Tensor to scalar ratio of perturbation amplitudes and inflaton dynamics

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For the inflaton perturbations it is shown that the evolution of the difference between the spectral indices can be translated into information on the scale dependence of the tensor to scalar amplitudes ratio, \( r \), and how the scalar field potential can be derived from that information. Examples are given where \( r \) converges to a constant value during inflation but dynamics are rather different from the power–law model. Cases are presented where a constant \( r \) is not characteristic of the inflationary dynamics through the resulting perturbation spectra are consistent with the CMB and LSS data. The inflaton potential corresponding to \( r \) given by a \( n \)-th order polynomial of the e–folds number is derived in quadratures expressions. Since the observable difference between the spectral indices evaluated at a pivot scale yields information about the linear term of that polynomial, the first order case is explicitly written down. The solutions show features beyond the exponential form corresponding to power–law inflation and can be matched with current observational data.

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I. INTRODUCTION

During the early phase of the cosmological evolution the universe could experience a period of accelerated expansion known as inflation. The simplest scenario with a time dependent equation of state yielding the required negative pressure is that of a single real scalar field, the inflaton, with dynamics dominated by its potential energy. For extensive details and references on this scenario and on the other topics mentioned in this Introduction see book \[1\] and Ref. \[2\].

The inflaton quantum fluctuations would be stretched by the expansion beyond the radius within which causal interactions take place. These curvature perturbations could reenter the causal horizon in much later epochs and, through gravitational collapse, lead to anisotropies in the cosmic microwave background (CMB) temperature and to cosmological large scale structure (LSS). During inflation primordial gravitational waves are also produced. These tensor perturbations induce a curled polarization in the CMB radiation and increase the overall amplitude of its anisotropies at large angular scales.

A hint about the physics in the very early universe could be obtained by fitting to data the result of analytical calculations of the CMB and density spectra. These calculations depend on the values of some parameters which are the ones to be pinned down. The initial conditions for the evolution of the thermal anisotropies are also characterized by several quantities. These primordial parameters are often given as the multipoles of

\[
\ln A^2(k) = \ln A^2(k_*) + n(k_*) \ln \frac{k}{k_*} + \frac{1}{2} \frac{d n(k)}{d \ln k} \bigg|_{k = k_*} \ln^2 \frac{k}{k_*} + \cdots, \tag{1}
\]

where \( A \) stands for the normalized amplitudes of the scalar (\( A_S \)) or tensor (\( A_T \)) perturbations, the corresponding spectral indices, \( n \), are defined by,

\[
n_S - 1 = \frac{d \ln A_S^2}{d \ln k}, \tag{2}
\]

\[
n_T = \frac{d \ln A_T^2}{d \ln k}, \tag{3}
\]

\( k = aH \) is the comoving wavenumber corresponding to the wavelength matching the Hubble horizon during inflation and \( k_* \) is a pivot scale.

Initially, only \( A_S(k_*) \) was fitted. Since inflation predicts nearly scale–invariant spectra, the tilt given by the scalar spectral index was then taken into account. This reduces the primordial spectrum to a power–law function of the scale \( k \). The only single field model exactly yielding such a spectrum is power–law inflation \[3\] where the cosmic scale factor, \( a \), behaves like a power–law of the cosmic time, \( t \), and the inflaton potential is an exponential function. Signals of nonzero ‘running’ of the scalar index, \( dn_S/d \ln k \), have already been reported \[4\] and must be refined by near future observations, allowing, this way, to move beyond the power–law approximation.

The role of the tensor perturbations deserves also attention when determining the best–fit values of the cosmological parameters from CMB and LSS spectra. That is motivated in part by the possibility of measuring the cosmic background polarization, allowing the tensorial contribution to be indirectly determined. This contribution can be parametrized in terms of the relative amplitudes of the tensor and scalar perturbations,

\[
r = \frac{A_T^2}{A_S^2}, \tag{4}
\]

where \( \alpha \) is a constant. Presently, due to measurement limitations, a constant value of \( r \) is fitted.

Nevertheless, in Ref. \[5\] it was shown that few inflationary models produce an exactly constant tensor to
scalar ratio, and, in order to be proper scenarios of inflation, they must be observationally indistinguishable from power–law inflation, where \( r = \text{constant} \). Since power–law inflation is just one of many suitable final stages of the inflaton dynamics, it implies that, either a constant value for \( r \) may be no characteristic of the underlying inflationary evolution or that conclusions about the inflaton potential beyond an exponential form may be no possible to be drawn. This last factor could be a strong limitation for programs of the inflaton potential reconstruction.

Taking the above into account, the aim of this letter is to analyze the relation between the functional form of the tensor to scalar ratio and the inflationary dynamics. After introducing in Sec. II the relevant equations, the analysis is done in Sec. III by means of several examples. Finally, the results are discussed in Sec. IV and it is concluded that the difference between the spectral indices can be a very useful quantity because it yields information on the scale dependence of the tensor to scalar ratio, hence allowing to observe features of the inflaton potential different from the exponential form.

**II. KEY EQUATIONS**

The horizon–flow functions proved to be useful quantities while describing the inflationary dynamics and perturbations. They were defined in Ref. [3] as

\[
\epsilon_0 = \frac{d_H(N)}{d_H}, \quad \epsilon_{m+1} = \frac{d \ln |\epsilon_m|}{dN}, \quad m \geq 0, \tag{5}
\]

where \( d_H \equiv a / \dot{a} \) denotes the Hubble distance, \( N \equiv \ln(a / a_i) \) the number of e–folds since time \( t_i \), and \( d_H(t_i) \). During inflation, \( 0 \leq \epsilon_1 < 1 \). For \( m > 1 \), \( \epsilon_m \) may take any real value, though usually \( |\epsilon_m| < 1 \) for the inflationary spectra to be weakly scale dependent.

For instance, in Ref. [3] the CMB constrains \( \epsilon_1 < 0.1 \) and \( |\epsilon_2| < 0.3 \) were found for fixed \( \epsilon_3 = 0 \).

To next–to–leading order in an expansion in terms of the horizon–flow functions the spectral indices \( \epsilon_1 \) and \( \epsilon_2 \) can be written as [4] [10]

\[
\begin{align*}
n_S - 1 &= -2\epsilon_1 - \epsilon_2 + 2\epsilon_1^2 - 2(C + 3)\epsilon_1 \epsilon_2 - C\epsilon_2\epsilon_3, \tag{6} \\
n_T &= -2\epsilon_1 - 2\epsilon_1^2 - 2(C + 1)\epsilon_1 \epsilon_2,
\end{align*}
\]

where \( C \approx -0.7293 \). If these expressions are analyzed as a system of differential equations (where the differentiation, according with definitions [4] is done with respect to \( N \)), then it is not a closed system. Even after differentiating Eq. (4) and substituting the result and Eq. (6) itself into Eq. (1), one still has to deal with three independent variables, namely, \( \epsilon_1, n_S \) and \( n_T \) (see Refs. [4] [11] for more detailed discussion). There are not other independent equations relating these variables which could be used to close the system of equations (4) and (6).

Therefore, information on the functional forms of the observables \( n_S \) and \( n_T \) it is necessary in order to describe the dynamics of \( \epsilon_1 \). This is the basic philosophy behind the Stewart–Lyth inverse problem [11].

It follows from definitions (3), (3) and (4) that

\[
\frac{d \ln r}{d \ln k} = \Delta n \equiv n_T - (n_S - 1). \tag{8}
\]

This way, any information on the evolution of both spectral indices can be used as information on the scale dependence of the tensor to scalar ratio. Using expression (3), after subtracting Eq. (6) to Eq. (7) yields

\[
\frac{d \ln r}{d \ln k} = \epsilon_2 + \epsilon_1 \epsilon_2 + C\epsilon_2\epsilon_3. \tag{9}
\]

Equality \( k = aH \) implies,

\[
\frac{dN}{d \ln k} = \frac{1}{1 - \epsilon_1}. \tag{10}
\]

Then, changing variables in Eq. (3) it is obtained,

\[
C\frac{d \epsilon_2}{dN} + \epsilon_2 = \frac{d \ln r}{dN}. \tag{11}
\]

where Eqs. (3) were used and terms with order higher than the second one were neglected consistently with the approximations applied to derive Eqs. (6) and (7).

Taking into account the definition (3) for \( \epsilon_2 \), Eq. (11) can be rewritten as,

\[
C \frac{d \ln \epsilon_1}{dN} + \ln \epsilon_1 = \ln \frac{r}{r_0}, \tag{12}
\]

where \( r_0 \) is an integration constant. Now, the number of variables to solve for has been reduced to two by comprising in \( r \) the information on \( n_S \) and \( n_T \). Remains only one equation to deal with, thus, it is still compulsory to find out how \( r \) behaves to describe the dynamics of \( \epsilon_1 \).

According with definitions (4),

\[
H^2(N) = H_0^2 \exp \left(-2 \int \epsilon_1(N) dN \right), \tag{13}
\]

where \( H_0 \) is an integration constant. On the other hand using the definition (4) for \( \epsilon_1 \) and the Friedmann equation for the inflaton cosmology [4] [12],

\[
\frac{d \phi}{dN} = \sqrt{\frac{2}{\kappa}} \sqrt{\epsilon_1}, \tag{14}
\]

where \( \phi \) is the inflaton, \( \kappa \equiv 8\pi / m_{Pl}^2 \) and \( m_{Pl} \) is the Planck mass. Hence, given a solution for \( \epsilon_1(N) \) the corresponding potential as function of the inflaton field is,

\[
V(\phi) = \begin{cases} 
\phi(N) = \sqrt{\frac{2}{\kappa}} \sqrt{\epsilon_1} dN - \phi_0, \\
V(N) = H_0^2 \frac{N}{\kappa} [3 - \epsilon_1] \exp \left(-2 \int \epsilon_1 dN \right),
\end{cases} \tag{15}
\]

where the expression for the potential as function of \( N \) is derived from the Einstein equations for the scalar field cosmology [4] [13]. In this way, the inflaton potential can, in principle, be determined from just the information on the functional form of the tensor to scalar ratio or, equivalently, on the evolution of the difference between the tensor and scalar spectral indices.
III. SOME SOLUTIONS

Current analysis of CMB and LSS observations yields $r < 1$ \[1\], hence solutions satisfying $\ln r < 0$ shall be searched for. Eq. (12) can be written as,

$$ r = r_0 \varepsilon_1 \exp(C\varepsilon_2). \tag{16} $$

Expanding this expression to next–to–leading order and comparing with the corresponding expansion for $A_T/A_S$ \[16\] it can be determined that $r_0 = 16 \[16\]$. For large values of $|\varepsilon_2|$, models with $\varepsilon_2 > 0$ are favored. For the next–to–leading order approximation to proceed it must be required the minimum value, $\varepsilon_2 = -0.3$. Together with the constrain $r < 1$, this lead to $\varepsilon_1 < 0.05$. These values coincides with the $2\sigma$ bounds reported in Ref. \[1\].

Both, Eqs. (11) and (12) are in the class of first order linear differential equations like,

$$ y' + \alpha y = g(x), \tag{17} $$

with a prime denoting differentiation with respect to $x$ and $\alpha$ a constant. Using the ansatz $y = \exp(-\alpha x)f(x)$, the general solution of Eq. (17) is obtained as,

$$ y = \exp(-\alpha x) \left[ B + \int_{x_0}^{x} g(x)\exp(\alpha x)dx \right], \tag{18} $$

where $B$ is the constant arising from the integration of the homogeneous equation and the integration limits are determined by the range of $x$ where $g(x)$ is continuous. Thus, one has the option of solving any of Eqs. (11) or (12), regarding the complexity of the corresponding left hand side expression. If $g(x) \to \beta = \text{const.}$ while $x \to \infty$, then, for $\alpha > 0$, applying the L’Hopital’s rule it can be determined that, for any lower integration limit, the solution converges to $\beta/\alpha$. However, for $\alpha < 0$ the only solution converging to $\beta/\alpha$ is,

$$ y = -\exp(-\alpha x) \int_{x}^{\infty} g(x)\exp(\alpha x)dx. \tag{19} $$

Since in Eqs. (11) and (12), $\alpha = 1/C < 0$, this implies an attractor given by the particular solution $\varepsilon_1 = r_*/16$ for $r = r_* = \text{const.}$ (or $\Delta n = 0$ from Eq. (3)). This corresponds to power–law inflation and it is just a special case (that for $B = 0$), even if $r \to r_*$ while $N \to \infty$. For instance, if it is assumed

$$ \ln r = b \exp(-aN) + \ln r_\infty, \tag{20} $$

with $r_\infty$, $a > 0$ and $b$ some constants then, according with Eq. (19), the asymptotic solution for $\varepsilon_1$ will be

$$ \varepsilon_1 = \varepsilon_{1\infty} \exp \left[ -\frac{b}{Ca - 1} \exp(-aN) \right], \tag{21} $$

where $\varepsilon_{1\infty} = r_\infty/16$. This solution converges to $\varepsilon_1 = \varepsilon_{1\infty}$ while $N \to \infty$. For $b > 0$ ($b < 0$), this happen starting from values larger (smaller) than $\varepsilon_{1\infty}$. If $b = 0$, corresponding to $r = r_\infty = \text{const.}$, then the outcome is just the power–law solution given by $\varepsilon_1(N) = \varepsilon_{1\infty}$. This is the kind of behavior observed in Ref. \[13\] for models with constant tensorial index and also in the first case analyzed in Ref. \[12\] for models with weakly scale dependent spectral indices.

According to Eq. (19), the more general solution with $r$ given by Eq. (20) is,

$$ \varepsilon_1 = \varepsilon_{1\infty} \exp \left[ B \exp(-1/CN) \exp \left[ -\frac{b}{Ca - 1} \exp(-aN) \right] \right]. \tag{22} $$

Now the dynamics is different. If $B > 0$ ($B < 0$) the solution increases (decreases) very fast starting from,

$$ \varepsilon_1 = \varepsilon_{1\infty} \exp(B) \exp \left[ -\frac{b}{Ca - 1} \right]. \tag{23} $$

If $b = 0$, this behavior is modified only in the rate of change of $\varepsilon_1$, consistently with the solution in Ref. \[1\] for models with $r = \text{const.}$. Thus, $B$ is a bifurcation parameter for the inflationary dynamics described by Eq. (22).

Functional forms of $r$ converging to a constant value are motivated by difficulties of measuring the primordial gravitational waves background \[1\]. Maybe the unique for a while alternative for determining the contribution of tensor modes to CMB is the derivation of the tensor to scalar ratio from the background radiation polarization, the more optimistic expectations being measuring a constant central value of $r \[1\]$. However, there are other behaviors of $r$ consistent with current constrains on the primordial spectra. For example, if in Eq. (24) $a$ is changed for $-a$, the general solution (22) is modified in exactly the same way, i.e., $a$ for $-a$. Nevertheless, now we have richer dynamics as for $r$ as for $\varepsilon_1$, depending on the values of the different parameters involved. For $B = 0$ and $b < 0$, $r \to 0$ with $N \to \infty$ but there are two different behaviors for $\varepsilon_1$: if $0 < a < 1/C$, $\varepsilon_1 \to \infty$, else $\varepsilon_1 \to 0$. Interestingly is the case $B = 0$ and $b > 0$, since $r \to \infty$ with $N \to \infty$ but for $0 < a < 1/C$, $\varepsilon_1 \to 0$. According to Eq. (3), this example can be made consistent with current analysis of CMB and LSS observations \[2, 3, 10\] by tuning the values of $a$ and $b$ to control the behavior of $\varepsilon_2$ and $\varepsilon_3$. For $B \neq 0$ more possibilities arise depending on whether $B$ is positive or negative, and greater or less than $b/(1 + aC)$. Solutions with $r \to \infty$, $\varepsilon_1 \to 0$ and $|\varepsilon_2| < 1$, $|\varepsilon_3| < 1$ while $N \to \infty$, can be easily found.

Equally satisfactory, from the observational point of view, are functional forms of $r$ exhibiting a transient period of almost power–law behavior during a sufficiently large numbers of $N$. For example, a proper choice of the constants $a$, $b$, $\alpha$, $\beta$ ensures that to be the case for

$$ r = r_0 \left\{ a \tan[\alpha(N - N_0)] - b \right\} \times \exp \left\{ \frac{\beta}{\cos^2[\alpha(N - N_0)]} (a \tan[\alpha(N - N_0)] - b) \right\} \tag{24} $$

and the corresponding solution for $\varepsilon_1$,

$$ \varepsilon_1 = \varepsilon_{1(0)} \left\{ a \tan[\alpha(N - N_0)] - b \right\}. \tag{24} $$
This result is similar to case 2 in Ref. [12] for models with weakly scale dependent spectral indices.

Now, if \( g(x) \) in Eq. (17) is given precise enough by,
\[
g(x) = \sum_{p=0}^{n} a_p (x - x_0)^p, \tag{25}
\]
with \( n \) some finite integer, then solution (18) reads,
\[
y(x) = B_0 \exp(-\alpha x) + \sum_{p=0}^{n} \sum_{q=0}^{p} a_p (-1)^q \frac{p!}{(p-q)!} \alpha^{(q+1)} (x - x_0)^{p-q}
\]
for \( p \geq q \). The solution of Eq. (12) is then,
\[
\epsilon_1(N) = \epsilon_{1(0)} \exp \left[ B_0 \exp \left( \frac{1}{C} N \right) \right] \times \prod_{p=0}^{n} \prod_{q=0}^{p} \exp \left[ b_{p,q} (N - N_0)^{p-q} \right], \tag{26}
\]
where \( b_{p,q} = a_p (-1)^q C^{(q+1)} p!/(p-q)! \). The general expression for the inflaton potential corresponding to \( r \) given by a \( n \)-th order polynomial of the e–folds number, \( N \), can be derived in terms of quadratures expressions for \( \epsilon_1 \), by substituting solution (26) into Eq. (13).

Unfortunately, the future prospects for obtaining information from observations for higher order coefficients in expansion (23) are strongly limited. However, it is timely to recall that \( dr/d\ln k = r \Delta n \) which, according to Eqs. (9) and (16) is a next–to–leading order expression, hence \( a_1 \approx a_0 \Delta n(N_0) \). With these regards, it becomes important to analyze in details case,
\[
\ln \frac{r}{16} = a_0 + a_1(N - N_0), \tag{27}
\]
where the corresponding solution for \( \epsilon_1 \) is,
\[
\epsilon_1(N) = \epsilon_{1(0)} \exp \left[ B \exp \left( \frac{N}{C} \right) \right] \exp(AN), \tag{28}
\]
with \( \epsilon_{1(0)} = \epsilon_{1(0)} \exp \{a_0 + (N_0 - C) a_1 C\} \) and \( A = a_1 C \). The asymptotes of this solution for \( B \neq 0 \) will be mainly determined by the value and sign of \( B \). However, in the same fashion as it was shown in Ref. [5] for \( A = 0 \) (i.e., \( r/16 = a_0) \), if the model yielding \( \epsilon_1 \) given by Eq. (23) is expected to be compatible with current data, \( B \) has to be chosen an extraordinarily small number, making the corresponding scenario very difficult to be observationally distinguished from power–law inflation. More interesting is the case \( B = 0 \). From Eq. (15), the corresponding inflaton potential is obtained as,
\[
V = V_0 \left[ 3 - \frac{A^2}{4} \psi^2 \right] \exp \left( -\frac{A}{2} \psi^2 \right), \tag{29}
\]
where \( \psi \equiv \sqrt{\kappa/2} (\phi + \phi_0) \). In Fig. 1 sectors of this potential are plotted for \( A = -0.0073 \) (\( V_I \)) and \( A = 0.0073 \) (\( V_{II} \)).

\[\text{FIG. 1: Sectors of the inflaton potential given by Eq. (29) for } A = -0.0073 \text{ (} V_I \text{) and } A = 0.0073 \text{ (} V_{II} \text{).} \]

In Fig. 2 are presented the corresponding to this model scalar and tensorial spectral indices,
\[
n_S = 1 - A + [2 + (2C + 3) A] L_W \left[ -\epsilon_{1(i)} \left( \frac{k}{k_*} \right)^A \right] - 2 L_W^2 \left[ -\epsilon_{1(i)} \left( \frac{k}{k_*} \right)^A \right], \tag{30}
\]
\[
n_T = 2 [1 + (C + 1) A] L_W \left[ -\epsilon_{1(i)} \left( \frac{k}{k_*} \right)^A \right] - 2 L_W^2 \left[ -\epsilon_{1(i)} \left( \frac{k}{k_*} \right)^A \right], \tag{31}
\]
where Eqs. (4), (7) and (10) were used and \( L_W[x] \) is the Lambert \( W \) function. For a large set of \( A \) and \( \epsilon_{1(i)} \) values, these spectra agree with CMB and LSS observations, \( n_S = 1.033 \pm 0.066, n_T = 0.09 \pm 0.16 \), \(-0.05 < \frac{dn_S}{d\ln k} < 0.02 \), \( \epsilon_1 < 0.05 \) and \( 12.5 \epsilon_1 < r < 1 \).
IV. DISCUSSION

The analysis of the general solution [13] confirms that the power–law inflationary scenario is just one among many suitable final stages for the inflaton dynamics as it was previously discussed in Ref. [6]. Moreover, as the examples in the previous section indicate, a tensor to scalar ratio $r$ converging to a constant value while the early universe inflates is neither a necessary nor a sufficient condition for the corresponding inflationary model to yield perturbation spectra consistent with current CMB and LSS data in the range of observable scales.

From these examples it can also be concluded that radically different inflationary dynamics can yield the same asymptotic functional form for the tensor to scalar ratio. Therefore, the derivation of the inflaton potential from the information on the functional form of $r$ will hardly be unique. Here it is important to note that by ‘different dynamics’ it is implied solution behaviors, which can be quite different for several values of the involved parameters, even if the corresponding analytical expression for the potential, with unvalued parameters, is unique.

The above conclusion must be regarded together with the fact that it seems not possible to obtain information on the inflaton potential beyond the exponential form, using exclusively the observational information on values of the spectral indices ($n_S$ and $n_T$) and the running of the scalar index ($\Delta n_S/d\ln k$) evaluated at the pivot scale, or on a constant central value of the tensor to scalar ratio $r$. This would be a serious handicap for any program of reconstruction of the inflaton potential.

A way to improve the situation described in the above paragraphs, could be to combine the information on $\Delta n$ (the difference between the spectral indices) and the value of $r$, with the two first horizon–flow functions. $\Delta n$ can be derived from observations and, according to Eq. (8), gives information on the scale dependence of $r$. With this information evaluated at the pivot scale, the underlying inflaton potential can be promptly sketched substituting expression (28) into Eq. (15). Moreover, here all three cases, power–law inflation, $B \neq 0$ (if imitating power–law inflation) and $B = 0$ can be differentiated. In the case of power–law inflation, $\Delta n$ and $\epsilon_2$ will be zero. If $B \neq 0$ then, as discussed in the previous section, its absolute value must be extraordinarily small what, after ten or more e–foldings, leads to $\epsilon_3 \approx -1/C \approx 1.3712$, a distinctively large, hardly suitable, value which will be the dominant contribution to $\Delta n$ according to Eq. (3). Finally, for $B = 0$, $\epsilon_2 = C r \Delta n = A$, $\epsilon_3 = 0$ and the potential will be given by Eq. (29). As shown in Fig. 8, realizations of this potential resemble the cases of monomial potentials with even order ($V_I$, $\epsilon_2 < 0$), and inflation near a maximum ($V_{II}$, $\epsilon_2 > 0$) (see Ref. [4] for examples of such inflationary scenarios) allowing, therefore, to observe features of the inflaton potential beyond the exponential form characteristic of power–law inflation.

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