Next-to-Leading-Order Staggered Chiral Perturbation Theory *

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We have extended staggered chiral perturbation theory (\chiPT) to \ord{a^2 p^2}, \ord{a^4}, and \ord{a^2 m}, the orders necessary for a full next-to-leading order (NLO) calculation of pseudo-Goldstone boson (PGB) masses and decay constants including taste-symmetry violations. We present predictions relating \so{4} taste-breaking splittings in masses, pseudoscalar decay constants, and dispersion relations. These can be used to test the $\sqrt{\det}$ trick.

1. Introduction

Lattice simulations with staggered fermions\cite{1} are fast compared to those with other standard fermion discretizations\cite{2}. They therefore allow QCD calculations on a reasonable time scale with light dynamical quarks. In addition to this computational advantage, staggered fermions also possess a remnant $U(1)$ chiral symmetry, even at zero lattice spacing. Their primary disadvantage is that each continuum staggered fermion flavor comes in four degenerate tastes, and the continuum $SU(4)$ flavor symmetry associated with each flavor is almost completely broken by the lattice, leaving only a discrete subgroup. While taste-breaking effects only enter at \ord{a^2}, they turn out to be important numerically at current lattice spacings\cite{3}.

In order to extract physical quantities from lattice simulations, one must perform both continuum and chiral extrapolations using functional forms determined in chiral perturbation theory\cite{4}. Thus, forms applied to staggered simulations must account for taste violations. Near the continuum limit, where there is a controlled expansion in powers of the lattice spacing, \chiPT systematically describes taste violations in the PGB sector. Reference \cite{5} determined the \ord{a^2} staggered potential for a single flavor and showed that even at tree-level it correctly predicts the mass degeneracy pattern observed among the different tastes of PGBs\cite{6,7,8,9}. References \cite{10} and \cite{11} extended the potential to multiple staggered flavors and calculated the 1-loop mass and decay constant for the taste Goldstone ($\xi_5$) meson. However, it is well known from continuum \chiPT that one must work to at least NLO for adequate chiral extrapolation of lattice data\cite{12}. Thus, the utility of \chiPT is limited without higher-order operators. We have therefore determined all linearly-independent \ord{a^2 p^2}, \ord{a^4}, and \ord{a^2 m} operators in the staggered chiral Lagrangian for a general multiple-flavor, partially quenched theory, including source terms for left- and right-handed currents and scalar and pseudoscalar densities.\footnote{The full list of NLO taste-breaking operators is given elsewhere\cite{13}.} The full NLO staggered chiral Lagrangian consists of these operators, which incorporate discretization effects, plus the standard continuum \ord{p^4}, \ord{p^2 m}, and \ord{m^2} operators.

2. NLO Staggered Chiral Lagrangian

Determination of the staggered chiral Lagrangian requires two steps. First one constructs the quark level effective action, including explicit nonzero $a$ effects. Following the method of Symanzik\cite{14}, \sff consists of all local continuum lattice operators of dimension $\leq 6$ invariant under the lattice symmetries. In practice it is easiest to first determine all allowed lattice operators of dimension $\leq 6$, as in Refs. \cite{15,16}, and then map these onto continuum quark-level operators\cite{15}. Next one maps the continuum quark-level operators onto chiral operators describing the PGB sector. To be most general, we consider the partially quenched (PQ) theory – unquenched QCD can be recovered by taking an appropriate limit. \chiPT uses the following power counting scheme: $p^2/\lqcd^2 \approx m/\lqcd \approx a^2 \lqcd^2$, which applies to current staggered lattice simulations. Thus the LO staggered potential is of \ord{a^2}\cite{10,11}, while...
the NLO taste-violating operators in the staggered chiral Lagrangian are of $O(a^2 p^2)$, $O(a^4)$, and $O(a^2 m)$ [13].

The NLO chiral operators can be divided into two types according to how badly they break the continuum $SU(4)$ taste symmetry. Those of type A are rotationally invariant and preserve an $SO(4)$ subgroup of the continuum $SU(4)$ taste symmetry. In contrast, type B operators are only invariant under certain combined spin and taste rotations, and maximally break the continuum symmetry down to the lattice symmetry group. Because all operators in the LO chiral Lagrangian are of type A, the tree-level PGB masses are split into irreps of $SO(4)$-taste: $\xi_I$, $\xi_5$, $\xi_\mu$, $\xi_\bar{\mu}$, and $\xi_{\mu\nu}$. However, analytic NLO contributions due solely to the B-type operators can cause further mass and decay constant splittings among the tastes.

3. NLO Relations for PGB Properties

Because there are 234 new NLO taste-breaking operators, determining all of their coefficients does not seem possible. Nevertheless, there is one set of quantities to which only a few contribute and for which predictions are possible — those which are nonzero only because $SO(4)$-taste symmetry is broken at NLO.

We first consider $SO(4)$-taste and rotational symmetry violation in the pion dispersion relations. Only eight $O(a^2 p^2)$ operators contribute:\footnote{Operators of $O(a^2 m)$ only occur in type A, while type B $O(a^4)$ operators contract into $SO(4)$-invariant ones for two-PGB processes.}

$$a^2 \sum_\mu \sum_{\nu \neq \mu} \left\{ C_2 \text{Str}(\bar{\partial}_\mu \Sigma^\dagger \xi_{\mu\nu} \partial_\mu \Sigma \xi_{\nu\mu}) \\ + C_7 \text{Str}(\Sigma \partial_\mu \Sigma^\dagger \xi_{\mu\nu}) \text{Str}(\Sigma^\dagger \partial_\mu \Sigma \xi_{\nu\mu}) \\ + C_{10} \left[ \text{Str}(\Sigma \partial_\mu \Sigma^\dagger \xi_{\mu\nu} \partial_\mu \Sigma \xi_{\nu\mu}) + \text{p.c.} \right] \\ + C_{13} \left[ \text{Str}(\Sigma \partial_\mu \Sigma^\dagger \xi_{\mu\nu} \partial_\mu \Sigma \xi_{\nu\mu}) + \text{p.c.} \right] \right\} \\ + a^2 \sum_\mu \left\{ C_{44V} \text{Str}(\partial_\mu \Sigma^\dagger \xi_{\mu\nu} \partial_\mu \Sigma \xi_{\nu\mu}) + \text{p.c.} \right\} \\ + C_{44A} \left[ \text{Str}(\partial_\mu \Sigma^\dagger \xi_{\mu\nu} \partial_\mu \Sigma \xi_{\nu\mu}) + \text{p.c.} \right] \\ + C_{47V} \left[ \text{Str}(\partial_\mu \Sigma^\dagger \xi_{\mu\nu}) \right]^2 + \text{p.c.} \\ + C_{47A} \left[ \text{Str}(\partial_\mu \Sigma^\dagger \xi_{\mu\nu}) \text{Str}(\partial_\mu \Sigma \xi_{\nu\mu}) + \text{p.c.} \right] \right\} \right\} \right\}. \tag{1}

One can see that these operators break $SO(4)$-taste and Euclidean rotational symmetry from the fact that they have more than two repeated indices. This property is common to all B-type operators in the staggered chiral Lagrangian. Note that four of these operators contain two supertraces; they produce hairpin correlators which only contribute to properties of flavor-singlet PGBs. Taste $\xi_{\mu\nu}$ and $\xi_5$ hairpins were not present in the LO Lagrangian, but do occur at NLO.

As expected, we observe rotational symmetry violation in the pion dispersion relations:\footnote{Dispersion relations for tastes 1 and 5 are Euclidean rotation invariant at this order.}

$$E_k^2 = (p_i^2 + p_j^2) + p_k^2 \left( 1 + \delta_k - \delta_4 \right) + m_{\mu}^2 \left( 1 + \delta_k \right),$$

$$E_k^4 = \tilde{p}^2 \left( 1 + \delta_4 - \delta_k \right) + m_{\mu}^2 \left( 1 + \delta_4 \right),$$

$$E_{k5}^2 = (p_i^2 + p_j^2) + p_k^2 \left( 1 + \delta_k - \delta_4 \right) + m_{\mu\nu}^2 \left( 1 + \delta_k \right),$$

$$E_{k5}^4 = \tilde{p}^2 \left( 1 + \delta_4 - \delta_k \right) + m_{\mu\nu}^2 \left( 1 + \delta_4 \right),$$

$$E_{lm}^2 = (p_i^2 + p_j^2) \left( 1 + \delta_{lm} - \delta_4 \right) + p_k^2 + m_{\mu\nu}^2 \left( 1 + \delta_{lm} \right),$$

$$E_{k4}^2 = (p_i^2 + p_j^2) \left( 1 + \delta_k - \delta_4 \right) + p_k^2 + m_{\mu\nu}^2 \left( 1 + \delta_k \right). \tag{2}$$

Here $E_F$ is the energy of a taste $F$ pion extracted from the exponential fall-off of the 2-point function along the Euclidean time ($4$) direction. The masses, $m_F$, are the full NLO masses excluding the contributions from the type B operators enumerated above, so they fall into irreps of $SO(4)$. The corrections, $\delta F = \delta m_F^2 / m_F^2$, arise from the additional $SO(4)$-breaking operators, and further split the tastes into irreps of the $\text{timeslice group}$, the rest frame symmetry group of the staggered action [14]. As this is the maximum possible splitting among tastes on a lattice with zero spatial momentum, these degeneracy classes hold to all orders in $\text{S}^\text{PT}$. A testable prediction can be made in terms of mass and energy splittings among the $SO(4)$ irreps $\xi_I$, $\xi_{\mu\bar{\mu}}$, and $\xi_{\mu\nu}$:

$$\frac{E_k^2 - E_4^2}{m_k^2 - m_4^2} = 1 + \frac{p_i^2 + p_j^2 + 2p_k^2}{(m_k^2 + m_4^2)^2},$$

$$\frac{E_{k5}^2 - E_{45}^2}{m_{k5}^2 - m_{45}^2} = 1 + \frac{p_i^2 + p_j^2 + 2p_k^2}{(m_{k5}^2 + m_{45}^2)^2}.$$
\[
\frac{E_{lm}^2 - E_{k4}^2}{m_{lm}^2 - m_{k4}^2} = 1 + 2 \left( \frac{p_l^2 + p_m^2}{m_{lm}^2 + m_{k4}^2} \right)^2.
\]

These are now the full NLO masses which are directly measurable on the lattice.

The only Euclidean rotational and SO(4) taste symmetry-violating contributions to both PGB masses and pseudoscalar decay constants are from the same \( \mathcal{O}(a^3 p^2) \) operators, through wavefunction renormalization. Thus, the splittings within SO(4) irreps in these two quantities are related:

\[
\begin{align*}
\frac{f_{lm}^P - f_{k4}^P}{f_{lm}^P + f_{k4}^P} & = \frac{1}{2} \left( \frac{2m_{lm}^2 - m_{k4}^2}{m_{lm}^2 + m_{k4}^2} \right), \\
\frac{f_{lm}^P - f_{k4}^P}{f_{lm}^P + f_{k4}^P} & = \frac{1}{2} \left( \frac{2m_{lm}^2 - m_{k4}^2}{m_{lm}^2 + m_{k4}^2} \right), \\
\frac{f_{lm}^P - f_{k4}^P}{f_{lm}^P + f_{k4}^P} & = \frac{1}{2} \left( \frac{2m_{lm}^2 - m_{k4}^2}{m_{lm}^2 + m_{k4}^2} \right).
\end{align*}
\]

This is the simplest prediction of NLO S\( \chi \)PT that one can test on the lattice. It is essential for these expressions that the \( Z \)-factors are SO(4)-invariant, and therefore identical for both tastes in the expression. Thus they can be tested using bare lattice operators, thereby avoiding \( \mathcal{O}(a^3) \) ambiguities in matching lattice and continuum operators that could destroy the relationship. Because, in the \( \delta s \), there are three independent coefficients, and there are three splittings, there are no predictions relating mass splittings among different SO(4) irreps.

We note that, in the talk at Lattice 2004, we presented a different prediction involving the axial decay constant:

\[
\frac{f_A^A - f_A^{A \nu}}{f_A^A + f_A^{A \nu}} = -\left( \frac{m_{lm}^2 - m_{k4}^2}{m_{lm}^2 + m_{k4}^2} \right),
\]

and so forth. We now realize that this is incorrect for two reasons. First, it does not properly take into account wavefunction renormalization when calculating \( f_A^A \). Moreover, even if one correctly calculates the contributions from the listed \( \mathcal{O}(a^3 p^2) \) operators, this relationship is broken by additional operators containing sources for left- and right-handed currents. These operators, in which the covariant derivative acts on a taste spurion, rather than a \( \Sigma \)-field, affect the axial current, but not the masses or pseudoscalar density.

4. Conclusion

We have used NLO S\( \chi \)PT to predict quantitative relationships between PGB masses, decay constants, and dispersion relations. We emphasize that, although the underlying lattice symmetries tell us that the approximate degeneracies among the SO(4) irreps must be split into the true lattice irreps, they do not predict any particular relationships among the SO(4) splittings in various PGB properties. These come strictly from S\( \chi \)PT. These relationships therefore provide a simple test of whether the effective field theory applies at all to the lattice data, which is based on the assumption that the underlying lattice “theory” using the \( \sqrt{\text{Det}} \) trick is not sick. Testing them will provide empirical evidence either for or against the validity of this trick.

REFERENCES

1. L. Susskind, Phys. Rev. D 16 (1977) 3031.
2. C. Bernard et al., Nucl. Phys. Proc. Suppl. 106, 199 (2002).
3. C. T. H. Davies et al. [HPQCD Collaboration], Phys. Rev. Lett. 92, 022001 (2004).
4. C. Aubin et al., arXiv:hep-lat/0407028.
5. C. Aubin et al. [MILC Collaboration], arXiv:hep-lat/0407028.
6. W. Lee and S. Sharpe, Phys. Rev. D 60, 114503 (1999).
7. N. Ishizuka et al., Nucl. Phys. B411 (1994) 875.
8. S. Aoki et al., Phys. Rev. D 62, 094501 (2000).
9. K. Orginos et al., Phys. Rev. D 59, 014501 (1999).
10. C. Aubin and C. Bernard, Phys. Rev. D 68, 034014 (2003).
11. C. Aubin and C. Bernard, Phys. Rev. D 68, 074014 (2003).
12. C. Bernard et al., Nucl. Phys. Proc. Suppl. 119, 170 (2003).
13. S. R. Sharpe and R. S. Van de Water, in preparation.
14. K. Symanzik, Nucl. Phys. B226 (1983) 187, 205.
15. Y. Luo, Phys. Rev. D 57, 1998.
16. M. Golterman, Nucl. Phys. B273, 663 (1986).