SUPERSYMMETRIC YANG-MILLS THEORY
ON A FOUR-MANIFOLD

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ABSTRACT

By exploiting standard facts about $N = 1$ and $N = 2$ supersymmetric Yang-Mills theory, the Donaldson invariants of four-manifolds that admit a Kahler metric can be computed. The results are in agreement with available mathematical computations, and provide a powerful check on the standard claims about supersymmetric Yang-Mills theory.

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1. Introduction

In four-dimensional supersymmetric Yang-Mills theory formulated on flat $\mathbb{R}^4$, certain correlation functions are independent of spatial separation and are hence effectively computable by going to short distances. This is the basis for one of the most fruitful techniques for studying dynamics of those theories [1,2], and gains even more power when combined with analysis of instanton corrections to superpotentials [3].

The facts used to establish such results on flat $\mathbb{R}^4$ also have global manifestations, especially in the case of $N = 2$ and $N = 4$ super Yang-Mills theory. For example, when $N = 2$ super Yang-Mills theory is formulated (in a suitable twisted fashion which we will recall) on an arbitrary four-manifold $M$, one can define certain correlation functions which are independent of the metric of $M$ [4]. These correlation functions are thus, crudely speaking, topological invariants of $M$. Actually, since the Yang-Mills action contains derivatives, its formulation requires a differentiable structure on $M$. Hence, the special correlation functions are invariants of $M$ as a differentiable manifold. In fact [4,5], they are equivalent to the celebrated Donaldson invariants [6,7], which are the basis of much of what is known about smooth four-manifolds.

The realization of the Donaldson invariants as correlation functions of a physical theory leads one to wonder whether physical methods could be used to calculate them, or conversely, whether topological computations can be used to constrain the behavior of physical models. In this paper, I will present some results in this direction. To be specific, I will show how standard properties of $N = 1$ supersymmetric Yang-Mills theory can be used to determine the Donaldson invariants of those four manifolds that admit a Kahler metric. This is an area that has been much investigated (for reviews see [8,9]), and the formulas that we will obtain agree with the known mathematical formulas. In fact, comparison with some of those formulas – especially the formulas of O’Grady [10] for K3 surfaces and of Kronheimer and Mrowka [11] for so-called four-manifolds of simple type – helped
considerably in working out the argument presented in this paper.

As in a recent study of the Verlinde algebra and the cohomology of the Grassmannian [12], the basic idea will be to exploit the way a topological theory arises as part of a physical theory. The topological correlation functions of interest will arise as specific correlation functions in a model (essentially $N = 1$ supersymmetric Yang-Mills theory) which has asymptotic freedom, chiral symmetry breaking, and a dynamically generated mass gap; the mass gap in particular severely constrains the topological correlation functions.

To make the subject as accessible as possible, the paper will be organized as follows. The problem and its solution are described in a relatively non-technical way in §2. Many technical details are deferred for §3.

There have been a couple of other recent papers considering supersymmetric Yang-Mills theory on Kahler manifolds. Park [13] described many of the relevant general features and also adapted the two-dimensional construction of [14] to four dimensions – where unfortunately this approach does not immediately give much computational power. Johansen [15] formally wrote down the twisting of $N = 1$ models on Kahler manifolds.

2. Outline Of The Argument

First let us recall some of the basic properties of $N = 1$ and $N = 2$ super Yang-Mills theory in four dimensions, with gauge group $G$. In the minimal $N = 1$ theory, without extra matter multiplets, the fields are a gauge field $A_m$ and a gluino field $\lambda$ (a Majorana spinor with values in the adjoint representation). The Lagrangian is

$$L = \frac{1}{e^2} \int d^4 x \sqrt{g} \left( -\frac{1}{4} F_{mn}F^{mn} - \overline{\lambda} i\sigma_{\alpha\dot{\alpha}}^m i D_m \lambda^\alpha \right). \quad (2.1)$$

Our conventions are as in Wess and Bagger [16] and are explained at the beginning
of §3. Classically there is a $U(1)$ $R$ symmetry

$$\lambda_\alpha \rightarrow e^{i\gamma} \lambda_\alpha$$
$$\lambda_{\dot{\alpha}} \rightarrow e^{-i\gamma} \lambda_{\dot{\alpha}}.$$ (2.2)

Instantons reduce the symmetry group to a subgroup $\mathbb{Z}_{2h}$ of $U(1)$, where $h$ is the dual Coxeter number of $G$; for $G = SU(2)$, the main case that we will consider for illustration, the symmetry group is $\mathbb{Z}_4$.

This theory is asymptotically free, and its key properties are believed to be those of conventional QCD: confinement, a mass gap, and spontaneous chiral symmetry breaking. The chiral symmetry in question is the finite group $\mathbb{Z}_{2h}$ just described. The standard conjecture is that this symmetry group is broken down to a $\mathbb{Z}_2$ subgroup, whose unique non-trivial element is the transformation

$$\lambda_\alpha \rightarrow -\lambda_\alpha, \quad \lambda_{\dot{\alpha}} \rightarrow -\lambda_{\dot{\alpha}}.$$ (2.3)

This symmetry breaking produces an $h$-fold degeneracy of the vacuum, which is believed to be the only degeneracy.

In fact, the symmetry in (2.3) is equivalent to a $2\pi$ rotation in space-time, and so cannot be spontaneously broken if rotation invariance is unbroken. On the other hand, additional unbroken symmetries would forbid fermion masses. The standard conjectures stated in the last paragraph are thus in keeping with the experience of ordinary QCD: in strongly coupled theories of gauge bosons and fermions only, the unbroken symmetry group is a maximal subgroup of the global symmetry group that allows fermion masses.

To this minimal $N = 1$ theory, it is possible to add matter fields in the form of “chiral superfields.” Such a superfield is a bose-fermi pair $\Phi = (\phi, \psi)$, of spin $(0, 1/2)$, and transforming in an arbitrary representation of the gauge group. In addition to the gauge interactions, the chiral superfields can have additional interactions governed by a “superpotential.” The only special case of the superpotential
that will be important in this paper is the following: chiral superfields \( \Phi \) in a real representation of \( G \) can have a mass term
\[
\int d^4 x d^2 \theta m \Phi^2 + \text{h.c.}
\] (2.4)
compatible with supersymmetry.

An important real representation is the adjoint representation. If one adds to the \( N = 1 \) theory a chiral superfield \( \Phi \) in the adjoint representation, then the minimally coupled gauge theory (with vanishing superpotential) in fact has \( N = 2 \) supersymmetry. The Lagrangian is written at the beginning of §3. It is possible to add a bare mass term as in (2.4) for the \( \Phi \) multiplet, breaking \( N = 2 \) supersymmetry down to \( N = 1 \) supersymmetry.

Though the \( N = 2 \) theory is still asymptotically free, its behavior is quite different from that of the minimal \( N = 1 \) theory. The chiral symmetry group is now, at the classical level, a group \( U(2) \) acting on the pair \( (\lambda, \psi) \). Instantons break this to a subgroup whose connected component is isomorphic to \( SU(2) \); we will call this internal \( SU(2) \) symmetry group \( SU(2)_I \).

In contrast to what one might guess from the behavior of strongly coupled gauge theories without scalars, it is believed that \( SU(2)_I \) is not spontaneously broken and that there is no mass gap. In fact, this follows from the form of the classical potential
\[
V = \text{Tr} [\phi, \bar{\phi}]^2,
\] (2.5)
with \( \phi \) a complex scalar in the adjoint representation. This potential has “flat directions,” with \( \langle \phi \rangle \neq 0 \), breaking the gauge symmetry down to \( U(1)^r \) (\( r \) being the rank of \( G \)), and leaving a weakly coupled abelian theory, in which all charged fields are massive. This behavior holds in the quantum theory at least in the weakly coupled region of large \( \langle \phi \rangle \), since \( N = 2 \) supersymmetry permits no perturbations that would lift the vacuum degeneracy in the flat directions. The instanton contributions in the weakly coupled theory for large \( \langle \phi \rangle \) have been analyzed by Seiberg [17].
The goal in what follows is to use facts about the topology of four dimensions to retrieve the conclusion just stated that the minimal $N = 2$ theory does not have a mass gap, and conversely to use the mass gap and related properties of the $N = 1$ theory to deduce conclusions about the topology of four dimensions.

2.1. Twisting

First we will recall how one constructs from the $N = 2$ super Yang-Mills theory a twisted topological field theory. For more details, the reader can consult [4] and also §3 below.

The rotation group $K$ in four dimensional Euclidean space is locally $SU(2)_L \times SU(2)_R$. In addition, the connected component of the global symmetry group of the $N = 2$ theory is, as explained above, $SU(2)_I$. The theory, when formulated on a flat $\mathbb{R}^4$, therefore has a global symmetry group

$$H = SU(2)_L \times SU(2)_R \times SU(2)_I.$$  

(2.6)

The supercharges $Q^{\alpha i}$ and $Q_{\dot{\alpha} i}$ transform under $H$ as $(1/2, 0, 1/2)$ and $(0, 1/2, 1/2)$, respectively.

Apart from its standard embedding in $H$ (as $SU(2)_L \times SU(2)_R$), $K$ also has nonstandard embeddings. Let $SU(2)_R'$ be a diagonal subgroup of $SU(2)_R \times SU(2)_I$, and let

$$K' = SU(2)_L \times SU(2)_R'.$$  

(2.7)

Then $K'$ is isomorphic to $K$, and we can think of the $N = 2$ super Yang-Mills theory as a Poincaré invariant theory with rotations acting by $K'$. Use of $K'$ instead of $K$ to generate rotations is natural if one replaces the standard stress tensor $T$ of the theory by a modified stress tensor $T'$ (which differs from $T$ by addition of a derivative term that does not contribute to the translation generators).
But this substitution changes the physical interpretation of the theory considerably. All fields, commuting or anticommuting, have integer spin with respect to $K'$. In particular, this is so for the supercharges, which under $SU(2)_L \times SU(2)_R'$ transform as $(1/2, 1/2) \oplus (0, 1) \oplus (0, 0)$.

The $(0, 0)$ component of the supercharge is the rotation invariant object $Q = e^{\dot{a}i}Q_{\dot{a}i}$. By virtue of the supersymmetry algebra, $Q$ obeys $Q^2 = 0$; this follows from the fact that $Q^2$ transforms as $(0, 0)$ while the only bosonic operators in the supersymmetry algebra are the momenta, transforming as $(1/2, 1/2)$. Moreover, it is possible to find a component $S$ of the underlying supercurrent such that the modified stress tensor $T'$ can be written

$$T'_{\mu\nu} = \{Q, S_{\mu\nu}\}. \quad (2.8)$$

These facts ensure that with $Q$ regarded as a BRST operator, and $T'$ used as the stress tensor when coupling to gravity, the theory can be interpreted as a topological field theory.

Indeed [4], more or less because $Q$ is Lorentz invariant in the $K'$ sense, the $N = 2$ theory can be formulated on an arbitrary four manifold $M$ in such a way as to preserve the existence of the fermionic symmetry $Q$ and the basic relations $Q^2 = 0$ and (2.8). At this stage the twisted theory comes into its own. On flat $\mathbb{R}^4$, the twisted theory is merely a different way of looking at the usual physical theory; but as soon as we formulate the theory on a curved space-time, the twisted theory is really different, because of the use of a different stress tensor.

### 2.2. Operators

The important part of the BRST symmetry (in the sense that other fields do not enter in the construction of observables) is

$$\delta A_m = i\epsilon\lambda_m$$

$$\delta\lambda_m = -\epsilon D_m\phi$$

$$\delta\phi = 0. \quad (2.9)$$
Here $A$ is the gauge field, $\lambda_m$ is a linear combination of the fermions (the details will be described in §3), $\phi$ is a scalar, and $\epsilon$ is an anticommuting parameter. One easily verifies from this formula that, up to a gauge transformation, $Q^2 = 0$ in acting on these fields. (The multiplet (2.9) has a mathematical interpretation in terms of the equivariant cohomology of the gauge group acting on the space of connections, but that need not concern us here.)

Since $\phi$ is BRST invariant, but cannot be written as $\{Q, \ldots\}$, the obvious BRST invariant observables are of the form $P(\phi)$, with $P$ an invariant polynomial on the Lie algebra of $G$. For $G = SU(2)$, the basic invariant polynomial is the quadratic Casimir operator. (All others are polynomials in this and lead to nothing essentially new.) So the basic $Q$-invariant observable is

$$O(x) = O^{(0)}(x) = \frac{1}{8\pi^2} \text{Tr} \phi^2(x),$$  \hspace{1cm} (2.10)

with $\text{Tr}$ the trace in the fundamental two dimensional representation of $SU(2)$. More generally, for a Lie group of rank $r$, there are $r$ independent Casimir operators and accordingly $r$ independent operators generalizing $O^{(0)}(x)$.

So the first example of a topological correlation function is

$$\langle O(x_1)O(x_2) \ldots O(x_s) \rangle,$$  \hspace{1cm} (2.11)

with arbitrary distinct points $x_i$. (There is no singularity in the $O(x)O(y)$ operator product for $x \to y$, so one could in fact let some of the $x_i$ coincide, but it is more convenient to keep them distinct.) This correlation function is independent of the metric of $M$ because of (2.8); and so it must be independent of the choice of the $x_i$, because up to a change in metric on $M$, there is no invariant information in the choice of distinct points $x_1 \ldots x_s \in M$. To prove more explicitly that (2.11) is independent of the $x_i$, note that although $O$ cannot be written as $\{Q, \ldots\}$, its derivative can be:

$$\frac{\partial}{\partial x^m} O = \{Q, \frac{\text{Tr} \phi \lambda_m}{4\pi^2}\}.$$  \hspace{1cm} (2.12)

I write this equation as $dO^{(0)} = \{Q, O^{(1)}\}$, where $O^{(1)}$ is an operator-valued
one-form $\mathcal{O}^{(1)} = \text{Tr} \phi \lambda_m dx^m / 4\pi^2$. It is the beginning of a descent procedure. One recursively finds operator-valued $k$-forms $\mathcal{O}^{(k)}$ such that

$$d\mathcal{O}^{(k)} = \{Q, \mathcal{O}^{(k+1)}\}. \tag{2.13}$$

For full details on this, see [4]. (2.13) means that $\mathcal{O}^{(k)}$ is $Q$-invariant up to an exact form and is closed up to $\{Q, \ldots\}$. Accordingly, if $\Sigma$ is a $k$-dimensional submanifold of $M$ (or more generally a $k$-dimensional homology cycle), then

$$I(\Sigma) = \int_{\Sigma} \mathcal{O}^{(k)} \tag{2.14}$$

is BRST invariant and depends only on the homology class of $\Sigma$.

In practice Donaldson theory is most intensively studied on simply connected four manifolds $M$. In that case, $k$-dimensional homology cycles $\Sigma$ exist only for $k = 0, 2, 4$. For $k = 0$, $\Sigma$ is a point $x \in M$, and $I(\Sigma)$ is simply our friend $\mathcal{O}(x)$. For $k = 4$, $\Sigma$ must be $M$ itself. For $\mathcal{O}$ constructed from the quadratic Casimir operator, $I(M)$ turns out to be simply an elementary topological invariant, the instanton number $I(M) = \int_M \text{Tr} F \wedge F / 8\pi^2$. Therefore, for $G = SU(2)$, interesting quantum operators do not arise for $k = 4$. Interesting operators with $k = 4$ do arise for other groups, which have higher Casimir operators.

Hence for $SU(2)$, apart from $k = 0$, we focus on the case that $k = 2$ and that $\Sigma \subset M$ is an oriented two dimensional submanifold of $M$. For each such $\Sigma$, we get a BRST invariant operator $I(\Sigma)$, which up to $\{Q, \ldots\}$ depends only on the homology class of $\Sigma$. If we denote the components of the operator-valued two-form $\mathcal{O}^{(2)}$ as

$$Z_{mn}(x) = \frac{1}{4\pi^2} \text{Tr} (\phi F_{mn} - i\lambda_m \lambda_n) \tag{2.15}$$

then

$$I(\Sigma) = \int_{\Sigma} Z_{mn} d\sigma^{mn}. \tag{2.16}$$
The correlation functions

\[ \langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_r) I(\Sigma_1) \ldots I(\Sigma_s) \rangle \] (2.17)

are the celebrated Donaldson invariants of smooth four-manifolds.

There is actually a small subtlety here: the Donaldson invariants as usually defined by topologists are larger by a universal constant factor than the correlation functions just introduced. This factor is the number of elements in the center of the gauge group; we will call the center \( Z(G) \) and let \( \#Z(G) \) denote the number of its elements. Thus if \( \langle \rangle_T \) denotes the Donaldson invariants as usually defined topologically, then

\[ \langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_r) I(\Sigma_1) \ldots I(\Sigma_s) \rangle_T = \#Z(G) \cdot \langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_r) I(\Sigma_1) \ldots I(\Sigma_s) \rangle . \] (2.18)

The reason for this is that to recover the conventional topological definition of Donaldson’s invariants from the path integral (by a semi-classical evaluation of the path integral, discussed in [4,5]) one must delete a factor of \( 1/\#Z(G) \) that appears in the properly normalized path integral and is explained in [18, p. 159]. In fact, in the Fadde’ev-Popov definition of the path integral, one divides by the volume of the gauge group, taking account of the center of the group even though it acts trivially on the space of connections; but topologists usually omit to divide by the order of the center.

The Dimension Of Moduli Space

I will not attempt here a systematic account of the field theoretic approach to Donaldson theory. However, one point is so fundamental that it requires mention.

The classical \( N = 2 \) theory has a \( U(2) \) symmetry, acting on the two fermi fields \( (\lambda_\alpha, \psi_\alpha) \). So far we have exploited the anomaly-free \( SU(2) \), in twisting the model. The center \( U(1) \subset U(2) \) is anomalous; nevertheless it plays an essential role. Let us call this quantum number \( U \).
As explained in [4], on a given four-manifold and for a given instanton number, the total violation of $U$, which we will call $\Delta U$, equals the dimension of the Yang-Mills instanton moduli space $\mathcal{M}$. For $SU(2)$ this is

$$\Delta U = \dim \mathcal{M} = 8k - \frac{3}{2}(\chi + \sigma), \quad (2.19)$$

with $k$ the instanton number and $\chi$ and $\sigma$ the Euler characteristic and signature of $M$. (The quantity $(\chi + \sigma)/2$ is always integral.)

On the other hand, the operators $\mathcal{O}$ and $I(\Sigma)$ have $U = 4$ and $U = 2$, respectively, so a correlation function

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_r) I(\Sigma_1) \ldots I(\Sigma_s) \rangle \quad (2.20)$$

vanishes unless

$$4r + 2s = \dim \mathcal{M} = 8k - \frac{3}{2}(\chi + \sigma). \quad (2.21)$$

2.3. Role Of A Mass Gap

$N = 2$ supersymmetric Yang-Mills theory does not have a mass gap – for reasons explained earlier. Nevertheless, we will ultimately perform calculations by perturbing to a situation in which there is a mass gap. To see how the mass gap enters, we will first assume – counter-factually – that the $N = 2$ theory does have a mass gap, and see what computations become possible.

Actually, some of the arguments (those that involve $I(\Sigma)$ as well as $\mathcal{O}$) require a somewhat stronger assumption. We will assume that the ground state structure, on any space-time manifold, is the same as it is in the bulk theory. This means roughly that the vacua can be completely labeled by gauge-invariant local observables. The only known theories with mass gap in which this is not so are theories with unbroken and unconfined gauge symmetries, in which measurements of Wilson lines or global holonomies are needed to distinguish the vacua. Thus the assumption that we will
exploit is essentially that the $N = 2$ theory has a mass gap, and no unbroken, unconfined gauge symmetries.

We will also temporarily make a further simplification and assume that there is only one vacuum. This is a minor point and will eventually be corrected by introducing a sum over vacua.

**Consequences Of The Assumption**

With our assumptions, let us determine the Donaldson invariants. First we consider the zero point function – that is the partition function in the absence of any operator insertions. We will use the symbol $\langle X \rangle$ to refer to an unnormalized path integral on the four-manifold $M$ with an insertion of the operator $X$, so the partition function will be denoted as $(1)$. When we need to refer to the normalized vacuum expectation value of $X$ in the infinite volume limit (on a flat $\mathbb{R}^4$), we will write this as $\langle X \rangle_\Omega$, with $\Omega$ being the infinite volume vacuum state.

We use our liberty to pick a convenient metric on $M$. We begin with any metric $g$, and then scale it up by $g \to tg$, with $t$ a positive constant, and consider what happens for $t \to \infty$. The metric on $M$ becomes everywhere nearly flat. In a theory with a mass gap, the response to a background gravitational field is given by an effective action that can be expanded as a sum of local operators. In fact, the partition function is

$$\langle 1 \rangle = \exp(-L_{\text{eff}}), \quad (2.22)$$

where $L_{\text{eff}}$ has an expansion

$$L_{\text{eff}} = \int d^4x \sqrt{g} \left( u + vR + wR^2 + \ldots \right), \quad (2.23)$$

with constants $u, v, w, \ldots$ and local operators $1, R, R^2, \ldots$ ($R$ is the Ricci scalar of $M$). This is the general structure that would arise in any local theory with a mass gap, but in the case at hand the expansion is severely restricted by topological
invariance. The above expansion is valid for large $t$. If $U$ is an operator of dimension $n$, then $\int d^4x \sqrt{g}U$ scales as $t^{4-n}$, so the terms written explicitly in (2.23) make contributions of order $t^4, t^2,$ and 1. Topological invariance means that in the particular case at hand, the expansion must be independent of $t$, so only operators of dimension four can appear. The only topological invariants of a four-manifold that can be written as the integral of a local operator are the Euler characteristic $\chi$ and the signature $\sigma$, so

$$\langle 1 \rangle = \exp(a\chi + b\sigma)$$ (2.24)

with some universal constants $a$ and $b$. In the case of $SU(2)$ Donaldson theory, we will eventually determine $a$ and $b$ by comparing to computations on some particular four-manifolds.

Now we include the local operator $\mathcal{O}$ and consider its correlation functions

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_r) \rangle.$$ (2.25)

Upon scaling up the metric, we can assume that the $x_i$ are far apart from one another and inserted in a region of $M$ that is essentially flat on a scale much larger than any Compton wavelength in the theory. So via cluster decomposition, $\mathcal{O}$ can be replaced by its vacuum expectation value $\langle \mathcal{O} \rangle_\Omega$:

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_r) \rangle = \langle \mathcal{O} \rangle_\Omega^r \cdot \langle 1 \rangle.$$ (2.26)

This can conveniently be written

$$\langle \exp(\lambda \mathcal{O}) \rangle = \langle 1 \rangle \cdot \exp(\lambda \langle \mathcal{O} \rangle_\Omega),$$ (2.27)

with $\lambda$ a complex parameter.
The arguments become more interesting if we consider the operator $I(\Sigma)$. I will first show that given the assumptions, the one point function of this operator vanishes,

$$\langle I(\Sigma) \rangle = 0,$$  \hfill (2.28)

for every four manifold $M$ and every two dimensional surface (or homology cycle) $\Sigma \subset M$. In fact,

$$\langle I(\Sigma) \rangle = \int_\Sigma d\sigma^{mn} \langle Z_{mn} \rangle,$$  \hfill (2.29)

and the desired conclusion will follow from suitable properties of the one point function

$$\langle Z_{mn}(x) \rangle.$$  \hfill (2.30)

$Z_{mn}(x)$ is a well-defined, gauge-invariant local operator of the sort one usually studies in physical applications of Yang-Mills theory. Its one point function is a standard sort of Yang-Mills observable but is not a topological invariant; only integrated expressions like (2.29) are such invariants. In studying (2.30) as a prelude to analyzing (2.29), we are taking the main step in this paper: constraining the topological correlation functions by exploiting the way they arise as special correlators in a physical theory.

Clearly, on $\mathbb{R}^4$ with a flat metric, $\langle Z_{mn} \rangle = 0$ by Lorentz invariance. To determine what happens on a general four-manifold, we again consider an arbitrary metric $g$ scaled up by $g \to tg$ with $t \to \infty$. Since the area of $\Sigma$ grows like $t^2$ for $t \to \infty$, to show that the integrated expectation value (2.29) vanishes, it suffices to show that for every $x \in M$, $\langle Z_{mn}(x) \rangle$ vanishes for large $t$ faster than $1/t^2$. This is so as the mass gap lets us write an expansion for $\langle Z_{mn}(x) \rangle$ in terms of local invariants of the Riemannian geometry of $M$. The expansion looks like

$$\langle Z_{mn} \rangle = D_m R D_n D_s D^s R \mp \ldots$$  \hfill (2.31)

Since all possible terms have dimension considerably greater than two, the expec-
tation value vanishes faster than $1/t^2$.

Now, we move on to compute the two point function of $I(\Sigma)$. Let $\Sigma_1, \Sigma_2$ be two two-dimensional submanifolds of $M$. The desired two point function is

$$\langle I(\Sigma_1)I(\Sigma_2) \rangle = \int_{\Sigma_1 \times \Sigma_2} \langle Z_{mn}(x)Z_{pq}(y) \rangle \, d\sigma^{mn}(x)d\sigma^{pq}(y).$$ (2.32)

As before, we analyze this by scaling up the metric by $g \to tg$, with $t \to \infty$. For $x \neq y$, $\langle Z_{mn}(x)Z_{pq}(y) \rangle$ vanishes faster than $1/t^4$, by the same reasoning as above. The only possible surviving contribution would be a contribution that for $t \to \infty$ is localized where $x = y$, that is at the intersection points of $\Sigma_1$ and $\Sigma_2$.

By perturbing $\Sigma_1$ and $\Sigma_2$ slightly within their homology (and even homotopy) classes, we can assume that they meet transversely at finitely many points $w_1, \ldots, w_n$. We can also assume that the metric of $M$ is flat in a neighborhood of the $w_i$ and that the $\Sigma_a$ look locally like intersections of coordinate hyperplanes.

The contribution to (2.32) which for $t \to \infty$ is localized at one of the intersection points $w_i$ can only depend on local invariants of the behavior of the $\Sigma_a$ near $w_i$. The only such invariant is the relative orientation with which $\Sigma_1$ and $\Sigma_2$ meet at $w_i$. On symmetry grounds, the local contribution in (2.32) is proportional to this relative orientation. (The operator $I(\Sigma)$ is defined by integrating a differential form over $\Sigma$ and so changes sign if the orientation of $\Sigma$ is reversed; reversing the orientation of one of the $\Sigma_a$ changes the relative orientation with which $\Sigma_1$ and $\Sigma_2$ meet at $w_i$.) So – given our assumption – the sum of the local contributions in (2.32) is proportional to the algebraic intersection number of the $\Sigma$’s:

$$\langle I(\Sigma_1)I(\Sigma_2) \rangle = \eta \cdot \#(\Sigma_1 \cap \Sigma_2) \cdot \langle 1 \rangle.$$ (2.33)

Here $\eta$ is an unknown universal constant. This formula is valid even for $\Sigma_2 = \Sigma_1$; one sees this by perturbing $\Sigma_2$ within its homotopy class to a new surface $\Sigma_1'$ with only transverse intersections with $\Sigma_1$. 

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By a generalization of this reasoning, if $\Sigma_1 \ldots \Sigma_s$ are independent two-dimensional submanifolds of $M$, then the mass gap implies

$$\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) \right) \rangle = \exp \left( \frac{\eta}{2} \sum_{a,b} \alpha_a \alpha_b \#(\Sigma_a \cap \Sigma_b) \right) \cdot \langle 1 \rangle. \quad (2.34)$$

Indeed, the $\Sigma_a$ can be assumed to have only pairwise intersections; the right hand side of (2.34) comes from the local contributions of the intersection points. The $I(\Sigma_a)$ are analogous to quantum fields with Gaussian correlations controlled by the two point function (2.33).

No new phenomena occur if we consider correlators with $O$ and $I(\Sigma)$ together, since the points at which $O$ is inserted can be taken disjoint from the surfaces $\Sigma_a$ – and then cluster decomposition can be used, as before, to replace the $O$'s by their vacuum expectation values. So we can combine and (2.27) and (2.34) to get

$$\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) + \lambda O \right) \rangle = \exp \left( \frac{\eta}{2} \sum_{a,b} \alpha_a \alpha_b \#(\Sigma_a \cap \Sigma_b) + \lambda \langle O \rangle \right) \cdot \langle 1 \rangle. \quad (2.35)$$

Several Vacua

It remains to generalize this formula to take account of the possibility that the theory has not one vacuum state $\Omega$ but several. Assume existence of a discrete set of vacua $\Omega_\rho$, with $\rho$ ranging over some finite set $S$. To generalize (2.35), we simply write a sum over contributions of the $\Omega_\rho$, allowing for the fact that $\eta$, $\langle O \rangle$, and the constants $a$ and $b$ of equation (2.24) may depend on the choice of vacuum. We get

$$\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) + \lambda O \right) \rangle = \sum_{\rho \in S} C_\rho \exp \left( \frac{\eta}{2} \sum_{a,b} \alpha_a \alpha_b \#(\Sigma_a \cap \Sigma_b) + \lambda \langle O \rangle_\rho \right). \quad (2.36)$$

Here I have written the partition function $\langle 1 \rangle_\rho$ in the $\rho$ vacuum as $C_\rho$; by (2.24) it...
is of the form

\[ C_\rho = \exp (a_\rho \chi + b_\rho \sigma) \]  

(2.37)

with some universal constant \( a_\rho \) and \( b_\rho \).

2.4. Restriction To Kahler Manifolds And \( N = 1 \)

Upon comparison to mathematical computations of Donaldson invariants in special cases, one finds that these formulas are in fact false. Therefore, the \( N = 2 \) theory does not obey our assumptions; this comes as no surprise, in view of the flat directions in the classical potential which ensure that the \( N = 2 \) theory does not have a mass gap. Were (2.36) valid, Donaldson theory would detect only familiar, classical invariants of four-manifolds and so would not have its usual mathematical applications.

However, in certain respects, the formulas are nearly true at least for \( G = SU(2) \) where mathematical computations have been performed. For instance, they are true (with minor modification) for \( K3 \) surfaces according to results of [10]. For a large class of four manifolds (those of “simple type”), (2.36) gives the right asymptotic behavior for large values of \( \alpha_a \), according to results of Kronheimer and Mrowka [11]. (To be more precise, according to [11], the Gaussian function of the \( \alpha_a \) that appears on the right hand side must be multiplied by a sum of exponentials, which are “subleading” but contain all of the subtle information in the theory.) Why does our simple computation based on a faulty assumption enjoy these successes?

The thesis of the present paper will be that this is so because the assumptions are valid for \( N = 1 \) super Yang-Mills theory – and this is almost as good. To be more precise, in what follows we will show how \( N = 1 \) super Yang-Mills theory can be used to study Donaldson theory in the case that the four-manifold \( M \) is a Kahler manifold (that is, a manifold admitting a Kahler metric). This is in fact a very basic case in Donaldson theory and the main arena of actual computations.
For some time it was conjectured that the Donaldson invariants are zero except for four-manifolds admitting a Kahler metric; this is now known to be false [19], but the known counter-examples are still closely related to Kahler manifolds.

**The Twisted Theory On A Kahler Manifold**

For our purposes, the definition of a four-dimensional Kahler manifold is that it is a Riemannian four-manifold on which the holonomy group is not the usual $SU(2)_L \times SU(2)_R$, but is contained in $SU(2)_L \times U(1)_R$, with $U(1)_R$ being a subgroup of $SU(2)_R$ isomorphic to $U(1)$.

The two dimensional representation of $SU(2)$ has a single anti-symmetric invariant; given two doublets $v_\alpha, w_\beta$ this invariant can be written as $\epsilon^{\alpha\beta} v_\alpha w_\beta$. Upon restriction to a $U(1)$ subgroup, the same representation also admits a symmetric invariant, which we might write as $\delta^{\alpha\beta} v_\alpha w_\beta$. The implication of this for the twisted $N=2$ model is as follows. While on a general four-manifold the twisted theory has a single BRST operator $Q = \epsilon^{\dot{a}i} Q_{\dot{a}i}$, on a Kahler manifold there is also a second one $Q' = \delta^{\dot{a}i} Q_{\dot{a}i}$.

One way to formulate this is that on a Kahler manifold, one can write

$$Q = Q_1 + Q_2,$$  

(2.38)

where $Q_1$ is the part of $Q$ inside one $N = 1$ subalgebra and $Q_2$ is the part in the second. Exactly how the Kahler reduction in structure group makes possible the decomposition in (2.38) will be clarified in §3. These operators obey

$$Q_1^2 = Q_2^2 = \{Q_1, Q_2\} = 0,$$  

(2.39)

and the observables of Donaldson theory, such as $\mathcal{O}$ and $I(\Sigma)$, are annihilated by both $Q_1$ and $Q_2$.

In explicit computations via instantons, $Q$, $Q_1$, and $Q_2$ correspond to the de Rham exterior derivative $d$ and to the $\overline{\partial}$ and $\partial$ operators of instanton moduli space.
It is a familiar fact in cohomology of smooth, compact, Kahler manifolds that the (complex-valued) de Rham cohomology is isomorphic to the $\overline{\partial}$ cohomology. Somewhat analogously, in the Donaldson theory of Kahler manifolds, one can disregard the $Q_2$ symmetry and keep track of the topological behavior using $Q_1$ alone; this will be explained in §3.1.

Since $Q_1$ lies in an $N = 1$ subalgebra, perturbations that break $N = 2$ supersymmetry down to $N = 1$ can preserve $Q_1$ invariance. Is there such a perturbation that will simplify the problem?

2.5. The Mass

This question is no sooner asked than answered. Temporarily we work on a flat $\mathbb{R}^4$, with Euclidean coordinates $y^1, \ldots, y^4$. Of course, we pick a complex structure, determined say by complex coordinates $z_1 = y^1 + iy^2$, $z_2 = y^3 + iy^4$, to regard $\mathbb{R}^4$ as a Kahler manifold.

As was explained at the outset of this section, the $N = 2$ theory, regarded as an $N = 1$ theory, has the following multiplets: there is a gauge multiplet $(A, \lambda)$ and a chiral supermultiplet $\Phi = (\phi, \psi)$ in the adjoint representation. The chiral supermultiplet can have a bare mass

$$\Delta L = -m \int d^4xd^2\theta \mathrm{Tr} \Phi^2 - \mathrm{h.c.} \quad (2.40)$$

preserving $N = 1$ supersymmetry. This can be written in the general form*

$$\sum_a \alpha_a I(\Sigma_a) + \{Q_1, \ldots\}, \quad (2.41)$$

so up to $\{Q_1, \ldots\}$, by adding the mass term we are just shifting the Lagrangian by a linear combination of the observables that we want to study anyway.

* That is, it can be written this way in the case of a global Kahler manifold, as we will see in §3. On flat $\mathbb{R}^4$, the term $\sum_a \alpha_a I(\Sigma_a)$ should better be written $\int_{\mathbb{R}^4} \omega \wedge Z$, where $\omega = dz_1 \wedge dz_2$. Globally, the two formulations are equivalent up to $\{Q_1, \ldots\}$. 

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The $N = 2$ theory perturbed by the $N = 1$ invariant bare mass term (2.40) has a mass gap for the following reasons. The perturbation certainly gives a mass to the $\Phi$ multiplet, so we are reduced at low energies to a minimal $N = 1$ system. That system in turn has a dynamically generated mass gap, at a much lower scale if $m$ is large.

What are the global symmetries of the perturbed system? The minimal $N = 1$ model has a $Z_2$ symmetry described earlier. The only symmetry of the $\Phi$ multiplet is $\Phi \rightarrow -\Phi$, which generates a group that we will call $Z_2'$. The overall global symmetry group is hence $Z_{2h} \times Z_2'$. The pattern of symmetry breaking is easy to work out. $Z_{2h}$ is broken to $Z_2$, as explained earlier, while the symmetry $Z_2'$ that acts only on fields with a positive bare mass (which can be taken large) is unbroken. The unbroken symmetry group is therefore $Z_2 \times Z_2'$.

The Mass Term On A Curved Kahler Manifold

If we consider not flat $R^4$ but a general Kahler manifold $M$, an important subtlety arises in the description of the mass term. The subtlety arises because of the twisting by $U(1)_R$. The mass term in (2.40) is defined using a measure $d^4x \, d^2\theta$; the factor $d^2\theta$ carries $U(1)_R$ charge (in the twisted model) and there is no natural way to fix the space-time dependence of its phase on a curved Kahler manifold.

The problem can be remedied as follows. Write $d^4x \, d^2\theta = d^2 z \cdot d^2 \bar{z} d^2 \theta$. Then the factor $d^2 \bar{z} d^2 \theta$ is in fact naturally defined on any Kahler manifold (as should be clearer, along with some subsequent claims, in §3), so the question is to generalize $d^2 z$.

The factor $d^2 z$, or better $m \, d^2 z$, including the factor of $m$ from (2.40), can be interpreted as a holomorphic two-form on $R^4$. The generalization is immediate: on any Kahler manifold, pick a holomorphic two-form $\omega$ and replace $m \, d^2 z$ by $\omega$. The generalization of (2.40) to a general Kahler manifold is thus

$$\Delta L = - \int \omega \wedge d^2 \bar{z} d^2 \theta \, \text{Tr} \, \Phi^2 + \text{h.c.}$$

(2.42)

We will verify in §3 that this is of the form $\sum_a \alpha_a I(\Sigma_a) + \{Q_1, \ldots\}$. 20
At this stage, however, an important restriction arises. A non-zero holomorphic two-form $\omega$ exists not on arbitrary Kahler manifolds, but only on those on which $H^{2,0}(M) \neq 0$. So we are limited to Kahler manifolds obeying that condition. This is in fact natural. One of Donaldson’s earliest results [20] was a sort of anomaly: the topological correlation functions of the twisted $N = 2$ theory are in fact topological invariants only under a certain topological condition (the self-dual part of $H^2(M)$ should have dimension bigger than one); specialized to Kahler manifolds, the condition is precisely $H^{2,0}(M) \neq 0$. So the restriction required by our methods is in fact the “right” one.

2.6. Some Simple Computations

Now let us carry out some actual computations of the Donaldson invariants for Kahler manifolds. Since there is a mass gap, we can proceed much as before; but there are a few subtleties.

For simplicity we will specialize to gauge group $G = SU(2)$. So the global symmetry group is $\mathbb{Z}_4 \times \mathbb{Z}_2'$, spontaneously broken to $\mathbb{Z}_2 \times \mathbb{Z}_2'$; as a result there are two vacuum states, which we will call $|+\rangle$ and $|-\rangle$. The sum over vacuum states in formulas such as (2.36) therefore involves two terms. The constants $\eta$ and $\langle O \rangle$ that appear in that formula are odd under the broken symmetry, so $\eta_+ = -\eta_-$. We normalize the operators $O$ and $I(\Sigma)$ so that $\eta_{\pm} = \pm 1$ and $\langle O \rangle_{\pm} = \pm 2$. By comparing to special cases of Donaldson invariants that have been

† There are two caveats here: (1) There is one other exceptional case of a Kahler manifold, namely $\mathbb{CP}^2$, for which the Donaldson invariants are well-defined; that is, the correlation functions are independent of the metric. The construction of this paper does not apply and I have no reason to expect the correlation functions to be given by similar formulas. (2) Even when $H^{2,0}(M) = 0$ and the correlation functions of the twisted theory are not topological invariants, they still have very nice properties (which have been much exploited mathematically) and there may be a useful way to study them using physical techniques.

‡ Apart from issues I discuss below, one must consider the violation of Lorentz invariance of the twisted theory by the $N = 1$ invariant bare mass term. Lorentz invariance was used in analyzing the two-forms to arrive at (2.36). We will show in §3 how to derive at (2.36), by a slightly lengthier route, using properties that are preserved by the bare mass.
computed mathematically, one can verify that these normalizations agree with the standard topological normalizations.

A subtlety arises in using the broken symmetry to relate the contributions of the $|+\rangle$ and $|−\rangle$ vacua in formulas such as (2.36). The global symmetry group $\mathbb{Z}_4 \times \mathbb{Z}_2'$ of the theory is a subgroup of the classical symmetry group consisting of symmetries that are not explicitly broken by gauge instantons. However, gravitational instantons – in this case the effects of working on a four-manifold $M$ – might induce additional symmetry breaking. We should remember that the $\mathbb{Z}_4$ symmetry is generated by

$$\alpha : \lambda \rightarrow i\lambda, \quad \bar{\lambda} \rightarrow -i\bar{\lambda}. \quad (2.43)$$

In an instanton field of instanton number $k$, the net number of $\lambda$ minus $\bar{\lambda}$ zero modes is, from the index theorem,

$$\Delta = 4k - 3(1 - h^{1,0} + h^{2,0}), \quad (2.44)$$

with $h^{p,q}$ the dimension of $H^{p,q}(M)$. This can also be written in a way that makes sense for any four-manifold (but might not be integral in general):

$$\Delta = 4k - \frac{3}{4} (\chi + \sigma) = \frac{1}{2} \dim \mathcal{M}, \quad (2.45)$$

with $\chi$ and $\sigma$ the Euler characteristic and the signature of $M$, and $\mathcal{M}$ the moduli space of instantons.

Under the symmetry (2.43), the integration measure for the fermion zero modes transforms as $i^{-\Delta}$. Therefore, the underlying broken symmetry does not, as one might expect, cause the constants $C_+$ and $C_-$ in formulas such as (2.36) to be equal. Rather, the relation is

$$C_- = i^\Delta C_+. \quad (2.46)$$

Notice that the $k$-dependence drops out of this relation, because the spontaneously broken symmetry has no anomaly under Yang-Mills instantons. Of course, similar
formulas can be worked out for groups other than $SU(2)$ – the difference being that there are not two but $h$ vacua, each with its own constant $C$.

Using (2.45), we can also write

$$C_- = \exp \left( -\frac{3i\pi}{8} (\chi + \sigma) \right) C_+, \quad (2.47)$$

showing that the relation between $C_-$ and $C_+$ is compatible with the fact that each can be written as in (2.37).

**Hyper-Kahler Manifolds**

To work out actual formulas, we consider first the special case of hyper-Kahler manifolds. This is the case in which the holomorphic two-form $\omega$, which is responsible for the mass gap of the theory, has no zeroes, so the mass gap can be applied most simply.

Moreover, for a hyper-Kahler manifold, the holonomy is not $SU(2)_L \times U(1)_R$ (as for general Kahler manifolds) but simply $SU(2)_L$. This means that the twisting is trivial, and the physical model coincides with the topological model.

Actually, there are only two cases of compact, non-singular hyper-Kahler manifolds of dimension four: a four-torus $T^4$, and a $K3$ surface. For $T^4$, $h^{1,0} = 2$, $h^{2,0} = 1$, and $\Delta = 0$ modulo four, so $C_- = C_+$ according to (2.46). In fact, for the four-torus $\chi = \sigma = 0$, and therefore $C_+ = C_- = 1$, according to (2.37). Hence the sum in (2.36) takes the following form in this case:

$$\left\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) + \lambda \mathcal{O} \right) \right\rangle = \exp \left( \frac{1}{2} \sum_{a,b} \alpha_a \alpha_b \#(\Sigma_a \cap \Sigma_b) + 2\lambda \right)$$

$$+ \exp \left( -\frac{1}{2} \sum_{a,b} \alpha_a \alpha_b \#(\Sigma_a \cap \Sigma_b) - 2\lambda \right). \quad (2.48)$$

Unfortunately, mathematical computations of Donaldson invariants of the four-torus have not been performed, so there is nothing to compare (2.48) to. Inciden-
tally, it is straightforward to extend (2.48) to include the additional observables associated with the odd dimensional cohomology of the torus.

It remains to consider the case of $K3$. For $K3$, $h^{1,0} = 0$, $h^{2,0} = 1$, so from (2.46), $C_+ = -C_-$. Therefore, (2.48) is replaced by

$$\left\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) + \lambda O \right) \right\rangle = C \left( \exp \left( \frac{1}{2} \sum_{a,b} \alpha_a \alpha_b \#(\Sigma_a \cap \Sigma_b) + 2\lambda \right) \right. - \left. \exp \left( -\frac{1}{2} \sum_{a,b} \alpha_a \alpha_b \#(\Sigma_a \cap \Sigma_b) - 2\lambda \right) \right),$$

(2.49)

with an unknown constant $C$. According to computations of [10], this formula is in fact correct with $C = 1/4$.

Chiral symmetry breaking is the reason that the right hand sides of both (2.48) and (2.49) are given by not a single exponential but a sum of two exponentials, with a relative plus sign in one case and minus sign in the other case. These signs play the following role. For the four-torus, for dimensional grounds explained in (2.21), a correlation function

$$\langle O(x_1) \ldots O(x_r) I(\Sigma_1) \ldots I(\Sigma_s) \rangle$$

(2.50)

vanishes unless $4r + 2s$ is divisible by 8. For the $K3$ surface, the requirement is that $4r + 2s$ should be congruent to 4 modulo 8. A single exponential of the form in (2.48) or (2.49) would not obey these requirements, but the given sums of two exponential with plus or minus signs do obey them. Thus (given the mass gap of the $N = 1$ theory and the other constraints) chiral symmetry breaking is essential in making sense out of Donaldson theory. This fact (of which we will see subtler versions later, involving the canonical divisor) is a sort of global generalization of old insights [1] concerning supersymmetric Yang-Mills theory on $\mathbb{R}^4$.  

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2.7. GENERAL KAHLER MANIFOLDS

Now we consider the case of a general Kahler manifold $M$, with $H^{2,0} \neq 0$. The important new phenomena spring from the fact that (if $M$ is not hyper-Kahler) the canonical class of $M$ is non-trivial. This implies that the mass term

$$\Delta L = -\int_M \omega \wedge d^2z d^2\theta \Tr \Phi^2 - \text{h.c.}$$  \hspace{1cm} (2.51)

is not everywhere non-zero. The $\Phi$ mass vanishes where $\omega$ does; and $\omega$ vanishes on a divisor $C$ representing the canonical class of $M$. For simplicity we will assume – at least initially – that $C$ is the union of smooth, disjoint Riemann surfaces $C_y$ along which $\omega$ has simple zeroes.

The local model that we reduce to by scaling up the metric of $M$ is now the following. Each $C_y$ becomes a flat complex dimension one submanifold of a flat $\mathbb{R}^4 \cong \mathbb{C}^2$. One might think of the $C_y$ as world-sheets of cosmic strings. The strings in question are similar to “global strings” (their existence hinges on the vanishing of a gauge-invariant field $\omega$) rather than “local strings” (in the core of which there is an enhanced unbroken gauge symmetry).

The most important phenomenon associated with this particular type of cosmic string is that it can capture fermion zero modes. In fact, pick complex coordinates $z_1, z_2$ on $\mathbb{R}^4$, and suppose that $\omega = z_1 dz_1 \wedge dz_2$, with a simple zero on the Riemann surface $C$ defined by $z_1 = 0$. Any example (with non-singular $C_y$) reduces locally to this example when the metric is scaled up.

In this example, the mass term of the $\Phi$ multiplet is proportional to $z_1$, so it changes in phase by $2\pi$ in making a small circuit around the “core of the string” at $z_1 = 0$. This is the standard situation in which fermion zero modes are trapped near the core of a string. In fact, the Dirac equation for the motion of the $\psi$ field (the fermionic part of the $\Phi$ multiplet) in the $z_1$ plane has a normalizable zero mode, as we will discuss in more detail in §3. When one allows for the $z_2$
dependence of the $\psi$ wave-function, this “zero mode” becomes a two-dimensional quantum field, propagating on the string world-sheet, that is, the $z_2$ plane. We will call this two dimensional quantum field $\psi'$. $\psi'$ is a spin one-half fermion in the adjoint representation of $G$ (since those are the quantum numbers of $\psi$); it has definite chirality, which we will conventionally call left-handed, since the zero mode in the $z_1$ plane has definite chirality.

In explaining the origin of the effective two-dimensional field $\psi'$, the gauge couplings were not essential; the crucial zero mode develops because of the topology of the “Higgs field” $\omega$. When we consider the gauge fields, a new consideration arises: the coupling of gauge fields to two-dimensional chiral fermions $\psi'$ is anomalous. Since the underlying four-dimensional theory is anomaly-free, there must be additional fields trapped along the string that cancel the anomaly. To understand the details, one would need to know something about how the strongly coupled $N = 1$ supersymmetric Yang-Mills theory behaves near the core of the string. This appears out of reach at present. But somehow, extra modes must be trapped along the string in such a way that the effective theory along the string is a gauge-invariant, anomaly-free theory that we will call the cosmic string theory. It can be seen that the two-dimensional cosmic string theory has $(0, 2)$ supersymmetry, but we will not exploit that.

2.8. Symmetries And Vacuum Structure Of The Cosmic String Theory

Since we do not even know the fields and Lagrangian of the cosmic string theory, how can we proceed? We will make what reasonable conjectures we can based on the symmetries.

As we recall, the bulk theory had a $\mathbb{Z}_{2h} \times \mathbb{Z}_{2'}$ global symmetry, which we believe is spontaneously broken in bulk to $\mathbb{Z}_2 \times \mathbb{Z}_2'$. Along the string, the question of symmetry breaking must be re-examined; further symmetry breaking might occur in the core of the string.
In bulk, $\mathbb{Z}_2'$ couples only to particles of positive bare mass, and it was implausible that it would be spontaneously broken. Along the string, however, $\mathbb{Z}_2'$ couples to (and in fact only to) the massless chiral fermion $\psi'$, so in the cosmic string theory $\mathbb{Z}_2'$ is in fact a chiral symmetry. It is very plausible that such a chiral symmetry would be spontaneously broken.

On the other hand, a diagonal subgroup $\mathbb{Z}_2'' \subset \mathbb{Z}_2 \times \mathbb{Z}_2'$ is generated by the operator $(-1)^F$ that counts fermions modulo two; this is equivalent to a $2\pi$ rotation of the cosmic string, and cannot be spontaneously broken if rotation invariance is unbroken.

We will make for the cosmic string theory the same sort of assumption that we made in bulk. We will assume that the cosmic string theory has a mass gap and that the only vacuum degeneracy comes from symmetry breaking. (Thus, in particular, the vacuum degeneracy can be measured by local operators.) For the pattern of symmetry breaking, I will assume that $\mathbb{Z}_2 \times \mathbb{Z}_2'$ is spontaneously broken to $\mathbb{Z}_2''$ – the largest unbroken symmetry that permits fermion masses. Thus – regardless of the gauge group – I postulate that the cosmic string theory has two vacuum states (for a given choice of the vacuum in the bulk four-dimensional theory).

Hitherto we have considered the symmetries of the cosmic string theory which are (i) not explicitly broken by Yang-Mills instantons; (ii) not spontaneously broken in bulk. As in the bulk theory, for instance the discussion of (2.46), we must in addition worry about whether there is further explicit breaking of symmetries by gravitational instantons. What this means concretely is as follows. We consider the actually relevant case in which the string world sheet is not the $z_2$ plane but a compact Riemann surface $C_y$; we need to know whether the path integral measure along $C_y$ for the effective two-dimensional theory is even or odd under the broken $\mathbb{Z}_2'$ symmetry.

As in the bulk discussion, the question comes down to counting modulo two the $\psi'$ zero modes. If the number of zero modes is even, the measure is even under $\mathbb{Z}_2'$; if the number of zero modes is odd, the measure is odd. The number of zero
modes modulo two is a topological invariant, independent of the gauge fields (and all fields other than $\psi'$), so the problem can be treated as a problem of free fermions along $C_y$.

Let $d$ be the dimension of the gauge group. The $\psi'$ field has $d$ components; if it is treated as a free field, the number of zero modes is $d$ times what it would be for a single chiral fermion. If $d$ is even, there is no further issue; the number of zero modes, and the path integral measure, are even. More subtle is the case of an odd dimensional group, such as $SU(2)$. The result then depends on the spin structure. The number of zero modes is even or odd for chiral fermions coupled to a so-called even or odd spin structure. Let $\epsilon_y$ be 0 or 1 depending on whether the spin structure is even or odd. Then the number of zero modes along $C_y$ is $d\epsilon_y$, modulo two, so the path integral measure transforms under the broken $\mathbb{Z}_2'$ symmetry by a factor of

$$t_y = (-1)^{d\epsilon_y}.$$  \hfill (2.52)

The spin structure that arises here cannot be chosen arbitrarily but must be deduced from the underlying four-dimensional theory. This will be done in §3.3 with the following result. If $C_y \subset M$ is a smooth Riemann surface defined by a first order zero of a holomorphic two-form $\omega$, then the inverse of the normal bundle $N_y$ to $C_y$ in $M$ can be interpreted as a spin bundle of $C_y$. This (or actually its complex conjugate) is the spin bundle of which $\psi'$ is a section; its evenness or oddness determines (if $d$ is odd) whether $\epsilon_y$ is 0 or 1 and so whether the measure is even or odd under the broken symmetry.

**Computations**

Now we come to the computational stage. Imitating our considerations in bulk, the first step is to analyze the partition function of the cosmic string theory, with a given choice of the bulk vacuum state $|+\rangle$ or $|-\rangle$ and a given refinement to include the vacuum structure along the string. We will label the four vacuum states in the vicinity of a particular string as $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$, with the first
sign indicating the ambient behavior in bulk and the second sign indicating the behavior near the string.

As in the derivation of (2.24), the assumption of a mass gap implies that the partition function of the cosmic string theory on a Riemann surface $C$ must be of the form

$$
\langle 1 \rangle_C = \exp \left( - \int_C d^2 x \sqrt{g} W(x) \right)
$$

with $W(x)$ a local operator constructed from the Riemannian geometry. The only topological invariant of a Riemann surface that can be constructed in this way (indeed, its only topological invariant at all) is the Euler characteristic $\chi(C)$. It would appear, therefore, that the partition function of the four-dimensional theory, allowing for contributions from all of the $C_y$, would have a factor

$$
\prod_y \exp \left( -w \chi(C_y) \right) = \exp \left( -w \sum_y \chi(C_y) \right),
$$

with some universal coefficient $w$. However, using standard facts in complex geometry, $\sum_y \chi(C_y)$ is a linear combination of $\sigma(M)$ and $\chi(M)$, so this factor can be absorbed in adjusting the constants $a$ and $b$ in (2.24).

We also have to consider the local operators that arise in the cosmic string theory. The local operator $O(x)$ of the original four-dimensional theory can be inserted at points disjoint from the $C_y$, and so does not contribute to the cosmic string theory. However, the other operator $I(\Sigma)$ is constructed as an integral over a two-manifold $\Sigma$:

$$
I(\Sigma) = \int_{\Sigma} Z_{\text{mn}} d\sigma^{\text{mn}}.
$$

Topologically, intersections of $\Sigma$ with the $C_y$ may be unavoidable; they can, however, be taken to be transverse.
We recall that when the metric of $M$ is scaled up, the bulk contribution to the integral in (2.55) can be discarded. In our earlier analysis, we obtained local contributions at points of intersections of $\Sigma$’s. Similarly, local contributions will arise at intersections of the $\Sigma_a$ with the $C_y$. At every intersection point $P$ of $\Sigma_a$ and $C_y$, the integral (2.55) will contribute a local operator $V(P)$ in the cosmic string theory – times a sign depending on the relative orientation of $\Sigma_a$ and $C_y$. (As in the discussion of intersections of $\Sigma$’s, such a sign must appear, since $I(\Sigma)$ is odd under reversal of orientation of $\Sigma$.) So, if $\#(\Sigma \cap C_y)$ is the algebraic intersection number of $\Sigma$ and $C_y$, then one can make the replacement

$$I(\Sigma) \to \sum_y \#(\Sigma \cap C_y)V_y + \text{terms involving intersections of } \Sigma' \text{'s.} \quad (2.56)$$

Here $V_y$ denotes the operator $V$ inserted on $C_y$.

Inside correlation functions, the scalar operator $V$ of the cosmic string theory can be replaced by its vacuum expectation value – like any local operator in a theory with a mass gap. We must work out the expectation value of $V$ in the various vacuum states, which by our hypotheses are related by spontaneously broken symmetries. Let us therefore describe the symmetry and vacuum structure precisely. Let $\alpha$ and $\beta$ be the generators $\alpha : \lambda \to i\lambda, \Phi \to \Phi$ and $\beta : \lambda \to \lambda, \Phi \to -\Phi$ of the $\mathbb{Z}_4$ and $\mathbb{Z}_2'$ factors, respectively, of the global symmetry group $\mathbb{Z}_4 \times \mathbb{Z}_2'$. Then, as we will see in §3.3,

$$\alpha V = iV, \quad \beta V = -V. \quad (2.57)$$

In bulk there are two vacua, $|+\rangle$ and $|\rangle$, with

$$\alpha |+\rangle = |\rangle, \quad \alpha |\rangle = |+\rangle. \quad (2.58)$$

Allowing for the behavior near the string, these two states bifurcate into four, as
described earlier; we can choose the labeling of the vacua so that

\[ \alpha |++\rangle = |--\rangle \]
\[ \alpha |--\rangle = |--\rangle \]
\[ \alpha |--\rangle = |++\rangle. \]

(2.59)

The action of \( \beta \) need not be separately given as \( \alpha^2 \beta \) is unbroken and hence \( \beta \) acts on the vacua as \( \alpha^2 \). If therefore the expectation value of \( V \) in the state \( |++\rangle \) is \( v \), then altogether

\[ \langle V \rangle_{++} = v \]
\[ \langle V \rangle_{--} = -iv \]
\[ \langle V \rangle_{+-} = -v \]
\[ \langle V \rangle_{-+} = iv. \]

(2.60)

**Final Evaluation**

Finally, then, to obtain the topological correlation functions for Kahler manifolds obeying our assumptions, one must sum over vacuum states in bulk and along the \( C_y \). Relative phases in the contributions of different vacua come from (2.46) and (2.52). The parameters \( \Delta \) and \( t_y \) appearing in those equations are not independent, but are related by

\[ \Delta + \sum_y \epsilon_y \cong 0 \text{ modulo } 2, \]

(2.61)

(with \( t_y = (-1)^{\epsilon_y} \)), as we will explain in §3.4.

To compute correlation functions of \( I(\Sigma)'s \), one determines the cosmic string contributions as in (2.56), and replaces \( V \) by its expectation values just given.

In writing down the final result, I will adjust the constants \( a \) and \( b \) in (2.24) and the constant \( v \) in (2.60) to values chosen to agree with special cases of mathematical
computations. The requisite values turn out to be $a = (7/4) \ln 2$, $b = (11/4) \ln 2$, and $v = 1$. The constants $\eta$ and $\langle O \rangle$ were similarly adjusted earlier, so we have fixed five universal constants by comparing to particular calculations of Donaldson invariants.

We want the general formula for

$$\left\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) + \lambda O \right) \right\rangle.$$  \hfill (2.62)

It is convenient to set

$$\phi_y = \sum_a \alpha_a \#(\Sigma_a \cap C_y).$$  \hfill (2.63)

According to (2.56), the cosmic string contribution to the exponent in (2.62) is $\sum_y \phi_y V_y$. Replacing this with its expectation values in the various states, we see that the $|+\rangle$ vacuum, with its various bifurcations along the $C_y$, contributes

$$2^{\frac{1}{4}(7\chi + 11\sigma)} \exp \left( \sum_{a,b} \frac{\alpha_a \alpha_b}{2} \#(\Sigma_a \cap \Sigma_b) + 2\lambda \right) \prod_y \left( e^{\phi_y} + t_y e^{-\phi_y} \right).$$  \hfill (2.64)

The $|-\rangle$ vacuum, with its various bifurcations, contributes the chiral transform of this, or

$$i^{\Delta} 2^{\frac{1}{4}(7\chi + 11\sigma)} \exp \left( -\sum_{a,b} \frac{\alpha_a \alpha_b}{2} \#(\Sigma_a \cap \Sigma_b) - 2\lambda \right) \prod_y \left( e^{-i\phi_y} + t_y e^{i\phi_y} \right).$$  \hfill (2.65)

As a check, note that this is real by virtue of (2.61).

The final formula for the $SU(2)$ Donaldson invariants of Kahler manifolds (with $H^{2,0}(M) \neq 0$ and possessing a holomorphic two-form whose zero set is a union of smooth disjoint complex curves of multiplicity one) is obtained by summing these
contributions and is

\[
\left\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) + \lambda O \right) \right\rangle_T
= 2^{1+\frac{1}{4}(7\chi+11\sigma)} \exp \left( \sum_{a,b} \frac{\alpha_a \alpha_b}{2} \#(\Sigma_a \cap \Sigma_b) + 2\lambda \right) \prod_y \left( e^{\phi_y} + t_y e^{-\phi_y} \right)
+ i^\Delta 2^{1+\frac{1}{4}(7\chi+11\sigma)} \exp \left( -\sum_{a,b} \frac{\alpha_a \alpha_b}{2} \#(\Sigma_a \cap \Sigma_b) - 2\lambda \right) \prod_y \left( e^{-i\phi_y} + t_y e^{i\phi_y} \right).
\]

(2.66)

Here (as implied by the symbol \( \langle \ )_T \)) we use the normalization of topologists, and so include a factor of 2 – the order of the center of SU(2) – from (2.18).

Here is a check on (2.66). Using (2.45) and (2.21), the dimension of the moduli space of Yang-Mills instantons is \( 2\Delta \), and the correlation function

\[
\langle O(x_1) \ldots O(x_r) I(\Sigma_1) \ldots I(\Sigma_s) \rangle
\]

is zero unless

\[
4r + 2s = 2\Delta.
\]

(2.68)

Therefore, if we consider the \( \alpha_a \) to be of degree 2 and \( \lambda \) to be of degree 4, then the power series expansion of (2.66) should have only terms of degree congruent to \( 2\Delta \) modulo 8. It is easy to verify this.

Unlike the naive formula (2.36) that would arise if the \( N = 2 \) theory had a mass gap, the formula is not manifestly a topological (or differentiable) invariant. Indeed, the formula appears to depend on the complex structure of \( M \) (which determines the canonical divisor and thus the ultimately the \( \phi_y \)). The metric independence of the underlying twisted \( N = 2 \) theory ensures, however, that the formula really only depends on the structure of \( M \) as a differentiable manifold. The fact that the formula is a differentiable invariant but not obviously so is responsible for its mathematical interest.
Thus, a fairly typical mathematical application of Donaldson theory is as follows. In many instances one can construct pairs of Kahler four-manifolds $M_1$ and $M_2$ which have the same values of the obvious topological invariants but for which the formulas (2.66) do not coincide. Then one can infer that $M_1$ and $M_2$ are in fact not isomorphic as smooth manifolds. In practice, the data present in (2.66) beyond the classical invariants are the cohomology classes of the connected components $C_y$ of the canonical divisor and the types (even or odd) of the normal bundles to the $C_y$; $M_1$ and $M_2$ must have the same values of this data if they are to coincide as smooth four-manifolds.

**Simple Type Condition And Comparison To Known Formulas**

A four-manifold is said to be of simple type if $f = \langle \exp(\sum \alpha_a I(\Sigma_a) + \lambda \mathcal{O}) \rangle$ obeys the equation $\partial^2 f / \partial \lambda^2 = 4f$. (2.66) evidently implies that Kahler manifolds obeying our assumptions are of simple type. The derivation of (2.35) shows that the simple type condition is really a corollary of having a mass gap. Thus, the assumption that $H^{2,0}(M) \neq 0$, which permitted us to reduce to a theory that has a mass gap, was essential; we would not necessarily expect $\mathbb{CP}^2$, for instance, to obey a condition similar to the simple type condition.

In deriving (2.66) we assumed also that there exists a holomorphic two-form $\omega$ that vanishes precisely on a union of disjoint smooth curves. This is not relevant to the simple type condition, since regardless of the structure of the canonical divisor, one can assume that $\mathcal{O}$ is inserted at points disjoint from it. Therefore, only the bulk vacuum structure is relevant in determining the dependence of $f$ on $\lambda$, so (at a physical level of rigor) all Kahler manifolds with $H^{2,0} \neq 0$ are of simple type.

According to Kronheimer and Mrowka [11], on a four-manifold of simple type, $f$, as a function of the $\alpha_a$, is a Gaussian times a sum of exponentials. (In fact, they make more precise claims.) This was a clue suggesting the role of a mass gap and $N = 1$ supersymmetry, and (2.66) has the expected properties. We will say more about this presently.
2.9. More General Kahler Manifolds

In general, we should consider the case of a Kahler manifold $M$ with $H^{2,0} \neq 0$, but with a canonical divisor that does not have the simplifying properties that we have so far assumed. The zeroes of a holomorphic two-form $\omega$ will be a union of connected curves $C_y$; but – even if $\omega$ is chosen generically – the $C_y$ may not be smooth, and $\omega$ may vanish along components of $C_y$ with multiplicity greater than one.

No matter how badly behaved the canonical divisor may be, the mass gap ensures that the contribution of a given $C_y$ depends only on the local structure of $C_y$ and its intersections with the $\Sigma_a$. We recall that the $C_y$ were defined as the connected components of the zero locus of $\omega$. Each $C_y$ may in turn be a union of irreducible (but perhaps singular) Riemann surfaces $C_{y,\beta}$ (which intersect each other, perhaps at their singularities). The only homology invariants in the intersections of the $\Sigma_a$ and the $C_y$ are the algebraic intersection numbers $\#(\Sigma_a \cap C_{y,\beta})$, and the mass gap permits the correlation functions to depend on the natural generalization of (2.63):

$$\phi_{y,\beta} = \sum_a \alpha_a \#(\Sigma_a \cap C_{y,\beta}). \quad (2.69)$$

The possible generalization of (2.66) allowed by the mass gap is

$$\left\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) + \lambda O \right) \right\rangle_T = 2^{1+\frac{1}{4}(7\chi+11\sigma)} \exp \left( \sum_{a,b} \frac{\alpha_a \alpha_b}{2} \#(\Sigma_a \cap \Sigma_b) + 2\lambda \right) \prod_y F_y(\phi_{y,\beta})$$

$$+ i2^{1+\frac{1}{4}(7\chi+11\sigma)} \exp \left( -\sum_{a,b} \frac{\alpha_a \alpha_b}{2} \#(\Sigma_a \cap \Sigma_b) - 2\lambda \right) \prod_y F_y(-i\phi_{y,\beta}), \quad (2.70)$$

where $F_y(\phi_{y,\beta})$ is a universal function, depending only on the local structure of $\omega$ near $C_y$. $F_y$ is a function of all the $\phi_{y,\beta}$ for $C_{y,\beta}$ a component of $C_y$. 

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We can be somewhat more precise about the nature of the function $F_y$. Let us focus on a particular component $C_y$ of the canonical divisor; we will simply call it $C$, suppressing the $y$ index. Let $C_{\beta}$, $\beta = 1 \ldots w$ be the irreducible components of $C$, and deleting the $y$ index let $\phi_{\beta} = \sum a_\alpha \#(\Sigma_a \cap C_{\beta})$. Each $C_{\beta}$ has a multiplicity $n_{\beta}$ (the order of vanishing of the two-form $\omega$ in approaching a generic point of $C_{\beta}$). The vacuum state $|+\rangle$ of the bulk theory bifurcates along $C_{\beta}$ to a number of vacuum states $|\rho_{i_{\beta}}\rangle$, where $i_{\beta}$ runs over a finite set $S_{\beta}$. According to our assumptions above, if $n_{\beta} = 1$, then $S_{\beta}$ is a set with two elements. In what follows, I will make no assumption about the number of vacua for $n_{\beta} > 1$, but I will assume that the cosmic string theory has a mass gap and a finite vacuum degeneracy (which can be detected by local operators) for any value of $n_{\beta}$. A vacuum state of the cosmic string theory can be labeled by a $w$-plet $\vec{\rho} = (\rho_{i_1}, \ldots, \rho_{i_w})$. Indeed, once the generic behavior along the $C_{\beta}$ is given, all that remain are finitely many isolated singular points; no further issues of infinite volume and vacuum degeneracy arise in discussing the behavior near such isolated points.

The function $F$ is simply the expectation in the cosmic string theory associated with $C$ of the operator $\exp(\sum a_\alpha I(\Sigma_a))$. As in (2.56), the exponent can be replaced by $\sum_{\beta} \phi_{\beta} V_{\beta}$ where $V_{\beta}$ is the operator $V$ inserted at a generic point of $C_{\beta}$ and $\phi_{\beta} = \sum a_\alpha \#(\Sigma_a \cap C_{\beta})$. In sum, we must calculate the expectation value of the operator $\exp(\sum_{\beta} \phi_{\beta} V_{\beta})$ in the cosmic string theory. Scaling up the metric, this reduces in the usual way to a sum over the contributions of vacuum states,

$$F = \sum_{\vec{\rho}} \exp \left( \sum_{\beta} \phi_{\beta} V_{\beta} \right) |_{\vec{\rho}},$$

(2.71)

* In fact, a reasonable guess is that in general the cosmic string theory has $2^{n_0}$ vacua. The logic behind this is that a component $C_0$ with a given multiplicity $n_0$ arises from a two-form with an $n_0^{th}$ order zero along a complex curve, say $\omega = z_1^{n_0} dz_1 \wedge dz_2$. Such a two-form can be deformed to a two-form with simple zeroes on $n_0$ distinct curves, say $\omega' = \prod_{i=1}^{n_0} (z_1 - a_i) dz_1 \wedge dz_2$. After the perturbation, there are $2^{n_0}$ vacua, coming from a two-fold degeneracy on each of the $n_0$ components. If the mass gap holds uniformly during the process of deformation from $\omega$ to $\omega'$, then $2^{n_0}$ is also the number of vacua prior to the deformation.
with \( a_\tilde{\rho} \) being a constant measuring the amplitude with which the vacuum \( \tilde{\rho} \) appears. (Apart from the sort of effects considered in deriving (2.52) and (2.53), the \( a_\tilde{\rho} \) might receive contributions involving the structure of the singularities of \( C \). The singularities, being isolated, do not affect the bulk or vacuum structure along \( C \), but might contribute volume-independent multiplicative factors.) This sum can be factored as a product of contributions from the individual components \( C_\beta \):

\[
F = \sum_{\tilde{\rho}=(\rho_{i_1},...\rho_{i_w})} a_\tilde{\rho} \prod_{\beta=1}^{w} \exp \left( \phi_\beta \langle V_\beta \rangle_{i_\beta} \right) . \tag{2.72}
\]

Here \( \langle V_\beta \rangle_{i_\beta} \) is the expectation value of \( V_\beta \) in the vacuum \( \rho_{i_\beta} \). This is our final result.

So in particular, \( F \) is a sum of exponential functions of the \( \phi_\beta \). In view of how \( F \) enters in (2.70), this is in fact the general structure proved by Kronheimer and Mrowka [11] for four-manifolds of simple type. We actually have somewhat more precise information about the exponents that arise in \( F \). (These exponents are the “simple classes” in the language of Kronheimer and Mrowka.) For \( n_\beta = 1 \), the expectation value \( \langle V_\beta \rangle_{i_\beta} \) equals 1 or \(-1\), with our normalization above. For general \( n_\beta \), the possible values of \( \langle V_\beta \rangle_{i_\beta} \) depend only on \( n_\beta \). The simple classes are of the form

\[
\sum_{y,\beta} \langle V_\beta \rangle_{i_\beta} C_{y,\beta}, \tag{2.73}
\]

with \( i_\beta \in S_\beta \) and so are determined by the \( n_\beta \).

If one is willing to make the assumption of the last footnote, one can be more precise: the possible values of \( \langle V \rangle \) are of the form \( \pm 1 \pm 1 \ldots \pm 1 \) with a sum of \( n_\beta \) terms, which are the contributions of the \( n_\beta \) “ordinary” cosmic strings into which the multiplicity \( n_\beta \) string can be deformed. The expectation values are thus \( n_\beta, n_\beta - 2, n_\beta - 4, \ldots, -n_\beta \), and the simple classes are thus

\[
\sum_{y,\beta} a_{y,\beta} C_{y,\beta}, \tag{2.74}
\]
where the $a_{y,\beta}$ are integers no greater in absolute value than the multiplicity $n_{y,\beta}$ and congruent to $n_{y,\beta}$ modulo two. There is actually some additional evidence for this formula: by considering the special case of an elliptic surface with a multiple fiber, and comparing to a conjecture presented at the end of [11], one can see that if their conjecture is true, the expectation values and simple classes are as just stated. (The multiple fiber can have an arbitrary $n_\beta$, so a knowledge of the possible values of $\langle V_\beta \rangle$ in this case determine the general structure of the simple classes.) It may be that the list of simple classes is shorter than just claimed, as some of the $a_\vec{\rho}$ may vanish.

There remains the question of whether the constants $a_\vec{\rho}$ of equation (2.71) contain any deep information about four-manifolds. I expect the answer to be negative because the function $F$ is invariant under a rather crude equivalence relation. Because of the mass gap, in studying the cosmic string theory near $C$, one can restrict to a small open neighborhood of $U$; because of the underlying topological invariance, one can then deform the complex structure of $U$ and the two-form $\omega$. I expect that using this relation one can eliminate the singularities of $C$ and reduce to the case that a connected component $C$ of the canonical divisor is a smooth curve of multiplicity 1 or a genus one curve of multiplicity $n$ with normal bundle of order $n$. In the former case, from our above results, $F(\phi) = 2 \cosh \phi$ or $2 \sinh \phi$ depending on the type of the normal bundle; in the latter case, according to the conjecture of Kronheimer and Mrowka, $F(\phi) = \sinh(n + 1) \phi / \sinh \phi$.

Kahler Manifolds With $\pi_1 \neq 0$

To complete the discussion of Kahler manifolds, it remains to discuss the case of a Kahler manifold $M$ with $\pi_1(M) \neq 0$.

Actually, the fundamental group played no particular role in our considerations (given that we assume that there are no unbroken and unconfined gauge symmetries), and as long as one restricts to the operators $O(x)$ and $I(\Sigma)$, the correlation functions are still given by the above formulas. The only real novelty is the following. When the odd Betti numbers of $M$ are non-zero, one can consider additional
operators \( I(T) = \int_T \mathcal{O}^{(r)} \), with \( T \) an \( r \)-cycle in \( M \) for \( r = 1 \) or \( r = 3 \).

Correlation functions of the new operators can be analyzed with the same methods used for \( I(\Sigma) \) and \( \mathcal{O} \). The mass gap means that the correlation functions depend only on intersections of the various cycles with each other and with the canonical divisor. Just as in our discussion of two-cycles, the intersections that do not involve the canonical divisor detect only the classical intersection ring of \( M \). The interesting information comes from intersections involving the canonical divisor \( C \); the only such intersection involving odd-dimensional cycles is \( T \cap T' \cap C \) with \( T \) and \( T' \) being three-cycles. Since \( \Sigma = T \cap T' \) is a two-dimensional class, we are again studying intersections \( \Sigma \cap C \). The only novelty might be that the operator \( V \) of the above discussion would be replaced by another operator of the cosmic string theory, but I doubt that anything essentially new can be detected in this way.

2.10. Other Gauge Groups

Before discussing other gauge groups in general, let us discuss one particularly simple case: \( G' = SO(3) \). Because \( G' \) is locally isomorphic to \( G = SU(2) \), all local considerations are unchanged. The symmetries, the symmetry breaking pattern, the vacuum structure, and the constants \( a, b, \eta, \langle \mathcal{O} \rangle \), and \( \langle V \rangle \), are all determined by the structure on \( \mathbb{R}^4 \) (perhaps in the presence of an infinite straight cosmic string) and are unchanged in going from \( SU(2) \) to \( SO(3) \). Only a few global factors need modification.

The only difference between the \( SO(3) \) theory and the \( SU(2) \) theory is that there are \( SO(3) \) bundles that cannot be derived from \( SU(2) \) bundles. Given an \( SO(3) \) bundle \( E \), the obstruction to lifting its structure group to \( SU(2) \) is the second Stiefel-Whitney class \( w_2(E) \). Before proceeding, we must formulate precisely what we mean by the \( SO(3) \) theory. It is natural to define an \( SO(3) \) theory by summing over all isomorphism classes of \( SO(3) \) bundles, but to get the sharpest mathematical formulas, I will instead fix the second Stieffel-Whitney class and sum
over all bundles with a given value of \( w_2(E) \). Also, I will take the gauge group to be the “same” as the \( SU(2) \) gauge group in the sense of only allowing gauge transformations which, along any path on which the bundle \( E \) is trivialized, lift to \( SU(2) \) gauge transformations. (Otherwise we would have to divide by an extra factor equal to the number of components of the group of \( SO(3) \) gauge transformations; this is \( \#H^1(M, Z_2) \). Also, as \( SO(3) \) has no center, the factor of 2 in (2.18) in comparing different normalizations would not appear.)

The remaining global issues that entered the \( SU(2) \) analysis involved the action of the global symmetries on the space of vacua. Since the \( SO(3) \) theory coincides with the \( SU(2) \) theory on \( \mathbb{R}^4 \), it has after perturbing by the bare mass the familiar global symmetry group \( \mathbb{Z}_4 \times \mathbb{Z}_2' \). However, on a four-manifold \( M \), the path integral measure may transform differently. As we recall, the transformation of the path integral measure depends on index theorems that determine the number of fermion zero modes both in bulk and along the cosmic strings.

If \( M \) is simply-connected, \( w_2(E) \) can be lifted to an integral cohomology class (which we will also call \( w_2(E) \)) which is well-defined modulo two; hence \( w_2(E)^2 \) is well-defined modulo four. Even if \( M \) is not simply-connected, one can still make sense of \( w_2(E)^2 \) modulo four, as noted in [7, p. 41]. The expression \( c_2(E) = \int_M \text{Tr} F \wedge F/8\pi^2 \) (with \( \text{Tr} \) an invariant quadratic form on the Lie algebra normalized to agree with the trace in the two-dimensional representation of \( SU(2) \)) is not necessarily an integer if \( E \) cannot be derived from an \( SU(2) \) bundle. The relation is rather

\[
c_2(E) \cong -\frac{w_2(E)^2}{4} \mod \mathbb{Z}.
\]

Consequently, the formula (2.44) for the net number of \( \lambda \) zero modes is modified to

\[
\Delta_E = 4k - w_2(E)^2 - 3(1 - h^{1,0} + h^{2,0}),
\]

with \( k \in \mathbb{Z} \). The path integral measure therefore transforms under the generator
\( \alpha \) of \( \mathbb{Z}_4 \times \mathbb{Z}_2' \) not as \( i^\Delta \) but as \( i^{\Delta_E} \), with

\[
\Delta_E = \Delta - w_2(E)^2. \tag{2.77}
\]

Now let us consider what happens along a cosmic string component \( C_y \). Let \( S \) be the relevant spin bundle along \( C_y \). The number of fermion zero modes modulo two is a topological invariant of the bundle \( S \otimes E \). If \( w_2(E) \) is zero when restricted to \( C_y \), then we can trivialize \( E \) along \( C_y \); then \( S \otimes E \cong S \oplus S \oplus S \). The number of zero modes of fermions coupled to \( S \otimes E \) is hence congruent modulo two to the number \( \epsilon_y \) of zero modes of fermions coupled to \( S \). Hence the transformation law of the measure is \( t_{y,E} = (-1)^{\epsilon_y + (w_2(E),C_y)} \) if \( (w_2(E),C_y) = 0 \). On the other hand, if \( (w_2(E),C_y) \neq 0 \), then we can take \( E \) restricted to \( C_y \) to be \( O \oplus O(n) \oplus O(-n) \), where \( O(n) \) is any line bundle of odd degree \( n \) and \( O(-n) \) is its inverse. So \( S \otimes E \cong S \oplus S(n) \oplus S(-n) \); for large positive \( n \), the number of zero modes for fermions coupled to those three line bundles are \( \epsilon_y, n, \) and \( 0 \). Since \( n \) is odd, this is congruent to \( \epsilon_y + 1 \) modulo two, so the measure transforms as \( t_{y,E} = (-1)^{\epsilon_y + 1} \) in this case. In either case, we can write

\[
t_{y,E} = (-1)^{\epsilon_y + (w_2(E),C_y)}. \tag{2.78}
\]

Now we can assemble the ingredients. Pick a class \( x \in H^2(M, \mathbb{Z}_2) \), and let \( \langle \cdot \rangle_{T,x} \) denote a topologically normalized correlation function, summed over all isomorphism classes of \( SO(3) \) bundles \( E \) with \( w_2(E) = x \). Then

\[
\left\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) + \lambda O \right) \right\rangle_{T,x} = 2^{1+\frac{1}{4}(7\chi+11\sigma)} \exp \left( \sum_{a,b} \frac{\alpha_a \alpha_b}{2} \#(\Sigma_a \cap \Sigma_b) + 2\lambda \right) \prod_y \left( e^{-\phi_y} + (1)^{\langle x,C_y \rangle} t_y e^{-\phi_y} \right)
\]

\[
+ i^{\Delta - x^2} 2^{1+\frac{1}{4}(7\chi+11\sigma)} \exp \left( - \sum_{a,b} \frac{\alpha_a \alpha_b}{2} \#(\Sigma_a \cap \Sigma_b) - 2\lambda \right) \prod_y \left( e^{-i\phi_y} + (1)^{\langle x,C_y \rangle} t_y e^{i\phi_y} \right). \tag{2.79}
\]

Thus, while the \( SU(2) \) theory determines the cohomology classes of the \( C_y \) modulo
torsion, the $SO(3)$ theory determines the pairings of the $C_y$ with arbitrary mod two classes and thus is sensitive to the two-torsion. The additional information is trivial for simply-connected $M$, but in general the $SO(3)$ theory does contain limited additional information.

That the right hand side of (2.79) is real follows from (2.61) together with the following standard fact:

$$(x, x) + (x, C) = 0 \text{ modulo } 2,$$  \hspace{1cm} (2.80)

with $x$ any element of $H^2(M, \mathbb{Z}_2)$ and $C = \sum_y C_y$ the canonical divisor (which we are identifying with the Poincaré dual cohomology class). Indeed, since $C$ reduces mod 2 to the second Stieffel-Whitney class $w_2(M)$, this follows from the Wu formula, which asserts that $(x, x) = (x, w_2(M)) \text{ mod } 2$ for any $x \in H^2(M, \mathbb{Z}_2)$. See [7, p. 6] for a quick proof of the Wu formula for the case that $M$ is simply-connected.

Groups Of Higher Rank

Now, let us briefly discuss what happens for gauge groups of higher rank.

One modification is that the global symmetry of the $N = 2$ theory perturbed by a mass term that respects $N = 1$ invariance is not $\mathbb{Z}_4 \times \mathbb{Z}_2'$ but $\mathbb{Z}_{2h} \times \mathbb{Z}_2'$, with $h$ the dual Coxeter number. In bulk this is broken down to $\mathbb{Z}_2 \times \mathbb{Z}_2'$, so the number of vacuum states in bulk is $h$ instead of 2. Along the cosmic string it is most plausible that $\mathbb{Z}_2 \times \mathbb{Z}_2'$ is broken down to a diagonal $\mathbb{Z}_2$ subgroup (which permits fermion masses), giving a two-fold degeneracy.

A more substantial change comes from the fact that for the gauge groups of rank greater than one, there are higher Casimir operators. For instance, for $G = SU(n)$, (2.10) can be generalized to $O_r^{(0)}(x) = \frac{1}{(2\pi)^r r!} \text{Tr} \phi^{r+1}(x)$ with $1 \leq r \leq n - 1$. Applying the descent procedure, one gets operator valued $k$-forms $O_r^{(k)}(x)$, for $r = 1 \ldots n - 1$. The main novelty is the occurrence of non-trivial operator-valued
four-forms (in contrast to $\mathcal{O}_1^{(4)}$ which is related to a classical invariant, the instanton number). One can perturb the Lagrangian by

$$L \rightarrow L + \sum_{r=2}^{n-1} b_r \int_M \mathcal{O}_r^{(4)}, \quad (2.81)$$

with arbitrary coupling constants $b_r$; this gives a family of topological field theories.

We can think of the $b_r$ as infinitesimal variables in the sense that $b_r^{n_r} = 0$ for some large, unspecified values of $n_r$. This is sufficient for studying the correlation functions, which is all we really want. Thus in particular, turning on the $b_r$ does not modify the mass gap and symmetry breaking pattern, which are stable against small perturbations. After turning on the $b_r$, correlation functions of the $\mathcal{O}_r^{(d)}$ with $d < 4$ can be computed as we have done above for $SU(2)$ in terms of a certain number of universal quantities (in the $SU(2)$ case they are $\eta$, $\langle \mathcal{O} \rangle$, $\langle V \rangle$, $a$, and $b$; for higher rank there are more such constants because there are more operators). Once the $b_r$ are permitted to vary, the universal quantities are no longer “constants” but become instead functions of the $b_r$.

Thus, a somewhat more elaborate structure arises in the case of Lie groups of rank greater than one. However, the implications for topology of Kahler manifolds may be limited, since one can only hope to detect the structure of $C$, but most of the invariants of $C$ that possess the necessary properties to be detected by theories such as these ones are already detected by the $SU(2)$ and $SO(3)$ theories. Perhaps with other groups one can learn more about the torsion part of the cohomology classes of the $C_y$; so far we have detected only the two-torsion (via the $SO(3)$ theory).

\* One can see such a phenomenon in the two-dimensional analog of Donaldson theory – where it occurs already for $SU(2)$. The analog of the $b_r$ are the $\delta_i$ in eqn. (5.5) of [14]; the dependence on $\delta_i$ arises through a change of variables given in eqn. (5.9) as a result of which the “constants” that control correlation functions of $Q_{(0)}$ and $Q_{(1)}$ become functions of the $\delta_i$. 

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3. Properties Of Supersymmetric Yang-Mills Theories

The purpose of this section is to fill in details suppressed in §2 concerning the relevant properties of supersymmetric Yang-Mills theories. In doing so, we will generally use conventions of Wess and Bagger [16]. For instance, doublets of the $SU(2)_L$ (or $SU(2)_R$) rotation symmetries are represented by spinor indices $\alpha, \beta \ldots = 1, 2$ (or $\dot{\alpha}, \dot{\beta}, \ldots = 1, 2$). Doublets of the internal $SU(2)_I$ symmetry will be denoted by indices $i, j, \ldots = 1, 2$. These indices are raised and lowered with the antisymmetric tensor $\epsilon_{\alpha\beta}$ (or $\epsilon_{\dot{\alpha}\dot{\beta}}, \epsilon_{ij}$) with sign convention such that $\epsilon_{12} = 1 = \epsilon^{21}$.

Tangent vector indices to spacetime are denoted $m, n, \ldots = 1 \ldots 4$; the spinor and tangent vector indices are related by the tensor $\sigma^m_{\lambda \dot{\alpha}}$ described in appendix A of [16]; one similarly uses $\sigma^{mn}_{\lambda \dot{\alpha} \dot{\beta}} = (\sigma^m_{\lambda \dot{\alpha}} \sigma^n_{\dot{\alpha} \dot{\beta}} - m \leftrightarrow n) / 4$, etc.

The fields of the minimal $N = 2$ supersymmetric Yang-Mills theory are the following: a gauge field $A_m$, fermions $\lambda^i_{\alpha}$ and $\lambda^{\dot{i}}_{\dot{\alpha}}$ transforming as $(1/2, 0, 1/2)$ and $(0, 1/2, 1/2)$ under $SU(2)_L \times SU(2)_R \times SU(2)_I$, and a complex scalar $B$ — all in the adjoint representation of the gauge group. Covariant derivatives are defined by $D_m \Phi = (\partial_m + iA_m) \Phi$ and the Yang-Mills field strength is $F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n]$. The supersymmetry generators transform as $(1/2, 0, 1/2) \oplus (0, 1/2, 1/2)$; introducing infinitesimal parameters $\eta_{\alpha}^i$ and $\bar{\eta}_{\dot{\alpha}}^{\dot{i}}$, they can be written

\[
\begin{align*}
\delta A_m &= -i\bar{\lambda}_{\dot{\alpha}}^{\dot{i}} \sigma_{ma\dot{\alpha}} \eta_{\alpha}^i + i\bar{\eta}_{\dot{\alpha}}^{\dot{i}} i \sigma_{ma\dot{\alpha}} \lambda^i_{\alpha} \\
\delta \lambda^i_{\alpha} &= \sigma^{mn}_{\alpha \beta} \eta^j_{\beta} F_{mn} + i\eta_{\alpha}^i D + i\sqrt{2} \sigma^{m}_{a\dot{a}} D_m B e^{ij} \bar{\eta}^{\dot{i}}_{\dot{a}} \\
\delta \bar{\lambda}_{\dot{a}}^{\dot{i}} &= \sigma^{mn}_{\alpha \beta} \eta^j_{\beta} F_{mn} - i\eta_{\dot{a}}^{\dot{i}} D + i\sqrt{2} D_m B e^{ij} \eta^{\alpha j} \\
\delta B &= \sqrt{2} \eta^{\alpha i} \lambda^i_{\alpha} \\
\delta \bar{B} &= \sqrt{2} \bar{\eta}^{\dot{i} \dot{a}} i \bar{\lambda}_{\dot{a}}^{\dot{i}}
\end{align*}
\] (3.1)

with $D = [B, \overline{B}]$. The minimal Lagrangian is (with signature $-+++$)

\[
L = \frac{1}{e^2} \int_M d^4x \text{Tr} \left( -\frac{1}{4} F_{mn} F^{mn} - i\bar{\lambda}_{\dot{\alpha}}^{\dot{i}} \sigma^m_{a\dot{a}} D_m \lambda^i_{\alpha} - D_m \overline{B} D^m B - \frac{1}{2} [\bar{B}, B]^2 - \frac{i}{\sqrt{2}} B e^{ij} [\lambda_{\alpha}^i, \lambda_{\alpha}^j] + \frac{i}{\sqrt{2}} B e^{ij} [\bar{\lambda}_{\dot{a}}^{\dot{i}}, \bar{\lambda}_{\dot{a}}^{\dot{j}}] \right). \tag{3.2}
\]
These formulas have been adapted from the formulas of [16, p. 50] for $N = 1$ supersymmetry. Here $\text{Tr}$ is an invariant quadratic form on the Lie algebra which for $G = SU(N)$ we can conveniently take to be the trace in the $N$ dimensional representation.

To construct a topological field theory, one replaces $SU(2)_R$ by a diagonal subgroup of $SU(2)_R \times SU(2)_I$. This can be conveniently incorporated in the formalism by replacing the internal indices $i, j, \ldots$ by another $SU(2)_R$ index $\hat{\beta}$. To describe the topological transformation laws, one sets

$$\eta^{\alpha i} = 0,$$
$$\eta^{\hat{\alpha} \hat{\beta}} = -\epsilon^{\hat{\alpha} \hat{\beta}} \rho$$

(with $\rho$ an anticommuting parameter). This gives a one-component supersymmetry which is the BRST symmetry of the twisted topological theory; the corresponding charge will be called $Q$. The transformation laws are

$$\delta A_m = i\rho \sigma_{m \alpha \hat{\beta}} \lambda^{\alpha \hat{\beta}},$$
$$\delta \lambda_{\alpha \hat{\beta}} = i\rho \sqrt{2} \sigma_{m \alpha \hat{\beta}} D_mB,$$
$$\delta \lambda_{\hat{\alpha} \hat{\beta}} = \rho \sigma^{mn}_{\hat{\alpha} \hat{\beta}} F_{mn} + i\epsilon_{\hat{\alpha} \hat{\beta}} \rho D,$$
$$\delta B = 0$$
$$\delta \overline{B} = -\sqrt{2} \rho \epsilon_{\hat{\alpha} \hat{\beta}} \overline{\lambda}^{\hat{\alpha} \hat{\beta}}.$$

The square of this transformation vanishes, up to a gauge transformation. By requiring invariance under this topological symmetry, one can construct a topological field theory on an arbitrary four-manifold $M$.

Now let us construct the observables of this theory. If we set

$$\lambda_m = \sigma_m^{\alpha \hat{\beta}} \lambda_{\alpha \hat{\beta}},$$
$$B = \frac{i\phi}{2\sqrt{2}}.$$

(3.5)
then the topological transformation laws become, in part,

\[
\begin{align*}
\delta A_m &= i \rho \lambda_m \\
\delta \lambda_m &= -\rho D_m \phi \\
\delta \phi &= 0,
\end{align*}
\]  

(3.6)

which is a standard topological field theory multiplet (related mathematically to the equivariant cohomology of the gauge group acting on the space of connections). The other fields do not contribute to the BRST cohomology.

The most obvious invariant operators are of the form \( \mathcal{O}(x) = P(\phi(x)) \), with \( P \) an invariant polynomial on the Lie algebra of \( G \) and \( x \) a point in the four-manifold \( M \). For \( G = SU(2) \), the only essential choice of \( P \) is the quadratic Casimir operator, and then we can take

\[
\mathcal{O}(x) = \frac{1}{8\pi^2} \text{Tr} \phi^2 = -\frac{1}{\pi^2} \text{Tr} B^2.
\]  

(3.7)

Calling this operator \( \mathcal{O}^{(0)}(x) \) to stress that it can be regarded as a zero-form on \( M \), one next iteratively finds operator-valued \( k \)-forms \( \mathcal{O}^{(k)} \), \( 1 \leq k \leq 4 \), obeying

\[
d\mathcal{O}^{(k)} = \{Q, \mathcal{O}^{(k+1)}\}.
\]  

(3.8)

For full details see [4]. In the case of a simply-connected four-manifold, the most important of these operators is \( \mathcal{O}^{(2)} \) (as we explained in §2.2). The components of \( \mathcal{O}^{(2)} \) were called \( Z_{mn} \) in §2; explicitly

\[
Z_{mn} = -\frac{i}{4\pi^2} \text{Tr} \left( 2\sqrt{2} B F_{mn} + \lambda_m \lambda_n \right).
\]  

(3.9)

One now, as in §2, defines observables by

\[
I(\Sigma) = \int_{\Sigma} \mathcal{O}^{(2)},
\]  

(3.10)

with \( \Sigma \) a two-dimensional homology cycle. From (3.8) one deduces that \( I(\Sigma) \) is
Q-invariant, and up to \( \{Q, \ldots \} \) depends only on the homology class of \( \Sigma \). Equivalentlly, one can introduce a closed two form \( \theta \) and consider

\[
I(\theta) = \int_M \mathcal{O}^{(2)} \wedge \theta. \tag{3.11}
\]

The two formulations are essentially equivalent, with \( \theta \) being the Poincaré dual of \( \Sigma \).

### 3.1. Kähler Manifolds

Now we consider the further constructions that are possible if the metric on \( M \) is Kähler. This means that the holonomy is not \( SU(2)_L \times SU(2)_R \) but \( SU(2)_L \times U(1)_R \), with \( U(1)_R \) a subgroup of \( SU(2)_R \). The two-dimensional representation of \( SU(2)_R \) decomposes under \( U(1)_R \) as a sum of two one-dimensional representations (of “charge \( \pm 1/2 \)”). Accordingly, there are invariant projections of any doublet \( v^\dot{\beta} \) onto the components \( v^\dot{1} \) and \( v^\dot{2} \) transforming in definite \( U(1)_R \) representations. This permits one to define a complex structure on \( M \); one simply declares that the one-forms \( dx^m \sigma_{m\alpha\dot{2}} \) are of type \( (0, 1) \) while the one-forms \( dx^m \sigma_{m\alpha\dot{1}} \) are of type \( (1, 0) \).

Similarly, one can decompose the topological generator of equation (3.3) into two components, taking

\[
\begin{align*}
\eta_{\alpha\dot{i}} &= 0 \\
\bar{\eta}_{\dot{\alpha}\dot{1}} &= \rho_1 \epsilon_{\dot{\alpha}\dot{1}} \\
\bar{\eta}_{\dot{\alpha}\dot{2}} &= \rho_2 \epsilon_{\dot{\alpha}\dot{2}}
\end{align*} \tag{3.12}
\]

with anticommuting parameters \( \rho_1, \rho_2 \). The explicit \( \dot{1} \) in (3.12) means that the \( \rho_1 \) transformation is the part of the topological symmetry of (3.3) that comes from an \( N = 1 \) subalgebra of the underlying \( N = 2 \) supersymmetry; likewise the \( \rho_2 \) transformation comes from the second \( N = 1 \) subalgebra. Let \( Q_1, Q_2 \) be the charges corresponding to the \( \rho_1 \) and \( \rho_2 \) transformations. Then \( Q = Q_1 + Q_2 \), and \( Q_1^2 = Q_2^2 = \{Q_1, Q_2\} = 0 \).
The vertex operators $O^{(k)}$ are invariant under both the $\rho_1$ and $\rho_2$ transformations, modulo the equations of motion. The fact that the equations of motion enter here will be important presently.

In this paper, it will suffice to consider the $\rho_1$ symmetry, with $\rho_2 = 0$. The transformation laws are

\[
\begin{align*}
\delta A_m &= i\rho_1 \sigma_{ma_1} \lambda^{a_1} \\
\delta \lambda_{a_1} &= i\sqrt{2} \rho_1 \sigma^m_{a_1} D_mB \\
\delta \lambda_{a_2} &= 0 \\
\delta \xi_{a_1} &= \rho_1 (\sigma^{mn}_{a_1} F_{mn} - i\epsilon_{a_1} D) \\
\delta \xi_{a_2} &= 0 \\
\delta B &= 0 \\
\delta B &= -\sqrt{2} \rho_1 \xi_{i_2} .
\end{align*}
\]

Certain of these equations have the following interpretation. If we decompose $F$ into its parts $F^{2,0}$, $F^{1,1}$, and $F^{0,2}$ of the indicated types, then $\delta \xi_{i_1}$ is proportional to $F^{0,2}$, so a BRST-invariant configuration must have $F^{0,2} = 0$; hence it describes a holomorphic vector bundle. The first equation shows that the $(0,1)$ part of the connection $A$ is BRST-invariant; hence the holomorphic structure of the bundle is BRST-invariant.

To justify a claim made at the end of §2.4, the reason that $Q_1$-invariance is enough in analyzing the topological correlation functions is that the change in the Lagrangian under a change in the Kahler metric or coupling constant can be written in the form $\{Q_1, \ldots\}$. We can see this as follows. Since $Q_1$ lies in an $N = 1$ algebra, it is convenient to recall the standard way of writing the $N = 2$ theory in $N = 1$ superspace. The part involving the $\Phi$ multiplet is $\int d^4x d^2\theta d^2\bar{\theta} \text{Tr} \bar{\Phi} \Phi$. Since $Q_1$ acts (up to a total derivative) as $\int d\theta^\alpha$ for suitable $\alpha$, this is $\{Q_1, \ldots\}$, and its variation with respect to metric or coupling has the same property. The gauge kinetic energy is $\int d^4x d^2\theta \text{Tr} W^2 + \int d^4x d^2\bar{\theta} \text{Tr} \bar{W}^2$. The first term here is
\{Q_1, \ldots \} for the same reason as before, while (by a standard identity) the second coincides with the first up to a multiple of the instanton number. Hence a change in Kahler metric or gauge coupling induces at most a term of the form \{Q, \ldots \} plus a change in the \(\theta\) angle (which is the coupling that multiplies the instanton number). Using the familiar chiral anomaly of the \(U(1)\) \(R\)-current, a change in the \(\theta\) angle can be absorbed in a chiral rotation or rescaling of the fields \(I(\Sigma)\) and \(\mathcal{O}\) and (allowing for the gravitational part of the chiral anomaly) a redefinition of the constants called \(a\) and \(b\) in (2.24). The rescaling of \(I(\Sigma)\) and \(\mathcal{O}\) merely affect the values of the constants \(\eta, \langle \mathcal{O} \rangle\), and \(v\) that entered our formulas.

**The Mass Term**

Exploiting the fact that \(Q_1\) is the only essential symmetry, we want to introduce a mass term for some of the fields, preserving the \(Q_1\) symmetry. As was foreseen in §2, this construction depends on a choice of holomorphic two-form \(\omega\) on \(M\).

We consider adding to the Lagrangian \(I(\omega) = \int_M \mathcal{O}^{(2)} \wedge \omega\). This contains a term \(\int_M \text{Tr} BF \wedge \omega\). Since \(\omega\) is of type \((2, 0)\), only the \((0, 2)\) part of \(F\) enters here; as noted after equation (3.13), this is of the form \(\{Q_1, \ldots \}\). So modulo \(\{Q_1, \ldots \}\), the \(BF\) term can be dropped and \(I(\omega)\) is proportional to

\[
L_1 = -\frac{1}{2} \int_M \text{Tr} \lambda_\alpha \hat{\lambda}_\bar{\alpha} \sigma_{mn22} \cdot \omega_{kl} \epsilon^{mnlk} \sqrt{g} d^4 x. \tag{3.14}
\]

Now, in verifying the \(Q_1\) invariance of \(L_1\) (or equivalently, of \(I(\omega)\)) it is necessary to use the equations of motion. In fact, a small computation shows that

\[
\delta L_1 = -i \sqrt{2} \rho_1 \int_M \text{Tr} B \epsilon^{\hat{\alpha} \hat{\beta}} \sigma^m_{\alpha \bar{\alpha}} D_m \lambda_\alpha \hat{\lambda}_\bar{\alpha} \omega_{kl} \epsilon^{nplk} \sqrt{g} d^4 x. \tag{3.15}
\]

This vanishes by the equation of motion of \(\bar{\lambda}\), as one can see by considering the part of the original Lagrangian \(L\) that contains \(\bar{\lambda}\). This is

\[
L_0 = -\frac{i}{e^2} \int d^4 x \sqrt{g} \text{Tr} \left( \bar{\lambda} \sigma^m_{\alpha \bar{\alpha}} D_m \lambda + \sqrt{2} \bar{\lambda} [B, \lambda] \right). \tag{3.16}
\]

Instead of saying that \(\delta L_1\) is \(Q_1\) invariant modulo the equations of motion derived
from $L$, one can modify the $Q_1$ transformation laws so that the full Lagrangian $L + L_1$ is $Q_1$-invariant. The requisite correction to the transformation laws is that, instead of vanishing, $\delta \bar{\lambda}_{\dot{a}2}$ should be as follows:

$$\delta \bar{\lambda}_{\dot{a}2} = -\sqrt{2} e^2 \rho_1 B \epsilon^{\dot{a}2} \sigma_{np\dot{a}\dot{2}} \omega_{kl} \epsilon^{npkl}. \quad (3.17)$$

Since $\delta B = 0$, this correction to the transformation law does not affect the fact that $Q_1^2 = 0$. Nor does it affect the $Q_1$ invariance of the various observables of the theory (such as $O$ or $I(\Sigma)$), as they are independent of $\bar{\lambda}$.

While preserving $Q_1$ invariance, we could add to the Lagrangian any expression of the form $\{Q_1, V\}$. It is convenient to do so with the particular choice

$$V = -\frac{1}{\sqrt{2}} \int d^4 x \sqrt{g} \bar{B} \bar{\lambda}_{\dot{a}2} \epsilon^{\dot{a}2} \sigma_{np\dot{a}\dot{2}} \omega_{kl} \epsilon^{npkl}. \quad (3.18)$$

The corresponding addition to the Lagrangian is

$$L_2 = -\frac{1}{2} \int d^4 x \sqrt{g} \bar{\lambda}_{\dot{a}2} \sigma_{np\dot{a}\dot{2}} \omega_{kl} \epsilon^{npkl} - e^2 \int d^4 x \sqrt{g} \bar{B} \bar{B}. \quad (3.19)$$

So we consider the combined Lagrangian

$$\hat{L} = L + L_1 + L_2 = L + I(\omega) + \{Q_1, \ldots\}$$

$$= L - \frac{1}{2} \int_M d^4 x \text{Tr} \left( m \lambda_{\alpha} \bar{\lambda}_{\alpha2} \bar{\lambda}_{\dot{a}2} \sigma_{mn\dot{a}\dot{2}} \omega_{kl} \epsilon^{npkl} m \right) - e^2 \int_M d^4 x \sqrt{2m} \text{Tr} \bar{B} B. \quad (3.20)$$

with

$$m = \sigma_{mn\dot{a}\dot{2}} \omega_{kl} \epsilon^{mnkl}. \quad (3.21)$$

As promised in §2, we have realized a mass term for the $N = 1$ matter multiplet (which consists of $B$ and $\psi_{\alpha} = \lambda_{\alpha2}$) by adding to the Lagrangian a term of the form $I(\omega) + \{Q_1, \ldots\}$. The mass is proportional to the holomorphic two-form $\omega$, as expected. The $N = 1$ gauge multiplet, consisting of the gauge field $A_m$ and the gluino $\lambda_{\alpha} = \lambda_{\alpha1}$, remains massless.
Symmetries

For subsequent use, it will be important to determine the transformation law of the observables $\mathcal{O}$ and $I(\theta)$ under the global symmetries of the theory. (For a related analysis, see [13].)

As we discussed in §2, the twisted theory, perturbed by the mass term as in (3.20), has a global symmetry $\mathbb{Z}_4 \times \mathbb{Z}_2'$. We will describe the symmetries in the language of $N=1$ supersymmetry, the $N=1$ multiplets being $(A_m, \lambda)$ and $(B, \psi)$. The generator $\alpha$ of $\mathbb{Z}_4$ acts by

$$\alpha(\lambda) = i\lambda, \quad \alpha(\bar{\lambda}) = -i\bar{\lambda}, \quad \alpha(B) = iB,$$

(3.22)

with $A_m, \psi$ being invariant. (Such a symmetry, which acts differently on different components of a supermultiplet, is usually called an $R$ symmetry.) The non-trivial element $\beta$ of $\mathbb{Z}_2'$ acts by

$$\beta(B) = -B, \quad \beta(\psi) = -\psi,$$

(3.23)

with the gauge multiplet $(A_m, \lambda)$ being invariant.

Now let us determine the transformation laws of the observables. $\mathcal{O}(x) = -\text{Tr} B^2/\pi^2$ evidently transforms as $(-1, 1)$ under $\alpha$ and $\beta$. Now let us consider the quantum numbers of $I(\theta)$ with $\theta$ a $(p, q)$-form (with $p+q = 2$); we will schematically call this operator $I^{p,q}$. $I^{1,1}$ is of the general form $\int_M \theta \wedge (\lambda \psi + BF)$; therefore, from the transformation laws presented in the last paragraph, $I^{1,1}$ transforms as $(i, -1)$.

What about $I^{2,0}$? This is analyzed most efficiently by noting that $I^{2,0}$ is present in the Lagrangian, so it must be invariant, transforming as $(1, 1)$, modulo $\{Q_1, \ldots\}$. If one tries to verify this explicitly, one has schematically $I^{2,0} \cong \psi \psi + BF$ in the original twisted topological field theory. The two terms transform differently under the symmetries, but the second is $\{Q_1, \ldots\}$, a fact exploited
above, and so can be dropped. It appears that we can take $I^{2,0} \cong \psi\psi$, which is invariant, as expected. This is the correct structure of $I^{2,0}$ in the absence of the mass perturbation introduced above, but that perturbation brings about the following change. We recall that $I^{2,0}$ was only $Q_1$-invariant modulo the $\lambda$ equations of motion. In adding a perturbation $L_2 = \{Q_1, V\}$ to the Lagrangian to complete the mass term (3.20), we have disturbed the $\lambda$ equations of motion, and accordingly an extra term must be added to $I^{2,0}$. The result can be found most simply by noting that (before or after addition of $L_2$) $I^{2,0}$ can be interpreted as $\partial L / \partial m$, with $m$ as in (3.21). This gives an extra term $\partial L_2 / \partial m$, and finally $I^{2,0} \cong \psi\psi + \omega \mathcal{B}B$.

Finally, we consider $I^{0,2}$, which is of the general form $\int_M \theta \wedge (\lambda \lambda + BF)$. Again, the two terms transform differently under the symmetries. The second term can be written as $\{Q_1, \ldots\}$ using the correction (3.17) to the transformation laws, and so is again inessential. So $I^{0,2}$ transforms as $(-1, 1)$.

To summarize this discussion, the quantum numbers of the observables are as follows:

\begin{align*}
\mathcal{O} : & \quad (-1, 1) \\
I^{2,0} : & \quad (1, 1) \\
I^{1,1} : & \quad (i, -1) \\
I^{0,2} : & \quad (-1, 1).
\end{align*}

Moreover modulo $\{Q_1, \ldots\}$, we have schematically

\begin{align*}
I^{1,1} & \cong \lambda \psi + BF \\
I^{0,2} & \cong \lambda \lambda. \tag{3.25}
\end{align*}

**Gluino Condensation**

Let us now analyze in bulk the dynamical effects of the mass perturbation in (3.20). This means that we work on a flat $\mathbb{R}^4$ and take $\omega = dz_1 \wedge dz_2$; of course

\footnote{This is actually only true for $\theta = \omega$, but that is good enough for probing Donaldson theory as $\omega$ was in any case an arbitrary holomorphic two-form.}
any example, suitably scaled up, looks like this locally as long as one keeps away
from the zeroes of $\omega$.

Seen as an $N = 1$ theory, the $N = 2$ supersymmetric Yang-Mills theory has
a gauge multiplet $(A_m, \lambda)$ and a matter multiplet $\Phi = (B, \psi)$. The perturbation
in (3.20) is in bulk simply the usual $N = 1$ invariant bare mass term for the $\Phi$
multiplet. Its addition to the Lagrangian leaves at low energies the pure $N = 1$
supersymmetric gauge theory of $(A_m, \lambda)$. This theory has a $\mathbb{Z}_4$ symmetry, generated
by the transformation $\alpha$ described in (3.22). It is believed that this symmetry is
broken to its $\mathbb{Z}_2$ subgroup, by “gluino condensation,” that is, by the expectation
value of $\text{Tr} \lambda^a \lambda_\alpha$, which is essentially our friend $\mathcal{I}^{0,2}$. We will abbreviate $\text{Tr} \lambda^a \lambda_\alpha$
as $\lambda$.

It will be necessary therefore for us to discuss some standard properties of
gluino condensation. Let us recall the basic structure of the renormalization group.
If a quantum theory is defined using a renormalization point $\mu$ and with a coupling
constant $g$, then the effective coupling constant $\bar{g}$ obeys

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}\right) \bar{g} = 0, \quad (3.26)$$

with $\beta(g)$ the beta function. If $\bar{g}$ is defined by measurements at an energy $E$,
then $\bar{g}$ is a function only of the dimensionless variables $E/\mu$ and $g$. If for weak
coupling the beta function looks like $\beta = -bg^3 + \ldots$ (with $b > 0$ for asymptotically
free theories) then the renormalization group equation can be written in the weak
coupling regime as

$$\left(\mu \frac{\partial}{\partial \mu} - 2b(g^2)^2 \frac{\partial}{\partial g^2}\right) \bar{g}^2(E/\mu, g) = 0. \quad (3.27)$$

The solution is

$$\frac{1}{\bar{g}^2(E/\mu)} = \frac{1}{g^2} - 2b \ln(\mu/E). \quad (3.28)$$
Strong coupling arises for $\mathcal{g} \cong 1$ or

$$E \cong \mu \exp(-1/2bg^2). \quad (3.29)$$

Even though a weak coupling approximation was used in solving (3.26), (3.29) gives correctly the singular behavior of $E$ for small $g$ which is in fact dominated by the behavior in the weakly coupled regime.

Now, we are interested in the case in which at some mass scale $m$, an $N = 2$ theory is explicitly broken down to an $N = 1$ theory. The $N = 2$ theory has a beta function $\beta = -bg^3 + \ldots$, and the $N = 1$ theory has a beta function $\beta = -b'g^3 + \ldots$. The relation between them is

$$b = \frac{2b'}{3}. \quad (3.30)$$

(In fact, the beta function of supersymmetric Yang-Mills theory of general $N$ is proportional to $N - 4$ and has a well-known zero for $N = 4$.) We suppose that $m << \mu$ and is large enough to be in the weak coupling regime. In that case, as long as $E \geq m$, we can use the solution (3.28) for the effective coupling. However, for $E \leq m$, we must evolve the effective coupling using the beta function of the low energy theory. In doing this, we use (3.28) to determine the initial conditions for the renormalization group evolution of the low energy theory. In fact, for the renormalization scale $\mu_L$ and coupling $g_L$ of the low energy theory, we take the values $\mu_L = m$ and $g_L = \mathcal{g}(m/\mu, g)$ that are appropriate for the microscopic $N = 2$ theory at the scale $m$. The subsequent evolution of the effective coupling to energies below $m$ is carried out using an equation just like (3.26), with $\mu, g, \mathcal{g}$, and $\beta$ replaced by $m, \mathcal{g}(m/\mu, g)$, the effective coupling $\mathcal{g}_L$ of the low energy theory, and the beta function of the low energy theory. So we get the analog of (3.28):

$$\frac{1}{\mathcal{g}_L^2(E/m, \mathcal{g}^2(m/\mu, g))} = \frac{1}{\mu^2} - 2b' \ln(m/E). \quad (3.31)$$

These equations combine to give

$$\frac{1}{\mathcal{g}_L^2} = \frac{1}{g^2} - 2b \ln(\mu/m) - 2b' \ln(m/E). \quad (3.32)$$
Gluino condensation occurs for $\mathcal{J}_L$ of order one, which occurs for

$$E \cong \exp(-1/2b'g^2)m^{1-(b/b')}\mu^{b/b'}.$$  \hfill (3.33)

In our particular case, using (3.30), the dependence of the energy scale on $m$ and $\mu$ is

$$E \cong m^{1/3}\mu^{2/3}. \hfill (3.34)$$

Since the gluino condensate has dimension three and so scales as $E^3$, we get

$$\langle \lambda\lambda \rangle \cong m\mu^2, \hfill (3.35)$$

a result that will be essential later. This estimate of the $m$ dependence of $\langle \lambda\lambda \rangle$ is reliable for $m$ in the weak coupling regime – a harmless restriction in the topological theory as the coupling constant dependence of the Lagrangian is $\{Q, \ldots\}$. It is possible to show that the $m$ dependence given in (3.35) is exact using arguments of holomorphicity, as in [2,21].

Of course, in (3.35) we have determined only the $m$ dependence of the gluino condensate, not the overall constant multiplying this. Because of the basic structure of $\mathbb{Z}_4$ broken to $\mathbb{Z}_2$, there are really two vacua with $\langle \lambda\lambda \rangle \cong \pm m\mu^2$, times a constant independent of $m$ and $\mu$.

3.2. Evaluation Of Correlation Functions In Bulk

Apart from gluino condensation, the $N = 1$ theory is also believed to have a mass gap. In §2.3, we sketched how the topological correlation functions are determined by the mass gap. I will here fill in various details omitted there, involving the quantum numbers of the operators, the precise role of gluino condensation, and the lack of Lorentz invariance (even in bulk) of the mass term that breaks $N = 2$ down to $N = 1$. The reason for the latter statement is that, in the twisted theory, the mass term comes from a choice of holomorphic two-form (say $\omega = dz_1 \wedge dz_2$ on a flat $\mathbb{R}^4$) which breaks the $SO(4)$ rotation symmetry of $\mathbb{R}^4$ down to $SU(2)$. 

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First we consider correlation functions of the operator $O = -\text{Tr} B^2/\pi^2$. This operator transforms as $(-1, 1)$, so it can get an expectation value once the global symmetry $\mathbb{Z}_4 \times \mathbb{Z}_2'$ is spontaneously broken to $\mathbb{Z}_2 \times \mathbb{Z}_2'$; this expectation value will, of course, be odd under the broken symmetry. To see how this comes about more explicitly, we integrate out the massive $(B, \psi)$ multiplet to express $O$ purely in terms of light fields. This can be done via the one loop Feynman diagram of figure (1), with the result that $O$ can be replaced by $\lambda\lambda/m$, times a constant that I will not compute here.\footnote{The factor of $1/m$ is needed on dimensional grounds: $O$ and $\lambda\lambda$ have dimension 2 and 3, respectively. By examining the diagram, it is easy to see that one gets $1/m$ and not $1/m^{1-\alpha\pi^2}$ for $\alpha \neq 0$; this actually follows from general principles of holomorphicity.} On the other hand, according to (3.35), the expectation value of $\lambda\lambda$ scales as $m\mu^2$. So we get

$$\langle O \rangle = \pm \mu^2 C, \quad (3.36)$$

with the $\pm$ sign depending on the sign of the gluino condensate and $C$ a universal constant; $\mu^2 C$ is the universal constant $\langle O \rangle$ of §2.

**Two-Forms**

As a prelude to incorporating the two-form operators, let us compare correlation functions $\langle \rangle$ of topological observables in the twisted theory to correlation functions $\langle \rangle_1$ in the theory perturbed by a mass term. The relation is

$$\langle A_1 A_2 \ldots A_n \rangle_1 = \langle A_1 A_2 \ldots A_n e^{I(\omega)} \rangle, \quad (3.37)$$

since the mass term was $-I(\omega) + \{Q_1 \ldots \}$.

The main difference between the present problem and the case considered in §2.3 is the lack of Lorentz invariance in the bulk theory, once a mass term is introduced via a choice of holomorphic two-form $\omega$. In §2.3, Lorentz invariance was used to prove that the vacuum expectation value of $I(\Sigma)$ or equivalently $I(\theta)$ vanishes. This will no longer be true.
If we look at the quantum numbers in (3.24), and take into account the fact that $\mathbb{Z}_4 \times \mathbb{Z}_2'$ is spontaneously broken down to $\mathbb{Z}_2 \times \mathbb{Z}_2'$, it is clear that $I^{1,1}$ cannot have a vacuum expectation value, and it appears that either $I^{0,2}$ or $I^{2,0}$ might have expectation values.

In fact, the unbroken supersymmetry of the $N = 1$ theory implies that $\langle 1 \rangle_1 = 0$ in bulk and hence that

$$\langle \exp I(\omega) \rangle = 0$$

for $\omega$ a holomorphic two-form. Thus, $I^{2,0}$ has no vacuum expectation value.

However, according to (3.25), $I^{0,2}$ is proportional to our friend the gluino bi-linear $\lambda \lambda$ and thus has a vacuum expectation value due to gluino condensation. According to (3.35), the expectation value of $\lambda \lambda$ is proportional to the mass $m$ or in other words to the two-form $\omega$. The operator $I^{0,2}(\theta) = \int_M Z \wedge \theta$ is of course also proportional to the two-form $\theta$. So one gets in fact

$$\langle I(\theta) \rangle_1 = \eta_0 \int_M \theta \wedge \omega,$$

with $\eta_0$ a universal constant that is proportional to the gluino condensate and therefore odd under the spontaneously broken symmetry. This is equivalent to

$$\langle I(\theta)e^{I(\omega)} \rangle = \eta_0 \int_M \theta \wedge \omega.$$ 

A consequence of (3.39) is as follows. If $\Sigma$ is any Riemann surface in $M$ (not necessarily holomorphically embedded) then

$$\langle I(\Sigma) \rangle_1 = \eta_0 \int_{\Sigma} \omega.$$ 

This arises upon expanding the Poincaré dual of $\Sigma$ in classes of type $(2,0)$, $(1,1)$, and $(0,2)$, recalling that $I^{p,q}$ has an expectation value only for $(p,q) = (0,2)$, and using (3.39).
Now let us consider the general case of several two-forms. We still, however, temporarily keep away from zeros of the holomorphic two-form $\omega$; those will be considered later.

As in §2.3, to exploit the mass gap it is convenient to introduce $s$ Riemann surfaces $\Sigma_1, \ldots, \Sigma_s$ in space-time, which we can assume to intersect only pairwise. (The $\Sigma_a$ are not necessarily holomorphically embedded.) As we saw in §2.3, using Lorentz invariance of the bulk theory and assuming that the mass gap holds everywhere, one gets

$$
\left< \exp \left( \sum_a \alpha_a I(\Sigma_a) \right) \right>_1 = \exp \left( \eta \sum_{a,b} \alpha_a \alpha_b \# I(\Sigma_a \cap \Sigma_b) \right).
$$

(3.42)

with some universal constant $\eta$. We recall that in the derivation of (3.42), one uses the fact that (by Lorentz invariance) the expectation value of $I(\Sigma_a)$ vanishes in bulk. In the present case, that is no longer so; the expectation values are given in (3.41). Including the appropriate extra contributions, we get

\[
\left< \exp \left( \sum_a \alpha_a I(\Sigma_a) \right) \right>_1 = \exp \left( \eta \sum_{a,b} \alpha_a \alpha_b \# I(\Sigma_a \cap \Sigma_b) + \eta_0 \sum_a \alpha_a \int_{\Sigma_a} \omega \right),
\]

(3.43)

just as if $I(\Sigma)$ is a free or Gaussian field with a one point function given in (3.41) and a two point function given in (2.33).

This is equivalent to

\[
\left< \exp \left( \sum_a \alpha_a I(\Sigma_a) + I(\omega) \right) \right>_1 = \exp \left( \eta \sum_{a,b} \alpha_a \alpha_b \# I(\Sigma_a \cap \Sigma_b) + \eta_0 \sum_a \alpha_a \int_{\Sigma_a} \omega \right).
\]

(3.44)

It must be the case that $\eta_0 = \eta$, since otherwise the formula is not invariant under a change in the unnatural splitting of the exponent on the left hand side between
$I(\omega)$ and the $I(\Sigma_a)$. For $\eta_0 = \eta$, we can reexpress (3.44) in a more invariant form by expanding $I(\omega)$ in terms of the $I(\Sigma_a)$ (plus $\{Q_1, \ldots\}$):

$$\left\langle \exp \left( \sum_{a} \alpha_a I(\Sigma_a) \right) \right\rangle = \exp \left( \eta \sum_{a,b} \alpha_a \alpha_b \# I(\Sigma_a \cap \Sigma_b) \right). \quad (3.45)$$

Thus, we have recovered the key formula (2.35) of §2.3, though in a more roundabout way because of the lack of manifest Lorentz invariance in the $N = 1$ formulation.

Before leaving this subject, I will comment briefly on the computation of $\eta$. Near intersections of $\Sigma$’s, one can assume that locally the $\Sigma$’s look like holomorphically embedded Riemann surfaces. For instance, locally $\Sigma_1$ could be the locus $z_1 = 0$ and $\Sigma_2$ the locus $z_2 = 0$ in $\mathbb{R}^4 \cong \mathbb{C}^2$. In this case, $I(\Sigma_1) = \int_{z_1=0} I^{1,1}$ and similarly for $I(\Sigma_2)$. Since $I^{1,1} \cong \text{Tr}(\lambda \psi + BF)$, we have to evaluate something like

$$\int_{\Sigma_1 \times \Sigma_2} \text{Tr}(\lambda \psi + BF)(z_1) \cdot \text{Tr}(\lambda \psi + BF)(z_2). \quad (3.46)$$

The role of the mass gap is here very explicit since $B, \psi$ have bare masses in the $N = 1$ theory. Integrating out these massive fields, the integral in (3.46) can be expressed in terms of operators of the low energy theory – that is, the $N = 1$ theory of the gauge multiplet $A_m, \lambda$. The leading contribution comes from integrating out the $\psi$ field in (3.46) and is simply the local operator $\lambda \lambda / m$ evaluated at $z_1 = z_2 = 0$. Since the gluino condensate is of the general form $\eta m$ with $\eta$ a constant independent of $m$, the factors of $m$ cancel out and the local contribution to the expectation value of (3.46) is independent of $m$, as expected. Of course, this contribution is odd under the broken symmetry, as promised in §2; the underlying reason for this is that the operator $I^{1,1}$ transforms as $(i, -1)$ according to (3.24).
3.3. Behavior Near Cosmic Strings

Now we want to analyze the behavior near a cosmic string. We work on flat $\mathbb{R}^4$ with Euclidean coordinates $y^1 \ldots y^4$ and complex coordinates $z_1 = y^1 + iy^2$, $z_2 = y^3 + iy^4$. With the conventions of Wess and Bagger [16] (rotated from Minkowski space to Euclidean space by setting $y^0 = -iy^4$), the Dirac operator is

$$\sigma^m D_m = 2 \begin{pmatrix} D_{\bar{z}_2} & D_{z_1} \\ D_{\bar{z}_1} & -D_{z_2} \end{pmatrix}. \quad (3.47)$$

We consider the cosmic string defined by the holomorphic two-form $\omega = z_1 dz_1 \wedge d\bar{z}_2$, which vanishes on the curve $C$ defined by $z_1 = 0$. Away from $C$, the field $\psi^\alpha = \lambda^\alpha \psi$ has a bare mass. This mass vanishes in the core of the string.

As we claimed in §2.7, there is a normalizable zero mode for the $\psi$ field in the $z_1$ plane. In fact, the equations of motion of $\psi^1$ and $\bar{\psi}^\dagger$, allowing for the kinetic energy and the coupling to $\omega$, are

$$-i \frac{\partial \psi^1}{\partial z_1} + z_1 \bar{\psi} = 0$$

$$-i \frac{\partial \bar{\psi}^\dagger}{\partial \bar{z}_1} + z_1 \psi^1 = 0. \quad (3.48)$$

The zero mode is $\psi^1 = \exp(-|z_1|^2)$, $\bar{\psi}^\dagger = i \exp(-|z_1|^2)$, with $\psi^2 = \bar{\psi}^\dagger = 0$.

Now we wish to consider the motion in the $z_2$ direction. To do so, we write $\psi^1 = \exp(-|z_1|^2) \psi'(z_2, \bar{z}_2)$, $\bar{\psi}^\dagger = i \exp(-|z_1|^2) \psi'(z_2, \bar{z}_2)$, with $\psi'$ a two-dimensional fermi field, defined on $C$, and with values in the adjoint representation. With this ansatz, the kinetic energy of the $\psi$ field reduces to

$$\int_C dz_2 d\bar{z}_2 \text{ Tr } \psi' D_{\bar{z}_2} \psi'. \quad (3.49)$$

This shows that, as claimed in §2.7, $\psi'$ behaves like a chiral fermi field along $C$. 

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To determine the spin structure to which $\psi'$ is coupled, we consider a global situation in which $C$ is a Riemann surface in a four-manifold $M$ defined by the vanishing of a holomorphic two-form $\omega$. Locally one can write $\omega = z_1 dz_1 \wedge dz_2$, with $z_1$ and $z_2$ being local coordinates normal and tangent to $C$. The existence of a global two-form of the stated behavior implies that $dz_1$ transforms as $(dz_2)^{-1/2}$.

To say this in a more sophisticated way, the normal bundle $N_C$ to $C$ obeys $N_C \otimes 2 \cong K_C^{-1}$, with $K_C$ the canonical bundle to $C$. Thus a section of $N_C$ can be interpreted as a $(-1/2, 0)$ form on $C$. Hence $N_C^{-1}$ — whose sections transform as $\partial/\partial z_1$ — is a spin bundle of $C$, that is, a bundle of $(1/2, 0)$ forms. The complex conjugate bundle $\overline{N_C}^{-1}$ is a bundle of $(0, 1/2)$ forms or an antiholomorphic spin bundle.

On the other hand, the formula (3.13) shows that $\psi'$ transforms as $DB/D\bar{z}_1$, to which it is related by the symmetry. Thus, $\psi'$ transforms as a section of the antiholomorphic spin bundle $\overline{N_C}^{-1}$, as was claimed in §2.8.

Quantum Numbers Of $V$

The last claim made about cosmic strings in §2 was that the operator $V$ has the quantum numbers claimed in (2.57).

In fact, as in equation (2.56), $V$ is the contribution to integrals such as $I(\theta) = \int_M \theta \wedge Z$ coming from the intersection with a cosmic string. As the Poincaré dual of the cosmic string is a class of type $(1, 1)$, only the $(1, 1)$ part of $\theta$ contributes; hence $V$ transforms under the symmetry as does $I^{1,1}$, and (2.57) is a consequence of (3.24).

3.4. A Useful Formula

Finally, we must establish the formula (2.61) that was needed to show that the final expression for the Donaldson invariants is real.

We let $\omega$ be a holomorphic two-form on $M$ that vanishes on a divisor $C$. Let $K$ be the canonical bundle of $M$ and $K|_C$ its restriction to $C$. There is an exact
sequence of sheaves

\[ 0 \rightarrow \mathcal{O} \xrightarrow{\omega} K \rightarrow K|_C \rightarrow 0. \tag{3.50} \]

The first map is \( f \rightarrow f \cdot \omega \). This leads to an exact sequence

\[ 0 \rightarrow H^0(M, \mathcal{O}) \xrightarrow{\omega} H^0(K) \rightarrow H^0(C, K|_C) \rightarrow H^1(M, \mathcal{O}) \xrightarrow{\omega^1} H^1(M, K) \rightarrow \ldots \tag{3.51} \]

We truncate this to

\[ 0 \rightarrow H^0(M, \mathcal{O}) \xrightarrow{\omega} H^0(K) \rightarrow H^0(C, K|_C) \rightarrow \ker(\omega^1) \rightarrow 0. \tag{3.52} \]

The existence of such an exact sequence implies that

\[ \dim H^0(M, \mathcal{O}) - \dim H^0(M, K) + \dim H^0(C, K|_C) - \dim \ker(\omega^1) = 0. \tag{3.53} \]

The map \( \omega^1 \) can be interpreted as the map from \( \alpha \in H^1(M, \mathcal{O}) \cong H^{0,1}(M, \mathcal{C}) \) to \( \alpha \wedge \omega \in H^1(M, K) \cong H^{2,1}(M, \mathcal{C}) \). A natural antisymmetric bilinear form on \( H^{0,1}(M) \) can be defined by

\[ \langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta \wedge \omega. \tag{3.54} \]

By Poincaré duality, \( \alpha \wedge \omega \) vanishes as an element of \( H^{2,1}(M, \mathcal{C}) \) if and only if \( \langle \alpha, \beta \rangle = 0 \) for all \( \beta \).

Now in general, if \( V \) is a vector space of finite dimension and \( \langle \, , \, \rangle \) is an antisymmetric bilinear form on \( V \), then the kernel of \( \langle \, , \, \rangle \) is defined as the set of \( x \in V \) such that \( \langle x, y \rangle = 0 \) for all \( y \in V \); simple linear algebra shows that the dimension of the kernel is congruent to the dimension of \( V \) modulo two. The facts stated in the last paragraph mean that \( \ker \omega^1 \) is the same as the kernel of \( \langle \, , \, \rangle \).
and hence that
\[ \dim \ker \omega^1 \cong \dim H^{0,1}(M) \mod 2. \] (3.55)

This lets us rewrite (3.53) in the form
\[ \dim H^0(C, K|_{C}) \cong 1 - h^{0,1}(M) + h^{0,2}(M) \mod 2. \] (3.56)

(We recall that \( h^{p,q}(M) = \dim H^{p,q}(M) \).)

On the other hand, if \( C \) is the union of connected components \( C_y \), then
\[ H^0(C, K|_{C}) = \bigoplus_y H^0(C_y, K|_{C_y}). \]
Above we showed in essence that \( K|_{C_y} \) is the spin bundle of the fermions that propagate along \( C_y \) (actually, the chiral fermions \( \psi' \) are coupled to the antiholomorphic spin bundle that is complex conjugate to \( K|_{C_y} \)). So the number of fermion zero modes along \( C_y \) is \( \epsilon_y = \dim H^0(C_y, K|_{C_y}) \).

Taking account of the formula for \( \Delta \) in (2.44), it follows then that (3.56) is equivalent to
\[ \Delta \cong \sum_y \epsilon_y \mod 2. \] (3.57)

This is the desired formula used in §2.
1) A one loop diagram by which the operator $\mathcal{O}$ can be expressed as an effective operator in the low energy theory.
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