3d quantum gravity coupled to matter

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Abstract

We investigate the phase structure of three-dimensional quantum gravity coupled to an Ising spin system by means of numerical simulations. The quantum gravity part is modelled by the summation over random simplicial manifolds, and the Ising spins are located in the center of the tetrahedra, which constitute the building blocks of the piecewise linear manifold. We find that the coupling between spin and geometry is weak away from the critical point of the Ising model. At the critical point there is clear coupling, which however does not seem to change the first order transition between the “hot” and “cold” phase of three dimensional simplicial quantum gravity observed earlier.

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1 Introduction

In the recent years there has been a remarkable progress in our understanding of two-dimensional quantum gravity coupled to matter. Although it is still somewhat unclear what kind of theory we deal with when we consider pure 2d quantum gravity (how is the “average” intrinsic geometry to be characterized etc.), we get definite answers, if we ask for the modification of critical exponents of conformal field theories when they are coupled covariantly to 2d quantum gravity. From the point of view of string theory it is important to study the two-dimensional models, but if we want to consider possible theories of quantum gravity outside the context of string theories (and there are good reasons for taking such a point of view) they are to be considered only as toy models for higher dimensional gravity. In this letter we attempt to take a first step in the direction of the study of higher dimensional gravity coupled to matter. We will confine ourselves to three dimensions. There are several reasons for this. Firstly we do not have available the same powerful analytical methods as in two dimensions and we have to rely on numerical simulations. These are quite a lot easier in three dimensions than in four dimensions. In addition we expect three dimensional gravity to be placed in between the solvable two-dimensional case and four-dimensional gravity in the following sense: Three-dimensional (classical) gravity has no dynamical degrees of freedom by itself, as is also the case for two-dimensional gravity. But as for two-dimensional gravity this does not imply that it cannot couple in a non-trivial way to the matter fields. On the other hand the theory shares with the four-dimensional theory the feature that the Einstein-Hilbert term in the action is non-trivial, not renormalizable and (if not regularized) unbounded from below. From this point of view we might get important hints which can be used in four dimensions from a theory, where we can hopefully use some of the intuition we have now gathered in the two-dimensional studies. Furthermore it is much easier to find non-trivial statistical models in three- than in four dimensions. In fact the simplest example is the Ising model which we are going to use.

2 The model

Three-dimensional quantum gravity modelled on random triangulations was introduced in refs. [1, 2, 3] and numerical simulations first performed in refs. [4, 5, 6, 7, 8]. Let us briefly describe the model and summarize the results.

The continuum action of Einstein-Hilbert gravity in three dimensions can be written as

\[ S[g] = \lambda \int d^3 \xi \sqrt{g} - \frac{1}{16\pi G} \int d^3 \xi \sqrt{g} R. \]  \hspace{1cm} (1)
In the dynamical triangulated approach the functional integral over metrics is replaced by a summation over all possible triangulations. Here, as in two dimensions, it is important that we restrict ourselves to a fixed topology (which we take to be that of $S^3$). The building blocks which we glue together in order to form the piecewise linear manifolds are regular tetrahedra. Consequently we will get (see for instance [7] for a careful discussion)

\[ \int d^3 \xi \sqrt{g} \sim N_3 \]

\[ \int d^3 \xi \sqrt{g} R \sim c N_1 - 6 N_3 \]

where $N_0, ..., N_3$ denote the number of vertices, links, triangles and tetrahedra which constitute the simplicial manifold. The number $c = 2\pi / \arccos(1/3)$ is chosen such that $R = 0$ corresponds to flat three-dimensional space. This means that the discretized action can be written as

\[ S[T] = k_3 N_3 - k_1 N_1 \]

and the recipe for going from the continuum functional to the discretized one will be:

\[ \int \mathcal{D}[g_{\mu\nu}] e^{-S[g]} \to \sum_T e^{-S[T]} . \]

The result of the numerical simulations is as follows: For small $k_1$ (i.e. for large bare gravitational coupling constant) the system seems to be in a phase of large Hausdorff dimension (denoted the “hot” phase). Even for large systems consisting of 28,000 tetrahedra the linear extension (the average geodesic distance) of the system is small and increases only slowly with volume. For $k_1 \approx 4.0$ there is a phase transition to a state where the system is quite extended. The extension grows linear with the volume, showing that the Hausdorff dimension is one. This phase is probably a lattice artifact which signals the dominance of the conformal mode of gravity for a small gravitational constant. The transition between the two phases is a first order transition since pronounced hysteresis is observed. We conclude that if we restrict ourselves to actions of the form (4) it seems not possible to define a continuum limit of the lattice model in the usual sense, with a divergent correlation length. This is not an undesirable situation since the presence of a divergent correlation length would force us to identify at least one massless field in three-dimensional quantum gravity. But we know such fields are not present in classical three-dimensional gravity.

Let us couple the above defined model to Ising spins. The coupling is done in the same way as for discretized two-dimensional quantum gravity, where we know
the dynamical triangulated model coupled to Ising spins leads to critical exponents which agree with the ones calculated using continuum formalism. To each tetrahedron \( i \) we associate an Ising spin \( \sigma_i \) and the partition function for the combined system is

\[
Z(\beta, k_1, k_3) = \sum_{N_3} e^{-k_3 N_3} \sum_{T \sim N_3} \prod_{[\sigma]} e^{k_1 N_1} e^\beta \sum_{<i,j>} (\delta_{\sigma_i \sigma_j} - 1).
\]  

(6)

In this formula \( T \sim N_3 \) signifies the summation over all piecewise linear manifolds which can be formed by gluing \( N_3 \) (regular) tetrahedra together such that the topology is that of \( S^3 \). \( \sum_{[\sigma]} \) means the summation over all spin configurations while \( \sum_{<i,j>} \) stands for the summation over all neighbour pairs of tetrahedra. One annoying aspect of the above formalism is that we are forced to perform a grand canonical simulation where \( N_3 \) is not fixed. The reason is that we (contrary to two dimensions) have no ergodic updating algorithm which preserves the volume \( N_3 \). In practice it is however possible to perform the measurements at a fixed \( N_3 \) and the important coupling constants will then be \( \beta \) and \( k_1 \). We refer to [7] for a detailed discussion.

The spin updating is performed by the single cluster variant of the Swendsen-Wang algorithm developed by Wolff [9]. The cluster updating algorithms have been successfully applied to the Ising model coupled to 2d gravity [10, 11, 12, 13] and to the ordinary three dimensional Ising model [14].

3 Numerical results

As mentioned above three-dimensional simplicial quantum gravity has two phases depending on the value of \( k_1 \). The first statement we can make is that this is unchanged by the coupling to Ising spin.

In the “hot” phase \((k_1 \leq 4.0)\) where the Hausdorff dimension is large the numerical value of the magnetization

\[
|\sigma| \equiv \frac{1}{N_3} \sum_{i=1}^{N_3} |\sigma_i|
\]

(7)

is shown in fig.1 as a function of \( \beta \). We see a clear signal indicating a phase transition from a disordered phase (small \( \beta \)) where \( |\sigma| \approx 0 \) to an ordered phase (large \( \beta \)) where \( |\sigma| \approx 1 \). The transition becomes sharper with increased volume \( N_3 \) and seems to be a second order transition. This situation is contrasted by the magnetization curve in the “cold” phase shown in fig.2. Here is only a gradual cross over to \( |\sigma| \approx 1 \) for large \( \beta \), and the cross over is weakened for increased volume \( N_3 \). The situation is precisely as one would expect in the case of a one-dimensional system where there is no spontaneous magnetization. We conclude that the Hausdorff dimension \( d_H \approx 1 \)
measured in the pure gravity case seems to reflect correctly the dimension relevant for coupling to matter.

In the $k_1 - \beta$ plane we have the phase-diagram shown in fig.3. If $\beta$ is away from the critical value $\beta_c(k_1)$ (which has only a weak dependence on $k_1$) the coupling between the fluctuations in geometry and spin seems weak and of course it vanishes in the limits $\beta \to \infty$ and $\beta \to 0$. In these limits we therefore have a strong first order transition between the “hot” and the “cold” phase of three-dimensional quantum gravity, precisely as is the case in the absence of spins \footnote{\textcolor{blue}{1}}. In the “hot” phase, where the Ising system has a second order transition, we have seen an increased coupling between geometry and spins when we approach the critical $\beta_c(k_1)$. This is shown in fig.4 where we plot the average curvature $\langle R \rangle$ as a function of $\beta$. A clear peak is seen at $\beta_c$. This enhanced coupling between geometry and spins at the critical point is qualitatively in agreement with the 2d results, where we have a change in the string susceptibility exponent $\gamma_{\text{string}}$ (not to be confused with the magnetic susceptibility exponent $\gamma_{\text{mag}}$) from the pure gravity value $-1/2$ to $-1/3$, precisely when $\beta = \beta_c$.

Unfortunately it is not clear that the entropy exponent analogous to $\gamma_{\text{string}}$ exists in the hot phase of three-dimensional quantum gravity \footnote{\textcolor{blue}{2}} so we have no obvious exponent with which we can compare the effect of the spin coupling, but the enhanced coupling between spin and geometry leaves open the possibility that the transition between the “hot” and “cold” phase changes from a first order to a second order transition. We have looked for hysteresis when changing $k_1$ and adjusting $\beta$ to the critical value $\beta_c(k_1)$. While the hysteresis is indeed weaker when measured this way, we still see a clear hysteresis (fig.5) and we conclude that there is never a second order transition in geometry.

Let us make the following remark concerning the determination of the phase diagram shown in fig.3: Due to the strong hysteresis it is somewhat ambiguous. We have used the following procedure: Well inside the “hot” phase the system follows a unique path when changing $k_1$ and keeping $\beta$ fixed as illustrated in fig.5. The precise location depends on the value of $\beta$. We have extrapolated these paths until they intersect the parts of the hysteresis curves which correspond to the “cold” phase.

In fig.6 we have shown the spin-spin correlation as a function of geodesic distance. To be precise there are two obvious candidates for geodesic distances (see \footnote{\textcolor{blue}{3}} for a discussion in the context of four-dimensional simplicial quantum gravity). We can define the geodesic distance $d_1$ between two vertices as the length of the shortest path along links connecting the two vertices. Alternatively we could have defined the geodesic distance $d_2$ between two tetrahedra as the length of the shortest path connecting the two tetrahedra, moving from center to center in neighbour tetrahedra.
which have a triangle in common. The first definition is clearly much closer to the “correct” definition obtained by considering the manifolds as piecewise linear, with the curvature attached to the links. The other definition corresponds to moving along links in the dual graph, which is a $\phi^4$ graph. For a single manifold the two distances can differ a lot, but when an ensemble average is taken it seems as if one can consider them as proportional. A similar result is true in four-dimensional gravity ([15]). Clearly $d_2$ is most convenient for our purpose and starting from a given tetrahedron $i_0$ we define the volume $V_2(r)$ inside a ball of $d_2$-geodesic radius $r$ around $i_0$ as the number of tetrahedra within this distance. Further the differential volume is $dV_2(r) \equiv V_2(r) - V_2(r - 1)$. We can now define a spin-spin correlation function as

$$g(r) \equiv \left\langle \frac{1}{dV_2(r)} \sum_{i \in dV_2(r)} \sigma_i \sigma_{i_0} \right\rangle. \quad (8)$$

An alternative correlation function would be

$$G(r) \equiv \left\langle \sum_{i \in dV_2(r)} \sigma_i \sigma_{i_0} \right\rangle. \quad (9)$$

which is related to the magnetic susceptibility $\chi(\beta)$ by

$$\sum_r G(r) = \chi(\beta) \sim |\beta - \beta_c(k_1)|^{-\gamma} \quad \text{for} \quad \beta \rightarrow \beta_c(k_1) \quad (10)$$

In fig.6 we have shown $g(r)$ for two different values of $\beta$. In principle one can extract the mass gap $m(\beta)$ from the exponential fall off of $g(r)$ and in this way determine the critical exponent $\nu$ defined by $m(\beta) = |\beta - \beta_c|^\nu$. This seems however difficult to do in a reliable way, in accordance with the experience in two-dimensional quantum gravity coupled to Ising spins, and it is maybe understandable if one keeps in mind that not only is a precise determination of $\beta_c$ needed in order to extract $\nu$. In addition our data are folded into the distribution $dV_2(r)$ which determines the Hausdorff dimension, a quantity which by itself is very difficult to measure in a reliable way.

In the same way we can construct $\chi(\beta)$ from $G(r)$, but it does not lead to a precise determination of the critical exponent $\gamma$ (again in agreement with the experience from two dimensions).

4 Discussion

We have shown that the phase structure of three-dimensional simplicial quantum gravity, as described in [8], is not modified by the presence of matter, at least in the simplest case of coupling to Ising spins. In the so-called “hot” phase the Ising spin
system has a second order transition, while it has no transition in the “cold” phase in agreement with the effective one-dimensional nature of this phase. The existence of two phases in three-dimensional gravity is caused by the Einstein-Hilbert term in the action. In two dimensions this term is absent (for a fixed topology) and we have only one phase of pure gravity. This phase seems to have most in common with the “hot” phase of three-dimensional gravity and it is natural to expect that the critical properties of the matter theories covariantly coupled to 3d gravity could be changed in a non-trivial way when we are in this “hot” phase, simply by analogy with the two-dimensional models.

One way to investigate the possible non-trivial scaling of the Ising model in the “hot” phase is by means of finite size scaling. In two-dimensional gravity this approach seems to work somewhat better than the direct attempts to measure $\nu$ and $\gamma$ mentioned above. The disadvantage of the method is that it only gives us certain combinations of the exponents.

If a given thermodynamic function $F$ has a critical behaviour

$$F(\beta) \sim (\beta - \beta_c)^{-x}$$

one expects in ordinary flat space a finite size dependence of the form

$$F(\beta, L) = L^x f(|\beta - \beta_c|L^{1/\nu})$$

where $L$ denotes the linear size of the system and the exponent $\nu$ is determined by the divergence of the correlation length $\xi(\beta)$:

$$\xi(\beta) \sim |\beta - \beta_c|^{-\nu}.$$ (13)

By a measurement of $F(\beta_c, L) \sim f(0)L^x$ as a function of $L$ we can determine the combination $x/\nu$, while measurements away from $\beta_c$ would give us $x$ directly for sufficiently large $L$.

If we want to use these formulae for systems coupled to quantum gravity we must identify the divergent correlation length $\xi$ in terms of geodesic distances, as was already discussed in the last section. We further have to identify the linear extension $L$. If the system has a finite Hausdorff dimension $d_H$ it is tempting to define

$$L \sim N^{1/d_H}.$$ (14)

These ideas have been used with some success in 2d-quantum gravity \cite{11, 12, 13}, but could be spoiled if the Hausdorff dimension is infinite. Since the $d_H$ from (very tentative) direct measurements of $V_2(r)$ seems large in the “hot” phase it is tempting to conjecture that the critical exponents of the Ising model in the “hot” phase take
their mean-field values. A reliable determination of the various exponents along the lines discussed above requires a considerable amount of computer time, since it is already quite demanding in two dimensions, but we hope to be able to address the question in a future publication.

Note added: While completing this article we received a paper by Baillie [16], who investigates the same system. There seems to be little overlap with our work since he did not explore the phase structure in $k_1 - \beta$-plane. In fact this is not possible with the size of systems he uses.

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Figure Captions

Fig.1 The magnetization $|\sigma|$ (defined by (7)) as a function of $\beta$ in the “hot” phase for $N_3 = 4000$ (triangles) and $N_3 = 10000$ (circles).

Fig.2 The magnetization $|\sigma|$ (defined by (7)) as a function of $\beta$ in the “cold” phase for $N_3 = 4000$ (triangles) and $N_3 = 10000$ (circles).

Fig.3 The phase diagram for 3d quantum gravity coupled to matter. Filled circles are results obtained for $N_3 = 10000$.

Fig.4 The average curvature $\langle R \rangle$ as a function of $\beta$ for $N_3 = 4000$ (circles) and $N_3 = 10000$ (squares). The position of the peak coincides with the value of $\beta_c$ determined from the magnetization curve.

Fig.5 The hysteresis curve for pure gravity (triangles) and in the case where the Ising spin system is critical i.e. where it couples in a maximal way to gravity (circles). $N_3 = 10000$.

Fig.6 The spin-spin correlation function $g(r)$ (defined by (8)) as a function of the geodesic distance $r$ for $\beta = 0.5$ (full drawn curve) and $\beta = 0.8$ (dotted curve), $k_1 = 3.7$ and $N_3 = 4000$. The best estimate of the critical value of $\beta$ is: $\beta_c = 0.85$.