LARGE AMPLITUDE SOLUTIONS IN $L^p_t L^\infty_x$ TO THE BOLTZMANN EQUATION FOR SOFT POTENTIALS

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Abstract. In this paper we consider the Cauchy problem on the angular cutoff Boltzmann equation near global Maxwillians for soft potentials either in the whole space or in the torus. We establish the existence of global unique mild solutions in the space $L^p_t L^\infty_x L^\infty_\omega$ with polynomial velocity weights for suitably large $p \leq \infty$, whenever for the initial perturbation the weighted $L^p_t L^\infty_x L^\infty_\omega$ norm can be arbitrarily large but the $L^1_t L^\infty_x$ norm and the defect mass, energy and entropy are sufficiently small. The proof is based on the local in time existence as well as the uniform a priori estimates via an interplay in $L^p_t L^\infty_x L^\infty_\omega$ and $L^\infty_t L^\infty_x L^1_\omega$.

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1. INTRODUCTION

We are concerned with the Cauchy problem on the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \quad F(0, x, v) = F_0(x, v),$$

(1.1)
where $F(t, x, v) \geq 0$ is the density distribution function of gas particles with position $x \in \Omega = \mathbb{R}^3$ or $\mathbb{T}^3$ and velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The bilinear collision operator $Q$ acting only on velocity variable is given by

$$Q(G, F)(v) = \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \theta) [G(u') F(v') - G(u) F(v)] d\omega du.$$

In this paper, we consider soft potentials under the Grad’s angular cutoff assumption. Thus, the collision kernel $B(v - u, \theta)$ takes the form of

$$B(v - u, \theta) = |v - u|^{\gamma} b(\theta),$$

(1.2)
where $-3 < \gamma < 0$ and $0 \leq b(\theta) \leq C |\cos \theta|$ for some positive constant $C$ with $\cos \theta = \frac{(v - u) \cdot \omega}{|v - u|}$. The post-collision velocities $v'$ and $u'$ satisfy

$$v' = v - [(v - u) \cdot \omega] \omega, \quad u' = u + [(v - u) \cdot \omega] \omega,$$

$$u' + v' = u + v, \quad |v'|^2 + |u'|^2 = |v|^2 + |u|^2.$$  

(1.3)

Let the global Maxwillian $\mu$ be denoted by

$$\mu(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp \left( -\frac{|v|^2}{2} \right).$$
Moreover, we assume that the following conservation laws and the entropy inequality hold for any solution $F(t, x, v)$ to (1.1) respectively:

\[
M_0 := \int_{\Omega} \int_{\mathbb{R}^3} \{F(t, x, v) - \mu(v)\} dvdx = \int_{\Omega} \int_{\mathbb{R}^3} \{F_0(x, v) - \mu(v)\} dvdx, \quad (1.4)
\]

\[
J_0 := \int_{\Omega} \int_{\mathbb{R}^3} v\{F(t, x, v) - \mu(v)\} dvdx = \int_{\Omega} \int_{\mathbb{R}^3} v\{F_0(x, v) - \mu(v)\} dvdx,
\]

\[
E_0 := \int_{\Omega} \int_{\mathbb{R}^3} |v|^2\{F(t, x, v) - \mu(v)\} dvdx = \int_{\Omega} \int_{\mathbb{R}^3} |v|^2\{F_0(x, v) - \mu(v)\} dvdx, \quad (1.5)
\]

and

\[
\int_{\Omega} \int_{\mathbb{R}^3} \{F(t, x, v) \log F(t, x, v) - \mu(v) \log \mu(v)\} dvdx \\
\leq \int_{\Omega} \int_{\mathbb{R}^3} \{F_0(x, v) \log F_0(x, v) - \mu(v) \log \mu(v)\} dvdx. \quad (1.6)
\]

For given initial data $F_0(x, v)$ we call $M_0$, $J_0$, $E_0$ and $\int (F_0 \ln F_0 - \mu \ln \mu)$ by the defect mass, momentum, energy and entropy, respectively. Using the similar notations as [8], we define

\[
\mathcal{E}(F(t)) := \int_{\Omega} \int_{\mathbb{R}^3} \{F(t, x, v) \log F(t, x, v) - \mu(v) \log \mu(v)\} dvdx + \left(\frac{3}{2} \log(2\pi) - 1\right)M_0 + \frac{1}{2}E_0,
\]

with the initial datum $\mathcal{E}(F_0) := \mathcal{E}(F(0))$. Note that it can be verified that $\mathcal{E}(F(t)) \geq 0$ for any $t \geq 0$, in particular, $\mathcal{E}(F_0) \geq 0$.

The Boltzmann equation, which is a fundamental mathematical model in collisional kinetic theory, describes the behavior of rarefied gas in non-equilibrium state. There are extensive literatures for the initial and/or boundary value problems of the Boltzmann equation, e.g. [5, 26] and the references therein. The well-known global existence result of renormalized solutions for general $L^1_{x,v}$ initial data with finite mass, energy and entropy was proved by DiPerna-Lions [6] where the uniqueness of such solutions remains unknown. In the perturbation framework near global Maxwellians, Grad [10] studied the linearized operator and Ukai [23] developed the spatially inhomogeneous well-posedness theory by the spectral analysis and the bootstrap argument, see also [17, 19, 25]. For the enormous works of the linearized operator, interested readers may also refer to Ellis-Pinsky [7], Baranger-Mouhot [1] and the references therein. The energy method in Sobolev spaces was developed through the macro-micro decomposition by Liu-Yang-Yu [16] and Guo [12].

In contrast with the hard potentials, the collision frequency $\nu(v) \sim (1 + |v|)^{\gamma}$ in case of soft potentials $-3 < \gamma < 0$ has no strictly positive lower bound and we are lack of the spectral gap of the linearized operator. For $-1 < \gamma < 0$, based on the decay in time for the linearized equation and the bootstrap argument on the nonlinear equation, Caffiisch [3, 4] studied the global existence and large-time behavior of the solutions in $\mathbb{T}^3$. In $\mathbb{R}^3$, the global solution and large-time behavior were solved through the semi-group theory, which was established by Ukai-Asano [24]. When $-3 < \gamma < 0$, Guo [11] constructed the global classical solutions and Guo-Strain [21, 22] proved the large-time behavior.

Among the works in perturbation framework mentioned above, the initial data should have small oscillations near the global Maxwellian. In the large amplitude situation, Duan-Huang-Wang-Yang [8] developed an $L^\infty_x L^1_v \cap L^\infty_{x,v}$ approach to obtain the global existence and uniqueness of mild solutions in $\mathbb{R}^3$ or $\mathbb{T}^3$ for $-3 < \gamma \leq 1$ in the condition that both $\mathcal{E}(F_0)$ and the $L^1_v L^\infty_x$ norm of $(F_0 - \mu)/\sqrt{\mu}$ are small enough, while the $L^\infty_{x,v}$ norm of $\langle v \rangle^\beta (F_0 - \mu)/\sqrt{\mu}$ is only required to be bounded for suitably large $\beta$. The smallness in $L^\infty_{x,v}$ is replaced by the smallness in $L^1_v L^\infty_x$ so that the initial data is allowed to have large amplitude around the global Maxwellian with respect to space variable. Motivated by [8] and [14], Nishimura [13] obtained the global existence for hard potentials in $L^p_x L^\infty_{x,v}$ for large $p$ in order to reduce $L^\infty_x$ to $L^p_x$ with finite $p$. However, the well-posedness theory in such spaces for soft potentials seems still left open.
Now we prepare to state the main results of this paper. Since we need to consider the solutions around the global Maxwillian, we define the perturbation function

$$f(t, x, v) = \frac{F(t, x, v) - \mu(v)}{\sqrt{\mu(v)}}.$$

Substituting it into \([1.1]\), we obtain a Cauchy problem for \(f(t, x, v)\) of the form

$$\partial_t f + v \cdot \nabla_x f + \nu(v) f - K f = \Gamma(f, f), \quad f(0, x, v) = f_0(x, v),$$

where the collision frequency \(\nu(v)\), the operator \(K\) and the nonlinear term \(\Gamma\) are respectively given by

$$\nu(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \mu(u) d\omega du \sim (1 + |v|)^\gamma,$$

$$(K f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \sqrt{\mu(u)} \left( \sqrt{\mu(u')} f(v') + \sqrt{\mu(v')} f(u') - \sqrt{\mu(v)} f(u) \right) d\omega du.$$

$$\Gamma(f, f) = \Gamma_+(f, f) - \Gamma_-(f, f), \quad \Gamma_\pm(f, f) = \frac{1}{\sqrt{\mu}} Q_\pm(\sqrt{\mu} f, \sqrt{\mu} f),$$

with

$$Q_+(f, g) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) f(v') g(u') d\omega du, \quad Q_-(f, g) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) f(v) g(u) d\omega du.$$

The velocity weight function is denoted by \(w_\beta(v) = (1 + |v|^2)^\frac{\beta}{2} \sim (1 + |v|)^\beta\). Since our results and proofs do not rely on the derivatives of the weighted function, both forms of \(w_\beta(v)\) are equivalent. Then from \([1.4]\), by integrating along the backward trajectory, we obtain the mild form

$$f(t, x, v) = e^{-\nu(v)t} f_0(x - v t, v) + \int_0^t e^{-\nu(v)(t-s)} (K f)(s, x - v(t-s), v) ds$$

$$+ \int_0^t e^{-\nu(v)(t-s)} \Gamma(f, f)(s, x - v(t-s), v) ds.$$

Given two functions \(f = f(t, x, v)\) and \(f_0 = f_0(x, v)\), for any \(0 \leq T_0 \leq T\), the \(L^p_x L^\infty_{t_0, t} L^\infty_x\) norm, \(L^p_x L^\infty_x\) norm and \(L^1_x L^\infty_x\) norm are respectively defined by

$$\|f\|_{L^p_x L^\infty_{t_0, t} L^\infty_x} := \left\{ \int_{\mathbb{R}^3} \left[ \sup_{t \in [T_0, T]} \sup_{x \in \Omega} |f(t, x, v)|^p \right]^\frac{1}{p} dv \right\}^{\frac{1}{p}},$$

$$\|f_0\|_{L^p_x L^\infty_x} := \left\{ \int_{\mathbb{R}^3} \sup_{x \in \Omega} |f_0(x, v)|^p dv \right\}^{\frac{1}{p}},$$

$$\|f_0\|_{L^1_x L^\infty_x} := \int_{\Omega} \left( \sup_{v \in \mathbb{R}^3} |f_0(x, v)| \right) dx.$$

If \(T_0 = 0\), we write \(\|f\|_{L^p_x L^\infty_x} \) instead of \(\|f\|_{L^p_x L^\infty_{t_0} L^\infty_x}\). In this paper, we consider solutions in \(L^p_x L^\infty_x\). In the following sections, we will prove the local existence for bounded \(L^p_x L^\infty_x\) initial data and establish the \(L^p_x L^\infty_x \cap L^\infty_x L^1_v\) estimates in order to extend the obtained local solution to a global solution for small \(L^1_x L^\infty_x\) initial data with small \(E(F_0)\).

Throughout the paper, if a constant \(C\) depends on some parameters \(\beta_1, \beta_2, \ldots\), then we denote it by \(C(\beta_1, \beta_2, \ldots)\) to emphasize the explicit dependence. The main two results of the paper are stated below.

**Theorem 1.1 (Local existence).** Assume \([1.2]\) with \(-3 < \gamma < 0\). Let \(p > \max\{6/(5 + \gamma), 4/(3 - \gamma), 3/(3 + \gamma), (2 - \gamma)/2\}\) and \(\beta > \max\{3/p', 36, 6 - 2\gamma\}\), where \(\frac{1}{p} + \frac{1}{p'} = 1\). Assume \(F_0(x, v) := \mu + \sqrt{\mu} f_0 \geq 0\) with \(\|w_\beta f_0\|_{L^p_x L^\infty_x} < \infty\). Then there exists a constant \(C_1 = C_1(\beta, \gamma) > 0\) and a positive time

$$T_1 := \frac{1}{6 C_1 (1 + \|w_\beta f_0\|_{L^p_x L^\infty_x})} > 0,$$

(1.9)
such that the Cauchy problem on the Boltzmann equation \( \text{(1.1)} \) admits a unique mild solution \( F(t, x, v) = \mu + \sqrt{\nu} f(t, x, v) \geq 0, (t, x, v) \in [0, T_1] \times \Omega \times \mathbb{R}^3, \) in the sense of \( \text{(1.5)} \), satisfying
\[
\|w_\beta f\|_{L_{\beta}^2 L_t^\infty L_x^\infty} \leq 2 \|w_\beta f_0\|_{L_{\beta}^2 L_x^\infty}. \tag{1.10}
\]

**Theorem 1.2** (Global existence). Let all the assumptions in \( \text{Theorem 1.1} \) be satisfied. There is a constant \( C_2 = C_2(\gamma, \beta) > 0 \) such that for any constant \( M \geq 1 \) that can be arbitrarily large, there exists a constant \( \varepsilon = \varepsilon(\gamma, \beta, M) > 0 \) such that if it holds that \( \|w_\beta f_0\|_{L_{\beta}^2 L_x^\infty} \leq M \) and
\[
\max\{E(F_0), \|f_0\|_{L_{\beta}^1 L_x^\infty}\} \leq \varepsilon,
\]
then the Cauchy problem on the Boltzmann equation \( \text{(1.1)} \) admits a unique global mild solution \( F(t, x, v) = \mu + \sqrt{\nu} f(t, x, v) \geq 0, (t, x, v) \in [0, \infty) \times \Omega \times \mathbb{R}^3, \) in the sense of \( \text{(1.5)} \), satisfying
\[
\|w_\beta f\|_{L_{\beta}^2 L_t^\infty L_x^\infty} \leq C_2 M^2,
\tag{1.11}
\]
for any \( T \geq 0 \).

The proof of Theorem 1.1 is based on the fixed point theorem. We first construct an approximation sequence using the perturbed equation. Then we prove that it is a Cauchy sequence in \( L_{\beta}^2 L_t^\infty L_x^\infty \) provided \( p \) is large enough and \( T \) is small enough. The difficulty is due to the nonlinear term \( \Gamma(f^n, f^n) \). We need to prove the norm of \( \int_0^1 \|w_\beta \Gamma(f^n, f^n)\|_1(s, x_1, v)ds \) is bounded by \( CT \|w_\beta f^n\|_{L_{\beta}^2 L_t^\infty L_x^\infty}^2 \). When \( p = \infty \) as in \( \text{[5]} \), we can directly obtain \( \|w_\beta f\|_{L_{\beta}^2 L_t^\infty L_x^\infty}^2 \) from \( \Gamma(f^n, f^n) \) and the rest of the integral can be bounded by \( CT \). However, when we consider \( L_p \) instead of \( L_{\infty} \) for some \( p \in \mathbb{R} \), it is not straightforward to obtain \( \|w_\beta f\|_{L_{\beta}^2 L_t^\infty L_x^\infty}^2 \) from the point-wise estimate of the nonlinear term. Moreover, the gain term contains \( u' \) and \( v' \) as variables and the whole integral is taken with respect to \( v \). In this paper, we use the transformation \( z_0 = (u - v) \cdot \omega, z_\perp = z - z_0 \) as well as multiple integral inequalities to get the \( L_p \) \( L_t^\infty L_x^\infty \) norm of \( w_\beta f \) from the nonlinear term. At last we can obtain the estimates
\[
\|w_\beta f^{n+1}\|_{L_{\beta}^p L_t^\infty L_x^\infty} \leq 2 \|w_\beta f_0\|_{L_{\beta}^2 L_x^\infty}
\]
and
\[
\|w_\beta f^{n+2} - w_\beta f^{n+1}\|_{L_{\beta}^p L_t^\infty L_x^\infty} \leq \frac{1}{2} \|w_\beta f^{n+1} - w_\beta f^n\|_{L_{\beta}^2 L_t^\infty L_x^\infty}.
\]
Then the approximation sequence is a Cauchy sequence. After taking the limit, we yield a unique local solution which is bounded by the initial data.

Next we sketch the proof of Theorem 1.2. To establish the global \( L_{\infty} \) bound, in the previous works such as \( \text{[13-15, 20, 25]} \), the following inequality is applied to estimate \( \Gamma(f, f) \)
\[
\|w_\beta \Gamma(f, f)\|_1(t, x, v) \leq C\nu(v) \|w_\beta f(t)\|_{L_{\infty}}^2. \tag{1.12}
\]
We can infer from the above inequality that the \( L_{\infty} \) smallness of the initial data is necessary. In order to deal with large initial data, as in \( \text{[5]} \), we can improve the inequality \( \text{(1.12)} \) to be
\[
\|w_\beta \Gamma(f, f)\|_1(t, x, v) \leq C\nu(v) \|w_\beta f(t)\|_{L_{\infty}} \left( \int_{\mathbb{R}^3} |f(t, x, v)|dv \right)^{2-\tau}, \tag{1.13}
\]
for some \( 0 \leq \tau \leq 1 \). Then due to the hyperbolicity of the Boltzmann equation, one can prove that if \( \mathcal{E}(F_0) \) and \( \|f_0\|_{L_{\beta}^1 L_x^\infty} \) are small enough, \( \int_{\mathbb{R}^3} |f(t, x, v)|dv \) will be small uniformly in \( x \) for \( t \geq T_1 \), where \( T_1 \) is a positive number. Then we can obtain the estimate in \( L_{\infty} \) without assuming the initial data to be small. For hard potentials in \( L_p \) spaces, a similar idea as \( \text{[13]} \) is established in \( \text{[18]} \), which can be applied to yield global solutions.

For soft potentials, it is difficult to have a good decay property for the operator \( K \) after taking integration in \( v \). We will introduce a cut-off function as in \( \text{[22]} \) to avoid this inconvenience. Also, the point-wise inequality \( \epsilon e^{-\frac{|w^2|}{2}} |v - u|^{\gamma} \leq C(1 + |v|)^{\gamma} \) in \( \text{[15]} \) does not hold anymore. We need to use various integral inequalities and transformations to control the nonlinear term. Moreover, there are terms like \( \int_0^1 e^{-\nu(v/|t-s|)} (w_\beta \Gamma)(f, f)(s, x - v(t - s), v)ds \) which will cause troubles for our analysis, since it is hard to get \( \|w_\beta f\|_{L_p} \) from those terms if we take the \( L_p \) norm. Then we point
out that the order for taking $L^p_T$ norm and $L^\infty_T$ norm will matter. If we take $L^\infty_T$ first, we can escape from the difficulty stated above. In this way, we establish the inequality
\[ \|w_{\beta-\Gamma}(f,f)\|_{L^p_T L^\infty_T} \leq C \|f\|_{L^\infty_T}^{a+} \|f\|_{L^1_T}^{a-} \|w_{\beta-\Gamma}f\|_{L^p_T L^\infty_T}, \]
for some $0 \leq a \leq 1$. Then we will show that $\|f\|_{L^\infty_T}^{a+} \|f\|_{L^1_T}^{a-}$ is small under the smallness condition of $E(F_0)$ and $\|f_0\|_{L^1_T}$. Finally, (1.11) follows since we can close our a priori assumption.

As for the organization of the paper, in Section 2, we will give some useful properties of the operator $K$ and introduce some notations. In Section 3, we prove theorem 1.1 which is the local solution result. In Section 4, we deduce the $L^p_T L^\infty_x \cap L^\infty_T L^1_x$ estimate and use it to prove Theorem 1.2.

2. Preliminaries

We will need the following properties of the operator $K$. Details of the proof can be found in [29].

**Lemma 2.1.** For $-3 < \gamma < 0$, $(Kf)(v)$ can be written as
\[ (Kf)(v) = \int_{\mathbb{R}^3} k(v,\eta) f(\eta) d\eta, \]
with
\[ |k(v,\eta)| \leq C|v-\eta|^{\gamma} e^{-\frac{|v|^2}{2}} e^{-\frac{|\eta|^2}{2}} + \frac{C(\gamma)}{|v-\eta|^{\frac{3-\gamma}{2}}} e^{-\frac{|v-n|^{2}}{8|v-\eta|^{2}}}, \]
where $C(\gamma)$ is a constant depending only on $\gamma$. For $\beta \in \mathbb{R}$, we have the estimate
\[ \int_{\mathbb{R}^3} |k(v,\eta) \cdot \frac{w_{\beta}(v)}{w_{\beta}(\eta)}| d\eta \leq C(\gamma)(1 + |v|)^{-1}. \]
The above inequality still holds after replacing $k(v,\eta)$ by $k(\eta,v)$ since $k(v,\eta) = k(\eta,v)$.

In order to yield the global existence, it is necessary to get more decay in $|v|$ from $K$. We introduce a smooth cut-off function $\chi_m = \chi_m(\tau)$ as in [22] with $0 \leq m \leq 1$, $0 \leq \chi_m \leq 1$. Let $\chi_m(\tau) = 1$ for $\tau \leq m$ and $\chi_m(\tau) = 0$ for $\tau \geq 2m$. Then $K$ can be split into $K^m + K^c$ where
\[ (K^m f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u,\theta) \chi_m(|v-u|) \sqrt{\mu(u)} \left( \sqrt{\mu(w)} f(w') + \sqrt{\mu(v)} f(v') - \sqrt{\mu(v)} f(u) \right) d\omega du. \]
For $K^c = K - K^m$, we have the following lemma, which provides the decay we need. The proof is given in the appendix of [8].

**Lemma 2.2.** Let $-3 < \gamma < 0$ and $\beta \in \mathbb{R}$. There is a function $l(v,\eta)$ such that
\[ (K^c f)(v) = \int_{\mathbb{R}^3} l(v,\eta) f(\eta) d\eta \]
with
\[ \int_{\mathbb{R}^3} |l(v,\eta) \cdot \frac{w_{\beta}(v)}{w_{\beta}(\eta)}| d\eta \leq C(\gamma)m^{-1} \frac{\nu(v)}{(1 + |v|)^2}, \]
\[ \int_{\mathbb{R}^3} |l(v,\eta) \cdot \frac{w_{\beta}(v)}{w_{\beta}(\eta)}| e^{-\frac{|\eta|^2}{2m}} d\eta \leq C e^{-\frac{|v|^2}{8m}}, \]
\[ \int_{\mathbb{R}^3} |l(v,\eta) \cdot \frac{w_{\beta}(v)}{w_{\beta}(\eta)}| e^{-\frac{|v-n|^2}{8m}} d\eta \leq C(\gamma)m^{-1} \frac{\nu(v)}{(1 + |v|)^2}. \]
Furthermore, $l(v,\eta)$ also has the same properties as $k(v,\eta)$ that
\[ \int_{\mathbb{R}^3} |l(v,\eta) \cdot \frac{w_{\beta}(v)}{w_{\beta}(\eta)}| d\eta \leq C(\gamma)(1 + |v|)^{-1}, \]
and
\[ |l(v, \eta)| \leq C|v - \eta|^r e^{-\frac{|v|^2}{2}} e^{-\frac{|\eta|^2}{2}} + \frac{C(\gamma)}{|v - \eta|^{\frac{1}{2}}} e^{-\frac{|v - \eta|^2}{8v - \eta^2}}. \tag{2.7} \]

All the inequalities hold after changing \(l(v, \eta)\) to \(l(\eta, v)\).

Moreover, we need the following smallness property for \(K^m\) when \(0 < m \ll 1\).

**Lemma 2.3.** For \(-3 < \gamma < 0\), \(p > 3/(3 + \gamma)\) and \(\frac{1}{p} + \frac{1}{p'} = 1\), we have the following pointwise bound of \(K^m\),
\[ |(K^m f)(v)| \leq Cm^{\gamma + \frac{1}{p}} e^{-\frac{|u|^2}{2m}} \left[ \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v'|^2}{2}} |f(v')|^p du \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|u'|^2}{2}} |f(u')|^p du \right)^{\frac{1}{p}} \right] \]
\[ + \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v'|^2}{2}} |f(v')|^p du \right)^{\frac{1}{p}}, \tag{2.8} \]

where \(v', u'\) are given in (1.3). The three terms on the right-hand side of (2.8) are obtained from the corresponding three terms on the right-hand side of (2.7).

**Proof.** From the definition of \(K_m\) (2.2), it is direct to see that
\[ |(K^m f)(v)| \leq \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \chi_m(|v - u|) \sqrt{|\mu(u)|} \left( \left| \sqrt{|\mu(u')|} f(v') \right| + \left| \sqrt{|\mu(v')|} f(u') \right| + \left| \sqrt{|\mu(v)|} f(u) \right| \right) d\omega du. \]

We prove for the first term on the right-hand side above which contains \(\sqrt{|\mu(u)|} f(v')\). Noticing the fact that \(e^{-\frac{|u|^2}{2m}} \leq Ce^{-\frac{|u|^2}{2m}}\) for \(|v - u| \leq 2m\), it holds that
\[ \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \chi_m(|v - u|) \sqrt{|\mu(u)|} \mu(u') f(v') d\omega du \]
\[ \leq Ce^{-\frac{|u|^2}{2m}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^{\gamma} e^{-\frac{|v'|^2}{2}} \chi_m(|v - u|) du \right)^{\frac{1}{p'}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|u'|^2}{2}} |f(v')|^p du \right)^{\frac{1}{p}}. \tag{2.9} \]

We have \(\gamma > -3\) by our assumption that \(p > 3/(3 + \gamma)\), which yields that
\[ \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^{\gamma} e^{-\frac{|v'|^2}{2}} \chi_m(|v - u|) du \right)^{\frac{1}{p'}} \]
\[ \leq C \left( \int_{\mathbb{R}^3} |v - u|^{\gamma} \chi_m(|v - u|) du \right)^{\frac{1}{p'}} \]
\[ \leq C \left( \int_{\mathbb{R}^3} |u|^{\gamma} \chi_m(|u|) du \right)^{\frac{1}{p'}} \]
\[ \leq Cm^{\gamma + \frac{1}{p'}}. \]

Then together with (2.3), it follows that
\[ \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \chi_m(|v - u|) \sqrt{|\mu(u)|} \mu(u') f(v') d\omega du \]
\[ \leq Cm^{\gamma + \frac{1}{p'}} e^{-\frac{|u|^2}{2m}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|u'|^2}{2}} |f(v')|^p du \right)^{\frac{1}{p'}}. \]

The second and third terms in the right-hand side of (2.8) can be estimated similarly. \(\Box\)

The following lemma will be used frequently in Section 4. For the proof, see [8] Lemma 2.7 and [14].
Lemma 2.4. Let $F(t,x,v)$ satisfy (1.4), (1.5) and (1.6), we have

\[
\int_\Omega \int_{\mathbb{R}^3} \left| F(t,x,v) - \mu(v) \right|^2 \frac{\chi_{\{|F(t,x,v)-\mu(v)| \leq \mu(v)\}} + \left| F(t,x,v) - \mu(v) \right| \chi_{\{|F(t,x,v)-\mu(v)| \geq \mu(v)\}}}{\mu(v)} dv dx \\
\leq 4 \left( \int_\Omega \int_{\mathbb{R}^3} \{ F_0 \log F_0 - \mu \log \mu \} dv dx + \left( \frac{3}{2} \log(2\pi) - 1 \right) M_0 + \frac{1}{2} E_0 \right) := 4\mathcal{E}(F_0)
\]

In order to simplify our calculations, we define some notations. For given functions $f = f(t,x,v)$ and functions $l(v,\eta)$ which is defined in (2.3)

\[
\| f(t,v) \|_{L^\infty_x} := \sup_{x \in \Omega} \| f(t,x,v) \|, \quad \| f(v) \|_{L^\infty_t L^\infty_x} := \sup_{t \in [T_0,T]} \sup_{x \in \Omega} \| f(t,x,v) \|
\]

\[
\| f(t,x) \|_{L^1_v} := \int_{\mathbb{R}^3} \| f(t,x,v) \| dv,
\]

\[
\| g(v) \|_{L^\infty_v} := \sup_{v \in \Omega} \| g(x,v) \|
\]

(2.10)

When $T_0 = 0$, $\| f(v) \|_{L^\infty_t L^\infty_x} := \| f(v) \|_{L^\infty_t L^\infty_x}$ and $\| f \|_{L^\infty_t L^\infty_x L^1_v} := \| f \|_{L^\infty_t L^\infty_x L^1_v}.$

3. Local-in-time Existence

In this section we consider the local existence of (1.1) with bounded $L^p_t L^\infty_x$ initial data. Firstly, rewrite the perturbed equation (1.7) as

\[
\partial_t f + v \cdot \nabla_x f + \nu(v)f - \Gamma_-(f,f) = Kf + \Gamma_+(f,f).
\]

Recall that

\[
\Gamma_-(f,f)(t,x,v) = \frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu} f, \sqrt{\mu} f)(t,x,v) = \int_{\mathbb{R}^3} \int_{S^2} B(v-u,\theta) (\sqrt{\mu} f) (t,x,v) f(t,x,v) dw du.
\]

(3.2)

Notice that from (2.2) and the fact that $\nu(v) = \int_{\mathbb{R}^3} \int_{S^2} B(v-u,\theta) \mu(u) dw du$, we have

\[
\{ \nu f + \Gamma_-(f,f) \} (t,x,v) = f(t,x,v) \int_{\mathbb{R}^3} \int_{S^2} B(v-u,\theta) \{ \mu(u) + (\sqrt{\mu} f) (t,x,v) \} dw du.
\]

After integrating along the backward trajectory, we can construct our approximation sequence $\{ f_n \}_{n=1}^\infty$ from (3.4) as following,

\[
f_{n+1}^0(t,x,v) = e^{-\int_0^t g^n_\tau \cdot (\tau-x-v(t-\tau),\tau) dr} f_0(x-vt,v)
\]

\[
+ \int_0^t \int_{\rho^2} e^{-\int_0^s g^n_\tau \cdot (\tau-x-v(t-\tau),\tau) ds} (Kf^n)(s,x-v(t-s),v) dv ds
\]

\[
+ \int_0^t \int_{\rho^2} e^{-\int_0^s g^n_\tau \cdot (\tau-x-v(t-\tau),\tau) ds} \Gamma_+(f^n,f^n)(s,x-v(t-s),v) dv ds,
\]

(3.3)

where $g^n(\tau,y,v) = \int_{\mathbb{R}^3} \int_{S^2} B(v-u,\theta) \left( \mu(u) + (\sqrt{\mu} f^n) (\tau,y,u) \right) dw du$, $f^{n+1}(0,x,v) = f_0(x,v)$ and $f^0(t,x,v) = 0$. If we define $F^n = \mu + \sqrt{\mu} f^n$, we can write down the corresponding equation for $F^n$ that

\[
F^{n+1}(t,x,v) = e^{-\int_0^t g^n_\tau \cdot (\tau-x-v(t-\tau),\tau) dt} F_0(x-vt,v)
\]

\[
+ \int_0^t e^{-\int_0^s g^n_\tau \cdot (\tau-x-v(t-\tau),\tau) ds} Q_+(F^n,F^n)(s,x-v(t-s),v) dv ds,
\]

with $F^{n+1}(0,x,v) = F_0(x,v)$ and $F^0(t,x,v) = \mu(v) \geq 0$. If we assume that $F^n \geq 0$, then $g^n(\tau,y,v) \geq 0$ and $Q_+(F^n,F^n)(s,x-v(t-s),v) \geq 0$, which yields $F^{n+1} \geq 0$. By induction on $n$, we have $F^n \geq 0$ for $n = 1, 2, \cdots$. Then it holds that $g^n(\tau,y,v) = \int_{\mathbb{R}^3} \int_{S^2} B(v-u,\theta) F^n(\tau,y,u) dw du \geq 0$.

Once we have the approximation sequence, we can prove that it is uniformly bounded and also a Cauchy sequence. Then after taking the limit, we will obtain a local solution. The uniqueness can be deduced similarly as how we prove the approximation sequence is Cauchy sequence.
For \((t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^3\), the following inequality holds directly from (3.3),

\[
|w_\beta(v)f^{n+1}(t, x, v)| \leq |w_\beta(v)f_0(x - vt, v)| + \int_0^t |w_\beta(v)(Kf^n)(s, x - v(t - s), v)| \, ds
\]

\[
= \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^\gamma b(\theta)w_\beta(v)e^{-\frac{|v|^2}{4}} f^n(s, x, u') f^n(s, x_1, u') \, d\omega du \, ds
\]

\[
\leq CT \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^\gamma \cos \theta w_\beta(v) e^{-\frac{|v|^2}{4}} |f^n(u') f^n(v')| \, d\omega du. \quad (3.9)
\]

Since \(|v|^2 \leq |v'|^2 + |v''|^2\), either \(|v|^2 \leq 2|v'|^2\) or \(|v|^2 \leq 2|v''|^2\). Then there exists a strictly positive constant \(C\) such that \(w_\beta(v) \leq w_\beta(v)\chi_{\{v|v|^2 \leq 2|v'|^2\}} + w_\beta(v)\chi_{\{v|v|^2 \leq 2|v''|^2\}} \leq C (w_\beta(u') + w_\beta(v'))\). By
this inequality, (3.9) and the fact that we can exchange $u'$ and $v'$ by a rotation, we have

\[
I_2(t, x, v) \leq CT \int_{\mathbb{R}^3} \int_{S^2} |v - u|^\gamma |\cos \theta| (w_\beta(u') + w_\beta(v')) e^{\frac{1}{2} |u'|^2} \|f^n(u') f^n(v')\|_{L^\infty_t L^2_x} d\omega du \\
\leq CT \int_{\mathbb{R}^3} \int_{S^2} |v - u|^\gamma |\cos \theta| w_\beta(u') e^{\frac{1}{2} |u'|^2} \|f^n(u')\|_{L^\infty_t L^2_x} \|(w_\beta f^n)'(u')\|_{L^\infty_t L^2_x} d\omega du \\
\leq CT \left( \int_{\mathbb{R}^3} \int_{S^2} |v - u|^\gamma |\cos \theta| \left( \frac{1}{(1 + |v'|)^{\beta'}} e^{\frac{1}{2} |u'|^2} \right)^{\frac{1}{p'}} \right) d\omega du \\
\times \left( \int_{\mathbb{R}^3} \|(w_\beta f^n)(u')\|_{L^p_{t} L^\infty_x} \|(w_\beta f^n)'(v')\|_{L^p_{t} L^\infty_x} du \right)^{\frac{1}{p'}}. \tag{3.10}
\]

We define

\[
\tilde{I}_1 := \int_{\mathbb{R}^3} \int_{S^2} |v - u|^\gamma |\cos \theta| \left( \frac{1}{(1 + |v'|)^{\beta'}} e^{\frac{1}{2} |u'|^2} \right)^{\frac{1}{p'}} d\omega du.
\]

Then it follows from (3.10) that

\[
I_2(t, x, v) \leq CT \left( \tilde{I}_1 \right)^{\frac{1}{p'}} \left( \int_{\mathbb{R}^3} \|(w_\beta f^n)(u')\|_{L^p_{t} L^\infty_x} \|(w_\beta f^n)'(v')\|_{L^p_{t} L^\infty_x} du \right)^{\frac{1}{p'}}. \tag{3.11}
\]

Denote $z = u - v$, $z_n = (u - v) \cdot \omega$, $z_\perp = z - z_n$. We assume that $p > \max\{6 / (5 + \gamma), 3 / (3 + \gamma), 4 / (3 + \gamma)\}$ which implies $\frac{5}{2} p > -3$, $\frac{2}{3} p' - 2 > -3$ and $\frac{2}{3} p' - 2 < 0$ respectively. Here $3 / (3 + \gamma)$ can be replaced by $2 / (3 + \gamma)$, but we use $3 / (3 + \gamma)$ because of (2.9). Also we require $\beta > 3 / p'$, then it holds that

\[
\tilde{I}_1 \leq \int_{\mathbb{R}^3} \int_{z_\perp} \left( \frac{|z_n|^2}{|z|^2} \right)^{\frac{2}{p' - 2}} \frac{1}{|z|^2} e^{\frac{1}{2} |x + z_n|^2} \left( \frac{(1 + |v + z_n|)^{\beta p'}}{(1 + |v|)^{\beta p'}} \right)^{\frac{1}{p'}} d\omega dz_\perp dz_n \\
\leq \int_{\mathbb{R}^3} \int_{z_\perp} |z_n|^{\frac{2}{p' - 2}} e^{\frac{1}{2} |x + z_n|^2} dz_\perp \leq C (1 + |y|)^{\frac{2}{p' - 2} - 1} \left( \frac{1}{(1 + |y|)^{\beta p'}} \right)^{\frac{1}{p'}} dy \quad (y = v + z_n). \tag{3.12}
\]

It follows from our assumption $-3 < \frac{2}{3} p' < 0$ that

\[
\int_{z_\perp} \frac{1}{|z|^2} e^{\frac{1}{2} |x + z_n|^2} dz_\perp \leq C (1 + |y|)^{\frac{2}{p' - 2}} \leq C
\]

for some constant $C$. Thus, substituting the inequality above into (3.12), we have

\[
\tilde{I}_1 \leq C \int_{\mathbb{R}^3} |y - v|^{\frac{2}{p' - 2}} \frac{1}{(1 + |y|)^{\beta p'}} dy \\
\leq C (1 + |v|)^{\frac{2}{p' - 2}} \leq C. \tag{3.13}
\]

The second equality above holds since $\frac{2}{3} p' - 2 > -3$, $\beta > 3 / p'$. For the last inequality in (3.13), we use the condition $\frac{2}{3} p' - 2 < 0$. By (3.11), (3.13), and $dudv = dv'dv'$, after taking $L^p_t L^\infty_x$ norm, we deduce that

\[
\|I_2\|_{L^p_t L^\infty_x} \leq CT \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \|(w_\beta f^n)(u')\|_{L^p_t L^\infty_x} \|(w_\beta f^n)'(v')\|_{L^p_t L^\infty_x} dudv \right)^{\frac{1}{p}} = CT \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \|(w_\beta f^n)(u)\|_{L^p_t L^\infty_x} \|(w_\beta f^n)(v)\|_{L^p_t L^\infty_x} dudv \right)^{\frac{1}{p}} \leq CT \|w_\beta f^n\|_{L^p_t L^\infty_x}^2. \tag{3.14}
\]
According to the observation above, we can obtain the upper bound of $w_β f^n$. It follows from (3.4), (3.8) and (3.14) that

$$\|w_β f^{n+1}\|_{L^p_{\Omega} L^\infty_{x, v}} \leq \|w_β f_0\|_{L^p_{\Omega} L^\infty_{x, v}} + C_1 T \left( \|w_β f^n\|_{L^p_{\Omega} L^\infty_{x, v}} + \|w_β f^n\|_{L^p_{\Omega} L^\infty_{x, v}}^2 \right),$$

for some constant $C_1 > 1$. We set

$$T_1 = \frac{1}{6C_1(1 + \|w_β f_0\|_{L^p_{\Omega} L^\infty_{x, v}})},$$

then it holds from (3.15) and (3.16) that

$$\|w_β f^{n+1}\|_{L^p_{\Omega} L^\infty_{x, v}} \leq 2\|w_β f_0\|_{L^p_{\Omega} L^\infty_{x, v}}.$$  

With this uniform upper bound, we can prove the approximation sequence is a Cauchy sequence. By taking the difference between $w_β f^{n+2}$ and $w_β f^{n+1}$ and recalling the definition of $f^n$ (3.9), it holds that

$$w_β(f^{n+2} - f^{n+1})(t, x, v)$$

$$= w_β(v) f_0(x - vt, v) \left( e^{-f_0' g^{n+1}(\tau, x-v(t-\tau), v)} - e^{-f_0' g^n(\tau, x-v(t-\tau), v)} \right)$$

$$+ \int_0^t w_β(v) \left( K f^{n+1} \right)(s, x - v(t-s), v) \left( e^{-f_0' g^{n+1}(\tau, x-v(t-\tau), v)} - e^{-f_0' g^n(\tau, x-v(t-\tau), v)} \right) ds$$

$$+ \int_0^t w_β(v \Gamma_+(f^{n+1}, f^{n+1}))(s, x - v(t-s), v)$$

$$\times \left( e^{-f_0' g^{n+1}(\tau, x-v(t-\tau), v)} - e^{-f_0' g^n(\tau, x-v(t-\tau), v)} \right) ds$$

$$+ \int_0^t e^{-f_0' g^n(\tau, x-v(t-\tau), v)} \left( K f^{n+1} - K f^n \right)(s, x - v(t-s), v) ds$$

$$+ \int_0^t e^{-f_0' g^n(\tau, x-v(t-\tau), v)} \left( \Gamma_+(f^{n+1}, f^{n+1}) - \Gamma_+(f^n, f^n) \right)(s, x - v(t-s), v) ds,$$

for $(t, x, v) \in [0, T_1] \times \Omega \times \mathbb{R}^3$. Noticing $g^n \geq 0$ for $n = 1, 2, \cdots$ and $|e^{-a} - e^{-b}| \leq |a - b|$ for any $a, b \geq 0$, we have the following inequality for $s \in [0, t]$,

$$\left| e^{-f_0' g^{n+1}(\tau, x-v(t-\tau), v)} - e^{-f_0' g^n(\tau, x-v(t-\tau), v)} \right| \leq \int_s^t |(g^{n+1} - g^n)(\tau, x-v(t-\tau), v)| d\tau.$$  

Obviously we also have $\left| e^{-f_0' g^n(\tau, x-v(t-\tau), v)} \right| \leq 1$. Hence we obtain the pointwise bound

$$\left| w_β(f^{n+2} - f^{n+1})(t, x, v) \right| \leq \tilde{F}_1(t, x, v) + \tilde{F}_2(t, x, v),$$

where

$$\tilde{F}_1(t, x, v) := |w_β(v) f_0(x - vt, v)| \int_0^t \left| (g^{n+1} - g^n)(\tau, x-v(t-\tau), v) \right| d\tau$$

$$+ \int_0^t \left| w_β(v) \left( K f^{n+1} \right)(s, x - v(t-s), v) \int_s^t \left| (g^{n+1} - g^n)(\tau, x-v(t-\tau), v) \right| d\tau ds$$

$$+ \int_0^t \left| w_β(v \Gamma_+(f^{n+1}, f^{n+1}))(s, x - v(t-s), v) \int_s^t \left| (g^{n+1} - g^n)(\tau, x-v(t-\tau), v) \right| d\tau ds$$

$$+ \int_0^t \left| w_β(v \left( K f^{n+1} - K f^n \right)(s, x - v(t-s), v) \right| ds,$$

and

$$\tilde{F}_2(t, x, v) := \int_0^t \left| w_β(v \left( \Gamma_+(f^{n+1}, f^{n+1}) - \Gamma_+(f^n, f^n) \right)(s, x - v(t-s), v) \right| ds.$$
Recall that $g^n(\tau, y, v) = \int_{B} B(v-u, \theta) \left[ \mu(u) + \left( \sqrt{L^n} \right) (\tau, y, u) \right] du$. Since for $p > 3/(3+\gamma)$, $p'\gamma > -3$, by similar arguments as in (3.10), one gets that

$$
\int_t^s |(g^{n+1} - g^n)(\tau, x - v(t - \tau), v)| d\tau \leq \int_t^s |(g^{n+1} - g^n)(\tau, x - v(t - \tau), v)| d\tau \leq CT_1 \int_{R^3} \int_{S^2} |v - u| |\cos \theta| e^{-\frac{|v-u|^2}{4r^2}} \|f^{n+1}(u) - f^n(u)\|_{L^2_{T_1} L^\infty_x} du
$$

$$
\leq CT_1 \left( \int_{R^3} \left( \int_{S^2} |v - u| \right) |\cos \theta| e^{-\frac{|v-u|^2}{4r^2}} |w| \right) \left( \int_{R^3} \|f^{n+1}(u) - f^n(u)\|_{L^p_{T_1} L^\infty_x} du \right)^{\frac{1}{p}} \leq CT_1 \|f^{n+1} - f^n\|_{L^p_{T_1} L^\infty_x}.
$$

Also for the last term on the right-hand side of (3.19), using similar arguments as in (3.10), (3.16) and (3.21), we have

$$
\int_0^t \left| w_\beta(v) \left( Kf^{n+1} - Kf^n \right) (s, x - v(t - s), v) \right| ds
$$

$$
= \int_0^t \left| \int_{R^3} k(v, \eta) \frac{w_\beta(v)}{w_\beta(\eta)} (w_\beta f^{n+1} - w_\beta f^n) (s, x - v(t - s), \eta) d\eta \right| ds
$$

$$
\leq \int_0^t \left( \int_{R^3} |k(v, \eta)| \left| \frac{w_\beta(v)}{w_\beta(\eta)} \right| d\eta \right)^{\frac{1}{p'}} \left( \int_{R^3} \|k(v, \eta)\| \|w_\beta f^{n+1} - w_\beta f^n\|_{L^p_x} d\eta \right)^{\frac{1}{p}} ds
$$

$$
\leq CT_1 \left( \int_{R^3} \|k(v, \eta)\| \|w_\beta f^{n+1} - w_\beta f^n\|_{L^p_x} d\eta \right)^{\frac{1}{p}}.
$$

It follows from (3.19), (3.21) and (3.22) that

$$
\tilde{F}_1(t, x, v) \leq CT_1 \|w_\beta f^{n+1} - w_\beta f^n\|_{L^p_{T_1} L^\infty_x} \times \left( |w_\beta(v)f_0(x - vt, v)| \right. + \int_0^t \left. \left| w_\beta(v)(Kf^{n+1})(s, x - v(t - s), v) \right| ds \right.
$$

$$
\left. + \int_0^t \left| w_\beta(v)\Gamma_+(f^{n+1} - f^n)(s, x - v(t - s), v) \right| ds \right).
$$

After taking $L^p_{T_1} L^\infty_x$ norm, by (3.17), (3.23) and similar arguments as how we estimate the right-hand side of (3.3), we can bound $L^p_{T_1} L^\infty_x$ norm of $\tilde{F}_1(t, x, v)$ as follows:

$$
\|\tilde{F}_1\|_{L^p_{T_1} L^\infty_x} \leq CT_1 \left( 1 + \|w_\beta f_0\|_{L^p_x} \right) \|w_\beta f^{n+1} - w_\beta f^n\|_{L^p_{T_1} L^\infty_x}.
$$

Next we need to estimate $\tilde{F}_2(t, x, v)$. It is direct to see

$$
\tilde{F}_2(t, x, v) \leq \int_0^t \left| w_\beta(v)\Gamma_+(f^{n+1} - f^n)(s, x - v(t - s), v) \right| ds
$$

$$
\left. + \int_0^t \left| w_\beta(v)\Gamma_+(f^{n+1} - f^n)(s, x - v(t - s), v) \right| ds \right).
$$

(3.25)
We firstly focus on the integral containing $\Gamma_+(f^{n+1}, f^n)$. By using the similar arguments in (3.25) and (3.26), we have
\[
\left| w_\beta(v) \Gamma_+(f^{n+1}, f^n)(s, x - v(t - s), v) \right| 
\leq C \int_{R^3} \int_{S^2} |v - u| \cos \theta |w_\beta(v)e^{-\frac{|v|^2}{2}}| \|f^{n+1}(u') (f^{n+1} - f^n)(v')\|_{L_{\infty}^{s}L_{\infty}^{2}} d\omega du 
\leq C \int_{R^3} \int_{S^2} |v - u| \cos \theta |e^{-\frac{|v|^2}{2}}| \|w_\beta f^{n+1}(u') (f^{n+1} - f^n)(v')\|_{L_{\infty}^{s}L_{\infty}^{2}} d\omega du 
+ C \int_{R^3} \int_{S^2} |v - u| \cos \theta |e^{-\frac{|v|^2}{2}}| \|f^{n+1}(u') (w_\beta f^{n+1} - w_\beta f^n)(v')\|_{L_{\infty}^{s}L_{\infty}^{2}} d\omega du 
\leq C \left( \int_{R^3} \int_{S^2} \left| |(w_\beta f^n)(u')\right|^p \left\| (w_\beta f^{n+1} - w_\beta f^n)(v')\right\|_{L_{\infty}^{s}L_{\infty}^{2}}^p d\omega du \right)^{\frac{1}{p}}. 
\]
We can treat $|w_\beta(v) \Gamma_+(f^{n+1} - f^n, f^n)(s, x - v(t - s), v)|$ in the same way, then we conclude that
\[
\int_0^t \left| w_\beta(v) \left( \Gamma_+(f^{n+1}, f^n) - \Gamma_+(f^n, f^n) \right)(s, x - v(t - s), v) \right| ds 
\leq CT_1 \left( \int_{R^3} \left\| (w_\beta f^n)(u')\right\|_{L_{\infty}^{s}L_{\infty}^{2}}^p \left\| (w_\beta f^{n+1} - w_\beta f^n)(v')\right\|_{L_{\infty}^{s}L_{\infty}^{2}}^p d\omega du \right)^{\frac{1}{p}}. 
\text{(3.26)}
\]
It follows from (3.25) and (3.26) that
\[
\| \tilde{F}_2 \|_{L_{\infty}^{s}L_{\infty}^{s}L_{\infty}^{2}} \leq CT_1 \| w_\beta f_0 \|_{L_{\infty}^{s}L_{\infty}^{2}} \| w_\beta f^{n+1} - w_\beta f^n \|_{L_{\infty}^{s}L_{\infty}^{2}}. 
\text{(3.27)}
\]
where $\tilde{F}_2$ is defined in (3.20). Using (3.18), (3.24), (3.27) and recalling that $T_1 = \frac{1}{6C_1(1 + \|w_\beta f_0\|_{L_{\infty}^{s}L_{\infty}^{2}})}$ from (3.16), we yield
\[
\| w_\beta f^{n+2} - w_\beta f^{n+1} \|_{L_{\infty}^{s}L_{\infty}^{s}L_{\infty}^{2}} \leq \left\| \tilde{F}_1 \right\|_{L_{\infty}^{s}L_{\infty}^{s}L_{\infty}^{2}} + \left\| \tilde{F}_2 \right\|_{L_{\infty}^{s}L_{\infty}^{s}L_{\infty}^{2}} 
\leq CT_1 \left( 1 + \| w_\beta f_0 \|_{L_{\infty}^{s}L_{\infty}^{2}} \right) \| w_\beta f^{n+1} - w_\beta f^n \|_{L_{\infty}^{s}L_{\infty}^{s}L_{\infty}^{2}} 
\leq C \left\| w_\beta f^{n+1} - w_\beta f^n \right\|_{L_{\infty}^{s}L_{\infty}^{s}L_{\infty}^{2}} 
\leq \frac{1}{2} \left\| w_\beta f^{n+1} - w_\beta f^n \right\|_{L_{\infty}^{s}L_{\infty}^{s}L_{\infty}^{2}},
\]
by choosing $C_1$ large enough such that $\frac{C}{6C_1} \leq \frac{1}{2}$. Then we have proved that the approximation sequence is a Cauchy sequence. After taking the limit, we can see the limit function is a local-in-time solution of (1.11) and satisfies the conservation laws and entropy inequality. (1.10) follows from letting $n$ tend to infinity in (3.17). The uniqueness can be obtained in the same way as how we estimate (3.18). Up to now, we finish the proof of the local existence.

4. Global-in-time Existence

In order to obtain the global existence, we rewrite the mild form (1.8) as
\[
f(t, x, v) = e^{-\nu(v)t} f_0(x - vt, v) + \int_0^t e^{-\nu(v)(t-s)} (K^m f)(s, x - v(t - s), v) ds 
+ \int_0^t e^{-\nu(v)(t-s)} (K^c f)(s, x - v(t - s), v) ds 
+ \int_0^t e^{-\nu(v)(t-s)} \Gamma(f, f)(s, x - v(t - s), v) ds,
\text{(4.1)}
\]
where $K^m$ is defined in (2.2) and $K^c = K - K^m$.

4.1. Estimates on $\Gamma$. We first introduce the following lemma in order to estimate $\Gamma$.

**Lemma 4.1.** Let $\gamma$, $\beta$ and $p$ satisfy the assumption in Theorem 1.1 and $3/(3+\gamma) < q < p$. Then for any positive $T_0$, $\bar{T}$ with $0 \leq T_0 \leq \bar{T}$, there is a strictly positive constant $C$ such that

\[
\|w_{3-\gamma} \Gamma^-(f, f)\|_{L_t^p L_x^\infty} L_x^r \leq C \|w_{3-\gamma}^f\|_{L_t^p L_x^\infty}^{1+\frac{q(\gamma-1)}{p-q}} \|f\|_{L_t^{p_0} L_x^\infty} \|f\|_{L_t^{p_0} L_x^\infty} \quad (4.2)
\]

\[
\|w_{3-\gamma} \Gamma^+(f, f)\|_{L_t^p L_x^\infty} L_x^r \leq C \|w_{3-\gamma}^f\|_{L_t^p L_x^\infty}^{1+\frac{q(\gamma-1)}{p-q}} \|f\|_{L_t^{p_0} L_x^\infty} \|f\|_{L_t^{p_0} L_x^\infty},
\]

where $r = p - \frac{p-q}{q}$.

**Proof.** Assume $T_0 \leq t \leq \bar{T}$. We first prove inequality (4.2). Denote $q' = \frac{q}{q-1}$. By Hölder’s inequality, we have

\[
|w_{3-\gamma}^f(t, x, v)| = \left| (1 + |v|)^{-\gamma} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^{-\gamma} b(\theta) w_{3-\gamma}^f(t, x, v) e^{-i\omega \cdot v} f(t, x, v) d\omega du \right|
\]

\[
\leq C(1 + |v|)^{-\gamma} \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^{-\gamma} \cos \theta |e^{-i\omega \cdot v}| q' \right)^{\frac{1}{q'}}
\]

\[
\times \left( \int_{\mathbb{R}^3} |f(t, x, u)|^q |(w_{3-\gamma}^f)(t, x, v)| q' du \right)^{\frac{1}{q'}}.
\]

Notice that we require $q > 3/(3+\gamma)$, which implies $\gamma q' > -3$. Then it holds that

\[
\left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left| \frac{|v - u|^{-\gamma} \cos \theta |e^{-i\omega \cdot v}|}{q'} \right| \right)^{\frac{1}{q'}} \leq C \left( \int_{\mathbb{R}^3} \left| v - u \right|^{-\gamma} e^{-i\omega \cdot v} \right)^{\frac{1}{q'}} du
\]

\[
\leq C(1 + |v|)^{\gamma}.
\]

We substitute this inequality into (4.4) and obtain

\[
|w_{3-\gamma}^f(t, x, v)| \leq C'(w_{3-\gamma}^f(t, x, v) \left( \int_{\mathbb{R}^3} |f(t, x, u)|^q du \right)^{\frac{1}{q'}}
\]

\[
\leq C'(w_{3-\gamma}^f(t, x, v) \left( \int_{\mathbb{R}^3} |f(t, x, u)| du \right)^{\frac{p-q}{p-q-1}} \left( \int_{\mathbb{R}^3} |(w_{3-\gamma}^f)(t, x, u)|^q du \right)^{\frac{(q-1)}{q(q-1)}}
\]

by the interpolation inequality in Lebesgue spaces $\|f\|_{L_q} \leq \|f\|_{L_1}^{\frac{p-q}{p-q-1}} \|f\|_{L_{p_0}}^{\frac{(q-1)}{q(q-1)}}$ for $1 < q < p$. Then the inequality (4.2) follows from (4.3) by taking the $L_t^{p_0} L_x^\infty L_x^r$ norm.
Next we prove the inequality (4.3), noticing that we can exchange $u'$ and $v'$ by a rotation and $w_\beta(v) \leq C (w_\beta(v') + w_\beta(u'))$ for some constant $C$, similar arguments as (4.4) yield that

$|w_{\beta-\gamma}(v) \Gamma_+ (f, f)(t, x, v)|$

\[
\leq C (1 + |v|)^{-\gamma} \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^{\gamma'} \cos \theta |\gamma' e^{-\frac{|u|^2}{4} d\omega du} \right)^{\frac{1}{\gamma'}}
\]

\[
\times \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|u|^2}{4}} |(w_\beta f)(t, x, u')|^{\gamma'} |f(t, x, v')|^{\gamma' d\omega du} \right)^{\frac{1}{\gamma'}}.
\]

Write $|f(t, x, v')|^{\gamma} = |f(t, x, v')|^{\varrho v(t, x, v')} |f(t, x, v')|^{\gamma - \frac{\beta}{4\pi}}$. Applying Hölder’s inequality to the last term above, it holds that

$|w_{\beta-\gamma}(v) \Gamma_+ (f, f)(t, x, v)|$

\[
\leq C \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|u|^2}{4}} |f(t, x, v')|^{\varrho} d\omega du \right)^{\frac{1}{\varrho}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|u|^2}{4}} |(w_\beta f)(t, x, u')|^{\gamma} |f(t, x, v')|^{\gamma d\omega du} \right)^{\frac{1}{\gamma}},
\]

(4.6)

where $r = p - \frac{p - \varrho p}{\varrho} \leq p$. For convenience, we define

$\tilde{I}_2 := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|u|^2}{4}} |f(t, x, v')|^{\frac{1}{\varrho}} d\omega du$. Using transformation $z = u - v$, $z_\perp = (v - u) \cdot \omega$, $z_\parallel = z - z_\perp$, as (3.12), we have

$\tilde{I}_2 \leq \int_{\mathbb{R}^3} \int_{z_\perp} e^{-\frac{|z_\perp|^2}{4}} \left( \int_{z_\parallel} f(t, x, \eta) \right)^{\frac{1}{\varrho}} \left( \frac{1}{|\eta - v|^2} \right) d\eta$. (4.7)

It is direct to get $\beta/12 > 3$ from our assumption that $\beta > 36$. Then $\int_{\mathbb{R}^3} (1 + |\eta|)^{-\frac{\beta}{4\pi}} |\eta - v|^{-\frac{2}{\pi} d\eta}$ will be uniformly bounded in $v$. By $|f(t, x, \eta)|^{\frac{1}{\varrho}} \leq |f(t, x, \eta)|^{\frac{1}{\varrho}} |w_{\beta/2} f(t, x, \eta)|^{\frac{1}{\varrho}} (1 + |\eta|)^{-\frac{\beta}{4\pi}}$, Hölder’s inequality and (4.4), we obtain

$\tilde{I}_2 \leq C \int_{\mathbb{R}^3} |f(t, x, \eta)|^{\frac{1}{\varrho}} |w_{\beta/2} f(t, x, \eta)|^{\frac{1}{\varrho}} \left( \frac{1 + |\eta|}{{\frac{\beta}{4\pi}}} \right)^{\frac{\beta}{4\pi}} d\eta$

\[
\leq C \left( \int_{\mathbb{R}^3} |f(t, x, \eta)|^{\frac{1}{\varrho}} |w_{\beta/2} f(t, x, \eta)|^{\frac{1}{\varrho}} d\eta \right)^{\frac{1}{\varrho}} \left( \int_{\mathbb{R}^3} \left( \frac{1 + |\eta|}{{\frac{\beta}{4\pi}}} \right)^{\frac{\beta}{4\pi}} d\eta \right)^{\frac{\beta}{4\pi}}.
\]

(4.8)

Using the relation

$\left( \int_{\mathbb{R}^3} |(w_{\beta/2} f)(t, x, \eta)| d\eta \right)^{\frac{1}{\varrho}}$

\[
\leq \left( \int_{\mathbb{R}^3} \left( \frac{1}{(1 + |v|)^{\frac{2}{\varrho}}} \right)^{\varrho'} \right)^{\frac{1}{\varrho'}} \left( \int_{\mathbb{R}^3} |(w_{\beta} f)(t, x, \eta)|^{\varrho} d\eta \right)^{\frac{1}{\varrho}}
\]

\[
\leq C \left( \int_{\mathbb{R}^3} |(w_{\beta} f)(t, x, \eta)|^{\varrho} d\eta \right)^{\frac{1}{\varrho}},
\]

(4.9)
we have from (4.8) that
\[ \tilde{J}_2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} e^{-\frac{|w|^2}{2}} |f(t, x, v')|^2 d\omega dv \]
\[ \leq C \left( \int_{\mathbb{R}^3} |f(t, x, \eta)| d\eta \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \left| (w_\beta f)(t, x, \eta) \right|^p d\eta \right)^{\frac{1}{2}} \]
\[ \leq C \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| (w_\beta f)(t, x, \eta) \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}}. \]
Together with (4.6), after taking the $L^p_{tL^m_{mT}L^1_x}$ norm and by $dudv = du'dv'$, (4.9) follows from the fact that
\[ \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}} \leq C \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| (w_\beta f)(t, x, \eta) \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}}, \]
\[ \times \left( \int_{\mathbb{R}^3} \left| (w_\beta f)(u') \right|^{p} \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| (w_\beta f)(u') \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \right)^{\frac{1}{2}} \]
\[ \leq C \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| (w_\beta f)(u) \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}}, \]
\[ \times \left( \int_{\mathbb{R}^3} \left| f(v) \right|^{p} \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \right)^{\frac{1}{2}} \]
\[ \leq C \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| (w_\beta f)(u) \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}}, \]
\[ \leq C \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| (w_\beta f)(u) \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}}, \]
\[ \times \left( \int_{\mathbb{R}^3} \left| f(v) \right|^{p} \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \right)^{\frac{1}{2}} \]
\[ \leq C \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| (w_\beta f)(u) \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}}, \]
In the last inequality above, we use the inequality
\[ \left( \int_{\mathbb{R}^3} \left| f(v) \right|^{p} \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \right)^{\frac{1}{2}} \]
\[ \leq \left( \int_{\mathbb{R}^3} \left( \frac{1}{|1 + |v||^{\gamma}} \right)^{\frac{p'}{p'}} \right)^{\frac{1}{p'}} \left( \int_{\mathbb{R}^3} \left| w_\beta f(v) \right|^{p} \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \right)^{\frac{1}{2}} \]
\[ \leq C \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}} \cdot \| (w_\beta f)(u) \|_{L^p_{tL^m_{mT}L^1_x}}. \]
We have completed the proof of Lemma 4.1.

4.2. Global $L^p_{tL^m_{mT}L^1_x}$ Estimate. Now we can deduce the following result, which allows us to bound the $L^p_{tL^m_{mT}L^1_x}$ norm of $w_\beta f$ by the initial data, $\mathcal{E}(F_0)$ and the product of $\| f \|_{L^p_{tL^m_{mT}L^1_x}}$ and $\| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}$.

**Lemma 4.2.** Let all the assumptions in Theorem 4.1 be satisfied. It holds that
\[ \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}} \leq C_2 \left\{ \| w_\beta f_0 \|_{L^p_{tL^m_{mT}L^1_x}} + \| w_\beta f_0 \|^2 + \mathcal{E}(F_0) + \mathcal{E}(F_0) \right\} \]
\[ + C_2 \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{1+\frac{1}{2}} + C_2 \| w_\beta f \|_{L^p_{tL^m_{mT}L^1_x}}^{\frac{1}{2}} \cdot \| f \|_{L^p_{tL^m_{mT}L^1_x}}^{1+\frac{1}{2}} \]
\[ \text{for } T_1 \text{ defined in (1.9), } T \geq T_1 \text{ and some constant } C_2 > 1. \]

**Proof.** By the mild form (4.1), for $(t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^3$, it is noted that
\[ (w_\beta f(t, x, v) = e^{-\nu(t-s)}(w_\beta f_0)(x - vt, v) + \int_0^t e^{-\nu(t-s)}(w_\beta K^m f)(s, x - v(t-s), v)ds \]
\[ + \int_0^t e^{-\nu(t-s)}(w_\beta K^c f)(s, x - v(t-s), v)ds \]
\[ + \int_0^t e^{-\nu(t-s)}(w_\beta \Gamma)(f, f)(s, x - v(t-s), v)ds \]
\[ = J_0(t, x, v) + J_1(t, x, v) + J_2(t, x, v) + J_3(t, x, v). \]

(4.10)
We define
\[
\tilde{J}(v) := \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} e^{-\frac{|v|^2}{\nu}} \|f(v')\|^p_{L^p_x L^\infty_t} \, dv \, du \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} e^{-\frac{|v|^2}{\nu}} \|f(u')\|^p_{L^p_x L^\infty_t} \, dv \, du \right)^{\frac{1}{p}}.
\]

It follows from Lemma 2.3 that
\[
|J_1(t, x, v)| \leq \int_0^t e^{-\nu(v)(t-s)} |(w_\beta K^m f)(s, x - v(t-s), v)| \, ds
\leq C m^{\gamma + \frac{p}{p-1}} \tilde{J}(v) w_\beta(v) e^{-\frac{|v|^2}{\nu}} \int_0^t e^{-\nu(v)(t-s)} \, ds
\leq C m^{\gamma + \frac{p}{p-1}} \tilde{J}(v).
\]

In that last inequality above, we use the fact that \(w_\beta(v) e^{-\frac{|v|^2}{\nu}} \int_0^t e^{-\nu(v)(t-s)} \, ds = \frac{w_\beta(v)}{\nu(v)} e^{-\frac{|v|^2}{\nu}} \leq C\). Then after taking the \(L^p_x L^\infty_t \) norm, by \(dudu = du'dv'\) and the definition of \(\tilde{J}(v)\) (4.11), we have
\[
\|J_1\|_{L^p_x L^\infty_t} \leq C m^{\gamma + \frac{p}{p-1}} \|\tilde{J}\|_{L^p_x}.
\]

Next we consider \(J_3\). It is noted that
\[
|J_3(t, x, v)| = \left| \int_0^t e^{-\nu(v)(t-s)} (w_\beta \Gamma)(f, f)(s, x - v(t-s), v) \, ds \right|
\leq \left| \int_0^t e^{-\nu(v)(t-s)} \nu(v) \, ds \right| \|w_\beta - \gamma \Gamma(f, f)(v)\|_{L^p_0 T^1 L^\infty_x}
\leq \|w_\beta - \gamma \Gamma(f, f)(v)\|_{L^p_x L^\infty_t}. \tag{4.12}
\]

We observe the fact that
\[
\|w_\beta - \gamma \Gamma(f, f)\|_{L^p_x L^\infty_t} \leq C \|w_\beta - \gamma \Gamma(f, f)\|_{L^p_T L^\infty_x} + C \|w_\beta - \gamma \Gamma(f, f)\|_{L^p_T L^\infty_x} \tag{4.14}
\]
for some strictly positive constant \(C\). Then it follows from (4.12) and (4.14) that
\[
\|w_\beta - \gamma \Gamma(f, f)\|_{L^p_x L^\infty_t} \leq C \|f\|_{L^p_T L^\infty_x} \|w_\beta f\|_{L^p_x L^\infty_t} + C \|f\|_{L^p_T L^\infty_x} \|w_\beta f\|_{L^p_x L^\infty_t} \tag{4.15}
\]
By Hölder's inequality, one gets that
\[
\|f(t, x)\|_{L^1_x} \leq C \|w_\beta f(t, x)\|_{L^p_x} \leq C \|w_\beta f\|_{L^p_T L^\infty_x} \tag{4.16}
\]
for \(t \in [0, T_1]\), \(\beta > 3\). Then by (4.10), (4.10) and the fact that \(\frac{p-1}{q(p-1)} + \frac{p-1}{q(p-1)} = 2\), we have
\[
\|f\|_{L^{(p-1)/2}_T L^{(p-1)/q(p-1)}_x} \|w_\beta f\|_{L^{1+\frac{p(q-1)}{q(p-1)}}_T L^\infty_x} \leq C \|w_\beta f\|_{L^{1+\frac{p(q-1)}{q(p-1)}}_T L^\infty_x} \|w_\beta f\|_{L^p_T L^\infty_x} \leq C \|w_\beta f\|_{L^p_T L^\infty_x} \tag{4.17}
\]
It holds from (4.15), (4.17) that
\[
\|w_\beta - \gamma \Gamma - (f, f)\|_{L^p_x L^\infty_t} \leq C \|w_\beta f_0\|_{L^p_L L^\infty_x} + C \|w_\beta f\|_{L^p_T L^\infty_x} \|f\|_{L^{(p-1)/q(p-1)}_T L^{(p-1)/q(p-1)}_x} \tag{4.18}
\]
Using similar arguments as (4.12), (4.13), (4.16), (4.17) and the fact that \( \frac{1}{q} \left( \frac{1}{q} - \frac{1}{p} \right) = 2 \), one gets the estimate for \( \Gamma_+ \) from (4.3) that

\[
\|w_{\beta^+} \Gamma_+(f, f)\|_{L^q_T L^q_x L^\infty_t} \leq C \|w_{\beta^+} f_0\|_{L^p_x L^\infty_t}^2 + C \|w_{\beta^+} f\|_{L^p_x L^\infty_t} \|f\|_{L^\infty_T L^\infty_x L^1_t}^{\frac{1}{q} \left( \frac{1}{q} - \frac{1}{p} \right) + 1 + \frac{\beta}{q}}, \tag{4.19}
\]

Then it follows from (4.13), (4.18) and (4.19) that

\[
\|J_3\|_{L^p_T L^q_x L^\infty_t} \leq C \|w_{\beta^+} f_0\|_{L^p_x L^\infty_t}^2 + C \|w_{\beta^+} f\|_{L^p_x L^\infty_t} \|f\|_{L^\infty_T L^\infty_x L^1_t}^{\frac{1}{q} \left( \frac{1}{q} - \frac{1}{p} \right) + 1 + \frac{\beta}{q}} \|f\|_{L^\infty_T L^\infty_x L^1_t} \tag{4.20}
\]

Obviously it holds that \( \|J_0\|_{L^p_T L^q_x L^\infty_t} \leq C \|w_{\beta^+} f_0\|_{L^p_x L^\infty_t} \). Together with (4.12) and (4.20), we have

\[
\|J_0 + J_1 + J_3\|_{L^p_T L^q_x L^\infty_t} \leq C \|w_{\beta^+} f_0\|_{L^p_x L^\infty_t} + C \|w_{\beta^+} f_0\|_{L^p_x L^\infty_t} + C \|w_{\beta^+} f\|_{L^p_x L^\infty_t} \|f\|_{L^\infty_T L^\infty_x L^1_t}^{\frac{1}{q} \left( \frac{1}{q} - \frac{1}{p} \right) + 1 + \frac{\beta}{q}} \|f\|_{L^\infty_T L^\infty_x L^1_t} \tag{4.21}
\]

We need to treat \( J_2(t, x, v) \) carefully. Let \( x_1 = x - v(t - s) \). Recall from (2.3) that \( (K^c g)(v) = \int_{R^3} l(v, \eta) g(\eta) d\eta \) and \( w_{\beta^+}(v, \eta) = l(v, \eta) w_{\beta^+}(\eta) \). Using the mild form (4.1), we can rewrite \( J_2(t, x, v) \) as

\[
J_2(t, x, v) = \int_0^t e^{-\nu(v)(t-s)} (w_{\beta^+} K^c f)(s, x_1, v) ds
\]

\[
= \int_0^t e^{-\nu(v)(t-s)} \int_{R^3} w_{\beta^+}(v) l(v, \eta) f(s, x_1, \eta) d\eta ds
\]

\[
= \int_0^t e^{-\nu(v)(t-s)} \int_{R^3} l(v, \eta) e^{-\nu(v)s} (w_{\beta^+} f_0)(x_1 - \eta s, \eta) d\eta ds
\]

\[
+ \int_0^t e^{-\nu(v)(t-s)} \int_{R^3} l(v, \eta) \int_0^s e^{-\nu(\eta)(s-s_1)} (w_{\beta^+} K^m f)(s_1, x_1 - \eta(s - s_1), \eta) ds_1 d\eta ds
\]

\[
+ \int_0^t e^{-\nu(v)(t-s)} \int_{R^3} \int_{R^3} l(v, \eta) l(w_{\beta^+}(\eta, \xi)) \times \int_0^s e^{-\nu(\eta)(s-s_1)} (w_{\beta^+} f)(s_1, x_1 - \eta(s - s_1), \xi) ds_1 d\eta d\xi ds
\]

\[
+ \int_0^t e^{-\nu(v)(t-s)} \int_{R^3} l(w_{\beta^+}(v, \eta)) \times \int_0^s e^{-\nu(\eta)(s-s_1)} (w_{\beta^+} \Gamma(f, f))(s_1, x_1 - \eta(s - s_1), \eta) ds_1 d\eta ds.
\]

We take the absolute value of \( J_2(t, x, v) \) to obtain

\[
|J_2(t, x, v)| \leq |J_{20}(t, x, v)| + |J_{21}(t, x, v)| + |J_{22}(t, x, v)| + |J_{23}(t, x, v)|,
\]
where

\[
J_{20}(t, x, v) := \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \left| l_{\beta, \delta}(v, \eta) e^{-\nu(\eta)s} (w_{\beta} f_0)(x_1 - \eta s, \eta) \right| \, d\eta \, ds
\]
\[
J_{21}(t, x, v) := \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \left| l_{\beta, \delta}(v, \eta) \right| \times \int_0^s \left| e^{-\nu(\eta)(s-s_1)} (w_{\beta} K_{\eta} f)(s_1, x_1 - \eta(s - s_1), \eta) \right| \, ds \, d\eta \, ds
\]
\[
J_{22}(t, x, v) := \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left| l_{\beta, \delta}(v, \eta) l_{\beta, \delta}(\eta, \xi) \right| \times \int_0^s \left| e^{-\nu(\eta)(s-s_1)} (\eta f(s_1, x_1 - \eta(s - s_1), \xi) \right| \, ds \, d\eta \, d\xi \, ds
\]
\[
J_{23}(t, x, v) := \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \left| l_{\beta, \delta}(v, \eta) \right| \times \int_0^s \left| e^{-\nu(\eta)(s-s_1)} (w_{\beta} \Gamma(f, f))(s_1, x_1 - \eta(s - s_1), \eta) \right| \, ds \, d\eta \, ds.
\]

We bound the above four terms \( \{J_{2i}\}_{i=0}^3 \) one by one. Using the property (2.4) and H"older's inequality, we obtain

\[
J_{20}(t, x, v) = \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \left| l_{\beta, \delta}(v, \eta) e^{-\nu(\eta)s} (w_{\beta} f_0)(x_1 - \eta s, \eta) \right| \, d\eta \, ds
\]
\[
\leq \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \left| l_{\beta, \delta}(v, \eta) e^{-\nu(\eta)s} \right| \left| (w_{\beta} f_0)(x_1 - \eta s, \eta) \right| \, d\eta \, ds
\]
\[
\leq \int_0^t e^{-\nu(v)(t-s)} \left( \int_{\mathbb{R}^3} l(v, \eta) \left| \frac{w_{\beta}(v)}{w_{\beta}(\eta)} \right|^p \, d\eta \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^3} l(v, \eta) \left| (w_{\beta} f_0)(x_1 - \eta s, \eta) \right|^p \, d\eta \right)^{\frac{1}{p}} \, ds
\]
\[
\leq C_m \frac{\nu(v)}{(1 + |v|)^2} \left( \int_{\mathbb{R}^3} l(v, \eta) \left| (w_{\beta} f_0)(x_1 - \eta s, \eta) \right|^p \, d\eta \right)^{\frac{1}{p}} \, ds. \tag{4.22}
\]

We observe that \( \left| (w_{\beta} f_0)(x_1 - \eta s, \eta) \right| \leq \| (w_{\beta} f_0)(\eta) \|_{L^\infty} \), then \( \int_{\mathbb{R}^3} l(v, \eta) \left| (w_{\beta} f_0)(\eta) \right|^p \, d\eta \) does not depend on \( s \), which together with \( \int_0^t e^{-\nu(v)(t-s)} \, ds \leq \frac{1}{\nu(v)} \) and (4.22) yield that

\[
J_{20}(t, x, v) \leq C_m \frac{\nu(v)}{(1 + |v|)^2} \left( \int_{\mathbb{R}^3} l(v, \eta) \left| (w_{\beta} f_0)(\eta) \right|^p \, d\eta \right)^{\frac{1}{p}} \, ds
\]
\[
\leq C_m \left( \int_{\mathbb{R}^3} l(v, \eta) \left| (w_{\beta} f_0)(\eta) \right|^p \, d\eta \right)^{\frac{1}{p}}.
\]

In the last inequality above, we use the fact that \( \frac{\nu(v)}{(1 + |v|)^2} \left( \int_{\mathbb{R}^3} l(v, \eta) \left| (w_{\beta} f_0)(\eta) \right|^p \, d\eta \right)^{\frac{1}{p}} = (1 + |v|)^{\frac{2p - 2}{p}} \leq C \) since \( p > (2 - \gamma)/2 \) from our assumption. Then similar as (5.3), taking \( \| \cdot \|_{L^p L^\infty} \), we have

\[
\| J_{20} \|_{L^p L^\infty} \leq C_m \| w_{\beta} f_0 \|_{L^p L^\infty}. \tag{4.23}
\]
\( J_{21}(t, x, v) \) can be estimated in such way. Denote \( \eta' = \eta + [(\eta_\ast - \eta) \cdot \omega] \omega \), \( \eta_\ast = \eta - [(\eta_\ast - \eta) \cdot \omega] \omega \), and recall from (4.11) that

\[
\tilde{J}(\eta) = \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{\beta}{2} |f(\eta)|^2} \frac{\|f(\eta')\|^p_{L^p_x L^\infty_\nu}}{\nu_{\ast}^p} d\omega d\nu \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{\beta}{2} |f(\eta')|^2} \frac{\|f(\eta')\|^p_{L^p_x L^\infty_\nu}}{\nu_{\ast}^p} d\omega d\nu \right)^{\frac{1}{p}}.
\]

By our assumption \( p > \frac{3}{(3 + \gamma)} \), using Lemma (2.3), we obtain the following pointwise bound of \( J_{21}(t, x, v) \),

\[
J_{21}(t, x, v) \leq \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) \right| \int_0^s e^{\nu(\eta)(s-s_1)} \left| (w_\beta K^m f)(s_1, x_1 - \eta(s-s_1), \eta) \right| ds_1 \eta d\eta ds
\]

\[
\leq C m^{-\frac{\gamma + \frac{\beta}{p}}{\gamma - \frac{\beta}{p}}} \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) \right| e^{-\frac{\beta}{2} \nu^2} \int_0^s e^{\nu(\eta)(s-s_1)} ds_1 \tilde{J}(\eta) d\eta ds
\]

\[
\leq C m^{-\frac{\gamma + \frac{\beta}{p}}{\gamma - \frac{\beta}{p}}} \int_0^t e^{-\nu(v)(t-s)} ds \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) \right| \tilde{J}(\eta) d\eta.
\]

By similar arguments in (4.22), it holds that

\[
J_{21}(t, x, v) \leq C m^{-\frac{\gamma + \frac{\beta}{p}}{\gamma - \frac{\beta}{p}}} \frac{1}{\nu(v)} \left( \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) \right| \eta d\eta \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) \right| \tilde{J}(\eta) d\eta \right)^{\frac{1}{p}}
\]

\[
\leq C m^{-\frac{\gamma + \frac{\beta}{p}}{\gamma - \frac{\beta}{p}}} \frac{1}{\nu(v)} \left| \nu(v) \right| \left( \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) \right| \tilde{J}(\eta) d\eta \right)^{\frac{1}{p}}
\]

\[
\leq C m^{-\frac{\gamma + \frac{\beta}{p}}{\gamma - \frac{\beta}{p}}} \left( \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) \right| \tilde{J}(\eta) d\eta \right)^{\frac{1}{p}}.
\]

Then recalling \( \| \tilde{J} \|_{L^p_x} \leq C \| w_\beta f \|_{L^p_x L^\infty_\nu} \) in (4.12), it follows from (4.24) that

\[
\| J_{21} \|_{L^p_x L^\infty_\nu L^\infty} \leq C m^{-\frac{\gamma + \frac{\beta}{p}}{\gamma - \frac{\beta}{p}}} \| w_\beta f \|_{L^p_x L^\infty_\nu L^\infty}
\]

We turn to \( J_{23} \) now, similar as above,

\[
J_{23}(t, x, v) \leq \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) \right| \int_0^s e^{\nu(\eta)(s-s_1)} \left| (w_\beta \Gamma(f, f))(s_1, x_1 - \eta(s-s_1), \eta) \right| ds_1 \eta d\eta ds
\]

\[
\leq C \left( \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) \right| d\eta \right)^{\frac{1}{p}} \left[ \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) \right| (w_\beta \Gamma(f, f))(\eta) d\eta \right]^{\frac{1}{p}}
\]

\[
\leq C \left[ \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) \right| (w_\beta \Gamma(f, f))(\eta) d\eta \right]^{\frac{1}{p}}.
\]

Then by our estimate (4.18) and (4.19), taking the \( L^p_x L^\infty_\nu L^\infty \) norm, we obtain

\[
\| J_{23} \|_{L^p_x L^\infty_\nu L^\infty} \leq C \| w_\beta f_0 \|^2_{L^p_x L^\infty} + C \| w_\beta f \|_{L^p_x L^\infty}^{1 + \frac{(p-\gamma)}{p} + \frac{(p-\gamma)}{p}} \| f \|_{L^p_x L^\infty}^{2 \frac{(p-\gamma)}{p}} + C \| w_\beta f \|_{L^p_x L^\infty}^{\frac{1}{2} \left( \frac{1}{p} - \frac{\beta}{p} \right) + 1 + \frac{\beta}{p}} \| f \|_{L^p_x L^\infty}^{\frac{1}{2} \left( \frac{1}{p} - \frac{\beta}{p} \right)}
\]

\[
(4.26)
\]

At last we consider \( J_{22}(t, x, v) \). Recall

\[
J_{22}(t, x, v) = \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) l_{w_\beta}(\eta, \xi) \right| ds_1 \eta d\eta ds.
\]

We divide \( J_{22}(t, x, v) \) into four cases as (4.14).
Case 1. \(|v| \geq N\). A direct calculation shows that
\[
J_{22}(t, x, v) \leq \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) l_{w_\beta} \right| d\eta ds d\xi d\eta ds .
\]

We first integrate with respect to \(s_1\) first, then integrate with respect to \(s\).
\[
J_{22}(t, x, v) \leq \int_0^t e^{-\nu(v)(t-s)} ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left| l_{w_\beta}(v, \eta) l_{w_\beta} \right| \frac{1}{\nu(\eta)} \| (w_\beta f) \|_{L^\infty_{T} L^\infty_x} d\eta d\xi ds . \tag{4.27}
\]

Recalling \(l_{w_\beta}(v, \eta) = l(\eta) \| w_\beta(\eta) \|_{L^\infty_{T} L^\infty_x} \| \) from the notation we define in (2.10), then by Lemma 2.2 and Hölder's inequality, one obtains that
\[
J_{22}(t, x, v) \leq \frac{C_m}{\nu(v)} \left( \int_{\mathbb{R}^3} l(v, \eta) l(\eta, \xi) \frac{1}{\nu(\eta)} \| (w_\beta f) \|_{L^\infty_{T} L^\infty_x} d\eta d\xi ds \right) . \tag{4.28}
\]

Since \(p > (2 - \gamma)/2\) and \(|v| \geq N\), then \(\frac{2}{p} + \frac{2}{p} > 0\) and
\[
\frac{C_m}{\nu(v)} \left( \int_{\mathbb{R}^3} l_{w_\beta}(v, \eta) \frac{\nu(\eta)}{(1 + |\eta|)^2} \frac{1}{\nu(\eta)} \| (w_\beta f) \|_{L^\infty_{T} L^\infty_x} d\eta \right) ^{\frac{1}{p'}} \leq \frac{C_m}{\nu(v)} \left( \int_{\mathbb{R}^3} l_{w_\beta}(v, \eta) d\eta \right) ^{\frac{1}{p'}} \leq \frac{C_m}{\nu(v)} \left( \frac{\nu(v)}{(1 + |v|)^2} \right) ^{\frac{1}{p'}} \leq \frac{C_m}{(1 + |v|)^{\frac{2}{p} + \frac{2}{p}}} \leq \frac{C_m}{N^{\frac{2}{p} + \frac{2}{p}}} .
\]

Substituting the above inequality into (4.28), it holds that
\[
J_{22}(t, x, v) \leq \frac{C_m}{N^{\frac{2}{p} + \frac{2}{p}}} \left( \int_{\mathbb{R}^3} l(v, \eta) l(\eta, \xi) \frac{1}{\nu(\eta)} \| (w_\beta f) \|_{L^\infty_{T} L^\infty_x} d\eta d\xi \right) ^{\frac{1}{p'}} ,
\]
which yields that

\[ \|J_{22}\|_{L^p_v L^\infty_x L^\infty_x} \leq \frac{C_m}{N^{\frac{1}{p} + \frac{1}{q}} + \frac{1}{p}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} l(v, \eta) \nu(v, \eta) \left| \frac{1}{\nu(\eta)} \right| \|w_\beta f(\xi)\|_{L^p_v L^\infty_x L^\infty_x} d\eta d\xi \right)^{\frac{1}{p}} \]

\[ \leq \frac{C_m}{N^{\frac{1}{p} + \frac{1}{q}} + \frac{1}{p}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} l(v, \eta) \frac{\nu(\eta)}{(1 + |\eta|)^2} \left| \frac{1}{\nu(\eta)} \right| \|w_\beta f(\xi)\|_{L^p_v L^\infty_x L^\infty_x} d\eta d\xi \right)^{\frac{1}{p}} \]

\[ \leq \frac{C_m}{N^{\frac{1}{p} + \frac{1}{q}} + \frac{1}{p}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} l(v, \eta) \|w_\beta f(\xi)\|_{L^p_v L^\infty_x L^\infty_x} d\eta d\xi \right)^{\frac{1}{p}} \]

\[ \leq \frac{C_m}{N^{\frac{1}{p} + \frac{1}{q}} + \frac{1}{p}} \|w_\beta f\|_{L^p_v L^\infty_x L^\infty_x} . \quad (4.29) \]

**Case 2.** \(|v| \leq N, |\eta| \geq 2N\) or \(|\eta| \leq 2N, |\xi| \geq 3N\). Then either \(|\eta - v| \geq N\) or \(|\xi - \eta| \geq N\), again by (2.5) and (1.27), similar as (1.28), when \(|\eta - v| \geq N\), we have

\[ J_{22}(t, x, v) \leq \frac{1}{\nu(v)} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |w_{\beta, \nu}(v, \eta)w_{\beta, \nu}(\eta, \xi)| \left| \frac{1}{\nu(\eta)} \right| \|w_\beta f(\xi)\|_{L^p_v L^\infty_x L^\infty_x} d\eta d\xi \right)^{\frac{1}{p}} \]

\[ \leq \frac{1}{\nu(v)} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left| w_{\beta, \nu}(v, \eta) e^{-\nu(v)|\eta|^2} \int_{\mathbb{R}^3} l(v, \eta, \xi) \left| \frac{1}{\nu(\xi)} \right| \|w_\beta f(\xi)\|_{L^p_v L^\infty_x L^\infty_x} d\eta d\xi \right)^{\frac{1}{p}} \]

\[ \leq C_m e^{-\frac{\nu(v)^2}{2}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left| l(v, \eta) l(\eta, \xi) \left| \frac{1}{\nu(\eta)} \right| \|w_\beta f(\xi)\|_{L^p_v L^\infty_x L^\infty_x} \right| d\eta d\xi \right)^{\frac{1}{p}} \quad (4.30) \]

The case when \(|\xi - \eta| \geq N\) can be estimated in the same way. Then we obtain that

\[ \|J_{22}\|_{L^p_v L^\infty_x L^\infty_x} \leq C_m e^{-\frac{\nu(v)^2}{2}} \|w_\beta f\|_{L^p_v L^\infty_x L^\infty_x} . \quad (4.31) \]

**Case 3.** \(|v| \leq N, |\eta| \leq 2N, |\xi| \leq 3N, s - s_1 \leq 1\). Since \(e^{-\nu(\eta)(s-s_1)} \leq 1\) and \(\int_0^t e^{-\nu(v)(t-s)} ds \leq \frac{1}{\nu(v)}\), one has that

\[ J_{22}(t, x, v) = \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |w_{\beta, \nu}(v, \eta)l_{\beta, \nu}(\eta, \xi)| \]

\[ \times \int_{s_1}^{s-s_1} e^{-\nu(\eta)(s-s_1)} \left| (w_\beta f)(s_1, x_1 - \eta(s - s_1), \xi) \right| ds_1 d\eta d\xi ds \]

\[ \leq \frac{1}{\nu(v)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |w_{\beta, \nu}(v, \eta)l_{\beta, \nu}(\eta, \xi)| \|w_\beta f(\xi)\|_{L^p_v L^\infty_x L^\infty_x} d\eta d\xi \]

\[ \leq C_{m, N, \lambda} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |l(v, \eta)| \left| \frac{1}{\nu(\eta)} \right| \|w_\beta f(\xi)\|_{L^p_v L^\infty_x L^\infty_x} d\eta d\xi \right)^{\frac{1}{p}} , \quad (4.32) \]

which yields

\[ \|J_{22}\|_{L^p_v L^\infty_x L^\infty_x} \leq C_{m, N, \lambda} \|w_\beta f\|_{L^p_v L^\infty_x L^\infty_x} . \quad (4.33) \]

**Case 4.** \(|v| \leq N, |\eta| \leq 2N, |\xi| \leq 3N, s - s_1 \geq 1\).

Recall from (2.7) that

\[ |l(v, \eta)| \leq \frac{C_{\gamma}}{|v - \eta|^{\gamma + 1}} e^{-\nu(v)|\eta|^2} e^{-\frac{|\eta|^2}{16|\nu(v)\eta|^2}} + C|v - \eta|^\gamma e^{-\frac{|\eta|^2}{16}} e^{-\frac{|\eta|^2}{16}} . \]

Since \(p > 3/(3 + \gamma), p' \gamma > -3\) and \(\frac{3(\gamma + 1)}{\gamma + 2} < 3\), then \(\sup_{v \in \mathbb{R}^3} \left| \int_{\mathbb{R}^3} |w_{\beta, \nu}(v, \eta)|^{p'} d\eta \right|^{\frac{1}{p'}} < \infty.\)
We can approximate $l_{w,\beta}$ by a smooth function $l_N$ with compact support such that

\[
\sup_{|v| \leq 3N} \left| \frac{d}{d\eta} \right|_{L^p}^{L^p} \leq \frac{C_m}{N^{10}} \tag{4.34}
\]

We can rewrite $l_{w,\beta}(v,\eta)l_{w,\beta}(\eta,\xi) = (l_{w,\beta}(v,\eta) - l_N(v,\eta)) l_{w,\beta}(\eta,\xi) + (l_{w,\beta}(\eta,\xi) - l_N(\eta,\xi)) l_N(v,\eta) + l_N(v,\eta)l_N(\eta,\xi)$. A direct calculation shows that

\[
J_{22}(t, x, v) \leq \int_0^t e^{-\nu(v)(t-s)} \int_{|\eta| \leq 2N, |\xi| \leq 3N} |l_{w,\beta}(v,\eta) - l_N(v,\eta)| |l_{w,\beta}(\eta,\xi)|
\times \int_0^{s-\lambda} e^{-\nu(\eta)(s-s_1)} (w_\beta f)(s_1, x_1 - \eta(s-s_1), \xi) \, ds_1 \, \eta d\xi ds.
\]

Splitting the right-hand side above into three parts, we have

\[
J_{22}(t, x, v) \leq \int_0^t e^{-\nu(v)(t-s)} \int_{R^3} |l_{w,\beta}(v,\eta) - l_N(v,\eta)| |l_{w,\beta}(\eta,\xi)|
\times \int_0^{s-\lambda} e^{-\nu(\eta)(s-s_1)} ds_1 \| (w_\beta f)(\xi) \|_{L^p_x L^\infty_t} \, \eta d\xi ds
+ \int_0^t e^{-\nu(v)(t-s)} \int_{R^3} |l_{w,\beta}(\eta,\xi) - l_N(\eta,\xi)| |l_N(v,\eta)|
\times \int_0^{s-\lambda} e^{-\nu(\eta)(s-s_1)} ds_1 \| (w_\beta f)(\xi) \|_{L^p_x L^\infty_t} \, \eta d\xi ds
+ \int_0^t e^{-\nu(v)(t-s)} \int_{|\eta| \leq 2N, |\xi| \leq 3N} |l_N(v,\eta)l_N(\eta,\xi)|
\times \int_0^{s-\lambda} e^{-\nu(\eta)(s-s_1)} \| (w_\beta f)(s_1, x_1 - \eta(s-s_1), \xi) \|_{L^p_x L^\infty_t} ds_1 \, \eta d\xi ds. \tag{4.35}
\]

For the first two terms on the right-hand side of (4.35), we first integrate with respect to $s_1$, then integrate with respect to $s$ to get

\[
J_{22}(t, x, v) \leq \frac{1}{\nu(v)} \int_{|\eta| \leq 2N, |\xi| \leq 3N} |l_{w,\beta}(v,\eta) - l_N(v,\eta)| \| (w_\beta f)(\xi) \|_{L^p_x L^\infty_t} \, \eta d\xi
+ \frac{1}{\nu(v)} \int_{|\eta| \leq 2N, |\xi| \leq 3N} |l_{w,\beta}(\eta,\xi) - l_N(\eta,\xi)| \| l_N(v,\eta) \|_{L^p_x L^\infty_t} \, \eta d\xi
+ \int_0^t e^{-\nu(v)(t-s)} \int_{|\eta| \leq 2N, |\xi| \leq 3N} |l_N(v,\eta)l_N(\eta,\xi)|
\times \int_0^{s-\lambda} e^{-\nu(\eta)(s-s_1)} (w_\beta f)(s_1, x_1 - \eta(s-s_1), \xi) \, ds_1 \, \eta d\xi ds
= J_{221}(t, x, v) + J_{222}(t, x, v) + J_{223}(t, x, v). \tag{4.36}
\]

By the approximation (4.34) and the fact that

\[
\frac{1}{\nu(v)} \frac{1}{\nu(\eta)} \leq CN^6
\]
for $|v| \leq N$, $|\eta| \leq 2N$, we yield our estimate for the first term on the right-hand side of \((4.36)\),

\[
J_{221}(t, x, v) = \frac{1}{\nu(v)} \iint_{|\eta| \leq 2N, |\xi| \leq 3N} |l_{w_{\beta}}(v, \eta) - l_N(v, \eta)| \left| \frac{1}{\nu(\eta)} \| (w_{\beta}(\eta, \xi)) \|_{L^2 \rightarrow L^\infty} \right| d\eta d\xi
\]

\[
\leq C N^6 \iint_{|\eta| \leq 2N, |\xi| \leq 3N} |l_{w_{\beta}}(v, \eta) - l_N(v, \eta)| \left| \frac{1}{\nu(\eta)} \| (w_{\beta}(\eta, \xi)) \|_{L^2 \rightarrow L^\infty} \right| d\eta d\xi
\]

\[
\leq C N^6 \left( \int_{|\eta| \leq 2N} \int_{|\xi| \leq 3N} |l_{w_{\beta}}(v, \eta) - l_N(v, \eta)| \| (w_{\beta}(\eta, \xi)) \|_{L^2 \rightarrow L^\infty} d\eta d\xi \right)^{\frac{1}{p}}
\]

\[
\times \left( \int_{|\xi| \leq 3N} \int_{|\eta| \leq 2N} \| (w_{\beta}(\eta, \xi)) \|_{L^2 \rightarrow L^\infty} d\eta d\xi \right)^{\frac{1}{q}}
\]

\[
\leq C N^9 \left( \int_{|\eta| \leq 2N} \int_{|\xi| \leq 3N} |l_{w_{\beta}}(v, \eta) - l_N(v, \eta)| \| (w_{\beta}(\eta, \xi)) \|_{L^2 \rightarrow L^\infty} d\eta d\xi \right)^{\frac{1}{p}}
\]

\[
\leq C_m \| w_{\beta} \|_{L^\infty \rightarrow L^\infty}.
\]

Similarly we have

\[
J_{222}(t, x, v) \leq C_m \| w_{\beta} \|_{L^\infty \rightarrow L^\infty}.
\]

We turn to $J_{223}$ now, denoting $\nu_N = \inf_{|v| \leq 3N} |\nu(v)| > 0$, it holds that

\[
J_{223} = \int_0^t e^{-\nu(v)(t-s)} \int_{|\eta| \leq 2N, |\xi| \leq 3N} |l_N(v, \eta)l_N(\eta, \xi)|
\]

\[
\times \int_0^{s-\lambda} e^{-\nu(\eta)(s-s_1)} (w_{\beta}(\eta))(s_1, x_1 - \eta(s-s_1), \xi) \, ds_1 d\eta d\xi ds
\]

\[
\leq \int_0^t e^{-\nu_N(t-s)} \int_0^{s-\lambda} e^{-\nu_N(s-s_1)} \int_{|\eta| \leq 2N, |\xi| \leq 3N} |l_N(v, \eta)l_N(\eta, \xi)|
\]

\[
\times |(w_{\beta}(\eta))(s_1, x_1 - \eta(s-s_1), \xi) \, ds_1 d\eta d\xi ds
\]

\[
\leq C_{m, N} \int_0^t e^{-\nu_N(t-s)} \int_0^{s-\lambda} e^{-\nu_N(s-s_1)}
\]

\[
\times \int_{|\eta| \leq 2N, |\xi| \leq 3N} |(w_{\beta}(\eta))(s_1, x_1 - \eta(s-s_1), \xi) \, ds_1 d\eta d\xi ds.
\]

We are able to control $J_{223}$ by $\left( \lambda^{-\frac{4}{3}} \sqrt{E(F_0)} + \lambda^{-3} E(F_0) \right)$ in the following way,

\[
\iint_{|\eta| \leq 2N, |\xi| \leq 3N} |(w_{\beta}(\eta))(s_1, x_1 - \eta(s-s_1), \xi) \, d\eta d\xi
\]

\[
\leq C_N \int_{|\eta| \leq 2N, |\xi| \leq 3N} \left( \frac{|F - \mu|}{\sqrt{\mu}} \right) \, (s_1, x_1 - \eta(s-s_1), \xi) \, \chi\{|F(s_1, x_1 - \eta(s-s_1), \xi) - \mu(\xi)\leq \mu(\xi)\} \, d\eta d\xi
\]

\[
+ C_N \int_{|\eta| \leq 2N, |\xi| \leq 3N} |(F - \mu)(s_1, x_1 - \eta(s-s_1), \xi) \, \chi\{|F(s_1, x_1 - \eta(s-s_1), \xi) - \mu(\xi)\geq \mu(\xi)\} \, d\eta d\xi
\]

\[
\leq C_N \frac{1}{(s-s_1)^{\frac{3}{2}}} \left\{ \int_{|\xi| \leq 3N} \left( \frac{|F - \mu|^2}{\mu} \right) \, (s_1, y, \xi) \, \chi\{|F(s_1, y, \xi) - \mu(\xi)|\leq \mu(\xi)\} \, dy d\xi \right\}^{\frac{1}{2}}
\]

\[
+ C_N \frac{1}{(s-s_1)^3} \int_{|\xi| \leq 3N} |(F - \mu)(s_1, y, \xi) \, \chi\{|F(s_1, y, \xi) - \mu(\xi)|\geq \mu(\xi)\} \, dy d\xi.
\]
In the last step above we use the transformation \( y = x_1 - \eta(s-s_1) \). Substituting (4.40) into (4.39) and using Lemma 2.4, we obtain

\[
J_{22} \leq C_{m,N} \left( \lambda^{-\frac{2}{3}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3} \mathcal{E}(F_0) \right)
\]  
(4.41)

Then by (4.37), (4.38), (4.41), one has

\[
\|J_{22}\|_{L^p_t L^q_x} \leq \frac{C_m}{N} \|w_\beta f\|_{L^p_t L^q_x L^\infty_x} + C_{m,N} \left( \lambda^{-\frac{2}{3}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3} \mathcal{E}(F_0) \right).
\]  
(4.42)

In summary of all the four cases, by (4.29), (4.31), (4.33) and (4.42), we obtain

\[
\|J_{22}\|_{L^p_t L^q_x L^\infty_x} \leq \left( C_{m,N} \lambda + \frac{C_m}{N} \right) \|w_\beta f\|_{L^p_t L^q_x L^\infty_x} + C_{m,N} \left( \lambda^{-\frac{2}{3}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3} \mathcal{E}(F_0) \right).
\]  
(4.43)

Combining our estimates on \( J_{20} \), (4.22), \( J_{21} \), (4.25), \( J_{22} \), (4.43), \( J_{23} \), (4.20), one gets that

\[
\|J_2\|_{L^p_t L^q_x L^\infty_x} \leq C_m \|w_\beta f_0\|_{L^p_t L^q_x} + \left( C_m \gamma^{\frac{1}{p}-\frac{1}{p}} + C_{m,N} \lambda + \frac{C_m}{N} \right) \|w_\beta f\|_{L^p_t L^q_x L^\infty_x} + \left( C \|f\|_{L^p_t L^q_x L^\infty_x} \right)
\]
\[
+ C \|w_\beta f_0\|_{L^p_t L^q_x} + C_{m,N} \left( \lambda^{-\frac{2}{3}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3} \mathcal{E}(F_0) \right).
\]  
(4.44)

It follows from (1.1), (2.21), (4.44) that

\[
\|w_\beta f\|_{L^p_t L^q_x L^\infty_x} \leq C_m \|w_\beta f_0\|_{L^p_t L^q_x} + \left( C_m \gamma^{\frac{1}{p}} + C_{m,N} \lambda + \frac{C_m}{N} \right) \|w_\beta f\|_{L^p_t L^q_x L^\infty_x} + \left( C \|f\|_{L^p_t L^q_x L^\infty_x} \right)
\]
\[
+ C \|w_\beta f_0\|_{L^p_t L^q_x} + C_{m,N} \left( \lambda^{-\frac{2}{3}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3} \mathcal{E}(F_0) \right).
\]  
(4.45)

Finally (1.9) holds by first choosing small \( m \), then choosing small \( \lambda \) and large \( N \). \( \square \)

4.3. Smallness of \( \|f\|_{L^p_{T_1} L^q_x L^1_t} \). We also need the following lemma, which implies that no matter how large \( \|w_\beta f_0\|_{L^p_t L^q_x} \) is, we can choose very small \( \mathcal{E}(F_0) \), \( \|w_\beta f_0\|_{L^p_t L^q_x} \) such that \( \|f\|_{L^p_{T_1} L^q_x L^1_t} \) will be small.

**Lemma 4.3.** Let \( \gamma, \beta \) and \( p \) satisfy the assumption in Theorem 1.1 \( 3/(3+\gamma) < q < p \), and \( T_1 \) is the constant given in Theorem 1.1. Then for any \( T > T_1 \), it holds that

\[
\int_{\mathbb{R}^3} |f(t,x,v)| \, dv \leq \int_{\mathbb{R}^3} e^{-\nu(v)}\, |f_0(x,v)| \, dv + \left( C_m \gamma^{\frac{1}{p}} + C\lambda + \frac{C_m}{N} \right) \|w_\beta f\|_{L^p_t L^q_x L^\infty_x} \]
\[
+ C \left( \lambda + \frac{1}{N^{\frac{2}{3}-3}} \right) \|w_\beta f\|_{L^p_t L^q_x L^\infty_x} \]
\[
+ C_N \left( \lambda^{-\frac{2}{3}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3} \mathcal{E}(F_0) \right)^{\frac{3}{2}} \|w_\beta f\|_{L^p_t L^q_x L^\infty_x} \]
\[
+ C_N \left( \lambda^{-\frac{2}{3}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3} \mathcal{E}(F_0) \right)^{\frac{3}{2}} \|w_\beta f\|_{L^p_t L^q_x L^\infty_x};
\]  
(4.45)

for any \((t,x) \in [T_1,T] \times \Omega\), where \( r = p - \frac{p-q}{4q} \).

**Proof.** Let \((t,x) \in [T_1,T] \times \Omega\). Using (1.1), we have

\[
\int_{\mathbb{R}^3} |f(t,x,v)| \, dv \leq \int_{\mathbb{R}^3} e^{-\nu(v)}\, |f_0(x,v)| \, dv + G_1(t,x) + G_2(t,x) + G_3(t,x),
\]  
(4.46)
where

\[ G_1(t, x) := \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} |(K^m f)(s, x - v(t-s), v)| \, dv \, ds \]

\[ G_2(t, x) := \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} |(K^c f)(s, x - v(t-s), v)| \, dv \, ds \]

\[ G_3(t, x) := \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} |\Gamma(f, f)(s, x - v(t-s), v)| \, dv \, ds. \]

Here \( G_1(t, x) \) can be estimated as (4.24). Indeed, by the arguments in (4.12) and our definition for \( \tilde{J}(v) \) in (4.11) and noticing that \( \frac{1}{\nu(v)} e^{-\frac{|v|^2}{2\nu(v)}} \leq C e^{-\frac{|v|^2}{2\nu(v)}} \), one gets that

\[
G_1(t, x) = \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} |(K^m f)(s, x - v(t-s), v)| \, dv \, ds \\
\leq C m^{\frac{1}{p'}} \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} ds \, e^{-\frac{|v|^2}{2\nu(v)}} \tilde{J}(v) \, dv \\
\leq C m^{\frac{1}{p'}} \int_{\mathbb{R}^3} e^{-\frac{|v|^2}{2\nu(v)}} \tilde{J}(v) \, dv \\
\leq C m^{\frac{1}{p'}} \left( \int_{\mathbb{R}^3} e^{-\frac{|v|^2}{2\nu(v)}} \, dv \right)^{\frac{1}{p'}} \left( \int_{\mathbb{R}^3} |\tilde{J}(v)|^p \, dv \right)^{\frac{1}{p'}} \\
\leq C m^{\frac{1}{p'}} \|w_\beta f\|_{L^p_T L^p_x L^p_T}.
\]

Consider \( G_2(t, x) \) in four cases like \( J_{22} \). Recall

\[
G_2(t, x) = \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} |(K^c f)(s, x - v(t-s), v)| \, dv \, ds \\
= \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} \left| \int_{\mathbb{R}^3} l(v, \eta) f(s, x - v(t-s), \eta) \, d\eta \right| \, dv \, ds.
\]

**Case 1.** \( t - \lambda \leq s \leq t \). By similar arguments as in (4.32), we have

\[
G_2(t, x) = \int_{t-\lambda}^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} |(K^c f)(s, x - v(t-s), v)| \, dv \, ds \\
\leq \lambda \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{w_\beta(v)} \left| \int_{\mathbb{R}^3} \frac{w_\beta(v)}{w_\beta(\eta)} \, d\eta \right| \|w_\beta f\|_{L^p_T L^p_x} \, dv \, ds \\
\leq \lambda \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} l(v, \eta) \frac{w_\beta(v)}{w_\beta(\eta)} \, d\eta \right)^{\frac{1}{p'}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{w_\beta(\eta)} \, d\eta \right)^{\frac{1}{p'}} \\
\leq C \lambda \|w_\beta f\|_{L^p_T L^p_x L^p_T}.
\]

**Case 2.** \( |\eta| \geq N \). Recall our assumption that \( \beta > 36 \), \( \frac{1}{\nu(v)w_{\beta/2}(v)} \) is bounded. We can obtain \( 1 \) from our property of \( l(v, \eta) \) in (2.6). Taking the \( L^\infty_T L^\infty_x \) first and integrating with respect to \( s \) like (4.24), it holds that

\[
G_2(t, x) = \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} \left| \int_{|\eta| \geq N} l(v, \eta) f(s, x - v(t-s), \eta) \, d\eta \right| \, dv \, ds \\
\leq \int_{|\eta| \geq N} \int_{\mathbb{R}^3} \frac{1}{\nu(v)w_{\beta/2}(v)} \left| \int_{|\eta| \geq N} l(v, \eta) \frac{w_{\beta/2}(v)}{w_{\beta/2}(\eta)} \, d\eta \right| \|w_{\beta/2} f\|_{L^p_T L^p_x} \, d\eta.
\]
and further one has

\[ G_2(t, x) \leq \frac{C}{N} \int_{|\eta| \geq N} \| (w_{\beta/2} f)(\eta) \|_{L_\infty^p L_\infty^q} d\eta \]

\[ \leq \frac{C}{N} \left( \int_{R^3} \left( \frac{1}{1 + |\eta|^2} \right)^{\frac{\beta}{p'}} d\eta \right) \left( \int_{|\eta| \geq N} \| (w_{\beta/2} f)(\eta) \|_{L_\infty^p L_\infty^q}^\frac{p'}{\beta} d\eta \right) \]

\[ \leq \frac{C}{N} \| w_{\beta/2} f \|_{L_\infty^p L_\infty^q} \cdot 
\]

**Case 3.** \(||\eta| \leq N, |v| \geq 2N\). Then \(|v - \eta| \geq N\), similar as (4.30), by \(e^{-\frac{N^2}{2}} \leq \frac{C}{N}\) and

\[ \int_0^1 e^{-\nu(v)(t-s)} ds \leq \frac{1}{m(v)}, \]

we obtain

\[ G_2(t, x) \leq \int_{|v-\eta| \geq N} \frac{1}{\nu(v) w_{\beta/2}(v)} d(v, \eta) \frac{w_{\beta/2}(v)^{\beta}}{w_{\beta/2}(\eta)^{\beta}} e^{-\frac{N^2}{2\nu(v)}} \frac{d\eta}{\| (w_{\beta/2} f)(\eta) \|_{L_\infty^p L_\infty^q}} d\eta dv \]

\[ \leq \frac{C_m}{N} \int_{R^3} \| (w_{\beta/2} f)(\eta) \|_{L_\infty^p L_\infty^q} d\eta \]

\[ \leq \frac{C_m}{N} \| w_{\beta/2} f \|_{L_\infty^p L_\infty^q}. \] (4.48)

**Case 4.** \(||\eta| \leq N, |v| \leq 2N, 0 \leq s \leq t - \lambda\). Approximate \(l_{w, \lambda}\) by \(l_N\) as (4.34). Using the similar arguments in (4.37), (4.39) and (4.40), one gets that

\[ G_2(t, x) \leq \int_0^{t-\lambda} \int_{R^3} e^{-\nu(v)(t-s)} \left( \int_{R^3} \| l_{w, \lambda}(v, \eta) - l_N(v, \eta) \|_2 \| f(s, x - v(t-s), \eta) \|_2 d\eta \right) dv ds \]

\[ + \int_0^{t-\lambda} \int_{R^3} e^{-\nu(v)(t-s)} \left( \int_{R^3} \| l_N(v, \eta) \|_2 \| (w_{\beta} f)(s, x - v(t-s), \eta) \|_2 d\eta \right) dv ds \]

\[ \leq \frac{C_m}{N} \| w_{\beta} f \|_{L_\infty^p L_\infty^q} + C_m \int_{|\eta| \leq N, |v| \leq 2N} \| (w_{\beta} f)(s, x - v(t-s), \eta) \|_2 d\eta dv \]

\[ \leq \frac{C_m}{N} \| w_{\beta} f \|_{L_\infty^p L_\infty^q} + C_m \lambda \left( \lambda^{-\frac{\beta}{2}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3} \mathcal{E}(F_0) \right). \]

Then \(G_2(t, x)\) satisfies

\[ G_2(t, x) \leq C_m \left( \lambda + \frac{1}{N} \right) \| w_{\beta} f \|_{L_\infty^p L_\infty^q} + C_m \lambda \left( \lambda^{-\frac{\beta}{2}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3} \mathcal{E}(F_0) \right). \] (4.49)

At last we need to bound \(G_3(t, x)\) which can be divided into three parts. Choose \(q\) such that \(3/(3 + \gamma) < q < p\), denote \(x_1 = x - v(t-s)\). Recall

\[ G_3(t, x) = \int_0^t \int_{R^3} e^{-\nu(v)(t-s)} \| \Gamma(f, f)(s, x - v(t-s), v) \|_2 dv ds \]

\[ \leq C \int_0^t \int_{R^3} e^{-\nu(v)(t-s)} \int_{R^3} \int_{S^2} |v - u|^{\gamma} e^{-\frac{|u|^2}{4}} \]

\[ \times \left( |f(t, x_1, u') f(t, x_1, v')| + |f(t, x_1, u) f(t, x_1, v)| \right) d\omega du dv ds. \] (4.50)

**Case 1.** \(-t - \lambda \leq s \leq t\). It is straightforward to see that

\[ |f(t, x_1, u') f(t, x_1, v')| + |f(t, x_1, u) f(t, x_1, v)| \leq \| f(u') f(v') \|_{L_\infty^p L_\infty^q} + \| f(u) f(v) \|_{L_\infty^p L_\infty^q}. \]

We now have

\[ G_3(t, x) \leq C \int_{-t-\lambda}^t \int_{R^3} e^{-\nu(v)(t-s)} \int_{R^3} \int_{S^2} |v - u|^{\gamma} e^{-\frac{|u|^2}{4}} \]

\[ \times \left( \| f(u') f(v') \|_{L_\infty^p L_\infty^q} + \| f(u) f(v) \|_{L_\infty^p L_\infty^q} \right) d\omega du dv ds, \]
so it hold that
\[
G_3(t, x) \leq C \lambda \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^\gamma e^{-\frac{|v|^2}{4}} \left( \|f(u')f(v')\|_{L^\infty_{x}L^\infty_v} + \|f(u)f(v)\|_{L^\infty_{x}L^\infty_v} \right) d\omega dudv.
\]
(4.51)

We observe that \( w_\beta(v) \leq C \omega_\beta(u')w_\beta(v') \) and
\[
\|f(u')f(v')\|_{L^\infty_{x}L^\infty_v} + \|f(u)f(v)\|_{L^\infty_{x}L^\infty_v}
\leq \frac{1}{w_\beta/2(v)} \left( Cw_\beta/2(v)\|f(u')f(v')\|_{L^\infty_{x}L^\infty_v} + \|f(u)(w_\beta/2f)(v)\|_{L^\infty_{x}L^\infty_v} \right)
\leq \frac{1}{w_\beta/2(v)} \left( Cw_\beta/2(u')w_\beta/2(v')\|f(u')f(v')\|_{L^\infty_{x}L^\infty_v} + \|f(u)(w_\beta/2f)(v)\|_{L^\infty_{x}L^\infty_v} \right)
\leq \frac{C}{w_\beta/2(v)} \left( \|f(w_\beta/2f)(u')w_\beta/2(v')\|_{L^\infty_{x}L^\infty_v} + \|f(w_\beta/2f)(u')(w_\beta/2f)(v)\|_{L^\infty_{x}L^\infty_v} \right).
\]

Applying the above inequality to (4.51) and using Hölder’s inequality as (4.4), by \( dudv = dv'dv' \), we yield
\[
G_3(t, x) \leq C \lambda \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{|v - u|^\gamma}{w_\beta/2(v)} e^{-\frac{|v|^2}{4}} \left( \|f(w_\beta/2f)(u')(w_\beta/2f)(v')\|_{L^\infty_{x}L^\infty_v}
+ \|f(w_\beta/2f)(u)(w_\beta/2f)(v)\|_{L^\infty_{x}L^\infty_v} \right) d\omega dudv
\leq C \lambda \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \|f(w_\beta/2f)(u')(w_\beta/2f)(v')\|_{L^\infty_{x}L^\infty_v} d\omega dudv \right)^{\frac{1}{q}}
+ C \lambda \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \|f(w_\beta/2f)(u)(w_\beta/2f)(v)\|_{L^\infty_{x}L^\infty_v} d\omega dudv \right)^{\frac{1}{q'}}
\leq C \lambda \|f\|_{L_{x}^qL_{v}^\infty}^2 .
\]
(4.52)

Case 2. \(|u| \geq N \) or \(|v| \geq N \). Set \( q' = \frac{q}{1 - q} \). It follows from similar arguments as (4.48) and (4.52) that
\[
G_3(t, x) \leq C \int_{|u| \geq N} \int_{|v| \geq N} \int_{\mathbb{S}^2} \frac{|v - u|^\gamma}{\nu(v)w_\beta/2(v)} e^{-\frac{|v|^2}{4}} \left( \|f(w_\beta/2f)(u')(w_\beta/2f)(v')\|_{L^\infty_{x}L^\infty_v}
+ \|f(w_\beta/2f)(u)(w_\beta/2f)(v)\|_{L^\infty_{x}L^\infty_v} \right) d\omega dudv
\leq C \left( \int_{|u| \geq N} \int_{|v| \geq N} \int_{\mathbb{S}^2} \|f(w_\beta/2f)(u')(w_\beta/2f)(v')\|_{L^\infty_{x}L^\infty_v} d\omega dudv \right)^{\frac{1}{q'}}
\times \left( \int_{|u| \geq N} \int_{|v| \geq N} \frac{|v - u|^\gamma e^{-\frac{|v|^2}{4}}}{\nu(v)w_\beta/2(v)} d\omega dudv \right)^{\frac{1}{q'}}. 
\]
(4.53)

Consider \(|v| \geq N \) first. We note that \( q > 3/(3 + \gamma) \), \( \gamma q' > -3 \), which yields
\[
\int_{|v| \geq N} \frac{|v - u|^\gamma e^{-\frac{|v|^2}{4}}}{\nu(v)w_\beta/2(v)} dudv \leq C \int_{|u| \geq N} \frac{1}{(1 + |v|)^{\beta q/2}} dv \leq \frac{C}{N^{\frac{3q}{3q' - 3}}}. 
\]
(4.54)

Then we turn to \(|u| \geq N \). Since \( e^{-\frac{|v|^2}{4}} \) can be controlled by \( \frac{1}{(1 + |u|)^\alpha} \) for any \( \alpha > 0 \),
\[
\int_{|u| \geq N} \frac{|v - u|^\gamma e^{-\frac{|v|^2}{4}}}{\nu(v)w_\beta/2(v)} dudv \leq C \int_{|u| \geq N} (1 + |u|)^{\gamma q'} e^{-\frac{|v|^2}{4}} dv \leq \frac{C}{N^{\frac{3q'}{3q' - 3}}}. 
\]
(4.55)
Hence after taking $L^p_t L^q_x$ norm, by (1.52), (1.54), (1.55) and the assumption that $\beta > 36 > -2\gamma$, we have
\begin{equation}
G_3(t, x) \leq \frac{C}{N^{2-\frac{2\gamma}{\beta}}} \left\| w^\beta f \right\|_{L^p_t L^q_x}^2 \leq \frac{C}{N^{2-\frac{2\gamma}{\beta}}} \left\| w^\beta f \right\|_{L^p_t L^q_x}^2.
\end{equation}

Case 3. If $|u| \leq N$ and $|v| \leq N$, $0 \leq s \leq t - \lambda$, $x_1 = x - v(t - s)$. Our first estimate (1.50) for $G_3(t, x)$ shows that
\begin{align*}
G_3(t, x) &\leq \int_0^{t-\lambda} \int_{|v| \leq N} e^{-\nu v(t-s)} \left( \int_{|u| \leq N} \int_{\mathbb{R}^2} |v - u|^{-\frac{\gamma}{1+2\nu}} |f(s, x_1, u)f(s, x_1, v)| \, du \right) \, dv \, ds \\
&\quad + \int_0^{t-\lambda} \int_{|v| \leq N} e^{-\nu v(t-s)} \left( \int_{|u| \leq N} \int_{\mathbb{R}^2} |v - u|^{-\frac{\gamma}{1+2\nu}} |f(s, x_1, u)f(s, x_1, v)| \, du \right) \, dv \, ds \\
&= G_{31}(t, x) + G_{32}(t, x).
\end{align*}

We focus on $G_{31}(t, x)$ first, denote $\nu_N = \inf_{|v| \leq 3N} |\nu(v)| > 0$. It follows from the similar arguments in (1.3) and (1.5) that
\begin{align*}
G_{31}(t, x) &\leq \int_0^{t-\lambda} \int_{|v| \leq N} e^{-\nu_N(t-s)} \left( \int_{|u| \leq N} \int_{\mathbb{R}^2} |v - u|^{-\frac{\gamma}{1+2\nu}} |f(s, x_1, u)f(s, x_1, v)| \, du \right) \, dv \, ds \\
&\quad \leq C \int_0^{t-\lambda} e^{-\nu_N(t-s)} \left( \int_{|u| \leq N, |v| \leq N} |f(s, x_1, u)|^\varphi |f(s, x_1, v)|^{\varphi} \, du \right) \, dv \, ds \\
&\quad \leq C \| w^\beta f \|_{L^p_t L^q_x}^2 \int_0^{t-\lambda} e^{-\nu_N(t-s)} \int_{|u| \leq N, |v| \leq N} |f(s, x_1, u)| |f(s, x_1, v)| \, du \, dv \, ds.
\end{align*}

Also by (4.40) and Lemma 2.3 using Hölder’s inequality repeatedly, we obtain
\begin{align*}
&\int \int_{|u| \leq N, |v| \leq N} |f(s, x - v(t-s), u)| |f(s, x - v(t-s), v)| \, du \, dv \\
&\quad \leq \left( \int \int_{|u| \leq N, |v| \leq N} |f(s, x - v(t-s), u)| \, du \right)^\frac{1}{\varphi} \\
&\quad \times \left( \int \int_{|u| \leq N, |v| \leq N} \|f(u)\|_{L^p_t L^q_x} \|f(v)\|_{L^p_t L^q_x}^p \, du \right)^{\frac{1}{p}} \\
&\quad \leq C_N \| w^\beta f \|_{L^p_t L^q_x}^2 \left( \int_{|u| \leq N} \|f(u)\|_{L^p_t L^q_x} \, du \right)^\frac{1}{\varphi} \left( \lambda^{\frac{2}{\varphi}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3}\mathcal{E}(F_0) \right)^\frac{1}{p} \\
&\quad \leq C_N \| w^\beta f \|_{L^p_t L^q_x}^{1 + \frac{2}{\varphi}} \left( \lambda^{\frac{2}{\varphi}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3}\mathcal{E}(F_0) \right)^\frac{1}{p}.
\end{align*}

Thus, after substituting the above inequality into (4.57) and integrating with respect to $s$, we have
\begin{equation}
G_{31}(t, x) \leq C_N \| w^\beta f \|_{L^p_t L^q_x}^{1 + \frac{2}{\varphi}} \left( \lambda^{\frac{2}{\varphi}} \sqrt{\mathcal{E}(F_0)} + \lambda^{-3}\mathcal{E}(F_0) \right)^\frac{1}{p}.
\end{equation}

Finally we turn to $G_{32}(t, x)$, as how we treat $G_{31}$ in (4.57),
\begin{equation}
G_{32}(t, x) \leq C \int_0^{t-\lambda} e^{-\nu_N(t-s)} \left( \int \int_{|u| \leq N, |v| \leq N} \int_{\mathbb{R}^2} e^{-\frac{|u|^2}{2\nu}} |f(s, x_1, u')|^\varphi |f(s, x_1, v')|^\varphi \, du \, dv \right) \, ds.
\end{equation}
Notice that differently from (4.57), this time we keep $e^{-\frac{|\omega|^2}{2}}$ inside the integral. A similar argument as (4.6) shows that

\[
\left( \iint_{\{|u| \leq N, |v| \leq N\}} e^{-\frac{|\omega|^2}{2}} |f(s, x_1, u')|^q |f(s, x_1, v')|^q d\omega dv \right)^\frac{1}{q} 
\leq \left( \iint_{\{|u| \leq N, |v| \leq N\}} e^{-\frac{|\omega|^2}{2}} |f(s, x_1, v')|^q d\omega dv \right)^\frac{1}{q} \cdot \left( \iint_{\{|u| \leq N, |v| \leq N\}} e^{-\frac{|\omega|^2}{2}} |f(s, x_1, u')|^q d\omega dv \right)^\frac{1}{q} 
\leq C \|w_\beta f\|_{L_t^p L_x^q}^{1+\frac{q}{p}} \left( \int_{\{|u| \leq N, |v| \leq N\}} e^{-\frac{|\omega|^2}{2}} |f(s, x_1, v')|^q d\omega dv \right)^\frac{1}{q}.
\]

Since we have $v' = v + [(u - v) \cdot \omega, \omega, |v'| \leq 3N, x_1 = x - v(t - s)$, it holds that

\[
\iint_{\{|u| \leq N, |v| \leq N\}} e^{-\frac{|\omega|^2}{2}} |f(s, x_1, v')|^q d\omega dv 
\leq C_N \left( \iint_{\{|u| \leq 3N, |v| \leq N\}} |f(s, x_1, \eta)| \left( \int_{|\eta - v|^2 \geq \frac{1}{4} |\eta|^2} e^{-\frac{|\omega|^2}{2}} d\omega \right) d\eta dv \right)^\frac{1}{q} + C_N \left( \iint_{\{|u| \leq 3N, |v| \leq N\}} |f(s, x_1, \eta)| \left( \int_{|\eta - v|^2 \leq \frac{1}{4} |\eta|^2} e^{-\frac{|\omega|^2}{2}} d\omega \right) d\eta dv \right)^\frac{1}{q} 
\leq C_N \left( \lambda^{-\frac{q}{2}} \mathcal{E}(F_0) + \lambda^{-3} \mathcal{E}(F_0) \right)^\frac{1}{q}.
\]

Together with (4.59), we get

\[
G_{32}(t, x) \leq C_N \left( \lambda^{-\frac{q}{2}} \mathcal{E}(F_0) + \lambda^{-3} \mathcal{E}(F_0) \right)^\frac{1}{q} \|w_\beta f\|_{L_t^p L_x^q}^{1+\frac{q}{p}}. \tag{4.60}
\]

From (4.58) and (4.60), for Case 3, we have

\[
G_3(t, x) \leq C_N \left( \lambda^{-\frac{q}{2}} \mathcal{E}(F_0) + \lambda^{-3} \mathcal{E}(F_0) \right)^\frac{1}{q} \|w_\beta f\|_{L_t^p L_x^q}^{1+\frac{q}{p}} + C_N \left( \lambda^{-\frac{q}{2}} \mathcal{E}(F_0) + \lambda^{-3} \mathcal{E}(F_0) \right)^\frac{1}{q} \|w_\beta f\|_{L_t^p L_x^q}^{1+\frac{q}{p}}. \tag{4.61}
\]

Using (4.32), (4.59), (4.61) we obtain the estimate for $G_3(t, x)$ that

\[
G_3(t, x) \leq C \left( \lambda + \frac{1}{N^{\frac{q}{2} - 3}} \right) \|w_\beta f\|_{L_t^p L_x^q}^2 + C_N \left( \lambda^{-\frac{q}{2}} \mathcal{E}(F_0) + \lambda^{-3} \mathcal{E}(F_0) \right)^\frac{1}{q} \|w_\beta f\|_{L_t^p L_x^q}^{1+\frac{q}{p}} + C_N \left( \lambda^{-\frac{q}{2}} \mathcal{E}(F_0) + \lambda^{-3} \mathcal{E}(F_0) \right)^\frac{1}{q} \|w_\beta f\|_{L_t^p L_x^q}^{1+\frac{q}{p}}. \tag{4.62}
\]

According to (4.40), (4.37), (4.29), (4.62), the estimate (4.35) follows. This completes the proof of Lemma 4.13. \qed

4.4. Global existence. With all the discussions above, we can prove Theorem 1.2 now. Including the assumptions of Theorem 1.1 and Theorem 1.2 we make the a priori assumption

\[
\|w_\beta f\|_{L_t^p L_x^q} \leq 2A = 2C_2 \left( M^2 + \sqrt{\mathcal{E}(F_0)} + \mathcal{E}(F_0) \right),
\]
where $M > 1$, $\|w_\beta f_0\|_{L^p_\nu L^\infty_\nu} < M$ and $C_2$ is defined in Lemma 4.2. Then by Lemma 4.2 one gets that

$$
\|w_\beta f\|_{L^p_\nu L^3_\nu L^\infty_\nu} \leq A + C_2 (2A)^{\frac{1}{\lambda} + \frac{1}{N^2-3}} \|f\|_{L^p_\nu L^3_\nu L^\infty_\nu} + C_2 (2A)^{\frac{\lambda}{\lambda} + \frac{1}{N} + \frac{3}{N^2-3}} \|f\|_{L^p_\nu L^3_\nu L^\infty_\nu}.
$$

It follows from Lemma 4.3 that

$$
\int_{\mathbb{R}^3} |f(t,x,v)| dv \leq \int_{\mathbb{R}^3} e^{-\nu(t) t} |f_0(x - vt,v)| dv + \left( Cm^{\gamma + \frac{\beta}{\gamma}} + CL + Cn \right) (2A) \\
+ C \left( \lambda + \frac{1}{N^2-3} \right) (2A)^2 \\
+ C N \left( \frac{\lambda}{\lambda} \sqrt{E(F_0)} + \lambda^{-3} E(F_0) \right) (2A)^{\frac{1}{\lambda} + \frac{2\gamma}{\gamma(\beta-1)}} \\
+ C N \left( \frac{\lambda}{\lambda} \sqrt{E(F_0)} + \lambda^{-3} E(F_0) \right) (2A)^{\frac{\gamma}{\gamma} + \frac{1}{\gamma}}.
$$

Also recall from Theorem 1.1 that $T_1 = \frac{1}{6\epsilon_1 (1 + \|w_\beta f_0\|_{L^p_\nu L^\infty_\nu})} > \frac{C}{M}$. We consider the case that $t \geq T_1$. If $\Omega = \mathbb{R}^3$,

$$
\int_{\mathbb{R}^3} e^{-\nu(t) t} |f_0(x - vt,v)| dv \leq \frac{1}{T_1} \|f_0\|_{L^1_\nu L^\infty_\nu} \leq CM^3 \|f_0\|_{L^1_\nu L^\infty_\nu}.
$$

If $\Omega = \mathbb{T}^3$, by $\int_{\{\epsilon \leq M_1\}} |f_0(x - vt,v)| dv \leq \int_{\{\epsilon \geq M_1\}} |f_0(x - vt,v)| dv + \int_{\{\epsilon \leq M_1\}} |f_0(x - vt,v)| dv$

$$
\leq \int_{\{\epsilon \geq M_1\}} |f_0(x - vt,v)| dv + C \left\{ M_1^3 \|f_0\|_{L^1_\nu L^\infty_\nu} + M^3 \|f_0\|_{L^1_\nu L^\infty_\nu} \right\}
$$

$$
\leq M_1^{\frac{3}{\gamma} + \beta} \|w_\beta f_0\|_{L^p_\nu L^3_\nu L^\infty_\nu} + CM_1^3 \|f_0\|_{L^1_\nu L^\infty_\nu} + CM^3 \|f_0\|_{L^1_\nu L^\infty_\nu}.
$$

By choosing $M_1 = \left( \frac{\|w_\beta f_0\|_{L^p_\nu L^3_\nu L^\infty_\nu}}{\|f_0\|_{L^1_\nu L^\infty_\nu}} \right)^{3 + \frac{3}{\gamma} + \beta}$, we have

$$
\int_{\mathbb{R}^3} e^{-\nu(t) t} |f_0(x - vt,v)| dv \leq C \|w_\beta f_0\|_{L^p_\nu L^3_\nu L^\infty_\nu} \|f_0\|_{L^1_\nu L^\infty_\nu} \|f_0\|_{L^1_\nu L^\infty_\nu} \\
+ C \left\{ \frac{1}{\gamma} \|f_0\|_{L^1_\nu L^\infty_\nu} \right\} + CM^3 \|f_0\|_{L^1_\nu L^\infty_\nu}.
$$

Then we can first choose $\epsilon$, $\lambda$, $N$ large, and then let $\max\{E(F_0), \|f_0\|_{L^1_\nu L^\infty_\nu}\} \leq \epsilon$ for some $\epsilon$ which depends on $\beta$, $\gamma$, $M$ such that

$$
2C_2 (2A)^{\frac{\beta}{\gamma} + \frac{1}{\gamma(\beta-1)}} \|f\|_{L^p_\nu L^3_\nu L^\infty_\nu} + 2C_2 (2A)^{\frac{\gamma}{\gamma} + \frac{1}{\gamma}} \|f\|_{L^p_\nu L^3_\nu L^\infty_\nu} \leq \frac{1}{2}.
$$

Using (4.63), (4.64), we directly obtain that

$$
\|w_\beta f\|_{L^p_\nu L^3_\nu L^\infty_\nu} \leq \frac{3}{2} A.
$$

We have closed the a priori assumption. Naturally the estimate (1.11) holds. Hence, the proof of Theorem 1.2 is finished.

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