Exact nonequilibrium solutions of the
Einstein–Boltzmann equations. II

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Abstract.
We find exact solutions of the Einstein–Boltzmann equations with relaxational collision term in FRW and Bianchi I spacetimes. The kinematic and thermodynamic properties of the solutions are investigated. We give an exact expression for the bulk viscous pressure of an FRW distribution that relaxes towards collision–dominated equilibrium. If the relaxation is toward collision–free equilibrium, the bulk viscosity vanishes – but there is still entropy production. The Bianchi I solutions have zero heat flux and bulk viscosity, but nonzero shear viscosity. The solutions are used to construct a realisation of the Weyl Curvature Hypothesis.

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1. Introduction

In Paper I [1], we derived exact properties of the Einstein–Boltzmann equations with a relaxation–time model of collisions. An exact truncated distribution was used to derive transport equations similar in form to the Israel-Stewart thermodynamics. We also found an exact truncated solution for massless particles in a flat FRW spacetime. This paper extends Paper I by considering exact non–truncated Einstein–Boltzmann solutions in FRW and Bianchi I spacetimes.

In Section 2 we present exact anisotropic solutions for a flat FRW spacetime, and investigate their properties. In particular we show that the bulk viscosity of these solutions is always zero, since they are relaxing towards free–streaming equilibrium. For solutions that relax toward hydrodynamic behaviour, we give an exact formula for the bulk viscous pressure. Our solutions generalise the equilibrium anisotropic solutions given by [2]. Isotropic non–equilibrium solutions are only possible for massless particles, and they have the surprising property that entropy is generated despite the vanishing of bulk viscosity, heat flow, shear viscosity – and of all non-scalar moments of the distribution. This illustrates the point that the standard dissipative quantities (bulk viscosity, heat flux, shear viscosity) cannot provide a complete or exact description of non–equilibrium states.

In Section 3 we find exact anisotropic solutions in Bianchi I spacetimes. The solutions have zero heat flux and bulk viscosity but nonzero shear viscosity. In Section 4 we use the results of the previous sections to construct a model in which an Einstein–Boltzmann solution in FRW spacetime evolves into an Einstein–Boltzmann solution in Bianchi spacetime. The basic idea, due to Matravers and Ellis [3], is that anisotropy in the FRW distribution is communicated to the geometry via the Einstein field equations when the gas cools sufficiently for massive particles to become non–relativistic. We provide an explicit dissipative realisation of the Ellis–Matravers model, which is in accord with Penrose’s Weyl Curvature Hypothesis, i.e. that the universe is initially conformally flat and that anisotropy, inhomogeneity and entropy production develop as the universe expands [4], [5].

The distribution \( f(x, p) \) satisfies the Boltzmann equation with relaxational collision term, i.e. the BGK equation [1]

\[
L[f] \equiv \frac{df}{dv} \equiv p^i \frac{\partial f}{\partial x^i} - \Gamma^i_{jk} p^j p^k \frac{\partial f}{\partial p^i} = \gamma(x, E)(\bar{f} - f)
\]  

(1)

where \( \gamma \) encodes microscopic interaction information in a linear, macroscopic approximation [1]. \( \bar{f} \) is the distribution towards which (or away from which if \( \gamma < 0 \)) \( f \) is relaxing, and \( E = -u_i p^i \) is the particle energy relative to the average four–velocity \( u^i \) associated with \( \bar{f} \). If \( \bar{f} \) is a dynamic (or ‘global’) equilibrium distribution [1], \( L[\bar{f}] = 0 \),
then the solution of (1) is
\[ f = \bar{f} + h e^{-\Gamma} \] where \( L[h] = 0 \), \( \Gamma(x(v), E(v)) = \int^v \gamma(x(u), E(u)) \, du \) (2)

The standard form for \( \gamma \) is the Anderson–Witting (AW) form [6]:
\[ \gamma(x, E) = \frac{E}{\tau(x)} \] (3)

where \( \tau \) is the mean interaction time. This form includes the case of radiative transfer with isotropic scattering [7]. Using \( dv = dt/p^0 = dt/E \) and (3), the relaxation factor in (4) becomes a spacetime scalar:
\[ \Gamma = \int \frac{dt}{\tau} \] (4)

The distribution may be covariantly decomposed relative to \( u^i \) [8]:
\[ f(x, p) = F(x, E) + F_i(x, E)e^i + F_{ij}(x, E)e^i e^j + \ldots \] (5)

where the covariant multipoles \( F_{ij} \ldots \) are isotropic, spatial, trace-free and symmetric, and \( e^i \) is the unit spatial projection of \( p^i \).

In order to avoid unnecessary details, we will not summarise the basic equations and results of Paper I, but refer where necessary directly to the equations in that paper in the form (I:n), where \( n \) is the equation number in Paper I.

2. Exact non–equilibrium Einstein–BGK solution in FRW spacetime

In Paper I we found an Einstein–BGK solution for a truncated form of the distribution function. This solution was used to derive a set of exact thermodynamic laws. In this section we present the full non–truncated solution in flat FRW spacetime with natural coordinates \( x^i = (t, x^\nu) \):
\[ ds^2 = -dt^2 + R^2(t)(dx^2 + dy^2 + dz^2) \] (6)

By homogeneity, the spatial momenta \( p_\nu \) are constants of the motion, i.e. \( L[p_\nu] = 0 \). Thus homogeneous distributions are of the form \( f(t, p_\nu) \), while isotropic and homogeneous distributions are of the form \( f(t, w) \) where
\[ w^2 \equiv (p_1)^2 + (p_2)^2 + (p_3)^2 = (E^2 - m^2)R^2 \] (7)

and where \( u^i = \delta^i_0 \). If the homogeneous \( f \) is relaxing toward the homogeneous and isotropic equilibrium, \( \bar{f} \), then by (3) and (4)
\[ f(t, p_\nu) = \bar{f}(w) + h(p_\nu)e^{-\Gamma(t)} \], \( \Gamma(t) = \int_0^t \frac{dt'}{\tau(t')} \) (8)

Clearly \( f \) is spatially homogeneous but dynamically anisotropic, and it must depend explicitly on the cosmic time (i.e. \( \dot{\Gamma} \neq 0 \)) if it is non–equilibrium. Note that since
\[ L[f] = 0, \quad \bar{f} \text{ can only be a collision–dominated Maxwell–Boltzmann equilibrium if } m = 0; \text{ for } m > 0, \quad \bar{f} \text{ must be a collision–free equilibrium distribution.} \]

In this case the BGK solution \( f \) represents a distribution relaxing towards free–streaming isotropic equilibrium.

In order to investigate the properties of the solutions, we use the covariant harmonic decomposition (5):

\[
F(t, w) = \bar{F}(w) + H(w)e^{-\Gamma(t)}, \quad F_{\nu\kappa}(t, w) = H_{\nu\kappa}(w)e^{-\Gamma(t)}
\]

The kinematics and the dynamics of the solution (9) are determined by the particle four–current \( N_i = n u_i + k_i \) and energy–momentum tensor \( T_{ij} = \mu u_i u_j + p h_{ij} + \pi_{ij} + q_i u_j + q_j u_i \), where \( n \) is the number density, \( k_i \) is the number flux, \( \mu \) is the energy density, \( p \) is the isotropic pressure, \( h_{ij} = g_{ij} + u_i u_j \) is the spatial projector, \( \pi_{ij} \) is the anisotropic pressure tensor and \( q_i \) is the heat flux. These quantities follow from (I:40) and (9):

\[
n = \bar{n} + 4\pi e^{-\Gamma(t)} \int_0^\infty w^2 H(w)dw
\]

\[
k_\nu = \bar{k}_\nu + 4\pi e^{-\Gamma(t)} \int_0^\infty w^3[w^2 + m^2 R^2(t)]^{-1/2} H_\nu(w)dw
\]

\[
\mu = \bar{\mu} + 4\pi e^{-\Gamma(t)} \int_0^\infty w^2[w^2 + m^2 R^2(t)]^{1/2} H(w)dw
\]

\[
p = \bar{p} + 4\pi e^{-\Gamma(t)} \int_0^\infty w^4[w^2 + m^2 R^2(t)]^{-1/2} H(w)dw
\]

\[
q_\nu = 4\pi e^{-\Gamma(t)} \int_0^\infty w^3 H_\nu(w)dw
\]

\[
\pi_{\nu\kappa} = 8\pi e^{-\Gamma(t)} \int_0^\infty w^4[w^2 + m^2 R^2(t)]^{-1/2} H_{\nu\kappa}(w)dw
\]

with \( k_0 = 0, \quad q_0 = 0 \) and \( \pi_{0i} = 0 \). A non–zero number flux gives a particle drift that is out of keeping with FRW symmetry, and although it is possible to satisfy the field equations for \( k_i \neq 0 \) (see the fluid solutions of Calvao and Salim [4]), we regard this as unnatural. From equation (11) it is clear that in order to get a zero number flux, which gives a non–tilted kinematic average 4–velocity, \( H_\nu \) must vanish for \( m > 0 \). It is possible to find nonzero \( H_\nu \) if \( m = 0 \):

\[
\int_0^\infty w^2 H_\nu(w)dw = 0
\]

The full Boltzmann collision term is based on microscopic conservation so that the macroscopic conservation of momentum and energy are identically satisfied. This is not the case for the BGK collision model, and the conditions imposed by the conservation equations require separate investigation [1]. The conservation of particle number, energy and momentum are given by equation (I:48). On using the FRW BGK solution (1), we
find the following condition for the conservation of particles:
\[
\int_{0}^{\infty} w^2 H(w) dw = 0
\] (17)
which by (10) implies \( n = \bar{n} \), i.e. the number density is matched to that of the limiting equilibrium.

The condition for the conservation of energy gives
\[
\int_{0}^{\infty} w^2 [w^2 + m^2 R^2(t)]^{1/2} H(w) dw = 0
\] (18)
If \( m > 0 \), this forces \( H = 0 \), but \( H \) may be nonzero for \( m = 0 \). In all cases, (18) in (12) implies \( \mu = \bar{\mu} \), so that the energy density matches the equilibrium value.

Momentum conservation gives
\[
\int_{0}^{\infty} w^3 H \nu(w) dw = 0
\] (19)
Condition (19) is automatically satisfied for \( m > 0 \), since \( H \nu = 0 \) from \( k \nu = 0 \). For \( m = 0 \), it is a further condition. By (14), we see that (19) leads to \( q \nu = \bar{q} \nu = 0 \), so that \( u^i \) is also the energy–frame four–velocity (it is already the particle–frame four–velocity by \( k \nu = 0 \), so the two four–velocities coincide in these solutions).

The entropy density \( s \) is given by (I:40g). For the FRW solution (\( 3 \)):
\[
s = \frac{4\pi}{R^3(t) \tau(t)} \int_{0}^{\infty} w^2 \{ [\bar{F}(w) + H(w) e^{-\Gamma(t)}] \left[ 1 - \ln[\bar{F}(w) + H(w) e^{-\Gamma(t)}] \right] \\
- \frac{1}{6} [\bar{F}(w) + H(w) e^{-\Gamma(t)}]^{-1} e^{-2\Gamma(t)} H \nu(w) H' \nu(w) + \cdots \} dw
\] (20)
The entropy production rate (I:40i) becomes
\[
S^i;_i = \frac{4\pi e^{-\Gamma(t)}}{R^3(t) \tau(t)} \int_{0}^{\infty} w^2 \{ H(w) \ln[\bar{F}(w) + H(w) e^{-\Gamma(t)}] \\
+ \frac{1}{6} e^{-\Gamma(t)} H \nu(w) H' \nu(w) \left[ \bar{F}(w) + H(w) e^{-\Gamma(t)} \right]^{-2} \times \\
\times [2\bar{F}(w) + H(w) e^{-\Gamma(t)}] + \cdots \} dw
\] (21)
For the Boltzmann collision term, \( S^i;_i \geq 0 \) follows identically, but this is not true for the BGK collision term (\( 4 \)). It is not clear in general from (21) whether the \( H \)-theorem, \( S^i;_i \geq 0 \), is satisfied without restrictions for the BGK solution (\( 4 \)). This needs to be checked for each specific solution. Note that for \( m > 0 \), when \( H = 0 = H \nu \), the only contribution to \( S^i;_i \) is from the quadrupole and higher moments. For \( m = 0 \), \( H \) is in general nonzero and there is a monopole contribution to the entropy production:
\[
S^i;_i = \frac{4\pi e^{-\Gamma(t)}}{R^3(t) \tau(t)} \int_{0}^{\infty} w^2 H(w) \ln[\bar{F}(w) + H(w) e^{-\Gamma(t)}] dw + \cdots
\] (22)
In summary:
\[
H = 0 = H \nu \quad \text{for} \ m > 0
\] (23)
\[
\int_{0}^{\infty} w^r H(w) dw = 0 = \int_{0}^{\infty} w^r H \nu(w) dw \quad \text{for} \ m = 0 \ (r = 2, 3)
\] (24)
With (23) and (24), conservation of particle number and energy–momentum is satisfied and we have
\[ n = \bar{n}, \mu = \bar{\mu}, k_\nu = 0, q_\nu = 0 \] (25)

Now we impose the Einstein field equations (I:61) (note that the conservation equations are already satisfied):
\[ q_\nu = 0 = \pi_{\nu\kappa} \] (26)
\[ \mu = 3\frac{\dot{R}^2}{R^2} \] (27)

The heat flux is already zero by (25). Vanishing anisotropic stress requires, by (15)
\[ \int_0^\infty w^4[w^2 + m^2R^2(t)]^{-1/2}H_{\nu\kappa}(w)dw = 0 \] (28)

For \( m > 0 \) (28) forces \( H_{\nu\kappa} = 0 \). For \( m = 0 \), \( H_{\nu\kappa} \) is subject to
\[ \int_0^\infty w^3H_{\nu\kappa}(w)dw = 0 \] (29)

Using (12) in the Friedmann equation (27) we can write it as
\[ \dot{R}(t) = \left[ \frac{4\pi}{3R^2(t)} \int_0^\infty w^2[w^2 + m^2R^2(t)]^{1/2}\bar{F}(w)dw \right]^{1/2} + \frac{4\pi e^{-\Gamma(t)}}{3R^2(t)} \int_0^\infty w^2[w^2 + m^2R^2(t)]^{1/2}H(w)dw \] (30)

By (23) and (24) the term containing \( H(w) \) in (30) vanishes for \( m \geq 0 \) and we can re-arrange this equation and give the solution explicitly as
\[ t = \frac{\sqrt{3}}{2} \int_0^{R^2} \left[ \frac{4\pi}{3R^2} \int_0^\infty w^2[w^2 + m^2u]^{1/2}\bar{F}(w)dw \right]^{-1/2}du \] (31)

which is the same as for an equilibrium Einstein solution [4]. By specifying \( \bar{F} \), \( R(t) \) can be determined in principle, and the metric (6) will be known – completing the Einstein solution. This is analogous to specifying an equation of state in a fluid model.

The reason that \( R \) has the same form as for an equilibrium viscous solution lies in the vanishing of the bulk viscous pressure \( \Pi = p - \bar{p} \). By (13)
\[ \Pi = \frac{4\pi e^{-\Gamma}}{3R^4} \int_0^\infty w^4[w^2 + m^2R^2]^{-1/2}H(w)dw \] (32)

It follows from (23) and (24) that \( \Pi = 0 \) for \( m \geq 0 \). For \( m = 0 \), this is in accord with the approximation schemes used to derive transport equations [10]. For \( m > 0 \), the vanishing of bulk viscosity is a consequence of our choice of \( \bar{f} \) satisfying \( L[\bar{f}] = 0 \). This means that \( \bar{f} \) cannot be a Maxwell–Boltzmann distribution (unless the universe is static) [1], and therefore \( f \) cannot be approaching the hydrodynamic regime where
the standard approximation schemes are applied and predict $\Pi \neq 0$. Instead $f$ must be collision–free, and if $f$ is near to free–streaming, our results show that, unsurprisingly, there is no bulk viscosity.

If we drop the restriction $L[f] = 0$ and take $\bar{f}$ to be a local Maxwell–Boltzmann distribution, we have

$$\bar{f} = \exp \left[ \alpha(t) - \frac{E}{T(t)} \right]$$

(33)

where $T$ is the temperature and $\alpha$ the chemical potential. Using the general (i.e. $L[\bar{f}] \neq 0$) BGK solution (I:44), we find the following exact formula for $\Pi$:

$$\Pi = -\frac{4\pi}{3} e^{-\Gamma(t)} \int_0^t dt e^{\Gamma(t)} \int_0^\infty dE (E^2 - m^2)^{3/2} e^{-E/T(t)} \times$$

$$\times \left\{ \dot{\alpha} + \frac{E}{T(t)} \left[ \dot{T} + \frac{\dot{R}}{R(t)} \left( 1 - \frac{m^2}{E^2} \right) \right] \right\}$$

(34)

In the case $m = 0$, when $T \propto R^{-1}$ and $\dot{\alpha} = 0$, we have $\Pi = 0$, as expected.

What is remarkable about the solutions with $L[f] = 0$ is this: despite the vanishing of viscosity and heat flux, there is dissipation, since $L[f] \neq 0$. The simplest case is the isotropic massless solution ($\alpha$ constant)

$$f = \bar{f} + H e^{-\Gamma(t)}, \quad \bar{f} = e^{\alpha - w}$$

(35)

first presented in [1]. By (22) and (24) the entropy production rate is

$$S^i_{\,;i} = \frac{4\pi e^{-\Gamma(t)}}{R(t) \tau(t)} \int_0^\infty w^2 H(w) \ln[1 + H(w) e^{w - \Gamma(t) - \alpha}] dw$$

(36)

and this is clearly positive since $f$ is. The point is that the bulk viscosity, heat flux and shear viscosity are only approximate indicators of non–equilibrium states, and even if they vanish there can still be dissipation. Non–equilibrium states in general cannot be completely, and certainly not exactly, described by these standard dissipative quantities. Usually this is understood in terms of the effect of multipoles higher than the quadrupole, which are neglected in the standard approximation schemes. However, the isotropic solution (35) has no multipoles beyond the scalar monopole, and yet it is out of equilibrium.

3. Exact non–equilibrium Einstein–BGK solutions in Bianchi I spacetime

Bianchi I spacetime,

$$ds^2 = -dt^2 + X^2(t)dx^2 + Y^2(t)dy^2 + Z^2(t)dz^2$$

(37)

is distinguished kinematically from FRW spacetime by non–zero shear, whose evolution is given by

$$\dot{\sigma}_{ij} - \sigma_{ik} \sigma^{k}_{\,j} + \frac{1}{2} h_{ij} \sigma^{kl} \sigma_{kl} - \frac{1}{2} \pi_{ij} + E_{ij} = 0$$

(38)
where $E_{ik} = C_{ijkl}u^ju^l$ is the electrical part of the Weyl tensor $C_{ijkl}$. By symmetry, the trace–free spatial tensors have the form

$$A^i_j = \text{diag} \left( 0, A^1_1, A^2_2, -A^1_1 - A^2_2 \right)$$  \hspace{1cm} (39)

where $A_{ij} = \sigma_{ij}, \pi_{ij}, E_{ij}$ or $F_{ij}$.

The Einstein field equations for this metric are \[\ddot{X} + \ddot{Y} + \ddot{Z} = -\frac{1}{2}(\mu + 3p) \] \hspace{1cm} (40)
and

$$q_j = 0 \hspace{1cm} (41)$$

$$\ddot{X} + \ddot{Y} \dddot{X} + \dddot{Z} \ddot{X} = \frac{1}{2}(\mu - p) + \pi_1^1 \hspace{1cm} (42)$$

$$\dddot{Y} + \ddot{Y} \dddot{X} + \dddot{Z} \ddot{Y} = \frac{1}{2}(\mu - p) + \pi_2^2 \hspace{1cm} (43)$$

$$\ddot{Z} + \ddot{Z} \dddot{X} + \dddot{Z} \ddot{Y} = \frac{1}{2}(\mu - p) - \pi_1^1 - \pi_2^2 \hspace{1cm} (44)$$

The conservation of energy–momentum reduces to $(\Theta = u^i_{,i})$

$$\dot{\mu} + (\mu + p)\Theta + \pi_{ij}\sigma^{ij} = 0$$

which is identically satisfied if $(40) - (44)$ are satisfied.

By homogeneity, the spatial momenta $p_\nu$ are constants of the motion. Thus homogeneous distributions are of the form $f(t, p_\nu)$, including the special case $f(t, w)$ where

$$w^2 = (p_1)^2 + (p_2)^2 + (p_3)^2$$
$$= X^4(t)(p^1)^2 + Y^4(t)(p^2)^2 + Z^4(t)(p^3)^2 \hspace{1cm} (45)$$

Contrary to the FRW case, $w$ is not isotropic, i.e. it is not a function only of $E$ in momentum space, where

$$E = [m^2 + X^{-2}(t)(p_1)^2 + Y^{-2}(t)(p_2)^2 + Z^{-2}(t)(p_3)^2]^{1/2} \hspace{1cm} (46)$$

Both $w$ and $E$ are spatially homogeneous. However, there is no simple relation between the anisotropic constant of motion $w$ and the isotropic non–constant energy $E$, unlike the FRW case. This happens since $h_{ij}$ is anisotropic:

$$\lambda^2 \equiv E^2 - m^2 = h^{\mu\nu}p_\mu p_\nu$$

$$w^2 = \delta^{\mu\nu}p_\mu p_\nu$$

In the FRW case $h^{\mu\nu} = R^{-2}(t)\delta^{\mu\nu}$.

If the homogeneous, anisotropic $f$ is relaxing toward the homogeneous, anisotropic $\bar{f}$, where $L[\bar{f}] = 0$, then the BGK solution has the same form (8) as in the FRW case, but
with \( w \) given by (45) (and anisotropic). It follows that the Bianchi I solution matches the FRW solution (48) in the limit \( R(t) = X(t) = Y(t) = Z(t) \). The covariant harmonic decomposition (49) takes the form

\[
f(t, p_\nu) = F(t, E) + F_\kappa(t, E)e^\kappa + F_{\kappa\rho}(t, E)e^\kappa e^\rho + \cdots
\]

where

\[
F_{\nu \cdots \kappa}(t, E) = \tilde{F}_{\nu \cdots \kappa}(t, E) + e^{-\Gamma(t)}H_{\nu \cdots \kappa}(t, E)
\]

Note that since \( \tilde{f} \) is anisotropic, its higher order multipoles cannot be neglected in the decomposition. Although the solution and its decomposition do not take the convenient form of the FRW case, it gives us the necessary tools to investigate the conditions for an Einstein–BGK solution.

Because of the nonvanishing shear in the Bianchi I geometry, the covariant multipoles in (47) are no longer independent. The relationship between the multipoles is determined by the Boltzmann equation. By attaching an orthonormal tetrad to \( u^i \), the Liouville operator \( L \) can be written in the covariant harmonic form (I:10). This allows one to write the Boltzmann equation as a set of coupled differential equations in the multipoles. The first two Boltzmann multipole equations for a homogeneous distribution function [11] become, for a Bianchi I geometry and a BGK AW collision term:

\begin{align}
2 \frac{15}{15} \lambda^{-1} & \frac{\partial}{\partial E} \left( \lambda^3 \sigma^{\nu \kappa} F_{\nu \kappa} \right) + \frac{1}{3} \lambda^2 \Theta \frac{\partial F}{\partial E} \\
- E \frac{\partial F}{\partial t} & = \tau^{-1} E(F - \tilde{F}) \\
- \frac{6}{35} \lambda^{-2} & \frac{\partial}{\partial E} \left( \lambda^4 \sigma^{\kappa \rho} F_{\kappa \rho} \right) + \frac{2}{5} \lambda^{1/2} \frac{\partial}{\partial E} \left( \lambda^{3/2} \sigma_{\nu \kappa} F^{\kappa} \right) \\
+ \frac{1}{3} \lambda^2 \Theta \frac{\partial F_{\nu}}{\partial E} & = \tau^{-1} E(F_{\nu} - \tilde{F}_{\nu})
\end{align}

where we have used the Bianchi I symmetry to simplify the expressions given in (48). The higher order multipole equations [11] (p492) together with (48), (49) show that if \( F, F_\nu \) are specified, then the multipole equations place a chain of restrictions on the quadrupole and higher moments. The important point is that if the shear is non–zero, the multipoles are no longer independent.

From (I:40), the kinematic and dynamic quantities of the solution (47) are

\begin{align}
n & = \bar{n} + 4\pi e^{-\Gamma(t)} \int_m^\infty E \lambda H(t, E) dE \\
k_\nu & = \bar{k}_\nu + \frac{4\pi}{3} e^{-\Gamma(t)} \int_m^\infty \lambda^2 H_\nu(t, E) dE \\
\mu & = \bar{\mu} + 4\pi e^{-\Gamma(t)} \int_m^\infty E^2 \lambda H(t, E) dE \\
p & = \bar{p} + \frac{4\pi}{3} e^{-\Gamma(t)} \int_m^\infty \lambda^3 H(t, E) dE
\end{align}
\begin{align}
q_\nu &= \bar{q}_\nu + \frac{4\pi}{3} e^{-\Gamma(t)} \int_m^\infty E \lambda^2 H_\nu(t, E) dE \quad (54) \\
\pi_{\nu \kappa} &= \bar{\pi}_{\nu \kappa} + \frac{8\pi}{15} e^{-\Gamma(t)} \int_m^\infty \lambda^3 H_{\nu \kappa}(t, E) dE \quad (55)
\end{align}

The particle flux $\bar{k}_i$, energy flux $\bar{q}_i$ and anisotropic stress $\bar{\pi}_{ij}$ of the equilibrium distribution $\bar{f}$ are in general nonzero because $\bar{f}$ is anisotropic. These quantities do not reflect any dissipation, but are part of the measure of deviation from isotropy (compare [12], [13]).

Given that the BGK collision model does not guarantee macroscopic conservation, we impose the conditions for the conservation of particle number, energy and momentum (I:40g):

\begin{align}
\int_m^\infty E \lambda H dE &= 0 \quad (56) \\
\int_m^\infty E^2 \lambda H dE &= 0 \quad (57) \\
\int_m^\infty E \lambda^2 H_\nu dE &= 0 \quad (58)
\end{align}

With (50), (52) and (54) these give the matching conditions:

\begin{align}
n = \bar{n}, \quad \mu = \bar{\mu}, \quad q_\nu = \bar{q}_\nu \quad (59)
\end{align}

The Bianchi I solution has equilibrium particle number density, energy density and energy flux. In general, since $H$ and $H_\nu$ depend explicitly on time, (56) – (58) require

\begin{align}
H = 0 = H_\nu \quad (60)
\end{align}

The Einstein field equations are imposed next. Equation (41) along with (54) and (58) implies

\[
\int_m^\infty E^2 \lambda^2 \bar{F}_\nu dE = 0
\]

Again, in general this requires

\[
\bar{F}_\nu = 0 \quad (61)
\]

Equations (39) and (55) require

\[
\int_m^\infty \lambda^3 F_{\nu \kappa}(t, E) dE = \frac{15}{8\pi} \text{diag} \left( \pi_1^1, \pi_2^2, -\pi_1^1 - \pi_2^2 \right) \quad (62)
\]

Thus $F_{ij}$ has at most two independent components. Then (48), (49), (54), (61) and (62) are the restrictions on the harmonics for an Einstein–BGK solution. Once $F$ and $F^{\nu}_{\nu}$ (no sum) are specified, the remaining field equations in principle yield a solution $\{X(t), Y(t), Z(t)\}$ (this is analogous to specifying equations of state for $p$ and $\pi_{ij}$ in fluid models).
By (53) the bulk viscous pressure $\Pi$ is given by
\[ \Pi = \frac{4\pi}{3} e^{-\Gamma(t)} \int_{m}^{\infty} \lambda^3 H(t, E) dE \]
(63)

It follows from (59) and (60) that $\Pi = 0$ for $m \geq 0$. Again the bulk viscosity vanishes when $m > 0$ because $f$ is relaxing toward free–streaming.

The restrictions (60), (61) and (62) on the multipoles may for example be satisfied by the choice:
\[ F = \bar{F}, \quad F_{\nu} = 0 = \bar{F}_{\nu}, \quad H_{\nu} = \text{diag} (V_1, V_2, -V_1 - V_2) \]
(64)

where
\[ \pi_{\nu} = \frac{8\pi}{15} e^{-\Gamma(t)} \int_{m}^{\infty} \lambda^3 V_{\nu}(t, E) dE \quad \text{(no sum)} \]

This choice implies that the equilibrium distribution $\bar{f}$ has perfect fluid behaviour, i.e.
\[ \bar{k}_{\nu} = 0, \quad \bar{q}_{\nu} = 0, \quad \bar{\pi}_{\nu} = 0 \]
and by (51), there is no particle flux, i.e. $k_{\nu} = 0$.

The BGK multipole equations (48) and (49) reduce to
\[ 2e^{-\Gamma} \frac{\partial}{\partial E} (\lambda^3 \sigma_{\nu\kappa} H_{\nu\kappa}) + 5\lambda^3 \Theta \frac{\partial \bar{F}}{\partial E} - 15\lambda E \frac{\partial \bar{F}}{\partial t} = 0 \]
(65)
\[ \frac{\partial}{\partial E} (\lambda^4 \sigma^{\kappa\rho} F_{\nu\kappa\rho}) = 0 \]
(66)

The higher order BGK multipole equations become conditions on the fourth and higher multipoles which may always be satisfied, since these multipoles are otherwise unrestricted. The condition (66) gives
\[ \sigma^{\kappa\rho} F_{\nu\kappa\rho} = 0 \]
while (65) is a constraint on $\bar{F}$ and $V_{\nu}$. Once these are specified subject to (65), the remaining field equations determine $g_{\nu\nu}(t)$ in principle, thus completing the Einstein–BGK solution. Using the fact that
\[ \sigma_{\nu\kappa} = \text{diag} \left( X, -\frac{1}{3} \Theta, Y, -\frac{1}{3} \Theta, Z, -\frac{1}{3} \Theta \right) \]
(65) leads to the new result
\[ 2e^{-\Gamma} \left( \frac{\dot{X}}{X} - \frac{\dot{Z}}{Z} \right) \frac{\partial}{\partial E} (\lambda^3 V_1) + 2e^{-\Gamma} \left( \frac{\dot{Y}}{Y} - \frac{\dot{Z}}{Z} \right) \frac{\partial}{\partial E} (\lambda^3 V_2) \]
\[ + 5 \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) \lambda^3 \frac{\partial \bar{F}}{\partial E} - 15\lambda E \frac{\partial \bar{F}}{\partial t} = 0 \]
(67)

It is clearly possible to find $\bar{F}, V_{\nu}$ that satisfy the single linear equation (67).
4. Anisotropy generation in FRW cosmologies

Using the FRW and Bianchi I results we now construct a model in which an Einstein–Boltzmann solution in FRW spacetime evolves into an Einstein–Boltzmann solution in Bianchi I spacetime. It is assumed that the universe initially has FRW geometry with a matter distribution that is described by an anisotropic distribution function which is also compatible with FRW geometry. Ellis and Matravers \[14], \[15], \[3] propose two mechanisms whereby the anisotropy of the matter distribution is communicated to the universe. The first applies to a universe that is initially FRW with an equilibrium particle distribution that is inhomogeneous and anisotropic. The anisotropy and inhomogeneity is not communicated to the spacetime geometry. The particle distribution is effectively collision–free, because the particles are assumed to enjoy asymptotic freedom under the high temperature conditions of the early universe. As the universe expands, the temperature drops and the collisions become significant, allowing the anisotropy to be communicated to the geometry. In the second mechanism, while the temperature is very high the particles effectively behave as if their rest mass is zero. As the universe expands and cools, and the threshold energies of different particles are reached, their rest mass becomes significant. Again, this effective change in the zero rest mass allows the inhomogeneity and anisotropy to be communicated. We show that the generation of anisotropy can be modelled starting from an anisotropic and non–equilibrium BGK solution. A non–equilibrium model allows us to overcome some of the drawbacks of the equilibrium model – for example, the difficulty in motivating a collision–free early–universe phase.

We assume that the universe initially \((t \leq t_0)\) has FRW geometry and that the particle distribution is given by \[9\]. This distribution function is dynamically anisotropic and has Bianchi I symmetry, i.e \(f = f(t, p_\nu)\). For the high temperature conditions in the early universe, the particle rest mass is insignificant and an Einstein–BGK solution may be chosen that is consistent with the Bianchi I solution \[14\]. Furthermore, the local equilibrium \(\bar{f}\) to which \(f\) is close is a collision–dominated Planckian distribution, in keeping with standard physics of the early universe (and not requiring asymptotic freedom or other exotic processes).

By \[18\], \[24\], \[29\] we can take

\[
H = 0, \quad H_\nu = 0, \quad \int_0^\infty w^3 H_{\nu\kappa}(w)dw = 0
\]  

(68)

where \(w = R(t)E\). Thus under high temperature conditions for which \(m = 0\), it is possible to find distribution functions with non–zero quadrupole, but for which the condition \(\pi_{\nu\kappa} = 0\) is satisfied. We need non–zero \(H_{\nu\kappa}\) in order to ‘switch on’ \(\pi_{\nu\kappa}\) (the mechanism for this will be described below). Then by \[38\] the shear anisotropy \(\sigma_{ij}\) will emerge and the geometry will evolve to a Bianchi I phase. By \[64\] and \[68\], we could
take $H_{\nu\kappa}$ of the form

$$H^\nu{}_{\kappa}(w) = \text{diag} [V_1(w), V_2(w), -V_1(w) - V_2(w)]$$

(69)

where $V_\nu$ are non-zero but satisfy

$$\int_0^\infty w^3 V_\nu(w) dw = 0$$

(70)

Thus for $t \leq t_0$ the universe is considered FRW, while for $t \geq t_0$ it is Bianchi I. On the hypersurface $t = t_0$, the geometry and the distribution function symmetry must satisfy both the conditions for FRW and Bianchi I spacetimes. This is the case if the following matching conditions are satisfied [3]:

$$\dot{X}(t_0) = \dot{Y}(t_0) = \dot{Z}(t_0)$$

(71)

As discussed in [3] and [15], these matching conditions always have a solution. Then this solution becomes the initial conditions for the Bianchi I solution, which is governed by existence and uniqueness theorems for the Einstein–Boltzmann equations (see [15]). The transition from effectively massless to effectively massive behaviour will take place over a cosmic time $\delta t$ which is much less than the expansion time. Evolution away from FRW is triggered as soon as the mass becomes dynamically significant, i.e. after time $\delta t$. In practice we are treating the transition as instantaneous, in a similar way to standard models of electron–positron annihilation. Our simple model illustrates that in principle, finely tuned anisotropy in the matter distribution can be unlocked dynamically to generate anisotropy in the spacetime geometry.

Note that the evolution away from FRW occurs entirely within the Bianchi I phase, as the shear grows from zero. At the transition, $\sigma_{ij}(t_0) = 0$, and $\sigma_{ij}$ remains zero unless $\pi_{ij}$ becomes non-zero to force the universe to evolve to the Bianchi I geometry. In the Bianchi I phase $t \geq t_0$, the solution is given by (47) and it clearly reduces to the FRW solution at $t = t_0$ as a result of the matching conditions (71). Because $w$ becomes isotropic and reduces to the FRW form with the application of the matching conditions, the Bianchi I equilibrium solution $\bar{f}$ reduces to the FRW equilibrium solution at $t_0$.

In summary, the mathematical model is the following. The collision model is AW:

$$\Gamma(t) = \int_0^t \frac{dt'}{\tau(t')} \text{ for all } t \geq 0$$

(72)

The particle energy $E = -u_i p^i = p^0$ is

$$E = \begin{cases} R^{-1}(t) \left[(p_1)^2 + (p_2)^2 + (p_3)^2\right]^{1/2} & (m = 0) \\ \left[m^2 + X^{-2}(t)(p_1)^2 + Y^{-2}(t)(p_2)^2 + Z^{-2}(t)(p_3)^2\right]^{1/2} & t \geq t_0 \end{cases}$$

(73)

with $w^2 = (p_1)^2 + (p_2)^2 + (p_3)^2$ for all $t \geq 0$. (Note that $w = R(t)E$ for $t \leq t_0$.)
The particle distribution function is given by

\[ f(t, p_\nu) = \bar{f}(w) + h(p_\nu) e^{-\Gamma(t)} \]

where

\[ \bar{f} = \begin{cases} \bar{F}(R(t)E) & t \leq t_0 \\ \bar{F}(t, E) + \bar{F}_\nu(t, E)e^\nu + \cdots & t \geq t_0 \end{cases} \] (74)

with \( \bar{F}_\nu(t_0, E) = 0, \bar{F}(t_0, E) = \bar{F}(R(t_0)E) \), and

\[ h(p_\nu) = H(t, E) + H_\nu(t, E)e^\nu + \cdots \]

with \( H_\nu(t, E) = H_\nu(R(t)E) \) for \( t \leq t_0 \). Note that in the FRW phase \( t \leq t_0 \) each \( \bar{F}_\nu \) and \( H_\nu \) is a Liouville solution (hence functions of \( w \) only), because the multipoles decouple in \( L[f] = 0 \) due to \( \sigma_{ij} = 0 \). This is no longer true in the Bianchi I phase \( t \geq t_0 \) and hence \( \bar{F}_\nu \) and \( H_\nu \) are functions of \( t \) and \( E \).

The distribution is specified to yield an Einstein–BGK solution for \( t \geq 0 \):

\[ H(t, E) = 0 = H_\nu(t, E) = 0 \quad \bar{F}_\nu(t, E) = 0 = \bar{F}_{\nu\kappa}(t, E) \]

The resultant solution is

\[ f(t, p_\nu) = \bar{F}(t, E) + e^{-\Gamma(t)} H_{\kappa\rho}(t, E)e^\kappa e^\rho + \cdots \] (75)

with \( \bar{F} = \bar{F}(R(t)E) \), \( H_{\nu\kappa} = H_{\nu\kappa}(R(t)E) \) for \( t \leq t_0 \), and \( H_{\nu\kappa} \) obeys

\[ H^\nu_{\ \kappa}(t, E) = \text{diag} \left[ V_1(t, E), V_2(t, E), -V_1(t, E) - V_2(t, E) \right] \]

where for \( t \leq t_0 \):

\[ V_\nu(t, E) = V_\nu(R(t)E) \] (76)

\[ \int_0^\infty w^2 V_\nu(w) dw = 0 \]

and for \( t \geq t_0 \):

\[ \pi^\nu_{\ \nu}(t) = \frac{8\pi}{15} e^{-\Gamma(t)} \int_m^\infty (E^2 - m^2)^{3/2} V_\nu(t, E) dE \quad \text{(no sum)} \] (77)

Finally, \( \bar{F} \) and \( V_\nu \) are subject for \( t \geq t_0 \) to the constraint (67):

\[ 2e^{-\Gamma} \left( \frac{\dot{X}}{X} - \frac{\dot{Z}}{Z} \right) \frac{\partial}{\partial E} \left( \lambda^3 V_1 \right) + 2e^{-\Gamma} \left( \frac{\dot{Y}}{Y} - \frac{\dot{Z}}{Z} \right) \frac{\partial}{\partial E} \left( \lambda^3 V_2 \right) \]

\[ + 5 \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) \lambda^3 \frac{\partial \bar{F}}{\partial E} - 15 \lambda E \frac{\partial \bar{F}}{\partial t} = 0 \] (78)
It is clearly possible to find $\bar{F}$, $V_\nu$ such that (76) – (78) are satisfied (and therefore all field and conservation equations will be satisfied). The solution has zero number flux, bulk viscosity and energy flux for all $t \geq 0$.

The early universe temperature is high enough that the particles have $m = 0$, and the matter distribution is described by (75), consistent with an FRW spacetime geometry. The conditions (67), (76) and (77) can be satisfied by appropriate $H_{\nu\kappa} \neq 0$ and thus the field and conservation conditions are satisfied for both the FRW and Bianchi I phases. As the universe expands and cools below the threshold energy (at $t = t_0$) for the particles under consideration, the distribution is no longer effectively of zero rest mass particles and the condition (76) is no longer satisfied. Condition (28) is required to ensure vanishing anisotropic stress in the FRW phase (i.e. the field equation $\pi_{\nu\kappa} = 0$ is satisfied). For the AW collision model (76) is satisfied by $H_{\nu\kappa} \neq 0$ only if $m = 0$, by (28). Thus, as soon as the particle rest mass is no longer effectively zero, the anisotropic stress becomes non-zero ($\pi_{\nu\kappa} \neq 0$) and is given by (77). As a result shear anisotropy emerges, forcing the universe to evolve away from the FRW to the Bianchi I phase.

During the phase $t \leq t_0$ the collision rate is high and therefore $e^{-\Gamma}$ may become small. This forces the non-equilibrium, anisotropic distribution function $f(t, p_\nu)$ to approach the isotropic, collision-dominated equilibrium distribution $\bar{f}(w)$. However, as long as $e^{-\Gamma} > 0$ (even if it is very small) the model presented here works. The model therefore represents a high temperature situation where the matter distribution is nearly isotropic (forced by the high collision rate). The distribution function has Bianchi I symmetry but satisfies all the conditions for an FRW universe. As the universe cools the particle threshold energy is reached and the FRW condition is no longer satisfied. The remnant of the initial anisotropy therefore acts as the seed for the change to Bianchi I geometry when the particle mass becomes significant, communicating the anisotropy of the distribution function to the spacetime geometry. Hence, this model suggests a mechanism by which anisotropy present in the radiation era could act as the seed for anisotropy generation in the spacetime geometry. The small anisotropy in the microwave background radiation could be a physical example of the remnant anisotropy considered here (see [16] for a related discussion).

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