Some estimation about Taylor-Maclaurin coefficients of generalized subclasses of bi-univalent functions

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Abstract — Our objective in this paper is to introduce and investigate comprehensive-constructed subclasses of normalized analytic and bi-univalent functions on the unit open disc. Bounds for the second and third Taylor-Maclaurin coefficients of functions belonging to this subclasses were investigated. Furthermore, some improvement and connections to some of the previous known results are also pointed out.

Keywords: Analytic functions; Univalent and bi-univalent functions; Maclaurin series; Coefficient bounds.

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1. Introduction, Definitions and Notations

Let \( \mathcal{A} \) denote the class of all analytic functions \( f \) defined in the open unit disk \( \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} \) and normalized by the condition \( f(0) = f'(0) - 1 = 0 \). Thus each \( f \in \mathcal{A} \) has a Taylor-Maclaurin series expansion of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1.1}
\]

Further, let \( \mathcal{S} \) denote the class of all functions \( f \in \mathcal{A} \) which are univalent in \( \mathbb{U} \) (for details, see [9]; see also some of the recent investigations [2,3]). Two of the important and well-investigated subclasses of the analytic and univalent function class \( \mathcal{S} \) are the class \( \mathcal{S}^*(\alpha) \) of starlike functions of order \( \alpha \) in \( \mathbb{U} \) and the class \( \mathcal{K}(\alpha) \) of convex functions of order \( \alpha \) in \( \mathbb{U} \). By definition, we have

\[
\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}, \tag{1.2}
\]

and

\[
\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}. \tag{1.3}
\]

It is clear from the definitions (1.2) and (1.3) that \( \mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \). Also we have

\[
f(z) \in \mathcal{K}(\alpha) \quad \text{iff} \quad zf'(z) \in \mathcal{S}^*(\alpha),
\]

and

\[
f(z) \in \mathcal{S}^*(\alpha) \quad \text{iff} \quad \int_{0}^{\infty} \frac{f(t)}{t} \, dt = F(z) \in \mathcal{K}(\alpha).
\]
It is well-known [9] that every function \( f \in \mathcal{S} \) has an inverse map \( f^{-1} \) that satisfies the following conditions:

\[
f^{-1}(f(z)) = z \quad (z \in \mathbb{D}),
\]

and

\[
f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \frac{1}{4}).
\]

In fact, the inverse function is given by

\[
f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.
\]

(A.1.4)

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{D} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathbb{D} \). Let \( \Sigma \) denote the class of bi-univalent functions in \( \mathbb{D} \) given by (A.1.1). For a brief history and some interesting examples of functions and characterization of the class \( \Sigma \), see Srivastava et al. [19], Frasin and Aouf [10], and Mageesh and Yamini [15]. Examples of functions in the class \( \Sigma \) are

\[
\frac{z}{1 - z} - \log(1 - z) \quad \text{and} \quad \frac{1}{2} \log\left(\frac{1 + z}{1 - z}\right),
\]

and so on. However, the familiar Koebe function is not a member of \( \Sigma \). Other common examples of functions in \( \mathcal{S} \) such as

\[
z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1 - z^2}
\]

are also not members of \( \Sigma \).

In 1967, Lewin [13] investigated the bi-univalent function class \( \Sigma \) and showed that \( |a_2| < 1.51 \). Subsequently, Brannan and Clunie [5] conjectured that \( |a_2| \leq \sqrt{2} \). Later, Netanyahu [17] showed that \( \max \{a_2\} = \frac{3}{4} \) if \( f \in \Sigma \). Brannan and Taha [6] introduced certain subclasses of a bi-univalent function class \( \Sigma \) similar to the familiar subclasses \( \mathcal{S}(\alpha) \) and \( \mathcal{K}(\alpha) \) of starlike and convex functions of order \( \alpha \) (\( 0 \leq \alpha < 1 \)), respectively (see [4]). Thus, following the works of Brannan and Taha [6], for \( 0 \leq \alpha < 1 \), a function \( f \in \Sigma \) is in the class \( \mathcal{S}_\alpha(\alpha) \) of bi-starlike functions of order \( \alpha \); or \( \mathcal{K}_\alpha(\alpha) \) of bi-convex functions of order \( \alpha \) if both \( f \) and \( f^{-1} \) are respectively starlike or convex functions of order \( \alpha \). Recently, many researchers have introduced and investigated several interesting subclasses of the bi-univalent function class \( \Sigma \) and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_n| \) for each \( f \in \Sigma \) given by (A.1.1) is still an open problem.

This work is concerned with the coefficient bounds for the Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \). Furthermore, we modify the definition of the classes introduced by Srivastava et al. [22] and modify estimations. Finally, several connections to some of the previous results are pointed out.

**Definition 1.1.** For \( 0 \leq \lambda \leq 1, 0 \leq \delta \leq 1, 0 \leq \mu \leq 1, 0 \leq \gamma \leq 1, 0 < \alpha \leq 1 \) and \( \tau \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). let \( f \in \Sigma \) given by (A.1.1), then \( f \) is said to be in the class \( \mathcal{S}_\Sigma^\alpha(\tau, \delta, \lambda, \gamma) \) if it satisfy the following conditions

\[
\left| \arg\left(1 + \frac{1}{\tau} \left[ \frac{(1 - \delta) f(z) + \delta f'(z) + \mu z^2 f''(z)}{(1 - \lambda)z + \lambda(1 - \gamma)f(z) + \lambda z f'(z)} - 1 \right] \right) \right| < \frac{\alpha \pi}{2},
\]

and

\[
\left| \arg\left(1 + \frac{1}{\tau} \left[ \frac{(1 - \delta) g(w) + \delta w g'(w) + \mu w^2 g''(w)}{(1 - \lambda)w + \lambda(1 - \gamma)g(w) + \lambda w g'(w)} - 1 \right] \right) \right| < \frac{\alpha \pi}{2},
\]

for all \( z \in \mathbb{D}, g = f^{-1} \in \Sigma \) given by (A.1.4).
Definition 1.2. For $0 \leq \gamma \leq 1, 0 \leq \mu \leq 1, 0 \leq \beta < 1$, and $\tau \in \mathbb{C} = \mathbb{C}\{0\}$, let $f \in \Sigma$ given by (1.1), then $f$ is said to be in the class $S_{\Sigma}(\tau, \delta, \mu, \lambda, \gamma; \beta)$ if it satisfies the following conditions

\[
\Re \left( 1 + \frac{1}{\tau} \left[ \frac{(1 - \delta)f(z) + \delta zf'(z) + \mu z^2 f''(z)}{(1 - \lambda)z + \lambda(1 - \gamma)f(z) + \lambda \gamma z f'(z)} - 1 \right] \right) > \beta,
\]

for all $z \in \mathbb{U}$, $g = f^{-1} \in \Sigma$ given by (1.4).

**Remark 1.** If we put $\delta = 1$ in the definitions 1.1 and 1.2, we modified the definitions of the classes $\mathcal{H}_\Sigma(\tau, \mu, \lambda, \gamma; \alpha)$ and $\mathcal{H}_\Sigma(\tau, \mu, \lambda, \gamma; \beta)$ respectively which were introduced by Srivastava et al. [22].

**Remark 2.** For special choices of the parameters $\tau, \delta, \mu, \lambda, \gamma$, we can obtain the following subclasses as a special case of our main classes defined above:

1. $S_{\Sigma}^{\alpha}(\lambda, \delta, 0, \gamma) = N_{\Sigma}(\alpha, \lambda, \delta)$ which introduced by Serap Bulut [7].
2. $S_{\Sigma}^{\alpha}(1, 1, 0, \lambda, 0, \beta) = S_{\Sigma}(1, 1, 0, \lambda, 0; \beta) = M_{\Sigma}^{\alpha, 1, 0}(\beta, \lambda)$ which introduced by Srivastava et al. [20].
3. $S_{\Sigma}(1, 1, 0, 1, \lambda) = \mathcal{G}_\Sigma(\alpha, \lambda)$ and $S_{\Sigma}(1, 1, 0, 1, \lambda; \beta) = \mathcal{M}_\Sigma(\lambda, \beta)$ which introduced by Murugusundaramoorthy et al. [16].
4. $S_{\Sigma}(1, 1, 1, 1, \lambda) = \mathcal{B}_\Sigma(\alpha, \lambda)$ and $S_{\Sigma}(1, 1, 1, 1, \lambda; \beta) = \mathcal{N}_\Sigma(\beta, \lambda)$ which introduced by Keerthi and Raja [12].
5. $S_{\Sigma}(1, 1, 0, 0, \gamma) = \mathcal{B}_\Sigma(\alpha, \lambda)$ and $S_{\Sigma}(1, 1, 0, 0, \gamma; \beta) = \mathcal{B}_\Sigma(\beta, \lambda)$ which introduced by Frasin and Aouf [10].
6. $S_{\Sigma}^{\alpha}(1, 1, 1, 0, \gamma) = \mathcal{H}_\Sigma(\alpha, \beta)$ and $S_{\Sigma}(1, 1, 1, 0, \gamma; \gamma) = \mathcal{H}_\Sigma(\gamma, \beta)$ which introduced by Frasin [11].
7. $S_{\Sigma}^{\alpha}(1, 1, 0, 0, \gamma) = \mathcal{H}_\Sigma(\alpha, \beta)$ and $S_{\Sigma}(1, 1, 0, 0, \gamma; \beta) = \mathcal{H}_\Sigma(\beta, \gamma)$ which introduced by Srivastava et al. [19].

**Lemma 1.3.** [9] If $h \in \mathcal{P}$, then the estimates $|c_n| \leq 2, n = 1, 2, 3, \ldots$ are sharp, where $\mathcal{P}$ is the family of all functions $h$ which are analytic in $\mathbb{U}$ for which $h(0) = 1$ and $\Re(h(z)) > 0(z \in \mathbb{U})$ where

\[
h(z) = 1 + c_1 z + c_2 z^2 + \ldots, z \in \mathbb{U}.
\]

2. **Coefficient bounds for the function class $S_{\Sigma}^{\alpha}(\tau, \delta, \lambda, \gamma)$**

In this section, we establish coefficient bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of the function $f \in S_{\Sigma}^{\alpha}(\tau, \delta, \lambda, \gamma)$.

**Theorem 2.1.** Let $f(z)$ defined by (1.1) belonging to the class $S_{\Sigma}^{\alpha}(\tau, \delta, \lambda, \gamma)$, then

\[
|a_2| \leq \frac{2\tau |\tau|}{\sqrt{2\alpha \tau \Omega + (1 - \alpha)(1 + \delta + 2\mu - \lambda - \gamma \lambda)^2}} \tag{2.1}
\]

and

\[
|a_3| \leq \min \left\{ \frac{4\tau |\tau|^2}{(1 + \delta + 2\mu - \lambda - \gamma \lambda)^2} + \frac{2\tau |\tau|}{|1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda|^2}, \frac{2\tau |\tau|}{|\Omega|^2} \right\} \tag{2.2}
\]

where

\[
\Omega = 1 + 2\delta + 6\mu - 2\lambda - 3\gamma \lambda - \lambda \delta - 2\mu \lambda + \lambda^2 + 2\gamma \lambda^2 - \delta \gamma \lambda - 2\mu \gamma \lambda + \gamma^2 \lambda^2. \tag{2.3}
\]
Proof. Since \( f, g \in S^*(\tau, \delta, \lambda, \gamma) \), then there exist two functions \( h_1(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \) and \( h_2(w) = 1 + \sum_{n=1}^{\infty} q_n w^n \) with positive real part in the unit disc such that

\[
1 + \frac{1}{\tau} \left[ (1 - \delta)f(z) + \delta z f'(z) + \mu z^2 f''(z) \right] = (h_1(z))^\alpha
\]

(2.4)

and

\[
1 + \frac{1}{\tau} \left[ (1 - \delta)g(w) + \delta w g'(w) + \mu w^2 g''(w) \right] = (h_2(w))^\alpha
\]

(2.5)

Now by comparing the coefficients of powers \( z, z^2, w \) and \( w^2 \) in boss sides of equations (2.4) and (2.5), we obtain

\[
\frac{(1 + \delta + 2\mu - \lambda - \gamma \lambda)}{\tau} = \alpha p_1
\]

(2.6)

\[
\frac{(1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda) a_3 - (1 + \delta + 2\mu - \lambda - \gamma \lambda)(\lambda + \gamma \lambda) a_2^2}{\tau} = \alpha p_2 + \frac{\alpha (\alpha - 1)}{2} p_1^2
\]

(2.7)

\[
- \frac{(1 + \delta + 2\mu - \lambda - \gamma \lambda)}{\tau} = \alpha q_1
\]

(2.8)

\[
\frac{(1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda)(2a_2^2 - a_3) - (1 + \delta + 2\mu - \lambda - \gamma \lambda)(\lambda + \gamma \lambda) a_2^2}{\tau} = \alpha q_2 + \frac{\alpha (\alpha - 1)}{2} q_1^2
\]

(2.9)

From equations (2.6) and (2.8), we deduce

\[
p_1 = -q_1,
\]

(2.10)

and

\[
a_2 = \frac{\alpha \tau p_1}{1 + \delta + 2\mu - \lambda - \gamma \lambda}.
\]

(2.11)

By adding equation (2.7) to (2.9), we obtain

\[
2\frac{\Omega}{\tau} a_2^2 = \alpha (p_2 + q_2) + \frac{\alpha (\alpha - 1)}{2} (p_1^2 + q_1^2)
\]

(2.12)

where \( \Omega \) are given by (2.3).

Using equation (2.10) and substituting the value of \( a_2 \) from equation (2.11) into equation (2.12), we deduce

\[
p_1^2 = \frac{(p_2 + q_2)(1 + \delta + 2\mu - \lambda - \gamma \lambda)^2}{2\alpha \tau \Omega + (1 - \alpha)(1 + \delta + 2\mu - \lambda - \gamma \lambda)^2}
\]

(2.13)

therefore, by applying Lemma 1 in the equation (2.13), then

\[
|p_1| \leq \frac{2|1 + \delta + 2\mu - \lambda - \gamma \lambda|}{\sqrt{2\alpha \tau \Omega + (1 - \alpha)(1 + \delta + 2\mu - \lambda - \gamma \lambda)^2}}.
\]

(2.14)

By substituting from equation (2.14) into (2.11) and using Lemma 1, we conclude the desired estimate of \( a_2 \) given by (4.1).

On the other hand, to investigate the bounds of \( |a_3| \), subtracting equation (2.9) from (2.7) we obtain

\[
2\frac{(1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda)}{\tau} (a_3 - a_2^2) = \alpha (p_2 - q_2) + \frac{\alpha (\alpha - 1)}{2} (p_1^2 - q_1^2)
\]

(2.15)
By using equation (2.10) into (2.15), we obtain

\[ a_3 = a_2^2 + \frac{\alpha \tau}{2(1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda)} (p_2 - q_2), \]  

(2.16)

Using Lemma 1 and substituting of the value of \( a_2 \) from (2.11) into (2.16), we conclude one of the desired estimates of \( |a_3| \).

Now, by substituting the value of \( a_2^2 \) from equation (2.12) into (2.16), we obtain

\[ a_3 = \frac{\alpha \tau}{2} \left[ p_2 \left( \frac{1}{\Omega} + \frac{1}{(1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda)} \right) + q_2 \left( \frac{1}{\Omega} - \frac{1}{(1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda)} \right) \right], \]  

(2.17)

Applying Lemma 1 for the coefficients \( p_2, q_2 \), we conclude the other desired estimation of \( |a_3| \). □

3. Coefficient bounds for the function class \( S_\Sigma(\tau, \delta, \lambda, \gamma; \beta) \)

In this section, we establish coefficient bounds for the Maclaurin coefficients \( |a_2| \) and \( |a_3| \) of the function \( f \in S_\Sigma(\tau, \delta, \lambda, \gamma; \beta) \).

**Theorem 3.1.** Let \( f(z) \) defined by (1.1) belonging to the class \( S_\Sigma(\tau, \delta, \lambda, \gamma; \beta) \), then

\[ |a_2| \leq \min \left\{ \frac{2(1 - \beta) |\tau|}{|1 + \delta + 2\mu - \lambda - \gamma \lambda|}, \sqrt{\frac{2|\tau| (1 - \beta)}{|\Omega|}} \right\}. \]  

(3.1)

and

\[ |a_3| \leq \min \left\{ \frac{4(1 - \beta)^2 |\tau|^2}{|1 + \delta + 2\mu - \lambda - \gamma \lambda|^2}, \frac{2(1 - \beta) |\tau|}{|1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda|^2}, \frac{2(1 - \beta) |\tau|}{|\Omega|} \right\}. \]  

(3.2)

**Proof.** Since \( f, g \in S_\Sigma(\tau, \delta, \lambda, \gamma; \beta) \), then there exist two functions \( P(z) = \sum_{n=1}^{\infty} p_n z^n \) and \( Q(w) = \sum_{n=1}^{\infty} q_n w^n \) with positive real part in the unit disc such that

\[ 1 + \frac{1}{\tau} \left[ (1 + \delta) f(z) + \delta z f'(z) + \mu z^2 f''(z) \right] = \beta + (1 - \beta) P(z) \]  

(3.3)

and

\[ 1 + \frac{1}{\tau} \left[ (1 + \delta) g(w) + \delta w g'(w) + \mu w^2 g''(w) \right] = \beta + (1 - \beta) Q(w) \]  

(3.4)

Now by comparing the coefficients of powers \( z, z^2, w \) and \( w^2 \) in boss sides of equations (3.3) and (3.4), we obtain

\[ \frac{1 + \delta + 2\mu - \lambda - \gamma \lambda}{\tau} = (1 - \beta) p_1 \]  

(3.5)

\[ \frac{(1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda) a_3 - (1 + \delta + 2\mu - \lambda - \gamma \lambda)(\lambda + \gamma \lambda) a_2^2}{\tau} = (1 - \beta) p_2 \]  

(3.6)

\[ - \frac{(1 + \delta + 2\mu - \lambda - \gamma \lambda)}{\tau} = (1 - \beta) q_1 \]  

(3.7)

\[ \frac{(1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda)(2a_2^2 - a_3) - (1 + \delta + 2\mu - \lambda - \gamma \lambda)(\lambda + \gamma \lambda) a_2^2}{\tau} = (1 - \beta) q_2. \]  

(3.8)
From equations (3.5) and (3.7), we deduce
\[ p_1 = -q_1, \]  
and
\[ a_2 = \frac{(1 - \beta)\tau p_1}{1 + \delta + 2\mu - \lambda - \gamma \lambda}. \]  

By adding equation (3.6) to (3.8), we obtain
\[ \frac{2\Omega a_2^2}{\tau} = (1 - \beta)(p_2 + q_2), \]  
where \( \Omega \) are given by (2.3). Applying Lemma 1, we obtain
\[ |a_2| \leq \sqrt{\frac{2|\tau|(1 - \beta)}{|\Omega|}}. \]  

On the other hand, from equation (3.10) we can deduce also that
\[ |a_2| \leq \frac{2(1 - \beta)|\tau|}{|1 + \delta + 2\mu - \lambda - \gamma \lambda|}. \]  

Combining this with inequality (3.12), we obtain the desired estimate on the coefficient \( |a_2| \) which given by (4.2).

In order to deduce an estimation of bounds of \( |a_3| \), subtracting equation (3.8) from (3.6), we get
\[ a_3 = a_2^2 + \frac{\tau(1 - \beta)(p_2 - q_2)}{2(1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda)}. \]  
By substituting the value of \( a_2 \) from equation (3.10) with applying Lemma 1 for the coefficients \( p_1, p_2 \) and \( q_2 \), we obtain
\[ |a_3| \leq \frac{4(1 - \beta)^2|\tau|^2}{(1 + \delta + 2\mu - \lambda - \gamma \lambda)^2} + \frac{2|\tau|(1 - \beta)}{|1 + 2\delta + 6\mu - \lambda - 2\gamma \lambda|}. \]  

Now, the value of \( a_2 \) from equation (3.11) with applying Lemma 1 for the coefficients \( p_2, q_2 \), we obtain the other estimation of \( |a_3| \) which given by (4.2).

4. Some corollaries and consequences

In this section, we introduced and modified some previous known results as an immediate consequences of Theorem 2.1 and 3.1.

Remark 3. Putting \( \delta = 1 \), we modified the results considered by Srivastava et al. [22, Theorem 1 and 2].

Corollary 4.1. Let \( f(z) \) given by (1.1) be in the class \( \mathcal{H}_2(\tau, \mu, \lambda; \alpha) \) then,
\[ |a_2| \leq \frac{2\alpha|\tau|}{\sqrt{|2\alpha\tau \tilde{\Omega} + (1 - \alpha)(2 + 2\mu - \lambda - \gamma \lambda)^2|}}. \]
and

\[ |a_3| \leq \min \left\{ \frac{4\alpha^2|\tau|^2}{(2 + 2\mu - \lambda - \gamma\lambda)^2} + \frac{2\alpha|\tau|}{|\Omega|}, \frac{2\alpha|\tau|}{3 + 6\mu - \lambda - 2\gamma\lambda}, \frac{2\alpha|\tau|}{|\Omega|} \right\} \]

where

\[ \Omega = \gamma^2\lambda^2 + 2\gamma\lambda^2 - 2\gamma\lambda\mu - 4\gamma\lambda + \lambda^2 - 2\lambda\mu - 3\lambda + 6\mu + 3 \]

**Corollary 4.2.** Let \( f(z) \) given by (1.1) be in the class \( \mathcal{H}_C(\tau, \mu, \lambda, \gamma; \beta) \) then,

\[ |a_2| \leq \min \left\{ \frac{2(1 - \beta)|\tau|}{|2 + 2\mu - \lambda - \gamma\lambda|}, \sqrt{\frac{2|\tau|(1 - \beta)}{|\Omega|}} \right\} \]

and

\[ |a_3| \leq \min \left\{ \frac{4(1 - \beta)^2|\tau|^2}{(2 + 2\mu - \lambda - \gamma\lambda)^2} + \frac{2(1 - \beta)|\tau|}{|3 + 6\mu - \lambda - 2\gamma\lambda|}, \frac{2(1 - \beta)|\tau|}{|\Omega|} \right\} \]

**Remark 4.** Putting \( \tau = 1, \lambda = 0 \) and replace \( \delta \) by \( \lambda \) and also \( \mu \) by \( \delta \), we obtain the result considered by Bulut [7, Theorem 5].

**Corollary 4.3.** Let \( f(z) \) given by (1.1) be in the class \( \mathcal{N}_C(\beta, \lambda, \delta) \) then,

\[ |a_2| \leq \min \left\{ \frac{2(1 - \beta)}{1 + \lambda + 2\delta}, \sqrt{\frac{2(1 - \beta)}{1 + 2\lambda + 6\delta}} \right\} \]

\[ |a_3| \leq \frac{2(1 - \beta)}{1 + 2\lambda + 6\delta} \]

**Remark 5.** Putting \( \tau = \delta = \lambda = 1, \mu = 0 \) and replace \( \gamma \) by \( \lambda \), we obtain the result considered by Murugusundaramoorthy et al. [16, Theorem 4 and 5].

**Corollary 4.4.** Let \( f(z) \) given by (1.1) be in the class \( \mathcal{G}_C(\alpha, \lambda) \) then,

\[ |a_2| \leq \frac{2\alpha}{(1 - \lambda)\sqrt{1 + \alpha}}, \]

\[ |a_3| \leq \min \left\{ \frac{4\alpha^2}{(1 - \lambda)^2} + \frac{\alpha}{1 - \lambda}, \frac{2\alpha}{(1 - \lambda)^2} \right\}. \]

**Corollary 4.5.** Let \( f(z) \) given by (1.1) be in the class \( \mathcal{M}_C(\beta, \lambda) \) then,

\[ |a_2| \leq \frac{\sqrt{2(1 - \beta)}}{1 - \lambda}, \]

\[ |a_3| \leq \min \left\{ \frac{2(1 - \beta)}{(1 - \lambda)^2}, \frac{2(1 - \beta)^2}{(1 - \lambda)^2} + \frac{1 - \beta}{1 - \lambda} \right\}. \]

**Remark 6.** Putting \( \tau = \delta = 1 \) and \( \mu = \gamma = 0 \), we obtain the result considered by Srivastava et al. [20, Theorem 2.1 and 3.1 with \( a = c, b = 1 \)].

**Corollary 4.6.** Let \( f(z) \) given by (1.1) be in the class \( \mathcal{S}^{\alpha,1,\beta}_\lambda(\alpha, \lambda) \) then,

\[ |a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(\lambda^2 - 3\lambda + 3) + (1 - \alpha)(2 - \lambda)^2}}, \]

\[ |a_3| \leq \min \left\{ \frac{4\alpha^2}{(2 - \lambda)^2} + \frac{2\alpha}{3 - \lambda}, \frac{2\alpha}{\lambda^2 - 3\lambda + 3} \right\}. \]
Corollary 4.7. Let \( f(z) \) given by (1.1) be in the class \( M_{\Sigma}^{1,\alpha}(\beta, \lambda) \) then,

\[
|a_2| \leq \sqrt{\frac{2(1 - \beta)}{\lambda^2 - 3\lambda + 3}},
\]

\[
|a_3| \leq \min\left\{ \frac{2(1 - \beta)}{\lambda^2 - 3\lambda + 3}, \frac{4(1 - \beta)^2}{(2 - \lambda)^2} + \frac{2(1 - \beta)}{3 - \lambda} \right\}.
\]

Remark 7. Putting \( \tau = \delta = \lambda = 1 \) and replace \( \mu, \gamma \) by \( \lambda \), we obtain the result considered by Keerthi and Raja [12, Corollary 2.3 and 3.4].

Corollary 4.8. Let \( f(z) \) given by (1.1) be in the class \( B_{\Sigma}(\alpha, \lambda) \) then,

\[
|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1 + 2\lambda) + (1 - 3\alpha)(1 + \lambda)^2}},
\]

\[
|a_3| \leq \min\left\{ \frac{4\alpha^2}{(1 + \lambda)^2} + \frac{\alpha}{1 + 2\lambda}, \frac{2\alpha}{1 + 2\lambda - \lambda^2} \right\}.
\]

Corollary 4.9. Let \( f(z) \) given by (1.1) be in the class \( N_{\Sigma}(\beta, \lambda) \) then,

\[
|a_2| \leq \sqrt{\frac{2(1 - \beta)}{1 + 2\lambda - \lambda^2}},
\]

\[
|a_3| \leq \min\left\{ \frac{2(1 - \beta)}{1 + 2\lambda - \lambda^2}, \frac{4(1 - \beta)^2}{(1 + \lambda)^2} + \frac{1 - \beta}{1 + 2\lambda} \right\}.
\]

Remark 8. Putting \( \tau = 1, \lambda = \mu = 0 \) and replace \( \delta \) by \( \lambda \), we obtain the result considered by Frasin and Aouf [10, Theorem 2.2 and 3.2].

Corollary 4.10. Let \( f(z) \) given by (1.1) be in the class \( B_{\Sigma}(\alpha, \lambda) \) then,

\[
|a_2| \leq \frac{2\alpha}{\sqrt{\alpha(1 + 2\lambda - \lambda^2) + (1 + \lambda)^2}},
\]

\[
|a_3| \leq \frac{2\alpha}{1 + 2\lambda}.
\]

Corollary 4.11. Let \( f(z) \) given by (1.1) be in the class \( B_{\Sigma}(\beta, \lambda) \), then

\[
|a_2| \leq \sqrt{\frac{2(1 - \beta)}{1 + 2\lambda}},
\]

\[
|a_3| \leq \frac{2(1 - \beta)}{1 + 2\lambda}.
\]

Remark 9. Putting \( \tau = \delta = 1, \lambda = 0 \) and replace \( \mu \) by \( \beta \), we obtain the result considered by Frasin [11, Theorem 2.2 and 3.2].

Corollary 4.12. Let \( f(z) \) given by (1.1) be in the class \( H_{\Sigma}(\alpha, \beta) \) then,

\[
|a_2| \leq \frac{2\alpha}{\sqrt{2(2 + \alpha) + 4\beta(\alpha + \beta + 2 - \alpha\beta)}},
\]

\[
|a_3| \leq \frac{2\alpha}{3(1 + 2\beta)}.
\]
Corollary 4.13. Let \( f(z) \) given by (1.1) be in the class \( \mathcal{H}_\Sigma(\gamma, \beta) \), then

\[
|a_2| \leq \sqrt{\frac{2(1 - \gamma)}{3(1 + 2\gamma)}},
\]

\[
|a_3| \leq \frac{2(1 - \gamma)}{3(1 + 2\beta)}.
\]

Remark 10. Putting \( \tau = \delta = 1 \) and \( \mu = \lambda = 0 \), we obtain the result considered by Srivastava et al. [19, Theorem 1 and 2].

Corollary 4.14. Let \( f(z) \) given by (1.1) be in the class \( \mathcal{H}_\alpha^c \) then,

\[
|a_2| \leq \alpha \sqrt{\frac{2}{2 + \alpha}},
\]

\[
|a_3| \leq \frac{2\alpha}{3},
\]

Corollary 4.15. Let \( f(z) \) given by (1.1) be in the class \( \mathcal{H}_\Sigma(\beta) \), then

\[
|a_2| \leq \sqrt{\frac{2(1 - \beta)}{3}},
\]

\[
|a_3| \leq \frac{2(1 - \beta)}{3}.
\]

Remark 11. Finally, we introduce some results improved in our paper,

1. The estimation of \(|a_3|\) in Corollaries 4.10, 4.11 are improvement of the estimates obtained by Frasin and Aouf [10, Theorem 2.2 and 3.2].

2. The estimation of \(|a_3|\) in Corollaries 4.12, 4.13 are improvement of the estimates obtained by Frasin [11, Theorem 2.2 and 3.2].

3. The estimation of \(|a_3|\) in Corollaries 4.14, 4.15 are improvement of the estimates obtained by Srivastava et al. [19, Theorem 1 and 2].

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