DIRECT AND INVERSE THEOREMS IN THE THEORY OF APPROXIMATION BY THE RITZ METHOD

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Abstract. For an arbitrary self-adjoint operator \( B \) in a Hilbert space \( \mathcal{H} \), we present direct and inverse theorems establishing the relationship between the degree of smoothness of a vector \( x \in \mathcal{H} \) with respect to the operator \( B \), the rate of convergence to zero of its best approximation by exponential-type entire vectors of the operator \( B \), and the \( k \)-modulus of continuity of the vector \( x \) with respect to the operator \( B \). The results are used for finding a priori estimates for the Ritz approximate solutions of operator equations in a Hilbert space.

1. Introduction

Let \( B \) be a closed linear operator with dense domain of definition \( \mathcal{D}(B) \) in a separable Hilbert space \( \mathcal{H} \) over the field of complex numbers.

Let \( C^\infty(B) \) denote the set of all infinitely differentiable vectors of the operator \( B \), i.e.,

\[
C^\infty(B) = \bigcap_{n \in \mathbb{N}_0} \mathcal{D}(B^n), \quad \mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}.
\]

For a number \( \alpha > 0 \), we set

\[
\mathcal{E}^\alpha(B) = \{ x \in C^\infty(B) \mid \exists c = c(x) > 0 \forall k \in \mathbb{N}_0 \|B^k x\| \leq c \alpha^k \}.
\]

The set \( \mathcal{E}^\alpha(B) \) is a Banach space with respect to the norm

\[
\|x\|_{\mathcal{E}^\alpha(B)} = \sup_{n \in \mathbb{N}_0} \frac{\|B^n x\|}{\alpha^n}.
\]

Then \( \mathcal{E}(B) = \bigcup_{\alpha > 0} \mathcal{E}^\alpha(B) \) is a linear locally convex space with respect to the topology of the inductive limit of the Banach spaces \( \mathcal{E}^\alpha(B) \):

\[
\mathcal{E}(B) = \lim_{\alpha \to \infty} \ind \mathcal{E}^\alpha(B).
\]

Elements of the space \( \mathcal{E}(B) \) are called exponential-type entire vectors of the operator \( B \). The type \( \sigma(x, B) \) of a vector \( x \in \mathcal{E}(B) \) is defined as the number

\[
\sigma(x, B) = \inf \{ \alpha > 0 : x \in \mathcal{E}^\alpha(B) \} = \limsup_{n \to \infty} \|B^n x\|^{\frac{1}{n}}.
\]

In what follows, we always assume that the operator \( B \) is self-adjoint in \( \mathcal{H} \), and \( E(\Delta) \) is its spectral measure.

Let \( G(\cdot) \) be an almost everywhere finite measurable function on \( \mathbb{R} \). A function \( G(B) \) of the operator \( B \) is understood as follows:

\[
G(B) := \int_{-\infty}^{\infty} G(\lambda)dE(\lambda).
\]

As shown in [1], one has \( \mathcal{E}^\alpha(B) = E([-\alpha, \alpha])\mathcal{H} \) for every \( \alpha > 0 \).

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According to [2], we set
\[ \omega_k(t, x, B) = \sup_{0 < t \leq t} \| \Delta_k x \|, \quad k \in \mathbb{N}, \]  
(1)
where
\[ \Delta_k = (U(h) - 1)^k = \sum_{j=0}^{k} (-1)^{k-j} C_k^j U(jh), \quad k \in \mathbb{N}_0, \quad h \in \mathbb{R} \quad (\Delta_0 \equiv 1, \quad h \in \mathbb{R}_+), \]  
(2)
and \( U(h) = \exp(ihB) \) is the group of unitary operators in \( \mathfrak{H} \) with generator \( iB \).

The definition of \( \omega_k(t, x, B) \) implies that the following assertions are true \( k \in \mathbb{N} \):

1. \( \omega_k(0, x, B) = 0 \);
2. for fixed \( x \), the function \( \omega_k(t, x, B) \) does not decrease on \( \mathbb{R}_+ = [0, \infty) \);
3. \( \omega_k(\alpha t, x, B) \leq [1 + \alpha]^k \omega_k(t, x, B) \) \( (\alpha, t > 0) \);
4. for fixed \( t \in \mathbb{R}_+ \), the function \( \omega_k(t, x, B) \) is continuous in \( x \).

Further, we establish an inequality of the Bernstein Nikolskïi type.

**Lemma 1.1.** Let \( G(\lambda) \) be a nonnegative even function on \( \mathbb{R} \) that is nondecreasing on \( \mathbb{R}_+ \), let \( x \in \mathcal{E}(B) \) and let \( \sigma(x, B) \leq \alpha \). Then
\[ \| \Delta_k G(B)x \| \leq h^k \alpha^k G(\alpha) \| x \|, \quad h > 0, \quad k \in \mathbb{N}_0. \]  
(3)

**Proof.** Since \( \sigma(x, B) \leq \alpha \) and \( |1 - e^{i\lambda h}|^2 = 4^2 \sin^2 \frac{\lambda h}{2} \leq \lambda^2 h^2, \quad \lambda \in \mathbb{R} \), on the basis of operational calculus for the operator \( B \) we get
\[ \| \Delta_k G(B)x \|^2 = \int_{-\alpha}^{\alpha} |(1 - e^{i\lambda h})^k|^2 G^2(\lambda) d(E_\lambda x, x) \leq \int_{-\alpha}^{\alpha} \lambda^{2k} G^2(\lambda) d(E_\lambda x, x) \leq h^{2k} \alpha^{2k} G^2(\alpha) \| x \|^2. \]  
(4)
\[ \square \]

For \( k = 0 \) Lemma 1.1 yields
\[ \| G(B)x \| \leq G(\alpha) \| x \|. \]  
(5)

**Corollary 1.1.** Under the conditions of Lemma 1.1 with respect to \( x \) and \( \sigma(x, B) \), the following relation is true:
\[ \| \Delta_k x \| \leq h^k \cdot \alpha^k \cdot \| x \|, \quad h \geq 0. \]

**Proof.** For the proof of this statement, it suffices to take \( G(\cdot) \equiv 1, \lambda \in \mathbb{R} \), in Lemma 1.1. \[ \square \]

If \( \mathfrak{H} = L_2([0, 2\pi]) \) and \( (Bx)(t) = ix'(t) \),
\[ D(B) = \{ x(t) | x \in W_1^2([0, 2\pi]), \quad x(0) = x(2\pi) \}, \]
where \( W_1^2([0, 2\pi]) \) is a Sobolev space, then \( \mathcal{E}(B) \) coincides with the set of all trigonometric polynomials, \( \sigma(x, B) \) is the degree of the polynomial \( x \), \( \mathcal{E}^{\alpha}(B) \) is the set of all trigonometric polynomials whose degrees do not exceed \( \alpha \); \( (U(h)x)(t) = \tilde{x}(t + h), \omega_k(t, x, B) \) is the \( k \)th modulus of continuity of the function \( x(t) \), and inequality (3) for \( G(\lambda) = |\lambda^m| \) and \( k = 0 \) turns into a Bernstein-type inequality in the space \( L_2[0, 2\pi] \) (here \( \tilde{x}(t) \) is understood as the \( 2\pi \)-periodic extension of the function \( x(t) \)).
For an arbitrary $x \in \mathcal{H}$ following [5, 6], we set
\[
E_r(x, B) = \inf_{y \in \mathcal{E}(B) : \sigma(y, B) \leq r} \|x - y\|, \quad r > 0,
\]
i.e., $E_r(x, B)$ is the best approximation of the element $x$ by exponential-type entire vectors $y$ of the operator $B$ for which $\sigma(y, B) \leq r$. For fixed $x$, $E_r(x, B)$ does not increase and $E_r(x, B) \to 0$, $r \to \infty$. It is clear that
\[
E_r(x, B) = \|x - E([-r, r])x\| = \|x - F([0, r])x\|,
\]
where $F(\Delta)$ is the spectral measure of the operator $|B| = \sqrt{B^*B}$.

**Theorem 1.1.** Suppose that $G(\lambda)$ satisfies the conditions of Lemma [7]. Then, for any $x \in \mathcal{D}(G(B))$ the following relation is true:
\[
\forall k \in \mathbb{N} \quad E_r(x, B) \leq \frac{\sqrt{k + 1}}{2^k G(r)} \omega_k \left( \frac{\pi}{r}, G(B)x, B \right), \quad r > 0.
\]

**Proof.** Using the spectral representation for the operator $B$ and the monotonicity of the function $G(\lambda)$, we obtain
\[
\omega_k^2(t, G(B)x, B) = \sup_{0 < \tau \leq t} \|(e^{i\tau B} - 1)^k G(B)x\|^2 \geq \|(e^{i\tau B} - 1)^k G(B)x\|^2 = \int_{-\infty}^{\infty} |e^{tB} - 1|^2 G^2(\lambda) d(E_\lambda x, x) = 2^k \int_{\mathbb{R}} (1 - \cos \lambda t)^k G^2(\lambda) d(E_\lambda x, x) \geq 2^k G^2(r) \int_{|\lambda| \geq r} (1 - \cos \lambda t)^k d(E_\lambda x, x).
\]
We fix $r > 0$ and take $t : 0 \leq t \leq \frac{\pi}{r}$. Then $\sin rt \geq 0$. We multiply both sides of the above inequality by $\sin rt$ and integrate the result with respect to $t$ from 0 to $\frac{\pi}{r}$. Then
\[
\int_0^{\pi/r} \omega_k^2(t, G(B)x, B) \sin rt dt \geq 2^k G^2(r) \int_0^{\pi/r} \int_{|\lambda| \geq r} (1 - \cos \lambda t)^k \sin rt d(E_\lambda x, x) dt = 2^k G^2(r) \int_{|\lambda| \geq r} \left( \int_0^{\pi/r} (1 - \cos \lambda t)^k \sin rt dt \right) d(E_\lambda x, x).
\]
Since the function $\omega_k^2(t, G(B)x, B)$ is monotonically nondecreasing, we have
\[
\int_0^{\pi/r} \omega_k^2(t, G(B)x, B) \sin rt dt \leq \int_0^{\pi/r} \omega_k^2 \left( \frac{\pi}{r}, G(B)x, B \right) \sin rt dt = \frac{2}{r} \omega_k^2 \left( \frac{\pi}{r}, G(B)x, B \right).
\]
Using the inequality (see [7])
\[
\int_0^{\pi} (1 - \cos \theta t)^k \sin t dt \geq \frac{2^{k+1}}{k+1}, \quad \theta \geq 1, \quad k \in \mathbb{N}
\]
and relations (7) and (8), we get
\[
\frac{2}{r} \omega_k^2 \left( \frac{\pi}{r}, G(B)x, B \right) \geq 2^k G^2(r) \int_{|\lambda| \geq r} \left( \frac{1}{r} \frac{2^{k+1}}{k+1} \right) d(E_\lambda x, x) = \frac{2^{2k+1} G^2(r)}{r(k+1)} E_r^2(x, B),
\]
which is equivalent to (5). □

For $G(\lambda) = |\lambda|^m$, $\lambda \in \mathbb{R}$, $m > 0$ Theorem [1,4] yields the following corollary:
Corollary 1.2. Let \( x \in D(\|B\|^m) , \ m > 0 \). Then, for any \( k \in \mathbb{N} \)
\[
E_r(x, B) \leq \frac{\sqrt{k + 1}}{2^{k+1}} \omega_k \left( \frac{\pi}{r} \|B\|^m x, B \right) , \quad r > 0 .
\] (11)

For the case where \( B \) is the operator of differentiation with periodic boundary conditions in the space \( \mathcal{H} = L_2([0, 2\pi]) \), i.e., \( (Bx)(t) = ix'(t) \) and \( D(B) = \{ x(t) \mid x \in W^1_2([0, 2\pi]), x(0) = x(2\pi) \} \), inequality (11) is presented in [8] for \( k = 1 \) and in [7] for arbitrary \( k \in \mathbb{N} \).

We now formulate the inverse theorem in the case of approximation of a vector \( x \) by exponential-type entire vectors of the operator \( B \).

Theorem 1.2. Let \( \omega(t) \) be a function of the type of a modulus of continuity for which the following conditions are satisfied:

1): \( \omega(t) \) is continuous and nondecreasing for \( t \in \mathbb{R}_+ \);
2): \( \omega(0) = 0 \);
3): \( \exists c > 0 \forall t > 0 \) \( \omega(2t) \leq c \omega(t) \);
4): \( \int_0^1 \frac{\omega(t)}{t} dt < \infty \).

Also assume that the function \( G(\lambda) \) is even, nonnegative, and nondecreasing for \( \lambda \geq 0 \), and, furthermore, \( \sup_{\lambda \geq 0} \frac{G(\lambda)}{G(1)} < \infty \).

If, for \( x \in \mathcal{H} \), there exists \( m > 0 \) such that
\[
E_r(x, B) < \frac{m}{G(1)} \omega \left( \frac{1}{r} \right) , \quad r > 0,
\] (12)
then \( x \in D(G(B)) \) and, for every \( k \in \mathbb{N} \), there exists a constant \( m_k > 0 \) such that
\[
\omega_k(t, G(B)x, B) \leq m_k \left[ \frac{t^k}{k+1} \int_0^t \frac{\omega(\tau)}{\tau} d\tau + \int_0^t \frac{\omega(\tau)}{\tau} d\tau \right] , \quad 0 < t \leq \frac{1}{2} .
\] (13)

First, we prove the following statement:

Lemma 1.2. Suppose that the function \( \omega(t) \) satisfies conditions 1),2),3) of Theorem 1.2.
If, for \( x \in \mathcal{H} \), there exists \( c > 0 \) such that
\[
E_r(x, B) < m \omega \left( \frac{1}{r} \right) , \quad r > 0
\] (14)
then, for every \( k \in \mathbb{N} \), there exists a constant \( c_k > 0 \) such that
\[
\omega_k(t, x, B) \leq c_k \cdot t^k \int_0^t \frac{\omega(\tau)}{\tau} d\tau , \quad 0 < t \leq \frac{1}{2} .
\] (15)

Proof. It follows from condition (14) that there exists a sequence \( \{ u_{2i} \}_{i=0}^\infty \) of exponential-type entire vectors such that \( \sigma(u_{2i}, B) \leq 2^i \) and
\[
\|x - u_{2i}\| \leq m \cdot \omega \left( \frac{1}{2^i} \right) .
\] (16)

We take an arbitrary \( h \in (0, \frac{1}{4}) \) and choose a number \( N \) so that \( \frac{1}{2^N} < h \leq \frac{1}{2^{N+1}} \). Inequality (14) yields
\[
\Delta_h^k x = \Delta_h^k u_1 + \sum_{j=1}^N \Delta_h^k (u_{2j} - u_{2j-1}) + \Delta_h^k (x - u_{2N})
\] (17)
By virtue of (12) there exists a sequence $u_{2j} - u_{2j-1} \leq \|u_{2j} - u_{2j-1}\| \leq m \cdot \omega \left( \frac{1}{2^j} \right) + m \cdot \omega \left( \frac{1}{2^{j-1}} \right) \leq 2m \cdot \omega \left( \frac{1}{2^j} \right) \leq 2cm \cdot \omega \left( \frac{1}{2^i} \right). \tag{18}

By virtue of the monotonicity of $\omega(t)$, we have

$$2^k \int_{1/2^j}^{1/2^{j-1}} \frac{\omega(u)}{u^{k+1}} du \geq 2^k \omega \left( \frac{1}{2^j} \right) \int_{1/2^j}^{1/2^{j-1}} \frac{1}{u^{k+1}} du = \frac{2^k}{k} \omega \left( \frac{1}{2^j} \right) \left( 2^k - 1 \right) \geq 2^k \omega \left( \frac{1}{2^j} \right). \tag{19}$$

Since $\sigma(u_{2j}, B) \leq 2^j$ and $\sigma(u_1, B) \leq 1$, according to Corollary 1.1 we get

$$\|\Delta_h^k u_1\| \leq h^k \cdot \|u_1\|,$$

$$\|\Delta_h^k (u_{2j} - u_{2j-1})\| \leq h^k \cdot (2^j)^k \cdot \|u_{2j} - u_{2j-1}\|.$$

Relations (16), (18) and (19) yield

$$\|\Delta_h^k (u_{2j} - u_{2j-1})\| \leq 2cmh^k \cdot 2^{k^2} \omega \left( \frac{1}{2^j} \right) \leq 2^{k^2+1} cmh^k \int_{1/2^j}^{1/2^{j-1}} \frac{\omega(u)}{u^{k+1}} du$$

and

$$\|\Delta_h^k (x - u_{2N})\| \leq (\|e^{(k+1)x}\| + 1)^k \cdot \|x - u_{2N}\| \leq 2^k \cdot \|x - u_{2N}\| \leq 2^k \cdot \omega \left( \frac{1}{2^N} \right).$$

Using these inequalities, we obtain

$$\|\Delta_h^k x\| = \left| \Delta_h^k u_0 + \sum_{j=1}^{N} \Delta_h^k (u_j - u_{j-1}) + \Delta_h^k (x - u_N) \right| \leq$$

$$\leq h^k \|u_0\| + 2^{k+1} cmh^k \sum_{j=1}^{N} \int_{1/2^j}^{1/2^{j-1}} \frac{\omega(u)}{u^{k+1}} du + 2^k cm \cdot \omega \left( \frac{1}{2^N} \right) \leq$$

$$\leq h^k \|u_0\| + 2^{k+1} cmh^k \int_{1/2}^{1/2} \frac{\omega(u)}{u^{k+1}} du + 2^k cm \cdot \omega(2h) \leq$$

$$\leq h^k \|u_0\| + 2^{k+1} cmh^k \int_{h}^{1} \frac{\omega(u)}{u^{k+1}} du + 2^k cm \cdot \omega(h) =$$

$$= h^k \left( \|u_0\| + 2^{k+1} cm \int_{h}^{1} \frac{\omega(u)}{u^{k+1}} du + 2^k cm \frac{k}{1-h^k} \int_{h}^{1} \frac{\omega(u)}{u^{k+1}} du \right) \leq$$

$$\leq c_k \cdot h^k \int_{h}^{1} \frac{\omega(u)}{u^{k+1}} du,$$

where $c_k = \int_{1/2}^{1} \frac{\omega(u)}{u^{k+1}} du + 2^{k+1} cm + 2^k cm \frac{k}{1-h^k}$. \hfill \Box

**Remark 1.1.** As follows from the proof, the lemma remains true under somewhat weaker conditions than those formulated in the theorem, namely, it is sufficient that, for an element $x \in \mathcal{F}$, there exist at least one sequence $\{u_{2j}\}_{j=0}^{\infty}$, such that

$$\sigma(u_{2j}, B) \leq 2^j \quad \text{and} \quad \forall j \in \mathbb{N} \quad \|x - u_{2j}\| \leq m \cdot \omega \left( \frac{1}{2^j} \right).$$

**Proof of Theorem.** By virtue of (12) there exists a sequence $\{u_{2^n}\}_{n=1}^{\infty}$ such that $\sigma(u_{2^n}) \leq 2^n$ and

$$\|x - u_{2^n}\| \leq \frac{c}{G(2^n)} \omega \left( \frac{1}{2^n} \right), \quad n \in \mathbb{N}. \tag{20}$$
It follows from inequality (20) and conditions 1), 2) of the theorem that \( \|x - u_{2n}\| \to 0 \) as \( n \to \infty \), and, therefore, the vector \( x \) can be represented in the form

\[
x = u_1 + \sum_{k=1}^{\infty} (u_{2k} - u_{2k-1}).
\]

Since \( \sigma(u_{2k} - u_{2k-1}, B) \leq 2^k, k \in \mathbb{N} \) taking (5) into account we obtain

\[
\|G(B)u_{2k} - G(B)u_{2k-1}\| \leq G(2^k) \|u_{2k} - u_{2k-1}\| \leq G(2^k) (\|x - u_{2k}\| + \|x - u_{2k-1}\|) \leq
\]

\[
\leq G(2^k) \left( \frac{m}{G(2^k)} \omega \left( \frac{1}{2^k} \right) + \frac{m}{G(2^k-1)} \omega \left( \frac{1}{2^{k-1}} \right) \right) \leq
\]

\[
\leq 2G(2^k) \cdot \frac{m}{G(2^k-1)} \omega \left( \frac{1}{2^k-1} \right) \leq 2cc_1m \cdot \omega \left( \frac{1}{2^k} \right) \leq \frac{2cc_1m}{\ln 2} \int_{2^{-k+1}}^{2^{-k+1}} \frac{\omega(u)}{u} du,
\]

where \( c_1 \) denotes \( \sup \frac{G(2\lambda)}{\lambda} \) for \( \lambda > 0 \). Therefore, the series \( \sum_{k=1}^{\infty} (G(B)u_{2k} - G(B)u_{2k-1}) \) converges. The closedness of the operator \( G(B) \) implies that \( x \in \mathcal{D}(G(B)) \) and

\[
G(B)x = G(B)u_1 + \sum_{k=1}^{\infty} (G(B)u_{2k} - G(B)u_{2k-1}).
\]

This yields

\[
\|G(B)x - G(B)u_2\| \leq \sum_{k=j+1}^{\infty} \|G(B)u_{2k} - G(B)u_{2k-1}\| \leq 2cc_1m \sum_{k=j+1}^{\infty} \omega(2^{-k}) \leq
\]

\[
\leq 2cc_1m \int_0^{2^{-j}} \frac{\omega(u)}{u} du =: \tilde{c} \Omega(2^{-j}), \quad j \in \mathbb{N}
\]

where

\[
\tilde{c} := 2cc_1m \quad \text{and} \quad \Omega(t) := \int_0^t \frac{\omega(u)}{u} du
\]

It is easy to verify that the function \( \Omega(t) \) possesses the following properties:

1: \( \Omega(t) \) is continuous and monotonically nondecreasing;

2: \( \Omega(0) = 0 \);

3: for \( t > 0 \), the following relation is true:

\[
\Omega(2t) = \int_0^{2t} \frac{\omega(u)}{u} du = \int_0^t \frac{\omega(2u)}{u} du \leq c_2 \int_0^t \frac{\omega(u)}{u} du = c_2 \Omega(t).
\]

Therefore, setting \( \omega(t) := \Omega(t) \) in Lemma 1.2 and taking Remark 1.1 into account, we get

\[
\omega_k(G(B)x, t, B) \leq c_k \cdot t^k \int_t^1 \frac{\Omega(u)}{u^{k+1}} du = \frac{c_k \cdot t^k}{k} \left( \Omega(u) \frac{1}{u^k} \right)_1^t + \int_t^1 \frac{\omega(u)}{u^{k+1}} du \leq
\]

\[
\leq m_k \left( t^k \int_t^1 \frac{\omega(u)}{u^{k+1}} du + \int_0^t \frac{\omega(u)}{u} du \right). \quad \square
\]

Theorem 1.2 shows that, in the case where \( \omega(t) = t^\alpha, \ t \geq 0, \ \alpha > 0 \) and \( \mathcal{E}_t(x, B) = O \left( \frac{1}{t^\alpha} \right) \), one has

\[
\omega_k(t, x, B) = \begin{cases} O \left( t^k \right) & k < \alpha \\ O \left( t^k \ln t \right) & k = \alpha \\ O \left( t^\alpha \right) & k > \alpha \end{cases}
\]

2. Consider the equation

\[
Ax = y,
\]

(21)
where $A$ is a positive-definite self-adjoint operator with discrete spectrum, $y \in \mathfrak{F}$, $x \in \mathcal{D}(A)$ is the required solution of Eq. [21]. Let $\mathfrak{F}_+$ denote the completion of the set $\mathcal{D}(A)$ with respect to the norm $\|\cdot\|_+$, generated by the scalar product

$$(x,y)_+ = (Ax,y).$$

Under the conditions imposed above on the operator $A$, Eq. [21] has a unique solution $x \in \mathcal{D}(A)$ and, according to the Dirichlet principle [9], the determination of this solution is equivalent to the determination of the vector $u \in \mathcal{D}(A)$, on which the functional

$$F(z) = (Az,z) - 2Re(y,z),$$

defined on $\mathcal{D}(A)$ attains its minimum.

Let $\{e_k\}_{k=1}^\infty$ be a complete linearly independent system of vectors from $\mathcal{D}(A)$ (so-called coordinate system), and let

$$\mathcal{H}_n = \ldots \{e_1, \ldots, e_n\}.$$ 

By $x_n$ we denote the vector on which $F(z)$ attains its minimum on $\mathcal{H}_n$. The vector $x_n$ is called the Ritz approximate solution of Eq. [21]. As is known, independently of the choice of a coordinate system, the sequence $x_n$ converges to $x$ in the space $\mathfrak{F}_+$ (and, hence, in $\mathfrak{F}$). The residual $R_n = \|Ax_n - y\|$ does not always tend to zero in $\mathfrak{F}$. However, if the coordinate system $\{e_k\}_{k=1}^\infty$ is chosen so that it forms an orthonormal proper basis of some positive-definite self-adjoint operator $B$ related to $A$ in the sense that $\mathcal{D}(A) = \mathcal{D}(B)$, then $R_n \to 0$ as $n \to \infty$ (see [2]), and, therefore, the quantities $r_n = \|x_n - x\|_+$ also tend to zero as $n \to \infty$. However, the investigation of the behavior of these quantities, which depend on the choice of $\{e_k\}_{k=1}^\infty$ and on the right-hand side of Eq. [21], at infinity turned out to be a rather difficult problem and remains unsolved. Some particular results for operators generated by boundary-value problems for ordinary differential equations were obtained in numerous papers by many authors (see the survey [10]). For the abstract case, some particular situations were considered in [11]). In [6], direct and inverse theorems were established for the first time under the condition that $x \in C^\infty(B)$ and estimates for the quantity $R_n$ were obtained in the case where the smoothness of the vector $x$ is finite, i.e., $x \in \mathcal{D}(B^k)$. Below, we completely characterize the quantity $r_n$ for $x \in \mathcal{D}(B^k)$.

In what follows, we assume that the following conditions are satisfied:

1°: The operator $A$ is self-adjoint and positive definite.

2°: The coordinate system in the Ritz method is an orthonormal basis of a positive-definite self-adjoint operator $B$ with discrete simple spectrum $(Be_k = \lambda_k e_k)$ that is related to $A$.

Let $x_n$ denote the Ritz approximate solution of Eq. [21] with respect to the coordinate system $\{e_k\}_{k=1}^\infty$. We set

$$\overline{x}_n = \sum_{k=1}^n (x,e_k)e_k.$$ 

Since the operators $A$ and $B$ are positive definite and self-adjoint and $\mathcal{D}(A) = \mathcal{D}(B)$, it follows from the Heinz inequality [12] that $\mathcal{D}(A^\alpha) = \mathcal{D}(B^\alpha)$ for any $\alpha \in (0,1)$, and, therefore, the operators $B^{\frac{1}{2}}A^{-\frac{1}{2}}$ and $A^{\frac{1}{2}}B^{-\frac{1}{2}}$ are defined and bounded on the entire space $\mathfrak{F}$, and, for any $x \in \mathcal{D}(A)$, one has

$$c_1^{-1}|||x|||_+ \leq ||x||_+ \leq c_2|||x|||_+, \quad (22)$$

where $|||x|||_+ = \|B^{1/2}x\|$, $c_1 = \|B^{1/2}A^{-1/2}\|$ and $c_2 = \|A^{1/2}B^{-1/2}\|$.

**Lemma 1.3.** For any $n \in \mathbb{N}$ and $x \in \mathcal{D}(B)$, the following inequality is true:

$$|||x - \overline{x}_n|||_+ \leq |||x - x_n|||_+ \leq c_3|||x - \overline{x}_n|||_+, \quad (23)$$

where $c_3 = \|B^{1/2}A^{-1/2}\| \|A^{1/2}B^{-1/2}\|$.
Proof. Since
\[ B^{1/2} \left( \sum_{k=1}^{n} (x, e_k) e_k \right) = \sum_{k=1}^{n} \left( B^{1/2} x, e_k \right) e_k, \]
we have
\[ \| x - \tilde{x}_n \|_+ = \left\| B^{1/2} \left( x - \sum_{k=1}^{n} (x, e_k) e_k \right) \right\| = \left\| B^{1/2} x - \sum_{k=1}^{n} \left( B^{1/2} x, e_k \right) e_k \right\| \leq \left\| B^{1/2} x - B^{1/2} x \right\| = \| x - x_n \|_+ \]

Taking into account that the Ritz approximation \( x_n \) is the best approximation of a vector \( x \) in the norm \( \| \cdot \|_+ \), we get
\[ \| x - x_n \|_+ = \left\| B^{1/2} (x - x_n) \right\| \leq \left\| B^{1/2} A^{-1/2} \right\| \left\| A^{1/2} (x - x_n) \right\| = c_1 \| x - x_n \|_+ \leq c_1 \| x - \tilde{x}_n \|_+ = c_1 \left\| A^{1/2} (x - \tilde{x}_n) \right\| \leq c_1 c_2 \left\| B^{1/2} (x - \tilde{x}_n) \right\| = c_3 \| x - \tilde{x}_n \|_+ \]

Taking into account the relations
\[ E_{\lambda_n}(B^{1/2} x, B) = \| x - \tilde{x}_n \|_+ \]
and
\[ E_{\lambda_n}(B^{1/2} x, B) = E_{\lambda_n+\eta}(B^{1/2} x, B), \quad 0 < \eta < \lambda_{n+1} - \lambda_n, \]

inequalities (22) and (23), and Theorem 1.1 with \( G(\lambda) = |\lambda|^{\alpha - \frac{1}{2}}, \alpha \geq 1 \), we establish the following result:

**Theorem 1.3.** If \( x \in D(B^\alpha) \), \( \alpha \geq 1 \), then the following relation holds for every \( \forall k \in \mathbb{N} \):
\[
\| x - x_n \|_+ \leq c_0 \frac{\sqrt{k + 1}}{2^k \lambda_{n+1}^{\alpha - \frac{1}{2}}} \omega_k \left( \frac{\pi}{\lambda_{n+1}}, B^\alpha x, B \right),
\]
where \( c_0 = c_2 c_3 \), and \( c_2 \) and \( c_3 \) are the constants from inequalities (22) and (23).

Since, for \( x \in D(B^\alpha) \)
\[
\omega_k \left( \frac{\pi}{\lambda_{n+1}}, B^\alpha x, B \right) \to 0, \quad n \to \infty,
\]
we conclude that, for \( x \in D(B^\alpha) \)
\[
\lim_{n \to \infty} \lambda_{n+1}^{\alpha - \frac{1}{2}} \| x - x_n \|_+ = 0 \quad \text{(24)}
\]

We now give examples of operators \( A \) and \( B \) for which equality (24) for \( \alpha > 1 \) does not yield the inclusion \( x \in D(B^\alpha) \). We set
\[
\delta = L_2([0, \pi]), \quad A = B = -\frac{d^2}{dt^2}, \quad D(A) = D(B) = \{ x(t) \mid x \in W^2_2([0, \pi]), x(0) = x(\pi) = 0 \},
\]
\[
\lambda_k(B) = k^2, \quad e_k = \sqrt{\frac{2}{\pi}} \sin kt, \quad x = x(t) = \sqrt{\frac{2}{\pi}} \sum_{k=2}^{\infty} x_k \sin kt,
\]
where \( x_k = \frac{1}{k^{2\alpha + \frac{1}{2}} \ln^{\frac{1}{2}} k}, k \in \mathbb{N}\setminus\{1\} \). The equality
\[
\sum_{k=2}^{\infty} \frac{k^{4\alpha}}{k^{4\alpha + 1} \ln k} = \sum_{k=2}^{\infty} \frac{1}{k \ln k} = \infty
\]
shows that \( x \notin \mathcal{D}(B^\alpha) \). However, since

\[
\|x - x_n\|^2_+ = \|x - \bar{x}_n\|^2_+ = \sum_{k=n+1}^{\infty} \frac{k^2}{k^{4\alpha-1} \ln k} \leq \frac{1}{\ln(n+1)} \int_n^{\infty} \frac{1}{t^{4\alpha-1}} dt = \frac{1}{(4\alpha - 2)n^{4\alpha-2} \ln(n+1)}
\]

we have

\[
\lim_{n \to \infty} \lambda_n^{\frac{\alpha}{2}}(B) \|x - x_n\|_+ \leq \lim_{n \to \infty} n^{2\alpha-1} \frac{1}{\sqrt{4\alpha - 2}} \frac{1}{\sqrt{n^{2\alpha-1}}} = 0
\]

It follows from Theorem 1.3 inequality (22) and Lemma 1.3 that the following statement is true:

**Theorem 1.4.** Suppose that \( \omega(t) \) satisfies the conditions of Theorem 1.2. If, for \( x \in \mathcal{D}(B), n \in \mathbb{N} \) and \( \alpha > 1 \) one has

\[
\|x - x_n\|_+ \leq \frac{c}{\lambda_{n+1}^{\alpha/2}} \omega\left(\frac{1}{\lambda_{n+1}}\right),
\]

where \( c \equiv \text{const.} \), then \( x \in \mathcal{D}(B^\alpha) \).

Note that, by virtue of inequality (22), \( \|\cdot\|_+ \) in Theorems 1.3 and 1.4 can be replaced by \( \|\cdot\|_+ \).

The same theorem immediately yields the following corollary:

**Corollary 1.3.** Suppose that the following inequality holds for \( x \in \mathcal{D}(B), n \in \mathbb{N}, \alpha > 1 \) and \( \varepsilon > 0 \)

\[
\|x - x_n\|_+ \leq \frac{c}{\lambda_{n+1}^{\alpha\varepsilon}}.
\]

Then \( x \in \mathcal{D}(B^\alpha) \).

**Remark 1.2.** If, as the Ritz approximate solution of (21), one takes the vector \( x_n \) on which the functional \( F(z) \) attains its minimum on \( \mathcal{H}_n = \mathcal{H}_\lambda \oplus \mathcal{H}_\lambda \oplus \cdots \oplus \mathcal{H}_\lambda \), where \( \mathcal{H}_\lambda \) is the eigensubspace of the operator \( B \) corresponding to the eigenvalue \( \lambda \), then, under assumption \( \psi \) one can omit the condition of the simplicity of the spectrum.

3. We set \( \mathcal{H} = L_2(0, \pi), \mathcal{D}(A) = \{ x \in W_0^2[0, \pi], x'(0) = x'(\pi) = 0 \} \) and

\[
(Ax)(t) = -x''(t) + q(t)x(t), \quad q(t) > 0, \quad q \in C([0, \pi]).
\]

We define an operator \( B \) as follows:

\[
\mathcal{D}(B) = \mathcal{D}(A), \quad Bx = -x'' + x.
\]

The operators \( A \) and \( B \) are self-adjoint and positive definite in \( L_2(0, \pi) \). The spectrum of \( B \) consists of the eigenvalues \( \lambda_k(B) = k^2 + 1, k \in \mathbb{N} \), corresponding to the eigenfunctions \( \sqrt{\frac{2}{\pi}} \cos(kt) \), which form an orthonormal basis in the space \( L_2(0, \pi) \).

Let \( k \in \mathbb{N} \) and \( q(t) \in C^{2k}[0, 2\pi] \). It is easy to verify that \( \mathcal{D}(A^{k+1}) = \mathcal{D}(B^{k+1}) \) if and only if \( q^{2j+1}(0) = q^{2j+1}(\pi) = 0, j = 0, \ldots, k \). If \( y(t) \in C^{2(k-1)}[0, 2\pi] \) and \( y^{2j+1}(0) = y^{2j+1}(\pi) = 0, j = 0, \ldots, k \), then \( y(t) \in \mathcal{D}(A^k) \). Therefore, the solution of the problem

\[
-x''(t) + q(t)x(t) = y(t) \quad (25)
\]

\[
x'(0) = x'(&) = 0
\]

belongs to the set \( \mathcal{D}(A^{k+1}) = \mathcal{D}(B^{k+1}) \) and relation (24) directly yields the following statement:
Theorem 1.5. If \( g(t) \in C^{2k}[0,\pi], \) \( g^{(2j+1)}(0) = g^{(2j+1)}(\pi) = 0, \) \( j = 0, \ldots, k, \) and \( y(t) \in C^{2(k-1)}[0,2\pi], \) \( y^{(2j+1)}(0) = y^{(2j+1)}(\pi) = 0, \) \( j = 0, \ldots, k-1, \) then the Ritz approximate solution of problem \((23)-(26)\) satisfies the relation
\[
\|x_n - x\|_{W^2_2[0,\pi]} = o\left(\frac{1}{n^{2k+1}}\right).
\]

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