Effective potential for classical field theories subject to stochastic noise

David Hochberg
Carmen Molina-París
Juan Páez-Mercader
Matt Visser

Laboratorio de Astrofísica Espacial y Física Fundamental, Apartado 50727, 28080 Madrid, Spain
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA
Physics Department, Washington University, Saint Louis, Missouri 63130-4899, USA

Centro de Astrobiología, INTA, Ctra. Ajalvir, Km. 4, 28850 Torrejón Madrid, Spain

7 April 1999; 18 June 1999; 28 November 2000; LATEX-ed August 13, 2018

Classical field theories coupled to stochastic noise provide an extremely powerful tool for modeling phenomena as diverse as turbulence, pattern-formation, and the structural development of the universe itself. In this Letter we sketch out the field theory via the most direct route, avoiding the Martin–Siggia–Rose construction (with its extra unphysical conjugate fields used for book-keeping purposes). We will assume that the partial differential equation, plus initial and boundary conditions, is well-posed. Given a particular realization of the noise, the differential equation is assumed to have a unique solution \( \phi_{\text{soln}}(x,t|\eta) \). For any function \( Q(\phi) \) of the field \( \phi \) we can define the stochastic average, (over the noise), as

\[
\langle Q(\phi) \rangle \equiv \int (D\eta) \ P[\eta] \ Q(\phi_{\text{soln}}(x,t|\eta)).
\]

Here \( P[\eta] \) is the probability density functional of the noise. It is normalized to 1, but is otherwise completely arbitrary. We next use a functional delta-function to write the following identity

\[
\phi_{\text{soln}}(x,t|\eta) \equiv \int (D\phi) \ \phi \ \delta[\phi - \phi_{\text{soln}}(x,t|\eta)]
= \int (D\phi) \ \phi \ \delta[D\phi - F[\phi] - \eta] \ \sqrt{J} J^\dagger.
\]

The Jacobian functional determinant, \( J \), is defined by

\[
J \equiv \det \left( D - \frac{\delta F}{\delta \phi} \right).
\]

The above is the functional analogue of a standard delta-function result. If \( f(x) = 0 \) has a unique solution at \( x = x_0 \), then \( x_0 = \int dx \ x \ \delta(x-x_0) = \int dx \ x \ \delta(f(x)) |f'(x)| = \int dx \ x \ \delta(f(x)) \sqrt{f'(x)[f'(x)]^*} \). It is easy to see that one also has the identity

\[
Q(\phi_{\text{soln}}(x,t|\eta)) \equiv \int (D\phi) \ Q(\phi) \ \delta[D\phi - F[\phi] - \eta] \ \sqrt{J} J^\dagger.
\]

The ensemble average over the noise becomes

\[
\langle Q(\phi) \rangle = \int (D\eta) \ P[\eta] \times \int (D\phi) \ Q(\phi) \ \delta[D\phi - F[\phi] - \eta] \ \sqrt{J} J^\dagger.
\]
The noise integral is easy to perform, with the result that
\[ \langle Q(\phi) \rangle = \int (D\phi) \, P[D\phi - F[\phi]] \, Q(\phi) \sqrt{\mathcal{J}\mathcal{J}^\dagger}. \quad (7) \]

The effect of the noise appears in the stochastic average only through its probability distribution \( P[D\phi - F[\phi]] \). The presence of the functional determinant is essential; it is often discarded without comment in the literature, but in general it must be kept to ensure proper counting of the solutions to the original SPDE. A particularly useful quantity is the characteristic functional. With an obvious condensed notation, \( dx = d^d\vec{x} \, dt \), we define
\[ Z[J] = \left\langle \exp \left( \int J(x) \phi(x) \, dx \right) \right\rangle \]
\[ = \int (D\phi) \, P[D\phi - F[\phi]] \exp \left( \int J \, \phi \, dx \right) \sqrt{\mathcal{J}\mathcal{J}^\dagger}. \quad (8) \]

When there is no risk of confusion we will suppress the \( dx \) completely. This key result will enable us to calculate the effective action and the effective potential in a direct way. At this stage we make some assumptions about the noise: We assume the noise is Gaussian with zero mean (so that the only non-vanishing cumulant is the second order one). If the noise has a non-zero mean, one can always redefine the forcing term \( F[\phi] \) to make the noise be of mean zero. We still do not need to assume translation invariance, nor do we need to assume the noise is white, power-law, or colored. Arbitrary Gaussian noise is enough since it allows us to write the noise probability distribution as
\[ P[\eta] = \frac{1}{\sqrt{\text{det}(2\pi G_\eta)}} \exp \left[ -\frac{1}{2} \int dx \, dy \, \eta(x) \, G_\eta^{-1}(x,y) \, \eta(y) \right]. \quad (9) \]

The characteristic function is thus
\[ Z[J] = \frac{1}{\sqrt{\text{det}(2\pi G_\eta)}} \int (D\phi) \sqrt{\mathcal{J}\mathcal{J}^\dagger} \exp \left( \int J \phi \right) \]
\[ \exp \left[ -\frac{1}{2} \int (D\phi - F[\phi])G_\eta^{-1}(D\phi - F[\phi]) \right]. \quad (10) \]

This characteristic function contains all the physics describable by \( Z[J] \). The noise \( \eta \) has been completely eliminated and survives only through the explicit appearance of its two-point function \( G_\eta \). Since the characteristic function is now given as a functional integral over the physical field \( \phi \), all the standard machinery of statistical and quantum field theory can be brought to bear. This formula for the characteristic function demonstrates that (modulo Jacobian determinants) all of the physics of any stochastic differential equation can be extracted from a functional integral based on the “classical action” (note: this is not yet the effective action)
\[ S[\phi] = \frac{1}{2} \int (D\phi - F[\phi])G_\eta^{-1}(D\phi - F[\phi]) \quad (11) \]

This action \( S[\phi] \) can be viewed as a generalization of the Onsager–Machlup action \( \mathcal{J} \). Onsager and Machlup dealt with stochastic mechanics rather than field theory, and limited attention to white noise. As they developed their formalism with the notions of linear response theory in mind, they tacitly assumed the “forcing term” \( F[\phi] \) to be linear, so that both the noise and the field fluctuations were Gaussian.

The Jacobian functional determinant is sometimes (but not always) constant. This is a consequence of the causal structure of the theory as embodied in the fact that we are only interested in retarded Green functions. The situation here is in marked contrast to that in QFT, where the relativistic nature of the theory forces the use of Feynman Green functions (+i\( \epsilon \) prescription) \[ \mathcal{J} \]. In order to avoid enumerating special cases, and to have a formalism that can handle both constant and field-dependent Jacobian factors, we exponentiate the determinant via the introduction of Fadeev–Popov ghost fields \[ \mathcal{J} \]
\[ \mathcal{J} = \frac{1}{\text{det}(2\pi I)} \int (D[g, g^\dagger]) \exp \left[ -\frac{1}{2} \int g^\dagger \left( D - \frac{\delta F}{\delta \phi} \right) g \right]. \quad (12) \]

The \( g \) and \( g^\dagger \) fields are known as scalar ghost fields. They are defined in terms of anticommuting complex variables and behave in a manner similar to an ordinary complex scalar field except that there is an extra minus sign for each ghost loop. In all cases we are interested in the operator \( (D - \delta F/\delta \phi) \), even if not self-adjoint, is real and non-singular. This allows us to replace \( \sqrt{\mathcal{J}\mathcal{J}^\dagger} \) by \( \mathcal{J} \), so that the characteristic function \( Z[J] \) can be written as
\[ Z[J] = \frac{1}{\sqrt{\text{det}(2\pi)^d G_\eta)} \int (D[\phi, g, g^\dagger]) \exp \left( \int J \phi \right) \]
\[ \exp \left[ -\frac{1}{2} \int (D\phi - F[\phi])G_\eta^{-1}(D\phi - F[\phi]) \right] \]
\[ \exp \left[ -\frac{1}{2} \int g^\dagger \left( D - \frac{\delta F}{\delta \phi} \right) g \right]. \quad (13) \]

This procedure trades off the functional determinants for two extra functional integrals. In order to develop a Feynman diagram expansion we treat the driving term \( F[\phi] \) as the perturbation and expand around the free-field theory defined by setting \( F = 0 \). There are two propagators, one for the \( \phi \) field, and one for the ghost fields. For simplicity, we will now take the noise to be translation invariant \( \mathcal{J} \). Translation-invariance lets us take simple Fourier transforms in the difference variable.
\[G_{\text{field}}(\vec{k}, \omega) = \frac{G_\eta(\vec{k}, \omega)}{D(-\vec{k}, -\omega) D(\vec{k}, \omega)} \tag{14}\]

\[G_{\text{ghost}}(\vec{k}, \omega) = \frac{1}{D(\vec{k}, \omega)}. \tag{15}\]

The generating function for connected correlation functions is defined by \[^{11,12}\]

\[W[J] = +\mathcal{A} (\ln Z[J] - \ln Z[0]), \tag{22}\]

and the effective action by \[^{11,12}\]

\[\Gamma[\phi; \phi_0] = -W[J] + \int \phi J; \quad \frac{\delta \Gamma[\phi; \phi_0]}{\delta \phi} = J, \tag{23}\]
Here \( S_2 = \frac{\delta^2 S}{\delta \phi(x) \delta \phi(y)} \) is the matrix of second functional derivatives of the action \( S[\phi] \). The notation \( S[\phi_0] \) is shorthand for \( S[\phi(J = 0)] \). When we consider field theories based on SPDEs, the loop-counting parameter \( a \) becomes \( \mathcal{A} \), and the bare action is replaced by \( \mathcal{J} \).

\[
S[\phi] \rightarrow S[\phi] - \frac{1}{2} \mathcal{A} \left( \ln \mathcal{J} + \ln \mathcal{J}^\dagger \right). \tag{26}
\]

The noise at this stage is translation-invariant and Gaussian. Inserting equation (26) into the formula for the one-loop effective action, we obtain the general result (applicable to all SPDEs)

\[
\Gamma[\phi; \phi_0] = S[\phi] + \frac{1}{2} \mathcal{A} \left\{ \ln \det(S_2[\phi]) - \ln \mathcal{J}[\phi] - \ln \mathcal{J}^\dagger[\phi] \right\} - (\phi \rightarrow \phi_0) + O(\mathcal{A}^2). \tag{27}
\]

It is often useful to restrict to constant (spacetime independent) fields. For such field configurations the effective action reduces to the effective potential defined by

\[
\mathcal{V}[\phi; \phi_0] \defeq \frac{\Gamma[\phi; \phi_0]}{\Omega}, \tag{28}
\]

where \( \Omega = \int \frac{d^d \vec{x} \, dt}{(2\pi)^d} \) is the volume of spacetime. It is, in fact, sufficient to have both \( D \phi = 0 \) and \( F[\phi] = \) constant, the reason for this qualification will become clear when we discuss the effective potential for the KPZ equation (32). A straightforward calculation leads to

\[
\mathcal{V}[\phi] = \frac{1}{4} F^2[\phi] + \frac{1}{4} \mathcal{A} \int \frac{d^d \vec{k} \, d\omega}{(2\pi)^d+1} \ln \left[ 1 + \frac{\tilde{g}_2(\vec{k}, \omega) F[\phi]}{\frac{\delta^2 F}{\delta \phi^4}} \right]
\]

\[
\times \frac{\frac{\delta^2 F}{\delta \phi^4}}{\left( D^\dagger(\vec{k}, \omega) - \frac{\delta F}{\delta \phi} \right) \left( D(\vec{k}, \omega) - \frac{\delta F}{\delta \phi} \right)} - (\phi \rightarrow \phi_0) + O(\mathcal{A}^2). \tag{29}
\]

The above formula contains the most important result of this Letter. This result is qualitatively very reminiscent of the effective potential for scalar QFT [12]:

\[
\mathcal{V}[\phi] = V(\phi) + \frac{1}{4} \mathcal{A} \int \frac{d^d \vec{k} \, d\omega}{(2\pi)^d+1} \ln \left[ 1 + \frac{\delta^2 V}{\delta \phi^4} \right]
\]

\[
\times \frac{\frac{\delta^2 V}{\delta \phi^4}}{\omega^2 + \vec{k}^2 + m^2} - (\phi \rightarrow \phi_0) + O(h^2). \tag{30}
\]

The major difference is the fact that the scalar QFT propagator is replaced with a more complicated and typically nonrelativistic propagator. Notice also that for SPDEs one can naturally adapt the noise to be both the source of fluctuations and the regulator to keep the Feynman diagram expansion finite. This follows immediately by inspection of (13) which shows that the (momentum and frequency dependent) noise shape function \( \tilde{g}_2 \) will affect the momentum and frequency behavior of the one-loop integral. The finiteness, divergence structure, and renormalizability of this integral will very much depend on the functional form of \( \tilde{g}_2 \). Renormalizing such infinities on a case by case basis is relatively simple since we are only working to one-loop [3,4].

In this Letter we have shown how to set up a “direct” functional formalism to explicitly calculate an effective action and a corresponding effective potential for a general class of non-linear SPDE’s with additive noise. The effective action and effective potential have the same structural definition as they do in quantum field theory, though here they can be calculated even when the SPDE does not follow from a Hamiltonian or Lagrangian variational principle. At the one-loop level in the noise amplitude we have derived a compact expression for the effective potential reminiscent of, but radically distinct from, the effective potential of scalar QFT. The effective potential for stochastic phenomena provides a powerful tool for exploring the influence of noise on a system, the onset of pattern formation, and noise-induced dynamical symmetry breaking.

We hope to have convinced the reader that the “direct approach” described in this Letter, and developed more fully in [4], is both useful and complementary to the more traditional MSR formalism [4]. Some questions can more profitably be asked and answered in this “direct” formalism. For instance, our general formula for the one-loop effective potential for arbitrary SPDEs subject to translation-invariant Gaussian noise appears difficult to extract from the MSR formalism.

---

Electronic mail: hochberg@laeff.esa.es
Electronic mail: carmen@t6-serv.lanl.gov
Electronic mail: mercader@laeff.esa.es
Electronic mail: visser@kiwi.wustl.edu

[1] P.C. Martin, E.D. Sigia, and H.A. Rose, Phys. Rev. A8 (1973) 423–508.
[2] J. Zinn–Justin, Quantum field theory and critical phenomena (Oxford University Press, Oxford, England, 1996).
[3] U. Frisch, Turbulence (Cambridge University Press, Cambridge, England, 1995).
[4] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. 56 (1986) 889–892.
[5] A. Berera and L-Z. Fang, Phys. Rev. Lett. 72 (1994) 458–461.
[6] D. Hochberg and J. Pérez–Mercader, Gen. Rel. Gravit. 28 (1996) 1427.
[7] T. Goldman, D. Hochberg, R. Laflamme, and J. Pérez–Mercader, Phys. Lett. A222 (1996) 177.
[8] J.F. Barbero, A. Domínguez, T. Goldman, and J. Pérez–Mercader, Europhys. Lett. 38 (1997), 637–642.
[9] D. Hochberg, C. Molina–París, J. Pérez–Mercader, and M. Visser, Effective action for stochastic partial differential equations, Phys. Rev. E60 (1999) 6343–6360.
[10] L. Onsager and S. Machlup, Phys. Rev. 91, 1505 (1953).

[11] S. Weinberg, The quantum theory of fields I,II (Cambridge University Press, Cambridge, England, 1996).

[12] R. J. Rivers, Path integral methods in quantum field theory (Cambridge University Press, Cambridge, England, 1987).

[13] D. Hochberg, C. Molina-París, J. Pérez-Mercader, and M. Visser, Effective potential for the massless KPZ system, Physica A280 (2000) 437-455 [cond-mat/9904391].

[14] D. Hochberg, C. Molina–París, J. Pérez–Mercader, and M. Visser, Effective potential for the reaction-diffusion-decay system, J. Statist. Phys. 99 (2000) 903-941 [cond-mat/9909078].