KRULL DIMENSION FOR LIMIT GROUPS III:
SCOTT COMPLEXITY AND ADJOINING ROOTS TO FINITELY
GENERATED GROUPS

LARSEN LOUDER

ABSTRACT. This is the third paper in a sequence on Krull dimension for limit
groups, answering a question of Z. Sela. We give generalizations of the well
known fact that a nontrivial commutator in a free group is not a proper power
to both graphs of free groups over cyclic subgroups and freely decomposable
groups. None of the paper is specifically about limit groups.

1. INTRODUCTION

This and the companion paper [Lou08c] contain an analysis of sequences of
limit groups obtained through a certain systematic process of adjoining roots and
passing to limit group quotients. An analysis of arbitrary finitely generated groups
obtained through adjoining roots is, of course, not practical, however something
can be said if only limit groups are considered.

This paper is preceded by [Lou08a, Lou08b], and is followed up by [Lou08c].
In [Lou08b] we reduce the problem of finite Krull dimension to a statement about
adjoining roots to limit groups. The argument needed divides neatly into two parts.
The first, dealt with in this paper, is a study of groups obtained by adjoining roots
to almost all finitely generated groups. We handle the case of arbitrary limit groups
in the last.

Definition 1.1 ([Sco73, Swa04]). Let $G$ be a finitely generated group with Grushko
decomposition $G_1 * \cdots * G_p * F_q$. The Scott complexity of $G$ is the ordered pair
$\text{sc}(G) = (q - 1, p)$.

Let $G$ be a group. An element $g \in G$ is indivisible if it’s not a proper power, i.e.,
if $h^k = g$ then $k = \pm 1$. The group $\langle G, \gamma'_i | (\gamma'_i)^{k_i} = \gamma_i, i = 1 \ldots n \rangle$ is denoted
$G \langle \sqrt[\gamma_i]{\gamma'_i} \rangle$. For notational convenience we usually suppress the “$k_i$” from the nota-
tion. This is justified by the fact that the degree of the root is incidental as long as
$\gamma_i$ indivisible in $G$, as we see in Lemma 2.5. The letter $F$ represents a nonabelian
free group. The corank of a finitely generated group is the maximal rank of a free
group it maps onto.

Date: July 29, 2018.
2000 Mathematics Subject Classification. Primary: 20F65; Secondary: 20E05, 20E06.
Key words and phrases. limit groups, krull dimension, Stallings’s folding.

Most of this research was done while at the University of Utah. The author also gratefully ac-
knowledges support from the National Science Foundation.
Theorem 1.2 (Scott complexity and adjoining roots to groups). Suppose that \( \phi: G \hookrightarrow H \) and \( H \) is a quotient of \( G' = G\langle \sqrt[n]{\gamma_i} \rangle \), \( \gamma_i \) a collection of distinct conjugacy classes of indivisible elements of \( G \) such that \( \gamma_i \neq \gamma_j^{-1} \) for all \( i, j \) and \( \gamma_i \in \gamma_i \). Then \( \text{sc}(G) \geq \text{sc}(H) \). If equality holds and \( H \) has no \( \mathbb{Z}_2 \) free factors, there are presentations of \( G \) and \( H \) as

\[
G \cong G_1 \ast \cdots \ast G_p \ast F^G_q, \quad H \cong H_1 \ast \cdots \ast H_p \ast F^H_q
\]

a partition of \( \{\gamma_i\} \) into subsets \( \gamma_{j,i} \), \( j = 0, \ldots, p, i = 1, \ldots, i_p \), representatives \( \gamma_{j,i} \in G_j, \gamma_{j,i}, i \geq 1, \gamma_{0,i} \in G^G_q, \gamma_{0,i} \), such that with respect to the presentations of \( G \) and \( H \):

- \( \phi(G_i) < H_i \)
- \( G_j \langle \sqrt[j]{\gamma_{j,i}} \rangle \to H_j \)
- \( \phi(F^G_q) < F^H_q \)
- \( F^G_q = \langle \gamma_{0,1} \rangle \ast \cdots \ast \langle \gamma_{0,i_0} \rangle \ast F \)
- \( F^H_q = \langle \sqrt[0]{\gamma_{0,1}} \rangle \ast \cdots \ast \langle \sqrt[0]{\gamma_{0,i_0}} \rangle \ast F \)
- \( G' \cong G_1 \langle \sqrt[1]{\gamma_{1,i}} \rangle \ast \cdots \ast G_p \langle \sqrt[p]{\gamma_{p,i}} \rangle \ast \langle \sqrt[0]{\gamma_{0,1}} \rangle \ast \cdots \ast \langle \sqrt[0]{\gamma_{0,i_0}} \rangle \ast F \)

All homomorphisms are those suggested by the presentations.

Example 1.3 (No \( \mathbb{Z}_2 \) free factors of \( H \) is a necessary hypothesis). Let \( G = G_1 \ast G_2, H = H_1 \ast \mathbb{Z}_2 \), where \( H_1 = G_1 \ast G_2 / \langle \langle \alpha_1 = \alpha_2 \rangle \rangle \), \( \alpha_i \neq 1 \in G_i \), and \( x \) generates \( \mathbb{Z}_2 \). Then \( H \) is a quotient of \( G\langle \sqrt[n]{\alpha_1 \alpha_2} \rangle \). The inclusion \( G \hookrightarrow H \) maps \( G_1 \to H_1 \) and \( G_2 \) to \( xG_2x < xH_1x \). Then \( \alpha_1 \to \alpha_1 \) and \( \alpha_2 \to x\alpha_1x \) and \( \alpha_1 \alpha_2 \to (x\alpha_1x) \).

In the case \( \text{sc}(G) = \text{sc}(H) = (g - 1, 0) \), and adjoining a root to a single nontrivial element, Theorem 1.2 reduces to a theorem of Shenitzer [She53] which states that if an amalgamation \( F_n \ast (t) \ F_m \) is free (necessarily of rank \( n + m - 1 \)), then \( t \) is a basis element in at least one vertex group. If \( G\langle \sqrt[n]{\gamma} \rangle \) is free then Theorem 1.2 reduces to Shenitzer’s theorem: \( \gamma \) is not a basis element of \( \langle \sqrt[n]{\gamma} \rangle \), therefore \( \gamma \) is a basis element of \( F \).

A modern proof of Shenitzer’s Theorem goes as follows: If \( F_n \ast (t) \ F_m \) is free, then each map of a vertex group into \( F_n \ast (t) \ F_m \) may be represented by an immersion \( t_{n|m}: \Gamma_{n|m} \to R_{n+m-1} \). Represent \( t \) by immersions of \( S^1 \hookrightarrow \Gamma_{n|m} \), and build a graph of spaces \( X \) by gluing the boundary components of an annulus to \( \Gamma_n \sqcup \Gamma_m \) along the immersions of \( S^1 \). Extend \( t_{n|m} \) to a map \( \eta: X \to R_{n+m-1} \). Pull back midpoints of edges of \( R_{n+m-1} \) to produce embedded graphs in \( X \) transverse to both \( \Gamma_n \) and \( \Gamma_m \). The preimage graphs must be forests, otherwise \( F_n \ast (t) \ F_m \hookrightarrow F_{n+m-1} \) has nontrivial kernel. This implies that the representation of \( t \) as an immersion, in at least one of \( \Gamma_{n|m} \), must cover some edge only one time, i.e., it is a basis element in one of the factors.

If we drop the hypothesis that \( F_n \ast (t) \ F_m \) is free the means to conclude that preimages of midpoints of edges are forests disappears. The next theorem shows that this hypothesis can be weakened.
Theorem 1.4 (Improved Shenitzer’s Theorem). Let \( G = F_1 *_{(t)} F_2, F_1 \) and \( F_2 \) free of rank \( n \) and \( m \), respectively. If \( t \) is indivisible in at least one vertex group \( G \) has corank \( n + m - 1 \), then \( G \) is free.

Our approach to Theorem 1.4 is to carry out a more careful analysis of the space constructed to prove Shenitzer’s theorem. Before moving on to this, we give a short outline of some existing work on ranks of subgroups generated by solutions to equations defined over \( \mathbb{F} \).

The first equation over the free group to receive much attention was Vaught’s equation \( \Omega = \{ a^2b^2c^2 = 1 \} \). Lyndon showed that if \( a^2b^2c^2 = 1 \) in a free group, then \( a, b, \) and \( c \) commute. This characterization of solutions amounts to the fact that a commutator in a free group isn’t a square. Lyndon and Schützenberger [LS62] extended this fact to equations defined over \( \mathbb{F} \).

More generally, Baumslag showed that if \( (y_1, \ldots, y_{n+1}) \) is a solution to \( \omega(x_1, \ldots, x_n) = x_{n+1}^k, k > 1 \) in \( \mathbb{F} \), and \( \omega \) is neither a proper power nor a basis element in \( \langle x_1, \ldots, x_n \rangle \), then the subgroup of \( \mathbb{F} \) generated by \( \{y_1, \ldots, y_{n+1}\} \) has rank less than \( n \).

In a remark, Baumslag seems to suggest a conjugacy separability problem for elements of the free group: If \( \alpha \) and \( \beta \) are nonconjugate elements of \( \mathbb{F} \), then either (without loss) there is a free factorization \( \mathbb{F} \cong \langle \alpha' \rangle \) such that \( \alpha \) is a power of \( \alpha' \) and \( \beta \) is conjugate into \( F \), or \( \langle \mathbb{F}, t | t \alpha t^{-1} \beta^{-1} \rangle \) has corank less than \( n \). The next corollary is a resolution of this question.

Corollary 1.5. Let \( F \) be a free group of rank \( n > 1 \), and let \( Z_i < F \) be finitely many distinct conjugacy classes of maximal cyclic subgroups of \( F \). Let \( t_j \) be stable letters, \( \gamma_1^j, \gamma_2^j \) elements of \( \bigcup_i Z_i \setminus \{1\} \). Let \( G \) be the group \( F * \langle t_j \rangle / \langle t_j \gamma_1^j t_j^{-1} = \gamma_2^j \rangle \). Let \( \sim \) be the equivalence relation generated by

\[
Z_i \sim Z_i' \iff \text{There exists } j \text{ such that } \gamma_1^j \in Z_i \text{ and } \gamma_2^j \in Z_i'
\]

If \( \sim \) has no singleton equivalence classes and \( G \) has corank \( n \), then for some \( i \) there is a free factorization of \( F \) as \( F \cong F_1 \ast Z_i \) such that every element \( \gamma_1^s \) is conjugate into either \( F_1 \) or \( Z_i \) and \( F_1 \) contains a conjugate of some \( \gamma_2^s \).

This corollary should be compared to the criterion for malnormality of rank two subgroups of free groups given in [FMR02]. Say that a subgroup \( H \) of a free group \( \mathbb{F} \) is isolated if it is closed under taking roots, isolated on generators if it is closed under taking roots of generators, and malnormal on generators if, for all \( g \in \mathbb{F} \setminus H \) and basis elements \( h \) of \( H \), \( g h g^{-1} \notin H \). Fine, Myasnikov, and Rosenberger prove that if a rank two subgroup of a free group is isolated and malnormal on generators, then it is malnormal. By Theorem 1.2 a rank two subgroup of a free group is isolated if and only if it is isolated on generators, and, if not, is not malnormal on generators. If malnormality on generators fails for some reason other than isolation, by Corollary 1.5 any two elements nonconjugate in \( H \) but conjugate in \( \mathbb{F} \) must be conjugate to a basis of \( H \). A non-malnormal subgroup must contain such a pair of elements, and so malnormality is implied by malnormality on generators. The author suspects that the paper of Fine et al. contains a proof of Corollary 1.5 for the case considered in this paragraph.
Acknowledgments. It’s hard to overstate my gratitude to Mark Feighn for his careful reading of more versions of this paper than I can count. Many thanks must also go to Mladen Bestvina for pointing out the proof of Shenitzer’s theorem and teaching me how to fold, many moons ago.

2. Injections, Immersions, and Graphs of Spaces

To analyze homomorphisms of the type in 1.2, 1.4, and 1.5, we need to construct spaces which efficiently represent injections of groups. Given an injection $F_1 \hookrightarrow F_2$ of free groups, Stallings constructs graphs $\Gamma_1$ and $\Gamma_2$ and a map $\Gamma_1 \to \Gamma_2$ which, under suitable identifications of $\pi_1(\Gamma_1)$ with $F_1$, $\pi_1(\Gamma_2)$ with $F_2$, represents given homomorphism. In this section we generalize his construction to spaces which represent injections of groups which aren’t necessarily freely indecomposable, but which are strong enough to promote Stallings’ type results from free groups to Grushko free factorizations of freely decomposable groups. Our maps give absolutely no information about restrictions to freely indecomposable free factors.

We give a brief review of immersions before we generalize immersions to “relative graphs,” or relative one-complexes, a class of spaces slightly larger than that of graphs.

2.1. Immersions. A map of graphs $\varphi: \Gamma_1 \to \Gamma_2$ is combinatorial if

- $\varphi(\Gamma_1^{(0)}) \subset \Gamma_2^{(0)}$
- If $e$ is an edge of $\Gamma_1$ then there is an edge $f$ of $\Gamma_2$ so that $\varphi|_e : e \to f$ is a homeomorphism.

A combinatorial map induces, for each vertex $v$ of $\Gamma$, a map $\varphi_v : \text{lk}(v) \to \text{lk}(\varphi(v))$. If each $\varphi_v$ is injective then $\varphi$ is an immersion. Our goal is to represent an injection $G = G_1 * \cdots * G_{P_G} * \mathbb{F}_{q_G} \hookrightarrow H = H_1 * \cdots * H_{P_H} * \mathbb{F}_{q_H}$ as an ‘immersion’ of suitable cell complexes with fundamental groups $G$ and $H$. We first translate the link condition into local relative $\pi_1$ injectivity, then we extend the translation to relative graphs.

Let $\varphi: \Gamma_1 \to \Gamma_2$ be combinatorial, $\pi_1$ injective, and suppose that the stars of vertices of $\Gamma_2$ are embedded. For each vertex $v$ there is an induced map

$$\pi_1(\varphi_v) : \pi_1(\text{St}(v), \partial \text{St}(v)) \to \pi_1(\text{St}(\varphi(v)), \partial \text{St}(\varphi(v)))$$

If $\varphi$ is an immersion then $\pi_1(\varphi_v)$ is injective for all vertices $v$ of all combinatorial representatives of the topological realization of $\varphi$.

If $\Box(X)$ is a collection of subspaces of a space $X$ denote the disjoint union of elements of $\Box(X)$ by $\Box_\bullet(X)$.

Definition 2.1 (Relative Graph). A relative graph is a topological space $W$ with the structure of a CW-pair of cell complexes $(X, \mathcal{Y}(X))$ where

- The topological realization of $X$ is homeomorphic to $W$.
- $\mathcal{Y}(X)$ is a collection of connected subcomplexes of $X$. If $Y$ and $Y' \in \mathcal{Y}(X)$ meet, then they meet in finitely many vertices. If it’s clear which relative graph we’re referring to, the ‘$(X)$’ of ‘$\mathcal{Y}(X)$’ will be suppressed.
- If $e$ is a cell not contained in $\mathcal{Y}_\bullet$ then $e$ is at most one-dimensional
There is a (not necessarily connected) graph $\Gamma_X$, and for each valence one or zero vertex $v$ of $\Gamma_X$, there are finitely many maps $\eta_{v,i} : v \to \mathcal{Y}_*(X)$.

- $W$ is homeomorphic to the quotient space $\Gamma_X \cup \mathcal{Y}_*(X) / v \sim \eta_{v,i}(v)$.
- Each $Y \in \mathcal{Y}(X)$ has nontrivial fundamental group.

Let $(X, \mathcal{Y}(X))$ be a relative graph. The zero skeleton of $(X, \mathcal{Y}(X))$ is the set of zero cells not contained in any element of $\mathcal{Y}(X)$, along with the connected components of the union of the elements of $\mathcal{Y}(X)$ in $X$. Pick an element $v$ of the zero skeleton, let $e_1, \ldots, e_n$ be the oriented edges of $\Gamma_X$ such that $\tau(e_i) \in v$, and identify the edges $e_j$ with intervals $I_1, \ldots, I_n$. Then the star of $v$ is the complex $St(v) = v \cup \bigcup I_j / (1 \in I_j \sim \tau(e_j))$. There is a map $St(v) \to X$ such that $v \to v$ is the identity map and $I_j \to e_j$ is simply the prescribed identification.

The essential zero skeleton of a relative graph $X, X^E$ is obtained by removing all valence two vertices from the zero skeleton. The inessential zero skeleton consists of the valence two vertices of $\Gamma_X$, and is denoted $X'$.

A relative graph $X$ is admissible if every map $St(v) \to X$ is an embedding and, for all $v, w \in X^E$, $St(v)$ and $St(w)$ don’t share any edges of $\Gamma_X$.

**Definition 2.2.** Let $X$ and $X'$ be admissible relative graphs. A map $\varphi : X \to X'$ is combinatorial if the preimage of the interior of an edge of $X'$ is the union of interiors of edges from $X$. The restriction of $\varphi$ to the interior of an edge of $X$ is a homeomorphism onto the corresponding interior in $X'$.

**Definition 2.3.** Let $\varphi : X' \to X$ be a combinatorial map of relative graphs, and suppose that $(X, \mathcal{Y}(X))$ is admissible. We say that $\varphi$ is an immersion if neither of the following conditions holds:

- If, for some vertex $v \in \Gamma_X \setminus \mathcal{Y}_*(X)$ and $w \in \varphi^{-1}(v)$,
  \[ \pi_1(\varphi_w) : \pi_1(St(w), \partial St(w)) \to \pi_1(St(v), \partial St(v)) \]
  is not injective then $\varphi$ is not an immersion.
- Suppose $\pi_1(\varphi_w)$ is injective for all such $w$ and $v$ as in 1. If, for some $Y \in \mathcal{Y}(X)$ and connected component $N$ of $\varphi^{-1}(St(Y))$, the map
  \[ \pi_1(\varphi_N) : \pi_1(N, \partial N) \to \pi_1(St(Y), \partial St(Y)) \]
  is not injective then $\varphi$ is not an immersion.

Definition 2.3 requires some explanation. Suppose $\varphi$ doesn’t satisfy the first bullet, i.e., it is an immersion at ordinary vertices. Now consider $Y \in \mathcal{Y}(X)$, $St(Y)$, and a connected component $N$ of $\varphi^{-1}(St(Y))$. The preimage $N$ is the union of elements $Y'_i \in \mathcal{Y}(X')$ and edges of $\Gamma_{X'}$. If $v \in \varphi^{-1}(\partial St(Y)) \cap Z$ then at most one edge of $\Gamma_{X'} \cap N$ meets $v$, since if there were two, then the map on $St(v)$ wouldn’t be injective, contrary to hypothesis. Thus $N = St(V)$ for some subcomplex $V$ of $X'$, and $\partial(N) = \varphi^{-1}(\partial St(Y)) \cap N$. Thus the second condition has the same form as the first in the event that the first doesn’t hold. Stallings calls paths representing elements of $\ker(\pi_1(\varphi_N))$ binding ties [$\text{Sta65}$].

An edge path in a relative graph is a combinatorial map of a subdivided interval. An edge path which is an immersion is reduced. An edge path is homotopic,
relative to its endpoints, to a reduced edge path, and any two reduced edge paths homotopic relative to endpoints are equivalent, as combinatorial objects, via homotopies supported on those subsegments of the interval with image contained in some \( \mathcal{Y}(X) \subset X \).

A fold of a relative graph \( X \) is a map of the following type: Let \( e \) be an edge of \( \Gamma_X \) and identify \( e \) with the unit interval so that \( \iota(e) \sim 0 \) and \( \tau(e) \sim 1 \). Let \( p: e \to X \) be a reduced edge path with \( p(0) = \iota(e) \), \( p^{-1}(\tau(e)) = 0 \). The fold of \( X \) at \( e \) along \( p \) is the space obtained by crushing any edges of \( X' = X/(t \sim p(t)) \) which meet a valence one vertex of \( \Gamma_{X'} \).

**Lemma 2.4** ([Sta83]). If \( \varphi: (X, \mathcal{Y}(X)) \to (X', \mathcal{Y}(X')) \) is \( \pi_1 \)-injective and combinatorial, elements of \( \mathcal{Y}(X') \) aspherical, then there is a relative graph \( \mathcal{Y}(X', X) \), \( \mathcal{Y}(X') \equiv \mathcal{Y}(X) \), a homotopy equivalence \( F: X \to X' \), and an immersion \( \overline{\varphi}: X \to X' \) such that \( \overline{\varphi} \circ F \) is homotopic to \( \varphi \) and \( F \) is a composition of folds.

The proof of Lemma 2.4 is an easy riff on folding. Note that if \( \varphi: X \to X' \) is \( \pi_1 \)-injective then it is homotopic to a combinatorial \( \varphi': (X, \mathcal{Y}(X)) \to (X', \mathcal{Y}(X')) \). This is accomplished by first homotoping \( \varphi \) so that \( \mathcal{Y}(X) \) has image in \( \mathcal{Y}(X') \). This can be done since each \( Y \in \mathcal{Y}(X) \) has freely indecomposable fundamental group, \( \varphi \) is \( \pi_1 \)-injective, and each \( Y' \in \mathcal{Y}(X') \) is aspherical. Then homotope \( \varphi \) so that every vertex of \( \Gamma_X \) has image in \( \Gamma_{X'} \). Now subdivide the edges of \( \Gamma_X \) and homotope \( \varphi \) to a combinatorial map.

**Proof:** Suppose \( \varphi \) isn’t an immersion and satisfies condition 1 of Definition 2.3. If \( v \) maps to \( w \) and \( \pi_1(\varphi_w) \) isn’t injective, then perform an ordinary Stallings fold.

Suppose \( \varphi \) doesn’t satisfy condition 1, but does satisfy condition 2, and let \( Y, St(Y), N, \) and \( V \) be as in Definition 2.3. Let \( b_1 \) and \( b_2 \) be two vertices of \( \partial N \) with a path \( p: [0, 1] \to N \) such that \( p(0) = b_1 \) and \( p(1) = b_2 \), and such that \(|\varphi \circ p|\) is trivial in \( \pi_1(St(Y), \partial St(Y)) \).

Since \( \varphi \) is \( \pi_1 \)-injective, \( b_1 \neq b_2 \). Adjacent to \( b_1 \) and \( b_2 \) are unique distinct edges \( e_1, e_2 \subset N, \Gamma_X \) so that \( \iota(e_1) = b_1 \). The path \( p \) can be homotoped to the composition of two paths, the first traversing \( e_1 \), and the second a path \( p': [0, 1] \to N \setminus (e_1 \cap \partial e_1) \) satisfying \( p'^{-1}(\tau(e_1)) = \{1\} \) and \( p'(1) = \iota(e_1) = b_2 \). Homotope the restriction of \( \varphi \) to \( e_1 \) so it agrees with \( (p')^{-1} \). Now let \( X'' \) be the fold of \( X \) at \( e_1 \) along the path \( p' \), with \( F \) the quotient map. There is an obvious map \( \varphi': X'' \to X' \) and the composition \( \varphi' \circ F \) is homotopic to \( \varphi \). Rinse and repeat. The process must terminate since the number of edges of \( \Gamma_{X''} \) is strictly less than the number of edges of \( \Gamma_X \).

One important property of immersions of relative graphs is that if \( p \) is a reduced edge path then the composition of \( p \) and an immersion is also a reduced edge path. More generally, compositions of immersions are immersions.

### 2.2. Graphs of Spaces.

#### 2.2.1. Graphs of Free Groups.

In the next sub-subsection Theorem 1.2 will be reduced to an analysis of spaces arising by adjoining roots to free groups. We begin with a slightly more general construction than the one we need since the
analysis will give easy proofs of Theorem 1.4 and Corollary 1.5. We now state the main theorem of this subsection. In conjunction with Theorem 2.9, Theorem 2.5 implies Theorem 1.2.

**Theorem 2.5.** Let $X$ be a 2-covered graph of spaces arising from adjunctions of roots to non-conjugate, indivisible elements $\gamma_i$ of a free group. Furthermore, suppose that $\gamma_i$ and $\gamma_j^{-1}$ are nonconjugate for all $i \neq j$. If $\chi(\Gamma(X)) = \chi(\Gamma_U(X))$ then the edge spaces of $X$ are trees.

Definitions follow.

An edge of the graph underlying a graph of spaces is denoted by a lower case letter, and the space associated to that edge is denoted by the same letter uppercased. Edges of graphs of spaces are oriented, and the edge map associated to the preferred orientation is typically denoted “$\tau$”.

**Definition 2.6** (Graph, Graph of Spaces). A graph is a set $W$ with an involution $\bar{\phantom{w}}$ and retractions $\tau, \iota: W \to \text{Fix}(\phantom{w})$ compatible with $\bar{\phantom{w}}$: $\iota(\bar{w}) = \tau(w)$, $\tau(\bar{w}) = \iota(w)$

The elements of $W \setminus \text{Fix}(\phantom{w})$ are the oriented edges of the graph. The fixed set of $\phantom{w}$ is the set of vertices, and the maps $\tau$ and $\iota$ are the terminal and initial vertices of oriented edges, respectively. We say that an edge $e$ is incident to $v$ if $\tau(e) = v$.

Note that a graph in this sense is a special kind of category. A (ordinary) graph of spaces is a functor from a graph $(W, \iota, \tau, \bar{\phantom{w}})$ to $\text{Top}$. Let $\mathcal{G}$ be the category of simplicial graphs whose maps are combinatorial immersions. For us, a graph of spaces is a functor from a graph $(W, \tau, \iota, \bar{\phantom{w}})$ to $\mathcal{G}$. We’ll be mostly interested in graphs of spaces which satisfy a rather restrictive criterion on collections of edges incident to vertices.

Members of a graph will be referred to with lower case variables, and their images in $\mathcal{G}$ will have capital variable names. If an edge $e$ is incident to $v$, then we say that $E$ is incident to $V$, similarly for variables with subscripts.

**Definition 2.7** (2-Covered). A finite graph $V$ is 2-Covered by $\{E_i\}$ if, for every $i$, there is an immersion $\tau_i: E_i \rightarrow V$ and each edge $f$ of $V$ is the image under $\bigsqcup \tau_i$ of exactly two edges from $\bigsqcup E_i$.

For the remainder of this section a graph of spaces $X$ will satisfy the condition that if $e_1, \ldots, e_n$ are incident to $v$, then $V$ is 2-covered by $\{\tau_i: E_i \rightarrow V\}$.

We fix some notation for graphs of spaces.

- An underlying graph $\Gamma_U$ that the graph of spaces is built on, i.e., if $X$ is a graph of spaces, then $X$ really corresponds to a functor $\Gamma_U(X) \to \mathcal{G}$.
- vertex spaces are connected graphs $V_i$.
- Edge spaces in the topological realization are products of intervals with connected “edge-graphs” $E_j$. Edge graphs may be points. Each edge space $E_j \times I$ in the topological realization of $X$ has an embedded copy of $E_j$, $E_j \times \{\frac{1}{2}\}$. 

For a graph of spaces, there is a natural (not necessarily connected) subcomplex \( \Gamma(X) \) consisting of horizontal edges: \( \Gamma(X) = X(0), \Gamma(X) = \bigcup_j F_j(0) \times I \), with identifications induced by the immersions \( \tau \). The horizontal subgraph \( \Gamma(X) \) is the realization of the graph of spaces induced by restricting to the zero skeleton of the vertex and edge spaces. (Note that we cheated a little. When we defined a graph of spaces, we insisted on having connected vertex and edge spaces. Zero skeleta are rarely connected, but the definition makes sense just the same.) Let \( \Gamma_{\infty}(X) \) be the subset of \( \Gamma(X) \) consisting of the connected components not homeomorphic to \( S^1 \). Let \( \Gamma_{\infty}(X) \) be the subset consisting of components homeomorphic to \( S^1 \).

For a graph of free groups over cyclic subgroups with a homomorphism to a free group which embeds the vertex groups, there is a natural complex \( \Gamma^*(X) \) consisting of components homeomorphic to \( S^1 \). Let \( \Gamma^*(X) \) be the subset consisting of components homeomorphic to \( S^1 \).

For a graph of spaces, there is a natural (not necessarily connected) subcomplex \( \Gamma(X) \) consisting of horizontal edges: \( \Gamma(X) = X(0), \Gamma(X) = \bigcup_j F_j(0) \times I \), with identifications induced by the immersions \( \tau \). The horizontal subgraph \( \Gamma(X) \) is the realization of the graph of spaces induced by restricting to the zero skeleton of the vertex and edge spaces. (Note that we cheated a little. When we defined a graph of spaces, we insisted on having connected vertex and edge spaces. Zero skeleta are rarely connected, but the definition makes sense just the same.) Let \( \Gamma_{\infty}(X) \) be the subset of \( \Gamma(X) \) consisting of the connected components not homeomorphic to \( S^1 \). Let \( \Gamma_{\infty}(X) \) be the subset consisting of components homeomorphic to \( S^1 \).

For a graph of free groups over cyclic subgroups with a homomorphism to a free group which embeds the vertex groups, there is a natural complex \( \Gamma^*(X) \) consisting of components homeomorphic to \( S^1 \). Let \( \Gamma^*(X) \) be the subset consisting of components homeomorphic to \( S^1 \).

Let \( G = \Delta(F_i, Z_j) \) be a graph of free groups \( F_i \) (not necessarily nonabelian!) over nontrivial cyclic subgroups \( Z_j \). If \( \phi: G \rightarrow \mathbb{F}_n \) is a homomorphism which embeds each \( F_i \), then we can build a nice graph of spaces representing \( \phi \). For each \( i \), choose an immersion \( \varphi_i: \Gamma_i \rightarrow R_n \), where \( R_n \) is the rose with \( n \) petals, and fundamental group \( \mathbb{F}_n \) and \( \varphi_i(\Gamma_i) = F_i \). Each cyclic edge group \( Z_j \) must embed in \( \mathbb{F}_n \), so for each \( j \), choose an immersion \( \varphi_j: S^1_j \rightarrow R_n \) representing the image of \( Z_j \). If \( Z_j \rightarrow F_i \) then \( \varphi_j \) lifts to an immersion \( \varphi_{i,j}: S^1_j \rightarrow \Gamma_i \) (There may be more than one possibility for \( \varphi_{i,j} \), corresponding to a monogon in \( \Delta \). Choose two, one for each orientation of the edge.) Use the data \( \varphi_{i,j} \) to attach annuli, one for each edge of \( \Delta \), to the graphs \( \Gamma_i \), to build a graph of spaces \( X \). Our original homomorphism \( \phi \) induces a map \( \varphi: X \rightarrow R_n \). Restricted to an annulus \( S^1_j \times I \), the map is projection to the first factor, followed by the immersion \( \varphi_i \). Let \( b \) be the basepoint of \( R_n \). Now regard \( X \) as a 2-covered graph of spaces by setting \( \{ V_p \} \) to be the connected components of \( \varphi^{-1}(b) \), and edge graphs connected components of preimages of midpoints of edges of \( R_n \). The homomorphism \( \phi|_{F_i} \) factors through the inclusion \( \Gamma_i \rightarrow X \). See the bottom two rows of Figure [15]

The graph of groups \( G \) has corank at most

\[
\sum_{i=1}^l \text{rk}(F_i) - l + 1
\]

If \( n = - \sum \chi(\Gamma_i) + 1 \), then \( \chi(\Gamma(X)) = 1 - n = \sum \chi(\Gamma_i) = \chi(\Gamma(X)) \).

**Lemma 2.8.** If \( G = \Delta(F_i, Z_j) \) is a graph of free groups over nontrivial cyclic subgroups and \( G \rightarrow \mathbb{F}_m \) then

\[
m \leq 1 - \sum \chi(F_i)
\]

A homomorphism \( \phi: \Delta(F_i, Z_j) \rightarrow \mathbb{F} \) such that the inequality of Lemma 2.8 is an equality has maximal corank.

2.2.2. **Non-free groups.** In this subsection we consider the situation of Theorem [1,2]. The objective of this section is to prove the following theorem.
Theorem 2.9. Suppose that $\phi: G \hookrightarrow H$ and $H$ is a quotient of $G' = G[\sqrt{\gamma}]$, $\gamma_i$, classes of indivisible elements of $G$ such that $\gamma_i \neq \gamma_j^{ \pm 1}$ for all $i \neq j$ and $\gamma_i \in \gamma_i$. Then $\text{sc}(G) \geq \text{sc}(H)$. If equality holds and $H$ has no $\mathbb{Z}_2$ free factors then there is a partition of $\{\gamma_i\}$ into subsets $\gamma_{j,i}$, $j = 0, \ldots, p$, $i = 1, \ldots, i_j$, such that $\gamma_{j,i} \in G_j, \gamma_{j,i}, j \geq 1$, and a $2$-covered graph of spaces $X$ such that:

- $\chi(\Gamma(X)) = \chi(\Gamma(X))$.
- $\Gamma_\infty(X)$ is connected and has a collection of connected subgraphs $R_Y$, $Y \in \mathcal{Y}(X_G)$, which have pairwise disjoint image under the map $\Gamma(X) \to \Gamma_U(X)$.
- $H_j$ is a quotient of $G_j[\sqrt{\gamma_{j,i}}]$.
- For each $Y$ there is an attaching map $\psi_Y: R_Y \to Y$. The attaching maps of those components of $X$ with $\chi(\Gamma_U) = 0$ are the boundaries of mapping cylinders corresponding to $\gamma_{j,i}, j \geq 0$. The boundary of the mapping cylinder associated to $\gamma_{j,i}$ is attached to $Y_j$ along $\gamma_{j,i}$.
- The space $X_G = (\Gamma_\infty(X) \sqcup \bigcup R_Y)/(x \sim \psi_Y(x))$ has fundamental group $G$.
- The space $\bar{X} = (X \sqcup \bigcup R_Y)/(x \sim \psi_Y(x))$ has fundamental group $G'$.
- The attaching maps $\gamma_{0,i} \to X_G$ factor through $\Gamma_\infty(X)$, and in fact $X$ is the union of $\Gamma_\infty(X)$ and the mapping cylinders for $\gamma_{0,i}$.

The first and second bullets are key. We now prove Theorem 1.2 assuming Theorems 2.9 and 2.5.

Proof of Theorem 1.2. The proof is by induction on $\text{sc}(G)$. Assume the conclusions of Theorem 2.9. By Theorem 2.5 every edge space of $X$ is a tree. Since the graphs $R_Y$ have disjoint image in the projection from $X$ to $\Gamma_U(X)$ there is an edge $e$ of $\Gamma_U(X)$ not in the image of any $R_Y$. The edge space associated to $E$ is a tree, hence there is an edge of $\Gamma_\infty(X)$ not attached to any $Y \in \mathcal{Y}(X_G)$ and which is crossed exactly once by the representative of one of $\gamma_{0,i}$, say $\gamma_{0,1}$. Then $G' \cong G_1 \ast \langle \gamma_{0,1} \rangle$, with all $\gamma_i \neq \gamma_{0,1}$ conjugate into $G_1$. Let $G_1' = G_1[\sqrt{\gamma}]$, $\gamma_i \neq \gamma_{0,1}$ and let $H_1$ be the image of $G_1'$. Then $H \cong H_1 \ast \langle k_{\gamma_0,1} \rangle$ and $\text{sc}(G_1) = \text{sc}(H_1) = (q_{G} - 2, p_G)$. Repeating this procedure for all $\gamma_{0,i}$, reduce to the case where each $\gamma$ is conjugate into some $G_j$.

Then $G_{i_0} = G_1 \ast \cdots \ast G_p \ast F$ and all leftover $\gamma_i$ are elements of some $\gamma_{i,j}$, $i \geq 0$, and $G_{i_0}' = G_1[\sqrt{\gamma_{1,i}]} \ast \cdots \ast G_p[\sqrt{\gamma_{p,i}]}] \ast F$.

Passing to the image of $G_j[\sqrt{\gamma_{j,i}]}$ in $H$, we see immediately that $H_{i_0} \cong \text{Im}_H(G_1[\sqrt{\gamma_{1,i}]}]) \ast \cdots \ast \text{Im}_H(G_p[\sqrt{\gamma_{p,i}]}]) \ast F$.

Reassembling the free factors split off by Theorem 2.5 proves the theorem. \hfill \Box

Remark 2.10. Simply knowing that some $\gamma_{0,i}$ is a basis element in $\pi_1(\Gamma(X))$ isn’t sufficient to imply that $\langle \gamma_{0,i} \rangle$ is a free factor of $G$, thus Theorem 1.2 cannot be deduced from Baumslag’s theorem.
For the remainder of this section, fix an inclusion $\phi: G \hookrightarrow H$ of finitely generated groups which lifts to an epimorphism $\bar{\phi}: G' \twoheadrightarrow H$. Before we begin, replace $H$ by a group $H'$ as follows: Let $\phi: L \rightarrow L'$ be an epimorphism of groups. Then

$$sc(\phi) = \max \{ sc(L'') \mid \phi \text{ factors through an epimorphism } L'' \rightarrow L' \}$$

Let $H'$ be a group such that $G' \rightarrow H$ factors through $H'$ and such that $H'$ achieves $sc(G' \rightarrow H)$. Clearly the freely indecomposable free factors of $H'$ have freely indecomposable image in $H$.

**Lemma 2.11.** Let $H' \hookrightarrow H$ be as above. Then

$$(q_{H'}, p_{H'}) \geq (q_H, p_H)$$

If a freely indecomposable free factor of $H'$ has trivial image in $H$ then the inequality is strict.

**Proof.** Let $\mathbb{F}_H$ be the free part of $H$. Then every freely indecomposable free factor of $H'$ is in the kernel of $H' \rightarrow \mathbb{F}_H$ and we see that $\mathbb{F}_H \rightarrow \mathbb{F}_H$. Suppose that some freely indecomposable free factor of $H$ does not contain the image of a freely indecomposable free factor of $H'$. Let $H_0$ be this freely indecomposable free factor. Then, by the reasoning above, $\mathbb{F}_H$ maps onto $\mathbb{F}_H * H_0$. Since $\mathbb{F}_H$ has the same rank as $\mathbb{F}_H$, $H_0$ must be trivial. Thus, $p_H \leq p_{H'}$. 

Rather than work with the inclusion $G \hookrightarrow H$, we work with $G \hookrightarrow H'$, suppressing the $'$ for convenience.

Our first task is to find a suitable way to represent $\phi$ and $\bar{\phi}$ as maps of cell complexes. We start by representing $\phi$ as an immersion $\varphi: X_G \rightarrow X_H$ given by Lemma 2.4. Once this is done, we build a nice space $\bar{X}$ with fundamental group $G[\sqrt[n]{\gamma_i}]$. This space is equipped with a well behaved map to $X_H$ and we use this map to endow $\bar{X}$ with a new graph of spaces structure transverse to the old one.

An admissible relative graph of spaces is *minimal* if it has no valence one vertices, and for every valence two vertex $v$ of $\Gamma_X \setminus \Upsilon_s$, the relative graph obtained by unsubdividing $\Gamma_X$ at $v$ is inadmissible, and for every valence one vertex $v$ of $\Gamma_X$, $v$ is contained in some $Y \in \Upsilon(X)$. Suppose $X$ is admissible and minimal. Let $\mathcal{N}(X) = \{N_i\}$ be the collection of all closures of connected components of $X \setminus X^I$. If $N$ contains some element $v \in X^E$ then $N$ is simply $\text{St}(v)$. If $N \in \mathcal{N}(X)$ then define $\partial N = N \cap X^I$. A minimal admissible relative graph is illustrated in Figure 1.

Choose a relative graph of spaces $X_H$ with fundamental group $H$, and whose components $Y_1, \ldots, Y_p \in \Upsilon(H)$ are aspherical and have fundamental group $\pi_1(Y_i) \cong H_i$. Choose a relative graph $X_G$ with fundamental group $G$ and an immersion $\varphi: X_G \rightarrow X_H$ provided by Lemma 2.4 representing $\phi$. We now build a space $\bar{X}$ with fundamental group $G[\sqrt[n]{\gamma_i}]$. Choose, for each $\gamma_i$, an immersion $\gamma_i: S^1 \hookrightarrow X_G$ representing the conjugacy class $[\gamma_i]$. If $\gamma$ is conjugate into some $G_i$ then the immersion $\gamma: S^1 \rightarrow X_G$ has image in the connected component of $\Upsilon_s(X_G)$ representing $G_i$. Let $\{\gamma_i\}$ be the subcollection of $\{\gamma_i\}$ consisting of elements conjugate
into $G_i$. Let $\{\alpha_1, \ldots, \alpha_m\}$ be the subcollection of $\{\gamma_i\}$ which are not conjugate into any $G_i$.

Let $M_i$ be the mapping cylinder of the $k_i$ fold cover $S_i^1 \rightarrow S^1$, and let $r_i$ be the core curve (the range $S_i^1$) of $M_i$. Now glue the $M_i$ along $S_i^1$ to $X_G$ using the immersions $\gamma_i$ as attaching maps to form a space $\widetilde{X}$. For each $r_i$, choose an immersion $r_i \rightarrow X_H$ representing the conjugacy class of $\psi_i \gamma_i \in H$. The map $\varphi: X_G \rightarrow X_H$ lifts to a continuous map $\tilde{\varphi}: \widetilde{X} \rightarrow X_H$ which agrees with the immersions $r_i$ and $\varphi$.

Each mapping cylinder $M_i$ is a quotient space of an annulus $A_i$. Let $\mathcal{A}$ be this collection of annuli and let $i_a: \mathcal{A} \rightarrow \widetilde{X}$ be the disjoint union of the maps $A_i \rightarrow M_i \rightarrow \widetilde{X}$.

The space $\widetilde{X}$ is a quotient of the disjoint union of $\mathcal{A}$, $X_G$, and the core curves $r_j$ of the mapping cylinders $M_j$ (which are built out of annuli from $\mathcal{A}$ and said core curves).). In analogy with a 2-covered graph of spaces, set $\Gamma(\widetilde{X}) = X_G \sqcup \bigsqcup r_i$. Homotope $\tilde{\varphi}$, relative to $\Gamma(\widetilde{X})$, so that $\mathcal{A} \rightarrow X_H$ is transverse to $X_H^1$. Then $i_a^{-1}(\mathcal{B}_*) = V$ is an embedded 1-submanifold of $\mathcal{A}_*$. Using an innermost disc argument and asphericity of each component of $X_H^1$, homotope $\tilde{\varphi}$ so that the submanifold contains no simple closed curves bounding disks. If an (innermost) arc $\alpha \subset V$ has endpoints in only one boundary component of $\mathcal{A}$ then one of $X_G \rightarrow X_H$ or some $r \rightarrow X_H$ fails to be an immersion, contrary to hypothesis. If $V$ contains an $S^1$ which does not bound a disk then some $\gamma_k$ vanishes in $H$, contrary to hypothesis. Thus every connected component of $V$ is an arc connecting distinct boundary components of some $A \in \mathcal{A}$. If $\gamma_i$ is conjugate into some $G_j$ then the immersions $\varphi \circ \gamma_i$ and $r_i$ have images in $\mathcal{Y}_*(H)$ and any component of $V$ contained in $M_i$ is a circle which does not bound a disk. A circle contained in an annulus is homotopic to each boundary component, hence if such a circle

Figure 1. A minimal admissible relative graph.
Lemma 2.16 or Figure 2. Possibilities for $S$ mapping to $N \in \mathcal{N}(X_H)$.

existence then $\gamma_i$ would have to be trivial in $G$, a contradiction. Thus, for such $M$, $V \cap M = \emptyset$.

The first step in our analysis of $\tilde{X}$ is to resolve $\tilde{\varphi}$, giving $\tilde{X}$ a graph of spaces structure transverse to the graph of spaces decomposition $X_G \sqcup \bigsqcup M_i / \sim$. Let $Z$ be the collection of connected components of $\tilde{\varphi}^{-1}(N)$, $N \in \mathcal{N}(X_H)$, and let $\mathcal{B}$ be the collection of connected components of preimages of $X_H^I$. By transversality $i_a^{-1}(\mathcal{B}_*)$ is a one manifold with boundary contained in $\partial \mathcal{A}_*$, and thus $\mathcal{B}_*$ is a graph contained in $\tilde{X}$. For each $B \in \mathcal{B}$ there are two embeddings $B \hookrightarrow Z_* \sqcup \mathcal{W}_*$. By transversality $i_a^{-1}(\mathcal{B}_*)$ is a one manifold with boundary contained in $\partial \mathcal{A}_*$, and thus $\mathcal{B}_*$ is a graph contained in $\tilde{X}$. For each $B \in \mathcal{B}$ there are two embeddings $B \hookrightarrow Z_* \sqcup \mathcal{W}_*$. Note that if $\gamma_i$ is conjugate into some $G_j$ then $M_i$ is completely contained in some $Z$.

The boundary of $Z \in \mathcal{Z}$ is the set $\mathcal{B}_* \cap Z$, and coincides with the set of points of $Z$ mapping to $X_H^I$. Inclusions $B \hookrightarrow Z$ are simply inclusions of boundary components.

Some possibilities for $Z \cap X_G$ are illustrated in Figure 2.

Each mapping cylinder $M_j$ is either completely contained in $Z_*$ or has nontrivial intersection with $\mathcal{B}_*$. If $M$ has nontrivial intersection with $\mathcal{B}_*$, then $\tau \cap Z$ (recall that $\tau$ is the core curve of $M$) is a collection of closed intervals, even in number. The preimage $i_a^{-1}(\mathcal{B}_*)$ slices an annulus $A \in \mathcal{A}$ into rectangles $R^{+}_1, R^{-}_1, \ldots, R^{+}_m, R^{-}_m$, where the $R^{+}_z$ have images in $\mathcal{N}_*(X_H)$ and $R^{-}_z$ have images in $\mathcal{E}_*(X_H)$.

The connected components of the intersections of the relative graph $\Gamma(\tilde{X})$ and elements of $\mathcal{Z}$ or $\mathcal{W}$ are relative graphs with distinguished valence one vertices $\{Z|W\} \cap \mathcal{B}_*$. Let $\mathcal{S}(Z)$ be the collection of connected components of $\Gamma(\tilde{X}) \cap Z$. If $S \in S$ set $\partial S = S \cap \mathcal{B}_*$. For a fixed $Z \in \mathcal{Z}$ let $\mathcal{R}(Z)$ be the subcollection of all $R^{+}_z$ (coming from all annuli $A \in \mathcal{A}$) which are contained in $Z$.

The boundary of each rectangle $R$ is composed of two types of arcs, $\partial^\pm R = R \cap \partial \mathcal{A}_*$ and $\partial^\pm R = R \cap i_a^{-1}(\mathcal{B}_*)$. The former shall be known as horizontal boundary arcs and the latter as vertical boundary arcs.

Let $\partial^+ R$ be a horizontal boundary arc of some $R \in \mathcal{R}(Z)$. Then $\varphi^+ = i_a|_{\partial^+ R}$ has image in some connected component $S$ of $\mathcal{S}_*(Z)$ and $\partial \varphi^+ R$ maps to $\partial S = S \cap \mathcal{W}_*$.
\( B_\bullet \subset \partial Z \). We define \( \varphi^- \) similarly. If \( S \) is a tree then this path is an embedding and connects distinct boundary components of \( S \). If \( S \) contains a relative space \( Y \) of \( X_G \), then it is a reduced edge path since the maps representing \( \alpha_i \) were immersions.

We reconstruct \( Z \) by gluing the rectangles \( \mathcal{R}(Z) \) to \( \mathcal{S}_\bullet(Z) \) via the attaching maps \( \varphi^\pm : (\partial^\pm R, \partial \partial^\pm R) \to \bigsqcup (S, \partial S) \in \mathcal{S}(Z) \).

![Figure 3](image1.png)

The boundary of \( Z \) is the union of vertical boundary arcs of the rectangles comprising it, along with all valence one vertices of \( \partial \mathcal{S}_\bullet(Z) \) not contained in some vertical boundary arc of a rectangle. By construction, \( Z \) is a connected component of \( \varphi^{-1}(N) \) for some \( N \in \mathcal{N}(X_H) \). Suppose \( N \) is a star of some \( v \in X^E_H \). Let \( \{b_1, \ldots, b_k\} \) be the valence one vertices comprising \( \partial N \), with incident edges \( e_i \subset N \) so that \( \tau(e_i) = b_i \). If \( \psi : Z \to N \) is the restriction of \( \tilde{\varphi} \) then \( \psi^{-1}(e_i) \) is a collar neighborhood of \( \psi^{-1}(b_i) \in B \), a boundary component of \( Z \). The restriction \( \psi \) factors through the map which projects each collar onto the \( I \) factor. Call the resulting quotient space \( \mathring{Z} \). If \( N \) contains elements of \( X^E_H \) then \( Z \) is homeomorphic to a product \( B \times I \), for some \( B \in \mathcal{B} \), let \( Z \to \mathring{Z} \) be the projection to the \( I \) factor. For each \( B \) in \( \mathcal{B} \) let \( \mathring{B} \) be the quotient space consisting of a single point.

The lift \( \tilde{\varphi} : X \to X_H = X^E_H \) factors through the graph of spaces \( X_K, \pi_1(X_K) = K \), obtained by reassembling the collection \( \{\mathring{Z} | Z \in Z\} \) If \( B \subset Z_1, Z_2 \), then identify the images \( \mathring{B} \in \mathring{Z_1} \) and \( \mathring{B} \in \mathring{Z_2} \). Construction of \( X_K \) is illustrated in Figure 4.

![Figure 4](image2.png)

Let \( S_1, \ldots, S_n \) be the connected components of \( \mathcal{S}(Z) \), and let \( R_1, \ldots, R_k \) be the rectangles from \( \mathcal{R}(Z) \). For each rectangle let \( \varphi^\pm_j : \partial^\pm R_j \to \mathcal{S}_\bullet(Z) \) be the
attaching maps for the horizontal boundary arcs of $R_j$. Then $Z$ is the quotient space

$$S_\bullet(Z) \sqcup R_\bullet(Z) / (x \sim \varphi^\pm_j(x))$$

The boundary of $Z$ consists is the union $\bigcup \partial^j L R_j \cup \bigcup_i \partial S_i = B_\bullet \cap Z \subset Z$.

**Lemma 2.12.** The spaces $\overline{Z}$ have freely indecomposable (or trivial!) fundamental groups.

**Proof.** Suppose not. Then $\pi_1(\overline{X}) \twoheadrightarrow H$ factors through a group with strictly higher Scott complexity. Recall that $H$ is the new name for $H'$, chosen to achieve $\text{sc}(G' \twoheadrightarrow H)$. If some $\overline{Z}$ had freely decomposable fundamental group then $\text{sc}(K) > \text{sc}(H')$. □

Recall that $q_H \leq q_G$, and that if equality holds then no $S \in S(Z)$ has fundamental group with nontrivial free part, thus we may assume that no $S$ has $\overline{Z}$ as a free factor of its fundamental group.

Let $G = G(Z)$ be the graph with vertex set $S(Z)$ and edge set $R(Z)$. The endpoints of an edge $R$ are the boundary components $\partial^\pm R$, and an endpoint $\partial^\pm R$ is attached to $S$ if the image of $\varphi^\pm$ is contained in $S$. Let $T$ be a maximal tree in $G$. Build a space $Z_T$ by restricting to the tree $T$.

Since no $S \in S(Z)$ has nontrivial free part and $G$ embeds in $H$, the components $Z \in \overline{Z}$ fall into three classes:

1. $\pi_1(\overline{Z})$ is trivial. Such $Z$ contain no $Y \in \mathcal{Y}(X)$.
2. $\pi_1(\overline{Z})$ is nontrivial and $Z$ contains no $Y \in \mathcal{Y}(X)$
3. $\pi_1(\overline{Z})$ is nontrivial and $Z$ contains some $Y \in \mathcal{Y}(X)$

Let $\mathcal{Z}_i$ be the subset of $\overline{Z}$ containing all $Z$ of the $i$–th type. For each $Z$, let

$$\Delta^-_\overline{Z}(Z) = \frac{1}{2} (\# \partial Z_T - \# \overline{Z})$$

and for $Z \in \mathcal{Z}_3$ let

$$\Delta^+_\overline{Z}(Z) = \frac{1}{2} b_1(\partial Z_T)$$

and for $Z \in \mathcal{Z}_{1|2}$ set $\Delta^+_\overline{Z}(Z) = 0$.

If $S$ is a relative graph with no loops (no contribution to $q_G$), set $\kappa(S, \partial S) = \frac{1}{2} \# \partial S - 1$, and observe that

$$\sum_{Z \in \overline{Z}} \sum_{S \subset S(Z)} \kappa(S, \partial S) = q_G - 1$$

and

$$\sum_{Z \in \overline{Z}} \kappa(\overline{Z}, \partial \overline{Z}) = q_H - 1$$

The complexity $\kappa$ is intended to be a stand-in for curvature. Beware the sign convention we’ve chosen.

If $Z \in \mathcal{Z}_1$ then define $\Delta^+_p(\overline{Z}) = \Delta^-_\overline{Z}(Z) = 0$. If $Z \in \mathcal{Z}_2$ then define $\Delta^+_p(\overline{Z}) = 1$ and $\Delta^-_p(\overline{Z}) = 0$. If $Z \in \mathcal{Z}_3$ and $Y_1, \ldots, Y_{k+1}$ are the components of $\mathcal{Y}(X_G)$
contained in $Z$, then $\Delta^-_p(Z) = k$ and $\Delta^+_p(Z) = 0$. We now give three lemmas relating the quantities $\Delta^\pm_{\rho|\rho'}$ to one another.

**Lemma 2.13.**

$$\kappa(Z, \partial Z) - \sum_{S \in S(Z)} \kappa(S, \partial S) = \Delta^+_q(Z) - \Delta^-_q(Z)$$

If $Z \in Z_3$ then $\Delta^-_p(Z) \geq b_1(\partial Z_T) = 2\Delta^+_q(Z)$.

**Proof.** To show the first equality we only need establish that

$$\frac{1}{2} \# \partial Z_T - 1 = \sum_{S \in S(Z)} \left( \frac{1}{2} \# \partial S - 1 \right) + \Delta^+_q(Z)$$

To this end, let $T \subset G(Z)$ be a maximal tree, and let $S_1, \ldots, S_k$ be an enumeration of $S(Z)$ such that $S_{i+1}$ is connected to $S_0 \cup R_0 \cup S_1 \cup R_1 \cdots S_i$ by an edge $R_i \subset T$. Assume that $R_i$ is oriented so that $\partial^+ R_i$ is attached to $S_{i+1}$. Let $Z_i$ be the union of $S_1, \ldots, S_i$ and $R_1, \ldots, R_{i-1}$.

The boundary of $\partial^+ | - R_i$ consists of two points. Suppose that for at least one of $+$ or $-$, the image $\partial^+ | - R_i$ is contained in two distinct boundary components of

![Figure 5. Illustration for Lemma 2.13](image-url) Each independent loop in $\partial Z_T$ must contribute at least 1 to $\Delta^-_p$. The tree in this example is a tripod, there is one boundary component of $Z_T$, and $\Delta^-_p = 2$. A similar example with only two components $S$ would have $\Delta^-_p = 1$. 

![Diagram of a tripod tree](image-url)
at least one of $Z_i$ or $S_{i+1}$. If this is the case then

$$\kappa(Z_{i+1}, \partial Z_{i+1}) = \frac{1}{2} \# \partial Z_{i+1} - 1$$

$$= \frac{1}{2} (\# \partial Z_i + \# S_{i+1} - 2) - 1$$

$$= \kappa(Z_i, \partial Z_i) + \kappa(S_{i+1}, \partial S_{i+1})$$

If both $\partial^+ R_i$ and $\partial^− R_i$ have image in the same boundary component of $S_{i+1}$, $Z_i$, respectively, then $Z_i$ contains at least $b_1(\partial Z_i) + 1$ elements of $\mathcal{Y}$ and $S_{i+1}$ must contain a new element of $\mathcal{Y}$. We then have

$$\kappa(Z_{i+1}, \partial Z_{i+1}) = \frac{1}{2} \# \partial Z_{i+1} - 1$$

$$= \frac{1}{2} (\# \partial Z_i + \# S_{i+1} - 1) - 1$$

$$= \kappa(Z_i, \partial Z_i) + \kappa(S_{i+1}, \partial S_{i+1}) + \frac{1}{2}$$

Each such rectangle makes a contribution of $+1$ to $b_1(\partial Z_T)$, $+1/2$ to $\Delta^+_\varphi(Z)$, and only such rectangles make such contributions, thus

$$\kappa(Z_T, \partial Z_T) = \sum_{S \in S(Z)} \kappa(S, \partial S) + \Delta^+_\varphi(Z)$$

We now need to compare $\Delta^+_\varphi(Z)$ to $\Delta^-_\varphi(Z)$. If $\partial \partial^\pm R$ maps to a single boundary component of $S \in S(Z)$ then, since $\partial^\pm R \to S$ is an immersion, $S$ must have nontrivial fundamental group, and since the free part of $S$ is trivial, it must contain some element $Y$ of $\mathcal{Y}(X_G)$.

Choose the exhaustion of $T$ so that $S_0$ has an incident edge $R$, $\partial^− R \to S_0$ such that $\partial \partial^− R$ maps to a single boundary vertex of $S_0$. By the reasoning above, $S_1$ contains some element $Y_0 \in \mathcal{Y}(X_G)$. Let $R_{i_1}, \ldots, R_{i b_1(\partial X_T)}$ be the rectangles such that $\partial \partial^+ R_{i_j}$ maps to a single boundary component of $S_{i_j+1}$. Then each $S_{i_j+1}$ contains some element $Y_j \in \mathcal{Y}(X_G)$. Since the $S_{i_j}$ are distinct, $Z$ contains at least $b_1(\partial Z_T)$ elements of $\mathcal{Y}(X_G)$, i.e., $\Delta^-_\varphi(Z) \geq b_1(\partial Z_T)$.

Not all spaces which abstractly resemble $Z$’s occur as $Z$’s. We now give a definition for a certain class of useful spaces resembling them.

**Definition 2.14.** A union of trees is a graph of spaces $Z$ whose vertex spaces are relative trees $S_1, \ldots, S_n$, edge spaces are intervals $I_1, \ldots, I_m$, and whose attaching maps $(I, \partial I) \to (S, \partial S)$ are reduced edge paths. The boundary of $Z$ is the union

$$\partial Z = \bigcup \partial S_i \cup \bigcup (\partial I_j \times I)$$

**Lemma 2.15.** If $\Delta^-_\varphi(Z) = 0$ then $Z$ has the following form: There are subcollections $\mathcal{R}_i \subset \mathcal{R}(Z)$ such that the restriction $Z_i$ of $Z$ to the rectangles $\mathcal{R}_i$ is homeomorphic to a product $G_i \times I$, $G_i$ a graph, and $Z$ is recovered by gluing components of $\cup(G_i^{(0)} \times I)$ to $S_\bullet(Z)$. The graph with vertex set $\{Z_i\} \cup S(Z)$ and
an edge between \( Z_i \) and \( S \) if \( v \times I \subset Z_i \) is identified with an edge path in \( S \) is a tree.

If \( Z \in \mathbb{Z}_2 \) then \( \Delta_q^{-}(Z) \geq \frac{1}{2} \). If equality holds then \( \pi_1(Z) \cong \mathbb{Z}_2 \).

**Proof.** First suppose that \( \Delta_q^{-}(Z) = 0 \). As before, let \( G(Z) \) be the graph with vertex set \( S(Z) \), edge set \( \mathcal{R}(Z) \), and maximal tree \( T \subset G \). Let \( Z_T \) be the graph of spaces obtained by restricting to \( T \) and let \( \mathcal{R}'(Z) \) be the subset of \( \mathcal{R}(Z) \) consisting of rectangles not contained in \( T \).

The boundary of \( Z_T \) is a forest in the boundary of \( Z \). Let \( \sim \) be the equivalence relation on \( \mathcal{R} \) generated by \( R_1 \sim R_2 \) if

\[
[\partial^+ R_1 \rightarrow S(Z)] = [\partial^+ R_2 \rightarrow S(Z)]
\]

Let \( \sim' \) be the same equivalence relation restricted to \( Z_T \). For each \( \sim' \)-equivalence class \([R]\) let \( Z_{[R]} \) be the subspace of \( Z_T \) obtained by restricting to rectangles in \([R]\). Let \( B \) be a boundary component of \( Z_{[R]} \). Then \( Z_{[R]} \) is homeomorphic to the product \( B \times I \).

We claim that the map from \( \sim' \) equivalence classes to \( \sim \) equivalence classes is an injection. Consider a rectangle \( R \in \mathcal{R}' \), let \( S^+(S^-) \) be the member of \( S \) containing the image of \( \partial^+ R (\partial^- R) \), and let \( R_1, \ldots, R_n \) be the path in \( T \) from \( S^- \) to \( S^+ \). The configuration of \( R_i \) and \( S^\pm \) has the form illustrated in the following figure. Boundary components are bold.

The boundary components of \( S^\pm \cup R_i \) are contained in disjoint boundary components of \( Z' \). Attaching \( R \) to this configuration, we see that to satisfy \( \Delta_q^{-}(Z) = 0 \), \( \partial^+ R \) must be attached to \( \partial^+ R_{n_i} \), \( \partial^- R \) must be attached to \( \partial^- R_1 \), for every \( i \), \( R_i \sim R_{i+1} \), and in fact the induced orientations of \( \partial^\pm R \) must be coherent. From this it is easy to see that \( \sim \) equivalence classes must be homeomorphic to the products \( B \times I \) above. To construct \( Z \), collar neighborhoods of boundary components of \( Z \) are crushed to intervals. From the characterization of \( \sim \) equivalence classes above, we see that \( Z \) is homotopy equivalent to a wedge of spheres, hence has trivial fundamental group.

The decomposition of \( Z \) as a union of products follows immediately.

Now suppose that \( \Delta_q^{-}(Z) > 0 \). Clearly \( \Delta_q^{-}(Z) \geq \frac{1}{2} \). Suppose \( \Delta_q^{-}(Z) \) is \( \frac{1}{2} \). Let \( Z_T \) be as before, and attach a rectangle \( R \) to \( \hat{Z}_T \) such that \( \#\partial Z_T \cup R = \#\partial Z_T - 1 \). Let \( R_1, \ldots, R_n \), \( S^\pm \) be a path in \( T \) as before. Then \( R \) and \( R_i \) must be in the configuration illustrated in Figure 6. The boundary is in bold. Adding rectangles, maintaining \( \Delta_q^{-}(Z) = \frac{1}{2} \), does not change the fundamental group of \( Z \), which is clearly \( \mathbb{Z}_2 \). In particular,

\[
\sum_{Z \in \mathbb{Z}_2} \Delta_q^{-}(Z) \geq \| \mathbb{Z}_2 \|
\]
Since \( q_H \leq q_G \) we see immediately that if \( q_H = q_G \) then
\[
\sum_{Z \in \mathcal{Z}_3} \Delta^+_q(Z) = \sum_{Z \in \mathcal{Z}} \Delta^-_q(Z)
\]
Then there are at most \( 2 \sum_{Z \in \mathcal{Z}_3} \Delta^+_q(Z) \) elements \( Z \in \mathcal{Z}_2 \) since each such component has \( \Delta^-_q(Z) \geq \frac{1}{2} \). Thus \( \|Z_2\| = \sum_{Z \in \mathcal{Z}_3} \Delta^-_q(Z) \leq \sum_{Z \in \mathcal{Z}_3} \Delta^+_q(Z) \) and we conclude that \( p_{H'} \leq p_G \). If equality holds then for each \( Z \in \mathcal{Z}_2, \Delta^-_q(Z) = \frac{1}{2} \), and for each \( Z \in \mathcal{Z}_3, \Delta^+_q(Z) = 1 \). Moreover, by Lemma 2.15 each element \( Z \) has fundamental group \( \mathbb{Z}_2 \), and we conclude that \( p_H' \leq p_G \).

**Lemma 2.16.** If \( q_{H'} = q_G \) then \( p_{H'} \leq p_G \). For all \( Z \in \mathcal{Z}_2, \Delta^-_q(Z) = \frac{1}{2} \). In particular, every such \( Z \) has fundamental group \( \mathbb{Z}_2 \).

We now revert to \( H = H \) and \( H' = H' \).

**Lemma 2.17.** If \( H \) has no \( \mathbb{Z}_2 \) free factors and \( (q_G, p_G) = (q_{H'}, p_{H'}) \) then \( \mathcal{Z}_2 \) is empty, all members of \( \mathcal{Z}_3 \) contain exactly one element of \( \mathcal{Y}(X_G) \), and \( \Delta^{\pm}_p(Z) = 0 \) for all \( Z \).

**Proof.** By Lemma 2.11 \( \text{sc}(H) \leq \text{sc}(H') \), and by Lemma 2.16 \( \text{sc}(G) \geq \text{sc}(H') \). Suppose \( \mathcal{Z}_2 \) is nonempty. Then \( H' \) has a freely indecomposable free factor \( \mathbb{Z}_2 \). Since \( H \) has no \( \mathbb{Z}_2 \) free factors, this implies \( \text{sc}(H) < \text{sc}(G) \), contrary to hypothesis.

Thus \( \Delta^+_p(Z) = 0 \) for all \( Z \). Let \( Z \in \mathcal{Z}_3 \). Since \( Z \) is freely indecomposable and \( \pi_1(Z) \) maps to a freely indecomposable free factor of \( H \), to have equality in the second coordinate, each such \( Z \in \mathcal{Z}_3 \) must contain exactly one \( Y \in \mathcal{Y}(X_G) \). \( \square \)

**Definition 2.18.** An union of trees is *treelike* if \( \Delta^-_q(Z) = 0 \), as is an element of \( \mathcal{Z}_3 \) if \( \Delta^-_q(Z) = \Delta^+_q(Z) = 0 \).

Let \( Z \) be a treelike union of trees, and express \( Z \) as a union of \( \bigcup (B_i \times I) \cup S(Z) \) modulo attaching maps. For each \( B \times I \), let \( \pi_B \) be the projection onto the \( I \) coordinate. Give \( B \times I \) the foliation whose leaves are \( \pi^{-1}(x), x \in I \), and give each \( S \in S(Z) \) the foliation whose leaves are simply the points of \( S \). Then define \( F(Z) \) to be the foliation on \( Z \) induced by the foliations on \( B \times I \) and \( S \). Define \( \Gamma_Z \)
to be the leaf space of $\mathcal{F}(Z)$, and denote the quotient map by $\pi_Z$. The following lemma is obvious from the construction.

**Lemma 2.19.** The following facts about $\Gamma_Z$ are true:

- $\Gamma_Z$ is a finite tree.
- Each $S \in S(Z)$ embeds in $\Gamma_Z$ under the quotient map. Any two images intersect in at most an interval.
- Each valence one vertex of $\Gamma_Z$ is the image of exactly one boundary component of $Z$.
- $\kappa(\Gamma_Z, \partial \Gamma_Z) = \kappa(Z, \partial Z)$
- Point preimages under $\pi_Z$ are connected.

The following extension property also holds. Let $f: Z \to A$ be a continuous map to an aspherical space $A$. If every boundary component of $Z$ is mapped to a point and $g: (I, \partial I) \to S_\bullet$ is a reduced edge path, then there exists a lift $\tilde{f}: \Gamma_Z \to A$ such that $\tilde{f} \circ \pi_Z$ is homotopic to $f$ via a homotopy which is constant on $\partial Z$ and $I$.

The space $\tilde{X}$ was constructed by adjoining mapping cylinders $M$ along their boundaries to elements represented by immersions $\gamma: S^1 \to X_G$. Any interesting homotopies of $\gamma$ are supported on arcs $I = [a, b] \subset S^1$ such that $\gamma(I) \subset \mathcal{Y}_\bullet(X_G)$. In fact, up to such homotopies, $\gamma$ is essentially unique: if an immersion $\gamma'$ is chosen, rather than $\gamma$, to represent the conjugacy class of $\gamma \in G$ (again, conflating immersions and conjugacy classes), the space $\tilde{X}'$ constructed differs from $\tilde{X}$ only in that for some (possibly more than one) $Z \in Z$, the attaching map of a rectangle $\partial^\pm R \to S$ is altered by a homotopy supported on an arc contained in $\partial^\pm R$ and having image in some $Y \in \mathcal{Y}$ (contained in $S$).

**Convention 2.20.** For convenience, we choose, for every homotopy class $[p]$ in $\pi_1(S, \partial S), S \in S(Z), Z \in Z$, a unique representative reduced edge path $p: (I, \partial I) \to (S, \partial S)$, such that $[p]^{-1}$ is represented by $t \mapsto p(1-t)$ unless $[p]$ represents a two-torsion element, in which case we choose $p$ to represent $[p]^{-1}$. Construct $\tilde{X}$ so that every attaching map $\partial^\pm R \to S_\bullet$ agrees with the chosen representative in its homotopy class.

For each $Z$, under the hypothesis that $(q_H, p_H) = (q_G, p_G)$, there is at most one element $S_0 \in S(Z)$ containing an element $Y \in \mathcal{Y}(X_G)$. Without loss of generality, we may assume that $S_0$ is the star of $Y$ in $X_G$ and $\Gamma_{X_G} \cap S_0$ is a single point. See Figure 7. For such $S_0$ let $e_i$ be the oriented edges of $\Gamma_{X_G}$ such that $\tau(e_i) = \tau(e_j) = b \in Y$ for all $i, j$. Then $S_0$ takes the form

**Lemma 2.21.** Suppose $Z$ is treelike and contains only one element $Y$ of $\mathcal{Y}(X_G)$. Let $S_0$ be the element of $S$ containing $Y$. Then there are treelike unions of trees $Z_1, \ldots, Z_n$, $S_i \in S(Z_i)$, $S_i \cong I$, and reduced edge paths $h_i: (S_i, \partial S_i) \to (S_0, \partial S_0)$ (which are in the fixed list of representatives of homotopy classes $\partial^\pm R \to S_\bullet$) such that

$$Z = ((S_0 \sqcup \bigsqcup Z_i)/(x \sim h_i(x)))$$

The proof of Lemma 2.21 will resemble the proof of Lemma 2.15.
Proof. As in Lemma 2.15, construct $G(Z)$. Let $G_1, \ldots, G_n$ be the closures of the connected components of $G \setminus \{ S_0 \}$, and let $T_i$ be a maximal tree in $G_i$ which meets $S_0$ only once. Construct $Z'_i$ by restricting $Z$ to $T_i$, and consider what happens when a rectangle is attached to $Z'_i$ (it must be attached along both horizontal boundary arcs to $Z'_i$).

If $R$ is not attached to $S_0$, we carry out the same analysis done in Lemma 2.15 which we now revisit. Consider $R \in G_i \setminus T_i$ which is attached along $\partial^- R$ to $S_0$, and construct the path $R_1, \ldots, R_n$ from $S^-$ to $S^+$, and assume that $S^- = S_0$. The union of the $R_i$ and $S^\pm$ has one of the forms illustrated in Figure 8.

Orient each $\partial^\pm R_i$ so that $\partial^+ R_i$ and $\partial^- R_{i-1}$ are oriented coherently, and so that under the map $Z \to N$ (recall that $Z$ is a connected component of a preimage of $N \in \mathcal{N}(X_H)$) each $\partial^\pm R_i$ maps to the image of $e_{k(i)} g_i e_{k(i)}$, in the first case, and maps to $e_{k(i)} g_i \overline{f_{l(i)}}$ in the second.

In the second case, without loss, suppose that $\partial^+ R$ and $\partial^+ R_n$ are oriented the same way. By the same argument (Lemma 2.15) used to show that in a union of trees $Z$, the graph of spaces associated to a maximal tree in $G(Z)$ is treelike, we know that $Z'_i$ is treelike, and in order for $Z$ to be treelike, we must have that $\partial^- R$ is attached to $S_0$ along a path $e_{k(i)} g_i \overline{f_{l(i)}}$, otherwise the number of boundary components must strictly decrease. In $N$, $e_{k(i)} g_i \overline{f_{l(i)}}$ is homotopic to $e_{k(i)} g' \overline{f_{l(i)}}$, and since $Y \to N$ is $\pi_1$ injective, we have that $g_i$ and $g'$ represent the same element of $\pi_1(Y)$. By Convention 2.20 $g_i = g'$.
In the first case, assume again that $\partial^+ R_\alpha$ and $\partial^+ R$ are oriented the same way. Then for the same reason that $\partial^- R$ is attached to a path $e_{k(i)}g_{j(i)}$ in the previous case, $\partial^- R$ is attached along a path $e_{k(i)}g'_{k(i)}$ in $S_0$. As before, $g_i = g'$ because $\pi_1(Y)$ embeds in $H$.  

In either case, every rectangle in $G_i$ is attached along the same path $e_{k(i)}g_{j(i)}$ or $e_{k(i)}g_{j(i)}$. For each $i$, introduce a relative graph $S_i \cong I$ with edge path $S_i \to S_0$ agreeing with $e_{k(i)}g_{j(i)}$ or $e_{k(i)}g_{j(i)}$, as the case may be. Each rectangle $R \in G_i$ which meets $S_0$ has attaching map $\partial^- R \to S_0$ which factors through a map $\psi_R: \partial^- R \to S_i$. Let $Z_i$ be the graph of spaces $(G_i \setminus \{S_0\}) \cup \{S_i\}$ with attaching maps $\psi_R$ for appropriate $R$ or attaching maps agreeing with the original attaching maps if $R$ does not meet $S_0$. 

We recover $Z$ by attaching each $Z_i$ along $S_i$ to $S_0$ via $h_i = e_{k(i)}g_{j(i)}$ (where $f_{i(i)} = e_{k(i)}$ in the first case). Each $Z_i$ is a union of trees (Definition 2.14) which, if one were not treelike, would imply that $Z$ is not treelike.  

Before we proceed, perform a homotopy of $\tilde{\varphi}$ which will simplify the analysis a little: Fix $Z$ containing a single $Y$, and let $\{Z_i\}$, $\{S_i\}$, and $S_0$ be as in the previous lemma. The map $\varphi$ carries $(Z, \partial Z)$ to some $(N, \partial N) \in \mathcal{N}(X_H)$. Consider the map $Z_i \to \Gamma Z_i$ constructed after Definition 2.14 Since each element of $\mathcal{N}(X_H)$ was chosen to be aspherical, by Lemma 2.19 we may choose a homotopy of $\tilde{\varphi}$, supported on $Z_i$, fixed on $\partial Z_i$ and $S_i$, and so that the restrictions $\varphi|_{Z_i}$ factor through maps $(\Gamma Z_i, \partial \Gamma Z_i) \to (N, \partial N)$. 

Given $Z$ containing some $Y$, Build a rose $R_Y$ with basepoint $b$ and a petal $p_i$ for each $Z_i$. Define a map $\psi_Y: R_Y \to Y$ such that the $i$'th petal maps to the path $g_i$ in $\pi_1(Y, b)$ such that the edge path $h_i: S_i \to S_0$ that $Z_i$ is attached to $S_0$ along is precisely the path $e_{k(i)}g_{j(i)}$, for the appropriate $k(i), l(i)$. 

Let $\Gamma Z$ be the graph $R_Y \cup_{b = \tau(e_i)} e_i$. There is an obvious map $\Gamma Z \to S_0$, mapping $\Gamma Z \supset e_i \to e_i \subset S_0$ and $R_Y \to Y$ via $\psi_Y$. Attach each $Z_i$ along $S_i$ to $\Gamma Z$ along the path $h_i = e_{l(i)}p_i e_{k(i)}$ to build a graph of spaces $V(Z)$, $\partial V(Z) = \bigcup \partial Z_i \subset V(Z) \cong \partial Z$. Then $Z$ is recovered by attaching $V(Z)$ to $S_0$ by the map $\Gamma Z \to S_0$. 

Define 

$$S(V(Z)) = \{\Gamma Z\} \cup \bigcup (S(Z_i) \setminus S_i)$$

The relationship between $\{S_i\}$, $\Gamma Z$, $\{Z_i\}$, $V(Z)$ and $Z$ is illustrated in the following triple of pushouts. 

\[
\begin{array}{ccc}
\Pi S_i & \to & \Gamma Z \\
\downarrow & & \downarrow \\
\Pi Z_i & \to & V(Z) \\
\downarrow & & \downarrow \\
& & S_0 \\
\end{array}
\]

\[\text{It makes no difference if the collection } \{Z_i\} \text{ is empty or not.}\]
For each $Z_i$ we have the quotient map $\pi_{Z_i}: Z_i \to \Gamma_Z$. The attaching map $h_i: S_i \to S_0$ factors through $\tilde{h}_i: S_i \to \Gamma_Z$. Since $S_i \in \mathcal{S}(Z_i)$, by Lemma 2.19 $S_i$ embeds in $\Gamma_Z$ under $\pi_{Z_i}$, thus there is an induced map

$$\tilde{h}_i: \text{Im}_{\pi_{Z_i}}(S_i) \to \Gamma_Z$$

For this reason we call the image of $S_i$ in $\Gamma_Z$ $S_i$ as well. Define a graph $\Gamma_U(V(Z))$ as the pushout in Figure 9.

\[\begin{array}{ccc}
I I Z_i & \longrightarrow & I I \Gamma_{Z_i} \\
\downarrow & & \downarrow \\
I I S_i & \longrightarrow & \Gamma_U(V(Z)) \\
\downarrow & & \downarrow \\
\Gamma_Z & \longrightarrow & \Gamma_Z \\
\end{array}\]

**Figure 9.** The diagram defining $\Gamma_U(V(Z))$

Since the restriction of $\tilde{\varphi}$ to $Z_i$ factors through $\Gamma_{Z_i}$ (after the homotopy provided by Lemma 2.19), and $\Gamma_U(V(Z))$ is the union of $\Gamma_Z$ and $\Gamma_{Z_i}$ along $S_i$, and since $\tilde{\varphi}$ is recovered by attaching $V(Z)$ along $S_i$ to $S_0$, the restriction of $\tilde{\varphi}$ to $V(Z)$ factors through the restriction to $\Gamma_U(V(Z))$. On the other hand, $S_0$ is simply $Y_Z \cup \Gamma_Z$, identified along $R_Z$. Hence, $\tilde{\varphi}|_Z$ is the composition of the projection of $V(Z)$ to $\Gamma_U(V(Z))$ followed by the restriction to $S_0$.

One key property of $V(Z)$ is that $\partial V(Z)$ is precisely $\partial Z$. For $Z$ not containing $Y \in \mathcal{Y}(X_G)$, set $V(Z) = Z$. Let $\mathcal{V}$ be the collection of all $V(Z)$. Recall the definition of $B$. For each $B \in B$ there are two inclusions, each a homeomorphism with a boundary component of $Z$. Let $\mathcal{B}$ be a point, as in the construction of $Z'$. Since each $B$ maps to some boundary component of some $Z$, and each boundary component of a $Z$ is a boundary component of $V(Z)$, and since each $\tilde{\varphi}|_{B[V(Z)}$ factors through $\{\mathcal{B}|_\Gamma U(Z)\}$, we may sensibly form the pushouts in Figure 10.

\[\begin{array}{ccc}
I I B(0) & \longrightarrow & I I \mathcal{S}(V(Z)) \longrightarrow \Gamma(\mathcal{X}) \\
\downarrow & & \downarrow \\
I I B & \longrightarrow & I I V(Z) \longrightarrow \mathcal{X} \\
\downarrow & & \downarrow \\
I I \mathcal{B} & \longrightarrow & I I \Gamma_U(V(Z)) \longrightarrow \Gamma_U(\mathcal{X}) \\
\end{array}\]

**Figure 10.** Pushouts defining $\Gamma(\mathcal{X})$, $\mathcal{X}$, and $\Gamma_U(\mathcal{X})$. Vertical arrows in the top row are inclusions, vertical arrows in the bottom row are projections.
We see from the construction that $\Gamma(X)$ and $\Gamma_U(X)$ are graphs, and, since point preimages of $V(Z) \to \Gamma_Z$ are connected, that $\pi_1(X) \to \pi_1(\Gamma_U(X))$ is onto. We can give an alternate description of $X$ as $\Gamma_\infty(X) \cup \{x_{0,i}\}$. For each component $Y \in \mathcal{Y}(X_G)$, there is exactly one $Z \in \mathcal{Z}$ containing it. Let $R_Y$ be the rose contained in $\Gamma_Z$. Then we recover $\bar{X}$ by gluing each $R_Y \subset \Gamma_U(V(Z))$ to $Y$ via the map $\psi_Y$. The images of each $R_Y$ are disjoint under $\chi \to \Gamma_U(X)$. It follows easily from the fact that each $\Gamma_Z$, embeds in $\Gamma_Z$ under $\pi_Z$ that the map $\Gamma(X) \to \Gamma_U(X)$ is an immersion.

Each $\Gamma_U(V(Z))$ is a graph with distinguished boundary vertices. Declare that every point of $\Gamma_U(V(Z))$ whose complimentary components number at least three to be vertices of $\Gamma_U(X)$, and let $V_1, \ldots, V_{n_2}$ be the leaves of $\mathcal{F}(Z)$ (These are the only leaves which don’t necessarily have neighborhoods which are products.) which map to the vertices of $\Gamma_U(V(Z))$. Each such leaf is a finite graph. Setting all such leaves, along with components of $B_\bullet \subset X$, to be vertex spaces, it is easily seen that $X$ is a 2-covered graph of spaces.

Lemma 2.22. $\chi(\Gamma(X)) = \chi(\Gamma_U(X))$

Proof. If $\Gamma$ is a graph with distinguished valence one vertices $\partial \Gamma$, define

$$\kappa(\Gamma, \partial \Gamma) = -\chi(\Gamma) + \frac{1}{2} \# \partial \Gamma$$

as was previously done for graphs with no loops. Observe that

$$\sum_{Z \in \mathcal{Z}} \sum_{S \in \mathcal{S}(V(Z))} \kappa(S, \partial S) = -\chi(\Gamma(X))$$

and

$$\sum_{Z \in \mathcal{Z}} \kappa(\Gamma_U(V(Z)), \partial \Gamma_U(V(Z))) = -\chi(\Gamma_U(X))$$

Thus we only need to check that $\kappa(\Gamma_U(V(Z)), \partial V(Z)) = \sum_{S \in \mathcal{S}(V(Z))} \kappa(S, \partial S)$ for all $Z$. This follows easily from $\chi(\Gamma_U(V(Z))) = \chi(\Gamma_Z)$, $\Delta_Z(Z_i) = 0$, and the fact that $\partial S_i$ maps to distinct boundary components of $\Gamma_Z$, as in the computation carried out in Lemma 2.13. \hfill \Box

3. Moves on Graphs of Spaces

A 2-covered graph of spaces is generally an ugly beast, but can be converted to a more amenable object by folding, reducing, and collapsing. We handle them in reverse order.

Definition 3.1 (Collapse). If $X$ is a graph of spaces and $e$ is an edge of $\Gamma_U(X)$ with $\tau(e) \neq \iota(e)$, and if $\tau: E \to V_{\tau(e)}$ is an embedding, then we can collapse $X$ to $X_E$ by crushing the edge space $e \times E$ to $\iota(E)$. In the topological realization of $X$, collapse $E \times I$ to $E \times \{0\}$. The resulting vertex is $E_{\iota(e)} \cup E_{\tau(e)}/\tau(w) \sim \iota(w)$, $w \in E$. The edge maps incident to the new vertex are still immersions, and it’s easy to check that the quotient map is a homotopy equivalence which respects $\pi_1(\Gamma(X)) \to \pi_1(X)$. 

**Definition 3.2** (Weight). The weight of a graph is the number of edges.

**Definition 3.3** (Reduced). Some graphs of spaces admit trivial simplifications. For instance, the topological realization of a graph with a valence two vertex can be given a simpler description by un-subdividing an edge. A similar statement holds for our 2-covered graphs: If \( V \) is a vertex space in \( X \) and \( E_1 \) and \( E_2 \) are the only incident edges, then if both maps \( E_i \to V \) are graph isomorphisms, then \( X \) is reducible. By collapsing one of the incident edges, the number of reducible vertices strictly decreases. If \( X \) has no reducible vertices, and all valence one vertices have nonzero weight, then it is reduced.

**Definition 3.4** (Folding). Fix a 2-covered graph of spaces \( X \). Given a set of edges, indexed by \( J \), we define a new graph of spaces \( X_J \), called a fold of \( X \). We say that \( X_J \) is obtained from \( X \) by folding.

How to fold: Let \( V \) be a vertex of a graph of spaces \( X \) as above. Let \( \{ (E_i, \tau_i) \}_{i \in I} \) be the oriented edges whose terminal vertex is \( V \). For \( J \subset I \), define

\[
V_J = \bigcup_{j \in J} \tau_j(E_j)
\]

Let \( \{ V_{J,p} \}_{p=1..l_p} \) be the connected components of \( V_J \) and \( \{ V_{I\setminus J,q} \}_{q=1..l_q} \) the connected components of \( V_{I\setminus J} \), and \( \{ E_{J,r} \}_{r=1..l_r} \) the connected components of \( V_J \cap V_{I\setminus J} \).

For each index \( p, q, r \), introduce new vertices \( v_{J,p}, v_{I\setminus J,q} \), and oriented edges \( e_{J,r} \) with \( \tau(e_{J,r}) \) the member of \( \{ V_{J,p} \} \) that \( E_{J,r} \) is contained in, and \( \iota(e_{J,r}) \) the member of \( \{ V_{I\setminus J,q} \} \) that \( E_{J,r} \) is contained in. Define \( \iota: e_{I\setminus J} \to \cdots \to \tau : e_{J,r} \to \cdots \), where \( \cdots \) represents the appropriate component \( V_{J,p} \) or \( V_{I\setminus J,q} \).

This data, along with the (undisturbed) data from the rest of the graph of spaces \( X \) defines a new graph of spaces (in the 2-Covered sense) \( X_J \) with the vertex space \( V \) split apart.

Folding is illustrated in Figure 11. Note that \( J \) may consist of a single element, yet the split space may still be distinct from the original space. Also, beware that it’s possible for the underlying graph’s complexity to increase: the subgraph of \( \Gamma_U(X_J) \) spanned by \( e_{I\setminus J} \) may not be a tree.

**Definition 3.5** (Unfoldable). A vertex \( v \in \Gamma_U(X) \) is unfoldable if for all \( J \subset I \), where \( I \) is the indexing set of the incident edges \( E_i \), one of

\[
V_J \simeq \coprod_{j \in J} E_j \quad \text{or} \quad V_{I \setminus J} \simeq \coprod_{i \in I \setminus J} E_i
\]

holds. If a vertex isn’t unfoldable, then it is foldable.

Unfoldable vertices are particularly nice. Not only do they fall into two basic simple types, folding an unfoldable vertex doesn’t change the graph of spaces.

**Lemma 3.6** (Structure of Unfoldable Vertices). A reduced unfoldable vertex \( v \) has the form
• There is a distinguished edge $e_0$ adjacent to $v$. The rest of the edges $e_1, \ldots, e_m$ are undistinguished.

• The map $E_0 \looparrowright V$ is not an embedding.

• The maps $E_i \looparrowright V$, $i \neq 0$, are embeddings with pairwise disjoint images.

or

• $v$ has valence three and all incident edge maps are embeddings. The image of every incident edge space meets every other. There is a vertex $w$ of $V$ which is in the image of every incident edge space.

A fold of an unfoldable vertex in $X$ recovers $X$.

Definition 3.7. An unfoldable vertex with a distinguished edge $e_0$ such that $E_0 \looparrowright V$ is not an embedding is degenerate. A vertex that has valence three and whose incident edges embed is nondegenerate.

Proof of Lemma 3.6. Let $v$ be an unfoldable vertex.

If an incident edge $E_0 \looparrowright V$ isn’t an embedding, then it’s clear we’re in the first case of the lemma. Take $J = \{0\}$. Then the graph covered by the remaining incident edge graphs is homeomorphic to their disjoint union.

Thus we need to show that the second case of the lemma holds, assuming every incident edge map is an embedding. Suppose that the valence of $v$ is at least four. Either there is a chain of incident edge graphs $E_{i_1}, i = 1, 2, 3, 4$, such that $\text{Im}(E_i) \cap \text{Im}(E_{i+1}) \neq \emptyset$ or there is an incident edge $E_1$ whose image meets every other incident edge graph. In the first case, we may take $J = \{1, 2\}$.

In the second case, if $E_{i_1}, i, j, k \neq 1, i, j, k$ distinct, whose images meet $E_1$, then they must have disjoint images since there is no chain of length four. For example, if $\text{Im}(E_2) \cap \text{Im}(E_3)$, then the sequence $(E_4, E_1, E_2, E_3)$ is a chain of length four. Since $V$ is connected, there is an edge $f$ of $V$, contained in the image of $E_1$, which isn’t covered by any $E_i$, $i \neq 1$, thus $f$ is covered twice by $E_1$, a contradiction.

Let $V' = \text{Im}(E_1) \cup \text{Im}(E_2)$. If $f$ is an edge of $V$ meeting $V'$ and $f$ isn’t contained in $V'$ then $f$ is covered by $E_3$. The endpoint of $f$ contained in $V'$ is contained in the image of every incident edge space.
If $E_{J,r}$ is the set of edges introduced by folding a set of incident edges $\{E_j\}_{j \in J}$, then the original graph of spaces is recovered by collapsing $\{E_{J,r}\}$.

Let $\varphi: X_J \rightarrow X$ be the collapsing map. Then $\varphi$ is a homotopy equivalence. Let $\Gamma$ be a connected component of $\Gamma(X)$. If $\Gamma_J$ is the associated connected component of $\Gamma(X_J)$, $\varphi_J$ is the collapsing map restricted to $\Gamma_J$, and $\varphi_U$ is induced map on underlying graphs, then

\[
\begin{array}{ccc}
\Gamma_J & \xrightarrow{z} & X_J \\
\downarrow{\varphi_J} & & \downarrow{\varphi} \\
\Gamma & \xrightarrow{z} & X \\
\downarrow{\varphi_U} & & \downarrow{\varphi_U}
\end{array}
\]

commutes. Collapsing restricted to the horizontal subgraph crushes forests, thus $\varphi_J$ is a homotopy equivalence, $\varphi_U$ is an epimorphism, and the unlabeled arrows are the natural epimorphisms

\[\text{graph of spaces} \rightarrow \text{underlying graph}\]

Given a graph of 2-covered graphs, there is a reduced space $X \mapsto X^R$ obtained by trimming trees and removing all valence two vertices for which both incident edge maps are graph isomorphisms.

4. SIMPLIFYING GRAPHS OF SPACES

Under certain favorable conditions a folded space admits further simplification. There is a complexity, which, when minimized through folding and collapsing, gives an optimal graph of spaces equivalent to a given one. The structure of the vertex and incident edge spaces of a space minimal with respect to this complexity is considerably simpler than that of a nonminimal graph of spaces.

**Definition 4.1** (Complexity of Graphs of Spaces). Let $k(X)$ be the maximal valence of a vertex in $\Gamma_U(X)$, $m_l(x)$ the number of vertices of valence $l$, $m_2^{red}(X)$ is the number of reducible valence two vertices, and $m_2^{deg}(X)$ is the number of degenerate valence two vertices in $\Gamma_U(X)$. If $X$ is reduced, and $k(X) \geq 3$, then the complexity of $X$ is the tuple

\[c(X) = (-b_1(\Gamma_U(X)), k(X), m_k(X), \ldots, m_3(X), m_2^{red}(X), -m_2^{deg}(X))\]

If $k(X) = 2$, then the entries $m_k(X), \ldots, m_3(X)$ don’t appear. The order is the lexicographic one.

$X'$ is obtained from $X$ by folding if there is a sequence of folds $X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_k = X'$. Let $\text{Folds}(X)$ be the set of graphs of spaces which can be obtained by folding.

**Lemma 4.2** (Minima of $c$). Let $X \in \text{Folds}(Y)$. If $c(X)$ is minimal then $X$ is reduced and every vertex of $X$ is unfoldable.

The conclusions of Lemma 4.2 are illustrated in Figure 12.

**Proof.** If $X$ can be reduced, then reducing decreases $c$. The idea here is that folding takes a vertex in $\Gamma_U(X)$ and blows it up to a bipartite graph. Then either $b_1(\Gamma_U(X))$
If $\Gamma_U(X)$ has a foldable vertex then $X$ isn’t a minimum of $c$: Let $v$ be a foldable vertex, and let $\{E_j\}_{j \in J}$ such that neither
\[ V_J \simeq \bigcup_{j \in J} E_j \quad \text{nor} \quad V_{I \setminus J} \simeq \bigcup_{i \in I \setminus J} E_i \] (♦)
holds.

Suppose $J = \{1, 2\}$. Let $v_J$ be the additional vertex corresponding to $V_J$. Let $v_1, \ldots, v_q$ be the vertices corresponding to connected components of $V_{I \setminus J}$, and let $e_1, \ldots, e_r$ be the edges corresponding to connected components of $V_J \cap V_{I \setminus J}$. Let $s$ be the valence of $v$.

If $r > q$ then $b_1(\Gamma_U(X_J)) > b_1(\Gamma_U(X))$, thus $c(X)$ isn’t minimal.

Thus we may assume that $r = q$. There are $s - 2$ edges incident to $v_1, \ldots, v_q$. Each vertex $v_j, j = 1..q$ has valence at most $s - 1$ and $v_J$ has valence at most $s$.

If $v_J$ has valence $s$, then $r = q = s - 2$ and the edges $E_i, i \in I \setminus J$, have pairwise disjoint images, therefore at least one must be an immersion but not an embedding, increasing $m_2^{deg}$.

If $J = \{1\}$, a similar argument works. If the valence of $v_J$ is $s$, then $r = q = s - 1$, implying that the images $\text{Im}(E_i), i \in I \setminus J$, are pairwise disjoint, therefore at least one of them is an immersion but not an embedding, increasing $m_2^{deg}$. If a
Lemma 3.6. Let \( v \) be a vertex on a graph. If \( v \) is a foldable vertex, then \( \chi(\Gamma(X)) \) decreases. 

The Euler characteristic of a graph can be computed by adding the “curvatures” of its vertices:

\[
\chi(\Gamma) = \sum_{v \in \Gamma^{(0)}} \kappa(v), \quad \kappa(v) = 1 - \frac{1}{2} \text{valence}(v)
\]

For each vertex \( v \) of \( \Gamma \), let \( \tau(v) \) be the set of vertices of \( \Gamma \) which map to \( v \). Let \( \kappa(\tau) = \sum_{w \in \tau} \kappa(w) \), and so \( \chi(\Gamma) = \sum \kappa(\tau) \).

There are two cases to consider. Recall the structure of unfoldable vertices from Lemma 3.6.

**v is degenerate:** Let \( k + 1 \) be the valence of \( v \). Let \( E_0 \) be the immersed edge graph, and \( E_i, i = 1..k \), the embedded edge graphs. Every vertex of \( V \) is the image of at least two distinct vertices of \( \bigcup E_i \), hence has valence at least two.

Let \( V_1 \) be the union of edges of \( V \) covered twice by \( E_0 \). The vertex graph is the union \( \cup_{i \neq 0} \text{Im}(E_i) \cup V_1 \).

Suppose \( V_1 \) is nonempty. Let \( V_1^o = (\text{Im}(E_0) \setminus \cup_{i \neq 0} \text{Im}(E_i)) \subset V_1 \) be the subgraph of \( V \) covered by \( E_0 \) but not covered by any other edge space. Then \( V_1 \) is the interior of \( V_1 \). A vertex in \( V_1^o \) contributes a vertex with valence at least two to \( \Gamma \). If \( w \) is a vertex in \( V_1 \), it has an incident edge \( f \), \( \tau(f) = w \), which, since \( \tau \) is an immersion, is the image of two distinct oriented edges \( f_1 \) and \( f_2 \), \( \tau(f_1) \neq \tau(f_2) \), from \( E_0 \). The terminal vertices of \( f_i \) map to \( u \) under \( \tau \), thus the vertex \( w \), regarded as a vertex of \( \Gamma(X) \), has valence at least two in \( \Gamma(X) \).

We now handle the vertices \( \cup_{i \neq 0} \text{Im}(E_i) \cap V_1 \). Edges not contained in \( V_1 \) are each covered once by \( E_0 \) and once by \( \cup_{i \neq 0} E_i \). Since \( V \) is connected, there are oriented edges \( f_i \subset V_1 \) which meet \( \text{Im}(E_i) \) at their terminal vertices \( w_i \). Each \( f_i \) is the image of distinct oriented edges \( f_1 \neq f_2 \subset E_0 \) with distinct (since \( E_0 \) immerses in \( V \)) terminal vertices \( w^1_i \) and \( w^2_i \). Thus \( w_i \) is the image of a vertex in \( E_i \), and the image of two vertices in \( E_0 \), hence \( w_i \) has valence at least three vertex in \( \Gamma(X) \).

If \( V_1 \) is empty, then there is a vertex \( w \in V = \text{Im}(E_0) \) which is the image of two vertices in \( E_0 \). Then \( w \) is the image of a vertex in \( E_1 \) as well. Then \( w \), regarded as a vertex of \( \Gamma(X) \), has valence at least three.

In all cases \( \kappa(\tau) \leq -1/2 \cdot k < \kappa(v) = (2 - (k + 1))/2 = 1/2 - 1/2 \cdot k \). The inequality is strict.

**v is nondegenerate:** There are three edges incident to \( V \): \( E_1 \), \( E_2 \), and \( E_3 \), and all incident edge maps are embeddings.
Suppose $E_1$ is a point. Since $\tau_2 : E_2 \to V$ and $\tau_3 : E_3 \to V$ are both embeddings, and every edge is covered once by $E_2$ and once by $E_3$, both are surjective. A surjective immersion of graphs is an isomorphism, hence both maps are graph isomorphisms. Let $w$ be the image of $E_1$. The incident edge maps map $w_2 \in E_2$ and $w_3 \in E_3$ in $E_2$ to $w$. Since $w$ is also the image of $E_1$, it is a valence three vertex of $\Gamma(X)$. Every other vertex in $\tilde{v}$ has valence two in $\Gamma(X)$, hence contributes nothing to $\kappa(\tilde{v})$. Thus $\kappa(\tilde{v}) = \kappa(v) = -1/2$.

We’re left with the possibility of three nontrivial embeddings, i.e., $E_{1|2|3}$ aren’t points. Every vertex of $\tilde{v}$ has, by the previous arguments, valence at least two. Since the incident edge maps are embeddings, every vertex is covered at most once by each incident edge, i.e., every vertex in $\tilde{v}$, regarded as a vertex of $\Gamma(X)$, has valence at most three. Since $v$ is nondegenerate there exists a point of triple intersection, hence

- $\kappa(\tilde{v}) \leq \kappa(v) = -1/2$
- If $\kappa(\tilde{v}) = \kappa(v)$ then there is exactly one point of triple intersection of incident edge graphs, otherwise $\kappa(\tilde{v}) \leq -1 < \kappa(v) = -1/2$

By the cases above, we conclude the inequality

$$\chi(\Gamma(X)) = \sum \kappa(\tilde{v}) \leq \sum \kappa(v) = \chi(\Gamma_U(X))$$

(♠) for minima of $c$.

If $X$ isn’t a minimum of $c$, then let $X_c$ be a member of $\text{Folds}(X)$ with minimal complexity. Since $b_1(\Gamma_U(X_c)) \geq b_1(\Gamma_U(X))$

$$\chi(\Gamma_U(X)) \geq \chi(\Gamma_U(X_c)) \geq \chi(\Gamma(\Gamma(X_c))) = \chi(\Gamma(X))$$

(♣)

We’ll be interested in graphs of spaces whose horizontal subgraphs have the same Euler characteristic as their underlying graphs. When this happens, the space can be folded so that all vertex spaces have the simplest form possible.

**Lemma 4.4** $(\chi(\Gamma(X)) = \chi(\Gamma_U(X)))$. Suppose $X$ is a minimum of $c$ and $\chi(\Gamma(X)) = \chi(\Gamma_U(X))$. Then every vertex has valence three. If $V$ is a vertex with incident edge spaces $E_i$, $i = 1, 2, 3$, then $\cap E_i$ is a single point.

**Proof.** Suppose $\chi(\Gamma(X)) = \chi(\Gamma_U(X))$. By (♣), every minimum $X_c$ of $c$ obtained by folding satisfies $b_1(\Gamma_U(X_c)) \geq b_1(\Gamma_U(X))$. If this inequality is strict, then, by Lemma 4.3, $\chi(\Gamma(X)) < \chi(\Gamma_U(X))$. Thus $\chi(\Gamma(X)) = \chi(\Gamma_U(X))$, and for every minimum $X_c$, $\chi(\Gamma(U(X_c))) = \chi(\Gamma_U(X_c))$.

Let $X_c$ have minimal $c$ out of all members of $\text{Folds}(X)$. If $X_c$ had a degenerate vertex, then the inequality (♣) would be strict, thus every vertex is unfoldable, has valence three, and is nondegenerate. By the argument used to prove Lemma 4.3 there is exactly one point of triple intersection of edges incident to every vertex graph.

**Lemma 4.5.** A graph $V$, 2-covered by connected subgraphs $E_i \hookrightarrow V$, $i = 1, 2, 3$, such that $\cap \text{Im}(E_i)$ is a single vertex $w$, has one of the following forms:

- $E_i$ are all points.
\begin{itemize}
  \item $E_1$ is a point, and $E_{2|3} \cong V$. $W(E_{2|3}) > 0$.
  \item $V = \text{Im}(E_1) \vee \text{Im}(E_2)$, $E_3 \cong V$. $W(E_{1|2|3}) > 0$.
  \item $V$ is the union of three subgraphs $V_{1|2|3}$ which meet at a single vertex $w \in V \cong \bigvee_w V_i$. $E_i \cong V_{i+1} \vee w V_{i+2}$. $W(E_{1|2|3}) > 0$.
\end{itemize}

**Definition 4.6** (Separable, Trivial, Splittable). A vertex $v$ of a graph of spaces such that $V$ satisfies Lemma 4.5 is called separable. If $V$ satisfies one of the first two bullets $v$ is trivial. Otherwise it is nontrivial. If $V$ satisfies the third bullet $v$ is splittable. If $V$ satisfies the fourth, $v$ is separable, but unsplittable.

**Proof of Lemma 4.5** Let $w = w_v$ be the sole point of triple intersection. Let $\mathcal{P}$ be the set of edge-paths starting at $w$ and that terminate if they meet $w$ again. Let $\{E_i\}_{i=0,1,2}$ be the edge graphs incident to $V$. We divide $\mathcal{P}$ into three subclasses $\mathcal{P}_{j,k}, j \neq k$. A path $p$ lies in $\mathcal{P}_{j,k}$ if the image of $p$ is contained in $\text{Im}(E_j) \cap \text{Im}(E_k)$. Let $V_i = \bigcup_{p \in \mathcal{P}_{i+1,i+2}} \text{Im}(p)$. At most one of $V_i$ can be a point. Every point/edge of $V$ lies in one of $V_i$, which all meet at $w$, the sole point of triple intersection. An incident edge $E_i$ is then isomorphic to $V_{i+1} \cup V_{i+2}$. □

See Figure 13 for an illustration. In virtue of Lemmas 4.4 and 4.5 we make the following definition.

**Definition 4.7.** If $X$ satisfies $\chi(\Gamma_U(X)) = \chi(\Gamma(X))$ and $X$ is a minimum of $c$, since every vertex of $\Gamma_U(X)$ is separable, we say that $X$ is separable.

## 5. Separable Graphs of Spaces

In this section we consider only separable graphs of spaces.

The next two lemmas give us the means to analyze the separable graphs of spaces.

**Definition 5.1.** A graph of spaces is irreducible if it has no trivial edge spaces, i.e., there are no “obvious” free product decompositions of its fundamental group. The removal of interiors of weight 0 edges and leftover vertices from $X$ yields graphs of spaces $X_i$ which are the irreducible components of $X$.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure13}
\caption{$w$ separates $V$. $p$ is a path from the proof of Lemma 4.5.}
\end{figure}
Every vertex of a separable graph of spaces turns each edge space into a (possibly trivial) wedge of subgraphs. We would like to push this structure around the graph of spaces to give each edge graph the coarsest treelike structure compatible with all decompositions forced upon it. This lets us express a two-covered graph of spaces as a union of cylinders. Under certain circumstances a graph of spaces can be repeatedly collapsed and folded to what is essentially a wedge of cylinders.

We start by defining the cylinders of a graph of spaces \( X \). Roughly speaking, a cylinder \( C \) is a graph of spaces whose underlying graph is a circle, has a map to \( X \) compatible with edge maps, and if the map \( C \to X \) factors through a similar such map \( C' \to X \), then \( C \cong C' \).

**Definition 5.2 (Cylinder).** A graph of spaces is a cylinder if its underlying graph is a circle and has only reducible valence two vertices. A cylinder is homeomorphic to the mapping torus of a combinatorial automorphism of a graph.

Let \( S(X) \) be a set of indivisible (not factoring through a covering map \( S^1 \to S^1 \)), unoriented, closed, immersed edge paths in \( \Gamma(X) \) uniquely representing every conjugacy class of maximal cyclic subgroup of \( \pi_1(\Gamma(X)) \) as an immersion \( \iota: S^1 \to \Gamma(X) \). There is an immersion \( S(\iota): S(X) \hookrightarrow \Gamma(X) \). A graph of spaces \( X \) is a union of annuli and Möbius bands \( \{ A_j \} \) and \( \Gamma(X) \) along boundary maps \( \varphi_j: \partial A_j \hookrightarrow \Gamma(X) \). Each annulus is a union of squares and the map \( \varphi_j \) is a pair (or a singleton, if \( A_j \) is a Möbius band) of edge paths in \( \Gamma(X) \). The maps \( \varphi_j \) factor through \( S(X) \), i.e., there are lifts

\[
\widetilde{\varphi}_j: \partial A_j \hookrightarrow S(X)
\]

such that \( S(\iota) \circ \widetilde{\varphi}_j = \varphi_j \). This is because all edge maps \( E \to V \) are immersions, hence the maps \( \partial A_j \to \Gamma(X) \) are immersions.

A graph of spaces \( X \) is the union \( \Gamma(X) \cup_{\varphi_j} A_j \). Define a new graph of spaces \( \overline{X} \) to be \( S(X) \cup_{\varphi_j} A_j \). The set of cylinders of \( X \), denoted \( \text{Cyl}(X) \), is the collection of connected components of \( \overline{X} \) containing an annulus or Möbius band.

The boundary of a cylinder \( C, \partial_X C \), is the subgraph of \( \Gamma(C) \) corresponding to elements of \( S(X) \) whose images are contained in \( \Gamma(\infty)(X) \). The boundary map \( S(\iota)|_{\partial_X C} \) is denoted \( \varphi_C \). The inclusion map of a cylinder (which isn’t really an inclusion, but we ignore this technicality) \( C \to X \) is denoted \( \psi_C \).

The space \( X \) is recovered by forming the quotient space \( \Gamma(\infty)(X) \cup_{\varphi_C} C \in \text{Cyl}(X) \).

A transverse graph of a cylinder \( C \in \text{Cyl}(X) \) is an edge space or a vertex space of \( C \). A transverse graph, when it doesn’t matter if it’s an edge space or vertex space, is typically denoted \( F \). Choose an orientation on each edge of \( \Gamma_U(C) \) such that the edges of \( \Gamma_U(C) \) are \( e_0, \ldots, e_{n-1} \) and \( \tau(e_i) = \iota(e_{i+1 \text{ mod } n}) \), and with vertices \( v_i \) such that \( \iota(v_i) = v_i \). Let \( \alpha_C \) be the map

\[
\prod_i (\tau_i \sqcup \iota_i^{-1}) : \text{EdgeGraphs}(C) \sqcup \text{VertexGraphs}(C) \to \text{VertexGraphs}(C) \sqcup \text{EdgeGraphs}(C)
\]
Lemma 5.5 (Separating Subgraphs, Structure of Vertex/Edge Spaces). Let \( \alpha_C \) be a cylinder of \( X \), then \( \alpha_C^2 \) represents one \( n \)-th of a rotation of \( C \). Also, \( \alpha_C^{2n} = \text{id} \).

Let \( X \) be a graph of spaces. If \( X_1, \ldots, X_n \) are the irreducible components of \( X \), then each cylinder has image contained in one \( X_i \). The essential boundary, \( \partial_X^{\text{ess}} C \) is \( \partial X_i \) if \( C \) has image in \( X_i \).

If \( C \) is a cylinder of \( X, F \) a transverse graph of \( C \), and \( |F \cap \partial_X^{\text{ess}}(C)| > 1 \), then the cylinder is \textit{good}. Otherwise it is \textit{bad}. Note that an irreducible component that consists of a single cylinder is automatically bad since \( \Gamma_\infty \) of a cylinder is empty.

Definition 5.3. Let \( \mathcal{F}(E) \) be the set of edge spaces

\[
\{ F \in \bigcup_{C \in \text{Cyl}(X)} \text{EdgeGraphs}(C) \mid \psi_C(F) \subset E \}
\]

An element \( F \in \mathcal{F}(E) \) is a peripheral element of \( E \) if it contains a vertex \( w \in \Gamma_\infty(X) \cap E \) and if \( w \in F' \in \mathcal{F}(E) \) then \( F = F' \). The vertex \( w \) is a boundary vertex of \( E \).

To get the ball rolling we need a way to take a peripheral element \( F \) of the set of cylinder cross sections \( \mathcal{F}(E) \) and a boundary element \( \Gamma_\infty(X) \cap F \) and push it around the graph of spaces until a splitting vertex is discovered.

Definition 5.4 (Pushing). A subset of a graph of spaces is vertical if it lies in a fiber of the map \( \pi: X \to \Gamma_U(X) \). Let \( E \times I \) be an edge space of \( X \). Say that \( x \) and \( y \) are equivalent if \( x \) and \( y \) have the same \( E \) coordinate. Horizontality is the equivalence relation generated by the relations on the edge spaces.

If \( Y \) is a vertical subset of \( X \) then \( Y \) pushes along a path \( p: [0, a] \to \Gamma_U(X) \) if there is a function \( P \) such that for each \( y \in Y \), \( P(y, [0, a]) \) is horizontal, and the following diagram commutes

\[
\begin{array}{ccc}
Y \times [0, a] & \xrightarrow{P(u, t)} & X \\
\downarrow & & \downarrow \pi \\
[0, a] & \xrightarrow{p(t)} & \Gamma_U(X)
\end{array}
\]

Since the diagram commutes, for each \( t \), the set \( P(Y, t) \) is vertical.

If \( C \) is a cylinder of \( X \) and \( F \) is a transverse graph of \( C \) then \( C \) is a vertical subset of \( C \) and \( \psi_C(F)(=F) \) is a vertical subset of \( X \). If a connected vertical subset \( Y \) of \( X \) containing \( F \) pushes along every path \( p \) that \( F \) pushes along then \( Y = F \). The rotation by \( t \) of \( C \) is a one parameter family of homeomorphisms \( \alpha: C \times \mathbb{R} \to C \). If \( F \) is a transverse graph of \( C \) then pushing \( \psi_C(F) \) around \( X \) can be realized by the composition \( P = \psi_C \circ \alpha \).

Lemma 5.5 (Separating Subgraphs, Structure of Vertex/Edge Spaces). Suppose \( X \) is irreducible and separable.

If \( C \) is a cylinder of \( X \), and \( F \) is a transverse subgraph of \( C \), then \( \psi_C \) embeds \( F \). Every nonzero weight edge or vertex space of \( X \) is a union of images of edge or vertex spaces, respectively, of cylinders of \( X \).

If \( E \) is an edge space and \( w \in \Gamma_\infty(X) \cap E \) is not a cutpoint of \( E \) then \( \mathcal{F}(E) \) has a peripheral element containing \( w \).
$\psi_C(E_C) \in \Gamma_\infty(X) \cap E$ 

**Figure 14.** Edges are treelike. $\psi_C(E_C)$ is the image of an edge space $E_C$ of a cylinder $C$ of $X$. The same picture holds for vertex spaces.

**Proof.** Let $F$ be an edge space of $C$. Suppose there are vertices $p$ and $q$ such that $\psi_C(p) = \psi_C(q)$. First, note that $p$ and $q$ must be contained in $\partial X(C)$. There are subgraphs $\Gamma_p$ and $\Gamma_q$ of $\Gamma(C)$ containing $p$ and $q$, respectively. Suppose $\psi_C(\alpha_C^k(p)) = \psi_C(\alpha_C^k(q))$ for all $k$. Then $\psi_C(\Gamma_p)$ and $\psi_C(\Gamma_q)$ must represent the same element of $\mathbb{S}(X)$, thus $\Gamma_p = \Gamma_q$ as sets, but this implies that $\Gamma_p$ must represent a periodic path in $\Gamma_\infty(X)$. The other possibility is that there exist $p$ and $q$ such that $\psi_C(p) \neq \psi_C(q)$, but $\psi_C(\alpha_C(p)) = \psi_C(\alpha_C(q))$. This is clearly impossible since edge maps of $X$ are injective. Thus $\psi_C$ embeds vertex and edge spaces. The collection $\bigcup \psi_C$ is clearly injective on the collection of edges of vertex and edge spaces. Since every edge of an edge or vertex space comes from an annulus in $X$, we have the first part of the lemma.

Suppose $F, F' \in \mathcal{F}(E)$ with vertices $w_1, w_2 \in F \cap F'$. Clearly $w_1, w_2 \in \Gamma_\infty(X)$. If $F$ is an edge graph of $C$ and $F'$ is an edge graph of $C'$, let $\Gamma_i^{(0)}$ be the component of $\Gamma_\infty(C^{(0)})$ containing $\psi_C^{-1}(w_i)$. Since $\Gamma_\infty(X) \cap E$ doesn’t separate $F$ and $F'$, we must have $\Gamma_i = \Gamma_i^{(0)}$, contradicting the construction of $\text{Cyl}(X)$. □

We’re now ready to prove Theorem 2.5. If we represent maximal corank homomorphisms as immersions $\Gamma(X) \to \Gamma_U(X)$ factoring through $\pi_1(X) \to \pi_1(\Gamma_U(X))$, then we may choose an optimal representation: by Lemma 4.2 we may fold $X$ to a space $X_\epsilon$ which minimizes $\epsilon$. Since $\chi(\Gamma_U(X_\epsilon)) = \chi(\Gamma_U(X)) = \chi(\Gamma(X)) = \chi(\Gamma(X_\epsilon))$, by Lemma 4.4, $X_\epsilon$ is separable. Also note that the rest of the diagram in Figure 15 commutes. $\psi$ is the homotopy equivalence given by the sequence of folds and collapses to $X_\epsilon$.

**Proof of Theorem 2.5** Let $\phi : F = \mathbb{F}_n \hookrightarrow \mathbb{F}_n$ be a map that extends to a surjection $\overline{\phi} : G = F \left\{ \gamma_i^{1/k_i} \right\}_{i=1..m} \to \mathbb{F}_n$, with $\gamma_i$ pairwise nonconjugate, indivisible, and
Let $M_i$ be the mapping cylinder of the $k_i$–fold cover $S^1 \to S^1$ corresponding to adding the $k_i$–th root to $\gamma_i$. The domain $S^1$ has an immersion $\gamma_i: S^1 \to \Gamma_\infty(X_c)$ representing the conjugacy class of $\gamma_i$. Since $\gamma_i$ is indivisible, $\gamma_i$ is an element of $S(X_c)$. The range $S^1$ represents the $k_i$–th root of $\gamma_i$ and is called $r_i$. There is a map $\psi_{M_i}: M_i \to X_c$ which factors through some cylinder inclusion $\psi_{C\ast}$. This map gives $M_i$ the structure of a 2-covered graph of spaces.

First, note that since $\gamma_i$ is indivisible, $M_i$ embeds in the cylinder $C$. If $C$ was the union of more than one mapping cylinder, then some pair $\gamma_i$ and $\gamma_j$ would have to be conjugate, thus $C = M_i$ and our separable space $X_c$ is the union $\Gamma_\infty(X_c) \cup \gamma_i M_i$.

To complete the analysis of the cylinders, note that we must have $|F \cap \partial X_c(M_i)| = k_i$ for any transverse graph $F$, otherwise the immersion $\gamma_i$ must be a proper power. Thus every $F$ is a tree, and by Lemma 5.5, every cycle in $E$ is contained in some element of $F(E)$, $E$ is a tree.

Remark 5.6. We can now deduce Theorem 1.2 in the case that $sc(G) = sc(H) = (q - 1, 0)$. Since edge spaces of $X$ are trees, there is some element $\gamma_1$ which crosses an edge of $\Gamma(X)$ only once. Thus $G$ can be written as $F \ast \langle \gamma_1 \rangle$.

6. SPLITTING GRAPHS OF SPACES

Peripheral elements of $F(E)$ and boundary vertices play an essential role in finding moves which simplify graphs of spaces. It is not enough that a graph of spaces merely have splittable vertices. The notion that suffices since that of a splittable vertex does not is that of a splitting vertex. A splitting vertex has the property that one can collapse the “outgoing” edge adjacent to the vertex, and strategically fold two of the edges in the resulting graph of spaces, producing a new graph of spaces which still has a splitting vertex. Pushing and the treelike structure of edge spaces are used to produce splitting vertices.

**Definition 6.1 (Splitting Vertex).** A vertex $v$ of $\Gamma_U(X)$ is splitting if

- $v$ is splittable.
\begin{itemize}
  \item $V \cong E_1 \lor E_2$. The edge $e$ such that $E \cong V$ is the outgoing edge of $v$.
  \item If $w$ is the valence three vertex of $V$ and $e_1$ and $e_2$ are the other edges incident to $v$, for at least one of $E_1$ or $E_2$, say $E_1$, there is a peripheral element $F \in \mathcal{F}(E_1)$ with boundary vertex $w \in F$ such that $\tau_1(w)$ is the valence three vertex of $\iota(e)$.
\end{itemize}

The edges $e_1$ and $e_2$ are the incoming edges. Numbered as in the bullets, $e_1$ is the primary incoming edge.

The relationship between pushing, peripheral elements of $\mathcal{F}(E)$, and splitting edges of $\Gamma_U(X)$ is what allows us to take a separable graph of spaces and convert it to one with a bad cylinder. We first show that edge spaces have peripheral elements.

**Lemma 6.2.** If $X$ is an irreducible, separable graph of spaces, $\chi(\Gamma_U(X)) < 0$, all of whose cylinders are good, then $X$ has a splitting vertex.

**Proof.** Let $\pi$ be the quotient map $X \to \Gamma_U(X)$.

Note that by Lemma 5.5 every edge space $E$ contains a vertex $w \in \Gamma_\infty(X) \cap E$ such that $w$ is contained in exactly one member $F$ of $\mathcal{F}(E)$. Choose such an edge $e$ not contained in $\cup \Gamma_i$ and regard $F, w$, as subsets of $E \times \{1/2\}$.

Let $p : [0, a] \to \Gamma_U(X)$ be the shortest path such that $p(0) = \pi(F)$ and $P(w, a)$ is a valence three vertex $w'$ of $\Gamma_\infty(X)$ in the vertex space $V$. Let $e_0, \ldots, e_n$ be the sequence of edges that $p$ traverses. At integer values of $t$, $F_t = P(F, t) \subset E_t \times \{1/2\}$. By the construction of the cylinders, $F_t \in \mathcal{F}(E_t)$, and it is obvious that $F_t$ is a peripheral element of the associated edge space and $P(w, t)$ is a boundary vertex of $E_t$ for the appropriate $t$. Since $P(w, a)$ is the valence three vertex of $V$. By construction $v$ is splitting. \hfill \Box

Now we need to know how to proceed when a separable graph of graphs has a splitting vertex. There is a move, called splitting, which takes as input a separable graph of spaces which has a splitting edge and outputs a “simpler” graph of spaces which is either reducible or has a splitting edge and lower complexity.

**Definition 6.3 (Splitting).** Suppose $X$ is separable and has a splitting vertex $V$. A splitting of $X$ is a graph of spaces $X_s$ obtained as follows: $V$ is splittable, so we can express $V$ as a wedge $V = L \lor_w R$, with incident edge graphs homeomorphic to $L, R,$ or $V$.

Define $e = e(v)$ to be the edge of $\Gamma_U(X)$ such that $E \cong V$. Let $\overline{X}$ be the space obtained by collapsing $e$. Suppose $e_1$ and $e_2$ are the (oriented) edges other than $e$ incident to $v$. Let $v'$ be the other endpoint of $e$. Note that $v'$ is distinct from $v$ since $\mathcal{W}(V) > \mathcal{W}(L), \mathcal{W}(R)$. Let $e_3$ and $e_4$ be the (oriented) edges other than $e$ incident to $v'$. In the collapsed space, let $\overline{\tau_i}, i = 1, 2, 3, 4$, be the image of $e_i$. A splitting of $X$ is a nontrivial fold of $\overline{X}$ obtained by folding with $J = \{1, 3\}$ or $J = \{1, 4\}$.

A splittable vertex $v$ of $X$ determines an edge $e(v) \in \Gamma_U(X)$ with $\iota(e) = v \neq \tau(e) = v'$. Let $w$ and $w'$ be the valence three vertices of $V$ and $V'$, respectively, and let $\pi : X \to \overline{X}$ be the map which collapses the edge $e(v)$. 


Let $v_i$, $i = 1, \ldots, n$, be the vertices of $\Gamma_U(X)$. The relative weight of a vertex $v_i$ is the quantity
\[
\mathcal{W}(X \setminus v_i) = \sum_{j \neq i} \mathcal{W}(V_j)
\]

**Lemma 6.4** (Splitting Decreases Relative Weights). If $v$ is a splitting vertex of an irreducible, separable, graph of spaces $X$, then there is a collapse $X$, a fold $X_s$ of $X$, and if $X_s$ is irreducible, there is a splitting vertex $v_s$ of $X_s$ such that $\mathcal{W}(X \setminus v) > \mathcal{W}(X_s \setminus v_s)$.

If $X$ is separable, then there is a sequence of collapses and folds to a space with no splitting vertices.

**Proof.** Let $v = \tau(e)$, and let $g$ be the edge, not equal to $e$, such that $V \cong E \vee G$. Let $v'$ be the terminal endpoint of $f$, and let $h$ and $i$ be the two additional edges incident to $v' = \tau(f)$. Also, let $w$ be the separating vertex of $V$ and $w'$ be the separating vertex of $V'$. Since $\mathcal{W}(F) > \mathcal{W}(E)$, $\mathcal{W}(G)$, $f$ is embedded, thus we can collapse $f$ to obtain a space $X$ with vertex $\pi$, $V \cong V'$, and incident edge spaces $E, G, H,$ and $I$.

First, write $V$ as $A \vee B \vee C$ such that $H \cong A \vee B$ and $I \cong B \vee C$. Let $\pi: X \to \overline{X}$ be the quotient map. There are two cases to consider.

$\pi(w) = \pi(w')$: In this case, since $\pi(w)$ separates, and $\pi(E)$ has only one element $F \in \mathcal{F}(E)$ such that $\pi(F)$ meets $\pi(w)$, $\pi(E)$ is contained in, without loss, $A$. Folding $\pi$ and $\overline{h}$ together creates two new vertices, one of which is homeomorphic to $H$, is splitting, has an incident edge $e_s$ such that the pair $(e_s, h_s)$ is either splitting or such that $X_s$ has a weight 0 edge. In the event that $(e_s, h_s)$ is splitting, the other vertex has weight $\mathcal{W}(V') - \mathcal{W}(E)$, i.e., $\mathcal{W}(X_s \setminus \tau(e_s)) < \mathcal{W}(X \setminus \tau(e))$.

$\pi(w) \neq \pi(w')$: This case splits into two sub-cases. If $\pi(w') \subset \pi(G)$ then, without loss, $\pi(E) \subset \pi(H)$. Folding $\overline{h}$ and $\overline{f}$ together as in the previous case shows the lemma.

We’re left with the case $\pi(w') \subset \pi(E)$. Without loss, $\pi(G) \subset \pi(I)$. Folding $\overline{g}$ and $\overline{f}$ together creates a new splitting vertex with space isomorphic to $I$ and with incident edges isomorphic to $G$ and $B \cup (C \cap E)$. The vertex incident edge $g$ is splitting. This case is illustrated in the bottom row of Figure 19.

To see the second part of the lemma, suppose $X$ is irreducible and has a splitting pair. By the previous part of the lemma, we can split and fold to a space $X_s$ with a weight 0 edge. Let $Y_i$ be the irreducible components of $X_s$. Each component $Y_i$ is seen to be separable. Now induct on $\chi(\Gamma_U(Y_i))$. □

If an irreducible component of a graph of spaces has a bad cylinder then there is no guarantee the space can be further simplified. The next theorem shows that one can convert a graph of spaces to a “minimal” one, where minimal means that no sequence of collapses and folds ever leads to the creation of a weight 0 edge. Combining Lemmas 6.2 and 6.4 we have the following theorem.

**Theorem 6.5** (Splitting to bad cylinders). If $X$ is separable, there is a sequence of splittings to a space $X_0$ such that every irreducible component has a bad cylinder.
As a consequence of this and the analysis of edge spaces from the previous section, we can now establish the conjugacy separability result stated in the introduction.

**Proof of Corollary 1.5.** As before Theorem 2.5, represent the homomorphism $\tilde{\phi}: G \to \mathbb{F}_n$ as a homomorphism $X_c \to \Gamma_U(X_c)$. Since $\tilde{\phi}$ has maximal corank, $X_c$ is separable.

We prove the theorem by observing that the hypothesis that $\sim$ has no singleton equivalence classes implies that either all cylinders are good or the theorem holds. What are the cylinders of $X_c$? The maximal abelian subgroups $Z_i$ of $F$ can be represented as elements of $\mathbb{S}(X_c)$. The stable letters $t_j$ from $G$ give $\gamma_j^1 \in Z_i$ and

\[ \text{primary} \]

\[ \text{primary} \]

**Figure 16.** Illustration of case $\pi(w) \neq \pi(w')$ of Lemma 6.4.

**Figure 17.** Splitting when $\pi(w) = \pi(w')$. 
\[ \gamma_j \in \mathbb{Z}_i \]
and, for each \( j \), an annulus \( A_j \) glued between \( \mathbb{Z}_i \) and \( \mathbb{Z}_{i'} \) as elements of \( \mathbb{S}(X_c) \). Then the cylinders of \( X_c \) are represented precisely by the equivalence classes of \( \sim \) from the statement of the theorem. Since there are no singleton equivalence classes the boundary \( \partial X_c(C) \) of every cylinder \( C \) has more than one component. The key thing to notice is that an edge space \( E \) of \( C \) meets every component of \( \partial X_c(C) \) at least one time. Thus a cylinder is bad if and only if \( \partial^{\text{ess}} X_c(C) \) has only one component. A cylinder is illustrated in Figure 18.

By Theorem 6.5 we may replace \( X_s \) by a separable graph of spaces \( X_b \) whose irreducible components each contain a bad cylinder. Choose a bad cylinder \( C \) and a component \( Z_b \subset \partial C \setminus \partial^{\text{ess}} C \). All edges of \( \Gamma_\infty(X_b) \) which meet \( Z_b \) have weight 0. This collection of edges can be folded together to give \( F \) a free factorization \( Z_b * F' \) satisfying the theorem.

\[ \square \]

REFERENCES

[Bau65] Gilbert Baumslag, *Residual nilpotence and relations in free groups*, J. Algebra 2 (1965), 271–282. MR MR0179239 (31 #3487)

[FMR02] Benjamin Fine, Alexei Myasnikov, and Gerhard Rosenberger, *Malnormal subgroups of free groups*, Comm. Algebra 30 (2002), no. 9, 4155–4164. MR MR1936462 (2004c:20040)

[Lou08a] Larsen Louder, *Krull dimension for limit groups I: Bounding strict resolutions*, 2008, http://arXiv.org/abs/math/0702115v3

[Lou08b] Larsen Louder, *Krull dimension for limit groups II: aligning JSJ decompositions*, 2008, http://arXiv.org/abs/0805.1935v2

[Lou08c] Larsen Louder, *Krull dimension for limit groups IV: Adjoining roots*, 2008, http://arXiv.org/abs/math/NUMBER

[LS62] R. C. Lyndon and M. P. Schützenberger, *The equation \( a^M = b^N c^P \) in a free group*, Michigan Math. J. 9 (1962), 289–298. MR MR0162838 (32 #5723)

[Sco73] G. P. Scott, *Finitely generated 3-manifold groups are finitely presented*, J. London Math. Soc. (2) 6 (1973), 437–440. MR MR0380763 (52 #1660)

[She55] Abe Shenitzer, *Decomposition of a group with a single defining relation into a free product*, Proc. Amer. Math. Soc. 6 (1955), 273–279. MR MR0069174 (16,995c)

[Sta65] John R. Stallings, *A topological proof of Grushko’s theorem on free products*, Math. Z. 90 (1965), 1–8. MR MR0188284 (32 #5723)

[Sta83] , *Topology of finite graphs*, Invent. Math. 71 (1983), no. 3, 551–565. MR MR695906 (85m:05037a)

[Swa04] G. A. Swarup, *Delzant’s variation on Scott complexity*, January 2004.
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1043, USA

E-mail address, Larsen Louder: llouder@umich.edu, lars@d503.net