Abstract. Let \( p \) be a prime number, let \( G \) be a finite group, let \( N \) be a normal subgroup of \( G \), and let \( \theta \) be a \( G \)-invariant irreducible character of \( N \). In Rizo (J Algebra 514:254–272, 2018), we introduced a canonical partition of the set \( \text{Irr}(G|\theta) \) of irreducible constituents of the induced character \( \theta^G \), relative to the prime \( p \). We call the elements of this partition the \( \theta \)-blocks. In this paper, we construct a canonical basis of the complex space of class functions defined on \( \{ x \in G \mid x_p \in N \} \), which supersedes previous non-canonical constructions. This allows us to define \( \theta \)-decomposition numbers in a natural way. We also prove that the elements of the partition of \( \text{Irr}(G|\theta) \) established by these \( \theta \)-decomposition numbers are the \( \theta \)-blocks.

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1. Introduction

Let \( p \) be a prime number, and let \( M \) be a maximal ideal of the ring \( \mathbb{R} \) of algebraic integers in \( \mathbb{C} \) containing \( p \). For every finite group \( G \), and only depending on \( M \), R. Brauer constructed a basis of the space \( \mathfrak{c}f(G^p) \) of complex class functions defined on the set \( G^p \) of the \( p \)-regular elements of \( G \), namely the set \( \text{IBr}(G) \) of the irreducible Brauer characters of \( G \). Suppose now that \( N \) is a fixed normal subgroup of \( G \), and consider \( G^0 = \{ g \in G \mid g_p \in N \} \), where \( g_p \) is the \( p \)-part of \( g \in G \), and let \( \mathfrak{c}f(G^0) \) denote the space of complex class functions defined on \( G^0 \). If \( \chi \) is any complex class function defined on \( G \), then \( \chi^0 \) denotes the restriction of \( \chi \) to the set \( G^0 \).

If \( N \) is a \( p \)-group, Navarro constructed in Ref. [6] a canonical basis \( \text{IBr}(G, N) \) of \( \mathfrak{c}f(G^0) \) (depending only on \( M \)) satisfying that whenever \( \chi \in \) The author acknowledges support by Ministerio de Ciencia e Innovación PID2019-103854GB-I00 and FEDER funds.
\[ \chi^0 = \sum_{\varphi \in \text{IBr}(G,N)} d_{\chi \varphi} \varphi \]

for some uniquely defined non-negative integers \( d_{\chi \varphi} \). The \( N \)-projective characters

\[ \Phi_{\varphi} = \sum_{\chi \in \text{Irr}(G)} d_{\chi \varphi} \chi \]

were previously studied by Külshammer and Robinson in Ref. [4].

In the not-so-widely available [7], Navarro constructed a similar basis of \( cf(G^0) \) for any normal subgroup \( N \) of \( G \), where \( N \) was not necessarily a \( p \)-group. However, his methods did not allow him to prove that this basis was canonical (that is, depending only on \( M \)). In the first main result of this paper, we construct such a canonical basis.

When studying the complex space \( cf(G^0) \), it is well known that standard Clifford reductions (that we shall review below) allow us to fix a \( G \)-invariant character \( \theta \in \text{Irr}(N) \) and only consider the subspace \( cf(G^0|\theta) \) which is the span of \( \{ \chi^0 | \chi \in \text{Irr}(G|\theta) \} \), where, as usual, \( \text{Irr}(G|\theta) \) is the set of irreducible constituents of the induced character \( \theta^G \).

**Theorem A.** Suppose that \( p \) is a prime, \( G \) is a finite group, \( N \) is a normal subgroup of \( G \), and \( \theta \in \text{Irr}(N) \) is \( G \)-invariant. Then, there is a canonical basis \( \text{IBr}(G|\theta) \) of the space \( cf(G^0|\theta) \) (depending only on the choice of the maximal ideal \( M \)), such that whenever \( \chi \in \text{Irr}(G|\theta) \), then:

\[ \chi^0 = \sum_{\varphi \in \text{IBr}(G|\theta)} d_{\chi \varphi} \varphi \]

for some uniquely determined non-negative integers \( d_{\chi \varphi} \). If \( N = 1 \), then \( \text{IBr}(G|1) = \text{IBr}(G) \).

We call the elements of this basis the \( \theta \)-Brauer characters and the non-negative integers \( d_{\chi \varphi} \) the \( \theta \)-decomposition numbers. If \( N \) is a \( p' \)-group, it is easy to see that \( \text{IBr}(G|\theta) \) is exactly the set of irreducible Brauer characters of \( G \) lying over \( \theta \) (as a Brauer character).

The existence of \( \text{IBr}(G|\theta) \) allows us to define a new natural linking in \( \text{Irr}(G|\theta) \): we say that \( \chi, \psi \in \text{Irr}(G|\theta) \) are \( \theta \)-linked if there is \( \varphi \in \text{IBr}(G|\theta) \), such that \( d_{\chi \varphi} \neq 0 \neq d_{\psi \varphi} \). In fact, we shall prove that the \( \theta \)-blocks defined in Ref. [10] can be characterized by this linking.

**Theorem B.** Suppose that \( p \) is a prime, \( G \) is a finite group, \( N \) is a normal subgroup of \( G \), and \( \theta \in \text{Irr}(N) \) is \( G \)-invariant. Then, the connected components of the graph defined in \( \text{Irr}(G|\theta) \) by \( \theta \)-linking are the \( \theta \)-blocks.

Once \( \theta \)-Brauer characters and \( \theta \)-decomposition numbers are defined, of course, many natural questions arise. However, our main interest in these new objects is to study if they can be of any help to understand classical problems on blocks, defect groups, Brauer characters, or decomposition numbers. For instance, as suggested by Navarro, perhaps, the \( \theta \)-decomposition numbers
can be bounded by a function of the size of \( \theta \)-defect groups (as defined in Ref. [10]), and the presence of normal subgroups might facilitate reductions to simple groups.

In the last part of this paper, we give analogs to classical results on blocks for the \( \theta \)-blocks: a \( \theta \)-version of Brauer’s first main theorem and a \( \theta \)-version of block orthogonality, which we believe might have independent interest.

This paper is structured as follows: in Sect. 2, we recall the definition and some properties of the \( \theta \)-blocks that will be used later on. In Sect. 3, we prove Theorem A, and in Sect. 4, we prove Theorem B. In Sect. 5, we prove the \( \theta \)-versions of Brauer’s first main theorem and block orthogonality.

2. Preliminaries

Let us start by recalling here the definition of \( \theta \)-blocks given in Ref. [10]. The notion of character triple isomorphism (see Definition 11.23 of [3]), as well as the theory of projective representations, is essential to define and to deal with \( \theta \)-blocks.

Recall that a complex projective representation of a finite group \( G \) is a map:

\[ P : G \to \text{GL}_n(\mathbb{C}), \]

such that for every \( x, y \in G \), there is some \( \alpha(x, y) \in \mathbb{C}^\times \) satisfying:

\[ P(x)P(y) = \alpha(x, y)P(xy). \]

The function \( \alpha : G \times G \to \mathbb{C}^\times \) is called the factor set of \( P \).

If \( G \) is a finite group, \( N \triangleleft G \), and \( \theta \in \text{Irr}(N) \) is \( G \)-invariant, then we say that \( (G, N, \theta) \) is a character triple. The theory of character triples and their isomorphisms were developed by I. M. Isaacs, and we refer to Chapter 11 of [3] for their properties (see also Chapter 5 of [8]). Character triples have associated projective representations in the following way. If \( (G, N, \theta) \) is a character triple, a projective representation \( P \) of \( G \) is associated with \( \theta \) if:

(a) \( P_N \) is an ordinary representation of \( N \) affording \( \theta \), and
(b) \( P(ng) = P(n)P(g) \) and \( P(gn) = P(g)P(n) \) for \( g \in G \) and \( n \in N \).

A fundamental fact about projective representations associated with character triples is that, given a character triple, one can always find a projective representation associated with it, \( P \), such that its factor set \( \alpha \) has roots of unity values (see, for instance, Theorem 8.2 of [2] or Theorem 5.5 of [8]). Using such a projective representation \( P \), it is possible to associate with each character triple \( (G, N, \theta) \) a new finite group \( \hat{G} \) constructed as follows. Let \( Z \) be a finite subgroup of the multiplicative group of the field of the complex numbers containing the values of \( \alpha \). As a set, \( \hat{G} \) is the set of pairs \( \{(g, z) \mid g \in G, z \in Z\} \), and we give to \( \hat{G} \) the following product:

\[ (g, z)(h, t) = (gh, \alpha(g, h)zt). \]

It turns out that \( \hat{G} \) is a finite group that contains \( N \) as a normal subgroup and \( Z \subseteq Z(\hat{G}) \). Moreover, if we define \( \tau(g, z) = z\text{trace}(P(g)) \), we have that \( \tau \)
is an irreducible character of $\hat{G}$ extending $\theta$. For a more detailed explanation, see Theorem 11.28 of [3] or Theorem 5.6 of [8]. We care to remark that $\hat{G}$ depends on the choice of $Z$. However, we will not emphasize this in the notation, since it is not going to matter for our purposes.

One of the keys in the construction of the $\theta$-blocks is the following. Suppose that $(G, N, \theta)$ is a character triple, let $\hat{G}$ be as above, and write $\hat{N} = N \times Z$. Then, we have that $\hat{N}$ is normal in $\hat{G}$, $\hat{N}/N$ is central in $\hat{G}/N$ and there is a uniquely defined linear character $\hat{\lambda}$ of $\hat{N}/N$, such that $(G, N, \theta)$ and $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ are isomorphic character triples (see Theorem 11.28 of [3] or Corollary 5.9 of [8]). In particular, this means that there is a bijection:

$$ * : \text{Irr}(G|\theta) \to \text{Irr}((\hat{G}/N|\hat{\lambda}). $$

Since we are going to use how this bijection is explicitly constructed, we review this construction here. Let $\chi \in \text{Irr}(G|\theta)$ and let $\pi : \hat{G} \to G$ be the onto group homomorphism $(g, z) \mapsto g$, which has kernel $Z$. Since $\pi$ induces an isomorphism $\hat{G}/Z \to G$, there is a unique $\chi^\pi \in \text{Irr}(\hat{G})$, such that $\chi^\pi(g, z) = \chi(g)$ for all $g \in G, z \in Z$. Since $\chi$ lies over $\theta$ notice that $\chi^\pi$ lies over $\theta$. Let $\tau \in \text{Irr}(\hat{G})$ be the character extending $\theta$ mentioned above. By Gallagher’s Corollary 6.17 of [3], there exists a unique $\chi^* \in \text{Irr}(\hat{G}/N)$, such that $\chi^\pi = \chi^* \tau$. (Recall that we view the characters of $H/N$ as characters of $H$ that contain $N$ in its kernel.) Now, evaluating in $(1, z)$ for $z \in Z$, we easily check that $\chi^* \in \text{Irr}(\hat{G}/N|\hat{\lambda}).$

**Notation 2.1.** We say that the group $\hat{G}$ constructed above is a representation group associated with $(G, N, \theta)$ and $\mathcal{P}$ (and $Z$). We usually write $G^* = \hat{G}/N$, $N^* = \hat{N}/N$ and $\theta^* = \hat{\lambda}$ and we say that the character triple $(G^*, N^*, \theta^*)$ is a standard isomorphic character triple for $(G, N, \theta)$ given by $\mathcal{P}$. Also, we call the bijective map $* : \text{Irr}(G|\theta) \to \text{Irr}(G^*|\theta^*)$ constructed above the standard bijection.

We recall now the definition of $\theta$-blocks and $\theta$-defect groups given in Ref. [10].

**Definition 2.2.** Let $(G, N, \theta)$ be a character triple. Let $\hat{G}$ be a representation group for $(G, N, \theta)$ and let $\pi : \hat{G} \to G$ be the canonical homomorphism $(g, z) \mapsto g$ with kernel $Z$. Let $* : \text{Irr}(G|\theta) \to \text{Irr}(\hat{G}/N|\hat{\lambda})$ be the associated standard bijection. We say that a non-empty subset $B_\theta \subseteq \text{Irr}(G|\theta)$ is a $\theta$-block of $G$ if there exists a $p$-block $\hat{B}$ of $\hat{G}/N$, such that:

$$ B_\theta^* = \{ \chi^* \mid \chi \in B_\theta \} = \text{Irr}(\hat{B}|\hat{\lambda}) $$

where $\chi^*$ denotes the image of $\chi$ through the standard bijection. If $\hat{D}/N$ is a defect group of $\hat{B}$, then we say that $\pi(\hat{D})/N$ is a $\theta$-defect group of $B_\theta$.

Of course, as we have introduced it here the definition of $\theta$-blocks seems to depend on the choice of the projective representation associated with $\theta$. The same happens with the $\theta$-defect groups. However, one of the main results in Ref. [10] is to prove that they are canonically defined (Theorem 4.3 of [10]).
3. Theorem A

Suppose again that $N$ is a normal subgroup of $G$ and consider the normal set $G^0 = \{ x \in G \mid x_p \in N \}$ and the complex space $\text{cf}(G^0)$ of complex class functions defined on $G^0$. If $\delta \in \text{cf}(G)$, we denote by $\delta^0$ the restriction of $\delta$ to $G^0$. The space $\text{cf}(G^0)$ can naturally be decomposed as a direct sum of subspaces. Indeed, for a given $\theta \in \text{Irr}(N)$, we define $\text{cf}(G^{\theta})$ to be the $\mathbb{C}$-span of $\text{Irr}(G^{\theta})$, and we let:

$$\text{cf}(G^0|\theta) = \text{cf}(G|^0\theta) = \{ \delta^0 \mid \delta \in \text{cf}(G^{\theta}) \} .$$

Of course, $\text{cf}(G^0|\theta) = \text{cf}(G^0|\theta^g)$ for $g \in G$. In fact, it is not difficult to prove (see Lemma 2.1 of [6]) that if $\Theta$ is a complete set of representatives of the $G$-action on $\text{Irr}(N)$, then:

$$\text{cf}(G^0) = \bigoplus_{\theta \in \Theta} \text{cf}(G^0|\theta) .$$

The strategy now is to fix $\theta \in \text{Irr}(N)$ and focus on $\text{cf}(G^0|\theta)$. The next natural step is to prove that if $T = G_\theta$ is the stabilizer of $\theta$ in $G$, then induction $\psi \mapsto \psi^G$ defines a linear isomorphism:

$$\text{cf}(T^0|\theta) \to \text{cf}(G^0|\theta) .$$

This is done in Lemma 2.2 of [6]. Using induction, this allows us to work with a $G$-invariant $\theta$, that is, with a character triple $(G, N, \theta)$.

Suppose now that a set $\text{IBr}(G)$ of irreducible Brauer characters of $G$ is given (in other words, that we have chosen a maximal ideal $M$ containing $p$ in the ring of algebraic integers $\mathbb{R}$ of the complex numbers.) If $N$ is a $p$-group, Navarro constructed in Ref. [6] a natural basis of $\text{cf}(G^0)$ depending just on $\text{IBr}(G)$. Since we are using this construction explicitly, we review it here for the reader’s convenience.

Suppose that $\theta \in \text{Irr}(N)$ is $G$-invariant and $N$ is a $p$-group. We define $\hat{\theta} \in \text{cf}(G^0|\theta)$ as follows. If $x \in G^0$, then $x_p \in N$ and $N \langle x \rangle / N$ is a $p'$-group. Since $N$ is a $p$-group, there is a canonical extension $\hat{\theta}_x \in \text{Irr}(N \langle x \rangle)$ (Corollary 8.16 of [3]). This is the unique extension of $\theta$ to $N \langle x \rangle$ whose determinantal order is a power of $p$. Now, we define $\hat{\theta}(x) = \hat{\theta}_x(x)$. If $\eta$ is any class function defined on the $p$-regular elements of $G$, we define:

$$(\theta \star \eta)(x) = \hat{\theta}(x)\eta(x_{p'})$$

for $x \in G^0$. One of the main results in Ref. [6] (Theorem 4.3 of [6]) is that:

$$\text{IBr}(G|\theta) = \{ \theta \star \eta \mid \eta \in \text{IBr}(G) \}$$

is a basis of $\text{cf}(G^0|\theta)$ and that if $\chi \in \text{Irr}(G|\theta)$, then:

$$\chi^0 = \sum_{\varphi \in \text{IBr}(G|\theta)} d_{\chi|\varphi} \varphi$$

for some (uniquely defined) non-negative integers $d_{\chi|\varphi}$.

What happens when $N$ is not necessarily a $p$-group? This case is solved (in a non-canonical way) in Ref. [7]. Suppose first that $N$ is central. Hence,
In Ref. [7], it is proved that:

$$\chi_NIBr(G)$$

with way of choosing a canonical (set of irreducible Brauer characters of \(G\) for some (uniquely defined) non-negative integers that are obtained through this process coincide.

if we choose two standard isomorphic triples, then the corresponding bases to \(\theta\) for some uniquely determined non-negative integers \(\theta\) (that is, if \(\theta \in IBr(N)\) (that is, if \(N = N_{p'}\)), then \(\alpha = 1\), \(\beta = \theta\), and \(IBr(G|\theta)\) is exactly the set of irreducible Brauer characters of \(G\) lying over \(\theta\).

Finally, it is shown in Lemma 2.1 of [7] that if \((G, N, \theta)\) and \((G^*, N^*, \theta^*)\) are isomorphic character triples, then there is natural isomorphism of the vector spaces:

\[
* : cf(G^0|\theta) \rightarrow cf((G^*)^0|\theta^*),
\]

such that

\[
(\chi^*)^0 = (\chi^0)^*
\]

for \(\chi \in cf(G|\theta)\). This easily shows that, if \(N^*\) is central in \(G^*\), then the inverse image of:

\[
IBr(G^*|\theta^*) = \{\theta^* \ast \eta^* | \eta \in IBr(G^*|\theta^*)\}
\]

where \(\theta^* = \theta^*_p \times \theta^{/p}\), is a basis of \(cf(G^0|\theta)\). Since this basis depends on the choice of the isomorphic character triple \((G^*, N^*, \theta^*)\), we denote it by \(B_{(G^*, N^*, \theta^*)}\). Hence, if \(\chi \in Irr(G|\theta)\), then:

\[
\chi^0 = \sum d_{\chi, \varphi} \varphi
\]

for some uniquely determined non-negative integers \(d_{\chi, \varphi}\), where \(\varphi\) runs over \(B_{(G^*, N^*, \theta^*)}\). The problem with this construction is that there is no known way of choosing a canonical \((G^*, N^*, \theta^*)\) with \(N^*\) central that is isomorphic to \((G, N, \theta)\). Our main theorem in this section solves this by proving that if we choose two standard isomorphic triples, then the corresponding bases that are obtained through this process coincide.

Assume that \((G^*, N^*, \theta^*)\) is any character triple isomorphic to \((G, N, \theta)\), with \(N^*\) central, and again write \(N^* = N^*_p \times N^*_{p'}\), where \(N^*_p \in Syl_p(N^*)\) and hence \(\theta^* = \theta^*_p \times \theta^*_p\), with \(\theta^*_p \in Irr(N^*_p)\) and \(\theta^*_{p'} \in Irr(N^*_{p'})\). By Theorem 2.4 of [7], the set:

\[
IBr(G^*|\theta^*) = \{\theta^* \ast \varphi^* | \varphi^* \in IBr(G^*|\theta^*)\}
\]

is a basis of \(cf((G^*)^0|\theta^*)\). Since \(cf(G^*|\theta^*)\) is the \(\mathbb{C}\)-span of \(Irr(G^*|\theta^*)\), for each \(\varphi^* \in IBr(G^*|\theta^*)\), we can write:

\[
\theta^* \ast \varphi^* = \sum_{\chi^* \in Irr(G^*|\theta^*)} a_{\varphi^* \chi^*} (\chi^*)^0
\]
for some complex numbers \(a_{\varphi^*}\chi^* \in \mathbb{C}\). (We remark that these numbers are not necessarily unique, but for our purposes this is not going to matter.) Since \((G, N, \theta)\) and \((G^*, N^*, \theta^*)\) are isomorphic character triples, we know that there exists a bijection \(* : \text{Irr}(G|\theta) \rightarrow \text{Irr}(G^*|\theta^*)\), and by Lemma 2.1 of [7], the map \(\Psi^* \mapsto (\Psi^*)^0\) from \(\text{cf}(G|\theta)^0 \rightarrow \text{cf}(G^*|\theta^*)^0\) is an isomorphism of vector spaces. Hence, the basis of \(\text{cf}(G|\theta)^0\) described in Ref. [7] is:

\[
\mathcal{B}_{(G^*,N^*,\theta^*)} = \left\{ \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi^*}\chi^* \varphi^* \mid \varphi^* \in \text{IBr}(G^*|\theta^*) \right\}.
\]

If \((G, N, \theta)\) is a character triple and \((G_1, N_1, \theta_1)\) and \((G_2, N_2, \theta_2)\) are standard isomorphic character triples, we wish to prove that \(\mathcal{B}_{(G_1,N_1,\theta_1)} = \mathcal{B}_{(G_2,N_2,\theta_2)}\). As usual, if \(\chi \in \text{cf}(G)\), we denote by \(\chi^{p'}\) the restriction of \(\chi\) to \(G^{p'}\). We need the following easy observation.

**Lemma 3.1.** Let \(N \triangleleft G\), \(\theta \in \text{Irr}(N)\) linear, and \(\alpha, \beta \in \text{cf}(G) \cup \text{cf}(G^{p'})\), then:

1. \(\theta \ast (\alpha + \beta) = \theta \ast \alpha + \theta \ast \beta\),
2. if \(\theta \ast \alpha = \theta \ast \beta\), then \(\alpha^{p'} = \beta^{p'}\).

**Proof.** (a) is straightforward. Now, suppose that \(\theta \ast \alpha = \theta \ast \beta\), and take \(y \in G^{p'}\). Notice that \(y \in G^\circ\). Since \(\theta\) is linear, we have that \(\alpha(y) = \theta(1)\alpha(y) = (\theta \ast \alpha)(y) = (\theta \ast \beta)(y) = \theta(1)\beta(y) = \beta(y)\) and (b) follows. \(\Box\)

We will also use also the following non-trivial result of [7].

**Theorem 3.2.** Suppose that \(N \subseteq Z(G)\), and let \(\theta = \alpha\beta \in \text{Irr}(N)\), where \(\alpha \in \text{Irr}(N)\) has \(p\)-power order and \(\beta \in \text{Irr}(N)\) has \(p'\)-order. Let \(\chi \in \text{cf}(G|\theta)\), and then, \(\chi^\circ = \alpha \ast \chi\).

**Proof.** See Theorem 2.4 of [7]. \(\Box\)

Recall that if \(\alpha : \hat{G} \rightarrow G\) is a surjective group homomorphism with kernel \(Z\) and \(\psi \in \text{Irr}(G)\), we denote by \(\psi^\alpha\) the unique irreducible character of \(\hat{G}\), such that \(\psi^\alpha(x) = \psi(\alpha(x))\) for \(x \in \hat{G}\). Now, note that if \(x \in G^{p'}\), then \(\alpha(x) \in G^{p'}\). Hence, if \(\varphi \in \text{IBr}(G)\), we denote by \(\varphi^\alpha\) the unique irreducible Brauer character of \(\hat{G}\), such that \(\varphi^\alpha(x) = \varphi(\alpha(x))\) for \(x \in \hat{G}^{p'}\). Notice that \(Z \subseteq \ker(\varphi^\alpha)\).

We recall here the following result of [10], which will be essential to prove Theorem A.

**Lemma 3.3.** Let \((G, N, \theta)\) be a character triple. Let \(P_1, P_2\) be projective representations of \(G\) associated with \(\theta\), with factor sets \(\alpha_1\) and \(\alpha_2\), respectively, whose values are roots of unity. Let \(Z_i\) be the subgroup of the multiplicative group of the field of complex numbers generated by the values of \(\alpha_i\). Let \(\hat{G}_i\) be the representation group associated with \(P_i\) and let \((\hat{G}_1/N, \hat{N}_1/N, \hat{\lambda}_1)\) and \((\hat{G}_2/N, \hat{N}_2/N, \hat{\lambda}_2)\) be the standard isomorphic character triples given by \(P_1\) and \(P_2\), respectively. Let \(\hat{G} = G \times Z_1 \times Z_2\) (as a set) and define in \(\hat{G}\) the product:

\[
(g, z_1, z_2)(h, z'_1, z'_2) = (gh, \alpha_1(g, h)z_1z'_1, \alpha_2(g, h)z_2z'_2).
\]

Then, the following holds.
Theorem 3.4. Let $\hat{G}$ be a finite group and $N \times 1 \times 1$ is a normal subgroup of $\hat{G}$ (which we identify with $N$).

(b) The maps $\rho_1 : \hat{G} \to \hat{G}_1$ and $\rho_2 : \hat{G} \to \hat{G}_2$ given by $(g, z_1, z_2) \mapsto (g, z_1)$ and $(g, z_1, z_2) \mapsto (g, z_2)$ are surjective group homomorphisms with kernels $Z_2$ and $Z_1$, respectively.

(c) Let $\chi \in \text{Irr}(G \theta)$ and let $\chi_i \in \text{Irr}(\hat{G}_i/N | \hat{\lambda}_i)$ be the image of $\chi$ under the standard bijection. Let $\hat{\chi}_i = \chi_i^{\theta_i} \in \text{Irr}(\hat{G}/N)$. Then, there exists a linear character $\beta \in \text{Irr}(\hat{G}/N)$, such that $\beta \hat{\chi}_1 = \hat{\chi}_2$.

Proof. This is part of Theorem 4.2 of [10].

Finally, we prove the following result which, as we mentioned above, will be key to prove Theorem A.

Theorem 3.4. Let $(G, N, \theta)$ be a character triple and let $(G_1, N_1, \theta_1)$ and $(G_2, N_2, \theta_2)$ be standard isomorphic character triples. Then:

$$\mathcal{B}(G_1, N_1, \theta_1) = \mathcal{B}(G_2, N_2, \theta_2).$$

Proof. Let $P_1$ and $P_2$ be projective representations associated with $\theta$ arising $(G_1, N_1, \theta_1)$ and $(G_2, N_2, \theta_2)$, respectively (that is, $G_i = \hat{G}_i/N$, $N_i = \hat{N}_i/N$ and $\theta_i = \hat{\lambda}_i$ in the notation of Lemma 3.3). If $\chi \in \text{Irr}(G|\theta)$, we write $\chi_i \in \text{Irr}(G_i|\theta_i)$ for the image of $\chi$ through the respective standard bijections. Now:

$$\text{IBr}(G_1|\theta_1) = \{ \theta_1 \ast \varphi_1 \mid \varphi_1 \in \text{IBr}(G_1|(|\theta_1|_p')) \}$$

$$= \left\{ \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi_1^\circ \mid \varphi_1 \in \text{IBr}(G_1|(|\theta_1|_p')) \right\}$$

and

$$\text{IBr}(G_2|\theta_2) = \{ \theta_2 \ast \varphi_2 \mid \varphi_2 \in \text{IBr}(G_2|(|\theta_2|_p')) \}$$

$$= \left\{ \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_2 \chi_2} \chi_2^\circ \mid \varphi_2 \in \text{IBr}(G_2|(|\theta_2|_p')) \right\}.$$

Hence:

$$\mathcal{B}(G_1, N_1, \theta_1) = \left\{ \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi_1^\circ \mid \varphi_1 \in \text{IBr}(G_1|(|\theta_1|_p')) \right\}$$

and

$$\mathcal{B}(G_2, N_2, \theta_2) = \left\{ \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_2 \chi_2} \chi_2^\circ \mid \varphi_2 \in \text{IBr}(G_2|(|\theta_2|_p')) \right\}.$$

Since $\mathcal{B}(G_1, N_1, \theta_1)$ and $\mathcal{B}(G_2, N_2, \theta_2)$ are basis of $\text{cf}(G|\theta)^\circ$, we have that $|\mathcal{B}(G_1, N_1, \theta_1)| = |\mathcal{B}(G_2, N_2, \theta_2)|$. Therefore, we just need to prove that:

$$\mathcal{B}(G_1, N_1, \theta_1) \subseteq \mathcal{B}(G_2, N_2, \theta_2).$$
Let
\[ \psi = \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi^\circ \in \mathcal{B}(G_1, N_1, \theta_1). \]

To prove that \( \psi \in \mathcal{B}(G_2, N_2, \theta_2) \), we need to show that:
\[ \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi_2^\circ \in \text{IBr}(G_2|\theta_2). \]

In other words, we need to prove that:
\[ \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi_2^\circ = \theta_2 \ast \varphi_2 \]

for some \( \varphi_2 \in \text{IBr}(G_2|\theta_2)^p \). By Theorem 3.2, we have that \( \chi_2^\circ = \theta_2 \ast \chi_2 \).

Then, by Lemma 3.1 (a):
\[ \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi_2^\circ = \theta_2 \ast \left( \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi_2 \right). \]

Write:
\[ \varphi_2 = \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} (\chi_2)^{p'}. \]

Then, \( \varphi_2 \in \text{cf}(G_2^{p'}) \), and for \( g \in G_2^\circ \), we have that:
\[ (\theta_2 \ast \varphi_2)(g) = \theta_2(gp)\varphi_2(gp') = \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi_2^\circ(g). \]

To the end, we just need to prove that \( \varphi_2 \in \text{IBr}(G_2|\theta_2)^{p'} \).

Write \( \hat{G} = G \times Z_1 \times Z_2 \) as in Lemma 3.3, and let \( \rho_1 : \hat{G} \to \hat{G}_1 \) and \( \rho_2 : \hat{G} \to \hat{G}_2 \) be the maps defined in Lemma 3.3(b). Write \( \hat{\chi}_i = \chi_i^{p_1} \in \text{Irr}(\hat{G}) \) and let \( \beta \) be the linear character of \( \hat{G}/N \), such that \( \beta \hat{\chi}_1 = \hat{\chi}_2 \) (Lemma 3.3(c)). Since \( \beta \) is linear and \( N \subseteq \ker(\beta) \), we have that \( \beta^{p'} \in \text{IBr}(\hat{G}/N) \).

Since \( \psi \in \mathcal{B}(G_1, N_1, \theta_1) \), we have that:
\[ \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi_1^\circ \in \text{IBr}(G_1|\theta_1), \]

and hence:
\[ \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi_1^\circ = \theta_1 \ast \varphi_1 \]

for some \( \varphi_1 \in \text{IBr}(G_1|\theta_1)^{p'} \). Now, again by Theorem 3.2 and Lemma 3.1(a), we have that:
\[ \theta_1 \ast \varphi_1 = \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi_1^\circ = \theta_1 \ast \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} \chi_1. \]

Since \( \theta_1 \) is linear, by Lemma 3.1(b), we obtain that:
\[ \varphi_1 = \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1} (\chi_1)^{p'}. \]
Hence:
\[ \varphi_1^{o_1} = \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1}(\hat{\chi}_1)^{p'}. \]

Now, since \( \beta \hat{\chi}_1 = \hat{\chi}_2 \), we have that:
\[ \eta := \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1}(\beta \hat{\chi}_1)^{p'} = \sum_{\chi \in \text{Irr}(G|\theta)} a_{\varphi_1 \chi_1}(\beta \hat{\chi}_1)^{p'} = \beta^{p'} \varphi_1^{o_1} \in \text{IBr}(\hat{G}/N). \]

Let \( \hat{\rho}_2 : \hat{G}/Z_1 \rightarrow \hat{G}_2 \) be the isomorphism induced by \( \rho_2 \). Since \( Z_1 \subseteq \text{ker}(\hat{\chi}_2) \), we have that \( \eta = \varphi_2^{\hat{\rho}_2} \), and hence, \( \varphi_2 \in \text{IBr}(\hat{G}_2) \). Since \( N \subseteq \text{ker}(\varphi_2) \), we obtain \( \varphi_2 \in \text{IBr}(G_2|((\theta_2)_p') \) as desired.

As a consequence, we obtain Theorem A.

**Corollary 3.5.** Suppose that \( p \) is a prime, \( G \) is a finite group, \( N \) is a normal subgroup of \( G \), and \( \theta \in \text{Irr}(N) \) is \( G \)-invariant. Then, there is a canonical basis \( \text{IBr}(G|\theta) \) of the space \( \text{cf}(G_0|\theta) \) (depending only on the choice of the maximal ideal \( M \)), such that whenever \( \chi \in \text{Irr}(G|\theta) \), then:
\[ \chi^0 = \sum_{\varphi \in \text{IBr}(G|\theta)} d_{\chi \varphi} \varphi \]
for some uniquely determined non-negative integers \( d_{\chi \varphi} \). If \( N = 1 \), then \( \text{IBr}(G|1) = \text{IBr}(G) \).

**Proof.** Let \( \text{IBr}(G|\theta) = B_{(G^*,N^*,\theta^*)} \), where \( (G^*,N^*,\theta^*) \) is a standard isomorphic character triple. In Corollary 2.5 of [7], it is proved that \( \text{IBr}(G|\theta) \) is a \( \mathbb{C} \)-basis of the space \( \text{cf}(G|\theta)^{o} \). By Theorem 3.4, we know that this basis is independent of the choice of isomorphic character triples, and hence, it is canonical, once we have fixed \( M \).

\[ \square \]

### 4. Theorem B

As we said in the Introduction, we call the elements of the basis \( \text{IBr}(G|\theta) \) the \( \theta \)-Brauer characters. If \( \chi \in \text{Irr}(G|\theta) \) and \( \varphi \in \text{IBr}(G|\theta) \), we denote the coefficient of \( \varphi \) in \( \chi^{o} \) by \( d_{\chi \varphi}^{o} \). Recall that we call the numbers \( d_{\chi \varphi} \) the \( \theta \)-decomposition numbers. Note that if \( N \) is a \( p \)-group, these \( \theta \)-decomposition numbers are the same that Navarro gives in [6].

Let \( \chi, \psi \in \text{Irr}(G|\theta) \). We say that \( \chi \) and \( \psi \) are \( \theta \)-linked if there exists \( \varphi \in \text{IBr}(G|\theta) \), such that:
\[ d_{\chi \varphi} \neq 0 \neq d_{\psi \varphi}. \]

The connected components of the graph defined by \( \theta \)-linking define a partition in \( \text{Irr}(G|\theta) \). We call the elements of this partition the blocks defined by \( \theta \)-decomposition numbers. We shall prove that these blocks are, in fact, the \( \theta \)-blocks.

If \( \chi \) and \( \psi \) are irreducible characters of \( G \), we say that \( \chi \) and \( \psi \) are linked if there exists \( \varphi \in \text{IBr}(G) \), such that:
\[ d_{\chi \varphi} \neq 0 \neq d_{\psi \varphi}, \]
where \( d_{\chi \varphi} \) and \( d_{\psi \varphi} \) are the classical decomposition numbers.
Lemma 4.1. Let \((G, N, \theta)\) be a character triple with \(N \subseteq \mathbb{Z}(G)\). Let \(\chi, \psi \in \text{Irr}(G|\theta)\). Then, \(\chi\) and \(\psi\) are linked if and only if they are \(\theta\)-linked.

Proof. Write \(N = N_p \times N_p'\), with \(N_p \in \text{Syl}_p(N)\), and \(\theta = \theta_p \times \theta_p'\), with \(\theta_p \in \text{Irr}(N_p)\) and \(\theta_p' \in \text{Irr}(N_p')\). By Theorem 3.2, we have that \(\chi = \theta \ast \chi = \theta \ast \chi p'\). Since \(\chi \in \text{Irr}(G|\theta p')\), it is clear that all the Brauer irreducible constituents of \(\chi p'\) lie over \(\theta p'\). Now, using Lemma 3.1, we have that:

\[
\chi p' = \sum_{\varphi \in \text{IBr}(G|\theta p')} d_{\chi \varphi} \varphi,
\]

if and only if

\[
\chi = \sum_{\varphi \in \text{IBr}(G|\theta p')} d_{\chi \varphi} (\theta \ast \varphi).
\]

\[\square\]

Note that from Lemma 4.1, we deduce that in the case that \(N\) is central, the \(\theta\)-decomposition numbers and the classical decomposition numbers coincide. This agrees with the results obtained by in Ref. [11].

The key to prove that the blocks defined by \(\theta\)-decomposition numbers and the \(\theta\)-blocks coincide is the following result of [10].

Theorem 4.2. Suppose that \(N \subseteq \mathbb{Z}(G)\) and let \(\theta \in \text{Irr}(N)\). Let \(B\) be a Brauer \(p\)-block of \(G\), such that \(\text{Irr}(B|\theta)\) is not empty. Then, the matrix \(D_{B, \theta} = (d_{\chi \varphi})\), where \(\chi \in \text{Irr}(B|\theta)\), \(\varphi \in \text{IBr}(B)\) and \(d_{\chi \varphi}\) are the classical decomposition numbers, is not of the form:

\[
\begin{pmatrix}
* & 0 \\
0 & *
\end{pmatrix},
\]

for any ordering of the rows and columns.

Proof. See Theorem 6.2 of [10]. \[\square\]

Using Lemma 4.1 and Theorem 4.2, we easily obtain the following.

Theorem 4.3. Let \((G, N, \theta)\) be a character triple with \(N \subseteq \mathbb{Z}(G)\). Then, the blocks defined by \(\theta\)-decomposition numbers are exactly the sets \(\text{Irr}(B|\theta)\) where \(B\) runs over the \(p\)-blocks of \(G\).

Proof. Let \(B_\theta\) be a block defined by \(\theta\)-decomposition numbers. By Lemma 4.1, we know that \(B_\theta \subseteq \text{Irr}(B|\theta)\) for some \(p\)-block \(B\). We prove now that \(\text{Irr}(B|\theta) \subseteq B_\theta\).

Let \(D = (d_{\chi \varphi})\) be the decomposition matrix of \(G\) and write \(D_{B, \theta}\) for the submatrix of \(D\) whose rows and columns are indexed by elements in \(\text{Irr}(B|\theta)\) and \(\text{IBr}(B)\), respectively. By Theorem 4.2, we know that \(D_{B, \theta}\) is not of the form:

\[
\begin{pmatrix}
* & 0 \\
0 & *
\end{pmatrix}
\]
for any ordering of the rows and columns. Hence, if \( \chi, \psi \in \text{Irr}(B|\theta) \), there exists \( \chi = \chi_1, \chi_2, \ldots, \chi_k = \psi \) and \( \varphi_1, \varphi_2, \ldots, \varphi_{k-1} \) with \( \chi_i \in \text{Irr}(B|\theta) \) and \( \varphi_i \in \text{IBr}(B) \), such that:

\[
d_{\chi_i \varphi_i} \neq 0 \neq d_{\chi_{i+1} \varphi_i}.
\]

By Lemma 4.1, the \( \theta \)-decomposition numbers are the classical decomposition numbers. This completes the proof. \( \square \)

The following is Theorem B.

**Theorem 4.4.** Let \( (G, N, \theta) \) be a character triple. The blocks defined by \( \theta \)-decomposition numbers are exactly the \( \theta \)-blocks of \( G \).

**Proof.** Let \( (G^*, N^*, \theta^*) \) be a standard isomorphic character triple and let \( * : \text{Irr}(G|\theta) \to \text{Irr}(G^*|\theta^*) \) be the standard bijection. Let \( \chi, \psi \in \text{Irr}(G|\theta) \).

By Lemma 2.1 of [7], we have that the map \( \Xi \mapsto (\Xi^*)^\circ \) from \( \text{cf}(G|\theta)^\circ \to \text{cf}(G^*|\theta^*)^\circ \) is an isomorphism of vector spaces. Therefore, \( \chi, \psi \in \text{Irr}(G|\theta) \) lie in the same block defined by \( \theta \)-decomposition numbers if and only if \( \chi^*, \psi^* \) lie in the same block defined by \( \theta^* \)-decomposition numbers. Since \( N^* \subseteq Z(G^*) \), using Theorem 4.3, we have that \( \chi^*, \psi^* \) lie in the same block defined by \( \theta^* \)-decomposition numbers if and only if \( \chi^*, \psi^* \) lie in the same \( p \)-block of \( G^* \), that is, if and only if \( \chi \) and \( \psi \) lie in the same \( \theta \)-block. \( \square \)

5. **More Results on \( \theta \)-Blocks**

If \( \text{Bl}(G|D) \) denotes the set of \( p \)-blocks of \( G \) having defect group \( D \), Brauer’s first main theorem asserts that there is a natural bijection:

\[
\text{Bl}(G|D) \to \text{Bl}(N_G(D)|D),
\]

where if \( B \mapsto b \), then \( b \) is known as the Brauer correspondent of \( B \). We prove an analog of this result for \( \theta \)-blocks.

If \( P/N \) is a \( p \)-subgroup of \( G/N \), then we will denote by \( \text{Bl}_\theta(G|P/N) \) the set of \( \theta \)-blocks of \( G \) having \( P/N \) as a \( \theta \)-defect group. If \( \chi \) is a character of \( G \), we denote by \( \text{bl}(\chi) \) the \( p \)-block of \( G \) containing \( \chi \).

**Theorem 5.1.** Let \( (G, N, \theta) \) be a character triple and let \( P/N \) be a \( p \)-subgroup of \( G/N \). Then, there exists a natural bijection:

\[
\text{Bl}_\theta(G|P/N) \to \text{Bl}_\theta(N_G(P)|P/N).
\]

**Proof.** Let \( (G^*, N^*, \theta^*) \) be a standard isomorphic character triple and let \( * : \text{Irr}(G|\theta) \to \text{Irr}(G^*|\theta^*) \) be the standard bijection. Let \( \mathcal{P} \) be the projective representation of \( G \) associated with \( \theta \) affording the representation group \( \hat{G} \), such that \( G^* = \hat{G}/N \) and \( N^* = \hat{N}/N \) (see Notation 2.1). Let \( \pi : \hat{G} \to G \) be the canonical onto group homomorphism \( (g, z) \mapsto g \). Recall that if \( B_\theta \) is a \( \theta \)-block of \( G \) having defect group \( P/N \), then there exists a block of \( G^* \), \( B^* \), such that \( B^*_\theta = \text{Irr}(B^*|\theta^*) \) and there exists \( P^* = \hat{P}/N \), a defect group of \( B^* \), such that \( P/N = \pi(\hat{P})/N \).
Since $N_{G^*}(P^*) \leq N_{G^*}(P^*N^*)$, it is not difficult to see that block induction defines a bijection:

$$\text{Bl}(N_{G^*}(P^*N^*)|P^*) \rightarrow \text{Bl}(G^*|P^*).$$

If $H^* \leq G^*$, write $\text{Bl}(H^*|P^*, \theta^*)$ to denote the set of blocks $c^*$ of $H^*$ having $P^*$ as a defect group and such that $|\text{Irr}(c^*|\theta^*)| \neq 0$. We claim that induction defines a bijection:

$$\text{Bl}(N_{G^*}(P^*N^*)|P^*, \theta^*) \rightarrow \text{Bl}(G^*|P^*, \theta^*).$$

Notice that we just need to prove that given $b^* \in \text{Bl}(N_{G^*}(P^*N^*)|P^*)$, then $|\text{Irr}(b^*|\theta^*)| \neq 0$ if and only if $|\text{Irr}((b^*)G^*|\theta^*)| \neq 0$. Let $b^* \in \text{Bl}(N_{G^*}(P^*N^*)|P^*, \theta^*)$ and write $B^* = (b^*)G^*$. Let $\psi^* \in \text{Irr}(b^*|\theta^*)$. By Corollary 6.4 of [5], we have that there exists $\chi^* \in \text{Irr}(B^*)$ over $\psi^*$. Thus, $\chi^* \in \text{Irr}(b^*|\theta^*)$ and $|\text{Irr}(b^*|\theta^*)| \neq 0$. Now, let $B^* \in \text{Bl}(G^*|P^*, \theta^*)$ and write $B^* = (b^*)G^*$, where $b^* \in \text{Bl}(N_{G^*}(P^*N^*)|P^*)$. Let $\psi^* \in \text{Irr}(b^*)$ and let $\varphi^* \in \text{Irr}(N^*)$ lying under $\psi^*$, so $\psi^* \in \text{Irr}(b^*|\varphi^*)$. By Corollary 6.4 of [5], there exists $\xi^* \in \text{Irr}(B^*)$ over $\psi^*$, and hence over $\varphi^*$. By Theorem 9.2 of [5], $B^*$ covers $\text{bl}(\varphi^*)$. Since $|\text{Irr}(B^*|\theta^*)| \neq 0$, by Theorem 9.2 of [5], we have that $B^*$ also covers $\text{bl}(\theta^*)$. By Corollary 9.3, we have that $\text{bl}(\theta^*)$ and $\text{bl}(\varphi^*)$ are $G^*$-conjugate. Since $N^*$ is central, we have that $\text{bl}(\theta^*) = \text{bl}(\varphi^*)$ and, therefore, $b^*$ covers $\text{bl}(\theta^*)$. By Theorem 9.4 of [5], there exists $\mu^* \in \text{Irr}(b^*|\theta^*)$. This proves the claim.

Consider the character triple $(N_{G^*}(P), N, \theta)$. We claim that $(N_{G^*}(P^*N^*), N^*, \theta^*)$ is a standard isomorphic character triple to $(N_G(P), N, \theta)$. Write $\hat{G} = G_{\hat{H}}(\hat{N})$. Since $\hat{N} \leq \hat{H} \leq \hat{G}$, we can write (as a set) $\hat{H} = \pi(\hat{H}) \times Z$. Write $H = \pi(\hat{H})$ and notice that $N \leq H \leq G$. We first show that $H \leq N_G(P)$. Indeed, let $h \in H$. Then:

$$P^h = (PN)^h = \pi(\hat{P}\hat{N})^{\pi(h,1)} = \pi(\hat{P}\hat{N}^{(h,1)}) = \pi(\hat{P}\hat{N}) = PN = P.$$

Notice that $\mathcal{P}_H$ is a projective representation of $H$ associated with $\theta$ with factor set $\beta = \alpha_{H \times H}$. Since $Z$ contains the values of $\beta$, we obtain that $(\hat{H}/N, \hat{N}/N, \theta^*)$ is standard isomorphic to the character triple $(H, N, \theta)$. Observe that:

$$N_{G^*}(P^*N^*) = \hat{H}/N,$$

and hence:

$$N_{G^*}(P^*N^*)/N^* \cong H/N.$$

Therefore:

$N_G(P)/N = vN_{G^*}(P/N) \cong N_{G^*}(P^*N^*/N^*) = N_{G^*}(P^*N^*)/N^* \cong H/N.$

Since $H \leq N_{G^*}(P)$, we obtain $H = N_G(P)$. Therefore, $(N_{G^*}(P^*N^*), N^*, \theta^*)$ is standard isomorphic to the character triple $(N_G(P), N, \theta)$. Now, if $c_\theta$ is a $\theta$-block of $N_G(P)$ with defect group $P/N$ and $c^*$ is the corresponding block in $N_{G^*}(P^*N^*)$ arising $c_\theta$, the map $c_\theta \mapsto c^*$ defines a bijection:

$*: \text{Bl}_\theta(N_G(P)|P/N) \rightarrow \text{Bl}(N_{G^*}(P^*N^*)|P^*, \theta^*).$
In the same way, the map $B_{\theta} \mapsto B^*$ defines a bijection:

$^* : \text{Bl}_{\theta}(G|P/N) \rightarrow \text{Bl}(G^*|P^*, \theta^*)$.

We conclude that there exists a bijection:

$\text{Bl}_{\theta}(N_G(P)|P/N) \rightarrow \text{Bl}_{\theta}(G|P/N)$.

□

Of course, the definition of $\theta$-blocks is related to projective representations, and hence with blocks of twisted group algebras. These have been studied before by many authors and, in particular, there is a version of Brauer’s first main theorem for twisted group algebras by Conlon [1] and Reynolds [9].

Our next goal is to prove a $\theta$-version of the following classical result.

**Theorem.** (Block orthogonality) Let $g, h \in G$ be such that $g_p$ and $h_p$ are not $G$-conjugate. If $B$ is a block of $G$, then:

$$\sum_{\chi \in \text{Irr}(B)} \chi(g) \overline{\chi(h)} = 0.$$ 

**Proof.** See, for instance, Corollary 5.11 of [5]. □

Recall that, given a $p$-element $x \in G$, the $p$-section of $x$ is the set:

$$S(x) = \{ z \in G | z_p \text{ is conjugate in } G \text{ to } x \}.$$ 

The following is a result on classical blocks that we believe of independent interest.

**Theorem 5.2.** Suppose that $B$ is a $p$-block of $G$, $N \lhd G$ and let $\theta \in \text{Irr}(N)$ be $G$-invariant. Let $g, h \in G$. Suppose that $(Ng)_p$ and $(Nh)_p$ are not $G/N$-conjugate. Then:

$$\sum_{\chi \in \text{Irr}(B|\theta)} \chi(g) \overline{\chi(h)} = 0.$$ 

**Proof.** Define the class function on $G$:

$$\Omega = \sum_{\chi \in \text{Irr}(G|\theta)} \overline{\chi(h)} \chi.$$ 

Let $x = g_p$. If $y \in C_G(x)$ is $p$-regular, we claim that $\Omega(xy) = 0$. First, notice that $Nx$ and $Nh$ are not $G/N$-conjugate. Indeed, suppose that there exists $Nw \in G/N$, such that $(Nx)^{Nw} = Nh$. Since $Nx$ is a $p$-element, $Ny$ is a $p'$-element, and they commute that we have that $(Nx)_p = Nx = (Ng)_p$. Then:

$$Nh_pNh_{p'} = Nh = (Nx)^{Nw} = (Nx)^{Nw}(Ny)^{Nw},$$

and we have that $(Nx)^{Nw} = Nh_p$ a contradiction, since $(Ng)_p$ and $(Nh)_p$ are not $G/N$-conjugate. Now, by Knörr’s theorem (see, for instance, Theorem 5.21 of [8]), we have that:

$$\Omega(xy) = \sum_{\chi \in \text{Irr}(G|\theta)} \overline{\chi(h)} \chi(xy) = 0.$$
Next, we show that $\Omega$ vanishes on the $p$-section of $x$, $S(x)$. Indeed, let $z \in S(x)$, and let $u \in G$ be such that $z_p^u = x$. Then, $z_p^{u'}$ is a $p$-regular element in $C_G(x)$ and, therefore, $\Omega(z^u) = \Omega(xz_p^{u'}) = 0$. Since $\Omega$ is a class function, we obtain $\Omega(z) = 0$.

By Theorem 5.10 of [5], we obtain:

$$0 = \Omega_B(g) = \sum_{\chi \in \text{Irr}(B)} [\Omega, \chi]\chi(g)$$

$$= \sum_{\chi \in \text{Irr}(B)} \left( \sum_{\psi \in \text{Irr}(G|\theta)} \overline{\psi(h)}\psi, \chi \right)\chi(g)$$

$$= \sum_{\chi \in \text{Irr}(B)} \sum_{\psi \in \text{Irr}(G|\theta)} \overline{\chi(h)}[\psi, \chi]\chi(g)$$

$$= \sum_{\chi \in \text{Irr}(B|\theta)} \overline{\chi(h)}\chi(g),$$

as wanted. $\Box$

As a consequence, we obtain the $\theta$-version of the block orthogonality theorem.

**Corollary 5.3.** Let $(G, N, \theta)$ be a character triple and suppose that $B_\theta$ is a $\theta$-block of $G$. Let $g, h \in G$. Suppose that $(Ng)_p$ is not $G/N$-conjugate to $(Nh)_p$. Then:

$$\sum_{\chi \in \text{Irr}(B_\theta)} \chi(g)\overline{\chi(h)} = 0.$$

**Proof.** Let $(G^*, N^*, \theta^*)$ be a standard isomorphic character triple and let $^* : \text{Irr}(G|\theta) \to \text{Irr}(G^*|\theta^*)$ be the standard bijection. Write also $^* : G/N \to G^*/N^*$ for the isomorphism induced by the canonical onto group homomorphism $\pi : \hat{G} \to G$ defined by $(x, z) \mapsto x$, and write $(Nx^*) = N^*x^*$. Recall that (see explanation above Notation 2.1):

$$\chi(x) = \chi^\pi(x, 1) = \tau(x, 1)\chi^*(x^*)$$

for all $x \in G$. Then:

$$\sum_{\chi \in \text{Irr}(B_\theta)} \chi(g)\overline{\chi(h)} = \sum_{\chi^* \in \text{Irr}(B^*|\theta^*)} \tau(g, 1)\chi^*(g^*)\overline{\tau(h, 1)}\chi^*(h^*)$$

$$= \tau(g, 1)\overline{\tau(h, 1)} \sum_{\chi^* \in \text{Irr}(B^*|\theta^*)} \chi^*(g^*)\overline{\chi^*(h^*)}.$$ 

Since $(Ng)_p$ is not $G/N$-conjugate to $(Nh)_p$, we have that $(N^*g^*)_p$ is not $G^*/N^*$-conjugate to $(N^*h^*)_p$, and by Theorem 5.2, we have that:

$$\sum_{\chi^* \in \text{Irr}(B^*|\theta^*)} \chi^*(g^*)\overline{\chi^*(h^*)} = 0,$$

as wanted. $\Box$
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