Acoustic scattering by periodic arrays of air-bubbles

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Abstract

This paper considers acoustic scattering by and propagation through line and plane arrays of air-bubbles in liquid media. The self-consistent method is used to derive the effective scattering function of a single bubble embedded in the arrays, incorporating all multiple scattering processes. For the line case, an exact result is derived. In the plane array situation, only an approximate analytic result is possible. Numerical computations have been carried out to show the multiple scattering effects on wave scattering. It is shown that depending on the distance between bubbles the resonance peak of a single bubble can either be broadened or narrowed due to multiple scattering and it shows an oscillatory behavior as the distance changes. Meanwhile, the peak scattering amplitude is also be either enhanced or reduced. The previously predicted strong enhancement, however, is not evident. For plane arrays, the usual resonant scattering of a single bubble in absence of other bubbles can be suppressed by multiple scattering when the distance between bubbles is sufficiently small. As the distance increases, the resonant scattering starts to appear, and the resonance peak position is alternately shifted towards higher and lower values. Moreover, it is predicted that wave propagation through a plane bubble array can be significantly inhibited in a range of frequencies slightly higher than the natural frequency of a single bubble, possibly a useful feature for noise screening. The ambiguities in the previous results are pointed out.
1 Introduction

When propagating through media containing many scatterers, waves will be scattered by each scatterer. The scattered wave will be again scattered by other scatterers. Such a process will be repeated to establish an infinite recursive pattern of rescattering between scatterers, forming a course of multiple scattering. Because of multiple scattering, the overall scattering effect in the system may not be represented simply by the sum of the effects of individual scatterers in isolation. It has now become well-known that multiple scattering of waves is responsible for a wide range of fascinating phenomena. This includes, on large scales, twinkling light in the evening sky, modulation of ambient sound at ocean surfaces [1], and acoustic scintillation from turbulent flows[2] and fish schools[3]. On smaller scales, phenomena such as white paint, random laser[4], electrical resistivity, photonic[5] and sonic bandgaps[6] in periodic structures also find their roots in wave multiple scattering. Even more interesting, perhaps, multiple scattering may lead to a phase transition in wave propagation, that is, due to multiple scattering propagating waves may be trapped in space and will remain confined in the neighborhood of initial site until dissipated. In the meantime, individual scatterers shows an amazing collective behavior; such a collective behavior effectively prevent waves from propagation and yields the phenomenon of Anderson wave localization[7, 8].

Tremendous effort has been devoted to the study of multiple scattering of waves and a large body of literature exists (Refer to, for example, the monograph by Ishimaru[9]). The work of Poldy[10], Lax[11], Waterman et al.[12], Twersky[13], and many others serves as a cornerstone to the subject and provides various schemes describing multiple scattering processes in a number of situations of interest. In a series of articles, Foldy and Twersky described the multiple scattering of waves in media containing arbitrary scatterers by a set of self-consistent equation. If not impossible, the exact solution to such a set of coupled equations is difficult to obtain. Certain approximations, such as the perturbation series in the diagrammatic method[15], generally have to be resorted to.

An exact description of multiple scattering is only possible in several simple systems. Multiple scattering of acoustic waves by a finite number of air-filled bubbles in liquids has been one of such rare systems and poses a useful model system to study wave multiple scattering[14]. A rigorous treatment of multiple scattering not only provides definite but also new insight into phenomena associated with multiple scattering. For example, the recent numerical investigation[7, 8, 16] has shed further light on the aforementioned phenomenon of wave localization which could not be possibly obtained within the realm of approximations. Despite the success, however, the research is purely numerical and has been
limited to the case of a finite number of air-bubbles. It would be desirable to explore more complicated and practical situations involving an infinite number of scatterers and pursue the conditions in which an analytic description of multiple scattering can be obtained, thus rendering a justification of numerical results extended for infinite scatterers. This article presents one of such attempts.

The problem considered here is multiple scattering of acoustic waves by regular arrays of air-bubbles in a liquid. Two situations will be deliberated: linear and planar arrays. It is shown that the two cases have the closed form solutions. These two simple situations are chosen so as to display the physical essence in a most explicit form. There are a number of earlier works on these circumstances. Weston[17] first considered the frequency response of air-bubbles forming linear and planar arrays. Weston derived approximate formulas for sound scattering by an air-bubble embedded in the arrays and predicted that the line array of air-bubbles behaves like a cylindrical bubble: the sharp sphere resonance, a well-known feature for a single spherical bubble[18], is suppressed and a broader resonance at a lower frequency appears. He further showed that a plane array of air-bubbles behaves like a plane screen of gas - there is no resonance at all. Later, bubbles in linear arrays were further numerically studied in [19] for the case that the incident wave is perpendicular to the linear axis. In contrast to Weston, his results indicate that depending on the spacing between bubbles, a line of bubbles need not necessarily lead to an increase in damping for the ensemble. However, the results in [19] are not only too limited but are distracted by errors. Tolstoy and Tolstoy[20, 21] also considered line and plane arrays of air-bubbles. They predicted that pronounced partial resonances can be observed in both systems. Their results, however, have been questioned[22, 23, 24]. In short, the previous results are in discrepancy. A definite and careful investigation is clearly needed.

This paper considers further the problem of line and plane arrays of resonant monopole scatterers like the air-bubbles. The self-consistent approach from Foldy[10] and Twersky[13] will be followed to derive a set of coupled equations for which exact analytic solutions are obtained. It will be shown that the total acoustic scattering by arrays of bubbles can be expressed in terms of the scattering from individual bubbles. The effects of multiple scattering is represented by an effective scattering function of a single bubble embedded in the array. Both line and plane line arrays will be considered. Wherever appropriate, comparison with the previous results will be made. We note that the present investigation is limited to the linear response of air-bubbles. When the stimulation field is too strong, therefore the interaction between bubbles can be very large, the linear response approximation may fail.
2 Theory

The problem considered here is illustrated by Figs. 1 and 2. In Fig. 1, a unit plane wave $e^{ik \cdot \vec{r}}$ is incident on a line array of identical air-bubbles. The incident wave makes an angle of $\theta$ with the line of the array. The distance between two neighboring bubbles is $d$. The bubble radius is taken as $a$. Fig. 2 shows that the identical bubbles form a square lattice in the $x - y$ plane with lattice constant $d$. The incident wave is in the direction denoted by $\theta$ and $\phi$ in the spherical coordinates.

In absence of other bubbles, the scattered wave from a single bubble can be written as

$$p_s^i = p_0(\vec{r}_i) f e^{ik|\vec{r} - \vec{r}_i|},$$  \hspace{1cm} (1)

in which $p_0(\vec{r}_i)$ is the incident wave at the bubble located at $\vec{r}_i$ and $f$ is the scattering function of the single bubble. It has been found numerically[25] that when $ka < 0.35$, to which the following discussion is restricted, the scattering function is isotropic and given by

$$f = \frac{a}{\omega_0^2 - 1 - i\delta},$$  \hspace{1cm} (2)

where $\omega_0$ is the natural frequency of the bubble and $\delta$ is the damping factor of the bubble including radiation, thermal exchange, and viscosity effects[Refer to Appendix 6 in Ref. [18]].

When many bubbles are present, the scattered wave from the $i$-th bubble is a linear response to the total incident wave and the scattered wave from other bubbles, and therefore can be written as

$$p_s^i(\vec{r}) = f \left( p_0(\vec{r}_i) + \sum_{j \neq i}^\infty p_s^j(\vec{r}_i) \right) e^{ik|\vec{r} - \vec{r}_i|}.$$  \hspace{1cm} (3)

We define an effective scattering function for each bubble as

$$p_s^i(\vec{r}) = p_0(\vec{r}_i) F_i e^{ik|\vec{r} - \vec{r}_i|}.$$  \hspace{1cm} (4)

Due to the symmetry, all bubbles have the same effective scattering function, i.e. $F_1 = F_2 = \cdots = F_i = \cdots = F$.

Substituting Eq. (4) into Eq. (3), we obtain

$$F = \frac{1}{f^{-1} - \sum_{j \neq i}^\infty e^{ik|\vec{r} - \vec{r}_i|}/|\vec{r} - \vec{r}_i|}. $$  \hspace{1cm} (5)

The total scattered wave will be

$$p_s(\vec{r}) = \sum_{i=1}^N p_s^i(\vec{r}) = F \sum_{i=1}^\infty p_0(\vec{r}_i) e^{ik|\vec{r} - \vec{r}_i|}/|\vec{r} - \vec{r}_i|. $$  \hspace{1cm} (6)
2.1 Infinite line arrays

Effective scattering function. In the line array case, we have from Eq. (5)

\[ F = \frac{1}{f^{-1} - 2k \sum_{n=1}^{\infty} \frac{e^{ikdn}}{nkd} \cos(kdn \cos \theta)}, \]  

(7)

where \( n \) takes positive integers. Equation (7) is equivalent to the result previously encountered by Weston[17]. Define

\[ I \equiv 2 \sum_{n=1}^{\infty} \frac{e^{ikdn}}{nkd} \cos(kdn \cos \theta). \]  

(8)

This can be evaluated as

\[
I = \frac{1}{kd} \left\{ - \ln(1 - \cos(kd(1 + \cos \theta)))(1 - \cos[kd(1 - \cos \theta)]) \\
+ i(2\pi - [kd(1 + \cos \theta)] - [kd(1 - \cos \theta)]) \right\},
\]

(9)

where \([x] \) means \( 2\pi \) modulo of \( x \); therefore \([kd(1 - \cos \theta)] + [kd(1 + \cos \theta)] \neq 2kd\). Writing \( x = 2n\pi + x' \) with \( n \) being an integer and \( x' \) limited to \((0, 2\pi)\), then \([x] = x'\). With (9), Eq. (7) represents the exact solution. The result in Eq. (9) differs from the previously published result [20] by a factor of \( \sqrt{2} \) in the logarithm and by the modulo values.

Total scattered field. The total scattered wave can be evaluated in the far field limit, \( r >> d \) with \( r \) being the perpendicular distance from the field point to the line. In this limit, the summation in Eq. (6) can be converted into an integral. The resulting formula is

\[
p_s(r) = i\pi \frac{F}{d} H_0^{(1)}(kr \sin \theta),
\]

(10)

where \( H_0^{(1)}(x) \) is the zero-th order type one Hankel function.

With Eq. (7), we see from Eq. (10) that the scattered wave is isotropic in the plane perpendicular to the line and it depends on the incident angle and on the perpendicular distance. Eq. (10) also indicates that the scattering by the array is characterized by the effective scattering function of each individual bubble. This may partially resolve the debate between Tolstoy et al. and Twersky[20, 22, 23, 24]. Tolstoy et al. used the effective scattering function to study the superresonance behavior of a bubble array. Twersky alternatively defined a scattering amplitude of the whole array and claimed that this is only observable. He further stated that an individual scattering amplitude is not observable and numerical computations for an individual scattering function do not represent physically observable data. Tolstoy et al. argued that the characteristics of a single bubble can be inferred at near field. From the present approach, we see that in the case of an infinite line array, the individual effective
scattering function does provide useful information and represent observable data at far field as well. In fact, Eq. (10) will be valid as long as \( r >> d \) and \( Nd >> r \) with \( N \) being the total number of the scatterers. Therefore even for a finite number of scatterers, the individual scattering function may still delineate the observables.

**Modified natural frequency and damping rate.** With Eq. (9), the scattering function \( F \) is solved as

\[
F = \frac{1}{\omega^2 - 1 - i\delta - kaI}.
\]

Thus the new resonance peak and the damping rate \((\delta_R)\) are determined from

\[
\frac{\omega_0^2}{\omega^2} - 1 - kaI_R = 0, \quad \delta_R = \delta + kaI_M,
\]

respectively. Here \( I_R \) and \( I_M \) represent the real and the imaginary parts of \( I \) separately:

\[
I_R = -\frac{1}{2kd} \ln(1 - \cos[kd(1 + \cos \theta)])(1 - \cos[kd(1 - \cos \theta)]), \quad \text{and}
\]

\[
I_M = \frac{1}{2kd}(2\pi - [kd(1 + \cos \theta)] - [kd(1 - \cos \theta)]).
\]

### 2.2 Plane arrays

**Effective scattering function.** In this case, the origin can be fixed at the position of one of the bubbles. The effective scattering function of a single bubble will be

\[
F = \frac{1}{f - 1 - \sum_j' \frac{e^{ik|\vec{r}_j|}}{|\vec{r}_j|} e^{i\vec{k} \cdot \vec{r}_j}},
\]

where the summation runs over all the bubbles except the one at the origin. Unfortunately, the summation in Eq. (15) cannot yet be made into a simple closed form and in general has to be evaluated numerically. For simplicity, we consider the case \( \phi = 0 \) [Fig. 2], i.e. the incidence is in the \( x - z \) plane. Define a quantity \( I \):

\[
I \equiv \sum_j' \frac{e^{ik|\vec{r}_j|}}{|\vec{r}_j|} e^{i\vec{k} \cdot \vec{r}_j}.
\]

The summation can be approximated as

\[
I \approx \frac{1}{kd^2} \int_{d}^{\infty} d^2 r \frac{e^{ik|x|}}{r} e^{-ik|\vec{r}|} = \frac{2\pi}{(kd)^2} \left( \frac{i}{\sin \theta} - \int_0^{kd} dx e^{ix} J_0(x \cos \theta) \right),
\]

where \( J_0 \) is the zero-th order Bessel function of the first kind. For normal incidence, \( \theta = \pi/2 \), we get approximately

\[
I = \frac{2\pi i}{(kd)^2} e^{ikd}.
\]
The effective scattering function is thus

\[ F = \frac{1}{f^{-1} - kI} = \frac{1}{f^{-1} - \frac{2\pi kF}{(kd)^2} e^{ikd}.} \]  

(19)

As will be shown later, the approximated result in Eq. (19) is reasonable.

**Natural frequency and damping rate.** Similar to the line case, the resonance frequency incorporating multiple scattering is determined by

\[ \frac{\omega_0^2}{\omega^2} - 1 - kaI_R = 0, \]  

(20)

which is reduced to

\[ \frac{\omega_0^2}{\omega^2} - 1 + \frac{2\pi ka}{(kd)^2} \sin(kd) = 0, \]  

(21)

for the normal incidence. The corresponding damping rate is

\[ \delta_R = \delta + \frac{2\pi ka}{(kd)^2} \cos(kd). \]  

(22)

**Transmission through bubble screens.** Consider the normal incidence. The transmitted wave through one plane array of bubbles

\[ p_F = e^{ik\mathbf{r}} + \sum_{i=1}^{N} F e^{ik|\mathbf{r} - \mathbf{r}_i|}. \]  

(23)

Assume that the incident wave is along the z-axis and we use the cylindrical coordinates; here the incident direction is the axis of symmetry. Considering the far field limit as in the line case, we obtain

\[ p_F = \left( 1 + \frac{2\pi ikF}{(kd)^2} \right) e^{ikz}. \]  

(24)

The backscattered wave is

\[ p_B = \frac{2\pi ikF}{(kd)^2} e^{-ikz}. \]  

(25)

In deriving Eq. (24), we used the following result

\[ \int_{0}^{\infty} dt e^{it_0 \sqrt{1 + t^2}} = \frac{1}{i t_0} e^{it_0}, \]  

(26)

which can be obtained by the method of change of variables.

**Validation of the approximation and energy conservation.** As we can see from the above, in the derivation of Eq. (17), we used the approximation which converts the summation into the integral. A way to verify the approximation is to apply the energy conservation law. Write

\[ T = \frac{2\pi ikF}{(kd)^2}. \]  

(27)
Obviously $|1 + T|$ and $|T|$ represent the transmission and reflection coefficients respectively. Note that the present reflection coefficient differs from that of Weston (Refer to Eq. (29) in Ref. [17]). In the limit of $kd << 1$, Weston obtained

$$|T|^2 = \frac{1}{1 + \left(\frac{\omega_0^2 kd^2}{2\pi a}\right)^2}, \tag{28}$$

while the present approach yields

$$|T|^2 = \frac{1}{1 + \left(\frac{\omega_0^2 - 1}{\omega_0^2 - 1}\right)\frac{kd^2}{2\pi a}}. \tag{29}$$

1. There are some further approximations in Weston’s derivation. His result is more valid for frequencies significantly below the nature frequency of the bubbles.

The energy conservation law requires,

$$|1 + T|^2 + |T|^2 + A = 1, \tag{30}$$

where $A$ denotes the absorption due to the thermal exchange and viscosity:

$$A = \frac{4\pi}{kd^2} \text{Im}[F] - \frac{4\pi}{d^2} |F|^2, \tag{31}$$

where ‘Im’ means taking the imaginary part. In deriving Eq. (31), we used the optical theorem[9]. It can be shown that the approximate result from Eq. (19) satisfies the energy conservation law to a few percentage, hereby providing a justification of the approximation.

3 Numerical results

The above theoretical results will be illustrated by numerical examples in this section. We consider that bubbles arrays are placed in water. The following parameters are used in the computation: the sound speeds of the air inside the bubble and of the surrounding water are 340 m/s and 1500 m/s, the mass densities for air and water are 1.29 and 1000 kg/m³ respectively. The thermal and viscosity coefficients are taken from Appendix 6 in [18].

3.1 Line arrays

In Fig. 3, the scattering strength $|F|$ of a single bubble from Eq. (7) is plotted versus $ka$ for various bubble separations, with comparison to the scattering strength of the single bubble in absence of other bubbles. The incidence is perpendicular to the line. The bubble radius is 1 mm. Figures (b), (d),
and (f) are respectively replots of (a), (c), and (e) in the normalized scales, i.e. normalized by the maximum of the scattering strength. From these figures, we see that (1) when the separation between bubbles is small, the multiple scattering heavily suppresses the scattering strength of the single bubble; (2) the resonance peak is red shifted (shifted towards lower values); (3) the resonance peak can either broadened or narrowed, depending on the separation; (4) the multiple scattering effects are negligible when the separation exceeds a certain value.

Fig. 4 plots the effective scattering strength as a function of \( ka \) for three incident angles. Here we see that as the incidence angle deviates from the normal direction, the resonance peak will be further reduced and the peak position is more shifted towards lower frequencies. The previously predicted quasi-resonance [21] is not evident. As expected, when the separation is sufficiently large, the results reduce to that of a single bubble without the presence of other bubbles.

Fig. 5 plots the relative natural frequency and damping shifts with respect to the case without multiple scattering as a function of the bubble separation. Two bubble sizes and two incident angles are assumed. The results are shown to be almost the same for the two bubble sizes. While the frequency is always shifted to a lower value, the damping shift shows an interesting regular oscillatory feature. As the incidence is deviated from the normal direction, the oscillation period is reduced. At certain ranges of bubble separations, the damping is reduced by the multiple scattering, indicating that the resonance peak is narrowed. The special cases considered in Ref. [19] fall in these ranges.

The relative peak scattering amplitude is plotted as a function of bubble separation in Fig. 6 for two bubble sizes and two incident angles. Again, the oscillatory features appear, in line with that shown in Fig. 5. For most separations, the peak amplitude is reduced by multiple scattering. In certain ranges of separation, however, the peak scattering strength is enhanced for the larger bubble case, but not as much as predicted by Tolstoy[21]. For the smaller bubbles, the peak amplitude is always reduced by multiple scattering. It is worth noting that the thermal and viscosity effects, which are more important for smaller bubbles, are not considered by Tolstoy. This may explain partially the discrepancy.

\section*{3.2 Plane array}

In this situation, only normal incidence is considered. The effective scattering strength \(|F|\) is plotted against \( ka \) for various lattice constants \((d/a)\) in Fig. 7. In the plots, the dashed lines refer to the scattering strength of a single free bubble. The multiple scattering effects are stronger than in the
line case. As we can see, the resonance scattering of the single bubble is fully suppressed when the bubbles are closed packed, in agreement with Weston’s results[17]. As the bubble separation increases, however, the resonance picture starts to appear and the multiple scattering effects decreases.

Consider the cases that the resonant scattering appears. In contrast to the line array case, the natural frequency is shifted to higher values when the separation is less than certain amount. In some ranges of separation, the resonance peak is shifted to a lower value slightly. This is shown by Fig. 8 for bubble radii 1 and 0.1 mm separately. In this figure, the effects of multiple scattering on the damping is also presented. For \( d/a \) smaller than about 100, the damping is significantly increased by multiple scattering, implying peak broadening. There is a range of bubble separations, however, the damping is in fact reduced by multiple scattering; this occurs for \( d/a \) between 100 and 400.

The relative scattering amplitude at resonance with respect to that of a single free bubble is plotted versus \( d/a \) for two bubble sizes in Fig. 9. Again, an oscillatory property surfaces. The reduction and enhancement in scattering strength alternate, as the bubble separation varies. The larger the bubble, the larger is the enhancement factor; it can be as large as 1.4 for \( a = 1 \) mm.

We also studied the transmission through the plane array. The transmission coefficient \( |1 + T| \) is shown in Fig. 10 as a function of \( ka \) for four bubble separations for the with and without thermal and viscosity dissipation. As shown in the plots, the propagation can be significantly blocked by the plane array for frequencies slightly above the resonance of a single free bubble; the vertical bar indicates the resonance position of the single free bubble. As the separation increases, the range of frequencies in which the transmission is inhibited decreases, until disappears. Such an inhibition property may be useful for utilizing bubble layers as a noise screen. The inhibition also play a role in defining acoustic wave localization in bubbly liquids[16]. When the thermal or viscosity damping is not included, the transmission is shown to be enhanced at certain bubble separations, as illustrated by Fig. 10(c). This is against the energy conservation, indicating the failure of the formula (27) in this case.

4 Summary

In summary, we considered acoustic scattering from regular arrays of air-bubbles for low frequencies, i.e. \( ka << 1 \). The self-consistent method is used to derive the effective scattering function of a single bubble embedded in the arrays, including all multiple scattering processes. The total scattered wave is expressed in term of this effective scattering function. An exact solution is presented for the case of line arrays. For the plane arrays, an approximate result is given. The approximation is justified in view of
energy conservation. The numerical results show that depending on the distance between bubbles the resonance peak of a single free bubble can either be broadened or narrowed due to multiple scattering and shows an oscillatory feature as the distance is increased. In the same spirit, the peak scattering amplitude can also be enhanced or reduced. The enhancement is less than the previous prediction. Furthermore, wave propagation through a plane bubble array can be significantly inhibited in a range of frequencies slightly higher than the natural frequency of a single bubble. The results from this paper can also be extended to scattering by multiple plane arrays of bubbles.

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References

[1] Farmer, D. M. and Vagle, S.
    Waveguide propagation of ambient sound in the ocean-surface bubble layer
    J. Acoust. Soc. Am.
    86
    1989
    1897-1908

[2] Farmer, D. M., Clifford S. F. and Verall, J. A.
    Scintillation structure of a turbulent tidal flow
    J. Geophys. Res.
    92, (C5)
    1985
    5368-5382

[3] Ye, Z. and Curran, T. and Lemon, D.
    Fish detection by the acoustic scintillation technique
    ICES J. Mar. Sci.
    53
[4] Lawandy, N. M. and Balachandran, R. M. and Gomes, A. S. L. and Sauvain, E.
Laser action in strongly scattering media
Nature 368
1994
436-438

[5] Robertson, W. M., Arjavalingam, G., Meade, R. D., Brommer, K. D., Rappe, A. M., Joonnopoulos, J. D.
Measurement of the photon dispersion relation in two-dimensional ordered dielectric arrays
J. Opt. Soc. Am.
B 10
1993
322-327

[6] Ye Z. and Hoskinson, E.
Bandgaps and localization in acoustic propagation in water with air-cylinders
Appl. Phys. Lett.
77
2000
4428-4430

[7] Ye, Z. and Hsu, H. and Hoskinson, E. and Alvarez, A.
On localization of acoustic waves
Chin. J. Phys.
37
1999
345-353

[8] Hoskinson, E. and Ye, Z.
Phase transition in acoustic propagation in 2D random liquid media
Phys. Rev. Lett.
83
1999
2734-2737

[9] Ishimaru, A.
Wave Propagation and Scattering in Random Media
Academic Press
New York
1978
Vols. 1 and 2.

[10] Foldy, L. L.
The multiple scattering of waves
Phys. Rev.
67
1945
107-119

[11] Lax, M.
Multiple scattering of waves
Rev. Mod. Phys.
23
1951
287-310

[12] Waterman P. C. and Truell, R.
Multiple scattering of waves
J. Math. Phys. (N.Y.)
2
1961
512-537

[13] Twersky, V.
On scattering of waves by random distributors
J. Math. Phys. (N.Y.)
3
1962
700-715

[14] Ilinskii, Yu. a. and Zabolotskaya, E. A.
Cooperative radiation and scattering of acoustic waves by gas bubbles in liquids
J. Acoust. Soc. Am.
92
1992
2837-2841

[15] Ye, Z. and Ding, L.
Acoustic dispersion and attenuation relations in bubbly mixture
J. Acoust. Soc. Am.
98
1995
1629-1636

[16] Ye, Z. and Alvarez, A.
Acoustic localization in bubbly liquids
Phys. Rev. Lett.
80
1998
3503-3506

[17] Weston, D. E.
Acoustic interaction effects in arrays of small spheres
J. Acoust. Soc. Am.
39
1966
316-322

[18] Clay, C. S. and Medwin, H.
Acoustical Oceanography
John-Wiley & Sons
[19] Feuillade, C.
Scattering from collective modes of air bubbles in water and the physical mechanism of superresonances
J. Acoust. Soc. Am.
98
1995
1178-1190

[20] Tolstoy, I. Superresonant systems of scatterers I
J. Acoust. Soc. Am.
80
1986
282-194
Tolstoy, I. and Tolstoy, A.
Superresonant systems of scatterers II
J. Acoust. Soc. Am.
83
1988
2086-2096

[21] Tolstoy, I. and Tolstoy, A.
Line and plane arrays of resonant monopole scatterers
J. Acoust. Soc. Am
87
1990
1038-1043

[22] Twersky, V.
Multiple scattering by finite regular arrays of resonators
J. Acoust. Soc. Am.
87
[23] Tolstoy, I.
Comment on ‘Multiple scattering by finite regular arrays of resonators’ [J. Acoust. Soc. Am. 87, 25-41 (1990)]
J. Acoust. Soc. Am.
88
1990
1178-1179

[24] Twersky, V.
Comments on resonant system of scatterers
J. Acoust. Soc. Am. 88
1990
1179-1180

[25] Ye, Z.
Low-frequency acoustic scattering by gas-filled prolate spheroids in liquids
J. Acoust. Soc. Am.
101
1997
1945-1952
Figure Captions

**Fig. 1** The geometry for a line array of bubbles. The bubble radius is \( a \) and the separation between bubbles is \( d \). The incident wave makes an angle \( \theta \) with the line array.

**Fig. 2** The geometry for a plane array. The array is in the \( x - y \) plane and forms a square lattice with constant \( d \). The incident wave makes angles \( \theta \) and \( \phi \) in the spherical coordinates. In the numerical computation, we consider the case that the wave propagates along the \( z \)-axis.

**Fig. 3** Line array: Scattering and effective scattering strength with respect to a single free bubble versus frequency for various bubble separations. The bubble radius is 1 mm. Here the notation \( f \) can either refer to the scattering function of a single bubble or the effective scattering function \( F \) of a single bubble embedded in a line array of bubbles.

**Fig. 4** Line array: Scattering strength as a function of \( ka \) for different bubble separations and incidence angles. The bubble radius is 1 mm.

**Fig. 5** Line array: Relative frequency and damping shifts as a function of \( d/a \) for two bubble sizes and two incident angles \( (\theta = \pi/2 \text{ and } \pi/4) \). Here \( \delta f \) is shift in peak position due to multiple scattering between bubbles, \( f_0 \) is the resonance frequency of a single bubble in isolation \( (f_0 = \omega_0/2\pi) \).

**Fig. 6** Line array: Relative peak scattering amplitude as a function of \( d/a \) for two bubble sizes and two incident angles.

**Fig. 7** Plane array: Scattering strength versus frequency for various bubble separations. The bubble radius is 1 mm. Here the notation \( f \) can either refer to the scattering function of a single bubble or the effective scattering function \( F \) of a single bubble embedded in a line array of bubbles.

**Fig. 8** Plane array: Relative frequency and damping shifts as a function of \( d/a \) for two bubble sizes.

**Fig. 9** Plane array: Relative peak scattering amplitude as a function of \( d/a \) for two bubble sizes.

**Fig. 10** Plane array: Relative transmission versus \( ka \) for four bubble separations. Without the plane array, the transmission is normalized as one.
Incident wave
(a) Scattering strength: $a = 1$ mm, $d/a = 2$

(b) Relative strength: $a = 1$ mm, $d/a = 2$

(c) Scattering strength: $a = 1$ mm, $d/a = 100$

(d) Relative strength: $a = 1$ mm, $d/a = 100$
(e) Scattering strength: $a = 1$ mm, $d/a = 350$

(f) Relative strength: $a = 1$ mm, $d/a = 350$
(a) Scattering strength: \( a = 1\text{mm}, \ d/a = 2 \)

(b) Scattering strength: \( a = 1\text{mm}, \ d/a = 10 \)

(c) Scattering strength: \( a = 1\text{mm}, \ d/a = 50 \)

(d) Scattering strength: \( a = 1\text{mm}, \ d/a = 100 \)
(a) Frequency shift: $a = 1 \text{ mm}$

(b) Damping shift: $a = 1 \text{ mm}$

(c) Frequency shift: $a = 0.1 \text{ mm}$

(d) Damping shift: $a = 0.1 \text{ mm}$
(a) Relative peak amplitude: $a = 1$ mm

(b) Relative peak amplitude: $a = 0.1$ mm
(a) Scattering strength: $a = 1$ mm, $d/a = 2$

(b) Scattering strength: $a = 1$ mm, $d/a = 10$

(c) Scattering strength: $a = 1$ mm, $d/a = 20$

(d) Scattering strength: $a = 1$ mm, $d/a = 50$
(a) Frequency shift: $a = 1\text{ mm}$

(b) Damping shift: $a = 1\text{ mm}$

(c) Frequency shift: $a = 0.1\text{ mm}$

(d) Damping shift: $a = 0.1\text{ mm}$
(a) Relative peak amplitude: $a = 1 \text{ mm}$

(b) Relative peak amplitude: $a = 0.1 \text{ mm}$
(a) Transmission: $a = 0.1$ mm, $d/a = 20$

(b) Transmission: $a = 0.1$ mm, $d/a = 50$

(c) Transmission: $a = 0.1$ mm, $d/a = 200$

(d) Transmission: $a = 0.1$ mm, $d/a = 10^3$