Moments of moments of primes in arithmetic progressions

R. de la Bretèche\(^1\) | D. Fiorilli\(^2\)

\(^1\)Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université Paris Cité, Sorbonne Université, Paris CEDEX 13, France

\(^2\)CNRS, Université Paris-Saclay, Laboratoire de mathématiques d’Orsay, Orsay, France

Correspondence
D. Fiorilli, CNRS, Université Paris-Saclay, Laboratoire de mathématiques d’Orsay, 91405, Orsay, France.
Email: daniel.fiorilli@universite-paris-saclay.fr

Funding information
NSERC

Abstract
We establish unconditional \(\Omega\)-results for all weighted even moments of primes in arithmetic progressions. We also study the moments of these moments and establish lower bounds under the Generalized Riemann Hypothesis (GRH). Finally, under GRH and the Linear Independence Hypothesis (LI), we prove an asymptotic for all moments of the associated limiting distribution, which, in turn, indicates that our unconditional and GRH results are essentially best possible. Using our probabilistic results, we formulate a conjecture on the moments with a precise associated range of validity, which we believe is also best possible. This last conjecture implies a \(q\)-analog of the Montgomery-Soundararajan conjecture on the Gaussian distribution of primes in short intervals. The ideas in our proofs include a novel application of positivity in the explicit formula and the combinatorics of arrays of characters that are fixed by certain involutions.

MSC 2020
11N13 (primary), 11M26 (secondary)

1 | INTRODUCTION

In addition to having a rich history spanning decades, the study of moments of arithmetical sequences in residue classes is experiencing accelerating and major recent progress. For instance, starting from the works of Liu [71, 72], Perelli [84], Friedlander and Goldston [35], Hooley [52, 54, 55], and culminating with the work of Harper–Soundararajan [42] and its generalization by

© 2023 The Authors. The publishing rights in this article are licensed to the London Mathematical Society under an exclusive licence.
Mastrostefano [74], we now have a general unconditional lower bound for the variance of primes and generalized divisor functions up to $x$ in arithmetic progressions modulo $q$, on average over $q \leq Q$ and in the range $x^{3+\varepsilon} \leq Q \leq x$. These questions have their origin in the foundational work of Barban [11], Davenport–Halberstam [28], Montgomery [77], and Hooley [45], as well as in the earlier work of Selberg [96] and others on primes in short intervals. There is a myriad of other related works which would be relevant to mention here, including work on variations of the Montgomery–Hooley asymptotic formula [37, 40, 45–55, 67, 97–99], relations with pair correlation and other statistics on zeros of $L$-functions [17, 20, 73], spaced moduli [8–10, 16, 30, 76], other arithmetic functions [12–14, 27, 38, 57, 68, 75, 80, 82, 83, 86, 91, 103, 107–109], as well as function field analogs [39, 58, 59, 92].

In parallel, we have been experiencing intensive ongoing work on moments of $L$-functions in families, guided by the random matrix theory analogs [60, 61] and the conjectures [21, 22, 24, 25]. In this direction, we mention the fundamental work of Ramachandra [89, 90] and Heath-Brown [43], as well as the relatively recent Rudnick–Soundararajan breakthrough [93, 94], which inspired much further work [1, 3, 18, 87, 101] on lower bounds. As for upper bounds, we mention Soundararajan’s landmark result [102] which was sharpened to the conjectured order of magnitude by Harper [41], as well as some of the many other papers on the subject [19, 23, 44, 62, 63, 81, 88, 110].

Much less is known about higher moments of arithmetical sequences in residue classes, with a notable family of exceptions. Indeed, in the influential work [34], Fouvry, Ganguly, Kowalski and Michel have computed all moments of the classical divisor function in a certain range. This work has inspired further developments including generalizations to coefficients of $GL(N)$ automorphic forms [65, 69, 100, 106, 111]. Moreover, very recently, Nunes has established upper bounds on higher moments of squarefree integers [83]. As for the sequence $\Lambda(n)$ (the von Mangoldt function), the third moment has been computed by Hooley [51], as well as by Vaughan [105] with a major arcs approximation. Hooley has also formulated a conjecture on higher moments [50] on average over $q \leq Q$ in the restricted range $x/(\log x)^4 \leq Q = o(x/\log x)$. Other than this conjecture, we are not aware of any results on higher moments of $\Lambda(n)$ in arithmetic progressions.

In this paper, we put forward a new technique which will allow us, through positivity and involved combinatorics, to establish lower bounds on average for all weighted moments of $\Lambda(n)$ in arithmetic progressions. More precisely, we will establish lower bounds on all moments of moments. The conceptual idea of studying moments of moments has its roots in Cramer’s result [26] on the variance of $\psi(x) - x$, Hooley’s bound [49] on the $t$-average of the variance of $\psi(e^t; q, a)$, Vaughan’s upper bound [104] on the higher $q$-moments of the variance of $\psi(x; q, a)$, the second author’s work [31] on the moments of the limiting distribution of the variance, as well as the very recent work of Assiotis, Bailey, and Keating [4–7] on moments of moments in the context of families of $L$-functions and of characteristic polynomials of large random matrices. The approach in the present paper is to study the higher moments of $\psi(e^t; q, a)$ via its $t$-moments for a given modulus $q$; in other words, we consider a double moment, over $a \pmod{q}$ and over $t \leq T$. As a side result, we are also able to study the fixed residue class $a = 1$, once more on average over $t \leq T$. Note also that all our results apply to individual moduli $q$, rather than to the more usual average over $q$.

As pointed out by Hooley [53], it is somewhat paradoxical that the GRH allows one to obtain an asymptotic for $\psi(x; q, a)$ for individual moduli $q \leq x^{3-\varepsilon}$, while asymptotics for the variance of $\psi(x; q, a)$ require that $x^{3+\varepsilon} \leq q \leq x$. This last barrier has recently been broken over function fields in the influential work of Keating and Rudnick [58]. However, as far as we know, it still
stands over \( \mathbb{Q} \). Interestingly, with the current methods, we were able to obtain lower bounds of the conjectured order of magnitude for moduli that are of size \( x^{\delta} \) for some small enough \( 0 < \delta < \frac{1}{2} \) (in fact, \( x^{\delta} \) can be replaced by a smaller function tending to infinity, no matter how slowly).

The moments of \( \Lambda(n) \) are strongly linked with the spacing properties of zeros of \( L \)-functions (see, e.g., [20, 26, 36]), and a fortiori to the Diophantine properties of imaginary parts of zeros of \( L(s, \chi) \). An analogous relation was uncovered in the Monach–Montgomery determination of the exact order of magnitude of the error term in the prime number theorem [79, p. 484], which depends on an effective version of the linear independence hypothesis on zeros of \( \xi(s) \). Having to assume such a hypothesis is a major drawback in the application of the explicit formula to this context. Nevertheless, in the current work, we were able to overcome this hypothesis through an application of positivity. As a result, we obtain optimal lower bounds that are independent of the Diophantine properties of zeros of \( L \)-functions. Moreover, by combining these results with an approach pioneered by Littlewood [70], we obtain unconditional and conjecturally optimal \( \Omega \)-results on all even moments.

Before we state our results, we need to introduce some notation. For \( \eta \in L^1(\mathbb{R}) \), we recall the usual definition of the Fourier transform

\[
\hat{\eta}(\xi) := \int_{\mathbb{R}} \eta(t) e^{-2\pi i \xi t} dt.
\]

As in the work of Fouvry, Ganguly, Kowalski, and Michel [34], we will count \( \Lambda(n) \) with a weight function. More precisely, for any fixed \( \delta > 0 \), we define \( S_{\delta} \subset L^1(\mathbb{R}) \) to be the set of all nontrivial differentiable even \( \eta : \mathbb{R} \to \mathbb{R} \) such that for all \( t \in \mathbb{R} \),

\[
\eta(t), \eta'(t) \ll e^{-\left(\frac{1}{2} + \delta\right)|t|},
\]

and moreover, for all \( \xi \in \mathbb{R} \), we have that

\[
0 \leq \hat{\eta}(\xi) \ll (|\xi| + 1)^{-1}(\log(|\xi| + 2))^{-2-\delta}.
\]

(1.1)

For \( a, q \in \mathbb{N} \), we define

\[
\psi_\eta(x; q, a) := \sum_{\substack{n \geq 1 \\ n \equiv a \pmod{q}}} \frac{\Lambda(n)}{n^{1/2}} \eta(\log(n/x)); \quad \psi_\eta(x, \chi_0, q) := \sum_{\substack{n \geq 1 \\ (n, q) = 1}} \frac{\Lambda(n)}{n^{1/2}} \eta(\log(n/x)).
\]

The effect of this choice of test function is the concentration of the mass on those \( n \) for which \( n \approx x \). As an example of element of \( S_{\delta} \), one can take \( \eta_K(t) = e^{-K|t|} \) with \( K \geq \frac{1}{2} + \delta \), for which \( \hat{\eta}(\xi) = 2K/(K^2 + (2\pi \xi)^2) \). Note also that the classical prime counting functions correspond to the choice \( \eta_0(t) := e^{\pi t} 1_{t \geq 0} \notin S_{\delta} \), for which \( \psi_{\eta_0}(x; q, a) = \psi(x; q, a)/\sqrt{x} \) and \( \hat{\eta}_0(\xi) = (1/2 - 2\pi i \xi)^{-1} \).

---

\[\footnote{The upper bound on \( \hat{\eta}(\xi) \) is a quite mild condition given the differentiability of \( \eta \); going through the proof of the Riemann–Lebesgue lemma, we see, for instance, that a stronger bound holds as soon as \( \eta' \) is monotous. (A stronger bound holds if \( \eta \) is twice differentiable.) As for the positivity condition, we can take, for example, \( \eta = \eta_1 \ast \eta_1 \) for some smooth and rapidly decaying \( \eta_1 \).} \]
The object of study in this paper is the moment\footnote{Note that the moments are usually not divided out by $\varphi(q)$; this normalization should be kept in mind while comparing our results to those in the literature. Moreover, since $\psi_n(x;\chi_0,q)$ is of order $x^{\frac{1}{2}}$, in a sense we have also normalized by $x^{\frac{n}{2}}$.} \\

$$M_n(x, q; \eta) = \frac{1}{\varphi(q)} \sum_{\substack{a \pmod{q} \in T_{\eta} \setminus \{0\}}} \left( \frac{\psi_n(x; q, a) - \psi_\eta(x, \chi_0 q)}{\varphi(q)} \right)^n. \quad (1.2)$$

We define the constants $\alpha(h) := \hat{h}(0)$ and \\

$$\beta_q(h) := -\hat{h}(0) \left( \log(8\pi) + \gamma_0 + \sum_{p \mid q} \frac{\log p}{p-1} \right) + \frac{1}{2} \int_0^\infty \frac{e^{-x^2}}{1-e^{-x}} \left( 2\hat{h}(0) - \hat{h}(x) - \hat{h}(-x) \right) dx,$$

where $\gamma_0$ is the Euler–Mascheroni constant. Here are our main unconditional results.

**Theorem 1.1.** Let $\delta > 0$, $\eta \in S_\delta$, fix $\varepsilon > 0$, and let $g : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 3}$ be any increasing function tending to infinity for which $g(u) \leq e^u$. For a positive proportion of moduli $q$, there exists an associated value $x_q$ such that $g(c_1(\varepsilon, \delta, \eta) \log x_q) \leq q \leq g(c_2(\varepsilon, \delta, \eta) \log x_q)$, where the $c_j(\varepsilon, \delta, \eta) > 0$ are constants, and \\

$$M_2(x_q, q; \eta) \geq (1 - \varepsilon) \frac{\alpha(\hat{\eta}^2) \log q}{\varphi(q)}. \quad (1.4)$$

If, moreover, $g(u) \leq u^{\frac{1}{M}}$ for some $M \geq 4\varepsilon^{-1}$, then we can choose $x_q$ in such a way that \\

$$g((\log x_q)^{1 - \frac{3}{2M^2}}) \leq q \leq g(\log x_q)$$

and \\

$$M_2(x_q, q; \eta) \geq \frac{\alpha(\hat{\eta}^2) \log q + \beta_q(\hat{\eta}^2)}{\varphi(q)} + \frac{1}{q^{\frac{3}{2} + \varepsilon}}. \quad (1.5)$$

**Remark 1.1.**

1. The expression $(\alpha(\hat{\eta}^2) \log q + \beta_q(\hat{\eta}^2))/\varphi(q)$ is the expected main term for $M_2(x, q; \eta)$ (see Conjecture 1.8 below).
2. One can obtain an $\Omega$-result for $M_2(x_q, q; \eta) - (\alpha(\hat{\eta}^2) \log q + \beta_q(\hat{\eta}^2))/\varphi(q)$, which is slightly stronger than (1.5), but with undetermined sign. More precisely, the term $1/q^{\frac{3}{2} + \varepsilon}$ can be replaced with a constant times $\log q/\varphi(q)^{\frac{3}{2}}$.
3. Goldston and Vaughan [37] have shown\footnote{Note that in [37, Theorem 1.1], $-c_xQ$ should be replaced by $+c_xQ$.} that under GRH and in the range $x^{\frac{5}{7} + \varepsilon} \leq Q \leq x^{1-\varepsilon}$,

$$\frac{1}{Q^2} \sum_{q \leq Q} \left( \sum_{\substack{a \pmod{q} \in \mathbb{Z} \setminus \{0\}}} \frac{\Lambda(n) - x}{\varphi(q)} \right)^2 \log q - \beta_q(\hat{\eta}_0^2) = \Omega_{\pm} \left( x^{-\frac{3}{7}} Q^{-\frac{1}{2}} \right),$$

where $\beta_q(\hat{\eta}_0^2) = -(\gamma_0 + \log(2\pi) + \sum_{p \mid q} \frac{\log p}{p-1})$. \hspace{1cm} (1.6)
In a certain range, we have similar results for all even moments. Here and throughout, we will denote the \( n \)th moment of the Gaussian by

\[
\mu_n := \begin{cases} 
\frac{(2m)!}{2^m m!} & \text{if } n = 2m \text{ for some } m \in \mathbb{N}, \\
0 & \text{otherwise.}
\end{cases}
\] (1.6)

**Theorem 1.2.** Let \( \delta > 0, \eta \in S_5 \), and fix \( m \geq 2 \). For any \( B \in \mathbb{R}_{>0} \), there exists a real number \( \lambda \in \left[ \frac{1}{m-1+B}, \frac{1}{B} \right] \) with the following property. For any fixed \( \varepsilon > 0 \), there exists a sequence of moduli \( \{q_i\}_{i \geq 1} \) and associated values \( \{x_i\}_{i \geq 1} \) such that \( q_i = (\log x_i)^{1+o(1)} \) and

\[
M_{2m}(x_i, q_i; \eta) \geq (1 - \varepsilon)\mu_{2m} \frac{\left( \alpha(\hat{\eta}^2) \log q_i \right)^m}{\varphi(q_i)^m}.
\] (1.7)

If moreover \( g : \mathbb{R}_{>1} \to \mathbb{R}_{>3} \) is an increasing function such that \( g(x) \leq x^{1/m} \) with \( M \geq 2\varepsilon^{-1} \), then for a positive proportion of \( q \), there exists \( x_q \) such that \( g((\log x_q)^{1 - \frac{1}{m}}) \leq q \leq g((\log x_q)^{1 - \frac{1}{m}}) \) and

\[
M_{2m}(x_q, q; \eta) \geq \mu_{2m} \left( \frac{\left( \alpha(\hat{\eta}^2) \log q_i + \beta q_i(\hat{\eta}^2) \right)}{\varphi(q_i)} \right)^m + \frac{1}{q_i^{m+\frac{1}{2}+\varepsilon}}.
\] (1.8)

**Remark 1.2.**

1. Under the additional condition \( q \geq (\log_2 x)^{1+\varepsilon} \), we believe that Theorems 1.1 and 1.2 are essentially best possible (see Conjecture 1.8). However, in the range \( q \leq \varepsilon \log_2 x \), we believe that \( M_{2m}(x, q; \eta) \) can be larger (see Remark 1.6).

2. Under GRH, we will show in Section 7 that analogs of Theorems 1.1 and 1.2 are valid for all moduli \( q \). For instance, it follows from Lemma 7.4 that for every large enough \( q \), there exists an associated value \( x_q \) such that \( \log x_q \asymp_{\varepsilon, \eta} \log q \) and

\[
M_2(x_q, q; \eta) \geq (1 - \varepsilon)\frac{\alpha(\hat{\eta}^2)}{\varphi(q)} \log q.
\]

Moreover, one can obtain a version of Theorem 1.2 (still unconditional) which holds for a positive proportion of moduli \( q \) in the range \( \log q / \log \log x \asymp_m 1 \).

3. Under GRH, one can also obtain \( \Omega \)-results for the odd moments \( M_{2m+1}(x, q; \eta) \) (using this time even moments, that is the lower bound on \( \mathcal{V}_{2r, 2m+1}(T, q; \eta, \Phi) \) in Theorem 1.3 below).

Theorems 1.1 and 1.2 are applications of our estimates for higher moments of moments, which we now describe. We define the expected value

\[
m_n(q; \eta) := \lim_{T \to \infty} \frac{1}{T} \int_0^T M_n(e^t, q; \eta) dt,
\] (1.9)

whenever the limit exists. We consider the set \( \mathcal{U} \subset L^1(\mathbb{R}) \) of nontrivial even integrable functions \( \Phi : \mathbb{R} \to \mathbb{R} \) such that \( \hat{\Phi} \geq 0 \). As a consequence, we have that \( \Phi(0) > 0 \). For \( s, n \in \mathbb{N} \) such that (1.9)

\(^{\dagger}\) We will see in Lemma 4.3 that GRH implies the existence of this limit.
exists, we define
\[ \mathcal{V}_{s,n}(T, q; \eta, \Phi) := \frac{1}{T} \int_0^\infty \Phi \left( \frac{t}{T} \right) \left( M_n(e^t, q; \eta) - m_n(q; \eta) \right)^s \, dt. \] (1.10)

We also define the quantities
\[ \nu_n := n!^2 \sum_{k \geq 2, m \geq 0, n = k + 2m} \frac{1}{k! 2^m m! 2}; \quad V_n(q; \eta) := \frac{\nu_n \left( \alpha(\hat{\eta}^2) \log q \right)^n}{\varphi(q)^{n+1}}. \] (1.11)

We are ready to state our main theorem.

**Theorem 1.3.** Assume GRH, and let \( \delta > 0 \), \( \eta \in S_\delta \), \( \Phi \in \mathcal{V}' \), \( q \geq 3 \), \( T \geq 1 \), \( n \geq 2 \), \( k, r \geq 1 \). Then, the limit (1.9) exists. Moreover, if \( q \) is large enough in terms of \( \delta \) and \( \eta \), then in the ranges \( n \leq \log q / \log 2 q \) and \( r \leq \log q \), the moment of moment \( \mathcal{V}_{s,n}(T, q; \eta, \Phi) \) satisfies the lower bounds
\[ \mathcal{V}_{2r,n}(T, q; \eta, \Phi) \geq \mu_{2r} V_n(q; \eta)^r \left( 1 + O_{\delta, \eta} \left( n \frac{\log_2 q}{\log q} \right) \right)^r + O_{\delta, \eta} \left( \frac{r}{\log q} \right)^r + O_{\Phi} \left( \frac{(C_{\delta, \eta} \log q)^{2r} n}{T \varphi(q)^{2r}} \right); \]
\[ (-1)^n \mathcal{V}_{2r+1,n}(T, q; \eta, \Phi) \geq (2r + 1)! \frac{\delta_{n,2r+1} V_n(q; \eta)^{2r+1}}{2^{r-1} (r-1)! \varphi(q)^{1/2}} \]
\[ \times \left( 1 + O_{\delta, \eta} \left( n \frac{\log_2 q}{\log q} \right) \right)^r + O_{\delta, \eta} \left( \frac{r}{\log q} \right)^r + O_{\Phi} \left( \frac{(C_{\delta, \eta} \log q)^{(2r+1)n}}{T \varphi(q)^{2r+1}} \right). \]

Here, \( \delta_{n,2r+1} = 0 \) if \( n \) is odd, and otherwise \( \delta_{n,2r+1} := \nu_n^{-3} (\nu'_n + \nu''_n) \), where \( \nu'_n \) and \( \nu''_n \) are constants of the same nature as \( \nu_n \), which are defined in (6.17) and (6.18), respectively. Moreover, the constant \( C_{\delta, \eta} > 0 \) depends only on \( \delta, \eta \). Finally, the mean of the \( n \)th moment \( m_n(q; \eta) \) satisfies the bounds
\[ m_{2k+1}(q; \eta) \leq 0 \] and
\[ m_{2k}(q; \eta) \geq \mu_{2k} \left( \frac{\alpha(\hat{\eta}^2) \log q + \beta_q(\hat{\eta}^2)}{\varphi(q)} \right)^k + O \left( \frac{(C_{\delta, \eta} \log q)^k}{\varphi(q)^{k+1}} \right). \]

**Remark 1.3.**

1. In Proposition 7.9, we also obtain lower bounds for the moments of \( \psi^0(e^t; q, 1) - \psi^0(e^t, \chi_{0,q})/\varphi(q) \); in other words, we may fix the individual residue class 1 (mod \( q \)).
2. The conditions \( \hat{\gamma}, \hat{\Phi} \geq 0 \) are crucial here (recall the definitions of \( S_\delta \) and \( \mathcal{V}' \)). If they were to be dropped, then as discussed earlier, in order to obtain a lower bound of the strength of Theorem 1.3 using the techniques of this paper, one would need to assume an effective version of the linear independence hypothesis on Dirichlet \( L \)-functions. More precisely, it would be sufficient to assume that if \( \gamma_1, \ldots, \gamma_n \) are imaginary parts of zeros of height at most \( S \geq 1 \) of

---

\(^\dagger\) It is crucial here that we subtract \( m_n(q; \eta) \) rather than the empirical mean \( \frac{1}{T} \int_0^T M_n(e^t, q; \eta) \); in the latter case, the resulting combinatorial expression is not well suited for an application of positivity of \( \hat{\gamma}, \hat{\Phi} \).

\(^\ddagger\) As we will see in (1.18), \( V_n(q; \eta) \) is the (conjectural) asymptotic variance of \( M_n(e^t, q, \eta) \).
L(s, \chi_1), \ldots L(s, \chi_n), \text{ respectively, with } \chi_j \neq \chi_{0,q}, \text{ then either } |\gamma_1 + \cdots + \gamma_n| \geq \exp(-(qS)^{1+\varepsilon}), \text{ or } n = 2m \text{ and for each } j, \text{ there exists } \ell_j \neq j \text{ for which } \chi_j = \chi_{\ell_j} \text{ and } \gamma_j = -\gamma_{\ell_j}. \text{ This is the } q\text{-analog of the Montgomery–Monach conjecture [79, (15.24)] on zeros of } \zeta(s). \text{ However, this approach would impose the extra condition } q \leq (\log T)^{1-\varepsilon} \text{ (in other words, when translating back to } M_n(x, q; \eta), \text{ we would need } q \leq (\log x)^{1-\varepsilon}).

As another application of our results, we will establish \Omega\text{-results for the usual prime counting functions (without smooth weights).

Corollary 1.4. Assume GRH, and fix } g : \mathbb{R}_{\geq 1} \rightarrow \mathbb{R}_{\geq 3} \text{ an increasing function tending to infinity for which } g(u) \leq e^{u}. \text{ For each large enough modulus } q, \text{ there exists } a (\mod q) \text{ with } (a, q) = 1 \text{ and values } x_q \text{ for which } g(c_1 \log x_q) \leq q \leq g(c_2 \log x_q), \text{ where } 0 < c_1 < c_2 < \frac{1}{2} \text{ are absolute, and

\[ \sum_{n \leq x_q \atop n \equiv a (\mod q)} \Lambda(n) - \frac{1}{\varphi(q)} \sum_{n \leq x_q \atop (n,q)=1} \Lambda(n) \gg \frac{(x_q \log q)^{\frac{1}{2}}}{\varphi(q)^{\frac{1}{2}}}. \]

Unconditionally, the same holds for a positive proportion of moduli } q.

Our method also allows to isolate the individual arithmetic progression 1 (mod q). We will show in Proposition 7.9 that under GRH, for any } \eta \in S_\delta \text{ and } \Phi \in \mathcal{U}, \text{ and in the range } m \leq \log q/(2 \log_2 q),

\[ \frac{1}{T} \int_0^\infty \Phi \left( \frac{t}{T} \right) \left( \psi_\eta(e^t; 1, q) - \frac{\psi_\eta(e^t; \chi_0, q)}{\varphi(q)} \right)^{2m} \, dt \geq \mu_{2m} \left( \frac{\alpha(\delta^2) \log q + \beta_q(\delta^2)}{\varphi(q)} \right)^{m} \Phi \left( \frac{C_{\delta,\eta} \log q^{m}}{\varphi(q)^{m+1}} + \frac{(K_{\delta,\eta} \log q)^{2m}}{T} \right), \]

where } C_{\delta,\eta}, K_{\delta,\eta} > 0 \text{ are constants. We deduce the following.

Corollary 1.5. Assume GRH, and fix } \varepsilon > 0 \text{ small enough and } M > 1 + \varepsilon. \text{ Let } g : \mathbb{R}_{\geq 1} \rightarrow \mathbb{R}_{\geq 3} \text{ be an increasing function tending to infinity such that } g(u) \leq u^{\frac{1}{M}}. \text{ Then, for each large enough modulus } q, \text{ there exists an associated value } x_q \text{ such that } g((\log x_q)^{1-\frac{1}{M}-2\varepsilon}) \leq q \leq g((\log x_q)^{1-\frac{1}{M}}) \text{ and

\[ \sum_{n \leq x_q \atop n \equiv 1 (\mod q)} \Lambda(n) - \frac{1}{\varphi(q)} \sum_{n \leq x_q \atop (n,q)=1} \Lambda(n) \gg \frac{(x_q \log q)^{\frac{1}{2}}}{\varphi(q)^{\frac{1}{2}}}. \] (1.12)

In the next theorem, we will show under GRH that } M_n(e^t, q; \eta) \text{ has a limiting distribution as } t \rightarrow \infty \text{ (which we will denote by } H_n(q; \eta)), \text{ and estimate the corresponding moments.

Definition 1.6. For } n \geq 1, \text{ we say that } E : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n \text{ has limiting distribution } \mu \text{ if } \mu \text{ is a probability distribution on } \mathbb{R} \text{ such that for any bounded continuous } f : \mathbb{R}^n \rightarrow \mathbb{R}, \text{ we have that

\[ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(E(t)) \, dt = \int_{\mathbb{R}^n} f \, d\mu. \] (1.13)
We will establish lower bounds for all centered moments

$$\mathbb{V}_{s,n}(q; \eta) := \mathbb{E}[(H_n(q; \eta) - \mathbb{E}[H_n(q; \eta)])^s].$$

(1.14)

Under the additional hypothesis LI below, we will show that these bounds are sharp.

**Hypothesis LI**: For any $q \geq 3$, the multiset

$$Z(q) := \{ \Im(m(\varphi_\chi)) \geq 0 : \ell(q, \chi) = 0, \chi \pmod{q}, \Re(\varphi_\chi) \geq 1 \}$$

is linearly independent over $\mathbb{Q}$.

**Remark 1.4.** LI implies that the zeros of $L(s, \chi)$ are simple, and that $L(\frac{1}{2}, \chi) \neq 0$.

**Theorem 1.7.** Assume GRH, and let $\delta > 0$, $\eta \in S_\delta$, $q \geq 3$, $n \geq 2$, $r, m \geq 1$. Then, $M_n(e^i, q; \eta)$ possesses a limiting distribution of as $t \to \infty$, which will be denoted by $H_n(q; \eta)$. If moreover $q$ is large enough in terms of $\delta$ and $\eta$, $n \leq \log q / \log q$ and $r \leq \log q$, then the variance defined in (1.14) satisfies the lower bounds

$$\mathbb{V}_{2r,n}(q; \eta) \geq \mu_{2r} V_n(q; \eta)^r \left( 1 + O_{\delta, \eta} \left( n \frac{\log_2 q}{\log q} \right) \right)^r + O_{\delta, \eta} \left( \frac{r}{\log q} \right);$$

$$\mathbb{V}_{2r+1,n}(q; \eta) \geq (-1)^n \frac{(2r + 1)!}{2^r (r - 1)!} \frac{\vartheta_{n,2r+1}}{\varphi(q)^\frac{1}{2}} V_n(q; \eta)^{2r+1} \times \left( 1 + O_{\delta, \eta} \left( n \frac{\log_2 q}{\log q} \right) \right)^r + O_{\delta, \eta} \left( \frac{r}{\log q} \right).$$

Assuming in addition that LI holds, we have that $\mathbb{E}[H_{2m+1}(q; \eta)] = 0$ and

$$\mathbb{E}[H_{2m}(q; \eta)] = \mu_{2m} \left( \alpha(\hat{\eta}^2) \log q + \beta_\chi(\hat{\eta}^2) \right)^m + O_m \left( \frac{\log q}{\varphi(q)^{m+1}} \right),$$

(1.17)

and for higher moments, we have the asymptotic estimates

$$\mathbb{V}_{2r,n}(q; \eta) = \mu_{2r} V_n(q; \eta)^r \left( 1 + O_{n,r} \left( \frac{\log_2 q}{\log q} \right) \right);$$

$$\mathbb{V}_{2r+1,n}(q; \eta) = (-1)^n \frac{(2r + 1)!}{2^r (r - 1)!} \frac{\vartheta_{n,2r+1}}{\varphi(q)^\frac{1}{2}} V_n(q; \eta)^{2r+1} \left( 1 + O_{n,r} \left( \frac{\log_2 q}{\log q} \right) \right).$$

(1.18)

(1.19)

**Remark 1.5.** In the specific case $m = 1$, one can obtain an upper bound of the correct order of magnitude in (1.17) assuming GRH only, by using Hooley’s arguments [49, Theorem 1]. However, for higher moments, applying the same approach would yield an upper bound which is $1 / \varphi(q)^{1+o(1)}$.

In other words, as $q \to \infty$, $H_n(q; \eta)$ has variance asymptotic to $V_n(q; \eta)$, and the normalized random variable $(H_n(q; \eta) - \mathbb{E}[H_n(q; \eta)]) / \sqrt{\mathbb{V}[H_n(q; \eta)]}$ has Gaussian moments up to a certain point. This does not persist for very large moments (in terms of $q$); $H_n(q; \eta)$ is bounded by $K_{\delta, \eta} \log q^n / \varphi(q)$ for some $K_{\delta, \eta} > 0$, and thus $\mathbb{V}_{s,n}(q; \eta) \ll (2K_{\delta, \eta} \log q)^n / \varphi(q)^s$. The presence of the constant $\nu_n$ (rather than the variance of the $n$th power of the Gaussian) in the variance $V_n(q; \eta)$...
can be explained by the fact that the values of $\psi_\eta(e^t, q; a) - \psi_\eta(e^t, \chi_{0,q})/\varphi(q)$ are not independent as $a$ runs over the invertible residues modulo $q$. Indeed, a calculation similar to [66, Lemma 2.1] shows that the covariances are nonzero.

We deduce the following conjecture which generalizes and refines Hooley’s conjecture [50].

**Conjecture 1.8.** Let $\varepsilon > 0$, $n \geq 1$, $\delta > 0$, and $\eta \in S_\delta$. In the range $(\log_2 x)^{1+\varepsilon} \leq q \leq x^{1-\varepsilon}$ and for fixed values of $n$, we have the asymptotic

$$M_n(x, q; \eta) = (\mu_n + o_{q, x \to \infty}(1)) \left( \frac{\alpha(\eta^2) \log q}{\varphi(q)} \right) \frac{n}{2}.$$

(1.20)

**Remark 1.6.** We also conjecture that (1.20) holds in the range $(\log_2 x)^{1+\varepsilon} \leq q \leq x^{1-\varepsilon}$ with the function $\eta_0(t) = e^{t^2} 1_{t \leq 0}$, that is with the classical weight (since then $\eta_0(\log(n/x))/n^\frac{1}{2} = x^{-\frac{1}{2}} 1_{n \leq x}$). This generalizes [31, Conjecture 1.1]. Moreover, the lower bound in the range $(\log_2 x)^{1+\varepsilon} \leq q \leq x^{1-\varepsilon}$ is essentially best possible. Indeed, by [33, Theorem 1.2] and Hölder’s inequality, for $\varepsilon \log_2 x / \log_2 x \leq q \leq \varepsilon \log_2 x$, we have that the moments $M_{2m}(x, q; \eta_0)$ can be as large as $\left( \frac{\log q}{\varphi(q)} \right)^m \left( \frac{\log_2 x}{q} \right)^m$. Moreover, one can show that (1.20) does not hold for $q$ very close to $x$ (indeed, in this range, $M_{2m}(x, q; \eta)$ can be as large as $(\log q)^{2m-1}/\varphi(q)^m$).

Conjecture 1.8 suggests the following subsequent conjecture about the distribution of primes in arithmetic progressions.

**Conjecture 1.9.** Let $\varepsilon > 0$. In the range $(\log_2 x)^{1+\varepsilon} \leq q \leq x^{1-\varepsilon}$ and for fixed $V \in \mathbb{R}$,

$$\frac{1}{\varphi(q)} \left\{ a \pmod{q} : \psi(x; q, a) - \frac{\psi(x, \chi_{0,q})}{\varphi(q)} > V \frac{(x \log q)^{\frac{1}{2}}}{\varphi(q)^{\frac{1}{2}}} \right\} \leq \int_V^\infty e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}} + o_{q, x \to \infty}(1).$$

Conjectures 1.8 and 1.9 are $q$-analogues of the conjectures of Montgomery and Soundararajan on primes in short intervals [78, Conjectures 1.2] (see also the recent article [15]).

## 2. WEIL EXPLICIT FORMULA

The goal of this short section is to express $M_n(x, q; \eta)$ as a sum over zeros of Dirichlet $L$-functions. For $\eta \in S_\delta$, we will study the moments of $\psi_\eta(x; q, a)$ via the explicit formula for

$$\psi_\eta(x, \chi) := \sum_{n=1}^\infty \Lambda(n) \chi(n) n^\frac{1}{2} \eta(\log(n/x)),$$

where $\chi \pmod{q}$ is a Dirichlet character of conductor $q \chi \geq 3$. By orthogonality, we have the identity

$$M_n(x, q; \eta) = \frac{1}{\varphi(q)^n} \sum_{\substack{\chi_1, \ldots, \chi_n \neq \chi_{0,q} \\chi_1, \ldots, \chi_n \neq \chi_{0,q}}} \psi_\eta(x, \chi_1) \cdots \psi_\eta(x, \chi_n).$$

(2.1)

We recall that $\eta$ is even and real-valued, and hence for all $\xi \in \mathbb{R}$, $\hat{\eta}(-\xi) = \hat{\eta}(\xi) = \hat{\eta}(\xi)$. We begin with the following bound on “ramified primes.”
Lemma 2.1. Let $\delta > 0$ and $\eta \in S_\delta$. Let $\chi$ be a nonprincipal character modulo $q$ with associated primitive character $\chi^*$. For any $t \geq 0$ and $q \geq 3$, we have the bound

$$\psi_\eta(e^t, \chi) - \psi_\eta(e^t, \chi^*) \ll_{\delta, \eta} e^{-\frac{t}{2}} \log q.$$ 

Proof. By the triangle inequality,

$$|\psi_\eta(e^t, \chi) - \psi_\eta(e^t, \chi^*)| \leq \sum_{(n,q)>1} \frac{\Lambda(n)}{n^{\frac{1}{2}}} |\eta(t - \log n)|$$

$$= \sum_{p|q} \log p \sum_{k \geq 1} \frac{1}{p^k} |\eta(t - k \log p)|.$$ 

As $\eta(u) \ll e^{-\left(\frac{1}{2} + \delta\right)|u|}$, we split the sum over $k$ into two parts, according to whether $k \log p \geq t$ or not. A straightforward calculation shows that both of these sums are $\ll e^{-\frac{t}{2}}$. The claimed bound follows.

We are ready to apply the explicit formula.

Lemma 2.2. Let $q \geq 3$, let $\chi \pmod{q}$ be a nonprincipal Dirichlet character, and let $\delta > 0$ and $\eta \in S_\delta$. Then for $t \geq 0$, we have the formula

$$\psi_\eta(e^t, \chi) = -\sum_{\varphi_\chi} e^{\left(\varphi_\chi - \frac{1}{2}\right)it} \hat{\eta}\left(\frac{\varphi_\chi - \frac{1}{2}}{2\pi i}\right) + O_{\delta, \eta}(e^{-\frac{t}{2}} \log q), \tag{2.2}$$

where $\varphi_\chi$ runs over the nontrivial zeros of $L(s, \chi)$.

Proof. Let $\chi^*$ be the primitive character modulo $q_\chi$ which induces $\chi$, and define as usual $a(\chi) = 0$ when $\chi(-1) = 1$ and $a(\chi) = 1$ when $\chi(-1) = -1$. We will evaluate the sum over zeros on the right-hand side of (2.2). We can replace $L(s, \chi)$ by $L(s, \chi^*)$, since these have the same nontrivial zeros. We apply Weil’s explicit formula for $L(s, \chi^*)$. Taking $F(x) := \eta(2\pi x + t)$ in [79, Theorem 12.13], we find that $\hat{F}(\varphi_\chi - \frac{1}{2}) = \frac{1}{2\pi} e^{\left(\varphi_\chi - \frac{1}{2}\right)it} \hat{\eta}\left(\frac{\varphi_\chi - \frac{1}{2}}{2\pi i}\right)$, and it follows that

$$\sum_{\varphi_\chi} e^{\left(\varphi_\chi - \frac{1}{2}\right)it} \hat{\eta}\left(\frac{\varphi_\chi - \frac{1}{2}}{2\pi i}\right) = \left(\log \frac{q_\chi}{\pi} + \frac{1}{\Gamma}\left(\frac{1}{4} + \frac{a(\chi)}{2}\right)\right) \eta(t)$$

$$- \sum_{n=1}^{\infty} \Lambda(n) \left(\frac{n}{q_\chi} \eta(t - \log n) + \chi^*(n)\eta(t + \log n)\right) \tag{2.3}$$

$$+ \int_{0}^{\infty} e^{-\left(\frac{1}{2} + a(\chi)\right)x} \frac{1}{1 - e^{-2x}} (2\eta(t) - \eta(x + t) - \eta(-x + t)) \, dx.$$ 

Since $\eta \in S_\delta$, the first and third terms are clearly $\ll_{\eta} e^{-\frac{t}{2}} \log q$. For the term involving $\chi^*(n)$, the triangle inequality yields that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\frac{1}{2}}} \chi^*(n)\eta(t + \log n) \ll_{\delta, \eta} e^{-\frac{t}{2}}.$$ 

The proof is completed by applying Lemma 2.1. \qed
Throughout the paper, we will use the Riemann–von Mangoldt formula (see, e.g., [79, Corollary 14.7])

\[ N(T, \chi) := \# \{ \varphi_\chi : |\Im m(\varphi_\chi)| \leq T \} = \frac{T}{\pi} \log \left( \frac{q_\chi T}{2\pi e} \right) + O(\log(q_\chi(T + 2))) \quad (T \geq 1). \quad (2.4) \]

The following slightly weaker version of GRH will appear naturally in this paper.

**Hypothesis GRH \( \hat{\eta} \).** For all \( q \geq 3 \), if \( \chi \) is any primitive character to the modulus \( q \) and \( \varphi_\chi \) is a nontrivial zero of \( L(s, \chi) \), then either \( \Re e(\varphi_\chi) = \frac{1}{2} \), or \( \hat{\eta}(\varphi_\chi - \frac{1}{2} \frac{1}{2\pi i}) = 0 \).

We observe that Hypothesis GRH \( \hat{\eta} \) is independent of the Riemann Hypothesis for \( \zeta(s) \) (since we have excluded the principal character).

As a corollary of Lemma 2.2, we obtain the following uniform bounds.

**Corollary 2.3.** Let \( q \geq 3 \) and \( n \geq 2 \), let \( \chi \) (mod \( q \)) be a nonprincipal Dirichlet character, and let \( \delta > 0 \) and \( \eta \in S_\delta \). Then, under GRH \( \hat{\eta} \), for \( t \geq 0 \) and for \( q \) large enough, we have the bounds

\[ |\psi_\eta(e^t, \chi)| \ll_{\delta, \eta} \log q; \quad (2.5) \]

\[ M_n(e^t, q; \eta) \ll \frac{(C_{\delta, \eta} \log q)^n}{\varphi(q)}, \quad (2.6) \]

where \( C_{\delta, \eta} \) is a positive constant depending only on \( \delta \) and \( \eta \).

**Proof.** The bound (2.5) is obtained by applying (1.1) to the sum over zeros in Lemma 2.2, followed by a summation by parts using (2.4). The bound on \( M_n(e^t, q; \eta) \) follows from inserting this bound into (2.1). \( \square \)

### 3 | Conductors and Convergent Sums Over Zeros

In this section, we fix \( \delta > 0 \) and consider the set \( T_\delta \) of nontrivial measurable functions \( h : \mathbb{R} \to \mathbb{R} \) having the following properties. We require that \( \hat{\xi} \leftrightarrow \xi h(\xi) \) is integrable, and that, for all \( \xi \in \mathbb{R} \), we have the bounds

\[ 0 \leq h(\xi) \ll (1 + |\xi|)^{-1}(\log(2 + |\xi|))^{-2-2\delta}. \]

Moreover, for all \( t \in \mathbb{R} \), we have that\(^\dagger\)

\[ \hat{h}(t), \hat{h}'(t) \ll e^{-\left(\frac{1}{2} + \frac{\delta}{2}\right)|t|}. \]

Note that if \( \eta \in S_\delta \) is nontrivial, then \( h_\eta := \hat{\eta}^2 \in T_{\delta/2} \). We extended \( h \) to \( s \in \mathbb{C} : |\Im m(z)| \leq \frac{1}{4\pi} \) by writing

\[ h(s) := \int_{\mathbb{R}} e^{2\pi i s \xi} \hat{h}(\xi)d\xi. \quad (3.1) \]

\(^\dagger\) The integrability of \( \xi \mapsto \xi h(\xi) \) implies that \( \hat{h} \) is differentiable (see [64, p. 430]).
Our goal is to compute various sums involving the quantity

\[ b(\chi; h) := \sum_{\varphi \chi} h\left(\frac{\varphi - \frac{1}{2}}{2\pi i}\right) \]  

(3.2)

with \( h \in T_\delta \), where \( \chi \) is a nonprincipal character modulo \( q \geq 3 \), and where the sum is running over the nontrivial zeros \( \varphi \chi \) of \( L(s, \chi) \). To do so, we will first need to bound the sum

\[ b_2(\chi; h) := -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\frac{3}{2}}} (\chi^*(n)\hat{h}(\log n) + \overline{\chi^*(n)}\hat{h}(-\log n)), \]  

(3.3)

where \( \chi^* \) is the primitive character modulo \( q_\chi \) inducing \( \chi \).

**Lemma 3.1.** Let \( \delta > 0 \) and \( h \in T_{\delta/2} \). For \( q \geq 3 \) and \( \chi \equiv 0 \pmod{q} \) nonprincipal, we have the estimate

\[ \frac{1}{\varphi(q)} \sum_{\chi \equiv 0 \pmod{q}} b_2(\chi; h) \ll \frac{\log q}{\varphi(q)}. \]  

(3.4)

**Proof.** First, since \( h \) is real-valued,

\[ b_2(\chi; h) = -2\Re\left( \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\frac{3}{2}}} \chi^*(n)\hat{h}(\log n) \right). \]

We use the same method as [31, Lemma 3.2]. To do so, we apply [32, Proposition 3.4], and deduce that

\[ \sum_{\chi \equiv 0 \pmod{q}} \left( b_2(\chi; h) + 2\Re\left( \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\frac{3}{2}}} \chi(n)\hat{h}(\log n) \right) \right) \ll \sum_{p^{\nu} \equiv 1 \pmod{q/p^{\nu}} \nu, e \geq 1} \frac{\log p}{p^{e(1+\frac{\delta}{2})}} \varphi\left( \frac{q}{p^{\nu}} \right) \]

\[ \ll \log q, \]

where the second inequality follows from an argument similar to that in the proof of [31, Lemma 3.2]. Hence, we can apply the orthogonality relations and obtain the estimate

\[ \sum_{\chi \equiv 0 \pmod{q}} b_2(\chi; h) = -2\Re\left( \varphi(q) \sum_{n \equiv 1 \pmod{q}} \frac{\Lambda(n)}{n^{\frac{3}{2}}} \hat{h}(\log n) \right) + O_h(\log q). \]

The proof is finished by noting that

\[ \sum_{n \equiv 1 \pmod{q}} \frac{\Lambda(n)}{n^{1+\frac{\delta}{2}}} \ll \sum_{j=1}^{\infty} \frac{\log(1 + qj)}{(1 + qj)^{1+\frac{\delta}{2}}} \ll \delta \frac{\log q}{q^{1+\frac{\delta}{2}}}. \]

We are now ready to estimate the average of \( b(\chi; h) \). The proof will involve the first moment of \( \log q_\chi \), which was computed in [32].
Proposition 3.2. Let $\delta > 0$, $h \in T_{5/2}$, and $q \geq 3$. We have the average estimate

$$
\frac{1}{\varphi(q)} \sum_{\chi \pmod{q} \neq \chi_0,q} b(\chi; h) = \alpha(h) \log q + \beta_q(h) + O_{\delta,h}(\log q/\varphi(q)),
$$

(3.5)

where $\alpha(h)$ and $\beta_q(h)$ are defined in (1.3). Moreover, we have the pointwise estimate

$$
b(\chi; h) = \alpha(h) \log q \chi + O_{\delta,h}(1).
$$

(3.6)

Proof. We generalize [31, Lemma 3.2] by applying the explicit formula [79, Theorem 12.13] with $F(x) := 2\pi \hat{\chi}(-2\pi x)$. For $\chi \pmod{q}$ nonprincipal, this yields that

$$
b(\chi; h) = b_1(\chi; h) + b_2(\chi; h) + b_3(\chi; h),
$$

(3.7)

where $b_2(\chi; h)$ is defined in (3.3) and

$$
b_1(\chi; h) := \left( \log \left( \frac{q \chi}{\pi} \right) + \frac{\Gamma'(\frac{1}{4} + \frac{1}{2}a(\chi))}{\Gamma(\frac{1}{4} + \frac{1}{2}a(\chi))} \right) \hat{\chi}(0);
$$

$$
b_3(\chi; h) := \int_0^\infty \frac{e^{-(\frac{1}{2} + a(\chi)) x}}{1 - e^{-2x}} \left( 2\hat{\chi}(0) - \hat{\chi}(x) - \hat{\chi}(-x) \right) dx,
$$

(3.8)

with $a(\chi)$ defined as in the proof of Lemma 2.2. We first prove (3.6). We clearly have that $b_1(\chi; h) = \alpha(h) \log q \chi + O_h(1)$. As for $b_3(\chi; h)$, after recalling that $h \in T_{5/2}$ implies that $\hat{\chi}$ is differentiable, we deduce that $b_3(\chi; h) \ll \sup_{t \in [0,1]} |\hat{\chi}'(t)| + O_h(1) \ll_h 1$. Finally,

$$
b_2(\chi; h) \ll_h \sum_{n \geq 1} \frac{\Lambda(n)}{n^{1 + \delta/2}} \ll \delta^{-1}.
$$

The claimed estimate follows.

We now move to (3.5). The formulas [79, (C.15), (C.16)] imply that

$$
b_1(\chi; h) = \alpha(h) \left( \log \left( \frac{q \chi}{8\pi} \right) - \gamma_0 - \frac{\pi}{2} \chi(-1) \right).
$$

(3.9)

Now, by [32, Proposition 3.3], we have that

$$
\frac{1}{\varphi(q)} \sum_{\chi \pmod{q} \neq \chi_0,q} \log q \chi = \log q - \sum_{p | q} \log \frac{p}{p-1},
$$

(3.10)

and hence,

$$
\frac{1}{\varphi(q)} \sum_{\chi \pmod{q} \neq \chi_0,q} b_1(\chi; h) = \alpha(h) \left( \log \left( \frac{q}{8\pi} \right) - \sum_{p | q} \log \frac{p}{p-1} - \gamma_0 \right) + O_h\left( \frac{1}{\varphi(q)} \right).
$$

(3.11)

The computation of the average of $b_2(\chi; h)$ was done in Lemma 3.1, and that of $b_3(\chi; h)$ is straightforward. \hfill $\square$
Remark 3.1. In a subsequent article, we will show how the techniques of this section allow to obtain an estimate for

\[ b_{2n}(\chi) := \sum_{\chi} \frac{1}{|\chi|^2n}. \]

We will also need to estimate incomplete sums over zeros. To handle these, we will establish a pair correlation result. We first need to estimate the following sum over characters.

Lemma 3.3. Let \( m, n, q \in \mathbb{N} \) with \( q \geq 3 \), let \( \chi \) (mod \( q \)) be a fixed character, and define

\[ S_q(m, n) := \sum_{\chi \chi_2 (mod \ q)} \chi_1(n)\chi_2(n). \]

Then, for any primes \( p_1, p_2 \) and integers \( e_1, e_2 \geq 1 \) and \( \nu_1, \nu_2 \geq 0 \) such that \( p_1^{\nu_1}, p_2^{\nu_2} \parallel q \), we have the formula

\[
S_q(p_1^{e_1}, p_2^{e_2}) = \begin{cases} 
\chi(p_1^{e_1})\varphi(q)1_{p_1^{e_1} = p_2^{e_2} (mod \ q)} & \text{if } (p_1, q) = 1, \\
\chi(p_2^{e_2})\varphi(q/p_1^{\nu_1})1_{p_1^{e_1} = p_2^{e_2} (mod \ q/p_1^{\nu_1})} & \text{if } p_1 \mid q, p_2 \nmid q, \\
\chi(p_1^{e_1})\varphi(q/p_2^{\nu_2})1_{p_1^{e_1} = p_2^{e_2} (mod \ q/p_2^{\nu_2})} & \text{if } p_1 \nmid q, p_2 \mid q, \\
1_{\text{cond}(\chi)/q/p_1^{\nu_1} r(p_1^{e_1} + q/p_1^{\nu_1})\varphi(q/p_1^{\nu_1})1_{p_1^{e_1} = p_2^{e_2} (mod \ q/p_1^{\nu_1})} & \text{if } p_1 \mid q, p_2 = p_1, \\
\chi(1)(p_2^{e_2} + q/p_2^{\nu_2})\chi(2)(p_1^{e_1})\varphi(q/p_1^{\nu_1}) & \text{if } p_1, p_2 \mid q, p_1 \neq p_2,
\end{cases}
\]

where by definition \( \text{cond}(\chi) \) is the conductor of \( \chi \), and for \( p_1 \neq p_2 \), \( \chi = \chi^{(1)}\chi^{(2)}\chi^{(3)} \) with \( \chi^{(1)} \) (mod \( p_1^{\nu_1} \)), \( \chi^{(2)} \) (mod \( p_2^{\nu_2} \)) and \( \chi^{(3)} \) (mod \( q / (p_1^{\nu_1} p_2^{\nu_2}) \)).

Proof. The easiest case is when \( (p_1 p_2, q) = 1 \), in which we have that

\[ S_q(p_1^{e_1}, p_2^{e_2}) = \sum_{\chi_1 (mod \ q)} \chi_1(p_1^{e_1})\chi_1(p_2^{e_2}) = \chi(p_2^{e_2})\varphi(q)1_{p_1^{e_1} = p_2^{e_2} (mod \ q)} \]

If \( p_1 \mid q \) but \( p_2 \nmid q \), then

\[ S_q(p_1^{e_1}, p_2^{e_2}) = \sum_{\chi_1, \chi_2 (mod \ q)} \chi_1(p_1^{e_1})\chi_2(p_2^{e_2}) = \sum_{\chi_1 (mod \ q)} \chi_1(p_1^{e_1})(\chi_1(p_2^{e_2})), \]

which by [32, Proposition 3.4] is equal to \( \chi(p_2^{e_2})\varphi(q/p_1^{\nu_1})1_{p_1^{e_1} = p_2^{e_2} (mod \ q/p_1^{\nu_1})} \).

Let us now assume that \( p_1, p_2 \mid q \). First, observe that

\[ S_q(p_1^{e_1}, p_2^{e_2}) = \sum_{\chi_1 (mod \ q/p_1^{\nu_1}) \chi_2 (mod \ q/p_2^{\nu_2})} \chi_1(p_1^{e_1})\chi_2(p_2^{e_2}). \]

We claim that given \( \chi \) (mod \( q \)) and \( \chi_1 \) (mod \( q/p_1^{\nu_1} \)), the condition \( \forall x : (x, q) = 1, \chi_1\chi_2(x) = \chi(x) \) uniquely determines \( \chi_2 \) (mod \( q/p_2^{\nu_2} \)). Indeed, we have that for all \( f \geq 1 \), \( \chi_2(p_2^f) = \chi_2(p_2^f +
\(q/p_2^{\nu_2}\), and \((p_2^T + q/p_2^{\nu_2}, q) = 1\). Moreover, if \(\text{cond}(\chi \chi_1) | q/p_2^{\nu_2}\), then such a \(\chi_2\) always exists. As a result, \(S_q(p_1^{\nu_1}, p_2^{\nu_2})\) is equal to

\[
\sum_{\chi_1 \mod q/p_1^{\nu_1}} \chi_1(p_1^{\nu_1})(\chi \chi_1)(p_2^{\nu_2} + q/p_2^{\nu_2}) = \chi(p_2^{\nu_2} + q/p_2^{\nu_2}) \sum_{\chi_1 \mod q/p_2^{\nu_2}} \chi_1(p_1^{\nu_1})\chi_1(p_2^{\nu_2} + q/p_2^{\nu_2}).
\]

Let us now assume that \(p_1 \neq p_2\), in which case \(\chi \mod q\) can be decomposed as \(\chi = \chi^{(1)} \chi^{(2)} \chi^{(3)}\), with \(\chi^{(1)} \mod p_1^{\nu_1}\), \(\chi^{(2)} \mod p_2^{\nu_2}\) and \(\chi^{(3)} \mod q/(p_1^{\nu_1} p_2^{\nu_2})\). Similarly, we can decompose any \(\chi_1 \mod q/p_1^{\nu_1}\) as \(\chi_1 = \chi^{(2)} \chi^{(3)}\), with \(\chi^{(2)} \mod p_2^{\nu_2}\) and \(\chi^{(3)} \mod q/(p_1^{\nu_1} p_2^{\nu_2})\).

The condition \(\text{cond}(\chi \chi_1) | q/p_2^{\nu_2}\) is equivalent to saying that \(\chi^{(2)} \chi^{(2)} = \chi^{(0)} p_2^{\nu_2}\). In other words,

\[
S_q(p_1^{\nu_1}, p_2^{\nu_2}) = \chi(p_2^{\nu_2} + q/p_2^{\nu_2}) \chi^{(2)}(p_1^{\nu_1}) \chi^{(2)}(p_2^{\nu_2} + q/p_2^{\nu_2}) \sum_{\chi_1 \mod q/(p_1^{\nu_1} p_2^{\nu_2})} \chi_1(p_1^{\nu_1}) \chi_1(p_2^{\nu_2} + q/p_2^{\nu_2}),
\]

which by the orthogonality relations is equal to

\[
\chi(p_2^{\nu_2} + q/p_2^{\nu_2}) \chi^{(2)}(p_1^{\nu_1}) \chi^{(2)}(p_2^{\nu_2} + q/p_2^{\nu_2}) \varphi(q/(p_1^{\nu_1} p_2^{\nu_2})) \sum_{\chi_1 \mod q/(p_1^{\nu_1} p_2^{\nu_2})} \chi_1(p_1^{\nu_1}) \chi_1(p_2^{\nu_2} + q/p_2^{\nu_2}).
\]

The final case to consider is \(p_1 = p_2 \mid q\), in which the condition \(\text{cond}(\chi \chi_1) | q/p_1^{\nu_1}\) is equivalent to \(\text{cond}(\chi) | q/p_1^{\nu_1}\). The sum \(S_q(p_1^{\nu_1}, p_2^{\nu_2})\) is then equal to

\[
1_{\text{cond}(\chi) \mod q/p_1^{\nu_1}} \chi(p_1^{\nu_1} + q/p_1^{\nu_1}) \sum_{\chi_1 \mod q/p_1^{\nu_1}} \chi_1(p_1^{\nu_1}) \chi_1(p_2^{\nu_2}) = 1_{\text{cond}(\chi) \mod q/p_1^{\nu_1}} \chi(p_1^{\nu_1} + q/p_1^{\nu_1}) \varphi(q/p_1^{\nu_1}) \sum_{\chi_1 \mod q/p_1^{\nu_1}} \chi_1(p_1^{\nu_1}) \chi_1(p_2^{\nu_2}),
\]

by the orthogonality relations.

Here is the pair correlation result we will need. We will assume GRH\(_h\) for brevity; however, it is very possible that this assumption can be removed.

**Proposition 3.4.** Let \(q \geq 3\), and fix \(\chi \mod q\) and \(z \in \mathbb{R}\). Let \(h \in \mathcal{T}_{3/2}\), and assume GRH\(_h\). Let \(\Phi \in L^1(\mathbb{R})\) be even and supported in \([-1, 1]\). Then in the range \(1 \leq L \leq \log q - \log_2 q\), we have the bound

\[
P(h; \chi, z) := \sum_{\chi_1, \chi_2 \neq \chi \chi_1} \sum_{\chi_1 \chi_2 = \chi} h\left(\frac{\gamma_1}{2\pi}\right) h\left(\frac{\gamma_2}{2\pi}\right) \hat{\Phi}\left(\frac{L}{2\pi}(z - \gamma_1 - \gamma_2)\right) \ll_{\delta, h} \varphi(q) \left(\log q\right) \left(1 + \frac{\log q}{L}\right),
\]

where \(\gamma_j = \frac{1}{2} + i\gamma_j\) runs through the zeros of \(L(s, \chi_j)\). The implied constant is independent of \(\chi\) and \(z\).

**Proof.** We apply (2.3) with \(\eta := \hat{h}\); we obtain the identity

\[
b(\chi; h, t) := \sum_{\gamma_\chi} h\left(\frac{\gamma_\chi - \frac{1}{2}}{2\pi i}\right) e^{i\gamma_\chi - \frac{1}{2} t} = b_1(\chi; h, t) + b_2(\chi; h, t) + b_3(\chi; h, t) + b_4(\chi; h, t),
\]
where

\[
b_1(\chi; h, t) := \left( \log \frac{q_{\chi}}{\pi} + \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{a(\chi)}{2} \right) \right) \hat{h}(t);
\]

\[
b_2(\chi; h, t) := - \sum_{n \geq 1} \frac{\Lambda(n)}{n^2} \chi^n(n) \hat{h}(t - \log n);
\]

\[
b_3(\chi; h, t) := \int_0^{\infty} \frac{e^{-\left( \frac{1}{2} + a(\chi) \right)y}}{1 - e^{-2y}} \left( \frac{2}{\pi} \right) \hat{h}(t) - \hat{h}(t + y) - \hat{h}(t - y) \right) \, dy;
\]

\[
b_4(\chi; h, t) := - \sum_{n \geq 1} \frac{\Lambda(n)}{n^2} \chi^n(n) \hat{h}(t + \log n).
\]

We have the bounds \( b_1(\chi; h, t) \ll |\hat{h}(t)| \log q \leq \hat{h}(0) \log q \); \( b_3(\chi; h, t), b_4(\chi; h, t) \ll h \); \( b(\chi; h, t) \ll h \log q \), and as a result, \( b_2(\chi; h, t) \ll h \log q \) as well. We deduce that

\[
P(h; \chi, z) = \int_{\mathbb{R}} \Phi(\xi) e^{-iLz\xi} \sum_{\chi_1, \chi_2 \neq \chi_0, q} \sum_{\chi_1 \chi_2 = \chi} b_2(\chi_2, h, L\xi) b_2(\chi_2, h, L\xi) \, d\xi + O_{\delta, h} \left( \varphi(q) (\log q)^2 + \varphi(q) \log q \right).
\]

The main term is equal to

\[
P_M(h, z, L) := \sum_{m,n \geq 1} \frac{\Lambda(m)\Lambda(n)}{(mn)^{\frac{1}{2}}} I_{m,n}(h, z, L) \sum_{\chi_1, \chi_2 \neq \chi_0, q} \sum_{\chi_1 \chi_2 = \chi} \chi'_1(m) \chi'_2(n),
\]

where

\[
I_{m,n}(h, z, L) := \int_{\mathbb{R}} \Phi(\xi) e^{-iLz\xi} \hat{h}(\log m - L\xi) \hat{h}(\log n - L\xi) \, d\xi.
\]

We note that \( I_{m,n}(h, z, L) \ll L^{-1} \), and moreover for \( m \leq n \),

\[
I_{m,n}(h, z, L) \ll \begin{cases}
(mn)^{-\frac{1}{2} - \delta} L^{-1} e^{L(1+2\delta)} & \text{if } e^L \leq m \leq n, \\
\left( \frac{m}{n} \right)^{2+\delta} \left( \frac{1}{L} + 1 - \frac{\log m}{L} \right) & \text{if } m \leq e^L \leq n, \\
\left( \frac{m}{n} \right)^{2+\delta} \left( \frac{1}{L} + \frac{\log(n/m)}{L} \right) & \text{if } m \leq n \leq e^L.
\end{cases}
\]

(3.13)

In particular, if \( m \leq n \) and \( m \leq e^L \), we have the bound \( I_{m,n}(h, z, L) \ll \left( \frac{m}{n} \right)^{\frac{1}{2} + \delta}. \)
We now apply Lemma 3.3. The contribution to $P_M(h; \chi, z)$ of those $n, m$ for which $(mn, q) = 1$ is given by

$$P_{(mn,q)=1} := \sum_{m,n \geq 1 \atop (mn,q)=1} \frac{\Lambda(m)\Lambda(n)}{(mn)^{\frac{1}{2}}} I_{m,n}(h, z, L)(\chi(m)\varphi(q)\chi_{n \equiv m \pmod{q}} - \chi(m) - \chi(n) + 1)$$

$$\ll J + \varphi(q)L + e^L,$$

where the contribution of the nondiagonal terms in the part of the sum involving the factor $\chi(m)\varphi(q)\chi_{n \equiv m \pmod{q}}$ is bounded by a constant times

$$J := \varphi(q) \sum_{m,k \geq 1} \frac{\Lambda(m)\Lambda(m+kq)}{(m+m+kq)^{\frac{1}{2}}} I_{m,m+kq}(h, z, L) \ll \varphi(q) \frac{e^L (\log q + L)}{q}.$$

In other words, we have shown that $P_{(mn,q)=1}$ is an admissible error term.

For the remaining terms in $P_M(h, z, L)$, we first note that we may drop the conditions $\chi_1, \chi_2 \neq \chi_{0,q}$ in (3.12) at the cost of the error term $O(e^L)$; we will denote the resulting sum by $Q(h; \chi, z)$. By Lemma 3.3, the contribution to $Q(h; \chi, z)$ of those $n, m$ for which either $(m, q) > 1$ and $(n, q) = 1$, or $(n, q) > 1$ and $(m, q) = 1$, is then

$$\ll \sum_{n \geq 1 \atop (n, q) = 1} \sum_{\nu, \epsilon \geq 1} \frac{\Lambda(n) \log p}{(np^\nu)^{\frac{1}{2}}} I_{n, p^\nu}(h, z, L) \varphi(q/p^\nu) \ll \varphi(q) \log_2 q.$$

As for the contribution of those $m, n$ for which $m = p_1^{\nu_1}, n = p_2^{\nu_2}$ with $p_1 \neq p_2$ and $p_1, p_2 | q$, it is

$$\ll \sum_{p_1 \mid q} \sum_{\nu_1, \epsilon_1 \geq 1, \nu_2, \epsilon_2 \geq 1} \frac{\log p_1 \log p_2}{(p_1^{\nu_1} p_2^{\nu_2})^{\frac{1}{2}}} I_{p_1^{\nu_1}, p_2^{\nu_2}}(h, z, L) \varphi(q/(p_1^{\nu_1} p_2^{\nu_2})) \ll_{\epsilon} \varphi(q) \log_2 q.$$

To show this last bound, we used the inequalities

$$\sum_{\nu_1, \nu_2 \geq 1} \varphi(q/(p_1^{\nu_1} p_2^{\nu_2})) \leq \sum_{\nu_1, \nu_2 \geq 1} \frac{\varphi(q)}{\varphi(p_1^{\nu_1} p_2^{\nu_2})} \ll \frac{\varphi(q)}{p_1 p_2},$$

and applied the bounds (3.13).

Finally, the contribution of those $n, m$ for which $m = p^{\nu_1}, n = p^{\nu_2}$ with $p^{\nu} | q$ and $\nu \geq 1$ is

$$\ll \sum_{p^{\nu} \mid q} \frac{(\log p)^2}{p^{\nu+2}} I_{p^{\nu_1}, p^{\nu_2}}(h, z, L) \varphi(q/p^{\nu}) \ll \frac{1}{L} \sum_{p^{\nu} \mid q} \frac{(\log p)^2}{p} \varphi(q/p^{\nu}) \ll \varphi(q).$$

The proof is finished. \qed

The next step is to estimate the variance of $\log q$. This was done in [31, Lemma 3.3], which we refine and rectify here.
Lemma 3.5. For $q \geq 3$, we have the estimate

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}, \chi \neq \chi_0,q} \left( \log q_{\chi} - \log q + \sum_{p|q} \frac{\log p}{p-1} \right)^2 = \sum_{p|q} \frac{(\log p)^2}{p} + O(1) \ll (\log_2 q)^2.$$  

Moreover, for $h \in T_{\delta/2}$, we have that

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}, \chi \neq \chi_0,q} (b(\chi; h) - \alpha(h) \log q)^2 \ll (\alpha(h) \log_2 q)^2 + O_{\delta,h}(1).$$

Proof. Performing a computation analogous† to that in [31, Lemma 3.3], we see that

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}, \chi \neq \chi_0,q} (\log q_{\chi} - \log q)^2$$

$$= \sum_{p' \parallel q} \frac{p(\log p)^2}{(p-1)^2} \left( 1 + \frac{1}{p} - \frac{2}{p^r} \right) + 2 \sum_{p_1, p_2|q \atop p_1 < p_2} \frac{(\log p_1)(\log p_2)}{(p_1-1)(p_2-1)} + O\left( \frac{(\log q)^2}{\varphi(q)} \right)$$

$$= \left( \sum_{p|q} \frac{\log p}{p-1} \right)^2 + \sum_{p' \parallel q} \frac{p(\log p)^2}{(p-1)^2} \left( 1 - \frac{2}{p^r} \right) + O\left( \frac{(\log q)^2}{\varphi(q)} \right).$$

We conclude that

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}, \chi \neq \chi_0,q} \left( \log q_{\chi} - \log q + \sum_{p|q} \frac{\log p}{p-1} \right)^2 = \sum_{p' \parallel q} \frac{p(\log p)^2}{(p-1)^2} \left( 1 - \frac{2}{p^r} \right) + O\left( \frac{(\log q)^2}{\varphi(q)} \right)$$

$$= \sum_{p|q} \frac{(\log p)^2}{p} + O(1) \ll (\log_2 q)^2,$$

and the first claimed estimate follows. For the second, we combine the last bound with (3.6). ☐

We now establish the final lemma of this section, which will be useful in our computation of higher moments.

Lemma 3.6. Let $\delta > 0$, $h \in T_{\delta/2}$, and let $k \geq 2$. If $q$ is large enough in terms of $\delta$ and $h$ and $\chi$ is a character modulo $q$, then in the range $k \leq \log q / \log_2 q$, we have the estimate

$$\sum_{\chi_1, \ldots, \chi_k \pmod{q}, \chi_1, \ldots, \chi_k \neq \chi_0,q, \chi_1 \cdots \chi_k = \chi} b(\chi_1; h) \cdots b(\chi_k; h) = \varphi(q)^{k-1}(\alpha(h) \log q)^k \left( 1 + O\left( k \frac{\log_2 q}{\log q} \right) \right).$$ (3.14)
Moreover, if we assume GRH, then uniformly for \( z \in \mathbb{R} \), we have the bound

\[ \sum_{X_1, \ldots, X_k \equiv \chi \pmod{q}} h\left( \frac{\gamma_{X_1}}{2\pi} \right) \cdots h\left( \frac{\gamma_{X_k}}{2\pi} \right) = o_{\delta, h}(q^{k-1}(\alpha(h) \log q)^{k-1}), \quad (3.15) \]

where we used the shorthand \( \gamma_{X_j} := (\varphi_{X_j} - \frac{1}{2})/i \).

**Proof.** We first establish (3.15). The left-hand side of this equation is

\[ \leq \sum_{X_3, \ldots, X_k \equiv \chi \pmod{q}} \sum_{X_3, \ldots, X_k \neq \chi_0, q} h\left( \frac{\gamma_{X_3}}{2\pi} \right) \cdots h\left( \frac{\gamma_{X_k}}{2\pi} \right) \]

\[ \times \sup_{z' \in \mathbb{R}} \sup_{\chi \equiv \chi_0, q} \left| \sum_{\chi \neq \chi_0, q} \sum_{\chi_1, \chi_2} h\left( \frac{\gamma_{\chi_1}}{2\pi} \right) h\left( \frac{\gamma_{\chi_2}}{2\pi} \right) \right| \Phi_0\left( \frac{\log q}{3\pi} (z' - \gamma_1 - \gamma_2) \right), \]

where \( \Phi_0(x) := \max(0, 1 - |x|) \). Here, we replaced the sum over \( \chi = (X_3 \cdots X_k)^{-1} \) with \( z' = z - \gamma_3 - \cdots - \gamma_k \) by the supremum over \( z' \) and \( \chi \), and deleted the condition \( \gamma_1 + \gamma_2 = z' \) by adding the factor involving \( \Phi_0 \).

The claimed bound follows from Proposition 3.4 and (3.6).

We now move to (3.14). By expanding the product \( a_1 \cdots a_k = (a_1 - b_1 + b_1)(a_2 - b_2 + b_2) \cdots (a_k - b_k + b_k) \), we deduce the identity

\[ a_1 \cdots a_k - b_1 \cdots b_k = \sum_{(\varepsilon_1, \ldots, \varepsilon_k) \neq (0, \ldots, 0)} c(\varepsilon_1) \cdots c(\varepsilon_k), \]

where

\[ c(\varepsilon_j) := \begin{cases} a_j - b_j & \text{if } \varepsilon_j = 1 \\ b_j & \text{otherwise.} \end{cases} \]

We observe that for \( \ell \geq 2 \),

\[ C_\ell(q; \chi) := \left| \left\{ (X_1, \ldots, X_\ell) : X_1 \cdots X_\ell = \chi \cdot X_j \neq \chi_0, q \right\} \right| = (\varphi(q) - 1)^{\ell-1} + O(\varphi(q)^{\ell-2}). \quad (3.16) \]

and hence by setting \( a_j := b(\chi_j; h) \) and \( b_j := \alpha(h) \log q \), it follows that

\[ \sum_{X_1, \ldots, X_k \equiv \chi \pmod{q}} b(\chi_1; h) \cdots b(\chi_k; h) - C_k(q; \chi)(\alpha(h) \log q)^k \]

\[ = \sum_{\emptyset \neq I \subset \{1, \ldots, k\}} (\alpha(h) \log q)^{k-|I|} \sum_{X_1, \ldots, X_k \equiv \chi \pmod{q}} \prod_{j \in I} (b(\chi_j; h) - \alpha(h) \log q). \]
Now, noting that $C_1(q; \chi) \in \{0, 1\}$, the triangle and Cauchy–Schwarz inequalities and the estimate (3.16) imply that for any fixed $J \subseteq \{1, \ldots, k\}$,

$$\left| \sum_{\chi_1 \cdots \chi_k \pmod q \atop \chi_1 \cdots \chi_k \neq \chi_0, a} \prod_{j \in J} (b(\chi_j; h) - \alpha(h) \log q) \right| \leq \varphi(q) \left( \prod_{j \in J} \left| \sum_{\chi \neq \chi_0, a} b(\chi; h) - \alpha(h) \log q \right| \right)^{|J|/2} \leq \varphi(q)^{k-1} (\alpha(h) \log_2 q + c_{\delta, h})^{|J|},$$

by Lemma 3.5, where $c_{\delta, h} > 0$ depends on $\delta$ and $h$. When $J = \{1, \ldots, k\}$, this is $\ll \varphi(q)^{k-1} (\alpha(h) \log_2 q + c_{\delta, h})^{k-1} \cdot (\alpha(h) \log q + c_{\delta, h})$, by the estimate (3.6) and the trivial bound $\log q \chi \leq \log q$. Consequently, we have the bound

$$\sum_{\chi_1 \cdots \chi_k \pmod q \atop \chi_1 \cdots \chi_k \neq \chi_0, a} b(\chi_1; h) \cdots b(\chi_k; h) - C_k(q)(\alpha(h) \log q)^k \ll \varphi(q)^{k-1} \sum_{j=1}^{k-1} \binom{k}{j} (\alpha(h) \log q)^{k-j} (\alpha(h) \log_2 q + c_{\delta, h})^j$$

$$+ \varphi(q)^{k-1} (\alpha(h) \log_2 q + c_{\delta, h})^{k-1} (\alpha(h) \log q + c_{\delta, h})$$

$$= \varphi(q)^{k-1} (\alpha(h) \log q)^k \left( \left( 1 + \frac{\log_2 q + c_{\delta, h} \alpha(h)^{-1}}{\log q} \right)^k - 1 + O \left( \left( \frac{\log_2 q}{\log q} \right)^{k-1} \right) \right)$$

$$\ll k \frac{\log_2 q + c_{\delta, h} \alpha(h)^{-1}}{\log q} \varphi(q)^{k-1} (\alpha(h) \log q)^k \left( 1 + \frac{\log_2 q + c_{\delta, h} \alpha(h)^{-1}}{\log q} \right)^{k-1}.$$

The claim follows.

## 4 THE PROBABILISTIC MODEL

The two goals of this section are to show that under GRH (see the beginning of Section 2 for the definition of this hypothesis), the function $M_n(e^t, q; \eta)$ has a limiting distribution, and moreover, we can compute its moments. We begin with the following lemma.
Lemma 4.1. For \( n, m \in \mathbb{N} \), let \( E = (u_{\mu,j}, v_{\mu,j})_{1 \leq \mu \leq m} \) be an array of functions \( u_{\mu,j}, v_{\mu,j} : \mathbb{R}_{\geq 0} \to \mathbb{R} \) such that \( E \) has a limiting distribution in \( \mathbb{R}^{2mn} \). Then, the function

\[
F(t) := \Re e \left( \sum_{\mu=1}^{m} \prod_{j=1}^{n} (u_{\mu,j}(t) + iv_{\mu,j}(t)) \right)
\]

has a limiting distribution in \( \mathbb{R} \).

Proof. This follows from the definition of limiting distribution. Defining \( G : \mathbb{R}^{2mn} \to \mathbb{R} \) by

\[G((x_{\mu,j}, y_{\mu,j})_{\mu,j}) := \Re e(\sum_{\mu=1}^{m} \prod_{j=1}^{n} (x_{\mu,j} + iy_{\mu,j})) \]

one can check that the measure defined by

\[\mu_F(f) := \mu_E(f \circ G)\]

is a limiting distribution for \( F \). In other words, \( \mu_F \) is the pushforward measure \( G_\ast \mu_E \).

We can now show the existence of the limiting distribution.

Lemma 4.2. Let \( \delta > 0, \eta \in S_\delta \), and assume GRH \( \hat{\eta} \). Then for any \( n \geq 2 \) and \( q \geq 3 \), \( M_n(e^t, q; \eta) \) has a limiting distribution.

Proof. In the expression (2.1), we order the set of vectors of characters \((\chi_1, \ldots, \chi_n)\) modulo \( q \) such that \( \chi_j \neq \chi_{0,q}, \chi_1 \cdots \chi_n = \chi_{0,q} \), as \( \{(\psi_{\mu,1}, \ldots, \psi_{\mu,n}) : 1 \leq \mu \leq C_n(q)\} \), where \( C_n(q) \) is defined in (3.16). It follows that

\[M_n(e^t, q; \eta) = \sum_{\mu=1}^{m} \prod_{j=1}^{n} (u_{\mu,j}(t) + iv_{\mu,j}(t)),\]

where \( m = C_n(q) \), \( u_{\mu,j}(t) := \Re e(\psi_{\eta}(e^t, \psi_{\mu,j})) \), \( v_{\mu,j}(t) := \Im e(\psi_{\eta}(e^t, \psi_{\mu,j})) \). By Lemma 2.2, we have that

\[
u_{\mu,j}(t) = -\Re e \left( \sum_{\gamma_{\psi_{\mu,j}}} e^{it\gamma_{\psi_{\mu,j}} \hat{\eta}_{\xi}(\gamma_{\psi_{\mu,j}}/2\pi)} \right) + O(e^{-t/2} \log q);
\]

where \( \varphi_{\psi_{\mu,j}} = \frac{1}{2} + i\gamma_{\psi_{\mu,j}} \) is running over the nontrivial zeros \( \varphi_{\psi_{\mu,j}} \) of \( L(s, \psi_{\mu,j}) \), and where the \( b \) over the sum indicates that we are summing over zeros \( \varphi_{\psi_{\mu,j}} \) for which \( \hat{\eta}((\varphi_{\psi_{\mu,j}} - \frac{1}{2})/(2\pi i)) \neq 0 \). In particular, since we are assuming GRH \( \hat{\eta} \), we are only summing over zeros on the critical line; in other words, \( \gamma_{\psi_{\mu,j}} \in \mathbb{R} \). We now apply [2, Theorem 1.4] (see also [29]), whose conditions are satisfied thanks to the fact that \( \eta \in S_\delta \) and the Riemann–von Mangoldt formula (2.4). It follows that

\[E(t) := (u_{\mu,j}(t), v_{\mu,j}(t))_{1 \leq \mu \leq m} \]

admits a limiting distribution on \( \mathbb{R}^{2mn} \), and the same is true for \( M_n(e^t, q; \eta) \) by Lemma 4.1.

Our next goal is to give an expression for the moments of \( H_n(q; \eta) \).
Lemma 4.3. Let $\delta > 0$ and $\eta \in S_\delta$, and assume GRH$_{\hat{\eta}}$. Let $s \in \mathbb{N}$, $n \geq 2$, $q \geq 3$, and let $H_n(q; \eta)$ be the random variable associated to the limiting distribution of $M_n(e^t, q; \eta)$. Then we have the formulas

\[
\mathbb{E}[H_n(q; \eta)^s] = \lim_{T \to \infty} \frac{1}{T} \int_0^T M_n(e^t, q; \eta)^s dt;
\]

\[
\mathbb{E}[(H_n(q; \eta) - \mathbb{E}[H_n(q; \eta)])^s] = \lim_{T \to \infty} \frac{1}{T} \int_0^T (M_n(e^t, q; \eta) - \mathbb{E}[H_n(q; \eta)])^s dt.
\]

Proof. The proof is direct since $M(e^t, q; \eta)$ is bounded by Corollary 2.3, and thus, $H_n(q; \eta)$ is compactly supported. Hence, with $B(q; \eta) \coloneqq \sup_{t \geq 0} |M(e^t, q; \eta)|$, one can take

\[
f(x) = \begin{cases} 
  x^s & \text{if } |x| \leq B(q; \eta); \\
  x^s(B(q; \eta) + 1 - |x|) & \text{if } B(q; \eta) < |x| \leq B(q; \eta) + 1; \\
  0 & \text{if } |x| > B(q; \eta) + 1
\end{cases}
\]

in (1.13), and deduce that

\[
\lim_{T \to \infty} \frac{1}{T} \int_{t \leq T} M(e^t, q; \eta)^s dt = \int_{|x| \leq B(q;\eta)} x^s d\mu_{H_n(q;\eta)}(x) = \mathbb{E}[H_n(q; \eta)^s].
\]

The proof is similar for centered moments.

In order to give an explicit description of the moments of $H_n(q; \eta)$, we apply the explicit formula in Lemma 2.2. The following notation will simplify the resulting expression. For $s, n \in \mathbb{N}$, we introduce the set of arrays of characters

\[
X_{s,n} := \left\{ \chi = (\chi_{\mu,j})_{1 \leq \mu \leq s}^{1 \leq j \leq n} : \begin{array}{l}
\chi_{\mu,j} \neq \chi_{0,q} \quad (1 \leq j \leq n), \\
\chi_{\mu,1} \cdots \chi_{\mu,n} = \chi_{0,q} \quad (1 \leq \mu \leq s)
\end{array} \right\}.
\]

(4.1)

For $\chi \in X_{s,n}$, we define the set of associated arrays of nontrivial zeros

\[
\Gamma(\chi) := \left\{ \gamma = (\gamma_{\mu,j})_{1 \leq \mu \leq s}^{1 \leq j \leq n} : L(\frac{1}{2} + i\gamma_{\mu,j}, \chi_{\mu,j}) = 0, |\Re m(\gamma_{\mu,j})| < \frac{1}{2} \right\},
\]

(4.2)

as well as the subset

\[
\Gamma_0(\chi) := \left\{ \gamma \in \Gamma(\chi) : \sum_{1 \leq \mu \leq s}^{1 \leq j \leq n} \gamma_{\mu,j} = 0 \right\}.
\]

For $\gamma \in \Gamma(\chi)$, we will use the notations

\[
\hat{\gamma}(\chi) := \prod_{1 \leq \mu \leq s}^{1 \leq j \leq n} \hat{\gamma}(\frac{\gamma_{\mu,j}}{2\pi}); \\
\Sigma_{\gamma} := \frac{1}{2\pi} \sum_{1 \leq \mu \leq s}^{1 \leq j \leq n} \gamma_{\mu,j}.
\]

(4.3)

Note that under GRH$_{\hat{\eta}}$, for each $\gamma \in \Gamma(\chi)$, we have that either $\gamma \in \mathbb{R}^s$, or $\hat{\gamma}(\gamma) = 0$. 
Lemma 4.4. Let $\delta > 0$, $\eta \in S_5$, $\Phi \in L^1(\mathbb{R})$, $\Phi$ even, and assume GRH$_\eta$. For $s \in \mathbb{N}$, $n \geq 2$, $T \geq 1$ and $q \geq 3$, we have the formula

$$
\frac{1}{T} \int_0^\infty \Phi\left(\frac{t}{T}\right) M_n(e^t, q; \eta)^s dt = \frac{(-1)^{sn}}{2\varphi(q)^{sn}} \sum_{\chi \in X_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) \hat{\Phi}(T\Sigma_{\gamma}) + O\left(\frac{(K_{\delta,\eta} \log q)^{sn}}{T\varphi(q)^s}\right),
$$

where $K_{\delta,\eta} > 0$ is a constant. In particular,

$$
\mathbb{E}[H_n(q; \eta)^s] = \frac{(-1)^{sn}}{2\varphi(q)^{sn}} \sum_{\chi \in X_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma).
$$

Proof. By Lemma 2.2, we have that, for any nonprincipal character $\chi$ of modulus $q$ and $t \geq 0$,

$$
\psi_\eta(e^t, \chi) = -\sum_{\gamma} e^{i\gamma \chi t} \hat{\eta}\left(\frac{\gamma \chi}{2\pi}\right) + O(e^{-\frac{t}{2} \log q}),
$$

(4.4)

where the sum is taken over the zeros $\rho_\chi = \frac{1}{2} + i\gamma_\chi$ of $L(s, \chi)$. In particular, under GRH$_\eta$, either $\gamma_\chi \in \mathbb{R}$ or $\hat{\eta}\left(\frac{\gamma \chi}{2\pi}\right) = 0$. Taking complex conjugates, it follows that

$$
\psi_\eta(e^t, \chi) = -\sum_{\gamma} e^{-i\gamma \chi t} \hat{\eta}\left(\frac{\gamma \chi}{2\pi}\right) + O(e^{-\frac{t}{2} \log q}).
$$

(4.5)

We see that

$$
\frac{(-1)^{sn}}{\varphi(q)^{sn}} \sum_{\chi \in X_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) \hat{\Phi}(T\Sigma_{\gamma}) = \frac{1}{T} \int_\mathbb{R} \Phi\left(\frac{t}{T}\right) \frac{(-1)^{sn}}{\varphi(q)^{sn}} \sum_{\chi \in X_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) e^{-2\pi i t \Sigma_{\gamma}} dt
$$

$$
= \frac{1}{T} \int_\mathbb{R} \Phi\left(\frac{t}{T}\right) \left(\frac{(-1)^n}{\varphi(q)^n} \sum_{\chi \in X_{1,n}} \sum_{\gamma_1, \ldots, \gamma_n} \hat{\eta}\left(\frac{\gamma_{\chi_1}}{2\pi}\right) \cdots \hat{\eta}\left(\frac{\gamma_{\chi_n}}{2\pi}\right) e^{-it(\gamma_{\chi_1} + \cdots + \gamma_{\chi_n})}\right)^s dt.
$$

We split the integral as $\int_\mathbb{R} = \int_0^\infty + \int_{-\infty}^0$. In the first of these, an application of (4.5) and (2.5) shows that it is

$$
= \frac{1}{T} \int_0^\infty \Phi\left(\frac{t}{T}\right) \left(\frac{1}{\varphi(q)^n} \sum_{\chi \in X_{1,n}} \psi_\eta(e^t, \chi) \cdots \psi_\eta(e^t, \chi_n)\right)^s dt + O\left(\frac{(K_{\delta,\eta} \log q)^{sn}}{T\varphi(q)^s}\right)
$$

$$
= \frac{1}{T} \int_0^\infty \Phi\left(\frac{t}{T}\right) M_n(e^t, q; \eta)^s dt + O\left(\frac{(K_{\delta,\eta} \log q)^{sn}}{T\varphi(q)^s}\right),
$$

by (2.1) and since $M_n(e^t, q; \eta)$ is real. Making the change of variables $u = -t$ and applying (4.4), we see that the same holds for the integral between $-\infty$ and $0$, and the claimed estimate follows.

As for the claimed identity, we specialize to $\Phi = \Phi_0 = 1_{[-1,1]}$. Taking $T \to \infty$ and applying Lemma 4.3, we obtain that

$$
\mathbb{E}[G_n(q; \eta)^s] = \lim_{T \to \infty} \frac{(-1)^{sn}}{2\varphi(q)^{sn}} \sum_{\chi \in X_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) \hat{\Phi_0}(T\Sigma_{\gamma}).
$$
Now, \( \hat{\Phi}_0(\xi) = \sin(2\pi \xi) / \pi \xi \), and by dominated convergence (note that for any \( \eta \in S_5 \), the sum over \( \gamma \) converges absolutely), we can interchange the limit and the sum over \( \gamma \). The claimed identity follows.

5 | EXPECTED VALUE OF MOMENTS

This section serves as a warm-up for the following one. We recall that \( \mathcal{C} \subset L^1(\mathbb{R}) \) is the set of nontrivial even integrable functions \( \Phi : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \hat{\Phi} \geq 0 \). Our goal is to estimate the first moment of the \( n \)th moment

\[
\mathcal{M}_{1,n}(T, q; \eta, \Phi) := \frac{1}{T} \int_0^T \Phi\left(\frac{t}{T}\right) M_n(e^t, q; \eta) dt.
\]

We begin with the following combinatorial lemma.

**Lemma 5.1.** Let \( m \geq 1, \delta > 0, \eta \in S_5 \), and assume GRH. Then, for \( q \geq 3 \), we have the lower bound

\[
\frac{1}{\varphi(q)^2} \sum_{\gamma_j \in \chi_{1,2m}} \sum_{\gamma_j \in \gamma_{\chi_1,\chi_{2m}}} \hat{\gamma}(\gamma_{\chi_1} / 2\pi) \cdots \hat{\gamma}(\gamma_{\chi_{2m}} / 2\pi) \geq \mu_{2m} \left( \frac{\alpha(\hat{\eta}^2) \log q + \beta_q(\hat{\eta}^2)}{\varphi(q)} \right)^m + O\left( \frac{\mu_{2m} (C_{\delta,\eta} \log q)^m}{\varphi(q)^{m+1}} \right),
\]

where the \( \gamma_j \) are running over the nontrivial zeros of \( L(s, \chi_j) \), and \( C_{\delta,\eta} > 0 \) is a constant.

**Proof.** By positivity of \( \hat{\gamma}(\gamma_j / 2\pi) \) (GRH is crucial in this step), we can restrict the sum over \( (\gamma_j) \) to those ordinates of zeros for which \( \forall i, \exists j \neq i \text{ s.t. } \gamma_j = -\gamma_i \) and \( \gamma_j = \gamma_i \). We can further restrict the sum such that the associated pairs \( \{i_k, j_k\} \) are disjoint, and such that \( \chi_{i_k} \not\in \{\chi_{i'\ell}, \chi_{i'\ell'} \forall i' \neq i_k, j_k\} \). We may also take \( i_k, j_k \) such that for any \( k \), the index \( i_k \) satisfies

\[
i_k = \min\{i_{\ell'}, j_{\ell'} \ (\ell' \geq k)\}.
\]

In particular, \( i_1 = 1 \). The number of such \( i_1, j_1, \ldots, i_k, j_k, \ldots, i_m, j_m \) is

\[
= (2m - 1)(2m - 3) \cdots 1 = \mu_{2m}.
\]

For a given choice of sets \( \{i_k, j_k\} \), the contribution to the left-hand side of (5.2) is

\[
\geq \frac{1}{\varphi(q)^2 \mu_{2m}} \sum_{\chi_{i_1} \neq \chi_{0,q}} b^+(\chi_{i_1}; \hat{\eta}^2) \sum_{\chi_{i_2} \not\in \gamma_{0,q}, \gamma_{\chi_1}, \gamma_{\chi_1}} b^+(\chi_{i_2}; \hat{\eta}^2) \cdots \sum_{\chi_{i_m} \not\in \gamma_{0,q}, \gamma_{\chi_1}, \gamma_{\chi_1}, \ldots, \gamma_{\chi_{m-1}, \gamma_{\chi_1}, \gamma_{\chi_1}, \ldots, \gamma_{\chi_{m-1}}}} b^+(\chi_{i_m}; \hat{\eta}^2),
\]

where denoting by \( \sum^* \) a sum without multiplicities,

\[
b^+(\chi; \hat{\eta}^2) := \sum^*_{\gamma_{\chi}} \left| \frac{1}{2 + i\gamma_{\chi}} L(s, \chi) \right|^2 \hat{\gamma}(\gamma_{\chi} / 2\pi) \geq b(\chi; \hat{\eta}^2).
\]
After applying this lower bound, we can forget the conditions

\[ \chi_{i_k} \notin \{ \chi_{i_1}, \ldots, \chi_{i_{k-1}}, \overline{\chi_{i_1}}, \ldots, \overline{\chi_{i_k}} \} \]

by adding an error term bounded by \( \ll m^2 \varphi(q)^{-m-1}(K_{\delta, \eta} \log q)^m \). This yields that the left-hand side of (5.2) is

\[ \geq \mu_{2m} \frac{1}{\varphi(q)^2m} \left( \sum_{\chi \pmod{q}} b(\chi; \tilde{\eta}^2) \right)^m + O \left( m^2 \mu_{2m} \frac{(K_{\delta, \eta} \log q)^m}{\varphi(q)^{m+1}} \right), \]

which by Proposition 3.2 is

\[ \geq \mu_{2m} \left( \frac{\alpha(\tilde{\eta}^2) \log q + \beta_q(\tilde{\eta}^2)}{\varphi(q)} \right)^m + O \left( \mu_{2m} \frac{(2K_{\delta, \eta} \log q)^m}{\varphi(q)^{m+1}} \right). \]

The claimed lower bound follows. \( \square \)

We deduce the following inequalities on the mean of moments \( \mathcal{M}_{1, 2m}(T, q; \eta, \Phi) \), which is defined in (5.1).

**Proposition 5.2.** Let \( m \geq 1, \Phi \in \mathcal{U}, \delta > 0, \eta \in S_\delta, \) and assume GRH\( \tilde{\eta} \). For any \( T \geq 1 \) and \( q \geq 3 \), we have that

\[ \mathcal{M}_{1, 2m}(T, q; \eta, \Phi) \geq \mu_{2m} \left( \frac{(K_{\delta, \eta} \log q)^{2m+1}}{T \varphi(q)} \right), \]

where \( K_{\delta, \eta}, C_{\delta, \eta} > 0 \) are constants.

**Proof.** We apply positivity in Lemma 4.4, noting that \( \tilde{\eta}, \hat{\Phi} \geq 0 \). We obtain the lower bound

\[ (\neg 1)^n M_{1, n}(T, q; \eta, \Phi) \geq \frac{1}{\varphi(q)^n} \sum_{\chi \in \chi_{1, n}} \sum_{\gamma_{X_1}, \ldots, \gamma_{X_n}} \hat{\eta} \left( \frac{\gamma_{X_1}}{2\pi} \right) \cdots \hat{\eta} \left( \frac{\gamma_{X_n}}{2\pi} \right) + O \left( \frac{(K_{\delta, \eta} \log q)^n}{T \varphi(q)} \right). \]

If \( n = 2m + 1 \), then the claimed bound follows at once from positivity of \( \hat{\eta} \left( \frac{\gamma_{X_1}}{2\pi} \right) \) (hypothesis GRH\( \tilde{\eta} \) is crucial here). If \( n = 2m \), then we apply Lemma 5.1 and obtain the desired result. \( \square \)

We are ready to prove the second part of Theorem 1.3.

**Proof of Theorem 1.3, second part.** The claimed bounds follow from positivity in Lemma 4.4 (recall also Lemma 4.3), and from Proposition 5.2. \( \square \)
6  |  HIGHER MOMENTS OF MOMENTS

The goal of this section is to prove Theorem 1.3. We first apply the inclusion–exclusion principle in order to evaluate the main terms which we will obtain later. For \( s \in \mathbb{N}, \sigma = (\sigma_1, \ldots, \sigma_s) \in \mathbb{R}^s \), \( T \in \mathbb{R}_{>0} \) and \( \Phi \in \mathcal{U} \), we define the quantities

\[
\Delta_s(\sigma) := \sum_{I \subset \{1, \ldots, s\}} (-1)^{|I|} \delta_0 \left( \sum_{\mu \in I} \sigma_\mu \right) \prod_{\mu \notin I} \delta_0(\sigma_\mu); \tag{6.1}
\]

\[
\Delta_s(\sigma; \Phi, T) := \frac{1}{\hat{\Phi}(0)} \sum_{I \subset \{1, \ldots, s\}} (-1)^{|I|} \hat{\Phi} \left( T \sum_{\mu \in I} \sigma_\mu \right) \prod_{\mu \notin I} \delta_0(\sigma_\mu), \tag{6.2}
\]

where

\[
\delta_0(x) := \begin{cases} 
1 & \text{if } x = 0; \\
0 & \text{otherwise.}
\end{cases}
\]

Note that

\[
\lim_{T \to +\infty} \Delta_s(\sigma; \Phi, T) = \Delta_s(\sigma).
\]

**Lemma 6.1.** Let \( s \in \mathbb{N}, \sigma \in \mathbb{R}^s, T \in \mathbb{R}_{>0} \) and \( \Phi \in \mathcal{U} \). The function \( \Delta_s(\sigma) \) is the indicator function of the \( \sigma \) such that \( \sum_{\mu=1}^s \sigma_\mu = 0 \) and \( \sigma_\mu \neq 0 \) for all \( 1 \leq \mu \leq s \). Moreover,

\[
\Delta_s(\sigma; \Phi, T) \geq \Delta_s(\sigma). \tag{6.3}
\]

**Proof.** We observe that

\[
\hat{\Phi}(0) \Delta_s(\sigma; \Phi, T) = \sum_{I \subset \{1, \ldots, s\}: \forall \mu \notin I, \sigma_\mu = 0} (-1)^{|I|} \hat{\Phi} \left( \sum_{\mu=1}^s \sigma_\mu \right) \prod_{\mu=1}^s \left( 1 - \delta_0(\sigma_\mu) \right).
\]

We then deduce that

\[
\Delta_s(\sigma; \Phi, T) = \begin{cases} 
\frac{1}{\hat{\Phi}(0)} \hat{\Phi} \left( T \sum_{\mu=1}^s \sigma_\mu \right) & \text{if } \sigma_\mu \neq 0 \text{ for all } \mu, \\
0 & \text{otherwise.}
\end{cases} \tag{6.4}
\]

The claimed bound follows. \[ \Box \]

We recall the definition (1.10) of \( V_{s,n}(T, q; \eta, \Phi) \), that of \( \Gamma(\chi) \) in (4.2), and moreover for \( \gamma \in \Gamma(\chi) \), we define

\[
\sigma_\gamma := \left( \frac{1}{2\pi} \sum_{j=1}^n \gamma_{\chi,\mu,j} \right)_{1 \leq \mu \leq s} \in \mathbb{C}^s. \tag{6.5}
\]

We also recall the definition (4.1). Our next goal is to express \( V_{s,n}(T, q; \eta, \Phi) \) as a sum over zeros of Dirichlet \( L \)-functions.
Lemma 6.2. Assume GRH, and let $\delta > 0$, $\eta \in S_\delta$, $\Phi \in \mathcal{L}^1(\mathbb{R})$, $\Phi$ even, $s \geq 1$ and $n \geq 2$. For $T \geq 1$ and $q \geq 3$, we have the formula

$$V_{s,n}(T, q; \eta, \Phi) = \frac{(-1)^{sn}}{\varphi(q)^{sn}} \sum_{\chi \in X_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) \Delta_\gamma(\sigma; \Phi, T) + O_{\Phi} \left( \frac{(K_{\delta, \eta} \log q)^{sn}}{T \varphi(q)^s} \right), \quad (6.6)$$

where $\hat{\eta}(\gamma)$ is defined in (4.3), $\Gamma(\chi)$ is defined in (4.2) and $K_{\delta, \eta} > 0$ is a constant.

Proof. The proof follows the lines of that of Lemma 4.4. We see that

$$L_{s,n}(T, q; \eta, \Phi) := \frac{(-1)^{sn}}{\varphi(q)^{sn}} \sum_{\chi \in X_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) \Delta_\gamma(\sigma; \Phi, T)$$

$$= \frac{1}{T \hat{\Phi}(0)} \int_{\mathbb{R}} \Phi \left( \frac{t}{T} \right) \frac{(-1)^{sn}}{\varphi(q)^{sn}} \sum_{I \subseteq \{1, \ldots, s\}} (-1)^{s-|I|} \sum_{\chi \in X_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) e^{-it \sum_{\gamma \in \Gamma(\chi)} \psi_{\chi}^I} \, dt$$

$$= \frac{1}{T \hat{\Phi}(0)} \int_{\mathbb{R}} \Phi \left( \frac{t}{T} \right) \frac{(-1)^{sn}}{\varphi(q)^{sn}} \sum_{I \subseteq \{1, \ldots, s\}} (-1)^{s-|I|} \sum_{\chi_{\mu} \neq \chi_{0,q} \mu \in \Gamma} \sum_{\gamma_{\mu} \neq \gamma_{0,q} \mu \in \Gamma} \hat{\eta}(\gamma) e^{-it \sum_{\gamma \in \Gamma(\chi)}} \, dt.$$ 

Now, the sums over $\chi_{\mu,j}$ depend on $|I|$ rather than on $I$ itself, and thus, we deduce that

$$L_{s,n}(T, q; \eta, \Phi) = \frac{1}{T \hat{\Phi}(0)} \int_{\mathbb{R}} \Phi \left( \frac{t}{T} \right) \frac{(-1)^{sn}}{\varphi(q)^{sn}} \sum_{k=0}^{s} (-1)^{s-k} \left( \sum_{\chi \in X_{1,n}} \prod_{j=1}^{n} \hat{\eta} \left( \frac{\gamma_{\chi,j}}{2\pi} \right) e^{-it \gamma_{\chi,j}} \right)^k \times$$

$$\sum_{\chi \in X_{1,n}} \prod_{j=1}^{n} \hat{\eta} \left( \frac{\gamma_{\chi,j}}{2\pi} \right) \, dt.$$ 

By Lemmas 4.3 and 4.4, the second term in parentheses is equal to $(-1)^n \varphi(q)^n m_n(q; \eta)$. We split the integral as $\int_{\mathbb{R}} = \int_{0}^{\infty} + \int_{-\infty}^{0}$. In the first of these, an application of (4.5) and (2.5) shows that it is

$$= \frac{1}{T \hat{\Phi}(0)} \int_{0}^{\infty} \Phi \left( \frac{t}{T} \right) \frac{1}{\varphi(q)^{sn}} \sum_{k=0}^{s} (-1)^{s-k} \left( \sum_{\chi \in X_{1,n}} \prod_{j=1}^{n} \psi_{\eta}(e^{\chi_1}, \ldots, e^{\chi_n}) \right)^k \times$$

$$\left( \varphi(q)^n m_n(q; \eta) \right)^{s-k} \, dt + O_{\Phi} \left( \frac{(K_{\delta, \eta} \log q)^{sn}}{T \varphi(q)^s} \right).$$

\[ \frac{1}{T \Phi(0)} \int_0^\infty \Phi \left( \frac{t}{T} \right) \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} M_n(e^t, q; \eta)^k m_n(q; \eta)^s \frac{dt}{T \Phi(q)^s} + O(\Phi(K \delta, \log q)^s), \]

by (2.1). Once more, the same holds for the integral between \(-\infty\) and 0. An application of the binomial theorem achieves the proof. □

We will establish a lower bound for \(\gamma_{s,n}(T, q; \eta, \Phi)\) through positivity of \(\hat{\eta}\) and \(\hat{\Phi}\). This will involve the following combinatorial object.

**Definition 6.3.** For \(r \in \mathbb{N}\) and \(n \geq 2\), we define \(F_{2r,n}\) to be the set of involutions \(\pi : \{1, \ldots, 2r\} \times \{1, \ldots, n\} \to \{1, \ldots, 2r\} \times \{1, \ldots, n\}\) having no fixed point and with the following two properties. First, for each fixed \(1 \leq \mu \leq 2r\), there exists a unique \(1 \leq \nu \leq 2r, \nu \neq \mu\) such that \(J_{\mu,\nu}(\pi) \neq \emptyset\), where

\[ J_{\mu,\nu}(\pi) := (\{\mu\} \times \{1, n\}) \cap \pi(\{\nu\} \times \{1, n\}) \quad (6.7) \]

is set of elements on the “row” \(\mu\) which are sent to some element on the “row” \(\nu\) through \(\pi\). Second, we require that for each \(1 \leq \mu \leq 2r\),

\[ \sum_{1 \leq \nu \leq 2r \atop \nu \neq \mu} |J_{\mu,\nu}(\pi)| \geq 2. \]

Given an involution \(\pi \in F_{2r,n}\) and an array \(\chi = (\chi_{\mu,j})_{1 \leq \mu \leq 2r \atop 1 \leq j \leq n}\) of characters modulo \(q\), we will write

\[ \pi(\chi) := (\chi_{\pi(\mu,j)})_{1 \leq \mu \leq 2r \atop 1 \leq j \leq n}. \]

One can identify \(F_{2r,n}\) with the set \(\mathcal{P}_{2r,n}\) of pairs \((\mu, j), (\nu, i)\) with \(1 \leq \mu, \nu \leq 2r, 1 \leq j, i \leq n, (\mu, j) \neq (\nu, i)\), and such that for each fixed \(\mu\),

\[ \left| \left\{ \nu \in [1, 2r] \setminus \{\mu\} \mid \exists j, i : ((\mu, j), (\nu, i)) \in \mathcal{P}_{2r,n} \right\} \right| = 1, \]

\[ \sum_{1 \leq \nu \leq 2r \atop \nu \neq \mu} \left| \left\{ j \in [1, n] \mid \exists i : ((\mu, j), (\nu, i)) \in \mathcal{P}_{2r,n} \right\} \right| \geq 2. \]

We note that for any \(\pi \in F_{2r,n}\), we have that \(|J_{\nu,\mu}(\pi)| = |J_{\mu,\nu}(\pi)|\), and moreover, for any fixed \(1 \leq \mu \leq 2r\),

\[ |J_{\mu,\mu}(\pi)| \equiv 0 \pmod{2}, \quad \sum_{\nu=1}^{2r} |J_{\mu,\nu}(\pi)| = n. \quad (6.8) \]

We begin with the following technical lemma, whose proof gives an idea on how we will count characters.

**Lemma 6.4.** Let \(r \in \mathbb{N}, n \geq 2, q \geq 3\) and let \(\pi \in F_{2r,n}\) be such that \(J_{\nu,\mu}(\pi) \neq \emptyset\) if either \(\mu = \nu\) or \(\{\mu, \nu\} = \{2u-1, 2u\}\) for some \(1 \leq u \leq r\), and \(J_{\mu,\nu}(\pi) = \emptyset\) otherwise. Let also \(1 \leq u \leq r\), and let \((\mu_0, j_0), (\mu_1, j_1)\) be such that \(1 \leq \mu_1 \leq \mu_0, \mu_0 \in \{2u-1, 2u\}, 1 \leq j_0, j_1 \leq n, \) and \(\pi(\mu_0, j_0) \neq (\mu_1, j_1)\).
Then, given a fixed array of characters \((\chi_{\mu,j})_{1 \leq \mu \leq 2u-2}\), we have the bound
\[
\sum_{1 \leq j \leq n} X_{2u-1,1} \cdots X_{2u-1,n} X_{2u,1} \cdots X_{2u,n} 
\leq q \sum_{1 \leq j \leq n} \chi_{\mu, j} \in \{\chi_\nu, i \mid i \leq n, \nu \leq \mu, (i, \nu) \notin \{(j, \mu), \pi(j, \mu)\}\},
\]
where \(\Theta_{\pi}(2u) = \prod_{\nu \neq \mu} \chi_\mu, \nu \). The implied constant is absolute.

**Proof.** The second relation of (6.8) applied for \(\mu = 2u\) and \(\mu = 2u - 1\) gives
\[
|J_{2u,2u}| + |J_{2u,2u-1}| + |J_{2u-1,2u-1}| + |J_{2u-1,2u}| = |J_{2u,2u}| + 2|J_{2u,2u-1}| + |J_{2u-1,2u-1}| = 2n.
\]
The pairs \((\mu_0, j_0), (\mu_1, j_1)\) are elements of either \(J_{2u-1,2u}, J_{2u,2u-1}, J_{2u,2u}\) or \(J_{2u-1,2u-1}\). First, we will treat the case where \((\mu_0, j_0), (\mu_1, j_1)\) \(\in J_{2u-1,2u-1}\), and thus, \((\mu_0, j_0), (\mu_1, j_1) = (2u - 1, j_1)\) and \(\pi(\mu_0, j_0) = (2u - 1, j_0')\), \(\pi(\mu_1, j_1) = (2u - 1, j_1')\) with \(j_0', j_1', j_1''\) all distinct. We have that
\[
\sum_{\chi_{\mu, j}(\mu, j) \in J_{2u-1,2u-1} \cup J_{2u,2u} \setminus X_{2u,1}} 1 \leq q |J_{2u-1,2u}| + |J_{2u-1,2u-1}| - 1.
\]
Indeed, the character \(\chi_{2u-1, j_0}\) is determined by the value of \(\chi_{2u-1, j_1}\) through the relation \(\chi_{2u-1, j_0} = \chi_{2u, j_1}\). Moreover, we claim that
\[
\sum_{\chi_{\mu, j}(\mu, j) \in J_{2u,2u-1}} 1 \leq q |J_{2u-1,2u}| - 1.
\]
Indeed, in this sum, we may only sum over the characters \(\chi_{\mu, j}\) for \((\mu, j) \in J_{2u,2u-1}\), as these determine the values of the characters for \((\mu, j) \in J_{2u,2u-1}\) thanks to the relation \(\chi_{\mu, j} = \chi_{\mu, j}\). Finally, one of \(\chi_{\mu, j}\) for \((\mu, j) \in J_{2u,2u-1}\) is determined by the other characters \(\chi_{\mu, l}\) for \((\mu, l) \in J_{2u,2u-1}\) thanks to the relation \(\prod_{j \in J_{2u-1,2u}} \chi_{2u-1, j} = \chi_0, q\). This implies the claimed upper bound. The bound for the case \((\mu_0, j_0) \in J_{2u-1,2u-1}, (\mu_1, j_1) \in J_{2u,2u}\) follows along the same lines.

Second, we treat the case where \((\mu_0, j_0), (\mu_1, j_1) \in J_{2u-1,2u-1}\) (in particular this implies that \(\mu_0 = \mu_1 = 2u - 1\)); the proof for the other cases is similar. If \(\pi(2u - 1, j_0) = (2u, j_0')\) and \(\pi(2u - 1, j_1) = (2u, j_1')\), then the sum we are interested in is actually equal to
\[
\sum_{\chi_{2u-1,1} \cdots \chi_{2u-1,n}, \chi_{2u,1} \cdots \chi_{2u,n}} 1,
\]
where \(\pi'\) is equal to \(\pi\) except for the values \(\pi'(2u - 1, j_0) = (2u - 1, j_1)\) and \(\pi'(2u, j_0') = (2u, j_1')\). Thus, we have reduced the problem to the case treated in the first part of the proof, and the result follows.
We continue to estimate sums over characters in the following lemma.

**Lemma 6.5.** Let \( r \in \mathbb{N}, n \geq 2, q \geq 3 \) and let \( \pi \in F_{2r,n} \). We have the estimate
\[
C_{2r,n}(q; \pi) := \sum_{\chi \in X_{2r,n}} 1 = \varphi(q)^{(n-1)r} \left( 1 + O \left( \frac{n^2 r^2}{\varphi(q)} \right) \right). \tag{6.9}
\]

If moreover \( q \) is large enough in terms of \( \delta \) and \( \eta \), then in the ranges \( n \leq \log q / \log_2 q \) and \( r \leq c \varphi(q)^{1/2}/n \) with \( c > 0 \) a small enough absolute constant, we have the estimate
\[
\sum_{\chi \in X_{2r,n}} \prod_{1 \leq \mu \leq 2r, 1 \leq j \leq n} b(\chi_{\mu,j}; \eta^2)^{1/2} = \varphi(q)^{(n-1)r} \left( \alpha \eta^2 \log q \right)^{nr} \left( 1 + O \left( \frac{n \log_2 q}{\log q} \right) \right)^r. \tag{6.10}
\]

Under the additional assumption of GRH\( \eta \), if \( 1 \leq \mu_0 \neq \nu_0 \leq 2r \) are such that \( J_{\mu_0,\nu_0}(\pi) \neq \emptyset \), then we have the bound
\[
\sum_{\chi \in X_{2r,n}} \sup_{z \in \mathbb{R}} \left\{ \sum_{\varphi_{\mu_0,1} \cdots \varphi_{\mu_0,n} \neq 0} \prod_{1 \leq \mu \leq 2r, 1 \leq j \leq n} b(\chi_{\mu,j}; \eta^2)^{1/2} \right\} \ll_{\delta, \eta} \varphi(q)^{(n-1)r} \left( \alpha \eta^2 \log q \right)^{nr-1}. \tag{6.11}
\]

**Proof.** We begin with (6.9). Fix \( \pi \in F_{2r,n} \). In the sum defining \( C_{2r,n}(q; \pi) \) and by definition of \( F_{2r,n} \), we can change the indexing in the variable \( \mu \) in such a way that \( J_{\mu,\nu}(\pi) \neq \emptyset \) (recall (6.7)) if \( \{\mu, \nu\} \in \{2u-1, 2u\} \) for some \( 1 \leq u \leq r \), and \( J_{\mu,\nu}(\pi) = \emptyset \) otherwise.

We distinguish two different sets of indices:
\[
J_\pi := \bigcup_{\mu=1}^{2r} J_{\mu,\mu}(\pi) = \{\mu, j \in \{\mu\} \times [1, n]\};
\]
\[
I_\pi := \bigcup_{1 \leq \mu \neq \nu \leq 2r} J_{\mu,\nu}(\pi) = \{\mu, j \in \{\mu\} \times [1, n]\};
\]
and we recall the relations (6.8). For each fixed \( 1 \leq u \leq r \), the conditions \( X_{2u-1,1} \cdots X_{2u-1,n} = X_{2u,1} \cdots X_{2u,n} = X_0,q \) are equivalent to \( \Theta_{\pi}(2u) := \prod_{1 \leq j \leq n} X_{2u,j} = X_0,q \). Note also that the condition \( \chi_{\mu,j} = \chi_{\nu,i} \Rightarrow (\mu, j) \in \{(v, i), \pi(v, i)\} \) implies that for \( (v, i) \notin \{(\mu, j), \pi(\mu, j)\}, \chi_{v,i} \neq \chi_{\mu,j} \).
The choice of $\pi$ implies that we can write $\pi(\chi) = (\pi_2(\chi_2), \pi_4(\chi_4), \ldots, \pi_{2r}(\chi_{2r}))$ where $\chi = (\chi_2, \chi_4, \ldots, \chi_{2r})$ and $\chi_{2u} = (\chi_{\mu, j})_{\mu \in \{2u-1, 2u\}}$. Then, we can write

$$C^{2r, n}(q; \pi) = \sum_{\chi_2 \in X_{2,n}} \sum_{\chi_4 \in X_{2,n}} \ldots \sum_{\chi_{2r} \in X_{2,n}} 1,$$

where the star means that in each of the sums, we have the extra condition

$$\chi_{\mu, j} \notin \{ \chi_{\nu, i} | i \leq n, \nu \leq \mu, (i, \nu) \notin \{(\mu, j), \pi(\mu, j)\} \} \quad (\mu = 2u - 1, 2u; 1 \leq j \leq n). \quad (6.12)$$

We first treat $C^{2r, n}(q; \pi)$ as if there were no star on the sums; we will come back to the original sums later. For any fixed $1 \leq u \leq r$, we have that

$$\sum_{\chi_{2u} \in X_{2,n}} 1 = \sum_{\chi_{2u} \in X_{2,n}} 1 \cdot \sum_{\chi_{2u} \in X_{2,n}} 1.$$

The second factor on the right-hand side is equal to

$$(\varphi(q) - 1)^{1/2} |J_{2u,2u-1}|-1 \left(1 + O \left( \frac{1}{\varphi(q)} \right) \right),$$

since $|J_{2u,2u-1}| \geq 2$. As for the first, it is equal to the square of

$$\sum_{\chi_{2u} \in X_{2,n}} 1 = (\varphi(q) - 1)^{1/2} |J_{2u,2u}|-1.$$

The analog of the estimate (6.9), with no star on the sums, follows. We now indicate how to remove the condition (6.12) using Lemma 6.4. This is done by summing over the $\leq (2rn)^2$ possibilities for the pair of pairs $(\mu_0, j_0), (\mu_1, j_1)$ for which $\chi_{\mu_0, j_0} = \chi_{\mu_1, j_1}$ and $\pi(\mu_0, j_0) \neq (\mu_1, j_1)$; in each of these sums, we can apply Lemma 6.4. The claimed bound follows.

We now move to (6.10). Following the steps above, we may once more assume that $J_{\mu, \nu}(\pi) \neq \emptyset$ (recall (6.7)) if $\{\mu, \nu\} \in \{2u - 1, 2u\}$ for some $1 \leq u \leq r$, and $J_{\mu, \nu}(\pi) = \emptyset$; otherwise, this implies that the left-hand side of (6.10) is equal to

$$L^{2r, n}(q; \pi) := \sum_{\chi_2 \in X_{2,n}} \sum_{\chi_4 \in X_{2,n}} \ldots \sum_{\chi_{2r} \in X_{2,n}} \prod_{1 \leq \mu \leq 2r} b(\chi_{\mu, j}; \hat{\eta})^{1/2}.$$

As before, we may remove the star by introducing an admissible error term; this is possible thanks to Proposition 3.2 in which we use the bound $q_\chi < q$. Now, for any fixed $1 \leq u \leq r$, keeping in mind
that $J_{2u,2u-1} \neq \emptyset$, we notice that by Lemma 3.6,

$$\sum_{\chi_{2u,j} \neq \chi_{0,q}} \prod_{j \in J_{2u,2u-1}} b(\chi_{2u,j}; \hat{\eta}^2) = \varphi(q)^{|J_{2u,2u-1}|-1}(\alpha(\hat{\eta}^2) \log q)^{|J_{2u,2u-1}|} \left(1 + O\left(\frac{\log q}{\log q}\right)\right).$$

The argument is similar for the factors involving $J_{2u,2u}$ and $J_{2u-1,2u-1}$, and this concludes the proof. The proof of (6.11) is similar, thanks to Lemma 3.6.

□

We are ready to obtain our first lower bound on $V_{2r,n}(T, q; \eta, \Phi)$ in terms of $|F_{2r,n}|$.

Lemma 6.6. Let $\delta > 0$, $\eta \in S_{\delta}$, $\Phi \in U$, $r \geq 1$, and $n \geq 2$, and assume GRH $\hat{\eta}$. For $T \geq 1$, $q$ large enough in terms of $\delta$ and $\eta$, and in the ranges $n \leq \log q / \log_2 q$ and $r \leq \log q$, we have the lower bound

$$V_{2r,n}(T, q; \eta, \Phi) \geq \frac{V_n(q; \eta)^r}{v_n^r} |F_{2r,n}| \left(1 + O_{\delta, \eta} \left(n \log_2 q \frac{\log q}{\log q}\right)^r + O_{\delta, \eta} \left(\frac{r}{\log q}\right)^r\right) + O_{\Phi} \left((K_{\delta, \eta} \log q)^{2rn} \frac{T \varphi(q)^{2r}}{s}\right),$$

where $K_{\delta, \eta} > 0$ is a constant.

Proof. We recall that we can view the $\pi \in F_{2r,n}$ as involutions on the set of arrays of characters $\chi = (\chi_{\mu,j})_{1 \leq \mu \leq 2r, 1 \leq j \leq n}$ by setting $\pi(\chi) = (\chi_{\pi(\mu,j)})_{1 \leq \mu \leq 2r, 1 \leq j \leq n}$.

By Lemma 6.1 and by positivity of $\hat{\eta}$ and of $\Delta_{2r}(\sigma, \Phi, T)$, we can restrict the sum on the right-hand side of (6.6) as follows:

$$V_{2r,n}(T, q; \eta, \Phi) \geq \frac{1}{\varphi(q)^{2rn}} \sum_{\chi \in F_{2r,n}} \sum_{\pi \in \chi_{2r,n}} \hat{\eta}(\chi) \Delta_{2r}(\sigma, \Phi, T) + O_{\Phi} \left((K_{\delta, \eta} \log q)^{sm} \frac{T \varphi(q)^{2r}}{s}\right)$$

$$\geq \frac{1}{\varphi(q)^{2rn}} \sum_{\pi \in F_{2r,n}} \sum_{\chi \in \chi_{2r,n}} \hat{\eta}(\chi) \Delta_{2r}(\sigma, \Phi, T) + O_{\Phi} \left((K_{\delta, \eta} \log q)^{sm} \frac{T \varphi(q)^{2r}}{s}\right).$$

(6.14)

Now, we may bound the inner sum as follows:

$$\sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) \geq \sum_{\gamma \in \Gamma(\chi)} \prod_{1 \leq \mu \leq 2r, 1 \leq j \leq n} \left(\frac{\text{ord}_{x=\frac{1}{2}+iy_{\mu,j}} L(s, \chi_{\mu,j})}{1/2}\right)^{1/2}.$$
Applying Lemma 6.5, we can remove the conditions $\sum_{j=1}^{n} Y_{\mu,j} \neq 0$ at the cost of an admissible error term. The resulting sum is handled with (6.10), and the claimed estimate follows. □

The final step is to compute $|F_{2r,n}|$.

**Lemma 6.7.** For $r \in \mathbb{N}$ and $n \geq 2$, we have the formula

$$|F_{2r,n}| = \mu_{2r} v_n,$$

where $F_{2r,n}$ was defined in Definition 6.3 and $v_n$ in (1.11).

**Proof.** We can view the elements of $F_{2r,n}$ as pairs $\{(\mu, j), (\nu, i)\}$, where $1 \leq \mu, \nu \leq 2r$, $1 \leq j, i \leq n$, and $\pi(\mu, j) = (\nu, i)$. Clearly, $|F_{2r,n}| = \mu_{2r} |G_{2r,n}|$, where $G_{2r,n}$ is the set of involutions $\pi : \{1, \ldots, 2r\} \times \{1, \ldots, n\} \to \{1, \ldots, 2r\} \times \{1, \ldots, n\}$ having no fixed point and such that (recall (6.7))

$$\{(\mu, \nu) \in [1, 2r]^2 : J_{\mu,\nu}(\pi) \neq \emptyset\} = \{(2u, 2u - 1), (2u - 1, 2u) : 1 \leq u \leq r\},$$

and $|J_{2u,2u-1}| \geq 2$. For $\pi \in G_{2r,n}$ and $1 \leq u \leq r$, we let $k_u(\pi) := |J_{2u-1,2u-1}(\pi)|$ (recall (6.7)), and $k(\pi) = (k_u(\pi))_{1 \leq u \leq r}$. We recall (6.8); in particular, $\mathcal{E}(\pi) := k(\pi)/2 \in (\mathbb{Z}_{\geq 0})^r$. For any given $\mathcal{E} \in \{0, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor\}^r$, there are exactly $\prod_{1 \leq u \leq r} \left(\frac{n}{2\mathcal{E}_u} \right)^2 (n - 2\mathcal{E}_u)! (\frac{(2\mathcal{E}_u)!}{2^u \mathcal{E}_u!})^2$ involutions $\pi \in G_{2r,n}$ for which $\mathcal{E}(\pi) = \mathcal{E}$. The claim follows. □

We now work with odd values $s = 2r + 1$. The combinatorics are more complicated here, and we need to change the definition of the set $F_{2r,n}$ accordingly.

**Definition 6.8.** For $r \in \mathbb{N}$ and $n \geq 2$, we define $F_{2r+1,n}$ to be the set of involutions $\pi : \{1, \ldots, 2r + 1\} \times \{1, \ldots, n\} \to \{1, \ldots, 2r + 1\} \times \{1, \ldots, n\}$ having no fixed point and such that there exists a set $N_\pi \subset \{1, \ldots, 2r + 1\}$ of cardinality 3 for which for each fixed $\mu \in S_\pi := \{1, \ldots, 2r + 1\} \setminus N_\pi$, there exists a unique $1 \leq \nu < 2r + 1$, $\nu \neq \mu$ such that $J_{\mu,\nu} \neq \emptyset$, and moreover $\{|(\mu, \nu) : \mu, \nu \in N_\pi, \mu < \nu, J_{\mu,\nu}(\pi) \neq \emptyset\| \in \{2, 3\}$. In other words, $\pi$ is such that there is a set of “rows” $S_\pi$ that are paired two by two through $\pi$, and its complement $N_\pi$ is such that its rows are either paired in a 3-cycle, or for which there is one line which is paired with the other two. Moreover, we require that for each $1 \leq \mu \leq 2r + 1$,

$$\sum_{1 \leq \nu \leq 2r + 1 \atop \nu \neq \mu} |J_{\mu,\nu}| \geq 2.$$

Note that the relations (6.8) (with $2r + 1$ in place of $2r$) hold for any $\pi \in F_{2r+1,n}$. We first prove an analog of Lemma 6.5.

**Lemma 6.9.** Let $r \in \mathbb{N}$, $n \geq 2$ and $q \geq 3$, and let $\pi \in F_{2r+1,n}$. We have the estimate

$$C_{2r+1,n}(q; \pi) := \sum_{X \in X_{2r+1,n} \atop \pi(X) = \overline{X}} 1 = \varphi(q)^{(n-1)2r+1} \frac{1}{2} \left(1 + O\left(\frac{n^2 r^2}{\varphi(q)}\right)\right). \quad (6.15)$$
If moreover $q$ is large enough in terms of $\delta$ and $\eta$, then in the ranges $n \leq \log q / \log 2$ and $r \leq \varphi(q)^{1/2}/n$, we have the estimate

$$\sum_{\chi \in \chi_{2r+1,n}} \prod_{1 \leq \mu \leq 2r+1, 1 \leq j \leq n} b(\chi_{\mu,j}; \eta^2)^{1/2} = \varphi(q)^{(n-1)\frac{2r+1}{2} - \frac{1}{2}} (\alpha(\eta^2) \log q)^n \frac{2r+1}{2} \times \left(1 + O\left(\frac{n \log q}{\log q}\right)^{r+1}\right).$$

Finally, the analog of (6.11) holds with $\varphi(q)^{(n-1)\frac{2r+1}{2} - \frac{1}{2}} (\alpha(\eta^2) \log q)^n \frac{2r+1}{2} - 1$ in place of $\varphi(q)^{(n-1)r(\alpha(\eta^2) \log q)^n r - 1}$.

**Proof.** Fix $\pi \in F_{2r+1,n}$. In the sum defining $C_{2r+1,n}(q; \pi)$ and by definition of $F_{2r+1,n}$, we can change the indexing in the variable $\mu$ in such a way that $J_{\mu,\nu}(\pi) \neq \emptyset$ if either $\{\mu, \nu\} \subseteq \{1, 2, 3\}$ or $\{\mu, \nu\} = \{2u, 2u + 1\}$ for some $2 \leq u \leq r$, and $J_{\mu,\nu}(\pi) = \emptyset$ otherwise. For $\chi \in X_{2r+1,n}$, let us write $\chi = (\chi_1, \chi_2)$, where $\chi_1 = (\chi_{\mu,j})_{1 \leq \mu \leq 3, 1 \leq j \leq n} \in X_{3,n}$ and $\chi_2 = (\chi_{\mu,j})_{4 \leq \mu \leq 2r+1, 1 \leq j \leq n} \in X_{2r-2,n}$. With this notation, we can define $\pi_1$ and $\pi_2$ in such a way that $\pi(\chi) = (\pi_1(\chi_1), \pi_2(\chi_2))$. Then the condition $\pi(\chi) = \chi$ is equivalent to $\pi_1(\chi_1) = \chi_1$ and $\pi_2(\chi_2) = \chi_2$. This is possible since $\pi_1(\chi_1)$ does not depend on $\chi_2$ and $\pi_2(\chi_2)$ does not depend on $\chi_1$. Consequently, we may write

$$C_{2r+1,n}(q; \pi) = \sum_{\chi_1 \in X_{3,n}} \sum_{\chi_2 \in X_{2r-2,n}} 1.$$

The innermost sum can be computed using the same arguments as in Lemma 6.5, and we deduce that it is sufficient to prove (6.15) for $r = 1$. Recalling Definition 6.8, this means that $N_{\pi} = \{1, 2, 3\}$ and $S_{\pi} = \emptyset$.

We first treat the case where $|(\mu, \nu) : \mu < \nu, J_{\mu,\nu}(\pi) \neq \emptyset| = 2$; by reordering $\mu$ in the array of characters $\chi = (\chi_{\mu,j})_{1 \leq \mu \leq 3}$, we may assume that $J_{1,2}(\pi), J_{1,3}(\pi) \neq \emptyset$, and $J_{2,3}(\pi) = \emptyset$. Under the condition $\pi(\chi) = \chi$, the relations $\prod_{j=1}^n \chi_{\mu,j} = \chi_{0,q}$ ($\mu = 1, 2, 3$) are equivalent to

$$\Theta_{\mu,1}(\pi) := \prod_{j \in J_{\mu,1}(\pi)} \chi_{\mu,j} = \chi_{0,q} (\mu = 2, 3)$$

(since the remaining relation is automatically satisfied). We reformulate the relations for $j \in J_{\mu,\mu}$ which follow from the equality $\pi(\chi) = \chi$, with $\chi \in X_{3,n}$. To do so, we denote $\chi_{\mu,j} := (\chi_{\mu,j})_{j \in J_{\mu,\mu}}$ and $\pi_{\mu}(\chi_{\mu,j}) := (\pi(\chi_{\mu,j}))_{j \in J_{\mu,\mu}}$. Then, the condition $\pi(\chi) = \chi$ implies that $\pi_{\mu}(\chi_{\mu,j}) = \chi_{\mu,j}$ for $1 \leq \mu \leq 3$. This condition also implies that the values of $\chi_{1,j}$ with $j \notin J_{1,1}$ are determined by those of $\chi_{\mu,j}$ with $\mu \in \{2, 3\}$. We deduce that

$$C_{3,n}(q; \pi) = \sum_{\chi_{2,1,\ldots,\chi_{2,n},\chi_{3,1,\ldots,\chi_{3,n}} \neq \chi_{0,q}} 1 \prod_{j \in J_{1,1}(\pi)} (1 \cdot \chi_{0,q,j}) \prod_{\mu \in \{2, 3\}} \pi_{\mu}(\chi_{\mu,j})^{\varphi(q)/2} \cdot \varphi(q)^{\varphi(q)/2} \prod_{\mu \in \{2, 3\}} \varphi(q)^{\varphi(q)/2} \cdot \varphi(q)^{\varphi(q)/2} \left(1 + O\left(\frac{n^2}{\varphi(q)}\right)\right).$$
Applying (6.8), we deduce that the exponent of \( \varphi(q) \) is equal to
\[
\frac{|J_{2,2}|}{2} + \frac{|J_{3,3}|}{2} + \frac{|J_{1,1}|}{2} - 1 + \frac{|J_{2,1}|}{2} + \frac{|J_{1,2}|}{2} + \frac{|J_{1,3}|}{2} + \frac{|J_{2,3}|}{2} + \frac{|J_{3,1}|}{2} - 1 + \frac{|J_{2,2}|}{2} + \frac{|J_{3,3}|}{2} - |J_{1,1}| - 2.
\]

By our hypothesis on \( \pi \), we have that \( n - |J_{1,1}| = n - |J_{2,2}| + n - |J_{3,3}| \), which concludes the proof in this case.

We now move to the case where \( |\{(\mu, \nu) : \mu < \nu, J_{\mu,\nu}(\pi) \neq \emptyset\}| = 3 \). Given an array of characters \( \chi = (\chi_{\mu,j})_{\mu,j} \) such that \( \pi(\chi) = \overline{\chi} \) and keeping the notation
\[
\Theta_{\mu,\nu}(\pi) := \prod_{j \in J_{\mu,\nu}} \chi_{\mu,j},
\]
we can rewrite the conditions \( \prod_{j=1}^n \chi_{\mu,j} = \chi_{0,q} (1 \leq \mu \leq 3) \) as
\[
\Theta_{1,2}(\pi) = \Theta_{1,3}(\pi); \quad \Theta_{1,3}(\pi) = \Theta_{2,3}(\pi).
\]

This new set of conditions is now minimal, since \( J_{1,3} \cap J_{2,3} = \emptyset \). We can now write
\[
C_3, n(q; \pi) = \sum_{\chi_{1,1} \neq \chi_{0,q}} \sum_{\chi_{2,2} \neq \chi_{0,q}, (j \in J_{2,3} \cup J_{2,3})} \sum_{\chi_{3,3} \neq \chi_{0,q}, (j \in J_{3,3})} \frac{1}{\pi_1(\chi_1) = \pi_2(\chi_2) = \pi_3(\chi_3) = \pi(\pi_1(\chi_1), \pi_2(\chi_2), \pi_3(\chi_3))} \chi_{\mu,j} = \chi_{\nu,i} \Rightarrow (\mu, j) \in \{(\nu, i), \pi(\nu, i)\}
\]
\[
= \varphi(q) \left( \frac{|J_{1,1}|}{2} + |J_{2,2}| + |J_{3,3}| - 1 \right) \cdot \varphi(q) \left( \frac{|J_{1,2}|}{2} + |J_{2,3}| - 1 \right) \cdot \varphi(q) \left( \frac{|J_{3,3}|}{2} \right) \left( 1 + O \left( \frac{n^2}{\varphi(q)} \right) \right).
\]

The exponent of \( \varphi(q) \) is now
\[
\frac{|J_{1,1}| + |J_{2,2}| + |J_{3,3}|}{2} + |J_{1,2}| + |J_{2,3}| + |J_{3,3}| - 2,
\]
which is seen to be equal to \( \frac{3}{2}n - 2 \) by adding the equations \( n = |J_{\mu,1}| + |J_{\mu,2}| + |J_{\mu,3}| \) for \( 1 \leq \mu \leq 3 \) and using the relation \( \sum_{\mu,j} = |J_{\nu,i}| \).

The second and third claimed estimates follow as in the proof of Lemma 6.5. \( \square \)

**Lemma 6.10.** Assume GRH_{h_0}, and let \( \delta > 0, \eta \in S_\delta, \Phi \in U \), and \( r \in \mathbb{N}, n \geq 2 \), with \( n \) even. For \( T \geq 1, q \) large enough in terms of \( \delta \) and \( \eta \) and in the ranges \( n \leq \log q / \log_2 q \) and \( r \leq \log q \), we have the lower bound
\[
\gamma_{2r+1,n}(T, q; \eta, \Phi) \geq \frac{1}{\varphi(q)^{\frac{1}{2}}} \left( \frac{V_n(q; \eta)}{V_n} \right)^{\frac{2r+1}{2}} \left| F_{2r+1,n} \right|^r \left( 1 + O_{\delta, \eta} \left( \frac{n \log_2 q}{\log q} \right) \right)^r + O_{\delta, \eta} \left( \frac{r}{\log q} \right).
\]

(6.16)
Moreover, for all $q \geq 3$ and all $n \geq 1$, we have the weaker bound

$$(-1)^n \mathcal{V}_{2r+1,n}(T, q; \eta, \Phi) \geq O_\Phi \left( \frac{(K_{\delta,\eta} \log q)^{(2r+1)n}}{T \varphi(q)^{2r+1}} \right).$$

Here, $K_{\delta,\eta} > 0$ is a constant.

**Proof.** The estimate for odd values of $n$ is obtained by applying Lemma 6.2 and discarding the sum over zeros using positivity. For even values of $n$, we bound the sum over zeros in the same way as in Lemmas 6.5 and 6.6, using this time Lemma 6.9. \hfill \Box

In order to state the last combinatorial lemma of this section, we define the constants

$$\nu'_n := \sum_{(\ell, k) \in \mathbb{Z}_+^2} \frac{n!^3}{k_1!k_2!2\ell_1!\ell_2!\ell_3!};$$

(6.17)

$$\nu''_n := \frac{1}{3} \sum_{(\ell, k) \in \mathbb{Z}_+^2} \frac{n!^3}{k_1!k_2!k_3!2\ell_1!\ell_2!\ell_3!};$$

(6.18)

**Lemma 6.11.** For $r \in \mathbb{N}$ and $n \in 2\mathbb{N}$, we have the formula

$$|F_{2r+1,n}| = \frac{(2r + 1)!}{2^r(r - 1)!} \nu'^{-1} \nu'' + \nu'_n + \nu''_n.$$

**Proof.** Clearly,

$$|F_{2r+1,n}| = \frac{(2r + 1)!}{2^r(r - 1)!} \left( |A_{2r+1,n}| + \frac{1}{3} |B_{2r+1,n}| \right),$$

where $A_{2r+1,n}, B_{2r+1,n}$ are sets of involutions $\pi : \{1, \ldots, 2r + 1\} \times \{1, \ldots, n\} \to \{1, \ldots, 2r + 1\} \times \{1, \ldots, n\}$ having no fixed point, for which for each $1 \leq \mu \leq 2r + 1$,

$$\sum_{1 \leq \nu \leq 2r+1 \atop \nu \neq \mu} |J_{\mu,\nu}(\pi)| \geq 2,$$

(6.19)

such that for $\pi \in A_{2r+1,n}$,

$$\{\{\mu, \nu\} : J_{\mu,\nu}(\pi) \neq \emptyset\} = \{\{1, 2\}, \{1, 3\}\} \cup \{\{2u, 2u + 1\} : 2 \leq u \leq r\},$$

and for $\pi \in B_{2r+1,n}$,

$$\{\{\mu, \nu\} : J_{\mu,\nu}(\pi) \neq \emptyset\} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\} \cup \{\{2u, 2u + 1\} : 2 \leq u \leq r\}.$$
We let

\[ k_u(\pi) := \begin{cases} |J_{u,1}(\pi)| & \text{if } 1 \leq u \leq 3, \\ |J_{2u-4,2u-4}(\pi)| & \text{if } 4 \leq u \leq r + 2, \end{cases} \]

and \( k(\pi) = (k_u(\pi))_{1 \leq u \leq r+2} \). Note that \( \ell(\pi) := k(\pi)/2 \in (\mathbb{Z}_{\geq 0})^{r+2} \).

For \( \pi \in A_{2r+1,n} \) and \( 1 \leq u \leq r + 1 \), we observe that

\[ n - 2\ell_1(\pi) = |J_{1,2}(\pi)| + |J_{1,3}(\pi)|, \quad n - 2\ell_2(\pi) = |J_{1,2}(\pi)|, \quad n - 2\ell_3(\pi) = |J_{1,3}(\pi)|, \]

and hence \( n + 2\ell_1(\pi) = 2\ell_2(\pi) + 2\ell_3(\pi) \). For any given \( \ell \in \{0, \ldots, \lfloor n/2 \rfloor - 1\}^{r+2} \) such that \( n + 2\ell_1 = 2\ell_2 + 2\ell_3 \), there are exactly

\[ \left( \frac{n - 2\ell_1}{n - 2\ell_2} \right) (n - 2\ell_2)! (n - 2\ell_3)! \prod_{1 \leq u \leq 3} \left( \begin{array}{c} n \\ 2\ell_u \end{array} \right)! \prod_{4 \leq u \leq r+2} \left( \begin{array}{c} n \\ 2\ell_u \end{array} \right)! \left( \begin{array}{c} 2\ell_u \\ 2\ell_u \end{array} \right)! \]

involutions \( \pi \in A_{2r+1,n} \) for which \( \ell(\pi) = \ell \).

As for \( B_{2r+1,n} \), we let \( m_1(\pi) := |J_{1,2}(\pi)|, m_2(\pi) := |J_{2,3}(\pi)| \) and \( m_3(\pi) := |J_{3,1}(\pi)| \). For any given \( \ell \in \{0, \ldots, \lfloor n/2 \rfloor - 1\}^{r+2} \) and \( m \in \{1, \ldots, n-1\}^3 \) such that \( m_u + m_{u+2} + 2 \ell_u = n \) for \( 1 \leq u \leq 3 \) (with the convention that \( m_4 = m_1, m_5 = m_2 \)), there are

\[ \prod_{u=1}^3 \left( \frac{n - 2\ell_u}{m_u} \right) m_u! \left( \begin{array}{c} n \\ 2\ell_u \end{array} \right)! \prod_{4 \leq u \leq r+2} \left( \begin{array}{c} n \\ 2\ell_u \end{array} \right)! \left( \begin{array}{c} 2\ell_u \\ 2\ell_u \end{array} \right)! \]

involutions \( \pi \in B_{2r+1,n} \) (note that \( m_u(\pi) \geq 1 \) implies the condition \( (6.19) \)) for which \( \ell(\pi) = \ell \) and \( m(\pi) = m \). The claim follows from adding all the different possible outcomes.

We are now ready to prove the main results of this section.

**Proof of Theorem 1.3.** The lower bounds for \( \mathbb{V}_{s,n}(T, q; \eta, \Phi) \) follow from Lemmas 6.6 and 6.10, combined with Lemmas 6.7 and 6.11.

We also establish the lower bounds in Theorem 1.7.

**Proof of Theorem 1.7, first part.** First note that under GRH, Lemma 6.2 implies that

\[ \mathbb{V}_{s,n}(q; \eta) = \lim_{T \to \infty} \mathbb{V}_{s,n}(T, q; \eta, 1_{[-1,1]}) = \frac{(-1)^{sn}}{\varphi(q)^{sn}} \sum_{\chi \in \chi_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) \Delta_s(\sigma_\gamma), \]

by dominated convergence. For \( s = 2r \), we argue as in the proof of Lemma 6.6 and use positivity of \( \hat{\eta} \) to deduce that

\[ \mathbb{V}_{2r,n}(q; \eta) \geq \left( 1 + O_{\delta,\eta} \left( n \log_2 q \over \log q \right) \right)^{r} + O_{\delta,\eta} \left( r \over \log q \right) \frac{V_n(q; \eta)^r}{n^r} |F_{2r,n}|. \]

Thanks to Lemma 6.7, this implies (1.15). The proof of (1.16) is similar, following this time Lemmas 6.10 and 6.11.
The goal of this section is to prove Theorems 1.1 and 1.2. We will separate the proof into two parts. On the one hand, we will show that it follows from GRH, and, on the other hand, we will show that it also follows from its negation.

We first need to adapt a result of Hooley [49, Theorem 1].

**Lemma 7.1.** Let \( \delta > 0 \) and \( \eta \in S_\delta \), and assume GRH. Then, in the range \( q \geq 3 \) and \( \log q \leq S \), we have the bound

\[
\frac{1}{S} \int_0^S M_2(e^t, q; \eta) dt \ll_{\delta, \eta} \frac{\log q}{\varphi(q)}.
\]

**Proof.** We will show that in the range \( 2 \log q \leq T \), we have the bound

\[
\mathcal{V}_{1,2}(T, q; \eta, \Phi_0) = \frac{2}{T} \int_0^\infty \Phi_0\left(\frac{t}{T}\right) M_2(e^t, q; \eta) dt \ll_{\delta, \eta} \frac{\log q}{\varphi(q)},
\]

where

\[
\Phi_0(x) := \begin{cases} 1 - |x| & \text{if } |x| \leq 1, \\ 0 & \text{otherwise}, \end{cases}
\]

and \( \hat{\Phi}_0(\xi) = \frac{\sin(\pi \xi)}{\pi \xi} \geq 0 \). This clearly implies our claim.

By Lemma 6.2, we have the estimate

\[
\mathcal{V}_{1,2}(T, q; \eta, \Phi_0) = \frac{2}{\varphi(q)^2} \sum_{\chi \neq \chi_0, q} \sum_{\gamma, \lambda \chi} \widehat{\eta}\left(\frac{\gamma}{2\pi}\right) \widehat{\eta}\left(\frac{\lambda}{2\pi}\right) \widehat{\Phi}_0\left(\frac{T}{2\pi}(\gamma - \lambda)\right) + O\left(\frac{(K_{\delta, \eta} \log q)^2}{\varphi(q)^2 T}\right),
\]

where \((\gamma, \lambda)\) runs over the pairs of imaginary parts of zeros of \( L(s, \chi) \) on the critical line. We will consider three distinct ranges of \(|\gamma - \lambda|\).

First, the bound (3.6) implies that

\[
\sum_{|\gamma - \lambda| \geq 1} \widehat{\Phi}_0\left(\frac{T}{2\pi}(\gamma - \lambda)\right) \ll \frac{(\log q)^2}{T^2},
\]

whose contribution to \( \mathcal{V}_{1,2}(T, q; \eta, \Phi_0) \) is \( \ll \frac{(\log q)^2}{T^2 \varphi(q)} \).

Second, in the range \( w \leq |\gamma - \lambda| \leq 2w \) (which we denote by \(|\gamma - \lambda| \sim w\)) with \( \frac{1}{3\log q} \leq w \leq 1 \), we will use the fact that for \(|t| \in [0, \frac{9}{10}]\), \( \Phi_0(t) \geq \frac{1}{10} \). We deduce that

\[
\sum_{\chi \neq \chi_0, q} \sum_{|\gamma - \lambda| \sim w} \widehat{\Phi}_0\left(\frac{T}{2\pi}(\gamma - \lambda)\right) \ll \frac{1}{(wT)^2} \sum_{\chi \neq \chi_0, q} \sum_{|\gamma - \lambda| \sim w} \widehat{\eta}\left(\frac{\gamma}{2\pi}\right) \widehat{\eta}\left(\frac{\lambda}{2\pi}\right) \widehat{\Phi}_0\left(\frac{\gamma - \lambda}{5\pi w}\right) \ll \frac{\varphi(q)(\log q)^2}{wT^2},
\]
by Proposition 3.4. As a result, decomposing the sum over $\gamma_x$ into dyadic intervals, we obtain the bound
\[
\frac{2}{\varphi(q)^2} \sum_{X \neq X_{0,q}} \sum_{\gamma_x, \lambda_x, \psi \psi' \neq \psi} \hat{\eta} \left( \frac{\gamma_x}{2\pi} \right) \hat{\eta} \left( \frac{\lambda_x}{2\pi} \right) \Phi_0 \left( \frac{T}{2\pi} (\gamma_x - \gamma_{\psi'}) \right) \ll \frac{(\log q)^2}{\varphi(q) T^2}.
\]
Finally, we are left to bound
\[
\frac{2}{\varphi(q)^2} \sum_{X \neq X_{0,q}} \sum_{\gamma_x, \lambda_x, \psi \psi' \neq \psi} \hat{\eta} \left( \frac{\gamma_x}{2\pi} \right) \hat{\eta} \left( \frac{\lambda_x}{2\pi} \right) \Phi_0 \left( \frac{T}{2\pi} (\gamma_x - \gamma_{\psi'}) \right)
\leq \frac{2}{\varphi(q)^2} \sum_{X \neq X_{0,q}} \sum_{\gamma_x, \lambda_x, \psi \psi' \neq \psi} \hat{\eta} \left( \frac{\gamma_x}{2\pi} \right) \hat{\eta} \left( \frac{\lambda_x}{2\pi} \right)
\ll \frac{1}{\varphi(q)^2} \sum_{X \neq X_{0,q}} \sum_{\gamma_x, \lambda_x, \psi \psi' \neq \psi} \hat{\eta} \left( \frac{\gamma_x}{2\pi} \right) \hat{\eta} \left( \frac{\lambda_x}{2\pi} \right) \Phi_0 \left( \frac{\log q}{4\pi} \cdot (\gamma_x - \lambda_x) \right).
\]
Applying Proposition 3.4, we see that this term is $\ll (\log q) / \varphi(q)$. The proof is finished. 

We now show that $M_m(e^t, q; \eta)$ takes large values by using the bounds obtained in Section 5.

**Lemma 7.2.** Let $m \geq 1$, $\delta > 0$, $\eta \in S_{\delta}$, and assume that GRH$_{\eta}$ holds. Fix $\varepsilon > 0$ and let $C_{\varepsilon, \delta, \eta}$ be a large enough constant, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be any function such that for all $t \geq 3$, $f(t) \geq 1$. Then, for all $q \geq (C_{\varepsilon, \delta, \eta} m)^m$, there exists an associated value $x_m(q)$ for which $\log(x_m(q)) \in [f(q), f(q) \varphi(q)^m \mu^{-1} m]^{1\over 2m}$ and such that
\[
M_{2m}(x_m(q), q; \eta) \geq (1 - \varepsilon) \mu_{2m} (1 - \alpha \hat{\eta}^2) \log q)^m \varphi(q)^{m-1}.
\]

**Proof.** We can clearly assume that $\alpha(\hat{\eta}^2) = 1$. We apply Proposition 5.2 with $\Phi = \Phi_0$ where $\Phi_0$ is defined in (7.1). We deduce that for $C_{\varepsilon, \delta, \eta}$ large enough, $\varphi(q) \geq (C_{\varepsilon, \delta, \eta})^m$ and $T \geq (C_{\varepsilon, \delta, \eta} \log q)^m \mu^{-1}_{2m}$, \( \frac{\mu_{2m} \log q}{\varphi(q)^{m-1}} \),
\[
1 \int_0^\infty \Phi_0 \left( \frac{t}{T} \right) M_{2m}(e^t, q; \eta) \, dt \geq \left( \frac{1}{2} - \frac{\varepsilon}{3} \right) \mu_{2m} \log q)^m \varphi(q)^m.
\]
We claim that there exists $t_m(q) \in \left[ \frac{\mu_{2m} T}{\varphi(q)^{m-1}(C_{\varepsilon, \delta, \eta} \log q)_m}, T \right]$ (where we might need to enlarge the value of the constant $C_{\varepsilon, \delta, \eta} > 0$) such that
\[
\Phi_0 \left( \frac{t_m(q)}{T} \right) M_{2m}(e^{t_m(q)}, q; \eta) \geq (1 - \varepsilon) \mu_{2m} \log q)^m \varphi(q)^m.
\]
To show this, assume otherwise that for all $t \in [S, T]$, where we used the shorthand $S = T\mu_{2m}/\varphi(q)^{m-1}(C_{\varepsilon,\delta,\eta} \log q)^m$,

$$\Phi_0\left(\frac{t}{T}\right)M_{2m}(e^t, q; \eta) \leq (1 - \varepsilon)\mu_{2m}\frac{(\log q)^m}{\varphi(q)^m}.$$  

Applying Corollary 2.3, we deduce that

$$\int_0^\infty \Phi_0\left(\frac{t}{T}\right)M_{2m}(e^t, q; \eta)\, dt = \int_S^T \Phi_0\left(\frac{t}{T}\right)M_{2m}(e^t, q; \eta)\, dt + \int_0^S \Phi_0\left(\frac{t}{T}\right)M_{2m}(e^t, q; \eta)\, dt$$

$$\leq \left(\sup_{S \leq t \leq T} M_{2m}(e^t, q; \eta)\Phi_0\left(\frac{t}{T}\right)\right) \int_S^T \, dt + O\left(S \frac{(K_{\delta,\eta} \log q)^{2m}}{\varphi(q)}\right).$$  

When $C_{\varepsilon,\delta,\eta}$ is large enough, this contradicts (7.3). In summary, for all $q \geq 3$ such that $\varphi(q) \geq (C_{\varepsilon,\delta,\eta})^m$ (by enlarging once more the constant $C_{\varepsilon,\delta,\eta}$, we may replace $\varphi(q)$ with $q$) and for any $U \geq 1$, there exists $x_m(q)$ such that $\log(x_m(q)) \in [U, U\varphi(q)^{m-1}(C_{\varepsilon,\delta,\eta} \log q)^m\mu_{2m}^{-1}]$ and such that

$$M_{2m}(x_m(q), q; \eta) > (1 - \varepsilon)\mu_{2m}\frac{(\log q)^m}{\varphi(q)^m}.$$  

The proof is finished. □

In the particular case $n = 2$, we can localize the large values of $M_n(e^t, q; \eta)$ more precisely thanks to Lemma 7.1.

**Lemma 7.3.** Let $\delta > 0$ and $\eta \in S_\delta$, and assume that GRH$_{\hat{\eta}}$ holds. Fix $\varepsilon > 0$, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be any function such that for all $x \geq 3$, $f(x) \geq \log x$. Then, for all $q$ large enough in terms of $\delta, \eta$ and $\varepsilon$, there exists an associated value $x(q)$ for which $f(q) \approx_{\delta,\eta,\varepsilon} \log(x(q))$ and such that

$$M_2(x(q), q; \eta) \geq (1 - \varepsilon)\mu_{2m}\frac{(\log q)^m}{\varphi(q)^m}. \quad (7.5)$$

**Proof.** The proof is almost identical to that of Lemma 7.2 except that one can apply Lemma 7.1 and deduce that the left-hand side of (7.4) is

$$\leq (1 - \varepsilon)T\frac{\log q}{\varphi(q)} + O_{\delta,\eta}\left(S \frac{\log q}{\varphi(q)}\right).$$  

□

In the next lemma, we apply our results on higher moments of $M_{2m}(x, q; \eta)$ and deduce a stronger $\Omega$-result. However, our localization of the large values of $M_{2m}(x, q; \eta)$ is less precise than in Lemma 7.2.

**Lemma 7.4.** Let $m, r \geq 1$, $\delta > 0$, $\eta \in S_\delta$, and assume that GRH$_{\hat{\eta}}$ holds. Fix $\varepsilon > 0$ and let $C_{\varepsilon,\delta,\eta,m,r}$ be a large enough constant, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be any function such that for all $t \geq 3$, $f(t) \geq 1$. Then, for all $q \geq C_{\varepsilon,\delta,\eta,m,r}$, there exists an associated value $x_m(q)$ for which $\log(x_m(q)) \in$
\[ f(q), f(q)C_{\varepsilon, \delta, \eta, m, r} \varphi(q)^{2mr+m-r}(\log q)^{m(2r+1)} \] and such that

\[
M_{2m}(x_q, q; \eta) > \mu_{2m} \left( \frac{\alpha(\hat{\eta}^2) \log q + \beta_q(\hat{\eta}^2)}{\varphi(q)} \right)^m + (1 - \varepsilon)E_{r,m} \left( \frac{\gamma_{2m}^2(\alpha(\hat{\eta}^2) \log q)^m}{\varphi(q)^{m+1} + \frac{1}{\varphi(q)^{2r+1}}} \right),
\]

where

\[
E_{r,m} := \left( \frac{(2r + 1)!}{2^{r+1}(r - 1)!} \theta_{2m,2r+1} \right)^{\frac{1}{2r+1}}.
\]

**Proof.** The proof starts as that of Lemma 7.2, with the test function \( \Phi_0 \) defined in (7.1). From Theorem 1.3 (in which GRH \( \hat{\eta} \) is clearly sufficient), we deduce that for \( \varphi(q) \geq C_{\varepsilon, \eta, n, r} \) and \( T \geq C_{\varepsilon, \delta, \eta, m, r}(\log q)^m(2r+1)\varphi(q)^{2mr+m-r} \),

\[
\frac{1}{T} \int_{\mathbb{R}} \Phi_0 \left( t \right) \left( M_{2m}(e^t, q; \eta) - m_{2m}(q; \eta) \right)^{2r+1} dt \\
\geq \left( 1 - \frac{\varepsilon}{3} \right)^{2r+1} \frac{(2r + 1)!}{2^{r+1}(r - 1)!} \frac{\theta_{2m,2r+1}}{\varphi(q)^{\frac{1}{2}}} V_{2m}(q; \eta)^{2r+1} \frac{1}{2^{2r+1}}.
\]

Similarly as in the proof of Lemma 7.2 (note that the fact that \( 2r + 1 \) is odd is crucial here), we deduce that there exists \( t_m(q) \in [D_{\varepsilon, \delta, \eta, m, r} \varphi(q)^{m(2r+1)-r}(\log q)^{m(2r+1)}, T] \), where \( D_{\varepsilon, \delta, \eta, m, r} > 0 \) is a small enough constant, such that

\[
M_{2m}(e^t, q; \eta) - m_{2m}(q; \eta) > \left( 1 - \frac{\varepsilon}{2} \right) \left( \frac{(2r + 1)!}{2^{r+1}(r - 1)!} \right) \frac{1}{\varphi(q)^{\frac{1}{2}}} V_{2m}(q; \eta)^{\frac{1}{2}}.
\]

Finally, we apply the lower bound on \( m_{2m}(q; \eta) \) given in Theorem 1.3.

\[ \square \]

**Corollary 7.5.** Let \( \delta > 0 \) and \( \eta \in S_\delta \), and assume that GRH \( \hat{\eta} \) holds. Fix \( \varepsilon > 0 \) small enough, and let \( g : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 3} \) be an increasing function tending to infinity such that \( g(u) \leq e^u \). For each modulus \( q \gg \varepsilon, \delta, \eta \), there exists an associated value \( x_q \) such that \( g(c_1(\varepsilon, \delta, \eta) \log x_q) \leq q \leq g(c_2(\varepsilon, \delta, \eta) \log x_q) \), where the \( c_j(\varepsilon, \delta, \eta) > 0 \) are constants, and

\[
M_2(x_q, q; \eta) \geq (1 - \varepsilon) \frac{\alpha(\hat{\eta}^2) \log q}{\varphi(q)}.
\]

If moreover \( g(u) \leq u^{\frac{1}{M}} \) for some \( M \geq 4\varepsilon^{-1} \) and for all \( u \), then we can find \( x_q \) such that \( g((\log x_q)^{1-\frac{1}{M\varepsilon}}) \leq q \leq g((\log x_q)^{\frac{1}{M\varepsilon}}) \) and

\[
M_2(x_q, q; \eta) \geq \frac{\alpha(\hat{\eta}^2) \log q + \beta_q(\hat{\eta}^2)}{\varphi(q)} + \frac{1}{\varphi(q)^{\frac{1}{2} - \varepsilon}}.
\]

**Proof.** For the first statement, we apply Lemma 7.3 with \( f = g^{-1} \), which implies that \( f(q) \leq \log q \).
For the second statement, we apply Lemma 7.4 with \( f = g^{-1} \), which now implies that \( f(q) \geq q^M \). Hence, taking \( r = \lfloor \frac{\varepsilon^{-1}}{4} \rfloor \), we have that (7.8) holds for some \( x_q \) in the range \( f(q) \leq \log(x_q) \leq f(q)C_{\varepsilon,1,r} \varphi(q)^{r+1} (\log q)^{2r+1} \leq f(q)^{1+\frac{1-\varepsilon}{4}+2} \), and the proof follows. Here, we have used the bound \( f(q) \geq q^M \) which implies that \( \log q < M \log f(q) \leq f(q)^{\varepsilon^2} \).

\[ \square \]

**Corollary 7.6.** Let \( \delta, \varepsilon > 0 \) and \( \eta \in S_2 \), assume that GRH\(_\hat{\eta} \) holds, and fix \( m \geq 2 \). For any \( B \in \mathbb{R}_{>0} \), there exists a real number \( \lambda \in [\frac{1}{m-1+B}, \frac{1}{B}] \) with the following property. There exists a sequence of moduli \( \{ q_i \}_{i \geq 1} \) and associated values \( \{ x_i \}_{i \geq 1} \) such that

\[ q_i = (\log x_i)^{\lambda + o(1)} \quad \text{and} \quad M^{2m}(x_i, q_i; \eta) \geq (1-\varepsilon)\mu_{2m}(\alpha(\hat{\eta}^2 \log q_i) \varphi(q_i)^m). \]  

(7.9)

Let moreover \( g : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 3} \) be an increasing function such that as \( x \to \infty \), \( g(x) \leq x^{1+m\delta} \), with \( M \geq 2e^{-1} \). Then, for each modulus \( q \geq C_{\varepsilon,\delta,\eta,m} \), there exists an associated value \( x_q \) such that

\[ g((\log x_q)^{1-\frac{1}{m\delta}}) \leq q \leq g(\log x_q) \quad \text{and} \quad (7.6) \text{ holds.} \]

Proof. The proof of the second part is similar to that of Corollary 7.5. In the first part, we pick \( f(q) = (q \log q)^B \), and apply Lemma 7.2. The resulting sequence \( x_m(q) \) is such that

\[ B + o(1) \leq \frac{\log \log x_m(q)}{\log q} \leq B + m - 1 + o(1), \]

and we can extract a subsequence having the desired property. \( \square \)

We now work under the negation of GRH\(_{\hat{\eta}} \). It follows from the work of Kaczorowski and Pintz [56] that if \( L(s, \chi) \) does not have real nontrivial zeros, then \( \psi_\eta(x, \chi) \approx \log X \) large oscillations in the interval \([1, X]\). Moreover, Pintz [85] and Schlage-Puchta [95] have obtained stronger results for \( \psi(x) - x \). A weaker but unconditional result will be sufficient for our purposes.

**Lemma 7.7.** Let \( \delta > 0 \) and \( \eta \in S_2 \), and fix \( \varepsilon > 0 \). For \( q \geq 3 \), let \( \chi \) be a nonprincipal character \((\mod q)\), and let \( \Theta_{\chi,\eta} \) be the supremum of the real parts of the zeros \( \varphi_X \chi \) of \( L(s, \chi) \) such that \( \hat{\eta}(\frac{\varphi_X - 1}{2\pi i}) \neq 0 \). Then, for every large enough \( X \in \mathbb{R} \) (in terms of \( \chi, \varepsilon \) and \( \eta \)), there exists \( x \in [X^{1-\varepsilon}, X] \) such that

\[ |\psi_\eta(x, \chi)| > x^{\Theta_{\chi,\eta}-\frac{1}{2}-\varepsilon}. \]

Proof. We distinguish two cases, depending on the value of

\[ \gamma_{\chi,\eta} := \inf\{|\gamma| : \gamma \in \mathbb{R}, L(\Theta_{\chi,\eta} + i\gamma, \chi) = 0, \hat{\eta}(\frac{\Theta_{\chi,\eta} - 1}{2\pi i}) \neq 0\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}. \]

If \( \gamma_{\chi,\eta} > 0 \), then we apply [56, Theorem 1] with \( f(x) := 2\Re(\psi_\eta(x, \chi)) = \psi_\eta(x, \chi) + \psi_\eta(x, \overline{\chi}) \). We have that

\[ \int_0^\infty f(x)x^{-s-1}dx = -\hat{\eta}(\frac{s}{2\pi i})(L(s + \frac{1}{2}, \chi) + L(s + \frac{1}{2}, \chi)). \]  

(7.10)
By definition of $\Theta_{\chi,\eta}$, the function $L'(s + \frac{1}{2}, \chi)/L(s + \frac{1}{2}, \chi)$ has a pole $s_0$ of residue $\geq 1$ in the half plane $\Re(s) > \Theta_{\chi,\eta} - \frac{1}{2} - \varepsilon$, and moreover $\hat{\eta}(\frac{s_0}{2\pi i}) \neq 0$. Since $L(s + \frac{1}{2}, \chi)$ is entire, it follows that the poles of $L'(s + \frac{1}{2}, \chi)/L(s + \frac{1}{2}, \chi)$ have positive residues, and we conclude that the right-hand side of (7.10) is meromorphic on $\mathbb{C}$ and has a pole $s_0$ in the half plane $\Re(s) > \Theta_{\chi,\eta} - \frac{1}{2} - \varepsilon$. Hence, the conditions of [56, Theorem 1] are satisfied and the conclusion follows.

We now assume that $\gamma_{\chi,\eta} = 0$, which implies that $L(\Theta_{\chi,\eta}, \chi) = 0$ and $\hat{\eta}(\frac{\Theta_{\chi,\eta} - 1}{2\pi i}) \neq 0$. In this case, for $1 < T_1 < T_2$, we consider the average

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} e^{-t(\Theta_{\chi,\eta} - \frac{1}{2})} \Re(\psi_{\eta}(e^t, \chi)) dt,$$

which by Lemma 2.2 is

$$\Re e \sum_{\varphi_{\chi} \neq \Theta_{\chi,\eta}} \frac{e^{T_2(\varphi_{\chi} - \Theta_{\chi,\eta})} - e^{T_1(\varphi_{\chi} - \Theta_{\chi,\eta})}}{(T_2 - T_1)(\varphi_{\chi} - \Theta_{\chi,\eta})} \hat{\eta}\left(\frac{\varphi_{\chi} - \frac{1}{2}}{2\pi i}\right)$$

$$- \Re \left(\hat{\eta}\left(\frac{\Theta_{\chi,\eta} - \frac{1}{2}}{2\pi i}\right)\right) \text{ord}_{s=\Theta_{\chi,\eta}} L(s, \chi) + O_{X}\left(\frac{e^{-\Theta_{\chi,\eta}T_1}}{T_2 - T_1}\right).$$

We have a similar identity for the average of $\Im m(\psi_{\eta}(e^t, \chi))$. Then, taking $T_1 = (1 - \varepsilon) \log X$ and $T_2 = \log X$, we claim that (7.11) takes the form

$$\frac{1}{\varepsilon \log X} \int_{(1 - \varepsilon) \log X}^{\log X} e^{-t(\Theta_{\chi,\eta} - \frac{1}{2})} \Re(\psi_{\eta}(e^t, \chi)) dt$$

$$= -\Re \left(\hat{\eta}\left(\frac{\Theta_{\chi,\eta} - \frac{1}{2}}{2\pi i}\right)\right) \text{ord}_{s=\Theta_{\chi,\eta}} L(s, \chi) + O_{X}\left(\frac{1}{\varepsilon \log X}\right).$$

Indeed, we have that $|e^{T_j(\varphi_{\chi} - \Theta_{\chi,\eta})}| \leq 1$. Moreover, we have the bound

$$\hat{\eta}(s) = \frac{1}{-2\pi is} \int_{\mathbb{R}} e^{-2\pi ist} \eta'(t) dt \ll \frac{1}{|s|} \int_{\mathbb{R}} e^{2\pi |\Im m(s)| |t|} |\eta'(t)| dt,$$

which is $\ll 1/|s|$ whenever $\Im m(s) \leq \frac{1}{4\pi}$ by definition of $S_q$. This is the case with $s = (\varphi_{\chi} - \frac{1}{2})/(2\pi i)$, and hence, the infinite sum over zeros in (7.11) converges absolutely.

Coming back to (7.12) and its analog for $\Im m(\psi_{\eta}(e^t, \chi))$, we recall that $\hat{\eta}\left(\frac{\Theta_{\chi,\eta} - \frac{1}{2}}{2\pi i}\right) \neq 0$, and deduce that for all large enough $X$, there exists $t \in [(1 - \varepsilon) \log X, \log X]$ such that either $|e^{-t(\Theta_{\chi,\eta} - \frac{1}{2})} \Re(\psi_{\eta}(e^t, \chi))| \gg_{\chi,\eta} 1$, or $|e^{-t(\Theta_{\chi,\eta} - \frac{1}{2})} \Im m(\psi_{\eta}(e^t, \chi))| \gg_{\chi,\eta} 1$. This implies the claimed assertion. 

**Corollary 7.8.** Let $\delta > 0$ and $\eta \in S_{\delta}$, and assume that $\text{GRH}_{\hat{\eta}}$ does not hold. Then, there exists an absolute (ineffective) number $\kappa > 0$ such that the following holds. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 3}$ be an increasing function tending to infinity such that $f(x) \leq x^{\kappa}$. For a positive (ineffective) proportion of moduli $q$,
there exists an associated value \( x_q \) such that \( f(x_q^{1-\kappa}) \leq q \leq f(x_q) \) and for each \( m \geq 1 \),

\[
M_{2m}(x_q, q; \eta) \gg \frac{x_q^{m\kappa}}{\varphi(q)^m}.
\]

The implied constant in this bound is also ineffective.

Proof. We first treat the case \( m = 1 \). Let \( \chi_\varepsilon \) be a primitive character of least modulus \( q_\varepsilon \) for which there exists a zero \( \varepsilon_\varepsilon = \Theta_\varepsilon + i\gamma_\varepsilon \) of \( L(x, \chi_\varepsilon) \) with \( \Theta_\varepsilon > \frac{1}{2} \), and such that \( \hat{\eta}(\frac{\varepsilon_\varepsilon - \frac{1}{2}}{2\pi}) \neq 0 \). Fix \( \varepsilon < \Theta_\varepsilon - \frac{1}{2}, \) and pick any \( \kappa < \Theta_\varepsilon - \frac{1}{2} - \varepsilon < \frac{1}{2} \). The set of moduli we will consider is the set of large enough multiples of \( q_\varepsilon \); those form a positive proportion of all moduli. If \( q \) is such a multiple, we apply Lemma 7.7, and find that when \( f^{-1}(q) \) is large enough in terms of \( \kappa, \varepsilon, \eta \) and \( \chi_\varepsilon \), there exists \( x_q \in [f^{-1}(q), (f^{-1}(q))^{1+\kappa}] \) such that

\[
|\psi_\eta(x_q, \chi_\varepsilon)| > x_q^{\Theta_\varepsilon - \frac{1}{2} - \varepsilon}.
\]

We deduce that

\[
M_2(x_q, q; \eta) = \frac{1}{\varphi(q)^2} \sum_{\chi(\bmod q) \neq \chi_0} |\psi_\eta(x_q, \chi)|^2 \geq \frac{1}{\varphi(q)^2} |\psi_\eta(x_q, \chi_\varepsilon)|^2 \gg \frac{x_q^{2\Theta_\varepsilon - 1 - \varepsilon}}{\varphi(q)^2},
\]

which is the required estimate.

For \( m \geq 2 \), we apply Hölder’s inequality and deduce that for all \( x \geq 0 \),

\[
M_{2m}(x, q; \eta) \geq M_2(x, q; \eta)^m.
\]

The result follows.

We are now ready to prove our main unconditional results.

Proof of Theorems 1.1 and 1.2. If GRH_\eta is false, then the claimed \( \Omega \)-results follow from Corollary 7.8. Otherwise, they follow from Corollaries 7.5 and 7.6.

Proof of Corollary 1.4. We argue by contradiction. Assume that for some \( q \) and \( K > 0 \) and for all \( a (\bmod q) \) and \( y \in [\exp(c_2^{-1} g^{-1}(q)), \exp(c_1^{-1} g^{-1}(q))] \), we have the bound

\[
\left| \sum_{n \leq y \atop n \equiv a (\bmod q)} \Lambda(n) - \frac{1}{\varphi(q)} \sum_{n \leq y \atop (n, q) = 1} \Lambda(n) \right| \leq K \frac{(y \log q)^{\frac{1}{2}}}{\varphi(q)^{\frac{1}{2}}}.
\]

We let \( \eta = \eta_0 \now \eta_0 \), where \( \eta_0 \) is any nontrivial smooth even function supported in \([-1, 1]\). Applying summation by parts, we deduce that in the range \( x \in [\exp(c_2^{-1} g^{-1}(q) + 2), \exp(c_1^{-1} g^{-1}(q) - 2)] \),
\[ \psi_\eta(x; q, a) - \frac{\psi_\eta(x, \chi_0, q)}{\varphi(q)} = \int_{0}^{\infty} \frac{\eta(\log(\frac{x}{y}))}{y^{\frac{1}{2}}} \left( \frac{\psi(y; q, a) - \psi(y, \chi_0, q)}{\varphi(q)} \right) dy \]

\[ = - \int_{e^{-2x}}^{e^2x} \frac{\eta'(\log(\frac{y}{x})) - \frac{1}{2} \eta(\log(\frac{y}{x}))}{y^{\frac{3}{2}}} \left( \frac{\psi(y; q, a) - \psi(y, \chi_0, q)}{\varphi(q)} \right) dy \]

\[ \ll K \left( \frac{\log q}{\varphi(q)} \right)^{\frac{1}{2}}. \]

This yields the upper bound \( M_2(x; q; \eta) \ll K^2 \log q / \varphi(q) \). Under GRH, when \( K \) is small enough and for some choice of \( 0 < c_1, c_2 < \frac{1}{2} \), this contradicts Lemma 7.3 as soon as \( q \) is large enough. Under the negation of GRH, this contradicts Lemma 7.8 as soon as \( q \) is in a well-chosen set (given by Lemma 7.8) of positive (ineffective) density.

Finally, we end this section by proving Corollary 1.5. This will be achieved through the following proposition, which we believe is of independent interest.

**Proposition 7.9.** Assume GRH, and let \( \delta > 0, \eta \in S_\delta, T \geq 1 \) and \( \Phi \in U \). For \( m \in \mathbb{N}, q \geq 3 \), and in the range \( m \leq \log q / (2 \log_2 q) \), we have the bound

\[ \frac{1}{T} \int_0^\infty \Phi \left( \frac{t}{T} \right) \left( \psi_\eta(e^t; q, 1) - \frac{\psi_\eta(e^t, \chi_0, q)}{\varphi(q)} \right)^{2m} dt \]

\[ \geq \mu_{2m} \left( \frac{\alpha(\hat{\eta}^2) \log q + \beta_2(\hat{\eta}^2)}{\varphi(q)} \right)^m + O_\Phi \left( \mu_{2m} \frac{(C_\delta, \eta \log q)^m}{\varphi(q)^{m+1}} + \left( K_\delta, \eta \log q \right)^{2m} \right), \]

where \( C_\delta, \eta, K_\delta, \eta > 0 \) are constants.

**Proof.** The proof follows the lines of that of Lemma 6.2. Applying the orthogonality relations and the explicit formula in Lemma 2.2, we see that

\[ \frac{1}{T} \int_0^\infty \Phi \left( \frac{t}{T} \right) \left( \psi_\eta(e^t; q, 1) - \frac{\psi_\eta(e^t, \chi_0, q)}{\varphi(q)} \right)^{2m} dt \]

\[ = \frac{1}{T} \int_0^\infty \Phi \left( \frac{t}{T} \right) \frac{1}{\varphi(q)^{2m}} \sum_{\chi_1, \ldots, \chi_{2m} \neq \chi_0, q} \psi_\eta(e^t, \chi_1) \cdots \psi_\eta(e^t, \chi_{2m}) dt \]

\[ = \frac{1}{T} \int_0^\infty \Phi \left( \frac{t}{T} \right) \frac{1}{\varphi(q)^{2m}} \sum_{\chi_1, \ldots, \chi_{2m} \neq \chi_0, q} \prod_{j=1}^{2m} \hat{\eta}(\gamma_{x_j} / 2\pi) e^{-it\gamma_{x_j}} dt + O_\Phi \left( \frac{(K_\delta, \eta \log q)^{2m}}{T} \right) \]

\[ = \frac{1}{2\varphi(q)^{2m}} \sum \sum \hat{\Phi} \left( \frac{T}{2\pi} (\gamma_{x_1} + \cdots + \gamma_{x_{2m}}) \right) \prod_{j=1}^{2m} \hat{\eta}(\gamma_{x_j} / 2\pi) + O_\Phi \left( \frac{(K_\delta, \eta \log q)^{2m}}{T} \right). \]
At this point, we apply positivity. Note that it is crucial here that the residue class we are working with is \( a = 1 \), since otherwise we would not be able to handle the signs of \( \overline{\chi}(a) \) in the explicit formula. The claimed result follows from Lemma 5.1.

**Proof of Corollary 1.5.** After applying Proposition 7.9, the rest of the proof is similar to that of Corollary 1.4, the only major difference being that we apply the uniform bound

\[
\psi_\eta(e^i; q, 1) - \frac{\psi_\eta(e^i, \chi_0, q)}{\varphi(q)} \ll \log q
\]

rather than Lemma 7.1. This is responsible for our weaker control on the size of \( x_q \) in terms of \( q \). □

## 8 APPLICATIONS OF LI

The goal of this section is to study the probabilistic model \( H_n(q; \eta) \) Defined in Lemma 4.3, for some \( \eta \in S_\delta \), under GRH and LI. To begin, we will see that the relation \( \gamma_1 + \cdots + \gamma_n = 0 \) in the definition of \( \Gamma_0(\chi) \) (see Section 4) has a very simple set of solutions.

**Lemma 8.1.** Let \( q \geq 3 \), and let \( \gamma_1, \ldots, \gamma_n \) be ordinates of nontrivial zeros of the functions \( L(s, \chi_1), \ldots, L(s, \chi_n) \), respectively, where \( \chi_1, \ldots, \chi_n \) are nonprincipal characters \( (\mod q) \). Under GRH and LI, the relation \( \gamma_1 + \cdots + \gamma_n = 0 \) implies that \( n = 2m \) for some \( m \in \mathbb{N} \), and moreover, there exists \( i_1, \ldots, i_m \) and \( j_1, \ldots, j_m \) such that

\[
\{i_1, \ldots, i_m, j_1, \ldots, j_m\} = \{1, \ldots, 2m\}
\]

and

\[
\gamma_{i_k} + \gamma_{j_k} = 0, \quad \overline{\chi_{j_k}} = \chi_{i_k}.
\]

**Proof.** Clearly, LI implies that \( L(1/2, \chi) \neq 0 \), that is, \( \gamma_\chi \neq 0 \). We argue by induction over \( n \). The statement is clear for \( n = 1 \). For \( n = 2 \), note that \( \gamma_1 + \gamma_2 = 0 \) implies that \( \gamma_1 \gamma_2 < 0 \), say \( \gamma_1 > 0 \) and \( \gamma_2 < 0 \). Also, \( 1/2 - i \gamma_2 \) is a zero of \( L(s, \overline{\chi_2}) \), and \( \gamma_1 = -\gamma_2 \) implies by LI that \( \chi_1 = \overline{\chi_2} \).

Let \( n \geq 3 \) and assume that the statement holds for all \( 1 \leq r < n \). We can reorder the \( \gamma_i \) in such a way that \( \gamma_1, \ldots, \gamma_\ell > 0 \), and \( \gamma_{\ell+1}, \ldots, \gamma_n < 0 \), for some \( 1 \leq \ell \leq n - 1 \) (clearly, the \( \gamma_i \) are not all of the same sign). Then,

\[
\gamma_1 + \cdots + \gamma_\ell = (-\gamma_{\ell+1}) + \cdots + (-\gamma_n), \tag{8.1}
\]

and \( 1/2 - i \gamma_{\ell+1}, \ldots, 1/2 - i \gamma_n \) are zeros of \( L(s, \overline{\chi_j}) \) with \( \gamma_j > 0 \). We claim that

\[
\{\gamma_1, \ldots, \gamma_\ell\} = \{-\gamma_{\ell+1}, \ldots, -\gamma_n\}.
\]

Indeed, if, for example, \( \gamma_1 \not\in \{-\gamma_{\ell+1}, \ldots, -\gamma_n\} \), then the formula (8.1) implies that \( \gamma_1 \) is a \( \mathbb{Q} \)-linear combination of positive imaginary parts of other zeros, which contradicts LI. We conclude that there exists \( j_\ell \geq \ell + 1 \) such that \( -\gamma_{j_\ell} = \gamma_\ell \); by LI, we also need to have \( \chi_\ell = \overline{\chi_{j_\ell}} \). Hence, our induction hypothesis for \( r = n - 2 \) implies that \( r \) is even, \( \ell - 1 = n - \ell - 1 \), and moreover, we
can find distinct indices \( j_1, \ldots, j_{\ell-1} \in \{ \ell + 1, \ldots, n \} \setminus \{ j_\ell \} \) such that for \( 1 \leq k \leq \ell - 1 \), \(-\gamma_{j_k} = \gamma_k\) and \( \chi_k = \overline{\chi_{j_k}} \).

A direct consequence of Lemma 4.4 is the following.

**Lemma 8.2.** Let \( \delta > 0, \eta \in S_\delta, r, m \in \mathbb{N} \) and \( q \geq 3 \). Under GRH and LI, we have that

\[
\mathbb{E}[H_{2m+1}(q; \eta)^{2r+1}] = 0.
\]

**Proof.** By Lemma 8.1, if \( \gamma_{\mu,j} \) are imaginary parts of zeros of \( L(s, \chi_{\mu,j}) \), then \( \gamma_{\mu,1} + \cdots + \gamma_{\mu,2m+1} \neq 0 \). Hence, Lemma 4.4 implies that \( \mathbb{E}[H_{2m+1}(q; \eta)^{2r+1}] \) is equal to an empty sum. \( \square \)

Taking \( \Phi = 1_{[-1,1]} \) and \( T \to \infty \) in Lemma 6.2 and applying Lemma 4.3, we obtain the following corollary of Lemma 8.1.

**Corollary 8.3.** Let \( \delta > 0 \) and \( \eta \in S_\delta \), and assume GRH and LI. Let \( \chi = (\chi_{\mu,j})_{1 \leq \mu \leq \ell, 1 \leq j \leq n} \) be an array of characters, and for \( 1 \leq \mu \leq \ell \) and \( 1 \leq j \leq n \), denote by \( \gamma_{\mu,j} \) the imaginary part of a generic non-trivial zero of \( L\left(s, \chi_{\mu,j}\right) \). Then, there exists an involution \( \pi \) of the set \( \{1, \ldots, \ell\} \times \{1, \ldots, n\} \) having no fixed points such that \( \pi(\chi) = \overline{\chi} \) and, for \( 1 \leq \mu \leq \ell \) and \( 1 \leq j \leq n \), \( \gamma_{\mu,j} + \gamma_{\pi(\mu),j} = 0 \). Moreover, if the product \( sn \) is even, then we have the upper bound

\[
\forall_{s,n}(q; \eta) \leq \frac{1}{\varphi(q)^{sn}} \sum_{\pi \in I_{s,n}} \sum_{\chi \in k_{s,n}} \sum_{\gamma \in \chi(\chi)} \hat{\eta}(\gamma) \Delta_s(\sigma_\gamma),
\]

where \( I_{s,n} \) is the set of involutions \( \pi \) of the set \( \{1, \ldots, \ell\} \times \{1, \ldots, n\} \) having no fixed points and such that \( \sum_{1 \leq \mu \leq \ell} |J_{\mu,\nu}(\pi)| \equiv 0 \pmod{2}; \sum_{1 \leq \mu \leq \ell} |J_{\mu,\nu}(\pi)| \geq 2 \), \( \nu \neq \mu \)

where the set \( J_{\mu,\nu}(\pi) \) is defined in (6.7).

Note that this upper bound on \( \forall_{s,n}(q; \eta) \) is not an equality, since there are arrays of characters associated to more than one involution \( \pi \in I_{s,n} \). However, we will see in Lemma 8.5 that the overcount is negligible. The following definition will be useful.

**Definition 8.4.** For \( r \geq 1 \) and \( n \geq 2 \), let \( G_{2r,n} \) be the set of involutions \( \pi : \{1, \ldots, 2r\} \times \{1, \ldots, n\} \to \{1, \ldots, 2r\} \times \{1, \ldots, n\} \) having no fixed points for which (8.2) holds, and

\[
\left| \left\{ (\mu, \nu) : 1 \leq \mu < \nu \leq 2r, J_{\mu,\nu}(\pi) \neq \emptyset \right\} \right| \geq r + 1.
\]

In particular, recalling Definition 6.3, we have that

\[
I_{2r,n} = F_{2r,n} \cup G_{2r,n}.
\]

\(^1\) Actually, the first condition in (8.2) is automatic since \( \pi \) is an involution having no fixed points.
Lemma 8.5. Let \( r \in \mathbb{N}, \ n \geq 2, \ q \geq 3, \) and let \( \pi \in G_{2r,n}. \) We have the upper bound

\[
C_{2r,n}(q; \pi) := \sum_{\chi \in X_{2r,n}} 1 \leq \varphi(q)^{(n-1)r-1}.
\]

Proof. For an array of characters \( \chi = (\chi_{\mu,j})_{1 \leq \mu \leq s \atop 1 \leq j \leq n} \) such that \( \pi(\chi) = \overline{\chi}, \) we will use the notation

\[
\Theta_{\mu,\nu}(\chi, \pi) := \prod_{(\mu,j) \in J_{\mu,\nu}} \chi_{\mu,j}.
\]

Note that \( \Theta_{\mu,\mu}(\chi, \pi) = X_{0,q} \) and \( \Theta_{\mu,\nu}(\chi, \pi) = \Theta_{\nu,\mu}(\chi, \pi). \) The relations

\[
\chi_{\mu,1} \cdots \chi_{\mu,n} = X_{0,q} \quad (1 \leq \mu \leq 2r)
\]

are equivalent to

\[
\prod_{\nu \neq \mu} \Theta_{\mu,\nu}(\chi, \pi) = X_{0,q} \quad (1 \leq \mu \leq 2r). \tag{8.3}
\]

We consider

\[
E_1(\pi) := \{ (\mu, \nu) : 1 \leq \mu < \nu \leq 2r, J_{\mu,\nu}(\pi) \neq \emptyset \}.
\]

We let \( E(\pi) \subset E_1(\pi) \) be a set of least cardinality for which

\[
\bigcup_{(\mu, \nu) \in E(\pi)} \{ \mu, \nu \} = \{1, \ldots, 2r\}, \tag{8.4}
\]

that is, every \( 1 \leq \mu \leq 2r \) is paired with some \( 1 \leq \nu \leq 2r \) through \( E(\pi). \) The existence of \( E(\pi) \) follows from the assumption (8.2). Clearly, \( |E(\pi)| \geq r. \)

We consider two distinct cases. First, we assume that \( |E(\pi)| = r. \) Then, by reordering the array \( \chi, \) we may assume that \( E(\pi) = \{(1,2), (3,4), \ldots, (2r-1,2r)\} \) and \( (1,3) \in E_1(\pi). \) For a given value of \( \Theta_{\mu,\nu}(\chi, \pi) \) for all \((\mu, \nu) \in \{ (\mu, \nu) : 1 \leq \mu < \nu \leq 2r \} \setminus (E(\pi) \cup \{(1,3)\}), \) we claim that the relations (8.3) have a unique solution, that is, the \( \Theta_{\mu,\nu}(\chi, \pi) \) with \((\mu, \nu) \in E(\pi) \cup \{(1,3)\}\) are determined. This is clearly the case for \( \mu_0 = 5, 7, \ldots, 2r-1 \) since exactly one of the \( \Theta_{\mu,\nu}(\chi, \pi) \) with \((\mu, \nu) \in E(\pi) \cup \{(1,3)\}\) appears in the relations (8.3) with \( \mu = \mu_0. \) Now, for \((\mu_0, \nu_0) \in \{(1,2), (1,3), (3,4)\}, \) we have the relations

\[
\begin{align*}
\Theta_{1,2}(\chi, \pi)\Theta_{2,3}(\chi, \pi)\Theta_{3,4}(\chi, \pi) &= \prod_{5 \leq \nu \leq 2r} \Theta_{2,\nu}(\chi, \pi), \\
\Theta_{1,3}(\chi, \pi)\Theta_{2,3}(\chi, \pi)\Theta_{3,4}(\chi, \pi) &= \prod_{5 \leq \nu \leq 2r} \Theta_{3,\nu}(\chi, \pi), \\
\Theta_{1,4}(\chi, \pi)\Theta_{2,4}(\chi, \pi)\Theta_{3,4}(\chi, \pi) &= \prod_{5 \leq \nu \leq 2r} \Theta_{4,\nu}(\chi, \pi),
\end{align*}
\]

which imply once more a unique choice for \( \Theta_{\mu,\nu}(\chi, \pi) \) with \((\mu, \nu) \in \{(1,2), (1,3), (3,4)\}. \) Now, \( \Theta_{\mu,\nu} \) is a product of \( |J_{\mu,\nu}| \) characters; hence, the total number of summands in \( C_{2r,n}(q; \pi) \) is at most

\[
\exp \left( \log(\varphi(q)) \left( \sum_{\mu=1}^{2r} \frac{|J_{\mu,\mu}|}{2} + \sum_{1 \leq \mu < \nu \leq 2r \atop (\mu, \nu) \in E(\pi) \cup \{(1,3)\}} |J_{\mu,\nu}| \right) \right) \leq \varphi(q)^{(n-1)r-1}.
\]
We now consider the second case where $|E(\pi)| \geq r + 1$. We claim that for every $(\mu_0, v_0) \in E(\pi)$, we either have that $\mu_0 \notin \{\mu, v\}$ for all $(\mu, v) \in E(\pi) \setminus \{(\mu_0, v_0)\}$, or $v_0 \notin \{\mu, v\}$ for all $(\mu, v) \in E(\pi) \setminus \{(\mu_0, v_0)\}$. Indeed, the contrary would contradict minimality of $E(\pi)$, since $E(\pi) \setminus \{(\mu_0, v_0)\}$ would have the property (8.4). Moreover, for a given value of $\Theta_{\mu_0, v_0}(\chi, \pi)$ for all $(\mu, v) \in \{(\mu, v) : 1 \leq \mu < v \leq 2r\} \setminus E(\pi)$, we claim that the relations (8.3) have a unique solution, that is, the $\Theta_{\mu_0, v_0}(\chi, \pi)$ with $(\mu, v) \in E(\pi)$ are determined. To show this, we fix $(\mu_0, v_0) \in E(\pi)$, and recall that either $\mu_0 \notin \{\mu, v\}$ for all $(\mu, v) \in E(\pi) \setminus \{(\mu_0, v_0)\}$, or $v_0 \notin \{\mu, v\} \setminus \{(\mu_0, v_0)\}$ for all $(\mu, v) \in E(\pi)$. In the first case, taking $\mu = \mu_0$ in (8.3) shows that $\Theta_{\mu_0, v_0}(\chi, \pi)$ is uniquely determined. In the second, this follows from taking $\mu = v_0$. As before, we conclude that

$$C_{2r,n}(q; \pi) \leq \varphi(q)^{(n-1)r-1}.$$

We now move to the case where $s = 2r + 1$ is odd. As in the even case, we use the notation

$$E_1(\pi) := \{(\mu, v) : 1 \leq \mu < v \leq 2r + 1, J_{\mu, v}(\pi) \neq \emptyset\}.$$

**Definition 8.6.** Let $r \in \mathbb{N}$, $n \geq 2$. We define $G_{2r+1,n}$ to be the set of involutions $\pi : \{1, \ldots, 2r + 1\} \times \{1, \ldots, n\} \rightarrow \{1, \ldots, 2r + 1\} \times \{1, \ldots, n\}$ having no fixed points such that (8.2) holds for $1 \leq \mu \leq 2r + 1$, and for which either $|E_1(\pi)| \geq r + 2$, or $|E_1(\pi)| = r + 2$ and there does not exist $1 \leq \mu_1 < \mu_2 < \mu_3 \leq 2r + 1$ such that $\{(\mu_1, \mu_2), (\mu_2, \mu_3), (\mu_1, \mu_3)\} \subset E_1(\pi)$.

We remark that (8.2) implies that $|E_1(\pi)| \geq r + 1$, and moreover, we have that

$$I_{2r+1,n} = F_{2r+1,n} \cup G_{2r+1,n}.$$

**Lemma 8.7.** Let $r \in \mathbb{N}$, $n \in 2\mathbb{N}$, $q \geq 3$, and let $\pi \in G_{2r,n}$. We have the bound

$$C_{2r+1,n}(q; \pi) := \sum_{\chi \in \chi_{2r+1,n} \pi(\chi) = \chi} 1 \leq \varphi(q)^{(n-1)\frac{2r+1}{2} - \frac{3}{2}}.$$

**Proof.** We will use the same notation as in the proof of Lemma 8.5. We separate the proof into two distinct cases. First, we assume that $|E(\pi)| \geq r + 2$. As in the proof of Lemma 8.5, for a given value of $\Theta_{\mu, v}(\chi, \pi)$ for all $(\mu, v) \in \{(\mu, v) : 1 \leq \mu < v \leq 2r + 1\} \setminus E(\pi)$, the relations (8.3) have a unique solution. We conclude that

$$C_{2r+1,n}(q; \pi) \leq \varphi(q)^{(n-1)\frac{2r+1}{2} - \frac{3}{2}}.$$

Second, we assume that $E(\pi) = r + 1$. After reordering $\chi$, we may also assume that

$$E(\pi) = \{(1, 2), (1, 3), (4, 5), (6, 7), \ldots, (2r, 2r + 1)\}.$$

Now, since $E_1(\pi) \geq r + 2$, we may assume that one of $(1,4),(2,4)$, or $(4, 6)$, which we will denote by $(\mu_e, v_e)$, is an element of $E_1(\pi)$ (recall the definition of $G_{2r+1,n}$). Arguing as before, we see that for a given value of $\Theta_{\mu, v}(\chi, \pi)$ for all $(\mu, v) \in \{(\mu, v) : 1 \leq \mu < v \leq 2r + 1\} \setminus (E(\pi) \cup \{(\mu_e, v_e)\})$, the relations

$$\prod_{v \neq \mu} \Theta_{\mu, v}(\chi, \pi) = \chi_{0,q} \quad (1 \leq \mu \leq 2r + 1) \quad (8.5)$$
determine \( \Theta_{\mu,\nu}(\chi, \pi) \) for \((\mu, \nu) \in E(\pi) \cup \{(\mu_e, \nu_e)\}, \mu \geq 8.\) In the case where \((\mu_e, \nu_e) = (1,4)\), \(\Theta_{6,7}(\chi, \pi)\) is also determined, and moreover, we have that

\[
\begin{align*}
\begin{cases}
\Theta_{1,2}(\chi, \pi) \Theta_{2,3}(\chi, \pi) = \prod_{6 \leq \nu \leq 2r + 1} \Theta_{2, \nu}(\chi, \pi), \\
\Theta_{1,3}(\chi, \pi) \Theta_{2,3}(\chi, \pi) = \prod_{6 \leq \nu \leq 2r + 1} \Theta_{3, \nu}(\chi, \pi), \\
\Theta_{1,4}(\chi, \pi) \Theta_{2,4}(\chi, \pi) = \prod_{6 \leq \nu \leq 2r + 1} \Theta_{4, \nu}(\chi, \pi), \\
\Theta_{1,5}(\chi, \pi) \Theta_{2,5}(\chi, \pi) = \prod_{6 \leq \nu \leq 2r + 1} \Theta_{5, \nu}(\chi, \pi),
\end{cases}
\end{align*}
\]

which shows that \(\Theta_{1,2}(\chi, \pi), \Theta_{1,3}(\chi, \pi), \Theta_{4,5}(\chi, \pi),\) and \(\Theta_{1,4}(\chi, \pi)\) are also determined. The same argument works in the case \((\mu_e, \nu_e) = (2,4)\). Finally, for \((\mu_e, \nu_e) = (4,6)\) we have that

\[
\begin{align*}
\begin{cases}
\Theta_{1,2}(\chi, \pi) \Theta_{2,3}(\chi, \pi) = \prod_{4 \leq \nu \leq 2r + 1} \Theta_{2, \nu}(\chi, \pi), \\
\Theta_{1,3}(\chi, \pi) \Theta_{2,3}(\chi, \pi) = \prod_{4 \leq \nu \leq 2r + 1} \Theta_{3, \nu}(\chi, \pi), \\
\Theta_{4,5}(\chi, \pi) \Theta_{5,6}(\chi, \pi) \Theta_{5,7}(\chi, \pi) = \prod_{\nu \notin \{4,7\}} \Theta_{5, \nu}(\chi, \pi), \\
\Theta_{4,6}(\chi, \pi) \Theta_{5,6}(\chi, \pi) \Theta_{6,7}(\chi, \pi) = \prod_{\nu \notin \{4,7\}} \Theta_{6, \nu}(\chi, \pi), \\
\Theta_{4,7}(\chi, \pi) \Theta_{5,7}(\chi, \pi) \Theta_{6,7}(\chi, \pi) = \prod_{\nu \notin \{4,7\}} \Theta_{7, \nu}(\chi, \pi),
\end{cases}
\end{align*}
\]

which once more show that \(\Theta_{1,2}(\chi, \pi), \Theta_{1,3}(\chi, \pi), \Theta_{4,5}(\chi, \pi), \Theta_{4,6}(\chi, \pi),\) and \(\Theta_{6,7}(\chi, \pi)\) are determined.

We conclude that

\[
C_{2r+1,n}(q; \pi) \leq \varphi(q)^{(n-1)}2^{r+1/2 - 3/2}.
\]

We are now ready to finish the proof of Theorem 1.7.

**Proof of Theorem 1.7, second part.** We first prove the estimate for \(\mathbb{E}[H_{2m}(q; \eta)]\). Lemma 4.4 implies that

\[
\mathbb{E}[H_{2m}(q; \eta)] = \frac{1}{\varphi(q)^{2m}} \sum_{\chi \in X_{1, n}} \sum_{\gamma_1, \ldots, \gamma_{2m}} \hat{\eta} \left( \frac{\gamma_1 \chi_1}{2 \pi} \right) \cdots \hat{\eta} \left( \frac{\gamma_{2m} \chi_{2m}}{2 \pi} \right),
\]

which combined with Lemma 8.1 take the form (recall (3.2), and note that under LI, the zeros of \(L(s, \chi)\) are simple)

\[
\begin{align*}
\mathbb{E}[H_{2m}(q; \eta)] &= \frac{1}{\varphi(q)^{2m}} \sum_{\{i_1, j_1, \ldots, i_m, j_m\}} \sum_{\chi_{i_1}, \ldots, \chi_{i_m} \neq 0} \sum_{\gamma_{i_1}, \ldots, \gamma_{i_m} = 0} \left| \hat{\eta} \left( \frac{\gamma_{i_1} \chi_{i_1}}{2 \pi} \right) \right|^2 \cdots \left| \hat{\eta} \left( \frac{\gamma_{i_m} \chi_{i_m}}{2 \pi} \right) \right|^2 \\
&\quad + O \left( \mu_{2m} \frac{m^2}{\varphi(q)^{2m}} \sum_{\chi_{i_1} \neq 0} b(\chi; \eta)^2 \left( \sum_{\chi_{i_1} \neq 0} b(\chi; \eta)^2 \right)^{m-2} \right) \\
&= \frac{\mu_{2m}}{\varphi(q)^m} \sum_{\chi \mod q \atop \chi \neq \chi_{0,q}} b(\chi; \eta)^2 \left( \sum_{\chi \neq \chi_{0,q}} b(\chi; \eta)^2 \right)^{m-2} + O \left( \mu_{2m} \frac{m^2(K_{\delta, q} \log q)^m}{\varphi(q)^{m+1}} \right),
\end{align*}
\]

by Proposition 3.2. The claimed estimate follows from a second application of Proposition 3.2.
Moving on to higher moments, by taking $T \to \infty$ in Lemma 6.2 and applying Lemma 4.3, we obtain that

\[
\mathbb{V}_{s,n}(q; \eta) = \frac{(-1)^{sn}}{\phi(q)^{sn}} \sum_{\chi \in \chi_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) \Delta_s(\sigma_\gamma)
\]

\[
= \frac{(-1)^{sn}}{\phi(q)^{sn}} \sum_{\chi \in \chi_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) \Delta_s(\sigma_\gamma),
\]

by Lemma 8.1. The inner sum may be further restricted by requiring that $\gamma_{\chi_{\mu,j}} + \gamma_{\chi_{\nu,i}} = 0$, where $(\mu, j)$ and $(\nu, i)$ are such that $\chi_{\mu,j} = \chi_{\nu,i}$. We will first discard characters in the outer sum in such a way to change the condition $\forall (\mu, j), \exists (\nu, i) : \chi_{\mu,j} = \chi_{\nu,i}$ into $\forall (\mu, j), \exists! (\nu, i) : \chi_{\mu,j} = \chi_{\nu,i}$. The sum of the extra terms is

\[
\ll (K_{s,n} \log q)^{sn} \sum_{\chi \in \chi_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) \Delta_s(\sigma_\gamma),
\]

where we have used the bound

\[
\sum_{\gamma} |\hat{\eta}(\gamma)|^2 \ll \delta_{s,n} \log q,
\]

which is a consequence of the decay of $\hat{\eta}$ combined with the Riemann–von Mangoldt formula. By Lemmas 8.5 and 8.7, the total contribution of the $\pi \in G_{s,n}$ is

\[
\ll |G_{s,n}| \phi(q)^{sn} \frac{\delta_{s,n}}{2} \left( K_{s,n} \log q \right)^{sn},
\]

where $\delta_{s,n} = 1$ when $2 \nmid s$, and is zero otherwise. If $s = 2r$ and $\pi \in F_{2r,n}$, then after reordering $\chi$ in the inner sum, we can assume that $J_{2u-1,2u}(\pi) \neq \emptyset$ for $1 \leq u \leq r$ and that $J_{\mu,\nu}(\pi) = \emptyset$ otherwise. There are two subcases. If there exists

\[X_{2u-1,j} \in \{ \chi_{\nu,i} : \nu \notin \{ 2u - 1, 2u \}, i \leq n \},\]

then arguing as in the proof of Lemma 6.5, one shows that the inner sum is $\ll n^2 r^2 \phi(q)^{(n-1)r-1}$. Otherwise, there must exist

\[X_{2u-1,j} \in \{ \chi_{\nu,i} : \nu \notin \{ 2u - 1, 2u \}, i \leq n \} \setminus \{ \chi_{2u-1,j} \}, \chi_{2u-1,j} \in F_{2r,n} \}
\]

Once more, the inner sum is $\ll n^2 r^2 \phi(q)^{(n-1)r-1}$.
We conclude that
\[
\psi_{s,n}(q; \eta) = \frac{(-1)^{sn}}{\varphi(q)^{sn}} \sum_{\pi \in I_{s,n}} \sum_{\chi \in \chi_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma) \Delta_s(\gamma)
\]
\[
\times \sum_{\forall (\mu,j), \exists !(\nu,i) \neq (\mu,j) : \chi \mu,j = \chi \nu,i} \sum_{\forall (\mu,j), \gamma \nu,i + \gamma \pi(\mu,j) = 0} \left( |G_{s,n}| + n^2 r^2 |F_{s,n}| \right) \sigma(q)^{sn} \frac{(K_{s,n} \log q)^{sn/2}}{\varphi(q)^{n+1 + \delta_{2|s}}}.
\]

Now, by Lemmas 8.5 and 8.7, the sum over \( \pi \in G_{s,n} \) is absorbed in the error term. The remaining terms are
\[
= \frac{(-1)^{sn}}{\varphi(q)^{sn}} \sum_{\pi \in F_{s,n}} \sum_{\chi \in \chi_{s,n}} \sum_{\gamma \in \Gamma(\chi)} \hat{\eta}(\gamma). \]

Applying Lemmas 6.5 and 6.9 yields the claimed estimate on \( \psi_{s,n}(q; \eta) \).

\[\square\]

**ACKNOWLEDGEMENTS**

The work of the second author was supported at the University of Ottawa by an NSERC discovery grant.

**JOURNAL INFORMATION**

The *Proceedings of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

**REFERENCES**

1. A. Akbary and B. Fodden, *Lower bounds for power moments of L-functions*, Acta Arith. **151** (2012), no. 1, 11–38.
2. A. Akbary, N. Ng, and M. Shahabi, *Limiting distributions of the classical error terms of prime number theory*, Q. J. Math. **65** (2014), no. 3, 743–780.
3. J. Andrade, *Rudnick and Soundararajan’s theorem for function fields*, Finite Fields Appl. **37** (2016), 311–327.
4. T. Assiotis and J. P. Keating, *Moments of moments of characteristic polynomials of random unitary matrices and lattice point counts*, Random Matrices Theory Appl. **10** (2021), no. 2, Paper No. 2150019, 16 pp.
5. T. Assiotis, E. C. Bailey, and J. P. Keating, *On the moments of the moments of the characteristic polynomials of Haar distributed symplectic and orthogonal matrices*, arXiv:1910.12576, 2019.
6. E. C. Bailey and J. P. Keating, *On the moments of the moments of the characteristic polynomials of random unitary matrices*, Commun. Math. Phys. **371** (2019), 689–726.
7. E. C. Bailey and J. P. Keating, *On the moments of the moments of \( \zeta(\frac{1}{2} + it) \)*, arxiv:2006.04503, 2020.
8. R. C. Baker, *Primes in arithmetic progressions to spaced moduli*, Acta Arith. **153** (2012), no. 2, 133–159.
9. R. C. Baker, *Primes in arithmetic progressions to spaced moduli. II*, Q. J. Math. **65** (2014), no. 2, 597–625.
10. R. C. Baker and T. M. Freiberg, *Sparser variance for primes in arithmetic progression*, Monatsh. Math. **187** (2018), no. 2, 217–236.
11. M. B. Barban, *The large sieve method and its applications in the theory of numbers*, Uspekhi Mat. Nauk **21** (1966), 51–102 (Russian), Russian Math. Surveys **22** (1966) 49–103.
12. V. Blomer, *The average value of divisor sums in arithmetic progressions*, Q. J. Math. **59** (2008), no. 3, 275–286.
13. P. Le Boudec, *On the distribution of squarefree integers in arithmetic progressions*, Math. Z. 290 (2018), no. 1-2, 421–429.

14. R. de la Bretèche and D. Fiorilli, *Major arcs and moments of arithmetical sequences*, Amer. J. Math. 142 (2020), no. 1, 45–77.

15. R. de la Bretèche and D. Fiorilli, *On a conjecture of Montgomery and Soundararajan*, Math. Ann. 381 (2021), no. 1, 575–591.

16. J. Brüdern and T. D. Wooley, *Sparse variance for primes in arithmetic progressions*, Q. J. Math. 62 (2011), no. 2, 289–305.

17. H. M. Bui, J. P. Keating, and D. J. Smith, *On the variance of sums of arithmetic functions over primes in short intervals and pair correlation for L-functions in the Selberg class*, J. Lond. Math. Soc. (2) 94 (2016), no. 1, 161–185.

18. V. Chandee and X. Li, *Lower bounds for small fractional moments of Dirichlet L-functions*, Int. Math. Res. Not. IMRN 2013 (2013), no. 19, 4349–4381.

19. V. Chandee and X. Li, *The sixth moment of automorphic L-functions*, Algebra Number Theory 11 (2017), no. 3, 583–633.

20. V. Chandee, Y. Lee, S-C. Liu, and M. Radziwill, *Simple zeros of primitive Dirichlet L-functions and the asymptotic large sieve*, Q. J. Math. 65 (2014), no. 1, 63–87.

21. J. B. Conrey and D. W. Farmer, *Mean values of L-functions and symmetry*, Int. Math. Res. Not. 2000 (2000), no. 17, 883–908.

22. J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, *Integral moments of L-functions*, Proc. Lond. Math. Soc. (3) 91 (2005), no. 1, 33–104.

23. J. B. Conrey, H. Iwaniec, and K. Soundararajan, *The sixth power moment of Dirichlet L-functions*, Geom. Funct. Anal. 22 (2012), no. 5, 1257–1288.

24. J. B. Conrey and A. Ghosh, *A conjecture for the sixth moment of the Riemann zeta-function*, Int. Math. Res. Not. IMRN 15 (1998), 775–780.

25. J. B. Conrey and S. Gonek, *High moments of the Riemann zeta-function*, Duke Math. J. 107 (2001), no. 3, 577–604.

26. H. Cramér, *Ein Mittelwertsatz in der Primzahltheorie*, Math. Z. 12 (1922), 147–53.

27. M. J. Croft, *Square-free numbers in arithmetic progressions*, Proc. Lond. Math. Soc. (3) 30 (1975), 143–159.

28. H. Davenport and H. Halberstam, *Primes in arithmetic progressions*, Michigan Math. J. 13 (1966), 485–489.

29. L. Devin, *Chebyshev's bias for analytic L-functions*, Math. Proc. Cambridge Philos. Soc. 169, no. 1, 103–140.

30. P. D. T. A. Elliott, *Primes in short arithmetic progressions with rapidly increasing differences*, Trans. Amer. Math. Soc. 353 (2001), no. 7, 2705–2724.

31. D. Fiorilli, *The distribution of the variance of primes in arithmetic progressions*, Int. Math. Res. Not. 2015 (2015), no. 12, 4421–4448.

32. D. Fiorilli and G. Martin, *Inequities in the Shanks-Rényi prime number race: an asymptotic formula for the densities*, J. Reine Angew. Math. 676 (2013), 121–212.

33. D. Fiorilli and G. Martin, *A disproof of Hooley's conjecture*, Preprint.

34. É. Fouvry, S. Ganguly, E. Kowalski, and P. Michel, *Gaussian distribution for the divisor function and Hecke eigenvalues in arithmetic progressions*, Comment. Math. Helv. 89 (2014), no. 4, 979–1014.

35. J. B. Friedlander and D. A. Goldston, *Variation of distribution of primes in residue classes*, Quart. J. Math. Oxford Ser. (2) 47 (1996), no. 187, 313–336.

36. D. A. Goldston and H. L. Montgomery, *Pair correlation of zeros and primes in short intervals*, Analytic number theory and diophantine problems, Proc. Conf., Stillwater/Okla. 1984, Prog. Math. 70 (1987), 183–203.

37. D. A. Goldston and R. C. Vaughan, *On the Montgomery-Hooley asymptotic formula*, Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), London Math. Soc. Lecture Note Ser., vol. 237, Cambridge University Press, Cambridge, 1997, pp. 117–142, 11N13 (11P05)

38. O. Gorodetsky, K. Matomäki, M. Radziwiłł, and B. Rodgers, *On the variance of squarefree integers in short intervals and arithmetic progressions*, arXiv:2006.04060, 2020.

39. C. Hall, J. P. Keating, and E. Roditty-Gershon, *Variance of arithmetic sums and L-functions in $\mathbb{F}_q[t]$*, Algebra Number Theory 13 (2019), no. 1, 19–92.

40. G. Harman, *The Montgomery-Hooley theorem in short intervals*, Mathematika 59 (2013), no. 1, 129–139.

41. A. J. Harper, *Sharply conditional bounds for moments of the Riemann zeta function*, arXiv:1305.4618.
42. A. J. Harper and K. Soundararajan, Lower bounds for the variance of sequences in arithmetic progressions: primes and divisor functions, Q. J. Math. 68 (2017), no. 1, 97–123.
43. D. R. Heath-Brown, Fractional moments of the Riemann zeta function, J. Lond. Math. Soc. (2) 24 (1981), no. 1, 65–78.
44. D. R. Heath-Brown, Fractional moments of Dirichlet L-functions, Acta Arith. 145 (2010), no. 4, 397–409.
45. C. Hooley, On the Barban-Davenport-Halberstam theorem. I. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, III, J. Reine Angew. Math. 274/275 (1975), 206–223.
46. C. Hooley, Fractional moments of the Riemann zeta function, J. Lond. Math. Soc. (2) 9 (1974/75), 625–636.
47. C. Hooley, Fractional moments of Dirichlet L-functions, Acta Arith. 145 (2010), no. 4, 397–409.
48. C. Hooley, On the distribution of sequences in arithmetic progressions, Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), vol. 1, pp. 357–364. Canad. Math. Congress, Montreal, Que., 1975.
49. C. Hooley, On the distribution of sequences in arithmetic progressions, Acta Arith. 94 (2000), no. 1, 53–86.
50. C. Hooley, On the distribution of sequences in arithmetic progressions, Proc. Lond. Math. Soc. (3) 33 (1976), no. 3, 535–548.
51. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Lond. Math. Soc. (2) 9 (1974/75), 625–636.
52. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 499 (1998), 1–46.
53. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
54. C. Hooley, On the distribution of sequences in arithmetic progressions, Proc. Lond. Math. Soc. (3) 33 (1976), no. 3, 535–548.
55. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
56. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
57. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
58. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
59. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
60. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
61. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
62. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
63. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
64. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
65. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
66. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
67. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
68. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
69. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
70. C. Hooley, On the distribution of sequences in arithmetic progressions, J. Reine Angew. Math. 509 (1999), 1–46.
73. H.-Q. Liu, *Barban-Davenport-Halberstam average sum and exceptional zero of L-functions*, J. Number Theory **128** (2008), no. 4, 1011–1043.
74. D. Mastrostefano, *A lower bound for the variance of generalized divisor functions in arithmetic progressions*, arxiv:2004.05602.
75. Z. Meng, *Some new results on k-free numbers*, J. Number Theory **121** (2006), no. 1, 45–66.
76. H. Mikawa and T. P. Peneva, *Primes in arithmetic progressions to spaced moduli*, Arch. Math. (Basel) **84** (2005), no. 3, 239–248.
77. H. L. Montgomery, *Primes in arithmetic progressions*, Michigan Math. J. **17** (1970), 33–39.
78. H. L. Montgomery and K. Soundararajan, *Primes in short intervals*, Comm. Math. Phys. **252** (2004), no. 1–3, 589–617.
79. H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory. I. Classical theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2007.
80. Y. Motohashi, *On the distribution of the divisor function in arithmetic progressions*, Acta Arith. **22** (1973), 175–199.
81. M. Munsch, *Shifted moments of L-functions and moments of theta functions*, Mathematika **63** (2017), no. 1, 196–212.
82. R. M. Nunes, *Squarefree numbers in arithmetic progressions*, J. Number Theory **153** (2015), 1–36.
83. R. M. Nunes, *Moments of the distribution of k-free numbers in short intervals and arithmetic progressions*, arXiv:2010.03696, 2020.
84. A. Perelli, *The L^1 norm of certain exponential sums in number theory: a survey*, Rend. Sem. Mat. Univ. Politec. Torino **53** (1995), no. 4, 405–418. Number theory, II (Rome, 1995).
85. J. Pintz, *Oscillatory properties of the remainder term of the prime number formula*, Studies in Pure Mathematics, vol. 551–560, Birkhäuser, Basel, 1983.
86. P. Pongsriiam and R. C. Vaughan, *The divisor function on residue classes II*, Acta Arith. **182** (2018), no. 2, 133–181.
87. M. Radziwiłł and K. Soundararajan, *Continuous lower bounds for moments of zeta and L-functions*, Mathematika **59** (2013), no. 1, 119–128.
88. M. Radziwiłł and K. Soundararajan, *Moments and distribution of central L-values of quadratic twists of elliptic curves*, Invent. Math. **202** (2015), no. 3, 1029–1068.
89. K. Ramachandra, *Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. I*, Hardy-Ramanujan J. **1** (1978), 15 pp.
90. K. Ramachandra, *Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. II*, Hardy-Ramanujan J. **3** (1980), 1–24.
91. B. Rodgers and K. Soundararajan, *The variance of divisor sums in arithmetic progressions*, Forum Math. **30** (2018), no. 2, 269–293.
92. E. Roditty-Gershon, *Square-full polynomials in short intervals and in arithmetic progressions*, Res. Number Theory **3** (2017), Paper No. 3, 18 pp.
93. Z. Rudnick and K. Soundararajan, *Lower bounds for moments of L-functions*, Proc. Natl. Acad. Sci. USA **102** (2005), no. 19, 6837–6838.
94. Z. Rudnick and K. Soundararajan, *Lower bounds for moments of L-functions: symplectic and orthogonal examples*, Multiple Dirichlet series, automorphic forms, and analytic number theory, Proc. Sympos. Pure Math., vol. 75, Amer. Math. Soc., Providence, RI, 2006, pp. 293–303.
95. J. -C. Schlage-Puchta, *Oscillations of the error term in the prime number theorem*, Acta Math. Hungar. **156** (2018), no. 2, 303–308.
96. A. Selberg, *On the normal density of primes in small intervals, and the difference between consecutive primes*, Arch. Math. Naturvid. **47** (1943), no. 6, 87–105.
97. E. Smith, *A generalization of the Barban-Davenport-Halberstam theorem to number fields*, J. Number Theory **129** (2009), no. 11, 2735–2742.
98. E. Smith, *A Barban-Davenport-Halberstam asymptotic for number fields*, Proc. Amer. Math. Soc. **138** (2010), no. 7, 2301–2309.
99. E. Smith, *A variant of the Barban-Davenport-Halberstam theorem*, Int. J. Number Theory **7** (2011), no. 8, 2203–2218.
100. P. Song, W. Zhai, and D. Zhang, *Power moments of Hecke eigenvalues for congruence group*, J. Number Theory **198** (2019), 139–158.
101. K. Sono, Continuous lower bounds for the moments of Dedekind zeta-functions, J. Number Theory 188 (2018), 335–356.
102. K. Soundararajan, Moments of the Riemann zeta function, Ann. of Math. (2) 170 (2009), no. 2, 981–993.
103. R. C. Vaughan, On a variance associated with the distribution of general sequences in arithmetic progressions. I, II, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. 356 (1998), no. 1738, 781–791, 793–809.
104. R. C. Vaughan, On a variance associated with the distribution of primes in arithmetic progressions, Proc. Lond. Math. Soc. (3) 82 (2001), no. 3, 533–553.
105. R. C. Vaughan, Moments for primes in arithmetic progressions. II, Duke Math. J. 120 (2003), no. 2, 385–403.
106. P. Xi, Gaussian distributions of Kloosterman sums: vertical and horizontal, Ramanujan J. 43 (2017), no. 3, 493–511.
107. R. Warlimont, Squarefree numbers in arithmetic progressions, J. London Math. Soc. (2) 22 (1980), no. 1, 21–24.
108. W. Yao, A Barban-Davenport-Halberstam theorem for integers with fixed number of prime divisors, Sci. China Math. 57 (2014), no. 10, 2103–2110.
109. W. Yao, A generalization of the Montgomery-Hooley theorem, J. Number Theory 162 (2016), 496–517.
110. M. P. Young, The fourth moment of Dirichlet L-functions, Ann. Math. (2) 173 (2011), no. 1, 1–50.
111. D. Zhang and Y. Wang, Higher-power moments of Fourier coefficients of holomorphic cusp forms for the congruence subgroup $\Gamma_0(N)$, Ramanujan J. 47 (2018), no. 3, 685–700.