PRICING EUROPEAN OPTIONS UNDER STOCHASTIC VOLATILITY MODELS: CASE OF FIVE-PARAMETER GAMMA-VARIANCE PROCESS

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Abstract. We consider a $\Gamma(\alpha, \theta)$ Ornstein-Uhlenbeck process and build a continuous sample path Variance-Gamma (VG) Process with five parameters ($\mu, \delta, \sigma, \alpha, \theta$): location ($\mu$), symmetric ($\delta$), volatility ($\sigma$), and shape ($\alpha$) and scale ($\theta$). We investigate the associated Lévy process and show that the Lévy density belongs to the KoPoL family of order $\nu = 0$, intensity $\alpha$ and steepness parameters $\frac{\delta}{\sigma^2} - \sqrt{\frac{\delta^2}{\sigma^4} + \frac{1}{\theta^2}}$ and $\frac{\delta}{\sigma^2} + \sqrt{\frac{\delta^2}{\sigma^4} + \frac{1}{\theta^2}}$. The associated Lévy process is also shown to converge in distribution to a Lévy process driven by a Normal distribution with mean ($\mu + \alpha \theta \delta$) and variance $\alpha (\theta^2 \delta^2 + \sigma^2 \theta)$. The data used as illustrations comes from the fitting of the five-parameter Variance Gamma (VG) model to the SPY ETF daily price by Nzokem (Nzokem 2021 J. Phys.: Conf. Ser. 2090 012094). The five-parameter Variance Gamma (VG) process is used subsequently to investigate how well the VG model fits the European option price compare to the Black-Scholes option. The Esscher martingale of the VG model is shown to be another VG model with adjusted parameter. The closed form of the VG option price is provided. The numerical solution is computed using Fractional Fast Fourier (FRFT) algorithm. Compare to the Black-Scholes (BS) model, we find that the VG option is overvalued for deep out of the money (OTM) options. The error sign changes for Deep in the money (ITM) and long term time to maturity, where the VG option is undervalued.

1. Introduction

Black-Scholes (BS) model [1] is consider the cornerstone of the option pricing theory, since the model relies on the fundamental assumption that the asset returns are normally distributed with known mean and variance. However, empirical studies have revealed that the Black-Scholes (BS) model is inconsistent with a set of well established stylized features [2]. Due to subsequent development of the option pricing theory, new class of models have emerged in the literature in order to address the stylized characteristics of the markets. These class of models have the fundamental probabilistic properties of infinity divisibility and belong to the family of Lévy processes [3]. Among the class of models, we have the subclass of Jump-Diffusion models and the subclass of Stochastic Volatility models.

The Jump-Diffusion process is modelled as an independent Brownian motion plus a compensated compound Poisson Process. The popular models in the literature are Merton’s jump-diffusion model [4] and Kou’s jump-diffusion model [5], where the random jump size follows respectively a normal distribution and an asymmetric double exponential distribution. Stochastic volatility

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arXiv:2201.03378v1 [q-fin.PR] 6 Jan 2022
(SV) models are an extension of the standard geometric Brownian motion (GBM) model, where the observed volatility is modelled as a stochastic process. In a stochastic volatility framework [6], the constant volatility ($\sigma$) in a standard geometric Brownian motion (GBM) model is replaced by a deterministic function of a stochastic process ($\sigma(Y_t)$) where $Y_t$ represents the solution of stochastic differential equation (SDE). This implies that stochastic volatility model has two sources of randomness, which can be either correlated or not. In the literature, we have two main types of SV models: Diffusion-based SV models and the non-Gaussian Ornstein-Uhlenbeck based SV models. In the popular diffusion-based SV models, $Y_t$ follows a Feller’s square root process [7] or a Log-normal process [8] and the deterministic function is a squared root of the stochastic process ($\sigma(Y_t) = \sqrt{Y_t}$). The non-Gaussian Ornstein-Uhlenbeck based SV models have been introduced and thoroughly studied in [9][12]. The SV model with Ornstein-Uhlenbeck type process are mathematically tractable, have many appealing features and important implications for their use in option pricing.

From the perspective of derivative asset analysis, we will build a five-parameter VG model as Stochastic volatility model using $\Gamma(\alpha, \theta)$ Ornstein-Uhlenbeck process to represent volatility; While there is a great number of studies on option pricing under VG Model with three parameters [13–16], certainly due to the technical issues of fitting high parametric models to the marginal distribution of asset returns; the amount of literature considering GV Model with five parameters is rather limited. However, using five-parameter VG Model achieves a better trade-off between flexibility and tractability. In fact, a five-parameter VG Model is flexible enough to capture empirically stylized facts; and remains tractable at the same time.

In the option pricing theory, the challenge is often the existence of the Equivalent Martingale Measure (EMM) and whether it preserves the structure of the Variance-gamma measure. In fact, the Variance Gamma process is not a Gaussian process and the market is incomplete; therefore, the Equivalent Martingale Measure is not unique. The Esscher transform of the historic measure is considered optimal with respect to some optimisation criterion [17]. It was shown [18] the Esscher martingale measure coincides with the minimal entropy martingale measure for Lévy processes. The Esscher transform of the five-parameter VG model will be used to produce a closed-form formula for European option. The five parameters estimations come from [19,20], where the maximum likelihood was used to fit the five-parameter Variance Gamma model to the daily SPY ETF Price data from January 04, 2010 to December 30, 2020.

The remainder of this paper is organized as follows. We begin a short introduction of the Lévy process framework; section 2.2 and 2.3 will be devoted to build a five-parameter Gamma Variance process from a $\Gamma(\alpha, \theta)$ Ornstein-Uhlenbeck type process, and perform simulations of the VG process. Section 2.4 will investigate Lévy density and the asymptotic distribution of the VG process. Equivalent Martingale Measure (EMM) are discussed in section 3.1, and the Esscher transform measure is determined. In section 3.2, we Extended Black-Scholes framework and provide a closed-form solution for European option. Section 3.3 uses the characteristic exponent of the VG model and derives the integral representation for the option price; and the numerical results are compare with the BS closed formula.
2. Variance - Gamma Process: Stochastic Volatility Model

2.1 Lévy Framework and Asset Pricing.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a filtered probability space, with \(\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t\) and \(\{\mathcal{F}_t\}_{t \geq 0}\) is a filtration. \(\mathcal{F}_t\) is a \(\sigma\)-algebra included in \(\mathcal{F}\) and for \(\tau < t\), \(\mathcal{F}_\tau \subseteq \mathcal{F}_t\).

A stochastic process \(Y = \{Y_t\}_{t \geq 0}\) is a Lévy process, if it has the following properties

1. \((L1): Y_0 = 0\) a.s
2. \((L2): Y_t\) has independent increments
   i.e for \(0 < t_1 < t_2 < \cdots < t_n\), the random variables \(Y_{t_1}, Y_{t_1} - Y_{t_2}, \ldots, Y_{t_n} - Y_{t_{n-1}}\) are independent;
3. \((L3): Y_t\) has stationary increments
   for any \(t_1 < t_2 < +\infty\), the probability distribution of \(Y_{t_1} - Y_{t_2}\) depends only on \(t_1 - t_2\)
4. \((L4): Y_t\) is stochastically continuous: for any \(t\) and \(\varepsilon > 0\), \(\lim_{s \to t} \mathbb{P}(|Y_s - Y_t| > \varepsilon) = 0\);
5. \((L5): \text{cadlag paths}\)
   \(t \mapsto Y_t\) is a.s right continuous with left limits;

Given a Lévy process \(Y = \{Y_t\}_{t \geq 0}\) on the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), we can model an asset value as follows.

\[ S_t = S_0 e^{Y_t} \]

Where \(S = \{S_t\}_{t \geq 0}\) is the asset value process

**Theorem 2.1 (Lévy–Khintchine representation)**

Let \(Y = \{Y_t\}_{t \geq 0}\) be a Lévy process on \(\mathbb{R}\). Then it characteristic exponent admits the following representation.

\[ \phi(\xi) = -\log \left( \mathbb{E} e^{i \xi Y_1} \right) = -i \gamma \xi + \frac{1}{2} \sigma^2 \xi^2 + \int_0^t \left( e^{i \xi y} - 1 - y \xi 1_{|y| \leq 1} \right) \Pi(dy) \]  

(2.1)

where \(\gamma \in \mathbb{R}\), \(\sigma \geq 0\) and \(\Pi\) is a \(\sigma\)-finite measure called the Lévy measure of \(Y\), satisfying the property

\[ \int_{-\infty}^{+\infty} \min(1, |y|^2) \Pi(dy) < +\infty \]

For the theorem proof, see [21-23].

Each Lévy process is uniquely determined by the Lévy–Khintchine triplet \((\gamma, \sigma^2, \Pi)\). The terms of this triplet suggest that a Lévy process can be seen as having three independent components: a linear drift, a Brownian motion, and a Lévy jump process. When the diffusion term \(\sigma = 0\), we have a Lévy jump process; in addition, if \(\gamma = 0\), we have a pure jump process.

2.2 \(\Gamma(\alpha, \theta)\) Ornstein-Uhlenbeck process.

The Ornstein-Uhlenbeck process is a diffusion process that was introduced by Ornstein and Uhlenbeck [24] to model the stochastic behavior of the velocity of a particle undergoing Brownian motion. The Ornstein-Uhlenbeck diffusion \(\sigma^2 = \{\sigma^2(t), t \geq 0\}\) is the solution of the Langevin...
Stochastic Differential Equation (SDE) (2.2)
\[ d\sigma^2(t) = -\lambda \sigma^2(t) dt + dB(\lambda t) \quad (2.2) \]

Where \( \lambda > 0 \) and \( B = \{ B_t, t \geq 0 \} \) is a Brownian motion. In recent years, the Ornstein-Uhlenbeck process has been used in finance for capturing important distributional deviations from Gaussianity and for modelling of dependence structures. The extension of the Ornstein-Uhlenbeck processes was obtained by replacing the Brownian motion in (2.2) by \( z(t) \), a background driving Lévy process (BDLP) \([9, 10, 25]\). The SDE (2.2) becomes
\[ d\sigma^2(t) = -\lambda \sigma^2(t) dt + dz(\lambda t) \quad \lambda > 0 \quad (2.3) \]

Where the process \( z(t) = \{ z(t), t \geq 0, z(0) = 0 \} \) is subordinator; a process with non-negative, independent and stationary increments, which implies \( \sigma^2(t) \geq 0 \). Correspondingly \( z(t) \) moves up entirely by jumps and then tails off exponentially \([12]\).

**Lemma 2.2** The general form of the stationary process \( \sigma^2(t) \), solution of (2.3) is given by :
\[ \sigma^2(t) = -\int_{-\infty}^{+\infty} e^{-s} dz(\lambda t - s) \quad \lambda > 0 \quad (2.4) \]

**Proof.**
\[ \sigma^2(t) = -\int_{-\infty}^{+\infty} e^{-s} dz(\lambda t - s) = -\int_{0}^{+\infty} e^{-\lambda s} dz(\lambda (t - s)) = \int_{-\infty}^{t} e^{-\lambda (t-s)} dz(\lambda s) \quad \lambda > 0 \quad (2.5) \]

By using the variable changing method, we can have different expressions of (2.4).
\[ \sigma^2(t) = \int_{-\infty}^{t} e^{-\lambda (t-s)} dz(\lambda s) \implies d\sigma^2(t) = -\lambda \sigma^2(t) dt + dz(\lambda t) \quad (2.6) \]

(2.4) can be written as follows:
\[ \sigma^2(t) = \int_{-\infty}^{t} e^{-\lambda (t-s)} dz(\lambda s) = e^{-\lambda t} \sigma^2(0) + \int_{0}^{t} e^{-\lambda (u-s)} dz(\lambda s) \quad (2.7) \]

with
\[ \sigma^2(0) = \int_{-\infty}^{0} e^{\lambda s} dz(\lambda s) \]

**Theorem 2.3**
We supposed that a Lévy process \( z(t) = \sum_{k=1}^{N(t)} \xi_k \), which is generated by a compound poison process: \( N(t) \) is poisson process with instantaneous rate \( \alpha \), and \( \xi_k \) follows an exponential distribution with rate \( \theta \).

The stationary marginal distribution of \( \sigma^2(t) \) is Gamma distribution \( \Gamma(\alpha, \theta) \)

**Proof.**
\[ \sigma^2(t + u) = \int_{-\infty}^{t+u} e^{-\lambda (t+u-s)} dz(\lambda s) = e^{-\lambda u} \sigma^2(t) + \int_{0}^{u} e^{-\lambda (u-s)} dz(\lambda s) \quad u \geq 0 \quad (2.8) \]
The stationary solution $\sigma^2(t)$ of (2.3) can be written as in (2.8). Because of the stationarity, we have

$$\vartheta(\xi) = \vartheta(\xi e^{-\lambda u}) \Phi(u, \xi)$$  \hspace{1cm} (2.9)$$

$\vartheta(\xi)$ is the characteristic function of the stationary distribution of $\sigma^2(t)$ and $\Phi(u, \xi)$ is the characteristic function of $\int_0^\infty e^{-\lambda(u-s)} dz(\lambda s)$. We have $0 \leq e^{-\lambda u} \leq 1$ for $u \geq 0$, and the relation (2.8) shows that $\sigma^2(t)$ is self-decomposable.

$z(t)$ is a compound poison process with the function characteristic.

$$g(\xi) = E(e^{i\xi z(1)}) = exp \left\{ \int_0^\infty (e^{i\xi x} - 1) \alpha f(x) dx \right\} = exp(\rho(\xi)) \hspace{1cm} \rho(\xi) = \frac{i\xi \alpha}{\theta - i\xi}$$  \hspace{1cm} (2.10)$$

It was shown in [11] that $\Phi(u, \xi)$ can be expressed as follows

$$\Phi(u, \xi) = \exp \left\{ \frac{\lambda}{\theta - i\xi} \int_0^u \rho(\xi e^{-\lambda(u-s)}) ds \right\}$$  \hspace{1cm} (2.11)$$

$$= \exp \left\{ i\xi \int_0^\xi \frac{\rho(w)}{w} dw \right\}$$  \hspace{1cm} (2.12)$$

By replacing, $\frac{\rho(w)}{w} = \frac{i\alpha}{\theta - iw}$, we have

$$\Phi(u, \xi) = \left( \frac{\theta - i\xi e^{-\lambda u}}{\theta - i\xi} \right)^\alpha$$  \hspace{1cm} (2.13)$$

$\vartheta(\xi)$ is continuous at zero and we have:

$$\vartheta(\xi) = \lim_{u \to \infty} \vartheta(\xi e^{-\lambda u}) \Phi(u, \xi) = \left( \frac{1}{1 - i\frac{1}{\theta} \xi} \right)^\alpha = (1 - i\theta^{-1} \xi)^{-\alpha}$$  \hspace{1cm} (2.14)$$

From (2.14), $\vartheta(\xi)$ is the function characteristics of the gamma distribution; and the stationary marginal distribution of $\sigma^2(t)$ is the $\Gamma(\alpha, \theta)$ Gamma distribution.

Another method developed in [9, 10, 12, 25, 26] uses the relationship between the $\xi(t)$ Lévy density $w(x)$ and the Lévy density $u(x)$ of $\sigma^2(t)$.

$$u(x) = \int_1^\infty w(xr) dr$$  \hspace{1cm} (2.15)$$

The Lévy-Khintchine representation theorem [2.1] provides the meaning of the Lévy density. From (2.10), we have the Lévy density $W(x) = \alpha f(x) = \alpha \theta e^{-\theta x}$ and the Lévy density $u(x)$ of $\sigma^2(t)$ can be deduced as follows.

$$u(x) = \int_1^\infty \alpha \theta e^{-\theta x} dr = \frac{\alpha}{x} e^{-\theta x}$$  \hspace{1cm} (2.16)$$

$u(x)$ is the Lévy density of Gamma distribution $\Gamma(\alpha, \theta)$.

We can integrate the stationary non-negative process $\sigma^2(t)$.

$$\sigma^{2*}(t) = \int_0^t \sigma^2(s) ds$$  \hspace{1cm} (2.17)$$
By integration by part method, \((2.17)\) becomes
\[
\sigma^2(t) = \lambda^{-1} \sigma^2(0)(1 - e^{-\lambda t}) + \lambda^{-1} \int_0^t \left(1 - e^{-\lambda(t-s)}\right) dz(\lambda s) \tag{2.18}
\]
\[
= \lambda^{-1} \left(-\sigma^2(t) + z(\lambda t) + \sigma^2(0)\right) \tag{2.19}
\]

It results from \((2.19)\) that the process \(\sigma^2(t)\) is continuous as \(z(\lambda t)\) and \(\sigma^2(t)\) co-break \([9,26]\). In addition, the shape of \(\sigma^2(t)\) is determined by \(z(\lambda t)\). In fact, \(\sigma^2(t)\) and \(z(\lambda t)\) co-integrate. The co-integration can be shown by transforming the equation \((2.19)\) into \((2.20)\). \(\lambda \sigma^2(t) - z(\lambda t)\) is a stationary process.
\[
\lambda \sigma^2(t) - z(\lambda t) = -\sigma^2(t) + \sigma^2(0) \tag{2.20}
\]

For \(\lambda = 1\) and \(\sigma^2(0) = 0\), the compound poison process in \((2.21)\), the \(\Gamma(\alpha, \theta)\) Ornstein-Uhlenbeck process in \((2.22)\), and \(\sigma^2(t)\) in \((2.23)\) were simulated and the results are in Fig 12a, Fig 12b, and Fig 12c respectively.

\[
z(t) = \sum_{k=1}^{N(t)} \xi_k \tag{2.21}
\]
\[
\sigma^2(t) = \sigma^2(0)e^{\lambda t} + \sum_{k=1}^{N(t)} \exp(-\lambda(t-a_k))\xi_k \tag{2.22}
\]
\[
\sigma^2(t) = \int_0^t \sigma^2(s) ds \tag{2.23}
\]

**Figure 1.** Simulations: \(\hat{\alpha} = 0.8845, \hat{\theta} = 0.9378\)

The estimations of the gamma distribution parameter \(\Gamma(\alpha, \theta)\) was performed by the FRFT Maximum likelihood on the daily SPY prices \([19]\).
2.2.1 Variance - Gamma Process: Semi-Martingale.

Let \( Y^* = \{Y^*_t\} \), a stochastic process, used to model the log of an asset price.

\[
Y^*_t = A_t + M_t
\]

(2.24)

where \( A_t \) and \( M_t \) are the drift parameters, \( t \) represents the continuous time clock, and \( W(t) \) is the standard brownian motion and independent of \( \sigma^2(t) \).

\[
\sigma(t) = \sqrt{\sigma^2(t)} \quad \sigma^2(t) = \int_0^t \sigma^2(s) ds
\]

(2.27)

\( \sigma(t) \) is the spot or instantaneous volatility and \( \sigma^2(t) \) is the chronometer or the integrated variance of the process. As shown in Fig 12c, the Gamma process \( (\sigma^2(t)) \) is a strictly increasing process of the stationary process \( (\sigma^2(t)) \).

The mean process \( A_t \) is a predictable process with locally bounded variation. In fact, \( A_t \) is continuous and differentiable because of \( \sigma^2(t) \).

\( M_t \) is a local martingale. The derivative of \( M_t \) in (2.26) can be written as a Stochastic Differential Equation (SDE) (2.28)

\[
dM_t = \sigma \sigma(t)dW(t)
\]

(2.28)

\( Y^*_t \) is a special semi-martingale \([9, 27]\) and the decomposition \( Y^*_t = A_t + M_t \) is unique.

(a) Simulations: \( Y^* = \{Y^*_t\} \)

(b) Simulations versus SPY ETF data

**FIGURE 2.** AVG Model: \( \hat{\mu} = 0.0848, \hat{\delta} = -0.0577, \hat{\sigma} = 1.0295, \hat{\alpha} = 0.8845, \hat{\theta} = 0.9378 \)

Fig 2a and Fig 2b display, in bleu color, the simulation of the logarithmic of the asset price \( (y^*) \) in (3.34). The simulation is compared with the daily S&P 500 ETF (SPY) historical return data from January 4, 2010 to December 30, 2020. The real data is displayed in red color in Fig 2a.
2.3 Variance - Gamma Process: Parameter Estimations.

The stochastic process in (3.34) is the solution of the following Stochastic Differential Equation (SDE):

\[ dY_t^* = (\beta + \delta \sigma^2(t))dt + \sigma \sigma(t)dW(t) \quad (2.29) \]

Given an interval of length \( \Delta \), we define \( \sigma^2_n \) and \( Y_n \) over the interval \( [(n-1)\Delta; n\Delta] \).

\[ \sigma^2_n = \int_{(n-1)\Delta}^{n\Delta} d\sigma^2(s) = \sigma^2_{n\Delta} - \sigma^2_{(n-1)\Delta} \quad (2.30) \]

\[ Y_n = \int_{(n-1)\Delta}^{n\Delta} dY^*_s = Y^*_{n\Delta} - Y^*_{(n-1)\Delta} \quad (2.31) \]

The volatility component can be transformed into a normal distribution variable \( X(1) \)

\[ \int_{(n-1)\Delta}^{n\Delta} \sigma(t)dW(t) \overset{d}{=} N\left(0, \int_{(n-1)\Delta}^{n\Delta} \sigma^2(s)ds\right) \]

\( N(0, 1) \) is a standard normal distribution

\[ \overset{d}{=} N\left(0, \sigma^2_n\right) \]

\[ \overset{d}{=} \sigma_n N(0, 1) \]

\[ \overset{d}{=} \sigma_n X(1) \]

We have a normal distribution variable with mean 0 and variance \( \sigma^2_n \)

\[ \int_{(n-1)\Delta}^{n\Delta} \sigma(t)dW(t) = \sigma_n X(1) \quad (2.32) \]

By integrating the instantaneous return rate \( (2.29) \) per component, we have:

\[ \int_{(n-1)\Delta}^{n\Delta} dY^*_s = \beta \Delta + \delta \int_{(n-1)\Delta}^{n\Delta} d\sigma^2(s) + \sigma \int_{(n-1)\Delta}^{n\Delta} \sigma(t)dW(t) \]

Based on (2.30), (2.31) and (2.32), we have the following equation over the interval \( [(n-1)\Delta; n\Delta] \)

\[ Y_n = \mu + \delta \sigma^2_n + \sigma \sigma_n X(1) \quad (2.33) \]

Where \( \mu = \beta \Delta, X(1) \overset{d}{=} N(0, 1) \) and \( \sigma^2_n \overset{d}{=} \Gamma(\alpha, \theta) \).

In case \( \Delta \) is a daily length, \( Y_n \) becomes the daily return rate. The equation (2.33) was analysed in [19, 20] as a daily return rate and the parameters were estimated. Based on the fundamental probabilistic relationships of infinite divisibility, the Fractional Fourier Transform (FRFT) was used to implement the maximum likelihood method in [19, 20], which produces the estimations of five parameters \( (\mu, \delta, \alpha, \theta, \sigma) \). The data in [19, 20] came from the daily SPY historical data (adjustment for splits and dividends), SPDR S&P 500 ETF (SPY); the period spans from January
4, 2010 to December 30, 2020. For more details on the methodology and the results, see [19, 20, 28].

### Table 1. FRFT Maximum Likelihood AVG Parameters Estimations

| Model | Parameters | Statistics |
|-------|------------|------------|
|       | $\hat{\mu} = 0.0848$ | $E(\hat{Y}) = 0.0369$ |
| AVG   | $\hat{\delta} = -0.0577$ | $\text{Var}(\hat{Y}) = 0.8817$ |
|       | $\hat{\sigma} = 1.0295$ | $\text{Skew}(Y) = -0.173$ |
|       | $\hat{\alpha} = 0.8845$ | $\text{Kurt}(Y) = 6.412$ |
|       | $\hat{\theta} = 0.9378$ |

Source: Nzokem(2021) [19, 20]

Table 1 presents the estimation results of Asymmetric Variance-Gamma (AVG) model, labelled AVG.

### Table 2. Results of AVG Model Parameter Estimations

| Iterations | $\mu$ | $\delta$ | $\sigma$ | $\alpha$ | $\theta$ | Log(ML) | $\frac{d\text{Log}(ML)}{d\psi}$ |
|------------|------|--------|--------|--------|--------|--------|-------------------|
| 1          | 0    | 0      | 1      | 1      | 1      | -3582.8388 | 598.743251 |
| 2          | 0.05905599 | -0.009445 | 1.03195903 | 0.9130208 | 1.03208412 | -3561.5099 | 447.807305 |
| 3          | 0.06949925 | 0.00400035 | 1.04104444 | 0.88478895 | 1.05131996 | -3559.5656 | 498.289445 |
| 4          | 0.07514039 | 0.00557711 | 1.17577397 | 0.67326429 | 1.17786666 | -3560.6221 | 211.365781 |
| 5          | 0.08928373 | -0.0263176 | 1.03756321 | 0.83426661 | 0.94304967 | -3554.4434 | 498.289445 |
| 6          | 0.08676498 | -0.0521887 | 1.0337015 | 0.85591875 | 0.95066351 | -3550.6419 | 204.467192 |
| 7          | 0.086995 | -0.0608517 | 1.02788937 | 0.87382621 | 0.95054954 | -3549.8465 | 66.8039738 |
| 8          | 0.08542912 | -0.058547 | 1.02705241 | 0.88258411 | 0.94321299 | -3549.7023 | 15.3209117 |
| 9          | 0.08478622 | -0.0576654 | 1.02995166 | 0.88447791 | 0.93670036 | -3549.6921 | 1.14764198 |
| 10         | 0.0847798 | -0.057736 | 1.02922308 | 0.8849072 | 0.9381041 | -3549.692 | 0.72787708 |
| 11         | 0.08476475 | -0.0577217 | 1.02960343 | 0.88450434 | 0.93755549 | -3549.692 | 0.07850459 |
| 12         | 0.08477094 | -0.057488 | 1.02942608 | 0.8844984 | 0.93790784 | -3549.692 | 0.3723941 |
| 13         | 0.08476804 | -0.057736 | 1.02950397 | 0.88450117 | 0.93774266 | -3549.692 | 0.7132146 |
| 14         | 0.0847694 | -0.0577434 | 1.02947043 | 0.88449987 | 0.93781995 | -3549.692 | 0.00814365 |
| 15         | 0.08476876 | -0.0577411 | 1.02948868 | 0.88450048 | 0.93778375 | -3549.692 | 0.00380345 |
| 16         | 0.08476906 | -0.0577422 | 1.02948014 | 0.88450019 | 0.9378007 | -3549.692 | 0.00178206 |
| 17         | 0.08476892 | -0.0577417 | 1.02948414 | 0.88450033 | 0.93779276 | -3549.692 | 0.00083415 |
| 18         | 0.08476898 | -0.0577419 | 1.02948226 | 0.88450026 | 0.93779555 | -3549.692 | 8.56E-05 |
| 19         | 0.08476895 | -0.0577418 | 1.02948314 | 0.88450029 | 0.93779474 | -3549.692 | 4.01E-05 |

As shown in Table 2, the maximization procedure convergences after respectively 20 iterations for AVG1.
2.4 Variance - Gamma Process: Probability versus Lévy Density.

It was shown previously that a Variance-Gamma Process \(Y = \{Y_t\}\) with five parameters \((\mu, \delta, \sigma, \alpha, \theta)\) can have the following representation.

\[
Y_t = \mu t + \delta \sigma^2 s(t) + \sigma \int_0^t \sigma(s) dW(s) \tag{2.34}
\]

where \(\mu\) and \(\delta\) are the drift parameters, \(t\) represents the continuous time clock, and \(W(t)\) is the standard brownian motion and independent of \(\sigma^2(t)\). \(\sigma(t)\) is the spot or instantaneous volatility, \(\sigma^2(t)\) is the spot or instantaneous variance and \(\sigma^{2s}(t)\) is the chronometer or the integrated variance of the process.

\[
\sigma(t) = \sqrt{\sigma^2(t)} \quad \sigma^{2s}(t) = \int_0^t \sigma^2(s) ds \tag{2.35}
\]

We considerate the characteristic function of the Variance-Gamma process \(Y = \{Y_t\}\)

\[
E \left[ e^{itY_t} \right] = E \left[ e^{it(\mu t + \delta \sigma^2 s(t) + \sigma \int_0^t \sigma(s) dW(s))} \right] = e^{it\mu \xi} E \left[ e^{i\xi (\delta \sigma^2 s(t) + \sigma \int_0^t \sigma(s) dW(s))} \right] \tag{2.36}
\]

\(\int_0^t \sigma(s) dW(s)\) is the Itô integral with respect to the Brownian motion, and we have the following expression

\[
\int_0^t \sigma(s) dW(s) \overset{d}{=} N \left(0, \int_0^t \sigma^2(s) ds\right) \tag{2.37}
\]

\[
\overset{d}{=} N \left(0, \sigma^{2s}(t)\right) \tag{2.38}
\]

where \(N(0, 1)\) is a standard normal distribution.

From (2.36) and (2.38) we have

\[
E \left[ e^{i\xi (\delta \sigma^2 s(t) + \sigma \int_0^t \sigma(s) dW(s))} \right] = E \left[ e^{i\xi N(\delta \sigma^2 s(t), \sigma^2 \sigma^{2s}(t))} \right] \tag{2.39}
\]

\[
= E \left[ E \left[ e^{i\xi N(\delta \sigma^2 s(t), \sigma^2 \sigma^{2s}(t))} \right] | \sigma^{2s}(t) \right] \tag{2.40}
\]

\[
= E \left[ e^{i(\delta \xi - \frac{1}{2} \sigma^2 \xi^2) \sigma^{2s}(t)} \right] \tag{2.41}
\]

\(\sigma^{2s}(t)\) is a Lévy process generated by the Gamma distribution \(\Gamma(\alpha, \theta)\), \(\sigma^{2s}(t) \overset{d}{=} \Gamma(t\alpha, \theta)\) and we have

\[
E \left[ e^{i(\delta \xi - \frac{1}{2} \sigma^2 \xi^2) \sigma^{2s}(t)} \right] = \frac{1}{(1 + \frac{i}{2} \theta \sigma^2 \xi^2)^{t\alpha}} E \left[ e^{i\delta \xi W} \right] \overset{d}{=} \Gamma(t\alpha, \frac{1}{2} \sigma^2 \xi^2 + \frac{1}{\theta}) \tag{2.42}
\]

\[
= \frac{1}{(1 - i\delta \theta \xi + \sigma^2 \theta^2 \xi^2)^{t\alpha}} \tag{2.43}
\]

From (2.36), (2.41) and (2.43), we have

\[
E \left[ e^{i\mu \xi} \right] = \frac{e^{i\mu \xi}}{(1 - i\delta \theta \xi + \sigma^2 \theta^2 \xi^2)^{t\alpha}} \tag{2.44}
\]
With

\[ E \left[ e^{i Y \xi} \right] = \left( \phi(\xi) \right)^t \]  
\[ \phi(\xi) = \frac{e^{i \mu \xi}}{(1 - i \delta \theta \xi + \sigma^2 \theta \xi^2)\alpha} \]  
\[ \phi(\xi, t) = -\text{Log}(E \left[ e^{i Y \xi} \right]) = -t \text{Log}(\phi(\xi)) \]  

\subsection*{2.4.1 Lévy measure and the structure of the jumps.}

**Lemma 2.4** *(Frullani integral)* \( \forall \alpha, \beta > 0 \) and \( \forall z \in \mathbb{C} \) with \( \Re(z) \leq 0 \), we have

\[ \frac{1}{(1 - \frac{z}{a})^\beta} = e^{- \int_0^\infty (1-e^{zx})^\beta e^{-\alpha x} dx} \]

For lemma proof, see [29]

**Theorem 2.5** *(Variance-Gamma Model representation)*

Let \( Y = Y_t \) be a Lévy process on \( \mathbb{R} \) generated by the VG model with parameter \( (\mu, \delta, \sigma, \alpha, \theta) \). The characteristic exponent of the Lévy process admits the following representation.

\[ \phi(\xi, 1) = -\text{Log} \left( E e^{i Y_1 \xi} \right) = i \mu \xi + \int_{-\infty}^{+\infty} \left( 1 - e^{-i \xi u} \right) \Pi(u) du \]  

\( \Pi(u) \) is the Lévy density of \( Y \):

\[ \Pi(u) = \alpha \left( \frac{1_{\{u>0\}}}{u} e^{-x_1 u} + \frac{1_{\{u<0\}}}{|u|} e^{-x_2 u} \right) \]

With

\[ x_1 = \frac{\delta}{\sigma^2} + \sqrt{\frac{\delta^2}{\sigma^4} + \frac{2}{\sigma^2 \theta^2}} \]
\[ x_2 = \frac{\delta}{\sigma^2} - \sqrt{\frac{\delta^2}{\sigma^4} + \frac{2}{\sigma^2 \theta^2}} \]

and \( \Pi(u) \) satisfies the properties

\[ \int_{-\infty}^{+\infty} \Pi(u) du = +\infty \quad \text{and} \quad \int_{-\infty}^{+\infty} \text{Min}(1, |u|) \Pi(u) du < +\infty \]  

Proof.

We consider the characteristic function \( \phi(\xi) \) in (2.52) of the VG model with parameter \( (\mu, \delta, \sigma, \alpha, \theta) \) developed previously

\[ \phi(\xi) = \frac{e^{i \mu \xi}}{(1 - i \delta \theta \xi + \sigma^2 \theta \xi^2)\alpha} \]  

We factor the quadratic function in (2.52),

\[ \left( 1 + \frac{1}{2} \theta \sigma^2 x^2 - i \delta \theta x \right)^\alpha = \left( \frac{1}{2} \theta \sigma^2 \right)^\alpha (x - i x_1)^\alpha (x - i x_2)^\alpha \]  

(2.53)
With
\[ x_1 = \frac{\delta}{\sigma^2} + \sqrt{\frac{\delta^2}{\sigma^4} + \frac{2}{\theta \sigma^2}} \quad x_2 = \frac{\delta}{\sigma^2} - \sqrt{\frac{\delta^2}{\sigma^4} + \frac{2}{\theta \sigma^2}} \]

We apply the lemma 2.4 on each factor of the quadratic function (2.53)
\[ \left( \frac{1}{2} \theta \sigma^2 \right)^{\alpha} (x - ix_1)^{\alpha} (x - ix_2)^{\alpha} = \left( 1 + \frac{ix}{x_1} \right)^{\alpha} \left( 1 + \frac{ix}{x_2} \right)^{\alpha} = e^{\int_0^\infty \left( 1 - e^{-iu} \frac{\alpha}{u} e^{-x_1u} du \right)} \]
\[ = e^{\int_0^\infty \left( 1 - e^{-iu} \frac{\alpha}{u} e^{-x_1u} du \right)} \]
\[ = e^{\int_0^\infty \left( 1 - e^{-iu} \frac{\alpha}{u} e^{-x_1u} du + \int_{-\infty}^0 \left( 1 - e^{-iu} \frac{\alpha}{u} e^{-x_1u} du \right) \right)} \]
\[ = e^{\int_{-\infty}^\infty \left( 1 - e^{-iu} \right) \Pi(u) du} \]

Taking into account (2.53), we have
\[ \left( 1 + \frac{1}{2} \theta \sigma^2 x^2 - i \delta \theta x \right)^{\alpha} = e^{\int_{-\infty}^\infty \left( 1 - e^{-iu} \right) \Pi(u) du} \quad (2.54) \]

Where \( \Pi(u) \) is:
\[ \Pi(u) = \alpha \left( \frac{1}{u} \right)^{\alpha} \left( 1 + \frac{1}{\left| u \right|} \right) \]
\[ = \alpha \left( \frac{1}{u} \right)^{\alpha} \left( 1 + \frac{1}{\left| u \right|} \right) \quad (2.55) \]

From (2.47) and (2.55), we have:
\[ \varphi(\xi, t) = -t \log(\varphi(\xi)) = -it \mu \xi + t \log \left( 1 + \frac{1}{2} \theta \sigma^2 x^2 - i \delta \theta x \right)^{\alpha} \]
\[ = -it \mu \xi + t \log \left( 1 + \frac{1}{2} \theta \sigma^2 x^2 - i \delta \theta x \right)^{\alpha} \]
\[ = -it \mu \xi + \int_{-\infty}^{+\infty} (1 - e^{-iu}) t \Pi(u) du \]

We have
\[ \varphi(\xi, t) = -t \log(\varphi(\xi)) = -it \mu \xi + \int_{-\infty}^{+\infty} (1 - e^{-iu}) t \Pi(u) du \quad (2.56) \]

For \( t = 1 \), we have (2.48)

We can check some properties of \( \Pi(u) \)
\[ \int_{-\infty}^{+\infty} \Pi(u) du = +\infty \quad \text{in fact} \quad \lim_{|u| \to 0} \Pi(u) = +\infty \quad (2.57) \]
\[
\int_{-\infty}^{+\infty} \min(1, |u|) \Pi(du) = \int_{-1}^{1} \min(1, |u|) \Pi(du) + \int_{1}^{+\infty} \min(1, |u|) \Pi(du) + \int_{-\infty}^{-1} \min(1, |u|) \Pi(du)
\]
\[
= \alpha \left( \frac{1 - e^{-x_1}}{x_1} + \frac{1 - e^{-x_2}}{-x_2} + \Gamma(0, x_1) + \Gamma(0, -x_2) \right) \quad \text{with} \quad \Gamma(s, u) = \int_{u}^{+\infty} u^{s-1} e^{-u} du
\]

And we have:

\[
\int_{-\infty}^{+\infty} \min(1, |u|) \Pi(du) < +\infty \tag{2.58}
\]

The results in (2.57) show that the Variance - Gamma process is not a finite activities process. In fact, any finite interval does not have a finite number of jumps. Therefore, the process can not be written as a Compound Poisson process [9]. The Variance - Gamma process is rather an infinite activities process with an infinite number of jumps in any given time interval. The arrival rate of jumps of all sizes in the VG process is defined by the Lévy density (2.59)

\[
\Pi(x) = \begin{cases} 
\frac{\alpha}{u} e^{-x_1 u}, & \text{if } x > 0 \\
\frac{\alpha}{|u|} e^{-x_2 u}, & \text{if } x < 0 
\end{cases} \tag{2.59}
\]

As shown in Fig 3a, the highly arrival rates of jumps are concentrated around the origin 0. In fact, the smaller is the jump size, the higher is the arrival rate for the Gamma-Variance model. The steepness parameters \(-x_2\) and \(x_1\), defined the rate of exponential decay of the tails of the Lévy density in each side. As shown in Fig 3a and (2.59), the Lévy density is asymmetric and the left tail is heavier as \(-x_2 < x_1\). On the other hand, the results in (2.58) proofs that the VG process is a finite variation process, which is contrary to the Brownian motion process. The Gamma distribution Parameter (\(\alpha\)), which is called the process intensity [17], plays an important role in the Lévy density. In fact, the intensity of the process (\(\alpha\)) has a similar role as the variance parameter in the Brownian motion process. The Lévy density function (2.59) is different for negative and positive jump size \(x\). The difference has led [15] to see the VG process as the difference of two increasing processes, with each process provides separately the market up and down moves.

Using the AVG model data in table [1][19][20], we have \(x_1 = 1.4775\) and \(x_2 = -1.3640\). Fig 3a and Fig 3b display the Lévy and the probability densities. As shown in Fig 3, the shape of the distribution of the densities are different. Even-though, both densities are linked by the same characteristic function.

Variance Gamma Process can be described as a subfamily of KoBoL family, which is the extension of Koponen’s family by Boyarchenko and Levendorskii [17]. The KoBoL family is also called CGMY-model (named after Carr, German, Madan and Yor) [30]. Under the KoBoL familly,
the Lévy density has the following general form: see [17] for more details

$$\Pi(x) = \begin{cases} C_+ u^{-\nu-1} e^{-\lambda_+ u}, & \text{if } x > 0 \\ C_- |u|^{-\nu-1} e^{\lambda_- u}, & \text{if } x < 0 \end{cases}$$ (2.60)

Where $C_+ > 0$, $C_- > 0$, $\nu > 0$ and $\lambda_- < 0 < \lambda_+$. As a subfamily of KoBoL family, the variance Gamma process belongs to the process class of order $\nu = 0$, intensity $C_+ = C_- = \alpha$ and steepness parameters $\lambda_- = -x_2 = -\frac{\delta}{\sigma^2} + \sqrt{\frac{\delta^2}{\sigma^4} + \frac{2}{\theta \sigma^2}}$ and $\lambda_+ = x_1 = \frac{\delta}{\sigma^2} + \sqrt{\frac{\delta^2}{\sigma^4} + \frac{2}{\theta \sigma^2}}$.

### 2.5 Variance - Gamma Process: Asymptotic distribution.

**Theorem 2.6** (Variance Gamma process probability density)

Let $Y = Y_t$ be a Lévy process on $\mathbb{R}$ generated by the VG model with parmeter $(\mu, \delta, \sigma, \alpha, \theta)$. The probability density function can be written as follows

$$f(y, t) = \frac{1}{\sigma \Gamma(t \alpha) \theta^\alpha} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y - t \mu - \delta v)^2}{2\sigma^2}} v^\alpha - 1 e^{-v \frac{\theta}{\sigma^2}} dv \quad t \geq 0 \quad y \in \mathbb{R}$$ (2.61)

**Proof:**

$\phi(\xi, t)$ in (2.56) provides the relation between the characteristic exponent and the Lévy density. The expression is used as follows:

$$\phi(\xi, t) = -\log \left( E e^{\beta y} \right) = -i \mu \xi + \int_{-\infty}^{\infty} (1 - e^{-i\xi u}) t \Pi(u) du$$

$$t \Pi(u) = t \alpha \left( \frac{1_{[u>0]}}{u} e^{-x_1 u} + \frac{1_{[u<0]}}{|u|} e^{x_2 |u|} \right)$$

$$\mu_t = t \mu \quad \alpha_t = t \alpha$$ (2.62)
It was shown in [19] that the probability density of a VG model with parameter \((\mu, \delta, \sigma, \alpha, \theta)\) can be written

\[
f(y) = \frac{1}{\sigma \Gamma(\alpha)} \theta^\alpha \int_0^{+\infty} e^{-\frac{(y-\mu-\delta v)^2}{2\sigma^2 v}} v^{\alpha-1} e^{-\frac{v}{\theta}} dv \tag{2.63}
\]

By replacing the parameters (2.62) in (2.63), we have the result in theorem 2.6. □

As shown in (2.62), the dynamic of the probability density \(f(y,t)\) is carried by two parameters: \(t\mu\) and \(t\alpha\). \(f(y,t)\) can be compared to the histogram of the SPY ETF return data and the probability density of the AVG Model (in table 1) adjusted to the following period: Quarterly \((\tau = 0.25)\) and Semi-Annual \((\tau = 0.5)\). Fig 4 shows the histogram of the SPY ETF return data and the probability densities of the AVG Model (in table 1) for short periods: Quarterly and Semi-annually. As illustrated in Fig 4a and Fig 4b, the AVG probability density (in red) and the probability density generated by the Lévy Process (in black) match perfectly the SPY ETF return data when the time frame is not large, we have \(\tau = 0.25\) and \(\tau = 0.5\) in the graphs.

**Theorem 2.7 (Asymptotic distribution of Variance Gamma process)**

Let \(Y = Y_t\) be a Lévy process on \(\mathbb{R}\) generated by the VG model with parameter \((\mu, \delta, \sigma, \alpha, \theta)\). Then \(Y_t\) converges in distribution to a Lévy process driving by a Normal distribution with mean \(a = \mu + \alpha \theta \delta\) and variance \(\sigma^2 = \alpha (\theta^2 \delta^2 + \sigma^2 \theta)\)

\[Y_t \overset{d}{\sim} N(ta, t\sigma^2) \quad \text{as} \quad t \to +\infty\]

**Proof:**

Let us have:

\[b_t = \sqrt{tb} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad a_t = ta\]

\[b = \sqrt{\alpha (\theta^2 \delta^2 + \sigma^2 \theta)} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad a = \mu + \alpha \theta \delta\]
\( \phi(\xi, t) \) is the characteristic function of the Lévy process \( Y = \{Y_t\} \), we use the expression (2.44).

\[
\phi(\xi, t) = E\left[ e^{i\xi Y_t} \right] = \frac{e^{i\mu \xi}}{\left(1 - i\delta \theta \xi + \sigma^2 \theta \xi^2 \right)^{it}}
\]

\( \phi^T(\xi, t) \) is the characteristic function of the stochastic process \( \frac{Y_t - at}{b_t} \) and we have

\[
\phi^T(\xi, t) = E\{e^{i\xi \frac{Y_t - at}{b_t}}\} = e^{-i\frac{\alpha \theta \xi}{b_t}} \phi \left( \frac{\xi}{b_t}, t \right)
\]

\[
= e^{i\alpha \theta \delta \frac{\xi}{b_t}} \left( 1 + \frac{1}{2} \theta \sigma^2 \frac{\xi^2}{b_t^2} - i\delta \theta \xi \frac{\xi}{\sqrt{b_t}} \right)^{-it
\}
\]

Let us have:

\[
u(t) = \frac{1}{2} \theta \sigma^2 \frac{\xi^2}{tb^2} - i\delta \theta \xi \frac{\xi}{\sqrt{tb}} \quad \lim_{t \to +\infty} \nu(t) = 0
\]

We can use the Taylor expansions of \( \ln(1 + u) \)

\[
\ln(1 + \frac{1}{2} \theta \sigma^2 \frac{\xi^2}{tb^2} - i\delta \theta \xi \frac{\xi}{\sqrt{tb}}) = \frac{1}{2} (\theta \sigma^2 + \delta^2 \theta^2) \frac{\xi^2}{tb^2} - i\delta \theta \xi \frac{\xi}{\sqrt{tb}} + o \left( \frac{1}{t \sqrt{t}} \right)
\]

\[
\lim_{t \to +\infty} o \left( \frac{1}{t \sqrt{t}} \right) = 0
\]

The characteristic function, \( \phi^T(\xi, t) \), developed previously becomes:

\[
\phi^T(\xi, t) = e^{i\alpha \theta \delta \frac{\xi}{b_t}} \left( 1 + \frac{1}{2} \theta \sigma^2 \frac{\xi^2}{tb^2} - i\delta \theta \xi \frac{\xi}{\sqrt{tb}} \right)^{-it
\} = e^{i\alpha \theta \delta \frac{\xi}{b_t}} e^{-it \ln(1 + \frac{1}{2} \theta \sigma^2 \frac{\xi^2}{tb^2} - i\delta \theta \xi \frac{\xi}{\sqrt{tb}})}
\]

\[
= e^{-\frac{1}{2} \alpha (\theta \sigma^2 + \delta^2 \theta^2) \frac{\xi^2}{b_t^2}} + o \left( \frac{1}{\sqrt{t}} \right)
\]

We have:

\[
\lim_{t \to +\infty} \phi^T(\xi, t) = \lim_{t \to +\infty} E\{e^{i\frac{Y_t - at}{b_t} \xi}\} = e^{-\frac{1}{2} \xi^2}
\]

By applying the limit in (2.64), we produce the cumulant-generating function \([31]\) of the Normal distribution. We have the following convergence in distribution

\[
\frac{Y_t - at}{b_t} \overset{d}{\sim} N(0, 1) \quad \text{as} \quad t \to +\infty
\]
However, when the period becomes large, for example Thrid Quaterly and Annually periods in Fig 5a and Fig 5b respectively, the VG probability density generated by the Lévy Process misses to capture some stylized facts (peakedness, skewness and kurtosis) in the data. In fact, as shown in Fig 5 for large period of time, the probability density generated by the Lévy Process (in black) fails to capture the heavy tail of the real data distribution. In contrast, the probability densities of the AVG Model (in red) performs well annually.

![Figure 5a](image1.png) ![Figure 5b](image2.png)

(a) Third Quarterly SPY ETF Return  
(b) Annually SPY ETF Return

**Figure 5.** $f(y,t): \hat{\mu} = 0.0848, \hat{\delta} = -0.0577, \hat{\sigma} = 1.0295, \hat{\alpha} = 0.8845, \hat{\theta} = 0.9378

The discrepancy between the histogram and the probability density generated by the Lévy process can be explained by the Asymptotic distribution in Theorem 2.7. As shown (in black) in Fig 5, we have a Lévy process driven by a normal distribution.
3. Variance - Gamma Process: Pricing European Options

3.1 Variance - Gamma Process: Risk Neutral Esscher Measure.

The method of Esscher transform was introduced by [32] as an efficient technique for pricing derivative securities if the logarithms of the underlying asset prices are modelling by a Lévy process. An Esscher transform of a stock-price process provides an equivalent martingale measure; under such measure, the price of any derivative security is simply calculated as the expectation of the discounted payoffs. Some measures of the Esscher transform preserve the structure of the initial measure. Examples of commonly conservative measure in the literature are Normal, Compound Poisson, Gamma, Exponential, and Inverse Gaussian distributions. The existence of the equivalent Esscher transform measure is not always guaranteed; and the issue of unicity of the pound Poisson, Gamma, Exponential, and Inverse Gaussian distributions. The existence of the initial measure. Examples of commonly conservative measure in the literature are Normal, Com-

\[
M(h, t) = \phi(-ih)^\prime = \frac{e^{ith}}{(1 - \frac{1}{2} \theta \sigma^2 h^2 - \delta \theta h)^\alpha} = M(h, 1)^\prime \quad \text{with } h_1 < h < h_2 \quad (3.1)
\]

\[
M(h, 1) = \frac{e^{ith}}{(1 - \frac{1}{2} \theta \sigma^2 h^2 - \delta \theta h)^\alpha} \quad \text{with } h_1 < h < h_2 \quad (3.2)
\]

\[
h_1 = -\frac{\delta}{\sigma^2} - \sqrt{\frac{\delta^2}{\sigma^4} + \frac{2}{\theta \sigma^2}} \quad h_2 = -\frac{\delta}{\sigma^2} + \sqrt{\frac{\delta^2}{\sigma^4} + \frac{2}{\theta \sigma^2}} \quad (3.3)
\]

Under the Esscher transform with parameter h, the probability density of \( Y = \xi \) becomes:

\[
\hat{f}(x, t, h) = \frac{e^{hx} f(x, t)}{M(h, t)} \quad \text{with } h_1 < h < h_2 \quad (3.4)
\]

The Moment generating function of the Esscher transform VG model

\[
M(z, t, h) = E^h [e^{\xi_t}] = \int_0^\infty \int_0^{+\infty} e^{\xi_t} \hat{f}(x, t, h) dx = \int_0^{+\infty} \frac{e^{(h+z)x} f(x, t)}{M(x, t)} dx \quad (3.5)
\]

\[
= \frac{M(h+z, t)}{M(h, t)} \quad \text{with } h_1 < h < h_2 - z \quad (3.6)
\]

\[
= \left( \frac{M(h+z, 1)}{M(h, 1)} \right)^t = M(z, 1, h)^t \quad \text{with } h_1 < h < h_2 - z \quad (3.7)
\]

with

\[
M(z, 1, h) = \frac{M(h+z, 1)}{M(h, 1)} = e^{\mu_z (M_{ss}(z, 1, h))^\alpha} \quad M_{ss}(z, 1, h) = \frac{1 - \frac{1}{2} \theta \sigma^2 h^2 - \delta \theta h}{1 - \frac{1}{2} \theta \sigma^2 (h+z)^2 - \delta \theta (h+z)} \quad (3.8)
\]

\( \hat{f}(x, t, h) \) is the modified probability density of \( f(x, t) \) defined in (2.61). \( \exp(x) \) is a strictly increasing function, the probability measure generated by \( \hat{f}(x, t, h) \) is equivalent to the original probability measure generated by \( f(x, t) \). In fact, both probability measures have the same null
sets \( \{ \text{sets of probability measure zero} \} \). Given the process \( \{ e^{-r \tau} S(\tau) \} \), with \( r \) the constant risk-free rate of interest. We look into the conditions to have \( h = h^* \) such as

\[
E^h \left[ e^{-r \tau} S(\tau) \right] = S(0)
\]

(3.9)

we have \( S(\tau) = S(0)e^{Y_\tau} \), with \( Y_\tau \) is the Variance - Gamma process. The equation (3.9) becomes

\[
e^{r\tau} E^h \left[ e^{Y_\tau} \right] = M(1, 1, h^*)^\tau = \left( \frac{M(h^* + 1, 1)}{M(h^*, 1)} \right)^\tau \quad \text{with} \quad h_1 < h < h_2 - 1
\]

(3.10)

the first condition is that

\[
h_1 - h_2 < 1 \quad \frac{\delta^2}{\sigma^4} + \frac{2}{\theta \sigma^2} < \frac{1}{4}
\]

The equation (3.10) is equivalent to (3.11) and the existence of \( h^* \) is shown by Fig 6 for AVG Model in Table 1.

\[
e^{r-\mu} = M_{**}(1, 1, h^*) = M(h^* + 1, 1) = \frac{1 - \frac{1}{2} \theta \sigma^2 h^* - \delta \theta h^*}{1 - \frac{1}{2} \theta \sigma^2 (h^* + 1)^2 - \delta \theta (h^* + 1)}
\]

(3.11)

More general, we consider the function

\[
g(h) = \frac{1 - \frac{1}{2} \theta \sigma^2 h^2 - \delta \theta h}{1 - \frac{1}{2} \theta \sigma^2 (h + 1)^2 - \delta \theta (h + 1)}
\]

\[
dg{h} = \frac{1/2 \theta \sigma^4 h^2 + \delta \theta^2 (1/2 \sigma^2 + \delta) h + \delta \theta^2 (1/2 \sigma^2 + \delta)}{(1 - \frac{1}{2} \theta \sigma^2 (h + 1)^2 - \delta \theta (h + 1))^2}
\]

we have

\[
\frac{dg}{dh} (h) > 0 \quad h_1 < h < h_2 - 1 \quad \lim_{h \to h_1} g(h) = 0 \quad \lim_{h \to h_2 - 1} g(h) = +\infty
\]

(3.12)

(3.12) shows the existence and the unicity of \( h^* \) in \([h_1, h_2 - 1]\) such that

\[
e^{r-\mu} = g(h^*)
\]

For AVG Model in Table 1 [19, 20]: \( \hat{\mu} = 0.0848 \), \( \hat{\delta} = -0.0577 \), \( \hat{\sigma} = 1.0295 \), \( \hat{\alpha} = 0.8845 \), \( \hat{\theta} = 0.9378 \). We have \( h_1 = -1.3651 \) and \( h_2 = 1.4740 \). Over the interval \([h_1; h_2 - 1]\), \( g(h) \) is an increasing function, as shown in Fig 6a. Fig 6b provides the solution \( h^* \) of equation (3.11) for free interest rate less than 10%. It is important to note that the solution \( h^* \) increases with the free interest rate \( r \).
From the Esscher transform, we have the Equivalent Martingale Measure (EMM) $Q$, which can be written as the Radon-Nikodym derivative:

$$
\frac{dQ}{dP} = e^{hY_{\tau}} = e^{hY_{\tau} - \log(M(h^*, \tau))}
$$

(3.13)

and $E^Q$ for expectation with respect to $Q$

$$
E^Q [e^{-r\tau}S(\tau)] = E^P \left[ e^{-r\tau}S(\tau) \frac{dQ}{dP} \right] = S(0)E^P \left[ e^{(1+h^*)Y_{\tau} - \log(M(h^*, \tau)) - r\tau} \right]
$$

$$
= S(0)M(h^*, \tau) e^{-r\tau} \left( \frac{M(h^* + 1, \tau)}{M(h^*, \tau)} \right)
$$

(3.10)

$$
E^Q [e^{-r\tau}S(\tau)] = S(0)
$$

(3.14)

**Theorem 3.1 (Variance-Gamma Esscher transform distribution)**

Esscher transform of Variance-Gamma process $Y = \{Y_t\}_{t \geq 0}$ with parameter $(t\mu, \delta, \sigma, t\alpha, \theta)$ is also a Variance-Gamma process with parameter $(t\mu, \hat{\delta}, \sigma, t\alpha, \hat{\theta})$

$$
\hat{\delta} = \delta + h\sigma^2 \quad \hat{\theta} = \frac{\theta}{1 - \frac{1}{2}t\sigma^2 h^2 - \delta \theta h}
$$

(3.15)
Proof:
From (3.8), we have
\[
M^{**}(z, 1, h) = \frac{1 - \frac{1}{2} \theta \sigma^2 h^2 - \delta \theta h}{1 - \frac{1}{2} \theta \sigma^2 (h + z)^2 - \delta \theta (h + z)}
\]  
(3.16)
we can divide the denominator by the numerator of the function \(M^{**}(z, 1, h)\) in (3.16), and rearrange the resulting expression.
\[
M^{**}(z, 1, h) = \frac{1}{1 - \frac{1}{2} \tilde{\theta} \sigma^2 z^2 - \tilde{\delta} \tilde{\theta} z}
\]  
(3.17)
\[
\tilde{\theta} = \frac{\theta}{1 - \frac{1}{2} \tilde{\theta} \sigma^2 h^2 - \tilde{\delta} \theta h}
\tilde{\delta} = \delta + h \sigma^2
\]  
(3.18)
\[
M(z, 1, h) \quad \text{in (3.8) becomes}
\]
\[
M(z, 1, h) = e^{\mu z} \left( 1 - \frac{1}{2} \tilde{\theta} \sigma^2 z^2 - \tilde{\delta} \tilde{\theta} z \right)^{-\alpha}
\]  
(3.19)
(3.20)
Using the Esscher transform method, the moment generating function for Variance - Gamma process \(Y = (Y_t)_{t \geq 0}\) becomes:
\[
M(z, t, h) = E^h [e^{X_t}] = M(z, 1, h)^t = e^{\mu z} \left( 1 - \frac{1}{2} \tilde{\theta} \sigma^2 z^2 - \tilde{\delta} \tilde{\theta} z \right)^{t \alpha} \quad \text{with} \quad \tilde{h}_1 < z < \tilde{h}_2
\]  
(3.21)
(3.22)
We have a new Variance - Gamma process with parameter \((t \mu, \tilde{\delta}, \sigma, t \alpha, \tilde{\theta})\) \(\square\)
The Esscher transform method preserves the structure of the five-parameter VG process; it introduces an addition symmetric parameter \((h \sigma^2)\) and inflates the Gamma scale parameter by \(\frac{1}{1 - \frac{1}{2} \theta \sigma^2 h^2 - \delta \theta h}\) factor.

3.2 Variance - Gamma Model : Extended Black-Scholes Formula.

Corollary 3.2 (Extended Black-Scholes)
Let \(r\), a continuously compounded risk-free rate of interest; \(Y = Y_t\), a Variance-Gamma Process with parameter \((\mu, \delta, \sigma, \alpha, t\theta)\); and \((S(0)e^{X_T} - K)^+\), the terminal payoff for a contingent claim with the expiry date \(T\). Then at time \(t < T\), the arbitrage price of the European call option with the strike price \(K\) can be written as follows.
\[
F_{\text{call}}(S_t, t) = S(t) \left[ 1 - \hat{F}(\log\left(\frac{K}{S(t)}\right), \tau, h^*) - Ke^{-\tau r} \left[ 1 - \hat{F}(\log\left(\frac{K}{S(t)}\right), \tau, h^*) \right] \right]
\]  
(3.22)
\[
\hat{F}(\log\left(\frac{K}{S(t)}\right), \tau, h^*) = \int_{-\infty}^{\log(\frac{K}{S(t)})} \hat{f}(\xi, \tau, h^*) d\xi \quad \hat{f}(\xi, \tau, h^*) \quad \text{in (3.4)}
\]  
(3.23)
Where $\tau = T - t$, and $\hat{F}(k, \tau, h^*)$ and $\hat{F}(k, \tau, h^* + 1)$ are the cumulative distribution of VG Model with parameter $(\tau \mu, \tilde{\delta}, \sigma, \tau \alpha, \tilde{\theta})$ and parameter $(\tau \mu, \tilde{\delta} + \sigma^2, \sigma, \tau \alpha, \tilde{\theta}e^{-\tau \delta})$ respectively.

Proof:

$$f(Y_T, K) = (S(0)e^{Y_T} - K)^+ = S(t)(e^{Y_T} - k)^+$$
$$Y_T = Y_T + Y_t = S(t)g(Y_T)$$ \quad $S(t) = S(0)e^{Y_t}$ and $k = \frac{K}{S(t)}$

Under the Equivalent Martingale Measure (EMM), $\hat{F}(\xi, \tau, h^*)$ is the probability density of VG model with parameter $(\tau \mu, \tilde{\delta}, \sigma, \tau \alpha, \tilde{\theta})$. We note $k = \frac{K}{S(t)}$

$$S(t)e^{-rT} \int_{\log(k)}^{+\infty} e^z \hat{f}(\xi, \tau, h^*)d\xi = S(t)e^{-rT} \int_{\log(k)}^{+\infty} e^z \frac{e^{h^*\xi}f(\xi, \tau)}{M(h, t)}d\xi \hat{f}(\xi, \tau, h^*) \text{ in (3.4)}$$
$$= S(t)e^{-rT} \int_{\log(k)}^{+\infty} e^{(1+h^*)\xi}f(\xi, \tau)M(h^*, t)d\xi e^{rT} = \frac{M(h^*+1, \tau)}{M(h^*, \tau)} \text{ in (3.10)}$$
$$= S(t) \int_{\log(k)}^{+\infty} \frac{e^{(1+h^*)\xi}f(\xi, \tau)}{M(1+h^*, t)}d\xi$$
$$= S(t) \int_{\log(k)}^{+\infty} \hat{f}(\xi, \tau, h^* + 1)d\xi$$

We can now show the relation in (3.22)

$$F_{call}(S_t, t) = S(t)e^{-rT}E[\hat{Q}[g(X_T)] = S(t)e^{-rT} \int_{-\infty}^{+\infty} \hat{f}(\xi, \tau, h^*)(e^z - k)^+ dy$$
$$= S(t)e^{-rT} \int_{\log(k)}^{+\infty} e^z \hat{f}(\xi, \tau, h^*)d\xi - Ke^{-rT} \int_{\log(k)}^{+\infty} \hat{f}(\xi, \tau, h^*)d\xi$$
$$= S(t) \int_{\log(k)}^{+\infty} \hat{f}(\xi, \tau, h^* + 1)d\xi - Ke^{-rT} \int_{\log(k)}^{+\infty} \hat{f}(\xi, \tau, h^*)d\xi$$
$$= S(t) \left[1 - \hat{F}(\log(k), \tau, h^* + 1)\right] - Ke^{-rT} \left[1 - \hat{F}(\log(k), \tau, h^*)\right]$$

with

$$\hat{F}(\log(k), \tau, h^*) = \int_{-\infty}^{\log(k)} \hat{f}(\xi, \tau, h^*)d\xi \quad \hat{F}(\log(k), \tau, h^* + 1) = \int_{-\infty}^{\log(k)} \hat{f}(\xi, \tau, h^* + 1)d\xi$$

From theorem 2.6 and theorem 3.1, $\hat{f}(\xi, \tau, h^*)$ is the probability density of the VG model with parameter $(\tau \mu, \tilde{\delta}, \sigma, \tau \alpha, \tilde{\theta})$.

$$\hat{f}(\xi, \tau, h^*) = \frac{1}{\sigma \Gamma(\tau \alpha) \tilde{\theta}^{\tau \alpha}} \int_{0}^{+\infty} \frac{1}{\sqrt{2\pi v}} e^{-\frac{(\xi - \tilde{\delta} - \theta h^*)^2}{2v\sigma^2}} v^{\tau \alpha - 1} e^{-\frac{v}{\tilde{\theta}}} dv \quad (3.24)$$
$$\tilde{\delta} = \delta + h^* \sigma^2 \quad \tilde{\theta} = \frac{\theta}{1 - \frac{1}{2} \theta \sigma^2 h^* - \delta h^*} \quad (3.25)$$
Following the same methodology, \( \hat{f}(\xi, \tau, h^*) \) is the probability density of the VG model with parameter \((\tau\mu, \tilde{\delta}', \sigma, \tau\alpha, \tilde{\theta}')\). we have
\[
\tilde{\delta}' = \tilde{\delta} + \sigma^2 \\
\tilde{\theta}' = \tilde{\theta} e^{r\mu / \tilde{\alpha}} \tag{3.26}
\]
In fact, as in 3.25, we have:
\[
\tilde{\delta}' = \tilde{\delta} + (h^* + 1)\sigma^2 = \tilde{\delta} + \sigma^2
\]
And
\[
\tilde{\theta}' = \frac{\theta}{1 - \frac{1}{2} \tilde{\theta} \sigma^2 (h^* + 1)^2 - \delta \theta (h^* + 1)} = \frac{\theta}{1 - \frac{1}{2} \tilde{\theta} \sigma^2 h^* - \delta \theta h^*} e^{r\mu / \tilde{\alpha}} = \tilde{\theta} e^{r\mu / \tilde{\alpha}}
\]
We have
\[
\hat{f}(\xi, \tau, h^*) = \frac{1}{\sigma \Gamma(\tau \alpha)} \int_0^{+\infty} \frac{1}{\sqrt{2\pi v}} e^{-\frac{(y - \mu v)^2}{2\sigma^2}} v^{\tau \alpha - 1} e^{-\frac{\tilde{\theta} y}{\alpha}} dv \tag{3.27}
\]
### 3.2.1 Equivalent Martingale Measure (EMM) Computation.

Under the Equivalent Martingale Measure (EMM), \( \hat{f}(\xi, \tau, h^*) \) is the probability density of VG model with parameter \((\tau\mu, \tilde{\delta}, \sigma, \tau\alpha, \tilde{\theta})\). The Fourier transform is:
\[
\mathcal{F}[\hat{f}] (y, \tau, h^*) = E \left[ e^{-i\mu y} \right] = \left( \frac{e^{-i\mu y}}{1 + \frac{1}{2} \tilde{\theta} \sigma^2 y^2 + i\tilde{\delta} \tilde{\theta} y} \right)^\tau = \phi(y)^\tau = e^{i\mu y + 1} \log(\phi(y)) = e^{-\tau \phi(y)} \tag{3.28}
\]
\( \phi(y) \) and \( \varphi(y) \) are respectively the Fourier Transform of the probability density and the characteristic exponent of VG model with parameter \((\mu, \tilde{\delta}, \sigma, \alpha, \tilde{\theta})\)

\[
\hat{f}(\xi, \tau, h^*) \text{ can be written as the inverse Fourier Transform from (3.28)}
\]
\[
\hat{f}(\xi, \tau, h^*) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi z} \mathcal{F}[\hat{f}](z, \tau, h^*) dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi z - \tau \varphi(z)} dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi z - \tau \varphi(z)} dz
\]
It was shown in [19] that we can have
\[
\mathcal{F}[\hat{F}](\xi, \tau, h^*) = \frac{\mathcal{F}[\hat{f}](\xi, \tau, h^*)}{i\xi} + \pi \mathcal{F}[\hat{f}](0) \delta(\xi) \tag{3.29}
\]
Based on (3.29), we deduce
\[
\hat{F}(\xi, \tau, h^*) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi z} \mathcal{F}[\hat{f}](z, \tau, h^*) dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi z - \tau \varphi(z)} \frac{dz}{iz} + \frac{1}{2}
\]
We have the probability density and cumulative functions
\[
\hat{f}(\xi, \tau, h^*) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi z - \tau \varphi(z)} dz \\
\hat{F}(\xi, \tau, h^*) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\xi z - \tau \varphi(z)}}{iz} dz + \frac{1}{2} \tag{3.30}
\]
For AVG Model in Table 1 [19, 20]: \( \hat{\mu} = 0.0848, \hat{\delta} = -0.0577, \hat{\sigma} = 1.0295, \hat{\alpha} = 0.8845, \hat{\theta} = 0.9378 \); and a risk-free rate of interest 6%, we have a Esscher transform parameter 
\( h^* = -0.5136 \) in (3.10). \( \hat{f}(\xi, \tau, h^*) \) and \( \hat{f}(\xi, \tau, h^* + 1) \) can be computed by the Fractional Fast Fourier

(FRFT) [19] as shown in Fig 8a. Using the numerical integration technique, \( \hat{f}(\xi, \tau, h^*) \) and \( \hat{f}(\xi, \tau, h^* + 1) \) can be also computed through the 12-point rule Composite Newton-Cotes Quadrature Formulas [33, 34].

\[
\hat{f}(\xi, \tau, h) \approx \frac{1}{n} \sum_{p=0}^{n-1} \sum_{j=0}^{Q} W_j g(x_{Qp+j}, \tau, h)
\]

(3.31)

\[
g(x, \tau, h) = \frac{1}{\sigma^\tau \Gamma(\tau \alpha) \sqrt{2\pi}} e^{-\frac{(x - \tau \mu - \hat{\delta})^2}{2\sigma^2}} v^{\tau \alpha - \frac{1}{2}} e^{-\hat{\theta}}
\]

(3.32)

In order to compute (3.31) for \( h = h^* \) and \( h = h^* + 1 \), the following parameter values are used:
\( a = 0, b = 20, Q = 12, n = 5000Q, n_0 = 5000 \) and the weights \( \{W_j\}_{0 \leq j \leq Q} \) values come from Table 1 [33] and table 4.1 in [34].

![Figure 7](image-url)

(a) FRFT Martingale Measure  
(b) Newton Cote Martingale Measure  
(c) FRFT versus Newton Cote Methods  
(d) FRFT versus Newton Cote Methods

**Figure 7.** Estimations: \( \hat{f}(\xi, \tau, h^*) \) versus \( \hat{f}(\xi, \tau, h^* + 1), \tau = 0.25 \)
3.3 Variance - Gamma Model : Generalized Black-Scholes Formula.

**Theorem 3.3**

Let \( r \), a continuously compounded risk-free rate of interest; \( Y = Y_t \), a Variance-Gamma Process with parameter \((\mu, \delta, \sigma, \alpha, \theta)\); and \((S(0)e^{Y_t} - K)^+\), the terminal payoff for a contingent claim with the expiry date \( T \). Then at time \( t < T \), the arbitrage price of the European call option with the strike price \( K \) can be written as follows.

\[
F_{\text{call}}(S_t, t) = \frac{K}{2\pi} \int_{-\infty+iq}^{\infty+iq} e^{\left(i\xi \log\left(\frac{S(0)}{K}\right) - \tau(r + \phi(\xi))\right)} \frac{d\xi}{i\xi(i\xi - 1)} \tag{3.33}
\]

Where \( \phi(z) \) is the characteristic exponent of VG model, \( \tau = T - t \) and \( q < -1 \).

**Proof:**

\[
(S(0)e^{Y_T} - K)^+ = S(t)(e^{Y_T} - k)^+ \quad Y_T = Y_t + Y_t
\]

\[
= S(t)g(Y_t, k) \quad S(t) = S(0)e^{Y_t} 	ext{ and } k = \frac{K}{S(t)} \tag{3.34}
\]
(S(0)e^{Y_r} - K)^+ is the payoff of the call option. The Fourier transform can be computed as follows
\[
\hat{g}(y, k) = \mathcal{F}[g](y, k) = \int_0^{\infty} e^{-i\xi x} g(x, k) dx = \int_0^{\infty} e^{-i\xi x} (e^x - k)^+ dx
\]
\[
= \int_{\log(k)}^{\infty} e^{-i\xi x} (e^x - k) dx
\]
\[
= \int_{\log(k)}^{\infty} e^{(1-i\xi)x} dx - k \int_{\log(k)}^{\infty} e^{-i\xi x} dx
\]
\[
= \frac{1}{1 - iy} \left[ e^{(1-i\xi)x} \right]_{\log(k)}^{+\infty} + k \left[ e^{-i\xi x} \right]_{\log(k)}^{+\infty}
\]
\[
= \frac{ke^{-iy\log(k)}}{iy(iy - 1)} \quad \text{for } \Re(y) < -1
\]

We have the Fourier transform of the call payoff
\[
\hat{g}(y, k) = \mathcal{F}[g](y, k) = \frac{ke^{-iy\log(k)}}{iy(iy - 1)} \quad \text{for } \Re(y) < -1 \quad (3.36)
\]

It is shown in (3.30) and in (3.28) that \( \hat{f}(\xi, \tau, h^*) \) and \( \varphi(y) \) can be written as follows
\[
\hat{f}(\xi, \tau, h^*) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi z - \tau \varphi(z)} dz \quad \varphi(y) = -i\mu y + \alpha \log(1 + \frac{1}{2} \theta \sigma^2 y^2 - i\delta \theta y) \quad (3.37)
\]
with \((\delta, \theta)\) defines in (3.15)
\[
\delta = \delta + h^* \sigma^2 \quad \theta = \frac{\theta}{1 - \frac{1}{2} \theta \sigma^2 h^* - \delta \theta h^*}
\]

\( F_{\text{call}}(S_t, t) \) is the call function under the Equivalent Martingale Measure (EMM), and we can now find a good expression of the function
\[
F_{\text{call}}(S_t, t) = e^{-rT} E^{h^*} [(S(0)e^{X_r} - K)^+] = S(t)e^{-rT} E^{h^*} [g(X_t)] \quad \text{recall (3.35)}
\]
\[
= S(t) e^{-rT} \int_{-\infty}^{\infty} \hat{f}(y, \tau, h^*) g(y, k) dy
\]
\[
= S(t) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\xi z - \tau \varphi(z)} g(y, k) dy dz \quad \text{recall (3.36)}
\]
\[
= S(t) \int_{-\infty}^{+\infty} e^{-\tau (r + \varphi(z))} \hat{g}(z, k) dz
\]
\[
= \frac{K}{2\pi} \int_{-\infty+iq}^{+\infty+iq} \exp \left[ -iz \log(k) - \tau (r + \varphi(z)) \right] dz \quad \Re(z) \leq q < -1 \quad \text{and } k = \frac{K}{S(t)}
\]
\[
= \frac{K}{2\pi} \int_{-\infty+iq}^{+\infty+iq} \frac{iz \log(S(t))/k - \tau (r + \varphi(z))}{i(z + 1)} dz
\]
We have the formula (3.33)
\[
F_{\text{call}}(S_t, t) = \frac{K}{2\pi} \int_{-\infty+iq}^{+\infty+iq} \frac{iz \log(S(t))/k - \tau (r + \varphi(z))}{i(z + 1)} dz \quad (3.38)
\]
3.4 European Option Pricing by Fractional Fast Fourier (FRFT).

3.4.1 Parameter q Evaluation.
Let us consider the stock or index price $S = S_0 e^Y$ and the strike price $K$, it was shown in (3.36) that the Fourier transform of the call payoff can be written as follows.

$$\hat{g}(y, k) = \mathcal{F}[g](y, k) = \frac{ke^{-iy\log(k)}}{iy(iy - 1)} \quad \text{for } \Im(y) < -1$$

We can recover the call payoff from the inverse of Fourier in (3.36)

$$\check{g}(x, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyx} \mathcal{F}[g](y, k) dy \quad \text{for } q < -1 (3.39)$$

The payoff in (3.39) depends on the parameter $(q)$. As shown in Fig 9a for $q = -2$, the inverse of Fourier in (3.39) produces poor results; in fact, the curve in red is fluctuating around real call payoff $(e^Y - k)^+$. For $q = -1.002$, the inverse of Fourier over-estimates the call payoff.

In order to find $q$ value with a high level of accurate, we define the error function $(ER(k, q))$ between the real call payoff and the inverse Fourier payoff, with $k$ (strike price) and the parameter $q$ as inputs.

$$ER(k, q) = \sqrt{\frac{1}{m} \sum_{j=1}^{m} [(e^{x_j} - k)^+ - \check{g}(x_j, k)]^2} \quad \text{with } -M \leq x_j \leq M (3.40)$$

At the money (ATM) option, the strike price $k = 1$, and the $ER(k, q)$ can be analysed as a function of one variable $q$. Fig 9b displays the error function $(ER)$ graph as a function of $q$. ER is a convex function, decreases and increases over the interval $]-\infty, -1[$. ER reaches a minimum at $q = -1.0086$.

![Figure 9](image_url)

(a) $(e^x - k)^+$ versus $\check{g}(x, k)$ (ATM)  
(b) $ER(k, q)$ and $q$ (ATM)

**Figure 9.** Optimal $q$ parameter

Fig 10a displays the $ER(k, q)$ minimum value as a function of the strike price $k$; and the correspondent optimal parameter $q$ as a function of the strike price $k$ (Fig 10b). Both graphs display almost a constant function with respect to the strike price.
3.4.2 Calculating the Fourier Integral by FRFT.

The method of Fourier transform [13] provides a valuable and powerful tools for option pricing under a class of Lévy processes when the characteristic function is much simpler than their density function. We compute the value of the call option on the SPY ETF with the Fractional Fast Fourier (FRFT).

For \( x = \log \left( \frac{S(t)}{K} \right) \), \( F_{\text{call}}(S(t)T) \) is the price per one dollar of the strike price. We have

\[
F_{\text{call}}^G(S(t), \tau) = \frac{1}{2\pi} \int_{-\infty+iq}^{\infty+iq} \frac{e^{iz\log \left( \frac{S(t)}{K} \right)} - \tau(r + \varphi(z))}{iz(iz - 1)} dz
\]

we assume

\[
f(\xi) = \frac{\exp \left[ -\tau(r + \varphi(z)) \right]}{i\xi(i\xi - 1)} \quad F(x) = \frac{1}{2\pi} \int_{-\infty+iq}^{\infty+iq} e^{ix\xi} f(\xi) d\xi
\]

Following the notation of the Fractional Fast Fourier (FRFT) developed in [19] (Section 2 and Appendix A.1.)

\[
F(x_k) = \frac{1}{2\pi} \int_{-\infty+iq}^{\infty+iq} e^{ix_k \xi} f(\xi) d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\xi + iq)\xi_k} f(\xi + iq) d\xi = e^{-qx_k} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi_k \xi} f(\xi + iq) d\xi
\]

\[
\approx e^{-qx_k} \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\xi_k \xi} f(\xi + iq) d\xi
\]

\[
\approx \gamma e^{-(q + \varphi(\xi)\xi_k)} G_k(f(\xi + iq)e^{\pi jn\delta}, -\delta)
\]

\[
F(x_k) \approx \gamma e^{-(q + \varphi(\xi)\xi_k)} G_k(f(\xi + iq)e^{\pi jn\delta}, -\delta)
\]  \hspace{1cm} (3.42)
3.5 Empirical Analysis.

The empirical Analysis aims at examining the effect of the FRFT evaluation method and the effect of the underlying Gamma–Variance (VG) model compared to Black–Scholes model on the option price. The option prices are computed and the pricing errors are analysed across maturity and strike price; the FRFT VG model prices is compared to the closed-form Black-Scholes prices. The closed-form Black–Scholes prices are computed under the risk-neutral measure using the formula \[ F_{\text{BS}}^{\text{call}}(S_t, \tau) = S_t N(d_1) - Ke^{-r\tau}N(d_2) \] (3.43)

Where
\[ d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \quad d_2 = d_1 - \sigma\sqrt{\tau} \quad N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt \] (3.44)

The variance \( \sigma^2 \) of the Black–Scholes prices is defined in theorem 2.7. The five–parameter VG model estimations from Table 1 \[ [19, 20] \] will be used as data; combined with a risk-free interest rate of 6%, the Esscher transform parameter \( h^* = -0.5136 \) was computed in (3.10). Moneyness describes the intrinsic value of an option in its current state; it is an indicator as to whether the option would make money if it was exercised immediately. Option moneyness can be classified into three categories: At The Money (ATM) option \( (k = \frac{S_t}{K} = 1) \), Out of The Money (OTM) option \( (k = \frac{S_t}{K} < 1) \), and In The Money (ITM) option \( (k = \frac{S_t}{K} > 1) \).

In order to compare the numerical VG option price with the closed form BS option price, the error was computed as follows.

\[ Error(k, \tau) = \frac{F_{\text{call}}^{\text{GV}}(S_t, \tau)}{K} - \frac{F_{\text{call}}^{\text{BS}}(S_t, \tau)}{K} \] (3.45)

The error in (3.45) is a function of the time to maturity \( \tau \) and the option moneyness \( k \). Holding constant the time to maturity \( \tau \), the VG option prices are slightly higher than the BS option prices when option is deep Out of The Money (OUT). As shown in Fig 11, for \( \tau = 0.25 \) and \( \tau = 0.75 \), the errors are positive when option is deep Out of The Money (OTM). However, the pattern changes when option is deep In The Money (ITM). In fact, when time to maturity \( \tau = 0.75 \) becomes long term, the BS option prices become overpriced.
On the other side, the VG option is underpriced when the time to maturity ($\tau$) is short term or long term. When we have medium term time to maturity, VG option is overpriced. As illustrated in Fig 12 for ATM and ITM options, the errors are negative in the short and long terms, and positive in the medium term.

![Figures](a) Error($k=1, \tau$)  (b) Error($k=1.5, \tau$)  (c) ATM option

**FIGURE 12.** Effect of time to Maturity ($\tau$)

Fig 13 graphs the error of European Call option price as a function of time to maturity and moneyness option. Independently to maturity, the VG option is overpriced in the deep Out of the money (OTM) option. As shown the red color in Fig 13b.

![Figures](a) Error($k, \tau$)  (b) Error($k, \tau$) (top view)

**FIGURE 13.** Combined Effects of time to Maturity ($\tau$) and Moneyness Option ($k = \frac{S_t}{K}$)

As shown in Fig 13b when the option is near the money, the VG option is overpriced in the medium term time to maturity. In the case of deep In the money (ITM) option, the BS option is overprice in the long term time to mature and underpriced in the short term time to maturity.

On August 04, 2021, the SPY ETF market price closes at $438.98. we obtain the VG call option price on SPY ETF through equation (3.45) and the BS price through the closed form (3.45). Table 3 is organised by period and option moneyness or strike price given a constant spot price.
### Table 3. The price of a call on SPY ETF

| Period | Strike Price | Moneyless | BSM Price | VGM price | Period | Strike Price | Moneyless | BSM Price | VGM price |
|--------|--------------|-----------|-----------|-----------|--------|--------------|-----------|-----------|-----------|
| 0.25   | 365.817      | 20%       | 180.740   | 182.437   | 0.5    | 365.817      | 20%       | 232.442   | 236.419   |
| 0.25   | 373.600      | 18%       | 177.625   | 179.116   | 0.5    | 373.600      | 18%       | 230.066   | 234.225   |
| 0.25   | 381.722      | 15%       | 171.224   | 172.318   | 0.5    | 381.722      | 15%       | 222.066   | 226.700   |
| 0.25   | 390.204      | 13%       | 167.921   | 168.834   | 0.5    | 390.204      | 13%       | 222.145   | 227.352   |
| 0.25   | 399.073      | 10%       | 164.558   | 165.308   | 0.5    | 399.073      | 10%       | 219.839   | 224.955   |
| 0.25   | 408.353      | 8%        | 161.141   | 161.752   | 0.5    | 408.353      | 8%        | 214.375   | 220.011   |
| 0.25   | 418.076      | 5%        | 157.651   | 158.154   | 0.5    | 418.076      | 5%        | 211.510   | 217.425   |
| 0.25   | 428.273      | 2%        | 154.070   | 154.503   | 0.5    | 428.273      | 2%        | 208.585   | 214.793   |
| 0.25   | 438.980      | 0%        | 150.431   | 150.839   | 0.5    | 438.980      | 0%        | 205.606   | 212.121   |
| 0.25   | 441.353      | -3%       | 146.743   | 147.179   | 0.5    | 441.353      | -3%       | 202.527   | 209.367   |
| 0.25   | 452.756      | -5%       | 142.965   | 143.490   | 0.5    | 452.756      | -5%       | 199.358   | 206.542   |
| 0.25   | 464.157      | -8%       | 139.108   | 139.792   | 0.5    | 464.157      | -8%       | 196.108   | 203.657   |
| 0.25   | 475.558      | -10%      | 135.184   | 136.101   | 0.5    | 475.558      | -10%      | 192.761   | 199.358   |
| 0.25   | 487.059      | -13%      | 131.179   | 132.414   | 0.5    | 487.059      | -13%      | 189.303   | 196.108   |
| 0.25   | 498.560      | -15%      | 127.087   | 128.728   | 0.5    | 498.560      | -15%      | 185.771   | 192.761   |
| 0.25   | 510.061      | -18%      | 123.087   | 125.081   | 0.5    | 510.061      | -18%      | 182.240   | 189.303   |
| 0.25   | 521.562      | -20%      | 119.087   | 121.081   | 0.5    | 521.562      | -20%      | 178.771   | 185.771   |

### 4. Conclusion

In the study, a $\Gamma(\alpha, \theta)$ Ornstein-Uhlenbeck type process was used to build a continuous sample path of a five-parameter Variance Gamma (VG) process ($\mu, \delta, \sigma, \alpha, \theta$). The data parameters from [19,20] was used to simulate the gamma process ($\sigma^2(t)$) and the continuous sample path of the subordinator process ($\sigma^2*(t)$). Both simulations were used as inputs in order to simulate the continuous sample path of the VG process, which is, in fact, a sample path of a SPY EFT daily return. The Lévy density of the VG process was derived and shown to belong to a KoPoL family of order $\nu = 0$, intensity $\alpha$ and steepness parameters $\delta \sigma^2 - \frac{\theta^2}{\sigma^2} + \frac{\alpha \delta^2}{\sigma^4}$ and $\delta \sigma^2 + \frac{\theta^2}{\sigma^4} + \frac{\alpha \delta^2}{\sigma^4}$. In the same token, it was shown that the VG process converges in distribution to a Lévy process driven by a Normal distribution with mean $(\mu + \alpha \theta \delta)$ and variance $\alpha (\theta^2 \delta^2 + \sigma^4 \theta)$. Regarding the Equivalent Martingale Measure (EMM), the Esscher transform of the five-parameter VG process ($\mu, \delta, \sigma, \alpha, \theta$) is shown to be another VG process with parameter ($\mu, \delta, \sigma, \alpha, \theta$). A closed-form solution for the European call option was provided. The EMM probability densities were computed with the Fractional Fast Fourier (FRFT) and 12-point Newton Cote methods. The characteristic exponent of the VG Esscher transform was used to provide a numerical solution of the European option price and the results were compare with the black-scholes (BS) closed formula. It was observed that the VG option is overpriced in the deep Out of the money (OTM) option. The VG option is also overpriced when the option is near the money and time to maturity is medium term. In the case of deep In the money (ITM) option, the BS option is overprice in the long term time to mature and underpriced in the short term time to maturity.
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