A MODIFIED CONJUGATE GRADIENT METHOD WITH GLOBAL-
CONVERGENCE FOR UNCONSTRIANED OPTIMIZATION PROBLEMS

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(Received: July 10, 2021; Accepted for Publication: October 19, 2021)

ABSTRACT

In this paper a new conjugate gradient method for unconstrained optimization is suggested, the new method is based on Conjugate Descent (CD) formula which is a modified of the CD formula and which is also sufficiently descent and globally convergent. Numerical evidence shows that this new conjugate gradient algorithm is considered as one of the competitive conjugate gradient methods.

1. INTRODUCTION

As it is shown in the following unconstrained optimization problem

\[ \min \{ f(x) : x \in \mathbb{R}^n \} \]  \hspace{1cm} (1)

where \( f \) is a function of continuously differentiability of \( n \) real variables with gradient \( g = \nabla f \). Methods of conjugate gradient are a class of significant ways for solving (1), particularly for problems with large scale, which have the following form:

\[ x_{k+1} = x_k + \alpha_k d_k \]

in the sense that \( \alpha_k \) is the step size, \( d_k \) is the conjugate search direction. The direction of the search is usually defined by:

\[ d_k = \begin{cases} -g_k & k = 1 \\ -g_k + \beta_k d_{k-1} & k > 1 \end{cases} \]

where \( g(x) = \nabla f(x) \) and \( \beta_k \) is scalar parameter. Certain choices of the parameter \( \beta_k \) are given which are in chronological order.

One of the formulae for \( \beta_k \) is Conjugate Descent (CD) of Fletcher [5, 6]:

\[ \beta_k^{CD} = \frac{-\| g_k \|}{d_{k-1} g_{k-1}} \]  \hspace{1cm} (3)

where \( y_k = g_{k+1} - g_k \) and \( \| . \| \) denoted the Euclidean norm. The step size \( \alpha_k \) is obtained through the exact linear search or inexact linear search \( \alpha_k \) should satisfy the conditions of strong Wolfe [1]:

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where $0 < \rho < \sigma < 1$.

Conjugate Descent ensures a descent direction for general functions if the line search satisfies the strong Wolfe conditions (4), (5) with $\sigma < 1$. But the global convergence of the CD method is proved, see [2].

In 1999, Dai and Yuan [1] proposed the DY conjugate gradient method using $\beta_k$ defined by

$$
\beta_k^\text{DY} = \frac{\|g_k\|^2}{d_{k-1}^T y_{k-1}}.
$$

In 2001 [3], they proposed an updated form of $\beta_k$ with three parameters, which may be regarded as a convex combination of several earlier choices of $\beta_k$ listed in [7]; but the three parameters are restricted in small intervals.

In [4], Dai and Yuan proposed a family of globally convergent conjugate methods, in which $\beta_k$ introduced by

$$
\beta_k = \frac{\|g_k\|^2}{\lambda |g_{k-1}|^2 + (1 - \lambda)d_{k-1}^T y_{k-1}}' \in [0,1]
$$

where $\lambda \in [0,1]$ is a parameter.

In this note, we present a new formula which is a modified of the CD method defined by

$$
\beta_k^\text{New} = \frac{\|g_k\|^2}{\mu|d_{k-1}^T g_k| - d_{k-1}^T g_{k-1}}.
$$

Note that if we use the exact line search, our new algorithm reduces to the standard CD algorithm. In this paper, however, we consider general nonlinear functions and an inexact line search.

The rest of the paper is organized as follows. In Section 2, the new suggested algorithm is presented. The sufficient descent and Global convergence properties of the proposed methods are analysed in Section 3. Some numerical results are reported in Section 4. Finally, we draw a conclusion in Section 5.

1. ALGORITHM FOR NEW METHOD

In this section, we state the steps of our algorithm.

step 1: Set $k = 1$, select initial point $x_1 \in \mathbb{R}^n$, a very small positive $\varepsilon > 0$

step 2: $g_k = \nabla f(x_k)$, if $g_k = 1$ then stop

else set $d_1 = -g_1$

step 3: Compute step length $\alpha_k$ to minimize $f(x_k)$

step 4: $x_{k+1} = x_k + \alpha_k d_k$
step 5: $g_{k+1} = \nabla f(x_{k+1})$

step 6: Compute the parameter $\beta_k$ by (6).

step 7: $d_{k+1} = -g_{k+1} + \beta_k d_k$.

step 8: If $k = i$, or if $\|g_k\| < \varepsilon$ is satisfied then go to step 2

else $k = k + 1$ then go to step 3

We use the same algorithm in [1] which is restated here for the sake of convenience.

The convergence properties of the new method are stated in the following theorem.

2. CONVERGENCE OF THE NEW METHOD

In this section, we give the search direction $d_k^T$ satisfies the sufficient descent condition. We can notice by (5) that

$$d_{k-1}^T (g_k - g_{k-1}) \geq \sigma d_{k-1}^T g_{k-1} - d_{k-1}^T g_{k-1} = (\sigma - 1)d_{k-1}^T g_{k-1}$$

(7)

Lemma 1: If $\mu > 1$, then

$$g_k^T d_k < -(1 - \frac{1}{\mu}) \|g_k\|^2 < 0 \text{ for } k = 1, 2, \ldots.$$

Proof: If $k = 1$ then $d_1 = -g_1$ and since $\mu > 1$ we get

$$g_1^T d_1 = -\|g_1\|^2 < -(1 - \frac{1}{\mu}) \|g_1\|^2 < 0.$$

Assume by induction that $g_{k-1}^T d_{k-1} = -\|g_{k-1}\|^2 < -(1 - \frac{1}{\mu}) \|g_{k-1}\|^2 < 0$. By (2), (5), (6) and (7), we have

$$g_k^T d_k = -\|g_k\|^2 + \beta_k^N d_k^T g_{k-1}$$

$$= -\|g_k\|^2 + \frac{\mu |d_{k-1}^T g_k| - d_{k-1}^T g_{k-1}}{\|g_k\|^2} g_k^T d_{k-1}$$

$$\leq \|g_k\|^2 + \frac{\mu |d_{k-1}^T g_k| - d_{k-1}^T g_{k-1}| g_k^T d_{k-1}|}{\|g_k\|^2}$$

$$\leq \|g_k\|^2 + \frac{\|g_k\|^2 |g_k^T d_{k-1}|}{\mu |d_{k-1}^T g_k|}$$

$$= -(1 - \frac{1}{\mu}) \|g_k\|^2$$
we remark that our Lemma 1 implies that $d_k$ is a sufficient descent direction in (6).

**Lemma 2:** Once the sequence $\{x^k\}$ is clearly generated by (1) and (2), the step size $\alpha_k$ satisfies

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty.$$  

The proof of this Lemma is given in [1].

Global convergence of our method will be established and numerical evidence will be listed to support.

**Theorem 1 (Global convergence):** If $\mu > 1$ in (6), $\tilde{f}$ is bounded and $g(x)$ is Lipschitz in the level set, then our algorithm either terminates at a stationary point or

$$\lim_{k \to \infty} \|g_k\| = 0.$$

**Proof:** in case the conclusion never holds, then a real number does exist $\varepsilon > 0$ such that

$$\|g_k\| > \varepsilon, \text{ for all } k = 1, 2, 3, \ldots$$

since $d_k + g_k = \beta_k d_{k-1}$, we have:

$$\|d_k\|^2 = \beta_k^2 \|d_{k-1}\|^2 - \|g_k\|^2 - 2g_k^T d_k$$

(9)

Now

$$g_k^T d_k = -\|g_k\|^2 + \beta_k^{\text{New}} g_k^T d_{k-1}$$

$$= -\|g_k\|^2 + \frac{\|g_k\|^2}{\mu |d_{k-1}^T g_k| - d_{k-1}^T g_{k-1}} g_k^T d_{k-1}$$

$$\leq \frac{d_{k-1}^T g_{k-1} + g_k^T d_{k-1}}{\mu |d_{k-1}^T g_k| - d_{k-1}^T g_{k-1}} \|g_k\|^2$$

There exist $\sigma$ such that:

$$\sigma g_{k-1}^T d_{k-1} \leq g_{k-1}^T d_{k-1} \leq -\sigma g_{k-1}^T d_{k-1}$$

Then

$$\sigma g_{k-1}^T d_{k-1} \leq \frac{d_{k-1}^T g_{k-1} - \sigma g_{k-1}^T d_{k-1}}{\mu |d_{k-1}^T g_k|} \|g_k\|^2$$

$$= \frac{(1 - \sigma)d_{k-1}^T g_{k-1}}{\mu |d_{k-1}^T g_k|} \|g_k\|^2$$

Since $d_{k-1}^T g_{k-1} < 0$ and $d_k^T g_k < 0$ we see that

$$\|g_k\|^2 \leq \frac{\mu |d_{k-1}^T g_k|}{(1 - \sigma)|d_{k-1}^T g_{k-1}|} d_k^T g_k$$

That is,
\[ \beta_k^{New} = \frac{\|g_k\|^2}{\mu |d_{k-1}^T g_k| - d_{k-1}^T g_{k-1}} \]

\[ \leq \frac{\|g_k\|^2}{\mu |d_{k-1}^T g_k|} \leq \frac{|d_k^T g_k|}{(1 - \sigma)|d_{k-1}^T g_{k-1}|} \]

Replace \( \beta_k \) in (9) with \( \beta_k^{New} \), we get

\[ \frac{\|d_k\|^2}{(g_k^T d_k)^2} \leq \frac{\|d_{k-1}\|^2}{(g_{k-1}^T d_{k-1})^2} - \frac{\|g_k\|^2}{(g_{k-1}^T d_{k-1})^2} - 2 \frac{1}{g_k^T d_k} \]

\[ = \frac{\|d_{k-1}\|^2}{(g_{k-1}^T d_{k-1})^2} - \left( \frac{\|g_k\|}{g_{k-1}^T d_{k-1}} + \frac{1}{\|g_k\|} \right)^2 + \frac{1}{\|g_k\|^2} \]

\[ \leq \frac{\|d_{k-1}\|^2}{(g_{k-1}^T d_{k-1})^2} + \frac{1}{\|g_k\|^2} \leq \frac{|d_{k-1}^T g_{k-1}|}{(g_{k-1}^T d_{k-1})^2} + \frac{1}{\varepsilon^2} \]

Since \( d_1 = -g_1 \), so that

\[ \frac{\|d_k\|^2}{(g_k^T d_k)^2} < \frac{\|d_1\|^2}{(g_1^T d_1)^2} + \frac{k-1}{\varepsilon^2} = \frac{1}{\|g_1\|^2} + \frac{k-1}{\varepsilon^2} < \frac{1}{\varepsilon^2} + \frac{k-1}{\varepsilon^2} = \frac{k}{\varepsilon^2} \]

Thus

\[ \sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} > \sum_{k=1}^{\infty} \frac{\varepsilon^2}{k} = +\infty \]

which is contrary to Lemma 2. Hence \( \sum_{k=1}^{\infty} \frac{(g_{k+1}^T d_{k+1})^2}{\|d_{k+1}\|^2} < \infty \)

3. NUMERICAL RESULTS

As it is shown in Table (1), a comparison between the new introduced algorithm and the standard CD method will be done. Such the comparative tests consist of well-known non-linear problems with 8 various functions for \( n = 4, 100, 1000, 5000 \). Moreover, we wrote the code in Fortran 95 language and we take \( \mu > 1 \), for all cases the stopping condition \( \|g_{k+1}\| \leq 1 \times 10^{-5} \). On the other hand, the comparative results demonstrated in Table (1) which contains number of iteration (NOI) and number function evaluations (NOF). The experimental results are given in Table (1) verifying that the new conjugate gradient algorithm \( \beta_k^{New} \) is superior to standard (CD) with respect to (NOI) and (NOF).
Table (1): Comparing the two algorithms standard (CD) New algorithm performance

| Test functions | n     | New Algorithm | Standard CD method |
|----------------|-------|---------------|--------------------|
|                |       | NOI | NOF | NOI | NOF |
| SUM            | 4     | 3   | 11  | 3   | 11  |
|                | 100   | 14  | 85  | 14  | 85  |
|                | 500   | 21  | 119 | 22  | 114 |
|                | 1000  | 26  | 144 | 26  | 126 |
|                | 5000  | 33  | 162 | 38  | 198 |
| FRED           | 4     | 8   | 23  | 8   | 23  |
|                | 100   | 8   | 23  | 9   | 25  |
|                | 500   | 8   | 23  | 9   | 25  |
|                | 1000  | 8   | 23  | 9   | 25  |
|                | 5000  | 8   | 23  | 9   | 25  |
| GWood          | 4     | 25  | 60  | 28  | 65  |
|                | 100   | 25  | 60  | 28  | 65  |
|                | 500   | 26  | 62  | 29  | 68  |
|                | 1000  | 26  | 62  | 29  | 68  |
|                | 5000  | 26  | 62  | 29  | 68  |
| ROSEN          | 4     | 27  | 76  | 30  | 85  |
|                | 100   | 29  | 81  | 30  | 85  |
|                | 500   | 29  | 81  | 30  | 85  |
|                | 1000  | 29  | 81  | 30  | 85  |
|                | 5000  | 29  | 81  | 30  | 85  |
| NON-DIAGONAL   | 4     | 20  | 57  | 23  | 61  |
|                | 100   | 27  | 74  | 27  | 73  |
|                | 500   | 27  | 71  | 27  | 73  |
|                | 1000  | 27  | 71  | 27  | 73  |
|                | 5000  | 27  | 71  | 27  | 73  |
| MIELE          | 4     | 34  | 98  | 51  | 171 |
|                | 100   | 40  | 130 | 68  | 246 |
|                | 500   | 46  | 160 | 68  | 246 |
|                | 1000  | 46  | 160 | 68  | 246 |
|                | 5000  | 52  | 191 | 74  | 284 |
| CUBIC          | 4     | 12  | 37  | 13  | 38  |
|                | 100   | 13  | 39  | 14  | 40  |
|                | 500   | 13  | 39  | 15  | 44  |
|                | 1000  | 13  | 39  | 15  | 44  |
|                | 5000  | 13  | 39  | 15  | 44  |
| Gcantrel       | 4     | 14  | 88  | 12  | 67  |
|                | 100   | 18  | 139 | 18  | 142 |
|                | 500   | 19  | 152 | 23  | 210 |
|                | 1000  | 19  | 152 | 23  | 210 |
|                | 5000  | 22  | 195 | 28  | 278 |
| Total          |       | 910 | 3344 | 1076 | 4079 |

In Table (2), the rate of improvement in the new algorithm with the standard CD method is demonstrated.
Table (2): Comparison of the rate of improvement between the standard CD method and the new algorithm

| Tools | Standard CD method | New algorithm  |
|-------|-------------------|----------------|
| NOI   | 100%              | 84.5724%       |
| NOF   | 100%              | 81.9808%       |

5. CONCLUSION

The numerical performance of the new algorithm is better than the standard algorithm, as it is noticed that (NOI), (NOF) of the standard method are around 100%, that reflects the new algorithm has improved as compared to standard with (15.43%) in (NOI) and (18.02%) in (NOF).

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