Tripartite thermal correlations in an inhomogeneous spin–star system

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Abstract
We exploit the tripartite negativity to study the thermal correlations in a tripartite system, that is, the three outer spins interacting with the central one in a spin–star system. We analyse the dependence of such correlations on the homogeneity of interactions, starting from the case where central–outer spin interactions are identical and then focusing on the case where the three coupling constants are different. We single out some important differences between the negativity and the concurrence.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
The notion of thermal entanglement relies on the amount of entanglement possessed by a physical system when it had undergone a thermalization process that has brought it into a thermal state [1]. In fact, in the presence of interaction between different parts of a compound system, even after a complete thermalization, the system can exhibit appreciable quantum correlations since the Hamiltonian eigenstates are, in general, entangled states. The establishment of a relation between temperature and entanglement has quickly brought the idea of using the entanglement as an order parameter in quantum phase transitions [2, 3]. Moreover, it has given a stronger stimulus for searching quantum correlations in the macroscopic world [4–6].

Thermal entanglement has been studied in connection with many possible applications and in different physical systems: detailed analysis in spin chains described by Heisenberg models has been given [7] as well as in atom–cavity systems [8] and in simple molecular models [9]. The quantumness of correlations has been singled out in thermalized spin chains [10] also in connection with non-local effects [11], and the application of thermal entanglement in optimal quantum teleportation protocols has been proposed [12].

Spin systems have been studied in depth not only in the neighbour-interaction configuration, leading to the Heisenberg model, but also in star networks. This kind of systems can be realized in many physical contexts, from Josephson junctions [13] to trapped ions [14] to solid state physics [15]. In 2004, Hutton and Bose analysed a physical system made of a central spin interacting with a set of N outer spins at zero temperature, bringing to light interesting properties which are immediately traceable back to the features of the ground state of this spin–star network [16]. They showed that the evenness or oddness of the number of outer spins determines the law the entanglement amount scales with. The same spin configuration, in a simplified version involving three outer spins only, has been recently considered by Wan-Li et al [17] in the special case where all the interactions between the central spin and the outer ones are identical. Exploiting the high degree of symmetry of the system, they concentrate on appropriate concurrences [18] to extract information about the presence of quantum correlations. However, the disclosure of quantum correlations in tripartite systems could require tools and concepts more adequate than the simple concurrences. Sabin and Garcia-Alcaine have contributed to solve this still very open problem1 introducing the notion of tripartite negativity [20], which is an effective tool to reveal the existence of quantum correlations traceable back to the impossibility of separating any of the three subsystems from the other two.

In this paper, we consider a spin–star system where three outer spins are coupled to the central one with different strengths, and analyse thermal entanglement in different situations. In the following section we present the model

1 For a review on entanglement in multipartite systems, see [19].
under scrutiny, which is characterized by anisotropic spin–
spin interactions due to the absence of longitudinal \( z \)-\( z \) 
couplings. In the third section we consider the system in the 
homogeneous case, showing the presence of sharp changes 
in the amount of entanglement at zero temperature due to 
abrupt variations of the ground state. In the fourth section we 
analyse the inhomogeneous model, focusing on two types of 
inhomogeneity. Finally, in the last section some comments 
and conclusive remarks are given.

2. Physical scenario

System and Hamiltonian. In this section we present the spin–
star system we have analysed, which is pictured in figure 1. 
Numbers 1–3 indicate the three outer spin 1/2, while the 
central one is indicated with a latin capital letter C. With \( c_1 \), \( c_2 \) 
and \( c_3 \) we indicate the coupling constants of the central spin 
with the outer spins 1, 2 and 3, respectively. Moreover, the 
whole system is immersed in a constant uniform magnetic field 
of modulus \( B_0 \) and directed along the \( z \)-axis. The Hamiltonian 
model is then given by (with \( \hbar = 1 \))

\[
\begin{align*}
H &= H_1 + H_2, \\
H_1 &= -\frac{\gamma_i}{2} B_0 \sum_i \sigma_i^z = \frac{\alpha_0}{2} \sum_i \sigma_i^z, \\
H_2 &= \sum_{i=1,2,3} c_i (\sigma_i^+ \sigma_i^- + \sigma_i^- \sigma_i^+),
\end{align*}
\]

where \( \sigma_i^+ = |+i\rangle\langle+| - |-i\rangle\langle-| \), \( \sigma_i^- = |+i\rangle\langle-| - |-i\rangle\langle+| \), 
\( \alpha_0 \) is the unperturbed Bohr frequency and \( c_i \)'s are coupling 
constants.

The term \( H_1 \) describes the interaction of the system with 
the magnetic field, while the second one \( H_2 \) arises from the 
dipole-like interactions between the central spin and the outer 
ones. The absence of \( z \)-\( z \) interaction reflects a certain degree of 
anisotropy.

Assuming that the system is in a thermal state, its density 
operator takes the form

\[
\rho = \frac{1}{Z} e^{-\frac{H}{\kappa B T}} = e^{-\frac{H}{\kappa B T}} Z^{-1}.
\]

Figure 1. The physical system. The spins labelled 1–3 are coupled 

to the central (C) one. The magnetic field \( B_0 \) is orthogonal to the 
plane where the four spins lie.

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Assuming that the system is in a thermal state, its density 
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\[
\rho = \frac{1}{Z} e^{-\frac{H}{\kappa B T}} = e^{-\frac{H}{\kappa B T}} Z^{-1}.
\]

so that \( \rho \) and \( H \) share the same eigenstates. The Hamiltonian 
model given in equation (1) takes into account possible 
inhomogeneities; nevertheless, it is surely of interest studying 
it in the homogeneous case \( (c_1 = c_2 = c_3 = c) \) which 
turns out to be simpler from a mathematical point of view, 
paving the way to the study of the more complicated 
inhomogeneous model. Wan-Li et al [17] have analysed 
the homogeneous model taking advantage of the concurrence 
to quantify entanglement. In addition, they have assumed a 
more general anisotropic interaction. Since in real situations 
the construction of a system with perfectly homogeneous 
interactions could be quite difficult, in this paper we investigate 
the features of thermal quantum correlations in the presence of 
inhomogeneity.

Tripartite negativity. Remark ing that the analysis is 
performed on the outer subsystem, which is made of three 
parts, we have the need to study quantum correlations in 
tripartite systems. There are various proposals of tripartite 
entanglement quantifiers [21–24] and witnesses [26–28], 
the latter ones based on the key assertions in [25]. Nevertheless, up 
to now and to the best of our knowledge, there is not a definitive 
answer to the request of quantifying tripartite entanglement.

For instance, it has been shown recently that three-tangle [29] 
could be an improper tool for such a purpose [30]. The 
situation is much more complicated when we have to consider 
mixed states. Indeed, all the recipes mentioned in [21–24, 29] 
refer to pure states and fail when applied to non-pure states.

Since we are studying thermal correlations, therefore dealing 
with highly mixed states, we need a tool to investigate quantum 
correlations in the mixed states of tripartite systems. Again, 
to the best of our knowledge, the tripartite negativity introduced 
by Sabin and Garcia-Alcaine [20] is a good quantity that allows 
us to study quantum correlations in tripartite systems, even in 
the case of mixed states. Other quantities are very much 
related to the specific structure of the analysed system, since 
observables that assume values in a certain range when the state 
is entangled are considered. Sabin and Garcia-Alcaine apply 
bipartite negativity to all the three possible bipartitions that 
can be singled out isolating one subsystem and considering 
the other two as a whole; then they consider the geometric 
mean of these three quantities. According to the definition 
of negativity [31, 32], the partial negativity related to the 
bipartition \( I = (JK) \) (in which the two subsystems \( J \) and 
\( K \) are considered as a whole) is given by

\[
\mathcal{N}_{I-JK} = \sum_i |\sigma_i(T_{IJ})| - 1,
\]

where \( \sigma_i(T_{IJ}) \) is the \( i \)-th eigenvalue of \( T_{IJ} \), which is the 
partial transpose related to subsystem \( I \), of the total \( (JK) \) density 
matrix. The parameter used to study correlations in tripartite 
systems is then

\[
\mathcal{N}_{123} = \sqrt{\mathcal{N}_{1-23}\mathcal{N}_{2-13}\mathcal{N}_{3-12}},
\]

which is the above-mentioned tripartite negativity. In [20] 
the following properties have been proven:

(i) \( \tau \) separable or simply bi-separable \( \Rightarrow \mathcal{N}_{123} = 0; \)
(ii) invariance of \( \mathcal{N}_{123} \) under LU operators;
(iii) monotonicity of \( \mathcal{N}_{123} \) under LOCC operators.
Figure 2. Tripartite negativity $N_{123}$ for the homogeneous model as a function of the coupling constant ($c$) and temperature ($k_BT$), both in units of $\omega_0$. Both the diminishing with increasing temperature and the low-temperature transitions with respect to $c$ are well visible.

Though $N_{123}$ is not a sufficient condition to single out tripartite entanglement, it is an effective tool to study quantum correlations in tripartite systems. In fact, finding $N_{123} \neq 0$ implies that none of the three subsystems is separable, hence revealing correlations involving all the three subsystems. It is also important to note that the negativity $N_{IJK}$ does involve all the three spins, and establishes the existence of a correlation between $I$ and the entire couple made of $J$ and $K$, which is a very different operation from tracing over one spin, say $K$, and evaluating the degree of correlation between the other two, say $I$ and $J$.

## 3. Homogeneous model

Let us consider our model in the special case $c_1 = c_2 = c_3 = c$ that we have already addressed as the homogeneous case. The complete diagonalization of this Hamiltonian model is given in the appendix. In order to compute the tripartite negativity we first need to find the explicit form of the outer-spin density operator $\rho_{C23}$, tracing over the degrees of freedom of the central spin.

It is quite clear that because of homogeneity, $N_{I_{2-3}} = N_{2-13} = N_{3-12}$, so the geometric mean reduces to one of these three quantities. In figure 2 we plot $N_{123}$ as a function of temperature ($T$) and coupling constant ($c$). One can see some interesting features of $N_{123}$, for example, its diminishing with increasing $T$, and the presence of abrupt transitions at low temperature due to fast variations of the ground state. In figure 3, where $N_{123}(k_BT, c = 6\omega_0)$ is plotted, this behaviour is better shown which is due to the fact that the more temperature increases the more $\rho_{C123}$ approaches the maximally mixed states ($\frac{1}{310}$), making also $\rho_{123}$ maximally mixed. A more interesting trend of $N_{123}$ is observed at low temperature where abrupt variations are present, as shown in figure 4. At $k_BT = 0.01\omega_0$, fast entanglement variations are well visible at $c \approx 0.6\omega_0$ and $c \approx 3.7\omega_0$. The reason for this occurrence is that each of these points the system undergoes a level crossing involving its two lowest levels (ground and first excited states). Let us call $A_1$, $A_2$, $A_3$ the three plateaux of figure 4, corresponding to $(c < 0.6\omega_0)$, $(0.6\omega_0 < c < 3.7\omega_0)$ and $(c > 3.7\omega_0)$, respectively. The entanglement in these areas is the same as the entanglement possessed by the ground state. In $A_1$ entanglement is evidently zero and, in fact, the ground state of the whole system is $|g^{(A_1)}\rangle = |000\rangle$, corresponding to $\rho_{123}^{(A_1)} = 0$. In the regions $A_2$ and $A_3$, the ground states of four spin systems are $|g^{(A_2)}\rangle = (|0100\rangle + |0010\rangle + |0001\rangle - \frac{1}{\sqrt{3}}|1000\rangle)/\sqrt{6}$ and $|g^{(A_3)}\rangle = (|0011\rangle + |0101\rangle + |0110\rangle - (|1100\rangle + |1010\rangle + |1001\rangle))/\sqrt{6}$, and the respective density matrices are $\rho_{123}^{(A_2)} = (|W\rangle\langle W|+|000\rangle\langle 000|)/2$ and $\rho_{123}^{(A_3)} = (|W\rangle\langle W|+|\tilde{W}\rangle\langle \tilde{W}|)/2$, where $|W\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$ and $|\tilde{W}\rangle = (|110\rangle + |101\rangle + |111\rangle)/\sqrt{3}$. 

Figure 3. Tripartite negativity $N_{123}$ of the homogeneous model versus $k_BT$, at $c = 10\omega_0$. Temperature is in units of $\omega_0$. In the area where $k_BT < \omega_0/2$ we observe that the negativity is almost constant.

Figure 4. Tripartite negativity $N_{123}$ of the homogeneous model versus coupling constants (in units of $\omega_0$), at $k_BT = 0.01\omega_0$. Abrupt variations at $c \approx 0.6\omega_0$ and $c \approx 3.7\omega_0$ are well visible. Three plateaux, $A_1$ ($c < 0.6\omega_0$), $A_2$ ($0.6\omega_0 < c < 3.7\omega_0$) and $A_3$ ($c > 3.7\omega_0$), are quite evident.
4. Inhomogeneous model

Once we have analysed the homogeneous model, we turn to the study of the lack of homogeneity in the three interactions. For the sake of simplicity we shall concentrate on two kinds of inhomogeneities that we call type A (c_1 = c, c_2 = c x, c_3 = c) and type B (c_1 = c, c_2 = c x, c_3 = c x^2), in both of which the inhomogeneity is characterized by a suitable dimensionless real parameter x. In both cases we study the thermal quantum correlations of the tripartite outer subsystem as a function of temperature (T), coupling parameter (c) and parameter of inhomogeneity (x). It is worth noting that in the homogeneous model the structure of eigenstates of \( H \) is independent of the coupling constant (c), so that the features of the correlations are essentially determined by the structure of the eigenvalues only. Instead, in the inhomogeneous case the eigenvalues are c and x dependent and the eigenstates depend on x; therefore, the thermal correlations are affected both by the crossing of levels and by the modification of the structure of the eigenstates. We will see that while the change of the eigenstates produces smooth changes of \( \mathcal{N}_{123} \), on the other hand a level crossing, especially at low temperature, can produce a very sharp variation of the negativity.

4.1. Model with inhomogeneity of type A

In spite of the lack of some symmetry, the inhomogeneous model of type A is still easily solvable and its diagonalization is given in the appendix.

Since one can see that in the area A_3 of figure 4 negativity assumes its greatest value, we start the analysis of the inhomogeneous model of type A showing in figure 5 the complete dependence of \( \mathcal{N}_{123} \) on temperature and inhomogeneity parameter for c = 6000 (this specific value of c belongs indeed to the area A_3 of the homogeneous model). It is quite obvious that the physical reason for the decreasing of \( \mathcal{N}_{123} \) with respect to T is the same as for the homogeneous case, i.e. the high degree of mixedness. At low temperature, in the 0 < x < 1 area, the smaller the x, the closer to zero the negativity. In fact, when x is much smaller than 1, c_2 = c x is much smaller than c_1 = c_3 = c, and hence, roughly speaking, spin 2 can be considered almost decoupled, and then separable. Moreover, at x_0 \( \approx 0.43 \), this negligibility of the coupling between spins C and 2 is concomitant to a level crossing between the two lowest energy levels, \( E_1'(c = 6000, x) = -3000(x + \sqrt{8 + x^2}) \) and \( E_2'(c = 6000, x) = -(6\sqrt{2 + x^2} + 1)000 \), which causes a sharp change at \( x = x_0 \), where \( E_1' = E_2' \).

Still, at low temperature, for x greater than 1, \( \mathcal{N}_{123} \) decreases as x becomes increasing greater than 1, because the coupling with spin 2 becomes stronger than the other two, making spins 1 and 3 almost separable from the spin 2. Around x = 5.5 a level crossing occurs. Because of the diminishing of \( \mathcal{N}_{123} \) for x \( \gg 1 \) and x \( \ll 1 \), there must be a maximum in the intermediate region. Unexpectedly, this maximum is reached at x \( = x_M \approx 2.46 \), instead of x = 1. We think this is an important result because there is no big difference in the values of negativity in a wide contour of the maximum (let us say from x = 0.5 to x = 5.5). In fact, it is conceptually important that the maximum of correlations is not reached in the homogeneous case, where the high symmetry of the system could lead to the idea of stronger correlations between its parts.

Evaluating the eigenvalues of the Hamiltonian one can find that the ground state in the nearby of this maximum is \( \ket{\Psi_2^-} \), which becomes \( \ket{\Psi_M^{(3)}} \approx 0.2073\ket{0011} + \ket{1000} + 0.2073\ket{1010} + 0.1001 \) for x = x_M.

In this region, the values of \( \mathcal{N}_{123} \) and in particular the appearance of a maximum are essentially determined by the dependence of the structure of the state \( \ket{\Psi_2^-} \) on the inhomogeneity parameter. The relevant density operator is a mixture of two Werner-like states, and can be written in the following way:

\[
\rho^{2-}(x) = \frac{1}{2} \ket{w_1(x)}\bra{w_1(x)} + \frac{1}{2} \ket{w_2(x)}\bra{w_2(x)},
\]

with

\[
\ket{w_1(x)} = \mathcal{N}\ket{001} + \ket{110} + \frac{\sqrt{8 + x^2} - x}{2}\ket{101},
\]

\[
\ket{w_2(x)} = \mathcal{N}\ket{100} + \ket{001} + \frac{\sqrt{8 + x^2} - x}{2}\ket{010},
\]

where

\[
\mathcal{N} = (2 + ((\sqrt{8 + x^2} - x)/2)^2)^{-1/2}.
\]

The relevant negativity can be evaluated and it turns out to be

\[
\mathcal{N}(\rho^{2-}(x)) = \frac{1}{2^{1/3}} \left\{ -\frac{1}{(8 + x^4)^{1/3}}(x + \sqrt{8 + x^2}) - \sqrt{2(20 + x(x + \sqrt{8 + x^2}))} |x - 3\sqrt{8 + x^2} + 8\sqrt{8 + x^2} \times (|\lambda_1(x)| + |\lambda_2(x)| + |\lambda_3(x)|)^2 \right\}^{1/3},
\]

Figure 5. Negativity \( \mathcal{N}_{123} \) versus temperature (\( k_B T \), in units of \( \omega_0 \)) and inhomogeneity parameter (x), at c = 6000, for the inhomogeneous model of type A. The trend with respect to temperature is decreasing. The dependence of \( \mathcal{N}_{123} \) on the inhomogeneity parameter, at low temperature, exhibits different trends: initially, smooth changes, then fast falling around x \( \approx 5.5 \) and zero value from there on.
with $\lambda_i(x)$'s being the roots of the algebraic equation in $\lambda$:
\[
(-10x + x^3 + (2 + x^2)\sqrt{8 + x^2})
+ (-32x - 4x^3 - 48\sqrt{8 + x^2} - 4x^2\sqrt{8 + x^2})\lambda
+ (128x + 16x^3 - 384\sqrt{8 + x^2} - 48x^2\sqrt{8 + x^2})\lambda^2
+ (1024\sqrt{8 + x^2} + 128x^2\sqrt{8 + x^2})\lambda^3 = 0,
\]
where the coefficients, and hence the solutions, depend on $x$. The behaviour of $N(\rho^2(x))$ is shown in figure 6, where the negativity of the thermal state for $c = 60\omega_0$ and $T \approx 0$ versus $x$ has also been plotted. It is well visible that the behaviour of $N(\rho^2(x))$ perfectly reproduces the edge of the top visible in figure 5 in the low-temperature region and for $x$ belonging to the region where $|\Psi_T^+\rangle$ is the ground state of the system.

It could be interesting to compare the behaviour of the tripartite negativity with the values of the three concurrences related to the three couples of spins that can be extracted tracing over one of them: $C_{23} = C(tr_1 \rho^{2-})$, $C_{12} = C(tr_2 \rho^{2-})$, $C_{13} = C(tr_3 \rho^{2-})$. Figure 7 shows that around $x = 0.5$ and $x = 5.5$ there are abrupt changes in the values of the concurrences (in the same points where the tripartite negativity exhibits the same behaviour), due to rapid changes of the ground state. It is worth noting that in the regions 3.5–5.5 the concurrences $C_{12}$ and $C_{13}$ are vanishing, while the tripartite negativity (and hence all three relevant bipartite negativity functions) is not. This apparent contradiction reflects the very different meanings of these quantities. Indeed, for example, while the concurrence $C_{12}$ measures the entanglement between spins 1 and 2, without considering the spin 3 that has been preliminarily traced over, the negativity $N_{1-23}$ takes into account the correlations between spin 1 and the whole couple made of the two spins 2 and 3. Therefore, in that region, we can talk about correlations between spin 1 and the couple (2, 3) without correlations between spin 1 and either of the other two spins, after the third has been traced over. We think this is an important result, since it underlines the difference between $N$ and $C$.

Figure 8 shows the complete dependence of $N_{123}$ on temperature and coupling parameter, for $x = 3$. We observe (by performing other plots, which we are not reporting here) that the qualitative behaviour of $N_{123}$ is independent of the specific value of $x$, provided it is neither too small nor too big. It is quite evident that the trend with respect to temperature is the usual one. At low temperature the dependence of $N_{123}$ on $c$ exhibits only the step trend due to level crossings. The highest step involves the energy levels $E_4^- (c, x = 3) = -c + \sqrt{11 - \omega_0}$ and $E_5^- (c, x = 3) = -(3 + \sqrt{17})c/2$ (see the appendix), and it is quite easy to find that the crossing occurs at $c_1 \approx 4.08\omega_0$, where $E_4^- = E_2^-$ while $E_5^- < E_2^-$ for $c < c_1$ and $E_2^- < E_4^-$ for $c > c_1$.

To conclude the analysis of the inhomogeneous model of type A, in figure 9 we have plotted the complete dependence of $N_{123}$ on coupling and inhomogeneity parameters, fixing the value of temperature at $k_B T = 0.01\omega_0$. The dependence on the coupling parameter is qualitatively the same as for the previous case (see figure 8). Indeed, we observe a step trend caused by a level crossing between the two lowest energy levels. As
a function of \( x \), \( N_{123} \) exhibits both smooth decreasing, due to increasing separability of spins 1 and 3, and fast variations traceable back to a level crossing between the two lowest energy levels \( E_2 \) and \( E_1 \). In fact, we verified that the big step observed in figure 9 corresponds to the locus of the points of the plane \((c, x)\) in which the two lowest levels have the same energy. Thanks to the simple form of the eigenvalues of our model, one can find a simple analytical expression of this curve:

\[
c(x) = \frac{x^2 + 4 + \alpha(x)}{4x} + \frac{1}{2} \sqrt{\frac{x^2 + 5 + \alpha(x)}{2}},
\]

where \( \alpha(x) \) is

\[
\alpha(x) = \sqrt{16 + 10x^2 + x^4}.
\]

### 4.2. Model with inhomogeneity of type B

The second type of inhomogeneity we decided to study is characterized by the coupling constants \( c_1 = c \), \( c_2 = cx \) and \( c_3 = cx^2 \). Also for this model we have considered the dependence on \( T, c \) and \( x \). Surprisingly, what we have found is that the behaviour of negativity in the presence of the inhomogeneity of type B is not different from what we have found in the presence of the inhomogeneity of type A. The diminishing with increasing temperature, the abrupt changes at low temperature in connection with level crossings and the presence of a maximum for \( x \neq 1 \) are all present also in this model of inhomogeneity. Since for type B we do not have an explicit analytical diagonalization of the Hamiltonian, we also miss an analytical expression of the state of the system in the nearby of the maximum. In fact, we have only numerical results, according to which, for example, for \( T = 0.01 \text{\ } k_B \) and \( c = 6 \text{\ } \omega_0 \) the maximum of negativity occurs at \( x = x_M \approx 1.91 \), where the ground state of the system is approximately

\[
\Psi_M^{(B)} = 0.292(1100 - |0111\rangle) + 0.471(|1001\rangle - |0110\rangle) + 0.439(|1010\rangle - |0101\rangle).
\]

### 5. Conclusions

In this paper we have analysed the thermal correlations in a spin–star system consisting of a central spin interacting with three outer ones, all immersed in a magnetic field. In the absence of a tripartite entanglement measure, we have decided to exploit the tripartite negativity, which is at least able to put in evidence the lack of separability and simple bi-separability of a mixed quantum state of a tripartite system.

We started from considering the homogeneous model in which the three peripheral spins interact in the same way with the central one, showing the appearance of a significant degree of inseparability revealed by a non-vanishing tripartite negativity \( N_{123} \). Typical behaviour of thermal correlations reflects the behaviour of \( N_{123} \), which decreases to zero value as temperature increases and, instead, exhibits both appreciable values and abrupt changes at very low temperature. These occurrences show in a very clear way the role of thermal entanglement mediator played by the central spin. Furthermore, we have considered the presence of inhomogeneity, meant as differences in the coupling constants between the outer spins and the central one, focusing on two special cases. In the first case (inhomogeneity of type A), two spins interact in the same way with the central one and the third has a different coupling constant. In the second case (inhomogeneity of type B), all three outer spins are characterized by different coupling strengths. Though some differences are visible, the qualitative behaviour of \( N_{123} \) is quite similar for the two inhomogeneous models and even for the homogeneous one. This suggests the idea that the thermal quantum correlations mediated by the central spin are not very much damaged by a certain lack of homogeneity that could characterize a more realistic situation, provided the degree of inhomogeneity is not high.

A remarkable point is that we have singled out the presence of the maxima of the tripartite negativity corresponding to inhomogeneous models, i.e. for the values of the inhomogeneous parameter much larger than 1. Though the differences between a maximum value of negativity and its values nearby (even up to the homogeneous case, i.e. up to \( x = 1 \)) is not very big, we think that the fact that a higher degree of symmetry in the system does not guarantee a higher degree of correlations between all its parts is conceptually important.

Another important point is that we have found some regions of the parameter space where it happens that two concurrences, say \( C_{12} \) and \( C_{23} \), are vanishing, meaning that there is no entanglement between 1 and 2 and between 2 and 3, while the relevant negativity \( N_{2 \rightarrow 13} \) is non-vanishing, meaning that there is a correlation between 2 and the couple made of 1 and 3. This fact points out in a very transparent way the different meanings of concurrence and negativity.

### Appendix

In this appendix, we give the eigenvalues and eigenvectors of the Hamiltonian of the inhomogeneous model of type A

\[
(c_1 = c, \ c_2 = c \ x, \ c_3 = c),
\]

as functions of \( x > 0, \ c > 0, \)
\( \omega_0 \). The special case \( x = 1 \) gives the solutions for the homogeneous model.

Energies

\[
E_{1}^{\pm} = \pm c x, \quad |\Psi_{1}^{\pm}\rangle = \frac{1}{\sqrt{2}}[(|0111\rangle \pm |1100\rangle),
\]

\[
E_{2}^{\pm} = \frac{c}{2}[x+(8+x^2)^{\frac{1}{2}}], \quad |\Psi_{2}^{\pm}\rangle = \frac{1}{K_1}\left[\begin{array}{c}
(|0011\rangle \\
\pm |1100\rangle)
\end{array}\right],
\]

\[
E_{3}^{\pm} = \frac{c}{2}[x-(8+x^2)^{\frac{1}{2}}], \quad |\Psi_{3}^{\pm}\rangle = \frac{1}{K_1}\left[\begin{array}{c}
(|0011\rangle \\
\pm |1100\rangle)
\end{array}\right],
\]

\[
E_{4}^{\pm} = \pm c(2+x^2)^{\frac{1}{2}} + \omega_0, \quad |\Psi_{4}^{\pm}\rangle = \frac{1}{K_2}\left[\begin{array}{c}
\sqrt{2+x^2}(0111) \\
\pm x|0011\rangle + |1100\rangle
\end{array}\right],
\]

\[
E_{5}^{\pm} = \pm c(2+x^2)^{\frac{1}{2}} - \omega_0, \quad |\Psi_{5}^{\pm}\rangle = \frac{1}{K_2}\left[\begin{array}{c}
(|0100\rangle + |0010\rangle) \\
\pm \sqrt{2+x^2}|1000\rangle,
\end{array}\right]
\]

\[
E_{6} = -\omega_0, \quad |\Psi_{6}^{A}\rangle = \frac{1}{K_3}\left[\begin{array}{c}
\frac{1}{x}(|0001\rangle + |0010\rangle),
\end{array}\right]
\]

\[
E_{7} = \omega_0, \quad |\Psi_{7}^{A}\rangle = \frac{1}{K_3}\left[\begin{array}{c}
\frac{1}{x}(|1011\rangle + |1100\rangle),
\end{array}\right]
\]

\[
E_{8} = -2\omega_0, \quad |\Psi_{8}\rangle = |0000\rangle,
\]

\[
E_{9} = 2\omega_0, \quad |\Psi_{9}\rangle = |1111\rangle,
\]

where \( K_1, K_2 \) and \( K_3 \) are

\[
K_1^2 = 4 + 2\left(\frac{\sqrt{8 + x^2} - x}{2}\right)^2,
\]

\[
K_2^2 = 2(2 + x^2),
\]

\[
K_3^2 = 2 + 3 + x^2.
\]

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