A NEW CHARACTERIZATION OF PRINCIPAL IDEAL DOMAINS

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Abstract. In 2008 N. Q. Chinh and P. H. Nam characterized principal ideal domains as integral domains that satisfy the following two conditions: (i) they are unique factorization domains, and (ii) all maximal ideals in them are principal. We improve their result by giving a characterization in which each of these two conditions is weakened. At the same time we improve a theorem by P. M. Cohn which characterizes principal ideal domains as atomic Bézout domains. We will also show that every PC domain is AP and that the notion of PC domains is incomparable with the notion of pre-Schreier domains (hence with the notions of Schreier and GCD domains as well).

1. Introduction and preliminaries

The goal of this paper is to improve the 2008 result of N. Q. Chinh and P. H. Nam [2, Corollary 1.2.] in which they gave a characterization of principal ideal domains as integral domains that satisfy the following two conditions: (i) they are unique factorization domains, and (ii) all maximal ideals in them are principal. Our main result is a new characterization of principal ideal domains obtained by weakening each of the conditions in the Chinh and Nam’s result. At the same time we improve the so called Cohn’s theorem which characterizes principal ideal domains as atomic Bézout domains. In order to state our improvement, we will introduce a new condition for integral domains and prove that that new condition is indeed weaker than the corresponding conditions in the two mentioned theorems.

We begin by recalling some definitions and statements. All the notions that we use but not define in this paper can be found in the

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classical reference books \cite{3} by P. M. Cohn, \cite{6} by R. Gilmer, \cite{7} by I. Kaplansky, and \cite{8} by D. G. Northcott.

In this paper all rings are integral domains, i.e., commutative rings with identity in which \( xy = 0 \) implies \( x = 0 \) or \( y = 0 \). A non-zero non-unit element \( x \) of an integral domain \( R \) is said to be irreducible (and called an atom) if \( x = yz \) with \( y, z \in R \) implies that \( y \) or \( z \) is a unit. A non-zero non-unit element \( x \) of an integral domain \( R \) is said to be prime if \( x \mid yz \) with \( y, z \in R \) implies \( x \mid y \) or \( x \mid z \). Every prime element is an atom, but not necessarily vice-versa. Two elements \( x, y \in R \) are said to be associates if \( x = uy \), where \( u \) is a unit. We then write \( x \sim y \).

An integral domain \( R \) is said to be atomic if every non-zero non-unit element of \( R \) can be written as a (finite) product of atoms. An integral domain \( R \) is called a principal ideal domain (PID) if every ideal of \( R \) is principal. The condition for integral domains that every ideal is principal is called the PID condition. An integral domain \( R \) is called a unique factorization domain (UFD) if it is atomic and for every non-zero, non-unit \( x \in R \), every two factorizations of \( x \) into atoms are equal up to order and associates. An integral domain \( R \) is called an ACCP domain if every increasing sequence of principal ideals of \( R \) stabilizes. It is well-known that every PID is a UFD, every UFD is an ACCP domain, and every ACCP domain is atomic.

An integral domain \( R \) is called a Bézout domain if every two-generated ideal of \( R \) is principal. (An ideal \( I \) of \( R \) is said to be two-generated if \( I = (a, b) \) for some \( a, b \in R \).) The condition for integral domains that every two-generated ideal is principal is called the Bézout condition. Obviously, every PID is a Bézout domain. The converse is not true.

**Proposition 1.1** (\cite[9.4, Exercise 5, pages 306-307]{5}). Bézout condition for integral domains is strictly weaker than the PID condition. More concretely, \( R = \mathbb{Z} + \mathbb{Q}[X] \) is a Bézout domain which is not a PID.

Note that the notation \( R = \mathbb{Z} + \mathbb{Q}[X] \) means that \( R \) consists of all the polynomials from \( \mathbb{Q}[X] \) whose constant term is from \( \mathbb{Z} \).

We call the PIP condition the condition for integral domains that every prime ideal is principal. We call the MIP condition the condition for integral domains that every maximal ideal is principal. The MIP domains are the domains which satisfy the MIP condition. Clearly, the PID condition implies the PIP condition and the PIP condition implies the MIP condition. More precise relations between these conditions are given in the next proposition and Corollary \cite[5]{3}.
Proposition 1.2 ([5] 8.2, Exercise 6, page 283). The PID condition for integral domains is equivalent to the PIP condition. In other words, if every prime ideal of an integral domain \( R \) is principal, then \( R \) is a PID.

The final item that we cover in this introduction is the notion of a monoid ring for a commutative monoid \( M \), written additively. The elements of the monoid ring \( F[X; M] \), where \( F \) is a field and \( X \) is a variable, are the polynomial expressions, also called polynomials,\(^{(1)}\)

\[ f(X) = a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n}, \]

where \( n \geq 0, \ a_1, \ldots, a_n \in F, \ \alpha_1, \ldots, \alpha_n \in M \). The polynomials \( f(X) = a, \ a \in F, \) are called the constant polynomials. The addition and the multiplication of the polynomials are naturally defined. We say that \( M \) is cancellative if for any elements \( a, b, c \in M, \ a + b = a + c \) implies \( b = c \). The monoid \( M \) is torsion-free if for any \( n \in \mathbb{N} \) and \( a, b \in M, \ na = nb \) implies \( a = b \). All the monoids that we use in this paper are cancellative and torsion-free, hence the monoid rings \( F[X; M] \) are integral domains.

2. A NEW CONDITION FOR INTEGRAL DOMAINS

We introduce a new condition for integral domains, that we haven’t met in the literature.

Definition 2.1. We call the principal containment condition (PC) the condition for integral domains that every proper two-generated ideal is contained in a proper principal ideal. We say that an integral domain is a PC domain if it satisfies the PC condition.

Clearly, Bézout condition implies the PC condition, and the MIP condition implies the PC condition.

Proposition 2.2. There exists a Bézout domain which is not a MIP domain.

Proof. Consider the monoid ring \( R = F[X; \mathbb{Q}_+] \) (\( F \) a field), consisting of all the polynomials of the form

\[ f(X) = a_0 + a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n} \]

with \( a_0, a_1, \ldots, a_n \in F \) and \( 0 < \alpha_1 < \cdots < \alpha_n \) from \( \mathbb{Q}_+ \). Let \( m \) be the maximal ideal of \( R \) consisting of all the polynomials in \( R \) whose constant term is 0. Consider the localization \( D = R_m \). The units of \( D \) have the form

\[ \frac{a_0 + a_1X^{\alpha_1} + \cdots + a_mX^{\alpha_m}}{b_0 + b_1X^{\beta_1} + \cdots + b_nX^{\beta_n}}, \]
where the $a_i$ and $b_j$ are from $F$ with $a_0, b_0$ non-zero. Hence every non-zero element of $D$ has the form $uX^\alpha$, where $u$ is a unit in $D$ and $\alpha \in \mathbb{Q}_+$. The maximal ideal $mR_m$ of $D$ consists of all $uX^\alpha$ with $\alpha > 0$ and is not finitely generated. So $D$ does not satisfy the MIP condition. However, for any two elements $uX^\alpha, vX^\beta$ of $D$ with $\alpha \leq \beta$ we have $uX^\alpha | vX^\beta$ and so $D$ is Bézout.

We will show later (see Proposition 3.4) that there also exists a MIP domain which is not a Bézout domain. Thus the notion of a PC domain is strictly weaker than each of the notions MIP and Bézout. Finally in Proposition 3.3 we show that the notion of a PC domain is not “just a union” of the notions of Bézout and MIP domains, i.e., that there is a PC domain which is neither Bézout, nor MIP.

Consider now the following diagram.

![Diagram showing the relationships between atomic domain, UFD, PID, PC, Bézout, MIP, PIP, and PIP: every prime ideal principal.]

There is one equivalence in the diagram, the rest are implications (and all of them are strict) and non-implications. The higher the condition is (in each of the two parts of the diagram), the weaker it is. One can try to characterize PIDs by combining one condition from the left part of the diagram with one condition from the right part of the diagram.

The next two theorems are characterizations of PIDs of that type. The first one (Cohn’s Theorem) is Theorem 2.3 that was first stated in [4, Proposition 1.2]. (Cohn remarks in [4] that it is easy to prove
that Bezout’s domains which satisfy ACCP are PIDs, however, ACCP
is not equivalent to atomicity, as it was later shown.) The proof can be
seen in Cohn’s book [3, 10.5 Theorem 3]. The second one is Theorem
proofed in 2008 by Chinh and Nam in [2].

**Theorem 2.3** (Cohn’s Theorem). *If* \( R \) *is an atomic Bézout domain,
then \( R \) *is a PID.*

**Theorem 2.4** ([2, Corollary 1.2.]). *If* \( R \) *is a UFD in which
every maximal ideal is principal, then \( R \) *is a PID.*

Our next theorem improves both of the above theorems. It weakens
one of the conditions in Cohn’s theorem and both conditions in the
Chinh and Nam’s theorem.

**Theorem 2.5.** Let \( R \) *be an atomic domain which satisfies the PC
condition. Then \( R \) *is a PID.*

**Proof.** By Proposition [1,2] it is enough to show that every prime ideal
is principal. Let \( P \) be a nonzero prime ideal of \( R \). Let \( x \neq 0 \) be an
element of \( P \). Since \( R \) is atomic, we can write \( x = p_1 p_2 \cdots p_n \), where
the \( p_i \)'s are atoms. As \( p_1 p_2 \cdots p_n \in P \) and \( P \) is prime, at least one of
the \( p_i \)'s, say \( p_1 \), is in \( P \). We claim that \( P = (p_1) \). Let \( y \) be an element
of \( P \). Since \( R \) satisfies PC, \( (p_1, y) \subset (c) \) for some proper principal ideal
\( (c) \). From \( p_1 = ct \) for some \( t \in R \), we have \( t \sim 1 \) (as \( p_1 \) is an atom and
\( c \sim 1 \)). Now from \( y = cr \) for some \( r \in R \), we get \( y = p_1 t^{-1} r \), hence
\( y \in (p_1) \). Thus \( P = (p_1) \). \( \square \)

3. **Merging the diagrams and making them more detailed**

In this section we will merge the diagrams from the previous section
and make them more detailed. That will illustrate the importance of
the notion of a PC domain that we introduced in the previous section.
We first need to give some definitions.

An integral domain is called a *GCD domain* if every two elements
of it have a greatest common divisor (see [1 page p.4]). An element \( c \)
of an integral domain \( D \) is called *primal* if for any \( a, b \in D \) we have:
\( c \mid ab \Rightarrow c = c_1 c_2 \) where \( c_1 \mid a \) and \( c_2 \mid b \). This notion was introduced
in [4], where a new version of the definition of Schreier domains is
also given: an integral domain \( D \) is *Schreier* if it is integrally closed
and each of its elements is primal. The notion of pre-Schreier domains
is introduced in [9]: an integral domain is *pre-Schreier* if each of its
elements is primal. Clearly every Schreier domains is pre-Schreier, but
not conversely. A new proof of the well-known result that every GCD
domain is Schreier was given in [4]. The converse is not true. Also,
every Bézout domain is GCD, but not conversely (see [4]). An integral
domain is called an *AP domain* if each of its atoms is prime, i.e., if
the notions of an atom and of a prime element in it coincide. Every
pre-Schreier domain is an AP domain, but not vice-versa (see [9]). It is
well-known that an integral domain is a UFD if and only if it is atomic
and AP.

Let us say a few words about the importance of the notion of PC
domains. An old result of Skolem from 1939 states that an integral
domain is a UFD if and only if it is atomic and GCD. However, weaker conditions were found which, together with atomicity, imply the UFD condition, namely, an integral domain is UFD if and only if it is atomic and AP (or pre-Schreier, or Schreier, or GCD). An analogous situation is with the conditions which, together with atomicity, imply the PID condition (see the previous diagram). Cohn’s 1968 theorem ([4]) states an integral domain is PID if and only if it atomic and Bézout. The result of Chinh and Nam ([2]) states that an integral domain is a PID if and only if it is UFD and MIP, which is, as a consequence of our theorem 2.5, equivalent with atomic and MIP. Our notion of PC domains provides a condition which is weaker than each of the conditions Bézout and MIP, however, it is still strong enough to be, together with atomicity, equivalent with the PID condition. That is the main value of this notion.

We will now justify the previous diagram.

**Proposition 3.1.** Every PC domain is an AP domain.

*Proof.* Let $R$ be a PC domain and let $a$ be an atom of $R$. Suppose $a \mid xy$ for some $x, y \in R$, but $a \nmid x$ and $a \nmid y$. Then $x, y$ are not units. The ideal $(a, x)$ is proper, otherwise $ra + sx = 1$ for some $r, s \in R$, hence $rya + sx = y$, hence $rya + sta = y$ for some $t \in R$, hence $a \mid y$, a contradiction. Since $R$ is PC, there is a proper ideal $(b)$ containing $(a, x)$. But then $a \in (b)$, so $b \mid a$, hence (since $a$ is an atom and $b$ is a non-unit) $b \sim a$. Also $x \in (b)$, so $b \mid x$, hence $a \mid x$ (as $b \sim a$), a contradiction. □

**Proposition 3.2.** There exists an AP domain which is not a PC domain.

*Proof.* Consider the additive monoid $M = \mathbb{N}_0 \times \mathbb{N}_0$ and the associated monoid domain $R = F[X; M]$, where $F$ is a field. The polynomials $f \in R$ whose constant term is 0 form a maximal ideal, say $m$, of $R$. Let $D = R_m$ be the localization of $R$ at $m$. The elements of $D$ have the form

$$x = \frac{X^{(r,s)} \cdot (a_0 + a_1 X^{(m_1,n_1)} + \ldots + a_k X^{(m_k,n_k)})}{1 + b_1 X^{(p_1,q_1)} + \ldots + b_l X^{(p_l,q_l)}},$$

where $k, l \geq 0$, $a_i, b_j \in F$ ($0 \leq i \leq k$, $1 \leq j \leq l$), $a_0 \neq 0$, and $(m_1,n_1), \ldots, (m_k,n_k)$ (pairwise distinct), $(p_1,q_1), \ldots, (p_l,q_l)$ (pairwise distinct), $(r,s)$ are elements of $\mathbb{N}_0 \times \mathbb{N}_0$. Hence $x \sim X^{(r,s)}$ and so the only atoms of $D$ are $X^{(0,1)}$ and $X^{(1,0)}$, and they are both prime. Thus $D$ is an AP domain. Th ideal $(X^{(0,1)}, X^{(1,0)})$ is proper, but it is not
contained in a proper principal ideal as no \(X^{(r,s)}\) can divide both \(X^{(1,0)}\) and \(X^{(0,1)}\) unless it is a unit. Thus \(D\) is not a PC domain. \(\Box\)

**Proposition 3.3.** There exists a PC domain which is neither pre-Schreier (hence not Bézout), nor MIP.

**Proof.** Let \(i\) be an irrational number such that \(0 < i < 1\). Let \(q\) be a rational number such that \(19 < q < 20\). Consider the additive submonoid

\[
M = ([0, 5 + \frac{i}{2}] \cap \mathbb{Q}) \cup (5 + \frac{i}{2}, \infty)
\]

of \(\mathbb{R}_+\). Since \(5 < 5 + \frac{i}{2} < 5.5\), we have

\[
8 < q - 10 - i < 10,
\]

so that \(q - 10 - i \in M\). Let \(r\) be a rational number from \((10, 10 + i)\). Then \(8 < q - r < 10\). We claim that it is impossible to find four numbers \(\alpha, \beta, \alpha', \beta' \in M\) such that the following relations hold (at the same time):

\[
\begin{align*}
\alpha + \beta &= 10 + i, \\
\alpha + \alpha' &= r, \\
\beta + \beta' &= q - r.
\end{align*}
\]

Suppose to the contrary. Then by (4) at least one of the elements \(\alpha, \alpha'\) is \(\leq \frac{r}{2}\), hence \(< 5 + \frac{i}{2}\), hence rational. Since \(\alpha + \alpha'\) is rational, the other element is rational too. Thus \(\alpha\) is rational. In the same way \(\beta\) is rational. However, by the equation (3) \(\alpha + \beta\) is irrational, a contradiction.

Let now \(R = F[X; M]\), where \(F\) is a field. Then the polynomials \(f \in R\) whose constant term is 0 form a maximal ideal, say \(\mathfrak{m}\), of \(R\). Let \(D = R_\mathfrak{m}\), the localization of \(R\) at \(\mathfrak{m}\). The elements of \(D\) have the form

\[
x = \frac{X^\gamma (a_0 + a_1 X^{\gamma_1} + \cdots + a_m X^{\gamma_m})}{1 + b_1 X^{\delta_1} + \cdots + b_m X^{\delta_m}},
\]

where \(m, n \geq 0, a_i, b_j \in F\) \((0 \leq i \leq m, 0 \leq j \leq n)\), and \(\gamma, \gamma_1, \ldots, \gamma_m, \delta_1, \ldots, \delta_n\) are elements of \(M\) with \(0 < \gamma_1 < \cdots \gamma_m, 0 < \delta_1 < \cdots \delta_n\).

We can write \(x = X^\gamma u\), where \(u\) is a unit in \(D\), \(\gamma \in M\). The element \(x\) is a unit if and only if \(\gamma = 0\). Since \(q - 10 - i \in M\), we have

\[
X^{10+i} | X^q = X^r X^{q-r}.
\]

We show that it is not possible to find two elements \(y, z \in D\) such that \(y | X^r, z | X^{q-r}\), and \(yz = X^{10+i}\). Suppose to the contrary. Then we can assume \(y = X^\alpha\) and \(z = X^\beta\) for some \(\alpha, \beta \in M\), such that there
are $\alpha', \beta' \in M$ satisfying the relations (3), (4), and (5). However, we showed above that that is not possible. Hence $D$ is not pre-Schreier. In particular, $D$ is not Bézout.

Note that the maximal ideal $mR_m$ of $D$ is not finitely generated since for any $X^{\gamma_1}, \ldots, X^{\gamma_t}$, with $\gamma_i > 0$ ($i = 1, \ldots, t$) elements of $M$, there is a $\gamma \in M$ such that $0 < \gamma < \min\{\gamma_1, \ldots, \gamma_t\}$, so that $X^\gamma \notin (X^{\gamma_1}, \ldots, X^{\gamma_t})$. Thus $D$ is not MIP.

However, $D$ is a PC domain since for any $X^{\gamma_1}, X^{\gamma_2} \in D$ (with $\gamma_1, \gamma_2 > 0$ elements of $M$) there is a sufficiently small positive rational number $\gamma \in M$ such that $X^\gamma \mid X^{\gamma_1}$ and $X^\gamma \mid X^{\gamma_2}$. Hence $D \supset (X^\gamma) \supset (X^{\gamma_1}, X^{\gamma_2})$.

**Proposition 3.4.** There exists a MIP domain which is not pre-Schreier (hence not Bézout).

**Proof.** Let the numbers $i, q, r$, and the monoid $M$ be like in Proposition 3.3. Consider the submonoid

$$N = (\mathbb{Z} \times (M \setminus \{0\})) \cup \mathbb{N}_0$$

of the additive monoid $\mathbb{Z} \times \mathbb{R}_+$. Let $R = F[X; N]$, where $F$ is a field. The polynomials $f \in R$ whose constant term is 0 form a maximal ideal, say $m$, of $R$. Let $D = R_m$ be the localization of $R$ at $m$. The elements of $D$ have the form

$$x = \frac{a_0 X^{(k_0, \alpha_0)} + \cdots + a_m X^{(k_m, \alpha_m)}}{1 + b_1 X^{(l_1, \beta_1)} + \cdots + b_n X^{(l_n, \beta_n)}},$$

where $m, n \geq 0$, $a_i, b_j \in F$ ($0 \leq i \leq m$, $1 \leq j \leq n$), and $(k_0, \alpha_0), \ldots, (k_m, \alpha_m)$, $(l_1, \beta_1), \ldots, (l_n, \beta_n)$ are elements of $N$. We assume that $\alpha_0 \leq \cdots \leq \alpha_m$ and the $(k_i, \alpha_i)$ are pairwise distinct, as well as that $0 < \beta_1 \leq \cdots \leq \beta_n$ and the $(l_j, \beta_j)$ are pairwise distinct. Let $\nu$ be the largest element of $\{0, 1, \ldots, m\}$ such that $\alpha_0 = \cdots = \alpha_\nu$. Then we denote

$$x^* = a_0 X^{(k_0, \alpha_0)} + \cdots + a_\nu X^{(k_\nu, \alpha_\nu)}.$$

Note that for any $x, y \in D$ we have

$$(xy)^* = x^* y^*.$$

Suppose also that $k_0 < k_1 < \cdots < k_\nu$. We consider two cases.

1st case: $\alpha_0 = 0$. Then we factor out $X^{(k_0, 0)}$ from the numerator in (7) and have

$$x = (X^{(1,0)})^{k_0} \cdot \frac{a_0 X^{(k_1 - k_0, 0)} + \cdots + a_\nu X^{(k_\nu - k_0, 0)} + \cdots + a_m X^{(k_m, \alpha_m)}}{1 + b_1 X^{(l_1, \beta_1)} + \cdots + b_n X^{(l_n, \beta_n)}},$$

so that either

$$x = u \quad (\text{if } k_0 = 0),$$
or,
\[(10)\]
\[x = (X^{(1,0)})^{k_0} u \quad \text{(if } k_0 \geq 1),\]
where \(u\) is a unit in \(D\).

2nd case: \(\alpha_0 > 0\). Then we factor out any \(X^{(k,0)} \ (k \in \mathbb{N}_0)\) from the numerator in (7) and we have
\[(11)\]
\[x = (X^{(1,0)})^k \cdot \frac{a_0 X^{(k_0-k,\alpha_0)} + \cdots + a_m X^{(k_m-k,\alpha_m)}}{1 + b_1 X^{(l_1,\beta_1)} + \cdots + b_n X^{(l_n,\beta_n)}}.\]

Denote \(n = (X^{(1,0)})\), the ideal of \(D\) generated by \(X^{(1,0)}\). It follows from (2), (10), and (11) that \(n = mR_m\), the maximal ideal of \(D\), and that in the 1st case \(x\) is an element of \(n^{k_0} \setminus n^{k_0+1} \ (k_0 \geq 0)\), and in the 2nd case \(x\) is an element of \(n^\omega = \cap_{k=1}^\infty n^k\). Since the maximal ideal is principal, \(D\) is a MIP domain.

We now show that \(D\) is not pre-Schreier. By (6) from Proposition 3.3
\[(12)\]
\[X^{(0,10+i)} \mid X^{(0,q)} = X^{(0,0)}X^{(0,q-r)}.\]

We show that it is not possible to find two elements \(y, z \in D\) such that
\[
y \mid X^{(0,r)},
\]
\[
z \mid X^{(0,q-r)},
\]
\[(13)\]
\[yz = X^{(0,10+i)}.
\]

Suppose to the contrary. Then
\[(14)\]
\[yy' = X^{(0,0)}X^{(0,r)},
\]
\[(15)\]
\[zz' = X^{(0,q-r)}.
\]

for some \(y', z' \in D\). Let \(\alpha, \beta, \alpha', \beta'\) be the second coordinate of the exponents that appear in \(y^*, z^*, y'^*,\) and \(z'^*\), respectively. Then from (13), (14), and (15), using (8), we get
\[
\alpha + \beta = 10 + i,
\]
\[
\alpha + \alpha' = r,
\]
\[
\beta + \beta' = q - r.
\]

However, this is not possible as we have seen in the proof of Proposition 3.3. □

**Corollary 3.5.** The MIP condition is strictly weaker than the PIP condition.

**Proof.** Otherwise every MIP domain would be a PID, hence pre-Schreier, contradicting the previous proposition. □
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