The optimal Leray-Trudinger inequality

GIUSEPPINA DI BLASIO*

giuseppina.diblasio@unicampania.it

GIOVANNI PISANTE*

giovanni.pisante@unicampania.it

GEORGIOS PSARADAKIS*

georgios.psaradakis@unicampania.it

*Dipartimento di Matematica e Fisica
Università degli Studi della Campania “L. Vanvitelli”
Viale Lincoln 5, 81100 Caserta, Italy

August 11, 2022

Abstract

We fill the gap left open in [MT], regarding the minimum exponent on the logarithmic correction weight so that the Leray-Trudinger inequality (see [PsSp]) holds. Instead of the representation formula used in [PsSp] and [MT], our proof uses expansion in spherical harmonics as in [VZ].
1 Introduction

Let $\Omega$ be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1, 2\}$. The classical Sobolev inequality asserts that

$$\sup_{u \in D_n(\Omega)} \int_{\Omega} |u|^{2n/(n-2)} \, dx < \infty,$$

where we have set

$$D_n(\Omega) := \left\{ u \in C^1_c(\Omega) \mid \int_{\Omega} |\nabla u|^2 \, dx \leq 1 \right\}.$$

It is well known that the exponent $2n/(n - 2)$ cannot be increased. At least when $\Omega$ has finite Lebesgue measure, this may suggest that functions in $D^2(\Omega)$ could be bounded. Standard examples show that this is not the case and Neil Trudinger in [Tr] has established the optimal embedding in this case. More precisely, Trudinger’s inequality (see [Pe], [Po] and [Y] for prior results and [M] for the best constant) says that there exists a positive constant $c$ such that

$$\sup_{u \in D_2(\Omega)} \int_{\Omega} e^{\alpha u^2} \, dx < \infty \quad \forall \, \alpha < c,$$

and the exponent in the power $u^2$ cannot be increased.\footnote{In [T] and throughout this paper $\int_{\Omega} f \, dx$ stands for $(\mathcal{L}^n(\Omega))^{-1} \int_{\Omega} f \, dx$.}

Consider now the following higher dimensional Hardy-type inequality

$$\int_{\Omega} |\nabla u|^2 \, dx - \left(\frac{n-2}{n}\right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} \, dx \geq 0 \quad \forall \, u \in C^1_c(\Omega).$$

Leray seems to be the first to have provided a proof of (2) for $n = 3$ (see [Le, pg 47]). Note that in case $0 \in \Omega$, the constant $\left(\frac{n-2}{n}\right)^2$ turns out to be the best possible. Nevertheless, already in his paper, Leray proved a substitute inequality for the case $n = 2$ (see [Le, pg 49]). Of interest in this paper is the following version of Leray’s inequality: if $\Omega \subset \mathbb{R}^2$ is a bounded domain that contains the origin, then we have

$$I_2[u] := \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{4} \int_{\Omega} \frac{|u|^2}{|x|^2} X_1^2 \left(\frac{|x|}{R_\Omega}\right) \, dx \geq 0 \quad \forall \, u \in C^1_c(\Omega),$$

where $R_\Omega := \sup_{x \in \Omega} |x|$ and

$$X_1(t) := (1 - \log t)^{-1}, \quad t \in (0, 1], \quad X_1(0) := 0.$$

Moreover, the constant $1/4$ is the best possible and the power 2 on $X_1$ cannot be decreased (see for example [BFT1, Theorems 4.3 & 5.4] with $k = N$ there).
Now let $u \in C^1_c(\Omega) \setminus \{0\}$ where $\Omega \subset \mathbb{R}^2$ is as above and suppose further that $I_2[u] \leq 1$. Setting

$$v = uX_1^{1/2},$$

and using integration by parts, one easily obtains (see \cite[Proposition 2.6]{PsSp})

$$\int_\Omega |\nabla v|^2 \frac{X_1}{X^{1-1}(|x|/R_{\Omega})} \, dx = I_2[u]. \tag{4}$$

Since $X_1(t) \leq 1$ for all $t \in [0, 1]$, equality (4) readily implies $\int_\Omega |\nabla v|^2 \, dx \leq I_2[u] \leq 1$. Hence

$$\int_\Omega e^{\alpha u} X_1(|x|/R_{\Omega}) \, dx \leq \sup_{v \in D_2(\Omega)} \int_\Omega e^{\alpha u} \, dx < \infty,$$

because of Trudinger’s inequality \cite{Trudinger}. Consequently, with $c$ as in \cite{Trudinger} and

$$I_2(\Omega) := \{ u \in C^1_c(\Omega) \mid I_2[u] \leq 1 \},$$

we have the following combination of (3) and (1)

$$\sup_{u \in I_2(\Omega)} \int_\Omega e^{\alpha u} X_1(|x|/R_{\Omega}) \, dx < \infty \quad \forall \alpha < c. \tag{5}$$

In this paper we establish the optimal version of (5). More precisely, it has been shown in \cite{PsSp} that estimate (5) is far from being optimal. In fact, \cite[Theorem 1.1]{PsSp} says that given $\varepsilon > 0$, there exists a positive constant $c = c(\varepsilon)$ such that

$$\sup_{u \in I_2(\Omega)} \int_\Omega e^{\alpha u} X_1(|x|/R_{\Omega}) \, dx < \infty \quad \forall \alpha < c,$$

and that such an estimate fails to hold for all $\alpha > 0$ when $\varepsilon = 0$. Inspired by a result of Calanchi and Ruf \cite{CR}, further understanding on the problem was provided by Mallick and Tintarev in \cite{MT}. They showed that for any $\gamma \geq 2$ there exists a positive constant $c$, not depending on $\gamma$, such that

$$\sup_{u \in I_2(\Omega)} \int_\Omega e^{\alpha u} X_\gamma^2(|x|/R_{\Omega}) \, dx < \infty \quad \forall \alpha < c, \tag{6}$$

where

$$X_\gamma(\cdot) := X_1(X_1(\cdot)).$$

Moreover, such an estimate fails to hold for all $\alpha > 0$ when $\gamma < 1$. As observed in \cite{MT}, by \cite[Lemma 5]{CR}, inequality (6) is true for $\gamma = 1$ when restricted to $I_2^{\text{rad}}$, i.e. radially symmetric functions of $I_2(\mathbb{R}^2)$. In §3.1, we extend this result to the multi-dimensional case. Furthermore, in §3.2 we take away the radial restriction, proving thus the following optimal Leray-Trudinger inequality:

$^2\mathbb{B}^n$ stands for the unit ball of $\mathbb{R}^n$ having center at the origin.
Theorem 1.1. Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, be a bounded domain that contains the origin and set $R_{\Omega} := \sup_{x \in \Omega} |x|$. There exists a positive constant $c = c(n)$ such that

$$\sup_{u \in I_n(\Omega)} \int_{\Omega} e^{\alpha \left[|u|X_2^{1/n}(|x|/R_{\Omega})\right]^{n/(n-1)}dx} < \infty \quad \forall \alpha < c,$$

where $I_n(\Omega) := \{ u \in C^1_c(\Omega) \mid I_n[u] \leq 1 \}$ with

$$I_n[u] := \int_{\Omega} \nabla u|/n dx - \left(\frac{n-1}{n}\right)^n \int_{\Omega} \frac{|u|^n}{|x|^n} X_1^n\left(\frac{|x|}{R_{\Omega}}\right) dx.$$  \hspace{1cm} (8)

Moreover, the exponent $1/n$ on $X_1$ cannot be increased.

Note that $I_n[u]$ is always nonnegative (an elementary proof based on integration by parts and Hölder’s inequality can be found in [BFT1, Theorem 4.2] or [PsSp, Theorem 2.1]). For the optimality of the exponent on $X_1$ and of the constant which appear in (8), we refer to [BFT1, Theorem 5-(i)].

It is well known that the proof of the $n$-dimensional Trudinger’s inequality:

$$\sup_{u \in D_n(\Omega)} \int_{\Omega} e^{\alpha |u|^{n/(n-1)}} dx < \infty \quad \forall \alpha < c,$$

for some positive $c = c(n)$, is based on finding the sharp growth of the $L^q$-norm of $W^{1,n}_0$ functions, as $q \to \infty$. The key estimate to prove (9) is indeed

$$\left(\int_{\Omega} |u|^q dx\right)^{1/q} \leq c(n) q^{1-1/n} \|\nabla u\|_{L^n(\Omega)} \quad \forall q > n,$$

whenever $u \in C^1_c(\Omega)$. Following a similar path, to establish Theorem 1.1, we analogously prove

$$\left(\int_{\Omega} \left(\frac{|u|X_2^{1/n}(|x|/R_{\Omega})}{|x|^n} \right)^q dx\right)^{1/q} \leq c(n) q^{1-1/n} \left(I_n[u]\right)^{1/n} \quad \forall q > n,$$

whenever $u \in C^1_c(\Omega)$.

For partial results on the corresponding problem dealing with the Hardy inequality that involves the distance to the boundary of convex or mean convex domains, we refer to [WY], [FP] and [dBPP].

2 Preliminary estimates

2.1 Lower estimates on $I_n[u; \Omega]$

Notation. From now on we write $X_1, X_2$ instead of $X_1(|x|/R_{\Omega}), X_2(|x|/R_{\Omega})$.

We recall a known lower estimate for the Hardy-Leray difference
Proposition 2.1 ([PsSp] Proposition 2.6). Set $\lambda_n := 2^{n-1} - 1$. Whenever $u \in C^1_c(\Omega)$ we have
\[
\int_{\Omega} |\nabla v|^n X_1^{-n+1} \, dx \leq \lambda_n I_n[u], \tag{12}
\]
where $v := X_1^{1-1/n} u$. For $n = 2$ we have equality in (12).

In order to prove theorem 1.1, we are now going to establish one more lower estimate on $I_n[u; \Omega]$. This estimate (see (13) below) will be enough to prove our main theorem for radial functions. However, in the next section, it will be combined with Proposition 2.1 to remove the radiality assumption; see [GkPs]. Observe that for $n = 2$, estimate (13) agrees with (12).

Proposition 2.2. Set $\kappa_n := \lambda_n \left( \frac{2n}{n-1} \right)^{n-2}$. Whenever $u \in C^1_c(\Omega)$ we have
\[
\int_{\Omega} |x|^{2-n} |v|^{n-2} |\nabla v|^2 X_1^{-1} \, dx \leq \kappa_n I_n[u], \tag{13}
\]
where $v := X_1^{1-1/n} u$. For $n = 2$ we have equality in (13).

Proof. It suffices to consider $u \in C^1_c(\Omega \setminus \{0\})$. Setting $u = X_1^{-1+1/n} v$ we compute
\[
|\nabla u|^n = \left| X_1^{-1+1/n} \nabla v - \frac{n-1}{n} X_1^{1/n} \frac{v}{|x|} \right|^n.
\]
Applying the vectorial inequality (see also [Li])
\[
|b - a|^n - |a|^n \geq \frac{1}{\lambda_n^{2n-2}} |a|^{n-2} |b|^2 - n |a|^{n-2} a \cdot b, \tag{14}
\]
we get
\[
|\nabla u|^n - \left( \frac{n-1}{n} \right)^n \frac{|v|^n}{|x|^n} X_1 \geq \frac{1}{\kappa_n} |x|^{2-n} |v|^{n-2} |\nabla v|^2 X_1^{-1}
- \left( \frac{n-1}{n} \right)^{n-1} |x|^{-n} \nabla \left( |v|^n \right) \cdot x.
\]
This means
\[
I_n[u; \Omega] \geq \frac{1}{\kappa_n} \int_{\Omega} |x|^{2-n} |v|^{n-2} |\nabla v|^2 X_1^{-1} \, dx
+ \left( \frac{n-1}{n} \right)^{n-1} \int_{\Omega} |v|^n \, \text{div} \left\{ |x|^{-n} x \right\} \, dx.
\]
Since $\text{div} \left\{ |x|^{-n} x \right\} = 0$ in $\Omega \setminus \{0\}$, we deduce (13). Note that the proof of (12) in [PsSp] follows the same argument but uses the vectorial inequality
\[
|b - a|^n - |a|^n \geq \frac{1}{\lambda_n} |b|^n - n |a|^{n-2} a \cdot b, \tag{15}
\]
instead (see also [Li]).
2.2 An identity for the improved $L^2$-Hardy difference

One more ingredient we will use is the following equality, originally due to Filippas and Tertikas; see [FT] equality (6.7). We include the proof for the convenience of the reader.

**Proposition 2.3.** Let $n \in \mathbb{N}$ and let $U \subset \mathbb{R}^n$ be a bounded domain containing the origin. For any $f \in C^1_c(U \setminus \{0\})$ we have

$$\int_U |\nabla f|^2 \, dx - \left(\frac{n-2}{2}\right)^2 \int_U \frac{|f|^2}{|x|^2} \, dx - \frac{1}{4} \int_U \frac{|f|^2}{|x|^2} X_1^2 \, dx$$

$$= \int_U |x|^{2-n} |\nabla g|^2 X_1^{-1} \, dx,$$

(16)

where $g$ is defined through $f = |x|^{1-n/2} X_1^{-1/2} g$.

**Proof.** We compute first

$$\nabla f = -\frac{n-2}{2} |x|^{-n/2} X_1^{-1/2} g \frac{x}{|x|} - \frac{1}{2} |x|^{-n/2} X_1^{1/2} g \frac{x}{|x|} + |x|^{1-n/2} X_1^{-1/2} \nabla g,$$

in $U \setminus \{0\}$. Expanding now the square, we obtain

$$|\nabla f|^2 = \left(\frac{n-2}{2}\right)^2 \frac{|f|^2}{|x|^2} + \frac{1}{4} \frac{|f|^2}{|x|^2} X_1^2 + |x|^{2-n} X_1^{-1} |\nabla g|^2$$

$$+ \frac{n-2}{2} \frac{g^2}{|x|^n} - \frac{n-2}{2} |x|^{1-n} X_1 X_1^{-1} \nabla g \cdot \frac{x}{|x|} - \frac{1}{2} |x|^{1-n} \nabla g \cdot \frac{x}{|x|},$$

in $U \setminus \{0\}$. But this readily says that the difference between the left hand side and right hand side of (16) is given by

$$\frac{n-2}{2} \int_U \frac{g^2}{|x|^n} \, dx - \frac{1}{2} \int_U \left(n-2 + X_1 \right) |x|^{1-n} X_1^{-1} \nabla g \cdot \frac{x}{|x|} \, dx$$

$$= \frac{n-2}{2} \int_U \frac{g^2}{|x|^n} \, dx + \frac{1}{2} \int_U \text{div} \left\{ \left(n-2 + X_1 \right) |x|^{1-n} X_1^{-1} \frac{x}{|x|} \right\} g^2 \, dx$$

$$= 0,$$

because $\text{div} \left\{ \left(n-2 + X_1 \right) |x|^{1-n} X_1^{-1} \frac{x}{|x|} \right\} = -(n-2) |x|^{-n}$ in $U \setminus \{0\}$. 

\[\square\]

3 Proof of the main result

In this section we give the detailed proof of Theorem 1.1. Our proof is reminiscent of the arguments from [GKP], that were used to prove the optimal $L^p$-Hardy-Sobolev inequality for $2 < p < n$. 

6
3.1 The case of radial functions

We start with an elementary lemma which substitutes an end point case of [Ps Lemma 6.1] (see also [CR Lemma 5]).

**Lemma 3.1.** For any \( g \in AC([0, 1]) \) with \( g(1) = 0 \) there holds

\[
\sup_{r \in [0, 1]} \left\{ |g(r)| X_2^{1/2}(r) \right\} \leq \left( \int_0^1 t |g'(t)|^2 X_1^{-1}(t) \, dt \right)^{1/2}.
\]  

(17)

**Proof.** Let \( 0 \leq r < 1 \). Since \( f(1) = 0 \) we have

\[
|g(r)| \leq \int_r^1 |g'(t)| \, dt = \int_r^1 \left\{ t^{-1/2} X_1^{1/2}(t) \right\} \left\{ t^{1/2} |g'(t)| X_1^{-1/2}(t) \right\} \, dt \leq \left( \int_r^1 t^{-1} X_1(t) \, dt \right)^{1/2} \left( \int_r^1 t |g'(t)|^2 X_1^{-1}(t) \, dt \right)^{1/2}.
\]

Since \( (\log X_1(t))' = t^{-1} X_1(t) \), the left integral is easily computed. We deduce

\[
|g(r)| \leq \left( - \log X_1(r) \right)^{1/2} \left( \int_0^1 t |g'(t)|^2 X_1^{-1}(t) \, dt \right)^{1/2}.
\]  

(18)

The definition of \( X_2 \) implies

\[
- \log X_1(r) = \frac{1 - X_2(r)}{X_2(r)} \leq \frac{1}{X_2(r)},
\]

which when inserted in (18) readily gives (17). \( \square \)

The proof of theorem 1.1 for radial functions is based on the following key proposition.

**Proposition 3.2.** Let \( q \geq 1 \). For any radial function \( f \in C^1_c(\mathbb{R}^n) \) we have

\[
\left( \int_{\mathbb{R}^n} \left( |f|^{2/n} X_1^{1-1/n} X_2^{1/n} \right)^q \, dx \right)^{1/q} \leq \frac{e^{n/q}}{n} \Gamma^{1/q} \left( 1 + q \frac{n-1}{n} \right) \left( \int_{\mathbb{R}^n} |x|^{2-n} \left| \nabla f \right|^2 X_1^{-1} \, dx \right)^{1/n},
\]

(19)

where \( \Gamma \) is the gamma function.
Proof. Write $\tilde{f}$ for the radial profile of $f$. From Lemma 3.1 with $g = \tilde{f}$ we obtain
\[
|\tilde{f}(r)|^{2/n} X_2^{1/n}(r) \leq \left( \int_0^1 t |\tilde{f}'(t)|^2 X_1^{-1}(t) \, dt \right)^{1/n}.
\]
This is the same as
\[
|f(x)|^{2/n} X_2^{1/n}(|x|) \leq \left( \frac{1}{n \omega_n} \int_{\mathbb{B}^n} |x|^{2-n} |\nabla f|^2 X_1^{-1} \, dx \right)^{1/n}.
\]
Multiplying both sides with $X^{-1+1/n}(|x|)$ and taking $L^q(\mathbb{B}^n)$-norms we arrive at
\[
\left( \int_{\mathbb{B}^n} \left( |f|^{2/n} X_1^{-q(1-1/n)} X_2^{1/n} \right)^q \, dx \right)^{1/q} \leq n^{-1/n} \left( \int_{\mathbb{B}^n} X_1^{-q(1-1/n)} \, dy \right)^{1/q} \left( \int_{\mathbb{B}^n} |x|^{2-n} |\nabla f|^2 X_1^{-1} \, dx \right)^{1/n}.
\]
(20)

Next we use elementary calculus to estimate the first integral on the right of (20). Clearly,
\[
\int_{\mathbb{B}^n} X_1^{-q(1-1/n)} \left( |y| \right) \, dy = n \int_0^1 t^{n-1} X_1^{-q(1-1/n)}(t) \, dt \quad = \frac{e^n}{n^{q(1-1/n)}} \int_{\mathbb{B}^n} e^{-r q(1-1/n)} \, dr ,
\]
where we have performed the change of variables $t \mapsto e^{1-t/n}$ to reach the last expression. From this we readily get
\[
\int_{\mathbb{B}^n} X_1^{-q(1-1/n)} \left( |y| \right) \, dy \leq \frac{e^n}{n^{q(1-1/n)}} \Gamma \left( 1 + q \frac{n-1}{n} \right).
\]
(21)
The proof of (19) follows by inserting (21) in (20).

Remark 3.3. We note at this point that applying Stirling’s formula and elementary estimates, we deduce from (19) that:
\[
\left( \int_{\mathbb{B}^n} \left( |f|^{2/n} X_1^{-1-q(1/n)} X_2^{1/n} \right)^q \, dx \right)^{1/q} \leq c(n) \frac{q^{1-1/n}}{n^{q(1/n)}} \left( \int_{\mathbb{B}^n} |x|^{2-n} |\nabla f|^2 X_1^{-1} \, dx \right)^{1/n},
\]
(22)
for any radial function $f \in C^1_c(\mathbb{B}^n)$. This is a form of (19) in the spirit of (10). In fact, it is (22) we are utilizing in the next section to treat the spherical mean of a general function $f \in C^1_c(\mathbb{B}^n)$ (not necessarily radial).
Proof of theorem 1.1 for radial functions. Let $u \in \mathcal{I}_n(\mathbb{R}^n) \setminus \{0\}$ be radially symmetric and define $v$ through $u = vX_1^{-1+1/n}$. The function

$$w := |v|^{n/2},$$

is then radially symmetric and also $w \in C^1_c(\mathbb{B}^n)$. Therefore, by taking $f = w$ in proposition 3.2 we obtain

$$\left( \int_{\mathbb{B}^n} \left( |v|X_1^{-1-1/n} X_2^{1/n} \right)^q \, dx \right)^{1/q} \leq \left( \frac{n}{2} \right)^{2/n} \frac{e^{n/q}}{n} \Gamma^{1/q} \left( 1 + q \frac{n - 1}{n} \right) \left( \int_{\mathbb{B}^n} |x|^{2-n} |v|^{n-2} |\nabla v|^{2} X_1^{-1} \, dx \right)^{1/n}.$$

Taking into account (12) with the hypothesis $I_n[u; \mathbb{B}^n] \leq 1$, this becomes

$$\left( \int_{\mathbb{B}^n} \left( uX_2^{1/n} \right)^q \, dx \right)^{1/q} \leq e^{n/q} \left( \frac{\kappa_n}{4\omega_n n^{n-2}} \right)^{1/n} \Gamma^{1/q} \left( 1 + q \frac{n - 1}{n} \right). \quad (23)$$

The rest of the proof is standard: Taking $q = ln/(n - 1)$, $l \in \mathbb{N}$, in (23) we get

$$\int_{\mathbb{B}^n} \left( \left[ u(x)X_2^{1/n} (|x|) \right]^{n/(n-1)} \right)^l \, dx \leq e^n \left( \frac{\kappa_n}{4\omega_n n^{n-2}} \right)^{l/(n-1)} \Gamma(1+l).$$

Note that this holds true also for $l = 0$. Now we multiply both sides by $a^l / l!$, note that $\Gamma(1+l) = l!$ and add for all integers $l \in [0, m]$, $m \in \mathbb{N}$, to arrive at

$$\int_{\mathbb{B}^n} \sum_{l=0}^{m} \frac{1}{l!} \left[ a \left( |u(x)|X_2^{1/n} (|x|) \right)^n \right]^l \, dx \leq e^n \sum_{l=0}^{m} \left[ a \left( \frac{\kappa_n}{4\omega_n n^{n-2}} \right)^{1/(n-1)} \right]^l.$$

The series on the right converges if and only if

$$a < \left( \frac{4\omega_n n^{n-2}}{\kappa_n} \right)^{1/(n-1)}.$$

Hence for any such $a$ the proof is completed by letting $m \to \infty$ and using the monotone convergence theorem. \[\square\]

**Remark 3.4.** Since $\kappa_2 = 1$, for $n = 2$, estimate (24) reads $a < 4\pi$; that is $a$ has to be strictly less than Moser’s sharp constant. We don’t know if it still holds true for $a = 4\pi$. 

9
3.2 Proof of Theorem 1.1

Hence it is enough to establish (11) for Ω = BR0(0). Furthermore, being scaling invariant, it is enough to consider only the case RΩ = 1. Finally, it is enough to assume that u ∈ C1c(Bn \ {0}) (since W1,0n(Bn \ {0}) = W1,0(Bn); see for instance [HKM, Theorem 2.43]). Pick u ∈ Ln(Bn \ {0}) \ {0} and consider the function v defined through the transformation u = vX \ \frac{1}{1+1/n} 1. Following [VZ], we use spherical coordinates x = (r, θ) (r = |x| and θ = x/|x|) to decompose v into spherical harmonics. For this purpose, let \{h_l\} \in \mathbb{N} \cup \{0\} be the orthonormal basis of L2(Sn−1) that is comprised of eigenfunctions of the Laplace-Beltrami operator −∆Sn−1 (the angular part of the Laplacian when expressed in spherical coordinates). This has corresponding eigenvalues λ_l = l(l + n − 2), l ∈ \mathbb{N} \cup \{0\} (see [Schn, Appendix]). Thus for all l, m ∈ \mathbb{N} \cup \{0\} we have

\[ -\Delta_{S^{n-1}} h_l = \lambda_l h_l \text{ on } S^{n-1} \quad \text{and} \quad \int_{S^{n-1}} h_l(\theta)h_m(\theta) \, d\sigma(\theta) = \delta_{lm}. \]

With these definitions we have the decomposition of v in its spherical harmonics

\[ v(x) = \sum_{l=0}^{\infty} v_l(r)h_l(\theta). \]

In particular h_0(\theta) = 1 and the first term in the above decomposition is given by the spherical mean of v on ∂B_r(0), that is

\[ v_0(r) = \int_{\partial B_r(0)} v(x) \, d\sigma(x) = \int_{S^{n-1}} v(r\theta) \, d\sigma(\theta). \]

Now let q > n. Minkowski’s inequality implies

\[ \left( \int_{\mathbb{B}^n} \left( |u|X_2^{1/n} \right)^q \, dx \right)^{1/q} \leq \left( \int_{\mathbb{B}^n} \left( |u_0|X_2^{1/n} \right)^q \, dx \right)^{1/q} + \left( \int_{\mathbb{B}^n} \left( |u - u_0|X_2^{1/n} \right)^q \, dx \right)^{1/q}, \]

(25)

where u_0 := v_0X_1^{−1+1/n}. Note that since X_1 is a radial function, u_0 is just the spherical mean of u on ∂B_r(0); a radial function.

Define w := |v|^{n/2} and consider w_0; that is, the spherical mean of w on ∂B_r(0). Hölder’s
inequality implies $|v_0| \leq w_0^{2/n}$. Indeed,

$$|v_0(r)|^{n/2} = \left| \int_{\partial B_r(0)} v(y) \, d\sigma(y) \right|^{n/2} \leq \int_{\partial B_r(0)} |v(y)|^{n/2} \, d\sigma(y) = \int_{\partial B_r(0)} w(y) \, d\sigma(y) = w_0(r).$$

Using this, we see for the first term in (25) that

$$\left( \int_{\mathbb{R}^n} \left| u_0 \right|^{X_2^{1/n}} \frac{q}{d} \right)^{1/q} = \left( \int_{\mathbb{R}^n} \left( \left| u_0 \right| X_1^{-1+1/n} X_2^{1/n} \right)^q \, dx \right)^{1/q} \leq \left( \int_{\mathbb{R}^n} \left( \left| u_0 \right| X_2^{-1+1/n} X_2^{1/n} \right)^q \, dx \right)^{1/q} \leq c_1(n) \frac{q}{1-1/n} \left( \int_{\mathbb{R}^n} \left| x \right|^{2-n} \left| \nabla w_0 \right|^{2X_1^{-1}} \, dx \right)^{1/n}, \quad (26)$$

because of (22) with $f = w_0$.

For the second term of (25) we use first the fact that $X_2 \leq 1$ and then (10) to the function $u - u_0 \in W^{1,n}(\mathbb{R}^n)$ to get

$$\left( \int_{\mathbb{R}^n} \left( \left| u - u_0 \right| X_2^{1/n} \right)^q \, dx \right)^{1/q} \leq c_2(n) \frac{q}{1-1/n} \left( \int_{\mathbb{R}^n} \left| \nabla (u - u_0) \right|^{n} \, dx \right)^{1/n} = c_2(n) \frac{q}{1-1/n} \left( \int_{\mathbb{R}^n} \left| \nabla [(u - u_0) X_1^{-1+1/n}] \right|^{n} \, dx \right)^{1/n}. \quad (27)$$

Claim: The assumption $I_n[u] \leq 1$ is enough for each of the two integrals on the right of (26) and (27) to be bounded by a constant not depending on $u$. In particular we will show that

$$\int_{\mathbb{R}^n} \left| x \right|^{2-n} \left| \nabla w_0 \right|^{2X_1^{-1}} \, dx \leq c_3(n) \int_{\mathbb{R}^n} \left| x \right|^{2-n} \left| u \right|^{n-2} \left| \nabla v \right|^{2X_1^{-1}} \, dx, \quad (28)$$

and that

$$\int_{\mathbb{R}^n} \left| \nabla [(u - u_0) X_1^{-1+1/n}] \right|^{n} \, dx \leq c_4(n) \int_{\mathbb{R}^n} \left| \nabla v \right|^{n} X_1^{-n+1} \, dx. \quad (29)$$
Because of (12) and (13), estimates (28) and (29) will readily imply our claim.

Proof of (28): Set \( \zeta := |x|^{1-n/2} X_1^{-1/2} w \), then by proposition 2.3 with \( U = \mathbb{B}^n \) and \( f = \zeta_0 \), we have the following equality

\[
\int_{\mathbb{B}^n} |x|^{2-n}|\nabla w_0|^2 X_1^{-1} \, dx = \int_{\mathbb{B}^n} |\nabla \zeta_0|^2 \, dx - \left( \frac{n-2}{2} \right)^2 \int_{\mathbb{B}^n} \frac{|\zeta_0|^2}{|x|^2} \, dx - \frac{1}{4} \int_{\mathbb{B}^n} \frac{|\zeta_0|^2}{|x|^2} X_1^2 \, dx. \tag{30}
\]

From [FT, eq. (7.6)] we know that

\[
\int_{\mathbb{B}^n} |\nabla \zeta_0|^2 \, dx - \left( \frac{n-2}{2} \right)^2 \int_{\mathbb{B}^n} \frac{|\zeta_0|^2}{|x|^2} \, dx - \frac{1}{4} \int_{\mathbb{B}^n} \frac{|\zeta_0|^2}{|x|^2} X_1^2 \, dx \leq \int_{\mathbb{B}^n} |x|^{2-n}|\nabla w|^2 X_1^{-1} \, dx - \left( \frac{n-2}{2} \right)^2 \int_{\mathbb{B}^n} \frac{|\zeta_0|^2}{|x|^2} \, dx - \frac{1}{4} \int_{\mathbb{B}^n} \frac{|\zeta_0|^2}{|x|^2} X_1^2 \, dx \]

the last equality again because of proposition 2.3 with \( U = \mathbb{B}^n \), but \( f = \zeta \) this time. We have just showed that

\[
\int_{\mathbb{B}^n} |x|^{2-n}|\nabla w_0|^2 X_1^{-1} \, dx \leq \int_{\mathbb{B}^n} |x|^{2-n}|\nabla w|^2 X_1^{-1} \, dx,
\]

and the proof of (28) follows at once since \( w := |v|^{n/2} \).

Proof of (29): We start from the right hand side of (29), by noticing that

\[
\int_{\mathbb{B}^n} |\nabla v|^n X_1^{-n+1} \, dx \]

\[
= \int_0^1 X_1^{-n+1}(r) r^{n-1} \int_{S^{n-1}} \left( (\partial_r v)^2 + \frac{1}{r^2} |\nabla \theta v|^2 \right)^{n/2} \, d\sigma(\theta) \, dr \]

\[
\geq \int_0^1 X_1^{-n+1}(r) r^{n-1} \int_{S^{n-1}} |\partial_r v|^n \, d\sigma(\theta) \, dr \]

\[
+ \int_0^1 X_1^{-n+1}(r) r^{-1} \int_{S^{n-1}} |\nabla \theta v|^n \, d\sigma(\theta) \, dr, \tag{32}
\]

by the fact that \((\kappa + \lambda)^q \geq \kappa^q + \lambda^q\), for all \( \kappa, \lambda \geq 0 \) and any \( q \geq 1 \), and also because
$X_1^{-n+1}(r) \geq 1$ for all $r$. To estimate the first term on the right of (32) we use (15) to get

$$\int_{\mathbb{S}^{n-1}} |\partial_r v|^n \, d\sigma(\theta)$$

$$\geq \int_{\mathbb{S}^{n-1}} |\partial_r v|^n \, d\sigma(\theta) + \frac{1}{\lambda_n} \int_{\mathbb{S}^{n-1}} |\partial_r (v - v_0)|^n \, d\sigma(\theta)$$

$$+ n \int_{\mathbb{S}^{n-1}} |\partial_r v_0|^{n-2} (\partial_r v_0) \partial_r (v - v_0) \, d\sigma(\theta). \quad (33)$$

But since $\{v_l\}_{l \in \mathbb{N} \cup \{0\}}$ are radial

$$\int_{\mathbb{S}^{n-1}} |\partial_r v_0|^{n-2} (\partial_r v_0) \partial_r (v - v_0) \, d\sigma(\theta)$$

$$= |v_0'(r)|^{n-2} v_0'(r) \int_{\mathbb{S}^{n-1}} \partial_r (v - v_0) \, d\sigma(\theta)$$

$$= |v_0'(r)|^{n-2} v_0'(r) \sum_{l=1}^{\infty} v_l'(r) \int_{\mathbb{S}^{n-1}} h_l(\theta) \, d\sigma(\theta) = 0,$$

and so

$$\int_{\mathbb{S}^{n-1}} |\partial_r v|^n \, d\sigma(\theta) \geq \frac{1}{\lambda_n} \int_{\mathbb{S}^{n-1}} |\partial_r (v - v_0)|^n \, d\sigma(\theta),$$

where we have cancel also the first term on the right hand side of (33). Plugging this to (32) we deduce

$$\int_{\mathbb{B}^n} |\nabla v|^n X_1^{-n+1} \, dx$$

$$\geq \frac{1}{\lambda_n} \int_0^1 X_1^{-n+1}(r) r^{n-1} \int_{\mathbb{S}^{n-1}} \left( |\partial_r (v - v_0)|^n + \frac{1}{r^n} |\nabla v|^n \right) \, d\sigma(\theta) \, dr$$

$$+ \left( 1 - \frac{1}{\lambda_n} \right) J$$

$$\geq \frac{1}{\lambda_n} \int_0^1 X_1^{-n+1}(r) r^{n-1} \int_{\mathbb{S}^{n-1}} \left( |\partial_r (v - v_0)|^2 + \frac{1}{r^2} |\nabla v|^2 \right)^{n/2} \, d\sigma(\theta) \, dr$$

$$+ \left( 1 - \frac{1}{\lambda_n} \right) J$$

$$= \frac{1}{\lambda_n} \int_{\mathbb{B}^n} |\nabla (v - v_0)|^n X_1^{-n+1} \, dx + \left( 1 - \frac{1}{\lambda_n} \right) J, \quad (34)$$

by the fact that $\kappa^q + \lambda^q \geq 2^{1-q}(\kappa + \lambda)^q$, for all $\kappa, \lambda \geq 0$ and any $q \geq 1$. To estimate $J$ observe first that

$$\int_{\mathbb{S}^{n-1}} (v - v_0) \, d\sigma(\theta) = \sum_{l=1}^{\infty} v_l(r) \int_{\mathbb{S}^{n-1}} h_l(\theta) \, d\sigma(\theta) = 0,$$
so that utilizing once more the fact that $v_0$ is radial, we may use the Poincaré inequality on $\mathbb{S}^{n-1}$ (see for example [H])

\[
\int_{\mathbb{S}^{n-1}} |\nabla_\theta v|^n \, d\sigma(\theta) = \int_{\mathbb{S}^{n-1}} |\nabla_\theta (v - v_0)|^n \, d\sigma(\theta) \geq C_\mathcal{P}(n) \int_{\mathbb{S}^{n-1}} |v - v_0|^n \, d\sigma(\theta).
\]

Inserting this in the definition of $J$, we get from (34) the existence of a positive constant $C(n)$ such that

\[
\int_{\mathbb{B}^n} |\nabla v|^n X_1^{-n+1} \, dx \\
\geq C(n) \left( \int_{\mathbb{B}^n} |\nabla (v - v_0)|^n X_1^{-n+1} \, dx + \int_{\mathbb{B}^n} |x|^{-n} |v - v_0|^n X_1^{-n+1} \, dx \right)
\]

Since $X_1 \leq 1$ we replace $X_1^{-n+1}$ by $X_1$ in the second integral to deduce

\[
\int_{\mathbb{B}^n} |\nabla v|^n X_1^{-n+1} \, dx \\
\geq C(n) \left( \int_{\mathbb{B}^n} |\nabla (v - v_0)|^n X_1^{-n+1} \, dx + \int_{\mathbb{B}^n} |x|^{-n} |v - v_0|^n X_1 \, dx \right) \\
\geq C'(n) \int_{\mathbb{B}^n} |\nabla [(v - v_0) X_1^{-1+1/n}]|^n \, dx
\]

as required.

Acknowledgements 1. G. di Blasio and G. Pisante are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) whose support is gratefully acknowledged. The research has also been supported by project Vain-Hopes within the program VALERE: VAntiViteLli pEr la RicEerca.

References

[BFT1] G. Barbatis, S. Filippas, A. Tertikas, A unified approach to improved $L^p$ Hardy inequalities with best constants, Trans. Amer. Math. Soc. 356 (2004) 2169-2196.

[dBPP] G. di Blasio, G. Pisante, G. Psaradakis, A weighted anisotropic Sobolev type inequality and its applications to Hardy inequalities, Math Ann. 379, (2021) 1343-1362.

[CR] M. Calanchi, B. Ruf, Trudinger-Moser type inequalities with logarithmic weights, J. Differential Equations 258 (2015) 1967-1989.
S. Filippas, G. Psaradakis, *The Hardy–Morrey and Hardy–John–Nirenberg inequalities involving distance to the boundary* J. Differ. Equ. 261 (2016) 3107–3136.

S. Filippas, A. Tertikas, *Optimizing improved Hardy inequalities*, J. Funct. Anal. 192 (2002) 186-233.

K. T. Gkikas, G. Psaradakis, *Optimal non-homogeneous improvements for the series expansion of Hardy’s inequality*, Commun. Contemp. Math. ?? (2021) 2150031.

E. Hebey, Nonlinear analysis on manifolds: Sobolev spaces and inequalities. *Courant Lect. Notes Math.* Vol. 5 (Amer. Math. Soc. 1999).

J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear potential theory of degenerate elliptic equations. Unabridged republication of the 1993 original. Dover 2006.

J. Leray, *Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l’Hydrodynamique*, J. Math. Pures Appl. 12 (1933) 1-82.

P. Lindqvist, *On the equation* \( \text{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0 \), Proc. Amer. Math. Soc. 109 (1990) 157-164.

A. Mallick, C. Tintarev, *An improved Leray-Trudinger inequality*, Commun. Contemp. Math. 20 (2017) 1750034.

A. Mercaldo, M. Sano, F. Takahashi, *Finsler Hardy inequalities*, Math. Nachrichten 2020 https://doi.org/10.1002/mana.201900117

J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. 20 (11) (1971) 1077-1092.

J. Peetre, *Espaces d’interpolation et théorème de Soboleff*, Ann. Inst. Fourier (Grenoble) 16 (1966) 279-317.

S. Pohozaev, *On S. L. Sobolev’s embedding theorem in the case* \( p\ell = n \), Proc. Tech. Sci. Conf. on Advances of Sci. Res., Math. Section, Moskov. Énerget. Inst., Moscow, 1965, 158-170.

G. Psaradakis, *An optimal Hardy-Morrey inequality*, Calc. Var. Partial Differential Equations 45 (2012) 421-441.

G. Psaradakis, D. Spector, *A Leray-Trudinger inequality*, J. Funct. Anal. 269 (2015) 215-228.

R. Schneider, Convex bodies: the Brunn-Minkowski theory. 2nd expanded edition. *Encyclopedia Math. Appl.* Vol. 151 (Cambridge Univ. Press 2014).

N. S. Trudinger, *On embeddings into Orlicz spaces and some applications*, J. Math. Mech. (Indiana Univ. Math. J.) 17 (1967) 473-483.
[VZ] J. L. Vázquez and E. Zuazua *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential*, J. Funct. Anal. 173 (2000) 103-153.

[WY] G. Wang, D. Ye *A Hardy–Moser–Trudinger inequality*, Adv. Math. 230 (2012) 294–320.

[Y] V. I. Yudovich, *Some estimates connected with integral operators and with solutions of elliptic equations*, Dokl. Akad. Nauk SSSR 138 (1961) 805-808.