LIPSCHITZ STABILITY IN DETERMINATION OF COEFFICIENTS IN A TWO-DIMENSIONAL BOUSSINESQ SYSTEM BY ARBITRARY BOUNDARY OBSERVATION

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Abstract. In this paper, we consider the inverse problem of determining two spatially varying coefficients appearing in the two-dimensional Boussinesq system from observed data of velocity vector and the temperature in a given arbitrarily subboundary. Based on Carleman estimates, we prove a Lipschitz stability result.

1. Introduction and main result.

1.1. Statement of the problem. Let us consider the Boussinesq system in bounded domain $\Omega$ of $\mathbb{R}^2$ with connected and $C^\infty$ boundary $\Gamma = \partial \Omega$. Given $T > 0$, we consider the following problem of the nonlinear Boussinesq system

$$\begin{align*}
\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \pi - \text{div}(\nu(x)\nabla \mathbf{v}) &= \theta \mathbf{e}_2 + k(x, t) \quad \text{in } Q := \Omega \times (0, T), \\
\theta_t + \mathbf{v} \cdot \nabla \theta - \text{div}(\kappa(x)\nabla \theta) &= h(x, t) \quad \text{in } Q, \\
\text{div} \mathbf{v} &= 0 \quad \text{in } Q.
\end{align*}$$

(1)

Throughout this paper, $t$ and $x = (x_1, x_2)$ denote the time variable and the spatial variable, respectively, and $\mathbf{v} = (v_1, v_2)$ denotes the unknown velocity vector, $\theta = \theta(x, t)$ stands for the temperature in a gravity field, $\pi$ is the unknown scalar pressure where $\nabla \pi$ models the acceleration, $\mathbf{e}_2 := (0, 1)$ is a given direction, in which $\theta \mathbf{e}_2$ represents the buoyancy force, $(k, h)$ stand an external force, and the positive and unknown parameters $\nu$ and $\kappa$ denote respectively the viscosity and the thermal diffusivity such that $\nu, \kappa \in C^2(\overline{\Omega})$ and $\nu(x), \kappa(x) > 0$ for all $x \in \overline{\Omega}$.

Additionally, we attach the following boundary conditions to system (1)

$$\mathbf{v} \cdot n = 0, \quad \text{curl } \mathbf{v} = 0, \quad \theta = 0 \quad \text{on } \Sigma := \Gamma \times (0, T),$$

(2)

where $n = (n_1, n_2)$ denotes the outward unit normal vector to $\Gamma$. The boundary condition for the velocity $\mathbf{v}$ in (2) is the free boundary condition of the Navier boundary condition.

Furthermore, if we add suitable initial data $(\mathbf{v}_0, \theta_0)$ satisfying $\text{div}\mathbf{v}_0 = 0$ and the compatibility condition (2), then we can prove, see Appendix A and [15, 23, 24, 26, 2020 Mathematics Subject Classification. Primary: 35R30, 35M33; Secondary: 76M21.

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that the initial boundary value problem corresponding to the system (1) possesses a unique solution \((v, \theta, \pi)\) such that
\[
\begin{align*}
v &\in C([0, T]; H^3(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\
v_t &\in L^2(0, T; L^2(\Omega)), \\
\theta &\in C([0, T]; H^3(\Omega)), \\
\pi &\in L^2(0, T; H^1(\Omega)).
\end{align*}
\]

The standard 2D Boussinesq equations have attracted considerable attention recently due to their physical applications and mathematical significance. The Boussinesq system (1) models geophysical flows such as atmospheric fronts and ocean circulations and it can be used to study the influence of the convection phenomenon in a viscous or inviscid fluid that takes place in atmosphere, ocean and inside earth or stars (see, e.g \[28\]).

The Boussinesq equations have been widely studied and the one main focus of recent research on the 2D Boussinesq equations has been the global well-posedness and the regularity issue. The main subject of this paper is the inverse problem of determining stably the two spatially varying coefficients \(\nu\) and \(\kappa\), in the Boussinesq model (1)-(2), from observed data of velocity vector \(v\) and the temperature \(\theta\) on an arbitrarily subboundary \(\Gamma_0 \subset \Gamma\) and the observation data of \(v\) and \(\theta\) at given time \(t_0 \in (0, T)\).

Before dealing with the inverse problem under consideration, let us state in brief some of the results that are relevant to this problem.

1.2. **Inverse coefficients problem.** Inverse coefficients problems for partial differential equations is one of the most rapidly growing mathematical research area, and have attracted many attention over the last decades. As for theoretical methods, we consider two types of formulations: infinitely many measurements by Dirichlet-to-Neumann map, while the second called a finite measurement by Carleman estimates. In this paper, our inverse problem is formulated with finite boundary measurement. In fact, as for a different formulation of inverse problems with a single measurement, Bukhgeim and Klibanov [10] proposed a remarkable method based on a Carleman estimate and established the uniqueness for inverse problems of determining spatially varying coefficients for scalar partial differential equations in which we can refer to Bellassoued and Yamamoto [5, 7], Bukhgeim, Cheng, Isakov, and Yamamoto [9], Imanuvilov and Yamamoto [21], and Klibanov and Yamamoto [25].

As we said previously our method is based on the \(L^2\)-weighted inequality called Carleman estimate and was introduced in 1939 by Carleman [11] as an exponential weight energy inequalities in order to prove a uniqueness result for elliptic partial differential equations with regular two-dimensional coefficients. This type of inequality has been generalized by Hörmander [17] and other authors for large classes of differential operators in any dimension. Since then, the use of these inequalities has largely exceeded their original application. They not only give quantitative results of single extension but they are also used for the study of inverse problems as well as in control theory for PDEs. Moreover the Carleman estimates and their applications to inverse problems are considered by many authors and there are many works that are relevant to this topic [1, 2, 3]. We list some works for the well-known equations in mathematical physics. For hyperbolic equation, Bellassoued and Yamamoto in [4] discussed the derivation of Carleman estimates and their application to inverse problems of determining spatially varying coefficients or source terms. For parabolic equations, Yuan and Yamamoto in [34] determine some coefficients of the principal part of a parabolic equation by boundary observations. Yamamoto
in [33] has given a great survey by summarizing different types of Carleman estimates and methods for applications to some inverse problems. Moreover, Choulli, Imanuvilov, Puel and Yamamoto [12] have worked on the inverse source problem for linearized Navier-Stokes equations with data in arbitrary subdomain.

In [13], Fan, Di Cristo, Jiang and Nakamura consider the inverse problem of determining the viscosity coefficient in the Navier-Stokes equation with Dirichlet boundary conditions by the observation data in a neighborhood of the whole boundary and they proved a Lipschitz stability by using the Carleman estimates. In this paper thanks to the boundary conditions that we considered by taking \( \mathbf{v} \cdot \mathbf{n} = 0 \) and \( \text{curl} \mathbf{v} = 0 \), we can relax the stability estimate by observing the solution of the Boussinesq system (1) only in a given arbitrarily subboundary.

Furthermore, for applications of the Bukhgeim-Klibanov method to the systems, there are some papers devoted to inverse hyperbolic-parabolic systems, we can refer to [6] in which Bellassoued and Yamamoto established Carleman estimates with second large parameter for a parabolic-hyperbolic system, a thermoelastic plate system and a thermoelasticity system with residual stress, and they prove a Hölder stability estimate in [5] for the inverse problem of determining the heat source term. Then, in a thermoelastic model, Wu and Liu [32] study the inverse problem of determining two spatially varying coefficients by proving the Lipschitz stability and the uniqueness for this inverse problem and based on Carleman estimates. In addition Bellassoued and Riahi in [3] proved the uniqueness and a Hölder stability in determining spatially varying coefficients for coupled system of mixed hyperbolic-parabolic type, which describes the Biot consolidation model in poro-elasticity. For elasticity, we refer to Isakov and Kim in [22] and Imanuvilov, Isakov and Yamamoto in [20].

While, mathematical studies on the two-dimensional inverse coefficients problem of Boussinesq system have not been very developed. We can cite the work of Fan, Jiang and Nakamura [14] in which they proved two results of stability estimate: one is a Lipschitz stability by identifying a spatially external force with observation data in an arbitrary sub-domain over a time interval and the other one is a conditional stability estimate of identifying the two initial conditions with a single observation on a sub-domain.

Our inverse coefficients problem can be described as follows:

Let \( \Gamma_0 \subset \Gamma \) be given non-empty arbitrary subboundary. We consider a function \( \eta \in C^2(\Omega) \) satisfying

\[
\eta(x) > 0 \quad \forall x \in \Omega, \quad |\nabla \eta(x)| > 0 \quad \forall x \in \overline{\Omega},
\]

\[
\partial_n \eta \leq 0 \quad \forall x \in (\Gamma \setminus \Gamma_0), \quad \partial_n \eta \geq 0 \quad \forall x \in \Gamma_0.
\]

(3)

For the existence of such function, we can see [18, 16].

Throughout this paper, let us consider the admissible set \( \mathcal{A} = \mathcal{A}(M, \kappa_0, \nu_0) \) of unknown coefficients \( \kappa \) and \( \nu \) given by

\[
\mathcal{A} = \{ (\kappa, \nu) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega}), \| \kappa \|_{C^2(\overline{\Omega})} + \| \nu \|_{C^2(\overline{\Omega})} \leq M, \ k \geq k_0, \ \nu \geq \nu_0, \ x \in \overline{\Omega} \},
\]
where $M, \kappa_0, \nu_0 > 0$ are given.

Before stating the main result, we first introduce the following assumptions which we shall use repeatedly in the sequel.

**Assumptions.**

(A.1) Let $\eta \in C^2(\overline{\Omega})$ satisfying (3) and $\omega_0 \subset \Omega$ a given subdomain such that $\partial \omega_0 \supset \Gamma$. For $(\kappa, \nu) \in \mathcal{A}$, we assume that there exists $t_0 \in (0, T)$ and $m_0 > 0$ such that

$$|\nabla \eta(x, t) \cdot \nabla \eta(x)| \geq m_0 \quad \text{for all} \quad x \in \Omega \setminus \omega_0,$$

$$|\Delta(\nabla \omega(x, t_0))| \geq m_0, \quad \text{for all} \quad x \in \Omega.$$

Here $(\mathbf{v}, \theta)$ is the solution of (1) and the function $\eta$ satisfying (3).

(A.2) We assume that the solution $(\mathbf{v}, \theta)$ of (1)-(2) satisfies the a priori boundedness

$$\|\mathbf{v}\|_{W^{2, \infty}(0, T; W^{3, \infty}(\Omega))} + \|\theta\|_{W^{2, \infty}(0, T; W^{3, \infty}(\Omega))} \leq M_1,$$

for some given constant $M_1 > 0$.

(A.3) We assume that

$$\sum_{|\alpha|=1, 2} |\partial^{\alpha} \nu_1(x) - \partial^{\alpha} \nu_2(x)| \leq \epsilon_0 |\nu_1(x) - \nu_2(x)|, \quad x \in \Omega,$$

for $\epsilon_0$ sufficiently small.

**Theorem 1.1.** Let $(\kappa_j, \nu_j) \in \mathcal{A}$, $j = 1, 2$, such that $\kappa_1 = \kappa_2$ in $\omega_0$, and we assume that (A.1), (A.2) and (A.3) be held. Then there exists a positive constant $C > 0$ such that

$$\|\nu_1 - \nu_2\|_{L^2(\Omega)} + \|\kappa_1 - \kappa_2\|_{H^1(\Omega)} \leq C \left( \|\mathbf{v}_1(\cdot, t_0) - \mathbf{v}_2(\cdot, t_0)\|_{H^1(\Omega)} + \|	heta_1(\cdot, t_0) - \theta_2(\cdot, t_0)\|_{H^1(\Gamma_0)} + \|\partial_n(\nabla \mathbf{v}_1 - \mathbf{v}_2)\|_{H^2(0, T; L^2(\Gamma_0))} + \|\partial_n(\theta_1 - \theta_2)\|_{H^2(0, T; L^2(\Gamma_0))} \right),$$

where $(\mathbf{v}_j, \theta_j)$, $j = 1, 2$, are the corresponding solutions of (1)-(2) for $(\kappa, \nu) = (\kappa_j, \nu_j)$, $j = 1, 2$. Here $C$ depending on $\Omega, T$ and $M$.

As an immediate consequence of Theorem 1.1, we have the following uniqueness result.

**Corollary 1.** Under the same assumptions as in Theorem 1.1 and if

$$\mathbf{v}_1(\cdot, t_0) = \mathbf{v}_2(\cdot, t_0), \quad \theta_1(\cdot, t_0) = \theta_2(\cdot, t_0), \quad \text{in} \ \Omega,$$

and

$$\mathbf{v}_1 \cdot \tau = \mathbf{v}_2 \cdot \tau, \quad \partial_n(\nabla \mathbf{v}_1) = \partial_n(\nabla \mathbf{v}_2), \quad \partial_n \theta_1 = \partial_n \theta_1, \quad \text{on} \ \Sigma_0.$$

Then $\nu_1 = \nu_2$ and $\kappa_1 = \kappa_2$, in $\Omega$.

1.4. **Comments on the assumptions.** The hypotheses (A.1)-(A.2) and (A.3) are commonly used in the study of inverse problems for equations or parabolic systems, we can cite [13, 14, 20, 21, 33, 34].

1. In order to guarantee the assumption (A.1) for a suitable initial data $(\mathbf{v}_0, \theta_0)$ and a source term $(\mathbf{k}, h)$, we fix a trajectory $(\dot{\mathbf{v}}, \dot{\theta})$, together with certain
pressure $\hat{\pi}$, which will be a regular enough solution to the following Boussinesq system with intermediate data at $t = t_0$

\[
\begin{align*}
\dot{\mathbf{v}} + \nabla \cdot \nabla \mathbf{v} + \nabla \hat{\pi} - \text{div}(\nu(x)\nabla \mathbf{v}) &= \hat{\Omega} \mathbf{c_2} \quad \text{in } Q, \\
\dot{\theta} + \nabla \cdot \nabla \theta - \text{div}(\kappa(x)\nabla \theta) &= 0 \quad \text{in } Q, \\
\text{div} \mathbf{v} &= 0 \quad \text{in } Q, \\
\mathbf{v} \cdot n &= 0, \quad \text{curl } \mathbf{v} = 0, \quad \dot{\theta} = 0 \quad \text{on } \Sigma, \\
\mathbf{v}(x, t_0) &= \mathbf{r}(x), \quad \dot{\theta}(x, t_0) = \chi(x) \eta(x) \quad \text{in } \Omega,
\end{align*}
\]

where $\chi \in C^\infty_0(\Omega)$, $\chi = 1$ in $\overline{\Omega \setminus \omega_0}$, and $\mathbf{r} \in H^3(\Omega)$ such that $\text{div} r = 0$, in $\Omega$, and $|\Delta(\text{curl } \mathbf{r}(x))| \geq m_0$ in $\overline{\Omega}$ form some $m_0 > 0$ and satisfying the compatibility conditions:

\[
r \cdot n = 0, \quad \text{and } \text{curl } \mathbf{r} = 0 \quad \text{on } \Gamma.
\]

We denote

\[
\mathbf{v}_0(x) = \mathbf{v}(x, 0), \quad \dot{\theta}_0(x) = \dot{\theta}(x, 0), \quad x \in \Omega.
\]

Since system (1) satisfies the local exact controllability, see Imannuvilov [19], that is we can find a control $(\mathbf{k}, h) \in L^2(Q)$ and $\delta > 0$ such that for any initial data $(\mathbf{v}_0, \theta_0) \in H^2(\Omega) \times H^1(\Omega)$ satisfies the compatibility condition (2) and

\[
\|\mathbf{v}_0 - \mathbf{v}_0\|_{H^2(\Omega)} + \|\theta_0 - \theta_0\|_{H^1(\Omega)} < \delta
\]

the solution $(\mathbf{v}, \theta)$ of (1) exist and satisfies

\[
\mathbf{v}(x, t_0) = \mathbf{v}(x, t_0) = \mathbf{r}(x), \quad \theta(x, t_0) = \dot{\theta}(x, t_0) = \chi(x) \eta(x), \quad x \in \Omega.
\]

Then we deduce that

\[
|\nabla \theta(x, t_0) \cdot \nabla \eta(x)| \geq m_0, \quad x \in \overline{\Omega \setminus \omega_0},
\]

and

\[
|\Delta(\text{curl } \mathbf{v}(x, t_0))| = |\Delta(\text{curl } \mathbf{r}(x))| \geq m_0, \quad x \in \overline{\Omega}.
\]

2. An example of a function $\mathbf{r}$ satisfies the compatibility condition (6), $\text{div} \mathbf{r} = 0$, and $|\Delta(\text{curl } \mathbf{r}(x))| \geq m_0$: Take $\Omega$ the unit disc of $\mathbb{R}^2$, and let

\[
\mathbf{r}(x) = (2x_2 - x_2^3 - x_2 x_1^2, -2x_1 + x_1^3 + x_1 x_2^2), \quad |x| < 1.
\]

Then we have

\[
\mathbf{r}(x) \cdot n = \mathbf{r}(x) \cdot x = 0, \quad \text{curl } \mathbf{r}(x) = 0 \quad \text{on } \Gamma,
\]

and

\[
\text{div} \mathbf{r}(x) = 0, \quad |\Delta(\text{curl } \mathbf{r}(x))| = 8, \quad |x| \leq 1.
\]

3. If the initial data $(\mathbf{v}_0, \theta_0)$ and the source term $(\mathbf{k}, h)$ are sufficiently smooth and small enough or $T$ is small enough, then the condition (A.2) is satisfied.

4. A similar condition to (A.3) is assumed in [13] by Fan, Di Cristo, Jiang and Nakamura. This assumption is physically acceptable. (A.3) can be replaced by other conditions using many measurements. For example, in this case we obtain an algebraic system of $(\nu, \nabla \nu, \Delta \nu)$. We can assume this system is solvable and prove a similar stability theorem.

Finally we give the following Proposition, which guarantee the assumption (A.1) for a strong solution to (1).
Proposition 1. Let \((v, \theta) \in C([0, T], H^5(\Omega)) \times C([0, T], H^3(\Omega))\) be a solution to (1), with initial data \((v_0, \theta_0) \in H^5(\Omega) \times H^3(\Omega)\) satisfying

\[
|\Delta(\text{curl } v_0(x))| \geq m, \quad x \in \Omega, \quad |\nabla \theta_0(x) \cdot \nabla \eta(x)| \geq m, \quad x \in \Omega \setminus \omega_0,
\]

form some positive constant \(m > 0\). Then there exists \(t_0 \in (0, T)\) (small) such that the Assumption (A.1) holds true.

Proof. By the Sobolev embedding theorem, we have \(\Delta(\text{curl } v(x, t)) \in C([0, T] \times \Omega)\) and \(\nabla \theta(x, t) \in C([0, T] \times \Omega)\). Then for any \(\varepsilon > 0\), we can find \(t_0 \in (0, T)\) such that

\[
|\Delta(\text{curl } v(x, t_0)) - \Delta(\text{curl } v_0(x))| \leq \varepsilon, \quad |(\nabla \theta(x, t_0) - \nabla \theta_0(x)) \cdot \nabla \eta(x)| \leq \varepsilon, \quad \forall x \in \Omega.
\]

Taking \(\varepsilon > 0\) sufficiently small, we get

\[
|\Delta(\text{curl } v(x, t_0))| \geq m - \varepsilon \geq \frac{m}{2}, \quad \forall x \in \Omega; \quad |(\nabla \theta(x, t_0)| \geq m - \varepsilon \geq \frac{m}{2}, \quad \forall x \in \Omega \setminus \omega_0.
\]

This completes the proof. \(\square\)

Inspired by the work of Bellassoued and Yamamoto [8] and following the same strategy as Fian, Jiang and Nakamura in [14], we intend to apply Carleman estimates for the Boussinesq system (1)-(2) and then give the stability inequality in the recovery of the unknown coefficients \(\nu\) and \(\kappa\) via single measurement and from data of the solutions in a subboundary over a time interval.

The paper is organized as follows: In Section 2 we derive a few technical Lemmas and Carleman estimates in which they play a crucial role in the proof of the main result. Section 3 is devoted to linearize our inverse problem. In Section 4, following the Bukhgeim-Klibanov method, we prove the desired stability result.

2. Preliminary estimates. In this section, we derive some results in which they will play a crucial role in proving the main result. So we shall begin with the following Carleman estimates for solutions of system (1)-(2). These estimates are the main tool needed for the derivation of Theorem 1.1. So, we give Carleman estimates for parabolic and elliptic equations with the singular weight function with special properties, in which we can refer to Fursikov and Imanuvilov [16]. Let us define

\[
\ell(t) = t(T - t), \quad t \in (0, T).
\]

We note that \(\eta\) satisfies (3). Then, we set the weight function

\[
\alpha(x, t) := \frac{e^{\lambda \eta(x)} - e^{2\lambda \|\eta\|_{\infty}}}{\ell(t)}, \quad (x, t) \in Q, \quad (7)
\]

where \(\lambda > 0\) is a parameter.

We use usual functions spaces, \(H^k(Q)\) and

\[
H^{1,2}(Q) = H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)).
\]

In order to formulate our Carleman estimate, we consider the following linear parabolic system

\[
\begin{align*}
y_t - \sum_{i,j=1}^{2} \partial_i (a_{ij}(x) \partial_j y) + \sum_{i=1}^{2} b_i(x, t) \partial_i y &= g(x, t) \quad \text{in } Q, \\
y &= 0 \quad \text{on } \Sigma.
\end{align*}
\]
Lemma 2.1. There exists a constant $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, there exist $C_1 > 0$ and $s_0 := s_0(\lambda) > 0$ such that the following inequality holds

$$
\int_Q \left( |y|^2 + |\Delta y|^2 + \frac{s^2}{\ell^2(t)} |\nabla y|^2 + \frac{s^4}{\ell^4(t)} |y|^2 \right) e^{2s\alpha} \, dx \, dt \leq C_1 \left( \int_Q \frac{s}{\ell(t)} |y|^2 e^{2s\alpha} \, dx \, dt + \int_{\Sigma_0} \frac{s^2}{\ell^2(t)} |\partial_n y|^2 e^{2s\alpha} \, d\sigma dt \right),
$$

for all $s \geq s_0$ and $y \in H^{1,2}(Q)$ satisfying the system (8), where $C_1$ depend on $\Omega, \Gamma_0, T$ and $\lambda$. The proof of this Carleman estimate is similar to that in Imanuvilov and Yamamoto [21].

Next, we state the second key Carleman estimate for Laplace operator with a time singular weight function given by (7).

We recall the following Carleman estimate for Laplace operator.

Lemma 2.2. There exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, there exist $C_2 > 0$ and $s_0 := s_0(\lambda) > 0$ such that the following inequality holds

$$
\int_Q \left( \frac{s^3}{\ell^3(t)} |\nabla \psi|^2 + \frac{s^5}{\ell^5(t)} |\psi|^2 \right) e^{2s\alpha} \, dx \, dt 
\leq C_2 \left( \int_Q \frac{s^2}{\ell^2(t)} |\Delta \psi|^2 e^{2s\alpha} \, dx \, dt + \int_{\Sigma_0} \frac{s^3}{\ell^3(t)} |\partial_n \psi|^2 e^{2s\alpha} \, d\sigma dt \right),
$$

for all $s \geq s_0$ and $\psi \in L^2(0, T; H^1_0(\Omega) \cap H^2(\Omega))$, where $C_2$ depend on $\Omega, T$ and $\lambda$. The proof of this Carleman estimate is similar to that of the previous Lemma given by [21].

Moreover we need the following Carleman estimate for a first order operator. Firstly, we consider the first order partial differential operator

$$
A(x, D)v = \sum_{j=1}^2 \gamma_j(x) \partial_j v + \gamma_0(x) v, \quad x \in \Omega, \quad v \in H^1(\Omega),
$$

where $\gamma_0 \in C(\overline{\Omega}), \gamma = (\gamma_1, \gamma_2) \in \left( C^1(\overline{\Omega}) \right)^2$ such that $\|\gamma_0\|_{C(\overline{\Omega})} \leq M$ and $\|\gamma_j\|_{C^1(\overline{\Omega})} \leq M$, $1 \leq j \leq 2$. We assume that

$$
|\gamma(x) \cdot \nabla \eta(x)| \geq C_0, \quad x \in \overline{\Omega},
$$

for some positive constant $C > 0$.

In the sequel and without loss of generality, we may assume that $t_0 = T/2$ and to simplify the notations, we note $\alpha(x, t_0) := \alpha_0(x), x \in \Omega$.

Lemma 2.3. For any $\lambda > 0$, there exist constants $s_0 > 0$ and $C > 0$ such that the following estimate holds true

$$
s \int_\Omega |v(x)|^2 e^{2s\alpha_0} \, dx \leq C \int_\Omega |A(x, D)v(x)|^2 e^{2s\alpha_0} \, dx,
$$
Lemma 2.4. The following estimate holds
\[ \int_{\Omega} |z(x, t)|^2 dx \leq \int_Q \left( s|z(x, t)|^2 + s^{-1}|z_t(x, t)|^2 \right) dx dt, \]
for any $s > 0$ and $z \in H^1_0(0, T; L^2(\Omega))$. \hfill \Box
Proof. We have
\[
\int_\Omega |z(x,t_0)|^2 \, dx = \int_0^{t_0} \frac{d}{dt} \left( \int_\Omega |z(x,t)|^2 \, dx \right) dt = 2 \int_0^{t_0} \int_\Omega z(x,t) z_t(x,t) \, dx \, dt.
\]
Then by Cauchy-Schwartz inequality, we obtain
\[
\int_\Omega |z(x,t_0)|^2 \, dx \leq \int_Q (s|z(x,t)|^2 + s^{-1}|z_t(x,t)|^2) \, dx \, dt.
\]
This completes the proof of the Lemma. \(\square\)

Finally we have the following Lemma.

**Lemma 2.5.** There exist a constant \(C > 0\), \(s_0 > 0\) such that the following estimate holds true
\[
\int_Q \frac{s}{t(t)} |g|^2 e^{2s\alpha} \, dx \, dt \leq C \sqrt{s} \int_\Omega |g|^2 e^{2s\alpha} \, dx,
\]
for all \(g \in L^2(\Omega)\), and any \(s > s_0\).

For the proof, we can follow the same calculations as that in [21].

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3. **Linearized inverse problem.** Let us begin with a simple but useful remark. The main difficulties of the Boussinesq equation connected firstly with the pressure term \(\pi\) which we don’t have any information on it spatially on the boundary and secondly with the non-linearity of the inverse problem for this we cannot use the method in [12]. To overcome these difficulties, we follow the same method and steps as in [14] to get a linearized problem. So we use the stream function \(\psi(\cdot,t)\), (see [19]), \(t \in (0,T)\), as follows
\[
\begin{align*}
\nabla^\perp \psi(x,t) &= \mathbf{v}(x,t) \quad \text{in } \Omega, \\
\psi(x,t) &= 0 \quad \text{on } \Gamma,
\end{align*}
\]
which, as is well known, \(\nabla^\perp\) is the orthogonal gradient defined by:
\[
\nabla^\perp = \left( \partial_2, -\partial_1 \right)^T.
\]

Let \(w = \text{curl } \mathbf{v}\). Then we apply curl to the equation (14), we get
\[
\begin{align*}
-\Delta \psi(x,t) &= w(x,t) \quad (x,t) \in Q, \\
\psi(x,t) &= 0 \quad (x,t) \in \Sigma.
\end{align*}
\]
So the Boussinesq system (1)-(2) will be as follows
\[
\begin{align*}
\mathbf{w}_t + \mathbf{v} \cdot \nabla w - \text{div}(\nu \nabla \mathbf{w}) &= \partial_1 \theta + \sum_{i=1}^2 \partial_i (\nabla \nu \wedge \partial_i \mathbf{v}) + \text{curl } \mathbf{k} \quad \text{in } Q, \\
\theta_t + \mathbf{v} \cdot \nabla \theta - \text{div}(\kappa \nabla \theta) &= h \quad \text{in } Q, \\
\text{div } \mathbf{v} &= 0 \quad \text{in } Q, \\
w &= 0 \quad \text{on } \Sigma,
\end{align*}
\]
where we have used
\[
\text{curl } (\text{div}(\nu \nabla \mathbf{v})) = \text{div}(\nu \nabla w) + \sum_{i=1}^2 \partial_i (\nabla \nu \wedge \partial_i \mathbf{v}).
\]
We consider two sets of coefficients \((\kappa_j, \nu_j) \in \mathcal{A}, j = 1,2\), and the corresponding solutions \((\mathbf{v}_j, \theta_j, \psi_j, w_j), j = 1,2\), of (14), (15) and (16) and we define
\[
\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2, \quad \theta = \theta_1 - \theta_2, \quad \psi = \psi_1 - \psi_2, \quad w = w_1 - w_2, \quad \nu = \nu_1 - \nu_2, \quad \kappa = \kappa_1 - \kappa_2.
\]
Thus, rewriting inequality (22) in terms of (23), leads to

\[
\begin{aligned}
    w_t + \nu_1 \cdot \nabla w - \text{div}(\nu_1 \nabla \theta) &= f(x, t) & \text{in } Q, \\
    \theta_t + \nu_1 \cdot \nabla \theta - \text{div}(\kappa_1 \nabla \theta) &= g(x, t) & \text{in } Q, \\
    w = \theta = 0 & \text{on } \Sigma,
\end{aligned}
\]

where

\[
f(x, t) = \partial_t \theta + \text{div}(\nu \nabla w_2) - \nu \cdot \nabla w_2 + \sum_{i=1}^{2} \partial_i (\nabla \nu_1 \wedge \partial_i \nu) + \sum_{i=1}^{2} \partial_i (\nabla \nu \wedge \partial_i \nu_2),
\]

and

\[
g(x, t) = \text{div}(\kappa \nabla \theta_2) - \nu \cdot \nabla \theta_2.
\]

We have the following Lemma.

**Lemma 3.1.** Let \( \phi_1(x) = \text{div}(\kappa(x) \nabla \theta_2(x, t_0)) \). Then there exist \( s_0 > 0, C > 0 \) such that the following estimate holds

\[
\int_{\Omega} (|\nabla \phi_1(x)|^2 + |\phi_1(x)|^2)e^{2s_0} \, dx \\
\leq C \int_{Q} \left( s^{-1}(|\theta_t|^2 + |\nabla \theta_t|^2) + \frac{s}{\ell^2(t)}(|\theta_t|^2 + |\nabla \theta_t|^2) \right)e^{2s} \, dx \, dt \\
+ C \left( \|\nu(\cdot, t_0)\|_{H^1(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^3(\Omega)}^2 \right),
\]

for all \( s > s_0 \).

**Proof.** We start with the estimate of \( \phi_1(x) \), by the second equation of (17), (18) and (19), we have

\[
\phi_1(x) = \text{div}(\kappa \nabla \theta_2(x, t_0)) = g(x, t_0) + \nu(x, t_0) \cdot \nabla \theta_2(x, t_0) \\
= \theta_t(x, t_0) + \nu_1(x, t_0) \cdot \nabla \theta(x, t_0) - \text{div}(\kappa_1 \nabla \theta(x, t_0)) + \nu(x, t_0) \cdot \nabla \theta_2(x, t_0).
\]

From Assumption (A.2), there exists \( C > 0 \) such that

\[
\int_{\Omega} |\phi_1(x)|^2 e^{2s_0} \, dx \leq C \left( \int_{\Omega} |\theta_t(x, t_0)|^2 e^{2s_0} \, dx + \int_{\Omega} |\nabla \theta(x, t_0)|^2 e^{2s_0} \, dx \\
+ \int_{\Omega} |\Delta \theta(x, t_0)|^2 e^{2s_0} \, dx + \int_{\Omega} |\nu(x, t_0)|^2 e^{2s_0} \, dx \right).
\]

Thus, we obtain

\[
\int_{\Omega} |\phi_1(x)|^2 e^{2s_0} \, dx \leq C \int_{\Omega} |\theta_t(x, t_0)|^2 e^{2s_0} \, dx + C \left( \|\nu(\cdot, t_0)\|_{H^1(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^3(\Omega)}^2 \right).
\]

Applying Lemma 2.4, with \( z = \ell(t) \theta_t e^{s_0} \) and using the fact that \( |\alpha_t| \leq C \ell^{-2}(t) \) with \( \ell(t) \leq T^2/4 \), we get

\[
\int_{\Omega} |\theta_t(x, t_0)|^2 e^{2s_0} \, dx \leq C \int_{Q} \left( s^{-1}|\theta_t|^2 + \frac{s}{\ell^2(t)}|\theta_t|^2 \right)e^{2s} \, dx \, dt.
\]

Thus, rewriting inequality (22) in terms of (23), leads to

\[
\int_{\Omega} |\phi_1(x)|^2 e^{2s_0} \, dx \leq C \int_{Q} \left( s^{-1}|\theta_t|^2 + \frac{s}{\ell^2(t)}|\theta_t|^2 \right)e^{2s} \, dx \, dt \\
+ C \left( \|\nu(\cdot, t_0)\|_{L^2(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^2(\Omega)}^2 \right).
\]
Now, let us estimate $\nabla \phi_1$. Firstly, by (21), we find
\[
\partial_t \phi_1(x) = \partial_t \left( \text{div}(\kappa \nabla \theta_2(x, t_0)) \right)
\]
\[
= \partial_t \theta_1(x, t_0) + \partial_v \nu_1(x, t_0) \cdot \nabla \theta(x, t_0) + \nu_1(x, t_0) \cdot \nabla \partial_t \theta(x, t_0) - \partial_i \text{div}(\kappa \nabla \theta(x, t_0))
\]
\[
+ \partial_i \nu(x, t_0) \cdot \nabla \theta_2(x, t_0) + \nu(x, t_0) \cdot \nabla \partial_t \theta_2(x, t_0), \quad i = 1, 2.
\]
Then, we integrate over $\Omega$ and taking account Assumption (A.2), we conclude that
\[
\int_\Omega |\nabla \phi_1(x)|^2 e^{2s\alpha_0} dx \leq \int_\Omega |\nabla \theta_1(x, t_0)|^2 e^{2s\alpha_0} dx
\]
\[
+ \int_\Omega |\nabla \nu(x, t_0)|^2 e^{2s\alpha_0} dx + \int_\Omega |\nu(x, t_0)|^2 e^{2s\alpha_0} dx.
\]
Therefore
\[
\int_\Omega |\nabla \phi_1(x)|^2 e^{2s\alpha_0} dx \leq C \int_\Omega |\nabla \theta_1(x, t_0)|^2 e^{2s\alpha_0} dx
\]
\[
+ C \left( \|\nu(\cdot, t_0)\|_{L^2(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{L^2(\Omega)}^2 \right). \tag{25}
\]
To estimate the first term in the right-hand-side of (25), we apply again Lemma 2.4 by taking $z = \ell(t) \nabla \theta_1 e^{\alpha t}$ and using that $|\alpha_1| \leq C \ell^{-2}(t)$ and $\ell(t) \leq T^2/4$, we get
\[
\int_\Omega |\nabla \theta_1(x, t_0)|^2 e^{2s\alpha_0} dx \leq C \int_Q \left( s^{-1} |\nabla \theta_1|^2 + \frac{s}{\ell^2(t)} |\nabla \theta_1| e^{2s\alpha_0} \right) dx dt.
\]
From (25), we deduce
\[
\int_\Omega |\nabla \phi_1(x)|^2 e^{2s\alpha_0} dx \leq C \int_Q \left( s^{-1} |\nabla \theta_1|^2 + \frac{s}{\ell^2(t)} |\nabla \theta_1| e^{2s\alpha_0} \right) dx dt
\]
\[
+ C \left( \|\nu(\cdot, t_0)\|_{L^2(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{L^2(\Omega)}^2 \right). \tag{26}
\]
Finally, we combine (24) and (26) to complete the proof. \hfill \Box

4. Proof of main result. This section is devoted to the proof of Theorem 1.1 thanks to the application of preliminary estimates and combinations of Carleman estimates given in section 3.

Firstly, let $(w, \theta)$ solves system (17) and let $\psi$ solution of system (15). For simplicity, we consider the following notations, for $j = 0, 1, 2$, let
\[
w^{(j)} = \partial_t^j w, \quad \theta^{(j)} = \partial_t^j \theta, \quad \psi^{(j)} = \partial_t^j \psi \quad \text{and} \quad f^{(j)} = \partial_t^j f, \quad g^{(j)} = \partial_t^j g, \quad (x, t) \in Q.
\]
By a simple computation, we obtain for $j = 0, 1, 2$, $(w^{(j)}, \theta^{(j)})$ satisfy
\[
\begin{cases}
w_t^{(j)} + \nu_1 \cdot \nabla w^{(j)} - \text{div}(\nu_1(x, t) \nabla w^{(j)}) = F^{(j)}(x, t) & \text{in} \ Q, \\
\theta_t^{(j)} + \nu_1 \cdot \nabla \theta^{(j)} - \text{div}(\kappa_1(x, t) \nabla \theta^{(j)}) = G^{(j)}(x, t) & \text{in} \ Q, \\
w^{(j)} = \theta^{(j)} = 0 & \text{on} \ \Sigma,
\end{cases} \tag{27}
\]
with
\[
F^{(0)}(x, t) = f(x, t), \quad F^{(1)}(x, t) = f^{(1)}(x, t) - \nu_1^{(1)} \cdot \nabla w, \\
F^{(2)}(x, t) = f^{(2)}(x, t) - 2\nu_1^{(1)} \cdot \nabla w^{(1)} - \nu_1^{(2)} \cdot \nabla w, \tag{28}
\]
and
\[ G^{(0)}(x, t) = g(x, t), \quad G^{(1)}(x, t) = g^{(1)}(x, t) - \mathbf{v}_1^{(1)} \cdot \nabla \theta, \]
\[ G^{(2)}(x, t) = g^{(2)}(x, t) - 2\mathbf{v}_1^{(1)} \cdot \nabla \theta^{(1)} - \mathbf{v}_1^{(2)} \cdot \nabla \theta. \]
Moreover since \( \psi^{(j)} = \partial_t \psi, j = 0, 1, 2 \), solves the following system
\[
\begin{align*}
-\Delta \psi^{(j)} &= w^{(j)} \quad \text{in } Q, \\
\psi^{(j)} &= 0 \quad \text{on } \Sigma.
\end{align*}
\] (30)

We have the following Lemma.

**Lemma 4.1.** There exist constants \( s_0 \) and \( C > 0 \) such that for any \( s \geq s_0 \) the following estimate holds
\[
\sum_{j=0}^{2} \int_{Q} \left( \frac{s^2}{\ell^2(t)} (|\nabla w^{(j)}|^2 + |\nabla \theta^{(j)}|^2) + \frac{s^4}{\ell^4(t)} (|w^{(j)}|^2 + |\theta^{(j)}|^2) \right) e^{2s\alpha} \, dx dt
\leq C \int_{Q} \frac{s}{\ell(t)} \left( |\nu(x)|^2 + |\kappa(x)|^2 + |\nabla \kappa(x)|^2 \right) e^{2s\alpha} \, dx dt
+ C \sum_{j=0}^{2} \int_{\Sigma_0} \left( \frac{s^2}{\ell^2(t)} (|\partial_n w^{(j)}|^2 + |\partial_n \theta^{(j)}|^2) + \frac{s^3}{\ell^3(t)} |\partial_n \psi^{(j)}|^2 \right) e^{2s\alpha} \, d\sigma dt.
\] (31)

**Proof.** We start by applying parabolic Carleman estimate (9) to the first equation of (27) by taking \( y = w^{(j)}, j = 0, 1, 2 \), we get
\[
\sum_{j=0}^{2} \int_{Q} \left( |w^{(j)}|^2 + |\Delta w^{(j)}|^2 + \frac{s^2}{\ell^2(t)} |\nabla w^{(j)}|^2 + \frac{s^4}{\ell^4(t)} |w^{(j)}|^2 \right) e^{2s\alpha} \, dx dt
\leq C \left( \sum_{j=0}^{2} \int_{Q} \frac{s}{\ell(t)} |F^{(j)}|^2 e^{2s\alpha} \, dx dt + \sum_{j=0}^{2} \int_{\Sigma_0} \frac{s^2}{\ell^2(t)} |\partial_n w^{(j)}|^2 e^{2s\alpha} \, d\sigma dt \right).
\] (32)

Since Assumption (A.2) and (28), we get
\[
\sum_{j=0}^{2} \int_{Q} \frac{s}{\ell(t)} |F^{(j)}|^2 e^{2s\alpha} \, dx dt \leq \sum_{j=0}^{2} \int_{Q} \frac{s}{\ell(t)} |f^{(j)}|^2 e^{2s\alpha} \, dx dt
+ C \sum_{j=0}^{1} \int_{Q} \frac{s}{\ell(t)} |\nabla w^{(j)}|^2 e^{2s\alpha} \, dx dt.
\]

In terms of (18) we have
\[ f^{(1)}(x, t) = \partial_t \theta^{(1)}(x, t) + \text{div}(\nu \nabla w_2^{(1)}(x, t)) - \mathbf{v}^{(1)}(x, t) \cdot \nabla w_2(x, t) \]
\[ - \mathbf{v}(x, t) \cdot \nabla w_2^{(1)}(x, t) + \sum_{i=1}^{2} \partial_i(\nabla \nu_1 \wedge \partial_i \mathbf{v}^{(1)}) + \sum_{i=1}^{2} \partial_i(\nabla \nu \wedge \partial_i \mathbf{v}_2^{(1)}), \]
and
\[ f^{(2)}(x, t) = \partial_t \theta^{(2)}(x, t) + \text{div}(\nu \nabla w_2^{(2)}(x, t)) - \mathbf{v}^{(2)} \cdot \nabla w_2(x, t) - \mathbf{v} \cdot \nabla w_2^{(2)}(x, t) \]
\[ - 2\mathbf{v}^{(1)}(x, t) \cdot \nabla w_2^{(1)}(x, t) + \sum_{i=1}^{2} \partial_i(\nabla \nu_1 \wedge \partial_i \mathbf{v}^{(2)}) + \sum_{i=1}^{2} \partial_i(\nabla \nu \wedge \partial_i \mathbf{v}_2^{(2)}). \]
We obtain, by Assumption (A.2) and (4)
\[
\sum_{j=0}^{2} \int_{\Omega} \frac{s}{l(t)} |F^{(j)}|^2 e^{2\sigma t} \, dx \, dt \leq C \int_{\Omega} \frac{s}{l(t)} |v|^2 e^{2\sigma t} \, dx \, dt \\
+ C \sum_{j=0}^{2} \int_{\Omega} \frac{s}{l(t)} |\nabla w^{(j)}|^2 + |\nabla \theta^{(j)}|^2 + |\nabla \psi^{(j)}|^2 \rangle e^{2\sigma t} \, dx \, dt,
\]
where we have used \( \Delta \psi^{(j)} = -\text{curl} w^{(j)}, j = 0, 1, 2 \). Thus, inserting (33) in the right-hand-side of (32), we obtain
\[
\sum_{j=0}^{2} \int_{\Omega} \left( |w^{(j)}|^2 + |\Delta w^{(j)}|^2 + \frac{s^2}{l^2(t)} |\nabla w^{(j)}|^2 + \frac{s^4}{l^4(t)} |w^{(j)}|^2 \right) e^{2\sigma t} \, dx \, dt \\
\leq C \int_{\Omega} \frac{s}{l(t)} |v|^2 e^{2\sigma t} \, dx \, dt + C \sum_{j=0}^{2} \int_{\Omega} \left( |\nabla w^{(j)}|^2 + |\nabla \theta^{(j)}|^2 + |\nabla \psi^{(j)}|^2 \rangle e^{2\sigma t} \, dx \, dt \\
+ C \sum_{j=0}^{2} \int_{\Omega} \frac{s^2}{l^2(t)} |\partial_{\eta} w^{(j)}|^2 e^{2\sigma t} \, dx \, dt.
\]
We apply again Lemma 2.1 to the second equation of (27) by taking \( y = \theta^{(j)}, j = 0, 1, 2 \), it yields for \( s \) sufficiently large
\[
\sum_{j=0}^{2} \int_{\Omega} \left( |\theta^{(j)}|^2 + |\Delta \theta^{(j)}|^2 + \frac{s^2}{l^2(t)} |\nabla \theta^{(j)}|^2 + \frac{s^4}{l^4(t)} |\theta^{(j)}|^2 \right) e^{2\sigma t} \, dx \, dt \\
\leq C \sum_{j=0}^{2} \int_{\Omega} \frac{s}{l(t)} |G^{(j)}|^2 e^{2\sigma t} \, dx \, dt + C \sum_{j=0}^{2} \int_{\Omega} \frac{s^2}{\ell^2(t)} |\partial_{\eta} \theta^{(j)}|^2 e^{2\sigma t} \, dx \, dt.
\]
From Assumption (A.2) and (29), we get
\[
\sum_{j=0}^{2} \int_{\Omega} \frac{s}{l(t)} |G^{(j)}|^2 e^{2\sigma t} \, dx \, dt \leq \sum_{j=0}^{2} \int_{\Omega} \frac{s}{l(t)} |g^{(j)}|^2 e^{2\sigma t} \, dx \, dt \\
+ C \sum_{j=0}^{1} \int_{\Omega} \frac{s}{l(t)} |\nabla \theta^{(j)}|^2 e^{2\sigma t} \, dx \, dt.
\]
Taking into account
\[
g^{(1)} = \text{div}(\kappa \nabla \theta^{(1)}) - v^{(1)} \cdot \nabla \theta - v \cdot \nabla q^{(1)},
\]
and
\[
g^{(2)} = \text{div}(\kappa \nabla \theta^{(2)}) - v^{(2)} \cdot \nabla \theta - v \cdot \nabla \theta^{(2)} - 2v^{(1)} \cdot \nabla \theta^{(1)},
\]
we deduce, from Assumption (A.2)
\[
\sum_{j=0}^{2} \int_{\Omega} \frac{s}{l(t)} |G^{(j)}|^2 e^{2\sigma t} \, dx \, dt \leq C \int_{\Omega} \frac{s}{l(t)} (|\kappa|^2 + |\nabla \kappa|^2) e^{2\sigma t} \, dx \, dt \\
+ C \sum_{j=0}^{2} \int_{\Omega} \frac{s}{l(t)} (|\nabla \theta^{(j)}|^2 + |\nabla \psi^{(j)}|^2) e^{2\sigma t} \, dx \, dt.
\]
So, by inserting (36) in (35), we obtain
\[
\sum_{j=0}^{2} \int_{Q} \left( |\theta(t)|^2 + |\Delta \theta(t)|^2 + \frac{s^2}{\ell^2(t)} |\nabla \theta(t)|^2 + \frac{s^4}{\ell^4(t)} |\theta(t)|^2 \right) e^{2s\alpha} \, dxdt
\leq C \int_{Q} \frac{s}{\ell(t)} \left( |\kappa|^2 + |\nabla \kappa|^2 \right) e^{2s\alpha} \, dxdt + C \sum_{j=0}^{2} \int_{Q} \frac{s^2}{\ell^2(t)} \left( |\nabla \psi(t)|^2 + |\nabla \psi(t)|^2 \right) e^{2s\alpha} \, dxdt
+ C \sum_{j=0}^{2} \int_{\Sigma_{\alpha}} \frac{s^2}{\ell^2(t)} |\partial_{\alpha} \psi(t)|^2 e^{2s\alpha} \, d\sigma dt. \quad (37)
\]

Now, we apply the elliptic Carleman estimate (10) to the solutions $\psi(t)$, $j = 0, 1, 2$, of (30), we obtain
\[
\sum_{j=0}^{2} \int_{Q} \left( \frac{s^3}{\ell^4(t)} |\nabla \psi(t)|^2 + \frac{s^5}{\ell^6(t)} |\psi(t)|^2 \right) e^{2s\alpha} \, dxdt \leq C \sum_{j=0}^{2} \int_{Q} \frac{s^2}{\ell^2(t)} |w(t)|^2 e^{2s\alpha} \, dxdt
+ C \sum_{j=0}^{2} \int_{\Sigma_{\alpha}} \frac{s^3}{\ell^3(t)} |\partial_{\alpha} \psi(t)|^2 e^{2s\alpha} \, d\sigma dt. \quad (38)
\]

Then, collecting (34), (37) and (38), we get
\[
\sum_{j=0}^{2} \int_{Q} \left( |w_j(t)|^2 + |\Delta w_j(t)|^2 + |\theta(t)|^2 + |\Delta \theta(t)|^2 + \frac{s^2}{\ell^2(t)} \left( |\nabla w_j(t)|^2 + |\nabla \theta(t)|^2 \right) \right.
+ \frac{s^3}{\ell^3(t)} |\nabla \psi(t)|^2 + \frac{s^4}{\ell^4(t)} \left( |w(t)|^2 + |\theta(t)|^2 \right) + \frac{s^5}{\ell^5(t)} |\psi(t)|^2 \right) e^{2s\alpha} \, dxdt
\leq C \int_{Q} \frac{s}{\ell(t)} \left( |\nu|^2 + |\kappa|^2 + |\nabla \kappa|^2 \right) e^{2s\alpha} \, dxdt
+ C \sum_{j=0}^{2} \int_{Q} \left( \frac{s}{\ell(t)} \left( |\nabla w_j(t)|^2 + |\nabla \theta(t)|^2 + |\nabla \psi(t)|^2 \right) + \frac{s^2}{\ell^2(t)} |w(t)|^2 \right) e^{2s\alpha} \, dxdt
+ C \sum_{j=0}^{2} \int_{\Sigma_{\alpha}} \left( \frac{s^2}{\ell^2(t)} \left( |\partial_{\alpha} w_j(t)|^2 + |\partial_{\alpha} \theta(t)|^2 \right) + \frac{s^3}{\ell^3(t)} |\partial_{\alpha} \psi(t)|^2 \right) e^{2s\alpha} \, d\sigma dt.
\]

Finally, taking $s$ large enough and noting that $\ell^{-1}(t) \geq 4/T^2$, we deduce that
\[
\sum_{j=1}^{2} \int_{Q} \left( \frac{s^2}{\ell^2(t)} \left( |\nabla w_j(t)|^2 + |\nabla \theta_j(t)|^2 \right) + \frac{s^4}{\ell^4(t)} \left( |w_j(t)|^2 + |\theta_j(t)|^2 \right) \right) e^{2s\alpha} \, dxdt
\leq C \int_{Q} \frac{s}{\ell(t)} \left( |\nu(x)|^2 + |\kappa(x)|^2 + |\nabla \kappa(x)|^2 \right) e^{2s\alpha} \, dxdt
+ C \sum_{j=0}^{2} \int_{\Sigma_{\alpha}} \left( \frac{s^2}{\ell^2(t)} \left( |\partial_{\alpha} w_j(t)|^2 + |\partial_{\alpha} \theta_j(t)|^2 \right) + \frac{s^3}{\ell^3(t)} |\partial_{\alpha} \psi(t)|^2 \right) e^{2s\alpha} \, d\sigma dt.
\]

This completes the proof of the Lemma.

Now, we complete the proof of Theorem 1.1 by the use of the results we have already obtained in the previous. Let us first start by identify the first coefficient $\kappa$.  

Lemma 4.2. There exist constants $s_0$ and $C > 0$ such that for any $s \geq s_0$ the following estimate holds

$$s \int_{\Omega} (|\nabla \kappa|^2 + |\kappa|^2) e^{2s\alpha_0} \, dx$$

$$\leq C \int_{Q} \left( s^{-1} (|\theta|^{(2)}|^2 + |\nabla \theta^{(2)}|^2) + \frac{s}{t^2} (|\theta|^{(1)}|^2 + |\nabla \theta^{(1)}|^2) \right) e^{2s\alpha} \, dxdt$$

$$+ C \left( \|v(\cdot, t_0)\|_{H^1(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^2(\Omega)}^2 \right).$$  \hspace{1cm} (39)

Proof. The identification of $\kappa$ needs to use Lemma 2.3 since $\partial^a \kappa_1 = \partial^a \kappa_2$; $|\alpha| \leq 1$ on the boundary $\Gamma$ in the fact of Assumption (A.1) and with the choice of $\gamma(x) = \nabla \theta_2(x, t_0), v = \kappa$ and $\nu = \partial_i \kappa$.

In virtue of

$$\text{div}(\partial_i \kappa \nabla \theta_2(x, t_0)) = \partial_i (\text{div}(\kappa \nabla \theta_2(x, t_0))) - \text{div}(\kappa \nabla \partial_i \theta_2(x, t_0)), \quad \forall \, i = 1, 2.$$

(40) Thus in terms of (40), immediately we have

$$\int_{\Omega} \left( |\text{div}(\partial_i \kappa \nabla \theta_2(x, t_0))|^2 + |\text{div}(\kappa \nabla \theta_2(x, t_0))|^2 \right) e^{2s\alpha_0} \, dx$$

$$\leq \int_{\Omega} \left( |\partial_i \phi_1(x)|^2 + |\text{div}(\kappa \nabla \partial_i \theta_2(x, t_0))|^2 + |\phi_1(x)|^2 \right) e^{2s\alpha_0} \, dx,$$

where $\phi_1$ is given in Lemma 3.1. Since

$$\text{div}(\kappa \nabla \partial_i \theta_2(x, t_0)) = \kappa \text{div}(\nabla \partial_i \theta_2(x, t_0)) + \nabla \kappa \cdot \nabla \partial_i \theta_2(x, t_0),$$

we deduce that

$$\int_{\Omega} \left( |\text{div}(\partial_i \kappa \nabla \theta_2(x, t_0))|^2 + |\text{div}(\kappa \nabla \theta_2(x, t_0))|^2 \right) e^{2s\alpha_0} \, dx$$

$$\leq C \int_{\Omega} \left( |\nabla \phi_1(x)|^2 + |\phi_1(x)|^2 \right) e^{2s\alpha_0} \, dx + C \int_{\Omega} |\kappa \text{div}(\nabla \partial_i \theta_2(x, t_0))|^2 e^{2s\alpha_0} \, dx$$

$$+ C \int_{\Omega} \left| \nabla \kappa \cdot \nabla \partial_i \theta_2(x, t_0) \right|^2 e^{2s\alpha_0} \, dx.$$  \hspace{1cm} (41)

Applying Lemma 2.3 to the left-hand-side of the above inequality, we obtain

$$s \int_{\Omega} (|\partial_i \kappa|^2 + |\kappa|^2) e^{2s\alpha_0} \, dx$$

$$\leq C \int_{\Omega} \left( |\text{div}(\partial_i \kappa \nabla \theta_2(x, t_0))|^2 + |\text{div}(\kappa \nabla \theta_2(x, t_0))|^2 \right) e^{2s\alpha_0} \, dx.$$  \hspace{1cm} (42)

Then, inserting (42) into the left-hand-side of (41), using Assumption (A.1) and choosing $s > s_0$ large, we get

$$s \int_{\Omega} (|\nabla \kappa|^2 + |\kappa|^2) e^{2s\alpha_0} \, dx \leq \int_{\Omega} (|\nabla \phi_1|^2 + |\phi_1|^2) e^{2s\alpha_0} \, dx.$$  \hspace{1cm} (43)

Finally, we insert (20) into the right-hand-side of (43) we deduce the following estimate

$$s \int_{\Omega} (|\nabla \kappa|^2 + |\kappa|^2) e^{2s\alpha_0} \, dx$$

$$\leq C \int_{Q} \left( s^{-1} (|\theta|^{(2)}|^2 + |\nabla \theta|^{(2)}|^2) + \frac{s}{t^2} (|\theta|^{(1)}|^2 + |\nabla \theta|^{(1)}|^2) \right) e^{2s\alpha} \, dxdt$$
This completes the proof of the Lemma. \hfill \square

The next step is to identify the second coefficient \( \nu \).

**Lemma 4.3.** There exist constants \( s_0 \) and \( C > 0 \) such that for any \( s \geq s_0 \) the following estimate holds

\[
\int_{\Omega} |\nu(x)|^2 e^{2s\alpha_0} \, dx \leq C \int_{Q} \left( \frac{s^{-1} |\omega(2)|^2}{\ell^2(t)} + \frac{s}{\ell^2(t)} |w(1)|^2 \right) e^{2s\alpha} \, dx\, dt
\]

\[
+ C \left( \|w(\cdot, t_0)\|_{H^{3}(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^{3}(\Omega)}^2 \right).
\]

**Proof.** Firstly, we have

\[
\Delta (\text{curl } v_2(x, t_0)) + \nabla \nu(x) \cdot \nabla w_2(x, t_0).
\]

So, it yields

\[
\int_{\Omega} |\nu(x)|^2 e^{2s\alpha_0} \, dx \leq m_0^{-2} \int_{\Omega} |\nu(x)|^2 |\Delta (\text{curl } v_2(x, t_0))|^2 e^{2s\alpha_0}
\]

\[
\leq C \int_{\Omega} |\text{div}(\nu \nabla w_2(x, t_0))|^2 e^{2s\alpha_0} \, dx + \epsilon_0^2 \int_{\Omega} |\nu(x)|^2 e^{2s\alpha_0} \, dx.
\]

Taking \( \epsilon_0 \) sufficiently small, we get

\[
\int_{\Omega} |\nu(x)|^2 e^{2s\alpha_0} \, dx \leq C \int_{\Omega} |\text{div}(\nu \nabla w_2(x, t_0))|^2 e^{2s\alpha_0} \, dx
\]

On the other hand, we have

\[
\text{div}(\nu \nabla w_2(x, t_0)) = w_1(x, t_0) + v_1(x, t_0) \cdot \nabla w(x, t_0) - \text{div}(\nu \nabla w(x, t_0)) - \partial_t \theta(x, t_0)
\]

\[
+ v(x, t_0) \cdot \nabla w_2(x, t_0) - \sum_{i=1}^2 \partial_i (\nabla \nu_1 \wedge \partial_i v(x, t_0)) - \sum_{i=1}^2 \partial_i (\nabla \nu \wedge \partial_i v_2(x, t_0)).
\]

Inserting (46) in (45), we get

\[
\int_{\Omega} |\nu(x)|^2 e^{2s\alpha_0} \, dx \leq C \int_{\Omega} |w_1(x, t_0)|^2 e^{2s\alpha_0} \, dx + C \epsilon_0^2 \int_{\Omega} |\nu(x)|^2 e^{2s\alpha_0} \, dx
\]

\[
+ \left( \|w(\cdot, t_0)\|_{H^{3}(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^{3}(\Omega)}^2 \right).
\]

Applying Lemma 2.4 by taking \( z = \ell(t) w_1 e^{s\alpha} \) and using that \( |\alpha_1| \leq C \ell^{-2}(t) \) to the left-hand-side of the above inequality and taking \( \epsilon_0 \) sufficiently small, we obtain

\[
\int_{\Omega} |\nu(x)|^2 e^{2s\alpha_0} \, dx \leq C \int_{Q} \left( s^{-1} |\omega(2)|^2 + \frac{s}{\ell^2(t)} |w(1)|^2 \right) e^{2s\alpha} \, dx\, dt
\]

\[
+ C \left( \|w(\cdot, t_0)\|_{H^{3}(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^{3}(\Omega)}^2 \right).
\]

This completes the proof. \hfill \square
Thus, for \( \ell \) large enough and in terms of \( \ell^{-2} \), we deduce

\[
\int_{\Omega} (|\nu|^2 + s(|\nabla \kappa|^2 + |\kappa|^2)) e^{2s\alpha} \, dx \leq C \int_{Q} \left( s^{-1} \left( |w(2)|^2 + |\theta(2)|^2 + |\nabla \theta(2)|^2 \right) + \frac{8}{\ell^2(t)} \left( |w(1)|^2 + |\theta(1)|^2 + |\nabla \theta(1)|^2 \right) \right) e^{2\alpha} \, dx dt + C \left( \|v(\cdot, t_0)\|_{H^3(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^2(\Omega)}^2 \right).
\]

(47)

Inserting (31) in the right-hand-side of (47), for \( s > 1 \) large enough and in terms of \( \ell^{-2}(t) \leq C \ell^{-4}(t) \), we deduce

\[
\int_{\Omega} (|\nu|^2 + s(|\nabla \kappa|^2 + |\kappa|^2)) e^{2s\alpha} \, dx \leq C \int_{Q} \left( s^{-1} \left( |w(2)|^2 + |\theta(2)|^2 + |\nabla \theta(2)|^2 \right) + \frac{8}{\ell^2(t)} \left( |w(1)|^2 + |\theta(1)|^2 + |\nabla \theta(1)|^2 \right) \right) e^{2\alpha} \, dx dt + C \left( \|v(\cdot, t_0)\|_{H^3(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^2(\Omega)}^2 \right) + C \left( \|v(\cdot, t_0)\|_{H^3(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^2(\Omega)}^2 \right).
\]

(48)

By Lemma 2.5, we deduce that

\[
\int_{Q} \frac{1}{\ell(t)} \left( |\nu|^2 + |\kappa|^2 + |\nabla \kappa|^2 \right) e^{2\alpha} \, dx dt \leq \frac{C}{\sqrt{s}} \int_{\Omega} \left( |\nu|^2 + |\kappa|^2 + |\nabla \kappa|^2 \right) e^{2\alpha} \, dx. \quad (49)
\]

Thus, for \( s \) is large and by inserting (49) in the right-hand-side of (48), we deduce

\[
\int_{\Omega} \left( |\nu|^2 + s(|\nabla \kappa|^2 + |\kappa|^2) \right) e^{2s\alpha} \, dx \\
\leq C_1 \left( \|\partial_n w\|_{H^2(0,T;L^2(\Gamma_0))}^2 + \|\partial_n \psi\|_{H^2(0,T;L^2(\Gamma_0))}^2 + \|\partial_n \theta\|_{H^2(0,T;L^2(\Gamma_0))}^2 \right) + C_2 \left( \|v(\cdot, t_0)\|_{H^3(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^2(\Omega)}^2 \right).
\]

Finally, since

\[
\partial_n \psi = \nabla \psi \cdot n = \partial_1 \psi n_1 + \partial_2 \psi n_2 \\
= -v_1 n_2 + v_2 n_1 = \nu \cdot \tau,
\]

where \( \tau = (-n_2, n_1) \) is the vector tangential to the \( \Gamma \), we deduce (5).

The proof of Theorem 1.1 is completed.

**Appendix A. Well-posedness of the direct problem of Boussinesq system.**

The purpose of this section is to give some regularity properties of the direct problem of the 2D Boussinesq system and we introduce some material and method which can be used to find the assumption (A.2). By taking time derivative to the following system (50)-(54) and proceeding as the proof of Proposition 2 in below and [29], we can extend the regularity of the solution, provided the regularity of the initial
data, in order to guarantee the assumption (A.1). Our analysis here is close to [30, 15, 24, 35, 27, 29, 26] and we will keep many arguments proved in these papers.

Let us consider the Boussinesq system in bounded domain $\Omega$ of $\mathbb{R}^2$ with connected and $C^\infty$ boundary $\Gamma = \partial \Omega$. Given $T > 0$, we consider the following problem of the nonlinear Boussinesq system

\[
\begin{align*}
 v_t + v \cdot \nabla v + \nabla \pi - \text{div}(\nu \nabla v) &= \theta \varepsilon_2 & \text{in } Q, \\
 \theta_t + v \cdot \nabla \theta - \text{div}(\kappa \nabla \theta) &= 0 & \text{in } Q, \\
 \text{div} v &= 0 & \text{in } Q, \\
 v \cdot n &= 0, \quad \text{curl } v = 0, \quad \theta = 0 & \text{on } \Sigma, \\
 v(x, 0) &= v_0(x), \quad \theta(x, 0) = \theta_0(x) & \text{in } \Omega.
\end{align*}
\]

(50)

(51)

(52)

(53)

(54)

As usual, for global existence of smooth solutions, we require the following compatibility conditions:

\[
v_0 \cdot n = 0, \quad \text{curl } v_0 = 0, \quad \theta_0 = 0, \quad \text{div}(\kappa \nabla \theta_0) = 0, \quad \text{on } \Gamma.
\]

(55)

**Proposition 2.** Let $v_0 \in H^3(\Omega)$ with $\text{div} v_0 = 0$, $\theta_0 \in H^3(\Omega)$ and satisfies the compatibility conditions (55), and let $T > 0$. Then there exist a unique solution $(v, \theta)$ to (50)-(54) and a positive constant $C > 0$ such that

\[
\|v\|_{C([0, T); H^3(\Omega))} \leq C, \quad \|v_t\|_{L^2(0, T; L^2(\Omega))} + \|v\|_{L^2(0, T; H^3(\Omega))} \leq C,
\]

(56)

and

\[
\|\theta\|_{C([0, T); H^3(\Omega))} \leq C, \quad \|\theta_t\|_{L^2(0, T; L^2(\Omega))} + \|\theta\|_{L^2(0, T; H^3(\Omega))} \leq C.
\]

(57)

**Proof.** Since it is well known that the solution $(v, \theta)$ is smooth see [26]-[24], we need to show estimates (56) and (57). First, by maximum principle, it follows from (51) that

\[
\|\theta\|_{L^\infty(0, T; L^3(\Omega))} \leq \|\theta_0\|_{L^\infty(\Omega)} \leq C \|\theta_0\|_{H^2(\Omega)}.
\]

Here we have used the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$.

Multiplying (51) by $\theta$ and integrating the result equation by parts over $\Omega$ and using (53), we have

\[
\frac{1}{2} \frac{d}{dt} \|\theta(\cdot, t)\|_{L^2(\Omega)}^2 + \|\sqrt{\kappa} \nabla \theta(\cdot, t)\|_{L^2(\Omega)}^2 = 0,
\]

which imply

\[
\|\theta\|_{L^\infty(0, T; L^2(\Omega))} + \|\theta\|_{L^2(0, T; H^1(\Omega))} \leq C.
\]

Multiplying (50) by $v$, integrating the result equation by parts over $\Omega$, using (52), (53) and Cauchy-Schwartz inequality, we derive

\[
\frac{1}{2} \frac{d}{dt} \|v(\cdot, t)\|_{L^2(\Omega)}^2 + \|\sqrt{\nu} \nabla v(\cdot, t)\|_{L^2(\Omega)}^2 \leq \|\theta(\cdot, t)\|_{L^2(\Omega)} \|v(\cdot, t)\|_{L^2(\Omega)},
\]

which gives

\[
\|v\|_{L^\infty(0, T; L^2(\Omega))} \leq C, \quad \|v\|_{L^2(0, T; H^1(\Omega))} \leq C.
\]

Applying curl to (50), we obtain for $w = \text{curl } v$

\[
w_t + v \cdot \nabla w - \text{div}(\nu \nabla w) = \partial_t \theta + \sum_{i=1}^2 \partial_i (\nabla \nu \cdot \partial_i v) \quad \text{in } Q.
\]

(58)
Multiplying (58) by $w$ and integrating the result equation by parts over $\Omega$, we find for any $\varepsilon > 0$

\[
\frac{1}{2} \frac{d}{dt} \left\| w(\cdot, t) \right\|_{L^2(\Omega)}^2 + \left\| \sqrt{\kappa} \nabla w(\cdot, t) \right\|_{L^2(\Omega)}^2 \\
= \int_{\Omega} \partial_t \theta(x, t) w(x, t) dx - \sum_{i=1}^{2} \int_{\Omega} (\nabla \nu \cdot \partial_i \nabla \theta) \partial_i w(x, t) dx \\
\leq \left\| \theta(\cdot, t) \right\|_{H^1(\Omega)} \left\| w(\cdot, t) \right\|_{L^2(\Omega)} + C_{\varepsilon} \left\| \nabla w(\cdot, t) \right\|_{H^1(\Omega)} + \varepsilon \left\| \nabla w(\cdot, t) \right\|_{L^2(\Omega)}^2.
\]

Taking $\varepsilon$ sufficiently small and using the fact that

\[
\left\| \nabla w(\cdot, t) \right\|_{H^1(\Omega)} \leq C \left\| w(\cdot, t) \right\|_{L^2(\Omega)},
\]

we obtain

\[
\frac{1}{2} \frac{d}{dt} \left\| w(\cdot, t) \right\|_{L^2(\Omega)}^2 + \frac{\mu_0}{2} \left\| \nabla w(\cdot, t) \right\|_{L^2(\Omega)}^2 \leq C + C \left\| w(\cdot, t) \right\|_{L^2(\Omega)}^2. \tag{59}
\]

Applying the Grönwall inequality to (59), we get

\[
\left\| \nabla \right\|_{L^\infty(0, T; H^1(\Omega))} \leq \left\| w \right\|_{L^\infty(0, T; L^2(\Omega))} \leq C
\]

Using again (59), we obtain

\[
\left\| \nabla \right\|_{L^2(0, T; H^2(\Omega))} \leq C.
\]

Multiplying (51) by $-\text{div}(\kappa \nabla \theta)$, and integrating the result equation by parts over $\Omega$, we find

\[
\frac{1}{2} \frac{d}{dt} \left\| \sqrt{\kappa} \nabla \theta(\cdot, t) \right\|_{L^2(\Omega)}^2 + \left\| \text{div}(\kappa \nabla \theta(\cdot, t)) \right\|_{L^2(\Omega)}^2 = \int_{\Omega} \nabla \theta(\cdot, t) \cdot \nabla \text{div}(\kappa \nabla \theta(\cdot, t)) dx \\
\leq C \left\| \nabla \theta(\cdot, t) \right\|_{L^4(\Omega)} \left\| \nabla \text{div}(\kappa \nabla \theta(\cdot, t)) \right\|_{L^2(\Omega)} \\
\leq C \left\| \nabla \theta(\cdot, t) \right\|_{L^4(\Omega)} \left\| \theta(\cdot, t) \right\|_{H^1(\Omega)}
\]

Applying the Gagliardo-Nirenberg inequality, we get

\[
\left\| \nabla \theta(\cdot, t) \right\|_{L^4(\Omega)} \leq C \left\| \nabla \theta(\cdot, t) \right\|_{L^2(\Omega)} \left\| \theta(\cdot, t) \right\|_{H^1(\Omega)}.
\]

Using the Hölder inequality, we arrive at, for any $\varepsilon > 0$,

\[
\frac{d}{dt} \left\| \sqrt{\kappa} \nabla \theta(\cdot, t) \right\|_{L^2(\Omega)}^2 + \frac{\kappa_0}{2} \left\| \Delta \theta(\cdot, t) \right\|_{L^2(\Omega)}^2 \leq \left\| \nabla \theta(\cdot, t) \right\|_{L^2(\Omega)}^2 \left\| \Delta \theta(\cdot, t) \right\|_{L^2(\Omega)}^2 \\
+ C \left\| \nabla \theta(\cdot, t) \right\|_{L^2(\Omega)} \left\| \Delta \theta(\cdot, t) \right\|_{L^2(\Omega)} \\
\leq C_{\varepsilon} \left\| \nabla \theta(\cdot, t) \right\|_{L^2(\Omega)}^2 + \varepsilon \left\| \Delta \theta(\cdot, t) \right\|_{L^2(\Omega)}^2.
\]

Which leads to

\[
\left\| \nabla \theta \right\|_{L^\infty(0, T; L^2(\Omega))} \leq C, \quad \left\| \theta \right\|_{L^2(0, T; H^2(\Omega))} \leq C.
\]

Multiplying (51) by $\theta_t$ and integrating the result equation by parts over $\Omega$, we obtain, for any $\varepsilon > 0$,

\[
\left\| \theta_t(\cdot, \cdot) \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \left\| \sqrt{\kappa} \nabla \theta(\cdot, \cdot) \right\|_{L^2(\Omega)}^2 = \int_{\Omega} \theta_t \cdot \nabla \theta dx \leq \varepsilon \left\| \theta_t \right\|_{L^2(\Omega)}^2 + C \left\| \nabla \theta(\cdot, \cdot) \right\|_{L^2(\Omega)}^2 \\
\leq \frac{1}{2} \left\| \theta_t(\cdot, \cdot) \right\|_{L^2(\Omega)}^2 + C \left\| \nabla \theta(\cdot, \cdot) \right\|_{L^2(\Omega)}^2 + \varepsilon \left\| \Delta \theta(\cdot, \cdot) \right\|_{L^2(\Omega)}^2 \\
\leq \frac{1}{2} \left\| \theta_t(\cdot, \cdot) \right\|_{L^2(\Omega)}^2 + \varepsilon \left\| \Delta \theta(\cdot, \cdot) \right\|_{L^2(\Omega)}^2 + C \left\| \nabla \theta(\cdot, \cdot) \right\|_{L^2(\Omega)}^2 + C \left\| \nabla \theta(\cdot, \cdot) \right\|_{L^2(\Omega)}^2.
\]
Rewriting Equation (51) as follows
\[ \text{div}(\kappa \nabla \theta(t,x)) = \theta_t(t,x) + \mathbf{v}(t,x) \cdot \nabla \theta(t,x) \]
and using the elliptic estimate, we can get, for any \( \varepsilon > 0 \)
\[
\|\theta(t,\cdot)\|_{H^2(\Omega)}^2 \leq C \left( \|\theta_t(t,\cdot)\|_{L^2(\Omega)}^2 + \|\nabla \theta(t,\cdot)\|_{L^2(\Omega)}^2 + \|\theta(t,\cdot)\|_{H^1(\Omega)}^2 \right)
\]
\[
\leq C \left( \|\theta_t(t,\cdot)\|_{L^2(\Omega)}^2 + \|\nabla \theta(t,\cdot)\|_{L^2(\Omega)}^2 \|\nabla \theta(t,\cdot)\|_{L^1(\Omega)}^2 + \|\theta(t,\cdot)\|_{H^1(\Omega)}^2 \right)
\]
\[
\leq C \left( \|\theta_t(t,\cdot)\|_{L^2(\Omega)}^2 + \|\nabla \theta(t,\cdot)\|_{L^2(\Omega)}^2 \theta(t,\cdot)\|_{H^2(\Omega)}^2 + \varepsilon'^2 \|\theta(t,\cdot)\|_{H^2(\Omega)}^2 + \|\theta(t,\cdot)\|_{H^1(\Omega)}^2 \right).
\]
Hence, choosing \( \varepsilon' \) small, we obtain, for some constant \( C > 0 \)
\[
\|\theta(t,\cdot)\|_{H^2(\Omega)}^2 \leq C \left( \|\theta_t(t,\cdot)\|_{L^2(\Omega)}^2 + \|\theta(t,\cdot)\|_{H^1(\Omega)}^2 \right).
\]  
(61)
Inserting (61) into (60), and taking \( \varepsilon' \) small, we get
\[
\|\theta_t(t,\cdot)\|^2 + \frac{1}{2} \frac{d}{dt} \|\sqrt{\kappa} \nabla \theta(t,\cdot)\|^2 \leq C \|\nabla \theta(t,\cdot)\|^2 \leq C \|\sqrt{\kappa} \nabla \theta(t,\cdot)\|^2.
\]
Which implies
\[
\|\theta_t\|_{L^2(0,T;L^2(\Omega))} + \|\theta\|_{L^\infty(0,T;H^1(\Omega))} \leq C.
\]  
(62)
Multiplying (50) by \( \mathbf{v}_t \) and integrating the result equation by parts over \( \Omega \), we have
\[
\|\mathbf{v}_t(t,\cdot)\|^2 + \frac{1}{2} \frac{d}{dt} \|\sqrt{\kappa} \nabla \mathbf{v}(t,\cdot)\|^2 = -\int_{\Omega} \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{v}_t(t,x)dx + \int_{\Omega} \theta(t,x)e_2 \cdot \mathbf{v}_t(t,x)dx.
\]
Using the Gagliardo-Nirenberg inequality, the Hölder inequality, and proceed as [24]-[26], we obtain the following estimate
\[
\|\mathbf{v}_t(t,\cdot)\|^2 + \frac{1}{2} \frac{d}{dt} \|\sqrt{\kappa} \nabla \mathbf{v}(t,\cdot)\|^2 \leq C_1 \|\nabla \mathbf{v}(t,\cdot)\|^2 + C \leq C \|\sqrt{\kappa} \nabla \mathbf{v}(t,\cdot)\|^2 + C.
\]
Which gives
\[
\|\mathbf{v}_t\|_{L^2(0,T;L^2(\Omega))} + \|\mathbf{v}\|_{L^\infty(0,T;H^1(\Omega))} \leq C.
\]
Applying \( \partial_t \) to (51), we see that
\[
\theta_{tt}(t,x) + \mathbf{v} \cdot \nabla \theta_t(t,x) - \Delta \theta_t(t,x) = -\mathbf{v}_t \cdot \nabla \theta(t,x).
\]
Multiplying the above equation by \( \theta_t \), using (52)-(53), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\theta_t(t,\cdot)\|^2 + \|\text{div}(\kappa \nabla \theta_t(t,\cdot))\|^2 \leq C \|\theta(t,\cdot)\|_{L^\infty(\Omega)} \|\mathbf{v}_t(t,\cdot)\|_{L^2} \|\nabla \theta_t(t,\cdot)\|_{L^2}
\]
\[
\quad \leq \varepsilon \|\nabla \theta_t(t,\cdot)\|_{L^2(\Omega)}^2 + C \|\mathbf{v}_t(t,\cdot)\|_{L^2(\Omega)}^2.
\]
Which implies
\[
\|\theta_t\|_{L^2(0,T;H^2(\Omega))} + \|\theta_t\|_{L^\infty(0,T;L^2(\Omega))} \leq C.
\]
Equations (51) and (54) can be rewritten as
\[
\Delta \theta(t,x) = f(t,x) := \partial_t \theta + \mathbf{v} \cdot \nabla \theta \quad \text{in } Q,
\]
\[
\theta(t,x) = 0 \quad \text{on } \Sigma,
\]
and then by the elliptic regularity, using (62) and (61), we get
\[
\|\theta(t,\cdot)\|_{H^2(\Omega)}^2 \leq C \|f(t,\cdot)\|_{L^2(\Omega)}^2 \leq C \left( 1 + \|\nabla \theta(t,\cdot)\|_{L^2(\Omega)} \|\theta(t,\cdot)\|_{H^2(\Omega)} \right).
\]
Which implies
\[
\|\theta\|_{L^\infty(0,T;H^2(\Omega))} \leq C.
\]
Similarly, we have
$$\|\theta\|_{L^2(0,T;H^3(\Omega))} \leq \|f\|_{L^2(0,T;H^1(\Omega))} \leq C,$$
which yields
$$\|\theta\|_{L^2(0,T;W^{1,\infty}(\Omega))} \leq C \|\theta\|_{L^2(0,T;H^3(\Omega))} \leq C.$$  \hfill (64)

Using (53), (see [24]), we infer that
$$v \cdot \nabla \theta(t, x) = 0, \quad (t, x) \in \Sigma.$$  \hfill (65)

It follows from (65) and (53), that
$$\text{div}(\kappa \nabla \theta(t, x)) = 0, \quad (t, x) \in \Sigma.$$  \hfill (66)

Applying \text{div}(\kappa \nabla \cdot) to (51) and multiplying by \text{div}(\kappa \nabla \theta(t, x)) using (66), (63), (64) and (66), we derive
$$\|\theta\|_{L^\infty(0,T;H^3(\Omega))} \leq C.$$  \hfill (67)

Similarly, we proceed as in [24], we find
$$\|v\|_{L^\infty(0,T;H^3(\Omega))} + \|v\|_{L^2(0,T;H^4(\Omega))} \leq C.$$  \hfill (68)

This completes the proof.

\hfill \Box

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