SOME REMARKS ON LOCAL CLASS FIELD THEORY
OF SERRE AND HAZEWINKEL

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ABSTRACT. We give a new approach for local class field theory of Serre and Hazewinkel. In the case of characteristic zero, we also show a D-module version of this theory. Two-dimensional local class field theory is discussed in this framework.

1. Introduction

First we use the terminology of [Ser60], [Ser61], and [DG70] to state the first main theorem (Theorem 1.1) of this paper. Let $k$ be a perfect field of characteristic $p \geq 0$ and $K = k((T))$. We fix an algebraic closure $\overline{K}$ of $K$. All the algebraic extensions of $K$ are taken inside $\overline{K}$, for example, the separable closure $K_s$, the perfect closure $K_p$, the maximal abelian extension $K^{ab}$, the maximal unramified extension $K^{ur}$. The group of units of $K$ can be viewed as a proalgebraic group over $k$ in the sense of [Ser60]; we denote this group by $U_K$. For each perfect $k$-algebra $R$ (perfect means that the $p$-th power map is an isomorphism) we have the group

$$U_K(R) = \left\{ \sum_{i=0}^{\infty} a_i T^i \bigg| a_i \in R, \ a_0 \in R^\times \right\}$$

of $R$-rational points. We consider the $K_p$-rational point $-T + \mathbf{T}$ of $U_K$ and the corresponding morphism $\varphi: \text{Spec } K_p \to U_K$. We denote by $\eta$ the composite map

$$I(K^{ab}/K) \hookrightarrow \text{Gal}(K^{ab}/K) \overset{\sim}{\to} \pi_1^{\text{et}}(\text{Spec } K_p)^{ab} \cong \pi_1^{\text{et}}(U_K)^{ab} \to \pi_1^{k-\text{sp}}(U_K).$$

Here we denote by $I(K^{ab}/K)$ the inertia group of the extension $K^{ab}/K$, by $\pi_1^{\text{et}}(\cdot)^{ab}$ the maximal abelian quotient of the étale fundamental group, and by $\pi_1^{k-\text{sp}}$ the first left derived functor of the functor taking the maximal proconstant quotient in the category of commutative proalgebraic groups over $k$. Then we state the first main theorem of this paper:

**Theorem 1.1.** The above defined map $\eta: I(K^{ab}/K) \to \pi_1^{k-\text{sp}}(U_K)$ is an isomorphism. Moreover, if $k$ is either a finite field, an algebraic closure of a finite field, or a field of characteristic zero, then the inverse of $\eta$ coincides with the isomorphism $\theta: \pi_1^{k-\text{sp}}(U_K) \to I(K^{ab}/K)$ of Serre-Hazewinkel (Ser60, DG70).

Next we assume that $\text{char}(k) = 0$ (hence $K_p = K$) and use the notion of $D$-module (cf. [HTT08]) to state the second main theorem (Theorem 1.2) of this paper. Let $n \geq 0$ be an integer and $U_K^{n+1}$ be the proalgebraic group of $(n+1)$-th principal units. We say that a $D$-module $M$ on the $k$-scheme $U_K/U_K^{n+1}$ with $O$-rank $1$ is compatible with group structure if $\mu^* M \cong \text{pr}_1^* M \otimes \text{pr}_2^* M$, where $\mu: U_K/U_K^{n+1} \times U_K/U_K^{n+1} \to U_K/U_K^{n+1}$ is the multiplication and $\text{pr}_i$ is the $i$-th projection $(i = 1, 2)$. A $D$-module $N$ on the $k$-scheme $\text{Spec } K$ with $O$-rank $1$ is said to have irregularity $n$ if its connection form with respect to some (hence any) $K$-basis of $N$ has a form $f dT/T$ for some $f \in K^\times$ with valuation $-n$. With these terminologies the second main theorem of this paper is stated as follows:

**Theorem 1.2.** Assume that $\text{char}(k) = 0$. The map $\varphi: \text{Spec } K \to U_K$ induces, by pulling back, an equivalence of categories between the category $\mathcal{C}$ of $D$-modules of $O$-rank $1$ on the $k$-scheme $U_K/U_K^{n+1}$ which are
compatible with group structure and the category $C'$ of $D$-modules of $O$-rank $1$ on the $k$-scheme $\text{Spec} K$ with irregularity $\leq n$.

We also discuss a two-dimensional analogue of the above theory.

We give a couple of comments on literatures. First, the above defined map $\varphi: \text{Spec} K_p \to U_K$ have also been defined by Contou-Carrère ([CC94]). Second, the existence of an equivalence of categories between $C$ and $C'$ may be known for the specialists of the geometric Langlands correspondence (cf. [Fre07], [Bei16]).

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2. PROOF OF THEOREM 1.1

2.1. Proof of the part “$\eta$ is an isomorphism”. Both groups $\pi_1^{k\text{-gp}}(U_K)$ and $I(K^{ab}/K)$ are profinite abelian groups. Thus it is enough to show that $\eta$ induces an isomorphism between the Pontryagin dual groups. The Pontryagin dual of $\pi_1^{k\text{-gp}}(U_K)$ is canonically isomorphic to $\text{Ext}^1_k(U_K, \mathbb{Q}/\mathbb{Z})$ which is defined as the direct limit of the groups $\text{Ext}^1_k(U_K, \mathbb{Z}/m\mathbb{Z})$, $m \geq 1$, of extension classes of proalgebraic groups over $k$ (cf. [Ser60, §5.4]). Therefore the problem is equivalent to showing that the dual map $\eta^\vee: \text{Ext}^1_k(U_K, \mathbb{Q}/\mathbb{Z}) \to H^1(I(K^{ab}/K), \mathbb{Q}_l/\mathbb{Z}_l)$ is an isomorphism for each prime number $\ell$. Since $U_K \cong G_m \times U^1_K$, we have

$$\text{Ext}^1_k(U_K, \mathbb{Q}_l/\mathbb{Z}_l) \cong \text{Ext}^1_k(G_m, \mathbb{Q}_l/\mathbb{Z}_l) \oplus \text{Ext}^1_k(U^1_K, \mathbb{Q}_l/\mathbb{Z}_l).$$

2.1.1. The case $\ell \neq p$. We compute the groups $\text{Ext}^1_k(U_K, \mathbb{Q}_l/\mathbb{Z}_l)$, $H^1(I(K^{ab}/K), \mathbb{Q}_l/\mathbb{Z}_l)$ for $\ell \neq p$.

Lemma 2.1. If $p = 0$, the usual exponential map $\prod_{n \geq 1} G_a \to U^1_K$ sending $(a_n)_{n \geq 1} \in \prod_{n \geq 1} G_a$ to $\prod_{n \geq 1} \exp(a_n T^n) \in U^1_K$ is an isomorphism of proalgebraic groups. If $p > 0$, the Artin-Hasse exponential map $\prod_{p^n \geq 1} W \to W^1_K$ sending $a = (a_n)_{p^n \geq 1} \in \prod_{p^n \geq 1} W$ with $a_n = (a_{n0}, a_{n1}, \ldots) \in W$ to $\prod_{p^n \geq 1, m \geq 0} F(a_{nm}T^{p^m}) \in U^1_K$ is an isomorphism of proalgebraic groups. Here we denote by $W$ the additive group of Witt vectors and set $F(t) = \exp(-\sum_{e \geq 0} t^{p^e}/p^e) \in \mathbb{Z}_p[[t]]$.

Proof. See [Ser88] Chapter V, §3, 15 and 16. \[\square\]

Lemma 2.2. The group $\text{Ext}^1_k(U^1_K, \mathbb{Q}_l/\mathbb{Z}_l)$ is zero. The group $\text{Ext}^1_k(G_m, \mathbb{Q}_l/\mathbb{Z}_l)$ is generated by the extension classes given by

$$0 \to \mathbb{Z}/\ell^d\mathbb{Z} \to G_m \xrightarrow{\ell^d} G_m \to 0,$$

where $d$ runs through the integers such that $k^\times$ contains all the $\ell^d$-th roots of unity and the map $\mathbb{Z}/\ell^d\mathbb{Z} \to G_m$ corresponds to the choice of a primitive $\ell^d$-th root of unity.

Proof. First we show that $\text{Ext}^1_k(U^1_K, \mathbb{Q}_l/\mathbb{Z}_l) = 0$. Lemma 2.1 shows that the $\ell$-th power map induces an automorphism on $U^1_K$. Since $\mathbb{Q}_l/\ell^d\mathbb{Z}_l$ is $\ell$-power torsion, we have $\text{Ext}^1_k(U^1_K, \mathbb{Q}_l/\mathbb{Z}_l) = 0$.

Next we compute $\text{Ext}^1_k(G_m, \mathbb{Q}_l/\mathbb{Z}_l)$. Since this group is isomorphic to the group of characters of $\ell$-power order of $\pi_1^{k\text{-gp}}(U_K)$, it is a union of subgroups $\text{Ext}^1_k(G_m, Z/\ell^d\mathbb{Z})$ for $d' \geq 1$. Taking the long exact sequence of the exact sequence

$$0 \to \mu_{\ell^{d'}} \to G_m \xrightarrow{\ell^{d'}} G_m \to 0$$

we have an exact sequence

\[1\]

$$\text{Hom}_k(G_m, Z/\ell^d\mathbb{Z}) \to \text{Hom}_k(\mu_{\ell^{d'}} Z/\ell^d\mathbb{Z}) \to \text{Ext}^1_k(G_m, Z/\ell^d\mathbb{Z}) \xrightarrow{\ell^{d'}} \text{Ext}^1_k(G_m, Z/\ell^d\mathbb{Z}).$$

Since $G_m$ is connected and $Z/\ell^d\mathbb{Z}$ is discrete, the first term of $\text{1}$ is zero. Since $\mathbb{Z}/\ell^d\mathbb{Z}$ is killed by $\ell^{d'}$, the third map of $\text{1}$ is a zero map. Thus we have an isomorphism $\text{Hom}_k(\mu_{\ell^{d'}} Z/\ell^d\mathbb{Z}) \cong \text{Ext}^1_k(G_m, Z/\ell^d\mathbb{Z})$. If $d$ is the maximal integer less than $d'$ such that $k^\times$ contains all the $\ell^{d'}$-th roots of unity, then any morphism $\mu_{\ell^d} \to Z/\ell^d\mathbb{Z}$ factors through the maximal constant quotient $\mu_{\ell^{d'}}$ of $\mu_{\ell^d}$. Thus we have $\text{Hom}_k(\mu_{\ell^d} Z/\ell^d\mathbb{Z}) = \text{Hom}_k(\mu_{\ell^{d'}} Z/\ell^d\mathbb{Z})$. If $d = d'$, the group $\text{Hom}_k(\mu_{\ell^{d'}}, Z/\ell^d\mathbb{Z})$ is a cyclic group generated by an isomorphism $\mu_{\ell^d} \cong Z/\ell^d\mathbb{Z}$ corresponding to the choice of a primitive $\ell^d$-th root of unity. This
generator corresponds to the desired extension class via the above defined isomorphism \(\text{Hom}_k(\mu_{\ell^d}, \mathbb{Z}/\ell^d\mathbb{Z}) \cong \text{Ext}^1_k(G_m, \mathbb{Z}/\ell^d\mathbb{Z})\) because there is a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mu_{\ell^d} & \longrightarrow & G_m & \longrightarrow & 0 \\
\downarrow{\cong} & & \downarrow{\text{id}} & & \downarrow{\text{id}} & & \\
0 & \longrightarrow & \mathbb{Z}/\ell^d\mathbb{Z} & \longrightarrow & G_m & \longrightarrow & 0,
\end{array}
\]

where the map \(\mathbb{Z}/\ell^d\mathbb{Z} \to G_m\) is the inverse of the isomorphism \(\mu_{\ell^d} \cong \mathbb{Z}/\ell^d\mathbb{Z}\) followed by the inclusion \(\mu_{\ell^d} \hookrightarrow G_m\).

\[\square\]

**Lemma 2.3.** The group \(H^1(I(K^{ab}/K), \mathbb{Q}_\ell/\mathbb{Z}_\ell)\) is generated by the characters given by

\[
\sigma \mapsto \psi(\sigma((-T)^{1/\ell^d})/(-T)^{1/\ell^d}),
\]

where \(d\) runs through the integers such that \(k^\times\) contains all the \(\ell^d\)-th roots of unity and \(\psi: \mu_{\ell^d} \cong \mathbb{Z}/\ell^d\mathbb{Z}\) is an isomorphism.

**Proof.** Note that \(I(K^{ab}/K) = \text{Gal}(K^{ab}/K^{ab} \cap K^{ur}) \cong \text{Gal}(K^{ab}K^{ur}/K^{ur})\). For each integer \(d \geq 1\), the field \(K^{ur}\) has a unique Galois extension of degree \(\ell^d\), namely \(K^{ur}((-T)^{1/\ell^d})\). This field is contained in \(K^{ab}K^{ur}\) if and only if \(k^\times\) contains all the \(\ell^d\)-th roots of unity.

Now we show that \(\eta^\vee: \text{Ext}^1_k(U_K, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1(I(K^{ab}/K), \mathbb{Q}_\ell/\mathbb{Z}_\ell)\) is an isomorphism for \(\ell \neq p\). The extension class given in Lemma 2.2 gives an isogeny \(G_m \to G_m\) with kernel \(\mathbb{Z}/\ell^d\mathbb{Z}\). The map \(\varphi: \text{Spec} K_p \to U_K\) followed by the projection \(U_K \to G_m\) corresponds to the rational point \(-T\). Taking the fiber product of these maps we have

\[
\begin{array}{cccccc}
\text{Spec} K_p((-T)^{1/\ell^d}) & \longrightarrow & G_m \\
\downarrow & & \downarrow{\text{id}} \\
\text{Spec} K_p & \longrightarrow & G_m.
\end{array}
\]

Thus, in view of the above lemmas, we know that the map \(\eta^\vee: \text{Ext}^1_k(U_K, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1(I(K^{ab}/K), \mathbb{Q}_\ell/\mathbb{Z}_\ell)\) is an isomorphism for \(\ell \neq p\).

**2.1.2. The case \(\ell = p\).** We have to treat the groups of characters of \(p\)-power order. We reduce the problem to that of order \(p\).

**Lemma 2.4.** Let \(f: A \to B\) be a homomorphism between abelian groups \(A\) and \(B\). If both \(A\) and \(B\) are \(p\)-divisible and \(p\)-power torsion, and \(f\) induces an isomorphism between the \(p\)-torsion part of \(A\) and that of \(B\), then \(f\) is an isomorphism.

**Lemma 2.5.** The group \(\text{Ext}^1_k(G_m, \mathbb{Q}_p/\mathbb{Z}_p)\) is zero. The group \(\text{Ext}^1_k(U^1_K, \mathbb{Q}_p/\mathbb{Z}_p)\) is \(p\)-divisible.

**Proof.** First we show that \(\text{Ext}^1_k(G_m, \mathbb{Q}_p/\mathbb{Z}_p) = 0\). The \(p\)-th power map induces an automorphism on \(G_m\) since we work in the category of quasi-algebraic groups in the sense of [Ser60]. Since \(\mathbb{Q}_p/\mathbb{Z}_p\) is \(p\)-power torsion, we have \(\text{Ext}^1_k(G_m, \mathbb{Q}_p/\mathbb{Z}_p) = 0\).

Next we show the \(p\)-divisibility of \(\text{Ext}^1_k(U^1_K, \mathbb{Q}_p/\mathbb{Z}_p)\). Since this group is isomorphic to the group of characters of \(p\)-power order of \(\pi_1^{k^{sp}}(U^1_K)\), it is a union of subgroups \(\text{Ext}^1_k(U^1_K, \mathbb{Z}/p^d\mathbb{Z})\) for \(d \geq 1\). We show that \(\text{Ext}^1_k(U^1_K, \mathbb{Z}/p^d\mathbb{Z})\) is canonically isomorphic to \(\bigoplus_{d \geq 1} W_d(k)\) and that the natural injection \(\text{Ext}^1_k(U^1_K, \mathbb{Z}/p^d\mathbb{Z}) \hookrightarrow \text{Ext}^1_k(U^1_K, \mathbb{Z}/p^{d+1}\mathbb{Z})\) corresponds to the map \(\bigoplus_{d \geq 1} W_d(k) \hookrightarrow \bigoplus_{d \geq 1} W_{d+1}(k)\) of multiplication by \(p\), which implies the \(p\)-divisibility of \(\text{Ext}^1_k(U^1_K, \mathbb{Q}_p/\mathbb{Z}_p)\). Since \(U^1_K \cong \bigotimes_{d \geq 1} W_d\), we have \(\text{Ext}^1_k(U^1_K, \mathbb{Z}/p^d\mathbb{Z}) \cong \bigoplus_{d \geq 1} \text{Ext}^1_k(W_d, \mathbb{Z}/p^d\mathbb{Z})\). Taking the long exact sequence of the exact sequence

\[
0 \longrightarrow W \overset{p^d}{\longrightarrow} W \longrightarrow W_d \longrightarrow 0
\]
we have an exact sequence

\[(2) \quad \text{Hom}_k(W, \mathbb{Z}/p^d\mathbb{Z}) \longrightarrow \text{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z}) \longrightarrow \text{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}.\]

Since \(W\) is connected and \(\mathbb{Z}/p^d\mathbb{Z}\) is discrete, the first term of \((2)\) is zero. Since \(\mathbb{Z}/p^d\mathbb{Z}\) is killed by \(p^d\), the third map of \((2)\) is a zero map. Thus we have an isomorphism \(\text{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z}) \cong \text{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z})\). There is a canonical element \(\varepsilon_d \in \text{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z})\) corresponding to the Artin-Schreier-Witt isogeny \(\varphi\). Each element \(a \in W_d(k)\) gives, by multiplication, an endomorphism on \(W_d\), hence an endomorphism \(a^*\) on \(\text{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z})\). The map \(W_d(k) \to \text{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z}), a \mapsto a^*\varepsilon_d\), is an isomorphism (DG70, Chapter V, §3, 6.10). Thus we get isomorphisms

\[(3) \quad \text{Ext}^1_k(U_{K^1}, \mathbb{Z}/p^d\mathbb{Z}) \cong \bigoplus_{p^n \geq 1} \text{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z}) \cong \bigoplus_{p^n \geq 1} \text{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z}) \cong \bigoplus_{p^n \geq 1} W_d(k).\]

The natural injection \(\text{Ext}^1_k(U_{K^1}, \mathbb{Z}/p^d\mathbb{Z}) \hookrightarrow \text{Ext}^1_k(U_{K^1}, \mathbb{Z}/p^{d+1}\mathbb{Z})\) corresponds, on each direct summand of the third term of \((3)\), to the map \(R^*p_*: \text{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z}) \to \text{Ext}^1_k(W_{d+1}, \mathbb{Z}/p^{d+1}\mathbb{Z})\), where \(R: W_{d+1} \to W_d\) is the projection. The following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}/p^d\mathbb{Z} & \longrightarrow & W_d & \longrightarrow & 0 \\
p & & \downarrow & & \downarrow & & p \\
0 & \longrightarrow & \mathbb{Z}/p^{d+1}\mathbb{Z} & \longrightarrow & W_{d+1} & \longrightarrow & 0
\end{array}
\]

shows that \(p_\ast\varepsilon_d = p^*\varepsilon_{d+1}\). Hence \(R^*p_*\varepsilon_d = R^*a^*p^*\varepsilon_{d+1} = (pa)^*\varepsilon_{d+1}\). Thus the map \(R^*p_*: \text{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z}) \to \text{Ext}^1_k(W_{d+1}, \mathbb{Z}/p^{d+1}\mathbb{Z})\) corresponds to the multiplication \(p: W_d(k) \hookrightarrow W_{d+1}(k)\) via the third isomorphism of \((3)\), as desired.

\[\square\]

**Lemma 2.6.** The group \(H^1(I(K^{ab}/K), \mathbb{Q}_p/\mathbb{Z}_p)\) is \(p\)-divisible.

**Proof.** The largest pro-\(p\) quotient of \(\text{Gal}(K_s/K)\) is pro-\(p\) free ([Ser02, Chapter I, §2.2, Corollary 1]). Thus \(H^1(\text{Gal}(K_s/K), \mathbb{Q}_p/\mathbb{Z}_p)\) is \(p\)-divisible. Since \(H^1(I(K^{ab}/K), \mathbb{Q}_p/\mathbb{Z}_p)\) is a quotient of \(H^1(\text{Gal}(K_s/K), \mathbb{Q}_p/\mathbb{Z}_p)\), the group \(H^1(I(K^{ab}/K), \mathbb{Q}_p/\mathbb{Z}_p)\) is also \(p\)-divisible.

\[\square\]

We calculate the groups of characters of order \(p\).

**Lemma 2.7.** As a special case \((d = 1)\) of the isomorphism \((3)\), we have isomorphisms

\[\text{Ext}^1_k(U_{K^1}, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p^n \geq 1} \text{Ext}^1_k(W, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p^n \geq 1} \text{Ext}^1_k(G_\alpha, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p^n \geq 1} k.\]

The map \(k \to \text{Ext}^1_k(G_\alpha, \mathbb{Z}/p\mathbb{Z})\) sends an element \(a \in k^\times\) to the extension class given by

\[
0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow G_\alpha \xrightarrow{a^{-1}\varphi} G_\alpha \longrightarrow 0,
\]

where \(\varphi\) is the Artin-Schreier isogeny.

**Proof.** This is immediate from the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & G_\alpha & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & G_\alpha & \longrightarrow & 0
\end{array}
\]

\[\square\]
Lemma 2.8. The map defined by
\[ \bigoplus_{p|n \geq 1} kT^{-n} \to H^1(I(K^{ab}/K), \mathbb{Z}/p\mathbb{Z}), \]
\[ aT^{-n} \mapsto \left( \sigma \mapsto \sigma(\varphi^{-1}(aT^{-n})) - \varphi^{-1}(aT^{-n}) \right) \]
is an isomorphism.

Proof. Since the natural surjection \( \text{Gal}(K_s/K) \to \text{Gal}(k_s/k) \) admits a section ([Ser02 Chapter II, §4.3, Exercises]), we know that the sequence
\[ 0 \to H^1(\text{Gal}(k_s/k), \mathbb{Z}/p\mathbb{Z}) \to H^1(\text{Gal}(K_s/K), \mathbb{Z}/p\mathbb{Z}) \to H^1(I(K^{ab}/K), \mathbb{Z}/p\mathbb{Z}) \to 0 \]
is exact. The first and the second term of this sequence is calculated by Artin-Schreier theory. Thus the third term also is calculated. The result is the desired form.

Thus we are reduced to show that the map
\[ \eta^\vee : \operatorname{Ext}^1_b(U, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p|n \geq 1} k \to \bigoplus_{p|n \geq 1} kT^{-n} \cong H^1(I(K^{ab}/K), \mathbb{Z}/p\mathbb{Z}) \]
is an isomorphism. We need to calculate the following map:
\[ \text{Spec } K_p \xrightarrow{\sim} U^1_K/(U^1_K)^p \cong \prod_{p|n \geq 1} W/pW \cong \prod_{p|n \geq 1} G_a. \]
The map \( \text{Spec } K_p \to U^1_K/(U^1_K)^p \) corresponds to the \( K_p \)-rational point \( 1 - T^{-1} \mathbf{T} \) of \( U^1_K/(U^1_K)^p \). The isomorphism \( \prod_{p|n \geq 1} G_a \cong U^1_K/(U^1_K)^p \) sends each element \( (a_n)_{p|n \geq 1} \) of the left hand side to \( \prod_{p|n \geq 1} F(a_n \mathbf{T}^n) \) of the right hand side.

Proposition 2.9. (1) The inverse of the isomorphism \( \prod_{p|n \geq 1} G_a \cong U^1_K/(U^1_K)^p \) is given by the map
\[ U^1_K/(U^1_K)^p \xrightarrow{\text{dlog}} \prod_{p|n \geq 1} G_a \mathbf{T}^n \xrightarrow{\alpha} \prod_{p|n \geq 1} G_a, \]
where \( \text{dlog}(g) = (g'/g)\mathbf{T} \) and \( \alpha(\sum_{n \geq 1} b_n \mathbf{T}^n \text{dlog} \mathbf{T}) = (-b_n/n)_{p|n \geq 1} \).

(2) The rational point \( 1 - T^{-1} \mathbf{T} \) corresponds to \( (1/(nT^n))_{p|n \geq 1} \) via the isomorphism \( U^1_K/(U^1_K)^p(K_p) \cong \prod_{p|n \geq 1} G_a(K_p) \).

(3) The map \( \text{Spec } K_p \to \prod_{p|n \geq 1} G_a \) gives the \( K_p \)-rational point \( (1/(nT^n))_{p|n \geq 1} \) of \( \prod_{p|n \geq 1} G_a \).

Proof. (1): Using the identity \( \text{dlog} F(t) = - \sum_{e \geq 0} tp^e \text{dlog} t \), we have
\[ \text{dlog} \left( \prod_{p|n \geq 1} F(a_n \mathbf{T}^n) \right) = - \sum_{e \geq 0} (a_n \mathbf{T}^n)^p \text{dlog}(a_n \mathbf{T}^n) = \sum_{e \geq 0} (-n)(a_n \mathbf{T}^n)^p \text{dlog} \mathbf{T}. \]

Thus the map \( \alpha \circ \text{dlog} \) sends \( \prod_{p|n \geq 1} F(a_n \mathbf{T}^n) \) to \( (a_n)_{p|n \geq 1} \), as desired. (2): A simple calculation shows that \( (\alpha \circ \text{dlog})(1 - T^{-1} \mathbf{T}) = (1/(nT^n))_{p|n \geq 1} \). (3): This follows from (2).

Now we calculate \( \eta^\vee \). Let \( n \geq 1 \) be an integer prime to \( p \) and \( a \neq 0 \) be an element of \( k \) regarded as an element of \( \bigoplus_{p|n \geq 1} k \) by the \( n \)-th inclusion \( k \hookrightarrow \bigoplus_{p|n \geq 1} k \). The corresponding extension of \( G_a \) is given in Lemma 2.7. We have a cartesian diagram
\[ \begin{array}{ccc} \text{Spec } K_p & \xrightarrow{\varphi^{-1}(a/nT^n)} & G_a \\ \downarrow & & \downarrow \alpha^{-1} \psi \\ \text{Spec } K_p & \xrightarrow{\sim} & G_a. \end{array} \]

Thus \( \eta^\vee : \bigoplus_{p|n \geq 1} k \to \bigoplus_{p|n \geq 1} kT^{-n} \) preserves the direct factors and the map on the \( n \)-th factor is given by multiplication by \( 1/n \). This shows that \( \eta : I(K^{ab}/K) \to \pi_1^{k\text{-gp}}(U_K) \) is an isomorphism.
2.2. Proof of the part “η^{-1} = \theta for some cases”. First we show that \eta^{-1} = \theta for the case where \k is a finite field of \q elements.

**Proposition 2.10.** There is a cartesian diagram

\[
\begin{array}{ccc}
\text{Spec}(K^\text{ram}_T)_p & \longrightarrow & \text{U}_K \\
\downarrow & & \downarrow f^{-1} \\
\text{Spec } K_p & \varphi \longrightarrow & \text{U}_K.
\end{array}
\]

Here \(K^\text{ram}_T\) is the field \K adjoining all the \(T^m\)-torsion points (where \(m\) runs through the integers \(\geq 1\)) of the Lubin-Tate formal group \(F_f\) (cf. [Iwa60]) whose equation of formal multiplication by \(T\) is equal to \(f(X) = TX + X^q\). The morphism \(\text{Spec}(K^\text{ram}_T)_p \to \text{U}_K\) corresponds to the rational point \(\sum_{m=0}^{\infty} \alpha_m T^m\), where \(\alpha_m\) is a generator of the module of \(T^m\)-torsion points of \(F_f\). The map \(F\) is the \(q\)-th power relative Frobenius morphism (hence \(F^{-1}\) is the Lang isogeny over \(\k\)). The induced isomorphism \(\text{Gal}(K^\text{ram}_T/K) \cong \text{U}_K\) coincides with the one given by Lubin-Tate theory.

**Proof.** We calculate the geometric fiber of \(-T + T\). Let \(g = \sum a_m T^m\) be an element of \(\text{U}_K(\overline{\k})\). The equation \(F(g)/g = -T + T\) is equivalent to the system of equations \(f(a_0) = 0, f(a_{m+1}) = a_m, m \geq 0\). Thus, for each \(m \geq 0\), \(a_m\) is a generator of the module of \(T^{m+1}\)-torsion points of \(F_f\). This proves the existence of the above cartesian diagram. Next we calculate the action of \(\text{Gal}(K^\text{ram}_T/K)\) on the fiber of \(-T + T\). The Lubin-Tate group \(F_f\) for \(f(X) = TX + X^q\) is the formal completion \(\hat{\G}_a\) of the additive group with the formal multiplication of each element \(\sum b_m T^m\) of \(\O_K\) given by the power series \(\sum b_m f^m(X) \in \text{End} \hat{\G}_a\), where \(f^m\) is the \(m\)-th iteration of \(f\). Thus, if \(\sigma\) corresponds to \(u(T) = \sum b_m T^m\) via the isomorphism \(\text{Gal}(K^\text{ram}_T/K) \cong \text{U}_K\) of Lubin-Tate theory, we have

\[
\sigma \left( \sum_{m=0}^{\alpha} a_{m+1} T^m \right) = \sum_{m=0}^{\infty} \sigma(a_{m+1} T^m) = \sum_{0 \leq k \leq m < \infty} b_k \alpha_{m+1-k} T^m = u(T) \sum_{m \geq 0} \alpha_{m+1} T^m.
\]

Thus the action of \(\sigma\) on the fiber of \(-T + T\) is given by multiplication by \(u(T)\), as required. \(\square\)

Thus the map \(\eta: I(K^{ab}/K) \to \pi_1^{k, \text{gp}}(\text{U}_K)\) factors through the isomorphism of Lubin-Tate theory:

\[I(K^{ab}/K) \to \text{Gal}(K^\text{ram}_T/K) \cong \text{U}_K \cong \pi_1^{k, \text{gp}}(\text{U}_K).\]

Since the isomorphism \(\theta\) of Serre-Hazewinkel for finite \(k\) coincides with the one given by Lubin-Tate theory, the equality \(\eta^{-1} = \theta\) for such \(k\) follows.

**Remark.** The above proposition, combined with the fact that \(\eta: I(K^{ab}/K) \to \pi_1^{k, \text{gp}}(\text{U}_K)\) is an isomorphism, which was proved in the previous subsection, gives another proof of the local Kronecker-Weber theorem for Lubin-Tate extensions: we have just been proved that the canonical surjection \(I(K^{ab}/K) \twoheadrightarrow \text{Gal}(K^\text{ram}_T/K)\) is an isomorphism, that is, \(K^{ab} = K^\text{ram}_T K^{ur}\).

Next we show that \(\eta^{-1} = \theta\) for the case where \(k\) is an algebraic closure of a finite field. We put \(K_n = \mathbb{F}_{p^n}((T))\). Then we have

\[\text{Gal}(K^{ab}/K) = \text{Gal}((\cup K_n)^a/\cup K_n) = \lim I(K^{ab}_n/K_n).\]

Also by [DG70] Chapter V, [3, 2.3] we have \(\pi_1^{k, \text{gp}}(\text{U}_K) = \lim_{\rightarrow} \pi_1^{k, \text{gp}}(\text{U}_{K_n})\). Since the maps \(\varphi: \text{Spec}(K_n)_p \to \text{U}_{K_n}\) are compatible with base extension, the equality \(\eta^{-1} = \theta\) is reduced to the finite residue field case.

Finally we treat the case \(\text{char}(k) = 0\). Let \(L/K\) be a totally ramified abelian extension of degree \(n\). Kummer theory and the exponential map show that the inclusion \(\text{U}_K \hookrightarrow \text{U}_L\) induces an isomorphism \(\text{U}_K \cong \text{U}_L/\text{V}_L/K\), where \(\text{V}_L/K\) is a subgroup of \(\text{U}_K\) generated by \((\sigma - 1)\text{U}_L\) for various \(\sigma \in \text{Gal}(L/K)\). The composite of this isomorphism and the norm map \(\text{N}_L/K: \text{U}_L/\text{V}_L/K \to \text{U}_K\) is the \(n\)-th power endomorphism
on $U_K$, which is an automorphism on the subgroup $U_K^1$. Thus we have the following diagram whose two squares are both cartesian:

$$
\begin{array}{ccc}
\text{Spec } K^w((-T)^{1/n}) & \longrightarrow & U_L/V_{L/K} \\
\downarrow & & \downarrow N_{L/K} \\
\text{Spec } K^w & \longrightarrow & U_K \\
\end{array}
\xrightarrow{n} \begin{array}{c}
G_m \\
\end{array}
$$

Then the equality $\eta^{-1} = \theta$ follows.

### 3. Proof of Theorem 1.2

First we describe the category $\mathcal{C}$. Write $U_K/U_K^{n+1} \cong G_m \times G_m^n = \text{Spec } k[T_0^\pm, T_1, \ldots, T_n]$ and put $A = k[T_0^\pm, T_1, \ldots, T_n]$, $D_A = A[\partial T_0, \ldots, \partial T_n]$. Since $A$ is a UFD, any line bundle on $U_K/U_K^{n+1}$ can be trivialized. Let $M = A e^N$ be a $D_A$-module of $A$-rank 1 with a basis $e^M$ and a connection form $\omega^M = f_0^M dT_0/T_0 + \sum_{1 \leq i \leq n} f_i^M dT_i$, where $f_i^M \in A$. For $M$ to be compatible with group structure, it is necessary and sufficient that $f_i^M$ is equal to a constant $a_i^N \in k$ for each $i$. Let $N = A e^N$ be another $D_A$-module of $A$-rank 1 with a connection form $\omega^N = t_0^N dT_0/T_0 + \sum_{1 \leq i \leq n} a_i^N dT_i$, with $a_i^N \in k$. We determine the space of $D_A$-homomorphisms $\text{Hom}_{D_A}(M, N)$. Since both $M$ and $N$ are $A$-rank 1, this space can be viewed as a $k$-subspace of $A$. An element $g \in \text{Hom}_{D_A}(M, N) \subset A$ should satisfy a system of differential equations

$$
\partial_{T_0} g = \frac{a_0^M - a_0^N}{T_0}, \quad \partial_{T_i} g = (a_i^M - a_i^N) g, \quad 1 \leq i \leq n.
$$

This system has a non-zero solution $g$ in $A$ if and only if $a_0^M - a_0^N \in \mathbb{Z}$ and $a_i^M = a_i^N$ for $1 \leq i \leq n$. If these conditions are satisfied, the space of solutions is a 1-dimensional $k$-vector space spanned by $T_0^{a_0^M - a_0^N}$.

In particular the isomorphism classes of objects of the category $\mathcal{C}$ is classified by the space

$$
\frac{(k/\mathbb{Z}) dT_0}{T_0} \oplus \bigoplus_{1 \leq i \leq n} k dT_i
$$

by taking the connection form.

Next we describe the category $\mathcal{C}'$. If $M = K e^M$ (resp. $N = K e^N$) is a $D_K = K[\partial T]$-module with irregularity $\leq n^M$ (resp. $\leq n^N$) with a connection form $f^M dT/T = \sum_{-\infty \leq i \leq n^M} a_i^M T^{-i} dT/T$ (resp. $f^N dT = \sum_{-\infty \leq i \leq n^N} a_i^N T^{-i} dT/T$), then the space $\text{Hom}_{D_K}(M, N)$ is zero unless $a_0^M - a_0^N \in \mathbb{Z}$ and $a_i^M = a_i^N$ for $i \geq 1$. If these conditions are satisfied, $\text{Hom}_{D_K}(M, N)$ is a 1-dimensional $k$-vector space spanned by

$$
T_0^{a_0^M - a_0^N} \exp \left( \sum_{i \leq 0} \frac{a_i^M - a_i^N}{T^{-i}} T^{-i} \right).
$$

In particular the isomorphism classes of objects of the category $\mathcal{C}'$ is classified by the space

$$
\left( \frac{(k/\mathbb{Z})}{T} \oplus \bigoplus_{1 \leq i \leq n} k T^{-i} \right) dT
$$

by taking the connection form.

Now we describe the functor of pulling back by $\varphi: \text{Spec } K \rightarrow U_K/U_K^{n+1}$. The map $\varphi$ followed by the isomorphism $U_K/U_K^{n+1} \cong G_m \times G_m^n$ given in Lemma 1.2 corresponds to a rational point $(-T, (-T^{-i}/i)_i)$. If $M$ is an object of $\mathcal{C}$ with a connection form $\omega^M = a_0^M dT_0/T_0 + \sum_{1 \leq i \leq n} a_i^M dT_i$, then the pullback $\varphi^* M$ has a connection form $\varphi^* \omega^M = \sum_{0 \leq i \leq n} a_i^M T^{-i} dT/T$. Using this description and the above classification we know that the functor of pulling back by $\varphi$ is fully faithful and essentially surjective. Thus we get Theorem 1.2.
4. AN AUXILIARY RESULT

The following proposition is a refinement of Proposition 2.10.

**Proposition 4.1.** Assume that $k$ is either a finite field, an algebraic closure of a finite field, or a field of characteristic 0. Then, for any finite totally ramified abelian extension $L/K$, there is a map $\text{Spec} \, L_p^{ur} \to U_L/V_{L/K}$ and a cartesian diagram

\[
\begin{array}{ccc}
\text{Spec} \, L_p^{ur} & \longrightarrow & U_L/V_{L/K} \\
\downarrow & & \downarrow N_{L/K} \\
\text{Spec} \, K_p^{ur} & \xrightarrow{\varphi} & U_K.
\end{array}
\]

The induced isomorphism $\text{Gal}(L/K) \cong \text{Ker}(N_{L/K})$ coincides with $\theta$.

We prove this proposition below. Note that the group $U_L(K)$ is equipped with two different actions of $\text{Gal}(K_s/K)$, namely the one induced by the action of $\text{Gal}(L/K)$ on the proalgebraic group $U_L$ and the one induced by the action on the coefficient field $K$. For $g \in U_L(K)$ and $\sigma \in \text{Gal}(K_s/K)$ we denote by $g^{[\sigma]}$ (resp. $g^\sigma$) the action of $\sigma \in \text{Gal}(K_s/K)$ on $g \in U_L(K)$ in the former (resp. the latter) sense.

**Lemma 4.2.** Assume that $k$ is finite. Let $m \geq 1$ be an integer and $L = K_T^m$ be the field $K$ adjoining all the $T^m$-torsion points of the Lubin-Tate group $F_j$. For any $g \in U_L$, the image of $N_{L/K}g$ in $U_K/U_K^m$ depends only on the image of $g^{T^{-1}} = g^F/g$ in $U_L/V_{L/K}$. Thus we obtain a map $N_{L/K} \circ (F^{-1})^{-1} : U_L/V_{L/K} \to U_K/U_K^m$. This map makes the following diagram commutative:

\[
\begin{array}{c}
0 \longrightarrow \text{Gal}(L/K) \longrightarrow U_L/V_{L/K} \xrightarrow{N_{L/K}} U_K \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 \longrightarrow U_K/U_K^m \longrightarrow U_K/U_K^{m(F^{-1})^{-1}} \longrightarrow 0.
\end{array}
\]

Here the map $\text{Gal}(L/K) \to U_L/V_{L/K}$ is given by $\sigma \mapsto [\alpha_m]^{|\sigma|^{-1}} = [\alpha_m]^{\sigma m} (\alpha_m$ is defined similarly to $T$) and the map $\text{Gal}(L/K) \to U_K/U_K^m$ is the isomorphism of local class field theory. All other unnamed maps are the canonical ones.

**Proof.** The well-definedness of $N_{L/K} \circ (F^{-1})^{-1}$: The kernel of the endomorphism $F^{-1}$ of $U_L/V_{L/K}$ is equal to $U_LV_{L/K}/V_{L/K}$; its image by $N_{L/K}$ is contained in $N_{L/K}(U_L) = U_K^{m}$. This proves the well-definedness. The commutativity of the left square: See [Ser79, Chapter XIII, §5].

**Proof of Proposition 4.1.** First we prove Proposition 4.1 for the case where $k$ is finite and $L = K_T^m$. By Proposition 2.10 we have a cartesian diagram

\[
\begin{array}{ccc}
\text{Spec} \, L_p^{ur} & \longrightarrow & U_K/U_K^m \\
\downarrow & & \downarrow F^{-1} \\
\text{Spec} \, K_p^{ur} & \xrightarrow{\varphi} & U_K/U_K^m.
\end{array}
\]

Combining this diagram with Lemma 4.2 we have the following diagram whose two squares are both cartesian:

\[
\begin{array}{ccc}
\text{Spec} \, L_p^{ur} & \longrightarrow & U_L/V_{L/K} \xrightarrow{(F^{-1})^{-1}} U_K/U_K^m \\
\downarrow & & \downarrow N_{L/K} \\
\text{Spec} \, K_p^{ur} & \xrightarrow{\varphi} & U_K \longrightarrow U_K/U_K^m.
\end{array}
\]

This diagram induces isomorphisms $\text{Gal}(L/K) \cong \text{Ker}(N_{L/K}) \cong U_K/U_K^m$. Since this induced isomorphism $\text{Gal}(L/K) \cong U_K/U_K^m$ (resp. $\text{Ker}(N_{L/K}) \cong U_K/U_K^m$) coincides with the one given by local class field theory by Proposition 2.10 (resp. by Lemma 4.2), so is $\text{Gal}(L/K) \cong \text{Ker}(N_{L/K})$. 

Now let $L/K$ be an arbitrary finite totally ramified abelian extension. By the local Kronecker-Weber theorem there exists an integer $m$ such that $L^{ur} \subset (K_m^{ur})^{ur}$. Consider the following diagram whose two squares are both cartesian:

$$
\begin{array}{ccc}
\text{Spec}(K_m^{ur})_L & \longrightarrow & X & \longrightarrow & \text{Spec } K^{ur}_L \\
\downarrow & & \downarrow & & \downarrow \\
U_{K_m^L}/V_{K_m^L}/K & \longrightarrow & U_L/V_L/K & \longrightarrow & U_K.
\end{array}
$$

Since the maps $\text{Spec}(K_m^{ur})_L \to X \to \text{Spec } K^{ur}_L$ are finite étale, the scheme $X$ is of the form $\text{Spec } L'$ for some intermediate extension $L'$ of $(K_m^{ur})_L/K^{ur}$. We show that $L' = L_G^{ur}$. Let $g$ be an element of the fiber of $N_{K_m^L}/K$ over $-T + T$ and put $h = N_{K_m^L}/Lg$ and $a = N_{K_m^L}/L\alpha_m$. Then $h$ is the rational point corresponding the map $\text{Spec } L' \to U_L/V_L/K$ and $a$ is a prime element of $L$. For any $\sigma \in \text{Gal}(L/K)$ the equality $g^{\sigma - 1} = \alpha_m^{[\sigma]}$ holds in $U_L/V_L/K$ by Proposition 1.1 for the extension $K_m^L/K$. Taking $N_{K_m^L}/L$ on both side of this equality we have $h^{\sigma - 1} = \alpha^{[\sigma]}$. Since $\sigma|_L = 1$ (resp. $\sigma|_L = 1$) is equivalent to $h^{\sigma - 1} = 1$ (resp. $\alpha^{[\sigma]} = 1$), we have $L' = L$. This proves Proposition 1.1 for finite $k$.

The proof of Proposition 1.1 for the case where $k$ is an algebraic closure of a finite field is reduced to the finite case by the similar argument used in the proof of $\eta^{-1} = \theta$ for such $k$. The case char($k$) = 0 is already treated in the proof of $\eta^{-1} = \theta$ for the characteristic 0 case.

\[ \square \]

5. A TWO-DIMENSIONAL ANALOGUE

In this section we discuss an analogue of the above theory for the field $K = k((S))((T))$. We denote by $K_2$ the functor of the second algebraic $K$-group (Basz). For each perfect $k$-algebra $R$ we have an abelian group $K_2(R[[S, T]])$. This gives a group functor which we denote by $K_2[[S, T]]$. The $K_2$-rational point

$$
\{ -S + S, -T + T \} \in K_2[[S, T]]((K_p) = K_2(k((S))((T))_{(p)[[S, T]]})
$$

gives a morphism $\varphi: \text{Spec } K_p \to K_2[[S, T]]$, where $\{,\}$ denotes the symbol map. This is an analogue of the map $\text{Spec } k((T))_{(p)} \to U_{k((T))}$ previously defined and studied.

When $k$ is a finite field $\mathbb{F}_q$, the field $K = k((S))((T))$ is called a two-dimensional local field (FK00) of positive characteristic. For each perfect $k$-algebra $R$ there is a $k$-automorphism of $R[[S, T]]$ which maps each element of $R$ to its $q$-th power and fixes $S$ and $T$. This $k$-automorphism induces an $k$-automorphism on $K_2[[S, T]]$ which we denote by $F$. Consider the following cartesian diagram:

$$
\begin{array}{ccc}
X & \longrightarrow & K_2[[S, T]] \\
\downarrow & & \downarrow F^{-1} \\
\text{Spec } K_p & \longrightarrow & K_2[[S, T]].
\end{array}
$$

Then we expect that $K_2[[S, T]]$ can be viewed as a kind of “algebraic group over $k$” and the equation $x^{F^{-1}} = \{ -S + S, -T + T \}$ gives a two-dimensional analogue of Lubin-Tate theory so that $X$ is the Spec of the perfect closure of a large totally ramified abelian extension of $K$ (cf. Proposition 2.10).

To avoid some technical difficulties and prove a rigorous statement, we use the space of 2-forms instead of $K_2[[S, T]]$. For each perfect $k$-algebra $R$ we have the space of 2-forms $\Omega^2_{R[[S, T]]/R}$. This functor is represented by a proalgebraic group over $k$ isomorphic to an infinite product of $G_a$ with coordinate $z_{ij} := S^iT^j dS \wedge dT$, $i, j \geq 0$. We denote this group by $\Omega[[S, T]]$. The dlog map $K_2[[S, T]] \to \Omega[[S, T]]$ is defined. There is a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & K_2[[S, T]] \\
\downarrow & & \downarrow \text{dlog} \\
\text{Spec } K_p & \longrightarrow & \Omega[[S, T]].
\end{array}
$$
We put $\varphi' = \text{dlog} \circ \varphi$.

**Proposition 5.1.** There is a cartesian diagram

$$
\begin{array}{ccc}
\text{Spec } A_p & \longrightarrow & \Omega[[S,T]] \\
\downarrow & & \downarrow F-1 \\
\text{Spec } K_p & \longrightarrow & \Omega[[S,T]].
\end{array}
$$

Here we denote by $A$ the ring $K[x_{ij} \mid i, j \geq 0]/(x_{ij}^q - x_{ij} - S^{-i-1}T^{-j-1})$ and by $A_p$ the direct limit of the $p$-th power maps $A \rightarrow A \rightarrow \cdots$.

**Proof.** The map $\varphi' : \text{Spec } K_p \rightarrow \Omega[[S,T]]$ corresponds to a rational point

$$
\text{dlog}(-S + S, -T + T) = \frac{d(-S + S)}{-S + S} \wedge \frac{d(-T + T)}{-T + T} = \sum_{i,j \geq 0} S^{-i-1}T^{-j-1}S^{ij}dS \wedge dT.
$$

If $\sum x_{ij}S^{ij}dS \wedge dT \in \Omega[[S,T]](\overline{K})$ lies in the geometric fiber of $F - 1$ over this rational point, it should satisfy

$$(F - 1) \sum_{i,j \geq 0} x_{ij}S^{ij}dS \wedge dT = \sum_{i,j \geq 0} (a_{ij}^q - x_{ij})S^{ij}dS \wedge dT = \sum_{i,j \geq 0} S^{-i-1}T^{-j-1}S^{ij}dS \wedge dT.
$$

Thus we get the proposition. \qed

Next we assume $\text{char}(k) = 0$ and calculate the pullback of a $D$-module on $\Omega[[S,T]]$ which is compatible with group structure in analogy with Theorem 1.2. Let $n, m \geq 0$ be integers. The group $\Omega[[S,T]]$ has the algebraic quotient $G$ with coordinate $z_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq m$. Any $D$-module $M$ on $G$ of $O$-rank 1 which is compatible with group structure has a connection form of the form $\sum a_{ij}dz_{ij}$ with $a_{ij} \in k$. We have

$$(\varphi')^* \sum_{0 \leq i \leq n \atop 0 \leq j \leq m} a_{ij}dz_{ij} = \sum_{0 \leq i \leq n \atop 0 \leq j \leq m} a_{ij}d(S^{-i-1}T^{-j-1})$$

$$= - \sum_{0 \leq i \leq n \atop 0 \leq j \leq m} (i + 1)a_{ij}S^{-i-2}T^{-j-1}dS - \sum_{0 \leq i \leq n \atop 0 \leq j \leq m} (j + 1)a_{ij}S^{-i-1}T^{-j-2}dT.
$$

This is a connection form of the pullback of $M$ by $\varphi'$.

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