Adler-Bell-Jackiw anomaly in VSR electrodynamics

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Abstract

In this paper, we examine the problem of anomalies of the fermionic currents in the context of the very special relativity (VSR). We consider the VSR contributions to the triangle amplitude for the case of the axial-vector vertex $\langle J^5 J^\mu J^\nu \rangle$, which allows the evaluation of the vector and axial Ward identities. Actually, we observe that the VSR nonlocal effects renders a novel anomaly in the vector Ward identity, and also contribute in a very interesting and unique way for the Adler-Bell-Jackiw anomaly.

1 Introduction

In the development of models in quantum field theories, one necessarily enforces a symmetry principle when establishing the physical degrees of freedom (fields) present in the physical system [1]. A desired symmetry would leave the classical action invariant under a given transformation of the fields. As we know, the local gauge symmetries have been the keystone for the establishment of the standard model of the particle physics, as well as most of the phenomenological models. On the other hand, there are symmetries related to the physical content of the fields and their couplings, depending on the given model, and have a crucial role in their definition. These two types of symmetries, when preserved at the classical (tree) level, have an interesting feature in the quantum realm, because they are not necessarily preserved in the transition to the quantum theory, which are known as anomalous symmetries [2–4].

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The study of anomalies in the gauge theories is extremely important in demonstrating its consistency and physical properties, such as unitarity and renormalizability, and they are expected to cancel. For instance, these cancellations are of central importance in constraining the fermionic content of the standard model. In order to verify whether a symmetry is preserved at the quantum level, one must analyze the respective Ward identity (or Slavnov-Taylor identity in the non-Abelian case).

Remarkably, the full structure of the anomaly can be calculated exactly at the one-loop order, and in the case of the chiral anomaly, it is given by the famous Adler-Bell-Jackiw (ABJ) anomaly, justifying the $\pi_0 \to \gamma \gamma$ decay \[2,5–7\]. As another physical application of the anomalies, we can refer to the detected signatures of the chiral anomaly in the magneto-transport measurements in Weyl semimetals \[8\].

Naturally, the models describing the physics beyond the standard model should fulfill such strong consistency requirements as well. In this paper, we shall consider the problem of the anomalies in a particular class of the models presenting Lorentz symmetry violation \[9–12\]. Contrary to the common lore, Lorentz violating effects are not necessarily related to Planck scale physics, it is also possible to formulate such class of models from a phenomenological group theory point of view.

A Lorentz violating framework, preserving all the basic elements of special relativity, is the Cohen and Glashow very special relativity (VSR) \[13,14\]. The main aspect of the VSR proposal is that the laws of physics are invariant under the (kinematical) subgroups of the Poincaré group. In (3 + 1)-dimensional spacetime, there are two VSR subgroups, SIM(2) and HOM(2), that preserve the direction of a lightlike four-vector $n_\mu$ by scaling, transforming as $n \to e^{\epsilon} n$ under boost in the z direction. This feature implies that there is a preferred direction in the Minkowski spacetime, where Lorentz violating terms can be constructed as ratios of contractions of the vector $n_\mu$ with other kinematical vectors, for instance $n^\mu/(n.p)$ \[13\].

The simplest example of VSR models is the free scalar massless field, whose action is given by

$$S = \int d^4x \, \bar{\phi} \tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi, \quad (1.1)$$

where the field equation reads $\tilde{\Box} \phi = (\Box - m^2) \phi$, with the wiggle derivative operator defined by $\tilde{\partial}_\mu = \partial_\mu + \frac{1}{2} m^2 n_\mu$. We observe that the Lorentz violation appears in a nonlocal form and the parameter $m$ sets the scale for the VSR effects. The VSR approach has been extended to the gauge theories, where many interesting theoretical and phenomenological aspects of VSR effects have been extensively discussed \[15–22\].

Hence, since VSR gauge theories possess very interesting features, we formulate in the present work a gauge invariant Lagrangian density in order to examine the ABJ anomaly in the VSR electrodynamics. There were some works investigating the ABJ anomaly in the different Lorentz violating models \[23–27\], but not yet in the VSR context. Although, some aspects of the axial anomaly have been examined in the VSR-inspired Schwinger model \[28\].

We start Sec. 2 by reviewing the main aspects of the VSR gauge invariance, and the related classical conserved currents in the VSR electrodynamics. The study of the chiral anomaly in the VSR setup can be well motivated due to engendering nonlocal couplings as well as generating a massive mode in the propagator, which possibly will render nontrivial modifications. Thus, it is valid to ask how the chiral anomaly behaves in the presence of such massive mode coming from Lorentz
violating effects. In Sec. 3, we consider the axial anomaly in terms of the triangle amplitude. We first evaluate the VSR contributions to the vector Ward identity. Next, we discuss the VSR effects in the axial Ward identity, related to the ABJ anomaly. In the case of the ABJ anomaly, we use the ’t Hooft-Veltman rule to perform some algebraic manipulations with $\gamma_5$ within the dimensional regularization method. In Sec. 4, we summarize the results, and present our final remarks.

2 Anomalies in the VSR electrodynamics

In order to discuss the ABJ anomaly in the context of VSR electrodynamics, we shall consider the following Lagrangian density

$$\mathcal{L}_{\text{vsr}} = \overline{\psi} (i\gamma^\mu \nabla_\mu - m_e) \psi,$$

(2.1)

which establishes the dynamics of the fermions minimally coupled to an external electromagnetic potential $A_\mu$. The VSR gauge coupling is defined by means of the covariant derivative as below

$$\nabla_\mu \psi = D_\mu \psi + \frac{1}{2} m^2 n_\mu \left( 1 - \frac{1}{n.D} \right) \psi,$$

(2.2)

written in terms of the ordinary covariant derivative $D_\mu = \partial_\mu - ieA_\mu$, and it reproduces the wiggle derivative $\tilde{\partial}_\mu = \partial_\mu + \frac{1}{2} m^2 n_\mu \left( 1 - \frac{1}{n.D} \right)$ in the noninteracting limit.

In our analysis of the problem of anomalies, in terms of the Ward identity, we will first study the conserved currents in the classical level. In this model, we have two conserved currents related to the gauge and chiral symmetries. We can obtain these currents from the fermionic field equations

$$(i\gamma^\mu \nabla_\mu - m_e) \psi = 0,$$

(2.3)

and its conjugated, from straightforward manipulations, resulting into a vector and axial-vector currents, respectively [28]

$$J^\mu = \overline{\psi} \gamma^\mu \psi + \frac{m^2}{2} \left( \frac{1}{n.D} \nabla \psi \right) \nabla^\mu \left( \frac{1}{n.D} \psi \right),$$

(2.4)

$$J_5^\mu = \overline{\psi} \gamma^\mu \gamma_5 \psi + \frac{m^2}{2} \left( \frac{1}{n.D} \nabla \psi \right) \nabla^\mu \gamma_5 \left( \frac{1}{n.D} \psi \right).$$

(2.5)

These two gauge invariant currents are classically conserved in the usual sense, i.e. $\partial_\mu J^\mu = 0$ and $\partial_\mu J_5^\mu = 0$, in the chiral limit $m_e \to 0$. The vector current (2.4) is invariant under the local transformations $\psi \to e^{ie\chi(x)} \psi$, $\bar{\psi} \to e^{-ie\chi(x)} \bar{\psi}$, $A_\mu \to A_\mu - \partial_\mu \chi(x)$, while the axial-vector current (2.5) under the global chiral transformation $\psi \to e^{i\alpha \gamma_5} \psi$, $\bar{\psi} \to \bar{\psi} e^{i\alpha \gamma_5}$.

An interesting point is that, since VSR effects can generate a massive mode for the fermionic field, how the chiral symmetry stands in this case, is it preserved or violated? For that matter, we shall verify the validity of the chiral current conservation law, by means of the Ward identity, in the VSR framework.
In summary, we will study the ABJ anomaly of these two currents in terms of the triangle amplitude \(^{1}\)

\[
\langle J_5^\lambda (q) J^\mu (k_1) J^\nu (k_2) \rangle,
\]

(2.6)

by computing the vector Ward identity \(k_1^\mu \langle J_5^\lambda (q) J^\mu (k_1) J^\nu (k_2) \rangle =?\), as well as the axial-vector Ward identity \(q_\lambda \langle J_5^\lambda (q) J^\mu (k_1) J^\nu (k_2) \rangle =?\). Besides engendering massive modes for the dynamical fields, VSR effects are also present in the minimal coupling between the gauge field and the fermionic fields in terms of the Lagrangian density (2.1). For this reason, the currents (2.4) and (2.5) are modified by the VSR nonlocal terms, and will be used to construct the triangle amplitude (2.6).

Hence, the necessary Feynman rules to be used in our analysis can be readily determined from (2.1), and the currents (2.4) and (2.5), where the fermionic propagator reads

\[
S(p) = \frac{i(p + m_e)}{p^2 - m_e^2},
\]

(2.7)

and the vertex Feynman rules are presented as below

- The vector vertex \(\langle \bar{\psi}(p) \psi(p') A_\mu (k) \rangle\)

\[
\Lambda^\mu (p, p') = -ie\gamma^\mu - ie\frac{m^2}{2} \frac{n^\mu n'}{(n.p)(n.p')} \equiv -ie\gamma^\alpha \mathcal{R}_\alpha^\mu (p, p');
\]

(2.8)

- The vector-axial vertex function

\[
\Phi_5^\mu (p, p') = -i\gamma^\mu \gamma_5 - i\frac{m^2}{2} \frac{n^\mu n_5}{(n.p)(n.p')} \equiv -i\gamma^\alpha \gamma_5 \mathcal{R}_\alpha^\mu (p, p');
\]

(2.9)

- The 4-point vertex function \(\langle \bar{\psi}(p) \psi(p') A_\mu (k) A_\nu (k') \rangle\)

\[
\Gamma^{\mu\nu} = -ie^2 \frac{m^2}{2} \frac{n^\mu n_\nu}{(n.p)(n.p')} \left[ \frac{1}{n.(p + k)} + \frac{1}{n.(p + k')} \right].
\]

(2.10)

We have also introduced an useful notation in Eqs. (2.8) and (2.9) to encode the VSR nonlocal factor

\[
\mathcal{R}_\alpha^\mu (p, p') = \delta_\alpha^\mu + \frac{m_\alpha n^\mu}{2(n.p)(n.p')}.
\]

(2.11)

In the next section, based on these Feynman rules, we shall proceed to the analysis and evaluation of a series of Ward identities. All of these are related to the triangle graph \(\langle J_5^\lambda (q) J^\mu (k_1) J^\nu (k_2) \rangle\). Although all the amplitudes that we will investigate involve a \(\gamma_5\) matrix, only the analysis of the axial anomaly makes use of an algebraic manipulation involving \(\gamma_5\) within the dimensional regularization, making then necessary to use the ’t Hooft-Veltman rule [30,31].

\(^1\)It is important to emphasize that the full structure of the anomaly is given by the triangle graph [29]. This model is used to discuss the decay of \(\pi_0 \to 2\gamma\) with proton-antiproton virtual pairs.
3 Ward identities for the vector-axial vertex

We proceed now to the discussion of the amplitude related to the vector-axial vertex, which yields the so-called ABJ anomaly. In this case, we shall consider two Ward identities, the vector and the axial one, related to the classical currents (2.4) and (2.5), respectively. At one-loop order, we have three diagrams contributing to the amplitude \( \langle J^\alpha_\gamma(q) J^\mu(k_1) J^\nu(k_2) \rangle \), where the crossing symmetry is considered. These contributions are depicted in Fig. 1, in which the outgoing momenta of the photons are labeled by \( k_1 \) and \( k_2 \). Hence, the Feynman expression of the relevant total amplitude is given by

\[
\mathcal{A}_{\mu\nu}^\lambda(q, k_1, k_2) = -i \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ S(p) \Phi^\lambda_5(p - q, p) S(p - q) \Lambda^\nu(p - k_1, p - q) \times S(p - k_1) \Lambda^\mu(p, p - k_1) \right] + \left( \begin{array}{c} \mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2 \end{array} \right)
\]

\[
- \frac{i m^2}{2} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ S(p) \Phi^\lambda_5(p - q, p) S(p - q) \Gamma^{\mu\nu}(p, p - q, -k_1, -k_2) \right], \quad (3.1)
\]

where the transferred momentum is \( q = k_1 + k_2 \). Moreover, in order to discuss the desired Ward identities, we can express the amplitude (3.1) in its explicit form

\[
\mathcal{A}_{\mu\nu}^\lambda(q, k_1, k_2) = -ie^2 \int \frac{d^2q}{(2\pi)^2} \text{Tr} \left[ \left( \not{q} + m_e \right) \gamma^\alpha \gamma_5 \left( \left( \not{p} - \not{q} \right) + m_e \right) \gamma^5 \left( \left( \not{p} - \not{k_1} \right) + m_e \right) \right] \left( p^2 - \mu^2 \right) \left( (p - q)^2 - \mu^2 \right) \left( (p - k_1)^2 - \mu^2 \right)
\]

\[
\times \mathcal{R}_\alpha^\lambda(p, p - q, p) \mathcal{R}_{\nu}^\mu(p - k_1, p - q) \mathcal{R}_{\nu}^\mu(p, p - k_1) + \left( \begin{array}{c} \mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2 \end{array} \right)
\]

\[
- \frac{i e^2 m^2}{2} \int d^2q \text{Tr} \left[ \left( \not{q} + m_e \right) \gamma^\alpha \gamma_5 \left( \left( \not{p} - \not{q} \right) + m_e \right) \not{q} \right] \left( p^2 - \mu^2 \right) \left( (p - q)^2 - \mu^2 \right) \left( (p - k_1)^2 - \mu^2 \right)
\]

\[
\times \frac{n^{\mu_1} n^{\mu_2} \mathcal{R}_\alpha^\lambda(p - q, p)}{(n. p)(n. (p - q))} \left[ \frac{1}{n. (p - k_1)} + \frac{1}{n. (p - k_2)} \right]. \quad (3.2)
\]

Some technical aspects that are worth recalling before the computation: the first step in solving the amplitude (3.2) is the evaluation of the trace part. For this, we should remember that the axial trace with 2 or an odd number of \( \gamma \) matrices vanishes, and also that \( \text{Tr} \left[ \gamma_5 \gamma^\rho \gamma^\alpha \gamma^\beta \gamma^\gamma \right] = 4i \epsilon^{\rho\alpha\beta\gamma} \). Another important identity that we shall use is

\[
\text{Tr} \left( \gamma^\mu_1 \gamma^\mu_2 \ldots \gamma^\mu_{n-1} \gamma^\mu_n \right) = (-1)^n \text{Tr} \left( \gamma^\mu_0 \gamma^\mu_{n-1} \ldots \gamma^\mu_2 \gamma^\mu_1 \right), \quad (3.3)
\]

that follows from the charge conjugation invariance, \( C^{-1} \gamma^\mu C = - (\gamma^\mu)^T \), and it is valid for any number of \( \gamma \) matrices.
3.1 Vector Ward Identity

Let us evaluate the vector Ward identity $k_{1\mu}A_{5}^{\mu\nu}$, which consists in the contraction of $k_{1\mu}$ in the expression (3.2). Under this consideration, we obtain

$$
k_{1\mu}A_{5}^{\mu\nu} = -ie^2 \int \frac{d^\omega p}{(2\pi)^2} \text{Tr} \left[ (\not{p} + m_e) \gamma^\sigma \gamma_5 \left( (\not{\rho} - \not{q}) + m_e \right) \gamma^\rho \left( (\not{\rho} - \not{k}_1) + m_e \right) \gamma^\theta \right] \times \mathcal{R}_\sigma (p-q,p) \mathcal{R}_\rho (p-k_1,p-q) \left( k_{1\mu} \mathcal{R}_\nu (p,p-k_1) + \left( \begin{array}{c} \mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2 \end{array} \right) \right)
$$

$$
- \frac{ie^2 m^2}{2} \int \frac{d^\omega p}{(2\pi)^2} \text{Tr} \left[ (\not{p} + m_e) \gamma^\sigma \gamma_5 \left( (\not{\rho} - \not{q}) + m_e \right) \gamma^\rho \right] \times \frac{n^\nu (n.k_1)}{(n.p) (n.(p-q))} \frac{1}{n.(p-k_1)} + \frac{1}{n.(p-k_2)} .
$$

The complicated trace parts present in Eq. (3.4) can be casted into simpler forms, by using the following identities

$$
k_1' \left( \frac{1}{\not{\rho} - m_e} \right) = 1 - (\not{\rho} - \not{k}_1) \left( \frac{1}{\not{\rho} - m_e} \right) - \frac{m^2}{2} \not{\gamma} \left( \frac{1}{\not{\rho} - m_e} \right) \left( \frac{1}{n.(p-k_1)} - \frac{1}{n.(p-k_2)} \right) + \frac{m_e}{\not{\rho} - m_e} .
$$

and also

$$
\frac{1}{(\not{\rho} - \not{q}) - m_e} \left( \frac{1}{(\not{\rho} - \not{k}_2) - m_e} \right) - \frac{m_e}{(\not{\rho} - \not{q}) - m_e}
$$

$$
+ \frac{m^2}{2} \frac{1}{(\not{\rho} - \not{q}) - m_e} \not{\gamma} \left( \frac{1}{n.(p-k_2)} - \frac{1}{n.(p-k_1)} \right)
$$

We observe that the last term in the above identities, $m^2$ dependent $(1/(n.p))$ terms, are additional terms due to VSR in comparison to the usual algebra. This follows because of $(\not{\rho} + \not{q}) \neq (\not{\rho} + \not{q})$, remember that $\tilde{p}_\mu = p_\mu - m^2 \frac{n_\mu}{2 (n.p)}$. 6
Finally, gathering all these results, and making some algebraic manipulations, we find the simplified expression

\[
k_{1\mu}A^\mu_5 = -ie^2m^2n^\nu \int \frac{d^d p}{(2\pi)^d} \frac{Tr \left[ \gamma_5 \gamma^\lambda (p' - k_2) \gamma^\nu (p + k_1) \right]}{\left( (p + k_1)^2 - \mu^2 \right) \left( (p - k_2)^2 - \mu^2 \right) (n.p) (n. (p - k_2))} \times \frac{1}{n. (p - q)} \left[ \frac{1}{n. (p - k_1)} + \frac{1}{n. (p - k_2)} \right].
\]  

(3.7)

We should remark that since we are interested in elucidating the VSR effects (which appears both as massive modes in the fermionic propagator and also as nonlocal couplings), we considered the limit \( m_e = 0 \) in the numerator of the expression (3.7) as a simplification.

The remaining terms can now be readily evaluated by computing the trace parts. To simplify the momentum integrals, it is convenient to use the following property

\[
\frac{1}{n. (p + k_i) n. (p + k_j)} = \frac{1}{n. (k_i - k_j)} \left( \frac{1}{n. (p + k_j)} - \frac{1}{n. (p + k_i)} \right).
\]

(3.8)

Hence, taking all of these steps into consideration, and solving the momentum integrals with the results provided in the Appendix A, the final expression for the vector Ward identity is given by

\[
k_{1\mu}A^\mu_5 = \frac{ie^2m^2n_\sigma n^\nu e^{\xi c n_\sigma} k_2 k_1 x}{4\pi^2 (n.q)} \int dx \frac{1}{n. u} \left[ 1 - \frac{u^2 - M^2}{u^2} \ln \left[ 1 - \frac{u^2}{M^2 + u^2} \right] \right],
\]

(3.9)

where we have defined the variables

\[
u = -qx - k_1 (1 - x), \quad M^2 = \mu^2 - (1 - x) k_1^2 - xq^2.
\]

(3.10)

We see from the expression (3.9) that the VSR induces an anomaly from the vector Ward identity. Moreover, the tensor structure is also interesting because it is a mixture of the ABJ terms with the VSR null vector \( n_\alpha \). By means of illustration of the vector anomaly (3.9), we can consider the low-energy limit, \(^2\) so that the vector anomaly reads

\[
k_{1\mu}A^\mu_5 \approx -\frac{ie^2m^2 e^{\xi c n_\sigma} n^\nu n_\sigma k_2 k_1 x}{4\pi^2 (n.q) (n.k_2)} \ln \left[ 1 + \frac{(n.k_2)}{(n.k_1)} \right].
\]

(3.11)

We see that the vector Ward identity (3.11) is not satisfied for the VSR axial vertex, in contrast to the usual QED result; and this anomaly is a purely VSR effect since it vanishes as \( m^2 \to 0 \). This is

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\(^2\)By low-energy regime, we means that the fermionic mass \( \mu^2 \) is much larger than any other energy scale of the problem, i.e. \( \mu^2 \gg (q^2, k_1^2, k_2^2) \). This consideration allows us to write a well behaved perturbative series in \( m^2/\mu^2 \).
already an interesting result, because it shows that the nonlocal effects, arising from VSR, lead to a departure from the Lorentz invariant cases. Hence, it can help us to understand the modifications of the conservation laws in this context. We proceed next to the evaluation of the axial Ward identity, and shall compute the VSR contributions to the ABJ anomaly.

3.2 Axial Ward identity

In the discussion of the axial Ward identity $q_\lambda A^{\mu\lambda}_5$, we will face some issues in manipulating the $\gamma_5$ matrix in the trace computations \cite{31}. These issues originate from the fact that we use the dimensional regularization method to evaluate the VSR invariant momentum integrals, where shifts of integration variables are allowed by definition. Hence, in order to handle the problem of the definition of $\gamma_5$ in higher dimensions, we shall follow the ’t Hooft-Veltman rule \cite{30,31}. This consists in the splitting of the $\omega$ dimensional spacetime into two parts: a 4-dimensional (physical) and a $(\omega - 4)$-dimensional subspace

$$\int d^\omega p \to \int d^4 p \int d^{\omega-4} L.$$ (3.12)

In this case, the internal momentum is expressed as below

$$\mathcal{P} = \mathcal{P} + \mathcal{L} = (\gamma^0 p_0 + ... + \gamma^3 p_3) + (\gamma^4 L_4 + ... + \gamma^{\omega-1} L_{\omega-1}),$$ (3.13)

where we have denoted the internal momentum $L$ for the remaining $\omega - 4$ components.

Within the ’t Hooft-Veltman rule, the $\gamma_5$ algebra is written as

$$\{\gamma_5, \gamma^\mu\} = 0, \quad \mu = 0, 1, 2, 3$$ \hspace{1cm} (3.14)

$$[\gamma_5, \gamma^\mu] = 0, \quad \mu = 4, ..., \omega - 1,$$ \hspace{1cm} (3.15)

and all other familiar rules are still valid, including the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \mu, \nu = 0, 1, ..., \omega - 1,$$ \hspace{1cm} (3.16)

with the metric tensor components $g_{\mu\nu} = \text{diag}(+1, -1, ..., -1)$.

We observe that all the external momenta and VSR vector, $(q, k_1, k_2, n)$, remain 4-dimensional. This implies that the nonlocal factors are not affected by this variable separation, i.e. $\tilde{\mathcal{P}} = (\mathcal{P} + \mathcal{L}) = \mathcal{P} + \mathcal{L}$ since $(n.\mathcal{P}) = (n.\mathcal{P})$. Moreover, we see that $\mathcal{L}^2 = -L^2$ and $\mathcal{P}^2 + \mathcal{P}\mathcal{L} = 0$.

Hence, the axial Ward identity comes from the contraction of (3.2) with $q_\lambda$ in the axial vertex,
so that
\[ q_\lambda A^\mu\nu_5 = -ie^2 \int \frac{d^4 P}{(2\pi)^4} \text{Tr} \left[ \left( \tilde{P} + m_e \right) \gamma^\sigma \gamma_5 \left( \left( \tilde{P} - q \right) + m_e \right) \gamma^\rho \left( \left( \tilde{P} - k_1 \right) + m_e \right) \gamma^\theta \right] \]
\[ \times \left[ q_\lambda R^\lambda_\sigma \left( P - q, P \right) \right] R^\nu_\sigma \left( P - k_1, P - q \right) R^\mu_\sigma \left( P, P - k_1 \right) + \left( \mu \leftrightarrow \nu \right) \]
\[ - \frac{1}{2} \int \frac{d^4 P}{(2\pi)^4} \text{Tr} \left[ \left( \tilde{P} + m_e \right) \gamma^\sigma \gamma_5 \left( \left( \tilde{P} - q \right) + m_e \right) \gamma^\rho \left( \left( \tilde{P} - k_1 \right) + m_e \right) \gamma^\theta \right] \]
\[ \times \frac{n^\mu n^\nu \left( q_\lambda R^\lambda_\sigma \left( P - q, P \right) \right)}{n. (P - k_1) n. (P - q)} \left[ \frac{1}{n. (P - k_1)} + \frac{1}{n. (P - k_2)} \right]. \] (3.17)

In order to simplify the trace parts of the expression (3.17), we shall consider the following identity
\[ \tilde{q}_5 = \gamma_5 \left( \left( \tilde{P} - q \right) - m_e \right) + \left( \tilde{P} - m_e \right) \gamma_5 - 2 \kappa \gamma_5 + 2m_e \gamma_5 + \frac{m^2}{2} \left( \frac{1}{n. (P - q)} - \frac{1}{n. P} \right) \gamma_5 \] (3.18)

that takes into account the 't Hooft-Veltman rule for the \( \gamma_5 \) algebra and spacetime splitting, and also some properties from VSR. After some straightforward manipulation, we find that
\[ \frac{1}{\tilde{P} - m_e} \frac{1}{\left( \tilde{P} - q \right) - m_e} = \frac{1}{\tilde{P} - m_e} \gamma_5 + \frac{1}{\left( \tilde{P} - q \right) - m_e} - 2 \frac{1}{\tilde{P} - m_e} \frac{1}{\gamma_5} \frac{1}{\left( \tilde{P} - q \right) - m_e} \]
\[ + \frac{m^2}{2} \frac{1}{\tilde{P} - m_e} \left( \gamma_5 \right) \frac{1}{\left( \tilde{P} - q \right) - m_e} \left( \frac{1}{n. (P - q)} - \frac{1}{n. P} \right) \]
\[ + 2m_e \frac{1}{\tilde{P} - m_e} \gamma_5 \frac{1}{\left( \tilde{P} - q \right) - m_e}. \] (3.19)

Hence, by considering theses results, we are able to rewrite the axial Ward identity (3.17) in a simplified form
\[ q_\lambda A^\mu\nu_5 = -ie^2 \int \frac{d^4 P}{(2\pi)^4} \text{Tr} \left[ \left( \tilde{P} + k_1 \right) \gamma_5 \gamma^\beta \tilde{P} \gamma^\alpha \right] R^\mu_\alpha (p + k_1, p) R^\nu_\beta (p, p - k_2) + \left( \mu \leftrightarrow \nu \right) \]
\[ - ie^2 \int \frac{d^4 P}{(2\pi)^4} \text{Tr} \left[ \gamma_5 (\tilde{P} - k_2) \gamma^\beta \tilde{P} \gamma^\alpha \right] R^\mu_\alpha (p + k_1, p) R^\nu_\beta (p, p - k_2) + \left( \mu \leftrightarrow \nu \right) \]
\[ + 2ie^2 \int \frac{d^4 P}{(2\pi)^4} \text{Tr} \left[ \left( \tilde{P} + k_1 \right) \gamma_5 (\tilde{P} - k_2) \gamma^\beta \tilde{P} \gamma^\alpha \right] \]
\[ \times R^\mu_\alpha (p + k_1, p) R^\nu_\beta (p, p - k_2) + \left( \mu \leftrightarrow \nu \right) \]. (3.20)
Here, we have already considered $m_e = 0$ in the numerator as a simplification. This consideration comes from the fact that we are not interested in the usual relation of the violation of chiral symmetry in the presence of $m_e \neq 0$, but actually on the VSR leading order effects in $m^2$ into the ABJ anomaly. Moreover, we notice that some integrals vanished, in our development from Eq. (3.17) to (3.20), since it is impossible to form a two-index pseudotensor which depends on only one vector.

The evaluation of the trace parts and momentum integration of (3.20) is a lengthy, tedious, but straightforward task. To this end, one should apply first the spacetime decomposition as presented in the Eq. (3.13), then use the $\gamma$ matrices identity

$$\gamma^\beta \gamma^\theta \gamma^\lambda = \eta^{\beta\theta} \gamma^\lambda - \eta^{\beta\lambda} \gamma^\theta + \eta^{\theta\lambda} \gamma^\beta - ie^{\beta\theta\lambda\rho} \gamma_\rho \gamma_5, \quad (3.21)$$

as well as $\text{Tr}(\gamma_5 \gamma^\xi \gamma^\alpha \gamma^\chi \gamma^\beta) = 4 i e^{\xi\alpha\beta}$ to compute the trace parts. Furthermore, the integrals in the $(p, L)$ momentum variables are evaluated with aid of the results mentioned in the Appendix A.

Finally, we gather all the leading $m^2$ contributions to the complete expression of the axial Ward identity as

$$q_\lambda A^{\mu\lambda}_5 = \frac{ie^2 m_2 e^{\xi\nu\lambda} n_\lambda k_2}{4\pi^2} \int dx \ln \left[ \frac{\Delta_2 + r_2^2 - 2 (n \cdot r_2) (\bar{n} \cdot r_2)}{\Delta_1 + r_1^2 - 2 (n \cdot r_1) (\bar{n} \cdot r_1)} \right] \frac{\Delta_1 + x^2 k_1^2}{\Delta_1 + x^2 k_1^2} \left[ \Delta_1 + x^2 k_1^2 - 2 x^2 (n \cdot k_1) (\bar{n} \cdot k_1) \right]$$

$$+ \frac{ie^2 m_2 e^{\xi\nu\lambda} n_\lambda k_1}{4\pi^2} \int dx \left(1 - \frac{1}{x}\right) \ln \left[ \frac{\Delta_1 + x^2 k_1^2 - 2 x^2 (n \cdot k_1) (\bar{n} \cdot k_1)}{\Delta_1 + x^2 k_1^2} \right] \frac{\Delta_2 + r_2^2 - 2 (n \cdot r_2) (\bar{n} \cdot r_2)}{\Delta_2 + r_2^2} \left[ \Delta_2 + x^2 k_2^2 - 2 x^2 (n \cdot k_2) (\bar{n} \cdot k_2) \right]$$

$$- \frac{ie^2 m_2 e^{\xi\nu\lambda} n_\lambda (k_1 - k_2)_{\alpha}}{4\pi^2} \frac{1}{(n \cdot q)} \int dx \frac{1}{x} \ln \left[ \frac{\Delta_3 + r_3^2 - 2 (n \cdot r_3) (\bar{n} \cdot r_3)}{\Delta_3 + r_3^2} \right] \frac{\Delta_3 + r_3^2 - 2 (n \cdot r_3) (\bar{n} \cdot r_3)}{\Delta_3 + r_3^2} \left[ \Delta_3 + x^2 k_3^2 - 2 x^2 (n \cdot k_3) (\bar{n} \cdot k_3) \right]$$

$$+ \frac{ie^2 \gamma^{\lambda\mu\nu}}{2\pi^2} \lambda_k k_{2\lambda} k_{1\sigma}, \quad (3.22)$$

with

$$r_1 = x k_1, \quad r_2 = -x k_2, \quad r_3 = -x q, \quad (3.23)$$

$$\Delta_{1,2} = \mu^2 - k_{1,2}^2, \quad \Delta_3 = \mu^2 - q^2. \quad (3.24)$$

It is interesting to observe that the expression (3.22) shows that the VSR contributes to the chiral anomaly in very distinct way. Moreover, the tensor structure is quite different from the ABJ term, presenting a novel contractions of the external 4-momentum $(k_1, k_2)$ with the VSR null vector $n_\alpha$.

In order to illustrate the leading VSR contributions to the ABJ anomaly in (3.22), we consider the low-energy limit of the axial Ward identity, which yields

$$q_\lambda A^{\mu\lambda}_5 \approx \frac{ie^2}{2\pi^2} \gamma^{\lambda\mu\nu} k_{2\lambda} k_{1\sigma} + \mathcal{O}(m^4), \quad (3.25)$$

which corresponds to the standard result in QED. It is a surprising result that the leading VSR contributions to the ABJ anomaly (3.20) vanish in the low-energy limit. Another interesting aspect of this vanishing contribution is that, contrary to the vector Ward identity, the VSR effects respect the classical conserved axial-vector current (2.5).
4 Final remarks

In this paper, we have analyzed the Adler-Bell-Jackiw anomaly for Dirac fermions in the context of very special relativity. The main goal of the proposed study was to examine the behavior of known conservation laws in the quantum realm of a Lorentz violating model, by analyzing the radiative corrections related to the VSR nonlocal gauge couplings.

We started by discussing the classical conserved currents, vector and vector-axial ones, that are modified by VSR contributions. Since the anomaly is fully determined by the triangle graphs, we have established the amplitude $\langle J^\lambda_5(q) J^{\mu}(k_1) J^{\nu}(k_2) \rangle$ in the VSR framework, and considered the respective Ward identities. In the case of the vector Ward identity, we observe that it yields an anomaly, that follows exclusively from VSR effects, unlike the usual QED. On the other hand, in the case of the axial Ward identity, it required the use of the’t Hooft-Veltman rules, since it involved the algebraic manipulation of the $\gamma_5$ in the context of dimensional regularization. Surprisingly, the VSR contributions vanish in the low-energy limit, resulting that the ABJ anomaly is not modified by VSR effects in the leading order of the parameter $m^2$.

Hence, although VSR changes both Ward identities, its effect in the leading order upon the vector and axial Ward identity is different: the two currents present an anomaly in the quantum level, the vector anomaly is an entirely VSR effect, while the axial anomaly is the usual ABJ expression. It is interesting to observe that the VSR nonlocal effects contribute in a very interesting way in the evaluation of the vector anomaly. These facts show that the infrared effects (originating from the nonlocal VSR factors) are very distinct in the quantum regime, changing significantly the behavior of the triangle amplitudes, rendering anomalous symmetries.

Since there is a close relation among the ABJ anomaly with the $\pi_0 \to \gamma \gamma$ decay \cite{6,7}, it is a natural extension to examine the VSR effects upon this type of anomalous processes and find how the VSR nonlocal effects contribute to its decay width. This is currently under development and will be reported elsewhere.

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A Useful integrals

In order to cope with the momentum integral in VSR, we use the Mandelstam-Leibbrandt prescription extended to VSR \cite{20}

$$\int \frac{1}{(q^2 + 2q.p - m^2)^a (n.q)^b} = (-1)^{a+b}i\pi \frac{(-2)^b}{\Gamma(a)\Gamma(b)} (\bar{n}.p)^b \int_0^1 dt \frac{1}{\Delta^{a+b-2}},$$  \hspace{1cm} (A.1)

where $\Delta = m^2 + p^2 - 2(n.p)(\bar{n}.p)t$, and $\bar{n}$ is a new null vector ($\bar{n}^2 = 0$) with the property $(n.\bar{n}) = 1$, whose explicit form must be provided in the VSR framework \cite{20}. Taking into account the
properties such as reality, right scaling \((n, \bar{n}) \rightarrow (\lambda n, \lambda^{-1} \bar{n})\) and being dimensionless [20], we find \(\bar{n}_\mu = \frac{p_\mu}{(n.p)} - \frac{p^2 n_\mu}{2(n.p)^2}\). The remaining useful integrals with \(q^\mu\) and \(q^\mu q^\nu\) in the numerator can be obtained by direct derivation of (A.1) in relation to \(p_\mu\) and \(p_\mu p_\nu\), respectively.

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