Constraints and Evolution in Cosmology

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Abstract

We review some old and new results about strict and non strict hyperbolic formulations of the classical Einstein equations.

1 Introduction

The cosmos of general relativity is a pseudo-Riemannian manifold \((V, g)\) of Lorentzian signature \((-+,\ldots,+)\). The Einstein equations link its Ricci tensor with a phenomenological tensor which describes the stresses and energy of the sources. They read:

\[
\text{Ricci}(g) = \rho,
\]

that in local coordinates \(x^\lambda, \lambda = 0, 1, 2, \ldots, n\), where \(g = g_{\lambda\mu} dx^\lambda dx^\mu\) (classical physics \(n = 3\)),

\[
R_{\alpha\beta} = \frac{\partial}{\partial x^\lambda} \Gamma^\lambda_{\alpha\beta} - \frac{\partial}{\partial x^\alpha} \Gamma^\lambda_{\beta\lambda} + \Gamma^\lambda_{\alpha\beta} \Gamma^\mu_{\beta\lambda} - \Gamma^\lambda_{\alpha\mu} \Gamma^\mu_{\beta\lambda} = \rho_{\alpha\beta},
\]

(1.1)

where the \(\Gamma\)'s are the Christoffel symbols:

\[
\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu}(\frac{\partial}{\partial x^\alpha} g_{\beta\mu} + \frac{\partial}{\partial x^\beta} g_{\alpha\mu} - \frac{\partial}{\partial x^\mu} g_{\alpha\beta}),
\]

and \(\rho\) is a symmetric 2-tensor given in terms of the stress energy tensor \(T\) by,

\[
\rho_{\alpha\beta} \equiv T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \text{tr} T, \quad \text{with} \quad \text{tr} T \equiv g^{\lambda\mu} T_{\lambda\mu}.
\]

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Due to the Bianchi identities the left hand side of the Einstein equations satisfies the identities, with $\nabla_\alpha$ the covariant derivative in the metric $g$,

$$\nabla_\alpha(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta} R) \equiv 0, \quad R \equiv g^{\lambda\mu} R_{\lambda\mu}$$

The stress energy tensor of the sources satisfies the conservation laws which make the equations compatible,

$$\nabla_\alpha T^{\alpha\beta} = 0.$$

In vacuum the stress energy tensor is identically zero. The presence of sources brings up new problems specific to various types of sources.

Since very little is known about the global properties of the universe, it is legitimate to study arbitrary manifolds and metrics. Also, in modern attempts to unify all the fundamental interactions, manifolds of dimension $N$ greater than four, endowed with metrics with $(N-4)$-dimensional isometry groups are considered. We will therefore not restrict our study to four-dimensional manifolds, when possible.

Since we will treat only non-quantum fields, it seems that a first problem to look at is the problem of classical dynamics, i.e., the problem of evolution of initial data. The Einstein equations are a geometric system, invariant by diffeomorphisms of $\mathcal{V}$, the associated isometries of $g$, and transformation of the sources. From the analyst’s point of view they constitute, for the metric, a system of second order quasi-linear partial differential equations. The system is determined because the characteristic determinant is identically zero (a property linked with diffeomorphism invariance). It is overdetermined because the Cauchy data are not arbitrary (which must be interpreted geometrically).

To study the Cauchy problem we must split space and time. The evolution will be formulated for time dependent space tensors.

2 Moving frame formulas

2.1 Frame and coframe

A moving frame in a subset $\mathcal{U}$ of a differentiable $(n+1)$-dimensional manifold $\mathcal{V}$ is a set of $(n+1)$ vector fields on $\mathcal{U}$ linearly independent in the tangent space $T_x\mathcal{V}$ at each point $x \in \mathcal{U}$. A coframe on $\mathcal{U}$ is a set of $(n+1)$ 1-forms $\theta^\alpha$ linearly independent at each $x \in \mathcal{U}$ in the dual $T^*_x\mathcal{V}$. In the domain $\mathcal{U}$ of a chart a coframe is defined by $(n+1)$ linearly independent differential
1-forms,

\[ \theta^\alpha \equiv a^\alpha_\beta dx^\beta, \quad (2.1) \]

with \( a^\alpha_\beta \) functions on \( \mathcal{U} \).

The metric is Lorentzian if the quadratic form is of Lorentzian signature.

**Remark 2.1** The splitting \( \mathcal{V} = \mathcal{M} \times \mathbb{R} \), with \( \mathcal{M} \) an orientable 3-manifold, implies the existence of a global coframe (but not of global coordinates!).

The coframe defined by the 1-forms \( \theta^\alpha \) is called the natural frame if \( \theta^\alpha \equiv dx^\alpha \).

In a general frame the differentials of the 1-forms \( \theta^\alpha \) do not vanish; they are given by the 2-forms,

\[ d\theta^\alpha \equiv -\frac{1}{2} C^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma. \quad (2.2) \]

The functions \( C^\alpha_{\beta\gamma} \) on \( \mathcal{U} \) are called the structure coefficients of the frame.

The Pfaff derivative \( \partial_\alpha \) of a function on \( \mathcal{U} \) is such that,

\[ df \equiv \frac{\partial f}{\partial x^\alpha} dx^\alpha \equiv \partial_\alpha f \theta^\alpha. \quad (2.3) \]

We denote by \( A \) with elements \( A^\beta_\alpha \) of the matrix inverse of \( a \) with elements \( a^\beta_\alpha \). It holds that:

\[ \partial_\alpha f \equiv A^\beta_\alpha \frac{\partial f}{\partial x^\beta}. \quad (2.4) \]

Pfaff derivatives do not commute. One deduces from (2.2) and the identity \( d^2 f \equiv 0 \) that,

\[ d^2 f \equiv \frac{1}{2} \{ \partial_\beta \partial_\gamma f - \partial_\gamma \partial_\beta f - C^\alpha_{\beta\gamma} \partial_\alpha f \} \theta^\beta \wedge \theta^\gamma \equiv 0, \quad (2.5) \]

hence,

\[ (\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha) f \equiv C^\gamma_{\alpha\beta} \partial_\gamma f. \quad (2.6) \]

### 2.2 Metric

A metric on \( \mathcal{U} \) is a nondegenerate quadratic form of the \( \theta^\alpha \)'s:

\[ g \equiv g_{\alpha\beta} \theta^\alpha \theta^\beta. \quad (2.7) \]

A frame is called orthonormal for the metric \( g \) if \( g_{\alpha\beta} = \pm 1 \). In the case of a Lorentzian metric we will denote by \( \theta^0 \) the timelike (co)axis and \( \theta^i \) the space (co)axis, then in an orthonormal frame, \( g_{00} = -1 \) and \( g_{ij} = \delta_{ij} \), the Euclidean metric.
2.3 Connection

A linear connection on $\mathcal{V}$ permits the definition of an intrinsic derivation of vectors and tensors. It is defined in the domain $\mathcal{U}$ by a matrix-valued 1-form $\omega$ i.e., by a set of matrices $\omega^\beta_\gamma$ linked to functions $\omega^\beta_\alpha$ by the identities,

$$\omega^\beta_\gamma \equiv \omega^\beta_\alpha \theta^\alpha. \tag{2.8}$$

The covariant derivative of a vector $v$ with components $v^\alpha$ is,

$$\nabla_\alpha v^\beta \equiv \partial_\alpha v^\beta + \omega^\beta_\alpha v^\gamma. \tag{2.9}$$

An analogous formula holds for a covariant vector now with a minus sign in front of $\omega^\beta_\alpha$.

**Definition 2.1** The connection $\omega$ is called the Riemannian connection of $g$ if

- It has vanishing torsion, i.e.,

$$d\theta^\gamma + \omega^\gamma_\alpha \theta^\alpha \wedge \theta^\beta = 0, \tag{2.10}$$

that is

$$\omega^\alpha_\beta - \omega^\alpha_\gamma = C^\alpha_\beta_\gamma. \tag{2.11}$$

- The covariant derivative of the metric is zero, i.e.,

$$\partial_\alpha g^\beta_\gamma - \omega^\lambda_\alpha g^\beta_\lambda - \omega^\lambda_\beta g^\alpha_\lambda = 0. \tag{2.12}$$

These two conditions imply by straightforward computation that,

$$\omega^\beta_\alpha \equiv \Gamma^\beta_\alpha_\gamma + g^\beta_\mu \bar{\omega}_{\alpha\gamma\mu}, \tag{2.13}$$

with

$$\bar{\omega}_{\alpha\gamma\mu} \equiv \frac{1}{2}(g_{\mu\lambda} C^\lambda_\alpha_\gamma - g_{\lambda\gamma} C^\lambda_\alpha_\mu - g_{\alpha\lambda} C^\lambda_\gamma_\mu), \tag{2.14}$$

$$\Gamma^\beta_\alpha_\gamma \equiv \frac{1}{2} g^\beta_\mu (\partial_\alpha g_{\gamma\mu} + \partial_\gamma g_{\alpha\mu} - \partial_\mu g_{\alpha\gamma}). \tag{2.15}$$

The quantities $\Gamma$ are called the Christoffel symbols of the metric $g$. The connection coefficients reduce to them in the natural frame. They are zero for an orthonormal frame.

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1. The interpretation of this condition is that the second covariant derivatives of scalar functions commute, namely

$$\nabla_\alpha \partial_\beta f - \nabla_\beta \partial_\alpha f \equiv 0.$$

In particular in the natural frame $\omega^\alpha_\beta$ is symmetric in $\beta$ and $\gamma$. 

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2.4 Curvature

2.4.1 Definition

The non-commutativity of covariant derivatives is a geometric property of the metric. It signals its curvature. The Riemann curvature tensor is defined as an exterior 2-form with value a linear map in the tangent plane to \( \mathcal{V} \) by the following identity:

\[
(\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda)v^\alpha \equiv R^\alpha_{\lambda\mu, \beta} v^\beta,
\]

which gives by straightforward identification,

\[
R^\alpha_{\lambda\mu, \beta} \equiv \partial_\lambda \omega^\alpha_{\mu\beta} - \partial_\mu \omega^\alpha_{\lambda\beta} + \omega^\alpha_{\lambda\rho} \omega^\rho_{\mu\beta} - \omega^\alpha_{\mu\rho} \omega^\rho_{\lambda\beta} - \omega^\alpha_{\rho\beta} C^\rho_{\lambda\mu}.
\]

2.4.2 Symmetries and antisymmetries

The Riemann tensor is a symmetric double 2-form: it is antisymmetric in its first two indices, and in its last two indices written in covariant form. It is invariant by the interchange of these two pairs.

2.4.3 Bianchi identities

It holds that (vanishing of the covariant differential of the curvature 2-form):

\[
\nabla_\alpha R_{\beta\gamma,\lambda\mu} + \nabla_\beta R_{\gamma\alpha,\lambda\mu} + \nabla_\gamma R_{\alpha\beta,\lambda\mu} \equiv 0.
\]

2.4.4 Ricci tensor, scalar curvature, Einstein tensor

The Ricci tensor is defined by

\[
R_{\alpha\beta} \equiv R^\lambda_{\alpha, \lambda\beta}.
\]

The scalar curvature is,

\[
R \equiv g^{\alpha\beta} R_{\alpha\beta}.
\]

The Einstein tensor is,

\[
S_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R.
\]
2.4.5 Conservation identity

Contracting the Bianchi identities gives that,
\[ \nabla_\alpha \R_{\beta\gamma\mu} - \nabla_\beta \R_{\gamma\mu} + \nabla_\gamma \R_{\beta\mu} \equiv 0, \]
and a further contraction gives the following identity satisfied by the Einstein tensor:
\[ \nabla_\alpha S^{\alpha\beta} \equiv 0. \]
This identity implies that the sources must satisfy the so-called conservation laws,
\[ \nabla_\alpha T^{\alpha\beta} = 0. \]

3 (n+1)-splitting adapted to space slices

3.1 Definitions

We consider a spacetime with manifold \( V = \mathcal{M} \times \mathbb{R} \) and hyperbolic metric \( g \) such that the submanifolds \( \mathcal{M}_t = \mathcal{M} \times \{t\} \) are spacelike. We take a frame with space axis \( e_i \) tangent to the space slice \( \mathcal{M}_t \) and time axis \( e_0 \) orthogonal to it. Such a frame is particularly adapted to the solution of the Cauchy problem and will be called a Cauchy adapted frame. The dual coframe is:
\[ \theta^i = dx^i + \beta^i dt, \]
\[ \beta^i \text{ is a time dependent vector tangent to } \mathcal{M}_t \text{ called the shift.} \]
The 1-form \( \theta^0 \) does not contain \( dx^i \). We choose:
\[ \theta^0 = dt. \]
The pfaffian derivatives (action of the vector basis \( e_\alpha \)) with respect to the adapted coframe are,
\[ \partial_0 = \partial_t - \beta^j \partial_j, \quad \partial_i = \partial/\partial x^i, \quad \text{with} \quad \partial_t = \partial/\partial t. \]
In the coframe \( \theta^\alpha \) the metric reads,
\[ ds^2 = -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j. \]
The function \( N \) is called the lapse. We shall assume throughout \( N > 0 \) and the space metric \( \bar{g} \) induced by \( g \) on \( \mathcal{M}_t \) properly Riemannian. An overbar will denote a spatial tensor or operator, i.e., a \( t \)-dependent tensor or operator on \( \mathcal{M} \). Note that \( \bar{g}_{ij} = g_{ij} \) and \( \bar{g}^{ij} = g^{ij} \).
3.2 Structure coefficients

The structure coefficients of a frame of a Cauchy adapted frame are found to be, 

\[ C_{0j}^i = -C_{j0}^i = \partial_j \beta^i, \]  

(3.5) 

and all other structure coefficients are zero.

3.3 Splitting of the connection

We denote by \( \bar{\nabla} \) covariant derivatives in the space metric \( \bar{g} \). Using the general formulas (2.12), (2.13), (2.14) we find that,

\[ \omega^{i}_{jk} = \Gamma^{i}_{jk} = \bar{\Gamma}^{i}_{jk}, \]  

(3.6) 

\[ \omega^{0}_{0i} = Ng^{ij} \partial_j N, \quad \omega^{0}_{i0} = \omega^0_{i0} = N^{-1} \partial_i N, \quad \omega^{00}_{00} = N^{-1} \partial_0 N, \]

and

\[ \omega^{0}_{ij} = \frac{1}{2}N^{-2}\{\partial_0 g_{ij} + g_{hj}C_{i0}^h + g_{ih}C_{j0}^h\}, \]  

(3.7) 

that is, 

\[ \omega^{0}_{ij} = \frac{1}{2}N^{-2}\{\partial_0 g_{ij} - g_{hj}\partial_i \beta^h - g_{ih}\partial_j \beta^h\}. \]  

(3.8)

Using the expression (3.3) of \( \partial_0 \) we obtain that,

\[ \omega^{0}_{ij} = \frac{1}{2}N^{-2}\partial_0 g_{ij}, \]  

(3.9) 

where the operator \( \hat{\partial}_0 \) is defined on any \( t \)-dependent space tensor \( T \) by the formula,

\[ \hat{\partial}_0 = \frac{\partial}{\partial t} - \bar{L}_\beta, \]  

(3.10) 

where \( \bar{L}_\beta \) is the Lie derivative on \( M_t \) with respect to the spatial vector \( \beta \). Note that \( \hat{\partial}_0 \) \( T \) is a \( t \)-dependent space tensor of the same type as \( T \).

The extrinsic curvature \( K_{ij} \) (second fundamental tensor) of \( M_t \) is classically defined as the projection on \( M_t \) of the covariant derivative of the unit normal \( \nu \), past-oriented, that is

\[ K_{ij} \equiv \frac{1}{2}(\nabla_i \nu_j + \nabla_j \nu_i), \]  

(3.11) 

with, in our coframe,

\[ \nu_i = 0, \quad \nu_0 = N. \]  

(3.12)
Lemma 3.1 The following identity holds,

$$\hat{\partial}_0 g_{ij} \equiv -2N K_{ij}. \quad (3.13)$$

**Proof.** It holds that,

$$\nabla_i \nu_j = -\omega_{ij}^0 \nu_0 = -\frac{1}{2} N^{-1} \hat{\partial}_0 g_{ij} = K_{ij}. \quad (3.14)$$

The remaining connection coefficients are found to be,

$$\omega_{ij}^0 = -N K_{ij}^i + \partial_j \beta^i, \quad \omega_{0j}^i = -N K_{j}^i, \quad (3.15)$$

and this completes the proof of the Lemma. ■

### 3.4 Splitting of the Riemann tensor

We deduce from the general formula giving the Riemann tensor and the splitting of the connection the following identities,

$$R_{ij,kl} = \bar{R}_{ij,kl} + K_{ik}K_{lj} - K_{il}K_{kj}, \quad (3.16)$$

$$R_{0i,kj} = N(\tilde{\nabla}_j K_{ki} - \tilde{\nabla}_k K_{ji}), \quad (3.17)$$

$$R_{0i,0j} = N(\hat{\partial}_0 K_{ij} + N K_{ik} K_{kj}^k + \nabla_i \partial_j N). \quad (3.18)$$

From these formulae one obtains the following ones for the Ricci curvature:

$$NR_{ij} = NR_{ij} - \hat{\partial}_0 K_{ij} + N K_{ij} K_{hk}^h - 2N K_{ik} K_{kj}^k - \nabla_i \partial_j N, \quad (3.19)$$

$$N^{-1} R_{0j} = \partial_j K_{hk}^h - \nabla_h K_{jh}^h, \quad (3.20)$$

$$R_{00} = N(\hat{\partial}_0 K_{hk}^h - N K_{ij} K^{ij} + \tilde{\Delta} N). \quad (3.21)$$

Also,

$$g^{ij} R_{ij} = \bar{R} - N^{-1} \hat{\partial}_0 K_{hk}^h + (K_{hk}^h)^2 - N^{-1} \tilde{\Delta} N, \quad (3.22)$$

$$S_{00} \equiv R_{00} - \frac{1}{2} g_{00} R \equiv \frac{1}{2}(R_{00} + g^{ij} R_{ij}), \quad (3.23)$$

hence,

$$2N^{-2} S_{00} \equiv -2S_{0}^0 \equiv \bar{R} - K_{ij} K^{ij} + (K_{hk}^h)^2, \quad (3.24)$$

with $\bar{R} = g^{ij} \bar{R}_{ij}$. 

8
4 Constraints and evolution

We see in the above decomposition of the Ricci tensor that none of the Einstein equations contains the time derivatives of the lapse \( N \) and shift \( \beta \). One is thus led to consider the Einstein equations as a dynamical system for the two fundamental forms \( \bar{g} \) and \( K \) of the space slices \( \mathcal{M}_t \). This dynamical system splits as follows.

**Constraints**

The restriction to \( \mathcal{M}_t \) of the right hand side of the identities (3.19) and (3.23) contains only the metric \( g_{ij} \) and the extrinsic curvature \( K_{ij} \) of \( \mathcal{M}_t \) as tensor fields on \( \mathcal{M}_t \). When the Einstein equations are satisfied, i.e., when,

\[
S_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = T_{\alpha\beta} \equiv \rho_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\rho,
\]  

(4.1)

these identities lead to the following equations called constraints:

Momentum constraint

\[
C_i \equiv \frac{1}{N}(R_{0i} - \rho_{0i}) \equiv -\nabla_h K^h_i + \nabla_i K^h_h - N^{-1}\rho_{0i},
\]  

(4.2)

Hamiltonian constraint

\[
C_0 \equiv \frac{2}{N^2}(S_{00} - T_{00}) \equiv \bar{R} - K^i_j K^j_i + (K^h_h)^2 + 2T^0_0.
\]  

(4.3)

These constraints are transformed into a system of elliptic equations on each submanifold \( \mathcal{M}_t \), in particular on \( \mathcal{M}_0 \), for \( \bar{g} = g_0, K = K_0 \), by the conformal method (cf. C-B and York 1980, Isenberg 1995, C-B, Isenberg and York 2000).

**Evolution**

The equations,

\[
R_{ij} \equiv \bar{R}_{ij} - \frac{\hat{\partial}_0 K_{ij}}{N} - 2K_{jh}K^h_i + K_{ij}K^h_h - \frac{\nabla_j \partial_i N}{N} = \rho_{ij},
\]

together with the definition,

\[
\hat{\partial}_0 g_{ij} = -2NK_{ij},
\]

determine the derivatives transversal to \( \mathcal{M}_t \) of \( \bar{g} \) and \( K \) when these tensors are known on \( \mathcal{M}_t \) as well as the lapse \( N \) and shift \( \beta \), and source \( \rho_{ij} \). It is natural to look at these equations as evolution equations determining \( \bar{g} \) and \( K \), while \( N \) and \( \beta \), projections of the tangent to the
time line respectively on $e_0$ and the tangent space to $\mathcal{M}$, are considered as gauge variables. This point of view is supported by the following theorem (Anderson and York 1997, previously given for sources in C-B and Noutcheueme 1988):

**Theorem 4.1** When $R_{ij} - \rho_{ij} = 0$ the constraints satisfy a linear homogeneous first order symmetric hyperbolic system, they are satisfied if satisfied initially.

**Proof.** When $R_{ij} - \rho_{ij} = 0$ we have, in the Cauchy adapted frame, with $\rho \equiv g^{\alpha\beta} \rho_{\alpha\beta}$,

$$R - \rho = -N^2(R^{00} - \rho^{00}),$$

hence,

$$S^{00} - T^{00} = \frac{1}{2}(R^{00} - \rho^{00}) \quad \text{and} \quad R - \rho = -2N^2(S^{00} - T^{00}) = 2(S_0^0 - T_0^0),$$

and

$$S^{ij} - T^{ij} = -\frac{1}{2}\bar{g}^{ij}(R - \rho) = -\bar{g}^{ij}(S_0^0 - T_0^0),$$

the Bianchi identities give therefore a linear homogeneous system for $\Sigma_0^i \equiv S_0^i - T_0^i$ and $\Sigma_0^0 \equiv S_0^0 - T_0^0$ with principal parts,

$$N^{-2} \partial_0 \Sigma_0^0 + \bar{g}^{ij} \partial_j \Sigma_0^0, \quad \text{and} \quad \partial_0 \Sigma_0^0 + \partial_i \Sigma_0^i.$$

This system is symmetrizable hyperbolic, it has a unique solution, zero if the initial values are zero. The characteristic which determines the domain of dependence is the light cone. ■

### 5 Analytic Cauchy problem

Geometrical initial data for the Einstein equations are a triple $(\mathcal{M}, \bar{g}_0, K_0)$ with $\bar{g}_0$ a properly Riemannian metric on the n-dimensional manifold $\mathcal{M}$, and $K_0$ a symmetric 2-tensor on $\mathcal{M}$. A solution of the Cauchy problem for the Einstein equations with these initial data is an $(n+1)$-dimensional pseudo-Riemannian manifold $(\mathcal{V}, g)$ which we shall suppose of signature $(-, +, \ldots, +)$, such that $\mathcal{M}$ can be identified with a submanifolds $\mathcal{M}_0$ of $\mathcal{V}$, with $\bar{g}_0$ the metric induced by $g$ on $\mathcal{M}_0$ and $K_0$ the extrinsic curvature of $\mathcal{M}_0$ as submanifold of $(\mathcal{V}, g)$. The manifold $(\mathcal{V}, g)$ is called an *Einsteinian development* of the data. When the sources are given, for example zero (vacuum case), these geometrical initial data cannot be chosen arbitrarily, they must satisfy on $\mathcal{M}$ the constraints (4.2), (4.3).
The evolution equations read:

\[
\begin{align*}
\partial_t g_{ij} &= -2NK_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i, \\
\partial_t k_{ij} &= N\{\bar{R}_{ij} - 2K_{ih}K^h_j + K_{ij}K^h_h\} - \bar{\nabla}_i \partial_j N + \beta^h \bar{\nabla}_j k_{ij} + k_{ih} \bar{\nabla}_j \beta^h + k_{hj} \bar{\nabla}_i \beta^h - N\rho_{ij}.
\end{align*}
\]

(5.1) (5.2)

No equation contains the time derivatives of the lapse \( N \) and shift \( \beta \). We suppose that these quantities are given on \( \mathcal{V} \). The system is of the Cauchy-Kovalevski type, therefore we have the following theorem:

**Theorem 5.1** If the initial data are analytic on \( \mathcal{M}_0 \) while the sources, the shift and the lapse are analytic in a neighborhood of \( \mathcal{M}_0 \) then there exists a neighborhood of \( \mathcal{M}_0 \) in \( \mathcal{M} \times \mathbb{R} \) such that the evolution equations have a solution in this neighborhood taking these Cauchy data.

We deduce from this Theorem the following one:

**Theorem 5.2** If the sources satisfy the conservation laws, in particular are zero, on \( \mathcal{V} \) the solution of the evolution equations satisfies the full Einstein equations if the initial data satisfy the constraints.

Since the conservation laws depend also on the metric the application of these theorems requires further study, except in the case where the sources are zero. In this vacuum case we can enunciate:

**Corollary 5.3** Analytic initial data satisfying the vacuum constraints admit a vacuum Einsteinian development.

**Proof.** Take an arbitrary analytic shift \( \beta \) and \( N > 0 \).

We will come back later to the geometric uniqueness problem.

6 Non-strict hyperbolicity of \( R_{ij} = 0 \)

An evolution part of Einstein equations should exhibit causal propagation, i.e., with domain of dependence determined by the light cone of the spacetime metric.

The equations \( R_{ij} = 0 \) are, when \( N \) and \( \beta \) are known, a second order differential system for \( g_{ij} \). The hyperbolicity of a quasi-linear system is defined through the linear differential operator
obtained by replacing in the coefficients the unknown by given values. In our case and for given $N, \beta$ and $g_{ij}$, the principal part of this operator acting on a symmetric 2-tensor $\gamma_{ij}$ is,

$$\frac{1}{2}\{(N^{-2}\partial_{00}^2 - g^{hk}\partial_{hk}^2)\gamma_{ij} + \partial_k \partial_j \gamma_{ik} + \partial_i \partial_k \gamma_{jk} - g^{hk}\partial_i \partial_j \gamma_{hk}\}.$$ 

The characteristic matrix $\mathcal{M}$ at a point of spacetime is the linear operator obtained by replacing the derivation $\partial$ by a covariant vector $\xi$. The characteristic determinant is the determinant of this linear operator. For simplicity we compute it in the classical case of space dimension $n = 3$. We find,

$$\text{Det} \mathcal{M} = \xi^a_0 (g^\alpha \beta \xi_\alpha \xi_\beta)^3.$$ 

The characteristic cone is the dual of the cone defined in the cotangent plane by annulation of the characteristic polynomial. For our system the characteristic cone splits into the light cone of the given spacetime metric and the normal to its space slice. Since these characteristics appear as multiple and the system is non-diagonal, it is not hyperbolic in the usual sense. One can prove by diagonalization of the system the following theorem (C-B 2000).

**Theorem 6.1** When $N > 0$ and $\beta$ are given, arbitrary, the system $R_{ij} = 0$ is a non-strict hyperbolic system in the sense of Leray-Ohya for $g_{ij}$, in the Gevrey class $\gamma = 2$, as long as $g_{ij}$ is properly Riemannian. If the Cauchy data as well as $N$ and $\beta$ are in such a Gevrey class, the Cauchy problem has a local in time solution with domain of dependence determined by the light cone.

### 7 Wave equation for $K$, Hyperbolic system

Various hyperbolic systems have been obtained in recent years for the evolution of the dynamical variables $(g_{ij}, K_{ij})$ by linear combination of $R_{ij}$ with the constraints. The first of these hyperbolic systems has been obtained in C.B and Ruggeri 1983, zero shift, extended in C.B and York 1995 to an arbitrary shift. It uses a quasi-diagonal system of wave equations for $K_{ij}$, shown to hold modulo a gauge condition for $N$, called now ‘densitizing the lapse’. It works as follows.

We use the expressions for $R_{0i}$ and $R_{ij}$, together with $\hat{\partial}_0 g_{ik} = -2NK_{ik}$ which imply,

$$\hat{\partial}_0 \Gamma^h_{ij} = -g^{hk}\{\bar{\nabla}_i(NK_{jk}) + \bar{\nabla}_j(NK_{ik}) - \bar{\nabla}_k(NK_{ij})\}.$$ 

(7.1)
to obtain the identity, with \( f_{ij} = f_{ji} \), and \( H \equiv \text{tr}(K) \equiv K^j_i, \)

\[
\Omega_{ij} \equiv \hat{\partial}_0 R_{ij} - \nabla (i R_{j0}) \equiv -\hat{\partial}_0 (N^{-1} \hat{\partial}_0 K_{ij}) + \nabla h \nabla^h (NK_{ij}) - \hat{\partial}_0 (N^{-1} \nabla^j \partial_i N) - N \nabla^i \partial_j H + \hat{\partial}_0 (H K_{ij} - 2 K_{im} K^m_j) - \nabla (i (K_j) \partial^h N) - 2 \hat{\partial}_0 N \nabla^i \partial_j H - \hat{\partial}_0 (N^{-1} \nabla^j \partial_i N) - N \hat{\partial}_0 (K^m_{ij}) + H \nabla^j \partial_i N. \tag{7.2}
\]

This identity shows that for a solution of the Einstein equations,

\[
R_{\alpha\beta} = \rho_{\alpha\beta},
\]

the extrinsic curvature \( K \) satisfies a second order differential system which is quasi-diagonal with principal part the wave operator, except for the terms \(-N \nabla^i \partial_j H\). The unknown \( \bar{g} \) appears at second order, as well as \( N \) except for the term \(-\hat{\partial}_0 (N^{-1} \nabla^j \partial_i N)\). It holds that,

\[
\hat{\partial}_0 (N^{-1} \nabla^j \partial_i N) + N \nabla^j \partial_i H \equiv N^{-1} \nabla_j \partial_i (\partial_0 N + N^2 H) + X_{ij}, \tag{7.3}
\]

where \( X_{ij} \) is only of first order in \( \bar{g} \) and \( K \).

We then densitize the lapse, that is we set,

\[
N = \alpha (\det \bar{g})^{1/2}, \tag{7.4}
\]

where \( \alpha \) is an arbitrary positive tensor density of weight \(-1\) called the ‘densitized lapse’. Then, using the formula for the derivative of a determinant and the identity from Lemma 3.1, we find that:

\[
\partial_0 N = -N^2 H + \hat{\partial}_0 \alpha (\det \bar{g})^{1/2}. \tag{7.5}
\]

We see that \( \partial_0 N + N^2 H \) is an algebraic function of \( \bar{g} \). Hence, we have proved the following theorem.

**Theorem 7.1** Let \( \alpha > 0 \) and \( \beta \) be an arbitrary space tensor density and space tensor, depending on \( t \). Set \( N = \alpha (\det \bar{g})^{1/2} \), then:

1. The equations

\[
\Omega_{ij} = 0, \tag{7.6}
\]

are a quasi-diagonal system of wave equations for \( K \).

2. The equations above together with

\[
\hat{\partial}_0 g_{ij} = -2 N K_{ij}, \tag{7.7}
\]

are a hyperbolic Leray system for \( \bar{g} \) and \( K \).
Proof. Part 1 has already been proved. To prove part 2, we give to the equations and unknown the following weights:

\[ m(K) = 2, \quad m(\bar{g}) = 2, \quad n(\Omega = 0) = 0, \quad n(\partial \bar{g} = 2NK) = 1. \]  

(7.8)

The principal operator is then a matrix diagonal by blocks. Each block, corresponding to a pair \((ij)\) of indices, is given by,

\[
\begin{pmatrix}
-N^2 \partial^2_{00} + g^{ij} \partial_i \partial_j & X \\
0 & \partial_0
\end{pmatrix}
\]

with \(X\) a second order operator. The characteristic determinant is \((-N^2 \xi^2 + g^{ij} \xi_i \xi_j)\xi_0\), it is a hyperbolic polynomial.

8 Hyperbolic-elliptic system.

An alternative method to the densitization of the lapse is to consider \(H\) as a given function \(h\) on space time, i.e., imposing given mean extrinsic curvature on the space slices. The second order equation for \(K\) obtained above reduces again to a quasi-diagonal system with principal part the wave operator. This gauge condition was used by Christodoulou and Klainerman, with \(h = 0\), in the asymptotically Euclidean case, in the general case by C.B and York 1996. The lapse \(N\) is then determined through the equation \(R^0_0 = \rho^0_0\) which now reads in the general (non-vacuum) case,

\[
\bar{\nabla}^i \partial_i N - (K_{ij} K^{ij} - \rho^0_0)N = -\partial_0 h.
\]

This equation is an elliptic equation for \(N\) when \(\bar{g}, K\) and \(\rho\) are known.

Note that for energy sources satisfying the energy condition we have \(-\rho^0_0 \geq 0\) as well as \(|K|^2 \equiv K_{ij} K^{ij} \geq 0\), an important property for the solution of the elliptic equation. The mixed hyperbolic-elliptic system that we have constructed will determine the unknowns \(N\) and \(\bar{g}\) in a neighborhood of \(M\) in \(M \times \mathbb{R}\) when the shift \(\beta\) is chosen.

9 Local existence and uniqueness

Known properties of hyperbolic systems on manifolds and the use of the Bianchi identities to show the preservation of constraints under evolution, lead to the following theorem.

Theorem 9.1 Let \((\mathcal{M}, \bar{g}_0, K_0)\) be an initial data set satisfying the vacuum constraints, where \(\mathcal{M}\) is a smooth \(n\)-dimensional manifold endowed with a smooth, Sobolev regular, Riemannian
metric $e$, and where $\bar{g}_0 \in H^{u,loc}_s$, $K_0 \in H^{u,loc}_{s-1}$ are a properly Riemannian metric and a symmetric 2-tensor on $M$ respectively. Suppose arbitrarily given on $M \times I$, $I$ an interval of $\mathbb{R}$, the gauge variables $\alpha, \beta \in C^0(I, H^{u,loc}_s) \cap C^1(I, H^{u,loc}_{s-1}) \cap C^2(I, H^{u,loc}_{s-2})$. Then if $s > \frac{n}{2} + 1$ there exists an interval $J \subset I$ and a Lorentzian metric,

$$g = -N^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad N = (\alpha \det \bar{g})^{1/2},$$

(9.1)

solution of the given Cauchy problem on $M_{t_0}$, $t_0 \in J$, for the vacuum Einstein equations. For a given pair $\alpha, \beta$ this solution is unique.

**Notation 9.2** The spaces $H^{u,loc}_s$ are spaces of tensors which have generalized covariant derivatives in the metric $e$ of order up to $s$ which are square integrable on each open set of some given covering of $M$, with norm uniformly bounded (i.e., for a given tensor, independent of the subset). The manifold $(M, e)$ is called Sobolev regular if the covering can be chosen so that these Sobolev spaces satisfy the usual embedding and multiplication properties. This is always the case if $(M, e)$ is complete.

Two isometric space times $(V, g)$ and $(\tilde{V}, \tilde{g})$ are considered as identical. One can prove the following theorem.

**Theorem 9.3 (Physical uniqueness)**

1. Let $(\tilde{M} \times \tilde{J}, \tilde{g})$ be another solution of the Cauchy problem of Theorem 9.1, with a different gauge choice. There exists an isometry of $(\tilde{M} \times \tilde{J}', \tilde{g})$ onto $(M \times J', g)$, with $J' \subset J$ and $\tilde{J}' \subset \tilde{J}$.

2. The solution of the Cauchy problem for the vacuum Einstein equations is geometrically unique (i.e., up to isometries) in the class of globally hyperbolic space times.

For the definition of globally hyperbolic spacetimes see S. Cotsakis’ lectures, this volume.

10 First order hyperbolic systems

Such systems are supposed to be more amenable to numerical computation.

10.1 FOSH systems

A first order system of $N$ equations which reads,

$$M^{\alpha IJ}(u)\partial_\alpha u_I + f^J(u) = 0,$$

(10.1)
where $u_I$, $I = 1, \ldots, N$, are a set of unknowns (for instance the components of one or several tensors), is called symmetric if the matrices $M^\alpha$ are symmetric. Such a symmetric system is hyperbolic for the space slices $\mathcal{M}_t$ if the matrix $\tilde{M}^i$, coefficient of $\partial/\partial t$ is positive definite. The energy associated to such systems is straightforward to write in a Cauchy adapted frame. In this case $\tilde{M}^i = M^0$, while $\tilde{M}^i = M^i - \beta^i M^0$. Equations (10.1) imply,

$$u_J M^{\alpha IJ}(u) \partial_\alpha u_I + u_J f^J(u) \equiv \frac{1}{2} \partial_\alpha (M^{\alpha IJ} u_I u_J) + F(u) = 0. \quad (10.2)$$

We express the $\partial_\alpha$’s as linear combinations of the usual time and space partial derivatives. Then, Eq. (10.2) takes the form,

$$u_J \tilde{M}^{\alpha IJ}(u) \frac{\partial}{\partial x^\alpha} u_I + u_J f^J(u) \equiv \frac{1}{2} \frac{\partial}{\partial x^\alpha} (\tilde{M}^{\alpha IJ} u_I u_J) + F(u) = 0. \quad (10.3)$$

Integrating this equation on a strip $\mathcal{M} \times [0, T]$ leads to an energy equality and an energy inequality (see later “Bel-Robinson energy”).

Anderson, C-B and York 1995 have written the system (7.6), (7.7) together with the gauge condition (7.5) as a FOSH system, using also the equation $R_{00} = 0$. A FOSH system had also been obtained by Frittelli and Reula 1994 just by combination of $R_{ij}$ with the constraints and densitization of the lapse. They used it to discuss the Newtonian approximation. A number of variants has been written since then, and their quality for numerical computation discussed (see in particular the Einstein-Christoffel system of Anderson and York 1999).

### 10.2 Other first order hyperbolic systems

Another criterion than symmetry has been recently used to test hyperbolicity of first order systems: It is the number of linearly independent eigenvectors associated to a multiple characteristic. If this number is equal to the multiplicity, and modulo conditions of uniformity, the system is hyperbolic. The verification of such a property involves heavy computations. Kidder, Scheel and Teukolsky 2001 write a whole family of such systems. They introduce as new unknowns the partial derivatives $\partial_\alpha g_{ij}$. They show that densitization of the lapse is a necessary condition for the hyperbolicity of the obtained systems. They evolve some of them numerically in the case of a one-black-hole spacetime and discuss their accuracy, i.e., how well the constraints are preserved.
11 Bianchi-Einstein equations

The Riemann tensor is the geometric object which intrinsically defines gravitational effects. The following results bring nothing new for the local existence and unicity theorems but they are useful for the global-in-time studies.

11.1 Wave equation for the Riemann tensor

The Riemann tensor of a pseudo-Riemannian metric, \( R_{\alpha\beta,\lambda\mu} \), is antisymmetric in its pair of first indices as well as in its pair of last indices. We call it a symmetric double 2-form because it possesses the symmetry,

\[
R_{\alpha\beta,\lambda\mu} \equiv R_{\lambda\mu,\alpha\beta}.
\] (11.1)

The Riemann tensor satisfies the Bianchi identities

\[
\nabla_\alpha R_{\beta\gamma,\lambda\mu} + \nabla_\gamma R_{\alpha\beta,\lambda\mu} + \nabla_\beta R_{\gamma\alpha,\lambda\mu} \equiv 0.
\] (11.2)

These identities imply by contraction,

\[
\nabla_\alpha R^{\gamma\alpha,\lambda\beta} + \nabla_\gamma R^{\alpha\beta,\lambda\mu} + \nabla_\beta R^{\gamma\alpha,\lambda\mu} \equiv 0.
\] (11.3)

Using the symmetry (11.1) gives the identities:

\[
\nabla_\alpha R^{\alpha\beta,\lambda\mu} + \nabla_\mu R^{\lambda\beta,\alpha\mu} - \nabla_\lambda R^{\mu\beta,\alpha\mu} \equiv 0.
\] (11.4)

If the Ricci tensor \( R_{\alpha\beta} \) satisfies the Einstein equations,

\[
R_{\alpha\beta} = \rho_{\alpha\beta},
\] (11.5)

then the previous identities imply the equations (Bel, Lichnerowicz)

\[
\nabla_\alpha R^{\alpha\beta,\lambda\mu} = \nabla_\lambda \rho_{\mu\beta} - \nabla_\mu \rho_{\lambda\beta}.
\] (11.6)

Equations (11.2) and (11.6) are analogous to the Maxwell equations for the electromagnetic 2-form \( F \):

\[
dF = 0, \quad \delta F = J,
\] (11.7)

where \( J \) is the electric current.

Theorem 11.1 The Riemann tensor of an Einsteinian spacetime of arbitrary dimension satisfies a quasi-diagonal, semilinear system of wave equations.
Proof. One deduces from (11.2) and the Ricci identity, an identity of the form:

$$\nabla^\alpha \nabla_\alpha R_{\beta\gamma,\lambda\mu} + \nabla_\gamma \nabla^\alpha R_{\alpha\beta,\lambda\mu} + \nabla_\beta \nabla^\alpha R_{\gamma\alpha,\lambda\mu} + S_{\beta\gamma,\lambda\mu} \equiv 0,$$

where $S_{\beta\gamma,\lambda\mu}$ is an homogeneous quadratic form in the Riemann tensor,

$$S_{\beta\gamma,\lambda\mu} \equiv \{ R_{\gamma \rho} R_{\rho \beta,\lambda\mu} + [ (R^\alpha_{\gamma \lambda} R^\alpha_{\lambda \beta,\rho\mu} - (\lambda \to \mu)) ] - \{ \beta \to \gamma \} \}.$$

Using equations (11.6), when the Ricci tensor satisfies the Einstein equations, gives equations of the form,

$$\nabla^\alpha \nabla_\alpha R_{\beta\gamma,\lambda\mu} + S_{\beta\gamma,\lambda\mu} = J_{\beta\gamma,\lambda\mu},$$

with $J_{\beta\gamma,\lambda\mu}$ depending on the sources $\rho_{\alpha\beta}$ and being zero in vacuum:

$$J_{\beta\gamma,\lambda\mu} \equiv \nabla_\gamma (\nabla_\mu \rho_{\lambda\beta} - \nabla_\lambda \rho_{\mu\beta}) - (\beta \to \gamma),$$

and this completes the proof. ■

11.2 Case n=3, FOS system

In a coframe $\theta^0, \theta^i$ where $g_{0i} = 0$, equations (11.2) with $\{\alpha\beta\gamma\} = \{ijk\}$ and equations (11.6) with $\beta = 0$ do not contain derivatives $\partial_0$ of the Riemann tensor. We call them ‘Bianchi constraints’. The remaining equations, called from here on ‘Bianchi equations,’ read as follows:

$$\nabla_0 R_{hk,\lambda\mu} + \nabla_k R_{0h,\lambda\mu} + \nabla_h R_{k0,\lambda\mu} = 0,$$

$$\nabla_0 R^0_{i,\lambda\mu} + \nabla_k R^h_{i,\lambda\mu} = \nabla_\lambda \rho_{\mu i} - \nabla_\mu \rho_{\lambda i} \equiv J_{\lambda\mu i},$$

where the pair $(\lambda\mu)$ is either $(0j)$ or $(jl)$, with $j < l$. There are 3 of one or the other of these pairs if the space dimension $n$ is equal to 3.

Equations (11.2) and (11.6) are, for each given pair $(0j)$, a first order system for the components $R_{hk,0j}$ and $R_{0h,0j}$. If we choose at a point of the spacetime an orthonormal frame the principal operator is diagonal by blocks, each block corresponding to a choice of a pair $(\lambda\mu, \lambda < \mu)$, is a symmetric 6 by 6 matrix which reads:

$$\begin{pmatrix}
\partial_0 & 0 & 0 & \partial_2 & -\partial_1 & 0 \\
0 & \partial_0 & 0 & 0 & \partial_3 & -\partial_2 \\
0 & 0 & \partial_0 & -\partial_3 & 0 & \partial_1 \\
\partial_2 & 0 & -\partial_3 & \partial_0 & 0 & 0 \\
-\partial_1 & \partial_3 & 0 & 0 & \partial_0 & 0 \\
0 & -\partial_2 & \partial_1 & 0 & 0 & \partial_0
\end{pmatrix}.$$
We have proved:

**Theorem 11.2** *The Bianchi evolution equations are a FOS (first order symmetrizable) system.*

The Bianchi equations depend on the choice of frame, as does their hyperbolicity.

### 11.3 Cauchy adapted frame

The numerical valued matrix $\mathcal{M}_t$ of coefficients of the operator $\partial/\partial t$ corresponding to the Bianchi equations relative to the Cauchy adapted frame is proportional to the unit matrix, with coefficient $N^{-2}$, hence is positive definite and the following theorem holds.

**Theorem 11.3** *The Bianchi equations associated to a Cauchy adapted frame are a FOSH system, with space sections $\mathcal{M}_t$.*

We will give an explicit expression of the full system after introducing two 'electric' and two 'magnetic' space tensors associated with the double 2-form $R$. They are the gravitational analogs of the electric and magnetic vectors associated with the electromagnetic 2-form $F$. That is, we define the 'electric' tensors by,

$$E_{ij} \equiv R^0_{i,0j}, \quad (11.14)$$

$$D_{ij} \equiv \frac{1}{4} \eta_{ikl} \eta_{0lm} R^{hk,lm}, \quad (11.15)$$

while the 'magnetic' tensors are given by,

$$H_{ij} \equiv \frac{1}{2} N^{-1} \eta_{ik} R^{hk,0j}, \quad (11.16)$$

$$B_{ji} \equiv \frac{1}{2} N^{-1} \eta_{ik} R^{0j,hk}. \quad (11.17)$$

In these formulae, $\eta_{ijk}$ is the volume form of the space metric $\bar{g} \equiv g_{ij} dx^i dx^j$.

**Lemma 11.4**

1. The electric and magnetic tensors are always such that

$$E_{ij} = E_{ji}, \quad D_{ij} = D_{ji}, \quad H_{ij} = B_{ji} \quad (11.18)$$

2. If the Ricci tensor satisfies the vacuum Einstein equations with cosmological constant

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \quad (11.19)$$

then the following additional properties hold

$$H_{ij} = H_{ji} = B_{ij} = B_{ji}, \quad E_{ij} = D_{ij} \quad (11.20)$$
Proof. (1) The Riemann tensor is a symmetric double 2-form, the electric and magnetic 2-tensors associated to it by the relations possess obviously the given symmetries.

(2) The Lanczos identity for a symmetric double two-form, with a tilde representing the spacetime double dual, gives

\[ \tilde{R}_{\alpha\beta,\lambda\mu} + R_{\alpha\beta,\lambda\mu} \equiv C_{\alpha\lambda}g_{\beta\mu} - C_{\alpha\mu}g_{\beta\lambda} + C_{\beta\mu}g_{\alpha\lambda} - C_{\beta\lambda}g_{\alpha\mu}, \]  

(11.21)

with

\[ C_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{4}Rg_{\alpha\beta}. \]  

(11.22)

It implies that \( \tilde{R}_{\alpha\beta,\lambda\mu} + R_{\alpha\beta,\lambda\mu} = 0 \) if \( C_{\alpha\beta} = 0 \), in particular for an Einsteinian vacuum spacetime with possibly a cosmological constant. The relations (11.20) can then be proved by a straightforward calculation that employs the relation \( \eta_{0ijk} = N\eta_{ijk} \) between the spacetime and space volume forms.

In order to extend the treatment to the non-vacuum case and to avoid introducing unphysical characteristics in the solution of the Bianchi equations, we will keep as independent unknowns the four tensors \( E, D, B, \) and \( H \), which will not be regarded necessarily as symmetric. The symmetries will be imposed eventually on the initial data and shown to be conserved by evolution.

We now express the Bianchi equations in terms of the time-dependent space tensors \( E, H, D, \) and \( B \). We use the following relations, found by inverting the definitions (11.14)-(11.17),

\[ R_{0k,0j} = -N^2E_{ij}, \quad R_{hk,0j} = N\eta^i_{hk}H_{ij}, \]  

(11.23)

\[ R_{hk,lm} = \eta^i_{hk}\eta^j_{lm}D_{ij}, \quad R_{0j,lm} = N\eta^i_{lm}B_{ji}. \]  

(11.24)

We will express spacetime covariant derivatives of the Riemann tensor in terms of space covariant derivatives \( \nabla \) and time derivatives, \( \partial_0 \), of \( E, H, D, \) and \( B \) by using the connection coefficients in (3+1)-form as given in Section 3.

The first Bianchi equation with \( [\lambda\mu] = [0j] \) has the form,

\[ \nabla_0R_{hk,0j} + \nabla_kR_{0h,0j} - \nabla_hR_{0k,0j} = 0. \]  

(11.25)

A calculation incorporating previous definitions, then grouping derivatives using \( \partial_0 \) and \( \nabla_i \), gives to the first pair of Bianchi equations, with \( [\lambda\mu] = [0j] \), the following forms:

\[ \partial_0E_{ij} - N\eta^h_{ij}\nabla_hH_{ij} + (L_2)_{ij} = J_{0ji}, \]  

(11.26)
where $J$ is zero in vacuum and

$$
\dot{\eta}^i_{hk} H_{ij} + 2N \nabla[h E_k]_j + (L_1)_{hk,j} = 0,
$$

with,

$$(L_2)_{ij} \equiv -N(\text{tr} \mathbf{K}) E_{ij} + NK^k_j E_{ik} + 2NK^k_i E_{kj}$$

$$- (\nabla_h N) \eta^{kl} h_{lj} + NK^k_h \eta^{kl} E_{ik} + 2NK^k_i \eta^{lk} E_{kj} + (\nabla k N) \eta^l_{kj} B_{il},$$

$$(L_1)_{hk,j} \equiv NK^l_j \eta^i_{hk} H_{il} + 2(\nabla[h N] E_k)_{lj} + 2N\eta^i_{lj} K^k_{[k} B_{h]}$$

$$- (\nabla_l N) \eta^i_{hk} \eta^{lj} D_{lm}.$$

We see that the non-principal terms $L_1$ and $L_2$ are linear in $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$, with coefficients linear in the geometrical elements $\mathbf{K}$ and $\nabla N$. The characteristic matrix of the principal terms is symmetrizable. The unknowns $E_{i(j)}$ and $H_{i(j)}$, with fixed $j$ and $i = 1,2,3$ appear only in the equations with given $j$. The other unknowns appear in non-principal terms. The characteristic matrix is composed of three blocks around the diagonal, each corresponding to one given $j$.

The $j^{th}$ block of the characteristic matrix in an orthonormal frame for the space metric $\bar{g}$, with unknowns listed horizontally and equations listed vertically ($j$ is suppressed), is given by,

$$
\begin{pmatrix}
\xi_0 & 0 & 0 & 0 & N\xi_3 & -N\xi_2 \\
0 & \xi_0 & 0 & -N\xi_3 & 0 & N\xi_1 \\
0 & 0 & \xi_0 & N\xi_2 & -N\xi_1 & 0 \\
0 & -N\xi_3 & N\xi_2 & \xi_0 & 0 & 0 \\
N\xi_3 & 0 & -N\xi_1 & 0 & \xi_0 & 0 \\
-N\xi_2 & N\xi_1 & 0 & 0 & 0 & \xi_0
\end{pmatrix}.
$$

This matrix is symmetric and its determinant is the characteristic polynomial of the $\mathbf{E}$, $\mathbf{H}$ system. It is given by,

$$-N^6(\xi_0 \xi^2_0)(\xi_0 \xi^2_0)^2.$$

The characteristic matrix is symmetric in an orthonormal space frame and the coefficient matrix $M^t$ is positive definite (it is the unit matrix). Therefore, the first order system is symmetrizable hyperbolic with respect to the space sections $M_t$. We do not have to compute the symmetrized form explicitly because one can obtain energy estimates directly by using the contravariant associates $E^{ij}$, $H^{ij}$, ... of the unknowns.
The second pair of Bianchi equations, for \( D_{ij} \) and \( B_{ij} \), obtained for \([\lambda \mu] = [lm]\) is analogous. The characteristic matrix for the \([lm]\) equations, with unknowns \( D_{ij} \) and \( B_{ij} \), \( j \) fixed, with an orthonormal space frame, is the same as the matrix found above.

If the spacetime metric \( g \) is considered as given, as well as the sources, the Bianchi equations form a linear symmetric hyperbolic system with domain of dependence determined by the light cone of \( g \). The coefficients of the terms of order zero are \( \nabla N \) or \( NK \). The system is homogeneous in vacuum (zero sources).

### 11.4 FOSH system for \( u \equiv (E, H, D, B, \bar{g}, K, \bar{\Gamma}) \)

The Bianchi equations depend on the metric. Our problem is to find a system for determining the metric from the Riemann tensor (through eventually other auxiliary unknowns), which together with the Bianchi equations, constitute a well-posed system. It is possible to construct a FOSH system linking the metric and the connection to our Bianchi field (Anderson, C.B and York 1997), if we again densitize the lapse, i.e., set \( N = \alpha(\det \bar{g})^{1/2} \). This system is inspired by an analogous one constructed in conjunction with the Weyl tensor by H. Friedrich 1996.

### 11.5 Elliptic - hyperbolic system

Instead of determining the metric from the curvature through hyperbolic equations, one can try to do so by elliptic equations on space slices. Well-posed problems for such equations are essentially global ones and depend on the global geometric properties of the space manifolds.

#### 11.5.1 Determination of \( K \)

We deduce from the identities (3.17), which read,

\[
N^{-1}R_{0i,jk} \equiv \nabla_j K_{ki} - \nabla_k K_{ji},
\]

and the Ricci identity, that

\[
\tilde{\nabla}^j (N^{-1}R_{0i,jk}) \equiv \tilde{\nabla}^j \nabla_j K_{ki} - \tilde{\nabla}_k \tilde{\nabla}^j K_{ji} - \tilde{R}^j_{...kjh}K_{hi} - \tilde{R}^j_{...khi}K_{ji}.
\]

As a gauge condition we now suppose that \( H \equiv K^i_h \) is a given function \( h \) on the spacetime, \( N \) then satisfies on each slice the equation deduced from the Einstein equation \( R_{00} = 0 \):

\[
\Delta N - K^{ij} K_{ij}N = -\partial_0 h
\]
The use of the momentum constraint on each slice gives for $K$ (when $N$ and the Riemann tensor of spacetime are known) a quasi-diagonal semilinear system, elliptic if $\bar{g}$ is properly Riemannian, namely,
\[
\bar{\nabla}_j \bar{\nabla}_j K_{ki} - \bar{R}^j_{...kjh} K_{hi} - \bar{R}^j_{...khj} K_{ji} = \bar{\nabla}_h \partial_i h + \bar{\nabla}^j (N^{-1} R_{0ijhk}),
\]
(11.37)
where (cf. (3.16)),
\[
\bar{R}_{ij,kl} = R_{ij,kl} - K_{ik} K_{lj} + K_{il} K_{kj}.
\]
(11.38)

The global solvability of the equation for $K$ can be proved under some conditions, for instance in a neighborhood of Euclidean space, perhaps also for a Robertson-Walker spacetime with compact space of negative curvature.

### 11.5.2 Determination of $\bar{g}$

The equation $\hat{\partial}_0 \bar{g}_{ij} = -2N K_{ij}$ determines as before $\bar{g}$ when $N, \beta$ and $K$ are known. However it does not improve the regularity on $\mathcal{M}_t$ of $\bar{g}$ over the regularity of $K$.

A better result can be sought through the identity which gives the Ricci tensor of $\bar{g}$ in terms of the Riemann tensor of spacetime and $K$ by,
\[
\bar{R}_{ij} \equiv \bar{g}^{hk} R_{ih,jk} + H K_{ij} - K^h_i K_{jh}.
\]
(11.39)

Various methods have been devised to determine a Riemannian metric from its Ricci tensor by elliptic equations (see in particular Andersson and Moncrief, to appear, for the case of compact manifolds with negative curvature).

### 12 Bel-Robinson energy

The new formulation brings nothing new for the local existence and uniqueness theorem. It is useful in obtaining geometrical energy estimates (Bel-Robinson energy) leading possibly to global existence theorems. Such estimates have been used by Christodoulou and Klainerman 1989 in the case of asymptotically Euclidean manifolds. They are used by Andersson and Moncrief (to appear) for compact manifolds with negative curvature.
12.1 Bel-Robinson energy in a strip

Multiply (11.27) by $\frac{1}{2} \eta^{hk} H^{ij}$, and recall that $\eta^{hk} \eta^{i}_{hk} = 2 \delta^{i}_{t}$, $\eta_{lrk} \eta^{ihk} = \delta^{i}_{r} \delta^{h}_{t}$, and $\hat{\partial}_{0} g^{ij} = 2 N K^{ij}$. Then we find that,

$$\frac{1}{2} \eta^{hk} H^{ij} \hat{\partial}_{0} (\eta^{i}_{hk} H_{ij}) = \frac{1}{2} \hat{\partial}_{0} (H_{ij} H^{ij}) - M_{1}, $$

$$M_{1} \equiv \frac{1}{4} \eta^{i}_{rs} H_{lm} \eta^{i}_{hk} H_{ij} \hat{\partial}_{0} (g^{hr} g^{ks} g^{jm})$$

$$= N (K_{h}^{i} H^{ij} - K^{i}_{t} H^{lj} + K^{j}_{i} H^{ti}) H_{ij}. $$

Likewise, multiply (11.26) by $E^{ij}$ to obtain,

$$E^{ij} \hat{\partial}_{0} E_{ij} = \frac{1}{2} \hat{\partial}_{0} (E_{ij} E^{ij}) - M_{2}, $$

$$M_{2} \equiv N (K^{i}_{t} E^{ij} + K^{j}_{i} E^{ti}) E_{ij}. $$

Multiplication by appropriate factors (cf. Anderson, C.B and York 1997) of the second pair of Bianchi equations leads to analogous results. The sum of the expressions so obtained from the four Bianchi equations gives an expression where the spatial derivatives add to form an exact spatial divergence, just as for all symmetric systems. Indeed, we obtain,

$$\frac{1}{2} \hat{\partial}_{0} \left(|E|^{2} + |H|^{2} + |D|^{2} + |B|^{2}\right) + \nabla_{h} (NE^{ij} \eta^{ih}_{j} H_{ij})$$

$$- \nabla_{h} (NB^{ij} \eta^{jh}_{i} D_{ij}) = Q(E, H, D, B) + S, $$

where we have denoted by $|\cdot|$ the pointwise $g$ norm of a space tensor, and where $Q$ is a quadratic form with coefficients $\nabla N$ and $K$. The source term $S$, zero in vacuum, is,

$$S \equiv J_{0ij} E^{ij} - \frac{1}{2} N J_{lm} \eta^{lm} B^{th}. $$

We define the Bel-Robinson energy at time $t$ of the field $(E, H, D, B)$, called a ‘Bianchi field’ when it satisfies the Bianchi equations, to be the integral,

$$\mathcal{E}(t) \equiv \frac{1}{2} \int_{\mathcal{M}_{t}} (|E|^{2} + |H|^{2} + |D|^{2} + |B|^{2}) \mu_{g^{t}}. $$

We will prove the following.

**Theorem 12.1** Suppose that $g$ is $C^{1}$ on $\mathcal{M} \times [0, T]$ and that the $\bar{g}$ norms of $\nabla N$ and $K$ are uniformly bounded on $\mathcal{M}_{t}$, $t \in [0, T]$. Denote by $\pi(t)$ the supremum

$$\pi(t) = \text{Sup}_{\mathcal{M}_{t}} (|\nabla N| + |K|). $$

(12.8)
Suppose the matter source $J \in L^1([0, T], L^2(M_t))$, then the Bel energy of a $C^1$ Bianchi field with compact support in space satisfies for $0 \leq t \leq T$ the following inequality,

$$B(t)^{1/2} \leq (B(0)^{1/2} + \frac{C}{2} \int_0^t \|J\|_{L^2(M, r)} d\tau) \exp\left(C \int_0^t \pi(\tau) d\tau\right),$$

(12.9)

where $C$ is a given positive number.

**Proof.** We integrate the identity (12.5) above on the strip $M \times [0, t]$ with respect to the volume element $\mu_{\bar{g}} d\tau$. If the Bianchi field has support compact in space the integral of the space divergence term vanishes. The integration of $\partial_0 f$ on a strip of spacetime with respect to the volume form $d\tau \mu_{\bar{g}}$ goes as follows, for an arbitrary function $f$,

$$\int_0^t \int_{M_t} \partial_0 f \mu_{\bar{g}} d\tau = \int_0^t \int_{M_t} (\partial_t - \beta^i \partial_i) f \mu_{\bar{g}} d\tau.$$

It holds that,

$$\int_{M_t} \partial_0 f \mu_{\bar{g}} = \int_{M_t} \{\partial_t (f \mu_{\bar{g}}) - f \partial_t \mu_{\bar{g}} - \tilde{\nabla}_i (\beta^i f) - f \tilde{\nabla}_i \beta^i\} \mu_{\bar{g}}.$$

(12.10)

Using the expression for the derivative of a determinant and the relation between $\bar{g}$ and $K$, we find that,

$$\partial_t \mu_{\bar{g}} = (-N \text{tr} K + \tilde{\nabla}_i \beta^i) \mu_{\bar{g}}.$$

(12.11)

and therefore if $f$ has compact support in space,

$$\int_{M_t} \partial_0 f \mu_{\bar{g}} = \partial_t \int_{M_t} f \mu_{\bar{g}} + \int_{M_t} f N \text{tr} K \mu_{\bar{g}}.$$

(12.12)

The integration on a strip leads therefore to the equality,

$$B(t) = B(0) + \int_0^t \int_{M_t} (\tilde{Q} + S) \mu_{\bar{g}} d\tau,$$

(12.13)

with

$$\tilde{Q} = Q + \frac{1}{2} N \text{tr} K (|E|^2 + |H|^2 + |D|^2 + |B|^2).$$

We deduce from this equality, and the expression indicated in (12.5) for $Q$, the following inequality, with $C$ some number,

$$B(t) \leq B(0) + C \left\{ \int_0^t \sup_{M_t} (|\tilde{\nabla} N| + |K|) B(\tau) + \|J\|_{L^2(M, r)} B(t)^{1/2} \right\} d\tau.$$

(12.14)

This inequality and the resolution of the corresponding equality imply the result. □

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2 One can take advantage of the decomposition of $\tilde{Q}$ to obtain better estimates in the case $\text{Tr} K \leq 0$. 

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Remark 12.2 The quantities $-\nabla_k N$ and $2K_{ij}$ are respectively the $(0k)$ and $(ij)$ components of the Lie derivative $L_n g$ of the spacetime metric $g$ with respect to the unit normal $n$ to $M_t$ (its (00) component is identically zero). The Bel-Robinson energy is therefore conserved if this Lie derivative is zero.

The estimate of the Bel-Robinson energy is only an intermediate step in global existence proofs since it depends on the metric which itself depends on curvature.

12.2 Local energy estimate

We take as a domain $\Omega$ of spacetime the closure of a connected open set whose boundary $\partial \Omega$ consists of three parts: A domain $\omega_t$ of $M_t$, a domain $\omega_0$ of $M_0$, and a lateral boundary $L$. We assume $L$ is spacelike or null and ‘ingoing’, that is timelike lines entering $\Omega$ at a point of $L$ are past-directed. We also assume that the boundary $\partial \Omega$ is regular in the sense of Stokes formula. We use the identity previously found and integrate this identity on $\Omega$ with respect to the volume form $\mu_{\bar{g}} d\tau$. It can be proved that the integral on $L$ resulting from the application of Stokes formula is nonnegative. The Bel-Robinson energy on $\omega_t$ satisfies therefore the same type of inequality as found before on $M_t$. In particular, we have $B(\omega_t) = 0$, if $B(\omega_0) = 0$ and $J = 0$ (vacuum case). Then $E = H = D = B = 0$ in $\Omega$ if they vanish on the intersection of $M_0$ with the past of $\Omega$ (result found by York 1987). Note that such a result is not sufficient to prove the propagation of gravitation with the speed of light because it treats only curvature tensors that are zero in some domain, not the difference of nonzero curvature tensors. The Bianchi equations are not by themselves sufficient to estimate such differences because their coefficients depend on the metric, which itself depends on the curvature.

13 (n+1)-splitting in a time-adapted frame

13.1 Metric and coframe

We choose the time axis to be tangent to the time lines, i.e., the cobasis $\theta$ is such that $\theta^i$ does not contain $dx^0$. We set,

$$\theta^i = a^i_j dx^j, \quad \theta^0 = Ud^0 + b_i dx^i. \quad (13.1)$$

We will call such a frame a CF- (Cattaneo-Ferrarese) frame. The Pfaff derivatives $\partial_\alpha$ in the CF-frame are linked to the partial derivatives $\partial/\partial x^\alpha$ by the relations,

$$\partial_0 = U^{-1} \frac{\partial}{\partial x^0}, \quad \partial_i = A^j_i \left[ \frac{\partial}{\partial x^j} - U^{-1} b_j \frac{\partial}{\partial x^0} \right].$$
with $A^i_j$ the matrix inverse of $a^i_j$. The structure coefficients of the coframe are found to be,

$$c^0_{0i} = N^{-1} \partial_i N - A^i_j \partial_0 b_j = -c^0_{i0},$$

and with $f_{[ij]} \equiv f_{ij} - f_{ji}$,

$$c^0_{ij} = U A^h_i \partial_j (N^{-1} b_h), \quad c^i_{00} = A^i_j \partial_0 a^j_i, \quad c^i_{hk} = A^j_{[i} \partial_{j]} a^j_i.$$

**Remark 13.1** If the time lines are not hypersurface orthogonal (i.e., if $b_i \neq 0$), the coefficients $c^i_{ij}$ are different from the structure coefficients of the space frame $\theta^i$.

Choosing the frame to be orthonormal the metric reads,

$$g = - (\theta^0)^2 + \sum_{i=1}^{3} (\theta^i)^2. \quad (13.2)$$

### 13.2 Splitting of connection

We deduce from the general formulas,

$$\omega^0_{00} = \omega^0_{i0} = 0,$$

$$Y_i \equiv \omega_{00,i} = \omega^0_{0i} = \omega^i_{00} = -c_{0i,0} = c^0_{0i} = U^{-1} \partial_i U - A^i_j \partial_0 b_j,$$

and we know that $\omega^i_{0i} = \omega_{0i,j}$ is antisymmetric in $i$ and $j$. We set,

$$\omega_{0i,j} \equiv f_{ij} = \frac{1}{2} \{A^h_i \partial_0 a^j_h - A^h_j \partial_0 a^i_h + A^h_{[i} \partial_{j]} b_h \}.$$ 

Let $e_\alpha \equiv \partial_\alpha$ be the frame dual to $\theta^\alpha$, i.e., such that the vector $e_\alpha$ has components $\delta^\lambda_\alpha$. Then,

$$\nabla_\beta e^\lambda_\alpha = \omega^\lambda_{\beta\alpha},$$

in particular, $\omega_{0i,j}$ is the projection on $e_{(j)}$ of the derivative of $e_{(i)}$ in the direction of $e_{(0)}$. We have $f_{ij} = 0$ if the frame is Fermi-transported along the time line. We will make this hypothesis to simplify the formulas.

The connection coefficient $\omega_{i0,j}$ is the sum of a term symmetric in $i$ and $j$ and an antisymmetric one and therefore we have,

$$X_{ij} \equiv \omega_{i0,j} = \omega^j_{i0} = \frac{1}{2} \{A^h_j \partial_0 a^i_h + A^h_i \partial_0 a^j_h + A^h_{[i} \partial_{j]} b_h \}.$$ 

The antisymmetric term vanishes if the time lines are hypersurface orthogonal ($b_i = 0$).

The coefficients $\omega^h_{ij}$ are also linear expressions in terms of the first derivatives of the frame coefficients, they are identical to the connection constructed with the $a^i_j$ at fixed $t$ only if $b_i = 0$. 27
13.3 Splitting of curvature

Using the general formulas we find in the chosen frame,
\[ R^{\iota\iota\iota}_{0h...}j \equiv \partial_{0}^{\omega_{hj}} + X^{\iota\iota\iota}_{h} + Y^{i}X^{\iota\iota\iota}_{hj} - Y_{j}X^{i}_{h} \]  \hspace{1cm} (13.3)

we denote by \( \tilde{\nabla} \) is the pseudo-covariant derivative constructed with \( \partial_{i} \) and \( \omega_{ij}^{h} \) (Cataneo-Ferrarese transversal derivative). We have,
\[ R^{\iota\iota\iota}_{hj...}0 \equiv \tilde{\nabla}_{h}Y^{i} - \partial_{0}X^{\iota\iota\iota}_{h} - X^{\iota\iota\iota}_{h}X^{\iota\iota\iota}_{j} \]  \hspace{1cm} (13.4)
\[ R^{\iota\iota\iota}_{hk...}j \equiv \tilde{R}^{\iota\iota\iota}_{hk...}j + X^{\iota\iota\iota}_{h}X^{\iota\iota\iota}_{j} - X^{\iota\iota\iota}_{jk}X^{\iota\iota\iota}_{h}, \]  \hspace{1cm} (13.5)

where \( \tilde{R}^{\iota\iota\iota}_{hk...}j \) denotes the expression formally constructed as a Riemann tensor with the coefficients \( \omega_{ij}^{h} \) and,
\[ R^{\iota\iota\iota}_{kh...}0 \equiv \tilde{\nabla}_{k}X^{\iota\iota\iota}_{hj} - \tilde{\nabla}_{h}X^{\iota\iota\iota}_{jk} - Y_{j}(X^{\iota\iota\iota}_{kh} - X^{\iota\iota\iota}_{hk}). \]  \hspace{1cm} (13.6)

\textbf{Remark 13.2} The symmetry \( R_{kh,0j} = R_{0j,kh} \) results from the expression of the connection in terms of frame coefficients.

We deduce from the splitting of the Riemann tensor the following identities:
\[ R_{00} \equiv \tilde{\nabla}_{i}Y^{i} - \partial_{0}X^{i}_{i} - X^{\iota\iota\iota}_{h}X^{\iota\iota\iota}_{j}, \]  \hspace{1cm} (13.7)
\[ R_{h0} \equiv \tilde{\nabla}_{j}X^{\iota\iota\iota}_{hj} - \tilde{\nabla}_{h}X^{\iota\iota\iota}_{j} - Y^{j}(X^{\iota\iota\iota}_{jh} - X^{\iota\iota\iota}_{hj}). \]

13.4 Bianchi equations (case n=3)

13.4.1 Bianchi quasi constraints

The Bianchi identities and their contraction contain, as in a Cauchy adapted frame, equations which do not contain the derivative \( \partial_{0} \) of the Riemann tensor, namely,
\[ \nabla_{i}R_{jh,\lambda\mu} + \nabla_{h}R_{ij,\lambda\mu} + \nabla_{j}R_{hi,\lambda\mu} \equiv 0, \]  \hspace{1cm} (13.8)
\[ \nabla_{\alpha}R^{\alpha}_{0,\lambda\mu} = -\nabla_{\mu}\rho_{\lambda0} + \nabla_{\lambda}\rho_{\mu0}. \]  \hspace{1cm} (13.9)

We call these equations Bianchi quasi-constraints.
13.4.2 Bianchi evolution system

The remaining Bianchi equations can be written, as in the case of a Cauchy adapted frame, as a FOS (first order symmetric) system for two pairs of ‘electric’ and ‘magnetic’ 2-tensors. This system cannot be said to be hyperbolic in the usual sense: The principal matrix $M^0$ is the unit matrix hence positive definite but the operators $\partial/\partial t$ appears also in the matrices $M^i$. We say that the system is a quasi-FOSH system. It is a usual FOSH system with $t$ as a time variable if the matrix of coefficients of $\partial/\partial t$ is positive definite. It can be proved that this is the case if the metric induced on the $t = \text{constant}$ submanifolds,

$$\bar{g}_{ij} = \sum_h a^h_i a^h_j - b_i b_j,$$  \hspace{1cm} (13.10)

is positive definite and $U > 0$.

13.4.3 Quasi-FOSH system for connection and frame

When the Riemann tensor is known the identities which express it become equations for the connection. Some of them do not contain the derivative $\partial_0$, we call them connection quasi-constraints. Identities linking connection and frame become first order equations for the frame coefficients. No equation gives the evolution of $U$. It can be considered as a gauge variable fixing the time parameter.

13.5 Vacuum case

In vacuum we give arbitrarily on the spacetime $\mathcal{V}$ the scalar $U$, length of the tangent vector $\partial/\partial i$ to the time line, together with the projection $Y_i$ of $\nabla_{e_0} e_0$ on $e_i$. The quantities $f^i_j$ being chosen zero, the identities previously written give, when the Riemann tensor is known, equations with principal operator the dragging along the time lines of $\omega^i_{hj}$ and $X_{ij}$, and when the connection is known equations for the dragging of the frame coefficients. These equations together with the Bianchi evolution equations constitute a quasi-FOSH system. This system is a FOSH system with respect to $t$ as long as $\bar{g}_{ij}$ is positive definite.

13.6 Perfect fluid

In the presence of fluid sources one can obtain a quasi-FOSH system for the gravitational and fluid variables by taking as time lines the flow lines and proceeding as follows (Friedrich 1998).
13.6.1 Fluid equations

The stress energy tensor of a perfect fluid is,

\[ T_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + p g_{\alpha\beta}, \]

Then,

\[ \rho_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + \frac{1}{2}(\mu - p)g_{\alpha\beta}, \]

and one supposes that the matter energy density \( \mu \) is a given function of the pressure \( p \).

The Euler equations of the fluid, which express the generalized conservation law \( \nabla_\alpha T^{\alpha\beta} = 0 \), are equivalent to the equations,

\[ (\mu + p)u^\alpha \nabla_\alpha u^\beta + (u^\alpha u^\beta + g^{\alpha\beta})\partial_\alpha p = 0, \quad \text{with} \quad u^\alpha u_\alpha = -1, \]

and,

\[ (\mu + p)\nabla_\alpha u^\alpha + u^\alpha \partial_\alpha \mu = 0. \]

In our coframe they read,

\[ (\mu + p)Y_i + \partial_i p = 0, \quad Y_i \equiv \omega^i_{00}, \]

\[ \partial_0 \mu + (\mu + p)X^i_i = 0. \]  \[ (13.11) \]

Using the index \( F \) of the fluid defined by,

\[ F(p) = \int \frac{dp}{\mu(p) + p}, \]

we have,

\[ Y_i = -\partial_i F, \]  \[ (13.12) \]

and,

\[ \mu^i_j \partial_0 F + X^i_i = 0. \]  \[ (13.13) \]

The commutation relation between Pfaff derivatives and the definitions give that,

\[ (\partial_0 \partial_i - \partial_i \partial_0)F = c^\alpha_{0i} \partial_\alpha F = Y_i \partial_0 F - X^j_i \partial_j F, \]
and therefore,
\[
\mu'_p \left[ \partial_0 Y_i + Y_i \partial_0 F + (f^i_j - X^i_j) \partial_j F \right] - \partial_i \mu'_p \partial_0 F - \partial_i X^h_i = 0.
\] (13.14)

The use of previous identities replaces \( \partial_\alpha F \) by functions of \( Y, X \) and \( p \). The derivatives \( \partial_\alpha \mu'_p \) are functions of \( Y \) and \( p \) since.

\[
\partial_i \mu'_p = \mu'_p \partial_i p.
\]

Following H. Friedrich, we replace \( \partial_i X_h^i \) by its expression deduced from the equation,
\[
R_{i0} \equiv \nabla_h X^i_i - \nabla_i X^h_i - Y^h (X_{hi} - X_{ih}),
\]
and changing names of indices we obtain,
\[
\mu'_p \partial_0 Y_h - \nabla_j X^j_h - Y^h (X_{hi} - X_{ih}) + Y_h \partial_0 F +
- X^j_h \partial_j F \partial_0 F + \partial_h \mu'_p \partial_0 F = 0.
\] (13.15)

Replacing \( \nabla_h Y_i \) by \( \nabla_i Y_h + c^0_{hi} Y_0 \) where,
\[
Y_0 \equiv - \partial_0 F \equiv - (\mu'_p)^{-1} X^i_i,
\]
\[
\xi^0_{ih} \equiv \omega^0_{hi} - \omega^0_{ih} \equiv - \omega_{hi,0} + \omega_{ih,0} \equiv X_{ih} - X_{hi},
\] (13.16)

we obtain,
\[
\partial_0 X_{ih} - \nabla^i Y_h + (\mu'_p)^{-1} X^i_i (X_{ih} - X_{hi}) - Y_h Y^i + X^j_i X^j_i X_{hi} + X^j_i X^j_i = -R_{hi}^{ij}. \] (13.17)

The principal operator on the unknowns \( Y \) and \( X \) in the above equations is diagonal by blocks and symmetric. The \( h \)-block reads:

\[
\begin{pmatrix}
\mu'_p \partial_0 & -\partial_1 & -\partial_2 & -\partial_3 \\
-\partial_1 & \partial_0 & 0 & 0 \\
-\partial_2 & 0 & \partial_0 & 0 \\
-\partial_3 & 0 & 0 & \partial_0
\end{pmatrix}
\]

If \( \mu'_p > 0 \) the matrix \( M^0 \) is positive definite in the CF-frame. The system is a quasi-FOSH system for the pairs \( Y_h, X^j_h \).

**Remark 13.3** The characteristic determinant associated with the system (13.15), (13.17) is,
\[
\left\{ \xi^2_0 (\mu'_p \xi^2_0 - \sum_{i=1,2,3} \xi^2_i) \right\}^3.
\]

The roots of \( \mu'_p \xi^2_0 - \sum_{i=1,2,3} \xi^2_i = 0 \) correspond to sound waves. Their speed is at most 1 (speed of light) if and only if \( \mu'_p \geq 1 \).
13.6.2 Sources of the Bianchi equations

In our frame the source tensor $\rho$ reduces to,

$$\rho_{00} = \frac{1}{2}(\mu + 3p), \quad \rho_{0i} = 0, \quad \rho_{ij} = \frac{1}{2}(\mu - p)\delta_{ij}.$$  

We have seen that $\partial_\alpha p$ and $\partial_\alpha \mu$ are smooth functions of $p, Y$ and $X$. The same property holds for $J_{i,0j}$ and $J_{i,hk}$.

13.7 Conclusion

Assembling the results of the previous subsections we find the following theorem.

**Theorem 13.4** The Einstein equations with source a perfect fluid give a quasi-FOSH system for the Riemann curvature tensor, the frame and connection coefficients, and the density of matter, when the flow lines are taken as timelines, $U$ and $f^i_j \equiv \omega^i_{0j}$ given arbitrarily.

**Corollary 13.5** The EEF (Einstein-Euler-Friedrich) system is a FOSH system relatively to $t = \text{constant}$ slices as long as the quadratic form,

$$g_{jh} = \sum_{i=1,2,3} a^i_j a^i_h - b_j b_h,$$  \hspace{1cm} (13.18)

is positive definite, $U > 0$ and $\mu'_p \geq 1$.

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