GAUGE SYMMETRY AND NEURAL NETWORKS

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We propose a new model of neural network. It consists of spin variables to describe the state of neurons as in the Hopfield model and new gauge variables to describe the state of synapses. The model possesses local gauge symmetry and resembles lattice gauge theory of high-energy physics. Time dependence of synapses describes the process of learning. The mean field theory predicts a new phase corresponding to confinement phase, in which brain loses ability of learning and memory.

1 Introduction

The Hopfield model of neural network successfully explains some basic functions of human brain such as associated memory. However, to be a more realistic model, at least the following points should be taken into account:

(1) Effects of external stimulations through eyes and ears on neurons.
(2) Effects of time variations of synapses on neurons.

The point (2) is essential to describe the function of learning, since the possible patterns to memorize are completely determined according to the strengths of synapse connections among neurons as long as they are time independent. Their time dependence induces the process of learning itself.

In Sect.2, we review the Hopfield model briefly. In Sect.3, we propose a new model of neural network, in which the strengths of synapse connections are regarded as gauge connections and vary in time according to the gauge principle. By using the mean field theory, we see that the model predicts a new state of brain in which both learning and memory are impossible. In Sect.4, we present future outlook.

2 Hopfield model

Let us review the framework of the Hopfield model briefly. Its energy $E_H(\{S_i\})$ is given by

$$E_H(\{S_i\}) = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} S_i S_j,$$

(1)
where $S_i = \pm 1$ is the Ising spin variable to describe the state of $i$-th neural cell ($i = 1, 2, \ldots, N$) as $S_i = 1$; excited, $S_i = -1$; unexcited. $J_{ij}$ is a given real constant that expresses the strength of synapse connection for the signal propagating from the $j$-th cell to the $i$-th cell.

The time evolution of $S_i(t)$ for every discrete time interval $\epsilon$ (often set unity) is governed by the following equation:

$$S_i(t + \epsilon) = \text{sgn}[-\frac{\partial E_H}{\partial S_i}(t)] = \text{sgn}[\sum_j J_{ij} S_j(t)].$$

(2)

Thus, $J_{ij} > 0$ tries to proliferate (un)excited cells, while $J_{ij} < 0$ prefers mixtures of excited and unexcited ones. If the system converges into certain configuration of $\{S_i\}$ after a sufficiently long time, it corresponds to recalling certain pattern. Such a configuration should be a stationary point of $E_H$, i.e., $\partial E_H/\partial S_i = 0$ for every $i$. All these configurations are determined once $J_{ij}$ are given.

Practically speaking, the rule (2) may not necessarily hold all the time interval due to an unavoidable error in signal propagations. Such a situation may be simulated by adding random noises $\eta_i(t)$ into the square bracket in the right-hand side of (2), whose strength can be identified as a fictitious "temperature" $T$. If $T$ is large, the error in signal propagations occurs frequently. Thus it is interesting to study statistical mechanics of the system $E_H$ by using Boltzmann distribution. The partition function $Z_H$ is given by

$$Z_H = \prod_i \sum_{S_i = \pm 1} \exp(-\beta E_H), \quad \beta \equiv 1/T.$$ 

(3)

In case that all $J_{ij}$ are positive, the system has two phases;

- Ferromagnetic phase below certain critical temperature $T_c$, $T < T_c$, in which there is a long-range order and the average $\langle S_i \rangle \neq 0$.

- Paramagnetic phase above $T_c$, $T > T_c$, in which $S_i$ are random and $\langle S_i \rangle = 0$.

The ferromagnetic phase corresponds to the state of clear memory, while in the paramagnetic phase no definite patterns can persist. If $J_{ij}$ is complicated, there arises a spin-glass phase as we shall see.

Explicitly, let us fix $J_{ij}$ according to the Hebb’s rule as

$$J_{ij} = \frac{1}{N} \sum_{\alpha=1}^{M} \xi_i^\alpha \xi_j^\alpha,$$

(4)
where we prepare $M$ patterns $S_i = \xi_i^\alpha$ ($\alpha = 1, \cdots, M$) to recall. The replica method gives rise to the phase diagram shown in Fig. 1. Each phase is explained in Table 1.

Table 1. Phases of the Hopfield model.

| Phase     | $\sum_i S_i$ | $\sum_i S_i^2$ | Property     |
|-----------|--------------|----------------|--------------|
| Ferromagnetic | $\neq 0$     | $\neq 0$       | memory       |
| Spin glass    | 0             | $\neq 0$       | false memory |
| Paramagnetic  | 0             | 0              | no memory    |

3 New model with local gauge symmetry

As pointed out in Sect. 1, to incorporate the function of learning, one needs the time variation of $J_{ij}$. There are various approaches for this point. Below, we regard both $S_i$ and $J_{ij}$ as dynamical variables and treat them on an equal footing. Let us assume that their time dependence is controlled so as to reach a local minimum of the new energy $E(\{S_i, J_{ij}\})$. To determine $E$, we impose the condition that $E$ is local gauge invariant under the following gauge transformation:

$$S_i \rightarrow S'_i \equiv V_i S_i, \quad J_{ij} \rightarrow J'_{ij} \equiv V_i J_{ij} V_j, \quad E(\{S'_i, J'_{ij}\}) = E(\{S_i, J_{ij}\}), \quad (5)$$

where $V_i = \pm 1$ is the $Z(2)$ variable associated with $i$-th cell. Since $J_{ij}$ describes the state of the synapse connecting $i$-th and $j$-th cells, it is natural to regard it as the connection of gauge theory. The neural network may possess certain conservative quantity in association with the long-term memory. The local gauge symmetry we address may respect such a conservation law. This point
will be reported in detail in a separate publication. It is often stressed that the connections $J_{ij}$ and $J_{ji}$ are independent (asymmetric). Then a general form of $E(\{S_i, J_{ij}\})$ and the partition function $Z$ may be given by

$$
E = -\frac{1}{2} \sum_{i,j} S_i J_{ij} S_j + \frac{g_2}{2} \sum_{i,j} J_{ij} J_{ji} \\
+ \frac{g_3}{3!} \sum_{i,j,k} J_{ij} J_{jk} J_{ki} + \frac{g_4}{4!} \sum_{i,j,k,l} J_{ij} J_{jk} J_{kl} J_{li} + \cdots
$$

$$
Z = \prod_i \sum_{S_i = \pm 1} \prod_{\mu \neq j} \int dJ_{ij} \exp(-\beta E), \quad \beta \equiv 1/T.
$$

Each term of $E$ is gauge invariant since $V^2_i = 1$, and depicted in Fig.2.

$E$ takes a form very similar to the lattice gauge theory in particle physics, where $S_i$ corresponds to a matter field and $J_{ij}$ to an exponentiated gauge field. If the parameters $g_2, g_3, g_4, \ldots$ are set zero, $E$ reduces to $E_H$ of (1).

### 3.1 Model I

To be explicit, we need to specify the model further. Let us first consider the $Z(2)$ Higgs gauge model on a 3D cubic lattice,

$$
E_I = -\lambda \sum_x \sum_{\mu=1}^3 S_{x+\mu} J_{x\mu} S_x - \frac{1}{g^2} \sum_x \sum_{\mu<\nu} J_{x\mu} J_{x+\mu,\nu} J_{x+\nu,\mu} J_{x\nu},
$$

$$
Z_I = \prod_x \sum_{S_x = \pm 1} \prod_{\mu, \mu' \neq \pm 1} \exp(-\beta E_I) \equiv \exp(-\beta F_I),
$$

where $x$ denotes the lattice site on which $S_x$ lives, and $\mu (= 1, 2, 3)$ denotes both the direction and the unit vector. We consider only the connections between the nearest-neighbor sites $(x, x + \mu)$ and treat them as symmetric $Z(2)$ variable on a link $(x, x + \mu)$; $J_{x, x + \mu} = J_{x + \mu, x} \equiv J_{x\mu} = \pm 1$. The $\lambda$ term and the $1/g^2$-term is depicted in Fig.3.
The time evolution of \( J_{x\mu} \) may be given by the similar rule as (3),

\[
S_x(t + \epsilon) = \text{sgn}\left[ -\frac{\partial E_I}{\partial S_x}(t) + \eta_x(t) \right],
\]

\[
J_{x\mu}(t + \alpha \epsilon) = \text{sgn}\left[ -\frac{\partial E_I}{\partial J_{x\mu}}(t) + \zeta_{x\mu}(t) \right],
\]

where \( \alpha \) sets the ratio of the two time scales for \( S_x \) and \( J_{x\mu} \). We report our study of (8) elsewhere.

Below we study the phase diagram of \( E_I \) by using the mean field theory, which is formulated as a variational principle as follows. Let us introduce a variational energy \( E_0 \). Then the Jensen-Peierls inequality gives rise to

\[
F_I \leq F_0 + \langle E_I - E_0 \rangle_0,
\]

\[
Z_0 = \text{Tr} \exp(-\beta E_0) \equiv \exp(-\beta F_0),
\]

\[
\langle O \rangle_0 = Z_0^{-1} \text{Tr} O \exp(-\beta E_0),
\]

where \( \text{Tr} \) implies \( \prod_x \sum_{S_x = \pm 1} \prod_{x,\mu} \sum_{J_{x\mu} = \pm 1} \). We choose the variational parameters in \( E_0 \) so that the right-hand-side of inequality reaches the minimum. For \( E_0 \) we assume the translational invariance of mean fields and employ the single-site and single-link energy,

\[
E_0 = -\sum_x \sum_{\mu} W_{x\mu} J_{x\mu} - \sum_x h_x S_x,
\]

with the two variational parameters, \( W_{x\mu} = W \) and \( h_x = h \). The result is given in Table 2 and Fig. 4.

Table 2. Phases of Model I of (7).

| Phase  | \( \langle S_x \rangle \) | \( \langle J_{x\mu} \rangle \) | Memory | Learning | Hopfield Model |
|--------|----------------|----------------|--------|----------|---------------|
| Higgs  | \( \neq 0 \) | \( \neq 0 \) | yes  | yes  | Ferromagnetic |
| Coulomb | 0 | \( \neq 0 \) | no  | yes  | Paramagnetic |
| Confinement | 0 | 0 | no  | no  | not available |
As shown in the Table 2, one may take $\langle S_x \rangle$ as an order parameter to judge whether the system succeeds to recall definite patterns, and $\langle J_{x\mu} \rangle$ to judge whether the system is able to learn some new patterns. In the confinement phase, neither memory nor learning is possible. This phase is missing in the Hopfield model.

We note that Monte Carlo simulations of the 3D $\mathbb{Z}(2)$ Higgs gauge model exhibit these three phases, but the phase boundary of Higgs and confinement phases does not continue to $\beta/g^2 = 0$ but terminates at some finite value. These two phase can be reached each other smoothly by contouring the end point. This "complementarity" reflects that $|J_{x\mu}| = 1$ and is proved by rigorous treatment, but not predicted correctly in our variational treatment.

To what extent are these results trustworthy? To answer this question, we introduce and study other two models, Model II and Model III.

### 3.2 Model II

The framework and the energy $E_{II}$ of Model II is same as Model I, but we allows additional state $J_{x\mu} = 0$, which describes the possibility that the connection between $x$ and $x + \mu$ is missing:

$$E_{II} = E_I, \quad Z_{II} = \prod_x \sum_{S_x = \pm 1} \prod_{x, \mu} \sum_{J_{x\mu} = 0, \pm 1} \exp(-\beta E_{II}).$$

The phase diagram calculated by the similar mean field theory is shown in Fig.5. The global structure remains the same as Fig.4, although the region of the confinement phase is enlarged as expected since the added states clearly favor this phase.
3.3 Model III

In Model III, we introduce two independent $Z(2)$ variables $J_{x\mu}$ and $\bar{J}_{x\mu}$ for the synapse between $x$ and $x + \mu$ to take their independence into account as

$$J_{x\mu} \equiv J_{x,x+\mu}, \quad \bar{J}_{x\mu} \equiv J_{x+\mu,x}. \quad (12)$$

We also define $J_{x,-\mu} \equiv J_{x-\mu,x}$. The energy $E_{III}$ is then given by

$$E_{III} = -\lambda \sum_x \left( \sum_{\pm \mu} J_{x,\pm \mu} S_{x,\pm \mu} \right) \left( \sum_{\pm \nu} J_{x,\pm \nu} S_{x,\pm \nu} \right) - \frac{1}{g^2} \sum_x \sum_{\mu<\nu} [J_{x\mu}\bar{J}_{x+\mu,\nu}J_{x+\nu,\mu}\bar{J}_{x\nu} + (\mu \leftrightarrow \nu)]. \quad (13)$$

We note that the expression $J_{ij}S_iS_j$ in $E_{II}, I, II$ washes out the asymmetry $J_{ij} \neq J_{ji}$, while the first term of (13) reflects it. Each term in $E$ is depicted in Fig.6.

Figure 6. Graphical representation of Model III of (13). (a) $J_{x\mu}$ and $\bar{J}_{x\mu}$; (b) $\lambda$ term; (c) $g^{-2}$ term.
The phase diagram in mean field theory is shown in Fig. 7. The global structure still remains unchanged, although the region of the confinement phase is diminished considerably. This may be understood since the first term in $E_{III}$ is bilinear in $J_{ij}$ in contrast with $E_{H, I, II}$, and favors nonvanishing $J_{ij}$.

4 Summary and outlook

Our results may be summerized with some future outlook as follows;

- Due to the dynamical variables $J_{ij}$, a new (confinement) phase appears at high temperatures, which describes the new state of no ability of learning and memory.

- To describe the spin-glass phase, further study of long-range correlation and/or frustrations is necessary.

- Relaxing of $J_{ij} = \pm 1(, 0)$ to $-\infty < J_{ij} < \infty$ may be interesting, but requires a detailed form of the energy.

- Study of the time evolution of $J_{ij}$ and $S_i$ may describe the mechanism of learning such as the process to forget the patterns.

- Study of the effect of local gauge symmetry on brain function on a "quantum" level is interesting. Introduction of gauged versions of quantum brain dynamics \cite{k} and cellular automata with Penrose’s idea \cite{p} may be the first step.
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