On the continuous Zauner conjecture

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Denote by $H = \mathbb{C}^d$ a complex Hilbert space of dimension $d$.

**Zauner’s (weak) conjecture:**
For any $d \geq 2$ there exist $d^2$ unit vectors $\{|x_i\rangle\}_{i=1}^{d^2} \in H$ such that

1. the frame is tight: $\sum_{i=1}^{d^2} \langle x_i | x_i \rangle = dl$
2. it’s equiangular: $|\langle x_i | x_j \rangle|^2 = \text{const} = \frac{1}{d+1}$ for $i \neq j$
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Such a set $\{|x_i\rangle\}$ is a maximal ETF. It’s also called a SIC, since $\{\frac{1}{d} |x_i\rangle\langle x_i|\}$ forms a symmetric, informationally complete, positive operator-valued measure.
Symmetric and asymmetric subspaces

The symmetric subspace of $H \otimes H$ is defined by

$$H_{\text{sym}} = \text{span}\{|\phi\rangle|\phi\rangle\}_{\phi \in H}.$$

It’s the $(+1)$ eigenspace of the swap (flip) operator $U_{\text{sw}}$,

$$U_{\text{sw}}|i\rangle|j\rangle = |j\rangle|i\rangle, \quad \forall i, j \in [d].$$
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It’s complement, the asymmetric subspace $H_{\text{asym}}$, is the $(-1)$ eigenspace of $U_{\text{sw}}$,

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$$H_{\text{asym}} = \text{span}\{\ket{\phi}\bra{\psi} - \ket{\psi}\bra{\phi}\} | \phi, \psi \in H \}.$$

Their dimensions are

$$\dim H_{\text{sym}} = \frac{d(d + 1)}{2}, \ \dim H_{\text{asym}} = \frac{d(d - 1)}{2}.$$

By $\Pi_{\text{sym}}, \Pi_{\text{asym}}$ we denote the projectors onto $H_{\text{sym}}, H_{\text{asym}}$ respectively.
A set of unit vectors $\{|x_i\rangle\}_{i=1}^n \in \mathbb{C}^d$ is a weighted projective 2-design if

$$\sum_{i=1}^n w_i (|x_i\rangle\langle x_i|)^\otimes 2 = \frac{2}{d(d+1)} \Pi_{\text{sym}} = \mathbb{E}_\phi (|\phi\rangle\langle \phi|)^\otimes 2,$$

where $w_i \geq 0$, $\sum_i w_i = 1$.

It’s a projective 2-design if $w_i = \frac{1}{n}$, $i \in [n]$.

For a weighted projective 2-design we have $n \geq d^2$. 
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\]

where \( w_i \geq 0, \sum_i w_i = 1. \)

It’s a projective 2-design if \( w_i = \frac{1}{n}, i \in [n]. \)

For a weighted projective 2-design we have \( n \geq d^2. \)

A SIC is a minimal projective 2-design.

Conversely\(^1\), a weighted projective 2-design with \( n = d^2 \) elements is a SIC (which means the weights must be equal).

\(^1\)Scott, 2006
A linear map \( \Phi : \mathbb{C}^{d \times d} \to \mathbb{C}^{r \times r} \) is a quantum channel if it is

1. trace preserving: \( \text{Tr}(\Phi(X)) = \text{Tr}(X) \) for any \( X \)
2. completely positive: \( \mathcal{I}_n \otimes \Phi \) is a positive map for any \( n \), i.e. \( (\mathcal{I}_n \otimes \Phi)(\rho) \geq 0 \) for any \( \rho \geq 0 \) from \( \mathbb{C}^{n \times n} \otimes \mathbb{C}^{d \times d} \).
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The Choi matrix of a linear map $\Phi$ is defined by

$$
\mathcal{C}(\Phi) = (\mathcal{I}_d \otimes \Phi)(\frac{1}{d} \sum_{ij} E_{ij} \otimes E_{ij}) = \frac{1}{d} \sum_{ij} E_{ij} \otimes \Phi(E_{ij}).
$$

Note that the map $\mathcal{C}$ is linear and invertible.
A linear map $\Phi : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^{r \times r}$ is a quantum channel if it is

1. trace preserving: $\text{Tr}(\Phi(X)) = \text{Tr}(X)$ for any $X$

2. completely positive: $\mathcal{I}_n \otimes \Phi$ is a positive map for any $n$, i.e. $\left(\mathcal{I}_n \otimes \Phi\right)(\rho) \succeq 0$ for any $\rho \succeq 0$ from $\mathbb{C}^{n \times n} \otimes \mathbb{C}^{d \times d}$.

The Choi matrix of a linear map $\Phi$ is defined by

$$
C(\Phi) = (\mathcal{I}_d \otimes \Phi)\left(\frac{1}{d} \sum_{i,j} E_{ij} \otimes E_{ij}\right) = \frac{1}{d} \sum_{ij} E_{ij} \otimes \Phi(E_{ij}).
$$

Note that the map $C$ is linear and invertible.

The following are equivalent$^2$:

- a map $\Phi$ is completely positive
- $C(\Phi) \succeq 0$
- there exists a Kraus decomposition

$$
\Phi(X) = \sum_k A_k X A_k^\dagger, \quad A_k \in \mathbb{C}^{r \times d}.
$$

---

$^2$Choi, 1972
A state $\rho \in \mathbb{C}^{d \times d} \otimes \mathbb{C}^{r \times r}$ is separable if there exists a decomposition

$$\rho = \sum_{k=1}^{m} \lambda_k \rho_k^{(1)} \otimes \rho_k^{(2)},$$

where $\lambda_k \geq 0$, $\sum_k \lambda_k = 1$, and $\rho_k^{(1)}$, $\rho_k^{(2)}$ are states on the corresponding subsystems. Otherwise $\rho$ is called entangled.
A state \( \rho \in \mathbb{C}^{d \times d} \otimes \mathbb{C}^{r \times r} \) is separable if there exists a decomposition

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where \( \lambda_k \geq 0, \sum_k \lambda_k = 1 \), and \( \rho_k^{(1)}, \rho_k^{(2)} \) are states on the corresponding subsystems. Otherwise \( \rho \) is called entangled.

Clearly, any separable \( \rho \) also has a pure separable decomposition, where all \( \rho_k^{(1)}, \rho_k^{(2)} \) are pure states. The length of separability, denoted by \( \text{len}(\rho) \), is the minimum number of summands in a pure separable decomposition of \( \rho \). An immediate consequence is that \( \text{len}(\rho) \geq \text{rank}(\rho) \). It follows from Caratheodory’s theorem that \( \text{len}(\rho) \leq d^2 r^2 \).
A quantum channel $\Phi$ is entanglement breaking if $(\mathcal{I}_n \otimes \Phi)(\rho)$ is separable for any $n$ and any state $\rho$ on $\mathbb{C}^{n \times n} \otimes \mathbb{C}^{d \times d}$.
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The following are equivalent$^3$:

- a channel $\Phi$ is entanglement breaking
- $\mathcal{C}(\Phi)$ is separable
- there exists a Kraus decomposition

$$
\Phi(X) = \sum_k A_k X A_k^\dagger, \quad A_k \in \mathbb{C}^{r \times d},
$$

where Kraus operators $A_k$ are rank one matrices.

---

$^3$Horodecki, Shor, Ruskai, 2003
Consider the maps $\Phi_t : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^{d \times d}$ defined by

$$\Phi_t(X) = t \cdot X + (1 - t) \cdot \text{Tr}(X) \frac{1}{d} I_d, \quad t \in \mathbb{R}. $$

In other words, $\Phi_t$ is a linear combination of

$$\Phi_0(X) = \text{Tr}(X) \frac{1}{d} I_d, \quad \Phi_1(X) = X.$$
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In other words, $\Phi_t$ is a linear combination of

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Maps $\Phi_t$ are quantum channels for $t \in \left[-\frac{1}{d^2 - 1}, 1\right]$. But entanglement breaking only for $t \in \left[-\frac{1}{d^2 - 1}, \frac{1}{d+1}\right]$. 

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Entanglement breaking rank\(^4\) of EB map \(\Phi\), denoted by \(\text{ebr}(\Phi)\), is the minimum number of summands in the Kraus decomposition of \(\Phi\), where \(A_k\) are rank one.

\(^4\)Pandey, Paulsen, Prakash, Rahaman 2020; Paulsen CodEx Talk 21.07.2020
Entanglement breaking rank\textsuperscript{4} of EB map $\Phi$, denoted by $\text{ebr}(\Phi)$, is the minimum number of summands in the Kraus decomposition of $\Phi$, where $A_k$ are rank one.

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**Theorem (PPPR, 2020):** Zauner’s conjecture is equivalent to the statement that \(\text{ebr}(\Phi, \frac{1}{d+1}) = d^2\) for any \(d \geq 2\).

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**Conjecture:** \(\text{ebr}(\Phi_t) = d^2\) for any \(d \geq 2\) and \(t \in \left[-\frac{1}{d^2-1}, \frac{1}{d+1}\right]\).

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Proved for \(d = 2\) and \(d = 3\) via explicit construction of Kraus decompositions, where rank one \(A_k\) are continuous over \(t\).

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**Conjecture:** \(\text{ebr}(\Phi_t) = d^2\) for any \(d \geq 2\) and \(t \in [-\frac{1}{d^2-1}, \frac{1}{d+1}]\).

Proved for \(d = 2\) and \(d = 3\) via explicit construction of Kraus decompositions, where rank one \(A_k\) are continuous over \(t\).

Since \(\text{ebr}\) is lower semi-continuous it’s enough to prove the conjecture only for \(t \in [-\frac{1}{d^2-1}, \frac{1}{d+1}]\).

\(^4\)Pandey, Paulsen, Prakash, Rahaman 2020; Paulsen CodEx Talk 21.07.2020
**Lemma:** $\Phi(X) = |a\rangle\langle b| X |b\rangle\langle a| \iff C(\Phi) = (|b\rangle\langle b|)^T \otimes |a\rangle\langle a|.$

**Corollary:** If $\Phi$ is entanglement breaking then

$$\text{ebr}(\Phi) = \text{len}(C(\Phi)).$$
Lemma: $\Phi(X) = |a\rangle\langle b| X |b\rangle\langle a| \iff C(\Phi) = (|b\rangle\langle b|)^T \otimes |a\rangle\langle a|.$

Corollary: If $\Phi$ is entanglement breaking then $\text{ebr}(\Phi) = \text{len}(C(\Phi)).$

The partial transpose map $I \otimes T$ is a linear map defined by $(I \otimes T)(A \otimes B) = A \otimes B^T.$

Partial transposition of a separable state is separable. Moreover, the length of separability remains the same.
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Partial transposition of a separable state is separable. Moreover, the length of separability remains the same.

In total, we have

$$\text{ebr}(\Phi) = \text{len}(C(\Phi)) = \text{len}((I \otimes T)C(\Phi)) = \text{len}((T \otimes I)C(\Phi)).$$
Werner and isotropic states

For $t \in \left[ -\frac{1}{d^2-1}, \frac{1}{d+1} \right]$ the states

$$C(\Phi_t) = t \cdot \frac{1}{d} \sum_{ij} E_{ij} \otimes E_{ij} + (1 - t) \cdot \frac{1}{d^2} I_{d^2}$$

are known as separable isotropic states$^5$.

$^5$Watrous, TQI, 2018
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are known as separable isotropic states\(^5\). Their partial transpose

$$(I \otimes T)C(\Phi_t) = t \cdot \frac{1}{d} U_{sw} + (1 - t) \cdot \frac{1}{d^2} I_{d^2}$$

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Their partial transpose

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(l \otimes T)C(\Phi_t) = t \cdot \frac{1}{d} U_{sw} + (1 - t) \cdot \frac{1}{d^2} I_{d^2}
\]

are known as separable Werner states — linear combinations of \( \Pi_{\text{sym}} \) and \( \Pi_{\text{asym}} \). In particular,

\[
(l \otimes T)C\left(\Phi_\frac{1}{d+1}\right) = \frac{2}{d(d+1)} \Pi_{\text{sym}} \quad \implies \quad \text{ebr}(\Phi_\frac{1}{d+1}) = \text{len}(\Pi_{\text{sym}}).
\]

\(^5\)Watrous, TQI, 2018
Let $\{|x_i\rangle\}_{i=1}^n \in H$ be a weighted projective 2-design, i.e.

$$
\sum_{i=1}^n w_i \langle x_i| x_i \rangle \otimes 2 = \frac{2}{d(d+1)} \Pi_{\text{sym}},
$$

where $w_i \geq 0$, $\sum_i w_i = 1$. It means $\text{len}(\Pi_{\text{sym}}) \leq n$. 

The size of minimal weighted projective 2-designs
Let \( \{ |x_i\rangle \}_{i=1}^n \in H \) be a weighted projective 2-design, i.e.

\[
\sum_{i=1}^{n} w_i (|x_i\rangle\langle x_i|)^\otimes 2 = \frac{2}{d(d+1)} \Pi_{\text{sym}},
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where \( w_i \geq 0, \sum_i w_i = 1 \). It means \( \text{len}(\Pi_{\text{sym}}) \leq n \).

**Theorem**: The size of a minimal weighted projective 2-design equals \( \text{ebr}(\Phi_{\frac{1}{d+1}}) \).

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\[\text{Iverson, King, Mixon, 2021; King CodEx Talk 02.02.2021} \]
Let \(|x_i\rangle\}_{i=1}^{n} \in H\) be a weighted projective 2-design, i.e.

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\sum_{i=1}^{n} w_i (|x_i\rangle\langle x_i|) \otimes 2 = \frac{2}{d(d + 1)} \Pi_{\text{sym}},
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**Theorem**: The size of a minimal weighted projective 2-design equals \( \text{ebr}(\Phi_{\frac{1}{d+1}}) \).

**Proof sketch**: 1. Use \( \text{ebr}(\Phi_{\frac{1}{d+1}}) = \text{len}(\Pi_{\text{sym}}) \).

2. Let \( \text{len}(\Pi_{\text{sym}}) = m \), that is \( \exists \{|x_i\rangle, \{y_i\rangle, w_i \geq 0, \sum_i w_i = 1 \}

\[
\sum_{i=1}^{m} w_i |x_i\rangle\langle x_i| \otimes |y_i\rangle\langle y_i| = \frac{2}{d(d+1)} \Pi_{\text{sym}}.
\]

It follows \( |x_i\rangle|y_i\rangle \in H_{\text{sym}} \implies |x_i\rangle|y_i\rangle = |x'_i\rangle|x'_i\rangle \).

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6Iverson, King, Mixon, 2021; King CodEx Talk 02.02.2021
### Choi’s map cheat sheet

| Linear $\Phi : \mathbb{C}^{d \times d} \to \mathbb{C}^{r \times r}$ | $\mathcal{C}(\Phi) \in \mathbb{C}^{d \times d} \otimes \mathbb{C}^{r \times r}$ |
|---|---|
| $\Phi$ is completely positive | $\mathcal{C}(\Phi) \succeq 0$ |
| $\Phi$ is a quantum channel | $\mathcal{C}(\Phi)$ is a state |

**Assuming $\Phi$ is a quantum channel:**

| Kraus rank of $\Phi$ | $\iff$ | rank of $\mathcal{C}(\Phi)$ |
|---|---|---|
| $\Phi$ is entanglement breaking | $\iff$ | $\mathcal{C}(\Phi)$ is separable |
| $\text{ebr}(\Phi)$ | $\iff$ | $\text{len}(\mathcal{C}(\Phi))$ |
| $T \circ \Phi, \Phi \circ T$ | $\iff$ | $(I \otimes T)\mathcal{C}(\Phi), (T \otimes I)\mathcal{C}(\Phi)$ |
| EB depolarising channels $\Phi_t$ | $\iff$ | separable isotropic states |
| EB channels $T \circ \Phi_t = \Phi_t \circ T$ | $\iff$ | separable Werner states |
| $T \circ \Phi \frac{1}{d+1}$ | $\iff$ | $\Pi_{\text{sym}} \cdot \frac{2}{d(d+1)}$ |
Mutually unbiased frames (kind of)

Let $ebr(\Phi_t) = d^2$ for some $t \in \left[\frac{-1}{d^2-1}, \frac{1}{d+1}\right]$. Equivalently, $\text{len}((I \otimes T)C(\Phi_t)) = d^2$, which means there exist unit frames $\{|x_i\rangle\}$, $\{|y_i\rangle\}$ and $w_i \geq 0$, $\sum_i w_i = 1$ for $i \in [d^2]$ such that

$$\sum_{i=1}^{d^2} w_i |x_i\rangle \otimes |y_i\rangle = (I \otimes T)C(\Phi_t) = t \cdot \frac{1}{d} U_{sw} + (1 - t) \cdot \frac{1}{d^2} I_{d^2}.$$
Mutually unbiased frames (kind of)

Let $ebr(\Phi_t) = d^2$ for some $t \in \left[\frac{-1}{d^2-1}, \frac{1}{d+1}\right]$. Equivalently, $\text{len}((I \otimes T)C(\Phi_t)) = d^2$, which means there exist unit frames $\{|x_i\rangle\}$, $\{|y_i\rangle\}$ and $w_i \geq 0, \sum_i w_i = 1$ for $i \in [d^2]$ such that

$$\sum_{i=1}^{d^2} w_i |x_i\rangle\langle x_i| \otimes |y_i\rangle\langle y_i| = (I \otimes T)C(\Phi_t) = t \cdot \frac{1}{d} U_{sw} + (1-t) \cdot \frac{1}{d^2} I_{d^2}.$$ 

Then\textsuperscript{7} all $w_i$ are equal $1/d^2$ and the frames $\{|x_i\rangle\}$, $\{|y_i\rangle\}$ are

1. tight,
2. informationally-complete if $t \neq 0$,
3. kind of mutually unbiased: $|\langle x_i|y_j \rangle|^2 = \begin{cases} \frac{t(d^2-1)+1}{d}, & i = j, \\ \frac{1-t}{d}, & i \neq j, \end{cases}$
4. reciprocal: $|\langle x_i|y_j\rangle\otimes|y_i\rangle| = |t|, \quad i \neq j.$

\textsuperscript{7}D.Y., 2022
Conversely, let for \( t \in \left[ \frac{-1}{d^2-1}, \frac{1}{d+1} \right] \) unit frames \( \{|x_i\rangle\}_{i=1}^{d^2}, \{|y_i\rangle\}_{i=1}^{d^2} \) are tight, informationally-complete, and kind of mutually unbiased:

\[
|\langle x_i \mid y_j \rangle|^2 = \begin{cases} 
\frac{t(d^2-1)+1}{d}, & i = j, \\
\frac{1-t}{d}, & i \neq j.
\end{cases}
\]
Conversely, let for $t \in \left[\frac{-1}{d^2-1}, \frac{1}{d+1}\right]$ unit frames $\{|x_i\rangle\}_{i=1}^{d^2}$, $\{|y_i\rangle\}_{i=1}^{d^2}$ are tight, informationally-complete, and kind of mutually unbiased:

$$|\langle x_i | y_j \rangle|^2 = \begin{cases} \frac{t(d^2-1)+1}{d}, & i = j, \\ \frac{1-t}{d}, & i \neq j. \end{cases}$$

Then

$$\frac{1}{d^2} \sum_{i=1}^{d^2} |x_i \rangle \langle x_i | \otimes |y_i \rangle \langle y_i | = t \cdot \frac{1}{d} U_{sw} + (1 - t) \cdot \frac{1}{d^2} l_{d^2}, \quad (*)$$

i.e. $\text{len}((I \otimes T)C(\Phi_t)) = \text{ebr}(\Phi_t) = d^2$. 
Conversely, let for \( t \in \left[ \frac{-1}{d^2-1}, \frac{1}{d+1} \right] \) unit frames \( \{|x_i\rangle\}_{i=1}^{d^2}, \{|y_i\rangle\}_{i=1}^{d^2} \) are tight, informationally-complete, and kind of mutually unbiased:

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|\langle x_i | y_j \rangle|^2 = \begin{cases} 
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\frac{1-t}{d}, & i \neq j.
\end{cases}
\]

Then

\[
\frac{1}{d^2} \sum_{i=1}^{d^2} |x_i \rangle \langle x_i| \otimes |y_i \rangle \langle y_i| = t \cdot \frac{1}{d} U_{sw} + (1 - t) \cdot \frac{1}{d^2} I_{d^2}, \quad (*)
\]

i.e. \( \text{len}((I \otimes T)C(\Phi_t)) = \text{ebr}(\Phi_t) = d^2. \)

**Corollary:** the PPPR conjecture is equivalent to the existence of such a pair of frames (except the case \( t = 0 \)).
Let $\omega = e^{2\pi i/d}$, $\tau = -e^{\pi i/d}$. Define clock and shift matrices by

$$Z = \sum_{i=0}^{d-1} w^i |i\rangle\langle i|, \quad X = \sum_{i=0}^{d-1} |i+1\rangle\langle i|, \quad |d\rangle := |0\rangle.$$ 

For $a = (a_1, a_2) \in \mathbb{Z}_d^2$ unitaries $D_a = \tau^{a_1 a_2} X^{a_1} Z^{a_2}$ form a projective representation of $\mathbb{Z}_d^2$. A frame $\{D_a |v\rangle\}_{a \in \mathbb{Z}_d^2}$ is called WH-covariant.
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For \( d = 2 \) and \( d = 3 \) there are WH-covariant solutions to Eq. (*) that is,

\[
|x_a(t)\rangle = D_a |x(t)\rangle, \quad |y_a(t)\rangle = D_a |y(t)\rangle,
\]

where \( |x(t)\rangle, |y(t)\rangle \) are fiducial vectors of both frames. Moreover, \( |x(t)\rangle, |y(t)\rangle \) are continuous and differentiable over \( t \).
The case where $t = 0$

The equality for WH-covariant frames becomes

$$\sum_{a \in \mathbb{Z}_d^2} D_a^{\otimes 2}(|x(0)\rangle \langle x(0)| \otimes |y(0)\rangle \langle y(0)|) D_a^{\dagger \otimes 2} = I_{d^2}$$
The equality for WH-covariant frames becomes

\[ \sum_{a \in \mathbb{Z}^2_d} D_a \otimes^2 (|x(0)\rangle \langle x(0)| \otimes |y(0)\rangle \langle y(0)|) D^\dagger \otimes^2 = I_{d^2} \]

There is a nice looking solution to this:

\[ |x(0)\rangle = |0\rangle, \quad |y(0)\rangle = F |0\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle, \]

where \( F = \frac{1}{\sqrt{d}} \sum_{ij} \omega^{ij} |i\rangle \langle j| \) is the Fourier transform.
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where $F = \frac{1}{\sqrt{d}} \sum_{i,j} \omega^{ij} |i\rangle \langle j|$ is the Fourier transform.

It turns out that this solution is not a good starting point for a series of WH-covariant solutions of Eq. (*) for $t \in [0, \varepsilon]$ because it can’t be differentiable at $t = 0^+$. 

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On the continuous Zauner conjecture
For a WH SIC with fiducial $|x\rangle$ we have

\[
\frac{1}{d^2} \sum_{a \in \mathbb{Z}_d^2} D_{a}^{\otimes 2} (|x\rangle \langle x| \otimes |x\rangle \langle x|) D_{a}^{\dagger \otimes 2} = \frac{2}{d(d+1)} \Pi_{\text{sym}}.
\]

Thus the frame $(D_{a} |x\rangle)^{\otimes 2}$ is tight on $H_{\text{sym}}$. 
WH bases

For a WH SIC with fiducial $\vert x \rangle$ we have

$$\frac{1}{d^2} \sum_{a \in \mathbb{Z}_d^2} D_a \otimes^2 (\vert x \rangle \langle x \vert \otimes \vert x \rangle \langle x \vert) D_a^\dagger \otimes^2 = \frac{2}{d(d + 1)} \prod_{\text{sym}}.$$ 

Thus the frame $(D_a \vert x \rangle) \otimes^2$ is tight on $H_{\text{sym}}$.

By Naimark’s theorem there exists an ONB $\{ \vert b_a \rangle \}$ on $H \otimes H$ such that

$$(D_a \vert x \rangle) \otimes^2 = \sqrt{\frac{2d}{d + 1}} \prod_{\text{sym}} \vert b_a \rangle.$$
For a WH SIC with fiducial $|x\rangle$ we have

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By Naimark’s theorem there exists an ONB $\{|b_a\rangle\}$ on $H \otimes H$ such that

$$(D_a |x\rangle) \otimes^2 = \sqrt{\frac{2d}{d+1}} \Pi_{\text{sym}} |b_a\rangle.$$

But in fact\(^8\), we can always find a WH-covariant type of such a basis, that is $|b_a\rangle = D_a \otimes^2 |b\rangle$. Equivalently,

$$|x\rangle |x\rangle = \sqrt{\frac{2d}{d+1}} \Pi_{\text{sym}} |b\rangle,$$

where $|b\rangle \in H \otimes^2$ is a fiducial basis vector.

\(^{8}\) Ostrovsky, D.Y., “Geometric properties of SIC-POVM tensor square”, 2022
This result can be extended. Let for some \( t \in \left[ \frac{-1}{d^2-1}, \frac{1}{d+1} \right] \):

\[
\frac{1}{d^2} \sum_{\mathbf{a} \in \mathbb{Z}_d^2} D_{\mathbf{a}}^\otimes^2 (|x(t)\rangle\langle x(t)| \otimes |y(t)\rangle\langle y(t)|) D_{\mathbf{a}}^{\dagger \otimes^2} = t \cdot \frac{1}{d} U_{\text{sw}} + (1-t) \cdot \frac{1}{d^2} I_d^2.
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This result can be extended. Let for some $t \in \left[\frac{-1}{d^2-1}, \frac{1}{d+1}\right]$: 

$$\frac{1}{d^2} \sum_{a \in \mathbb{Z}_d^2} D_a^{\otimes 2}(|x(t)\rangle \langle x(t)| \otimes |y(t)\rangle \langle y(t)|) D_a^{\dagger \otimes 2} = t \cdot \frac{1}{d} U_{sw} + (1-t) \cdot \frac{1}{d^2} I_d^2.$$ 

Let $M_t = d \cdot \sqrt{t \cdot \frac{1}{d} U_{sw} + (1-t) \cdot \frac{1}{d^2} I_d^2} \geq 0$. Then there exists a WH-covariant basis $|b_a(t)\rangle = D_a^{\otimes 2} |b(t)\rangle$ of $H^{\otimes 2}$ such that 

$$|x(t)\rangle \langle y(t)| = M_t |b(t)\rangle.$$
the PPPR conjecture is equivalent to the existence of a specific pair of mutually unbiased frames
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Conclusions

- the PPPR conjecture is equivalent to the existence of a specific pair of mutually unbiased frames.
- It is probably not true for $d \geq 4$ except $t = 0$, $t = \frac{1}{d+1}$.
- It’s still interesting to find $\text{ebr}$ (equivalently, $\text{len}$) for maps $\Phi_t$ and EB maps in general.
Thank you!