1. Introduction

This paper is concerned with bounding

\[ M(x) = \sum_{n \leq x} \mu(n). \]

J.E. Littlewood [6] proved that if the Riemann Hypothesis (RH) is true then, for any fixed \( \epsilon > 0 \), \( 1/\zeta(1/2 + \epsilon + it) \ll |t|^\epsilon \). It follows by Perron’s formula that

\[ M(x) \ll x^{\frac{1}{2}+\epsilon}. \]

Conversely, the estimate \( M(x) \ll x^{\frac{1}{2}+\epsilon} \) implies, by partial summation, the convergence of the series \( \sum_{n=1}^{\infty} \mu(n)n^{-s} = 1/\zeta(s) \) for any \( \sigma > 1/2 \), and therefore RH. Subsequently, E. Landau [5] showed that, assuming RH, (1) is valid with \( \epsilon \ll \log \log x / \log \log \log x \), and E.C. Titchmarsh [13] improved this to \( \epsilon \ll 1/\log \log x \). H. Maier and H.L. Montgomery [7] obtained a substantial improvement over these results, and established that

\[ M(x) \ll x^{\frac{1}{2}} \exp\left(C (\log x)^{\frac{39}{61}} \right). \]

They comment that the limit of their method would be an exponent in (2) slightly smaller than 39/61. In this paper, we introduce some new ideas which permit the following better result.

**Theorem 1.** Assume RH. For large \( x \) we have

\[ M(x) \ll \sqrt{x} \exp((\log x)^{\frac{1}{2}} (\log \log x)^{14}). \]

The main ingredient in our proof is a result on the frequency with which ordinates of the zeros of \( \zeta(s) \) can cluster in short intervals, which may be of independent interest. Let

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$N(T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with ordinate $\gamma$ lying in $[0, T]$. Recall that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),$$

where $\pi S(T) = \arg \zeta\left(\frac{1}{2} + iT\right)$ and the argument is obtained by continuous variation from 2 (where the argument is zero) to $2 + iT$ to $\frac{1}{2} + iT$. It is easy to show that $S(T) \ll \log T$, and on RH Littlewood established that $S(T) \ll \log T / \log \log T$. Recently D. Goldston and S. Gonek [3] put Littlewood’s bound into the elegant form $|S(T)| \leq \left(\frac{1}{2} + o(1)\right) \log T / \log \log T$.

Building on their work, we quantify here the frequency of large values of $S(t + h) - S(t - h)$; equivalently, the frequency with which the interval $[t - h, t + h]$ contains an unusual number of ordinates of zeros of $\zeta(s)$.

**Theorem 2.** Assume RH. Let $T$ be large, and let $0 \leq h \leq \sqrt{T}$, and $(\log \log T)^2 \leq V \leq \log T / \log \log T$ be given. The number of well-spaced points $T \leq t_1 < t_2 < \ldots < t_R \leq 2T$ with $t_{j+1} - t_j \geq 1$ and such that

$$\left| N(t_j + h) - N(t_j - h) - \frac{h}{\pi} \log \frac{t_j}{2\pi} \right| > V$$

satisfies the bound

$$R \ll T \exp \left( -V \log \frac{V}{\log \log T} + 3V \log \log V \right).$$

In [10, 11] A. Selberg established unconditionally that $\pi S(t)$ has a Gaussian distribution with mean 0 and variance $\frac{1}{2} \log \log T$. This suggests a better bound for $R$ than that furnished by Theorem 2. Namely, perhaps the bound $R \ll T \exp(-CV^2 / \log \log T)$ holds for some absolute positive constant $C$, uniformly in $V$. This is in keeping with the recent conjecture of D.W. Farmer, Gonek and C.P. Hughes [2] that $S(t) \ll \sqrt{\log T \log \log T}$. By adapting the ideas in [12] it would be possible to establish the conjectured bound for $R$ (assuming RH) in the range $V(\log \log T)$ log log log $T$. A more detailed analysis of such results is the focus of my ongoing joint work with Chris Hughes and Nathan Ng.

Using Theorem 2 we shall establish an estimate for the frequency with which small values of $|\zeta(s)|$ are attained. The main result of my paper [12] deals with corresponding estimates for the frequency with which large values of $|\zeta(1/2 + it)|$ are attained. To state our results conveniently we require a definition.

**Definition 3.** Let $T$ be large and let $(\log \log T)^2 \leq V \leq \log T / \log \log T$ be given. We say that a point $t \in [T, 2T]$ is $V$-typical if the following three conditions hold; if one of these criteria fails, we say that the point is $V$-atypical.

(i). Let $x = T^{1/V}$. For all $\sigma \geq \frac{1}{2}$ we have

$$\left| \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma + it} \log n} \frac{\log(x/n)}{\log x} \right| \leq 2V.$$

(ii). Every sub-interval of $(t - 1, t + 1)$ of length $2\pi V / \log T$ contains at most $3V$ ordinates of zeros of $\zeta(s)$.

(iii). Every sub-interval of $(t - 1, t + 1)$ of length $2\pi V / ((\log V) \log T)$ contains at most $V$ ordinates of zeros of $\zeta(s)$. 
Proposition 4. Assume RH. Let $T$ be large. Any point $t \in [T, 2T]$ is $V$-typical provided $V \geq \left( \frac{1}{2} + o(1) \right) \log T / \log \log T$. Given $(\log \log T)^2 \leq V \leq \log T / \log \log T$, the number of well-spaced $V$-atypical points $T \leq t_1 \leq \ldots \leq t_R \leq 2T$ with $t_{j+1} - t_j \geq 1$ satisfies

$$R \ll T \exp \left( -V \log \frac{V}{\log \log T} + 4V \log \log V \right).$$

Proposition 5. Assume RH. Let $T$ be large, and suppose $t \in [T, 2T]$ is $V$-typical for some $(\log \log T)^2 \leq V \leq \log T / \log \log T$. Put $\sigma_0 = \frac{1}{2} + \frac{V}{\log T}$. For $2 \geq \sigma \geq \sigma_0$ we have

$$\log |\zeta(\sigma + it)| \geq -V \log \log V,$$

and for $\frac{1}{2} < \sigma \leq \sigma_0$ we have

$$\log |\zeta(\sigma + it)| \geq -V \log \left( \frac{\sigma_0 - 1/2}{\sigma - 1/2} \right) - 8V \log \log V.$$

We will describe in §5 below how our main result, Theorem 1, follows from Propositions 4 and 5, and a careful application of Perron’s formula. Just as we expect that the true bound for $R$ in Theorem 2 should be much smaller, we may expect a corresponding improvement of Proposition 4. Perhaps the better bound $R \ll T \exp(-CV^2 / \log \log T)$ holds, for some positive constant $C$. If such were the case, then our method would yield $M(x) \ll x^{\frac{1}{4}} \exp(C(\log \log x)^3)$ for some positive constant $C$. Even this is far from the conjectured maximal order of magnitude for $M(x)$: Gonek (unpublished, but see N. Ng [8]) has conjectured that

$$\infty > \limsup_{x \to \infty} \frac{M(x)}{\sqrt{x(\log \log \log x)^5}} > 0 > \liminf_{x \to \infty} \frac{M(x)}{\sqrt{x(\log \log \log x)^5}} > -\infty.$$  

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2. Preliminary Lemmas

We collect here three familiar results that we shall need below. These are Selberg’s construction of good approximations to characteristic functions of intervals, the explicit formula connecting primes and zeros, and a version of the large sieve.

Lemma 6. Let $h > 0$ and $\Delta > 0$ be given. Let $\chi_{[-h,h]}$ denote the characteristic function of the interval $[-h,h]$. There exist even analytic functions $F_-$, and $F_+$ satisfying the following properties.

(i) $F_-(u) \leq \chi_{[-h,h]}(u) \leq F_+(u)$ for real $u$. 

(ii) We have
\[ \int_{-\infty}^{\infty} |F_\pm(u) - \chi_{[-h,h]}(u)| du \leq 1/\Delta. \]

(iii) \( \hat{F}_\pm(x) = 0 \) for \( |x| \geq \Delta \) where \( \hat{F}_\pm(x) = \int_{-\infty}^{\infty} F_\pm(u)e^{-2\pi i xu} du \) denotes the Fourier transform. Also,
\[ \hat{F}_\pm(x) = \frac{\sin(2\pi hx)}{\pi x} + O\left(\frac{1}{\Delta}\right). \]

(iv) If \( z = x + iy \) is a complex number with \( |z| \geq 2h \) then
\[ |F \pm(z)| \ll \frac{e^{2\pi \Delta|y|}}{|\Delta||z|^2}. \]

Proof. Such functions were constructed by Selberg (see [9]), using Beurling’s approximation to the signum function. We give a brief description; for a detailed discussion see J.D. Vaaler [15]. Set \( K(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2 \sum_{n=-\infty}^{\infty} \frac{\text{sgn}(n)}{(z-n)^2 + 2/z} \),
where \( \text{sgn}(x) \) is the sign function taking values 1 for positive \( x \), \(-1\) for negative \( x \), and 0 for \( x = 0 \). Beurling showed that \( H(x) - K(x) \leq \text{sgn}(x) \leq H(x) + K(x) \), and that
\[ \int_{-\infty}^{\infty} |H(x) \pm K(x) - \text{sgn}(x)| dx = 1. \]

The desired functions \( F_\pm \) are given by
\[ F_\pm(z) = \frac{1}{2} \left( H(\Delta(x+h)) \pm K(\Delta(x+h)) + H(\Delta(h-x)) \pm K(\Delta(h-x)) \right). \]

Properties (i)-(iii) are well-known, and it is not difficult to check the bound in (iv).

Lemma 7. Let \( h(s) \) be analytic in the strip \( |\text{Im}(s)| \leq \frac{1}{2} + \epsilon \) for some \( \epsilon > 0 \), taking real values on the real line, and satisfying \( |h(s)| \ll (1 + |s|)^{-1-\delta} \) for some \( \delta > 0 \). Then, with \( \rho = \frac{1}{2} + i\gamma \) denoting the non-trivial zeros of \( \zeta(s) \),
\[ \sum_{\rho} h(\gamma) = h\left(\frac{1}{2}i\right) + h\left(-\frac{1}{2}i\right) + \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u)\left( \text{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) - \log \pi \right) du \]
\[ - \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left( h\left(\frac{\log n}{2\pi}\right) + h\left(-\frac{\log n}{2\pi}\right) \right). \]

Proof. This is the explicit formula; see for example Lemma 1 in [3], or Chapter 5 of H. Iwaniec and E. Kowalski [4].
Lemma 8. Let \( A(s) = \sum_{p \leq N} a(p)p^{-s} \) be a Dirichlet polynomial. Let \( T \) be large and suppose \( s_r = \sigma_r + it_r \) \((r = 1, \ldots, R)\) be points with \( T < t_1 < t_2 < \ldots < t_R \leq 2T \) and \( t_{r+1} - t_r \geq 1 \), and \( \sigma_r \geq \alpha \). For any \( k \) with \( N^k \leq T \) we have

\[
\sum_{r=1}^{R} |A(s_r)|^{2k} \ll T(\log T)^{2k} \left( \sum_{p \leq N} |a(p)|^2 p^{-2\alpha} \right)^k.
\]

Proof. This large sieve type inequality may be found as Lemma 5 in Maier and Montgomery [7].

3. Proof of Theorem 2

We use Lemma 6 to approximate the characteristic function of \([-h, h]\), taking there \( \Delta = (1 + \eta)(\log T)/(2\pi V) \) with \( \eta = 1/\log V \). Let \( F_{\pm} \) denote the functions produced in Lemma 6. We now appeal to the explicit formula, Lemma 7, taking \( h(s) = F_{\pm}(s - t) \) where \( T < t < 2T \). Observe that \( \hat{h}(x) = \hat{F}_{\pm}(x)e^{-2\pi ixt} \). Therefore, the explicit formula gives

\[
\sum_{\rho} F_{\pm}(\gamma - t) = F_{\pm}\left(\frac{1}{2i} - t\right) + F_{\pm}\left(-\frac{1}{2i} - t\right) - \frac{1}{\pi} \operatorname{Re} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{1/2 + it}} \hat{F}_{\pm}\left(\frac{\log n}{2\pi}\right) + \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{\pm}(u) \left( \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{t + u}{2}\right) - \log \pi \right) du.
\]

(5)

Using Stirling’s formula we may readily check that for \( 0 < h \leq \sqrt{T} \) (or see equation (13) of [3])

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} F_{\pm}(u) \left( \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{t + u}{2}\right) - \log \pi \right) du = \frac{1}{2\pi} \log \frac{t}{2\pi} \hat{F}_{\pm}(0) + O(1).
\]

Note that with the + choice of sign the LHS of (5) is at least \( N(t + h) - N(t - h) \), while with the – choice of sign it is at most \( N(t + h) - N(t - h) \). Moreover \( \hat{F}_{+}(0) \leq 2h + 1/\Delta \), and \( \hat{F}_{-}(0) \geq 2h - 1/\Delta \). These observations lead to

\[
\left| N(t + h) - N(t - h) - \frac{h}{\pi} \log \frac{t}{2\pi} \right| \leq \frac{1}{\Delta} \frac{\log T}{2\pi} + \max_{\pm} \left( \left| F_{\pm}\left(\frac{1}{2i} - t\right)\right| + \left| F_{\pm}\left(-\frac{1}{2i} - t\right)\right| \right) + \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{1/2 + it}} \hat{F}_{\pm}\left(\frac{\log n}{2\pi}\right) + O(1).
\]

(6)

Now \( \log T/(2\pi \Delta) = V (1 - \eta + O(\eta^2)) \) and by part (iv) of Lemma 6 the contribution of \( F_{\pm}(\pm 1/2i - t) \) terms is \( \ll T^{-1} \). Therefore if the LHS of (6) exceeds \( V \) then we must have

\[
\max_{\pm} \left| \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{1/2 + it}} \hat{F}_{\pm}\left(\frac{\log n}{2\pi}\right) \right| \geq 2\eta V.
\]
Since $\hat{F}_\pm(x) = 0$ for $|x| \geq \Delta$ the sums above may be restricted to $n \leq \exp(2\pi \Delta) = T^{(1+\eta)/V}$. Moreover, the contribution of prime cubes and higher powers is $O(1)$. Thus we have either

$$\max_{\pm} \left| \sum_{p \leq T^{(1+\eta)/V}} \frac{\log p}{p^\pm \pi} \hat{F}_\pm \left( \frac{\log p}{2\pi} \right) \right| \geq \eta V, \quad \text{or} \quad \max_{\pm} \left| \sum_{p \leq T^{(1+\eta)/2V}} \frac{\log p}{p^{1+2it}} \hat{F}_\pm \left( \frac{\log p}{\pi} \right) \right| \geq \eta V.$$

We conclude, for our sequence of $R$ well-spaced points $t_j$, and any positive integer $k$ that

$$R(\eta V)^{2k} \leq \sum_{j=1}^{R} \sum_{\pm} \left| \sum_{p \leq T^{(1+\eta)/V}} \frac{\log p}{p^\pm \pi} \hat{F}_\pm \left( \frac{\log p}{2\pi} \right) \right|^{2k} + \left| \sum_{p \leq T^{(1+\eta)/2V}} \frac{\log p}{p^{1+2it}} \hat{F}_\pm \left( \frac{\log p}{\pi} \right) \right|^{2k}.$$ 

Suppose that $k \leq V/(1+\eta)$, so that Lemma 8 applies. In that case we obtain that

$$R(\eta V)^{2k} \ll T(\log T)^2 k^k \left( \sum_{p \leq T^{(1+\eta)/V}} \frac{\log^2 p}{p^\pm} \hat{F}_\pm \left( \frac{\log p}{2\pi} \right)^2 \right)^k + \left( \sum_{p \leq T^{(1+\eta)/2V}} \frac{\log^2 p}{p^{2\pi}} \hat{F}_\pm \left( \frac{\log p}{\pi} \right)^2 \right)^k.$$ 

Using (iii) of Lemma 6 we conclude that the above is

$$\ll T(\log T)^2 (Ck \log \log T)^k,$$

for some positive constant $C$. Hence

$$R \ll T(\log T)^2 \left( \frac{Ck \log \log T}{\eta^2 V^2} \right)^k,$$

and the Theorem follows upon recalling that $\eta = 1/\log V$, and taking the largest permissible value for $k$, namely $[V/(1+\eta)]$.

4. LOWER BOUNDS FOR $|\zeta(s)|$: PROOF OF PROPOSITIONS 4 AND 5

Proof of Proposition 4. If $V \geq (\frac{1}{2} + \epsilon) \log T/\log \log T$ then $x = T^{1/V} \leq (\log T)^{2-\epsilon}$ so that criterion (i) of Definition 3 is met. Moreover, Goldston and Gonek’s estimate (see Theorem 1 of [3]) that for large $t$ and $0 < h \leq \sqrt{T}$ one has $|N(t+h) - N(t) - \frac{h}{2\pi} \log \frac{t}{2\pi}| \leq (\frac{1}{2} + o(1)) \log t/\log \log t$, readily shows that criteria (ii) and (iii) are also met. Therefore $t$ is $V$-typical for $V \geq (\frac{1}{2} + o(1)) \log T/\log \log T$.

We now obtain the bound for the number $R$ of well-spaced $V$-atypical points. If a point is $V$-atypical then one of the criteria (i), (ii), or (iii) must be violated. Appealing to Lemma 8 we may show (arguing exactly as in our proof of Theorem 2 above) that the number of well-spaced points for which condition (i) fails is $\ll T(\log T)^2 \exp(-2 + o(1)) V \log(V/\log \log T))$. Theorem 2 shows that the number of well-spaced points for
which (ii) fails is \( \ll T \exp(-(2 + o(1))V \log(V \log \log T)) \) as well. Theorem 2 also shows that the number of well-spaced points for which condition (iii) fails is
\[
\ll T \exp \left( -V \log \frac{V}{\log \log T} + 4V \log \log V \right).
\]
Hence the bound for \( R \) claimed in Proposition 4 follows.

**Proof of Proposition 5.** Suppose that \( t \) is \( V \)-typical, so that conditions (i)-(iii) of Definition 3 hold. We must now establish the estimates (3) and (4). For \( s = \sigma + it \) we write
\[
F(s) = \sum_{\rho} \Re \frac{1}{s - \rho} = \sum_{\rho} \frac{(\sigma - 1/2)}{(\sigma - 1/2)^2 + (t - \gamma)^2}.
\]
By Stirling’s formula and Hadamard factorization we have (see (2.12.7) of Titchmarsh [14], or Chapter 12 of H. Davenport [1])
\[
\Re \frac{\zeta'}{\zeta}(s) = F(s) - \frac{1}{2} \log T + O(1).
\]

**Lemma 9.** Let \( T \leq t \leq 2T \) be \( V \)-typical. For \( \frac{1}{2} < \sigma \leq \sigma_0 = \frac{1}{2} + \frac{V}{\log T} \), we have
\[
\log |\zeta(\sigma + it)| \geq \log |\zeta(\sigma_0 + it)| - V \log \frac{(\sigma_0 - \frac{1}{2})}{(\sigma - \frac{1}{2})} - 7V \log \log V.
\]

**Proof.** Using (7) we see that
\[
\log |\zeta(\sigma_0 + it)| - \log |\zeta(\sigma + it)| = \int_{\sigma}^{\sigma_0} \Re \frac{\zeta'}{\zeta}(u + it)du \leq \int_{\sigma}^{\sigma_0} F(u + it)du
\]
\[
= \frac{1}{2} \sum_{\gamma} \log \frac{(\sigma_0 - \frac{1}{2})^2 + (t - \gamma)^2}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2}.
\]

We split the sum over \( \gamma \) into various intervals. First we have the range where \(|t - \gamma|\) is below \( 2\pi V/((\log V) \log T) \). Second we have the intervals \( 2\pi(n + 1/\log V) V/\log T \leq |t - \gamma| \leq 2\pi(n + 1 + 1/\log V) V/\log T \) for \( 0 \leq n \leq N = [(\log T)/(4\pi V)] \). Finally there is the range \(|t - \gamma| > 2\pi(N + 1 + 1/\log V) V/\log T \). Using condition (iii) of Definition 3, we see that the first range contributes to (8) an amount \( \leq V \log((\sigma_0 - \frac{1}{2})/(\sigma - \frac{1}{2})) \). In the second range we use condition (ii) of Definition 3, and conclude that the contribution of such terms to (8) is
\[
\leq 3V \sum_{n=0}^{N} \log 1 + \left( \frac{n + 1/\log V}{n + 1/\log V} \right)^2 \leq 6V \log \log V + 10V.
\]

Splitting into intervals of length 1, we see easily that the final range contributes
\[
\leq \frac{1}{2} \sum_{|t - \gamma| > 1/2} \frac{(\sigma_0 - \frac{1}{2})^2}{|t - \gamma|^2} = o(V).
\]

Putting everything together we obtain the Lemma.

From Lemma 9, estimate (4) would follow once (3) is established. In other words, we now need to deal with \( \sigma \geq \sigma_0 \). For this we need the following Lemma.
Lemma 10. Let $t$ be large and let $T \leq t \leq 2T$. Uniformly for $\frac{1}{2} < \sigma \leq 2$, and $2 \leq x \leq T$ we have
\[
\log |\zeta(\sigma + it)| \geq \text{Re} \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma+it} \log n} \frac{\log(x/n)}{\log x} - \left(1 + \frac{x^{\frac{1}{2}-\sigma}}{(\sigma - \frac{1}{2}) \log x}\right) F(\sigma + it) \log x + O(1).
\]

Proof. Let $z$ have imaginary part $t$ and real part lying in $(\frac{1}{2}, 2]$. Consider, for $c > \frac{1}{2}$
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{-\zeta'(z + w)}{\zeta(z + w)} x^w w^2 dw = \sum_{n \leq x} \frac{\Lambda(n)}{n^z} \log(x/n),
\]
upon integrating term by term using the Dirichlet series expansion of $-\frac{\zeta'}{\zeta}(z + w)$. On the other hand, moving the line of integration to the left and calculating residues this equals
\[
-\frac{\zeta'}{\zeta}(z) \log x - \left(\frac{\zeta'}{\zeta}(z)\right)' - \sum_{\rho} \frac{x^{\rho-z}}{\rho(z - \rho)} + O\left(\frac{1}{T}\right).
\]
Integrating from $z = \sigma + it$ to $z = 2 + it$ we obtain that
\[
\sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma+it} \log n} \log(x/n) + O(1) = (\log x) \log \zeta(\sigma + it) + \frac{\zeta'}{\zeta}(\sigma + it) - \sum_{\rho} \int_{\sigma}^{2} \frac{x^{\rho-u-it}}{(\rho - u - it)^2} du.
\]
The sum over zeros above is bounded in magnitude by
\[
\sum_{\rho} \frac{1}{|\rho - \sigma - it|^2} \int_{\sigma}^{2} \frac{x^{\frac{1}{2}-u}}{\log x} \sum_{\rho} \frac{1}{|\rho - \sigma - it|^2} = \frac{x^{\frac{1}{2}-\sigma}}{(\sigma - \frac{1}{2}) \log x} F(\sigma + it).
\]
Combining these remarks with (7), the Lemma follows.

Lemma 11. Let $T \leq t \leq 2T$ be $V$-typical. There exists a constant $C$ such that for $2 \geq \sigma \geq \sigma_0(= \frac{1}{2} + \frac{V}{\log T})$ we have
\[
\log |\zeta(\sigma + it)| \geq -CV.
\]

Proof. Taking $x = T^{1/V}$ in Lemma 10 and using condition (i) of Definition 3 we obtain
\[
\log |\zeta(\sigma + it)| \geq \text{Re} \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma+it} \log n} \frac{\log(x/n)}{\log x} - \frac{2V}{\log T} F(\sigma + it) \geq -2V - \frac{2V}{\log T} F(\sigma + it).
\]
To bound $F(\sigma + it)$, we divide the ordinates $\gamma$ into the ranges $2\pi nV/\log T \leq |t - \gamma| < 2\pi(n + 1)V/\log T$ for $0 \leq n \leq N = \lfloor (\log T)/(4\pi V) \rfloor$, and the remaining range for $\gamma$. The first kind of zeros contribute, using (ii) of Definition 3,
\[
\ll V \sum_{0 \leq n \leq N} \frac{(\sigma - \frac{1}{2})}{(\sigma - \frac{1}{2})^2 + (2\pi nV/\log T)^2} \ll \log T.
\]
The remaining zeros contribute
\[\sum_{|t-\gamma|>1/2} \frac{(\sigma - \frac{1}{2})}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2} \ll \log T.\]

The Lemma follows.

Lemma 11 establishes a stronger form of the desired estimate (3), and as noted earlier, the estimate (4) follows from (3) and Lemma 9. This completes our proof of Proposition 5.

5. Proof of Theorem 1

We may assume that \(x\) has fractional part half. A standard application of Perron’s formula (see §17 of [1]) gives, with \(c = 1 + \frac{1}{\log x}\),

\[M(x) = \frac{1}{2\pi i} \int_{c-i[x]}^{c+i[x]} \frac{x^s}{s\zeta(s)} ds + O(\log x).\] (9)

We now deform the contour of integration, replacing the line segment from \(c - i[x]\) to \(c+i[x]\) with a piecewise linear path connecting these points and comprising of a number of horizontal and vertical line segments. We will describe shortly the vertical line segments of this contour. The horizontal line segments shall connect neighboring vertical segments, with two end horizontal segments connecting the end vertical segments to \(c-i[x]\) and \(c+i[x]\). Set \(x_0 = \exp(\sqrt{\log x})\); one vertical segment shall join \(\frac{1}{2} + \frac{1}{\log x} - ix_0\) to \(\frac{1}{2} + \frac{1}{\log x} + ix_0\). For an integer \(x_0 \leq n \leq \lceil x \rceil - 1\) we let \(V_n\) denote the least integer lying in the interval \([(\log \log n)^2, \log n/\log \log n]\) such that all points in \([n, n+1]\) are \(V_n\)-typical. Notice that the existence of \(V_n\) is guaranteed by the first assertion of Proposition 4. There shall be a vertical line segment joining \(\frac{1}{2} + \frac{V_n}{\log x} + in\) to \(\frac{1}{2} + \frac{V_n}{\log x} + i(n+1)\), and its complex conjugate shall also be one of our vertical segments. This completes our definition of the contour.

No pole is encountered in deforming our contour, and it remains to estimate the integral on these various horizontal and vertical lines. For the vertical segment from \(\frac{1}{2} + \frac{1}{\log x} - ix_0\) to \(\frac{1}{2} + \frac{1}{\log x} + ix_0\) we use that (see (14.14.2) of [14])

\[|\zeta(\frac{1}{2} + \frac{1}{\log x} + it)| \gg (|t| + 2)^{-\log \log x}\]

so that

\[\left| \int_{\frac{1}{2} + \frac{1}{\log x} + ix_0}^{\frac{1}{2} + \frac{1}{\log x} - ix_0} \frac{x^s}{s\zeta(s)} ds \right| \ll x^{\frac{1}{2}} \exp((\log x)^{\frac{1}{2}} \log \log x).\] (10)

Now suppose \(x_0 \leq n \leq \lceil x \rceil - 1\). The corresponding vertical integral is, using Proposition 5,

\[\ll \frac{x^{\frac{1}{2}}}{n} \exp(V_n) \exp \left( V_n \log \frac{\log x}{\log n} + 8V_n \log \log V_n \right).\] (11)
Naturally, the same bound applies to the complex conjugate vertical line segment. Now consider the horizontal line segment going from \( \frac{1}{2} + \frac{V_n}{\log x} + i(n+1) \) to \( \frac{1}{2} + \frac{V_{n+1}}{\log x} + i(n+1) \) (if \( n = [x] - 1 \) then the horizontal line segment goes from \( \frac{1}{2} + \frac{V_n}{\log x} + i[x] \) to \( c + i[x] \)). This contributes an amount

\[
\ll \frac{x^{1/2}}{n} \left( \exp \left( V_n \log \frac{\log x}{\log n} + 9V_n \log \log V_n \right) + \exp \left( V_{n+1} \log \frac{\log x}{\log(n+1)} + 9V_{n+1} \log \log V_{n+1} \right) \right).
\]

We split the range for \( n \) into dyadic blocks. Suppose \([T, 2T]\) is such a dyadic block. Summing the estimates (11, 12) over elements \( n \) in this dyadic block we obtain

\[
\ll \frac{x^{1/2}}{T} \sum_{\frac{\log T}{\log \log T} \geq V \geq (\log \log T)^2} \exp \left( V \log \frac{\log x}{\log T} + 9V \log \log V \right) \# \{ T \leq n \leq 2T : V_n = V \}.
\]

The terms \( V \leq 2(\log \log T)^2 \) contribute an amount

\[
\ll x^{1/2} \exp \left( 2(\log \log T)^2 \log \frac{\log x}{\log T} + 18(\log \log T)^2 \log \log \log T \right) \ll x^{1/2} \exp((\log \log x)^4),
\]

which is acceptable. Consider now the contribution of larger values of \( V \). If \( V_n = V \) then by the minimality of \( V_n \), it follows that some point in \([n, n+1]\) is \((V_n - 1)\)-atypical. Appealing to Proposition 4 (pick points from every other interval in order to ensure well-spacing) we conclude that the number of such \( n \) is \( \ll T \exp(-(V - 1) \log(V/\log \log T) + 4V \log \log V) \). Therefore the quantity in (13) is

\[
\ll x^{1/2} \exp((\log \log x)^4) + x^{1/2} (\log x)^2 \sum_{\frac{\log T}{\log \log T} \geq V \geq (\log \log T)^2} \exp \left( V \log \frac{\log x}{\log T} - V \log \frac{V}{\log \log T} + 13V \log \log V \right).
\]

A little calculus shows that this is

\[
\ll x^{1/2} \exp((\log \log x)^4) + x^{1/2} \exp \left( \frac{(\log x) \log \log T}{\log T} \left( \log \frac{\log x}{\log T} \right)^{13} \right).
\]

Since \( x_0 \leq T \leq x \), we conclude that the contribution of these horizontal and vertical line segments is \( \ll x^{1/2} \exp((\log x)^{1/2} (\log \log x)^{14}) \). Combining this with (9) and (10) we have established the Theorem.

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