THE CHORD CONSTRUCTION

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Abstract. Let $F$ be a smooth plane curve of degree 3. Let $\beta$ be an element in $\text{Pic}(F)[2] - \{0\}$. Let us define

$$F' := \{p(p + \beta) | p \in F\} \subset (\mathbb{P}^2)^*.$$ 

In this note we show that $F'$ is a smooth embedding of $F/\beta$. Moreover, let $\beta'$ be the generator of $\text{Pic}(F)/\beta$, and let $p \in F$ be a flex, then $p(p + \beta) + \beta'$ is a flex on $F'$. We present two proofs.

1. Notation. Let $F$ be a smooth curve of genus 1, $\beta \in \text{Pic}(F)[2]$. Let us denote $F' := F/\beta$. Let $\pi$ be the quotient map $F \to F'$ and let $\beta'$ be the image of the non-zero element in $\text{Pic}(F)[2]/\beta$ in $\text{Pic}(F')$. Let $H$ be an element in $\text{Pic}^3(F)$, and $i : F \to \mathbb{P}^2$ the map associated to $H$. This notation will hold in the next definition and couple of lemmas.

2. Definition (The chord construction (c.c.)). Let us define

$$i' : F/\beta \to \mathbb{P}^2^* \quad \{p, p + \beta\} \mapsto i(p)i(p + \beta)$$

3. Theorem. The c.c is an embedding.

Proof. Let us identify $F$ with its image $i(F)$. 

The c.c. is 1-1: Let $H$ be the line corresponding to a point $\{p, \beta + p\} \in F/\beta$. Then $F \cap H$ consists of three points:

$$\{p, p + \beta, H - (2p + \beta)\}.$$ 

If $(H - (2p + \beta)) - \beta \neq p, p + \beta$, then $i'(H)$ consist of the unique point $\{p, p + \beta\}$, so it is 1-1. If $(H - (2p + \beta)) - \beta = p$ then $H - (2p + \beta) = p + \beta$, so $i'$ is 1-1 here too. The remaining case is similar. (in the last two cases $p$ (respectively $p + \beta$) is a flex). For $p \in \mathbb{P}^2$, let $p^*$ be the line it defines in $\mathbb{P}^2^*$. If $F$ does not have a CM then for a generic $p \in F$, $p^*$ is transversal to $F'$: Suppose it is not. By upper-semicontinuity, it is never transversal, so $F$ is the dual curve of $i'(F')$. Since $i'(F')$ is not a line (it is an immersion of $F'$), this induces a degree 1 map $i(F') \to F$, and therefor a degree 1 map $F' \to F$. This means that $F$ has a CM.

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The c.c is of degree 3: Assume first that $F$ does not have a CM. Choose a generic $p \in F$. The degree of $i'$ is the number of lines through $p$ which belong to $i'(F')$. There are exactly two types of such lines:

- The line is $p(p + \beta)$.
- The line is the one passing through a pair of the four solutions of $x + (x + \beta) = H - p$.

Now suppose $F$ has a CM. Then $F$ and the c.c. are a flat limit of curves without a CM, so the degree of $i'$ is constant. (see [Ha] proposition III.9.8)

4. Theorem. If $i(p)$ is a flex, then so is $i'(\pi(p) + \beta')$.

Proof. We are going to drop the $i, i'$ notation, and identify both curves with their images.

Involutions on $F \subset \mathbb{P}^2$: Let $p$ be a flex on $F$. Let $\alpha$ be the unique involution of $\mathbb{P}^2$ that

- sends $F$ to itself.
- fixes $p$.

Note that this involution has a fixed line $L$, where $L \cap F$ consists of the translates of $p$ by points in $\text{Pic}(F)[2] - \{0\}$. All the involutions of $\mathbb{P}^2$ that send $F$ to itself are of this form (they have to send flexes to flexes, and there are 9 flexes, so one of them is fixed). The same holds for $F'$.

The dual involution: Given $\alpha$, there is a dual involution $\alpha'$ on $\mathbb{P}^2_\ast$. By the definition of the c.c, $\alpha'$ sends $F'$ to itself. The fixed line and point of $\alpha'$ are $p^\ast, L^\ast$ respectively. There is exactly one pair of points in $F$ with difference $\beta$ on $L$. This is the pair $\{p + \gamma, p + \gamma + \beta\}$ where $\gamma$ is some element in $\text{Pic}(F)[2] - \{0, \beta\}$.

5. We develop a formula for the c.c. Theorem [13] (which states the formula) also gives an alternative proof for theorems [3,4].

6. Notation. Let us denote homogeneous coordinates on $\mathbb{P}^2$ by $[X; Y; Z]$. Let us denote the dual coordinates on $\mathbb{P}^2_\ast$ by $[U; V; W]$. Let us also define

$$x := X/Z, y := Y/Z,$$

and denote the $\mathbb{A}^2$ coordinates by $(x, y)$ Let $F$ as the closure of the locus of $y^2 = f$ where

$$f(x) = x^3 + ax^2 + bx.$$

Let us identify $F$ with $\text{Pic}(F)$ by choosing $O := [0; 1; 0]$ be the zero. Let $\beta$ be the point $(0, 0) = [0; 0; 1]$.

7. Lemma. Let $p = (x(p), y(p))$ be a point on $F$. Then

$$p + \beta = (bx(p), -by(p)/x(p)^2)$$
Proof. There are exactly three points in
\[ F \cap \beta p, \]
these are \( p, \beta, -p - \beta \). Since \( \beta p \) is given by the nulls of \( y/x = y(p)/x(p) \), They are the solutions of
\[
(x \frac{y(p)}{x(p)})^2 = x^3 + ax^2 + bx.
\]
One of the solutions of this equation is 0. the other two are the roots of
\[
0 = x^2 + (a - \frac{y(p)}{x(p)})^2 x + b.
\]
Since one of the roots is \( x(p) \), we see that the other is
\[
b/x(p) = x(p + \beta).
\]
We also deduce that
\[
y(p + \beta) = -y(-p - \beta) = -\frac{y(p)}{x(p)} b/x(p) = -by(p)/x(p)^2.
\]

8. Lemma. The lines of the c.c. are given in \( \mathbb{P}^2 \) by
\[
[y(p)(x(p)^2 + b); bx(p) - x(p)^3; -2bx(p)y(p)],
\]
where \( p \) are the points on \( F \).

Proof. Suppose the line \( p(\beta + p) \) is given by
\[
(1) \quad sx + c = y.
\]
In this case we have:
\[
s = \frac{y(p) - y(p + \beta)}{x(p) - x(p + \beta)} = \frac{y(p) + by(p)/x(p)^2}{x(p) - b/x(p)} = \frac{y(p)(x(p)^2 + b)}{x(p)^3 - bx(p)}
\]
\[
C = y(p) - sx(P).
\]
Simplifying
\[
[s; -1; c],
\]
We get the result.

9. Lemma. Assuming the c.c. is of degree 3, it satisfies the equation
\[
e(U - \mu^{-1}W)V^2 = W^3 - \frac{2\mu b}{\mu^2 - b} (U - \mu^{-1}W)W^2 - \frac{\mu^2 b}{\mu^2 - b} (U - \mu^{-1}W)^2 W.
\]
for some \( e \), where \( \mu = 2b/a \).
Proof. Let us consider the involution \((x, y) \mapsto (x, -y)\) on \(\mathbb{P}^2\). It has a fixed line \(l\): the nulls of \(y = 0\), and a unique fixed point out of \(l\): \(O\). If we consider the dual involution, we see that it has a fixed line at the dual of \(O\), to be denoted \(O^*\). It also has a unique fixed point out of \(O^*\). This is the point dual to \(l\), to be denoted \(l^*\). This means that \(l^*\) is a flex of the c.c. and its translations by elements of order 2, all sit on \(O^*\). As in the proof of lemma \(3\), there are two types of such lines:

- The line is \(O\beta\).
- The line is the one passing through a pair of the four solutions of \(2x = \beta\).

In the first type, this is the line \(x = 0\). In the second type, we are looking for the solutions of

\[
x \neq 0, y^2 = f(x), x \frac{dy}{dx} = y \quad \iff \quad x \neq 0, x \frac{df}{dx} = f \\
3x^2 + 2ax + b = 2x^2 + 2ax + 2b \iff x^2 = b.
\]

We have to identify the tangent to the flex. A tangent to the flex (on a curve of degree 3) is a line that intersect the curve only in the flex. In dual coordinates: It is the unique point \(q = (\mu, 0)\) on the line \(y = 0\), such that \(y = 0\), is the ONLY c.c line on which \(q\) sits. The \(x\) coordinates of the intersection between a chord and \(y = 0\), can be obtained from equation \(1\), letting \(y = 0\). i.e. they are:

\[-c/s = \frac{y(p) - sx(p)}{s} = y(p)/s - x(p) = \frac{x(p)^3 - bx(p)}{x(p)^2 + b} - x(p) = \frac{-2bx(p)}{x(p)^2 + b}.
\]

We see that each such point is given by only two \(x(P)\). In \(q\), these \(x(p)\) are given by the roots of

\[x^2 + ax + b.
\]

Whence,

\[\mu = 2b/a.
\]

Let us apply the projective transformation

\[U' = U - 1/\mu W, V' = V, W' = W,
\]

and define

\[w' = W'/U', v' = V'/U'.
\]

In these coordinates we have

\[ev'^2 = w'(w' - \mu \sqrt{b}/(\mu - \sqrt{b}))(w' + \mu \sqrt{b}/(\mu + \sqrt{b})) = w'(w'^2 - \frac{2\mu b}{\mu^2 - b}w' - \frac{\mu^2 b}{\mu^2 - b})
\]
for some $e$. Going back to our original coordinates, we get
\[ e(U - \mu^{-1}W)V^2 = W^3 - \frac{2\mu b}{\mu^2 - b}(U - \mu^{-1}W)W^2 - \frac{\mu^2 b}{\mu^2 - b}(U - \mu^{-1}W)^2 W. \]

10. **Theorem.** The c.c is of degree 3, and $e = \frac{-8b^3}{a^2 - 4b}$

**Proof.** This is a trivial check, best done by a computer. One has to define:
\[ U := y(x^2 + b), V := bx - x^3, W := -2bxy, \mu := \frac{2b}{a}, \]
and verify that
\[ \frac{W^3 - \frac{2\mu b}{\mu^2 - b}(U - \mu^{-1}W)W^2 - \frac{\mu^2 b}{\mu^2 - b}(U - \mu^{-1}W)^2 W}{(U - \mu^{-1}W)V^2} = -\frac{8b^3y^2}{(a^2 - 4b)f(x)} \]

11. **Remark.** If $F$ is as before, and $\beta \in \text{Pic}^0(F) - \text{Pic}^0(F)[2]$, the chord construction gives a map of degree 1. The image of this map is of degree 6.

**References**

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