Stochastic Timed Games Revisited

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Abstract. Stochastic timed games (STGs), introduced by Bouyer and Forajt, naturally generalize both continuous-time Markov chains and timed automata by providing a partition of the locations between those controlled by two players (Player Box and Player Diamond) with competing objectives and those governed by stochastic laws. Depending on the number of players—2, 1, or 0—subclasses of stochastic timed games are often classified as 2\textsuperscript{1}2-player, 1\textsuperscript{1}2-player, and 1\textsuperscript{0}2-player games where the \(\frac{1}{2}\) symbolizes the presence of the stochastic “nature” player. For STGs with reachability objectives it is known that 1\textsuperscript{0}2-player one-clock STGs are decidable for qualitative objectives, and that 2\textsuperscript{1}2-player three-clock STGs are undecidable for quantitative reachability objectives. This paper further refines the gap in this decidability spectrum. We show that quantitative reachability objectives are already undecidable for 1\textsuperscript{2}2-player four-clock STGs, and even under the time-bounded restriction for 2\textsuperscript{1}2-player five-clock STGs. We also obtain a class of 1\textsuperscript{1}2, 2\textsuperscript{1}2-player STGs for which the quantitative reachability problem is decidable.

1 Introduction

Two-player zero-sum games over finite state-transition graphs are a natural framework for controller synthesis for discrete event systems. In this setting two players—say Player Box and Player Diamond (after necessity and possibility operators)—represent the controller and the environment, and control-program synthesis corresponds to finding a winning (or optimal) strategy of the controller for some given performance objective. Finite graphs, however, often do not satisfactorily model real-time safety-critical systems as they disregard not only the continuous dynamics of the physical environment but also the presence of stochastic behavior. Stochastic behavior in such systems stems from many different sources, e.g., faulty or unreliable sensors or actuators, uncertainty in timing delays, the random coin flips of distributed communication and security protocols.

Timed automata \cite{1} were introduced as a formalism to model asynchronous real-time systems interacting with a continuous physical environment. Timed automata and their two-player counterparts \cite{2} provide an intuitive and semantically unambiguous way to model non-stochastic real-time systems, and a number of case-studies \cite{23} demonstrate their application in the design and analysis of real-time systems. On the other hand, classical formalisms (discrete-time and continuous-time) Markov decision processes (MDPs) and stochastic games \cite{22,14} naturally model analysis and synthesis problems for stochastic systems, and have been applied in control theory, operations research, and economics.

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For the formal analysis of stochastic real-time systems, a number of recent works considered a combination of stochastic features with timed automata, e.g. probabilistic timed automata [17], continuous probabilistic timed automata [16] and stochastic timed automata [8]. Probabilistic timed automata, respectively continuous probabilistic and stochastic timed automata can be considered as generalizations of timed automata with the features of discrete-time Markov decision processes, respectively continuous-time Markov chains [4] (or even generalized semi-Markov processes [12]). Stochastic timed games [11] form the most general formalism for studying controller-synthesis for stochastic real-time systems. These games can be considered as interactions between three players—Player Box, Player Diamond and the stochastic player (Nature)—such that Player Box and Player Diamond are adversarial and choose their delay and action so as to maximize and minimize probability to reach a given set of target states, while the stochastic player plays according to a given probability distribution. A key verification problem in this setting is that of games with reachability objectives, where the goal of Player Diamond is to reach a set of target states, while the goal of the Player Box is to avoid it.

Related Work. Probabilistic timed automata [17] and games [15] can be considered as subclasses of stochastic timed games where all of the locations controlled by stochastic players are urgent (no time delay allowed), while the decision-stochastic timed automata of [9] can be seen as a subclass of 1½-player STGs where the locations of the rational players are urgent. The quantitative reachability problem for probabilistic timed automata is known to be decidable [17] with any number of clocks, while the best known decidability result for the quantitative reachability problem for 1½-player STGs is using a single clock. 1½-player STGs, also called stochastic timed automata (STA) [8], have also received considerable attention: an abstraction based on the region abstraction has been proposed, which allows to solve the qualitative reachability problem under a fairness assumption on the STA (several subclasses of STAs have been proven to be fair). For quantitative reachability, the only decidability result is for a subclass of single-clock STA [7], but a recent approximability result has been shown in [6] for the class of fair STA.

Other variants of stochastic timed automata have been studied in the past. The model in [16] uses “countdown clocks” (which decrease from a set value) unlike the more timed-automata style of clock variables used in our model. The model in [10] (which is also called stochastic timed automata; we shall refer to them here as Modest-STA) is very general and encompasses most models with time and probabilities (and in particular the STA of [8]). However, Modest-STA is more aimed at capturing general languages (and providing a tool-set to simulate their runs) and less with decidability issues, and hence is orthogonal to our approach.

Contributions. The scope of this paper is to investigate decidability of the reachability problem in STGs as defined in [11], for which the decidability picture is far from complete. In [11], the authors showed the decidability of qualitative reachability problem on 1-clock 1½-player STGs, and the undecidability of quantitative reachability problem on STGs (with 2½-players). This leaves a wide gap in the decidability horizon of STGs. In this paper, we study 1½, 2½-player games and contribute to a better understanding of the decidability status of STGs with quantitative reachability objectives.

Table[1] summarizes the results presented in this paper. We show that the quantitative reachability problem is already undecidable for 1½-player games for systems with 4 or more clocks and for 2½-player games the quantitative reachability problem remains undecidable even under the time-bounded restriction with 5 or more clocks. Another key contribution of this paper is the characterization of a previously unexplored subclass of stochastic timed
games for which we recover decidability of quantitative reachability game for $1\frac{1}{2}$ (and even $2\frac{1}{2}$)-player stochastic timed games. We call a 1-clock stochastic timed game initialized if (i) all the transitions from non-stochastic states to stochastic states reset the clock, and (ii) in every bounded cycle, the clock is reset. The definition can be generalized to multiple clocks using the notion of strong reset where one resets all the clocks together. For some of the gaps in this spectrum, we provide our best conjectures as justified in the Discussion section:–the undecidability of time-bounded quantitative reachability for $1\frac{1}{2}$-player STG, and the decidability of qualitative reachability of 1-clock $2\frac{1}{2}$-player STG. Due to lack of space, details of some proofs can be found in the Appendix.

### 2 Stochastic Timed Games

We use standard notations for the set of reals ($\mathbb{R}$), rationals ($\mathbb{Q}$), and integers ($\mathbb{Z}$), and add subscripts to indicate additional constraints (for instance $\mathbb{R}_{>0}$ is for the set of non-negative reals). Let $\mathcal{C}$ be a finite set of real-valued variables called clocks. A valuation on $\mathcal{C}$ is a function $\nu : \mathcal{C} \rightarrow \mathbb{R}_{>0}$. We assume an arbitrary but fixed ordering on the clocks and write $x_i$ for the clock with order $i$. This allows us to treat a valuation $\nu$ as a point $(\nu(x_1), \nu(x_2), \ldots, \nu(x_n)) \in \mathbb{R}^{|\mathcal{C}|}$. Abusing notations slightly, we use a valuation on $\mathcal{C}$ and a point in $\mathbb{R}^{|\mathcal{C}|}$ interchangeably. For a subset of clocks $X \subseteq \mathcal{C}$ and valuation $\nu \in \mathbb{R}^{|\mathcal{C}|}$, we write $\nu[X:=0]$ for the valuation where $\nu[X:=0](x) = 0$ if $x \in X$, and $\nu[X:=0](x) = \nu(x)$ otherwise. For $t \in \mathbb{R}_{>0}$, write $\nu + t$ for the valuation defined by $\nu(x) + t$ for all $x \in X$. The valuation $0 \in \mathbb{R}^{|\mathcal{C}|}$ is a special valuation such that $0(x) = 0$ for all $x \in \mathcal{C}$. A clock constraint over $\mathcal{C}$ is a subset of $\mathbb{R}^{|\mathcal{C}|}$ defined by a (finite) conjunction of constraints of the form $x \bowtie k$, where $k \in \mathbb{Z}_{>0}$, $x \in \mathcal{C}$, and $\bowtie \in \{<, \leq, =, >, \geq\}$. We write $\varphi(\mathcal{C})$ for the set of clock constraints. For a constraint $g \in \varphi(\mathcal{C})$, and a valuation $\nu$, we write $\nu \models g$ to represent the fact that valuation $\nu$ satisfies constraint $g$ (defined in a natural way).

A timed automaton (TA) $\mathcal{A} = (L, \mathcal{C}, E, I)$ such that (i) $L$ is a finite set of locations, (ii) $\mathcal{C}$ is a finite set of clocks, (iii) $E \subseteq L \times \varphi(\mathcal{C}) \times 2^L \times L$ is a finite set of edges, (iv) $I : L \rightarrow \varphi(\mathcal{C})$ assigns an invariant to each location. A state $s$ of a timed automaton is a pair $s = (\ell, \nu) \in L \times \mathbb{R}^{|\mathcal{C}|}$ such that $\nu \models I(\ell)$ (the clock valuation should satisfy the invariant of the location). If $s = (\ell, \nu)$, and $t \in \mathbb{R}_{>0}$, we write $s + t$ for the state $(\ell, \nu + t)$. A transition $(t, e)$ from a state $s = (\ell, \nu)$ to a state $s' = (\ell', \nu')$ is written as $s \xrightarrow{t, e} s'$ if $e = (\ell, g, C, \ell') \in E$, such that $\nu + t \models g$, and for every $0 \leq t' \leq t$ we have $v + t' \models I(\ell)$ and $\nu' = \nu + t[C:=0](x)$.

A run is a finite or infinite sequence of transitions $\rho = s_0 \xrightarrow{t_1, e_1} s_1 \xrightarrow{t_2, e_2} s_2 \ldots$ of states and

| Model   | Qualitative Results | Quantitative Results |
|---------|---------------------|----------------------|
| $\frac{1}{2}$ player | 1 clock | Dec. [3] | Dec. (some restrictions) [7] |
|         | $n$ clocks | Open in general | Dec. | Approx. (fair) [6] |
| $1\frac{1}{2}$ player | 1 clock | Dec. [11] | Dec. (Initialized, Theorem 5) |
|         | $n$ clocks | Open | Undec. (Theorem 3) |
| $2\frac{1}{2}$ player | 1 clock | Conj: Dec. | Dec. (Initialized, Corollary 9) |
|         | $n$ clocks | Open | Undec. (Time bounded, Theorem 6) |

Table 1 Results in bold are contributions from this paper. “Conj” are conjectures.
transitions. An edge $e$ is enabled from $s$ whenever there is a state $s'$ such that $s \xrightarrow{e} s'$.

Given a state $s$ of $A$ and an edge $e$, we define $I(s,e) = \{ t \in \mathbb{R}_{\geq 0} \mid s \xrightarrow{e} s' \}$ for some $s'$ and $I(s) = \bigcup_{e \in E} I(s,e)$. We say that $A$ is non-blocking iff for all states $s$, $I(s) \neq \emptyset$. Now we are ready to introduce stochastic timed games.

**Definition 1** (Stochastic Timed Games [11]). A stochastic timed game (STG) is a tuple $G = (A, (L_0, L_\infty, L_\text{box}), \omega, \mu)$ where

- $A = (L, C, E, \mathcal{I})$ is a timed automaton;
- $L_0, L_\infty$ and $L_\text{box}$ form a partition of $L$ characterizing the set of locations controlled by
  players $\Box$ and $\Diamond$ and the stochastic player, respectively;
- $\omega : E(L_\infty) \to \mathbb{Z}_{\geq 0}$ assigns some positive weight to each edge originating from $L_\infty$
  (notation $E(L_\infty)$);
- $\mu$ is a function assigning a measure over $I(s)$ to all states $s \in L_\infty \times \mathbb{R}_{\geq 0}^{|C|}$ satisfying the
  properties that $\mu(s)(I(s)) = 1$ and for Lebesgue measure $\lambda$, if $\lambda(I(s)) > 0$ then for each
  measurable set $B \subseteq I(s)$ we have $\lambda(B) = 0$ if and only if $\mu(s)(B) = 0$.

The timed automaton $A$ is said equipped with uniform distributions over delays if for every
state $s$, $I(s)$ is bounded, and $\mu(s)$ is the uniform distribution over $I(s)$. The timed
automaton $A$ is said equipped with exponential distributions over delays whenever, for every
state $s$, either $I(s)$ has Lebesgue measure zero, or $I(s) = \mathbb{R}_{\geq 0}$ and for every location $l$,
there is a positive rational $\alpha_l$ such that $\mu(s)(I(s)) = \int_{t \in \mathbb{R}_{\geq 0}} \alpha_l e^{-\alpha_l t} dt$. For $s \in L_\infty \times \mathbb{R}_{\geq 0}^{|C|}$ both delays and discrete moves will be chosen probabilistically: from $s$, a delay $t$ is chosen following the probability distribution over delays $\mu(s)$. Then, from state $s + t$, an enabled edge is selected following a discrete probability distribution that is given in a usual way with the
weight function $w$: in state $s + t$, the probability of edge $e$ (if enabled), denoted $p(s + t)(e)$ is $w(e)/\sum_{e'} \{ w(e') \mid e' \text{ is enabled in } s + t \}$. This way of probabilizing behaviours in timed
automata has been presented in [8].

If $L_\text{box} = \emptyset$ then the STGs are called $1^1$ STGs or $1^1$-player STGs while STGs with $L_\text{box} = L_0 = \emptyset$
are called $1^0$ STGs or $1^0$-player STGs or STAs. We often refer to $l \in L_\infty$ as stochastic nodes,
l $L_0$ as box (or $\Box$) nodes and $L_\infty$ as diamond (or $\Diamond$) nodes.

Fix a STG $G = (A, (L_0, L_\infty, L_\text{box}), \omega, \mu)$ with $A = (L, C, E, \mathcal{I})$ for the rest of this section.

**Strategies, Profiles, and Runs.** A strategy for Player $\Box$ (resp. $\Diamond$) is a function that maps
a finite run $\rho = s_0 \xrightarrow{t_1} s_1 \xrightarrow{t_2} s_2 \ldots s_n$ to a pair $(t, e)$ such that $s_n \xrightarrow{e} s'$ for some state
$s'$, whenever $s_n = (t_n, v_n)$ and $t_n \in L_\Box$ (resp. $t_n \in L_\Diamond$). In this work we focus on
deterministic strategies, though randomized strategies could also make sense; nevertheless
understanding the case of deterministic strategies is already challenging. A strategy profile is a pair $\Lambda = (\lambda_\Box, \lambda_\Diamond)$ where $\lambda_\Box, \lambda_\Diamond$ respectively are strategies of players $\Box$ and $\Diamond$. In order to measure probabilities of certain sets of runs, the following measurability condition
is imposed on strategy profiles $\Lambda = (\lambda_\Box, \lambda_\Diamond)$: for every finite sequence of edges $e_1, \ldots, e_n$ and every state $s$, the function $\kappa_s : (t_1, \ldots, t_n) \to (t, e)$ defined by $\kappa_s(t_1, \ldots, t_n) = (t, e)$ iff $\Lambda(s, e_1, \ldots, e_n) = (t, e)$, should be measurable.

Given a finite run $\rho$ ending in state $s_0$, and a strategy profile $\Lambda$, define $\text{Runs}(G, \rho, \Lambda)$ (resp. $\text{Runs}^\omega(G, \rho, \Lambda)$) to be the set of all finite (resp. infinite) runs generated by $\Lambda$ after prefix $\rho$; that is, the set of all runs of the automaton satisfying the following condition: If $s_i = (t_i, v_i)$ and $t_i \in L_\Box$ (resp. $t_i \in L_\Diamond$), then $\lambda_\Box$ (resp. $\lambda_\Diamond$) returns $(t_{i+1}, e_{i+1})$ when applied to $\rho \xrightarrow{t_i} s_i \xrightarrow{e_{i+1}} s_{i+1}$. Given a finite sequence $e_1, \ldots, e_n$ of edges, a symbolic path $\pi_\Lambda(\rho, e_1 \ldots e_n)$ is defined as

$$\pi_\Lambda(\rho, e_1 \ldots e_n) = \{ \rho' \in \text{Runs}(G, \rho, \Lambda) \mid \rho' = \rho \xrightarrow{t_1} s_1 \xrightarrow{t_2} s_2 \ldots \xrightarrow{t_n} s_n, \text{ with } t_i \in \mathbb{R}_{\geq 0} \}. $$
When $\Lambda$ is clear, we simply write $\pi(\rho, e_1 \ldots e_n)$.

**Probability Measure of a Strategy Profile.** Given a strategy profile $\Lambda = (\lambda_\Diamond, \lambda_\Box)$, and a finite run $\rho$ ending in $s = (t, v)$, a measure $P_\Lambda$ can be defined on the set $\text{Run}(G, \rho, \Lambda)$, following [11]: First, for the empty sequence $e$, $P_\Lambda(\pi(\rho, e)) = 1$, and

- If $t \in L_\Diamond$ (resp. $t \in L_\Box$), and $\lambda_\Diamond(\rho) = (t, e)$ (resp. $\lambda_\Box(\rho) = (t, e)$), then $P_\Lambda(\pi(\rho, e_1 \ldots e_n))$ equals $0$ if $e_1 \neq e$ and equals $P_\Lambda(\pi(\rho \xrightarrow{t} s', e_2 \ldots e_n))$, otherwise.
- If $t \in L_\Box$ then $P_\Lambda(\pi(\rho, e_1 \ldots e_n)) = \int_{s \in I(s, e_1)} p(s + t)(e_1) \cdot P_\Lambda(\pi(\rho \xrightarrow{t} s', e_2 \ldots e_n)) dt$.

where $s \xrightarrow{t} s'$ for every $t \in I(s, e_1)$.

The cylinder generated by a symbolic path is defined as follows: an infinite run $\rho''$ is in the cylinder generated by $\pi_\Lambda(\rho, e_1, \ldots, e_n)$ denoted $\text{Cyl}(\pi_\Lambda(\rho, e_1, \ldots, e_n))$ if $\rho'' \in \text{Runs}_\omega(G, \rho, \Lambda)$ and there is a finite prefix $\rho'$ of $\rho''$ such that $\rho' \in \pi_\Lambda(\rho, e_1, \ldots, e_n)$. It is routine to extend the above measure $P_\Lambda$ to cylinders, and thereafter to the generated $\sigma$-algebra; extending [8], one can show this is indeed a probability measure over $\text{Runs}_\omega(G, \rho, \Lambda)$.

**Example.** An example of a STG is shown in the adjoining figure. In this example all the locations belong to stochastic player (this is an $\frac{1}{2}$ STG) and there is only one clock named $x$.

We explain here the method for computing probabilities. We assume uniform distribution over delays at all states, and initial state $s_0 = (A, 0)$. Let $d_{\mu(A, 0)}$ be the uniform distribution over $[0,1]$ and $d_{\mu(B, 0)}$ uniform distribution over $[0,2]$. Then $P(\pi((A, 0), e_1e_2))$ equals

$$\int_0^1 \frac{P(\pi((B, 0), e_2))}{2} d_{\mu(A, 0)}(t) = \int_0^1 \frac{1}{2} \left( \int_1^2 \frac{1}{2} d_{\mu(B, 0)}(u) du \right) dt = \frac{1}{8}.$$ 

**Reachability Problem.** We study the reachability problem for STGs, stated as follows. Given a STG $G$ with a set $T$ of target locations, an initial state $s_0$ and a threshold $\triangleright p$ with $p \in [0,1] \cap Q$, decide whether there is a strategy $\lambda_\Diamond$ for Player $\Diamond$ such that for every strategy $\lambda_\Box$ for Player $\Box$, $P_\Lambda(\{\rho \in \text{Run}(G, s_0, \Lambda) \mid \rho \text{ visits } T\}) \triangleright p$, with $\Lambda = (\lambda_\Diamond, \lambda_\Box)$. There are two categories of reachability questions:

1. **Quantitative reachability:** The constraint on probability involves $0 < p < 1$.
2. **Qualitative reachability:** The constraint on probability involves $p \in \{0,1\}$.

The key results of the paper are the following:

**Theorem 2.** The quantitative reachability problem is

1. Undecidable for $1\frac{1}{2}$ STGs with $4$ or more clocks;
2. Undecidable for $2\frac{1}{2}$ STGs with $5$ or more clocks even under the time-bounded semantics;
3. Decidable for $1\frac{1}{2}$ and $2\frac{1}{2}$ initialized STGs with one clock.

**3 Undecidability Results for Quantitative Reachability**

In this section, we focus on the quantitative reachability problem for STGs. We strengthen the existing undecidability result, which holds for $2\frac{1}{2}$ STGs [11], in two distinct directions.
First, we show the undecidability of the quantitative reachability problem in $1\frac{1}{2}$ STGs, improving from $2\frac{1}{2}$. Second, we show the undecidability of the quantitative reachability problem for $2\frac{1}{2}$ STGs even in the time-bounded setting.

For both results, given a two-counter machine, we construct respectively, $1\frac{1}{2}$ and $2\frac{1}{2}$ STGs whose building blocks are the modules for the instructions in the two-counter machine. The objective of player $\Diamond$ is linked to a faithful simulation of various increment, decrement and zero-test instructions of the two-counter machine by choosing appropriate delays to adjust the clocks to reflect changes in counter values. However, the two proofs differ in how this verification is done and even in the problem from which the reduction is done, i.e., halting/non-halting for two-counter machines. This results in two quite different and non-trivial reductions as described in Subsection 3.1 and Subsection 3.2 respectively.

### 3.1 Quantitative reachability for $1\frac{1}{2}$ STGs

As mentioned above, in the case of $1\frac{1}{2}$ STGs we improve the corresponding result of [11] for $2\frac{1}{2}$ STGs. But unlike in [11], we reduce from the non-halting problem for two-counter machines to the existence of a winning strategy for Player $\Diamond$ with the desired objective. This crucial difference makes it possible for the probabilistic player to verify the simulation performed by player $\Box$.

**Theorem 3.** The quantitative reachability problem is undecidable for $1\frac{1}{2}$ STGs with $\geq 4$ clocks.

Let $M$ be a two-counter machine. Our reduction uses a $1\frac{1}{2}$ player STG $G$ with four clocks and uniform distributions over delays, and a set of target locations $T$ such that player $\Diamond$ has a strategy to reach $T$ with probability $\frac{1}{2}$ iff $M$ does not halt. Each instruction (increment, decrement and test for zero value) is specified using a module. The main invariant in our reduction is that upon entry into a module, we have that $x_1 = \frac{1}{2^3}, x_2 = \frac{1}{2^3}, x_3 = x_4 = 0$, where $c_1$ (resp. $c_2$) is the value of counter $C_1$ (resp. $C_2$) in $M$.

![Figure 1](image-url) The Increment $c_1$ module on the top and the GetProb gadget below
We outline the simulation of an increment instruction \( \ell_1 \): increment counter \( C_1 \) goto \( \ell_1 \) in Figure 4 (top). The module is entered with values \( x_1 = \frac{1}{2^i}, x_2 = \frac{1}{2^j}, x_3 = x_4 = 0 \). A time \( 1 - \frac{1}{2^i} \) is spent at location \( \ell_1 \), so that at location \( B \) we have \( x_1 = 0, x_2 = \frac{1}{2^j} + 1 - \frac{1}{2^i} \) (or \( \frac{1}{2^i} - \frac{1}{2^j} \), if \( c_2 > c_1 \) — we write in all cases \( \frac{1}{2^i} + 1 - \frac{1}{2^j} \) mod 1), \( x_3 = 1 - \frac{1}{2^i}, x_4 = 0 \). An amount of time \( t \in (0, \frac{1}{2^i-1}) \) is spent at \( B \), which is decided by Player \( \Diamond \). We rewrite this as \( t = \frac{1}{2^i-1} \pm \epsilon \) for \( -\frac{1}{2^i-1} < \epsilon < \frac{1}{2^i-1} \). This is because, ideally we want \( t \) to be \( \frac{1}{2^i-1} \) and want to consider any deviation as an error.

Now at \( C \), we have \( x_1 = t, x_2 = \frac{1}{2^j} + 1 - \frac{1}{2^i} + t \) mod 1, \( x_3 = 1 - \frac{1}{2^i} + t, x_4 = 0 \). The computation proceeds to \( D \) with probability \( \frac{1}{7} \), and the location \( \ell_j \) corresponding to the next instruction \( \ell_j \) is reached with \( x_1 = \frac{1}{2^i} - t, x_2 = \frac{1}{2^j} + t, x_3 = x_4 = 0 \). On the other hand, with probability \( \frac{6}{7} \), the gadget \( \text{GetProb} \) is reached. The gadget \( \text{GetProb} \) has 4 target locations \( T_1, T_2, T_3, T_4 \), which we will show are reached with probability \( \frac{1}{2} \) from the start location \( E_0 \) of \( \text{GetProb} \) iff \( t = \frac{1}{2^i-1} \). Thus, in this case when \( t = \frac{1}{2^i-1} \), we reach \( \ell_j \) with the values \( x_1 = \frac{1}{2^i-1}, x_2 = \frac{1}{2^j}, x_3 = x_4 = 0 \) which implies that \( c_1 \) has been incremented correctly according to our encoding. We now look at the gadget \( \text{GetProb} \).

**Lemma 4.** For any value \( \epsilon \in (-\frac{1}{2^i-1}, \frac{1}{2^i-1}) \), the probability to reach a target location in \( \text{GetProb} \) from \( E_0 \) is \( \frac{1}{2} (1 - 4\epsilon^2) \) \( (\leq \frac{1}{2}) \). Further this probability is equal to \( \frac{1}{2} \) iff \( \epsilon = 0 \).

**Proof.** Note that when the start location \( E_0 \) of \( \text{GetProb} \) is reached, we have \( x_1 = \frac{1}{2^i-1} + \epsilon, x_2 = 0, x_3 = 1 - \frac{1}{2^i-1} + \epsilon, x_4 = 0 \). A total of 2 time units can be spent at \( E_0 \). It can be seen that transitions to \( E_3 \) and \( E_4 \) are respectively enabled with the time intervals \([0, 1 - \frac{1}{2^i-1} - \epsilon]\) and \([1, 1 + \frac{1}{2^i-1} - \epsilon]\). Similarly, reaching \( E_1 \) and \( E_2 \) are enabled by the time intervals \([1 - \frac{1}{2^i-1} - \epsilon, 1]\) and \([1 + \frac{1}{2^i-1} - \epsilon, 2]\). The sum of probabilities of reaching either \( E_3 \) or \( E_4 \) is thus \( \frac{1}{2} (1 - 2\epsilon) \). Similarly, the sum of probabilities for reaching \( E_1 \) or \( E_2 \) is \( \frac{1}{2} (1 + 2\epsilon) \). The locations \( P_1, P_2 \) are then reached with the values \( x_1 = \frac{1}{2^i-1} + \epsilon, x_2 = 0, x_3 = 1 - \frac{1}{2^i-1} + \epsilon, x_4 = 0 \). The probability of reaching the target locations \( T_3 \) or \( T_4 \) (i.e., through \( P_1 \)) from \( E_0 \) is hence \( \frac{1}{2} (1 + 2\epsilon) \frac{1}{2} (1 - 2\epsilon) = \frac{1}{4} (1 - 4\epsilon^2) \), while the probability of reaching a target location \( T_1 \) or \( T_2 \) (i.e., through \( P_2 \)) from \( E_0 \) is \( \frac{1}{2} (1 + 2\epsilon) \frac{1}{2} (1 - 2\epsilon) = \frac{1}{4} (1 - 4\epsilon^2) \). Thus, the probability of reaching a target location (one of \( T_1, T_2, T_3, T_4 \)) in \( \text{GetProb} \) is \( \frac{1}{2} (1 - 4\epsilon^2) \), which is always \( \leq \frac{1}{2} \). This completes the first statement of the lemma. Further, from the expression, we immediately have that the probability to reach a target location in \( \text{GetProb} \) from \( E_0 \) is \( \frac{1}{2} \) iff \( \epsilon = 0 \).

The decrement \( c_1 \), increment \( c_2 \) as well as decrement \( c_2 \) modules are similar and these as well as the zero test modules can be found in the Appendix.

**Lemma 5.** Player \( \Diamond \) has a strategy to reach the (set of) target locations in \( \mathcal{G} \) with probability \( \frac{1}{2} \) iff the two-counter machine does not halt.

**Proof.** Suppose the two-counter machine halts (say in \( k \) steps). Then there are two cases: (a) the simulations of all instructions are correct in \( \mathcal{G} \). In this case, the target location can be reached in either of the first \( k \) steps. By Lemma 3, the probability of reaching a target location in the first \( k \) steps is the summation \( \frac{1}{2} \left( \frac{1}{2} \right)^2 \cdot \frac{1}{2} + \left( \frac{1}{2} \right)^3 \cdot \frac{1}{2} + \cdots + \left( \frac{1}{2} \right)^k \cdot \frac{1}{2} < \frac{1}{2} \). (b) Player \( \Diamond \) made an error in the computation in the first \( k \) steps. But then again by Lemma 3 the finite sum obtained is \( < \frac{1}{2} \) (since in the error step(s), the probability to reach target locations is \( \frac{1}{2} - 4\epsilon^2 < \frac{1}{2} \)). Thus, if the two-counter machine halts, under any strategy of \( \Diamond \) player, the probability to reach the target locations is \( < \frac{1}{2} \).
On the other hand, suppose the two-counter machine does not halt. Then, if Player $\Diamond$ chooses the strategy which faithfully simulates all instructions of the two-counter machine, the probability to reach the (set of) target locations is given by the infinite sum $\sum_{i=0}^{\infty} (\frac{1}{2})^{i} = \frac{1}{2}$. Any other strategy of Player $\Diamond$ corresponds to performing at least one error in the simulation. In this case, the infinite sum obtained has at least one term of the form $(\frac{1}{2})^k (\frac{1}{2} - 4\varepsilon^2)$, for $\varepsilon^2 > 0$. Clearly, such an infinite sum does not sum to $\frac{1}{2}$. This concludes the proof. \hfill $\Box$

The previous proof can be changed for other thresholds and to use unbounded intervals and exponential distributions.

### 3.2 Time-bounded quantitative reachability for $2\frac{1}{2}$ STGs

In this section, we tackle the time-bounded version of the quantitative reachability problem. This strengthens the definition of reachability by considering a given time bound $\Delta$, and requiring that $\mathcal{P}_c(\{\rho \in \text{Run}(G,s_0,\sigma) \mid \rho \text{ visits } T \text{ within } \Delta \text{ time units}\}) \geq p$.

In this new framework, we show the undecidability of the quantitative reachability problem for $2\frac{1}{2}$ STGs. We reduce from the halting problem for two-counter machines (unlike in the previous section, where our reduction was from the non-halting problem), using Player $\Box$ to verify the correctness of the simulation. The complication here is that the total time spent should be bounded and hence we cannot allow arbitrary time elapses. We will in fact show a global time bound of $\Delta = 5$ for this reduction.

**Theorem 6.** The time-bounded quantitative reachability problem is undecidable for $2\frac{1}{2}$ STGs with $\geq 5$ clocks.

**Proof.** Let $\mathcal{M}$ be a two-counter machine. We construct an STG with 5 clocks such that the two-counter machine $\mathcal{M}$ halts iff Player $\Diamond$ has a strategy to reach some desired locations with probability $\frac{1}{2}$, whatever Player $\Box$ does, and such that the total time spent is bounded by $\Delta = 5$ units.

The main idea behind the proof is that the total time spent in the simulation of the $k^{th}$ instruction will be $\frac{1}{2}$. We thus get a decreasing sequence of times $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ for simulating the instructions $1, 2, \ldots$ and so on. In total, we will use five clocks $x_1, x_2, z, a$ and $b$. The clocks $x_1$ and $x_2$ are used encode the counter values (along with the current instruction number) such that at the end of the $k^{th}$ instruction, if $k$ is even the values are encoded in $x_1$ and if $k$ is odd they are encoded in $x_2$ as follows:

- $(\text{enc}_{x_1})$: $k$ is even and $x_1 = \frac{1}{2^{x_1+1}3^{x_2+2}}, x_2 = 0, z = 1 - \frac{1}{2^k}, a = b = 0$;
- $(\text{enc}_{x_2})$: $k$ is odd and $x_2 = \frac{1}{2^{x_1+1}3^{x_2+2}}, x_1 = 0, z = 1 - \frac{1}{2^k}, a = b = 0$.

We start the simulation with $x_1 = 1, x_2 = z = 0 = a = b$ corresponding to the initial instruction ($k = 0$) and the fact that the values of $C_1, C_2$ are 0. Moreover, if $x_1 = \frac{1}{2^{x_1+1}3^{x_2+2}}$ at the end of the $k^{th}$ instruction, and if the $(k+1)^{th}$ instruction is an increment $C_1$ instruction, then at the end of the $(k+1)^{th}$ instruction, $x_2 = \frac{1}{2^{x_1+1}3^{x_2+2}}$. Clock $z$ keeps a separate track of the number of instructions simulated so far, by having a value $1 - \frac{1}{2^k}$ after completing the simulation of $k$ instructions. Clocks $a$ and $b$ are auxiliary clocks that we need for the simulation. We assume uniform distribution over delays in probabilistic locations. If no weight is written on an edge, it is assumed to be 1.

We outline the simulation of a increment instruction $\ell_i$, increment counter $C_1$, goto $\ell_j$. In Figure 5, assuming this is the $(k+1)^{th}$ instruction, where $k$ is even. Thus, at the end of the $k$ first instructions, we have $x_1 = \frac{1}{2^{x_1+1}3^{x_2+2}}, z = 1 - \frac{1}{2^k}$ and $a = b = x_2 = 0$ (the other
check case of odd \(k\), i.e., \((\text{enc}_{x_2})\) encoding is symmetric). At the end of this \((k + 1)\)th instruction’s simulation, the value of clock \(z\) should be \(z = 1 - \frac{1}{2^t}\) to mark the end of the \((k + 1)\)th instruction. Also, we must obtain \(x_2 = \frac{3}{2^t - 3} = \frac{3}{12}\), marking the successful increment of \(C_1\).

\[
\begin{align*}
\text{Check} & : a < 1 \quad \text{Check} \quad x_2 : 0 \\
\text{Check} & : b = 0 \quad \text{Check} \quad x_1, a := 0
\end{align*}
\]

Figure 2 Module for incrementing \(C_1\) (after an even number of steps)

Player \(\Diamond\) elapses times \(t_1, t_2\) in locations \(\ell_i, B\). When the player \(\Box\) location \(\text{Check}\) is reached, we have \(a = t_1 + t_2 = t\) and \(x_2 = t_2\), \(z = 1 - \frac{1}{2^t} + t_1 + t_2\). Player \(\Box\) has three possibilities: (1) to continue the simulation going to \(\ell_{k+2}\), (2) verify that \(t_2 = \frac{1}{2^t + 1 + 2^t + 1 + 2^t}\) by going to the widget ‘Check \(x_2\)’ or (3) verify that \(t_1 + t_2 = \frac{1}{2^t}\) by going to the widget ‘Check \(z\)’. These widgets are given in Figure 3. The probability of reaching a target location in widget ‘Check \(z\)’ is \(\frac{1}{2}(1 - t) + \frac{1}{2^t} = \frac{1}{2}\) iff \(t = \frac{1}{2^t}\). In widget ‘Check \(x_2\)’, the transitions from \(F_1\) to \(C_1\) and \(F_1\) to \(C_2\) are taken with probability \(\frac{1}{12}\) and \(\frac{1}{4}\), respectively since the weights of edges connecting \(F_1\) to \(C_1\) and \(F_1\) to \(C_2\) are respectively 1 and 11. With this, for \(n = \frac{1}{2} + \frac{1}{2^t} + \frac{1}{2^t + 1 + 2^t + 1 + 2^t}\), the probability of reaching a target location in ‘Check \(x_2\)’ is \(\frac{1}{2}(1 - t) + \frac{1}{2^t} = \frac{1}{2}\) iff \(t_2 = \frac{1}{12}\).

\[
\begin{align*}
\text{Check} & : b = 0 \quad \text{Check} \quad a > 1 \\
\text{Check} & : b \leq 1 \quad \text{Check} \quad z \leq 2 \\
\text{Check} & : b = 0 \quad \text{Check} \quad a \leq 1
\end{align*}
\]

Figure 3 Widgets ‘Check \(z\)’ (left) and ‘Check \(x_2\)’ (right)

**Time elapse for Increment.** If player \(\Box\) goes ahead with the simulation, the time elapse for the \((k + 1)\)th instruction is \(t_1 + t_2 = \frac{1}{2^t + 1}\). Consider the case when player \(\Box\) goes in to ‘Check \(z\)’. The time elapse till now is \(\frac{1}{2} + \cdots + \frac{1}{2^t + 1}\). The time spent in the ‘Check \(z\)’ widget is as follows: one unit is spent at location \(B_0\), one unit at location \(F_0\), and \(1 - t\) units at location \(E_0\). Thus, \(\leq 3\) units are spent at the ‘Check \(z\)’ widget. Similarly, the time spent in the ‘Check \(x_2\)’ widget is one unit at \(B_1\), \(1 - t\) units at \(C_1\), \(1 - n\) units at \(D_1\) and one unit at \(E_1\). Thus a time \(\leq 4\) is spent in ‘Check \(x_2\)’. Thus, the time spent till the \((k + 1)\)th instruction is \(\leq \frac{1}{2} + \cdots + \frac{1}{2^t + 1} + 4\) if player \(\Box\) goes in for a check, and otherwise it is \(\frac{1}{2} + \cdots + \frac{1}{2^t + 1}\).

**Other increment, decrement, zero-check Instructions.** The main module corresponding to increment \(C_2\) and decrement \(C_1, C_2\) is the same as in Figure 2. The only change needed is in the ‘Check \(x_2\)’ widget. While incrementing \(C_2\), we need \(x_2 = \frac{1}{2^t - 3} = \frac{1}{12}\). This is done by changing the weights on the outgoing edges from \(F_1\) to \(C_1\) and \(C_2\) to 1 and 17 respectively. Similarly, while decrementing \(C_1\), we need \(x_2 = \frac{1}{2}\). This is done by changing the weights on the outgoing edges of \(F_1\) to 1,2 respectively. Lastly, to decrement \(C_2\), we need \(x_2 = \frac{1}{2}\), and in this case the weights are 1 each.
The zero check module is a bit more complicated. The broad idea is that we use a diamond node to guess whether the current clock (say $C_1$) value is zero and branch into two sides (zero and non-zero). Then we use a box node on each branch to verify that the guess was correct. If correct, we proceed with the next instruction, if not, we check this by going to a special widget. In this widget, we can reach a target node with probability $\frac{1}{2}$ iff the guess is correct. The details of this widget and the proof that all these simulations can be done in time bounded by $\Delta \leq 5$ units is given in the Appendix.

4 Decidability results for quantitative reachability

We have seen in the previous section that the quantitative reachability problem is undecidable in $1\frac{1}{2}$ STGs with $\geq 4$ clocks. In this section we study the quantitative reachability problem in the setting of $1\frac{1}{2}$ STGs with a single clock. In [7], the quantitative reachability problem in $\frac{1}{2}$ STGs with a single clock, under certain restrictions, was shown to be decidable by reducing it to the quantitative reachability problem for finite Markov chains. In our case, we lift this to $1\frac{1}{2}$ STGs with a single clock, under similar restrictions, by reducing to the quantitative reachability problem in finite Markov decision processes (MDPs in short).

For the rest of this section, we consider a $1\frac{1}{2}$ STG $G = (A, (L_0, L_\infty), \omega, \mu)$ with a single clock denoted $x$. We write $c_{\max}$ for the maximal constant appearing in a guard of $G$.

We assume w.l.o.g. that target locations belong to player $\Diamond$ (a slight modification of the construction can be done if this is not the case). In the following, when we talk about regions, we mean the clock regions from the classical region construction for timed automata [1, 18]: since $G$ has a single clock, regions in this case are simply either singletons $\{c\}$ with $c \in \mathbb{Z}_{\geq 0} \cap ]0; c_{\max}]$, or open intervals $(c, c+1)$ with $c \in \mathbb{Z}_{\geq 0} \cap ]0; c_{\max}-1]$, or the unbounded interval $]c_{\max}, +\infty]$. While region automata are standardly finite automata, we build here from $G$ a region STG $G_R$, which has only clock constraints defined by regions (that is, either $x = c$ or $c < x < c + 1$ or $x > c_{\max}$), and such that each location of $G_R$ is indeed a pair $((\ell, R))$ where $\ell$ is a location of $G$ and $R$ a region (region $R$ is for the region which is hit when entering the location). While it is not completely standard, this kind of construction has been already used in [5, 7, 11], and questions asked on $G$ can be equivalently asked (and answered) on $G_R$. Now, we make the following restrictions on $G_R$ (which yields restrictions to $G$), which we denote $(*)$:

1. The TA $A$ is assumed to be structurally non-Zeno: any bounded cycle of $A$ (a cycle in which all edges have a non-trivial upper-bound) contains at least one location whose associated region is the zero region (i.e., edge leading to it, resets the clock).
2. For every state $s = ((\ell, r), v)$ of $G_R$ such that $\ell \in L_\Diamond$, $I(s) = \mathbb{R}_{\geq 0}$, and $\mu_s$ is an exponential distribution; Furthermore the rate of $\mu_s$ only depends on location $\ell$.
3. $G_R$ is initialized, that is, any edge from a non-stochastic location to a stochastic location resets the clock $x$.

While the first two assumptions are already made in [7], even in the $\frac{1}{2}$ player case, the third condition is new. In the following we denote $0$ for the region $\{0\}$ and $\infty$ for the unbounded region $]c_{\max}, +\infty[$.

We now show how to obtain an MDP from the STG $G_R$. The construction is illustrated on Figure 4.

A node $((\ell, R))$ of $G_R$ with $\ell \in L_\Diamond$ is deletable if $R$ is neither the region $0$ nor the region $\infty$. In Figure 4 (B, (0, 1)) and (A, (0, 1)) in $G_R$ are what we call deletable nodes. Then, we recursively remove all deletable nodes $G_R$ while labelling remaining paths with (finite) sequences of edges; each surviving edge is labelled by the probability of the (provably)
finite many sequences of edges appearing in the label. One can prove that this object is actually an MDP, which we denote $M_G$. Target states in $M_G$ are defined as the pairs $(\ell, R)$ where $\ell$ is a target location in $G$. We can prove that:

Lemma 7. If $G$ is an $1\frac{1}{2}$ player STG with one clock satisfying the hypotheses (*), then $M_G$ is an MDP such that: (a) for every strategy $\lambda_\diamond$ of player $\diamond$ in $G$, we can construct a strategy $\sigma_\diamond$ in $M_G$ such that the probability of reaching a target location in $G$ is the same as the probability of reaching a target state in $M_G$; and (b) for every strategy $\sigma_\diamond$ of player $\diamond$ in $M_G$, we can construct a strategy $\lambda_\diamond$ of player $\diamond$ in $G$ such that the probability of reaching a target location in $M_G$ is the same as the probability of reaching a target state in $G$.

This lemma allows to reduce the quantitative reachability problem from the $1\frac{1}{2}$ STG $G$ to the MDP $M_G$.

As an example, in Figure 4, we show a $1\frac{1}{2}$ player STG $G$, its region game graph $G_R$ and the MDP abstraction $M_G$. Note that all $\Diamond$ nodes remain, while only those stochastic nodes with regions $0$ and $\infty$ are retained in $M_G$. The stochastic nodes $(B, (0, 1))$ as well as $(C, (0, 1))$ are deleted in $M_G$. On deleting nodes from the region graph, the probability on the edges of $M_G$ is the probability of the respective paths from the region graph. For example, the edge from $(A, 0)$ to $(D, (0, 1))$ is labelled with $e_4e_7$ by deleting $(B, (0, 1))$.

Thus, the remaining thing that has to be addressed now is how to compute the probabilities and compare them with a rational threshold. The first thing to note is that the edges of the MDP are all labelled with polynomials over exponentials obtained using the delays from the underlying game with rational coefficients. For example, in Figure 4 in the MDP in the rightmost picture, we obtain: $P(e_1) = P(e_2) = P(e_5) = e^{-1}$, $P(e_6) = P(e_7) = P(e_8) = 1 - e^{-1}$, $P(e_4e_5) = e^{-1} - e^{-2}$, $P(e_4e_7) = 1 - 2e^{-1}$, $P(e_3e_4e_7) = 2 - 5e^{-1} + e^{-2}$, $P(e_3e_4e_5) = 1 - e^{-1} + e^{-2}$, and $P(e_3e_1) = \frac{1}{2}(1 - e^{-2})$. It can be seen that we can write each of these probabilities as a polynomial in $e^{-1}$. More generally, for any MDP with differing rates (of the exponential distribution) in each state, we get a set of rational functions in $e^{-\frac{q}{2}}$ for some $q \in \mathbb{Z}_{>0}$, where $q$ is obtained as a function of the rates in each state. Thus, using standard algorithms for MDPs [5], and as done for Markov chains in [7], we get that we can compute expressions for the probability of reaching the targets, and decide the threshold problem.

Theorem 8. Quantitative reachability for 1-clock $1\frac{1}{2}$-player STGs satisfying (*) is decidable.

We can lift this construction to include $\square$ player nodes, keeping the same initialized restriction with $\Box$ nodes as well. Then the region game graph $G_R$ includes $\Box$ nodes in the
obvious way, and we consider strategy profiles of $\Box$ and $\Diamond$. The question then is to check if $\Diamond$ has a strategy to reach a target with probability $\sim c$ against all possible strategies of $\Box$ in $M_G$. Hence we have that

**Corollary 9.** Quantitative reachability for 1-clock 2\$\frac{1}{2}\$ player STGs satisfying ($\ast$) is decidable.

## 5 Discussion

In this paper, we have refined the decidability boundaries for STGs as summarized in the table in Introduction. The significance of our undecidability results for quantitative reachability (via different two-counter machine reductions) lies in the fact that they introduce ideas which could potentially help in settling other open problems. We highlight these below:

- for 1\$\frac{1}{2}\$ player games, the crux is to cleverly encode the error $\epsilon$ made by player $\Diamond$ in such a way that it reflects as $\frac{1}{2} - \epsilon^2$ in the resulting probability. This ensures that the $\Diamond$ player can never cheat and the probability will be $< \frac{1}{2}$ as soon as there is an error (even when simulating a non-halting run of the two-counter machine). Indeed, this is why the reduction is from the non-recursively enumerable non-halting problem.
- for 2\$\frac{1}{2}\$ player games in the time-bounded setting, we obtain undecidability by showing a reduction from halting problem for two-counter machines. This is surprising, as time-boundedness restores decidability in several classical undecidable problems like the inclusion problem in timed automata [20] [21]. In the case of priced timed games [13], time-boundedness gives undecidability; however, this can be attributed to the fact that price variables are not clocks, and can grow at different rates in different locations.

Somehow, the combination of simple clocks and probabilities achieves the same.

Combining these ideas would, e.g., allow us to improve Theorem 6 by showing undecidability of time bounded, quantitative reachability in 1\$\frac{1}{2}\$ player STGs with a larger number of clocks. The main intricacy is to replace $\Box$ player nodes by stochastic nodes, and adapt the gadgets in such a way that, within a global time bound, the probability of reaching a target is $\frac{1}{2}$ iff all simulations are correct and the two-counter machine does not halt. As another example, if in the first item above, we obtain a probability of $1 - \epsilon^2$ (rather than $\frac{1}{2} - \epsilon^2$), this would settle the (currently open) qualitative reachability problem for 2\$\frac{1}{2}\$ games [11].

Coming to decidability results, we have for the first time characterized a family of 1\$\frac{1}{2}$, 2\$\frac{1}{2}$ player STGs for whom the quantitative reachability is decidable. The use of exponential distributions is mandatory to get a closed form expression for the probability. It is unclear if this construction can be extended to some larger classes of STGs. Figure 9 in [8] shows an example of a two-clock 1\$\frac{1}{2}$ player game for which the region abstraction fails to give any relevant information on the real “probabilistic” behaviour of the system (lack of so-called fairness); in particular it cannot be used for qualitative, and therefore quantitative, analysis of reachability properties. The decidability of qualitative reachability in 1\$\frac{1}{2}$, 2\$\frac{1}{2}$, multi-clock STG seems then hard due to the same problem of unfair runs. If one restricts to one clock, then the qualitative reachability of 1\$\frac{1}{2}$ STGs is decidable [11]. We conjecture that this can be extended to 2\$\frac{1}{2}$ STGs in the single clock case.

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Appendix

A Counter Machines

A two-counter machine $M$ is a tuple $(L, C)$ where $L = \{\ell_0, \ell_1, \ldots, \ell_n\}$ is the set of instructions—including a distinguished terminal instruction $\ell_n$ called HALT—and $C = \{c_1, c_2\}$ is the set of two counters. The instructions $L$ are one of the following types:

1. (increment $c$) $\ell_i : c := c + 1; \text{ goto } \ell_k$,
2. (decrement $c$) $\ell_i : c := c - 1; \text{ goto } \ell_k$,
3. (zero-check $c$) $\ell_i : \text{ if } (c > 0) \text{ then goto } \ell_k \text{ else goto } \ell_m$,
4. (Halt) $\ell_n : \text{HALT}$.

where $c \in C$, $\ell_i, \ell_k, \ell_m \in L$. A configuration of a two-counter machine is a tuple $(l, c, d)$ where $l \in L$ is an instruction, and $c, d$ are natural numbers that specify the value of counters $c_1$ and $c_2$, respectively. The initial configuration is $(\ell_0, 0, 0)$. A run of a two-counter machine is a (finite or infinite) sequence of configurations $(k_0, k_1, \ldots)$ where $k_0$ is the initial configuration, and the relation between subsequent configurations is governed by transitions between respective instructions. The run is a finite sequence if and only if the last configuration is the terminal instruction $\ell_n$. Note that a two-counter machine has exactly one run starting from the initial configuration. The halting problem for a two-counter machine asks whether its unique run ends at the terminal instruction $\ell_n$. It is well known ([19]) that the halting problem for two-counter machines is undecidable.

B Undecidability of Quantitative Reachability for $1\frac{1}{2}$ STGs

We complete the proof of the undecidability for qualitative reachability in $1\frac{1}{2}$ STGs. The simulation of an increment instruction was described in section 3.1. Here we describe the gadgets simulating decrement and zero test instructions. Figure 5 describes the gadget simulating the instruction $\ell_i : \text{if } C_1 > 0, \text{ then goto } \ell_j, \text{ else goto } \ell_k$. It can be seen that with probability $\frac{1}{2}$, the next instruction is simulated, while with probability $\frac{1}{2}$, we reach a target location.

![Figure 5 Zero Test Instruction](image-url)
Next, let us see the simulation of a decrement instruction $\ell_i$: decrement $C_1$, goto $\ell_j$. Figure 6 depicts this.

The decrement module has as its initial location $\ell_i$, which is entered with values $x_1 = \frac{1}{2^{7}}, x_2 = \frac{1}{2^{7}}, x_3 = x_4 = 0$. A non-deterministic time $t$ is spent at $\ell_i$. Ideally, $t = 1 - \frac{1}{2^{7-1}}$. At the stochastic node $B$, no time is spent. The simulation continues from the location $D$: $D$ is entered resetting $x_1$. At $D$ we thus have $x_1 = 0, x_2 = \frac{1}{2^{7}} + t, x_3 = t, x_4 = 0$. At $D$, a time $1 - t$ is spent, reaching $\ell_j$ with values $x_1 = t, x_2 = \frac{1}{2^{7}}, x_3 = x_4 = 0$.

Assume that the time spent at $\ell_i$, $t = 1 - \frac{1}{2^{7-1}} + \epsilon$. Now consider the case of going to the location $C$ from $B$ resetting $x_2$. At $C$, we have $x_1 = \frac{1}{2^{7}} + t = 1 - \frac{1}{2^{7}} + \epsilon, x_2 = 0, x_3 = 1 - \frac{1}{2^{7-1}} + \epsilon, x_4 = 0$. The gadget $GetProb$ is entered with values $x_1 = 0, x_2 = \frac{1}{2^{7}} - \epsilon, x_3 = 1 - \frac{1}{2^{7}}, x_4 = 0$. The initial location of $GetProb$ is $E_0$.

**Figure 6** The decrement $c_1$ module on the left and the $GetProb$ gadget on the right.
A total of 2 units of time can be spent at \( E_0 \). It can be seen that the time intervals \( [0, \frac{1}{2^1}] \) and \( [1, 2 - \frac{1}{2^1} + \epsilon] \) respectively are enabled to reach \( E_3 \) and \( E_4 \). Similarly, the time intervals \( [\frac{1}{2^1}, 1] \) and \( [2 - \frac{1}{2^1} + \epsilon, 2] \) respectively are enabled to reach \( E_1 \) and \( E_2 \). The probability of reaching \( E_3 \) or \( E_4 \) is thus \( \frac{1}{2}(1 + \epsilon) \) and the probability of reaching \( E_1 \) or \( E_2 \) is thus \( \frac{1}{2}(1 - \epsilon) \). The locations \( P_1, P_2 \) are reached with \( x_1 = 0, x_2 = \frac{1}{2^1} - \epsilon, x_3 = 1 - \frac{1}{2^1}, x_4 = 0 \). The probability of reaching a target location through \( P_1 \) (from \( E_0 \)) is hence \( \frac{1}{2}(1 + \epsilon) \frac{1}{2}(1 - \epsilon) = \frac{1}{4}(1 - \epsilon^2) \), while the probability of reaching a target location through \( P_2 \) (from \( E_0 \)) is \( \frac{1}{2}(1 + \epsilon) \frac{1}{2}(1 - \epsilon) = \frac{1}{4}(1 - \epsilon^2) \). The probability of reaching a target location in \( \text{GetProb} \) is thus, \( \frac{1}{2}(1 - 2\epsilon^2) \). Note that if we start with \( t = 1 - \frac{1}{2^{1-t}} - \epsilon \), we obtain exactly the same probability. Thus, the probability to reach a target location in \( \text{GetProb} \) is \( \frac{1}{2} \) iff \( \epsilon = 0 \).

### C Time-bounded quantitative reachability for \( 2\frac{1}{2} \) STGs

The details of the zero check (and the proof that it can be done in bounded time), which were missing in the main paper, due to lack of space, are given below. Let us consider \((wlog)\) the case when the \((k+1)th\) instruction checks whether counter \( C_1 \) is zero. Assume that after \( k \) instructions, we have \( x_1 = \frac{1}{2^{k+1}+\epsilon}, x_2 = 0, z = 1 - \frac{1}{2^t} \) and \( a = b = 0 \). The main module, given in Figure 7, can be divided into two parts.

![Figure 7 Zero Check C1(x1). x1 holds the value \( \frac{1}{2^{k+1}+\epsilon} \) on entering the module.](image)

1. First, we make sure that the instruction counter, i.e., Clock \( z \) is updated correctly: we spend times \( t_1, t_2 \) at locations \( l_{k+1}, B \) respectively, and check that \( t_1 + t_2 = \frac{1}{2^{t+1}} \) and \( t_2 = \frac{1}{2^{t+1}+\epsilon} \). For this it suffices to check that at location \( \text{Check} \) we have \( x_2 = \frac{1}{2^{k+1}+\epsilon}, x_1 = \frac{1}{2^{k+1}+\epsilon} + t_1 + t_2, z = 1 - \frac{1}{2^t} a = t_1 + t_2 \) and \( b = 0 \). This is done, as before, by the Player \( \Box \) using widgets \( \text{Check} \) \( z \) (given in Figure 3) and \( \text{Check} \) \( x \) similar to the widget \( \text{Check} \) \( x_2 \) in Figure 3 where one simply changes the weights on edges of \( F_1 \) to \( C_1 \) and \( C_2 \) to 1 and 5 respectively. Then, we proceed to \( D \).

2. At \( D \), player \( \Diamond \) guesses whether \( C_1 = 0 \) or not, by choosing an appropriate \( \Box \) location. From these, player \( \Box \) can either allow the simulation to continue, or check the correctness of \( \Diamond 's \) guess. This check is done in three steps:

   a. First, we eliminate \( k \) from \( \frac{1}{2^{k+1}+\epsilon} \) by multiplying by 6 for \( k \) times, and from \( a = \frac{1}{b} \) obtaining \( a = 1 \). Each time multiplication by 6 happens, the clocks \( x_1, x_2 \) alternate. The widgets \( \text{Rem}_k^1 \) and \( \text{Rem}_k^2 \) (Figure 3) are used alternately as long as \( a < 1 \), and
\(x_1, x_2\) alternately store values \(\frac{1}{2^{1+1+3^2+b}}\) \(\frac{1}{2^{1+1+3^2+b}}\) till \(\frac{1}{2^{1+1+3^2+b}}\) is obtained in one of \(x_1, x_2\).

b. Once \(\frac{1}{2^{1+1+3^2+b}}\) is obtained in \(x_1\) or \(x_2\), we further multiply by 3 for \(c_2\) times to obtain \(\frac{1}{2^1}\). This is done as represented in widgets Wid\(_{=0}\), Wid\(_{>0}\).

c. Finally, to check if player \(\Diamond\)'s guess is correct or not, we only need to check if \(x_1\) or \(x_2\) is 1 which corresponds to \(c_1 = 0\).

It can be seen that a target location is reached with probability \(\frac{1}{x}\) from Figure 7 iff (1) the \((k + 1)\)th instruction (zero check) is accounted for correctly, at locations \(l_{k+1}\) and \(B\) in figure 7. The widgets Check \(z\) and Check \(x\) check this. (2) Player \(\Diamond\) guesses correctly whether \(C_1\) is zero or not. If player \(\Box\) goes in for further checks, then player \(\Diamond\) must be faithful in the widgets Rem\(_k^z\) and Rem\(_k^x\), and also in widgets Wid\(_{=0}\) and Wid\(_{>0}\).

Figure 8 Rem\(_k^z\): Times \(t_1, t_2\) spent at \(A0, B2\) such that \(t_1 + t_2 = \frac{1}{x}\) and \(t_2 = \frac{1}{2^{1+1+3^2+b}}\). Note that \(k \geq 2\). Mul \(a\) checks on \(t_1 + t_2\) while Mul \(x_2\) checks on \(t_2\). Note that if Rem\(_k^z\) is entered from the > 0 \(\Box\) location of Figure 7 then the target in Rem\(_k^z\) is not reached with probability \(\frac{1}{x}\) when \(a = \frac{1}{x}\) and \(x_2 = \frac{1}{x}\) as this corresponds to the scenario where \(C_1 = C_2 = 0\) implying an incorrect guess by Player \(\Diamond\) that \(C_1 > 0\).

Figure 9 Mul \(a\) and Mul \(x_2\). On entry, \(x_1 = t_2, b = 0, x_2 = n + t_1 + t_2, z = t_1 + t_2, a = \frac{1}{2^{1+1+3^2+b}}\). Probability to reach a target in Mul \(a\) is \(\frac{1}{2^1} t_1 + t_2 + \frac{1}{2^1}(1 - \frac{1}{2^{1+1+3^2+b}})\), while that in Mul \(x_2\) is \(\frac{1}{2^1} (1 - n) + \frac{1}{2^1} t_2\). Thus, a target is reached in Mul \(a\) with probability \(\frac{1}{2^1}\) iff \(t_1 + t_2 = \frac{1}{2^{1+1+3^2+b}}\). That makes \(a = \frac{1}{2^1}\) at the end. Likewise, to get a probability \(\frac{1}{2}\) in Mul \(x_2\), we need \(t_2 = 6n\).
The time elapse in the main module for zero check is a. If player Q, the total time elapse till completion of Assume that the zero check is the b.

Figure 10 Wid:=0. A5 is the start node. J5 is entered with $a = t, x_1 = \frac{1}{2^{t+1}} + t$ and $b = 0, t$ is the time spent at A5. Probability to reach a target location from J5 is $\frac{1}{2} \cdot (1 + t) + \frac{1}{2^{t+1}}$ which is $\frac{1}{2}$ if $t = 2 \cdot \frac{1}{2^{t+1}}$. This verifies that value $\frac{1}{2^{t+1}}$ in $x_1$ is multiplied by 3 each time the loop is taken, since $x_1$ becomes $\frac{1}{2^{t+1}} + t = \frac{1}{2^{t+2}}$. Wid:=0 is obtained simply having a multiply by 2 module at R5.

**Time Elapse for Zero Check**

Let us start looking at the main module for zero check in Figure 7. Assume that this is the $(k + 1)$th instruction. A time $t_1 + t_2 = \frac{1}{2^{t+1}}$ is spent at locations $t_{k+1}, B$ in Figure 7. Following this, if player □ goes in for a check in widgets Check $z$ or Check $x$, the time elapse in these widgets is $< 4$ as seen in the Increment section. If not, control reaches one of the player □ locations = 0 or $> 0$. Here again, player □ can either go ahead, or enter the Rem $x$ widget.

The Rem $x$ widget is entered with $a = \frac{1}{2^{t+1}}, x_2 = \frac{1}{2^{t+1}} + t = 1, b = 0$. The time spent at A0, B2 is $\frac{1}{2^{t+1}}$. When control comes to the □ node, there are two possibilities:

1. player □ continues with the Rem $x$ widget, in which case a time $\frac{1}{2^{t+1}}$ is elapsed. This can continue till $a = \frac{1}{2}, t = \frac{1}{2^{t+1}} + \frac{1}{2^{t+2}} + \cdots + \frac{1}{2}$ is elapsed after which, the target $T$ is reached with probability $\frac{1}{2}$, or continues till $a = 1$ with time elapsed $\frac{1}{2^{t+1}} + \frac{1}{2^{t+2}} + \cdots + \frac{1}{2}$ and control goes into the widget Wid:=0 or Mul $a$ or Mul $x_2$. The time elapse in the Mul $a$, Mul $x_2$ widgets is $< 4$. In the case of Wid:=0, if the loop $B5 - A5$ is taken till $x_1 = 1$, a time $\frac{1}{2^{t+1}} + \frac{1}{2^{t+2}} + \cdots + \frac{1}{2^{t+1}}$ is spent till target $H5$ is reached. However, if player □ reaches out to the part from node J5, the time elapse is almost 2 to reach a target. Thus, summing up, the time elapse is

1. Assume that the zero check is the $(k + 1)$th instruction. The time that has elapsed till the start of this instruction is $\frac{1}{2} + \cdots + \frac{1}{2t}$.
2. The time elapse in the main module for zero check is $\frac{1}{2}\cdot t$. If player □ continues with the simulation, we are done.
3. If player □ enters any of the widgets (Check $x$, Check $z$, Rem $x$, Rem $x$, Wid:=0, Wid:=0), the time elapse is $< 4$ till a target is reached.
4. The total time elapse till completion of $(k + 1)$ instructions is thus $< \frac{1}{2} + \cdots + \frac{1}{2t} + \frac{1}{2^{t+1}} + 4$.

**Halting and Correctness of construction** The gadget corresponding to the halt instruction
is as follows: Once we reach the halt instruction, we go to a stochastic node \( A \) with no time delay. \( A \) has two outgoing edges, one which leads to a target node, and the other one to a non-target. With no delay at \( A \), the target is reached with probability \( \frac{1}{2} \). We quickly give an intuition behind the proof of correctness of this construction: Assume that the two counter machine halts. If Player \( \Diamond \) simulates all the instructions correctly, there are two possibilities:

1. Player \( \square \) allows simulation of the next instruction without entering any of the check gadgets. Then we will reach the halt location from where the probability to reach the target is indeed \( \frac{1}{2} \).

2. Player \( \square \) enters any of the check gadgets during the simulation of some instruction. As can be seen from our earlier detailed analysis, it is indeed the case that the probability to reach a target location is \( \frac{1}{2} \).

Assume now that the two counter machine does not halt. If Player \( \Diamond \) indeed simulates all the instructions correctly once again, then the only way to reach any target location is only by invoking a check gadget by Player \( \Box \). As said above, clearly, this probability will be \( \frac{1}{2} \) due to the correct simulation of Player \( \Diamond \). Again, note that the times spent during increment/decrement of the \((k + 1)\)th instruction is \( \frac{1}{2^{k+1}} \). This fact can be verified by the gadget \( \text{Check}_z \). In case of non-halting, therefore, the total time taken will converge to 1. Thus, the time taken to reach any target location is \( \leq 1 \) in case of non-halting and correct simulation by Player \( \Diamond \). Of course, if Player \( \Box \) never chooses to enter any of the check gadgets, then Player \( \Diamond \) can never reach a target location, and hence cannot win. The total elapse in case Player \( \Box \) enters a check gadget in the \((k + 1)\)th instruction is \( < 4 + \frac{1}{2} + \cdots + \frac{1}{2^k} < 5 \).

In both cases, if Player \( \Diamond \) does not simulate correctly the instruction, Player \( \Box \) can decide to check and the probability to reach a target location will be \( < \frac{1}{2} \). Hence, Player \( \Diamond \) has a winning strategy to ensure probability \( \frac{1}{2} \) for reaching a target location within \( \Delta = 5 \) time units if the two-counter machine halts.

### D Details for Section 4

#### D.1 Timed Region Graph

We begin with a formal definition of the timed region graph. Given a 1.5 STG \( \mathcal{G} = (\mathcal{A}, L, \omega, \mu) \), we define the timed region graph \( \mathcal{G}_R = (\mathcal{R}(\mathcal{A}), L \times \mathcal{R}, \omega^R, \mu^R) \) where \( \mathcal{R}(\mathcal{A}) \) has as its locations ordered pairs \((\ell, R)\) where \( \ell \in L \) and \( R \) is a classical region. The transitions of \( \mathcal{G}_R \) are defined as follows. We have a transition \((\ell, R) \xrightarrow{\text{guard}(R''), e, Y} (\ell', R')\) iff there exists an edge \( e = \ell \xrightarrow{R, Y} \ell' \) in \( \mathcal{A} \) such that there exists \( v \in R, t \in \mathbb{R} \) with \((\ell, v) \xrightarrow{t} (\ell', v + t), v + t \in R'' \), and \( v' = v''[Y \leftarrow 0] \in R' \). Here, \( \text{guard}(R'') \) represents the minimal guard that captures region \( R'' \). For instance, if region \( R'' \) is \((0, 1)\) then \( \text{guard}(R'') \) is \( \emptyset \). Also, \( Y \) is either the emptyset, or the single clock \( \{x\} \). The standard region automaton (Alur-Dill) can be recovered by labelling transitions of \( \mathcal{G}_R \) with only \( e \) rather than with \( \text{guard}(R''), e, Y \).

For every state \( s = (\ell, v) \) in \( \mathcal{A} \), there is a mapping \( i(s) \) which maps it to \((\ell, R)\) such that \( v \in R \). The probability measure for \( \mathcal{G}_R \) is defined such that \( \mu^R_{i(s)} = \mu_s \) and the weights of edges are also preserved. That is \( \omega^R(f) = \omega(e) \) where \( f = \text{guard}(R''), e, Y \) is the edge corresponding to \( e \), obtained from the map between states. For brevity, we decorate the transitions in Figure 4 with only \( e_i \) rather than with \( \text{guard}(R''), e, Y \).
A strategy $\sigma$ of $\Diamond$ in $\mathcal{G}$ is a function that maps a finite run $\rho = (l_0, v_0) \xrightarrow{d_0, e_0} (l_1, v_1) \xrightarrow{d_1, e_1} \ldots (l_n, v_n)$ to a transition $(d, e)$ where $d \in \mathbb{R}^+$ and $e$ is an edge, such that $(l_n, v_n) \xrightarrow{d, e} (l', v')$ for some $(l', v')$, whenever $l_n \in L_\circ$. For each such strategy $\sigma$ in $\mathcal{G}$, we have a corresponding strategy $i(\sigma)$ in $\mathcal{G}_R$ that maps the finite run $i(\rho) = (l_0, R_0) \xrightarrow{d_0} (l_1, R_1) \xrightarrow{d_1} \ldots (l_n, R_n)$ to a transition $f$ such that $(l_n, R_n) \xrightarrow{f} (l', R')$ for some $(l', R')$, whenever $l_n \in L_\circ$. Here, $f_i$ stands for $\text{guard}(R_i), e_i, Y$ such that $v_i + d_i \in \text{guard}(R_i^\rho)$. Moreover, $(l_i, R_i) = i(l_i, v_i)$ and $v_i \in R_i$ for all $i$. For every finite path $i((l, v), e_1 \ldots e_n)$ in $\mathcal{G}$, we have a finite set of paths $i((l, R), v, f_1 \ldots f_n)$ in $\mathcal{G}_R$, each one corresponding to a choice of regions passed. If $\rho$ is a run in $\mathcal{G}$, $i(\rho)$ stands for the unique image of the run in $\mathcal{G}_R$.

Lemma 10 (Strategy Mapping between $\mathcal{G}$ and $\mathcal{G}_R$). Let $\mathcal{G}$ be a $1\frac{1}{2}$ player STG. Then player $\Diamond$ has a strategy $\sigma$ in $\mathcal{G}$ to reach $(l, v)$ in $\mathcal{G}$ with probability $\sim c$ iff player $\Diamond$ has a strategy $i(\sigma)$ in $\mathcal{G}_R$ to reach $i(l, v)$ with the same probability.

Proof. The proof follows by construction of $\mathcal{G}_R$ from $\mathcal{G}$. Fix a strategy $\sigma$ in $\mathcal{G}$. At each $(l, v)$ such that $l \in L_\circ$, $\sigma$ chooses a time delay $d$ and an edge $e$ from $(l, v)$ based on the path $\rho$ seen so far, such that $(l, v)$ is the last state in $\rho$. Let $\rho = (l_0, v_0 = 0) \xrightarrow{d_0, e_0} (l_1, v_1) \xrightarrow{d_1, e_1} \ldots \xrightarrow{d_n-1, e_{n-1}} (l_n, v_n) = (l, v)$.

We induct on the number of stochastic nodes seen so far in $\rho$. Assume that in the path so far, we have witnessed exactly one stochastic node.

1. Assume $|\rho| = 1$ and $l_0$ is a stochastic node. In $\mathcal{G}_R$, we start with $(l_0, R_0)$ where $R_0 = 0$ is the initial region. To satisfy the guard on edge $e_0$ in $\mathcal{G}$, we can choose any appropriate delay $d_0$. In $\mathcal{G}_R$, the guard chosen is the minimal region which contains $d_0$. For each choice of $d_0$, we have an appropriate guard which captures the correct interval which contains it. This time interval determines the probability for the edge $e_0$ chosen in both $\mathcal{G}$ as well as $\mathcal{G}_R$ and is the same, by setting the limits of the integral.

If $l_0$ is not a stochastic node, then we simply continue mapping locations in $\mathcal{G}$ with those in $\mathcal{G}_R$, by mapping edges $e_i$ with $f_i$, until we reach a stochastic node. The first time we reach a stochastic node with valuation $v_i$, $(l_i, v_i)$ in $\rho$, we will reach in $\mathcal{G}_R$, the node $(l_i, R_i)$. At this point, as seen above, for a delay $d_i$ and an edge $e_i$, chosen in $\mathcal{G}$, we choose $f_i$ so that the minimal guard captures the precise time interval in which $d_i + v_i$ lies in. Since the minimal time interval containing $d_i + v_i$ determines the probability of $e_i$ in $\mathcal{G}$ and $f_i$ in $\mathcal{G}_R$, we have matched the probabilities till the first stochastic node.

2. Now assume that the probabilities are preserved till some $n - 1$ stochastic nodes seen, and we are going to see the $n$th stochastic node. The same argument as above applied to the $n$th stochastic node ensures that the probabilities incurred each time remain the same, and hence the probability of reaching some $(l, v)$ in $\mathcal{G}$ is same as that of reaching $i(l, v)$ in $\mathcal{G}_R$.

Lemma 11. If $\mathcal{G}$ is an initialized 1 clock $1\frac{1}{2}$ player STG, then $M_{\mathcal{G}}$ is a Markov decision process.

Proof sketch. Observe that since $\Diamond$ to $\bigcirc$ edges always reset the clock, we can compute the probability values on $\bigcirc$ nodes. We need to show that from any $\bigcirc$ node, the probability of the outgoing paths (and edges to $\Diamond$, $\bigcirc$ nodes) adds up to 1.

First observe that if $N = 0$, i.e., if $\mathcal{G}_R$ has no stochastic nodes $(l, a)$ s.t. $a \not\in \{0, \infty\}$, then $\mathcal{G}_R$ already defines an MDP, obtained by computing the discrete probability on the
edges (follows from the definition of an initialized STG: the absence of zero and unbounded regions in the stochastic nodes implies the absence of cycles in the STG).

Then, we recursively, remove all deletable nodes \( G_R \) to obtain new region graph STG \( G \) (with a new path-labeling alphabet on its edges), where the probabilities of any paths between nodes of \( G \) are the same as the probability of that path in \( G_R \). Thus, the sum of all probabilities of outgoing paths add to 1. Now, as all deletable nodes are removed, this gives an MDP.

We now elaborate on the construction of \( M_G \) given the STG \( G \). Let \( G = (A, (L^0, L^\omega), \omega, \mu) \) be an 1 clock 1+ player STG. Let us look at the region graph \( G_R \) corresponding to it. Further let \((l, R)\) be a deletable node in \( G_R \). Then we define \textit{remove}(l, R) which modifies the region graph \( G_R \) by

1. removing this node and all edges incoming to and outgoing from this node.
2. for each incoming edge \( e_1 \) from, say, \((l_1, R_1)\) to \((l, R)\) and each outgoing edge \( e_2 \) from \((l, R)\) to, say, \((l_2, R_2)\), we add a new direct edge from \((l_1, R_1)\) to \((l_2, R_2)\) with the new label \( e_1e_2 \).

Note that this operation is well-defined since, for every deletable node, there must exist an incoming edge (since the region is non-zero). Further, there must also exist an outgoing edge, since it is a stochastic node and hence the sum of probabilities on outgoing edges of stochastic nodes in \( G \) sums to 1. If there is a self-loop, then it must be reset (by the structural non-Zeno assumption) and then this node will not be deletable.

Let \( G_1 \) be the resulting structure obtained after the remove operation. The probability of these new edges labeled by paths in \( G_1 \) is the probability of the respective paths in \( G \).

**Lemma 12.** Suppose \( G_1 \) is obtained from \( G_R \) by performing \textit{remove}(l, R). Then, for each stochastic node in \( G_1 \) the sum of outgoing probabilities is 1.

**Proof.** Consider any node \((l', R')\) in \( G_1 \) such that \( l' \in L^0 \). There are two cases:

1. there is no edge in \( G_R \) from \((l', R')\) to \((l, R)\). Then the outgoing probabilities of \((l', R')\) do not change in \( G_1 \). As they summed to 1 in \( G \), they will continue to do so in \( G_1 \).
2. there is an edge \( e \) in \( G \) from \((l', R')\) to \((l, R)\). Then consider all outgoing edges from \((l, R)\) in \( G \), call them \( e_1, \ldots, e_k \). By stochasticity of \( G \), \( \sum_{i=1}^{k} \mathcal{P}(\pi((l, R), e_i)) = 1 \). Then in \( G_1 \) from \((l', R')\), we have exactly \( k \) outgoing edges labeled \( ee_1, ee_2, \ldots ee_k \). Now if \( E \) is the set of all other \((\neq e)\) edges outgoing from \((l', R')\), then the sum of probabilities of all outgoing edges from \((l', R')\) is given by \( \sum_{i=1}^{k} \mathcal{P}(\pi((l', R'), ee_i)) + \sum_{e' \in E} \mathcal{P}(\pi((l', R'), e')) \) which is

\[
\begin{align*}
&= \mathcal{P}(\pi((l', R'), e)) \cdot \sum_{i=1}^{k} \mathcal{P}(\pi((l, R), e_i)) + \sum_{e' \in E} \mathcal{P}(\pi((l', R'), e')) \\
&= \mathcal{P}(\pi((l', R'), e)) + \sum_{e' \in E} \mathcal{P}(\pi((l', R'), e')) = 1
\end{align*}
\]

This follows by linearity of the Lebesgue integral and stochasticity of \( G \).

Thus, \( G_1 \) is an (extended) STG in which edges are labeled by paths instead of edges and the probability of paths are computed as before. Thus by now repeatedly applying the remove operation on all deletable edges we obtain (after finitely many steps) an (extended) STG \( G_n \) in which there are no deletable edges. This implies that \( G_n \) is an MDP. Note that as an immediate consequence of the above lemma we also obtain that the probability of all paths are preserved.
Lemma 13 (Strategy Mapping between $G_R$ and $M_G$). Let $G_R$ be the timed region graph corresponding to a $1\downarrow 1$ player STG $G$. Then player $\diamond$ has a strategy $\sigma$ in $G_R$ to reach $(l, R)$ in $G_R$ with probability $\sim c$ iff $\diamond$ has a strategy $g(\sigma)$ in $M_G$ to reach $(l, R)$ with the same probability.

Proof. There are two parts to the proof.
(a) Let $(l, R)$ and $(l', R')$ be two nodes in $M_G$. Then for every path $\pi$ between $(l, R)$ and $(l', R')$ in $G_R$, we have a path $\pi'$ in $M_G$ and conversely. The probabilities of $\pi, \pi'$ are same in $G_R$ and $M_G$.
(b) Show that for every strategy $\sigma$ in $G_R$, there exists a strategy $g(\sigma)$ in $M_G$ that preserves probabilities.

We can prove (a) and (b) together. Consider $(l, R)$ and $(l', R')$ in $M_G$ such that $l \in L_\diamond$. Let $(l, R) \xrightarrow{f_1} (l_1, R_1) \xrightarrow{f_2} \cdots \xrightarrow{f_n} (l_n, R_n) \xrightarrow{f} (l', R')$ be a path $\pi$ in $G_R$, according to a strategy $\sigma$ in $G_R$. Let’s see what happens to this path in $M_G$. The operation of remove$(l, R)$ might remove some of the intermediate nodes of $\pi$ (excluding the first and last, since by assumption they are in $M_G$). Let $(l_i, R_i)$ be the first such node to be deleted. Then in $M_G$, we have all the nodes from $(l, R)$ to $(l_{i-1}, R_{i-1})$. According to strategy $\sigma$, $(l_i, R_i)$ has been selected based on the prefix till $(l_{i-1}, R_{i-1})$ whenever $l_{i-1} \in L_\diamond$. Clearly, in $M_G$, if all nodes until $(l_i, R_i)$ are carried forward, then the strategy chosen at all nodes $(l_j, R_j)$, $l_j \in L_\diamond$, $j < i$ is the same as $\sigma$.

If $(l_i, R_i)$ is deleted, clearly, $l_i \in L_\diamond$, and $R_i \neq 0, \infty$. Then, $l_{i-1} \notin L_\diamond$, by definition of initialized STG. Let $(l_i, R_i, k)$, $k < i$ be the last node from $L_\diamond$ before $(l_i, R_i)$. By the delete operation, we obtain the path $(l, R) \xrightarrow{f_i} (l_i, R_i) \xrightarrow{f_2} \cdots \xrightarrow{f_k} (l_k, R_k) \xrightarrow{f_{k+1}} \cdots (l_{i-1}, R_{i-1}) \xrightarrow{f_{i+1}f_i} (l_{i+1}, R_{i+1})$ till $(l_{i+1}, R_{i+1})$ in $G_R$. Continuing this, when we finish removing all deletable nodes, we obtain the path $\pi'$ in $M_G$ such that if nodes $(l_i, R_i), (l_{i+1}, R_{i+1}), \ldots, (l_n, R_n)$ are deleted, then we obtain the edge $(l_{i-1}, R_{i-1}) \xrightarrow{f_{i+1}f_i} (l_{i+1}, R_{i+1})$ in $M_G$. For any path $\pi$ in $G$, we obtain a unique path $\pi'$ in $M_G$. The strategy $g(\sigma)$ in $M_G$ is defined from strategy $\sigma$ in $G$ as follows:

- If $\sigma$ maps $(l_i, R_i)$ to $(l'_i, R'_i)$, choosing edge $f$ based on a path $\pi$ such that $(l_i, R_i)$ is the last node of $\pi$, then in $M_G$, $g(\sigma)$ maps $(l_i, R_i)$ to $(l'_i, R'_i)$ choosing edge $f$ based on the unique path $\pi'$ corresponding to $\pi$. Note here that the only change in the strategy $g(\sigma)$ as compared to $\sigma$ is the path $\pi'$ seen so far, obtained by deleting some nodes from $\pi$.

Given a path $\pi$ in $G_R$ as above, the probability of the path is obtained from the edges $f_1f_2 \cdots f_nf$. Since the sequence of labels on the path $\pi'$ are exactly same as $f_1f_2 \cdots f_nf$, the probability of $\pi$ and $\pi'$ are the same. Since this is true about all paths $\pi$ in $G_R$, we have the probability of reaching $(l', R')$ from $(l, R)$ in $G$ is same as the probability of reaching $(l', R')$ from $(l, R)$ in $M_G$, for any two nodes $(l', R'), (l, R)$ in $M_G$. $\square$

E Example of a 2-clock STA with unfair runs

This example has been taken from [3] to help the reader get an intuition of why two clocks or the uninitialized condition creates problems even in qualitative reachability. Our assumptions of 1-clock and initialized-ness circumvent these problems even for quantitative reachability.

In this example, one does not reach location $G$ almost surely, even though thats what one would conclude by working on the region graph. Every fair run using edges of non-zero probability indeed visits $G$ infinitely often. However, the problem is that the run $(e_4e_5e_6)^{\omega}$ has a non-zero probability. Thus, there is an unfair run in the automaton with a non-zero probability, and hence one cannot reach $G$ almost surely.
The interplay of the clocks $x,y$ is very useful here. In fact, if one starts in node $B$ with $x = 0, y = t_0$, then one reaches $E$ with $(2 - t_0, 0)$. The enabled interval for edge $e_4$ is $(1 - t_0, 2 - t_0)$, while that for $e_6$ is $t_1 \in (t_0, 1)$. Again, $e_4$ is enabled with time interval $(1 - t_1, 2 - t_1)$, while $e_6$ is enabled with $t_2 \in (t_1, 1)$ and so on.

$$P(\pi((B, (0, t_0)), (e_4 e_5 e_6))) = \frac{1}{2 - t_0} \int_{t=1-t_0}^{2-t_0} \frac{1}{1-t_0} \int_{t_1=t_0}^{1} dt_1 dt = \frac{1}{2 - t_0} \frac{1}{1-t_0} \int_{t_1=t_0}^{1} dt_1 dt$$

In particular, it can be shown that

$$P(\pi((B, (0, t_0)), (e_4 e_5 e_6)^n)) = \frac{1}{2 - t_0} \int_{t=1-t_0}^{2-t_0} \frac{1}{1-t_0} \int_{t_1=t_0}^{1} P(\pi((B, (0, t_1)), (e_4 e_5 e_6)^{n-1})) dt_1 dt = \frac{1}{2 - t_0} \frac{1}{1-t_0} \int_{t_1=t_0}^{1} P(\pi((B, (0, t_1)), (e_4 e_5 e_6)^{n-1})) dt_1 dt$$

By an inductive argument, [8] shows that $P(\pi((B, (0, t_0)), (e_4 e_5 e_6)^n)) = \frac{t_0}{2 - t_0} > 0$, and $P(\pi((B, (0, t_0)), (e_4 e_5 e_6)^\omega)) > 0$.

Note that this example is an uninitialized STA with 2 clocks. If one makes this example initialized, by resetting both $x,y$ on a transition (on $e_5, e_6$), then again it can be seen that the resulting automaton (Figures 10, 11 in [8]) also has unfair runs of non-zero probability.