Properties of Generalized Bessel Functions

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Abstract

The generalized Bessel function (GBF) extends the single variable Bessel function to several dimensions in a nontrivial manner. Two-dimensional GBFs have been studied extensively in the literature and have found application in laser physics, crystallography, and electromagnetics. In this article, we document several properties of $n$-dimensional GBFs including an underlying partial differential equation structure, asymptotics for simultaneously large order and argument, and analysis of generalized Neumann, Kapteyn, and Schlömilch series. We extend these results to mixed-type GBFs where appropriate.

1 Introduction

Bessel functions [1] are pervasive in mathematics and physics and are particularly important in the context of wave propagation. Bessel functions were first studied in the context of solutions to a second order differential equation known as Bessel’s equation:

$$x^2 f''(x) + xf'(x) + (x^2 - n^2)f(x) = 0$$ (1)

Solutions to this equation are known as Bessel functions of order $n$. Since equation (1) is a second order linear differential equation, there exist two linearly independent solutions. These solutions, denoted $J_n(x)$ and $Y_n(x)$, are referred to as Bessel functions of the first and second kind respectively, where $J_n(0)$ is finite and $Y_n(x)$ has a singularity at $x = 0$. These functions commonly arise in physical problems involving cylindrical equations including the Laplace, Helmholtz, and Schrödinger equations [2].

For integer order $n$, Bessel functions of the first kind admit an integral representation:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta - x \sin \theta} d\theta$$ (2)

Recognizing equation (2) as an inverse Fourier transform, we recover the following equation:

$$e^{-ix \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(x)e^{-in\theta}$$ (3)

In a signal processing context, the sinusoidal frequency modulated signal with modulation amplitude $x$ is represented by a complex Fourier series with Bessel function coefficients of varying order $n$ evaluated at $x$. Equation (3) is referred to as a Jacobi-Anger expansion [3].

Now consider a signal whose frequency is modulated by a Fourier sine series [4], represented as:

$$s(\theta) = \exp \left[ -i \sum_{k=1}^{\infty} x_k \sin k\theta \right]$$ (4)
Figure 1: One-dimensional Bessel functions of the first kind. These functions are odd or even depending on the parity of the order \( n \).

By leveraging a more general form of the Jacobi-Anger expansion, we may write the spectral decomposition of \( s(\theta) \) as

\[
s(\theta) = \sum_{n=-\infty}^{\infty} J^p_n(x) e^{-in\theta}
\]

where \( x : \{1, \ldots, m\} \to \mathbb{R} \), \( p : \{1, \ldots, m\} \to \mathbb{N} \) with \( p = \{p_1, \ldots, p_m\} \), and

\[
J^p_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left[ i \left( n\theta - \sum_{k=1}^{\infty} x_k \sin p_k \theta \right) \right] d\theta.
\]

We call \( J^p_n(x) \) an \( m \)-dimensional Generalized Bessel function. We say that \( p \) is the index of the GBF, and we require that \( \gcd(p) = 1 \) to avoid trivial simplifications. If any of the arguments in \( x \) are set to zero, then this reduces to a lower dimensional GBF. We can extend this definition to the mixed-type generalized Bessel functions (MGBF) such that the modulation function also includes cosines [5]:

\[
J_n(x; y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left[ i \left( n\theta - \sum_{k=1}^{\infty} (x_k \sin k\theta + y_k \cos k\theta) \right) \right] d\theta.
\]
An infinite-dimensional variant of the MGBF (and the GBF) can be constructed by requiring \( x, y \in \ell^2(\mathbb{N}) \) such that the integral converges.

![Plots of \( |J_n^{1,2}(x, y)| \) for \(-20 < x, y < 20\).](image)

Figure 2: Plots of \( |J_n^{1,2}(x, y)| \) for \(-20 < x, y < 20\). (Top Left) \( n = 0 \) (Top Right) \( n = 1 \) (Bottom Left) \( n = 2 \) (Bottom Right) \( n = 3 \). Musch like their one-dimensional counterparts, the GBFs are oscillatory in both \( x \) and \( y \) dimensions. They additionally display symmetries over the \( y \) axis.

Generalized Bessel functions of dimension \( m = 2 \) first appeared in the literature in the early 1900s as a simple but isolated extension of the Bessel functions [6]. These functions were re-examined and further generalized in the 1980s as more of an effort was made to connect the GBF to other notable special functions [7]. However, most analysis of GBFs is limited to the two dimensional case, and for a complete asymptotic development, even further limited to the case of \((p_1, p_2) = (1, 2)\) [8]. Using techniques from algebraic geometry [9], we describe a general method to analytically describe bifurcation surfaces for both GBFs with large argument, and GBFs with simultaneously large order and argument. These bifurcation surfaces are revealed to be algebraic.

While the Bessel function was defined as a solution to a differential equation, it is more difficult to discern whether the GBF satisfies a similar partial differential equation. Previous efforts [5] have showed that the \( J_n^{1,m} \) GBF satisfies an order \( m^2 \) linear PDE which does not bear any passing similarity to Bessel’s
differential equation. It was determined in [10] that the GBF does not solve a second order linear PDE (an incomplete conclusion), but certain modifications do; in particular the MBGF satisfies a first order linear Schrödinger type equation. However, this PDE only describes the simple cross sectional structure between variables $x_k$ and $y_k$ and ignores the more intricate interplay between the $x$ variables. In this paper, we demonstrate that it is possible to write a second order linear PDE for $J_{n}^{1,2}(x, y)$.

We conclude this paper with a discussion of various summations involving MGBFs. In [1], Neumann, Kapteyn, and Schlömilch series are developed for one-dimensional Bessel functions. We consider generalizations of these series where the MGBF replaces the Bessel function. It is possible to write closed forms for the moments of the Neumann and Kapteyn series, while for the Schlömilch series we can compute smoothness boundaries. Both the region of convergence for the Kapteyn series and the smoothness boundaries of the Schlömilch series are connected in an algebraic sense to the region of polynomial decay for MGBFs of increasing order and argument.

The rest of the paper is organized as follows; in section 2 we write linearly independent PDE representations for $J_{n}^{1,2}(x, y)$ and note a possible application to computing level sets. In section 3 we discuss asymptotics of the GBF and provide closed form results. We then develop closed forms for generalized Neumann and Kapteyn series in section 4, and also describe smoothness boundaries for the generalized Schlömilch series. We additionally describe extensions to the MGBF but often omit the equations as they are large and unwieldy.

2 Partial Differential Equations Representation

In this section, we demonstrate that the index $(1,2)$ GBF $J_{n}^{1,2}(x, y)$ solves a set of linearly independent partial differential equations. These equations can be derived from a combination of trigonometric identities and an integral identity. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function with period $P$, then

$$\int_{x_0}^{x_0 + P} f'(x) dx = 0$$

Let $h(\theta) = n\theta - x \sin \theta - y \sin 2\theta$, $f(x, y) = 2\pi J_{n}^{1,2}(x, y) = \int_{-\pi}^{\pi} e^{ih(\theta)} d\theta$, and $g(x, y) = \int_{-\pi}^{\pi} \cos \theta e^{ih(\theta)} d\theta$. To take derivatives of these expressions, we can interchange the derivative and integral operators; computing these derivatives involve integrating products of trigonometric functions and $e^{ih(\theta)}$. This allows us to exploit trigonometric identities to relate the derivatives of $f$ and $g$; for instance by $\sin 2\theta = 2\sin \theta \cos \theta$ we have $f_y = 2g_x$.

We apply equation (8) to $e^{ih(\theta)}$, $(\sin \theta)e^{ih(\theta)}$, and $(\cos \theta)e^{ih(\theta)}$ to generate the partial differential equations. The first identity becomes

$$0 = nf - xg - 2y \int_{-\pi}^{\pi} (\cos 2\theta)e^{ih(\theta)} d\theta$$

Using the identity $\cos 2\theta = 1 - 2\sin^2 \theta$, we have

$$0 = (n - 2y)f - xg - 4yf_{xx}$$

This allows us to represent $g$ in terms of $f$ and $f_{xx}$.

Applying equation (8) to $(\sin \theta)e^{ih(\theta)}$ leads to the following formula:

$$0 = g - nf_x + \frac{1}{2}xf_y - 2iy \int_{-\pi}^{\pi} (\sin \theta \cos 2\theta)e^{ih(\theta)} d\theta$$

Using the identity $\sin \theta \cos 2\theta = \sin 2\theta \cos \theta - \sin \theta$, we can write:

$$0 = g - (n + 2y)f_x + \frac{1}{2}xf_y + 2yy_y$$
By taking a derivative of this equation with respect to $x$ and using the identities $f_y = 2g_x$ and $f_{yy} = 2g_{xy}$, we can write the first partial differential equation:

$$(n + 2y)f_{xx} - yf_{yy} - \frac{x}{2}f_{xy} - f_y = 0$$  \hspace{1cm} (13)

We can derive the second partial differential equation by applying equation (8) to $(\cos \theta)e^{i\theta(\theta)}$. This leads us to the following formula:

$$0 = f_x - ng + x \int_{-\pi}^{\pi} (\cos^2 \theta) e^{i\theta(\theta)} d\theta + 2y \int_{-\pi}^{\pi} (\cos \theta \cos 2\theta) e^{i\theta(\theta)} d\theta$$  \hspace{1cm} (14)

By using trigonometric substitutions, we can write these integrals in terms of the derivatives of $f$ and $g$:

$$0 = xf_{xx} + f_x + xf - (n - 2)y + 2yf_{xy}$$  \hspace{1cm} (15)

Using the previous substitution for $g$ in equation (10), we have:

$$0 = [x^2 + 4y(n - 2)y]f_{xx} + 2xyf_{xy} + xf_x + [x^2 - (n - 2)y^2]g$$  \hspace{1cm} (16)

There are several promising similarities between equation (16) and Bessel’s differential equation in equation (1), particularly that the correct degree polynomial in $(x, y)$ multiplies the corresponding ordered derivative, and the dependence on $n^2$ in the coefficient of $f$. Moreover, substituting $y = 0$ returns Bessel’s differential equation.

We can use a similar procedure to find linearly independent PDEs which govern MGBFs of arbitrary dimension. Note that by $\sin^2 \theta + \cos^2 \theta = 1$, we have

$$\frac{\partial^2 J_n}{\partial x_k^2} + \frac{\partial^2 J_n}{\partial y_k^2} + J_n = 0$$  \hspace{1cm} (17)

This shows that every $(x_k, y_k)$ plane has circular level sets when all other variables are fixed. It was shown in [10] and CITE that the MGBF also satisfies a Schrodinger-type PDE:

$$nJ_n = i \sum_{k=1}^{m} k \left( x_k \frac{\partial J_n}{\partial y_k} - y_k \frac{\partial J_n}{\partial x_k} \right)$$  \hspace{1cm} (18)

A complete list of linearly independent second order PDE identities for the $J_n^{1,2}$ MGBF could be compiled using equation (8), however since there are four input variables we omit this list for brevity.

### 2.1 Level Sets

If we could find another second order linear PDE which $J_n^{1,2}(x, y)$ solves, then it would be possible to parameterize its vanishing level sets using a second order variant of the method of characteristics [11]. Suppose we have smooth functions $(x(t), y(t))$ such that $f(x(t), y(t)) = J_n^{1,2}(x(t), y(t)) = 0$. Then $x'(t)f_x + y'(t)f_y = 0$, and taking a second derivative of this equation gives

$$(x')^2f_{xx} + 2x'y'f_{xy} + (y')^2f_{yy} + x''f_x + y''f_y = 0$$  \hspace{1cm} (19)

This allows us to write the following matrix representation of the system:

$$\begin{bmatrix}
2(n + 2y) & -x & -2y & 0 & -2 \\
x^2 + 4y(n - 2)y & 2xy & 0 & x & 0 \\
(x')^2 & 2x'y' & (y')^2 & x'' & y'' \\
0 & 0 & 0 & x' & y'
\end{bmatrix}
\begin{bmatrix}
f_{xx} \\
f_{xy} \\
f_{yy} \\
f_x \\
f_y
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \\
0 \\
0 \end{bmatrix}$$  \hspace{1cm} (20)
We could then write a system of second order nonlinear ordinary differential equations with initial conditions \((x(0), y(0), x'(0), y'(0)) = (j_{n,k}, 0, 0, 1)\) or \((x(0), y(0), x'(0), y'(0)) = (0, j_{n/2,k}, 1, 0)\) for even \(n\) where \(j_{n,k}\) is the \(k\)-th root of the one-dimensional Bessel function \(J_n(x)\).

It may be possible to determine the topology of the level sets of \(J_n^2(x, y)\) by analyzing this ODE system. Note in figure 2 that for even \(n\), the zero surfaces intersecting the \(y\)-axis appear to be closed loops, while for \(n\) odd there appears to be a single infinite contour winding about the \(y\)-axis.

## 3 Asymptotic Properties

In this section, we characterize the bifurcation surfaces of the GBF using the method of stationary phase [12]. Consider the integral:

\[
I(t) = \int_a^b g(\theta) e^{itf(\theta)} d\theta
\]  

(21)

where \(g\) and \(f\) are smooth functions compactly supported in the interval. Consider the set \(S = \{x \in (a, b) : f'(x) = 0, f''(x) \neq 0\}\) whose members we refer to as points of stationary phase. If this set is nonempty, then we can make an approximation on \(I(t)\) for large \(t\):

\[
I(t) = \sum_{x \in S} g(x) e^{itf(x)} \sqrt{\frac{\pi}{t|f''(x)|}}(1 \pm i) + O(t^{-1})
\]  

(22)

Here, the phase of the summand is determined by the sign of \(f''(x)\). If the set of stationary phase points is empty, then \(I(t)\) decays superpolynomially, and under minor conditions, exponentially. Some care must be taken that there are no stationary phase points at the endpoints of the integral, and that the stationary phase points do not give the phase function a vanishing second derivative. The latter type of stationary phase points are referred to as critical points, and crossing over these points changes the structure of the approximation (i.e. changing the number of stationary phase points in the region of integration).

Compare the integral in equation (21) with our definition of the GBF in equation (6). For an \(m\)-dimensional GBF with fixed indices, there are \(m + 1\) terms which can vary (the order and \(m\) input arguments), any combination of which we can choose to be large and apply a stationary phase approximation to. In this section we will consider two cases; we first let the arguments be large relative to the order, and then we let both the arguments and the order be large. In practice, when the indices are arbitrary integers, these approximations can be difficult to analytically display as they involve solving high order polynomials. However, we can analytically find the locus of critical points which trace out bifurcation surfaces in \(m\)-dimensional space. For relatively small order these bifurcation surfaces become two linear \(m - 1\) dimensional surfaces, but for large order the bifurcation surfaces include an additional algebraic surface. The collection of bifurcation surfaces of the MGBF generally contains only higher order algebraic surfaces.

These more complex algebraic surfaces satisfy systems of polynomial equations which can be solved using the Sylvester matrix determinant. If we have a system of polynomial equations \(f(x) = 0\) and \(g(x) = 0\) with coefficients in the set \(\{a_1, ..., a_n\}\) and which have no common factors, then there exists a nontrivial multivariate polynomial called the resultant which satisfies \(\text{Res}(f, g; x) = F(a_1, ..., a_n) = 0\) [9]. If \(f(x) = \sum_{k=0}^n a_k x^k\) and \(g(x) = \sum_{k=0}^n b_k x^k\), the resultant may be written as the determinant of the Sylvester matrix:

\[
\text{Res}(f, g; x) = \det \begin{bmatrix}
a_n & a_{n-1} & \cdots & a_0 & 0 \\
b_n & b_{n-1} & \cdots & b_0 & 0 & \ddots \\
0 & a_n & \cdots & a_1 & a_0 & \ddots \\
0 & b_n & \cdots & b_1 & b_0 & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & & b_n & \cdots & b_0 \\
\end{bmatrix}
\]  

(23)
Suppose we have a system of \(m+1\) polynomial equations in \(m\) variables such that \(f_{1,k}(x_1,\ldots,x_m) = 0\) for \(k \leq m + 1\). Then we can generate a system of \(m\) polynomial equations \(f_{2,k}(x_1,\ldots,x_{m-1}) = \text{Res}(f_{1,k}, f_{1,m+1}; x_m)\) and continue iteratively until there is one polynomial with none of the \(\{x_k\}\) variables. This polynomial is the resultant of the entire system.

One caveat of this method is that the equation \(\text{Res}(f,g) = F(a_1,\ldots,a_n) = 0\) will include solutions \(\{a_1,\ldots,a_n\}\) which do not simultaneously satisfy \(f(x) = 0\) and \(g(x) = 0\) for some \(x\), whereas if the system is satisfied for some \(x\), the coefficients must solve \(F(a_1,\ldots,a_n) = 0\). For example, consider the system \(f(x) = ax + b + 1 = 0\) and \(g(x) = cx + d = 0\). For this system to be simultaneously satisfied for some \(x\), we must have \(F(a,b,c,d) = ad - bc - c = 0\). The trivial solution \(\{a,b,c,d\} = \{0,0,0,0\}\) satisfies this equation but clearly does not satisfy the system for any \(x\). Moreover, it will often occur that we would like for the simultaneous solution \(x\) to be real. Only a portion of the bifurcation curve will correspond to this situation, and this information is not encoded in the resultant. We will examine these cases as they arise.

We note that while we can solve the bifurcation surfaces analytically, these do not give a complete asymptotic description of the GBF. This would require us to solve for the points of stationary phase, which satisfy a polynomial of degree \(\max \mathbf{p}\) in \(\cos \theta\), which is typically not possible analytically.

### 3.1 Large Arguments

We first consider bifurcation curves of the GBF with large arguments relative to the order, as first elucidated by Korsch et. al. \([8]\). Otherwise stated, let us consider the function \(J^p_n(tx_1,\ldots,tx_m)\) for large values of \(t\). We write this function in a stationary phase form

\[
J^p_n(tx_1,\ldots,tx_m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} \exp \left[ -it \sum_{k=1}^{m} x_k \sin p_k \theta \right] d\theta
\]

(24)

where \(g(\theta) = e^{in\theta}\) and \(f(\theta) = -\sum_{k=1}^{m} x_k \sin p_k \theta\). If \((x_1,\ldots,x_m)\) is an element of the bifurcation curve, there must exist a \(\theta \in [-\pi, \pi]\) such that \(f'(\theta) = f''(\theta) = 0\). Otherwise,

\[
f'(\theta) = \sum_{k=1}^{m} x_k p_k \cos p_k \theta = 0, \quad f''(\theta) = \sum_{k=1}^{m} x_k p_k^2 \sin p_k \theta = 0.
\]

(25)

We note that these functions can be written as polynomials in \(\cos \theta\) and \(\sin \theta\). In particular, the Chebyshev polynomials of the first and second kinds respectively satisfy the identities

\[
T_n(\cos \theta) = \cos n \theta, \quad U_{n-1}(\cos \theta) = \frac{\sin n \theta}{\sin \theta}
\]

(26)

Using these identities, we can rewrite the system of equations (25):

\[
\sum_{k=1}^{m} x_k p_k T_{p_k}(\cos \theta) = 0, \quad \sin \theta \left[ \sum_{k=1}^{m} x_k p_k^2 U_{p_k-1}(\cos \theta) \right] = 0
\]

(27)

The right equation in (27) is trivially satisfied if \(\sin \theta = 0\), or when \(\theta = 0\) or \(\theta = \pi\). These two solutions lead to the following representations of bifurcation surfaces when plugged back into the left equation of (27):

\[
\sum_{k=1}^{m} x_k p_k = 0, \quad \sum_{k=1}^{m} x_k p_k (-1)^p_k = 0
\]

(28)

Notice that if all of the \(p_k\) are odd (they cannot all be even or else their greatest common denominator would be at least 2), then both equations of (27) denote the same surface. In this case, letting \(\theta = \frac{\pi}{2}\) or
\[ \theta = \frac{3\pi}{2} \] trivially satisfies the left equation of (27). Substituting these values into the right equation of (27) leads to the following alternate surface:

\[ \sum_{k=1}^{m} x_k p_k^2 (-1)^{p_k+1} = 0 \] (29)

We illustrate this dichotomy in behavior in Figure 4; notice that the bifurcation curves of \( J_{1.2}^n(x, y) \) and \( J_{2.3}^n(x, y) \) are symmetric about the coordinate axes, but the bifurcation curves of \( J_{1.3}^n(x, y) \) and \( J_{3.5}^n(x, y) \) have only rotational symmetry about the origin.

For the MGBF, there is no factor of \( \sin \theta \) in \( f''(\theta) \), so the bifurcation surfaces will be higher order algebraic surfaces. The bifurcation surface of the \( J_{1.2}^n(tx_1, tx_2; ty_1, ty_2) \) MGBF is an eighth order algebraic surface whose equation is too large for the page.

### 3.2 Large Order and Arguments

We now compute bifurcation curves of the GBF with simultaneously large order and large arguments, otherwise \( J_{tn}^p(x) \) for large values of \( t \). In these cases, there is a region of exponential decay containing the origin which grows linearly with \( n \). The bifurcation surfaces will bound this region.

In the stationary phase representation, we have \( f(\theta) = n\theta - \sum_{k=1}^{m} x_k \sin p_k \theta \) such that the bifurcation curves now satisfy:

\[ f'(\theta) = \sum_{k=1}^{m} x_k p_k \cos p_k \theta = n, \quad f''(\theta) = \sum_{k=1}^{m} x_k p_k^2 \sin p_k \theta = 0 \] (30)

As stated previously, we will be able to write a multivariate polynomial equation in \( \{x_k\} \), a subset of whose solutions satisfy equations (25) and (26) for some \( \theta \). By factoring out \( \sin \theta \) from the second equation, we are able to derive two linear bifurcation surfaces:

\[ \sum_{k=1}^{m} x_k p_k = n, \quad \sum_{k=1}^{m} x_k p_k (-1)^{p_k} = n \] (31)

We refer to these solutions as the trivial solutions. Notice that unlike the small order case of the previous section, these two surfaces are distinct even if \( p \) contains only odd indices, in which case they are parallel. If \( p = \{p_e, p_o\} \) corresponding to the even and odd indices respectively, then the intersection of the trivial surfaces can be represented as the intersection of two orthogonal surfaces:

\[ \sum_{k \in p_e} x_k p_k = 0, \quad \sum_{k \in p_o} x_k p_k = n \] (32)

The nontrivial bifurcation surfaces can be computed using the resultant as described at the beginning of the section. For instance, the nontrivial bifurcation curve of \( J_{tn}^{1.2}(tx, ty) \) becomes

\[ x^2 + 32y^2 + 16yn = 0 \] (33)

We plot the bifurcation curves of this GBF in figure 5. Notice here that the upper section of the ellipse as predicted by equation (33) does not actually behave as a bifurcation curve, i.e. it subdivides a coherent region of exponential decay. We remedy this discrepancy by noting that the simultaneous solutions of equation (30) must satisfy \( |\cos \theta| \leq 1 \), otherwise \( \theta \) is not in the region of integration. In this example, this implies that we must have \( \left| \frac{x}{8y} \right| \leq 1 \), or equivalently the section of the ellipse which lies above the ine \( y = -\frac{n}{6} \) is not truly part of the bifurcation curve.
Since previous asymptotic analysis of GBFs \[8\] \[13\] was dependent on explicitly solving for stationary phase points, those authors could only consider the $J_{1n}^2(tx,ty)$ GBF because the stationary phase points satisfy a quadratic function in $\cos \theta$. However, the resultant allows us to solve for bifurcation curves of MGBFs of arbitrary order. For example, the nontrivial bifurcation curve of $J_{1n}^{1,3}(tx,ty)$ becomes

$$(x - 9y)^3 + 81yn^2 = 0$$

(34)

This representation overstates the true bifurcation curve, and we can show that only the section of this curve which satisfies $\frac{1}{4} - \frac{x}{30y} \leq 1$ acts as a legitimate bifurcation curve, as shown in Figure 6.

We could additionally compute bifurcation surfaces of the MGBF in this way, but the equations would be far too large to display concisely.

4 MGBF Series

In this section, we compute various sums involving the MGBF as documented by Watson \[1\] for the one-dimensional Bessel function. The techniques used to compute these sums are not unique to the dimension of the Bessel function and are easily extended to the MGBF. For the generalized Neumann and Kapteyn series, we use generating functions to compute $k^{th}$ moments. These moments could then be used to approximate series involving arbitrary coefficients. The moments of the Schlömilch series are not so easily calculated even in the one-dimensional case. However, a distinguishing feature of this series in one dimension is that it is only piecewise smooth; the boundaries of these pieces occur at $x = 2\pi n$ for $n \in \mathbb{Z}$. We document a similar property in the GBF case and compute the smoothness boundaries using the resultant of the previous section.

4.1 Generalized Neumann Series

We first consider moments of the generalized Neumann series which we write as

$$\mu_k = \sum_{n=-\infty}^{\infty} n^k J_n(x;y)$$

(35)

Computing these sums becomes nearly trivial when we remember the Jacobi-Anger expansion of equation (4) and recognize that $\mu_k = i^k s(k)(0)$. If we let $X_k = \sum_{n=1}^{\infty} n^k x_n$ be the $k^{th}$ moment of $x$ and likewise for $Y_k$, then we can simply represent the first few moments as:

$$\mu_0 = e^{-iy_0}, \quad \mu_1 = X_1 e^{-iy_0}, \quad \mu_2 = (X_1^2 - iY_2)e^{-iy_0}, \quad \mu_3 = (X_3 - 3iX_1Y_2 + X_1^3)e^{-iy_0}$$

(36)

4.2 Generalized Kapteyn Series

We now compute moments of the generalized Kapteyn series, where the summation index is included both in the order and argument:

$$\mu_k = \sum_{n=-\infty}^{\infty} n^k J_n(nx;ny)$$

(37)

To compute these moments, we develop a generating function from a variant of the multi-dimensional feedback equation \[14\] \[15\]. Let $f(\theta) = \theta - \sum_{k=1}^{\infty} (x_k \sin k\theta + y_k \cos k\theta)$ and suppose we wish to invert this function, i.e. represent $\theta = f^{-1}(t)$ as a function of $t$. Notice that this inversion is only valid on the set $\Omega$ of coordinates $\{x, y\}$ on which $f(\theta)$ is monotone increasing. It can be shown that $\Omega$ corresponds to the region of exponential decay of the previous section with $n = 1$. This is intuitive as for $\{x, y\}$ outside this region, there will be stationary phase points in the integral representation such that $J_n(nx;ny) = O(n^{-1/2})$, therefore the series will not converge.
If we assume \( \{x, y\} \in \Omega \), then both \( f \) and \( f^{-1} \) are monotone increasing, continuous, bijective, and invertible. Since \( f(\theta + 2\pi) = f(\theta) + 2\pi \), we have

\[
f(f^{-1}(t + 2\pi)) = t + 2\pi = f(f^{-1}(t) + 2\pi)
\]

This implies that \( g(t) = f^{-1}(t) - t \) is 2\( \pi \) periodic such that \( g \) admits a Fourier expansion \( g(t) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} \) and the coefficients satisfy \( a_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-im\theta} dt \). By integration by parts, a change of variables to \( \theta \), and recognizing that \( \int_{-\pi}^{\pi} f'(\theta) e^{-imf(\theta)} d\theta = 0 \), we can write these coefficients as MGBFs:

\[
a_{-m} = -\frac{1}{2\pi mi} \int_{-\pi}^{\pi} e^{imf(\theta)} d\theta = -\frac{i}{m} J_m(mx; my)
\]

If we define \( f(\theta) = \theta - h(\theta) \), then we can write a generating function for the generalized Kapteyn series:

\[
h(\theta) = -i \sum_{m \neq 0} \frac{1}{m} J_m(mx; my) e^{im(\theta - h(\theta))}
\]

To solve for moments of the Kapteyn series, we must use the quantity \( \theta_0 \) which satisfies \( f(\theta_0) = 0 \). Since \( f \) is bijective, \( \theta_0 \) exists and is unique. Furthermore, if \( y = 0 \), then \( \theta_0 = 0 \), \( h^{(2n)}(\theta_0) = 0 \), and \( h^{(2n+1)}(\theta_0) = \sum k^{2n+1} x_k \). The moments arise by taking derivatives of both sides of equation (40) with respect to \( \theta \) and evaluating at \( \theta_0 \). We list the first few moments here:

\[
\mu_0 = \frac{h'('0)}{1 - h'('0)}, \quad \mu_1 = -\frac{i h''('0)}{(1 - h'('0))^3}, \quad \mu_2 = -\frac{h'''('0)}{(1 - h'('0))^3} - \frac{3h''('0)^2}{(1 - h'('0))^4}
\]

Versions of these equations for the GBF appear in Dattoli et. al. [14].

### 4.3 Generalized Schlömilch Series

We conclude this section with a discussion of generalized Schlömilch series which we write as

\[
\mu_k = \sum_{n=1}^{\infty} n^{-k} J_m(nx; ny)
\]

where \( m \) is fixed and \( k \) is a positive integer. In Watson [1], only special cases of the one-dimensional Schlömilch series admit a simple algebraic representation. However, it is noted that these summations are not smooth in neighborhoods about \( x = 2\pi n \) for \( n \in \mathbb{Z} \). We attempt to recover a similar result in the higher dimensional MGBF case.

For \( k \geq 2 \), we can interchange the sum and integral to write

\[
\mu_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{im\theta} \left[ \sum_{n=1}^{\infty} \frac{e^{-inh(\theta)}}{n^k} \right] d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{im\theta} \text{Li}_k(e^{-ih(\theta)})
\]

where \( \text{Li}_k(z) \) is the polylogarithm function [16]. The polylogarithm is valid for \( |z| < 1 \) but can be extended to an analytic function in \( \mathbb{C} \setminus [1, \infty) \). As such, the integrand of equation (43) passes through the branch point at \( z = 1 \) if there exist \( \theta \in [-\pi, \pi] \) such that \( h(\theta) = 2\pi n \) for some \( n \in \mathbb{Z} \). This creates a collection of surfaces in \( (x, y) \) space which we term smoothness boundaries; \( \mu_k \) is not smooth in neighborhoods of the smoothness boundaries and crossing over one of these boundaries changes the number of times the integrand passes through the branch point. Therefore, we index the smoothness boundaries by the simultaneous equations

\[
h(\theta) = 2\pi n, \quad h'(\theta) = 0
\]

Note that this closely relates to the discussion of bifurcation curves in the previous section. We can use similar methods to compute these smoothness boundaries. For \( J_n^{1,2}(x, y) \), these boundaries become:

\[
64(2\pi n)^4 y^2 + (2\pi n)^2 (x^4 - 80 x^2 y^2 - 128 y^4) + (64 y^6 - 48 x^2 y^4 + 12 x^4 y^2 - x^6) = 0
\]

We display these surfaces in Figure 7 along with the \( J_n^{1,3}(x, y) \) smoothness boundaries located at \( (2\pi n)^2 y - (x^3 + 9 x^2 y + 27 x y^2 + 27 y^3) = 0 \).
5 Conclusion

In this paper we have documented several novel properties of the generalized Bessel function. The properties listed here, while bearing similarity to the analogous results on one-dimensional Bessel functions, have a distinct algebraic geometry interpretation. It is the hope of the authors that the results in each of these sections can be improved upon with further research. Though it is likely that $J_{1,2}^n(x,y)$ does not solve another linearly independent partial differential equation, it may be possible to solve for the topology of its level sets via the ordinary differential equation system generated from equation (20). It would also be worthwhile to pursue more precise asymptotics for higher dimension GBFs beyond the bifurcation surface calculations presented here.

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Figure 3: Plots of $|J_n^{p,q}(x, y)|$ for $-20 < x, y < 20$. (Top Left) $n = 0, p = 1, q = 3$ (Top Right) $n = 1, p = 4, q = 5$ (Bottom Left) $n = 2, p = 2, q = 3$ (Bottom Right) $n = 3, p = 4, q = 7$. These GBFs with their more general indices $p$ and $q$ still possess the same oscillatory properties as their $J_n^{1,2}$ counterparts, with more complex symmetries.
Figure 4: Plots of $|J_0^{p,q}(x,y)|$ for $-20 < x, y < 20$. (Top Left) $p = 1, q = 2$ (Top Right) $p = 2, q = 3$ (Bottom Left) $p = 1, q = 3$ (Bottom Right) $p = 3, q = 5$. Notice that the top two figures, each containing an even index, have bifurcation curves which are symmetric about both coordinate axes. Meanwhile, the bottom two figures, both with odd indices, contain bifurcation curves with only rotational symmetry.
Figure 5: Plots of $|J_{40}^{1,2}(x, y)|$ for $-100 < x, y < 100$ (Left) Density plot (Right) Bifurcation curves. Notice that crossing the upper part of the ellipse does not lead to a difference in asymptotic behavior.

Figure 6: Plots of $|J_{40}^{1,3}(x, y)|$ for $-100 < x, y < 100$ (Left) Density plot (Right) Bifurcation curves. Notice that crossing the center part of the cubic curve does not lead to a change in asymptotic behavior.
Figure 7: (Left) Smoothness boundaries of the $J^{1,2}_n(x, y)$ Schlömilch Series (Right) Smoothness boundaries of the $J^{1,3}_n(x, y)$ Schlömilch series. Moments of the Schlömilch series are continuous in the domain but not smooth; they are smooth in regions bounded by these curves, however.