Limiting Behaviors for Longest Consecutive Switches in an IID Bernoulli Sequence

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Abstract In this paper we mainly discuss lower and upper bounds for the length of longest consecutive switches in IID Bernoulli sequences. This work is an extension of results for longest consecutive switches in unbiased coin-tossing, and might be applied to reliability theory, biology, quality control, pattern recognition, finance, etc.

Keywords Longest consecutive switches, lower and upper bounds, Borel–Cantelli lemma

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1 Introduction

A biased coin with two sides is tossed independently and repeatedly, where 0 is used to denote “tail” and 1 to denote “head”. In the rest of this paper, we let $\{X_i, i \geq 1\}$ be a sequence of independent Bernoulli trails with $P\{X_1 = 1\} = p$ and $P\{X_1 = 0\} = 1 - p =: q$ by assuming that $0 < p < 1$.

In unbiased case, [3, 4] studied the length of longest head-run’s limit behaviors. These limit theorems have been extended in many subsequent studies. We refer to [1, 12, 14, 19, 20]. [11] obtained a large deviation principle for longest head run. For more recent related references, we refer to [2, 8, 17].

In 2012, Anush posed the definition of “switch” at mathoverflow, and concerned the bounds for the number of coin tossing switches. According to [7], a “switch” is a tail followed by a head or a head followed by a tail. The exact probability distribution for the number of coin tossing switches was given by Joriki at mathematics stack exchange. However, limiting behaviors of the related distribution cannot be obtained directly due to its complexity. Therefore, probabilistic estimates are necessary for applications such as in molecular biology ([5, 16]), sensor networks ([15]), and image detection ([13]). [7] considered the number of switches in unbiased coin-tossing, and established the central limit theorem and the large deviation principle for the total number of switches. In [6], the authors obtained some limiting results for length of the longest consecutive switches in unbiased case with $p = q = \frac{1}{2}$. The purpose of the present note is...
to extend the results in [6] to IID Bernoulli sequences. As an application, it is reasonable to
deduce similar results in the paper for the Markov case (see e.g. in [9, 10, 18, 21]).

In the beginning, we introduce some notations. Following [7], for \( m, n \in \mathbb{N} \) define
\[
S_n^{(m)} := \sum_{i=m+1}^{n+m-1} (1 - X_{i-1}) X_i + X_{i-1} (1 - X_i)
\]
as the total number of switches in \( n \) trails \( \{X_m, X_{m+1}, \ldots, X_{m+n-1}\} \). Moreover, for \( m, N \in \mathbb{N} \)
and \( n = 1, \ldots, N \), define by
\[
H^{(N)}_{m,n} := \bigcup_{i=m}^{m+N-n+1} \{ S_n^{(i)} = n - 1 \}
\]
the set of consecutive switches of length \( n - 1 \) in \( N \) trails \( \{X_m, X_{m+1}, \ldots, X_{m+N-1}\} \). Then \( M_N^{(m)} \)
, defined as
\[
M_N^{(m)} := \max_{1 \leq n \leq N} \{ n - 1 \mid H^{(N)}_{m,n} \neq \emptyset \},
\]
stands for the length of longest consecutive switches in \( N \) trails \( \{X_m, X_{m+1}, \ldots, X_{m+N-1}\} \). For example, if one gets “11001011101”, then it includes 6 switches and the length of longest
consecutive switches is 3. When \( m = 1 \), the length of longest consecutive switches in \( N \) trails \( \{X_1, \ldots, X_N\} \) is particularly denoted by \( M_N \).

Throughout the paper, we use \( \lfloor a \rfloor \) to denote the largest integer which is no more than any
real number \( a \).

The rest of this paper is organized as follows. In Section 2 we present main results. The
proofs are given in Section 3.

2 Main Results
As discussed in [4] and [6], we mainly investigate lower and upper bounds for longest consecutive
switches. The main idea is similar. However, it is much more complicated when the coin-tossing
is biased. We have to discuss far more different situations to give precise calculations and
probabilistic estimates for longest consecutive switches (see Lemma 3.2 and Theorem 3.1).

Before giving the main results, we need the following limit (we’d like to thank one referee for
telling it to us),
\[
\lim_{N \to \infty} \left( \log_{1/\sqrt{pq}} \log_{1/\sqrt{pq}} \log_{1/\sqrt{pq}} N - \log_{1/\sqrt{pq}} \log_{1/\sqrt{pq}} e - \log_{1/\sqrt{pq}} \log \log N \right) = 0. \tag{2.1}
\]
Hereafter, we denote by “log” the logarithm with base \( e \).

In the first two theorems we give lower and upper bounds for longest consecutive switches.

**Theorem 2.1** Let \( \varepsilon \) be any positive number. Then for almost all \( \omega \in \Omega \), there exists a finite
\( N_0 = N_0(\omega, \varepsilon) \) such that for \( N \geq N_0 \)
\[
M_N \geq \left[ \log_{1/\sqrt{pq}} N - \log_{1/\sqrt{pq}} \log_{1/\sqrt{pq}} \log_{1/\sqrt{pq}} N + \log_{1/\sqrt{pq}} \log_{1/\sqrt{pq}} e - \log_{1/\sqrt{pq}} 2 - 1 - \varepsilon \right]
\]
:= \( \alpha_1(N) \), \tag{2.2}
where by (2.1), the function \( \alpha_1(N) \) can be replaced by
\[
\left[ \log_{1/\sqrt{pq}} \left( \frac{N}{2 \log \log N} \right) - 1 - \varepsilon \right]. \tag{2.3}
\]
Theorem 2.2 Let $\varepsilon$ be any positive number. Then for almost all $\omega \in \Omega$, there exists an infinite sequence $N_i = N_i(\omega)$ ($i = 1, 2, \ldots$) of integers such that

$$M_{N_i} < \left| \log_{1/\sqrt{pq}} N_i - \log_{1/\sqrt{pq}} \log_{1/\sqrt{pq}} \log_{1/\sqrt{pq}} N_i + \log_{1/\sqrt{pq}} \log_{1/\sqrt{pq}} e + 1 + \varepsilon \right|$$

$$=: \alpha_2(N_i), \quad (2.4)$$

where by (2.1), the function $\alpha_2(N)$ can be replaced by

$$\left| \log_{1/\sqrt{pq}} \left( \frac{N}{\log \log N} \right) + 1 + \varepsilon \right|. \quad (2.5)$$

According to Theorems 2.1 and 2.2, we know that the value of $M_N$ is larger than $\alpha_1$ but in general not larger than $\alpha_2$. In the next theorem we discuss the largest possible values of $M_N$.

Theorem 3.1 Let $\{\gamma_n\}$ be a sequence of positive numbers.

(i) If $\sum_{n=1}^{\infty} (pq)^{-\gamma_n} = \infty$, then for almost all $\omega \in \Omega$, there exists an infinite sequence $N_i = N_i(\omega, \{\gamma_n\})$ ($i = 1, 2, \ldots$) of integers such that $M_{N_i} \geq \gamma_{N_i} - 1$.

(ii) If $\sum_{n=1}^{\infty} (pq)^{-\gamma_n} < \infty$, then for almost all $\omega \in \Omega$, there exists a positive integer $N_0 = N_0(\omega, \{\gamma_n\})$ such that $M_N < \gamma_N - 1$ for all $N \geq N_0$.

The above theorem can be reformulated to estimate number of switches as follows:

Theorem 3.1* Let $\{\gamma_n\}$ be a sequence of positive numbers.

(i) If $\sum_{n=1}^{\infty} (pq)^{-\gamma_n} = \infty$, then for almost all $\omega \in \Omega$, there exists an infinite sequence $N_i = N_i(\omega, \{\gamma_n\})$ ($i = 1, 2, \ldots$) of integers such that $S_{\gamma_{N_i}}^{(N_i-\gamma_{N_i})} \geq \gamma_{N_i} - 1$.

(ii) If $\sum_{n=1}^{\infty} (pq)^{-\gamma_n} < \infty$, then for almost all $\omega \in \Omega$, there exists a positive integer $N_0 = N_0(\omega, \{\gamma_n\})$ such that $S_{\gamma_N}^{(N-\gamma_N)} < \gamma_N - 1$ for all $N \geq N_0$.

In the end, we give a limit result on the length of the longest consecutive switches.

Proposition 2.4 We have

$$\lim_{N \to \infty} \frac{M_N}{\log_{1/\sqrt{pq}} N} = 1 \quad a.s. \quad (2.6)$$

3 Proofs

3.1 Proofs of Theorems 2.1–2.3*

The basic idea comes from [4]. At first, we give an estimate for the length of longest consecutive switches, which is very useful in our proofs for Theorems 2.1–2.3*.

Theorem 3.1 Let $N, K \in \mathbb{N}$ with $N \geq 2K$. Then

$$(1 - (K + 1 - 2Kpq)(pq)^{K-1})^{1/2} \geq (1 - 2pq)(pq)^{K-1}$$

$$\leq P(M_N < K - 1)$$

$$\leq (1 - (K + 2)(pq)^{1/2} + 2(pq)^{K-1/2}(1/pq) - 1). \quad (3.1)$$

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2 Let $K, m \in \mathbb{N}$ and $M_{2K}^{(m)}$ is defined in (1.1). Then

$$P(M_{2K}^{(m)} \geq K - 1) = \begin{cases} (K + 2)(pq)^{1/2} - 2(pq)^K, & \text{if } K \text{ is even}, \\
(K + 1 - 2Kpq)(pq)^{K-1/2} - (1 - 2pq)(pq)^{K-1}, & \text{if } K \text{ is odd}. \end{cases} \quad (3.2)$$
Proof Since \( M_{2K}^{(m)} \), \( m \in \mathbb{N} \) are identically distributed, it is sufficient to consider the case where \( m = 1 \). For \( i = 1, \ldots, K + 1 \), denote

\[
F_i := \{(X_1, \ldots, X_{K+i-1}) \text{ is the first section of consecutive switches of length } K - 1 \text{ in the sequence } (X_1, \ldots, X_{2K})\}.
\]

It is equivalent to

\[
F_1 = \{M_K = K - 1\}, \quad F_i = \{M_{K}^{(i)} = K - 1, \ X_{j-1} = X_i\}, \quad i = 2, \ldots, K, \quad F_{K+1} = \{M_{K}^{(K+1)} = K - 1, \ X_{i-1} = X_i, \ M_K < K - 1\}.
\]

Then we have the decomposition

\[
\{M_{2K} \geq K - 1\} = \bigcup_{i=1}^{K+1} F_i, \quad (3.3)
\]

where \( F_i \cap F_j = \emptyset, \forall i \neq j \), and

\[
P(F_i) = \begin{cases} 
2(pq)^{[K/2]}, & \text{if } K \text{ is even,} \\
(pq)^{[K/2]}, & \text{if } K \text{ is odd,}
\end{cases} \quad (3.4)
\]

\[
P(F_{i+1}) = \begin{cases} 
(pq)^{[K/2]}, & \text{if } K \text{ is even,} \\
(p^2 + q^2)(pq)^{[K/2]}, & \text{if } K \text{ is odd,}
\end{cases} \quad (3.5)
\]

\[
P(F_{K+1}) = \begin{cases} 
(pq)^{[K/2]}(1 - 2(pq)^{[K/2]}), & \text{if } K \text{ is even,} \\
(pq)^{[K/2]}(p^2 + q^2)(1 - (pq)^{[K/2]}), & \text{if } K \text{ is odd.}
\end{cases} \quad (3.6)
\]

Plugging the above equalities into (3.3) and by \( p + q = 1 \) we obtain (3.2). \( \square \)

Proof of Theorem 3.1 Let \( N, K \in \mathbb{N} \) with \( N \geq 2K \). Denote

\[
B_j = \{M_{K}^{(j+1)} = K - 1\}, \quad j = 0, 1, \ldots, N - K.
\]

Then we have the decomposition

\[
C_l := \{M_{2K}^{(l+1)} \geq K - 1\} = \bigcup_{j=lK}^{(l+1)K} B_j, \quad l = 0, 1, \ldots, \left[\frac{N - 2K}{K}\right],
\]

\[
\{M_N \geq K - 1\} = \bigcup_{l=0}^{\left[\frac{N - 2K}{K}\right]} C_l.
\]

Let

\[
D_0 := C_0 \cup C_2 \cup \cdots \cup C_2\left[\frac{N-2K}{K}\right], \quad D_1 := C_1 \cup C_3 \cup \cdots \cup C_2\left[\frac{N-2K}{K}\right] - 1\] + 1 \quad (3.7)
\]

By the independence of events \( C_0, C_2, \ldots, C_2\left[\frac{N-2K}{K}\right] \), Lemma 3.2, and the independence of events \( C_1, C_3, \ldots, C_2\left[\frac{N-2K}{K}\right] + 1 \), we have

\[
P(D_0) = P(C_0)\left[\left(\frac{N-2K}{K}\right)\right] + 1, \quad P(D_1) = P(C_0)\left[\left(\frac{N-2K}{K}\right) - 1\right] + 1.
\]
Observing that \( \{M_N \geq K - 1\} = D_0 \cup D_1 \), it is obvious that
\[
P(M_N < K - 1) \leq \min\{P(D_0), P(D_1)\} \leq P(C_0)(\frac{K}{2})^{-1/2}. \tag{3.8}
\]
Moreover, it can be obtained that for any \( j = 0, 1, \ldots, N - K \),
\[
P(D_1 | B_j) \geq P(D_1).
\]
Similarly, for each even \( l_0 = 2, 4, \ldots, 2\left\lfloor \frac{N - 2K}{K} \right\rfloor \),
\[
P\left( D_1 \bigcup_{l\text{ even}, 0 \leq l \leq l_0 - 2} C_l | C_{l_0} \right) \geq P\left( D_1 \bigcup_{l\text{ even}, 0 \leq l \leq l_0 - 2} C_l \right),
\]
equivalently,
\[
P\left( \overline{D_1} \bigcap_{l\text{ even}, 0 \leq l \leq l_0 - 2} \overline{C_l} | C_{l_0} \right) \geq P\left( \overline{D_1} \bigcap_{l\text{ even}, 0 \leq l \leq l_0 - 2} \overline{C_l} \right),
\]
which implies that
\[
P\left( \overline{D_1} \bigcap_{l\text{ even}, 0 \leq l \leq l_0 - 2} \overline{C_l} \right) \geq P\left( \overline{D_1} \bigcap_{l\text{ even}, 0 \leq l \leq l_0 - 2} \overline{C_l} \right) P(C_{l_0}). \tag{3.9}
\]
Based on (3.9) and the independence of events \( C_0, C_2, \ldots, C_{2\left\lfloor \frac{N - 2K}{K} \right\rfloor} \)
\[
P(M_N < K - 1) = P(\overline{D_1} D_0) = P\left( \overline{D_1} \bigcap_{l\text{ even}, 0 \leq l \leq \left\lfloor \frac{N - 2K}{K} \right\rfloor} \overline{C_l} \right)
\]
\[
\geq P\left( \overline{D_1} \bigcap_{l\text{ even}, 0 \leq l \leq \left\lfloor \frac{N - 2K}{K} \right\rfloor} \overline{C_l} \right) P\left( C_2^{\left\lfloor \frac{N - 2K}{K} \right\rfloor} \right)
\]
\[
\geq \cdots
\]
\[
\geq P(\overline{D_1}) \prod_{l\text{ even}, 0 \leq l \leq \left\lfloor \frac{N - 2K}{K} \right\rfloor} P(\overline{C_l})
\]
\[
= P(\overline{D_1}) P(D_0)
\]
\[
\geq P(C_0)(\frac{K}{2})^{-1}. \tag{3.10}
\]
Besides, by \( pq \leq \frac{1}{4} \) we consider in (3.2) for any integer \( K \geq 2 \),
\[
(K + 2)(pq)^{\frac{K}{2}} - 2(pq)^K \leq (K + 1 - 2Kpq)(pq)^{\frac{K - 1}{2}} - (1 - 2pq)(pq)^{K - 1}.
\]
Plugging (3.2) into (3.8) and (3.10), we complete the proof. \qed

**Proof of Theorem 2.1** Let \( N_j \) be the smallest integer with \( \alpha_1(N_j) = j - 1 \). By the inequality on the right hand side of (3.1),
\[
\sum_{j=1}^{\infty} P(M_{N_j} < \alpha_1(N_j)) \leq \sum_{j=1}^{\infty} (1 - (j + 2)(pq)^{j/2} + 2(pq)^j(\frac{N_j}{2})^{-1/2})
\]
\[
\leq \sum_{j=1}^{\infty} (1 - j(pq)^{j/2})\frac{N_j}{pq}
\]
\[
= \sum_{j=1}^{\infty} e^{-j(pq)^{j/2}N_j},
\]
with \( e_j := (1 - j(pq)^{j/2})^{-j(pq)^{-j/2}} \) tends to \( e \) when \( j \) goes to \( \infty \). The denotation \( x \lesssim y \) means that there exists a constant \( C \) independent of all variables such that \( x \leq Cy \). Since \( \alpha_1(N_j) = j - 1 \), then by (2.3) we have \( j \leq \log_{1/\sqrt{pq}(N_j \log \log N_j)} - \varepsilon \), and by (2.2) we have \( j \leq \log_{1/\sqrt{pq}(N_j)} N_j \). We further estimate that

\[
\sum_{j=1}^{\infty} P(M_{N_j} < \alpha_1(N_j)) \lesssim \sum_{j=1}^{\infty} \exp \left( -\frac{N_j(pq)^{j/2}}{2} \right) \\
\lesssim \sum_{j=1}^{\infty} (\log N_j)^{-\varepsilon/2}
\lesssim \sum_{j=1}^{\infty} j^{-\varepsilon/2}.
\]

Considering that \( pq \in (0, \frac{1}{4}] \), and \( (pq)^{-\varepsilon/2} > 1 \) for any \( \varepsilon > 0 \), we obtain

\[
\sum_{j=1}^{\infty} P(M_{N_j} < \alpha_1(N_j)) < \infty.
\]

We complete the proof by the Borel–Cantelli lemma.

To prove Theorem 2.2, we need the following version of Borel–Cantelli lemma.

**Lemma 3.3 ([4, Lemma A])** Let \( A_1, A_2, \ldots \) be arbitrary events, satisfying the conditions

\[
\sum_{n=1}^{\infty} P(A_n) = \infty
\]

and

\[
\liminf_{n \to \infty} \frac{\sum_{1 \leq k < l \leq n} P(A_kA_l)}{\sum_{1 \leq k < l \leq n} P(A_k)P(A_l)} = 1.
\]

Then \( P(\limsup_{n \to \infty} A_n) = 1 \).

**Proof of Theorem 2.2** For \( \delta > 0 \), let \( N_j = N_j(\delta) \) be the smallest integer for which \( \alpha_2(N_j) = \lfloor j^{1+\delta} \rfloor - 1 \) with \( \alpha_2(N_j) \) given by (2.4). Set

\[
A_j = \{ M_{N_j} < \alpha_2(N_j) \}, \quad j \geq 1.
\]

For \( i < j \), we define

\[
B_{i,j} = \begin{cases} \\
\{ M_{N_i} < \alpha_2(N_j) \}, & \text{if } N_i \geq \alpha_2(N_j), \\
\Omega, & \text{otherwise},
\end{cases}
\]

\[
C_{i,j} = \{ M_{N_j-N_i}^{(N_j+1)} < \alpha_2(N_j) \}.
\]

We claim that

\[
P(A_j) = P(B_{i,j})P(C_{i,j})(1 - o(1)) \quad \text{as } i < j \to \infty.
\]

In fact, by the definitions of \( A_j, B_{i,j} \) and \( C_{i,j} \), we know that the events \( B_{i,j} \) and \( C_{i,j} \) are independent and

\[
A_j = B_{i,j} \cap C_{i,j} \cap \{ M_{2\alpha_2(N_j)}^{(N_j)} < \alpha_2(N_j) \}
\]
with $x^+ = x \vee 0$. By Lemma 3.2 we have

$$P\{M_{2\alpha_2(N_j)}^{(N_i, -\alpha_2(N_j)) + 1} < \alpha_2(N_j)\} = P\{M_{2\alpha_2(N_j)} < \alpha_2(N_j)\}$$

$$\geq P\{M_{2\alpha_2(N_j)} < \alpha_2(N_j) - 1\}$$

$$= 1 - o(1) \quad \text{as } j \to \infty. \quad (3.15)$$

Then (3.14) holds immediately by (3.15) and the independence of events $B_{ij}$ and $C_{ij}$.

In a similar way we also have

$$P(A_i A_j) = P(A_i) P(C_{i,j})(1 - o(1)) \quad \text{as } i < j \to \infty. \quad (3.16)$$

For $N_i \geq \alpha_2(N_j)$ and $i < j$, by Theorem 3.1 for sufficiently large $j$

$$P(B_{i,j}) \geq P(M_{N_j-1} < \alpha_2(N_j)) \geq (1 - W_j) \left(\frac{N_j - 1}{\alpha_2(N_j) + 1}\right)^{-1}$$

with $W_j := (1 - 2pq)^{\alpha_2(N_j)/2\alpha_2(N_j)} \to 0$ and $(1 - W_j)^{1/W} \to e^{-1}$ as $j \to \infty$. Define

$$0 < \varepsilon_0 < 1 \text{ satisfying for any } j \geq 1,$$

$$\lfloor j^{1+\delta} \rfloor = \alpha_2(N_j) + 1 > \varepsilon_0 \log_1/\sqrt{pq} N_j.$$

Then we have

$$\log_1/\sqrt{pq} \log_1/\sqrt{pq} N_j \leq (1 + \delta) \log_1/\sqrt{pq} j - \log_1/\sqrt{pq} \varepsilon_0. \quad (3.17)$$

The equality $\alpha_2(N_j) = \lfloor j^{1+\delta} \rfloor - 1$, (2.4), and (3.17) together imply

$$N_{j-1} \leq \log_1/\sqrt{pq} \log_1/\sqrt{pq} N_{j-1} \cdot \left(\frac{1}{\sqrt{pq}}\right)^{\lfloor (j-1)^{1+\delta} \rfloor} \leq \log_1/\sqrt{pq} \left\{ (j - 1)^{1+\delta} \frac{1}{\varepsilon_0} \right\} \cdot \left(\frac{1}{\sqrt{pq}}\right)^{\lfloor (j-1)^{1+\delta} \rfloor}$$

for any $j$. Hence

$$P(B_{i,j}) \geq e^{-\lim_{j \to \infty} W_j(\lfloor N_j^{-1}/\alpha_2(N_j) \rfloor - 1)} = e^{-(1-2pq) \lim_{j \to \infty}\frac{N_{j-1}}{(1/\sqrt{pq})(\log_1/\sqrt{pq})^{(j-1)^{1+\delta}/\varepsilon_0} + 1}}$$

$$\geq e^{-(1-2pq) \lim_{j \to \infty}\frac{\log_1/\sqrt{pq}^{(j-1)^{1+\delta}/\varepsilon_0}}{(1/\sqrt{pq})(\log_1/\sqrt{pq})^{(j-1)^{1+\delta}/\varepsilon_0} + 1}}$$

$$\to 1 \quad \text{as } j \to \infty. \quad (3.18)$$

According to (3.14), (3.16) and (3.18) we easily have

$$P(A_i A_j) = P(A_i) P(A_j)(1 + o(1)) \quad \text{as } i < j \to \infty. \quad (3.19)$$

By Theorem 3.1, we have that when $j$ is large enough,

$$P(C_{i,j}) = P(M_{N_j, N_j-1}^{(N_j, -\alpha_2(N_j))} < \alpha_2(N_j)) \geq P(M_{N_j} < \alpha_2(N_j))$$

$$\geq (1 - W_j) \left(\frac{N_j}{\alpha_2(N_j) + 1}\right)^{-1}$$

$$\to e^{2pq-1} \quad \text{as } j \to \infty. \quad (3.20)$$

Then by (3.14), (3.18) and (3.20), we obtain

$$\liminf_{n \to \infty} P(A_n) \geq e^{2pq-1}, \quad \text{and}$$

$$\sum_{n=1}^{\infty} P(A_n) = \infty. \quad (3.21)$$

Lemma 3.3 holds by (3.21) and (3.19). We complete the proof. □
Remark 3.4 Following our approach, the bounds given in (2.2) and (2.4) can’t be improved. But the lower and upper bounds can be further modified for specific $p$. See e.g. [6] with $p = 1/2$.

Proof of Theorem 2.3* Let $A_n = \{ S_{\gamma_n}^{(n-\gamma_n)} = \gamma_n - 1 \}$. Similar to (3.4) we have
\[ P(A_n) = P(M_{\gamma_n} = \gamma_n - 1) \sim (pq)^{\frac{\gamma_n}{2}}. \]
When $\sum_{n=1}^{\infty} (pq)^{\frac{\gamma_n}{2}} = \infty$, we have
\[ \sum_{n=1}^{\infty} P(A_n) = \infty. \]
Following the method in the proof of Theorem 2.2, we have
\[ \frac{P(A_i A_j)}{P(A_i) P(A_j)} = 1 + o(1) \quad \text{as} \quad j \to \infty. \]
Then (3.12) holds and by Lemma 3.3 we obtain Theorem 2.3*(i).

And when $\sum_{n=1}^{\infty} (pq)^{\frac{\gamma_n}{2}} < \infty$, we have
\[ \sum_{n=1}^{\infty} P(A_n) < \infty. \]
By the Borel–Cantelli lemma, we obtain Theorem 2.3*(ii).

3.2 Proof of Proposition 2.4

Theorem 2.1 implies
\[ \liminf_{N \to \infty} \frac{M_N}{\log_1/\sqrt{pq}N} \geq 1 \quad a.s. \]
We only need to prove
\[ \limsup_{N \to \infty} \frac{M_N}{\log_1/\sqrt{pq}N} \leq 1 \quad a.s. \quad (3.22) \]
For any $\varepsilon > 0$ and $N \in \mathbb{N}$, we introduce the following notations:
\[ u := \lceil (1 + \varepsilon) \log_1/\sqrt{pq}N \rceil + 1, \quad A_N = \bigcup_{k=1}^{N-u+1} \{ S_u^{(k)} = u - 1 \}. \]
Thus we have $P(A_N) \leq 2N(pq)^{\frac{\log_1/\sqrt{pq}}{1-pq}}$, and by the definition of $u$ it is easy to have
\[ P(A_N) \leq 2N(pq)^{-1} (pq)^{\frac{\log_1/\sqrt{pq}}{1-pq}}N = 2(pq)^{-1}N^{-\varepsilon}. \]
Let $T \in \mathbb{N}$ with $T\varepsilon > 1$. It holds that
\[ \sum_{k=1}^{\infty} P(A_{kT}) \leq 2(pq)^{-1} \sum_{k=1}^{\infty} \frac{1}{kT^\varepsilon} < \infty, \]
which, again by the Borel–Cantelli lemma
\[ P(A_{kT} \ i.o.) = P \left( \limsup_{k \to \infty} \bigcup_{i=1}^{kT-\tilde{u}+1} \{ S_u^{(i)} = \tilde{u} - 1 \} \right) = 0 \]
with $\tilde{u} := \lceil (1 + \varepsilon) \log_1/\sqrt{pq}kT \rceil + 1$. It follows by the definition of $\tilde{u}$ that
\[ \limsup_{k \to \infty} \frac{M_{kT}}{\log_1/\sqrt{pq}kT} \leq 1 + \varepsilon \quad a.s. \]
So for any $N \in \mathbb{N}$, we choose $k \in \mathbb{N}$ such that $k^T \leq N \leq (k+1)^T$ and obtain

$$M_N \leq M_{(k+1)^T} \leq (1 + \varepsilon) \log_{1/\sqrt{pq}}(k+1)^T \leq (1 + 2\varepsilon) \log_{1/\sqrt{pq}} k^T \leq (1 + 2\varepsilon) \log_{1/\sqrt{pq}} N$$

with probability 1 for all but finitely many $N$. Hence we get that $\limsup_{N \to \infty} \frac{M_N}{\log_{1/\sqrt{pq}} N} \leq 1 + 2\varepsilon$ a.s. (3.22) holds by the arbitrariness of $\varepsilon$.

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