Riemann bilinear form and Poisson structure in Hitchin-type systems.

D. Talalaev

Institute for Theoretical and Experimental Physics

Abstract. In this paper we reinterpret the Poisson structure of the Hitchin-type system in cohomological terms. The principal ingredient of a new interpretation in the case of the Beauville system is the meromorphic cohomology of the spectral curve, and the main result is the identification of the Riemann bilinear form and the symplectic structure of the model. Eventual perspectives of this approach lie in the quantization domain.

Contents

1 Model description
1.1 Hitchin-type systems ...................................................... 1
1.2 Rational matrices and separated variables .............................. 2

2 Canonical symplectic form. Hitchin system .............................. 5
2.1 Factorization .............................................................. 5
2.2 Splitting ................................................................. 8

3 Canonical symplectic form. Rational matrices .......................... 8
3.1 Linear dynamics on $Jac$ and Baker-Akhiezer function ............... 9
3.2 Deformations of the spectral curve .................................... 10
3.3 Riemann bilinear form and dynamics .................................. 10
Introduction

For the moment there are too many different examples of integrable systems. The methods of resolution whenever exist bear no resemblance. However the so-called Hitchin-type description fits into such systems as the Toda chain, the Neumann system, all versions of the spin Calogero-Moser system [1], the rational and elliptic Gaudin model with spin [2]. This observation encourage us to develop this language and use this type of coordinates in the quantization problem due to its relative generality.

This paper is organized as follows. In the first section we describe two types of considered models – the Hitchin system and the dynamical system on the space of rational matrices. We recall the construction of separated variables and “action-angle” variables. In this section we represent the Calogero-Moser system in terms of rational matrices. The second section is devoted to the symplectic structure of the Hichin system. We construct such a factorization of the tangent space to the phase space of the system which identify the symplectic form with the Serre pairing on the spectral curve between $H^0(\mathcal{K}_\Sigma)$ and $H^1(\mathcal{O}_\Sigma)$. In the third section we give the analogous but more explicit factorization of the tangent space to the phase space of the integrable system on the space of rational matrices. In this case we also relate the tangent space with the meromorphic cohomology group of the spectral curve and show that the symplectic structure coincides with the Riemann bilinear form. The conclusion contains some open questions in the problem of quantization.

Acknowledgments. The author is grateful to O. Babelon and F. Smirnov for fruitful discussions along the work on this paper; to I. Krichever for numerous advises, especially for the example of the lagrangian connection; to A. Chervov, A. Gorodentsev, A. Samohin and S. Loktev for useful remarks. A part of this work was done during the author’s staying at LPTHE (Université Paris 6). This work was partly supported by RFBR grant 01-01-00546, RFBR grant for support of young scientist 03-01-06236.

1 Model description

1.1 Hitchin-type systems

Originally the Hitchin system was defined in [3] as an integrable system on the cotangent bundle of the moduli space of stable holomorphic bundles of fixed rank $r$ and degree $d$ over the algebraic curve $\Sigma_0$. The Poisson structure is the canonical one on the cotangent bundle. The space of function in involution is constructed as follows: a cotangent vector at a point $E$ of the moduli space $\mathcal{M}$ is an element $\phi \in H^0(End(E) \otimes \mathcal{K})$. There is a well defined function $h_i : T^*\mathcal{M} \rightarrow H^0(\mathcal{K}^\otimes i)$ such that $h_i(E, \phi) = \frac{1}{i} tr \phi^i$. The map

$$h : T^*\mathcal{M} \longrightarrow \oplus_{i=1}^r H^0(\mathcal{K}^\otimes i)$$

is called Hitchin map and realize the algebraic integrability, which means that the fibers are abelian lagrangian varieties of half dimension. The crucial role is played by the
spectral curve that is defined as follows. One can define the bundle morphism (non-linear) $\text{char}(\phi) : \mathcal{K} \to \mathcal{K}^\otimes r$ by

$$\text{char}(\phi)(\mu) = \text{det}(\phi - \mu \ast \text{Id})$$

where $\mu$ is a local section of $\mathcal{K}$ and $\text{Id}$ is the identical global section of $\text{End}(E)$. The spectral curve is the preimage of the zero section in $\mathcal{K}^\otimes r$. It is an algebraic curve $\Sigma$ in the projectivization of the total space of $\mathcal{K}$.

The main problem in this description is the explicit parameterization of the moduli space. There are several achievements in this direction. Firstly one can parameterize holomorphic bundles by Čech cochains. A problematic step is an infinite dimensional reduction to cohomologies. Another approach works in some specific situations [4, 5] and relates to the Narasimhan-Ramanan parameterization [6, 7]. The Hecke-Turin coordinates [8, 9] provide another geometrical way to parameterize the moduli space.

The first level generalization of the Hitchin construction occurs when the base curve is singular. This remarkable situation provides the explicit parameterization of the moduli space in purely linear-algebraic terms. By the other hand it is rather general: such systems as the rational, trigonometric, and elliptic Calogero-Moser models, the rational and elliptic Gaudin systems, the Neumann system and the Toda chain arise in the framework of the Hitchin description on singular curves. This approach appeared in [10] and was elaborated in [11, 12].

The second level generalization crops up in the context of the space of Higgs pairs. A Higgs pair consists of a holomorphic bundle $E$ on $\Sigma_0$ and a bundles homomorphism $\phi : E \to E \otimes L$, where $L$ is some fixed line bundle on $\Sigma_0$ not necessarily the canonical class. The Poisson structure is less geometric in this case, it is given by the Kostant-Kirillov brackets on orbits of the loop group [13, 14, 15]. We show that both generalizations coincide in several examples and it is the subject of future works to establish their general correspondence.

### 1.2 Rational matrices and separated variables

In [16] the following description of integrable systems was proposed: let us consider a family $\text{Spec}$ of spectral curves $\Sigma_H$ defined by the equation

$$R(\lambda, \mu) = R_0(\lambda, \mu) + \sum_{i=1}^{g} H_i R_i(\lambda, \mu) = 0 \quad (1)$$

where $H_i$ are parameters and $R_i$ are fixed polynomials on two variables. An analog of the Hitchin fibration can be obtained in these terms: one takes the space $\mathbb{C}^{2g}$ which is the space of $g$-tuples of points $P_i$ in $\mathbb{C}^2$ with coordinates $(\lambda_i, \mu_i)$. A condition that a curve $\Sigma \in \text{Spec}$ contains $g$ points $P_i$ determines this curve

$$H = -B^{-1}V$$
where \( B_{ij} = R_j(\lambda_i, \mu_i) \) and \( V_i = R_0(\lambda_i, \mu_i) \). Additionally, \( g \) points \( P_i \) gives us a divisor \( D \) of degree \( g \) on this curve. Hence one has the following fibration

\[
\begin{array}{c}
\mathbb{C}^{2g} \\
\downarrow \text{Jac}(\Sigma) \\
\text{Spec}
\end{array}
\]

The total space is endowed with a family of Poisson structures \( \{ \lambda_i, \mu_j \} = \delta_{ij} f(\lambda_i, \mu_i) \). One of the propositions of [16] is that this construction provides an integrable system for all choices of such Poisson brackets, i.e. that the quantities \( H_i \) are in involution. In this case the problem of constructing “angle” variables can be solved by the following

**Lemma 1** The quantities

\[
\phi_i = \sum_k \int_{\lambda_k}^{\lambda_i} \frac{R_i(\lambda, \mu) d\lambda}{\partial_{\mu} R(\lambda, \mu) f(\lambda, \mu)}
\]

are conjugated to \( H_i \)

\[
\{ H_m, \phi_l \} = \delta_{ml}.
\]

**Proof.** Indeed, the equation (1) at the point \( P_j \) imply

\[
R_0(\lambda_j, \mu_j) + \sum_i H_i R_i(\lambda_j, \mu_j) = 0
\]

and

\[
\partial_{\mu} R(\lambda_j, \mu_j) \{ \lambda_k, \mu_j \} + \sum_i R_i(\lambda_j, \mu_j) \{ \lambda_k, H_i \} = 0.
\]

Introducing the notation

\[
C_{ik} = \{ \lambda_k, H_i \}, \quad \tilde{B}_{ji} = - \frac{R_i(\lambda_j, \mu_j)}{\partial_{\mu} R(\lambda_j, \mu_j) f(\lambda_j, \mu_j)}
\]

one obtains equivalently

\[
\tilde{B}C = I
\]

where \( I \) is the identity matrix. Now we calculate the Poisson bracket

\[
\{ H_m, \phi_l \} = - \sum_i \{ \lambda_i, H_m \} \frac{R_i(\lambda_i, \mu_i)}{\partial_{\mu} R(\lambda_i, \mu_i) f(\lambda_i, \mu_i)} = \sum_i C_{mi} \tilde{B}_{il} = \delta_{ml}
\]

due to uniqueness of an inverse matrix. □

Let us show that such kind of description in terms of separated variables arises from a generalization of the Hitchin system – the dynamical system on the space of Higgs pairs. We consider a specific case of the so-called Beauville system. Let us consider the rational curve \( \mathbb{C}P^1 \) and the degree \( n \) linear bundle \( \mathcal{O}(n) \) on it. The semistable bundles on \( \mathbb{C}P^1 \) can be represented in the form \( E = \mathcal{O}^r \otimes \mathcal{O}(m) \) where \( \mathcal{O}^r \) is the trivial holomorphic bundle
of rank $r$. A Higgs field is an element of $H^0(\mathcal{O}^r \otimes \mathcal{O}(n))$ due to the fact that $\text{End}(E)$ is trivial. On the space of such sections there is a family of Poisson structures given as follows: for a divisor $D$ of degree $n$ there is a realization of the space of holomorphic section $H^0(\mathcal{O}^r \otimes \mathcal{O}(n))$ by rational matrices with poles at $D = \sum_{i=1}^k n_i z_i$. This can be done by taking a trivialization of the bundle $\mathcal{O}^r \otimes \mathcal{O}(n)$ over the affine part $\mathbb{C}P^1/D$. On the space of rational matrices with prescribed poles

$$\Phi(z) = \sum_{i=1}^k \Phi_i(z) + \Phi(\infty); \quad \Phi_i(z) = \sum_{j=1}^{n_i} \frac{\Phi_j^i}{(z - z_i)^j},$$

there is a Poisson structure given by the Kostant-Kirillov form on co-adjoint orbits of the loop group represented by polar parts $\Phi_i(z)$. The involutive subalgebra of functions is spanned by coefficients of the characteristic polynomial. The casimirs of co-adjoint orbits are among them. There are too many integrals and that is why one fixes some of them to be central. Different choices of a divisor and central elements imply different integrable systems. Let us demonstrate it on examples.

**Example 1** Consider a divisor $D = z_1 + z_2 + z_3 - \infty$ of degree 2. The corresponding Higgs field is

$$\Phi(z) = \frac{\Phi_1}{z - z_1} + \frac{\Phi_2}{z - z_2} + \frac{\Phi_3}{z - z_3}$$

with the condition $\Phi_1 + \Phi_2 + \Phi_3 = 0$. Let us choose central elements to be $c_l = Tr(\Phi_1^l) + Tr(\Phi_2^l)$ and fix its value $c_l = 0$. These conditions mean that $\Phi_1$ and $-\Phi_2$ belong to the same orbit. In other words there exists such $\Lambda$ that $\Phi_1 \Lambda = -\Lambda \Phi_2$. Resolving the condition $\Phi_1 - \Lambda^{-1} \Phi_1 \Lambda + \Phi_3 = 0$ one obtains the traditional parameterization of the trigonometric Calogero-Moser system (see [11] for details).

One can easily see that the spectral curve $\Sigma$ of the model

$$\text{det}(\Phi(z) - k) = 0,$$

which is constructed as a covering of the rational curve $\mathbb{C}P^1$, can be considered as a covering of a singular curve with the double point. Indeed, the points of $\Sigma$ over $z_1$ and $z_2$ coincide, this implies that $k$ is a function on $\Sigma_{\text{sing}}$, where the singularities on $\Sigma_{\text{sing}}$ arise from the double point on $\mathbb{C}P^1$. The Higgs field in this case corresponds to an element of $H^0(\text{End}(E) \otimes \mathcal{K})$ on the rational curve with the double point for some holomorphic bundle $E$ over it given by $\Lambda$.

**Example 2** Consider a divisor $D = 2z_1 + z_2 - \infty$ of degree 2. The corresponding Higgs field is

$$\Phi(z) = \frac{\Phi_0}{(z - z_1)^2} + \frac{\Phi_1}{z - z_1} + \frac{\Phi_2}{z - z_2}$$

with the condition $\Phi_1 + \Phi_2 = 0$. Let us choose central elements to be $c_l = Tr(\Phi_0^l \Phi_1)$ and fix its values $c_l = 0$. It means that the orbit of $\Phi_1(z)$ contains an element of the form

$$A_0 \frac{1}{(z - z_1)^2}.$$
In other words there exists such $X$ that $\Phi_1 = [X, \Phi_0]$. Resolving the condition $\Phi_1 + \Phi_2 = 0$ one obtains the traditional parameterization of the rational Calogero-Moser system (see [11] for details).

By analogy with the trigonometric case the spectral curve $\Sigma$ of the model, which is constructed as a covering of the rational curve $\mathbb{C}P^1$, can be projected to a covering of a singular curve with the cusp singularity. The Higgs field in this case can be realized as an element of $H^0(\text{End}(E) \otimes \mathcal{K})$ on the rational curve with the cusp singularity for some holomorphic bundle $E$ over it given by $X$. Coincidence of the canonical form on co-adjoint orbits and the form in terms of separated variables was proved in [17].

2 Canonical symplectic form. Hitchin system

2.1 Factorization

We start with an identification between deformations of the spectral curve and holomorphic differentials on it. For a general algebraic curve this relationship can not be established, one has $(3g - 3)$-dimensional space of deformations and only $g$-dimensional space of holomorphic differentials. The spectral curves of considered systems are special; they are parameterized by a $g$-dimensional space. One has the Hitchin map

$$h : T^* \mathcal{M} \to \bigoplus_{i=1}^r H^0(\Sigma_0, \mathcal{K}^{\otimes i})$$

where the right hand side corresponds to the space of spectral curves $Spec$. It is a linear space and one can identify its tangent space $TSpec$ with itself. One also has that the direct image of the structure sheaf $\mathcal{O}_\Sigma$ of the spectral curve is

$$\pi_*(\mathcal{O}_\Sigma) = \mathcal{O}_0 \oplus \mathcal{K}_0^{-1} \oplus \ldots \oplus \mathcal{K}_0^{1-r}$$  \hspace{1cm} (2)

where we have denoted $\mathcal{O}_0 = \mathcal{O}_{\Sigma_0}$ and $\mathcal{K}_0 = \mathcal{K}_{\Sigma_0}$ and $\Sigma_0$ is the base curve. This expresses the fact that locally any function on $\Sigma$ is a polynomial of degree $< r$ on a fiber of the canonical class with coefficients which are functions on the base curve $\Sigma_0$. This consideration provides the following isomorphisms

$$H^0(\mathcal{O}_\Sigma) \cong H^0(\pi_*(\mathcal{O}_\Sigma)); \quad H^1(\mathcal{O}_\Sigma) \cong H^1(\pi_*(\mathcal{O}_\Sigma))$$

due to the fact that the spectral sequence $H^i(\Sigma_0, R^j \pi_* \mathcal{F})$ converges on the first term for coverings of curves. By virtue of Serre duality the second isomorphism is equivalent to the following

$$H^0(\mathcal{K}_\Sigma) \cong H^0(\pi_*(\mathcal{O}_\Sigma)^* \otimes \mathcal{K}_0) = H^0(\bigoplus_{i=1}^r \mathcal{K}_0^{\otimes i}) = TSpec.$$ \hspace{1cm} (3)

The differential of the Hitchin map $dh : T_{(E, \Phi)}(T^* \mathcal{M}) \to H^0(\mathcal{K}_\Sigma)$ has an explicit description occurred in the context of the Whitham hierarchy in [18]. As it was mentioned in the introduction the spectral curve $\Sigma$ lies in the compactification $\tilde{\mathbb{K}}_0$ of the total space of the canonical class of the base curve $\Sigma_0$. 

5
One has the tautological section \( y \in H^0(\widetilde{\mathcal{K}_0}, \pi^*(\mathcal{K}_0)) \). Its restriction to the spectral curve gives us a holomorphic differential over \( \Sigma \). The infinitesimal form of the Hitchin map becomes more clear in a local picture. Let \( U \) be an open set in \( \Sigma_0 \) and \( z \) – a local parameter on \( U \). The canonical class trivializes over \( U \) and the Higgs field \( \Phi \in H^0(\text{End}(E) \otimes \mathcal{K}_0) \) can be represented as \( \Phi(z) = L(z)dz \) where \( L(z) \) is a matrix valued function on \( z \). The spectral curve over \( U \) is a set of pairs \( (z, k) \) which solve the characteristic equation \( \det(L(z) - k) = 0 \). The tautological section in these terms is \( y = kdz \), which is a well defined holomorphic differential over the spectral curve. Now deforming \( \Phi \) one obtains a deformation \( \delta y := \delta kdz \) where by \( \delta k \) we denote a variation of eigenvalues of \( L \).

Furthermore one has the following

\textbf{Lemma 2} Let \( \delta \Phi \in T_{(E, \Phi)}(T^* \mathcal{M}) \) be a tangent vertical vector deforming \( \Phi \) then

\[ dh(\delta \Phi) = \delta y \]

under the identification \( \mathfrak{g} \).

\textbf{Proof.} Let us choose such a covering of the base curve \( \Sigma_0 \) with two charts \( U_1, U_2 \) that the intersection \( U = U_1 \cap U_2 \) does not contain brunching points. We choose also a local parameter \( z \) on \( U \). The canonical class \( \mathcal{K}_0 \) trivializes over \( U \) and we can choose a parameter \( k \) along fibers. The representation \( (2) \) means that \( \mathcal{O}_{\Sigma}(\pi^{-1}U) \) consists of functions \( f \) of the type

\[ f = \sum_{i=1}^{r} f_i(z)k^{i-1}, \quad (2) : f \mapsto \tilde{f} = \{ f_i | i = 1, \ldots, r \}. \]

We will prove that

\[ <dh(\delta \Phi), \tilde{f} > = <\delta y, f >, \]

where on both sides we use appropriate Serre pairings.

\[ <\delta y, f > = \sum_{k=1}^{r} \int_{\partial U_k} f\delta kdz = \sum_{k=1}^{r} \int_{\partial U_k} \sum_{i=1}^{r} f_i(z)k^{i-1}\delta kdz, \]

where \( U_k \) are distinct preimages of \( U : \pi^{-1}U = \sqcup_k U_k \). Using the fact that \( k \) takes the eigenvalues of \( L \) over \( U \) one obtains

\[ <\delta y, f > = \sum_{k=1}^{r} \int_{\partial U_k} \sum_{i=1}^{r} f_i(z)\frac{1}{k}\delta k^i\delta kdz = \int_{\partial U} \sum_{i=1}^{r} f_i(z)\delta \left( \frac{1}{i}TrL^i \right)dz. \]

Let us note that the expression on the right hand side is indeed the pairing between \( H^1(\mathcal{O}_0 \oplus \mathcal{K}_0^{-1} \oplus \ldots \oplus \mathcal{K}_0^{1-r}) \) and \( T \text{Spec} = H^0(\oplus_{i=1}^{r} \mathcal{K}_0^{\otimes i}) \)

Now we return to the Poisson description of the Hitchin system. Hamiltonian flows correspond to the dynamics of a line bundle \( \mathcal{L} \) over \( \Sigma \) constructed from the following exact sequence

\[ 0 \rightarrow \mathcal{L}(-R) \rightarrow \pi^*E \rightarrow \pi^*(\mathcal{E} \otimes \mathcal{K}_0) \rightarrow 0, \]

where \( R \) is the branching divisor of \( \Sigma \) over \( \Sigma_0 \). The line bundle \( \mathcal{L} \) varies over

\[ Jac_\Sigma = \frac{H^1(\mathcal{O}_\Sigma)}{H^1(\Sigma, \mathbb{Z})}. \]
Hence the phase space can be fibered over \( \text{Spec} \) by \( \text{Jac} \) where over a point \( \Sigma \in \text{Spec} \) the fiber is \( \text{Jac}_\Sigma \). A tangent vector to \( \text{Jac}_\Sigma \) is an element of \( H^1(O_\Sigma) \). Let us combine the natural imbedding of vectors tangent to fibers to the tangent space of \( T^*M \) at the point \((E, \Phi)\) and the differential of the Hitchin map to the exact sequence of vector spaces:

\[
0 \rightarrow H^1(O_\Sigma) \xrightarrow{\rho} T_{(E, \Phi)}(T^*M) \xrightarrow{dh} H^0(K_\Sigma) \rightarrow 0. \quad (4)
\]

**Lemma 3** The exact sequence (4) respects the special concordance between the canonical symplectic form \( \omega \) on \( T_{(E, \Phi)}(T^*M) \) and the Serre pairing

\[
< \cdot, \cdot > : H^1(O_\Sigma) \times H^0(K_\Sigma) \rightarrow \mathbb{C}.
\]

Namely, for an element \( \xi \in H^1(O_\Sigma) \) and \( X \in T_{(E, \Phi)}(T^*M) \) one has:

\[
\omega(\rho(\xi), X) = < \xi, dh(X) >.
\]

**Proof.** We consider the same covering as in the previous lemma. Let us call “vertical” a tangent vector \( v \in T_{(E, \Phi)}(T^*M) \) which projects to 0 on \( M \) by the differential of \( \tau : T^*M \rightarrow M \). One can see that each element of \( T_{(E, \Phi)}(T^*M) \) can be decomposed into the sum of a vertical vector field and a hamiltonian vector field at a point \((E, \Phi)\).

The statement is obvious when \( X \) is a value of a hamiltonian vector field. It is equivalent to the fact that the image \( \rho(H^1(O_\Sigma)) \) is lagrangian. Indeed, there is a \( g \)-dimensional space of hamiltonian vector fields which span the tangent space to the Jacobian at each point. Such fields are orthogonal because the hamiltonians are in involution.

Let \( X \) be a vertical tangent vector \( \delta \Phi = \delta Ldz \in T_{(E, \Phi)}(T^*M) \) and \( \delta \mathcal{L} \) be a cocycle deforming the line bundle on \( \Sigma \). The direct image gives us a deformation \( \delta E = \pi_* \delta \mathcal{L} \) of a holomorphic bundle \( E \) on the base curve \( \Sigma_0 \). Let us calculate the symplectic form:

\[
\omega(\rho(\delta \mathcal{L}), \delta \Phi) = \int_{\partial U} Tr(\delta E \delta \mathcal{L}) dz = \sum_i \int_{\partial U_i} \delta \mathcal{L} \delta k dz = \langle \delta \mathcal{L}, dh(\delta \Phi) \rangle,
\]

where the last equality is due to lemma 2. □

**Remark 1** We present here the interpretation of the previous lemma in terms of quasi-isomorphisms. Let us consider the diagram

\[
\begin{array}{ccc}
H^1(O_\Sigma) \oplus T_{(E, \Phi)}(T^*M) & \xrightarrow{p_1} & H^1(O_\Sigma) \oplus H^0(K_\Sigma) \\
\text{T}_{(E, \Phi)}(T^*M) & \xrightarrow{p_2} & \text{T}_{(E, \Phi)}(T^*M)
\end{array}
\]

where \( p_1 \) is a factorization subject to the diagonal embedding

\[
0 \rightarrow H^1(O_\Sigma) \xrightarrow{(id \oplus \rho) \Delta} H^1(O_\Sigma) \oplus T_{(E, \Phi)}(T^*M) \rightarrow \ldots
\]

and \( p_2 = id \oplus dh \). Let \( \omega' \) be the canonical symplectic form on \( H^1(O_\Sigma) \oplus H^0(K_\Sigma) \) defined by Serre duality. Then the lemma above says that

\[
p_1^* \omega = p_2^* \omega'.
\]
2.2 Splitting

One of principal questions in the context of the decomposition \(\mathcal{H}_0(\mathcal{K}_\Sigma)\rightarrow\mathcal{T}_{E,\Phi}(\mathcal{T}^\ast\mathcal{M})\) is a construction of a symplectic splitting of the exact sequence. This means an embedding \(j : \mathcal{H}_0(\mathcal{K}_\Sigma) \rightarrow \mathcal{T}_{E,\Phi}(\mathcal{T}^\ast\mathcal{M})\) such that \(dh \circ j = id|_{\mathcal{H}_0(\mathcal{K}_\Sigma)}\) and

\[
\rho \oplus j : \mathcal{H}_1(\mathcal{O}_\Sigma) \oplus \mathcal{H}_0(\mathcal{K}_\Sigma) \rightarrow \mathcal{T}_{E,\Phi}(\mathcal{T}^\ast\mathcal{M})
\]

is a symplectomorphism, where on the left hand side the symplectic form is defined by Serre duality.

**Lemma 4** The space of such splittings is an open subset in the space of lagrangian subspaces in \(\mathcal{T}_{E,\Phi}(\mathcal{T}^\ast\mathcal{M})\).

**Proof.** Let us note that the image \(\text{Im} j\) is a lagrangian subspace. Inversely, let \(V\) be a lagrangian subspace in \(\mathcal{T}_{E,\Phi}(\mathcal{T}^\ast\mathcal{M})\) such that \(V \cap \rho(\mathcal{H}_1(\mathcal{O}_\Sigma)) = 0\) (it is an open condition).

Then, choosing a basis \(\{e_1,\ldots,e_g\}\) in \(\mathcal{H}_0(\mathcal{K}_\Sigma)\) and the dual basis \(\{e^*_1,\ldots,e^*_g\}\) in \(\mathcal{H}_1(\mathcal{O}_\Sigma)\) let us construct the splitting \(j\) as follows:

\[
j(e_i) = f_i
\]

where \(\{f_1,\ldots,f_g\}\) is the basis in \(V \subset \mathcal{T}_{E,\Phi}(\mathcal{T}^\ast\mathcal{M})\) dual to \(\{\rho(e^*_1),\ldots,\rho(e^*_g)\}\). The condition \(dh \circ j = id|_{\mathcal{H}_0(\mathcal{K}_\Sigma)}\) fulfills due to lemma 3. \(\square\)

**Remark 2** In the rest of the paper we will consider several natural ways to construct such a splitting. We also hope that the main ingredient – “lagrangian connection” – plays a crucial role in the problem of quantization.

3 Canonical symplectic form. Rational matrices

Firstly let us show that there is a natural lagrangian connection in the case of Beauville system. The phase space is the space of \(g\) points \((\lambda_i, \mu_i)\) in \(\mathbb{C}^2\). There is an exact sequence similar to the case of the Hitchin system:

\[
\mathcal{T}\mathcal{J}ac_\Sigma \rightarrow \mathcal{T}\mathbb{C}^{2g} \xrightarrow{dh} \mathcal{T}_\Sigma\text{Spec.}
\]

The first injection is lagrangian. A splitting is an injection \(j\) inverting \(dh\):

\[
j : \mathcal{T}_\Sigma\text{Spec} \hookrightarrow \mathcal{T}\mathbb{C}^{2g}
\]

such that \(dh \circ j = id|_{\mathcal{T}_\Sigma\text{Spec}}\) and \(\text{Im} j\) is lagrangian. Let us fix \(\{\lambda_i\}\) and obtain a deformation of the spectral curve and the divisor on it by varying only the \(\mu\)-coordinates of points. This deformation is obviously lagrangian due to the description of the Poisson structure in terms of separated variables. This allows us to use the following identification

\[
\mathcal{T}\mathbb{C}^{2g} \cong \mathcal{H}_1(\mathcal{O}_\Sigma) \oplus \mathcal{H}_0(\mathcal{K}_\Sigma).
\]

In this section we also identify tangent vectors to the Jacobian with meromorphic differentials on the spectral curve by means of the Baker-Akhiezer function and construct the isomorphism

\[
\mathcal{T}\mathbb{C}^{2g} \cong \mathcal{H}_0(\Sigma, \Omega^1_{\mathfrak{m}}). \quad (6)
\]
where $\Omega^1_{\log}$ is the sheaf of meromorphic 1-forms of the second kind on the spectral curve.

We prove that the symplectic structure coincides with the Riemann bilinear form in terms of the right hand side of (6).

### 3.1 Linear dynamics on $Jac$ and Baker-Akhiezer function

**Remark 3** Here we demonstrate the correspondence between the Jacobian which is isomorphic to $Pic_0$ and the Čech realization of the group of holomorphic degree 0 line bundles in $H^1(\Sigma, O^*)$. We have to construct a representative cocycle for the point of $Jac$. The main technique is the Abel transform and its inverse.

For the first let us consider the situation of the dynamics linearized on the Jacobian by components of the standard Abel transform. It means that the coefficients $R_i$ define holomorphic differentials on the spectral curve $\Sigma$. The dynamics on $Jac$ can be represented by $X(t) = X(0) + tV$. The vector $V$ corresponds to the line bundle $L_0$ on $\Sigma$ of degree 0 constructed as follows. The line bundles on $\Sigma$ can be obtained by the inverse image of the Abel transform $A^*$ from the line bundles on $Jac$. The expression

$$s(X) = \frac{\theta(X + V)}{\theta(X)}$$

defines a line bundle on the Jacobian and has monodromy properties

$$s(X + l) = s(X), \quad s(X + Bl) = s(X) \exp(-2\pi i(l, V)),$$

where $l \in \mathbb{Z}^g$ and $B$ is the matrix of $b$-periods of holomorphic differentials. Now we fix a point $P_0$ on the spectral curve and consider the space of normalized meromorphic differentials of the second kind with poles at $P_0$. Its factor by exact differentials is $g$-dimensional. We can find such a differential $\Omega_0$ which has specific $b$-periods

$$\int_{b_i} \Omega_0 = V_i. \quad (7)$$

Now the expression

$$\psi(P) = s(A(P)) \exp(2\pi i \int^{P} \Omega_0)$$

is a well defined function on the spectral curve which has an essential singularity at $P_0$ and apart from $P_0$ its divisor is the Abel inverse of $V$. This function defines a line bundle on the spectral curve with the same divisor as follows: we define a covering $(U_0, U_1)$ of $\Sigma$ such that $U_1 = \Sigma/P_0$ and $U_0$ is a disk centered at $P_0$ which is sufficiently small to do not contain points of the divisor. We take the principal part $\psi_{ess}$ of the function $\psi$ at the point $P_0$ and declare it to be a transition function for the chosen covering. There is a meromorphic section $\psi$ on $U_1$ and $\psi/\psi_{ess}$ on $U_0$. It has the same divisor. We arrive to the classical lemma (see [19] and references therein)

**Lemma 5** The linear dynamics on the Jacobian can be represented in terms of the transition function $f_{01}$ subject to the chosen covering $(U_0, U_1)$ by follows

$$f_{01}(t) = f_{01}(0) \exp(t * \ln(\psi_{ess})). \quad (8)$$
This lemma finishes the construction of the correspondence between the tangent space of the Jacobian and meromorphic differentials of the second kind with poles at $P_0$. The choice of a family of spectral curves and a divisor of poles of meromorphic differentials characterizes different types of integrable systems. For example in the case of the Calogero-Moser system one needs to take meromorphic differentials with poles of the second order at points $P_i$ on the spectral curve which lie over $\infty \in \mathbb{C}P^1$. But in all cases there is the correspondence obtained above in terms of cohomologies.

### 3.2 Deformations of the spectral curve

Let us consider the special case of a spectral curve

$$R(\lambda, \mu) = R_0(\lambda, \mu) + \sum_i H_i R_i(\lambda, \mu) = 0$$

where the differentials

$$\tilde{\omega}_i = \frac{R_i(\lambda, \mu)d\lambda}{\partial_\mu R(\lambda, \mu)}$$

are holomorphic.

**Remark 4** Let us note that all mentioned systems can be represented in this form, at least birationally. An interesting question is geometric interpretation of the dynamics when the differentials linearizing Hamiltonian flows are meromorphic. The case of the spin generalization of the Calogero-Moser system gives us a partial response to this question. In fact one should consider the generalized Jacobian where the lattice is generated by the periods of holomorphic and some meromorphic differentials. And it can be identified with the Jacobian of a singular spectral curve obtained by singularization of the base rational curve.

In this case the identification between differentials and tangent vectors can be realized in a more direct way. The space of spectral curves is parameterized by coordinates $H_i$, one can identify $\tilde{\omega}_i$ with the tangent vector in $i$-th direction. This assignment is invariant: a linear change of coordinates $H_i$ involves an inverse linear change in terms of differentials due to the equation of the curve. This is in agreement with the tensor low for the tangent space. Hence a tangent vector deforming the spectral curve is identified with a holomorphic differential $w \in H^0(K_\Sigma)$.

### 3.3 Riemann bilinear form and dynamics

Another point of view on the phase space of the system on the space of rational matrices involves holomorphic and meromorphic differentials. As we have already seen the meromorphic differentials on the spectral curve correspond to the deformations of the line bundle and the holomorphic differentials give the deformations of the spectral curve. Using the splitting of the exact sequence we can establish an isomorphism between the tangent space to the phase space and the meromorphic cohomology space $H^0(\Sigma, \Omega^1_{\Sigma})$.  

10
where $\Omega_{2n}^1$ is the space of meromorphic differentials of the second kind. There is a natural bilinear form on $H^0(\Sigma, \Omega_{2n}^1)$

$$<\omega_1, \omega_2> = \omega_R(\omega_1, \omega_2) = \sum_{i=1}^g \left( \int_{a_i} \omega_1 \int_{b_i} \omega_2 - \int_{b_i} \omega_1 \int_{a_i} \omega_2 \right). \quad (9)$$

Now we choose the normalized bases of differentials: $\omega_i$ – holomorphic differentials and $\Omega_i$ – meromorphic differentials of the second kind with poles of order $i + 1$ at the chosen point $P_0$ such that

$$\int_{a_i} \omega_i = \delta_{ij}; \quad \int_{a_i} \Omega_j = 0.$$

In this basis the matrix of the Riemann bilinear form $[9]$ is

$$\omega_R = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}$$

where $B_{ij} = \int_{b_i} \Omega_j$.

**Theorem 1** The symplectic structure associated with the canonical Poisson structure on $T^*\mathbb{C}^{2g}$ viewed as a bilinear form on the space of meromorphic differentials coincides with the Riemann bilinear form $\omega_R$.

**Proof.** We proceed by evaluating the Hamiltonian map. As we have seen a flow corresponding to the Hamiltonian $H_k$ linearizes by $\int^P \tilde{\omega}_k$ where

$$\tilde{\omega}_k = \frac{R_k d\lambda}{\partial\mu R}.$$

One has a linear decomposition

$$\tilde{\omega}_k = \sum_i A_{ki}\omega_i.$$  

A tangent vector $V_k$ on the Jacobian defined as the hamiltonian flow of $H_k$ can be represented as the $k$-th column of $A^{-1}$ in the basis related to chosen holomorphic differentials $\omega_i$. Let us note that $dH_k$ has the same coordinates in terms of $d\tilde{H}_i$, where $\tilde{H}_i$ correspond to $\omega_i$. The meromorphic differential deforming the line bundle $\Omega_0$, defined by (7) for the $k$-th flow, can be given in coordinates associated with the chosen basis of differentials by $\Omega_0 = B_{ij}^{-1} A_{jk}^{-1} \Omega_i$. It demonstrates that the Hamiltonian vector field related to the meromorphic differential $\Omega_0$ is obtained by $\beta(dH_k)$, where $\beta$ is the bivector corresponding to the symplectic form $\omega_R$. This verifies that the off-diagonal blocks of comparing symplectic structures coincide. The diagonal blocks are zero in both cases. □

In the dual language this bilinear form can be represented by a very natural structure. The cotangent space to the phase space of considered system can be identified with $H_1(\Sigma, \mathbb{C})$.

**Lemma 6** The Poisson bivector is the intersection index form in terms of cycles $H_1(\Sigma, \mathbb{C})$.  

11
Proof. A bivector corresponding to the symplectic form $\omega$ is defined as follows: the symplectic form defines a map $\omega^\# : T(V) \to T^*(V)$

$$\omega^\#(v) = i(v)\omega.$$ 

Now a bivector is defined by

$$\beta(\omega^\#(X), \omega^\#(Y)) = \omega(X,Y).$$

A covector $\omega^\#(X)$ can be represented by

$$\omega^\#(X) = \sum_i \left( \int_{a_i} X \right) b_i - \sum_i \left( \int_{b_i} X \right) a_i;$$

one has a similar expression for $\omega^\#(Y)$. Inserting these covectors into $\beta$ for all $X, Y$ one obtains

$$\beta(b_i, b_j) = \beta(a_i, a_j) = 0; \quad \beta(a_i, b_j) = \delta_{ij}; \quad \beta(b_i, a_j) = -\delta_{ij}.$$ 

This bivector coincides with the intersection index.■

Remark 5 The proof of the theorem was given in the case of meromorphic differentials with one fixed pole of high order. We used this condition only to normalize differentials. The proof in the general case is strictly analogous.

Conclusion

- Lemma 6 gives us intuition of a geometric interpretation of the quantization: locally the classical Poisson algebra can be identified with a free commutative algebra generated by 1-cycles $H_1(\Sigma, \mathbb{Z})$ with the Poisson structure given by the intersection index. There is an obvious but rather tautological quantization of this Poisson algebra in geometric terms. One has a noncommutative multiplication on $H^*(\Sigma)$ which is just a wedge product. On the first cohomologies it restricts as

$$H^1(\Sigma) \times H^1(\Sigma) \to H^2(\Sigma).$$

Due to Poincaré duality one can identify $H^1(\Sigma, \mathbb{Z})$ with $H_1(\Sigma, \mathbb{Z})$ and $H^2(\Sigma, \mathbb{Z})$ with $H_0(\Sigma, \mathbb{Z})$. In fact this anticommutative multiplication on cycles coincides with the intersection index

$$c_1 * c_2 = \#(c_1, c_2).$$

It corresponds to the quantization condition up to the constant 2

$$c_1 * c_2 - c_2 * c_1 = 2\#(c_1, c_2).$$

A question here is to investigate different ways of quantizing this Poisson algebra and to understand its geometric meaning. Some recent advances in this direction were realized in [20] where the deformed abelian differentials were introduced and a noncommutative multiplication over them was constructed. Let us mention that this multiplication is far from being trivial, however it is anticommutative.
We hope that the appearance of the new object in the context of Hitchin-type systems – the lagrangian connection, which we used for splitting the factorization of the tangent space to the phase space, is not accidental. Eventually it is not significant on the classical level. However it contains some intrinsic geometrical data which could be responsible for quantum deformations.

References

[1] I. Krichever, O. Babelon, E. Billey, M. Talon, *Spin generalization of the Calogero-Moser system and the Matrix KP equation*. Preprint LPTHE 94/42. Amer. Math. Transl. **170** (1995) N2, 83-119.

[2] D. Talalaev *The elliptic Gaudin system with spin*. Theor.Math.Phys. **130** (2002) 361-374

[3] N. Hitchin, *Stable bundles and integrable systems*. Duke Math. Journal 1987 V **54** N1 91-114.

[4] Bert van Geemen, Emma Previato, *On the Hitchin System*. Duke Math. J. **85** (1996) 659-684.

[5] Krzysztof Gawedzki, Pascal Tran-Ngoc-Bich, *Hitchin Systems at Low Genera*. J.Math.Phys. **41** (2000) 4695-4712.

[6] M.S. Narasimhan, S. Ramanan, *Moduli of vector bundles on a compact Riemann surface*. Ann. of Math. **89** (1969), pp. 19-51.

[7] M.S. Narasimhan, S. Ramanan, *2Θ linear systems on Abelian varieties*. in Vector Bundles on algebraic varieties, Oxford Univ. Press, 1987, pp. 415-427.

[8] I. Krichever, *Vector bundles and Lax equations on algebraic curves*. Commun.Math.Phys. **229** (2002) 229-269; hep-th/0108110.

[9] B. Enriquez, V. Rubtsov, *Hecke-Tyurin parametrization of the Hitchin and KZB systems*. math.AG/9911087

[10] N. Nekrasov, *Holomorphic bundles and many-body systems*. PUPT-1534, Comm. Math. Phys.,**180** (1996) 587-604; hep-th/9503157.

[11] A. Chervov, D. Talalaev, *Hitchin system on singular curves I*. hep-th/0303069

[12] A. Chervov, D. Talalaev, *Hitchin system on syngular curves II. Schematic points to appear*

[13] A. Beauville, *Jacobienes des courbes spectrales et systèmes hamiltoniens complètement intégrables*. Acta Math. **164**(1990), 211-235.

[14] R. Donagi, E. Markman, *Spectral cover, algebraically completely integrable, Hamiltonian systems, and moduli of bundles*. alg-geom/9507017
[15] J. Harnad, J.C. Hurtubise, *Multi-hamiltonian structures for r-matrix systems.* [math-ph/0211076](math-ph/0211076).

[16] O. Babelon, M. Talon, *Riemann surfaces, separated variables and quantum integrability.* [hep-th/0209071](hep-th/0209071).

[17] O. Babelon, M. Talon, *The symplectic structure of rational Lax pair systems.* [solv-int/9812009](solv-int/9812009).

[18] I. Krichever, *The $\tau$-function of the universal Whitham hierarchy, matrix models and topological field theories.* [hep-th/9205110](hep-th/9205110), LPTENS–92/18.

[19] M.A. Semenov-Tian-Shansky, *Integrable systems and factorization problem.* [nlin.SI/0209057](nlin.SI/0209057).

[20] F.A. Smirnov, *Baxter equation and deformation of abelian differentials.* [math-ph/0302014](math-ph/0302014).