SCHATTEN CLASSES OF GENERALIZED HILBERT OPERATORS

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Abstract. Let \( D_v \) denote the Dirichlet type space in the unit disc induced by a radial weight \( v \) for which \( \hat{v}(r) = \int_1^r v(s) \, ds \) satisfies the doubling property \( \int_1^r v(s) \, ds \leq C \int_1^{1+r} v(s) \, ds \). In this paper, we characterize the Schatten classes \( S_p(D_v) \) of the generalized Hilbert operators

\[ H_g(f)(z) = \int_0^1 f(t)g'(tz) \, dt \]

acting on \( D_v \), where \( v \) satisfies the Muckenhoupt-type conditions

\[ \sup_{0 < r < 1} \left( \int_0^1 \frac{\hat{v}(s)}{(1-s)^2} \, ds \right)^{\frac{1}{2}} \left( \int_0^1 \frac{1}{\hat{v}(s)} \, ds \right)^{\frac{1}{2}} < \infty \]

and

\[ \sup_{0 < r < 1} \left( \int_0^r \frac{\hat{v}(s)}{(1-s)^4} \, ds \right)^{\frac{1}{2}} \left( \int_0^1 \frac{(1-s)^2}{\hat{v}(s)} \, ds \right)^{\frac{1}{2}} < \infty. \]

For \( p \geq 1 \), it is proved that \( H_g \in S_p(D_v) \) if and only if

\[ \int_0^1 \left( 1-r \right) \int_{-\pi}^{\pi} |g'(re^{i\theta})|^2 \, d\theta \frac{dr}{1-r} < \infty. \]

1. Introduction and main results

Let \( \mathbb{D} \) denote the open unit disk of the complex plane, and let \( H(\mathbb{D}) \) be the class of all analytic functions on \( \mathbb{D} \). A function \( v : \mathbb{D} \to (0, \infty) \), integrable over \( \mathbb{D} \), is called a weight. It is radial if \( v(z) = v(|z|) \) for all \( z \in \mathbb{D} \). The weighted Dirichlet space \( D_v \) consists of \( f \in H(\mathbb{D}) \) for which

\[ \|f\|_{D_v}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 v(z) \, dA(z) < \infty, \]

where \( dA(z) = \frac{dx \, dy}{\pi} \) is the normalized Lebesgue area measure on \( \mathbb{D} \). In this work, we will consider Dirichlet type spaces \( D_v \) induced by weights...
in the class $\hat{D}$ of the radial weights $v$ for which $\hat{v}(r) = \int_1^r v(s) \, ds$ satisfy $\sup_{0 < r < 1} \frac{\hat{v}(r)}{\hat{v}(\frac{r}{2})} < \infty$. The standard radial weights $v(z) = (1 - |z|)^\alpha$, $\alpha > -1$ meet this doubling property. We write $D_\alpha$ for the Dirichlet type space induced by the standard weight $(1 - |z|)^\alpha$. The Hardy space $H^2$ consists of $f \in H(D)$ for which $\|f\|_{H^2} = \lim_{r \to 1^-} M_2(r, f) < \infty$, where

$$M_2(r, f) = \left( \frac{1}{2\pi} \int_{-\pi}^\pi |f(re^{i\theta})|^2 \, d\theta \right)^{\frac{1}{2}}.$$

The classical Littlewood-Paley formula says that $H^2 = D_1$. We refer the reader to [6] for background information on this space. We denote by $A^2_\omega$ the Bergman space induced by a weight $\omega$ (see [12, Chapter 1]). Moreover, if $\omega$ is radial then $A^2_\omega = D_\omega^\star$, where

$$\omega^*(z) = \int_{|z|}^1 s \log \frac{s}{|z|} \omega(s) \, ds, \quad z \in D \setminus \{0\}.$$

See [12, Theorem 4.2] for the details.

Every $g \in H(D)$ induces an operator, that we call the generalized Hilbert operator $\mathcal{H}_g$, defined by

$$\mathcal{H}_g(f)(z) = \int_0^1 f(t)g'(tz) \, dt, \quad f \in H(D).$$

The sharp condition

$$\int_0^1 \frac{(1 - s)^2}{\hat{v}(s)} \, ds < \infty$$

ensures that the integral in (1.1) defines an analytic function for each $f \in D_v$ (see Lemma 7 below).

The choice $g(z) = \log \frac{1}{1 - z}$ in (1.1) gives an integral representation of the classical Hilbert operator $\mathcal{H}$. The Hilbert operator $\mathcal{H}$ is a model of Hankel operator, and has been the object of previous studies such as [1, 3, 5], where the authors dealt with questions related to the boundedness, the operator norm and the spectrum of $\mathcal{H}$. This has revealed a natural connection between $\mathcal{H}$ and other classical objects: the weighted composition operators, the Szegö projection and the Legendre functions of the first kind. The Hilbert operator is bounded on the classical Dirichlet type space $D_\alpha$ if and only if $\alpha \in (0, 2)$, as was shown in [4, 7]. In fact, if $\alpha \geq 2$ there is $f \in D_\alpha$, $f \geq 0$ on $[0, 1)$ such that $\int_0^1 f(t) \, dt = \infty$.

The generalized Hilbert operator $\mathcal{H}_g$ was introduced recently in [7], where it is provided, among other results, a description of the $g \in H(D)$ such that $\mathcal{H}_g$ is bounded, compact or Hilbert-Schmidt on $D_\alpha, \alpha \in (0, 2)$. In [11], the authors solve the question of when is $\mathcal{H}_g$ bounded or compact between weighted Bergman spaces $A^p_\omega$ and $A^q_\omega$, $1 < p, q < \infty$, induced by a large class of radial weights.
The primary aim of this paper is to determine the membership in Schatten ideals $S_p(\mathcal{D}_v)$ of generalized Hilbert operators $\mathcal{H}_g$ acting on Dirichlet type spaces $\mathcal{D}_v$, $v \in \hat{\mathcal{D}}$. This leads us to consider the following spaces. For $0 < p < \infty$, the mixed norm space $\mathcal{B}(2, p)$ consists of $g \in H(\mathbb{D})$ such that

$$
\|g\|_{\mathcal{B}(2, p)}^2 = |g(0)|^p + \int_0^1 M_2^2(r, g')(1-r)^{p-1} dr < \infty.
$$

Let us observe that $\mathcal{B}(2, 2)$ is nothing else but the classical Dirichlet space $\mathcal{D}_0 = \mathcal{D}$. The space $\mathcal{B}(2, \infty)$ consists of $g \in H(\mathbb{D})$ such that

$$
\|g\|_{\mathcal{B}(2, \infty)} = |g(0)| + \sup_{0 < r < 1} M_2(r, g')(1-r)^{\frac{1}{2}} < \infty.
$$

A classical result of Hardy and Littlewood [6, Chapter 5] asserts that $\mathcal{B}(2, \infty)$ coincides with the mean Lipschitz space $\Lambda(2, \frac{1}{2})$ of the $g \in H(\mathbb{D})$ having a non-tangential limit $g(e^{i\theta})$ almost everywhere and such that

$$
\omega_2(g, t) = O(t^{\frac{1}{2}}), \quad t \to 0,
$$

where

$$
\omega_2(g, t) = \sup_{0 < h \leq t} \left( \int_0^{2\pi} |g(e^{i(\theta+h)}) - g(e^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2}
$$

is the integral modulus of continuity of order 2.

The corresponding “little oh” mean Lipschitz space $b(2, \infty)$, usually denoted by $\lambda(2, \frac{1}{2})$, consists of $g \in H(\mathbb{D})$ such that

$$
\lim_{r \to 1^-} M_2(r, g')(1-r)^{1/2} = 0.
$$

The next theorem is the main result of this paper.

**Theorem 1.** Let $g \in H(\mathbb{D})$, $1 \leq p \leq \infty$ and $v \in \hat{\mathcal{D}}$ which satisfies the conditions

$$(1.3) \quad M_1(v) = \sup_{0 < r < 1} \left( \int_0^1 \frac{\hat{v}(s)}{(1-s)^2} ds \right)^{1/2} \left( \int_0^r \frac{1}{\hat{v}(s)} ds \right)^{1/2} < \infty,$$

and

$$(1.4) \quad M_2(v) = \sup_{0 < r < 1} \left( \int_0^r \frac{\hat{v}(s)}{(1-s)^4} ds \right)^{\frac{1}{2}} \left( \int_r^1 \frac{(1-s)^2}{\hat{v}(s)} ds \right)^{\frac{1}{2}} < \infty.$$

Then, the following conditions are equivalent:

(i) $\mathcal{H}_g \in S_p(\mathcal{D}_v)$;

(ii) $g \in \mathcal{B}(2, p)$.

Moreover,

$$
\|\mathcal{H}_g\|_{S_p(\mathcal{D}_v)} \asymp \|g - g(0)\|_{\mathcal{B}(2, p)}.
$$
It will also be proved (see Proposition 10 below) that \( \mathcal{H}_g \) is compact on \( \mathcal{D}_v \) if and only if \( g \in b(2, \infty) \). Notice that the conditions that characterize the Schatten classes do not depend on the weight defining the space.

If \( v(z) = (1-|z|)^\alpha \) is a standard weight, (1.3) holds if and only if \( \alpha > 0 \), and (1.4) is satisfied if and only if \( \alpha < 2 \), so both conditions hold simultaneously if and only if \( \alpha \in (0, 2) \). Therefore, in particular, Theorem 1 provides a characterization of Schatten classes of generalized Hilbert operators \( \mathcal{H}_g \) acting on the Hardy space \( H^2 (\alpha = 1) \) and standard Bergman spaces \( A^2_{\beta} \), \( \beta \in (-1, 0) \). Going further the next result follows from Theorem 1.

**Corollary 2.** Let \( g \in H(D) \), \( 1 \leq p \leq \infty \) and \( \omega \in \hat{D} \) which satisfies the condition

\[
\sup_{0 < r < 1} \left( \int_0^r \frac{\omega(t)}{(1-t)^2} dt \right)^{\frac{1}{2}} \left( \int_1^1 \frac{1}{\omega(t)} dt \right)^{\frac{1}{2}} < \infty. 
\]

Then, the following conditions are equivalent:

(i) \( \mathcal{H}_g \in S_p(A^2_{\omega}) \);
(ii) \( g \in B(2, p) \).

The Muckenhoupt-type condition (1.5) arises in the study of generalized Hilbert operators \( \mathcal{H}_g \) on weighted Bergman spaces in [11], where the authors describe the \( g \in H(D) \) such that \( \mathcal{H}_g \) is bounded, compact or Hilbert-Schmidt on \( A^2_{\omega} \) for the the subclass of \( \hat{D} \) consisting of regular weights.

From now on, for each radial weight \( v \) and \( x \in \mathbb{R} \) let us denote \( \hat{V}_x(r) = \frac{\hat{a}(r)}{(1-r)^x} \). Our approach to prove the case \( p = \infty \) of Theorem 1 reveals the role of Muckenhoupt type conditions (1.3) and (1.4). On one hand, (1.3) allows to prove that \( L^2_{\hat{V}_x} \) is a natural restriction of \( \mathcal{D}_v \) to functions defined on \( [0,1) \). On the other hand, the sublinear Hilbert operator defined by

\[
\tilde{\mathcal{H}}(f)(z) = \int_0^1 \frac{|f(t)|}{1-tz} dt
\]

behaves like a kind of maximal function for all generalized Hilbert operators \( \mathcal{H}_g \) such that \( g \in B(2, \infty) \), and hence, it will be essential to study its boundedness on \( L^2_{\hat{V}_x} \).

**Theorem 3.** Let \( v \in \hat{D} \) which satisfies the conditions (1.2) and (1.3). Then the following assertions are equivalent:

(i) \( \mathcal{H} : L^2_{\hat{V}_x} \rightarrow \mathcal{D}_v \) is bounded;
(ii) \( \tilde{\mathcal{H}} : L^2_{\hat{V}_x} \rightarrow \mathcal{D}_v \) is bounded;
(iii) \( v \) satisfies the Muckenhoupt type condition (1.4).

Moreover, if \( M_2(v) < \infty \), then

\[
\frac{M_2(v)}{M_1(v)} \lesssim \|\mathcal{H}\|_{(L^2_{\hat{V}_x}, \mathcal{D}_v)} \lesssim \|\tilde{\mathcal{H}}\|_{(L^2_{\hat{V}_x}, \mathcal{D}_v)} \lesssim M_1(v)M_2(v).
\]
We obtain as a byproduct the following result which extends several results in the literature [4, 11].

**Corollary 4.** Let $v \in \hat{D}$ which satisfies the conditions (1.3) and (1.4). Then, both the Hilbert operator $\mathcal{H}$ and the sublinear Hilbert operator $\tilde{\mathcal{H}}$ are bounded on $D_v$.

Throughout the paper the letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.

2. The Hilbert operator on $D_v$

2.1. Some results on weights. The following lemma provides useful characterizations of weights in $\hat{D}$. For a proof, see [13]. Given a radial weight $v$, we write $v_x = \int_0^1 s^x v(s) \, ds$, $x > -1$.

**Lemma 5.** Let $\omega$ be a radial weight. Then the following conditions are equivalent:

(i) $\omega \in \hat{D}$;

(ii) There exist $C = C(\omega) \geq 1$ and $\beta = \beta(\omega) > 0$ such that

\[ \hat{\omega}(r) \leq C \left( \frac{1-r}{1-t} \right)^\beta \hat{\omega}(t), \quad 0 \leq r \leq t < 1; \]

(iii) There exist $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that

\[ \int_0^t \left( \frac{1-t}{1-s} \right)^\gamma \omega(s) \, ds \leq C\hat{\omega}(t), \quad 0 \leq t < 1; \]

(iv) There exist $C = C(\omega) > 0$ and $\eta = \eta(\omega) > 0$ such that

\[ \omega_x \leq C \left( \frac{y}{x} \right)^\eta \omega_y, \quad 0 < x \leq y < \infty; \]

(v) There exists $C = C(\omega) > 0$ such that $\omega_n \leq C\omega_{2n}$ for all $n \in \mathbb{N}$;

(vi) $\omega_x \asymp \hat{\omega} \left( 1 - \frac{1}{x} \right)$, $x \in [1, \infty)$;

(vii) There exists $\lambda = \lambda(\omega) \geq 0$ such that

\[ \int_D \omega(z) \, dA(z) \asymp \frac{\hat{\omega}(\zeta)}{(1 - |\zeta|)^\lambda}, \quad \zeta \in D; \]

(viii) $\omega^*(z) \asymp \hat{\omega}(z)(1 - |z|)$ for $|z| \geq \frac{1}{2}$.

The following technical lemma will be used in the proof of Theorem 3.
Lemma 6. Let \( v \) be a radial weight. If (1.4) holds, then
\[
\sup_{0 \leq r < 1} \left( \int_0^r \frac{v(s)}{(1-s)^3} \, ds \right)^{\frac{1}{2}} \left( \int_r^1 \frac{1}{\hat{V}_2(s)} \, ds \right)^{\frac{1}{2}} < \infty.
\]

Proof. For \( 0 < r < 1 \), Fubini’s theorem gives
\[
\int_0^r \frac{\hat{v}(s)}{(1-s)^4} \, ds = \int_0^r \frac{v(s)[1 - (1-s)^3]}{3(1-s)^3} \, ds + \hat{v}(r) \frac{1 - (1-r)^3}{3(1-r)^3}.
\]

We may assume that \( r \in \left[ \frac{1}{2}, 1 \right) \), and then we have
\[
\int_0^r \frac{v(s)}{(1-s)^3} \, ds \leq C \int_0^{\frac{1}{2}} v(s) \, ds + C \int_{\frac{1}{2}}^r \frac{v(s)[1 - (1-s)^3]}{(1-s)^3} \, ds
\]
\[
\leq C \int_0^{\frac{1}{2}} v(s) \, ds + C \int_0^r \frac{\hat{v}(s)}{(1-s)^4} \, ds.
\]

By (1.4), there is \( C = C(v) \) such that \( \int_0^{\frac{1}{2}} v(s) \, ds \leq C_2(v) < \infty \). Therefore, the above inequality yields
\[
\sup_{\frac{1}{2} \leq r < 1} \left( \int_0^r \frac{v(s)}{(1-s)^3} \, ds \right)^{\frac{1}{2}} \left( \int_r^1 \frac{1}{\hat{V}_2(s)} \, ds \right)^{\frac{1}{2}}
\]
\[
\leq C \sup_{\frac{1}{2} \leq r < 1} \left( \int_0^{\frac{1}{2}} v(s) \, ds + \int_{\frac{1}{2}}^r \frac{\hat{v}(s)}{(1-s)^4} \, ds \right)^{\frac{1}{2}} \left( \int_0^1 \frac{1}{\hat{V}_2(s)} \, ds \right)^{\frac{1}{2}}
\]
\[
\leq C \int_0^{\frac{1}{2}} v(s) \, ds \int_0^1 \frac{1}{\hat{V}_2(t)} \, dt < \infty + C_2(v) \leq C_2(v) < \infty.
\]

This finishes the proof. \( \square \)

2.2. Hardy-Littlewood type inequalities. The first result in this subsection gives a sharp condition that ensures that \( \mathcal{H}_g \) is well defined on \( \mathcal{D}_v \).

Lemma 7. Let \( v \) be a radial weight which satisfies (1.2). Then there is a positive constant \( C(v) \) such that
\[
(2.1) \quad \int_0^1 |f(t)| \, dt \leq C(v) \| f \|_{\mathcal{D}_v}, \quad f \in H(\mathbb{D}).
\]

Proof. For any \( f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k \in H(\mathbb{D}) \), it is clear that
\[
(2.2) \quad \int_0^1 |f(t)| \, dt \leq |\hat{f}(0)| + \left( \sum_{k=1}^{\infty} k^2 |\hat{f}(k)|^2 v_{2k-1} \right)^{1/2} \left( \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)^2 v_{2k-1}} \right)^{1/2}.
\]
By Lemma 5(v)(vi)
\[ \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)^2v_{2k-1}} \leq C \sum_{k=1}^{\infty} \frac{1}{v(1- \frac{1}{k+1})^2} \int_{1- \frac{1}{k+1}}^{1} (1-s)^2 ds \]
\[ \leq C \int_{\frac{1}{2}}^{1} \frac{(1-s)^2}{v(s)} ds \leq C \int_{0}^{1} \frac{(1-s)^2}{v(s)} ds \leq C. \]
This together with (2.2) and the identity \( \| f \|_{D_v}^2 = |\hat{f}(0)|^2 + 2 \sum_{k=1}^{\infty} |k \hat{f}(k)|^2v_{2k-1}, \)
implies (2.1).

A classical result of Hardy-Littlewood ([6, Theorem 5.11]) says that
\[ \int_{0}^{1} M_v^2(r, f) \hat{\omega}(s) ds \leq C \| f \|_{H_v}^2. \]
See also the classical Féjer-Riesz inequality [6, Theorem 3.13]. Applying this inequality to dilated functions \( f_r(z) = f(rz), 0 < r < 1, \) and integrating respect to a radial weight \( \omega, \) it can be easily obtained that
\[ \int_{0}^{1} M_v^2(r, f) \hat{\omega}(r) dr \leq C \| f \|_{A_v^2}^2. \]
The next result shows a Hardy-Littlewood type inequality in a setting of weighted Dirichlet spaces.

**Lemma 8.** Let \( v \) be a radial weight which satisfies (1.3). Then, there exists \( C = C(v) > 0 \) such that
\[ \int_{0}^{1} M_v^2(s, f) \frac{\hat{v}(s)}{(1-s)^2} ds \leq C M_v^2(v) \| f \|_{D_v}^2, \quad f \in H(\mathbb{D}). \]

**Proof.** By condition (1.3) there is a constant \( C = C(v) > 0 \) such that \( \int_{0}^{1} \frac{\hat{v}(s)}{(1-s)^2} ds \leq C M_v^2(v) < \infty. \) Using [9, Theorem 1] with
\[ U^2(s) = \begin{cases} \frac{\hat{v}(s)}{(1-s)^2}, & 0 \leq s < 1 \\ 0, & s \geq 1 \end{cases} \]
and
\[ V^2(s) = \begin{cases} \hat{v}(s), & 0 \leq s < 1 \\ 0, & s \geq 1 \end{cases} \]
we obtain
\[ \int_{0}^{1} M_v^2(s, f) \frac{\hat{v}(s)}{(1-s)^2} ds \]
\[ \leq C M_v^2(v) |f(0)|^2 + \int_{0}^{1} \left( \int_{0}^{s} M_v(r, f') dr \right)^2 \frac{\hat{v}(s)}{(1-s)^2} ds \]
\[ \leq C M_v^2(v) \left( |f(0)|^2 + \int_{0}^{1} M_v^2(s, f') \hat{v}(s) ds \right). \]
Joining (2.3) and (2.5), we get (2.4). \( \square \)
It is worth mentioning that the inequality
\[ M_\infty(r, f') \leq C \frac{M_\infty(1 + r, f)}{1 - r}, \quad 0 < r < 1, \]
implies the reverse inequality of (2.5) for any \( f \in H(D) \) and \( v \in \hat{D} \).

2.3. Proof of Theorem 3. It is clear that (ii) \( \Rightarrow \) (i).

(i) \( \Rightarrow \) (iii). This part of the proof uses ideas from [9]. For \( r \in [0, 1) \), set \( \phi_r(t) = \frac{1}{V_2(t)} \chi_{[r, 1)}(t) \), so that \( \phi_r \in L^2_{V_2} \) by (1.2). Here, as usual, \( \chi_E \) stands for the characteristic function of a set \( E \). Bearing in mind Lemma 8, we deduce
\[ \|H(\phi_r)\|_{L^2_{V_2}} \lesssim M_1(v)\|H(\phi_r)\|_{D_v} \leq M_1(v)\|H\|_{(L^2_{V_2}, D_v)}\|\phi_r\|_{L^2_{V_2}}, \]
and hence
\[ \int_0^1 \hat{V}_2(s) \left( \int_r^1 \frac{1}{V_2(t)(1 - ts)} \frac{dt}{ds} \right)^2 ds \lesssim \int_r^1 \frac{1}{V_2(t)} dt. \tag{2.6} \]

On the other hand,
\[ \int_0^r \hat{V}_2(s) \left( \int_r^1 \frac{1}{V_2(t)(1 - ts)} \frac{dt}{ds} \right)^2 ds \geq \frac{1}{4} \left( \int_0^r \hat{V}_4(s) ds \right) \left( \int_r^1 \frac{1}{V_2(t)} dt \right)^2, \]
and this, together with (2.6), implies
\[ M_2(v) \lesssim M_1(v)\|H\|_{(L^2_{V_2}, D_v)} < \infty. \]

(iii) \( \Rightarrow \) (ii). For any \( \phi \in L^2_{V_2} \),
\[ (\widetilde{H}(\phi))'(z) = \int_0^1 t|\phi(t)| (1 - tz)^2 dt, \]
and so Minkowski’s inequality in continuous form yields
\[ M_2(r, (\widetilde{H}(\phi))') = \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^1 \frac{|\phi(t)|t}{1 - tre^{i\theta}} \frac{dt}{d\theta} \right|^2 d\theta \right)^\frac{1}{2} \leq \int_0^1 |\phi(t)| \left( \int_0^{2\pi} \frac{d\theta}{|1 - tre^{i\theta}|^4} \right)^\frac{1}{2} dt \approx \int_0^1 \frac{|\phi(t)|}{(1 - tr)^{3/2}} dt. \]

Hence, decomposing the range of variation of \( t \), we obtain
\[ (\widetilde{H}(\phi))^2 \lesssim I_1(r) + I_2(r) + |\widetilde{H}(\phi)(0)|^2 \]
where
\[ I_1(r) = \int_0^1 \left( \int_0^r \frac{|\phi(t)|}{(1 - t)^{3/2}} dt \right)^2 v(r) dr \]
and
\[ I_2(r) = \int_0^1 \left( \int_r^1 \frac{|\phi(t)|}{(1 - tr)^{3/2}} dt \right)^2 v(r) dr. \]
By (1.4)

\[
|\tilde{H}(\phi)(0)|^2 \leq \|\phi\|_{L^2_{\hat{V}_2}}^2 \int_0^1 \frac{1}{\hat{V}_2(s)} \, ds \leq C(v) M^2(v) \|\phi\|_{L^2_{\hat{V}_2}}^2 < \infty.
\]

The inequality

\[
I_1(r) \lesssim \|\phi\|_{L^2_{\hat{V}_2}}^2
\]

can be written as

\[
\int_0^1 \left( \int_0^r \Phi(t) \, dt \right)^2 U^2(r) \, dr \lesssim \int_0^1 \Phi^2(r)V^2(r) \, dr,
\]

where

\[
U^2(x) = \begin{cases} v(x), & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases},
\]
\[
V^2(x) = \begin{cases} (1 - x)\hat{v}(x), & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases},
\]

and \( \Phi(t) = \frac{|\phi(t)|}{(1-t)^{\frac{3}{2}}} \). From this, by [9, Theorem 1], (2.9) holds if and only if

\[
M_3(v) = \sup_{0 < r < 1} \hat{v}(r)^{\frac{1}{2}} \left( \int_0^r \frac{1}{(1-s)\hat{v}(s)} \, ds \right)^{\frac{1}{2}} < \infty.
\]

Using (1.3), we get

\[
\int_0^r \frac{1}{(1-s)\hat{v}(s)} \, ds \leq \frac{1}{1-r} \int_0^r \frac{1}{\hat{v}(s)} \, ds \leq \frac{M_1^2(v)}{(1-r)\int_r^1 \hat{V}_2(s) \, ds} \leq \frac{CM_1^2(v)}{\hat{v}(1+r)} \leq CM_1^2(v),
\]

which implies that \( M_3(v) \leq CM_1(v) \), and so by [9, Theorem 1]

\[
I_1(r) \leq C(v) M_1^2(v) \|\phi\|_{L^2_{\hat{V}_2}}^2.
\]

Moreover, by applying [9, Theorem 2] with

\[
U^2(x) = \begin{cases} \frac{v(x)}{(1-x)^3}, & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases},
\]

and

\[
V^2(x) = \begin{cases} \hat{V}_2(x), & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases},
\]

we deduce that

\[
I_2(r) \lesssim \int_0^1 \left( \int_r^1 \phi(t) \, dt \right)^2 \frac{v(r)}{(1-r)^3} \, dr \leq M_1^2(v) \|\phi\|_{L^2_{\hat{V}_2}}^2,
\]

holds whenever

\[
M_4(v) = \sup_{0 \leq r < 1} \left( \int_0^r \frac{v(s)}{(1-s)^3} \, ds \right)^{\frac{1}{2}} \left( \int_r^1 \frac{1}{\hat{V}_2(s)} \, ds \right)^{\frac{1}{2}} < \infty.
\]
By Lemma 6, \( M_4(v) \leq CM_2(v) < \infty \). This together with (2.7), (2.8) and (2.10) gives (iii) \( \Rightarrow \) (ii). Going further, since there is an absolute constant \( K > 0 \) such that \( \min\{M_1(v), M_2(v)\} > K \), we get
\[
\|\widehat{H}\|_{(L^2_{\tilde{\mathcal{D}}}, \mathcal{D}_v)} \lesssim M_1(v)M_2(v).
\]

\[\Box\]

Corollary 4 follows from Theorem 3 and Lemma 8.

3. PROOF OF THEOREM 1

The right choice of the norm used is in many cases a key to a good understanding of how a concrete operator acts in a given space. Here the spaces \( \mathcal{B}(2, p) \) will be equipped with an \( l^p \)-norm of the \( H^2 \) norms of dyadic blocks of the Maclaurin series. In fact, a calculation shows that
\[
\|g\|_{\mathcal{B}(2, \infty)}^2 \asymp |\widehat{g}(0)|^2 + \sup_{n \in \mathbb{N}} \left( \frac{1}{2^n} \sum_{k \in I(n)} k^2|\widehat{g}(k)|^2 \right),
\]
where \( g(z) = \sum_{k=0}^{\infty} \widehat{g}(k)z^k \) and \( I(n) = \{k \in \mathbb{N} : 2^n \leq k < 2^{n+1} - 1\}, \ n \in \mathbb{N}. \) The same techniques allow us to prove that
\[
g \in b(2, \infty) \iff \lim_{n \to \infty} \frac{1}{2^n} \sum_{k \in I(n)} k^2|\widehat{g}(k)|^2 = 0.
\]
Throughout this section, the expression \( \widehat{g}_k \) will denote \( k^2|\widehat{g}(k)|^2 \). Using [8, Theorem 1] it can also be proved that
\[
\|g\|_{\mathcal{B}(2, p)}^p \asymp |\widehat{g}(0)|^p + \sum_{n=0}^{\infty} 2^{-np} \left( \sum_{k \in I(n)} \widehat{g}_k \right)^{\frac{p}{2}}, \quad 0 < p < \infty.
\]

3.1. Boundedness and compactness.

**Proposition 9.** Let \( g \in H(\mathbb{D}) \) and \( v \in \tilde{\mathcal{D}} \) which satisfies the conditions (1.3) and (1.4). Then \( \mathcal{H}_g \) is bounded on \( \mathcal{D}_v \) if and only \( g \in \mathcal{B}(2, \infty) \). Moreover,
\[
\|\mathcal{H}_g\| \asymp \|g - g(0)\|_{\mathcal{B}(2, \infty)}.
\]

**Proof.** We use that the norm of \( \mathcal{H}_g(f) \) can be computed from the Taylor coefficients of \( g \) and the moments of \( f \chi_{[0,1]} \)
\[
\|\mathcal{H}_g(f)\|_{\mathcal{D}_v}^2 = \left| \widehat{g}(1) \int_0^1 f(t) \, dt \right|^2 + \sum_{j=1}^{\infty} j^2 \left| \int_0^1 t^j f(t) \, dt \right|^2 v_{2j-1}.
\]

Assume that \( g \in \mathcal{B}(2, \infty) \), and let us see first that
\[
\sum_{j=2}^{\infty} j^2 \left| \int_0^1 t^j f(t) \, dt \right|^2 v_{2j-1} \leq C\|g - g(0)\|_{\mathcal{B}(2, \infty)}^2 \|f\|_{\mathcal{D}_v}^2.
\]
Indeed, the left-hand side of the above can be decomposed in dyadic pieces in terms of the parameter $j$, and is therefore dominated by
\[
C \sum_{n=1}^{\infty} v_{2n+1-1} 2^{2n} \left( \int_0^1 t^{2n} |f(t)| \, dt \right)^2 \sum_{j \in I(n)} \tilde{g}_{j+1}
\]
\[
\leq C \|g-g(0)\|_{B(2,\infty)}^2 \sum_{n=1}^{\infty} v_{2n+1-1} 2^{3n} \left( \int_0^1 t^{2n} |f(t)| \, dt \right)^2
\]
\[
\leq C \|g-g(0)\|_{B(2,\infty)}^2 \sum_{m=0}^{\infty} \sum_{j \in I(m)} j^2 \left( \int_0^1 t^j |f(t)| \, dt \right)^2 v_{2j-1}
\]
\[
\leq C \|g-g(0)\|_{B(2,\infty)}^2 \|\tilde{H}(f)\|_{D_v}^2.
\]
By Corollary 4, the last quantity is less or equal than $C \|g-g(0)\|_{B(2,\infty)}^2 \|f\|_{D_v}^2$, showing the validity of (3.2).

Moreover, using Corollary 4 again, the remaining terms in (3.1) involving $\hat{g}(1)$ and $\hat{g}(2)$ can easily be controlled by $\|g-g(0)\|_{B(2,\infty)}^2 \|f\|_{D_v}^2$. This together with (3.1) and (3.2), implies that $H_g$ is bounded on $D_v$ with
\[
\|H_g\| \lesssim \|g-g(0)\|_{B(2,\infty)}.
\]

Reciprocally, assume that $H_g$ is bounded on $D_v$. For each $N \in \mathbb{N}$, denote $a_N = 1 - 2^{-N}$ and consider the function $f_N$ defined, for $z \in D$ as follows:
\[
f_N(z) = (1 - a_N)^{\frac{1}{2}} \hat{v}(a_N)^{-1/2} (1 - a_N z)^{1+\lambda/2},
\]
By Lemma 5(ii)(vii), $\lambda > 0$ can be chosen big enough so that
\[
(3.3) \quad \sup_N \|f_N\|_{D_v} < \infty.
\]
We are going to see now that
\[
\|H_g(f_N)\|_{D_v}^2 \geq C 2^{-N} \sum_{j \in I(N)} \tilde{g}_{j+1}, \quad N \in \mathbb{N}.
\]
By (3.1), the left hand side above is larger or equal than
\[
\sum_{n=0}^{\infty} 2^{2n} \left( \int_0^1 t^{2n+1} f_N(t) \, dt \right)^2 v_{2n+2-1} \sum_{j \in I(n)} \tilde{g}_{j+1}
\]
\[
\geq 2^{2N} \left( \int_0^1 t^{2N+1} f_N(t) \, dt \right)^2 v_{2N+2-1} \sum_{j \in I(N)} \tilde{g}_{j+1}
\]
\[
\geq C 2^{2N} \left( \int_{a_N}^1 t^{2N+1} f_N(t) \, dt \right)^2 \hat{v}(a_N) \sum_{j \in I(N)} \tilde{g}_{j+1}
\]
\[
\geq C 2^{-N} \sum_{j \in I(N)} \tilde{g}_{j+1}
\]
which together with (3.3) and the inequality \( \int_0^1 f_0(t) \, dt \geq C > 0 \) gives that
\[
\| g - g(0) \|_{b(2, \infty)} \lesssim \| \mathcal{H}_g \|.
\]

Now, we deal with the compactness of generalized Hilbert operators \( \mathcal{H}_g \).

**Proposition 10.** Let \( g \in H(\mathbb{D}) \) and \( v \in \mathcal{D} \) which satisfies the conditions (1.3) and (1.4). Then \( \mathcal{H}_g \) is compact on \( \mathcal{D}_v \) if and only \( g \in b(2, \infty) \).

We will need the following lemma, which can be easily proved by using (1.2), Hölder’s inequality and (2.4).

**Lemma 11.** Let \( v \) be a radial weight such that (1.2) and (1.3) are satisfied. Let \( \{f_j\}_{j=1}^\infty \) be a sequence in \( \mathcal{D}_v \) such that \( \sup_j \| f_j \|_{\mathcal{D}_v} = K < \infty \) and \( f_j \to 0 \), as \( j \to \infty \), uniformly on compact subsets of \( \mathbb{D} \). Then the following assertions hold:

(i) \( \lim_{j \to \infty} \int_0^1 |f_j(t)| \, dt = 0 \);
(ii) \( \mathcal{H}_g(f_j) \to 0 \), as \( j \to \infty \), uniformly on compact subsets of \( \mathbb{D} \) for each \( g \in H(\mathbb{D}) \).

We also remind the reader that the norm convergence in \( \mathcal{D}_v \), \( v \in \mathcal{D} \), implies the uniform convergence on compact subsets of \( \mathbb{D} \) by [13, Lemma 3.2]. With these tools and from the proof of Theorem 9, Proposition 10 can be shown using standard techniques. Therefore, its proof will be omitted. See [7, Section 7] or [11, Section 7] for further details.

### 3.2. Hilbert-Schmidt operators

First, we observe that
\[
\mathcal{B}(2, 2) \subset \mathcal{D}_v \quad \text{if} \quad v \in \mathcal{D}_v \quad \text{satisfies that} \quad \int_0^1 \tilde{V}_2(s) \, ds < \infty.
\]

In fact, by Lemma 5(vi)
\[
\sup_{j \in \mathbb{N}} (j + 1)v_{2j+1} \asymp \sup_{j \in \mathbb{N}} (j + 1)\tilde{v}
\left(1 - \frac{1}{2j + 1}\right)
\asymp \sup_{j \in \mathbb{N}} \int_{1 - \frac{1}{2j + 1}}^{1 - \frac{1}{2j + 2}} \tilde{V}_2(s) \, ds \leq \int_0^1 \tilde{V}_2(s) \, ds < \infty,
\]
which implies (3.4).

**Proposition 12.** Let \( g \in H(\mathbb{D}) \) and \( v \in \mathcal{D} \) which satisfies the conditions (1.3) and (1.4). Then \( \mathcal{H}_g \in S_2(\mathcal{D}_v) \) if and only if \( g \in \mathcal{B}(2, 2) \). Moreover,
\[
\| \mathcal{H}_g \|_{S_2(\mathcal{D}_v)} \asymp \| g - g(0) \|_{b(2, \infty)}.
\]

**Proof.** Denote \( e_0(z) = 1 \),
\[
e_n(z) = \frac{z^n}{\| z^n \|_{\mathcal{D}_v}} = \frac{z^n}{\sqrt{2n^2v_{2n-1}}}, \quad n \in \mathbb{N}\setminus\{0\},
\]
and consider the basis \{e_n\}_{n \in \mathbb{N}} of \mathcal{D}_v. If \( g(z) = \sum_0^\infty \hat{g}(k)z^k \in H(\mathbb{D}) \), since \( \mathcal{H}_g(e_0)(z) = \frac{g(z)-g(0)}{z} \), by (3.4)
\[
\|\mathcal{H}_g(e_0)\|_{\mathcal{D}_v}^2 \asymp \|g-g(0)\|_{\mathcal{D}_v}^2 \lesssim \|g-g(0)\|_{\mathcal{B}(2,2)}^2.
\]
On the other hand,
\[
\|\mathcal{H}_g(e_n)\|_{\mathcal{D}_v}^2 = \frac{|\hat{g}(1)|^2}{2(n+1)^2n^2v_{2n-1}} + \frac{1}{2n^2v_{2n-1}} \sum_{k=1}^\infty \frac{k^2\hat{g}_{k+1}}{(n+k+1)^2}v_{2k-1}, \quad n \in \mathbb{N}.
\]
Clearly,
\[
\sum_{n=1}^\infty \frac{1}{n^2(n+k+1)^2v_{2n-1}} \gtrsim \frac{1}{v_{2k+1}} \sum_{m=1}^\infty \sum_{n=m(k+1)} (m+1)(k+1) \frac{1}{n^2(n+k+1)^2} \gtrsim \frac{1}{(k+1)^3v_{2k+1}}, \quad k \in \mathbb{N}.
\]
The opposite inequality also holds, since \( v \in \hat{\mathcal{D}} \), from Lemma 5(vi) and (1.3), we get
\[
\sum_{n=1}^k \frac{1}{n^2(n+k+1)^2v_{2n-1}} \lesssim \frac{1}{(k+1)^2} \sum_{n=1}^k \frac{1}{\hat{v}(1 - \frac{1}{2n+1})} \int_1^{1-\frac{1}{(n+1)}} ds \lesssim \frac{1}{(k+1)^2} \int_0^{1-\frac{1}{(k+1)}} \hat{v}(s) ds \lesssim \frac{1}{(k+1)^2} \int_1^{1-\frac{1}{(k+1)}} \hat{v}_2(s) ds \lesssim \frac{1}{(k+1)^3v_{2k}}, \quad k \in \mathbb{N}.
\]
For the rest of the values of \( n \), using again Lemma 5, and (1.4)
\[
\sum_{n=k+1}^\infty \frac{1}{n^2(n+k+1)^2v_{2n-1}} \lesssim \sum_{n=k+1}^\infty \frac{1}{n^2v_{2n-1}} \int_1^{1-\frac{1}{n+1}} ds \lesssim \int_1^{1-\frac{1}{k+1}} V_2(s) ds \lesssim \int_0^{1-\frac{1}{k+1}} \hat{V}_4(s) ds \lesssim \frac{1}{(k+1)^3v_{2k}}, \quad k \in \mathbb{N}.
\]
From here, using Lemma 5(iv) and (3.5), we deduce
\[
\sum_{n=1}^\infty \|\mathcal{H}_g(e_n)\|_{\mathcal{D}_v}^2 = |\hat{g}(1)|^2 \sum_{n=1}^\infty \frac{1}{2(n+1)^2n^2v_{2n-1}} + \sum_{k=1}^\infty \frac{k^2\hat{g}_{k+1}v_{2k-1}}{2n^2(n+k+1)^2v_{2n-1}} \gtrsim |\hat{g}(1)|^2 \sum_{k=1}^\infty (k+1)|\hat{g}(k+1)|^2 \gtrsim \|g-g(0)\|_{\mathcal{B}(2,2)}^2,
\]
3.3. **Proof of Theorem 1.** (ii) ⇒ (i). In order to obtain this implication in Theorem 1 we use results from [10] on complex interpolation for the mixed norm spaces \( B(2, p) \) and from [14, Theorem 2.6] for Schatten classes. Given \((X_0, X_1)\) a compatible pair of Banach spaces, we denote by \((X_0, X_1)[\theta]\) the complex interpolating space of exponent \( \theta \in [0, 1] \). With the notation from [10] for \( D \) and \( A_{p,q}^{\delta,k} \), if one chooses the particular case \( D = \mathbb{D} \) in [10], then \( B(2, q) = A_{1,1}^{2,q} \). In this way, the following result on complex interpolation on the mixed norm space \( B(2, q) \) is a consequence of [10, Theorem 3.1] (see also [10, Theorem B]).

**Theorem A.** Let \( 0 < q_0 < q_1 \leq \infty \) and \( \theta \in (0, 1) \). If 
\[
\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},
\]
then
\[
(B(2, q_0), B(2, q_1))[\theta] = B(2, q).
\]

**Proposition 13.** Let \( 2 < p < \infty \) and \( v \in \hat{D} \) satisfying (1.3) and (1.4). If \( g \in B(2, p) \), then \( \mathcal{H}_g \in S_p(D_v) \) and \( \| \mathcal{H}_g \|_{S_p(D_v)} \lesssim \| g - g(0) \|_{B(2,p)} \).

**Proof.** Let us consider the linear operator \( T(g) = \mathcal{H}_g \). By Proposition 12 the operator \( T \) is bounded from \( B(2, 2) \) to \( S_2(D_v) \) with
\[
\| T(g) \|_{S_2(D_v)} = \| \mathcal{H}_g \|_{S_2(D_v)} \lesssim \| g - g(0) \|_{B(2,2)}.
\]
Analogously, by Proposition 9, \( T \) is bounded from \( B(2, \infty) \) to \( S_\infty(D_v) \) with
\[
\| T(g) \|_{S_\infty(D_v)} = \| \mathcal{H}_g \|_{S_\infty(D_v)} \lesssim \| g - g(0) \|_{B(2,\infty)}.
\]
So, the previous inequalities together with [2, Theorem 4.1.2, p. 88], Theorem A and [14, Theorem 2.6] imply that \( T : B(2, p) \to S_p(D_v) \) is bounded. This finishes the proof.

In order to deal with the case \( 0 < p < 2 \), we will need two technical lemmas.

**Lemma 14.** Let \( v \in \hat{D} \) satisfying (1.4). Then, for any \( q > 0 \) there is a constant \( C(q, v) \) such that
\[
\sum_{n=k}^{\infty} 2^{-3q_n} v_{2n+1}^{-q} \leq C(q, v) 2^{-3q_k} v_{2k+1}^{-q}.
\]
for all \( k \in \mathbb{N} \cup \{0\} \).

**Proof.** First, we prove that there exists \( \gamma = \gamma(v) \in (0, 1) \) such that
\[
\int_{\frac{1}{r+1}}^{1} \frac{(1-s)^2}{\gamma(v)} \, ds \leq \gamma \int_{r}^{1} \frac{(1-s)^2}{\gamma(v)} \, ds, \quad 0 < r < 1.
\]
Taking into account that $v \in \hat{D}$ and (1.4)

\begin{equation}
\int_{1+\tau}^{1} (1-s)^2 \frac{1}{\hat{v}(s)} ds \leq C(v) \left( \int_{0}^{1+\tau} \hat{V}_{4}(s) ds \right)^{-1} \\
\leq C(v) \frac{(1-r)^3}{\hat{v}(\frac{1+\tau}{2})} \leq C(v) \int_{r}^{1+\tau} (1-s)^2 \frac{1}{\hat{v}(s)} ds, \quad 0 < r < 1,
\end{equation}

which is equivalent to (3.6). Then, Lemma 5 and (3.7) yield

\begin{equation}
\sum_{n=1}^{\infty} 2^{-3q} v_{2^{n+1}} \leq C(q,v) \sum_{n=1}^{\infty} \left( \frac{1}{\hat{v} \left( 1 - \frac{1}{2^{n+1}} \right)} \int_{1-\frac{1}{2^{n+1}}}^{1} (1-s)^2 ds \right)^{q} \\
\leq C(q,v) \left( \int_{1-\frac{1}{2^{n+1}}}^{1} \frac{(1-s)^2}{\hat{v}(s)} ds \right)^{q} \sum_{n=1}^{\infty} \gamma^{n-k}.
\end{equation}

Since the last sum is convergent, all of the above is controlled by

\begin{equation}
C(q,v) \left( \frac{1}{2^{3k} \hat{v} \left( 1 - \frac{1}{2^{k+1}} \right)} \right)^{q} \leq C(q,v) \left( \frac{1}{2^{3k} v_{2^{k+1}}} \right)^{q}, \quad k \in \mathbb{N} \cup \{0\}.
\end{equation}

\Box

**Lemma 15.** Let $v \in \hat{D}$ satisfying (1.3). Then, for any $q > 0$ there is a constant $C(q,v)$ such that

\begin{equation}
\sum_{n=1}^{k} 2^{-qn} v_{2^{n+1}} \leq C(q,v) 2^{-qk} v_{2^{k+1}}
\end{equation}

for all $k \in \mathbb{N}$.

**Proof.** Now, we prove that there exists $\gamma = \gamma(v) \in (0,1)$ such that

\begin{equation}
\int_{1+\tau}^{1} \hat{V}_{2}(s) ds \leq \gamma \int_{r}^{1} \hat{V}_{2}(s) ds, \quad 0 < r < 1.
\end{equation}

By (1.3)

\begin{equation}
\int_{1+\tau}^{1} \hat{V}_{2}(s) ds \leq \frac{C(v)}{\int_{0}^{1+\tau} \frac{1}{\hat{v}(s)} ds} \leq \frac{C(v)}{\int_{r}^{1+\tau} \frac{1}{\hat{v}(s)} ds} \\
\leq C(v) \frac{\hat{v}(r)}{1-r} \leq C(v) \int_{r}^{1+\tau} \hat{V}_{2}(s) ds, \quad 0 < r < 1,
\end{equation}
which is equivalent to (3.8). So, by (3.9) and Lemma 5

\[
\sum_{n=1}^{k} 2^{-qn} v_{2^{n+1}} \leq C(q,v) \sum_{n=1}^{k} 2^{-qn} v \left(1 - \frac{1}{2^{n+1}}\right)^{-q} 
\]

\[
\leq C(q,v) \sum_{n=1}^{k} \left(\int_{1-\frac{1}{2^{n+1}}}^{1} \tilde{V}_2(s) \, ds\right)^{-q} 
\]

\[
\leq C(q,v) \left(\int_{1-\frac{1}{2^{n+1}}}^{1} \tilde{V}_2(s) \, ds\right)^{-q} \sum_{n=1}^{k} \gamma^{k-n} 
\]

\[
\leq C(q,v) 2^{-qk} v \left(1 - \frac{1}{2^{k+1}}\right)^{-q} \leq C(q,v) 2^{-qn} v_{2^{n+1}}, \quad k \in \mathbb{N}.
\]

\[\square\]

Now we are ready to prove the remaining case of (ii) \(\Rightarrow\) (i) in Theorem 1.

**Proposition 16.** Let \(v \in \mathring{D}\) satisfying (1.3) and (1.4). If \(0 < p < 2\) and 
\(g \in B(2,p)\), then \(H_g \in S_p(D_v)\) and \(\|H_g\|_{S_p(D_v)} \lesssim \|g - g(0)\|_{B(2,p)}\).

**Proof.** Let \(g(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k \in B(2,p)\). We use the orthonormal basis \(\{e_n\}_{n=0}^{\infty}\), where

\[
e_0(z) = 1 \quad \text{and} \quad e_n(z) = \frac{\sum_{k+1 \in I(n)} z^k}{\left(\sum_{k+1 \in I(n)} k^2 v_{2k-1}\right)^{1/2}}, \quad n \in \mathbb{N}.
\]

Since \(0 < p < 2\), \(B(2,p) \subset B(2,2)\). Thus, by Proposition 12, \(H_g \in S_2(D_v)\), and in particular \(H_g\) is a compact operator. Therefore, by [14, Theorem 1.26] and [14, Corollary 1.32] (applied to \(H^*H_g\))

\[
\|H_g\|_{S_p(D_v)}^p = \|H^*_g H_g\|_{S_{p/2}(D_v)}^{\frac{p}{2}} \leq \sum_{n=0}^{\infty} \langle H_g e_n, H_g e_n \rangle_{D_v}^{\frac{p}{2}}.
\]

Since \(H_g(e_0)(z) = \frac{g(z) - g(0)}{z}\), by (3.4)

\[
\|H_g(e_0)\|_{D_v}^2 \lesssim \|g - g(0)\|_{D_v}^2 \lesssim \|g - g(0)\|_{B(2,p)}^2.
\]

Moreover, for \(n \in \mathbb{N}\)

\[
H_g(e_n)(z) = \sum_{j=0}^{\infty} (j+1) \hat{g}(j+1) \left(\int_{0}^{1} t^j e_n(t) \, dt\right) z^j
\]

\[
= \left(\sum_{k+1 \in I(n)} (k+1)^2 v_{2k+1}\right)^{-1/2} \sum_{j=0}^{\infty} (j+1) \hat{g}(j+1) \left(\sum_{m+1 \in I(n)} \frac{1}{m+j+1}\right) z^j.
\]
So, it is enough to prove

\[(3.11)\]

\[
\sum_{n=0}^{\infty} \langle \mathcal{H}_g e_n, \mathcal{H}_g e_n \rangle_{\mathcal{D}_0}^{\frac{p}{2}} \lesssim \|g - g(0)\|_{B(2,p)}^p.
\]

By Lemma 5(v), \(\sum_{k+1 \in I(n)} (k+1)^2 v_{2k+1} \approx 2^{3n} v_{2^n+1}\), yielding

\[(3.12)\]

\[
\sum_{n=1}^{\infty} \langle \mathcal{H}_g e_n, \mathcal{H}_g e_n \rangle_{\mathcal{D}_0}^{\frac{p}{2}} \lesssim \sum_{n=1}^{\infty} 2^{-\frac{3n}{2}} v_{2^n+1}^{-\frac{p}{2}} \left( \sum_{m+1 \in I(n)} \frac{1}{m + j + 1} \right) \left( \sum_{j=1}^{\infty} j^2 \widetilde{g}_j \right) v_{2j-1}^{-\frac{p}{2}}
\]

\[
\lesssim J_1 + J_2 + J_3,
\]

where \(J_1\) is the first sum on the right hand side, and the second sum is decomposed in \(j \leq 2^{n+1} - 1\) (\(J_2\)) and \(j \geq 2^{n+1}\) (\(J_3\)). By Lemma 14,

\[(3.13)\]

\[
J_1 \lesssim |\widetilde{g}(1)|^p \sum_{n=1}^{\infty} 2^{-\frac{3n}{2}} v_{2^n+1}^{-\frac{p}{2}} \lesssim \|g - g(0)\|_{B(2,p)}^p.
\]

We now estimate for \(J_2\), which satisfies

\[(3.14)\]

\[
J_2 \approx \sum_{n=1}^{\infty} 2^{-\frac{3n}{2}} v_{2^n+1}^{-\frac{p}{2}} \left( \sum_{k=0}^{n} \sum_{j \in I(k)} j^2 \widetilde{g}_{j+1} v_{2j-1} \right)^{\frac{p}{2}}.
\]

Using Lemma 5 and Lemma 14, the above can be bounded by

\[(3.15)\]

\[
\sum_{n=1}^{\infty} 2^{-\frac{3n}{2}} v_{2^n+1}^{-\frac{p}{2}} \left( \sum_{k=0}^{n} 2^{2k} v_{2k+1} \sum_{j \in I(k)} \widetilde{g}_{j+1} \right)^{\frac{p}{2}}
\]

\[
\lesssim \sum_{n=1}^{\infty} 2^{-\frac{3n}{2}} v_{2^n+1}^{-\frac{p}{2}} \sum_{k=0}^{n} 2^{2k} v_{2k+1}^{p/2} \left( \sum_{j \in I(k)} \widetilde{g}_{j+1} \right)^{\frac{p}{2}}
\]

\[
\lesssim \sum_{k=0}^{\infty} 2^{2k} v_{2k+1}^{p/2} \left( \sum_{j \in I(k)} \widetilde{g}_{j+1} \right)^{\frac{p}{2}} \sum_{n=k}^{\infty} 2^{-\frac{3n}{2}} v_{2^n+1}^{-\frac{p}{2}}
\]

\[
\lesssim \sum_{k=0}^{\infty} 2^{-\frac{3k}{2}} \left( \sum_{j \in I(k)} \widetilde{g}_{j+1} \right)^{\frac{p}{2}} \lesssim \|g - g(0)\|_{B(2,p)}^p.
\]

To complete the estimation, we turn to \(J_3\), which satisfies

\[
J_3 \approx \sum_{n=1}^{\infty} 2^{-\frac{3n}{2}} v_{2^n+1}^{-\frac{p}{2}} \left( \sum_{k=n+1}^{\infty} 2^{-2k} \sum_{j \in I(k)} j^2 \widetilde{g}_{j+1} v_{2j-1} \right)^{\frac{p}{2}}.
\]
By Lemma 5 and Lemma 15, then $J_3$ is controlled by

$$\sum_{n=1}^{\infty} 2^{-\frac{mp}{p}} v_{2n+1} \left( \sum_{k=n+1}^{\infty} \sum_{j \in I(k)} \tilde{g}_{j+1} \right)^{\frac{p}{2}}$$

This together with (3.12), (3.13), (3.14) and (3.15) gives (3.11).

3.4. Proof of Theorem 1. (i) ⇒ (ii). In order to show the remaining part of the proof of Theorem 1 we will employ [14, Theorem 1.28].

**Proposition 17.** Let $g \in H(\mathbb{D})$ and $v \in \hat{D}$ satisfying (1.3) and (1.4). If $1 \leq p < \infty$ and $H_g \in S_p(D_v)$, then $g \in B(2, p)$ and

$$\|g - g(0)\|_{B(2, p)} \lesssim \|H_g\|_{S_p(D_v)}.$$  

**Proof.** Let $\{e_n\}_{n=0}^{\infty}$ be the orthonormal basis defined in (3.10). We write $M_0 = |\hat{g}(1)|$ and

$$M_n = M_n(g, v) = \left( \sum_{k+1 \in I(n)} k^2 \tilde{g}_{k+1} v_{2k-1} \right)^{1/2}, \quad n \in \mathbb{N}.$$  

Let us consider $N_g = \{n \in \mathbb{N} : M_n \neq 0\}$ and

$$\sigma_n = M_n^{-1} \sum_{k+1 \in I(n)} (k+1) \tilde{g}(k+1) z^k, \quad n \in N_g.$$  

The set \( \{ \sigma_n \}_{n \in \mathbb{N}_g} \) is orthonormal. Denote now \( h_k := k^2 v_{2k-1} \widetilde{g}_{k+1} \). By Lemma 5,

\[
|\langle H g e_n, \sigma_n \rangle_{D_v}| \leq \left( \sum_{k+1 \in I(n)} k^2 v_{2k-1} \right)^{-\frac{1}{2}} \left( \sum_{k+1 \in I(n)} h_k \right) \left( \sum_{m+1 \in I(n)} \frac{1}{m + k + 1} \right)
\]

\[
\leq \left( \sum_{k+1 \in I(n)} k^2 v_{2k-1} \right)^{-\frac{1}{2}} \left( \sum_{k+1 \in I(n)} h_k \right) \left( 2^{-n} \sum_{k+1 \in I(n)} \widetilde{g}_{k+1} \right) \geq \parallel g - g(0) \parallel_{B(2, p)}.
\]

Hence, by [14, Theorem 1.28]

\[
\infty > \parallel H g \parallel_{S_p(D_v)}^p \geq \sum_{n \in \mathbb{N}_g} |\langle H g e_n, \sigma_n \rangle_{D_v}|^p \geq \sum_{n \in \mathbb{N}_g} \left( 2^{-n} \sum_{k+1 \in I(n)} \widetilde{g}_{k+1} \right)^{\frac{p}{2}} \geq \parallel g - g(0) \parallel_{B(2, p)}^p.
\]

Finally, Theorem 1 follows from Propositions 9, 12, 13, 16 and 17.

### 3.5. Proof of Corollary 2.

Since \( \omega \in \widehat{D} \), by [12, Theorem 4.2] and Lemma 5,

\[
\| f \|_{A^2_{\omega}}^2 = \| f(0) \|_{\omega(\mathbb{D})}^2 + \| f' \|_{A^2_{\omega, v}}^2
\]

(3.16)

\[
\geq |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|) \hat{\omega}(|z|) dA(z)
\]

\[
= \| f \|_{A^2_{v, \omega}}^2, \quad f \in H(\mathbb{D}),
\]

where \( v(|z|) = (1 - |z|) \omega(|z|) \). Since \( \omega \in \widehat{D} \), we have that \( v \in \widehat{D} \) and \( \hat{v}(|z|) \approx (1 - |z|)^2 \omega(|z|), \ z \in \mathbb{D} \). So, using that \( \hat{\omega} \) is a non-decreasing function

\[
\sup_{0 < r < 1} \left( \int_0^1 \frac{\hat{v}(s)}{(1 - s)^2} ds \right) \left( \int_0^r \frac{1}{\hat{v}(s)} ds \right) \approx \sup_{0 < r < 1} \left( \int_0^1 \frac{\hat{\omega}(s)}{(1 - s)^2} ds \right) \left( \int_0^r \frac{1}{(1 - s)^2 \hat{\omega}(s)} ds \right) < \infty.
\]
Moreover, by (1.5)

\[
\sup_{0 < r < 1} \left( \int_0^r \frac{\hat{\omega}(s)}{(1-s)^4} \, ds \right)^{\frac{1}{2}} \left( \int_r^1 \frac{(1-s)^2}{\hat{\omega}(s)} \, ds \right)^{\frac{1}{2}} \\
\lesssim \sup_{0 < r < 1} \left( \int_0^r \frac{\hat{\omega}(s)}{(1-s)^2} \, ds \right)^{\frac{1}{2}} \left( \int_r^1 \frac{1}{\hat{\omega}(s)} \, ds \right)^{\frac{1}{2}} < \infty.
\]

Therefore, \(v\) satisfies both conditions, (1.3) and (1.4). This together with (3.16) and Theorem 1, finishes the proof.

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