THE BREADTH-DEGREE TYPE OF A FINITE $p$-GROUP

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Abstract. In the present paper we show that a stem finite $p$-group $G$ has size bounded by $\min (p^{8d-2 \log_2 d + b-4}(b+1)/2, p^{3b+4d-1}/2)$ where $b$ is the breadth of $G$ and $p^b$ is the maximum character degree of $G$. As a consequence there are only finitely many finite stem $p$-groups having breadth $b$ and maximum character degree $p^b$.

Introduction

The problem of finding all finite groups of a given order is quite old. In the case of finite $p$-groups (where $p$ is a prime number) the task is particularly hard as the number of isomorphism classes grows rapidly with the order. This particular difficulty led Philip Hall, aiming at achieving a systematic classification theory, as he himself writes in his famous paper [12], to introduce the concept of isoclinism, an equivalence relation among groups preserving the commutation structure. A group $T$ such that its center $Z(T)$ is contained in the derived subgroup $T'$ is said to be a stem group. Philip Hall showed that finite stem groups are exactly the ones of minimum order within an isoclinism class. He also pointed out that the ratio of the number of conjugacy classes of a given size of a finite group $G$ with respect to the order of $G$ is preserved by isoclinism. In [12] it is also stated that the ratio of the number of irreducible characters of $G$ of a given degree with respect to the order of $G$ is invariant under isoclinism. A proof of this result is given in [22, Theorem D]. Adding up over all the possible lengths of conjugacy classes (or over all the possible degrees of the irreducible characters) we see that also the commutativity degree $k(G)/|G|$ of a finite group $G$ is invariant under isoclinism. This number is known to be the probability that two randomly chosen elements commute (see [6, 10, 11, 20, 26]).

Another invariant under isoclinism which has been extensively studied is the breadth $b(G)$ of a finite $p$-group $G$. It is defined as the maximum integer $b$ such that there exists a conjugacy class in $G$ of size $p^b$. A finite $p$-group has breadth 1 if and only if its derived subgroup has order $p$. This result is usually attributed to H-G. Knoche [17], although Rolf Brandl recently made the authors aware that it was already known to W. Burnside [3, Chap VIII §99 p. 126, Ex. 1]. The breadth $b$ of a $p$-group $G$ gives information on the structure of $G$: in [27] M.R. Vaughan-Lee
showed that the derived subgroup of $G$ has order bounded by $p^{\frac{b(b+1)}{2}}$. A famous conjecture attributed to Peter M. Neumann stated that the nilpotency class $\text{cl}(G)$ of $G$ is at most $b(G) + 1$. Known bounds are $\text{cl}(G) \leq \frac{p^b}{p^b - 1} b(G) + 1$ by C. R. Leedham-Green, P. M. Neumann and J. Wiegold [19] and $\text{cl}(G) \leq \frac{5}{4} b(G) + 1$ for $p = 2$ by M. Cartwright [4]. Some counterexamples to the original conjecture have been provided by V. Felsch [7] and by W. Felsch, J. Neubüser and W. Plesken [8] and B. Eick et al [5]. A. Mann proved in [23] that the subgroup of a finite $p$-group generated by the elements having minimal positive breadth has nilpotency class at most 3. Finite $p$-groups of breadth 2 and 3 have been studied in [9, 24].

The breadth alone, however, is far from being a sufficient parameter in the attempt to enumerate the isoclinism classes of finite $p$-groups: indeed in Section 4, we see that, given an integer $b \geq 1$, it is not difficult to construct infinitely many isoclinism classes of finite $p$-groups having the same breadth $b$.

In a somewhat dual fashion one can consider the set $\text{Irr}(G)$ of irreducible characters of $G$ and the representation exponent $e(G)$ defined by the equality $p^{e(G)} = \max \{ \chi(1) \mid \chi \in \text{Irr}(G) \}$. This is also an isoclinism invariant. Trivially $e(G) = 0$ if and only if $G$ is abelian. The $p$-groups $G$ with $e(G) = 1$ have been characterized in [13] where it is shown that a necessary and sufficient condition for $e(G)$ to be equal to 1 is that either $G$ has a maximal subgroup which is abelian or that the center $Z(G)$ has index in $G$ at most $p^3$ (both conditions can hold at the same time, though). Incidentally this last condition implies that $G$ has breadth 2, whereas, on the contrary, there are $p$-groups of arbitrarily large breadth with a maximal subgroup which is abelian. The $p$-groups with representation exponent at most 2 have been studied in [25].

In Section 4 we show that the representation degree alone is not suitable for an enumeration attempt of the isoclinism classes of finite $p$-groups (see (4.1) below).

S.R. Blackburn showed in [2, Theorem 1] that the number of $p$-groups of given order within an isoclinism class is relatively small with respect to the number of all finite $p$-groups of that order, so that we expect to find “not too many” stem groups within an isoclinism class.

In this paper we define the breadth-degree type of a finite $p$-group $G$ to be the pair $(b(G), e(G))$. We approach the problem of giving a bound for a stem group of given breadth-degree type in two different ways giving two quadratic bounds respectively in Corollary 2.4 and in Theorem 3.7.

In Theorem 3.7 we prove that a stem $p$-group of given breadth-degree type $(b,d)$ has order bounded by $p^{\frac{b(b+4d-1)}{2}}$. It follows that there are only finitely many stem $p$-groups of given breadth-degree type. This suggests that the breadth-degree type is a parameter to be taken into account in any enumeration procedure (up to isoclinism) of classes of finite $p$-groups.

1. Notation and preliminaries

The groups considered in this paper will be finite $p$-groups where $p$ is a prime. We will use standard notation. In particular $\gamma_i(G)$ and $Z_i(G)$ will denote respectively the $i$-th term of the lower and the upper central series of $G$. If $x$ and $y$ are two elements of a group then the commutator $x^{-1}y^{-1}xy$ will be denoted by $[x,y]$ and when $H$ is a subset of $G$ the notation $[H,x]$ will be used to denote the set $\{ [h,x] \mid h \in H \}$. When $H \leq G$ is a subgroup and $x \in Z_2(G)$ we have that
[H, x] is a subgroup of the center of G. The set of irreducible characters of G will be denoted by Irr(G). The breadth \( b_G(g) \) of an element \( g \in G \) is defined by \( |g^G| = p^{b_G(g)} \) where \( g^G = \{ x^{-1}gx \mid x \in G \} \) is the conjugacy class of \( g \) in \( G \) and \( b(G) = \max \{ b_G(g) \mid g \in G \} \) is said to be the breadth of the group \( G \). The subscript is often omitted where there is no ambiguity and we will write \( b_G(g) \) to mean \( b_G(g) \). We denote by \( B_i(G) \) the subgroup of \( G \) generated by the elements whose breadth is at most \( i \). The representation exponent \( e(G) \) of \( G \) is defined by \( p^{e(G)} = \max \{ \chi(1) \mid \chi \in \text{Irr}(G) \} \). The number of the conjugacy classes of \( G \) is denoted by \( k(G) \). We shall say that a finite \( p \)-group \( G \) has breadth-degree type \( \text{bdt}(G) = (b, d) \) if \( b = b(G) \) and \( d = e(G) \).

We remind the reader that two finite groups \( G \) and \( H \) are said to be isoclinic if there are isomorphisms \( \alpha : G/Z(G) \rightarrow H/Z(H) \) and \( \beta : G' \rightarrow H' \) such that \( \beta([g, g']) = [h, h'] \) for all \( (g, g') \in G \times G \) and \( (h, h') \in H \times H \) satisfying the conditions \( hZ(H) = \alpha(gZ(G)) \) and \( h'Z(H) = \alpha(g'Z(G)) \). Isoclinism is an equivalence relation among finite groups. A group \( T \) is said to be stem if its center \( Z(T) \) is contained in the derived subgroup \( T' \) and stem groups are exactly the groups having minimum order in some isoclinism class. In the present paper we shall need to compare the order of a \( p \)-group \( G \) with the order of stem group in the isoclinism class of \( G \); the following remark will be useful to this purpose.

**Remark 1.1.** Let \( G \) be a finite group. Since \( |Z(G) \cap G'| = |G'| / |G'/Z(G)| = |G' / ([G'/Z(G)])'| \) we see that \( |Z(G) \cap G'| \) is invariant under isoclinism. Clearly \( |G| \geq |G : Z(G)| \cdot |Z(G) \cap G'| \) and the second member is invariant by isoclinism. Equality holds if and only if \( Z(G) \leq G' \), that is, \( G \) is a stem group. Thus, stem groups are precisely those of minimum order in their isoclinism class, hence \( |G : Z(G)| \cdot |Z(G) \cap G'| \) is the size of a stem group \( T \) in the isoclinism class of \( G \). In particular

\[
|G| = |G : Z(G)| \cdot |Z(G) : (Z(G) \cap G')| \cdot |Z(G) \cap G'|
\]

\[
= |T| \cdot |Z(G) : (Z(G) \cap G')|
\]

See also [1, Corollary 29.4].

It was already known to Philip Hall that the breadth and the representation degree of a finite group are invariant under isoclinism: two isoclinic \( p \)-groups have the same breadth-degree type.

We define the size \( \sigma(b, d) \) of the breadth-degree type \( (b, d) \) as

\[
\sigma(b, d) = \sup \{ |G| \mid G \text{ stem } p \text{-group, } \text{bdt}(G) = (b', d') \text{ with } b' \leq b \text{ and } d' \leq d \}.
\]

We will use, without further mention, the fact that if \( G \) is a finite \( p \)-group with \( \text{bdt}(G) = (b, d) \) and \( H \) is a quotient or a subgroup of \( G \) with \( \text{bdt}(H) = (b', d') \), then \( b' \leq b \) and \( d' \leq d \).

2. **Bounding the size of stem groups of fixed breadth-degree type**

We start with a lemma.

**Lemma 2.1.** Let the index of a subgroup \( H \) of a group \( G \) be finite and a product of \( k \) primes (counting multiplicities). If \( G \) is generated by \( H \) and a subset \( S \), then it is generated by \( H \) and \( k \) elements of \( S \).
Lemma 3.1. Let \( b \) be a subgroup. Assume that there exists an element \( g \) of breadth 1. To prove that \( \langle g, x \rangle \) contains \( H \) properly. Continuing in the same way, we construct a series \( H = H_0 < H_1 < \cdots < H_n = G \), such that for each \( i \) we have \( H_i = \langle H_{i-1}, x_i \rangle \), with \( x_i \in S \). Then \( G = \langle H, x_1, \ldots, x_n \rangle \), and \( n \leq k \).

\[
\text{Proof.} \quad \text{By Lemma 2.1,} \quad G \text{ contains a maximal abelian subgroup } H \text{ such that } |G : H| \text{ divides } p^{4d - \log_2 d - 2}. \quad \text{By Lemma 2.1,} \quad G \text{ is generated by } H \text{ and at most } 4d - \log_2 d - 2 \text{ elements } x_1, \ldots, x_n \text{ of breadth at most } b. \quad \text{Then } Z(G) = \cap_i C_G(x_i) \cap H \text{ has index at most } |G : H| \cdot |\Pi_i G : C_G(x_i)| \leq p^{(4d - \log_2 d - 2)(b + 1)}. \quad \Box
\]

Corollary 2.3. Under the assumptions of Theorem 2.2, assume also that \( G \) is a stem group. Then \( |G| \leq p^{n(n+1)/2} \) where \( n = (4d - \log_2 d - 2)(b + 1) \).

\[
\text{Proof.} \quad \text{By Lemma 2.1, a } p\text{-group } X \text{ in which } |X : Z(X)| = p^z \text{ satisfies the inequality } |X'| \leq p^{2(z-1)/2}. \quad \text{In our case } z \leq n \text{ and } Z(G) \leq G', \text{ hence } |G| \leq |G : Z(G)| |G'| \leq p^{(n(n+1)/2)}. \quad \Box
\]

Corollary 2.4. If \( G \) is a stem group of breadth-degree type \( (b, d) \), then \( |G| \leq p^{8d^2 - 2 \log_2 d + b - 4}(b + 1)/2 \). In other words \( \sigma(b, d) \leq p^{8d^2 - 2 \log_2 d + b - 4}(b + 1)/2 \).

\[
\text{Proof.} \quad \text{The proof mimics the one of the previous corollary by using the bound } |G'| \leq p^{b(b+1)/2} \text{ in 2.7.} \quad \Box
\]

Note that Corollary 2.4 requires that \( G \) has breadth \( b \) whereas Corollary 2.3 has the weaker hypothesis that \( G \) can be generated by elements of breadth at most \( b \).

3. A different bound

In this section we provide a bound for \( \sigma(b, d) \) different from the one in Corollary 2.4. This is done by finding a group \( K/L \), where \( L \triangleleft K \leq G \), such that \( b(K/L) < b(G) \), and applying induction.

Lemma 3.1. Let \( G \) be a non-abelian stem group and let \( b^* = \min_{g \in G \setminus Z(G)} b_G(g) \). The group \( G \) has a stem quotient of order \( \frac{|G|}{b^*} \), whose second centre contains an element of breadth 1.

\[
\text{Proof.} \quad \text{By Theorem 2.1, we can choose } g \text{ in } Z_2(G) \setminus Z(G) \text{ of breadth } b^*. \text{ Thus } [g, G] \text{ is a subgroup of } Z(G) \text{ of size } p^{b^*} \text{ and if we take a subgroup } N \text{ of } [g, G] \text{ of size } p^{b^* - 1}, \text{ the quotient } G/N \text{ has the requested size and } gN \text{ is an element of the second center of breadth 1. To prove that } G/N \text{ is stem, note that if } xN \text{ is in the centre of } G/N, \text{ then } [x, G] \leq N \text{ so that } b_G(x) < b^*. \text{ By the minimality of } b^*, \text{ we find that } x \in Z(G). \text{ Thus } x \in G' \text{ and } xN \in (G/N)'. \quad \Box
\]

Lemma 3.2. Let \( G \) be a finite group of breadth \( b \) and let \( N \) be a minimal normal subgroup. Assume that there exists an element \( g \in G \) such that \( [g, G] = N \). If \( M = C_G(g) \) then \( M \) is a maximal subgroup of \( G \) containing \( N \) and \( b(M/N) < b \). Moreover \( b_G(NyN) < b_G(y) \leq b \) for every \( y \in G \setminus M \) and \( [h, G] \geq N \) if \( b_G(h) = b \) and \( C_G(h) \not\leq M \).
Proof. Clearly $N$ is central and $b(g) = 1$, so that $M$ is maximal and contains $N$. Since $N$ is central, $[y,G] \cap N$ is a subgroup of $N$ for every $y \in G$. Provided that $[y,G] \cap N \neq 1$, this implies that $[y,G] \supseteq N$ and, consequently, $b_{G/N}(yN) < b_g(y) \leq b$. Note that this happens when $y \in G \setminus M$ for $[y,g]$ is a non-trivial element of $N$.

Let now $x$ be an element of $M$ and let $C = C_G(x)$. If $C \leq M$ then $b_{M/N}(xN) \leq b_M(x) < b_G(x) \leq b$. Otherwise $G = CM$ so that $x^G = x^M$. Thus $(xy)^G \supseteq (xy)^M = x^M g = x^G g$ and working modulo $N$ we get $b_{G/N}(xyN) \geq b_{G/N}(xN)$. Given $c \in C \setminus M$, we have that $[x,g,c] = [g,c]$ is a non-trivial element of $[x,g,G] \cap N$ and as above $b_{M/N}(xN) \leq b_{G/N}(xN) \leq b_{G/N}(xyN) < b_G(xy) \leq b$. This shows that $b(M/N) < b$.

Suppose that $h \in G$ and that $b_G(h) = b$ and $C_G(h) \not\leq M$. If $h \not\in M$, we have already noted that $[h,G] \supseteq N$. If $h \in M$, then $G = C_G(h)M$ implies that $[h,G] = [h,M]$ so that $b_G(h) = b$. Since $b_{M/N}(hN) \leq b(M/N) < b$, and this is possible only if $[h,M] \supseteq N$.

Part of the proof of item 2 of the following lemma has been inspired by [14, Proposition 4.1].

Lemma 3.3. Let $G$ be a non-abelian p-group of breadth-degree type $(b,d)$. Let $M \leq G$ be a maximal subgroup containing $Z(G)$, let $N \leq Z(G)$ be of order $|N| = p$ and set $C = C_G(G/N)$ and $D = C_{G/M}$. Then

1. If $x$ is chosen of minimal breadth $t = b_G(x)$ in the set $G \setminus M$ then $|Z(M) : Z(G)| \leq p^4$ and $|Z(M)M' : M'| \leq p^{2t} \cdot |Z(G)G' : G'|$. In particular $|Z(M)M' : M'| \leq p^{2t}$ when $G$ is stem.

2. $|C : Z(G)| = |G : D| \leq \chi(1)^2$ for all $\chi \in \text{Irr}(G)$ such that $N \not\leq \text{ker}\, \chi$.

3. If $|G : D| = \chi(1)^2$ for some $\chi \in \text{Irr}(G)$ such that $N \not\leq \text{ker}\, \chi$ then $C \cap D = Z(G), N \cap D' = 1$, $G' \leq ND'$ and $G = CD$ is a central product. If $\chi(1) > 1$ then $C' \neq 1$. Moreover if $G$ is stem then $D/N$ is stem, $D$ is isoclinic to $D/N$ and, provided that $\chi(1) > 1$, we have also $b(D) < b$.

4. If $C$ is not abelian then there exists $g \in Z_2(G)$ such that $b(g) = 1$ and $\min \{ b(x) \mid x \in G \setminus C_G(g) \} = 1$.

5. If $C$ is abelian then $b(G/N) = b - 1$ or $|G : D| = |C : Z(G)| \leq p^{\min(b,2d-1)}$. In particular $|Z(M)M' : M'| \leq p^{2b}$ and $|C : Z(G)| = |G : D| \leq p^{2d}$ when $G$ is stem.

Proof. The map $\varphi : m \mapsto [m,x]$ is an endomorphism of $Z(M)$. Since $M$ is a maximal subgroup in $G$ we have that $G = \langle x \rangle M$ and that the kernel of $\varphi$ is $Z(G)$. If we compute the breadth of $x$ we find $p^t = b(x) = |[G,x]| = |M,x| \geq |Z(M),x| = |\varphi(Z(M))| = |Z(M) : \text{ker}\, \varphi| = |Z(M) : Z(G)|$. Note that $G' = [x,M]M'$ implies that $|G' : M'| \leq |[x,M]| = p^t$. We have

$$|Z(M)M' : M'| \leq |Z(M)G' : M'| = |Z(M)G' : Z(G)G'| \cdot |Z(G)G' : G'| \cdot |G' : M'| \leq |Z(M) : Z(G)| \cdot |Z(G)G' : G'| \cdot |G' : M'| \leq p^{2t} \cdot |Z(G)G' : G'|.$$

If $G$ is stem then $Z(G) \leq G'$ so that $|Z(M)M' : M'| \leq p^{2t}$. This proves item 1.

Since $[G',Z_2(G)] = 1$ and since $C \leq Z_2(G)$ we have $G' \leq D$. Also $[y, G'] = [y, G]^p = 1$ for all $y \in G$ and $c \in C$, so that $\Phi(G) = G^pG' \leq D$ and $G/D$ is elementary abelian. As $N$ is cyclic of order $p$, the map $cZ(G) \mapsto [c, c] \in \text{Hom}(G/D, N) \cong G/D$ is easily seen to be an injective homomorphism of $C/Z(G)$ into $G/D$. Dually
the map \( gD \to [\cdot, g] \in \text{Hom}(C/Z(G), N) \cong C/Z(G) \) defines an injective homomorphism of \( G/D \) into \( C/Z(G) \). We have shown that \( |C : Z(G)| = |G : D| \).

As \( \cap_{\chi \in \text{Irr}(G)} \ker \chi = 1 \) there exists \( \chi \in \text{Irr}(G) \) such that \( N \not\subseteq \ker \chi \), in particular \( N \cap \ker \chi = 1 \) since \( N \) has order \( p \). If \( 1 \neq z \in N \leq Z(G) \) then \( \chi(z) = \chi(1) \vartheta \), where \( 1 \neq \vartheta \in \mathbb{C} \) is a \( p \)-th root of 1. Let \( y \in G \). If \( y \notin D \) there exists \( c \in C \) such that \( 1 \neq \vartheta \in \mathbb{C} \) is a \( p \)-th root of 1. Let \( y \in G \). If \( y \notin D \) there exists \( c \in C \) such that

\[
\chi(y) = \chi(y^c) = \chi(y [y, c]) = \chi(yz) = \vartheta \chi(y).
\]

It follows that \( 0 = (1 - \vartheta) \chi(y) \), which in turn forces \( \chi(y) = 0 \). We have shown that \( \chi(y) \neq 0 \) implies \( y \in D \) and as a consequence

\[
1 = \frac{1}{|G|} \sum_{y \in G} \chi(y) \chi(y) = \frac{1}{|G|} \sum_{y \in D} \chi(y) \chi(y) \leq \frac{1}{|G|} \sum_{y \in D} \chi(1)^2 = \frac{|D|}{|G|} \chi(1)^2,
\]

which yields \( |G : D| \leq \chi(1)^2 \). This completes the proof of item 2.

We deal now with item 3. The previous inequality is strict if and only if \( D \not\subseteq Z(\chi) \), in which case \( |\chi(y)|^2 < \chi(1)^2 \) for some \( y \in D \), thus \( D \leq Z(\chi) \). As above it is easy to see that, if \( c \in C \setminus Z(G) \), the fact that \( c \) has a non-trivial commutator in \( N \) implies \( \chi(c) = 0 \). This forces \( C \cap D = Z(G) \). The group \( CD \) has order \( |C| |D|/|C \cap D| = |C| |D|/|Z(G)| = |G| \) by item 2, so that \( G = CD \). Since \( |C, D| = 1 \) the group \( G \) is a central product and we have that the map \( C \times D \to G \) defined by \( (c, d) \to cd \) is an epimorphism. We deduce that \( \chi \) can be inflated to an irreducible character of \( C \times D \), that is \( \chi(cd) = \psi(\zeta) \eta(d) \) for some \( \psi \in \text{Irr}(C) \) and \( \eta \in \text{Irr}(D) \). We claim that \( \eta \) is linear. Indeed \( D \subseteq Z(\chi) \), that is \( \psi(1)^2 \eta(1)^2 = \chi(1)^2 \neq \chi(1) \cdot \psi(1) \cdot \eta(1) \cdot \psi(1) \eta(1) = \psi(1)^2 \eta(1)^2 \) for all \( d \in D \), giving \( Z(\eta) = D \). Hence \( D/\ker \eta \) is cyclic and \( D' \leq \ker \eta \) is linear. In particular if \( \chi(1) \neq 1 \) then \( \psi \) is not linear and \( D' \neq 1 \). If \( 1 \neq \vartheta \in \mathbb{C} \) then \( \eta(z) = \vartheta \) and \( \psi(z) = \delta \psi(1) \), where \( \delta \) and \( \vartheta \) are roots of unity, and since \( z \), as an element of \( C \), is identified in the central product with \( z \) as an element of \( D \), we have \( \delta = \vartheta \). If \( z \in ker(\eta) \), then \( \delta = \vartheta = 1 \) and \( \chi(z) = \psi(z) = \chi(1) \), i.e. \( z \in ker(\chi) \), a contradiction. Thus \( N \not\subseteq \ker \eta \). It follows that \( N \cap D' \subseteq N \cap ker \eta = 1 \), as \( N \) is of order \( p \). Hence \( G' = C/D' \leq ND' \).

Assume now that \( \chi \) is stem and \( \chi \neq 1 \). Suppose by way of contradiction that there exists \( h \in D \) such that \( b_D(h) = b \). In particular the equality \( b_{C_G(h)}(h) = b_{D(h)} \) holds and \( G = DC_G(h) = M^* C_G(h) \). By Lemma 3.2 we have that \( N \subseteq [h, G] = [h, CD] = [h, D] \subseteq D' \), a contradiction. This implies \( b(D) < b(G) \), completing thus the proof of item 3.

In order to prove the remaining items we may assume \( C \neq Z(G) \).

If \( C \) is not abelian then there exist \( c_1, c_2 \in C \) such that \( 1 \neq [c_1, c_2] = \vartheta \in N \). Let \( g = c_1 \), since \( c_2 \notin C_G(g) \) we have \( \min \{ b(x) \mid x \in G \setminus C_G(g) \} \leq 1 \), and item 3 is proven.

To prove item 4 first note that if \( h \notin D \) then \( h \) has a non-trivial commutator in \( N \) so that \( b_{G \cap N}(hN) = b_G(h) - 1 \). Suppose now that \( b_{G \cap N}(hN) = b \), from the argument above we have that \( h \in D \). Let \( K = C_G(h) \), \( g \in C \setminus Z(G) \) and \( M^* = C_G(g) \). As in the proof of Lemma 3.2 if \( KM^* = G \) then \( N \subseteq [G, h] \), giving
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\begin{equation}
\begin{align*}
b_{G/N}(hN) &= b - 1, \text{ a contradiction. It follows that } K \leq D = \bigcap_{g \in C} C_G(g). \text{ This yields } p^b = \sigma^b = |G : K| \geq |G : D| = |C : Z(G)|.

\text{By item } 2\text{ we know that } |G : D| \leq \chi(1)^2 \leq p^{2d} \text{ for every } \chi \in \text{Irr}(G) \text{ such that } N \nsubseteq \ker(\chi). \text{ Since there is at least one irreducible character of } G \text{ whose kernel does not contain } N, \text{ the equality } |G : D| = p^{2d} \text{ would imply that every such characters } \chi \text{ would have degree } p^d > 1 \text{ as } N \leq G' \text{ and } \ker \chi \text{ does not contain } N. \text{ By item } 3, C' \neq 1, \text{ a contradiction.}
\end{align*}
\end{equation}

\textbf{Remark 3.4.} Suppose that there exist two non-commuting elements } h \text{ and } k \text{ of breadth 1. Since } C_G(h) \text{ is a maximal, hence normal, subgroup of } G, \text{ it contains } [h, k] \text{ which then commutes with } h \text{ and, similarly, with } k: \text{ clearly it also commutes with every element in } L = C_G(h) \cap C_G(k). \text{ As } G \text{ is generated by } h, k \text{ and } L, \text{ the commutator } [h, k] \text{ is central and thus } \langle [h, k] \rangle \leq [G, h]: \text{ since } [G, h] \text{ has size } p, \text{ equality holds and } [G, h] \text{ is central, that is } h \in Z_2(G). \text{ Similarly } \langle [h, k] \rangle = [G, k], \text{ so that } C_G(G/[G, h]) \text{ contains both } h \text{ and } k \text{ and then it is not abelian.}

\textbf{Proposition 3.5.} Let } H \text{ be a stem } p\text{-group of breadth-degree type } (b, d), \text{ with } b \geq 2, \text{ and of order } |H| = p^k. \text{ Suppose that } G \text{ is any stem quotient of } H \text{ of order at least } |G| \geq p^{k+b+1} \text{ and that } S = \{ g \in Z_2(G) \mid b(g) = 1 \} \text{ is not empty } \text{(} G \text{ exists by Lemma 3.7). For } g \in S \text{ set } N_g = [g, G], \text{ } C_g = C_G(G/N_g), \text{ } M_g = C_G(g) \text{ and } D_g = C_G(C_g).

\begin{enumerate}
\item If } B_1(G) \text{ is not abelian then } p^k \leq p^{b+2d}\sigma(b - 1, d).
\item If } B_1(G) \text{ is abelian then } p^k \leq \begin{cases} p^{b+2d-1}\sigma(b - 1, d) & \text{if } \min_{g \in S} b(G/N_g) \leq b - 1, \\ p^{3b-2}\sigma(b - 1, d) & \text{if } \min_{g \in S} b(G/N_g) = b. \end{cases}
\end{enumerate}

\textbf{In particular } p^k \leq p^{3b+2d-2}\sigma(b - 1, d) \text{ in each of the two cases if } b \geq 2 \text{ and } d \geq 1.

\textbf{Proof.} In the rest of the proof, for the sake of brevity of notation, we shall use the letters } N, M, D \text{ and } C \text{ to denote respectively } N_g, M_g, D_g \text{ and } C_g \text{ when } g \in S. \text{ We already saw that } N \text{ is a minimal normal subgroup of } G \text{ and that } M < G \text{ is a maximal subgroup. Note that that } C \neq Z(G) \text{ as } g \in C \setminus Z(G).

\text{We now apply item } 1\text{ of Lemma 3.3 to the group } G/N. \text{ We have}

\begin{align*}
|Z(M/N)(M/N)' : (M/N)'| &\leq p^{2t} \cdot |Z(G/N)(G/N)' : (G/N)'| \\
&= p^{2t} \cdot |Z(G/N) : (Z(G/N) \cap (G/N)')| \\
&= p^{2t} \cdot |C/N : ((C/N) \cap (G/N)')| \\
&= p^{2t} \cdot |C : (C \cap G')| \\
&\leq p^{2t} \cdot |C : Z(G)|,
\end{align*}

where } t = \min \{ b_{G/N}(xN) \mid x \in G \setminus M \} \leq b - 1 \text{ by Lemma 3.2.}

\text{Let us start with the case when } B_1(G) \text{ is not abelian. By Remark 3.4 there exists } g \in S \text{ such that } C \text{ is not abelian.}

\text{Suppose first that for some } g \in S \text{ such that } C \text{ is not abelian and some } \chi \in \text{Irr}(G) \text{ such that } N \nsubseteq \ker \chi \text{ we have } 1 \neq |G : D| = \chi(1)^2 \leq p^{2d}. \text{ By item } 2\text{ of Lemma 3.3 the quotient } D/N \text{ is stem and } 1 \neq N = C'. \text{ Also } p^{k-b+1}/(p^{2d} \cdot p) \leq |G|/(|G/D| \cdot |N|) = |D/N| \leq \sigma(b - 1, d), \text{ which implies that}

p^k \leq p^{b+2d}\sigma(b - 1, d)
as claimed.

Suppose now that for every $g \in S$ such that $C$ is not abelian and every $\chi \in \text{Irr}(G)$ such that $N \not\subseteq \ker \chi$ we have $|G : D| \neq \chi(1)^2 \leq p^{2d}$. Item 2 of Lemma 3.3 implies that $|C : Z(G)| = |G : D| \leq p^{2d-1}$. Lemma 3.2 yields $b(M/N) < b$. As $C \not\subseteq M$ then $b_{G/N}(yN) = 0$ for some $y \in G \setminus M$, so that $t = 0$. We also have $|M/N| = |G|/p^2 \geq p^{k-b-1}$. Applying (3.1) with $t = 0$ we find

$$|Z(M/N)(M/N)' : (M/N)'| \leq p^{2t} \cdot |C : Z(G)| \leq p^{2d-1}.$$  

By Remark 3.6 the group $M/N$ is isoclinic to a stem group of order at least

$$\frac{|M/N|}{|Z(M/N)(M/N)'/(M/N)'|} \geq \frac{|G/N|}{|Z(G)|} \geq p^{k-b-1}/p^{2d-1} = p^{k-b-2d}.$$  

Hence, as claimed, also in this case we have

$$p^k \leq p^{b+2d} \sigma(b-1, d).$$

Finally we have to deal with the case when $B_1(G)$ is abelian. If $g \in S$ then $C$ is abelian and we can use item 5 of Lemma 3.3. We have $|C : Z(G)| = |G : D| \leq p^{2d-1}$ as $|G : D| \neq \chi(1)^2$ for all $\chi \in \text{Irr}(G)$ such that $N \not\subseteq \ker \chi$. Suppose first that $b(G/N) \leq b-1$. By Remark 3.4 the group $G/N$ is isoclinic to a stem group whose order is

$$\frac{|G/N|}{|Z(G/N)/(G/N \cap (G/N)')|} \geq \frac{|G/N|}{|C/(C \cap G')|} \geq \frac{|G/N|}{|Z(G)|} \geq p^{k-b}/p^{2d-1} = p^{k-b-2d+1},$$

this gives

$$p^k \leq p^{b+2d-1} \sigma(b-1, d)$$

when $b(G/N) \leq b-1$.

The other possibility is that $b(G/N) = b$. In this case there exists $y \in G$ such that $b = b_G(y) = b_{G/N}(yN)$. From Lemma 3.5 it follows that $y \in M$ and that $b(M/N) \leq b-1$. As a consequence either $|M, y| \geq N$ or $b_M(y) = b-1$. Since $|G, y|$ does not contain $N$, the only possible case is the latter one, thus $b(M) = b-1$ and, by item 4 of Lemma 3.3

$$\frac{|M|}{|Z(M)M'/M'|} \geq p^{k-b}/p^{2(b-1)} = p^{k-3b+2}.$$  

Hence

$$p^k \leq p^{3b-2} \sigma(b-1, d).$$

and item 2 follows.

Remark 3.6. It is easy to see that a stem $p$-group of breadth-degree type $(1, d)$ is extra-special of order $p^{2d+1}$. Indeed it has derived subgroup of order $p$ containing the center $Z(G)$. This implies $Z(G) = G' = \Phi(G)$. Hence $G$ has order $p^{2n+1}$ and its character degrees are 1 and $p^n$. As a consequence $\sigma(1, d) = p^{2d+1}$.

Theorem 3.7. For any given pair $(b, d) \in \mathbb{N} \times \mathbb{N}$ the size $\sigma(b, d)$ of the breadth-degree type $(b, d)$ is finite and $\sigma(b, d) \leq p^{\frac{M(3b+4d-11)}{2}}$.
Proof. Proposition 3.5 yields $\sigma(b, d) \leq p^{3b+2d-2}\sigma(b - 1, d)$ for $b \geq 2$ and $d \geq 1$. By induction it follows that

$$\sigma(b, d) \leq p^{\frac{3b^2+4bd-4d-b-2}{2}}\sigma(1, d) = p^{\frac{b(3b+4d-1)}{2}}.$$ 

since $\sigma(1, d) = p^{1+2d}$ as we already noted.

4. Examples and lower bounds

Consider the additive group $H = \mathbb{Z}_p[\vartheta]$ of the ring of the $p$-adic integers extended by a $p$-th root of unity $\vartheta \neq 1$ and let $g$ be the automorphism of order $p$ of $H$ induced by the multiplication by $\vartheta$. It is well known that the semidirect product $M_\infty = \langle g \rangle \times H$ is a pro-$p$ group of maximal class, i.e. any normal subgroup of finite index of $M_\infty$ is either a maximal subgroup or a term $\gamma_i(M_\infty) = (\vartheta - 1)^{i-1}H$ of the lower central series of $M_\infty$ (see also [18, Example 7.4.14]). Also the finite quotient $M_i = M_\infty/\gamma_i(M_\infty)$ is of maximal class and has order $|M_i| = p^i$. Since $M_i$ has an abelian maximal subgroup $N$, the character degrees of $M_i$ are 1 and $p$. It is also easy to see that if $x \notin N$ the $x^{M_i} = x^{M_i'}$ so that $b(M_i) = \log_p(|M_i|) = i - 2$. We have that $M_i$ has breadth-degree type $(i - 2, 1)$ for $i \geq 3$. Let $b \geq d$ and consider the group $T_{b,d} = M_{b-d+3} \times \prod_{j=1}^{d-1} E$, where $E$ is an extra-special group of order $p^3$. It’s easy to see that $T_{b,d}$ is stem of order $p^{b+2d}$ and that it has breadth-degree type $(b, d)$. Hence $\sigma(b, d) \geq p^{b+2d}$ for $b \geq d$.

Let $H_{p^d}$ be the group of the unitriangular 3 × 3 matrices over the Galois field $\mathbb{F}_{p^d}$ and $N \leq Z(H_{p^d})$ be any subgroup of order $p^{d-b}$ of its center. The quotient $H_{p^d}/N$ provides an example of a stem $p$-group of breadth-degree type $(b, d)$ and of order $p^{b+2d}$. Hence $\sigma(b, d) \geq p^{b+2d}$ also in the case $d > b$.

It follows that for every given $b$ and $d$ we have

$$\sup_{s \geq 1} \sigma(s, d) = \sup_{t \geq 1} \sigma(b, t) = \infty. \quad (4.1)$$

When $p$ is odd another interesting example is the stem group $F_{b+1} = F/\gamma_3(F)F^p$ where $F = \langle x_1, \ldots, x_{b+1} \rangle$ is the free group on $b + 1$ generators. A direct computation shows that $F_{b+1}$ has breadth $b$ and derived subgroup $F'_{b+1} = Z(F_{b+1})$ of maximum possible order $p^{b+1}$. Let $d = \left\lfloor \frac{b+1}{2} \right\rfloor$. On the one hand, since the square of the degree of any irreducible characters of $F_{b+1}$ divides the index $p^{b+1}$ of the center $Z(F_{b+1})$ [16 Corollary 2.30], the representation exponent of the group $F_{b+1}$ is at most $d$. On the other hand, the group $H_{p^d}$ is a quotient of $F_{b+1}$, so that the representation exponent of $F_{b+1}$ is at least $d$. We deduce that the representation exponent of $F_{b+1}$ is exactly $d = \left\lfloor \frac{b+1}{2} \right\rfloor$. This yields the lower bound

$$\sigma \left( b, \left\lfloor \frac{b+1}{2} \right\rfloor \right) \geq |F_{b+1}| = p^{b^2+3b+2} (p \text{ odd}).$$

Since $d' \leq d$ and $b' \leq b$ imply that $\sigma(b', d') \leq \sigma(b, d)$, for $b \leq 2d - 1$ we have that

$$p^{\frac{b^2+3b+2}{2}} \leq \sigma(b, d) \leq p^{\frac{b(3b+4d-1)}{2}} \leq p^{10d^2-9d+2} (p \text{ odd}),$$

whereas for $b \geq 2d$ we have

$$p^{d(2d+1)} \leq p^{\frac{(2d-1)^2+3(2d-1)+2}{2}} \leq \sigma(b, d) \leq p^{\frac{b(3b+4d-1)}{2}} \leq p^{\frac{b^2+3b}{2}} (p \text{ odd}).$$
References

[1] Yakov Berkovich. Groups of prime power order. Vol. 1, volume 46 of De Gruyter Expositions in Mathematics. Walter de Gruyter GmbH & Co. KG, Berlin, 2008. With a foreword by Zvonimir Janko.

[2] S. R. Blackburn. Enumeration within isoclinism classes of groups of prime power order. J. London Math. Soc. (2), 50(2):293–304, 1994.

[3] W. Burnside. Theory of groups of finite order. Dover Publications, Inc., New York, 1955. unabridged republication of the 2d ed. first published in 1911.

[4] Mark Cartwright. Class and breadth of a finite p-group. Bull. London Math. Soc., 19(5):425–430, 1987.

[5] Bettina Eick, M. F. Newman, and E. A. O’Brien. The class-breadth conjecture revisited. J. Algebra, 300(1):384–393, 2006.

[6] A. Erfanian, R. Rezaei, and P. Lescot. On the relative commutativity degree of a subgroup of a finite group. Comm. Algebra, 35(12):4183–4197, 2007.

[7] Volkmar Felsch. The computation of a counterexample to the class-breadth conjecture for p-groups. In The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), volume 37 of Proc. Sympos. Pure Math., pages 503–506. Amer. Math. Soc., Providence, R.I., 1980.

[8] Waltraud Felsch, Joachim Neubüser, and Wilhelm Plesken. Space groups and groups of prime-power order. IV. Counterexamples to the class-breadth conjecture. J. London Math. Soc. (2), 24(1):113–122, 1981.

[9] Norberto Gavioli, Avinoam Mann, Valerio Monti, Andrea Previtali, and Carlo M. Scoppola. Groups of prime power order with many conjugacy classes. J. Algebra, 202(2):129–141, 1998.

[10] R. M. Guralnick and G. R. Robinson. On the commuting probability in finite groups. J. Algebra, 300(2):509–528, 2006.

[11] W. H. Gustafson. What is the probability that two group elements commute? Amer. Math. Monthly, 80:1031–1034, 1973.

[12] P. Hall. The classification of prime-power groups. J. Reine Angew. Math., 182:130–141, 1940.

[13] I. M. Isaacs and D. S. Passman. Groups whose irreducible representations have degrees dividing \( p^e \). Illinois J. Math., 8:446–457, 1964.

[14] I. M. Isaacs and D. S. Passman. Groups with representations of bounded degree. Canad. J. Math., 16:299–309, 1964.

[15] I. M. Isaacs and D. S. Passman. A characterization of groups in terms of the degrees of their characters. Pacific J. Math., 15:877–903, 1965.

[16] I. Martin Isaacs. Character theory of finite groups. Dover Publications, Inc., New York, 1994. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)].

[17] Hans-Georg Knoche. Über den Frobenius’schen Klassenbegriff in nilpotenten Gruppen. Math. Z., 55:71–83, 1951.

[18] C. R. Leedham-Green and S. McKay. The structure of groups of prime power order, volume 27 of London Mathematical Society Monographs. New Series. Oxford University Press, Oxford, 2002. Oxford Science Publications.

[19] C. R. Leedham-Green, Peter M. Neumann, and James Wiegold. The breadth and the class of a finite p-group. J. London Math. Soc. (2), 1:409–420, 1969.

[20] Paul Lescot. Isoclinism classes and commutativity degrees of finite groups. J. Algebra, 177(3):847–869, 1995.

[21] Patrizia Longobardi, Mercede Maj, and Avinoam Mann. Minimal classes and maximal class in p-groups. Israel J. Math., 110:93–102, 1999.

[22] Avinoam Mann. Minimal characters of p-groups. J. Group Theory, 2(3):225–250, 1999.

[23] Avinoam Mann. Elements of minimal breadth in finite p-groups and Lie algebras. J. Aust. Math. Soc., 81(2):209–214, 2006.

[24] G. Parmeggiani and B. Stellmacher. p-groups of small breadth. J. Algebra, 213(1):52–68, 1999.

[25] D. S. Passman. Groups whose irreducible representations have degrees dividing \( p^2 \). Pacific J. Math., 17:475–496, 1966.

[26] David J. Rusin. What is the probability that two elements of a finite group commute? Pacific J. Math., 82(1):237–247, 1979.
[27] M. R. Vaughan-Lee. Breadth and commutator subgroups of $p$-groups. *J. Algebra*, 32:278–285, 1974.

[28] James Wiegold. Multiplicators and groups with finite central factor-groups. *Math. Z.*, 89:345–347, 1965.

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