Local Approximation Schemes for Topology Control

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ABSTRACT

This paper presents a distributed algorithm on wireless ad-hoc networks that runs in polylogarithmic number of rounds in the size of the network and constructs a linear size, lightweight, \((1 + \varepsilon)\)-spanner for any given \(\varepsilon > 0\). A wireless network is modeled by a \(d\)-dimensional \(\alpha\)-quasi unit ball graph (\(\alpha\)-UBG), which is a higher dimensional generalization of the standard unit disk graph (UDG) model. The \(d\)-dimensional \(\alpha\)-UBG model goes beyond the unrealistic “flat world” assumption of UDGs and also takes into account transmission errors, fading signal strength, and physical obstructions. The main result in the paper is this: for any fixed \(\varepsilon > 0\), \(0 < \alpha \leq 1\), and \(d \geq 2\) there is a distributed algorithm running in \(O(\log n \cdot \log^2 n)\) communication rounds on an \(n\)-node, \(d\)-dimensional \(\alpha\)-UBG \(G\) that computes a \((1 + \varepsilon)\)-spanner \(G'\) of \(G\) with maximum degree \(\Delta(G') = O(1)\) and total weight \(w(G') = O(w(MST(G)))\). This result is motivated by the topology control problem in wireless ad-hoc networks and improves on existing topology control algorithms along several dimensions. The technical contributions of the paper include a new, sequential, greedy algorithm with relaxed edge ordering and lazy updating, and clustering techniques for filtering out unnecessary edges.

Categories and Subject Descriptors: C.2.4 [Computer-Communication Networks]: Distributed Systems  
General Terms: Algorithms, Performance, Theory.  
Keywords: Spanners, Topology control, Wireless ad-hoc networks, Unit ball graphs.

1. INTRODUCTION

Let \(G = (V, E)\) be a graph with edge weights \(w : E \to \mathbb{R}^+\). For \(t \geq 1\), a \(t\)-spanner of \(G\) is a spanning subgraph \(G'\) of \(G\) such that for all pairs of vertices \(u, v \in V\), the length of a shortest \(uv\)-path in \(G'\) is at most \(t\) times the length of a shortest \(uv\)-path in \(G\). The problem of constructing a sparse \(t\)-spanner, for small \(t\), of a given graph \(G\) has been extensively studied by researchers in distributed computing and computational geometry and more recently by researchers in ad-hoc wireless networks. In this paper we present a fast distributed algorithm for constructing a linear size, lightweight \(t\)-spanner of bounded degree for any given \(t > 1\), on wireless networks. Below, we describe our result more precisely.

1.1 Network model

We model wireless networks using \(d\)-dimensional quasi unit ball graphs. For any fixed \(\alpha\), \(0 < \alpha \leq 1\) and integer \(d \geq 2\), a \(d\)-dimensional \(\alpha\)-quasi unit ball graph (\(\alpha\)-UBG, in short) is a graph \(G = (V, E)\) whose vertex set \(V\) can be placed in one-one correspondence with a set of points in the \(d\)-dimensional Euclidean space and whose edge set \(E\) satisfies the constraint: if \(|uv| \leq \alpha\) then \(\{u, v\} \in E\) and if \(|uv| > 1\) then \(\{u, v\} \notin E\). Here we use \(|uv|\) to denote the Euclidean distance between the points corresponding to vertices \(u\) and \(v\). The \(\alpha\)-UBG model does not prescribe whether a pair of vertices whose distance is in the range \((\alpha, 1)\] are to be connected by an edge or not. This is an attempt to take into account transmission errors, fading signal strength, and physical obstructions. Our algorithm does not need to know the locations of nodes of the \(\alpha\)-UBG in \(d\)-dimensional Euclidean space; just the pairwise Euclidean distances.

The \(\alpha\)-UBG model is a higher dimensional generalization of the somewhat simplistic unit disk graph (UDG) model of wireless networks that is popular in literature. Specifically, when \(\alpha = 1\) and \(d = 2\), a \(d\)-dimensional \(\alpha\)-UBG is just a UDG. UDGs are attractive due to their mathematical simplicity, but have been deservedly criticized for being unrealistic models of wireless networks [10]. In our view, \(d\)-dimensional \(\alpha\)-UBGs are a significant step towards a more realistic model of wireless networks. Two-dimensional \(\alpha\)-UBGs were proposed in [4] as a model of wireless ad-hoc networks with unstable transmission ranges and the difficulty of doing geometric routing in such networks was shown.

Our communication model is the standard synchronous message passing model that does not account for channel access and collision issues. In this communication model, time is divided into rounds. In each round, each node can send a different message to each of its neighbors, receive different messages from all neighbors and perform arbitrary (polynomial) local computation. The length of messages exchanged between nodes is logarithmic in the number of nodes. We measure the cost of our algorithm in terms of the number of communication rounds. Although this model is not widely considered to be realistic, it is nevertheless interesting because it demonstrates the locality of computations.
1.2 Our result

For any edge weighted graph \( J \), we use \( w(J) \) to denote the sum of the weights of all the edges in \( J \) and \( MST(J) \) to denote a minimum weight spanning tree of \( J \). For any fixed \( \varepsilon > 0 \), \( 0 < \alpha \leq 1 \), and \( d \geq 2 \) our algorithm runs in \( O(\log n \cdot \log^* n) \) communication rounds on an \( n \)-node, \( d \)-dimensional \( \alpha \)-UBG and computes a \((1 + \varepsilon)\)-spanner \( G' \) of \( G \) whose maximum degree \( \Delta(G') = O(1) \) and whose total weight \( w(G') = O(w(MST(G))) \). Since any spanner of \( G \) has weight bounded below by \( w(MST(G)) \), the weight of the output produced by the algorithm is within a constant times the optimal weight. As far as we know, our result significantly improves all known results of a similar kind along several dimensions. More on this further below.

1.3 Topology control

Our result is motivated by the topology control problem in wireless ad-hoc networks. For an overview of topology control, see the survey by Rajaraman [17]. Since an ad-hoc network does not come with fixed infrastructure, there is no topology to start with and informally speaking, the topology control problem is one of selecting neighbors for each node so that the resulting topology has a number of useful properties. More precisely, let \( V \) be a set of nodes that can communicate via wireless radios and for each \( v \in V \), let \( N(v) \) denote the set of all nodes that \( v \) can reach when transmitting at maximum power. The induced digraph \( G = (V,E) \), where \( E = \{ (u,v) \mid v \in N(u) \} \), represents the network in which every node has chosen to transmit at maximum power and has designated every node it can reach as its neighbor. The topology control problem is the problem of devising an efficient and local protocol \( P \) for selecting a set of neighbors \( N_P(v) \subseteq N(v) \) for each node \( v \in V \). The induced digraph \( G_P = (V,E_P) \), where \( E_P = \{ (u,v) \mid v \in N_P(u) \} \) is typically required to satisfy properties such as symmetry (if \( v \in N_P(u) \) then \( u \in N_P(v) \)), sparseness (\( |E_P| = O(|V|) \)) or bounded degree (\( |N_P(v)| \leq c \) for all nodes \( v \) and some constant \( c \)), and the spanner property. Sometimes stronger versions of connectivity such as \( k \)-vertex connectivity or \( k \)-edge connectivity (for \( k > 1 \)) are desired, both for providing fault-tolerance and for improving throughput [6] [7]. If the input graph consists of nodes in the plane, it is quite common to require that the output graph be planar [13] [14] [15] [16] [19]. This requirement is motivated by the existence of simple, memory-less, geometric routing algorithms that guarantee message delivery only when the underlying graph is planar [9].

Though the topology control problem is recent, there is already an extensive body of literature on the problem to which the above sample of citations do not do justice. However, many of the topology control protocols that provide worst case guarantees on the quality of the topology, assume that the network is modeled by a UDG. A recent example [15] presents a distributed algorithm that requires a linear number of communication rounds in the worst case to compute a planar \( t \)-spanner of a given UDG with \( t \approx 6.2 \) and in which each node has degree at most 25. These two constants can be slightly tuned – \( t \) can be brought down to about 3.8 with a significant increase in the degree bound. We improve on the result in [15] along several dimensions. As is generally known among practitioners in ad-hoc wireless networks, the “flat world” assumption and the identical transmission range assumption of UDGs are unrealistic [10]. By using an \( \alpha \)-UBG we significantly generalize our model of wireless networks, hopefully moving much closer to reality. For any \( \varepsilon > 0 \), our algorithm returns a \((1 + \varepsilon)\)-spanner; as far as we know, this is the first distributed algorithm that produces an arbitrarily good spanner for an \( \alpha \)-UBG model of wireless networks. We also guarantee that the total weight of the output is within constant times optimal – a guarantee that is not provided in [13]. Finally, using algorithmic techniques and distributed data structures that might be of independent interest, we ensure that our protocol runs in \( O(\log n \cdot \log^* n) \) communication rounds. We are not aware of any topology control algorithm that runs in poly-logarithmic number of rounds and provides anywhere close to the guarantees provided by our algorithm.

1.4 Spanners in computational geometry

Starting in the early 1990’s, researchers in computational geometry have attempted to find sparse, lightweight spanners for complete Euclidean graphs. Given a set \( P \) of \( n \) points in \( \mathbb{R}^d \), the tuple \( (P,E) \), where \( E \) is the set of line segments \( \{ (p,p') \mid p,p' \in P \} \), is called the complete Euclidean graph on \( P \). For any subset \( P' \subseteq P \), \((P,E')\) is called a Euclidean graph on \( P \). The specific problem that researchers in computational geometry have considered, is this. Given a set \( P \) of \( n \) points in \( \mathbb{R}^d \) and \( t > 1 \), compute a Euclidean graph on \( P \) that is a \( t \)-spanner of the complete Euclidean graph on \( P \), whose maximum degree is bounded by \( O(1) \) and whose weight is bounded by the weight of a minimum spanning tree on \( P \). For an early example, see [12] in which the authors show that there are “planar graphs almost as good as the complete graphs and almost as cheap as minimum spanning trees.” This was followed by a series of improvements [2] [5] [4] [6], with the most recent paper [2] presenting algorithms for constructing Euclidean subgraphs that provide the additional property of \( k \)-fault tolerance. Most of the papers mentioned above start with the following simple greedy algorithm.

Algorithm SEQ-GREEDY \((G = (V,E), t)\)

1. Order the edges in \( E \) in non-decreasing order of length.
2. \( E' \leftarrow \emptyset \), \( G' \leftarrow (V,E') \)
3. For each edge \( e = \{u,v\} \in E \) if there is no \( uv \)-path in \( G' \) of length at most \( t \cdot |uv| \)
   (a) \( E' \leftarrow E' \cup \{e\} \)
   (b) \( G' \leftarrow (V,E') \)

Output \( G' \).

It is well-known [4] that if the input graph \( G = (V,E) \) is the complete Euclidean graph, then the output graph \( G' = (V,E') \) produced by SEQ-GREEDY has the following useful properties: (i) \( G' \) is a \( t \)-spanner of \( G \), (ii) \( \Delta(G') = O(1) \), and (iii) \( w(G') = O(w(MST(G))) \). A naive implementation of SEQ-GREEDY takes \( O(n^3 \log n) \) time because a quadratic number of shortest path queries need to be answered on a dynamic graph with \( O(n) \) edges. Consequentially, papers in this area [4] [5] focus on trying to implement SEQ-GREEDY efficiently. For example, Das and Narasimhan [4] show how to use certain kind of graph clustering to answer shortest path queries efficiently, thereby reducing the running time of SEQ-GREEDY to \( O(n \log^2 n) \). One of the contributions of this paper is to show how a variant of the Das-Narasimhan
clustering scheme can be implemented and maintained efficiently, in a distributed setting.

1.5 Summary of our contributions

In obtaining the main result, our paper makes the following contributions.

1. We first show that sparse, lightweight \( t \)-spanners for arbitrarily small \( t \), not only exist for \( d \)-dimensional \( \alpha \)-UBGs, but can be computed using \textit{SEQ-GREEDY}. Note that sparse \( t \)-spanners for arbitrarily small values of \( t \) do not exist for general graphs. For example, there is a classical graph-theoretic result that shows that for any \( t \geq 1 \), there exist (infinitely many) unweighted \( n \)-vertex graphs for which every \( t \)-spanner needs \( \Omega(n^{1+1/(t+2)}) \) edges (see Page 179 in [10]).

2. We then consider a version of \textit{SEQ-GREEDY} in which the requirement that edges be considered in increasing order of length is relaxed. More precisely, the edges are distributed into \( O(\log n) \) bins \( B_0, B_1, B_2, \ldots \) such that edges in \( B_i \) are all shorter than edges in \( B_{i+1} \). It is then shown that any ordering of the edges in which edges in \( B_0 \) come first, followed by edges in \( B_1 \), followed by the edges in \( B_2 \), etc., is good enough for the correctness of \textit{SEQ-GREEDY}, even for \( d \)-dimensional \( \alpha \)-UBGs. More importantly, we show that the update step in \textit{SEQ-GREEDY} (Step 3(a)) need not be performed after each edge is queried. Instead, a more lazy update may be performed, after each bin is completely processed. Being able to perform a lazy update is critical for a distributed implementation; roughly speaking, we want the nodes to query all edges in a bin in parallel and not to have to rely on answers to queries on other edges in a bin.

3. We also use a clustering technique as a way to reduce the number of edges to be queried per node. Reducing the number of query edges per node, is critical to being able to guarantee that the output of our distributed version of \textit{SEQ-GREEDY} does not have too many edges incident on a node.

4. Our next contribution is to show that this relaxed version of \textit{SEQ-GREEDY} can be implemented in a distributed setting in \( O(\log n) \) phases — one phase corresponding to each bin — such that each phase requires \( O(\log^* n) \) communication rounds. Each phase requires the computation of maximal independent sets (MIS) on some derived graphs. We show that the derived graphs are unit ball graphs of constant doubling dimension [11] and use the \( O(\log^* n) \)-round MIS algorithm of Kuhn et al [11].

1.6 Extensions to our main result

Here we briefly report on extensions to our main result that we have obtained. They do not appear in this paper due to lack of space.

1. Let \( G = (V, E) \) be an edge-weighted graph. For any \( t > 1 \) and positive integer \( k \), a \( k \)-vertex fault-tolerant \( t \)-spanner of \( G \) is a spanning subgraph \( G' \) if for each subset \( S \) of vertices of size at most \( k \), \( G[V \setminus S] \) is a \( t \)-spanner of \( G[V \setminus S] \). A \( k \)-edge fault-tolerant \( t \)-spanner is defined in a similar manner. Using ideas from [2] we can extend our algorithm to produce a \( k \)-vertex (or a \( k \)-edge) fault-tolerant \( t \)-spanner in polylogarithmic number of communication rounds.

2. In this paper, we use Euclidean distances as weights for the edges of the input graph \( G \). However, if the metric \( c \cdot |w|^\gamma \), for positive constant \( c \) and \( \gamma \geq 1 \), is used in place of Euclidean distances \( |w| \), we can show that our algorithm still produces a spanner with all three desired properties. Relative Euclidean distances, such as the function mentioned above, may be used to produce energy spanners.

3. Let \( G = (V, E) \) be an edge-weighted graph. The \textit{power cost} of a vertex \( u \in V \) is \( |w(u)| = \max\{|w(u, v) | v \text{ is a neighbor of } u\} \). In other words, the power cost of a vertex \( u \) is proportional to the cost of \( u \) transmitting to a farthest neighbor. The \textit{power cost} of \( G \) is \( \sum_{u \in V} |w(u)| \) [3]. We can show that the output of our algorithm is not only lightweight with respect to the usual weight measure (sum of the weights of all edges) but also with respect to the power cost measure.

2. SEQUENTIAL RELAXED GREEDY ALGORITHM

Now we show that a relaxed version of \textit{SEQ-GREEDY} produces an output \( G' \) with all three desired properties, even when the input is not a complete Euclidean graph, but is a \( d \)-dimensional, \( \alpha \)-UBG for fixed \( d \) and \( \alpha \). Relaxing the requirement in \textit{SEQ-GREEDY} that the edges be totally ordered by length and allowing for the output to be updated lazily are critical to obtaining a distributed algorithm that runs in polylogarithmic number of rounds.

Let \( r > 1 \) be a constant to be fixed later and let \( W_i = r^i \alpha/n \) for each \( i = 0, 1, 2, \ldots \). Let \( I_0 = \{0, \alpha/n\} \) and for each \( i = 1, 2, \ldots \) let \( I_i = \{W_{i-1}, W_i\} \). Let \( m = \lceil \log_\alpha \frac{\alpha}{n} \rceil \). Then, since no edge has length greater than \( 1 \), the length of any edge in \( E \) lies in one of the intervals \( I_0, I_1, \ldots, I_m \). Let \( E_i = \{ \{u,v\} \in E : |uv| \in I_i \} \).

We now eliminate the restriction that edges within a set \( E_i \) be processed in increasing order by length. We run \textit{SEQ-GREEDY} in \( m + 1 \) phases: in phase \( i \), the algorithm processes edges in \( E_i \) in arbitrary order and adds a subset of edges in \( E_i \) to the spanner. For \( 0 \leq i \leq m \), we use \( G_i \) to denote the spanning subgraph of \( G \) consisting of edges \( E_0 \cup E_1 \cup \cdots \cup E_i \). Thus \( G_i \) is the portion of the input graph that the algorithm has processed in phase \( i \) and earlier. We use \( G'_i \) to denote the output of the algorithm at the end of phase \( i \). In other words, \( G'_i \) is the spanning subgraph of \( G \) consisting of edges of \( G \) that the algorithm has decided to retain in phases \( 0, 1, \ldots, i \). The final output of the algorithm is \( G' = G_m \).

The way \( E_0 \) is processed is different from the way \( E_i, i > 0 \) is processed. We now separately describe these two parts.

2.1 Processing Edges in \( E_0 \)

We start by stating a property of \( G_0 \) that follows easily from the fact that all edges in \( G_0 \) are small.

\textbf{Lemma 1.} Every connected component of \( G_0 \) induces a clique in \( G \).

The algorithm \textit{PROCESS–SHORT–EDGES} for processing edges in \( E_0 \) consists of three steps (i) determine the connected components of \( G_0 \), (ii) use \textit{SEQ-GREEDY} to compute a \( t \)-spanner
for each connected component (that is, a clique), and (iii) let \( G_0 \) be the union of the \( t \)-spanners computed in Step (2) and output \( G'_{t-1} \). The following theorem states the correctness of the \textsc{process-short-edges} algorithm. Its proof follows easily from the correctness of \textsc{seq-greedy}.

**Theorem 2.** \( G_0 \) satisfies the following properties. (i) For every edge \( \{u,v\} \in E_0 \), \( G_0 \) contains a \( uv \)-path of length at most \( t \cdot |uv| \), (ii) \( \Delta(G_0) = O(1) \), and (iii) \( w(G_0) = O(w(MST(G))) \).

### 2.2 Processing Long Edges

We now describe how edges in \( E_t \) are processed, for \( i > 0 \). The algorithm \textsc{process-long-edges} has five steps: (i) computing a cluster cover for \( G'_{i-1} \), (ii) selecting query edges in \( E_t \), (iii) computing a cluster graph \( H_{i-1} \) for \( G'_{i-1} \), (iv) answering shortest path queries for the query edges selected in Step (ii), and (v) removing redundant edges. These steps are described in the next five subsections.

#### 2.2.1 Computing a Cluster Cover for \( G'_{i-1} \)

At the beginning of phase \( i \) we compute a cluster cover of radius \( \delta W_{i-1} \), where \( \delta < 1 \) is a constant that will be fixed later. We start with an arbitrary vertex \( u \in V \) and run Dijkstra’s shortest path algorithm with source \( u \) on \( G'_{i-1} \), in order to identify nodes \( v \in V \) with the property that \( \text{sp}_{G'_{i-1}}(u,v) \leq \delta W_{i-1} \); each such node \( v \) gets included in the cluster \( C_u \). Once \( C_u \) has been identified, recurse on \( V \setminus C_u \) until all nodes belong to some cluster and we have a cluster cover of \( G'_{i-1} \) of radius \( \delta W_{i-1} \).

#### 2.2.2 Selecting Query Edges in \( E_t \)

As defined earlier, edges in \( E_t \) have weights in the interval \( I_t = [W_{i-1}, W_i] \), while the cluster cover for \( G'_{i-1} \) has radius \( \delta W_{i-1} \), with \( \delta < 1 \). This implies that each edge in \( E_t \) has endpoints in different clusters. Our goal is to select a unique query edge per pair of clusters. This will guarantee that there are a constant number of query edges incident on any node (see Lemma 3) and this fact will be critically used by the distributed version of our algorithm to guarantee the degree bound on the spanner that is constructed.

Let \( \theta \) be a quantity that satisfies \( 0 < \theta < \frac{\pi}{4} \) and \( t \geq 1/(\cos \theta - \sin \theta) \). For any value \( t > 1 \), no matter how small, there always exists a \( \theta \) that satisfies these restrictions. Define an edge \( e = \{u,v\} \in E_t \) to be a covered edge if there is a \( z \in V \) such that (i) \( \{u,z\} \in G'_{i-1} \), \( |vz| \leq \alpha \) and \( \angle vuz \leq \theta \) or (ii) \( \{v,z\} \in G'_{i-1} \), \( |uz| \leq \alpha \) and \( \angle uzx \leq \theta \). Any edge in \( E_t \) that is not covered is a candidate query edge. The motivation for these definitions is the following geometric lemma, due to Czumaj and Zhao [2].

**Lemma 3 (Czumaj and Zhao [2]).** Let \( 0 < \theta < \frac{\pi}{4} \) and \( t \geq \frac{\pi}{(\cos \theta - \sin \theta)^2} \). Let \( u, v, z \) be three points in \( \mathbb{R}^d \) with \( \angle uzv \leq \theta \). Suppose further that \( |uz| \leq |vz| \). Then the edge \( \{u,z\} \) followed by a \( t \)-spanner path from \( z \) to \( v \) is a \( t \)-spanner path from \( u \) to \( v \) (see Figure 1).

**Figure 1:** (a) Edge \( \{u,v\} \) is covered: \( \{u,z\} \) followed by a \( t \)-spanner \( zv \)-path is a \( t \)-spanner \( uv \)-path.

Now note that for each covered edge \( \{u,v\} \in E_t \), there exists \( z \) that satisfies the preconditions of Lemma 3 (by definition), and using this lemma we can show that \( G'_{i-1} \) already contains a \( uv \)-path of length at most \( t \cdot |uv| \). This suggests that covered edges need not be queried and therefore we can start with the complement of the set of covered edges as candidate query edges.

For each pair of clusters \( C_a \) and \( C_b \), let \( E_t[C_a, C_b] \) denote the subset of candidate query edges in \( E_t \) with one endpoint in \( C_a \) and the other endpoint in \( C_b \). Our algorithm selects a unique query edge \( \{x,y\} \) from each nonempty subset \( E_t[C_a, C_b] \). Assuming that \( x \in C_a \) and \( y \in C_b \), the edge \( \{x,y\} \) is selected so as to minimize

\[
 t \cdot |xy| - \text{sp}_{G'_{i-1}}(a,x) - \text{sp}_{G'_{i-1}}(b,y) \tag{1}
\]

The quantity in (1) is carefully chosen to guarantee that, if a \( t \)-spanner path between the endpoints of an edge \( \{x,y\} \) that minimizes (1) exists in \( G'_{i} \), then \( t \)-spanner paths between the endpoints of all edges in \( E_t[C_a, C_b] \) exist in \( G'_{i} \) (this property will later be shown in the proof of Theorem 4). This implies that, for each pair of clusters \( C_a \) and \( C_b \), querying the edge \( \{x,y\} \) in \( E_t[C_a, C_b] \) that minimizes (1) renders querying any other edge in \( E_t[C_a, C_b] \) redundant.

The following lemma shows that selecting query edges as described above filters all but a constant number of edges per cluster. The proof follows from two observations: (i) if a pair of cluster centers are connected by an edge in \( E_t \), then the clusters are not too far from each other in Euclidean space (in particular, no farther than \((4\delta + r)W_{i-1}\)), and (ii) the Euclidean distance between any pair of cluster centers is bounded from below by \( \delta W_{i-1}/t \), because they would otherwise be part of the same cluster.

**Lemma 4.** The number of query edges in \( E_t \) that are incident on any cluster is \( O(t^4(4\delta + r)^4) \), a constant.

#### 2.2.3 Computing a Cluster Graph

For each selected query edge \( \{x,y\} \in E_t \), we need to know if \( G'_{i-1} \) contains an \( xy \)-path of length at most \( t \cdot |xy| \). In general, the number of hops in a shortest \( xy \)-path in \( G'_{i-1} \) can be quite large and having to traverse such a path would mean that the shortest path query corresponding to edge \( \{x,y\} \) could not be answered quickly enough. To get around this problem, we use an idea from [3] in which the authors construct an approximation to \( G'_{i-1} \), called a cluster graph, and show that for any edge \( \{x,y\} \in E_t \), the shortest path query for \( \{x,y\} \) can be answered approximately on \( H_{i-1} \) in a constant number of steps. The goal of Das and Narasimhan [1] was to improve the running time of...
SEQ-GREEDY on complete Euclidean graphs, but we show that the Das-Narasimhan data structure can be constructed and maintained in a distributed fashion for efficiently answering shortest path queries for edges belonging to a α-UBG. In the following, we describe a sequential algorithm that starts with a cluster cover of \( G'_{i-1} \) of radius \( \delta W_{i-1} \), and builds a cluster graph \( H_{i-1} \) of \( G'_{i-1} \). This algorithm is identical to the one in Das and Narasimhan [4] and is included mainly for completeness.

The vertex set of \( H_{i-1} \) is \( V \) and the edge set of \( H_{i-1} \) contains two types of edges: intra-cluster edges and inter-cluster edges. An edge \( \{a, x\} \) is an intra-cluster edge if \( a \) is a cluster center and \( x \) is node in \( C_a \). Inter-cluster edges are between cluster centers. An edge \( \{a, b\} \) is an inter-cluster edge if \( a \) and \( b \) are cluster centers, and at least one of the following two conditions holds: (i) \( \text{sp}_{G'_{i-1}}(a, b) \leq W_{i-1} \), or (ii) there is an edge in \( G'_{i-1} \) with one endpoint in \( C_a \) and the other endpoint in \( C_b \). See Figure 2.

![Figure 2: Edges interior to disks are intra-cluster edges. Edge \( \{a, b\} \) is an inter-cluster edge because \( \text{sp}_{G'_{i-1}}(a, b) \leq W_{i-1} \), and \( \{b, c\} \) is an inter-cluster edge because \( \{u, v\} \) is in \( G'_{i-1} \). An xy-path in \( G'_{i-1} \), shown by the dashed curve may be approximated by the path \( x, a, b, y \) in \( H_{i-1} \).](image)

Regardless of the type of a cluster edge \( e = \{a, b\} \) (inter- or intra-), the weight of \( e \) is the value of \( \text{sp}_{G'_{i-1}}(a, b) \). The following lemma follows easily from the definition of inter-cluster edges.

**Lemma 5.** For any inter-cluster edge \( \{a, b\} \) in \( H_{i-1} \), we have that \( \text{sp}_{G'_{i-1}}(a, b) \leq (2^i + 1)W_{i-1} \).

The above upper bound also implies that \( |ab| \leq (2^i + 1)W_{i-1} \). Using this and arguments similar to those used for Lemma 4, we can show that the number of inter-cluster edges incident to a cluster center is \( O((5 + 1/\delta)^d) \), so we have the following lemma.

**Lemma 6.** The number of inter-cluster edges in \( H_{i-1} \) incident to a cluster center is \( O((5 + 1/\delta)^d) \), a constant.

The main reason for constructing the cluster graph \( H_{i-1} \) is that lengths of paths in \( H_{i-1} \) are close to lengths of corresponding paths in \( G'_{i-1} \) and shortest path queries for edges in \( E_i \) can be answered quickly in \( H_{i-1} \). The following lemma (whose proof appears in Das and Narasimhan [4]) shows that we can construct \( H_{i-1} \) such that path lengths in \( H_{i-1} \) approximate path lengths in \( G'_{i-1} \) to any desired extent, depending on the choice of \( \delta \).

**Lemma 7.** For any edge \( \{x, y\} \in E_i \), if there is a path between \( x \) and \( y \) in \( G'_{i-1} \) of length \( L_1 \), then there is a path between \( x \) and \( y \) in \( H_{i-1} \) of length \( L_2 \) such that \( L_1 \leq L_2 \leq \left[ \frac{1 + \alpha}{1 - 2\alpha} \right] L_1 \).

### 2.2.4 Answering Shortest Path Queries

For query edges \( \{x, y\} \in E_i \), we are interested in knowing whether \( G'_{i-1} \) has an \( xy \)-path of length at most \( t \cdot |xy| \). We ask this question on the cluster graph \( H_{i-1} \). If \( H_{i-1} \) contains an \( xy \)-path of length at most \( t \cdot |xy| \), we do not add \( \{x, y\} \) to \( G_i \); otherwise we do. If \( H_{i-1} \) contains an \( xy \)-path of length at most \( t \cdot |xy| \), then so does \( G'_{i-1} \) (by Lemma 8 since \( L_1 \leq L_2 \)). Therefore, not adding \( \{x, y\} \) to the spanner is not a dangerous choice. On the other hand, even if \( H_{i-1} \) does not contain an \( xy \)-path of length at most \( t \cdot |xy| \), \( G'_{i-1} \) might contain such a path and in this case adding edge \( \{x, y\} \) is unnecessary. Adding extra edges is of course not problematic for the \( t \)-spanner property. It will turn out that this is not a problem even for the requirement that the spanner should have bounded degree and small weight, given that paths in \( H_{i-1} \) can approximate paths in \( G'_{i-1} \) to an arbitrary degree.

Given the structure of the cluster graph, all but at most 2 edges in any simple \( xy \)-path are inter-cluster edges. Since the radius of each cluster is \( \delta W_{i-1} \), each inter-cluster edge has weight greater than \( \delta W_{i-1} \). We are looking for a path of length at most \( t \cdot |xy| \). Since \( |xy| \in ([W_{i-1}, W_i] \), we are looking for a path of length at most \( t \cdot W_i = t \cdot r \cdot W_{i-1} \). Any simple path in \( H_{i-1} \) of length at most \( t \cdot r \cdot W_{i-1} \) has at most \( 2 + \lceil |r|/\delta \rceil \) hops, which is a constant. This yields the following lemma.

**Lemma 8.** For any edge \( \{x, y\} \in E_i \), if \( \text{sp}_{H_{i-1}}(x, y) \leq t \cdot |xy| \), then \( H_{i-1} \) contains a shortest \( xy \)-path with \( O(1) \) hops (no more than \( 2 + \lceil |r|/\delta \rceil \) hops).

One issue we need to deal with, especially when attempting to construct and answer queries in \( H_{i-1} \) in a distributed setting, is that edges in \( H_{i-1} \) need not be present in the underlying network \( G \). Specifically, for an inter-cluster edge \( \{u, a\} \), where \( C_a \) is a cluster and \( u \in C_a \), it may be the case that \( |ua| > \alpha \) and \( \{u, a\} \) may be absent from \( G \). Similarly, an inter-cluster edge \( \{a, b\} \) in \( H_{i-1} \) may be absent in \( G \). However, for any edge \( \{x, y\} \in H_{i-1} \) (intra- or inter-cluster edge), we have the bound \( \text{sp}_{G'_{i-1}}(x, y) \leq (2^i + 1)W_{i-1} \). This follows from Lemma 8 and the fact that the radius of each cluster is \( \delta W_{i-1} \). Thus a shortest \( xy \)-path in \( G'_{i-1} \) lies entirely in a ball of radius \( (2^i + 1)W_{i-1} \) centered at \( x \). Since \( G'_{i-1} \) is a spanning subgraph of \( G \), this implies that there is a shortest \( xy \)-path \( P \) in \( G \) that lies entirely in the \( d \)-dimensional ball of radius \( (2^i + 1)W_{i-1} \) centered at \( x \). Since any two vertices in \( P \) that are two hops away from each other are at least \( \alpha \) apart (in the \( d \)-dimensional Euclidean space), \( P \) contains at most \( 2(2^i + 1)/\alpha \) hops. This argument yields the following theorem.

**Theorem 9.** For any edge \( \{x, y\} \in E_i \), if \( \text{sp}_{H_{i-1}}(x, y) \leq t \cdot |xy| \), then \( G \) contains a shortest \( xy \)-path with \( O(1) \) hops (no more than \( 2(2^i + 1)/\alpha \)).

This theorem implies that brute force search initiated from one of the endpoints, say \( x \), will be able to answer the shortest path query on edge \( \{x, y\} \) in \( O(1) \) rounds in a distributed setting.

### 2.2.5 Removing Redundant Edges

Let \( t_1 \) be such that \( 1 < t_1 < t \). Recall that shortest path queries for edges in \( E_i \) are answered on \( H_{i-1} \), and so updates
to $G'_i$ in phase $i$ do not influence subsequent shortest path queries in phase $i$. Thus it is possible that in phase $i$ two edges \{u, v\} and \{u', v'\} get added to $G_i$, yet both of the following hold:

(i) $s_{p_{H_{i-1}}}(u, u') + |u'v'| + s_{p_{H_{i-1}}}(v', v) \leq t_1 \cdot |uv|$

(ii) $s_{p_{H_{i-1}}}(u', u) + |uv| + s_{p_{H_{i-1}}}(v, v') \leq t_1 \cdot |u'v'|

Note that, since $s_{p_{G_{i-1}}}(x, y) \leq s_{p_{H_{i-1}}}(x, y)$ holds for any pair of nodes $x$ and $y$, and since $t_1 < t$, conditions (i) and (ii) above imply that $G'_i$ contains $t$-spanner paths from $u$ to $v$ and from $u'$ to $v'$. We call two edges \{u, v\} and \{u', v'\} satisfying conditions (i) and (ii) above mutually redundant: one of them could potentially be eliminated from $G_i$, without compromising the $t$-spanner property of $G_i$. In fact, such mutually redundant pairs of edges need to be eliminated from $G'_i$ because our proof that $G'$ has small weight (Theorem 13) depends on the absence of such pairs of edges.

To do this, we build a graph $J$ that has a node for each edge in a mutually redundant pair and an edge between every pair of nodes that correspond to a mutually redundant pair of edges in $G'_i$. We construct an MIS $I$ of $J$ and eliminate from $G'_i$ all edges associated with nodes in $J$ that do not appear in $I$.

2.3 The Three Desired Properties

Let $G' = G'_m$ be the spanner at the end of phase $m$. We now prove that $G'$ satisfies the three properties that the output of $\text{SEQ-GREEDY}$ was guaranteed to have. The proofs of these theorems form the technical core of the paper and are presented next in this section.

**Theorem 10.** For any $0 < \delta \leq \frac{t_1 \cdot \text{length}}{4}$, the output $G'$ is a $t$-spanner.

**Proof.** We first prove that the theorem holds for all query edges in $E$, then we extend the argument to non-query edges as well. Let \{x, y\} be an arbitrary query edge and let $i \geq 1$ be such that \{x, y\} $\in E_i$. Then either (i) \{x, y\} is added to the spanner in phase $i$, or (ii) $s_{p_{H_{i-1}}}(x, y) \leq t \cdot |xy|$. If the former is true and \{x, y\} is not a redundant edge, then the theorem holds. If \{x, y\} is a redundant edge but does not get removed from $G_i$, then again the theorem holds. If \{x, y\} is a redundant edge that gets removed from $G_i$, then at least one mutually redundant counterpart edge must remain in $G_i$ (since removed edges form an independent set), ensuring a $t$-spanner $xy$-path in $G_i$. If (ii) is true, then from Lemma 7 $s_{p_{G_{i-1}}}(x, y) \leq s_{p_{H_{i-1}}}(x, y)$ (first part of the inequality) and therefore $s_{p_{G_{i-1}}}(x, y) \leq t \cdot |xy|$.

For non-query edges, the proof is by induction on the length of edges in $G$. The base case corresponds to edges in $E_0$, for which $\text{SEQ-GREEDY}$ ensures that the theorem holds.

Assume that the theorem is true for any edge in $E$ of length no greater than some value $q$, and consider a smallest non-query edge \{x, y\} in $G$ of length greater than $q$. We prove that $s_{p_{G_i}}(x, y) \leq t \cdot |xy|$. Let $i$ be such that \{x, y\} $\in E_i$. We now consider two cases, depending on whether \{x, y\} is a candidate query edge in phase $i$ or not.

If \{x, y\} is not a candidate query edge, then it is a covered edge. That is, there exists an edge \{x, z\} in $G'_{i-1}$ such that $|yz| \leq \alpha$ and $\angle xyz \leq \theta$, or an edge \{y, z\} in $G'_{i-1}$ such that $|xz| \leq \alpha$ and $\angle xyz \leq \theta$. The two cases are symmetric and so without loss of generality, assume that the former is true. Here $\theta$ satisfies the hypothesis of the Czumaj-Zhao lemma (Lemma 3), that is, $0 < \theta < \frac{\pi}{4}$ and $t \geq \frac{\text{length}}{\text{r}^\theta}$.

Since $|yz| \leq \alpha$ and $G$ is an $\alpha$-UBG, this implies that \{y, z\} is an edge in $E$. Furthermore, since $0 < \theta < \frac{\pi}{4}$, we have $|yz| < |xy|$. Refer to Figure 3a. If \{y, z\} is a query edge, then by the argument above we have that $G'_i$ contains a $t$-spanner $yz$-path. Otherwise, if \{y, z\} is not a query edge, since its length is less than the length of \{x, y\}, by the inductive hypothesis we get that there is a $t$-spanner $yz$-path. In either case, Lemma 3 tells us that \{x, z\} followed by $p$ is a $t$-spanner path from $x$ to $y$, completing this case.

![Figure 3: (a) \{x, y\} is a covered edge (b) \{u, v\} is a query edge: if $G_i$ contains a $t$-spanner $uv$-path, then $G_i$ contains a $t$-spanner $xy$-path.](image)

We now consider the case when \{x, y\} is a candidate query edge in phase $i$, but not a query edge. Let $a$ and $b$ be such that $x \in C_a$ and $y \in C_b$, and let \{u, v\} be the query edge selected in phase $i$, with $u \in C_a$ and $v \in C_b$. Refer to Figure 3a. Due to the criteria for selecting \{u, v\}, we have

$$t \cdot |uv| - s_{p_{G'_{i-1}}}(a, u) - s_{p_{G'_{i-1}}}(b, v) \leq t \cdot |xy| - s_{p_{G'_{i-1}}}(a, x) - s_{p_{G'_{i-1}}}(b, y).$$

Recall that $G'_i$ is the partial spanner at the end of phase $i$. We show that $s_{p_{G'_{i-1}}}(x, y) \leq t \cdot |xy|$. We discuss two cases, depending on whether \{u, v\} was added to $G'_i$ or not.

Assume first that \{u, v\} was not added to $G'_i$. This means that $s_{p_{H_{i-1}}}(u, v) \leq t \cdot |uv|$. Note however that $s_{p_{H_{i-1}}}(u, v) = s_{p_{G'_{i-1}}}(a, u) + s_{p_{H_{i-1}}}(a, b) + s_{p_{G'_{i-1}}}(b, v) \leq t \cdot |uv|$. (3)

We now evaluate $s_{p_{G'_{i-1}}}(x, y) \leq s_{p_{G'_{i-1}}}(x, a) + s_{p_{G'_{i-1}}}(a, b) + s_{p_{G'_{i-1}}}(b, y) \leq s_{p_{G'_{i-1}}}(x, a) + s_{p_{H_{i-1}}}(a, b) + s_{p_{G'_{i-1}}}(b, y) \leq t \cdot |xy|$. (4)

This latter inequality involves simple substitutions that use inequalities (2) and (3), and completes this case.

Now assume that \{u, v\} was added to $G'_i$. Since $u \in C_a$ and $C_a$ has radius $\delta W_{i-1}$, we have that $s_{p_{G'_{i-1}}}(a, u) \leq \delta W_{i-1}$. Similarly, $s_{p_{G'_{i-1}}}(b, v) \leq \delta W_{i-1}$. These together with (2) yield

$$t \cdot |uv| - 25W_{i-1} \leq t \cdot |xy| - s_{p_{G'_{i-1}}}(a, x) - s_{p_{G'_{i-1}}}(b, y).$$

If the edge \{u, v\} turns out to be redundant and eliminated from $G_i$, the existence of a mutually redundant counterpart edge in $G'_i$ ensures that $s_{p_{G_i}}(u, v) \leq t_1 \cdot |uv|$. This enables us to construct in $G'_i$ a path from $a$ to $b$ of weight
Theorem 11. $G'$ has $O(1)$ degree.

Proof. Let $\theta$ be a quantity satisfying the conditions of Lemma 2. Fix a vertex $u$ and consider the $d$-dimensional unit radius ball centered at $u$. For some $T$ that depends only on $\theta$ and $d$, this ball can be partitioned into $T$ cones, each with apex $u$, such that for any $x, y$ in a cone, $\angle xuy < \theta$. Yao [20] shows how to construct such a partition with $T = O(d^{3/2} \cdot \sin^{-2}(\theta/2) \cdot \log(d \sin^{-1}(\theta/2)))$ cones. Place an infinite axis-parallel grid of $d$-dimensional cubes, each of dimension $\frac{1}{2d} \times \frac{1}{2d} \times \cdots \times \frac{1}{2d}$, on the plane. See Figure 3(c) for a 2-dimensional version of this picture. There are $O(1/\alpha^d)$ cells that intersect the unit ball centered at $u$, and therefore there are $O(1/\alpha^d)$ cells that intersect each cone in the cone partition of this unit ball. Thus the cones and the square cells together partition the unit ball centered at $u$ into $O(T/\alpha^d)$ regions. We show that in $G'$, $u$ has $O\left(\frac{\alpha^d(d+1)^d}{d^d}\right)$ neighbors in each region, which is a constant.

Let $v_1, v_2, \ldots, v_k$ be neighbors of $u$ in $G'$ that lie in a region $\chi$. Without loss of generality, assume that $|uv_1| \geq |uv_2|$, for $j = 2, \ldots, k$, and let $i$ be such that $\{u, v_i\} \in E_i$. Since $|uv_1| \leq |uv_j|$, we have that for all $j = 2, \ldots, k$, $\{u, v_j\} \in E_i$, with $i \leq j$. We now prove that $\{u, v_j\}$ is in fact in $E_i$ for all $j$. To derive a contradiction, assume that there is a $j > 1$ such that $\{u, v_j\} \in E_i$, with $i > j$. This means that just before edge $\{u, v_i\}$ is processed, $G'$ contains edge $\{u, v_j\}$. Also note that since $v_1$ and $v_j$ lie in the same region, $|v_1v_j| \leq \alpha$. But, this means that $\{u, v_i\}$ is a covered edge in phase $i$ and will not be queried. This contradicts the presence of edge $\{u, v_i\}$ in $G'$.

We have shown that $\{u, v_i\} \in E_i$ for all $j$. Recall that our algorithm picks a unique query edge per pair of clusters. This along with Lemma 2 proves that $k$ is constant.

In the next theorem, we show that the spanner produced by the algorithm has small weight. The proof relies on the line segments in the spanner satisfying a property known as the leapfrog property [21][2]. For any $t \geq t_2 > 1$, a set of line segments, denoted $F$, has the $(t_2, t)$-leapfrog property if for every subset $S = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_s, v_s\}\}$ of $F$

$$
t_2 \cdot |u_tv_1| < \sum_{i=2}^{s} \left( \left| \{u_iu_{i+1}\} \right| + |v_iu_i| \right).
$$

Informally, this definition says that if there exists an edge between $u_1$ and $v_1$, then any path not including $\{u_1, v_1\}$ must have length greater than $t_2 |u_tv_1|$ (see Figure 3(c) for an illustration of this definition). The following implication of the $(t_2, t)$-leapfrog property was shown by Das and Narasimhan [4].

Lemma 12. Let $t \geq t_2 > 1$. If the line segments $F$ in $d$-dimensional space satisfy the $(t_2, t)$-leapfrog property, then $wt(F) = O(wt(MST))$, where $MST$ is a minimum spanning tree connecting the endpoints of line segments in $F$. The constant in the asymptotic notation depends on $t_2, t$ and $d$.

Theorem 13. Let $0 < \delta < \min\{(t_1 - 1)/(6 + 2\delta), (t_1 - \alpha \theta)/\delta\}$. Let $t_3$ denote $t_1 \cdot (1 - 2\delta)/(1 + 6\delta)$. Let $1 < r < (t_1 + 1)/2$. When the relaxed greedy algorithm is run with these values of $\delta$ and $r$, the output $G'$ satisfies $w(G') = O(wt(MST(G)))$.

Proof. Let $\beta > 1$ be a constant picked as follows. When $\alpha t < 1$, pick $\beta$ satisfying $1 < \beta < \min\{2, (1/(1-\alpha t))\}$. Otherwise, pick $\beta$ satisfying $1 < \beta < 2$. Partition the edges of $G'$ into subsets $F_0, F_1, \ldots$ such that $F_0 = \{\{u, v\} \in G' | |uv| \leq \alpha \}$ and for each $j > 0$, $F_j = \{\{u, v\} \in G' | \alpha^{j-1} \leq |uv| \leq \alpha^j \}$. Let $t = \lceil \log_\alpha \frac{1}{\beta} \rceil$. Then every edge in $G'$ is in some subset $F_j$, $0 \leq j \leq t$. We will now show that each $F_j$ satisfies the $(t_j, t)$-leapfrog property, for any $t_j$ satisfying:

$$
1 \leq t_j \leq \frac{t_3}{2} \cdot \frac{t + 1}{r} \leq t_j - 1 - \frac{2t}{r} - \frac{2}{\beta} - \alpha t_\frac{1}{\beta}.
$$

It is easy to check that our choice for $\delta$, $r$, and $\beta$ guarantee that each quantity inside the min operator is strictly greater than 1. Showing the $(t_2, t)$-leapfrog property for $F_j$ would imply that $w(F_j) = O(w(MST(G)))$, and since the edges of $G'$ are partitioned into a constant number of subsets $F_j$, $w(G') = O(w(MST(G)))$.

Consider an arbitrary subset $S = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_s, v_s\}\} \subseteq F_0$. To prove inequality 13 for $S$, it suffices to consider the case when $\{u_1, v_1\}$ is a longest edge in $S$. We consider $F_0$ separately from $F_j$, $j > 0$.

The $F_0$ case. If for any $1 \leq k < s$, $|v_ku_{k+1}| > |u_1v_1|$ or $|v_1u_1| > |u_1v_1|$, then the leapfrog property holds. So we assume that for all $1 \leq k < s$, $|v_ku_{k+1}| \leq |u_1v_1|$ and $|v_1u_1| \leq |u_1v_1|$. Let $t$ be the phase in which $\{u_1, v_1\}$ gets processed, i.e. $\{u_1, v_1\} \in E_t$. Since $|u_1v_1| \leq \alpha$, it is the case that for all $1 \leq k < s$, $|v_ku_{k+1}| \leq \alpha$ and $|v_1u_1| \leq \alpha$. Hence, $\{\{v_k, u_{k+1}\} | 1 \leq k < s\}$ is a subset of edges of $G$ and each edge in this set gets processed in phase $i$ or earlier.

Assume first that at least one edge in the set $\{\{v_1, u_t\} | t \leq \alpha t < 1\}$ gets processed in phase $i$. Then

\[ \text{Figure 4: (a) Region } \chi \text{ contains two neighbors } v \text{ and } z \text{ of } u. \text{ (b) Definition of the } t \text{-leapfrog property with } S = \{\{u_1, v_1\}, \{u_2, v_2\}, \{u_3, v_3\}\}. \]
the right hand side of inequality (6) is at least $tW_{i-1}$, since edges in $E_i$ have weights in the interval $I_1 = (W_{i-1}, rW_{i-1}]$. Also since $t_2|u_1u_1| \leq t_2rW_{i-1}$, and since the inequality $t_2rW_{i-1} < tW_{i-1}$ is guaranteed by the values of $r$ and $t_2$ in (7), the leapfrog property holds for this case.

Assume now that all edges in $\{\{u_1, v_1\} \cup \{u_k, v_{k+1}\} \mid 1 \leq k < s\}$ have been processed in phase $i - 1$ or earlier, meaning that $t$-spanner paths between their endpoints exist in $G'_{i-1}$ at the time $\{u_1, v_1\}$ gets processed. For $1 \leq k < s$, let $P_k$ be a shortest $u_ku_{k+1}$-path in $G'_{i-1}$, and let $P$ be the following $u_1v_1$-path in $G'_{i-1}$. Let $P = P_1 \oplus (u_2, v_2) \oplus P_2 \oplus (u_3, v_3) \oplus \cdots \oplus P_s$. Here, we use $\oplus$ to denote concatenation. We distinguish three cases, depending on the size of the subset $S \cap E_i$.

(i) $|S \cap E_i| > 2$. Then, $w(P) \geq 2W_{i-1}$. We also have that $|u_1v_1| \leq rW_{i-1}$, since $\{u_1, v_1\} \in E_i$. It follows that $w(P) > t_2|u_1v_1|$ for any $t_2 < \frac{2}{3}$. Furthermore, $w(P)$ is no greater than the right hand side of the $(t_2, t)$-leapfrog inequality (6), so lemma holds for this case as well.

(ii) $|S \cap E_i| = 2$. In addition to $\{u_1, v_1\}$, assume that $\{u_k, v_k\} \in E_i$ for some $k$, $1 < k \leq s$. It the $(t_2, t)$-leapfrog inequality (6) holds, we are done and so let us assume the opposite of that:

$$t_2 \cdot |u_1v_1| \geq s|u_1v_1| + t \cdot \left( \sum_{i=1}^{s-1} |u_iu_{i+1}| + |u_1v_1| \right). \quad (8)$$

Since all edges $\{u_j, v_j\}, 1 \leq j \leq s$, except for $\{u_1, v_1\}$ and $\{u_k, v_k\}$ are in $G'_{i-1}$, and since $G'_{i-1}$ contains $t$-spanner $v_ju_{j+1}$-paths for all $j$, $1 \leq j < s$, and a $t$-spanner $v_1u_1$-path, the above inequality yields

$$t_2 \cdot |u_1v_1| \geq sp_{G'_{i-1}}(v_1, u_k) + |u_kv_k| + sp_{G'_{i-1}}(u_k, u_1).$$

Multiplying both sides by $(1 + 6\delta)/(1 - 2\delta)$ and using Lemma 7 we get

$$t_1 \cdot |u_1v_1| \geq sp_{H_{i-1}}(v_1, u_k) + |u_kv_k| + sp_{H_{i-1}}(u_k, u_1). \quad (9)$$

Let $\Delta = \sum_{i=1}^{s-1} |u_iu_{i+1}| + |u_1v_1|$. We now observe that

$$t_2 \cdot |u_kv_k| < \sum_{i=1}^{k-1} |u_iu_i| + \sum_{i=k+1}^{s} |u_iu_1| + t \cdot \Delta. \quad (10)$$

implies the $(t_2, t)$-leapfrog property. To see this use the fact that both $\{u_1, v_1\}$ and $\{u_k, v_k\}$ belong to $E_i$ and therefore $|u_1v_1| < r \cdot |u_kv_k|$, which substituted in (10) yields:

$$t_2 \cdot |u_kv_k| - (r - 1) \cdot |u_kv_k| < \sum_{i=2}^{s} |u_iu_1| + t \cdot \Delta.$$

We get the lower bound $t_2 \cdot |u_1v_1|$ on the left hand side of the above inequality by using $|u_kv_k| > |u_1v_1|/r$ again and our choice of $t_2 < (t_2 + 1)/r - 1$. This yields the $(t_2, t)$-leapfrog property. So we assume that inequality (10) does not hold, that is,

$$t_2 \cdot |u_kv_k| \geq \sum_{i=1}^{k-1} |u_iu_i| + \sum_{i=k+1}^{s} |u_iu_1| + t \cdot \Delta.$$
neighborhood along with pairwise distances between neighbors in one hop. Using this information, $u$ determines the connected component $C$ of $G_0$ that it belongs to. Then $u$ simply runs SEQ-GREEDY on $C$ and computes a $t$-spanner of $C$. Finally, $u$ identifies the edges of the $t$-spanner incident on itself and informs all its neighbors of this.

**Theorem 14.** The edges in $E_0$ can be processed in $O(1)$ rounds of communication.

### 3.2 Distributed Processing of Long Edges

In this section, we show how long edges, that is, edges in $E_i$, $i > 0$, can be processed in a distributed setting. The first step of this process is the computation of a cluster cover for the spanner $G_{i-1}'$ updated at the end of the previous phase.

#### 3.2.1 Distributed Cluster Cover for $G_{i-1}'$

Recall that in this step our goal is to compute a cluster cover $\{C_{i1}, C_{i2}, \ldots\}$ of $G_{i-1}'$ of radius $\delta W_{i-1}$. To do this, each node $u$ first identifies all nodes $v$ in $G$ satisfying $sp_{G_{i-1}'}(u, v) \leq \delta W_{i-1}$. Using arguments similar to those in Section 2.2.4, we can show that any node $v$ satisfying $sp_{G_{i-1}'}(u, v) \leq \delta W_{i-1}$ must be at most $2 \delta W_{i-1}/\alpha$ hops away from $u$. So each node $u$ constructs the subgraph of $G_{i-1}'$ induced by nodes that are at most $2 \delta W_{i-1}/\alpha$ hops away from it in $G$. Node $u$ then runs a (sequential) single source shortest path algorithm with source $u$ on the local view of $G_{i-1}'$ it has obtained and identifies all nodes $v$ satisfying $sp_{G_{i-1}'}(u, v) \leq \delta W_{i-1}$.

At the end of the above process, every node $u$ in the network is a cluster center. We now force some nodes to cease being cluster centers, so that all pairs of cluster centers are far enough from each other. Let $J$ be the graph with vertex set $V$ and whose edges $(x, y)$ are such that $x \in C_y$ (and by symmetry, $y \in C_x$).

**Lemma 15.** $J$ is a UBG that resides in a metric space of constant doubling dimension.

**Proof.** For any edge $(x, y)$ in $J$, we have that $x \in C_y$ and therefore $sp_{G_{i-1}'}(x, y) \leq \delta W_{i-1}$. Assign to every pair of nodes $(x, y)$ in $V$ a weight $w(x, y) = sp_{G_{i-1}'}(x, y)$. The weights $w$ form a metric simply because shortest path distances in any graph form a metric. Thus $J$ is a graph whose nodes reside in a metric space and whose edges connect pairs of nodes separated by distance of at most $\delta W_{i-1}$ (in the metric space). By scaling the quantity $\delta W_{i-1}$ up to one, we see that $J$ is a UBG in the underlying metric space defined by the weights $w$. Recall from [11] that the doubling dimension of a metric space is the smallest $\rho$ such that every ball can be covered by at most $2^\rho$ balls of half the radius. To see that the metric space induced by the weights $w$ has constant doubling dimension, start with a ball of $B$ radius $R$ centered at an arbitrary vertex $u$. Every vertex $v$ in ball $B$ satisfies $sp_{G_{i-1}'}(u, v) \leq R$. Now cover the vertices in $B$ using balls of radius $R/2$ as follows: repeatedly pick an uncovered vertex $v$ in $B$ and grow a ball of radius $R/2$ centered at $v$, until all vertices have been covered. We now show that the number of balls of radius $R/2$ is constant.

Let $a$ and $b$ be two arbitrary centers of different balls of radius $R/2$. Then $sp_{G_{i-1}'}(u, v) > R/2$, otherwise $a$ and $b$ would belong to the same ball of radius $R/2$. We distinguish three situations:

- $(a, b)$ is not an edge in $G$. This implies that $|ab| > \alpha$.
- $(a, b)$ is an edge in $G$ that has not been processed prior to phase $i$. This implies that $|ab| > W_{i-1} \geq 1/\delta$ (after scaling $\delta W_{i-1}$ up to 1).
- $(a, b)$ is an edge in $G$ that has been processed prior to phase $i$. This implies that $G_{i-1}'$ contains a $t$-spanner path from $a$ to $b$ and therefore $|ab| \geq sp_{G_{i-1}'}(a, b)/t \geq R/2t$.

We have established that $|ab| \geq \min\{\alpha, 1/\delta, R/2t\}$, so no two ball centers can be too close to each other. It follows that the number of balls of radius $R/2$ that fit inside $B$ is constant, proving the lemma true. \(\square\)

Let $I$ be an MIS of $J$ constructed using the MIS algorithm in [11]. This algorithm runs in $O(\log^* n)$ communication rounds on a UBG that resides in a metric space of constant doubling dimension. Then each node in $V \setminus I$ has one or more neighbors in $I$. Each node $u$ in $I$ is declared a cluster center, and each node $v$ in $V \setminus I$ attaches itself to the neighbor in $I$ with the highest identifier. This gives us the desired cluster cover of radius $\delta W_{i-1}$.

**Theorem 16.** A cluster cover of $G_{i-1}'$ of radius $\delta W_{i-1}$ can be computed in $O(\log^* n)$ rounds of communication.

#### 3.2.2 Distributed Query Edge Selection

Only nodes that are cluster heads need to participate in the process of selecting query edges. Each cluster head $a$ seeks to gather information on all edges in $E_i$ between the cluster $C_a$ and any other cluster $C_b$. Using the argument in Section 2.2.4, we know that every node in $C_a$ is at most $2 \delta W_{i-1}/\alpha$ hops away from $a$ in $G$. Therefore, if there is an edge $(u, v) \in E_i$, then $(u, v)$ is at most $1 + 2 \delta W_{i-1}/\alpha$ hops away from $a$. So $a$ gets information from nodes that are at most $1 + 2 \delta W_{i-1}/\alpha$ hops away from it and it identifies all edges in $E_i[C_a, C_b]$. Recall that this is the set of edges in $E_i$ which connect a node in $C_a$ and a node in $C_b$. Node $a$ then discards all covered edges from $E_i[C_a, C_b]$, leaving only candidate query edges in $E_i$ between $C_a$ and $C_b$. Finally, from among the candidate query edges, node $a$ selects an edge $(u, v)$ that minimizes $t \cdot |uv| - sp_{G_{i-1}'}(a, u) - sp_{G_{i-1}'}(b, v)$.

**Theorem 17.** Query edges from $E_i$ can be selected in $O(1)$ rounds of communication.

#### 3.2.3 Distributed Construction of the Cluster Graph

As in the query edge selection step, only the cluster heads need to perform actions to compute the cluster graph. Any member $u$ of a cluster $C_a$ lies at most $2 \delta W_{i-1}/\alpha$ hops away from $a$ in $G$. Thus $a$ can identify intra-cluster edges incident on it by gathering information from at most $2 \delta W_{i-1}/\alpha$ hops away. If $C_b$ is a cluster with $sp_{G_{i-1}'}(a, b) \leq W_{i-1}$, then node $a$ can identify the inter-cluster edge $(a, b)$ by gathering information from at most $2 \delta W_{i-1}/\alpha$ hops away. If $C_b$ is a cluster such that there is an edge $(u, v)$ in $G_{i-1}'$ with $u \in C_a$ and $v \in C_b$, then node $a$ can identify the inter-cluster edge $(a, b)$ by gathering information from at most $2(2\delta + 1)W_{i-1}/\alpha$ hops away. Note that the information that
a gathers contains a local view of $G_i'$ along with all pairwise distances. Using this information, node $a$ is able to run a single source shortest path algorithm with source $a$ and determine the weights of all inter-cluster and intra-cluster edges incident on $a$.

**Theorem 18.** Computing the cluster graph $H_{i-1}$ of $G'_{i-1}$ takes $O(1)$ communication rounds.

### 3.2.4 Answering Shortest Path Queries

Each node $u$ knows all the query edges incident on it. As proved in Section 2.2.3 node $u$ only needs to gather information from nodes that are at most a constant number of hops away, to be able to determine locally, for all incident query edges $(u, v) \in E_i$, whether $sp_{H_{i-1}}(u, v) \leq t \cdot |uv|$. Thus, after constant number of communication rounds, $u$ knows the subset of incident query edges $(u, v)$ for which $sp_{H_{i-1}}(u, v) > t \cdot |uv|$ and $u$ identifies these as the incident edges to be added to $G'_i$.

**Theorem 19.** Answering shortest path queries takes $O(1)$ communication rounds.

### 3.2.5 Distributed Removal of Redundant Edges

Two edges $(u, v)$ and $(u', v')$ in $G'_i$ are mutually redundant if (i) $sp_{H_{i-1}}(u, u') + |u'v'| + sp_{H_{i-1}}(v', v) \leq t_1 \cdot |uv|$ and (ii) $sp_{H_{i-1}}(u', u) + |uv| + sp_{H_{i-1}}(v, v') \leq t_1 \cdot |u'v'|$. Each node $u$ takes charge of all edges $(u, v)$ added to $G_i$ in phase $i$ and for which the identifier of $u$ is higher than the identifier of $v$. For each such edge $(u, v)$ that $u$ is in charge of, $u$ determines all edges $(u', v')$ such that $(u, v)$ and $(u', v')$ form a mutually redundant pair. Note that the nodes $u$ and $u'$ are a constant number of hops away from each other in $G_i$ and similarly for nodes $v$ and $v'$. Node $u$ then contributes to the construction of the graph $J$ by adding to $V(J)$ a vertex for each redundant edge $uv$ in charge of, and to $E(J)$ an edge connecting nodes in $V(J)$ that correspond to mutually redundant edges in $G_i$. We now show the following property of $J$:

**Lemma 20.** $J$ is a UBG that resides in a metric space of constant doubling dimension.

**Proof.** Let $a$ and $b$ be vertices in $J$ corresponding to edges $(u, v)$ and $(u', v')$ in $G'_i$. Assign to the vertex pair $(a, b)$ a weight equal to

$$d_J(a, b) = \min(sp_{H_{i-1}}(u, u), sp_{H_{i-1}}(v, v)), sp_{H_{i-1}}(u, v), sp_{H_{i-1}}(u, u), sp_{H_{i-1}}(v, v)).$$

First we show that the weights defined by $d_J$ form a metric. Clearly $d_J(a, a) = 0$ and $d_J(a, b) = d_J(b, a)$. To prove the triangle inequality, consider three vertices $a, b, c \in J$. Assume w.l.o.g. that

$$d_J(a, b) = sp_{H_{i-1}}(u, u) + sp_{H_{i-1}}(v, v),$$

$$d_J(b, c) = sp_{H_{i-1}}(u, u) + sp_{H_{i-1}}(v, v).$$

We identify two possible scenarios:

1. $d_J(a, c) = sp_{H_{i-1}}(u, u) + sp_{H_{i-1}}(v, v)$ (see Figure 5b). Since $sp$ is itself a metric, it follows immediately that $d_J(a, c) \leq d_J(a, b) + d_J(b, c)$.

2. $d_J(a, c) \geq sp_{H_{i-1}}(u, v) + sp_{H_{i-1}}(v, u)$ (see Figure 5b). Theorem 20. Then it must be that $d_J(a, c) \leq d_J(a, b) + d_J(b, c)$, cf. scenario (1).

We have shown that $d_J$ defines a metric. We now show that $J$ is a quasi-UBG residing in the metric space defined by $d_J$. For each edge $(a, b) \in J$, the following redundancy conditions hold:

(a) $sp_{H_{i-1}}(u, a) + sp_{H_{i-1}}(v, u) \leq t_1 \cdot |ua| - |ub|$

(b) $sp_{H_{i-1}}(u, a) + sp_{H_{i-1}}(v, u) \leq t_1 \cdot |vb| - |va|$

Recall that $(u, a)$ and $(v, b)$ are both in $E_i$, for some $i \geq 0$. This implies that their lengths differ by a factor of $r$ at the most: $W_{i-1} < |ua| - |vb| \leq r \cdot W_{i-1}$ and $W_{i-1} < |vb| - |ua| \leq r \cdot W_{i-1}$. Thus the right hand side of inequalities (a) and (b) above is a quantity that lies in the interval $((t_1 - r)W_{i-1}, (t_1 - 1)W_{i-1})$. By scaling $(t_1 - 1)W_{i-1}$ up to one we can say that $J$ is an $\frac{1-r}{1-t}$ - UBG in the underlying metric space defined by $d_J$.

It remains to show that the metric space defined by $d_J$ has constant doubling dimension. Throughout the rest of the proof we use $B_J(B_H)$ to denote a ball in the metric space defined by $d_J(sp_{H_{i-1}})$.

**Figure 5:** $d_J$ is a metric: (a) Nodes $a, b, c \in J$ correspond to edges $(u, a), (u, b), (u, c), (u, a) \in G'_i$. (b) $d_J(a, c) = sp_{H_{i-1}}(u, a) + sp_{H_{i-1}}(v, v)$ (c) $d_J(a, c) = sp_{H_{i-1}}(u, u) + sp_{H_{i-1}}(v, v)$. (c) $d_J(a, c) = sp_{H_{i-1}}(u, u) + sp_{H_{i-1}}(v, v)$.

**Figure 6:** (a) The metric space defined by $sp_{H_{i-1}}$ and (b) $d_J$, has doubling dimension.

Consider a ball $B_J(x, R)$ of radius $R$ centered at an arbitrary vertex $x \in J$ corresponding to edge $(u, v) \in H_{i-1}$. Let $a, b \in J$ be such that $d_J(a, b) > R/2$ (see Figure 6a). Suppose w.l.o.g. that $d_J(a, b) = sp_{H_{i-1}}(u, u) + sp_{H_{i-1}}(v, v)$. Then at least one of the following must be true:

(i) $sp_{H_{i-1}}(u, u) > R/4$.

(ii) $sp_{H_{i-1}}(v, v) > R/4$. 


We use these observations, along with the fact that $\text{sp}_H$ defines a metric space of constant doubling dimension, to show that $d_J$ defines a metric space of constant doubling dimension.

To cover all vertices in $B_J(x, R)$, do the following repeatedly: (i) pick an uncovered vertex $a \in J$ (ii) grow a ball $B_J(a, R/2)$, and (iii) grow two balls $B_H(u_a, R/4)$ and $B_H(v_a, R/4)$ in $H_{t-1}$, where $a = \{u_a, v_a\}$.

Arguments similar to the ones used in Lemma 15 show that, for any ball centers $u_a, v_b \in B_H(u_a, R)$, we have that $|u_a v_b| \geq \min\{\alpha, \frac{1}{\sqrt{t}}, \frac{R}{2}\}$. Since no two ball centers can be too close to each other, it follows that $B_H(u_a, R)$ gets covered by a constant number of balls of radius $R/4$, and similarly for $B_H(v_a, R)$. Conform observation (ii) above, corresponding to each uncovered $a \in J$, there is an uncovered vertex $u_a$ or $v_a$ in $H_{t-1}$. These together show that the number of balls covering $B_J(x, R)$ is constant, thus completing the proof.

Let $I$ be an MIS of $J$ constructed using the MIS algorithm in [11] that takes $O(\log^* n)$ communication rounds on a UBG that resides in a metric space of constant doubling dimension. Each node $u$ then removes from $G$, all incident edges in $V(J) \setminus I$.

Theorem 21. Removing redundant edges takes $O(\log^* n)$ communication rounds.

4. FUTURE WORK

The results presented in this paper apply to $\alpha$-UDGs embedded in constant-dimension Euclidean spaces, and do not directly generalize to doubling metric spaces. For low dimensional doubling metric spaces, we believe it possible to construct an $O(\log n \log^* n)$ distributed algorithm that produces a $(1 + \varepsilon)$-spanner with constant maximum degree. However, new techniques may be needed for lightweight spanners; the techniques presented in this paper use a key property (the leapfrog property) that does not seem to generalize to metrics of doubling dimension.

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