TOPOLOGICAL GROUPS THAT REALIZE HOMOGENEITY OF TOPOLOGICAL SPACES

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Abstract. We present results on simplifying an acting group preserving properties of actions: transitivity, being a coset space and preserving a fixed uniformity in case of a $G$-Tychonoff space.

1. Introduction

A topological space $X$ is homogeneous if for any $x, y \in X$ there is a homeomorphism $h$ of $X$ such that $h(x) = y$. In the study of topological homogeneity it is natural to ask about groups (or their classes) which acts continuously and transitively on a homogeneous space. The survey of A. V. Arhangel’skii and J. van Mill [4] can serve as a good introduction to the subject of the paper.

Let $G$ acts continuously and transitively on $X$. Using the conjugation of stabilizes of points from one orbit, a normal subgroup $K$ of the kernel of action can be chosen. The quotient group $G/K$ can be equipped with topology (weaker than the quotient topology) in which the naturally defined action is continuous. This leads to the possibility to define a continuous and transitive action on $X$ of a group $H$ such that $\chi(H) \leq \chi(X) \cdot \text{inv}(G)$ and $\text{ib}(H) \leq \text{ib}(G)$ (Theorem 3.9). Corollaries 3.10 and 4.11 show what restrictions we have on a pseudocharacter and an invariance number of a group which continuously, effectively and transitively acts on a space $X$ of character $\leq \tau$. Let us note that there is a continuous, effective and transitive action of a metrizable (discrete) group $G$ (of $\text{ib}(G) \geq |X|$) on a homogeneous space $X$.

We show how to replace transitive actions of $G$-range topological groups to topological groups from class $G$ (Theorem 3.3). As a class $G$ we can examine groups of weight or character $\leq \tau$. Transitive actions of $\tau$-narrow and $\tau$-balanced topological groups on spaces of character $\leq \tau$ can be replaced to transitive actions of groups of weight and character $\leq \tau$ respectively (Corollary 3.6). In particular, transitive actions of $\omega$-narrow and $\omega$-balanced topological groups on first countable spaces can be replaced to transitive actions of separable metrizable and metrizable groups respectively and transitive action of a subgroup of a product of Čech complete groups can be replaced to a transitive action of an inframetrizable group (Corollary 4.2). As a consequence we have: if an $\omega$-narrow group acts continuously and transitively on a space $X$ of countable character with Baire property, then $X$ is a separable metrizable space (Corollary 4.12); there is no continuous and transitive action of an $\omega$-narrow group on the homogenous space which is not a coset space.

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of Ford (Example 4.3), on the two-arrows space, on the homogenous first countable compactum which is not a coset space of Fedorchuk, on the Sorgenfrey line (Example 4.13).

A more strong form of homogeneity is when $X$ is a coset space of a topological group. Corollaries 3.7 and 4.4 show that if $X$ is a coset space of a $\tau$-balanced group (or a $\tau$-narrow group) and $\chi(X) \leq \tau$, then $X$ is coset space of a group of character $\leq \tau$ (or of weight $\leq \tau$). As a consequence we have: if a subgroup of a product of Čech complete groups acts continuously and transitively on a compactum $X$ of countable character, then $X$ is a metrizable compactum (Theorem 4.8); the Sorgenfrey line is not a coset space of an $\omega$-balanced group (Example 4.6); the two-arrows space is not a coset space of a subgroup of a product of Čech complete groups (Example 4.9).

If $X$ is a coset space of a topological group $G$, then for the natural action $\alpha$ of $G$ on $X$ by left translations the $G$-space $(G, X, \alpha)$ is $G$-Tychonoff and has equiuniformities. For continuous transitive actions it is not known.

**Question [21, Question 2.6].** Let $X$ be a Tychonoff $G$-space with the transitive action. Is it true that $X$ is $G$-Tychonoff?

Corollaries 3.8 and 4.5 show that if $X$ is a coset space of a $\tau$-narrow group and $\chi(X) \leq \tau$, then there is an equiuniformity on $X$ of weight $\leq \tau$.

The usage of equiuniformities allows to apply the replacing process preserving properties of actions to non transitive actions. It consists in equipment of a quotient group with respect to the kernel of action with the topology of uniform convergence. If $(G, X, \alpha)$ is a $G$-Tychonoff space and $\mathcal{U}$ is an equiuniformity on $X$, then we can replace the group $G$ on an action of a group $H$ such that $\chi(H) \leq w(\mathcal{U})$, $\text{ib}(H) \leq \text{ib}(G)$ and $\mathcal{U}$ is an equiuniformity (Theorem 3.11). As a consequence we have: there is no continuous and effective action on $X$ of a topological group $G$ with $\psi(G) > \tau$ for which $\mathcal{U}$, $w(\mathcal{U}) \leq \tau$, is an equiuniformity (Corollary 3.13). In Example 4.18 we make remarks about existence of complete metrics on a homogeneous Polish space which is not a coset space of J. van Mill.

## 2. Preliminaries

All spaces are assumed to be Tychonoff, maps are continuous and notations, terminology and designations are from [9]. Nbd(s) is an abbreviation of open neighborhood(s), $\text{cl}_X A$, $\text{int}_X A$ are the closure, interior of a subset $A$ of a space $X$, respectively. A metrizable space is called Polish if it is separable and has a complete metric. A map id is an identity map.

For information about topological groups see [3] and [23]. An *inframetrizable* group is a subgroup of a Čech complete group [23]. By $\text{ib}(G)$ the *index of narrowness* of a topological group $G$ is denoted and by $\text{inv}(G)$ the *invariance number* of $G$ is denoted, $\text{inv}(G) \leq \text{ib}(G)$, see [3, Chapter 5].

By a *separating family of homomorphisms* on a topological group we understand the family of homomorphisms which separates points and closed sets. By $N_G(e)$ we denote the family of nbds of the unit $e \in G$. If $\varphi : G \to H$ is an epimorphism then $\text{ib}(H) \leq \text{ib}(G)$. The *kernel of homomorphism* $\varphi : G \to H$ is denoted $\ker \varphi$.

A class $\mathcal{G}$ of topological groups is called $\tau$-*multiplicative* if the product of $\tau$ representatives of $\mathcal{G}$ is an element of $\mathcal{G}$. If $\tau = \sigma$ then we say $\sigma$-multiplicative. If the class is closed under arbitrary products then it is called *multiplicative*. 
A class $\mathcal{G}$ of topological groups is called hereditary if a subgroup of a representative of $\mathcal{G}$ is an element of $\mathcal{G}$.

2.1. $G$-spaces. By an action of a group $G$ on a set $X$ we mean a map $\alpha : G \times X \to X$ such that $\alpha(e, x) = x$, $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for any $x \in X$, $g, h \in G$ (where $e$ is the unit of the group $G$). We denote by $\alpha_x : G \to X$, $x \in X$, and $\alpha^g : X \to X$, $g \in G$, the maps defined by the rules $\alpha_x(g) = \alpha(g, x) = \alpha^g(x)$. If it is clear what action is considered, then we write $\alpha(g, x) := gx$. The action is transitive if the map $\alpha_x$ is a surjection. For a set $X$ with an action $\alpha$ of a group $G$ the subgroup $H = \{g \in G : gx = x, x \in X\}$ is called the kernel of action and is denoted $\ker \alpha$. If $\ker \alpha = \{e\}$, then the action is called effective.

For a family of subsets $\gamma = \{O_\alpha : \alpha \in \mathcal{A}\}$ and $g \in G$ we set $g \gamma = \{gO_\alpha : \alpha \in \mathcal{A}\}$. A family of covers $\Gamma = \{\gamma_s : s \in S\}$ is saturated if $g \gamma_s \in \Gamma$ for any $s \in S$, $g \in G$.

A $G$-space is a triple $(G, X, \alpha)$ of a space $X$ with a fixed continuous action $\alpha : G \times X \to X$ of a topological group $G$. The kernel $\ker \alpha$ of a continuous action is a closed subgroup.

Let $(G, X, \alpha_G)$ and $(H, Y, \alpha_H)$ be $G$-spaces. A pair of maps $(\varphi : G \to H$, $\, f : X \to Y)$ such that $\varphi : G \to H$ is a homomorphism and the diagram

$$
\begin{array}{ccc}
G \times X & \varphi \times f \downarrow & H \times Y \\
\downarrow \alpha_G & & \downarrow \alpha_H \\
X & \downarrow f & Y \\
\end{array}
$$

is commutative is called an equivariant pair of maps of $G$-spaces. Designation $(\varphi, f) : (G, X, \alpha_G) \to (H, Y, \alpha_H)$. The commutativity of the diagram may be written as the fulfillment of the following condition $f(gx) = \varphi(g)f(x)$ for any $x \in X$, $g \in G$.

If $f$ is an embedding then the equivariant pair $(\varphi, f)$ is an equivariant embedding of $(X, G_X, \alpha_X)$ into $(Y, G_Y, \alpha_Y)$. The composition of equivariant pairs of maps is an equivariant pair.

A $G$-space $(G, X, \alpha)$ is called $G$-Tychonoff, if there is an equivariant embedding $(\text{id}, f)$ of $(G, X, \alpha)$ into a $G$-space $(G, bX, \hat{\alpha})$ where $bX$ is a compactification of $X$ ($(G, bX, \hat{\alpha})$ (together with the embedding $(\text{id}, f)$) is a $G$-compactification of $(G, X, \alpha)$). The maximal $G$-compactification is denoted $(G, \beta_G X, \alpha_\beta)$.

2.2. $d$-Open actions. Information about $(d)$-open actions, their properties and natural uniform structures which are generated by $(d)$-open actions can be found in [13], [15] and [16]. For the convenience of the reader we remind some facts which are used below.

**Definition 2.1.** (see [13]) An action $\alpha : G \times X \to X$ is called

- open if $x \in \text{int}(Ox)$ for any point $x \in X$ and $O \in N_G(e)$;
- $d$-open if $x \in \text{int}(\text{cl}(Ox))$ for any point $x \in X$ and $O \in N_G(e)$.

**Remark 2.2.** Terminology in definition 2.1 is motivated by the fact that an action is “open” (“$d$-open”) iff maps $\alpha_x : G \to X$, $x \in X$, are open ($d$-open) [16] Remark 1.7].
F. Ancel called continuous open actions micro-transitive and continuous \(d\)-open actions weakly micro-transitive. If \((G, X, \alpha)\) is with a \(d\)-open action then \(X\) is a direct sum of clopen subsets (components of the action). Each component of the action is the closure of the orbit of an arbitrary point of this component. If an action is open then \(X\) is a direct sum of clopen subsets which are the orbits of the action [16, Remark 1.7]. For a \(G\)-space \(X\) with an open action and one component of action \(X\) is the quotient space of \(G\).

The following theorem is a generalization of a correspondent result of F. Ancel for complete metrizable groups [1].

Theorem 2.3. [15, Theorem 3] Let \((G, X, \alpha)\) be a \(G\)-space with a \(d\)-open action and let for any \(x \in X\) the coset space \(G/Gx\) be \(\check{\text{C}}\)ech complete. Then the action is open.

In particular, a \(d\)-open continuous action of a \(\check{\text{C}}\)ech complete group is open.

2.3. Uniform structures on \(G\)-spaces. Uniform structures on spaces are introduced by the families of covers [12] and are compatible with their topology. For covers \(u\) and \(v\) of \(X\) we denote \(u \succ v\), \(u^* \succ v\) if \(u\) refines, respectively star-refines \(v\).

A uniformity \(U\) on a \(G\)-space \(X\) is called equiuniformity (see, for example, [19]) if it is saturated and bounded (i.e. that for any \(u \in U\) there exist \(O \in N_G(e)\) and \(v \in U\) such that \(Ov = \{OV : V \in v\} \succ u\). The action on a \(G\)-Tychonoff space \((G, X, \alpha)\) with equiuniformity \(U\) can be extended to the continuous action of the Raikov completion \(\rho G\) of \(G\) on the completion \(\tilde{X}\) of \(X\) with respect to the equiuniformity \(U\) [19, Theorem 3.1]. By equivariant completion of \((G, X, \alpha)\) we understand a \(G\)-space \((G, \tilde{X}, \tilde{\alpha}_\rho)\) with the natural equivariant embedding \((\text{id}, i) : (G, X, \alpha) \rightarrow (G, \tilde{X}, \tilde{\alpha}_\rho)\).

Theorem 2.4. [13] If the action on a \(G\)-space \((G, X, \alpha)\) is \(d\)-open, then the family of covers \(\gamma_O = \{\text{int} (\text{cl}(Ox)) : x \in X\}, O \in N_G(e),\) is the base of the maximal equiuniformity on \(X\).

2.4. Topologizing groups using equiuniformities on \(G\)-Tychonoff spaces.

Theorem 2.5. [11, Theorem 2.2] If \(X\) possesses a uniform structure \(U\) under which every element of some homeomorphism group \(G\) is uniformly continuous, then \(G\) is a topological group relative to the uniform convergence notion induced by the uniformity.

A bijection \(f\) of uniform spaces is called a uniform equivalence if \(f\) and \(f^{-1}\) are uniformly continuous.

Remark 2.6. The sets \(O_u = \{g \in G : \forall x \in X \ g(x) \in \text{St}(x, u)\}, u \in U,\) are a base of nbds (not open) at \(e\).

The uniformity of uniform convergence in theorem 2.5 coincides with the right uniformity on \(G\) [23, Chapter 2, Exercise 2].

Proposition 2.7. [2] Remark 4.2 (c)] Uniformity \(U\) in theorem 2.5 is an equiuniformity and, hence, \((G, X, \alpha)\) is a \(G\)-Tychonoff space.

Corollary 2.8. [2, Corollary 4.5] If \(w(U) \leq \aleph_0\) for a totally bounded uniformity \(U\) on \(X\) then the group of uniform equivalences \(H(X, U)\) in the topology of uniform convergence is separable and metrizable.
The following theorem is a generalization of [2 theorem 4.6].

**Theorem 2.9.** \([\text{[17] Lemma 5]}\) Let \((G, X, \alpha)\) be a \(G\)-Tychonoff space with an effective action, \(U\) be an equiuniformity on \(X\). Then

1. each element of \(G\) is a uniform equivalence (with respect to \(U\)),
2. the topology of uniform convergence \(\tau_U\) on \(G\) is coarser than the original one,
3. \(((G, \tau_U), X, \alpha)\) is a \(G\)-Tychonoff space and \(U\) is an equiuniformity on \(X\).

This result shows, in fact, the following.

**Corollary 2.10.** Let \(U\) be an uniformity on \(X\). For any topological group \(G\) such that there is an effective action \(G \times X \to X\) for which \((G, X, \alpha)\) is a \(G\)-Tychonoff space and \(U\) is an equiuniformity on \(X\), the topology of uniform convergence on \(G\) is coarser than the original topology on \(G\).

2.5. Properties of equivariant pair of maps.

**Proposition 2.11.** Let \((\varphi, \text{id}) : (G, X, \alpha) \to (H, X, \gamma)\) be an equivariant pair of maps of \(G\)-spaces and \(\varphi : G \to H\) be an epimorphism. Then we have

(a) \(\ker \varphi \subset \ker \alpha\),
(b) \(\varphi(\ker \alpha) = \ker \gamma\) and if \(\alpha\) is effective then \(\gamma\) is effective;
(c) \(Gx = Hx\) for any \(x \in X\), in particular, if \(\alpha\) is transitive then \(\gamma\) is transitive;
(d) if \(\alpha\) is open (\(d\)-open) then \(\gamma\) is open (\(d\)-open) and components of actions \(\alpha\) and \(\gamma\) coincide;
(e) if \(U\) is an equiuniformity on \(X\) in \((H, X, \gamma)\) then \(U\) is an equiuniformity on \(X\) in \((G, X, \alpha)\), hence, if \((H, X, \gamma)\) is \(G\)-Tychonoff then \((G, X, \alpha)\) is \(G\)-Tychonoff.

**Proof.** Properties (a), (b) and (c) immediately follows from the equality \(\alpha(g, x) = \gamma(\varphi(g), x), x \in X, g \in G\).

In order to prove property (d) take \(x \in X\) and \(O \in N_H(e)\). For \(U = \varphi^{-1}(O) \in N_G(e)\)

\[
x \in \text{int} (Ux) = \text{int} (\varphi(U)x) = \text{int} (Ox) \text{ in case of an open action } \alpha,
\]

\[
x \in \text{int} (\text{cl}(Ux)) = \text{int} (\text{cl}(\varphi(U)x)) = \text{int} (\text{cl}(Ox)) \text{ in case of a } d\text{-open action } \alpha.
\]

Thus openness (\(d\)-openness) is preserved. Since \(\text{int} (\text{cl}(Gx)) = \text{int} (\text{cl}(Hx)), x \in X,\) the components of actions \(\alpha\) and \(\gamma\) coincide.

Property (e). If \(U\) is an equiuniformity on \(X\) in \((H, X, \gamma)\) then it is a uniformity on \(X\). From the equality \(\alpha(g, A) = \gamma(\varphi(g), A)\) for any \(A \subset X, g \in G\) it follows that \(U\) is a saturated uniformity in \((G, X, \alpha)\). Its boundedness follows from the equality \(\alpha(\varphi^{-1}(O), x) = \gamma(O, x), x \in X, O \in N_H(e)\).

The last statement in property (e) is a straightforward consequence. \(\square\)

**Remark 2.12.** If \(G\) is a discrete group then a \(G\)-space \((G, X, \alpha)\) is \(G\)-Tychonoff. This yields the existence of an example of a \(G\)-Tychonoff space which image under an equivariant pair of maps is not \(G\)-Tychonoff.

From proposition \([\text{2.11(e)}]\) we have.

**Corollary 2.13.** Let \((G, X, \alpha)\) be a \(G\)-Tychonoff space and \(U\) be the maximal equiuniformity on \(X\). If \((\varphi, \text{id}) : (G, X, \alpha) \to (H, X, \gamma)\) is an equivariant pair of maps and \(U\) is an equiuniformity on \(X\) in \((H, X, \gamma)\), then \(U\) is the maximal equiuniformity on \(X\) in \((H, X, \gamma)\).
3. Replacing acting group (general case)

Let \( \mathcal{G} \) be a class of topological groups. A topological group \( G \) is called range-\( \mathcal{G} \) if for any \( O \in \mathcal{N}_G(e) \) there exists a continuous homomorphism \( h \) of \( G \) to \( H \in \mathcal{G} \) such that \( h^{-1}(U) \subset O \) for some \( U \in \mathcal{N}_H(e) \) [3 § 3.4].

**Remark 3.1.** [3 Theorem 3.4.21] Let \( \mathcal{G} \) be a class of topological groups. The following conditions for a group \( G \) are equivalent:

- \( G \) is range-\( \mathcal{G} \);
- \( G \) has a separating family of homomorphisms to topological groups from \( \mathcal{G} \);
- \( G \) is topologically isomorphic to a subgroup of a product of a family of topological groups from \( \mathcal{G} \).

**Proposition 3.2.** Assume that \((G, X, \alpha)\) is a \( G \)-space and a group \( G \) is a subgroup of the product \( \Pi = \Pi \{G_s \in \mathcal{G} : s \in S\} \) of a family of topological groups. If

- a) the action \( \alpha \) is transitive and \( \chi(X) \leq \tau \) or
- b) \((G, X, \alpha)\) is \( G \)-Tychonoff, \( \mathcal{U} \) is an equivuniformity on \( X \) and \( w(\mathcal{U}) \leq \tau \),

then there exist: \( S' \subset S \) such that \( |S'| \leq \tau \);

an action \( \gamma : \text{pr}_{S'}(G) \times X \to X \), where \( \text{pr}_{S'} : \Pi \to \Pi_{S'} = \{G_s \in \mathcal{G} : s \in S'\} \) is a projection, such that \( \text{pr}_{S'}(G, X, \gamma) \) is a \( G \)-space with transitive action \( \gamma \) in case a) or \( \mathcal{U} \) is an equivuniformity on \( X \) in \((\text{pr}_{S'}(G), X, \gamma)\) in case b); and

an equivariant pair of maps \((\varphi = \text{pr}_{S'}|G, \text{id}) : (G, X, \alpha) \to (\text{pr}_{S'}(G), X, \gamma)\).

**Proof.** In case a) fix a point \( x \in X \) and its base of nbds \( \{U_t : t \in T\} \), \(|T| \leq \tau\). For each \( U_t \) take \( O_t \in \mathcal{N}_G(e) \) such that \( O_t U_t \subset U_t \) for some nbhd \( U_t \) of \( x \).

In case b) fix a base \( \{U_t : t \in T\} \), \(|T| \leq \tau\), of \( \mathcal{U} \). For each \( U_t \) take \( O_t \in \mathcal{N}_G(e) \) such that \( \{O_t x : x \in X\} \supset U_t \).

In both cases, without loss of generality, we may examine each \( O_t \) as the trace on \( G \) of a rectangular set in \( \Pi \) which depends on the finite number of coordinates \( S_t \), \( t \in T \). The cardinality of the set \( S' = \bigcup \{S_t : t \in T\} \) is \( \tau \).

Let \( \Pi_{S'} = \Pi \{G_s \in \mathcal{G} : s \in S'\} \), \( \text{pr}_{S'} : \Pi \to \Pi_{S'} \), \( H = \text{pr}_{S'}(G) \subset \Pi_{S'} \). Evidently, \( \varphi = \text{pr}_{S'}|G \) is a continuous homomorphism of \( G \) onto \( H \), and \( \varphi(O_t) \) is open in \( H \), \( g^{-1} \varphi^{-1}(\varphi(O_t)) = O_t \) for all \( t \in T \).

The straightforward checking allows to prove the following properties:

- for any \( O_t \) there is \( U(t) \in \mathcal{N}_H(e) \) such that \( \varphi^{-1}(U(t)) \subset O_t, t \in T \),
- for any \( U \in \mathcal{N}_H(e) \) and \( g \in G \) there is \( V \in \mathcal{N}_H(e) \) such that \( g^{-1} \varphi^{-1}(V) \subset \varphi^{-1}(U) \).

These properties yields that the action \( \gamma \) of \( H \) on \( X \) (for \( y \in X \), \( g \in H \) take \( h \in \varphi^{-1}(g) \) and put \( \gamma(g, y) = \alpha(h, y) \)) is well defined. In fact, by the definition of \( H \) for any \( h \in \ker \varphi \) we have \( \alpha(h, x) = x \). For \( y = gx \) and \( h \in \ker \varphi \) we have \( \alpha(h, gx) = \alpha(hg', x) = \alpha(g, h'x) = \alpha(g, x) = gx \) for \( h' = g^{-1}hg \in \ker \varphi \).

Further, \((H, X, \gamma)\) is a \( G \)-space in case a). In fact, every \( \gamma_h : X \to X \), \( h \in H \), is continuous. For \( y = gx \) and its nbhd \( W \) the set \( g^{-1}W \) is a nbhd of \( x \). There is a nbhd \( U \) of \( x \) and \( O = \varphi^{-1}(\varphi(O)) \) such that \( OU \subset g^{-1}W \). The set \( gU \) is a nbhd of \( gx \). There is \( V = \varphi^{-1}(\varphi(V)) \) such that \( g^{-1}Vg \subset O \). Then \( VgU \subset gOg^{-1}U = gOU \subset W \).

In case b) \((H, X, \gamma)\) is a \( G \)-Tychonoff space and \( \mathcal{U} \) is an equivuniformity on \( X \) in \((H, X, \gamma)\). In fact, every \( \gamma_h : X \to X \), \( h \in H \), is uniformly continuous with respect to \( \mathcal{U} \) and for every \( u, v \in \mathcal{U} \), \( v \gg u \), there is \( O \in \mathcal{N}_H(e) \) such that \( \{\gamma^{-1}(O)x : x \in X\} \gg v \) and \( Ov \gg u \). This yields that the action \( \gamma \) is continuous and bounded by \( \mathcal{U} \).
Evidently, \((\varphi, \text{id}) : (G, X, \alpha) \rightarrow (H, X, \gamma)\) is an equivariant pair of maps in both cases. Therefore the action \(\gamma\) is transitive in case a) \(2.11\) (c). \(\square\)

From proposition \(3.2\) and remark \(3.1\) we have.

**Theorem 3.3.** Assume that \(G\) is a \(\tau\)-multiplicative and hereditary class of topological groups and \(G\) is range-\(G\). Let \((G, X, \alpha)\) be

a) a \(G\)-space with a transitive action and \(\chi(X) \leq \tau\) or

b) a \(G\)-Tychonoff space with an equiuniformity \(U\) on \(X\) and \(w(U) \leq \tau\).

Then there exist: a \(G\)-space \((H, X, \gamma)\) with a transitive action \(\gamma\) in case a) or a \(G\)-Tychonoff space \((H, X, \gamma)\) with the equiuniformity \(U\) such that \(H \in G\); and

an equivariant pair of maps \((\varphi, \text{id}) : (G, X, \alpha) \rightarrow (H, X, \gamma)\), where \(\varphi\) is an epimorphism.

**Remark 3.4.** The following \(\tau\)-multiplicative and hereditary classes \(G\) of topological groups and corresponding range-\(G\) classes of groups are well-known \([3, \S\ 5.1]\).

| class \(G\) of groups | range-\(G\) class of groups |
|------------------------|-----------------------------|
| groups of weight \(\leq \tau\) | \(\tau\)-narrow groups (or groups \(G\) with \(\text{ib}(G) \leq \tau\)) |
| groups of character \(\leq \tau\) | \(\tau\)-balanced groups (or groups \(G\) with \(\text{inv}(G) \leq \tau\)) |

The classes of \(\tau\)-narrow and \(\tau\)-balanced group are multiplicative and hereditary. Every \(\tau\)-narrow group is \(\tau\)-balanced.

From property (c) of proposition \(2.11\) theorem \(3.3\) and remark \(3.4\) we have.

**Corollary 3.5.** Let \((G, X, \alpha)\) be a \(G\)-space, \(\alpha\) be a transitive action, \(\chi(X) \leq \tau\). If \(G\) is a \(\tau\)-narrow (respectively \(\tau\)-balanced) group, then there exists a topological group \(H\) such that \(w(H) \leq \tau\) (respectively \(\chi(H) \leq \tau\)) and \(H\) admits a continuous transitive action on \(X\).

From theorem \(3.3\), remark \(3.4\) and corollary \(2.13\) we have.

**Corollary 3.6.** Let \((G, X, \alpha)\) be a \(G\)-Tychonoff space and \(U\) be an equiuniformity (the maximal equiuniformity) on \(X\) such that \(w(U) \leq \tau\). Then there exist: a \(G\)-Tychonoff space \((H, X, \gamma)\) such that

(a) \(w(H) \leq \tau\) if \(G\) is a \(\tau\)-narrow group,

(b) \(\chi(H) \leq \tau\) if \(G\) is a \(\tau\)-balanced group,

\(U\) is an equiuniformity (the maximal equiuniformity) on \(X\) in \((H, X, \gamma)\); and

an equivariant pair of maps \((\varphi, \text{id}) : (G, X, \alpha) \rightarrow (H, X, \gamma)\), where \(\varphi\) is an epimorphism.

From property (d) of proposition \(2.11\) theorem \(3.3\) and remark \(3.4\) we have.

**Corollary 3.7.** Let \(\chi(X) \leq \tau\).

(a) If \(X\) is a coset space of a \(\tau\)-balanced group (or there is a transitive and \(d\)-open action on \(X\) of a \(\tau\)-balanced group), then \(X\) is a coset space of a topological group \(H\) (or there exists a transitive and \(d\)-open action of a group \(H\)) such that \(\chi(H) \leq \tau\).

(b) If \(X\) is a coset space of a \(\tau\)-narrow group (or there is a transitive and \(d\)-open action on \(X\) of a \(\tau\)-narrow group), then \(X\) is a coset space of a topological group \(H\) (or there exists a transitive and \(d\)-open action of a group \(H\)) such that \(w(H) \leq \tau\) and, hence, \(w(X) \leq \tau\).
Moreover, there exists an equivariant pair of maps \((\varphi, \text{id}) : (G, X, \alpha) \to (H, X, \gamma)\), where \(\varphi\) is an epimorphism.

**Corollary 3.8.** Let \(\chi(X) \leq \tau\). If \(X\) is a coset space of a \(\tau\)-balanced group \(G\) (or there is a transitive and \(d\)-open action on \(X\) of a \(\tau\)-balanced group \(G\)), then there exists an equiuniformity \(\mathcal{U}\) on \(X\) in \((G, X, \alpha)\) such that \(w(\mathcal{U}) \leq \tau\).

Let \(w(X) \leq \tau\). If \(X\) is a coset space of a \(\tau\)-balanced group \(G\) (or there is a transitive and \(d\)-open action on \(X\) of a \(\tau\)-balanced group \(G\)), then there exist:
- an equivariant extension \((\hat{G}, \hat{X}, \hat{\alpha})\) of \((G, X, \alpha)\) such that \(w(\hat{X}) \leq \tau\); and
- a complete equiuniformity \(\tilde{\mathcal{U}}\) on \(\hat{X}\) such that \(w(\tilde{\mathcal{U}}) \leq \tau\).

**Proof.** By corollary 3.7 there are: an open (or \(d\)-open) action on \(X\) of a group \(H\) such that \(\chi(H) \leq \tau\) and an equivariant pair of maps \((\varphi, \text{id}) : (G, X, \alpha) \to (H, X, \gamma)\), where \(\varphi\) is an epimorphism. By [14, Proposition 4] the weight of the maximal equiuniformity \(\mathcal{U}\) on \(X\) in \((H, X, \gamma)\) is \(\leq \tau\). By property (e) of proposition 2.11 \(\mathcal{U}\) is an equiuniformity on \(X\) in \((G, X, \alpha)\).

The proof of the second statement. Since \(\chi(X) \leq w(X) \leq \tau\) there exists an equiuniformity \(\mathcal{U}\) on \(X\) in \((G, X, \alpha)\) such that \(w(\mathcal{U}) \leq \tau\). The completion of \(X\) with respect to \(\mathcal{U}\) is the required equivariant completion \(\hat{X}\) of \(X\) of weight \(\tau\) and by [10, Theorem 3.1] there exists a natural extension \(\hat{\alpha} : G \times \hat{X} \to \hat{X}\) of \(\alpha\).

**Theorem 3.9.** Let \((G, X, \alpha)\) be a \(G\)-space, \(\alpha\) be a transitive action.

Then there exist a \(G\)-space \((H, X, \gamma)\) such that \(\gamma\) is a transitive action,
\[
\chi(H) \leq \chi(X) \cdot \text{iv}(G),
\]
\[
\text{ib}(H) \leq \text{ib}(G),
\]
\[
w(H) \leq \chi(X) \cdot \text{ib}(G),
\]
and an equivariant pair of maps \((\varphi, \text{id}) : (G, X, \alpha) \to (H, X, \gamma)\), where \(\varphi\) is an epimorphism.

**Proof.** The group \(G\) is a subgroup of the product \(\Pi = \Pi\{G_s \in G : s \in S\}\) of a family of topological groups of character \(\leq \text{iv}(G)\) [3, Theorem 5.1.9]. By proposition 3.2 there exist \(S' \subset S\) such that \(|S'| \leq \chi(X)|,
\((H = \text{pr}_{S'}(G), X, \gamma)\) is a \(G\)-space with a transitive action \(\gamma\) and an equivariant pair of maps \((\varphi, \text{id}) : (G, X, \alpha) \to (H, X, \gamma)\), where \(\varphi = \text{pr}_{S'}|G\).

It is easy to see that \(\chi(H) \leq \chi(X) \cdot \text{iv}(G)\) and \(\text{ib}(H) \leq \text{ib}(G)\). Hence,
\[
w(H) = \chi(H) \cdot \text{ib}(H) \leq \chi(X) \cdot \text{iv}(G) \cdot \text{ib}(G) = \chi(X) \cdot \text{ib}(G).\]

**Corollary 3.10.** Let \(\chi(X) \leq \tau\). There is no continuous, effective and transitive action on \(X\) of a topological group \(G\) with \(\psi(G) > \tau\), \(\text{iv}(G) \leq \tau\).

**Proof.** Let \(G\) acts continuously, effectively and transitively on \(X\) and \(\text{iv}(G) \leq \tau\). Then by theorem 3.9 there exists a group \(H\) such that \(\chi(H) \leq \tau\), \(H\) acts continuously, effectively and transitively on \(X\) and \(H\) is a continuous homomorphic image of \(G\). By property (a) of proposition 2.11 the map of \(G\) onto \(H\) is one-to-one. Then, \(\psi(G) \leq \tau\).

**Theorem 3.11.** Let \((G, X, \alpha)\) be a \(G\)-Tychonoff space, \(\mathcal{U}\) be an equiuniformity on \(X\).
Then there exist a $G$-Tychonoff space $(H, X, \gamma)$ such that $\mathcal{U}$ is an equiuniformity on $X$,

$$\chi(H) \leq w(\mathcal{U}),$$

$$\text{ib}(H) \leq \text{ib}(G),$$

$$w(H) \leq w(\mathcal{U}) \cdot \text{ib}(G),$$

and an equivariant pair of maps $(\varphi, \text{id}) : (G, X, \alpha) \rightarrow (H, X, \gamma)$, where $\varphi$ is an epimorphism.

**Proof.** Let us assume that the action $\alpha$ is effective. Otherwise, take a $G$-Tychonoff space $(G/\text{Ker} \alpha, X, \alpha')$.

From theorem 2.9 the topology of uniform convergence $\tau_\mathcal{U}$ on $G$ is weaker than the original topology, $H = (G, \tau_\mathcal{U})$ is a topological group by theorem 2.5, and the naturally defined action $H \times X \rightarrow X$ is continuous. Therefore, $(H, X, \gamma)$ is a $G$-Tychonoff space, $\mathcal{U}$ is an equiuniformity on $X$ and the equivariant pair of maps of $(G, X, \alpha)$ to $(H, X, \gamma)$ is well defined. Thus, $\text{ib}(H) \leq \text{ib}(G)$.

Since the sets $O_u = \{g \in G : \forall x \in X \; g(x) \in \text{St}(x, u)\}, \; u \in \mathcal{U}$, are a base of nbds (not open) at $e$, $\chi(H) \leq w(\mathcal{U})$. The last inequality follows from [3, Theorem 5.2.3].

**Remark 3.12.** Let us note that if the action $\alpha : G \times X \rightarrow X$ in theorem 3.11 is transitive, (d-)open, then the action $\gamma : H \times X \rightarrow X$ is transitive, (d-)open by proposition 2.11.

**Corollary 3.13.** Let $\mathcal{U}$ be a uniformity on $X$ such that $w(\mathcal{U}) \leq \tau$. There is no continuous and effective action on $X$ of a topological group $G$ with $\psi(G) > \tau$ for which $\mathcal{U}$ is an equiuniformity on $X$.

**Proof.** Let $G$ acts continuously and effectively on $X$, $\mathcal{U}$ be an equiuniformity on $X$ such that $w(\mathcal{U}) \leq \tau$. Then by theorem 3.11 there exists a group $H$ such that $\chi(H) \leq \tau$, $H$ acts continuously and effectively on $X$, $(H, X, \gamma)$ is a $G$-Tychonoff space, $\mathcal{U}$ is an equiuniformity on $X$ and $H$ is a continuous homomorphic image of $G$. By property (a) of proposition 2.11 the map of $G$ onto $H$ is one-to-one. Then $\psi(G) \leq \tau$. □

4. Replacing acting group (countable case)

**Remark 4.1.** The following $\sigma$-multiplicative and hereditary classes $\mathcal{G}$ of topological groups and corresponding range-$\mathcal{G}$ classes of topological groups are well-known [3 § 3.4], [23, Chapter 13].

| class $\mathcal{G}$ of groups | range-$\mathcal{G}$ class of groups |
|--------------------------------|-----------------------------------|
| groups of countable weight    | $\omega$-narrow groups            |
| (or separable metrizable groups) |                                   |
| groups of countable character | $\omega$-balanced groups          |
| (or metrizable groups)        |                                   |
| inframetrizable groups        | subgroups of products of Čech complete groups |
| (or subgroups of Čech complete groups) |                               |

The classes of $\omega$-narrow groups, $\omega$-balanced groups and subgroups of products of Čech complete groups are $\sigma$-multiplicative and hereditary.

Every $\omega$-narrow group is $\omega$-balanced. Every $\omega$-balanced group is a subgroup of a product of Čech complete groups.
From condition (c) of proposition 2.11, theorem 3.3 and remark 4.1 we have.

**Corollary 4.2.** Let \((G, X, \alpha)\) be a \(G\)-space, \(\alpha\) be a transitive action, \(\chi(X) \leq \aleph_0\). If \(G\) is an \(\omega\)-narrow group (respectively \(\omega\)-balanced group or a subgroup of a product of Čech complete groups), then there exists a separable metrizable (respectively inframetrizable) group \(H\) such that \(H\) admits a continuous transitive action on \(X\).

Moreover, in the case of a transitive action of an \(\omega\)-narrow group, \(X\) is a separable space with countable cellularity.

**Example 4.3.** It is easy to show that the cellularity of a homogeneous space \(X\) which is not a coset space from [11, § 5 Two examples] is uncountable and \(\chi(X) = \aleph_0\). If \(X\) admits a transitive continuous action of an \(\omega\)-narrow group then by corollary 4.2 \(X\) admits a transitive continuous action of a separable metrizable group and must have a countable cellularity. We obtain a contradiction. Hence, there is no continuous and transitive action of an \(\omega\)-narrow group on \(X\).

From condition (d) of proposition 2.11, theorem 3.3, remark 4.1 and [14, Corollary 4] (which, among other things, claims metrizability of a phase space \(X\) of a \(G\)-space \((G, X, \alpha)\), where \(G\) is metrizable and \(\alpha\) is a \(d\)-open action) we have.

**Corollary 4.4.** Let \(\chi(X) \leq \aleph_0\).

(a) If \(X\) is a coset space of a subgroup of a product of Čech complete groups (or there is a transitive and \(d\)-open action on \(X\) of a subgroup of a product of Čech complete groups), then \(X\) with the maximal equiuniformity is an inframetrizable space [10, Remark 2.6] as a coset space of an inframetrizable group (or since \(X\) admits a transitive and \(d\)-open action of an inframetrizable group).

(b) If \(X\) is a coset space of an \(\omega\)-balanced group (or there is a transitive and \(d\)-open action on \(X\) of an \(\omega\)-balanced group), then \(X\) is metrizable as a coset space of a metrizable group (or since \(X\) admits a \(d\)-open action of a metrizable group).

(c) If \(X\) is a coset space of an \(\omega\)-narrow group (or there is a transitive and \(d\)-open action on \(X\) of an \(\omega\)-narrow group), then \(X\) is separable metrizable as a coset space of a separable metrizable group (or since \(X\) admits a transitive \(d\)-open action of a separable metrizable group).

Applying theorem of extension of action to the completion of a phase space with respect to an equiuniformity [19, Theorem 3.1] and in the case of actions of \(\omega\)-narrow groups compactification theorem [18, Theorem 2.13] we have.

**Corollary 4.5.** Let \(\chi(X) \leq \aleph_0\). If \(X\) is a coset space of an \(\omega\)-balanced (respectively \(\omega\)-narrow) group \(G\) or there is a continuous transitive \(d\)-open action of an \(\omega\)-balanced (respectively \(\omega\)-narrow) group \(G\), then there exists an equiuniformity (respectively totally bounded equiuniformity) \(U\) on \(X\) such that \(w(U) \leq \aleph_0\).

Moreover, there exists an equivariant extension \((G, \tilde{X}, \tilde{\alpha})\) of \((G, X, \alpha)\) and a complete metric on \(\tilde{X}\) for which the induced uniformity is an extension of \(U\) (respectively \(G\)-compactification \((G, bX, \tilde{\alpha})\) of \((G, X, \alpha)\) where \(bX\) is a metrizable compactum).

**Example 4.6.** Since the Sorgenfrey line is not metrizable, there is no \(d\)-open transitive action of an \(\omega\)-balanced group on the Sorgenfrey line. Hence, the Sorgenfrey line is not a coset space of an \(\omega\)-balanced group.
The Sorgenfrey line is not Čech complete. Since a $d$-open action of a Čech complete group is open [15, Theorem 3] and a coset space of a Čech complete group is Čech complete [6, Theorem 2], there is no $d$-open transitive action of a Čech complete group on the Sorgenfrey line.

**Question 4.7.** Can the Sorgenfrey line be a coset space of a subgroup of a product of Čech complete groups (or, equivalently, of an inframetrizable group)?

**Theorem 4.8.** Let $X$ be a compact coset space of a subgroup of a product of Čech complete groups (or there is a continuous transitive $d$-open action of a subgroup of a product of Čech complete groups), $\chi(X) \leq \aleph_0$. Then $X$ is metrizable.

**Proof.** By item (a) of corollary [14] $X$ is a coset space of an inframetrizable group $G$ (or there is a continuous transitive $d$-open action of an inframetrizable group on $X$). Then $(G, X, \alpha)$ is a $G$-Tychonoff space with a natural open action $\alpha$ in the first case and $d$-open action in the latter case. By [19, Theorem 3.1] the extension $\check{\alpha} : \rho G \times X \to X$ of action $\alpha$ is well defined and $d$-open. The Raikov completion $\rho G$ is a Čech complete group and by [15, Theorem 3] the action $\check{\alpha}$ is open. Hence, $X$ is a coset space of a Čech complete group. By [8, Propositions 3.2.1, 3.2.2] $X$ is also a coset space of an $\omega$-narrow group. Item (c) of corollary [4.3] finishes the proof. ✷

**Example 4.9.** The two-arrows space [9, Exercises 3.10.C] is not a coset space of a subgroup of a product of Čech complete groups.

**Remark 4.10.** From the proof of theorem [22] we can deduce a positive answer on [16, Question 5.7] in case of compacta $X$ with $\chi(X) \leq \aleph_0$.

**Corollary 4.11.** Let $\chi(X) \leq \aleph_0$. There is no continuous, effective and transitive action on $X$ of an $\omega$-balanced group $G$ with $\psi(G) > \aleph_0$.

**Corollary 4.12.** Let $X$ be a homogeneous space with Baire property, $\chi(X) \leq \aleph_0$. If there is a continuous transitive action of an $\omega$-narrow group on $X$, then $X$ is a separable metrizable space.

**Proof.** In [24] it is proved that a transitive action $\alpha$ of an $\omega$-narrow group $G$ on a space $X$ with Baire property is $d$-open. Hence, $(G, X, \alpha)$ is a $G$-Tychonoff space with a transitive $d$-open action. By item (c) of corollary [4.3] $X$ admits a transitive $d$-open action of a separable metrizable group. In [24, Corollary 4] it is shown that a $d$-open action of a metrizable group yields that the space $X$ is metrizable. Moreover, if the acting group $G$ is separable and the action is transitive then the phase space $X$ is separable. ✷

**Example 4.13.** There is no continuous and transitive action of an $\omega$-narrow group on the two-arrows space and the homogeneous compactum $X$ of V. V. Fedorchuk [10] which is not a coset space and $\chi(X) = \aleph_0$ (see, also, [7]).

The first result can also be deduced from [24], since the two-arrows space is not dyadic.

The Sorgenfrey line has the Baire property. There is no continuous and transitive action of an $\omega$-narrow group on the Sorgenfrey line.

**Corollary 4.14.** Let $(G, X, \alpha)$ be a $G$-Tychonoff space and $\mathcal{U}$ be an equiuniformity on $X$ such that $w(\mathcal{U}) \leq \aleph_0$. Then there exist: a $G$-Tychonoff space $(H, X, \gamma)$ such that...
(a) $H$ is an inframetrizable group if $G$ is a subgroup of a product of Čech complete groups,
(b) $H$ is a metrizable group if $G$ is an $\omega$-balanced group,
(c) $H$ is a separable metrizable group if $G$ is an $\omega$-narrow group;

$U$ is an equiuniformity on $X$ in $(H, X, \gamma)$ and an equivariant pair of maps $(\varphi, \text{id}) : (G, X, \alpha) \to (H, X, \gamma)$, where $\varphi$ is an epimorphism.

Applying theorem of extension of action to the completion of a phase space with respect to an equiuniformity \[19\text{, Theorem 3.1}\] we have.

**Corollary 4.15.** Let $(G, X, \alpha)$ be a $G$-Tychonoff space and $U$ be an equiuniformity on $X$ such that $w(U) \leq \aleph_0$. Then there exist: an action $\gamma : H \times \tilde{X} \to \tilde{X}$, where $\tilde{X}$ is the (metrizable) completion of $X$ with respect to $U$;

(a) $H$ is a Čech complete group if $G$ is a subgroup of a product of Čech complete groups,
(b) $H$ is a complete metrizable group if $G$ is an $\omega$-balanced group,
(c) $H$ is a Polish group if $G$ is an $\omega$-narrow group,

such that $(H, \tilde{X}, \gamma)$ is a $G$-Tychonoff space, the extension $\tilde{U}$ of $U$ on $\tilde{X}$ is an equiuniformity in $(H, \tilde{X}, \gamma)$; and an equivariant embedding $(\varphi, \text{in}) : (G, X, \alpha) \to (H, \tilde{X}, \gamma)$, where $\varphi$ is an epimorphism and in is a natural embedding.

**Remark 4.16.** If in corollary $4.15$ uniformity $U$ is complete, then $\tilde{X} = X$.

**Corollary 4.17.** Let $(G, X, \alpha)$ be a $G$-Tychonoff space, $G$ be an $\omega$-narrow group and $U$ be an equiuniformity on $X$ such that $w(U) \leq \aleph_0$ and, hence, an equivariant compactification $(G, bX, \alpha)$ of $(G, X, \alpha)$, where $bX$ is a metrizable compactum.

**Example 4.18.** In \[22\] J. van Mill constructed a homogeneous Polish space $Z$ which is not a coset space and proved that no $\omega$-narrow group acts transitively on $Z$ by a separately continuous action \[22\text{, Corollary 3}\].

A) There is no complete metric on $Z$ for which the group $G$ of uniform equivalences in the topology of uniform convergence with respect to which acts $d$-openly and has one component of action. In fact, in this case the group $G$ is Raikov complete \[5\text{, Chapter X, § 3, Problem 16 c}] and, hence, is complete metrizable. Its action is open by \[15\text{, theorem 3}] and has one component of action. Hence, $Z$ is a coset space but this is a contradiction.

B) There is no totally bounded metric the group $G$ of uniform equivalences in the topology of uniform convergence with respect to which acts transitively. In fact, in this case $G$ is separable metrizable \[2\]. This is a contradiction with the properties of $Z$.

C) Let a subgroup $G$ of a product of Čech complete groups acts continuously, $d$-openly and with one component of action on $Z$. Then there is no complete equiuniformity with a countable base on $Z$. In fact, then by corollary $4.15$ and remark $4.16$ there is a $d$-open action on $Z$ with one component of action of a Čech complete group. Its action is open by \[15\text{, theorem 3}] and has one component of action. Hence, $Z$ is a coset space but this is a contradiction.
TOPOLOGICAL GROUPS THAT REALIZE HOMOGENEITY OF TOPOLOGICAL SPACES

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