On independent permutation separability criteria

Lieven Clarisse
Department of Mathematics, The University of York, Heslington, York YO10 5DD, U.K.

Paweł Wocjan
Computer Science Department & Institute for Quantum Information, California Institute of Technology, Pasadena, CA 91125, USA

Recently P. Wocjan and M. Horodecki [quant-ph/0503129] gave a characterization of combinatorially independent permutation separability criteria. Combinatorial independence is a necessary condition for permutations to yield truly independent criteria meaning that that no criterion is strictly stronger than any other. In this paper we observe that some of these criteria are still dependent and analyze why these dependencies occur. To remove them we introduce an improved necessary condition and give a complete classification of the remaining permutations. We conjecture that the remaining class of criteria only contains truly independent permutation separability criteria. Our conjecture is based on the proof that for two, three and four parties all these criteria are truly independent and on numerical verification of their independence for up to 8 parties. It was commonly believed that for three parties there were 9 independent criteria, here we prove that there are exactly 6 independent criteria for three parties and 22 for four parties.

I. INTRODUCTION

The field of quantum computation and information relies heavily on the existence of the entanglement phenomenon. Yet the basic question, whether a given multipartite quantum state is entangled or not, remains essentially open. Mathematically, a state is not entangled or separable if it can be expressed as a convex combination of product states \( \sum_i p_i \rho_i \otimes \cdots \otimes \rho_r \), with \( p_i > 0 \) and \( \sum_i p_i = 1 \). Much work has been done computationally in devising an efficient algorithm, which would tell whether the state is entangled or separable. Notable is the work by Doherty et al. [2, 3, 4], where a nested set of entanglement criteria is constructed, which in the limit of infinite tests detects every entangled state. When the state is separable, the algorithm never ends. Thus the convex set of separable states is iteratively approximated from the outside. The dual approach has been formulated in Ref. [5], which works from the inside, detecting separability. An independent two-way algorithm for detecting separability was devised in Ref. [6]. Yet, despite these advances, the separability problem is known to be NP-hard [7]. For the state of the art in these computational approaches the reader is referred to Ref. [8] and references therein.

Although these computational criteria solve the problem in principle, they are analytically hard to work with and lack a clear physical interpretation. Ideally we would also like to have a simple set of operational criteria detecting most of the entangled states, with a nice physical interpretation. The prime example of such a criterion is the partial transpose, originally introduced by Peres [8]. States not violating this criterion have been shown to be undistillable or bound entangled [10]. The partial transpose criterion is basically a transpose operation in one or more of the parties of the total system. Writing down the quantum state in a specific basis, this amounts to rearranging the matrix entries. The cross norm or realignment criterion [11, 12] works similarly and is independent of the partial transpose criterion. In particular it can detect bound entangled states. Some analytical properties of the realignment criterion have been studied in Ref. [13]. In Ref. [14] a unified approach to these two criteria was presented, which can be extended to multipartite systems. We refer to these criteria as permutation criteria.

Let us briefly recall the essential part of these criteria. For simplicity, consider a bipartite system and the operation \( T \) which performs a transpose of the second subsystem, that is

\[
|i\rangle \langle j| \otimes |k\rangle \langle l| \xrightarrow{T} |i\rangle \langle j| \otimes |l^*\rangle \langle k^*|.
\]

Where we denote vectors as \( |l^*\rangle \) for the complex conjugate of \( |l\rangle \). Due to linearity, this operation is well defined for arbitrary quantum states. A state \( \rho \) is entangled if the trace norm \( \|T(\rho)\| > 1 \). Note that the usual partial transpose criterion says that a state is entangled if \( T(\rho) \) has some negative eigenvalues, but as \( T(\rho) \) is Hermitian this is equivalent to saying that \( \|T(\rho)\| > 1 \). This was first observed in Ref. [15], where it was shown that the quantity \( \|T(\rho)\| \) gives rise to a good entanglement measure. The realignment criterion corresponds to the operation \( R \) which acts as

\[
|i\rangle \langle j| \otimes |k\rangle \langle l| \xrightarrow{R} |i\rangle \langle k^*| \otimes |j^*\rangle \langle l|.
\]

Again, we have that a state \( \rho \) is entangled if \( \|R(\rho)\| > 1 \). Other bipartite permutation criteria can be constructed,
but they turn out to be equivalent to either of these two.

For more than two parties the classification of inequivalent permutation criteria is an open problem. First steps towards such a classification have been made in Ref. \cite{14,16,17}. For three parties it was implied that there are 9 inequivalent criteria \cite{14,16,17}, and for four parties at most 34 inequivalent criteria \cite{16}.

The aim of this paper is to improve upon these results. The paper is structured as follows. In section 2 we review the work of Wojcik and Horodecki \cite{17} and introduce their graphical notation. Section 3 is devoted to our main results; we show that the class of the so-called combinatorially independent permutation criteria contains some equivalent criteria which always occur in pairs. We completely classify these criteria and give a new upper bound of the number of independent criteria (this is Theorem 3). In particular we find that there are at most 6 independent criteria for three parties and 22 for four parties. In section 4 we show that for 2, 3 and 4 parties the criteria from Theorem 3 are truly independent, in the sense that no criterion is strictly stronger than any other criterion. Finally, we discuss our results and argue that there are most likely no more equivalences in the criteria from Theorem 3.

II. NOTATION AND PREVIOUS RESULTS

We start this section with a formal definition of the permutation criteria. Consider an \( r \)-party state belonging to a Hilbert space \( H \), whose subsystems have the same dimension \( d \). A general state \( \rho \in H \) can be expanded as

\[
\rho = \sum_{i_1, i_2, \ldots, i_{2r}} \rho_{i_1 i_2 i_3 i_4 \ldots i_{2r-1} i_{2r}} |i_1 i_2 \ldots i_{2r-1} i_{2r} \rangle \langle i_1 i_2 \ldots i_{2r-1} i_{2r} |,
\]

where all indices run from 1 to \( d \).

Let \( S_{2r} \) denote the symmetric group with \( (2r)! \) elements, that is, the group of the permutations of the set \( \{1, 2, \ldots, 2r\} \). We define for each permutation \( \sigma \in S_{2r} \), a corresponding map \( \Lambda_\sigma \) on operators acting on \( H \) as

\[
[\Lambda_\sigma(\rho)]_{i_1 i_2 \ldots i_{2r-1} i_{2r}} = \rho_{i_\sigma(1) i_\sigma(2) \ldots i_\sigma(2r-1) i_\sigma(2r)}.
\]

We will represent permutations as \( [\sigma(1) \sigma(2) \ldots \sigma(2r)] \) or in disjoint cycles (an example below will make this clear). In Ref. \cite{14} it was shown that every permutation \( \sigma \in S_{2r} \) gives rise to an entanglement criterion. Namely, a state \( \rho \) is entangled, if \( \| \Lambda_\sigma(\rho) \| > 1 \) for any permutation \( \sigma \in S_{2r} \), where \( \| A \| = \text{Tr}(A A^\dagger)^{1/2} \) denotes the trace norm.

Let us illustrate these definitions for bipartite quantum states. With the notation introduced above the partial transpose criterion corresponds to the permutation \([1 2 4 3]\) = \( (3, 4) \) while the realignment criterion corresponds to the permutation \([1 3 2 4]\) = \( (2, 3) \).

**Definition 1 (Independent permutation criteria).** Let \( \sigma \) and \( \mu \) be two permutations in \( S_{2r} \). We call the corresponding entanglement criteria \( \sigma \) and \( \mu \) dependent if and only if

\[
\| \Lambda_\sigma(\rho) \| = \| \Lambda_\mu(\rho) \|,
\]

for all quantum states (that is, positive operators with trace 1). Else, the permutation criteria are called independent.

This definition is motivated by the fact that independence is a necessary condition for two permutations to yield truly independent entanglement criteria. This condition extends the necessary condition of combinatorial independence introduced in \cite{17}. The definition of combinatorial independence is very similar to the definition of independence. The decisive difference is that two permutations are combinatorially independent if and only if equality in (1) is achieved for all operators and not only quantum states. In this case, the maps \( \Lambda_\sigma \) and \( \Lambda_\mu \) are related by a norm-preserving map. A norm preserving map \( \Lambda \) is a map such that for any operator \( A \) (not only density operators), \( \| A \| = \| \Lambda(A) \| \). Moreover, it can be shown that if \( \Lambda_\sigma \) and \( \Lambda_\mu \) are related via a norm-preserving map \( \Lambda \), then \( \Lambda = \Lambda_\mu \) must come from some permutation \( \nu \). We call such a permutation norm-preserving permutation.

An example of such a norm-preserving permutation is the global quantum transpose (GQT), which transposes the complete system. It can be written as \( \tau = (1, 2)(3, 4) \cdots (2r-1, 2r) \). Another example of a norm preserving map \( \Lambda \) is the “unitary” transformation \( \Lambda(\rho) = U \rho V \), where \( U \) and \( V \) are unitary operators. Reordering the different parties in the density matrix representation is an example of such a “unitary” transformation. Consider for instance the transformation

\[
\rho = \sum_i \rho_i^A \otimes \rho_i^B \rightarrow \rho' = \sum_i \rho_i^B \otimes \rho_i^A.
\]

This mapping can be implemented by means of multiplication on the left and on the right by the swap operator \( T \).

To illustrate the necessary condition of combinatorial independence let us consider again a bipartite system. The operation \( R' \) induced by the permutation \( (1, 4) \) gives rise to the same criterion as \( R \) induced by the permutation \( (2, 3) \). Indeed, the permutation \( R' \tau \) (here and elsewhere permutations are evaluated from left to right) transforms \([1 2 3 4]\) into \([2 4 1 3]\), which up to reordering of the parties is equivalent to \([1 3 2 4]\). This is just the transformation defined by \( R \).

**Theorem 1 (Combinatorially independent criteria \cite{17}).** The group \( T \) of norm preserving permutations can be generated as

\[
T = \langle (2k, 2l), (2k - 1, 2l - 1), \tau \rangle,
\]

where \( 1 \leq k, l \leq r \) and \( \tau \) as before denotes the GQT. The combinatorially independent permutation criteria correspond to the right cosets \( S_{2r} / T \) of \( T \) in \( S_{2r} \). The number of independent permutation criteria is therefore not
larger than $\frac{1}{2}(\binom{r}{2})$. In this number, the class of trivial norm preserving criteria is also counted.

In the same paper, the authors also devised a graphical notion of the criteria, which leads to a way of selecting a simple representative for the right cosets. They decompose any permutation as a combination of 4 elementary permutations: the partial transpose, two types of realignment or reshuffling, and the identity. The corresponding graphical notations are loops from a subsystem to itself, arrows from one subsystem to another and free subsystems (no loops or arrows), as graphically depicted in Figure 1. We call $k$ the head and $l$ the tail if there is an arrow from $k$ to $l$. If there is a loop in $m$, then $m$ is both head and tail of the loop. We call the support of an arrow or a loop the set containing its head and tail. A configuration of arrows and loops is called disjoint, if the supports of all pairs and loops are disjoint.

**Theorem 2 (Canonical representation of combinatorially independent criteria)**. The right cosets $S_{2r}/T$ can always be represented by a disjoint configuration of arrows. All criteria that can be represented in this manner are combinatorially independent up to reversing the direction of all arrows and replacing loops by free subsystems and vice versa.

**III. MAIN RESULTS**

It was shown in Ref. [17] that two permutations $\mu$ and $\sigma$ are combinatorially dependent if there is a norm preserving permutation such $\Lambda$ that $\Lambda_\sigma(A) = \Lambda_\mu(A)$ for all operators $A$. One could assume that therefore Theorem 1 cannot be sharpened, that is, all combinatorially independent criteria are independent. This is true when one considers the permutation acting on arbitrary operators. In quantum mechanics however, we deal with positive operators, which are Hermitian. The following theorem exploits this fact and counts the new upper bound to the number of independent criteria.

**Theorem 3.** (i). The permutation criteria corresponding to the permutations $\sigma$ and $\tau\sigma$ are dependent. (ii). Let $Z := \{e, \tau\}$ be the subgroup of $S_{2r}$ generated by the QGT $\tau$. Define the action of $Z$ on the right cosets $S_{2r}/T$ by multiplication from the left of the cosets, that is, $e * \sigma T = e \sigma T$ and $\tau * \sigma T = \tau \sigma T$. The new upper bound on the number of independent criteria is the number of orbits under this action. It is given by $[21]$

$$\frac{1}{4} \left[ \binom{2r}{r} + 2^r + \binom{r}{r/2} \cdot \text{even}(r) \right], \quad (2)$$

where even($r$) = 1 if $r$ is even and 0 otherwise. This number includes the trivial criterion given by the identity permutation.

**Proof.** To prove (i), let us apply the permutation $\tau\sigma$ on an arbitrary quantum state $\rho$. We have

$$\|\Lambda_{\tau\sigma}(\rho)\| = \|\Lambda_{\sigma}(\Lambda_{\tau}(\rho))\| = \|\Lambda_{\sigma}(\rho^T)\| = \|\Lambda_{\sigma}(\rho)\| = \|\Lambda_{\sigma}(\rho)\|.$$  

Here we have used that $\rho^T = \bar{\rho}$ because $\rho$ is Hermitian and $\Lambda_{\sigma}(\bar{\rho}) = \Lambda_{\sigma}(\rho)$ because $\Lambda$ only permutes the entries of the matrix.

Observe that multiplying a coset $\sigma T$ by $\tau$ from the left is the same as conjugating it by $\tau$ because $\tau$ is contained in $T$, that is, we have $\tau \sigma T = \tau \sigma T \tau$.

It is readily verified that conjugation of a permutation by $\tau$ corresponds to exchanging heads (always odd numbers) and tails (always even numbers). Therefore, the direction of all (true) arrows is reversed and loops and free subsystems are not affected. An example is shown in Fig. 2. Following the four rules presented in [17], we call this Rule 5. The new rule either glues two criteria together or does not change them. More precisely, the orbits under the action of $Z$ have size 1 or 2. This is because $\tau$ is an involution

$$\sigma T \rightarrow \tau \sigma T \rightarrow \tau \tau \sigma T = \sigma T.$$  

(ii). There are $\frac{1}{2}(\binom{2r}{r})$ combinatorial independent criteria. With the new rule, there are at most

$$\frac{1}{2} \left( \binom{2r}{r} - \frac{1}{2} \left[ \binom{2r}{r} - K \right] \right),$$

independent criteria left. Here $K$ denotes the number of criteria not affected by conjugating by the new rule. Now note that the only criteria (represented as disjoint arrow configurations) not affected by conjugation with $\tau$ are

1. criteria with no arrows and
2. criteria containing only arrows and having no free subsystems.

If $r$ is odd, then situation (2) cannot occur. The number of these criteria are readily counted:

$$\binom{r}{0} + \binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{\lfloor (r/2) \rfloor} = 2^{r-1},$$

where we have used an identity of binomial coefficients. So that in the case of an odd number of subsystems, the number of criteria becomes (including the identity)

$$\frac{1}{4} \left[ \binom{2r}{r} + 2^r \right].$$

In the case $r$ is even we need to take care of situation (2). Now this number equals picking $r/2$ heads from $r$
choices, because exchanging tails does not matter. But we have to divide by two since exchanging all heads and tails does not matter either, so that the number of criteria satisfying (2) is given by
\[
\frac{1}{2} \left[ \binom{r}{r/2} \right].
\]
We conclude that in the case \(r\) is even, the number of criteria is given by (including the identity)
\[
\frac{1}{4} \left[ \binom{2r}{r} + 2^r + \binom{r}{r/2} \right].
\]
To complete the proof, we have to show that the criteria in the orbits of size 2 are combinatorially independent. Let \(\sigma\) be a permutation represented by a disjoint arrow configuration. Assume that there is an arrow from subsystem \(k\) to \(l\) in the disjoint configuration of \(\sigma\). Then there is an arrow from \(l\) to \(k\) in the disjoint configuration describing \(\tau \sigma \tau\). Loops and free subsystems are not affected. Using these observation we see that the configuration describing \(\sigma \tau \sigma \tau\) has a closed path from \(k\) to \(l\) and no loops. The closed path between \(k\) and \(l\) can be transformed into a loop on \(k\) and a loop on \(l\) with the help of Rule 3 (Exchanging heads) in Ref. 17. Now if we apply these arguments to all arrows of \(\sigma\) we see that the permutation \(\sigma \tau \sigma \tau\) is not norm-preserving. Consequently, the permutations \(\sigma\) and \(\tau \sigma \tau\) are combinatorial independent. This concludes the proof.

FIG. 2: Action of the new rule on a disjoint arrow configuration: the first equivalence corresponds to the new rule and the second corresponds to Rule 4 in [17].

IV. ILLUSTRATIONS

In this section we will illustrate the permutation criteria for two, three and four parties. The different criteria are shown graphically in Figure 3. Here loops (partial transpose) are depicted by a little circle, solid lines represent the first type of reshuffling and dotted lines, the second type of reshuffling. In this section we go further and show that the criteria from Theorem 3 are truly independent: no criteria detects strictly more states than any other criteria.

A. Two parties

For a quantum system consisting of two parties, there are only two non trivial inequivalent permutation criteria: the partial transpose in one of the subsystems and reshuffling between the two subsystems. For the low dimensional systems \(H \cong \mathbb{C}^2 \otimes \mathbb{C}^2\) and \(H \cong \mathbb{C}^2 \otimes \mathbb{C}^3\), the positivity of the partial transpose is a necessary and sufficient condition for separability [18], while this is not the case for the realignment criterion [12, 13]. For higher dimensional systems these criteria are truly independent. We have tested the realignment criterion on all known bound entangled states \(\rho \in H \cong \mathbb{C}^3 \otimes \mathbb{C}^3\) in the literature. The maximum value we have found for \(\| R(\rho) \|\) is 7/6 and is achieved for a particular chess-board state.
FIG. 3: Independent permutation criteria for a) two, b) three, and c) four particles. Picture adapted from Ref. [17].

For three parties we have proven that only 6 criteria are independent, previous work [16, 17] indicated that there were 9. The 6 criteria are the partial transposes (row QT) in the 3 subsystems and the 3 reshufflings (row R) between any of the two subsystems.

To show that all the criteria from row QT are independent, it is sufficient to note that there exist tripartite states which are separable with regard to two splits, but not to the third one (for an example with qubits, see Ref. [20]). It has been proven [13] that the trace norm of the realigned density matrix remains invariant when an uncorrelated ancilla is added. Now take a bipartite entangled state which violates the realignment criterion but not the partial transpose criterion (such as $\rho_c$). By adding an uncorrelated ancilla and reordering the systems, we can construct states that are detected by one criterion from row R only. These states are trivially not detected by any criterion from row QT as they are bound entangled.

Note that the realignment criterion, in contrast to the partial transpose criterion can detect genuine tripartite entangled states, that is, entangled states that are separable under any bipartite cut. This has been demonstrated in Ref. [14] with the tripartite bound entangled states from Ref. [21].

For four parties, there are at 22 non trivial independent permutation criteria. As for three parties, it is trivial to construct states that are only detected by one partial transpose criterion (that is a criterion with only loops). Again, each criterion with at least one realignment is truly independent of the transpose criteria because the trace norm of the realigned density matrix remains invariant when an uncorrelated ancilla is added.

To show true independence within the set of realignment criteria (rows R, R+QT, 2R and R+R'), let us first
consider the rows R and R+QT. Using states with a negative partial transpose, it is very easy to construct examples to show that these criteria are independent from each other and from the rows R and R’. We verified this using a random search over the state space on 4 qubits (using the algorithm outlined in Ref. [22]). In the same way it can be checked that the criteria from the rows 2R and R+R’ are independent from each other. To show that they are also independent from the rows R and R’, it can be verified that they detect states of the form
\[ \rho = (1 - \beta) \rho_c \otimes \rho_c + \beta I / 81 \]
for a larger parameter range of \( \beta > 0 \).

V. DISCUSSION

In Ref. [14] a powerful class of separability criteria was devised based on permutations. The class however contained many redundancies, and to give a complete characterization of the independent criteria is an open problem. In Ref. [17], a graphical representation for permutations together with rules for simplifying them were introduced based on a sufficient condition for two permutations to yield dependent criteria. This equivalence meant that two combinatorially dependent criteria yield the same value of the trace norm on all operators. Combinatorially, density operators have a prominent Hermitian symmetry, that is a global quantum transposition
\[ V \]  
binatorially, density operators have a prominent Hermitian symmetry, that is a global quantum transposition
\[ V \]  
Density operators differ from arbitrary operators also in the fact that they have positive eigenvalues. But since permutations only reorder matrix entries, it is not very likely that this positiveness would lead to more criteria to be dependent. We have verified the independence of the criteria numerically on a random state for 2 upto 8 parties.

In the same way as we illustrated for three and four parties, it is easy to see that the criteria with only loops (only partial transpositions) are independent. These criteria are independent from the ones having at least one realignment since those can detect bound entangled states. To prove the independence within the class of criteria with at least one realignment one could try to generalize the arguments from Section 4.

Acknowledgments

L. C. is supported by a WW Smith Scholarship. P. W. is supported by the National Science Foundation under the grant no. EIA 0086038. L. C. would like to thank Anthony Sudbery for careful reading of the manuscript and helpful suggestions. L. C. also thanks Christine Aronsen Storebø for very helpful discussions.

[1] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Physical Review A 40, 4277 (1989).
[2] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Distinguishing separable and entangled states, Physical Review Letters 88, 187904 (2002), quant-ph/0112007.
[3] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Complete family of separability criteria, Physical Review A 69, 022308 (2004), quant-ph/0308032.
[4] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Detecting multipartite entanglement, Physical Review A 71, 032333 (2005), quant-ph/0407143.
[5] F. Hulpke and D. Bruß, A two-way algorithm for the entanglement problem (2004), quant-ph/0407179.
[6] J. Eisert, P. Hyllus, O. Gühne, and M. Curty, Complete hierarchies of efficient approximations to problems in entanglement theory, Physical Review A 70, 062317 (2004), quant-ph/0407135.
[7] L. Gurvits, in Proceedings of the 35 ACM symposium on Theory of computing (New York, 2003), pp. 10–19 (see quant-ph/0303055 for the long version).
[8] P. Badziąż, P. Horodecki, and R. Horodecki, Towards ef- ficient algorithm deciding of distributed quantum states (2005), quant-ph/0504041.
[9] A. Peres, Separability criterion for density ma- trices, Physical Review Letters 77, 1413 (1996), quant-ph/9604005.
[10] M. Horodecki, P. Horodecki, and R. Horodecki, Mixed-state entanglement and distillation: is there a ‘bound’ entanglement in nature?, Physical Review Letters 80, 5239 (1998), quant-ph/9801069.
[11] O. Rudolph, Further results on the cross norm criterion for separability (2002), quant-ph/0202121.
[12] K. Chen and L. A. Wu, A matrix realignment method for recognizing entanglement, Quantum Information and Computation 3, 193 (2003), quant-ph/0205017.
[13] O. Rudolph, Some properties of the computable cross norm criterion for separability, Physical Review A 67, 032312 (2003), quant-ph/0212047.
[14] M. Horodecki, P. Horodecki, and R. Horodecki, Separabil- ity of mixed quantum states: linear contractions approach (2002), quant-ph/0206008.
[15] G. Vidal and R. F. Werner, A computable measure of entanglement (2001), quant-ph/0102117.
[16] H. Fan, A note on separability criteria for multipartite state (2002), quant-ph/0210168.
[17] P. Wocjan and M. Horodecki, Characterization of combinatorically independent permutation separability criteria (2005), quant-ph/0503129.
[18] M. Horodecki, P. Horodecki, and R. Horodecki, Separabil- ity of mixed states: necessary and sufficient conditions, Physics Letters A 223, 1 (1996), quant-ph/9605038.
[19] D. Bruß and A. Peres, Construction of quantum states with bound entanglement, Physical Review A 61, 30301 (2000), quant-ph/9911056.
[20] W. Dür and J. I. Cirac, Classification of multi-
qubit mixed states: separability and distillability properties, Physical Review A 61, 042314 (2000), quant-ph/9911044

[21] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Unextendible product bases and bound entanglement, Physical Review Letters 82, 5385 (1999), quant-ph/9808030

[22] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, On the volume of the set of mixed entangled states, Physical Review A 58, 883 (1998), quant-ph/9804024

[23] N. J. A. Sloane, The on-line encyclopedia of integer published electronically at: http://www.research.att.com/~njas/sequences/

[24] The integer sequence generated by the number of independent criteria equals the integer sequence A072377 from Ref. 23.