Braiding a flock: winding statistics of interacting flying spins

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When animal groups move coherently in the form of a flock, their trajectories are not all parallel, the individuals exchange their position in the group. In this Letter we introduce a measure of this mixing dynamics, which we quantify as the winding of the braid formed from the particle trajectories. Building on a paradigmatic flocking model we numerically and theoretically explain the winding statistics, and show that it is predominantly set by the global twist of the trajectories as a consequence of a spontaneous symmetry breaking.

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The collective behaviors observed in animal groups have attracted much attention in the biology, the mathematics and the physics communities over the last 20 years. Quantitative data analysis have established that the salient traits of collective motion are very well captured by the dynamics of flying spins: persistent random walkers endowed with interactions akin to ferromagnetic couplings between their velocities [1-4]. This framework has been extensively exploited to rationalise a variety of structural and dynamical orientational properties starting from the emergence of directed motion, to rapid (orientational) information transfer, see [4-9] and references therein. However, beyond these spectacular results, the entanglement and theoretically yet it is known to display non-trivial anomalous behaviors [10, 11].

In this Letter, we theoretically describe the mixing statistics of an archetypal flying-spin model. We first stress the intrinsic geometrical nature of the dynamics of particles in flock and map this problem to the braiding statistics of their trajectories. We evidence the nontrivial statistics of the winding between pairs of motile-particle trajectories, which is a robust measure of their entanglement. This quantity displays spatial correlations at the population scale. We single out the reason for the nontrivial statistics and show that the spontaneous breaking of a rotational symmetry causes the global twist of the trajectories, which we characterize in this paper.

The standard (metric) form of $\mathbf{F}_i$ used to account for the flocking dynamics is given by:

$$F_i = \frac{1}{\tau N_i} \sum_{j \in A_i} \hat{t}_j + \frac{1}{\tau N_i} \sum_{j \in B_i} f(r_{ij}) \hat{r}_{ij},$$

where $\tau$ is a relaxation time, which we henceforth set to $\tau = 1$, and $r_{ij} \equiv r_i - r_j$. The first term in Eq. (2) is the ferromagnetic term which promotes alignment of the $i^{th}$ particle with the mean direction of the $N_i$ neighbors lying in the sphere $A_i$ of radius $R_A = 1$. The second term corresponds to attractive and repulsive interactions within the sphere $B_i$ of radius $R_B = 5$ [13]. They are introduced to yield compact flocks. Following [6-14], we assume that these interactions are attractive above a distance $2r_c$, and repulsive when $r_c \leq r_{ij} < 2r_c$ with a divergent amplitude at $r_{ij} = r_c$. Here, we take $f(r_{ij}) = 1 - \left[ r_c/(r_{ij} - r_c) \right]^5$ with $r_c = 0.4$. Rotational diffusion is accounted for by the uncorrelated Gaussian white noises $\xi_i$, of variance $2D$. Eqs. (1) and (2) are first solved numerically using an explicit Euler scheme. Below a critical noise amplitude $D_c \sim 0.2$, the rotational symmetry of the particle orientation is spontaneously broken, which results in the formation of a compact polar flock moving along $\Pi(t) = \langle \hat{t}_i(t) \rangle_i$ as exemplified in Fig. 1(a) and in the supplementary video S1. Decreasing $D$ below $D_c$ the magnitude of the polarization $\Pi$ increases.
Quantifying the entanglement of the trajectories. We now introduce a quantity inspired by Lagrangian mixing in fluids which quantifies the entanglement of the trajectories [15–18]. Let us first introduce a convenient representation of the relative motion of the individuals irrespective of the global motion of the flock. We define a frame (\(\{\eta_1, \eta_2, \Pi\}\)) and the axis are parallel transported along the trajectory of \(G\). We then project the \(r_i\)s on the plane \((\hat{\eta}_1(t), \hat{\eta}_2(t))\), see Fig. 1(b) and supplementary movie S2. If the flock were undergoing a rigid-body motion, the projected positions would be stationary. In order to quantify how the dynamics deviates from this trivial behavior, we choose a reference axis, say \(\hat{\eta}_1\), and observe the relative positions of the particles along this line. As the group is rotationally invariant around \(\hat{\Pi}\), all directions of observation are equivalent. The mixing in the flock results in particle exchanges along \(\Pi\). We keep track of these crossings over time, and of the relative positions in the orthogonal direction \(\hat{\eta}_2\). This procedure is better understood when plotting the world lines of the particles, Fig. 1(b) and and supplementary movie S3. We assign a sign \(\epsilon = \pm 1\) to each crossing, as depicted in Fig. 1(c). Considering two particles \(i\) and \(j\), we can define their pair winding number, \(w_{ij}(T)\), as the linking number between their world lines: \(w_{ij}(T)\) is one half of the algebraic sum of the crossings involving particles \(i\) and \(j\), between time \(t_0\) and \(t_0 + T\). This quantity has a clear interpretation: it counts the number of turns of particle \(i\) around \(j\) (or, equivalently, of \(j\) around \(i\)), since a clockwise (resp. counter-clockwise) turn results in two negative (resp. positive) crossings. We finally average over the pairs and define the total winding between \(t_0\) and \(t_0 + T\), which is our measure of the entanglement of the world lines:

\[
W(T) = \frac{1}{N_p} \sum_{(i,j)} w_{ij}(T),
\]

where \(N_p = N(N-1)/2\) is the number of pairs. Importantly, \(W(T)\) does not depend on the distance between the particles, only the signs of the crossings and the times at which they occur matter. Therefore the world lines define a braid that can be drawn in the form of a normalized braid diagram, Fig. 1(d) [17]. Denoting by \(\epsilon_a\) the sign of the \(a^{th}\) crossing, we turn the sum over the pairs into a topological invariant of the braid: \(W(T) = (2/N_p) \sum_a \epsilon_a\). Practically we use the Artin representation of the braid word [16, 17, 19] and the braidlab library [20] to compute both the pair and the total winding numbers. The topological nature of \(W(T)\) makes it a very robust measure of the flock mixing. We now thoroughly investigate its statistics.

Winding statistics. The normalized distributions of the global winding \(W(T)\) are plotted in Fig. 2(a) for different values of the noise amplitude. For all the trajectory lengths, the total winding follows a Gaussian statistics with zero mean since the flock has no intrinsic chirality: clockwise and counter-clockwise windings are equally probable. The winding distribution is therefore fully characterized by its standard deviation \(\sqrt{\langle W^2(T) \rangle}\), where the brackets denote the average over different initial conditions. The winding fluctuations increase with the curvilinear length of the trajectories, Fig. 2(b). More quantitatively, the numerical data clearly display a crossover between a ballistic regime at short times where \(\sqrt{\langle W^2(T) \rangle} \propto T\) and a diffusive growth at long time where \(\sqrt{\langle W^2(T) \rangle} \propto \sqrt{T}\).

These first results would naively suggest a simple scenario. \(W(T)\) being the average of the crossing signs, if the crossing events were uncorrelated the Gaussian statistics would readily stem from the central limit theorem, and the variance of \(W(T)\) would obviously grow in a diffusive manner. However, this appealing explanation is inconsistent with a deeper analysis of the data. Let us carefully study the correlations between the pair windings, which

![FIG. 1: (a) Instantaneous positions and orientations of the particles in a compact polar flock (250 particles), and definition of the parallel-transported frame \((\hat{\eta}_1, \hat{\eta}_2, \Pi)\). \(D = 2.6 \times 10^{-2}\). (b) World lines of 5 particles in a polar flock. \(D = 2.6 \times 10^{-2}\). (c) Definition of the crossing signs. (d) Braid diagram associated to the world lines drawn in Fig. 1(b).](image-url)
are quantified by:

\[ C_{ww}(T) = \frac{1}{N_p(N_p-1)} \sum_{(i,j)} \sum_{(k,l) \neq (i,j)} \frac{\langle w_{ij}(t)w_{kl}(T) \rangle}{\langle w^2(T) \rangle}, \]

where \( \langle w^2(t) \rangle = N_p^{-1} \sum_{(i,j)} \langle w^2_{ij}(T) \rangle \) is the variance of the pair windings. Given this definition, \( C_{ww}(T) = 0 \) for uncorrelated \( w_{ij}s \), and \( C_{ww}(T) = 1 \) when the pair windings are fully correlated, which would happen e.g. if the flock were undergoing a rigid-body rotation. Unexpectedly, we find that the correlation between the pair windings does not vanish at long times. Conversely, it increases and plateaus at a finite value, Fig. 2(c), thereby ruling out the simple scenario sketched above. In order to check that this unexpected behavior is not a finite-size artifact, we first note that \( C_{ww}(T) \sim \langle W^2(T) \rangle / \langle w^2(T) \rangle \) in the large-\( N_p \) limit and plot the ratio \( \langle W^2(T) \rangle / \langle w^2(T) \rangle \) for flocks of different sizes \( N \), in Fig. 2(d). Whereas short trajectories have indeed winding correlations \( C_{ww}(T) \) that decay with the system size, as \( 1/N \), the winding correlations of the long trajectories do not display any significant variations with \( N \) when increasing the particle number by a factor of \( \sim 8 \). The crossover between these two regimes increases with the system size and will be discussed below. In order to gain more insight into these two opposite behaviors, we computed the same quantity for the wordlines of independent 2D random walkers confined in a circular box. \( C_{ww} \) follows the same \( 1/N \) nontrivial scaling observed for short flocking trajectories. We shall note that the finite time step of our numerical scheme regularizes the winding statistics of the random walkers and makes it possible to define its variance \[ \langle w^2(T) \rangle \] . This second set of observations confirms that the saturation of \( C_{ww}(T) \) at long time echoes spatial correlations of the crossing events at the entire flock-scale.

A twist in the statistics. We now elucidate the physical mechanism responsible for the winding correlations. As already alluded to, a global instantaneous rotation of the flock around its direction of motion \( \hat{\Pi} \) results in a fully correlated winding. We therefore separate the contribution of this global trajectory twist from the total winding. Denoting by \( \eta_i \) the position of particle \( i \) projected in the observation plane \( (G, \eta_1, \eta_2) \), the instantaneous rotation rate of the flock is:

\[ \Omega = \frac{1}{N} \sum_i \frac{1}{\eta_i^2} (\eta_i \times \dot{\eta}_i) \cdot \hat{\Pi}. \]

Integrating over time, we define the global twist that is the number of turns of the flock around \( \hat{\Pi} \):

\[ Tw(T) = \frac{1}{2\pi} \int_{t_0}^{t_0+T} dt \Omega(t). \]

We finally subtract this twist component from the total winding and introduce a measure of the total winding in the frame \( (G, \eta_1^*, \eta_2^*) \), obtained by rotating the parallel-transported frame \( (G, \eta_1, \eta_2) \) by an angle \( 2\pi x Tw \):

\[ W^* = W - Tw. \]

Hence \( W^* = N_p^{-1} \sum_{(i,j)} w^*_{ij} \), where \( w^*_{ij} = w_{ij} - Tw \) is the winding between particles \( i \) and \( j \) in this twisting frame. Within the braid picture, \( W^* \) is found by factorizing out the global twist of the braid word \[ \prod \].

In Fig. 3(a), we plot the standard deviation of \( W, Tw \) and \( W^* \) versus \( T \). The twist contribution dominates the total winding at long times and \( \langle W^2 \rangle \sim \langle Tw^2 \rangle \). This numerical fact is better understood by noting that the winding in the rotating frame qualitatively follows the behavior displayed by confined random walkers. Above the relaxation time \( \tau = 1 \) introduced in Eq. 4, \( W^*(T) \) follows the same diffusive evolution: \( \langle W^2(T) \rangle \sim T \). More importantly, we also find the same \( 1/N \) asymptotic scaling for \( C_{ww^*} \sim \langle W^2(T) \rangle / \langle w^2(T) \rangle \) showing that the spatial correlations of the displacements are short-ranged at all times, Fig. 3(b). This result contrasts with the behavior of the total winding in the parallel-transported frame where \( C_{ww} \sim \langle W^2(T) \rangle / \langle w^2(T) \rangle \) hardly depends
on $N$ at long times, Fig. 3(d). We can therefore propose the following scenario which correctly accounts for all our numerical findings: the long-range spatial correlations in the flock arise from the global rotation of the flock, hence from the global twist of the trajectories as clearly exemplified in the supplementary video S4.

**Hydrodynamic description of the trajectories’ twist.**

The very origin of the global rotation of the flock roots from the spontaneous breaking of the rotational symmetry of particle velocities. This symmetry breaking gives rise to a soft orientational mode which twists the trajectories in the parallel-transported frame at the entire-flock scale. We now lay out a more quantitative explanation by switching to an hydrodynamic description of the flock viewed as a polar active-fluid drop. We use the conventional hydrodynamic framework first introduced phenomenologically by Toner and Tu [10] and later derived from microscopic theories [2, 26, 27]. The fluid density and velocity fields are $\rho(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$. We focus on strongly polarized flocks in which all particles follow the same average direction. The momentum equation linearized around the homogeneously polarized state takes the simple form [2, 28]:

$$\partial_t (\rho \mathbf{v}) + \lambda (\mathbf{v} \cdot \nabla) \rho \mathbf{v} = -\nabla P(\rho) + \nabla \Gamma \mathbf{v}^2 (\rho \mathbf{v}) + \mathbf{f}, \quad (8)$$

where $P$ is the local pressure, and $\mathbf{f}(\mathbf{r}, t)$ is a Gaussian white noise with correlation $\langle f_\alpha(\mathbf{r}, t)f_\beta(\mathbf{r}', t') \rangle = 2\mathcal{D}_f \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')\delta_{\alpha\beta}$. In this continuous limit, the global rotation rate, Eq. [3], is given by:

$$\Omega = \frac{1}{N} \int d^3 \mathbf{r} \frac{1}{\eta^2} (\mathbf{v} \times \mathbf{v}) \cdot \hat{\Pi}. \quad (9)$$

Two comments are in order. Firstly, deep in the polarized phase, the linearity of Eq. (8) implies that the momentum fluctuations are Gaussian. After space and time integration, Eqs. (9) and (10) imply that the twist also follows a normal distribution, in agreement with our numerical findings reported in Fig. 3(c). Secondly, the damping of velocity fluctuations is set by the diffusive term $\Gamma \mathbf{v}^2 (\rho \mathbf{v})$, in Eq. (8) (see [2, 28] for more details). Hence the Fourier mode with wave-vector $\mathbf{q}$ decays in a time $\sim (\Gamma q^2)^{-1}$. While small-wavelength perturbations are quickly damped, the large-scale fluctuations that occur at the size of the flock, $q \sim 1/L_{\text{flock}}$, remain correlated over a time $T \sim L_{\text{flock}}^2/T$. This observation explains the time behavior of the trajectories’ twist fluctuations. At short times, $T < T_{\text{c}}$, the small-$q$ fluctuations result in finite time correlations in the rotation rate $\Omega$. Consequently, the twist fluctuations persist and undergo a “ballistic” growth: $\langle T^2(T) \rangle \propto T^2$. At long times, $T > T_{\text{c}}$, all the Fourier modes are relaxed, the correlations vanish, and we recover the observed diffusive behavior for the trajectory-twist fluctuations: $\langle T^2(T) \rangle \sim \mathcal{D}_T T$.

We further check this analysis, by studying how the diffusion coefficient $\mathcal{D}_T$ depends on the size of the flock. The temporal correlations of the rotation rate can be computed from Eq. (8) and (9). In the long-time limit, $|t - t'| \gg T_{\text{c}}$, we have $\langle \Omega(t)\Omega(t') \rangle \sim \mathcal{D}_T \delta(t - t')$, where the effective diffusivity scales as $\mathcal{D}_T \propto T^{-2} L_{\text{flock}}^2 N^{-2}$. This quantitative prediction is in excellent agreement with our numerical data, shown in Fig. 3(d). In addition, introducing the mean density $\rho_0$ of the flock, we note that $\mathcal{D}_T \propto \rho_0^{-5/3} N^{-1/3}$. Assuming that $\rho_0$ weakly depends on $N$, the variance of the twist is predicted to decay very slowly with the flock size as $N^{-1/3}$. This scaling confirms that the ratio between the twist and the local winding fluctuations $\langle (W^2)/(w^2) \rangle \sim N^{-1}$ diverges with the flock size. Given our limited range of flock sizes, the $-1/3$ scaling law could not be quantitatively checked. Nonetheless this weak decay is again consistent with the minute variations reported in Fig. 3(d). These analytical predictions unambiguously confirm that the soft rotational mode resulting in the global twist of the trajectories chiefly rules the winding statistics and hence the mixing of the self-propelled particles at long times. We stress that the prominence of the twist fluctuations is...
expected to be qualitatively robust to the detailed form of the interactions; it solely relies on the existence of a soft rotation mode which is generic to all the flying spin models.

We close with Letter from a potential experimental perspective. In most of the situations, in the wild, external perturbations and fields (e.g. gravity [11]) explicitly break the rotational symmetry. Measuring the winding statistics should be an effective mean to probe the response of the flock to these external bias which are difficult to quantify otherwise.

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