2-Meixner random variables and semi-quantum operators

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Abstract. The equivalence between the notions of “2-Meixner” and “classic Meixner” random variables is proved first. A derivation of the classic Meixner random variables, using double commutators involving the semi-quantum operators, is presented next.

1. Introduction
Since the discovery of the Meixner class of random variables in [11], many characterizations or derivations of this class have been given by various authors. A non-exhaustive list include papers [8], [10], [12], [17], and [18]. In [18], the Meixner random variables with infinite support, that means the binomial random variables were not included, were derived using the commutators of their quantum operators. A systematic study of a more general class of Meixner random variables, using the semi-quantum operators, was started in [14]. There, the notion of $n$-Meixner random variables was introduced, where $n$ represents the number of nested commutators used in the definition of these random variables. It was shown, in [14], that the 1-Meixner random variables are exactly those that are Gaussian and Gamma distributed. These two types, are only two of the six types of classic Meixner random variables.

In this paper we will show first that the 2-Meixner random variables are exactly the classic Meixner random variables. Then we will use the double commutator, involved in the definition of 2-Meixner random variables, to derive the classic Meixner random variables.

The paper is structured as follows. In section 2, we provide a brief background of the quantum and semi-quantum operators generated by a random variable having finite moments of all orders. In section 3, we prove that the classic Meixner random variables are exactly the 2-Meixner random variables. In section 4, we compute the derivative of the logarithm of the Laplace transform for each type of classic Meixner random variables. Finally, in section 2, we use the equivalence proven in section 3, to derive all classic Meixner random variables.

2. Background
We present now a short background of Quantum Probability in one dimension.
Let $(\Omega, F, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable (i.e., a measurable function).
For all $n \in \mathbb{N} \cup \{0\}$, let:

$$F_n := \{ f(X) \mid f \text{ is polynomial, } \deg(f) \leq n \},$$  \hfill (1)

where $\deg(f)$ denotes the degree of the polynomial $f$. The polynomial $f$ is assumed to have complex coefficients.

Since $X$ has finite moments of all orders, we have:

$$C \equiv F_0 \subset F_1 \subset F_2 \subset \cdots \subset L^2(\Omega, \mathcal{F}, P).$$  \hfill (2)

Because, for all $n \geq 0$, $F_n$ has finite dimension, $F_n$ is a closed subspace $L^2(\Omega, \mathcal{F}, P)$. Thus, we can orthogonalize the spaces $F_n$, for $n \geq 0$, and define $G_0 := F_0 \equiv \mathbb{C}$, and for all $n \geq 1$:

$$G_n := F_n \ominus F_{n-1},$$  \hfill (3)

where $F_n \ominus F_{n-1}$ denotes the orthogonal complement of $F_{n-1}$ into $F_n$. We also define the spaces: $G_{-1} = F_{-1} := \{0\}$.

For each $n \geq 0$, we call $G_n$ the $n$-th chaos space, and each random variable, $f(X)$, from $G_n$, a homogenous polynomial random variable of degree $n$.

We define the space of polynomial random variables as:

$$F := \{ f(X) \mid f \text{ is polynomial} \},$$  \hfill (4)

that means:

$$F = \bigcup_{n=0}^\infty F_n.$$  \hfill (5)

We can view now the random variable $X$ as the multiplication operator, denoted also by $X$, $X : F \to F$, defined by:

$$f(X) \mapsto Xf(X).$$  \hfill (6)

The following lemma can be easily checked, see [1]:

**Lemma 2.1** For all $n \geq 0$, we have:

$$XG_n \perp G_k,$$  \hfill (7)

for all $k \neq n-1, n, n+1$, where “$\perp$” means “orthogonal to”.

It follows from this lemma, that for a fixed polynomial random variable $f(X) \in G_n$, for some $n \geq 0$, since:

$$Xf(X) \in F_{n+1} = G_0 \oplus G_1 \oplus \cdots \oplus G_{n+1},$$  \hfill (8)

we have:

$$Xf(X) \in G_{n-1} \oplus G_n \oplus G_{n+1}.$$  \hfill (9)

That means, there exist three unique polynomial random variables $f_{n-1}(X) \in G_{n-1}$, $f_n(X) \in G_n$, and $f_{n+1}(X) \in G_{n+1}$, such that:

$$Xf(X) = f_{n-1}(X) + f_n(X) + f_{n+1}(X).$$  \hfill (10)

We define the operators:
\[ D^n : G_n \rightarrow G_{n-1}, \]  
(11)

\[ D^n f(X) := f_{n-1}(X). \]  
(12)

Since \( D^n \) decreases the degree of the homogenous polynomial \( f(X) \) by one unit, we call \( D^n \) an **annihilation operator**.

\[ D^0 : G_n \rightarrow G_n, \]  
(13)

\[ D^0 f(X) := f_n(X). \]  
(14)

Since \( D^0 \) does not change the degree of the homogenous polynomial \( f(X) \), we call \( D^0 \) a **creation operator**.

\[ D^+ : G_n \rightarrow G_{n+1}, \]  
(15)

\[ D^+ f(X) := f_{n+1}(X). \]  
(16)

Since \( D^+ \) increases the degree of the homogenous polynomial \( f(X) \) by one unit, we call \( D^+ \) a **creation operator**.

Lemma 2.1 can be written now as:

**Lemma 2.2** For all \( n \in \mathbb{N} \cup \{0\} \), we have:

\[ X|G_n = D^n + D^0_n + D^+_n, \]  
(17)

where \( X|G_n \) denotes the restriction of the multiplication operator \( X \) to the space \( G_n \).

We extend now the definition of the annihilation, preservation, and creation operators to the space of polynomial random variables in \( X, F \), in the following way. If \( f \in F \), then there exists \( N \in \mathbb{N} \cup \{0\} \), such that, \( f \in F_N \), and so, there exist and are unique \( f_0 \in G_0, f_1 \in G_1, \ldots, F_N \in G_N \), such that:

\[ f = f_0 + f_1 + \cdots + f_N. \]  
(18)

We define:

\[ a^- f = D^- f_0 + D^-_1 f_1 + \cdots + D^-_N f_N, \]  
(19)

\[ a^0 f = D^0 f_0 + D^0_1 f_1 + \cdots + D^0_N f_N, \]  
(20)

and

\[ a^+ f = D^+ f_0 + D^+_1 f_1 + \cdots + D^+_N f_N. \]  
(21)

We call \( a^- \) the **annihilation operator**, \( a^0 \) the **preservation operator**, and \( a^+ \) the **creation operator**. The annihilation, preservation, and creation operators, are called the **quantum operators** generated by the random variable \( X \).

Lemma 2.2 becomes now:
Theorem 2.3 Viewing $X$ as a multiplication operator, we have:

$$X = a^+ + a^0 + a^-.$$  \hfill (22)

In this equality the domain of the operators $X$, $a^-$, $a^0$, and $a^+$ is considered to be the space $F$ of all polynomial random variables in $X$.

Since $F_n$ is spanned by $1$, $X$, $X^2$, $X^n$, while $F_{n-1}$ is spanned by $1$, $X$, $X^2$, $X^{n-1}$, the codimension of $F_{n-1}$ into $F_n$ is at most $1$, for all $n \geq 1$. Thus, for all $n \geq 0$, the dimension of the space $G_n$ is at most $1$. In fact, if the probability distribution of $X$ has an infinite support, then for all $n \geq 1$, $\dim(G_n) = 1$, where \textit{“dim”} means dimension. On the other hand if $X$ takes on only finitely many values $x_1$, $x_2$, $\ldots$, $x_k$, with positive probabilities, then due to Lagrange interpolation polynomial, every function $f : \{x_1, x_2, \ldots, x_k\} \to \mathbb{C}$ is equal to a polynomial function of degree at most $k - 1$. Thus, in this case, we have:

$$F = F_{k-1},$$  \hfill (23)

and so, we have:

$$\dim(G_n) = \begin{cases} 1 & \text{if } n \leq k - 1 \\ 0 & \text{if } n \geq k. \end{cases}$$  \hfill (24)

If for some $n \geq 0$, $\dim(G_n) = 1$, then there exists a unique polynomial random variable $f_n(X)$ in $G_n$, with a leading coefficient equal to $1$. Since, we have:

$$Xf_n(X) \in G_{n+1} \oplus G_n \oplus G_{n-1},$$  \hfill (25)

there exist $\alpha_n$ and $\omega_n$ real numbers, such that:

$$Xf_n(X) = f_{n+1}(X) + \alpha_n f_n(X) + \omega_n f_{n-1}(X),$$  \hfill (26)

where the coefficient of $f_{n+1}(X)$ in the right-hand side of this equality is $1$, since the coefficient of $X^{n+1}$ in the left is $1$, while $f_n(X)$ and $f_{n-1}(X)$ do not contain $X^{n+1}$. The numbers $\{\alpha_n\}_{n \geq 0}$ and $\{\omega_n\}_{n \geq 0}$ are called the Szegö-Jacobi parameters of $X$. Since $f_{-1}(X) = 0$, we can choose $\omega_0$ however we want. We define $\omega_0 := 0$.

It follows from (26), that for all $n \geq 0$, we have:

$$a^+ f_n(X) = f_{n+1}(X),$$  \hfill (27)

$$a^0 f_n(X) = \alpha_n f_n(X),$$  \hfill (28)

and

$$a^- f_n(X) = \omega_n f_{n-1}(X).$$  \hfill (29)

It is not hard to see, that the square of the $L^2$-norm of the $n$-th orthogonal polynomial random variable, with leading coefficient $1$, $f_n(X)$, is:

$$E \left[ f_n(X)^2 \right] = \omega_1 \omega_2 \cdots \omega_n.$$  

It follows from here, that the probability distribution of $X$ has an infinite support, if and only if, for all $n \geq 1$:

$$\omega_n > 0.$$  \hfill (30)
On the other hand, if $X$ has a finite support of cardinality $k$, then for all $n \leq k - 1$, $\omega_n > 0$, while $\omega_k = 0$.

It is well-known that the creation operator $a^+$ is the adjoint of the annihilation operator $a^-$, while the preservation operator $a^0$ is symmetric, in the following sense:

For all $f(X)$ and $g(X)$ in $F$, that means, $f(X)$ and $g(X)$ are polynomial random variables in $X$, we have:

$$\langle a^- f(X), g(X) \rangle = \langle f(X), a^+ g(X) \rangle$$

and

$$\langle a^0 f(X), g(X) \rangle = \langle f(X), a^0 g(X) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(\mathbb{R}, F, P)$.

We define now the semi-annihilation operator $U : F \to F$,

$$U = a^- + \frac{1}{2} a^0$$

and semi-creation operator $V : F \to F$,

$$V = \frac{1}{2} a^0 + a^+.$$

We call $U$ and $V$ the semi-quantum operators of $X$.

It is clear that $V$ is the adjoint of $U$, and we have:

**Proposition 2.4** If we regard $X$ as a multiplication operator, then:

$$X = U + V.$$  \hspace{1cm} (35)

It follows from here, that the commutator of $U$ and $X$ is symmetric, since:

$$[U, X]^* = [U, U + V]^*$$

$$= [U, V]^*$$

$$= [V^*, U^*]$$

$$= [U, V]$$

$$= [U, U + V]$$

$$= [U, X].$$

3. Meixner random variables

In what follows, we assume that the support of the probability distribution of $X$ has at least two different points. Thus $\omega_1 > 0$.

The classic Meixner random variables, introduced for the first time in [11], are the random variables, having finite moments of all orders, whose Szegö-Jacobi parameters are of the form:

$$\alpha_n = \alpha n + \alpha_0$$

and

$$\omega_n = \beta n^2 + (t - \beta)n,$$  \hspace{1cm} (37)
for all \( n \in \mathbb{N} \cup \{0\} \). Here \( \alpha, \alpha_0, \beta, \) and \( t \) are real numbers, such that: \( t > 0 \) (due to the fact that \( \omega_1 = t \) and \( \omega_1 > 0 \)) and either \( \beta \geq 0 \) or, if \( \beta < 0 \), then, since \( \lim_{n \to \infty} [\beta n^2 + (t - \beta)n] = -\infty < 0 \) and \( \omega_n \) is not allowed to be negative, there exists \( N \in \mathbb{N} \), such that \( \omega_N = \beta N^2 + (t - \beta)N = 0 \). This means, if \( \beta < 0 \), then there exists \( N \in \mathbb{N} \), such that, \( t = \beta (1 - N) > 0 \), which is the same as saying that \( t \in -\mathbb{N} \beta \).

In [14], the definition of the \( n \)-Meixner random variables, using \( n \) nested commutators involving the semi-quantum operators, was introduced. More precisely, for \( n = 2 \), we say that a random variable \( X \), having finite moments of all orders, is a 2-Meixner random variable if:

\[
[[U;X],X] \in \mathbb{R} V + \mathbb{R} U + \mathbb{R} I,
\]

(38)

where \( V \) denotes the semi-creation operator, and \( U \) the semi-annihilation operator generated by \( X \), while \( I \) is the identity operator of the vector space \( F \). That means, there exist \( c, d, \) and \( e \) real numbers, such that:

\[
[[U;X],X] = cV + dU + eI.
\]

(39)

Moreover, since \([U;X] \) and \( X \) are both symmetric operators, their commutator \([[U;X],X] \) is an anti-symmetric operator. Thus, we must have:

\[
d = -c
\]

(40)

and

\[
e = 0.
\]

(41)

Therefore, \( X \) is a 2-Meixner random variable if and only if there exists \( c \in \mathbb{R} \), such that:

\[
[[U;X],X] = c(V - U)
\]

(42)

\[
= c(X - 2U),
\]

(43)

since

\[
X = U + V.
\]

We present now the first result of this paper.

**Proposition 3.1** If \( X \) is a random variable having finite moments of all orders, then \( X \) is a classic Meixner random variable if and only if \( X \) is a 2-Meixner random variable.

**Proof.** (\( \Rightarrow \)) Let us suppose that \( X \) is a classic Meixner random variable. Then its Szegö-Jacobi parameters are of the form:

\[
\alpha_n = \alpha n + \alpha_0
\]

(44)

and

\[
\omega_n = \beta n^2 + (t - \beta)n,
\]

(45)

for all \( n \geq 1 \), where \( \alpha, \alpha_0, \beta, \) and \( t \) are real numbers as explained above. Let \( \{f_n(X)\}_{0 \leq n < k} \), where \( k \) is the cardinality of the support of the probability distribution of \( X \), be the orthogonal polynomial random variables of \( X \) with the leading coefficient equal to 1. We have:

\[
X f_n(X) = f_{n+1}(X) + \alpha_n f_n(X) + \omega_n f_{n-1}(X),
\]

(46)
for all $0 \leq n < k$.

Let us introduce the operator $\mathcal{N} : F \to F$, defined by:

$$\mathcal{N} f_n(X) = n f_n(X), \quad (47)$$

for all $0 \leq n < k$, called the number operator.

**Claim 1.** The commutator of $a^-$ and $a^+$ is:

$$[a^-, a^+] = 2\beta \mathcal{N} + t I. \quad (48)$$

Indeed, for all $0 \leq n < k$, we have:

$$[a^-, a^+] f_n(X) = a^- a^+ f_n(X) - a^+ a^- f_n(X)$$
$$= a^- f_{n+1}(X) - a^+ (\omega_n f_{n-1}(X))$$
$$= (\omega_{n+1} - \omega_n) f_n(X)$$
$$= \{ [\beta (n+1)^2 + (t - \beta)(n+1)] - [\beta n^2 + (t - \beta)n] \} f_n(X)$$
$$= 2\beta n f_n(X) + t f_n(X)$$
$$= (2\beta \mathcal{N} + t I) f_n(X).$$

Since, we have:

$$a^0 = \alpha \mathcal{N} + \alpha_0 I, \quad (49)$$

it follows from this claim that:

$$[[a^-, a^+], a^0] = 0. \quad (50)$$

**Claim 2.** The commutator of $a^0$ and $a^+$ is:

$$[a^0, a^+] = \alpha a^+. \quad (51)$$

Indeed, for all $0 \leq n < k$, we have:

$$[a^0, a^+] f_n(X) = a^0 a^+ f_n(X) - a^+ a^0 f_n(X)$$
$$= a^0 f_{n+1}(X) - a^+ (\alpha_n f_n(X))$$
$$= (\alpha_{n+1} - \alpha_n) f_{n+1}(X)$$
$$= \{ [\alpha (n+1) + \alpha_0] - (\alpha n + \alpha_0) \} f_{n+1}(X)$$
$$= \alpha f_{n+1}(X)$$
$$= \alpha a^+ f_n(X).$$

Taking the adjoint in both sides of the formula from Claim 2, we obtain:

**Claim 3.** The commutator of $a^-$ and $a^0$ is:

$$[a^-, a^0] = \alpha a^-. \quad (52)$$

In particular, for $\alpha = 1$ and $\alpha_0 = 0$, we obtain the following commutators

$$[\mathcal{N}, a^+] = a^+ \quad (53)$$

and

$$[a^-, \mathcal{N}] = a^- \quad (54)$$
Thus, we have:

\[
\begin{align*}
[[U, X], X] & = [[U, U + V], X] \\
& = [[U, V], X] \\
& = \left[ a^- + \frac{1}{2} a^0, a^+ + \frac{1}{2} a^0 \right], X \\
& = [[a^-, a^+], X] + \frac{1}{2} [[a^-, a^0], X] + \frac{1}{2} [[a^0, a^0], X]. \tag{55}
\end{align*}
\]

Let us compute individually, each of the three double commutators involved in (55).

Since \(X = a^- + a^0 + a^+\), the first double commutator is:

\[
\begin{align*}
[[a^-, a^+], X] & = [[a^-, a^+], a^-] \\
& + [[a^-, a^+], a^0] \\
& + [[a^-, a^+], a^+] \\
& = [2\beta N + tI, a^-] + 0 + [2\beta N + tI, a^+] \\
& = 2\beta \{ [N, a^-] + [N, a^+] \} \\
& = 2\beta (-a^- + a^+) \\
& \in 2\beta (V - U) \tag{56}
\end{align*}
\]

\[
\begin{align*}
\in \mathbb{R} V + \mathbb{R} U. \tag{57}
\end{align*}
\]

The sum of the other two commutators involved in (55) (excepting the factor 1/2) is:

\[
\begin{align*}
[[a^-, a^0], X] + [[a^0, a^+], X] & = [\alpha a^-, X] + [\alpha a^+, X] \\
& = \alpha [a^- + a^+, X] \\
& = \alpha [X - a^0, X] \\
& = -\alpha [a^0, X] \\
& = \alpha [a^- + a^0 + a^+, a^0] \\
& = \alpha \{ [a^-, a^0] + [a^+, a^0] \} \\
& = \alpha^2 (a^- - a^+) \\
& = \alpha^2 (U - V) \tag{58}
\end{align*}
\]

\[
\begin{align*}
\in \mathbb{R} V + \mathbb{R} U. \tag{59}
\end{align*}
\]

It follows from (55), (57), and (59) that \(X\) is a 2-Meixner random variable. Moreover, we can conclude from from (55), (56), and (58) that:

\[
[[U, X], X] = \frac{4\beta - \alpha^2}{2} (V - U). \tag{60}
\]

The quantity:

\[
\Delta := \alpha^2 - 4\beta, \tag{61}
\]

which we will call the discriminant, plays an important role in distinguishing between various types of Meixner random variables.
Let us suppose now that $X$ is a 2-Meixner random variable. So, there exists $c \in \mathbb{R}$, such that:

$$[[U, X], X] = c (V - U).$$

(62)

We divide the double commutator $[[U, X], X]$ into five sums as follows:

$$[[U, X], X] = [[U, V], X] = \left[ [a^- + \frac{1}{2} a^0, a^+ + \frac{1}{2} a^0], a^- + a^0 + a^+ \right]$$

$$= \frac{1}{2} [[a^- , a^0], a^- ]$$

(63)

$$+ \frac{1}{2} [[a^- , a^0], a^0] + [[a^-, a^+], a^- ]$$

(64)

$$+ [[a^- , a^+], a^0] + \frac{1}{2} [[a^0, a^+], a^- ] + \frac{1}{2} [[a^- , a^0], a^+]$$

(65)

$$+ \frac{1}{2} [[a^0, a^+], a^0] + [[a^-, a^+], a^+]$$

(66)

$$+ \frac{1}{2} [[a^0, a^+], a^+]$$

(67)

Let us give a name to each of the five operators from (63), (64), (65), (66), and (67). We define:

$$S_1 := \frac{1}{2} [[a^- , a^0], a^- ],$$

(68)

$$S_2 := \frac{1}{2} [[a^-, a^0], a^0] + [[a^-, a^+], a^- ],$$

(69)

$$S_3 := [[a^-, a^+], a^0] + \frac{1}{2} [[a^0, a^+], a^- ] + \frac{1}{2} [[a^-, a^0], a^+]$$

(70)

$$S_4 := \frac{1}{2} [[a^0, a^+], a^0] + [[a^-, a^+], a^+]$$

(71)

and

$$S_5 := \frac{1}{2} [[a^0, a^+], a^+]$$

(72)

Thus, we have:

$$[[U, X], X] = S_1 + S_2 + S_3 + S_4 + S_5.$$  

(73)

Let $\{G_n\}_{1 \leq n < k}$ be the homogenous chaos spaces generated by $X$. Since:

$$V - U = a^+ - a^-,$$

(74)

it follows from (62) that, for all $0 \leq n < k$:

$$[[U, X], X] : G_n \rightarrow G_{n+1} \oplus G_{n-1}.$$  

(75)
On the other hand, since it is clear that, for all \( n \geq 0 \):

\[
S_5 : G_n \rightarrow G_{n+2},
\]

(76)

because \( G_{n+2} \) is orthogonal to \( G_{n+1} \oplus G_{n-1} \), we conclude from (73) that:

\[
S_5 = 0.
\]

(77)

Let \( \{\alpha_n\}_{0 \leq n < k} \) and \( \{\omega_n\}_{0 \leq n < k} \) be the Szegö-Jacobi parameters of \( X \). For each \( 0 \leq n < k \), let \( f_n(X) \) be the \( n \)-th orthogonal polynomial random variable, with a leading coefficient equal to 1, generated by \( X \). We have:

\[
S_5 f_n(X) = \left[ [a^0, a^+] , a^+ \right] f_n(X)
\]

\[
= [a^0, a^+] a^+ f_n(X) - a^+ [a^0, a^+] f_n(X)
\]

\[
= (a^0 a^+ - a^+ a^0) f_{n+1}(X) - a^+ (a^0 f_{n+1}(X) - \alpha_n a^+ f_n(X))
\]

\[
= a^0 f_{n+2}(X) - \alpha_{n+1} a^+ f_{n+1}(X) - \alpha_{n+1} a^+ f_{n+1}(X) + \alpha_n a^+ f_{n+1}(X)
\]

\[
= (\alpha_{n+2} - 2\alpha_{n+1} + \alpha_n) f_{n+2}(X).
\]

(78)

Since \( S_5 = 0 \), we conclude from (78) that, for all \( 0 \leq n < k \), such that \( n + 2 < k \), we have:

\[
\alpha_{n+2} - 2\alpha_{n+1} + \alpha_n = 0,
\]

(79)

which is equivalent to:

\[
\alpha_{n+2} - \alpha_{n+1} = \alpha_{n+1} - \alpha_n.
\]

(80)

Thus, the sequence \( \{\alpha_n\}_{0 \leq n < k} \) increases by a common difference, that we may call \( \alpha \). Therefore, \( \{\alpha_n\}_{0 \leq n < k} \) is an arithmetic progression with common difference \( \alpha \), and so, for all \( 0 \leq n < k \), we have:

\[
\alpha_n = \alpha n + \alpha_0,
\]

(81)

or equivalently:

\[
a^0 = \alpha N + \alpha_0 I.
\]

(82)

It follows from the last formula that:

\[
S_4 = \frac{1}{2} \left[ [a^0, a^+] , a^0 \right] + [a^-, a^+] a^+
\]

\[
= \frac{1}{2} \left[ [\alpha N + \alpha_0 I, a^+] , a^0 \right] + [a^-, a^+] a^+
\]

\[
= \frac{1}{2} \alpha \left[ [\alpha N, a^+] , a^0 \right] + [a^-, a^+] a^+
\]

\[
= \frac{1}{2} \alpha \left[ a^+, a^0 \right] + [a^-, a^+] a^+
\]

\[
= \frac{1}{2} \alpha \left[ a^+, \alpha N + \alpha_0 I \right] + [a^-, a^+] a^+
\]

\[
= -\frac{\alpha^2}{2} a^+ + [a^-, a^+] a^+.
\]

(83)
Moreover, since for all $0 \leq n < k$:

$$S_1 + S_2 + S_3 : G_n \to G_{n-2} \oplus G_{n-1} \oplus G_n$$  \hspace{1cm} (84)

and

$$S_4 : G_n \to G_{n+1},$$  \hspace{1cm} (85)

while

$$c(V - U) : G_n \to G_{n-1} \oplus G_{n+1},$$  \hspace{1cm} (86)

if we restrict the operatorial equality:

$$S_1 + S_2 + S_3 + S_4 = c(V - U)$$  \hspace{1cm} (87)

to the space $G_n$, and take the projection of the image of these operators on the space $G_{n+1}$, then we conclude that:

$$S_4 f_n(X) = c f_{n+1}(X).$$  \hspace{1cm} (88)

The last relation means:

$$\left( \left[ [a^-, a^+] , a^+ \right] - \frac{\alpha^2}{2} a^+ \right) f_n(X) = c f_{n+1}(X),$$  \hspace{1cm} (89)

for all $0 \leq n < k$ such that $n + 2 < k$. Doing absolutely the same computation as we did before to derive formula (78), with $\{ \omega_n \}_{0 \leq n < k}$ replacing $\{ \alpha_n \}_{0 \leq n < k}$, we conclude that for all $0 \leq n < k$ such that $n + 1 < k$, we have:

$$\omega_{n+2} - 2 \omega_{n+1} + \omega_n - \frac{\alpha^2}{2} = c.$$  \hspace{1cm} (90)

Moreover, if $k \in \mathbb{N}$, that means $k$ is a finite number, then if we set $n = k - 2$ in formula (89), since $f_{n+2}(X) = f_k(X) = 0$, the term $a^- a^+ a^+ f_n(X)$, that appears in the left-hand side of this formula after expanding the commutators, vanishes since:

$$a^- a^+ a^+ f_n(X) = a^- f_{n+2}(X) = a^- f_k(X) = a^- 0 = 0.$$  

Therefore, if $k$ is a finite number, and $n = k - 2$, the term $\omega_{n+2} = \omega_k$ no longer appears in the recursive formula (90). Thus, in this case, in order for formula (90) to be true for $n = k - 2$, we must have:

$$\omega_k := 0.$$  \hspace{1cm} (91)

This equality is equivalent to:

$$\delta_{n+1} - \delta_n = c + \frac{\alpha^2}{2},$$  \hspace{1cm} (92)
for all $0 \leq n < k$, where
\[ \delta_n := \omega_{n+1} - \omega_n, \]
that means, \( \{\delta_n\}_{1 \leq n < k} \) is the increment sequence generated by \( \{\omega_n\}_{0 \leq n < k} \).

Summing up the terms from both sides of formula (92), from \( n = 0 \) to \( n = N - 1 \), for some \( N < k \), we conclude that:
\[ \delta_N - \delta_0 = \left( c + \frac{\alpha^2}{2} \right) N. \]
Therefore, for all $0 \leq m < k$, we have:
\[ \omega_{m+1} - \omega_m = \left( c + \frac{\alpha^2}{2} \right) m + \delta_0 = \left( c + \frac{\alpha^2}{2} \right) m + \omega_1, \]
since, by convention, \( \omega_0 := 0 \).

Summing up the terms from both sides, from the last formula, from \( m = 0 \) to \( m = n - 1 \), and keeping in mind that, by convention \( \omega_0 = 0 \), we obtain:
\[ \omega_n = \sum_{m=0}^{n-1} (\omega_{m+1} - \omega_m) = \sum_{m=0}^{n-1} \left( \left( c + \frac{\alpha^2}{2} \right) m + \omega_1 \right) = \left( c + \frac{\alpha^2}{2} \right) n(n-1) + \omega_1 n = \frac{c + (\alpha^2/2)}{2} n^2 + \left( \omega_1 - \frac{c + (\alpha^2/2)}{2} \right) n. \]

If we define:
\[ \beta := \frac{c + (\alpha^2/2)}{2} \]
and
\[ t := \omega_1, \]
then the last formula becomes:
\[ \omega_n = \beta n^2 + (t - \beta) n, \]
for all $0 \leq n < k$.

If \( k \) is a finite natural number, \( k \geq 2 \), then formula (98) must hold also for \( n = k \). However, in this case \( \omega_k = 0 \), so that we have:
\[ 0 = \omega_k = \beta k^2 + (t - \beta) k. \]
Thus, we conclude that:

\[ t = (1 - k)\beta \]  

(100)

\[ \in -Nk. \]  

(101)

Therefore, \( X \) is a classic Meixner random variable. \( \square \)

We close this section by retaining the important formula that was derived in the proof of the above proposition, namely, if \( X \) is a classic Meixner random variable, then:

\[ [[U, X], X] = c(X - 2U), \]  

(102)

with:

\[ c = -\frac{\Delta}{2}, \]  

(103)

where \( \Delta = \alpha^2 - 4\beta \) denotes the discriminant of \( X \).

4. Derivatives of the logarithm of the characteristic functions of the classic Meixner distributions

In this section we compute the derivative of the logarithm of the Laplace transform of the classic Meixner random variables.

- If \( X \) is a Gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \), that means, \( X \) is given by the density function:

\[ f(x) = \frac{1}{\sqrt{2\pi\sigma}}e^{-(x-\mu)^2/(2\sigma^2)}d\mu(x), \]

then, for all \( t \in \mathbb{R} \), we have:

\[ E\left[e^{tX}\right] = \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} e^{tx}e^{-(x-\mu)^2/(2\sigma^2)}d\mu(x) \]

\[ = \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} e^{-(x-\mu-t\sigma^2)^2/(2\sigma^2)}d\mu(x)e^{(t^2\sigma^2/2)+\mu t} \]

\[ = e^{(t^2\sigma^2/2)+\mu t}. \]  

(104)

Defining:

\[ \psi(t) := \frac{d}{dt} \left[ \ln \left( E\left[e^{tX}\right] \right) \right], \]  

(105)

we obtain:

\[ \psi(t) = \sigma^2 t + \mu. \]  

(106)

In particular, if the mean \( \mu = 0 \), then:

\[ \psi(t) = \sigma^2 t. \]  

(107)

Moreover, if \( E[X^2] = 1 \), then \( \sigma = 1 \), and so:

\[ \psi(t) = t. \]  

(108)
If $X$ is Gamma distributed with parameter $\lambda > 0$, that means it is given by the density function:

$$f(x) = \frac{1}{\Gamma(\lambda)}x^{\lambda-1}e^{-x}1_{(0,\infty)}(x),$$

where $1_{(0,\infty)}$ denotes the characteristic function of the interval $(0, \infty)$, then its Laplace transform is:

$$E[e^{tX}] = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{tx}x^{\lambda-1}e^{-x}dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1}e^{-(1-t)x}dx$$

(let $u := (1-t)x$)  

$$= \frac{1}{(1-t)^\lambda} \cdot \frac{1}{\Gamma(\lambda)} \int_0^\infty u^{\lambda-1}e^{-u}du$$

$$= \frac{1}{(1-t)^\lambda}, \quad (109)$$

for all $t < 1$.

It follows from here, that for any $a$ and $b$ real numbers, if we define the random variable:

$$Y := aX + b,$$  

(we call $Y$ an affine transformation of $X$), then its Laplace transform is:

$$E[e^{tY}] = \frac{1}{(1-at)^\lambda} e^{bt}. \quad (110)$$

Therefore, we have:

$$\ln (E[e^{-tY}]) = bt - \lambda \ln(1-at). \quad (111)$$

Thus, the derivative of the logarithm of the Laplace transform of $Y$ is:

$$\psi(t) := \frac{d}{dt} (\ln (E[e^{tY}]))$$

$$= b + \frac{a\lambda}{1-at}$$

$$= -abt + a\lambda + b \quad \frac{1}{1-at}, \quad (112)$$

for all $t \in \mathbb{R}$, such that $at < 1$.

In particular, if $E[Y] = 0$, then since:

$$\frac{d}{dt} (E[e^{tY}])|_{t=0} = E[Y]$$

$$0 = E[Y]$$

we must have

$$\psi(0) = 0. \quad (113)$$
Thus, in this case $a \lambda + b = 0$, and:

$$\psi(t) = \frac{a^2 \lambda t}{1 - at}. \quad (117)$$

Moreover, if $E[X^2] = 1$, then $\psi'(0) = 1$. Since from (117), we conclude that:

$$\psi'(t) = \frac{a^2 \lambda}{(1 - at)^2}, \quad (118)$$

and $\psi'(0) = 1$, we must have:

$$a^2 \lambda = 1. \quad (119)$$

Therefore, in this case:

$$\psi(t) = \frac{t}{1 - at}. \quad (120)$$

- If $X$ is a Poisson random variable with mean $\lambda > 0$, then the probability distribution of $X$ is:

$$\mu = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \delta_{\{n\}} , \quad (121)$$

where $\delta_{\{a\}}$ denotes the Dirac delta measure at $a$, for all real numbers $a$.

The Laplace transform of $X$ is:

$$E[e^{tX}] = \sum_{n=0}^{\infty} e^{tn} \frac{\lambda^n}{n!} e^{-\lambda} \quad = \exp \left( \lambda \left( e^t - 1 \right) \right). \quad (122)$$

Thus, if $Y = aX + b$ is an affine transformation of $X$, then its Laplace transform is:

$$E[e^{tY}] = \exp \left( \lambda \left( e^{at} - 1 \right) + bt \right). \quad (123)$$

Therefore, we have:

$$\ln \left( E[e^{tY}] \right) = \lambda (e^{at} - 1) + bt, \quad (124)$$

and so, the derivative of the logarithm of the Laplace transform of $Y$ is:

$$\psi(t) := \frac{d}{dt} \ln \left( E[e^{tY}] \right) = a\lambda e^{at} + b. \quad (125)$$

In particular, if $E[Y] = 0$, then $\psi(0) = 0$, and so $b = -a\lambda$. Thus, we have:

$$\psi(t) = a\lambda (e^{at} - 1). \quad (126)$$

Moreover, if $E[X^2] = 1$, then $\psi'(0) = 1$. It follows from (126) that:

$$1 = \psi'(0) = a^2 \lambda. \quad (127)$$
Thus, we have:

\[
\psi(t) = \frac{1}{a} (e^{at} - 1)
\]

\[
= \frac{2}{a} \left( e^{a/2}t - e^{-(a/2)t} \right)
\]

\[
= \frac{2}{a} \left( \frac{\sinh((a/2)t)}{\cosh((a/2)t) - \sinh((a/2)t)} \right).
\]

(128)

- If \( X \) has a negative binomial (Pascal) distribution with parameter \( r > 0 \), that means its probability distribution is:

\[
\mu = \sum_{k=0}^{\infty} \frac{\Gamma(r+k)}{k!\Gamma(r)} p^r (1-p)^k \delta_{(k)},
\]

(129)

then its Laplace transform is:

\[
E[e^{tx}] = \sum_{k=0}^{\infty} \frac{\Gamma(r+k)}{k!\Gamma(r)} p^r (1-p)^k e^{tk}
\]

\[
= p^r \sum_{k=0}^{\infty} \frac{\Gamma(r+k)}{k!\Gamma(r)} [(1-p)e^t]^k.
\]

(130)

Since Gamma function satisfies the identity:

\[
\Gamma(x+1) = x\Gamma(x),
\]

(131)

for all \( x > 0 \), we have:

\[
\frac{\Gamma(r+k)}{k!\Gamma(r)} = \frac{(r+k-1)(r+k-2)\cdots r}{k!}.
\]

(132)

Thus, we have:

\[
E[e^{tx}] = p^r \sum_{k=1}^{\infty} \frac{(r+k-1)(r+k-2)\cdots r}{k!} [(1-p)e^t]^k
\]

\[
= p^r \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-r)(-r-1)\cdots(-r-k+1)}{k!} (-1)^k [(1-p)e^t]^k \right\}
\]

\[
= p^r \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-r)(-r-1)\cdots(-r-k+1)}{k!} (-1)^k [(1-p)e^t]^k \right\}
\]

\[
= p^r \left[ 1 - (1-p)e^t \right]^{-r},
\]

(133)

for \( t \) in a small neighborhood of 0, \((-\epsilon, \epsilon)\), such that \((1-p)e^t < 1\), due to Newton binomial formula:

\[
(1+x)^{-r} = 1 + \sum_{k=1}^{\infty} \frac{(-r)(-r-1)\cdots(-r-k+1)}{k!} x^k,
\]

(134)
for \(-1 < x < 1\). It follows from here that if \(Y = aX + b\) is an affine transformation of \(X\), then:

\[
E \left[ e^{tY} \right] = \left[ \frac{p}{1 - (1 - p)e^{at}} \right]^r e^{bt}.
\]  

(135)

Therefore, we obtain:

\[
\ln \left( E \left[ e^{tY} \right] \right) = r \ln(p) - r \ln \left( 1 - (1 - p)e^{at} \right) + bt,
\]

(136)

for all \(t \in (-\epsilon, \epsilon)\).

Thus, the derivative of the logarithm of the Laplace transform of \(Y\) is:

\[
\psi(t) := \frac{d}{dt} \left( E \left[ e^{tY} \right] \right)
\]

\[
= \frac{ra(1 - p)e^{at}}{1 - (1 - p)e^{at}} + b
\]

\[
= \frac{(ra - b)(1 - p)e^{at} + b}{1 - (1 - p)e^{at}}.
\]

(137)

In particular, if \(E[Y] = 0\), then \(b = -ra(1 - p)/p\), and so:

\[
\psi(t) = \frac{ra(1 - p)}{p} \cdot \frac{e^{at} - 1}{1 - (1 - p)e^{at}}.
\]

(138)

Moreover, if \(E[X^2] = 1\), then \(\psi'(0) = 1\). It follows from (138) that:

\[
1 = \psi'(0)
\]

\[
= \frac{ra^2(1 - p)}{p^2}.
\]

(139)

Thus, we have:

\[
\psi(t) = \frac{p}{a} \cdot \frac{e^{at} - 1}{1 - (1 - p)e^{at}}
\]

\[
= \frac{p}{a} \cdot \frac{e^{(a/2)t} - e^{-(a/2)t}}{e^{(a/2)t} - (1 - p)e^{(a/2)t}}
\]

\[
= \frac{2}{a} \cdot \frac{e^{(a/2)t} - e^{-(a/2)t}}{e^{(a/2)t} + e^{-(a/2)t} - [(2 - p)/p]e^{(a/2)t} - e^{-(a/2)t}}
\]

\[
= \frac{2}{a} \cdot \frac{\sinh((a/2)t)}{\cosh((a/2)t) - [(2 - p)/p] \sinh((a/2)t)}.
\]

(140)

- If \(X\) is a two parameter hyperbolic secant distributed random variable, then its density function is:

\[
f(x) = \frac{(\cos(a/2))^{2k}}{\sqrt{\pi \Gamma(k) \Gamma(k + (1/2))}} e^{ax} |\Gamma(k + ix)|^2,
\]

(141)

where \(k > 0\) and \(-\pi < \alpha < \pi\). The function \(f\) is Lebesgue integrable on \(\mathbb{R}\), since due to the Lerch estimate (see [6], page 15):

\[
|\Gamma(k + ix)| = \frac{\Lambda(k + 1)}{\sqrt{k^2 + x^2}} \sqrt{\frac{x}{\sinh(\pi x)}},
\]

(142)
for some $\lambda \in [1, \sqrt{1+t^2}]$, we can see that the even function $f(x) := |\Gamma(k+ix)|^2$, decreases, as $|x| \to \infty$, like $|x|/\exp(\pi|x|)$. This is the reason that we request $|\alpha| < \pi$.

The following formula is due to Ramanujan (see [4] and [16]):

$$\int_{-\infty}^{\infty} |\Gamma(k+ix)|^2 e^{-ix\xi} \, dx = \frac{\sqrt{\pi} \Gamma(k) \Gamma(k+1/2)}{\cosh(\xi/2)^{2k}}, \quad (143)$$

for all $\xi \in \mathbb{R}$. Due to the analyticity of the functions from both sides of (143), we can replace the real number $\alpha$ by the complex number $(\alpha + t)i$, for $t$ in a small neighborhood of 0, $(-\epsilon, \epsilon)$, and obtain:

$$E[e^{tX}] = \frac{\cosh(\alpha/2)^{2k}}{\sqrt{\pi} \Gamma(k) \Gamma(k+1/2)} \int_{\mathbb{R}} e^{(\alpha+t)x} |\Gamma(k+ix)|^2 \, dx$$

$$= \left[ \frac{\cos(\alpha/2)}{\cos((\alpha + t)/2)} \right]^{2k}. \quad (144)$$

We can see from here that, if $Y = aX + b$, for $a$ and $b$ in $\mathbb{R}$, then:

$$E[e^{tY}] = \left[ \frac{\cos(\alpha/2)}{\cos((\alpha + at)/2)} \right]^{2k} e^{bt}. \quad (145)$$

Therefore, we have:

$$\ln \left( E[e^{tY}] \right) = 2k \ln(\cos(\alpha/2)) - 2k \ln(\cos((\alpha + at)/2)) + bt. \quad (146)$$

Thus, the derivative of the logarithm of the Laplace transform of $Y$ is:

$$\psi(t) := \frac{d}{dt} \left( \ln \left( E[e^{tY}] \right) \right)$$

$$= ak \tan \left( \frac{\alpha + at}{2} \right) + b. \quad (147)$$

In particular, if $E[X] = 0$, then we have:

$$\psi(t) = ak \left[ \tan \left( \frac{\alpha + at}{2} \right) - \tan \left( \frac{\alpha}{2} \right) \right]. \quad (149)$$

Moreover, if $E[X^2] = 1$, then $\psi'(0) = 1$. It follows from (149) that:

$$1 = \psi'(0)$$

$$= \frac{a^2 k}{2} \sec^2(\alpha/2). \quad (150)$$

Thus, we have:

$$\psi(t) = \frac{2 \cos^2(\alpha/2) a}{a} \left[ \tan \left( \frac{\alpha + at}{2} \right) - \tan \left( \frac{\alpha}{2} \right) \right]$$

$$= \frac{2 \cos^2(\alpha/2) a}{a} \cdot \frac{\sin(at/2)}{\cos(\alpha/2) \cos((\alpha + at)/2)}$$

$$= \frac{2 \cos(\alpha/2) a}{a} \cdot \frac{\sin((a/2)t)}{\sin((\alpha/2) \cos((\alpha + at)/2) - \sin(\alpha/2) \sin((a/2)t))}$$

$$= \frac{2}{ai} \cdot \frac{\sinh((a/2)t)}{\cosh((a/2)t) + i \tan(\alpha/2) \sinh((a/2)t)}. \quad (151)$$
• If $X$ is a binomial random variable with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$, where $n$ is the number of independent trials and $p$ is the probability of a success, then the probability distribution of $X$ is:

$$
\mu = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta_k.
$$

(152)

Thus, for all $t \in \mathbb{R}$, we have:

$$
E[e^{tX}] = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}
$$

$$
= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k}
$$

$$
= [pe^t + (1-p)]^n.
$$

(153)

Therefore, if $Y = aX + b$, for $a$ and $b$ real numbers, then:

$$
E[e^{tY}] = [pe^{at} + (1-p)]^n e^{bt}.
$$

(154)

Taking a logarithm from both sides and differentiating, with respect to $t$, we obtain:

$$
\psi(t) := \frac{d}{dt} \left( \ln \left( E\left[ e^{tY} \right]\right) \right)
$$

$$
= \frac{nap e^{at}}{pe^{at} + (1-p)} + b
$$

$$
= \frac{p(na + b)e^{at} + (1-p)b}{pe^{at} + (1-p)}.
$$

(155)

In particular, if $E[Y] = 0$, then $\psi(0) = 0$, which is equivalent to $b = -nap$, and so, we have:

$$
\psi(t) = nap(1-p)\frac{e^{at} - 1}{pe^{at} + (1-p)}.
$$

(156)

Moreover, if $E[X^2] = 1$, then $\psi'(0) = 1$. It follows from (156) that:

$$
1 = \psi'(0)
$$

$$
= na^2 p(1-p).
$$

(157)

Thus, we have:

$$
\psi(t) = \frac{1}{a} \cdot \frac{e^{at} - 1}{pe^{at} + (1-p)}
$$

$$
= \frac{2}{a} \cdot \frac{e^{at} - 1}{2pe^{at} + (2-2p)}
$$

$$
= \frac{2}{a} \cdot \frac{e^{(a/2)t} - e^{-(a/2)t}}{e^{(a/2)t} + (2-2p)e^{-(a/2)t}}
$$

$$
= \frac{2}{a} \cdot \frac{e^{(a/2)t} - e^{-(a/2)t}}{e^{(a/2)t} - e^{-(a/2)t}}
$$

$$
= \frac{2}{a} \cdot \frac{e^{(a/2)t} + e^{-(a/2)t} - (1-2p)e^{(a/2)t} - e^{-(a/2)t}}{e^{(a/2)t} - e^{-(a/2)t}}
$$

$$
= \frac{2}{a} \cdot \frac{\sinh((a/2)t)}{\cosh((a/2)t) - (1-2p)\sinh((a/2)t)}.
$$

(158)
5. A derivation of the classic Meixner random variables from double commutators involving the semi-quantum operators

Let us assume that $X$ is a 2-Meixner random variable. That means, $X$ has finite moments of all orders, and if $U$ and $V$ denote its semi–annihilation and semi-creation operators, then there exists a real constant $c$, such that:

$$[[U, X], X] = c(X - 2U). \quad (159)$$

We assume that $X$ is not a constant random variable.

We can center $X$, by defining:

$$X' := X - E[X], \quad (160)$$

where $E[X]$ denotes the expectation of $X$. Then, if $a_X^-$, $a_X^0$, and $a_X^+$ denote the annihilation, preservation, and creation operators of $X$, since as a multiplication operator:

$$X = a_X^- + a_X^0 + a_X^+, \quad (161)$$

$X'$, viewed also as a multiplication operator, can be written as:

$$X' = X - E[X]I = a_X^- + (a_X^0 - E[X]I) + a_X^+, \quad (162)$$

where $I$ denotes the identity operator. It is not hard to see from here, that the quantum operators of $X'$ are:

$$a_{X'}^- = a_X^-, \quad (163)$$

$$a_{X'}^0 = a_X^0 - E[X]I, \quad (164)$$

and

$$a_{X'}^+ = a_X^+. \quad (165)$$

Thus, the semi-quantum operators of $X'$ are:

$$U_{X'} = U - \frac{1}{2}E[X]I \quad (166)$$

and

$$V_{X'} = V - \frac{1}{2}E[X]I. \quad (167)$$

Thus, we have:

$$[[U_{X'}, X'], X'] = [[[U - \frac{1}{2}E[X]I, X - E[X]I], X - E[X]I], X - E[X]I]$$

$$= [[U, X], X]$$

$$= c(X - 2U)$$

$$= c \left( X - 2 \left( U_{X'} + \frac{1}{2}E[X]I \right) \right)$$

$$= c \left( X' - 2U_{X'} \right).$$
So, we can see that the centered random variable $X'$ satisfies the same double commutator equation as $X$, with the same parameter $c$. Thus from now on we will assume that $X$ is centered, that means $E[X] = 0$.

If we multiply $X$ by the positive constant $k := 1/\|X\|_2$, where $\|X\|_2$ denotes the $L^2$-norm of $X$, then since:

$$kX = ka^+ + ka^0 + ka^-,$$

we can see that the semi-annihilation operator of $X' := kX$ is:

$$U_{X'} = kU,$$  \hspace{1cm} (169)

and so, we have:

$$[[U_{X'}, X'], U_{X'}] = k^3 [[U, X], X]$$

$$= k^3 c (X - 2U)$$

$$= k^2 (X' - 2U_{X'}).$$  \hspace{1cm} (171)

So, the double commutator, involving the re-scaled $U$ and $X$, satisfies the same type of equation as before, with a new constant $k^2c$ that has the same sign as the old constant $c$. Thus, from now on, we will also assume that $X$ is normalized, that means $E[X^2] = 1$.

Let $\phi := 1$ be the constant random variable equal to 1. Then for any $n \in \mathbb{N}$, the random variable $X^n$ can be obtained by applying repeatedly, $n$ times, the multiplication operator $X$, to $\phi$. Therefore, we have:

$$E[X^n] = \langle X^n \phi, \phi \rangle$$

$$= \langle (U + V)X^{n-1} \phi, \phi \rangle$$

$$= \langle UX^{n-1} \phi, \phi \rangle + \langle X^{n-1} \phi, U\phi \rangle,$$  \hspace{1cm} (172)

due to the fact that $V$ is the adjoint of $U$.

Since $\phi \in G_0$ and $a^- : G_0 \to G_{-1}$, but $G_{-1} = \{0\}$ is the null space, we have:

$$a^- \phi = 0.$$  \hspace{1cm} (173)

Also, since $E[X] = 0$, we have:

$$a^0 \phi = \langle X \phi, \phi \rangle$$

$$= E[X] \phi$$

$$= 0.$$  \hspace{1cm} (174)

Thus:

$$U\phi = a^- \phi + \frac{1}{2} a^0 \phi$$

$$= 0.$$  \hspace{1cm} (175)

Therefore, formula (172) reduces to:

$$E[X^n] = \langle UX^{n-1} \phi, \phi \rangle.$$  \hspace{1cm} (176)
Now, we are going to commute $U$ with each of the $n-1$ factors of $X$, from $X^{n-1}$, using Leibniz commutator formula:

\[
[U, X^{n-1}] = \sum_{i=0}^{n-2} X^{n-2-i} [U, X^i],
\]

until $U$ reaches $\phi$, in which case $U\phi = 0$. So, we have:

\[
E[X^n] = \langle UX^{n-1}\phi, \phi \rangle
= \langle X^{n-1}U\phi, \phi \rangle + \sum_{i_1=0}^{n-2} \langle X^{n-2-i_1}[U, X]X^{i_1}\phi, \phi \rangle
= \sum_{i_1=0}^{n-2} \langle X^{n-2-i_1}[U, X]X^{i_1}\phi, \phi \rangle.
\]

Let us compute $[U, X]\phi$ now. We have:

\[
[U, X]\phi = [U, U+V]\phi
= [U, V]\phi
= \left[ a^- + \frac{1}{2}a^0, a^+ + \frac{1}{2}a^0 \right]\phi
= \left[ a^-, a^+ \right]\phi + \frac{1}{2} [a^0, a^+] \phi
= a^-a^+\phi + \frac{1}{2}a^0a^+\phi.
\]

Since $a^0\phi = a^-\phi = 0$, and $X = a^+ + a^- + a^0$, we have:

\[
a^+\phi = X\phi
= X,
\]

where in the above two equalities, the first $X$, from $X\phi$, is viewed as a multiplication operator, while the second $X$ (that is alone, without $\phi$) is regarded as the random variable $X$.

Since $\|X\|_2 = 1$, the polynomial random variable $X$ forms an orthonormal basis of $G_1$. Thus, if $Pr_n$ denotes the orthogonal projection of $L^2(\Omega, F, P)$ onto $G_n$, for all $n \geq 0$, then we have:

\[
a^-a^+\phi = a^-X
= Pr_0 (X \cdot X)
= \langle X^2, \phi \rangle \phi
= E[X^2] \phi
= \phi.
\]

We also have:

\[
a^0a^+\phi = a^0X
= Pr_1 (X \cdot X)
= \langle X^2, X \rangle X
= E[X^3] X
= m_3 X,
\]
where:

\[ m_3 := E[X^3] \] (183)

denotes the third moment of \( X \).

We can see now from (179), (181), and (182) that:

\[ [U, X] \phi = \frac{m_3}{2} X + \phi. \] (184)

We are now going to center \([U, X]\), by defining the operator:

\[ M := [U, X] - \frac{m_3}{2} X - I. \] (185)

Since \( X \) commutes with \((m_3/2)X + I\), we have:

\[ [M, X] = [[U, X], X] = c(X - 2U). \] (186)

Moreover, it follows from (184) that:

\[ M \phi = 0. \] (187)

Let us come back now to formula (178). We have:

\[
\begin{align*}
E[X^n] &= \sum_{i_1=0}^{n-2} \langle X^{n-2-i_1} [U, X] X^{i_1} \phi, \phi \rangle \\
&= \sum_{i_1=0}^{n-2} \langle X^{n-2-i_1} \left( M + \frac{m_3}{2} X + I \right) X^{i_1} \phi, \phi \rangle \\
&= \frac{m_3}{2} \sum_{i_1=0}^{n-2} \langle X^{n-1} \phi, \phi \rangle + \sum_{i_1=0}^{n-2} \langle X^{n-2} \phi, \phi \rangle \\
&\quad + \sum_{i_1=0}^{n-2} \langle X^{n-2-i_1} MX^{i_1} \phi, \phi \rangle \\
&= \frac{m_3}{2} \binom{n-1}{1} E[X^{n-1}] + \binom{n-1}{1} E[X^{n-2}] \\
&\quad + \sum_{i_1=1}^{n-2} \langle X^{n-2-i_1} MX^{i_1} \phi, \phi \rangle ,
\end{align*}
\] (188)

since for \( i_1 = 0 \), we have:

\[ \langle X^{n-2-i_1} MX^{i_1} \phi, \phi \rangle = 0, \]

because \( M \phi = 0. \)

We commute now \( M \) with each of the \( i_1 \) factors of \( X \) from the last sum, using again Leibniz
commutator rule, until $M$ reaches $\phi$, and so, $M\phi = 0$. Thus we obtain:

\[
E[X^n] = \frac{m_3}{2} \binom{n-1}{1} E[X^{n-1}] + \binom{n-1}{1} E[X^{n-2}] + \sum_{i_1=1}^{n-2} \sum_{i_2=0}^{i_1} \langle X^{n-2-i_1} X^{i_1-1-i_2} [M, X] X^{i_2} \phi, \phi \rangle
\]

\[
= \frac{m_3}{2} \binom{n-1}{1} E[X^{n-1}] + \binom{n-1}{1} E[X^{n-2}] + \sum_{i_1=1}^{n-2} \sum_{i_2=0}^{i_1} \langle X^{n-2-i_1} X^{i_1-1-i_2} [M, X] X^{i_2} \phi, \phi \rangle
\]

\[
= \frac{m_3}{2} \binom{n-1}{1} E[X^{n-1}] + \binom{n-1}{1} E[X^{n-2}] + \sum_{i_1=1}^{n-2} \sum_{i_2=0}^{i_1} \langle X^{n-3-i_2} (X - 2U) X^{i_2} \phi, \phi \rangle
\]

\[
= \frac{m_3}{2} \binom{n-1}{1} E[X^{n-1}] + \binom{n-1}{1} E[X^{n-2}] + \sum_{i_1=1}^{n-2} \sum_{i_2=0}^{i_1} \langle X^{n-2-i_1} \phi, \phi \rangle
\]

\[
-2c \sum_{i_1=1}^{n-2} \sum_{i_2=0}^{i_1} \langle X^{n-3-i_2} U X^{i_2} \phi, \phi \rangle
\]

\[
= \frac{m_3}{2} \binom{n-1}{1} E[X^{n-1}] + \binom{n-1}{1} E[X^{n-2}] + \sum_{i_1=1}^{n-2} \sum_{i_2=0}^{i_1} \langle X^{n-3-i_2} U X^{i_2} \phi, \phi \rangle
\]

It must be mentioned that in the last sum, we may assume $i_2 \geq 1$ and accordingly $i_1 \geq 2$, since for $i_2 = 0$, we have $X^{n-3-i_2} U X^{i_2} \phi = 0$, because $U \phi = 0$.

We repeat now the whole procedure, in formula (190), by commuting first $U$ with each of the $i_2$ factors of $X$. Thus, the double sum indexed after $i_1$ and $i_2$, will become a triple sum $\sum_{i_1=2}^{n-2} \sum_{i_2=1}^{i_1-1} \sum_{i_3=0}^{i_2-1}$. After this, we center the commutator $[U, X]$, and the triple sum will yield the coefficient $\binom{n-1}{3}$. Finally, we commute $M$ with each of the $i_3$ factors of $X$, from $X^{i_3}$, giving rise to a quadruple sum, and obtain:

\[
E[X^n] = \frac{m_3}{2} \binom{n-1}{1} E[X^{n-1}] + \binom{n-1}{1} E[X^{n-2}] + c \binom{n-1}{2} E[X^{n-2}]
\]

\[
= \frac{m_3}{2} \binom{n-1}{1} E[X^{n-1}] + \binom{n-1}{1} E[X^{n-2}] + \binom{n-1}{2} E[X^{n-2}]
\]

\[
-2c \sum_{i_1=1}^{n-2} \sum_{i_2=0}^{i_1} \sum_{i_3=0}^{i_2} \sum_{i_4=0}^{i_3} \langle X^{n-3-i_2} U X^{i_2} \phi, \phi \rangle
\]

\[
= \frac{m_3}{2} \binom{n-1}{1} E[X^{n-1}] + \binom{n-1}{1} E[X^{n-2}] + \binom{n-1}{2} E[X^{n-2}]
\]

\[
-2c \sum_{i_1=1}^{n-2} \sum_{i_2=0}^{i_1} \sum_{i_3=0}^{i_2} \sum_{i_4=0}^{i_3} \langle X^{n-3-i_2} U X^{i_2} \phi, \phi \rangle
\]

\[
-2c \sum_{i_1=1}^{n-2} \sum_{i_2=0}^{i_1} \sum_{i_3=0}^{i_2} \sum_{i_4=0}^{i_3} \langle X^{n-3-i_2} U X^{i_2} \phi, \phi \rangle
\]

\[
-2c \sum_{i_1=1}^{n-2} \sum_{i_2=0}^{i_1} \sum_{i_3=0}^{i_2} \sum_{i_4=0}^{i_3} \langle X^{n-3-i_2} U X^{i_2} \phi, \phi \rangle
\]
We continue this procedure until there are no more factors of $X$ in between $U$ and $\phi$, and obtain:

$$E[X^n] = \frac{m_3}{2} \sum_{i=0}^{\left\lfloor(n-2)/2\right\rfloor} \binom{n-1}{2i+1}(-2c)^i E[X^{n-1-2i}] + \sum_{i=0}^{\left\lfloor(n-2)/2\right\rfloor} \binom{n-1}{2i+1}(-2c)^i E[X^{n-1-(2i+1)}] + c \sum_{i=0}^{\left\lfloor(n-3)/2\right\rfloor} \binom{n-1}{2i+2}(-2c)^i E[X^{n-1-(2i+1)}],$$

(192)

for all $n \in \mathbb{N}$.

It is not hard to prove from (192), by induction on $n$, that there exists a positive constant $K$, such that for all $n \geq 0$, we have:

$$|E[X^n]| \leq K \cdot n!. \quad (193)$$

This inequality allows, via Lebesgue dominated convergence theorem, to differentiate term by term the Taylor series:

$$E[e^{tX}] = \sum_{n=0}^{\infty} \frac{E[X^n]}{n!} t^n, \quad (194)$$

of the Laplace transform of $X$:

$$\varphi(t) := E[e^{tX}], \quad (195)$$

for $t$ in a neighborhood $V$ of 0.

We distinguish between two cases:

**Case I.** If $c = 0$, which is equivalent to the discriminant $\Delta = 0$, then formula (192) reduces to:

$$E[X^n] = \frac{m_3}{2} \binom{n-1}{1} E[X^{n-1}] + \binom{n-1}{1} E[X^{n-2}], \quad (196)$$

for all $n \geq 1$. Multiplying both sides of this relation by $t^{n-1}/(n-1)!$, and summing up the resulting relation from $n = 1$ to $\infty$, we obtain:

$$E[Xe^{tX}] = \frac{m_3}{2} t E[Xe^{tX}] + t E[e^{tX}], \quad (197)$$

which means:

$$\varphi'(t) = \frac{m_3}{2} t \varphi'(t) + t \varphi(t). \quad (198)$$

This is equivalent to:

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{t}{1 - (m_3/2)t}, \quad (199)$$

for all $t$ in a small neighborhood $V$ of zero. The left-hand side of this relation is the derivative of the logarithm of the Laplace transform of $X$, which was denoted in the previous section by $\psi(t)$. Thus we have:
• For $m_3 = 0$,

$$\psi(t) = t,$$

(200)

for all $t \in V$. Therefore, according to the discussion from the previous section, $X$ is a standard Gaussian random variable (see formula (108)).

• For $m_3 \neq 0$,

$$\psi(t) = \frac{t}{1 - (m_3/2)t},$$

(201)

for all $t \in V$. It follows from the previous section that $X$ is a shifted and re-scaled Gamma distributed random variable, with re-scaling parameter $a := m_3/2$ (see formula (120)).

We would like to mention that the Gaussian and (shifted and re-scaled) Gamma distributed random variables make up the class of 1-Meixner random variables, as it was shown in [14]. That means, they satisfy a one commutator equation:

$$[U, X] = aX + bI,$$

(202)

where $a$ and $b$ are real numbers. Moreover, the class of non-degenerate two dimensional 1-Meixner random vectors $(X, Y)$ was described completely in [15]. So the class of 1-Meixner random variables is a subclass of the class of 2-Meixner random variables.

Case 2. If $c \neq 0$, then let:

$$\gamma := \sqrt{-2c}$$

(203)

$$= \sqrt{\Delta},$$

(204)

with the agreement that, if $\Delta < 0$, then $\sqrt{\Delta}$ is the complex square of $\Delta$ whose imaginary part is positive.

Using Newton binomial formula, we can rewrite (192) as:

$$E[X^n] = \frac{m_3}{4\gamma} E\left[ X (X + \gamma)^{n-1} - X (X - \gamma)^{n-1} \right]$$

$$+ \frac{1}{2\gamma} E\left[ (X + \gamma)^{n-1} - (X - \gamma)^{n-1} \right]$$

$$+ \frac{1}{4} E\left[ X (X + \gamma)^{n-1} + X (X - \gamma)^{n-1} - 2X^n \right],$$

(205)

for all $n \geq 1$.

If we multiply both sides of (205) by $t^{n-1}/(n-1)!$, and sum up the resulting relation from $n = 1$ to $\infty$, then we get:

$$E[X e^{tX}] = \frac{m_3}{4\gamma} E\left[ X e^{(X+\gamma)} - X e^{(X-\gamma)} \right]$$

$$+ \frac{1}{2\gamma} E\left[ e^{t(X+\gamma)} - e^{t(X-\gamma)} \right]$$

$$+ \frac{1}{4} E\left[ X e^{t(X+\gamma)} + X e^{t(X-\gamma)} \right] + \frac{1}{2} E\left[ X e^{tX} \right],$$

(206)

(207)
for all $t$ in a small neighborhood $V$ of zero. 

If $\varphi$ denotes, as before, the Laplace transform of $X$, then the last relation can be written as:

$$\varphi'(t) = \frac{m_3}{2\gamma} \sinh(\gamma t) \varphi'(t) + \frac{1}{\gamma} \sinh(\gamma t) \varphi(t) - \frac{1}{2} \cosh(\gamma t) \varphi'(t) + \frac{1}{2} \varphi'(t),$$

for all $t \in V$.

It follows from here that:

$$\varphi'(t) \varphi(t) = \frac{2}{\gamma} \frac{\sinh(\gamma t)}{1 + \cosh(\gamma t) - (m_3/\gamma) \sinh(\gamma t)}$$

$$= \frac{2}{\gamma} \frac{2 \sinh((\gamma/2)t) \cosh((\gamma/2)t)}{2 \sinh((\gamma/2)t) \cosh((\gamma/2)t) - 2(m_3/\gamma) \sinh((\gamma/2)t) \cosh((\gamma/2)t)}$$

$$= \frac{2}{\gamma} \frac{\cosh((\gamma/2)t) - (m_3/\gamma) \sinh((\gamma/2)t)}{\cosh((\gamma/2)t) - (m_3/\gamma) \sinh((\gamma/2)t)},$$

for all $t \in V$.

We distinguish between four subcases:

- If $-2c < 0$, which is equivalent to $\Delta < 0$, then it follows from (209) and (151) that $X$ is a centered and re-scaled two parameter hyperbolic secant distributed random variable, with a re-scaling parameter $a = -i\gamma \in \mathbb{R}_+$. In this case the parameter $\alpha$ satisfies the equation:

$$\tan\left(\frac{\alpha}{2}\right) = \frac{m_3}{\sqrt{-\Delta}}.$$

It follows from (150) that the other parameter is:

$$k = \frac{2 \cos^2(\alpha/2)}{-\Delta}.$$

- If $\Delta > 0$ and $m_3 > \gamma := \sqrt{\Delta}$, then it follows from (209) and (140) that $X$ is a centered and re-scaled negative binomial random variable. The shifting parameter is $a = \gamma$. The probability of a success, $p$, satisfies the equation:

$$\frac{2 - p}{p} = \frac{m_3}{\gamma} \in (1, \infty).$$

It follows from (139) that the parameter $r$ is:

$$r = \frac{p^2}{\Delta(1 - p)}.$$

- If $\Delta > 0$ and $m_3 = \gamma$, then it follows from (209) and (128) that $X$ is a centered and re-scaled Poisson random variable. The shifting parameter is $a = \gamma$. It follows from (127) that the parameter $\lambda$ is:

$$\lambda = \frac{1}{a^2} = \frac{1}{\Delta}.$$
If $\Delta > 0$ and $-\gamma < m_3 < \gamma$, then it follows from (209) and (158) that $X$ is a centered and re-scaled binomial random variable. The probability of a success satisfies the equation:

$$1 - 2p = \frac{m_3}{\gamma}. \quad (215)$$

It follows from (157) that the number of independent repetitions of the experiment is:

$$n = \frac{1}{\Delta p(1-p)}, \quad (216)$$

and this must be a natural number.

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