Superspace Effective Actions for 4D Compactifications of Heterotic and Type II Superstrings

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Abstract

Two-dimensional sigma models are defined for the new manifestly spacetime supersymmetric description of four-dimensional compactified superstrings. The resulting target-superspace effective action is constrained by the way the spacetime dilaton couples to the worldsheet curvature: For the heterotic superstring, the worldsheet curvature couples to the real part of a chiral multiplet, and for Type II it couples to the real part of the sum of a vector multiplet and a tensor hypermultiplet.

For the Type II superstring, this contradicts the standard folklore that only a hypermultiplet counts string-loops, explains the peculiar dilaton coupling of Ramond-Ramond fields, and allows the effective action to be easily written in N=2 4D superspace. It also implies that vector multiplet interactions get no quantum corrections, while hypermultiplet interactions can only get corrections if mirror symmetry is non-perturbatively broken.
1. Introduction

There are two ways to construct low-energy effective actions in string theory. One can define a two-dimensional sigma model for the string in a curved background, use conformal invariance to determine the equations of motion, and look for an action which provides these equations of motion\[1\]. Alternatively, one can calculate on-shell S-matrix scattering amplitudes and look for an action which yields these amplitudes\[2\].

For the superstring in the RNS formalism, both of these methods are made clumsy by the complicated nature of the Ramond fields. For this reason, the part of the effective action coming from the Ramond fields is much less understood than the part coming from the Neveu-Schwarz sector. In light of recent conjectures relating non-perturbative states with the Ramond-Ramond sector of the type II superstring\[3\], this lack of understanding is especially bothersome. For example, the $F^2$ term for the graviphoton field strength appears to be independent of the dilaton $\varphi$, instead of having the expected $e^{-2\varphi}$ dependence of tree-level terms\[4\].

Recently, a new formalism for the superstring has been discovered with local N=2 worldsheet superconformal invariance\[5\]. This formalism is related to the RNS formalism by a field-redefinition\[6\], but has the advantage of being manifestly spacetime supersymmetric. It is especially well-suited for compactifications to four dimensions, where it allows manifestly 4D super-Poincaré invariant quantization\[7\].

In this paper, we use this new formalism to define a two-dimensional sigma model and construct a superspace effective action for 4D compactifications of heterotic and Type II superstrings. While the heterotic effective action in N=1 4D superspace has already appeared in the literature\[8\], the Type II effective action in N=2 4D superspace is new. Since spacetime-supersymmetry is manifest, there is no distinction between Ramond and Neveu-Schwarz fields and the previous confusion over the Ramond sector is easily resolved. Furthermore, the requirement that the Fradkin-Tseytlin term for dilaton coupling contains N=2 worldsheet supersymmetry implies certain non-renormalization theorems for the effective action. For those readers who are only interested in the new Type II superspace effective action and the resulting non-renormalization theorems, it may be useful to just read section 3 and section 5, and skip the derivation from the sigma model.
In section 2, we review the new manifestly spacetime supersymmetric formalism for the superstring. This formalism has critical $N=2$ worldsheet superconformal invariance, and for compactifications to four dimensions, superspace chirality is related to worldsheet chirality\cite{7}. Superstring scattering amplitudes can be computed by calculating correlation functions of BRST-invariant vertex operators on $N=2$ super-Riemann surfaces\cite{8}, and the amplitudes agree with those obtained using the RNS formalism. Since the massless vertex operators are related to the linearized interactions of the sigma model, these vertex operators will be reviewed.

The massless vertex operators are manifestly spacetime supersymmetric and are constructed from prepotentials of the spacetime superfields. For the heterotic superstring, these prepotentials describe $N=1$ conformal supergravity, a tensor multiplet, super-Yang-Mills multiplets, and chiral scalar multiplets which come from compactification moduli. (We assume throughout this paper that the compactification manifold has no isometries, so all moduli are described by four-dimensional scalars.) For the Type II superstring, the prepotentials describe $N=2$ conformal supergravity, a tensor hypermultiplet, and chiral or twisted-chiral multiplets from the compactification moduli. The superspace chirality of the compactification moduli superfields comes from the worldsheet chirality or twisted-chirality of the relevant $N=(2,2)$ primary fields. As will be discussed, Type II chiral multiplets can be interpreted as vector multiplets\cite{9} while twisted-chiral multiplets can be interpreted as tensor hypermultiplets\cite{10}. Note that tensor hypermultiplets contain the same on-shell component fields as scalar hypermultiplets, which are more commonly used to describe Type II compactification moduli.

Unlike the massless vertex operators, sigma models and effective actions are constructed from superspace gauge fields and field strengths rather than prepotentials. In section 3, we will review the torsion constraints which relate these gauge fields and field strengths to their prepotentials. To analyze these torsion constraints and to facilitate the construction of superspace actions, it will be useful to introduce conformal compensators. (In the bosonic string, the spacetime dilaton which couples to worldsheet curvature plays the role of a conformal compensator\cite{11}.) Although there are various types of compensators one can introduce for $N=1$ and $N=2$ supergravity, worldsheet supersymmetry of the
Fradkin-Tseytlin term in the sigma model uniquely determines the correct type. As discussed in reference [12], the correct type of compensator can also be determined by requiring factorization of closed superstring states involving worldsheet ghosts.

For the heterotic superstring, the compensator must be a chiral scalar multiplet, which identifies the off-shell theory as matter coupled to old minimal supergravity. For the Type II superstring, two compensators are required, one which is chiral (a vector multiplet) and one which is twisted-chiral (a tensor hypermultiplet). Normally the compensators are gauge-fixed to remove the unphysical spacetime invariances. However, in the superstring sigma model, spacetime conformal invariance is instead removed by gauge-fixing the non-compensator tensor multiplet. (This is similar to the bosonic string sigma model, where the physical scalar is gauge-fixed instead of the dilaton.) For the heterotic superstring, this leaves a spacetime $U(1)$ invariance (which is related to worldsheet $U(1)$ invariance), while for the Type II superstring, this leaves a spacetime $U(1) \times U(1)$ invariance (which is related to worldsheet $U(1) \times U(1)$ invariance).

In section 4, we use the above spacetime superfields to explicitly construct a sigma model for four-dimensional compactifications of heterotic and Type II superstrings. The sigma model for the heterotic superstring was constructed with the assistance of Jan de Boer, Peter van Nieuwenhuizen, Martin Roček, Ergin Sezgin, Kostas Skenderis, and Kellogg Stelle. Like the standard 4D GS sigma model[13], this sigma model contains a term proportional to $1/\alpha'$ where the superfields couple to their massless vertex operators. In the presence of torsion constraints (which do not put the superfields on-shell), this $1/\alpha'$ term is invariant under classical worldsheet superconformal transformations.

However, unlike the standard GS sigma model, this sigma model also contains a Fradkin-Tseytlin term where the spacetime compensators couple to worldsheet supercurvature. For the heterotic superstring, the spacetime chiral compensator couples to $N=(2,0)$ supercurvature, which is described by a worldsheet chiral superfield[14]. For the Type II superstring, the spacetime chiral and twisted-chiral compensators couple to $N=(2,2)$ supercurvature, which is described by a worldsheet chiral and twisted-chiral superfield. Quantum $N=2$ superconformal invariance of the combined $1/\alpha'$ and Fradkin-Tseytlin terms is expected to imply
the equations of motion for the spacetime superfields. This is currently being checked for the heterotic superstring by de Boer and Skenderis.

Finally in section 5, we use some simple properties of the sigma model to construct superspace effective actions. (For those readers only interested in effective actions, they can skip directly to this section, using section 3 as a reference.) Although the heterotic superspace effective action has already appeared in the literature\cite{8}, the Type II superspace effective action is new. This N=2 4D superspace action includes a chiral term for vector multiplet interactions and a twisted-chiral term for tensor hypermultiplet interactions. The twisted-chiral term is made SU(2) invariant by introducing harmonic-like variables\cite{15} \cite{16}. Since worldsheet Euler number couples in the Type II sigma model to the sum of the vector and tensor compensators, it is straightforward to determine the string-loop order of each term in the effective action. This type of dilaton coupling contradicts the standard folklore that loops are counted by just a hypermultiplet\cite{17}, and explains the dilaton coupling of Ramond-Ramond fields. (By dilaton, we always mean the field which couples to worldsheet curvature, and not the physical scalar which couples like the trace of the metric. The confusion in the literature was caused by the fact that the physical scalar, which does not count string loops, sits in a hypermultiplet.)

We then prove various non-renormalization theorems for the Type II effective action, including the theorem that the chiral term for vector multiplet interactions receives no quantum perturbative or non-perturbative corrections. Since mirror symmetry of the sigma model relates chiral and twisted-chiral terms in the effective action, hypermultiplet interactions can only receive quantum corrections if mirror symmetry were broken. This would seem to contradict Type II/heterotic string-duality conjectures, which require that Type II hypermultiplet interactions receive quantum corrections\cite{17}. However, the non-renormalization of hypermultiplet interactions is related to a Pecci-Quinn-like symmetry. If this Pecci-Quinn-like symmetry were non-perturbatively broken by spacetime instantons (which would imply the non-perturbative breaking of mirror symmetry), hypermultiplet interactions could receive quantum corrections.

In section 6, we summarize our results and discuss possible generalizations of this work for the superstring in more than four dimensions.
2. Review of the New Superstring Description

By twisting the ghost sector, any critical N=1 string can be “embedded” in a critical N=2 string without changing the physical theory\[18\]. After performing this embedding for the critical RNS superstring, a field-redefinition allows the resulting N=2 string to be made manifestly spacetime supersymmetric. This N=2 description of the superstring is especially elegant for compactifications to four dimensions, where the critical $c = 6$ matter sector splits into a $c = -3$ four-dimensional part and a $c = 9$ compactification-dependent part\[7\].

The four-dimensional part of the matter sector contains the spacetime variables, $x^m$ ($m = 0$ to $3$), the left-moving fermionic variables, $\theta^\alpha$ and $\bar{\theta}^{\dot{\alpha}}$ ($\alpha, \dot{\alpha} = 1$ to $2$), the conjugate left-moving fermionic variables, $p_\alpha$ and $\bar{p}_{\dot{\alpha}}$, and one left-moving boson $\rho$ (which takes values on a circle of radius $1$). For the heterotic GS superstring, one also has the right-moving fermions, $\zeta_q$ ($q = 1$ to $32 - 2r$), which describe the unbroken gauge degrees of freedom (e.g. the gauge group is $E_6 \times E_8$ when $r = 3$, and the gauge group is $SO(10) \times E_8$ when $r = 4$). For the Type II superstring, one needs the right-moving fermionic fields, $\hat{\theta}^\alpha, \hat{\bar{\theta}}^{\dot{\alpha}}, \hat{p}_\alpha, \hat{\bar{p}}_{\dot{\alpha}}$, and one right-moving boson $\hat{\rho}$.

The compactification-dependent part of the matter sector is described as in the RNS formalism by a $c = (9, 6+r)$ N=(2,0) superconformal field theory for the heterotic superstring, and a $c = (9, 9)$ N=(2,2) superconformal field theory for the Type II superstring. Note that because of manifest spacetime supersymmetry, there is no need to perform a GSO projection in either the four-dimensional or compactification-dependent sector.

2.1. The worldsheet action and N=2 stress-energy tensor

In N=2 superconformal gauge, the worldsheet action for these fields is:

$$\begin{align*}
\text{Heterotic} : & \quad \frac{1}{\alpha'} \int dz^+ dz^- \left[ \frac{1}{2} \partial_+ x^m \partial_- x_m + p_\alpha \partial_+ \theta^\alpha + \bar{p}_{\dot{\alpha}} \partial_+ \bar{\theta}^{\dot{\alpha}} + \zeta_q \partial_- \zeta_q \right. \\
& \quad \left. - \frac{\alpha'}{2} \partial_+ \rho D_- \rho + SC \right] \\
\text{Type II} : & \quad \frac{1}{\alpha'} \int dz^+ dz^- \left[ \frac{1}{2} \partial_+ x^m \partial_- x_m + p_\alpha \partial_+ \theta^\alpha + \bar{p}_{\dot{\alpha}} \partial_+ \bar{\theta}^{\dot{\alpha}} - \frac{\alpha'}{2} \partial_+ \rho D_- \rho + \right. \\
& \quad \left. \hat{p}_\alpha \partial_- \hat{\theta}^\alpha + \hat{\bar{p}}_{\dot{\alpha}} \partial_- \hat{\bar{\theta}}^{\dot{\alpha}} - \frac{\alpha'}{2} \partial_- \hat{\rho} D_+ \hat{\rho} + SC \right]
\end{align*}$$

(2.1) (2.2)
where $S_C$ is the action for the compactification-dependent superconformal field theory, $D_\rho = \partial_\rho + a_-$ and $D_+ = \partial_+ \hat{\rho} + \hat{a}_+$ are the worldsheet covariant derivatives ($e^\rho$ will carry U(1) charge), and $a_\pm, \hat{a}_\pm$ are the worldsheet U(1) gauge fields which in superconformal gauge satisfy $a_+ = \hat{a}_- = 0$. (For the Type II superstring, we use the U(1)xU(1) form of N=(2,2) supergravity which contains two independent U(1) gauge fields.) Note that the equations of motion for $a_-$ and $\hat{a}_+$ imply the chirality conditions for $\rho$ and $\hat{\rho}$. (We are ignoring subtleties associated with the propagation of $a_-$ and $\hat{a}_+$.)

The free-field OPE’s for these worldsheet variables are

$$x^m(y)x^n(z) \to -\alpha' \eta^{mn} \log |y - z|, \quad \rho(y)\rho(z) \to \log(y^- - z^-), \quad (2.3)$$

$$p_\alpha(y)\theta^\beta(z) \to \frac{\alpha' \delta^\beta_\alpha}{y^- - z^-}, \quad \bar{p}_\alpha(y)\bar{\theta}^\beta(z) \to \frac{\alpha' \delta^\beta_\alpha}{y^- - z^-}, \quad \zeta_q(y)\zeta_r(z) \to \frac{\alpha' \delta_qr}{y^+ - z^+},$$

$$\hat{p}_\alpha(y)\hat{\theta}^\beta(z) \to \frac{\alpha' \delta^\beta_\alpha}{y^+ - z^+}, \quad \hat{\bar{p}}_\alpha(y)\hat{\bar{\theta}}^\beta(z) \to \frac{\alpha' \delta^\beta_\alpha}{y^+ - z^+}, \quad \hat{\rho}(y)\hat{\rho}(z) \to \log(y^+ - z^+)$$

where $\eta^{mn} = (-1,1,1,1)$. Note that the chiral boson $\rho$ has a timelike signature and cannot be fermionized since $e^{i\rho(y)} e^{i\rho(z)} \to e^{2i\rho(z)/(y^- - z^-)}$ while $e^{i\rho(y)} e^{-i\rho(z)} \to (y^- - z^-)$. It has the same behavior as the negative-energy field $\phi$ that appears when bosonizing the RNS ghosts $\gamma = \eta e^{i\phi}$ and $\beta = \partial\xi e^{-i\phi}$. The strange $\alpha'$ dependence of $\rho$ in (2.1) will later be shown to be related to the Fradkin-Tseytlin term.

The left-moving $c = 6$ stress-energy tensor for this N=2 string is:

$$L = \frac{1}{2} \partial_+ x^m \partial_- x_m + p_\alpha \partial_- \theta^\alpha + \bar{p}_\alpha \partial_- \bar{\theta}^\alpha - \frac{\alpha'}{2} \partial_- \rho \partial_- \rho + L_C, \quad (2.4)$$

$$G = \frac{1}{\sqrt{\alpha'}} e^{i\rho}(d)^2 + G_C, \quad \bar{G} = \frac{1}{\sqrt{\alpha'}} e^{-i\rho}(\bar{d})^2 + \bar{G}_C, \quad J = i\alpha' \partial_- \rho + J_C,$$

where

$$d_\alpha = p_\alpha + i\sigma^{m}_{\alpha\beta} \bar{\theta}^\beta \partial_- x_m - \frac{1}{2} \bar{\theta}^2 \partial_- \theta^\alpha + \frac{1}{4} \theta^\alpha \partial_- (\bar{\theta})^2$$

$$\bar{d}_\alpha = \bar{p}_\alpha + i\sigma^{m}_{\alpha\beta} \theta^\beta \partial_- x_m - \frac{1}{2} \theta^2 \partial_- \bar{\theta}^\alpha + \frac{1}{4} \bar{\theta}^\alpha \partial_- (\theta)^2,$$

$(d)^2$ means $\frac{1}{2} \epsilon^{\alpha\beta} d_\alpha d_\beta$, and $[L_C, G_C, \bar{G}_C, J_C]$ are the left-moving generators of the $c = 9$ N=2 compactification-dependent stress-energy tensor. As was shown in reference [21], $d_\alpha$ and $\bar{d}_\alpha$ satisfy the OPE that $d_\alpha(y) d_\beta(z)$ is regular, $d^\alpha(y) \bar{d}^\beta(z) \to$
$2i\alpha' \sigma^m \Pi^m / (y^- - z^-)$ where $\Pi^m_\pm = \partial_\pm x^m - i\sigma^m_{\alpha\dot{\alpha}}(\theta^\alpha \partial_\pm \dot{\theta}^{\dot{\alpha}} + \dot{\theta}^{\dot{\alpha}} \partial_\pm \theta^\alpha)$, and $d_\alpha(y) \Pi^m_-(z) \to -2i\alpha' \sigma^m_\alpha \partial_- \theta^\alpha / (y^- - z^-)$.

For the heterotic superstring, the right-moving $c = 26$ N=0 stress-energy tensor is:

$$Heterotic: \quad \hat{L} = \frac{1}{2} \partial_+ x^m \partial_+ x_m + \zeta_q \partial_+ \zeta_q + \hat{L}_C$$

where $\hat{L}_C$ is the right-moving $c = 6 + r$ compactification-dependent stress-energy tensor. For the Type IIB superstring, the right-moving $c = 6$ N=2 stress-energy tensor is

$$TypeIIB: \quad \hat{L} = \frac{1}{2} \partial_+ x^m \partial_+ x_m + \hat{\bar{p}}_\alpha \partial_+ \theta^\alpha + \hat{\bar{p}}_{\dot{\alpha}} \partial_+ \dot{\theta}^{\dot{\alpha}} - \frac{\alpha'}{2} \partial_+ \hat{\bar{\rho}} \partial_+ \hat{\dot{\rho}} + \hat{L}_C,$$

where $\hat{\bar{d}}_\alpha$ and $\hat{\bar{d}}_\dot{\alpha}$ are obtained from (2.5) by using hatted variables and replacing $\partial_-$ with $\partial_+$. Since the mirror transformation [21] flips the sign of $\hat{\bar{\rho}}$, the right-moving stress-energy tensor for Type IIA compactifications is

$$TypeIIA: \quad \hat{L} = \frac{1}{2} \partial_+ x^m \partial_+ x_m + \hat{\bar{p}}_\alpha \partial_+ \theta^\alpha + \hat{\bar{p}}_{\dot{\alpha}} \partial_+ \dot{\theta}^{\dot{\alpha}} - \frac{\alpha'}{2} \partial_+ \hat{\bar{\rho}} \partial_+ \hat{\dot{\rho}} + \hat{L}_C,$$

where $\hat{\bar{d}}_\alpha$ and $\hat{\bar{d}}_\dot{\alpha}$ are obtained from (2.5) by using hatted variables and replacing $\partial_-$ with $\partial_+$. Since the mirror transformation [21] flips the sign of $\hat{\bar{\rho}}$, the right-moving stress-energy tensor for Type IIA compactifications is

$$TypeIIA: \quad \hat{L} = \frac{1}{2} \partial_+ x^m \partial_+ x_m + \hat{\bar{p}}_\alpha \partial_+ \theta^\alpha + \hat{\bar{p}}_{\dot{\alpha}} \partial_+ \dot{\theta}^{\dot{\alpha}} - \frac{\alpha'}{2} \partial_+ \hat{\bar{\rho}} \partial_+ \hat{\dot{\rho}} + \hat{L}_C,$$

where $\hat{\bar{d}}_\alpha$ and $\hat{\bar{d}}_\dot{\alpha}$ are obtained from (2.5) by using hatted variables and replacing $\partial_-$ with $\partial_+$. Since the mirror transformation [21] flips the sign of $\hat{\bar{\rho}}$, the right-moving stress-energy tensor for Type IIA compactifications is

$$TypeIIA: \quad \hat{L} = \frac{1}{2} \partial_+ x^m \partial_+ x_m + \hat{\bar{p}}_\alpha \partial_+ \theta^\alpha + \hat{\bar{p}}_{\dot{\alpha}} \partial_+ \dot{\theta}^{\dot{\alpha}} - \frac{\alpha'}{2} \partial_+ \hat{\bar{\rho}} \partial_+ \hat{\dot{\rho}} + \hat{L}_C,$$

The advantage of working with the variables $d_\alpha$ and $\Pi^m$ is that they commute with the spacetime supersymmetry generators,

$$q_\alpha = \int dz^- [p_\alpha - i\sigma^m_{\alpha\dot{\alpha}} \hat{\theta}^\dot{\alpha} \partial_- x_m - \frac{1}{4}(\hat{\theta})^2 \partial_- \theta_\alpha],$$
$$\bar{q}_\dot{\alpha} = \int dz^- [\bar{p}_{\dot{\alpha}} - i\sigma^m_{\alpha\dot{\alpha}} \theta^\alpha \partial_- x_m - \frac{1}{4}(\theta)^2 \partial_- \dot{\theta}_{\dot{\alpha}}].$$

When written in terms of the supersymmetric variables, the actions of (2.1) and (2.2) take the more familiar forms

$$Heterotic: \quad \frac{1}{\alpha'} \int dz^+ dz^- [\frac{1}{2} \Pi^m_+ \Pi^-_m + \Pi^-_m T^m_+ - \Pi^m_+ T^-_m + \zeta_q \partial_+ \zeta_q]$$
\[ +d_\alpha \partial_+ \theta^\alpha + \tilde{d}_\alpha \partial_+ \tilde{\theta}^\alpha - \frac{\alpha'}{2} \partial_+ \rho D_\rho + S_C \]

Type II:

\[
\frac{1}{\alpha'} \int dz^+ dz^- [\frac{1}{2} \Pi_+^m \Pi_{m-} + \Pi_m^m (T_{m+} + \hat{T}_{m+}) - \Pi_+^m (T_{m-} + \hat{T}_{m-}) \tag{2.11}
\]

\[
+ T_+^m \hat{T}_{m-} - T_-^m \hat{T}_{m+}
\]

\[ +d_\alpha \partial_+ \theta^\alpha + \tilde{d}_\alpha \partial_+ \tilde{\theta}^\alpha + \hat{d}_\alpha \partial_+ \hat{\theta}^\alpha + \tilde{\hat{d}}_\alpha \partial_+ \tilde{\hat{\theta}}^\alpha + S_C - \frac{\alpha'}{2} (\partial_+ \rho D_\rho + \partial_- \hat{\rho} D_\hat{\rho}) \]

where \( T_{m\pm} = \sigma_m^{\alpha \dot{\alpha}} (\theta_\alpha \partial_\pm \theta_{\dot{\alpha}} + \tilde{\theta}_{\dot{\alpha}} \partial_\pm \theta_\alpha) \) and \( \hat{T}_{m\pm} = \sigma_m^{\alpha \dot{\alpha}} (\hat{\theta}_\alpha \partial_\pm \tilde{\theta}_{\dot{\alpha}} + \tilde{\hat{\theta}}_{\dot{\alpha}} \partial_\pm \hat{\theta}_\alpha) \).

For the Type II superstring, \( \Pi_m^m = \partial_+ x^m + i \sigma_m^{\alpha \dot{\alpha}} (\theta_\alpha \partial_\pm \theta_{\dot{\alpha}} + \tilde{\theta}_{\dot{\alpha}} \partial_\pm \theta_\alpha + \hat{\theta}_\alpha \partial_\pm \tilde{\theta}_{\dot{\alpha}} + \tilde{\hat{\theta}}_{\dot{\alpha}} \partial_\pm \hat{\theta}_\alpha) \) and \( d_\alpha \) differs from (2.5) by terms which vanish on-shell. If the \( d_\alpha \) and \( \rho \) contributions are dropped, the actions in (2.10) and (2.11) are the standard heterotic and Type II Green-Schwarz four-dimensional actions [13].

Although one can formally write an N=2 worldsheet supersymmetric version of these actions as

\[
S = \int dz^+ dz^- \frac{1}{\alpha' \det e} [e_{+} I + e_{++} L + e_{-} \hat{L} + \xi_{+} G + \xi_{+} \hat{G} + a_{+} J \tag{2.12}
\]

\[ (+\tilde{\xi}_{-} \hat{G} + \tilde{\xi}_{-} \hat{G} + a_{-} \hat{J}) \]

where \( (e_{\pm}, \xi_{\pm}, \hat{\xi}_{\pm}, a_{\pm}, \tilde{\xi}_{\pm}, \tilde{\hat{\xi}}_{\pm}, \tilde{a}_{\pm}) \) are the worldsheet supergravity fields, the quantum behavior of the \( \rho \) field makes it difficult to make manifest the worldsheet supersymmetry. Nevertheless, the free-field OPE’s of (2.3) make it straightforward to check that \([L, G, \hat{G}, J]\) form a \( c = 6 \) N=2 superconformal algebra, thereby implying the quantum N=2 superconformal invariance of (2.12).

Note that for the heterotic superstring, if a spacetime superfield \( \Phi \) is worldsheet chiral (i.e. \( \hat{G} \) has no singularities with \( \Phi \)), it is automatically superspace chiral (i.e. \( \nabla_{\dot{\alpha}} \Phi = 0 \) where \( \nabla_{\dot{\alpha}} = \frac{\partial}{\partial \theta_{\dot{\alpha}}} + i \sigma_{\alpha \dot{\alpha}} \theta_\alpha \partial_m \) since \( \tilde{d}_{\dot{\alpha}} \) has no poles with \( \Phi \). Similarly for the the Type II superstring, if a spacetime N=2 superfield \( \Phi \) is worldsheet chiral (i.e. \( \hat{G} \) and \( \hat{G} \) have no singularities with \( \Phi \)) or worldsheet twisted-chiral (i.e. \( \hat{G} \) and \( \hat{G} \) have no singularities with \( \Phi \)), then it is automatically superspace chiral (i.e., \( \nabla_{\dot{\alpha}} \Phi = \hat{\nabla}_{\dot{\alpha}} \Phi = 0 \)) or superspace twisted-chiral (i.e. \( \nabla_{\dot{\alpha}} \Phi = \hat{\nabla}_{\dot{\alpha}} \Phi = 0 \)). Also note that \( \hat{G}_C \) goes with \( \hat{G} \) for Type IIB compactifications, while \( \hat{G}_C \) goes with \( \hat{G} \) for Type IIA compactifications. So for Type IIB
(or Type IIA) compactifications, right-moving worldsheet chirality is correlated (or anti-correlated) with right-moving chirality of the compactification-dependent $c = 9$ superconformal field theory.

Scattering amplitudes can be calculated by evaluating correlation functions of physical vertex operators on N=2 super-Riemann surfaces. All vertex operators of zero ghost-number are constructed from U(1)-neutral combinations of the worldsheet matter fields and must carry integer $J_C$ charge in order not to have branch cuts with $G$. The massless vertex operators are simpler than in the RNS formalism, and since they play an essential role in the construction of the two-dimensional sigma model, they will be reviewed in the following sub-section.

2.2. Massless vertex operators

For the heterotic superstring, all massless vertex operators which are independent of the compactification are constructed from the real spacetime superfields $V_I(x, \theta, \bar{\theta})$ and $V_m(x, \theta, \bar{\theta})$, where $V_I$ is the prepotential for super Yang-Mills ($I=1 \text{ to } d$ labels the group index) and $V_m$ is the prepotential for N=1 supergravity plus a tensor multiplet ($m = 0 \text{ to } 3$ is a spacetime vector index).

In integrated form, these vertex operators are given by

$$\int dz^+ dz^- \{ \bar{G}, [G, V_P] \} O^P$$

(2.13)

where $O^I = j^I$ for the super Yang-Mills vertex operator ($j^I$ is the right-moving current constructed from the $\zeta_q$’s) and $O^m = \Pi_+^m = \partial_+ x^m + i \sigma^m_{\alpha \dot{\alpha}} (\theta^\alpha \partial_+ \bar{\theta}^{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}} \partial_+ \theta^\alpha)$ for the supergravity/tensor vertex operator. Note that $[G, V]$ means the residue of the single pole in the OPE of $G$ and $V$ (for $V$ on-shell, there are no double poles).

Up to surface terms, (2.13) is equal to

$$\int dz^+ dz^- [\bar{d}^\alpha (\nabla)^2 \bar{\nabla} . - d^\alpha \nabla_\alpha (\bar{\nabla})^2 + \partial_- \bar{\theta}^{\dot{\alpha}} \bar{\nabla} . - \partial_- \theta^\alpha \nabla_\alpha$$

$$- \frac{i}{2} \Pi_m^\alpha \sigma_m^{\alpha \dot{\alpha}} [\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}] V_P(x, \theta, \bar{\theta}) O^P.$$  

(2.14)

Gauge transformations which leave this vertex operator invariant are

$$\delta V_P = (\nabla)^2 \Lambda_P + (\bar{\nabla})^2 \bar{\Lambda}_P + \delta_P^m \partial_m \Omega,$$
which can gauge-fix \(4d + 4d\) component fields in \(V_I\) and \(20+20\) component fields of \(V_m\). In Wess-Zumino gauge, the remaining \(4d + 4d\) component fields of \(V_I\) are described by

\[
V_I = A_{lm} \sigma_{\alpha \dot{\alpha}}^m \theta^\alpha \bar{\theta}^\dot{\alpha} + \psi_{I \alpha} \theta^\alpha (\bar{\theta})^2 + \bar{\psi}_{I \dot{\alpha}} (\theta)^2 \bar{\theta}^\dot{\alpha} + D_I (\theta)^2 (\bar{\theta})^2
\]

(2.15)

where \(A_{lm}\) are the gluons, \(\psi_{I \alpha}\) are the gluinos, and \(D_I\) is the real auxiliary field. In Wess-Zumino gauge, the remaining \(12+12\) component fields of \(V_m\) are described by

\[
V_m = (h_{mn} + b_{mn} + l \eta_{mn}) \sigma_{\alpha \dot{\alpha}}^n \theta^\alpha \bar{\theta}^\dot{\alpha} + \]

\[
(\chi_{ma} + \bar{\xi}^\alpha \sigma_{m \dot{\alpha}}^a \dot{\theta}^\alpha (\bar{\theta})^2 + (\bar{\chi}_{ma} + \xi^\alpha \sigma_{m \dot{\alpha}}^a) \theta^\alpha \bar{\theta}^\dot{\alpha} + D_m (\theta)^2 (\bar{\theta})^2.
\]

(2.16)

From the superspin 3/2 piece of \(V_m\) representing conformal supergravity, \(h_{mn}\) is the traceless graviton, \(D_m\) is the auxiliary U(1) gauge field, and \(\chi_{ma}\) is the gravitino \((\sigma_{\alpha \dot{\alpha}}^m \chi_{m}^\alpha = 0)\). From the superspin 1/2 piece of \(V_m\) representing the tensor multiplet, \(b_{mn}\) is the anti-symmetric tensor, \(l\) is the physical scalar, and \(\xi^\alpha\) is the dilatino.

These vertex operators are on-shell when \(V_P\) is an \(N=2\) primary field of weight zero, i.e. \((\nabla)^2 V_P = (\bar{\nabla})^2 V_P = \partial_m \partial^m V_P = \partial^m V_m = 0\). These imply the usual equations of motion and polarization conditions on the component fields.

The compactification-dependent massless vertex operators for the heterotic superstring are constructed from spacetime chiral superfields \(M^{(i)}(x + i \bar{\theta} \bar{\theta}, \theta)\) which couple to worldsheet chiral primaries \(\Omega^{(i)}\) of the \(c = (9, 6 + r)\) \(N=(2,0)\) superconformal field theory representing the compactification manifold \((i)\) labels the compactification moduli). Assuming the compactification has no isometries, the relevant \(\Omega^{(i)}\)‘s have worldsheet U(1) charge +1 and describe either scalars, vectors, or spinors of \(SO(16 - 2r)\).

For scalars, \(\Omega^{(i)}\) has dimension \((\frac{1}{2}, 1)\) and the vertex operator is

\[
\int dz^+ dz^- \left[ \{G, M^{(i)} \Omega^{(i)}\} + \{\bar{G}, \bar{M}^{(i)} \bar{\Omega}^{(i)}\} \right]
\]

(2.17)

\[
= \int dz^+ dz^- \left[ \frac{1}{\sqrt{\alpha'}} e^{i \rho d^\alpha (\nabla_\alpha M^{(i)}) \Omega^{(i)} + M^{(i)} \{G_C, \Omega^{(i)}\} + c.c.} \right]
\]

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where \(c.c.\) means complex conjugate and we are ignoring double poles in the OPE with \(G\) (these double poles vanish on-shell). For vectors, \(\Omega^{(i)}\) has dimension \((\frac{1}{2}, \frac{1}{2})\) and the vertex operator is

\[
\int dz^+ dz^- \{[G, M_q^{(i)} \Omega^{(i)}] + \{\bar{G}, \bar{M}_q^{(i)} \bar{\Omega}^{(i)}]\} \zeta_q
\]

\[
= \int dz^+ dz^- \left[ \frac{1}{\sqrt{\alpha'}} e^{i\alpha} d^\alpha (\nabla_\alpha M_q^{(i)}) \Omega^{(i)} + M_q^{(i)} \{G_C, \Omega^{(i)}\} + \text{c.c.}\right] \zeta_q
\]

where \(q = 1 \text{ to } 16 - 2r\). For spinors, \(\Omega^{(i)}\) has dimension \((\frac{1}{2}, \frac{1}{8})\) and the vertex operator is

\[
\int dz^+ dz^- \{[G, M_q^{(i)} \Omega^{(i)}] + \{\bar{G}, \bar{M}_q^{(i)} \bar{\Omega}^{(i)}]\} s^\gamma
\]

\[
= \int dz^+ dz^- \left[ \frac{1}{\sqrt{\alpha'}} e^{i\alpha} d^\alpha (\nabla_\alpha M_q^{(i)}) \Omega^{(i)} + M_q^{(i)} \{G_C, \Omega^{(i)}\} + \text{c.c.}\right] s^\gamma
\]

where \(s^\gamma = \exp(\sum_{q=1}^{8-r} \int z^2 \zeta_{2q-1} \zeta_{2q})\) is a dimension \(\frac{8-r}{8}\) spinor of \(SO(16 - 2r)\).

As was shown in reference \[22\], these \(SO(16 - 2r)\) representations combine into representations of the maximal unbroken subgroup of \(E_8\) (e.g. for \(r = 3\), they combine into representations of \(E_6\)).

The compactification-dependent vertex operators are on-shell when \(M^{(i)}\) is primary, i.e. \((\nabla)^2 M^{(i)} = 0\). In components, this implies \(\partial_m \partial^m a^{(i)} = \sigma_{\alpha \alpha'} \partial_m \xi^{\alpha(i)} = b^{(i)} = 0\) where \(M^{(i)} = a^{(i)}(x^+) + \theta_\alpha \xi^{\alpha(i)}(x^+) + (\theta)^2 b^{(i)}(x^+)\) and \(x^+ = x + i\theta \bar{\theta}\). (We will suppress \(SO(16 - 2r)\) indices from now on.)

For the Type II superstring, all compactification-independent massless vertex operators are constructed from a single real superfield, \(U(x, \theta, \bar{\theta}, \bar{\theta})\) \[23\]. In integrated form, these vertex operators are given by \[24\]

\[
\int dz^+ dz^- \{\hat{G}, [\hat{G}, [G, U]]\}
\]

\[
= \int dz^+ dz^- (\hat{d}^\alpha \nabla_\alpha (\hat{d})^2 \hat{\nabla}_\alpha - \hat{d}^\alpha \hat{\nabla}_\alpha (\hat{d})^2 + \partial_- \hat{\theta}^\alpha \hat{\nabla}_\alpha - \partial_+ \hat{\theta}^\alpha \hat{\nabla}_\alpha - \frac{i}{2} \Pi^m \sigma_m^{\alpha \alpha'} [\hat{\nabla}_\alpha, \hat{\nabla}_{\alpha'}])
\]

where all of the \(\nabla\)’s act on \(U\).
Gauge transformations which leave this vertex operator invariant are

$$\delta U = (\nabla)^2 \Lambda + (\bar{\nabla})^2 \bar{\Lambda} + (\nabla)^2 \hat{\Lambda} + (\bar{\nabla})^2 \bar{\hat{\Lambda}}$$

which can gauge-fix 96+96 component fields. In Wess-Zumino gauge, the remaining 32+32 component fields are described by

$$U = (h_{mn} + b_{mn} + l^{++} \eta_{mn}) \sigma_{\alpha \alpha}^m \sigma_{\beta \beta}^n \theta^\alpha \bar{\theta}^\alpha \bar{\theta}^\beta \bar{\theta}^\beta \tag{2.21}$$

$$+ (\hat{x}_{m \beta} + \hat{\xi}_{m \beta}) \sigma_{\alpha \alpha}^m \theta^\alpha \bar{\theta}^\alpha \hat{\theta}^\beta (\hat{\theta})^2 + (\chi_{m a} + \hat{\xi}_{m a}) \sigma_{\alpha \alpha}^m \theta^\alpha (\hat{\theta})^2 \hat{\theta}^\beta + \text{c.c.} +$$

$$T_{mn} (\sigma_{m \alpha \beta}^m \theta^\alpha (\hat{\theta})^2 \hat{\theta}^\beta + \bar{\sigma}_{m \alpha \beta}^m (\hat{\theta})^2 \bar{\theta}^\alpha \bar{\theta}^\beta) +$$

$$(A_{m -}^+ + \partial_m l^{+ +}) \sigma_{\alpha \alpha}^m \theta^\alpha (\hat{\theta})^2 \hat{\theta}^\beta + (A_{m -}^+ + \partial_m l^{--}) \sigma_{\alpha \alpha}^m \theta^\alpha (\hat{\theta})^2 \hat{\theta}^\beta +$$

$$y \epsilon_{\alpha \beta} \theta^\alpha (\hat{\theta})^2 \hat{\theta}^\beta + \bar{y} \epsilon_{\alpha \beta} \bar{\theta}^\alpha \bar{\theta}^\beta$$

$$(A_{m}^{U(1)} + A_{m -}^{U(1)}) \sigma_{\alpha \alpha}^m \theta^\alpha \bar{\theta}^\alpha (\hat{\theta})^2 + (A_{m}^{U(1)} - A_{m -}^{U(1)}) \sigma_{\beta \beta}^m \theta^\alpha \bar{\theta}^\beta (\hat{\theta})^2 +$$

$$\psi_\alpha \theta^\alpha (\hat{\theta})^2 (\hat{\theta})^2 + \bar{\psi}_\beta (\theta)^2 (\hat{\theta})^2 \hat{\theta}^\beta (\hat{\theta})^2 + \text{c.c.} +$$

$$D(\theta)^2 (\hat{\theta})^2 (\hat{\theta})^2.$$

From the superspin 1 piece of $U$ representing the conformal supergravity multiplet, $h_{mn}$ is the traceless graviton, $\chi_{m \alpha}$ and $\hat{x}_{m \beta}$ are the gravitinos, $A_{m}^{U(1)}$ is the U(1) gauge field, $A_{m}^{j \beta}$ are the three SU(2) gauge fields, $T_{mn}$ is the auxiliary tensor, $\psi_\alpha$ and $\bar{\psi}_\beta$ are the auxiliary fermions, and $D$ is the auxiliary scalar. From the superspin 0 piece of $U$ representing the tensor multiplet, $l^{j \beta}$ is the SU(2) triplet, $b_{mn}$ is the anti-symmetric tensor, $\xi^\alpha$ and $\hat{\xi}^\beta$ are the dilatinos, and $y$ is a complex auxiliary scalar.

This vertex operator is physical when $U$ satisfies the N=2 primary conditions

$$(\nabla)^2 U = (\bar{\nabla})^2 U = (\nabla)^2 U = (\bar{\nabla})^2 U = \partial_m \partial^m U = 0,$$

which imply the usual equations of motion and polarization conditions for the component fields.

The compactification-dependent massless vertex operators for the Type II superstring are constructed from spacetime chiral superfields $M_\mathcal{E}(x + i \theta \bar{\theta} + i \hat{\theta} \bar{\hat{\theta}}, \theta, \hat{\theta})$ and spacetime twisted-chiral superfields $M_{tc}(x + i \theta \bar{\theta} - i \hat{\theta} \bar{\hat{\theta}}, \theta, \hat{\theta})$, which couple to the $h^{2,1}$ worldsheet chiral primaries $\Omega_{\mathcal{E}}^{(i)}$ and the $h^{1,1}$ worldsheet twisted-chiral
primaries \( \Omega_{tc}^{(i)} \) of the \( c = (9, 9) \) N=(2,2) superconformal field theory representing the compactification manifold \( (h^{2,1} \) counts the number of complex moduli and \( h^{1,1} \) counts the number of Kähler moduli). The \( \Omega_{c}^{(i)} \)'s have U(1)×U(1) charge \((+1, +1)\) and dimension \((\frac{1}{2}, \frac{1}{2})\), while the \( \Omega_{tc}^{(i)} \)'s have U(1)×U(1) charge \((+1, -1)\) and dimension \((\frac{1}{2}, \frac{1}{2})\).

For Type IIB compactifications, the vertex operators are

\[
\int dz^+dz^- \left( \{ \hat{G}, [G, M_c^{(i)} \Omega_c^{(i)}] \} + \{ \hat{G}, [\bar{G}, \bar{M}_c^{(i)} \bar{\Omega}_c^{(i)}] \} \right) \quad (2.22)
\]

\[
= \int dz^+dz^- \left[ \left( \frac{1}{\sqrt{\alpha'}} \epsilon^{i\beta} d^\alpha \hat{\nabla}_\alpha + \hat{G}_C \right) \left( \frac{1}{\sqrt{\alpha'}} \epsilon^{i\beta} d^\alpha \nabla_\alpha + G_C \right) M_c^{(i)} \Omega_c^{(i)} + c.c. \right] ,
\]

\[
\int dz^+dz^- \left( \{ \hat{G}, [G, M_{tc}^{(i)} \Omega_{tc}^{(i)}] \} + \{ \hat{G}, [\bar{G}, \bar{M}_{tc}^{(i)} \bar{\Omega}_{tc}^{(i)}] \} \right) \quad (2.23)
\]

\[
= \int dz^+dz^- \left[ \left( \frac{1}{\sqrt{\alpha'}} \epsilon^{i\beta} d^\alpha \hat{\nabla}_\alpha + \hat{G}_C \right) \left( \frac{1}{\sqrt{\alpha'}} \epsilon^{i\beta} d^\alpha \nabla_\alpha + G_C \right) M_{tc}^{(i)} \Omega_{tc}^{(i)} + c.c. \right]
\]

where \( \nabla \)'s act only on \( M^{(i)} \)'s and \( G_C \)'s act only on \( \Omega^{(i)} \)'s. For Type IIA compactifications, \( \Omega_{c}^{(i)} \) is switched with \( \Omega_{tc}^{(i)} \) and \( \hat{G}_C \) is switched with \( \hat{G}_C \) in the above vertex operators.

These vertex operators are physical when \( M_c^{(i)} \) and \( M_{tc}^{(i)} \) satisfy the N=2 primary conditions \((\nabla)^2 M_c^{(i)} = (\hat{\nabla})^2 M_c^{(i)} = 0 \) and \((\nabla)^2 M_{tc}^{(i)} = (\hat{\nabla})^2 M_{tc}^{(i)} = 0 \).

For constructing superspace actions, it will be useful to separate these conditions into ”reality constraints” (which are satisfied off-shell) and equations of motion (which are only satisfied on-shell). The reality constraint for \( M_c^{(i)} \) is \((\nabla)^2 M_c^{(i)} = (\hat{\nabla})^2 M_c^{(i)} \), which implies the 8+8 component expansion of a vector multiplet field strength:

\[
M_c^{(i)} = w^{(i)} + \theta^\alpha \xi^{(i)} + \hat{\theta}^\beta \bar{\xi}^{(i)} + (\theta)^2 D^{(i)}_{++} + \theta^\alpha \hat{\theta}^\beta \epsilon_{\alpha \beta} D^{(i)}_{--} + (\hat{\theta})^2 D^{(i)}_{--} + \theta^\alpha \hat{\theta}^\beta F_{\alpha \beta}^{(i)}
\]

\[
+ \sigma_{m}^{\alpha \beta} \theta^\alpha (\hat{\theta})^2 \partial_m \xi^{(i)} + \sigma_{\beta}^{m} \hat{\theta}^\beta (\theta)^2 \partial_m \bar{\xi}^{(i)} + (\theta)^2 (\hat{\theta})^2 \partial_m \partial^m \bar{w}^{(i)} \quad (2.24)
\]

where \( D^{(i)}_{jk} \) is an auxiliary isotriplet, \( w^{(i)} \) is a complex scalar, \( F_{\alpha \beta}^{(i)} \) is a U(1) vector field strength \((\sigma_{m}^{\alpha \beta} \partial_m F_{\alpha \beta}^{(i)} = \sigma_{\beta}^{m} \partial_m F_{\alpha \beta}^{(i)} \), and \( \xi^{(i)}, \bar{\xi}^{(i)} \) are SU(2) spinors.

The reality constraint for \( M_{tc}^{(i)} \) is \((\nabla)^2 M_{tc}^{(i)} = (\hat{\nabla})^2 M_{tc}^{(i)} \), which implies the 8+8 component expansion of a tensor hypermultiplet field strength:

\[
M_{tc}^{(i)} = l^{(i)}_{++} + \theta^\alpha \lambda^{(i)}_{\alpha} + \hat{\theta}^\beta \bar{\lambda}^{(i)}_{\beta} + (\theta)^2 y^{(i)} + (\hat{\theta})^2 \bar{y}^{(i)} + \sigma_{m}^{\alpha \beta} \theta^\alpha \hat{\theta}^\beta (\partial_m l^{(i)}_{++} + \epsilon_{mpnq} H^{npq} ) \quad (2.25)
\]
\[ + \sigma_{\alpha\alpha}^m \theta^\alpha (\hat{\theta})^2 \partial_m \hat{\chi}^{\alpha(i)} + \sigma_{\beta\beta}^m \hat{\theta}^\beta (\theta)^2 \partial_m \hat{\chi}^{\beta(i)} + \theta^2 (\hat{\theta})^2 \partial_m \partial^m l^{(i)} \]

where \( l^{(i)} \) is a scalar isotriplet, \( y^{(i)} \) is a complex auxiliary scalar, \( H^{(i)}_{mnp} \) is a tensor field strength (\( \partial^m H^{(i)}_{mnp} = 0 \)), and \( \chi^{(i)}_\alpha, \hat{\chi}^{(i)}_\beta \) are SU(2) spinors.

The remaining primary conditions, \((\nabla)^2 M^{(i)}_c = -(\hat{\nabla})^2 \bar{M}^{(i)}_c \) and \((\nabla)^2 M^{(i)}_{tc} = -(\hat{\nabla})^2 \bar{M}^{(i)}_{tc} \), imply the usual polarization conditions and equations of motion for these component fields.

### 3. Superspace Description of the Massless Spectrum

From the previous section, we have seen that massless vertex operators which are independent of the compactification are constructed from superspace prepotentials. For the heterotic superstring, the N=1 supergravity and tensor multiplets are described by the superspin \( \frac{3}{2} \) and \( \frac{1}{2} \) parts of a real prepotential \( V_m \), and the super-Yang-Mills multiplet is described by a real prepotential \( V_I \). For the Type II superstring, the N=2 supergravity and tensor multiplets are described by the superspin 1 and 0 parts of a real prepotential \( U \).

#### 3.1. Gauge fields and field strengths

Although prepotentials are the most compact superspace representations for these multiplets, they are inconvenient for constructing super-reparameterization invariant quantities. In the sigma model and effective action, it will be more convenient to represent these multiplets with the vielbein \( E^A_M \), the anti-symmetric tensor \( B_{MN} \), and the super-Yang-Mills potentials \( A^I_M \), where \( A \) labels tangent-superspace indices and \( M \) labels curved-superspace indices. For the heterotic superstring, \( A = (a, \alpha, \hat{\alpha}) \) and \( M = (m, \mu, \hat{\mu}) \), while for the Type II superstring, \( A = (a, \alpha j, \hat{\alpha} j) \) and \( M = (m, \mu j, \hat{\mu} j) \) where \( j = \pm \) are SU(2)-indices which can be raised and lowered using the \( \epsilon_{jk} \) tensor. Comparing with the notation from the previous section,

\[ \theta^{\alpha+} = \theta^\alpha, \quad \theta^{\alpha-} = \hat{\theta}^\alpha, \quad \hat{\theta}^{\alpha+} = \hat{\theta}^\alpha, \quad \hat{\theta}^{\alpha-} = -\bar{\theta}^{\alpha}. \]  

(3.1)

Note that the complex conjugate of \( E^\alpha_M \) is \( \epsilon_{jk} E^\alpha_M \hat{\alpha}^k \).
The field strengths of these superspace gauge fields are obtained from the graded commutators

\[ [\nabla_A, \nabla_B] = T_{AB}^C \nabla_C + R_{ABC}^D M_D^C + F_{AB}^I T_I + f_{AB}^J Y_J, \quad H_{MNP} = \nabla_{[MBNP]}, \] (3.2)

where

\[ \nabla_A = E_A^M \partial_M + A_A^I T_I + w_{AB}^C M_C^B + \Gamma_A^J Y_J \] (3.3)

is the covariant derivative, \( w_{AB}^C \) is the spin connection, \( \Gamma_A^J \) are the U(N) connections (N=1 for heterotic and N=2 for Type II), \( M_D^C \) are the Lorentz generators, \( T_I \) are the Yang-Mills group generators, \( Y_J \) are the U(N) group generators, \( T_{AB}^C \) is the torsion, \( R_{ABC}^D \) is the supercurvature, \( F_{AB}^I \) is the super Yang-Mills field strength, \( f_{AB}^J \) is the U(N) field strength, and \( H_{MNP} \) is the tensor field strength. As in ordinary gravity, the connections \( w_{AB}^C \) and \( \Gamma_A^J \) are not independent superfields and are related to \( E_A^M \) by torsion constraints.

However, since \( E_A^M, B_{MN}, \) and \( A_M \) contain more component fields than the prepotentials, one needs to impose further torsion constraints to remove the additional degrees of freedom. These constraints will also be needed for worldsheet supersymmetry of the sigma model and are given explicitly as constraints 1) and 2) in equations (4.3) and (4.4). After imposing them, the above field strengths can be expressed in terms of the reduced field strengths of the following tables:

| Heterotic multiplet | tensor | vector | supergravity |
|---------------------|--------|--------|-------------|
| prepotentials       | \Xi_{\alpha}       | \ V_I | \ V_m        |
| potentials          | \ B_{MN}          | \ A_M^I | \ E_A^M       |
| unreduced field strengths | \ H_{ABC} | \ F_{AB}^I | \ T_{AB}^C, R_{ABC}^D, f_{AB} |
| reduced field strengths | \ L   | \ W_{\alpha}^I | \ | R, G_a, W_{\alpha\beta\gamma} |

where the N=1 reduced field strengths are defined by

\begin{align*}
H_{\alpha\beta\gamma} \cdot \ L &= \sigma_{\alpha\beta\gamma} \ L, \\
F_{\alpha\alpha}^I &= \sigma^{\alpha\alpha} W^I, \\
T_{\alpha\alpha}^\gamma &= \sigma_{\alpha\alpha} \ R, \\
T_{\alpha\alpha}^\gamma &= \sigma_{\alpha\alpha} \ W_{\alpha\beta\gamma}. 
\end{align*} (3.4)

| Type II multiplet | tensor | vector | supergravity |
|-------------------|--------|--------|-------------|
| prepotentials     | \Xi     | \ V^{jk} | \ U        |
| potentials        | \ B_{MN} | \ A_M | \ E_A^M       |
| unreduced field strengths | \ H_{ABC} | \ F_{AB} | \ T_{AB}^C, R_{ABC}^D, f_{AB} |
| reduced field strengths | \ L_{jk} | \ W  | \ S_{jk}, G_{\alpha}^{jk}, N_{\alpha\beta}, W_{\alpha\beta} |

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where the N=2 reduced field strengths are defined by

\[ H_{\alpha j, \beta k} = \sigma_{a \alpha \beta} L_{j k}, \quad F_{\alpha j, \beta k} = W_{\epsilon a \alpha \beta} \epsilon_{j k}, \] (3.5)

\[ T_{\alpha j} = i \delta^k_j (\sigma_{a \alpha \beta} \hat{W}_{\alpha \gamma} + \sigma_{a \alpha \gamma} N_{\alpha \beta}) + i \sigma_{a \alpha \beta} \hat{S}_{j}^{k}, \quad T_{a \alpha j} = -2i \sigma_{a \alpha \beta} \hat{G}_{j}^{k} \sigma_{\alpha \beta}^{c} . \]

Note that we have not discussed the chiral prepotential \( \Xi \) for N=1 and N=2 tensor multiplets. (\( V_m \) or \( U \) can only be used as the tensor prepotential in certain gauges.) The N=2 vector multiplet will appear when we discuss compensators and compactification-dependent states.

Bianchi identities imply various conditions on the above reduced field strengths. For example, for the heterotic superstring, \( L \) is a real N=1 linear superfield and \( W_{\alpha}^{I} \) is an N=1 chiral superfield satisfying

\[ ((\nabla)^2 + \bar{R}) L = ((\bar{\nabla})^2 + R) L = 0, \quad \bar{\nabla}_{\alpha} W_{\beta}^{I} = \nabla_{\alpha} \tilde{W}_{\beta}^{I} = \nabla^{\alpha} W_{\alpha}^{I} - \nabla^{\alpha} \tilde{W}_{\alpha}^{I} = 0. \] (3.6)

For the Type II superstring, Bianchi identities imply that \( L_{jk} \) is a real N=2 linear superfield and \( W \) is an N=2 chiral superfield satisfying

\[ \nabla_{\alpha (j} L_{k l)} = \bar{\nabla}_{\alpha (j} L_{k l)} = 0, \quad \nabla_{\alpha j} W = \nabla_{\alpha j} \tilde{W} = \nabla^{\alpha j} \nabla_{\alpha k} W - \nabla^{\alpha j} \bar{\nabla}_{\alpha k} \tilde{W} = 0. \] (3.7)

Note that \( \nabla_{\alpha+} = \nabla_{\alpha}, \quad \bar{\nabla}_{\alpha+} = \bar{\nabla}_{\alpha}, \quad \nabla_{\alpha-} = \nabla_{\alpha}, \quad \bar{\nabla}_{\alpha-} = -\bar{\nabla}_{\alpha} \), so \( L_{--} \) satisfies the twisted-chirality condition \( \bar{\nabla}_{\alpha} L_{--} = \nabla_{\alpha} L_{--} = 0 \). Implications of Bianchi identities for the reduced N=1 and N=2 supergravity field strengths can be found in references [25] and [26].

3.2. Compensators

For constructing superspace actions, it is useful to introduce compensator superfields which make the formalism invariant under spacetime scale, U(1), and for Type II compactifications, SU(2) transformations. Because of the form of the torsion constraints, it is easy to generalize a flat superspace action to curved superspace if the action contains these invariances. If the original action does not contain these invariances, one adds an appropriate power of the compensator superfield so that the trasformation of the integrand of the original action cancels against the transformation of the compensator superfield. This makes it possible to generalize arbitrary flat superspace actions to curved superspace. Of course,
the scale, U(1), and SU(2) invariances are not physical symmetries (unless they were already present in the original action), and can be removed by gauge-fixing the compensators to a constant.

Although it is not widely appreciated, the bosonic string also makes use of a conformal compensator field, $\varphi$, which couples to worldsheet curvature and will be called the dilaton. Recall that the physical massless vertex operator of the bosonic string is $\int dz^+ z^- (g_{mn} + b_{mn}) \partial_+ x^m \partial_- x^n$, and $g_{mn}$ splits at linearized level into a traceless piece $h_{mn} = g_{mn} - \frac{1}{D} \eta_{mn} \eta^{pq} g_{pq}$ representing conformal supergravity, and a physical scalar $l = \eta^{mn} g_{mn}$ which will not be called the dilaton. The vertex operator for the dilaton, on the other hand, is constructed from worldsheet ghosts which, when integrated out, give coupling to worldsheet curvature.

The usual version of the low-energy effective action for the bosonic string is

$$- \int d^D x \Phi (\Box + \frac{1}{12} H^{abc} H_{abc}) \Phi.$$  \hspace{1cm} (3.8)

(The relative coefficients of these terms can be determined by T-duality which transforms $g_{mn}$ into $b_{mn}$. In the supersymmetric cases, they are already fixed by supersymmetry.) Here $\Phi$ is related to the more common form of the dilaton field used in string theory by

$$\Phi = (-g)^{1/4} e^{-\varphi}$$ \hspace{1cm} (3.9)

(The fact that this is the T-duality invariant combination follows from the fact that $\Phi$ must soak up the $\sqrt{-g}$ measure, since $\sqrt{-g}$ is not T-duality invariant.) At this point we have not yet seperated out the trace of $g_{mn}$; this we now perform with full nonlinearity by the Weyl rescaling

$$g_{mn} \rightarrow l^2 g_{mn}$$ \hspace{1cm} (3.10)

(We do not scale $\Phi$, and leave $\Phi$-dependence out of the metric rescaling, so that $\Phi$ stays out of T-duality transformations.) The result is

$$S = - \int d^D x \left\{ \Phi^2 l^2 R + (D - 1) \Phi^2 (\nabla l)^2 ight\}$$ \hspace{1cm} (3.11)

$$- \left[ \Phi^{-1} \nabla (\Phi^2 l) \right]^2 + \frac{1}{12} \Phi^2 l^6 H^{abc} H_{abc} \right\}.$$
In general relativity one normally breaks the scale invariance introduced by this rescaling by gauge-fixing $\Phi = (-g)^{1/4}$ ($\varphi = 0$), or in this case the slightly modified gauge $\Phi = (-g)^{1/4}l^{-1}$, which produces an action with the standard form for the Einstein term, and the right sign for the scalar term:

$$-\int d^D x \sqrt{-g}[R + \frac{1}{12}l^4H^2 + (D - 2)(\nabla \ln l)^2]. \quad (3.12)$$

However, one can also break scale invariance by choosing the “string gauge” $l = 1$, which returns us to the form of the string effective action before Weyl rescaling. Although $\Phi$ naively appears to be a physical scalar in this string gauge, it is easily identified as a compensating scalar by its “wrong-sign” kinetic term. On the other hand, the physical scalar with a “right-sign” kinetic term has turned in string gauge into the determinant of the metric.

For N=1 and N=2 supergravity, there are various possible types of conformal compensators\[25\][27]. However, as will be shown in the following section, worldsheet supersymmetry of the Fradkin-Tseytlin term uniquely determines the correct type. (As discussed in \[12\], the correct compensator can also be determined by requiring that the closed superstring dilaton state factorizes into the product of open superstring ghost states.)

For the heterotic superstring, the conformal compensator is required to be a superspace chiral and anti-chiral superfield, $\Phi$ and $\Phi^c$, satisfying $\hat{\nabla}_\alpha \Phi = \nabla_\alpha \Phi = 0$. Using the U(1) connection $\Gamma_A$ and U(1) generator $Y$, superspace actions can be made conformally and U(1) invariant by introducing appropriate powers of $\Phi$ and $\Phi^c$. $\Phi$ will be defined to have U(1) weight $\frac{1}{2}$ (i.e. $[Y, \Phi] = \frac{1}{2}\Phi$ and $[Y, \Phi^c] = -\frac{1}{2}\Phi^c$), so by superspace rules, it must have conformal weight $\frac{3}{2}$. (To transform consistently under superconformal transformations, the conformal weight of a chiral superfield must be $(4 - N)/N$ times its U(1) weight where $N$ is the number of 4D supersymmetries.\[25\]) Note that the usual choice of U(1) weight for the chiral compensator is $\frac{1}{3}$, so we are defining $\Phi$ to be the usual chiral compensator raised to the $3/2$ power. (We are keeping the convention that $E_\alpha^M$ has conformal and U(1) weight $\frac{1}{2}$.)

For the Type II superstring, worldsheet supersymmetry of the Fradkin-Tseytlin term requires two different compensators. One type is described by superspace chiral and anti-chiral superfields, $\Phi_c$ and $\Phi^c_c$, satisfying $\hat{\nabla}_\alpha \Phi_c = \nabla_\alpha \Phi_c = 19$
\[ \nabla_\alpha \Phi_c = \hat{\nabla}_\alpha \Phi_c = 0. \]

After imposing the reality condition \((\nabla)^2 \Phi_c = (\hat{\nabla})^2 \Phi_c\), \(\Phi_c\) and \(\Phi_c\) can be identified with \(W^{(0)}\) and \(\bar{W}^{(0)}\) where \(W^{(0)}\) is the chiral field strength of a vector multiplet which satisfies equation (3.7). (This reality condition is not required by worldsheet supersymmetry of the sigma model, but is necessary for constructing superspace effective actions.)

The other Type II compensator is described by superspace twisted-chiral and twisted-anti-chiral superfields, \(\Phi_{tc}\) and \(\bar{\Phi}_{tc}\), satisfying \(\hat{\nabla}_\alpha \Phi_{tc} = \hat{\nabla}_\alpha \Phi_{tc} = \nabla_\alpha \Phi_{tc} = \hat{\nabla}_\alpha \Phi_{tc} = 0\). Although the twisted-chirality condition on \(\Phi_{tc}\) does not look \(SU(2)\)-covariant, it can be made covariant by identifying \(\Phi_{tc}\) and \(\bar{\Phi}_{tc}\) with \(L^{(0)}_{--}\) and \(L^{(0)}_{++}\) where \(L^{(0)}_{jk}\) is the linear field strength of a tensor hypermultiplet satisfying equation (3.7). Equation (3.7) also implies that \(\Phi_{tc}\) satisfies the reality condition \((\nabla)^2 \Phi_{tc} = (\bar{\nabla})^2 \Phi_{tc}\).

To write \(U(1) \times SU(2)\) invariant superspace actions, one introduces \(U(1) \times SU(2)\) connections \(\Gamma^{jk}_{A}\), \(U(1) \times SU(2)\) generators \(Y_{jk}\), and appropriate powers of \(W^{(0)}\) and \(L^{(0)}_{jk}\). Gauge invariance implies that the component field strength for the \(U(1)\) vector must carry conformal weight +2 and that the component field strength for the antisymmetric tensor must carry conformal weight +3. Since these field strengths sit in the \((\theta)^2\) components of \(W^{(0)}\) and \(L^{(0)}_{jk}\) (see equations (2.24) and (2.25) for component expansions), \(W^{(0)}\) must carry conformal weight +1 and \(L^{(0)}_{jk}\) must carry conformal weight +2. Note that \(W^{(0)}\) carries \(U(1)\) weight +1 and is an \(SU(2)\) singlet, while \(L^{(0)}_{jk}\) carries \(U(1)\) weight zero and is an \(SU(2)\) triplet.

### 3.3. Compactification-dependent states

Finally, we shall give the superfield description of the compactification dependent massless states, whose vertex operators are explicitly described in section 2. Since these moduli superfields describe deformations of the compactification manifold, we shall assume throughout this paper that the values of these moduli are small enough so that the massless spectrum is not modified.

For the heterotic superstring, these states couple to worldsheet chiral primary fields of the compactification-dependent \(c = 9\) \(N=(2,0)\) superconformal field theory. (We shall assume throughout this paper that the compactification manifold has no isometries, so all moduli are described by spacetime scalars.) As shown
in section 2, worldsheet and superspace chirality are correlated in the new description of the superstring, so the massless states coming from compactification moduli are described by spacetime chiral and anti-chiral superfields, $M^{(i)}$ and $\bar{M}^{(i)}$, satisfying $\nabla_\alpha M^{(i)} = \nabla_\alpha \bar{M}^{(i)} = 0$. ($(i)$ labels the compactification moduli, and we are suppressing possible Yang-Mills group indices.) These moduli superfields will be defined to carry zero conformal and U(1) weight.

For the Type II superstring, compactification-dependent massless states couple to the $h^{2,1}$ worldsheet chiral and $h^{1,1}$ worldsheet twisted-chiral primary fields of the $c = 9$ N=(2,2) superconformal field theory. ($h^{2,1}$ and $h^{1,1}$ count the number of complex and Kähler moduli of the compactification manifold.) As shown in section 2, left-moving compactification and superspace chirality are correlated in the new superstring description, while right-moving compactification and superspace chirality are correlated (or anti-correlated) for Type IIB (or Type IIA) compactifications. Therefore, for Type IIB (or Type IIA) compactifications, the corresponding massless states are described by $h^{2,1}$ (or $h^{1,1}$) spacetime chiral and anti-chiral superfields, $M_c^{(i)}$ and $\bar{M}_c^{(i)}$, and by $h^{1,1}$ (or $h^{2,1}$) spacetime twisted-chiral and twisted-anti-chiral superfields, $M_{tc}^{(i)}$ and $\bar{M}_{tc}^{(i)}$.

As was true for the Type II compensator superfields, it will be useful to impose a reality condition on these moduli superfields in order to construct superspace actions. Although one could use the same reality condition as for the compensators, this would force the moduli superfields to carry non-zero U(1) weight. A more convenient choice is to define $M_c^{(i)}$ and $M_{tc}^{(i)}$ such that

$$M_c^{(i)} = \log(W^{(i)}/W^{(0)}) , \quad M_{tc}^{(i)} = \log(L_{-\cdot}^{(i)}/L_{-\cdot}^{(0)})$$

(3.13)

where $W^{(i)}$ and $L_{jk}^{(i)}$ satisfy the conditions of equation (3.7) for vector and tensor field strengths. Although this means that $M_c^{(i)}$ and $M_{tc}^{(i)}$ satisfy the non-standard reality conditions, $(\nabla)^2(e^{M_c^{(i)}}W^{(0)}) = (\hat{\nabla})^2(e^{M_c^{(i)}}W^{(0)})$ and $(\nabla)^2(e^{M_{tc}^{(i)}}L_{++}^{(0)}) = (\hat{\nabla})^2(e^{M_{tc}^{(i)}}L_{++}^{(0)})$, it allows $M_c^{(i)}$ and $M_{tc}^{(i)}$ to have zero conformal and U(1) weight.

4. Sigma Models

Because we know the action in a flat background and the massless vertex operators from section 2, it is straightforward to construct a two-dimensional sigma model for the superstring in a curved background. This sigma model should
be spacetime super-reparameterization and gauge invariant, and at linearized level, its interactions should reproduce the massless vertex operators.

When the spacetime superfields satisfy their equations of motion, the sigma model action is expected to be $N=2$ superconformally invariant at the quantum level, i.e., the components of the $N=2$ stress-energy tensor form a $c = 6$ $N=2$ superconformal algebra. These superspace equations of motion are currently being computed for the heterotic superstring by de Boer and Skenderis, which will be the first covariant $\beta$-function computation in a fermionic background.

4.1. The classical term

The two-dimensional sigma model is constructed from the spacetime superfields of section 3 and splits into a classically worldsheet $N=2$ superconformally invariant term and a Fradkin-Tseytlin term. Using the action in a flat background and the vertex operators of section 2, it is easy to guess the classically worldsheet superconformally invariant term:

\[
\text{Heterotic} : \quad \frac{1}{\alpha'} \int dz^+ dz^- \left[ \frac{1}{2} \Pi_+^a \Pi_a^- + B_{AB} \Pi_+^A \Pi_-^B + \zeta_q \partial^- \zeta_q \right] + (A_B^I \Pi_+^B + W_\alpha^I d^\alpha - \bar{W}_\alpha^I \bar{d}^\alpha) j_I + d_\alpha \Pi_+^\alpha + \bar{d}_\alpha \Pi_-^\alpha + S_C + \{ G, M^{(i)} \Omega^{(i)} \} + \{ \bar{G}, \bar{M}^{(i)} \bar{\Omega}^{(i)} \} \]

\[
\text{Type IIB} : \quad \frac{1}{\alpha'} \int dz^+ dz^- \left[ \frac{1}{2} \Pi_+^a \Pi_a^- + B_{AB} \Pi_+^A \Pi_-^B \right] + d_\alpha \Pi_+^\alpha - \bar{d}_\alpha \Pi_-^\alpha + \hat{d}_a \Pi_+^a + \hat{\bar{d}}_a \Pi_-^a + d_\alpha P^{\alpha \beta} d_\beta + \bar{d}_\alpha \bar{P}^{\alpha \beta} d_\beta + \hat{d}_a Q^{\alpha \beta} d_\beta + \hat{\bar{d}}_a \bar{Q}^{\alpha \beta} d_\beta + S_C + \{ G, [ \hat{G}, M_c^{(i)} \Omega_c^{(i)} ] \} + \{ \bar{G}, [ \hat{G}, M_c^{(i)} \bar{\Omega}_c^{(i)} ] \}
\]

\[
+ \{ G, [ \hat{G}, M_{tc}^{(i)} \Omega_{tc}^{(i)} ] \} + \{ \bar{G}, [ \hat{G}, M_{tc}^{(i)} \bar{\Omega}_{tc}^{(i)} ] \}
\]

where $\Pi_\pm^A = E_M^A \partial_\pm Z^M$, $Z^M = (x^m, \theta^\mu, \bar{\theta}^{\bar{\mu}})$ for heterotic, and $Z^M = (x^m, \theta^\mu, \bar{\theta}^{\bar{\mu}}, \hat{\theta}^\mu, \hat{\bar{\theta}}^{\bar{\mu}})$ for Type II. In a flat background (i.e., $E_m^a = \delta_m^a$, $E_{\mu}^a = \sigma_{\mu}^a \hat{\theta}^{\bar{\mu}}$, $B_{\alpha a} = \sigma_{\alpha a \alpha} \hat{\theta}^a$ for heterotic; $E_m^a = \delta_m^a$, $E_{\mu}^j a = \sigma_{\mu}^a \hat{\theta}^\mu$, $B_{\alpha j a} = \sigma_{\alpha a \alpha} \hat{\theta}^\mu$, $B_{\alpha a + \beta +} = \theta_{\alpha} \hat{\theta}_{\beta}$ for Type II), it is easy to show that this action reproduces (2.1) and
Note that we are omitting the kinetic term for $\rho$ since it is zeroth order in $\alpha'$, and therefore goes with the Fradkin-Tseytlin term.

For the Type IIB sigma model, $M_c^{(i)}$ and $M_{tc}^{(i)}$ are the chiral and twisted-chiral moduli superfields defined in section 3. Although the reality constraints on $M_c^{(i)}$ and $M_{tc}^{(i)}$ are not implied by classical worldsheet superconformal invariance, these constraints (or a suitable modification) are expected to be implied by quantum superconformal invariance. $P^{\alpha\beta}$ and $Q^{\alpha\dot{\beta}}$ are chiral and twisted-chiral field strengths of $N=2$ conformal supergravity. From the $d^\alpha d^\alpha$ and $d^\alpha d^\alpha$ terms in the supergravity vertex operator of (2.21), one sees that at linearized level, $P^{\alpha\beta} = (\nabla)^2 \nabla^\alpha (\nabla)^2 \nabla^\beta U$ and $Q^{\alpha\dot{\beta}} = (\nabla)^2 \nabla^\alpha (\nabla)^2 \nabla^\beta U$. Note that the Type IIA sigma model action is obtained from the Type IIB action by switching $\Omega_c^{(i)}$ with $\Omega_{tc}^{(i)}$ and $\hat{G}_C$ with $\hat{G}_C$.

Up to field redefinitions (e.g. $E_A^M \rightarrow E_A^M + f_B^A E_B^M$ or $d^\alpha \rightarrow d^\alpha + g_A^\alpha \Omega_A^\alpha_{\pm}$), this is the most general sigma model action which is invariant under spacetime super-reparameterizations, under the gauge transformations ($\delta A_B^I = \nabla_B \Lambda_I$, $\delta W^I_\alpha = f^I_{JK} \Lambda^J W^K_\alpha$, $\delta j_I = f^I_{JK} \Lambda^J j_K$ and ($\delta B_{AB} = \nabla_A \Lambda_B - \nabla_B \Lambda_A$), and under classical $N=2$ worldsheet superconformal transformations.

Classical worldsheet superconformal transformations are defined by taking the Poisson bracket with the $N=2$ stress-energy tensor. (Unfortunately, the quantum nature of $\rho$ is an obstacle to making $N=2$ worldsheet supersymmetry manifest.) For example, from the Poisson bracket with $G$, $\delta Z^M = e^{i\rho} d^\alpha E^{M}_{\alpha}$, $\delta \tilde{d}^\alpha = e^{i\rho} \Pi^{\alpha}_{\pm} d^\alpha$, and $\delta \Pi^\alpha_{\pm} = e^{i\rho} d^\alpha T_{AB}^A \Pi^B_{\pm} + \delta_{\alpha}^A \partial_{\pm} (e^{i\rho} d^\alpha)$ where $T_{AB}^C$ is the torsion defined in equation (3.2). Under these transformations, (4.1) and (4.2) should be invariant up to terms proportional to the $N=2$ stress-energy tensor. (Terms proportional to the stress-energy tensor can be absorbed by transforming the worldsheet supergravity fields. We would like to thank Peter van Nieuwenhuizen for suggesting this method.) It is straightforward to check that classical invariance implies that $E_M^A$, $B_{AB}$, and $A_M$ satisfy the following constraints on their field strengths:

\[
\begin{align*}
\text{Heterotic} & : & 1) \quad & T^{c}_{\alpha\beta} = T^{c}_{\alpha\beta} \gamma = 0, \; F^{I}_{\alpha\beta} = 0; & (4.3) \\
& & 2) \quad & T^{c}_{\alpha\beta} = \sigma^c_{\alpha\beta}, \; T^{(bc)}_{\alpha} = 0, \; F^{I}_{\alpha\beta} \gamma = 0 \\
& & 3) \quad & T^{c}_{\alpha\beta} = 0, \; H^{c}_{\alpha\beta} = \sigma^c_{\alpha\beta}.
\end{align*}
\]
Type II: 1) $T_{\alpha j\beta k}^c = T_{\alpha j\beta k}^c \hat{\gamma} = 0$; 

2) $T_{\alpha j\beta k}^c = \epsilon_{j\beta k}^{\alpha\gamma} \sigma_{\alpha\beta}^c$, $T_{\alpha j(bc)} = 0$, $F_{\alpha\beta}^I = 0$; 

3) $T_{\alpha c}^{\alpha\beta} = \sigma_{ca\alpha}^{\alpha\beta} \hat{\gamma}$, $T_{\alpha c}^{\alpha\beta} = \sigma_{ca\alpha}^{\alpha\beta} \hat{\gamma}$, $T_{\alpha c}^{\alpha\beta} = T_{\alpha c}^{\alpha\beta} = 0$, $F_{\alpha\beta}^I = 0$; 

The first type of constraints are the usual representation preserving constraints which allow a consistent definition of chiral (and twisted-chiral) superfields. The second type are conventional constraints which define the vector components of the super-vierbein and super Yang-Mills gauge field, $E_a^M$ and $A_a^I$, in terms of the spinor components. Note that conventional constraints for the connections $w_{AB}^C$, $\Gamma_A$, and $\Gamma^{jk}_A$ can be defined arbitrarily since these superfields never appear in the sigma model action.

The third type of constraints are conformal-breaking constraints, which are necessary since the sigma model action is not invariant under the spacetime scale transformations that transform $\delta E_a^M = \Lambda E_a^M$. In their absence for the heterotic superstring, $T_{\alpha c}^{\alpha\beta}$ and $H_{\alpha c}^{\alpha\beta}$ would satisfy the equations $T_{\alpha c}^{\alpha\beta} = R_{ca}^{\alpha\beta}$ and $H_{\alpha c}^{\alpha\beta} = L_{ca}^{\alpha\beta}$ from equation (3.4). So these constraints break conformal invariance for the heterotic sigma model by gauge-fixing $L = 1$ (which implies $R = 0$ by the linear condition $((\nabla)^2 + R)L = 0$).

In the absence of conformal-breaking constraints for the Type II sigma model, $T_{\alpha c}^{\alpha\beta}$ and $H_{\alpha c}^{\alpha\beta}$ would satisfy the conditions of equation (3.5) [26]. Therefore conformal and $SU(2)/U(1)$ invariance is broken in the Type II sigma model by gauge-fixing $L_{jk} = \delta_{jk}$ (which as will be shown in the following paragraphs, implies $N_{\alpha\beta} = C_{jk}^{\alpha\beta} \delta_{jk} = S^{++} = S^{--} = 0$). The conformal-breaking constraint also implies that $P_{\alpha\beta} = W_{\alpha\beta} + \epsilon_{\alpha\beta} S^{++}$ and that $Q_{\alpha\beta} = G_{\alpha\beta}^{++}$.

In the tensor calculus approach to constructing actions for N=2 supergravity coupled to matter [27], a “minimal multiplet” is employed as a starting point, consisting of conformal supergravity coupled to a vector multiplet, in the scale (+U(1)) gauge $W^{(0)} = 1$. Since for string theory it is more useful to apply the string gauge $L_{jk} = \delta_{jk}$, it is instructive to compare the field content of the two gauges. As for N=1 [25], both gauges imply the vanishing of some of the torsions.
arising from the superconformal covariant derivatives. From (3.3), the dimension 1 torsions are originally \( W_{\alpha\beta}, S_{(jk)}, N_{\alpha\beta} \), and \( G_{j}^{k}{}_{a} \) (hermitian).

The basic procedure is: (1) Fix one of the two superfields \( W^{(0)} \) or \( L_{jk} \). (Although one could also fix the tensor compensator \( L_{jk}^{(0)} \), this would give the same field content as fixing the non-compensator \( L_{jk} \).) That breaks scale invariance (and more), but it doesn’t introduce a dimension 0 scalar or dimension \( \frac{1}{2} \) spinor (which come from fixing the second compensator), so you get no new tensors. (2) Since scale invariance is fixed, the torsions are now tensors. (Before, they were noncovariant under scale transformations.) Thus, at \( \theta = 0 \) they correspond to gauge invariant component fields. By a component analysis, we know what the component fields are, so match them up. The superspace is determined by looking at dimension 1 tensors, and there is no lower dimension stuff, so a linearized analysis is sufficient.

These are the relevant component fields: (1) From conformal supergravity, there is the auxiliary field \( W_{\alpha\beta} \), so that torsion will always survive. There are also the U(2) gauge fields \( G_{j}^{k}{}_{a} \), which will become gauge invariant only if they find scalars to eat. (2) A vector multiplet has a gauge vector, with the dimension 1 field strength \( F_{\alpha\beta} \), as well as the hermitian auxiliary fields \( D_{(jk)} \). It also has a complex physical scalar of dimension 0, whose real part is eaten by the conformal graviton, and whose imaginary part is eaten by the U(1) gauge vector. (3) A tensor multiplet has a dimension 1 field strength \( H_{a} \), and a complex scalar auxiliary field. It also has the hermitian physical fields \( L_{(jk)} \) of dimension 0, of which one is eaten by the graviton, the remaining two by the SU(2)/U(1) gauge vectors. (These dimensions differ from the original conformal weights, since scale invariance will be broken.)

The net result is then, from the original torsions \( W_{\alpha\beta}, S_{(jk)}, N_{\alpha\beta} \), and \( G_{j}^{k}{}_{a} \): (1) For the standard gauge \( W^{(0)} = 1 \), \( W_{\alpha\beta} \) survives as the conformal supergravity auxiliary, while \( N_{\alpha\beta} = F_{\alpha\beta} \). \( S_{(jk)} \) becomes hermitian, identified as the vector multiplet auxiliaries. The U(1) part of \( G_{j}^{k}{}_{a} \) survives, as the U(1) gauge field that ate the imaginary part of the vector multiplet scalar, but the SU(2) part does not, since it remains gauge. The net result is

\[
W^{(0)} = 1 \quad \Rightarrow \quad S_{jk} = \bar{S}_{jk}, \quad G_{(jk)a} = 0. \tag{4.5}
\]
(2) For the string gauge $L_{jk} = \delta_{jk}$, $W_{\alpha\beta}$ again survives, but there is no field to correspond to $N_{\alpha\beta}$. Since the tensor multiplet has only a complex scalar for auxiliaries, with vanishing weight under the unbroken U(1) subgroup of SU(2), it becomes $S_{+-}$ (still complex), while $S_{++}$ and $S_{--}$ vanish. The tensor field strength $H_a$ replaces the U(1) gauge field in the U(1) part of $G^k_{j\,a}$ (as in the analogous N=1 case), while $G_{++a}$ and $G_{--a}$ survive as the SU(2)/U(1) gauge vectors by eating the corresponding physical scalars of the tensor multiplet. But $\delta_{jk} G_{j\,a}$ dies for lack of a scalar to eat. The net result for this case is then

$$L_{jk} = \delta_{jk} \Rightarrow S_{++} = S_{--} = N_{\alpha\beta} = \delta_{jk} G_{j\,a} = 0.$$  \hspace{1cm} (4.6)

Actually, the third set of constraints for the heterotic superstring is slightly inconvenient. Because of anomalous transformations of the anti-symmetric tensor field, the linear superfield $L$ is not invariant under Yang-Mills and Lorentz gauge transformations. The appropriate gauge-invariant generalization of $L$ is

$$\tilde{L} = L + \alpha' (c_1 \Omega_{YM} + c_2 \Omega_{Lorentz})$$ \hspace{1cm} (4.7)

where $\Omega_{YM}$ is the super Yang-Mills Chern-Simons form satisfying $((\nabla)^2 + R)\Omega_{YM} = W^{I\alpha} W_{I\alpha}$, $\Omega_{Lorentz}$ is the super Yang-Mills Chern-Simons form satisfying $((\nabla)^2 + R)\Omega_{Lorentz} = W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}$, $c_1$ and $c_2$ are compactification-dependent constants, and $W_{\alpha\beta\gamma}$ is the chiral field strength of conformal supergravity.

Since $L = 1$ breaks Yang-Mills and Lorentz invariance, it is more convenient to choose the gauge $\tilde{L} = 1$, which implies $R = c_1 W^{I\alpha} W_{I\alpha} + c_2 W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}$ from the linear condition on $L$. Therefore, in order to preserve manifest Yang-Mills and Lorentz invariance, the third type of constraints in (4.3) should be modified for the heterotic superstring to

$$3) \; T_{\alpha c} = \alpha' \sigma_{\alpha c} (c_1 W^{I\gamma} W_{I\gamma} + c_2 W_{\gamma\delta\kappa} W^{\gamma\delta\kappa}), \; \tilde{H}_{\alpha\beta c} = \sigma_{\alpha\beta c}.$$ \hspace{1cm} (4.8)

where

$$\tilde{H}_{MNP} = H_{MNP}$$ \hspace{1cm} (4.9)

$$+ \alpha' Tr(c_1 (A_M \partial_N A_P) + 2 A_{[M} A_{N] A_P}) + c_2 (w_{[M} \partial_N w_{P]} + 2 w_{[M} w_{N] w_{P]})$$

includes the contribution of the Chern-Simons forms ($A^I_M$ is the Yang-Mills gauge field and $w_{MBC}$ is the Lorentz spin connection).
Because these modifications to the heterotic torsion constraints are higher-order in $\alpha'$, they cannot be checked using classical worldsheet superconformal invariance. To justify them, one should check that with these modified constraints, quantum $N=(2,0)$ superconformal invariance implies equations of motion which are invariant under Yang-Mills and Lorentz gauge transformations. Note that, with the exception of the representation-preserving constraints, modifications to the torsion constraints can always be absorbed by compensating transformations on the spacetime superfields. So any $\alpha'$ corrections to the non-representation-preserving constraints can be replaced by $\alpha'$ corrections to the relations between spacetime superfields in the sigma model action. This is similar to the situation in standard $N=2$ worldsheet supersymmetric sigma models, where worldsheet supersymmetry implies certain relations between coupling constants, but these relations may receive quantum corrections.

4.2. *The Fradkin-Tseytlin term*

So the classical term in the sigma model action is (4.1) or (4.2) with the constraints of (4.3) (and (4.8)) or (4.4), and one now needs to construct a Fradkin-Tseytlin term. As usual, this term is necessary because the massless vertex operator for the physical scalar, $\int dz^+ dz^- \partial_- x^m \partial_+ x_m$, couples to the determinant of the spacetime vierbein in the sigma model. So instead of coupling to this massless vertex operator, the spacetime dilaton couples to the N=2 supercurvature in a Fradkin-Tseytlin term.

It is often incorrectly stated that the dilaton must sit in the same supersymmetry multiplet as the anti-symmetric tensor, which leads to the erroneous conclusion that the dilaton must couple classically. When spacetime scale invariance is used to gauge-fix the non-compensator tensor multiplet (instead of the more conventional gauge-fixing of the conformal compensator), the physical anti-symmetric tensor sits in the multiplet of Poincaré supergravity. Although this version of Poincaré supergravity does contain a scalar field, the scalar plays the role of the determinant of the vierbein, rather than the role of the dilaton. So the dilaton must sit in the only remaining superfield, which is the conformal compensator $\Phi = e^{-\phi}$. (Note that $\phi = - \log \Phi$, rather than $\Phi$, will appear directly in the sigma model.)
For the heterotic superstring, the logarithm of the conformal compensator couples to N=(2,0) worldsheet supercurvature, which is described by a worldsheet chiral superfield Σ and its complex conjugate Σ. These worldsheet superfields are defined by 

$$[\overline{D}\overline{\kappa}, D^+] = \Sigma (m + iy)$$

and 

$$[D\kappa, D^+] = \Sigma (m - iy)$$

where m is the Lorentz generator, y is the U(1) generator, Dκ and D¯κ are the covariant fermionic derivatives, D+ is the covariant right-moving bosonic derivative, and {Dκ, D¯κ} = D− where D− is the covariant left-moving bosonic derivative. In components, Σ = χκ(r + if) and Σ = χκ(r - if) where κ and κ are the super-worldsheet anti-commuting parameters, r is the two-dimensional curvature, f is the field strength of the worldsheet U(1) gauge field, and χ, ¯χ are the field strengths of the worldsheet gravitini[28].

If and only if Φ = e−φ is superspace chiral (and therefore {G, φ} = 0), one can construct the worldsheet supersymmetric Fradkin-Tseytlin term

$$\int dz^+ dz^- (\int d\kappa e^{-1} \phi_\kappa \Sigma + \int d\bar{\kappa} e^{-1} \bar{\phi}_{\bar{\kappa}} \bar{\Sigma}) = \int dz^+ dz^- e^{-1} [(\phi + \bar{\phi})r + i(\phi - \bar{\phi})f + [G, \phi] \chi + [\bar{G}, \bar{\phi}] \bar{\chi}] =$$

$$\int dz^+ dz^- e^{-1} [(\phi + \bar{\phi})r + i(\phi - \bar{\phi})f + \frac{1}{\sqrt{\alpha'}} (e^{i\rho} d^\alpha \nabla_\alpha \phi \chi + e^{-i\rho} \bar{d}^\alpha \nabla_\alpha \bar{\phi} \bar{\chi})]$$

where ε is the super-worldsheet chiral density and e is the ordinary worldsheet density, we ignore possible double poles in the OPE of G with V (as in the vertex operators of section 2.2), and φκ is a worldsheet and spacetime superfield defined such that Qκφκ = [G, φκ] and Qκφκ = [G, φκ] where Qκ and Qκ are the worldsheet supersymmetry generators. (On a flat worldsheet, Qκ = ∂κ + κ∂− and Qκ = ∂κ + κ∂−, so φκ = φ + κ[G, φ] − κκ∂−φ satisfies this definition. It is easily checked that φκ is worldsheet chiral since ∂κφ = (κκ − κκ∂−)φ = 0.) Note that the Fradkin-Tseytlin term can be written independently of ΓA since

$$\nabla_\alpha \phi = E_\alpha^M \partial_M \phi + \Gamma_\alpha = E_\alpha^M \partial_M (\phi + \bar{\phi})$$

From the form of the Fradkin-Tseytlin term, it is reasonable to call the θ = ¯θ = 0 component of φ + ¯φ the dilaton, and the θ = ¯θ = 0 component of φ − ¯φ the axion. Just as the dilaton zero mode couples to the worldsheet Euler number, the axion zero mode couples to the worldsheet U(1) instanton number. Note that this axion is the compensating field for spacetime U(1) transformations, and not the dual of the anti-symmetric tensor.
When the heterotic Fradkin-Tseytlin term is in N=(2,0) superconformal
gauge, $\phi - \bar{\phi}$ couples to $a_-$ in the same way as the $\rho$ field in (2.1), so one can
combine these couplings into $\int dz^+ dz^- \partial_+ (\rho + i(\phi - \bar{\phi})) a_-$. This term is invariant
under the spacetime U(1) transformation

$$\delta \phi = \Lambda, \; \delta \bar{\phi} = -\Lambda, \; \delta \Gamma_M = \nabla_M \Lambda, \; \delta E_\alpha^M = \Lambda E_\alpha^M, \; \delta E_\dot{\alpha}^M = -\Lambda E_\dot{\alpha}^M$$

(4.11)

if one also transforms the worldsheet variables

$$\delta \rho = -2i \Lambda, \; \delta d^\alpha = \Lambda d^\alpha, \; \delta \bar{d}^{\dot{\alpha}} = -\Lambda \bar{d}^{\dot{\alpha}}.$$

Although it may look unusual for worldsheet variables to transform under a space-
time gauge transformation, it is similar to the transformation of heterotic chiral
fermions, $\zeta_q$, under a Yang-Mills gauge transformation.

However, the kinetic term for the $\rho$ variable, $\int dz^+ dz^- \partial_- \rho \partial_+ \rho$, is not in-
viant under the transformation of (4.11). We must therefore modify the kinetic
term to $\int dz^+ dz^- (\partial_- (\rho + i(\phi - \bar{\phi})) \partial_+ (\rho + i(\phi - \bar{\phi})))$, which combines with the
Fradkin-Tseytlin term in superconformal gauge to form the worldsheet and space-
time covariant expression

$$\int dz^+ dz^- [\partial_+ (\rho + i(\phi - \bar{\phi})) D_- (\rho + i(\phi - \bar{\phi}))$$

(4.12)

$$+(\phi + \bar{\phi}) r + \frac{1}{\sqrt{\alpha'}} (e^{i\rho} d^\alpha \nabla_\alpha \phi \chi + e^{-i\rho} \bar{d}^{\dot{\alpha}} \nabla_{\dot{\alpha}} \bar{\phi} \bar{\chi})].$$

(Although it may seem strange to have a term in the sigma model action which
is quadratic in $\phi$, this also occurs in the bosonic string if one couples the
dilaton $\varphi$ to worldsheet ghosts, and then integrates out the ghosts to obtain
$\int dz^+ dz^- (\partial_+ \varphi \partial_- \varphi + \varphi r)$) Note that the equation of motion for $a_-$ now
implies that the right-moving part of $\rho$ satisfies $\partial_+ \rho = -i \partial_+ (\phi - \bar{\phi})$.

For the Type II superstring, the logarithms of the conformal compensators
couple to N=(2,2) worldsheet supercurvature, which is described by a worldsheet
chiral superfield $\Sigma_c$ and its complex conjugate $\Sigma_c$, and by a worldsheet twisted-
chiral superfield $\Sigma_{tc}$ and its complex conjugate $\Sigma_{tc}$. (We use the U(1)$\times$U(1)
form of N=(2,2) supergravity which contains two independent U(1) gauge fields.[19])
These worldsheet superfields are defined by $\{\bar{D}_\bar{\kappa}, \hat{D}_\bar{\kappa}\} = \Sigma_c (m + iy)$, $\{D_\kappa, \hat{D}_\kappa\} =$
\( \Sigma_c(m - iy), \{ \bar{D}_{\kappa}, \hat{D}_{\kappa} \} = \Sigma_{tc}(m + iy), \{ D_{\kappa}, \hat{D}_{\kappa} \} = \Sigma_{tc}(m - iy) \), where \( m \) is the Lorentz generator, \( y \) and \( \bar{y} \) are the U(1) \( \times \) U(1) generators, and \( D_{\kappa}, \bar{D}_{\kappa}, \hat{D}_{\kappa}, \hat{\bar{D}}_{\kappa} \) are the covariant fermionic derivatives satisfying \( \{ D_{\kappa}, \bar{D}_{\kappa} \} = \bar{D}_{-} \) and \( \{ \hat{D}_{\kappa}, \hat{\bar{D}}_{\kappa} \} = \hat{D}_{+} \).

In components, \( \Sigma_c = b + \kappa(\chi + \psi) + \bar{\kappa}(\bar{\chi} + \bar{\psi}) + \kappa \bar{\kappa}(r + i(f + \hat{f}) + d) \) and \( \Sigma_{tc} = c + \kappa(\bar{\chi} - \bar{\psi}) + \bar{\kappa}(\bar{\chi} - \bar{\psi}) + \kappa \bar{\kappa}(r + i(f - \hat{f}) - d) \) where \( r \) is the two-dimensional curvature, \( f \) and \( \hat{f} \) are the right and left-moving worldsheet U(1) field strengths, \( \chi \) and \( \bar{\chi} \) are the complex gravitino field-strengths, \( b \) and \( c \) are complex weight 1 auxiliary fields, \( \psi \) and \( \bar{\psi} \) are complex weight \( \frac{3}{2} \) auxiliary fields, and \( d \) is a real weight 2 auxiliary field. These additional auxiliary fields are present since we are using the U(1) \( \times \) U(1) formulation of N=(2,2) supergravity.

If, and only if, \( \Phi_c = e^{-\phi_c} \) is superspace chiral and \( \Phi_{tc} = e^{-\phi_{tc}} \) is superspace twisted-chiral, one can construct the following Type II worldsheet Fradkin-Tseytlin term:

\[
\int dz^+ dz^- (\int d\kappa d\bar{\kappa} \epsilon_c^{-1} \phi_{\kappa \bar{\kappa}}^c \Sigma_c + \int d\kappa d\bar{\kappa} \epsilon_{tc}^{-1} \phi_{\kappa \bar{\kappa}}^{tc} \Sigma_{tc}) = (4.13)
\]

\[
\int d\kappa d\bar{\kappa} \epsilon_{tc}^{-1} \phi_{\kappa \bar{\kappa}}^{tc} \Sigma_{tc} + \int d\kappa d\bar{\kappa} \epsilon_{tc}^{-1} \phi_{\kappa \bar{\kappa}}^{tc} \Sigma_{tc}) = \int dz^+ dz^- e^{-1}[ (\phi_c + \bar{\phi}_c + \phi_{tc} + \bar{\phi}_{tc})r + i(\phi_c - \bar{\phi}_c + \phi_{tc} - \bar{\phi}_{tc})f \\
+ i(\phi_c - \bar{\phi}_c - \phi_{tc} + \bar{\phi}_{tc})\hat{f} + \frac{1}{\sqrt{\alpha'}} (e^{i\rho} d^\alpha \nabla_\alpha (\phi_c + \phi_{tc}) \bar{\chi} + e^{-i\rho} d^\alpha \hat{\nabla}_\alpha (\phi_c + \phi_{tc}) \bar{\chi} + \\
e^{i\rho} d^\alpha \hat{\nabla}_\alpha (\phi_c + \phi_{tc}) \bar{\chi} + e^{-i\rho} d^\alpha \hat{\nabla}_\alpha (\phi_c + \phi_{tc}) \bar{\chi}) + S_{aux}]
\]

where \( \epsilon_c \) and \( \epsilon_{tc} \) super-worldsheet chiral and twisted-chiral densities, \( \phi_{c}^c \) and \( \phi_{tc}^{tc} \) are defined as in the heterotic case, and \( S_{aux} \) describes the coupling of \( \phi \) to the worldsheet auxiliary fields.

Although classical worldsheet supersymmetry requires that \( \Phi_c = e^{-\phi_c} \) and \( \Phi_{tc} = e^{-\phi_{tc}} \) are chiral and twisted-chiral, it does not require that they are restricted superfields, i.e. they satisfy the reality constraints \( (\nabla)^2 \Phi_c = (\bar{\nabla})^2 \bar{\Phi}_c \) and \( (\nabla)^2 \Phi_{tc} = (\bar{\nabla})^2 \bar{\Phi}_{tc} \). As in the case of the superfields for the compactification moduli, quantum \( N = (2,2) \) superconformal invariance is expected to imply these reality conditions (or a suitable modification), as well as the equations of motion. Note that \( S_{aux} \) vanishes when \( \phi_c + \bar{\phi}_c = \phi_{tc} + \bar{\phi}_{tc} \) (i.e. when \( \Phi_c \bar{\Phi}_c = \Phi_{tc} \bar{\Phi}_{tc} \) since

30
d couples to \((\phi_c + \tilde{\phi}_c - \phi_{tc} - \tilde{\phi}_{tc})\), \(\psi\) couples to \(\nabla_{\alpha}(\phi_c - \phi_{tc}) = \nabla_{\alpha}(\phi_c + \tilde{\phi}_c - \phi_{tc} - \tilde{\phi}_{tc})\), and \(b\) couples to \(\nabla_{\alpha}\tilde{\nabla}_{\alpha}\phi_c = \nabla_{\alpha}\tilde{\nabla}_{\alpha}(\phi_c + \tilde{\phi}_c - \phi_{tc} - \tilde{\phi}_{tc})\). Because \(S_{aux}\) violates \(N=(2,2)\) superconformal invariance in a manner which does not appear to be cancelled by anomalies in the classical term of the sigma model, \(\Phi_c \tilde{\Phi}_c = \Phi_{tc} \tilde{\Phi}_{tc}\) is expected to be the on-shell superspace equation of motion (at least to lowest order in \(\alpha'\)). Note that when \(L_{jk} = \delta_{jk}\) and all CY fields are set to zero, \(\Phi_c \tilde{\Phi}_c = \Phi_{tc} \tilde{\Phi}_{tc}\) is indeed the equation of motion implied by varying the weight 2 conformal supergravity scalar in the low-energy Type II effective action at the end of section 5.

From the form of the Type II Fradkin-Tseytlin term, it is reasonable to call the \(\theta = \bar{\theta} = \bar{\theta} = \bar{\theta} = 0\) component of \(\phi_c + \tilde{\phi}_c + \phi_{tc} + \tilde{\phi}_{tc}\) the dilaton, and the \(\theta = \bar{\theta} = \bar{\theta} = \bar{\theta} = 0\) component of \(\phi_c - \tilde{\phi}_c \pm (\phi_{tc} - \tilde{\phi}_{tc})\) the axions. Just as the dilaton zero mode couples to the worldsheet Euler number, the axion zero modes couple to the worldsheet right and left-moving \(U(1)\) instanton number.

As in the heterotic case, one can combine the Type II Fradkin-Tseytlin term with the kinetic terms for \(\rho\) and \(\dot{\rho}\) to obtain

\[
\int \! dz^+ dz^- [\partial_+(\rho + i(\phi_c - \tilde{\phi}_c + \phi_{tc} - \tilde{\phi}_{tc}))D_-(\rho + i(\phi_c - \tilde{\phi}_c + \phi_{tc} - \tilde{\phi}_{tc}))] + \partial_-(\dot{\rho} + i(\phi_c - \tilde{\phi}_c - \phi_{tc} + \tilde{\phi}_{tc}))D_+(\dot{\rho} + i(\phi_c - \tilde{\phi}_c - \phi_{tc} + \tilde{\phi}_{tc})) + (\phi_c + \tilde{\phi}_c + \phi_{tc} + \tilde{\phi}_{tc}) r + \frac{1}{\sqrt{\alpha'}}(e^{i\rho} d^\alpha \nabla_\alpha (\phi_c + \phi_{tc}) \dot{\chi} + e^{-i\rho} d^\alpha \tilde{\nabla}_\alpha (\phi_c + \phi_{tc}) \dot{\tilde{\chi}} + e^{i\rho} d^\alpha \tilde{\nabla}_\alpha (\phi_c + \phi_{tc}) \dot{\chi} + e^{-i\rho} d^\alpha \nabla_\alpha (\phi_c + \phi_{tc}) \dot{\tilde{\chi}}) + S_{aux}]
\]

which is invariant under the spacetime \(U(1) \times U(1)\) transformations

\[
\delta \phi_c = \Lambda + \hat{\Lambda}, \quad \delta \tilde{\phi}_c = -\Lambda - \hat{\Lambda}, \quad \delta \phi_{tc} = \Lambda - \hat{\Lambda}, \quad \delta \tilde{\phi}_{tc} = -\Lambda + \hat{\Lambda}, \quad \delta \rho = -4i\Lambda, \quad \delta d^\alpha = \Lambda d^\alpha, \quad \delta d^\alpha = -\Lambda d^\alpha, \quad \delta \dot{\rho} = 2i\hat{\Lambda}, \quad \delta \dot{d}^\alpha = \hat{\Lambda} \dot{d}^\alpha, \quad \delta \dot{\tilde{d}}^\alpha = -\hat{\Lambda} \dot{d}^\alpha.
\]

Note that \(S_{aux}\) is separately invariant under these transformations since it only depends on the \(U(1) \times U(1)\) invariant combination \(\phi_c + \tilde{\phi}_c - \phi_{tc} - \tilde{\phi}_{tc}\).
5. Effective Actions

The most straightforward method for constructing the effective action is to use $\beta$-function methods to compute the low-energy equations of motion, and then look for an action which yields these equations. Although this $\beta$-function method is necessary for computing the explicit form of the action, we can learn a lot just by requiring that the action has the symmetries predicted by the sigma model. (By “effective action”, we always mean the true effective action, rather than the possibly anomalous Wilson effective action.) Since the superfields in the sigma model are conformally gauge-fixed by $\tilde{L} = 1$ or $L_{jk} = \delta_{jk}$, it will be convenient to first construct superspace actions in this conformal gauge, and then remove the gauge-fixing condition to obtain conformally-invariant superspace actions.

5.1. Heterotic superspace effective actions

From the form of the heterotic sigma model, the dimensionless coupling constant $\lambda$ (which couples to string loops) can be absorbed in the effective action by rescaling the conformal compensator $\Phi \rightarrow \lambda \Phi$. This means that string loops are counted not just by one component field, but by an entire superfield.

Similarly, the dimensionful coupling constant $\alpha'$ (which couples to the classical term in the sigma model) can be absorbed by rescaling $E_M^a \rightarrow (\alpha')^{1/2} E_M^a$ and $E_M^\alpha \rightarrow (\alpha')^{1/4} E_M^\alpha$. ($d^\alpha$ must also rescale to $(\alpha')^{3/4} d^\alpha$ in the sigma model, but this is just a redefinition of worldsheet variables.) Because the effective action is defined to be invariant under the conformal transformation $\delta E_M^a = -\Lambda E_M^a$, $\delta E_M^\alpha = -\frac{1}{2} \Lambda E_M^\alpha$, $\delta \tilde{L} = 2 \Lambda \tilde{L}$, $\delta \Phi = \frac{3}{4} \Lambda \Phi$, $\alpha'$ can be absorbed in the conformally-invariant action by rescaling $\tilde{L} \rightarrow \alpha' \tilde{L}$ and $\Phi \rightarrow (\alpha')^{3/4} \Phi$.

In terms of the conformally gauge-fixed superfields which appear in the heterotic sigma model, the most general N=1 superspace effective action which is U(1) invariant and satisfies the above properties is:

$$\sum_{g=0}^{\infty} \lambda^{2g-2} \int d^4 x (\int d^2 \theta d^2 \bar{\theta} E^{-1} \sum_{n_I = -\infty}^{\infty} \Phi^1{g+n_I} \bar{\Phi}^{1-g-n_I} \sum_{p=0}^{\infty} (\alpha')^{\frac{p-2}{2}} K_{g,n_I}^{p} (5.1)$$

$$+ \int d^2 \theta \Phi^2 - 2g(\alpha')^{\frac{3g-3}{2}} F_g + \int d^2 \bar{\theta} \bar{\Phi}^2 - 2g(\alpha')^{\frac{3g-3}{2}} \bar{F}_g$$

where $K_{g,n_I}^{p} = (K_{g,-n_I}^{p})^*$ is a general superfield with U(1) weight $n_I$ and conformal weight $p$, and $F_g$ is a chiral superfield with U(1) weight $g$ and conformal
weight $3g$ (the conformal weight of an N=1 chiral superfield must be three times its U(1) weight). We have explicitly separated out the $\Phi$ and $\bar{\Phi}$ zero modes in (5.1), so $K_{p,g,n,I}$ and $F_g$ are defined to be independent of these zero modes.

Note that while the real $d^2\theta d^2\bar{\theta}$ integral is written with a factor of $E^{-1} = sdet(E_A^M)^{-1}$ to make it a density, this real factor can not appear in the chiral $d^2\theta$ integrand. However, in a chiral basis where $\bar{\nabla}_{\dot{\alpha}} = \delta_{\dot{\alpha}}^{\dot{\alpha}} \partial_{\dot{\alpha}}$, $\Phi$ acts as the corresponding density. (We will always be using a chiral basis when defining chiral integrals.) In fact, all truly chiral superfields are such densities, with density weight corresponding directly to conformal weight. This simplifying feature of chiral integrals is one reason why their component evaluation is simpler than integrals over full superspace. A similar procedure can also be applied to non-supersymmetric theories, with gravity written as conformal gravity plus a compensating scalar: By writing all fields as densities, all factors of $\sqrt{-g}$ can be removed from the action. This procedure is useful for string theory, because it is the density form of the dilaton field, with the same weight as $(-g)^{3/2}$, that is invariant under T-duality. For example, the low-energy action of the bosonic closed string can be written with this dilaton field as the only density.

The effective action of (5.1) can be easily written in conformally-invariant form by simply inserting an appropriate power of $\tilde{L} = L + \alpha'(c_1 \Omega_{YM} + c_2 \Omega_{Lorentz})$ and removing the conformal gauge-fixing condition $\tilde{L} = 1$. Since the non-chiral measure carries conformal weight $-3$, the chiral measure carries conformal weight $-2$, $\Phi$ carries conformal weight $3/2$, and $\tilde{L}$ carries conformal weight $+2$, the conformally-invariant form of (5.1) is

$$\sum_{g=0}^{\infty} \lambda^{2g-2} \int d^4x \quad (5.2)$$

$$\left( \int d^2\theta d^2\bar{\theta} E^{-1} \sum_{n_I = -\infty}^{\infty} \Phi^{1-g+n_I} \bar{\Phi}^{1-g-n_I} \sum_{p=0}^{\infty} (\alpha')^{p/2} \tilde{L}^{3g-1-p} K_{p,g,n_I}^p \right)$$

$$+ \int d^2\theta \Phi^{2-2g}(\alpha')^{3g-3/2} F_g + \int d^2\bar{\theta} \bar{\Phi}^{2-2g}(\alpha')^{3g-3/2} \bar{F}_g).$$

(Note that $K_{p,g,n_I}^p$ may involve scale-invariant functions of $\tilde{L}$ such as $\partial_m (\log \tilde{L})$.)

Since the U(1) weight $g$ of $F_g$ must be absorbed by $\Phi^{2-2g}$, $F_g$ can only occur with coupling $\lambda^{2g-2}$. This means that $F_g$ cannot receive perturbative or
non-perturbative quantum corrections. In other words, after expanding $\Phi$ and $\tilde{L}$ around their vacuum expectation values (which can be set to one by rescaling $\lambda$ and $\alpha'$), $F_g$ appears only once in the perturbative expansion. Of course, this does not prevent quantum corrections if a chiral $F$-term can also be written as a non-chiral $D$-term, which could then receive quantum corrections through $U(1)$-neutral factors of $\Phi \bar{\Phi}$.

Note that any chiral $F$-term can be written as a non-chiral $D$-term if one allows the non-local operator $((\nabla)^2/\Box)$. However, non-local $1/\Box$ terms in the effective action come from anomalies, which are not expected to receive quantum corrections\[30\]. Furthermore, some chiral $F$-terms can be written as local non-chiral $D$-terms by pulling a factor of $((\nabla)^2/\Box)$ off a field-strength. Normally, this would not lead to quantum corrections, since adding $\Phi \bar{\Phi}$ to the $D$-term would break gauge invariance. However, in the heterotic effective action where the tensor multiplet transforms anomalously under Lorentz and Yang-Mills gauge transformations, this type of $D$-term can sometimes lead to quantum corrections.

For example, the $F$-term

$$\lambda^{2g-2} \alpha'^{\frac{3g-3}{2}} \int d^4x \int d^2\theta \, \Phi^{2-2g}(W^I_\alpha W^{\alpha I})^g \tau_g(M^{(i)})$$

(5.3)

can also be written as the non-chiral $D$-term

$$\lambda^{2g-2} \alpha'^{\frac{3g-3}{2}} \int d^4x \int d^2\theta d^2\bar{\theta} E^{-1} \Phi^{2-2g}(W^I_\alpha W^{\alpha I})^g \Omega_{YM} \tau_g(M^{(i)})$$

(5.4)

where $((\nabla)^2/\Box)\Omega_{YM} = W^I_\alpha W^{\alpha I}$ and $\tau_g(M^{(i)})$ is a chiral function of the compactification moduli. This non-chiral $D$-term can get corrections, for example, from

$$\lambda^{2(g+n)-2} \alpha'^{\frac{3g-3}{2}} \int d^4x \int d^2\theta d^2\bar{\theta}$$

(5.5)

$$E^{-1} \Phi^{2-2g-n} \bar{\Phi}^{-n}(W_\alpha W^{\alpha})(\Omega_{YM} \bar{\tau}_g(M^{(i)}, \bar{M}^{(i)}) \tilde{L}^{\frac{3g}{2}},$$

which would give quantum corrections to $\tau_g$ of $\lambda^{2n} \bar{\tau}_g(M^{(i)}, \bar{M}^{(i)})$. So the $F$-term is not protected against quantum corrections. Note, however, that because (5.5) is not gauge-invariant, it must occur in the combination

$$\lambda^{2g-2} \alpha'^{\frac{3g-5}{2}} \int d^4x \int d^2\theta d^2\bar{\theta} E^{-1} \Phi^{2-2g-n} \bar{\Phi}^{-n}$$

(5.6)
\[(W_\alpha W^\alpha)^{g^{-1}}(L + \alpha'(c_1 \Omega_{YM} + c_2 \Omega_{Lorentz})) \tilde{\tau}_g(M^{(i)}, \tilde{M}^{(i)}) L^{\frac{3g}{2}}.\]

This means that quantum corrections to the \(F\)-term of (5.3) are related by the proportionality constant \(c_1/c_2\) to quantum corrections of the \(F\)-term

\[\lambda^{2g-2} \alpha' \frac{e^{2g-2}}{2} \int d^4 x \int d^2 \theta \Phi^2 W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} (W_\delta W^\delta) g^{-1} \tau_g(M^{(i)}). \tag{5.7}\]

At low energies, only \(K_{0,0}^0\) contributes to the non-chiral \(D\)-term in the tree-level effective action. \((n_I = 0\) since \(U(1)\)-charged superfields have non-zero conformal weight.) Since only the compactification moduli have conformal weight zero, \(K_{0,0}^0\) must be some function \(K_0(M^{(i)}, \tilde{M}^{(i)})\). The tree-level low-energy effective action is therefore

\[\lambda^{-2} \alpha'^{-1} \int d^4 x \left[ \int d^2 \theta d^2 \bar{\theta} \Phi \bar{\Phi} \left( L + \alpha'(c_1 \Omega_{YM} + c_2 \Omega_{Lorentz}) \right)^{-\frac{1}{2}} K_0(M^{(i)}, \tilde{M}^{(i)}) + \right. \tag{5.8}\]

\[\left. \alpha'^{-\frac{1}{2}} \int d^2 \theta \Phi^2 F_0(M^{(i)}) + \alpha'^{-\frac{1}{2}} \int d^2 \bar{\theta} \bar{\Phi}^2 F_0(\tilde{M}^{(i)}) \right].\]

The supergravity, tensor, and super-Yang-Mills actions come from the \(D\)-term, while the cosmological constant and Yukawa couplings come from the \(F\)-term. Note that the kinetic term for Yang-Mills fields gets tree-level contributions from the \(D\)-term \(\lambda^{-2} \int d^4 x \int d^2 \theta d^2 \bar{\theta} c_1 \Omega_{YM} K(M^{(i)}, \tilde{M}^{(i)})\) and gets one-loop contributions from the \(F\)-term \(\int d^4 x \int d^2 \theta W_\alpha W^\alpha F_1(M^{(i)})\). Although many aspects of the heterotic superspace effective action have already been discussed in reference [8], the Type II superspace effective action has not yet appeared in the literature.

5.2. Type II superspace effective action

From the form of the Type II sigma model, the dimensionless coupling constant \(\lambda\) (which couples to string loops) can be absorbed in the effective action by rescaling the chiral and twisted-chiral compensators \(\Phi_c \rightarrow \lambda \Phi_c\) and \(\Phi_{tc} \rightarrow \lambda \Phi_{tc}\). As explained in section 3, \(\Phi_c\) can be identified with a vector field strength \(W_0^{(0)}\) and \(\Phi_{tc}\) can be identified with a tensor field strength \(L_{jk}^{(0)}\). Therefore, string loops in the Type II effective action are counted by the superfields \(W_0^{(0)}\) and \(L_{jk}^{(0)}\). (This loop-counting assumes that all compactification moduli have been expressed in terms of the dimensionless superfields \(M_c^{(i)}\) and \(M_{tc}^{(i)}\).) Since the graviphoton field strength is one of the components of \(W_0^{(0)}\), its kinetic term (which is quadratic in \(W_0^{(0)}\) appears at order \(\lambda^{-2}\), as expected for a tree-level contribution.
As in the heterotic sigma model, the dimensionful coupling constant $\alpha'$ can be absorbed by rescaling $E_M{}^a \to (\alpha')^{\frac{1}{2}} E_M{}^a$ and $E_M{}^{\alpha j} \to (\alpha')^{\frac{1}{2}} E_M{}^{\alpha j}$. Because the effective action is defined to be invariant under the conformal transformation

$$\delta E_M{}^a = -\Lambda E_M{}^a, \quad \delta E_M{}^{\alpha j} = -\frac{1}{2} \Lambda E_M{}^{\alpha j}, \quad \delta L_{jk} = 2\Lambda L_{jk}, \quad \delta W^{(0)} = \Lambda W^{(0)}, \quad \delta L_{jk}^{(0)} = 2\Lambda L_{jk}^{(0)},$$

$\alpha'$ can be absorbed in the conformally-invariant action by rescaling $L_{jk} \to \alpha' L_{jk},$ $W^{(0)} \to (\alpha')^{\frac{1}{2}} W^{(0)},$ and $L_{jk}^{(0)} \to \alpha' L_{jk}^{(0)}$.

For the Type II superstring, mirror symmetry further restricts the superspace effective action. Mirror transformations relate Type IIA compactifications to Type IIB compactifications by switching $\hat{G}_C$ with $\bar{G}_C$ (the right-moving fermionic generators for the compactification $N=2$) and switching $\Omega^i_c$ with $\Omega^i_{tc}$ (the worldsheet chiral and twisted-chiral primary fields). From the form of the Type II sigma model of (4.2) and (4.14), this switch can be undone by switching $E_M{}^{\alpha^-} \leftrightarrow E_M{}^{\alpha^+},$ $\Phi_c$ with $\Phi_{tc},$ and $M_c^i$ with $M_{tc}^i$ (one must also switch $\tilde{\rho}$ with $-\tilde{\rho},$ $\tilde{d}_\alpha$ with $-\tilde{d}_\alpha,$ and $\Pi_i^0$ with $-\Pi_i^0,$ but this is just a redefinition of worldsheet variables). So after gauge-fixing $L_{jk} = \delta_{jk}$ and imposing the standard reality conditions on $\Phi_c$ and $\Phi_{tc},$ mirror symmetry relates Type IIA effective actions to Type IIB effective actions by switching $E_M{}^{\alpha^-} \leftrightarrow E_M{}^{\alpha^+},$ $W^{(0)} \leftrightarrow L^{(0)}_{+},$ $\bar{W}^{(0)} \leftrightarrow L^{(0)}_{-},$ $M_c^i \leftrightarrow M_{tc}^i.$ (Note that $E_M{}^{\alpha^-}$ and $E_M{}^{\alpha^+}$ will always appear in combinations which allow this shift to preserve Lorentz invariance.)

Furthermore, mirror symmetry implies that the tensor multiplets $L_{jk}^{(0)}$ and $L_{jk}^{(i)}$ only appear through $\Phi_{tc}$ and $M_{tc}^i,$ and therefore, $L^{(0)}_{+}$ and $L^{(i)}_{+}$ never appear explicitly in the sigma model. Although $L^{(0)}_{+}$ and $L^{(i)}_{+}$ appear implicitly in $L^{(0)}_{++}$ and $M_{tc}^i$ through the linear constraint, their lowest component, $l^{(0)}_{+}$ and $l^{(i)}_{+},$ always appears with derivatives. Therefore, after gauge-fixing $L_{jk} = \delta_{jk},$ the Type II superspace effective action must be invariant under the Pecci-Quinn-like shifts

$$\delta L_{+}^{(0)} = c^{(0)}, \quad \delta L_{+}^{(i)} = c^{(i)},$$

where $c^{(0)}$ and $c^{(i)}$ are independent constants. This implies that before gauge-fixing $L_{jk},$ the Type II effective action must be invariant under

$$\delta L_{jk}^{(0)} = c^{(0)} L_{jk}, \quad \delta L_{jk}^{(i)} = c^{(i)} L_{jk},$$

which is the unique conformally and SU(2)-invariant generalization of (5.9).
In terms of the conformally and SU(2)/U(1) gauge-fixed superfields appearing in the Type II sigma model, the most general N=2 superspace action which is U(1)×U(1) invariant and satisfies the above properties is:

\[
\sum_{g=0}^{\infty} \lambda^{2g-2} \int d^4x
\]

\[
\left( \int d^2\theta d^2\bar{\theta} d^2\bar{\theta} E^{-1} \sum_{(\ell,\ell')=-\infty}^{\infty} \Phi_c^{\frac{1}{2}(1-\ell+\ell'+s)} \Phi_c^{\frac{1}{2}(1-\ell-\ell'-s)} \right)
\]

\[
+ \int d^2\theta d^2\bar{\theta} \Phi_c^{2-2g}(\alpha')^{-1} F_g^c + \int d^2\theta d^2\bar{\theta} \Phi_c^{2-2g}(\alpha')^{-1} F_g^c
\]

\[
+ \int d^2\theta d^2\bar{\theta} \Phi_c^{2-2g}(\alpha')^{-1} F_{tc}^c + \int d^2\theta d^2\bar{\theta} \Phi_c^{2-2g}(\alpha')^{-1} F_{tc}^c
\]

where \(K_{g,n_1,n_1,s}^p = (K_{g,-n_1,-n_1,s})^*\) is a general superfield with U(1)×U(1) weight \([n_1 + \tilde{n}_1, n_1 - \tilde{n}_1]\) and conformal weight \(p\) (s is unrestricted), \(F_g^c\) is a chiral superfield with U(1)×U(1) weight \([2g, 0]\) and conformal weight \(2g\) (the conformal weight of an N=2 chiral superfield must be equal its U(1) weight), and \(F_{tc}^c\) is a twisted-chiral superfield with U(1)×U(1) weight \([0, 2g]\) and conformal weight \(2g\) (although the conformal weight of an N=2 twisted-chiral superfield is unrestricted when \(L_{jk}\) is gauge-fixed, mirror symmetry forces \(F_{tc}^c\) to have the same conformal weight as \(F_g^c\)). We have explicitly separated out the \(\Phi\) zero modes in (5.11), so \(K_{g,n_1,n_1,s}^p\), \(F_g^c\), and \(F_{tc}^c\) are defined to be independent of these zero modes (however, they can depend on the moduli \(M_{c}^{(i)} = \log(W^{(i)}/\Phi_c)\) and \(M_{tc}^{(i)} = \log(L_{-}^{(i)}/\Phi_{tc})\)).

As in the heterotic superstring, \(F_g^c\) and \(F_{tc}^c\) do not receive perturbative quantum corrections since they must appear in the effective action only at order \(\lambda^{2g-2}\) so that their U(1)×U(1) charge is compensated by the U(1)×U(1) charge of \(\Phi_c\) or \(\Phi_{tc}\). In other words, after expanding \(\Phi_c\) and \(\Phi_{tc}\) around their vacuum expectation values, \(F_g^c\) and \(F_{tc}^c\) appear only once in the perturbative expansion. Furthermore, mirror symmetry implies that \(F_g^c\) for Type IIA compactifications is related to \(F_{tc}^c\) for Type IIB compactifications by switching \(E_M^{\alpha-}\) with \(E_M^{\alpha+}\) and \(M_{c}^{(i)}\) with \(M_{tc}^{(i)}\).
We are assuming that the chiral and twisted-chiral $F$-terms in (5.11) can not be written as non-chiral $D$-terms, which could then receive quantum corrections from $U(1)$-neutral combinations of $\Phi_c$ and $\Phi_{tc}$. (See the previous subsection for why such $D$-terms are unlikely.) Note that if a twisted-chiral $F$-term can be written as such a $D$-term, then mirror symmetry implies that the mirror chiral $F$-term can also be written as such a $D$-term. So in the unlikely event that $F_{tc}^g$ receives perturbative corrections, so does $F_c^g$. However, as will be discussed at the end of this section, if mirror symmetry were non-perturbatively broken, $F_{tc}^g$ might receive non-perturbative corrections without $F_c^g$ receiving corrections.

One type of $g$-loop chiral and twisted-chiral term is

$$\lambda^{2g-2}(\alpha')^{g-1} \int d^4x \left[ \int d^2\theta d^2\bar{\theta} \Phi_c^{2-2g}(P_{\alpha\beta} P^{\alpha\beta})^g \tau_c^g(M_c^{(i)}) + \text{c.c.} \right]$$

$$+ \int d^2\theta d^2\bar{\theta} \Phi_{tc}^{2-2g}(Q_{\alpha\beta} Q^{\alpha\beta})^g \tau_{tc}^g(M_{tc}^{(i)}) + \text{c.c.}$$

where $P_{\alpha\beta}$ and $Q_{\alpha\beta}$ are the chiral and twisted-chiral supergravity field strengths which appear in the sigma model of (4.2). These terms describe the scattering of two gravitons with either $2g - 2$ graviphotons or $2g - 2$ hypermultiplets, and occur only at $g$ string-loops in the perturbative $S$-matrix $[31][24]$. $\tau_c^g$ and $\tau_{tc}^g$ are functions which depend only on topological properties of the compactification manifold, and since mirror symmetry exchanges $P_{\alpha\beta} P^{\alpha\beta}$ with $Q_{\alpha\beta} Q^{\alpha\beta}$, $\tau_c^g$ for Type IIA compactifications is equal to $\tau_{tc}^g$ for Type IIB compactifications, and $\tau_{tc}^g$ for Type IIA compactifications is equal to $\tau_c^g$ for Type IIB compactifications.

By removing the gauge-fixing condition on $L_{jk}$, the non-chiral and chiral terms in (5.11) can easily be written in conformally and SU(2)-invariant form as

$$\sum_{g=0}^{\infty} \lambda^{2g-2} \frac{1}{2} (1-g+n_l+n_{\bar{l}}+s) \left( (W^{0})_{j}^{l} \frac{1}{2} (1-g-n_{l}+n_{\bar{l}}+s) (W^{0})_{j}^{l} (1-g-n_{l}+n_{\bar{l}}+s) \right)$$

$$\sum_{g=0}^{\infty} \lambda^{2g-2} \frac{1}{2} (1-g+n_l+n_{\bar{l}}-s) \left( (y^{j} y^{k} L_{jk}^{0}) \frac{1}{2} (1-g-n_{l}-n_{\bar{l}}-s) (y^{j} y^{m} L_{lm}^{0}) \frac{1}{2} (1-g-n_{l}-n_{\bar{l}}-s) \right)$$

$$\sum_{p=0}^{\infty} \lambda^{2g-2} \frac{1}{2} (L^{jk} L_{jk}) \frac{1}{2} (3g-3+s-2p) K_{g,n_{l},n_{\bar{l}},s}^{p}(y,\bar{y})$$

$$+ \int d^4\theta (W^{0})^{2-2g}(\alpha')^{g-1} F_c^g + \int d^4\bar{\theta} (W^{0})^{2-2g}(\alpha')^{g-1} F_{\bar{c}}^g$$
where $y^j \bar{y}^k$ is defined as $\epsilon^{jk} + \frac{L^{jk}}{\sqrt{L_{lm}L^{lm}}}$ in the non-chiral integrand, and $K_{g,n_1,n_2}^p(y, \bar{y})$ is defined by contracting all SU(2) + indices with $y$'s and all SU(2) − indices with $\bar{y}$'s. Note that there are an equal number of $y$'s and $\bar{y}$'s in the integrand because of U(1) invariance.

However, in order to write the twisted-chiral term of (5.11) in conformally and SU(2)-invariant form, one needs to introduce two independent complex variables, $u^+$ and $u^-$. Using $u^\pm$ and any linear superfield $h_{j_1 \ldots j_n}$ satisfying

$$\nabla_\alpha(j_0 h_{j_1 \ldots j_n}) = \nabla_{\bar{\alpha}}(j_0 h_{j_1 \ldots j_n}) = 0,$$

one can define a “harmonic” superfield of degree $n$, $\tilde{h}(u^+, u^-) = h_{j_1 \ldots j_n} u^1 \ldots u^n$, which satisfies

$$u^j \nabla_{\alpha_j} \tilde{h} = u^j \nabla_{\bar{\alpha}_j} \tilde{h} = 0. \quad (5.15)$$

Note that the product of two harmonic superfields of degree $n_1$ and $n_2$ is a harmonic superfield of degree $n_1 + n_2$.

Using a harmonic superfield $\tilde{h}$ of degree 2 (which has conformal weight 2 by N=2 superspace rules), one can then define the conformally and SU(2)-invariant action

$$\int d^4x \oint u^j du^j \int d^4\theta \tilde{h} = \int d^4x \oint u^j du^j \int (v^j d\theta_j)^2 \int (\bar{v}^k d\bar{\theta}_k)^2 (v^l u_l)^4 \tilde{h} \quad (5.16)$$

where $v^j$ is arbitrary (because $\tilde{h}$ is harmonic, the integral is independent of $v^j$) and $\oint du$ is defined as a contour integral in $CP1$ around a pole of $\tilde{h}$. (Note that the action is invariant under the complex projective transformation $u^j \rightarrow qu^j$.)

In order to reproduce the twisted-chiral term of (5.11), $\tilde{h}$ should be defined as

$$\tilde{h} = \lambda^{2g-2}(\alpha')^{g-1} \frac{(\bar{L}^{(0)})^{2-2g} \bar{F}_g}{\bar{L}} \quad (5.17)$$

where $\bar{L}^{(0)} = u^j u^k L_{jk}^{(0)}$, $\bar{L} = u^j u^k L_{jk}$, and $\bar{F}_{tc}$ is a harmonic superfield of degree $4g$ which satisfies $\bar{F}_g(u^- = 1, u^+ = 0) = F_{tc}^g$ when $L_{jk}$ is gauge-fixed to $\delta_{jk}$. (Note that $\nabla - \bar{F}_g = \nabla - \bar{F}_g = 0$ when $u^- = 1$ and $u^+ = 0$, so $F^{tc}_g$ is a twisted-chiral superfield of U(1)×U(1) charge $[0, 2g]$.) With this choice of $\tilde{h}$, (5.16) is

$$\lambda^{2g-2}(\alpha')^{g-1} \int d^4x \oint u^j du^j \int d^4\theta \frac{(\bar{L}^{(0)})^{2-2g} \bar{F}_g}{\bar{L}} \quad (5.18)$$

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where the contour integral is performed around the pole of \( \tilde{L} = w^j u^k L_{jk} \).

When \( L_{jk} \) is gauge-fixed to \( \delta_{jk} \), the action of (5.18) becomes

\[
\lambda^2 g^{-2} (\alpha')^{g-1} \int d^4x \int u_k du^k \int d^4\theta \frac{(\tilde{L}(0))^{2-2g} \tilde{F}_g}{u^+ u^-} \int d^4x \int d^2\theta d^2\tilde{\theta} d\xi (\tilde{L}(0))^{2-2g}(u^- = 1, u^+ = \xi) \tilde{F}_g (u^- = 1, u^+ = \xi)
\]

where we have chosen the \( CP^1 \) basis \((u^-, u^+) = (1, \xi)\) and performed a contour integral around \( \xi = 0 \). Since \( \Phi_{tc} = L^{(0)}_{-\dot{-}} \), (5.18) and its complex conjugate reproduce the twisted-chiral and twisted-anti-chiral terms of (5.11).

Note that (5.18) is invariant under the Pecci-Quinn-like shifts

\[
\delta L_{jk}^{(0)} = c^{(0)} L_{jk}, \quad \delta L_{jk}^{(i)} = c^{(i)} L_{jk}
\]

where \( c^{(0)} \) and \( c^{(i)} \) are independent constants, since these shifts eliminate the pole when \( \tilde{L} \) vanishes. However, it is plausible that this shift symmetry is broken by spacetime non-perturbative instantons, just as other types of Pecci-Quinn symmetries can be broken. (This would imply that mirror symmetry is broken non-perturbatively since mirror symmetry implies that linear superfields appear only through \( \Phi_{tc} \) and \( M^{(i)}_{tc} \) in the effective action.)

If the shift symmetry were non-perturbatively broken, one could consider harmonic actions of the type

\[
\lambda^2 g^{-2} (\alpha')^{g-1} \int d^4x \int u_k du^k \int d^4\theta \frac{(\tilde{L}(0))^{2-2g} y(\tilde{L}(0)/\lambda \tilde{L}) \tilde{f}}{L} \int d^4x \int d^2\theta d^2\tilde{\theta} d\xi (\tilde{L}(0))^{2-2g} \tilde{F}_g (u^- = 1, u^+ = \xi)
\]

where \( \tilde{f} = f_{j_1...j_{4g}} u^{j_1} ... u^{j_{4g}} \) is a harmonic superfield of degree \( 4g \) and \( y(\tilde{L}(0)/\lambda \tilde{L}) \) is an arbitrary function (e.g., \( y(\tilde{L}(0)/\lambda \tilde{L}) = \exp(-\tilde{L}(0)/\lambda \tilde{L}) \)).

Expanding around the background expectation value of \( \tilde{L}(0)/\tilde{L} \), this term would give corrections to hypermultiplet interactions of the form \( \delta F_g^{tc} = y(\frac{1}{\lambda}) f_{-...-} \). Note, however, that vector multiplet interactions can not receive non-perturbative corrections since, as in the heterotic superstring effective action, the linear superfield can not appear in a chiral action.

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Finally, the tree-level low-energy effective action for the Type II superstring is given by

\[
\frac{1}{\lambda^2 \alpha'} \int d^4x \left[ \int d^4\theta (W^{(0)})^2 F_0^{c} (M_c^{(i)}) + c.c. \right. \\
+ \oint u_j du^j \int d^4\theta \left( \frac{\tilde{L}^{(0)}}{L} \right)^2 F_0^{tc} (\tilde{M}_{tc}^{(i)}) + c.c. \right] 
\]  

(5.21)

where \( \tilde{M}_c^{(i)} = W^{(i)}/W^{(0)} \) and \( \tilde{M}_{tc}^{(i)} = \tilde{L}^{(i)}/\tilde{L}^{(0)} \). Although the chiral part of this low-energy action agrees with the component action of \([27]\), the “harmonic” part of (5.21) allows more general tensor hypermultiplet couplings than the “improved tensor” action of \([27]\). Note that unlike the improved tensor action of \([27]\), (5.21) contains the Pecci-Quinn-like symmetry of (5.10).

Although N=2 supersymmetry also allows the cosmological term

\[
\frac{1}{\lambda^2 \sqrt{\alpha'}} \int d^4x \int d^2\theta d^2\bar{\theta} d^2\bar{\theta} E^{-1} L_{jk}^{(0)} V^{jk(0)} 
\]  

(5.22)

where \( V^{jk(0)} \) is the prepotential of the vector compensator, this term breaks the Pecci-Quinn-like symmetry and is therefore perturbatively forbidden.

6. Conclusions

In this paper, we have constructed manifestly spacetime supersymmetric sigma models and effective actions for 4D compactifications of heterotic and Type II superstrings. For the heterotic superstring, the sigma model can be found in equations (4.1) and (4.12), and the effective action in (5.2) and (5.8). For the Type II superstring, the sigma model can be found in (4.2) and (4.14), and the effective action in (5.13), (5.18), and (5.21).

We have also proven various non-renormalization theorems for the superspace effective action, including the theorem that chiral \( F \)-terms receive no perturbative or non-perturbative corrections. For the Type II superstring, mirror symmetry implies that twisted-chiral \( F \)-terms (which describe hypermultiplet interactions) are also unrenormalized. However, it is plausible that mirror symmetry is non-perturbatively broken by spacetime instantons, which would allow hypermultiplet interactions to receive non-perturbative corrections.

Our results in this paper were based on the observation that string loops in the 4D heterotic superstring are counted by an N=1 chiral compensator, and
string loops in the $4D$ Type II superstring are counted by an N=2 vector compensator and an N=2 tensor compensator. This explains the coupling of Ramond-Ramond fields, and contradicts the standard folklore that Type II string loops are counted by just a hypermultiplet\cite{17}. The mistaken folklore was caused by confusing the physical scalar (which sits in a hypermultiplet and couples like the determinant of the metric) with the dilaton compensator (which couples to the worldsheet curvature).

Based on a similar incorrect reasoning, the standard folklore also claims that for the $4D$ heterotic superstring with $N = 2$ spacetime supersymmetry, string loops are counted by a vector multiplet\cite{17}. Although we are presently unsure of the precise supersymmetry multiplet for the dilaton compensator in this N=2 heterotic case, we believe that it is not just a vector multiplet, and therefore the standard folklore is again incorrect.

An obvious question is if our techniques can be generalized to the ten-dimensional uncompactified superstring. Although there does exist an N=2 worldsheet-supersymmetric description of the uncompactified superstring, this description is not manifestly SO(9,1) invariant\cite{5}. This makes it difficult to extend the worldsheet action in a flat superspace background to a sigma model action in a curved superspace background. Nevertheless, we conjecture that if such an extension were performed (probably using harmonic variables to make Lorentz invariance manifest), the $10D$ supergravity theory would contain a spacetime conformal compensator which couples to worldsheet supercurvature in the sigma model. Hopefully, a proper understanding of this compensator will lead to an off-shell superspace description of $10D$ supergravity.

A less ambitious question is if our techniques can be generalized to six-dimensional compactifications of the superstring (which, after toroidal compactification, would be useful for understanding the N=2 $4D$ heterotic superstring). Since there already exists an N=2 worldsheet-supersymmetric description of the superstring with manifest SO(5,1) invariance\cite{24}, the answer is probably yes. As was shown with Cumrun Vafa, this six-dimensional superstring actually contains N=4 worldsheet supersymmetry, which can be made manifest by introducing SU(2)/U(1) harmonic variables. It is likely that, as in references \cite{32} and \cite{15},
these harmonic variables will be useful for constructing manifestly spacetime supersymmetric sigma models and effective actions. A 6D superspace effective action would be very useful for studying the recent string-duality conjectures which relate the 6D heterotic and Type II superstrings\cite{3,4,17}.

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