Unique factor ordering in the continuum limit of LQC

William Nelson and Mairi Sakellariadou

King’s College London, Department of Physics, Strand WC2R 2LS, London, U.K.

We show that the factor ordering ambiguities associated with the loop quantisation of the gravitational part of the cosmological Hamiltonian constraint, disappear at the level of Wheeler-DeWitt equation only for a particular choice of lattice refinement model, which coincides with constraints imposed from phenomenological and consistency arguments.

PACS numbers: 04.60.Kz, 04.60.Pp, 98.80.Qc

I. INTRODUCTION

Loop Quantum Gravity (LQG) [1], a non-perturbative and background independent canonical quantisation of general relativity in four space-time dimensions, is one of the main approaches to quantising gravity. Even though the full theory of LQG is not yet complete, its successes lead us to apply LQG techniques in a simple setting where symmetry principles can be used. The application of LQG to the cosmological sector, known as Loop Quantum Cosmology (LQC) [2, 3], has recently made significant progress. The difference between LQC and other approaches of quantum cosmology, is that the input is motivated by a full quantum gravity theory. The simplicity of the setting (typically homogeneity and then sometimes also isotropy, although recent progress has been made towards inhomogeneous cosmologies [4]), combined with the discreteness of spatial geometry provided by LQG, render feasible the overall study of LQC dynamics.

Loop quantum cosmology is formulated in terms of SU(2) holonomies of the connection and triads. In LQC, the quantum evolution is described by a second order difference equation, instead of the second order differential equation of the Wheeler-DeWitt (WDW) approach to quantum cosmology. As the scale factor increases, the universe eventually enters the semi-classical regime, and the WDW differential equation describes, to a very good approximation, the subsequent evolution.

In the ‘old’ quantisation, the quantised holonomies were taken to be shift operators with a fixed magnitude, but later it was found [5, 6] that this leads to problematic instabilities in the continuum semi-classical limit. Indeed, as the universe expands, the Hamiltonian constraint operator creates new vertices of a lattice state, leading in LQC to a refinement of the discrete lattice. The effect of the lattice refinement has been modelled and the elimination of the instabilities in the continuum era has been explicitly shown [7]. Lattice refinement leads to new dynamical difference equations which, in general, do not have a uniform step-size. Thus, their study gets quite involved, particularly in the anisotropic cases. Recently, numerical techniques have been developed [8, 9] to address this issue.

The correct refinement model should be given by the full LQG theory. One would have, in principle, to use the full Hamiltonian constraint and find the way that its action balances the creation of new vertices as the volume increases. Instead, phenomenological arguments have been used, where the choice of the lattice refinement is constrained by the form of the
matter Hamiltonian. Here, we use another argument to specify, within a particular class (power law), the choice of the lattice refinement.

There are many equivalent ways of writing the Hamiltonian constraint in terms of the triad and the holonomies of the connection, since at the classical level holonomies commute. However, each of these factor ordering choices leads to a different factor ordering of the WDW equation in the continuum limit. In what follows we explicitly demonstrate that the ambiguities at the classical limit of LQC, which is precisely the WDW equation, disappear only for a particular choice of a lattice refinement model.

II. ELEMENTS OF LQC

To quantise the gravitational Hamiltonian for isotropic flat cosmologies, we restrict ourselves to an elementary (fiducial) cell $V$, with finite fiducial volume; only in this volume spatial integrations will be performed. This is the usual approach to regularise divergences appearing in a quantisation scheme based on a Hamiltonian framework within flat homogeneous models.

Introducing a flat fiducial metric $^{0}q_{ab}$, in which the volume of $V$ is $V_0$, the phase space variables $p, c$ of loop quantum cosmology read

$$|p| = V_0^{3/2}a^2 \frac{\omega}{4}; \quad c = V_0^{1/3}2\gamma \dot{a},$$

(with the lapse function set to 1) where $a$ is the cosmological scale factor and $\gamma$ is the Barbero-Immirzi parameter, labelling inequivalent quantum theories. The triad component $p$, determining the physical volume of the fiducial cell, is connected to the connection component $c$, determining the physical edge length of the fiducial cell, through the Poisson bracket

$$\{c, p\} = \frac{\kappa \gamma}{3},$$

(with $\kappa = 8\pi G$) which (for this choice of variables) is independent of the volume factor $V_0$.

The Hamiltonian formulation in the full LQG theory is based upon the Ashtekar variables, namely the connection $A_i^a$ and (density weighted) triad $E_i^a$, arising from a canonical transformation of the ADM variables. Note that $i$ refers to the Lie algebra index and $a$ is a spatial index, with $a, i = 1, 2, 3$. They are given by

$$A_i^a = c V_0^{-1/3} \omega_i^a; \quad E_i^a = p V_0^{-2/3} \sqrt{^0q} X_i^a,$$

where $^0q$ is the determinant of the fiducial background metric $^0g_{ab} = \omega_a^i \omega_b^i$, with $\omega_a^i$ a basis of left-invariant one-forms, and $X_i^a$ are the Bianchi I basis vectors $X_i^a = \delta_i^a$.

After quantisation, states in the kinematical Hilbert space can be expressed as (linear combinations of) eigenstates of $\hat{p}$, namely

$$\hat{p}|\mu\rangle = \frac{\kappa \gamma \hbar}{6} |\mu||\mu\rangle,$$

which are diagonal, i.e. $\langle \mu_1 | \mu_2 \rangle = \delta_{\mu_1 \mu_2}$.

Just as in full LQG, there is no operator corresponding to the connection, however the
action of its holonomy is well defined,\\n\\n$$\hat{h}_i|\mu\rangle = \left(\hat{C}_s \mathbb{1} + 2\pi \hat{S}_n\right)|\mu\rangle,$$  \hspace{1cm} (2.5)\\n\\nwhere $\hat{S}_n$ and $\hat{C}_s$ are given by
\\n$$\hat{S}_n|\mu\rangle = \frac{1}{2i} \left( e^{\frac{i\tilde{\mu}}{2}} - e^{-\frac{i\tilde{\mu}}{2}} \right) |\mu\rangle = -\frac{i}{2} \left( |\mu + \tilde{\mu} - |\mu - \tilde{\mu}| \right),$$  \\
$$\hat{C}_s|\mu\rangle = \frac{1}{2} \left( e^{-\frac{i\tilde{\mu}}{2}} + e^{\frac{i\tilde{\mu}}{2}} \right) |\mu\rangle = \frac{1}{2} \left( |\mu + \tilde{\mu} + |\mu - \tilde{\mu}| \right).$$  \hspace{1cm} (2.6)\\n
When $\tilde{\mu}$ is a constant (typically called $\mu_0$ is the ‘old’ quantisation), it is clear that the operator $\exp\left(i\hat{\mu}/2\right)$ acts as a simple shift operator, namely
\\n$$\exp\left(i\hat{\mu}/2\right)|\mu\rangle = \exp\left(\frac{-d}{d\mu}\right)|\mu\rangle = |\mu + \tilde{\mu}\rangle.$$  \hspace{1cm} (2.7)\\n
In the case of lattice refinement, $\tilde{\mu} = \tilde{\mu}(\mu)$ is not a constant and the shift interpretation is no longer valid. However, one can change variables from $\mu$ to $\nu$:\\n\\n$$\nu = k \int \frac{d\mu}{\tilde{\mu}(\mu)},$$  \hspace{1cm} (2.8)\\n
for which
\\n$$e^{\frac{i\tilde{\mu}}{d\mu}} = e^{\frac{d}{d\nu}},$$  \hspace{1cm} (2.9)\\n
with $k$ a constant. In these new variables the holonomies do act as simple shift operators, with parameter length $k$, for states labelled by $\nu$, defined as eigenvalues of $f(\tilde{\mu})$ (with $f$ the implicit function giving $\mu(\nu)$; it is obtained by solving Eq. (2.8)). Thus,
\\n$$e^{\frac{i\tilde{\mu}}{d\nu}}|\nu\rangle = e^{\frac{d}{d\nu}}|\nu\rangle = |\nu + k\rangle.$$  \hspace{1cm} (2.10)\\n
One should keep in mind, that the relationship between $\nu$ and geometric quantities, such as volume, is more complicated than their relationship with $\mu$. In what follows, we consider $\tilde{\mu}$ to be of the form $\tilde{\mu} = \mu_0\mu^{-A}$, where $\mu_0$ is some constant \[10\]. In this case, one can explicitly solve Eq. (2.8) to obtain
\\n$$\nu = \frac{k\mu_0^{1-A}}{\mu_0(1 - A)}.$$  \hspace{1cm} (2.11)\\n
This one-parameter family of lattice refinement models includes the ‘old’ ($A = 0$) and ‘new’ ($A = -1/2$) quantisations. Motivated by the full LQG theory, $A$ is expected to lie in the range between $-1/2 < A < 0$ \[6\]. There are several phenomenological and consistency arguments supporting $A = -1/2 \[10, 12\]$, nevertheless here we keep $A$ as an undetermined parameter and show that the choice $A = -1/2$ leads to important consequences for the factor ordering of the continuum limit of the theory.
III. FACTOR ORDERING

The classical gravitational part of the Hamiltonian constraint (with the lapse function set to unity) is given by

$$\mathcal{C}_{\text{grav}} = -\frac{1}{\gamma^2} \int_V \epsilon_{ijk} \frac{E^{ai}E^{bj}F_{ab}^{ik}}{\sqrt{|\det E|}},$$

(3.1)

where $$F_{ab}^{ik} = \partial_a A_b^i - \partial_b A_a^i + \epsilon_{ijk} A_a^j A_b^k$$ is the curvature of the connection. Writing this in terms of our quantisable variables ($p$ and the holonomies of $c$), it reads \[11\]

$$\hat{\mathcal{C}}_{\text{grav}} = \frac{2i}{\kappa^2 h^2 \gamma^3 k^3} \text{tr} \sum_{ijk} \epsilon_{ijk} \left( \hat{h}_i \hat{h}_j \hat{h}_k^{-1} \hat{h}_j^{-1} \hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right] \right).$$

(3.2)

There are many possible choices of factor ordering that could have been made at this point, since classically the action of the holonomies commute. Each of these possible choices will lead, in principle, to a different factor ordering of the resulting continuum WDW equation.

We consider only factor orderings of the form of cyclic permutations of the holonomy and volume operators within the trace. They have the advantage of being trivially equivalent with respect to the spin indices, while they remain within the irreducible representations of the holonomies, hence avoiding the spurious, ill-behaved, solutions, present for higher representations \[12\]. Whilst these holonomies do not represent a complete set of factor ordering choices, nevertheless they include the choices commonly made in the literature \[11, 12\]. Finally, using these factor orderings we can demonstrate how different lattice refinement models alter the factor ordering of the resulting continuum WDW equation.

Using Eqs. (2.5), (2.6), the action of the different factor ordering choices give us explicitly:

$$\epsilon_{ijk}\text{tr} \left( \hat{h}_i \hat{h}_j \hat{h}_k^{-1} \hat{h}_k^{-1} \hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right] \right) = -24\hat{\text{Sn}}^2 \hat{\text{Cs}}^2 \left( \hat{\text{Cs}} \hat{\text{V}} \hat{\text{Sn}} - \hat{\text{Sn}} \hat{\text{V}} \hat{\text{Cs}} \right),$$

(3.3)

$$\epsilon_{ijk}\text{tr} \left( \hat{h}_j \hat{h}_i \hat{h}_k^{-1} \hat{h}_j^{-1} \hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right] \right) = -24\hat{\text{Sn}}^2 \hat{\text{Cs}} \left( \hat{\text{Cs}} \hat{\text{V}} \hat{\text{Sn}} - \hat{\text{Sn}} \hat{\text{V}} \hat{\text{Cs}} \right) \hat{\text{Cs}},$$

(3.4)

$$\epsilon_{ijk}\text{tr} \left( \hat{h}_i^{-1} \hat{h}_j^{-1} \hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right] \hat{h}_j \hat{h}_i \right) = -12\hat{\text{Sn}}^2 \left( \hat{\text{Cs}} \hat{\text{V}} \hat{\text{Sn}} - \hat{\text{Sn}} \hat{\text{V}} \hat{\text{Cs}} \right) \hat{\text{Cs}}^2 -12\hat{\text{Cs}}^2 \left( \hat{\text{Cs}} \hat{\text{V}} \hat{\text{Sn}} - \hat{\text{Sn}} \hat{\text{V}} \hat{\text{Cs}} \right) \hat{\text{Sn}}^2,$$

(3.5)

$$\epsilon_{ijk}\text{tr} \left( \hat{h}_j \hat{h}_i \left[ \hat{h}_k^{-1}, \hat{V} \right] \hat{h}_i \hat{h}_j \hat{h}_i^{-1} \right) = -24\hat{\text{Cs}} \left( \hat{\text{Cs}} \hat{\text{V}} \hat{\text{Sn}} - \hat{\text{Sn}} \hat{\text{V}} \hat{\text{Cs}} \right) \hat{\text{Sn}}^2 \hat{\text{Cs}},$$

(3.6)

$$\epsilon_{ijk}\text{tr} \left( \hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right] \hat{h}_i \hat{h}_j \hat{h}_i^{-1} \hat{h}_j^{-1} \right) = -24 \left( \hat{\text{Cs}} \hat{\text{V}} \hat{\text{Sn}} - \hat{\text{Sn}} \hat{\text{V}} \hat{\text{Cs}} \right) \hat{\text{Sn}}^2 \hat{\text{Cs}}^2,$$

(3.7)

$$\epsilon_{ijk}\text{tr} \left( \hat{h}_i \hat{h}_j \hat{h}_i^{-1} \hat{h}_j^{-1} \hat{h}_k \right) = -24 \left( \hat{\text{Cs}} \hat{\text{V}} \hat{\text{Sn}} - \hat{\text{Sn}} \hat{\text{V}} \hat{\text{Cs}} \right) \hat{\text{Sn}}^2 \hat{\text{Cs}}^2,$$

(3.8)
where we have made extensive use of the trace identities:

\[
\begin{align*}
\text{tr} (\tau_i) &= -\frac{i}{2} \text{tr} (\sigma_i) = 0 \\
\text{tr} (\tau_i \tau_j) &= -\frac{1}{4} \text{tr} (\sigma_i \sigma_j) = -\frac{1}{2} \delta_{ij} \\
\text{tr} (\tau_i \tau_j \tau_k) &= \frac{i}{8} \text{tr} (\sigma_i \sigma_j \sigma_k) = -\frac{1}{4} \epsilon_{ijk} \\
\text{tr} (\tau_i \tau_j \tau_k \tau_l) &= \frac{1}{16} \text{tr} (\sigma_i \sigma_j \sigma_k \sigma_l) = -\frac{1}{8} \delta_{ik} + \frac{1}{4} \epsilon_{ijl} \epsilon_{kjl}.
\end{align*}
\]

(3.9)

Using Eqs. (2.6) and defining \( \hat{V} |\nu\rangle = V_\nu |\nu\rangle \), the action of each of these factor orderings on the basis \( |\nu\rangle \) can be calculated. For clarity, we will derive the continuum limit only for the factor ordering choice given in Eq. (3.3); the other ones follow along similar lines and for completeness have been included in an appendix (Appendix A).

Thus,

\[
\hat{S}^2 \hat{C}^2 \left( \hat{C} \hat{V} \hat{S}^2 - \hat{S}^2 \hat{C} \hat{V} \right) |\nu\rangle = \frac{-i}{32} \left( V_{\nu+k} - V_{\nu-k} \right) \left( |\nu + 4k\rangle - 2|\nu\rangle + |\nu - 4k\rangle \right). \tag*{(3.10)}
\]

Extending the above, one can obtain the action of the chosen factor ordering on a general state in the Hilbert space given by \( |\Psi\rangle = \sum_\nu \psi_\nu |\nu\rangle \). The explicit difference equations for the coefficients \( \psi_\nu \) read

\[
\epsilon_{ijk} \text{tr} \left( \hat{h}_i \hat{h}_j \hat{h}_k^{-1} \hat{h}_k^{-1} \hat{h}_{-k} \left[ \hat{h}_k^{-1}, \hat{V} \right] \right) |\Psi\rangle = \frac{-3i}{4} \sum_\nu \left[ \left( V_{\nu-3k} - V_{\nu-5k} \right) \psi_{\nu-4k} - 2 \left( V_{\nu+k} - V_{\nu-k} \right) \psi_{\nu} \right.
\]

\[
+ \left( V_{\nu+5k} - V_{\nu+3k} \right) \psi_{\nu+4k} \right] |\nu\rangle. \tag*{(3.11)}
\]

We can now take the continuum limit of these expressions by expanding \( \psi_\nu \approx \psi (\nu) \) as a Taylor expansion in small \( k/\nu \), i.e. in the limit that the discreteness scale \( (k) \) is much smaller than the scale of the universe (which is given by \( \nu \)). By noting that the volume is given by

\[
V_\nu |\nu\rangle \sim [\mu (\nu)]^{3/2} |\nu\rangle, \tag*{(3.12)}
\]

where \( \mu (\nu) \) is obtained by Eq. (2.11) and we are neglecting a constant factor \( (\kappa \gamma \hbar/6)^{3/2} \), we find

\[
V_{\nu \pm nk} \sim \left[ (\nu \pm nk) \alpha \right]^{3/[2(1-A)]} \tag*{(3.13)}
\]

where \( \alpha = \mu_0 (1 - A) / k \).

In general, the above needs also to be expanded in the \( k/\nu \to 0 \) limit and due to the \( k^{-3} \) factor in Eq. (3.2) it is necessary to go to third order in both this and the Taylor expansion. To expand Eq. (3.11) we need the difference between the volume eigenvalues evaluated on
different lattice points, given by

\[ V_{\nu+nk} - V_{\nu+mk} \sim \frac{3k}{2(1-A)} \alpha^{3/[2(1-A)]} \nu^{(1+2A)/[2(1-A)]} \left[ (n - m) + \frac{1 + 2A}{4(1-A)} k (n^2 - m^2) \right] \]

\[+ \frac{(1+2A)(4A-1)k^2}{24(1-A)^2} \nu^2 (n^3 - m^3) + O(\frac{k^3}{\nu^3}) \]. \quad (3.14)

Performing a Taylor expansion of Eq. (3.11) we get the large scale continuum limit of the Hamiltonian constraint:

\[ \lim_{k/\nu \to 0} \epsilon_{ijk} \text{tr} \left( \hat{h}_i \hat{h}_j \hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right] \right) |\Psi\rangle \sim -36i \frac{\alpha^{3/[2(1-A)]} k^3}{1-A} \sum_{\nu} \nu^{(1+2A)/[2(1-A)]} \left[ \frac{d^2\psi}{d\nu^2} + \frac{1 + 2A}{1-A} \frac{\nu}{\nu} \frac{d\psi}{d\nu} + \frac{(1 + 2A)(4A-1)}{(1-A)^2} \frac{1}{4\nu^2} \psi (\nu) \right] |\nu\rangle, \quad (3.15)\]

Taking \( A = 0 \) reproduces the large scale factor ordering associated with the ‘old’ quantisation, as expected \[7\]. Notice that \( A = -1/2 \), which corresponds to the ‘new’ quantisation, leads to the following very simple form of the evolution equation in the continuum limit:

\[ \lim_{k/\nu \to 0} C_{\text{grav}} |\Psi\rangle = \frac{72\mu_0/k}{\kappa^2 \hbar^3} \left( \frac{\kappa \gamma \hbar}{6} \right)^{3/2} \sum_{\nu} \frac{d^2\psi}{d\nu^2} |\nu\rangle, \quad (3.16)\]

where we have used \( \alpha = 3\mu_0/(2k) \) and reintroduced all the constants.

For \( \mu_0 = k \) we arrive at the following final result for the continuum limit of the WDW equation:

\[ \lim_{k/\nu \to 0} C_{\text{grav}} |\Psi\rangle = \frac{72}{\kappa^2 \hbar^3} \left( \frac{\kappa \gamma \hbar}{6} \right)^{3/2} \sum_{\nu} \frac{d^2\psi}{d\nu^2} |\nu\rangle. \quad (3.17) \]

Repeating this tedious, but straightforward calculation for the other factor ordering choices given in Eqs. (3.4)-(3.8) results in the following differential equations in the continuum limit (see the Appendix A for details),

\[ \lim_{k/\nu \to 0} \epsilon_{ijk} \text{tr} \left( \hat{h}_i \hat{h}_j^{-1} \hat{h}_k^{-1} \hat{V} \hat{h}_k \right) |\Psi\rangle \sim \frac{-36i}{1-A} \alpha^{3/[2(1-A)]} \]

\[ \times k^3 \sum_{\nu} \nu^{(1+2A)/[2(1-A)]} \left[ \frac{d^2\psi}{d\nu^2} + \frac{1 + 2A}{1-A} \frac{\nu}{\nu} \frac{d\psi}{d\nu} + \frac{(1 + 2A)(4A-1)}{(1-A)^2} \frac{1}{4\nu^2} \psi (\nu) \right] |\nu\rangle, \quad (3.18) \]
Once again we see that \( A = -1/2 \) results in a particular simplification, in that all of the considered factor ordering choices reduce to Eq. (3.17). This is to be expected since quantum factor ordering ambiguities should disappear at the classical level. The crucial finding is that the LQC ambiguities, associated with the factor ordering in the Hamiltonian constraint disappear at the continuum described by the WDW equation, only for a lattice refinement power law model with \( A = -1/2 \). Thus, only this model has a non-ambiguous continuum limit.

In addition to this lattice refinement model providing a unique choice of factor ordering for the continuum limit equations, the action of the volume operator is greatly simplified,

\[
V_{\nu+nk} - V_{\nu+mk} = (n - m) \frac{3\mu_0}{2}.
\] (3.21)

This is no accident since the initial motivation for this quantisation procedure was that the volume, rather than the area, should get quantised. The consequence is that quantum corrections to this classical equation (i.e. quantum corrections to general relativity) enter only in the Taylor approximation \( \psi(\nu + nk) \approx \psi(\nu) + \ldots \) and not in the expansion of the volume terms, i.e. Eq. (3.14) requires no approximation.

Finally, to relate this result to more usual variables, we can use \( \mu \sim p = a^2 \) and \( \nu \sim \mu^{3/2} \), to find that the factor ordering of the Wheeler-DeWitt equation predicted by the large scale limit of LQC reads

\[
C_{\text{grav}} \sim \frac{d^2\psi}{d\nu^2} \sim a^{-2} \frac{d}{da} \left( a^{-2}\frac{d\psi}{da} \right),
\] (3.22)

where constants, but no factors on \( a \), have been dropped.
IV. CONCLUSIONS

Just as in the quantisation of standard fields, loop quantum cosmology results in factor ordering ambiguities. One expects that the classical limit should unambiguously result to the original classical equation. Here, we have shown that in general this is not true at the level of the Wheeler-DeWitt equation, which is the classical limit of LQC (it is obtained when the discreteness scale is set to zero). This ordering ambiguity disappears however for the particular lattice refinement model given by $A = -1/2$, which is typically called ‘new’ or ‘improved’ quantisation.

The work presented here can be viewed in two ways: One could accept the phenomenological [7, 10] and consistency [13] requirements indicating that $A = -1/2$ is the correct quantisation approach. In this way, we have shown that LQC predicts a unique factor ordering of the Wheeler-DeWitt equation in its continuum limit (at least for the particular class of factor ordering considered here). Alternatively, one could require that factor ordering ambiguities in LQC should disappear at the level of the Wheeler-DeWitt equation. In this case, we have shown that the lattice refinement model should be $A = -1/2$. In either case, we have clearly demonstrated that there is a strong link between factor ordering and lattice refinement in LQC; these two ambiguities are closely related. In fact, we have shown that by specifying a particular lattice refinement model we can uniquely determine the factor ordering of the equation in the continuum limit, and vice versa.

In conclusion, it is remarkable that the requirement for the Wheeler-DeWitt factor ordering to be unique, is precisely the same requirement reached by physical considerations of large scale physics and consistency of the quantisation structure. In particular, it has been previously shown [10] that for LQC to generically support inflation and other matter fields without the onset of large scale quantum gravity corrections, $A$ should be equal to $-1/2$. It has been recently shown [13] that physical quantities depend on the choice of the elementary cell used to regulate the spatial integrations, unless one chooses $A = -1/2$, and that sensible effective equations exist only for this choice. Taking this together with the uniqueness of the factor ordering of the Wheeler-DeWitt equation, which as we have shown also requires $A = -1/2$, it is clear that several vastly different unrelated approaches have converged on the same restriction of the theory. It is possible that this restriction can be used to improve our understanding of the underlying full loop quantum gravity theory.

Acknowledgments

This work is partially supported by the European Union through the Marie Curie Research and Training Network UniverseNet (MRTN-CT-2006-035863).

[1] C. Rovelli, Quantum Gravity (Cambridge University Press, Cambridge, 2004).
[2] A. Ashtekar, M. Bojowald and J. Lewandowski, Adv. Theor. Math. Phys. 7 (2003) 233 [arXiv:gr-qc/0304074].
[3] M. Bojowald, Class. Quant. Grav. 19 (2002) 2717 [arXiv:gr-qc/0202077].
[4] C. Rovelli and F. Vidotto, [arXiv:0805.4585].
APPENDIX A

The action of the different factor ordering possibilities considered in Eqs. (3.4)-(3.8) on a basis state $|\nu\rangle$ is given by

$$
\hat{S}_n^2 \hat{C}_s \left( \hat{C}_s \hat{V} \hat{S}_n - \hat{S}_n \hat{V} \hat{C}_s \right) \hat{C}_s |\nu\rangle = \frac{-i}{32} \left[ (V_{\nu+2k} - V_{\nu}) \left( |\nu + 4k\rangle - |\nu + 2k\rangle - |\nu\rangle + |\nu - 2k\rangle \right) \\
+ (V_{\nu} - V_{\nu-2k}) \left( |\nu + 2k\rangle - |\nu\rangle - |\nu - 2k\rangle + |\nu - 4k\rangle \right) \right], \quad (A1)
$$

$$
\hat{S}_n^2 \left( \hat{C}_s \hat{V} \hat{S}_n - \hat{S}_n \hat{V} \hat{C}_s \right) \hat{C}_s^2 |\nu\rangle = \frac{-i}{32} \left[ (V_{\nu+3k} - V_{\nu+k}) \left( |\nu + 4k\rangle - 2|\nu + 2k\rangle + |\nu\rangle \right) \\
+ 2(V_{\nu+k} - V_{\nu-k}) \left( |\nu + 2k\rangle - 2|\nu\rangle + |\nu - 2k\rangle \right) \\
+ (V_{\nu-k} - V_{\nu-3k}) \left( |\nu\rangle - 2|\nu - 2k\rangle + |\nu - 4k\rangle \right) \right], \quad (A2)
$$

$$
\hat{C}_s^2 \left( \hat{C}_s \hat{V} \hat{S}_n - \hat{S}_n \hat{V} \hat{C}_s \right) \hat{S}_n^2 |\nu\rangle = \frac{-i}{32} \left[ (V_{\nu+3k} - V_{\nu+k}) \left( |\nu + 4k\rangle + 2|\nu + 2k\rangle + |\nu\rangle \right) \\
- 2(V_{\nu+k} - V_{\nu-k}) \left( |\nu + 2k\rangle + 2|\nu\rangle + |\nu - 2k\rangle \right) \\
+ (V_{\nu-k} - V_{\nu-3k}) \left( |\nu\rangle + 2|\nu - 2k\rangle + |\nu - 4k\rangle \right) \right], \quad (A3)
$$
\[ \begin{aligned}
\hat{C}_s \left( \hat{C}_s \hat{V} \hat{S}_n - \hat{S}_n \hat{V} \hat{C}_s \right) \hat{S}_n^2 \hat{C}_s |\nu\rangle &= \frac{-i}{32} \left[ (V_{\nu+4k} - V_{\nu+2k}) \left( |\nu + 4k\rangle + |\nu + 2k\rangle \right) \\
&\quad - (V_{\nu+2k} - V_{\nu}) \left( |\nu + 2k\rangle + |\nu\rangle \right) \\
&\quad - (V_{\nu} - V_{\nu-2k}) \left( |\nu\rangle + |\nu - 2k\rangle \right) \\
&\quad + (V_{\nu-2k} - V_{\nu-4k}) \left( |\nu - 2k\rangle + |\nu - 4k\rangle \right) \right], \quad (A4)
\end{aligned} \]

\[ \begin{aligned}
\left( \hat{C}_s \hat{V} \hat{S}_n - \hat{S}_n \hat{V} \hat{C}_s \right) \hat{S}_n^2 \hat{C}_s^2 |\nu\rangle &= \frac{-i}{32} \left[ (V_{\nu+5k} - V_{\nu+3k}) |\nu + 4k\rangle - 2 (V_{\nu+k} - V_{\nu-k}) |\nu\rangle \\
&\quad + (V_{\nu-3k} - V_{\nu-5k}) |\nu - 4k\rangle \right]. \quad (A5)
\end{aligned} \]

Extending these to a general state in the Hilbert space given by \(|\Psi\rangle = \sum_\nu \psi_\nu |\nu\rangle\) gives,

\[ \begin{aligned}
\epsilon_{ijk} \text{tr} \left( \hat{h}_j \hat{h}_i^{-1} \hat{h}_j^{-1} \hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right] \hat{h}_i \right) |\Psi\rangle &= \frac{-3i}{4} \sum_\nu \left( (V_{\nu-2k} - V_{\nu-4k}) \psi_{\nu-4k} - (V_{\nu} - V_{\nu-2k}) \psi_{\nu-2k} \\
&\quad - (V_{\nu+2k} - V_{\nu}) \psi_{\nu} + (V_{\nu+4k} - V_{\nu+2k}) \psi_{\nu+2k} \\
&\quad + (V_{\nu-2k} - V_{\nu-4k}) \psi_{\nu-2k} - (V_{\nu} - V_{\nu-2k}) \psi_{\nu} \\
&\quad - (V_{\nu+2k} - V_{\nu}) \psi_{\nu+2k} + (V_{\nu+4k} - V_{\nu+2k}) \psi_{\nu+4k} \right) |\Psi\rangle \quad (A6)
\end{aligned} \]

\[ \begin{aligned}
\epsilon_{ijk} \text{tr} \left( \hat{h}_i^{-1} \hat{h}_j^{-1} \hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right] \hat{h}_i \hat{h}_j \right) |\Psi\rangle &= \frac{-3i}{4} \sum_\nu \left( (V_{\nu-k} - V_{\nu-3k}) \psi_{\nu-4k} + (V_{\nu+3k} - V_{\nu+k}) \psi_{\nu} \\
&\quad - 4 (V_{\nu+k} - V_{\nu-k}) \psi_{\nu} + (V_{\nu-k} - V_{\nu-3k}) \psi_{\nu} \\
&\quad + (V_{\nu+3k} - V_{\nu+k}) \psi_{\nu+4k} \right) |\nu\rangle, \quad (A7)
\end{aligned} \]
\[ \epsilon_{ijk} \text{tr} \left( \hat{h}_j^{-1} \hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right] \hat{h}_i \hat{h}_j \hat{h}_i^{-1} \right) |\Psi\rangle = \frac{-3i}{4} \sum_{\nu} \left( \left( V_{\nu} - V_{\nu-2k} \right) \psi_{\nu-4k} + \left( V_{\nu+2k} - V_{\nu} \right) \psi_{\nu-2k} \right. \\
\left. - \left( V_{\nu} - V_{\nu-2k} \right) \psi_{\nu-2k} - \left( V_{\nu+2k} - V_{\nu} \right) \psi_{\nu} \right) \langle \nu \right), (A8) \]

Performing the Taylor expansion of these as in Eq. (3.15) we get,

\[ \lim_{k/\nu \to 0} \epsilon_{ijk} \text{tr} \left( \hat{h}_k \left[ \hat{h}_i^{-1}, \hat{V} \right] \hat{h}_i \hat{h}_j \hat{h}_i^{-1} \hat{h}_j^{-1} \right) |\Psi\rangle \sim \frac{-36i}{1 - A} \frac{\alpha^{3/(2(1-A))}}{2} k^3 \sum_{\nu} \nu^{(1+2A)/(2(1-A))} \left( \frac{d^2 \psi}{d\nu^2} + \frac{1 + 2A}{1 - A} \frac{1}{\nu} \frac{d\psi}{d\nu} + \frac{(1 + 2A)(4A - 1)}{(1 - A)^2} \frac{1}{4\nu^2} \psi(\nu) \right) |\nu\rangle, (A10) \]

\[ \lim_{k/\nu \to 0} \epsilon_{ijk} \text{tr} \left( \hat{h}_i \hat{h}_j \hat{h}_i^{-1} \hat{h}_j^{-1} \left[ \hat{h}_k^{-1}, \hat{V} \right] \hat{h}_i \hat{h}_j \right) |\Psi\rangle \sim \frac{-36i}{1 - A} \frac{\alpha^{3/(2(1-A))}}{2} k^3 \sum_{\nu} \nu^{(1+2A)/(2(1-A))} \left( \frac{d^2 \psi}{d\nu^2} + \frac{1 + 2A}{1 - A} \frac{1}{2\nu} \frac{d\psi}{d\nu} + \frac{(1 + 2A)(4A - 1)}{(1 - A)^2} \frac{1}{8\nu^2} \psi(\nu) \right) |\nu\rangle, (A11) \]
\[
\lim_{k/\nu \to 0} \epsilon_{ijk} \text{tr} \left( \hat{h}^{-1}_j \hat{h}_k \left[ \hat{h}^{-1}_k, \hat{V} \right] \hat{h}_i \hat{h}_j \hat{h}^{-1}_i \right) |\Psi\rangle = \\
\lim_{k/\nu \to 0} \epsilon_{ijk} \text{tr} \left( \hat{h}_k \left[ \hat{h}^{-1}_k, \hat{V} \right] \hat{h}_i \hat{h}_j \hat{h}^{-1}_i \hat{h}^{-1}_j \right) |\Psi\rangle = \\
\lim_{k/\nu \to 0} \epsilon_{ijk} \text{tr} \left( \left[ \hat{h}^{-1}_k, \hat{V} \right] \hat{h}_i \hat{h}_j \hat{h}^{-1}_i \hat{h}^{-1}_j \right) |\Psi\rangle \sim \\
\frac{-36i}{1 - A^{3/(2(1-A))} k^3} \sum_{\nu} \nu^{(1+2A)/(2(1-A))} \frac{d^2 \psi}{d\nu^2} |\nu\rangle .
\]