QUANTUM PROBABILITY MEASURES AND TOMOGRAPHIC PROBABILITY DENSITIES

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Abstract. Introduced recently approach based on tomographic probability distribution of quantum states is shown to be closely related with the known notion of the quantum probability measures discussed in quantum information theory and positive operator valued measures approach. Partial derivative of the distribution function of quantum probability measure associated with the homodyne quadrature (symplectic quantum measure) is shown to be equal the tomogram of the quantum state. Analogous relation of the spin tomogram to quantum probability measure associated with spin state is obtained. Star-product of symplectic quantum measures is studied. Evolution equation for symplectic quantum measures is derived.

1. Introduction

Recently \cite{1, 2, 3} the tomographic probability distributions were found to be related to Wigner functions \cite{4}. The tomoraphic probability distribution of \cite{1, 2} was used in optical tomography scheme \cite{5, 6} to reconstruct the Wigner function of photon states by measuring the homodyne quadrature distributions and applying the Radon transform \cite{7} to find the Wigner function. In \cite{3} an extension of optical tomography to symplectic tomography \cite{8} was suggested. In framework of the symplectic tomography scheme the Wigner function and density operator can be reconstructed using Fourier-like integral (instead of Radon integral) of the symplectic tomogram. In \cite{9} the tomographic approach was shown to be connected with well-known star-product quantization procedure \cite{10}.

In quantum information theory and positive operator valued measures approach (in the context of quantum measurements) \cite{11} the notion of quantum probability measure for a generic variable is a basic concept. Namely, to each pair (\(\hat{\rho}, \hat{M}\)) consisting of the quantum state (the density operator) \(\hat{\rho}\) and the positive operator valued measure \(\hat{M}\), which can be identified with the quantum observable, it is associated
axiomatically (in the way of [12]) the probability measure $\mu_{\hat{\rho}}^{\hat{M}}$ which determines the readings of a classical measuring instrument. Taking into account the ensemble $\{\mu_{\hat{\rho}}^{\hat{M}}\}$ for the fixed measure $\hat{M}$ and all states $\hat{\rho}$, it is possible to reconstruct $\hat{M}$. In quantum information theory [13], the states $\hat{\rho}$ play a role of quantum probability measures used to encode the information. After the states $\hat{\rho}$ are transmitted through the quantum channel, one uses $\hat{M}$ as decision rules allowing to decode the data from the channel output. It is worthy to point out that one can connect a state with a probability distribution density for arbitrary observable $\hat{a}$ using the characteristic function techniques (see [14]). This construction is related to the construction given by von Neumann in [15], which is used in quantum information theory [11, 13].

The aim of our work is to establish a new connection of tomographic probability with such well studied mathematical object as quantum probability theory in addition to known connection with the star-product. Namely, we show that the symplectic tomogram can be obtained as a partial derivative of the distribution function of quantum probability measure. Till now the explicit relations between the tomographic probability distribution (tomogram) of quantum states and quantum probability measure concept (for the specific observable) used in quantum information theory were not known.

The paper is organized as follows: In Section 2 we review the quantum probability theory. In Section 3 and 4 symplectic tomography and spin tomography respectively will be discussed. In Section 5 connection of tomogram with quantum probability measure will be obtained. In Section 6 new concept of star-product for symplectic quantum measures will be introduced. In Section 7 evolution equation for symplectic quantum measures will be obtained. Section 8 is devoted to conclusions and perspectives.

2. Basic concepts of quantum probability

Let $H$ be a separable Hilbert space. A positive linear operator $\hat{\rho}$ in $H$ is said to be a state (density operator) if $Tr\hat{\rho} = 1$. We denote $\mathcal{L}(H)$, $\mathcal{L}_+(H) \subset \mathcal{L}(H)$, $\sigma(H)$ and $\hat{I}$ the sets of hermitian operators (quantum observables), positive operators, states and the identity operator in $H$ correspondingly. A map $\hat{M}$ transmitting each Borel subset $\Omega \subset \mathbb{R}$ to a positive operator $\hat{M}(\Omega) \in \mathcal{L}_+(H)$ is said to be a positive operator valued measure (POVM) if $\sum_i \hat{M}(\Omega_i) = \hat{I}$ for any finite or countable fragmentation $\mathbb{R} = \bigcup \Omega_i$ in the sense of the strong operator convergence of the series. The POVM $\hat{M}$ is said to be orthogonal if $\hat{M}(\Omega)^2 = \hat{M}(\Omega)$. 
By means of the spectral theorem, given \( \hat{a} \in \mathcal{L}(H) \) there exists an orthogonal POVM \( \hat{M} \) such that

\[
\hat{a} = \int_{\mathbb{R}} X d\hat{M}((-\infty, X]) = \int_{\mathbb{R}} X \delta(\hat{a} - X) dX,
\]

where \( \delta(\hat{a} - X) \) is a density of the measure \( \hat{M} \). It means that the measure \( \hat{M} \) is expressed in terms of the Heaviside function

\[
\hat{M}((-\infty, X]) = \theta(\hat{a} - X) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{k} e^{-ik(\hat{a} - X)} dk.
\]

The presentation of the spectral theorem by means of the operator-valued delta function can be understood as one can check that matrix elements of last two parts of (1) in eigenvector basis of the operator \( \hat{a} \) coincide if we use standard Dirac delta function properties. In this way, it is possible to identify observables \( \hat{a} \in \mathcal{L}(H) \) with orthogonal POVM. We shall say that \( \hat{a} \in \mathcal{L}(H) \) is associated with the POVM \( \hat{M} \) appearing in its spectral representation. Taking \( \hat{\rho} \in \sigma(H) \) and the POVM \( \hat{M} \) one can put

\[
\mathcal{M}_{\hat{\rho}}(\hat{M})(\Omega) = Tr(\hat{\rho} \hat{M}(\Omega))
\]

for any Borel set \( \Omega \subset \mathbb{R} \). Then \( \mathcal{M}_{\hat{\rho}}(\hat{M}) \) is a classical probability measure on \( \mathbb{R} \). Calculating the matrix \( \rho(x, x') \) of the density operator \( \hat{\rho} \) in the basis in which \( \hat{a} \) is diagonal, we can represent (2) in the form

\[
\mathcal{M}_{\hat{\rho}}(\hat{M})(\Omega) = \int_{\Omega} \rho(x, x) dx.
\]

If the POVM \( \hat{M} \) is associated with some observable \( \hat{a} \in \mathcal{L}(H) \), we shall write \( \mathcal{M}_{\hat{\rho}}^{\hat{M}} \) as well as \( \mathcal{M}_{\hat{\rho}}^{\hat{a}} \). In quantum probability theory \([1, 4]\), it is postulated that \( \mathcal{M}_{\hat{\rho}}^{\hat{M}} = \mathcal{M}_{\hat{\rho}}^{\hat{a}} \) defined by the formula (2) gives us a distribution of the readings of a classical instrument measuring the quantum observable \( \hat{a} \) associated with POVM \( \hat{M} \) in the state \( \hat{\rho} \). Notice that for the orthogonal POVM the approach introduced above appeared firstly in \([15]\). The same probability distribution can be obtained by means of the characteristic functions techniques \([14]\). Alternatively, in quantum information theory \([13]\) \( \hat{M} \) determines certain decision rule allowing to decode the information containing in the state \( \hat{\rho} \).

For example, consider the position operator \( \hat{x} \in \mathcal{L}(H) \) acting in the Hilbert space \( H = L^2(\mathbb{R}) \) by the fromula \( (\hat{x}f)(x) = xf(x), \ f \in H \).
Denote \(\chi_{\Omega}\) the characteristic (indicator) function of the Borel set \(\Omega \subset \mathbb{R}\) such that
\[
\chi_{\Omega}(x) = \begin{cases} 
1, & x \in \Omega \\
0, & x \notin \Omega
\end{cases}
\]
Involve the operator \(\hat{\chi}_{\Omega}\) acting by the formula \((\hat{\chi}_{\Omega} f)(x) = \chi_{\Omega}(x)f(x), f \in H\). Then the map \(\hat{M}\) defined for a Borel set \(\Omega\) by the condition \(\hat{M}(\Omega) = \hat{\chi}_{\Omega}\) is an orthogonal POVM on \(\mathbb{R}\). One can see that \(\hat{M}\) gives us the spectral decomposition of the operator \(\hat{x}\) such that
\[
\hat{x} = \int_{\mathbb{R}} X d\hat{\chi}_{(-\infty, x]}.
\]
Denote \(|X><X|\) the density of the orthogonal POVM \(\hat{\chi}\), then (4) transforms to the following,
\[
\hat{x} = \int_{\mathbb{R}} X|X><X|dX,
\]
\[
\int_{\mathbb{R}} |X><X|dX = \hat{I}.
\]
In this way, the probability measure determined by the state \(\hat{\rho}\) can be written as
\[
\mathcal{M}_{\hat{\rho}}^{\hat{x}}(\Omega) = Tr(\hat{\rho}\hat{\chi}_{\Omega}) = \int_{\Omega} \rho(X, X)dX,
\]
where \(\rho(X, X')\) is a matrix of the density operator \(\hat{\rho}\) in the basis consisting of generalized eienvectors \(|X><X|\). For a pure vacuum state \(\hat{\rho}_0 = |\psi_0><\psi_0|\) in the Schrodinger representation
\[
<x|\psi_0> = \frac{1}{(\pi)^{1/4}} exp(-\frac{x^2}{2})
\]
the formula (5) yields
\[
\mathcal{M}_{\hat{\rho}_0}^{\hat{x}}(\Omega) = \frac{1}{\sqrt{\pi}} \int_{\Omega} exp(-x^2)dx,
\]
which is a Gaussian measure with the zero mean and the variance equal to 1/2.
Let \(\hat{J}_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\) be an operator of spin projection on the \(z\)-axis for the particle with the total spin \(J = \frac{1}{2}\). Picking up the Euler angles
\(\phi, \psi, \theta\) one can define the rotation matrix by the formula

\[
R(\phi, \psi, \theta) = \begin{pmatrix}
\cos \frac{\theta}{2} e^{i(\phi+\psi)/2} & \sin \frac{\theta}{2} e^{-i(\phi-\psi)/2} \\
\sin \frac{\theta}{2} e^{i(\phi-\psi)/2} & \cos \frac{\theta}{2} e^{-i(\phi+\psi)/2}
\end{pmatrix}.
\]

Notice that (7) determines irreducible representation of the group \(SU(2)\). It is straightforward to check that a spectral decomposition of the operator \(\hat{a} = R(\phi, \psi, \theta) \hat{J}_z R(\phi, \psi, \theta)^{-1}\) is given by the formula

\[
\hat{a} = \left( \frac{1}{2} \cos \theta e^{-i\psi} - \frac{i}{2} \sin \theta e^{i\psi} \right) \left( \frac{1}{2} \sin \theta e^{i\psi} - \frac{i}{2} \cos \theta e^{-i\psi} \right).
\]

Using (8) we obtain for the probability measure (2) which is discrete in the state \(\hat{\rho} = |\psi><\psi|\) with \(|\psi> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) the distribution function as

\[
F(x) = M^{\rho}_y((-\infty, x]) = \begin{cases} 
0, & x < -\frac{1}{2} \\
\cos^2 \frac{\theta}{2}, & -\frac{1}{2} \leq x < \frac{1}{2} \\
1, & x \geq \frac{1}{2}
\end{cases}
\]

The function (9) gives us the Bernoulli distribution concentrated in two points \(x_1 = -\frac{1}{2}\) and \(x_2 = \frac{1}{2}\) with the probabilities \(p = \cos^2 \frac{\theta}{2}\) and \(1 - p = \sin^2 \frac{\theta}{2}\) correspondingly.

3. Tomographic representations for continuous variables

Tomographic probability (tomogram) determining a quantum state is introduced by relation

\[
w(X, \mu, \nu) = \langle \delta(X - \mu \hat{x} - \nu \hat{p}) \rangle_{\hat{\rho}},
\]

where \(\hat{x}\) and \(\hat{p}\) are position and momentum operators and the density operator \(\hat{\rho}\) defines the averaging for arbitrary observable \(\hat{a}\) by

\[
\langle \hat{a} \rangle_{\hat{\rho}} = Tr(\hat{\rho} \hat{a}).
\]

The Dirac \(\delta\)-function in (10) is defined by its Fourier decomposition as

\[
\delta(\hat{a}) = \frac{1}{2\pi} \int e^{ik\hat{a}} dk.
\]

According to \(\delta\) the tomogram (10) is related to the density matrix in position representation \(\rho(y, y')\) by

\[
w(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \int \int \rho(y, y') e^{i(\mu y - \nu y')} dy dy'.
\]
and with the Wigner function defined in [4] as
\begin{equation}
W(q, p) = \int \rho(q + \frac{u}{2}, q - \frac{u}{2}) e^{-ipu} du
\end{equation}
via the relation
\begin{equation}
w(X, \mu, \nu) = \int \int W(q, p) \delta(X - \mu q - \nu p) \frac{dq dp}{2\pi}.
\end{equation}
The tomogram determines the Wigner function as
\begin{equation}
W(q, p) = \frac{1}{2\pi} \int \int \int w(X, \mu, \nu) e^{i(X - \mu q - \nu p)} dXd\mu d\nu.
\end{equation}
It means that the tomogram can be used to describe the quantum states completely. In terms of a wave function \(\psi(x)\) the tomogram reads for the pure state \(\rho = |\psi><\psi|\)
\begin{equation}
w(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \int \psi(y) e^{-i\mu y} e^{\frac{i\nu y}{\nu}} dy^2.
\end{equation}
Put
\[\hat{\rho}_n = |\psi_n><\psi_n|, \ n = 0, 1, 2, \ldots,\]
\begin{equation}
< X|\psi_n >= \frac{1}{(\pi)^{1/4}} \frac{1}{\sqrt{2^n n!}} \text{H}_n(X) \exp\left(-\frac{X^2}{2}\right)
\end{equation}
are wave functions of the excited state of an oscillator. Here \(\text{H}_n, \ n = 0, 1, 2, \ldots\) are the Hermite polynomials. Then [17] gives us the tomogram for the state \(\hat{\rho}_n\) as
\begin{equation}
w_n(X, \mu, \nu) = \frac{1}{\sqrt{\pi} 2^n n!} \frac{1}{\sqrt{\mu^2 + \nu^2}} \text{H}_n\left(\frac{X}{\sqrt{\mu^2 + \nu^2}}\right) \exp\left(-\frac{X^2}{\mu^2 + \nu^2}\right).
\end{equation}
Let us involve the wave function of a coherent state \(|\psi_\alpha><\psi_\alpha|, \ \alpha \in \mathbb{C}\),
\begin{equation}
< X|\psi_\alpha >= \frac{1}{(\pi)^{1/4}} \exp\left(-\frac{X^2}{2} + \sqrt{2}\alpha X - \frac{\alpha^2}{2} - \frac{\alpha^2}{2}\right).
\end{equation}
Then the tomogram of the state \(|\psi_\alpha><\psi_\alpha|\) is given by
\begin{equation}
w_\alpha(X) = \frac{1}{\sqrt{\pi(\mu^2 + \nu^2)}} \exp\left(-\frac{(X - \sqrt{2\text{Re}\alpha \mu} - \sqrt{2\text{Im}\alpha \nu})^2}{\mu^2 + \nu^2}\right).
\end{equation}
The most important property of the tomogram is that it is a standard density of the probability distribution function on \(\mathbb{R}\), i.e.
\begin{equation}
w(X, \mu, \nu) \geq 0
\end{equation}
and

\[(22) \quad \int w(X, \mu, \nu) dX = 1.\]

The physical meaning of the real variable \(X\) is that this variable is equal to position of a particle measured in rotated and scaled reference frame of the particle phase space, in which we get

\[(23) \quad X = \mu q + \nu p, \quad \mu = e^\lambda \cos \phi, \quad \nu = e^{-\lambda} \sin \phi,\]

where \(\phi\) and \(\lambda\) are an angle of the rotation and a real squeezing parameter correspondingly.

4. Tomography of spin

Here we review the tomogram of discrete variables (spin). The tomogram of a spin state is defined by the relation

\[(24) \quad w^{(j)}(m, \theta, \phi) = \langle \delta(m - R^{(j)}(\phi, \psi, \theta) \hat{J}_z R^{(j)}(\phi, \psi, \theta)^{-1}) \rangle_\rho,\]

where \(\delta\) is the Kronecker operator delta function which for arbitrary hermitian operator \(\hat{u}\) with integer eigenvalues is given by the Fourier integral

\[\delta(\hat{u}) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\phi \hat{u}} d\phi,\]

a nonnegative half-integer \(j\) is the total spin, \(m = -j, -j + 1, \ldots, j\) is the spin projection on \(z\)-axis, \(\hat{J}_z\) is a operator of spin projection on the \(z\)-axis and \(R^{(j)}(\phi, \psi, \theta)\) is the matrix of irreducible representation of the group \(SU(2)\) in standard basis, elements of the group are parametrized by the Euler angles \(\phi, \psi, \theta\). The state of spin is determined by the hermitian nonnegative \(2j + 1 \times 2j + 1\) - density matrix with unit trace. One can show [17, 18] that the density operator can be found from \(24\). In fact, the formula \(24\) can be considered as a linear system of equations determining the density matrix if the tomogram is a known function.

5. Quantum probability and tomograms

By the definition of the orthogonal POVM \(\hat{M}\) given by the spectral decomposition of the operator \(\hat{a}\), we get \(\frac{d}{dx} \hat{M}((\infty, X)) = \delta(X - \hat{a})\). Suppose that \(\hat{M}\) is determined by the spectral decomposition of \(\mu \hat{x} + \nu \hat{p}\). Comparing [10, 11] and \(2\) we get

\[(25) \quad w(X, \mu, \nu) = \frac{d}{dx} M_{\rho}^{\mu \hat{x} + \nu \hat{p}}((\infty, X)).\]
Using the relation (25) one can calculate the distribution function of the measure $\mathcal{M}_{\rho_n}^{\hat{x}+\hat{p}}$ for the states $\hat{\rho}_n$ (18) in which the tomograms $w_n$ are given by the formula (19) such that

\[
\mathcal{M}_{\rho_n}^{\hat{x}+\hat{p}}((\infty, X]) = \frac{1}{\sqrt{\pi (\mu^2 + \nu^2)}} \int_{-\infty}^{X} H_n^2 \left( \frac{y}{\sqrt{\mu^2 + \nu^2}} \right) \exp \left( - \frac{y^2}{\mu^2 + \nu^2} \right) dy.
\]

Take the function

\[
\phi_\alpha(x) = \frac{1}{(\pi)^{1/4}} \exp \left( - \frac{x^2}{2} + \sqrt{2\alpha x} - \frac{\alpha^2}{2} \right)
\]

and decompose it into the series over the excited wave functions of an oscillator $\psi_n(x)$ (18), then

\[
\phi_\alpha(x) = \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}} \psi_n(x).
\]

It follows that

\[
\sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{\alpha^n \beta^m}{\sqrt{n!} \sqrt{m!}} \int_{-\infty}^{X} \psi_n(y) \psi_m(y) dy = \int_{-\infty}^{X} \phi_\alpha(y) \phi_\beta(y) dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{X} \exp \left( - y^2 + \sqrt{2(\alpha + \beta)y - \frac{\alpha^2 + \beta^2}{2}} \right) dy = \frac{1}{2} e^{\alpha \beta} \left( 1 + \text{erf} \left( \frac{X}{\sqrt{\mu^2 + \nu^2}} \right) \right).
\]

Using (27) we obtain for the probability measure (26)

\[
\mathcal{M}_{\rho_0}^{\hat{x}+\hat{p}}((\infty, X]) = \frac{1}{2} (1 + \text{erf} \left( \frac{X}{\sqrt{\mu^2 + \nu^2}} \right)),
\]

which is in accordance with (6) if one put $\mu = 1, \nu = 0$. 

In this way, using the tomogram it is possible to obtain the probability measure $\mathcal{M}_\rho^{\mu \hat{x} + \nu \hat{p}}$ without a calculation of the spectral decomposition for the operator $\mu \hat{x} + \nu \hat{p}$. We shall call $\mathcal{M}_\rho^{\mu \hat{x} + \nu \hat{p}}$ by a symplectic quantum probability measure.

Analogously, suppose that the spectral decomposition of the operator $\hat{a} = R^{(j)}(\phi, \psi, \theta) \hat{J}_z R^{(j)}(\phi, \psi, \theta)^{-1}$ has the following form

$$\hat{a} = \sum_{k=-j}^{j} k \hat{M}_k,$$

where $\hat{M}_k$, $-j \leq k \leq j$, form the orthogonal POVM. It follows from (24) that

$$w^{(j)}(m, \theta, \phi) = \sum_{k=-j}^{j} \delta_{km} \text{Tr}(\hat{\rho} \hat{M}_k) =$$

$$\sum_{k=-j}^{j} \delta_{km} \mathcal{M}_\rho^{\hat{a}}((k-1, k]) = \mathcal{M}_\rho^{\hat{a}}((m-1, m]],$$

where $\delta_{km}$ is the Kronecker symbol, $m = -j, -j + 1, \ldots, j$.

6. Star-product for symplectic quantum measures

To formulate quantum mechanics using a map $\hat{a} \to f_\hat{a}(\vec{x})$ between the set of hermitian operators and the set of functions on appropriate space, one needs to involve a new multiplication rule for the functions which should be associative. This multiplication is said to be a star-product (see [10]) and, in particular, it can be involved for the functions defined by the map $(\vec{x} = (X, \mu, \nu))$

$$f_\hat{a}(\vec{x}) = \text{Tr}(\hat{a} \delta(X - \mu \hat{x} - \nu \hat{p}))$$

in the way of [9] such that

$$f_\hat{a} * f_\hat{b}(\vec{x}) = \int K(\vec{x}_1, \vec{x}_2, \vec{x}) f_\hat{a}(\vec{x}_1) f_\hat{b}(\vec{x}_2) d\vec{x}_1 d\vec{x}_2,$$

where the kernel $K(\vec{x}_1, \vec{x}_2, \vec{x})$ is written as

$$K(\vec{x}_1, \vec{x}_2, \vec{x}) = \text{Tr}(\hat{D}(\vec{x}_1) \hat{D}(\vec{x}_2) \delta(X - \mu \hat{x} - \nu \hat{p}))$$

and $d\vec{x} = dX d\mu d\nu$. Here the operator $\hat{D}(\vec{x})$ is defined by the formula

$$\hat{D}(\vec{x}) = \frac{1}{2\pi} e^{iX} e^{-i\mu \hat{x} - i\nu \hat{p}}.$$
The operator \( \hat{a} \) can be reconstructed from the function \( f_\hat{a}(\vec{x}) \) by the formula

\[
\hat{a} = \int \int \int f_\hat{a}(\vec{x}) \hat{D}(\vec{x}) dX d\mu d\nu.
\]

In particular, if \( \hat{a} = \hat{\rho} \in \sigma(H) \) is a state, then (28) yields a number of the probability distribution densities associated with \( \hat{\rho} \).

Now consider the map

\[
\hat{\rho} \rightarrow M_\hat{\rho}(\vec{x}) = M_{\hat{\rho}^{\mu+\nu}}((\infty, X])
\]

from the set of states \( \sigma(H) \) to the set of probability measures on the real line. The spectral theorem determines a representation of arbitrary observable \( \hat{a} \) as an integral over pure states. Hence it is possible to extend the map (31) from the states \( \hat{\rho} \) to all observables \( \hat{a} \) by means of a linearity. Taking the spectral decomposition \( \mu \hat{x} + \nu \hat{p} = \int X d\hat{M}((\infty, X]) \) we get

\[
M_\hat{a}(\vec{x}) = Tr(\hat{a} \hat{M}((\infty, X])).
\]

Here for an arbitrary observable \( \hat{a} \) we obtain the family \( M_\hat{a}(\vec{x}) \) determining a non-positive Borel measure on the real line for any fixed \( \mu \) and \( \nu \). We shall call them by symplectic quantum measures. Comparing (30), (25) and (32) we obtain

\[
\frac{d}{dX} M_\hat{a}(\vec{x}) = f_\hat{a}(\vec{x})
\]

and the operator \( \hat{a} \) can be reconstructed from the family of symplectic quantum measures \( M_\hat{a} \) such that

\[
\hat{a} = \int \int \int \hat{D}(\vec{x}) dM_\hat{a}(X) d\mu d\nu.
\]

Here \( dM_\hat{a}(X) \) denotes the measure on the real line determined by \( M_\hat{a}(\vec{x}) \) for the fixed \( \mu \) and \( \nu \). In the formula (34) and below we claim that, at first, the integration is done over \( dM_\hat{a}(X) \) for the fixed \( \mu, \nu \) and then we integrate over \( d\mu \) and \( d\nu \). The formula (34) allows to define an associative multiplication for the symplectic quantum measures \( M_\hat{a} \) which can be derived from (29),

\[
M_\hat{a} \ast M_\hat{b}(\vec{x}) = \int \int \int K(\vec{x}_1, \vec{x}_2, \vec{x}) dM_\hat{a}(X_1) d\mu_1 d\nu_1 dM_\hat{b}(X_2) d\mu_2 d\nu_2,
\]

where the kernel \( K(\vec{x}_1, \vec{x}_2, \vec{x}) \) defined by the formula

\[
K(\vec{x}_1, \vec{x}_2, \vec{x}) = Tr(\hat{D}(\vec{x}_1) \hat{D}(\vec{x}_2) \hat{M}((\infty, X])),
\]
\( \vec{x}_1 = (X_1, \mu_1, \nu_1), \vec{x}_2 = (X_2, \mu_2, \nu_2) \). By a construction the multiplication (35) is in accordance with the obvious multiplication on the set of observables, such that

\[
\mathcal{M}_{\hat{a}\hat{b}}(\vec{x}) = \mathcal{M}_{\hat{a}} \ast \mathcal{M}_{\hat{b}}(\vec{x}).
\]

It follows from (36) that the symplectic probability measures \( \mathcal{M}_\rho \) associated with the pure states \( \hat{\rho} = |\psi> <\psi| \) are idempotents with respect to the multiplication \( \ast \), i.e.

\[
\mathcal{M}_{|\psi><\psi|} \ast \mathcal{M}_{|\psi><\psi|} = \mathcal{M}_{|\psi><\psi|}.
\]

7. Evolution for symplectic quantum measures

It was shown in [19] that if the quantum state satisfies the time-evolution equation

\[
\partial_t \hat{\rho} = -i[\hat{H}, \hat{\rho}],
\]

then for systems with Hamiltonian of the form \( \hat{H} = \frac{\hat{p}^2}{2} + V(\hat{q}) \) the evolution of the quantum tomogram (10) by virtue of (37) is given as

\[
\frac{\partial w}{\partial t} - \mu \frac{\partial w}{\partial \nu} - \frac{\partial w}{\partial X} = 0.
\]

It is straightforward to check that the equation (38) determines the evolution of the functions (28) and, therefore, by means of the connection (33), of the distribution functions \( \mathcal{M}_{\hat{a}}(X) \) of symplectic quantum measures (32). Integrating (38) over \( X \) we obtain the following form of evolution,

\[
\frac{\partial \mathcal{M}_{\hat{a}}(X)}{\partial t} - \mu \frac{\partial \mathcal{M}_{\hat{a}}(X)}{\partial \nu} - \frac{\partial \mathcal{M}_{\hat{a}}(X)}{\partial X} = 0.
\]

Note that we have derived the linear evolution equation valid only for symplectic quantum measures. The general equation for continuous nondemolition measurement introduced in [20, 21] is nonlinear.
8. Conclusion

To conclude we summarise the main results of the paper. We established relation (see Equ. (25)) of symplectic tomograms to distribution functions of symplectic quantum probability measures, which gives a possibility to use results of the quantum probability theory to study properties of quantum tomograms. On the base of the established relation the evolution equation for the symplectic (non-positive in general) quantum measures is obtained (see Equ. (39)). The star product of symplectic quantum measures is introduced (see Equ. (35)). The approach is illustrated by example of several different states of harmonic oscillator. In addition, the quantum probability measures and its connection with tomogram are constructed for such discrete observable as spin.

References

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