Seiberg-Witten Equations on $\mathbb{R}^8$

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Abstract

We show that there are no nontrivial solutions of the Seiberg-Witten equations on $\mathbb{R}^8$ with constant standard $\text{spin}^c$ structure.
1. Introduction

The Seiberg-Witten equations are meaningful on any even-dimensional manifold. To state them, let us recall the general set-up, adopting the terminology of the forthcoming book by D. Salamon ([1]).

A spin$^c$-structure on a $2n$-dimensional real inner-product space $V$ is a pair $(W, \Gamma)$, where $W$ is a $2n$-dimensional complex Hermitian space and $\Gamma : V \to \text{End}(W)$ is a linear map satisfying

$$\Gamma(v)^* = -\Gamma(v), \quad \Gamma(v)^2 = -\|v\|^2$$

for $v \in V$. Globalizing this defines the notion of a spin$^c$-structure $\Gamma : TX \to \text{End}(W)$ on a $2n$-dimensional (oriented) manifold $X$, $W$ being a $2n$-dimensional complex Hermitian vector bundle on $X$. Such a structure exists iff $w_2(X)$ has an integral lift. $\Gamma$ extends to an isomorphism between the complex Clifford algebra bundle $C_c(TX)$ and $\text{End}(W)$. There is a natural splitting $W = W^+ \oplus W^-$ into the $\pm i^n$ eigenspaces of $\Gamma(e_2n, e_{2n-1} \cdots e_1)$ where $e_1, e_2, \cdots, e_{2n}$ is any positively oriented local orthonormal frame of $TX$.

The extension of $\Gamma$ to $C_2(X)$ gives via the identification of $\Lambda^2(T^*X)$ with $C_2(X)$ a map

$$\rho : \Lambda^2(T^*X) \to \text{End}(W)$$

given by

$$\rho\left(\sum_{i<j} \eta_{ij} e_i^* \wedge e_j^*\right) = \sum_{i<j} \eta_{ij} \Gamma(e_i)\Gamma(e_j).$$

The bundles $W^\pm$ are invariant under $\rho(\eta)$ for $\eta \in \Lambda^2(T^*X)$. Denote $\rho^\pm(\eta) = \rho(\eta)|_{W^\pm}$. The map $\rho$ (and $\rho^\pm$) extends to

$$\rho : \Lambda^2(T^*X) \otimes \mathbb{C} \to \text{End}(W).$$

(If $\eta \in \Lambda^2(T^*X) \otimes \mathbb{C}$ is real-valued then $\rho(\eta)$ is skew-Hermitian and if $\eta$ is imaginary-valued then $\rho(\eta)$ is Hermitian.)

A Hermitian connection $\nabla$ on $W$ is called a spin$^c$ connection (compatible with the Levi-Civita connection) if

$$\nabla_v(\Gamma(w)\Phi) = \Gamma(w)\nabla_v\Phi + \Gamma(\nabla_v w)\Phi$$

where $\Phi$ is a spinor (section of $W$), $v$ and $w$ are vector fields on $X$ and $\nabla_v w$ is the Levi-Civita connection on $X$. $\nabla$ preserves the subbundles $W^\pm$. 

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There is a principal $Spin^c(2n) = \{ e^{i\theta} x | \theta \in \mathbb{R}, x \in Spin(2n) \} \subset C^c(\mathbb{R}^{2n})$ bundle $P$ on $X$ such that $W$ and $TX$ can be recovered as the associated bundles
\[ W = P \times_{Spin^c(2n)} C^{2n}, \quad TX = P \times_{Ad} \mathbb{R}^{2n}, \]
$Ad$ being the adjoint action of $Spin^c(2n)$ on $\mathbb{R}^{2n}$. We get then a complex line bundle $L_{\Gamma} = P \times_{\delta} \mathbb{C}$ using the map $\delta : Spin^c(2n) \to S^1$ given by $\delta(e^{i\theta} x) = e^{2i\theta}$.

There is a one-to-one correspondence between $spin^c$ connections on $W$ and $spin^c(2n) = Lie(Spin^c(2n)) = spin(2n) \oplus i\mathbb{R}$-valued connection-1-forms $\hat{A} \in \mathbf{A}(P) \subset \Omega^1(P, spin^c(2n))$ on $P$.

Now consider the trace-part $A$ of $\hat{A}$: $A = \frac{1}{2n} trace(\hat{A})$. This is an imaginary valued 1-form $A \in \Omega^1(P, i\mathbb{R})$ which is equivariant and satisfies
\[ A_p(p \cdot \xi) = \frac{1}{2n} trace(\xi) \]
for $v \in T_pP, g \in Spin^c(2n), \xi \in spin^c(2n)(where p \cdot \xi$ is the infinitesimal action). Denote the set of imaginary valued 1-forms on $P$ satisfying these two properties by $\mathbf{A}(\Gamma)$. There is a one-to-one correspondence between these 1-forms and $spin^c$ connections on $W$. Denote the connection corresponding to $A$ by $\nabla_A$. $\mathbf{A}(\Gamma)$ is an affine space with parallel vector space $\Omega^1(X, i\mathbb{R})$. For $A \in \mathbf{A}(\Gamma)$ the 1-form $2A \in \Omega^1(P, i\mathbb{R})$ represents a connection on the line bundle $L_{\Gamma}$. Because of this reason $A$ is called a virtual connection on the virtual line bundle $L_{\Gamma}^{1/2}$. Let $F_A \in \Omega^2(X, i\mathbb{R})$ denote the curvature of the 1-form $A$. Finally, let $D_A$ denote the Dirac operator corresponding to $A \in \mathbf{A}(\Gamma)$,
\[ C^\infty(X, W^+) \to C^\infty(X, W^-) \]
defined by
\[ D_A(\Phi) = \sum_{i=1}^{2n} \Gamma(e_i) \nabla_{A,e_i}(\Phi) \]
where $\Phi \in C^\infty(X, W^+)$ and $e_1, e_2, \cdots, e_{2n}$ is any local orthonormal frame.

The Seiberg-Witten equations can now be expressed as follows. Fix a $spin^c$ structure $\Gamma : TX \to End(W)$ on $X$ and consider the pairs $(A, \Phi) \in \mathbf{A}(\Gamma) \times C^\infty(X, W^+)$. The SW-equations read
\[ D_A(\Phi) = 0, \quad \rho^+(F_A) = (\Phi\Phi^*)_0 \]
where $(\Phi\Phi^*)_0 \in C^\infty(X, End(W^+))$ is defined by $(\Phi\Phi^*)(\tau) = \langle \Phi, \tau > \Phi$ for $\tau \in C^\infty(X, W^+)$ and $(\Phi\Phi^*)_0$ is the traceless part of $(\Phi\Phi^*)$.  

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In dimension $2n = 4$, $\rho^+(F_A) = \rho^+(F_A^+) = \rho(F_A^+)$ (where $F^+$ is the self-dual part of $F$ and the second equality understood in the obvious sense), and therefore self-duality comes intimately into play. The first problem in dimensions $2n > 4$ is that there is not a generally accepted notion of self-duality. Although there are some meaningful definitions ([2],[3],[4],[5],[6]) (Equivalence of self-duality notions in [2],[3],[5],[6] has been shown in [7], making them more relevant as they separately are), they do not assign a well-defined self-dual part to a given 2-form. Even though $\rho^+(F_A)$ is still meaningful, it is apparently less important due to the lack of an intrinsic self-duality of 2-forms in higher dimensions.

The other serious problem in dimensions $2n > 4$ is that the SW-equations as they are given above are overdetermined. So it is improbable from the outset to hope for any solutions. We verify below for $2n = 8$ that there aren’t indeed any solutions.

In dimension $2n = 4$ it is well-known that there are no finite-energy solutions ([1]), but otherwise whole classes of solutions are found which are related to vortex equations ([8]). In the physically interesting case $2n = 8$ we will suggest a modified set of equations which is related to generalized self-duality referred to above. These equations include the 4-dimensional Seiberg-Witten solutions as special cases.

2. Seiberg-Witten Equations on $R^8$

We fix the constant spin$^c$ structure $\Gamma : R^8 \rightarrow C^{16 \times 16}$ given by

$$\Gamma(e_i) = \begin{pmatrix} 0 & \gamma(e_i) \\ -\gamma(e_i)^* & 0 \end{pmatrix}$$

($e_i, i = 1, 2, ..., 8$ being the standard basis for $R^8$), where

$$\gamma(e_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma(e_2) = \begin{pmatrix} i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix}.$$
\[
\gamma(e_3) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
\gamma(e_4) = \begin{pmatrix}
0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0
\end{pmatrix},
\]

\[
\gamma(e_5) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix},
\]

\[
\gamma(e_6) = \begin{pmatrix}
0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\
0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i & 0 & 0 & 0 & 0
\end{pmatrix},
\]
\[
\gamma(e_7) = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
\end{pmatrix},
\]
\[
\gamma(e_8) = \begin{pmatrix}
0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
\end{pmatrix}.
\]

(We obtain this spin\(^c\) structure from the well-known isomorphism of the complex Clifford algebra \(C^c(\mathbb{R}^{2n})\) with \(\text{End}(\Lambda^* \mathbb{C}^n)\).)

In our case \(X = \mathbb{R}^8, W = \mathbb{R}^8 \times \mathbb{C}^{16}, W^+ = \mathbb{R}^8 \times \mathbb{C}^8\) and \(L_\Gamma = L_{\Gamma^{1/2}} = \mathbb{R}^8 \times \mathbb{C}\). Consider the connection 1-form

\[
A = \sum_{i=1}^{8} A_i dx_i \in \Omega^1(\mathbb{R}^8, i\mathbb{R})
\]
on the line bundle \(\mathbb{R}^8 \times \mathbb{C}\). Its curvature is given by

\[
F_A = \sum_{i<j} F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbb{R}^8, i\mathbb{R})
\]

where \(F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\). The spin\(^c\) connection \(\nabla = \nabla_A\) on \(W^+\) is given by

\[
\nabla_i \Phi = \frac{\partial \Phi}{\partial x_i} + A_i \Phi
\]

\((i = 1, \ldots, 8)\) where \(\Phi : \mathbb{R}^8 \to \mathbb{C}^8\).

\[
\rho^+ : \Lambda^2(T^*X) \otimes \mathbb{C} \to \text{End}(W^+)
\]
is given by
Proof: Trivial but tedious manipulation with the linear system.

Acknowledgement

There are no nontrivial solutions of the Seiberg-Witten equations on \( R^8 \) with constant standard spin\(^c \) structure, i.e.

\[ \rho^+(F_A) = (\Phi \Phi^*)_0 \] (alone) implies \( F_A = 0 \) and \( \Phi = 0 \).

Proof: Trivial but tedious manipulation with the linear system.

\[ (\Phi \Phi^*)_0 = \begin{pmatrix} \phi_1 \bar{\phi}_1 - 1/8 \sum \phi_i \bar{\phi}_i & \phi_1 \bar{\phi}_2 & \ldots & \phi_1 \bar{\phi}_8 \\ \phi_2 \bar{\phi}_1 & \phi_2 \bar{\phi}_2 - 1/8 \sum \phi_i \bar{\phi}_i & \ldots & \phi_2 \bar{\phi}_8 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_8 \bar{\phi}_1 & \phi_8 \bar{\phi}_2 & \ldots & \phi_8 \bar{\phi}_8 - 1/8 \sum \phi_i \bar{\phi}_i \end{pmatrix} \]

It was remarked by Salamon([1],p.187) that \( \rho^+(F_A) = 0 \) implies \( F_A = 0 \) (i.e.reducible solutions of 8-dim. SW-equations are flat.)

It can be explicitly verified that all solutions are reducible and flat:

**Proposition:** There are no nontrivial solutions of the Seiberg-Witten equations on \( R^8 \) with constant standard spin\(^c \) structure, i.e.

\[ \rho^+(F_A) = (\Phi \Phi^*)_0 \] (alone) implies \( F_A = 0 \) and \( \Phi = 0 \).

The above work is based on a talk given at the 5th Gökova Geometry-Topology Conference held at Akyaka-Muğla, Turkey during May, 1996.
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