Overactivated transport in the localized phase of the superconductor-insulator transition

V. Humbert 1,2, M. Ortuño3, A. M. Somoza4, L. Bergé4, L. Dumoulin4, C. A Marrache-Kikuchi4

1 Université Paris-Saclay, CNRS/IN2P3, CSNSM, 91405 Orsay, France 2 Unité Mixte de Physique, CNRS, Thales, Université Paris-Saclay, 91767, Palaiseau, France 3 Departamento de Física - CIoyN, Universidad de Murcia, Murcia 30071, Spain 4 Université Paris-Saclay, CNRS/IN2P3, UCLab, 91405 Orsay, France

Beyond a critical disorder, two-dimensional (2D) superconductors become insulating. In this Superconductor-Insulator Transition (SIT), the nature of the insulator is still controversial. Here, we present an extensive experimental study on insulating Nb$_3$Si$_{1-x}$ close to the SIT, as well as corresponding numerical simulations of the electrical conductivity. At low temperatures, we show that electronic transport is activated and dominated by charging energies. The sample thickness variation results in a large spread of activation temperatures, fine-tuned via disorder. We show numerically and experimentally that the localization length varies exponentially with thickness. At the lowest temperatures, overactivated behavior is observed in the vicinity of the SIT and the increase in the activation energy can be attributed to the superconducting gap. We derive a relation between the increase in activation energy and the temperature below which overactivated behavior is observed. This relation is verified by many different quasi-2D systems.

The recent interest in low dimensional systems primarily stems from the many exotic electronic phases they exhibit, due to their extreme sensitivity to any emerging order. The Superconductor-Insulator Transition (SIT) in two-dimensional disordered materials is one such famous example. There, superconductivity competes with localization and Coulomb interactions to give rise to unusual electronic phases. Despite several decades of investigation of the SIT, the nature of the insulator is still a subject of debate. Some argue that Coulomb interactions and disorder prevail so that the insulator is fermionic, whereas others claim that localized Cooper pairs exist even if global superconducting coherence is suppressed. Moreover, there is a controversy on whether the system is electronically homogeneous, granular or fractal. Especially intriguing are the reports of very strong temperature dependences of the resistivity which have prompted fierce debate as to their possible interpretation. Arrhenius, or activated, behavior is found in the electronic transport of many insulators. In this case, the temperature dependence of the resistance is of the form:

\[ R = R_0 \exp \left( \frac{T_0}{T} \right)^\alpha, \]

where \( T_0 \) is the activation temperature. It is usually associated either with a band gap or with nearest neighbor hopping. In disordered materials, the insulating behavior originates from charge carriers being spatially localized and activated behavior usually takes place at relatively high temperature. At low temperatures, electrons have to compromise between tunneling to close neighbors at the price of an energy mismatch, or traveling further where the hopping energy difference may be smaller. This process is known as variable-range hopping (VRH) and results in a temperature dependence of the resistance of the form:

\[ R = R_0 \exp \left( \frac{T_0}{T} \right)^\alpha. \]

For non-interacting systems, Mot26 predicted an exponent \( \alpha = D/(D + 1) \), with \( D \) the system dimensionality. Efros and Shklovskii27 extended the argument to Coulomb interacting systems and obtained an exponent of 1/2, independent of dimensionality. In both cases, the temperature dependence is subactivated, i.e. \( \alpha < 1 \). Experimentally, however, in systems close to the SIT, activated28 and, in some cases, overactivated dependencies are found at very low temperatures. This unfamiliar behavior calls for investigation and is the subject of the present work.

Here, we investigate, both experimentally and numerically, transport properties of quasi-2D systems on the insulating side of the SIT. We have measured over 80 insulating Nb$_3$Si$_{1-x}$ films down to 7 mK to span several orders of magnitude in activation temperatures. We perform Monte Carlo simulations of VRH conductivity on quasi-2D systems, so as to reproduce the experimental situation. We analyze at which conditions activated behavior is present at very low temperatures and investigate the different scenarios leading to overactivated behavior. We compare our experimental results with the outcome of our simulations to extract the main physics of conduction mechanisms in the insulating state of the SIT. Our main conclusion is that, close to the SIT, appearance of Cooper pairing enhances the activation temperature at low temperature.

1 Experimental system. Thin metal alloy films constitute model systems to study the zero-magnetic field superconductor-insulator transition. In these compounds, the SIT can be driven either by a reduction of the sample thickness or by a variation in stoichiometry which directly affects the amount of disorder in the system. In amorphous Nb$_x$Si$_{1-x}$ (a-NbSi) which we consider, thermal treatment is yet another parameter that enables us to very finely and progressively tune a single sample disorder30. The combination of these experimental tuning parameters allows us to study the effects of thickness and disorder independently.

We have considered over 80 insulating a-NbSi thin films with compositions (\( x = 9 \% \) and 13.5 %), thicknesses (\( d \in [20, 2500] \) Å) and heat treatments (\( \theta_{ht} \in [70, 160]^{\circ} \)) that allowed us to investigate localized regimes with activation temperatures spanning over four decades. The as-deposited sample conductivity has been studied from room temperature down to 7 mK (see Materials and Methods). The corresponding temperature dependence of the samples sheet resistance is
shown in figure 1 (panel (a) $x = 13.5\%$ batch, panel (c) $x = 9\%$ batch). The films considered in this work are all insulators and, as expected, a thickness reduction drives the system deeper into the insulating regime. All $x = 13.5\%$ samples have been progressively annealed from 70 to 160°C, to slightly change the amount of disorder in the films. Panels (d) to (f) of figure 1 show the effect of successive heat treatments on a single sample: in our case, the higher the heat treatment temperature $\theta_{ht}$, the more insulating the film becomes.

For all films and all heat treatments, the low temperature sheet resistance follows an Arrhenius-type law as given by equation (1). This is particularly visible in figure 1 (panels (b) and (c)). The $x = 9\%$ samples show a single activated behavior, while the $x = 13.5\%$ samples deviate from this law at the lowest temperatures. Previous works have reported such a behavior which have been referred to as overactivated and associated with the presence of superconductivity in the system.[23, 24] However, this intriguing feature has not been systematically addressed experimentally. As can be seen from figure 1, the resistance actually undergoes a crossover from an activated regime to an overactivated regime at very low temperatures, with a slightly larger characteristic energy $T_0$. A quantitative understanding of both regimes is one of the main goals of this work.

2 Activated behavior. Let us first study the activated regime and derive the expression for its characteristic temperature. Close to the SIT, one expects thin films to have very large localization lengths $\xi_{loc}$ and therefore very high dielectric constants $\kappa$. As a consequence, the electric field lines will tend to stay within the film, resulting in an overactivated regime at very low temperatures, with a slightly larger characteristic energy $T_0$. All samples then end up with very similar activation temperatures, of the order of $kT_0 = E_0$, where $k$ is Boltzmann constant.

Logarithmic interactions in disordered systems imply the existence of charging energies, and so effective electronic granularity, irrespective of the film morphology. We can think of our samples as formed by effective grains with a charging energy given by Eq. (4).

3 Exponential dependence of the localization length with thickness. Let us come back to the experimental situation. In figure 2, the activation temperature $T_0$ is plotted as a function of sample thickness for $x = 13.5\%$ and all heat treatment temperatures. The first striking feature of this plot is that $T_0$ varies over almost four orders of magnitude. Except for the thinnest sample ($d = 20 \, \text{Å}$), $T_0$ presents an exponential dependence on sample thickness ($T_0 \propto \xi_{loc}$), with $\zeta$ that depends on the heat treatment temperature, i.e. the degree of disorder.

According to Eqs. (4) and (5), when the localization length is large, $T_0 \sim (kd)^{-1} \sim \xi_{loc}^{-2}.d^{-1}$. To explain the exponential dependence of $T_0$ on thickness observed experimentally, $\xi_{loc}$ must increase exponentially with thickness. To investigate this dependence, we have calculated $\xi_{loc}$ for square samples of finite thickness in an Anderson model, through a one-parameter scaling analysis of their conductance. The main results of these simulations are presented in figure 2d where we plot $\xi_{loc}.a$ as a function of $W/ t$. ($W$ is the disorder level and $t$ the kinetic energy of the electrons) for thicknesses ranging from $d = 1$ to 7 layers.

For small disorder levels, one can appreciate the strong dependence of $\xi_{loc}$ on both $W$ and $d$. More quantitatively, our numerical simulations establish that in this regime the localization length is of the form (Fig. 2e):

$$\xi_{loc} \approx A \exp \left( \frac{\eta d}{W^2} \right)$$

with $A$ and $\eta$ approximately constants, but non-universal. The small shift between data for different thicknesses can be taken into account with a thickness-dependent prefactor.

Although the exponential dependence of the localization length with thickness has been observed in many systems, it is often attributed to a thickness-induced change in the amount of the disorder $W$. We are here able to distinguish the quite distinct effects of both parameters. Equation (3) is in line with the self-consistent theory of Anderson localization[40] that predicts this exponential dependence of the localization length $\xi$ on disorder $W$ for 2D systems. It is also in agreement with analytical self-consistent results in weakly localized 2D systems[42] and with numerical simulations performed on quasi-1D systems.[43] An exponential dependence of the localization length with thickness was observed in Mo-C films[44] in VRH regimes.

The activation temperature therefore exponentially decreases with thickness as shown by the straight lines in Fig. 2b. Moreover, we expect each $\theta_{ht}$ to correspond to a single value of $W$, so that, the disorder-induced spread in $T_0$ should linearly depend on the thickness, which is experimentally the case in our films. For small thicknesses, the contribution of the host dielectric constant $\kappa_0$ becomes non-negligible and $T_0$ is smaller than the exponential behavior, as shown by the solid line.
Figure 1 | Low temperature resistance measurements. (a) Temperature dependence of the sheet resistance for the $x = 13.5\%$ as-deposited batch. (b) and (c) Sheet resistance as a function of $1/T$ on a semilogarithmic scale to highlight the activated behavior for the $x = 13.5\%$ (b) and $x = 9\%$ (c) as-deposited samples. (d) to (f) Effect of successive heat treatments on the sheet resistance of the $13.5\%$ 40 Å (d), 35 Å (e) and 30 Å (f) samples. The insets show the $R_{\square}(T)$, while the main panels display the same data as a function of $1/T$. (g) At low temperature, at about 0.9 K for the $13.5\%$ 30 Å as-deposited film, there is a crossover between two activated regimes.

Figure 2 | (a) A sketch of the model used for numerical simulations. The electrode connected to the left edge is grounded, the electrode at the right edge has a potential $V$. The bottom panel shows a lateral view of the model where the capacitors $C_0$ connecting the system to the gate can be seen. (b) Numerical results of the resistance as a function of inverse temperature. Each color corresponds to a different value of $C_0$. The solid lines correspond to the clean case whereas the dashed lines correspond to the case where disorder is added (5\% of the nodes have fixed charges $\pm 1$ at random). (c) Characteristic temperature $T_0$ as a function of the sample thickness (in Å) for different heat treatments ($x = 13.5\%$ sample). The solid line corresponds to equations $1$-$3$ with $\kappa_0 \approx 250$. For each thickness, the disorder level was averaged over the different heat treatments. Each dashed line corresponds to the exponential thickness dependence for a given heat treatment (as-deposited and $\theta_{ht} = 160^\circ$ C). (d) Localization length on a logarithmic scale as a function of the disorder level $W$ (relative to the kinetic energy $t$) for thicknesses ranging from 1 to 7 layers. (e) Exponential dependence of $\xi_{loc}$ with $d/W^2$. 
4 Overactivation. We will now turn to the overactivated regime, i.e. the increase in the activation energy at the lowest temperatures, in the immediate vicinity of the SIT.

The thinnest \((d = 20 \, \text{Å})\) samples do not exhibit any overactivated regime. However, the rest of the samples do. For \(d = 45 \, \text{Å}\) the whole activated regime is actually overactivated. To appreciate the systematicity of the overactivated behavior, we scaled the high temperature activated regime for the 13.5% 35 ˚A-thick sample (figure 3a) for the different heat treatment temperatures (i.e. disorder). By doing so, the overactivated regimes cannot be made to scale. Moreover, the ratio of the overactivated characteristic temperature over the activated one decreases with the disorder level.

We have extracted \(T_0^\times\), whenever an overactivated was observed, by fitting the low temperature data to equation [1]. The transition temperatures \(T_{\text{cross}}\) between the two activated regimes were also extracted from the crossing point between the two fits. The corresponding results are plotted in figure 3d as a function of the disorder level, quantified by the high temperature (200 K) resistance \(R_{200\,\text{K}}\). We note that these temperatures vary over almost three orders of magnitude. Moreover, \(T_{\text{cross}}\) gets closer to \(T_0\) as we approach the SIT, at about \(R_{200\,\text{K}} \approx 3450 \, \Omega\). Our aim is now to understand the relation between these characteristic temperatures since they shine light on the conduction mechanisms.

Since the overactivated regime is only found close to the SIT, it is reasonable to assume it is related to superconductivity. Let us examine what will happen if, due to the proximity to the SIT, superconductivity sets in locally in some (effective) grains. In this case, Cooper pairs will be phase coherent on a short range, but there will not be any global phase coherence. There are then two different possibilities. The first one is that conduction is still dominated by one-electron processes. Then, when a single electron enters a superconducting grain, it has to pay an extra energy penalty associated with the superconducting gap. The second possibility is that conduction between superconducting grains is ensured by tunneling of Cooper pairs. In this case the energy penalty for a pair entering a superconducting grain is four times the charging energy, since the grain becomes doubly charged. In both cases, the crossover temperature to the overactivated regime must be the order of the pairing energy. In our experimental case, \(T_{\text{cross}}\) is much smaller than the activation energy and since \(T_0 \sim E_0\), we conclude that the pairing energy is much smaller than the charging energy, so that conduction by single electrons is more likely to be the main conduction mechanism.

We have performed Monte Carlo simulations of the conductivity on our random capacitor model to quantify the increase in activation energy when the grains become superconducting. We consider the model with only charging energies and no long-range interactions, since, as we have seen, both cases produce similar results, and the higher computational efficiency of the former is convenient in this highly demanding calculation.

We assumed that grains become superconducting at the critical temperature \(T_{\text{cross}}\). Below this temperature, the new energy penalty to charge a superconducting grain is:

\[
E_0' = E_0 + \Delta(T),
\]

where \(\Delta\) is the superconducting gap. Assuming a BCS ratio of 2, we have [23]

\[
E_0' = E_0 + 2kT_c \sqrt{\frac{T_{\text{cross}} - T}{T_{\text{cross}}}}. \tag{8}
\]

The results of the simulations are shown in figure 3b, where we plot \(R\) versus \(1/T\) for several values of \(T_{\text{cross}}\) (in units of \(T_0\)). The black curve corresponds to the non-superconducting case (pure activated behavior) and the rest to different values of \(T_{\text{cross}}/T_0\), chosen to reproduce the experimental situation. At temperatures below \(T_{\text{cross}}\), we obtain a new roughly activated regime, as experimentally observed. As expected, the transition temperature between both regimes coincides with \(T_{\text{cross}}\).

In figure 3c we compare our numerical simulations for \(T_{\text{cross}} = 0.14T_0\) with the experimental data for the 35 Å sample (\(\theta_{\text{ht}} = 110^\circ\)) by scaling the theoretical curve so that both coincide in the activated regime. The overactivated behavior is fairly well reproduced by our simulations.

Extracting characteristic temperatures from our numerical results in the same way as from the experimental data, we have established the relation:

\[
T_0^\times - T_0 = \gamma T_{\text{cross}} \tag{9}
\]

and found \(\gamma = 1.8\). For each experimental point, we derived the value of \(\gamma\) given by Eq. (9) and represent it as an orange triangle (bottom panel of figure 3c). On average, we find \(\gamma = 1.5\) (black line). The agreement with theoretical predictions is fairly good, clearly indicating that the overactivated regime is a consequence of electrons having to pay the superconducting gap penalty, proportional to \(T_{\text{cross}}\). Close to the SIT, uncertainties on \(\gamma\) are large, since the activated regime expands on a very narrow temperature range.

We have also reanalyzed the overactivated behavior found in the literature and compared the experimental data to our prediction. This can be seen in figure 3e where we represent \(\gamma\) obtained from equation (9). The value of \(\gamma\) is fairly constant for all systems, and of the order of unity. This is remarkable given the variety of systems considered, and the three orders of magnitude over which \(T_0\) extends.

Let us note that the theoretical prediction depends both on the BCS ratio linking \(T_c\) and \(\Delta\), which we took to be 2 following STM data for a-NbSi [23] and on the proportion of superconducting grains. If, instead of having all grains superconducting, only a fraction \(p\) of them are, \(T_0^\times - T_0\) should be proportional to \(p\). From the agreement between the simulations and the experiment, we can conclude that a large fraction of the grains become superconducting.

We can explain the dependence of \(T_{\text{cross}}\) on disorder (Figure 3d) by assuming that the superconducting gap is of the form

\[
\Delta = \frac{\Delta_{\text{SC}}}{d} \left(1 + \frac{\xi_{\text{loc}}}{\xi_{\text{SC}}}\right) \sim T_{\text{cross}} \tag{10}
\]

Indeed, close to the SIT, equation (10) is in agreement with the experimental dependence of \(T_{\text{cross}}\) with \(d^{-1/2}\). As the system moves away from the SIT, equation (10) reflects how the superconducting gap increases in small grains as the inverse of the volume over which Cooper pairs are forced to be confined, along similar lines to ref. [53]. At high disorder, the dependence of \(T_{\text{cross}}\) and \(T_0\) on the fastest changing parameter, \(\xi_{\text{loc}}\), is similar (\(T_0 \propto \xi_{\text{loc}}^{-2}\)) through equations (4) and (5)). Strictly speaking, \(\xi_{\text{loc}}\) in Eq. (10) should be the Cooper pair localization length, that, due to interactions, may be larger than the one particle localization length, but this difference is small in the regime considered and we have assumed that both were equal.

According to (10), \(T_{\text{cross}}\) should be of the form \(a/d + b\, T_0\). As can be seen figure 3d, there is a good agreement between the experimental \(T_{\text{cross}}\) and \(1.8/d [\times 0.15 \times T_0\) (green open squares)]. However, close to the SIT, the temperature range over which \(T_0\) is observed vanishes (it is the case for our 45 Å samples). Since \(T_0\) and \(T_0^\times\) have similar dependencies, \(T_{\text{cross}}\) should then be of the form \(a/d + b'\, T_0\). This is experimentally the case (green full squares). In this regime, \(T_0^\times\) is almost constant because the charging energy is smaller than the superconducting gap which is constant and equal to \(\xi_{\text{SC}}/d\) \(\Delta_0\).

5 Conclusion. We have shown that the activated transport behavior observed in thin films close to the SIT is due to transport dominated by charging energies. In homogeneous systems, the electronic granularity is a consequence of a diverging localization length. We have
Figure 3 | (a) Scaled resistance as a function of $T_0/T$ so that the activated regime overlaps for all disorder levels (tuned by the successive heat treatments) for the 35 Å-thick 13.5% sample. (b) Simulation results for the resistance as a function of $1/T$, on a semilogarithmic scale. The black line corresponds to activated behavior, with no superconductivity, and the other curves to overactivation due to superconductivity for several values of the ratio $T_{cross}/T_0$. (c) Resistance as a function of the inverse temperature, $1/T$, on a semilogarithmic scale, for the 13.5% 35 Å-thick sample (heat treatment of 110°C, black dots). The continuous blue curve is our simulation without superconductivity effects, while the red curve is with grain superconductivity for the case $T_{cross}/T_0 = 0.14$. (d) Characteristic temperatures $T_0$, $T_{0}'$, and $T_{cross}$ for all measured samples. The bottom panel shows the value of $\gamma = (T'_{0} - T_0)/T_{cross}$ as given by equation (9). The black line then corresponds to the average $\gamma = 1.5$. The green squares correspond to $T_{cross}$ given by equation (10). Far from the SIT, $T_{cross} \approx 1.8/d [\AA] + 0.15 \times T_0$ (green open squares), and closer to the SIT, $T_{cross} \approx 1.8/d [\AA] + 0.20 \times T_0$ (green full squares). (e) Characteristic temperatures $T_0$, $T_{0}'$, and $T_{cross}$ of the activated regimes in several different experiments. The bottom panel gives the value of $\gamma$, determined by equation (9). The black line corresponds to $\gamma = 1.5$. The experimental data are from Ref. [15] for Sn and Pb, [16] for Bi, [32] for Al, [17] for InOx/In, [46] for InOx, and [22] for $\beta'$-(BEDT-TTF)$_2$ICl$_2$. 
established that, for a fixed amount of disorder, $\xi_{\text{loc}}$ depends exponentially on the film thickness. The overactivated regime observed close to the SIT is a cross-over to a regime governed both by the charging energy and the superconducting gap. Our results indicate that the latter depends critically on the grains size. The overall conclusion is that, in the insulating regime close to the SIT, localized Cooper pairs exist but electronic transport is still dominated by single electrons.

6 Acknowledgments. The authors thank Zvi Ovadyahu for valuable discussions and guidance, and Dan Shahar’s group for sharing their experimental data on InO$_x$. This work has been partially supported by the ANR (grant No. ANR 2010 BLAN 0403 01). M.O. and A.S. acknowledge support by Fundación Séneca grant 19907/GERM/15 and AEI (Spain) grant PID2019-104272RB-C52. V.H., L.B., L.D. and C.A.M.K acknowledge the support of the ANR (grant ANR 2010 BLAN 0403 01).

7 Materials and Methods.

Experimental details. The a-NbSi films have been grown at room temperature by e-beam co-deposition of Niobium and Silicon under ultra-high vacuum (the chamber pressure during the deposition was typically of a few $10^{-8}$ mbar). The film composition was fixed by the respective evaporation rates of Nb and Si (both of the order of 1 Å.s$^{-1}$) and monitored in situ by a set of dedicated piezoelectric quartz crystals. The sample thickness was determined by the duration of the deposition. Both parameters have been checked ex situ by Rutherford Backscattering Spectroscopy (RBS).

The samples have been deposited on sapphire substrates coated with a 25-nm-thick SiO$_2$ underlayer designed to smooth the substrate surface. They were also protected from oxidation by a 25-nm-thick SiO$_2$ underlayer. The samples have been deposited onto sapphire substrates coated with a 25-nm-thick SiO$_2$ underlayer designed to smooth the substrate surface. They were also protected from oxidation by a 25-nm-thick SiO$_2$ underlayer. a-NbSi films of similar compositions and thicknesses have been measured to be continuous, amorphous and homogeneous at least down to a thickness 2.5 nm.

The transport characteristics of a-NbSi thin films are mainly determined by their composition $x$ and their thickness $d$. However, an additional thermal treatment can also microscopically modify the system disorder while keeping $x$ and $d$ constant. a-NbSi becomes more insulating as the heat treatment temperature increases without any change in the sample morphology. Thermal treatments and the film composition have an analogous effect on the disorder level: films of similar sheet resistance $R_{\text{sheet}}$ have the same transport characteristics. On the other hand, we have shown that the effect of the thickness is distinct.

In the present case, we have considered as-deposited films which parameters are listed in Table 1. The composition $x$ has been chosen so that the samples are close to the SIT ($x = 13.5 \%$) or further within the insulating regime ($x = 9 \%$). The sample thickness has been varied between $d = 20$ and $50$ Å for $x = 13.5 \%$. For this stoichiometry, the critical thickness at which the system undergoes a thickness-tuned SIT is about 140 Å. a-Nb$_{1-x}$Si$_x$ is not superconducting, even for bulk samples, and in this work, we considered thicknesses of 125 and 250 Å. Each batch having a constant composition is considered to have the same disorder level ($W$ constant). We can therefore directly evaluate the effect of the thickness within each batch.

The samples resistances have been measured in a dilution refrigerator with a base temperature of 7 mK. In the case of the $x = 13.5\%$ batch, two regions of each sample, labeled left and right, could be probed independently. We used standard low noise transport measurement techniques to ensure the samples were measured without electrical heating of the electronic bath. In other words, we made sure via appropriate filtering that the sample electron temperature was the same as its phononic temperature. Moreover, we have checked that the applied bias was sufficiently low for the resistance measurement to be in the ohmic regime.

| Name         | $x$ (%) | $d$ (Å) | $R_{4K}$ (Ω) |
|--------------|--------|--------|-------------|
| CKSAS43 α left | 13.5%  | 20     | 111 Ω       |
| CKSAS43 α right | 13.5%  | 20     | n.d         |
| CKSAS61 α left | 13.5%  | 30     | 162 kΩ      |
| CKSAS61 α right | 13.5%  | 30     | 151 kΩ      |
| CKSAS61 β left | 13.5%  | 35     | 18.8 kΩ     |
| CKSAS61 β right | 13.5%  | 35     | 17.8 kΩ     |
| CKSAS61 γ left | 13.5%  | 40     | 10.8 kΩ     |
| CKSAS61 γ right | 13.5%  | 40     | 10.5 kΩ     |
| CKSAS61 δ left | 13.5%  | 45     | 7.73 kΩ     |
| CKSAS61 δ right | 13.5%  | 45     | 7.50 kΩ     |

Table 1 | Characteristics of the different a-NbSi samples: composition $x$, thickness $d$, low temperature sheet resistance $R_{4K}$ evaluated at 4.2 K except for the 20 Å-thick sample for which they have been measured at 5 K.

Capacitor model. To calculate the hopping conductance of quasi-2D high dielectric constant disordered systems, we consider a 2D random capacitor model in order to get a consistent set of energies. Sites (nodes) are randomly distributed and the corresponding junction network without crossings can be constructed using a Delaunay triangulization algorithm. Capacitors with randomly chosen capacitances were placed at the links between adjacent nodes, as shown in Fig. 3.a. The capacitances $C_{i,j}$ assume random values drawn from the distribution $C_{i,j} = C_0 e^\varphi$, where $\varphi \in [-B/2, B/2]$. We have chosen $C_0$ (in units of $\epsilon_0 a$, where $a$ is the lattice constant) and $B = 2$. The left bank is connected to the ground, while the right bank is at the potential $V$, and there are periodic boundary conditions in the lateral direction. In order to take into account the leakage of field lines to outside the 2D system due to the finite value of $a$, we introduce capacitances to the ground, $C_0$, as shown in the bottom panel of Fig. 3.a.

The plates of each capacitor carry opposite charges and the total charge on each site $Q_i$ is the sum of the charges on the plates of the capacitors connected to it, $Q_i = \sum_j q_{i,j}$. Here, $q_{i,j}$ is the charge on the plates of the capacitor connecting nodes $i$ and $j$. We construct a vector $Q$, whose components are the charges of the nodes in the sample, and a vector $V$, whose components are the node
potentials. Both vectors are related by $Q = CV$ where the capacitance matrix $C$ has components equal to

$$[C]_{i,j} = \left( \sum_{k \neq i} C_{i,k} \right) \delta_{i,j} - C_{i,j}. \quad (12)$$

The total electrostatic energy of this system can be obtained in a compact form through the inverse of the capacitance matrix $C^{-1}$

$$H = \frac{1}{2} Q C^{-1} Q^T. \quad (13)$$

The matrix $C^{-1}$ plays the role of an interaction matrix. In the continuous limit, the average interaction between two charges separated by a distance $r$ is proportional to the modified Bessel function $K_0(r/L)$, where the screening length is $\Lambda = \tau_0 \sqrt{C/C_0^{\dagger}}$. In the limit of small $r$ ($r_0 \ll r \ll \Lambda$) we have $K_0(x) \approx -\ln(x/2)$, and the effective interaction is given by Eq. (3) provided that we identify the screening length with $\Lambda = \kappa d$, and choose $C = \epsilon_0 \kappa d$ and $C_0 = \epsilon_0 \tau_0^2/(\kappa d)$. In the simulation of hopping transport, $\tau_0$ is our unit of distance, which for random nodes is defined by $\tau_0 = L/N^{1/2}$, $N$ being the number of nodes, $L$ the size of the system, and $\epsilon^2/(\tau_0 \kappa)$ is our unit of energy. The charging energy, i.e., the energy cost for putting a unit charge at a given site $i$, is equal $C_{i-i}/2$.

Carriers hop from site to site transferring a quantized unit charge. They can hop over many sites with a probability that exponentially decays with distance, thus modeling charge transfer in transition systems mediated by the cotunneling mechanism. The transition rate between sites $i$ and $j$ can be expressed as:

$$\Gamma_{i,j} = \tau_0^{-1} e^{-2r_{i,j}/\xi_{loc}} e^{-\Delta_{i,j}/kT} \quad (14)$$

where $\tau_0^{-1}$ is the phonon frequency, $r_{i,j}$ the hopping distance, $\xi_{loc}$ the localization length and $\Delta_{i,j}$ the transition energy, given by our capacitor model. To simulate hopping conductivity in the system of interacting electrons, we employ a kinetic Monte Carlo method. The allowed node charges are $0$ and $\pm 1$. At each Monte Carlo step, the algorithm chooses a pair of sites $(i, j)$, with the probability proportional to $\exp(-2r_{i,j}/\xi)$, Ref. [18]. Doing so, the time step associated with a hop attempt per site is $\tau_0/\sum_{i,j} \exp(-2r_{i,j}/\xi)$, where $\tau_0$ is the inverse phonon frequency. The algorithm first checks whether the transfer of a unit charge from site $i$ to $j$ is compatible with the allowed node charges. Then it calculates the transition energy

$$\Delta_{i,j} = \epsilon_j - \epsilon_i - C_{i,j}^{-1} \frac{1}{2} \left( C_{i,j}^{-1} + C_{j,i}^{-1} \right), \quad (15)$$

where $\epsilon_{i,j} = \sum_j C_{i,j}^{-1} \epsilon_j$ is the potential at $i$, and the hop is performed when $\Delta_{i,j}$ is negative or with probability $\exp(-\Delta_{i,j}/T)$ otherwise. All site potentials $\epsilon_i$ are recalculated after each successful transition. The last term in (15) is the charging energy of the nodes involved. The electric current is generated by the potential difference $V_{lead}$ between the leads at the opposite sides of the sample, which is reflected in the site potentials by extending the $Q$ vector to include nodes $j$ in the lead connected to nodes $i$ in the sample and associating a charge $-C_{i,j} V_{lead}$ with each of them.

The algorithm starts from an initial random charge configuration and follows the dynamics at a given temperature. Once the system is in a stationary situation, the conductivity of each sample is calculated from the number of electrons crossing to one of the leads. The number of Monte Carlo steps performed in this calculation drastically increases with decreasing $T$, and it is determined by the condition that the net charge crossing one of the leads is on the order of 1000. The number of samples considered is 100. Finally, we averaged $\ln \sigma$ over the set of samples (an ensemble averaging).

The main results of the simulations were shown in Fig. 2b. From them, we concluded that, in the presence of disorder, the activation energy is fairly independent of the screening length or, equivalently, of the ratio $C_0/C$. To simulate the overactivated regime, we have to use the most efficient numerical procedure in order to reach very low temperatures. To this end, in Fig. 2 we compare the resistance for the exact interacting potential in the $C_0 = 1$ case (thick curves) with the results including only the charging energy (thin curves). Again, thick lines correspond to the case without site disorder, while thin lines reflect samples with 5% of the nodes having fixed random charges. We see that the activation energies are similar and conclude that a model with charging energies only, without the logarithmic contribution, is adequate to simulate conductivity in disordered samples.

**Localization length.** For the calculation of the localization length, we consider the standard Anderson Hamiltonian for spinless particles

$$H = \sum_i \epsilon_i n_i + \sum_{i,j} \left( t_c c^\dagger_i c_j + c^\dagger_j c_i \right) \quad (16)$$

where $c_i^\dagger$ is the creation operator on site $i$ and $n_i = c_i^\dagger c_i$ is the number operator. We consider a transfer energy $t = -1$, which sets our unit of energy, and a disordered site energy $\epsilon_i \in [-W/2, W/2]$. The double sum runs over nearest neighbors. We have considered square samples of finite thickness with dimensions $L \times L \times d$. All calculations are done at an energy equal to 0.1, to avoid possible specific effects associated with the center of the band.

We have studied numerically the zero temperature conductance $g$ proportional to the transmission coefficient $T$ between two semi-infinite leads attached to opposite sides of the sample,

$$g = \frac{2e^2}{h} T, \quad (17)$$

where the factor of 2 comes from spin. We measure the conductance in units of $2e^2/h$. We calculate the transmission coefficient from the Green function, which is obtained propagating layer by layer with the recursive Green function method. We can solve samples with lateral section up to $L \times d = 400$. The number of different realizations employed is $10^4$ for most values of the parameters. We have considered wide leads with the same section as the samples, represented by the same Hamiltonian as the system, but without diagonal disorder. We use cyclic periodic boundary conditions in the long direction perpendicular to the leads, and hard wall conditions in the narrow traversal direction.

According to single-parameter scaling, the conductance is a function of the ratio of the two relevant lengths of our problem,

$$g = g_0 f(L/\xi_{loc}(W,d)), \quad (18)$$

where $L$ is the lateral system size and $\xi$ the correlation length, which carries the dependence on $W$ and $d$. Data for different $W$ and $d$ can be made to overlap by an adequate choice of $\xi_{loc}(W,d)$. We use this idea in both the diffusive and the localized regimes. In the former the conductance depends logarithmically on system size

$$g = g_0 - \frac{2}{\pi} \log \frac{L}{\xi_{loc}}, \quad (19)$$

where the factor $2/\pi$ has been obtained by diagrammatic perturbation theory. In the strongly localized regime, the conductance depends exponentially on the system size

$$g = c \exp \left(-\frac{2L}{\xi_{loc}}\right). \quad (19)$$

The results for $\xi_{loc}(W,d)$ are shown in Fig. 2d.

To analyze quantitatively the overall behavior of $\xi_{loc}$, we have performed a new one-parameter scaling analysis of the data plotted.
The idea is to plot $\log(\xi_{loc}/d)$ as a function of $d$ and shift the data horizontally by the disorder-dependent amount that best overlaps the data. This shifting quantity corresponds to the three-dimensional localization length $\xi_{loc}$ if it is delocalized (Fig. 3). The two branches therefore correspond to the well-known three-dimensional localization-delocalization transition at a critical disorder of about $W_c \approx 16.5$. The corresponding scaling is shown in the inset of Fig. 5. For small (resp. large) disorder levels, the system is in the upper (resp. lower) branch and the corresponding 3D system is delocalized (resp. localized): $\xi_{loc}$ increases (resp. decreases) faster than thickness. The transition between those two regimes, for our model, occurs when:

$$\log\left(\frac{\xi_{loc}}{d}\right) \approx 0.3. \tag{20}$$

This implies that for a given thickness and amount of disorder $\xi_{loc} > 2d$, the system will tend to metallic when its thickness increases and to an insulator when $\xi_{loc} < 2d$.

Usually in experimental situations, one does not know with enough precision the relation between $d$ and $\xi_{loc}$ to decide if a sample for a given disorder will become extended or localized when its thickness increases according to criteria $\xi_{loc}/d$. We can establish a new criteria to predict the 3D character of a system. If the 3D system corresponding to a given disorder value is extended, as its thickness decreases, the ratio $\xi_{loc}/d$ decreases, meaning that the localization length decreases faster than the thickness:

$$\frac{\xi_{loc,1}}{\xi_{loc,2}} > \frac{d_1}{d_2}. \tag{21}$$

if $d_1 > d_2$. Conversely, if the 3D system corresponding to a given disorder value is localized:

$$\frac{\xi_{loc,1}}{\xi_{loc,2}} < \frac{d_1}{d_2}. \tag{22}$$

for $d_1 > d_2$. Let us apply criterion $\xi_{loc}/d$ to our experimental samples.

For the most disordered $x = 13.5\%$ samples ($\theta_{ht} = 150^\circ C$), one can estimate the ratio of the localization lengths through the relation $\xi_{loc} \propto T^0_{01}/2$ (equations $\xi_{loc}^2$ and $\xi_{loc}^3$):

$$\xi_{loc,1}^2 \approx \sqrt{\frac{T_{02}}{T_{01}}} = 6 \times \frac{d_1}{d_2} = \frac{4}{3} \tag{23}$$

for $d_1 = 40\, \text{Å}$ and $d_2 = 30\, \text{Å}$ for instance. The corresponding 3D system is therefore on the metallic side of the Metal-Insulator Transition. All considered $x = 13.5\%$ samples exhibit $\xi_{loc} \gg d$. The $x = 9\%$ samples are also extended, but closer to the transition: $\xi_{loc,1}/\xi_{loc,2} \approx 2.6$ for $d_1/d_2 = 2$.

References:

1. Service, R. F. Beyond graphene, Science 348, 490-492 (2015).
2. Phillips, P. Not Just a Phase, Nat. Phys. 12, 206 (2016).
3. Ren, Y., Qiao, Z. & Niu, Q. Topological Phases in Two-Dimensional Materials: a Review, Rep. Prog. Phys. 79, 066501 (2016).
4. Feigel’man, M. V., Ioffe, L. B., Kravtsov, V. E. & Cuevas, E. Fractal Superconductivity near Localization Threshold, Ann. Phys. 325, 1390-1478 (2010).
5. Kaputlinik, A., Kivelson, S. A., & Spivak, B. Colloquium: anomalous metals: failed superconductors, Reviews of Modern Physics, 91, 011002 (2019).
6. Vinokur, V. M., Baturina, T. I., Fistul, M., Mironov, A. Y., Baklanov, M. R. & Strunk, C. Superinsulator and quantum synchronization, Nature 452, 615-615 (2008).
7. Couñado, F., Crauste, O., Drillien, A. A., Humbert, V., Bergé, L., Marrache-Kikuchi, C. A. & Dumoulin, L. Dissipative Phases across the Superconductor-to-Insulator Transition, Sc. Rep. 6, 35834 (2016).
8. Saclépé, B., Dubouchet, T., Chapelier, C., Sanquer, M., Ovadia, M., Shafrar, D., Feigel’man, M. & Ioffe, L. Localization of Preformed Cooper Pairs in Disordered Superconductors, Nat. Phys. 7, 239-244 (2011).
9. Boudaïm, K., Loh, Y. L., Randeria, M. & Trivedi, N. Single- and Two-Particle Energy Gaps across the Disorder-Driven Superconductor-Insulator Transition, Nat. Phys. 7, 884 (2011).
10. Finkelstein, A. M. Influence of Coulomb interaction on the properties of disordered metals, JETP Lett. 61, 85-89 (1995).
11. Szabó, P., Samuely, T., Hašková, V., Kačmarčík, J., Zemlička, M., Grajcar, M., Rodrigo, J. G. & Samuely, P. Fermionic scenario for the destruction of superconductivity in ultrathin MoC films evidenced by STM measurements, Physical Review B 93, 014505 (2016).
12. Yang, C., Liu, Y., Wang, Y., Feng, L., He, G., Sun, J., Tang, Y., Wu, C., Xiong, J., Zhang, W. & al. Intermediate bosonic metallic state in the superconductor-insulator transition, Science 366, 1505 (2019).
13. Eley, S., Gopalakrishnan, S., Goldbart, P. M. & Mason, N. Approaching zero-temperature metallic states in mesoscopic superconductor-normal-superconductor arrays, Nature Physics 8, 59 (2012).
14. Kamalapure, A., Das, T., Ganguli, S. C., Parmar, J. B., Bhattacharyya, S., & Raychaudhuri, P. Emergence of nanoscale inhomogeneity in the superconducting state of a homogeneously disordered conventional superconductor, Scientific reports 3, 2979 (2013).
15. Dynes, R. C., Garno, J. P., & Rowell, J. M. Two-dimensional electrical conductivity in quench-condensed metal films, Physical Review Letters 40, 479 (1978).
16. Paredo, K. A., Tan, K. S. B. & Goldman, A. M. Evolution towards superconductivity in granular films of bismuth, Physical Review B 76, 110508 (2007).
17. Kim, J. J., Kim, J. & Lee, H. J. Superconductivity effect on electrical conduction in insulating granular films, Physical Review B 48, 11709 (1992).
18. Sambandamurthy, G., Engel, L. W., Johansson, A., Peled, E., & Shafrar, D. Experimental evidence for a collective insulating state in two-dimensional superconductors, Phys. Rev. Lett. 94, 01703 (2005).
19. Baturina, T. Y. I., Mironov, A. Y., Vinokur, V. M., Baklanov, M. R. & Strunk, C. Hyperactivated resistance in TiN films on the insulating side of the disorder-driven superconductor-insulator transition, JETP Lett. 88, 752 (2008).
20. Baturina, T. I., Mironov, A. Y., Vinokur, V. M., Baklanov, M. R. & Strunk, C. Localization superconductivity in the quantum-critical region of the disorder-driven superconductor-insulator transition in TiN thin films, Phys. Rev. Lett. 99, 257003 (2007).
21. Adkins, C. J., Thomas, J. M. D. & Young, M. D. Increased resistance below the superconducting transition in granular metals, J. Phys. C: Solid St. Phys. 13, 3427 (1980).
22. Tajima, N., Kato, R. & Taniguchi, H. Transport properties of an organic Mott insulator β-(BEDT-TTF) 2CuI2Cl2, Europhys. Lett. 83, 27008 (2008).
23. Eltros, A. L. & Pollák, M. Electron-electron interactions in disordered systems (Elsevier, 2012).
