INJECTIVE PROPERTY RELATIVE TO NONSINGULAR
EXACT SEQUENCES

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Abstract. We investigate modules \( M \) having the injective property relative to nonsingular modules. Such modules are called "\( N \)-injective modules." It is shown that \( M \) is an \( N \)-injective \( R \)-module if and only if the annihilator of \( Z_2(R_R) \) in \( M \) is equal to the annihilator of \( Z_2(R_R) \) in \( E(M) \). Every \( N \)-injective \( R \)-module is injective precisely when \( R \) is a right nonsingular ring. We prove that the endomorphism ring of an \( N \)-injective module has a von Neumann regular factor ring. Every (finitely generated, cyclic, free) \( R \)-module is \( N \)-injective, if and only if \( R \) is an \( N \)-injective, if and only if \( R \) is right \( t \)-semisimple. The \( N \)-injective property is characterized for right extending rings, semilocal rings and rings of finite reduced rank. Using the \( N \)-injective property, we determine the rings whose all nonsingular cyclic modules are injective.

1. Introduction

To describe the content of the paper we first state some notations and recall a few relevant results. Throughout, all rings are associative with unity and all modules are unitary right modules. For a subset \( K \) of an \( R \)-module \( M \), we denote \( r_R(K) = \{ r \in R : Kr = 0 \} \), and for a subset \( I \) of \( R \) we denote \( l_M(I) = \{ m \in M : mI = 0 \} \). Recall that the singular submodule \( Z(M) \) of a module \( M \) is the set of \( m \in M \) such that \( mI = 0 \) for some essential right ideal \( I \) of \( R \), or equivalently, \( r_R(m) \leq_e R_R \) (the notation \( \leq_e \) denotes an essential submodule). The Goldie torsion (or second singular) submodule \( Z_2(M) \) of a module \( M \) is defined by \( Z_2(M/Z(M)) = Z(M/Z(M)) \). The following facts are well known: \( Z_2(M/Z_2(M)) = 0 \). If \( f : M \to N \) is a homomorphism, then \( f(Z_2(M)) \leq Z_2(N) \). Moreover, \( Z_2(M) \cap A = Z_2(A) \) for every submodule \( A \) of \( M \), and \( Z_2(\bigoplus \lambda M_\lambda) = \bigoplus \lambda Z_2(M_\lambda) \) for every class of \( R \)-modules \( M_\lambda \). A module \( M \) is called singular if \( Z(M) = M \) and nonsingular if \( Z(M) = 0 \), or equivalently, \( Z_2(M) = 0 \). The module \( M \) is called \( Z_2 \)-torsion if \( Z_2(M) = M \).

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Clearly, a submodule $A$ of $M$ is $\mathbb{Z}_2$-torsion if and only if $A \leq \mathbb{Z}_2(M)$. The class of $\mathbb{Z}_2$-torsion modules is closed under submodules, factor modules, direct sums, and extensions. In [2], a submodule $A$ of $M$ is called $t$-essential in $M$ (written by $A \leq_{tes} M$) if for every submodule $B$ of $M$, $A \cap B \leq \mathbb{Z}_2(M)$ implies that $B \leq \mathbb{Z}_2(M)$. Using this notion, it is easy to see that $\mathbb{Z}_2(M)$ is the set of $m \in M$ such that $mI = 0$ for some $t$-essential right ideal $I$ of $R$, or equivalently, $r_R(m) \leq_{tes} R_R$. Following [2], a submodule $C$ of $M$ is said to be $t$-closed in $M$ if $C \leq_{tes} C' \leq M$ implies that $C = C'$; and a module $M$ is called $t$-extending if every $t$-closed submodule of $M$ is a direct summand. In fact, $t$-extending modules are precisely the modules $M$ for which every closed submodule of $M$ containing $\mathbb{Z}_2(M)$ is a direct summand of $M$.

Over the last 50 years numerous mathematicians have investigated rings over which certain cyclic modules have a homological property. Among these, determining the rings whose certain cyclic modules are injective has been of interest. Osofsky [12] proved that every cyclic $R$-module is injective, if and only if every $R$-module is injective, if and only if $R$ is semisimple. A cyclic $R$-module is called proper cyclic if it is not isomorphic to $R$. A ring $R$ is called a right PCI-ring if every proper cyclic $R$-module is injective. Faith [5] proved that a right PCI-ring is either a semisimple ring or a simple right semihereditary right Ore domain. An excellent reference for a thorough study of these rings is [8]. The rings for which every singular module is injective were studied by Goodearl [6]. He called them right SI-rings and characterized such rings as those nonsingular ones for which $R/I$ is semisimple for every essential right ideal $I$ of $R$. Osofsky and Smith [13] showed that every singular cyclic $R$-module is injective if and only if $R$ is a right SI-ring. More results on such rings can be found in [4] and [14]. Motivated by these, a natural question is: “What are the rings whose all nonsingular cyclic modules are injective?” In [3] the rings whose all nonsingular modules are injective were studied. Such rings are called right $t$-semisimple rings. It was shown that $R$ is right $t$-semisimple, if and only if every nonsingular $R$-module is semisimple, if and only if $R/Z_2(R_R)$ is a semisimple ring, if and only if $R$ is a direct product of two rings, one is semisimple and the other is right $Z_2$-torsion. By [3, Example 4.15], the class of right $t$-semisimple rings is properly contained in that of rings $R$ for which every nonsingular cyclic $R$-module is injective. This raises another question: “Under which condition(s) the class of rings $R$ for which every nonsingular cyclic $R$-module is injective coincides with that of right $t$-semisimple rings?” But, it is a fact, obtained by Baer’s criterion, that a nonsingular $R$-module $M$ is injective precisely when $M$ is injective relative to the nonsingular $R$-module $R/Z_2(R_R)$. This leads us to investigate the modules $M$ which are injective relative to nonsingular modules for finding the answers of the above questions.

Let $M$ and $L$ be $R$-modules. Recall that $M$ is said to be $L$-injective (or, injective relative to $L$) if for every monomorphism $f : K \to L$ and every homomorphism $g : K \to M$, there is a homomorphism $h : L \to M$ such that $hf = g$. We say that an $R$-module $M$ is $N$-injective if $M$ is injective relative
to every nonsingular $R$-module; in other words, $M$ is injective relative to every nonsingular exact sequence $0 \to K \to L$. (Note that every submodule of a nonsingular module is nonsingular.) Section 2 is devoted to study $\mathcal{N}$-injective modules. Every injective module and every module over a right $t$-semisimple ring are $\mathcal{N}$-injective. It is proved that $M$ is $\mathcal{N}$-injective, if and only if $M$ is injective relative to $R/Z_2(R_R)$, if and only if $l_M(Z_2(R_R)) = l_{E(M)}(Z_2(R_R))$, if and only if $M = Z_2(M) \oplus M'$, where $Z_2(M)$ is $\mathcal{N}$-injective and $M'$ is injective (Theorem 2.2). A nonsingular module is $\mathcal{N}$-injective if and only if it is injective (Corollary 2.3(i)). For a module $M$,

$$\text{injective} \Rightarrow \mathcal{N}\text{-injective} \Rightarrow t\text{-extending},$$

but none of these implications is reversible (Corollary 2.3(ii)). The classes of injective $R$-modules and $\mathcal{N}$-injective $R$-modules coincide if and only if $R$ is a right nonsingular ring (Proposition 2.7). We prove that if $M$ is an $\mathcal{N}$-injective module, then $S/T$ is a von Neumann regular ring, where $S = \text{End}(M)$ and $T = \{ \varphi \in S : \varphi M \leq Z_2(M) \}$ (Theorem 2.9). This implies that $R/Z_2(R_R)$ is a von Neumann regular ring whenever $R$ is $\mathcal{N}$-injective (Corollary 2.10).

In Section 3, we give several characterizations obtained by the $\mathcal{N}$-injective property. It is proved that $R$ is a right $t$-semisimple ring, if and only if every (finitely generated, cyclic, free) $R$-module is $\mathcal{N}$-injective, if and only if $R^{(3)}$ is $\mathcal{N}$-injective (Theorem 3.1). This, in particular, implies that a semilocal ring is $\mathcal{N}$-injective precisely when $R$ is right $t$-semisimple (Corollary 3.2). In the sequel, it is shown that $R$ is $\mathcal{N}$-injective if and only if $Z_2(R_R)$ is $R/Z_2(R_R)$-injective and every nonsingular cyclic $R$-module is injective and projective (Proposition 3.6). A right extending ring $R$ is $\mathcal{N}$-injective if and only if $R/Z_2(R_R)$ is a right self-injective ring (Theorem 3.7). Moreover, if $R$ is a ring of finite reduced rank, then $R$ is $\mathcal{N}$-injective if and only if $R$ is right $t$-semisimple (Proposition 3.8).

By the obtained results, we find some answers to the above mentioned questions: i) The rings whose every nonsingular cyclic module is injective are characterized. In fact, $R$ is such a ring if and only if $R/Z_2(R_R)$ is a right self-injective ring, and if $R$ is right extending, these are equivalent to $R$ being right $\mathcal{N}$-injective (Theorem 3.7). ii) The class of rings $R$ for which every nonsingular cyclic $R$-module is injective coincides with that of right $t$-semisimple rings whenever $R$ is either semilocal or of finite reduced rank (Corollary 3.10).

2. $\mathcal{N}$-injective modules

We say that an $R$-module $M$ is $\mathcal{N}$-injective if $M$ is injective relative to every nonsingular $R$-module. Clearly, every injective $R$-module is $\mathcal{N}$-injective. The following example shows that the class of $\mathcal{N}$-injective $R$-modules properly contains that of injective $R$-modules. More examples of $\mathcal{N}$-injective modules will be given in Examples 2.6.

Example 2.1. Let $R_1$ be a right $Z_2$-torsion ring (e.g., $R_1 = \mathbb{Z}/p^2\mathbb{Z}$, where $p$ is a prime number), $R_2$ be a semisimple ring (e.g., $R_2 = D$ is a division
ring), and $R = R_1 \times R_2$. Assume that $M$ is an $R$-module, $f : A \to B$ is an $R$-monomorphism where $B$ is a nonsingular $R$-module, and $g : A \to M$ is an $R$-homomorphism. By [3, Theorems 3.2(4) and 3.8(3)], $A$ is a direct summand of $B$, and hence $g$ can be extended to an $R$-homomorphism $h : B \to M$. This shows that $M$ is $\mathcal{N}$-injective.

The next result gives several equivalent conditions for an $\mathcal{N}$-injective module.

**Theorem 2.2.** The following statements are equivalent for an $R$-module $M$.

1. $M$ is $\mathcal{N}$-injective.
2. $M$ is $R/Z_2(R_R)$-injective.
3. $l_M(Z_2(R_R)) = l_{E(M)}(Z_2(R_R))$.
4. $l_M(Z_2(R_R))$ is an injective $R/Z_2(R_R)$-module.
5. $M = Z_2(M) \oplus M'$, where $Z_2(M)$ is $\mathcal{N}$-injective and $M'$ is injective.
6. For every monomorphism $f : A \to B$ of $R$-modules where $A$ is nonsingular, and every $R$-homomorphism $g : A \to M$, there exists an $R$-homomorphism $h : B \to M$ such that $hf = g$.

**Proof.** (1) $\Rightarrow$ (6). Let $f : A \to B$ be a monomorphism of $R$-modules where $A$ is nonsingular, and $g : A \to M$ be a homomorphism. Assume that $\pi : B \to B/Z_2(B)$ is the natural epimorphism. Since $A$ is nonsingular, $\pi f : A \to B/Z_2(B)$ is a monomorphism. So by hypothesis, there exists a homomorphism $\theta : B/Z_2(B) \to M$ such that $\theta \pi f = g$. Set $h = \theta \pi$.

(6) $\Rightarrow$ (5). Let $C$ be a complement of $Z_2(M)$ in $M$, and $f : C \to E(C)$ be the inclusion map, where $E(C)$ is the injective hull of $C$. Moreover, assume that $g : C \to M$ is the inclusion map. By hypothesis, there exists a homomorphism $h : E(C) \to M$ such that $hf = g$. Since $g$ is a monomorphism and $C \leq E(C)$, we conclude that $h$ is a monomorphism. Thus $h(E(C)) \cong E(C)$ is injective, and so $h(E(C))$ is a direct summand of $M$, say $M = K \oplus h(E(C))$. Since $C$ is nonsingular we conclude that $E(C)$ is nonsingular, and so $h(E(C))$ is nonsingular. Thus $Z_2(M) \leq K$. On the other hand, $c = g(c) = hf(c) = h(c)$, for every $c \in C$. Thus $C \leq h(E(C))$. Hence $Z_2(M) \oplus C \leq M$ implies that $Z_2(M) \leq K$. But $Z_2(M)$ is closed, and so $Z_2(M) = K$. Since $M$ satisfies (6) and $Z_2(M)$ is a direct summand of $M$, it is easy to see that $Z_2(M)$ also satisfies (6). Thus $Z_2(M)$ is $\mathcal{N}$-injective. Now by setting $M' = h(E(C))$, the desired decomposition is obtained.

(5) $\Rightarrow$ (2). Since $Z_2(M)$ and $M'$ are $R/Z_2(R_R)$-injective, so is $M$.

(2) $\Rightarrow$ (4). Let $\overline{R} = R/Z_2(R_R)$, and $\overline{I}$ be a right ideal of $\overline{R}$. Moreover, assume that $g : \overline{I} \to l_M(Z_2(R_R))$ is an $\overline{R}$-homomorphism. By hypothesis $g$ can be extended to an $R$-homomorphism $h : \overline{R} \to M$. But clearly, $h(\overline{I}) \leq l_M(Z_2(R_R))$, and so $g$ can be extended to the $\overline{R}$-homomorphism $h : \overline{R} \to l_M(Z_2(R_R))$. Thus by Baer’s criterion, $l_M(Z_2(R_R))$ is an injective $\overline{R}$-module.

(4) $\Rightarrow$ (3). Set $\overline{R} = R/Z_2(R_R)$, and $K = l_M(Z_2(R_R))$. By [7, Exercise 5J], $l_{E(K)}(Z_2(R_R)) = E(K_M)$. Now we show that $l_{E(K)}(Z_2(R_R)) = l_{E(M)}(Z_2(R_R))$. Clearly, $E(K)$ is a direct summand of $E(M)$, say $E(K) \oplus D = E(M)$. Let
\(x \in l_{E(M)}(Z_2(R))\) and \(x = e + d\), where \(e \in E(K)\) and \(d \in D\). Obviously, \(e \in l_{E(K)}(Z_2(R))\) and \(d \in l_D(Z_2(R))\). If \(d \neq 0\), then there exists \(r \in R\) such that \(0 \neq dr \in M\). Thus \(dr Z_2(R) \leq dZ_2(R) = 0\), and so \(dr \in K \cap D = 0\) which is impossible. Hence \(d = 0\) and \(x = e \in l_{E(K)}(Z_2(R))\). This shows that \(l_{E(K)}(Z_2(R)) = l_{E(M)}(Z_2(R))\), as desired. Therefore \(E(K\underline{\lambda}) = l_{E(M)}(Z_2(R))\). Since \(K\underline{\lambda}\) is injective we conclude that \(l_M(Z_2(R)) = l_{E(M)}(Z_2(R))\).

(3) \(\Rightarrow\) (1). First note that \(l_{E(M)}(Z_2(R))\) is an injective \(R/Z_2(R)\)-module. In fact, let \(\underline{\tau} = R/Z_2(R)\), and \(\tau\) be a right ideal of \(\underline{\tau}\). Moreover, let \(\varphi : \tau \rightarrow l_{E(M)}(Z_2(R))\) be an \(\underline{\tau}\)-homomorphism. Then \(\varphi\) can be extended to an \(R\)-homomorphism \(\psi : \underline{\tau} \rightarrow E(M)\). But clearly, \(\psi(\underline{\tau}) \leq l_{E(M)}(Z_2(R))\), and so \(\varphi\) is extended to the \(\underline{\tau}\)-homomorphism \(\psi : \underline{\tau} \rightarrow l_{E(M)}(Z_2(R))\). Thus by Baer’s criterion we conclude that \(l_{E(M)}(Z_2(R))\) is an injective \(\underline{\tau}\)-module, as desired.

Now let \(N\) be a nonsingular \(R\)-module, \(f : A \rightarrow N\) be an \(R\)-monomorphism and \(g : A \rightarrow M\) be an \(R\)-homomorphism. Since \(A\) is nonsingular, \(AZ_2(R) = 0\), and hence \(g(A) \leq l_M(Z_2(R))\). But, by hypothesis and what we have shown above \(l_M(Z_2(R))\) is an injective \(\underline{\tau}\)-module. So there exists an \(\underline{\tau}\)-homomorphism \(h : N \rightarrow l_M(Z_2(R))\) such that \(hf = g\). Clearly, \(h : N \rightarrow M\) is an \(R\)-homomorphism. This shows that \(M\) is \(N\)-injective. \(\square\)

**Corollary 2.3.** (i) A nonsingular module \(M\) is \(N\)-injective if and only if \(M\) is injective.

(ii) If \(M\) is an \(N\)-injective module, then \(M\) is \(t\)-extending.

**Proof.** (i) This follows from Theorem 2.2(5).

(ii) This is obtained by Theorem 2.2(5) and [2, Theorem 2.11(3)]. \(\square\)

The converse implication of Corollary 2.3(ii) is not always true. For example, \(\mathbb{Z}\) is an extending module which is not injective, hence it is not \(N\)-injective by Corollary 2.3(i).

**Corollary 2.4.** The following statements are equivalent for a ring \(R\).

1. \(R/Z_2(R)\) is a right Noetherian ring.
2. \(M^{(N)}\) is \(N\)-injective, for every \(N\)-injective module \(M\).
3. Every direct sum of \(N\)-injective modules is \(N\)-injective.

**Proof.** (1) \(\Rightarrow\) (3). Let \(M = \bigoplus_{\lambda \in \Lambda} M_{\lambda}\), where each \(M_{\lambda}\) is \(N\)-injective. By Theorem 2.2(4), \(l_{M_{\lambda}}(Z_2(R))\) is an injective \(R/Z_2(R)\)-module. Hence \(l_M(Z_2(R)) = \bigoplus_{\lambda \in \Lambda} l_{M_{\lambda}}(Z_2(R))\) is an injective \(R/Z_2(R)\)-module since \(R/Z_2(R)\) is right Noetherian. Thus by Theorem 2.2(4), \(M\) is \(N\)-injective.

(3) \(\Rightarrow\) (2). This implication is clear.

(2) \(\Rightarrow\) (1). By [11, Theorem 7.48(4)], it suffices to show that \(M^{(N)}\) is an injective \(R/Z_2(R)\)-module, for every injective \(R/Z_2(R)\)-module \(M\). Since \(M\)
is \( R/Z_2(R_R) \)-injective as an \( R \)-module. Theorem 2.2(2) implies that \( M \) is \( \mathcal{N} \)-injective. Thus by hypothesis, \( M^{(3)} \) is \( \mathcal{N} \)-injective, hence \( R/Z_2(R_R) \)-injective. So \( M^{(3)} \) is an injective \( R/Z_2(R_R) \)-module. \( \square \)

A ring \( R \) is called a right \( V \)-ring (or right co-semisimple) if every simple \( R \)-module is injective.

**Corollary 2.5.** The following statements are equivalent for a ring \( R \).

(1) Every simple \( R \)-module is \( \mathcal{N} \)-injective.

(2) \( R/Z_2(R_R) \) is a right \( V \)-injective.

**Proof.** (1) \( \Rightarrow \) (2). Let \( S \) be a simple \( R/Z_2(R_R) \)-module. Clearly, \( S \) is a simple \( R \)-module, and so as an \( R \)-module, \( S \) is \( \mathcal{N} \)-injective, hence \( R/Z_2(R_R) \)-injective. Thus \( S \) is an injective \( R/Z_2(R_R) \)-module.

(2) \( \Rightarrow \) (1). Let \( S \) be a simple \( R \)-module. Clearly, \( l_S(Z_2(R_R)) \) is \( S \) or \( 0 \). So by hypothesis, \( l_S(Z_2(R_R)) \) is an injective \( R/Z_2(R_R) \)-module. Hence \( S \) is \( \mathcal{N} \)-injective by Theorem 2.2(4). \( \square \)

In the following we give more examples of \( \mathcal{N} \)-injective modules.

**Examples 2.6.** (i) Let \( U \) be a right \( Z_2 \)-torsion ring (e.g., \( U = \mathbb{Z}/p^2\mathbb{Z} \) for a prime number \( p \)). Then \( T = \left( \begin{array}{l} 0 \\ 0 \end{array} \right) \) is a right \( Z_2 \)-torsion ring; see [3, Proposition 3.11]. Set \( R = T \times \mathbb{Z} \), and \( M = T \times \mathbb{Q} \). Since \( T \) is right \( Z_2 \)-torsion, every \( T \)-module \( X \) is \( Z_2 \)-torsion (note that \( XZ_2(T_R) \leq Z_2(X) \)), and hence every \( T \)-module is \( \mathcal{N} \)-injective. On the other hand, \( \mathbb{Q} \) is an injective \( \mathbb{Z} \)-module. Therefore \( T \) is an \( \mathcal{N} \)-injective \( R \)-module and \( \mathbb{Q} \) is an injective \( R \)-module. But, \( Z_2(M) = T \), and so by Theorem 2.2(5), \( M \) is an \( \mathcal{N} \)-injective \( R \)-module.

(ii) Let \( R_1 \) be a right \( Z_2 \)-torsion ring (e.g., \( R_1 = \prod_p \mathbb{Z}/p^2\mathbb{Z} \), where \( p \) runs through the set of prime numbers), \( R_2 \) a right nonsingular right Noetherian ring (e.g., \( R_2 = \left( \begin{array}{l} 0 \\ 0 \end{array} \right) \)), where \( D \) is a division ring), and \( R = R_1 \times R_2 \). By [3, Lemma 3.10], \( Z_2(R_R) = R_1 \), and so \( R/Z_2(R_R) \approx R_2 \) is right Noetherian. Now let \( M \) be an \( R \)-module and \( \Lambda \) be a set. By Corollary 2.4, \( E(M)^{(\Lambda)} \) is an \( \mathcal{N} \)-injective \( R \)-module.

(iii) Let \( R_1 \) be a right \( Z_2 \)-torsion ring (e.g., \( R_1 = \prod_\Lambda \mathbb{Z}/p^2\mathbb{Z} \), where \( p \) is a prime number and \( \Lambda \) is a set), \( R_2 \) a right nonsingular right \( V \)-ring (e.g., \( R_2 \) is a field), and \( R = R_1 \times R_2 \). Then \( Z_2(R_R) = R_1 \), and so \( R/Z_2(R_R) \approx R_2 \) is a right \( V \)-ring. Thus by Corollary 2.5, \( R/L \) is an \( \mathcal{N} \)-injective \( R \)-module, for every maximal right ideal \( L \) of \( R \).

The following result shows that the classes of \( \mathcal{N} \)-injective \( R \)-modules and injective \( R \)-modules coincide if and only if \( R \) is a right nonsingular ring.

**Proposition 2.7.** The following statements are equivalent for a ring \( R \).

(1) Every \( \mathcal{N} \)-injective \( R \)-module is injective.

(2) \( R \) is right nonsingular.

**Proof.** The implication (2) \( \Rightarrow \) (1) follows from Theorem 2.2. For (1) \( \Rightarrow \) (2), set \( A = l_R(Z_2(R_R)) \). We show that \( A \) is an essential right ideal of \( R \). Let \( I \) be a
right ideal of $R$ such that $A \cap I = 0$. So $I_{K}(Z_{2}(R)) = 0$ for every $R$-submodule $K$ of $I$. Thus by Theorem 2.2(4), $K$ is $N$-injective, and so by hypothesis it is injective. This implies that $I$ is a semisimple direct summand of $R$. On the other hand, if $J$ is a nonsingular right ideal of $R$, then $JZ_{2}(R) \leq Z_{2}(J) = 0$, and so $J \leq A$. Hence by the semisimple property of $I$ we conclude that $I$ is singular. But $R$ cannot contain a nonzero singular direct summand, and so $I = 0$. This shows that $A$ is an essential right ideal of $R$. Thus $E(A) = E(R)$. By Theorem 2.2(4), $I_{E(A)}(Z_{2}(R))$ is an injective $R/Z_{2}(R)$-module, and so it is $N$-injective as an $R$-module. Thus by hypothesis, $I_{E(A)}(Z_{2}(R))$ is an injective $R$-module. But $A \leq I_{E(A)}(Z_{2}(R))$, and so $I_{E(A)}(Z_{2}(R)) = E(A)$. Thus $Z_{2}(R) = RZ_{2}(R) \leq E(R)Z_{2}(R) = E(A)Z_{2}(R) = 0$. Hence $R$ is right nonsingular.

**Corollary 2.8.** The following statements are equivalent for a ring $R$.

1. Every $N$-injective $R$-module is projective.
2. $R$ is semisimple.

**Proof.** It suffices to show that (1) $\Rightarrow$ (2). By hypothesis, every injective $R$-module is projective. So $R$ is quasi-Frobenius, and hence every projective $R$-module is injective; see [11, Theorems 7.55 and 7.56(2)]. Thus hypothesis implies that every $N$-injective $R$-module is injective. Hence $R$ is right nonsingular by Proposition 2.7. So by [3, Corollary 4.6], $R$ is semisimple.

We end this section by proving that the endomorphism ring of an $N$-injective module has a von Neumann regular factor ring. It will be observed that the endomorphism ring of an $N$-injective module is not necessarily von Neumann regular; see Remark 3.5.

**Theorem 2.9.** Let $M$ be a module, $S = \text{End}(M)$, and $T = \{ \varphi \in S : \varphi M \leq Z_{2}(M) \}$. If $M$ is $N$-injective, then $S/T$ is a von Neumann regular ring.

**Proof.** First we show that $T$ is a two-sided ideal of $S$. Let $\varphi \in T$ and $\psi \in S$. Since $\varphi \in T$ we conclude that $\varphi^{-1}(Z_{2}(M)) = M$. But clearly, $\varphi^{-1}(Z_{2}(M)) \leq (\psi \varphi)^{-1}(Z_{2}(M))$, hence $(\psi \varphi)^{-1}(Z_{2}(M)) = M$. So $\psi \varphi \in T$. On the other hand, $(\psi \varphi)^{-1}(Z_{2}(M)) = (\psi^{-1}(Z_{2}(M))) = \psi^{-1}(M) = M$. Hence $\psi \varphi \in T$. This shows that $T$ is a two-sided ideal of $S$.

Now we show that $S/T$ is von Neumann regular. Let $\psi \in S$. By Corollary 2.3(ii), $M$ is $t$-extending. So by [2, Theorem 2.11(5)], there exists a direct summand $D$ of $M$, say $M = D \oplus E$, such that $\psi^{-1}(Z_{2}(M)) \leq_{t} D$. Assume that ‘bar’ denotes the image in $M/Z_{2}(M)$. Since $Z_{2}(M) \leq \psi^{-1}(Z_{2}(M))$ we conclude that $\overline{M} = \overline{D} \oplus \overline{E}$. Moreover, $\overline{\psi} : \overline{E} \to \overline{E}$ defined by $\overline{x} = \overline{\psi x}$ is an isomorphism ($\overline{\psi}$ is one-to-one, since $\psi x \in Z_{2}(M)$ implies that $x \in \psi^{-1}(Z_{2}(M)) \cap E \leq D \cap E = 0$). But $\overline{M}$ is injective by Theorem 2.2(5), and so $\overline{M}$ has $C_{2}$ condition. Thus $\overline{\psi} E$ is a direct summand of $\overline{M}$, say $\overline{M} = \overline{\psi E} \oplus K$. This implies that $M = \psi E \oplus (K + Z_{2}(M))$; in fact, it is enough to show that $\psi E \cap (K + Z_{2}(M)) = 0$. Let $\psi x = k + z$, where $x \in E$, $k \in K$ and $z \in Z_{2}(M)$. Then $\psi x + Z_{2}(M) =$
The following statements are equivalent for a ring \( R \) of finite reduced property. For right extending rings, semilocal rings and rings of finite reduced \( \mathbb{Z} \)-injection property. Hence by hypothesis and Theorem 2.2(5), the module \( (\psi - \psi \psi)^{-1}(Z_2(M)) = M \). Hence \( \psi - \psi \psi \in T \), and so \( S/T \) is von Neumann regular.

**Corollary 2.10.** Let a ring \( R \) be \( N \)-injective.

(i) \( R/Z_2(R_R) \) is a von Neumann regular ring.

(ii) \( \text{Rad}(P) \leq Z_2(P) \) for every projective \( R \)-module \( P \).

**Proof.** (i) Let \( r \in R \), and \( f_r \) be the endomorphism of \( R \) defined by \( f_r(x) = rx \).

If \( r \in Z_2(R_R) \), then \( f_r(R) \leq Z_2(R_R) \). If \( f_r(R) \leq Z_2(R_R) \), then \( f_r(1) = r \in Z_2(R_R) \). Therefore under the ring isomorphism \( \Phi : R \to S = \text{End}(R_R) \) defined by \( \Phi(r) = f_r \), the ideal \( Z_2(R_R) \) is isomorphic to \( T = \{ \varphi \in S : \varphi R \leq Z_2(R_R) \} \).

Hence \( R/Z_2(R_R) \cong S/T \), and so by Theorem 2.9, \( R/Z_2(R_R) \) is a von Neumann regular ring.

(ii) Since the Jacobson radical of a von Neumann regular ring is zero, (i) implies that \( \text{Rad}(R) \leq Z_2(R_R) \). Hence \( \text{Rad}(P) = PRad(R) \leq PZ_2(R_R) \leq Z_2(P) \).

### 3. More characterizations

In this section we give several characterizations obtained by the \( N \)-injective property. For right extending rings, semilocal rings and rings of finite reduced rank, the \( N \)-injective property is characterized. Moreover, we determine the rings \( R \) for which every nonsingular cyclic \( R \)-module is injective. Recall that a ring \( R \) is right \( t \)-semisimple if and only if \( R/Z_2(R_R) \) is a semisimple ring.

**Theorem 3.1.** The following statements are equivalent for a ring \( R \).

1. Every free (projective) \( R \)-module is \( N \)-injective.
2. Every cyclic \( R \)-module is \( N \)-injective.
3. Every \( R \)-module is \( N \)-injective.
4. \( R \) is right \( t \)-semisimple.
5. \( R^{(S)} \) is \( N \)-injective.
6. \( [R/Z_2(R_R)]^{(S)} \) is an injective \( R/Z_2(R_R) \)-module.

**Proof.** (1) \( \Rightarrow \) (4). Let \( [R/Z_2(R_R)]^{(A)} \) be a free \( R/Z_2(R_R) \)-module. Since \( Z_2(R^{(A)}) = Z_2(R_R)^{(A)} \) we conclude that \( [R/Z_2(R_R)]^{(A)} \cong R^{(A)}/Z_2(R_R)^{(A)} \).

Hence by hypothesis and Theorem 2.2(5), the module \( [R/Z_2(R_R)]^{(A)} \) is an injective \( R \)-module, and so it is an injective \( R/Z_2(R_R) \)-module. Thus \( R/Z_2(R_R) \) is a right \( \Sigma \)-injective ring, and so it is quasi-Frobenius by [4, 18.1]. On the other hand, \( R/Z_2(R_R) \) is a right nonsingular ring. Thus by [3, Corollary 4.6], \( R/Z_2(R_R) \) is a semisimple ring.
Let $M$ be a cyclic $R/Z_2(R_R)$-module. Then $M$ is a cyclic $R$-module, and so by hypothesis, $M$ is $R/Z_2(R_R)$-injective. Hence $M$ is an injective $R/Z_2(R_R)$-module. Thus $R/Z_2(R_R)$ is a semisimple ring.

(4) $\Rightarrow$ (3). Assume that $B$ and $M$ are $R$-modules, and $A$ is a nonsingular submodule of $B$. By [3, Theorem 3.2(4)], $A$ is a direct summand of $B$. So clearly, every $R$-homomorphism $g : A \to M$ can be extended to an $R$-homomorphism $h : B \to M$. Thus by Theorem 2.2(6), $M$ is $N$-injective.

(3) $\Rightarrow$ (1), (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (5). These implications are obvious.

(5) $\Rightarrow$ (6). Clearly, $l_{R}(Z_2(R_R)) = [l_{R}(Z_2(R_R))]$. Thus by Theorem 2.2(4), $[l_{R}(Z_2(R_R))]$ is an injective $R/Z_2(R_R)$-module.

(6) $\Rightarrow$ (1). Let $R^{(A)}$ be a free $R$-module. By hypothesis, $[l_{R}(Z_2(R_R))]$ is an injective $R/Z_2(R_R)$-module. Thus by [1, Theorem 25.1], $[l_{R}(Z_2(R_R))]^{(A)}$ is an injective $R/Z_2(R_R)$-module. So by Theorem 2.2(4), $R^{(A)}$ is $N$-injective. □

A ring $R$ is called semilocal if $R/Rad(R)$ is semisimple. Semiperfect rings (hence right and left perfect rings, semiprimary rings, right and left Artinian rings, and local rings) are semilocal. The next result determines the $N$-injective semilocal rings. Moreover, by Corollary 2.10, if $R$ is $N$-injective, then $Rad(R) \leq Z_2(R_R)$. The converse implication is not necessarily true even though $R$ is right Noetherian; e.g., $R = Z$. The next result shows that the converse implication holds for semilocal rings.

**Corollary 3.2.** Let $R$ be a semilocal ring. The following statements are equivalent.

1. $R$ is $N$-injective.
2. $R$ is right $t$-semisimple.
3. $Rad(R) \leq Z_2(R_R)$.

If $R$ is local, the above statements are equivalent to

4. $R$ is right $Z_2$-torsion.

**Proof.** (3) $\Rightarrow$ (2). If $R$ is semilocal, then $R/Rad(R)$ is semisimple. Thus by hypothesis, $R/Z_2(R_R)$ is semisimple, and so $R$ is right $t$-semisimple.

(2) $\Rightarrow$ (1). This follows from Theorem 3.1.

(4) $\Rightarrow$ (2). This is clear by [3, Theorem 2.3].

Now assume that $R$ is a local ring. We show that (3) $\Rightarrow$ (4). Since $R$ is local, $Rad(R)$ is essential in $R$. So by [2, Proposition 2.2(4)], $R/Rad(R)$ is $Z_2$-torsion. Moreover, by hypothesis, $Rad(R)$ is $Z_2$-torsion. Therefore $R$ is right $Z_2$-torsion. □

Recall that a ring $R$ is called quasi-Frobenius if $R$ is right (or left) Artinian and right (or left) self-injective.

**Corollary 3.3.** A ring $R$ is quasi-Frobenius if and only if $R$ is right $t$-semisimple and $R^{(N)}$ is $Z_2(R_R)$-injective.

**Proof.** ($\Rightarrow$) Since $R^{(N)}$ is injective, it is $Z_2(R_R)$-injective. Moreover, by [3, Proposition 4.5], $R$ is right $t$-semisimple.
By Theorems 3.1(3) and 2.2(5), \( Z_2(R_R) \) is a direct summand of \( R \). Moreover, by Theorem 3.1(5), \( R^{(5)} \) is \( R/Z_2(R_R) \)-injective. Thus by hypothesis, \( R^{(5)} \) is \( R \)-injective, so \( R^{(5)} \) is injective. Hence \( R \) is quasi-Frobenius by [4, 18.1(b)] and [1, Theorem 25.1]. □

Recall that \( R \) is called a right pseudo-Frobenius ring if \( R \) is an injective cogenerator in \( \text{Mod-} R \). Every quasi-Frobenius ring is right pseudo-Frobenius; see [9, Theorem 19.25]. The next result shows that a right pseudo-Frobenius ring for which the second singular ideal is Noetherian is quasi-Frobenius.

**Corollary 3.4.** Let \( R \) be a ring.

1. If \( R \) is right pseudo-Frobenius, then \( R \) is right \( t \)-semisimple.
2. \( R \) is quasi-Frobenius if and only if \( R \) is right pseudo-Frobenius and \( Z_2(R_R) \) is Noetherian (Artinian).
3. \( R \) is quasi-Frobenius if and only if \( R \) is right Kasch and \( Z_2(R_R) \) is injective and Noetherian (Artinian).

**Proof.** (1) Since \( R \) is right pseudo-Frobenius, \( R \) is right self-injective and semi-perfect. Hence Corollary 3.2 implies that \( R \) is right \( t \)-semisimple.

(2) Let \( R \) be right pseudo-Frobenius and \( Z_2(R_R) \) be Noetherian (Artinian). By (1), \( R \) is right \( t \)-semisimple, and so \( R/Z_2(R_R) \) is Noetherian (Artinian). Thus \( R \) is Noetherian (Artinian), and hence \( R \) is quasi-Frobenius. The converse is clear.

(3) Let \( R \) be quasi-Frobenius. Then \( Z_2(R_R) \) is injective and Noetherian (Artinian). Moreover, \( R \) is right pseudo-Frobenius, and so by [9, Theorem 19.25], \( R \) is right Kasch. The converse implication follows from [15, Theorem 5] and (2). □

**Remark 3.5.** (i) The endomorphism ring of an \( N \)-injective module has a von Neumann regular factor ring (Theorem 2.9), but itself is not necessarily von Neumann regular. In fact, by Theorem 3.1(5) and [10, Proposition 2.17], if \( R \) is a right \( t \)-semisimple ring which is not semisimple, then \( R^{(5)} \) is \( N \)-injective and \( \text{End}(R^{(5)}) \) is not von Neumann regular.

(ii) Recall that every injective \( R \)-module is projective if and only if every projective \( R \)-module is injective (and these are equivalent to \( R \) being quasi-Frobenius). However, Corollary 2.8 and Theorem 3.1 show that this equivalence does not hold if we replace injective by \( N \)-injective.

**Proposition 3.6.** The following statements are equivalent for a ring \( R \).

1. \( R \) is \( N \)-injective.
2. \( Z_2(R_R) \) is \( R/Z_2(R_R) \)-injective and every finitely generated (cyclic) nonsingular \( R \)-module is injective and projective.

**Proof.** (1) \( \Rightarrow \) (2). By Theorem 2.2(5), \( Z_2(R_R) \) is \( R/Z_2(R_R) \)-injective. Let \( M \) be a finitely generated nonsingular \( R \)-module. There exists a finitely generated free \( R \)-module \( F \) such that \( M \cong F/C \) for some submodule \( C \) of \( F \). By [2, Proposition 2.6(6)], \( C \) is a \( t \)-closed submodule of \( F \). On the other hand, \( F \) is
$\mathcal{N}$-injective, and so by Corollary 2.3(ii), $F$ is $t$-extending. Thus $C$ is a direct summand of $F$, and so $M$ is isomorphic to a direct summand of $F$. This implies that $M$ is projective and $\mathcal{N}$-injective which implies that $M$ is injective by Corollary 2.3(i).

(2) $\Rightarrow$ (1). By Theorem 2.2(2), $Z_2(R_R)$ is $\mathcal{N}$-injective. Since $R/Z_2(R_R)$ is projective by hypothesis, $Z_2(R_R)$ is a direct summand of $R$, say $R = Z_2(R_R) \oplus R'$. But, $R' \cong R/Z_2(R_R)$ is injective by hypothesis, and so by Theorem 2.2(5), $R$ is $\mathcal{N}$-injective. □

The following result characterizes the rings over which every cyclic (finitely generated) nonsingular module is injective. Moreover, this result determines that when a right extending ring is $\mathcal{N}$-injective.

**Theorem 3.7.** The following statements are equivalent for a ring $R$.

(1) Every cyclic (finitely generated) nonsingular $R$-module is injective.

(2) $R/Z_2(R_R)$ is a right self-injective ring.

If $R$ is right extending, then the above statements are equivalent to

(3) $R$ is $\mathcal{N}$-injective.

**Proof.** (1) $\Rightarrow$ (2). By hypothesis, $R/Z_2(R_R)$ is an injective $R$-module, and hence, a right self-injective ring.

(2) $\Rightarrow$ (1). Let $M$ be a finitely generated nonsingular $R$-module. Then $M$ is a finitely generated nonsingular $R/Z_2(R_R)$-module. But, $R/Z_2(R_R)$ is a right self-injective ring, and by Proposition 3.6, every finitely generated nonsingular module over a right self-injective ring is injective. So $M$ is an injective $R/Z_2(R_R)$-module. Therefore Baer’s criterion implies that $M$ is an injective $R$-module.

(3) $\Rightarrow$ (1). This follows from Proposition 3.6.

Now assume that $R$ is right extending. We show that (1) $\Rightarrow$ (3). Since $R$ is right extending, $Z_2(R_R)$ is a direct summand of $R$, say $R = Z_2(R_R) \oplus R'$. By [4, 7.11], $Z_2(R_R)$ is $R'$-injective. Hence $Z_2(R_R)$ is $R/Z_2(R_R)$-injective. On the other hand, $R'$ is injective since $R'$ is a cyclic nonsingular $R$-module. Thus by [2, Theorem 2.11(3)], $R^{(a)} = Z_2(R_R)^{(a)} \oplus R'^{(a)}$ is $t$-extending. So by hypothesis and [2, Remark 3.14], every finitely generated nonsingular $R$-module is injective and projective. Thus by Proposition 3.6, $R$ is $\mathcal{N}$-injective. □

A ring $R$ is called of finite (Goldie) reduced rank if the uniform dimension of $R/Z_2(R_R)$ is finite. Every ring of finite uniform dimension is of finite reduced rank; see [9, (7.35)].

**Proposition 3.8.** The following statements are equivalent for a ring $R$ of finite reduced rank.

(1) $R$ is $\mathcal{N}$-injective.

(2) $R$ is right $t$-semisimple.

(3) Every nonsingular principal right ideal of $R$ is injective.

(4) Every nonsingular principal right ideal of $R$ is a direct summand.
Proof. The implication (2) ⇒ (1) follows from Theorem 3.1, the implication (1) ⇒ (3) follows from Proposition 3.6, and the implication (3) ⇒ (4) is clear.

(4) ⇒ (2). By [3, Theorem 2.3(4)], it suffices to show that a nonsingular right ideal \( K \) of \( R \) is a direct summand. Since \( R \) is of finite reduced rank, so is \( K \). Hence \( K \) is of finite uniform dimension as it is nonsingular. Thus by [9, Proposition (6.30)'] and [1, Proposition 10.14], \( K \) is a finite direct sum of indecomposable right ideals. So by hypothesis, \( K \) is a finite direct sum of minimal right ideals, say \( K = a_1R ⊕ a_2R ⊕ \cdots ⊕ a_nR \). If \( n = 1 \), then \( K \) is a direct summand of \( R \). Let \( n > 1 \). By induction, assume that \( a_2R ⊕ \cdots ⊕ a_nR = eR \) for some idempotent \( e \in R \). Since \( (1 - e)a_1R \) is a submodule of \( K \), it is nonsingular. Hence by hypothesis, \( (1 - e)a_1R = e'R \) for some idempotent \( e' \in R \). However, \( K = eR + e'R \) and \( ee' = 0 \). Therefore \( e'' = e + e' - e'e \) is an idempotent and \( K = e''R \) is a direct summand of \( R \), as desired.

Following [2], a ring \( R \) is called right \( \Sigma \)-t-extending if every free \( R \)-module is \( t \)-extending.

Corollary 3.9. A ring \( R \) is right \( t \)-semisimple if and only if \( R \) is \( N \)-injective and right \( \Sigma \)-t-extending.

Proof. (⇒) This follows from Theorem 3.1 and [3, Corollary 3.6].

(⇐) Let \( R^{(\lambda)} \) be a free \( R \)-module. By [2, Theorem 2.11(3)], \( [R/Z_2(R_R)]^{(\lambda)} \cong R^{(\lambda)}/Z_2(R^{(\lambda)}) \) is an extending \( R \)-module. Thus \( [R/Z_2(R_R)]^{(\lambda)} \) is an extending \( R/Z_2(R_R) \)-module. So \( R/Z_2(R_R) \) is a right \( \Sigma \)-extending ring. Thus by [4, 12.21((d) ⇔ (e))], \( R/Z_2(R_R) \) is an Artinian ring. So \( R \) is of finite reduced rank. Thus by Proposition 3.8, \( R \) is right \( t \)-semisimple.

Our last result shows that a ring \( R \) for which every nonsingular cyclic \( R \)-module is injective is precisely a right \( t \)-semisimple ring, whenever \( R \) is either semilocal or of finite reduced rank; see [3, Example 4.15].

Corollary 3.10. Let \( R \) be a ring which is either semilocal or of finite reduced rank. Then every cyclic (finitely generated) nonsingular \( R \)-module is injective if and only if \( R \) is right \( t \)-semisimple.

Proof. The implication (⇐) is obtained by [3, Theorem 3.2(4)]. For (⇒), set \( \overline{R} = R/Z_2(R_R) \). By Theorem 3.7, \( \overline{R} \) is right self-injective. So \( \text{Rad}(\overline{R}) \leq Z_2(\overline{R}) \) by Corollary 2.10(ii). But \( Z_2(\overline{R}) = 0 \), hence \( \text{Rad}(\overline{R}) \leq Z_2(R_R) \). Moreover, \( \overline{R} \) is von Neumann regular by Corollary 2.10(i). So by [3, Lemma 4.12], every nonsingular cyclic right ideal of \( R \) is a direct summand. Thus Corollary 3.2(3) and Proposition 3.8(4) imply that \( R \) is right \( t \)-semisimple.

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