Generalizing Virtual Values to Multidimensional Auctions: a Non-Myersonian Approach

Song Zuo†

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Abstract

We consider the revenue maximization problem of a monopolist via a non-Myersonian approach that could generalize to multiple items and multiple buyers. Although such an approach does not lead to any closed-form solution of the problem, it does provide some insights to this problem from different angles. In particular, we consider both Bayesian (Bayesian Incentive Compatible + Bayesian Individually Rational) and Dominant-Strategy (Dominant-Strategy Incentive Compatible + ex-post Individually Rational) implementations, where all the buyers have additive valuations and quasi-linear utilities and all the valuations are independent across buyers (not necessarily independent across items).

The main technique of our approach is to formulate the problem as an LP (probably with exponential size) and apply primal-dual analysis. We observe that any optimal solution of the dual program naturally defines the virtual value functions for the primal revenue maximization problem in the sense that any revenue maximizing auction must be a virtual welfare maximizer (cf. Myerson’s auction for single item [1981]).

Based on this observation, we have the following results (most of them are previously unknown for the multi-item multi-buyer setting):

1. We characterize a sufficient and necessary condition for BIC = DSIC, i.e., the optimal revenue of Bayesian implementations equals to the optimal revenue of dominant-strategy implementations ($BRev = DRev$). The condition is if and only if the optimal DSIC revenue $DRev$ can be achieved by a DSIC and ex-post IR virtual welfare maximizer with buyer-independent virtual value functions (buyer i’s virtual value is independent with other buyers’ valuations).

2. In light of the characterization, we further show that when all the valuations are i.i.d., it is further equivalent to that separate-selling is optimal. In particular, it respects one result from the recent breakthrough work on the exact optimal solutions in the multi-item multi-buyer setting by Yao [2016].

3. We also observe that dual programs can be interpreted as the optimal transport problem. This result is previously shown by Daskalakis et al. [2013, 2015] for the single buyer setting. Thus we automatically obtain a generalized version for the multi-buyer setting.

4. We provide an alternative proof of Myerson’s auction. In particular, we can directly start with solving the optimal DSIC and ex-post IR auction instead of first solving the optimal Bayesian implementation then showing that it is also a dominant-strategy implementation.

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†Institute for Interdisciplinary Information Sciences, Tsinghua University, Beijing, China. songzuo.z@gmail.com.

1It is in fact implied by Cai et al. [2012], while our results stand on the exact form of the virtual value functions.

2For the multi-buyer setting, the dual program is a generalized version of optimal transport problem.
1 Introduction

Roadmap. We introduce our notations and the common definitions in Section 2. We apply the dual analysis and define virtual values in Section 3. In Section 4 we present our main characterization and we then show that in Section 5 for the i.i.d. setting, such a characterization implies that DSIC = BIC if and only if separate-selling is optimal.

2 Notations

Throughout this manuscript, we use subscripts $i$ to indicate buyers and superscripts $j$ to indicate items. We also use boldface (without subscript) notations for vectors across all the buyers (e.g., matrices for allocation $x$ and value $v$ while vectors for payments $p$) and notations without superscripts (but with subscripts) for vectors across all the items for some certain buyer (e.g., allocation $x_i$ and value $v_i$ of buyer $i$, while both are vectors). As a general convention, we use subscripts $-i$ for the vectors without the element(s) for buyer $i$ and $[n] = 1, \ldots, n$ for the set of buyers and $[m] = 1, \ldots, m$ for the set of items. We also use $·$ to emphasize the inner product of between two vectors.

As we will consider both Bayesian and dominant-strategy implementations, we use variables with $\bar{\cdot}$ for Bayesian implementations (e.g., $\bar{x}_i(v)$ and $\bar{p}_i(v)$) while those without $\bar{\cdot}$ for dominant-strategy implementations (e.g., $x_i(v)$ and $p_i(v)$).

We consider the case where the buyers have independent values with each other (yet the values of the same buyer for different items might be correlated). The values are additive and the utilities are quasi-linear. We will formalize the definitions later.

For ease of using linear programs, we consider discrete distributions with finite supports.\(^3\)

2.1 Direct Auctions

A direct auction $M = (x, p)$ consists of the allocation $x : \mathbb{R}^{n \times m}_+ \rightarrow [0, 1]^{n \times m}$ and the payment $p : \mathbb{R}^{n \times m}_+ \rightarrow \mathbb{R}_+^n$. The utility of each buyer $i$ is

$$u_i(v) = v_i \cdot x_i(v) - p_i(v) = \sum_{j \in [m]} v_i^j x_i^j(v) - p_i(v).$$

For any value profile $v \in \mathbb{R}^{n \times m}_+$, the allocation and payment must satisfy the following feasibility constraint, $\forall j \in [m], v \in \mathbb{R}^{n \times m}_+$,

$$1 \cdot x_i^j(v) = \sum_{i \in [n]} x_i^j(v) \leq 1.$$

We use $\mu$ to denote the probability measure of the common prior knowledge on the private values. In particular, since the values are independent across buyers, $\mu(v)$ can be written as

$$\mu(v) = \mu_1(v_1) \mu_2(v_2) \cdots \mu_n(v_n).$$

Let $V_i \subseteq \mathbb{R}^m_+$ be any finite support of the prior distribution of buyer $i$, namely,

$$\forall v_i \in V_i, \mu_i(v_i) \geq 0, \text{ and } \sum_{v_i \in V_i} \mu_i(v_i) = 1.$$

Write $\mathcal{V} = V_1 \times V_2 \times \cdots \times V_n$. In particular, we will assume that $\forall v_i \neq 0, \mu_i(v_i) > 0$ to simplify the discussion of corner cases.

\(^3\)Generalization to arbitrary distribution (if possible) would require linear programs for infinite dimensions.
2.2 Bayesian Implementation

A direct mechanism $\bar{M} = (\bar{x}, \bar{p})$ is Bayesian Incentive Compatible (BIC), if $\forall v_i, v_i' \in \mathbb{R}_+^m$,

$$\mathbb{E}_{v_{-i}} [v_i \cdot \bar{x}_i(v) - \bar{p}_i(v)] \geq \mathbb{E}_{v_{-i}} [v_i \cdot \bar{x}_i(v', v_{-i}) - \bar{p}_i(v', v_{-i})];$$

(BIC)

Bayesian Individually Rational (BIR), if $\forall v_i \in \mathbb{R}_+^m$,

$$\mathbb{E}_{v_{-i}} [v_i \cdot \bar{x}_i(v) - \bar{p}_i(v)] \geq 0.$$  

(BIR)

By restricting to the support space $\mathcal{V}$, we can define the optimal Bayesian direct mechanism as the following linear program:

$$\begin{align*}
\text{max} & \quad \sum_v \mu(v) \sum_i \bar{p}_i(v) \\
\text{s.t.} & \quad \sum_v \mu(v) (v_i \cdot \bar{x}_i(v) - \bar{p}_i(v)) \geq \sum_v \mu(v) (v_i \cdot \bar{x}_i(v', v_{-i}) - \bar{p}_i(v', v_{-i})), \forall i \in [n], v_i, v_i' \in \mathcal{V}_i \\
& \quad \sum_v \mu(v) (v_i \cdot \bar{x}_i(v) - \bar{p}_i(v)) \geq 0, \forall i \in [n], v_i \in \mathcal{V}_i \\
& \quad \sum_i \bar{x}_i(v) \leq 1, \forall j \in [m], v \in \mathcal{V} \\
& \quad \bar{x}_i(v), \bar{p}_i(v) \geq 0, \forall i \in [n], j \in [m], v \in \mathcal{V}.
\end{align*}$$

(BLP)

Although any feasible solution to this linear program only defines the allocation and payments for the value profiles in the support space and the BIC and BIR properties are only guaranteed within the support space, there is a standard extension method to recover the Bayesian implementation that is (i) defined on the full value space $\mathbb{R}_+^{nxm}$, (ii) BIC and BIR (on the full value space), and (iii) consistent with the given feasible solution on support space $\mathcal{V}$.

Lemma 2.1 (Bayesian Extension). Given any feasible solution $(\bar{x}, \bar{p})$ to BLP, the extended direct mechanism $(\bar{x}', \bar{p}')$ defined as follows satisfies BIC and BIR.

$$\begin{align*}
\bar{x}'(v) &= \bar{x}(v), \quad \bar{p}'(v) = \bar{p}(v') \\
\text{where} & \quad \forall i \in [n], \text{ if } v_i \in \mathcal{V}_i, v_i' = v_i \\
& \quad \text{otherwise, } v_i' = \arg \max_{v_i \in \mathcal{V}_i} \mathbb{E}_{v_{-i}} [v_i' \cdot \bar{x}_i(v_i, v_{-i}) - \bar{p}_i(v_i, v_{-i})].
\end{align*}$$

2.3 Dominant-Strategy Implementation

A direct mechanism $M = (x, p)$ is Dominant-Strategy Incentive Compatible (DSIC), if $\forall v \in \mathbb{R}_+^{nxm}, v_i' \in \mathbb{R}_+^m$,

$$v_i \cdot x_i(v) - p_i(v) \geq v_i \cdot x_i(v', v_{-i}) - p_i(v', v_{-i});$$

(DSIC)

Ex-post Individually Rational (epIR), if $\forall v \in \mathbb{R}_+^{nxm}$,

$$v_i \cdot x_i(v) - p_i(v) \geq 0.$$  

(epIR)
Similarly, we have the following linear program for optimal dominant-strategy direct mechanisms:

\[
\text{max } \sum_{v} \mu(v) \sum_{i} p_i(v) \tag{DSLP}
\]

s.t.

\[
v_i \cdot x_i(v) - p_i(v) \geq v_i \cdot x_i(v_i', v_{-i}) - p_i(v_i', v_{-i}), \forall i \in [n], v \in \mathcal{V}, v_i' \in \mathcal{V}_i
\]

\[
v_i \cdot x_i(v) - p_i(v) \geq 0, \forall i \in [n], v_i \in \mathcal{V}_i
\]

\[
\sum_{i} x_i^j(v) \leq 1, \forall j \in [m], v \in \mathcal{V}
\]

\[
x_i^j(v), p_i(v) \geq 0, \forall i \in [n], j \in [m], v \in \mathcal{V}
\]

Again, any feasible solution to this linear program is only a limited dominant-strategy implementation, while the following lemma (first by Dobzinski et al. [2011]) provides an extension method similar to the Bayesian case.

**Lemma 2.2** (Dominant-Strategy Extension). Given any feasible solution \((x, p)\) to DSLP, the extended direct mechanism \((x', p')\) defined as follows satisfies DSIC and epIR.

\[
x_i'(v) = x_i(v') \quad p_i'(v) = p_i(v')
\]

where \(\forall i \in [n], \text{ if } v \in \mathcal{V}, v' = v\)

else if \(v_{-i} \in \mathcal{V}_{-i}, v_i' = \arg\max_{v_i \in \mathcal{V}_i} v_i' \cdot x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})\)

otherwise, \(v_i' = 0\).

Therefore, from now on, we will only focus on the value space \(\mathcal{V}\).

### 2.4 Separate Selling

By separate-selling, we mean to sell each of the item independently via the Myerson’s auction. We use SRev to denote the revenue of separate-selling.

### 3 Dual Programs and Virtual Values

Now we write down the corresponding dual programs. In particular, we will omit the “for-all” quantifiers on the free variables in the rest of the paper.

#### 3.1 Dominant-Strategy Implementation

##### 3.1.1 Duality and Complementary Slackness

First for dominant-strategy implementation, let \(\zeta_i(v, v_i'; v_{-i}), \eta_i(v),\) and \(\xi^j(v)\) be the corresponding multipliers of the constraints. By reorganizing DSLP into the standard form, we obtain,
Primal Dominant-Strategy

\[ x_i^j(v), p_i(v) \geq 0 \]

\[
\max \sum_v \mu(v) \sum_i p_i(v) \\
\text{subject to} \quad -v_i \cdot x_i(v) + p_i(v) + v_i \cdot x_i(v') - p_i(v') \leq 0 \\
\quad -v_i \cdot x_i(v) + p_i(v) \leq 0 \\
\quad \sum_i x_i^j(v) \leq 1
\]

Hence the dual program is

Dual Dominant-Strategy

\[ \zeta_i(v_i, v'_i; v_{-i}), \eta_i(v), \xi_i^j(v) \geq 0 \]

\[
\min \sum_v \sum_j \xi_i^j(v) \\
\text{subject to} \quad \xi_i^j(v) - \left( \eta_i(v)v_i^j + \sum_{v'_i} \left( \zeta_i(v_i, v'_i; v_{-i})v_i^j - \zeta_i(v'_i, v_i; v_{-i})v'_i^j \right) \right) \geq 0 \quad x_i^j(v) \\
\eta_i(v) + \sum_{v'_i} \left( \zeta_i(v_i, v'_i; v_{-i}) - \zeta_i(v'_i, v_i; v_{-i}) \right) \geq \mu(v) \quad p_i(v)
\]

We then use \( P \) to denote the polytope of all feasible solutions of the primal linear program, and similarly \( D \) for the polytope of the dual linear program. For ease of notation, we use the following abbreviations:

For primal: \( \Delta u_i(v_i, v'_i; v_{-i}) = v_i \cdot x_i(v'_i, v_{-i}) - p_i(v'_i, v_{-i}) - (v_i \cdot x_i(v) + p_i(v)) \)

\( u_i(v) = v_i \cdot x_i(v) + p_i(v) \)

\( s^j(v) = \sum_i x_i^j(v) \)

For dual: \( \phi_i^j(v) = \eta_i(v)v_i^j + \sum_{v'_i} \left( \zeta_i(v_i, v'_i; v_{-i})v_i^j - \zeta_i(v'_i, v_i; v_{-i})v'_i^j \right) \)

\( \psi_i(v) = \eta_i(v) + \sum_{v'_i} \left( \zeta_i(v_i, v'_i; v_{-i}) - \zeta_i(v'_i, v_i; v_{-i}) \right) \)

Then the primal and dual look like

| Primal | Dual |
|--------|------|
| \[ x_i^j(v), p_i(v) \geq 0 \] | \[ \zeta_i(v_i, v'_i; v_{-i}), \eta_i(v), \xi_i^j(v) \geq 0 \] |
| \[
\max \sum_v \mu(v) \sum_i p_i(v) \\
\text{subject to} \quad \Delta u_i(v_i, v'_i; v_{-i}) \leq 0 \\
\quad -u_i(v) \leq 0 \\
\quad s^j(v) \leq 1
\] | \[
\min \sum_v \sum_j \xi_i^j(v) \\
\text{subject to} \quad \xi_i^j(v) - \phi_i^j(v) \geq 0 \\
\quad \psi_i(v) \geq \mu(v) \\
\quad p_i(v)
\] |

Clearly, both \( P \) and \( D \) are always nonempty, and \( P \) is bounded. Now, suppose \( \pi^* = (x^*_{i,j}(v), p_i^*(v)) \in P \) is an optimal solution of the primal, and \( \delta^* = (\zeta_i^*(v_i, v'_i; v_{-i}), \eta_i^*(v), \xi_i^*(v)) \in D \) is an optimal solution of
the dual. By strong duality theorem, we know that \( \text{obj}(\pi^*) = \text{obj}(\delta^*) \), which is the optimal revenue of any dominant-strategy implementation, denoted by \( \text{DRev} \):

\[
\text{DRev} = \text{obj}(\pi^*) = \sum_{v} \mu(v) \sum_{i} p_i^*(v) = \text{obj}(\delta^*) = \sum_{v} \sum_{j} \xi^{*j}(v).
\]

Finally, we add slack variables to both primal and dual:

**Primal**

\[
x^j_i(v), p_i(v) \geq 0
\]

variables

\[
a_i(v_i, v'_i; v_{-i}), b_i(v), c^j_i(v) \geq 0
\]

slack variables

max \[
\sum_{v} \mu(v) \sum_{i} p_i(v)
\]

objective

s.t. \[
\Delta u_i(v_i, v'_i; v_{-i}) + a_i(v_i, v'_i; v_{-i}) = 0
\]

\[
- u_i(v) + b_i(v) = 0
\]

\[
\xi_i(v_i, v'_i; v_{-i})
\]

\[
\eta_i(v)
\]

\[
s^j_i(v) + c^j_i(v) = 1
\]

\[
\xi^j_i(v)
\]

**Dual**

\[
\xi_i(v_i, v'_i; v_{-i}), \eta_i(v), \xi^j_i(v) \geq 0
\]

variables

\[
\alpha_i^j(v), \beta_i(v) \geq 0
\]

slack variables

min \[
\sum_{v} \sum_{j} \xi^j_i(v)
\]

objective

s.t. \[
\xi^j_i(v) - \phi^j_i(v) - \alpha^j_i(v) = 0
\]

\[
\psi_i(v) - \beta_i(v) = \mu(v)
\]

\[
p_i(v)
\]

In what follows, we will abuse the notation \( P \) and \( D \) as the feasible polytopes for both normal variables and slack variables for primal and dual, respectively.

**Complementary Slackness**

For any feasible primal solution \( \pi \in P \) and dual solution \( \delta \in D \), they are the optimal solution for primal and dual if and only if:

\[
a_i(v_i, v'_i; v_{-i}) \xi_i(v_i, v'_i; v_{-i}) = 0, \quad b_i(v) \eta_i(v) = 0, \quad c^j_i(v) \xi^j_i(v) = 0
\]

\[
\alpha_i^j(v) x^j_i(v) = 0, \quad \beta_i(v) p_i(v) = 0.
\]

In particular, the gap between primal and dual objectives equals the sum of all the products.

\[
\text{obj}(\delta) - \text{obj}(\pi) = \sum_{i,v,v'_i \neq v_i} a_i(v_i, v'_i; v_{-i}) \xi_i(v_i, v'_i; v_{-i}) + \sum_{i,v} b_i(v) \eta_i(v)
\]

\[
+ \sum_{j,v} c^j_i(v) \xi^j_i(v) + \sum_{i,j,v} \alpha_i^j(v) x^j_i(v) + \sum_{i,v} \beta_i(v) p_i(v).
\]

(DSCS)

We then focus on interpreting the complementary slackness conditions.

**3.1.2 Virtual Values**

For any fixed optimal solutions \( \pi^* \in P \) and \( \delta^* \in D \), since \( \alpha_i^j(v) x^j_i(v) = 0 \), we conclude that:

\[
x^j_i(v) > 0 \implies 0 = \alpha_i^j(v) x^j_i(v) = \xi^j_i(v) - \phi^j_i(v).
\]
Theorem 3.2 defines the virtual values in our setting as analog to common virtual values (e.g., Myerson’s virtual value).

\[ \forall i' \in [n], \ \phi_{i'}^{\ast j}(v) = \xi_{i'}^{\ast j}(v) \geq \phi_{i'}^{\ast j}(v), \]

where the inequality is the first constraint of the dual program \((\xi^{\ast j}(v) - \phi_{i'}^{\ast j}(v) \geq 0)\).

Moreover, by \(c^{\ast j}(v)\xi^{\ast j}(v) = 0\), we obtain:

\[ 1 - \sum_{i} x_{i}^{j} = c^{\ast j}(v) > 0 \implies 0 = \xi^{\ast j}(v) \geq \phi^{\ast j}_{i'}(v), \ \forall i' \in [n], \]

which implies that item \(j\) is not fully allocated only if for all buyer \(i'\), \(\phi^{\ast j}_{i'}(v)\) is not strictly positive.

**Definition 3.1 (Expected Virtual Values).** For any optimal dual solution \(\delta^{\ast}\), \(\phi_{i'}^{\ast j}(v)\) defines the expected virtual values in the sense that any optimal auction must maximize the expected virtual welfare.

Note that the expected virtual values are different from the virtual values commonly used in revenue maximization literatures. For example, virtual values are not always well-defined for some extreme cases (such as discrete/unbounded distributions and distributions with point masses), while the expected virtual values are always explicitly defined by a dual optimal solution. Later in **Theorem 3.2**, we will formally define the virtual values in our setting as analog to common virtual values (e.g., Myerson’s virtual value).

**Theorem 3.2 (Virtual Values).** For any optimal dual solution \(\delta^{\ast}\) satisfying certain regularization conditions (defined later), there exists corresponding virtual value functions \(\varphi_{i}^{j} : \mathcal{V} \to \mathbb{R} \cup \{-\infty\}\) such that

\[ \varphi_{i}^{j}(v)\mu(v) = \phi_{i}^{\ast j}(v). \]

In particular,

\[ \varphi_{i}^{j}(v) = \begin{cases} \phi_{i}^{\ast j}(v)/\mu(v), & \text{if } \mu(v) > 0; \\ \text{some real number in } \mathbb{R}, & \text{if } \mu_{i}(v_{-i}) = 0; \\ -\infty, & \text{if } v_{i} = 0, \ \mu_{i}(0) = 0, \ \mu_{-i}(v_{-i}) > 0. \end{cases} \]

Moreover, any optimal auction (optimal primal solution) \(\pi^{\ast}\) must be a virtual welfare maximizer:

1. Each item is only allocated to the buyer(s) with the highest and non-negative virtual value on this item;
2. The highest virtual value for any unallocated (or partially allocated) item must be non-positive (for partially allocated items, the highest virtual value must be zero).

The basic idea is to simply define \(\varphi_{i}^{j}(v) = \phi_{i}^{\ast j}(v)/\mu(v)\), while it only works when \(\mu(v) > 0\). For those \(v \in \mathcal{V}\) such that \(\mu(v) = 0\), the virtual value \(\varphi_{i}^{j}(v)\) can be defined only if \(\phi_{i}^{\ast j}(v) = 0\) as well.

Hence the first regularization condition is:

\[ \forall v \in \mathcal{V}, \ i \in [n], \ \mu_{-i}(v_{-i}) = 0 \implies \phi_{i}^{\ast j}(v) = 0. \]  \(1\)

Besides, we will require two more regularization conditions to simplify the discussion in upcoming sections, which are:

\[ \forall v \in \mathcal{V}, \ i \in [n], \ v_{i} \neq 0, \ \eta_{i}^{j}(v) = 0 \text{ and } \eta_{i}^{j}(0, v_{-i}) = \mu_{-i}(v_{-i}), \]  \(2\)

\[ \forall v \in \mathcal{V}, \ i \in [n], \ \beta_{i}^{j}(v) = \psi_{i}^{j}(v) - \mu(v) = 0. \]  \(3\)

By our previous interpretations on some of the complementary slackness conditions, we remain to prove the following lemma:
Lemma 3.3 (Regular Dual OPT). There always exists an optimal dual solution $\delta^*$ satisfying the regularization condition (1), (2), and (3).\footnote{We note that some of the optimal dual solution $\delta^*$ may not satisfy these regularization, but there always exist the ones satisfy them.}

**Proof of Theorem 3.2.** Directly implied by Lemma 3.3. \hfill \Box

Furthermore, by Lemma 3.3, we can reformulate $\phi^{*I}(v)$ as follows,

$$
\phi^{*I}(v) = \psi_i(v)v_i + \sum_{v_i'} \zeta_i(v_i', v_i; v_{-i})(v_i' - v_i') = \mu(v)v_i + \sum_{v_i'} \zeta_i(v_i', v_i; v_{-i})(v_i' - v_i')
$$

$$
\Rightarrow \varphi_i(v) = v_i + \sum_{v_i'} \zeta_i(v_i', v_i; v_{-i})(v_i' - v_i') / \mu(v).
$$

(DSVV)

### 3.2 Bayesian Implementation

#### 3.2.1 Duality and Complementary Slackness

Now, for Bayesian implementation, let $\tilde{\zeta}(v_i, v_i')$, $\tilde{\eta}(v_i)$, and $\tilde{\xi}(v)$ be the corresponding multipliers of the constraints. By reorganizing BLP into the standard form, we obtain,

**Primal Bayesian**

$$
\begin{align*}
\hat{x}_i(v), \bar{p}_i(v) & \geq 0 \quad \text{variables} \\
\max \quad & \sum_v \mu(v) \sum_i \bar{p}_i(v) \quad \text{objective} \\
\text{s.t.} \quad & \sum_{v_i} \mu_{-i}(v_{-i})(-v_i \cdot \bar{x}_i(v) + \bar{p}_i(v) + v_i \cdot \bar{x}_i(v_i', v_{-i}) - \bar{p}_i(v_i', v_{-i})) \leq 0 \quad \tilde{\zeta}_i(v_i, v_i') \\
& \sum_{v_i} \mu_{-i}(v_{-i})(-v_i \cdot x_i(v) + \bar{p}_i(v)) \leq 0 \quad \tilde{\eta}_i(v_i) \\
& \sum_i \hat{x}_i(v) \leq 1 \quad \tilde{\xi}(v)
\end{align*}
$$

Hence the dual program is

**Dual Bayesian**

$$
\begin{align*}
\tilde{\zeta}_i(v_i, v_i'), \tilde{\eta}_i(v_i), \tilde{\xi}(v) & \geq 0 \quad \text{variables} \\
\min \quad & \sum_v \sum_i \tilde{\xi}(v) \quad \text{objective} \\
\text{s.t.} \quad & \tilde{\xi}(v) - \mu_{-i}(v_{-i}) \left( \tilde{\eta}_i(v_i)v_i + \sum_{v_i'} \left( \tilde{\zeta}_i(v_i, v_i')v_i' - \tilde{\zeta}_i(v_i', v_i)v_i' \right) \right) \geq 0 \quad \hat{x}_i(v) \quad \text{(Dual BLP)} \\
& \mu_{-i}(v_{-i}) \left( \tilde{\eta}_i(v_i) + \sum_{v_i'} \left( \tilde{\zeta}_i(v_i, v_i') - \tilde{\zeta}_i(v_i', v_i) \right) \right) \geq \mu(v) \quad \bar{p}_i(v)
\end{align*}
$$
Similarly, we then use $\hat{P}$ to denote the polytope of all feasible solutions of the Bayesian primal linear program, $\hat{D}$ for the polytope of the Bayesian dual linear program, and the following abbreviations:

For primal:  
\[
\Delta \bar{u}_i(v_i, v'_i) = \sum_{v_{-i}} \mu_{-i}(v_{-i}) \left( v_i \cdot \bar{x}_i(v'_{-i}) - \bar{p}_i(v'_{-i}) - (v_i \cdot \bar{x}_i(v) + \bar{p}_i(v)) \right) \\
\bar{u}_i(v_i) = \sum_{v_{-i}} \mu_{-i}(v_{-i})(v_i \cdot \bar{x}_i(v) + \bar{p}_i(v)) \\
\bar{s}^i(v) = \sum_i \bar{x}_i(v)
\]

For dual:  
\[
\bar{\phi}_i(v_i) = \bar{\eta}_i(v_i) \bar{v}_i + \sum_{v'_i} \left( \bar{\xi}_i(v_i, v'_i) v'_i - \bar{\xi}_i(v'_i, v_i) \right) \\
\bar{\psi}_i(v_i) = \bar{\eta}_i(v_i) + \sum_{v'_i} \left( \bar{\xi}_i(v_i, v'_i) - \bar{\xi}_i(v'_i, v_i) \right)
\]

Clearly, both $\hat{P}$ and $\hat{D}$ are always nonempty, and $\hat{P}$ is bounded. Now, suppose $\bar{\pi}^* = \langle \bar{x}_i^*(v), \bar{\eta}_i^*(v) \rangle \in \hat{P}$ is an optimal solution of the primal, and $\bar{\delta}^* = \langle \bar{\xi}_i^*(v_i, v'_i), \bar{\xi}_i^*(v_i) \rangle \in \hat{D}$ is an optimal solution of the dual. By strong duality theorem, we know that $\text{obj}(\bar{\pi}^*) = \text{obj}(\bar{\delta}^*)$, which is the optimal revenue of any Bayesian implementation, denoted by $\text{BREV}$:

\[
\text{BREV} = \text{obj}(\bar{\pi}^*) = \sum_v \mu(v) \sum_i \bar{p}_i^*(v) = \text{obj}(\bar{\delta}^*) = \sum_v \sum_j \bar{\xi}_j^i(v).
\]

Again, we add slack variables to both primal and dual:

**Primal**  
\[
\begin{align*}
\bar{s}^i(v), \bar{p}_i(v) & \geq 0 & \text{variables} \\
\bar{a}_i(v_i, v'_i), \bar{b}_i(v_i), \bar{c}^i(v) & \geq 0 & \text{slack variables} \\
\text{max} & \sum_v \mu(v) \sum_i \bar{p}_i(v) & \text{objective} \\
\text{s.t.} & \Delta \bar{u}_i(v_i, v'_i) + \bar{a}_i(v_i, v'_i) = 0 & \bar{\xi}_i(v_i, v'_i) \\
& - \bar{u}_i(v_i) + \bar{b}_i(v_i) = 0 & \bar{\eta}_i(v_i) \\
& \bar{s}^i(v) + \bar{c}^i(v) = 1 & \bar{\xi}_i^i(v)
\end{align*}
\]

**Dual**  
\[
\begin{align*}
\bar{\xi}_i(v_i, v'_i), \bar{\eta}_i(v_i), \bar{\xi}_i^i(v) & \geq 0 & \text{variables} \\
\bar{\alpha}_i^i(v), \bar{\beta}_i(v) & \geq 0 & \text{slack variables} \\
\text{min} & \sum_v \sum_j \bar{\xi}_j^i(v) & \text{objective} \\
\text{s.t.} & \bar{\xi}_i^i(v) - \mu_{-i}(v_{-i}) \bar{\phi}_i(v_i) - \bar{\alpha}_i^i(v) = 0 & \bar{\xi}_i^i(v) \\
& \mu_{-i}(v_{-i}) \bar{\psi}_i(v_i) - \bar{\beta}_i(v) = \mu(v) & \bar{p}_i(v)
\end{align*}
\]

As we did for dominant-strategy implementation, we will abuse the notation $\hat{P}$ and $\hat{D}$ as the feasible polytopes for both normal variables and slack variables of primal and dual, respectively.

**Complementary Slackness**  
For any feasible primal solution $\bar{\pi} \in \hat{P}$ and dual solution $\bar{\delta} \in \hat{D}$, they are the optimal solution for primal and dual if and only if:

\[
\begin{align*}
\bar{a}_i(v_i, v'_i) \bar{\xi}_i(v_i, v'_i) = 0, & \quad \bar{b}_i(v_i) \bar{\eta}_i(v_i) = 0, \quad \bar{c}^i(v) \bar{\xi}_i^i(v) = 0 \quad \bar{\alpha}_i^i(v) \bar{\xi}_i^i(v) = 0, \quad \bar{\beta}_i(v) \bar{p}_i(v) = 0.
\end{align*}
\]
In particular, the gap between primal and dual objectives equals the sum of all the products.

\[
\text{obj}(\delta) - \text{obj}(\bar{\pi}) = \sum_{i,v_i \neq \bar{v}_i} \bar{a}_i(v_i, v'_i) \bar{c}_i(v_i, v'_i) + \sum_{i,v_i} \bar{b}_i(v_i) \bar{\eta}_i(v_i)
\]

\[
+ \sum_{j,v} \bar{c}^j(v) \bar{c}^j(v) + \sum_{i,j,v} \bar{a}^j_i(v) \bar{c}^j_i(v) + \sum_{i,v} \bar{b}^j_i(v) \bar{\eta}_i(v).
\]  

(BCS)

3.2.2 (Bayesian) Virtual Values

We then repeat the interpreting of the complementary slackness conditions as we did for the dominant-strategy implementation. In particular, we can conclude that \( \mu_{-i}(v_{-i}) \delta^* / \psi^*_i(v) \) is the expected virtual value in Bayesian setting:

**Definition 3.4 (Expected (Bayesian) Virtual Values).** For any optimal dual solution \( \delta^* \), \( \mu_{-i}(v_{-i}) \delta^* / \psi^*_i(v) \) defines the expected (Bayesian) virtual values in the sense that any optimal auction must maximize the expected virtual welfare.

Similarly, we can define virtual values for Bayesian implementations.

**Theorem 3.5 ((Bayesian) Virtual Values).** For any optimal dual solution \( \delta^* \) satisfying certain regularization conditions (defined later), there exists corresponding virtual value functions \( \bar{\varphi}^j_i(v_i) : \mathcal{V}_i \rightarrow \mathbb{R} \cup \{-\infty\} \) such that

\[
\bar{\varphi}^j_i(v_i) \mu(v) = \phi^j_i(v_i) \mu_{-i}(v_{-i}).
\]

In particular,

\[
\bar{\varphi}^j_i(v_i) = \begin{cases} 
\phi^j_i(v_i) / \mu_i(v_i), & \text{if } \mu(v) > 0; \\
\text{some real number in } \mathbb{R}, & \text{if } \mu_{-i}(v_{-i}) = 0; \\
-\infty, & \text{if } v_i = 0, \mu_i(0) = 0, \mu_{-i}(v_{-i}) > 0.
\end{cases}
\]

Moreover, any optimal auction (optimal primal solution) \( \bar{\pi}^* \) must be a virtual welfare maximizer:

1. Each item is only allocated to the buyer(s) with the highest and non-negative virtual value on this item;
2. The highest virtual value for any unallocated (or partially allocated) item must be non-positive (for partially allocated items, the highest virtual value must be zero).

The corresponding regularization conditions are as follows:

\[
\forall v \in \mathcal{V}, i \in [n], \mu_{-i}(v_{-i}) = 0 \implies \phi^j_i(v_i) = 0 \tag{4}
\]

\[
\forall v \in \mathcal{V}, i \in [n], v_i \neq 0, \bar{\eta}^+_i(v_i) = 0 \text{ and } \bar{\eta}^+_i(0) = 1, \tag{5}
\]

\[
\forall v \in \mathcal{V}, i \in [n], \bar{\bar{\beta}}^j_i(v) = \mu_{-i}(v_{-i}) \bar{\phi}^j_i(v_i) - \mu(v) = 0. \tag{6}
\]

By our previous interpretations on some of the complementary slackness conditions, we remain to prove the following lemma:

**Lemma 3.6 (Regular Dual OPT).** There always exists an optimal dual solution \( \delta^* \) satisfying the regularization condition (4), (5), and (6).\(^5\)

\(^5\)We note that some of the optimal dual solution \( \delta^* \) may not satisfy these regularization, but there always exist the ones satisfy them.
Proof of Theorem 3.5. Directly implied by Lemma 3.6.

Furthermore, by Lemma 3.6, we can reformulate $\hat{\phi}^i_j(v_i)$ as follows,

$$\hat{\phi}^i_j(v_i) = \hat{\psi}^i_j(v_i) v_i^j + \sum_{v'_i} \hat{\xi}^i(v_i, v_i)(v'_i - v^j_i) = \mu_i(v_i) v_i^j + \sum_{v'_i} \hat{\xi}^i(v_i, v_i)(v'_i - v^j_i)$$

$$\implies \varphi^i_j(v_i) = v_i^j + \sum_{v'_i} \tilde{\xi}^i(v_i, v_i)(v'_i - v^j_i) / \mu_i(v_i). \quad \text{(BVV)}$$

\section{Characterization}

In this section, we present the sufficient and necessary characterization of BIC = DSIC.

\subsection{A sufficient and necessary condition of BIC = DSIC}

In previous sections, we defined two types of virtual values, i.e., dominant-strategy virtual values (DSVV) and Bayesian virtual values (BVV). In particular, the Bayesian virtual values for buyer $i$, $\varphi^i_i(v_i)$, are independent of the values of other buyers by the construction, while the dominant-strategy virtual values for buyer $i$, $\varphi^i_i(v_i)$, depend on the values of other buyers as well.

We say the (regular) dominant-strategy virtual values are \textit{agent-independent}, if for each buyer $i$, her virtual values (and related dual variables, $\eta_i(v)$ and $\zeta_i(v_i, v'_i; v_{-i})$) are independent of the values of other buyers:

$$\forall v_i \in V_i, v_{-i}, v'_{-i} \in V_{-i}, \quad \eta_i(v_i, v_{-i}) = \eta_i(v_i, v'_{-i}), \quad \varphi_i(v_i, v_{-i}) = \varphi_i(v_i, v'_{-i}), \quad \zeta_i(v_i, v'_i; v_{-i}) = \zeta_i(v_i, v'_i; v'_{-i}). \quad \text{(AI)}$$

Then our first main result is the following characterization:

\textbf{Theorem 4.1.} \textbf{BIC = DSIC if and only if that there is an optimal DSIC auction that is induced by agent-independent virtual values.}

Proof of Theorem 4.1. \textit{``$\implies$''}:

One key observation is that any solution of Dual BLP induces a solution of Dual DSLP. In particular, let $\hat{\delta}^* = (\hat{\xi}^i_j(v'_i, v_i), \hat{\eta}^i_i(v_i), \hat{\xi}^j_j(v)) \in \hat{\mathcal{D}}$ be an optimal solution to Dual BLP. The following $\hat{\delta}$ constructed from $\hat{\delta}^*$ is a feasible solution to Dual DSLP:

$$\hat{\delta} = (\hat{\xi}^i_j(v_i, v'_i; v_{-i}), \hat{\eta}^i_i(v), \hat{\xi}^j_j(v))$$

$$\hat{\xi}^i_j(v_i, v'_i; v_{-i}) = \hat{\xi}^i_j(v'_i, v_i) \mu_{-i}(v_{-i})$$

$$\hat{\eta}^i_i(v) = \hat{\eta}^i_i(v_i) \mu_{-i}(v_{-i})$$

$$\hat{\xi}^j_j(v) = \hat{\xi}^j_j(v).$$

We omit the verification of $\hat{\delta} \in \mathcal{D}$, which is directly implied by the definition of Dual BLP and Dual DSLP (as well as the fact that $\mu_{-i}(v_{-i}) \geq 0$).

In the meanwhile, note that the objective value of $\hat{\delta}^*$ in Dual BLP is the same as the objective value of $\hat{\delta}$ in Dual DSLP, we conclude that:

$$\text{obj}(\hat{\delta}^*) = \text{obj}(\hat{\delta}) \geq \text{obj}(\hat{\delta}^*),$$

where $\delta^*$ is an optimal solution of Dual DSLP and the last inequality is from the optimality of $\hat{\delta}^*$. 

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On the other hand, by the hypothesis that BIC = DSIC, i.e., \( \text{obj}(\delta^\ast) = \text{obj}(\delta) \), the constructed solution \( \delta^\ast \) is in fact an optimal solution to Dual DSLP.

Since \( \delta^\ast \) is an arbitrary optimal solution to Dual BLP, we can further assume that it is regular (Lemma 3.6). The corresponding \( \hat{\delta}^\ast \) then is also regular according to the definition in Lemma 3.3 (we omit the verification here, which is straightforward by the definitions). Therefore, \( \hat{\delta} \) defines the virtual values for an optimal DSIC auction (DSVV):

\[
\hat{\psi}_i^j(v) = v_i^j + \sum_{v_i'} \hat{\xi}_i(v_i', v_i; v_{-i})(v_i^j - v_i'^j)/\mu_i(v_i).
\]

In particular, the virtual values are the same as (BVV) and are agent-independent. \( (\hat{\eta}_i(v)/\mu_i(v_{-i}) \) and \( \hat{\zeta}_i(v_i, v_i'; v_{-i})/\mu_i(v_{-i}) \) are also invariant in \( v_{-i} \).

“\( \Leftarrow \)”:

By the hypothesis that there exist agent-independent virtual values \( \delta^\ast = (\zeta^\ast_i(v_i', v_i; v_{-i}), \eta^\ast_i(v), \xi^\ast_j(v)) \) inducing an optimal DSIC auction, we can construct the following \( \hat{\delta} \), which is a feasible solution to Dual BLP:

\[
\hat{\delta} = (\hat{\zeta}_i(v_i', v_i; v_{-i}), \hat{\eta}_i(v), \hat{\xi}_i^j(v)) \quad \hat{\zeta}_i(v_i', v_i) = \zeta^\ast_i(v_i', v_i; v_{-i})/\mu_i(v_{-i}) \quad \hat{\eta}_i(v_i) = \eta^\ast_i(v)/\mu_i(v_{-i}) \quad \hat{\xi}_i^j(v) = \xi^\ast_i^j(v).
\]

Note that according to the definition of agent-independence (AI), the construction of \( \hat{\delta} \) is consistent for all \( v_{-i} \). In particular, if \( \mu_i(v_{-i}) = 0 \), for all \( v_i' \in V_i \), by (AI),

\[
\zeta^\ast_i(v_i', v_i; v_{-i}) = \xi^\ast_i(v_i', v_i; v_{-i}) = 0.
\]

Then \( \zeta^\ast_i(v_i', v_i; v_{-i}) \) must be zero, as there exists \( v'_{-i} \) such that \( \mu_i(v'_{-i}) > 0 \). Hence in such special cases, we can safely define \( \hat{\zeta}_i(v_i', v_i) = 0 \) and similarly \( \hat{\eta}_i(v_i) = 0 \).

In the meanwhile, such a construction ensures that (i) its objective value in BLP being the same as the objective value of \( \delta^\ast \) in DSLP

\[
\text{obj}(\hat{\delta}) = \text{obj}(\delta^\ast),
\]

and (ii) \( \hat{\delta} \) is also a feasible solution to Dual BLP (we omit the further verification here, which is straightforward by the definitions)

\[
\hat{\zeta}_i(v_i', v_i)\mu_i(v_{-i}) = \zeta^\ast_i(v_i', v_i; v_{-i}) \quad \hat{\eta}_i(v_i)\mu_i(v_{-i}) = \eta^\ast_i(v).
\]

Therefore, we conclude that DSIC = BIC:

\[
\text{obj}(\hat{\delta}) \geq \text{obj}(\delta^\ast) \geq \text{obj}(\delta) = \text{obj}(\hat{\delta}),
\]

where \( \delta^\ast \) is any optimal solution to Dual BLP and the last inequality is due to the fact that BIC \( \geq \) DSIC. \( \Box \)
5 The I.I.D. Setting

In this section, we further show that if the value distributions are i.i.d. and 0 is in the supports, then the previous characterization implies that $\text{DREV} = \text{SRev}$. In the meanwhile, since separate selling employees agent-independent virtual values, $\text{DREV} = \text{SRev}$ directly implies that $\text{BREV} = \text{DREV} = \text{SRev}$. In other words, although $\text{BREV} \geq \text{DREV} \geq \text{SRev}$ in general, any two of them being equal implies that all of them are equal:

**Corollary 5.1.** For $n \geq 3$,

$$\text{BREV} = \text{DREV} \quad \text{or} \quad \text{DREV} = \text{SRev} \quad \text{or} \quad \text{SRev} = \text{BREV} \quad \implies \quad \text{BREV} = \text{DREV} = \text{SRev}.$$ 

In fact, we have the following theorem:

**Theorem 5.2.** In the i.i.d. value setting with $n \geq 3$, if the optimal DSIC auction is induced by agent-independent virtual values, there exist item-independent (and agent-independent) virtual values inducing an optimal DSIC auction.

The agent-independent virtual values are called *item-independent*, if

$$\forall v_i, v'_i \in V_i, \quad \varphi^j_i(v_i, v'_{i \setminus j}) = \varphi^j_i(v_i, v'_{i \setminus j}) \quad \text{or} \quad \varphi^j_i(v_i, v'_{i \setminus j}), \varphi^j_i(v_i, v'_{i \setminus j}) \leq 0.$$ 

In particular, **Corollary 5.1** directly follows from **Theorem 5.2**:

- if virtual value $\varphi^j_i$ is restricted to depending on $v'_i$ only, separate selling via Myerson’s auction would be the optimal (hence $\text{BREV} = \text{DREV} \implies \text{BREV} = \text{DREV} = \text{SRev}$);
- if $\text{DREV} = \text{SRev}$, then the optimal DSIC auction can be induced by agent- and item-independent virtual values, implying DSIC = BIC (hence $\text{DREV} = \text{SRev} \implies \text{BREV} = \text{DREV} = \text{SRev}$);
- if $\text{SRev} = \text{BREV}$, then by $\text{BREV} \geq \text{DREV} \geq \text{SRev}$, all of them must be equal (hence $\text{SRev} = \text{BREV} \implies \text{BREV} = \text{DREV} = \text{SRev}$).

We then move to the proof of **Theorem 5.2**, which relies on the following lemma:

**Lemma 5.3** (Upper Bounded Virtual Values). $\varphi^j_i \leq v'_i$.

*Proof of Theorem 5.2.* We prove by contradiction. Assume that the virtual values are not item-independent. Note that the valuations are i.i.d., hence, without loss of generality, we assume that the virtual values for the agents are the same and the allocations are symmetric. Hence we also omit the subscripts of virtual values through the proof.

In particular, let $v^j$ denote the maximum value of the $j$-th item in the support and $v'_0$ denote the value profile with all maximum value except that the value of the $j$-th item being 0:

$$v'_0 = (v^1, \ldots, v^{j-1}, 0, v^{j+1}, \ldots, v^m).$$

Consider $v_i, v'_i$ and $j$ such that $\varphi^j_i(v'_i, v'_{i \setminus j})$ and $\varphi^j_i(v'_i, v'_{i \setminus j})$ are different, i.e.,

$$\varphi^j_i(v'_i, v'_{i \setminus j}) < \varphi^j_i(v'_i, v'_{i \setminus j}) \quad \text{and} \quad \varphi^j_i(v'_i, v'_{i \setminus j}) > 0. \quad (7)$$

Let $v = v_i$ and $v'_{(-j)} = (v'_i, v'_{i \setminus j})$. Pick an arbitrary buyer $i' \neq i$ and fix her values being $v'_{i'}$. Then fix the values of all the remaining buyers (except for $i$ and $i'$) being $v$, i.e.,

$$v_{i'} = v, \forall i' \neq i, i'.$$
Note that the virtual values of buyer $i'$ are already determined. According to the definition of agent-independent virtual values:

$$\phi^j_i(v_i) = v^j_i + \sum_{v_i'} \zeta_i^+(v_i', v_i)(v_i' - v^j_i)/\mu_i(v_i),$$

we have that $\phi^j_i(v^0_i) \leq 0$ and $\phi^j_i(v_i') \geq \bar{v}^j$ for $j' \neq j$. Combining with Lemma 5.3, we further have that $\phi^j_i(v^0_i) = \bar{v}^j$ for $j' \neq j$.

Then consider the two cases where $v_i$ is either $v$ or $v^{(-j)}$.

- In both cases, the allocations of item $j' \neq j$ won’t change, because either item $j'$ is always allocated to buyer $i'$ or always allocated uniformly at random.
- $v_i = v$. In this case, all the buyers except $i'$ have the same value and hence the same allocation. In particular, they will get $1/(n - 1)$ of item $j$.
- $v_i = v^{(-j)}$. In this case, according to the assumption (7), buyer $i$ has the highest (positive) virtual value on item $j$ and will be allocated the entire item. To ensure that buyer $i$ in this case won’t have incentive to misreport her values as $v$, she will be charged $v^j_i(n - 2)/(n - 1)$ for the extra $(n - 2)/(n - 1)$ fraction of item $j$ comparing with the previous case.

Given the previous analysis, if buyer $i$ has any value $v'$ with $v^j_i > v^j_i$, misreporting her value as $v$ is strictly dominated by misreporting as $v^{(-j)}$. Due to the complementary slackness condition (DSCS),

$$\zeta^+(v', v) = 0,$$

which implies that

$$\phi^j_i(v) \geq v^j_i.$$

By the assumption (7) and Lemma 5.3, we get a contradiction:

$$v^j_i \leq \phi^j_i(v) < \phi^j_i(v^{(-j)}) \leq v^j_i.$$

\[\square\]

6 Missing Proofs

Proof of Lemma 3.3. For condition (1), if $\mu_{-i}(v_{-i}) = 0$, then $\mu(v) = 0$, and we can simply let

$$\forall v_i, v'_i \in V_i, \ z^+_i(v_i, v'_i; v_{-i}) = \eta^+_i(v) = 0.$$

By doing so, $\delta^*$ is still a feasible solution to the dual program, and the objective value does not change. Hence the optimality is preserved. Moreover,

$$\phi^+_i(v) = \eta^+_i(v)v^j_i + \sum_{v'_i} \left( \zeta^+_i(v_i, v'_i; v_{-i})v^j_i - \zeta^+_i(v'_i, v_i; v_{-i})v^j_i \right) = 0,$$

as desired by (1).

In addition, when $v_i = 0$, $\mu_i(0) = 0$, and $\mu_{-i}(v_{-i}) > 0$,

$$\phi^+_i(v) = \eta^+_i(v)v^j_i + \sum_{v'_i} \left( \zeta^+_i(v_i, v'_i; v_{-i})v^j_i - \zeta^+_i(v'_i, v_i; v_{-i})v^j_i \right) = -\sum_{v'_i} \zeta^+_i(v'_i, v_i; v_{-i})v^j_i \leq 0.$$

For condition (2) and (3), we do the following changes:
\[\forall v \in V, i \in [n], v_i \neq 0,\]
\[
\hat{\eta}^*_i(v) = 0 \quad \text{and} \quad \hat{\eta}^*_i(0, v_{-i}) = \mu_{-i}(v_{-i}),
\]
\[
\hat{\zeta}^*_i(v_i, 0; v_{-i}) = \zeta^*_i(v_i, 0; v_{-i}) + \eta^*_i(v)
\quad \text{and} \quad \hat{\zeta}^*_i(0, v_i; v_{-i}) = \zeta^*_i(0, v_i; v_{-i}) + \beta^*_i(v).
\]

- All others remain the same.

Then we verify one by one that (i) the constructed \(\tilde{\delta}^*\) is still feasible (\(\tilde{\delta}^* \in \mathcal{D}\)), (ii) regularization condition (2) and (3) are satisfied, and (iii) the objective remains the same (\(\text{obj}(\tilde{\delta}^*) = \text{obj}(\delta^*)\)).

1. \(\tilde{\delta}^* \in \mathcal{D}\): clearly, all the variables in \(\tilde{\delta}^*\) are still non-negative.

Then we show that for \(v_i \neq 0\), \(\tilde{\phi}^j_i(v) = \phi^j_i(v)\):
\[
\tilde{\phi}^j_i(v) = \hat{\eta}^*_i(v)v^j_i + \sum_{v' \neq 0} \left( \hat{\zeta}^*_i(v_i, v'_i; v_{-i})v^j_i - \hat{\zeta}^*_i(v'_i, v_i; v_{-i})v^j_i \right)
\]
\[
= (\hat{\eta}^*_i(v) + \hat{\zeta}^*_i(v_i, 0; v_{-i}))v^j_i + \sum_{v' \neq 0} \left( \hat{\zeta}^*_i(v_i, v'_i; v_{-i})v^j_i - \hat{\zeta}^*_i(v'_i, v_i; v_{-i})v^j_i \right) - \hat{\zeta}^*_i(0, v_i; v_{-i})0
\]
\[
= (\hat{\eta}^*_i(v) + \hat{\zeta}^*_i(v_i, 0; v_{-i}))v^j_i + \sum_{v' \neq 0} \left( \hat{\zeta}^*_i(v'_i, v_i; v_{-i})v^j_i - \hat{\zeta}^*_i(v'_i, v_i; v_{-i})v^j_i \right) - \hat{\zeta}^*_i(0, v_i; v_{-i})0
\]
\[
= \phi^j_i(v); \quad \tilde{\phi}^j_i(0, v_{-i}) \leq \phi^j_i(0, v_{-i});
\]

for \(v_i \neq 0\), \(\tilde{\psi}^*_i(v) = \mu(v)\):
\[
\tilde{\psi}^*_i(v) = \hat{\eta}^*_i(v) + \sum_{v' \neq 0} \left( \hat{\zeta}^*_i(v_i, v'_i; v_{-i}) - \hat{\zeta}^*_i(v'_i, v_i; v_{-i}) \right)
\]
\[
= \hat{\eta}^*_i(v) + \hat{\zeta}^*_i(v_i, 0; v_{-i}) + \sum_{v' \neq 0} \left( \hat{\zeta}^*_i(v_i, v'_i; v_{-i}) - \hat{\zeta}^*_i(v'_i, v_i; v_{-i}) \right) - \hat{\zeta}^*_i(0, v_i; v_{-i})
\]
\[
= \hat{\eta}^*_i(v) + \hat{\zeta}^*_i(v_i, 0; v_{-i}) + \sum_{v' \neq 0} \left( \hat{\zeta}^*_i(v'_i, v_i; v_{-i}) - \hat{\zeta}^*_i(v'_i, v_i; v_{-i}) \right) - \hat{\zeta}^*_i(0, v_i; v_{-i}) - \beta^*_i(v)
\]
\[
= \psi^*_i(v) - \beta^*_i(v) = \mu(v);
\]
and finally if $\mu_{-i}(v_{-i}) = 0$, by previous constructions, $\tilde{\psi}^*_{i}(0, v_{-i}) = 0 = \mu_{-i}(v_{-i})$; otherwise,

$$\tilde{\psi}^*_{i}(0, v_{-i}) = \tilde{\eta}^*_{i}(0, v_{-i}) + \sum_{v'_i} \left( \tilde{\zeta}^*_{i}(0, v'_i; v_{-i}) - \tilde{\zeta}^*_{i}(v'_i, 0; v_{-i}) \right)$$

$$= \mu_{-i}(v_{-i}) + \sum_{v'_i} \left( \xi^*_{i}(0, v'_i; v_{-i}) + \beta^*_{i}(v'_i, v_{-i}) - \xi^*_{i}(v'_i, 0; v_{-i}) - \eta^*_{i}(v'_i, v_{-i}) \right)$$

$$= \mu_{-i}(v_{-i}) + \sum_{v'_i} \left( \xi^*_{i}(0, v'_i; v_{-i}) - \xi^*_{i}(v'_i, 0; v_{-i}) \right)$$

$$+ \sum_{v_i \neq 0} \left( \sum_{v'_i} \left( \xi^*_{i}(v_i, v'_i; v_{-i}) - \xi^*_{i}(v'_i, v_i; v_{-i}) - \mu(v) \right) \right)$$

$$= \mu_{-i}(v_{-i}) - \sum_{v_i \neq 0} \mu(v) = \mu(0, v_{-i}).$$

2. Regularization condition (2): directly implied by construction, and (3): implied by $\tilde{\psi}^*_{i}(v) = \mu(v)$, which is proved above.

3. The objective remains the same because we did not change variables $\xi^{*j}(v)$ at all.

Proof of Lemma 3.6. Similar to the proof of Lemma 3.3, omitted.

Proof of Lemma 5.3. Omitted.

7 Interpreta tions for the Duals

In this section, we provided some interpretations of the dual problems of the revenue maximization problem under the dominant-strategy implementation and the Bayesian implementation. In particular, they can be thought as an extended version of the interpretation by Daskalakis et al. [2013, 2015].

To be added.

8 Recover Myerson’s Result for the Single-Item Setting

To be added.

9 Future Work

We plan to extend our results for Section 5 to either (i) independent but non-identical cases, or (ii) continuous distribution cases.

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