Generalized polarizabilities of the pion in chiral perturbation theory

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Abstract

We present a calculation of the virtual Compton scattering amplitude for \( \gamma^* + \pi \to \gamma + \pi \) in the framework of chiral perturbation theory at \( \mathcal{O}(p^4) \). We explicitly derive expressions for generalized electromagnetic polarizabilities and discuss alternative definitions of these quantities.

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I. INTRODUCTION

Compton scattering of real photons (RCS) is one of the simplest reactions for obtaining information on the structure of a stable composite system. When expanded in the frequency of the photon, the leading-order term of the low-energy scattering amplitude is specified by the model-independent Thomson limit in terms of the charge and the mass of the target. Genuine structure effects first appear at second order and can be parametrized in terms of the electric and magnetic polarizabilities (for an overview see, e.g., Refs. [1–3]). As there is no stable pion target, the empirical information on the electromagnetic polarizabilities has been extracted from high-energy pion-nucleus bremsstrahlung [4,5] and radiative pion photoproduction off the nucleon [6]. In principle, the electromagnetic polarizabilities of the pion also enter into the crossed process \( \gamma \gamma \rightarrow \pi \pi \). However, there is some debate concerning the accuracy of extracting these quantities from the crossed channel [7–11].

From a theoretical point of view, a precise determination of the pion polarizabilities is of great importance, since (approximate) chiral symmetry allows one to predict the electromagnetic polarizabilities of the charged pion in terms of the radiative decay \( \pi^+ \rightarrow e^+ \nu_e \gamma \) [12]. Corrections to the leading-order PCAC result have been calculated at \( \mathcal{O}(p^6) \) in chiral perturbation theory and turn out to be rather small [13]. New experiments are presently being carried out [14] or have been proposed [15,16] to significantly reduce the uncertainties in the empirical results and thus subject the predictions of chiral symmetry to a stringent test.

Clearly, the possibilities to investigate the structure of the target increase substantially if virtual photons are used, because energy and three-momentum can be varied independently and, furthermore, the longitudinal component of the transition current can be explored. In particular, virtual Compton scattering (VCS) off the nucleon, as tested in the reaction \( e^- + p \rightarrow e^- + p + \gamma \), has attracted considerable interest (see, e.g., Refs. [17,18]). The pion-VCS amplitude of \( \gamma^* + \pi \rightarrow \gamma + \pi \) can, in principle, be studied through the inelastic scattering of high-energy pions off atomic electrons, \( \pi^- + e^- \rightarrow \pi^- + e^- + \gamma \). Such events are presently analyzed as part of the SELEX E781 experiment [19].

In this paper, we will investigate the VCS reaction \( \gamma^* + \pi \rightarrow \gamma + \pi \) in the framework of chiral perturbation theory at \( \mathcal{O}(p^4) \). We will first give a short survey of chiral perturbation theory and then define our conventions for the VCS invariant amplitude. We then discuss the result for the soft-photon and residual amplitudes, respectively. Finally, the model-dependent residual amplitude is analyzed in terms of alternative definitions of generalized polarizabilities.

II. THE CHIRAL LAGRANGIAN

Chiral perturbation theory (ChPT) [20,22] is based on the chiral \( SU(2)_L \times SU(2)_R \) symmetry of QCD in the limit of vanishing \( u- \) and \( d- \)quark masses. The assumption of spontaneous symmetry breaking down to \( SU(2)_V \) gives rise to three massless pseudoscalar Goldstone bosons with vanishing interactions in the limit of zero energies. These Goldstone bosons are identified with the physical pion triplet, the nonzero pion masses resulting from
an explicit symmetry breaking in QCD through the quark masses. The effective Lagrangian of the pion interaction is organized in a so-called momentum expansion,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \cdots,$$

(1)

where the subscripts refer to the order in the expansion. Interactions with external fields, such as the electromagnetic field, as well as explicit symmetry breaking due to the finite quark masses, are systematically incorporated into the effective Lagrangian. Covariant derivatives and quark-mass terms count as $\mathcal{O}(p)$ and $\mathcal{O}(p^2)$, respectively. Weinberg’s power counting scheme [21] allows for a classification of the Feynman diagrams by establishing a relation between the momentum expansion and the loop expansion. The most general chiral Lagrangian at $\mathcal{O}(p^2)$ is given by

$$\mathcal{L}_2 = \frac{F^2}{4} \text{Tr} \left[ D_\mu U(D^\mu U)^\dagger + \chi U^\dagger + U\chi^\dagger \right],$$

(2)

where $U$ is a unimodular unitary $(2 \times 2)$ matrix, transforming as $V_R U V_L^\dagger$ for $(V_L, V_R) \in \text{SU}(2)_L \times \text{SU}(2)_R$. As a parametrization of $U$ we will use

$$U(x) = \frac{\sigma(x) + i \vec{r} \cdot \vec{\pi}(x)}{F}, \quad \sigma^2(x) + \vec{r}^2(x) = F^2,$$

(3)

where $F$ denotes the pion-decay constant in the chiral limit: $F_\pi = F[1 + \mathcal{O}(\hat{m})] = 92.4$ MeV. We will work in the isospin-symmetric limit $m_u = m_d = \hat{m}$. The quark mass is contained in $\chi = 2B_0 \hat{m} = m^2_3$ at $\mathcal{O}(p^2)$, where $B_0$ is related to the quark condensate $\langle \bar{q}q \rangle$. The covariant derivative $D_\mu U = \partial_\mu U + \frac{i}{2} \epsilon A_\mu [\tau_3, U]$ contains the coupling to the electromagnetic field $A_\mu$. The most general structure of $\mathcal{L}_4$, first obtained by Gasser and Leutwyler (see Eq. (5.5) of Ref. [22]), reads, in the standard trace notation,

$$\mathcal{L}_4^{\text{GL}} = \frac{l_1}{4} \left\{ \text{Tr}[D_\mu U(D^\mu U)^\dagger] \right\}^2 + \frac{l_2}{4} \text{Tr}[D_\mu U(D_\nu U)^\dagger] \text{Tr}[D^\mu U(D^{\nu} U)^\dagger] + \frac{l_3}{16} \left[ \text{Tr}(\chi U^\dagger + U\chi^\dagger) \right]^2 + \frac{l_4}{4} \text{Tr}[D_\mu U(D^\mu \chi)^\dagger + D_\mu \chi(D^\mu U)^\dagger] + l_5 \left[ \text{Tr}(F^{\mu \nu}_L F^{\nu \mu}_L U^\dagger) - \frac{1}{2} \text{Tr}(F^{\mu \nu}_L F^{\mu \nu}_R) \right] + \frac{l_6}{2} \text{Tr}[F^{R \mu}_L D^\mu U(D^{\nu} U)^\dagger + F^{L \mu}_L (D^\mu U)^\dagger D^{\nu} U] - \frac{l_7}{16} \left[ \text{Tr}(\chi U^\dagger - U\chi^\dagger) \right]^2 + \cdots,$$

(4)

where three terms containing only external fields have been omitted. For the electromagnetic interaction, the field-strength tensors are given by $F^{\mu \nu}_L = F^{\mu \nu}_R = -\frac{1}{2} \tau_3 (\partial^\mu A^\nu - \partial^\nu A^\mu)$.

**III. CONVENTIONS**

In the following, we will discuss the VCS amplitude for $\gamma^*(q, \epsilon) + \pi^i(p_i) \rightarrow \gamma(q', \epsilon') + \pi^j(p_f)$ $(q^2 \leq 0, q'^2 = 0, q' \cdot \epsilon' = 0)$. Throughout the calculation we use the conventions of Bjorken and Drell [23] with $e^2/4\pi \approx 1/137, c > 0$. For the isospin decomposition of the invariant amplitude we use

$$\mathcal{M}_{ij} = \delta_{ij} A + (\delta_{ij} - \delta_{i3}\delta_{j3}) B,$$

(5)
where \( i \) and \( j \) denote the cartesian isospin indices of the initial and final pions, respectively. With the definition

\[
|\pi^\pm(p)\rangle = \frac{1}{\sqrt{2}}[a_1^\dagger(p) \pm ia_2^\dagger(p)]|0\rangle, \quad |\pi^0(p)\rangle = a_3^\dagger(p)|0\rangle,
\]

we may express the physical amplitudes in terms of the isospin amplitudes

\[
\mathcal{M}_{\pi^+} = \mathcal{M}_{\pi^-} = \frac{1}{2}(\mathcal{M}_{11} + \mathcal{M}_{22}) = \mathcal{A} + \mathcal{B},
\]

\[
\mathcal{M}_{\pi^0} = \mathcal{M}_{33} = \mathcal{A}.
\]

We split the contributions to \( \mathcal{M}_{ij} \) into a pole piece \( (P) \) and a one-particle-irreducible, residual part \( (R) \), \( \mathcal{A} = \mathcal{A}_P + \mathcal{A}_R \), \( \mathcal{B} = \mathcal{B}_P + \mathcal{B}_R \) (see Fig. 1). Since the \( \pi^0 \) is its own antiparticle, the electromagnetic vertex \( \pi^0\pi^0\gamma^* \) vanishes due to charge-conjugation invariance and hence \( \mathcal{A}_P \equiv 0 \). In general, the pole piece \( \mathcal{B}_P \) and the one-particle-irreducible piece \( \mathcal{B}_R \) are not separately gauge invariant.

**IV. SOFT-PHOTON AMPLITUDE**

According to Weinberg’s power counting, a calculation of the \( s \)- and \( u \)-channel pole terms at \( \mathcal{O}(p^4) \) involves the renormalized irreducible vertex at \( \mathcal{O}(p^4) \),

\[
\Gamma^\mu(p',p) = (p' + p)^\mu F(q^2) + (p' - p)^\mu \frac{p'^2 - p^2}{q^2} [1 - F(q^2)], \quad q = p' - p,
\]

where \( F(q^2) \) is the prediction for the electromagnetic form factor of the pion (see Eq. (15.3) of Ref. [21]). To that order, the renormalized propagator is simply given by

\[
i\Delta_R(p) = \frac{i}{p^2 - m_\pi^2 + i0^+},
\]

with \( m_\pi^2 \) the \( \mathcal{O}(p^4) \) result for the pion mass squared (see Eq. (12.2) of Ref. [21]). Note that Eqs. (8) and (9) satisfy the Ward-Takahashi identity [24,25]

\[
q^\mu \Gamma^\mu(p',p) = \Delta_R^{-1}(p') - \Delta_R^{-1}(p).
\]

With these ingredients the result for \( \mathcal{B}_P \) at \( \mathcal{O}(p^4) \) reads

\[
\mathcal{B}_P = -is^2 \left\{ F(q^2) \left[ \frac{2p_f \cdot \epsilon^* (2p_i + q) \cdot \epsilon}{s - m_\pi^2} + \frac{(2p_f - q) \cdot \epsilon 2p_i \cdot \epsilon^*}{u - m_\pi^2} \right] + 2q \cdot \epsilon q \cdot \epsilon^* \frac{1 - F(q^2)}{q^2} \right\},
\]

which is easily seen not to be gauge invariant by itself. The set of one-particle-irreducible diagrams is shown in Fig. 2 and gives rise to a residual part of the form

\[
\mathcal{B}_R = is^2 \left[ 2\epsilon \cdot \epsilon^* + 2(q^2 \epsilon \cdot \epsilon^* - q \cdot \epsilon q \cdot \epsilon^*) \frac{F(q^2) - 1}{q^2} \right] + \bar{\mathcal{B}}_R.
\]

We combine Eqs. (10) and (11) into the form
\[ \mathcal{B} = \tilde{\mathcal{B}}_P + \tilde{\mathcal{B}}_R, \]  

\[ \tilde{\mathcal{B}}_P = -ie^2 F(q^2) \left[ \frac{2p_f \cdot \epsilon^* (2p_i + q) \cdot \epsilon}{s - m_e^2} + \frac{(2p_f - q) \cdot \epsilon 2p_i \cdot \epsilon^*}{u - m_e^2} - 2\epsilon \cdot \epsilon^* \right], \]  

with the result that \( \tilde{\mathcal{B}}_P \) and \( \tilde{\mathcal{B}}_R \) are now separately gauge invariant. In particular, \( \tilde{\mathcal{B}}_P \) has the form of the soft-photon result obtained in Eq. (10) of Ref. [27]. A somewhat different approach for obtaining the soft-photon result can be found in Ref. [27].

\[ \text{V. RESIDUAL AMPLITUDES} \]

As has been discussed in detail in Ref. [28], a gauge-invariant parametrization of the residual amplitude for \( \gamma^* + \pi \to \gamma + \pi \) can be written in terms of three invariant functions \( f_i(q^2, q \cdot q', q \cdot P) \), where \( P = p_i + p_f \). At \( \mathcal{O}(p^4) \), the result for the residual isospin amplitudes \( \mathcal{A}_R \) and \( \bar{\mathcal{A}}_R \) reads:

\[ \mathcal{A}_R = -ie^2 (q' \cdot q \cdot \epsilon^* - q' \cdot q' \cdot \epsilon^*) \frac{m_i^2 + 2q \cdot q' - q^2}{8\pi^2 F^2 q \cdot q'} G(q^2, q \cdot q'), \]

\[ \bar{\mathcal{A}}_R = -ie^2 (q' \cdot q \cdot \epsilon^* - q' \cdot q \cdot \epsilon^*) \left[ \frac{4(2l_5^r - l_0^r)}{F^2} - \frac{2m_i^2 + 2q \cdot q' - q^2}{16\pi^2 F^2 q \cdot q'} G(q^2, q \cdot q') \right], \]

where the combination \( 2l_5^r - l_0^r = (2.85 \pm 0.42) \times 10^{-3} \) is determined through the decay \( \pi^+ \to e^+ \nu e\gamma \). In Eqs. (14) and (15) we have introduced the abbreviation

\[ G(q^2, q \cdot q') = 1 + \frac{m_i^2}{q \cdot q'} \left[ J^{(-1)}(a) - J^{(-1)}(b) \right] - \frac{q^2}{2q \cdot q'} \left[ J^{(0)}(a) - J^{(0)}(b) \right], \]

where

\[ J^{(n)}(x) := \int_0^1 dy y^n \ln[1 + x(y^2 - y)] \]

and

\[ a := \frac{q^2}{m_i^2}, \quad b := \frac{q^2 - 2q \cdot q'}{m_i^2}. \]

The one-loop integrals \( J^{(0)} \) and \( J^{(-1)} \) are given by (see Appendix C of Ref. [29])

\[ J^{(0)}(x) = \begin{cases} 
-2 - \sigma \ln \left( \frac{\sigma - 1}{\sigma + 1} \right) & (x < 0), \\
-2 + 2 \sqrt{\frac{1}{x} - 1} \arccot \left( \sqrt{\frac{1}{x} - 1} \right) & (0 \leq x < 4), \\
-2 - \sigma \ln \left( \frac{1 - \sigma}{1 + \sigma} \right) - i\pi \sigma & (4 < x), 
\end{cases} \]

\[ J^{(-1)}(x) = \begin{cases} 
\frac{1}{2} \ln^2 \left( \frac{\sigma - 1}{\sigma + 1} \right) & (x < 0), \\
-\frac{1}{2} \arccos^2 \left( 1 - \frac{x}{2} \right) & (0 \leq x < 4), \\
\frac{1}{2} \ln^2 \left( \frac{1 - \sigma}{1 + \sigma} \right) - \frac{\pi^2}{2} + i\pi \ln \left( \frac{1 - \sigma}{1 + \sigma} \right) & (4 < x), 
\end{cases} \]

\[ ^1 \text{In reproducing these results we found Refs. [30] and [31] useful.} \]
with
\[ \sigma(x) = \sqrt{1 - \frac{4}{x}}, \quad x \notin [0, 4]. \]

A comparison of Eqs. (14) and (13) with Eq. (18) of Ref. [28] shows that, at \( \mathcal{O}(p^4) \), only one of the three functions \( f_i(q^2, q \cdot q', q \cdot P) \) contributes, i.e., \( f_2 = f_3 = 0 \). Furthermore, at this order in the chiral expansion, the function \( f_1 \) does not depend on \( q \cdot P = q' \cdot P \). Our result for \( A_R \) is in agreement with Ref. [32], where the photoproduction of neutral pion pairs in the Coulomb field of a nucleus was studied.

VI. GENERALIZED POLARIZABILITIES OF GUICHON, LIU, AND THOMAS

In order to discuss the generalized polarizabilities, we expand the function \( \mathcal{G} \) of Eq. (16) for negative \( q^2 \) around \( q \cdot q' = 0 \),
\[
\mathcal{G}(q^2, q \cdot q') = -\frac{q \cdot q'}{m^2_\pi} J^{(0)\prime}(q^2 \frac{q^2}{m^2_\pi}) \frac{dJ^{(0)}(x)}{dx}, \quad J^{(0)\prime}(x) = \frac{1}{x} \left[ 1 + 2 \frac{\ln (\sigma - 1)}{\sigma + 1} \right] = \frac{1}{x} \left[ 1 + 2 J^{(-1)\prime}(x) \right], \quad x < 0.
\]

For the charged and neutral pion we obtain, respectively,
\[
f^{0 \pm}_1(q^2, q \cdot q', q \cdot P) = -\frac{4(2l^r_5 - l^r_6)}{F^2} + \frac{2q \cdot q'}{16\pi^2 F^2 q \cdot q'} J^{(0)\prime}(q^2 \frac{q^2}{m^2_\pi}) \mathcal{G}(q^2, q \cdot q')
\]
\[
= -\frac{2q \cdot q'}{16\pi^2 F^2 m^2_\pi} J^{(0)\prime}(q^2 \frac{q^2}{m^2_\pi}) + \mathcal{O}(q \cdot q'),
\]
\[
f^{0 \sigma}_1(q^2, q \cdot q', q \cdot P) = -\frac{1}{8\pi^2 F^2} \left( 1 - \frac{q^2}{m^2_\pi} \right) J^{(0)\prime}(q^2 \frac{q^2}{m^2_\pi}) + \mathcal{O}(q \cdot q').
\]

We will first discuss the generalized polarizabilities as defined in Ref. [33], where the residual amplitude was analyzed in the photon-pion center-of-mass frame in terms of a multipole expansion. Only terms linear in the frequency of the final photon were kept, and the result was parametrized in terms of “generalized polarizabilities.” The connection with the covariant approach was established in Ref. [28], where it was also found that only two of the three polarizabilities \( P^{(01,01)0}, P^{(11,11)0}, \) and \( P^{(01,11)0} \) of Ref. [33] are independent, once the constraints due to charge conjugation are combined with particle-crossing symmetry. According to Eqs. (35) and (36) of Ref. [28] we define generalized electric and magnetic polarizabilities \( \alpha(|q|^2) \) and \( \beta(|q|^2) \), respectively, as
\[
\alpha(|q|^2) \equiv -\frac{e^2}{4\pi} \sqrt{\frac{3}{2}} P^{(01,01)0}(|q|) = \frac{e^2}{8\pi m_\pi} \sqrt{\frac{m_\pi}{E_i}} \left[ -f_1(\omega_0^2 - |q|^2, 0, 0) + 2m_\pi \frac{|q|^2}{\omega_0} \right] f_2(\omega_0^2 - |q|^2, 0, 0),
\]
\[
\beta(|q|^2) \equiv -\frac{e^2}{4\pi} \sqrt{\frac{3}{8}} P^{(11,11)0}(|q|) = \frac{e^2}{8\pi m_\pi} \sqrt{\frac{m_\pi}{E_i}} f_1(\omega_0^2 - |q|^2, 0, 0),
\]
where $\omega_0 = q_0|_{\omega' = 0} = m_\pi - \sqrt{m_\pi^2 + |q|^2}$. A few remarks are in order at this point.

1. In our present work, we strictly stick to the convention of Ref. [23]. This is why Eqs. (21) and (22) differ by an overall factor $1/2m_\pi$ from Ref. [28], where in Eq. (1) an additional factor $2m_\pi$ was introduced for the spin-0 case.

2. The variable $q^2$ only appears in the combination $q^2/m_\pi^2$, resulting in
\[
\left. \frac{q^2}{m_\pi^2} \right|_{\omega' = 0} = \frac{2m_\pi - E_i}{m_\pi}, \quad E_i = \sqrt{m_\pi^2 + |q|^2}.
\]

3. The factor $\sqrt{m_\pi/E_i}$ originates from an additional normalization factor $\mathcal{N}$ in Eq. (32) of Ref. [33], such that
\[
\frac{2m_\pi}{\sqrt{4E_iE_f}} \rightarrow \sqrt{\frac{m_\pi}{E_i}}.
\]

Using the results of Eqs. (19) and (20) together with $f_2 = 0$, we then obtain
\[
\alpha_{\pi^0}(\bar{q})^2 = -\beta_{\pi^0}(\bar{q})^2 = \frac{e^2}{8\pi m_\pi} \sqrt{\frac{m_\pi}{E_i}} \left[ \frac{4(2l_5^r - l_6^r)}{F_\pi^2} - 2 \frac{m_\pi - E_i}{m_\pi} \frac{1}{(4\pi F_\pi)^2} J^{0\prime}(2 \frac{m_\pi - E_i}{m_\pi}) \right],
\]
\[
\alpha_{\pi^0}(\bar{q})^2 = -\beta_{\pi^0}(\bar{q})^2 = \frac{e^2}{4\pi} \left( \frac{1}{(4\pi F_\pi)^2 m_\pi} \right) \left( 1 - 2 \frac{m_\pi - E_i}{m_\pi} \right) J^{0\prime}(2 \frac{m_\pi - E_i}{m_\pi}).
\]

At the one-loop level, the $|\bar{q}|^2$ dependence is entirely given in terms of the pion mass $m_\pi$ and the pion-decay constant $F_\pi$, i.e., no additional $\mathcal{O}(p^4)$ low-energy constant enters. At $|\bar{q}|^2 = 0$, Eqs. (23) and (24) reduce to the RCS polarizabilities [1]
\[
\bar{\alpha}_{\pi^\pm} = \bar{\beta}_{\pi^\pm} = \frac{e^2}{4\pi} \frac{2}{m_\pi F_\pi^2} (2l_5^r - l_6^r) = (2.68 \pm 0.42) \times 10^{-4} \text{ fm}^3,
\]
\[
\bar{\alpha}_{\pi^0} = \bar{\beta}_{\pi^0} = -\frac{e^2}{4\pi} \frac{1}{96\pi^2 F_\pi^2 m_\pi} = -0.50 \times 10^{-4} \text{ fm}^3,
\]

where we made use of $J^{0\prime}(0) = -\frac{1}{6}$. At $\mathcal{O}(p^6)$, the RCS predictions for the charged pion read $\bar{\alpha}_{\pi^\pm} = (2.4 \pm 0.5) \times 10^{-4} \text{ fm}^3$ and $\bar{\beta}_{\pi^\pm} = (-2.1 \pm 0.5) \times 10^{-4} \text{ fm}^3$ [13]. The corresponding corrections amount to a 12% (24%) change of the $\mathcal{O}(p^4)$ result, indicating a good convergence. We also note that the original degeneracy $\bar{\alpha} = -\bar{\beta}$ is lifted at $\mathcal{O}(p^6)$. The predictions of ChPT have to be compared with the empirical results $\bar{\alpha}_{\pi^\pm} = (6.8 \pm 1.4) \times 10^{-4} \text{ fm}^3$ [1], $\bar{\alpha}_{\pi^\pm} = (20 \pm 12) \times 10^{-4} \text{ fm}^3$ [1], and $\bar{\beta}_{\pi^\pm} = (-7.1 \pm 4.6) \times 10^{-4} \text{ fm}^3$ [1]. Clearly, an improved accuracy is required to test the chiral predictions. For the neutral pion, the $\mathcal{O}(p^6)$ corrections turn out to be much larger, $\bar{\alpha}_{\pi^0} = (-0.35 \pm 0.10) \times 10^{-4} \text{ fm}^3$ and $\bar{\beta}_{\pi^0} = (1.50 \pm 0.20) \times 10^{-4} \text{ fm}^3$ [23].
VII. ALTERNATIVE DEFINITION OF THE GENERALIZED DIPOLE POLARIZABILITIES

Another generalization of the RCS polarizabilities is obtained by parametrizing the invariant amplitude as

\[ -i\mathcal{M} = B_1 F^\mu\nu F'_{\mu\nu} + \frac{1}{4} B_2 (P_\mu F^\mu\nu') (P^\sigma F'_{\rho\sigma}) + \frac{1}{4} B_3 (P^\mu q^\sigma F'_{\mu\nu} (P^\sigma q^\rho F'_{\rho\sigma}), \tag{27} \]

where \( F^\mu\nu \) and \( F'_{\mu\nu} \) refer, respectively, to the gauge-invariant combinations

\[ F^\mu\nu = -i q^\mu \epsilon^\nu + i q^\nu \epsilon^\mu, \quad F'_{\mu\nu} = i q^\mu \epsilon'^\nu - i q^\nu \epsilon'^\mu. \]

The functions \( B_1, B_2, \) and \( B_3 \) are even functions of \( P \). Introducing the suggestive notation

\[ \vec{E} = i (q_0 \vec{c} - \vec{q} \epsilon_0), \quad \vec{B} = i \vec{q} \times \vec{c}, \quad \vec{E}' = -i (q_0' \vec{c}' - \vec{q}' \epsilon'_0), \quad \vec{B}' = -i \vec{q}' \times \vec{c}' \]

the structures of Eq. \((27)\) are particularly simple when evaluated in the pion Breit frame (p.B.f.) defined by \( \vec{P} = 0 \),

\[ F^\mu\nu F'_{\mu\nu} = \left[ -2 \vec{E} \cdot \vec{E}' + 2 \vec{B} \cdot \vec{B}' \right] \text{p.B.f.,} \]

\[ P_\mu F^\mu\nu P^\nu F'_{\rho\sigma} = \left[ -P_0^2 \vec{E} \cdot \vec{E}' \right] \text{p.B.f.,} \]

\[ P^\nu q^\mu F'_{\mu\nu} P^\rho q^\sigma F'_{\sigma\rho} = \left[ P_0^2 \vec{q} \cdot \vec{E} \right] \text{p.B.f.} \]

Note that by definition \( [P_0^2]_{\text{p.B.f.}} = P^2 \). In the p.B.f., Eq. \((27)\) can thus be expressed as

\[ -i\mathcal{M} = \left[ 2B_1 \vec{B} \cdot \vec{B}' - \left( 2B_1 + \frac{P^2}{4} B_2 \right) \vec{E} \cdot \vec{E}' + \frac{P^2}{4} B_3 \vec{q} \cdot \vec{E} \right] \text{p.B.f.} \tag{28} \]

Since \( \vec{E} = \vec{E}_T + \vec{E}_L \), \( \vec{E} \cdot \vec{E}' \) contains both transverse and longitudinal components with respect to \( \vec{q} \), for which reason we will introduce the quantities \( \alpha_T \) and \( \alpha_L \) below:

\[ -i\mathcal{M} = \left[ 2B_1 \vec{B} \cdot \vec{B}' - \left( 2B_1 + \frac{P^2}{4} B_2 \right) \vec{E}_T \cdot \vec{E}' + \frac{P^2}{4} B_3 |\vec{q}|^2 - \left( 2B_1 + \frac{P^2}{4} B_2 \right) \right] \vec{E}_L \cdot \vec{E}' \text{p.B.f.} \tag{29} \]

We now consider the limit \( \omega' \to 0 \) of the residual amplitudes, for which \( B_1' \to b_1'(q^2) \), and define three generalized dipole polarizabilities in terms of the invariants of Eq. \((27)\),

\[ 8\pi m_\pi \beta(q^2) \equiv 2b_1'(q^2), \tag{30} \]

\[ 8\pi m_\pi \alpha_T(q^2) \equiv -2b_1'(q^2) - \left( M^2 - \frac{q^2}{4} \right) b_2'(q^2), \tag{31} \]

\[ 8\pi m_\pi \alpha_L(q^2) \equiv -2b_1'(q^2) - \left( M^2 - \frac{q^2}{4} \right) [b_2'(q^2) + q^2 b_3'(q^2)], \tag{32} \]

\[ ^2 \text{A detailed discussion will be given in Ref. \[27\].} \]
the superscript \( r \) referring to the residual amplitudes beyond the soft-photon result. In
general, the transverse and longitudinal electric polarizabilities \( \alpha_T \) and \( \alpha_L \) will differ by a
term, vanishing however in the RCS limit \( q^2 = 0 \). Comparing with Eq. (28), the generalized
dipole polarizabilities are seen to be defined such that they multiply the structures \( \vec{B} \cdot \vec{B}' \),
\( \vec{E}_T \cdot \vec{E}'_T \), and \( \vec{E}_L \cdot \vec{E}'_L \), respectively, as \( \omega' \to 0 \). We note that \([\vec{B} \cdot \vec{B}']_{p,B.f.}\) and \([\vec{E}_L \cdot \vec{E}'_L]_{p,B.f.}\) are
of \( \mathcal{O}(\omega') \) whereas \([\vec{E}_T \cdot \vec{E}'_T]_{p,B.f.} = \mathcal{O}(\omega'^2) \), i.e., that different powers of \( \omega' \) have been kept.

At \( q^2 = 0 \), the usual RCS polarizabilities are recovered,

\[
\beta(0) = \tilde{\beta}, \quad \alpha_L(0) = \alpha_T(0) = \bar{\alpha}.
\]

The connection to the generalized polarizabilities of Guichon et al. \cite{33} can either be estab-
lished by direct comparison or via the results of Ref. \cite{28},

\[
\alpha(|\vec{q}|^2) = -\frac{e^2}{4\pi} \sqrt{\frac{3}{2}} P^{(01,01)0}(|\vec{q}|) = \sqrt{\frac{m_\pi}{E_i}} \alpha_L(\omega_0^2 - \vec{q}^2), \tag{34}
\]

\[
\beta(|\vec{q}|^2) = -\frac{e^2}{4\pi} \sqrt{\frac{3}{8}} P^{(11,11)0}(|\vec{q}|) = \sqrt{\frac{m_\pi}{E_i}} \beta(\omega_0^2 - \vec{q}^2), \tag{35}
\]

with \( \omega_0 = q_0|\omega' = 0 = m_\pi - E_i, \ \omega_0^2 - \vec{q}^2 = 2m_\pi(m_\pi - E_i), \) and \( E_i = \sqrt{m_\pi^2 + \vec{q}^2} \), and all
variables referring to the cm frame. Using Eqs. (35) - (37) of Ref. \cite{28}, we find that the
transverse electric dipole polarizability is part of a second-order contribution in \( \omega' \) beyond
the approximation of Guichon et al.,

\[
\alpha_T(q^2) = \alpha_L(q^2) + \frac{e^2}{4\pi} (4M^2 - q^2)q^2 \tilde{f}_3(q^2, 0, 0), \tag{36}
\]

where \( q \cdot P \tilde{f}_3 \equiv f_3 \).

At \( \mathcal{O}(p^4), f_2 = f_3 = 0 \), with the result of particularly simple expressions for the general-
dized dipole polarizabilities,

\[
\alpha_L^{\pi\pm}(q^2) = \alpha_T^{\pi\pm}(q^2) = -\beta^{\pi\pm}(q^2) = \frac{e^2}{8\pi m_\pi} \left[ \frac{4(2l_{5T}^e - l_{6T}^e)}{F_\pi^2} - \frac{q^2}{m_\pi^2} \frac{1}{(4\pi F_\pi)^2} J^{(0)}(\frac{q^2}{m_\pi^2}) \right], \tag{37}
\]

\[
\alpha_L^{\pi\mp}(q^2) = \alpha_T^{\pi\mp}(q^2) = -\beta^{\pi\mp}(q^2) = \frac{e^2}{4\pi} \left( \frac{1}{(4\pi F_\pi)^2 m_\pi} \right) \left[ 1 - \frac{q^2}{m_\pi^2} \right] \frac{1}{J^{(0)}}(\frac{q^2}{m_\pi^2}). \tag{38}
\]

The results for the generalized dipole polarizabilities are shown in Fig. \ref{fig:3}. Even though
chiral perturbation theory is only applicable for small external momenta, for the sake of
completeness we also quote the asymptotic behavior as \( q^2 \to -\infty, \)

\[
\alpha_L^{\pi\pm}(q^2) \to \bar{\alpha}_{\pi\pm} + 3\bar{\alpha}_{\pi\mp} = 1.18 \times 10^{-4} \text{ fm}^3, \tag{39}
\]

\[
\alpha_L^{\pi\mp}(q^2) \to 6\bar{\alpha}_{\pi\mp} = -3.0 \times 10^{-4} \text{ fm}^3. \tag{40}
\]

As in the case of real Compton scattering, we expect the degeneracy \( \alpha_L(q^2) = \alpha_T(q^2) = -\beta(q^2) \) to be lifted at the two-loop level.
VIII. SUMMARY

We have calculated the invariant amplitudes for virtual Compton scattering off the pion, \( \gamma^* + \pi \to \gamma + \pi \), at the one-loop level, \( \mathcal{O}(p^4) \), in chiral perturbation theory. For the charged pion, the result may be decomposed into a gauge-invariant soft-photon amplitude involving the electromagnetic form factor of the pion and a gauge-invariant residual amplitude. For the neutral pion, the soft-photon amplitude vanishes. We have analyzed the low-energy behavior of the residual amplitudes in terms of generalized polarizabilities. In this context we have introduced two alternative definitions of the generalized polarizabilities, a first one based on a multipole expansion in the center-of-mass frame, and a second one based on a covariant approach interpreted in the pion Breit frame. The connection between the different approaches has been established. In the framework of ChPT at \( \mathcal{O}(p^4) \), the momentum dependence of the generalized polarizabilities is entirely predicted in terms of the pion mass and the pion-decay constant, i.e., no additional counter-term contribution appears. As in the case of real Compton scattering, the results at \( \mathcal{O}(p^4) \) show a degeneracy of the polarizabilities, \( \alpha_L(q^2) = \alpha_T(q^2) = -\beta(q^2) \), which we expect to be lifted at the two-loop level.

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REFERENCES

[1] B. R. Holstein, Comments Nucl. Part. Phys. 19, 239 (1990).
[2] A. I. L’vov, Int. J. Mod. Phys. A 8, 5267 (1993).
[3] S. Scherer, Real and Virtual Compton Scattering at Low Energies, Lectures at the 11th Indian-Summer School on Intermediate Energy Physics, Prague, Czech Republic, September 1998, to appear in Czech. J. Phys., Mainz University Report No. MKPH-T-99-1, nucl-th/9901056, 1999.
[4] Yu. M. Antipov et al., Phys. Lett. 121B, 445 (1983).
[5] Yu. M. Antipov et al., Z. Phys. C 26, 495 (1985).
[6] T. A. Aibergenov et al., Czech. J. Phys. B36, 948 (1986).
[7] D. Babusci, S. Bellucci, G. Giordano, G. Matone, A. M. Sandorfi, and M. A. Moinester, Phys. Lett. B 277, 158 (1992).
[8] A. E. Kaloshin and V. V. Serebryakov, Phys. Lett. B 278, 198 (1992).
[9] J. F. Donoghue and B. R. Holstein, Phys. Rev. D 48, 137 (1993).
[10] A. E. Kaloshin and V. V. Serebryakov, Z. Phys. C 64, 689 (1994).
[11] J. Portolèes and M. R. Pennington, in The Second DAΦNE Physics Handbook, edited by L. Maiani, G. Pancheri, and N. Paver (SIS, Frascati, Italy, 1995).
[12] M. V. Terent’ev, Sov. J. Nucl. Phys. 16, 87 (1973).
[13] U. Bürgi, Phys. Lett. B 377, 147 (1996); Nucl. Phys. B479, 392 (1996).
[14] J. Ahrens et al., Few-Body Syst. Suppl. 9, 449 (1995).
[15] M. A. Moinester and V. Steiner, Pion and Kaon Polarizabilities and Radiative Transitions, in Proceedings of the Workshop Chiral Dynamics: Theory and Experiment, Mainz, Germany, September 1997, edited by A. M. Bernstein, D. Drechsel, and Th. Walcher (Springer, Berlin, 1998).
[16] T. Gorringe (spokesman), TRIUMF Expt. E838 (1998).
[17] N. d’Hose, Experiments on Nucleon Polarizabilities, in Proceedings of the Workshop Chiral Dynamics: Theory and Experiment, Mainz, Germany, September 1997, edited by A. M. Bernstein, D. Drechsel, and Th. Walcher (Springer, Berlin, 1998).
[18] P. A. M. Guichon and M. Vanderhaeghen, Prog. Part. Nucl. Phys. 41, 125 (1998).
[19] M. A. Moinester et al. (The SELEX Collaboration), Tel Aviv University Report No. TAUP-2568-99, hep-ex/9903039, 1999.
[20] S. Weinberg, Physica 96A, 327 (1979).
[21] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) 158, 142 (1984).
[22] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985).
[23] J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964).
[24] J. C. Ward, Phys. Rev. 78, 182 (1950).
[25] Y. Takahashi, Nuovo Cim. 6, 371 (1957).
[26] H. W. Fearing and S. Scherer, Few-Body Syst. 23, 111 (1998).
[27] A. I. L’vov, S. Scherer, B. Pasquini, C. Unkmeir, and D. Drechsel, in preparation.
[28] D. Drechsel, G. Knöchlein, A. Metz, and S. Scherer, Phys. Rev. C 55, 424 (1997).
[29] S. Bellucci, J. Gasser, and M. E. Sainio, Nucl. Phys. B423, 80 (1994); ibid. B431, 413 (1994).
[30] R. Barbieri, J. A. Mignaco, and E. Remiddi, Nuovo Cim. 11 A, 824 (1972).
[31] G. ’t Hooft and M. Veltman, Nucl. Phys. B153, 365 (1979).
[32] A. A. Bel’kov, M. Dillig, and A. V. Lanyov, J. Phys. G 23, 823 (1997).
[33] P.A.M. Guichon, G.Q. Liu, and A.W. Thomas, Nucl. Phys. A591, 606 (1995).
FIGURES

FIG. 1. Compton scattering amplitude: Hatched and cross-hatched vertices denote one-particle-reducible and one-particle-irreducible contributions, respectively. All building blocks are renormalized.

FIG. 2. Diagrams contributing to the one-particle-irreducible residual amplitude at $O(p^4)$. Vertices derived from $\mathcal{L}_{2n}$ are denoted by $2n$ in the interaction blobs. $Z$ denotes the wave function renormalization factor corresponding to the Lagrangian of Eq. (3) and the pion field of Eq. (3). At $O(p^4)$, only the contribution from $\mathcal{L}_2$ has to be multiplied by $Z$. 
FIG. 3. $\mathcal{O}(p^4)$ prediction for the generalized dipole polarizabilities $\alpha_L(-Q^2)$ of the charged pion (solid curve) and the neutral pion (dashed curve) as function of $Q^2$ [see Eqs. (37) and (38)]. At $\mathcal{O}(p^4)$, $\alpha_L(q^2) = \alpha_T(q^2) = -\beta(q^2)$. 