On Generalized Grahaml Numbers

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Abstract

In this paper, we introduce the generalized Grahaml sequences and we deal with, in detail, three special cases which we call them Grahaml, Grahaml-Lucas and modified Grahaml sequences. We present Binet’s formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

Keywords: Fibonacci numbers; Grahaml numbers; Grahaml-Lucas numbers; 3-primes numbers; Lucas 3-primes numbers; Tribonacci numbers.

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1 Introduction

In this paper, we investigate the generalized Grahaml sequences and we investigate, in detail, three special cases which we call them Grahaml, Grahaml-Lucas and modified Grahaml sequences.

The sequence of Fibonacci numbers \( \{F_n\} \) and the sequence of Lucas numbers \( \{L_n\} \) are defined by

\[
F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \quad F_0 = 0, \quad F_1 = 1,
\]

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respectively. The generalizations of Fibonacci and Lucas sequences lead to several nice and interesting sequences.

The generalized Tribonacci sequence \( \{W_n(W_0, W_1, W_2; r, s, t)\}_{n \geq 0} \) (or shortly \( \{W_n\}_{n \geq 0} \)) is defined as follows:
\[
W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1}
\]
where \( W_0, W_1, W_2 \) are arbitrary complex (or real) numbers and \( r, s, t \) are real numbers.

This sequence has been studied by many authors, see for example \([1,2,3,4,5,6,7,8,9,10,11,12,13]\).

The sequence \( \{W_n\}_{n \geq 0} \) can be extended to negative subscripts by defining
\[
W_n = -stW_{n+1} - rtW_{n+2} + tW_{n+3}
\]
for \( n = 1, 2, 3, \ldots \) when \( t \neq 0 \). Therefore, recurrence (1) holds for all integer \( n \).

As \( \{W_n\} \) is a third order recurrence sequence (difference equation), it’s characteristic equation is
\[
x^3 - rx^2 - sx - t = 0 \tag{2}
\]
whose roots are
\[
\alpha = \alpha(r, s, t) = \frac{r}{3} + A + B \\
\beta = \beta(r, s, t) = \frac{r}{3} + \omega A + \omega^2 B \\
\gamma = \gamma(r, s, t) = \frac{r}{3} + \omega^2 A + \omega B
\]
where
\[
A = \left( \frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta} \right)^{1/3}, \quad B = \left( \frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta} \right)^{1/3}
\]
\[
\Delta = \Delta(r, s, t) = \frac{r^3t}{27} - \frac{r^2s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \quad \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)
\]

Note that we have the following identities
\[
\alpha + \beta + \gamma = r, \\
\alpha \beta + \alpha \gamma + \beta \gamma = -s, \\
\alpha \beta \gamma = t.
\]

If \( \Delta(r, s, t) > 0 \), then the Eqn. (2) has one real (\( \alpha \)) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers \( n \), using Binet’s formula
\[
W_n = \frac{b_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{3}
\]
where
\[
b_1 = W_2 - (\beta + \gamma)W_1 + \beta \gamma W_0, \quad b_2 = W_2 - (\alpha + \gamma)W_1 + \alpha \gamma W_0, \quad b_3 = W_2 - (\alpha + \beta)W_1 + \alpha \beta W_0.
\]
Note that the Binet form of a sequence satisfying (2) for non-negative integers is valid for all integers \( n \), for a proof of this result see [14]. This result of Howard and Saidak [14] is even true in the case of higher-order recurrence relations.

In this paper we consider the case \( r = 2 \), \( s = 3 \), \( t = 5 \) and in this case we write \( V_n = W_n \).

A generalized Grahaml sequence \( \{V_n\}_{n \geq 0} = \{V_n(V_0, V_1, V_2)\}_{n \geq 0} \) is defined by the third-order recurrence relations

\[
V_n = 2V_{n-1} + 3V_{n-2} + 5V_{n-3}
\]  

(4)

with the initial values \( V_0 = c_0, V_1 = c_1, V_2 = c_2 \) not all being zero.

The sequence \( \{V_n\}_{n \geq 0} \) can be extended to negative subscripts by defining

\[
V_{-n} = -\frac{3}{5}V_{-(n-1)} - \frac{2}{5}V_{-(n-2)} + \frac{1}{5}V_{-(n-3)}
\]

for \( n = 1, 2, 3, \ldots \). Therefore, recurrence (4) holds for all integer \( n \).

(3) can be used to obtain Binet formula of generalized Grahaml numbers. Binet formula of generalized Grahaml numbers can be given as

\[
V_n = \frac{b_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)}
\]

where

\[
b_1 = V_2 - (\beta + \gamma)V_1 + \beta\gamma V_0, \quad b_2 = V_2 - (\alpha + \gamma)V_1 + \alpha\gamma V_0, \quad b_3 = V_2 - (\alpha + \beta)V_1 + \alpha\beta V_0.
\]

(5)

Here, \( \alpha, \beta \) and \( \gamma \) are the roots of the cubic equation \( x^3 - 2x^2 - 3x - 5 = 0 \). Moreover

\[
\alpha = \frac{2}{3} + \frac{205}{54} + \sqrt{\frac{1231}{108}}^{1/3} + \frac{205}{54} - \sqrt{\frac{1231}{108}}^{1/3}
\]

\[
\beta = \frac{2}{3} + \omega \left( \frac{205}{54} + \sqrt{\frac{1231}{108}}^{1/3} \right) + \omega^2 \left( \frac{205}{54} - \sqrt{\frac{1231}{108}}^{1/3} \right)
\]

\[
\gamma = \frac{2}{3} + \omega^2 \left( \frac{205}{54} + \sqrt{\frac{1231}{108}}^{1/3} \right) + \omega \left( \frac{205}{54} - \sqrt{\frac{1231}{108}}^{1/3} \right)
\]

where

\[
\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)
\]

Note that

\[
\alpha + \beta + \gamma = 2, \quad \alpha\beta + \alpha\gamma + \beta\gamma = -3, \quad \alpha\beta\gamma = 5.
\]

The first few generalized Grahaml numbers with positive subscript and negative subscript are given in the following Table 1.
The sequences subscripts: $G_n$ for $n$ in $(5)$, where $n$ is an integer. Now we define three special cases of the sequence Lucas sequence $L_n$. The first few values of the special third-order numbers with positive and negative subscripts are shown in Table 2.

Table 2. The first few values of the special third-order numbers with positive and negative subscripts

| $n$ | $V_n$ | $V_{-n}$ |
|-----|-------|----------|
| 0   | $V_0$ |          |
| 1   | $V_1$ |          |
| 2   | $V_2$ |          |
| 3   | $5V_0 + 3V_1 + 2V_2$ | $V_0$ |
| 4   | $10V_0 + 11V_1 + 7V_2$ | $2V_1 - V_0$ |
| 5   | $35V_0 + 31V_1 + 25V_2$ | $6V_2 - 5V_1 + V_0$ |
| 6   | $125V_0 + 110V_1 + 81V_2$ | $2V_0$ |
| 7   | $405V_0 + 368V_1 + 272V_2$ | $17V_2 - 12V_1 + V_0$ |
| 8   | $1360V_0 + 1221V_1 + 912V_2$ | $30V_0 + 25V_1 - 11V_2$ |

Now we define three special cases of the sequence $\{V_n\}$. Graham-Lucas sequence $\{G_n\}_{n \geq 0}$, Graham-Lucas sequence $\{H_n\}_{n \geq 0}$ and modified Graham sequence $\{E_n\}_{n \geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+3} = 2G_{n+2} + 3G_{n+1} + 5G_n, \quad G_0 = 0, G_1 = 1, G_2 = 2,$$

and

$$H_{n+3} = 2H_{n+2} + 3H_{n+1} + 5H_n, \quad H_0 = 3, H_1 = 2, H_2 = 10,$$

for all integer $n$. The sequences $\{G_n\}_{n \geq 0}$, $\{H_n\}_{n \geq 0}$ and $\{E_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = \frac{3}{5}G_{-(n+1)} - \frac{2}{5}G_{-(n+2)} + \frac{1}{5}G_{-(n+3)},$$

and

$$H_{-n} = \frac{3}{5}H_{-(n+1)} - \frac{2}{5}H_{-(n+2)} + \frac{1}{5}H_{-(n+3)},$$

and

$$E_{-n} = \frac{3}{5}E_{-(n+1)} - \frac{2}{5}E_{-(n+2)} + \frac{1}{5}E_{-(n+3)}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (8), (9) and (10) hold for all integer $n$. Note that the sequences $G_n$, $H_n$ and $E_n$ are not indexed in [15] yet. Next, we present the first few values of the Graham, Graham-Lucas and modified Graham numbers with positive and negative subscripts:

Table 3. The first few values of the Graham, Graham-Lucas and modified Graham numbers

| $n$ | $G_n$ | $H_n$ | $E_n$ |
|-----|-------|-------|-------|
| 0   | $V_0$ | $V_0$ | $V_0$ |
| 1   | $V_1$ | $V_1$ | $V_1$ |
| 2   | $V_2$ | $V_2$ | $V_2$ |
| 3   | $5V_0 + 3V_1 + 2V_2$ | $H_0$ | $H_0$ |
| 4   | $10V_0 + 11V_1 + 7V_2$ | $H_1$ | $H_1$ |
| 5   | $35V_0 + 31V_1 + 25V_2$ | $H_2$ | $H_2$ |
| 6   | $125V_0 + 110V_1 + 81V_2$ | $H_3$ | $H_3$ |
| 7   | $405V_0 + 368V_1 + 272V_2$ | $H_4$ | $H_4$ |
| 8   | $1360V_0 + 1221V_1 + 912V_2$ | $H_5$ | $H_5$ |

For all integers $n$, Graham, Graham-Lucas and modified Graham numbers (using initial conditions in (5)) can be expressed using Binet’s formulas as

$$G_n = \alpha^n + \beta^n + \gamma^n$$

where

$$\alpha = \frac{-1 + \sqrt{5}}{2}, \quad \beta = \frac{-1 - \sqrt{5}}{2}, \quad \gamma = \frac{-1 \pm \sqrt{1 - 4\alpha}}{2}.$$
and
\[ H_n = \alpha^n + \beta^n + \gamma^n, \]
and
\[ E_n = \frac{(\alpha - 1)\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{(\beta - 1)\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{(\gamma - 1)\gamma^n}{(\gamma - \alpha)(\gamma - \beta)}, \]
respectively.

## 2 Generating Functions

Next, we give the ordinary generating function \( \sum_{n=0}^{\infty} V_n x^n \) of the sequence \( V_n \).

**Lemma 2.1.** Suppose that \( f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n \) is the ordinary generating function of the generalized Grahaml sequence \( \{V_n\}_{n \geq 0} \). Then, \( \sum_{n=0}^{\infty} V_n x^n \) is given by
\[
\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2}{1 - 2x - 3x^2 - 5x^3}.
\]

**Proof.** Using the definition of generalized Grahaml numbers, and subtracting \( 2x \sum_{n=0}^{\infty} V_n x^n \), \( 3x^2 \sum_{n=0}^{\infty} V_n x^n \) and \( 5x^3 \sum_{n=0}^{\infty} V_n x^n \) from \( \sum_{n=0}^{\infty} V_n x^n \) we obtain
\[
(1 - 2x - 3x^2 - 5x^3) \sum_{n=0}^{\infty} V_n x^n = \sum_{n=0}^{\infty} V_n x^n - 2x \sum_{n=0}^{\infty} V_n x^n - 3x^2 \sum_{n=0}^{\infty} V_n x^n - 5x^3 \sum_{n=0}^{\infty} V_n x^n
\]
\[
= \sum_{n=0}^{\infty} V_n x^n - 2 \sum_{n=0}^{\infty} V_n x^{n+1} - 3 \sum_{n=0}^{\infty} V_n x^{n+2} - 5 \sum_{n=0}^{\infty} V_n x^{n+3}
\]
\[
= \sum_{n=0}^{\infty} V_n x^n - 2 \sum_{n=1}^{\infty} V_{n-1} x^n - 3 \sum_{n=2}^{\infty} V_{n-2} x^n - 5 \sum_{n=3}^{\infty} V_{n-3} x^n
\]
\[
= (V_0 + V_1 x + V_2 x^2) - 2(V_0 x + V_1 x^2) - 3V_0 x^2
\]
\[
+ \sum_{n=3}^{\infty} (V_n - 2V_{n-1} - 3V_{n-2} - 5V_{n-3}) x^n
\]
\[
= V_0 + V_1 x + V_2 x^2 - 2V_0 x - 2V_1 x^2 - 3V_0 x^2
\]
\[
= V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2.
\]

Rearranging above equation, we obtain
\[
\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2}{1 - 2x - 3x^2 - 5x^3}.
\]

The previous Lemma gives the following results as particular examples.

**Corollary 2.1.** Generated functions of Grahaml, Grahaml-Lucas and modified Grahaml numbers are
\[
\sum_{n=0}^{\infty} G_n x^n = \frac{x}{1 - 2x - 3x^2 - 5x^3},
\]
and
\[
\sum_{n=0}^{\infty} H_n x^n = \frac{3 - 4x - 3x^2}{1 - 2x - 3x^2 - 5x^3}.
\]
and 
\[ \sum_{n=0}^{\infty} E_n x^n = \frac{x - x^2}{1 - 2x - 3x^2 - 5x^3}, \]
respectively.

3 Obtaining Binet Formula from Generating Function

We next find Binet formula of generalized Graham numbers \( \{V_n\} \) by the use of generating function for \( V_n \).

**Theorem 3.1.** *(Binet formula of generalized Graham numbers)*

\[
V_n = \frac{d_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{d_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{d_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)}
\]

where
\[
\begin{align*}
    d_1 &= V_0 \alpha^2 + (V_1 - 2V_0)\alpha + (V_2 - 2V_1 - 3V_0), \\
    d_2 &= V_0 \beta^2 + (V_1 - 2V_0)\beta + (V_2 - 2V_1 - 3V_0), \\
    d_3 &= V_0 \gamma^2 + (V_1 - 2V_0)\gamma + (V_2 - 2V_1 - 3V_0).
\end{align*}
\]

**Proof.** Let
\[ h(x) = 1 - 2x - 3x^2 - 5x^3. \]
Then for some \( \alpha, \beta \) and \( \gamma \) we write
\[ h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x), \]
\[ \text{i.e.,} \quad 1 - 2x - 3x^2 - 5x^3 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x) \]
Hence \( \frac{1}{\alpha}, \frac{1}{\beta}, \) and \( \frac{1}{\gamma} \) are the roots of \( h(x) \). This gives \( \alpha, \beta, \) and \( \gamma \) as the roots of
\[ h(\frac{1}{x}) = 1 - \frac{2}{x} - \frac{3}{x^2} - \frac{5}{x^3} = 0. \]
This implies \( x^3 - 2x^2 - 3x - 5 = 0 \). Now, by (11) and (13), it follows that
\[ \sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)}. \]
Then we write
\[ \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)} = \frac{A_1}{1 - \alpha x} + \frac{A_2}{1 - \beta x} + \frac{A_3}{1 - \gamma x}. \]
So
\[ V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2 = A_1(1 - \beta x)(1 - \gamma x) + A_2(1 - \alpha x)(1 - \gamma x) + A_3(1 - \alpha x)(1 - \beta x). \]
If we consider \( x = \frac{1}{\alpha} \), we get \( V_0 + (V_1 - 2V_0)\frac{1}{\alpha} + (V_2 - 2V_1 - 3V_0)\frac{1}{\alpha^2} = A_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha}) \). This gives
\[ A_1 = \frac{\alpha^2(V_0 + (V_1 - 2V_0)\frac{1}{\alpha} + (V_2 - 2V_1 - 3V_0)\frac{1}{\alpha^2})}{(\alpha - \beta)(\alpha - \gamma)} = \frac{V_0 \alpha^2 + (V_1 - 2V_0)\alpha + (V_2 - 2V_1 - 3V_0)}{(\alpha - \beta)(\alpha - \gamma)}. \]
Similarly, we obtain
\[ A_2 = \frac{V_0 \beta^2 + (V_1 - 2V_0)\beta + (V_2 - 2V_1 - 3V_0)}{(\beta - \alpha)(\beta - \gamma)}, \quad A_3 = \frac{V_0 \gamma^2 + (V_1 - 2V_0)\gamma + (V_2 - 2V_1 - 3V_0)}{(\gamma - \alpha)(\gamma - \beta)}. \]
Thus (14) can be written as
\[ \sum_{n=0}^{\infty} V_n x^n = A_1 (1 - \alpha x)^{-1} + A_2 (1 - \beta x)^{-1} + A_3 (1 - \gamma x)^{-1}. \]
This gives
\[ \sum_{n=0}^{\infty} V_n x^n = A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n = \sum_{n=0}^{\infty} (A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n) x^n. \]
Therefore, comparing coefficients on both sides of the above equality, we obtain
\[ V_n = A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n \]
where
\[
\begin{align*}
A_1 &= \frac{V_0 \alpha^2 + (V_1 - 2V_0) \alpha + (V_2 - 2V_1 - 3V_0)}{(\alpha - \beta)(\alpha - \gamma)}, \\
A_2 &= \frac{V_0 \beta^2 + (V_1 - 2V_0) \beta + (V_2 - 2V_1 - 3V_0)}{(\beta - \alpha)(\beta - \gamma)}, \\
A_3 &= \frac{V_0 \gamma^2 + (V_1 - 2V_0) \gamma + (V_2 - 2V_1 - 3V_0)}{(\gamma - \alpha)(\gamma - \beta)},
\end{align*}
\]
and then we get (12).

Note that from (5) and (12) we have
\[
\begin{align*}
V_2 - (\beta + \gamma)V_1 + \beta \gamma V_0 &= V_0 \alpha^2 + (V_1 - 2V_0) \alpha + (V_2 - 2V_1 - 3V_0), \\
V_2 - (\alpha + \gamma)V_1 + \alpha \gamma V_0 &= V_0 \beta^2 + (V_1 - 2V_0) \beta + (V_2 - 2V_1 - 3V_0), \\
V_2 - (\alpha + \beta)V_1 + \alpha \beta V_0 &= V_0 \gamma^2 + (V_1 - 2V_0) \gamma + (V_2 - 2V_1 - 3V_0).
\end{align*}
\]

Next, using Theorem 3.1, we present the Binet formulas of Grahaml, Grahaml-Lucas and modified Grahaml sequences.

**Corollary 3.2.** Binet formulas of Grahaml, Grahaml-Lucas and modified Grahaml sequences are
\[
G_n = \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)},
\]
and
\[
H_n = \alpha^n + \beta^n + \gamma^n,
\]
and
\[
E_n = \frac{(\alpha - 1)\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{(\beta - 1)\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{(\gamma - 1)\gamma^n}{(\gamma - \alpha)(\gamma - \beta)}.
\]
respectively.

We can find Binet formulas by using matrix method with a similar technique which is given in [16]. Take \(k = i = 3\) in Corollary 3.1 in [16]. Let
\[
\Lambda = \begin{pmatrix}
  \alpha^2 & \alpha & 1 \\
  \beta^2 & \beta & 1 \\
  \gamma^2 & \gamma & 1
\end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix}
  \alpha^{n-1} & \alpha & 1 \\
  \beta^{n-1} & \beta & 1 \\
  \gamma^{n-1} & \gamma & 1
\end{pmatrix},
\]
\[
\Lambda_2 = \begin{pmatrix}
  \alpha^2 & \alpha^{n-1} & 1 \\
  \beta^2 & \beta^{n-1} & 1 \\
  \gamma^2 & \gamma^{n-1} & 1
\end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix}
  \alpha^2 & \alpha & \alpha^{n-1} \\
  \beta^2 & \beta & \beta^{n-1} \\
  \gamma^2 & \gamma & \gamma^{n-1}
\end{pmatrix}.
\]
Then the Binet formula for Grahaml numbers is

\[
G_n = \frac{1}{\det(\Lambda)} \sum_{j=1}^{3} G_{4-j} \det(\Lambda_j) = \frac{1}{\Lambda} (G_3 \det(\Lambda_1) + G_2 \det(\Lambda_2) + G_1 \det(\Lambda_3))
\]

\[
= \frac{1}{\det(\Lambda)} (7 \det(\Lambda_1) + 2 \det(\Lambda_2) + \det(\Lambda_3))
\]

\[
= \left( \begin{array}{ccc} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{array} \right) + 2 \left( \begin{array}{ccc} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{array} \right) / \left( \begin{array}{ccc} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{array} \right).
\]

Similarly, we obtain the Binet formula for Grahaml-Lucas and modified Grahaml numbers as

\[
H_n = \frac{1}{\Lambda} (H_3 \det(\Lambda_1) + H_2 \det(\Lambda_2) + H_1 \det(\Lambda_3))
\]

\[
= \left( \begin{array}{ccc} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{array} \right) + 10 \left( \begin{array}{ccc} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{array} \right) / \left( \begin{array}{ccc} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{array} \right).
\]

and

\[
E_n = \frac{1}{\Lambda} (E_3 \det(\Lambda_1) + E_2 \det(\Lambda_2) + E_1 \det(\Lambda_3))
\]

\[
= \left( \begin{array}{ccc} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{array} \right) + 2 \left( \begin{array}{ccc} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{array} \right) / \left( \begin{array}{ccc} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{array} \right).
\]

respectively.

4 Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence \(\{F_n\}\), namely,

\[F_{n+1}F_{n-1} - F_n^2 = (-1)^n\]

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

\[
\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.
\]

The following theorem gives generalization of this result to the generalized Grahaml sequence \(\{V_n\}_{n \geq 0}\).

Theorem 4.1 (Simson Formula of Generalized Grahaml Numbers). For all integers \(n\), we have

\[
\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \end{vmatrix} = 5^n \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \end{vmatrix}.
\]

(15)

Proof. (15) is given in Soykan [17].

The previous theorem gives the following results as particular examples.
Corollary 4.2. For all integers $n$, Simson formula of Grahaml, Grahaml-Lucas and modified Grahaml numbers are given as

\[
\begin{vmatrix}
G_{n+2} & G_{n+1} & G_n \\
G_{n+1} & G_n & G_{n-1} \\
G_n & G_{n-1} & G_{n-2}
\end{vmatrix} = -5^{n-1},
\]

and

\[
\begin{vmatrix}
H_{n+2} & H_{n+1} & H_n \\
H_{n+1} & H_n & H_{n-1} \\
H_n & H_{n-1} & H_{n-2}
\end{vmatrix} = -1231 \times 5^{n-2},
\]

and

\[
\begin{vmatrix}
E_{n+2} & E_{n+1} & E_n \\
E_{n+1} & E_n & E_{n-1} \\
E_n & E_{n-1} & E_{n-2}
\end{vmatrix} = -9 \times 5^{n-2},
\]

respectively.

5 Some Identities

In this section, we obtain some identities of Grahaml, Grahaml-Lucas and modified Grahaml numbers. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 5.1. The following equalities are true:

\[
\begin{align*}
125H_n &= 138G_{n+4} - 331G_{n+3} - 379G_{n+2}, \\
25H_n &= -11G_{n+3} + 7G_{n+2} + 138G_{n+1}, \\
5H_n &= -3G_{n+2} + 21G_{n+1} - 11G_n, \\
H_n &= 3G_{n+1} - 4G_n - 3G_{n-1}, \\
H_n &= 2G_n + 6G_{n-1} + 15G_{n-2},
\end{align*}
\]

and

\[
\begin{align*}
6155G_n &= 18H_{n+4} - 166H_{n+3} + 461H_{n+2}, \\
1231G_n &= -26H_{n+3} + 103H_{n+2} + 18H_{n+1}, \\
1231G_n &= 51H_{n+2} - 60H_{n+1} - 130H_n, \\
1231G_n &= 42H_{n+1} + 23H_n + 255H_{n-1}, \\
1231G_n &= 107H_n + 381H_{n-1} + 210H_{n-2},
\end{align*}
\]

Proof. Note that all the identities hold for all integers $n$. We prove (16). To show (16), writing

\[
H_n = a \times G_{n+4} + b \times G_{n+3} + c \times G_{n+2}
\]

and solving the system of equations

\[
\begin{align*}
H_0 &= a \times G_4 + b \times G_3 + c \times G_2, \\
H_1 &= a \times G_5 + b \times G_4 + c \times G_3, \\
H_2 &= a \times G_6 + b \times G_5 + c \times G_4
\end{align*}
\]

we find that $a = \frac{138}{125}, b = -\frac{331}{125}, c = -\frac{379}{125}$. The other equalities can be proved similarly.
Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between \( \{G_n\} \) and \( \{E_n\} \).

**Lemma 5.2.** The following equalities are true:
\[
\begin{align*}
125E_n &= -14G_{n+4} + 68G_{n+3} - 63G_{n+2}, \\
25E_n &= 8G_{n+3} - 21G_{n+2} - 14G_{n+1}, \\
5E_n &= -G_{n+2} + 2G_{n+1} + 8G_n, \\
E_n &= G_n - G_{n-1},
\end{align*}
\]
and
\[
\begin{align*}
45G_n &= -4E_{n+4} + 13E_{n+3} + 7E_{n+2}, \\
9G_n &= E_{n+3} - E_{n+2} - 4E_{n+1}, \\
9G_n &= E_{n+2} - E_{n+1} + 5E_n, \\
9G_n &= E_{n+1} + 8E_n + 5E_{n-1}, \\
9G_n &= 10E_n + 8E_{n-1} + 5E_{n-2}.
\end{align*}
\]

Thirdly, we give a few basic relations between \( \{H_n\} \) and \( \{E_n\} \).

**Lemma 5.3.** The following equalities are true:
\[
\begin{align*}
225H_n &= 134E_{n+4} - 233E_{n+3} - 572E_{n+2}, \\
45H_n &= 7E_{n+3} - 34E_{n+2} + 134E_{n+1}, \\
9H_n &= -4E_{n+2} + 31E_{n+1} + 7E_n, \\
9H_n &= 23E_{n+1} - 5E_n - 20E_{n-1}, \\
9H_n &= 41E_n + 49E_{n-1} + 115E_{n-2}.
\end{align*}
\]
and
\[
\begin{align*}
30775E_n &= 371H_{n+4} + 2H_{n+3} + 4518H_{n+2}, \\
6155E_n &= 148H_{n+3} + 681H_{n+2} - 371H_{n+1}, \\
1231E_n &= 77H_{n+2} - 163H_{n+1} + 148H_n, \\
1231E_n &= -9H_{n+1} + 83H_n + 385H_{n-1}, \\
1231E_n &= 65H_n + 358H_{n-1} - 45H_{n-2}.
\end{align*}
\]

We now present a few special identities for the modified Graham sequence \( \{E_n\} \).

**Theorem 5.1.** (Catalan’s identity) For all integers \( n \) and \( m \), the following identity holds
\[
E_{n+m}E_{n-m} - E_n^2 = (G_{n+m} - G_{n+m-1})(G_{n-m} - G_{n-m-1}) - (G_n - G_{n-1})^2
\]
\[
= (G_n(G_m - G_{m+1}) + G_{n-1}(-G_m + G_{m-2} + G_{n-2}(-G_m + G_{m-1})))
\]
\[
+ (G_n(G_{n-m} - G_{1-m}) + G_{n-1}(-G_{n-m} + G_{n-m-3}) + G_{n-2}(-G_{n-m} + G_{n-m-1})))
\]
\[
- (G_n - G_{n-1})^2
\]

Proof. We use the identity \( E_n = G_n - G_{n-1} \)
and the identity (22).
Note that for \( m = 1 \) in Catalan’s identity, we get the Cassini identity for the modified Graham-Lucas sequence

**Corollary 5.2.** (Cassini’s identity) For all integers numbers \( n \) and \( m \), the following identity holds

\[
E_{n+1}E_{n-1} - E_n^2 = (G_{n+1} - G_n)(G_{n-1} - G_{n-2}) - (G_n - G_{n-1})^2.
\]

The d’Ocagne’s, Gelin-Cesàro’s and Melham’ identities can also be obtained by using \( E_n = G_n - G_{n-1} \). The next theorem presents d’Ocagne’s, Gelin-Cesàro’s and Melham’ identities of modified Graham-Lucas sequence \( \{E_n\} \).

**Theorem 5.3.** Let \( n \) and \( m \) be any integers. Then the following identities are true:

(a) (d’Ocagne’s identity)

\[
E_{m+1}E_n - E_mE_{n+1} = (G_{m+1} - G_m)(G_n - G_{n-1}) - (G_m - G_{m-1})(G_{n+1} - G_n).
\]

(b) (Gelin-Cesàro’s identity)

\[
E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (G_{n+2} - G_{n+1})(G_{n+1} - G_n)(G_{n-1} - G_{n-2})(G_{n-2} - G_{n-3}) - (G_n - G_{n-1})^4.
\]

(c) (Melham’s identity)

\[
E_{n+3}E_{n+2}E_{n+1}E_{n-1} - E_n^6 = (G_{n+3} - G_n)(G_{n+2} - G_{n+1})(G_{n+1} - G_n)(G_{n+3} - G_{n+2}) - (G_{n+3} - G_{n+2})^3.
\]

Proof. Use the identity \( E_n = G_n - G_{n-1} \).

6 Linear Sums

The following proposition presents some formulas of generalized Graham-Lucas numbers with positive subscripts.

**Proposition 6.1.** If \( r = 2, s = 3, t = 5 \) then for \( n \geq 0 \) we have the following formulas:

(a) \( \sum_{k=0}^{n} V_k = \frac{1}{5} (V_{n+3} - V_{n+2} - 4V_{n+1} - V_2 + V_1 + 3V_0) \).

(b) \( \sum_{k=0}^{n} V_2k = \frac{1}{10} (2V_{2n+2} + 11V_{2n+1} + 35V_{2n} + 2V_2 - 11V_1 + 10V_0) \).

(c) \( \sum_{k=0}^{n} V_2k+1 = \frac{1}{10} (7V_{2n+2} + 29V_{2n+1} - 10V_{2n} - 7V_2 + 10V_1 + 10V_0) \).

Proof. Take \( r = 2, s = 3, t = 5 \) in Theorem 2.1 in [18], see also [19].

As special cases of above proposition, we have the following three corollaries. First one presents some summing formulas of Graham numbers (take \( V_n = G_n \) with \( G_0 = 0, G_1 = 1, G_2 = 2 \)).

**Corollary 6.1.** For \( n \geq 0 \) we have the following formulas:

(a) \( \sum_{k=0}^{n} G_k = \frac{1}{2} (G_{n+3} - G_{n+2} - 4G_{n+1} - 1) \).

(b) \( \sum_{k=0}^{n} G_{2k} = \frac{1}{10} (-2G_{2n+2} + 11G_{2n+1} + 35G_{2n} - 7) \).

(c) \( \sum_{k=0}^{n} G_{2k+1} = \frac{1}{10} (7G_{2n+2} + 29G_{2n+1} - 10G_{2n} + 2) \).

Second one presents some summing formulas of Graham-Lucas numbers (take \( V_n = H_n \) with \( H_0 = 3, H_1 = 2, H_2 = 10 \)).

**Corollary 6.2.** For \( n \geq 0 \) we have the following formulas:

(a) \( \sum_{k=0}^{n} H_k = \frac{1}{2} (H_{n+3} - H_{n+2} - 4H_{n+1} + 4) \).
For any $n$, we have the following formulas:

(a) $\sum_{k=0}^{n} H_{2k} = \frac{1}{15}(-2H_{2n+2} + 11H_{2n+1} + 35H_{2n} + 28)$.

(b) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{15}(7H_{2n+2} + 29H_{2n+1} - 10H_{2n} - 8)$.

(c) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{15}(7H_{2n+2} + 29H_{2n+1} - 10H_{2n} + 9)$.

Third one presents some summing formulas of modified Graham-Lucan numbers (take $V_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

**Corollary 6.3.** For $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} E_{2k} = \frac{1}{7}(E_{n+3} - E_{n+2} - 4E_{n+1})$.

(b) $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{7}(-2E_{2n+2} + 11E_{2n+1} + 35E_{2n} - 9)$.

(c) $\sum_{k=0}^{n} E_{2k+2} = \frac{1}{7}(7E_{2n+2} + 29E_{2n+1} - 10E_{2n} + 9)$.

The following proposition presents some formulas of generalized Graham-Lucan numbers with negative subscripts.

**Proposition 6.2.** If $r = 2, s = 3, t = 5$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=0}^{n} G_{r-k} = \frac{1}{7}(-10G_{n-1} - 8G_{n-2} - 5G_{n-3} + V_2 - V_1 - 4V_0)$.

(b) $\sum_{k=0}^{n} G_{s-k} = \frac{1}{7}(-2G_{2n+1} + 16G_{2n} + 10G_{2n-1} - 2V_2 + 11V_1 - 10V_0)$.

(c) $\sum_{k=0}^{n} G_{t-k} = \frac{1}{7}(2V_{2n+1} - 11V_{2n} - 35V_{2n-1} + 7V_2 - 16V_1 - 10V_0)$.

Proof. Take $r = 2, s = 3, t = 5$ in Theorem 3.1 in [18], see also [19].

From the above proposition, we have the following corollary which gives sum formulas of Graham-Lucan numbers (take $V_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

**Corollary 6.4.** For $n \geq 1$, Graham-Lucan numbers have the following properties.

(a) $\sum_{k=0}^{n} G_{r-k} = \frac{1}{7}(-10G_{n-1} - 8G_{n-2} - 5G_{n-3} + 1)$.

(b) $\sum_{k=0}^{n} G_{s-k} = \frac{1}{7}(-2G_{2n+1} + 16G_{2n} + 10G_{2n-1} + 7)$.

(c) $\sum_{k=0}^{n} G_{t-k} = \frac{1}{7}(2G_{2n+1} - 11G_{2n} - 35G_{2n-1} - 2)$.

Taking $V_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in the last proposition, we have the following corollary which presents sum formulas of Graham-Lucan numbers.

**Corollary 6.5.** For $n \geq 1$, Graham-Lucan numbers have the following properties.

(a) $\sum_{k=0}^{n} H_{r-k} = \frac{1}{7}(-10H_{n-1} - 8H_{n-2} - 5H_{n-3} - 4)$.

(b) $\sum_{k=0}^{n} H_{s-k} = \frac{1}{7}(-2H_{2n+1} + 16H_{2n} + 10H_{2n-1} - 28)$.

(c) $\sum_{k=0}^{n} H_{t-k} = \frac{1}{7}(2H_{2n+1} - 11H_{2n} - 35H_{2n-1} + 8)$.

From the above proposition, we have the following corollary which gives sum formulas of modified Graham-Lucan numbers (take $V_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

**Corollary 6.6.** For $n \geq 1$, modified Graham-Lucan numbers have the following properties.

(a) $\sum_{k=0}^{n} E_{r-k} = \frac{1}{7}(-10E_{n-1} - 8E_{n-2} - 5E_{n-3})$.

(b) $\sum_{k=0}^{n} E_{s-k} = \frac{1}{7}(-2E_{2n+1} + 16E_{2n} + 10E_{2n-1} + 9)$.

(c) $\sum_{k=0}^{n} E_{t-k} = \frac{1}{7}(2E_{2n+1} - 11E_{2n} - 35E_{2n-1} - 9)$. 

53
7 Matrices Related with Generalized Grahaml Numbers

Matrix formulation of $W_n$ can be given as

\[
\begin{pmatrix}
W_{n+2} \\
W_{n+1} \\
W_n
\end{pmatrix} =
\begin{pmatrix}
r & s & t \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}^n
\begin{pmatrix}
W_2 \\
W_1 \\
W_0
\end{pmatrix}.
\] (17)

For matrix formulation (17), see [20]. In fact, Kalman gave the formula in the following form

\[
\begin{pmatrix}
W_n \\
W_{n+1} \\
W_{n+2}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
r & s & t
\end{pmatrix}^n
\begin{pmatrix}
W_2 \\
W_1 \\
W_0
\end{pmatrix}.
\]

We define the square matrix $A$ of order 3 as:

\[
A = \begin{pmatrix}
2 & 3 & 5 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

such that $\det A = 5$. From (4) we have

\[
\begin{pmatrix}
V_{n+2} \\
V_{n+1} \\
V_n
\end{pmatrix} =
\begin{pmatrix}
2 & 3 & 5 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}^n
\begin{pmatrix}
V_{n+1} \\
V_n \\
V_{n-1}
\end{pmatrix}.
\] (18)

and from (17) (or using (18) and induction) we have

\[
\begin{pmatrix}
V_{n+2} \\
V_{n+1} \\
V_n
\end{pmatrix} =
\begin{pmatrix}
2 & 3 & 5 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}^n
\begin{pmatrix}
V_2 \\
V_1 \\
V_0
\end{pmatrix}.
\]

If we take $V = G$ in (18) we have

\[
\begin{pmatrix}
G_{n+2} \\
G_{n+1} \\
G_n
\end{pmatrix} =
\begin{pmatrix}
2 & 3 & 5 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}^n
\begin{pmatrix}
G_{n+1} \\
G_n \\
G_{n-1}
\end{pmatrix}.
\] (19)

We also define

\[
B_n = \begin{pmatrix}
G_{n+1} & 3G_n + 5G_{n-1} & 5G_n \\
G_n & 3G_{n-1} + 5G_{n-2} & 5G_{n-1} \\
G_{n-1} & 3G_{n-2} + 5G_{n-3} & 5G_{n-2}
\end{pmatrix}
\]

and

\[
C_n = \begin{pmatrix}
V_{n+1} & 3V_n + 5V_{n-1} & 5V_n \\
V_n & 3V_{n-1} + 5V_{n-2} & 5V_{n-1} \\
V_{n-1} & 3V_{n-2} + 5V_{n-3} & 5V_{n-2}
\end{pmatrix}
\]

**Theorem 7.1.** For all integer $m, n \geq 0$, we have

(a) $B_n = A^n$

(b) $C_n A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

**Proof.**
(a) By expanding the vectors on both sides of (19) to 3-columns and multiplying the obtained on the right-hand side by $A$, we get

$$B_n = AB_{n-1}.$$  

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$  

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of $C_1$, (b) follows.

(c) We have

$$AC_{n-1} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_n & 3V_{n-1} + 5V_{n-2} & 5V_{n-1} \\ V_{n-1} & 3V_{n-2} + 5V_{n-3} & 5V_{n-2} \\ V_{n-2} & 3V_{n-3} + 5V_{n-4} & 5V_{n-3} \end{pmatrix}$$

$$= \begin{pmatrix} V_{n+1} & 3V_n + 5V_{n-1} & 5V_n \\ V_n & 3V_{n-1} + 5V_{n-2} & 5V_{n-1} \\ V_{n-1} & 3V_{n-2} + 5V_{n-3} & 5V_{n-2} \end{pmatrix} = C_n,$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^n A^{m-1}C_1 = A^{n-1} A^m C_1 = A^{n-1} C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

Some properties of matrix $A^n$ can be given as

$$A^n = 2A^{n-1} + 3A^{n-2} + 5A^{n-3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 5^n$$

for all integer $m$ and $n$.

**Theorem 7.2.** For $m, n \geq 0$ we have

$$V_{n+m} = V_n G_{m+1} + V_{n-1}(3G_m + 5G_{m-1}) + 5V_{n-2}G_m$$

$$= V_n G_{m+1} + (3V_{n-1} + 5V_{n-2}) G_m + 5V_{n-3}G_{m-1}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of $C_{n+m}$ is the product of row $C_n$ and a column $B_m$. From the last equation we say that an element of $C_{n+m}$ is the product of a row $C_n$ and column $B_m$. We just compare the linear combination of the 2nd row and 1st column entries of the matrices $C_{n+m}$ and $C_n B_m$. This completes the proof.

**Remark 7.1.** By induction, it can be proved that for all integers $m, n \leq 0$, (20) holds. So for all integers $m, n$, (20) is true.

**Corollary 7.3.** For all integers $m, n$, we have

$$G_{n+m} = G_n G_{m+1} + G_{n-1}(3G_m + 5G_{m-1}) + 5G_{n-2}G_m,$$

$$H_{n+m} = H_n G_{m+1} + H_{n-1}(3G_m + 5G_{m-1}) + 5H_{n-2}G_m,$$

$$E_{n+m} = E_n G_{m+1} + E_{n-1}(3G_m + 5G_{m-1}) + 5E_{n-2}G_m.$$
8  Conclusions

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. We introduce the generalized Grahaml sequence (it’s three special cases, namely, Grahaml, Grahaml-Lucas and modified Grahaml sequences) and we present Binet’s formulas, generating functions, Simson formulas, the summation formulas, some identities and matrices for these sequences. Generalized Grahaml sequence (and it’s three special cases: Grahaml, Grahaml-Lucas and modified Grahaml sequences) can also be called (named) as generalized 3-primes sequence (3-primes, Lucas 3-primes and modified 3-primes sequences, respectively).

Competing Interests
Author has declared that no competing interests exist.

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