LONG-TIME DYNAMICS FOR A CLASS OF EXTENSIBLE BEAMS WITH NONLOCAL NONLINEAR DAMPING\textsuperscript{*}

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Abstract. In this paper we consider new results on well-posedness and long-time dynamics for a class of extensible beam/plate models whose dissipative effect is given by the product of two nonlinear terms. The addressed model contains a nonlocal nonlinear damping term which generalizes some classes of dissipations usually given in the literature, namely, the linear, the nonlinear and the nonlocal frictional ones. A first mathematical analysis of such damping term is presented and represents the main novelty in our approach.

1. Introduction. This article addresses global well-posedness and long-time dynamics to the following extensible beam model subject to a nonlocal nonlinear damping

\[ u_{tt} + \Delta^2 u - \kappa M(\|\nabla u\|_2^2)\Delta u + N(\|\nabla u\|_2^2)g(u_t) + f(u) = h \quad \text{in} \quad \Omega \times (0, \infty), \tag{1} \]

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial \Omega$, $M(\|\nabla u\|_2^2)$ and $N(\|\nabla u\|_2^2)$ represent nonlocal coefficients, where $\| \cdot \|_2$ stands for the norm in $L^2(\Omega)$, $g$ and $f$ are real nonlinear functions, $h$ is an external force and the parameter $\kappa \geq 0$ is related to the extensibility of the beam. The assumptions on the nonlinear terms $M$, $N$, $g$ and $f$ will be given in Sections 2 and 3. We consider two types of boundary conditions, namely, clamped boundary condition

\[ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \tag{2} \]

where $\nu$ is the unit exterior normal to $\partial \Omega$, or hinged boundary condition

\[ u = \Delta u = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty). \tag{3} \]

\textsuperscript{*} Dedicated to the memory of Professor Igor Chueshov.

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The initial conditions associated with displacement $u$ are given by

$$u(\cdot, 0) = u_0(\cdot), \quad u_t(\cdot, 0) = u_1(\cdot) \quad \text{in} \quad \Omega. \quad (4)$$

As reported by the authors in [15, 16], equation (1) is an $n$-dimensional abstract version related to vibrations of nonlinear plate and extensible beam models. In [16] it is provided an extensive survey on references concerning to extensible models closely to (1)-(4) and their results on existence, uniqueness, asymptotic stability and long-time behavior.

In this work our main goal is to present a first analysis on well-posedness and long-time behavior when problem (1)-(4) is under the influence of the following nonlocal nonlinear damping term

$$N(\|\nabla u\|_2^2)g(u_t), \quad (5)$$

which is given by the product of two nonlinear components. Indeed, $N$ is assumed to be any positive $C^1$-function and $g$ is a nonlinear $C^1$-function with polynomial growth, for instance $g(u_t) \approx |u_t|^\gamma u_t$, $\gamma > 0$. See Assumptions (A2)-(A3) in Section 2. This kind of nonlocal dissipative effect constitutes a generalization of nonlocal weak damping $N(\|\nabla u\|_2^2)u_t$ which was first introduced by Lange and Perla Menzala [23] with respect to plate models and subsequently studied by Cavalcanti et al. [3] in a viscoelastic context. In that occasion, the authors in [23] derived the plate model by taking the imaginary part of a Schrödinger equation with nonlocal term $a(u(t)) := N(\|\nabla u(t)\|_2^2)$ and, consequently, the term $N(\|\nabla u\|_2^2)u_t$ arises as a type of nonlocal Kirchhoff damping. Moreover, the damping (5) can be seen as generalization of a dissipation studied more recently in [15, 16]. Indeed, when the growth of $g$ is linear ($g(u_t) \approx u_t$), then the damping term in (5) turns into the simpler case $N(\|\nabla u\|_2^2)u_t$, which makes all computations easier than the case treated in the present paper. We also refer to [8, 20, 21] where it is studied long-time dynamics of a class of plate models with state-dependent nonlinear weak damping $\sigma(u)u_t$.

As far as we know, another kind of nonlocal fractional damping is given by

$$N(\|\nabla u\|_2^2) (-\Delta)^\theta u_t, \quad 0 \leq \theta \leq 1. \quad (6)$$

Chueshov and Kolbasin [9] studied long-time behavior for a class of abstract equations encompassing 2-dimensional Berger plate models ($M$ being linear in (1)) with damping given by (6) and power $\theta$ covering the range $0 < \theta \leq 1$. In [9] the chosen structure of the damping also approaches viscoelastic Kelvin-Voigt, structural and viscous damping. When $N$ is constant, for example $N \equiv 1$, then Coti Zelati [13] considered a structural damping $A^\theta u_t$ with $0 \leq \theta \leq 1$ and Biazutti and Crippa [2] assumed power $0 < \theta \leq 1$. In the latter, the condition $\theta > 0$ is crucial since the multiplier $A^{\theta/4}u_t$ is required (see [2, Lemma 2.2]). But these two last papers are concerned with fully linear damping which goes away from the scope of this work. To our best knowledge the only paper which deals with damping like (6) by moving $\theta$ uniformly on $[0, 1]$ and, at the same time, considering a nonlocal damping coefficient $N > 0$ is given in [16]. In what concerns beam models in a 1-dimensional framework, replacing the gradient by the Laplacian operator as the argument of the function $N$, then the case $\theta = 1$ in (6) can be also seen as a kind of nonlocal damping related to flight structures, see e.g. Balakrishnan and Taylor [1]. Although nonlocal, we observe that the damping term (6) is just nonlinear in one of its components which does not address nonlinearities like (5). It is worth noting that the case $\theta = 0$ in (6) characterizes a particular case of the damping (5) when $g(u_t) \approx u_t$. 
We also refer to [6, 7, 24, 29] for results on existence and long-time behavior for a class Kirchhoff wave models with nonlocal fractional damping (6).

On the other hand, a more natural damping term in the existing literature for beam models like (1) is given by \( g(u_t) \), say \( N \equiv 1 \) in (5), as a generalization of linear frictional dissipations \( u_t \). For this kind of damping there are a lot of papers which provide studies on well-posedness and long-time dynamics for beam/plate models related to (1). See for instance [4, 5, 12, 22, 17, 18, 27, 28, 35, 36, 37] and reference therein. However, the class of functions \( N \) allowed in (5) encompasses the constant case \( N \equiv 1 \) and, therefore, our results are given in a more general framework. An example of this situation appears when we deal with the difference of two trajectories \( u_1 - u_2 \) of problem (1)-(4). In such case the following term arises

\[
N(\|\nabla u_1\|_2^2)g(u_1^t) - N(\|\nabla u_2\|_2^2)g(u_2^t),
\]

whose controllability is worse than the standard case \( g(u_1^t) - g(u_2^t) \) appearing in the above mentioned works. Hence, a new mathematical analysis is necessary for the term (7) in all crucial results of this paper, see e.g. the proofs of Theorem 2.1-(iv), Proposition 2 and Proposition 4.

The main results of this paper are Theorem 2.1 (see Section 2) which ensures the well-posedness of problem (1)-(4) and Theorems 3.1 and 3.2 (stated at the end of Section 3) which establish the existence of attractors to the dynamical system associated with problem (1)-(4) as well as their qualitative properties with respect to geometrical characterization, finite dimension, regularity of trajectories and exponential attractor. To their proofs new mathematical arguments (estimates) are provided with respect to nonlocal nonlinear damping term (5). When compared with the existing literature, our results complement and generalize all the above mentioned works dealing with beam/plate models under frictional dissipations. In other words, the nonlocal nonlinear damping term generalizes (at least) three classes of usual damping in the literature, namely, the linear \( u_t \), the nonlinear \( g(u_t) \), and the nonlocal frictional one \( N(\|\nabla u\|_2^2)u_t \).

The remaining paper is organized as follows. In Section 2 we first provide the notations and assumptions used throughout the paper and then the well-posedness result to problem (1)-(4) with respect to strong and weak solutions. In Section 3 we consider the dynamical system corresponding to problem (1)-(4) and several results on its long-time dynamic properties up to the main results on attractors. A final Appendix A ends this work with examples of functions \( f, g \) and \( M \) satisfying their respective assumptions required a priori.

2. Well-posedness. We begin this section with the introduction of the function spaces which shall be used throughout this paper:

\[
W_0 = L^2(\Omega), \quad W_1 = H^1_0(\Omega), \quad W_2 = \begin{cases} H^2(\Omega) \cap H^1_0(\Omega) & \text{for (2)}, \\ H^2(\Omega) & \text{for (3)}. \end{cases}
\]

For \( m = 3, 4 \), we consider

\[
W_m = \begin{cases} H^m(\Omega) \cap H^2_0(\Omega) & \text{for (2)}, \\ \{u \in H^m(\Omega) \cap H^2_0(\Omega); \; \Delta u \in H^1_0(\Omega)\} & \text{for (3)}, \end{cases}
\]

and for \( \gamma \geq 0 \),

\[
W_{4,\gamma} = \begin{cases} W^{4,\frac{\gamma+2}{\gamma+1}}_\gamma(\Omega) \cap H^2_0(\Omega) & \text{for (2)}, \\ \{u \in W^{4,\frac{\gamma+2}{\gamma+1}}_\gamma(\Omega) \cap H^2_0(\Omega); \; \Delta u \in H^1_0(\Omega)\} & \text{for (3)}. \end{cases}
\]
When $\gamma = 0$ note that $W_{4,0} = W_4$. Here the notation $(\cdot, \cdot)$ stands for $L^2$-inner product and $\| \cdot \|_p$ denotes $L^p$-norm. Thus, $\| \nabla \cdot \|_2$ and $\| \Delta^2 \cdot \|_2$ represent the norms in $W_1$ and $W_2$, respectively. When there is no possibility of confusion we shall use the same notation $(\cdot, \cdot)$ to represent the duality pairing between any Banach space $W$ and its dual $W^*$. Denoting by $\lambda_1 > 0$ the first eigenvalue of the bi-harmonic operator $\Delta^2$ with boundary condition (2) or (3), then

$$\lambda_1 \|u\|_2^2 \leq \|\Delta u\|_2^2, \quad \lambda_1^{1/2} \|\nabla u\|_2 \leq \|\Delta u\|_2, \quad \forall u \in W_2. \quad (8)$$

We also consider the following phase space

$$\mathcal{H} = W_2 \times W_0, \quad \|(u, v)\|_\mathcal{H}^2 = \|\Delta u\|_2^2 + \|v\|_2^2, \quad (u, v) \in \mathcal{H},$$

where the analysis of the asymptotic behavior of solutions shall be done.

Now we introduce the assumptions on the functions $f$, $g$, $M$, and $N$ as follows:

(A$_1$) $f$ is a $C^1$-function on $\mathbb{R}$ satisfying

$$|f'(s)| \leq c_f(1 + |s|^\rho), \quad \forall s \in \mathbb{R}, \quad (9)$$

$$-c_f - \frac{\alpha_1}{2} s^2 \leq \hat{f}(s) := \int_0^s f(\tau) d\tau \leq f(s) + \frac{\alpha_1}{2} s^2, \quad \forall s \in \mathbb{R}, \quad (10)$$

where we consider $c_f > 0$, $c_f \geq 0$, $0 \leq \alpha_1 < \lambda_1$, and

$$\rho > 0 \quad \text{if} \quad 1 \leq n \leq 4 \quad \text{or} \quad 0 < \rho \leq \frac{4}{n-4} \quad \text{if} \quad n \geq 5; \quad (11)$$

(A$_2$) $g$ is a $C^1$-function on $\mathbb{R}$ fulfilling

$$c_g |s|^{\gamma} \leq g'(s) \leq C_g'(1 + |s|^\gamma), \quad \forall s \in \mathbb{R}, \quad (12)$$

where we assume $c_g > 0$, $C_g' > 0$, and

$$\gamma > 0 \quad \text{if} \quad 1 \leq n \leq 4 \quad \text{or} \quad 0 < \gamma \leq \frac{8}{n-4} \quad \text{if} \quad n \geq 5; \quad (13)$$

(A$_3$) $M$ and $N$ are $C^1$-functions on $[0, +\infty)$ with

$$N(s) > 0, \quad \kappa M(s) \geq -\alpha_2, \quad \forall s \in [0, +\infty), \quad (14)$$

where $0 \leq \alpha_2 < \lambda_1^{1/2}$;

(A$_4$) $\alpha$ and $\alpha_2$ are chosen so that

$$\alpha := 1 - \left(\frac{\alpha_1}{\lambda_1} + \frac{\alpha_2}{\lambda_1^{1/2}}\right) > 0. \quad (15)$$

Remark 1. Assumptions (A$_1$)-(A$_4$) deserve some a priori comments as follows.

1. Without loss of generality we can consider $f(0) = g(0) = 0$. Indeed, if $f(0) = f_0 \neq 0$ or $g(0) = g_0 \neq 0$, we define $\hat{f}(s) = f(s) - f_0$ or $\hat{g}(s) = g(s) - g_0$, respectively. Therefore, $\hat{f}(0) = \hat{g}(0) = 0$ with $\hat{f}$ satisfying (9)-(10) for some constants $\hat{c}_f \geq 0$ and $0 \leq \hat{\alpha}_1 < \lambda_1$, and $\hat{g}$ fulfilling (12);

2. Inequalities (11) and (13) imply the following embeddings $W_2 \hookrightarrow L^{2(n+1)}(\Omega)$ and $W_2 \hookrightarrow L^{n+1}(\Omega)$, respectively, for any dimension $n \geq 1$. In addition, the cases $\rho = 0$ in (11) and $\gamma = 0$ in (13) are obviously allowed. However, the latter ones describe simpler cases of linear growth for $f$ and $g$;

3. The second condition in (14) implies

$$\kappa \hat{M}(s) := \kappa \int_0^s M(\tau) d\tau \geq -\alpha_2 s, \quad \forall s \in [0, +\infty). \quad (16)$$
Theorem 2.1, well as in the subsequent section dealing with long time behavior. To conclude the continuous dependence of solutions as they always have the same sign, which can be easily verified analyzing the cases.

The last term in the above expression is nonnegative since both terms within parentheses always have the same sign, which can be easily verified analyzing the cases |s| > |r| and |s| < |r|. Thus, inequality (18) holds true. Such inequality will be a critical estimate in order to conclude the continuous dependence of solutions as well as in the subsequent section dealing with long time behavior.

The well-posedness of problem (1)-(4) is given by the following result.

**Theorem 2.1 (Well-Posedness).** Let T > 0 be arbitrary, h ∈ W₀ and κ ≥ 0. Under hypotheses (A₁)-(A₄) we have:

(i) **(Strong solutions)** If initial data (u₀, u₁) ∈ W₄ × (W₂ ∩ L²⁺(Ω)), then problem (1)-(4) possesses a strong solution u in the class $u \in L^∞(0, T; W_{4,γ}), \quad u_{t} \in L^∞(0, T; W_{2}), \quad u_{tt} \in L^∞(0, T; W_{0}). \quad (19)$

(ii) **(Weak solutions)** If initial data (u₀, u₁) ∈ H, then problem (1)-(4) possesses a weak solution u in the class $(u, u_{t}) \in L^∞(0, T; H) ∩ C([0, T], H). \quad (20)$
In addition, it is possible to find that
\[ u_t \in L^{γ+2}(0,T;L^{γ+2}(Ω)) \quad \text{and} \quad u_{tt} \in L^{γ+2}(0,T;W^2_0). \]  
(21)

(iii) (Energy inequality) Both strong and weak solutions satisfy the following energy inequality
\[ E_n(t) + C_N \int_s^t \|u_t(τ)\|_{γ+2}^2 \, dτ \leq E_n(s), \quad ∀ \, t > s \geq 0, \]  
(22)
for some constant \( C_N = C_N(\|(u_0, u_1)\|_H) > 0 \), where \( E_n(t) := E_n(u(t), u_t(t)) \) stands for the energy solution
\[ E_n(t) = \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \|Δu(t)\|_2^2 + \frac{κ}{2} [\|∇u(t)\|^2_2] + (\tilde{f}(u(t)), 1) - (u(t), h). \]  
(23)

(iv) (Continuous dependence) Both strong and weak solutions depend continuously on the initial data in \( H \). More precisely, if \( z^1 = (u_0, u_1), z^2 = (v_0, v_1) \) are two solutions corresponding to initial data \( z_0^1 = (u_0, u_1), z_0^2 = (v_0, v_1) \), respectively, then
\[ \|z^1(t) - z^2(t)\|_H \leq e^{C_0 \int_0^t \beta(s) \, ds} \|z_0^1 - z_0^2\|_H, \quad ∀ \, t \in [0,T], \]  
(24)
where \( C_0 = C_0(\|z_0^1\|_H, \|z_0^2\|_H) > 0 \) and \( β(t) = 1 + \|v(t)\|_{γ+2}^2 \). In particular, problem (1)-(4) has uniqueness of solution in both cases.

The proof on existence relies on the Faedo-Galerkin method. The nonlocal nonlinear damping in (1) needs a careful analysis during the whole proof as we shall see in the sequel. Theorem 2.1 is proved in some steps as follows.

2.1. Proof Theorem 2.1 (i): We start with the following approximate problem
\[ (u_{\text{tt}}^m(t), \omega_j) + (Δu^m(t), Δ\omega_j) - κM (\|∇u^m(t)\|_2^2) (Δu^m(t), \omega_j) + N (\|∇u^m(t)\|_2^2) g(u_t^m(t), \omega_j) + (f(u^m(t)), \omega_j) = (h, \omega_j), \]  
(25)
\[ u^m(0) = u_0^m, \quad u_t^m(0) = u_1^m, \]  
(26)
for \( j = 1, \ldots, m \), which has a local solution
\[ u^m(t) = \sum_{i=1}^m y_{im}(t) ω_i \in \text{Span}\{ω_1, \ldots, ω_m\}, \]  
on \([0, T_m] \), \( m \in \mathbb{N} \), given by a standard method in ODE theory, where \( (ω_j)_j \in \mathbb{N} \) is an orthonormal basis in \( W_0 \) given by eigenfunctions of \( Δ^2 \) with boundary condition (2) or (3). In what follows a priori estimates are shown in order to extend the local solution to the interval \([0,T]\) and then conclude the existence of strong and weak solutions for (1)-(4).

We first consider the problem (25)-(26) with
\[ (u_0^m, u_1^m) \to (u_0, u_1) \]  
strongly in \( W_4 \times (W_2 \cap L^{2(γ+1)}(Ω)). \)  
(27)

A Priori Estimate I. Replacing \( ω_j \) by \( u_t^m(t) \) in the approximate equation (25) yields
\[ \frac{d}{dt} E_κ^m(t) + N (\|∇u^m(t)\|_2^2) \int_Ω g(u^m_t(t)) u^m_t(t) \, dx = 0, \]  
(28)
where \( E_κ^m(t) \) is the functional energy (23) for Galerkin’s solutions. Besides, using (17) we obtain
\[ \int_Ω g(u^m_t(t)) u^m_t(t) \, dx ≥ \frac{cg}{γ+1} \|u^m(t)\|_{γ+2}^{γ+2}. \]  
(29)
Combining (28) with (29) and integrating the resulting expression from 0 to $t \leq T_m$, we have

$$E^m_\kappa(t) + \frac{c_\gamma'}{\gamma + 1} \int_0^t N(\|\nabla u^m(s)\|_{L^2}^2, \|u^m_t(s)\|_{L^2}^\gamma + 2) ds \leq E^m_\kappa(0).$$  \hspace{1cm} (30)

On the other hand, from assumptions (10), (14)-(15), observing (8) and (16), and applying Young inequality with $\varepsilon = \alpha \lambda_1/4$, then

$$E^m_\kappa(t) \geq \frac{1}{2} \|u^m_t(t)\|_{L^2}^2 + \frac{\alpha}{2} \|\Delta u^m(t)\|_{L^2}^2 - \frac{\varepsilon}{\lambda_1} \|\Delta u^m(t)\|_{L^2}^2 - c_f |\Omega| - \frac{1}{4\varepsilon} \|h\|_{L^2}^2
$$

$$\geq \frac{1}{2} \|u^m_t(t)\|_{L^2}^2 + \frac{\alpha}{4} \|\Delta u^m(t)\|_{L^2}^2 - c_f |\Omega| - \frac{1}{\alpha \lambda_1} \|h\|_{L^2}^2,$$

that is,

$$\frac{1}{2} \|u^m_t(t)\|_{L^2}^2 + \frac{\alpha}{4} \|\Delta u^m(t)\|_{L^2}^2 \leq E^m_\kappa(t) + c_f |\Omega| + \frac{1}{\alpha \lambda_1} \|h\|_{L^2}^2.$$  \hspace{1cm} (31)

From (30)-(31), since $N > 0$ on $[0, +\infty)$, and $E^m_\kappa(0)$ is bounded when assumptions (10)-(14) are taken in place, then

$$\|u^m_t(t)\|_{L^2}^2 + \|\Delta u^m(t)\|_{L^2}^2 \leq C_1, \quad \forall t \in [0, T_m],$$

where $C_1 = C_1(\|\Delta u_0\|_{L^2}, \|u_1\|_{L^2}, \|h\|_{L^2}, |\Omega|) > 0$. This is sufficient to extend all approximate solutions on $[0, T]$. In addition, using again that $N$ is strictly positive, then there exists a constant $c_N = c_N(\|(u_0, u_1)\|_{H}) > 0$ such that

$$N(\|\nabla u^m(t)\|_{L^2}^2) \geq c_N > 0, \quad \forall t \in [0, T].$$  \hspace{1cm} (32)

Going back to (30) one has

$$E^m_\kappa(t) + \frac{c_\gamma c_\gamma'}{\gamma + 1} \int_0^t \|u^m_t(s)\|_{L^2}^\gamma + 2 ds \leq E^m_\kappa(0).$$  \hspace{1cm} (33)

Combining (31) and (33) we conclude

$$\|u^m_t(t)\|_{L^2}^2 + \|\Delta u^m(t)\|_{L^2}^2 + \int_0^t \|u^m_t(s)\|_{L^2}^\gamma + 2 ds \leq C_1,$$

for any $t \in [0, T]$ and $m \in \mathbb{N}$, and some constant $C_1 > 0$ depending on initial data in $\mathcal{H}$.

Therefore, from (34) we obtain

$$u^m \text{ is bounded in } L^\infty(0, T; W_2),$$

$$u^m_t \text{ is bounded in } L^\infty(0, T; W_0),$$

$$u^m_{tt} \text{ is bounded in } L^{\gamma + 2}(0, T; L^{\gamma + 2}(\Omega)).$$  \hspace{1cm} (35) (36) (37)

A Priori Estimate II. Differentiating the approximate equation (25) with respect to time and taking the multiplier $u^m_{tt}(t)$ in the resulting expression we have

$$\frac{1}{2} \frac{d}{dt} \left(\|u^m_t(t)\|_{L^2}^2 + \|\Delta u^m(t)\|_{L^2}^2\right) + N(\|\nabla u^m(t)\|_{L^2}^2) \int_\Omega g'(u^m_t(t))[u^m(t)]^2 dx
$$

$$= -2\kappa M'(\|\nabla u^m(t)\|_{L^2}^2)(u^m(t), \Delta u^m_t(t)) \int_\Omega \Delta u^m(t) u^m_t(t) dx
$$

$$+ \kappa M(\|\nabla u^m(t)\|_{L^2}^2) \int_\Omega \Delta u^m_t(t) u^m(t) dx$$

$$= -2\kappa M'(\|\nabla u^m(t)\|_{L^2}^2)(u^m(t), \Delta u^m_t(t)) \int_\Omega \Delta u^m(t) u^m_t(t) dx
$$

$$+ \kappa M(\|\nabla u^m(t)\|_{L^2}^2) \int_\Omega \Delta u^m_t(t) u^m(t) dx.$$  \hspace{1cm} (38)
Returning to (38), using that \( M, N \in C^1([0, \infty)) \), the estimates (32) and (34), and also Cauchy-Schwarz and Young inequalities, we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| u_{tt}^m(t) \|_2^2 + \| \Delta u_{tt}^m(t) \|_2^2 \right) + C \int \Delta u_{tt}^m(t) | u_{tt}^m(t) |^2 \, dx \leq C \left( \| u_{tt}^m(t) \|_2^2 + \| \Delta u_{tt}^m(t) \|_2^2 \right) + C \int |g(u_{tt}^m(t))| | u_{tt}^m(t) | \, dx
\]

for some constant \( C > 0 \) depending on the initial data in \( \mathcal{H} \). Let us estimate the right hand side of (39). From now on we use the same parameter \( C \) to designate different positive constants which may depend on weak initial data but not on \( t \) and \( m \).

Using (12), Hölder inequality, embedding \( W_2 \hookrightarrow W_0 \) and Young inequality with \( \epsilon > 0 \), we infer

\[
\int |g(u_{tt}^m(t))| | u_{tt}^m(t) | \, dx \leq C \int (| u_{tt}^m(t) | + | u_{tt}^m(t) |^{\gamma + 1}) | u_{tt}^m(t) | \, dx
\]

\[
\leq C \| u_{tt}^m(t) \|_2 \| u_{tt}^m(t) \|_2 + C \int | u_{tt}^m(t) |^{\gamma + 1} | u_{tt}^m(t) |^{\gamma + 2} | u_{tt}^m(t) | \, dx
\]

\[
\leq C \| u_{tt}^m(t) \|_2 \| u_{tt}^m(t) \|_2 + C \| u_{tt}^m(t) \|_2 \gamma + 2 \left[ \int | u_{tt}^m(t) |^{\gamma + 2} | u_{tt}^m(t) |^2 \, dx \right]^{\gamma + 2}
\]

\[
\leq C (\| u_{tt}^m(t) \|_2^2 + \| \Delta u_{tt}^m(t) \|_2^2) + C \| u_{tt}^m(t) \|_2 \gamma + 2
\]

\[
+ \epsilon \int \left[ \partial_t (| u_{tt}^m(t) |^{\gamma + 2} u_{tt}^m(t) ) \right]^2 \, dx.
\]

Now applying (9), Hölder inequality with \( \frac{p}{2(p+1)} + \frac{1}{2(p+1)} + \frac{1}{2} = 1 \), embedding \( W_2 \hookrightarrow L^{2(p+1)}(\Omega) \), estimate (34) and Young inequality, we obtain

\[
\int |f'(u^m(t))| | u_{tt}^m(t) | | u_{tt}^m(t) | \, dx \leq C \int (1 + | u^m(t) |^p | u_{tt}^m(t) |) \, dx
\]

\[
\leq C \left( \| u^m(t) \|_{2(p+1)} + \| u^m(t) \|_{2(p+1)}^{p} \| u_{tt}^m(t) \|_2 \right)
\]

\[
\leq C (\| \Delta u_{tt}^m(t) \|_2^2 + \| u_{tt}^m(t) \|_2^2).
\]
Inserting these two last estimates in (39), and choosing \( \epsilon = \frac{c_N \epsilon_0'}{2(2+1)} > 0 \), we get
\[
\frac{d}{dt} \left( \|u^m_i(t)\|^2_2 + \|\Delta u^m_i(t)\|^2_2 \right) + \int_{\Omega} \left[ \partial_t \left( |u^m_i(t)|^2 u^m_i(t) \right) \right]^2 dx \leq C \|u^m_i(t)\|^2_2 + C \left( \|\Delta u^m_i(t)\|^2_2 + \|u^m_i(t)\|^2_2 \right),
\]
for every \( t \in [0, T] \) and some constant \( C > 0 \). In addition, estimate (34) implies that \( \|u^m_i(t)\|^2_2 \leq C_0 \) in \( L^1(0, T) \), and a standard estimate (similar to Cavalcanti et al. [4]), shows that \( \|u^m_0\|_2 \leq C \), where \( C > 0 \) is a constant which depends on initial data \((u_0, u_1) \in W_4 \times (W_2 \cap L^2(\gamma+1)) \)) but is independent of \( m \). Thus, integrating (40) on \([0, t]\), using Gronwall’s inequality and (27), we arrive at
\[
\|u^m_i(t)\|^2_2 + \|\Delta u^m_i(t)\|^2_2 + \int_0^t \int_{\Omega} \left[ \partial_t \left( |u^m_i(t)|^2 u^m_i(t) \right) \right]^2 dx ds \leq C_2,
\]
for any \( t \in [0, T] \) and \( m \in \mathbb{N} \), where \( C_2 = C_2(\|\Delta u_0\|_2, \|\Delta u_1\|_2, \|u_1\|_{2(\gamma+1)}, \Omega, T) > 0 \). From estimate (41) we conclude
\[
(u^m_i) \text{ is bounded in } L^\infty(0, T; W_2), \tag{42}
(u^m_{11}) \text{ is bounded in } L^\infty(0, T; W_0), \tag{43}
(\partial_t (|u^m_i|^2 u^m_i)) \text{ is bounded in } L^2(0, T; W_0). \tag{44}
\]
Finally, the boundedness (35)-(37) and (42)-(44) are sufficient to pass the limit on the approximate equation (25) and achieve a function \( u \) satisfying (19) and \( u_{tt} + \Delta^2 u - \kappa M (|\nabla u(t)|^2) \Delta u + N (|\nabla u(t)|^2) g(u) + f(u) = h \) in \( L^\infty(0, T; L^{\frac{2+2}{\gamma+1}}) \). The initial conditions in (4) are obtained by using the equation and (27). Therefore, problem (1)-(4) has a strong solution. \( \square \)

**Remark 2.** With respect to boundary condition (3) we can also make another a priori estimate by replacing \( \omega_j \) by \( -\Delta u^m_i \) in the approximate equation (25). Then performing analogous calculations as given by the authors in [16] it is possible to find that strong solutions also satisfy
\[
(u^m_i) \text{ is bounded in } L^\infty(0, T; W_4), \tag{45}
(\partial_{\nu} (|u^m_i|^2 u^m_i)) \text{ is bounded in } L^2(0, T; W_0),
\]
where \( W_3 = \{ u \in H^3(\Omega) \cap H^2_0(\Omega); \Delta u \in H^2(\Omega) \} \) in this case.

**Remark 3.** The only limit we need to watch out is about the nonlocal nonlinear term given by the product of two nonlinearities. Actually, such limit needs to be clarified since it is not provided by previous literature so far. Thereby, a more accurate analysis of this convergence is made below for weak solutions and so it holds true in the stronger case.

### 2.2. Proof Theorem 2.1 (ii): Given \((u_0, u_1) \in \mathcal{H}\) there exists a sequence \((u^m_0, u^m_1)\) in \(W_4 \times (W_2 \cap L^2(\gamma+1))\) such that
\[
(u^m_0, u^m_1) \to (u_0, u_1) \text{ strongly in } \mathcal{H}.
\]
In addition, for each regular initial data \((u^m_0, u^m_1)\), there exists a solution in the class \(u^m \in L^\infty(0, T; W_4), u^m_1 \in L^\infty(0, T; W_2), u^m_{11} \in L^\infty(0, T; L^{\frac{2+2}{\gamma+1}}(\Omega)), \forall T > 0,\)
to the following problem
\begin{align}
   u''_m + \Delta^2 u_m - \kappa M (||\nabla u_m(t)||_2^2) \Delta u_m + N(||\nabla u_m(t)||_2^2) g(u_m) + f(u_m) &= h, \\
   u_m(0) = u'_m(0) = u_0' \
\end{align}
(46)

Taking the multiplier \( u''_m \) in (46) and proceeding similar to a priori estimate I, observing (45), then estimate (34) remains true, namely,
\begin{equation}
   \| u''_m(t) \|_2^2 + \| \Delta u_m(t) \|_2^2 + \int_0^t \| u''_m(s) \|_2^{\gamma+2} ds \leq C, \quad \forall \ t \in [0, T],
\end{equation}
(48)

for some constant \( C = C(||(u_0, u_1)||_{\mathcal{H}}) > 0 \).

Besides, as we shall see in the proof of Theorem 2.1 (iv), the difference of strong solutions \( u^m(t) - u^k(t) := w(t) \), for \( m, k \in \mathbb{N} \), satisfies the following inequality
\begin{equation}
   \frac{d}{dt} \left( \| \Delta w(t) \|_2^2 + \| w(t) \|_2^2 \right) + C_N \int_{\Omega} \left[ |u_k(t)|^\gamma + |v_k(t)|^\gamma \right]|w(t)|^2 dx \leq C \left( 1 + \| u^k(t) \|_2^{\gamma+2} + \| w(t) \|_2^2 \right),
\end{equation}
(49)

where \( C_N, C > 0 \) are constants depending only on \((u_0, u_1) \in \mathcal{H} \). Applying Gronwall’s inequality we obtain, in view of (48), that
\begin{equation}
   \| \Delta w(t) \|_2^2 + \| w(t) \|_2^2 \leq C_T \left( \| \Delta w(0) \|_2^2 + \| w(0) \|_2^2 \right), \quad \forall \ t \in [0, T],
\end{equation}
(50)

where \( C_T = C_T(||(u_0, u_1)||_{\mathcal{H}}, T) > 0 \). From (49) and (45) it follows that \((u^m, u''_m)\) is a Cauchy sequence in \( C([0, T], \mathcal{H}) \). Thereby, applying (48) and (49) there exists a subsequence \((u^k)\) of \((u^m)\) such that
\begin{align}
   u^k &\to u \quad \text{weak star in } L^\infty(0, T; W^2_2), \\
   u^k_t &\to u_t \quad \text{weak star in } L^\infty(0, T; W^0_0), \\
   u^k &\to u \quad \text{strongly in } C([0, T], W^2_2), \\
   u^k_t &\to u_t \quad \text{strongly in } C([0, T], W^0_0), \\
   u^k &\to u \quad \text{weak in } L^{7+2}(0, T; L^{7+2}(\Omega)).
\end{align}
(51)-(54)

The above limits (50)-(54) along with (45) are enough to pass the limit on the system (46)-(47) and show that (20)-(21) are satisfied with
\begin{equation}
   u_{tt} + \Delta^2 u - \kappa M (||\nabla u(t)||_2^2) \Delta u + N(||\nabla u(t)||_2^2) g(u) + f(u) = h \quad \text{in } L^{7+2}(0, T; W^2_2),
\end{equation}
(55)

with initial conditions (4). In accordance with Remark 3 it remains to analyze the limit of the nonlocal nonlinear term \( N(||\nabla u^k(t)||_2^2) g(u^k(t)) \). From Aubin-Lions compactness theorem, see e.g. [25], there exists a subsequence of \((u^k)\), still denoted by \((u^k)\), such that
\begin{equation}
   u^k \to u \quad \text{strongly } L^{7+2}(0, T; W^1_2).
\end{equation}
(56)
On the other hand, using again the Mean Value Theorem and assumption (12) with \( g(0) = 0 \), we obtain
\[
|g(u^k_t(t))| = |g'(\theta u^k_t(t))u^k_t(t)| \leq C_{g'}[1 + |u^k_t(t)|^\gamma]|u^k_t(t)|, \quad \text{for some} \quad \theta \in (0, 1),
\]
and so
\[
\int_0^T \int_\Omega |g(u^k_t(t))|^{\frac{\gamma+2}{\gamma+1}} \, dx \, dt \leq C_{g'} \int_0^T \int_\Omega (|u^k_t(t)| + |u^k_t(t)|^{\gamma+1})^{\frac{\gamma+2}{\gamma+1}} \, dx \, dt
\]
\[
\leq 4C_{g'} \int_0^T \int_\Omega |u^k_t(t)|^{\frac{\gamma+2}{\gamma+1}} \, dx \, dt
\]
\[
+ 4C_{g'} \int_0^T \int_\Omega |u^k_t(t)|^{\gamma+2} \, dx \, dt.
\]
From (48) and using that \( L^{\gamma+2}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{\frac{\gamma+2}{\gamma+1}}(\Omega) \), results
\[
g(u^k_t) \quad \text{is bounded in} \quad L^{\frac{\gamma+2}{\gamma+1}}(0, T; L^{\frac{\gamma+2}{\gamma+1}}(\Omega)).
\]
Then, using that \( g \in C^1(\mathbb{R}) \) and uniqueness of the weak limit we infer
\[
g(u^k_t) \rightharpoonup g(u_t) \quad \text{weak in} \quad L^{\frac{\gamma+2}{\gamma+1}}(0, T; L^{\frac{\gamma+2}{\gamma+1}}(\Omega)) = [L^{\gamma+2}(0, T; L^{\gamma+2}(\Omega))]',
\]
that is,
\[
\int_0^T \int_\Omega g(u^k_t(t))w(t) \, dx \, dt \rightharpoonup \int_0^T \int_\Omega g(u_t(t))w(t) \, dx \, dt, \quad \forall w \in L^{\gamma+2}(0, T; L^{\gamma+2}(\Omega)).
\]
In particular, taking \( w = \omega_j \theta \) with \( \theta \in L^{\gamma+2}(0, T) \) and \( j \in \mathbb{N}, \, j \leq k \), it follows that
\[
\int_0^T \left( \int_\Omega g(u^k_t(t)) \omega_j \, dx \right) \theta(t) \, dt \rightharpoonup \int_0^T \left( \int_\Omega g(u_t(t)) \omega_j \, dx \right) \theta(t) \, dt.
\]
Thus,
\[
g(u^k_t(t), \omega_j) \rightharpoonup (g(u_t(t)), \omega_j) \quad \text{weak in} \quad L^{\frac{\gamma+2}{\gamma+1}}(0, T).
\]
(57)
Hence, combining (56) and (57) with classical results in functional analysis we are able to pass the limit on the nonlocal nonlinear term in (46) to achieve the desired weak solution (and also in (25) for regular solutions).

2.3. **Proof Theorem 2.1 (iii):** Taking the multiplier \( u_t \) in (1), then a straightforward computation implies that \( E_N(t) \) given in (23) satisfies
\[
\frac{d}{dt} E_N(t) + N \left( \|\nabla u(t)\|_2^2 \right) \int_\Omega g(u_t(t)) u_t(t) \, dx = 0,
\]
for strong solutions, where the derivative is understood in the distributional sense, see for instance Yang [37, Lemma 3.2]. Regarding the same arguments as in (29) and (32), and denoting by \( C_N = \frac{c_N}{\gamma+1} > 0 \), then
\[
\frac{d}{dt} E_N(t) + C_N \|u_t(t)\|^{\gamma+2}_{\gamma+2} \leq 0.
\]
(58)
Therefore, the energy inequality (22) holds true by integrating (58) on \( [s, t] \) for any \( 0 \leq s < t \). In addition, (22) is also ensured for weak solutions using standard density arguments.
2.4. Proof Theorem 2.1 (iv): Let \( z^1 = (u,u_t) \) and \( z^2 = (v,v_t) \) be two strong (or weak) solutions of (1)-(4) with initial data \( z_0^1 = (u_0,u_1) \) and \( z_0^2 = (v_0,v_1) \), respectively. Setting \( w = u - v \), the difference \( z^1 - z^2 = (w,w_t) \) solves the following problem in the strong (or weak) sense

\[
w_{tt} + \Delta^2 w + N(\|\nabla u(t)\|_2^2) [g(u_t) - g(v_t)] \\
= \kappa M(\|\nabla u(t)\|_2^2) \Delta w + \kappa [M(\|\nabla u(t)\|_2^2) - M(\|\nabla v(t)\|_2^2)] \Delta v \\
- [N(\|\nabla u(t)\|_2^2) - N(\|\nabla v(t)\|_2^2)] g(v_t) - [f(u) - f(v)],
\]

with initial condition \( (w(0),w_t(0)) = z_0^1 - z_0^2 \).

We first consider the estimates below for strong solutions. Taking the multiplier \( w_t \) in (59) we infer

\[
\frac{1}{2} \frac{d}{dt} \left( \|w_t(t)\|_2^2 + \|\Delta w(t)\|_2^2 \right) + I_0 = \sum_{k=1}^4 I_k,
\]

where

\[
I_0 = N(\|\nabla u(t)\|_2^2) \int_\Omega \left[ g(u_t(t)) - g(v_t(t)) \right] w_t(t) \, dx,
\]

\[
I_1 = \kappa M(\|\nabla u(t)\|_2^2) \int_\Omega \Delta w(t) w(t) \, dx,
\]

\[
I_2 = \kappa [M(\|\nabla u(t)\|_2^2) - M(\|\nabla v(t)\|_2^2)] \int_\Omega \Delta v(t) w_t(t) \, dx,
\]

\[
I_3 = - [N(\|\nabla u(t)\|_2^2) - N(\|\nabla v(t)\|_2^2)] \int_\Omega g(v_t(t)) w_t(t) \, dx,
\]

\[
I_4 = - \int_\Omega [f(u(t)) - f(v(t))] w_t(t) \, dx.
\]

In addition, we note that estimate (34), which holds for any strong (or weak) solution, implies that \( N(\|\nabla u(t)\|_2^2) \geq c_N > 0 \) for every \( t > 0 \), where \( c_N = c_N(\|z_0^1\|_H) > 0 \). From this and condition (18) we get

\[
I_0 \geq \frac{c_N c_{\alpha'}}{2(\gamma + 1)} \int_\Omega \left[ |u_t(t)|^\gamma + |v_t(t)|^\gamma \right] |w_t(t)|^2 \, dx.
\]

Inserting (61) in (60) and denoting by \( c_1 = \frac{c_N c_{\alpha'}}{2(\gamma + 1)} > 0 \) we deduce

\[
\frac{1}{2} \frac{d}{dt} \left( \|w_t(t)\|_2^2 + \|\Delta w(t)\|_2^2 \right) + c_1 \int_\Omega \left[ |u_t(t)|^\gamma + |v_t(t)|^\gamma \right] |w_t(t)|^2 \, dx = \sum_{k=1}^4 I_k,
\]

Now let us estimate the right-hand side of (62). For simplicity, the parameter \( C > 0 \) shall denote several constants which depend on the initial data in \( H \), but not on \( t > 0 \).

First of all, since \( M,N \in C^3([0,\infty)) \), we have from (34), the Main Value Theorem, and embedding \( W_2 \hookrightarrow W_1 \), that

\[
\max_{\tau \in [0,\lambda_1^{-1/2}C_1]} \{|M(\tau)|,|M'(\tau)|,|N(\tau)|,|N'(\tau)|\} \leq C < \infty,
\]

and

\[
|\frac{1}{2} M(\|\nabla u(t)\|_2^2) - M(\|\nabla v(t)\|_2^2)|, |N(\|\nabla u(t)\|_2^2) - N(\|\nabla v(t)\|_2^2)| \leq C \|\nabla w(t)\|_2,
\]

where \( C = C(\|z_0^1\|_H,\|z_0^2\|_H) > 0 \).
Using Young inequality it follows directly
\[ |\mathcal{I}_3| \leq C \left( \|\Delta w(t)\|_2^2 + \|w_t(t)\|_2^2 \right), \]
for some \( C > 0 \). Applying assumption (12), embedding \( W_2 \hookrightarrow W_1 \), Hölder and Young inequalities, one has
\[ |\mathcal{I}_4| \leq C \|\nabla w(t)\|_2 \int_\Omega \left[ |v_t(t)| + |v_t(t)|^{\gamma + 1} \right] |w_t(t)| \, dx \]
\[ \leq C \|\Delta w(t)\|_2 \|v_t(t)\|_2 \|w_t(t)\|_2 + C \|\Delta w(t)\|_2 \int_\Omega \left[ |v_t(t)|^{\frac{2}{\gamma + 2}} + |v_t(t)| \right] |w_t(t)| \, dx \]
\[ \leq C \|\Delta w(t)\|_2 \|v_t(t)\|_2 \|w_t(t)\|_2 \]
\[ \quad + \frac{c_1}{2} \int_\Omega \left[ |w_t(t)|^\gamma + |v_t(t)|^\gamma \right] |w_t(t)|^2 \, dx \]
\[ \leq C (1 + \|v_t(t)\|_2^{\gamma + 2})(\|\Delta w(t)\|_2^2 + \|w_t(t)\|_2^2) \]
\[ \quad + \frac{c_1}{2} \int_\Omega \left[ |w_t(t)|^\gamma + |v_t(t)|^\gamma \right] |w_t(t)|^2 \, dx. \]

Further, using assumption (9), Hölder inequality with \( \frac{2}{\gamma + 1} + \frac{1}{2} = 1 \), embedding \( W_2 \hookrightarrow L^2(\gamma + 1)(\Omega) \), and Young inequality, one gets
\[ |\mathcal{I}_4| \leq C \int_\Omega \left[ 1 + |u(t)|^\rho + |v(t)|^\rho \right] |w(t)| |w_t(t)| \, dx \]
\[ \leq C \left[ \Omega \frac{\rho}{\gamma + 1} + \|u(t)\|_{L^2(\gamma + 1)}^\rho + \|v(t)\|_{L^2(\gamma + 1)}^\rho \right] \|w(t)\|_{L^2(\gamma + 1)} \|w_t(t)\|_2 \]
\[ \leq C \left( \|\Delta w(t)\|_2^2 + \|w_t(t)\|_2^2 \right). \]

Replacing these last four estimates in (62) results
\[ \frac{1}{2} \frac{d}{dt} \left( \|\Delta w(t)\|_2^2 + \|w_t(t)\|_2^2 \right) + \frac{c_1}{2} \int_\Omega \left[ |w_t(t)|^\gamma + |v_t(t)|^\gamma \right] |w_t(t)|^2 \, dx \]
\[ \leq C (1 + \|v_t(t)\|_2^{\gamma + 2})(\|\Delta w(t)\|_2^2 + \|w_t(t)\|_2^2), \] (65)
for any \( t \in [0, T] \) and some \( C = C(\|z_0\|_H, \|z_0^2\|_H) > 0 \). Again from estimate (34) we see that \( \beta(t) = 1 + \|v_t(t)\|_2^{\gamma + 2} \) has the property \( \beta \in L^1(0, T) \). Thus, integrating (65) on \([0, t]\) and applying Gronwall’s inequality we arrive at
\[ \|\Delta w(t)\|_2^2 + \|w_t(t)\|_2^2 \leq C \int_0^t \beta(s) \, ds \left( \|\Delta w(0)\|_2^2 + \|w_t(0)\|_2^2 \right), \] (66)
for some constant \( C = C(\|z_0\|_H, \|z_0^2\|_H) > 0 \). Thereby, estimate (24) follows for strong solutions keeping in mind that \( (w, w_t) = z_1^1 - z_2^1 \) and taking \( C_0 = C/2 \). In particular, we have uniqueness of strong solution by taking \( z_0^1 = z_0^2 \).

Employing density arguments the same conclusion also holds true for the difference of weak solutions since they were obtained by the limit transition from strong solutions. Indeed, given weak initial conditions \( z_0^1 = (u_0, u_1), z_0^2 = (v_0, v_1) \in H \), let us consider a sequence of regular initial data \( z_0^{1,m} = (u_0^m, u_1^m), z_0^{2,m} = (v_0^m, v_1^m) \in H \).
\[ W_4 \times (W_2 \cap L^{2(\gamma+1)}(\Omega)) \]

such that

\[ (z_0^{1,m}, z_0^{2,m}) \rightarrow (z_0^1, z_0^2) \text{ strongly in } \mathcal{H} \times \mathcal{H}, \quad (67) \]

and the respective regular solutions \( z^{1,m} = (u^m, u_t^m), z^{2,m} = (v^m, v_t^m) \) converging to the weak solutions \( z^1 = (u, u_t), z^2 = (v, v_t) \) as in (52)-(53), namely,

\[ (z^{1,m}, z^{2,m}) \rightarrow (z^1, z^2) \text{ strongly in } C([0, T], \mathcal{H} \times \mathcal{H}). \quad (68) \]

Since (24) holds for strong solutions we conclude

\[ \|z^{1,m}(t) - z^{2,m}(t)\|_\mathcal{H} \leq C_0 \int_0^t \beta^m(s) \, ds \|z_0^{1,m} - z_0^{2,m}\|_\mathcal{H}, \quad \forall \, t \in [0, T], \ m \in \mathbb{N}, \quad (69) \]

where \( C_0 = C_0(\|z_0^1\|_\mathcal{H}, \|z_0^2\|_\mathcal{H}) > 0 \) and \( \beta^m(s) = 1 + ||v_t^m(s)||_{\mathcal{H}}^2 \). Therefore, (24) is ensured for weak solutions after passing the limit in (69) and using (67)-(68).

In addition, since the energy identity (60) remains valid for weak solutions using smooth mollifiers, see e.g. Lions and Magenes [26, Lemma 8.3] (see also Yang [37, Lemma 3.2]), then we also have uniqueness of weak solution performing the same estimates as above.

The proof of Theorem 2.1 is now complete.

**Remark 4.** The same conclusions (i)-(iv) of Theorem 2.1 can be shown by changing assumption (\( A_1 \)) to the following:

\[ (A_5) \, f \text{ is a } C^2 \text{-function on } \mathbb{R} \text{ satisfying} \]

\[ |f''(s)| \leq c f^\rho(1 + |s|^{\rho-1}), \quad \forall \, s \in \mathbb{R}, \quad (70) \]

\[ f'(s) \geq -\alpha_1, \quad \hat{f}(s) \leq f(s) + \frac{\alpha_1}{2} s^2, \quad \forall \, s \in \mathbb{R}, \quad (71) \]

where we consider \( c f^\rho > 0, \ 0 < \alpha_1 < \lambda_1 \), and

\[ \gamma \geq \rho \geq 1 \text{ if } 1 \leq n \leq 4 \text{ or } 1 \leq \rho \leq \gamma \leq \frac{8}{n-4} \text{ if } n \geq 5. \quad (72) \]

Condition (72) shows that we can extend the growth \( \rho \) of the source \( f(u) \) up to the growth \( \gamma \) corresponding to the damping term \( g(u_t) \). However, in spite of providing a better range for \( \rho \), it is worth noting that \((A_5)\) requires hypotheses on the second derivative of \( f \) with growth dominated by the growth of \( g \). Summarizing, assumptions \((A_2)-(A_5)\) are sufficient to conclude well-posedness to problem (1)-(4) as stated in Theorem 2.1. The proof can be done under minor modifications only in the proof of \textit{A Priori Estimate II} and item (iv) of Theorem 2.1. The precise details are similar to those ones given by Yang [37, Theorem 2.1].

3. Long-time dynamics.

3.1. \textbf{Generation of a dynamical system}. Let us define the family of nonlinear evolution operators \( S_\kappa(t) : \mathcal{H} \rightarrow \mathcal{H} \) given by

\[ S_\kappa(t)(u_0, u_t) = (u(t), u_t(t)), \quad t \geq 0, \quad (73) \]

where \((u, u_t)\) is the unique weak solution of (1)-(4) given by Theorem 2.1. Thus, in view of assumptions \((A_1)-(A_4)\) or \((A_2)-(A_5)\) we have a well defined family of dynamical systems \((\mathcal{H}, S_\kappa(t))\), \( \kappa \geq 0 \), possessing the following properties:

\textbf{• (Gradient System)} the energy relation (22) implies that \((\mathcal{H}, S_\kappa(t)), \, \kappa \geq 0 \), is a gradient dynamical system with strict Lyapunov functional given by the corresponding energy \( E_\kappa(t) \) defined in (23);

\textbf{• (Lipschitz Property)} from (24) the evolution semigroup \( S_\kappa(t) \) satisfies a locally Lipschitz property on the phase space \( \mathcal{H} \) for every parameter \( \kappa \geq 0 \).
In what follows our main goal is to provide additional properties to the dynamical system defined in (73). Then we establish our results on attractors and their qualitative properties. To do so we first show some technical results on stability to the solution of problem (1)-(4).

3.2. Technical results. We also assume the following assumption on the nonlocal coefficient \( M \) of extensibility.

(A\(_6\)) \( M \) and \( \hat{M} \) satisfy the relation

\[
\kappa \hat{M}(s) \leq 2\kappa M(s)s + \alpha_2 s, \quad \forall \ s \in [0, +\infty).
\]  

(74)

Proposition 1. Under assumptions of Theorem 2.1, we additionally suppose that (A\(_6\)) is in place and define

\[
\hat{E}_k(t) = E_k(t) + \frac{R}{8}, \quad t \geq 0, \quad R := 8 \left( c_f |\Omega| + \frac{1}{\alpha_1} \|h\|_0^2 \right).
\]  

(75)

If we consider a weak solution \( z = (u, u_t) \) of (1)-(4) corresponding to initial data \( (u_0, u_1) \in B \), where \( B \subset H \) is an arbitrary bounded set, then there exists a constant \( \mu = \mu_B > 0 \) (depending on the size of \( B \)) such that

\[
\hat{E}_k(t) \leq \left( \frac{\gamma}{2\mu} (t - 1)^+ + [\hat{E}_k(0)]^{-\gamma/2} \right)^{2/\gamma} + R \quad \text{if} \quad \gamma > 0,
\]  

(76)

and

\[
\hat{E}_k(t) \leq \hat{E}_k(0) \left( \frac{1 + \mu}{\mu} \right) e^{-\theta t} + R \quad \text{if} \quad \gamma = 0,
\]  

(77)

for any \( t > 0 \), where we denote \( s^+ = (s + |s|)/2 \) and \( \theta = \ln \left( \frac{1+\mu}{\mu} \right) > 0 \).

Proof. We first note that a similar procedure as shown in (31) leads to

\[
\hat{E}_k(t) \geq \frac{1}{2} \|u_t(t)\|_2^2 + \frac{\alpha}{4} \|\Delta u(t)\|_2^2 \geq \frac{\alpha}{4} \|z(t)\|_{\hat{H}}^2, \quad t \geq 0.
\]  

(78)

Further, in view of (58) and observing that \( \frac{d}{dt} E_k(t) = \frac{d}{dt} \hat{E}_k(t) \), we get

\[
\frac{d}{dt} \hat{E}_k(t) + C_N \int_\Omega |u_t(t)|^{\gamma+2} dx \leq 0, \quad t > 0,
\]  

(79)

where \( C_N = C_N(B) > 0 \). Integrating (79) from \( t \) to \( t+1 \), and defining the following auxiliary functional

\[
[W(t)]^2 := \hat{E}_k(t) - \hat{E}_k(t+1) = E_k(t) - E_k(t+1),
\]  

(80)

we obtain

\[
\int_t^{t+1} \int_\Omega |u_t(s)|^{\gamma+2} dx ds \leq \frac{1}{C_N} [W(t)]^2.
\]  

(81)

From Hölder inequality with \( \frac{1}{\gamma+2} + \frac{2}{\gamma+2} = 1 \) and (81), we deduce

\[
\int_t^{t+1} \int_\Omega |u_t(s)|^2 dx ds \leq |\Omega|^{\frac{2}{\gamma+2}} \left( \int_t^{t+1} \int_\Omega |u_t(s)|^{\gamma+2} dx ds \right)^{\frac{2}{\gamma+2}} \leq \left( |\Omega|^{\gamma} C_N^{-2} \right)^{\frac{1}{\gamma+2}} [W(t)]^{\frac{1}{\gamma+2}},
\]  

(82)
and using the Mean Value Theorem for integrals there exist \( t_1 \in [t, t + \frac{1}{4}] \) and \( t_2 \in [t + \frac{3}{4}, t + 1] \) such that
\[
\|u(t_i)\|_2^2 \leq 4 \left( \frac{\lambda_1}{\Omega}\right)^{\frac{1}{p+1}} W(t_i)^{\frac{2}{p+1}}, \quad i = 1, 2. \tag{83}
\]

On the other hand, taking the multiplier \( u \) in (1) we infer
\[
\|\Delta u(t)\|_2^2 + \kappa M(\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2 + \int_{\Omega} f(u(t))u(t) ~dx - \int_{\Omega} hu(t) ~dx \tag{84}
\]
\[
= \|u(t)\|_2^2 - \frac{d}{dt} \int_{\Omega} u(t)u(t) ~dx - N(\|\nabla u(t)\|_2^2) \int_{\Omega} g(u(t))u(t) ~dx.
\]

Using second inequality of (10) (which corresponds to second one in (71)), condition (74), and observing (8) and (15), we have
\[
\|\Delta u(t)\|_2^2 + \kappa M(\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2 + \int_{\Omega} f(u(t))u(t) ~dx \geq \frac{\kappa}{2} M(\|\nabla u(t)\|_2^2) + \int_{\Omega} \hat{f}(u(t)) ~dx + \left(1 + \frac{\alpha}{2}\right)\|\Delta u(t)\|_2^2.
\]

Replacing it in (84) and noting that \( \alpha > 0 \) we deduce
\[
\frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{\kappa}{2} M(\|\nabla u(t)\|_2^2) + \int_{\Omega} \hat{f}(u(t)) ~dx - \int_{\Omega} hu(t) ~dx \leq \|u(t)\|_2^2 - \frac{d}{dt} \int_{\Omega} u(t)u(t) ~dx - N(\|\nabla u(t)\|_2^2) \int_{\Omega} g(u(t))u(t) ~dx. \tag{85}
\]

Adding \( \frac{1}{2}\|u(t)\|_2^2 + \frac{R}{8} \) in both sides of (85) and using definitions given in (23) and (75) we obtain
\[
\hat{E}_\kappa(t) \leq \frac{3}{2}\|u(t)\|_2^2 - \frac{d}{dt} \int_{\Omega} u(t)u(t) ~dx - N(\|\nabla u(t)\|_2^2) \int_{\Omega} g(u(t))u(t) ~dx + \frac{R}{8}.
\]

Integrating the above inequality on \([t_1, t_2]\), and since \( t_2 - t_1 < 1 \), it follows that
\[
\int_{t_1}^{t_2} \hat{E}_\kappa(s) ~ds \leq \frac{3}{2}\int_{t_1}^{t_2} \|u(s)\|_2^2 ~ds + J_1 + J_2 + \frac{R}{8}, \tag{86}
\]

where
\[
J_1 = -\int_{\Omega} u(t_2)u(t_2) ~dx + \int_{\Omega} u(t_1)u(t_1) ~dx,
\]
\[
J_2 = -\int_{t_1}^{t_2} N(\|\nabla u(s)\|_2^2) \int_{\Omega} g(u(s))u(s) ~dx ~ds.
\]

Now let us estimate the right hand side of (86) in terms of the functional \( W(t) \) defined in (80). Indeed, using Cauchy-Schwarz inequality and (83), and then (78) and Young inequality with \( \epsilon = 1/8 \), we have
\[
|J_1| \leq \frac{4}{\lambda_1^{1/2}} \left( \frac{\Omega}{\lambda_1^{1/2}} \right)^{\frac{1}{p+1}} W(t)^{\frac{2}{p+1}} \left[ \sup_{t \leq s \leq t+1} \|\Delta u(s)\|_2 \right]^{\frac{2}{p+1}} \left[ \sup_{t \leq s \leq t+1} \hat{E}_\kappa(s) \right]^{1/2}
\]
\[
\leq \frac{8}{\Omega} \left( \frac{\Omega}{\lambda_1^{1/2}} \right)^{\frac{1}{p+1}} W(t)^{\frac{2}{p+1}} \left[ \sup_{t \leq s \leq t+1} \hat{E}_\kappa(s) \right]^{1/2}
\]
\[
\leq \mu_0 W(t)^{4/4} + \frac{1}{8} \sup_{t \leq s \leq t+1} \hat{E}_\kappa(s),
\]
where \( \mu_0 = \mu_0(B) > 0 \) depends on the size of \( B \subset \mathcal{H} \). From (8), (78)-(79) and since \((u_0, u_1) \in B\), then \( \|\nabla u(t)\|_2^2 \leq C(B) < \infty \) for all \( t \geq 0 \), and from continuity of the function \( N \) results

\[
N \left( \|\nabla u(t)\|_2^2 \right) \leq C_0(B) < \infty.
\]

Using assumption (12), Hölder inequality, estimates (81)-(82), embeddings \( W_2 \hookrightarrow L^{\gamma + 2}(\Omega) \) and \( W_2 \hookrightarrow L^2(\Omega) \) along with (78), and Young inequality with \( \epsilon = 1/8 \), we obtain

\[
|\mathcal{J}_2| \leq C_0C_{\phi'} \int_{t_1}^{t_2} \int_{\Omega} |u_t(s)|^{\gamma + 1} |u(s)| \, dx \, ds + C_0C_{\phi'} \int_{t_1}^{t_2} \int_{\Omega} |u_t(s)||u(s)| \, dx \, ds
\]

\[
\leq C_0C_{\phi'} \left( \int_{t_1}^{t_2} \int_{\Omega} |u_t(s)|^{\gamma + 2} \, dx \, ds \right)^{\frac{\gamma + 2}{\gamma + 3}} \left( \int_{t_1}^{t_2} \int_{\Omega} |u(s)|^{\gamma + 2} \, dx \, ds \right)^{\frac{\gamma + 2}{\gamma + 3}}
\]

\[
+ C_0C_{\phi'} \left( \int_{t_1}^{t_2} \int_{\Omega} |u_t(s)|^2 \, dx \, ds \right)^{\frac{\gamma + 2}{2}} \left( \int_{t_1}^{t_2} \int_{\Omega} |u(s)|^2 \, dx \, ds \right)^{\frac{\gamma + 2}{2}}
\]

\[
\leq C_0C_{\phi'} \frac{[W(t)]^\frac{\gamma + 2}{\gamma + 3}}{C_N} \sup_{t \leq s \leq t + 1} \|u(s)\|_{\gamma + 2}
\]

\[
+ C_0C_{\phi'} \left( \frac{\Omega}{\gamma} C_N^{-2} \right)^\frac{\gamma + 2}{2(\gamma + 2)} [W(t)]^\frac{\gamma + 2}{2} \sup_{t \leq s \leq t + 1} \|u(s)\|_2
\]

\[
\leq \mu_1 \left( [W(t)]^\frac{4(\gamma + 1)}{\gamma + 2} + [W(t)]^\frac{4}{\gamma + 2} \right) + \frac{1}{8} \sup_{t \leq s \leq t + 1} \hat{E}_k(s),
\]

for some constant \( \mu_1 = \mu_1(B) > 0 \) depending on the size of \( B \subset \mathcal{H} \). Replacing the above estimates for \( |\mathcal{J}_1| \) and \( |\mathcal{J}_2| \) in (86), and using again (82), we obtain

\[
\int_{t_1}^{t_2} \hat{E}_k(s) \, ds \leq \mu_2 \left( [W(t)]^\frac{\gamma + 2}{\gamma + 2} + [W(t)]^\frac{4(\gamma + 1)}{\gamma + 2} \right) + \frac{1}{4} \sup_{t \leq s \leq t + 1} \hat{E}_k(s) + \frac{R}{8},
\]

(87)

for some constant \( \mu_2 = \mu_2(B) > 0 \) depending on \( B \subset \mathcal{H} \). Now, applying again the Mean Value Theorem there exists \( \tau \in [t_1, t_2] \) such that

\[
\int_{t_1}^{t_2} \hat{E}_k(s) \, ds = \hat{E}_k(\tau)(t_2 - t_1) \geq \frac{1}{2} \hat{E}_k(t + 1),
\]

where we use that \( \hat{E}_k(t) \) is non-increasing as well as \( E_k(t) \), and from (80) we arrive at

\[
\hat{E}_k(t) \leq W(t)^2 + 2 \int_{t_1}^{t_2} \hat{E}_k(s) \, ds.
\]

(88)

Combining (87) and (88) we get

\[
\hat{E}_k(t) \leq 2[W(t)]^2 + 4\mu_2 \left( [W(t)]^\frac{\gamma + 2}{\gamma + 2} + [W(t)]^\frac{4(\gamma + 1)}{\gamma + 2} \right) + \frac{R}{2},
\]

(89)

where we use again that \( \hat{E}_k(t) \) is non-increasing. In addition, observing that \( \frac{4(\gamma + 1)}{\gamma + 2} < 2 \) (with equality if \( \gamma = 0 \)), we rewrite (89) as

\[
\hat{E}_k(t) \leq [W(t)]^\frac{\gamma + 2}{\gamma + 2} \left( 4\mu_2 + 2[W(t)]^\frac{2}{\gamma + 2} + 4\mu_2[W(t)]^\frac{4}{\gamma + 2} \right) + \frac{R}{2}.
\]

(90)

In view of (79)-(80), and since \((u_0, u_1) \in B\), there exists a constant \( \mu_3 = \mu_3(B) > 0 \) depending on \( B \subset \mathcal{H} \) such that

\[
\left( 4\mu_2 + 2[W(t)]^\frac{2}{\gamma + 2} + 4\mu_2[W(t)]^\frac{4}{\gamma + 2} \right) \leq \mu_3(B) < \infty, \quad t \geq 0.
\]
From this and using again the identity (80), then (90) can be rewritten as
\[
[E_k(t)]^{1+\frac{2}{\gamma}} \leq \mu(B)[E_k(t) - E_k(t + 1)] + R^{1+\frac{2}{\gamma}},
\]
where \(\mu = (2\mu_0)^{\frac{2}{\gamma}}\). Thus, applying Nakao’s Lemma (cf. [32, Lemma 2.1]), we have
\[
E_k(t) \leq \left(\frac{\gamma}{2\mu}(t - 1)^+ + [E_k(0)]^{-\gamma/2}\right)^{-2/\gamma} + R \quad \text{if} \quad \gamma > 0,
\]
and
\[
E_k(t) \leq E_k(0) \left(\frac{1 + \mu}{\mu}\right)e^{-\theta t} + R \quad \text{if} \quad \gamma = 0,
\]
for any \(t > 0\), where \(\mu = \mu(B) > 0\), \(\theta = \ln\left(\frac{1+\mu}{\mu}\right) > 0\). This concludes the proof of the desired estimates (76) and (77).

**Corollary 1. (Energy decay)** Under assumptions of Proposition 1 with \(h = 0\) and \(c_f = 0\) in (10), then the energy \(E_k(t)\) defined in (23) satisfies the following decay rates:

(i) if \(\gamma > 0\), then there exists a constant \(C = C(||(u_0, u_1)||_{\mathcal{H}}) > 0\) such that
\[
E_k(t) \leq C(1 + t)^{-\frac{\theta}{\gamma}} \forall \ t > 0;
\]
(ii) if \(\gamma = 0\), then there exist constants \(C, \beta > 0\) depending on \(||(u_0, u_1)||_{\mathcal{H}}||\) such that
\[
E_k(t) \leq Ce^{-\beta t}, \quad \forall \ t > 0.
\]

**Proof.** The proof is an immediate consequence of Proposition 94 by noting that (91) holds with \(R = 0\) and \(E_k(t) = E_k(t)\) when one takes \(h = 0\) and \(c_f = 0\). Therefore, conclusions (92)-(93) follows from Nakao’s Lemma (cf. Nakao [30, 31]).

**Corollary 2. (Absorbing set)** Under assumptions of Proposition 1, let us consider any bounded set \(B \subset \mathcal{H}\). If \((u_0, u_1) \in B\), then there exists \(t_B > 0\) such that
\[
||(u(t), u_t(t))||_{\mathcal{H}} \leq r, \quad \forall \ t > t_B,
\]
where \((u(t), u_t(t)) = S_k(t)(u_0, u_1)\) is the weak solution of problem (1)-(4) and \(r > 0\) is a constant independent of \((u_0, u_1)\). In particular, the set
\[
B = \{(u, v) \in \mathcal{H}; ||(u, v)||_{\mathcal{H}} \leq r\}
\]
is a bounded absorbing set for \(S_k(t)\) defined in (73). In other words, the dynamical system \((\mathcal{H}, S_k(t))\) is dissipative.

**Proof.** For initial data \((u_0, u_1)\) lying in \(B\) we obtain from estimates (76) and (77) that there exists \(t_B > 0\) dependent of \(B \subset \mathcal{H}\) such that
\[
E_k(t) \leq 2R, \quad \forall \ t > t_B.
\]
From (78) one sees that (94) follows by taking \(r = 2 \left[\frac{2R}{\alpha}\right]^{1/2} > 0\).

**Remark 5.** From (20) and (94) one sees that the solution \((u, u_t)\) of (1)-(4) corresponding to an initial data \((u_0, u_1)\) lying in bounded sets \(B \subset \mathcal{H}\) is globally bounded in \(\mathcal{H}\), that is,
\[
||(u(t), u_t(t))||_{\mathcal{H}} \leq C_B, \quad \forall \ t \geq 0,
\]
for some constant $C_B > 0$ depending on $B$. Moreover, going back to (22) we infer
\[
\limsup_{t \to +\infty} \left[ \int_0^t \|u_i(\tau)\|^2 \gamma_2^2 d\tau \right] \leq C_B, \quad \gamma \geq 0,
\] (96)
for some constant $C_B > 0$.

**Proposition 2.** Under assumptions of Proposition 1, let us consider a bounded set $B \subset \mathcal{H}$ and $z_i(t) = (u^i(t), u^i_1(t)), i = 1, 2$, two weak solutions of problem (1)-(4) with $z_1(0) = (u_0^1, u_1^1) \in B$. Let us also denote $w(t) := u^1(t) - u^2(t)$ and
\[
E_w(t) := \|w_1(t)\|^2 + \|\Delta w(t)\|^2 + \kappa M(\|\nabla u^1(t)\|_2^2)\|\nabla w(t)\|_2^2.
\] (97)
(i) If $\gamma > 0$, then there exist constants $C_B, C'_B > 0$ (depending on $B$) such that
\[
\alpha \|z^1(t) - z^2(t)\|_H^2 \leq \left( \frac{\gamma}{2C_B} - (t - 1)^+ + \left( \sup_{0 \leq s \leq 1} E_w(s) \right)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} + C'_B \sup_{t \leq s \leq t+1} \left[ \|\nabla w(s)\|_2^2 + \|w(s)\|_2^2 \right]^{\frac{1}{2}},
\] (98)
for any $t > 0$, where $s^+ = (s + |s|)/2$ and $\alpha$ is given in (15).
(ii) If $\gamma = 0$, then there exist constants $C_B, C'_B > 0$ (depending on $B$) such that
\[
\alpha \|z^1(t) - z^2(t)\|_H^2 \leq \left( \frac{1 + C_B}{C_B} \right) \sup_{0 \leq s \leq 1} E_w(s) e^{-\theta t} + C'_B \sup_{t \leq s \leq t+1} \left[ \|\nabla w(s)\|_2^2 + \|w(s)\|_2^2 \right]^{\frac{1}{2}},
\] (99)
for any $t > 0$, where $\theta = \ln \left( \frac{1 + C_B}{C_B} \right) > 0$ and $\alpha$ is given in (15).

**Proof.** We start by noting the following equivalence
\[
E_w(t) \sim \|z_1(t) - z_2(t)\|_H^2.
\]
Indeed, from condition (14) it follows that
\[
E_w(t) \geq \|w_1(t)\|^2 + \alpha \|\Delta w(t)\|^2 \geq \alpha \|z_1(t) - z_2(t)\|_H^2.
\] (100)
On the other hand, using (63) and embedding $W_2 \hookrightarrow W_1$, we infer
\[
E_w(t) \leq \|w_1(t)\|^2 + C_{0,B} \|\Delta w(t)\|^2 \leq C_{1,B} \|z_1(t) - z_2(t)\|_H^2.
\] (101)
In what follows we proof (98)-(99) for $E_w(t)$ and then use the above equivalence. Firstly, proceeding as in the proof of Theorem 2.1 (iv), the difference $z_1 - z_2 = (w, w_t)$ satisfies
\[
w_{tt} + \Delta^2 w - \kappa M(\|\nabla u^1(t)\|_2^2)\Delta w + N(\|\nabla u^1(t)\|_2^2) \left[ g(u^1_t) - g(u^2_t) \right]
= \kappa \Psi_M \Delta u^2 - \Psi_N g(u^2_t) - f(u^1) - f(u^2),
\] (102)
with initial condition $(w(0), w_t(0)) = z_0^1 - z_0^2$, where we denote
\[
\Psi_J = J(\|\nabla u^1(t)\|_2^2) - J(\|\nabla u^2(t)\|_2^2) \quad \text{for} \quad J = M, N.
Taking the multiplier \( w_t \) in (102) we get
\[
\frac{1}{2} \frac{d}{dt} E_w(t) + I = \kappa M' (\| \nabla u^1(t) \|_2^2) (\Delta u^1(t), u^1_t(t)) \| \nabla w(t) \|_2^2 \\
+ \kappa \Psi_M \int_\Omega \Delta u^2(t) w_t(t) \, dx - \Psi_N \int_\Omega g(u^2_t(t)) w_t(t) \, dx
\]
\[
- \int_\Omega [f(u^1(t)) - f(u^2(t))] w_t(t) \, dx,
\]
where
\[
I = N (\| \nabla u^1(t) \|_2^2) \int_\Omega [g(u^1_t(t)) - g(u^2_t(t))] w_t(t) \, dx.
\]
Using assumptions (A2)-(A3) and proceeding analogously to (61) we have
\[
I \geq c_N c_\gamma c_\gamma \int_\Omega [\| u^1_t(t) \|_\gamma + \| u^2_t(t) \|_\gamma] |w_t(t)|^2 \, dx \geq C_0 \int_\Omega \| w_t(t) \|_{\gamma+2}^2 \, dx,
\]
where we use \(|a| \gamma + |b| \gamma \geq 2^{-\gamma} |a - b| \gamma \) in the last inequality and take \( C_0 = 2^{-\gamma} c_N c_\gamma c_\gamma > 0 \) (which depends on initial data). Replacing (104) in (103) we obtain
\[
\frac{1}{2} \frac{d}{dt} E_w(t) + C_0 \| w_t(t) \|_{\gamma+2}^\gamma \leq \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4,
\]
where
\[
\Theta_1 = \kappa \Psi_M \int_\Omega \Delta u^2(t) w_t(t) \, dx,
\]
\[
\Theta_2 = \kappa M' (\| \nabla u^1(t) \|_2^2) (\Delta u^1(t), u^1_t(t)) \| \nabla w(t) \|_2^2,
\]
\[
\Theta_3 = - \Psi_N \int_\Omega g(u^2_t(t)) w_t(t) \, dx,
\]
\[
\Theta_4 = - \int_\Omega [f(u^1(t)) - f(u^2(t))] w_t(t) \, dx.
\]
Now we estimate the terms \( \Theta_1, \Theta_2, \Theta_3, \Theta_4 \) as follows. To do so we consider \( \eta > 0 \) given. From (63)-(64), embedding \( L^{\gamma+2}(\Omega) \hookrightarrow W_0 \) and Young inequality with \( \gamma+2 \) and \( \frac{\gamma+2}{\gamma+2} = 1 \), we have
\[
|\Theta_1| \leq C_{2,B} \| \nabla w(t) \|_2 \| w_t(t) \|_{\gamma+2} \leq C_{\eta,B} \| \nabla w(t) \|_2 \| w_t(t) \|_{\gamma+2}^\gamma + \eta \| w_t(t) \|_{\gamma+2}^\gamma.
\]
Also,
\[
|\Theta_2| \leq C_{3,B} \| \nabla w(t) \|_2^2 \leq C_{3,B} \| \nabla w(t) \|_2 \| \nabla w(t) \|_2 \| w_t(t) \|_{\gamma+2}^\gamma \leq C_{4,B} \| \nabla w(t) \|_2 \| w_t(t) \|_{\gamma+2}^\gamma.
\]
In addition, using (63)-(64), assumption (12), Hölder and Young inequalities, we get
\[
|\Theta_3| \leq C''_{\eta,B} (1 + \| u^2_t(t) \|_{\gamma+2}^\gamma) \| \nabla w(t) \|_2 \| w_t(t) \|_{\gamma+2}^\gamma + \eta \| w_t(t) \|_{\gamma+2}^\gamma.
\]
Further, using Hölder inequality with \( \frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1 \), embedding \( W_2 \hookrightarrow L^{2(\rho+1)}(\Omega) \) and Young inequality, we obtain
\[
|\Theta_4| \leq C_5 [1 + \| u^1_t(t) \|_{2(\rho+1)}^\rho + \| u^2(t) \|_{2(\rho+1)}^\rho] \| w(t) \|_{2(\rho+1)} \| w_t(t) \|_2 \| w_t(t) \|_2 \| w_t(t) \|_{\gamma+2}^\gamma
\leq C_{6,B} \| w(t) \|_{2(\rho+1)} \| w_t(t) \|_2 \| w_t(t) \|_{\gamma+2}^\gamma
\leq C''_{\rho,B} \| w(t) \|_{2(\rho+1)} \| w(t) \|_2 \| w_t(t) \|_{\gamma+2}^\gamma + \eta \| w_t(t) \|_{\gamma+2}^\gamma.
Replacing these four last estimates in (105) and choosing \( \eta = C_0 \), we obtain
\[
\frac{d}{dt} E_w(t) + C_0 \| w_i(t) \|_{\gamma+2}^{\gamma+2} \leq C_{7,B} \left( 1 + \| u_i^2(t) \|_{\gamma+2}^{\gamma+2} \right) \left[ \| \nabla w(t) \|_2^{\gamma+2} + \| w(t) \|_{2(\rho+1)}^{\gamma+2} \right].
\]
(106)

Integrating (106) from \( t \) to \( t+1 \), results
\[
C_0 \int_t^{t+1} \| w_i(s) \|_{\gamma+2}^{\gamma+2} ds \leq E_w(t) - E_w(t+1)
+ C_{7,B} \int_t^{t+1} \left( 1 + \| u_i^2(s) \|_{\gamma+2}^{\gamma+2} \right) ds \sup_{t \leq s \leq t+1} \left[ \| \nabla w(s) \|_2^{\gamma+2} + \| w(s) \|_{2(\rho+1)}^{\gamma+2} \right].
\]
Using that \( 1 + \| u_i^2 \|_{\gamma+2}^{\gamma+2} \in L^1(0,T) \), we have
\[
C_0 \int_t^{t+1} \| w_i(s) \|_{\gamma+2}^{\gamma+2} ds \leq E_w(t) - E_w(t+1)
+ C_{8,B} \sup_{t \leq s \leq t+1} \left[ \| \nabla w(s) \|_2^{\gamma+2} + \| w(s) \|_{2(\rho+1)}^{\gamma+2} \right] := [W(t)]^2.
\]
(107)

From (107) and Hölder inequality, we obtain
\[
\int_t^{t+1} \int_\Omega |w_i(t)|^2 dxdt \leq C_9 [W(t)]^{\frac{4}{\gamma+2}},
\]
(108)
with \( C_9 = |\Omega|^{1/\gamma+2}/C_0^{\frac{4}{\gamma+2}} \). From the Mean Value Theorem for integrals there exist \( t_1 \in [t, t + \frac{1}{2}] \) and \( t_2 \in [t + \frac{3}{4}, t+1] \) such that
\[
\| w_i(t_1) \|_{\Omega}^2 \leq 4C_9 [W(t)]^{\frac{4}{\gamma+2}}, \quad i = 1, 2.
\]
(109)

Now, multiplying (102) by \( w(t) \) and integrating over \( \Omega \times [t_1, t_2] \), we get
\[
\int_{t_1}^{t_2} \left[ |\Delta w_i(s)|_2^2 ds + \kappa M (|\nabla u^1(s)|_2^2) |\nabla w_i(s)|_2^2 \right] ds = \int_{t_1}^{t_2} \| w_i(s) \|_2^2 ds + J_1 + J_2 + J_3 + J_4 + J_5,
\]
(110)
where
\[
\begin{align*}
J_1 &= \int_\Omega [w_i(t_1)w(t_1) - w_i(t_2)w(t_2)] dx, \\
J_2 &= - \int_{t_1}^{t_2} N(|\nabla u^1(s)|_2^2) \int_\Omega \left[ g(u^1_i(s)) - g(u^1_i(s)) \right] w(s)dxds, \\
J_3 &= - \int_{t_1}^{t_2} \Psi_N \int_\Omega g(u^1_i(s))w(s)dxds, \\
J_4 &= - \int_{t_1}^{t_2} \int_\Omega \left[ f(u^1(s)) - f(u^2(s)) \right] w(s)dxds, \\
J_5 &= \kappa \int_{t_1}^{t_2} \Psi_M \int_\Omega \Delta u^2(s)w(s)dxds.
\end{align*}
\]
Let us estimate the terms \( J_1, \ldots, J_5 \). Initially, from (109), we have
\[
J_1 \leq C_{10} [W(t)]^{\frac{4}{\gamma+2}} + \frac{1}{12} \sup_{t \leq s \leq t+1} E_w(s).
\]
From the Mean Value Theorem, Hölder inequality with \( \frac{2}{\gamma+2} + \frac{1}{\gamma+2} + \frac{1}{\gamma+2} = 1 \) and embedding \( W_2 \hookrightarrow L^{\gamma+2}(\Omega) \), we obtain

\[
|J_2| \leq C_{11,B} \left( \int_{t_1}^{t_2} \int_{\Omega} |w|^{\gamma+2} dx ds \right)^{\frac{1}{\gamma+2}} \left( \int_{t_1}^{t_2} \int_{\Omega} |w|^{\gamma+2} dx ds \right)^{\frac{1}{\gamma+2}} \\
\leq C_{12,B}[W(t)]^{\frac{2}{\gamma+2}} \sup_{t \leq s \leq t+1} \|w(s)\|_{\gamma+2} \\
\leq C_{13,B}[W(t)]^{\frac{2}{\gamma+2}} \sup_{t \leq s \leq t+1} \|\Delta w(s)\|_2 \\
\leq C_{14,B}[W(t)]^{\frac{1}{\gamma+2}} + \frac{1}{12} \sup_{t \leq s \leq t+1} E_w(s)
\]

and

\[
|J_3| \leq C_{15,B} \sup_{t \leq s \leq t+1} \|\nabla w(s)\|_2 \left( \int_{t_1}^{t_2} \int_{\Omega} |w|^{\gamma+2} dx ds \right)^{\frac{1}{\gamma+2}} \\
\leq C_{16,B} \sup_{t \leq s \leq t+1} \|\nabla w(s)\|_2^2 + \frac{1}{12} \sup_{t \leq s \leq t+1} E_w(s).
\]

In addition, from Hölder inequality with \( \frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1 \) and embedding \( L^{2(\rho+1)}(\Omega) \hookrightarrow W_0 \), we have

\[
|J_4| \leq C_{17,B} \int_{t_1}^{t_2} \left[ \|u\|_{2(\rho+1)}^\rho + \|v\|_{2(\rho+1)}^\rho \right] \|\Delta w\|_2 \|w\|_2 ds \\
\leq C_{18,B} \sup_{t \leq s \leq t+1} \|w(s)\|_{2(\rho+1)}^2.
\]

Finally, since \( M \in C^1([0, +\infty)) \) and \( W_1 \hookrightarrow W_0 \), we infer

\[
|J_5| \leq C_{19,B} \int_{t_1}^{t_2} \|\nabla w(s)\|_2 \|w(s)\|_2 ds \leq C_{20,B} \sup_{t \leq s \leq t+1} \|\nabla w(s)\|_2^2.
\]

Substituting the above last five estimates for \( J_1, \ldots, J_5 \) in (110) and noting that

\[
\|w\|_{2\rho+1}^2 \leq C_B \|w\|_{2(\rho+1)}^{\frac{2+\rho}{\rho+1}}, \quad \|\nabla w\|_2^2 \leq C_B \|\nabla w\|_2^{\frac{2+\rho}{\rho+1}},
\]

we arrive at

\[
\int_{t_1}^{t_2} \left[ \|\Delta w(s)\|_2^2 ds + \kappa M(\|\nabla u_1(s)\|_2^2) \|\nabla w(s)\|_2^2 \right] ds \\
\leq C_{21,B}[W(t)]^{\frac{2}{\gamma+2}} + \frac{1}{4} \sup_{t \leq s \leq t+1} E_w(s) \\
+ C_{22,B} \sup_{t \leq s \leq t+1} \left[ \|\nabla w(s)\|_2^{\frac{2+\rho}{\rho+1}} + \|w(s)\|_{2(\rho+1)}^{\frac{2+\rho}{\rho+1}} \right],
\]

and from definition of \( E_w(t) \) it follows that

\[
\int_{t_1}^{t_2} E_w(s) ds \leq C_{21,B}[W(t)]^{\frac{2}{\gamma+2}} + \frac{1}{4} \sup_{t \leq s \leq t+1} E_w(s) \\
+ C_{22,B} \sup_{t \leq s \leq t+1} \left[ \|\nabla w(s)\|_2^{\frac{2+\rho}{\rho+1}} + \|w(s)\|_{2(\rho+1)}^{\frac{2+\rho}{\rho+1}} \right].
\]
From the Mean Value Theorem, there exists $t^* \in [t_1, t_2]$ such that
\begin{align}
E_w(t^*) &\leq 2C_{21,B}[W(t)]^{\frac{1}{\gamma + 2}} + \frac{1}{2} \sup_{t \leq s \leq t+1} E_w(s) \\
&+ 2C_{22,B} \sup_{t \leq s \leq t+1} \left[ \|\nabla w(s)\|_2^{\frac{\gamma + 2}{\gamma + 1}} + \|w(s)\|_2^{\frac{\gamma + 2}{(\rho + 1)} (\gamma + 1)} \right].
\end{align}
(111)

Let $\zeta \in [t, t+1]$ such that $E_w(\zeta) = \sup_{t \leq s \leq t+1} E_w(s)$. Integrating (106) over $[t, \zeta]$ and $[t^*, t+1]$, then a direct computation leads to
\begin{align}
\sup_{t \leq s \leq t+1} E_w(s) \leq E_w(t^*) + [W(t)]^2 + 2C_{8,B} \sup_{t \leq s \leq t+1} \left[ \|\nabla w(s)\|_2^{\frac{\gamma + 2}{\gamma + 1}} + \|w(s)\|_2^{\frac{\gamma + 2}{(\rho + 1)}} \right].
\end{align}
(112)

Inserting (111) in (112) we obtain
\begin{align}
\sup_{t \leq s \leq t+1} E_w(s) \leq 2[W(t)]^{\frac{1}{\gamma + 2}} \left[ 2C_{21,B} + [W(t)]^{2-\frac{1}{\gamma + 2}} \right] \\
+ C_{23,B} \sup_{t \leq s \leq t+1} \left[ \|\nabla w(s)\|_2^{\frac{\gamma + 2}{\gamma + 1}} + \|w(s)\|_2^{\frac{\gamma + 2}{(\rho + 1)}} \right] \\
\leq C_{24,B}[W(t)]^{\frac{4}{\gamma + 2}} + C_{23,B} \sup_{t \leq s \leq t+1} \left[ \|\nabla w(s)\|_2^{\frac{\gamma + 2}{\gamma + 1}} + \|w(s)\|_2^{\frac{\gamma + 2}{(\rho + 1)}} \right].
\end{align}
(113)

From (113) and using that $[a + b]^{\frac{\gamma + 2}{\gamma + 2}} \leq 2^{\frac{\gamma + 2}{\gamma + 2}} [a^{\frac{\gamma + 2}{\gamma + 2}} + b^{\frac{\gamma + 2}{\gamma + 2}}]$, results
\begin{align}
\sup_{t \leq s \leq t+1} [E_w(s)]^{\frac{\gamma + 2}{\gamma + 2}} \leq C_{25,B}[W(t)]^2 + C_{27,B} \sup_{t \leq s \leq t+1} \left[ \|\nabla w(s)\|_2^{\frac{\gamma + 2}{\gamma + 1}} + \|w(s)\|_2^{\frac{\gamma + 2}{(\rho + 1)}} \right],
\end{align}
where
\begin{align}
C_{27,B} &= C_{26,B} \sup_{t \leq s \leq t+1} \left[ \|\nabla w(s)\|_2^{\frac{\gamma + 2}{\gamma + 1}} + \|w(s)\|_2^{\frac{\gamma + 2}{(\rho + 1)}} \right].
\end{align}
(114)

Last, from (107) and (114) we conclude
\begin{align}
\sup_{t \leq s \leq t+1} [E_w(s)]^{\frac{\gamma + 2}{\gamma + 2}} &\leq C_B [E_w(t) - E_w(t+1)] \\
&+ C_B' \sup_{t \leq s \leq t+1} \left[ \|\nabla w(s)\|_2^{\frac{\gamma + 2}{\gamma + 1}} + \|w(s)\|_2^{\frac{\gamma + 2}{(\rho + 1)}} \right],
\end{align}
where $C_B = C_{25,B}$ and $C_B' = (C_{8,B} + C_{27,B})$. Therefore, regarding estimates (100)-(101) and applying Nakao’s Lemma (cf. [32, Lemma 2.1]), we conclude (98) and (99). This completes the proof of Proposition 2. \qed

As a consequence of Proposition 2, we show below that the dynamical system $(H, S_\kappa(t))$, $\kappa \geq 0$, defined in (73) is asymptotically smooth. To this end we use the following compactness criteria, see e.g. [10, 11, 19].

**Proposition 3** ([11], Theorem 7.1.11). Let $(H, S(t))$ be a dissipative dynamical system in a complete metric space $H$. Assume that for any bounded positively invariant set $B \subset H$ and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B) > 0$ such that
\begin{align}
\|S(T)z^1 - S(T)z^2\|_H \leq \varepsilon + \phi_T(z^1, z^2), \quad \forall z^1, z^2 \in B,
\end{align}
(115)
where $\phi_T : B \times B \to \mathbb{R}$ satisfies
\begin{align}
\liminf_{n \to \infty} \liminf_{m \to \infty} \phi_T(z^n, z^m) = 0,
\end{align}
(116)
for any sequence \((z^n)\) in \(B\). Then \((H, S(t))\) is an asymptotically smooth dynamical system.

**Corollary 3** (Asymptotic smoothness property). Let assumptions of Proposition 2 be in force. Then, in both cases \(\gamma > 0\) or \(\gamma = 0\), the dynamical system \((\mathcal{H}, S_n(t))\) is asymptotically smooth.

**Proof.** Given a bounded positively invariant set \(B \subset \mathcal{H}\), we denote by \(C_B > 0\) different constants depending on the size of \(B\) but not on \(t\). For \(z_1, z_2 \in B\) we need to show that \(S_n(t)z_i = (u^i(t), u^i_2(t))\), \(i = 1, 2\), satisfies (115)-(116). Indeed, given \(\varepsilon > 0\), from Proposition 2, inequalities (98)-(99), we can choose \(T > 0\) large enough such that

\[
|S_n(T)z_1 - S_n(T)z_2|_{\mathcal{H}} 
\leq \varepsilon + C_B \begin{cases} 
\sup_{0 \leq s \leq T} \left[ \|\nabla (u^1(s) - u^2(s))\|_2^{\frac{3+2}{\gamma}} + \|u^1(s) - u^2(s)\|_2^{\frac{3+2}{\gamma}} \right]^{\frac{\gamma}{3+2}}, \\
\sup_{0 \leq s \leq T} \left[ \|\nabla (u^1(s) - u^2(s))\|_2^{\frac{2}{\gamma}} + \|u^1(s) - u^2(s)\|_2^{\frac{2}{\gamma}} \right]^{\frac{\gamma}{2}}, 
\end{cases}
\]

(117)

for some constant \(C_B > 0\) depending on \(B\). Let us estimate the right-hand side of (117). Applying interpolation theorem and (95) it follows that

\[
\|\nabla (u^1(t) - u^2(t))\|_2 \leq C_{\theta_1} \|\Delta (u^1(t) - u^2(t))\|_2^{1-\theta_1} \|u^1(t) - u^2(t)\|_2^{\theta_1},
\]

for some constant \(C_B > 0\) and \(\theta_1 = 1/2\). Also, taking \(\theta_2 = \frac{n}{4} \left(1 - \frac{1}{\rho+1}\right)\),

\[
\|u^1(t) - u^2(t)\|_2^{\rho+1}, \quad \|u^1(t) - u^2(t)\|_2^{\rho+1},
\]

(118)

Taking \(\vartheta = \min\{\theta_1, \theta_2\}\), and noting that \(\|u^1(t)\|_2\) and \(\|u^2(t)\|_2\) are uniformly bounded, there exists a constant \(C_B > 0\) such that

\[
\|\nabla (u^1(t) - u^2(t))\|_2^{\frac{3+2}{\gamma}} + \|u^1(t) - u^2(t)\|_2^{\frac{3+2}{\gamma}} \leq C_B \|u^1(t) - u^2(t)\|_2^{\frac{3+2}{\gamma}}, \quad \gamma \geq 0.
\]

Replacing (118) in (117) we get

\[
|S_n(T)z_1 - S_n(T)z_2|_{\mathcal{H}} \leq \varepsilon + \Psi_T(z_1, z_2),
\]

where \(\Psi_T : \mathcal{H} \times \mathcal{H} \to \mathbb{R}\) is given by

\[
\Psi_T(z_1, z_2) = C_B \begin{cases} 
\sup_{0 \leq s \leq T} \|u^1(s) - u^2(s)\|_2^{\frac{\vartheta}{\gamma}}, \quad \gamma > 0, \\
\sup_{0 \leq s \leq T} \|u^1(s) - u^2(s)\|_2^{\frac{\vartheta}{2}}, \quad \gamma = 0.
\end{cases}
\]

Now let us show that \(\Psi_T\) satisfies (116). In fact, given a sequence of initial data \(z_m = (u^m_0, u^m_2) \in B\), as before, we write \(S_n(t)z_m = (u^m(t), u^m_2(t))\). Since \(B\) is invariant by \(S_n(t)\), \(t \geq 0\), it follows that \((u^m(t), u^m_2(t))\) are uniformly bounded in \(\mathcal{H} = W_2 \times W_0\). Hence,

\[
(u^m, u^m_2) \text{ is bounded in } C([0, T], W_2 \times W_0), \quad T > 0.
\]
Since $W_2$ is compactly embedded in $W_0$, then in view of Aubin’s Lemma (see Simon [34, Corollary 4]) there exists a subsequence of $(u^m)$, still denoted by $(u^m)$, such that

$$(u^m) \text{ converges strongly in } C([0, T], W_0), \quad T > 0,$$

which is enough to conclude

$$\lim_{k \to \infty} \lim_{m \to \infty} \Psi_T(z_k, z_m) = 0.$$  

From Proposition 3, $(H, S_+)$ is an asymptotically smooth dynamical system. □

**Proposition 4.** Under assumptions $\textbf{(A}_2\textbf{-A}_6\textbf{)}$ with $g'(0) > 0$, let us consider a bounded set $B \subset H$ and $z_i(t) = (u_i(t), u_i^2(t))$, $i = 1, 2$, two weak solutions of problem (1)-(4) with $z_i(0) = (u_i^0, u_i^1) \in B$. Then, there exist constants $\sigma_B, C_B, C'_B > 0$ depending on $B$ such that

$$\|z_1(t) - z_2(t)\|_H^2 \leq C_B e^{-\sigma_B t}\|z_1(0) - z_2(0)\|_H^2 + C'_B \int_0^t e^{-\sigma_B (t-s)}\|\nabla w(s)\|_2^2 ds, (119)$$

for any $t > 0$, where we set $w = u^1 - u^2$.

**Proof.** For convenience, we first define

$$\Psi_J = J(\|\nabla u^1(t)\|_2^2) - J(\|\nabla u^2(t)\|_2^2) \quad \text{for} \quad J = M, N.$$

The function $(w, w_t) = z_1 - z_2$ solves the problem

$$w_{tt} + \Delta^2 w = -\kappa M(\|\nabla u^1(t)\|_2^2) \Delta w + N(\|\nabla u^1(t)\|_2^2) [g(u_i^1) - g(u_i^2)] + [f(u^1) - f(u^2)] = -\kappa\Psi_M \Delta w - \Psi_N g(u_i^1),$$

with initial condition $(w(0), w_t(0)) = z_1(0) - z_2(0)$. Taking the multiplier $w_i$ in (120) we obtain

$$\frac{d}{dt} V_e(t) + \mathcal{I} + \mathcal{J} = \kappa\Psi_M \int_\Omega \Delta u^2(t) w_i(t) \, dx - \Psi_N \int_\Omega g(u_i^2(t)) w_i(t) \, dx$$

$$+ \kappa M'(\|\nabla u^1(t)\|_2^2)(\Delta u^1(t), u_i^1(t)) \|\nabla w(t)\|_2^2, (121)$$

where

$$V_e(t) = \frac{1}{2}\|w(t)\|_2^2 + \frac{1}{2}\|\Delta w(t)\|_2^2 + \kappa M(\|\nabla u^1(t)\|_2^2)\|\nabla w(t)\|_2^2, (122)$$

and

$$\mathcal{I} = N(\|\nabla u^1(t)\|_2^2) \int_\Omega [g(u_i^1(t)) - g(u_i^2(t))] w_i(t) \, dx,$$

$$\mathcal{J} = \int_\Omega [f(u^1(t)) - f(u^2(t))] w_i(t) \, dx.$$  

Now we consider proper estimates for $\mathcal{I}$ and $\mathcal{J}$. Since $g \in C^1(\mathbb{R})$ and $g'(0) > 0$, then there exists a constant $\delta > 0$ such that

$$g'(s) > 0 \quad \text{for all} \quad |s| < 2\delta,$$

and so, noting that $g' > 0$ on $\mathbb{R}$, there exists a constant $m_0 > 0$ such that

$$g'(s) \geq m_0 > 0 \quad \text{for all} \quad |s| \leq \delta,$$

In addition, condition (12) also implies

$$g'(s) \geq c g' \delta > 0 \quad \text{for all} \quad |s| \geq \delta.$$
Thus, choosing $m = \min\{m_0, c \delta^\gamma\} > 0$, we infer
\[ g'(s) \geq m, \quad \forall \, s \in \mathbb{R}. \quad (123) \]

From (123) and (18) we obtain
\[ I = \frac{T}{2} + \frac{T}{2} \geq \varrho_0 \int_{\Omega} |w_t(t)|^2 \, dx + \varrho_1 \int_{\Omega} \left[ |u_1^2(t)|^\gamma + |u_2^2(t)|^\gamma \right] |w_t(t)|^2 \, dx, \quad (124) \]
where we set $\varrho_0 = \frac{c m}{2} > 0$ and $\varrho_1 = \frac{c N m}{\delta^\gamma \gamma + 1} > 0$, which are constants depending on $B$.

Besides, from Mean Value Theorem there exists $\vartheta \in (0, 1)$ such that
\[ f(u^1) - f(u^2) = f'(\vartheta u^1 + (1 - \vartheta)u^2)w = f'(\xi_\vartheta)w, \quad (125) \]
where we denote $\xi_\vartheta := \vartheta u^1 + (1 - \vartheta)u^2$. Then, we can rewrite $J$ as
\[ J = \int_{\Omega} f'(\xi_\vartheta(t))w(t)w_t(t) \, dx = \int_{\Omega} f'(\xi_\vartheta(t))\frac{1}{2} \frac{d}{dt} |w(t)|^2 \, dx \]
\[ = \frac{1}{2} \frac{d}{dt} \int_{\Omega} f'(\xi_\vartheta(t))|w(t)|^2 \, dx - \frac{1}{2} \int_{\Omega} f''(\xi_\vartheta(t))\partial_t(\xi_\vartheta(t))|w(t)|^2 \, dx. \quad (126) \]

Replacing (124) and (126) in (121), we obtain
\[ \frac{d}{dt} F_\kappa(t) + \varrho_0 \|w_t(t)\|_2^2 + \varrho_1 \int_{\Omega} \left[ |u_1^2(t)|^\gamma + |u_2^2(t)|^\gamma \right] |w_t(t)|^2 \, dx \leq \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4, \quad (127) \]
where
\[ F_\kappa(t) = V_\kappa(t) + \frac{1}{2} \int_{\Omega} f'(\xi_\vartheta(t))|w(t)|^2 \, dx, \quad (128) \]
and
\[ \Theta_1 = \kappa \Psi M \int_{\Omega} \Delta u^2(t)w_t(t) \, dx, \]
\[ \Theta_2 = \kappa M' \left( \|\nabla u^1(t)\|_2^2 \right) (\Delta u^1(t), u_1^2(t)) \|\nabla w(t)\|_2^2, \]
\[ \Theta_3 = - \Psi N \int_{\Omega} g(u_2^2(t))w_t(t) \, dx, \]
\[ \Theta_4 = \frac{1}{2} \int_{\Omega} f''(\xi_\vartheta(t))\partial_t(\xi_\vartheta(t))|w(t)|^2 \, dx. \]

From Assumptions (A3) and (A5) it is easy to see that
\[ F_k(t) \sim \|z_1(t) - z_2(t)\|_{\mathcal{H}}^2. \]

Indeed, taking into account conditions (14) and (71) we have
\[ \frac{\alpha}{2} \|z_1(t) - z_2(t)\|_{\mathcal{H}}^2 \leq \frac{1}{2} \|w_t(t)\|_2^2 + \frac{\alpha}{2} \|\Delta w(t)\|_2^2 \leq F_k(t). \quad (129) \]

On the other hand, using that $M$ is continuous, assumption (70) and immersion $W_2 \hookrightarrow L^{\rho+2}(\Omega)$ we also have
\[ F_\kappa(t) \leq \frac{1}{2} \|w_t(t)\|_2^2 + C_{0,B} \|\Delta w(t)\|_2^2 \leq C_{1,B} \|z_1(t) - z_2(t)\|_{\mathcal{H}}^2, \quad (130) \]
for some constants $C_{0,B}, C_{1,B} > 0$ depending on $B$.

In what follows we will estimate the terms $\Theta_1, \cdots, \Theta_4$ on the right side hand of (127). To this end we use the same parameter $C_B > 0$ to designate different
constants depending on \( B \). Since \( M, N \in C^1([0, \infty)) \), from estimates (64) and (95), Young and Hölder inequalities and condition (12), we have

\[
|\Theta_1| \leq C_B \|
abla w(t)\|_2 \|w_t(t)\|_2 \leq \frac{\theta_0}{4} \|w_t(t)\|_2^2 + \frac{C_B^2}{\theta_0} \|\nabla w(t)\|_2^2,
\]

\[
|\Theta_2| \leq C_B \|
abla w(t)\|_2^2,
\]

and

\[
|\Theta_3| \leq C_B \|
abla w(t)\|_2 \|w_t(t)\|_2 + C_B \|
abla w(t)\|_2 \|u^2(t)\|_{\gamma+2}^2 \left( \int_{\Omega} |u^2_t(t)|^\gamma |w_t(t)|^2 \, dx \right)^{1/2}
\]

\[
\leq \frac{\theta_0}{4} \|w_t(t)\|_2^2 + \frac{C_B^2}{\theta_0} \|\nabla w(t)\|_2^2
\]

\[
+ \frac{C_B^2}{2 \theta_1 \lambda_1} \|u^2_t(t)\|^2_{\gamma+2} \|u^2(t)\|_{\gamma+2} \left( \int_{\Omega} |u^2_t(t)|^\gamma |u^2_t(t)|^\gamma \, dx \right) + \frac{\theta_1}{2} \int_{\Omega} \left[ |u^1_t(t)|^\gamma + |u^2_t(t)|^\gamma \right] |w_t(t)|^2 \, dx.
\]

In addition, from assumption (70), Hölder inequality with \( \frac{\rho-1}{\rho+2} + \frac{1}{\rho+2} + \frac{2}{\rho+2} = 1 \) and embeddings \( W_2 \hookrightarrow L^{\rho+2}(\Omega) \), \( L^{\rho+2}(\Omega) \hookrightarrow L^{\rho+2}(\Omega) \) (see (72)), we obtain

\[
|\Theta_4| \leq C \left( 1 + \|u^1(t)\|_{\rho+2}^{-1} + \|u^2(t)\|_{\rho+2}^{-1} \right) \left( \|u^1_t(t)\|_{\rho+2} + \|u^2_t(t)\|_{\rho+2} \right) \|w(t)\|_{\rho+2}^2
\]

\[
\leq C_B \left( \|u^1_t(t)\|_{\gamma+2} + \|u^2_t(t)\|_{\gamma+2} \right) \|\Delta w(t)\|_2^2,
\]

for some constants \( C > 0 \) depending on \( \Omega \) and \( C_B > 0 \) depending on \( B \). Substituting estimates (131)-(134) in (127) we deduce

\[
\frac{d}{dt} F_k(t) \leq -\frac{\theta_0}{2} \|w_t(t)\|_2^2 - \frac{\theta_1}{2} \int_{\Omega} \left[ |u^1_t(t)|^\gamma + |u^2_t(t)|^\gamma \right] |w_t(t)|^2 \, dx
\]

\[
+ C_B \left[ \|u^2_t(t)\|_{\gamma+2}^2 + \sum_{j=1}^2 \|u^j_t(t)\|_{\gamma+2} \right] \|\Delta w(t)\|_2^2 + C_B \|\nabla w(t)\|_2^2,
\]

for some constant \( C_B > 0 \). Now we consider the functional

\[
F^\varepsilon_k(t) = F_k(t) + \varepsilon \chi(t) \quad \text{with} \quad \chi(t) = \int_{\Omega} w_t(t) w(t) \, dx,
\]

where \( \varepsilon > 0 \) will be chosen later. Deriving \( \chi(t) \) and using the expressions (120) and (125) we get

\[
\frac{d}{dt} \chi(t) = \|w_t(t)\|_2^2 = \|\Delta w(t)\|_2^2 - \kappa M \|\nabla u^1(t)\|_2^2 \|\nabla w(t)\|_2^2 - \int_{\Omega} f'(\xi_0(t)) |w(t)|^2 \, dx + \Phi_1 + \Phi_2 + \Phi_3,
\]

where

\[
\Phi_1 = \kappa \Psi_M \int_{\Omega} \Delta u^2(t) w(t) \, dx,
\]

\[
\Phi_2 = -\Psi_N \int_{\Omega} g(u^2_t(t)) w(t) \, dx,
\]

\[
\Phi_3 = -N \|\nabla u^1(t)\|_2^2 \int_{\Omega} \left[ g(u^1_t(t)) - g(u^2_t(t)) \right] w(t) \, dx.
\]

Let us estimate \( \Phi_1, \Phi_2, \Phi_3 \). From estimates (64) and (95) it follows readily

\[
|\Phi_1| \leq C_B \|\nabla w(t)\|_2^2,
\]

(138)
for some constant $C_B > 0$. Also using condition (12), immersions $W_2 \hookrightarrow W_1$, $W_2 \hookrightarrow L^{\gamma+2} (\Omega)$, and Hölder inequality, results

$$|\Phi_2| \leq C_B \|
abla w(t)\|_2^2 + C_B \|\nabla w(t)\|_2 u_2^2(t) \|u_2^1(t)\|^{\gamma+1}_{\gamma+2} \|w(t)\|_{\gamma+2}$$

$$\leq C_B \|\nabla w(t)\|_2^2 + C_B \|u_2^1(t)\|^{\gamma+1}_{\gamma+2} \|\Delta w(t)\|_2^2,$$  \hspace{0.5cm} (139)

and

$$|\Phi_3| \leq C_B \int_\Omega [1 + (|u_1^1(t)| + |u_1^2(t)|)^\gamma] \|w(t)\|_2 \|w(t)\|_{\gamma+2} \|\nabla w(t)\|_2$$

$$+ C_B \int_\Omega \left[(|u_1^1(t)| + |u_1^2(t)|)^{\frac{\gamma}{2}} \|w(t)\|_{\gamma+2} \right] \left[(|u_1^1(t)| + |u_1^2(t)|)^{\frac{\gamma}{2}} \|w(t)\|_{\gamma+2}\right] \|w(t)\|_{\gamma+2} \|\nabla w(t)\|_2$$

$$\leq \|w(t)\|^2_2 + \frac{C_B^2}{4} \|\nabla w(t)\|^2_2 + \frac{C_B^2}{4} \|u_1^1(t)\|^\gamma_{\gamma+2} + \|u_1^2(t)\|^\gamma_{\gamma+2} \|\Delta w(t)\|_2^2,$$  \hspace{0.5cm} (140)

for some constant $C_B > 0$. Going back to (137) and replacing (138)-(140) we obtain

$$\frac{d}{dt} \chi(t) \leq \frac{\epsilon}{2} \|\Delta w(t)\|_2^2 - \int_\Omega f'(\xi(t)) \|w(t)\|^2_2 \ dx$$

$$+ C_B \left[\|u_1^2(t)\|^{\gamma+1}_{\gamma+2} + \sum_{j=1}^2 \|u_1^j(t)\|^{\gamma}_{\gamma+2}\right] \|\Delta w(t)\|_2^2$$

$$+ \frac{2}{\alpha} \|w(t)\|_2^2 + \int_\Omega \left[(|u_1^1(t)| + |u_1^2(t)|)^\gamma \right] \|w(t)\|_2^2 \ dx + C_B \|\nabla w(t)\|_2^2,$$  \hspace{0.5cm} (141)

for some constant $C_B > 0$. Deriving $F^\kappa_\varepsilon(t)$ in (136), using estimates (135) and (141), we arrive at

$$\frac{d}{dt} F^\kappa_\varepsilon(t) \leq \left[\frac{\alpha}{2} - \varepsilon\right] \|w(t)\|^2_2 - \varepsilon \|\Delta w(t)\|_2^2 - \varepsilon \kappa M \left(\|\nabla u_1^1(t)\|_2\right) \|\nabla w(t)\|_2^2$$

$$- \varepsilon \int_\Omega f'(\xi(t)) \|w(t)\|^2_2 \ dx + C_B \Pi(t) \|\Delta w(t)\|_2^2$$

$$- \left[\frac{\alpha}{2} - \varepsilon\right] \left[\|u_1^1(t)\|^\gamma + \|u_1^2(t)\|^\gamma\right] \|w(t)\|^2_2 \ dx + C_B (1 + \varepsilon) \|\nabla w(t)\|^2_2,$$

where

$$\Pi(t) = \left[\|u_1^2(t)\|^{\gamma+1}_{\gamma+2} + \varepsilon \|u_1^2(t)\|^{\gamma+1}_{\gamma+2} + \sum_{j=1}^2 \|u_1^j(t)\|^{\gamma}_{\gamma+2} + \sum_{j=1}^2 \|u_1^j(t)\|^{\gamma}_{\gamma+2}\right].$$

Choosing $0 < \varepsilon \leq \min \left\{\frac{\alpha}{2}, \frac{\alpha}{2}, 1, \frac{1}{2} \alpha \lambda_1^{\gamma/2}\right\}$, noting that $\|\Delta w(t)\|_2^2 \leq \frac{\alpha}{2} F^\kappa_\varepsilon(t)$ by (129) and regarding definitions in (122) and (128), we obtain

$$\frac{d}{dt} F^\kappa_\varepsilon(t) \leq -\varepsilon F^\kappa_\varepsilon(t) + \frac{2C_B}{\alpha} \Pi(t) F^\kappa_\varepsilon(t) + 2C_B \|\nabla w(t)\|_2^2, \quad t > 0,$$  \hspace{0.5cm} (142)

with

$$\Pi(t) = \left[\|u_1^2(t)\|^{\gamma+1}_{\gamma+2} + \|u_1^2(t)\|^{\gamma+1}_{\gamma+2} + \sum_{j=1}^2 \|u_1^j(t)\|^{\gamma}_{\gamma+2} + \sum_{j=1}^2 \|u_1^j(t)\|^{\gamma}_{\gamma+2}\right].$$
Applying Gronwall inequality in (144) we deduce that
\[ \int the dynamical system \quad (H_{\kappa}) \]
whenever a sequence \( x \) dynamical system.

make this paper more self-contained, we introduce the concept of a quasi-stable also gives another way to conclude asymptotic smoothness property. In order to Lasiecka \[11\] it is introduced concept of quasi-stability property that, in particular,
\[ \tilde{C}_B > 0 \text{ depending on } B. \]
Finally, combining (146) with (129)-(130), we conclude that the stability inequality (119) holds true. Therefore, the proof of Proposition 4 is complete.

The above Proposition 4 shall allow us to achieve richer qualitative properties to the dynamical system \((H, S_{\kappa}(t))\), \(\kappa \geq 0\), defined in (73). In fact, in Chueshov and Lasiecka \[11\] it is introduced concept of quasi-stability property that, in particular, also gives another way to conclude asymptotic smoothness property. In order to make this paper more self-contained, we introduce the concept of a quasi-stable dynamical system.

We recall that a seminorm \( n_X(\cdot) \) defined on a Banach space \( X \) is compact if whenever a sequence \( x_j \to 0 \) weakly in \( X \) one has \( n_X(x_j) \to 0 \). Let \( X, Y, Z \) be three
reflexive Banach spaces with $X$ compactly embedded in $Y$ and put $H = X \times Y \times Z$, where the case with trivial space $Z = \{0\}$ is allowed. Consider the dynamical system $(H, S(t))$ given by an evolution operator

$$S(t)z = (u(t), u_t(t), \zeta(t)), \quad z = (u_0, u_1, \zeta_0) \in H,$$

(147)

where the functions $u$ and $\zeta$ have regularity

$$u \in C(\mathbb{R}^+; X) \cap C^1(\mathbb{R}^+; Y), \quad \zeta \in C(\mathbb{R}^+; Z).$$

(148)

Then one says that $(H, S(t))$ is quasi-stable on a set $B \subset H$ if there exists a compact seminorm $n_X$ on $X$ and nonnegative scalar functions $a(t)$ and $c(t)$ locally bounded in $[0, \infty)$, and $b(t) \in L^1(\mathbb{R}^+)$ with $\lim_{t \to \infty} b(t) = 0$, such that,

$$\|S(t)z^1 - S(t)z^2\|^2_H \leq a(t)\|z^1 - z^2\|^2_H,$$

(149)

and

$$\|S(t)z^1 - S(t)z^2\|^2_H \leq b(t)\|z^1 - z^2\|^2_H + c(t) \sup_{0 < s < t} [n_X(u^1(s) - u^2(s))]^2,$$

(150)

for any $z^1, z^2 \in B$. Inequality (150) is called stabilizability inequality.

**Corollary 4** (Quasi-stability property). Let assumptions of Proposition 4 be in force. Then the dynamical system $(\mathcal{H}, S_{\kappa}(t))$ is quasi-stable on any bounded positively invariant set $B \subset \mathcal{H}$. In particular, it is also asymptotically smooth.

*Proof.* We need to show that $(\mathcal{H}, S_{\kappa}(t))$ satisfies (147)-(150). Since $(\mathcal{H}, S_{\kappa}(t))$ is defined in (73), then Theorem 2.1 implies that conditions (147) and (148) hold with $X = W_2, Y = W_0, Z = \{0\}$ and $\mathcal{H} = W_2 \times W_0$. Thus we only need to verify the Lipschitz condition (149) and the stabilizability inequality (150). As before, let us consider a bounded positively invariant set $B \subset \mathcal{H}$ and $z_0^i = (u_0^i, u_1^i) \in B$. From (24) and (96) it follows that (149) holds true with $a(t) = e^{\int_0^t (1 + u^2(s)) \frac{\gamma^2}{2} ds}$ locally bounded on $[0, \infty)$. Thus, it remains to prove (150). Indeed, denoting by

$$b(t) = C_Be^{-\sigma_B t}, \quad c(t) = C_B \int_0^t e^{-\sigma_B (t-s)} ds \quad \text{and} \quad n_{W_2}(u) = \|\nabla u\|_2,$$

then, in view of (119), the semigroup solution $S_{\kappa}(t)z_0^i = (u^i(t), u_1^i(t)), i = 1, 2,$ satisfies

$$\|S_{\kappa}(t)z_0^1 - S_{\kappa}(t)z_0^2\|^2_H \leq b(t)\|z_0^1 - z_0^2\|^2_H + c(t) \sup_{0 < s < t} [n_{W_2}(u^1(s) - u^2(s))]^2,$$

for any $z_0^i \in B$ and $t > 0$. Since $B$ is bounded one sees that $b(t) \in L^1(\mathbb{R}^+)$ with $\lim_{t \to +\infty} b(t) = 0$ and $c(t)$ is locally bounded on $[0, +\infty)$. Moreover, form definition of $c(t)$ it is ensured that

$$c_{\infty} = \sup_{t \in \mathbb{R}^+} c(t) < \infty.$$

(151)

In addition, from compact embedding $W_2 \hookrightarrow W_1$, we conclude that $n_{W_2}(\cdot)$ is a compact seminorm on the space $W_2$. Therefore, the stabilizability estimate (150) also holds true. This completes the proof of the quasi-stability property to the dynamical system $(\mathcal{H}, S_{\kappa}(t))$. In particular, from [11, Proposition 7.9.4], $(\mathcal{H}, S_{\kappa}(t))$ is asymptotically smooth. \qed
3.3. Main results on attractors. Now we are in conditions to state our main results on attractors with respect to the dynamical system \((\mathcal{H}, S_\kappa(t))\), \(\kappa \geq 0\), defined in (73).

**Theorem 3.1** (Attractor - Case I). Under assumptions of Proposition 2 we have:

(i) **(Attractor)** The dynamical system \((\mathcal{H}, S_\kappa(t))\) associated with problem (1)-(4) possesses a unique global attractor \(\mathfrak{A}_\kappa \subset \mathcal{H}\), which is compact and connected.

(ii) **(Geometrical structure)** The global attractor is precisely the unstable manifold \(\mathfrak{A}_\kappa = W^u(N_\kappa)\), emanating from the set of stationary solutions

\[ N_\kappa = \{(u,0) \in \mathcal{H}; \; \Delta^2 u - \kappa M (\|\nabla u\|_2^3) \Delta u + f(u) = h\}, \quad \kappa \geq 0. \quad (152) \]

Moreover, any trajectory stabilizes to the set \(N_\kappa\), namely,

\[ \lim_{t \to +\infty} \text{dist}_H(S_\kappa(t)z, N_\kappa) = 0, \quad \forall \; z \in \mathcal{H}. \]

In particular, for every \(\kappa \geq 0\), the set of stationary solutions \(N_\kappa\) constitutes a global minimal attractor to the dynamical system \((\mathcal{H}, S_\kappa(t))\).

**Proof.** From Corollaries 2 and 3, the dynamical system \((\mathcal{H}, S_\kappa(t))\), \(\kappa \geq 0\), is dissipative and asymptotically smooth. Then item (i) follows from [11, Theorem 7.2.3]. Since \((\mathcal{H}, S_\kappa(t))\) is a gradient dynamical system and has a compact global attractor \(\mathfrak{A}_\kappa\), then item (ii) follows promptly from [10, Theorem 2.28] and [11, Theorem 7.5.10]. \(\square\)

**Theorem 3.2** (Attractor - Case II). Under assumptions of Proposition 4 we have:

(i) **(Attractor and geometrical structure)** The dynamical system \((\mathcal{H}, S_\kappa(t))\) possesses a unique compact global attractor \(\mathfrak{A}_\kappa \subset \mathcal{H}\), which is characterized by the unstable manifold \(\mathfrak{A}_\kappa = M^u(N_\kappa)\), emanating from the set \(N_\kappa\) defined in (152).

(ii) **(Finite dimensionality)** The compact global attractor \(\mathfrak{A}_\kappa\), \(\kappa \geq 0\), has finite fractal and Hausdorff dimension in \(\mathcal{H}\).

(iii) **(Regularity)** Any full trajectory \(\Gamma = \{(u(t); u_t(t)); \; t \in \mathbb{R}\}\) from the attractor \(\mathfrak{A}_\kappa\), \(\kappa \geq 0\), has the following regularity

\[ (u_t, u_{tt}) \in L^\infty(\mathbb{R}; W_2 \times W_0). \]

Moreover, there exists a constant \(R > 0\) such that

\[ \sup_{\Gamma \subset \mathfrak{A}_\kappa} \sup_{t \in \mathbb{R}} (\|u_{tt}(t)\|_2^2 + \|u_{tt}(t)\|_2^2) \leq R^2. \]

(iv) **(Exponential attractor)** The dynamical system \((\mathcal{H}, S_\kappa(t))\) possesses a generalized fractal exponential attractor \(\mathfrak{A}_\kappa^\text{exp}\) with finite fractal dimension in the extended space

\[ \mathcal{H}_{-1} := W_0 \times W'_2. \]

In addition, from interpolation theorem, there exists a generalized fractal exponential attractor with finite fractal dimension in a smaller extended space \(\mathcal{H}_{-\delta}\), where

\[ \mathcal{H} \subset \mathcal{H}_{-\delta} \subset \mathcal{H}_{-1}, \quad 0 < \delta \leq 1. \]

**Proof.** (i) The proof is similar to items (i)-(ii) of Theorem 3.1 by using Corollary 4 instead of Corollary 3.

(ii)-(iii) From Corollary 4, \((\mathcal{H}, S_\kappa(t))\) is quasi-stable on any bounded positively invariant set \(B \subset \mathcal{H}\). In particular it is quasi-stable on the attractor \(\mathfrak{A}_\kappa\). From [11, Theorem 7.9.6] we conclude that \(\mathfrak{A}_\kappa\) has finite fractal dimension. The finiteness of
Hausdorff dimension follows since it is bounded by the fractal one, see [14, Chapter 2]. In addition, in view of (151), we can also conclude that the regularity properties (153)-(154) hold true by applying [11, Theorem 7.9.8].

(iv) The proof follows applying verbatim similar arguments as in the proof of Theorem 2.3-(vi) given by authors in [16]. See also [33, Theorem 4.5].

Appendix A. Some examples. We finish this work by giving some different examples of functions $f$, $M$ and $g$ fulfilling assumptions ($A_1$)-($A_6$).

Example 1. We first consider classical functions $f$ satisfying ($A_1$) or ($A_5$).

a) $f(s) = |s|^\rho s$, $\rho > 0$;

b) $f(s) = a|s|^\rho s + b$, $a, \rho > 0$, $b \in \mathbb{R}$;

c) $f(s) = a|s|^\rho s + bs$, $a, \rho > 0$, $b \in \mathbb{R}$;

d) $f(s) = |s|^\rho s \pm |s|^\delta s$, $\rho > \delta > 0$;

e) $f(s) = a \sin(s)$, $a > 0$;

f) $f(s) = a \ln(s^2 + 1)$, $a > 0$;

g) $f(s) = -a \arctan(s)$, $a > 0$;

h) $f(s) = \frac{s}{|s|^2 + 1}$.

To prove ($A_1$) or ($A_5$), the parameters $a$ and $b$ are are given in terms of $\alpha_1, c_f, c_f'$, $c_f''$.

Example 2. Now we consider some examples of functions $M$ satisfying assumptions ($A_3$) and ($A_6$).

a) $M(s) = -\frac{k}{2}, k > 0$;

b) $M(s) = s^p - k$, $p > -\frac{1}{2}$, $k > 0$;

c) $M(s) = b \ln(s + e^{-k})$, $b \in [0, 1]$, $k > 0$;

d) $M(s) = e^s - (1 + k)$, $k > 0$;

e) $M(s) = b \sinh(s) - k$, $b \geq 0$, $k > 0$.

Assumptions ($A_3$) and ($A_6$) are fulfilled by taking $k = \frac{\alpha_2}{\kappa} > 0$, $\kappa \neq 0$.

Example 3. We finally give two standard examples of polynomial functions $g$.

a) $g(s) = |s|^\gamma s$, $\gamma \geq 0$, satisfies ($A_2$);

b) $g(s) = |s|^\gamma s + as$, $\gamma \geq 0$, $a > 0$, complies with ($A_2$) and the additional condition required in Proposition 4.

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