An efficient explicit approach for predicting the Covid-19 spreading with undetected infectious: The case of Cameroon

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Abstract. This paper considers an explicit numerical scheme for solving the mathematical model of the propagation of Covid-19 epidemic with undetected infectious cases. We analyze the stability and convergence rate of the new approach in $L^\infty$-norm. The proposed method is less time consuming. Furthermore, the method is stable, at least second-order convergent and can serve as a robust tool for the integration of general systems of ordinary differential equations. A wide set of numerical evidences which consider the case of Cameroon are presented and discussed.

Keywords: mathematical model of SARS-Cov-2 spreading, a two-level explicit scheme, stability analysis, convergence rate, numerical experiments.

AMS Subject Classification (MSC). 65M10, 65M05.

1 Introduction and motivation

Deterministic models are important decision tools that can be useful to forecasting different scenarios. The first motivation of studying such models is based on the use of the theory of ordinary/partial differential equations and a low computational complexity which can permit a better calibration of the model characteristics. Furthermore, deterministic approaches are the only suitable methods that can be used when modeling a new problem with few data. For more details, we refer the readers to [2, 30, 11, 49] and references therein. The use of the mathematical models as a predictive tool in the simulation of complex problems arising in a broad range of practical applications in biology, environmental fluid mechanic, chemistry and applied mathematics (for example: mathematical model in population biology and epidemiology, mixed Stokes-Darcy model, Navier-Stokes equations, nonlinear time-dependent reaction-diffusion problem, heat conduction equation and unsteady convection-diffusion-reaction equations) represents a good candidate for developing efficient numerical schemes in the approximate solutions of such problems [10, 34, 9, 10, 39, 38, 36, 4], [14, 32, 41, 45, 53]. For parabolic partial differential equations (PDEs) which present strong steep gradients (for instance: shallow water flow and advection-diffusion equations), numerical algorithms are needed with good resolution of steep gradients [12, 8, 20, 22, 31, 41, 47, 29].

Early in an epidemic, the quality of the data on infections, deaths, tests and other factors often are limited by undetection or inconsistent detention of cases, reporting delays, and poor documentation, all of which affect the quality of any model output. Simpler models may provide less valid predictions since they cannot capture complex and unobserved human mixing patterns and other time-varying parameters of infectious disease spread. Also, complex models may be no more reliable than simpler ones if they miss key aspects of the biological entities (either ions, molecules, proteins or cells) [1]. At a time when numbers of cases and deaths from coronavirus 2019 (Covid-19) pandemic continue to increase with alarming speed,
accurate forecasts from mathematical models are increasingly important for physicians, politicians, epidemiologists, the public and most importantly, for authorities responsible of organizing care for the populations they serve. Given the unpredictable behavior of severe acute respiratory syndrome Covid-19, it is worth mentioning that efficient numerical approaches are the best tools that can be used to predict the spread of the disease with reasonable accuracy. These predictions have crucial consequences regarding how quickly and strongly the government of a country mores to curb a pandemic. However, assuming the worst-case scenario at state and national levels will lead to inefficiencies (such as: the competition for beds and supplies) and may compromise effective delivery and quality of care, whereas supposing the best-case scenario can conduct to disastrous underpreparation.

Covid-19 is a rapidly spreading infectious disease caused by the novel coronavirus SARS-Cov-2, a betacoronavirus which has provided a global epidemic. Up today, no drug to treat the Covid-19 disease is officially available (approved by the World Health Organization (WHO)) and a vaccine will not be available for several months at the earliest. The only approaches widely used to slow the spread of the pandemic are those of classical epidemic control such as: physical distancing, contact tracing, hygiene measure, quarantine and case of isolation. However, the primary and most effective use of the epidemiologic models is to estimate the relative effect of various interventions in reducing disease burden rather than to produce precise quantitative predictions about extent on duration of disease burdens. Nevertheless, consumers of such models including the media, the publics and politicians sometimes focus on the quantitative predictions of infectious and mortality estimates. Such measures of potential disease burden are also necessary for planners who consider future outcomes in light of health care capacity. The big challenge consists to assess such estimates.

In this paper, we develop an efficient numerical scheme for solving a mathematical model well adapted to Covid-19 pandemic subjected to special characteristics (effect of undetected infected cases, effect of different sanitary and infectiousness conditions of hospitalized people and estimation of the needs of beds in hospitals) and considering different scenarios [19]. Specifically, the proposed technique should provide the numbers of detected infected and undetected infected cases, numbers of deaths and needs of beds in hospitals in countries (for example, in Cameroon) where Covid-19 is a very serious health problem. It is Worth noticing that the model of Covid-19 considered in this work has been obtained under the assumption of "only within-country disease spread" for territories with relevant number of people infected by SARS-Cov-2, where local transmission is the major cause of the disease spread (for instance: case of Cameroon). Furthermore, the parameters of the model used in this note are taken from the literature [24, 25, 19]. Our study also relates the disease fatality rate with the percentage of detected cases over the real total infected cases which allows to analyze the importance of this percentage on the impact of Covid-19. In addition, to demonstrate the efficiency and validity of the new approach when applied to the mathematical problem of coronavirus 2019 epidemic, we consider the case of Cameroon, the country of the central Africa where one can observe the highest number of people infected by the new virus SARS-Cov-2. We compare the results produced by the numerical method to the data obtained from this country and those provided by the World Health Organization in its reports [46]. Finally, it important to mention that the considered area (Cameroon) in the numerical experiments can be replaced by any territory worldwide.

This paper is organized as follows: Section 2 considers some preliminaries together with the mathematical formulation of Covid-19 spreading. In section 3 we provide a full description of the two-level explicit scheme for solving the problem indicated in section 2. Section 4 analyzes the stability and the convergence rate of the new procedure while a large set of numerical experiments are presented and critically discussed in Section 5. We draw in section 6 the general conclusion and we provided our future investigations.

2 Preliminaries and mathematical model of SARS-Cov-2 infectiousness

We use a mathematical formalism [19] that describes how infectiousness varies as a function of time since infections for a representative cohort of infected persons. We assume that transmission of SARS-Cov-2 is contagious from person to person and not point source. Furthermore, it is also assumed that, at the initial
phase of Covid-19 disease, the proportion of the population with immunity to SARS is negligible \[\text{[3,[16],4,[48]}\]. At the beginning of a contagious epidemic, a small number of infected people start passing the disease to a large population. Individuals can go through nine states. They start out susceptible (\(X_1\): the person is not infected by the disease pathogen), exposed (\(X_2\): the person is in the incubation period after being infected by the disease pathogen, but has no clinical signs), infected (\(X_3\): the person has finished the incubation period, may infected other people and start developing the clinical signs. Here, people can be taken in charge by sanitary authorities of this country (hospitalized persons) or not detected by the authorities and in quarantine at home (\(X_5\): the person is in hospital or in quarantine at home, can still infect other people, but will recover), hospitalized but will die (\(X_6\): the person is hospitalized and can infect other people, but will die), recovered after being previously infectious but undetected (\(X_8\): the person was not previously detected as infectious, survived the disease, is no longer infectious and has developed a natural immunity to the disease), recovered after being previously detected as infectious (\(X_7\): the person survived the disease, is no longer infectious and has developed a natural immunity to the virus, but she/he remains in hospital for a convalescence period \(d_0\) days), dead by SARS-Cov-2 (\(X_9\)).

The proposed model is based on thirteen parameters.

- \(R_0\) denotes basis reproductive number, that is, the expected number of new infectious cases per infectious case,
- \(N\) is the number of persons in a considered country before the starting of the pandemic,
- \(\mu_1 \in [0,1]\), designates the natality rate \((\text{day}^{-1})\) in the considered country (the number of births per day and per capita),
- \(\mu_2 \in [0,1]\), represents the mortality rate \((\text{day}^{-1})\) in the considered country (the number of deaths per day and per capita),
- \(w(t) \in [\underline{w}, \overline{w}] \subset [0,1]\), denotes the case fatality rate in the considered territory at time \(t\) (the proportion of deaths compared to the total number of infectious people (detected or undetected). Here, \(\underline{w}\) and \(\overline{w}\) are the minimum and maximum case fatality rates in the country, respectively),
- \(\theta(t) \in [\underline{\theta}, 1]\), means the fraction of infected people that are detected and reported by the authorities in the country at time \(t\). For the convenience of writing, we assume that all the deaths due to Covid-19 are detected and reported, so \(\theta(t) \geq \overline{\theta}\),
- \(\beta_{X_j} \in \mathbb{R}^+\), for \(j = 2, 3, \ldots, 6\), are the disease contact rates \((\text{day}^{-1})\) of a person in the corresponding compartment \(X_j\), in the country (without taking into account the control measures),
- \(\beta_{X_4}(\theta) \in \mathbb{R}^+\), represents the disease contact rates \((\text{day}^{-1})\) of a person in compartment \(X_4\), in the country (without taking into account the control measures), where the fraction of infected individuals that are detected is \(\theta(t)\),
- \(\gamma_{X_3} \in (0, \infty)\), designates the transition rate \((\text{day}^{-1})\) from compartment \(X_2\) to compartment \(X_3\). It’s the same in all the countries,
- \(\gamma_{X_3}(t) \in (0, \infty)\), is the transition rate \((\text{day}^{-1})\) from compartment \(X_3\) to compartments \(X_4\), \(X_5\) or \(X_6\) at time \(t\). It can change from a country to another,
- \(\gamma_{X_4}(t), \gamma_{X_3}(t)\) and \(\gamma_{X_4}(t) \in (0, \infty)\), denote the transition rate \((\text{day}^{-1})\) from compartments \(X_4\), \(X_5\) or \(X_6\) to compartments \(X_7\), \(X_8\) and \(X_9\), respectively, in the considered country at time \(t\),
- \(m_{X_j}(t) \in [0,1]\), for \(j = 2, 3, \ldots, 6\), are functions representing the efficiency of the control measures applied to the corresponding compartment in the considered country at time \(t\),
- \(\tau_1\) is the person infected that arrives in the territory from other countries per day. \(\tau_2\) is the person infected that leaves the territory from other countries per day. Both can be modeled following the between-country spread part of the Be-CoDis model, see \([18]\).
The control measures applied by the government to curb the Covid-19 spread are those provided by the WHO in [7, 17]:

- isolation: infected people are isolated from contact with other persons. Only sanitary professionals are in contact with them. Isolated patients receive an adequate medical treatment that reduces the Covid-19 fatality rate,
- quarantine: movement of people in the area of origin of an infected person is restricted and controlled (for instance: quick sanitary check-points at the airports) to avoid that possible infected people spread the disease,
- tracing: the aim of tracing is to identify potential infectious contacts which may have infected an individual or spread SARS-Cov-2 to other people. Increase the number of tests in order to increase the percentage of detected infected persons,
- increase of sanitary resources: number of operational beds and sanitary personal available to detect and treat affected people is increased, producing a decrease in the infectious period for the compartment $X_3$.

Furthermore, the mathematical model of coronavirus 2019 epidemic considers the following assumptions:

(a) the population at risk is large enough and time period of concern is short enough that over the time period of interest, very close to 100% of the population is susceptible,

(b) the pandemic is at the early stage and has not reached the point where the susceptible population decreases so much due to death or post-infection immunity that the average number of secondary cases falls,

(c) unprotected contact results in infection,

(d) the epidemic in the population of interest begins with a single host (note that the equations used in computing cases and deaths are easily modified if this is not the case),

(e) infectivity occurs during the incubation period only,

(f) the models are deterministic, that is, the thirteen parameters of Covid-19 spread cited above are constant values.

Under these assumptions, the mathematical formulation of Covid-19 disease is given by the following system of nonlinear ordinary differential equations:

$$\frac{dX_1}{dt} = -\frac{X_1}{N} \left[mX_2(t)\beta X_2(t)X_2 + mX_3(t)\beta X_3(t)X_3 + mX_4(t)\beta X_4(\theta)X_2 + mX_5(t)\beta X_5(t)X_5 + mX_6(t)\beta X_6(t)X_6\right]$$
$$- \mu_m X_1 + \mu_n [X_1 + X_2 + X_3 + X_4 + X_7 + X_8], \quad (1)$$

$$\frac{dX_2}{dt} = \frac{X_1}{N} \left[mX_2(t)\beta X_2(t)X_2 + mX_3(t)\beta X_3(t)X_3 + mX_4(t)\beta X_4(\theta)X_2 + mX_5(t)\beta X_5(t)X_5 + mX_6(t)\beta X_6(t)X_6\right]$$
$$- \mu_m X_2 - \gamma X_2(t)X_2 + \tau(t) - \tau_2(t), \quad (2)$$

$$\frac{dX_3}{dt} = \gamma X_2(t)X_2 - (\mu_m + \gamma X_3(t)) X_3, \quad \frac{dX_4}{dt} = (1 - \theta(t))\gamma X_3(t)X_3 - (\mu_m + \gamma X_4(t)) X_4, \quad (3)$$

$$\frac{dX_5}{dt} = (\theta(t) - w(t))\gamma X_3(t)X_3 - \gamma X_5(t)X_5, \quad \frac{dX_6}{dt} = w(t)\gamma X_3(t)X_3 - \gamma X_6(t)X_6, \quad (4)$$

$$\frac{dX_7}{dt} = \gamma X_5(t)X_5 - \mu_m X_7, \quad \frac{dX_8}{dt} = \gamma X_4(t)X_4 - \mu_m X_8 \quad \text{and} \quad \frac{dX_9}{dt} = \gamma X_6(t)X_6, \quad (5)$$

with the initial conditions

$$X_j(t_0) = X_j^0 \in (0, \infty), \quad \text{for} \quad j = 1, 2, \cdots, 9, \quad (6)$$
where all the unknowns $X_j$ dependent on the time $t \in [t_0, T_{max}]$. Setting $X(t) = (X_1, X_2, \ldots, X_9)^T$ and $F(t, X(t)) = (F_1(t, X(t)), F_2(t, X(t)), \ldots, F_9(t, X(t)))^T$, where

$$F_1(t, X(t)) = -\frac{X_1}{N} [m_{X_2}(t)\beta_{X_2}(t)X_2 + m_{X_3}(t)\beta_{X_3}(t)X_3 + m_{X_4}(t)\beta_{X_4}(\theta)X_2 + m_{X_5}(t)\beta_{X_5}(t)X_5$$
$$+ m_{X_6}(t)\beta_{X_6}(t)X_6] - \mu_m X_1 + \mu_n [X_1 + X_2 + X_3 + X_4 + X_7 + X_8],$$

$$F_2(t, X(t)) = -\frac{X_1}{N} [m_{X_2}(t)\beta_{X_2}(t)X_2 + m_{X_3}(t)\beta_{X_3}(t)X_3 + m_{X_4}(t)\beta_{X_4}(\theta)X_2 + m_{X_5}(t)\beta_{X_5}(t)X_5$$
$$+ m_{X_6}(t)\beta_{X_6}(t)X_6] - \mu_m X_2 - \gamma_{X_2}(t)X_2 + \tau_1(t) - \tau_2(t),$$

$$F_3(t, X(t)) = \gamma_{X_2}(t)X_2 - (\mu_m + \gamma_{X_3}(t))X_3,$$

$$F_4(t, X(t)) = (1 - \theta(t))\gamma_{X_3}(t)X_3 - (\mu_m + \gamma_{X_4}(t))X_4,$$

$$F_5(t, X(t)) = (\theta(t) - w(t))\gamma_{X_3}(t)X_3 - \gamma_{X_5}(t)X_5,$$

$$F_6(t, X(t)) = w(t)\gamma_{X_5}(t)X_3 - \gamma_{X_6}(t)X_6,$$

$$F_7(t, X(t)) = \gamma_{X_5}(t)X_5 - \mu_m X_7,$$

$$F_8(t, X(t)) = \gamma_{X_4}(t)X_4 - \mu_m X_8$$
and
$$F_9(t, X(t)) = \gamma_{X_6}(t)X_6,$$

the system of nonlinear equations (11-15) is equivalent to

$$\frac{dX}{dt} = F(t, X).$$

**Remark.** In the modeling point of view, the term $\frac{w(t)}{dt}$ corresponds to the apparent fatality rate of the disease (obtained by considering only the detected cases) in the considered area at time $t$, whereas $w(t)$ is the real fatality rate of coronavirus 2019 disease.

Since the mathematical model of Covid-19 provided by the system of equations (11-15) is too complex and because both natality and mortality (not from SARS-Cov-2) do not seem to be useful factors for this pandemic (at least for relatively short periods of time), we assume in the rest of this paper that

$$\mu_m = \mu_n = 0.$$

It is worth mentioning that the aim of this paper is to compute the following Covid-19 characteristics:

1) the model cumulative of coronavirus 2019 cases at day $t$ given by

$$c_m(t) = X_5(t) + X_6(t) + X_8(t) + X_9(t),$$

2) the model cumulative number of deaths (due to Covid-19) at day $t$, which is given by $X_9(t)$,

3) $R_e(t)$ which is the effective reproductive number of Covid-19,

4) the number of people in hospital is estimated by the following equation

$$Host(t) = X_6(t) + p(t)[X_5(t) + (X_7(t) - X_7(t - d_0))],$$

where $p(t)$ represents the fraction, at time $t$, of people in compartment $X_5$ that are hospitalized and $d_0$ days is the period of convalescence (i.e., the time a person is still hospitalized after recovering from Covid-19). This function can help to estimate and plan the number of clinical beds needed to treat all the SARS-Cov-2 cases at time $t$,

5) the maximum number of hospitalized persons at the same time in the territory during the time interval $[t_0, T_{max}]$, which is defined as

$$MaxHost = \max_{t_0 \leq t \leq T_{max}} Host(t).$$

$MaxHost$ can help to estimate and plan the number of clinical beds needed to treat all the coronavirus 2019 cases over the interval $[t_0, T_{max}]$,.
the number of people infected during the time interval $[t_0, T_{max}]$, by contact with people in compartments $X_2$, $X_4$ and $X_{10} = X_5 + X_6$, respectively. They are defined as

$$
\Gamma_{X_2}(t) = \frac{1}{N} \int_{t_0}^{T_{max}} m_{X_2}(s) \beta_{X_2}(s) X_2(s) X_1(s) ds,
$$

(17)

$$
\Gamma_{X_4}(t) = \frac{1}{N} \int_{t_0}^{T_{max}} m_{X_4}(s) \beta_{X_4}(s) X_4(s) X_1(s) ds,
$$

(18)

$$
\Gamma_{X_{10}}(t) = \frac{1}{N} \int_{t_0}^{T_{max}} (m_{X_5}(s) \beta_{X_5}(s) X_5(s) + m_{X_6}(s) \beta_{X_6}(s) X_6(s)) X_1(s) ds.
$$

(19)

We recall that the basis reproduction number $R_0$ is defined as the number of cases an infected individual generates on average over the course of its infectious period, in an otherwise uninfected population and without special control measures. It depends on the considered population, but does change during the spread of the disease, while the effective reproduction number $R_e(t)$ is defined as the number of cases one infected person generates on average over the course of its infectious period. A part of the population can be already infected and/or special control measures that have been implemented. It depends on the spread of the disease. In addition, $R_e(t_0) = R_0$, and the evolution of the epidemic slow down when $R_e(t) < 1$.

Now, applying the next generation method [50] to the nonlinear system (1)-(5) to get

$$
R_0 = \{ \gamma_{X_6}((\beta_{X_4}(1 - \theta)\gamma_{X_5} + \beta_{X_5}\gamma_{X_4}(\theta - w))\gamma_{X_5} + \beta_{X_5}\gamma_{X_4}\gamma_{X_5}\gamma_{X_6}) 
+ w\beta_{X_6}\gamma_{X_5}\gamma_{X_4}\gamma_{X_5}\gamma_{X_6}\} (\gamma_{X_2}\gamma_{X_5}\gamma_{X_4}\gamma_{X_6})^{-1},
$$

(20)

and

$$
R_e(t) = \frac{X_1(t)}{N} \{ \gamma_{X_6}((m_{X_4}\beta_{X_4}(1 - \theta)\gamma_{X_5} + m_{X_5}\beta_{X_5}\gamma_{X_4}(\theta - w))\gamma_{X_5} + m_{X_5}\beta_{X_5}\gamma_{X_4}\gamma_{X_5}\gamma_{X_6})\gamma_{X_2} 
+ m_{X_5}\beta_{X_4}\gamma_{X_5}\gamma_{X_4}\gamma_{X_6}\} (\gamma_{X_2}\gamma_{X_5}\gamma_{X_4}\gamma_{X_6})^{-1},
$$

(21)

where for the sake of simplicity of notations, all previous coefficients correspond to their particular values at times $t_0$ and $t$, respectively.

In the literature [26, 52], it is established that the observed patterns of Covid-19 are not completely consistent with the hypothesis that high absolute humidity may limit the survival and transmission of the virus, whereas the lower is the temperature, the greater is the survival period of the SARS-Cov-2 outside the host. Since there is no scientific evidence of the effect of the humidity and the temperature on the SARS-Cov-2, these factors are not included in our model.

Focusing on the application on the control strategies, the efficiency of these measures indicated in [23] satisfies equations

$$
m_{X_j}(t) := m(t) = \begin{cases} 
(m_l - m_{l+1}) \exp[-k_{l+1}(t - \lambda_l)], & t \in [t_l, \lambda_{l+1}), l = 0, 1, \ldots, q - 1, \\
(m_{q-1} - m_q) \exp[-k_q(t - \lambda_{q-1})], & t \in [t_{q-1}, \infty),
\end{cases}
$$

(22)

for $j = 1, 2, \ldots, 6$, where $m_l \in [0, 1]$ measures the intensity of the control measures (greater value implies lower value of disease contact rates), $k_l \in [0, \infty)$ (in day$^{-1}$) simulates the efficiency of the control strategies (greater value implies lower value of disease contact rates) and $\lambda_l \in [0, \infty)$, $l = 1, 2, \ldots, q$, denotes the first day of application of each control strategy. $\lambda_0 \in [t_0, \infty)$ is the first day of application of a control measure that was being used before $t_0$, if any. In this work, $q \in \mathbb{N}$ represents the number of changes of control strategy. In general, the values of $\lambda_l$ are typically taken in the literature (using dates when the countries implement special control measures). It is important to remind that some of the values of $m_l$ can be also sometimes known. The rest of the parameters needed to be calibrated.
In the following, we assume that the case fatality rate \( w(t) \), depends on the considered country, time \( t \), and it can be affected by the application of the control measures (such as, earlier detection, better sanitary condition, etc...)\[18\]. Thus, it satisfies equation
\[
w(t) = m(t)\overline{w} + (1 - m(t))\overline{w},
\]
where \( \overline{w} \in [0, 1] \) is the case fatality rate when no control measures are applied (i.e., \( m(t) = 1 \)) and \( w \in [0, 1] \) is the case fatality rate when implemented control measures are fully applied \( (m(t) = m_q) \).

Denoting by \( d_{X_3}, d_{X_4}, d_{X_5} \) and \( d_{X_6} \), be the "average" duration in days of a person in compartment \( X_3, X_4, X_5 \) and \( X_6 \), respectively, without the application of control strategies, we assume as in [24, 25] that
\[
\begin{align*}
\gamma(t) := \gamma_{X_3}(t) &= \frac{1}{d_{X_3} - g(t)}, \quad (day^{-1}) \\
\rho(t) := \gamma_{X_4}(t) &= \gamma_{X_3}(t) = \frac{1}{d_{X_4} + g(t)}, \quad (day^{-1}) \\
\psi(t) := \gamma_{X_6}(t) &= \gamma_{X_5}(t) = \frac{1}{d_{X_4} + g(t) + \delta}, \quad (day^{-1})
\end{align*}
\]
where \( g(t) = d_g(1 - m(t)) \) represents the decrease of the duration of the function \( d_{X_3} \) due to the application of the control measures at time \( t \), \( d_g \) is the maximum number of days that \( d_{X_3} \) can be decreased due to the control measures.

Finally, the disease contact rate \( \beta_{X_4}(\theta) \) is defined by
\[
\beta_{X_4}(\theta) = \begin{cases} 
\overline{\beta}_{X_3}, & \text{if } \theta = \overline{w}, \\
\overline{\beta}_{X_3}, & \text{if } \theta = \overline{w}, \\
\overline{\beta}_{X_3}, & \text{if } \theta = \overline{w}.
\end{cases}
\]

where \( \overline{\beta}_{X_3} \) and \( \overline{\beta}_{X_3} \) are suitable lower and upper bounds, respectively. For the convenience of writing, we assume that \( \overline{\beta}_{X_3} = \beta_{X_3} := \beta \). In addition, the people in compartments \( X_2, X_4, X_5 \) and \( X_6 \) are less infectious than people in compartment \( X_3 \) (due to their lower virus load or isolation measures). This fact results in
\[
\beta_{X_2} = c_{X_2}\beta_{X_3}, \quad \beta_{X_4} = \beta_{X_5} = c_{X_{45}}(t)\beta_{X_3}, \quad \beta_{X_6} = c_u\beta_{X_3},
\]
where \( c_{X_2}, c_{X_{45}}, c_u \in [0, 1] \).
3 Construction of the two-level explicit numerical scheme

In this section, we develop the robust two-level explicit scheme for solving the mathematical problem (1)-(5) modeling the spread of Covid-19 with undetected cases.

Let \( h := \Delta t = \frac{T_{\text{max}} - t_0}{M} \) be the step size, \( M \) is a positive integer. Set \( t_n = t_0 + nh \), \( t_{n-\frac{1}{2}} = \frac{t_{n} + t_{n-1}}{2} \) for \( n = 1, 2, ..., M \) and \( \Omega_h = \{ t_n, 0 \leq n \leq M \} \) be a regular partition of \([t_0, T_{\text{max}}]\). Let \( \mathcal{F}_h = \{ X^n_i, n = 0, 1, ..., M ; 1 \leq i \leq 9 \} \), be the grid functions space defined on \( \Omega_h \times \mathbb{R}^3 \subset I \times \mathbb{R}^3 := [t_0, T_{\text{max}}] \times \mathbb{R}^3 \).

Define the following norms

\[
\| X^n \|_\infty = \max_{1 \leq i \leq 9} | X^n_i | \quad \text{and} \quad \| X \|_{L^2(t)} = \left( \frac{M}{h} \sum_{n=0}^{M} \| X^n \|^2_{\infty} \right)^{\frac{1}{2}}
\]

(29)

where \( | \cdot | \) designates the norm defined on the field of complex numbers \( \mathbb{C} \). Furthermore, denote

\[
F_j^{(i)}(t, X(t)) = \sum_{l} F_i(t_l, X(t_l)) L_i(t),
\]

(30)

where the function \( L_i(t) \) is given by

\[
L_i(t) = \prod_{q \neq l} \frac{t - t_q}{t_l - t_q},
\]

(31)

be a polynomial of degree \( j \) interpolating the function \( F_i(t_l, X(t_l)) \) at the node points \( t_l, F_i(t_l, X(t_l)) \). According to equations (30)-[31], it’s important to remind that \( F_j^{(i)}(t, X(t)) \) is not necessarily the interpolation polynomial of degree \( j \) of the function \( F_i(t, X(t)) \) at the node points \( t_l, F_i(t_l, X(t_l)) \).

Now, integrating both sides of equation (12) at the node points \( t_n \) and \( t_{n+\frac{1}{2}} \), this yields

\[
X(t_{n+\frac{1}{2}}) - X(t_n) = \int_{t_n}^{t_{n+\frac{1}{2}}} F(t, X) dt,
\]

which is equivalent to

\[
X(t_{n+\frac{1}{2}}) = X(t_n) + \int_{t_n}^{t_{n+\frac{1}{2}}} F(t, X) dt.
\]

(32)

For \( j = 1, F_1^{(i)}(t, X(t)) \) is a linear polynomial approximating the function \( F_i(t, X(t)) \) at the points \( t_n, F_i(t_n, X(t_n)) \) and \( t_{n+\frac{1}{2}}, F_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) \). Using equations (30) and (31), it is easy to observe that

\[
F_1^{(i)}(t, X(t)) = F_i(t_n, X(t_n)) \frac{t - t_{n+\frac{1}{2}}}{t_n - t_{n+\frac{1}{2}}} + F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) \frac{t - t_n}{t_{n+\frac{1}{2}} - t_n} - \frac{2}{h} \left[ F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) - F_i(t_n, X(t_n)) \right] t + t_{n+\frac{1}{2}} F_i(t_n, X(t_n)) - t_n F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right),
\]

(33)

where the error term is given by

\[
F_i(t, X(t)) - F_1^{(i)}(t, X(t)) = \frac{1}{2} (t - t_n)(t - t_{n+\frac{1}{2}}) \frac{d^2 F_i}{dt^2} (t, X(t)) := O(h^2),
\]

(34)

where \( t_n \) (respectively, each component \( X_i(t_n) \) of \( X(t) \)) is an unknown function which is between the maximum and the minimum of the numbers \( t_n, t_{n+\frac{1}{2}} \) and \( t \) (respectively: \( X_i(t_n), X_i(t_{n+\frac{1}{2}}) \) and \( X_i(t) \)). But equation (34) can be rewritten as

\[
F_i(t, X(t)) = F_1^{(i)}(t, X(t)) + O(h^2), \quad \text{for} \quad i = 1, 2, ..., 9.
\]

(35)
Substituting approximation (35) into the ith equation of the system (32), we obtain
\[
X_i(t_{n+\frac{1}{2}}) = X_i(t_n) + \int_{t_n}^{t_{n+\frac{1}{2}}} P_1^{(i)}(t, X) dt + O(h^3), \quad \text{for } i = 1, 2, \ldots, 9,
\] (36)
which is equivalent to the following system
\[
X(t_{n+\frac{1}{2}}) = X(t_n) + \int_{t_n}^{t_{n+\frac{1}{2}}} Q_1(t, X) dt + O(h^3),
\] (37)
where \(Q_1(t, X(t)) = (P_1^{(1)}(t, X(t)), P_1^{(2)}(t, X(t)), \ldots, P_1^{(9)}(t, X(t)))^T\) and \(O(h^3) = (O(h^3), O(h^3), \ldots, O(h^3))^T\).

The integration of both sides of equation (33) provides
\[
\int_{t_n}^{t_{n+\frac{1}{2}}} P_1^{(i)}(t, X) dt = \frac{1}{h} \left[ \left( F_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) - F_i(t_n, X(t_n)) \right) (t_{n+\frac{1}{2}}^2 - t_n^2) + 2 \left[ t_{n+\frac{1}{2}} F_i(t_n, X(t_n)) - t_n F_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) \right] (t_{n+\frac{1}{2}} - t_n) \right]
+ \frac{h}{4} \left[ F_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) + F_i(t_n, X(t_n)) \right],
\] (38)
where the last two equalities come from the identities \(t_{n+\frac{1}{2}}^2 - t_n^2 = (t_{n+\frac{1}{2}} - t_n)(t_{n+\frac{1}{2}} + t_n)\), \(t_{n+\frac{1}{2}} - t_n = \frac{L}{2}\) and \(t_{n+\frac{1}{2}} + t_n = 2t_n + \frac{L}{2}\). To get the desired first-level of the new algorithm, we should approximate the sum \(F_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) + F_i(t_n, X(t_n))\) by the term \(a_1 F_i(t_n, X(t_n)) + a_2 F_i(t_n + p_1 h, X(t_n) + p_2 h F(t_n, X(t_n)))\), in which the coefficients \(a_1\), \(a_2\), \(p_1\) and \(p_2\), are real numbers and are chosen so that the Taylor expansion
\[
\frac{X_i(t_{n+\frac{1}{2}}) - X_i(t_n)}{h/2} - \frac{1}{2} \left[ F_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) + F_i(t_n, X(t_n)) \right] = O(h^2).
\]
The application of the Taylor series expansion for \(X_i\) and \(F_i\) about \(t_n\) and \((t_n, X(t_n))\), respectively, with step size \(h/2\) using forward difference representations gives
\[
X_i(t_{n+\frac{1}{2}}) = X_i(t_n) + \frac{h}{2} F_i(t_n, X(t_n)) + \frac{h^2}{8} \left[ \partial_t F_i(t_n, X(t_n)) + \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial_{x_k} F_i(t_n, X(t_n)) \right] + O(h^3),
\] (39)
and
\[
F_i(t_n + p_1 h, X(t_n) + p_2 h F(t_n, X(t_n))) = F_i(t_n, X(t_n)) + h p_1 \partial_t F_i(t_n, X(t_n)) +
\]
\[
\frac{h^2}{8} \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial_{x_k} F_i(t_n, X(t_n)) + O(h^2),
\] (40)
where \(\partial_t F_i\) denotes \(\frac{\partial F_i}{\partial t}\) and \(\partial_{x_k} F_i\) represent \(\frac{\partial F_i}{\partial x_k}\), for \(k = 1, 2, \ldots, 9\).

Combining equations (39)-(40), direct calculations yield
\[
\frac{X_i(t_{n+\frac{1}{2}}) - X_i(t_n)}{h/2} - \frac{1}{2} \left[ F_i(t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}})) + F_i(t_n, X(t_n)) \right] = \left( 1 - \frac{a_1 + a_2}{2} \right) F_i(t_n, X(t_n)) +
\]
\[
\frac{h}{2} \left[ \left( \frac{1}{2} - a_2 p_1 \right) \partial_t F_i(t_n, X(t_n)) + \left( \frac{1}{2} - a_2 p_2 \right) \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial_{x_k} F_i(t_n, X(t_n)) \right] + O(h^2),
\]
which equals \(O(h^2)\) if and only if \(a_1 + a_2 = 2a_2 p_1 = \frac{1}{2}\) and \(a_2 p_2 = \frac{1}{2}\). But the last two equations require \(a_2 \neq 0\), \(p_1 \neq 0\) and \(p_2 \neq 0\). Take for instance
\[
p_1 = p_2 = \frac{1}{2}, \quad \text{so} \quad a_1 = a_2 = 1.
\] (41)
Plugging equation (40) and relation (41), it is not hard to observe that the sum $F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) + F_i(t_n, X(t_n))$ is approximate as

$$F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) + F_i(t_n, X(t_n)) = 2F_i(t_n, X(t_n)) + \frac{h}{2}[\partial_t F_i(t_n, X(t_n)) +$$

$$\sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial_k F_i(t_n, X(t_n))] + O(h^3).$$

(42)

A combination of equations (38) and (42) results in

$$\int_{t_n}^{t_{n+\frac{1}{2}}} p^{(i)}_1(t, X)dt = \frac{h}{2}F_i(t_n, X(t_n)) + \frac{h^2}{8}\left[\partial_t F_i(t_n, X(t_n)) + \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial_k F_i(t_n, X(t_n))\right] + O(h^3).$$

(43)

Substituting equation (43) into (36), this provides

$$X_i(t_{n+\frac{1}{2}}) = X_i(t_n) + \frac{h}{2}F_i(t_n, X(t_n)) + \frac{h^2}{8}\left[\partial_t F_i(t_n, X(t_n)) + \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial_k F_i(t_n, X(t_n))\right] + O(h^3).$$

(44)

Tracking the infinitesimal term $O(h^3)$, equation (44) can be approximate as

$$X_i^{n+\frac{1}{2}} = X^n_i + \frac{h}{2}F_i(t_n, X^n) + \frac{h^2}{8}\left[\partial_t F_i(t_n, X^n) + \sum_{k=1}^{9} F_k(t_n, X^n) \partial_k F_i(t_n, X^n)\right], \text{ for } i = 1, 2, ..., 9. \quad (45)$$

The difference equations provided by relation (45) represent the first-level of the new approach.

To develop the second-level of the desired algorithm, we should integrate both sides of system (42) at the node points $t_{n+\frac{1}{2}}$ and $t_{n+1}$. Thus

$$X(t_{n+1}) - X(t_{n+\frac{1}{2}}) = \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} F(t, X)dt,$$

which can be rewritten as

$$X(t_{n+1}) = X(t_{n+\frac{1}{2}}) + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} F(t, X)dt.$$  

This implies

$$X_i(t_{n+1}) = X_i(t_{n+\frac{1}{2}}) + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} F_i(t, X)dt, \text{ for } i = 1, 2, ..., 9. \quad (46)$$

Replacing $F_i(t, X)$ by the linear interpolating polynomial $p^{(i)}_1(t, X)$ at the node points $(t_n, F_i(t_n, X^n))$ and $(t_{n+\frac{1}{2}}, F_i(t_{n+\frac{1}{2}}, X^n))$, and using (54), equation (46) is approximate as

$$X_i(t_{n+1}) = X_i(t_{n+\frac{1}{2}}) + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} p^{(i)}_1(t, X)dt + O(h^3).$$

(47)

Plugging (44) and (46), simple computations give

$$X_i(t_{n+1}) = X_i(t_{n+\frac{1}{2}}) + \frac{2}{h} \left[ \frac{1}{2} \left( F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) - F_i(t_n, X(t_n)) \right) \right] (t_{n+1}^2 - t_{n+\frac{1}{2}}^2) + |t_{n+1}F_i(t_n, X(t_n)) -$$

$$t_{n+\frac{1}{2}}F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right)] (t_{n+1} - t_{n+\frac{1}{2}}) + O(h^3) = X_i(t_{n+\frac{1}{2}}) +$$

$$\frac{h}{2} \left[ 3F_i \left( t_{n+\frac{1}{2}}, X(t_{n+\frac{1}{2}}) \right) - F_i(t_n, X(t_n)) \right] + O(h^3).$$

(48)
Omitting the error term $O(h^3)$, we obtain the second-level of the new method which is defined as

$$X_i^{n+\frac{1}{2}} = X_i^n + \frac{h}{2} F_i (t_n, X^n) + \frac{h^2}{8} \left[ \partial_t F_i (t_n, X^n) + \sum_{k=1}^{9} F_k (t_n, X^n) \partial_k F_i (t_n, X^n) \right], \quad \text{for } i = 1, 2, \ldots, 9. \quad (49)$$

Under the assumption given by equations (63), (22), (23), (25), and (28), the functions $F_i (t, X)$, $(i = 1, 2, \ldots, 9)$ defined by relations (7)-(11) becomes

$$F_1 (t, X(t)) = -\frac{m(t)X_1}{N} [\beta X_2 (t) X_2 + \beta X_3 (t) X_3 + \beta X_4 (t) X_1 + C X_5 (t) \beta X_5 (t) X_6], \quad (50)$$

$$F_2 (t, X(t)) = \frac{m(t)X_1}{N} [\beta X_3 (t) X_2 + \beta X_3 (t) X_3 + \beta X_4 (t) X_2 + C X_5 (t) \beta X_2 (t) X_5 + C X_5 (t) \beta X_3 (t) X_6] - \alpha X_2, \quad (51)$$

$$F_3 (t, X(t)) = \alpha X_2 - \gamma (t) X_3, \quad F_4 (t, X(t)) = (1 - \theta(t)) \gamma (t) X_3 - \rho(t) X_4, \quad (52)$$

$$F_5 (t, X(t)) = (\theta(t) - w(t)) \gamma (t) X_3 - \rho(t) X_5, \quad F_6 (t, X(t)) = w(t) \gamma (t) X_3 - \psi(t) X_6, \quad (53)$$

$$F_7 (t, X(t)) = \rho(t) X_5, \quad F_8 (t, X(t)) = \rho(t) X_4 \text{ and } F_9 (t, X(t)) = \psi(t) X_6, \quad (54)$$

By straightforward computations, it is not hard to observe that

$$\partial_t F_1 = \frac{1}{N} \{ C_{X_2} \beta X_2 [(m(t) F_1 + \dot{m}(t) X_1) X_2 + m(t) X_1 X_2] + \beta [(m(t) F_1 + \dot{m}(t) X_1) X_3 + m(t) X_1 F_3]$$

$$\partial_t F_1 = \frac{1}{N} [m(t) \beta X_2 [(m(t) F_1 + \dot{m}(t) X_1) X_2 + m(t) X_1 F_1] + \beta [m(t) F_1 + \dot{m}(t) X_1] X_3 + m(t) X_1 F_3]$$

$$\partial_t F_1 = \frac{1}{N} [m(t) \beta X_2 [(m(t) F_1 + \dot{m}(t) X_1) X_2 + m(t) X_1 F_1] + \beta [m(t) F_1 + \dot{m}(t) X_1] X_3 + m(t) X_1 F_3]$$

$$\partial_t F_1 = \frac{1}{N} [m(t) \beta X_2 [(m(t) F_1 + \dot{m}(t) X_1) X_2 + m(t) X_1 F_1] + \beta [m(t) F_1 + \dot{m}(t) X_1] X_3 + m(t) X_1 F_3]$$

$$\partial_t F_1 = \frac{1}{N} [m(t) \beta X_2 [(m(t) F_1 + \dot{m}(t) X_1) X_2 + m(t) X_1 F_1] + \beta [m(t) F_1 + \dot{m}(t) X_1] X_3 + m(t) X_1 F_3]$$

$$\partial_t F_1 = \frac{1}{N} [m(t) \beta X_2 [(m(t) F_1 + \dot{m}(t) X_1) X_2 + m(t) X_1 F_1] + \beta [m(t) F_1 + \dot{m}(t) X_1] X_3 + m(t) X_1 F_3]$$

$$\partial_t F_1 = \frac{1}{N} [m(t) \beta X_2 [(m(t) F_1 + \dot{m}(t) X_1) X_2 + m(t) X_1 F_1] + \beta [m(t) F_1 + \dot{m}(t) X_1] X_3 + m(t) X_1 F_3]$$

$$\partial_t F_1 = \frac{1}{N} [m(t) \beta X_2 [(m(t) F_1 + \dot{m}(t) X_1) X_2 + m(t) X_1 F_1] + \beta [m(t) F_1 + \dot{m}(t) X_1] X_3 + m(t) X_1 F_3]$$

$$\partial_t F_1 = \frac{1}{N} [m(t) \beta X_2 [(m(t) F_1 + \dot{m}(t) X_1) X_2 + m(t) X_1 F_1] + \beta [m(t) F_1 + \dot{m}(t) X_1] X_3 + m(t) X_1 F_3]$$

To provide a full description of the two-level explicit formulation for solving the mathematical problem (1)-(11) to predicting the spread of Covid-19 model, we should put together relations (11a), (11b) and the initial condition given by equation (6). Specifically, for $n = 1, 2, \ldots, M - 1,$

$$X_i^{n+\frac{1}{2}} = X_i^n + \frac{h}{2} F_i (t_n, X^n) + \frac{h^2}{8} \left[ \partial_t F_i (t_n, X^n) + \sum_{k=1}^{9} F_k (t_n, X^n) \partial_k F_i (t_n, X^n) \right], \quad \text{for } i = 1, 2, \ldots, 9; \quad (70)$$
\[ X_i^{n+1} = X_i^{n+\frac{1}{2}} + \frac{h}{2} \left[ 3F_i \left( t_i^{n+\frac{1}{2}}, X_i^{n+\frac{1}{2}} \right) - F_i(t_i^n, X_i^n) \right], \quad \text{for } i = 1, 2, ..., 9; \]  
subject to the initial condition
\[ X_i(t_0) = X_i^0, \quad \text{for } i = 1, 2, ..., 9; \]  
where the functions \( F_i \) (\( i = 1, 2, ..., 9 \)) and its partial derivatives are given by equations (50)-(69).

Using the tools provided in section 3 we are ready to analyze the stability and the convergence rate of the two-level explicit procedure \((70)-(72)\) for predicting the spread of SARS-Cov-2 epidemic modeled by equations \((1)-(6)\).

### 4 Stability analysis and convergence rate of the new algorithm

In this section we wish to examine the stability and convergence rate of the new technique \((70)-(72)\) applied to the initial-value problem \((1)-(6)\).

Firstly, we define the strip \( S = \{(t, X), t_0 \leq t \leq T_{\max}, X \in \mathbb{R}^9\} \) in which both exact and computed solutions of problem \((1)-(6)\) should lie. It comes from equations (50)-(69) that the functions \( F_i \) and their partial derivatives are continuous on the strip \( S \), but the partial derivatives are unbounded on this set. Thus it follows from the Henrici result [17] that the system of equations \((1)-(6)\) admits a unique solution \( X(t) \) defined in a certain neighborhood \( U(t_0) \subset [t_0, T_{\max}] \) of the initial point \( t_0 \). Without loss of this constraint, we assume in the following that \( U(t_0) = [t_0, T_{\max}] \) (indeed, we are dealing with a real world problem which, in reality should have a unique solution defined over the interval \([t_0, T_{\max}]\)). This shows the existence and uniqueness of the solution of the initial-value problem \((1)-(6)\).

Let now introduce the functions \( \Delta_i(t_i, X_i(t_i)) \) and \( \delta_i(t_i, X_i(t_i)) \) for \( l = n, n+\frac{1}{2} \) be the difference quotient of the exact solution \( X_i(t_i) \) of equations \((1)-(6)\) at time \( t_i \) and the difference quotient of the approximate solution \( X_i^l \) of problem \((70)-(72)\) obtained at time \( t_i \), respectively. Moreover, \( \Delta_i(t_i, X_i(t_i)) \) and \( \delta_i(t_i, X_i(t_i)) \) are given by

\[ \Delta_i(t_i, X_i(t_i)) = \begin{cases} \frac{X_i(t_i+\frac{1}{2})-X_i(t_i)}{h/2}, & \text{if } h \neq 0, \\ F_i(t_i, X_i(t_i)), & \text{for } h = 0, \end{cases} \]  
and

\[ \delta_i(t_i, X_i(t_i)) = \begin{cases} \frac{X_i^{l+\frac{1}{2}}-X_i^l}{h/2}, & \text{if } h \neq 0, \\ F_i(t_i, X_i^l), & \text{for } h = 0. \end{cases} \]  

It is worth mentioning that \( \delta_i(t_i, X_i^l(t_i)) = \delta_i(t_i, X_i^l) \). The local discretization error at the point \((t_i, X_i(t_i))\) of the considered scheme is defined as

\[ \sigma_i(t_i, X_i(t_i)) = \Delta_i(t_i, X_i(t_i)) - \delta_i(t_i, X_i(t_i)), \quad l = n \text{ or } n + \frac{1}{2}, \quad \text{for } i = 1, 2, ..., 9, \]  
indicates how well the exact solution of the differential equations \((1)-(6)\) obeys the difference equations \((70)-(72)\) provided by the two-level explicit formulation.

The following result (Theorem 4.1) analyzes the stability and gives the convergence rate of the proposed approach \((70)-(72)\).

**Theorem 4.1. (Stability analysis and convergence rate)**

Let \( e^n = X(t_n) - X^n \) be the global discretization error provided by algorithm \((70)-(72)\), where \( X(t_n) \) is the solution of system \((1)-(6)\) obtained at time \( t_n \) and \( X^n \) is the one provided by \((70)-(72)\) at time \( t_n \). Thus, it holds

\[ \|X^n\|_{\infty} = \max_{1 \leq i \leq 9} |X_i^n| \leq C_1, \]  
which implies \( \|X\|_{L^2(I)} = \left( h \sum_{n=0}^{M} \|X^n\|_{\infty}^2 \right)^{\frac{1}{2}} \leq C_1 \).
where \( C_1 \) is a positive constant independent of the step size \( h \) and \( X \) denotes the approximate solution. Furthermore

\[
\|e^n\|_\infty = \max_{1 \leq i \leq 9} |e^n_i| \leq C_2h^2, \quad \text{which implies} \quad \|e^n\|_{L^2(I)} = \left( \sum_{n=0}^{M} \|e^n\|_\infty^2 \right)^{\frac{1}{2}} \leq C_2h^2
\]

where \( C_2 \) is a positive constant that does not depend on the step size \( h \). In the following we represent the analytical solution by \( X(\cdot) \).

The proof of Theorem 4.1 requires the following intermediate result (namely Lemmas 4.1).

**Lemma 4.1.** Suppose the vectors \( q_j \in \mathbb{C} \) satisfy the estimates of the form

\[
\|q_{j+1}\|_{L^\infty(\mathbb{C})} \leq (1 + \epsilon)\|q_j\|_{L^\infty(\mathbb{C})} + \xi, \quad \epsilon > 0 \quad \text{and} \quad \xi > 0,
\]

for \( j = 0, 1, ..., m \). Then

\[
\|q_m\|_{L^\infty(\mathbb{C})} \leq \exp(m \epsilon)\|q_0\|_{L^\infty(\mathbb{C})} + \frac{\exp(m \epsilon) - 1}{\epsilon} \xi.
\]

**Proof.** We should prove inequality (77) by mathematical induction. Since \( \exp(\epsilon) \geq 1 + \epsilon \) and \( 1 \leq \frac{\exp(\epsilon) - 1}{\epsilon} \), for any \( \epsilon \geq 0 \), using the assumption of Lemma 4.1 it is not difficult to see that

\[
\|q_i\|_{L^\infty(\mathbb{C})} \leq (1 + \epsilon)\|q_0\|_{L^\infty(\mathbb{C})} + \exp(\epsilon)\|q_0\|_{L^\infty(\mathbb{C})} + \frac{\exp(\epsilon) - 1}{\epsilon} \xi.
\]

Now, let assume that

\[
\|q_{m-1}\|_{L^\infty(\mathbb{C})} \leq \exp((m - 1)\epsilon)\|q_0\|_{L^\infty(\mathbb{C})} + \frac{\exp((m - 1)\epsilon) - 1}{\epsilon} \xi.
\]

Combining estimates: \( 1 + \epsilon \leq \exp(\epsilon) \) and \( 1 \leq \frac{\exp(\epsilon) - 1}{\epsilon} \), together with inequality (76) provided by the assumption of Lemma 4.1 and (78), direct calculations give

\[
\|q_m\|_{L^\infty(\mathbb{C})} \leq (1 + \epsilon)\|q_{m-1}\|_{L^\infty(\mathbb{C})} + \xi \leq (1 + \epsilon) \left[ \exp((m - 1)\epsilon)\|q_0\|_{L^\infty(\mathbb{C})} + \frac{\exp((m - 1)\epsilon) - 1}{\epsilon} \xi \right] + \xi \leq \exp(\epsilon) \left[ \exp((m - 1)\epsilon)\|q_0\|_{L^\infty(\mathbb{C})} + \frac{\exp((m - 1)\epsilon) - 1}{\epsilon} \xi \right] + \xi \leq \exp(m\epsilon)\|q_0\|_{L^\infty(\mathbb{C})} + \frac{\exp(m\epsilon) - 1}{\epsilon} \xi.
\]

The last inequality comes the estimate \( \epsilon - \exp(\epsilon) \leq -1 \). This completes the proof of Lemma 4.1. \( \square \)

**Proof.** (of Theorem 4.1).

First of all, introduce the domains

\[
D_1 = \{(t, x) : t \in [t_0, T_{\text{max}}], \ x \in \mathbb{R}, \ |X_i(t) - x| \leq v \}, \quad S_0 = \{(t, x) : t_0 \leq t \leq T_{\text{max}}, \ x \in \mathbb{R} \},
\]

and

\[
D = \{(t, X) : t \in [t_0, T_{\text{max}}], \ X \in \mathbb{R}^3, \ \|X(t) - X\|_\infty \leq v \},
\]

where \( v \) is a positive constant independent of the step size \( h \) and \( X_i(t) \) denotes the ith component of the exact solution \( X(t) \) of the initial-value problem (1)-(3).

Plugging approximations (70)-(71) and equation (73), it is not hard to observe that

\[
\delta_i(t_n, X_i(t_n)) = F_i(t_n, X^n) + \frac{h}{4} \left[ \partial_t F_i(t_n, X^n) + \sum_{k=1}^{9} F_k(t_n, X^n) \partial_h F_i(t_n, X^n) \right],
\]

and

\[
\delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) = 3F_i(t_{n+\frac{1}{2}}, X^n+\frac{1}{2}) - F_i(t_n, X^n).
\]
Hence, the functions $\delta_i$ given by equation (74) and their partial derivatives are continuous on the domain $D_i$. Since $D_i$ is a compact subset of the strip $S_0$, their partial derivatives are bounded on $D_i$. Applying the Mean-Value Theorem, there exists a positive constant $L_i$ which does not depend on the step size $h$ so that

$$|\delta_i(t, X_i) - \delta_i(t, Y_i)| \leq L_i|X_i - Y_i|,$$

(82)

for every $(t, X_i)$ and $(t, Y_i)$ in $D_i$. Furthermore, a simple manipulation of (44) results in

$$\frac{X_i(t_{n+\frac{1}{2}}) - X_i(t_n)}{h/2} = F_i(t_n, X(t_n)) + \frac{h}{4} \left[ \partial_i F_i(t_n, X(t_n)) + \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial_k F_i(t_n, X(t_n)) \right] + O(h^2),$$

which is equivalent to

$$\frac{X_i(t_{n+\frac{1}{2}}) - X_i(t_n)}{h/2} - \left[ F_i(t_n, X(t_n)) + \frac{h}{4} \left( \partial_i F_i(t_n, X(t_n)) + \sum_{k=1}^{9} F_k(t_n, X(t_n)) \partial_k F_i(t_n, X(t_n)) \right) \right] = O(h^2).$$

Utilizing equations (73), (75) and (80), this becomes

$$\sigma_i(t, X_i(t)) = \Delta_i(t, X_i(t)) - \delta_i(t, X_i(t)) = O(h^2),$$

which implies

$$|\sigma_i(t, X_i(t))| \leq C_{i4} h^2, \quad \text{for } i = 1, 2, ..., 9,$$

(83)

where $C_{i4}$ are positive constants that do not depend on the step size $h$. Setting $C_4 = \max_{1 \leq i \leq 9} C_{i4}$; $\sigma(t_n, X(t_n)) = (\sigma_1(t_n, X_1(t_n)), ..., \sigma_9(t_n, X_9(t_n)))^T$ and taking the maximum of both sides of estimate (83) over $i$, to get

$$\|\sigma(t_n, X_i(t_n))\|_{L^\infty} \leq C_4 h^2.$$  

(84)

In a similar manner, combining approximation (49), equations (73), (74) and (75), straightforward calculations provide

$$\frac{X_i(t_{n+1}) - X_i(t_{n+rac{1}{2}})}{h/2} - \left[ 3F_i(t_{n+rac{1}{2}}, X(t_{n+rac{1}{2}})) - F_i(t_n, X(t_n)) \right] = O(h^2),$$

which can be rewritten as

$$\Delta_i(t_{n+rac{1}{2}}, X_i(t_{n+rac{1}{2}})) - \delta_i(t_{n+rac{1}{2}}, X_i(t_{n+rac{1}{2}})) = O(h^2).$$

Utilizing the definition of $\sigma_i(t_{n+rac{1}{2}}, X_i(t_{n+rac{1}{2}}))$, this implies

$$|\sigma_i(t_{n+rac{1}{2}}, X_i(t_{n+rac{1}{2}}))| \leq C_{i5} h^2,$$

where $C_{i5}$ ($1 \leq i \leq 9$) are positive constants independent of $h$. Taking the maximum over $i$, this yields

$$\|\sigma(t_{n+rac{1}{2}}, X(t_{n+rac{1}{2}}))\|_{L^\infty} \leq C_5 h^2,$$

(85)

where $C_5 = \max_{1 \leq i \leq 9} C_{i5}$; $\sigma(t_{n+rac{1}{2}}, X(t_{n+rac{1}{2}})) = (\sigma_1(t_{n+rac{1}{2}}, X_1(t_{n+rac{1}{2}})), ..., \sigma_9(t_{n+rac{1}{2}}, X_9(t_{n+rac{1}{2}})))^T$.

We consider the functions $\hat{\delta}_i(t, x)$ defined in the strip $S_0 = \{(t, x) : t_0 \leq t \leq T_{\max}, x \in \mathbb{R} \}$, as

$$\hat{\delta}_i(t, x) = \begin{cases} \delta_i(t, x), & \text{if } (t, x) \in D_i, \\ \delta_i(t, x_i(t) + v), & \text{for } t \in [t_0, T_{\max}] \text{ and } x > X_i(t) + v, \\ \delta_i(t, X_i(t) - v), & \text{if } t \in [t_0, T_{\max}] \text{ and } x < X_i(t) - v. \end{cases}$$

(86)

We remind that $X_i(t)$ is the $i$th component of the exact solution $X(t)$ of the initial-value problem (1)-(6). Now, using relation (86), it is not hard to observe that each function $\hat{\delta}_i$ satisfies the "Lipschitz requirement" in the strip $S_0$, that is

$$|\hat{\delta}_i(t, x) - \hat{\delta}_i(t, y)| \leq L_i|x - y|,$$

(87)
for all \((t,x)\) and \((t,x)\) in \(S_0\), where the positive constant is given by relation (32). Thus, each \(\delta_i(t)\) is continuous on \(S_0\). Indeed. Let \((t,x)\) and \((t,y)\) be two elements of \(S_0\). If \((t,x)\) and \((t,y)\) lie in \(D_i\), then estimate (37) holds thanks to inequality \(\delta_i(t)\), so the function \(\delta_i(t)\) is continuous and satisfies the "Lipschitz requirement" on \(D_i\). Otherwise, either \((t,x)\) or \((t,y)\) do not lie in \(D_i\). This corresponds to three cases: (a) \((t,x)\) lies in \(D_i\) and \((t,y)\) does not lie in \(D_i\), (b) \((t,x)\) does not lie in \(D_i\) and \((t,y)\) does lie in \(D_i\) and (c) \((t,x)\) and \((t,y)\) do not lie in \(D_i\). Here we should prove only one case, for instance \((t,x)\) does not lie in \(D_i\) and \((t,y)\) does lie in \(D_i\), the proof of the two others cases are similar.

\[
(t,x) \notin D_i \Leftrightarrow |x - X_i(t)| > \upsilon \Leftrightarrow x - X_i(t) > \upsilon \Leftrightarrow -(x - X_i(t)) > \upsilon \Leftrightarrow x > X_i(t) + \upsilon \text{ or } x < X_i(t) - \upsilon.
\]

So,

\[
\hat{\delta}(t,x) - \hat{\delta}(t,y) = \begin{cases} 
|\delta_i(t,X_i(t) + \upsilon) - \delta_i(t,y)|, & \text{for } t_0 \leq t \leq T_{\text{max}} \text{ and } x > X_i(t) + \upsilon, \\
|\delta_i(t,X_i(t) - \upsilon) - \delta_i(t,y)|, & \text{if } t \in [t_0,T_{\text{max}}] \text{ and } x < X_i(t) - \upsilon,
\end{cases}
\]

\[
\leq \begin{cases} 
L_i|X_i(t) + \upsilon - y|, & \text{if } t \in [t_0,T_{\text{max}}] \text{ and } x > X_i(t) + \upsilon, \\
L_i|X_i(t) - \upsilon - y|, & \text{for } t_0 \leq t \leq T_{\text{max}} \text{ and } x < X_i(t) - \upsilon.
\end{cases}
\]

But

\[
(t,y) \in D_i \Leftrightarrow |X_i(t) - y| \leq \upsilon \Leftrightarrow X_i(t) - y \leq \upsilon \text{ and } -X_i(t) + y \leq \upsilon.
\]

Using this, it is easy to see that

\[
x > X_i(t) + \upsilon \Leftrightarrow x - y > X_i(t) + \upsilon - y \geq 0 \Leftrightarrow |X_i(t) - y| > |X_i(t) + \upsilon - y|,
\]

and

\[
x < X_i(t) - \upsilon \Leftrightarrow x - y < X_i(t) - \upsilon - y \leq 0 \Leftrightarrow |X_i(t) - y| > |X_i(t) - \upsilon - y|.
\]

This fact together with estimate (38) results in

\[
|\hat{\delta}(t,x) - \hat{\delta}(t,y)| \leq L_i|x - y|.
\]

This ends the proof of the first case (a). Thus, \(\hat{\delta}_i\) satisfies the "Lipschitz condition" on the strip \(S_0 = \{(t,x) : t_0 \leq t \leq T_{\text{max}}, \ x \in \mathbb{R}\}\). In addition, \(\hat{\delta}_i\) is also continuous on \(S_0\).

Since, \((t,X_i(t)) \in D_i\), so \(\hat{\delta}_i(t,X_i(t)) = \delta_i(t,X_i(t))\). A combination of (84) - (85) provides

\[
|\Delta_i(t_n, X_i(t_n)) - \hat{\delta}_i(t_n, X_i(t_n))| = |\Delta_i(t_n, X_i(t_n)) - \delta_i(t_n, X_i(t_n))| \leq C_4 h^2,
\]

and

\[
|\Delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) - \hat{\delta}_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}}))| = |\Delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) - \delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}}))| \leq C_5 h^2.
\]

We recall that the two-level explicit method (71), (72) is generated by the function \(\delta_i\). In fact, plugging equations (70), (71) and (74), \(\delta_i\) is explicitly defined as

\[
\delta_i(t_n, X_i(t_n)) = \begin{cases} 
F_i(t_n, X^n) + \frac{h}{4} \left( \partial_i F_i(t_n, X^n) + \sum_{k=1}^{9} F_k(t_n, X^n) \partial_k F_i(t_n, X^n) \right), & \text{if } h \neq 0, \\
F_i(t_n, X^n), & \text{if } h = 0,
\end{cases}
\]

and

\[
\delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) = 3F_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})) - F_i(t_n, X^n).
\]

Thus, approximations (70) and (71) become

\[
X_i^{n+\frac{1}{2}} = X_i^n + \frac{h}{2} \delta_i(t_n, X_i(t_n)) \quad \text{and} \quad X_i^{n+1} = X_i^{n+\frac{1}{2}} + \frac{h}{2} \delta_i(t_{n+\frac{1}{2}}, X_i(t_{n+\frac{1}{2}})).
\]
Analogously, the two-level numerical scheme generated by \( \hat{\delta}_i \) should furnish the approximate solutions which satisfy

\[
\hat{X}_i^{n+\frac{1}{2}} = \hat{X}_i^n + \frac{h}{2} \hat{\gamma}_i(t_n, \hat{X}_i(t_n))
\]

and

\[
\hat{X}_i^{n+1} = \hat{X}_i^{n+\frac{1}{2}} + \frac{h}{2} \hat{\delta}_i(t_{n+\frac{1}{2}}, \hat{X}_i(t_{n+\frac{1}{2}})).
\]

In view of equation (93), we have

\[
X_i(t_{l+\frac{1}{2}}) = X_i(t_l) + \frac{h}{2} \Delta_i(t_n, X_i(t_l)), \quad \text{for} \quad l = n, n + \frac{1}{2}.
\]

Plugging relations (91)-(93), and because \( \hat{\gamma}_i^l = X_i(t_l) - X_i^l \), direct calculations result in

\[
\hat{\epsilon}_i^{n+\frac{1}{2}} = \hat{\epsilon}_i^n + \frac{h}{2} \left( \Delta_i(t_n, X_i(t_n)) - \hat{\delta}_i(t_n, X_i(t_n)) \right) = \hat{\epsilon}_i^n + \frac{h}{2} \left[ (\hat{\delta}_i(t_n, X_i(t_n)) - \hat{\delta}_i(t_n, X_i(t_n))) 
+ (\Delta_i(t_n, X_i(t_n)) - \hat{\delta}_i(t_n, X_i(t_n))) \right].
\]

Taking the absolute value, it is easy to see that

\[
\left| \hat{\epsilon}_i^{n+\frac{1}{2}} \right| \leq \left| \hat{\epsilon}_i^n \right| + \frac{h}{2} \left[ \left| \hat{\delta}_i(t_n, X_i(t_n)) - \hat{\delta}_i(t_n, X_i(t_n)) \right| + \left| \Delta_i(t_n, X_i(t_n)) - \hat{\delta}_i(t_n, X_i(t_n)) \right| \right] \leq \left| \hat{\epsilon}_i^n \right| + \frac{h}{2} \left[ \left| \Delta_i(t_n, X_i(t_n)) - \hat{\delta}_i(t_n, X_i(t_n)) \right| \right] \leq \left( 1 + \frac{Lh}{2} \right) \left| \hat{\epsilon}_i^n \right| + C_4 h^3,
\]

where \( L = \max_{1 \leq i \leq 9} L_i \). The last two estimates follows from inequalities (87) and (89). Taking the maximum over \( i \), estimate (94) gives

\[
\left\| \hat{\epsilon}_i^{n+\frac{1}{2}} \right\|_{\infty} \leq \left( 1 + \frac{Lh}{2} \right) \left\| \hat{\epsilon}_i^n \right\|_{\infty} + C_4 h^3.
\]

In a similar way, utilizing relations (92), (93), (77) and (90), one easily shows that

\[
\left\| \hat{\epsilon}^{n+1} \right\|_{\infty} \leq \left( 1 + \frac{Lh}{2} \right) \left\| \hat{\epsilon}^{n+\frac{1}{2}} \right\|_{\infty} + \frac{C_5}{2} h^3.
\]

Substituting estimate (95) into (96) to obtain

\[
\left\| \hat{\epsilon}^{n+1} \right\|_{\infty} \leq \left( 1 + \frac{Lh}{2} \right)^2 \left\| \hat{\epsilon}^n \right\|_{\infty} + \left[ \left( 1 + \frac{Lh}{2} \right) \frac{C_4}{2} + \frac{C_5}{2} \right] h^3 = (1 + \alpha_{1h}) \left\| \hat{\epsilon}^n \right\|_{\infty} + \alpha_{2h} h^3,
\]

where \( \alpha_{1h} = L \left( 1 + \frac{Lh}{2} \right) \) and \( \alpha_{2h} = \left( 1 + \frac{Lh}{2} \right) \frac{C_4}{2} + \frac{C_5}{2} \). To guarantee the convergence of the algorithm, the step size should satisfy \( 0 < h \leq 1 \). This restriction allows to write: \( \alpha_{2h} \leq \left( 1 + \frac{Lh}{4} \right) \frac{C_4}{2} + \frac{C_5}{2} := \alpha_2 \). This fact, together with inequality (87) yield

\[
\left\| \hat{\epsilon}^{n+1} \right\|_{\infty} \leq \left( 1 + \frac{Lh}{2} \right)^2 \left\| \hat{\epsilon}^n \right\|_{\infty} + \alpha_2 h^3.
\]

Applying Lemma 1.1 it holds

\[
\left\| \hat{\epsilon}^n \right\|_{\infty} \leq \exp(n \alpha_{1h}) \left\| \hat{\epsilon}^0 \right\|_{\infty} + \frac{\exp(n \alpha_{1h}) - 1}{\alpha_{1h}} \alpha_{2h} h^3.
\]

But it comes from the initial condition that \( \hat{\epsilon}^0 = 0 \). Using this, relation (98) becomes

\[
\left\| \hat{\epsilon}^n \right\|_{\infty} \leq \frac{\exp[n Lh(1 + \frac{Lh}{4})] - 1}{L(1 + \frac{Lh}{4})} \alpha_2 h^3.
\]
Since $h = \frac{T_{max} - t_0}{n}$ and $t_n = t_0 + nh$, then $nh = t_n - t_0 \leq T_{max}$. Furthermore, $1 < 1 + \frac{L}{4} h \leq 1 + \frac{L}{4}$ (indeed, $0 < h \leq 1$). Utilizing this fact, we have

$$\| \hat{e}^n \|_\infty \leq \frac{\exp[\alpha T_{max}(1 + \frac{L}{4})] - 1}{L} e^{2h^2},$$

which can be rewritten as

$$|X_i(t_n) - \hat{X}_i^n| \leq \frac{\alpha}{L} \left[ \exp[\alpha T_{max}(1 + \frac{L}{4})] - 1 \right] h^2 < \frac{\alpha}{L} \left[ \exp[\alpha T_{max}(1 + \frac{L}{4})] - 1 \right], \text{ for } i = 1, 2, ..., 9, \quad (100)$$

since $0 < h \leq 1$. Setting $v = \frac{\alpha}{L} \left[ \exp[\alpha T_{max}(1 + \frac{L}{4})] - 1 \right] > 0$, estimate (100) indicates that $(t_n, \hat{X}_i^n) \in D_i$, for $i = 1, 2, ..., 9$.

Now, according to the definition of $\hat{\delta}_i$, we should have $X_i^n = \hat{X}_i^n$, $e_i^n = \hat{e}_i^n$ and $\hat{\delta}_i(t_n, X_i(t_n)) = \hat{\delta}_i(t_n, \hat{X}_i(t_n))$. This fact and estimates (99), (100) provide

$$\|X(t_n) - X^n\|_\infty \leq v \text{ and } \|e^n\|_\infty \leq vh^2.$$  (101)

It comes from the inequality $\|u\|_\infty - \|v\|_\infty \leq \|u - v\|_\infty$ (for any $u, v \in \mathbb{R}^n$) and estimate (101) that

$$\|X^n\|_\infty \leq \|X(t_n)\|_\infty + v.$$  (102)

The analytical solution $X(\cdot)$ is bounded on the interval $[t_0, T_{max}]$ because $(t, X(t)) \in D$, where the domain $D$ is given by relation (73). It follows from relation (102) that the approximate solution $X$ is also bounded over the interval $[t_0, T_{max}]$. Hence, the proposed approach (70)-(72) for solving the initial-value problem (1)-(3) is stable.

Finally, the second estimate in relation (101) suggests that the new method is at least second-order convergent. This completes the proof of Theorem 4.1.

5 Numerical experiments and Convergence rate

In this section, we use MatLab R2007b and we present a broad range of numerical evidences to illustrate and demonstrate the efficiency of the proposed approach applied to the mathematical model of the Covid-19 spreading. We stress that in this situation, we obtain satisfactory results, so our algorithm performances are not worse for multidimensional problems. We consider the particular case of the SARS-Cov-2 in Cameroon where the data are available in [6] and we analyze and discuss the obtained results. The other real data used in this study are taken from the literature [24, 25, 6]. More precisely, the number of people in this country is approximately $N = 25,000,000$, the fraction at time $t$, of people in compartment $X_5$ that are hospitalized is assumed equals 1 ($\rho(t) = 1$, because of the decision made by WHO it was decided to hospitalized all detected cases to reduce the transmission of the SARS-Cov-2 virus [51]), $d_{X_2} = 5.5$ days, $d_{X_3} = 6$ days, $d_{X_4} = 7.3$ days (that is, $\gamma_{X_1} = \frac{1}{7.3}$ (day$^{-1}$)), $C_u = 0.3$ days, $T_{max} = 55$ days, $t_0 = 0$ (which corresponds to 06 March 2020), $h = 1$ (the step size), $w = 0.00133$, $w = 0.0667$, $\beta_{X_2} = 0.1125$, $\beta_{X_3} = 0.375$, $d_0 = 14$ (the period of convalescence, we recall that this corresponds to the time a person is still hospitalized after recovering from Covid-19), $\gamma_{X_2} = 0.1818$, $\gamma_{X_3} = 0.78895$, $C_{X_2} = 0.3$, $k = 0.13$ (efficiency of control measures), $\lambda = 12$ (first day of the application of the control measures which corresponds to 17 March 2020 in Cameroon), $\delta_{X_3} = C_u \times \beta_{X_3} = 0.1125$ and $\gamma_{X_3} = \beta_{X_3} = 0.375$.

To prevent the transmission of the Coronavirus 2019 in the other cities of the country, the authorities decided to suspend all fights in the country and to restrict the number of passengers in all public transportation and outbound trains in the cities of Yaounde, Douala, Bafoussam and municipalities on 17 March 2020 [6]. Hence, the following implementation of the control measures is considered in order to indicate the real situation of the measured imposed

$$m(t) = \begin{cases} 
1, & \text{if } t \in [05 \text{ March } 2020, \lambda); \\
\exp[-k(t - \lambda)], & \text{if } t \in [\lambda, T_{max}),
\end{cases}$$

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where $\lambda$ corresponds to 17 March 2020. Furthermore, we assume that the fraction of detected infected people ($\theta(t)$) is a linear function and the disease contact rate of a person in compartment $X_4$ denoted $w(t)$ (without taking into account the control measures) in this territory is a continuous function and are given by

$$\theta(t) = \overline{\theta} + \frac{1 - \overline{\theta}}{T_{\text{max}}} t$$ and $\beta_{X_4}(\theta(t)) = \beta_{X_3} + \left(\beta_{X_4} - \beta_{X_3}\right) \frac{1 - \theta(t)}{1 - w(t)}$.

We recall that the initial data used in our experiments are taken in [6]. Specifically, as on 06 March 2020, there were two cases in which the first one was imported on 24 February 2020 and the other case was occurring by contact with the first case reported. By devoting resources, on 08 March 2020, 108 of the 176 identified individuals of local transmission were traced back to their presumed exposure, either to a known case or to a location linked to spread [6]. From this observation, we set $X_1(t_0) = N - 176$, $X_2(t_0) = 172$, $X_3(t_0) = X_4(t_0) = 1$, $X_5(t_0) = 2$, $X_j(t_0) = 0$, for $j = 6, 7, 8, 9$. Using the above tools together with the data provided in [6], we perform a wide set of numerical experiments to demonstrate the robustness of the new approach. More specifically:

In Table 1 we show the evolution of the model cumulative number of cases in Cameroon from 06 March 2020 to 30 April 2020 subjects to various values of the fraction of detected infected people that are documented (\(\theta\)), whereas Figure 1 (Figure 1) deals with the evolution of the predicted number of cases in the same country during the considered time interval (here, $T_{\text{max}} = 55$ days), taking into account different values of $\theta$. The model cumulative number of reported deaths and computed ones are presented in both Table 2 and Figure 1 (Figure 2). Also in this case, we consider different values of $\theta$. Table 3 and Figure 1 (Figure 3) indicate the model cumulative number of hospitalized people using various values of $\theta$ while both Table 4 and Figure 1 (Figure 4) suggest the number of infected individuals, who are not expected to be detected yet, may infect other persons and start to developing clinical signs. In both Table 5 and Figure 2 (Figure 5), we present the expected number of people that will recover, but can still infect other persons. In Table 6 and Figure 2 (Figure 6), we show the evolution of the cumulative number of people who recovered after being previously infected but were undetected and documented by the government and are no longer infectious. The model cumulative number of individuals exposed to the Covid-19 disease is indicated in both Table 7 and Figure 2 (Figure 7). Both Table 8 and Figure 2 (Figure 8) present the evolution of the number of persons infected by contact with people in compartment $X_2$, while Table 9 together with Figure 2 (Figure 9) show the model cumulative number of people infected by contact with individuals in compartment $X_4$. In Table 10 and Figure 2 (Figure 10), we present the number of persons infected by contact with people in compartment $X_{10} = X_5 + X_6$. Each analysis described above considers various values of $\theta$. Finally, we draw in Table 11 and sketch in Figure 3 (Figure 11) the effective reproduction number of SARS-Cov-2 for Cameroon.

Now, focusing on the values taken by $\theta$, we observe from the tables and figures that for a greater value of $\theta$ (considered in this analysis) the predicted results reproduce quite accurately the evolution of the number of cases, number of deaths and number of hospitalized people. In particular, for $\theta = 0.3212$ the predicted values on 30 April 2020 of cases, deaths and people in hospital are approximately 2236.0, 111.2786 and 47.4915, respectively, while these values become overestimated: 3260.6 cases, 166.1457 deaths and 90.1832 hospitalized people when $\theta = \overline{\theta} = 0.0667$ (the smallest value taken by $\theta$ in $[\overline{\theta}, 1]$). Furthermore, for various values of the parameter $\theta$ we observe that the maximum number of undetected cases: 116.0502 people (for $\theta = 0.0667$), 55.1727 people (for $\theta = 0.1515$), 74.3269 people (for $\theta = 0.3212$) and 56.5865 people (for $\theta = 0.7455$) estimated by the new approach represent: 5.5% (for $\theta = 0.0667$), 4.2257% (for $\theta = 0.1515$), 4.0439% (for $\theta = 0.3212$) and 3.6912% (for $\theta = 0.7455$) of the number of total cases obtained on 30 March 2020. Interestingly, the results provided by our method suggest that, despite the relative control of Covid-19 pandemic in Cameroon, they may still exist an undetected source of infected persons that could cause the increase of the disease in a near future, if the implementation of the control measures are significantly relaxed like the government decided at the beginning of May 2020. In addition, Figure 1 (Figure 3) indicates that the peak of the persons hospitalized in this country at the same time should be reached around 05 April 2020, with approximately 800 hospitalized patients. This number is associated to the smallest value of $\theta = 0.0667$ considered in this study. However, the obtained results slightly overestimate the observed values by around 693. Focusing on the recovered people, the considered technique suggests a maximum number of 3037.7 people, 2300.8 people, 2073.5 people, and 1686.0 people corresponding to various values of $\theta : 0.0667, 0.1515,
0.3212 and 0.7455, respectively. For $\theta = 0.1515$, the final number of 241.0999 people is very close to the real observation (around 244 hospitalized individuals on 15 April 2020). This shows that the proposed method is a reasonable decision tool to estimate the number of beds in hospital during a pandemic or an epidemic. Also, it is worth mentioning that our approach is able to detect early a reasonable expected date of this peak.

Finally, we observe from both Table 11 and Figure XI (Figure 11) that the value of the effective reproduction number ($R_e$) decreases since the application of the control measures and it becomes less than 1 after the peak is attained (30 March 2020).

Tables 1. Model cumulative number of cases with various values of $\theta$

|               | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|---------------|---------------|---------------|---------------|---------------|---------------|
| NC(real data) | 2             | 15            | 142           | 848           | 2014          |
| CM(0.0667)    | 2             | 696.3508      | 2149.8        | 3245.3        | 3322.0        |
| CM(0.1515)    | 2             | 681.7603      | 2048.7        | 2467.4        | 2481.5        |
| CM(0.3212)    | 2             | 647.5699      | 1861.4        | 2223.8        | 2236.0        |
| CM(0.7455)    | 2             | 583.5342      | 1538.2        | 1808.4        | 1817.5        |

Tables 2. The evolution of number of deaths with different values of $\theta$

|               | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|---------------|---------------|---------------|---------------|---------------|---------------|
| ND(real data) | 0             | 1             | 6             | 14            | 61            |
| X_9(0.0667)   | 0             | 17.8300       | 79.6390       | 141.2279      | 166.1457      |
| X_9(0.1515)   | 0             | 17.5511       | 73.8069       | 108.6332      | 121.8038      |
| X_9(0.3212)   | 0             | 16.8994       | 68.3353       | 99.5074       | 111.2786      |
| X_9(0.7455)   | 0             | 15.6638       | 58.6885       | 83.6826       | 93.0934       |

Tables 3. The cumulative number of hospitalized people

|               | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|---------------|---------------|---------------|---------------|---------------|---------------|
| NH (real obs.)| 2             | 8             | 61            | 244           | 328           |
| Host(0.0667)  | 2             | 335.5282      | 668.8886      | 476.0395      | 86.9377       |
| Host(0.1515)  | 2             | 325.1239      | 673.6724      | 241.0999      | 44.0635       |
| Host(0.3212)  | 2             | 300.8178      | 583.1121      | 209.3954      | 47.4915       |
| Host(0.7455)  | 2             | 255.0717      | 432.0405      | 158.2955      | 54.7413       |

These numbers should help to expect the number of beds in hospitals during an epidemic.

Tables 4. Cumulative number of infected individuals who are not expected to be detected yet, but start developing clinical signs

|               | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|---------------|---------------|---------------|---------------|---------------|---------------|
| X_3(0.0667)   | 1             | 70.3135       | 116.0502      | 15.4822       | 0.2363        |
| X_3(0.1515)   | 1             | 67.8561       | 85.1727       | 2.6167        | 0.0608        |
| X_3(0.3212)   | 1             | 62.0955       | 74.3269       | 2.2831        | 0.0530        |
| X_3(0.7455)   | 1             | 51.8569       | 56.5865       | 1.7375        | 0.0404        |

Tables 5. Model cumulative number of infected people that will recover but can still infect other persons

|               | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|---------------|---------------|---------------|---------------|---------------|---------------|
| X_5(0.0667)   | 2             | 307.1123      | 608.9473      | 486.4522      | 106.7015      |
| X_5(0.1515)   | 2             | 297.3910      | 626.9163      | 268.6669      | 52.8153       |
| X_5(0.3212)   | 2             | 274.6855      | 547.2620      | 232.7084      | 45.7222       |
| X_5(0.7455)   | 2             | 231.8949      | 413.7484      | 173.1906      | 33.9913       |
Table 6. Evolution of the number of undetected infected people who recovered but are no longer infectious

|                | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|----------------|---------------|---------------|---------------|---------------|---------------|
| $X_5(0.0667)$  | 0             | 342.9927      | 1402.0        | 2582.4        | 3037.7        |
| $X_5(0.1515)$  | 0             | 339.0853      | 1299.3        | 2070.9        | 2300.8        |
| $X_5(0.3212)$  | 0             | 329.8487      | 1202.1        | 1874.4        | 2073.5        |
| $X_5(0.7455)$  | 0             | 312.7987      | 1030.5        | 1537.8        | 1686.0        |

Table 7. Cumulative number of people exposed to Covid-19 disease

|                | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|----------------|---------------|---------------|---------------|---------------|---------------|
| $X_2(0.0667)$  | 172           | 317.5650      | 495.4672      | 52.1206       | 0.7125        |
| $X_2(0.1515)$  | 172           | 306.0850      | 289.9048      | 8.6553        | 0.2011        |
| $X_2(0.3212)$  | 172           | 279.0561      | 252.9417      | 7.5517        | 0.1754        |
| $X_2(0.7455)$  | 172           | 231.3889      | 192.4997      | 5.7472        | 0.1335        |

Table 8. Model cumulative number of people infected by contact with persons in compartment $X_2$

|                | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|----------------|---------------|---------------|---------------|---------------|---------------|
| $X_2(0.0667)$  | 19.3499       | 35.7245       | 37.7351       | 0.6432        | 0.0013        |
| $X_2(0.1515)$  | 19.3499       | 34.4331       | 0.0000        | 0.0000        | 0.0000        |
| $X_2(0.3212)$  | 19.3499       | 31.3926       | 0.0000        | 0.0000        | 0.0000        |
| $X_2(0.7455)$  | 19.3499       | 26.0303       | 0.0000        | 0.0000        | 0.0000        |

Table 9. Model cumulative number of people infected by contact with persons in compartment $X_4$

|                | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|----------------|---------------|---------------|---------------|---------------|---------------|
| $X_4(0.0667)$  | 0.3750        | -9.1174       | -12.2623      | -1.5011       | -0.0477       |
| $X_4(0.1515)$  | 0.3641        | -8.0160       | 0.0000        | 0.0000        | 0.0000        |
| $X_4(0.3212)$  | 0.3368        | -5.6330       | 0.0000        | 0.0000        | 0.0000        |
| $X_4(0.7455)$  | 0.2686        | -1.5867       | 0.0000        | 0.0000        | 0.0000        |

Table 10. Model cumulative number of people infected by contact with persons in compartment $X_{10} = X_5 + X_6$

|                | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|----------------|---------------|---------------|---------------|---------------|---------------|
| $X_{10}(0.0667)$ | 0.0044        | 0.7379        | 0.9748        | 0.1191        | 0.0038        |
| $X_{10}(0.1515)$ | 0.0058        | 0.9354        | 0.0000        | 0.0000        | 0.0000        |
| $X_{10}(0.3212)$ | 0.0069        | 1.0414        | 0.0000        | 0.0000        | 0.0000        |
| $X_{10}(0.7455)$ | 0.0076        | 0.9745        | 0.0000        | 0.0000        | 0.0000        |

Table 11. Effective reproduction number of SARS-Cov-2 for Cameroon

|                | 06 March 2020 | 17 March 2020 | 30 March 2020 | 15 April 2020 | 30 April 2020 |
|----------------|---------------|---------------|---------------|---------------|---------------|
| $\text{Re}(0.0667)$ | 1.1111        | 1.1111        | 0.7546        | 0.1229        | 0.0175        |
| $\text{Re}(0.1515)$ | 1.1164        | 1.1163        | 0.0000        | 0.0000        | 0.0000        |
| $\text{Re}(0.3212)$ | 1.1209        | 1.1208        | 0.0000        | 0.0000        | 0.0000        |
| $\text{Re}(0.7455)$ | 1.1236        | 1.1236        | 0.0000        | 0.0000        | 0.0000        |

6 General conclusion and future works

We have developed an efficient two-level explicit method for estimating the propagation of Covid-19 disease with undetected infectious cases. The analysis has shown that the new algorithm is stable, at least
second-order accuracy and can serve as a fast and robust tool for integrating of general systems of ordinary differential equations. Numerical results based on the case of Cameroon reproduce quite accurately the evolution of the number of cases (detected or undetected), number of deaths, number of people in hospitals, number of infected detected persons who recovered and number of infected undetected individuals who recovered by natural immunity from 06 March 2020 to 30 April 2020. The approach presented in this work can help to estimate the number of beds in hospitals during a pandemic. Furthermore, the proposed technique can be considered as a fundamental tool for detecting early a reasonable expected date of the peak during an epidemic. Our future works will consider the numerical solution of a more complex system of ordinary differential equations using the new two-level explicit approach.

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Model cumulative number of cases, deaths, hospitalized people and infected persons who are not detected yet.

Figure 1: Number of cases, deaths, hospitalized people and undetected infected individual
Cumulative number of recovered infected, undetected infected who recovered, people exposed to Covid-19, infected by contact with $X_2$.

Figure 5: Cumulative number of infected people that will recover with various values of $\theta$.

Figure 6: Cumulative number of undetected infected people who recovered with various values of $\theta$.

Figure 7: Cumulative number of people exposed to Covid-19 with different values of $\theta$.

Figure 8: Cumulative number of people infected by contact with $X_2$ with various $\theta$.

Figure 2: Number of recovered infected, recovered undetected infected, exposed to Covid-19, infected by contact with $X_2$. 
Model cumulative number of people infected by contact with $X_4$, $X_{10}$ and effective reproduction number.

Figure 9: Cumulative number of people infected by contact with $X_4$ with various $\theta$.

Figure 10: Cumulative number of people infected by contact with $X_{10}$ with various $\theta$.

Figure 11: Effective reproduction number of Covid-19 with various values of $\theta$.

Figure 3: Number of infected by contact with $X_4$, $X_{10}$ and effective reproduction number $Re$. 

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