Deformed Statistics Formulation of the Information Bottleneck Method

Abstract—The theoretical basis for a candidate variational principle for the information bottleneck (IB) method is formulated within the ambit of the generalized nonadditive statistics of Tsallis. Given a nonadditivity parameter \( q \), the role of the additive duality of nonadditive statistics \( (q^* = 2 - q) \) in relating Tsallis entropies for ranges of the nonadditivity parameter \( q < 1 \) and \( q > 1 \) is described. Defining \( X, \tilde{X}, X, Y \) to be the source alphabet, the compressed reproduction alphabet, and, the relevance variable respectively, it is demonstrated that minimization of a generalized IB (gIB) Lagrangian defined in terms of the generalized IB method are derived. These results generalize critical features of the IB method to the case of Tsallis statistics.

I. INTRODUCTION

Rate distortion (RD) theory [1,2] is a major branch of information theory which provides the theoretical foundations for lossy data compression. RD theory addresses the problem of determining the minimal amount of entropy (or information) \( R \) that should be communicated over a channel, so that the source (input signal/source alphabet/codebook) \( X \in \mathcal{X} \) can be approximately reconstructed at the receiver (output signal/reproduction alphabet/quantized codebook) \( \tilde{X} \in \tilde{\mathcal{X}} \) without exceeding a given expected distortion \( D \). Note that calligraphic fonts are used to denote sets. In turn, the information bottleneck (IB) method is a technique introduced by Tishby, Pereira, and Bialek [3, 4] for finding the best tradeoff between accuracy and complexity (compression) when summarizing (e.g. clustering) a discrete random variable \( X \), given a joint probability distribution between \( X \) and a relevance variable \( Y \in \mathcal{Y} \), i.e. \( p(x, y) \). In this regard, the IB method represents a significant qualitative improvement over RD theory. The IB method has acquired immense utility in machine learning theory. For example, the IB method and its modifications have successfully been employed in applications in diverse areas such as genome sequence analysis, astrophysics, and, text mining [4]. q-Deformed (or Tsallis) statistics [5,6] has recently been shown to yield interesting improvements concerning RD theory [7]. The present paper extends analogous ”q-“considerations to the IB method.

The generalized (nonadditive) statistics of Tsallis’ has recently been the focus of much attention in statistical physics, and allied disciplines. Note that the terms generalized statistics, \( q—deformed \) statistics, nonadditive statistics, and nonextensive statistics are used interchangeably. Nonadditive statistics, which generalizes the Boltzmann-Gibbs-Shannon (B-G-S) statistics, has recently found much utility in a wide spectrum of disciplines ranging from complex systems and condensed matter physics to financial mathematics. A continually updated bibliography of works in nonadditive statistics may be found at http://tsallis.cat.cbpf.br/biblio.htm.

Since the work on nonextensive source coding by Landsberg and Vedral [8], a number of studies on the information theoretic aspects of generalized statistics pertinent to coding related problems have been performed [9-12]. Most recently, the nonadditive statistics of Tsallis [5,6] has been utilized to develop a generalized statistics RD theory [7]. This paper [7] investigates nonadditive statistics within the context of RD theory in lossy data compression. The generalized statistics RD model performs variational minimization of the nonadditive RD Lagrangian employing a method developed [13] to ”rescue” the linear constraints originally employed by Tsallis [5]. Nonadditive statistics possesses a number of constraints having different forms [14-16].

RD theory is now briefly described as a precursor to introducing the leitmotif for the IB method. For a source alphabet \( X \in \mathcal{X} \) and a reproduction alphabet \( \tilde{X} \in \tilde{\mathcal{X}} \), the mapping of \( x \in \mathcal{X} \) to \( \tilde{x} \in \tilde{\mathcal{X}} \) is characterized by the quantizer \( p(\tilde{x}|x) \). The RD function is obtained by minimizing the generalized mutual information (GMI) \( I_q(X;\tilde{X}) \) (defined in Section 2, [7]) over all normalized \( p(\tilde{x}|x) \). In RD theory \( I_q(X;\tilde{X}) \) is known as the compression information (see Section 3, [7]). Here, \( q \) is the nonadditivity parameter [5, 6]. A significant feature of the nonadditive RD model [7] is that the threshold for the compression information is lower than that encountered in RD models derived from B-G-S statistics. This feature augurs well for utilizing Tsallis statistics in data compression applications.

1The absence of a definitive nonadditive channel coding theorem sometimes prompts the use of the term generalized mutual entropy instead of GMI [10].
By definition, the nonadditive RD function is \[ R_q(D) = \min_{p(\tilde{x}|x):d(x,\tilde{x}) \leq D} I_q \left( X; \tilde{X} \right) : 0 < q < 1, \] where, \( R_q(D) \) is the minimum of the compression information. The distortion measure is denoted by \( d(x,\tilde{x}) \) and is taken to be the Euclidean square distance for most problems in science and engineering [1,2]. Given \( d(x,\tilde{x}) \), the partitioning of \( X \) induced by \( p(\tilde{x}|x) \) has an expected distortion \( D = \langle d(x,\tilde{x}) \rangle > p(x,\tilde{x}) \).

Note that in this paper, \( \langle \cdot \rangle_{p(\cdot)} \) denotes the expectation with respect to the probability \( p(\cdot) \). RD theory a-priori specifies the nature of the distortion measure, which is tantamount to an a-priori specification of the features of interest in the source alphabet \( X \) to be contained in compressed representation \( \tilde{X} \).

Thus, the crux of the IB method is to simultaneously minimize the compression information \( I_q(X;\tilde{X}) \) and maximize the relevant information \( I_q(\tilde{X};Y) \). More specifically, the IB method extracts structure from the source alphabet via data compression, followed by a quantification of the information contained in the extracted structure with respect to a relevance variable. Consequently, the IB method "squeezes" the information between \( X \) and \( Y \) through a bottleneck \( \tilde{X} \). The IB method is compactly described by the Markov condition [3,4]

\[ \tilde{X} \leftrightarrow X \leftrightarrow Y. \]

As discussed in [7], the un-normalized GMI in Tsallis statistics acquires different forms in the regimes \( 0 < q < 1 \) and \( q \geq 1 \), respectively. For example, for \( 0 < q < 1 \), the GMI is of the form \( I_{q<1}(X;\tilde{X}) = -\sum_{x,\tilde{x}} p(x,\tilde{x}) \ln_q \left( \frac{p(\tilde{x}|x)}{p(x,\tilde{x})} \right) \).

For \( q > 1 \), the GMI is defined by \( I_{q>1}(X;\tilde{X}) = S_q(X) + S_q(\tilde{X}) - S_q(X,\tilde{X}) \), where \( S_q(X) \) and \( S_q(\tilde{X}) \) are the marginal Tsallis entropies for the random variables \( X \) and \( \tilde{X} \), and, \( S_q(X,\tilde{X}) \) is the joint Tsallis entropy [7]. Unlike the B-G-S case, \( I_{0<q<1}(X;\tilde{X}) \) can never acquire the form of \( I_{q>1} \), and vice versa [7, 9, 10], the reason being that the sub-additivities \( S_q(X,\tilde{X}) \leq S_q(X) \) and \( S_q(X,\tilde{X}) \leq S_q(\tilde{X}) \) are not generally valid when \( 0 < q < 1 \). While the form of \( I_{0<q<1}(X;\tilde{X}) \) is important in a number of practical interest in coding theory and learning theory where it is desirable that the GMI be expressed as the generalized Kullback-Leibler divergence (K-Ld) between the joint probability \( p(x,\tilde{x}) \) and the marginal probabilities \( p(x) \) and \( p(\tilde{x}) \) [1], un-normalized Tsallis entropies for \( q > 1 \) possess a number of important properties such as the \textit{generalized data processing inequality} and the \textit{generalized Fano inequality} [10]. The different forms of the GMI for \( 0 < q < 1 \) and \( q > 1 \) are reconciled by invoking the \textit{additive duality of nonadditive statistics} [17]. This entails a re-parameterization of the nonadditivity parameter \( q^* = 2 - q \), resulting in \textit{dual Tsallis entropies}.

This paper derives a theoretical basis for the generalised IB (gIB) method, which is a fundamental qualitative extension of the seminal work of Tishby, Pereira, and Bialek [3]. This analysis commences with the minimization of the gIB Lagrangian

\[ L^g_{IB}(p(\tilde{x}|x)) = I_q(X;\tilde{X}) - \tilde{\beta}_{gIB} I_q(\tilde{X};Y) : q > 1, \]

subject to the normalization of \( p(\tilde{x}|x) \). Here, \( \tilde{\beta}_{gIB} \) is the gIB tradeoff parameter for the simultaneous minimization and maximization described by (3). From (3), it is easily shown that: \( \delta L^g_{IB}(p(\tilde{x}|x)) = 0 \Rightarrow \frac{\delta I_q(X;\tilde{X})}{\delta I_q(\tilde{X};Y)} = \frac{1}{\tilde{\beta}_{gIB}} \). Thus, by increasing \( \tilde{\beta}_{gIB} \), convex curves akin to the RD curves [1,2,7], may be constructed in the "information plane" (\( I_q(X;\tilde{X}), I_q(\tilde{X};Y) \)). These are called relevance-compression curves [4].

Apart from its ability to model \textit{long-range interactions} when performing clustering of complex data sets, the gIB method also facilitates the analysis of the IB method within the context of predictability [18]. Predictability may be viewed as an excursion from the extensive B-G-S statistics, and is inherently nonextensive (nonadditive).

II. TSALLIS ENTROPIES AND DUAL TSALLIS ENTROPIES

The un-normalized Tsallis entropy, conditional Tsallis entropy, the jointly convex generalized K-Ld, and, the GMI may thus be written as [19,20]

\[ S_q(X) = -\sum_x p(x)^q \ln_q p(x), S_q(\tilde{X}) = -\sum_{\tilde{x}} p(\tilde{x})^q \ln_q p(\tilde{x}) \]

\[ S_q(X,\tilde{X}) = -\sum_{x,\tilde{x}} p(x,\tilde{x})^q \ln_q p(x,\tilde{x}) = S_q(X) + S_q(\tilde{X}) - S_q(X,\tilde{X}) \]

\[ D^q_{K-L}(p(X) || r(X)) = -\sum_p p(x) \ln_q \frac{r(x)}{p(x)} \]

\[ I_{0<q<1}(X;\tilde{X}) = -\sum_{x,\tilde{x}} p(x,\tilde{x}) \ln_q \frac{p(\tilde{x}|x)}{p(x,\tilde{x})} = D^q_{K-L}(p(X,\tilde{X}) || p(X)p(\tilde{X})) \]

respectively. The \textit{q-deformed logarithm} and the \textit{q-deformed} exponential are defined for \( 0 < q < 1 \) as [21]

\[ \ln_q(x) = \sum_{k=0}^{\infty} \frac{1-q}{1-q} \frac{x^k}{k!} \]

and,

\[ \exp_q(x) = \left\{ \begin{array}{ll} 1 + (1 - q)x & ; 1 + (1 - q)x \geq 0, \\ 0; & otherwise \end{array} \right. \]

The operations of \textit{q-deformed} relations are governed by \( q - \text{algebra} \) and \( q - \text{calculus} \) [21]. Apart from providing an analogy to equivalent expressions derived from B-G-S statistics, \( q - \text{algebra} \) and \( q - \text{calculus} \) endow generalized statistics with a unique information geometric structure. Salient results of \( q\text{-algebra} \) employed in this paper involving
the $q$--deformed addition ($\oplus_q$) and subtraction ($\ominus_q$), are [21]
\[ x \oplus_q y = x + y + (1 - q)xy, \]
\[ x \ominus_q y = \frac{x - y}{1 + (1-q)y}, \]
where $\ominus_q y = \frac{y}{y - 1}$.

\[ \ln_q (xy) = \ln_q (x) + \ln_q (y), \]
\[ \ln_q (x/y) = \ln_q (x) - \ln_q (y), \]
\[ \ln_q (x^{y}) = y \ln_q (x). \]

Given two independent variables $X$ and $Y$, one of the fundamental consequences of nonadditivity of the Tsallis entropy is the pseudo-additivity relation
\[ S_q (XY) = S_q (X) + S_q (Y) + (1 - q)S_q (X)S_q (Y). \]  
(7)

Re-parameterizing (5) via the additive duality $q^* = 2 - q$, yields the dual deformed logarithm and exponential
\[ \ln_{q^*} (x) = -\ln_q \left( \frac{1}{x} \right), \text{ and, } \exp_{q^*} (x) = \frac{1}{\exp_q (1/x)}. \]  
(8)

A dual Tsallis entropy defined by
\[ S_{q^*} (X) = -\sum p (x) \ln_{q^*} p (x). \]  
(9)

The dual Tsallis joint entropy obeys the relation
\[ S_{q^*} (X, \tilde{x}) = S_{q^*} (X) + S_{q^*} (\tilde{x} | X), \]
where,
\[ S_{q^*} (\tilde{x} | X) = -\sum \sum p (x, \tilde{x}) \ln_{q^*} p (\tilde{x} | x). \]  
(10)

Here, $\ln_{q^*} (x) = \frac{x^{1-q^*} - 1}{1-q^*}$. The dual Tsallis entropies acquire a form identical to the B-G-S entropies, with $\ln_q (\bullet)$ replacing $\log (\bullet)$. The GMI's $I_{q>1} (X; \tilde{X})$ and $I_{0<q<1} (X; \tilde{X})$ defined by the nonadditivity parameters $q > 1$ and $0 < q^* < 1$ respectively, relate to each other as (Theorem 3, [7])
\[ I_{q^*} (X; \tilde{X}) = -\sum \sum p (x, \tilde{x}) \ln_{q^*} \left( \frac{p(x)p(\tilde{x})}{p(x,\tilde{x})} \right) \]
\[ = (q^* = q) S_q (X) + S_{q^*} \left( \tilde{x} | X \right) \]
\[ = (q^* = q) I_{q^*} (\tilde{x} | X). \]  
(11)

Here, "$q^* \to q$" is a re-parameterization from $q^*$ to $q$, and,"$q \to q^*$" is a re-parameterization from $q$ to $q^*$.

III. GENERALIZED INFORMATION BOTTLENECK VARIATIONAL PRINCIPLE

A. Self-consistent equations

Depending upon the "upstream" and "downstream" variables in the Markov condition (2), the total probability may be expressed as
\[ \tilde{X} \leftarrow X \leftarrow Y \Rightarrow p (x, \tilde{x}, y) = p (x, y) p (\tilde{x} | x), \]
\[ \tilde{X} \leftarrow X \rightarrow Y \Rightarrow p (x, \tilde{x}, y) = p (x, \tilde{x}) p (y | x). \]  
(12)

The Markov condition $\tilde{X} \leftarrow X \leftarrow Y$ yields [3]
\[ p (y | \tilde{x}) = \frac{1}{p (\tilde{x})} \sum x p (y | x) p (\tilde{x} | x) p (x). \]  
(13)

Since $\tilde{X} \leftarrow X \leftarrow Y \leftarrow X$, the Markov condition yields through application of Bayes rule and consistency [3]
\[ p (y | \tilde{x}) = \sum x p (y | x) p (x | \tilde{x}). \]  
(14)

Thus
\[ p (\tilde{x}) = \sum x p (\tilde{x}, \tilde{y}) = \sum x p (x) p (\tilde{x} | x), \]
and,
\[ p (\tilde{x}, y) = \sum x p (x, \tilde{y}) = \sum x p (y) p (\tilde{x} | x). \]  
(15)

From (15), the following relations are obtained
\[ \frac{\delta p (\tilde{x})}{\delta p (\tilde{x} | x)} = p (x), \text{ and, } \frac{\delta p (\tilde{x}, y)}{\delta p (\tilde{x} | x)} = p (x | y). \]  
(16)

B. The variational principle

The gIB Lagrangian (3) cannot be expressed in terms of the generalized K-Ld. As discussed in [7], the additive duality is required to express the GMI for $q > 1$ in terms of the generalized K-Ld. This is required to formulate nonadditive numerical schemes akin to the EM algorithm [22], using the alternating minimization method based on the Csiszár-Tusnády theory [23]. The gIB Lagrangian in $q^*$--space is
\[ L_{gIB}^q [p (\tilde{x} | x)] = I_{q^*} (X; \tilde{X}) - \beta_{gIB} I_{q^*} (\tilde{X}; Y); 0 < q^* < 1, \]
contingent to the normalization of $p(\tilde{x} | x)$. Here, $I_{q^*} (X; \tilde{X})$ and $I_{q^*} (\tilde{X}; Y)$ are obtained from $I_q (X, \tilde{X})$ and $I_{q^*} (X; \tilde{X})$ employing (11). Variational minimization of (17) [7, 13] yields
\[ \frac{\delta}{\delta p (\tilde{x}, y)} L_{gIB}^q [p (\tilde{x}) | x] = \frac{\partial}{\partial y} \left( \frac{p (\tilde{x})}{p (y | \tilde{x})} \right)^{1-q^*} \]
\[ + \beta_{gIB} \sum_y p (y | \tilde{x}) \ln_{q^*} \left( \frac{p (y)}{p (y, \tilde{x})} \right) \]
\[ = \frac{\delta}{\delta p (\tilde{x}, y)} L_{gIB}^q [p (\tilde{x}) | x] = \frac{\partial}{\partial y} \left( \frac{p (\tilde{x})}{p (y | \tilde{x})} \right)^{1-q^*} \]
\[ + \beta_{gIB} \sum_y p (y | \tilde{x}) \ln_{q^*} \left( \frac{p (y)}{p (y, \tilde{x})} \right) \]
\[ = 0. \]  
(18)

Here, (a) is from Bayes’ theorem $p (y | \tilde{x}) = \frac{p (\tilde{x} | y) p (y)}{p (\tilde{x})}$ and
\[ \frac{p (\tilde{x} | y)}{p (\tilde{x})} = \frac{p (y | \tilde{x})}{p (y)} \]  
and, (16). The term $p (x)$ is canceled out. A $\ln_{q^*} (\bullet)$ term is introduced in (18), by adding and subtracting $\beta_{gIB} \sum_y p (y | x) \ln_{q^*} \left( \frac{p (y)}{p (y, \tilde{x})} \right)$, to yield
\[ \frac{1}{q^* - 1} \left( \frac{p (\tilde{x})}{p (y | \tilde{x})} \right)^{1-q^*} + \beta_{gIB} \sum_y p (y | \tilde{x}) \ln_{q^*} \left( \frac{p (y)}{p (y, \tilde{x})} \right) \]
\[ = 0. \]  
(19)

In (19), $-\lambda (1) (x) = -\lambda (x) / p (x) + \beta_{gIB} \sum_y p (y | x) \ln_{q^*} \left( \frac{p (y | x)}{p (y | \tilde{x})} \right)$, which is only dependent on $x$. The second term in (19) is expressed as:
\[ \beta_{gIB} \sum_y p (y | x) \ln_{q^*} \left( \frac{p (y | x)}{p (y | \tilde{x})} \right) \]
employing $q^*$--deformed
subtraction and addition (6), yielding

\[
\frac{1}{q-1} \left( \frac{p(x)}{p(\tilde{x}|x)} \right) ^{1-q^*} + \tilde{\beta}_{gIB} \sum_y p(y|x) \ln_q \left( \frac{p(y)}{p(y|x)} \right) \Theta > q^* \left( \frac{p(y)}{p(y|x)} \right) \\
- \lambda^{(1)}(x) + \tilde{\beta}_{gIB} \sum_y p(y|x) \ln_q \left( \frac{p(y)}{p(y|x)} \right) \Theta > q^* = 0 \\
\Rightarrow \frac{1}{q-1} \left( \frac{p(x)}{p(\tilde{x}|x)} \right) ^{1-q^*} + \tilde{\beta}_{gIB} \sum_y p(y|x) \ln_q \left( \frac{p(y)}{p(y|x)} \right) \\
- \lambda^{(2)}(x) = 0.
\]

Here, \( -\lambda^{(2)}(x) = -\left[ \tilde{\beta}_{gIB} I_{q^*} (x; Y) \right] \Theta > q^* \lambda^{(1)}(x), I_{q^*} (x; Y) = \sum_y p(y|x) \ln_q \left( \frac{p(y)}{p(y|x)} \right). \) Multiplying (20) by \( p(\tilde{x}|x) \) and summing over \( \tilde{x} \), yields

\[
\lambda^{(2)}(x) = \sum_{\tilde{x}} p(\tilde{x}|x) \left[ \left( \frac{p(\tilde{x}|x)}{p(x)} \right) ^{1-q^*} \right. \\
+ \tilde{\beta}_{gIB} \left( \sum_y p(y|x) \ln_q \left( \frac{p(y)}{p(y|x)} \right) \right) \left. p(\tilde{x}|x) \right].
\]

(21)

Defining \( \mathcal{G}_{gIB}(x) = (q^* - 1) \lambda^{(2)}(x) \), (20) yields

\[
p(\tilde{x}|x) = \left\{ \left. \left[ 1 - (q - 1) \right] \tilde{\beta}_{gIB} \sum_y p(y|x) \ln_q \left( \frac{p(y)}{p(y|x)} \right) \right] \right\} ^{\frac{1}{q-1}} \\
p(\tilde{x}|x) = \mathcal{G}_{gIB}(x) \Rightarrow \tilde{\beta}_{gIB}(x) \Theta > q^* \lambda^{(2)}(x).
\]

Setting \( \tilde{\beta}_{gIB}(x) = \tilde{\beta}_{gIB}(x) \), and, invoking the additive duality in the numerator \( (q^* = 2 - q) \), (22) yields the canonical transition probability

\[
p(\tilde{x}|x) = p(\tilde{x}) \exp\left[ -\tilde{\beta}_{gIB}(x) D^q_{K-L}[p(y|x)|p(y|\tilde{x})] \right] \\
\Rightarrow \tilde{Z} \left( x, \tilde{\beta}_{gIB}(x) \right) = \mathcal{G}_{gIB}(x) \Theta > q^* \lambda^{(2)}(x).
\]

(23)

In (23), \( \tilde{\beta}_{gIB}(x) \) is the gIB tradeoff parameter evaluated for each source alphabet \( x \in X \), and, \( \tilde{Z} \left( x, \tilde{\beta}_{gIB}(x) \right) \) is the partition function. The effective distortion measure has been self consistently obtained via the variational principle to be \( D^q_{K-L}[p(y|x)|p(y|\tilde{x})] \), without any a-priori assumptions. In the limit \( q \rightarrow 1 \) the B-G-S statistics result [3,4] is recovered. Solutions of (23) are valid only for \( \left\{ 1 - (1 - q) \tilde{\beta}_{gIB}(x) D^q_{K-L}[p(y|x)|p(y|\tilde{x})] \right\} > 0 \). The condition \( \left\{ 1 - (1 - q) \tilde{\beta}_{gIB}(x) D^q_{K-L}[p(y|x)|p(y|\tilde{x})] \right\} < 0 \) is called the Tsallis cut-off condition [5], and requires setting \( p(\tilde{x}|x) = 0 \) and stopping the iteration at the given \( \tilde{\beta}_{gIB} \).

C. Free energy of the system

The \( q^* - \text{deformed} \) nonadditive free energy of the system is [3]

\[
F^q_{gIB} \left[ p(\tilde{x}|x) ; p(x); p(y|\tilde{x}) \right] = \mathcal{F}^q_{gIB}(x) \\
= - \ln_q \left[ \tilde{Z} \left( x, \tilde{\beta}_{gIB}(x) \right) \right] p(x) = \left[ \frac{\tilde{\beta}_{gIB}(x) - 1}{q^* - 1} \right] p(x).
\]

(24)

Here, \([ \bullet ]\) denotes the arguments \([ p(\tilde{x}|x) ; p(x); p(y|\tilde{x}) ] \) of the free energy. Note \( F^q_{gIB} \) is the \( q^* - \text{deformed} \) gIB Helmholtz free energy. Invoking (4) and (21), (24) yields

\[
F^q_{gIB}(x) = I_{q^*} \left( x; \tilde{X} \right) + \tilde{\beta}_{gIB} \left( \sum_y p(y|x) \ln_q \left( \frac{p(y)}{p(y|x)} \right) \right) p(x, x) \\
\Rightarrow \left\{ I_{q^*} \left( x; \tilde{X} \right) - \tilde{\beta}_{gIB} \left( \sum_x p(x, x, y) \ln_q \left( \frac{p(x,y)}{p(\tilde{x}, x)} \right) \right) \right\} \\
\Rightarrow D^q_{K-L} \left[ p(\tilde{X}, X) \left\| p(\tilde{X}) \left\| p(X) \right. \right\| p(\tilde{X}) \right] p(x, x) \\
+ \tilde{\beta}_{gIB} D^q_{K-L} \left[ p(\tilde{X}, X) \left\| p(X) \left\| p(X) \right. \right\| p(\tilde{X}) \right] p(x, x). \\
\]

(25)

Here, (a) invokes the additive duality in the second term in order to introduce the expected effective distortion in \( q^* - \text{space} \), and, (b) invokes (12) to obtain the total probability \( p(x, \tilde{x}, y) \). In (25), \( F^q_{gIB}(x) \) is the sum of two generalized K-Ld’s having nonadditivity parameters \( q^* \) and \( q \), where \( 0 < q^* < 1 \) and \( q > 1 \). From [24], it is readily follows that \( F^q_{gIB}(x) \) is nonnegative and convex. The expected effective distortion term in (25) is related to the relevant information as

\[
F^q_{gIB}(x) = I_{q^*} \left( x; \tilde{X} \right) + \tilde{\beta}_{gIB} \left( \sum_x p(x, \tilde{x}) \sum_y p(y|x) \ln_q \left( \frac{p(y)}{p(y|x)} \right) \right) \\
\Rightarrow I_{q^*} \left( x; \tilde{X} \right) - \tilde{\beta}_{gIB} \left( \sum_y \sum_{x, \tilde{x}} \ln_q \left( \frac{p(y)}{p(y|x)} \right) \right) \\
\Rightarrow D^q_{eff} \left[ p(\tilde{X}, X) \left\| p(\tilde{X}) \left\| p(X) \right. \right\| p(\tilde{X}) \right] p(x, x, y) \\
\Rightarrow D_{eff}(x) = I_{q^*} \left( x; Y \right) - I_{q^*} \left( \tilde{X}; Y \right).
\]

(26)

Note \( I_{q^*} \left( \tilde{X}; Y \right) \leq I_{q^*} \left( X; Y \right) \) by the generalized data processing inequality [10]. Here, (a) employs \( \ln_q(x/y) = y^{q-1}(\ln_q(x) - \ln_q(y)) \) in (6), (b) adds and subtracts \( S_q(Y) \), and, invokes (11) and the symmetry of the GMI: \( I_q(Y; X) = I_q(X; Y) \); \( I_q(Y; \tilde{X}) = I_q(X; Y) \). From (22)-(24), an empirical criterion equivalent to the Tsallis cut-off condition, described in terms of the gIB free energy for any \( x \in X \), is: \( 1 + (q^* - 1) F^q_{gIB}(x) \) \( \leq 0 \).

IV. THE UPDATE EQUATIONS

Lemma 1: Given a joint distribution \( p(x)p(\tilde{x}|x) \), the distribution \( p(\tilde{x}) \) that minimizes \( D^q_{K-L}[p(X)p(X|X)||p(X)p(X)] \) is the marginal \( p^*(\tilde{x}) = \sum_x p(x)p(\tilde{x}|x) \), i.e.

\[
D^q_{K-L} \left[ p(X) \left\| p(X) \left\| p(X) \right. \right\| p(X) \right] = \min_{p(\tilde{x})} D^q_{K-L} \left[ p(X) \left\| p(X) \left\| p(X) \right. \right\| p(X) \right].
\]

(27)
Also
\[
\langle D^q_{K-L} [p(\tilde{x} | x)] p^*(\tilde{x}) \rangle_{p(x)} = 
\min_{p(\tilde{x})} \langle D^q_{K-L} [p(\tilde{x} | x)] p(\tilde{x}) \rangle_{p(x)}.
\]

(28)

Proof: The positivity condition for (27) is proven in Lemma 1 of [7], with \( q^* \) replacing \( q \). The positivity condition for (28) is
\[
- \sum_{x, \tilde{x}} p(x, \tilde{x}) \ln_q \left( \frac{p(\tilde{x})}{p(x)} \right) + \sum_{x, \tilde{x}} p(x, \tilde{x}) \ln_q \left( \frac{p^*(\tilde{x})}{p(x)} \right)
\]
(a) \[
= - \sum_{x, \tilde{x}} p(x, \tilde{x}) \ln_q^* \left( \frac{p(\tilde{x})}{p(x)} \right) + \sum_{x, \tilde{x}} p(x, \tilde{x}) \ln_q^* \left( \frac{p^*(\tilde{x})}{p(x)} \right)
\]
(b) \[
= D^q_{K-L} [p^*(\tilde{x}) \| p(\tilde{x})]
\]
\[
\times \left[ 1 + (q - 1) \left( D^q_{K-L} [p(\tilde{x} | x)] p(\tilde{x}) \right)_{p(x)} \right] > 0;
\]
\[
\forall 0 < q^* < 1, q > 1.
\]

Here, (a) invokes the additive duality, (b) employs the \( q \)-deformed algebra definition for \( \ln_q^* \) from (6) [21] by multiplying and dividing by \[
\sum_{x, \tilde{x}} p(x, \tilde{x}) (1 - q^*) \sum_{x, \tilde{x}} p(x, \tilde{x}) \ln_q^* \left( \frac{p(\tilde{x})}{p(x)} \right)
\]
and, establishes \[ p^*(\tilde{x}) = \sum_{x, \tilde{x}} p(x, \tilde{x}) p(\tilde{x} | x) \] after subjecting the term within brackets \( (\bullet) \) in (29) to the additive duality \( (q = 2 - q^*) \).

The free energy \( F_{gIB}[\bullet] \) is convex only when independently evaluated with respect to one of the convex distribution sets \( \{ p(\tilde{x} | x) \} \), \( \{ p(x, \tilde{x}) \} \), and \( \{ p(y | \tilde{x}) \} \). The update equations which minimize the free energy are obtained by projecting the free energy onto each convex distribution while keeping the other two arguments constant.

**Theorem 1:** Equations (14), (15) and (23) are satisfied at the minima of the free energy (24) for each argument of the free energy as
\[
\min_{p(\tilde{x} | x)} \min_{p(x, \tilde{x})} \min_{p(y | \tilde{x})} F_{gIB}[p(\tilde{x} | x); p(x, \tilde{x}) ; p(y | \tilde{x})].
\]

(30)

Denoting the iteration level as \( (\tau) \), minimization is performed independently by alternating between convex iterations
\[
p^{(\tau+1)}(\tilde{x} | x) \leftarrow p^{(\tau)}(\tilde{x}) \exp_a \left[ -\tilde{\beta}_{gIB}(x) D^q_{K-L} [p(y | \tilde{x}) | p^{(\tau)}(y | \tilde{x})] \right] \]
where \( \tilde{\beta}_{gIB}(x) = \frac{\tilde{\beta}_{gIB}(x)}{Z(\tau+1)(x, \tilde{x})} = \frac{\tilde{\beta}_{gIB}(x)}{Z(\tau)(x, \tilde{x})} \),
\[
p^{(\tau+1)}(x) \leftarrow \sum_{x, \tilde{x}} p(x, \tilde{x}) p^{(\tau+1)}(\tilde{x} | x)
\]
\[
p^{(\tau+1)}(y | \tilde{x}) \leftarrow \sum_{x, \tilde{x}} p(x, y) p^{(\tau+1)}(\tilde{x} | x)
\]

(31)

Proof: The outline of the proof is given herein owing to space constraints. Defining \( F^q_1[\bullet] = F_{gIB}[\bullet] + \tilde{\lambda}(x)(p(\tilde{x} | x) - 1) \), and following the procedure in Section III.B. of this paper, \( \frac{\delta F^q_1[\bullet]}{\delta p(\tilde{x} | x)} = 0 \) exactly yields (26). Minimization with respect to \( p(y | \tilde{x}) \) affects only \( D_{eff} \) in (25). Defining \( F^q_2[\bullet] = F_{gIB}[\bullet] + \tilde{\lambda}(\tilde{x})(p(\tilde{x} | x) - 1) \) and invoking \( D_{eff} = I_q(X; Y) - I_q(X; Y) \) from (26), employing (4), (11), (12), and (15) yields
\[
- \tilde{\beta}_{gIB}(x) \sum_{x, \tilde{x}} p(x, \tilde{x}, y) \frac{q^{\lambda}(y | \tilde{x})}{p(y | \tilde{x})} + \tilde{\lambda}(\tilde{x}) = 0
\]
\[
\Rightarrow - \frac{1}{p(y | \tilde{x})} \tilde{\beta}_{gIB}(x) \sum_{x, \tilde{x}} p(x, y, \tilde{x}) + \tilde{\lambda}(\tilde{x}) = 0; \tilde{\lambda}(\tilde{x}) = \frac{\tilde{\lambda}(\tilde{x})}{\tilde{\beta}_{gIB}}
\]

(32)

From (27), it may be shown that \( p(\tilde{x}) \) minimizes \( I_q(X; \tilde{X}) \). Since, \( D_{eff} \) is the expectation of a generalized K-Ld, (28) is applied to demonstrate that \( p(\tilde{x}) \) is a minimizer of \( D_{eff} \). Note that the gIB update equations are not globally convergent.

V. CONCLUSIONS AND DISCUSSIONS

Akin to the RD theory, the degree of compression may be assessed by the compression information \( I_q(X; \tilde{X}) \). However, while the RD method is upper bounded by an a-priori chosen optimal expected distortion \( D \), the gIB method is lower bounded by the relevant information \( I_q(X; Y) \). It has been demonstrated that in lossy compression, \( I_q(X; \tilde{X}) \) is always lower than its counterpart obtained using B-G-S statistics [7]. This observation implies that gIB relevance-compression curves will tend to traverse the forbidden region of an equivalent IB method based on B-G-S statistics. Future work casts the gIB model within the framework of Bregman divergences [25].

ACKNOWLEDGMENT

RCV gratefully acknowledges support from RAND-MSR contract CSM-DI & S-QIT-101155-03-2009.

REFERENCES

[1] T. Cover and J. Thomas *Elements of Information Theory*, John Wiley & Sons, New York, NY, 1991.
[2] T. Berger T. *Rate Distortion Theory*, Prentice-Hall, Englewood Cliffs, 1971.
[3] N. Tishby N. F. C. Pereira and W. Bialek, "The information bottleneck method", Proceedings of the 37th Annual Allerton Conference on Communication Control and Computing Eds B Haken and R S Sreenivas 368, University of Illinois, Urbana, IL, 1999.
[4] N. Slomn *The Information Bottleneck: Theory and Applications* PhD thesis, Hebrew University, Jerusalem, 2003.
[5] C. Tsallis "Possible generalizations of Boltzmann-Gibbs statistics" *J. Phys. A* 542, 1978.
[6] M. Gell-Mann and C. Tsallis, Eds. *Nonextensive Entropy: Interdisciplinary Applications*, Oxford University Press, Oxford, 2004.
[7] R. C. Venkatesan and A. Plastino "Generalized statistics framework for rate distortion theory" , *Physica A*, 388, 12, 2337, 2009.
[8] P. T. Landsberg and V. Nedral, "Distributions and channel capacities in generalized statistical mechanics", *Phys. Lett. A*, 247, 211, 1998.
[9] T. Yamano, "Information theory based on nonadditive information content", *Phys. Rev. E*, 63, 046105, 2001.
[10] S. Fujiuchi, "Information theoretical properties on Tsallis entropies", *J. Math. Phys.*, 47, 023302, 2006.
[11] H. Suyari, "Source coding theorem based on a nonadditive information content", *IEEE Trans. on Inform. Theory*, 50,8, 1783, 2004.
[12] T. Yamano, "Source coding theorem based on a nonadditive information content", *Physica A*, 305, 190 (2002).
[13] G. L. Ferri, S. Martinez S and A. Plastino A "Equivalence of the four versions of Tsallis statistics" *J. Stat. Mech.: Theory and Experiment 200504* P04009, 2005.
[14] E. M. F. Curado and C. Tsallis "Generalized statistical mechanics: connection with thermodynamics" *J. Phys. A: Math Gen.* 24 L69, 1991.
[15] C. Tsallis, R. S. Mendes and A. R. Plastino "The role of constraints within generalized nonextensive statistics" *Physica A* 261 534, 1998.
[16] S. Martínez, F. Nicolás, F. Pennini and A. Plastino "Generalized statistical mechanics: connection with thermodynamics" *Physica A* **286** 489, 2000.

[17] J. Naudts "Deformed exponentials and logarithms in generalized thermostatistics" *Physica A* **340** 32, 2004.

[18] W. Bialek, I. Nemenman I and N. Tishby, "Predictability, Complexity, and Learning Predictability, complexity, and learning", *Neural Comp.* **13** 2409, 2001.

[19] C. Tsallis "Generalized entropy-based criterion for consistent testing" *Phys. Rev. E* **58** 1442, 1998.

[20] L. Borland, A. Plastino and C. Tsallis "Information gain within nonextensive thermostatistics" *J. Math. Phys.* **39** 6490, 1998.

[21] E. Borges "A possible deformed algebra and calculus inspired in nonextensive thermostatistics" *Physica A* **340** 95, 2004.

[22] G. J. McLachlan and T. Krishnan *The EM Algorithm and Extensions*, John Wiley & Sons, New York, NY, 1996.

[23] I. Csiszár and G. Tusnády "Information geometry and alternating minimization procedures" *Statistics and Decisions* **1** 205, 1984.

[24] S. Furuichi "On uniqueness theorems for Tsallis entropy and Tsallis relative entropy" *IEEE Trans Inform. Theory* **51** 3638, 2005.

[25] P. Harremoës and N. Tishby "The information bottleneck revisited or how to choose a good distortion measure" *Proceedings of the IEEE Int. Symp. on Information Theory 2007* 566, 2007.