Characterizations of symmetric convex sets

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Abstract

In this work we prove that either a sequence of axis of symmetry or a sequence of hyperplanes of symmetry of a convex body \( K \) in the Euclidian space \( \mathbb{E}^d, n \geq 3 \), are enough to guarantee that \( K \) is either a generalized body of revolution or a sphere.

1. Introduction.

Let \( K \subset \mathbb{E}^d, d \geq 3 \), be a convex body and let \( p \in \mathbb{E}^d \). In dimension 3, an axis of symmetry of \( K \) is a line \( L \) such that \( K \) remains invariant after a rotation with axis \( L \) by angle \( \pi \). Naturally the notion axis of symmetry of a convex body is easily generalizable (Section 2). It is clear that if the body \( K \) satisfies one of the following two properties: A) all the lines \( L \) passing through \( p \) are axis of symmetry of \( K \), B) all the hyperplanes passing through \( p \) are hyperplanes of symmetry of \( K \), implies that the body \( K \) is a sphere with center at \( p \). For instance, Condition A) implies that all the sections of \( K \) are centrally symmetric and it is well known that such condition implies that \( K \) is an ellipsoid (see for example [3], [7], [8]) and, on the other hand, the unique ellipsoid with an infinite number of axis of symmetry is the sphere.

It is natural to try to reduce the quantity of axis of symmetry or hyperplanes of symmetry of a convex body \( K \subset \mathbb{E}^d \) in order to characterize either a
convex body of revolution or sphere. As an example, in [6] was proven, in \( \mathbb{E}^3 \), that if there exist a plane \( H \) and a point \( p \) such that if every line \( L \subset H \) through \( p \) is an axis of symmetry of \( K \) then \( K \) is a body of revolution. In this work we generalize the aforesaid result in two direction: 1) with respect to dimension, since there is a natural generalization of the notion of axis of symmetry in \( d \) dimensions, \( d \geq 3 \), we obtained Theorem 1, but in order to formulate it, and prove it, we required two generalizations, on the one hand, of the notion of \textit{body of revolution} (Definition 1) and, on the other hand, of a characterization of the ellipsoid, proved in [8] in three dimension, which was used as tool to provide a short proof of the \textit{False Centre Theorem} (Proposition 2); 2) Since is also possible a variations of the notion of axis of symmetry (Definition of \( k \)-axis of symmetry), we have obtained Theorem 3. Furthermore, we have reduced even more the quantity of axis of symmetry of a convex body in such a way that we have proved that either a sequence of axis of symmetry, Theorem 2, or a sequence of hyperplanes of symmetry, Theorem 3, of a convex body \( K \) are enough to guarantee that \( K \) is either a generalized body of revolution or a sphere. Regarding the later, a result have been presented in [2] (Proposition 1), in connection to Mahler’s conjecture, however, the authors explicitly avoid the case where the family of hyperplanes of symmetry of the convex body share a subspace. Therefore, using Proposition 1 we have completed the characterization of a generalized convex body of revolution or sphere in terms of the existence of a sequence of hyperplanes of symmetry of a convex body (Theorem 3).

The three dimensional case of Theorem 3 follows immediately from the fact that it is well understood the classification of the finite subgroups of \( SO(3) \), the special orthogonal group in \( \mathbb{E}^3 \) (See for example Theorem in 19.2 [1]). However, we have obtained a proof which involves the Süss-Schneider’s characterization of the sphere in terms of concurrent congruent sections [10], [11].

2. The results.

In order to establish our results we need the followings definitions.

Let \( L \subset \mathbb{E}^3 \) be a line and let \( \theta \) be an angle in \([0, 2\pi]\). We denote by \( R_{(L,\theta)} : \mathbb{E}^3 \to \mathbb{E}^3 \) the rotation with axis \( L \) and with angle \( \theta \). We just denote by \( R_L \) and \( R_{L,n} \), respectively, the maps \( R_{(L,\pi)}, R_{(L,\frac{2\pi}{n})} \).
Let $K \subset \mathbb{E}^3$ be a convex body, $L \subset \mathbb{E}^3$ be a line and $n$ be an integer, $n \geq 2$. The line $L$ is said to be an $n$-axis of symmetry of $K$ if the following relation

$$R_{L,n}(K) = K$$

holds. In the case $n = 2$, a 2-axis of symmetry of the convex body $K$ will just called axis of symmetry of $K$. For instance, if $C = [-1, 1] \times [-1, 1] \times [-1, 1]$ is a cube, centered at the origen, it has three types of axis of symmetry, namely, lines determined by the centres of parallel faces, the diagonals and the lines determined by the mid points of parallel edges, not in the same face, corresponding to the numbers 2, 3 and 4.

As a second example, we observe that if $M \subset \mathbb{E}^3$ is a body of revolution with axis $L$ and with a plane of symmetry $\Pi$, perpendicular to $L$, then every line $\Gamma$ contained in $\Pi$, through the point $\Pi \cap L$, is an axis of symmetry of $K$. It follows from the fact that for every line $\Gamma \subset \Pi$, such that $\Pi \cap L \in \Gamma$, are passing two perpendicular planes of symmetry of $K$. In Theorem 1 we prove, for every dimension $d$, $d \geq 3$, the converse of the aforesaid statement. Such characterization of a body of revolution was used in [6], in dimension 3, in order to prove that a body with a false axis of revolution is a sphere.

In order to generalize these notions, we need to recall the well known concept of hyperplane of symmetry.

**Hyperplane of symmetry.** Let $K \subset \mathbb{E}^d$ be a convex body, $d \geq 3$ and let $\Pi$ be a hyperplane. We denote by $S_\Pi : \mathbb{E}^d \to \mathbb{E}^d$ the reflexion with respect to $\Pi$. The hyperplane $\Pi$ is said to be a hyperplane of symmetry of $K$ if the relation

$$S_\Pi(K) = K$$

holds.

**Axis of rotation.** Let $M \subset \mathbb{E}^d$ be a convex body, $d \geq 3$ and let $\Gamma$ be a $(n - 2)$-flat. We denote by $R$ the map $S_{\Pi_2} \circ S_{\Pi_1}$, where $\Pi_1, \Pi_2$ are two $(d - 1)$-flat containing $\Gamma$ and whose unit normal vector $u_1, u_2$ makes and angle of $\pi/k$. $\Gamma$ is said to be a $(d - 2)$-axis of rotation of order $k$ of $M$ if

$$M = R(M)$$

holds. The map $R$ can be considered as a rotation with axis $\Gamma$ by an angle $2\pi/k$ (see, for example, [5] Pag. 6 and 23). It is clear that $R^k = \text{Id}$. 

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**Axis of symmetry.** Let $K \subset \mathbb{E}^d$ be a convex body, $d \geq 3$, and let $L \subset \mathbb{E}^d$ be a line. We denote by $R_L : \mathbb{E}^d \to \mathbb{E}^d$ the element of $O(d)$, the orthogonal group corresponding to $\mathbb{E}^d$, such that it acts as the identity on the line $L$ and on the hyperplane $L^\perp$, perpendicular to $L$, in such a way that $x \to -x$. $L$ is said to be a **axis of symmetry of $K$** if the relation

$$R_L(K) = K.$$ holds.

It is clear that if $L$ is an axis of symmetry of $K$, for one hand, all the sections of $K$, obtained with hyperplanes orthogonal to $L$, are centrally symmetric with center at $L$ and, on the other hand, all sections of $K$, with planes containing $L$, have $L$ as a line of symmetry. This observation motivates the following generalization of axis of symmetry of a convex body.

Let $K \subset \mathbb{E}^d$ be a convex body, $d \geq 3$, and let $\Gamma$ be a $k$-flat, $1 \leq k \leq d - 2$. $\Gamma$ is said to be a **$k$-axis of symmetry of $K$** if for all $(k + 1)$-flat $\Lambda$ containing $\Gamma$, the section $\Lambda \cap K$ has $\Gamma$ as a $k$-flat of symmetry.

**Definition 1 (Bodies of revolution.)** Let $\Omega \subset \mathbb{E}^d$ be a compact set, $d \geq 3$, and let $k$ be an integer, $1 \leq k < d$. The set $\Omega$ is said to be a **$k$-body of revolution** if there exists a decomposition of $\mathbb{E}^d$ in the form $\mathbb{E}^d = \mathbb{E}^{d-k} \oplus \mathbb{E}^k$, $\mathbb{E}^{d-k}$ orthogonal to $\mathbb{E}^k$, such that $\partial \Omega \cap \Gamma$ is an sphere of dimension $k - 1$ for all affine $k$-flat $\Gamma$ parallel to $\mathbb{E}^k$ and the centres are in $\mathbb{E}^{n-k}$.

We observe that a 1-body of revolution is just a convex body which is symmetric with respect to a hyperplane.

We are ready now to formulate our results.

**Theorem 1** Let $K \subset \mathbb{E}^d$ be a convex body, $d \geq 3$, and let $p \in \mathbb{E}^d$ be a point and let $k$ be a positive integer, $2 \leq k \leq d - 1$. Suppose that there exist a $k$-flat $\Lambda$ such that every line $L$, passing through $p$ and contained in $\Lambda$, is an axis of symmetry of $K$. Then $K$ is a $k$-body of revolution.

**Theorem 2** Let $K \subset \mathbb{E}^d$ be a convex body, $d \geq 3$. Suppose that there exists a sequence $\{L_n\} \subset \mathbb{E}^d$ of axis of symmetry of $K$ such that:

1) there exists an integer $k$, $2 \leq k \leq d$, so that $\text{aff}\{L_1, L_2, \ldots\} = k$. 

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ii) If \( i \neq j \), \( L_i \) is not perpendicular to \( L_j \).

Then, if \( k \neq d \), \( K \) is a \( k \)-body of revolution and, if \( k = d \), \( K \) is a sphere.

The following result was given in [2] in the context of Mahler’s conjecture and, since we will use it order to prove Theorem 3, we have decided to presente it formally.

**Proposition 1** Let \( K \subset \mathbb{E}^d \) be a convex body, \( d \geq 3 \), and let \( \{ \Pi_i \} \) be a sequence of hyperplanes in \( \mathbb{E}^d \). Suppose that \( \Pi_i \) is a hyperplane of symmetry of \( K \) for \( i = 1, 2, \ldots \). If, in addition, does not exist an integer \( k \), \( 1 \leq k \leq d-2 \), and a \( k \)-flat \( \Gamma \) such that \( \Gamma \subset \Pi_i \) for \( i = 1, 2, \ldots \), then \( K \) is an \((d-1)\)-sphere.

**Theorem 3** Let \( K \subset \mathbb{E}^d \) be a convex body, \( d \geq 3 \), and let \( \{ \Pi_i \} \) be a sequence of hyperplanes in \( \mathbb{E}^d \). Suppose that \( \Pi_i \) is a hyperplane of symmetry of \( K \) for \( i = 1, 2, \ldots \), there exists an integer \( k \), \( 1 \leq k \leq d-2 \), and a \( k \)-flat \( \Gamma \) such that

\[
\Pi_i \subset \Gamma \quad \text{for} \quad i = 1, 2, \ldots,
\]

and \( k \) is maximal with this property (i.e., if the integer \( l \) is bigger than \( k \), then there are no an \( l \)-flat \( \Lambda \) and an infinite subsequence \( \{ \Pi_i \} \) of \( \{ \Pi_i \} \) such that \( \Pi_i \subset \Lambda \) for \( i = 1, 2, \ldots \)). Then \( K \) is a \((d-k)\)-body of revolution.

**Theorem 4** Let \( K \subset \mathbb{E}^d \) be a convex body, \( d \geq 3 \), let \( p \in \mathbb{E}^d \) be a point and let \( k \) a positive integer, \( k \geq 3 \). Suppose that every \((n-2)\)-flat passing through \( p \) is a \((d-2)\)-axis of rotation of order \( k \) of \( K \), then \( K \) is a sphere with centre at \( p \).

**Theorem 5** Let \( K \subset \mathbb{E}^d \) be a strictly convex body, \( d \geq 3 \), let \( p \in \mathbb{E}^d \) be a point and let \( k \) be an integer \( 1 \leq k \leq d-2 \). Suppose that there exists \((k+1)\)-flat \( \Lambda \) such that \( p \in \Lambda \) and every \( k \)-flat \( \Gamma \subset \Lambda \), \( p \in \Gamma \), is a \( k \)-axis of symmetry of \( K \). Then all the \((k+1)\) sections of \( K \) parallel to \( \Lambda \) are spheres and the centres are situated in a \((d-(k+1))\)-flat orthogonal to \( \Lambda \) passing through \( p \), i.e., \( K \) is a \((k+1)\)-body of revolution.

**Preliminaries.**

Let \( \mathbb{E}^d \) be the Euclidean space of dimension \( d \) endowed with usual interior product \( \langle \cdot , \cdot \rangle : \mathbb{E}^d \times \mathbb{E}^d \to \mathbb{R} \). We take an orthogonal coordinate system \((x_1, \ldots, x_d)\) for \( \mathbb{E}^d \). Let \( B(d) = \{ x \in \mathbb{E}^d : ||x|| \leq 1 \} \) be the unit \( d \)-ball,
centered at the origin, and let \( S^{d-1} = \{ x \in \mathbb{E}^d : ||x|| = 1 \} \) its boundary. 
For the set \( A \subset \mathbb{E}^d \), by \( \text{aff}\{A\} \) we will denote the affine hull of \( A \). A \( k \)-flat, \( 1 \leq k \leq d-1 \), is a set \( \Gamma \subset \mathbb{E}^d \) such that there exits a subspace \( \Lambda \subset \mathbb{E}^d \), of dimension \( k \), and a point \( p \in \mathbb{E}^d \) such that \( \Gamma = p + \Lambda \). Let \( \Lambda \subset \mathbb{E}^d \) be a \( k \)-flat and let \( p \in \Lambda \) be a point. We denote by \( \Lambda^\perp \) the \((d-k)\)-flat orthogonal to \( \Lambda \) and passing through \( p \).

The set \( K \subset \mathbb{E}^d \) is said to be a **convex body** if it is a compact convex set with non empty interior. A **convex hypersurface** is the boundary of a convex body \( K \) in \( \mathbb{E}^d \) and it will be denote by \( \text{bd} \, K \). As usual, we will denote by \( \text{int} \, K \) the set \( K \setminus \text{bd} \, K \).

If \( \Gamma \) is a \( k \)-plane, \( 1 \leq k \leq n-1 \), then the **shadow boundary** \( S\partial(K, \Gamma) \) of \( K \) with respect to \( \Gamma \) is the set in \( \text{bd} \, K \), defined by

\[
S\partial(K, \Gamma) = \{ \Lambda \cap K : \Lambda \text{ is supporting } k\text{-plane of } K \text{ parallel to } \Gamma \}.
\]

The shadow boundary \( S\partial(K, \Gamma) \) of \( K \) with respect to \( \Gamma \) is said to be a **segment free** if there is not a line segment \( I \subset \text{bd} \, K \) contained in \( S\partial(K, \Gamma) \) parallel to \( \Gamma \).

In virtue that in this work we are going to consider non bounded subsets of \( \mathbb{E}^d \), in particular, flats, in order to give a metric on the non-empty closed sets of \( \mathbb{E}^d \), we will follow Busemann [4]. If \( A \) is a non-empty set, the distance from \( p \) to \( A \) is denoted by \( pA \) and defined as the greatest lower bound of \( px \) as \( x \) traverses \( A \) (\( px \) is the distance between \( p \) and \( x \)).

We select a point \( p \) and define the distance of the (non-empty) sets \( M, N \) as follows

\[
\delta_p(M, N) = \sup_{x \in \mathbb{E}^d} |xM - xN|e^{-px}.
\]  

(3)

With such distance the family of non-empty closed sets of \( \mathbb{E}^d \) is a metric space (see (3.8) of [4]). Furthermore, this metric space is a finitely compact space (Theorem (3.15) of [4]) and, precisely, such property play a fundamental role in several of our demostrations. We must observe (see Pag. 14 of [4]) that when we are dealing with bounded sets the factor \( e^{-px} \) may be omitted in (3) and distance then coincides with Hausdorff distance given as in (1.8.1) Pag. 48 of [10]. Therefore, from now on when we speak about convergence
of a sequence of subsets of $\mathbb{E}^d$, we will assume that the metric $\delta_p$, given by (3), is involve.

**Definition 2** A family of lines $\{L_1, ..., L_n\}$ is called a $n$-star of lines with apex $x_0$ if the lines $L_i$ are in a plane, are concurrent at $x_0$ and the angle between two consecutive lines is $\frac{2\pi}{n}$.

**Definition 3** Let $L_1$ and $L_2$ two axis of symmetry of the convex body $K$. The star determined by $L_1$ and $L_2$, which will be denoted by $\Sigma(L_1, L_2)$, is the family of lines $\{T_n\}$ constructed in the following way: $T_1 = L_1, T_2 = L_2$, and, in general, $T_k = R_{T_{k-1}}(T_{k-2})$.

(The map $R_L$, for a line $L \subset \mathbb{E}^d$, was defined when we gave the definition of axis of symmetry of a convex body).

We observe, for one hand, that each line in the family $\{T_n\}$ is an axis of symmetry of $K$ and, on the other hand, that $T_i \subset \text{aff}\{T_1, T_2\}$ for all $i$. If $L_1$ and $L_2$ are two lines with no empty intersection, we denote by $\Omega(L_1, L_2)$ the set of all lines contained in the plane $\text{aff}\{L_1, L_2\}$ and passing through the point $L_1 \cap L_2$.

**Remark 1** In function of the angle between $L_1$ and $L_2$ either $\Sigma(L_1, L_2)$ is a $n$-star of lines for some integer $n$ or $\Sigma(L_1, L_2)$ is a dense set in $\Omega(L_1, L_2)$.

The proofs of the following results are straightforward and are left to the reader.

I. Let $\Phi$ be a plane convex figure and let $\{L_1, ..., L_n\}$ be the collection of all its orthogonal lines of symmetry. Then, $\{L_1, ..., L_n\}$ is an $n$-starline.

II. Let $K \subset \mathbb{E}^d$ be a convex body and let $\{H_i\}$ be a sequence of hyperplanes that intersect int $K$. Suppose that $H_i \to H$, $L_i \subset H_i$ is a $(d - 2)$-plane of symmetry ($p_i \in H_i$ is a center of symmetry) of $H_i \cap K$, and $L_i \to L$ ($p_i \to p$); then, $L$ is a $(d - 2)$-plane of symmetry ($p$ is center of symmetry) of $H \cap K$.

III. Let $\{K_i\}$ be a sequence of convex figures such that $K_i \to K$ and, for every $i \in \mathbb{N}$, the figure $K_i$ has two lines of symmetry determining an angle $\theta_i$. If $\lim_{i \to \infty} \theta_i = 0$ Then $K$ is a circle.
The following result is well known, however, in virtue that did not found in the literature a proof for the case of a set with and infinite number of symmetries (For instance, in [5] there is proof for the finite case), we decided include here an elementary proof in terms of the unicity of the circumsphere of a convex body.

**Lemma 1** Let \( K \subset E^d \) be a convex body. We denote by \( \Omega \) the circumsphere of \( K \) and by \( o \) its centre. Then the planes of symmetry and the axis of symmetry of \( K \) are passing through the point \( o \).

**Proof.** Suppose that there exists a a hyperplane of symmetry \( \Gamma \) of \( K \) such that \( o \notin \Gamma \). Then \( K = S_\Gamma(K) \). Furthermore since \( S_\Gamma(\Omega) \neq \Omega \) and \( K \subset \Omega \), we have \( K = S_\Gamma(K) \subset S_\Gamma(\Omega) \). Hence, \( K \subset \Omega \cap S_\Gamma(\Omega) \). This implies that there exists a sphere of ratio lower than the ratio of \( \Omega \) and containing \( K \), however, this contradicts the assumption that \( \Omega \) is the circumsphere of \( K \).

On the other hand, let \( L \) be an axis of symmetry of \( K \). Then \( L \) is passing through \( o \). On the contrary, \( \Omega \neq R_L(\Omega) \). Since \( K \subset \Omega \) we have that

\[ K = R_L(K) \subset R_L(\Omega), \]

in virtue that \( L \) is an axis of \( K \). Thus \( K \subset \Omega \cap R_L(\Omega) \). Consequently, \( K \) is contained in a sphere the small ratio than \( \Omega \) which contradicts the choice of \( \Omega \). \( \blacksquare \)

**Lemma 2** Let \( K \subset E^d, d \geq 3 \), be a convex body. Suppose that \( \{L_n\} \subset E^d \) is a sequence of axis (hyperplanes) of symmetry of \( K \) and \( L \) is a line (hyperplane) such that \( L_n \to L \). Then \( L \) is an axis (hyperplane) of symmetry of \( K \).

**Proof.** Since \( L_n \to L \), by the Theorem 1.8.7 of [10], for all \( q \in L \cap K \), there exists \( q_n \in L_n \cap K \) such that \( q_n \to q \). We denote by \( \Gamma_n \) the orthogonal hyperplane to \( L_n \) and through \( q_n \) and by \( \Gamma \) the hyperplane orthogonal to \( L \) and through \( q \). Since \( L_n \) is an axis of symmetry of \( K \), \( \Gamma_n \cap K \) is centrally symmetric with center at \( q_n \). In virtue that \( L_n \to L \) and \( q_n \to q \), we have \( \Gamma_n \to \Gamma \). Thus \( \Gamma_n \cap K \to \Gamma \cap K \). From II, it follows that \( \Gamma \cap K \) is centrally symmetric with center at \( q \). Thus \( L \) is an axis of symmetry of \( K \). \( \blacksquare \)

The three dimensional case of the following proposition was proved in [8].

**Proposition 2** Let \( K \subset E^d, 3 \geq d \) be a convex body and let \( 0 \in \text{int } K \).
Suppose that, for every hyperplane \( H \) through 0, there is a line \( L_H \) such that \( \text{bd}(H \cap K) \subset S\partial(K, L_H) \). Then \( K \) is an ellipsoid.

**Proof.** First of all, we will assume that the Proposition holds for dimension \( d - 1 \). We are going to prove that \( K \) is an ellipsoid showing that, for every hyperplane \( \Pi \) through 0, the section \( \Pi \cap K \) is an ellipsoid (Theorem 16.14 in [1]). In order to prove that \( \Pi \cap K \) is an ellipsoid we will see that such section satisfies the conditions of the Proposition 2, i.e., we will prove that for every \((d - 2)\)-flat \( H' \subset \Pi \) through 0 there is a line \( L_{H'} \subset \Pi \) such that the relation

\[
\text{bd}(H' \cap (\Pi \cap K)) \subset S\partial((\Pi \cap K), L_{H'})
\]

holds and, in virtue of our initial assumption, it will follow that \( \Pi \cap K \) is an ellipsoid. Since the three dimensional case of the Proposition 2 was proved in [8], by the principle of induction, the proof of Proposition 2 would be complete.

Let \( \Pi \) be a hyperplane through 0 and let \( H' \subset \Pi \) be a \((d - 2)\)-flat through 0. We choose \( d - 2 \) points \( \{a_1, a_2, ..., a_{d-2}\} \subset \text{bd} K \) such that \( H' = \text{aff}\{a_1, a_2, ..., a_{d-2}\} \). Let \( \Pi' \) be a supporting hyperplane of \( K \) parallel to \( \Pi \), and let \( a_{d-1} \) be a point in \( \Pi' \cap \text{bd} K \). We denote by \( H \) the hyperplane \( \text{aff}\{a_1, a_2, ..., a_{d-2}, a_{d-1}\} \). By hypothesis there exist a line \( L_H \) such that \( \text{bd}(H \cap K) \subset S\partial(K, L_H) \). It follows that \( \text{bd}(H' \cap K) \subset S\partial(H' \cap K, L_H) \), i.e., the relation (4) holds for the line \( L_H \). First we suppose that \( a_{d-1} \) is a regular point of \( \text{bd} K \). By the regularity of the point \( a_{d-1} \), the line \( L_H \) is parallel to \( \Pi' \). Thus \( L_H \) is parallel to \( \Pi \) but this is just what we have claimed.

Now we suppose that \( a_{d-1} \) is a non regular boundary point of \( \text{bd} K \). Then either \( L_H \subset \Pi' \) or \( L_H \) is not contained in \( \Pi' \). In the first case occurs, we finish. Let us assume, then, that \( L_H \) is not contained in \( \Pi' \). We denote by \( \Delta_i \) a supporting \((d - 2)\)-flat of the body \( H' \cap K \) at the point \( a_i \) and contained in \( H' \), \( i = 1, 2, ..., d - 2 \). Then \( \Delta_1 \cap \Delta_2 \cap \cdots \Delta_{d-2} \subset H' \) is a line which will be denoted by \( \Gamma \). Since \( L_H \) is not contained in \( \Pi' \), \( L_H \) is not parallel to \( \Pi \) and \( \text{aff}\{\Gamma, L_H\} \) has dimension two. Hence there are supporting 2-flat of \( K \) parallel to \( \text{aff}\{\Gamma, L_H\} \) at the point \( a_i \), \( i = 1, 2, ..., d - 2 \). In virtue that \( \text{aff}\{\Gamma, L_H\} \cap \Pi' \) is a line, say \( L'_H \), there are supporting lines of \( K \) at the points \( a_i \), \( i = 1, 2, ..., d - 1 \), parallel to \( L'_H \). This implies that \( H \cap K \subset S\partial(K, L'_H) \). Consequently \( \text{bd}(H' \cap K) \subset S\partial(H' \cap K, L'_H) \) ■

The proofs.
Proof of Theorem. First we consider the case $k = d - 1$. We claim that $K$ is centrally symmetric. We denote by $\Lambda^\perp$ the line orthogonal to $\Lambda$ such that $p \in \Lambda^\perp$. For every hyperplane $\Omega$, $\Lambda^\perp \subset \Omega$, we consider the line $L \subset \Lambda$ orthogonal to $\Omega$ and passing through $p$. Since $L$ is an axis of symmetry of the body $K$, by hypothesis, it follows that $\Omega \cap K$ is centrally symmetric with centre at $p$. Varying $\Omega$, $\Lambda^\perp \subset \Omega$, we obtain that $K$ is centrally symmetric with centre at $p$.

Since $K$ is centrally symmetric, we take a co-ordinate system such that the centre of $K$ is the origin. Let $L \subset \Lambda$ be a line passing through $p$. We are going to show that

$$(L^\perp \cap \text{bd } K) \subset S\partial(K, L).$$

Let $\Phi$ be a hyperplane orthogonal to $L$ such that $\Phi \cap K \neq \emptyset$. We take $x \in \Phi \cap \text{bd } K$. We are going to show that there exists a line $\Delta$ parallel to $L$ such that $x \in \Delta$ and $\Delta \cap \text{bd } K = \{x, y\}$ where $y \in (-\Phi) \cap \text{bd } K$. Since $L$ is an axis of symmetry of the body $K$, $\Phi \cap \text{bd } K$ and $-(\Phi \cap \text{bd } K)$ have centres, say $c, -c$, respectively, and $c, -c \in L$. Since $-c + (c - x) = -x \in -(\Phi \cap \text{bd } K)$, we get $-c - (c - x) = x - 2c \in -(\Phi \cap \text{bd } K)$. Thus if we define $\Delta = x + L$ and $y = x - 2c$, we obtain what we have claimed. It yields that $S\partial(K, L)$ is contained in the region between $\Phi$ and $-\Phi$, except perhaps in the point of $S\partial(K, L)$ where there is a supporting line whose intersection with $\text{bd } K$ has dimension 1. Taking the limit $\Phi \to \Phi(L)$, it follows that $L^\perp \cap \text{bd } K \subset S\partial(K, L)$.

Now we claim that for every hyperplane $\Lambda'$, parallel to $\Lambda$ and such that $\Lambda' \cap \text{int } K \neq \emptyset$, the sections $\Lambda' \cap K$ is an ellipsoid with centre at $\Lambda' \cap \Lambda^\perp$. We denote by $p', K'$, respectively, the point $\Lambda' \cap \Lambda^\perp$ and the section $\Lambda' \cap K$. In order to prove our claim, we are going to show that, for every $(d - 2)$-plane $\Pi \subset \Lambda'$, $p' \in \Pi$, there exist a line $L_\Pi$ such that

$$K' \cap \Pi \subset S\partial(K', L_\Pi).$$

Once the relation (6) have been proved, by the Proposition 2 it will follow that $K'$ is an ellipsoid.

Let $L_\Pi \subset \Lambda$ be a line orthogonal to $\Pi$ and passing through $p$. It is clear that $\Pi \subset (L_\Pi)^\perp$ and $\Pi = \Lambda' \cap (L_\Pi)^\perp$. In virtue of (5)

$$\Lambda' \cap ((L_\Pi)^\perp \cap \text{bd } K) \subset \Lambda' \cap S\partial(K, L_\Pi).$$
Finally, in virtue that the angle between \( L S_\partial' \) due the equalities \( \Lambda a \) (i.e., \( -d \leq k \leq d - 1 \)) of symmetry of \( K \leq 1 \)-sphere with centre at \( p' \). Therefore, \( K \) is a \( (n - 1) \)-body of revolution. The proof of the case \( k = d - 1 \) is complete.

We consider now the case \( 2 \leq k \leq d - 2 \). We are going to prove that if \( \Lambda' \) is a \( k \)-flat parallel to \( \Lambda \) and such that \( \Lambda' \cap \text{int} K \neq \emptyset \), then \( \Lambda' \cap \text{int} K \) is a \((k - 1)\)-sphere with centre at \( \Lambda^\perp \), where \( \Lambda^\perp \) is the \((d - k)\)-flat orthogonal to \( \Lambda \) and passing through \( p \). We denote by \( \Delta \) the \((k + 1)\)-flat spanned by \( \Lambda \) and \( \Lambda' \), i.e., \( \Delta = \text{aff} \{\Lambda, \Lambda'\} \). Since every line \( L \) with \( p \in L \) and \( L \subset \Lambda \) is an axis of symmetry of \( K \), then every such \( L \) is an axis of symmetry of the body \( \Delta \cap K \). Thus, in virtue of case \( k = d - 1 \) of Theorem \( \Pi \), which already has been considered in the first part of the proof of Theorem \( \Pi \), \( \Delta \cap K \) is \( k \)-body of revolution, i.e., for every \( k \)-flat \( \Gamma \), parallel to \( \Lambda \), the section \( \Gamma \cap \text{int} K \) is a \((k - 1)\)-sphere with centre at the line \( \Lambda^\perp \cap \Delta \). Therefore, in particular, \( \Lambda' \cap \text{int} K \) is a \((k - 1)\)-sphere with centre at \( \Lambda^\perp \) as we have claimed. \( \square \)

**Lemma 3** Let \( K \subset \mathbb{E}^d, d \geq 3 \), be a convex body. Let \( \Pi \) be \( k \)-flat, \( 2 \leq k \leq d - 1 \), and let \( L' \subset \mathbb{E}^d \setminus \Pi \) be a line, both through the orien \( o \) of a coordinate system and \( L' \neq \Pi^\perp \). Suppose that every line \( L \subset \Pi \) through \( o \) is an axis of symmetry of \( K \) and, furthermore, \( L' \) is an axis of symmetry of the body \( K \). Then every line through \( o \) and contained in \( \text{aff} \{\Pi, L'\} \) is an axis of symmetry of \( K \).

**Proof.** Let \( v \) be a unit vector parallel to \( L' \). For each \( u \in \mathbb{S}^{k-1} \subset \Pi \), we denote by \( \alpha(u) \) the angle between the vector \( u \) and \( v \). Let \( \delta_1 \) and \( \delta_2 \) be the minimum and the maximum of \( \alpha(\cdot) \). Since \( L' \subset \mathbb{E}^d \setminus \Pi \) and \( L' \neq \Pi^\perp \) it follows that \( \delta_1 > 0 \) and \( \delta_2 < \pi/2 \). In virtue of the continuity of \( \alpha(u) \) and the compactness of \( \mathbb{S}^{k-1} \), we have that \( \alpha(\mathbb{S}^{k-1}) = [\delta_1, \delta_2] \). We denote by \( F \) the subset of irrational numbers in \( [\delta_1, \delta_2] \) and by \( E \) the set \( \alpha^{-1}(F) \). Since \( F \) is dense in \( [\delta_1, \delta_2] \), the set \( E \) is dense in \( \mathbb{S}^{k-1} \). For each \( u \in E \), we denote by \( \Pi(u) \) the plane \( \text{aff} \{u, v\} \). For \( u \in E \), we have that \( \alpha(u) \) is in \( F \) and, consequently, \( \Sigma(\text{lin}\{u\}, L') \) is a set of axis of symmetry of \( K \) in \( \Pi(u) \) dense in \( \Omega(\text{lin}\{u\}, L') \). Thus, by Lemma \( \Sigma \), every line in \( \Pi(u) \) through \( o \), is an axis of symmetry of \( K \).

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Let \( w \in S^k \). We are going to show that there exists an axis of symmetry \( L_w \) of \( K \) such that \( L_w = \text{aff}\{w\} \). Let \( e \in S^{k-1} \) such that \( \text{aff}\{e\} = \Pi \cap \text{aff}\{v, w\} \). Since \( E \) is dense in \( S^{k-1} \), there exists a sequence \( \{e_i\} \subset E \) such that \( e_i \to e \), when \( i \to \infty \). In virtue that \( e_i \to e \), when \( i \to \infty \), it follows that \( \Pi(e_i) \to \text{aff}\{v, w\} \). Hence we can find a sequence of lines \( \{\Delta_i\} \), \( \Delta_i \subset \Pi(e_i) \), such that \( \Delta_i \to L_w = \text{aff}\{w\} \subset \text{aff}\{v, w\} \), when \( i \to \infty \). By the first paragraph, every line \( \Delta_i \) is an axis of symmetry of the body \( K \). By Lemma 2, \( L_w \) is an axis of symmetry of \( K \). ■

Proof of Theorem \( 2 \): In virtue of Lemma 1 there exists \( p \in \mathbb{B}^d \) such that \( p \in L_n \), for all \( n \). We take a coordinate system such that the origin \( o \) is the point \( p \). We suppose that \( V = \text{aff}\{L_1, L_2, \ldots\} \) has dimension \( k \), \( k \geq 2 \).

First we consider the case when there exists a plane \( H \subset V \) and an infinite subsequence \( L_{n_i} \) of \( \{L_n\} \) such that \( \{L_{n_i}\} \subset H \), for all \( i \). We are going to prove that all line \( \Delta \subset H \), \( o \in \Delta \), is an axis of symmetry of \( K \). We observe that it is enough to find a sequence of axis of symmetry \( \{\Delta_n\} \subset H \) of \( K \) such that \( \Delta_n \to \Delta \) and, by Lemma 2 we will have that \( \Delta \) is an axis of symmetry of \( K \). Let \( \Delta \subset H \) be a line, \( o \in \Delta \). In virtue of the compactness of \( S^2 \), there exist an infinite subsequence of \( L_{n_i} \), which will be denote again by \( L_{n_i} \), and a line \( L \subset H \) through \( o \) such that \( L_{n_i} \to L \), when \( i \to \infty \). By Lemma 2, \( L \) is an axis of symmetry of \( K \). First of all, suppose that, for all \( i \), \( \Sigma(L, L_{n_i}) \) is a \( f(n_i) \)-star of line for some integer \( f(n_i) \). Since \( L_{n_i} \to L \), it follows that \( f(n_i) \to \infty \). Thus, given \( \epsilon > 0 \), there exists an integer \( n_i \) such that the difference between the angle of \( \Delta \) and the angle the some line in \( \Sigma(L, L_{n_i}) \), which will be denoted by \( \Delta_{k(n_i)} \), is lower than \( \epsilon \). It is clear that \( \Delta_{k(n_i)} \to \Delta \). On the other hand, if for some \( n_i \) the angles corresponding to the lines in \( \Sigma(L, L_{n_i}) \) determines a dense set in \([0, 2\pi]\), it follows immediately the existence of a sequence \( \{\Delta_n\} \subset \Sigma(L, L_{n_i}) \) of axis of symmetry of \( K \) such that \( \Delta_n \to \Delta \).

Now we suppose that there is not a plane \( H \subset V \) such that an infinite subsequence of \( \{L_n\} \) is contained in \( H \). Then the sequence of planes \( \{\Pi_n\} \) is infinite, where \( \Pi_n \) is the plane \( \text{aff}\{L, L_n\} \). Thus there exists a plane \( \Pi \subset V \) such that \( \Pi_n \to \Pi \) when \( n \to \infty \). If for some integer \( n_0 \), the angles corresponding to the lines in \( \Sigma(L, L_{n_0}) \) are a dense set in \([0, 2\pi]\), we have that for each line \( \Delta \subset \text{aff}\{L, L_{n_0}\} \) there exists a sequence \( \{\Delta_n\} \subset \Sigma(L, L_{n_0}) \) of axis of symmetry of \( K \) such that \( \Delta_n \to \Delta \). Consequently, by Lemma
Δ is an axis of symmetry of the body $K$. Thus all line in $Π$ is axis of symmetry of $K$. On the other hand, suppose that, for all $n$, $Σ(L, L_n)$ is a $f(n)$-star of lines for some integer $f(n)$. Since the sequence of planes $\{Π_n\}$ is infinite, the sequence $\{f(n)\}$ also is infinite. In virtue that $L_n → L$, it follows that $f(n) → ∞$. We are going to prove that all line in $Π$ is axis of symmetry of $K$. Let $Δ ⊂ Π$ be a line, $x ∈ Δ$. Again, we will show that there exists a sequence $\{Δ_n\}$ of axis of symmetry of $K$ such that $Δ_n → Δ$. Since $Π_n → Π$, there exists a sequence of lines $\{Γ_n\}$ such that $Γ_n ⊂ Π_n$ and $Γ_n → Δ$, i.e., given $ε > 0$, there exists $N_1$ such that if $n > N_1$, then $δ_p(Γ_n, Δ) < ε/2$. On the other hand, since $f(n) → ∞$, given $ε > 0$, there exists $N_2$ such that $n > N_2$, there exists a line of $Σ(L, L_n)$, say $Δ_k(n)$, such that $δ_p(Δ_k(n), Γ_n) < ε/2$. On the other hand, if $n > \max\{N_1, N_2\}$, then $δ_p(Δ_k(n), Δ) ≤ δ_p(Δ_k(n), Γ_n) + δ_p(Γ_n, Δ) < ε/2 + ε/2 = ε$, i.e., $Δ_k(n) → Δ$. If $k = 2$, we have seen, in the first part of the proof of Theorem 2, that every line $L ⊂ V$, through $o$, is an axis of symmetry of the body $K$. By Theorem 1 $K$ is a 2-body of revolution.

We suppose now $3 ≤ k ≤ d − 1$. We have proved that there exists a plane $Π ⊂ V$ such that every line in $Π$, through $o$, is an axis of symmetry of $K$. Since $k > 2$, there exists an integer $n_1$ such that $L_{n_1}$ is not contained in $Π$ and, by condition ii) of Theorem 2 $L_{n_1}$ is not perpendicular to $Π$. By Lemma 3 every line in $\aff\{Π, L_{n_1}\}$, through $o$, is an axis of symmetry of $K$. If $d = 3$, then $k = 3$ and $E^3 = \aff\{Π, L_{n_1}\}$ and, consequently, $K$ is a sphere (in particular, it follows that all the sections of $K$ are centrally symmetric and it is well know that such condition implies that $K$ is an ellipsoid and, on the other hand, the only ellipsoid with an infinite number of axis of symmetry is the sphere). If $d > 3$ and $k = 3$, then, by Theorem 1 $K$ is a 3-body of revolution. If $k > 3$, using repeatedly Lemma 3 we can assume the existence of axis of symmetry $\{L_{n_1}, L_{n_2}, ..., L_{n_{k−2}}\} ⊂ {L_n}$ of $K$ such that every line $L' \subset V = \aff\{Π, L_{n_1}, L_{n_2}, ..., L_{n_{k−2}}\}$ through $o$ is an axis of symmetry of $K$. Thus, if $k ≤ d − 1$, then $K$ is a $k$-body of revolution by Theorem 1. If $k = d$, then $K$ is a sphere and the proof of Theorem 2 is now complete. □

Proof of Theorem 3 In virtue of the hypothesis, there exists an integer $k$, $1 ≤ k ≤ d − 2$, and $k$-flat $Γ$ such that the hyperplanes of symmetry $Π_i$ of $K$ satisfies (2). Let $Δ$ be a $(d − k)$-flat ortogonal to $Γ$ and such that $Δ \cap \text{int} \ K \neq ∅$. By (2), $Δ$ is ortogonal to $Π_i$, $i = 1, 2, ...$. Consequently, $Δ \cap Π_i$.
is \((d-k-1)\)-flat of symmetry of \(\Delta \cap K\). We claim that there are no an integer \(t, 1 \leq t \leq d-k-1\), a \(t\)-flat \(\Psi \subset \Delta\) and subsequence \(\Delta \cap \Pi_{i_j}\) of \(\Delta \cap \Pi_i\) such that \(\Psi \subset \Pi_{i_j}, j = 1, 2, \ldots\) Otherwise, we would have that \(\text{aff}\{\Gamma, \Psi\} \subset \Pi_{i_j}\) and since \(\dim(\text{aff}\{\Gamma, \Psi\}) > k\) we would contradict the maximality of \(k\). By Proposition \[\text{I}\] \(\Delta \cap K\) is a sphere with centre at \(\Gamma\), i.e., \(K\) is a \((d-k)\)-body of revolution. \[\square\]

Proof of Theorem \[\text{I}\] Let \(\Pi\) be a hyperplane, \(p \in \Pi\). We are going to prove that for every hyperplane \(\Gamma\) with \(p \in \Gamma, \Gamma \neq \Pi\), the sections \(\Pi \cap K\) and \(\Gamma \cap K\) are congruent, i.e., there exists an orthogonal transformation \(\Omega : \mathbb{E}^d \to \mathbb{E}^d\) such that \(\Omega(\Pi \cap K) = \Gamma \cap K\). We denote by \(\Delta\) the subset of \(\mathbb{S}^{d-1}\) defined as follows: \(u \in \Delta\) if there exists a hyperplane \(\Sigma, p \in \Sigma\), whose corresponding unit normal vector is \(u\) and \(\Sigma \cap K\) and \(\Pi \cap K\) are congruent. We are going to show that \(\Delta = \mathbb{S}^{d-1}\). We denote by \(w\) the unit normal vector of \(\Pi\). Let \(L \subset \Pi\) be a \((d-2)\)-plane, \(p \in L\). In virtue that \(L\) is a \((d-2)\)-axis of rotation of order \(k\), there are hyperplanes \(\Sigma_1, \Sigma_2\) with unit normal vector \(v_1, v_2\), respectively, such that \(\Pi \cap K\) can be obtained from \(\Sigma_1 \cap K\) and \(\Sigma_2 \cap K\) after apply two rotations, both with axis \(L\), one by an angle \(2\pi/k\) and and the other by an angle \(2(k-1)\pi/k\). Since the boundary of \(K\) is a continuous surface, varying \(L\), always contained in \(\Pi\), we have that the collection of unit normal vector of the hyperplanes \(\Sigma\) is a \((d-2)\)-sphere \(\tau_w \subset \mathbb{S}^{d-1}\) with center at \(w\) and each vector \(v \in \tau_w\) makes an angle \(2\pi/k\) with \(w\). It is clear that \(\tau_w \subset \Delta\). Applying the same argument it follows that if \(v\) is in \(\Delta\), then \(\tau_v \subset \Delta\). Thus

\[
\bigcup_{v \in \tau_w} \tau_v \subset \Delta \quad (7)
\]

We denote by \(B_w\) the interior of the spheric cap with boundary \(\tau_w\) in \(\mathbb{S}^{d-1}\). In virtue that

\[
B_w = \bigcup_{v \in \tau_w} \tau_v,
\]

we get from \((7)\) that \(B_w \subset \Delta\). Finally we observe that the collection of sets \(\{B_v : v \in \tau_w\}\) covers the sphere. Hence \(\Delta = \mathbb{S}^{d-1}\), i.e., any two sections of \(K\) passing through \(p\) are congruent. In virtue of Süss-Schneider’s Theorem \(K\) is a sphere \([9],[11]\). \[\square\]

Lemma 4 Let \(K \subset \mathbb{E}^d\) be a convex body, \(d \geq 3, p \in \mathbb{E}^d\) be a point. Suppose that every \((d-2)\)-flat passing through \(p\) is a \((d-2)\)-axis of symmetry of \(K\), then \(K\) is a sphere with centre at \(p\).

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Proof. Let Λ be a hyperplane, \( p \in \Lambda \). In virtue of the hypothesis, each \((d - 2)\)-plane \( \Gamma \subset \Lambda, p \in \Gamma \), is a \((d - 2)\)-axis of \( K \) and, by definition, \( \Lambda \cap K \) has \( \Gamma \) as a \((d - 2)\)-flat of symmetry. By Proposition 1, \( \Lambda \cap K \) is a sphere. Thus \( K \) is a sphere. ■

Lemma 5 Let \( K \subset \mathbb{E}^d \) be a convex body, \( d \geq 3 \), and let \( \Gamma \) be a \( k \)-flat, \( 1 \leq k \leq d - 2 \). Then \( \Gamma \) is a \( k \)-axis of \( K \) if and only if, for all \((d - k)\)-flat \( \Omega \) orthogonal to \( \Gamma \) and such that \( \Omega \cup K \neq \emptyset \), the section \( \Omega \cap K \) is centrally symmetric with centre at the point \( \Omega \cap \Gamma \).

Proof. Let \( \Gamma \) be a \( k \)-axis and let \( \Omega \) be a \((n - k)\)-flat such that \( \Omega \) is orthogonal to \( \Gamma \) and \( \Omega \cup K \neq \emptyset \). For all \((k + 1)\)-flat \( \Pi, \Gamma \subset \Pi \), since \( \Gamma \) is a \( k \)-axis of \( K \), the section \( \Pi \cap \text{bd} K \) is symmetric respect to \( \Gamma \). Hence the line segment \( \Omega \cap (\Pi \cap \text{bd} K) \) has its mid point \( \Omega \cap \Gamma \). Varying \( \Pi, \Gamma \cap \Pi \), it follows that \( \Omega \cap K \) is centrally symmetric with centre at \( \Omega \cap \Gamma \). ■

Proof of Theorem 5 First we prove that \( K \) is centrally symmetric. For every \((n - k)\)-flat \( \Omega, \Lambda^\perp \subset \Omega \), we consider the \( k \)-flat \( \Omega^\perp \subset \Lambda \). By hypothesis \( \Omega^\perp \) is a \( k \)-flat of symmetry of \( K \). By Lemma 5, it follows that \( \Omega \cap K \) is centrally symmetric with centre at \( p \). Varying \( \Omega, \Lambda^\perp \subset \Omega \), we obtain that \( K \) is centrally symmetric with centre at \( p \).

Since \( K \) is centrally symmetric, we take a co-ordinate system such that the centre \( p \) of \( K \) is the origin. Let \( \Gamma \) be a \( k \)-flat, \( p \in \Gamma \subset \Lambda \). We are going to show that

\[
\Gamma^\perp \cap \text{bd} K \subset S\partial(K, \Gamma).
\]

Let \( \Phi \) be a \((n - k)\)-flat orthogonal to \( \Gamma \) such that \( \Phi \cap K \neq \emptyset \). We take \( x \in \Phi \cap \text{bd} K \). We are going to show that there exists a \( k \)-flat \( \Delta \) parallel to \( \Gamma \), such that \( x \in \Delta \) and \( \Delta \cap \text{bd} K = \{x, y\} \) where \( y \in (-\Phi) \cap \text{bd} K \). By Lemma 5 we know that \( \Phi \cap \text{bd} K \) and \(-\Phi \cap \text{bd} K\) have centres, say \( c, -c \), and \( c, -c \in \Gamma \). Since \(-c + (c - x) = -x \in (-\Phi \cap \text{bd} K) \), we get \(-c - (c - x) = x - 2c \in (-\Phi \cap \text{bd} K) \). Thus if we define \( \Delta = x + \Gamma \) and \( y = x - 2c \), we obtain what we have claimed. It yields that \( S\partial(K, \Gamma) \) is contained in the region between \( \Phi \) and \(-\Phi \), except perhaps in the point of \( S\partial(K, \Gamma) \) where there is a supporting \( k \)-flat whose intersection with \( \text{bd} K \) has dimension between 1 and \( k \). It follows, taking the limit \( \Phi \to \Gamma^\perp \), that \( \Gamma^\perp \cap \text{bd} K \subset S\partial(K, \Gamma) \).
For every two \((k+1)\)-flats \(\Lambda_1, \Lambda_2\) parallel to \(\Lambda\), the sections \(\Lambda_1 \cap K, \Lambda_2 \cap K\) are such that for all \((n-k)\)-flat \(\Omega, \Lambda^\perp \subset \Omega\), there are parallel supporting \(k\)-flats of \(\Lambda_1 \cap K, \Lambda_2 \cap K\) at the extreme points of the line segments \((\Lambda_1 \cap \Omega) \cap \text{bd} K, (\Lambda_2 \cap \Omega) \cap \text{bd} K\). From such property, it follows that the sections \(\Lambda_1 \cap K, \Lambda_2 \cap K\) are similar, similarly situated and have centres on \(\Lambda^\perp\). We omit the proof of this statement because the proof for two dimensional case of it (see lemma 9 in [7]) works, step by step, for our general case. To finish the proof of Theorem 5, we show that \(\Lambda \cap K\) is a sphere, we use the argument presented in the proof of Lemma 11. In virtue of the hypothesis, each \(k\)-flat \(\Gamma \subset \Lambda, p \in \Gamma\), is a \(k\)-axis of \(K\) and, consequently, \(\Lambda \cap K\) has \(\Gamma\) as a \(k\)-flat of symmetry. By Proposition 1, \(\Lambda \cap K\) is a sphere. □

Remark 2 We must observe that a \((d-2)\)-flat of symmetry of a convex body \(K\) is, at the same time, a \((d-2)\)-axis or rotation of order 2 of \(K\). Therefore Theorem 5, for the case \(k = d-2\), can be considered as a generalization of Theorem 1 for \((d-2)\)-axis or rotation of order 2.

Conjecture 1 Let \(K \subset \mathbb{E}^d\) be a convex body, \(d \geq 3\). Suppose that there exists a sequence \(\{\Gamma_n\} \subset \mathbb{E}^d\) of \((d-2)\)-axis of rotation of some order, say \(\phi(\Gamma_n)\), of \(K\) such that the sequence \(\{\phi(\Gamma_n)\}\) has an infinite number of terms different from 2. Then \(K\) is a sphere.

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