Slow foliation of a slow-fast stochastic evolutionary system

Guanggan Chen
College of Mathematics and Software Science,
Sichuan Normal University, Chengdu, 610068, China
E-mail: chenguanggan@hotmail.com

Jinqiao Duan
Institute for Pure and Applied Mathematics,
University of California, Los Angeles, CA 90095, USA
&
Department of Applied Mathematics,
Illinois Institute of Technology, Chicago, IL 60616, USA
E-mail: duan@iit.edu

Jian Zhang
College of Mathematics and Software Science,
Sichuan Normal University, Chengdu, 610068, China
E-mail: zhangjiancdv@sina.com

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Dedicated to Professor Zhiming Ma on the Occasion of his 65th Birthday

Abstract: This work is concerned with the dynamics of a slow-fast stochastic evolutionary system quantified with a scale parameter. An invariant foliation decomposes the state space into geometric regions of different dynamical regimes, and thus helps understand dynamics. A slow invariant foliation is established for this system. It is shown that the slow foliation converges to a critical foliation (i.e., the scale parameter is zero) in probability distribution, as the scale parameter tends to zero. The approximation of slow foliation is also constructed with error estimate in distribution. Furthermore, the geometric structure of the slow foliation is investigated: every fiber of the slow foliation parallels each other, with the slow manifold as a special fiber. In fact, when an arbitrarily chosen point of a fiber falls in the slow manifold, the fiber must be the slow manifold itself.

Key words: Invariant foliation; slow manifold; slow-fast stochastic evolutionary system; geometric structure

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1 Introduction

Random fluctuations may have delicate effects on dynamical evolution of complex systems ([1, 7, 10, 26]). The slow-fast stochastic evolutionary systems are appropriate mathematical models for various multi-scale systems under random influences.

We consider the following slow-fast stochastic evolutionary system

\[
\begin{align*}
\frac{dx^\varepsilon}{dt} &= Ax^\varepsilon + f(x^\varepsilon, y^\varepsilon) + \sigma_1 \dot{W}_1, \quad \text{in } H_s, \quad (1.1) \\
\frac{dy^\varepsilon}{dt} &= \frac{1}{\varepsilon} By^\varepsilon + \frac{1}{\varepsilon} g(x^\varepsilon, y^\varepsilon) + \frac{\sigma_2}{\sqrt{\varepsilon}} \dot{W}_2, \quad \text{in } H_f, \quad (1.2)
\end{align*}
\]

where \(\varepsilon\) is a small positive parameter \((0 < \varepsilon \ll 1)\). The Hilbert spaces \(H_s\) and \(H_f\), linear operators \(A\) and \(B\), nonlinearities \(f\) and \(g\), and mutually independent Wiener processes \(W_1\) and \(W_2\) will be specified in the next section. The white noises \(\dot{W}_1\) and \(\dot{W}_2\) are the generalized time derivatives of \(W_1\) and \(W_2\), respectively. The positive constants \(\sigma_1\) and \(\sigma_2\) are the intensities of white noises. Since the small scale parameter \(\varepsilon\) is such that \(\|\frac{dx}{dt}\|_{H_s} \ll \|\frac{dy}{dt}\|_{H_f}\), we usually say that \(x\) is the “slow” component and \(y\) is the “fast” component.

The main goal of this paper is to investigate state space decomposition for this system, by considering a slow invariant foliation, and examining its approximation and structure.

Invariant foliations and invariant manifolds play a significant role in the study of the qualitative dynamical behaviors, as they provide geometric structures to understand or reduce stochastic dynamics ([4, 5, 6, 11, 12, 13, 19, 20, 21]). An invariant foliation is about quantifying certain sets (called fibers or leaves) in state space for a dynamical system. A fiber consists of all those points starting from which the dynamical orbits are exponentially approaching each other, in forward time (“stable foliation”) or backward time (“unstable foliation”). These fibers are thus building blocks for understanding dynamics, as they carry dynamical information. Collectively they provide a decomposition of the state space.

For a system like (1.1)-(1.2), Schmalfuss and Schneider [22] studied the slow manifold in the finite dimensional case. Wang, Duan, and Roberts [24, 25] further studied the slow manifold, and a relation with averaging as quantified via large deviations and approximations. In the infinite dimensional setting, Fu, Liu and Duan [15] investigated the slow manifold and its approximation. These research works are at the level of geometric and global invariant sets.
In the context of analyzing individual sample solution paths, Freidlin [14] used large deviation theory to describe the dynamics, and Berglund and Gentz [3] showed that the sample solution paths are concentrated in a neighborhood of the critical manifold (also see [17]).

Although invariant foliation theory has been developed for deterministic systems in [2, 8, 9], it is still in infancy for stochastic evolutionary systems. Recently, Lu and Schmalfuss [18] studied the existence of random invariant foliation for a class of stochastic partial differential equations, and Sun, Kan and Duan [23] established the approximation of random invariant foliations.

We define that a slow foliation of a slow-fast system to be the foliation in which the fibers are parameterized or represented by slow variables, when the scale parameter $\varepsilon$ is sufficiently small. In a sense, the fast variables are eliminated. A critical foliation corresponds to the foliation with zero scale parameter. Furthermore, the slow foliation converges to the critical foliation, as the singular perturbation parameter $\varepsilon$ tends to zero.

For system (1.1)-(1.2), we establish the existence of slow foliation, which is a graph of a Lipschitz continuous map. The dynamical orbits of the slow-fast stochastic system are exponentially approaching each other in backward time only if they start from the same slow fiber. In addition, we show the slow foliation converges to a critical foliation in probability distribution, as $\varepsilon$ tends to zero. Furthermore, we examine the geometric structure of the slow foliation and show that fibers of the slow foliation parallel with each other. In fact, the slow manifold is one fiber of the slow foliation. When an arbitrarily chosen point of the slow foliation is in the slow manifold, the fiber passing through the point is just the slow manifold.

This paper is organized as follows. In the next section, we make hypotheses for the slow-fast system and recall basic concepts in random dynamical systems, including random slow manifolds. In §3, we present a motivating example about slow foliation. In §4, we prove the existence of slow foliation (Theorem 4.1), examine the geometric structure of the slow foliation, and analyze a relationship between the slow foliation and the slow manifold (Theorem 4.2). In §5, we establish the existence of a critical foliation (Theorem 5.1), prove the convergence of the slow foliation to the critical foliation in probability distribution as the scale parameter tends to zero (Theorem 5.2), and construct an approximation of slow foliation in probability distribution (Theorem 5.3).
2 Preliminaries

2.1 Basic setup

For the slow-fast system (1.1)-(1.2), let $H_s$ and $H_f$ be two separable Hilbert spaces with the norms $\| \cdot \|_{H_s}$ and $\| \cdot \|_{H_f}$, respectively. The space $H_s$ denotes the state space for slow variables, and $H_f$ the state space for fast variables. Henceforth, we use the subscripts or superscripts “$s$” and “$f$” to denote those spaces or quantities that are related to the slow variables and fast variables, respectively. We introduce the following hypotheses.

**Hypothesis H1** (Dichotomy condition): The linear operator $A$ generates a $C_0$-semigroup $e^{At}$ on $H_s$ satisfying

$$\|e^{At}x\|_{H_s} \leq e^{-\gamma_s t}\|x\|_{H_s}, \quad \text{for } t \leq 0,$$

and the linear operator $B$ generates a $C_0$-semigroup $e^{Bt}$ on $H_f$ satisfying

$$\|e^{Bt}y\|_{H_f} \leq e^{-\gamma_f t}\|y\|_{H_f}, \quad \text{for } t \geq 0,$$

where $\gamma_s < 0 < \gamma_f$.

**Hypothesis H2** (Lipschitz condition): The nonlinear functions

$$f : H_s \times H_f \rightarrow H_s,$$
$$g : H_s \times H_f \rightarrow H_f,$$

are $C^1$-smooth with $f(0,0) = 0$ and $g(0,0) = 0$, and satisfy a Lipschitz condition, i.e., there exists a positive constant $K$ such that for every $(x, y)^T \in H_s \times H_f$ and every $(\tilde{x}, \tilde{y})^T \in H_s \times H_f$, we have

$$\|f(x, y) - f(\tilde{x}, \tilde{y})\|_{H_s} \leq K(\|x - \tilde{x}\|_{H_s} + \|y - \tilde{y}\|_{H_f}),$$
$$\|g(x, y) - g(\tilde{x}, \tilde{y})\|_{H_f} \leq K(\|x - \tilde{x}\|_{H_s} + \|y - \tilde{y}\|_{H_f}).$$

Here and hereafter, the superscript “$T$” denotes the matrix transpose.

**Hypothesis H3** (Gap condition): The Lipschitz constant $K$ of the nonlinear functions $f$ and $g$ satisfies the condition $K < \frac{-\gamma_s - \gamma_f}{2\gamma_f - \gamma_s}$.

2.2 Random dynamical systems

We recall some basic concepts in random dynamical systems (12). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A flow $\theta$ of mappings $\{\theta_t\}_{t \in \mathbb{R}}$ is defined on the sample space $\Omega$ such that

$$\theta : \mathbb{R} \times \Omega \rightarrow \Omega, \quad \theta_0 = id, \quad \theta_{t_1}\theta_{t_2} = \theta_{t_1 + t_2},$$

(2.1)
for \( t_1, t_2 \in \mathbb{R} \). This flow is assumed to be \((B(\mathbb{R}) \otimes F, F)\)-measurable, where \( B(\mathbb{R}) \) is the \( \sigma \)-algebra of Borel sets on the real line \( \mathbb{R} \). To have this measurability, it is not allowed to replace \( F \) by its \( \mathbb{P} \)-completion \( \mathbb{F}^\mathbb{P} \); see Arnold [1, P547]. In addition, the measure \( \mathbb{P} \) is assumed to be ergodic with respect to \( \{\theta_t\}_{t \in \mathbb{R}} \). Then \( \Theta = (\Omega, \mathbb{F}, \mathbb{P}, \theta) \) is called a metric dynamical system.

For our purpose, we will consider a special but very important metric dynamical system induced by the Wiener process. Let \( W(t) \) be a two-sided Wiener process taking values in a Hilbert space \( H \). Its sample paths are in the space \( C_0(\mathbb{R}, H) \) of real continuous functions defined on \( \mathbb{R} \), taking zero value at \( t = 0 \). This set is equipped with the compact open topology. On this set we consider the measurable flow \( \theta = \{\theta_t\}_{t \in \mathbb{R}} \), defined by

\[
\theta_t \omega = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.
\]

The distribution of this process induces a probability measure on \( B(C_0(\mathbb{R}, H)) \) and it is called the Wiener measure. Note that this measure is ergodic with respect to \( \theta_t \); see [1, Appendix A].

We also consider, instead of the whole \( C_0(\mathbb{R}, H) \), a \( \{\theta_t\}_{t \in \mathbb{R}} \)-invariant subset \( \Omega \subset C_0(\mathbb{R}, H) \) of \( \mathbb{P} \)-measure one and the trace \( \sigma \)-algebra \( F \) of \( B(C_0(\mathbb{R}, H)) \) with respect to \( \Omega \). Recall that a set \( \Omega \) is called \( \{\theta_t\}_{t \in \mathbb{R}} \)-invariant if \( \theta_t \Omega = \Omega \) for \( t \in \mathbb{R} \). On \( F \), we consider the restriction of the Wiener measure and still denote it by \( \mathbb{P} \).

In general, the dynamics of a stochastic system on the state space \( H \) (often a Hilbert space) over the flow \( \theta \) is described by a cocycle. A cocycle \( \phi \) is a mapping:

\[
\phi : \mathbb{R} \times \Omega \times H \rightarrow H,
\]

which is \((B(\mathbb{R}) \otimes \mathbb{F} \otimes B(H), \mathbb{F})\)-measurable such that

\[
\phi(0, \omega, x) = x,
\]

\[
\phi(t_1 + t_2, \omega, x) = \phi(t_2, \theta_{t_1} \omega, \phi(t_1, \omega, x)),
\]

for \( t_1, t_2 \in \mathbb{R}, \omega \in \Omega \) and \( x \in H \). Then \( \phi \) together with the metric dynamical system \( \theta \) forms a random dynamical system.

A stable fiber and an unstable fiber of a foliation are defined as follows (also see [9]).

(i) \( W_{\beta_s}(x, \omega) \) is called a \( \beta \)-stable fiber passing through \( x \in H \) with \( \beta \in \mathbb{R}^- \), if \( \|\phi(t, \omega, x) - \phi(t, \omega, x)\|_H = O(e^{\beta t}), \forall \omega \in \Omega \) as \( t \to +\infty \) for all \( x, \bar{x} \in W_{\beta_s} \).

(ii) \( W_{\beta_u}(x, \omega) \) is called a \( \beta \)-unstable fiber passing through \( x \in H \) with \( \beta \in \mathbb{R}^+ \), if \( \|\phi(t, \omega, x) - \phi(t, \omega, x)\|_H = O(e^{\beta t}), \forall \omega \in \Omega \) as \( t \to -\infty \) for all \( x, \bar{x} \in W_{\beta_u} \).
Stable fibers form a stable foliation, while unstable fibers form an unstable foliation. Occasionally we use $W_\beta$ to denote either fibers. Furthermore, we say a foliation is invariant if the random dynamical system $\phi$ maps one fiber to another fiber in the following sense

$$\phi(t, \omega, W_\beta(x, \omega)) \subset W_\beta(\phi(t, \omega, x), \theta_t \omega).$$

### 2.3 A slow-fast random dynamical system

Let $\Theta_1 = (\Omega_1, \mathcal{F}_1, P_1, \theta^1)$ and $\Theta_2 = (\Omega_2, \mathcal{F}_2, P_2, \theta^2)$ be two independent metric dynamical systems as introduced in Section 2.2. Define

$$\Theta := \Theta_1 \times \Theta_2 = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1 \times P_2, (\theta^1, \theta^2)^T),$$

and

$$\theta_t \omega := (\theta_t^1 \omega_1, \theta_t^2 \omega_2)^T, \quad \text{for} \quad \omega := (\omega_1, \omega_2)^T \in \Omega_1 \times \Omega_2 := \Omega.$$

Let $W_1(t)$ and $W_2(t)$ be two mutually independent standard Wiener processes with values in $H_s$ and $H_f$, with covariances $Q_1 = \text{Id}_{H_s}$ and $Q_2 = \text{Id}_{H_f}$, respectively.

Consider the following linear stochastic evolutionary equations

$$d\delta(t) = A\delta dt + \sigma_1 dW_1,$$

$$d\eta^\varepsilon(t) = \frac{1}{\varepsilon} B\eta^\varepsilon dt + \frac{\sigma_2}{\sqrt{\varepsilon}} dW_2,$$  

and

$$d\xi(t) = B\xi dt + \sigma_2 dW_2.$$  

**Lemma 2.1** Assume that the Hypothesis H1 holds. Then equations (2.2), (2.3) and (2.4) have continuous stationary solutions $\delta(\theta^1 \omega_1)$, $\eta^\varepsilon(\theta^2 \omega_2)$ and $\xi(\theta^2 \omega_2)$, respectively. Furthermore, the stochastic process $\eta^\varepsilon(\theta^2 \omega_2)$ has the same distribution as the process $\xi(\theta^2 \omega_2)$.

Introduce new variables

$$X^\varepsilon = x^\varepsilon - \delta(\theta^1 \omega_1), \quad \text{and} \quad Y^\varepsilon = y^\varepsilon - \eta^\varepsilon(\theta^2 \omega_2).$$  

Then the slow-fast stochastic evolutionary equations (1.1)-(1.2) can be rewritten as

$$\frac{dX^\varepsilon}{dt} = AX^\varepsilon + F(X^\varepsilon, Y^\varepsilon, \theta^t \omega),$$  

(2.6)
\[ \frac{dY^\varepsilon}{dt} = \frac{1}{\varepsilon} BY^\varepsilon + \frac{1}{\varepsilon} G(X^\varepsilon, Y^\varepsilon, \theta^\varepsilon), \quad (2.7) \]

where
\[
F(X^\varepsilon, Y^\varepsilon, \theta^\varepsilon) := f(X^\varepsilon + \delta(\theta^1_1 \omega_1), Y^\varepsilon + \eta^\varepsilon(\theta^2_2 \omega_2)),
\]

and
\[
G(X^\varepsilon, Y^\varepsilon, \theta^\varepsilon) := g(X^\varepsilon + \delta(\theta^1_1 \omega_1), Y^\varepsilon + \eta^\varepsilon(\theta^2_2 \omega_2)).
\]

The state space for this system is \( \mathcal{H} = \mathcal{H}_s \times \mathcal{H}_f \).

Supplement the initial condition
\[ X^\varepsilon(0) = X_0, \quad \text{and} \quad Y^\varepsilon(0) = Y_0. \quad (2.8) \]

Under the Hypothesis H1-H3, by the classical evolutionary equation theory, system \((2.6)-(2.8)\) has a unique global solution for every \( \omega = (\omega_1, \omega_2)^T \in \Omega = \Omega_1 \times \Omega_2 \). No exceptional sets with respect to the initial conditions appear. Hence the solution mapping
\[
(t, \omega, (X_0, Y_0)^T) \mapsto \Phi^\varepsilon(t, \omega, (X_0, Y_0)^T) := (X^\varepsilon(t, \omega, (X_0, Y_0)^T), Y^\varepsilon(t, \omega, (X_0, Y_0)^T))^T
\]
generates a continuous random dynamical system. In fact, the mapping \( \Phi^\varepsilon \) is \((\mathcal{B}(\mathbb{R}) \otimes (\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes \mathcal{B}(\mathcal{H}_s \times \mathcal{H}_f), (\mathcal{F}_1 \otimes \mathcal{F}_2))\)-measurable.

As in Jones [16, p.49], a slow manifold of a slow-fast system is the manifold in which the fast variable is represented by the slow variable, when the scale parameter \( \varepsilon \) is sufficiently small. It also exponentially attracts other dynamical orbits. A critical manifold of a slow-fast system is the slow manifold corresponding to the zero scale parameter.

For \((2.6)-(2.7)\), similar to Fu, Liu and Duan [15] or Wang and Roberts [25], we have the following result about the slow manifold.

Consider the so-called Liapunov-Perron equation
\[ h^\varepsilon(\zeta, \omega) = \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{ds}{\varepsilon}} G(X^\varepsilon(s, \omega; \zeta), Y^\varepsilon(s, \omega; \zeta), \theta^\varepsilon_s \omega) ds, \quad \text{for any} \quad \zeta \in \mathcal{H}_s, \quad (2.9) \]

where \( X^\varepsilon(t, \omega; \zeta) \) and \( Y^\varepsilon(t, \omega; \zeta) \) are the solutions of system \((2.6)-(2.7)\) with the following forms
\[
\begin{pmatrix}
X^\varepsilon(t, \omega; \zeta) \\
Y^\varepsilon(t, \omega; \zeta)
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{\varepsilon} \int_{-\infty}^{t} e^{\frac{\delta(\theta^1_1 \omega_1)}{\varepsilon}} G(X^\varepsilon(s, \omega; \zeta), Y^\varepsilon(s, \omega; \zeta), \theta^\varepsilon_s \omega) ds \\
\frac{1}{\varepsilon} \int_{-\infty}^{t} e^{\frac{\delta(\theta^2_2 \omega_2)}{\varepsilon}} G(X^\varepsilon(s, \omega; \zeta), Y^\varepsilon(s, \omega; \zeta), \theta^\varepsilon_s \omega) ds
\end{pmatrix}.
\]
Then

$$M^\varepsilon(\omega) = \{(z, h^\varepsilon(z, \omega))^T | z \in H_s\} \quad (2.10)$$

is the slow manifold of the system (2.6)-(2.7). It is invariant in the following sense

$$\Phi^\varepsilon(t, \omega, M^\varepsilon(\omega)) \subset M^\varepsilon(\theta t, \omega), \quad \text{for} \quad t > 0.$$}

Furthermore, the slow manifold exponentially attracts other dynamical orbits.

In the rest of this paper, we use $W^\varepsilon_\beta((X_0, Y_0)^T, \omega)$ to denote a fiber of the slow foliation, and use $W^\theta_\beta((X_0, Y_0)^T, \omega)$ to denote a fiber of the critical foliation. According to Section 2.2, the slow foliation is essentially an unstable foliation.

### 3 A motivating example for slow foliation

Before presenting a general theory, we work out a simple example for slow foliation.

Consider the following slow-fast stochastic ordinary differential equations

$$\frac{dx^\varepsilon}{dt} = x^\varepsilon, \quad \text{in} \quad \mathbb{R}^1, \quad (3.1)$$

$$\frac{dy^\varepsilon}{dt} = -\frac{1}{\varepsilon} y^\varepsilon + \frac{1}{\varepsilon} (x^\varepsilon)^2 + \frac{1}{\sqrt{\varepsilon}} W_2, \quad \text{in} \quad \mathbb{R}^1, \quad (3.2)$$

where $W_2$ is a scalar Wiener process. It follows from (2.3) that the converted random system is as follows

$$\frac{dX^\varepsilon}{dt} = X^\varepsilon, \quad \text{in} \quad \mathbb{R}^1, \quad (3.3)$$

$$\frac{dY^\varepsilon}{dt} = -\frac{1}{\varepsilon} Y^\varepsilon + \frac{1}{\varepsilon} (X^\varepsilon)^2, \quad \text{in} \quad \mathbb{R}^1. \quad (3.4)$$

With the initial condition $X^\varepsilon(0) = X_0$ and $Y^\varepsilon(0) = Y_0$, the solution is

$$X^\varepsilon(t) = X_0 e^t, \quad \text{in} \quad \mathbb{R}, \quad (3.5)$$

$$Y^\varepsilon(t) = Y_0 e^{-\frac{t}{\varepsilon}} + \frac{1}{1 + 2\varepsilon} X_0^2 [e^{2t} - e^{-\frac{t}{\varepsilon}}], \quad \text{in} \quad \mathbb{R}, \quad (3.6)$$

where

$$X^\varepsilon(t) = X^\varepsilon(t, \omega_2, (X_0, Y_0)^T) = X^\varepsilon(t, \eta^\frac{1}{2}(\theta^2 \omega_2), (X_0, Y_0)^T) = x^\varepsilon,$$

$$Y^\varepsilon(t) = Y^\varepsilon(t, \omega_2, (X_0, Y_0)^T) = Y^\varepsilon(t, \eta^\frac{1}{2}(\theta^2 \omega_2), (X_0, Y_0)^T) = y^\varepsilon - \eta^\frac{1}{2} (\theta^2 \omega_2).$$
For every two points \((X_0, Y_0)^T\) and \((\tilde{X}_0, \tilde{Y}_0)^T\) in \(\mathbb{R}^1 \times \mathbb{R}^1\), we calculate the difference between two orbits
\[
I := |(X^\varepsilon(t, \omega_2, (X_0, Y_0)^T), Y^\varepsilon(t, \omega_2, (X_0, Y_0)^T))^T - (\tilde{X}^\varepsilon(t, \omega_2, (\tilde{X}_0, \tilde{Y}_0)^T), \tilde{Y}^\varepsilon(t, \omega_2, (\tilde{X}_0, \tilde{Y}_0)^T))^T| \leq |X_0 - \tilde{X}_0| \cdot e^t + \frac{1}{1+2\varepsilon} |(X_0^2 - \tilde{X}_0^2)| \cdot e^{-\varepsilon}.
\]

If the coefficient
\[
(Y_0 - \tilde{Y}_0) - \frac{1}{1+2\varepsilon} (X_0^2 - \tilde{X}_0^2) = 0,
\]
then the difference of two orbits is
\[
I = O(e^t), \quad \text{as} \quad t \to -\infty.
\]

Define
\[
\mathcal{W}^\varepsilon_1((X_0, Y_0)^T, \omega_2) = \{(\xi, \ell^\varepsilon(\xi, (X_0, Y_0)^T, \omega_2))^T \mid \xi \in \mathbb{R}^1 \}, \quad (3.8)
\]
where the function
\[
\ell^\varepsilon(\xi, (X_0, Y_0)^T, \omega_2) = Y_0 + \frac{1}{1+2\varepsilon} (\xi^2 - X_0^2), \quad \xi \in \mathbb{R}^1. \quad (3.9)
\]

Whenever an initial point \((\tilde{X}_0, \tilde{Y}_0)^T\) is in \(\mathcal{W}^\varepsilon_1((X_0, Y_0)^T, \omega_2)\), the condition \((3.7)\) holds. This immediately implies that the different dynamical orbits will be exponentially approaching each other as \(t \to -\infty\). Therefore, we say that \(\mathcal{W}^\varepsilon_1((X_0, Y_0)^T, \omega_2)\) is a fiber of the slow foliation. It is the graph of \(\ell^\varepsilon(\xi, (X_0, Y_0)^T, \omega_2)\). Different orbits of the slow-fast system \((3.3)-(3.4)\) are exponentially approaching each other in backward time, whenever they start from the same fiber.

As seen in \((3.9)\), the slow foliation of \((3.3)-(3.4)\) is a family of the parallel parabolic curves (i.e., fibers) in the state space.

In addition, from \((2.9)\) and \((2.10)\), we know that the slow manifold of \((3.3)-(3.4)\) is
\[
\mathcal{M}^\varepsilon(\omega_2) = \{(\xi, h^\varepsilon(\xi, \omega_2))^T \mid \xi \in \mathbb{R}^1 \}, \quad (3.10)
\]
where
\[
h^\varepsilon(\xi, \omega_2) = \frac{1}{1+2\varepsilon} \xi^2, \quad \xi \in \mathbb{R}^1. \quad (3.11)
\]
By comparing with \((3.9)\), it is clear that the slow manifold is a fiber of the slow foliation.
Now we consider another stochastic system independent of $\varepsilon$ as follows

$$\frac{dx^0}{dt} = 0, \quad \text{in } \mathbb{R}^1, \quad (3.12)$$

$$\frac{dy^0}{dt} = -y^0 + (x^0)^2 + \dot{W}_2, \quad \text{in } \mathbb{R}^1. \quad (3.13)$$

It follows from (2.3) that the converted random system is

$$\frac{dX^0}{dt} = 0, \quad \text{in } \mathbb{R}^1, \quad (3.14)$$

$$\frac{dY^0}{dt} = -Y^0 + (X^0)^2, \quad \text{in } \mathbb{R}^1. \quad (3.15)$$

The solution with initial condition $X^0(0) = X_0$ and $Y^0(0) = Y_0$ is

$$X^0(t) = X_0, \quad t \in \mathbb{R},$$
$$Y^0(t) = e^{-t}Y_0 + X_0^2(1 - e^{-t}), \quad t \in \mathbb{R},$$

where

$$X^0(t) = X^0(t, \omega_2, (X_0, Y_0)^T) = X^0(t, \xi(\theta_2^2 \omega_2), (X_0, Y_0)^T) = x^0,$$
$$Y^0(t) = Y^0(t, \omega_2, (X_0, Y_0)^T) = Y^0(t, \xi(\theta_2^2 \omega_2), (X_0, Y_0)^T) = y^0 - \xi(\theta_2^2 \omega_2).$$

By the same argument as above, a fiber of the foliation of (3.14)-(3.15) is

$$\mathcal{W}_1^0((X_0, Y_0)^T, \omega_2) = \{(\zeta, l^0(\zeta, (X_0, Y_0)^T, \omega_2))^T | \zeta \in \mathbb{R}^1\}, \quad (3.16)$$

where

$$l^0(\zeta, (X_0, Y_0)^T, \omega_2) = Y_0 + (\zeta^2 - X_0^2), \quad \zeta \in \mathbb{R}^1. \quad (3.17)$$

This is called the critical foliation for the system (3.3)-(3.4).

Observe that, by a time change $t \to \varepsilon t$, Equation (3.4) is transformed to Equation (3.15). Also notice that $\eta^{1/2}(\theta_2^2 \omega_2)$ has the same distribution as $\xi(\theta_2^2 \omega_2)$ by Lemma 2.1. Thus

$$l^\varepsilon(\zeta, (X_0, Y_0)^T, \omega_2) = Y_0 + \frac{1}{1 + 2\varepsilon}(\zeta^2 - X_0^2) \xrightarrow{d} Y_0 + (\zeta^2 - X_0^2) = l^0(\zeta, (X_0, Y_0)^T, \omega_2), \quad \text{as } \varepsilon \to 0,$$

for $\zeta \in \mathbb{R}^1$, where “$\xrightarrow{d}$” denotes the convergence in distribution. Therefore, the slow foliation converges in distribution to the critical foliation, as $\varepsilon$ tends to zero.

### 4 Slow foliation

In this section, we establish a theory of the slow foliation for the slow-fast system (1.1)-(1.2). We derive the existence of slow foliation for the corresponding random slow-fast system (2.6)-(2.7). The dynamical orbits of the system (2.6)-(2.7) in a given fiber are shown to exponentially
respectively. Define \( C^\beta \) with the norms 

\[
C^\beta_s^- = \{ \varphi : (-\infty, 0] \to H_s \mid \text{\varphi is continuous and } \sup_{t \leq 0} e^{-\beta t} \| \varphi \|_{H_s} < \infty \},
\]

\[
C^\beta_f^- = \{ \varphi : (-\infty, 0] \to H_f \mid \text{\varphi is continuous and } \sup_{t \leq 0} e^{-\beta t} \| \varphi \|_{H_f} < \infty \},
\]

with the norms

\[
\| \varphi \|_{C^\beta_s^-} = \sup_{t \leq 0} e^{-\beta t} \| \varphi \|_{H_s}, \quad \text{and} \quad \| \varphi \|_{C^\beta_f^-} = \sup_{t \leq 0} e^{-\beta t} \| \varphi \|_{H_f},
\]

respectively. Define \( C^\beta := C^\beta_s^- \times C^\beta_f^- \), with norm \( \|(X, Y)^T\|_{C^\beta} = \|X\|_{C^\beta_s^-} + \|Y\|_{C^\beta_f^-} \), for \((X, Y)^T \in C^\beta\).

Denote \( \Phi^\varepsilon(t, \omega, (X_0, Y_0)^T) = (X^\varepsilon(t, \omega, (X_0, Y_0)^T), Y^\varepsilon(t, \omega, (X_0, Y_0)^T))^T \) the solution of the slow-fast random system \((2.6)-(2.7)\) with the initial condition \( \Phi^\varepsilon(0, \omega, (X_0, Y_0)^T) = (X_0, Y_0)^T \).

Define the difference of two dynamical orbits

\[
\Psi^\varepsilon(t) = \Phi^\varepsilon(t, \omega, (\tilde{X}_0, \tilde{Y}_0)^T) - \Phi^\varepsilon(t, \omega, (X_0, Y_0)^T)
\]

\[
= (X^\varepsilon(t, \omega, (\tilde{X}_0, \tilde{Y}_0)^T) - X^\varepsilon(t, \omega, (X_0, Y_0)^T), Y^\varepsilon(t, \omega, (\tilde{X}_0, \tilde{Y}_0)^T) - Y^\varepsilon(t, \omega, (X_0, Y_0)^T))^T
\]

\[
:= (U^\varepsilon(t), V^\varepsilon(t))^T.
\]

(4.1)

Here the initial condition

\[
\Psi^\varepsilon(0) = (U^\varepsilon(0), V^\varepsilon(0))^T = (\tilde{X}_0 - X_0, \tilde{Y}_0 - Y_0)^T,
\]

and the solution

\[
X^\varepsilon(t, \omega, (\tilde{X}_0, \tilde{Y}_0)^T) = U^\varepsilon(t) + X^\varepsilon(t, \omega, (X_0, Y_0)^T),
\]

\[
Y^\varepsilon(t, \omega, (\tilde{X}_0, \tilde{Y}_0)^T) = V^\varepsilon(t) + Y^\varepsilon(t, \omega, (X_0, Y_0)^T).
\]

Moreover, \((U^\varepsilon, V^\varepsilon)^T\) satisfies

\[
\frac{dU^\varepsilon}{dt} = AU^\varepsilon + \Delta F(U^\varepsilon, V^\varepsilon, \theta_i^\varepsilon \omega),
\]

(4.2)

\[
\frac{dV^\varepsilon}{dt} = \frac{1}{\varepsilon} BV^\varepsilon + \frac{1}{\varepsilon} \Delta G(U^\varepsilon, V^\varepsilon, \theta_i^\varepsilon \omega),
\]

(4.3)

with nonlinearities

\[
\Delta F(U^\varepsilon, V^\varepsilon, \theta_i^\varepsilon \omega) = F(U^\varepsilon(t) + X^\varepsilon(t, \omega, (X_0, Y_0)^T), V^\varepsilon(t) + Y^\varepsilon(t, \omega, (X_0, Y_0)^T), \theta_i^\varepsilon \omega) - F(X^\varepsilon(t, \omega, (X_0, Y_0)^T), Y^\varepsilon(t, \omega, (X_0, Y_0)^T), \theta_i^\varepsilon \omega),
\]

(4.4)
and
\[
\Delta G(U^\varepsilon, V^\varepsilon, \theta^\varepsilon \omega) = G(U^\varepsilon(t) + X^\varepsilon(t, \omega, (X_0, Y_0)^T), V^\varepsilon(t) + Y^\varepsilon(t, \omega, (X_0, Y_0)^T), \theta^\varepsilon \omega) - G(X^\varepsilon(t, \omega, (X_0, Y_0)^T), Y^\varepsilon(t, \omega, (X_0, Y_0)^T), \theta^\varepsilon \omega),
\] (4.5)

and initial condition
\[
U^\varepsilon(0) = U_0 = \bar{X}_0 - X_0, V^\varepsilon(0) = V_0 = \bar{Y}_0 - Y_0.
\]

Define
\[
\mathcal{W}^\varepsilon_\beta((X_0, Y_0)^T, \omega) = \{(\bar{X}_0, \bar{Y}_0)^T \in H_s \times H_f | \Phi^\varepsilon(t, \omega, (X_0, Y_0)^T) - \Phi^\varepsilon(t, \omega, (\bar{X}_0, \bar{Y}_0)^T) \in C^-_\beta\}.
\] (4.6)

As we will show, \(\mathcal{W}^\varepsilon_\beta((X_0, Y_0)^T, \omega)\) is a fiber of the slow foliation for the slow-fast random system \([2.6] - [2.7]\).

Now we present some lemmas before our main results.

**Lemma 4.1** Assume that the Hypotheses H1-H3 hold. Take \(\beta\) as the positive real number \(-\gamma_2 / 2\). Then \((\bar{X}_0, \bar{Y}_0)^T\) is in \(\mathcal{W}^\varepsilon_\beta((X_0, Y_0)^T, \omega)\) if and only if there exists a function \(\Psi^\varepsilon(t) = (U^\varepsilon(t), V^\varepsilon(t))^T = (U^\varepsilon(t, \omega, (X_0, Y_0)^T); U^\varepsilon(0)), V^\varepsilon(t, \omega, (X_0, Y_0)^T); U^\varepsilon(0)) \in C^-_\beta\) such that
\[
\Psi^\varepsilon(t) = \left(\begin{array}{c} U^\varepsilon(t) \\ V^\varepsilon(t) \end{array}\right) = \left(\begin{array}{c} e^{At} U^\varepsilon(0) + \int_0^t e^{A(t-s)} \Delta F(U^\varepsilon(s), V^\varepsilon(s), \theta^\varepsilon \omega)ds \\ \frac{1}{\varepsilon} \int_{-\infty}^t e^{\beta(t-s)} \Delta G(U^\varepsilon(s), V^\varepsilon(s), \theta^\varepsilon \omega)ds \end{array}\right),
\] (4.7)

where \(\Delta F\) and \(\Delta G\) are defined in \([4.4]\) and \([4.5]\).

**Proof.** Let \((\bar{X}_0, \bar{Y}_0)^T \in \mathcal{W}^\varepsilon_\beta((X_0, Y_0)^T, \omega)\). By the variation of constants formula, we have
\[
U^\varepsilon(t) = e^{At} U^\varepsilon(0) + \int_0^t e^{A(t-s)} \Delta F(U^\varepsilon(s), V^\varepsilon(s), \theta^\varepsilon \omega)ds,
\] (4.8)
\[
V^\varepsilon(t) = e^{\frac{B(t-s)}{\varepsilon}} V^\varepsilon(\tau) + \frac{1}{\varepsilon} \int_{-\infty}^t e^{\frac{B(t-s)}{\varepsilon}} \Delta G(U^\varepsilon(s), V^\varepsilon(s), \theta^\varepsilon \omega)ds.
\] (4.9)

Since \(\Phi^\varepsilon(\cdot) \in C^-_\beta\), for \(\tau < 0\),
\[
\|e^{\frac{B(t-s)}{\varepsilon}} V^\varepsilon(\tau)|_{H_f} \leq \|V|_{C^-_\beta} \cdot e^{-\gamma_2 t / \varepsilon} \cdot e^{(-\gamma_1 t / \varepsilon - \beta)\cdot(-\tau)} \leq \|V|_{C^-_\beta} \cdot e^{-\gamma_2 t / \varepsilon} \cdot e^{(-\gamma_1 t / \varepsilon - \beta)\cdot(-\tau)} \to 0, \quad \text{as} \quad \tau \to -\infty,
\]

which implies that
\[
V^\varepsilon(t) = \frac{1}{\varepsilon} \int_{-\infty}^t e^{\frac{B(t-s)}{\varepsilon}} \Delta G(U^\varepsilon(s), V^\varepsilon(s), \theta^\varepsilon \omega)ds.
\] (4.10)
Therefore, it follows from (4.8)-(4.10) that (4.7) holds. By direct calculation, it is clear that the converse holds. This completes the proof of Lemma 4.1.

Lemma 4.2  Assume that the Hypotheses H1-H3 hold. Take $\beta$ as the positive real number $\frac{1}{2}$. For any given $U^\varepsilon(0) = \tilde{X}_0 - X_0 \in H_s$, there exists a sufficiently small positive constant $\varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_0)$, the system (4.7) has a unique solution $\Psi^\varepsilon(\cdot) = \Psi^\varepsilon(\cdot, \omega, (X_0, Y_0)^T; U^\varepsilon(0))$ in $C^{-}_\beta$.

**Proof.** Introduce two operators $J^\varepsilon_s : C^{-}_\beta \longrightarrow C^{-}_{s\varepsilon}$ and $J^\varepsilon_f : C^{-}_\beta \longrightarrow C^{-}_{f\varepsilon}$ satisfying

$$J^\varepsilon_s(\Psi^\varepsilon)[t] = e^{At}U^\varepsilon(0) + \int_0^t e^{A(t-s)}\Delta F(U^\varepsilon(s), V^\varepsilon(s), \theta^\varepsilon_s\omega)ds,$$

$$J^\varepsilon_f(\Psi^\varepsilon)[t] = \frac{1}{\varepsilon} \int_{-\infty}^t e^{B(t-s)}\Delta G(U^\varepsilon(s), V^\varepsilon(s), \theta^\varepsilon_s\omega)ds.$$

Also pose the operator $J^\varepsilon : C^{-}_\beta \longrightarrow C^{-}_\beta$ satisfying $J^\varepsilon(\Psi^\varepsilon) = (J^\varepsilon_s(\Psi^\varepsilon), J^\varepsilon_f(\Psi^\varepsilon))^T$. It is easy to verify that $J^\varepsilon_s$, $J^\varepsilon_f$ and $J^\varepsilon$ are well-defined in $C^{-}_{s\varepsilon}$, $C^{-}_{f\varepsilon}$ and $C^{-}_\beta$, respectively.

For any $\Psi^\varepsilon = (U^\varepsilon, V^\varepsilon)^T \in C^{-}_\beta$ and $\tilde{\Psi}^\varepsilon = (\tilde{U}^\varepsilon, \tilde{V}^\varepsilon)^T \in C^{-}_\beta$, then

$${\|J^\varepsilon_s(\Psi^\varepsilon) - J^\varepsilon_s(\tilde{\Psi}^\varepsilon)\|}_{C^{-}_{s\varepsilon}} \leq \sup_{t \leq 0} e^{-\gamma_s(t-s)}\|U^\varepsilon(s) - \tilde{U}^\varepsilon(s)\|_{H_s} + \|V^\varepsilon(s) - \tilde{V}^\varepsilon(s)\|_{H_f})ds$$

and

$${\|J^\varepsilon_f(\Psi^\varepsilon) - J^\varepsilon_f(\tilde{\Psi}^\varepsilon)\|}_{C^{-}_{f\varepsilon}} \leq \sup_{t \leq 0} e^{-\gamma_f(t-s)}\|U^\varepsilon(s) - \tilde{U}^\varepsilon(s)\|_{H_s} + \|V^\varepsilon(s) - \tilde{V}^\varepsilon(s)\|_{H_f})ds.$$
Put the constant
\[
\rho(\gamma_s, \gamma_f, K, \varepsilon) = \frac{K}{-\beta - \gamma_s} + \frac{K}{\gamma_f + \varepsilon \beta}.
\]  
(4.14)

Then
\[
\|J^\varepsilon(\Psi^\varepsilon) - J^\varepsilon(\tilde{\Psi}^\varepsilon)\|_{C_\beta^-} \leq \rho(\gamma_s, \gamma_f, K, \varepsilon)\|\Psi^\varepsilon - \tilde{\Psi}^\varepsilon\|_{C_\beta^-}.
\]  
(4.15)

Notice that the Hypothesis H3 holds, \(\beta = -\frac{\gamma_s}{2}\), and that
\[\rho(\gamma_s, \gamma_f, K, \varepsilon) \to \frac{K}{-\beta - \gamma_s} + \frac{K}{\gamma_f}, \text{ as } \varepsilon \searrow 0.\]

Therefore, there is a sufficiently small positive constant \(\varepsilon_0\) such that for \(\varepsilon \in (0, \varepsilon_0)\),
\[
0 < \rho(\gamma_s, \gamma_f, K, \varepsilon) < 1.
\]

Then the map \(J^\varepsilon(\Psi^\varepsilon)\) is contractive in \(C_\beta^-\) uniformly with respect to \((\omega, (X_0, Y_0)^T, U^\varepsilon(0))\). By the contraction mapping principle, we have that for each \(U^\varepsilon(0) \in H_s\), the mapping \(J^\varepsilon(\Psi^\varepsilon) = J^\varepsilon(\Psi^\varepsilon, \omega, (X_0, Y_0)^T; U^\varepsilon(0))\) has a unique fixed point, which still denoted by
\[
\Psi^\varepsilon(\cdot) = \Psi^\varepsilon(\cdot, \omega, (X_0, Y_0)^T; U^\varepsilon(0)) \in C_\beta^-.
\]

In other words, \(\Psi^\varepsilon(\cdot, \omega, (X_0, Y_0)^T; U^\varepsilon(0)) \in C_\beta^-\) is a unique solution of the system (4.7). ■

**Lemma 4.3** Assume that the Hypothesis H1-H3 hold. Take \(\beta\) as the positive real number \(\frac{-\gamma_s}{2}\). Let \(\Psi^\varepsilon(t) = \Psi^\varepsilon(t, \omega, (X_0, Y_0)^T; U^\varepsilon(0))\) be the unique solution of the system (4.7) in \(C_\beta^-\). For any \(U^\varepsilon(0)\) and \(\tilde{U}^\varepsilon(0)\) in \(C_\beta^-\), then there exists a sufficiently small positive constant \(\varepsilon_0\) such that for \(\varepsilon \in (0, \varepsilon_0)\), we have
\[
\|\Psi^\varepsilon(t, \omega, (X_0, Y_0)^T; U^\varepsilon(0)) - \Psi^\varepsilon(t, \omega, (X_0, Y_0)^T; \tilde{U}^\varepsilon(0))\|_{C_\beta^-} \leq \frac{1}{1 - \rho(\gamma_s, \gamma_f, K, \varepsilon)}\|U^\varepsilon(0) - \tilde{U}^\varepsilon(0)\|_{H_s},
\]  
(4.16)

where \(\rho(\gamma_s, \gamma_f, K, \varepsilon)\) is defined as (4.14).

Lemma 4.3 is easily deduced by using the same arguments as in the proof of Lemma 4.2. Here we omit it.

In the following, for every \(\zeta \in H_s\), we define
\[
U^\varepsilon(\zeta, (X_0, Y_0)^T, \omega) := Y_0 + \frac{1}{\varepsilon}\int_{-\infty}^0 e^{-\frac{s}{\varepsilon}} \Delta G(U^\varepsilon(s, \omega, (X_0, Y_0)^T; (\zeta - X_0)), V^\varepsilon(s, \omega, (X_0, Y_0)^T; (\zeta - X_0)), \theta^\varepsilon\omega) ds.
\]  
(4.17)

Now we give our main result.
Theorem 4.1 (Slow foliation) Assume that the Hypothesis H1-H3 hold. Take $\beta$ as the positive real number $\frac{\gamma_f}{2\rho}$. Then there exists a sufficiently small positive constant $\varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_0)$, the invariant foliation of the slow-fast random system (2.6)-(2.7) exists.

(i) Its one fiber is the graph of a Lipschitz function. That is

$$\mathcal{W}_\beta^\varepsilon((X_0, Y_0)^T, \omega) = \{(\zeta, l^\varepsilon(\zeta, (X_0, Y_0)^T, \omega))'^T | \zeta \in H_s\},$$

(4.18)

where $(X_0, Y_0)^T \in H_s \times H_f$, and the function $l^\varepsilon(\zeta, (X_0, Y_0)^T, \omega)$ is defined as (4.17). In addition, $l^\varepsilon(\zeta, (X_0, Y_0)^T, \omega)$ is Lipschitz continuous with respect to $\zeta$, whose Lipschitz constant $\text{Lipl}^\varepsilon$ satisfies

$$\text{Lipl}^\varepsilon \leq \frac{K}{\gamma_f + \varepsilon \beta} \cdot \frac{1}{1 - \rho(\gamma_s, \gamma_f, \varepsilon)},$$

where $\rho(\gamma_s, \gamma_f, K, \varepsilon)$ is defined as (4.14).

(ii) The dynamical orbits of (2.6)-(2.7) are exponentially approaching each other in backward time only if they start from the same fiber. That is, for any two points $(\tilde{X}_0^1, \tilde{Y}_0^1)^T$ and $(\tilde{X}_0^2, \tilde{Y}_0^2)^T$ in a same fiber $\mathcal{W}_\beta^\varepsilon((X_0, Y_0)^T, \omega)$,

$$\|\Phi^\varepsilon(t, \omega, (\tilde{X}_0^1, \tilde{Y}_0^1)^T) - \Phi^\varepsilon(t, \omega, (\tilde{X}_0^2, \tilde{Y}_0^2)^T)\|_{H_s \times H_f} \leq \frac{e^{\beta t}}{1 - \rho(\gamma_s, \gamma_f, \varepsilon)} \cdot \|\tilde{X}_0^1 - \tilde{X}_0^2\|_{H_s} = O(e^{\beta t}), \quad \forall t \to -\infty.$$

(4.19)

(iii) Its fiber is invariant, i.e.,

$$\Phi^\varepsilon(t, \omega, \mathcal{W}_\beta^\varepsilon((X_0, Y_0)^T, \omega)) \subset \mathcal{W}_\beta^\varepsilon(\Phi^\varepsilon(t, \omega, (X_0, Y_0)^T), \theta_t \omega).$$

Proof. (i) To prove a fiber of the slow foliation is the graph of a Lipschitz function.

It follows from (4.7) that

$$ \left( \begin{array}{c} \tilde{X}_0 - X_0 \\ \tilde{Y}_0 - Y_0 \end{array} \right) = \left( \begin{array}{c} \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{s}{\varepsilon}} \Delta G(U^\varepsilon(s), V^\varepsilon(s), \theta_s \omega) ds \\ \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{s}{\varepsilon}} \Delta G(U^\varepsilon(s, \omega, (X_0, Y_0)^T; U^\varepsilon(0)), V^\varepsilon(s, \omega, (X_0, Y_0)^T; U^\varepsilon(0)), \theta_s \omega) ds \end{array} \right), $$

which implies that

$$\tilde{Y}_0 = Y_0 + \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{s}{\varepsilon}} \Delta G(U^\varepsilon(s, \omega, (X_0, Y_0)^T; U^\varepsilon(0)), V^\varepsilon(s, \omega, (X_0, Y_0)^T; U^\varepsilon(0)), \theta_s \omega) ds$$

$$= Y_0 + \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{s}{\varepsilon}} \Delta G(U^\varepsilon(s, \omega, (X_0, Y_0)^T; \tilde{X}_0 - X_0)), V^\varepsilon(s, \omega, (X_0, Y_0)^T; \tilde{X}_0 - X_0)),$$

which just is $l^\varepsilon(\zeta, (X_0, Y_0)^T, \omega)$ if we take $\tilde{X}_0$ as $\zeta$ in $H_s$. Then from Lemma 4.1, Lemma 4.2, (4.6) and (4.17), we know that

$$\mathcal{W}_\beta^\varepsilon((X_0, Y_0)^T, \omega) = \{(\zeta, l^\varepsilon(\zeta, (X_0, Y_0)^T, \omega))^T | \zeta \in H_s\}. $$
Furthermore, for any $\zeta$ and $\widetilde{\zeta}$ in $H_s$, using Lemma 4.3,
\begin{align*}
\|f^\varepsilon(t, \zeta, (X_0, Y_0)^T, \omega) - f^\varepsilon(t, \widetilde{\zeta}, (X_0, Y_0)^T, \omega)\|_{H_f} & = \|V^\varepsilon(t, \omega, (X_0, Y_0)^T; \zeta - X_0) - V^\varepsilon(t, \omega, (X_0, Y_0)^T; \widetilde{\zeta} - X_0)\|_{H_f} \|_{t=0} \\
& \leq \|V^\varepsilon(-, \omega, (X_0, Y_0)^T; \zeta - X_0) - V^\varepsilon(-, \omega, (X_0, Y_0)^T; \widetilde{\zeta} - X_0)\|_{C^f_{\beta}} \\
& \leq \frac{K}{\gamma_f + \varepsilon_2} \|\Psi^\varepsilon(-, \omega, (X_0, Y_0)^T; \zeta - X_0) - \Psi^\varepsilon(-, \omega, (X_0, Y_0)^T; \widetilde{\zeta} - X_0)\|_{C^\beta_{\varepsilon}} \\
& \leq \frac{K}{\gamma_f + \varepsilon_2} \cdot \frac{1}{1 - \rho(\gamma_s, \gamma_f, K, \varepsilon)} \|\zeta - \widetilde{\zeta}\|_{H_s}.
\end{align*}

(ii) To prove the dynamical orbits are exponentially approaching each other in backward time only if they start from the same fiber.

From Lemma 4.1, using the same argument of the proof of Lemma 4.2, we easily got
\begin{align*}
\|\Psi^\varepsilon(-)\|_{C^\beta_{\varepsilon}} &= \|U^\varepsilon(-)\|_{C^\beta_{\varepsilon}} + \|V^\varepsilon(-)\|_{C^f_{\beta}} \\
&= \|e^{At}U^\varepsilon(0)\|_{C^\beta_{\varepsilon}} + \int_0^t \|e^{At-s}F(U^\varepsilon(s), V^\varepsilon(s), \theta^\varepsilon_s)\|_{C^\beta_{\varepsilon}} ds \\
&+ \|e^{\int_0^t e^{At-s}F(U^\varepsilon(s), V^\varepsilon(s), \theta^\varepsilon_s)ds}\|_{C^\beta_{\varepsilon}} \\
&\leq \|U^\varepsilon(0)\|_{H_s} + \frac{K}{\beta - \gamma_s} \|\Psi^\varepsilon(-)\|_{C^\beta_{\varepsilon}} + \frac{K}{\gamma_f + \varepsilon_2} \|\Psi^\varepsilon(-)\|_{C^\beta_{\varepsilon}} \\
&\leq \|U^\varepsilon(0)\|_{H_s} + \rho(\gamma_s, \gamma_f, K, \varepsilon) \|\Psi^\varepsilon(-)\|_{C^\beta_{\varepsilon}},
\end{align*}
where $\Psi^\varepsilon$ is defined as (4.1). Notice that the Hypothesis H3 holds, $\beta = \frac{-\gamma_s}{2}$, and that $\rho(\gamma_s, \gamma_f, K, \varepsilon) \longrightarrow \frac{K}{\gamma_f + \varepsilon_2} + \frac{K}{\gamma_f + \varepsilon_2}$ as $\varepsilon \downarrow 0$. Therefore, there exists a sufficiently small positive constant $\varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\rho(\gamma_s, \gamma_f, K, \varepsilon) < 1$. Then it follows from (4.20) that
\begin{align*}
\|\Psi^\varepsilon(-)\|_{C^\beta_{\varepsilon}} &\leq \frac{1}{1 - \rho(\gamma_s, \gamma_f, K, \varepsilon)} \|U^\varepsilon(0)\|_{H_s},
\end{align*}
which implies that
\begin{align*}
\|\Phi^\varepsilon(t, \omega, (\widetilde{X}_0, \widetilde{Y}_0)^T) - \Phi^\varepsilon(t, \omega, (X_0, Y_0)^T)\|_{H_s \times H_f} &\leq \frac{e^{\beta t}}{1 - \rho(\gamma_s, \gamma_f, K, \varepsilon)} \cdot \|U^\varepsilon(0)\|_{H_s}, \quad \forall \ t \leq 0.
\end{align*}

For any two points $(\widetilde{X}_0^1, \widetilde{Y}_0^1)^T$ and $(\widetilde{X}_0^2, \widetilde{Y}_0^2)^T$ in the same fiber $W^\varepsilon((X_0, Y_0)^T, \omega)$, from (4.21), we have
\begin{align*}
\|\Phi^\varepsilon(t, \omega, (\widetilde{X}_0^1, \widetilde{Y}_0^1)^T) - \Phi^\varepsilon(t, \omega, (X_0, Y_0)^T)\|_{H_s \times H_f} &\leq \frac{e^{\beta t}}{1 - \rho(\gamma_s, \gamma_f, K, \varepsilon)} \cdot \|U^\varepsilon(0)\|_{H_s}, \quad \forall \ t \leq 0,
\end{align*}
and
\begin{align*}
\|\Phi^\varepsilon(t, \omega, (\widetilde{X}_0^2, \widetilde{Y}_0^2)^T) - \Phi^\varepsilon(t, \omega, (X_0, Y_0)^T)\|_{H_s \times H_f} &\leq \frac{e^{\beta t}}{1 - \rho(\gamma_s, \gamma_f, K, \varepsilon)} \cdot \|U^\varepsilon(0)\|_{H_s}, \quad \forall \ t \leq 0,
\end{align*}
which immediately implies (4.19) holds.
(iii) To prove the fiber is invariant.

To see this, taking a fiber $W_\varepsilon^{\beta}(X_0, Y_0)^T$, we will show that the time $\tau$-map $\Phi^\varepsilon(\tau, \omega, \cdot)$ maps it into the fiber $W_\varepsilon^{\beta}(\Phi^\varepsilon(\tau, \omega, (X_0, Y_0)^T), \theta, \omega)$. Let $(\tilde{X}_0, \tilde{Y}_0)^T \in W_\varepsilon^{\beta}(X_0, Y_0)^T$. Then

$$
\Phi^\varepsilon(\tau, \omega, (\tilde{X}_0, \tilde{Y}_0)^T) - \Phi^\varepsilon(\tau, \omega, (X_0, Y_0)^T) \in C^{-\beta}_\varepsilon,
$$

which implies that

$$
\Phi^\varepsilon(\tau, \omega, (\tilde{X}_0, \tilde{Y}_0)^T) = \Phi^\varepsilon(\tau, \omega, (X_0, Y_0)^T).
$$

Thus by using the cocycle property

$$
\Phi^\varepsilon(\tau, \omega, (\tilde{X}_0, \tilde{Y}_0)^T) = \Phi^\varepsilon(\tau, \omega, (\tilde{X}_0, \tilde{Y}_0)^T),
$$

Then we have $\Phi^\varepsilon(\tau, \omega, (X_0, Y_0)^T) \in W_\varepsilon^{\beta}(\Phi^\varepsilon(\tau, \omega, (X_0, Y_0)^T), \theta, \omega)$. The proof is completed. ■

Remark 4.1 From Theorem 4.1, [25] and [15], we know that the invariance of the slow foliation means the dynamical system maps a fiber to another fiber, while the invariance of the slow manifold means the dynamical system preserve the dynamical orbits starting from the slow manifold still in the slow manifold.

Remark 4.2 For the negative time, from Theorem 4.1, we know that the slow foliation describes the dynamics of the system (2.6)-(2.7) in which the different dynamical orbits are exponential closed only if they starting from a same fiber. For the positive time, from [25] and [15], we know that slow manifold describes the dynamics of the system (2.6)-(2.7), which could exponentially attract other dynamical orbits. Therefore, the slow foliation and slow manifold are from different view of points to describe the dynamics of the slow-fast stochastic system.

Theorem 4.2 (Geometric properties of the slow foliation) Assume that the Hypothesis H1-H3 hold. Take $\beta$ as the positive real number $\frac{-\gamma_s}{2}$. Let $M^\varepsilon(\omega)$ and $W_\varepsilon^\beta((X_0, Y_0)^T, \omega)$ be the slow manifold and a fiber of the slow foliation for the slow-fast random system (2.6)-(2.7), respectively, which are well defined as (2.10) and (4.18). Put

$$
W_\varepsilon^m(\omega) := \{W_\varepsilon^\beta((X_0, Y_0)^T, \omega) | Y_0 - h^\varepsilon(X_0, \omega) := m \in H_f, (X_0, Y_0)^T \in H_s \times H_f\},
$$

where $h^\varepsilon(X_0, \omega)$ is defined as (2.9). Then the fiber $W_\varepsilon^m(\omega)$ parallels the fiber $W_\varepsilon^n(\omega)$ for any $m, n \in H_f$ and $m \neq n$. Especially, if $m = 0$, $W_\varepsilon^0(\omega)$ is just the slow manifold. Thus, the geometry constructor of the slow foliation is clear: every fiber of the slow foliation parallels each other, and the slow manifold is one fiber of the slow foliation.
Moreover, we have that

(i) when the arbitrary given point \((X_0, Y_0)^T\) of the slow foliation is in the slow manifold \(M^\varepsilon(\omega)\), the fiber \(W^\varepsilon_\beta((X_0, Y_0)^T, \omega)\) is just the slow manifold \(M^\varepsilon(\omega)\);

(ii) when the arbitrary given point \((X_0, Y_0)^T\) of the slow foliation is not in the slow manifold \(M^\varepsilon(\omega)\), the fiber \(W^\varepsilon_\beta((X_0, Y_0)^T, \omega)\) parallels the slow manifold \(M^\varepsilon(\omega)\).

Proof. From (4.7), for any \((\tilde{X}_0, \tilde{Y}_0)^T \in H_s \times H_f\), we have

\[
\tilde{Y}_0 - Y_0 = \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{B_s}{\varepsilon}} \Delta G(U^\varepsilon(s), V^\varepsilon(s), \theta^\varepsilon s, \omega) ds,
\]

which implies that

\[
\tilde{Y}_0 - \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{B_s}{\varepsilon}} G(X^\varepsilon(s, \omega; \tilde{X}_0), Y^\varepsilon(s, \omega; \tilde{X}_0), \theta^\varepsilon s, \omega) ds = Y_0 - \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{B_s}{\varepsilon}} G(X^\varepsilon(s, \omega; X_0), Y^\varepsilon(s, \omega; X_0), \theta^\varepsilon s, \omega) ds.
\]

In other words,

\[
\tilde{Y}_0 - h^\varepsilon(\tilde{X}_0, \omega) = Y_0 - h^\varepsilon(X_0, \omega),
\]

where \(h^\varepsilon(\cdot, \omega)\) is defined as (2.9).

For arbitrary given point \((X_0, Y_0)^T\) of the slow foliation, there exists \(m \in H_f\) such that

\[
Y_0 - h^\varepsilon(X_0, \omega) = m.
\]

If \(m = 0\), then \((X_0, Y_0)^T\) is in the slow manifold \(M^\varepsilon(\omega)\), which yields from (4.23) that

\[
\tilde{Y}_0 - h^\varepsilon(\tilde{X}_0, \omega) = 0, \quad \text{for any } \tilde{X}_0 \in H_s.
\]

Thus, \(W^\varepsilon_\beta(\omega) = M^\varepsilon(\omega)\).

If \(m \neq 0\), then \((X_0, Y_0)^T\) is not in the slow manifold \(M^\varepsilon(\omega)\). Then it immediately follows from (4.23) that

\[
\tilde{Y}_0 - h^\varepsilon(\tilde{X}_0, \omega) = m \neq 0, \quad \text{for any } \tilde{X}_0 \in H_s.
\]

Thus \((\tilde{X}_0, \tilde{Y}_0)^T\) is in the curve \(V^\varepsilon_\beta(\omega)\) that parallels the slow manifold \(M^\varepsilon(\omega) = V^\varepsilon_\beta(\omega)\). Furthermore, for \(m, n \in H_f\) and \(m \neq n\), the \(W^\varepsilon_\beta(\omega)\) parallels \(W^\varepsilon_\beta(\omega)\). The proof is completed.
5 Critical foliation

In this section, we will study the limiting case of the slow foliation for the slow-fast random system \((2.6)-(2.7)\) as the singular perturbation parameter \(\varepsilon\) tends to zero. Also, we delicately construct the approximation of slow foliation for sufficiently small \(\varepsilon\) in distribution.

Taking the time scaling \(t \to \varepsilon t\) for the system \((2.6)-(2.7)\), we have

\[
\frac{dX^\varepsilon}{dt} = \varepsilon AX^\varepsilon + \varepsilon F(X^\varepsilon, Y^\varepsilon, \theta_{\varepsilon}^t \omega), \quad (5.1)
\]

\[
\frac{dY^\varepsilon}{dt} = BY^\varepsilon + G(X^\varepsilon, Y^\varepsilon, \theta_{\varepsilon}^t \omega), \quad (5.2)
\]

where

\[
F(X^\varepsilon, Y^\varepsilon, \theta_{\varepsilon}^t \omega) := f(X^\varepsilon + \delta(\theta_{\varepsilon}^1 \omega_1), Y^\varepsilon + \eta^\varepsilon(\theta_{\varepsilon}^2 \omega_2)),
\]

\[
G(X^\varepsilon, Y^\varepsilon, \theta_{\varepsilon}^t \omega) := g(X^\varepsilon + \delta(\theta_{\varepsilon}^1 \omega_1), Y^\varepsilon + \eta^\varepsilon(\theta_{\varepsilon}^2 \omega_2)).
\]

Noticing that Lemma 2.1, we replace \(\eta^\varepsilon(\theta_{\varepsilon}^2 \omega_2)\) by \(\xi(\theta_{\varepsilon}^2 \omega_2)\) in \((5.1)-(5.2)\) to get a new random evolutionary system

\[
\frac{d\bar{X}^\varepsilon}{dt} = \varepsilon \bar{A} \bar{X}^\varepsilon + \varepsilon \bar{F}(\bar{X}^\varepsilon, \bar{Y}^\varepsilon, \bar{T}^t \omega), \quad (5.3)
\]

\[
\frac{d\bar{Y}^\varepsilon}{dt} = \bar{B} \bar{Y}^\varepsilon + \bar{G}(\bar{X}^\varepsilon, \bar{Y}^\varepsilon, \bar{T}^t \omega), \quad (5.4)
\]

where

\[
\bar{F}(\bar{X}^\varepsilon, \bar{Y}^\varepsilon, \bar{T}^t \omega) := f(\bar{X}^\varepsilon + \delta(\theta_{\varepsilon}^1 \omega_1), \bar{Y}^\varepsilon + \xi(\theta_{\varepsilon}^2 \omega_2)),
\]

\[
\bar{G}(\bar{X}^\varepsilon, \bar{Y}^\varepsilon, \bar{T}^t \omega) := g(\bar{X}^\varepsilon + \delta(\theta_{\varepsilon}^1 \omega_1), \bar{Y}^\varepsilon + \xi(\theta_{\varepsilon}^2 \omega_2)),
\]

with the initial condition \((\bar{X}^\varepsilon(0), \bar{X}^\varepsilon(0))^T = (X_0, Y_0)^T\), whose solution is denoted by

\[
\bar{\Phi}^\varepsilon(t, \omega, (X_0, Y_0)^T) = (\bar{X}^\varepsilon(t, \omega, (X_0, Y_0)^T), \bar{Y}^\varepsilon(t, \omega, (X_0, Y_0)^T))^T.
\]

Then the distribution of the solution \(\bar{\Phi}^\varepsilon(t, \omega, (X_0, Y_0)^T)\) of the system \((5.3)-(5.4)\) coincides with that of \((2.6)-(2.7)\) (also see \([22]\)).

Put

\[
\bar{\Phi}^\varepsilon(t, \omega, (\bar{X}_0, \bar{Y}_0)^T) = \bar{\Phi}^\varepsilon(t, \omega, (X_0, Y_0)^T) - (\bar{X}^\varepsilon(t, \omega, (\bar{X}_0, \bar{Y}_0)^T) - \bar{X}^\varepsilon(t, \omega, (\bar{X}_0, \bar{Y}_0)^T) - \bar{Y}^\varepsilon(t, \omega, (X_0, Y_0)^T))^T
\]

\[
= : (\bar{U}^\varepsilon(t), \bar{V}^\varepsilon(t))^T.
\]

Then \(\bar{U}^\varepsilon(t) = \bar{U}^\varepsilon(t, \omega, (X_0, Y_0)^T; (\bar{X}_0 - X_0))\) and \(\bar{V}^\varepsilon(t) = \bar{V}^\varepsilon(t, \omega, (X_0, Y_0)^T; (\bar{X}_0 - X_0))\) satisfies

\[
\begin{pmatrix}
\dot{\bar{U}}^\varepsilon(t) \\
\dot{\bar{V}}^\varepsilon(t)
\end{pmatrix} =
\begin{pmatrix}
e^{\varepsilon A t} \bar{U}^\varepsilon(0) + \varepsilon \int_0^t e^{\varepsilon A(t-s)} \Delta F(\bar{U}^\varepsilon(s), \bar{V}^\varepsilon(s), \bar{T}^t \omega) ds \\
\int_{-\infty}^t e^{B(t-s)} \Delta G(\bar{U}^\varepsilon(s), \bar{V}^\varepsilon(s), \bar{T}^t \omega) ds
\end{pmatrix}
\]

(5.5)
with $\bar{U}^\varepsilon(0) = \bar{X}_0 - X_0$. Here
\[
\Delta F(\bar{U}^\varepsilon(s), \bar{V}^\varepsilon(s), \bar{\theta}^\varepsilon) = -F(\bar{U}^\varepsilon(s) + \bar{X}^\varepsilon(s, \omega, (X_0, Y_0)^T), \bar{V}^\varepsilon(s) + \bar{Y}^\varepsilon(s, \omega, (X_0, Y_0)^T), \bar{\theta}^\varepsilon) - F(X^\varepsilon(s, \omega, (X_0, Y_0)^T), Y^\varepsilon(s, \omega, (X_0, Y_0)^T), \theta^\varepsilon),
\]
and
\[
\Delta G(\bar{U}^\varepsilon(s), \bar{V}^\varepsilon(s), \bar{\theta}^\varepsilon) = G(\bar{U}^\varepsilon(s) + \bar{X}^\varepsilon(s, \omega, (X_0, Y_0)^T), \bar{V}^\varepsilon(s) + \bar{Y}^\varepsilon(s, \omega, (X_0, Y_0)^T), \bar{\theta}^\varepsilon) - G(X^\varepsilon(s, \omega, (X_0, Y_0)^T), Y^\varepsilon(s, \omega, (X_0, Y_0)^T), \theta^\varepsilon).
\]

For every $\zeta \in H_s$, we define
\[
\tilde{\ell}^\varepsilon(\zeta, (X_0, Y_0)^T, \omega) := Y_0 + \int_{-\infty}^0 e^{-B^s} \Delta G(\bar{U}^\varepsilon(s, \omega, (X_0, Y_0)^T; (\zeta - X_0)), \bar{V}^\varepsilon(s, \omega, (X_0, Y_0)^T; (\zeta - X_0)), \bar{\theta}^\varepsilon) ds.
\] (5.6)

Using the same arguments as in Section 4, we can obtain the slow foliation of (5.3)- (5.4) as follows.

**Lemma 5.1** Assume that the Hypothesis H1-H3 hold. Take $\beta$ as the positive real number $\frac{\gamma}{2}$. Then there exists a sufficiently small positive constant $\varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_0)$, the foliation of the slow-fast random system (5.3)-(5.4) exists, whose one fiber is given by
\[
\mathcal{W}_\beta^\varepsilon((X_0, Y_0)^T, \omega) = \{(\zeta, \tilde{\ell}^\varepsilon(\zeta, (X_0, Y_0)^T, \omega))^T | \zeta \in H_s\},
\]
where $(X_0, Y_0)^T \in H_s \times H_f$, the function $\tilde{\ell}^\varepsilon(\zeta, (X_0, Y_0)^T, \omega)$ is defined as (5.6).

Furthermore, we obtain the relationship of foliation between of the system (2.6)-(2.7) and the system (5.3)-(5.4) as follows.

**Lemma 5.2** Assume that the Hypothesis H1-H3 hold. Take $\beta$ as the positive real number $\frac{\gamma}{2}$. Then there exists a sufficiently small positive constant $\varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_0)$, the foliation of the system (2.6)-(2.7) is the same as that of the system (5.3)-(5.4) in distribution, that is, for every $\zeta \in H_s$,
\[
l^\varepsilon(\zeta, (X_0, Y_0)^T, \omega) \overset{d}{=} \tilde{\ell}^\varepsilon(\zeta, (X_0, Y_0)^T, \omega),
\] (5.7)
where $\overset{d}{=}$ denotes equivalence in distribution.

**Proof.** For (4.11), taking the time scaling $s \to \varepsilon s$, and noticing that the solution of the system (2.6)-(2.7) has the same distribution as the solution of the system (5.3)-(5.4), we know
that for every $\zeta \in H_s$, 

$$
I^\varepsilon(\zeta, (X_0, Y_0)^T, \omega)
\begin{align*}
= & \ Y_0 + \frac{1}{\varepsilon} \int_0^1 e^{-\frac{B_s}{\varepsilon}} \Delta G(U^\varepsilon(s, \omega, (X_0, Y_0)^T; (\zeta - X_0)), V^\varepsilon(s, \omega, (X_0, Y_0)^T; (\zeta - X_0)), \theta_s^\varepsilon \omega)ds
\end{align*}
$$

Consider a new random evolutionary system

$$
\frac{dX^0}{dt} = 0, \quad \frac{dY^0}{dt} = BY^0 + G(X^0, Y^0, \theta^0_\omega),
$$

where

$$
G(X^0, Y^0, \theta^0_\omega) := g(X^0 + \delta(\omega_1), Y^0 + \xi(\theta^2_\omega)),
$$

with the initial condition $(X^0(0), Y^0(0))^T = (X_0, Y_0)^T$. Essentially the system (5.8)-(5.9) is the system (5.3)-(5.4) scaled by $\varepsilon t$ with zero singular perturbation parameter (i.e., the system (5.3)-(5.4) with $\varepsilon = 0$).

We denote the solution of the system (5.8)-(5.9) as follows

$$
\Phi^0(t, \omega, (X_0, Y_0)^T) = (X_0, Y^0(t, \omega, (X_0, Y_0)^T))^T.
$$

And put

$$
\Phi^0(t, \omega, (X_0, Y_0)^T) - \Phi^0(t, \omega, (X_0, Y_0)^T)
\begin{align*}
= & \ (\bar{X}_0 - X_0, \bar{Y}_0 - Y_0)^T - Y^0(t, \omega, (X, Y_0)^T))^T
\end{align*}
$$

Then $U^0(t) = U^0(0) = \bar{X}_0 - X_0$ and $V^0(t) = V^0(t, \omega, (X_0, Y_0)^T; (\bar{X}_0 - X_0))$ satisfies

$$
\begin{pmatrix} U^0(0) \\ V^0(0) \end{pmatrix} = \left( \int_{-\infty}^t e^{B(t-s)} \Delta G(U^0(0), V^0(s, \theta^0_\omega)ds \right),
$$

where

$$
\Delta G(U^0(0), V^0(s, \theta^0_\omega)) = G(U^0(0) + X_0, V^0(s, \omega, (X_0, Y_0)^T), \bar{Y}_0 - G(X_0, Y^0(s, \omega, (X_0, Y_0)^T, \theta^0_\omega)).
$$
For every $\zeta \in H_s$, we define
\[
l^0(\zeta, (X_0, Y_0)^T, \omega) := Y_0 + \int_{-\infty}^{0} e^{-Bs} \Delta G((\zeta - X_0), V^0(s, \omega, (X_0, Y_0)^T; (\zeta - X_0)), \theta^0)ds.
\] (5.11)

Again using the same arguments as in Section 4, we can obtain the invariant foliation of (5.8)-(5.9) as follows.

**Theorem 5.1 (Critical foliation)** Assume that the Hypothesis H1-H3 hold. Take $\beta$ as the positive real number $-\gamma_s^2$. Then the invariant foliation of the random system (5.8)-(5.9) exists.

(i) Its one fiber is the graph of a Lipschitz function. That is
\[
\mathcal{W}_\beta^0((X_0, Y_0)^T, \omega) = \{(\zeta, l^0(\zeta, (X_0, Y_0)^T, \omega))^T \mid \zeta \in H_s\},
\]
where $(X_0, Y_0)^T \in H_s \times H_f$, the function $l^0(\zeta, (X_0, Y_0)^T, \omega)$ is defined as (5.11). In addition, $l^0(\zeta, (X_0, Y_0)^T, \omega)$ is Lipschitz continuous with respect to $\zeta$, whose Lipschitz constant $\text{Lip}^0$ satisfies
\[
\text{Lip}^0 \leq \frac{K}{\gamma_f + \beta - K}.
\]

(ii) The dynamical orbits of (5.8)-(5.9) are exponentially approaching each other in backward time only if they start from the same fiber. That is, for any two points $(\tilde{X}_1^1, \tilde{Y}_1^1)^T$ and $(\tilde{X}_0^2, \tilde{Y}_0^2)^T$ in a same fiber $\mathcal{W}_\beta^0((X_0, Y_0)^T, \omega)$,
\[
\|\Phi^0(t, \omega, (\tilde{X}_1^1, \tilde{Y}_1^1)^T) - \Phi^0(t, \omega, (\tilde{X}_0^2, \tilde{Y}_0^2)^T)\|_{H_s \times H_f} = O(e^{\beta t}), \quad \forall \quad t \to -\infty.
\]

(iii) Its fiber is invariant, i.e.,
\[
\Phi^0(t, \omega, \mathcal{W}_\beta^0((X_0, Y_0)^T, \omega)) \subset \mathcal{W}_\beta^0(\Phi^0(t, \omega, (X_0, Y_0)^T), \theta_1^0\omega).
\]

**Remark 5.1** From the Hypothesis H1-H3, and $\beta = -\gamma_s^2$, we easily know that $\gamma_f + \beta - K$ is a positive constant.

**Remark 5.2** As we will show, the slow foliation of the system (2.6)-(2.7) converges to the foliation of the system (5.8)-(5.9) in distribution, as $\varepsilon$ tends to zero. We call the limiting status of the slow foliation as the critical foliation for the system (2.6)-(2.7).
Theorem 5.2 (Convergence in distribution to critical foliation) Assume that the Hypothesis H1-H3 hold. Take \( \beta \) as the positive real number \( \frac{2\omega}{\varepsilon} \). And assume that the nonlinear function \( f(x,y) \) is bounded in \( H_s \), that is, there exists a positive constant such that \( \| f(x,y) \|_{H_s} \leq C \). Then the slow foliation converges to the critical foliation of the system of the system (2.6)-(2.7) in distribution (i.e., the distribution of the slow foliation converges to the distribution of the critical foliation), as \( \varepsilon \) tends to zero. In other words,

\[
\ell^\varepsilon(\zeta, (X_0, Y_0)^T, \omega) = \ell^0(\zeta, (X_0, Y_0)^T, \omega) + O(\varepsilon), \quad \text{in } H_f \text{ as } \varepsilon \to 0. \tag{5.12}
\]

Proof. Noticing that Lemma 5.2, we only need to prove

\[
\tilde{\ell}^\varepsilon(\zeta, (X_0, Y_0)^T, \omega) \xrightarrow{d} \ell^0(\zeta, (X_0, Y_0)^T, \omega), \quad \text{in } H_f \text{ as } \varepsilon \to 0, \tag{5.13}
\]

for each given \( \zeta \in H_s \).

From (5.6) and (5.11), we know that

\[
\| \tilde{\ell}^\varepsilon(\zeta, (X_0, Y_0)^T, \omega) - \ell^0(\zeta, (X_0, Y_0)^T, \omega) \|_{H_f} = \| V^\varepsilon(t, \omega, (X_0, Y_0)^T; (\zeta - X_0)) - V^0(t, \omega, (X_0, Y_0)^T; (\zeta - X_0)) \|_{H_f} \tag{5.14}
\]

For \( V^\varepsilon(t) \) and \( V^0(t) \) with \( t \leq 0 \), we have

\[
\| V^\varepsilon(t, \omega, (X_0, Y_0)^T; (\zeta - X_0)) - V^0(t, \omega, (X_0, Y_0)^T; (\zeta - X_0)) \|_{H_f}
= \| \int_{-\infty}^{t} e^{B(t-s)} [\Delta G(\tilde{U}^\varepsilon(s), \tilde{V}^\varepsilon(s), \tilde{\theta}^\varepsilon_s \omega) - \Delta G(U^0(s), V^0(s), \tilde{\theta}^0_s \omega)] ds \|_{H_f}
= \| \int_{-\infty}^{t} e^{B(t-s)} [G(\tilde{U}^\varepsilon(s) + \tilde{X}^\varepsilon(s, \omega, (X_0, Y_0)^T), \tilde{V}^\varepsilon(s) + \tilde{Y}^\varepsilon(s, \omega, (X_0, Y_0)^T), \tilde{\theta}^\varepsilon_s \omega)
- G(U^0(s) + X_0, V^0(s) + Y^0(s, \omega, (X_0, Y_0)^T), \tilde{\theta}^0_s \omega)
- G(\tilde{X}^\varepsilon(s, \omega, (X_0, Y_0)^T), \tilde{Y}^\varepsilon(s, \omega, (X_0, Y_0)^T), \tilde{\theta}^\varepsilon_s \omega)
+ G(X_0, Y^0(s, \omega, (X_0, Y_0)^T), \tilde{\theta}^0_s \omega)] ds \|_{H_f}
\leq K \int_{-\infty}^{t} e^{B(t-s)} [\| \tilde{U}^\varepsilon(s) - U^0(s) \|_{H_s} + \| \tilde{V}^\varepsilon(s) - V^0(s) \|_{H_f}
+ 2 \| \tilde{X}^\varepsilon(s) - X_0 \|_{H_s} + 2 \| \tilde{Y}^\varepsilon(s) - Y^0(s) \|_{H_f}] ds
\]

for sufficiently small \( \varepsilon \).

To obtain the estimates of (5.15), we need to establish the a priori estimates of \( \| \tilde{X}^\varepsilon(t) - X_0 \|_{H_s} \), \( \| \tilde{Y}^\varepsilon(t) - Y^0(t) \|_{H_f} \), and \( \| \tilde{U}^\varepsilon(t) - U^0(t) \|_{H_s} \), respectively.

Step (i): To estimate \( \| \tilde{X}^\varepsilon(t) - X_0 \|_{H_s} \).

For the system (5.3)-(5.4), using the same argument of Lemma 4.1, we can write it in a integral form

\[
\begin{pmatrix}
\tilde{X}^\varepsilon(t) \\
\tilde{Y}^\varepsilon(t)
\end{pmatrix}
= 
\begin{pmatrix}
e^{\varepsilon A(t-s)}X_0 + \varepsilon \int_{0}^{t} e^{\varepsilon A(t-s)} F(\tilde{X}^\varepsilon(s), \tilde{Y}^\varepsilon(s), \tilde{\theta}^\varepsilon_s \omega) ds \\
\int_{-\infty}^{t} e^{B(t-s)} G(\tilde{X}^\varepsilon(s), \tilde{Y}^\varepsilon(s), \tilde{\theta}^\varepsilon_s \omega) ds
\end{pmatrix}. \tag{5.16}
\]
Then for any $t \leq 0$,
\[
\|\tilde{X}^\varepsilon(t) - X_0\|_{H_s} \\
\leq \|e^{\varepsilon At}X_0 - X_0\|_{H_s} + \varepsilon \int_t^0 e^{\varepsilon(-\gamma_s)(t-s)}\|F(\tilde{X}^\varepsilon(s), \tilde{Y}^\varepsilon(s), \vartheta^\varepsilon_\omega)\|_{H_s} ds \\
\leq \|e^{\varepsilon At}X_0 - X_0\|_{H_s} + \varepsilon \cdot C \cdot \int_t^0 e^{\varepsilon(-\gamma_s)(t-s)} ds \\
\leq \|\int_t^0 AX_0 e^{\varepsilon At} d\tau\|_{H_s} + \varepsilon \cdot C \cdot \int_t^0 e^{\varepsilon(-\gamma_s)(t-s)} ds \\
\leq \|AX_0\|_{H_s} \cdot \frac{\|\tilde{X}^\varepsilon\|_{L^\gamma}}{\gamma_s} + C \cdot C \cdot \frac{\|\tilde{Y}^\varepsilon\|_{L^\gamma}}{\gamma_s} \\
\leq C[1 - e^{-\gamma_s t}].
\]

Here and hereafter, we use $C$ to denote various positive constant independent of $\varepsilon$ and $t$.

**Step (ii): To estimate $\|\tilde{Y}^\varepsilon(t) - Y^0(t)\|_{H_f}$.**

For the system (5.8)-(5.9), using the same argument of Lemma 4.1, we also can write it in an integral form
\[
\begin{pmatrix}
X^0(t) \\
Y^0(t)
\end{pmatrix} = \begin{pmatrix}
X_0 \\
\int_{-\infty}^{t} e^{B(t-s)} G(X_0, Y^0(s), \vartheta^\varepsilon_\omega) ds
\end{pmatrix}.
\]

Then for any $t \leq 0$, using (5.16) and (5.17), we deduce that
\[
\|\tilde{Y}^\varepsilon(t) - Y^0(t)\|_{H_f} = \|\int_{-\infty}^{t} e^{B(t-s)} G(\tilde{X}^\varepsilon(s), \tilde{Y}^\varepsilon(s), \vartheta^\varepsilon_\omega) - G(X_0, Y^0(s), \vartheta^\varepsilon_\omega) ds\|_{H_f} \\
\leq K \int_{-\infty}^{t} e^{-\gamma_f(t-s)} (\|\tilde{X}^\varepsilon(s) - X_0\|_{H_s} + \|\tilde{Y}^\varepsilon(s) - Y^0(s)\|_{H_f}) ds \\
\leq K \int_{-\infty}^{t} e^{-\gamma_f(t-s)} [C(1 - e^{-\gamma_s t}) + \|\tilde{Y}^\varepsilon(s) - Y^0(s)\|_{H_f}] ds
\]

for sufficiently small $\varepsilon$.

Take a real number $\alpha$ satisfying
\[
\alpha \in \left(-\frac{2\gamma_f^2}{\gamma_s + 2\gamma_f}, 0\right).
\]

Then combining with the Hypothesis H1-H3, we know that
\[
\alpha + \gamma_f > 0, \quad -\alpha - \varepsilon\gamma_s > 0, \quad \text{and} \quad 0 < \frac{K}{\gamma_f + \alpha} < 1.
\]

It follows from (5.19) that
\[
\|\tilde{Y}^\varepsilon - Y^0\|_{C^\gamma_{\alpha}^-} = \sup_{t \leq 0} e^{-\alpha t}\|\tilde{Y}^\varepsilon(t) - Y^0(t)\|_{H_f} \\
\leq KC \sup_{t \leq 0} e^{-\alpha t} \int_{-\infty}^{t} e^{-\gamma_f(t-s)} [(1 - e^{-\gamma_s t}) ds + K \sup_{t \leq 0} e^{-\alpha t} \int_{-\infty}^{t} e^{-\gamma_f(t-s)}\|\tilde{Y}^\varepsilon(s) - Y^0(s)\|_{H_f} ds \\
\leq KC \sup_{t \leq 0} \left[\frac{1}{\gamma_f} e^{-\alpha t} \left(1 - \frac{1}{\gamma_s + \varepsilon_s} e^{-\alpha t - \varepsilon s t}\right) + \frac{K}{\gamma_f + \alpha} \cdot \|\tilde{Y}^\varepsilon - Y^0\|_{C^\gamma_{\alpha}^-} \right].
\]

(5.22)
Define a function

\[ p(t) := \frac{1}{\gamma_f} e^{-\alpha t} - \frac{1}{\gamma_f - \varepsilon \gamma_s} e^{-\alpha t - \varepsilon \gamma_s T}, \quad \text{for} \quad t \leq 0. \tag{5.23} \]

Then

\[ p'(t) = e^{-\alpha t} \left[ -\alpha - \frac{\alpha - \varepsilon \gamma_s e^{-\varepsilon \gamma_s T}}{\gamma_f - \varepsilon \gamma_s} \right] \geq e^{-\alpha t} \left[ -\alpha - \frac{\alpha - \varepsilon \gamma_s}{\gamma_f - \varepsilon \gamma_s} e^0 \right] \]

\[ \rightarrow 0, \quad \text{as} \quad \varepsilon \rightarrow 0, \]

which implies there exists a sufficient small \( \varepsilon_0 \) such that for \( \varepsilon \in (0, \varepsilon_0) \), \( p(t) \) is increasing with respect to the variable \( t \). Then we immediately get

\[ p(t) \leq p(0) = \frac{1}{\gamma_f} - \frac{1}{\gamma_f - \varepsilon \gamma_s}, \quad \text{for} \quad t \leq 0. \tag{5.24} \]

Thus, it follows from (5.22)–(5.24) that

\[ \| \hat{Y} - Y \|_{C_1} \leq \frac{KC}{1 - \frac{K}{\gamma_f + \alpha}} \left( \frac{1}{\gamma_f} - \frac{1}{\gamma_f - \varepsilon \gamma_s} \right). \tag{5.25} \]

Then

\[ \| \hat{Y}(t) - Y(t) \|_{H_1} \leq \frac{KC}{1 - \frac{K}{\gamma_f + \alpha}} \left( \frac{1}{\gamma_f} - \frac{1}{\gamma_f - \varepsilon \gamma_s} \right) \cdot e^{\alpha t}, \quad \text{for} \quad t \leq 0, \tag{5.26} \]

which is significative from (5.20) and (5.21).

**Step (iii): To estimate** \( \| \hat{Y}(t) - U(0) \|_{H_1} \).

It follows from (5.23) and (5.10) that

\[
\| \hat{Y}(t) - U(0) \|_{H_1} \leq \| e^{\alpha t} \hat{Y}(0) - U(0) \|_{H_1} + \| \int_0^t e^{\alpha (t-s)} \Delta F(\hat{Y}(s), \hat{Y}(s), \hat{V}(s), \hat{V}(s)) ds \|_{H_1}.
\]

Using the same argument as (i), we can get

\[ \| \hat{Y}(t) - U(0) \|_{H_1} \leq C[1 - e^{-\gamma \varepsilon t}]. \tag{5.27} \]

Now we go back to (5.15) to estimate \( \| \hat{V}(t) - V(0) \|_{H_f} \).

It follows from (5.15), (5.17), (5.26) and (5.27) that for any \( t \leq 0 \),

\[
\| \hat{V}(t, \omega, (X_0, Y_0)^T; (\zeta - X_0)) - V(0, \omega, (X_0, Y_0)^T; (\zeta - X_0)) \|_{H_f} \leq K \int_{-\infty}^t e^{-\gamma_f (t-s)} [3C[1 - e^{-\gamma \varepsilon s}] + 2 \frac{KC}{1 - \frac{K}{\gamma_f + \alpha}} \left( \frac{1}{\gamma_f} - \frac{1}{\gamma_f - \varepsilon \gamma_s} \right) \cdot e^{\alpha s} + \| \hat{V}(s) - V(0) \|_{H_f}] ds,
\]

\[ 25 \]
which implies that

\[
\|\tilde{V}^\varepsilon - V^0\|_{C^1_t,^\varepsilon} \leq K\|\tilde{V}^\varepsilon - V^0\|_{C^1_t,^\varepsilon} \cdot \sup_{t \leq 0} e^{-\alpha t} \int_{-\infty}^{0} e^{-\gamma_f(t-s)} e^{\alpha s} ds + KC \cdot \sup_{t \leq 0} e^{-\alpha t} \int_{-\infty}^{0} e^{-\gamma_f(t-s)} [(1 - e^{-\varepsilon^2 s}) + \left(\frac{1}{\gamma_f} - \frac{1}{\gamma_f - \varepsilon^2}\right) \cdot e^{\alpha s}] ds
\]

(5.28)

Again using (5.28) and (5.24), then it follows from (5.29) that

\[
\|\tilde{V}^\varepsilon - V^0\|_{C^1_t,^\varepsilon} \leq \frac{KC(1 + \frac{1}{\gamma_f})}{1 - \frac{K}{\gamma_f}} \cdot \left(\frac{1}{\gamma_f} - \frac{1}{\gamma_f - \varepsilon}\right) e^{\alpha t}, \text{ for } t \leq 0.
\]

(5.29)

which is also significative from (5.20) and (5.21). Then we immediately have that

\[
\|\tilde{V}^\varepsilon(t) - V^0(t)\|_{H_f} \leq \frac{KC(1 + \frac{1}{\gamma_f})}{1 - \frac{K}{\gamma_f}} \cdot \left(\frac{1}{\gamma_f} - \frac{1}{\gamma_f - \varepsilon}\right) e^{\alpha t}, \text{ for } t \leq 0.
\]

(5.30)

Hence, it finally follows from (5.14) and (5.30) that

\[
\|\bar{V}^\varepsilon(\zeta, (X_0, Y_0)^T, \omega) - \bar{V}^0(\zeta, (X_0, Y_0)^T, \omega)\|_{H_f} = \|\tilde{V}^\varepsilon(t, \omega, (X_0, Y_0)^T; (\zeta - X_0)) - V^0(t, \omega, (X_0, Y_0)^T; (\zeta - X_0))\|_{H_f}|_{t=0} \\
\leq \frac{KC(1 + \frac{1}{\gamma_f})}{1 - \frac{K}{\gamma_f}} \cdot \left(\frac{1}{\gamma_f} - \frac{1}{\gamma_f - \varepsilon}\right) \\
\rightarrow 0, \text{ as } \varepsilon \to 0,
\]

(5.31)

which implies (5.13) holds. This completes the proof.

\textbf{Theorem 5.3 (Approximation of slow foliation)} Assume that the Hypothesis H1-H3 hold. Take $\beta$ as the positive real number $\frac{\gamma_f}{2}$. And assume that the nonlinear function $f(x, y)$ is bounded in $H_s$. Then for sufficiently small $\varepsilon$, the slow foliation of the system (2.6)-(2.7) can be approximated in distribution as

\[
\mathcal{W}_{\varepsilon}^\beta((X_0, Y_0)^T, \omega) = \{(\zeta, l^\varepsilon(\zeta, (X_0, Y_0)^T, \omega)) | \zeta \in H_s\} \\
\overset{d}{=} \{(\zeta, l^0(\zeta, (X_0, Y_0)^T, \omega) + \varepsilon l^1(\zeta, (X_0, Y_0)^T, \omega) + O(\varepsilon^2)) | \zeta \in H_s\},
\]

(5.32)

where $l^0(\zeta, (X_0, Y_0)^T, \omega)$ is the critical foliation as (5.11), and $l^1(\zeta, (X_0, Y_0)^T, \omega)$ is well defined as (5.47).

\textbf{Proof}. From Lemma 5.2, it is only need to prove

\[
\bar{V}^\varepsilon(\zeta, (X_0, Y_0)^T, \omega) = l^0(\zeta, (X_0, Y_0)^T, \omega) + \varepsilon l^1(\zeta, (X_0, Y_0)^T, \omega) + O(\varepsilon^2), \text{ in } H_f.
\]

(5.33)
For the system (5.3)-(5.4), we write
\[
\dot{X}^\varepsilon(t) = \dot{X}^0(t) + \varepsilon X^1(t) + O(\varepsilon^2),
\]
\[
\dot{X}^\varepsilon(0) = X_0,
\]
and
\[
\dot{Y}^\varepsilon(t) = \dot{Y}^0(t) + \varepsilon Y^1(t) + O(\varepsilon^2),
\]
\[
\dot{Y}^\varepsilon(0) = Y_0,
\]
where \(\dot{X}^0(t), \dot{Y}^0(t), X^1(t)\) and \(Y^1(t)\) will be determined in the below. Also, notice that the relationship of \((\dot{X}^\varepsilon(t), \dot{Y}^\varepsilon(t))^T\) and \((\dot{U}^\varepsilon(t), \dot{V}^\varepsilon(t))^T\). We can write
\[
\dot{U}^\varepsilon(t) = \dot{U}^0(t) + \varepsilon U^1(t) + O(\varepsilon^2),
\]
\[
\dot{V}^\varepsilon(0) = \zeta - X_0,
\]
and
\[
\dot{V}^\varepsilon(t) = \dot{V}^0(t) + \varepsilon V^1(t) + O(\varepsilon^2),
\]
\[
\dot{V}^\varepsilon(0) = \bar{l}(\zeta, (X_0, Y_0)^T, \omega) - Y_0.
\]
Expanding \(F(\dot{X}^\varepsilon(t), \dot{Y}^\varepsilon(t), \bar{\theta}_f^T \omega)\) at \(\varepsilon = 0\) by Taylor formula, we infer that
\[
F(\dot{X}^\varepsilon(t), \dot{Y}^\varepsilon(t), \bar{\theta}_f^T \omega)
= f(\dot{X}^\varepsilon(t) + \delta(\theta^1 \omega_1), \dot{Y}^\varepsilon(t) + \xi(\theta^2 \omega_2))
= f(\dot{X}^0(t) + \delta(\omega_1), \dot{Y}^0(t) + \xi(\theta^2 \omega_2))
+ (\dot{X}^\varepsilon(t) - \dot{X}^0(t)) f_x(\dot{X}^0(t) + \delta(\omega_1), \dot{Y}^0(t) + \xi(\theta^2 \omega_2))
+ (\dot{Y}^\varepsilon(t) - \dot{Y}^0(t)) f_y(\dot{X}^0(t) + \delta(\omega_1), \dot{Y}^0(t) + \xi(\theta^2 \omega_2)) + O(\varepsilon^2)
\]
(5.38)
where \(f_x(\cdot, \cdot)\) and \(f_y(\cdot, \cdot)\) denote the partial derivative of \(f(x, y)\) with respect to the first variable
\(x\), and the second variable \(y\), respectively.

Similarly, we get
\[
G(\dot{X}^\varepsilon(t), \dot{Y}^\varepsilon(t), \bar{\theta}_f^T \omega)
= G(\dot{X}^0(t), \dot{Y}^0(t), \bar{\theta}_f^T \omega)
+ \varepsilon \cdot X^1(t) \cdot f_x(\dot{X}^0(t) + \delta(\omega_1), \dot{Y}^0(t) + \xi(\theta^2 \omega_2))
+ \varepsilon \cdot Y^1(t) \cdot f_y(\dot{X}^0(t) + \delta(\omega_1), \dot{Y}^0(t) + \xi(\theta^2 \omega_2)) + O(\varepsilon^2),
\]
(5.39)
where \(g_x(\cdot, \cdot)\) and \(g_y(\cdot, \cdot)\) denote the partial derivative of \(g(x, y)\) with respect to the first variable
\(x\), and the second variable \(y\), respectively. We also have
\[
F(\ddot{U}^\varepsilon(t) + \ddot{X}^\varepsilon(t), \ddot{V}^\varepsilon(t) + \ddot{Y}^\varepsilon(t), \bar{\theta}_f^T \omega)
= F(\ddot{U}^0(t) + \ddot{X}^0(t), \ddot{Y}^0(t) + \ddot{Y}^0(t), \bar{\theta}_f^T \omega)
+ \varepsilon \cdot (U^1(t) + X^1(t)) \cdot f_x(\ddot{U}^0(t) + \ddot{X}^0(t) + \delta(\omega_1), \ddot{Y}^0(t) + \ddot{Y}^0(t) + \xi(\theta^2 \omega_2))
+ \varepsilon \cdot (V^1(t) + Y^1(t)) \cdot f_y(\ddot{U}^0(t) + \ddot{X}^0(t) + \delta(\omega_1), \ddot{Y}^0(t) + \ddot{Y}^0(t) + \xi(\theta^2 \omega_2)) + O(\varepsilon^2),
\]
(5.40)
and

\[
G(\tilde{U}^\varepsilon(t) + \tilde{X}^\varepsilon(t), \tilde{V}^\varepsilon(t) + \tilde{Y}^\varepsilon(t), \theta_t^1 \omega) = G(\tilde{U}^0(t) + \tilde{X}^0(t), \tilde{V}^0(t) + \tilde{Y}^0(t), \theta_t^1 \omega) + \varepsilon \cdot (U^1(t) + X^1(t)) \cdot g_x(\tilde{U}^0(t) + \tilde{X}^0(t) + \delta(\omega_1), \tilde{V}^0(t) + \tilde{Y}^0(t) + \xi(\theta_t^2 \omega_2)) + \varepsilon \cdot (V^1(t) + Y^1(t)) \cdot g_y(\tilde{U}^0(t) + \tilde{X}^0(t) + \delta(\omega_1), \tilde{V}^0(t) + \tilde{Y}^0(t) + \xi(\theta_t^2 \omega_2)) + O(\varepsilon^2).
\]

Substituting (5.34) into (5.33), and substituting (5.35) into (5.4), then equating the terms with the same power of \(\varepsilon\), we deduce that

\[
\begin{align*}
\dot{X}^0(t) &= 0, \\
\dot{Y}^0(t) &= B\dot{Y}^0(t) + G(\dot{X}^0(t), \dot{Y}^0(t), \theta_t^0 \omega) \\
\end{align*}
\]

and

\[
\begin{align*}
X^1(t) &= A \dot{X}^0(t) + F(\dot{X}^0(t), \dot{Y}^0(t), \theta_t^1 \omega), \\
Y^1(t) &= BY^1(t) + X^1(t) \cdot g_x(\dot{X}^0(t) + \delta(\omega_1), \dot{Y}^0(t) + \xi(\theta_t^2 \omega_2)) + Y^1(t) \cdot g_y(\dot{X}^0(t) + \delta(\omega_1), \dot{Y}^0(t) + \xi(\theta_t^2 \omega_2)).
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
\dot{U}^0(t) &= 0, \\
\dot{V}^0(t) &= B\dot{Y}^0(t) + [G(\dot{U}^0(t) + \dot{X}^0(t), \dot{V}^0(t) + \dot{Y}^0(t), \theta_t^0 \omega) - G(\dot{X}^0(t), \dot{Y}^0(t), \theta_t^0 \omega)] \\
\end{align*}
\]

and

\[
\begin{align*}
U^1(t) &= A \dot{U}^0(t) + [F(\dot{U}^0(t) + \dot{X}^0(t), \dot{V}^0(t) + \dot{Y}^0(t), \theta_t^1 \omega) - F(\dot{X}^0(t), \dot{Y}^0(t), \theta_t^1 \omega)], \\
V^1(t) &= BV^1(t) + [(U^1(t) + X^1(t)) \cdot g_x(\dot{U}^0(t) + \dot{X}^0(t) + \delta(\omega_1), \dot{V}^0(t) + \dot{Y}^0(t) + \xi(\theta_t^2 \omega_2)) + Y^1(t) \cdot g_y(\dot{U}^0(t) + \dot{X}^0(t) + \delta(\omega_1), \dot{V}^0(t) + \dot{Y}^0(t) + \xi(\theta_t^2 \omega_2)) - X^1(t) \cdot g_x(\dot{V}^0(t) + \delta(\omega_1), \dot{Y}^0(t) + \xi(\theta_t^2 \omega_2)) - Y^1(t) \cdot g_y(\dot{V}^0(t) + \delta(\omega_1), \dot{Y}^0(t) + \xi(\theta_t^2 \omega_2))].
\end{align*}
\]

It immediately follows from (5.8), (5.10), (5.42) and (5.44) that

\[
\begin{align*}
\dot{X}^0(t) &= X^0(t), & \dot{Y}^0(t) &= Y^0(t), & \dot{U}^0(t) &= U^0(t), & \dot{V}^0(t) &= V^0(t).
\end{align*}
\]

In addition, using the contraction mapping principle as in Lemma 4.2, we can easily obtain the existence of \((X^1(t), Y^1(t))^T\) and \((U^1(t), V^1(t))^T\). Here, for simplicity, we omit it. Then we can define

\[
I^1(\zeta, (X_0, Y_0)^T, \omega) = \int_{-\infty}^{0} e^{-Bs} [(U^1(t) + X^1(t)) \cdot g_x(\zeta + \delta(\omega_1), V^0(t) + \dot{Y}^0(t) + \xi(\theta_t^2 \omega_2)) + (V^1(t) + Y^1(t)) \cdot g_y(\zeta + \delta(\omega_1), V^0(t) + \dot{Y}^0(t) + \xi(\theta_t^2 \omega_2)) - X^1(t) \cdot g_x(X_0 + \delta(\omega_1), Y^0(t) + \xi(\theta_t^2 \omega_2)) - Y^1(t) \cdot g_y(X_0 + \delta(\omega_1), Y^0(t) + \xi(\theta_t^2 \omega_2))] ds.
\]

(5.47)
For every \( \zeta \in H_s \), noticing that the initial condition \( X^0(0) = X_0 \) and \( U^0(0) = \zeta - X_0 \), it follows from (5.47) that Theorem 4.3 holds.

\[
\begin{align*}
\bar{\ell}^0(\zeta, (X_0, Y_0)^T, \omega) &= Y_0 + \int_{-\infty}^{0} e^{-B^* s} [G(\bar{U}^\varepsilon(s) + \bar{X}^\varepsilon(s) + \bar{Y}^\varepsilon(s), \theta_\varepsilon) \\
&\quad - G(\bar{X}^\varepsilon(s), \bar{Y}^\varepsilon(s), \theta_\varepsilon)] ds \\
&= Y_0 + \int_{-\infty}^{0} e^{-B^* s} \Delta G(U^0(0), V^0(0), \theta_0^s) ds \\
&\quad + \varepsilon \int_{-\infty}^{0} e^{-B^* s} [(U^1(t) + X^1(t)) \cdot g_x(U^0(t) + X^0(t) + \delta(\omega_1), V^0(t) + Y^0(t) + \xi(\theta_1^2 \omega_2)) \\
&\quad + (V^1(t) + Y^1(t)) \cdot g_y(U^0(t) + X^0(t) + \delta(\omega_1), V^0(t) + Y^0(t) + \xi(\theta_1^2 \omega_2)) \\
&\quad - X^1(t) \cdot g_x(X_0 + \delta(\omega_1), Y^0(t) + \xi(\theta_1^2 \omega_2)) \\
&\quad - Y^1(t) \cdot g_y(X_0 + \delta(\omega_1), Y^0(t) + \xi(\theta_1^2 \omega_2))] ds \\
&\quad + O(\varepsilon^2),
\end{align*}
\]

which immediately follows from (5.47) that Theorem 4.3 holds.

**Remark 5.3** In Section 4 and Section 5, the conditions for the general theory of the slow foliation are only sufficient condition, not the necessary condition.

**Example 5.1** Consider the following slow-fast stochastic evolutionary system

\[
\begin{align*}
\frac{dx^\varepsilon}{dt} &= x^\varepsilon + f(x^\varepsilon, y^\varepsilon) + \sigma_1 \dot{W}_1, \quad \text{in } H_s, \quad (5.49) \\
\frac{dy^\varepsilon}{dt} &= \frac{1}{\varepsilon} \Delta y^\varepsilon + \frac{1}{\varepsilon} g(x^\varepsilon, y^\varepsilon) + \frac{\sigma_2}{\sqrt{\varepsilon}} \dot{W}_2, \quad \text{in } H_f. \quad (5.50)
\end{align*}
\]

The system may model certain biological processes, for instance, the famous FitzHugh-Nagumo system, as a simplified version of the Hodgkin-Huxley model, which describes mechanisms of a neural excitability and excitation for macro-receptors.

Let \( A \) be \( \text{Id} \) (the identity operator) in \( H_s = L^2([0, \pi]) \). Then it is clear that \( \| e^{At} x \|_{H_s} \leq e^{t} \| x \|_{H_s} = e^{-\gamma_s t} \| x \|_{H_s} \) with \( \gamma_s = -1 \). Let \( B \) be \( \Delta \) with domain \( D = H^2([0, \pi]) \cap H_0^1([0, \pi]) \), whose eigenvalue are \( \lambda_k = -k^2 \) with the corresponding eigenfunction \( e_k = \sin kx \) (\( k = 1, 2, \ldots \)), generating a \( C_0 \)-semigroup \( \{ e^{Bt} : t \geq 0 \} \) on \( H_f = L^2([0, \pi]) \) satisfying \( \| e^{Bt} y \|_{H_f} \leq e^{-\gamma_f t} \| y \|_{H_f} \) with \( \gamma_f = 1 \).

Assume that nonlinear functions \( f : H_s \times H_f \to H_s \) and \( g : H_s \times H_f \to H_f \), which are
\(C^1\)-smooth with \(f(0, 0) = 0\) and \(g(0, 0) = 0\), and satisfy Lipschitz condition as

\[
\|f(x, y) - f(\tilde{x}, \tilde{y})\|_{H_s} \leq K(\|x - \tilde{x}\|_{H_s} + \|y - \tilde{y}\|_{H_f}),
\]

\[
\|g(x, y) - g(\tilde{x}, \tilde{y})\|_{H_s} \leq K(\|x - \tilde{x}\|_{H_s} + \|y - \tilde{y}\|_{H_f}),
\]

where \(K < \frac{1}{3}\). For example, \(f(x^\varepsilon, y^\varepsilon) = \frac{1}{4} \sin y^\varepsilon\) and \(g(x^\varepsilon, y^\varepsilon) = \frac{1}{4} \cos x^\varepsilon\).

Then for the system (5.49)-(5.50), taking \(\beta = -\gamma_2 = \frac{1}{2}\), as in Theorem 4.1 and Theorem 5.1, the slow foliation and the critical foliation exist. Their fibers are given as \(W^s_1((X_0, Y_0)^T, \omega)\) and \(W^q_1((X_0, Y_0)^T, \omega)\), respectively. Furthermore, the fiber \(W^s_1((X_0, Y_0)^T, \omega)\) of the slow foliation converges to the fiber \(W^q_1((X_0, Y_0)^T, \omega)\) of the critical foliation in distribution, as the singular perturbation parameter \(\varepsilon\) tends to zero. In addition, the slow manifold \(M^\varepsilon(\omega)\) as given by (2.10) is one fiber of the slow foliation, which parallels other fibers of the slow foliation.

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