REDUCTION OF THE SEMISTABILITY CONDITION FOR TENSORS

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Abstract. In this article, we study some feature of the semistability condition for tensors over smooth projective varieties. Moreover, we describe some related combinatorial problems, which arise from the description of a tensor \((E, \varphi)\) or, more precisely, a filtration of \(E\) as a \(d\)-dimensional matrix. Eventually, we study semistable tensors on the projective line.

1. Introduction

In this article we investigate the semistability condition for tensors. In particular, we manage to simplify the notion of semistability for these objects by reducing the number of inequalities involved in the condition.

A tensor consists, roughly speaking, of a (coherent) sheaf \(E\) over a variety \(X\), “decorated” with a morphism \(\varphi\) from \((E^\otimes a)^\oplus b\) to \((\text{det } E)^\otimes c \otimes D\) where \(D\) is a fixed torsion free sheaf over \(X\) (see Definition 1). A slightly different notion of sheaf decorated with a morphism was introduced by Schmitt, (see, for example \([5, 6, 7]\)) while the more general notion of tensor was introduced by Gomez and Sols in \([3]\). In both cases, such objects share the same semistability condition and gain their importance because they include many types of sheaves such as principal bundles, framed bundles, Higgs bundles, orthogonal and symplectic sheaves, and many others.

Recently, using this formalism, Gomez, Langer, Schmitt and Sols construct the moduli spaces of semistable principal bundles over smooth projective varieties over algebraically closed fields of positive characteristic \([4]\). The semistability notion is very important for sheaves and it plays a fundamental role in the construction of their moduli space. Unfortunately the semistability condition for tensors, as well as
the slope semistability condition, is quite complicated and has to be checked over all weighted filtrations, of any length, of the given sheaf $E$ (see Definition 5).

As main result of this paper we give a bound, which depends on the morphism $\varphi$, on the maximum length of the filtrations involved in the condition of semistability (Theorem 22). Moreover, in Section 3 we investigate some combinatorial problems rising from the behavior of the morphism $\varphi$ over a given filtration. Eventually, in Section 5 we study rank 3 semistable tensors on $\mathbb{P}^1$.

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2. Semistability conditions

Let $(X, \mathcal{O}_X(1))$ be an $n$ dimensional polarized, integral, separated scheme of finite type over an algebraically closed field $k$. A family $\{D_u\}_{u \in \mathcal{U}}$ of locally free sheaves over $X$ parametrized by a scheme $\mathcal{U}$ is a locally free sheaf $D$ on $X \times \mathcal{U}$, and for a given closed point $u \in \mathcal{U}$, we denote by $D_u$ the restriction to the slice $X \times u$.

From now on we fix a polynomial $P$ of degree $n$ and integer numbers $a, b, c, d$ and $r$ with $a, b, c, r \geq 0$.

Definition 1 (Definition 1.1). A tensor of type $(a, b, c, D, R)$ over $X$ is a triple $(E, \varphi, u)$ where $E$ is a coherent sheaf with Hilbert polynomial $P_E = P$, degree $\deg(E) = d$ and rank $\text{rk}(E) = r$, $R$ is a scheme, $D$ is a locally free sheaf over $X \times R$ and $\varphi$ is morphism

$$\varphi : (E^{\otimes a})^{\oplus b} \rightarrow (\text{det } E)^{\otimes c} \otimes D_u$$

not identically zero.

Sometimes we will simply call these objects tensors instead of tensors of type $\mathfrak{t} = (a, b, c, D, R)$ if the input data are clear by the context.

Remark 2. The notion of tensor generalizes the notion of decorated sheaves introduced by Schmitt ([7]) and studied by the authors in [1]. We recall the definition:

Definition 3. A decorated sheaf of type $\mathfrak{t} = (a, b, c, N)$ over $X$ is the datum of a torsion free sheaf $E$ over $X$ and a surjective morphism

$$\varphi : E_{a, b, c} \rightarrow (E^{\otimes a})^{\oplus b} \otimes (\text{det } E)^{\otimes c} \otimes N$$

where $N$ is a line bundle over $X$.

The morphism $\varphi : E_{a, b, c} \rightarrow N$ induces a morphism $E_{a, b} \rightarrow (\text{det } E)^{\otimes c} \otimes N$. By an abuse of notation, we still refer to the latter as $\varphi$. In this context is easy to see that a decorated sheaf of type $(a, b, c, N)$ corresponds (uniquely up to isomorphism) to a tensor of type $(a, b, c, D)$, where we have chosen $R = \{ \text{pt} \}$ and $D$ the pullback of $N$ over $X \times R$.

Remark 4. The category of tensors with fixed determinant $\text{det } E \simeq L$ of type $(a, b, c, D, R)$ is equivalent to the category of tensors with fixed determinant of type $(a, b, 0, \pi_X^* L^{\otimes c} \otimes D, R)$, where $\pi_X : X \times R \rightarrow X$ is the projection.
Since we are interested in studying the semistability condition of a given tensor, and not for families, from now on we will consider only tensors of type \((a, b, 0, D, R)\) with \(R = \{p_t\}\). Therefore, from now on, \(D\) will be regarded as a vector bundle over \(X\) and we will denote by \((E, \varphi)\) the triple \((E, \varphi, p_t)\) and by \((a, b, D)\) the quintuple \((a, b, 0, D, \{p_t\})\). The definitions of semistability and \(k\)-semistability that we are just about to introduce in short are the same both for tensors \((E, \varphi, u)\) of type \((a, b, c, D, R)\) both for tensors of type \((a, b, D)\). For these reasons and for simplicity’s sake we will give such definitions only for the latter.

Let \((E, \varphi)\) be a tensor of type \((a, b, D)\), consider the following filtration

\[
0 \subseteq E_{i_1} \subseteq \cdots \subseteq E_{i_s} \subseteq E = E
\]

of saturated subsheaves of \(E\), and let \(\underline{\alpha} = (\alpha_{i_1}, \ldots, \alpha_{i_s})\) be a vector of positive rational numbers. Finally let us denote by \(I = \{i_1, \ldots, i_s\}\) the set of indices appearing in the filtration and by \(|I|\) its cardinality. We will refer to the pair \((E^*, \underline{\alpha})_I\) as weighted filtration of \(E\) indexed by \(I\) or simply weighted filtration. A weighted filtration defines the following polynomial

\[
P_I(E^*, \underline{\alpha}) = \sum_{i \in I} \alpha_i \left( P_E \cdot \text{rk}(E_i) - \text{rk}(E) \cdot \text{rk}(E_i) \right),
\]

and the rational number

\[
L_I(E^*, \underline{\alpha}) = \sum_{i \in I} \alpha_i \left( \deg E \cdot \text{rk}(E_i) - \text{rk}(E) \cdot \deg E_i \right),
\]

where \(P_E\) denotes the Hilbert polynomial of \(E_i\) and \(d_i\) its degree. Finally we associate to \((E^*, \underline{\alpha})_I\) the following rational number also depending on \(\varphi\),

\[
\mu_I(E^*, \underline{\alpha}; \varphi) = - \min_{i_1, \ldots, i_s \in I} \left\{ \gamma_{i_1, i_1} + \cdots + \gamma_{i_s, i_s} \mid \varphi |_{E_{i_1} \otimes \cdots \otimes E_{i_s} \otimes b} \neq 0 \right\},
\]

where \(\mathcal{I} = I \cup \{r\}\)

\[
\gamma_i = (\gamma_{i, i}, \ldots, \gamma_{i, r}) \approx \sum_{i \in I} \alpha_i \frac{\text{rk}(E_i) - r, \cdots, \text{rk}(E_i) - r, \text{rk}(E_i), \ldots, \text{rk}(E_i)}}{\text{rk}(E_i)-\text{times}} \times \frac{\text{rk}(E_i)-\text{times}}{r-\text{rk}(E_i)}.
\]

The notion of semistability for a tensor depends on a stability parameter \(\delta\), which essentially measures how far a semistable tensor is from being semistable in the usual way. The parameter \(\delta\) is a rational polynomial \(\delta x^{n-1} + \delta_{n-2} x^{n-2} + \cdots + \delta_1 x + \delta_0\) with positive leading coefficient \(\delta \geq 0\).

**Definition 5 (Semistability).** Let \((E, \varphi)\) be a tensor of type \((a, b, D)\). Then \((E, \varphi)\) is \(\delta\)-(semi)stable if for any weighted filtration \((E^*, \underline{\alpha})_I\) the following inequality holds:

\[
P^\mu_I(E^*, \underline{\alpha}; \varphi) = P_I(E^*, \underline{\alpha}) + \delta \mu_I(E^*, \underline{\alpha}; \varphi) \geq 0.
\]

The tensor is slope \(\delta\)-(semi)stable if

\[
L^\mu_I(E^*, \underline{\alpha}; \varphi) = L_I(E^*, \underline{\alpha}) + \delta \mu_I(E^*, \underline{\alpha}; \varphi) \geq 0.
\]

Sometimes we will write \(P^\mu\) (resp. \(L^\mu\)) instead of \(P^\mu_I\) (resp. \(L^\mu_I\)) if the set of indices \(I\) is understood.
Notation. The notation \( > \) (resp. \( \succeq \)) means that \( > \) (resp. \( \succeq \)) has to be used in the definition of stable and \( \geq \) (resp. \( \preceq \)) in the definition of semistable.

Remark 6. As in the case of sheaves, we have the following chain of implications

\[
\text{δ-stability} \Rightarrow \text{δ-stability} \Rightarrow \text{δ-semistability} \Rightarrow \text{δ-semistability}
\]

Remark 7. (1) Let \((E^\bullet, \alpha)\) be a weighted filtration indexed by \(I\) and suppose that \(\mu_I = -(\gamma_{i_1, i_1} + \cdots + \gamma_{i_a, i_a})\), then there exists at least one permutation \(\sigma : \{i_1, \ldots, i_a\} \to \{i_1, \ldots, i_a\}\) such that \(\varphi|_{(E_{\sigma(i_1)} \otimes \cdots \otimes E_{\sigma(i_a)})^\oplus b} \neq 0\).

(2) From now on we will write

\[
\varphi|_{(E_{i_1} \otimes \cdots \otimes E_{i_a})^\oplus b} \neq 0
\]

if there exists at least one permutation \(\sigma : \{i_1, \ldots, i_a\} \to \{i_1, \ldots, i_a\}\) such that \(\varphi|_{(E_{\sigma(i_1)} \otimes \cdots \otimes E_{\sigma(i_a)})^\oplus b} \neq 0\).

Definition 8. Let \((E, \varphi)\) be a tensor of type \((a, b, D)\) and \((E^\bullet, \alpha)\) be a weighted filtration of \(E\) indexed by \(I\). For any \(i \in I\) let \((0 \subset E_i \subset E, \alpha_i)\) be the induced length 1 filtration. We will say that \((E^\bullet, \alpha)\) is non-critical if

\[
\mu_I(E^\bullet, \alpha; \varphi) = \sum_{i \in I} \mu_i(0 \subset E_i \subset E, \alpha_i; \varphi),
\]

and critical otherwise.

Now we will introduce another notion of semistability for tensors that will be useful in the future. This notion was already introduced and studied by the authors in the case of decorated bundles in [2].

Let \((E, \varphi)\) be as before and let \(F\) be a subsheaf of \(E\), then define

\[
k_{F,E} = k(F, E, \varphi) = \begin{cases} a & \text{if } \varphi|_{F_{a,b}} \neq 0 \\ k & \text{if } \varphi|_{F \otimes a_0 \otimes (a-k)} \neq 0 \text{ and } \varphi|_{F \otimes (a-k+1) \otimes (a-k-1)} = 0 \\ 0 & \text{otherwise.} \end{cases}
\]

Definition 9 (k-semistability). Let \((E, \varphi)\) be a tensor of type \((a, b, D)\) of positive rank; we will say that \((E, \varphi)\) is k-(semi)stable or slope k-(semi)stable if and only if for any proper subsheaf \(F\) the following inequalities hold.

\[
k-(semi)stable \quad \text{rk}(E)(P_E - \delta_{k,E}) \preceq \text{rk}(F)(P_F - a\delta)
\]

\[
slope k-(semi)stable \quad \text{rk}(E)(\deg(F) - \delta_{k,E}) \preceq \text{rk}(F)(\deg(E) - a\delta)
\]

If \(E\) is torsion free and \(F\) is a proper subsheaf let us define \(\mu^k(F) = \mu(F) - \frac{\delta_{k,E}}{\text{rk}(F)}\) and \(p_F^k = p_F - \frac{\delta_{k,E}}{\text{rk}(F)}\), where \(\mu(F) = \frac{\deg(F)}{\text{rk}(F)}\) and \(p_F = \frac{p_F}{\text{rk}(F)}\), then the above conditions become

\[
p_F^k \preceq p_E^k
\]

\[
\mu^k(F) \preceq \mu^k(E),
\]

respectively.
Remark 10. Let \((E, \varphi)\) and \(F\) be as before. A straightforward calculation shows that
\[
\mu(0 \subset F \subset E, 1; \varphi) = \text{rk}(E) k_{F,E} - a \text{rk}(F).
\]
Therefore the \(k\)-semistability condition coincides with the semistability condition for filtrations of length one. This clearly implies that
\[
(\text{semi})\text{stability} \Rightarrow k-(\text{semi})\text{stability}
\]
and
\[
slope (\text{semi})\text{stability} \Rightarrow \text{slope } k-(\text{semi})\text{stability}
\]
Therefore a tensor \((E, \varphi)\) is (semi)stable (resp. slope (semi)stable) if and only if it is \(k-(\text{semi})\text{stable} \) (resp. slope \(k\)-semistable) and condition \((7)\) (resp. \((8)\)) holds for any critical weighted filtration.

2.1. The associated matrix. Let \((E, \varphi)\) be a tensor of type \(\underline{a} = (a, b, D)\), and fix a weighted filtration \((E^*, \underline{\alpha})\) \(I = (0 \subset E_{i_1} \subset \cdots \subset E_{i_s} \subset E_r = E; \underline{\alpha} = (\alpha_i)_{i \in I})\) indexed by \(I = \{i_1, \ldots, i_s\}\).

Let \(M_1(E^*, \varphi)\) be the associated \(a\)-dimensional matrix, that is, the matrix defined by the following equation,
\[
m_{i_1 \cdots i_a} = \begin{cases} 1 & \text{if } \varphi_{(i_1a \cdots aE_{i_a})} \neq 0, \\ 0 & \text{otherwise,} \end{cases}
\]
where \(i_1, \ldots, i_a \in I\). Note that \(M_1(E^*, \varphi)\) is symmetric, that is, \(m_{i_1 \cdots i_a} = m_{\sigma(i_1) \cdots \sigma(i_a)}\) for any permutation \(\sigma\) in the symmetric group \(S_a\).

Definition 11. Let \(A^{\text{ord}} = \{(i_1, \ldots, i_a) \in I^a \mid i_1 \leq \cdots \leq i_a\}\) be the set of ordered \(a\)-tuples. We define a partial ordering \(\prec\) over \(A^{\text{ord}}\) in the following way. Let \(\underline{i} = (i_1, \ldots, i_a)\) and \(\underline{j} = (j_1, \ldots, j_a)\) be two elements of \(A^{\text{ord}}\), then
\[
(12) \quad \underline{i} \prec \underline{j} \iff i_s \geq j_s \text{ for any } s \in \{1, \ldots, a\}
\]
We will say that two elements \(\underline{i}, \underline{j} \in A^{\text{ord}}\) are comparable, and we will denote it by \(\underline{i} \sim \underline{j}\), and if only if \(\underline{i} \prec \underline{j}\) or \(\underline{j} \prec \underline{i}\). We will say that they are incomparable (and we will denote by \(\underline{i} \not\sim \underline{j}\)) otherwise.

Note that, if \(m_{i_1 \cdots i_a} = 1\) for a certain \(a\)-tupla \((i_1, \ldots, i_a)\), then it is easy to see that \(m_{j_1 \cdots j_a} = 1\) for any \((j_1, \ldots, j_a) \not\prec (i_1, \ldots, i_a)\). Conversely, if \(m_{i_1 \cdots i_a} = 0\), then \(m_{j_1 \cdots j_a} = 0\) for any \((j_1, \ldots, j_a) \not\succ (i_1, \ldots, i_a)\). Let \(A^{\text{ord}}_1 = \{(i_1, \ldots, i_a) \in A^{\text{ord}} \mid m_{i_1 \cdots i_a} = 1\}\), then the set \(P\) of \(\prec\)-maximal elements of \(A^{\text{ord}}_1\) uniquely determine \(M_1(E^*, \varphi)\). We will call these elements pivots and we will denote them by \(p\).

From now on we will identify \(M_1(E^*, \varphi)\) with the set of its pivots \(P = \{p_1, \ldots, p_p\}\), where \(p_i = (p_{i1}, \ldots, p_{ia}) \in I^a\).

Let \((E^*, \underline{\alpha})\) be a weighted filtration as before and let \(M_1(E^*, \varphi) = \{p_1, \ldots, p_p\}\), be the associated matrix. Define
\[
(13) \quad C_i = \frac{1}{r_i}P_{E_i} - rP_{E_i^*} - ar_i,
\]
\[
c_i = \deg E - r \deg E_i - ar_i
\]
and
\[
(14) \quad R_{1, \underline{\alpha}} = R_i(E^*, \underline{\alpha}; \varphi) = \max_{p_i \in M_1(E^*, \varphi)} \{R_{1, \underline{\alpha}}(p_i)\}
\]
where, for an element \( i = (i_1, \ldots, i_a) \in A_{\text{ord}}, \)
\[
R_{\ell, \alpha}(i) = \sum_{j=1}^{a} \left( \sum_{s \geq i_j, s \in I} \alpha_s \right).
\]

Note that
\[
\max_{p_i \in M_{\ell}(E^\bullet; \varphi)} \left\{ R_{\ell, \alpha}(p_i) \right\} = \max_{\ell \in A_{\text{ord}}} \left\{ R_{\ell, \alpha}(i) \mid \varphi_{(E_{i_1} \otimes \cdots \otimes E_{i_a}) \otimes b} \neq 0 \right\},
\]
indeed, if \( i \prec j \) then \( R_{\ell, \alpha}(i) \leq R_{\ell, \alpha}(j) \) and the pivots are exactly the \( \prec \)-maximal elements of \( A_{\text{ord}} \).

Using this formalism the (semi)stability condition (7) is equivalent to the following,
\[
\sum_{i \in I} \alpha_i C_i + r \delta R_{\ell, \alpha} \succ 0,
\]
while the slope (semi)stability condition (8) is equivalent to the following,
\[
\sum_{i \in I} \alpha_i e_i + r \delta R_{\ell, \alpha} \geq 0.
\]

Indeed, suppose that the minimum of \( \mu_1(E^\bullet; \alpha; \varphi) \) is attained in \((i_1, \ldots, i_a)\). Then \((i_1, \ldots, i_a)\) must coincide with a pivot \( p_j = (p_{j_1}, \ldots, p_{j_a}) \) of \( M_{\ell}(E^\bullet; \varphi) \) and
\[
\mu_1(E^\bullet; \alpha; \varphi) = - (\gamma_{1, i_1} + \cdots + \gamma_{1, i_a})
\]
\[
= - \left( \sum_{i \in I} \alpha_i r_i - \sum_{l \geq i_1} \alpha_l r_l + \cdots + \sum_{l \geq i_a} \alpha_l r_l - \sum_{l \geq i_a} \alpha_l r_l \right)
\]
\[
= - a \sum_{l \in I} \alpha_l r_l + r \left( \sum_{l \geq i_1} \alpha_l + \cdots + \sum_{l \geq i_a} \alpha_l \right)
\]
\[
= - a \sum_{l \in I} \alpha_l r_l + r \left( \sum_{j=1}^{a} \left( \sum_{l \geq p_{j_l}, l \in I} \alpha_l \right) \right)
\]
\[
= - a \sum_{l \in I} \alpha_l r_l + r R_{\ell, \alpha}(p_j).
\]

3. Combinatorial considerations

In this section we will treat some combinatorial problems related to the study of the semistability condition of tensors. Fix \( I = \{1, \ldots, s\} \) (since we are interested in combinatorial problems we can assume, without loss of generality, that \( I \) consists of the first \( s \) natural numbers), let us denote by \( \overline{I} = I \cup \{s+1\} \) and \( t = |\overline{I}| \). The cardinality of \( A_{\text{ord}} \) is well known, but we present a proof for the sake of completeness and introducing some notations.

**Proposition 12.** The cardinality of the set of ordered \( a \)-tuple \( A_{\text{ord}} \) is \( \nu_{a,t} = \binom{t+a-1}{a} \).

**Proof.** Consider the case in which \( a = 2 \), we want to count the number of all possible pairs \((i, j) \in \overline{I} \) with \( i \leq j \). If \( i = 1 \) we can choose \( j \) in \( t \) ways, if \( i = 2 \) can choose \( j \) in \( t-1 \) ways, and so on. Therefore, the set of all possible pairs has
cardinality $\nu_{2,t} = \binom{t+1}{2}$. Let $a = 3$, then all possible triples $(i, j, k)$ can be obtained in the following way. If $i = 1$ then we can choose $(j, k)$ in $\nu_{2,1}$ ways, if $i = 2$ than we can choose $(j, k)$ in $\nu_{2,1-1}$ ways and so on. In general,

$$
\begin{align*}
\nu_{a,t} &= \sum_{l=1}^{t} \nu_{a-1,l} \\
\nu_{2,t} &= \binom{t+1}{2}
\end{align*}
$$

We will prove by induction on $a$ that $\nu_{a,t} = \binom{t+a-1}{a}$. If $a = 2$ the above calculations shows that the first step is true. Suppose that the proposition holds true for $a - 1$. Then we have

$$
\nu_{a,t} = \sum_{l=1}^{t} \nu_{a-1,l} = \sum_{l=1}^{t} \left( \frac{l + a - 2}{a - 1} \right)
$$

$$
= \frac{1}{(a - 1)!} \sum_{l=1}^{t} (l(l+1) \ldots (l+a-2))
$$

$$
= \frac{1}{(a - 1)!} \frac{t(t+1) \ldots (t+a-1)}{a}
$$

$$
= \binom{t+a-1}{a}.
$$

Proposition 13. Let $I$, $s$ and $t$ as above. Fix $\mathbf{i} = (i_1, \ldots, i_a) \in A^{\text{ord}}$ then the cardinality of the set $A = \{ \mathbf{j} \in A^{\text{ord}} \mid \mathbf{j} \precsim \mathbf{i} \}$ is $\widetilde{m}_{a,t} (\mathbf{i})$, where

$$
\begin{align*}
\widetilde{m}_{a,t} (\mathbf{i}) &= \sum_{l=1}^{i_a-1} \widetilde{m}_{a-1,t-1} (i_2-l, \ldots, i_a-l) \\
\widetilde{m}_{2,t} (i_1, i_2) &= \frac{1}{2}(t+i_2-2i_1+2)(t-i_2+1)
\end{align*}
$$

where,

$$
\widetilde{m}_{a,t} (i_1-l, \ldots, i_a-l) = \widetilde{m}_{a,t} (\max\{i_1-l, 1\}, \ldots, \max\{i_a-l, 1\}).
$$

Note that, in particular $\widetilde{m}_{a,t} (1) = \nu_{a,t}$.

Proof. We will prove the theorem by induction on $a$. If $a = 2$, fix a pair $(i_1, i_2) \in \mathbb{T}^{\times 2}$ then we want to count the number of 2-tuples which are $\preceq$ to the given one. This number coincides with the number of $\bullet$’s plus one in the following matrix representation

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \bullet \\
\cdot & \cdot & 1 & \bullet \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Therefore,

$$
\widetilde{m}_{2,t} (i_1, i_2) = (t-i_1+1)+(t-i_1)+\cdots+(t-i_1+1-(t-i_2)) = \frac{1}{2}(t+i_2-2i_1+2)(t-i_2+1)
$$

Assume the result is true for all $a' \leq a - 1$. Let $\mathbf{i} = (i_1, \ldots, i_a) \in \mathbb{T}^{\times a}$ be an $a$-tupla, then the number of $a$-tuples $\mathbf{j}$ which are $\preceq$ to $\mathbf{i}$ are exactly the number of $(a-1)$-tuples which are $\preceq$ to $(i_2-i_1+1, \ldots, i_a-i_1+1)$ with respect to the smaller set of indices $\mathbb{T} \setminus \{1, \ldots, i_1\}$ in the layer $i_1$ plus the ones in the layer $i_1 + 1$ which
are \( \preceq \) of \((i_2 - i_1, \ldots, i_a - i_1)\) with respect to the set of indices \(\mathbf{1} \setminus \{1, \ldots, i_1 + 1\}\) and so on, the sum stops when \(i_a - l = 1\). The following figure shows the case \(a = 3\)

\[
\begin{array}{ccc}
\text{i_1-layer} & \cdots & (i_1+1)-layer \\
\cdot & \cdot & \cdot \\
1 & \cdot & 1 \\
(i_1+2)-layer \\
\cdot & \cdot & \cdot \\
1 & \cdot & 1
\end{array}
\]

Therefore,

\[
\tilde{m}_{a,t}(\mathbf{i}) = \tilde{m}_{a-i_a+1,t-i_1+1}(i_2 - i_1 + 1, \ldots, i_a - i_1 + 1) + \ldots + \tilde{m}_{a-i_a+2,t-i_a+2}(1, \ldots, 1, 2) + \nu_{a-i_a+1,t-i_1+1}.
\]

\[\text{Proposition 14.} \quad \text{Let } \mathbf{i}, s \text{ and } t \text{ be as above. Fix } \mathbf{i} = (i_1, \ldots, i_a) \in A^{ord}, \text{ then the cardinality of the set } A = \{ \mathbf{j} \in A^{ord} \mid \mathbf{i} \preceq \mathbf{j}\} \text{ is } \tilde{m}_{a,t}(\mathbf{i}), \text{ where}
\]

\[
\begin{align*}
\hat{m}_{a,t}(\mathbf{i}) &= \sum_{l=0}^{i_a-1} \hat{m}_{a-1,t-1}(i_2 - l, \ldots, i_a - l) \\
\hat{m}_{2,t}(i_1,i_2) &= \frac{1}{2}i_1(2i_2 - i_1 + 1)
\end{align*}
\]

where,

\[
\tilde{m}_{a,t}(i_1 - l, \ldots, i_a - l) = \tilde{m}_{a,t}\{\max\{i_1 - l, 1\}, \ldots, \max\{i_a - l, 1\}\}.
\]

Note that, in particular \(\tilde{m}_{a,t}(\mathbf{1}) = 0\).

Proof. If \(a = 2\) then the number of 2-tuples which are \(\preceq\) to \(\mathbf{i} = (i_1, i_2)\) corresponds to the number of \(\bullet\)'s plus one in the following figure,

\[
\begin{array}{ccc}
\bullet & \cdots & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

A calculation similar to the one done in the proof of Proposition 13 shows that the number of 2-tuple which are \(\preceq\) to \(\mathbf{i}\) are

\[
\hat{m}_{2,t}(\mathbf{i}) = i_2 + (i_2 - 1) + \ldots + (i_2 - i_1 + 1)
\]

\[
= \frac{1}{2}i_1(2i_2 - i_1 + 1).
\]

Using arguments similar to the ones used in the proof of Proposition 13 and proceeding by induction on \(a\) one gets the general formula. \(\diamondsuit\)

\[\text{Proposition 15.} \quad \text{Let } \mathbf{i}, s \text{ and } t \text{ be as above. Fix } \mathbf{i} = (i_1, \ldots, i_a) \in A^{ord} \text{ then the cardinality of the set of } a\text{-tuples which are incomparable with } \mathbf{i}, \text{ that is, the set}
\]

\[
A_{\mathbf{i}} = \{ \mathbf{j} \in A^{ord} \mid \mathbf{i} \preceq \mathbf{j} \text{ and } \mathbf{j} \preceq \mathbf{i}\}, \text{ is}
\]

\[
\nu_{a,t}(\mathbf{i}) = \sum_{l=0}^{i_a-2} \left( \tilde{m}_{a-1,t-1}(i_2 - l, \ldots, i_a - l) - 1 + \nu_{a-1,t-1}(i_2 - l, \ldots, i_a - l) \right)
\]

\[
+ \nu_{a,t-i_1+1}(i_2 - i_1 + 1, \ldots, i_a - i_1 + 1)
\]

\[\checkmark\]
where
\[ \nu_{a,t}(1, i_2, \ldots, i_a) = \sum_{l=0}^{i_a-2} \nu_{a-1,t-l}(i_2 - l, \ldots, i_a - l) \]
\[ \nu_{2,t}(i_1, i_2) = \frac{1}{2} \left( 2t(i_1 - 1) + (i_1 - i_2)^2 + i_1 + i_2 - 2i_1i_2 \right). \]

As before, we define
\[ \nu_{a,t}(i_1 - l, \ldots, i_a - l) = \nu_{a,t}(\max\{i_1 - l, 1\}, \ldots, \max\{i_a - l, 1\}) \]

Proof. We will prove the theorem by induction on \( a \). If \( a = 2 \), fix \( \mathbf{i} = (i_1, i_2) \in \mathbb{T}^{2} \) then we want to count the number of 2-tuples which are incomparable (not \( \preceq \) nor \( \succeq \)) with the given one, this number coincides with the number of \( \bullet \)'s in the following matrix representation

Therefore, \( \nu_{2,t}(\mathbf{i}) \) coincides with the total number of 2-tuples \( \nu_{2,t} \) minus the number of 2-tuples which are \( \preceq \) to \( \mathbf{i} \) minus the number of 2-tuple which are \( \succeq \) than \( \mathbf{i} \) plus 1, since we removed twice the \( a \)-tuple \( \mathbf{i} \). Therefore,
\[ \nu_{2,t}(\mathbf{i}) = \nu_{2,t} - \tilde{m}_{2,t}(\mathbf{i}) - \tilde{m}_{2,t}(\mathbf{i}) + 1 \]

Assume the result is true for all \( a' \leq a - 1 \). Let \( \mathbf{i} = (i_1, \ldots, i_a) \in \mathbb{T}^{a} \) be an \( a \)-tuple, then the number of \( a \)-tuples \( \mathbf{j} \) which are incomparable with \( \mathbf{i} \) coincides with the number of \((a - 1)\)-tuples which are incomparable with the \((a - 1)\)-tuples \((i_2 - l, \ldots, i_a - l)\) in the layer \( i_1 + l \) with respect the reduced set of indices \( \mathbb{T} \setminus \{1, \ldots, i_a + l\} \), plus the number of \( a \)-tuples which are incomparable with \( \mathbf{i} \) in the layers \( 1, \ldots, i_1 - 1 \). The value of the latter set of \( a \)-tuples coincides with number of \((a - 1)\)-tuples in the layer \( 1 + l \) which are \( \preceq \) or incomparable with the \((a - 1)\)-tuples \((i_2 - l, \ldots, i_a - l)\) minus one (because we do not want to count, in any layer, the \((a - 1)\)-tuple \((i_2, \ldots, i_a)\)) with respect the set of indices \( \mathbb{T} \setminus \{1, \ldots, l\} \) for \( l = 0, \ldots, a_1 - 1 \). The followin figure show the case \( a = 3 \)
Therefore,
\[
\nu_{a,t}(\underline{x}) = \sum_{l=0}^{i_t-2} \left( \tilde{m}_{a-1,t-1} (i_2 - l, \ldots, i_a - l) - 1 + \nu_{a-1,t-1}(i_2 - l, \ldots, i_a - l) \right) + \sum_{l=0}^{i_t-2} \nu_{a-1,t-1}(i_2 - l, \ldots, i_a - l).
\]

\[\blacktriangleleft\]

**Remark 16.** Note that Expression (21) is clearly equivalent to the following,
\[
\nu_{a,t}(\underline{x}) = \nu_{a,t} - \tilde{m}_{a,t}(\underline{x}) - \tilde{m}_{a,t}(\underline{x}) + 1
\]
Moreover, a calculation shows that, in the case of \(a = 3\), the formula of \(\nu_{3,t}(1, i, j)\) is the following,
\[
\nu_{3,t}(1, i, j) = \frac{1}{2}(j - 1) \left[ t(2i - 1 - j) + (i - j)^2 + (j - 1)^2 + i - 2ij + 1 \right]
\]

Let \(n, k\) be integers, denote with \(p_k(n)\) the number of partitions of \(n\) into exactly \(k\) parts. Then
\[
\left\{\begin{array}{ll}
p_k(n) &= p_k(n - k) + p_{k-1}(n - 1) \\
p_0(0) &= 1 \\
p_k(n) &= 0 \text{ if } n \leq 0 \text{ or } k \leq 0.
\end{array}\right.
\]

Is a well-known fact that \(p_1(n) = 1\) and that \(p_2(n) = \lfloor \frac{n}{2} \rfloor \).

**Lemma 17.** Let \(I\) and \(A^{\text{ord}}\) be as above and let \(\underline{i} = (i_1, \ldots, i_a)\) and \(\underline{j} = (j_1, \ldots, j_a)\) be two distinct elements of \(A^{\text{ord}}\). If \(\sum_{t=1}^{a} i_t = \sum_{t=1}^{a} j_t\) then \(\underline{i}\) and \(\underline{j}\) are incomparable (see Definition 11).

**Proof.** By hypothesis \(\sum_{t=1}^{a} i_t = \sum_{t=1}^{a} j_t\) then \(\underline{i}\) and \(\underline{j}\) are incomparable.
\[\blacktriangleleft\]

**Theorem 18.** Let \(I\) and \(A^{\text{ord}}\) be as above and fix a subset \(B\) of \(A^{\text{ord}}\) and let \(B^4\) be the subset of \(\preceq\)-maximal elements of \(B\). Denote by \(\text{mp}(a, t) = \max_{B \subseteq A^{\text{ord}}} |B^4|\), where \(|\cdot|\) denotes the cardinality of a finite set.

Then \(\text{mp}(1, t) = 1\), \(\text{mp}(2, t) = p_2(t + 1)\), otherwise
\[
\text{mp}(a, t) = \max_{a \leq x \leq at} f_{a,t}(x),
\]
where
\[
f_{a,t}(x) = p_a(x) - \sum_{l=a-1}^{x-t-1} p_{a-1}(l).
\]

**Proof.** If \(a = 1\) there is nothing to prove. If \(a = 2\) the maximum is clearly attained by taking \(B\) as the set of anti-diagonal elements, that is \(B = \{(1, r), (2, r - 2), \ldots, (\frac{t}{2}, \frac{t}{2} + 1)\}\) if \(t\) is even and \(B = \{(1, r), (2, r - 2), \ldots, (\frac{t+1}{2}, \frac{t+1}{2})\}\) if \(t\) is odd.

In any case any element of \(B\) is \(\preceq\)-maximal, that is \(B = B^4\), and the cardinality of \(B\) is \(\lfloor \frac{t + 1}{2} \rfloor\) which is exactly \(p_2(t + 1)\).
Let $a \geq 3$. We want to construct a subset, $B$ of $A_{\text{ord}}$ such that $B = B^3$ and which realizes the maximum. The idea for constructing $B$ is to start from an $a$-tuple $i_1 \in A_{\text{ord}}$, then adding $i_2 \in A_{\text{ord}}$ such that $i_1 \not< i_2$, then $i_3 \in A_{\text{ord}}$ such that $i_2 \not< i_3$ and $i_3 \not< i_2$ and so on, until is not possible to add any other $a$-tuple which is incomparable with all the previous ones. In this way we get a set $B$ with the property that $B = B^3$, but clearly the cardinality of $B$ depends on the choices made.

Claim. Starting from an $a$-tuple $i = (i_1, \ldots, i_a)$, in order to construct a set $B$ which is the largest possible, the best choice for adding an $a$-tuple (which is incomparable with the previous ones) is by adding an $a$-tuple $j = (j_1, \ldots, j_a)$ such that $\sum_{i=1}^a i_i = \sum_{i=1}^a j_i$.

Indeed, thanks to Lemma 17 the added $a$-tuples are incomparable with each other. Therefore we have to prove that the set $B$ constructed in this way is bigger than all others set constructed starting from the same $a$-tuple $i = (i_1, \ldots, i_a)$. Without loss of generality we can suppose that $i_1 = 1$. We will prove the claim by induction on $a$. If $a = 1$ there is nothing to prove. Suppose we proved the claim for $a - 1$. Starting from $i$, because of the inductive hypothesis, the best way in order to fill the first layer is adding all $a$-tuples having the first coordinate equal to 1 and the same sum of $i$. An $a$-tuple $j = (j_1, \ldots, j_a)$ such that $\sum_{i=1}^a j_i > \sum_{i=1}^a i_i$ is $\prec$ to at least one of the $a$-tuples already added. Therefore, the only possibility is adding an $a$-tuple $j$ with $\sum_{i=1}^a j_i = \sum_{i=1}^a i_i$. If the inequality is strict, it is easy to see that we are missing at least an $a$-tuple between $j$ and one of the previous ones.

Fix an $a$-tuple $i$; then we construct a set $B_i \subset A_{\text{ord}}$ as the set containing all $j \in A_{\text{ord}}$ such that $\sum_{i=1}^a i_i = \sum_{i=1}^a j_i$. By Lemma 17 we have that $B_i = B_i^3$; moreover, by the previous claim $\max_{B \subset A_{\text{ord}}} |B^3| \leq \max_{\mathcal{S} \subset A_{\text{ord}}} |B_\mathcal{S}|$.

Claim. Let $i = (i_1, \ldots, i_a)$ be an $a$-tuple and let $x = \sum_{i=1}^a i_i$ then $|B_i| = f_{a,t}(x)$.

Since $B_i$ contains all $a$-tuples $j = (j_1, \ldots, j_a)$ such that $\sum_{i=1}^a j_i = x$ then the cardinality of $B_i$ coincides with the number $a$-partitions of $x$ minus the partitions involving numbers greater or equal than $t + 1$. The cardinality of the former set is exactly $p_a(x)$ while the cardinality of the latter can be calculated as follows. The numbers greater or equal to $t + 1$ appearing in the partitions are numbers between $t + 1$ and $x - (a - 1)$. An integer $y \in [t + 1, x - (a - 1)]$ appears in the partitions as many times as the possible $a - 1$-partitions of $x - y$. Therefore, we have to remove $\sum_{y=t+1}^{x-1} p_{a-1}(y) = \sum_{l=0}^{x-t-1} p_{a-1}(l)$ of them.

By the previous claims, we have that

$$\max_{B \subset A_{\text{ord}}} |B^3| = \max_{\mathcal{S} \subset A_{\text{ord}}} |B_\mathcal{S}| = \max_{\mathcal{S} \subset A_{\text{ord}}} \left\{ f_{a,t} \left( \sum_{i=1}^a i_i \right) \right\} = \max_{a \leq x \leq at} \{ f_{a,t}(x) \}.$$

♦
Corollary 19. Let \( I \), \( A^{\text{ord}} \) and \( f_{a,t}(x) \) be as in Theorem 18. Then

\[
(24) \quad \nu_{a,t} = \sum_{x=a}^{at} f_{a,t}(x).
\]

Proof. Let \( i \neq j \) be two elements of \( A^{\text{ord}} \), then \( B_i \cap B_j = \emptyset \), where \( B_w = \{ w \in A^{\text{ord}} \mid \sum_{i=1}^{a} w_i = \sum_{i=1}^{a} v_j \} \) is defined as in the proof of Theorem 18. Since, as proved in Theorem 18, \( f_{a,t}(\sum_{i=1}^{a} v_j) = |B_w| \) and \( a \leq \sum_{i=1}^{a} v_j \leq at \), then

\[
\sum_{x=a}^{at} f_{a,t}(x) = |A^{\text{ord}}| = \nu_{a,t}.
\]

\[
\checkmark
\]

Corollary 20. Let \((E, \varphi)\) be a tensor of type \((a, b, D)\) and \((E^*, \varrho)\) a filtration indexed by \( I = \{i_1, \ldots, i_s\} \) and let \( t = s+1 \) be the cardinality of \( I \). Let \( M_I(E^*; \varphi) = \{p_1, \ldots, p_p\} \) be the associated matrix. Then \( p \leq mp(a, t) \).

Proof. This follows directly from Theorem 18 and the definition of \( M_I(E^*; \varphi) \).

\[
\checkmark
\]

Proposition 21. If \( a = 3 \) then the maximum of \( f_{3,t}(x) \) is attained for \( x = \lceil \frac{3t}{2} \rceil \).

Proof. If \( a = 3 \) it is well known that \( p_3(x) = \lceil \frac{1}{12} x^2 \rceil \), where \( \lceil \cdot \rceil \) is the nearest-integer function. Replacing \( p_3(x) \) with \( \lceil \frac{1}{12} x^2 \rceil \) and \( p_2(x) \) with \( \lfloor \frac{x}{2} \rfloor \) in Equation (23) and taking the derivative \( f'_{3,t}(x) = \frac{\partial f_{3,t}(x)}{\partial x} \) we get that \( f'_{3,t}(x) = 0 \) if \( x = \lfloor \frac{3t}{2} \rfloor \).

\[
\checkmark
\]

Conjecture. If \( a \) is even then the maximum number of pivots is \( |B^*_i| \) for \( i = \lfloor \frac{t}{2} \rfloor \), that is, the maximum of \( f_{a,t}(x) \) is attained at \( x = (t+1)p_2(a) \).

4. Main results

Let \((E, \varphi)\) be a tensor of type \((a, b, D)\). Let \((E^*, \varrho)\) be a filtration indexed by \( I = \{i_1, \ldots, i_s\} \). We call \((E^*, \beta)\) and \((E^*, \zeta)\) extended subfiltrations. An extended subfiltration \((E^*, \beta)\) is a proper subfiltration if there exists an \( i \in I \) such that \( \beta_i = 0 \). In this case we consider the filtration obtained by the previous one after having deleted the subsheaf \( E_i \).

If the maximum of the filtration \((E^*, \varrho)\) is achieved at the pivot \( p_j \) we have a linear system of inequalities \( \{S_i\}_{i=1, \ldots, p-1} \) of the form \( R_{i, \varrho}(p_j) \geq R_{i, \varrho}(p_k) \) for \( k \neq j \). We will denote the linear system of equations associated to \( \{S_i\}_{i=1, \ldots, p-1} \) by \( \{\Sigma_i\}_{i=1, \ldots, p-1} \). We will write \( S_i(\varrho) \geq 0 \) or \( S_i(\varrho) = 0 \), if we want to stress which weight we are considering.

Theorem 22. Let \((E, \varphi)\), \((E^*, \varrho)\), \((E^*, \beta)\), \((E^*, \zeta)\), \( I \) and \( M_I(E^*; \varphi) \) be as above. If \( s \geq p + 1 \) then there exist \( \beta, \zeta \in \mathbb{R}_+^s \) such that the subfiltrations \((E^*, \beta)\) and \((E^*, \zeta)\) are proper and the following equality holds:

\[
(25) \quad \sum_{i \in I} \alpha_i c_i + r\delta R_{4, \varrho} = \sum_{i \in I} \beta_i c_i + r\delta R_{4, \beta} + \sum_{i \in I} \zeta_i c_i + r\delta R_{4, \zeta},
\]
Proof. Let us consider the linear system of inequalities \( \{ S_i \}_{i=1,\ldots,p-1} \) associated to the filtration \( (E^*,\varpi) \) and the associated linear system of equations \( \{ S_i \}_{i=1,\ldots,p-1} \). If \( s \geq p + 1 \) we can always find a non-trivial solution \( \zeta \) for \( \{ S_i \} \) and we define \( \beta_i := \alpha_i - \zeta_i \), for any \( i \in I \). If the vector \( \alpha \) satisfies a linear inequality \( S_l(\alpha) \geq 0 \) then we have

\[ S_l(\beta) = S_l(\alpha - \zeta) = S_l(\alpha) - S_l(\zeta) = S_l(\alpha) \geq 0, \]

since \( S_l \) is linear and \( \zeta \) is a solution for \( \{ S_i \} \). Because of this, the linear system associated to the filtration \( (E^*,\beta) \) is the same of the one associated to \( (E^*,\varpi) \), id est, the maximum for the filtration \( (E^*,\beta) \) is achieved in the same pivot as in the filtration \( (E^*,\varpi) \). Moreover, the relation \( \beta_i := \alpha_i - \zeta_i \) implies

\[ \sum_{i \in I} \alpha_i c_i = \sum_{i \in I} \beta_i c_i + \sum_{i \in I} \zeta_i c_i. \]

So we need to show that

\begin{equation}
R_{l,\varpi} = R_{l,\beta} + R_{l,\zeta}.
\end{equation}

after replacing, for all \( i \in I \), \( \beta_i \) with \( \alpha_i - \zeta_i \), the terms \( \alpha_i \) will delete each others, hence Equation \( 26 \) becomes an equation in the variable \( \zeta_i \) which is equivalent to \( S_{l_0} \) for a certain \( l_0 \in \{ 1,\ldots,p \} \).

Finally, in order to get a decomposition in two proper filtrations we need to add the conditions \( \zeta_i = 0 \), and \( \zeta_j = \alpha_j \) (that is, \( \beta_j = 0 \)) for some \( i \neq j \). At the end, we have a system with \( p + 1 \) equations in \( s \) variables which admits a non-trivial solution if \( s \geq p + 1 \).

\begin{corollary}
Let \( (E,\varphi) \), \( (E^*,\varpi) \) and \( M_1(E^*;\varphi) = \{ p_1,\ldots,p_p \} \) as before. If \( p = 1 \) then exist \( \beta \in \mathbb{R}^*_+ \) such that

\begin{equation}
L_t^t(E^*,\varpi;\varphi) = \sum_{i \in I} L_t^t(0 \subset E_i \subset E,\beta_i;\varphi)
\end{equation}

\end{corollary}

Proof. It is sufficient to apply Theorem 22 and note that, in this case, the matrix associated to any subfiltration of \( (E^*,\varpi) \) always has one pivot; therefore, it is possible to repeat the procedure of Theorem 22 until the length of the subfiltration is \( < 2 \).

\begin{remark}
Now we want to give an example to show that the previous bound for \( s \) is sharp, that is, if \( s = p \) then the statement of the theorem does not hold.

Let \( 0 \subset E_1 \subset E_2 \subset E_3 \subset E \) be a filtration of \( E \) and, as usual, we denote by \( r \) and \( d \) the rank and degree of \( E \) and by \( r_i \) and \( d_i \) the rank and degree of \( E_i \); moreover, for convenience sake we assume that \( r \) is a multiple of \( 3 \). Eventually, we will fix the following invariants \( \delta = 1, a = 4, c_1 = -\frac{2}{3} r, c_2 = -2 r, c_3 = -\frac{10}{3} r \). Moreover, assume that the pivots are \((1,1,r,r),\) \((2,2,2,r)\) and \((3,3,3,3)\). In particular we get that \( k_{E_1,\varpi} = 2, k_{E_2,\varpi} = 3 \) and \( k_{E_3,\varpi} = 4 \), so the \( k \)-semistability conditions, \( c_i + k_{E_i,\varpi} \geq 0 \), are satisfied for \( i = 1, 2, 3 \). If we choose weights \( \alpha_1 = 4, \alpha_2 = 2 \) and \( \alpha_3 = 6 \) we get \( R_{1,\varpi} = 24 \) and the semistability condition becomes

\[ (\alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3 + 24)r = \left( \frac{8}{3} - 4 - 20 + 24 \right) r < 0, \]

so that the filtration destabilizes.
Now we will calculate the semistability conditions for the length 2 subfiltration and we will show that they do not destabilize.

- Filtration indexed by \{1, 2\}
  - case \(2\beta_1 > \beta_2\)
    \[
    \left( -\frac{2}{3}\beta_1 - 2\beta_2 + (2\beta_1 + 2\beta_2) \right) r = \left( \frac{4}{3}\beta_1 \right) r \geq 0
    \]
  - case \(2\beta_1 < \beta_2\)
    \[
    \left( -\frac{2}{3}\beta_1 - 2\beta_2 + (3\beta_2) \right) r = \left( \beta_2 - \frac{2}{3}\beta_1 \right) r \geq 0
    \]

- Filtration indexed by \{2, 3\}
  - case \(\beta_3 > 3\beta_2\)
    \[
    \left( -2\beta_2 - \frac{10}{3}\beta_3 + (4\beta_2) \right) r = \left( \frac{2}{3}\beta_3 - \beta_2 \right) r \geq 0
    \]
  - case \(\beta_3 < 3\beta_2\)
    \[
    \left( -2\beta_2 - \frac{10}{3}\beta_3 + (3\beta_2 + 3\beta_3) \right) r = \left( \beta_2 - \frac{1}{3}\beta_3 \right) r \geq 0
    \]

- Filtration indexed by \{1, 3\}
  - case \(\beta_1 > \beta_3\)
    \[
    \left( -\frac{2}{3}\beta_1 - \frac{10}{3}\beta_3 + (2\beta_1 + 2\beta_3) \right) r = \left( \frac{4}{3}\beta_1 - \frac{4}{3}\beta_3 \right) r \geq 0
    \]
  - case \(\beta_1 < \beta_3\)
    \[
    \left( -\frac{2}{3}\beta_1 - \frac{10}{3}\beta_3 + (4\beta_3) \right) r = \left( \frac{2}{3}\beta_1 - \frac{2}{3}\beta_3 \right) r \geq 0
    \]

So there are no destabilizing length 2 filtrations.

**Remark 25.** Since the proof of Theorem 22 does not depend on \(c_i\), replacing \(c_i\) with \(C_i\) one gets the same results, that is, if \(s \geq p + 1\), Theorem 22 provides \(\beta, \zeta \in \mathbb{R}_+\) such that the associated subfiltrations \((E^\bullet, \beta)_{\mathbb{I}}\) and \((E^\bullet, \zeta)_{\mathbb{I}}\) are proper and the following equality holds

\[
\sum_{i \in \mathbb{I}} \alpha_i C_i + r\delta R_{\mathbb{I}, \alpha} = \sum_{i \in \mathbb{I}} \beta_i C_i + r\delta R_{\mathbb{I}, \beta} + \sum_{i \in \mathbb{I}} \zeta_i C_i + r\delta R_{\mathbb{I}, \zeta}.
\]

**Proposition 26.** Let \((E, \varphi)\) be a tensor of type \((a, b, D)\), let \((E^\bullet, \alpha)\) be a weighted filtration indexed by \(\mathbb{I}\) and let \(M_{\mathbb{I}}(E^\bullet; \varphi) = \{p_1, \ldots, p_p\}\) be the associated matrix. If \(p = 1\) and \(s = |\mathbb{I}| \geq 2\) then there exists an index \(j_0 \in \mathbb{I}\) and a weight \(\beta_0 \in \mathbb{R}_+\) such that

\[
rr_{j_0} \beta_0 (\mu^k(E) - \mu^k(E_{j_0})) = L_{\mathbb{I}}(E^\bullet, \beta) + \delta \mu_{\mathbb{I}}(E^\bullet, \beta; \varphi),
\]

where \(r_{j_0} = rk(E_{j_0})\).
Proof. Since there is only one pivot the maximum is attained at \( p \) for any \( \beta \), this implies that the function \( R_1^+ \ni \beta \mapsto R_{1,\beta} \in \mathbb{R} \) is linear (while, usually, it is only piecewise linear). Therefore, the set

\[
H = \{ \beta \in R_1^+ \mid f(\beta) = 0 \}
\]

is a linear subspace of \( R_1^+ \) of dimension \( s - 1 \), where we put

\[
f : R_1^+ \longrightarrow \mathbb{R}, \quad \beta \longmapsto \sum_{i \in I} \beta_i c_i + r \delta(R_{1,\beta} - R_{1,\underline{\alpha}}).
\]

Let \( V_j \) be the coordinate hyperplane given by \( V_j = \{ \beta \in R_1^+ \mid \beta_j = 0 \} \), since \( H \cap V_j = \{ \beta \in R_1^+ \mid f(\beta) = 0 \) and \( \beta_j = 0 \} \) is a linear system of two equations in \( s \) variables, if \( s \geq 2 \), there exists \( j \) such that \( H \cap V_j \neq \emptyset \). So, for any \( \beta \) in \( H \cap V_j \), the filtration \((E^*, \delta)\) indexed by \( J = I \setminus \{ j \} \) is proper and

\[
L_1(E^*, \delta) + \delta \mu \delta(E^*, \delta; \varphi) = L_1(E^*, \underline{\delta}) + \delta \mu \delta(E^*, \underline{\delta}; \varphi).
\]

Note that the filtration and also the matrix \( M(E^*; \varphi) \) associated to the filtration \((E^*, \delta)\) indexed by \( J = I \setminus \{ j \} \) have an unique pivot \( p' \); therefore, we can iterate the procedure until the cardinality of the filtration is \( \geq 2 \). So in the end we get an index \( j_0 \in I \) and a weight \( \beta_0 \in \mathbb{R}_+ \) such that \( L_1(E_{j_0}, \beta_0) + \delta \mu \delta(E_{j_0}, \beta_0; \varphi) = L_1(E^*, \underline{\delta}) + \delta \mu \delta(E^*, \underline{\delta}; \varphi) \), but an easy calculation shows that \( L_1(0 \subset E_{j_0} \subset E, \beta_0) + \delta \mu \delta(0 \subset E_{j_0} \subset E, \beta_0; \varphi) \) is equal to \( r r_{j_0} \beta_0 (\mu^k(E) - \mu^k(E_{j_0})) \) and so we are done. \( \diamond \)

Remark 27. Let \((E, \varphi), (E^*, \alpha)\) and \( M(E^*; \varphi) = \{ p_1, \ldots, p_p \} \) be as before and suppose that \( L_1^+(E^*, \underline{\alpha}; \varphi) \leq 0 \) or \( P_1^+(E^*, \underline{\alpha}; \varphi) \leq 0 \). In the former case define

\[
H = \{ \beta \in R_1^+ \mid f(\beta) \leq 0 \}
\]

where

\[
f : R_1^+ \longrightarrow \mathbb{R}, \quad \beta \longmapsto \sum_{i \in I} (\beta_i - \alpha_i) c_i + r \delta(R_{1,\beta} - R_{1,\underline{\alpha}});
\]

while in the latter case define

\[
H = \{ (\beta, \zeta) \in H' \mid f(\beta, \zeta) \leq 0 \},
\]

where \( H' = \{ (\beta, \zeta) \in \mathbb{R}_+^{2s} \mid \beta_i + \zeta_i - \alpha_i = 0 \ \forall i \in I \} \) and

\[
f : \mathbb{R}_+^{2s} \longrightarrow \mathbb{R}, \quad (\beta, \zeta) \longmapsto \sum_{i \in I} (\beta_i + \zeta_i - \alpha_i) c_i + r \delta(R_{1,\beta} + R_{1,\zeta} - R_{1,\underline{\alpha}}).
\]

In any case \( f \) is a piecewise linear function (since the function \( \beta \mapsto R_{1,\beta} \) is so), while \( H' \) is a subspace of \( \mathbb{R}_+^{2s} \) of dimension \( s \). Denote by \( \partial H \) the set \( \ker f_{|H} \), then, in the former case, \( \partial H \) is a piecewise-linear subspace of \( R_1^+ \) of dimension \( s - 1 \), while in the latter case it is a piecewise-linear subspace of \( \mathbb{R}_+^{2s} \) of the same dimension. If exists an index \( i_0 \in I \) such that \( H \cap \{ \beta_{i_0} = 0 \} \neq \emptyset \) or, in the second case, exists indices \( i_0 \neq j_0 \in I \) such that \( H \cap \{ \beta_{i_0} = 0 \} \cap \{ \zeta_{j_0} = 0 \} \neq \emptyset \), then the (proper) subfiltration \((E^*, \beta)\) indexed by \( I \setminus \{ i_0 \} \) or, in the former case, the (proper) subfiltrations \((E^*, \beta)\) and \((E^*, \zeta)\) indexed respectively by \( I \setminus \{ i_0 \} \) and \( I \setminus \{ j_0 \} \) destabilize as \( E \). Therefore, the existence of a shorter filtration \((E^*, \beta)\) indexed
by \( J \subseteq I \) such that \( L_1^\mu(E^\bullet, \beta) \leq L_1^\mu(E^\bullet, \alpha) \) (or, in the former case, \( P_1^\mu(E^\bullet, \beta) \leq P_1^\mu(E^\bullet, \alpha) \)) depends on the resolubility of certain linear systems of inequalities.

5. Rank 3 tensor sheaves on the projective line

In this section we want to describe all degree 0 rank 3 slope semistable tensors \((E, \varphi)\) of type \((3, b, \Omega^{p_1})\) on \(\mathbb{P}^1\), where \(\Omega^{p_1}\) is the trivial bundle.

Since we are on \(\mathbb{P}^1\), the bundle \(E\) decompose as \(E = L_1 \oplus L_2 \oplus L_3\). We will denote by \(d_i\) the degree of \(L_i\), by \(k_i\) the number \(k_{L_i, E}\) and by \(d_{ij}, c_{ij}\) and \(k_i\) the corresponding invariants for the bundles \(L_i \oplus L_j\). Note that, in this setting, the bundles \(E^{\otimes 3}\) decomposes as:

\[
E^3 = L_1^{\otimes 3} \oplus L_1^{\otimes 3} \oplus L_1^{\otimes 3} \oplus 3(L_1^{\otimes 2} \otimes L_2) \oplus 3(L_1^{\otimes 2} \otimes L_3) \oplus 3(L_2^{\otimes 2} \otimes L_1) \oplus 3(L_3^{\otimes 2} \otimes L_1) \oplus 3(L_2^{\otimes 2} \otimes L_2) \oplus 6(L_1 \otimes L_2 \otimes L_3).
\]

As we have already remarked, if the matrix associated to a filtration has only one pivot, then the filtration splits (see Corollary 23) and so \(L^\mu_1(E^\bullet, \varphi) = \sum_{i \in I} L^\mu_1(\alpha_i)\) for \(0 \subset E_i \subset E, \alpha_i\). Thanks to Theorem 18, the maximum number of pivots in the case \(a = r = 3\) is \(\max_{3 \leq x \leq 9} f_{3, 3}(x)\). The maximum of \(f_{3, 3}(x)\) is attained for \(x = 5\) and \(f_{3, 3}(5) = 2\) therefore, in this case, the matrix associated to any filtration has one or at most two pivots, no matter what \((E, \varphi)\) is. A simple calculation shows that all possible matrices having two pivots are the following:

\[
\begin{align*}
(1) & \quad p_1 = (1, 1, 3) \text{ and } p_2 = (1, 2, 2) \\
(2) & \quad p_1 = (1, 1, 3) \text{ and } p_2 = (2, 2, 2) \\
(3) & \quad p_1 = (1, 2, 3) \text{ and } p_2 = (2, 2, 2) \\
(4) & \quad p_1 = (1, 3, 3) \text{ and } p_2 = (2, 2, 2) \\
(5) & \quad p_1 = (1, 3, 3) \text{ and } p_2 = (2, 2, 3)
\end{align*}
\]

Clearly the semistability condition depends on the morphism \(\varphi\), for example, if \(\varphi = 0\) then \(E\) must be a semistable bundle in the usual way hence we obtain \(d_1 = d_2 = d_3\). So, from now on, we will assume that \(\varphi\) is not identically zero. We start from the following easy but fundamental result.

**Lemma 28.** Let us assume that the semistability conditions for the tensor \((E, \varphi)\) hold for any filtration obtained starting from the line bundles \(L_i\), id est, all the filtrations

\[
0 \subset L_i \subset L_i \oplus L_j \subset E
\]

then the tensor \((E, \varphi)\) is semistable.

**Proof.** Let

\[
(E^\bullet, \alpha) = (0 \subset E_1 \subset E_2 \subset E, (\alpha_1, \alpha_2))
\]

any weighted filtration. We want to prove that there exist indices \(i, j \in \{1, 2, 3\}\) such that

\[
L^\mu(E^\bullet, \alpha; \varphi) \geq L^\mu(0 \subset L_i \subset L_i \oplus L_j \subset E, \alpha; \varphi).
\]

Since \(\text{rk}(E_1) = 1\), without loss of generality, we can assume that exists an injection \(g : E_1 \rightarrow L_1\). Clearly the cokernel of this morphism is a rank zero sheaf, hence, if we have a non zero morphism \(\varphi : L_1^{\otimes s} \otimes E^{\otimes a-s} \rightarrow \Omega^{p_1}\), then the restriction \(\varphi|_{E_1} : E_1^{\otimes s} \otimes E^{\otimes a-s} \rightarrow \Omega^{p_1}\) will be non zero as well, that is \(k_{E_1, E} = k_{E_1, E}\). A similar argument works also for the rank 2 sheaf \(E_2\) after considering the decomposition...
$E_2 = M_1 \oplus M_2$, where $M_i$ are rank one sheaves. More precisely, up to changing the order of $L_i$, we can assume that $M_i \subset L_i$. So we have proven that

$$\mu(E^*, \alpha; \varphi) = \mu(0 \subset L_i \subset L_i \oplus L_j \subset E, \alpha; \varphi)$$

Moreover since we have a non zero morphism $q : E_1 \rightarrow L_1$ we have that $\deg(E_1) \leq \deg(L_1)$, and $\deg(E_2) \leq \deg(L_1 \oplus L_2)$. So we have

$$L(E^*, \alpha) \geq L(0 \subset L_i \subset L_i \oplus L_j \subset E, \alpha)$$

which concludes the proof.

From now on, we will denote by $(i, ij)$ the following filtration

$$0 \subset L_i \subset L_i \oplus L_j \subset E,$$

indexed by $I = \{1, 2\}$. So, for example, the element $(1, 2)$ of the associated matrix describes the behaviour of $\varphi$ restricted to the bundle $L_i \otimes (L_i \oplus L_j) \otimes E$.

If $E$ is the trivial bundle $E = \mathcal{O}_X^3$, then the semistability conditions given for the subbundles $L_i = \mathcal{O}_{p_i}$ imply that $k_i \geq 1$. Let consider now the filtration $(i, ij)$. An easy computation shows that a such filtration is critical if and only if $k_j = 3$, and in this case the pivots are $(1, 2, 3)$ and $(2, 2, 2)$ and in both cases the semistability conditions are satisfied.

So now we assume that $E$ is not trivial, i.e., $E \neq \mathcal{O}_X^3$. We choose the indices in a such way that $d_1 \leq d_2 \leq d_3$, hence, since the degree of $E$ is $0$, all the possible cases are the following:

(i) $d_1 < 0 < d_2 < d_3$;
(ii) $d_1 < d_2 \leq 0 < d_3$;
(iii) $d_1 = d_2 < d_3$;
(iv) $d_1 < d_2 = d_3$.

Since $E$ is semistable, considering the semistability condition given by sub-sheaves, i.e., the slope $k$-semistability, we obtain the following conditions for $i = 1, 2, 3$:

$$d_i - k_i \overline{\theta} \leq \frac{d}{3} - \overline{\theta} = -\overline{\theta}$$

So in particular, $k_3 \geq 2$. However, since $d_3 > 0$ there are no non-zero morphisms $L_3^{\otimes 3} \rightarrow \mathcal{O}_{p_1}$, hence $k_3 = 2$, so, by definition of $k$, the restriction of $\varphi$ to $L_3^{\otimes 2} \otimes E$ gives a non-zero morphism $(L_3^{\otimes 2} \otimes E)^{\oplus b} \rightarrow \mathcal{O}_{p_1}$, that is a non-zero morphism:

$$((L_3^{\otimes 2} \otimes L_1) \oplus (L_3^{\otimes 2} \otimes L_2) \oplus L_3^{\otimes 3})^{\oplus b} \rightarrow \mathcal{O}_{p_1}$$

Since $\deg(L_3^{\otimes 3})$ and $\deg(L_3^{\otimes 2} \otimes L_2)$ are strictly positive, we get that the only non-zero component of the previous morphism must start from $(L_3^{\otimes 2} \otimes L_1)^{\oplus b}$ and this implies that $2d_3 + d_1 \leq 0$. Since $d_3 \geq d_2$ and by assumption $d_1 + d_2 + d_3 = 0$, the only possibility is that $d_2 = d_3 = -\frac{d_1}{2}$. Moreover, the existence of a non zero morphism $(L_3^{\otimes 2} \otimes L_1)^{\oplus b} \rightarrow \mathcal{O}_{p_1}$ implies that $k_1 \geq 1$.

Case $k_1 = 1$. The non-equivalent filtrations we have to consider are the filtrations $(1, 12)$, $(2, 12)$ and $(3, 23)$. Let us consider first the filtration $(1, 12)$, that is, the filtration:

$$0 \subset L_1 \subset L_1 \oplus L_2 \subset E.$$
Since $k_1 = 1$, then one pivot is $(1, 2, 2)$, and the only possibility for another pivot is in the position $(1, 1, 3)$; however this element of the matrix associated to $\varphi$ is zero; otherwise, $k_1$ should be 2, so the filtration is not critical. Now consider the filtrations $(2, 12)$ and $(3, 23)$, that is, the filtrations
\[ 0 \subset L_2 \subset L_1 \oplus L_2 \subset E \quad \text{and} \quad 0 \subset L_2 \subset L_2 \oplus L_3 \subset E. \]
In the first case one pivot is $(1, 1, 2)$ while, in the second case, it is $(1, 1, 3)$, but in either cases the filtrations are not critical.

Case $k_1 \geq 2$. In this case, one pivot of any of the previous filtrations is $(1, 1, 2)$; hence, there are no critical filtration.

References

[1] A. Lo Giudice, A. Pustetto, Stability of Quadric Bundles, Geometriae Dedicata, December 2013, DOI 10.1007/s10711-013-9939-x.
[2] A. Pustetto, Mehta-Ramanathan for and k-semistable Decorated Sheaves, preprint December 2013, arXiv:1312.7312v1
[3] T. Gómez and I. Sols, Stable tensors and moduli space of orthogonal sheaves, preprint (2003), arXiv:math/0103150
[4] T. Gómez, A. Langer, A. Schmitt, I. Sols, Moduli Spaces for Principal Bundles in Large Characteristic, “International Workshop on Teichmüller Theory and Moduli Problems”, Allahabad 2006 (India). Ramanujan Mathematical Society Lecture Notes Series 10 (2010), 281371
[5] A.H.W. Schmitt, Singular principal bundles over higher-dimensional manifolds and their moduli spaces, Int Math Res Notices, (2002) 2002 (23), pp 1183–1209.
[6] A.H.W. Schmitt, A closer look at semistability for singular principal bundles, Int Math Res Notices, (2004) Volume 2004, pp 3327–3366.
[7] A.H.W. Schmitt, Global boundedness for decorated sheaves, International Mathematics Research Notices, Volume 2004, number 68, pp 3637.