This paper shows that immersed totally geodesic \( m \)-dimensional suborbifolds of \( n \)-dimensional arithmetic hyperbolic orbifolds correspond to finite subgroups of the commensurator whenever \( m \geq \lfloor \frac{n}{2} \rfloor \). We call such totally geodesic suborbifolds finite centraliser subspaces (or fc-subspaces) and use them to formulate an arithmeticity criterion for hyperbolic lattices.

We show that a hyperbolic orbifold \( M \) is arithmetic if and only if it has infinitely many fc-subspaces, exhibiting examples of non-arithmetic orbifolds that contain non-fc subspaces of codimension one. We provide an algebraic characterisation of totally geodesically immersed suborbifolds of arithmetic hyperbolic orbifolds by analysing Vinberg’s commensurability invariants. This allows us to construct examples with the property that the adjoint trace field of the geodesic suborbifold properly contains the adjoint trace field of the orbifold. The case of special interest is that of exceptional trialitarian 7-dimensional orbifolds. We show that every such orbifold contains a totally geodesic arithmetic hyperbolic 3-orbifold of exceptional type.

Finally, we study arithmetic properties of orbifolds that descend to their totally geodesic suborbifolds, proving that all suborbifolds in a (quasi-)arithmetic orbifold are (quasi-)arithmetic.
1. Introduction

Let \( \mathbb{H}^n \) be the real hyperbolic \( n \)-space and \( G = \text{PO}_{n,1}(\mathbb{R}) = \text{Isom}(\mathbb{H}^n) \) be its isometry group. Here and below a hyperbolic lattice is a discrete subgroup of \( \text{Isom}(\mathbb{H}^n) \) having finite covolume with respect to the Haar measure or, equivalently, admitting a fundamental polytope \( P \subset \mathbb{H}^n \) of finite volume. In addition, a lattice is called uniform if it is cocompact or, equivalently, it admits a compact fundamental polytope. Otherwise, a lattice is called non-uniform.

Given a lattice \( \Gamma \subset G \), the associated quotient space \( M = \mathbb{H}^n/\Gamma \) is a finite volume hyperbolic orbifold. It is a manifold when \( \Gamma \) is torsion-free.

In this paper we attempt to give a more particular description of finite volume totally geodesic immersed suborbifolds (totally geodesic subspaces, for short) of hyperbolic orbifolds and manifolds \( \mathbb{H}^n/\Gamma \), with \( \Gamma \) being a lattice in \( G \). Our special interest lies with the cases of arithmetic, quasi-arithmetic and pseudo-arithmetic lattices [55, 19]. Their exact definitions, as well as other properties and results that we shall essentially use, will follow in Section 2.

We shall distinguish three types of lattices that provide an exhaustive description of all arithmetic lattices in \( \text{PO}_{n,1}(\mathbb{R}) \). Essentially, type I arithmetic lattices come from admissible quadratic forms, type II arithmetic lattices come from skew–Hermitian forms (or, equivalently, Hermitian forms) over quaternion algebras, and type III arithmetic lattices comprise one exceptional family in \( \text{PO}_{4,1}(\mathbb{R}) \) and another in \( \text{PO}_{7,1}(\mathbb{R}) \) (the so-called “trialitarian lattices”, cf. Section 3.4).
Let $\text{Comm}(\Gamma)$ be the commensurator of $\Gamma$ in $G$, and for a given group $F < \text{Comm}(\Gamma)$ let $\text{Fix}(F) = \{x \in \mathbb{H}^n | gx = x, \ \forall g \in F\}$ be the fixed point set of $F$ in $\mathbb{H}^n$.

**Definition 1.1.** An immersed totally geodesic suborbifold $N$ of a hyperbolic orbifold $M = \mathbb{H}^n/\Gamma$ is called a finite centraliser subspace (an fc–subspace for short) if there exists a finite subgroup $F < \text{Comm}(\Gamma)$ such that $U = \text{Fix}(F)$ is a totally geodesic subspace of $\mathbb{H}^n$ and $N = U/\text{Stab}_\Gamma(U)$.

As we prove in Theorem 1.9, an fc–subspace of dimension at least 2 of a finite volume hyperbolic orbifold $M$ is also a finite volume hyperbolic orbifold.

The definition of fc–subspaces is motivated by two classes of examples. Let $P$ be a finite-volume hyperbolic Coxeter $n$-polytope, $U \subset \mathbb{H}^n$ a totally geodesic subspace which supports a face $F$ of $P$ and denote by $\Gamma$ the Coxeter group generated by reflections in the facets of $P$. The quotient of $U$ under the action of its stabiliser in the group $\Gamma$ has finite covolume whenever $m = \dim(U) \geq 2 \ [1, 2]$. In a similar fashion, one can construct further examples of totally geodesic subspaces by considering the fixed-point set $U$ of a finite group $F$ of symmetries of a hyperbolic orbifold $M = \mathbb{H}^n/\Gamma$. In this setting the group $F$ is a subgroup of the normaliser of $\Gamma$ in $\text{Isom}(\mathbb{H}^n)$, and $U$ projects to an fc-subspace $N \subset M$ under the action of its stabiliser in $\Gamma$.

We shall assume throughout that the dimension of an fc–subspace is positive. Note that we do not exclude one–dimensional fc–subspaces. The central result of this paper is the following theorem.

**Theorem 1.2.** Let $M = \mathbb{H}^n/\Gamma$ be a finite volume hyperbolic $n$–orbifold. We have:

1. If $M$ is arithmetic, then it contains infinitely many fc–subspaces of positive dimension. Moreover, all totally geodesic suborbifolds of $M$ of dimension $m \geq \left\lceil \frac{n}{2} \right\rceil$ which are not 3-dimensional type III are fc–subspaces.
2. If $M$ is non–arithmetic, then it has finitely many fc–subspaces, their number being bounded above by $c \cdot \text{vol}(M)$, with a constant $c = \text{const}(n)$ depending only on $n$.

Notice that the condition that the totally geodesic subspace is not 3-dimensional type III in (1) is only needed when $4 \leq n \leq 7$. We can actually show that the condition is only needed for $6 \leq n \leq 7$. Indeed, the algebraic properties of 3-dimensional type III lattices imply that the corresponding orbifolds can only be geodesically immersed as suborbifolds of codimension $\geq 3$.

Let us put this theorem in a more general perspective. Parts (1) and (2) of the theorem show dichotomy for the fc–subspaces of arithmetic and non-arithmetic hyperbolic orbifolds. In a fundamental recent work by Bader,
Fisher, Miller, and Stover [3], and independently by Margulis and Moham-
madi [39] (for dimension $n = 3$ only), the arithmeticity of hyperbolic man-
ifolds is established in terms of the existence of infinitely many maximal
totally geodesic subspaces of dimension at least 2 (thereby excluding 1–
dimensional geodesics). These results provide a sufficient condition for the
arithmeticity of hyperbolic orbifolds, but do not detect arithmetic hyper-
bolic 2–orbifolds (obviously) and 3–dimensional orbifolds of type II and III,
as they contain no totally geodesic immersed 2–dimensional orbifolds. In the
3–dimensional case, we provide an alternative proof of a result of Lackenby–
Long–Reid [32, Proof of Theorem 1.2] (see also [15, Lemma 2.1]). This allows
us to detect the arithmeticity of these orbifolds by exploiting the existence
of infinitely many fc–geodesics:

**Corollary 1.3.** Let $M$ be a finite volume hyperbolic 3–orbifold. Then $M$ is
arithmetic if and only if $M$ contains infinitely many fc–geodesics. Moreover,
in the arithmetic case, all immersed totally geodesic surfaces and curves are fc–subspaces.

In this regard, part (1) of Theorem 1.2 can be also viewed as a general-
isation to arbitrary dimensions of the aforementioned result of Lackenby–
Long–Reid. Our methods also allow to detect arithmeticity of hyperbolic
surfaces:

**Corollary 1.4.** A finite area hyperbolic surface is arithmetic if and only if
all of its infinitely many closed geodesics are fc–subspaces.

In proving Theorem 1.2, we construct fc-subspaces of dimension $\geq 2$ in
all arithmetic hyperbolic orbifolds of dimension $\geq 4$. The most interesting
case is that of type III lattices in $\text{PO}_{7,1}(\mathbb{R})$ for which we are able to prove
the following property:

**Theorem 1.5.** Every 7-dimensional type III orbifold $M$ contains a 3-dimen-
sional type III totally geodesic fc-subspace.

This theorem implies that the main result in [3] becomes an arithmeticity
criterion detecting all arithmetic orbifolds of dimension $\geq 4$:

**Corollary 1.6.** Let $M = \mathbb{H}^n/\Gamma$ be a finite volume hyperbolic $n$–orbifold,
with $n \geq 4$. Then $M$ is arithmetic if and only if it contains infinitely many
maximal totally geodesic subspaces of dimension $\geq 2$.

The proofs in [3] and [39] are based on new powerful superrigidity the-
orems. In this regard our work only relies on the classical Margulis super-
rigidity that appears in his proof of the arithmeticity theorem [38, Chapter
IX]. On the other hand, we require a much more detailed analysis of the
algebraic structure of arithmetic subgroups. An advantage of fc-subspaces
compared to the general totally geodesic subspaces is that they are more
concrete and amenable for constructive arguments.

In his study of lattices in $\text{PO}_{n,1}(\mathbb{R})$ Vinberg introduced two commensu-
rability invariants: the adjoint trace field $k$, which is an algebraic number
field, and the ambient group, an algebraic $k$-group $G$ whose identity component $G^0$ is a $k$-form of the real group $\text{PO}_{n,1}$ if $n$ is even, or $\text{PSO}_{n,1}$ if $n$ is odd (see Section 3). We analyse the relation between the adjoint trace field of a hyperbolic orbifold $M = \mathbb{H}^n/\Gamma$ (i.e. the adjoint trace field of the lattice $\Gamma$) and the adjoint trace field of a totally geodesic immersed suborbifold $N = \mathbb{H}^m/\Lambda$, proving that (quasi–)arithmeticity is inherited by totally geodesic suborbifolds:

**Theorem 1.7.** Let $M$ be a quasi–arithmetic hyperbolic orbifold with adjoint trace field $k$, and $N \subset M$ be a finite volume totally geodesic suborbifold of dimension $m \geq 2$ with adjoint trace field $K$. Then $N$ is hyperbolic and quasi–arithmetic, and $k \subseteq K$. If $M$ is arithmetic, then $N$ is arithmetic as well.

The inclusion of trace fields expressed by Theorem 1.7 is slightly counterintuitive. If $M = \mathbb{H}^n/\Gamma$ and $U \subset H^m$ is a lift of $N$ to $H^m$, then the stabiliser $\text{Stab}_\Gamma(U)$ of $U$ in $\Gamma$ is a subgroup of $\Gamma$ and the field generated by the traces of the adjoint action of $\text{Stab}_\Gamma(U)$ on the Lie algebra of $O_{n,1}(\mathbb{R})$ is a subfield of the adjoint trace field $k$ of $M$. However, in order to compute the adjoint trace field $K$ of $N$ one has to factor out the action of $\text{Stab}_\Gamma(U)$ on the orthogonal bundle of the subspace $U$.

In other words, the stabiliser of $U$ in $O_{n,1}(\mathbb{R})$ is isomorphic to the product $O_{m,1}(\mathbb{R}) \times O_{n-m}(\mathbb{R})$, and the adjoint trace field $K$ of $N$ is obtained by extending $\mathbb{Q}$ with the traces of the projection to the Lie algebra of $O_{m,1}(\mathbb{R})$ of the adjoint action of $\text{Stab}_\Gamma(U)$. The proper inclusion of trace fields $k \subset K$ then follows from Borel’s Density Theorem and the analysis of the algebraic properties of the projection map $\text{Stab}_\Gamma(U) \to O_{m,1}(\mathbb{R})$.

There are several situations in which $k = K$, i.e. the trace field of $M$ coincides with that of the totally geodesic subspace $N$. For instance, this happens whenever $M$ is quasi–arithmetic and $N$ has codimension one. In the case of a quasi–arithmetic reflection group, Coxeter faces of the corresponding fundamental polytope are quasi–arithmetic over the same field of definition [9].

The work of Emery and Mila [19] shows that any Gromov–Piatetski-Shapiro manifold contains a totally geodesic subspace with a smaller adjoint trace field. Combining this with Theorem 1.7 we obtain an alternative proof of the non-arithmeticity of these manifolds (see Remark 5.1). This argument for verifying non-arithmeticity may apply to other locally symmetric spaces as well.

The fact that the adjoint trace field of a geodesic submanifold $N$ can be larger than the one of the arithmetic ambient manifold $M$ appears to be a previously unknown phenomenon and has some profound consequences which we explore thoroughly in the rest of this paper.

Following Theorem 1.7, we define a totally geodesic subspace $N$ of an arithmetic hyperbolic orbifold $M$ to be a subform space if its adjoint trace
field coincides with that of \( M \), and further refine the analysis of Vinberg’s
commensurability invariants in the arithmetic case by proving the following:

**Theorem 1.8.** Let \( N = \mathbb{H}^m/\Lambda \) be a totally geodesic subspace of an
arithmetic hyperbolic orbifold \( M = \mathbb{H}^n/\Gamma \). Suppose that \( N \) is not a 3-dimensional
type III orbifold and that \( [K : k] = d \geq 1 \), where \( K \) (resp. \( k \)) denotes the
adjoint trace field of \( \Lambda \) (resp. \( \Gamma \)). Then there exists a unique minimal subform space \( S \subseteq M \) of dimension \((m + 1) \cdot d - 1\) such that \( N \subseteq S \), and there
is no proper subform space of \( S \) which contains \( N \).

The assumption that the subspace \( N \) is not 3-dimensional type III in the
statement of Theorem 1.8 ensures that the field \( K \) is totally real. This is
only needed to determine the dimension of \( S \).

In order to prove Theorems 1.7 and 1.8 we analyse the relation between
the ambient group \( G \) of an arithmetic orbifold \( M \) and the ambient group
\( L \) of the totally geodesic suborbifold \( N \subset M \). Denote by \( U \) a lift of \( N \) to
the universal cover \( \mathbb{H}^n \) of \( M \). The group \( G \) contains a closed, admissible,
\( k \)-defined subgroup \( H \) such that \( H(\mathbb{R}) < \text{Stab}_{G(\mathbb{R})}(U) \), and \( H \) is \( k \)-isogenous
to a product \( C \times \text{Res}_{K/k}(L) \), where \( C \) is a \( k \)-group such that \( C(k) \) is compact
and \( \text{Res}_{K/k}(L) \) denotes the Weil restriction from \( K \) to \( k \) of the group \( L \).

This fact has as a consequence that a totally geodesic immersion \( N \subseteq M \)
of arithmetic hyperbolic orbifolds is a composition of two geodesic immersions
\( N \subseteq S \subseteq M \). The immersion \( N \subseteq S \) is determined by the factor
\( \text{Res}_{K/k}(L) \), and \( N \) is a Weil restriction subspace of \( S \), while the embedding
of \( S \) into \( M \) as a subform space depends on the compact factor \( C \). We provide
a description of totally geodesic immersions of arithmetic hyperbolic orbifolds obtained as either Weil restriction subspaces or subform spaces in
Section 4. In combination with Theorem 1.8, this gives a complete classification
of geodesic immersions between non-exceptional arithmetic hyperbolic
orbifolds.

The work of Bergeron and Clozel [7] actually implies that 7–dimensional
type III orbifolds do not admit totally geodesic immersions into higher
dimensional arithmetic hyperbolic orbifolds: if such a space were totally
geodesically immersed in a type I or type II space, then by passing to a suf-
ficiently large congruence cover one would create a non–zero first homology
class in the covering manifold (cf. [6, Corollary 1.8]), and by restriction (and
injectivity of the stable restriction map in cohomology) one would contra-
dict the main result of [7]. Theorem 1.5 shows that certain 3-dimensional
type III orbifolds immerse in 7–dimensional type III spaces. The question
remaining open is if there exist any other immersions between arithmetic
hyperbolic orbifolds which involves type III spaces.

The next theorem is akin to a well–known fact for uniform lattices in Lie
groups: see Lemma 4.4 in [57] or combine Theorem 1.13 with Lemma 1.14
in [46]:

**Theorem 1.9.** Let \( \Gamma < \text{Isom}(\mathbb{H}^n) \) be a (uniform) lattice and \( F < \text{Isom}(\mathbb{H}^n) \)
be a finite subgroup, such that \( H = \text{Fix}(F) \) is an \( m \)-dimensional plane in
with $m \geq 2$. If $F < \text{Comm}(\Gamma)$, then the stabiliser $\text{Stab}_\Gamma(H)$ of $H$ in $\Gamma$ is a (uniform) lattice acting on $H$.

It is worth stressing the fact that all codimension 1 totally geodesic sub-orbifolds in an arithmetic $n$–orbifold are fc–subspaces. In Section 6.4 we show how to build examples of non–fc subspaces (of high codimension) in arithmetic hyperbolic orbifolds. For non–arithmetical lattices we can give examples of totally geodesic subspaces of codimension 1 which are not fc–subspaces. Note that by [3] the number of such subspaces is always finite.

**Theorem 1.10.** Any non–arithmetic Gromov–Piatetski-Shapiro hyperbolic manifold contains a non–fc codimension 1 totally geodesic subspace.

The following fact follows from the recent results of Le–Palmer [33] and some previous work of Reid–Walsh [48].

**Theorem 1.11.** There exists a sequence of non–arithmetic hyperbolic 3–manifolds $M_k$, where $k = 2, 3, \ldots$, such that each $M_k$ contains exactly $k$ totally geodesic immersed surfaces and all of them are non-fc.

In the last section we apply the techniques introduced here to build some interesting examples of immersed totally geodesic subspaces of hyperbolic orbifolds. We begin by constructing an explicit example of a type I Weil-restriction subspace in a type I arithmetic lattice (Section 7.1). We then build an example of a type I Weil-restriction subspace in a type II arithmetic lattice (Section 7.2). Finally, we study totally geodesic subspaces of a particular non-arithmetic lattice generated by reflections in the facets of a non-compact Coxeter 5–simplex (Section 7.3). We show that this example contains a 2–dimensional arithmetic fc–subspace, arising as the fixed point set of its (unique) non-trivial symmetry.

The notion of fc-subspaces that we define and study in this paper for real hyperbolic orbifolds applies to the other locally symmetric spaces as well. In particular, it might be interesting to consider these subspaces in the complex hyperbolic case. For instance, in recent paper [16], Deraux carried out a detailed analysis of a set of examples of the fc-subspace of complex hyperbolic triangle orbifolds. As we observed in the real hyperbolic case, a systematic study of these subspaces may reveal new unexpected phenomena.

**Structure of the paper.** In Section 2 we recall some basic facts about algebraic groups, restriction of scalars, arithmetic lattices in semi–simple Lie groups and the definition of arithmetic, quasi–arithmetic and pseudo–arithmetic hyperbolic lattices. In Sections 3.1, 3.2, 3.3 and 3.4 we review the classification of arithmetic hyperbolic lattices and describe the involutions in their commensurators. In Section 4 we describe the two main techniques to construct totally geodesic immersion of arithmetic hyperbolic orbifolds: subform spaces and Weil restriction subspaces. The proofs of the main theorems are contained in Sections 5 and 6. In Sections 6.2 and 6.3 we exhibit
examples of non–arithmetic lattices containing codimension-one non–fc subspaces and in Section 6.4 we show how to build examples of non–fc subspaces in arithmetic hyperbolic orbifolds. Finally, in Section 7 we exhibit some interesting examples of totally geodesic immersions of hyperbolic orbifolds.

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**Notation.** Let us introduce the following standard notation for the whole paper (unless stated otherwise):

- If $k$ is a number field, then $k^\times$ denotes its multiplicative group, and $\mathcal{O}$ denotes its ring of integers. The algebraic closure of $k$ is denoted by $\overline{k}$.
- Bold capital letters $G$, $H$, $O_n$, etc., denote algebraic groups. By $G^\circ$ we denote the connected component of the identity element of $G$. If $R \subset \mathbb{C}$ is a ring and $G < \text{GL}_n(\mathbb{C})$, then $G(R)$ or $G_R$ denotes the group of $R$–points;
- If $k \subset \mathbb{C}$ is a field and $G < \text{GL}_n(\mathbb{C})$ is a $k$-group, then $\text{PG}$ denotes the adjoint group of $G$ and $\text{PG}_k$ denotes the $k$-points of $\text{PG}$ (now seen as a subgroup of $\text{GL}(g)$, where $g$ is the Lie algebra of $G$). In general the restriction $G_k \hookrightarrow \text{PG}_k$ of the adjoint map to the $k$-points is non-surjective;
- If $K/k$ is a field extension (both fields of characteristic zero) and $G$ is an algebraic $K$-group, $\text{Res}_{K/k}(G)$ denotes the algebraic $k$-group obtained from $G$ through the Weil restriction of scalars from $K$ to $k$. Its real points are denoted by $\text{Res}_{K/k}(G)_R$. The set of Galois embeddings $\sigma : K \rightarrow \mathbb{C}$ which restrict to the identity on $k$ is denoted by $S^\infty_{K/k}$.
- The capital letters $G$, $H$, $O_n$, etc., denote real Lie groups, and $G^\circ$ denotes the connected component of the identity of the group $G$;
- By $\Gamma, \Lambda, \text{etc.}$, we shall denote a lattice in a real Lie group $G$;
- A lattice $\Gamma$ is called uniform in $G$ if $G/\Gamma$ is compact;
2. Preliminaries

2.1. Algebraic groups. Some parts of this work require a considerable amount of the theory of algebraic groups. We give a short overview below and refer to [44] for a comprehensive introduction.

Let \( \Omega \) denote an algebraically closed field of characteristic zero. For the purpose of this work we may assume that \( \Omega \) is either \( \mathbb{C} \) or the field \( \mathbb{Q} \) of algebraic numbers. A linear algebraic group is a Zariski closed subgroup of the general linear group \( \text{GL}_n(\Omega) \). As such it is an algebraic subvariety \( G \) of \( \text{GL}_n(\Omega) \) such that the morphisms

\[
G \times G \ni (x, y) \to x \cdot y \in G, \\
G \ni x \to x^{-1} \in G,
\]

are algebraic and satisfy the group axioms. A morphism \( \phi : G \to H \) is a morphism of algebraic varieties which is also a group homomorphism. An isogeny is a surjective morphism with finite kernel.

If \( k \subset \Omega \) is a subfield and \( G < \text{GL}_n(\Omega) \) is a linear algebraic group, we say that \( G \) is defined over \( k \) (or that \( G \) is a \( k \)-group) if the ideal \( I \) of polynomial functions vanishing on \( G \) is generated by the intersection of \( I \) with the algebra of polynomials with coefficients in \( k \). A morphism \( \phi : G \to H \) of algebraic \( k \)-groups is defined over \( k \) (i.e. it is a \( k \)-morphism) if it can be expressed via polynomials with coefficients in \( k \). If \( G \) is an algebraic \( k \)-group and \( H \) is a normal \( k \)-subgroup of \( G \), then the quotient \( G/H \) is a \( k \)-subgroup and the quotient map \( G \to G/H \) is a \( k \)-morphism. A \( k \)-isogeny is an isogeny which is defined over \( k \).

If \( G < \text{GL}_n(\Omega) \) is a \( k \)-group, its group of \( k \)-points is the intersection

\[
G(k) = G \cap \text{GL}_n(k).
\]

If \( \Omega = \mathbb{C} \) and \( G \) is a linear algebraic group, then \( G < \text{GL}_n(\mathbb{C}) \) is endowed with a complex Lie group structure. If \( k \subset \mathbb{R} \) and \( G \) is a \( k \)-group, then

\[
G(\mathbb{R}) = G \cap \text{GL}_n(\mathbb{R})
\]

is a real Lie group.

Given field extensions \( k \subset K \subset \Omega \), we can regard a \( k \)-group \( G \) as a \( K \)-group \( \text{Ext}_{K/k} G \) which is said to be obtained from \( G \) via extension of scalars.
from $k$ to $K$. If $H$ is a $K$-group and $\text{Ext}_{K/k} G$ is $K$-isomorphic to $H$, we say that the $k$-group $G$ is a $K$-form of $H$.

Given an algebraic $k$-group $G < \text{GL}_n(\Omega)$, we denote the connected component of the identity (in the Zariski topology) by $G^0$. The identity component $G^0$ is a finite-index normal $k$-subgroup of $G$, and $G$ is connected if $G = G^0$. If $G$ is a $k$-group then $G^0(k)$ is Zariski dense in $G^0$.

A torus is a connected algebraic group $T$ for which there exists an isomorphism $T \cong (\mathbb{G}_m)^d$, where $\mathbb{G}_m \cong \text{GL}_1(\Omega)$ denotes the multiplicative group of $\Omega$ and $d = \dim(T)$ is the dimension of $T$. A character of a torus $T$ is a morphism $\chi : T \to \mathbb{G}_m$. The characters of a torus $T$ form a commutative group $X^*(T)$ under the operation $(\chi_1 + \chi_2)(g) = \chi_1(g) \cdot \chi_2(g)$. If $T$ has dimension $d$, then $X^*(T)$ is isomorphic to $\mathbb{Z}^d$. A $k$-defined torus $T$ that admits a $k$-defined isomorphism $T \cong (\mathbb{G}_m)^d$ is said to be $k$-split, and this is equivalent to all the characters in $X^*(T)$ being defined over $k$.

Let $G$ be a connected algebraic group. The maximal connected normal solvable subgroup of $G$ is called the radical of $G$ and is denoted by $R(G)$. The radical of a $k$-group is always defined over $k$. If $R(G) = \{e\}$ the group $G$ is said to be semisimple. The quotient of any connected $k$-group $G$ by its radical is a semisimple $k$-group. A disconnected algebraic group $G$ is semisimple if $G^0$ is semisimple.

All maximal tori in a connected semisimple algebraic group $G$ are conjugate under $G(\Omega)$ and thus all have the same dimension $d$, which we call the rank of $G$. If $G$ is a $k$-group all maximal $k$-split tori are conjugate under $G(k)$, and their dimension $s$ is the $k$-rank of $G$. A maximal $k$-split torus $S$ is always contained in a maximal $k$-defined torus $T$.

If $s = 0$ the $k$-split tori are trivial and the group $G$ is said to be anisotropic. If $s = d$ then there exists a $k$-defined maximal torus and $G$ is said to be split.

A Borel subgroup of a connected algebraic group $G$ is a maximal connected solvable subgroup $B < G$. All Borel subgroups are conjugate under $G(\Omega)$. It is not necessarily true that a $k$-group $G$ has a Borel subgroup defined over $k$. If there exists a $k$-defined Borel subgroup, then $G$ is said to be quasi-split.

A non-commutative connected algebraic group is absolutely almost simple if it has no nontrivial, connected, normal subgroups. To any connected semisimple group $G$ one can associate the (finite) set $\{G_1, \ldots, G_r\}$ of minimal connected normal subgroups of $G$. Each of these subgroups is absolutely almost simple and $G$ is an almost direct product of $G_1, \ldots, G_r$, meaning that the map

$$\prod_{i=1}^r G_i \ni (x_1, \ldots, x_r) \mapsto x_1 \cdot \ldots \cdot x_r \in G$$

(3)

is an isogeny.

A connected algebraic $k$-group $G$ is almost $k$-simple if it has no nontrivial, connected, $k$-defined normal subgroups. A connected $k$-group $G$ is
an almost direct product of its finitely many minimal connected normal \( k \)-defined subgroups, which are all almost \( k \)-simple. If \( G \) is almost \( k \)-simple, there exists a field \( k' \) containing \( k \), and an absolutely almost simple \( k' \)-group \( E \) such that \( G \) is isomorphic to the group \( \text{Res}_{k'/k}(E) \) obtained from \( E \) via \textit{restriction of scalars} from \( k' \) to \( k \) (see [54, §3.1.2.] and Section 2.2).

The Lie algebra \( g \) of an algebraic \( k \)-group \( G \) is defined as the algebra of left invariant derivations on the algebra of regular functions of \( G \) (Lie bracket given by the commutator). The group \( G \) acts by conjugation on \( g \) via Lie algebra automorphisms, yielding the adjoint representation

\[ \text{Ad} : G \to \text{GL}(g). \]

The kernel of the adjoint representation is the centraliser \( Z(G^0) \) of the identity component, which is a finite, normal \( k \)-subgroup of \( G \). A \( k \)-group is adjoint if the adjoint representation is faithful. The quotient map

\[ G \to G/Z(G^0) \]

is a \( k \)-isogeny of \( G \) onto the adjoint \( k \)-group \( \text{PG} = G/Z(G^0) \). If \( G \) is a connected semisimple adjoint group, it decomposes as a direct product of its absolutely simple factors, i.e. the isogeny in (3) is an isomorphism [44, Theorem 2.6.] and each factor of the direct product is simple. If moreover \( G \) is defined over \( k \), it decomposes as a direct product of its \( k \)-simple factors [54, §3.1.2].

2.1.1. \textit{Tits’ classification of semisimple algebraic groups} [54]. Suppose that \( G \) is a connected semisimple algebraic \( k \)-group, and denote by \( \mathcal{G} = \text{Gal}(\overline{k}/k) \) the absolute Galois group of \( k \). Let \( S < G \) be a maximal \( k \)-split torus and \( T < G \) be a maximal \( k \)-defined torus which contains \( S \). Denote by \( \Sigma \subset X^*(T) \) the set of all roots of \( G \) relative to \( T \), by \( N \) the normaliser of \( T \) in \( G \) and by \( W = N/T \) the \textit{Weyl group} of \( G \) relative to \( T \).

In \( X^*(T) \otimes \mathbb{R} \) we choose a scalar product invariant under the natural action of \( W \), endowing \( \Sigma \) with the structure of a root system. We also choose compatible orders in the character group \( X^*(S) \) and \( X^*(T) \), denote by \( \Delta \) the system of simple roots for \( G \) relative to \( T \), and by \( \Delta_0 \) the subsystem of those roots which vanish on \( S \).

The natural action of the group \( \mathcal{G} \) on \( X^*(T) \) induces an action by automorphisms of the root system \( \Sigma \). There is a splitting short exact sequence

\[ 1 \to W \to \text{Aut}(\Sigma) \to \Theta \to 1 \]

where \( \Theta = \{ \phi \in \text{Aut}(\Sigma) | \phi(\Delta) = \Delta \} \) is isomorphic to the group of automorphisms of the Dynkin diagram of the root system \( \Sigma \). This readily implies that \( \text{Aut}(\Sigma) \) is isomorphic to the semidirect product of \( W \) by \( \Theta \) [27, Section 12.2]. If the action of \( \mathcal{G} \) on \( \Sigma \) takes values in the Weyl group \( W \), the group \( G \) is called an \textit{inner form}. The most relevant properties of the action of \( \mathcal{G} \) on the root system \( \Sigma \) can be encoded in the \textit{Tits index} and in the \textit{Tits symbol}, which are defined as follows.
The action of $G$ on $\Sigma$ projects to an action, called the *-action, on the system $\Delta$ of simple roots and on the Dynkin diagram of the root system. The index of a group $G$ is the data of the Dynkin diagram, together with the *-action of the absolute Galois group $G$ on the diagram. The orbits of the vertices in $\Delta \setminus \Delta_0$ under the *-action are the so-called distinguished orbits and are circled.

We notice that it is implicit in Tits’ definition via the action of $G$ on conjugacy classes of maximal parabolic groups [54, §2.3] that the *-action of $G$ on the Dynkin diagram does not depend on the choice of a maximal $k$-torus $T$.

The symbol of a group $G$ is a symbol of the form $gX_{p,t}^{(t)}$, where $X_0$ determines the (absolute) type of the Dynkin diagram of $G$, $g$ denotes the order of the *-action of the absolute Galois group $G$ on the Dynkin diagram and $r$ is the relative rank of $G$, i.e. the number of distinguished orbits for the *-action. The $(t)$ symbol appears only for groups of classical type and it corresponds to the degree of a certain central division $k$-algebra involved in the definition of the corresponding group (see [54, pp. 55-61]).

The group $G$ is anisotropic if and only if $\Delta_0 = \Delta$ (equivalently, $r = 0$), while it is quasi-split if and only if $\Delta_0 = \emptyset$. In the quasi-split case all roots in the Tits index belong to a circled orbit and the action of the absolute Galois group $G$ on the root system $\Sigma$ preserves a system $\Delta_1 \subset \Sigma$ of simple roots.

Indeed, let us denote by $t$ the Lie algebra of $T$, by $b$ the Lie algebra of $B$ and by $L_\alpha = \{ x \in g \mid [t, x] = \alpha(t) \cdot x \text{ for all } t \in t \}$ the root space associated to a root $\alpha \in \Sigma$ (where $\alpha$ is now interpreted as an element of the dual $t^*$ of $t$). We have that

$$b = t \oplus \sum_{\alpha > 0} L_\alpha,$$

where the sum on the right hand side ranges over all positive roots with respect to the partial ordering induced by the choice of a set $\Delta'$ of simple roots [27, Section 16.4]. Since $T$ and $B$ are $k$-defined, it follows that the direct sum decomposition in (5) is preserved under the action of $G$. Hence, also the system of simple roots $\Delta'$ is preserved by the action of $G$. By choosing the ordering on $X^*(T)$ so that $\Delta = \Delta'$ we may assume that the action of $G$ on the Dynkin diagram of the root system $\Sigma$ is given by elements of the subgroup $\Theta < \text{Aut}(\Sigma)$.

If $G$ is a semisimple algebraic $k$-group, its Tits index and symbol are by definition those of the connected group $G^0$.

2.2. Weil restriction of scalars. In this section we briefly review a classical construction in algebraic geometry which will play an important role throughout the paper: Weil’s restriction of scalars.

Suppose that $K/k$ is a finite extension of algebraic number fields of degree $d$ and that $X$ is an algebraic variety over $K$ of dimension $n$. Then $X$ can be interpreted as an algebraic variety $\text{Res}_{K/k}(X)$ over $k$ of dimension $n \cdot d$. 
Such operation yields a covariant functor from the category of algebraic varieties over $K$ to the category of algebraic varieties over $k$, since for any $K$-morphism $f : X \to Y$ there exists an induced $k$-morphism

$$\text{Res}_{K/k}(f) : \text{Res}_{K/k}(X) \to \text{Res}_{K/k}(Y).$$

These functorial properties easily follow from the existence of a natural map $p : \text{Res}_{K/k}(X) \to X$ which is $K$-defined and has the following universal property (cf. [38, Section 1.7]): for any $k$-variety $Y$ and any $K$-morphism $f : Y \to X$, there exists a unique $k$-morphism $\phi : Y \to \text{Res}_{K/k}(X)$ such that $f = p \circ \phi$. The map $p$ induces a bijection between the $k$-points of $\text{Res}_{K/k}(X)$ and the $K$-points of $X$.

We shall be exclusively concerned with the case where $X$ is an affine variety, in which case restriction of scalars admits a fairly explicit description as we now explain. We notice that left multiplication is a $k$-linear map on the $k$-vector space $K$. By fixing a basis $\mathcal{B} = (b_1, \ldots, b_d)$, we construct the left-regular representation of $K$ as a (commutative) $k$-subalgebra $\mathcal{A}(k)$ of the algebra $M_{d \times d}(k)$. The equations that identify $\mathcal{A}(k)$ are $k$-linear in the coefficients $y_{ij}, i, j = 1, \ldots, d$, of the matrices in $M_{d \times d}(k)$.

Let us suppose that $X \subseteq \mathbb{A}_K^N$ is defined as the zero locus of a finite set of polynomials $p_1, \ldots, p_m \in K[x_1, \ldots, x_N]$. Using the left regular representation of $K$, we can associate to each equation of the form $p_l(x_1, \ldots, x_N) = 0$, $l = 1, \ldots, k$, a system of polynomial equations in $d^2N$ variables with coefficients in $k$. It is sufficient to interpret each of the coefficients of $p_l$ as a $d \times d$ matrix with coefficients in $k$ and each variable $x_h$ as a $d \times d$ matrix with entries given by variables $y_{ijh}$, and $i, j = 1, \ldots, d$. The operation of multiplication in $K$ now translates to row-by-column multiplication of $d \times d$ matrices in $\mathcal{A}(k)$.

Finally, we can form a system of polynomial equations with coefficients in $k$ by adjoining the equations coming from each polynomial $p_l$, $l = 1, \ldots, m$, with the linear equations involving the coefficients $y_{ijh}, i, j = 1, \ldots, d$ which define $\mathcal{A}(k)$ as a subalgebra of $M_{d \times d}(k)$ (we adjoin this set of equations for each variable $x_h$, $h = 1, \ldots, N$). Then the restriction of scalars

$$\text{Res}_{K/k}(X) \subseteq \mathbb{A}_k^{d^2N}$$

is the affine $k$-variety defined as the zero-locus of this system of equations. It follows from the construction that there is a one-to-one correspondence between the $K$-points of $X$ and the $k$-points of $\text{Res}_{K/k}(X)$.

In view of our need to review the connection between restriction of scalars and the construction of arithmetic lattices in semisimple Lie groups, we are particularly interested in describing the relation between the real points $X(\mathbb{R})$ of $X$ and the real points $\text{Res}_{K/k}(X)_\mathbb{R}$ of $\text{Res}_{K/k}(X)$.

In most cases we will apply restriction of scalars when the field $K$ is totally real, so we will assume that this is the case for the rest of this section. We denote by $S^\infty_{K/k}$ the set of Galois embeddings of $K$ which restrict to the identity on $k$. There are $d = [K : k]$ such embeddings, so that
\[ S_{K/k}^\sigma = \{ \text{id}, \sigma_1, \ldots, \sigma_{d-1} \}. \] For each \( \sigma \in S_{K/k}^\sigma \) and \( p \in K[x_1, \ldots, x_N] \), we denote by \( p^\sigma \) the polynomial obtained by applying \( \sigma \) to each coefficient of \( p \). Similarly, we denote by \( X^\sigma \) the affine algebraic \( \sigma(K) \)-variety defined as the zero locus of \( p_1^\sigma, \ldots, p_m^\sigma \in \sigma(K)[x] \).

By the primitive element theorem, the field \( K \) is equal to \( k(\alpha) \) for some \( \alpha \in K \) with minimal polynomial \( q(x) \in k[x] \) of degree \( d \). Therefore, as an abstract field \( K \) is isomorphic to \( k[x]/(q(x)) \). By extending coefficients to \( \mathbb{R} \), we see that \( K \otimes_k \mathbb{R} \) is isomorphic to \( \mathbb{R}[x]/(q(x)) \). Since \( K \) is totally real, all the roots of \( q(x) \) are real, and the map

\[
\mathbb{R}[x]/(q(x)) \ni [p(x)] \mapsto (p(\alpha), p(\sigma_1(\alpha)), \ldots, p(\sigma_{d-1}(\alpha)))
\]

is an isomorphism between \( K \otimes_k \mathbb{R} \) and \( \mathbb{R}^d \).

Notice that we have the following ring isomorphisms:

\[ K \cong \mathcal{A}(k), \quad K \otimes_k \mathbb{R} \cong \mathcal{A}(\mathbb{R}), \]

and thus, by composing with the isomorphism in (6), \( \mathcal{A}(\mathbb{R}) \) is naturally isomorphic to \( \mathbb{R}^d \). Finally, we interpret the variety \( X \) as being defined over the abstract field \( k[x]/(q(x)) \), while \( \text{Res}_{K/k}(X)_{\mathbb{R}} \) corresponds to \( X(\mathbb{R}[x]/(q(x))) \).

and due to the isomorphism (6) we have that

\[ \text{Res}_{K/k}(X)_{\mathbb{R}} \cong X(\mathbb{R}) \times X_{\sigma_1}(\mathbb{R}) \times \cdots \times X_{\sigma_{d-1}}(\mathbb{R}). \]

The natural map \( p : \text{Res}_{K/k}(X) \to X \) naturally extends to the morphism \( p : \text{Res}_{K/k}(X)_{\mathbb{R}} \to X(\mathbb{R}) \), and corresponds to the projection onto the first factor. The \( k \)-points of \( \text{Res}_{K/k}(X) \) correspond to elements of the form \((x, \sigma_1(x), \ldots, \sigma_{d-1}(x))\) where \( x \in X \).

Finally, we notice that if \( G \) is an algebraic \( K \)-group, then \( \text{Res}_{K/k}G \) is an algebraic \( k \)-group. Indeed, the group structure of \( G \) is defined by a \( K \)-polynomial map \( G \times G \to G \), and the functorial nature of restriction of scalars induces a \( k \)-polynomial map \( \text{Res}_{K/k}G \times \text{Res}_{K/k}G \to \text{Res}_{K/k}G \) which endows \( \text{Res}_{K/k}G \) with a group structure.

### 2.3. Arithmetic lattices.

Let \( G \) be a non–compact semi–simple real Lie group with finitely many connected components. Suppose that \( k \subset \mathbb{R} \) is a totally real algebraic number field and \( G \) is an admissible (for \( G \)) semi–simple algebraic \( k \)-group, i.e. there exists a surjective homomorphism

\[ \pi : G(k \otimes_{\mathbb{Q}} \mathbb{R})^0 = \prod_{\sigma : k \to \overline{\mathbb{R}}} G^\sigma(\overline{\mathbb{R}})^0 \to G^0 \]

with compact kernel. Here \( G^\sigma \) denotes the algebraic group defined over the Galois embedding \( \sigma : k \to \overline{\mathbb{R}} \) and obtained from an algebraic group \( G \) by applying \( \sigma \) to the coefficients of all polynomials that define \( G \). Thus, the group \( G(k \otimes_{\mathbb{Q}} \mathbb{R}) \) is simply the group \( \text{Res}_{k/\mathbb{Q}}(G)_{\overline{\mathbb{R}}} \), and \( G \) is admissible for \( G \) if and only if \( \text{Res}_{k/\mathbb{Q}}(G)_{\overline{\mathbb{R}}} \) surjects \( G^0 \) with compact kernel.

Since the group \( G(k \otimes_{\mathbb{Q}} \mathbb{R}) \) is defined over \( \mathbb{Q} \), it follows from the theorem of Borel and Harish-Chandra [13, Theorem 12.3] that \( G(\mathbb{O}) \) is isomorphic to a lattice in \( G(k \otimes_{\mathbb{Q}} \mathbb{R}) \) under the natural embedding of \( G(k) \) into...
Res}_{k/Q}(G)_\mathbb{R} \cong G(k \otimes \mathbb{Q} \mathbb{R})$. If $\Gamma$ is commensurable with $G(\mathcal{O})$, then $\Gamma$ is an arithmetic lattice in $G(k \otimes \mathbb{Q} \mathbb{R})$. From Godement’s compactness criterion [25] one obtains that $\Gamma$ is uniform if $k \neq \mathbb{Q}$. In general, both uniform and non–uniform lattices can exists if $k = \mathbb{Q}$.

If $G$ is a Lie group and $\pi : G(k \otimes \mathbb{Q} \mathbb{R}) \to G$ is a surjective homomorphism with compact kernel $K$, it is natural to extend the definition of arithmetic lattices by declaring a lattice in $G$ to be arithmetic if it is the image under $\pi$ of an arithmetic lattice in $G(k \otimes \mathbb{Q} \mathbb{R})$. Therefore all subgroups of the group $G = \pi(G(k \otimes \mathbb{Q} \mathbb{R}))$ commensurable with $\pi(G(\mathcal{O}))$ will be arithmetic lattices in $G$.

Finally, we require the notion of arithmeticity to be invariant under isomorphisms and isogenies of Lie groups. Every Lie group $G$ as above is isogenous to its identity component $G^0$, and thus in the setting of (7) we say that $\Gamma < G$ is an arithmetic lattice if $\Gamma \cap G^0$ is conjugate in the wide sense to $\pi(G(\mathcal{O}))^\sigma$. The field $k$ is called the field of definition (or the ground field) of $\Gamma$.

If $G$ is a simple non-compact real Lie group, then the above definitions can be reformulated in a more succinct way. Namely, let $\sigma_0 : k \hookrightarrow \mathbb{R}$ be a fixed identity embedding of a totally real field $k$, and $G$ a $k$-defined algebraic group such that $G^{\sigma_0}(\mathbb{R})^\sigma$ is isomorphic to $G^0$, while $G^\sigma(\mathbb{R})$ is compact for any other embedding $\sigma \neq \sigma_0$. In this case the algebraic $k$-group $G$ is admissible for $G$, and $\Gamma < G$ is arithmetic if it is commensurable in the wide sense with $G(\mathcal{O})$ for some totally real algebraic number field $k$ and some admissible (for $G$) algebraic $k$-group $G$.

If $G$ is an algebraic $k$-group and $\Gamma$ is commensurable with $G(\mathcal{O})$, then $\text{Comm}_G(G(\mathcal{O}))$ is easily seen to contain the group $G(k)$, which is dense in the identity component $G(\mathbb{R})^\sigma$. Moreover, there is the following arithmeticity criterion by Margulis: if $\Gamma$ is an irreducible lattice in a semi-simple real Lie group $G$, then $\Gamma$ is arithmetic if and only if $\text{Comm}_G(\Gamma)$ is dense in $G^0$. If $\Gamma$ is non–arithmetic, then $\text{Comm}_G(\Gamma)$ is the maximal (by subgroup inclusion) lattice of $G$ containing $\Gamma$ (cf. [36, Theorem 9] and [38, Chapter IX, Theorem B & Proposition 4.22]).

The celebrated Margulis’ Arithmeticity Theorem [37] states that if $\Gamma$ is an irreducible lattice in a semi-simple Lie group $G$ with $\text{rk}_\mathbb{R} G \geq 2$, then $\Gamma$ is arithmetic.

Regarding lattices in general, a fundamental theorem of Vinberg [56, Theorem 1] implies that if $\Gamma$ is a Zariski-dense lattice in a semi–simple adjoint real algebraic group $G$, then:

1. the adjoint trace field $k = \mathbb{Q}((\text{tr}(\text{Ad} \gamma) \mid \gamma \in \Gamma))$, where $\text{Ad} : G \to \text{GL}(\mathfrak{g})$ is the adjoint action of $G$ on its Lie algebra, is an invariant of the commensurability class of $\Gamma$;
(2) There exists a maximal \( k \)-defined algebraic group \( G \) such that
\[
G(\mathbb{R})^0 \cong G^0, \quad G(\mathbb{R}) < G, \quad \text{and} \quad \Gamma < G(k),
\]
up to conjugation.

It is called the ambient group of \( \Gamma \);
(3) The ambient group \( G \) is uniquely determined up to \( k \)-isomorphism by the commensurability class of \( \Gamma \).

We will at times refer to the adjoint trace field \( k \) and the ambient group \( G \) of a lattice \( \Gamma \) as the Vinberg invariants of \( \Gamma \). If
\[
G \cong H \quad \text{for a} \quad \mathbb{Q} \text{-defined algebraic group} \quad H
\]
and \( \Gamma \) is a locally rigid lattice in \( G \), then the adjoint trace field \( k \) is an algebraic number field. The same is true if \( G \) is isogenous to \( PSL_2(\mathbb{C}) \cong PSO_{3,1} \) and \( \Gamma \) is non–uniform (see [8, pp. 15-16]). In particular if \( G \cong PO_{n,1} \) with \( n \geq 3 \), then the adjoint trace field of a lattice \( \Gamma \) is an algebraic number field.

2.4. Hyperbolic lattices. We denote by \( \mathbb{H}^n \) the hyperbolic space, which is the unique simply connected complete Riemannian \( n \)-manifold with constant sectional curvature \(-1\). The hyperboloid model \( \mathbb{H}^n \) for hyperbolic space is defined as follows.

Consider the real vector space \( \mathbb{R}^{n+1} \) equipped with the standard quadratic form \( f \) of signature \((n, 1)\):
\[
f(x) = -x_0^2 + x_1^2 + \cdots + x_n^2.
\]

Let \( \mathcal{H} \) be the hyperboloid
\[
\mathcal{H} = \{ x \in \mathbb{R}^{n+1} \mid f(x) = -1 \} = \mathcal{H}^+ \cup \mathcal{H}^-,
\]
where
\[
\mathcal{H}^+ = \{ x \in \mathcal{H} \mid x_0 > 0 \} \quad \text{and} \quad \mathcal{H}^- = \{ x \in \mathcal{H} \mid x_0 < 0 \}.
\]

By equipping \( \mathcal{H}^+ \) with the Riemannian metric induced by restricting \( f \) to each tangent space \( T_p(\mathcal{H}^+) \), \( p \in \mathcal{H}^+ \), we obtain the hyperboloid model for the \( n \)-dimensional hyperbolic space \( \mathbb{H}^n \).

Let \( O_{n,1} = O(f, \mathbb{R}) \) be the orthogonal group of \( f(x) \), \( PO_{n,1} = PO(f, \mathbb{R}) \) be its projectivisation, and \( O_{n,1}^+ < O_{n,1} \) be the subgroup preserving \( \mathcal{H}^+ \). Thus, we can identify \( Isom(\mathbb{H}^n) \) with \( O_{n,1}^+ \).

Notice that \( O_{n,1}^+ \) is a semisimple real Lie group but is not realised as the group of \( \mathbb{R} \)-points of an algebraic group, since it is not a Zariski closed subset of \( O_{n,1} \). However \( O_{n,1}^+ \) is isomorphic to the \( \mathbb{R} \)-points of the adjoint real algebraic group \( PO_{n,1} = O_{n,1}/\{ \pm \text{id} \} \). The isomorphism is obtained by restricting the adjoint homomorphism \( \text{Ad} : O_{n,1} \to PO_{n,1} \) to the subgroup \( O_{n,1}^+ \).

A hyperbolic lattice is then defined as a lattice in \( PO_{n,1} \). If \( \Gamma < PO_{n,1} \) is a hyperbolic lattice, the quotient \( M = \mathbb{H}^n/\Gamma \) is a finite volume hyperbolic orbifold. If \( \Gamma \) is torsion-free, then \( M \) is a Riemannian manifold, and is called a hyperbolic manifold.

Suppose now that \( G = PO_{n,1} \) is the isometry group of the hyperbolic space \( \mathbb{H}^n \), and \( G \) is an adjoint admissible (for \( G \)) algebraic \( k \)-group, i.e.
$G(\mathbb{R})^\sigma$ is isomorphic to $G^\sigma$ and $G^\sigma(\mathbb{R})$ is a compact group for any non-identity embedding $\sigma: k \rightarrow \mathbb{R}$. Then any group $\Gamma$ commensurable with $G(O)$ is an arithmetic hyperbolic lattice. The commensurator $\text{Comm}_G(\Gamma)$ of such a lattice $\Gamma$ is precisely $G(k)$, since $G(\mathbb{C}) \cong \text{PO}_{n+1}(\mathbb{C})$ is centreless (see [41, Remark 5.2.5] and [11, Theorem 3(b)]).

Since the case $\text{rk}_R G = 1$ admits also irreducible non-arithmetic lattices, we introduce some weaker types of arithmeticity of hyperbolic lattices.

A lattice $\Gamma < G(\mathbb{R})$, where $G$ is an admissible algebraic $k$-group, is called quasi-arithmetic (with ground field $k$) if $\Gamma < G(k)$ and properly quasi-arithmetic if it is quasi-arithmetic, but not actually arithmetic [55]. The above mentioned Vinberg’s theorem shows that if $\Gamma' < G$ is commensurable with some quasi-arithmetic lattice $\Gamma < G(k)$, then $\Gamma'$ is also a subgroup of $G(k)$ and therefore quasi-arithmeticity is an invariant property of commensurability classes of hyperbolic lattices. Let us stress the fact that the notion of quasi-arithmeticity is distinct from the usual arithmeticity only for Lie groups of real rank 1, due to Margulis’ Arithmeticity Theorem [37].

Furthermore, a lattice $\Gamma < G(\mathbb{R})$ is called pseudo-arithmetic over $K/k$ if $\Gamma < G(K)$, $G$ is admissible over $k$, and $K$ is a multiquadratic extension of $k$ (i.e., $K = k(\sqrt{a_1}, \ldots, \sqrt{a_m})$, for some $a_1, \ldots, a_m \in k$). All the currently known examples of lattices in $\text{PO}_{n,1}$ for $n > 3$ are pseudo-arithmetic [19].

From the general classification of algebraic semi–simple groups by Tits [54], it follows that there exist three distinct types of arithmetic groups in $G = \text{PO}_{n,1}(\mathbb{R})$: type I is associated with quadratic forms (in all dimensions $n \geq 2$), type II is associated with unitary groups of skew–Hermitian forms with coefficients in quaternion algebras (in odd dimensions $n \geq 3$), and type III is related to the exceptional isomorphism in dimension $n = 3$ or to the triality phenomenon in dimension $n = 7$. Because of the exceptional isomorphism $\text{PSL}_2(\mathbb{C}) \cong \text{PO}_{3,1}^\sigma$ there is a classical description of arithmetic lattices acting on $\mathbb{H}^3$ as arithmetic lattices in $\text{PSL}_2(\mathbb{C})$. We shall discuss the types of arithmetic lattices and some of their specific properties in more detail in Section 3.

The group $\text{PO}_{n,1}$ is absolutely simple whenever $n \neq 3$, and by [45, Lemma 2.6] if $\Gamma$ is an arithmetic lattice in $G = \text{PO}_{n,1}(\mathbb{R})$, the adjoint trace field $k$ of $\Gamma$ and its field of definition coincide. This implies that $k$ of $\Gamma$ is totally real, the ambient group $G$ is admissible and $\Gamma$ is commensurable in the wide sense with the group $G(O)$ [19].

When $n = 3$, the group $\text{PO}_{3,1}$ is not absolutely simple. However, the group $\text{PO}_{3,1}(\mathbb{R})^\sigma = \text{PSO}_{3,1}(\mathbb{R})$ is isomorphic to $\text{PSL}_2(\mathbb{C})$, and $\text{PSL}_2$ is indeed absolutely simple. As a consequence of this fact there exist arithmetic lattices in $\text{PO}_{3,1}(\mathbb{R})$ whose adjoint trace field is not totally real, and whose ambient group is not admissible. We refer the reader to Section 3.3 for more details on this phenomenon.

2.4.1. Totally geodesic subspaces. Let $M = \mathbb{H}^n/\Gamma$ and $N = \mathbb{H}^m/\Lambda$, $m < n$, be finite-volume hyperbolic orbifolds. A map $i: N \rightarrow M$ is a totally geodesic
immersion if any of its lifts \( \tilde{i} : \mathbb{H}^m \to \mathbb{H}^n \) maps \( \mathbb{H}^m \) isometrically to an \( m \)-dimensional totally geodesic subspace \( U \subseteq \mathbb{H}^n \). Equivalently, there is an \( m \)-dimensional totally geodesic subspace \( U \subseteq \mathbb{H}^n \) such that its stabiliser \( \text{Stab}_\Gamma(U) < \text{PO}_{m,1}(\mathbb{R}) \times \text{O}_{n-m}(\mathbb{R}) \) acts as lattice on \( U \), and its projection to \( \text{PO}_{m,1}(\mathbb{R}) \) is isomorphic to \( \Lambda \). In this setting, we say that \( N \) is a \textit{totally geodesic subspace} of \( M \), and that \( \Lambda \) is a \textit{totally geodesic sublattice} of \( \Gamma \).

If the map \( i : N \to M \) above is an embedding (i.e. all the lifts of \( N \) to \( \mathbb{H}^n \) are pairwise disjoint) we say that \( i \) is a \textit{totally geodesic embedding}, and that \( N \) is a \textit{totally geodesic embedded subspace} of \( M \).

3. Arithmetic hyperbolic lattices and involutions

3.1. Arithmetic lattices of type I. Arithmetic lattices of type I in \( \mathbb{H}^n \) are also called arithmetic lattices of simplest type, and correspond to the following indices in Tits’ classification [54]: \( B_m \), if \( n \) is even, and either \( 1D_m^{(1)} \) or \( 2D_m^{(1)} \), if \( n = 2m - 1 \) is odd. Here \( 1D_m^{(1)} \) can only occur if the field of definition is \( \mathbb{Q} \) and \( m \) is odd [18, Proposition 13.6].

3.1.1. Admissible quadratic forms. Let \( k \subset \mathbb{R} \) be a totally real number field with the ring of integers \( \mathcal{O} \). Let \( f \) be a quadratic form defined over \( k \). We say that \( f \) is \textit{admissible} if it has signature \((n,1)\) and for any non-identity Galois embedding \( \sigma : k \to \mathbb{R} \) the form \( f^\sigma \) is positive definite. The fact that \( f \) is admissible as defined above clearly implies that the \( k \)-defined algebraic group \( \text{O}(f) \) is admissible for \( \text{O}_{n,1} \). It follows that the group \( \text{PO}(f, \mathcal{O}) \) is an arithmetic lattice in \( \text{PO}_{n,1} \) (see [13]).

A quadratic form \( f \) defined over \( k \) is called \textit{anisotropic} if it does not represent 0 over \( k \). Otherwise \( f \) is called \textit{isotropic}. It is easy to see that any admissible quadratic form \( f \) defined over \( k \neq \mathbb{Q} \) is anisotropic. Meyer’s theorem [51, Chapter 3.2, Corollary 2] implies that any quadratic form defined over \( \mathbb{Q} \) of signature \((n,1)\) with \( n \geq 4 \) is isotropic.

**Definition 3.1.** Any group \( \Gamma \) obtained as \( \text{PO}(f, \mathcal{O}) \) in the way described above, and any subgroup of \( \text{PO}_{n,1} \cong \text{PO}(f)_{\mathbb{R}} \) commensurable to such a group in the wide sense, is called an arithmetic lattice of simplest type, or an \textit{arithmetic lattice of type I}. The field \( k \) is called the \textit{field of definition} (or the ground field) of \( \Gamma \). A hyperbolic orbifold \( M = \mathbb{H}^n/\Gamma \) is said to be of type I if the group \( \Gamma \) is an arithmetic lattice of type I.

**Remark 3.2.** Given a totally real field \( k \), two admissible, \( k \)-defined quadratic forms \( f_1 \) and \( f_2 \) define the same commensurability class of hyperbolic lattices if and only if \( f_1 \) is equivalent over \( k \) to \( \lambda \cdot f_2 \) for some \( \lambda \in k^\times \).

Godement’s compactness criterion [25] implies that \( \Gamma \) is uniform if and only if \( f \) is anisotropic. Thus, for any totally real number field \( k \neq \mathbb{Q} \), the resulting orbifold \( \mathbb{H}^n/\Gamma \) is compact. If \( k = \mathbb{Q} \), then the orbifold \( \mathbb{H}^n/\Gamma \) is compact only if the quadratic form \( f \) does \textit{not} rationally represent 0. This can only happen if \( f \) has signature \((n,1)\) with \( n \leq 3 \), by Meyer’s theorem.
Remark 3.3. The classification of algebraic semi–simple groups by Tits [54] implies that all arithmetic lattices acting on $\mathbb{H}^{2n}$, $n \geq 1$, are of simplest type. The same applies to all non–uniform arithmetic lattices acting on $\mathbb{H}^n$, for $n \geq 2$.

3.1.2. Constructing $k$–involutions. We wish to build examples of $\mathbb{C}$–subspaces associated to a single involution in $\text{Comm}(\Gamma)$, where $\Gamma < \text{Isom}(\mathbb{H}^n)$ is an arithmetic hyperbolic lattice. If $k$ is the adjoint trace field of $\Gamma$ and $G$ is its ($k$–defined, admissible) ambient group, then $\text{Comm}(\Gamma) = G(k)$, so we are left with the task of analysing the fixed–point set of involutions in $G(k)$.

Here we examine the case of type I lattices: $G = \text{PO}(f)$, where $f$ is an admissible Lorentzian quadratic form of signature $(n, 1)$ over a totally real field $k$.

As a first step, we characterise all involutions in the group of $k$–points $O(f, k) = O(f)_k$. Let us fix a subspace $V_1 \subset k^{n+1}$ of dimension $m + 1$ such that $f|_{V_1}$ has signature $(m, 1)$, and let $V_1^\perp = V_{-1}$ be its orthogonal complement with respect to the form $f$. Clearly $k^{n+1}$ decomposes as a direct sum of $V_1$ and $V_{-1}$. Let $N$ denote the linear transformation which acts as the identity on $V_1$ and as multiplication by $-1$ on $V_{-1}$. It is easy to verify that $N$ belongs to $O(f, k)$ and that $N^2 = \text{id}$.

Also, all order 2 elements in $O(f, k)$ arise through the above construction. Suppose that $N \in O(f, k)$ has order 2. Then $N$ has eigenvalues 1 and $-1$ and the corresponding eigenspaces $V_1, V_{-1} \subset k^{n+1}$ are orthogonal with respect to the form $f$. Up to multiplication by $\pm \text{id}$, we may assume that the eigenspace $V_1$ intersects the upper sheet $\mathbb{H}^n$ of the hyperboloid. In this case, the restriction $f|_{V_1}$ of $f$ to $V_1$ will be an admissible Lorentzian quadratic form of signature $(m, 1)$, where $\dim(V_1) = m + 1$. If $\Gamma < \text{PO}(f)_{\mathbb{R}}$ is an arithmetic lattice and $i = [N] \in \text{PO}(f)_k$ is an involution with $N \in O(f, k)$ as above, then $\text{Stab}_\Gamma(V_1)$ corresponds to a totally geodesic type I arithmetic subspace of $\mathbb{H}^n/\Gamma$ associated to the admissible $k$–form $g = f|_{V_1}$.

If $n$ is even, then $\text{PO}(f)$ is $k$–isomorphic to $\text{SO}(f)$, and the latter is adjoint. Therefore all $k$–involutions in $\text{PO}(f)_k$ correspond to involutions in $\text{PO}(f, k)$. For odd $n \geq 3$, there might be involutions in $\text{PO}(f)_k$ which do not belong to $\text{PO}(f, k)$. Due to [20, Lemma 4.3], we have a complete description of all involutions in this case, too.

Let us denote by $A$ the matrix which represents the form $f$ with respect to the standard basis of $k^{n+1}$, and define the general orthogonal group of the form $f$ as

$$G\text{O}(f) = \{ M \in \text{GL}_{n+1}(\mathbb{C}) \mid M^tAM = \mu \cdot A, \text{ for some } \mu \in \mathbb{C} \}.$$

An involution $N \in \text{PO}(f)_k$ can be described as a linear transformation $N = (1/\sqrt{\mu}) \cdot M$, with $M \in G\text{O}(f, k) = G\text{O}(f)_k$ such that $M^2 = \mu \cdot \text{id}$. Here, both $M$ and $N$ are defined up to multiplication by $-\text{id}$. Notice that we always have $0 < \mu \in k^\times$. Moreover, $\mu$ is totally positive in the sense that $\sigma(\mu)$ is positive for all Galois embeddings $\sigma : k \to \mathbb{R}$.
Since we consider the projective model of $\mathbb{P}^n$, we may identify such $N \in \text{PO}(f)_k$ directly with $M$, as they describe the same projective transformation with the same (projective) fixed point set. In particular the matrix $N$ has eigenvalues $\pm 1$ while $M$ has eigenvalues $\pm \sqrt{\mu}$. Also, the eigenspaces for $M$ and $N$ coincide: the eigenspace $V_1$ for the eigenvalue $1$ of $N$ is called the positive eigenspace and coincides with the eigenspace relative to the eigenvalue $\sqrt{\mu}$ for $M$. Similarly, the eigenspace $V_{-1}$ for the eigenvalue $-1$ of $N$ (or $-\sqrt{\mu}$ of $M$) is called the negative eigenspace.

Let $K$ denote the field $k(\sqrt{\mu})$. Notice that $K$ is totally real due to the fact that $\mu$ is totally positive. The positive and negative eigenspaces are defined over $K$, in the sense that they are described as the set of solutions of a homogeneous linear system of equations with coefficients in $K$. Up to multiplication by $-\text{id}$, we can assume that the restriction $g$ of the form $f$ to the positive eigenspace has signature $(m, 1)$, for some $m > 0$, while the restriction $h$ of $f$ to the negative eigenspace is positive definite. Notice that the forms $g$ and $h$ are also defined over $K$.

We now claim that the form $g$ is admissible, and that if $\Gamma$ is an arithmetic lattice in $\text{O}(f)_R$, the stabiliser in $\Gamma$ of the positive eigenspace projects to an arithmetic lattice in $\text{O}(g)_R$. The claim follows by studying the restriction of scalars $\text{Res}_{K/k}\text{O}(g)$ of the orthogonal group $\text{O}(g)$, and proving that it is isomorphic to the centraliser $H$ of $N$ in $\text{O}(f)$. Notice that $H$ is the fixed point set of the $k$-automorphism of $\text{O}(f)$ given by conjugation by $N$, and so is indeed a $k$-subgroup of $\text{O}(f)$. An element of $\text{O}(f)$ commutes with $N$ if and only if it preserves the positive and negative eigenspaces, and therefore $H$ is $K$-isomorphic to $\text{O}(g) \times \text{O}(h)$. By the discussion in Section 2.2 we have the following $K$-defined isomorphism:

$$\text{Res}_{K/k}\text{O}(g) \cong \text{O}(g) \times \text{O}(g^\sigma),$$

where $g^\sigma$ is the form obtained by applying the non-trivial automorphism $\sigma \in \text{Gal}(K/k) \cong \mathbb{Z}_2$ to the coefficients of $g$. We note that $g$ is defined as the restriction of the form $f$ to the positive eigenspace for $N \in \text{GL}_{n+1}(K)$. The Galois automorphism $\sigma$ sends $N = 1/\sqrt{\mu}M$ to $N^\sigma = -1/\sqrt{\mu}M = -N$, therefore exchanging the positive and negative eigenspaces. Since the form $f$ is defined over $k$, we have that $f^\sigma = f$. We may thus identify the form $g^\sigma$ with the (positive definite) restriction $h$ of the form $f$ to the negative eigenspace of $N$ so that the group $\text{O}(g^\sigma)$ is $K$-isomorphic to $\text{O}(h)$. We remark that the discussion above implies that the positive and negative eigenspaces of $N$ have the same dimension, equal to $(n + 1)/2$.

We have thus established that

$$\text{Res}_{K/k}\text{O}(g) \cong \text{O}(g) \times \text{O}(h) \cong H$$

where all isomorphisms are defined over $K$. Now note that the group of $k$-points $H_k = \text{GL}_{n+1}(k) \cap H$ corresponds to pairs of the form $(M, M^\sigma)$, where $M \in \text{O}(g, K)$ and $M^\sigma \in \text{O}(h, K)$ is obtained from $M$ by applying to all of its entries the Galois automorphism $\sigma$. These are precisely the
$k$-points of $\text{Res}_{K/k} \mathcal{O}(g)$. As such the isomorphism (8) induces a one-to-one correspondence between the Zariski dense subgroups $\text{Res}_{K/k} \mathcal{O}(g)_k$ and $H_k$, implying that $\text{Res}_{K/k} \mathcal{O}(g)$ and $H$ are $k$-isomorphic. Therefore if $\Gamma < \mathcal{O}(f)_k$ is an arithmetic lattice, then

$$\text{Stab}_V = \Gamma \cap H_k$$

is an arithmetic subgroup of $\mathcal{O}(g)_\mathbb{R} \times \mathcal{O}(h)_\mathbb{R}$. By modding out the compact factor $\mathcal{O}(h)_\mathbb{R}$ we see that the projection $\Lambda$ of $\text{Stab}_V$ to $\mathcal{O}(g)_\mathbb{R}$ is an arithmetic subgroup of $\mathcal{O}(g)_\mathbb{R}$.

Finally, the admissibility of $g$ follows easily from the admissibility of $f$. Indeed, for each non-identity embedding $\eta : k \to \mathbb{R}$, we have that $(\text{Res}_{K/k} \mathcal{O}(g))_\mathbb{R}^\eta$ decomposes as the product of the restriction of the positive definite form $f^\eta$ to the positive and negative eigenspace for $N^\eta$, and therefore all the resulting factors are compact. We thus see that the involution $[N] \in \mathcal{P}(f)_k$ corresponds to a type I arithmetic $\mathbb{C}$-subspace of dimension $(n - 1)/2$ with adjoint trace field $K$ and ambient group $\mathcal{P}(g)$.

3.2. Arithmetic lattices of type II.

3.2.1. Quaternion algebras. Let $k$ be a field of characteristic $\neq 2$. A quaternion algebra over $k$ is a 4-dimensional central simple algebra $D$. Any quaternion algebra is isomorphic to the algebra $D(a, b)$ with $i^2 = a$, $j^2 = b$, and $ij = -ji = k$, for some choice of non-zero $a, b \in k$. These relations imply that $k^2 = -ab$. Over any field $k$, the quaternion algebra $D(1, 1)$ is isomorphic to the algebra $M_2(k)$ of $2 \times 2$ matrices with coefficients in $k$ through the following homomorphism:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As a corollary of Wedderburn’s structure theorem [35, Theorem 2.9.6], any quaternion algebra $D$ over $k$ is either isomorphic to the algebra $M_2(k)$ of $2 \times 2$ matrices (which always has zero divisors), or it is a division algebra (i.e. any non-zero element has a multiplicative inverse).

In any quaternion algebra $D$ there is an involutory anti-automorphism $q \to q^*$ (called the standard involution) whose fixed-point set coincides with the field $k$. It is obtained by multiplying $i, j$ and $k$ by $-1$ and satisfies $(pq)^* = q^*p^*$, for all $p, q \in D$. If $D \cong M_2(k)$, the standard involution can be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(9)

For any element $q \in D$, its norm is defined as $N(q) = qq^* \in k$ and its trace is defined as $\text{Tr}(q) = q + q^* \in k$. If $D \cong M_2(k)$, the norm and trace of an element are respectively the determinant and trace of the corresponding matrix. The set of quaternions $q \in D$ with unit norm forms a multiplicative group denoted by $\text{SL}_1(D)$, and this is the group of $k$-points of the algebraic.
$k$-group $\text{SL}_1(D)$, which is a $k$-form of $\text{SL}_2(\mathbb{K})$. If $D \cong M_2(k)$, then we have $\text{SL}_1(D) \cong \text{SL}_2(k)$.

Now, let $k$ be an algebraic number field, $\mathcal{O}$ its ring of integers and $D$ a quaternion algebra over $k$. An order in $D$ is an $\mathcal{O}$-submodule $O \subset D$ such that $\mathcal{O}$ is a subring containing 1 which generates $D$ over $k$. The subring $M_2(\mathcal{O})$ is always an order in $M_2(k)$. If $a, b \in \mathcal{O}$, the quaternions with coefficients in $\mathcal{O}$ always form an order in $D(a, b)$. Given an order $O \subset D$, its group of units is the group $\text{SL}_1(O) = \text{SL}_1(D) \cap O$. The groups of units of any two orders in a quaternion algebra $D$ are commensurable.

### 3.2.2. Arithmetic hyperbolic lattices from quaternion algebras.

Let $D$ be a quaternion algebra over the field $k$ with the ring of integers $\mathcal{O}$, and let

$$F(x, y) = \sum_{i,j=1}^{m} x_i^* a_{ij} y_j, \quad (a_{ij} \in D, a_{ij} = -a_{ji}^*)$$

be a non-degenerate skew–Hermitian form on the right $D$–module $D^m$. Let $U(F, D)$ denote the group of automorphisms of $D^m$ preserving the form $F$. This is the group $U(F)_k$ of $k$-points of the algebraic group $U(F)$. If $k$ is an algebraic number field and $O$ is some order in $D$, let $U(F, O)$ denote the subgroup of $U(F, D)$ preserving the $O$-lattice $O^m$.

If $D \cong M_2(k)$, then $U(F)$ is $k$–isomorphic to the orthogonal group $O(f)$ of a $k$–defined form $f$ of rank $2m$. Indeed, $D^m$ is a $4m$–dimensional vector space over $k$. Assuming that $D = D(1, 1)$, we set

$$D^m_\pm = \{x \in D^m \mid x \iota = \pm x\}.$$ 

Then $D^m = D^m_+ \oplus D^m_-$ and $D^m_\pm = D^m_+ \jmath$, so that $\dim D^m_+ = \dim D^m_- = 2m$.

One can notice that, if $x, y \in D^m_+$, then

$$F(x, y) = f(x, y)(\iota - 1)\jmath,$$

where $f$ is a symmetric non–degenerate bilinear form on $D^m_+$ with values in $k$. Indeed, for all $x, y \in D^m_+$ we have that

$$0 = F(x, y(\iota - 1)) = F(x, y)(\iota - 1), \quad \text{and} \quad 0 = F(x(\iota - 1), y) = (\iota - 1)F(x, y).$$

By a straightforward computation, the only quaternions $q \in D$ that satisfy $q(\iota - 1) = (\iota - 1)q = 0$ are precisely those of the form $\lambda(\iota - 1)\jmath$ for some scalar $\lambda \in k$ (the latter depending on $x, y \in D^m_\pm$). The fact that the form $f$ is symmetric follows from $F$ being skew–Hermitian.

Thus, the map $\Phi: A \to A|_{D^m_+}$ is an isomorphism

$$\Phi: U(F)_k = U(F, D) \to O(f, k) = O(f)_k,$$

where the form $f$ is uniquely determined (up to a scalar factor) by $F$. For $k = \mathbb{R}$, let us define the signature of $F$ to be the signature of $f$.

Now let $k \subset \mathbb{R}$ be a totally real number field and $D$ be a quaternion algebra such that $D^m \otimes \mathbb{R} \cong M_2(\mathbb{R})$ for all non–identity embeddings $\sigma: k \to \mathbb{R}$. The form $F$ regarded as a form on $(D \otimes \mathbb{R})^m$ is admissible if it has signature
Then $v$ is a form on the right division algebra over $k$ isomorphic to $M$.

Thus, for every order $O$ in $D$, we have that the projection $PU(F, O)$ of $U(F, O) = O(f) \cong O \cong O_{2m-1,1}$ is naturally identified with a subgroup of $O_{2m-1,1}$.

When $D \cong M_2(k)$, we have that $U(F, D)$ is $k$-isomorphic to $O(f, k)$, with $f$ being an admissible $k$-defined form. Thus we do not obtain anything new: in this case all arithmetic lattices in $PU(F)_k$ are actually type I lattices.

If $D$ is a division algebra we obtain an new class of hyperbolic arithmetic lattices.

**Definition 3.4.** Let $F$ be an admissible skew–Hermitian form defined over a division quaternion algebra $D$ over $k$, and $O$ be an order in $D$. All subgroups $\Gamma < PO_{2m-1,1}$ which are commensurable in the wide sense with a group of the form $PU(F, O)$ as above are called arithmetic lattices of type II. A hyperbolic orbifold $M = \mathbb{H}^{2m-1}/\Gamma$ is a type II orbifold if $\Gamma$ is an arithmetic lattice of type II.

**Remark 3.5.** Given a totally real field $k$ and a division algebra $D$ over $k$ satisfying the hypotheses above, two admissible skew–Hermitian forms $F_1$ and $F_2$ on $D^m$ define the same commensurability class of hyperbolic lattices if and only if $F_1$ is equivalent (up to change of basis in $D^m$) to $\lambda \cdot F_2$ for some $\lambda \in k^\times$.

**Remark 3.6.** If $k = \mathbb{Q}$ and $F$ is isotropic over $D$ then the orbifold $\mathbb{H}^{2m-1}/\Gamma$ is non-compact, otherwise $\Gamma$ is a uniform lattice, as follows from Godement’s criterion [25].

Also, in the non-uniform case the quaternion algebra $D$ is necessarily isomorphic to $M_2(\mathbb{Q})$, so that $\Gamma$ is a type I lattice. More generally, we prove the following.

**Proposition 3.7.** If $k$ is a totally real algebraic number field, $D$ is a quaternion division algebra over $k$ which splits over $\mathbb{R}$ and $F$ is a skew–Hermitian form on the right $D$–module $D^m$ of signature $(2m - 1, 1)$, then $F$ cannot represent 0.

**Proof.** Indeed, suppose that there exists a non–zero $v \in D^m$ such that $F(v, v) = 0$. Let us consider the real quaternion algebra $D \otimes \mathbb{R} \cong M_2(\mathbb{R})$. Then $v = x + yj$ for some $x, y \in (D \otimes \mathbb{R})_+^m$, and

$$0 = F(v, v) = f(x, x)(i - 1)j + f(x, y)(i - 1) + f(x, y)(i + 1) + f(y, y)(i + 1)j.$$  

(10)

Note that $\{i(j - 1), i - 1, i + 1, (i + 1)j\}$ is an $\mathbb{R}$-basis of $D \otimes \mathbb{R}$, and thus

$$f(x, x) = f(x, y) = f(y, y) = 0.$$  

Since $f$ is a form of signature $(2m - 1, 1)$, we have that $y = \mu x$ for some $\mu \in \mathbb{R}$, so that $v = x(1 + \mu j)$. 


If $\mu = \pm 1$, then we have $(1 + \mu j)(1 - \mu j) = 0$, and thus $v(1 - \mu j) = 0$, so that the coordinates of $v$ understood as elements of $(D \otimes \mathbb{R})^n_{\mathbb{R}}$ are zero divisors in $D \otimes \mathbb{R}$. Suppose that $q \in D$ is a non-zero coordinate of $v$. If $D$ were a division algebra, then $q$ would be invertible in $D$ and therefore also in $D \otimes \mathbb{R}$. This is a contradiction, since $D \otimes \mathbb{R}$ is a non-trivial associative algebra, and no element can be invertible and a zero divisor at the same time. If $\mu \neq \pm 1$, then $1 + \mu j$ is invertible, since $N(1 + \mu j) \neq 0$. Then $v(1 + \mu j)^{-1}(i - 1) = x(i - 1) = 0$, and again all the coordinates of $v$ are zero divisors. Thus, $D$ cannot be a division algebra.

**Remark 3.8.** Let us note that in the classification of algebraic semisimple groups by Tits [54], type II lattices as defined above belong to the indices $1D_{(2)}^{(2)} m_{0,0}$ or $2D_{(2)}^{(2)} m_{0,0}$, as also follows from Proposition 3.7 (see also [18, Proposition 13.6]). We would also like to mention the fact that the discussion in [58, p. 220] apparently does not take into account type II arithmetic lattices in dimension 3 (cf. Section 3.3).

3.2.3. Constructing $k$-involutions. Below we show how to construct all the $k$–involutions in any type II group of the form $\text{PU}(F)_k$ where, as in the previous section, $D$ is a quaternion division algebra over $k$ and $F$ is an admissible skew–Hermitian form on $D^m$ of signature $(2m - 1, 1)$.

Let $D_1$ be a right submodule of $D^m$. Since $D_1$ is a module over a division algebra, it has to be free, so that $D_1 \cong D^l$ for some $l < m$. We can restrict the form $F$ to a non-degenerate skew–Hermitian form $G$ on $D_1$. Let us suppose that the signature of $G$ is $(2l - 1, 1)$. Now, let us consider the orthogonal complement

$$D_1^\perp = \{ x \in D^m \mid F(x, y) = 0 \text{ for all } y \in D_1 \}.$$ 

Notice that since $F$ is skew–Hermitian, the derived orthogonality relation is symmetric.

We observe that $D^m = D_1 \oplus D_1^\perp$. Indeed, $D_1 \cap D_1^\perp$ is trivial by Proposition 3.7.

In order to prove that $D_1 + D_1^\perp = D^m$, we proceed as follows: first, fix a basis $(x_1, \ldots, x_l)$ of $D_1$ and complete it to a basis $(x_1, \ldots, x_m)$ of $D^m$. Next, apply the Gram–Schmidt orthogonalisation process by setting

$$\text{proj}_y(x) = y F(y, y)^{-1} F(y, x), \quad y_1 = x_1, \quad y_i = x_i - \sum_{j=1}^{i-1} \text{proj}_{y_j}(x_i), \ i = 2, \ldots, m.$$ 

Note that $\text{proj}_y(x)$ is well–defined whenever $y \in D^m$ is non–zero. Indeed $F(y, y)$ is non–zero by Proposition 3.7 and it is invertible since $D$ is a division algebra. Also, a straightforward check shows that $F(y, x - \text{proj}_y(x)) = 0$ for any $x \in D^m$.

We thus obtain an orthogonal basis $(y_1, \ldots, y_m)$ for $D^m$ such that the first $k$ vectors are a basis for $D_1$ and the remaining belong to $D_1^\perp$. This is clearly enough to conclude that $D_1 + D_1^\perp = D^m$, and moreover we see that $D_1^\perp \cong D^{m-l}$ and that the restriction of $F$ to $D_1^\perp$ has signature $(2(m - l), 0)$. 


Finally, we notice that if $N$ is a linear transformation of $D^m$ that acts as the identity on $D_1$ and as multiplication by $-1$ on $D_1^\perp$, then $N$ is an involution in $U(F, D) = U(F)_k$. Moreover, the fixed point set for the action of $\theta$ on $\mathbb{H}^{2m-1}$ is precisely a totally geodesic subspace of dimension $2l - 1$.

It is not difficult to see that all the $k$-involutions of $U(F, D)$ arise through the construction above. Indeed, let us suppose that $N \in U(F, D)$ has order 2. Since $D$ is a division algebra, the eigenvalues of $\theta$ as an element of $GL(D^m)$ can only be equal to $+1$ or $-1$. The corresponding right $D$-modules $D_1 = \{x \in D^m \mid \theta(x) = x\}$ and $D_{-1} = \{x \in D^m \mid \theta(x) = -x\}$ are orthogonal with respect to the form $F$ (i.e. $D_{-1} = D_1^\perp$). The involution $\theta$ corresponds to an element of $\text{Isom}(\mathbb{H}^n)$ if and only if the restriction $F|_{D_1}$ of the form $F$ to $D_1$ has signature $(2l - 1, 1)$ for $l = \dim D_1$.

Finally, suppose that $i = [N] \in PU(F, D)$, with $N$ as above, and that $\Gamma$ is a type II arithmetic lattice in $PU(F, D \otimes \mathbb{R}) = PU(F)_R$. By the discussion above, it follows that the fixed-point set of $i$ projects to a type II totally geodesic subspace in $M = \mathbb{H}^n/\Gamma$ with associated admissible skew-hermitian $k$-defined form $G = F|_{D_1}$.

We still need some work to classify the involutions in the group $PU(F)_k$ of $k$-points of the algebraic $k$-group $PU(F)$. The issue is the same that arises in the case of type I lattices acting on $\mathbb{H}^n$ with $n$ odd (see Section 3.1.2), namely that the group $U(F)$ is not adjoint. Fortunately, the argument from [20, Lemma 4.3] with small modifications applies to this case as well.

Notice that $D \otimes \mathbb{C} \cong M_2(\mathbb{C})$ so that the elements of $U(F, D \otimes \mathbb{C}) = U(F)$ can be represented by elements of $GL_{2m}(\mathbb{C})$. We now define the general unitary group of the form $F$:

$$GU(F) = \{B \in GL_{2m}(\mathbb{C}) \mid B^*FB = \mu F \text{ for some } \mu \in \mathbb{C}\},$$

where $F$ is now understood as a $2m \times 2m$ matrix with complex coefficients. The matrix $B^*$ is obtained from $B$ by transposing the $(D \otimes \mathbb{C})$-coefficients (each being represented by a $2 \times 2$ block) and then applying to each block the standard involution $(9)$. The corresponding group of $k$-points is

$$GU(F)_k = GU(F) \cap GL_{2m}(k).$$

Recall that by the argument of Section 3.2.2 there exists an isomorphism

$$\Phi : U(F)_R = U(F, D \otimes \mathbb{R}) \to O(f, \mathbb{R}) = O(f)_R,$$

where $f$ is an admissible $k$–form of signature $(2m - 1, 1)$. This isomorphism extends to an isomorphism $\Phi : GU(F) \to GO(f)$ of real algebraic groups, where $GO(f)$ is the general orthogonal group of $f$ [20, p. 7].

Notice that the element $\mu \in \mathbb{C}$ is uniquely determined by $B \in GU(F)$ so that one can unambiguously write $\mu = \mu(B)$. Moreover $B$ represents an equivalence between the forms $f$ and $\mu f$, which both have signature $(2m - 1, 1)$. By an exactly the same argument as in the type I case (see [20, p. 7]) we have that $\mu \in \mathbb{R}$ and $\mu > 0$. Moreover, if $m \geq 2$ then $\mu$ is totally positive and thus $K = k(\sqrt{\mu})$ is totally real.
Let us define the group of “scalar” $2m \times 2m$ matrices $C = \{ c \cdot \text{id} | c \in \mathbb{C}^* \}$. Then the projective general unitary group of $F$ is $\text{PGU}(F) = \text{GU}(F)/C$. The isomorphism $\Phi$ above naturally descends to an isomorphism $\text{PGU}(F) \rightarrow \text{PGO}(f)$, where $\text{PGO}(f) = \text{GO}(f)/C$ is the projective general orthogonal group of $f$, as defined in [20, p. 7].

Suppose that $B \in \text{GU}(F)_k$. We claim that $\mu(B) \in k_{>0}$. Notice that $B^*FB = \mu(B) \cdot F$, hence $\mu(B) \in D$. Moreover, $\mu(B) \in \mathbb{R}_{>0}$ and therefore belongs to the centre of $D \otimes \mathbb{R}$. These two facts together imply that $\mu(B)$ is in the centre of $D$, which is precisely $k$.

The following result is analogous to [20, Lemma 4.3] in the setting of type II lattices.

**Lemma 3.9.** Let $F$ be a skew–Hermitian form on the right $D$-module $D^m$, where $D$ is a quaternion algebra over a totally real number field $k$ satisfying the hypotheses of Definition 3.4. Let $\text{PU}(F)_k$ be the group of $k$–points of the adjoint group $\text{PU}(F)$. Then

$$\text{PU}(F)_k = \left\{ \frac{\pm 1}{\sqrt{\mu(B)}} B \mid B \in \text{GU}(F)_k \right\}.$$  

**Proof.** The proof follows closely the argument of [20, Lemma 4.3], with the main difference being that the groups $\text{O}(f)$ and $\text{GO}(f)$ are replaced by their type II counterparts $\text{U}(F)$ and $\text{GU}(F)$, respectively.

Let $\pi : \text{U}(F) \rightarrow \text{PU}(F)$ and $\eta : \text{GU}(F) \rightarrow \text{PGU}(F)$ be the quotient maps. Both $\pi$ and $\eta$ are $k$-homomorphisms of $k$-algebraic groups by [10, Theorem 6.8]. The inclusion map

$$\nu : \text{U}(F) \rightarrow \text{GU}(F)$$

is a $k$-homomorphism as well. By the universal mapping property [10, p. 94] the inclusion $\nu$ of $\text{U}(F)$ into $\text{GU}(F)$ induces a $k$-homomorphism $\overline{\nu} : \text{PU}(F) \rightarrow \text{PGU}(F)$ such that $\overline{\nu} \pi = \eta \nu$.

If $B \in \text{U}(F)$, then $\overline{\nu}(\pm B) = CB$ and so $\overline{\nu}$ is a monomorphism. Now, assume that $B \in \text{GU}(F)$. Then $B^*FB = \mu F$, and thus

$$(1/\sqrt{\mu} \cdot B)^* F (1/\sqrt{\mu} \cdot B) = F$$

which implies $\pm 1/\sqrt{\mu} \cdot B \in \text{U}(F)$. We have that $\overline{\nu}(\pm 1/\sqrt{\mu} \cdot B) = CB$ and thus $\overline{\nu}$ is an isomorphism.

There is a short exact sequence of algebraic $K$-groups

$$1 \rightarrow C \rightarrow \text{GU}(F) \xrightarrow{\eta} \text{PGU}(F) \rightarrow 1$$

which determines an exact sequence of Galois cohomology groups

$$1 \rightarrow C_k \rightarrow \text{GU}(F)_k \xrightarrow{\eta} \text{PGU}(F)_k \rightarrow H^1(k, C)$$

by [14, Prop. 1.17 and Corollary 1.23]. We have that $H^1(k, C) = 0$ by [52, p. 72, Prop. 1], therefore $\eta(\text{GU}(F)_k) = \text{PGU}(F)_k$, and $\text{PGU}(F)_k =$
\{CB \mid B \in \text{GU}(F)_k\}. Therefore
\[ \text{PU}(F)_k = \tau^{-1}(\text{PGU}(F)_k) = \left\{ \pm \frac{1}{\sqrt{\mu(B)}} B \mid B \in \text{GU}(F)_k \right\}. \]

Now that we have a complete description of \(k\)-involutions in the ad- joint \(k\)-group \(\text{PU}(F)\), we can discuss the structure of the corresponding \(fc\)-subspaces. The case where \(N\) belongs to \(\text{PU}(F,D)\) has already been treated, and we get that the corresponding \(fc\)-subspaces are arithmetic of type II, with the same adjoint trace field \(k\).

We now suppose that \(\pm N\) is an involution in \(\text{PU}(F)_k\) of the form \(\pm N = \pm \frac{1}{\sqrt{\mu(B)}} B\), with \(B \in \text{GU}(F)_k\) and \(\mu = \mu(B)\) is totally positive and not a square in \(k\). Let \(K = k(\sqrt{\mu})\). The discussion proceeds in very much the same way as for odd-dimensional type I lattices (see Section 3.1.2). Notice that both \(N\) and \(B\) are defined up to multiplication by \(-\text{id}\). Without loss of generality, we may assume that \(N = (1/\sqrt{\mu})B\). We define the positive eigenspace \(D_1\) of \(N\) and similarly the negative eigenspace \(D_{-1}\) of \(N\). Notice that now \(D_1\) and \(D_{-1}\) are right \((D \otimes K)\)-modules, and that the quaternion algebra \(D \otimes K\) may now split.

Up to multiplication by \(-\text{id}\), we may assume that the restriction \(G\) of the form \(F\) to \(D_1\) has signature \((n - 1,1)\), while the restriction \(H\) of \(F\) to \(D_{-1}\) has signature \((l,0)\), with \(n = \dim(D_1)\), \(l = \dim(D_{-1})\) and \(n + l = m\).

We claim that the form \(G\) is admissible. Notice that the nontrivial automorphism \(\sigma \in \text{Gal}(K/k)\) acts naturally on the elements of the quaternion algebra \(D \otimes K\) by sending \(q \otimes \lambda\) to \(q \otimes \lambda^\sigma\). By computing the restriction of scalars of \(U(G)\) we obtain the \(K\)-isomorphism:
\[ \text{Res}_{K/k} U(G) \cong U(G) \times U(G^\sigma), \]
where \(G^\sigma\) is obtained by applying \(\sigma\) to the coefficients of \(G\).

The action of \(\sigma\) sends \(N = (1/\sqrt{\mu})B\) to \(-N = -(1/\sqrt{\mu})B\), thus exchanging the positive and negative eigenspaces \(D_1\) and \(D_{-1}\). It follows that \(n = \dim(D_1) = \dim(D_{-1}) = l\) and that \(m = n + l\) is even. The form \(F\) is defined over \(k\), and because of this we have that \(F^\sigma = F\). We may therefore identify the form \(G^\sigma\) with the positive definite form \(H = F_{D_{-1}}\) so that \(U(G^\sigma)\) is isomorphic to group \(U(H)\). We obtain \(K\)-defined isomorphisms
\[ \text{Res}_{K/k} U(G) \cong U(G) \times U(H) \cong H, \quad (13) \]
where the \(k\)-group \(H\) is the centraliser of \(N\) in \(U(F)\). Moreover the isomorphism in (13) induces a bijection between the \(k\)-points of \(\text{Res}_{K/k} U(G)\) and those of \(H\), implying that these two groups are actually \(k\)-isomorphic. As in the case of type I lattices, the admissibility of \(G\) now follows easily from the admissibility of \(F\) over \(k\).

These facts together imply that if \(\Gamma < U(F)_k\) is a type II arithmetic lattice, then \(\text{Stab}_\Gamma(V_1) = \Gamma \cap H(k)\) is an arithmetic lattice in the group \(U(G)_R \times U(H)_R\). By modding out the the compact factor \(U(H)_R\) we obtain
that the projection $\Lambda$ of $\text{Stab}_\Gamma(V_1)$ to $U(G) \cong O_{m-1,1}$ is an arithmetic hyperbolic lattice with adjoint trace field $K$.

Finally, we want to understand whether this totally geodesic sublattice $\Lambda \subset U(p)G$ is of type I or type II. By the discussion in Section 3.2.2, if $D \otimes K \cong M_2(K)$ then $\Lambda$ is a type I lattice, while if $D \otimes K$ is a division algebra, $\Lambda$ is a type II lattice.

**Remark 3.10.** It is a well-known fact that all hyperbolic lattices acting on $\mathbb{H}^n$, with $n$ even, are type I lattices. To construct all arithmetic lattices acting on $\mathbb{H}^n$, where $n \neq 3, 7$ is an odd number, one has to consider both type I and type II lattices [58, p. 222]. In the remaining cases $n = 3, 7$, one has to add two exceptional families of arithmetic lattices which we now introduce.

### 3.3. Arithmetic hyperbolic lattices in dimension 3.

Due to the exceptional isomorphism between $\text{PSL}_2(\mathbb{C})$ and $\text{PSO}_{3,1}(\mathbb{R})$, there is a way to construct all arithmetic lattices acting on $\mathbb{H}^3$ through arithmetic lattices in $\text{PSL}_2(\mathbb{C})$, as follows:

1. Fix a complex number field $L$ with one complex place (up to complex conjugation) and a finite (possibly empty) set of real places.
2. Choose a quaternion algebra $A$ over $L$ such that $A$ is ramified at all real places of $L$.

If $O$ is an order in $A$, then the central quotient of the group of units of $O$, denoted by $\text{PSL}_1(O)$, is an arithmetic lattice acting on $\mathbb{H}^3$. Moreover, all arithmetic lattices acting on $\mathbb{H}^3$ are commensurable in the wide sense with a group of this form. In this setting both the field $L$ and the quaternion algebra $A$ are invariants of the commensurability class of the lattice $\Gamma$ and correspond to the so-called invariant trace field and the invariant quaternion algebra of $\Gamma$ [35, Section 3.3].

If the lattice is not uniform, then necessarily $L = \mathbb{Q}(\sqrt{-d})$, where $d > 0$ is square-free, $A = M_2(L)$, and the resulting lattice is commensurable with $\text{PSL}_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers of $L$. These are the so-called Bianchi groups. In all other cases $A$ is a division algebra [35, Section 8.2].

We now wish to understand how the invariant trace field of a lattice $\Gamma < \text{PSL}_2(\mathbb{C})$ relates to the adjoint trace field of the image of $\Gamma$ under the exceptional isomorphism $i : \text{PSL}_2(\mathbb{C}) \rightarrow \text{PSO}_{3,1}(\mathbb{R})$. As it is explained in [59], the invariant trace field is simply the adjoint trace field of $\Gamma$ regarded as a lattice in $\text{PSL}_2(\mathbb{C})$, i.e. $L = \mathbb{Q}(|\text{tr}(\text{Ad}(\gamma))|; \gamma \in \Gamma)$ where

$$\text{tr}(\text{Ad}(\gamma)) = \text{tr}(\gamma)^2 - 1 = \text{tr}(\gamma^2) + 1$$

is the trace of the adjoint representation of $\gamma \in \text{PSL}_2(\mathbb{C})$. The ambient group of $\Gamma < \text{PSL}_2(\mathbb{C})$ is $\text{PGL}_1(A)$: the central quotient of the group of invertible elements of the invariant quaternion algebra $A$. Notice that the group of $L$-points $\text{PGL}_1(A)_L$ is isomorphic the group $A^*/L^*$ defined as the quotient of the group of units $\text{GL}_1(A) = A^*$ of $A$ by the multiplicative action of $L^*$ [35, Theorem 8.4.4].
We now turn our attention to the exceptional isomorphism \( i : \text{PSL}_2(\mathbb{C}) \to \text{PSO}_{3,1}(\mathbb{R}) \), which is explicitly described in [35, Section 10.2]. For a complex number \( z = x + iy \), \( x, y \in \mathbb{R} \), denote by \( z^* \) its conjugate \( z^* = x - iy \). We prove the following:

**Proposition 3.11.** Let \( i \) denote the exceptional isomorphism from \( \text{PSL}_2(\mathbb{C}) \) to \( \text{PSO}_{3,1}(\mathbb{R}) \). For any element \( g \in \text{PSL}_2(\mathbb{C}) \), \( \text{tr}(\text{Ad}(i(g))) = \text{tr}(\text{Ad}(g)) + \text{tr}(\text{Ad}(g))^* \).

**Proof.** The isomorphism \( i \) can be conveniently interpreted as an isomorphism of real Lie groups between the \( \mathbb{R} \)-points of \( \text{Res}_{\mathbb{C}/\mathbb{R}}(\text{PSL}_2) \) and the group \( \text{PSO}_{3,1}(\mathbb{R}) \). Being an isomorphism of real Lie groups, it induces an isomorphism of the real Lie algebras and these induce an isomorphism of the corresponding adjoint representations. Thus, we opt to work directly in the group \( \text{Res}_{\mathbb{C}/\mathbb{R}}(\text{PSL}_2)_{\mathbb{R}} \) and compute the trace of the adjoint representation of its elements. In order to describe this group, we use the regular representation of complex number as 2 matrices with real coefficients of the form

\[
C \ni z = x + iy \to \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in M_2(\mathbb{R}),
\]

where \( x \) and \( y \) correspond respectively to the real and imaginary parts of \( z \in \mathbb{C} \). By doing so we can describe the Weil restriction of \( \text{SL}_2(\mathbb{C}) \) as:

\[
\text{Res}_{\mathbb{C}/\mathbb{R}}(\text{SL}_2)_{\mathbb{R}} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid AD - BC = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A, B, C, D \text{ of the form (14)} \right\},
\]

which is a 6-dimensional real Lie group. The group \( \text{Res}_{\mathbb{C}/\mathbb{R}}(\text{PSL}_2) \) is the quotient of \( \text{Res}_{\mathbb{C}/\mathbb{R}}(\text{SL}_2) \) by \( \{ \pm \text{id} \} \) and its Lie algebra is the Weil restriction from \( \mathbb{C} \) to \( \mathbb{R} \) of the Lie algebra of \( \text{SL}_2(\mathbb{C}) \), i.e. obtained by regarding the complex 3-dimensional Lie algebra \( \mathfrak{sl}_2 \) of as a 6-dimensional real Lie algebra \( \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathfrak{sl}_2) \).

As such, if \( B = (v_1, v_2, v_3) \) is a basis of \( \mathfrak{sl}_2 \), a basis of \( \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathfrak{sl}_2) \) is given by \( B' = (v_1, i \cdot v_1, v_2, i \cdot v_2, v_3, i \cdot v_3) \). Suppose that \( g \in \text{SL}_2(\mathbb{C}) \) and \( \text{Ad}(g) \in \text{GL}(\mathfrak{sl}_2) \) is represented with respect to the basis \( B \) by a matrix

\[
M = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \in \text{GL}_3(\mathbb{C})
\]

with trace \( \text{tr}(\text{Ad}(g)) = a_{1,1} + a_{2,2} + a_{3,3} \). We denote by \( \text{Res}_{\mathbb{C}/\mathbb{R}}(g) \) the element which corresponds to \( g \) in the group \( \text{Res}_{\mathbb{C}/\mathbb{R}}(\text{SL}_2) \). Its adjoint action \( \text{Ad}(\text{Res}_{\mathbb{C}/\mathbb{R}}(g)) \in \text{GL}(\text{Res}_{\mathbb{C}/\mathbb{R}}(\mathfrak{sl}_2)) \) is represented with respect to the basis \( B' \) by the matrix

\[
\text{Res}_{\mathbb{C}/\mathbb{R}}(M) = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix} \in \text{GL}_6(\mathbb{R}),
\]
where each $2 \times 2$ real submatrix $A_{i,j}$ is obtained from the corresponding coefficient of $M$ as in (14). We thus see that

$$\text{tr} (\text{Ad}(i(g))) = \text{tr} (\text{Ad}(\text{Res}_{\mathbb{C}/\mathbb{R}}(g))) = \text{tr}(A_{1,1}) + \text{tr}(A_{2,2}) + \text{tr}(A_{3,3}) =$$

$$= \text{tr}(\text{Ad}(g)) + \text{tr}(\text{Ad}(g))^*.$$

(17)

We note the following important corollary of Proposition 3.11:

**Corollary 3.12.** If $\Gamma \subset \text{PSL}_2(\mathbb{C})$ is an arithmetic lattice with invariant trace field $L$ that contains a subfield $k$ such that $[L : k] = 2$, then the adjoint trace field of $i(\Gamma) \subset \text{PSO}(3,1)$ is precisely $k$, and thus is totally real.

**Proof.** Since the field $L$ has one complex place, all of its subfields are totally real. Therefore $L$ is an imaginary quadratic extension of the totally real field $k$ and for any $z \in L$, $z + z^* \in k$. By Proposition 3.11, the adjoint trace field of $i(\Gamma)$ is

$$\mathbb{Q}((\text{tr}(\text{Ad}(i(\gamma))) \mid \gamma \in \Gamma)) = \mathbb{Q}((\text{tr}(\text{Ad}(\gamma))) + \text{tr}(\text{Ad}(\gamma))^* \mid \gamma \in \Gamma) \subset k.$$

By [59, Remark 7 (c)], there exists $\gamma \in \Gamma$ such that $\beta = \text{tr}(\text{Ad}(\gamma)) \in \mathbb{R}$ and $k = \mathbb{Q}(\beta)$. This implies that $\text{tr}(\text{Ad}(i(\gamma))) = 2\beta$, and therefore $k$ is precisely the adjoint trace field of $i(\Gamma)$.

It is natural to ask how to characterise type I and type II lattices among the 3-dimensional ones with the above description. As mentioned in [34, p. 366], type I and type II lattices correspond to the case where the field $L$ contains a totally real subfield $k$ such that the degree of the extension $[L : k]$ is 2, i.e. type I and type II lattices are those to which Corollary 3.12 applies.

We also record the following geometric characterisation: by [59, Remark 7(c)] type I and II lattices are precisely the arithmetic lattices acting on $\mathbb{H}^3$ which contain a pure translation along a geodesic. In this case, the distinction between type I and type II can be recovered as follows [50, pp. 199–200]: if the norm form $N_{L/k}(A)$ splits over $k$ then we obtain type I lattices, otherwise we get type II lattices. In the case of type I lattices, the corresponding admissible form $f$ defined over $k$ can be recovered explicitly from the data of the field $L$ and the quaternion algebra $D$ [35, Section 10.2].

The exceptional family of 3-dimensional hyperbolic lattices arises when the field $L$ does not contain a subfield $k$ such that $[L : k] = 2$. The corresponding arithmetic lattices are not of type I or II, thus we assign them to type III. These lattices contain no pure translation along a geodesic, i.e. all loxodromic elements act as a composition of a translation and a rotation along the same axis with the rotation having an angle $\theta$ which is not a rational multiple of $2\pi$. An explicit example of such a field $L$ as above is discussed for instance in [32, p. 22].
3.3.1. Type II lattices as arithmetic lattices in $\text{PSL}_2(\mathbb{C})$. Let us show how to construct a quaternion algebra $D$ over $k$, and the corresponding skew–Hermitian form $F$ on $D^2$ in the case of 3–dimensional type II–lattices. This argument is not new, however we could not find it anywhere in the literature.

We briefly recall the construction of the norm form (also called corestriction) $N_{L/k}(A)$ of the quaternion algebra $A$. Given a central simple algebra $A$ of degree $m$ over $L$, its conjugate algebra $A^c = \{a^c \mid a \in A\}$ is defined by the following operations:

$$a^c + b^c = (a + b)^c, \quad a^c \cdot b^c = (a \cdot b)^c, \quad \lambda \cdot a^c = (c(\lambda) \cdot a)^c,$$

where $a, b \in A$, $\lambda \in L$ and $c(\lambda)$ denotes the conjugate of $\lambda$. The switch map $s : A^c \otimes_L A \to A^c \otimes_L A$ defined by $a^c \otimes b \mapsto b^c \otimes a$ is $c$-semilinear over $L$ and is a $k$-algebra automorphism. The $k$-subalgebra

$$N_{L/k}(A) = \{z \in A^c \otimes_L A \mid s(z) = z\}$$

of elements fixed by $s$ is a central simple $k$-algebra of degree $m^2$. This construction induces a homomorphism of the respective Brauer groups [29, Proposition 3.13]:

$$N_{L/k} : \text{Br}(L) \to \text{Br}(k), \quad [A] \mapsto [N_{L/k}(A)].$$

Now, let $A$ be the quaternion algebra over a complex field $L$ associated with an arithmetic lattice $\Gamma$ in $\text{PSL}_2(\mathbb{C})$. Since $A$ is a quaternion algebra, it has order 1 (if $A = M_2(L)$) or 2 (if $A$ is a division algebra) in the Brauer group of $L$. There are two possible cases:

1. $[N_{L/k}(A)]$ has order 1 in $\text{Br}(k)$, $N_{L/k}(A) \cong M_4(k)$ and $\Gamma$ is a type I lattice.
2. $[N_{L/k}(A)]$ has order 2 in $\text{Br}(k)$ and $N_{L/k}(A) \cong M_2(D)$, where $D$ is a division quaternion algebra over $k$.

Notice that the case where $N_{L/k}(A)$ is a division algebra of degree 4 is excluded: since $k$ is an algebraic number field the order of $[N_{L/k}(A)]$ as an element of the Brauer group of $k$ is equal to the degree of the division algebra that is Brauer-equivalent to $N_{L/k}(A)$ [43, p. 359].

Suppose that we are in case (2) and let us fix a basis $B$ on the right $D$–module $D^2$. This choice specifies a $D$-algebra isomorphism between $N_{L/k}(A) \cong M_2(D)$ and the algebra $\text{End}_D(D^2)$ of $D^2$–endomorphisms. By [29, Proposition 4.1 and Theorem 4.2], there is a bijection between involutions of the first kind on $\text{End}_D(D^2)$ and non-degenerate skew–Hermitian forms on $D^2$ (up to multiplication in $k^\times$). If $\phi$ is an involution of the first kind on $\text{End}_D(D^2)$ and $f$ is the corresponding skew–Hermitian form, then $f$ and $\phi$ are related as follows:

$$f(x, g(y)) = f(\phi(g)(x), y) \quad (18)$$
for all $x$, $y$ in $D^2$ and $g \in \text{End}_D(D^2)$. The choice of the basis $\mathcal{B}$ allows us to rewrite (18) as

$$FG = \phi(G)^* F,$$

(19)

where $F$ (resp. $G$) denotes the matrix which represents the form $f$ (resp. the endomorphism $g$) with respect to the basis $\mathcal{B}$. The notation $M^*$ represents the matrix obtained from $M$ by transposing and applying the standard involution of $D$ to all its coefficients, so that $F^* = -F$.

Let $\sigma$ denote the standard involution on the quaternion algebra $A$. We can define an involution $\phi$ on $A^c \otimes_L A$ by setting $\phi(a^c \otimes b) = \sigma(a)^c \otimes \sigma(b)$. It is easy to see that $\phi$ is an involution of the second kind on $A^c \otimes_L A$ whose fixed point set is the field $k$. Moreover, it commutes with the switch map $s$, so that $\phi(N_{L/k}(A)) = N_{L/k}(A)$. Its restriction to $N_{L/k}(A)$, which we still denote by $\phi$ with a slight abuse of notation, is an involution of the first kind with associated skew–Hermitian form $f$ on $D^2$.

We now define a map $i$ from the quaternion $L$-algebra $A$ to the $k$-algebra $N_{L/k}(A) \cong M_2(D)$ as follows:

$$i(a) = a^c \otimes a.$$

The map $i$ is multiplicative and $i(1) = 1^c \otimes 1$, so that it becomes a group homomorphism when restricted to the subgroup $\text{SL}_1(A)$ of unit-norm elements of $A$. We claim that the map $i$ induces an isogeny between the group $\text{SL}_1(A)$ and $U(F)$, where $F$ is the skew–Hermitian matrix which represents the form $f$ with respect to the basis $\mathcal{B}$.

Let us denote by $G \in M_2(D)$ the element $i(a)$, where $a \in \text{SL}_1(A) = \text{SL}_1(A)_L$. Since $1 = a \cdot \sigma(a)$ we see that

$$i^2 = i(1) = i(\sigma(a) \cdot a) = i(\sigma(a)) \cdot i(a) = \phi(G) \cdot G,$$

(20)

where the last equality holds because

$$i(\sigma(a)) = (\sigma(a))^c \otimes \sigma(a) = \phi(a^c \otimes a) = \phi(i(a)).$$

By multiplying both sides of (19) on the left by $G^*$ and applying (20), we obtain

$$G^*FG = G^*\phi(G)^*F = (\phi(G) \cdot G)^*F = F,$$

and therefore $i(a) = G \in U(F, D) = U(F)_k$.

For any maximal order $O < A$, we have that $\text{SL}_1(O)$ is a lattice in $\text{SL}_1(A) \cong \text{SL}_2(\mathbb{C})$ and the image $i(\text{SL}_1(O))$ is a lattice in $U(F)_\mathbb{R} \cong O_{n,1}$, and thus Zariski-dense in $U(F)_\mathbb{R}$ by Borel’s density theorem. This gives rise to a surjective $k$-morphism $\text{Res}_{L/k}(\text{SL}_1(A)) \twoheadrightarrow U(F)_\mathbb{R}$. Moreover, ker$(i)$ is finite as it is given by the elements of $L$ with unit norm over $k$. Indeed, $L$ is an imaginary quadratic extension of $k$ and by Dirichlet’s unit theorem the rank of the group of units in the ring of integer elements over $k$ is zero. Thus, we have a $k$-isogeny between $\text{Res}_{L/k}(\text{SL}_1(A))$ and $U(F)_\mathbb{R}$.

Finally, we wish to go the other way around and express a type II arithmetic lattice $\Gamma < PU(F, D) \cong PU(F)_k$ acting on $\mathbb{H}^3$ as an arithmetic lattice in $\text{PSL}_2(\mathbb{C})$. Recall that the group $PU(F, D)$ is naturally identified
with a subgroup of $\text{PO}(f, \mathbb{R})$, for some symmetric bilinear form $f$ of signature $(3,1)$. Hence $\Gamma$ can be regarded as a lattice in $\text{PO}(f, \mathbb{R})$ and its index 2 subgroup $\Gamma' = \Gamma \cap \text{PSO}(f)_\mathbb{R}$ can be identified with an arithmetic lattice in $\text{PSL}_2(\mathbb{C})$ using the exceptional isomorphism between $\text{PSL}_2(\mathbb{C})$ and $\text{PSO}(f)_\mathbb{R} \cong \text{PO}_{3,1}$. We remark that by Proposition 3.13 below the invariant trace field of the resulting lattice in $\text{PSL}_2(\mathbb{C})$ is indeed an imaginary quadratic extension of a totally real field (see Remark 3.15).

We can therefore associate with $\Gamma'$ its invariant trace field $L$ and invariant quaternion algebra $A$, as defined in [35, Chapter 3]. Since $\Gamma'$ is an arithmetic lattice, we see that $L$ is a number field with one complex place (up to conjugation) and $A$ is ramified at all real places of $L$ [35, Theorem 8.3.2]. Moreover $\Gamma'$ will be commensurable in the wide sense with a group of the form $\text{PSL}_1(O)$, where $O$ is an order in $A$ [35, Corollary 8.3.3].

3.3.2. The Vinberg invariants of type III lattices in $\text{PSO}_{3,1}(\mathbb{R})$. Type III arithmetic lattices acting on $\mathbb{H}^3$ exhibit a peculiar behaviour when interpreted as lattices in $\text{PSO}_{3,1}(\mathbb{R})$. We begin by proving the following fact:

**Proposition 3.13.** Suppose that $\Gamma < \text{PSL}_2(C)$ is a type III arithmetic lattice, and denote by $i(\Gamma) < \text{PSO}_{3,1}(\mathbb{R})$ its image under the exceptional isomorphism. The adjoint trace field $k$ of $i(\Gamma)$ is not totally real.

**Proof.** Let $L$ denote the invariant trace field of $\Gamma$, and let $K = L \cap \mathbb{R}$. By [59, Remark 7(b)], there exists $\gamma \in \Gamma$ such that $\alpha = \text{tr}(\text{Ad}(\gamma)) \in \mathbb{C}$ and $L = K(\alpha)$. Since $\Gamma$ is a type III lattice, the extension $L/K$ is not purely imaginary and the minimal polynomial of $\alpha$ over $K$ has complex roots $\alpha$, $\alpha^*$ and a non-empty set of real roots $r_1, \ldots, r_k$.

In particular, the extension $L/K$ has both real and complex embeddings and thus is not a Galois extension. Let us denote by $\overline{L}$ the Galois closure of $L/K$. Since the extension $L/K$ is not quadratic we have that complex conjugation is not an automorphism of $L$. In particular $\alpha^*$ does not belong to $L$ and thus we have the following sequence of field extensions:

$$K \subsetneq L = K(\alpha) \subsetneq K(\alpha, \alpha^*) \subset \overline{L},$$

where $K(\alpha)$ is a proper subfield of $K(\alpha, \alpha^*)$. It follows that there is a non-trivial Galois embedding $\sigma : K(\alpha, \alpha^*) \to \overline{L}$ which is the identity on $L = K(\alpha)$. Following this fact, $\sigma(\alpha) = \alpha$ while $\sigma(\alpha^*)$ is a real number in the set $\{r_1, \ldots, r_k\}$ and $\sigma(\alpha + \alpha^*) = \sigma(\alpha) + \sigma(\alpha^*)$ is the sum of a non-real and a real number and is thus non-real. By Proposition 3.11, we have that $\text{tr}(\text{Ad}(i(\gamma))) = \alpha + \alpha^* \in k$, therefore the adjoint trace field $k$ is not totally real.

We mention the following Corollary of the Proposition 3.13:

**Corollary 3.14.** Suppose that $\Gamma < \text{PSL}_2(C)$ is a type III arithmetic lattice, and denote by $i(\Gamma) < \text{PSO}_{3,1}(\mathbb{R})$ its image under the exceptional isomorphism. The ambient group $G$ of $i(\Gamma)$ is not admissible.
Proof. By Proposition 3.13 the adjoint trace field $k$ of $i(\Gamma)$ is not totally real. Denote by $\sigma : k \to \mathbb{C}$ a non-real Galois embedding of $k$. The algebraic group $G$ is $k$-defined and non-compact at the identity embedding. We claim that the complex points of the “conjugate” group $G^\sigma(\mathbb{C})$ form a non-compact group, too. Arguing by contradiction, if $G^\sigma(\mathbb{C})$ is compact it has to be a compact complex algebraic group and so it is finite by [42, p. 134, Problem 3]. Since the conjugation map

$$G(k) \ni M \mapsto M^\sigma \in G^\sigma(\mathbb{C})$$

is injective, it follows that the group $G(k)$ is finite. This is impossible, since $G(k)$ is dense in the non–compact group $G(\mathbb{R})^o$.

Remark 3.15. As a consequence of Proposition 3.13, all type II arithmetic lattices in $\text{PSO}_3$ arise from an arithmetic lattice in $\text{PSL}_2(\mathbb{C})$ via the construction described in Section 3.3.1. This is due to the fact that type II lattices have a totally real adjoint trace field, and thus their invariant trace field as lattices in $\text{PSL}_2(\mathbb{C})$ is an imaginary quadratic extension of a totally real field.

3.3.3. Constructing $L$-involutions. As mentioned in the previous section, in the context of arithmetic lattices in $\text{PSL}_2(\mathbb{C})$ the notion of adjoint trace field has to be replaced by that of the invariant trace field $L$, and the role of the ambient group is now taken by the algebraic $L$-group $\text{PGL}_1(A)$, where $A$ denotes the invariant quaternion algebra. In order to construct the fc–subspaces we now have to describe the $L$-involutions in $\text{PGL}_1(A)$.

Notice that an element in $\text{PGL}_1(A)_L$ has order 2 if and only if it can be represented by an element $q \in A^*$ such that $q^2 \in L^*$. By taking the tensor product $A \otimes \mathbb{C}$ we see that $A^*$ is mapped injectively into $\text{GL}_2(\mathbb{C})$, and $q$ corresponds to a matrix $N \in \text{GL}_2(\mathbb{C})$ whose square is of the form $z \cdot \text{id}$ for some non-zero $z \in L$. In order for $N$ to be non–trivial in $\text{PGL}_2(\mathbb{C})$ we need that $N$ is not of the form $z \cdot \text{id}$. The latter is equivalent to $\text{tr} N = 0$, which means that $q^* = -q$ (i.e. $q$ is a pure quaternion). Hence $L$–involutions correspond to traceless elements of $\text{PGL}_1(A)$. Their geometric interpretation is that of a rotation of angle $\pi$ about a geodesic in $\mathbb{H}^3$ [22, Chapter V].

There are many traceless elements in the commensurator of an arithmetic lattice $\Gamma < \text{PSL}_2(\mathbb{C})$.

Proposition 3.16. Let $\Gamma < \text{PSL}_2(\mathbb{C})$ be an arithmetic lattice, and let $\gamma$ be a loxodromic element of $\Gamma$. There exists an involution in $\text{Comm}(\Gamma)$ that acts as rotation of angle $\pi$ about the axis of $\gamma$.

A geometric proof of this fact is provided in the proof of Theorem 1.2 in [32] and makes use of the so-called Jorgensen involutions, which are order 2 rotations around the common perpendicular to the geodesic axes of two loxodromic elements. However, the proof provided there requires a modified argument when $\Gamma$ is not cocompact. The problem in this case is that if $\gamma \in \Gamma$ is a rotation of angle $\pi$ around a geodesic $\alpha$, then there is no guarantee a
priori that $\alpha$ projects to a closed geodesic (see the proof of Theorem 1.9 for a discussion of this phenomenon). We provide here an alternative argument which only makes use of elementary linear algebra.

**Proof.** We denote by $L$ the invariant trace field of $\Gamma$, by $A$ its invariant quaternion algebra and by $\Gamma_{p_2q}^2$ the (finite-index) subgroup of $\Gamma$ generated by the squares of its elements. By [32, Theorem 1.2] the group $\Gamma_{p_2q}^2$ is derived from a quaternion algebra, i.e. is conjugate into a subgroup of $\text{PSL}_1(O)$, where $O$ is an order in the invariant quaternion algebra $A$. The element $\gamma^2$ obviously belongs to $\Gamma_{p_2q}^2$ and is loxodromic with the same axis as $\gamma$.

Suppose that $A = \left( \frac{m}{n} \right)$ for some $m, n \in L$. We can express the elements $\gamma^2$ as a linear combination

$$\gamma^2 = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k, \quad a, b, c, d \in L$$

of the standard basis $\{1, i, j, k\}$ of $A$ which is defined up to multiplication by $-1$. We now look for an invertible, traceless element

$$q = x \cdot 1 + y \cdot i + z \cdot j + w \cdot k \in A$$

which commutes with $\gamma^2$, corresponding to the required rotation of angle $\pi$ along the axis $\alpha$.

A manual computation allows to check that the condition $\gamma^2 \cdot q = q \cdot \gamma^2$ yields the following homogeneous linear system in the unknowns $x, y, z, w$:

$$\begin{cases} dz - cw = 0 \\ bw - dy = 0 \\ bz - cy = 0 \end{cases}$$

Notice that:

- The system does not depend on the values of $i^2 = m$ or $j^2 = n$ or on the real part $a$ of $\gamma^2$;
- The unknown $x$, which corresponds to the real part of $q$, appears in no equation;
- The $3 \times 3$ matrix built out of the coefficients of the unknowns $y, z, w$ has determinant $0$, independently of the choice of $b, c, d$.

This implies that there is always a non-zero solution with $x = 0$, which corresponds to a non-zero traceless quaternion $q$ which commutes with $\gamma^2$. If $A$ is a division algebra we can immediately conclude that $q$ is also invertible. However we also wish to account for the possibility that $A$ splits, which will happen if $\Gamma$ is a non co-compact lattice.

In this case we notice that $A \otimes \mathbb{C} \cong M_2(\mathbb{C})$ and $\gamma^2$ can be represented to a matrix $M$ in $\text{SL}_2(\mathbb{C})$ with eigenvalues $\lambda \neq 0$ and $\lambda^{-1}$. Suppose that $q$ is non-invertible and that it is represented by $N \in M_2(\mathbb{C})$. Since $N$ has trace $0$ it must have eigenvalue $0$ with multiplicity $2$. Now, the matrices $M$ and $N$ commute, as such can be brought in an upper triangular form by a simultaneous change of basis and without loss of generality we can assume that they are of the following form:
for some \( s, t \in \mathbb{C} \) with \( t \neq 0 \). The fact that \( M \) and \( N \) commute translates to the condition \( \lambda t = \lambda^{-1} t \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1 \). This implies that \( \gamma^2 \) is a parabolic isometry, which contradicts our assumption on \( \gamma \) being loxodromic.

We conclude by noticing that the quaternion \( q \) belongs to \( A^* \) and therefore its image in \( \text{PGL}_1(A) = \text{Comm}(\Gamma) \) is a rotation of angle \( \pi \) about the geodesic axis of \( \gamma \).

\[ M = \begin{pmatrix} \lambda & s \\ 0 & \lambda^{-1} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad (21) \]

3.4. **Exceptional arithmetic lattices in dimension 7.** As follows from the classification of semisimple algebraic groups by Tits [54], there exist anisotropic algebraic groups \( G \) defined over any number field \( k \) such that \( G = \text{Gal}(\overline{k}/k) \) induces an order three ("triality") outer automorphism of \( G(\overline{k}) \), where \( \overline{k} \) is the algebraic closure of the field \( k \). These groups can be described as groups of automorphisms of certain trialitarian algebras (see [24] or [29, Section 43]). We are interested in those examples where \( k \) is totally real and \( G \) is an admissible \( k \)-form of the real group \( \text{PSO}_{7,1} \).

\[ \begin{array}{cccc}
\alpha_2 & \alpha_1 & \alpha_3 & \alpha_4 \\
\tau & & & \\
\sigma & & & \\
\end{array} \]

**Figure 1.** The Tits index of \( ^6D_{4,0} \). The reader should notice that there are no circled roots as the group is totally anisotropic. The action of the absolute Galois group \( \mathcal{G} = \text{Gal}(\overline{k}/k) \) is induced by complex conjugation \( \sigma \), which exchanges \( \alpha_2 \) and \( \alpha_3 \), and the order 3 trialitarian automorphism \( \tau \) which permutes cyclically \( \alpha_1, \alpha_2 \) and \( \alpha_3 \).

It follows from [7] that, under the hypotheses above, the action of \( \mathcal{G} \) on the Dynkin diagram of the root system of \( G \) relative to a maximal \( k \)-torus is isomorphic to the symmetric group \( S_3 \). Moreover, the group \( G \) is always anisotropic [7]. It follows from this information that the Tits symbol of \( G \) is \( ^6D_{4,0} \) (for the Tits index, see Figure 1). Thus, by Godement’s compactness criterion [25], all type III arithmetic lattices in dimension 7 are uniform. For an explicit construction of an example see [6].

**Definition 3.17.** Let \( \Gamma < \text{PO}_{7,1}(\mathbb{R}) \) be a lattice commensurable with \( \text{G}(\mathcal{O}) \), for some triality algebraic \( k \)-group \( G \) as above. Then \( \Gamma \) is called an arithmetic lattice of type III. An orbifold \( M = \mathbb{H}^7/\Gamma \) is of type III if the group \( \Gamma \) is commensurable in the wide sense with an arithmetic lattice of type III.

We begin this section by proving the following result:
Proposition 3.18. Let $k$ be a totally real number field and $G$ a connected adjoint algebraic $k$-group with Tits index $6D_{4,0}$. Assume that the group $\text{Ext}_{\mathbb{R}/k}G$ obtained via extension of scalars from $k$ to $\mathbb{R}$ is isomorphic to $\text{PSO}_{7,1}$. There exists an order 2 element $\theta \in G(k)$ such that fixed point set for its action on $\mathbb{H}^7$ is a totally geodesic copy of $\mathbb{H}^3$.

We apply Proposition 3.18 to prove Theorem 1.5. Namely, we show that the quotient of $\mathbb{H}^3$ under the action of the centraliser in $G(k)$ of the involution $\theta$ constructed in the proposition is a 3-dimensional type III arithmetic orbifold $N$.

3.4.1. Constructing $k$-involutions. It is not difficult to construct order 2 elements in the $k$-points of an adjoint semisimple algebraic $k$-group $G$ of absolute type $D_n$, with $n \geq 2$. Indeed, let $T < G$ be a maximal $k$-torus. Then $T(\bar{k})$ is isomorphic to $(\mathbb{G}_m)^n$, where $\mathbb{G}_m$ denotes the multiplicative group of $\bar{k}$. Elements of the form $g = (\pm 1, \ldots, \pm 1)$ with at least one negative entry correspond to order 2 elements in the adjoint group $G$.

We have that conjugation by $g$, which we denote $\text{Inn}(g)$, is a $k$-automorphism of $G$ if and only if it commutes with the action of the absolute Galois group $G$. The automorphism $\text{Inn}(g)$ is the identity on $T$ and on its Lie algebra $t$, thus it acts as the identity on the root system $\Sigma$ of $G$ relative to $T$. It follows that, for every $\alpha \in \Sigma$, $\text{Inn}(g)$ preserves each root space $L_\alpha$, acting as an element of the form $\pm \text{id}$. Moreover by [27, Theorem, p. 75], the action on each root space is uniquely determined by the action on the root spaces for a system of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$.

Indeed, denote by $\Sigma_+$ (resp. $\Sigma_-$) the set of even (resp. odd) roots given by those $\alpha \in \Sigma$ for which $\text{Inn}(g)$ acts as $\text{id}$ (resp. $-\text{id}$) on the root space $L_\alpha$. Also, let $\Delta_+ = \Sigma_+ \cap \Delta$ and $\Delta_- = \Sigma_- \cap \Delta$. Every $\alpha \in \Sigma$ is expressed as a linear combination of the roots $\alpha \in \Delta$ with integer coefficients. Then the set $\Sigma_+$ (resp. $\Sigma_-$) is the set of roots which can be expressed as

$$\sum_{\alpha \in \Delta_+} n_\alpha \cdot \alpha + \sum_{\beta \in \Delta_-} m_\beta \cdot \beta$$

with $\sum_{\beta \in \Delta_-} m_\beta$ even (resp. odd). It follows that every partition of $\Delta = \Delta_+ \cup \Delta_-$ with non-empty $\Delta_-$ determines an involution of the form $\text{Inn}(g)$ as above, and all such involutions arise in this way.

Finally, we notice that $\text{Inn}(g)$ commutes with the action of the absolute Galois group $G$ if and only if this action preserves the partition $\Sigma = \Sigma_+ \cup \Sigma_-$ into the sets of even and odd roots. Equivalently, we require that the image of $\Delta_+$ (resp. $\Delta_-$) under the action of an element $\sigma \in G$ lies in $\Sigma_+$ (resp. $\Sigma_-$). In order to prove Proposition 3.18 we are left with the task of finding a maximal $k$-torus $T$ in a trialitarian $k$-form $G$ of $\text{PO}_{7,1}$ so that the action of $G$ is “small” enough to preserve a partition of $\Sigma$ into two sets of even and odd roots.
3.4.2. Proof of Proposition 3.18. Denote by $G' < G$ the kernel of the action of $G$ on the Dynkin diagram of $G$, and by $E$ the fixed field for $G'$. The field $E$ is a degree 6 extension of $k$, and is the smallest field such that the group $\text{Ext}_{E/k}(G)$ obtained via extension of scalars from $k$ to $E$ is an inner form.

By [44, Lemma 6.29], there exists a totally imaginary quadratic extension $L/k$ such that $\text{Ext}_{L/k} G$ is quasi-split. Moreover $L$ and $E$ are linearly disjoint over $k$, implying that the natural map $E \otimes_k L \to EL$ defined by $x \otimes y \to x \cdot y$ is an isomorphism.

Since $\text{Ext}_{L/k} G$ is quasi-split, it follows that $G$ contains an $L$-defined Borel subgroup. Denote by $\sigma$ the generator of $\text{Gal}(L/k) \cong \mathbb{Z}/2\mathbb{Z}$ (i.e., the complex conjugation). By [44, Lemma 6.17] there exists a Borel subgroup defined over $L$ such that $B \cap B^\sigma = T$ is a maximal $k$-torus in $G$, where $B^\sigma$ is the image of $B$ under the conjugation $\sigma$.

We now follow the discussion in [44, p. 374]. Denote by $\Sigma$ the root system of $G$ relative to $T$. The splitting field of the torus $T$ (i.e., the minimal field over which $T$ is isomorphic to $(\mathbb{G}_m)^4$) is the compositum $EL$, which is a degree 12 extension of $k$. Consider the Galois automorphism $\rho \in \text{Gal}(EL/k)$ which is the identity on $E$ and coincides with complex the conjugation $\sigma$ on $L$. Since $\rho$ belongs to $\text{Gal}(EL/E)$ and $G$ becomes an inner form over $E$, it follows that $\rho$ must act on $\Sigma$ via an element of the Weyl group $W$.

On the other hand, the automorphism $\rho$ extends to $EL$ the generator $\sigma$ of $\text{Gal}(L/k)$. Since $T = B \cap B^\sigma$, $\sigma$ has to take the positive roots associated to $B$ under (5) to the negative ones. The only element of the Weyl group which exchanges a system of positive roots with a system of negative roots is the antipodal map $\alpha$, which therefore corresponds to the action of $\rho$ on $\Sigma$.

The group $\text{Gal}(EL/k)$ is generated by $\rho$ and $\text{Gal}(EL/L) \cong \mathfrak{S}_3$. The group $\text{Gal}(EL/L)$ acts faithfully on the root system $\Sigma$ by preserving the system $\Delta$ of simple roots associated to $B$. It follows that the action of $\tilde{G} = \text{Gal}(\overline{K}/k)$ on $\Sigma$ factors through the action of $\text{Gal}(EL/k) \cong \mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}$, with $\mathfrak{S}_3$ acting on $\Delta$ and $\mathbb{Z}/2\mathbb{Z}$ acting via the antipodal map.

Now, suppose that $\Delta = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$, with $\alpha_0$ corresponding to the “central” root in the Dynkin diagram (the one connected by an edge to the other three roots, see Figure 1). Let us choose any partition $\Delta = \Delta_+ \cup \Delta_-$ (with non-empty $\Delta_-$) which is preserved by the action of the $\mathfrak{S}_3$ factor of $\text{Gal}(EL/k)$. There are only 3 possible choices:

$$\Delta_- = \{\alpha_0\}, \quad \Delta_+ = \{\alpha_1, \alpha_2, \alpha_3\}; \quad (23)$$
$$\Delta_+ = \{\alpha_0\}, \quad \Delta_- = \{\alpha_1, \alpha_2, \alpha_3\}; \quad (24)$$
$$\Delta_- = \Delta, \quad \Delta_+ = \emptyset. \quad (25)$$

Each choice will determine a partition of $\Sigma = \Sigma_+ \cup \Sigma_-$ into two sets of even and odd roots which is necessarily to be preserved also by the antipodal map $\alpha$. It follows that the whole action of $\tilde{G}$ preserves this partition, thus it commutes with $\text{Inn}(g)$. This implies that $\text{Inn}(g)$ is defined over $k$. The proof follows by taking $\theta = \text{Inn}(g)$. 

We are left with the task of verifying that the fixed point set for the action of $\theta$ on $\mathbb{H}^7$ is 3-dimensional. We notice first of all that $\theta$ corresponds to an element of $\text{PSO}_{7,1}$, i.e., to an orientation preserving isometry of $\mathbb{H}^7$. Given that $\theta^2 = \text{id}$, it follows that the action of $\theta$ on the Lie algebra $\mathfrak{g}$ is diagonalisable and has eigenvalues $1$ and $-1$ respectively. The multiplicities can be computed easily: $\theta$ acts as the identity on the 4-dimensional algebra $\mathfrak{t}$ and on each one-dimensional root space $L_{\alpha}$ for $\alpha \in \Sigma_+$ an even root. It also acts as $-\text{id}$ on each root space $L_{\alpha}$ for $\alpha \in \Sigma_-$ an odd root.

For all the possible choices $(23, 24, 25)$ for $\Delta_+$ and $\Delta_-$ we obtain that $\Sigma_+$ has 8 roots and $\Sigma_-$ has 16 roots. It follows that we have $m(1) = 12$, $m(-1) = 16$. Since the resulting involution $\theta$ has order 2, it can be represented by conjugation by a matrix $M \in \text{SO}_{7,1}$ such that $M^2 = \text{id}$. Up to conjugacy in $\text{SO}_{7,1}$ we may assume that $M$ is diagonal with $\pm 1$ entries on the diagonal. The only possibility such that the action on the Lie algebra has $m(1) = 12$, $m(-1) = 16$ is that $M$ has 4 entries equal to 1 and 4 entries equal to $-1$. Such an $M$ corresponds to a reflection along a 3 dimensional totally geodesic subspace in $\mathbb{H}^7$. \hfill \blacksquare

### 3.4.3. Proof of Theorem 1.5

Let $N$ denote the fc-subspace corresponding to the centraliser of the involution $\theta$. By [5, Proposition 15.2.2], $N$ is arithmetic. The rest of the proof is devoted to showing that the adjoint trace field of $N$ is not totally real. Proposition 3.13 then implies that $N$ is a 3-dimensional type III arithmetic hyperbolic orbifold.

We carry over the notation from the proof of Proposition 3.18. In particular, $E$ denotes the minimal field over which $\text{Ext}_{E/k}G$ becomes an inner form and $L$ is the field over which $\text{Ext}_{L/k}G$ becomes quasi-split. These fields are linearly independent over $k$, which implies that the map $\text{Gal}(E/k) \times \text{Gal}(L/k) \to \text{Gal}(EL/k)$ given by

$$(\phi, \eta)(x \cdot y) \to \phi(x) \cdot \eta(y)$$

for $\phi \in \text{Gal}(E/k)$, $\eta \in \text{Gal}(L/k)$, $x \in E$ and $y \in L$ is an isomorphism. Under this map the groups $\text{Gal}(E/k)$ and $\text{Gal}(L/k)$ are mapped to $\text{Gal}(EL/L)$ and $\text{Gal}(EL/E)$ respectively. We identify the root system of $G$ relative to $T$ with the 24 vectors in $\mathbb{R}^4$ obtained via permutations in the entries of $(\pm 1, \pm 1, 0, 0)$, and the system $\Delta = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ of simple roots corresponding to the $L$-defined Borel subgroup $B$ to

$$\alpha_0 = (0, 1, -1, 0), \ \alpha_1 = (1, -1, 0, 0), \ \alpha_2 = (0, 0, 1, -1), \ \alpha_3 = (0, 0, 1, 1).$$

(26)

As a first step, we analyse the action of complex conjugation $\sigma$ on the $D_4$ root system of $G$ relative to the torus $T$. Complex conjugation induces a nontrivial automorphism of both factors, since $E$ and $L$ are both imaginary fields. On the group $\text{Gal}(E/k) \cong \text{Gal}(EL/L) \cong \mathfrak{S}_3$, the Galois automorphism $\sigma$ corresponds to the permutation $p$ of the simple root system $\Delta$ that exchanges two non-central roots. We may assume without loss of generality
that these are $\alpha_2$ and $\alpha_3$, so that the associated map is a change in sign in the last coordinate. On $\text{Gal}(L/k) \cong \text{Gal}(EL/E)$, $\sigma$ corresponds to the antipodal map $a$. The full action of $\sigma$ on the $D_4$ root system is the composition $p \circ a$ which can be written as

$$\alpha \mapsto (x, -y, -z, w). \quad (27)$$

Concerning the “trialitarian” automorphism $\tau \in \text{Gal}(E/k) \cong \text{Gal}(EL/L)$, its action on the $D_4$ root system is given by a cyclic permutation of the roots $\alpha_1, \alpha_2, \alpha_3$:

$$\alpha \mapsto \frac{1}{2}(x+y+z+w, x+y-z-w, x-y+z-w, -x+y+z-w). \quad (28)$$

With these informations, we are able to describe the Tits index of the centraliser $H$ of the involution $\theta$, which is naturally a $k$-subgroup of $G$. Notice that the torus $T$ constructed in the proof of Proposition 3.18 is a maximal $k$-torus of $H$.

Let us assume that the choice of the partition $\Delta = \Delta_+ \cup \Delta_-$ is the one given in (23) (the other choices yield exactly the same results). The positive roots in $\Sigma_+$ are those of the form

$$\pm(1, 1, 0, 0), \quad (0, 0, 1, 1) \quad (29)$$

which form a root system of type $D_2 \times D_2 = A_1 \times A_1 \times A_1 \times A_1$ for $\mathbb{R}$. Its Dynkin diagram is given by the disjoint union of 4 vertices labeled $\alpha, \beta, \gamma$ and $\delta$, each vertex corresponding to a pair of opposite roots:

$$\alpha = \pm(0, 0, 1, -1), \quad \beta = \pm(0, 0, 1, 1), \quad \gamma = \pm(1, -1, 0, 0), \quad \delta = \pm(1, 1, 0, 0). \quad (30)$$

With the actions of complex conjugation and triality given in (27) and (28) we see that the action of $G = \text{Gal}(E/k)$ on the Dynkin diagram is as in Figure 2.

![Figure 2](image-url)

**Figure 2.** The Tits index (relative to the torus $T$) of the centraliser $H$ of the involution $\theta$. The arrows indicate the action on the Dynkin diagram of complex conjugation $\sigma$ and of the trialitarian automorphism $\tau \in \text{Gal}(E/k)$.

In particular, complex conjugation $\sigma$ preserves the two pairs of roots $\{\alpha, \beta\}$ and $\{\gamma, \delta\}. \quad \text{It follows that there exists an } \mathbb{R}-\text{defined isomorphism of} \quad (\dots)
connected adjoint algebraic groups $PH \cong L \times C$, where $PH$ denotes the adjoint group of $H$, the group $L$ corresponds to the $D_2$ root subsystem $(0,0,\pm1, \pm1)$ spanned by $\alpha$ and $\beta$, and the group $C$ corresponds to the root subsystem $(\pm1, \pm1, 0, 0)$ spanned by $\gamma$ and $\delta$. Since the roots $\alpha$ and $\beta$ are swapped by $\sigma$, it follows that $L$ has Tits symbol $^{2}D_{2}^{(1)}$, and is therefore isomorphic to $PSO_{3,1}$ by [54, p. 57]. Concerning the group $C$, we notice that complex conjugation $\sigma$ acts by exchanging each root with its opposite and therefore has Tits symbol $^{1}D_{2}^{(1)}$, implying that $C$ is isomorphic to $PSO_{4}$.

Recall that the minimal field $E$ such that $Ext_{E/k}G$ is an inner form is an imaginary Galois extension of the totally real field $k$ and $Gal(E/k)$ is isomorphic to the symmetric group $S_3$. Denote by $K$ the fixed field of complex conjugation $\sigma$, now interpreted as an element of $Gal(E/k)$. The field $K$ is a cubic extension of $k$ and the set $S_{K/k}^{c}$ of embeddings of $K$ relative to $k$ contains one real embedding (corresponding to $K$) and two complex conjugate embeddings, which we denote by $K^\tau$ and $K^{\tau^2}$. The action of the trialtarian automorphism $\tau$ permutes these 3 embeddings cyclically, while complex conjugation $\sigma$ fixes $K$ and exchanges $K^\tau$ and $K^{\tau^2}$.

It follows that the $D_2$ root systems spanned by $\{\alpha, \beta\}$ is preserved by the action of the absolute Galois group $Gal(\overline{k}/K)$. The minimal field of definition for the projection of $PH \to L$ is precisely $K$ and, as shown in the proof of Theorem 1.7, this is the adjoint trace field of $N$. Since $K$ is not totally real, we see that $N$ is a 3-dimensional type III arithmetic hyperbolic orbifold.

We conclude this section with the following supplementary result:

**Corollary 3.19.** The invariant trace field of the 3-dimensional type III orbifold $N = H^{3}/\Gamma$ of Theorem 1.5 is the complex embedding $K^{\tau}$ of the adjoint trace field $K$. In particular, it is a cubic extension of a totally real field.

**Proof.** The adjoint group $PH$ of $H$ decomposes over the algebraic closure $\overline{k}$ of $k$ as a direct product:

$$PH \cong PSL_2 \times PSL_2 \times PSO_4,$$

where the $PSL_2$-factors corresponds to the roots $\alpha$ and $\beta$ respectively, while the $PSO_4$-factor corresponds to the pair of roots $\{\gamma, \delta\}$. By the argument in the proof of Proposition 5.6, the invariant trace field of $N$ is the minimal field over which the projection $PH \to PSL_2$ onto the first factor is defined. The analysis of the Tits index of $PH$ (see Figure 2) shows that this field is precisely the fixed field for the automorphism $\tau\sigma\tau^{-1} \in Gal(E/k)$, i.e. the complex embedding $K^{\tau}$ of $K$. $\blacksquare$
4. Two kinds of totally geodesic subspaces

In this section, we introduce two techniques to construct totally geodesic immersions of arithmetic hyperbolic orbifolds into other arithmetic hyperbolic orbifolds. In Section 5 we shall show that any totally geodesic immersion of arithmetic hyperbolic orbifolds is obtained through a combination of these two techniques.

4.1. Subform spaces. We present here a simple generalisation of the method used in [30, Proposition 5.1] to construct codimension-one totally geodesic embeddings of type I arithmetic lattices.

Proposition 4.1. Let $\Lambda < \text{PO}(f, k)$ be a type I arithmetic lattice associated to an admissible form $f$ of signature $(m, 1)$ defined over a totally real field $k$. For any integer $n > m$, there exists an admissible $k$-defined form $g$ of signature $(n, 1)$ and a type I arithmetic lattice $\Gamma < \text{PO}(g, k)$ such that $\Lambda$ is a totally geodesic sublattice of $\Gamma$.

Proof. Consider any $k$-defined form $h$ with the property that $h^\sigma$ has signature $(n - m)$ for any Galois embedding $\sigma : k \to \bar{R}$. Define $g$ to be the orthogonal direct sum of the quadratic spaces associated to the forms $f$ and $h$. Then $g$ is an admissible $k$-form and we obtain an inclusion of $O(f)$ into $O(g)$. By applying [30, Proposition 2.1], we obtain that the arithmetic lattice $\Lambda < O(f, k)$ is realised as a totally geodesic sublattice of an arithmetic lattice $\Gamma < O(g, k)$. We conclude by projecting $\Lambda$ and $\Gamma$ to $\text{PO}(f, k)$ and $\text{PO}(g, k)$, respectively.

Remark 4.2. By varying the choice of the form $h$ in the argument above, we can obtain many projectively non-equivalent $k$-forms of signature $(n, 1)$, and thus infinitely many commensurability classes of type I lattices in $\text{Isom}(\mathbb{H}^n)$ of which $\Lambda$ is a totally geodesic sublattice.

Note that all the fc-subspaces arising as fixed point sets of involutions in $\text{PO}(f, k)$ as described in 3.1.2 fit into the description above.

The construction of subform spaces has a natural generalisation to type II lattices. It is very similar to the type I case, so we skip the details.

4.2. Subspaces via Weil restriction of scalars. The discussion at the end of Section 3.1.2 suggests a technique to construct further examples of totally geodesic immersions of type I lattices into other type I lattices, of perhaps much higher dimension.

Proposition 4.3. Let $\Lambda < \text{PO}(f, K)$ be a type I arithmetic lattice associated to an admissible form of signature $(n, 1)$ defined over a totally real field $K$. Let $k$ be a subfield of $K$ such that $[K : k] = d$. Then $\Lambda$ is a totally geodesic sublattice of a type I arithmetic lattice $\Gamma < \text{PO}(g, k)$ associated to an admissible $k$-defined form $g$ of signature $(d(n + 1) - 1, 1)$.
Proof. By the primitive element theorem we can suppose that $K = k(\alpha)$ and that $p(x) \in k[x]$ is the minimal polynomial of $\alpha$ over $k$, with $d$ distinct real roots $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{d-1}$. The field $K$ is then isomorphic to an abstract extension of $k$:

$$K \cong k[x]/(p(x)).$$

Let us denote by $S^\sigma_{K/k}$ the set of all Galois embeddings $\sigma: K \to \mathbb{R}$ which restrict to the identity on $k$. There are precisely $d = [K : k]$ such embeddings, so that $S^\sigma_{K/k} = \{\sigma_0, \sigma_1, \ldots, \sigma_{d-1}\}$. We assume that $\sigma_i(x) = \alpha_i$, so that $\sigma_0 = id|_K$ and let $\text{Gal}(\overline{K}/k)$ denote the Galois group of the Galois closure $\overline{K}$ of the extension $K/k$, which naturally acts freely and transitively on $S^\sigma_{K/k}$.

Let $(V, f)$ be a $K$–defined quadratic pair, i.e. $V$ is an $(n + 1)$–dimensional vector space over $K$ and $f$ is an admissible quadratic form of signature $(n, 1)$. By a slight abuse of notation, we denote by $f$ the associated symmetric bilinear form. By fixing a basis for $V$, we may assume that $V \cong K^{n+1}$.

For each embedding $\sigma \in S^\sigma_{K/k}$, we build a vector space $V^\sigma$ of dimension $n + 1$ over $\sigma(K)$ as follows:

- $V^\sigma = \{v^\sigma | v \in V\}$,
- $v^\sigma + w^\sigma = (v + w)^\sigma$,
- $\sigma(\lambda) \cdot v^\sigma = (\lambda \cdot v)^\sigma, \lambda \in K$,

where it is understood that $V = V^{\sigma_0}$. Note that $V^\sigma$ can be naturally interpreted as an $(n + 1)$–dimensional vector space over the field $k[x]/(p(x))$ by requiring that

$$(q(x)) \cdot v^\sigma = q(\sigma(x)) \cdot v^\sigma.$$

We now build a vector space $W$ over $k[x]/(p(x))$ of dimension $(n + 1) \cdot d$ by considering the direct sum

$$W = V \oplus V^{\sigma_1} \oplus \ldots \oplus V^{\sigma_{d-1}}. \quad (32)$$

Notice that the group $\text{Gal}(\overline{K}/k)$ acts linearly on the vector space $W$. If $\sigma \in \text{Gal}(\overline{K}/k)$ and $v^{\sigma_i} \in V^{\sigma_i}$, then

$$\sigma(v^{\sigma_i}) = v^{\sigma \circ \sigma_i} \in V^{\sigma \circ \sigma_i}.$$

Hence we can define the $k$–subspace $\text{Res}_{K/k}(V)$ of fixed points of the action of $\text{Gal}(K/k)$:

$$\text{Res}_{K/k}(V) = \{v + v^{\sigma_1} + \ldots + v^{\sigma_{d-1}} | v \in V\}.$$

Furthermore, for each $\sigma \in \text{Gal}(\overline{K}/k)$, a $\sigma(K)$–defined symmetric bilinear form $f^\sigma$ on $V^\sigma$ is given by

$$f^\sigma(v^\sigma, w^\sigma) = (f(v, w))^\sigma.$$

By endowing each factor of the direct sum decomposition (32) of $W$ with the corresponding form $f^{\sigma_i}$ and imposing the various factors to be pairwise orthogonal, we define a symmetric bilinear form $g$ on $W$ so that

$$(W, g) = (V, f) \oplus (V^{\sigma_1}, f^{\sigma_1}) \oplus \ldots \oplus (V^{\sigma_{d-1}}, f^{\sigma_{d-1}}).$$
We now claim that the restriction of the bilinear form $g$ to the $k$–subspace $\text{Res}_{K/k}(V)$ is a $k$-defined symmetric bilinear form, which we denote by $\text{Res}_{K/k}(f)$. Thus $(\text{Res}_{K/k}(V), \text{Res}_{K/k}(f))$ is a $k$-quadratic pair which we think of as the restriction of scalars of the $K$-defined quadratic pair $(V, f)$ to $k$. In order to prove the claim it is sufficient to notice that $g : W \times W \to k[x]/(p(x))$ is $k$-linear (where $W$ and $k[x]/(p(x))$ are now interpreted as $k$-vector spaces) and that for all $v, w \in V$:

$$g \left( \sum_{i=0}^{d-1} v^{\sigma_i}, \sum_{j=0}^{d-1} w^{\sigma_j} \right) = \sum_{i=0}^{d-1} f^{\sigma_i}(v^{\sigma_i}, w^{\sigma_i}) = \sum_{i=0}^{d-1} (f(v, w))^{\sigma_i} = \text{tr}(f(v, w)) \in k,$$

where $\text{tr} : K \to k$ is the trace of the field extension $K/k$.

We now note that the real vector space $\text{Res}_{K/k}(V) \otimes \mathbb{R}$, admits a direct sum decomposition as:

$$\text{Res}_{K/k}(V) \otimes_k \mathbb{R} = (V \otimes \mathbb{R}) \oplus (V^{\sigma_1} \otimes \sigma_1(\mathbb{R})) \oplus \cdots \oplus (V^{\sigma_{d-1}} \otimes \sigma_{d-1}(\mathbb{R})).$$ (33)

It is important to notice that the subspace $V^{\sigma_i} \otimes \mathbb{R}$ does not correspond to the tensor product of $\mathbb{R}$ with a $k$-subspace of $\text{Res}_{K/k}(V)$. The spaces $V^{\sigma_i}$, $i = 0, \ldots, d - 1$, are each defined over a different Galois embedding of $K$, so the direct sum decomposition (33) can be defined at best over the Galois closure of $K/k$.

The group of real points of $\text{Res}_{K/k} O(f)$ is isomorphic to $\text{O}(f, \mathbb{R}) \times \text{O}(f^{\sigma_1}, \mathbb{R}) \times \cdots \times \text{O}(f^{\sigma_{d-1}}, \mathbb{R}) \subset \text{O}(\text{Res}_{K/k}(f), \mathbb{R})$, which implies that the group $\text{Res}_{K/k}O(f)_{\mathbb{R}}$ corresponds to the subgroup of $O(\text{Res}_{K/k}(f))_{\mathbb{R}}$ that preserves the direct sum decomposition (33). This fact translates to a $k$-defined inclusion of $\text{Res}_{K/k} O(f)$ into $O(\text{Res}_{K/k}(f))$, and we can interpret the form $f$ as the restriction to $V$ of the $k$–defined form $\text{Res}_{K/k}(f)$ of signature $(d(n + 1) - 1, 1)$. The admissibility of $\text{Res}_{K/k}(f)$ is easy to check. Indeed, any Galois embedding $\eta : k \to \mathbb{R}$ extends to $K$. The form $(\text{Res}_{K/k}(f))^{\eta}$ can be represented over $\mathbb{R}$ as $f^{\eta} \oplus f^{\eta^{\sigma_1}} \oplus \cdots \oplus f^{\eta^{\sigma_{d-1}}}$, which will be positive definite for any $\eta \neq \text{id}|_k$ due to the fact that $f$ is admissible.

We thus obtain that $O(\text{Res}_{K/k}(f), O_k)$ is a type I arithmetic lattice with field of definition $k$. The stabiliser of the subspace $V \otimes \mathbb{R} \subset \text{Res}_{K/k}(V) \otimes \mathbb{R}$ is commensurable with $\text{Res}_{K/k} O(f)_{\mathbb{R}}$, and the projection of this group into $O(f)_{\mathbb{R}}$ is commensurable with $O(f)_{\mathbb{R}}$. More generally, by applying [30, Proposition 2.1] with $G = \text{Res}_{K/k} O(f)$ and $H = O(\text{Res}_{K/k}(f))$, we see that any arithmetic lattice $\Lambda < O(f, K)$ is realised as a totally geodesic sublattice of an arithmetic lattice $\Gamma < O(\text{Res}_{K/k}(f), k)$. We conclude by projecting $\Lambda$ and $\Gamma$ to $PO(f, K)$ and $PO(\text{Res}_{K/k}(f), k)$, respectively.

\begin{remark}
The statement of Proposition 4.3 contradicts Proposition 9.1 of [40]. Indeed the fields of definition of $N = \mathbb{H}^n/\Lambda$ and $M = \mathbb{H}^m/\Gamma$ are different, and $N$ is not a subform subspace of $M$ as defined in [40],
\end{remark}
Corollary 4.6. Let \( N = \mathbb{H}^m/\Lambda, \) \( m \geq 2 \) be a compact type I arithmetic hyperbolic orbifold such that \( \Lambda < \text{O}(f, K) \), with \( f \) being an admissible form of signature \((m,1)\) defined over a totally real algebraic number field \( K \) such that \( [K : \mathbb{Q}] = d > 1 \). Then \( N \) is realised as a totally geodesic immersed sub-orbifold in a non-compact type I arithmetic hyperbolic orbifold of dimension \( n = d \cdot (m + 1) - 1 \).

**Proof.** Apply Proposition 4.3 with \( k = \mathbb{Q} \) in order to build a totally geodesic immersion of \( N \) into \( M = \mathbb{H}^n/\Gamma \), where \( \Gamma < \text{O}(g, \mathbb{Q}) \) is arithmetic and \( g \) is an admissible, \( \mathbb{Q} \)-defined form of signature \((d \cdot (m+1) - 1,1)\). Since \( d \cdot (m + 1) \geq 6 \) the form \( g \) is isotropic by Meyer’s theorem, and thus \( M \) is non-compact. \( \blacksquare \)

We now turn our attention to the case of type II lattices and describe embeddings via Weil restriction of scalars in this setting.

**Proposition 4.7.** Let \( \Lambda < \text{PU}(F, D) \) be a type I or II arithmetic lattice associated to an admissible skew–Hermitian form \( F \) of signature \((2m-1,1)\) defined over a quaternion algebra \( D \) over a totally real number field \( K \). Let \( k \) be a subfield of \( K \) such that \( [K : k] = d \) and \( D \cong D' \otimes K \) for some quaternion algebra \( D' \) over \( k \). Then \( \Lambda \) is a totally geodesic sublattice of an arithmetic lattice \( \Gamma < \text{PU}(G, D') \) associated to an admissible \( k \)-defined skew–Hermitian form \( G \) of signature \((2dm - 1,1)\) defined over \( D' \).

Moreover, if \( D' \cong M_2(k) \) then \( \Gamma \) and \( \Lambda \) are type I lattices. Otherwise, \( D' \) is a division algebra and \( \Gamma \) is a type II lattice. In this case, \( \Lambda \) is a type II lattice if \( D \) is a division algebra, while it is a type I lattice if \( D \cong M_2(K) \).

**Proof.** We follow the same strategy as in the proof of Proposition 4.3, and carry over the notation. For each embedding \( \sigma \in \text{S}^{2m}_K/k \), we define the conjugate quaternion algebra \( D^\sigma \) over \( \sigma(K) \) as follows:

- \( D^\sigma = \{a^\sigma | a \in D\} \),
- \( a^\sigma + b^\sigma = (a + b)^\sigma, \ a^\sigma \cdot b^\sigma = (a \cdot b)^\sigma \),
- \( \sigma(\lambda) \cdot a^\sigma = (\lambda \cdot a)^\sigma \).

Notice that each algebra \( D^\sigma \) is isomorphic to \( D' \otimes \sigma(K) \), and \( (q \otimes \lambda)^\sigma = q \otimes \sigma(\lambda) \) for all \( q \in D', \lambda \in K \).

Also, each \( D^\sigma \) can be naturally interpreted as a quaternion algebra over the abstract field \( k[x]/(p(x)) \). Let us consider the direct sum

\[ D \oplus D^{\sigma_1} \oplus \cdots \oplus D^{\sigma_{d-1}} \]
which contains the $k$–subalgebra $\text{Res}_{K/k}(D)$ of elements invariant under the action of $\text{Gal}(\overline{K}/k)$ that maps $a^{\sigma_i} \in D^{\sigma_i}$ to $a^{\sigma \circ \sigma_i} \in D^{\sigma \circ \sigma_i}$ for each $\sigma \in \text{Gal}(\overline{K}/k)$. Observe that $\text{Res}_{K/k}(D)$ has the structure of a right $D'$–module of rank $d$.

Now let us build the right $D'$–module $(D')^m$ of rank $m$ and consider the direct sum

$$W = D^m \oplus (D')^m \oplus \ldots \oplus (D^{d-1})^m.$$  \hspace{1cm} (34)

We define the subset $\text{Res}_{K/k}(D^m) \subset W$ of fixed points under the action of $\text{Gal}(\overline{K}/k)$:

$$\text{Res}_{K/k}(D^m) = \{x + x^{\sigma_1} + \cdots + x^{\sigma_d} | x \in D^m\}.$$  

Note that $\text{Res}_{K/k}(D^m) \cong \text{Res}_{K/k}(D)^m$ is naturally a right $\text{Res}_{K/k}(D)$–module of rank $m$ and is thus a right $D'$–module of rank $d \cdot m$.

For each $\sigma \in \text{Gal}(\overline{K}/k)$, let $F^{\sigma}$ be the skew–Hermitian form on $(D')^m$ defined by

$$F^{\sigma}(a^{\sigma}, b^{\sigma}) = (F(a, b))^{\sigma}.$$  

By endowing each factor of the direct sum decomposition (34) of $W$ with the corresponding form and imposing the various factors to be pairwise orthogonal we define a skew–Hermitian form $G$ on $W$ with values in a quaternion algebra over the field $k[x]/(p(x))$.

We claim that the restriction of $G$ to the right $D'$–module $\text{Res}_{K/k}(D)^m$ is a skew–Hermitian form, which we interpret as the restriction of scalars of the form $F$ and denote by $\text{Res}_{K/k}(F)$. In order to prove the claim we proceed as follows. It is clear that $G$ is a sesquilinear form on the right $D'$-module $\text{Res}_{K/k}(D)^m$. We claim that it actually takes values in $D'$.

Let us define the following $k$–linear trace function: $\text{Tr} : D \to D'$ by setting $\text{Tr}(q \otimes \lambda) = q \otimes \text{tr}(\lambda)$. Let $\sigma_0 = \text{id}$, so that we have

$$G \left( \sum_{i=0}^{d-1} x^{\sigma_i} \sum_{j=0}^{d-1} y^{\sigma_j} \right) = \sum_{i=0}^{d-1} F^{\sigma_i}(x^{\sigma_i}, y^{\sigma_i}) = \sum_{i=0}^{d-1} (F(x, y))^{\sigma_i} = \text{Tr}(F(x, y)) \in D'.$$

Since the form $F$ is skew–Hermitian and $(q \otimes \lambda)^* = q^* \otimes \lambda$ for all $q \in D'$ and $\lambda \in K$, it follows that the form $G$ is also skew–Hermitian.

We now notice that $\text{Res}_{K/k}(D)^m \otimes \mathbb{R}$ is a right $D' \otimes \mathbb{R}$–module of rank $dm$ and it admits the following decomposition:

$$\text{Res}_{K/k}(D)^m \otimes_k \mathbb{R} = (D \otimes \mathbb{R})^m \oplus (D' \otimes \mathbb{R})^m \oplus \cdots \oplus (D^{d-1} \otimes \mathbb{R})^m.$$  

Therefore, the group of real points of $\text{Res}_{K/k}(U)(F)$ is isomorphic to $U(F, D \otimes \mathbb{R}) \times U(F^m, D' \otimes \mathbb{R}) \times \cdots \times U(F^{d-1}, D^{d-1} \otimes \mathbb{R}) \subset U(\text{Res}_{K/k}(F), D' \otimes \mathbb{R})$, and the inclusion $\text{Res}_{K/k}(U)(F) \subset U(\text{Res}_{K/k}(F))$ is defined over $k$. As in the case of type I lattices, the form $F$ is now interpreted as the restriction to $D^m$ of the form $\text{Res}_{K/k}(F)$, which clearly has signature $(2d - 1, 1)$ since $F^\sigma$ has signature $(2m, 0)$ for any non-identity $\sigma \in \text{Gal}(K/k)$. 

The admissibility of the form $\text{Res}_{K/k}(F)$ easily follows from the admissibility of $F$. Indeed, if $\eta : k \to \mathbb{R}$ is a non-identity Galois embedding of $k$ then it can be extended to $K$, and hence

$$
\text{Res}_{K/k}(F)^\eta = F^\eta \oplus F^\eta \sigma_1 \oplus \cdots \oplus F^\eta \sigma_{d-1}
$$

has signature $(2dm, 0)$, since every factor has signature $(2m, 0)$.

The conclusion is now straightforward: if $O$ is an order in $D_1$, then $O \otimes K$ is an order in $D$ and we see that $U(\text{Res}_{K/k}(F), O)$ is an arithmetic lattice defined over $k$. The stabiliser of the subspace $(D \otimes_K \mathbb{R})^m \subset \text{Res}_{K/k}(D)^m$ is commensurable with $\text{Res}_{K/k}(U(F)_{\mathcal{O}_k})$, and its projection into $U(F)_{\mathbb{R}} = U(F, D \otimes \mathbb{R})$ is commensurable with $U(F)_{\mathcal{O}_K}$. By applying [30, Proposition 2.1] with $\text{Res}_{K/k}(U(F) < U(\text{Res}_{K/k}(F))$, we see that any arithmetic lattice $\Lambda < U(F, D)$ is realised as a totally geodesic sublattice of an arithmetic lattice $\Gamma < U(\text{Res}_{K/k}(F), D')$. The proof of the first part of Proposition 4.7 follows by choosing the skew-hermitian form $G$ to be $\text{Res}_{K/k}(F)$ and by projecting $\Lambda$ and $\Gamma$ to $\text{PU}(F, K)$ and $\text{PU}(G, k)$, respectively.

For the second part of the statement, we notice that if $D' \cong M_2(k)$ then $D \cong D' \otimes \mathbb{K} \cong M_2(\mathbb{K})$. From the discussion in Section 3.2.2 it follows that both $\Lambda$ and $\Gamma$ are type I lattices. If $D'$ is a division algebra then $\Gamma$ is a type II lattice and there are two possible cases:

1. $D = D' \otimes \mathbb{K}$ is a division algebra and $\Lambda$ is a type II lattice;
2. $D \cong M_2(\mathbb{K})$ and $\Lambda$ is a type I lattice.

\[\blacksquare\]

**Remark 4.8.** As in the case of type I lattices, it is not very restrictive to suppose that the type II lattice $\Lambda < \text{PU}(F)_{\mathcal{K}}$ is contained in $\text{PU}(F, D)$. Indeed we have that the finite index subgroup $\Lambda^{(2)} < \Lambda$ is a subgroup of $\text{PU}(F, D)$ since, by Lemma 3.9, $g^2 \in \text{PU}(F, D)$ for all $g \in \Lambda$.

5. **Totally geodesic immersions of (quasi-)arithmetic hyperbolic lattices**

In this section we analyse the relation between the adjoint trace field and the ambient group of a (quasi-)arithmetic lattice and the adjoint trace field and ambient group of a totally geodesic sublattice (Theorem 1.7). This will allow us to generalise the examples of totally geodesic immersions constructed in Sections 4.1 and 4.2, define the notions of subform spaces and Weil restriction subspaces of arithmetic hyperbolic lattices, and prove Theorem 1.8. We also prove Theorem 1.9, which states that fc–subspaces have finite volume.

5.1. **Proof of Theorem 1.7.** We remark that arithmeticity of totally geodesic suborbifolds of arithmetic orbifolds is already known (see [5, Proposition 15.2.2]). Here we refine the analysis in order to control the behaviour of the adjoint trace fields.
Let $M = \mathbb{H}^n / \Gamma$ be an arithmetic hyperbolic $n$-orbifold with adjoint trace field $k$ and ambient group $G$. We are assuming that $M$ contains a totally geodesic suborbifold of dimension $\geq 2$ and therefore $M$ cannot be a 3-dimensional type III orbifold. As such its adjoint trace field $k$ is totally real and the ambient group $G$ is admissible (see Section 3.3 and Corollary 3.14).

We have that $\Gamma < G(k)$ and is commensurable with $G(O_k)$. Recall that the commensurator of $\Gamma$ in $G = G(\mathbb{R}) \cong \text{PO}_{n,1}(\mathbb{R})$ is precisely $G(k)$.

Let $m \geq 2$ and suppose that $N \subset M$ is a totally geodesic, finite volume $m$-dimensional suborbifold, obtained as the quotient of an $m$-dimensional totally geodesic subspace $U \subset \mathbb{H}^n$. We have that $\text{Stab}_\Gamma U$ acts as a lattice on $U$.

Let us consider the stabiliser $H = \text{Stab}_{G(k)}(U)$ of the subspace $U$ in the commensurator of $\Gamma$. We denote by $H$ its Zariski closure in $G$. Since the condition $M \subset \text{Stab}_{G(k)}(U)$ is polynomial in the coefficients of $M$, we see that $\text{Stab}_{G(k)}(U)$ is Zariski closed, and therefore it contains $H$. By repeating the argument from [5, Proposition 15.2.2] verbatim over the field $k$ we obtain that $H$ is $k$-defined, and is therefore an algebraic $k$-subgroup of $G$.

The natural projection map

$$\text{Stab}_{G(k)}(U) \cong \frac{\text{O}_{m,1}(\mathbb{R}) \times \text{O}_{n-m}(\mathbb{R})}{\langle (-\text{id}, -\text{id}) \rangle} \rightarrow \text{PO}_{m,1}(\mathbb{R})$$

restricts to a morphism

$$\pi : H(\mathbb{R}) \rightarrow \text{PO}_{m,1}(\mathbb{R})$$

which, by Borel’s density theorem [12], maps the connected component of the identity $H(\mathbb{R})^0$ surjectively onto $\text{PO}_{m,1}(\mathbb{R})^0$. We denote the kernel of $\pi$ by $C$. Notice that $C$ is a closed subgroup of the compact group

$$\text{Fix}(U) = \{ g \in G(\mathbb{R}) | g(x) = x \; \forall x \in U \} \cong \text{O}_{n-m}(\mathbb{R}),$$

and is therefore compact.

We notice first of all that the group $H$ is not necessarily connected, nor semisimple. We therefore denote by $R(H^0)$ the radical of $H^0$, i.e. the maximal connected normal solvable subgroup of $H^0$. It is a well-known fact that $R(H^0)$ is a $k$-subgroup of $H^0$ [44, p. 59]. The real points $R(H^0)_{\mathbb{R}}$ of the radical $R(H^0)$ are contained in the kernel of the projection map $\pi$. Indeed, the group $\pi(R(H^0)_{\mathbb{R}})$ is a connected normal solvable subgroup of the semisimple Lie group $\text{PO}_{m,1}(\mathbb{R})^0$ and thus it has to be trivial.

It follows that there is an induced surjective map with compact kernel

$$\pi' : H'(\mathbb{R}) \rightarrow \text{PO}_{m,1}(\mathbb{R})^0,$$

where $H'$ denotes the adjoint group of the connected semisimple $k$-group $H^0/R(G^0)$.

We now observe that the group $H'$ is admissible. Arguing by contradiction, if there were an embedding $\sigma : k \rightarrow \mathbb{R}$ such that $H^\sigma(\mathbb{R})$ is non-compact, we would have that the compact group $G^\sigma(\mathbb{R})$ contains a closed (and thus
A compact subgroup $H^\sigma(\mathbb{R})$ whose identity component surjects the noncompact group $H^\sigma(\mathbb{R})$.

Thus, if $\Gamma < G(k)$ is arithmetic then $\Lambda = \pi(\text{Stab}_T(U))$ is arithmetic in $\text{PO}_{m,1}(\mathbb{R})$. If $\Gamma < G(k)$ is only assumed to be quasi-arithmetic we still have that $\pi(\text{Stab}_T(U)) = \Lambda$ is a lattice contained in $\text{PO}_{m,1}(\mathbb{R})$, and thus is quasi-arithmetic.

We claim that the projection map $\pi$ in (35) admits a minimal field of definition $K$ such that $k \subseteq K$, and that this field is the adjoint trace field of the totally geodesic suborbifold $N$. Let us consider the decomposition of the adjoint group $H'$ as a direct product of $\mathbb{R}$-simple adjoint groups [54, p. 46]. The kernel of the map $\pi'$ is a normal subgroup of $H^\sigma(\mathbb{R})$, which implies that its intersection with a factor $F(\mathbb{R})$ of the decomposition is either trivial or all of $F(\mathbb{R})$. Therefore the kernel of $\pi'$ is the product of a finite set of compact simple factors $C_1(\mathbb{R}) \times \cdots \times C_l(\mathbb{R})$. Since the group $\text{PO}^\sigma_{m,1}$ is $\mathbb{R}$-simple, we see that it is a factor of the decomposition of $H'$ into $\mathbb{R}$-simple factors. In fact, it is the unique factor whose real points form a noncompact Lie group, and we obtain an isomorphism of algebraic $\mathbb{R}$-groups

$$H' \cong C_1 \times \cdots \times C_l \times \text{PO}^\sigma_{m,1},$$

with the map $\pi'$ being induced by the projection onto the last factor. The minimal field of definition of $\pi$ can easily be described as we now explain.

We notice first that the Dynkin diagram of $H'$ is the disjoint union of the Dynkin diagrams of the $\mathbb{R}$-simple factors of the decomposition (37). Consider the action of the absolute Galois group $\text{Gal}(\overline{k}/k)$ on the Dynkin diagram of $H'$, and define $\Phi < \text{Gal}(\overline{k}/k)$ as the subgroup which preserves the subdiagram corresponding to $\text{PO}^\sigma_{m,1}$.

We take as the field $K$ the fixed field for the group $\Phi$. Note that $K$ contains $k$ and that it is a subfield of the minimal field over which $H'$ becomes an inner form. The projection map $\pi'$ is $K$-defined, and cannot be defined over any field $k' \subset K$. The reason for this is that the group $\text{Gal}(\overline{k}/k')$ will contain $\Phi$ as a proper subgroup, implying that the $\text{PO}^\sigma_{m,1}$ factor is not invariant under the action of $\text{Gal}(\overline{k}/k')$.

We now show that the field $K$ is the adjoint trace field of the totally geodesic suborbifold $N \subset M$. Indeed, the Lie algebra $\mathfrak{h}$ of the group $H$ decomposes (as a vector space) over $k$ as the direct sum of its solvable radical $\mathfrak{r}$ with the semisimple Lie algebra $\mathfrak{h}'$ of $H'$. Moreover by Cartan’s criterion for semisimplicity, the Lie algebra $\mathfrak{h}'$ decomposes over $\mathbb{R}$ as the direct sum of the ideals corresponding to the $\mathbb{R}$-simple factors of (37):

$$\mathfrak{h}' = c_1 \oplus \cdots \oplus c_l \oplus \mathfrak{o}_{m,1}.$$

Notice that the $\mathfrak{o}_{m,1}$ factor is naturally endowed with the structure of a Lie algebra over $K$. The adjoint action of $H^\sigma(k)$ on $\mathfrak{h}'$ preserves each factor of this decomposition and is $k$-linear, while the projection map from $\mathfrak{h}'$ to $\mathfrak{o}_{m,1}$ is $K$-linear.
This implies that the trace of the projection of the adjoint action of $g \in H^o(k)$ to $\mathfrak{o}_{m,1}$ is an element of the field $K$, and therefore $K$ contains the adjoint trace field of the totally geodesic suborbifold $N$. The opposite inclusion follows easily from the characterisation of the adjoint trace field of $N = H^m/\Lambda$ as the minimal field over which the elements of $\Lambda$ can be simultaneously written down [56, Theorem 1, Corollary], and the observation that any $g \in H^o(k)$ acts $K$-linearly on the Lie algebra $\mathfrak{o}_{m,1}$.

**Remark 5.1.** The statement of Theorem 1.7 about the inclusion of adjoint trace fields cannot hold for totally geodesic immersions of pseudo-arithmetic hyperbolic orbifolds. Rather, there are examples where the opposite inclusion holds. As the work of Emery and Mila [19] shows, if $M$ is a Gromov–Piatetski-Shapiro non–arithmetic manifold, then its adjoint trace field $k$ is a multiquadratic extension of the adjoint trace field $K$ of its building blocks. Thus any connected component of the glueing locus of the blocks is a totally geodesic submanifold of $M$ whose adjoint trace field $K$ is a proper subfield of the adjoint trace field $k$ of $M$. By combining this fact with Theorem 1.7, we obtain a simple proof of non-arithmeticity of the Gromov–Piatetski-Shapiro manifolds.

Theorem 1.7 naturally suggests that the case where $k = K$, i.e. the adjoint trace field of the totally geodesic sublattice coincides with the adjoint trace field of the ambient lattice, is special. Motivated by this, we give the following definition:

**Definition 5.2.** Let $N = H^m/\Lambda$, $m \geq 2$, be a totally geodesic subspace of a quasi-arithmetic orbifold $M = H^n/\Gamma$. We say that $N$ is a subform space if the adjoint trace field $K$ of $\Lambda$ coincides with the adjoint trace field $k$ of $\Gamma$. In this setting, we say that $\Lambda$ is a subform lattice of $\Gamma$.

**Remark 5.3.** Our definition of subform space naturally extends the definition given by Meyer [40, Construction 4.11] for subspaces of type I orbifolds to subspaces of arithmetic orbifolds of any type.

We have already encountered many examples of subform spaces in arithmetic hyperbolic orbifolds: indeed all the involutions in $PO(f,k)$ described in Section 3.1.2 and the involutions in $PU(F,D)$ described in Section 3.2.3 give rise to subform lattices: the fixed point set $U$ for the action of the involution on $\mathbb{R}^{n+1}$ corresponds to a $k$-subspace (resp. a $D$-submodule) and the restriction of the quadratic form $f$ (resp. the skew-hermitian form $F$) to $U$ will be admissible and defined over $k$ (resp. defined on $D$, where $D$ is a quaternion algebra over $k$).

We begin by proving the following simple proposition:

**Proposition 5.4.** Let $N = H^m/\Lambda$ be a subform space of an arithmetic orbifold $M = H^n/\Gamma$. Then $N$ is an $fc$–subspace associated to a single involution in the commensurator of $\Gamma$. 
Proof. Denote by $G$ and $L$ the ambient groups of $M$ and $N$ respectively, which are both defined over the same adjoint trace field $k$. Notice that $G$ (resp. $L$) is a $k$-form of $PO_{n,1}$ (resp. $PO_{m,1}$). Here we opt to work with forms of the real group $O_{n,1}(R)$. Up to $k$-isomorphism there is a unique algebraic $k$-group $\tilde{G}$ in the $k$-isogeny class of $G$ whose real points are isomorphic to $O_{n,1}(R)$ [54, §2.6]. The group $\tilde{G}$ is obtained as a $\mathbb{Z}/2\mathbb{Z}$ central extension of $G$, and the nontrivial element of the center corresponds to $-id \in O_{n,1}$. If $G$ is of absolute type $B_n$, then $\tilde{G} = G \times \mathbb{Z}/2\mathbb{Z}$ is simply a direct product. On the other hand if $G$ is of absolute type $D_n$, the extension is induced by a non-trivial $k$-isogeny of $\tilde{G}^o$ onto $G^o$. The same conclusions hold for the group $\tilde{L}$.

We repeat the same argument as in the proof of Theorem 1.7 using the forms of $O_{n,1}$. Suppose that the totally geodesic suborbifold $N$ is the projection of a totally geodesic subspace $U \subset H^m$, and denote by $U$ the vector subspace of $\mathbb{R}^{n+1}$ such that $U \cap H^n = U$. We define $\tilde{H}$ as the Zariski closure of $\text{Stab}_{\tilde{G}(k)}(U)$ and notice that its real points form a subgroup of $\text{Stab}_{G(\mathbb{R})}(U)$.

The projection map

$$\text{Stab}_{G(\mathbb{R})}(U) \cong O_{m,1}(\mathbb{R}) \times O_{n-m}(\mathbb{R}) \to O_{m,1}(\mathbb{R}) \cong \tilde{L}(\mathbb{R})$$

restricts to a morphism $\tilde{\pi} : \tilde{H}(\mathbb{R}) \to \tilde{L}(\mathbb{R})$ which is again surjective on the respective identity components by Borel’s density theorem. Moreover we notice that the image under $\tilde{\pi}$ of $-id \in \tilde{H}$ is $-id \in \tilde{L}$.

The morphism $\tilde{\pi}$ is simply a lift of the morphism $\pi$ in (35) to $\tilde{H}$ and $\tilde{L}$. As such it is defined over the same field, which by hypothesis is precisely the common adjoint trace field $k$ of $M$ and $N$. In particular we obtain that $\tilde{H}$ decomposes over $k$ as a product $\tilde{H} = K \times \tilde{L}''$ where $K$ is the kernel of $\tilde{\pi}$ and $\tilde{L}'' \subset \tilde{L}$ is the image of $\tilde{\pi}$.

We conclude by noticing that the element $\theta = (id, -id)$ belongs to $\tilde{H}(k) < \tilde{G}(k)$. Its projection to $G$ lies in $G(k)$, and therefore corresponds to an order 2 element in the commensurator of the lattice $\Gamma$. Moreover $\theta$ corresponds to the identity element in $L(\mathbb{R}) = PO_{m,1}(\mathbb{R})$, meaning that it acts on the subspace $U$ by fixing it pointwise.

We now combine Proposition 5.4 with the characterisation of involutions in the ambient groups of type I and II arithmetic lattices given in Sections 3.1.2 and 3.2.3 to prove the following:

**Proposition 5.5.** Let $M$ be a type I (resp. type II) arithmetic hyperbolic orbifold, and $N \subset M$ a subform space in $M$ of dimension $\geq 2$. Then $N$ is a type I (resp. type II) arithmetic hyperbolic orbifold.

**Proof.** We deal with the case where $M$ is a type I arithmetic orbifold first. Let $k$ be its adjoint trace field and $G = PO(f)$ its ambient group, where $f$ is an admissible form defined over $k$. By Proposition 5.4, the subspace
$N$ is the projection of the fixed point set $U = U \cap \mathbb{H}^n$ of an involution $\theta \in \text{PO}(f)_k$.

By the discussion in Section 3.1.2, there are two possibilities:

1. the involution $\theta$ belongs to the image $\text{PO}(f,k)$ of $O(f,k)$ under the projection map $O(f) \to \text{PO}(f)$;
2. the involution $\theta$ belongs to $\text{PO}(f)_k \setminus \text{PO}(f,k)$.

Case (2) is not possible: by the discussion in Section 3.1.2, the adjoint trace field of $N$ would be a totally real quadratic extension of $k$ of the form $K = k(\sqrt{\mu})$, and thus $N$ would not be a subform space.

Therefore, $\theta$ belongs to $\text{PO}(f,k)$, the fixed-point-set $U$ of $\theta$ is a $k$-subspace, and the adjoint trace field of $N$ is $k$. The ambient group of the fc-subspace associated to $N$ is $\text{PU}(g)$, where $g$ is the $(k$-defined and admissible) restriction of $f$ to $U$.

The proof works similarly in the case where $M$ is a type II orbifold with adjoint trace field $k$ and ambient group of the form $\text{PU}(F)$. The subspace $N$ is the fixed point set of an involution $\theta \in \text{PU}(F)_k$. By the discussion in Section 3.2.3, the involution $\theta$ cannot belong to $\text{PU}(F)_k \setminus \text{PU}(F,D)$, or the adjoint trace field of $N$ would again be a totally real quadratic extension $K = k(\sqrt{\mu})$ of $k$. Thus $\theta$ belongs to $\text{PU}(F,D)$, its fixed-point-set is a right $D$-submodule $D_N$ of $D^m$, the adjoint trace field of $N$ is simply $k$ and its ambient group is $\text{PU}(G)$, where $G$ is the (admissible) restriction of $F$ to $D_N$.

Recall that if $\Lambda$ is an arithmetic lattice in $\text{PSO}_{3,1}(\mathbb{R}) \cong \text{PSL}_2(\mathbb{C})$, then one can associate to $\Lambda$ its invariant trace field $L$. We now analyse the relation between the adjoint trace field of an arithmetic orbifold and the invariant trace field of a 3-dimensional totally geodesic subspace.

**Proposition 5.6.** Let $N = \mathbb{H}^3/\Lambda$ be a totally geodesic suborbifold of an $n$-dimensional arithmetic orbifold $M = \mathbb{H}^n/\Gamma$. The adjoint trace field $k$ of $M$ is contained in the invariant trace field $L$ of $N$.

**Proof.** We bring forward the argument from the proof of Theorem 1.7. In particular, we define the algebraic $k$-group $H < G$ as the Zariski closure of $\text{Stab}_G(k)(U)$, where $U \cong \mathbb{H}^3 \subset \mathbb{H}^n$ is a geodesic subspace which projects down to $N$. By Borel’s density theorem, we obtain a surjective morphism with compact kernel

$$\pi : H^o(\mathbb{R}) \to \text{PSO}_{3,1}(\mathbb{R}).$$

(39)

After replacing $H^o$ with the adjoint group $H'$ of $H^o/R(H^o)$, we obtain a morphism of algebraic groups

$$\pi' : H'(\mathbb{R}) \to \text{PSO}_{3,1}(\mathbb{R}).$$

(40)

Notice that the real algebraic group $\text{PSO}_{3,1}$ is not absolutely simple. Its Dynkin diagram is the disjoint union of two vertices which are exchanged by the action of complex conjugation. This pair of vertices corresponds to a complex factor of $H'(\mathbb{R})$ isomorphic to $\text{PSL}_2(\mathbb{C})$. 

We now consider the decomposition of the adjoint semisimple group \( H' \) as a product of absolutely simple groups \([44, \text{Proposition 2.4, Theorem 2.6}]\) and notice that, just like in the proof of Theorem 1.7, the kernel of \( \pi_1 \) corresponds to the product of a finite set \( C_1, \ldots, C_l \) of factors whose real points are all compact. We thus obtain an isomorphism of complex algebraic groups

\[
H' \cong C_1 \times \cdots \times C_p \times \mathrm{PSL}_2 \times \mathrm{PSL}_2.
\]

Now consider any one of the two projections \( p : H' \to \mathrm{PSL}_2 \). We claim that it admits a minimal field of definition \( L \) which contains the field \( k \), and moreover is the invariant trace field of \( N \). Here it does not matter which particular copy of \( \mathrm{PSL}_2 \) (and corresponding projection map) we choose. The idea of the proof is just the same as in Theorem 1.7.

We first consider the action of the absolute Galois group \( \text{Gal}(\overline{k}/k) \) on the Dynkin diagram of \( H' \), define the subgroup \( \Phi \) which preserves the vertex corresponding to the chosen \( \mathrm{PSL}_2 \) factor, and take \( L \) to be the fixed field of \( \Phi \). Notice how \( L \) is necessarily a complex field, which properly contains the totally real field \( k \) and moreover is the minimal field of definition for the morphism \( p \).

Finally, we decompose the Lie algebra of \( H' \) over \( \mathbb{C} \) as a direct sum of ideals corresponding to the decomposition (41). The ideal \( \mathfrak{s}_2 \) corresponding to the chosen \( \mathrm{PSL}_2 \) factor is endowed with the structure of a Lie algebra over \( L \), and the trace of the adjoint action of \( g \in H'(k) \) takes values in \( L \). It follows that \( L \) contains the invariant trace field of \( N \). Moreover any \( g \in H'(k) \) acts \( L \)-linearly on the ideal \( \mathfrak{s}_2 \), which implies that \( L \) coincides with the invariant trace field of \( N \). \( \blacksquare \)

We finally have the tools to prove Theorem 1.8, which is meant to be an algebraic characterisation of totally geodesic immersions of arithmetic hyperbolic orbifolds. Here we make an essential use of Theorem 1.9 on the finiteness of volume of \( \mathbf{fc} \)-subspaces which is proved in Section 5.3.

5.2. Proof of Theorem 1.8. The last part of the statement is obvious: if there were a proper subform space \( S' \) of \( S \) which contains \( N \), then the adjoint trace fields of \( S' \) would be the same as that of \( M \) and thus \( S' \) would be a subform space of \( M \), contradicting the minimality of \( S \).

The subform space \( S \) can be constructed explicitly. In what follows, we carry over definitions and notations from the proof of Theorem 1.7. In particular \( k \) (resp. \( K \)) denotes the adjoint trace field of \( \Gamma \) (resp. \( \Lambda \)), while \( G \) (resp. \( L \)) denotes the ambient group.

Consider the adjoint semisimple \( k \)-group \( H' = H^\circ / R(H^\circ) \). By \([54, \S 3.1.2]\) it admits a unique decomposition as a direct product of \( k \)-simple adjoint groups whose factors can be recovered as follows.

Consider the set \( D \) of connected components of the Dynkin diagram of \( H' \). Its elements correspond to the factors of the decomposition of \( H' \) as a direct product of simple groups. Since these factors are all defined over \( \mathbb{Q} \), they are permuted by the action of the absolute Galois group \( \text{Gal}(\overline{\mathbb{Q}}/k) \).
The orbits of elements of $\mathcal{D}$ under this action correspond to the factors of the decomposition.

Let us label explicitly the factors of the decomposition for the group $H'$:

$$H' = C'_1 \times \cdots \times C'_q \times L'.$$

where $L'$ denotes the unique factor of the decomposition whose real points form a non-compact group. We claim that $L'$ is $k$-isomorphic to the group $\text{Res}_{K/k}(L)$.

To prove this claim we first notice that there exist a field $k'$ containing $k$ and an absolutely simple adjoint $k'$-group $E$ such that $L'$ is $k$-isomorphic to $\text{Res}_{k'/k}(E)$ [54, §3.1.2]. This means that the Dynkin diagram of $L'$ is the disjoint union of $[k' : k]$ connected components which are all of the same absolute type, in one–to–one correspondence with the set $S^\infty_{k'/k}$ of embeddings of $k'$ relative to $k$ and acted transitively upon by the group $\text{Gal}(\overline{L}/k)$, where $\overline{L}$ denotes the Galois closure of the extension $k'/k$.

If the dimension $m$ of the subspace $N$ is not 3 then its ambient group $L$ is absolutely simple. In this case we have that the group $E$ can be identified with the non-compact factor of the decomposition (37). As such its field of definition $k'$ is none other than the adjoint trace field $K$ of $N$ and $E$ is $K$-isomorphic to the ambient group $L$.

If the subspace $N$ has dimension $m = 3$ and is not of type III, we consider the decomposition (41) of $H'$ into a direct product of absolutely simple groups and identify the group $E$ with one of the two $\text{PSL}_2$ factors. This implies that $k'$ is the invariant trace field $L$ of $N$ and that $E$ is $L$-isomorphic to $\text{PGL}_1(A)$, where $A$ denotes the invariant quaternion algebra of $N$.

By the discussion in Section 3.3, we have that $L$ is an imaginary quadratic extension $L = K(\alpha)$ of the totally real adjoint trace field $K$ of $N$. Let $\overline{K}$ and $\overline{L}$ denote the Galois closures of the extensions $K/k$ and $L/k$ respectively. By [17, Proposition, p. 2], there is an integer $0 < \nu \leq [K : k]$ and a short exact sequence of groups:

$$1 \to (\mathbb{Z}/2\mathbb{Z})^\nu \to \text{Gal}(\overline{L}/k) \to \text{Gal}(\overline{K}/k) \to 1$$  \hspace{1cm} (43)

where the left (normal) factor contains complex conjugation $\sigma \in \text{Gal}(\overline{L}/k)$. Notice that [17] assumes that $k = \mathbb{Q}$ and that $L$ is a complex multiplication field, while in our case $L$ has a single complex place. However the same argument applies to our case without modifications.

It follows that the action of $\text{Gal}(\overline{L}/k)$ on the absolutely simple factors of $L'$ descends to an action of $\text{Gal}(\overline{K}/k)$ on the $\mathbb{R}$-simple factors, thus implying that the group $L' = \text{Res}_{L/k}(\text{PGL}_1(A))$ is isomorphic to $\text{Res}_{K/k}(L)$.

Let us substitute as in the proof of Proposition 5.4 the groups $L$ and $G$ with the corresponding forms $\hat{L}$ and $\hat{G}$ of $\text{O}_{m,1}$ and $\text{O}_{n,1}$ respectively. We denote by $\hat{H}$ the Zariski closure of $\text{Stab}_{\hat{G}}(U)$ and by $\hat{C}$ the kernel of the $k$-defined projection map $\hat{H} \to \text{Res}_{K/k}(\overline{L})$. 
We may assume that $\text{Res}_{K/k}(\mathcal{L})_{\mathbb{R}} \cong O_{m,1} \times \prod_{i=1}^{d-1} O_{m+1}$, where $d$ is the degree of $K/k$. Each factor corresponds to a Galois embedding $\sigma \in S^\infty_{K/k}$ and acts on an $(m + 1)$-dimensional subspace $U^\sigma$ of $\mathbb{R}^{n+1}$, equipped with the standard Minkowski form of signature $(n,1)$.

Each compact factor acts on a positive definite subspace $U^\sigma$, while the single non-compact factor acts precisely on the lift $U^\sigma_0 = U$ of the subspace $N$ to $\mathbb{R}^{n+1}$. The group $\tilde{C}(\mathbb{R})$ acts on some (possibly trivial) positive definite subspace of dimension $h$, and we get that $\mathbb{R}^{n+1}$ decomposes as an orthogonal direct sum of all these subspaces, i.e. $n + 1 = d \cdot (m + 1) + h$.

We obtain isomorphisms of algebraic $k$-groups

\begin{align}
\hat{H} &\cong \tilde{C} \times \text{Res}_{K/k}(\tilde{L}), \\
H &\cong \frac{\tilde{C} \times \text{Res}_{K/k}(\tilde{L})}{\langle (-\text{id}, -\text{id}) \rangle}
\end{align}

which completely characterise the arithmetic structure of the geodesic immersion of $N$ into $M$.

If the group $\tilde{C}$ is trivial, we simply set $S = M$. If $\tilde{C}$ is non-trivial, we notice that $i = [(-\text{id}, \text{id})]$ is an involution in $H(k) < G(k)$, and thus is an involution in the commensurator of $\Gamma$. We take $S$ to be the fc-subspace associated to this involution. Notice that $S$ has finite volume due to Theorem 1.7. In the first case, $S = M$ is trivially a subform space of $M$. In the second case, we can repeat the construction and define the corresponding groups for the immersion $S \subset M$. More specifically, we denote by

$$U_S = U \oplus U^{\sigma_1} \oplus \cdots \oplus U^{\sigma_{d-1}}$$

the subspace fixed by the involution $(-\text{id}, \text{id}) \in \hat{H}(k)$, by $\hat{H}_S \subset \hat{G}$ the Zariski closure of the group $\text{Stab}_{\hat{G}(k)}(U_S)$ (which is again defined over $k$) and by $L_S$ the ambient group of $S$.

We obtain that $\tilde{C}$ is contained in the kernel of the projection map

$$\pi_S : \hat{H}_S \to \hat{L}_S.$$
we obtain that the adjoint trace field of $S$ is equal to $k$ and therefore $S$ is a subform space.

The minimality of $S$ as a subform space containing $N$ can be proven as follows: the real points of the group $L_{S'}$ associated to a subform space $S'$ containing $N$ have to contain a closed group which is isomorphic to the group $\text{Res}_{K/k}(L)_{\mathbb{R}} \cong O_{m,1} \times \prod_{i=1}^{d-1} O_{m+1}$. and thus a lift $U_{S'}$ of $S'$ has to contain $U \oplus U^{\sigma_1} \oplus \cdots \oplus U^{\sigma_{d-1}} = U_S$. Consequently, we see that $S \subseteq S'$. ■

**Remark 5.7.** The statement about the inclusion of the adjoint trace fields in Theorem 1.7 also holds when the totally geodesic subspace $N$ is one-dimensional, provided that $M$ is not a 3-dimensional type III orbifold. In this particular setting we obtain that $L(\mathbb{R}) \cong \text{PO}_{1,1}$.

Indeed, to prove the inclusion $k \subseteq K$ we only use the surjectivity of the projection map $\pi : H(\mathbb{R})^0 \to L(\mathbb{R})^0$ and that the ambient group $G$ of $M$ is admissible (which implies that its $k$-defined subgroup $H$ is admissible). The first condition is always guaranteed by the fact that lattices in $\text{PO}_{1,1}$, which is not semi-simple, are Zariski dense. The second condition follows from the fact that $M$ is not 3-dimensional type III.

Consequently, the statement of Theorem 1.8 always applies when $N$ is one-dimensional and $M$ is not a 3-dimensional type III orbifold. In particular, we can speak of subform spaces of dimension one in type I, II and 7-dimensional type III lattices, and Proposition 5.4 applies in this case too.

The following two remarks concern the possible obstructions to extending Theorems 1.8 and 1.7 to the setting of quasi- or pseudo-arithmetic lattices.

**Remark 5.8.** We are not currently able to prove Theorem 1.8 under the assumption that the lattice $\Gamma$ is properly quasi-arithmetic. We do obtain the decomposition given in (44) for the $k$-group $H$. However, we cannot directly apply Theorem 1.9 to conclude that the minimal subform space $S$ has finite volume. The main obstruction here is that the involution $i = [(\text{id}, \text{id})]$ might not belong to the commensurator of $\Gamma$, which has infinite index in $G(k)$. Thus, $S$ might simply be the (infinite volume) quotient of $U_S = U \oplus U^{\sigma_1} \oplus \cdots \oplus U^{\sigma_{d-1}}$ under the discrete group $\text{Stab}_\Gamma(U_S)$.

**Remark 5.9.** We cannot prove that if $M = \mathbb{H}^n/\Gamma$ is pseudo-arithmetic and $N \subseteq M$ is a totally geodesic subspace, then $N$ has to be pseudo-arithmetic as well. The main problem here is proving that the group $H$, defined as the Zariski closure of $\text{Stab}(U) \cap \text{Comm}(\Gamma)$, is defined over the field of definition $k$ of the ambient group $G$ of $M$. Notice that the adjoint trace field $k'$ of $M$ is now a multiquadratic extension of $k$ and $\Gamma < \text{G}(k')$. It is indeed true that the Zariski closure of $\text{Comm}(\Gamma)$ is $k$-defined, since it is equal to $G$ by Borel’s density theorem. It could however be possible that the Zariski closure of $\text{Stab}(U)$ is only defined over $k'$.

We do remark however that by the angle rigidity theorem [23, Theorem 4.1] it follows that if $M$ is a Gromov - Piatetski-Shapiro manifold then all of its totally geodesic submanifolds are pseudo-arithmetic. In this setting
the ambient group $G = O(f)_K$, where $f$ is admissible over $k$ and $U$ is a $k$-subspace. It follows that both the Zariski closure of $\text{Stab}_U$ and the group $H$ are $k$-defined. Explicit examples of totally geodesic embeddings of Gromov–Piatetski–Shapiro manifolds are described in [31].

By Theorem 1.8, we see that $N$ is a subform space of $M$ precisely when $N = S$. At this point it becomes natural to consider the other possible extremal case, when $S = M$.

**Definition 5.10.** Let $N = \mathbb{H}^m/\Lambda$ be a totally geodesic subspace of an arithmetic orbifold $M = \mathbb{H}^n/\Gamma$. Suppose that $N$ is not 3-dimensional type III. We say that $N$ is a Weil restriction subspace of $M$ if the minimal subform space $S$ of $M$ which contains $N$ is precisely $M$. In this setting, we say that $\Lambda$ is a Weil restriction sublattice of $\Gamma$.

This definition also applies to one-dimensional subspaces of type I, II and 7-dimensional type III lattices by Remark 5.7. Notice how all the totally geodesic immersions built using Propositions 4.3 and 4.7 give rise to Weil restriction sublattices. Moreover, we have an explicit description of the ambient group of the lattice into which we construct the embedding: these are given by groups of the form $O(\text{Res}_{K/k}(f))$ (in the type I case) or $U(\text{Res}_{K/k}(F))$ (in the type II case). Theorems 1.7 and 1.8 simply state that the two techniques introduced in Section 4 are all that is needed to construct all totally geodesic immersions of arithmetic hyperbolic lattices, except when one of $M$ or $N$ is 3-dimensional type III. More precisely, if $M = \mathbb{H}^n/\Gamma$ and is an arithmetic hyperbolic orbifold not belonging to the exceptional family in dimension $n = 3$ and $N = \mathbb{H}^m/\Lambda$ is a totally geodesic subspace of $M$, then:

1. $N$ is arithmetic, its adjoint trace field $K$ is contains the adjoint trace field $k$ of $M$ and $[K : k] = d \geq 1$;
2. If $N$ is not 3-dimensional type III the field $K$ is totally real, the ambient group $L$ of $N$ is a $K$-form of $PO_{m,1}$, and the group $\text{Res}_{K/k}(L)$ is isogenous to a closed subgroup of an admissible $k$-form $L_S$ of $PO_{d(m+1)−1,1}$, which corresponds to an arithmetic hyperbolic orbifold $S$ of dimension $d(m + 1) − 1$ such that $N \subseteq S$;
3. $S$ is realised as a subform space in $M$, and $N$ is a Weil restriction subspace in $S$. In particular $\text{dim}(S) = d(m + 1) − 1 \leq n$.

We also see that all totally geodesic subspaces of low enough codimension in $M$ are subform spaces:

**Corollary 5.11.** Let $M = \mathbb{H}^n/\Gamma$ be an arithmetic hyperbolic orbifold with $M$ either type I, II or 7-dimensional type III. If $n$ is odd, then all totally geodesic subspaces of dimension $m \geq (n + 1)/2$ which are not 3-dimensional type III are subform spaces. If $n$ is even, then all totally geodesic subspaces of dimension $m \geq n/2$ which are not 3-dimensional type III are subform spaces.
Proof. The dimension $m$ of a subspace of $M$ which is not a subform space is subject to the constraint $d(m + 1) - 1 \leq n$, where $d = [K : k] > 1$. The maximum possible dimension is achieved when $d = 2$, so we get that $2(m + 1) - 1 \leq n$. For $n$ odd, this implies $m < (n + 1)/2$. For $n$ even, this implies $m < n/2$. ■

Let us notice that in the statement of Corollary 5.11 we need to care about the subspace not being 3-dimensional type III only for $n = 4, 5, 6$.

5.3. Proof of Theorem 1.9. First, we remark that the fixed point set of any collection of isometries acting on $\mathbb{H}^n$ is either empty or it is a totally geodesic subspace of $\mathbb{H}^n$ (possibly a point). Since by the hypothesis the group $F$ commensurates $\Gamma$, the group

$$\Gamma' = \bigcap_{g \in F} g \Gamma g^{-1}$$

is a finite index subgroup of $\Gamma$. Moreover, it is normalised by the finite group $F$. Hence, the group $\Gamma''$ generated by $\Gamma'$ and $F$ is a lattice in $\text{Isom}(\mathbb{H}^n)$, commensurable with $\Gamma$, and clearly $F < \Gamma''$. By [42, Lemma 4.4], the mapping $\phi : H/C_{\Gamma''}(F) \to \mathbb{H}^n/\Gamma''$ is proper, where $C_{\Gamma''}(F)$ denotes the centraliser of $F$ in $\Gamma''$, and $H = \text{Fix}(F)$ is a totally geodesic subspace of dimension $m$. Without loss of generality, we can suppose that $F = \{g \in \Gamma'' | gx = x \ \forall x \in H\}$.

Since $F$ is finite, the centraliser $C_{\Gamma''}(F)$ has finite index in the normaliser $N_{\Gamma''}(F)$, and the latter is easily seen to be the stabiliser $\text{Stab}_{\Gamma''}(H)$ of $H$ in $\Gamma''$. Hence, also the natural map $\phi' : H/\text{Stab}_{\Gamma''}(H) \to \mathbb{H}^n/\Gamma''$ is proper. This means that $H$ projects down to a properly immersed totally geodesic suborbifold $S$ in the orbifold $M = \mathbb{H}^n/\Gamma''$. Since the map $\phi'$ is proper, the cusps of $S$ correspond to cusps of $M$, and there are no accumulation points of $S$ inside of $M$.

Thus, once $M$ has finite volume, the orbifold $S$ has finite volume as well, except for the excluded case of $m = \text{dim}(H) = 1$, where $S$ could be an infinite geodesic. For one such example, let $\Gamma''$ be a reflection group in the facets of an ideal Coxeter polyhedron $\mathcal{P}$ in $\mathbb{H}^n$ and $F$ be the finite group of reflections in the facets of $\mathcal{P}$ which intersect along an edge $e$ having an ideal vertex. The length of the edge $e$ is clearly infinite.

Finally, by commensurability of the lattices $\Gamma$ and $\Gamma''$, we deduce that the stabilisers of $H$ in $\Gamma$ and $\Gamma''$ are commensurable, and therefore also $\text{Stab}_{\Gamma}(H)$ is a lattice acting on $H$. Finally, we notice that $\Gamma$ is uniform if and only if it does not contain parabolic elements, and in that case $\text{Stab}_{\Gamma}(H)$ does not contain parabolic elements either. ■

6. Finite centraliser subspaces and arithmeticity

In this section, we prove our arithmeticity criterion in terms of finiteness/infiniteness of fc–subspaces, and provide examples of nonarithmetic
hyperbolic orbifolds which contain non-fc–subspaces of codimension one. We also exhibit examples of non–fc subspaces in arithmetic orbifolds.

6.1. **Proof of Theorem 1.2.** (i). Let $\Gamma < PO_{n,1}(\mathbb{R})$, be an arithmetic group defined over a totally real field $k$. Then $\Gamma$ is either a type I, type II or type III lattice. According to the discussion in Sections 3.1.2, 3.2.3, 3.3.3 and by Proposition 3.18, there exists a $k$–involution $\theta \in \text{Comm}(\Gamma)$. In all cases, we can choose the involution $\theta$ in such a way that the fixed point set of its action on $\mathbb{H}^n$ has positive dimension. Conjugating by the elements of $\text{Comm}(\Gamma) = G(k)$, which is dense in $G(\mathbb{R})^o$ (cf. [41, Proposition 5.1.8]), we obtain countably many different fc–subspaces.

(ii). If $\Gamma$ is arithmetic and not 3-dimensional type III, by combining Corollary 5.11 with Proposition 5.4 we prove that if $M = H^n / \Gamma$ and $n$ is odd (resp. even), then all subspaces of dimension $m \geq (n+1)/2$ (resp. $m \geq n/2$) which are not 3-dimensional type III are fc-subspaces. However, this bound can be improved when $n$ is odd. We now consider this case more carefully.

Let $N$ be a Weil restriction subspace in $M$ and suppose that $r : K \rightarrow k$ where $K$ and $k$ denote the adjoint trace fields of $N$ and $M$ respectively. Notice that these conditions imply that $n = \text{dim}(M)$ is odd and that $\text{dim}(N) = m = (n+1)/2 - 1 = (n-1)/2$.

We claim that under the hypotheses above $N$ is necessarily an fc–subspace. Let us denote by $U$ a lift of $N$ to $H^n$ and by $H$ the Zariski closure of $\text{Stab}_{G(k)}(U)$ in the ambient group $G$ of $M$. From the proof of Theorem 1.8 we have that $H$ is $k$-defined and that

$$H \cong \frac{\text{Res}_{K/k}(\tilde{L})}{\langle \text{id} \rangle} \cong \frac{\tilde{L} \times \tilde{L}^\sigma}{\langle (\text{id}, -\text{id}) \rangle},$$

where $\tilde{L}$ denotes the $K$-form of $O(m,1)$ isogenous to the ambient group $L$ of $N$ and $\sigma : K \rightarrow K$ the nontrivial element of $\text{Gal}(K/k) \cong \mathbb{Z}/2\mathbb{Z}$. The action of the Galois automorphism $\sigma$ maps a pair $(A,B) \in \tilde{L}_K \times \tilde{L}_K^\sigma$ to the pair $(B^\sigma, A^\sigma)$. Since $(-\text{id}, -\text{id})$ is preserved by $\sigma$, this action descends to the group $H$. Moreover, in $H$ we have that $[(\text{id}, -\text{id})] = [(-\text{id}, \text{id})]$, therefore the element $[(\text{id}, -\text{id})] \in H(K)$ is fixed by the action of $\sigma$ and thus is an involution in $H(k) \subset G(k) = \text{Comm}(\Gamma)$ with $U$ as its fixed point set. This implies that $N$ is an fc–subspace, and thus all subspaces of $M$ of dimension $m \geq (n-1)/2$ which are not 3-dimensional type III are fc–subspaces.

In the case where $\Gamma$ is 3-dimensional type III, we only need to prove that all 1-dimensional totally geodesic subspaces are fc-subspaces. This follows by applying Proposition 3.16.

(iii). By Margulis’ theorem [38, Chapter IX, Th. B], if $\Gamma$ is non–arithmetic, then $\Gamma' = \text{Comm}(\Gamma)$ is the maximal lattice containing $\Gamma$ with finite index. Since $\Gamma'$ is a lattice, it contains only finitely many conjugacy classes of finite subgroups (this fact is well known for word hyperbolic groups, and hence for cocompact hyperbolic lattices; for a general result we refer to [49]). Now
assume that two finite subgroups $F_1, F_2 < \text{Comm}(\Gamma) = \Gamma'$ are conjugate in $\Gamma'$, $F_1 = \gamma F_2 \gamma^{-1}$. Then their fixed point sets $H_i = \text{Fix}(F_i)$ in $\mathbb{H}^n$ satisfy $H_1 = \gamma H_2$.

Let $\Gamma_i = \text{Stab}_\Gamma(H_i) = \{\alpha \in \Gamma \mid \alpha H_i = H_i\}, i = 1, 2$, be the corresponding stabilisers. Then we have $\alpha \in \Gamma_1, \alpha H_1 = H_1$, as well as $\alpha \gamma H_2 = \gamma H_2, \gamma^{-1} \alpha \gamma H_2 = H_2$. Hence $\Gamma_1 = \gamma \Gamma_2 \gamma^{-1}$ and therefore we have only finitely many $\Gamma'$–conjugacy classes of the stabilisers. Since $\Gamma$ has finite index in $\Gamma' = \text{Comm}(\Gamma)$, then we have finitely many $\Gamma$–conjugacy classes. Therefore, there are only finitely many $\text{fc}$–subspaces.

(iv). In [49], Samet obtained an effective upper bound for the number of conjugacy classes of finite subgroups of a lattice in terms of covolume. Let $\Gamma' = \text{Comm}(\Gamma)$ be the commensurator of the non-arithmetic lattice $\Gamma$. As discussed in part (iii), the $\text{fc}$–subspaces of $M = \mathbb{H}^n / \Gamma$ correspond to the strata in the natural orbifold stratification of $M' = \mathbb{H}^n / \Gamma'$ defined by

$$M'_F = H / \text{Stab}_\Gamma(H),$$

where $F < \Gamma'$ is a finite subgroup, $H = \text{Fix}(F)$, and any finite subgroup conjugate to $F$ in $\Gamma'$ gives the same stratum.

By [49, Theorem 1.3] the number of such strata is bounded above by $c \cdot \text{vol}(\mathbb{H}^n / \Gamma')$, with $c = \text{const}(n)$, a constant that depends on $n$ only. This quantity bounds the number of $\text{fc}$–subspaces of $M$ up to conjugation in $\Gamma'$. To count them up to conjugation in $\Gamma$ we need to multiply the former bound by the index $[\Gamma' : \Gamma]$, which gives $c \cdot \text{vol}(\mathbb{H}^n / \Gamma')[\Gamma' : \Gamma] = c \cdot \text{vol}(\mathbb{H}^n / \Gamma)$.

As for the main statements of the theorem, we have that (1) follows from (i)–(ii), and (2) follows from (iii)–(iv). $\blacksquare$

6.2. Examples of non–fc subspaces and proof of Theorem 1.10. By Theorem 1.2, all codimension 1 totally geodesic suborbifolds in an arithmetic orbifold are fc. For non–arithmetic lattices, Theorem 1.10 gives examples of maximal totally geodesic subspaces of codimension 1 which are not fc. Note that by [3] the number of such subspaces is always finite.

These examples come from the hybrid non–arithmetic lattices constructed by Gromov and Piatetski–Shapiro in [26]. We briefly recall the construction as described by Vinberg in [60] for the reader’s convenience.

Suppose that $\Gamma_1, \Gamma_2 < \text{Isom}(\mathbb{H}^n)$ are two lattices both containing a reflection $r$ in a hyperplane $H \subset \mathbb{H}^n$ which satisfy the following conditions:

(1) for any $\gamma \in \Gamma_i, i = 1, 2$, either $\gamma(H) = H$ or $\gamma(H) \cap H = \emptyset$,

(2) $N_{\Gamma_1}(r) = N_{\Gamma_2}(r) = \langle r \rangle \times \Gamma_0$, where $\Gamma_0$ leaves invariant the two half–spaces bounded by $H$. Here $N_{\Gamma_i}(r)$ denotes the normaliser of $r$ in $\Gamma_i$.

This means that $H$ projects to the same embedded fc–subspace in both quotient orbifolds of $\mathbb{H}^n$ by $\Gamma_1$ and by $\Gamma_2$.

Consider the set $\{\gamma(H) | \gamma \in \Gamma_i\}$ of translates of $H$ for a fixed $i$. This set decomposes the space $\mathbb{H}^n$ into a collection of closed pieces transitively
permuted by $\Gamma_i$, and each of these pieces is a fundamental domain for the action of $N_i = \langle \gamma r \gamma^{-1} \mid \gamma \in \Gamma_i \rangle$, the normal closure of $r$ in $\Gamma_i$. Let $D_i$ be one of these pieces, and let $\Delta_i = \{ \gamma \in \Gamma_i \mid \gamma(D_i) = D_i \}$. Then the group $\Gamma_i$ admits a decomposition as a semidirect product $\Gamma_i = N_i \rtimes \Delta_i$. Moreover, by choosing $D_1$ and $D_2$ to lie on opposite sides of the hyperplane $H$, we can ensure that $\Delta_1 \cap \Delta_2 = \Gamma_0$.

The following result from [60] shows that we can build a new hybrid lattice out of $\Gamma_1$ and $\Gamma_2$ as above.

**Theorem 6.1.** The group $\Gamma = \langle \Delta_1, \Delta_2 \rangle < \text{Isom}(\mathbb{H}^n)$ is a lattice. If $\Gamma_1$ and $\Gamma_2$ are incommensurable arithmetic groups, then the lattice $\Gamma$ is non-arithmetic.

As an abstract group, $\Gamma = \Delta_1 \ast_{\Gamma_0} \Delta_2$ (the amalgamated product of $\Delta_1$ and $\Delta_2$ along their common subgroup $\Gamma_0$). It is not difficult to build, for all $n \geq 3$, incommensurable torsion–free arithmetic lattices of simplest type $\Gamma_1, \Gamma_2 < \text{Isom}(\mathbb{H}^n)$ satisfying the conditions above (see for instance [26] or [30]).

It is fairly easy to check that, with the construction above, the hyperplane $H$ projects to a totally geodesic subspace $H/\Gamma_0$ in the orbifold $\mathbb{H}^n/\Gamma$. We argue that this is not an fc–subspace. Arguing by contradiction, we assume that the reflection $r$ in the hyperplane $H$ commensurates the lattice $\Gamma$. By setting $\Gamma' = r\Gamma r^{-1} \cap \Gamma$ and $\Gamma'' = \langle r, \Gamma' \rangle$, we obtain that $\Gamma''$ is commensurable with $\Gamma$.

Let $\Delta_i' = \Delta_i \cap \Gamma'$, and $\Gamma_0' = \Gamma_0 \cap \Gamma'$. The normaliser in $\Gamma''$ of the reflection $r$ is the group $N_{\Gamma''}(r) = \langle r \rangle \times \Gamma_0' = \langle r \rangle \times (\Delta_1' \cap \Delta_2')$. Let us denote by $N'$ the normal closure of $r$ in $\Gamma''$ (notice that $N'$ is generated by reflections $\gamma r \gamma^{-1}$ for all $\gamma \in \Gamma''$), and set $\Gamma'_1 = N' \rtimes \Delta_1'$. It is clear that $\Delta_1'$ is a finite index subgroup of $\Delta_i$, and that $\Gamma'_1$ is a finite index subgroup of $\Gamma_i$. Moreover, both $\Gamma'_1$ and $\Gamma'_2$ are sublattices of $\Gamma''$ and therefore have finite index in $\Gamma''$. This implies that $\Gamma_1$ and $\Gamma_2$ were commensurable to begin with, which is a contradiction. The proof of Theorem 1.10 is now complete.

**Remark 6.2.** As shown by Vinberg [60], the above construction of hybrid lattices can also be used to construct examples of finite-volume non–arithmetic Coxeter polytopes. The discussion above applies without modifications to these examples, providing non–arithmetic Coxeter lattices which contain codimension one non–fc sublattices.

**6.3. Proof of Theorem 1.11.** Here we provide examples of 3–dimensional, non–arithmetic hyperbolic manifolds $M_k$, $k \geq 1$, with the property that all their codimension one totally geodesic subspaces are non-fc and there are precisely $k$ of them. As far as we can see, unlike the examples by Gromov and Piatetski–Shapiro discussed above, these subspaces do not arise as the glueing locus of a hybrid lattice. These manifolds $M_k$ are obtained by Le and Palmer [33] as the $k$–sheeted cover of some hyperbolic 3-manifold $N_j$ described below.
Proposition 6.3 (Le and Palmer [33]). Let $N_j = S^3 \setminus K_j$ be the (hyperbolic) complement in $S^3$ of a twist knot $K_j$ with $j$ half-twists. If the number $j$ is an odd prime, then $N_j$ contains a unique immersed totally geodesic surface $S$ homeomorphic to a thrice–punctured sphere (see Fig. 3).

Notice that if $j = 2$, the manifold $N_j$ is the complement of the figure–eight knot, which is the only arithmetic hyperbolic knot complement by the work of Reid [47]. For all $j \neq 2$ the manifold $N_j$ is therefore non–arithmetic. We claim that in the latter case the surface $S$ is non-fc.

Proposition 6.4. Let $N_j$ be the (hyperbolic) complement in $S^3$ of a twist knot $K_j$ with $j$ half–twists, where $j$ is an odd prime. The totally geodesic thrice–punctured sphere $S$ is not an fc–subspace.

Proof. This follows from the work of Reid and Walsh [48, Theorem 3.1], which proves that if the complement in $S^3$ of a 2–bridge knot is non–arithmetic, then it admits no hidden symmetries. Translated in our terminology, this means that if $N = \mathbb{H}^3 / \Gamma$ is a non–arithmetic 2–bridge knot, then $\text{Comm}(\Gamma)$ coincides with the normaliser of $\Gamma$ in $\text{Isom}(\mathbb{H}^3)$. Equivalently, the elements of $\text{Comm}(\Gamma)$ correspond to symmetries of $N$ (rather than symmetries of some finite–index cover of $N$).

Twist knots are particular examples of 2–bridge knots, so the above result applies to $N = N_j$, $j \neq 2$. It therefore suffices to prove that there is no symmetry of $N_j$ that fixes $S$ pointwise. Note that such a symmetry would have to be a reflection in $S$, and therefore an orientation reversing element of $\text{Isom}(N_j)$.

However the Jones polynomial of the twist knot $K_j$, $j$ odd, is

$$V(q) = \frac{1 + q^{-2} + q^{-j} + q^{-j-3}}{q + 1}.$$

Since $V(q) \neq V(q^{-1})$, the twist knot complement $N_j$ is chiral, i.e. it admits no orientation–reversing symmetries.

As shown in [33], for any $N_j$ as above there exists a degree $k$ cover $M_k$ that contains exactly $k$ lifts of $S$, and these are the only totally geodesic surfaces in $M_k$. Then Theorem 1.11 follows.
Remark 6.5. All twist knot complements $N_j$ can be obtained through Dehn filling on one component of the Whitehead link complement $S^3 \setminus W$, which is hyperbolic and arithmetic. Moreover, in this case the thrice-punctured sphere $S$ does not intersect the core geodesic of the filling. The surface $S$ corresponds to a totally geodesic thrice-punctured sphere filling on one component of the Whitehead link complement $S$. Lattice $\Gamma$ is admissible and has signature

All twist knot complements $S^3 \setminus W$. We therefore obtain that the Dehn filling turns an fc–subspace into the non–fc subspace $S$.

6.4. Non-fc subspaces in arithmetic orbifolds. We conclude this section by providing examples of non-fc Weil restriction subspaces in arithmetic lattices of type I. We begin by choosing an extension $K/k$ of totally real fields of odd degree $d$ and an admissible quadratic form $f$ of signature $(m,1)$ defined over $K$. We denote by $\overline{K}$ the Galois closure of the extension $K/k$. By Proposition 4.3 we can choose an arithmetic lattice $\Lambda < \text{PO}(f,K)$ and realise it as a totally geodesic sublattice in an arithmetic lattice $\Gamma < \text{PO}(\text{Res}_{K/k}(f),k)$. Notice that the form $\text{Res}_{K/k}(f)$ is $k$-defined, admissible and has signature $(n,1)$ with $n = d \cdot (m + 1) - 1$. We claim that $N = \mathbb{H}^n/\Lambda$ is a non-fc subspace of $M = \mathbb{H}^n/\Gamma$.

Let us denote by $U \cong \mathbb{H}^m$ a lift of $N$ to $\mathbb{H}^n$, and suppose that $g \in \text{O}(\text{Res}_{K/k}(f))_\mathbb{R}$ commensurates $\Gamma$ and fixes $U$ pointwise. It follows that $[g] \in \text{PO}(\text{Res}_{K/k}(f))_k$ and moreover

$$[g] \in \frac{\text{O}(f) \times \text{O}(f^{\sigma_1}) \times \cdots \times \text{O}(f^{\sigma_{d-1}})}{\langle(-\text{id}, -\text{id}, \ldots, -\text{id})\rangle}$$

is invariant under the natural action of $\text{Gal}(\overline{K}/k)$ and is mapped by the projection onto the first factor to the identity element $[\pm \text{id}] \in \text{PO}(f)$. Since $\text{Gal}(\overline{K}/k)$ acts transitively on the various factors of the form $\text{O}(f^{\sigma})$ and $(\pm \text{id})^{\sigma} = \pm \text{id}$, it follows that necessarily

$$g = (\pm \text{id}, \pm \text{id}, \ldots, \pm \text{id}) \in \text{O}(f) \times \text{O}(f^{\sigma_1}) \times \cdots \times \text{O}(f^{\sigma_{d-1}}).$$

We claim that $g = \pm (\text{id}, \text{id}, \ldots, \text{id})$, and thus $[g] = \text{id} \in \text{PO}(\text{Res}_{K/k}(f))$. Indeed, denote by $n_+$ the number of $+\text{id}$ entries and by $n_-$ the number of $-\text{id}$ entries in (48). It is clear that the action of $\text{Gal}(\overline{K}/k)$ does not change the values of $n_+$ and $n_-$, as it simply permutes the entries, while multiplication by $(-\text{id}, -\text{id}, \ldots, -\text{id})$ changes $\text{id}$ to $-\text{id}$ and vice-versa, thus exchanging the values of $n_+$ and $n_-$. Since we require that $g^{\sigma}$ be equal to $\pm g$ for any $\sigma \in \text{Gal}(\overline{K}/k)$, then the only possibilities are the following:

1. $n_+ = d, n_- = 0, g^\sigma = g$ for all $\sigma \in \text{Gal}(K/k)$ and $[g] = \text{id} \in \text{PO}(\text{Res}_{K/k}(f))$;
2. $n_+ = 0, n_- = d, g^\sigma = g$ for all $\sigma \in \text{Gal}(K/k)$ and $[g] = \text{id} \in \text{PO}(\text{Res}_{K/k}(f))$;
3. both $n_+$ and $n_-$ are non-zero, and this implies that $g^\sigma = -g$ for some $\sigma \in \text{Gal}(\overline{K}/k)$.
Case (3) is impossible. Indeed, this would imply that $n_+ = n_-$, but in our case, $d = n_+ + n_-$ is odd. This proves the claim.

We conclude by noticing that $N$ is by construction a proper Weil restriction subspace of $M$. Moreover we have just proven that $\gamma \in \text{Comm}(\Gamma)$ fixes $U$ pointwise if and only if $\gamma = [g]$ is the identity element in the group $\text{PO}(\text{Res}_{K/k}(f)) \simeq \text{Isom}(\mathbb{H}^n)$. This implies that there is no proper fc-subspace of $M$ that contains $N$, and therefore $N$ itself is not an fc-subspace.

7. Examples

7.1. Involutions and quadratic extensions. We now provide explicit examples of involutions in $\text{PO}(f)_k$ which cannot be represented by an element of $O(f, k)$, as discussed in Section 3.1.2. Here $k = \mathbb{Q}$ and $f$ is a (trivially) admissible form of signature $(n, 1)$, with $n$ odd. An analogous example in the setting of arithmetic lattices in $\text{PSL}_2(\mathbb{C})$ is described in [30, p. 1314].

Let $k = \mathbb{Q}$, $V = \mathbb{Q}^{n+1}$ and consider the symmetric bilinear form whose matrix representation with respect to the standard basis $(e_0, e_1, \ldots, e_n)$ is given by the following block-diagonal matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus_m \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

where $m = (n - 1)/2$.

Now, consider the following matrix:

$$M = \begin{bmatrix} 0 & -1/\sqrt{5} \\ -\sqrt{5} & 0 \end{bmatrix} \oplus_m \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}.$$ 

The patient reader can manually check that $M$ is invertible and that conjugation of a matrix $N \in \text{GL}_{n+1}$ by $M$ is a polynomial function with rational coefficients in the entries of $N$. Moreover, $M^tAM = A$ and $M^2 = \text{id}$, therefore the matrix $M$ corresponds to an involution in $\text{PO}(f)_{\mathbb{Q}}$.

The positive eigenspace $V^+$ relative to the eigenvalue 1 for $M$ has dimension $(n + 1)/2$, with orthogonal basis $B_+$ given by:

$$B_+ = (e_0 - \sqrt{5}e_1, e_{2i} + (\sqrt{5} - 2)e_{2i+1}), \ i = 1, \ldots, m.$$ 

The negative eigenspace $V^-$ relative to the eigenvalue $-1$ has the same dimension $(n + 1)/2$, with orthogonal basis $B_-$ given by:

$$B_- = (e_0 + \sqrt{5}e_1, e_{2i} + (\sqrt{5} - 2)e_{2i+1}), \ i = 1, \ldots, m.$$ 

Notice how the vectors of $B_-$ are obtained from those of $B_+$ simply by applying to each coordinate the nontrivial Galois automorphism $\sigma$ of $\mathbb{Q}(\sqrt{5})$ which maps $\sqrt{5}$ to $-\sqrt{5}$.

The restriction $g$ of the form $f$ to $V^+$ is represented with respect to $B_+$ by the diagonal matrix with one entry equal to $-2\sqrt{5}$ and all other entries equal to $20 - 8\sqrt{5}$ (which is positive). Similarly, the restriction $h$ of the form $f$ to $V^-$ is represented with respect to $B_-$ by the diagonal matrix with one entry equal to $2\sqrt{5}$ and all other entries equal to $20 + 8\sqrt{5}$.
Thus we see that \( g \) has signature \( ((n-1)/2, 1) \) and \( h = g^\sigma \) is positive definite, so that \( g \) is admissible. Moreover, the group
\[
\text{Res}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(g)_{\mathbb{R}} = O(g)_{\mathbb{R}} \times O(h)_{\mathbb{R}}
\]
is realised as the subgroup of \( O(f, \mathbb{R}) \) which preserves the decomposition \( \mathbb{R}^{n+1} = (V^+ \otimes \mathbb{R}) \oplus (V^- \otimes \mathbb{R}) \). The space \( U = \mathbb{H}^n \cap (V^+ \otimes \mathbb{R}) \) projects to an arithmetic finite-volume totally geodesic subspace in \( \mathbb{H}^n/\text{PO}(f, \mathbb{Z}) \) which is a Weil restriction subspace with adjoint trace field \( \mathbb{Q}(\sqrt{5}) \) and ambient group \( \text{PO}(g) \).

7.2. An embedding of a type I lattice in a type II lattice. In this subsection, we construct an explicit example of a type I arithmetic hyperbolic orbifold realised as a Weil restriction subspace of a type II arithmetic hyperbolic orbifold.

Consider the rational quaternion algebra \( D' = \mathbb{Q}(\sqrt{-3}) \). By the discussion in [35, p. 88], \( D' \) is a division algebra. Let \( K = \mathbb{Q}(\sqrt{3}) \). Notice that the \( K \)-algebra \( D = D' \otimes K \) splits since 3 is a square in \( K \).

Let us now consider the admissible \( K \)-form of signature \((2m-1, 1)\) given by
\[
f(x) = -\sqrt{3}x_0^2 + x_1^2 + \ldots + x_{2m-1}^2.
\]
The form \( f \) can be interpreted as a form on \( D^m_+ = \{ x \in D^m | xi = x \} \), which is a \( K \)-subspace of dimension \( 2m \) of the right \( D \)-module \( D^m \) as in Section 3.2.2: it sufficient to fix a basis for \( D^m_+ \) in order to identify it with \( K^{2m} \).

We now extend the form \( f \) to a skew–Hermitian form \( F \) on \( D^m \) by setting
\[
F(x_1 + y_1 \mathbf{j}, x_2 + y_2 \mathbf{j}) = f(x_1, x_1)(i-1)\mathbf{j} + f(x_1, y_2)(i-1) + f(x_2, y_1)(i+1) + f(y_1, y_2)(i+1)\mathbf{j}
\]
for all \( x_1, y_1, x_2, y_2 \in D^m_+ \). The admissibility of \( F \) follows directly from the admissibility of the initial form \( f \).

Let us now consider an arithmetic lattice \( \Lambda < U(F, D) \). Since \( D \cong M_2(K) \), we have that \( \Lambda \) is a type I lattice by the discussion in Section 3.2.2. By applying Proposition 4.7 we realise \( \Lambda \) as a totally geodesic sublattice in \( \Gamma \cong U(G, D') \), where \( G = \text{Res}_{K/\mathbb{Q}}(F) \) is an admissible skew-hermitian form on \( (D')^{2m} \). It follows that \( \Gamma \) is a type II lattice and that the type I orbifold \( M = \mathbb{H}^{2m-1}/\Lambda \) is realised as a Weil restriction subspace in the type II orbifold \( N = \mathbb{H}^{4m-1}/\Gamma \).

7.3. One curious example: a 5–dimensional simplex. We turn our attention to the totally geodesic suborbifolds of the orbifold \( \mathbb{H}^5/\Gamma \) corresponding to the group \( \Gamma \) generated by reflections in the faces of a 5–dimensional hyperbolic simplex \( S \) with Coxeter diagram represented in Figure 4.

This simplex is non–compact and has 2 ideal vertices. In [55], Vinberg showed that the corresponding reflection group \( \Gamma \) is a non–arithmetic lattice defined over the field \( K = \mathbb{Q}(\sqrt{2}) \). More recently, it was shown that \( \Gamma \) is
not quasi-arithmetic either (it is a pseudo-arithmetic lattice, see [23] and [19]), and that it is not commensurable with any lattice obtained by glueing arithmetic pieces, such as GPS [26] or ABT [4] lattices.

Now we exhibit a 2-dimensional arithmetic fc–subspace of this orbifold. This embedding is interesting because it does not arise through the standard techniques described, for instance, in [30] and [31]. The first step consists in characterising the maximal lattice $\Gamma^1$ corresponding to the commensurator of $\Gamma$. Notice that there is an isometric involution of $S$ which acts on the facets as the permutation $\tau = (a, b)(c, d)(e, f)$.

**Proposition 7.1.** We have that $\Gamma' = \text{Comm}(\Gamma) = \langle \Gamma, \tau \rangle$.

**Proof.** The group $\Gamma$ can be easily shown to be a maximal co-finite reflection group in $\mathbb{H}^5$. Suppose that there exists a co-finite reflection group $\Lambda$ which contains $\Gamma$. Then, according to [21], $\Lambda$ must be a reflection group associated with a non–arithmetic hyperbolic Coxeter 5–simplex. However, $S$ is the only hyperbolic non–arithmetic Coxeter 5–simplex [28], therefore $\Lambda = \Gamma$. Then, it follows from [55] that $\Gamma' = \Gamma \rtimes \langle \tau \rangle$ is a maximal lattice. 

Notice that the discussion above implies that there is a splitting short exact sequence:

$$1 \rightarrow \Gamma \rightarrow \Gamma' \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1,$$

therefore $\Gamma'$ decomposes as a semidirect product $\Gamma \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ is generated by the isometric involution $\tau$ and acts by conjugation on $\Gamma$ through the permutation $\tau$ of the generators.

Now, consider any codimension $k$ face $F$ of the simplex $S$ which is not an ideal vertex. Since $S$ is a simple polytope, $F$ will lie at the intersection of $k$ facets $F_1, \ldots, F_k$ of $S$. The subgroup $G_F$ of $\Gamma$ generated by the reflections in these facets is finite and $\text{Fix}(F)$ is precisely the totally geodesic subspace $H$ of $\mathbb{H}^5$ which supports the face $F$. By Theorem 1.9, the stabiliser of $H$ in $\Gamma$ acts as a lattice on $H$, and therefore defines a totally geodesic sublattice of $\Gamma$.

By applying Theorem 1.9, it is easy to see that there is another totally geodesic sublattice in $\Gamma$ which corresponds to the fixed point set of the involution $\tau$. Let us prove the following fact.

**Figure 4.** The Coxeter diagram of a non–compact hyperbolic 5–simplex, with labels for its facets.
Subspace Stabilisers in Hyperbolic Lattices

Proposition 7.2. The fixed point set of the involution \( \tau \) is a hyperbolic plane \( H \cong \mathbb{H}^2 \subset \mathbb{H}^5 \). The stabiliser of \( H \) in \( \Gamma \) acts on \( H \) as the arithmetic \((2, 4, 8)\) triangle reflection group.

Proof. Realise the simplex \( S \) combinatorially as the projectivisation of the positive orthant in \( \mathbb{R}^6 \). Up to an appropriate identification of the vertices of \( S \) with the vectors of the standard basis of \( \mathbb{R}^6 \), we can suppose that the involution \( \tau \) is realised by the following permutation of the vectors of the standard basis:

\[
(e_1, e_2) \mapsto (e_3, e_4) \mapsto (e_5, e_6).
\]

The corresponding matrix has eigenvalue 1 with multiplicity 3, and the corresponding 3-dimensional eigenspace intersects the positive orthant in the subset

\[
\{ x_1 = x_2, x_3 = x_4, x_5 = x_6, x_i \geq 0 \}.
\]

By projecting onto the hyperboloid, we see that the fixed points set of the involution \( \tau \) in \( S \) is a hyperbolic triangle \( T \) whose sides \( s_1, s_2, s_3 \) lie respectively in \( a \cap b, c \cap d \) and \( e \cap f \). We denote by \( H \) the fixed points set in \( \mathbb{H}^5 \) of the involution \( \tau \): this will be a hyperbolic plane tessellated by copies of \( T \).

The centraliser \( C(\tau) \) of the involution \( \tau \) in \( \Gamma' \) coincides with the stabiliser of \( H \), and can easily be seen to be generated by \( \tau \) together with \( r_1 = abab \), \( r_2 = cdc \) and \( r_3 = efe \), where each \( r_i \) acts on \( H \) as a reflection in the side \( s_i \) of \( T \), and \( \tau \) acts as the identity. In order to describe both the geometry of \( T \) and the action of \( C(\tau) \) on \( H \) it is sufficient to find the order of the products of two reflections in the sides of \( T \). An easy computation with finite Coxeter groups shows that the orders of \( r_1 \cdot r_2 \), \( r_2 \cdot r_3 \), and \( r_1 \cdot r_3 \) are 8, 4, and 2, respectively. Thus, the stabiliser of \( H \) is the \((2, 4, 8)\)-triangle group, which is arithmetic by [53].

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