On the Klein-Gordon G"urses-oscillators and pseudo-G"urses-oscillators: vorticity-energy correlations and spacetime associated degeneracies

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Abstract: We discuss KG-oscillators in the (1+2)-dimensional G"urses spacetime and under position-dependent mass (PDM) settings. We observe that the KG-G"urses oscillators are introduced as a byproduct of the very nature of the G"urses spacetime structure. We report that the energy levels of such KG-G"urses oscillators admit vorticity-energy correlations as well as spacetime associated degeneracies (STAD). We discuss KG-G"urses oscillators’ results reported by Ahmed [14] and pinpoint his improper treatment of this model so that his results should be redirected to those reported in this study. Moreover, we introduce a new set of KG pseudo-G"urses oscillators that admits isospectrality and invariance with the KG-G"urses oscillators and inherits the same vorticity-energy correlations as well as STADs.

PACS numbers: 05.45.-a, 03.50.Kk, 03.65.-w

Keywords: Klein-Gordon oscillators, G"urses spacetime, position-dependent mass, vorticity-energy correlations, spacetime associated degeneracies.

I. INTRODUCTION

Klein-Gordon (KG) and Dirac oscillators [1–4] have received much attention over the years. KG-oscillators in G"odel-type spacetime (e.g., [1–3, 5–8]), in cosmic string spacetime and Kaluza-Klein theory backgrounds (e.g., [9, 10]), in Minkowski spacetime with space-like dislocation [12], in Som-Raychaudhuri [11], in (1+2)-dimensional G"urses spacetime backgrounds (e.g., [13–15]). The KG-oscillators in a (1+2)-dimensional G"urses spacetime described by the metric

\[ ds^2 = -dt^2 + dr^2 - 2\Omega r^2 dt d\theta + r^2 \left(1 - \Omega^2 r^2\right) d\theta^2 = g_{\mu\nu} dx^\mu dx^\nu; \ \mu, \nu = 0, 1, 2, \]

were investigated by Ahmed [14], using \( a_0 = b_0 = e_0 = 1, b_1 = c_0 = \lambda_0 = 0, \) and vorticity \( \Omega = -\mu/3, \) in the G"urses metric

\[ ds^2 = -\phi dt^2 + 2q dt d\theta + \frac{\hbar^2}{a_0} \psi - q^2 d\theta^2 + \frac{1}{\psi} dr^2 \] (2)

(i.e., as in Eq.(5) of [13]) where

\[ \phi = a_0, \ \psi = b_0 + \frac{b_1}{r^2} + \frac{3\lambda_0}{4} r^2, \ q = c_0 + \frac{e_0 \mu}{3} r^2, \ \hbar = e_0 r, \ \lambda_0 = \lambda + \frac{\mu^2}{27}. \] (3)

In this note, we shall show that there are more quantum mechanical features indulged in the spectroscopic structure of the KG-oscillators in the background of such a G"urses spacetime metric [11] than those reported by Ahmed [14], should this model be properly addressed. Throughout this note, such KG-oscillators shall be called KG-G"urses oscillators.

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We organize the current note in the following manner. In section 2, we revisit KG-oscillators in the (1+2)-dimensional Gürses spacetime of and present them in a more general form, that includes position-dependent mass (PDM, which is a metaphoric notion) settings along with Mirza-Mohadesi’s KG-oscillators recipe. We observe that the KG-Gürses oscillators are introduced as a byproduct of the very nature of the Gürses spacetime structure. This motivates us to first elaborate and discuss, in section 3, the effects of Gürses spacetime on the energy levels of the KG-Gürses oscillators, without the KG-oscillator prescription of Mirza-Mohadesi. Therein, we report that such KG-Gürses oscillators admit vorticity-energy correlations as well as spacetime associated degeneracies (STADs). In section 4, we discuss Ahmed’s model that includes Mirza-Mohadesi recipe and pinpoint Ahmed’s improper treatment of the model at hand. We consider the PDM KG-Gürses oscillators in section 5. We discuss and report KG pseudo-Gürses oscillators in section 6, where we observe that they admit isospectrality and invariance with the KG Gürses-oscillators and inherit the same vorticity-energy correlations as well as STADs. Our concluding remarks are given in section 7.

II. KG-GÜRSES OSCILLATORS AND PDM SETTINGS

The covariant and contravariant metric tensors corresponding to the (1+2)-dimensional Gürses spacetime of, respectively, read

\[
\begin{pmatrix} g_{\mu\nu} = \begin{pmatrix} -1 & 0 & -\Omega r^2 \\ 0 & 1 & 0 \\ -\Omega r^2 & 0 & (1 - \Omega^2 r^2) \end{pmatrix} \end{pmatrix} \iff g^{\mu\nu} = \begin{pmatrix} (\Omega^2 r^2 - 1) & 0 & -\Omega \\ 0 & 1 & 0 \\ -\Omega & 0 & 1/r^2 \end{pmatrix}; \ \ \ \det (g_{\mu\nu}) = -r^2. \tag{4}
\]

Then the corresponding KG-equation is given by

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \Psi \right) = m^2 \Psi. \tag{5}
\]

However, we shall now use the momentum operator

\[
p_{\mu} \rightarrow p_{\mu} + iF_{\mu}, \tag{6}
\]

so that it incorporates the KG-oscillator prescription of Mirza-Mohadesi as well as position-dependent mass (PDM) settings proposed by Mustafa. Where \( F_{\mu} = (0, F_r, 0) \) and our \( F_r = \eta r; \ \eta = m\omega \), of and \( F_r = \eta r + g'(r) / 4g(r) \) to also include PDM settings as in Mustafa. This would suggest that Ahmed’s model is retrieved when the positive-valued scalar multiplier \( g(r) = 1 \). Nevertheless, the reader should be aware that the regular momentum operator \( p_{\mu} \) is replaced by the PDM-momentum operator \( p_{\mu} + iF_{\mu} \) to describe PDM KG-particles in general (for more details on this issue the reader is advised to refer to).

Under such assumptions, the KG-equation would transform into

\[
\frac{1}{\sqrt{-g}} \left( \partial_{\mu} + F_{\mu} \right) \left[ \sqrt{-g} g^{\mu\nu} \left( \partial_{\nu} - F_{\nu} \right) \Psi \right] = m^2 \Psi, \tag{7}
\]

which consequently yields

\[
\left\{ -\partial_t^2 + \left( \Omega r \partial_t - \frac{1}{r} \partial_\theta \right)^2 + \partial_r^2 + \frac{1}{r} \partial_r - M(r) - m^2 \right\} \Psi = 0, \tag{8}
\]
where
\[ M(r) = \frac{\mathcal{F}_r}{r} + \mathcal{F}_r' + \mathcal{F}_r^2. \]  
(9)

We now substitute
\[ \Psi(t, r, \theta) = \exp(i [\ell \theta - Et]) \psi(r) = \exp(-i [\ell \theta - Et]) \frac{R(r)}{\sqrt{r}} \]  
(10)
to imply
\[ R''(r) + \left[ \lambda - \frac{(\ell^2 - 1/4)}{r^2} - \tilde{\omega}^2 r^2 - \tilde{M}(r) \right] R(r) = 0, \]  
(11)
where \( \ell = 0, \pm 1, \pm 2, \cdots \) is the magnetic quantum number,
\[ \tilde{M}(r) = -\frac{3}{16} \left( g'(r) \right)^2 + \frac{1}{4} \frac{g''(r)}{g(r)} + \frac{1}{4} \frac{g'(r)}{rg(r)} + \frac{1}{2} \frac{g(r)}{g(r)} \eta r, \]  
(12)
and
\[ \lambda = E^2 - 2 \Omega \ell E - 2 \eta - m^2; \quad \tilde{\omega}^2 = \Omega^2 E^2 + \eta^2. \]  
(13)

It is obvious that we retrieve Ahmed’s model \([14]\) when \( g(r) = 1 \). Moreover, we observe that the KG-Güres oscillators are introduced as a byproduct of the very nature of the Güres spacetime structure. This motivates us to first elaborate and discuss the effects of Güres spacetime on the energy levels of the KG-Güres oscillators, without the KG-oscillator prescription of Mirza-Mohadesi \([4]\) (i.e., with \( \eta = 0 \)).

III. KG-GÜRES OSCILLATORS: VORTICITY-ENERGY CORRELATIONS AND SPACETIME ASSOCIATED DEGENERACIES

It is obvious that KG-Güres oscillators are introduced by the very structure of Güres spacetime. That is, for \( \eta = 0 \), and \( g(r) = 1 \) our KG-equation \([11]\) collapses into the two-dimensional Schrödinger oscillator
\[ R''(r) + \left[ \lambda - \frac{(\ell^2 - 1/4)}{r^2} - \Omega^2 E^2 r^2 \right] R(r) = 0, \]  
(14)
which admits exact textbook solvability so that the eigenvalues and radial eigenfunctions, respectively, read
\[ \lambda = 2 |\Omega E| (2n_r + |\ell| + 1) \]  
(15)
and
\[ R(r) \sim r^{|\ell|+1/2} \exp \left( -\frac{|\Omega E| r^2}{2} \right) L_{n_r}^{|\ell|} \left( |\Omega E| r^2 \right) \iff \psi(r) \sim r^{|\ell|} \exp \left( -\frac{|\Omega E| r^2}{2} \right) L_{n_r}^{|\ell|} \left( |\Omega E| r^2 \right). \]  
(16)
where \( L_{n_r}^{|\ell|} \left( |\Omega E| r^2 \right) \) are the generalized Laguerre polynomials. Now with the help of \([13]\) and \([15]\) we obtain
\[ E^2 - 2 \Omega \ell E - m^2 = 2 |\Omega E| (2n_r + |\ell| + 1). \]  
(17)
This result should be dealt with diligently and rigorously, as mandated by the very nature of \( |\Omega E| = \Omega_{\pm} E_{\pm} \geq 0 \)
FIG. 1: The energy levels for KG-Gürses oscillators of (20) and (21) are plotted \( m = 1 \) (a) for \( n_r = 0, \ell = 0, \pm 1, \pm 2, \) and (b) for \( n_r = 3, \ell = 0, \pm 1, \pm 2, \pm 3. \)

or \( |\Omega E| = -\Omega \pm E_\pm \geq 0 \) (that secures the finiteness and square integrability of the radial wavefunction (16)), where \( \Omega_\pm = \pm |\Omega| \) and \( E_\pm = \pm |E|, \) That is, for \( |\Omega E| = \Omega_\pm E_\pm \) in (17) we obtain

\[
E_\pm^2 - 2 \Omega_\pm E_\pm \tilde{n}_+ - m^2 = 0; \quad \tilde{n}_+ = 2n_r + |\ell| + |\ell| + 1,
\]

(18)

and for \( |\Omega E| = -\Omega_\mp E_\pm \) we get

\[
E_\pm^2 + 2 \Omega_\pm E_\pm \tilde{n}_- - m^2 = 0; \quad \tilde{n}_- = 2n_r + |\ell| - |\ell| + 1.
\]

(19)

Which would allow us to cast

\[
E_\pm = \Omega_\pm \tilde{n}_+ \pm \sqrt{\Omega^2 \tilde{n}_+^2 + m^2} \rightarrow \begin{cases} E_+ = \Omega_\pm \tilde{n}_+ + \sqrt{\Omega^2 \tilde{n}_+^2 + m^2} \\ E_- = \Omega_\pm \tilde{n}_+ - \sqrt{\Omega^2 \tilde{n}_+^2 + m^2} \end{cases},
\]

(20)

for \( |\Omega E| = \Omega_\pm E_\pm \) and

\[
E_\pm = -\Omega_\mp \tilde{n}_- \pm \sqrt{\Omega^2 \tilde{n}_-^2 + m^2} \rightarrow \begin{cases} E_+ = -\Omega_- \tilde{n}_- + \sqrt{\Omega^2 \tilde{n}_-^2 + m^2} \\ E_- = -\Omega_+ \tilde{n}_- - \sqrt{\Omega^2 \tilde{n}_-^2 + m^2} \end{cases}.
\]

(21)

Consequently, one may rearrange such energy levels and cast them so that

\[
E_{\pm}^{(\Omega_+)} = \pm |\Omega| \tilde{n}_+ \pm \sqrt{\Omega^2 \tilde{n}_+^2 + m^2},
\]

(22)

for positive vorticity, and

\[
E_{\pm}^{(\Omega_-)} = \pm |\Omega| \tilde{n}_- \pm \sqrt{\Omega^2 \tilde{n}_-^2 + m^2}.
\]

(23)

for negative vorticity. Notably, we observe that \( \tilde{n}_\pm (\ell = \pm \ell) = \tilde{n}_\mp (\ell = \mp \ell) \) which would in effect introduce the so called vorticity-energy correlations so that \( E_{\pm}^{(\Omega_+)} (\ell = \pm \ell) = E_{\pm}^{(\Omega_-)} (\ell = \mp \ell). \) We have, therefore, four branches of energy levels so that the upper half (above \( E = 0 \) line) is represented by \( E_+ \) and the lower half (below \( E = 0 \) line) is represented by \( E_- \) in the correlations mentioned above. Yet for massless KG-Gürses oscillators we obtain \( E_{\pm}^{(\Omega_+)} = \pm 2 |\Omega| \tilde{n}_\pm \) and \( E_{\pm}^{(\Omega_-)} = \pm 2 |\Omega| \tilde{n}_\mp. \)
Moreover, in Figures 1(a) and 1(b) we observe yet a new type of degeneracies in each branch of the energy levels (i.e., in each quarter of the figures). That is, states with the irrational quantum number $\tilde{n}_+ = 2n_r + |\ell| + \ell + 1$ collapse into $\ell = 0$ state for $\forall \ell = -|\ell|$ and states with $\tilde{n}_- = 2n_r + |\ell| - \ell + 1$ collapse into $\ell = 0$ state for $\forall \ell = +|\ell|$. This type of degeneracies is introduced by the structure of spacetime (Gürses spacetime is used here) and therefore should be called, hereinafter, spacetime associated degeneracies (STADs).

IV. KG-GÜRSES PLUS MIRZA-MOHADESI'S OSCILLATORS

We now consider KG-Gürses plus Mirza-Mohadesi’s oscillators with $\eta \neq 0$, and $g(r) = 1$. In this case, our KG-equation (11) collapses again into the two-dimensional Schrödinger oscillator

$$R''(r) + \left[ \lambda - \frac{(\ell^2 - 1/4)}{r^2} - \tilde{\omega}^2 r^2 \right] R(r) = 0,$$

which admits exact textbook solvability so that the eigenvalues and radial eigenfunctions, respectively, read

$$\lambda = 2|\tilde{\omega}|(2n_r + |\ell| + 1) = 2|\Omega E| \sqrt{1 + \frac{\eta^2}{\Omega^2 E^2}}(2n_r + |\ell| + 1)$$

and

$$R(r) \sim r^{|\ell|+1/2} \exp\left(-\frac{|\tilde{\omega}| r^2}{2}\right) L_{n_n}^{|\ell|}(|\tilde{\omega}| r^2) \iff \psi(r) \sim r^{|\ell|} \exp\left(-\frac{|\tilde{\omega}| r^2}{2}\right) \frac{L_{n_n}^{|\ell|}(|\tilde{\omega}| r^2)}{L_{n_n}^{|\ell|}(|\tilde{\omega}| r^2)}.$$

Then, equation (13) along with (25) imply

$$E^2 - 2\Omega E\ell - 2|\Omega E| \sqrt{1 + \frac{\eta^2}{\Omega^2 E^2}}(2n_r + |\ell| + 1) - (m^2 + 2\eta) = 0.$$

It is obvious that for $\eta = 0$ in (27) one would exactly obtain the results for the KG-Gürses oscillators discussed above. In Figure 2(a), we notice that the vorticity-energy correlations as well as STADs are now only partially valid because of the energy shifts introduced by Mirza-Mohadesi’s [4] parameter $\eta$. In Figures 2(b) and 2(c) we can clearly observe such shifts in each quarter of the figures. That is, quarters 1 and 2 are for $\Omega = \Omega_+ = +|\Omega|$ (i.e., for $E_{\pm}^{(\Omega_+)}$), and 3 and 4 are for $\Omega = \Omega_- = -|\Omega|$ (i.e., for $E_{\pm}^{(\Omega_-)}$).

At this point, it should be pointed out that this equation was improperly treated by Ahmed [14], as he expressed the energies in terms of $\tilde{\omega}$ where $\tilde{\omega} = \sqrt{\Omega^2 E^2 + \eta^2}$ (see (16) vs (21) with (22) and (16) vs (35) with (36) of [14]). That is, the energies are given in terms of the energies and his results starting form his equation (21) to the end of his paper are rendered misleading, and are incorrect. His results should be redirected to the results reported in current note, therefore.

V. PDM KG-GÜRSES OSCILLATORS

In this section we consider PDM settings for KG-Gürses oscillators, where $g(r) = \exp(2\beta r^2) ; \beta \geq 0$. Under such settings, KG-equation (11) reads

$$R''(r) + \left[ \lambda - \frac{(\ell^2 - 1/4)}{r^2} - \bar{\Omega}^2 r^2 \right] R(r) = 0,$$
FIG. 2: The energy levels for KG-Gürses oscillators of (27) are plotted with $m = 1$ (a) for $\eta = 5$, $n_r = 1$, $\ell = 0, \pm 1, \pm 2$, (b) for $n_r = 2$, $\ell = 1$, $\eta = 0, 1, 3, 6, 9$ and (c) for $n_r = 2$, $\ell = -2$, $\eta = 0, 1, 3, 6, 9$.

with

$$\lambda = E^2 - 2 \Omega \ell E - 2 \beta - m^2 ; \quad \tilde{\Omega}^2 = \Omega^2 E^2 + \beta^2.$$  \hspace{1cm} (29)

In this case, the eigenvalues and radial wavefunctions, respectively, read

$$\lambda = 2 \left| \tilde{\Omega} \right| \left( 2n_r + |\ell| + 1 \right) = 2 |\Omega E| \sqrt{1 + \frac{\beta^2}{\Omega^2 E^2}} \left( 2n_r + |\ell| + 1 \right),$$

and

$$R(r) \sim r^{|\ell|+1/2} \exp \left( -\frac{|\tilde{\Omega}| r^2}{2} \right) L^{|\ell|}_{n_n} \left( |\tilde{\Omega}| r^2 \right) \iff \psi (r) \sim r^{|\ell|} \exp \left( -\frac{|\tilde{\Omega}| r^2}{2} \right) L^{|\ell|}_{n_n} \left( |\tilde{\Omega}| r^2 \right).$$  \hspace{1cm} (31)

Consequently, the energies are given by

$$E^2 - 2 \Omega \ell E - 2 \beta - m^2 = 2 |\Omega E| \sqrt{1 + \frac{\beta^2}{\Omega^2 E^2}} \left( 2n_r + |\ell| + 1 \right).$$

(32)

Obviously, the effect of $\beta$ on the energy levels is the same as that of the Mirza-Mohadesi's oscillators [4] parameter $\eta$. This would suggest that Mirza-Mohadesi's oscillators [4] may very well be considered as a special case of PDM KG-oscillators.

VI. KG PSEUDO-GÜRSES OSCILLATORS: VORTICITY-ENERGY CORRELATIONS AND SPACETIME ASSOCIATED DEGENERACIES

We now consider a spacetime described by the metric

$$ds^2 = -dt^2 + g(r) \, dr^2 - 2 \Omega Q(r) \, r^2 \, dt \, d\theta + Q(r) \, r^2 \left( 1 - \Omega^2 Q(r) r^2 \right) \, d\theta^2.$$  \hspace{1cm} (33)

Next, let us introduce a transformation of the radial part so that

$$\rho = \sqrt{Q(r)} r = \int \sqrt{g(r)} \, dr \implies \sqrt{g(r)} = \sqrt{Q(r)} \left[ 1 + \frac{Q'(r)}{2Q(r)} r \right].$$  \hspace{1cm} (34)
where \( \mathbb{R} \ni (\rho, r) \in [0, \infty] \), and hence \( Q(r) \in \mathbb{R} \) is a positive-valued dimensionless scalar multiplier (so is \( g(r) \)). In this case, our spacetime metric (33) now reads

\[
ds^2 = -dt^2 + d\rho^2 - 2\Omega\rho^2 dtd\theta + \rho^2 (1 - \Omega^2 \rho^2) \, d\theta^2.
\] (35)

This metric looks very much like that of Gürses [1] and consequently the KG-equation (14) that describes KG-Gürses oscillators is indeed invariant and isospectral with the corresponding KG pseudo-Gürses oscillators equation

\[
R''(\rho) + \left[ \lambda - \frac{(\ell^2 - 1/4)}{\rho^2} - \Omega^2 E^2 \rho^2 \right] R(\rho) = 0.
\] (36)

Hence, our KG pseudo-Gürses oscillators would copy the same energies for the KG-Gürses oscillators of (22) and (23) (discussed in section 3) so that

\[
E_{\pm}^{(\Omega \pm)} = \pm |\Omega| \, \tilde{n}_{\pm} \pm \sqrt{\Omega^2 \tilde{n}_{\pm}^2 + m^2},
\] (37)

for positive vorticity, and

\[
E_{\pm}^{(\Omega \pm)} = \pm |\Omega| \, \tilde{n}_{\mp} \pm \sqrt{\Omega^2 \tilde{n}_{\mp}^2 + m^2}.
\] (38)

for negative vorticity. However, the radial wavefunctions are now given by

\[
R(\rho) \sim \rho^{\ell + 1/2} \exp \left( -\frac{\Omega E}{2} \rho^2 \right) L_{\frac{\ell}{2}} \left( |\Omega E| \rho^2 \right) \iff \psi(\rho) \sim \rho^{\ell} \exp \left( -\frac{\Omega E}{2} \rho^2 \right) L_{\frac{\ell}{2}} \left( |\Omega E| \rho^2 \right).
\] (39)

The following notes on our spacetime metric (35) are unavoidable.

(a) The spacetime metric (35) looks very much like Gürses spacetime one of [1] and should be called, hereinafter, pseudo-Gürses spacetime, therefore.

(b) If we set \( \Omega = -\mu/3 \), \( a_0 = 1 \) in

\[
\phi = a_0, \; \psi = \frac{b_0}{\rho^2} + \frac{3\lambda}{4} \rho^2, \; q = c_0 + \frac{e_0 \mu}{3} \rho^2, \; h = e_0 \rho, \; \lambda_0 = \lambda + \frac{\mu^2}{27},
\] (40)

of (3) and use

\[
Q(r) = e_0 + \frac{3c_0}{\mu r^2} \iff g(r) = \frac{\mu e_0^2 r^2}{\mu e_0^2 r^2 + 3c_0},
\] (41)

(where the parametric values are adjusted so that \( (Q(r), g(r)) \in \mathbb{R} \) are positive-valued functions, i.e., \( c_0 < 0 \) ) we obtain

\[
q = c_0 + \frac{e_0 \mu}{3} r^2, \; \psi = \frac{1}{e_0} + \frac{3c_0}{\mu e_0^2 r^2}.
\] (42)

Which is yet another feasible structure for the Gürses spacetime of (2) and (3) with

\[
b_0 = \frac{1}{e_0}, \; b_1 = \frac{3c_0}{\mu e_0^2}, \; \lambda_0 = 0, \; h = e_0 r.
\] (43)

(c) As long as condition (34) is satisfied, all KG-pseudo-Gürses oscillators (including the one in (b) above) in the spacetime model of (35) admit isospectrality and invariance with the KG-Gürses oscillators (14) and inherit the same vorticity-energy correlations so that \( E_{\pm}^{(\Omega \pm)} (\ell = \pm \ell) = E_{\pm}^{(\Omega \pm)} (\ell = \mp \ell) \) as well as they inherit the spacetime associated degeneracies, discussed in section 3.
VII. CONCLUDING REMARKS

In the current proposal, we revisited KG-oscillators in the (1+2)-dimensional Gürses spacetime of [11] so that PDM settings and Mirza-Mohadesi’s KG-oscillators [4] are included. We have observed that KG-Gürses oscillators are introduced as a byproduct of the very nature of the Gürses spacetime structure. This has, in turn, motivated us to first elaborate and discuss the effects of Gürses spacetime on the energy levels of the KG-Gürses oscillators. We have found that such KG-Gürses oscillators admit vorticity-energy correlations as well as spacetime associated degeneracies (STADs) (documented in Figures 1(a) and 1(b)). However, for KG-Gürses plus Mirza-Mohadesi’s oscillators we have observed that the vorticity-energy correlations as well as STADs are only partially valid because of the energy shifts introduced by Mirza-Mohadesi’s parameter $\eta$ (documented in Figures 2(a), 2(b), and 2(c)). Nevertheless, this model was studied by Ahmed [14] who has reported improper treatment and incorrect results. Consequently, his reported results (starting from his equation (21) to the end of his paper) should be redirected to the ones reported in the current study. Moreover, we have shown that PDM setting may very well have the same effect on the spectrum as that reported for KG-Gürses plus Mirza-Mohadesi’s oscillators. Yet, a new set of the so called KG pseudo-Gürses oscillators is introduced and is shown to be invariant and isospectral with KG-Gürses oscillators. Therefore, such KG pseudo-Gürses-oscillators would inherit the vorticity-energy correlations as well as STADs of the KG-Gürses oscillators.

Data Availability Statement Authors can confirm that all relevant data are included in the article and/or its supplementary information files. The author confirms that there are no online supplementary files (for web only publication) in this article.

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