega_u := \omega + dd^c u \geq 0.
\]

We shall denote throughout, by \( B \) the “curvature” constant \( B > 0 \) satisfying

\[
- B \omega^2 \leq dd^c \omega \leq B \omega^2, \quad - B^3 \omega^3 \leq d\omega \wedge d^c \omega \leq B^3 \omega^3.
\]

For \( r \geq 1 \), its conjugate number will be denoted by \( r^* \), and we write \( L^r(X, \omega^n) \) for \( L^{r^*}(X, \omega^n) \), and

\[
\| . \|_r = \left( \int_X | . |^r \omega^n \right)^{\frac{1}{r}}, \quad \| . \|_\infty = \sup_X | . |.
\]

The volume of a Borel set \( E \subset X \) with respect to the metric \( \omega \) is given by

\[
\text{Vol}_\omega(E) := \int_E \omega^n.
\]
and without loss of generality we normalise the volume of $X$ to be 1, i.e.

$$Vol_{\omega}(X) = \int_X \omega^n = 1.$$ 

We always denote by $C$ a uniform positive constant that depends only on $X, \omega$ and the dimension $n$. It may differ from place to place.

Because $\omega$ is not closed, the Monge-Ampère operator does not preserve the volume of the manifold. Therefore, given $f$ on the right hand side of the Monge-Ampère equation the solution possibly exists for $cf$ ($c$ a positive constant). If $c = 1$ then we call $f$ MA-admissible. Thanks to results in [KN], [N15] we are able to get a priori bounds for the constant $c$.

**Lemma 2.1.** If $u, v \in PSH(\omega) \cap C(X)$ satisfy

$$\omega_u^n \leq c \omega_v^n$$

for some $c > 0$, then $c \geq 1$.

**Proof.** See [N15] Lemma 2.3, Corollary 2.4. □

The mixed form type inequality [Ko05], [Di09] has been extended to the Hermitian setting in [N15]. It will be used many times in the sequel.

**Lemma 2.2** (mixed form type inequality). Let $0 \leq f, g \in L^1(\omega^n)$ and $u, v \in PSH(\omega) \cap L^\infty(X)$. Suppose that $\omega_u^n \geq f \omega^n$ and $\omega_v^n \geq g \omega^n$ on $X$. Then for $k = 0, \ldots, n$

$$\omega_u^k \wedge \omega_v^{n-k} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} \omega^n \text{ on } X.$$

In particular, for $0 < \delta < 1$,

$$\omega_{\delta u + (1-\delta)v}^n \geq \left[\delta f^{\frac{1}{n}} + (1-\delta)g^{\frac{1}{n}}\right]^n \omega^n \text{ on } X.$$

**Proof.** See [N15] Lemma 1.9. □

The uniqueness of the Monge-Ampère equation in the realm of smooth functions is proven in [TW10b]. A priori there could exist some different continuous weak solutions as constructed in [KN]. However, if the right hand side is smooth, any continuous solution to is also smooth. It follows from the following lemma.

**Lemma 2.3.** Let $\rho \in PSH(\omega) \cap C(X)$. Assume that $\omega^n \leq \omega^\rho_n$ as two measures. Then $\rho$ is a constant.

**Proof.** Let $G \in C^\infty(X)$ be the Gauduchon function of the metric $\omega$, i.e $dd^c(\omega^n-1e^G) = 0$. The mixed forms type inequality (Lemma 2.2) implies that

$$\omega^\rho \wedge \omega^{n-1} \geq \omega^n.$$

Therefore, $\omega^\rho \wedge \omega^{n-1} e^G \geq \omega^n e^G$. It follows that $dd^c \rho \wedge \omega^{n-1} e^G \geq 0$. However, by the Stokes theorem and the Gauduchon condition

$$\int_X dd^c \rho \wedge \omega^{n-1} e^G = \int_X \rho dd^c(\omega^{n-1} e^G) = 0.$$

Hence,

$$dd^c \rho \wedge \omega^{n-1} e^G = 0 \Rightarrow dd^c \rho \wedge \omega^{n-1} = 0.$$

So $\rho \in PSH(\omega) \cap C(X)$ is a harmonic function on a compact manifold and therefore it is a constant. □
One of the steps to prove the stability estimate in Section 3 is the construction of the barrier function having large Monge-Ampère mass on some small set. The following result will help to do this. It is essentially contained in [KN], although not exactly in this form.

**Proposition 2.4.** Let $0 \leq f \in L^p(\omega^n)$, $p > 1$, be such that $\int_X f \omega^n > 0$. Suppose one can solve the complex Monge-Ampère equation

$$(\omega + dd^c w)^n = cf \omega^n, \quad \sup_X w = 0,$$

where $w \in PSH(\omega) \cap C(X)$ and $c > 0$. Then there exists a constant $V_{\text{min}} > 0$ depending only on $X, \omega, \|f\|_p$ such that whenever

$$\int_X f \omega^n \leq 2V_{\text{min}},$$

we have $c \geq 2^n$.

**Proof.** Suppose $c \leq 2^n$. We shall see that this leads to a contradiction for some positive $V_{\text{min}}$. So we have

$$\omega^n_w \leq 2^n f \omega^n.$$

Therefore, by Hölder’s inequality, for any Borel set $E \subset X$,

$$\int_E \omega^n_w \leq 2^n \|f\|_p [Vol_\omega(E)]^{\frac{1}{p^*}}.$$

The volume-capacity inequality [DK12, Corollary 2.4] or [KN, Proposition 5.1] implies that the Monge-Ampère measure $\omega^n_w$ satisfies the inequality [KN, Eq. (5.2)] for the admissible (in the sense used in [KN]) function

$$h(x) = \frac{C e^{ax}}{2^n \|f\|_p},$$

where $a > 0$ is a universal number depending only on $p, X, \omega$. Inequality [KN, Eq. (5.12)] for $0 < t \leq \frac{1}{3} \min\{\frac{1}{2n}, \frac{1}{2B}\}$ then gives:

$$t^n cap_\omega(\{w < S + t\}) \leq C \int_{\rho < S + 2t} \omega^n_w \leq C \int_X 2^n f \omega^n,$$

where $S = \inf_X w$ and $C > 0$ depends only on $n, B$. It implies that

$$\int_X f \omega^n \geq t^n h(t) \frac{1}{2nC}.$$
Define
\[ V_{\min} := \frac{t_0^n}{2n+2Ch(t_0)} > 0, \quad t_0 = \frac{1}{6} \min\{ \frac{1}{2n}, \frac{1}{2B} \}. \]

Then, the inequality \((2.4)\) and the assumptions would give a contradiction
\[ 2V_{\min} \geq \int_X f\omega^n \geq 4V_{\min} > 0. \]

Thus the proposition is proven. \(\square\)

The following minimum principle is inspired by [BT76, Theorem A], which was stated for a bounded open set in \(\mathbb{C}^n\). In the Hermitian setting it requires a different proof.

**Proposition 2.5** (minimum principle). Let \( U \subset \subset \mathbb{C}^n \) be an non-empty open set. Let \( u, v \in \text{PSH}(\omega) \cap C(X) \) be such that
\[ c\omega^n_u \leq \omega^n_v \text{ on } U \]
for some \( c > 1 \). Then,
\[ \min\{u(x) - v(x) : x \in \overline{U}\} = \min\{u(x) - v(x) : x \in \partial U\}. \]

**Proof.** Without loss of generality suppose that \( \min\{u(x) - v(x) : x \in \partial U\} = 0 \), i.e.
\[ v \leq u \text{ on } \partial U. \]

We need to show that \( \min\{u(x) - v(x) : x \in \overline{U}\} = 0 \). Suppose that it is not the case. Then there exists \( x_0 \in U \) such that \( u(x_0) < v(x_0) \). So,
\[ S := \min\{u(x) - v(x) : x \in U\} < 0. \]

Let us use the following notation
\[ S_\varepsilon := \min_{\overline{U}} \{u - (1 - \varepsilon)v\}, \]
for \( 0 < \varepsilon << 1 \). Then
\[ S - \varepsilon\|v\|_\infty \leq S_\varepsilon \leq S + \varepsilon\|v\|_\infty. \]

Therefore,
\[ U(\varepsilon, t) := \{u < (1 - \varepsilon)v + S_\varepsilon + t\} \cap U \subset \{u < v + S + 2\varepsilon\|v\|_\infty + t\} \cap U \subset \subset U, \]
for every \( 0 < t < \delta_0 \) with \( \delta_0 \) small enough, as \( u \geq v \) on \( \partial U \).

From the modified comparison principle [KN, Theorem 0.2] we have
\[ (2.6) \quad 0 \leq \int_{U(\varepsilon, t)} \omega^n_u(1 - \varepsilon)v \leq \left[ 1 + \frac{Ct}{\varepsilon^n} \right] \int_{U(\varepsilon, t)} \omega^n_u, \]
for every \( 0 < t < \frac{\varepsilon^3}{16B} \), where \( 0 < \varepsilon < \delta_0 \). It is easy to see that
\[ (2.7) \quad \omega^n_u(1 - \varepsilon)v \geq (1 - \varepsilon)^n\omega^n_u \geq (1 - \varepsilon)^n c\omega^n_u \]
on \( U \). As \( c > 1 \), we can choose \( 0 < \varepsilon < \delta_0 \) so that
\[ (2.8) \quad (1 - \varepsilon)^n c > 1. \]
It follows from \((2.6), (2.7)\) and \((2.8)\) that
\[ [\varepsilon^n(1 - \varepsilon)^n - 1] \int_{U(\varepsilon, t)} \omega^n_u \leq \frac{Ct}{\varepsilon^n} \int_{U(\varepsilon, t)} \omega^n_u. \]
Therefore, for fixed $0 < \varepsilon < \delta_0$ and for every $0 < t < \frac{\varepsilon^2}{16}$

$$[(1 - \varepsilon)^n c - 1] \leq \frac{C t}{\varepsilon^n}.$$ 

It is impossible. The proof is completed. □

We finish this section with a lemma from Riemannian geometry, which follows from Sobolev’s embedding theorem.

**Lemma 2.6.** Let $(X, \omega)$ be a compact $n$-dimensional Hermitian manifold. Let $\psi \in C(X) \cap W^{1,2}(X)$ be a real-valued function. Fix $d, \delta > 0$. Assume that

$$\text{Vol}_\omega(\{\psi < 0\}) \geq \delta, \quad \text{Vol}_\omega(\{\psi \geq d\}) \geq \delta.$$ 

The square of the norm of the gradient is given in local coordinates by

$$|\partial \psi|^2 = \sum g^{i\bar{j}}\partial_k \psi \partial_{\bar{l}} \psi, \quad \text{where} \quad \omega = \frac{i}{2} \sum g_{k\bar{l}} dz_k \wedge d\bar{z}_l.$$ 

Then,

$$\int_{\{0 < \psi < d\}} |\partial \psi| \omega^n \geq C d \delta^{\frac{4n-1}{2n}},$$ 

where $C > 0$ is a uniform constant depending only on $X, \omega$.

**Proof.** We apply Sobolev’s embedding theorem to the function

$$\Psi = \max(\psi, 0) - \max(\psi - d, 0) - M,$$

where $M = \int_X [\max(\psi, 0) - \max(\psi - d, 0)] \omega^n$. As $\int_X \Psi \omega^n = 0$, it gives

$$\|\Psi\|_{\frac{2n}{n-1}} \leq C \int_X |\partial \Psi| \omega^n.$$ 

It is easy to see that

$$\int_X |\partial \Psi| \omega^n = \int_{\{0 < \psi < d\}} |\partial \psi| \omega^n.$$ 

As $\text{Vol}_\omega(\{\psi \geq d\}) \geq \delta$, we have

$$M = \int_{\{0 < \psi < d\}} \psi \omega^n + d \int_{\{\psi \geq d\}} \omega^n \geq d \delta.$$ 

Moreover, as $\text{Vol}_\omega(\{\psi < 0\}) \geq \delta$,

$$\int_X |\max(\psi, 0) - \max(\psi - d, 0) - M| \frac{\omega^n}{2n-1} \geq \int_{\{\psi \leq 0\}} M \frac{\omega^n}{2n-1} \geq \int_{\{\psi \leq 0\}} (d \delta) \frac{\omega^n}{2n-1} \geq d \delta \delta^{\frac{2n-1}{2n-1}}.$$ 

Therefore, we conclude that $\int_{\{0 < \psi < d\}} |\partial \psi| \omega^n \geq C d \delta^{1 + \frac{2n-1}{2n}}$. □
3. Stability Estimates and Uniqueness

In this section we prove our main result: the stability and uniqueness of the Monge-Ampère equation with the positive right hand side in $L^p(\omega^n)$, $p > 1$.

**Theorem 3.1.** Let $0 \leq f, g \in L^p(\omega^n)$, $p > 1$, be such that $\int_X f \omega^n > 0$, $\int_X g \omega^n > 0$. Consider two continuous $\omega$-psh solutions of the complex Monge-Ampère equation

$$\omega_u^n = f \omega^n, \quad \omega_v^n = g \omega^n$$

with $\sup_X u = \sup_X v = 0$. Assume that $f$ is smooth and $f \geq c_0 > 0$.

Fix $0 < \alpha < \frac{1}{n+1}$. Then, there exists $C = C(c_0, \alpha, \|f\|_p, \|g\|_p)$ such that

$$\|u - v\|_\infty \leq C \|f - g\|_p^\alpha.$$

**Remark 3.2.** We shall remove the smoothness assumption on $f$ thus obtaining Theorem A from Introduction (see Remark 3.10).

**Remark 3.3.** It follows from Lemma 2.3 that the continuous $\omega$-psh function $u$ is automatically smooth as it coincides with the unique (normalised) smooth solution obtained by Tosatti and Weinkove [TW10b].

**Proof.** Short guide to the proof of Theorem 3.1. We shall use the notation:

$$\varphi = u - v, \quad t_0 = \min_{X} \varphi, \quad \Omega(t) = \{\varphi < t\}, \quad \|f - g\|_p \leq \varepsilon.$$

Assuming $0 < \varepsilon << 1$ we wish to show that $\|\varphi\|_\infty \leq C \varepsilon^n$ with a fixed $0 < \alpha < \frac{1}{n+1}$. The proof falls in two parts, where we consider, roughly, two situations. First, when the volume of a sublevel set is small, i.e. $\Vol_\omega(\Omega(t_1)) \leq V_{\min}$ (see Proposition 2.4, Lemma 3.4). Then one can construct a "barrier" $w$ having big Monge-Ampère mass on this set. Modifying $v$ by $sw$ with judiciously chosen, small $s$ we reach a contradiction with a minimum principle (Proposition 2.5). Although more technical, this part is similar to the proof in the Kähler case and does not require the assumption $f \geq c_0 > 0$. Then we consider sublevel sets of large volume, when $t_1 - t_0$ is of order $\varepsilon^n$. From upper and lower estimates of the energy of $\varphi$ in collars $\Omega(t') \setminus \Omega(t)$ one obtains (big) lower bounds on the Laplacian mass $\omega_u \wedge \omega^{n-1}$ in those sets (see Proposition 3.8). Then for collars with volumes large enough one would get a contradiction and this gives the desired bound on $\|\varphi\|_\infty$.

Since $\Vol_\omega(X) = 1$, it follows that $\|f - g\|_r \leq \|f - g\|_p \leq \varepsilon$ and

$$\int_E |f - g| \omega^n \leq \left(\int_E |f - g| r \omega^n\right)^{\frac{1}{r}} \leq \left(\int_E |f - g|^p \omega^n\right)^{\frac{1}{p}} \leq \varepsilon,$$

for every $1 \leq r \leq p$ and every Borel set $E \subset X$.

**Lemma 3.4.** Let $V_{\min} > 0$ be the constant from Proposition 2.4. Fix $t_1 > t_0$. If $\int_{\Omega(t_1)} f \omega^n \leq V_{\min}$, then

$$t_1 - t_0 \leq C \varepsilon^n,$$

where $0 < \alpha < \frac{1}{n+1}$ is fixed.
Proof. Consider the following sets
\[ \Omega_1 = \{ z \in \Omega(t_1) : f(z) \leq (1 + \varepsilon^\alpha)g \}, \quad \Omega_2 = \Omega(t_1) \setminus \Omega_1. \]

We have
\[
\int_{\Omega_2} f \omega^n \leq \int_{\Omega_2} |f - g| \omega^n + \int_{\Omega_2} g \omega^n \\
\leq \varepsilon + \varepsilon^{-\alpha} (f - g) \omega^n \\
\leq \varepsilon + \varepsilon^{1-\alpha} \leq 2\varepsilon^{1-\alpha}.
\]

Moreover,
\[
\int_{\Omega_2} f^p \omega^n \leq \int_{\Omega_2} 2^{p-1} (|f - g|^p + g^p) \omega^n \\
\leq \int_{\Omega_2} 2^{p-1} |f - g|^p \omega^n + \int_{\Omega_2} 2^{p-1} \varepsilon^{-\alpha} (f - g) \omega^n \\
\leq 2^{p-1} (\|f - g\|_p^p + \|f - g\|_p^{p\varepsilon^{-p\alpha}}) \\
\leq 2^{p-1} (\varepsilon^p + \varepsilon^{p\varepsilon^{-p\alpha}}) \leq 2^p \varepsilon^{p\varepsilon^{-p\alpha}}.
\]

Define for \(0 < \varepsilon << 1\) and \(A >> 1\) (to be chosen later)
\[
\hat{f}(z) = \begin{cases} 
  f(z) & \text{for } z \in \Omega_1, \\
  \varepsilon^{-n\alpha} f(z) & \text{for } z \in \Omega_2, \\
  \frac{1}{A} f(z) & \text{for } z \in X \setminus \Omega(t_1).
\end{cases}
\]

Then
\[
\int_X \hat{f} \omega^n = \int_{\Omega_1} f \omega^n + \int_{\Omega_2} \varepsilon^{-n\alpha} f \omega^n + \int_{X \setminus \Omega(t_1)} \frac{1}{A} f \omega^n \\
\leq V_{\min} + 2\varepsilon^{1-\alpha-n\alpha} + \frac{1}{A} \|f\|_1,
\]

and
\[
\int_X \hat{f}^p \omega^n = \int_{\Omega_1} f^p \omega^n + \int_{\Omega_2} \varepsilon^{-n\alpha p} f^p \omega^n + \int_{X \setminus \Omega(t_1)} \frac{1}{A^p} f^p \omega^n \\
\leq \|f\|_p^p + 2^p \varepsilon^{p\varepsilon^{-p\alpha}-n\alpha} + \frac{1}{A^p} \|f\|_p^p.
\]

Since \(0 < \alpha < \frac{1}{n+1}\), it follows from (3.4) and (3.5) that we can choose \(0 < \varepsilon << 1\) and \(A >> 1\) such that
\[
\max\{2\varepsilon^{1-(n+1)\alpha} \cdot \frac{1}{A} \|f\|_p \} \leq \min\{ \frac{V_{\min}}{4}, \frac{\|f\|_1}{2} \}.
\]

Notice that by our normalisation \(Vol(\omega)(X) = 1\), so we have
\[
\|f\|_r \leq \|f\|_p \quad \text{for } 1 \leq r \leq p.
\]

It implies that for such \(\varepsilon\) and \(A\),
\[
\int_X \hat{f} \omega^n \leq \frac{3}{2} V_{\min} \quad \text{and} \quad \|\hat{f}\|_p \leq 2\|f\|_p.
\]

By [KN, Theorem 0.1] there exists \(w \in PSH(\omega) \cap C(X)\) and \(\hat{c} > 0\) solving
\[
\omega^n_w = \hat{c} \hat{f} \omega^n, \quad \sup_X w = 0.
\]
It follows from Proposition 2.4 and (3.7) that
\begin{equation}
\hat{c} \geq 2^n.
\end{equation}
Moreover, as $\hat{f} \geq f/A$, using Lemma 2.1, we get that
\begin{equation}
\hat{c} \leq A.
\end{equation}
Let us define for $0 < s < 1$, $\psi_s = (1 - s)v + sw$. By the mixed form type inequality (Lemma 2.2) we have
\begin{equation}
\omega_\psi^n \geq [(1 - s)g^{\hat{\beta}} + s(\hat{\beta}f)^{\hat{\beta}}] \omega^n
= [(1 - s)\left(\frac{g}{f}\right)^{\hat{\beta}} + s\left(\frac{\hat{\beta}f}{f}\right)^{\hat{\beta}}] \omega^n
= [b(s)]^n f \omega^n.
\end{equation}
We shall see that $b(s) > 1$ for $2\varepsilon^\alpha \leq s \leq 1/2$ on $\Omega(t_1)$. Indeed, for $z \in \Omega_1$, we have $g/f \geq 1/(1 + \varepsilon^\alpha)$ and $\hat{\beta}f \geq 2^n f$. Therefore, on $\Omega_1$,
\begin{equation}
b(s) \geq \frac{1 - s}{1 + \varepsilon^\alpha} + 2s \geq \frac{1 - s}{1 + \varepsilon^\alpha} + 2s.
\end{equation}
Now, for $z \in \Omega_2$, we have $\hat{\beta}f \geq 2^n \varepsilon^{-\alpha}$. Therefore, on $\Omega_2$
\begin{equation}
b(s) \geq 2s \varepsilon^{-\alpha}.
\end{equation}
It follows from (3.10) and (3.11) that if
\begin{equation}
2\varepsilon^\alpha \leq s \leq \frac{1}{2},
\end{equation}
then
\begin{equation}
b(s) > 1 + \varepsilon^\alpha
\end{equation}
on $\Omega(t_1) = \Omega_1 \cup \Omega_2$. This means that for such a value of $s$,
\begin{equation}
\omega_\psi^n \geq (1 + \varepsilon^\alpha) f \omega^n \quad \text{on } \Omega(t_1).
\end{equation}

End of Proof of Lemma 3.4. Applying the minimum principle (Proposition 2.5) for $s = 2\varepsilon^\alpha$ and the function
\begin{equation}
u - [(1 - s)v + sw] = \varphi - s(w - v)
\end{equation}
on $\Omega(t_1)$, we get that
\begin{equation}
\min_{\Omega(t_1)} [\varphi(z) - s(w - v)(z)] = \min_{\partial\Omega(t_1)} [\varphi(z) - s(w - v)(z)].
\end{equation}
Therefore
\begin{equation}
t_1 - t_0 \leq 2s\|w - v\|_\infty \leq 4(\|v\|_\infty + \|w\|_\infty)\varepsilon^\alpha.
\end{equation}
Since $\hat{c} \leq A$, by [KN Corollary 5.6], we have
\begin{equation}\|w\|_\infty \leq A^{\hat{\beta}} H.
\end{equation}
where $H = H(\|f\|_p, \|g\|_p, X, \omega) > 0$ is a uniform bound for $u, v$, i.e.
\begin{equation}H \leq u, v \leq 0.
\end{equation}
Thus Lemma 3.4 follows.
We pass to the second part of the proof. The main ingredient here are a priori estimates for the Laplacian \( \omega_u \wedge \omega^{n-1} \) in collars \( \{a < \varphi < b\} \). Recall our notation:

\[
0 < c_0 \leq f \in C^\infty(X),
\]

and

\[
\omega_u^n = f \omega^n, \quad \omega_v^n = g \omega^n,
\]

where

\[
u \in C^\infty(X), \quad \omega + dd^c u > 0, \quad v \in PSH(\omega) \cap C(X).
\]

Moreover,

\[
t_0 = \min_X \varphi, \quad \|f - g\|_p \leq \varepsilon.
\]

**Lemma 3.5** (Laplacian mass on small collars). Let \( t_0 < a < b \) be two real numbers satisfying

\[
\varepsilon < b - a \quad \text{and} \quad a - t_0 = b - a.
\]

Assume that

\[
\text{Vol}_\omega(\{\varphi < a\}) \geq \delta \quad \text{and} \quad \text{Vol}_\omega(\{\varphi \geq b\}) \geq \delta,
\]

for some \( \delta > 0 \). Then,

\[
\int_{\{a < \varphi < b\}} \omega_u \wedge \omega^{n-1} \geq \delta_c > 0,
\]

where \( \delta_c = C c_0 \delta^{\frac{4n-4}{n}} > 0 \) (thus depends only on \( \delta, c_0, X, \omega \)).

**Proof.** We first estimate

\[
\int_{\{a < \varphi < b\}} d\varphi \wedge d^c \varphi \wedge \omega_u^{n-1}
\]

from above and then

\[
\int_{\{a < \varphi < b\}} \omega_u \wedge \omega^{n-1}, \quad \int_{\{a < \varphi < b\}} d\varphi \wedge d^c \varphi \wedge \omega_u^{n-1}
\]

from below. Let us start with the first one.

\[
\int_{\{a < \varphi < b\}} d\varphi \wedge d^c \varphi \wedge \omega_u^{n-1} \leq \int_{\{a < \varphi < b\}} d\varphi \wedge d^c \varphi \wedge T,
\]

where

\[
T = \sum_{k=0}^{n-1} \omega_u^k \wedge \omega_v^{n-1-k} \quad \text{and} \quad \omega_u^n - \omega_v^n = dd^c \varphi \wedge T.
\]

Then

\[
dd^c T = dd^c \omega \wedge T_1 + d\omega \wedge d^c \omega \wedge T_2
\]

where \( T_1, T_2 \) are positive currents (see \[DK12\]). Therefore, by the choice of the constant \( B > 0 \) (see \[2\]),

\[
-B(\omega^2 \wedge T_1 + \omega^3 \wedge T_2) \leq dd^c T \leq B(\omega^2 \wedge T_1 + \omega^3 \wedge T_2).
\]

It follows that for any Borel set \( E \subset X \) and a continuous function \( w \geq 0 \) on \( X \) we have (see e.g. \[N15\] Proposition 1.5))

\[
\left| \int_E wdd^c T \right| \leq B\|w\|_\infty \int_X (\omega^2 \wedge T_1 + \omega^3 \wedge T_2)
\]

\[
\leq CB\|w\|_\infty (1 + \|u\|_\infty)(1 + \|v\|_\infty)^n.
\]
To simplify notation, we write in what follows
\[ \psi := \varphi - a, \quad (\text{and then } \min_X \psi = t_0 - a). \]

Notice that this does not affect \( T \) and we still have \( dd^c \psi \wedge T = \omega^n_a - \omega^n_v \). The right hand side in the inequality (3.15) becomes

\[
(3.17) \quad \int_{\{0 < \psi < b-a\}} d\psi \wedge d^c \psi \wedge T.
\]

By the Stokes theorem,

\[
\int_{\{0 < \psi < b-a\}} d\psi \wedge d^c \psi \wedge T = \int_{\partial \{0 < \psi < b-a\}} \psi d^c \psi \wedge T - \int_{\{0 < \psi < b-a\}} (f - g) \omega^n - \int_{\{a < \psi < b\}} \psi d\psi \wedge d^c T
\]

\[
= (b - a) \int_{\{\psi = b-a\}} d^c \psi \wedge T - \int_{\{0 < \psi < b-a\}} (f - g) \omega^n - \int_{\{a < \psi < b\}} \psi d\psi \wedge d^c T
\]

\[
= (b - a) \left( \int_{\{\psi < b-a\}} dd^c \psi \wedge T + \int_{\{\psi < b-a\}} d^c \psi \wedge dT \right)
\]

\[
- \int_{\{0 < \psi < b-a\}} \psi d\psi \wedge d^c T - \int_{\{0 < \psi < b-a\}} (f - g) \omega^n
\]

\[
= (b - a)(J_{12} + J_{23} - J_3 - J_4),
\]

where we used \( dd^c \psi \wedge T = (f - g) \omega^n \) for the second inequality and the Stokes theorem once more for the fourth equality.

Our next step is to bound \( |J_k|, k = 1, 2, 3, 4 \). However, it is not hard to see from the computation below that one can deal with \( J_k \) and \(-J_k\), in the same way. So we only give the estimates from above for \( J_k \).

The bounds for \( J_1, J_4 \) are easy:

\[
(3.18) \quad J_1 = \int_{\{\psi < b-a\}} (\omega^n_a - \omega^n_v) \leq \int_X |f - g| \omega^n \leq \varepsilon,
\]

and

\[
(3.19) \quad J_4 = \int_{\{0 < \psi < b-a\}} (f - g) \omega^n \leq \int_{\{0 < \psi < b-a\}} |f - g| \omega^n \leq (b - a) \varepsilon.
\]

To estimate \( J_2, J_3 \) we will use the inequality (3.10) several times without mentioning it anymore. First, we consider \( J_2 \). Set

\[
J_2' = -J_2 - \int_{\{\psi \leq 0\}} d\psi \wedge d^c T.
\]
Then, using the Stokes theorem twice, we get that
\[ J_2 = \int_{\{0 < \psi < b-a\}} d\psi \wedge d^cT = \int_{\{0 < \psi < b-a\}} d(\psi d^cT) - \int_{\{0 < \psi < b-a\}} \psi dd^cT = (b-a) \int_{\{\psi = b-a\}} d^cT - \int_{\{0 < \psi < b-a\}} \psi dd^cT = (b-a) \int_{\{\psi < b-a\}} dd^cT - \int_{\{0 < \psi < b-a\}} \psi dd^cT \leq C(b-a). \]

Similarly, by the Stokes theorem,
\[ \int_{\{\psi \leq 0\}} d\psi \wedge d^cT = - \int_{\{\psi \leq 0\}} \psi dd^cT \leq C(a-t_0) = C(b-a), \]
where we used the fact that \( \min_{X} \psi = t_0 - a = -(b-a). \) This implies that
\[ |J_2| \leq C(b-a). \]

Again, using the Stokes theorem twice, we get that
\[ 2J_3 = \int_{\{0 < \psi < b-a\}} d\psi^2 \wedge d^cT = \int_{\{0 < \psi < b-a\}} d(\psi^2 d^cT) - \int_{\{0 < \psi < b-a\}} \psi^2 dd^cT = (b-a)^2 \int_{\{\psi = b-a\}} d^cT - \int_{\{0 < \psi < b-a\}} \psi^2 dd^cT = (b-a)^2 \int_{\{\psi < b-a\}} dd^cT - \int_{\{0 < \psi < b-a\}} \psi^2 dd^cT. \]

It follows that
\[ |J_3| \leq C(b-a)^2. \]

Combining the inequalities (3.15), (3.18), (3.19), (3.20) and (3.21) one obtains
\[ \int_{\{0 < \psi < b-a\}} d\psi \wedge d^c \psi \wedge \omega_u^{n-1} \leq C \left[ 2\varepsilon(b-a) + (b-a)^2 \right] \leq 4C(b-a)^2, \]
where \( 0 < \varepsilon \leq b-a \) is used in the second inequality. This is the desired upper bound. It remains to estimate from below the quantity
\[ I_u(a, b) := \int_{\{0 < \psi < b-a\}} \omega_u \wedge \omega_u^{n-1} \cdot \int_{\{0 < \psi < b-a\}} d\psi \wedge d^c \psi \wedge \omega_u^{n-1}. \]

Since \( u, v \in PSH(\omega) \cap C(X) \subset W^{1,2}(X) \), we have \( \psi \in W^{1,2}(X) \). Applying Lemma 2.6 for \( \psi, d = b-a \) and \( \delta \) we get that
\[ \int_{\{0 < \psi < b-a\}} |\partial \psi|^{\omega} \geq C(b-a)\delta^{\frac{4n-1}{4n}}. \]
Lemma 3.6. We have
\[ I_u(a, b) \geq \frac{c_0}{n^2} \left( \int_{\{0 < \psi < b-a\}} |\partial \psi| \omega^n \right)^2 \]
where
\[ |\partial \psi|^2 = \sum g_k \bar{l} \partial_k \psi \partial_l \psi, \quad \omega = \frac{i}{2} \sum g_k dz_k \land d\bar{z}_l \]
in a local coordinate chart.

Proof. As \( \psi \in C(X) \cap W^{1,2}(X) \), there exist \( \psi_j \in C^\infty(X) \) such that \( \psi_j \to \psi \) uniformly and \( \psi_j \to \psi \) in \( W^{1,2}(X) \). Therefore, upon applying the convergence theorem [BT82], [DK12], we may assume \( \psi \) is smooth. Then, the inequality is a consequence of the Cauchy-Schwarz inequality and the elementary point-wise inequality
\[ \frac{\omega_u \land \omega^{n-1}}{\omega^n}, \frac{\omega^{n-1} \land d\psi \land d^c\psi}{\omega^n} \geq \frac{1}{n} \frac{\omega_u^n}{\omega^n} \frac{d\psi \land d^c\psi \land \omega^{n-1}}{\omega^n} \geq \frac{c_0}{n^2} |\partial \psi|^2. \]
The first inequality can be checked by writing it down in normal coordinates at any given point. This is the sole place where we need to use the assumption on smoothness of \( f \) and \( u \). The approximation process would not work for the proof of the lemma because up to this point the uniqueness of continuous solutions is not yet asserted.

Thanks to (3.23) and Lemma 3.6, we get that
\[ (3.24) \quad I_u(a, b) \geq C c_0 (b - a)^2 \delta^{\frac{4n-1}{n}}. \]
This is the lower bound we needed. Combining the upper bound (3.22) and the lower bound (3.24), we obtain that
\[ \int_{\{0 < \psi < b-a\}} \omega_u \land \omega^{n-1} \geq \frac{Cc_0(b-a)^2 \delta^{\frac{4n-1}{n}}}{4C(b-a)^2} \]
\[ \geq Cc_0 \delta^{\frac{4n-1}{n}} =: \delta_e > 0. \]
Going back to the original notation \( \psi = \phi + a \) we get the desired inequality
\[ \int_{\{a < \phi < b\}} \omega_u \land \omega^{n-1} \geq \delta_e. \]
The lemma is proven.

Let us observe that from Lemma 3.5 and its proof, by the symmetry with respect to \( u \) and \( v \), one obtains also the following statement.

Remark 3.7. Let \( \hat{t}_0 = \max_X \phi \). Let \( a < b < \hat{t}_0 \) be two real numbers such that
\[ \varepsilon \leq b - a, \quad \text{and} \quad \hat{t}_0 - b = b - a. \]
Assume that
\[ \text{Vol}_\omega(\{\phi < a\}) \geq \delta, \quad \text{Vol}_\omega(\{\phi \geq b\}) \geq \delta \]
for some \( \delta > 0 \). Then,
\[ \int_{\{a < \phi < b\}} \omega_u \land \omega^{n-1} \geq \delta_e > 0 \]
where $\delta_e$ as in Theorem 3.7 depends only on $\delta, c_0, X, \omega$.

Thanks to a priori estimates of the Laplacian mass on small collars we get the following estimate for a larger collar (there is a similar statement corresponding to Remark 3.7).

**Proposition 3.8.** Assume that $t_1 - t_0 \geq \varepsilon$ and $\text{Vol}_\omega(\Omega(t_1)) \geq \delta$. Define for $k \geq 2$ integer,

$$t_k = 2^{k-1}(t_1 - t_0) + t_0.$$

If $\text{Vol}_\omega(X \setminus \Omega(t_N)) \geq \delta, N \geq 1$, then we have

$$\int_{\{t_0 < \varphi \leq t_N\}} \omega_u \wedge \omega^{n-1} \geq (N - 1)\delta_e,$$

where $\delta_e = Cc_0 \delta^\frac{2n-1}{n}$.

**Proof.** Observe that $t_k - t_{k-1} = t_{k-1} - t_0 \geq \varepsilon$ for $k \geq 2$. The assumptions of Lemma 3.5 are satisfied for $a = t_{k-1}, b = t_k, k = 2, \ldots, N$. Then, we get that

$$\int_{\{t_0 < \varphi \leq t_N\}} \omega_u \wedge \omega^{n-1} \geq \sum_{k=2}^{N} \int_{\{t_{k-1} < \varphi \leq t_k\}} \omega_u \wedge \omega^{n-1} \geq (N - 1)\delta_e$$

where $\delta_e = Cc_0 \delta^\frac{2n-1}{n}$. \hfill \Box

We have completed the first and second part of the outline. Now by combining them we will finish the proof the stability theorem.

*End of proof of Theorem 3.1.* We have that $t_0 = \min_X \varphi, \|f - g\|_p \leq \varepsilon$.

Set $-\hat{t}_0 = \min_X (-\varphi) = -\max_X \varphi$

Since $\sup_X u = \sup_X v = 0$, we have

$$(3.25) \quad \hat{t}_0 = \max_X \varphi \geq 0, \quad t_0 = \min_X \varphi \leq 0.$$

So

$$(3.26) \quad \|\varphi\|_\infty \leq \hat{t}_0 - t_0.$$

Our goal is to prove $\|\varphi\|_\infty \lesssim \varepsilon^n$. We consider two possibilities:

**Case 1:**

$$(3.27) \quad \min \left\{ \int_{\{\varphi < t_0 + \varepsilon\}} f \omega^n, \int_{\{\varphi > \hat{t}_0 - \varepsilon\}} f \omega^n \right\} \geq \frac{V_{\text{min}}}{4}.$$

**Case 2:**

$$(3.28) \quad \min \left\{ \int_{\{\varphi < t_0 + \varepsilon\}} f \omega^n, \int_{\{\varphi > \hat{t}_0 - \varepsilon\}} f \omega^n \right\} < \frac{V_{\text{min}}}{4}.$$

In the first case it follows from (3.27) and Hölder’s inequality that

$$(3.29) \quad \min \{\text{Vol}_\omega(\{\varphi < t_0 + \varepsilon\}), \text{Vol}_\omega(\{\varphi > \hat{t}_0 - \varepsilon\})\} \geq \delta_1,$$

where

$$\delta_1 = \frac{(V_{\text{min}}/4)^p}{\|f\|_p^p} > 0.$$
Define $t_N = 2^{N-1} \varepsilon + t_0$ for an integer $N \geq 1$. By (3.20), if $N$ satisfies $\varepsilon(2^{N-1} + 1) < \|\varphi\|_\infty$, then

$$t_N < t_0 - \varepsilon.$$  

Hence, applying Proposition 3.8 for $t_1 = t_0 + \varepsilon$ and $t_N$ as above, we get that

$$\int_{\{t_0 < \varphi \leq t_N\}} \omega_u \wedge \omega^{n-1} \geq (N - 1) \delta_{1e},$$  

where $\delta_{1e} = Cc_0 \delta_1^{-\alpha \frac{m-1}{4}}$. Let $G \geq 0$ to be the Gauduchon function of $\omega$, i.e. $dd^c(e^G \omega^{n-1}) = 0$. By the Stokes theorem

$$\int_X \omega_u \wedge \omega^{n-1} \leq \int_X e^G \omega_u \wedge \omega^{n-1} = \int_X e^G \omega^n.$$  

Choose

$$N_1 = \frac{\int_X e^G \omega^n}{\delta_{1e}} + 2.$$  

Then, by (3.30) and (3.31) we have

$$\int_X e^G \omega^n > ([N_1] - 1) \delta_{1e} > \int_X e^G \omega^n,$$  

where $[N_1]$ is the integer part of $N_1$. This is not possible, so for such a choice of $N_1$

$$\|\varphi\|_\infty \leq \varepsilon(1 + 2^{N_1}).$$  

Thus, in the first case the statement follows.

We turn now to the second case and assume (3.28). By the symmetry of the estimate for the Laplacian mass in large collars (Remark 3.7), without loss of generality, we may assume that

$$\int_{\{\varphi < t_0 + \varepsilon\}} f \omega^n \leq V_{\min}/4.$$  

Choose $t_1$ to be the supremum over $t$ for which

$$\int_{\Omega(t)} f \omega^n < V_{\min}.$$  

By (3.34) and Lemma 3.4 we have

$$\varepsilon \leq t_1 - t_0 \leq C \varepsilon^\alpha.$$  

Choose $\hat{t}_1$ to be infimum over $t$ for which

$$\int_{X \setminus \Omega(t)} g \omega^n \leq V_{\min}.$$  

By Lemma 3.4 applied for $(-\varphi)$ we have

$$\|\varphi\|_\infty \leq \varepsilon(1 + 2^{N_1}).$$  

Thus, in the first case the statement follows.

We turn now to the second case and assume (3.28). By the symmetry of the estimate for the Laplacian mass in large collars (Remark 3.7), without loss of generality, we may assume that

$$\int_{\{\varphi < t_0 + \varepsilon\}} f \omega^n \leq V_{\min}/4.$$  

Choose $t_1$ to be the supremum over $t$ for which

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By (3.34) and Lemma 3.4 we have

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Choose $\hat{t}_1$ to be infimum over $t$ for which

$$\int_{X \setminus \Omega(t)} g \omega^n \leq V_{\min}.$$  

By Lemma 3.4 applied for $(-\varphi)$ we have

$$\|\varphi\|_\infty \leq \varepsilon(1 + 2^{N_1}).$$  

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We turn now to the second case and assume (3.28). By the symmetry of the estimate for the Laplacian mass in large collars (Remark 3.7), without loss of generality, we may assume that

$$\int_{\{\varphi < t_0 + \varepsilon\}} f \omega^n \leq V_{\min}/4.$$  

Choose $t_1$ to be the supremum over $t$ for which

$$\int_{\Omega(t)} f \omega^n < V_{\min}.$$  

By (3.34) and Lemma 3.4 we have

$$\varepsilon \leq t_1 - t_0 \leq C \varepsilon^\alpha.$$  

Choose $\hat{t}_1$ to be infimum over $t$ for which

$$\int_{X \setminus \Omega(t)} g \omega^n \leq V_{\min}.$$  

By Lemma 3.4 applied for $(-\varphi)$ we have

$$\|\varphi\|_\infty \leq \varepsilon(1 + 2^{N_1}).$$  

Thus, in the first case the statement follows.
As in Proposition 3.8 we define

\begin{equation}
\delta_{2e} = C_0 \delta_2^{4n-1}
\end{equation}

and

\begin{equation}
t_N = 2^{N-1}(t_1 - t_0) + t_0.
\end{equation}

Take \( N \) to be the smallest integer which is larger than

\begin{equation}
N_2 := \frac{\int_X e^G \omega^n}{\delta_{2e}} + 2.
\end{equation}

If we had \( Vol_\omega(X \setminus \Omega(t_N)) \geq \delta_2 \), then Proposition 3.8 would lead to a contradiction

\[
\int_{\{t_0 < \varphi \leq t_N\}} \omega_u \wedge \omega^{n-1} \geq (N - 1) \delta_{2e} \int_X e^G \omega^n \geq \int_X \omega_u \wedge \omega^{n-1}.
\]

Therefore,

\[
Vol_\omega(\{ \varphi \geq t_N \}) = Vol_\omega(X \setminus \Omega(t_N)) \leq \delta_2.
\]

By the definition of \( \delta_2 \) it follows that

\[
\int_{X \setminus \Omega(t_N)} g \omega^n \leq \|g\|_p [Vol_\omega(X \setminus \Omega(t_N))]^{\frac{1}{p^*}} \leq V_{\text{min}}.
\]

Thus,

\begin{equation}
t_N \geq \hat{t}_1.
\end{equation}

We are ready to finish the proof in the second case. By (3.38) we have

\begin{equation}
|\varphi| \leq \hat{t}_0 - t_0.
\end{equation}

Furthermore, from (3.37) and (3.39) it follows that

\begin{equation}
\hat{t}_0 - t_0 = (\hat{t}_0 - t_N) + (t_N - t_0) \leq (\hat{t}_0 - \hat{t}_1) + 2^{N-1}(t_1 - t_0).
\end{equation}

Combine (3.38), (3.39) and (3.41) to get

\[
\hat{t}_0 - t_0 \leq C \varepsilon^\alpha + 2^{N-1}C \varepsilon^\alpha = C(1 + 2^{N-1})\varepsilon^\alpha.
\]

It follows from this, (3.38) and (3.40) that

\begin{equation}
\|\varphi\|_\infty \leq C(1 + 2^{N_2})\varepsilon^\alpha.
\end{equation}

The proof in the second case is completed. Finally, the desired stability estimate follows from (3.38) and (3.42). \( \square \)

The above stability result gives the uniqueness of continuous solutions.

**Corollary 3.9.** Suppose that \( 0 < c_0 \leq f \in L^p(\omega^n), p > 1 \). Then there is a unique \( u \in \text{PSH}(\omega) \cap C(X), \sup_X u = 0 \), and unique \( c > 0 \) such that

\[
\omega_u^n = cf \omega^n.
\]

**Proof.** By Lemma 2.4 we have that \( c \) is uniquely defined. Hence we may assume \( c = 1 \). Take a smooth sequence \( f_j \) such that \( f_j \geq \frac{\omega^n}{C} \) and \( f_j \) converges to \( f \) in \( L^p(\omega^n) \), as \( j \to +\infty \). By the theorem of Tosatti and Weinkove [TW10b] there exists a unique \( u_j \in C^\infty(X) \), \( \sup_X u_j = 0 \), and a unique constant \( c_j > 0 \) such that

\[
(\omega + dd^c u_j)^n = c_j f_j \omega^n, \quad \omega + dd^c u_j > 0.
\]

We first observe that \( c_j \to 1 \) as \( j \to +\infty \). Indeed, it follows from [KN] that

\[
\frac{1}{C} \leq c_j \leq C,
\]
where $C = C(\|f\|_p, X, \omega) > 0$. Suppose that there existed a subsequence $c_k \to c \neq 1$ as $k \to +\infty$. Consider
\begin{equation}
(\omega + dd^c u_k)^n = c_k f_k \omega^n.
\end{equation}
Since the family $\{u_k \in PSH(\omega) \cap C(X) : \sup_X u_k = 0\}$ is relatively compact in $L^1(\omega^n)$, after passing to a subsequence, still writing $u_k$, we may assume that $\{u_k\}$ is a Cauchy sequence in $L^1(\omega^n)$. [KN Corollary 5.10] implies that $\{u_k\}_k$ is a Cauchy sequence in $C(X)$. Thus we can write
\begin{equation}
(\omega + dd_c w)^n = cf \omega^n = c \omega^n,
\end{equation}
where $w$ is the limit of $\{u_k\}$. This contradicts the uniqueness of the constant (Lemma [2.1]). Thus we write
\begin{equation}
(\omega + dd^c u_j)^n = F_j \omega^n,
\end{equation}
where $F_j \geq c\frac{\alpha}{3}$ and $F_j$ converges in $L^p(\omega^n)$ as $j \to +\infty$. By Theorem 3.1 we have
\[ ||u_j - u||_\infty \leq C \|F_j - f\|_p \]
for a fixed $0 < \alpha < \frac{1}{n+1}$. Since $u_j$ are unique, we infer that $u$ is also unique. \hfill \Box

**Remark 3.10.** Having Corollary [3.4], the statement of Theorem 3.1 holds without the smoothness assumption on $f$. Thus we obtain Theorem A. It follows from the approximation argument as in the proof of Corollary [3.9].

One can prove a variant of the stability theorem, where there is $L^1$ norm on the right hand side, but then the exponent is worse by factor $1/n$.

**Theorem 3.11.** Let $0 \leq f, g \in L^p(\omega^n)$, $p > 1$, be such that $\int_X f \omega^n > 0$, $\int_X g \omega^n > 0$. Consider two continuous $\omega$-psh solutions of the complex Monge-Ampère equation
\[ \omega_u^n = f \omega^n, \quad \omega_v^n = g \omega^n \]
with $\sup_X u = \sup_X v = 0$. Assume that
\[ f \geq c_0 > 0. \]
Fix $0 < \alpha < \frac{1}{2n(n+1)}$. Then, there exists $C = C(c_0, \alpha, \|f\|_p, \|g\|_p)$ such that
\[ ||u - v||_\infty \leq C \|f - g\|^\alpha_1. \]

**Remark 3.12.** It is, in fact, enough to assume $v \in PSH(\omega) \cap L^\infty(X)$ as we are not going to use the minimum principle (Proposition 2.2) in the proof. Therefore, as in Corollary [3.8], we get the uniqueness of bounded $\omega$-psh solutions for the right hand side in $L^p$ and positive.

**Proof.** The theorem will follow as soon as we prove the following version of Lemma 3.3. We use again notation:
\[ \varphi = u - v, \quad \Omega(t) = \{ \varphi < t\}, \quad t_0 = \inf_X \varphi. \]

**Lemma 3.13.** Let $V_{\text{min}} > 0$ be the constant in Proposition 2.4. Fix $t_1 > t_0$. Assume that
\[ \|f - g\|_1 \leq \varepsilon \]
for $0 < \varepsilon << 1$. If $\int_{\Omega(t)} f \omega^n \leq V_{\text{min}}$, then
\begin{equation}
t_1 - t_0 \leq C \varepsilon^\alpha,
\end{equation}
where $0 < \alpha < \frac{1}{2n(n+1)}$ is fixed.
Proof. Define the sets: \[ \Omega_1 := \{ z \in \Omega(t_1) : f \leq (1 + \varepsilon^\alpha)g \} \quad \text{and} \quad \Omega_2 := \Omega(t_1) \setminus \Omega_1. \]

Since \( g < \varepsilon - \alpha (f - g) \) on \( \Omega_2 \), we have

\[
\int_{\Omega_2} f \omega^n \leq \int_{\Omega_2} |f - g| \omega^n + \int_{\Omega_2} g \omega^n
\leq \varepsilon + \varepsilon^{1-\alpha} - 2\varepsilon^{1-\alpha}.
\]

It follows that

\[
\int_{\Omega(t_1)} f \omega^n = \int_{\Omega_1} f \omega^n + \int_{\Omega_2} f \omega^n \leq \int_{\Omega_1} f \omega^n + 2\varepsilon^{1-\alpha} \leq V_{\min} + 2\varepsilon^{1-\alpha}.
\]

We construct a barrier function which is a bit different than the one in Lemma 3.4. Put

\[
\hat{f}(z) = \begin{cases} f(z) & \text{for } z \in \Omega(t_1), \\ \frac{1}{A} f(z) & \text{for } z \in X \setminus \Omega(t_1). \end{cases}
\]

As \( \int_{\Omega(t_1)} f \omega^n \leq V_{\min} \) we can choose \( A > 1 \) large enough so that

\[
\int_X \hat{f} \omega^n \leq \frac{3}{2} V_{\min}.
\]

Notice that \( \| \hat{f} \|_p \leq \| f \|_p \). By [KN, Theorem 0.1] we find \( w \in PSH(\omega) \cap C(X) \) and \( \hat{c} > 0 \) satisfying

\[
(\omega + dd^c w)^n = \hat{c} \hat{f} \omega^n, \quad \sup_X w = 0.
\]

By Proposition 2.4 we have

\[
2^n \leq \hat{c} \leq A,
\]

where the last inequality follows from (3.46) and Lemma 2.1. Hence,

\[
\hat{f} \geq 2^n f \quad \text{on } \Omega(t_1).
\]

Define for \( 0 < s < 1 \),

\[ \psi_s = (1 - s)v + sw. \]

It follows from the mixed form type inequality (Proposition 2.2) that

\[
(\omega + dd^c \psi_s)^n \geq \left[ (1 - s)g^{1/2} + s(\hat{c} f/f) f \right]^n \omega^n
= [(1 - s)(g/f)^{1/2} + s(\hat{c} f/f)^{1/2}]^n f \omega^n
:= [b(s)]^n f \omega^n.
\]

Therefore, on \( \Omega_1 \), we have

\[
b(s) \geq \frac{(1 - s)}{(1 + \varepsilon^\alpha)^{1/2}} + 2s \geq \frac{1 - s}{1 + \varepsilon^\alpha} + 2s.
\]

If \( 2\varepsilon^\alpha \leq s \leq 1 \), then

\[
b(s) \geq 1 + \varepsilon^\alpha \quad \text{on } \Omega_1.
\]

Let us use the notation:

\[ m_s := \inf_X (u - \psi_s) = \inf_X \{ u - v + s(v - w) \}. \]
We have that
\[ m_s \leq t_0 + s\|w\|_{\infty}. \]
Set for \(0 < \tau < 1\),
\[ m_s(\tau) := \inf_{X} [u - (1 - \tau)\psi_s]. \]
Then
\[ m_s(\tau) \leq m_s. \]
By the above definitions we have
\[ U(\tau, t) := \{u < (1 - \tau)\psi_s + m_s(\tau) + t\} \]
\[ \subset \{u < \psi_s + m_s + \tau\|\psi_s\|_{\infty} + t\} \]
\[ \subset \{u < v + t_0 + s(\|v\|_{\infty} + \|w\|_{\infty}) + \tau\|\psi_s\|_{\infty} + t\}. \]
We are going to show that
\[ t_1 - t_0 \leq 2s(\|v\|_{\infty} + \|w\|_{\infty}) + \tau\|\psi_s\|_{\infty}, \]
for \(s = 2\varepsilon^\alpha\) and \(\tau = \varepsilon^\alpha / 2\). Suppose it is false. By (3.50) we have
\[ U(\tau, t) \subset \{u < v + t_0 + (t_1 - t_0)\} = \Omega(t_1), \]
for \(0 < t < \frac{t_1 - t_0}{2}\). To go further we need to estimate the integrals:
\[ \int_{U(\tau, t)} f\omega^n \]
for \(0 < t << s, \tau\). By the modified comparison principle [KN, Theorem 0.2]
\[ \int_{U(\tau, t)} \omega^n_{(1-\tau)\psi_s} \leq \left(1 + \frac{Ct}{\tau^n}\right) \int_{U(\tau, t)} \omega^n_u, \]
for every \(0 < t < \min\{\frac{\varepsilon^3}{12}, \frac{t_1 - t_0}{2}\}\). Hence, a simple estimate from below gives
\[ (1 - \tau)^n \int_{U(\tau, t)} \omega^n_{\psi_s} \leq \left(1 + \frac{Ct}{\tau^n}\right) \int_{U(\tau, t)} \omega^n_u. \]
Using (3.49) for \(s = 2\varepsilon^\alpha\) we get
\[ (1 - \tau)^n(1 + \varepsilon^\alpha)^n \int_{U(\tau, t) \cap \Omega_1} f\omega^n \leq \left(1 + \frac{Ct}{\tau^n}\right) \int_{U(\tau, t)} f\omega^n. \]
If we write \(a(\varepsilon, \tau) = (1 - \tau)^n(1 + \varepsilon^\alpha)^n\), then
\[ a(\varepsilon, \tau) \geq (1 + \varepsilon^\alpha / 2 - \varepsilon^{2\alpha} / 2)^n > 1 + \varepsilon^\alpha / 4 \]
as we have \(\tau = \varepsilon^\alpha / 2\) and \(0 < \varepsilon^\alpha < 1 / 4\). Therefore, (3.52) implies that
\[ \left[a(\varepsilon, \tau) - \left(1 + \frac{2^n C t}{\varepsilon^\alpha}\right)\right] \int_{U(\tau, t) \cap \Omega_1} f\omega^n \leq \left(1 + \frac{2^n C t}{\varepsilon^\alpha}\right) \int_{\Omega_2} f\omega^n. \]
Thus, for \(0 < t \leq \varepsilon^{(n+1)\alpha} / 2^{n+3} C\),
\[ \frac{\varepsilon^\alpha}{8} \int_{U(\tau, t) \cap \Omega_1} f\omega^n \leq 2 \int_{\Omega_2} f\omega^n \leq 4\varepsilon^{1 - \alpha}, \]
where the last inequality used (3.45). Hence,
\[ \int_{U(\tau, t) \cap \Omega_1} f\omega^n \leq 32\varepsilon^{1 - 2\alpha}. \]
Altogether we get that for $0 < t \leq \varepsilon (n+1)\alpha / C$,
\begin{equation}
\int_{U(\tau, t)} f \omega^n \leq \int_{U(\tau, t) \cap \Omega_1} f \omega^n + \int_{\Omega_2} f \omega^n \leq C \varepsilon^{1-2\alpha}.
\end{equation}
This is the estimate we need. Now we are able make use of the results in [KN].
First, it follows from [KN, Theorem 5.3, Remark 5.5] that
\begin{equation}
\mathcal{h}(t/2) \leq \text{cap}_\omega(U(\tau, t/2)) \leq 2^n C t^n \int_{U(\tau, t)} f \omega^n
\end{equation}
for $0 < t \leq \varepsilon (n+1)\alpha / C$, where $\mathcal{h}(t)$ is the inverse of $\kappa(t)$ (see Proposition 2.4). It follows from (3.53) and (3.54) that
\[ \mathcal{h}(t) \leq \varepsilon^{1-2\alpha} t^n. \]
Then, take $t = \varepsilon (n+1)\alpha / C$ we obtain that
\begin{equation}
\mathcal{h}(\varepsilon (n+1)\alpha / C) \leq C \varepsilon^{1-2\alpha-n(n+1)\alpha} = : C \varepsilon^{\delta},
\end{equation}
where $\delta = 1 - [n(n+1) + 2|\alpha > 0$. However, we have that
\[ \mathcal{h}(t) \geq \left( \frac{1}{\alpha} \log \frac{C}{t} \right)^{-n} \]
for $0 < t < 1$ (see [N15, Eq. (3.23)]). As $\delta > 0$ is fixed, this leads to a contradiction in the inequality (3.55) for $\varepsilon > 0$ small enough. Thus we have proved that
\[ t_1 - t_0 \leq 4\varepsilon^{\alpha} (\|v\|_\infty + \|w\|_\infty + \|\psi_s\|_\infty), \]
for a fixed $0 < \alpha < \frac{1}{2+n(n+1)}$. The norms on the right hand side are controlled by $\|f\|_p, \|g\|_p, A, V\min$. So the lemma follows.

The rest of the proof of Theorem 3.11 is analogous to that of Theorem 3.1. Notice that the proof of Lemma 3.6, without the smoothness assumptions on $f, u$, follows by the approximation argument (as in Corollary 3.9) because the uniqueness of continuous solutions has been proven for $u$. Again, by the approximation argument, if $v \in PSH(\omega) \cap L^\infty(X)$ then it is enough to get the inequality (3.24) (see [BT87, Theorem 3.2]).

4. Hölder continuity

In this section we prove the Hölder continuity of solutions of the complex Monge-Ampère equation
\[ \omega^n_u = f \omega^n, \]
when $f \in L^p(X, \omega)$, $p > 1$ and $f > 0$. The Hölder exponent is explicitly given in terms of $p$ and the dimension $n$. In the Kähler case, with $f$ only nonnegative, the result is due to the first author [Ko08] (with exponent depending on the manifold), and to Demailly, Dinew, Guedj, Diep, Kołodziej, Zeriahi [DDGHKZ] in full generality. The new ingredient in the latter proof was the application of Demailly’s regularization method for $\omega$-psh functions, which uses the Riemann exponential map (De82) or the holomorphic part of this map (De94) - suitable for non-Kähler manifolds. We shall follow this scheme with necessary modifications. Namely, to apply Theorem A for a small perturbation of $f$ we need one more lemma (Lemma 4.3), since the perturbation defined as on Kähler manifold is, generically,
not MA-admissible. This makes the Hölder exponent worse, by factor $1/n$. Furthermore, the comparison principle used in the Kähler case is not available here, so we change the proof to apply just the minimum principle.

Following [De94] consider $\rho_\delta u$ the regularization of the $\omega$-psh function $u$ defined by

$$\rho_\delta u(z) = \frac{1}{2^n} \int_{\mathbb{T} \times X} u(\exp h_z(\zeta)) \rho\left(\frac{|\zeta|^2}{\delta^2}\right) dV(\zeta), \ \delta > 0;$$

where $\zeta \to \exp h_z(\zeta)$ is the (formal) holomorphic part of the Taylor expansion of the exponential map of the Chern connection on the tangent bundle of $X$ associated to $\omega$, and the smoothing kernel $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$\rho(t) = \begin{cases} \eta \left(1 - \frac{t}{\delta}\right)^2 \exp\left(\frac{1}{t - 1}\right) & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t > 1 \end{cases}$$

with a suitable constant $\eta$, such that

$$\int_{\mathbb{C}^n} \rho(\|z\|^2) dV(z) = 1$$

($dV$ being the Lebesgue measure in $\mathbb{C}^n$).

Of crucial importance is the following lemma from [De94, Proposition 3.8], and [BD12, Lemma 1.12].

**Lemma 4.1.** Fix any bounded $\omega$-psh function $u$ on a compact Hermitian manifold $(X, \omega)$. Define the Kiselman-Legendre transform with level $c$ by

$$U_{\delta, c} = \inf_{t \in [0, \delta]} (\rho_t u + Kt^2 - c \log \frac{t}{\delta} - K\delta^2),$$

Then for some positive constant $K$ depending on the curvature, the function $\rho_t u + Kt^2$ is increasing in $t$ and the following estimate holds:

$$\omega + dd^c U_{\delta} \geq -(Ac + K\delta^2) \omega,$$

where $A$ is a lower bound of the negative part of the Chern curvature of $\omega$.

We also need a lemma from [DDGHKZ] (which is actually stated for Kähler manifolds, but the same proof works for Hermitian ones after replacing the Riemannian tensor by the Chern curvature tensor used in [De94]) saying that for some constant $C$ depending on $\omega$ and $\|u\|_\infty$

$$\int_X \frac{\rho_t u - u}{t^2} \omega^n < C,$$

for any $\omega$-psh function $u$ and $t$ small enough.

**Theorem 4.2.** Consider the solution $u$ of the complex Monge-Ampère equation

$$\omega_n^u = f \omega^n,$$

on $(X, \omega)$ a compact Hermitian manifold with $f > c_0 > 0, \|f\|_p < \infty$. Then for any $\alpha < \frac{2}{p \cdot n(n+1)+1}$ the function $u$ is Hölder continuous, with Hölder exponent $\alpha$.

**Proof.** As explained in [Ko08] and [DDGHKZ] the result follows as soon as we show that

$$\rho_t u - u \leq ct^\alpha$$

for $t$ small enough.
Lemma 4.3. For $p > q > 1$ set $r = \frac{npq}{npq + p - q}$. Suppose $0 \leq f \in L^p(X)$ is MA-admissible and for a Borel set $E$ and sufficiently small positive $\delta$ one has $\int_E f^q \omega^n < \delta^q$ and $\int_E f^r \omega^n < \delta^q \frac{2(p-r)}{p}$. Consider a MA-admissible perturbation of $f$

\[(4.6)\]
\[g(z) = \begin{cases} 
0 & \text{for } z \in E \\
(1 + s) f(z) & \text{for } z \in X \setminus E.
\end{cases}\]

Then $\|f - g\|_r < C \delta^{1/n}$ for some $C > 0$ independent of $\delta$.

Proof. By Lemma 2.1 it is clear that $s \geq 0$. First, we are going to show that $s \leq \frac{C}{n} \delta^{1/n}$ with $C > 0$ independent of $\delta$. Indeed, we define for $N > 1$:

\[(4.7)\]
\[h(z) = \begin{cases} 
\delta^{-1} V_{\text{min}} f(z) & \text{for } z \in E \\
\frac{1}{N} f(z) & \text{for } z \in X \setminus E,
\end{cases}\]

where $V_{\text{min}}$ (defined in Proposition 2.4) corresponds to the norm of $f$ in $L^q(\omega^n)$. Using our assumptions and $\text{Vol}_\omega(X) = 1$ we get that

$$\int_X h \omega^n \leq V_{\text{min}} + \frac{1}{N} \|f\|_1,$$  
$$\int_X h^q \omega^n \leq V_{\text{min}}^q + \frac{1}{N^q} \|f\|_q^q.$$

Let $w \in PSH(\omega) \cap C(X)$, $c > 0$ solve $\omega_w^n = c h \omega^n$.

By Proposition 2.4 we have $c \geq 2^n$ for $N > 1$ large enough. Since $g$ is MA-admissible, there exists $v \in PSH(\omega) \cap C(X)$ such that $\omega_v^n = g \omega^n$.

Set for $0 < t < 1$,

$$\psi = (1 - t)v + tw.$$

By the mixed form type inequality

$$(\omega_\psi^n)^{1/n} \geq \frac{1}{1-t} g^{1/n} + t c^{1/n} h^{1/n}.$$

The right hand side exceeds

$$2t V_{\text{min}}^{1/n} \delta^{-1/n} f^{1/n}$$

on $E$ and

$$(1-t)(1+s)^{1/n} f^{1/n}$$
on $X \setminus E$. Thus, $(\omega_\psi^n)^{1/n}$ is bigger than $f^{1/n}$ on $X$ for

\[(4.8)\]
\[t \geq \delta^{1/n} V_{\text{min}}^{-1/n} \text{ and } (1-t)(1+s)^{1/n} > 1.
\]

The latter inequality holds if

$$t < \frac{(1+s)^{1/n} - 1}{(1+s)^{1/n}} = \frac{s}{\sum_{k=1}^n (1+s)^{k/n}}.$$

Note that $0 \leq s \leq C(\|f\|_p, X, \omega)$. Then, $t = s/C$ satisfies the second inequality in (4.8) for $C > 0$ large enough (independent of $\delta$). Therefore, we have

\[(4.9)\]
\[s < C \delta^{1/n} V_{\text{min}}^{-1/n},
\]
otherwise the inequalities in (4.8) for \( t = s/C \) would contradict Lemma 2.1. Hence (4.9) implies that
\[
\|f - g\|_r \leq \left( \int_E f^r \omega^n + s^{r} \|f\|_r \right) \leq C \max(\delta^{\frac{\alpha(q-p)}{r}}, \delta^{r/n})
\]
and by our choice of \( r \) the exponents on the right hand side are equal. Therefore \( \|f - g\|_r \leq C \delta^{1/n} \), which gives the statement. \( \square \)

Suppose that \( A - 1 > 0 \) is a bound for the curvature of \((X, \omega)\) from the statement of Lemma 4.1. By [KN] \( \rho \) is continuous, so assume that \( \min_X u = 1 \) and put \( b := 2 \max_X u \) and \( \theta := e^{-5Ab} \).

Fix \( \alpha < \frac{2}{p^* n(n+1)+1} \) and note that it is equivalent to (4.10)
\[
\alpha < \frac{2 - \alpha}{p^* n(n+1)}.
\]
Therefore one can choose \( q > 1 \) so close to 1 that the following inequality holds (4.11)
\[
\alpha < \frac{(2 - \alpha)(p - q)}{pq n(n+1)}.
\]

Let us set for \( \delta > 0 \):
\[
E(\delta) := \{(\rho_\delta u - u)(z) \geq Ab\delta^\alpha\}.
\]
By the definition of the Kiselman-Legendre transform at level \( \delta^\alpha \) (see (4.3))
\[
U_\delta = \inf_{t \in [0, \delta]} (\rho_t u + Kt^2 - \delta^\alpha \log \frac{t}{\delta} - K\delta^2),
\]
where \( K \) is chosen as in the formula (4.3). Recall from Lemma 4.1 that the functions \( \rho_\delta u + K\delta^2 \) are increasing in \( \delta \). By the same lemma
\[
\omega + dd^c U_\delta \geq -[(A - 1)\delta^\alpha + K\delta^2]\omega,
\]
for \( 0 < \delta < \delta_0 \), where \( \delta_0 > 0 \) is small enough. (We can also assume that \( A\delta_0^\alpha < 1, \delta_0^{2-\alpha} < Ab/K \), which will be used later.) Therefore
\[
U_\delta := \frac{1}{1 + A\delta^\alpha} U_\delta
\]
is \( \omega \)-psh on \( X \) and satisfies
\[
\omega + dd^c u_\delta \geq \frac{1}{2} \delta^\alpha \omega.
\]
Note that the definitions of \( U_\delta \) and \( \theta \) lead to
\[
\inf_{t \in [0, \delta]} (\rho_t u + Kt^2 - \delta^\alpha \log \frac{t}{\delta} - K\delta^2) \geq u - K\delta^2 + 5Ab\delta^\alpha,
\]
and therefore
\[
U_\delta \geq \min(\inf_{t \in [0, \delta]} (\rho_t u + Kt^2 - \delta^\alpha \log \frac{t}{\delta} - K\delta^2), u - K\delta^2 + 5Ab\delta^\alpha).
\]
By monotonicity of \( \rho_\delta u + K\delta^2 \) one infers
\[
U_\delta \geq \min(\rho_\theta u - K\theta^2, u - K\delta^2 + 5Ab\delta^\alpha).
\]

Thus, on the set
\[
F(\delta) = \{\rho_\theta u - u \geq 5Ab\delta^\alpha\}
\]
we have (4.13)
\[
U_\delta - u \geq 5Ab\delta^\alpha.
\]
To prove Hölder continuity of $u$ with the exponent $\alpha$ it is enough to show that $F(\delta)$ is empty for $\delta$ small enough. Reasoning by contradiction, we assume that $F(\delta) \neq \emptyset$. From (4.5) we have

$$\int_X |\rho \delta u - u| \omega^2 \leq c_1 \delta^2,$$

for $0 < \delta < \delta_0$ (decreasing $\delta_0$ if needed). Therefore

$$\int_{E(\delta)} \omega^2 \leq \frac{c_1}{Ab} \delta^{2-\alpha}.$$

Hence, by Hölder’s inequality,

$$\int_{E(\delta)} f^q \omega^2 < c_2 \delta^{(2-\alpha)(p-q)/p},$$

and

$$\int_{E(\delta)} f^r \omega^2 < c_2 \delta^{(2-\alpha)(p-r)/p},$$

for $r < p$. Let us define $g$ as in Lemma 4.3 with an open set $E$ containing $E(\delta)$ and such that

$$\int_E f \omega^2 < c_2 \delta^{(2-\alpha)(p-q)/p}$$

and

$$\int_E f^r \omega^2 < c_2 \delta^{(2-\alpha)(p-r)/p}.$$

Then Lemma 4.3 implies $\|f - g\|_r \leq c_3 \delta^{(2-\alpha)(p-q)/npq}$. Our stability theorem (and (4.11)) give, for $v$ defined by the equations

$$(\omega + dd^c v)^n = g \omega^n, \quad \max(u - v) = \max(v - u),$$

that

$$\|u - v\|_\infty \leq c_4 \delta^{\alpha'},$$

for some $\alpha' > \alpha$. The constants $c_j$ are independent of $\delta$. Decreasing $\delta_0$, we finally obtain

$$\|u - v\|_\infty \leq \frac{Ab}{2} \delta^\alpha, \quad \delta < \delta_0.$$

Observe that the choice of $b$ gives

$$U_\delta - u_\delta \leq \frac{Ab}{2} \delta^\alpha.$$

This inequality combined with (4.13) and (4.15), for $z \in F(\delta)$, lead to

$$(u_\delta - v)(z) = [(u_\delta - U_\delta) + (U_\delta - u) + (u - v)](z) \geq 3Ab \delta^\alpha.$$

On the other hand, if $g(z) > 0$, then $z \notin E$ and therefore $(U_\delta - u)(z) < Ab \delta^\alpha$. Again, applying (4.11), we obtain

$$(u_\delta - v)(z) \leq 2Ab \delta^\alpha.$$

The last two estimates prove that $\max(u_\delta - v)$ is attained within the open set $E$. However on this set $\omega^2_v = 0 < \omega^2_u$, which contradicts the minimum principle (Proposition 2.5). The theorem thus follows. $\square$
Székelyhidi and Tosatti [SZTo11] considered a weak solution to the equation
\[(\omega + dd^c u)^n = e^{-F(u,z)}\omega^n\]
on a compact \(n\)-dimensional Kähler manifold \((X,\omega)\). They proved that if \(F(x, z) \in C^\infty(\mathbb{R} \times X)\) and \(u \in PSH(\omega) \cap L^\infty(X)\), then \(u\) is smooth. An interesting corollary to this result says that if \(X\) is a Fano manifold and \(\omega\) represents the first Chern class, then setting \(F(u, z) = u - h\) with \(h\) satisfying \(dd^c h = Ric(\omega) - \omega\), one concludes that a Kähler-Einstein current with bounded potentials is smooth. Nie [Nie13] recently generalised the result of [SZTo11] proving the same for \((X,\omega)\) a compact Hermitian manifold satisfying
\[(5.1) \int_X \omega^n = \int_X (\omega + dd^c u)^n \quad \forall u \in PSH(\omega) \cap C^\infty(X) .\]
This is a restrictive assumption, but as remarked by Nie [Nie 13, Remark 4.1], the only missing ingredient to remove (5.1) is the stability theorem for the Monge-Ampère equation on compact Hermitian manifolds. Thanks to higher order estimates in [Nie13] and our results in Section 3 we obtain the proof of the Székelyhidi-Tosatti theorem on any compact Hermitian manifold.

**Theorem 5.1.** Let \((X,\omega)\) be a compact \(n\)-dimensional Hermitian manifold. Suppose that \(u \in PSH(\omega) \cap L^\infty(X)\) is a solution of the equation
\[(\omega + dd^c u)^n = e^{-F(u,z)}\omega^n\]
in the weak sense of currents, where \(F : \mathbb{R} \times X \to \mathbb{R}\) is smooth. Then \(u\) is smooth.

**Proof.** Without loss of generality we may assume that \(\sup_X u = 0\). As \(u\) is bounded the right hand side is bounded and strictly positive. Hence, by Remark 3.12 results in Sections 3 and 4 we know that \(u\) is Hölder continuous because it coincides with the unique continuous \(\omega\)-psh solution to the equation
\[(\omega + dd^c u)^n = e^{-F(u,z)}\omega^n, \quad \sup_X u = 0.\]
As stated in [Nie13] Remark 4.1] the argument in this paper gives the theorem as soon as the following stability estimate is proven.

**Lemma 5.2.** Let \(u_k\) be smooth functions such that
\[\lim_{k \to +\infty} \|u - u_k\|_\infty = 0.\]
By [TW10b] there exists a unique \(\psi_k \in PSH(\omega) \cap C^\infty(X)\) and a unique \(c_k > 0\) solving
\[(\omega + dd^c \psi_k)^n = c_k e^{-F(u_k, z)}\omega^n, \quad \sup_X \psi_k = 0.\]
Then,
\[\lim_{k \to +\infty} c_k = 1 \quad \text{and} \quad \lim_{k \to +\infty} \|\psi_k - u\|_\infty = 0.\]

**Proof.** Observe that \(e^{-F(u_k, z)}\) converges uniformly to \(e^{-F(u, z)}\). Therefore, the first assertion follows from the proof of Corollary 3.9. Thus, we have \(\lim_{k \to +\infty} c_k = 1\). It again implies that \(c_k e^{-F(u_k, z)} \in C^\infty(X)\) converges uniformly to \(e^{-F(u, z)}\). Therefore the second assertion follows from Theorem 3.1. \(\Box\)
Having this lemma the rest of the proof follows the lines of the proof in [Nie13, Sec. 4]. □

REFERENCES

[BT76] E. Bedford and B. A. Taylor, The Dirichlet problem for a complex Monge-Ampère operator. Invent. math. 37 (1976), 1–44.

[BT82] E. Bedford and B. A. Taylor, A new capacity for plurisubharmonic functions. Acta Math. 149 (1982), 1–40.

[BT87] E. Bedford and B. A. Taylor, Fine topology, Šilov boundary, and $(dd^c)^n$, J. Funct. Anal. 72 (1987), no. 2, 225–251.

[BD12] R. Berman and J.-P. Demailly, Regularity of plurisubharmonic upper envelopes in big cohomology classes, Proceedings of the Symposium “Perspectives in Analysis, Geometry and Topology” in honor of Oleg Viro (Stockholm University, May 2008), Eds. I. Itenberg, B. Jöricke, M. Passare, Progress in Math. 296, Birkhäuser/Springer, Boston (2012) 39–66.

[BEGZ] S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, Monge-Ampère equations in big cohomology classes, Acta Math. 205, (2010), 199–262.

[CDS15] X. X. Chen, S. Donaldson and S. Sun Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities, J. Amer. Math. Soc. 28 (2015), 183–197.

[Ch87] P. Cherrier, Équations de Monge-Ampère sur les variétés hermitiennes compactes. Bull. Sci. Math. 111 (2) (1987), 343–385.

[Chi13] I. Chiose, The Kähler rank of compact complex manifolds. preprint arXiv:1308.2043.

[De82] J.-P. Demailly, Estimations $L^2$ pour l’opérateur $\bar{\partial}$ d’un fibré vectoriel holomorphe semi-positif au-dessus d’une variété kählérienne complète, Ann. Sci. École Norm. Sup. 4e Sér. 15 (1982), 457-511.

[De94] J.-P. Demailly, Regularization of closed positive currents of type $(1,1)$ by the flow of a Chern connection, Aspects of Math. Vol. E26, Vieweg (1994), 105–126.

[DP04] J-P. Demailly and M. Paun, Numerical characterization of the Kähler cone of a compact Kähler manifold. Ann. of Math. (2) 159 (2004), no. 3, 1247-1274.

[DDGHKZ] J.-P. Demailly, S. Dinew, V. Guedj, P. Hiep, S. Kołodziej and A. Zeriahi, Hölder continuous solutions to Monge-Ampère equations, J. Eur. Math. Soc. (JEMS) 16 (2014), no. 4, 619–647.

[Din09] S. Dinew, An inequality for mixed Monge-Ampère measures. Math. Zeit. 262 (2009), 1–15.

[DK12] S. Dinew and S. Kołodziej, Pluripotential estimates on compact Hermitian manifolds. Adv. Lect. Math. (ALM), 21 (2012), International Press, Boston.

[DZh10] S. Dinew and Z. Zhang, On stability and continuity of bounded solutions of degenerate complex Monge-Ampère equations over compact Kähler manifolds. Adv. Math. 225 (2010), no. 1, 367–388.

[EGZ09] P. Eyssidieux, V. Guedj and A. Zeriahi, Singular Kähler-Einstein metrics. J. Amer. Math. Soc. 22 (2009), no. 3, 607-639.

[FY08] J.-X. Fu and S.-T. Yau, The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation. J. Diff. Geom. 98 (2008), 369–428.

[FWW10a] J. Fu, Z. Wang and D. Wu, Form-type Calabi-Yau equations, Math. Res. Lett. 17 (2010), no. 5, 887–903.

[FWW10b] J. Fu, Z. Wang and D. Wu, Form-type Calabi-Yau equations on Kähler manifolds of nonnegative orthogonal bisectional curvature, preprint, arXiv: 1010.2022.

[Gau77] P. Gauduchon, La théorème de l’excentricité nulle. C. R. Acad. Sci. Paris 285 (1977), 387–390.

[Gil11] M. Gill, Convergence of the parabolic complex Monge-Ampère equation on compact Hermitian manifolds. Com. Anal. Geom. 19 (2011), no. 2, 277–303.

[Gil13] M. Gill, The Chern-Ricci flow on smooth minimal models of general type. preprint arXiv: 1307.0060v1.

[GL10] B. Guan and Q. Li, Complex Monge-Ampère equations and totally real submanifolds. Adv. Math. 225 (2010), no. 3, 1185–1223.
S. Kołodziej, The Monge-Ampère equation on compact Kähler manifolds. Indiana Univ. Math. J. 52 (2003), no. 3, 667–686.

S. Kołodziej, The complex Monge-Ampère equation and pluripotential theory. Memoirs Amer. Math. Soc. 178 (2005), pp. 64.

S. Kołodziej, Hölder continuity of solutions to the complex Monge-Ampère equation with the right hand side in $L^p$. The case of compact Kähler manifolds. Math. Ann. 342 (2008), 379–386.

S. Kołodziej and N.-C. Nguyen, Weak solutions to the complex Monge-Ampère equation on Hermitian manifolds. preprint arXiv:1312.5491, to appear in Proceedings of the Conference in honor of Duong H. Phong. Contemp. Math. AMS.

S. Kołodziej and N.-C. Nguyen, The complex Monge-Ampère type equation on compact Hermitian manifolds and Applications, preprint, arXiv: 1501.00891.

X. Nie, Regularity of a complex Monge-Ampère equation on Hermitian manifolds. Comm. Anal. Geom. 22 (2014), no. 5, 833–856.

D. H. Phong, J. Song and J. Sturm, Complex Monge-Ampère equations. Surveys in geometric analysis. Vol. XVII, 327-410, Surv. Differ. Geom., 17, Int. Press, Boston, MA, 2012.

D. Popovici, Aeppli Cohomology Classes Associated with Gauduchon Metrics on Compact Complex Manifolds, preprint arXiv: 1310.3685.

J. Song, and G. Tian, The Kähler-Ricci flow on surfaces of positive Kodaira dimension. Invent. math. 170, (2007), 609–653.

J. Song and G. Tian, The Kähler-Ricci flow through singularities. arXiv:0904.4898

J. Song and G. Tian, Canonical measures and Kähler-Ricci flow, J. Amer. Math. Soc. 25 (2012), no. 2, 303–353.

G. Székelyhidi and V. Tosatti, Regularity of weak solutions of a complex Monge-Ampère equation, Anal. PDE 4 (2011), no. 3, 369–378.

G. Tian and Z. Zhang, On the Kähler-Ricci flow on projective manifolds of general type. Chinese Ann. Math. Ser. B 27 (2006), no. 2, 179–192.

V. Tosatti, Limits of Calabi-Yau metrics when the Kähler class degenerates, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 4, 755–776.

V. Tosatti, Adiabatic limits of Ricci-flat Kähler metrics, J. Differential Geom. 84 (2010), no. 2, 427–453.

V. Tosatti and B. Weinkove, Estimates for the complex Monge-Ampère equation on Hermitian and balanced manifolds. Asian J. Math. 14 (2010), no. 1, 19–40.

V. Tosatti and B. Weinkove, The complex Monge-Ampère equation on compact Hermitian manifolds. J. Amer. Math. Soc. 23 (2010), no. 4, 1187–1195.

V. Tosatti and B. Weinkove, On the evolution of a Hermitian metric by its Chern-Ricci form. preprint arXiv: 1201.0312, to appear in J. Differential Geom.

V. Tosatti and B. Weinkove, The Chern-Ricci flow on complex surfaces. Compos. Math. 149 (2013), no. 12, 2101–2138.

V. Tosatti and B. Weinkove, The Monge-Ampère equation for $(n-1)$- plurisubharmonic functions on a compact Kähler manifold, preprint arXiv: 1305.7511.

V. Tosatti and B. Weinkove and W. Yang, Collapsing of the Chern-Ricci flow on elliptic surfaces. preprint arXiv: 1302.6545, to appear in Math. Ann.

S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. Comm. Pure Appl. Math. 31 (1978), 339–411.

Faculty of Mathematics and Computer Science, Jagiellonian University 30-348 Kraków, Łojasiewicza 6, Poland;
e-mail: Slawomir.Kolodziej@im.uj.edu.pl

Faculty of Mathematics and Computer Science, Jagiellonian University 30-348 Kraków, Łojasiewicza 6, Poland;
e-mail: Nguyen.Ngoc.Cuong@im.uj.edu.pl