LARGE TIME BEHAVIOR OF SOLUTIONS TO THE NONLINEAR HYPERBOLIC RELAXATION SYSTEM WITH SLOWLY DECAYING DATA

Ikki Fukuda
Department of Mathematics, Hokkaido University

Abstract

We consider the large time asymptotic behavior of the global solutions to the initial value problem for the nonlinear damped wave equation with slowly decaying initial data. When the initial data decay fast enough, it is known that the solution to this problem converges to a self-similar solution to the Burgers equation called a nonlinear diffusion wave and its optimal asymptotic rate is obtained. In this paper, we focus on the case that the initial data decay more slowly than previous works and derive the corresponding asymptotic profile. Moreover, we investigate how the change of the decay rate of the initial values affect its asymptotic rate.

Keywords: nonlinear damped wave equation, hyperbolic relaxation system, asymptotic profile, second asymptotic profile, optimal decay estimate, slowly decaying data.

1 Introduction

In this paper, we consider the large time behavior of the global solutions to the initial value problem for the following system:

\[ \begin{align*}
  u_t + v_x &= 0, & v_t + u_x &= f(u) - v, & x \in \mathbb{R}, & t > 0, \\
  u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x \in \mathbb{R},
\end{align*} \tag{1.1} \]

where \( f : \mathbb{R} \to \mathbb{R} \) is a given smooth function. This system is a typical example of hyperbolic system of conservation laws with relaxation called Jin-Xin model, which arises as mathematical models in several physical phenomena, e.g. non-equilibrium gas dynamics, magnetohydrodynamics and viscoelasticity (see e.g. [5, 17]).

If we delete \( v \) from (1.1), we obtain the following damped wave equation with a nonlinear convection term:

\[ \begin{align*}
  u_{tt} - u_{xx} + u_t + (f(u))_x &= 0, & x \in \mathbb{R}, & t > 0, \\
  u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x), & x \in \mathbb{R},
\end{align*} \tag{1.2} \]

where the initial data \( u_1(x) = -\partial_x v_0(x) \). In the present paper, we consider (1.2) with the flux function \( f(u) \equiv au + \frac{b}{2}u^2 + \frac{c}{3}u^3 \), where \( |a| < 1, b \neq 0 \) and \( c \in \mathbb{R} \). In addition, we assume that

\[ \exists \alpha > 1, \ \exists C > 0 \ \text{s.t.} \ |u_0(x)| \leq C(1 + |x|)^{-\alpha}, \ x \in \mathbb{R}, \]
\[ \exists \beta > 1, \ \exists C > 0 \ \text{s.t.} \ |u_1(x)| \leq C(1 + |x|)^{-\beta}, \ x \in \mathbb{R}. \tag{1.3} \]

The purpose of our study is to obtain an asymptotic profile of the solution \( u(x, t) \) and to examine the optimality of its asymptotic rate to the asymptotic function.

First of all, we recall known results about the asymptotic behavior of the solutions to (1.2). Orive and Zuazua [14] studied the global existence and the asymptotic behavior of the solution of (1.2) with \( a = 0 \) when \( u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R}) \) and \( u_1 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \). Moreover, Ueda and Kawashima [16] constructed the solution to (1.2), provided the initial data \( u_0 \in W^{1,p}(\mathbb{R}) \cap L^1(\mathbb{R}) \) with

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and \( u_1 \in L^p(\mathbb{R}) \cap L^1(\mathbb{R}) \) for \( 1 \leq p \leq \infty \), and they showed that the solution of (1.2) converges to a nonlinear diffusion wave defined by

\[
\chi(x, t) \equiv \frac{1}{\sqrt{1 + t}} \chi_\ast \left( \frac{x - a(1 + t)}{\sqrt{1 + t}} \right), \quad x \in \mathbb{R}, \quad t > 0,
\]

(1.4)

where

\[
\chi_\ast(x) \equiv \frac{\sqrt{\pi}}{b} \frac{(e^{\frac{x^2}{4\mu}} - 1)e^{-\frac{x^2}{4\mu}}}{\sqrt{\pi} + (e^{\frac{x^2}{4\mu}} - 1) \int_x^{\infty} e^{-\eta^2} d\eta}, \quad M \equiv \int_\mathbb{R} (u_0(x) + u_1(x)) dx, \quad \mu \equiv 1 - a^2.
\]

(1.5)

More precisely, if \( u_0 \in W^{1,p}(\mathbb{R}) \cap L^1(\mathbb{R}) \), \( u_1 \in L^p(\mathbb{R}) \cap L^1(\mathbb{R}) \) and \( \|u_0\|_{W^{s,p}} + \|u_0\|_{L^1} + \|u_1\|_{L^p} + \|u_1\|_{L^1} \) is sufficiently small, then, for any \( \varepsilon > 0 \), we have

\[
\|\partial_x^l(u(\cdot,t) - \chi(\cdot,t))\|_{L^p} \leq C(1 + t)^{-1 + \frac{1}{2} - \frac{1}{p} + \varepsilon}, \quad t \geq 0, \quad l = 0,1.
\]

(1.6)

Here the weighted Lebesgue space \( L^1_1(\mathbb{R}) \) is defined by

\[
L^1_1(\mathbb{R}) \equiv \left\{ f \in L^1(\mathbb{R}); \|f\|_{L^1_1} \equiv \int_\mathbb{R} |f(x)|(1 + |x|) dx < \infty \right\}.
\]

(1.7)

Also, we note that \( \chi(x,t) \) is a solution of the following Burgers equation:

\[
\chi_t + \left( a \chi + \frac{b}{2} \chi^2 \right)_x = \mu \chi_{xx}, \quad \int_\mathbb{R} \chi(x,0) dx = M.
\]

(1.8)

Moreover, the optimality of the asymptotic rate to the nonlinear diffusion wave was obtained by Kato and Ueda [8] by constructing the second asymptotic profile which is the leading term of \( u - \chi \). Indeed, if \( u_0 \in W^{s,p}(\mathbb{R}) \cap W^{s,1}(\mathbb{R}) \cap L^1(\mathbb{R}), \ u_1 \in W^{s-1,p}(\mathbb{R}) \cap W^{s,1}(\mathbb{R}) \cap L^1(\mathbb{R}) \) for \( s \geq 2, \ 1 \leq p \leq \infty \), and \( \|u_0\|_{W^{s,p}} + \|u_0\|_{W^{s,1}} + \|u_1\|_{W^{s-1,p}} + \|u_1\|_{W^{s,1}} \) is sufficiently small, then we have

\[
\|\partial_x^l(u(\cdot,t) - \chi(\cdot,t) - V(\cdot,t))\|_{L^p} \leq C(1 + t)^{-1 + \frac{1}{2} - \frac{l}{p}}, \quad t \geq 1
\]

(1.9)

for \( 0 \leq l \leq s - 2 \), where

\[
V(x,t) \equiv -\kappa_d V_\ast \left( \frac{x - a(1 + t)}{\sqrt{1 + t}} \right) (1 + t)^{-1} \log(1 + t),
\]

(1.10)

and

\[
V_\ast(x) \equiv \frac{1}{\sqrt{4\pi d}} \int_{-\infty}^x \eta_\ast(y) e^{-\frac{y^2}{4d}} dy, \quad \eta_\ast(x) \equiv \exp \left( \frac{b}{2\mu} \int_x^{\infty} \chi_\ast(y) dy \right),
\]

\[
d \equiv \int_\mathbb{R} (\eta_\ast(y))^{-1}(\chi_\ast(y))^3 dy, \quad \kappa \equiv \frac{ab^2}{4\mu} + \frac{c}{3!}.
\]

(1.11)

From (1.8), the triangle inequality and (1.9), one can obtain the following optimal decay estimate:

\[
\|\partial_x^l(u(\cdot,t) - \chi(\cdot,t))\|_{L^p} = (\tilde{C} + o(1))(1 + t)^{-1 + \frac{1}{2} - \frac{l}{p}} \log(1 + t), \quad 0 \leq l \leq s - 2
\]

(1.12)

as \( t \to \infty \), where \( \tilde{C} \equiv |kd|\|\partial_x^l V\|_{L^p} \). Therefore, we see that the solution \( u(x,t) \) to (1.2) tends to the nonlinear diffusion wave \( \chi(x,t) \) at the rate of \( (1 + t)^{-1 + \frac{1}{2} - \frac{l}{p}} \log(1 + t) \) if \( M \neq 0 \) and \( \kappa \neq 0 \), i.e. we cannot take \( \varepsilon = 0 \) in (1.6). The similar estimates for (1.6) and (1.8) are obtained for Burgers type equations such as generalized Burgers equation, KdV-Burgers equation and BBM-Burgers equation (cf. [1, 2, 3, 4, 6, 7, 12]).

The above results [16, 8] are corresponding to the case where the decay rate of the initial data \( u_0 \) and \( u_1 \) are rapid because \( \|u_0\|_{L^1_1(\mathbb{R})} \) are realized when \( \alpha, \beta > 2 \) in (1.3). However for (1.2) in the case of \( 1 < \alpha \leq 2 \) or \( 1 < \beta \leq 2 \) in (1.3), it is not known that the optimal asymptotic rate to the nonlinear diffusion wave, as far as we know. On the other hand, it is studied that the asymptotic profile for the solution to the damped wave equation with power type nonlinearity for

\[
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\]
slowly decaying data in the supercritical case. Actually, Narazaki and Nishihara [13] studied the following equation when the initial data are not in $L^1$:

$$u_{tt} - u_{xx} + u_t = |u|^{p-1}u, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}.$$  \hspace{1cm} (1.13)

They assumed that the initial data satisfies the condition (1.3) with $\alpha = \beta = k$ and $0 < k \leq 1$ and showed that if $p > 1 + 2/k$ and the data $u_0 \in B^{1,k}(\mathbb{R})$, $u_1 \in B^{0,k}(\mathbb{R})$ are small, then the asymptotic profile is given by

$$\Psi(x, t) = c_k \int_{\mathbb{R}} \frac{1}{4\pi t} e^{-\frac{(x-y)^2}{4t}} (1 + |y|)^{-k} dy,$$  \hspace{1cm} (1.14)

provided that the data satisfies $\lim_{|x| \to \infty} (1 + |x|)^k (u_0 + u_1)(x) = c_k$. Here, we set $B^{m,k} \equiv \{ f \in C^m; (1 + |x|)^k \partial_x^j f \in L^\infty(0 \leq l \leq m) \}$. More precisely, they proved

$$\lim_{t \to \infty} a_k(t)\|u(\cdot, t) - \Psi(\cdot, t)\|_{L^\infty} = 0, \quad a_k(t) = \begin{cases} (1 + t)^{k/2}, & 0 < k < 1, \\ \frac{(1 + t)^{1/2}}{\log(1 + t)}, & k = 1. \end{cases}$$  \hspace{1cm} (1.15)

(In [13], (1.13) in two and three space dimensional cases are also studied.) However, as we mentioned in the above, the asymptotic profile of the solutions to (1.2) with slowly decaying data is not well known even if the data are in $L^1$. For this reason, we would like to analyze the asymptotic behavior of the solution to (1.2) in the case of $1 < \alpha \leq 2$ or $1 < \beta \leq 2$ in (1.3).

Now, we state our first main result which generalizes the result given in [16]:

**Theorem 1.1.** Assume the condition (1.3) holds with $1 < \min\{\alpha, \beta\} \leq 2$. Let $s$ be a positive integer and $1 \leq p \leq \infty$. Suppose that $u_0 \in W^{s,p}(\mathbb{R})$, $u_1 \in W^{s-1,p}(\mathbb{R})$ and $\|u_0\|_{W^{s,p}} + \|u_0\|_{L^1} + \|u_1\|_{W^{s-1,p}} + \|u_1\|_{L^1}$ is sufficiently small. Then (1.2) has a unique global solution $u(x, t)$ with

$$u \in \left\{ \bigcap_{k=0}^{2} C^k([0, \infty); W^{s-k,p}) \cap C([0, \infty); L^1), \quad 1 \leq p < \infty, \right. \left. \bigcap_{k=0}^{2} W^k([0, \infty); W^{s-k,\infty}) \cap C([0, \infty); L^1), \quad p = \infty, \right\}$$

where $\sigma = \min\{2, s\}$. Moreover, for any $\varepsilon > 0$, the estimate

$$\|u(\cdot, t) - \chi(\cdot, t)\|_{L^q} \leq C \begin{cases} (1 + t)^{-\min\{\alpha, \beta\}/2 + \frac{1}{q'}}, & t \geq 0, \quad 1 < \min\{\alpha, \beta\} < 2, \\ (1 + t)^{-1 + \frac{1}{q'} + \varepsilon}, & t \geq 0, \quad \min\{\alpha, \beta\} = 2 \end{cases}$$  \hspace{1cm} (1.16)

holds for any $q$ with $1 \leq q \leq \infty$, and the estimate

$$\|\partial_x^l \partial_t^k (u(\cdot, t) - \chi(\cdot, t))\|_{L^p} \leq C \begin{cases} (1 + t)^{-\min\{\alpha, \beta\}/2 + \frac{k}{p} - \frac{l}{p'}}, & t \geq 0, \quad 1 < \min\{\alpha, \beta\} < 2, \\ (1 + t)^{-1 + \frac{k}{p'} + \varepsilon}, & t \geq 0, \quad \min\{\alpha, \beta\} = 2 \end{cases}$$  \hspace{1cm} (1.17)

holds for $0 \leq k \leq 2$ and $l \geq 0$ with $0 \leq k + l \leq s$, where $\chi(x, t)$ is defined by (1.4).

Furthermore, we can show that the above asymptotic rate given in (1.16) is optimal with respect to the time decaying order in the $L^\infty$ sense by constructing the second asymptotic profile for the solution to (1.2). To state such a result, we define the following functions

$$Z(x, t) \equiv \int_{\mathbb{R}} \frac{c_{\alpha, \beta}(y) \partial_x (G_0(x - y, t) \eta(x, t))}{(1 + |y|)^{\min\{\alpha, \beta\} - 1}} dy, \quad c_{\alpha, \beta}(y) \equiv \begin{cases} c_{\alpha, \beta}^+, & y \geq 0, \\ c_{\alpha, \beta}^-, & y < 0 \end{cases}$$  \hspace{1cm} (1.18)

and

$$G_0(x, t) \equiv \frac{1}{\sqrt{4\pi k}} e^{-\frac{(x - y)^2}{4k}}, \quad \eta(x, t) \equiv \eta_{\alpha} \left( \frac{x - a(1 + t)}{\sqrt{1 + t}} \right) = \exp \left( \frac{b}{2\mu} \int_{-\infty}^{x} \chi(y, t) dy \right)$$  \hspace{1cm} (1.19)

with $\eta_{\alpha}(x)$ being defined by (1.10). Then, we have the following result:

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**Theorem 1.2.** Assume the condition (1.3) holds with 1 < \min\{\alpha, \beta\} ≤ 2, \(u_0 \in H^2(\mathbb{R}) \cap W^{2,1}(\mathbb{R})\) and \(u_1 \in H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})\). Suppose that \(\|u_0\|_{H^2} + \|u_0\|_{W^{2,1}} + \|u_1\|_{H^1} + \|u_1\|_{W^{1,1}}\) is sufficiently small. We set \(\chi_0(x) \equiv \chi(x, 0), \eta_0(x) \equiv \eta(x, 0)\) and

\[
z_0(x) \equiv \eta_0(x)^{-1} \int_{x}^{\infty} (u_0(y) + u_1(y) - \chi_0(y))dy.
\]

If there exists \(\lim_{x \to \pm \infty} (1 + |x|)^{\min\{\alpha, \beta\} - 1} z_0(x) \equiv c_{\alpha, \beta}^{\pm}\), then the solution to (1.2) satisfies

\[
\begin{align*}
\lim_{t \to \infty} (1 + t)^{\min\{\alpha, \beta\}} \|u(\cdot, t) - \chi(\cdot, t) - Z(\cdot, t)\|_{L^\infty} & = 0, \\
1 < \min\{\alpha, \beta\} & < 2, \\
\lim_{t \to \infty} (1 + t)^{\min\{\alpha, \beta\}} \|u(\cdot, t) - \chi(\cdot, t) - Z(\cdot, t) - V(\cdot, t)\|_{L^\infty} & = 0, \\
\min\{\alpha, \beta\} & = 2,
\end{align*}
\]

where \(\chi(x, t)\) and \(V(x, t)\) are defined by (1.4) and (1.9), respectively, while \(Z(x, t)\) and \(\eta(x, t)\) are defined by (1.6) and (1.9), respectively. Moreover, if \(M \neq 0\), there exist \(v_0 > 0\) and \(v_1 > 0\) independent of \(x\) and \(t\) such that

\[
\|Z(\cdot, t)\|_{L^\infty} \leq C(\max\{|c_{\alpha, \beta}^+, |c_{\alpha, \beta}^-|\}) (1 + t)^{-\min\{\alpha, \beta\} / 2} \|\sqrt{\nu_0}\|_{L^\infty}(1 + t)^{-\min\{\alpha, \beta\} / 2},
\]

holds for sufficiently large \(t\) with 1 < \(\min\{\alpha, \beta\}\) < 2 and

\[
\|Z(\cdot, t) + V(\cdot, t)\|_{L^\infty} \leq C(\max\{|c_{\alpha, \beta}^+, |c_{\alpha, \beta}^-|\} + |\kappa d| \|V_*\|_{L^\infty})(1 + t)^{-1} \log(1 + t),
\]

holds for sufficiently large \(t\) with \(\min\{\alpha, \beta\} = 2\), where

\[
\begin{align*}
\nu_0 & \equiv \sqrt{\mu(c_{\alpha, \beta}^+ - c_{\alpha, \beta}^-) \Gamma\left(3 - \min\{\alpha, \beta\}\right) - \frac{b\chi_0(0)(c_{\alpha, \beta}^+ + c_{\alpha, \beta}^-)}{2 - \min\{\alpha, \beta\}} \Gamma\left(2 - \min\{\alpha, \beta\}\right)}, \\
\nu_1 & \equiv \frac{c_{\alpha, \beta}^+ + c_{\alpha, \beta}^-}{2} - \kappa d, \\
\Gamma(s) & \equiv \int_0^\infty e^{-x} x^{s-1} dx, \quad s > 0,
\end{align*}
\]

while \(M, d\) and \(\kappa\) are defined by (1.5) and (1.11), respectively.

By virtue of Theorem 1.2, the optimality of the estimate (1.12) can be examined from (1.21), (1.22), (1.23) and (1.24). Actually, we have the following optimal estimates of \(u - \chi\):

**Corollary 1.3.** Under the same assumptions in Theorem 1.2, if \(v_0 \neq 0\) and \(v_1 \neq 0\), then the following estimates

\[
\|u(\cdot, t) - \chi(\cdot, t)\|_{L^\infty} \sim \begin{cases} (1 + t)^{-\min\{\alpha, \beta\} / 2}, & 1 < \min\{\alpha, \beta\} < 2, \\ (1 + t)^{-1} \log(1 + t), & \min\{\alpha, \beta\} = 2 \end{cases}
\]

holds for sufficiently large \(t\).

**Remark 1.4.** The similar result for Theorem 1.1 is obtained by Kitagawa [11] for the generalized Burgers equation. For Theorem 1.2, the author in [2] obtained the similar result for the generalized KdV-Burgers equation.

This paper is organized as follows. In Section 2, we introduce the global existence of solutions to (1.2) and prepare a couple of lemmas for an auxiliary problem. In Section 3, we drive the upper bound estimates of \(u - \chi\), i.e., we prove Theorem 1.1. Finally, we give the proof of our second main result Theorem 1.2. The proof of Theorem 1.2 is divided into two parts. We extract the second asymptotic profiles \(Z(x, t)\) and \(Z(x, t) + V(x, t)\) according to the decaying rate of the initial data in Section 4 and Section 5. It is the main novelty of this paper.
Notations. In this paper, for $1 \leq p \leq \infty$, $L^p(\mathbb{R})$ denotes the usual Lebesgue spaces. In the following, for $f, g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, we denote the Fourier transform of $f$ and the inverse Fourier transform of $g$ as follows:

$$
\hat{f}(\xi) \equiv \mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx, \quad \hat{g}(x) \equiv \mathcal{F}^{-1}[g](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} g(\xi) d\xi.
$$

Then, for $s \geq 0$, we define the Sobolev spaces by

$$
H^s(\mathbb{R}) \equiv \left\{ f \in L^2(\mathbb{R}); \|f\|_{H^s} \equiv \left( \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty \right\}.
$$

To express Sobolev spaces, for $1 \leq p \leq \infty$, we also set

$$
W^{m,p}(\mathbb{R}) \equiv \left\{ f \in L^p(\mathbb{R}); \|f\|_{W^{m,p}} \equiv \left( \sum_{n=0}^{m} \|\partial^m_x f\|^p_{L^p} \right)^{1/p} < \infty \right\}.
$$

Throughout this paper, $C$ denotes various positive constants, which may vary from line to line during computations. Also, it may depend on the initial data or other parameters. However, we note that it does not depend on the space and time variable $x$ and $t$. Finally, for positive functions $f(t)$ and $g(t)$, we denote $f(t) \sim g(t)$ if there exist positive constants $C_0$ and $C_0$ independent of $t$ such that $C_0g(t) \leq f(t) \leq C_0g(t)$ holds.

2 Preliminaries

In this section, we prepare a couple of lemmas to prove the main theorems. First, we shall mention the global existence and the decay estimates for the solutions to (1.2). Now, we consider the initial value problem for the linear damped wave equation:

$$
\begin{align*}
&u_{tt} - u_{xx} + u_t + au_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \\
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}.
\end{align*}
$$

(2.1)

By taking the Fourier transform for (2.1), it follows that

$$
\hat{u}(\xi, t) = \hat{G}(\xi, t)(\hat{u}_0(\xi) + \hat{u}_1(\xi)) + \partial_t \hat{G}(\xi, t)\hat{u}_0(\xi),
$$

where

$$
\hat{G}(\xi, t) \equiv \frac{1}{\lambda_1(\xi) - \lambda_2(\xi)} \left( e^{\lambda_1(\xi)t} - e^{\lambda_2(\xi)t} \right),
$$

(2.2)

$$
\lambda_1(\xi) \equiv \frac{1}{2}(-1 + \sqrt{1 - 4(\xi^2 + ai\xi)}), \quad \lambda_2(\xi) \equiv \frac{1}{2}(-1 - \sqrt{1 - 4(\xi^2 + ai\xi)}).
$$

Therefore, the solution of (2.1) can be expressed as follows:

$$
u(t) = G(t) * (u_0 + u_1) + \partial_t G(t) * u_0,
$$

(2.3)

where we set

$$
G(x, t) \equiv \mathcal{F}^{-1}[\hat{G}(\cdot, t)](x).
$$

For this function $G(x, t)$, we can show the following decay estimate (for the proof, see Ueda and Kawashima [16] and Kato and Ueda [8]).

Lemma 2.1. Let $1 \leq q \leq p \leq \infty$. Then the following $L^p - L^q$ estimates hold:

$$
\|G(t) * \phi\|_{L^p} \leq C(1 + t)^{-\frac{q}{2} + \frac{1}{p} - \frac{1}{q}}, \quad t \geq 0,
$$

(2.4)

$$
\|\partial^m_x \partial^l_t G(t) * \phi\|_{L^p} \leq C(1 + t)^{-\frac{q}{2} + \frac{1}{p} - \frac{m}{p} + \frac{l}{q}} \|\phi\|_{L^q} + Ce^{-c_0t} \|\phi\|_{W^{m+l-1,p}}, \quad t \geq 0,
$$

(2.5)

for $m + l \geq 1$, where $G(x, t)$ and $G_0(x, t)$ are defined by (2.3) and (1.19), respectively. Moreover, the solutions operator $G(t)*$ is approximated by $G_0(t)*$ in the following sense:

$$
\|(G - G_0)(t) * \phi\|_{L^p} \leq C(1 + t)^{-\frac{q}{2} + \frac{1}{p} - \frac{1}{q}} \|\phi\|_{L^q}, \quad t > 0,
$$

(2.6)

$$
\|\partial^m_x \partial^l_t (G - G_0)(t) * \phi\|_{L^p} \leq C(1 + t)^{-\frac{q}{2} + \frac{1}{p} - \frac{m}{p} + \frac{l}{q}} \|\phi\|_{L^q} + Ce^{-c_0t} \|\phi\|_{W^{m+l-1,p}}, \quad t > 0
$$

(2.7)

for $k + l \geq 1$. Here $c_0$ is a positive constant.
Applying the Duhamel principle to (1.2), we obtain
\[
    u(t) = G(t) * (u_0 + u_1) + \partial_t G(t) * u_0 - \int_0^t G(t - \tau) * (g(u, x)\gamma(\tau))d\tau,
\]
where \( g(u) \equiv \frac{\gamma}{2}u^2 + \frac{\gamma}{3}u^3 \). Therefore, by using Lemma 2.1, we obtain the global existence of the solutions to (1.2) as in the following proposition. The proof of it is given by a standard argument which is based on the contraction mapping principle (see e.g. Proposition 3.1 in [8]):

**Proposition 2.2.** Let \( s \) be a positive integer and \( 1 \leq p \leq \infty \). Suppose that \( u_0 \in W^{s,p}(\mathbb{R}) \cap L^1(\mathbb{R}) \), \( u_1 \in W^{s-1}(\mathbb{R}) \cap L^1(\mathbb{R}) \) and \( E_0^{(s,p)} \equiv \| u_0 \|_{W^{s,p}} + \| u_0 \|_{L^1} + \| u_1 \|_{W^{s-1,p}} + \| u_1 \|_{L^1} \) is sufficiently small. Then (1.2) has a unique global solution \( u(x,t) \) with
\[
    u(x,t) = \int_0^t \chi(x - \xi, t - \tau) G(\tau)\gamma(\tau) d\tau,
\]
where \( \gamma \) is a positive integer and \( \| \gamma \|_{L^1} \leq \| \gamma \|_{L^\infty} \) satisfies the following decay estimate (for the proof, see Lemma 4.3 in [8]):

**Lemma 2.3.** Let \( k, l \) and \( m \) be non-negative integers. Then, for \( |M| \leq 1 \) and \( p \in [1, \infty] \), we have
\[
    \| \partial_t^k \partial_x^l \gamma(x, t) \|_{L^p} \leq C|M|(1 + t)^{-\frac{1}{2} + \frac{1}{2p} - \frac{k + m}{2l}} \leq 0.
\]

On the other hand, we have the following estimates for \( G_0(x,t) \):

**Lemma 2.4.** Let \( k \) and \( l \) be nonnegative integers. Then, for \( p \in [1, \infty] \), we have
\[
    \| \partial_t^k \partial_x^l G_0(x,t) \|_{L^p} \leq C t^{-\frac{1}{2} + \frac{1}{2p} - \frac{k + l}{2}} \leq 0.
\]

Moreover, if \( \int_\mathbb{R} \phi(x)dx = 0 \) and
\[
    \exists \gamma > 1, \quad \exists C > 0 \quad \text{s.t.} \quad |\phi(x)| \leq C(1 + |x|)^{-\gamma}, \quad x \in \mathbb{R},
\]
then we have
\[
    \| \partial_t^k \partial_x^l G_0(x,t) * \phi \|_{L^p} \leq C \begin{cases} t^{-\frac{1}{2} + \frac{1}{2p} - \frac{k + l}{2}}, & t > 0, \quad 1 < \gamma < 2, \\ t^{-1 + \frac{1}{2p} - \frac{k + l}{2}} \log(2 + t), & t > 0, \quad \gamma = 2. \end{cases}
\]

**Proof.** We shall prove only (2.15) because (2.13) can be shown easily. Since \( \int_\mathbb{R} \phi(x)dx = 0 \), we see that
\[
    \partial_t^k \partial_x^l G_0(x,t) * \phi(x) = \int_\mathbb{R} \partial_t^k \partial_x^l G_0(x - y,t) \phi(y)dy \quad \text{and} \quad \int_\mathbb{R} \phi(y)dy
\]
\[
    = \int_\mathbb{R} \partial_t^k \partial_x^l G_0(x,y,t) - G_0(x,t))\phi(y)dy
\]
\[
    = \left( \int_{|y| \geq \sqrt{t}} + \int_{|y| \leq \sqrt{t}} \right) \partial_t^k \partial_x^l (G_0(x,y,t) - G_0(x,t))\phi(y)dy
\]
\[
    \equiv I_1(x,t) + I_2(x,t).
\]
In the proof of Theorem 1.2, we examine the second asymptotic profile of the solution to (1.2). To do so, we consider the function

\[ I(x, t) = \int_{|y| \leq \sqrt{t}} (\partial_t^k \partial_x^l G_0(x - \theta y) x(-y)) \phi(y) dy. \]

Therefore, we obtain from (2.13) and (2.14)

\[ \|I(\cdot, t)\|_{L^p} \leq \int_{|y| \leq \sqrt{t}} \|\partial_t^k \partial_x^l G_0(x - \theta y)\|_{L^p} (1 + |y|)^{-(\gamma - 1)} dy \]

\[ \leq Ct^{-\frac{k}{2} + \frac{l}{4} - \frac{k}{4} + \frac{l}{4}} \int_0^{\sqrt{t}} (1 + y)^{-(\gamma - 1)} dy \equiv Ct^{-\frac{k}{2} + \frac{l}{4} - \frac{k}{4} + \frac{l}{4}} J(t). \]

For \( J(t) \), we can easily show

\[ J(t) \leq \left\{ \begin{array}{ll}
\int_0^{\sqrt{t}} y^{-(\gamma - 1)} dy = \left[ \frac{1}{2 - \gamma} y^{2 - \gamma} \right]_0^{\sqrt{t}} = \frac{1}{2 - \gamma} t^{1 - \frac{\gamma}{2}}, & 1 < \gamma < 2, \\
\int_0^{\sqrt{t}} (1 + y)^{-(\gamma - 1)} dy = \log(1 + \sqrt{t}) \leq \log(2 + t), & \gamma = 2.
\end{array} \right. \]  \hspace{1cm} (2.19)

Thus, from (2.18) and (2.19) we have

\[ \|I(\cdot, t)\|_{L^p} \leq C \left\{ t^{-\frac{k}{2} + \frac{l}{4} - \frac{k}{4} + \frac{l}{4}}, \quad t > 0, \quad 1 < \gamma < 2, \\
t^{-1 + \frac{k}{2} - \frac{k}{4} + \frac{l}{4}} \log(2 + t), \quad t > 0, \quad \gamma = 2. \right. \]  \hspace{1cm} (2.20)

Combining (2.16), (2.17) and (2.20), we obtain (2.15).

In the rest of this section, let us prepare the ingredients to prove Theorem 1.2. First, we consider the function \( \eta(x, t) \) defined by (1.19) and an auxiliary problem. First, for the function \( \eta(x, t) \), we can easily obtain that

\[ \min\{1, e^{\frac{4M}{\lambda}}\} \leq \eta(x, t) \leq \max\{1, e^{\frac{4M}{\lambda}}\}, \]  \hspace{1cm} (2.21)

\[ \min\{1, e^{\frac{4M}{\lambda}}\} \leq \eta(x, t)^{-1} \leq \max\{1, e^{\frac{4M}{\lambda}}\}. \]  \hspace{1cm} (2.22)

Moreover, by using Lemma 2.3, we have the following \( L^p \) decay estimate (for the proof, see Corollary 2.3 in [7] or Lemma 5.4 in [8]).

**Lemma 2.5.** Let \( l \) be a positive integer and \( p \in [1, \infty] \). If \( |M| \leq 1 \), then we have

\[ \|\partial_x^l \eta(\cdot, t)\|_{L^p} + \|\partial_x^l (\eta(\cdot, t)^{-1})\|_{L^p} \leq C|M|(1 + t)^{-\frac{1}{2}(1 - \frac{k}{4}) - \frac{l}{4}}, \quad t \geq 0. \]  \hspace{1cm} (2.23)

In the proof of Theorem 1.2, we examine the second asymptotic profile of the solution to (1.2). To analyze the second asymptotic profile, we set \( \psi \equiv u + u_t - \chi \). Recalling \( \mu = 1 - a^2 \), the perturbation \( \psi(x, t) \) satisfies the following equation:

\[ \psi_t + a \psi_x + (b \chi \psi)_x - \mu \psi_{xx} = a \partial_x (\partial_t + a \partial_x) (u - \chi) + a \partial_x (\partial_t + a \partial_x) \chi - \frac{b}{2} \partial_x ((u - \chi)^2) - \frac{c}{3!} \partial_x (u^3) + \partial_x (b \chi u_t - \mu u_{xx}). \]
To analyze the above equation, we prepare the following auxiliary problem:

\[
z_t + a z_x + (b \chi z)_x - \mu z_{xx} = \partial_x \lambda(x, t), \quad x \in \mathbb{R}, \quad t > 0,
\]
\[
z(x, 0) = z_0(x), \quad x \in \mathbb{R}
\]  
(2.24)

where \(\lambda(x, t)\) is a given regular function decaying at spatial infinity. If we take the new valuable \(\tilde{x} \equiv x - at\), and set \(\tilde{z}(\tilde{x}, t) \equiv z(x, t), \quad \tilde{\chi}(\tilde{x}, t) \equiv \chi(x, t), \quad \tilde{\lambda}(\tilde{x}, t) \equiv \lambda(x, t)\) and \(\tilde{z}_0(\tilde{x}) \equiv z_0(\tilde{x})\), then (2.24) can be rewritten as follows:

\[
\tilde{z}_t + (b \tilde{\chi} \tilde{z})_{\tilde{x}} - \mu \tilde{z}_{\tilde{x} \tilde{x}} = \partial_{\tilde{x}} \tilde{\lambda}(\tilde{x}, t), \quad \tilde{x} \in \mathbb{R}, \quad t > 0,
\]
\[
\tilde{z}(\tilde{x}, 0) = \tilde{z}_0(\tilde{x}), \quad \tilde{x} \in \mathbb{R}
\]  
(2.25)

Therefore, if we set

\[
U[h](x, t, \tau) \equiv \int_{\mathbb{R}} \partial_x (G_0(x - y, t - \tau) \eta(x, t))(\eta(y, \tau))^{-1} \left( \int_{-\infty}^{y} h(\xi) d\xi \right) dy\]
\[
x \in \mathbb{R}, \quad 0 \leq \tau < t,
\]  
(2.26)

then, applying Lemma 3.3 in [1] or Lemma 3.1 in [7] to (2.25), we can deduce the following representation formula for (2.24):

**Lemma 2.6.** The solution of (2.24) is given by

\[
z(x, t) = U[z_0](x, t, 0) + \int_{0}^{t} U[\partial_x \lambda](x, t, \tau) d\tau, \quad x \in \mathbb{R}, \quad t > 0.
\]  
(2.27)

This explicit representation formula (2.27) plays an important role in the proof of Theorem 1.2. Especially, the following estimate can be obtained:

**Lemma 2.7.** Assume that \(|M| \leq 1\). Let \(1 \leq p, q \leq \infty\) and \(\frac{1}{p} + \frac{1}{q} = 1\). Then the following estimate

\[
\|U[\partial_x \lambda](\cdot, t, \tau)\|_{L^\infty} \leq C \sum_{n=0}^{1} (1 + t)^{-\frac{1}{4} + \frac{n}{q}} (t - \tau)^{-\frac{1}{2} + \frac{n}{q} - \frac{2}{q}} \|\lambda(\cdot, \tau)\|_{L^q}
\]  
(2.28)

holds for \(t > \tau\).

**Proof.** From (2.26), we obtain

\[
U[\partial_x \lambda](x, t, \tau) = \int_{\mathbb{R}} \partial_x (G_0(x - y, t - \tau) \eta(x, t))(\eta(y, \tau))^{-1} \lambda(y, \tau) dy
\]
\[
= \sum_{n=0}^{1} \partial_x^{1-n} \eta(x, t) \int_{\mathbb{R}} \partial_x^n G_0(x - y, t - \tau)(\eta(y, \tau))^{-1} \lambda(y, \tau) dy.
\]

By using Young’s inequality, Lemma 2.5, (2.21), (2.13) and (2.22), we obtain

\[
\|U[\partial_x \lambda](\cdot, t, \tau)\|_{L^\infty} \leq C \sum_{n=0}^{1} \|\partial_x^{1-n} \eta(\cdot, t)\|_{L^\infty} \|\partial_x^n G_0(t - \tau) \ast (\eta^{-1} \lambda)(\tau)\|_{L^\infty}
\]
\[
\leq C \sum_{n=0}^{1} (1 + t)^{-\frac{1}{4} + \frac{n}{q}} (t - \tau)^{-\frac{1}{2} + \frac{n}{q} - \frac{2}{q}} \|\lambda(\cdot, \tau)\|_{L^q}.
\]

\[\square\]
3 Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. In order to obtain the upper bound of \( u - \chi \), we rewrite the differential equations (1.2) and (1.7) as follows:

\[
\begin{align*}
\alpha (t) &= G_0(t) * \chi_0 - \frac{b}{2} \int_{0}^{t} G(t - \tau) * ((\chi^2)_x)(\tau) d\tau, \\
\end{align*}
\]

where \( g(u) = \frac{a}{2} u^2 + \frac{c}{3} u^3 \) and \( \chi_0(x) = \chi(x, 0) \). Therefore, if we set

\[
\phi(x, t) \equiv u(x, t) - \chi(x, t),
\]

then \( \phi(x, t) \) satisfies the following relation:

\[
\begin{align*}
\phi(t) &= (G - G_0)(t) * (u_0 + u_1) + G_0(t) * (u_0 + u_1 - \chi_0) \\
&\quad + \partial_t G(t) * u_0 - \frac{c}{3} \int_{0}^{t} G(t - \tau) * ((u^3)_x)(\tau) d\tau \\
&\quad - \frac{b}{2} \int_{0}^{t} (G - G_0)(t - \tau) * ((u^2)_x)(\tau) d\tau - \frac{b}{2} \int_{0}^{t} G_0(t - \tau) * ((u^2 - \chi^2)_x)(\tau) d\tau \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{align*}
\]

Now, let us prove Theorem 1.1. Our first step to show Theorem 1.1 is to derive the following proposition:

**Proposition 3.1.** Assume the same conditions on \( u_0 \) and \( u_1 \) in Theorem 1.1 are valid. Then, for any \( \varepsilon > 0 \), we have

\[
\| \phi(\cdot, t) \|_{L^1} \leq C \begin{cases} 
(1 + t)^{- \min\{\alpha, \beta\} + \frac{\varepsilon}{2}}, & t \geq 0, \ 1 < \min\{\alpha, \beta\} < 2, \\
(1 + t)^{- \frac{1}{2} + \varepsilon}, & t \geq 0, \ \min\{\alpha, \beta\} = 2,
\end{cases}
\]

where \( \phi(x, t) \) is defined by (3.3).

**Proof.** We set

\[
M(T) \equiv \begin{cases} 
\sup_{0 \leq s \leq T} (1 + t)^{- \gamma} \| \phi(\cdot, t) \|_{L^1}, & 1 < \gamma < 2, \\
\sup_{0 \leq s \leq T} (1 + t)^{\frac{1}{2} - \varepsilon} \| \phi(\cdot, t) \|_{L^1}, & \gamma = 2,
\end{cases}
\]

where \( \gamma \equiv \min\{\alpha, \beta\} \) and \( \varepsilon \) is any fixed constant such that \( 0 < \varepsilon < \frac{1}{2} \). It suffices to estimate the each term of the right hand side of (3.4). For the first term, from (2.6), we have

\[
\| I_1(\cdot, t) \|_{L^1} \leq C(1 + t)^{- \frac{\gamma}{2}} (\| u_0 \|_{L^1} + \| u_1 \|_{L^1}), \ t > 0.
\]

Also, since \( \int_{\mathbb{R}} (u_0 + u_1 - \chi_0) dx = 0 \), (1.3), (1.4) and (1.5), by using (2.15), it follows that

\[
\| I_2(\cdot, t) \|_{L^1} \leq C_0 \begin{cases} 
(1 + t)^{- \frac{\gamma}{2}}, & t \geq 1, \ 1 < \gamma < 2, \\
(1 + t)^{- \frac{1}{2} \log(2 + t)}, & t \geq 1, \ \gamma = 2,
\end{cases}
\]

where \( C_0 \) is a positive constant. For \( I_3 \), applying (2.5), we obtain

\[
\| I_3(\cdot, t) \|_{L^1} \leq C(1 + t)^{- \frac{1}{2}} \| u_0 \|_{L^1}, \ t \geq 0.
\]
Next, we evaluate $I_4$. Applying (2.5) and (2.9), we have

\[
\|I_4(\cdot,t)\|_{L^1} \leq C \int_0^t \|\partial_x G(t-\tau) * u^3(\tau)\|_{L^1} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-\frac{1}{2}} \|u^3(\cdot,\tau)\|_{L^1} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-\frac{1}{2}} \|u(\cdot,\tau)\|_\infty \|u(\cdot,\tau)\|_{L^1} d\tau \\
\leq CE_0^{(1,p)} \int_0^t (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{1}{2}} d\tau \\
\leq CE_0^{(1,p)} (1 + t)^{-\frac{1}{2}} \log(2 + t), \ t \geq 0.
\]

We note that $I_4$ does not appear if $c = 0$. For $I_5$, by using (2.7) and (2.9), similarly we have

\[
\|I_5(\cdot,t)\|_{L^1} \leq C \int_0^t \|\partial_x (G - G_0)(t-\tau) * u^2(\tau)\|_{L^1} d\tau \\
\leq C \int_0^t (t - \tau)^{-\frac{1}{2}} (1 + t - \tau)^{-\frac{1}{2}} \|u^2(\cdot,\tau)\|_{L^1} d\tau \\
\leq CE_0^{(1,p)} \int_0^t (t - \tau)^{-\frac{1}{2}} (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{1}{2}} d\tau \\
\leq CE_0^{(1,p)} (1 + t)^{-\frac{1}{2}} \log(2 + t), \ t \geq 1.
\]

Finally, we evaluate $I_6$. From Young’s inequality, (2.13), (2.9), (2.12), (3.3) and (3.6), we obtain

\[
\|I_6(\cdot,t)\|_{L^1} \leq C \int_0^t \|\partial_x G_0(t-\tau) * (u^2 - \chi^2)(\tau)\|_{L^1} d\tau \\
\leq C \int_0^t (t - \tau)^{-\frac{1}{2}} \|(u^2 - \chi^2)(\cdot,\tau)\|_{L^1} d\tau \\
\leq C \int_0^t (t - \tau)^{-\frac{1}{2}} (\|u(\cdot,\tau)\|_{L^\infty} + \|\chi(\cdot,\tau)\|_{L^\infty}) \|\phi(\cdot,\tau)\|_{L^1} d\tau \\
\leq CE_0^{(1,p)} M(T) \begin{cases} 
\int_0^t (t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{1}{2}} d\tau, & 1 < \gamma < 2, \\
\int_0^t (t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\gamma} d\tau, & \gamma = 2 
\end{cases} \\
\leq CE_0^{(1,p)} M(T) \begin{cases} 
(1 + t)^{-\frac{1}{2} + \varepsilon}, & t \geq 1, 1 < \gamma < 2, \\
(1 + t)^{-\frac{1}{2} + \varepsilon}, & t \geq 1, \gamma = 2.
\end{cases}
\]

Therefore, combining (3.4) and (3.7) through (3.12), we have

\[
\|\phi(\cdot,t)\|_{L^1} \leq CE_0^{(1,p)} (1 + t)^{-\frac{1}{2}} + CE_0^{(1,p)} (1 + t)^{-\frac{1}{2}} \log(2 + t) \\
+ C_0 \begin{cases} 
(1 + t)^{-\frac{1}{2} + \varepsilon}, & 1 \leq t \leq T, 1 < \gamma < 2, \\
(1 + t)^{-\frac{1}{2} + \varepsilon} \log(2 + t), & 1 \leq t \leq T, \gamma = 2 
\end{cases} \\
+ CE_0^{(1,p)} M(T) \begin{cases} 
(1 + t)^{-\frac{1}{2} + \varepsilon}, & 1 \leq t \leq T, 1 < \gamma < 2, \\
(1 + t)^{-\frac{1}{2} + \varepsilon}, & 1 \leq t \leq T, \gamma = 2.
\end{cases}
\]

For $0 \leq t \leq 1$, from (2.9), (2.12) and $|M| \leq E_0^{(1,p)}$, we obtain

\[
\|\phi(\cdot,t)\|_{L^1} \leq \|u(\cdot,t)\|_{L^1} + \|\chi(\cdot,t)\|_{L^1} \leq CE_0^{(1,p)}, \ 0 \leq t \leq 1.
\]

Since $\log(2 + t) \leq C(1 + t)^{\varepsilon}$, combining (3.13) and (3.14), we arrive at

\[
M(T) \leq CE_0^{(1,p)} + C_0 + C_1 E_0^{(1,p)} M(T),
\]
where $C_1$ is a positive constant. Therefore, we obtain the desired estimate
\[
M(T) \leq 2CE_0^{(1,p)} + 2C_0
\]
if $E_0^{(1,p)}$ is small that $C_1E_0^{(1,p)} \leq \frac{1}{2}$. This completes the proof. \hfill \Box

Next, we shall derive $L^\infty$-estimate of $u - \chi$. Actually, we have the following proposition:

**Proposition 3.2.** Assume the same conditions on $u_0$ and $u_1$ in Theorem 1.1 are valid. Then, for any $\varepsilon > 0$, we have
\[
\|\phi(\cdot, t)\|_{L^\infty} \leq C \begin{cases} 
(1 + t)^{-\min\{\alpha, \beta\}/2}, & t \geq 0, 1 < \min\{\alpha, \beta\} < 2, \\
(1 + t)^{-1+\varepsilon}, & t \geq 0, \min\{\alpha, \beta\} = 2,
\end{cases} \quad (3.15)
\]
where $\phi(x, t)$ is defined by (3.3).

**Proof.** We set
\[
N(t) = \begin{cases} 
\sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{\gamma}{2}} \|\phi(\cdot, \tau)\|_{L^\infty}, & 1 < \gamma < 2, \\
\sup_{0 \leq \tau \leq t} (1 + \tau)^{-\varepsilon} \|\phi(\cdot, \tau)\|_{L^\infty}, & \gamma = 2,
\end{cases} \quad (3.16)
\]
where $\gamma \equiv \min\{\alpha, \beta\}$ and $\varepsilon$ is any fixed constant such that $0 < \varepsilon < 1$. We evaluate the each term of the right hand side of (3.4). For $I_1$, from (2.6), we have
\[
\|I_1(\cdot, t)\|_{L^\infty} \leq C T^{-\frac{1}{2}} (1 + t)^{-\frac{1}{2}} (\|u_0\|_{L^1} + \|u_1\|_{L^1}) \leq CE_0^{(1,p)} (1 + t)^{-1}, \quad t \geq 1. \quad (3.17)
\]
Also, in the same way to get (3.8), applying (2.15), we see that
\[
\|I_2(\cdot, t)\|_{L^\infty} \leq C_0 \begin{cases} 
(1 + t)^{-\frac{1}{2}}, & t \geq 1, 1 < \gamma < 2, \\
(1 + t)^{-1} \log(2 + t), & t \geq 1, \gamma = 2,
\end{cases} \quad (3.18)
\]
where $C_0$ is a positive constant. For $I_3$, from (2.5), we get
\[
\|I_3(\cdot, t)\|_{L^\infty} \leq C(1 + t)^{-1} \|u_0\|_{L^1} + CE_0^{(1,p)} \|u_0\|_{L^1} \leq CE_0^{(1,p)} (1 + t)^{-1}, \quad t \geq 0. \quad (3.19)
\]
Here, we have used the Gagliardo-Nirenberg inequality $\|u_0\|_{L^\infty} \leq C \|u_0\|_{L^p}^{1-\frac{2}{p}} \|u_0'\|_{L^p}^{\frac{2}{p}} \leq CE_0^{(1,p)}$. Next, applying (2.5) and (2.9), we have
\[
\|I_4(\cdot, t)\|_{L^\infty} \leq C \int_0^t \|\partial_x G(t - \tau) * u_\tau(\cdot, \tau)\|_{L^\infty} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-1} \|u_\tau(\cdot, \tau)\|_{L^1} d\tau + C \int_0^t e^{-c_0(t - \tau)} \|u_\tau(\cdot, \tau)\|_{L^\infty} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-1} \|u_\tau(\cdot, \tau)\|_{L^1} d\tau + C \int_0^t e^{-c_0(t - \tau)} \|u_\tau(\cdot, \tau)\|_{L^\infty} d\tau \\
\leq CE_0^{(1,p)} \int_0^t (1 + t - \tau)^{-1} (1 + \tau)^{-1} d\tau + CE_0^{(1,p)} \int_0^t e^{-c_0(t - \tau)} (1 + \tau)^{-\frac{3}{2}} d\tau \\
\leq CE_0^{(1,p)} (1 + t)^{-1} \log(2 + t), \quad t \geq 0. \quad (3.20)
\]
For $I_5$, by using (2.6), Hölder’s inequality, (2.9) and (2.10), we have
\[
\|I_5(\cdot, t)\|_{L^\infty} \leq C \int_0^t \|G - G_0(t - \tau) * \partial_x (u^2)(\cdot, \tau)\|_{L^1} d\tau \\
\leq C \int_0^t (t - \tau)^{-\frac{1}{2}} (1 + t - \tau)^{-\frac{1}{2}} \|\partial_x (u^2)(\cdot, \tau)\|_{L^1} d\tau \\
\leq C \int_0^t (t - \tau)^{-\frac{1}{2}} (1 + t - \tau)^{-\frac{1}{2}} \|u(\cdot, \tau)\|_{L^q} \|\partial_x u(\cdot, \tau)\|_{L^p} d\tau \\
\leq CE_0^{(1,p)} \int_0^t (t - \tau)^{-\frac{1}{2}} (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-1} d\tau \\
\leq CE_0^{(1,p)} (1 + t)^{-1} \log(2 + t), \quad t \geq 1. \quad (3.21)
\]
Finally, we evaluate $I_0$. From Young’s inequality, (2.13), (2.9), (2.12), (3.3), (3.5) and (3.16), we obtain

$$
\|I_0(\cdot, t)\|_{L^\infty} \\
\leq C \int_0^t \|\partial_\tau G_0(t - \tau) * (u^2 - \chi^2)(\tau)\|_{L^\infty} d\tau \\
\leq C \int_0^{t/2} (t - \tau)^{-1}(u^2 - \chi^2)(\cdot, \tau) \|L^1 d\tau + C \int_{t/2}^t (t - \tau)^{-\frac{1}{2}}(u^2 - \chi^2)(\cdot, \tau) \|L^\infty d\tau \\
\leq C \int_0^{t/2} (t - \tau)^{-1}(u^2 - \chi^2)(\cdot, \tau) \|L^\infty d\tau + C \int_{t/2}^t (t - \tau)^{-\frac{1}{2}}(u^2 - \chi^2)(\cdot, \tau) \|L^\infty d\tau \\
+ C \int_{t/2}^t (t - \tau)^{-\frac{1}{2}}(\|u(\cdot, \tau)\|_{L^\infty} + \|\chi(\cdot, \tau)\|_{L^\infty}) \|\phi(\cdot, \tau)\|_{L^\infty} d\tau \\
\leq CE^{(1,p)}_0(t) \begin{cases}
\int_0^{t/2} (t - \tau)^{-1}(1 + \tau)^{-\frac{1}{2}} d\tau, & 1 < \gamma < 2, \\
\int_{t/2}^t (t - \tau)^{-\frac{1}{2}}(1 + \tau)^{-\frac{1}{2} + \varepsilon} d\tau, & \gamma = 2
\end{cases}
$$

(3.22)

Therefore, combining (3.4) and (3.17) through (3.22), we have

$$
\|\phi(\cdot, t)\|_{L^\infty} \leq CE^{(1,p)}_0(1 + t)^{-\frac{1}{2}} + CE^{(1,p)}_0(1 + t)^{-1} \log(2 + t) + C_0 \begin{cases}
(1 + t)^{-\frac{1}{2}}, & 1 \leq t \leq T, 1 < \gamma < 2, \\
(1 + t)^{-1} \log(2 + t), & 1 \leq t \leq T, \gamma = 2
\end{cases}
$$

(3.23)

For $0 \leq t \leq 1$, in the same way to get (3.14), we obtain

$$
\|\phi(\cdot, t)\|_{L^\infty} \leq \|u(\cdot, t)\|_{L^\infty} + \|\chi(\cdot, t)\|_{L^\infty} \leq CE^{(1,p)}_0, 0 \leq t \leq 1.
$$

(3.24)

Since $\log(2 + t) \leq C(1 + t)^2$, combining (3.23) and (3.24), it follows that

$$
N(T) \leq CE^{(1,p)}_0 + C_0 + C_1 E^{(1,p)}_0 N(T),
$$

where $C_1$ is a positive constant. Therefore, we obtain the desired estimate

$$
N(T) \leq 2CE^{(1,p)}_0 + 2C_0
$$

if $E^{(1,p)}_0$ is so small that $C_1 E^{(1,p)}_0 \leq \frac{1}{2}$. This completes the proof. \( \square \)

By virtue of Proposition 3.1, Proposition 3.2 and the interpolation inequality

$$
\|\phi(\cdot, t)\|_{L^q} \leq \|\phi(\cdot, t)\|_{L^\infty}^{1-1/q} \|\phi(\cdot, t)\|_{L^1}^{1/q}, \quad 1 \leq q \leq \infty,
$$

We have the following corollary:
Corollary 3.3. Assume the same conditions on \( u_0 \) and \( u_1 \) in Theorem 1.1 are valid. Then, for any \( \varepsilon > 0 \), we have

\[
\| \phi(\cdot, t) \|_{L^q} \leq C \begin{cases} 
(1 + t)^{-\min(\alpha, \beta) + \frac{1}{p} - \frac{1}{q}}, & t \geq 0, \ 1 < \min(\alpha, \beta) < 2, \\
(1 + t)^{-1 + \frac{1}{p} + \varepsilon}, & t \geq 0, \ \min(\alpha, \beta) = 2
\end{cases}
\] (3.25)

for any \( q \) with \( 1 \leq q \leq \infty \), where \( \phi(x, t) \) is defined by (3.3).

Next, we derive the following \( L^p \)-decay estimate of the spatial derivatives of \( u - \chi \):

Proposition 3.4. Assume the same conditions on \( u_0 \) and \( u_1 \) in Theorem 1.1 are valid. Then, for any \( \varepsilon > 0 \), we have

\[
\| \partial_x^l \phi(\cdot, t) \|_{L^p} \leq C \begin{cases} 
(1 + t)^{-\min(\alpha, \beta) + \frac{1}{p} - \frac{1}{q}}, & t \geq 0, \ 1 < \min(\alpha, \beta) < 2, \\
(1 + t)^{-1 + \frac{1}{p} + \varepsilon}, & t \geq 0, \ \min(\alpha, \beta) = 2
\end{cases}
\] (3.26)

for \( 0 \leq l \leq s \), where \( \phi(x, t) \) is defined by (3.3).

Proof. We have already shown (3.26) for \( l = 0 \). In the following, let us prove (3.26) for \( 1 \leq l \leq s \). We set

\[
L(T) \equiv \begin{cases} 
\sup_{0 \leq t \leq T} \sum_{l=1}^{s} (1 + t)^{-\frac{1}{p} + \frac{1}{q} + \frac{l}{p}}, & \text{if } 0 \leq \varepsilon, \\
\sup_{0 \leq t \leq T} \sum_{l=1}^{s} (1 + t)^{-1 + \frac{1}{p} + \varepsilon}, & \text{if } \varepsilon > 0,
\end{cases}
\] (3.27)

where \( \gamma \equiv \min(\alpha, \beta) \) and \( \varepsilon \) is any fixed constant such that \( 0 < \varepsilon < \frac{1}{2} \). We evaluate the each term of the right hand side of (3.4). For \( I_1 \), from (2.7), we have

\[
\| \partial_x^l I_1(\cdot, t) \|_{L^p} \leq C t^{-\frac{1}{2}(1 - \frac{1}{p}) - \frac{l}{p}} (1 + t)^{-\frac{1}{2}} \| u_0 + u_1 \|_{L^1} + C e^{-c\varepsilon t} \| u_0 + u_1 \|_{W^{1, p}} \\
\leq C E_0^{(s, p)} (1 + t)^{-1 + \frac{1}{p} - \frac{1}{q}}, \ t \geq 1.
\] (3.28)

Also, applying (2.15), it follows that

\[
\| \partial_x^l I_2(\cdot, t) \|_{L^p} \leq C_0 \begin{cases} 
(1 + t)^{-\frac{3}{2} + \frac{1}{p} - \frac{1}{q}}, & t \geq 1, \ 1 < \gamma < 2, \\
(1 + t)^{-1 + \frac{1}{p} - \frac{1}{q} - \frac{1}{p} \log(2 + t)}, & t \geq 1, \ \gamma = 2,
\end{cases}
\] (3.29)

where \( C_0 \) is a positive constant. For \( I_3 \), from (2.5), we get

\[
\| \partial_x^l I_3(\cdot, t) \|_{L^p} \leq C (1 + t)^{-1 + \frac{1}{p} - \frac{1}{q}} \| u_0 \|_{L^1} + C e^{-c\varepsilon t} \| u_0 \|_{W^{1, p}} \\
\leq C E_0^{(s, p)} (1 + t)^{-1 + \frac{1}{p} - \frac{1}{q}}, \ t \geq 0.
\] (3.30)

Next, applying (2.5), (2.9) and (2.10), we have

\[
\| \partial_x^l I_4(\cdot, t) \|_{L^p} \\
\leq C \int_{t/2}^{t/2} \| \partial_x^{l+1} G(t - \tau) * u^3(\tau) \|_{L^p} d\tau + C \int_{t/2}^{t} \| \partial_x G(t - \tau) * \partial_x^l (u^3(\tau)) \|_{L^p} d\tau \\
\leq C \int_{t/2}^{t/2} (1 + t - \tau)^{-1 + \frac{1}{p} - \frac{1}{q}} \| u^3(\cdot, \tau) \|_{L^1} d\tau + C \int_{t/2}^{t/2} e^{-c\varepsilon(t - \tau)} \| u^3(\cdot, \tau) \|_{W^{1, p}} d\tau \\
+ C \int_{t/2}^{t} (1 + t - \tau)^{-\frac{1}{2}} \| \partial_x^l (u^3(\cdot, \tau)) \|_{L^p} d\tau \\
\leq C E_0^{(s, p)} \int_{t/2}^{t/2} (1 + t - \tau)^{-1 + \frac{1}{p} - \frac{1}{q}} (1 + \tau)^{-\frac{1}{2}} d\tau + C E_0^{(s, p)} \int_{t/2}^{t/2} e^{-c\varepsilon(t - \tau)} (1 + \tau)^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} d\tau \\
+ C E_0^{(s, p)} \int_{t/2}^{t} (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} d\tau \\
\leq C E_0^{(s, p)} (1 + t)^{-1 + \frac{1}{p} - \frac{1}{q}} \log(2 + t), \ t \geq 0,
\] (3.31)
where we have used the estimate \( \| \partial_x^l (u^3(t)) \|_{L^p} \leq CE_0^{(s,p)} (1 + t)^{-\frac{3}{2} + \frac{3}{4} + \frac{5}{4}} \). For \( I_5 \), by using (2.7), (2.9) and (2.10), we have

\[
\| \partial_x^l I_5 (\cdot, t) \|_{L^p} \\
\leq C \int_0^{t/2} \| \partial_x^{l+1} (G - G_0) (t - \tau) * u^2 (\tau) \|_{L^p} d\tau + C \int_{t/2}^t \| \partial_x (G - G_0) (t - \tau) * \partial_x^l (u^2 (\tau)) \|_{L^p} d\tau \\
\leq C \int_0^{t/2} (t - \tau)^{-\frac{1}{2} + \frac{3}{4} + \frac{5}{4}} \| u^2 (\tau, \tau) \|_{L^1} d\tau + C \int_{t/2}^t \| u^2 (\cdot, \tau) \|_{W^{1,p}} d\tau \\
+ C \int_{t/2}^t (t - \tau)^{-\frac{1}{2} + \frac{3}{4} + \frac{5}{4}} \| \partial_x^l (u^2 (\cdot, \tau)) \|_{L^p} d\tau \\
\leq CE_0^{(s,p)} \int_0^{t/2} (t - \tau)^{-\frac{1}{2} + \frac{3}{4} + \frac{5}{4}} (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{1}{2}} d\tau \\
+ CE_0^{(s,p)} \int_{t/2}^t \| e^{-C_0(t-\tau)} (1 + \tau)^{-\frac{1}{2} + \frac{3}{4} + \frac{5}{4}} d\tau \\
+ CE_0^{(s,p)} \int_{t/2}^t (t - \tau)^{-\frac{1}{2} + \frac{3}{4} + \frac{5}{4}} (1 + \tau)^{-\frac{1}{2} + \frac{3}{4} + \frac{5}{4}} d\tau \\
\leq CE_0^{(s,p)} (1 + t)^{-\frac{1}{2} + \frac{3}{4} + \frac{5}{4}} \log (2 + t), \quad t \geq 1,
\]

(3.32)

where we have used the estimate \( \| \partial_x^l (u^2 (\cdot, t)) \|_{L^p} \leq CE_0^{(s,p)} (1 + t)^{-\frac{1}{2} + \frac{3}{4} + \frac{5}{4}} \). Finally, we evaluate \( I_6 \). We prepare the following estimate.

\[
\| \partial_x^l (u^2 - \chi^2 (\cdot, t)) \|_{L^p} = \| \partial_x^l ((u + \chi) (u - \chi)) (\cdot, t)) \|_{L^p} \\
\leq C (\| \partial_x^l u (\cdot, t) \|_{L^p} + \| \partial_x^l \chi (\cdot, t) \|_{L^p}) \| \phi (\cdot, t) \|_{L^\infty} \\
+ C \sum_{m=0}^{l-1} (\| \partial_x^m u (\cdot, t) \|_{L^\infty} + \| \partial_x^m \chi (\cdot, t) \|_{L^\infty}) \| \partial_x^{l-m} \phi (\cdot, t) \|_{L^p} \\
\leq CE_0^{(s,p)} (1 + t)^{-\frac{1}{2} + \frac{3}{4} + \frac{5}{4}} \begin{cases} \frac{1}{2}, & 1 < \gamma < 2, \\ \frac{1}{1 + \tau}, & \gamma = 2. \end{cases} \\
+ CE_0^{(s,p)} L(T) \sum_{m=0}^{l-1} (1 + t)^{-\frac{1}{2} + \frac{3}{4} + \frac{5}{4}} \begin{cases} \frac{1}{2}, & 1 < \gamma < 2, \\ \frac{1}{1 + \tau}, & \gamma = 2. \end{cases} \\
\leq CE_0^{(s,p)} (1 + L(T)) \begin{cases} \frac{1}{2}, & 1 < \gamma < 2, \\ \frac{1}{1 + \tau}, & \gamma = 2. \end{cases}
\]

(3.33)

Here, we have used (2.10), (2.12), (3.3), (3.25), (3.27) and the Gagliardo-Nirenberg inequality

\[
\| \partial_x^l u (\cdot, t) \|_{L^\infty} \leq C \| \partial_x^l u (\cdot, t) \|_{L^p}^{\frac{1}{2}} \| \partial_x^{l+1} u (\cdot, t) \|_{L^p}^{\frac{1}{2}}, \quad 0 \leq l \leq s - 1.
\]
Thus, combining (3.4), (3.28) through (3.32) and (3.34), we obtain

\[
\| \partial_x^k I_0(\cdot, t) \|_{L^p}
\leq C \int_0^{t/2} \| \partial_x^{k+1} G_0(t - \tau) * (u^2 - \chi^2)(\tau) \|_{L^p} d\tau + C \int_{t/2}^t \| \partial_x G_0(t - \tau) * \partial_x^k (u^2 - \chi^2)(\tau) \|_{L^p} d\tau
\]

\[
\leq C \int_0^{t/2} (t - \tau)^{-1} \frac{1}{\tau} \| (u^2 - \chi^2)(\cdot, \tau) \|_{L^1} d\tau + C \int_{t/2}^t (t - \tau)^{-1} \frac{1}{\tau} \| \partial_x^k ((u^2 - \chi^2)(\cdot, \tau)) \|_{L^p} d\tau
\]

\[
\leq CE_0^{(s,p)} (1 + L(T)) \left\{ \begin{array}{ll}
(1 + t)^{-1} \frac{1}{\tau} \frac{1}{p}, & 1 \leq t \leq T, 1 < \gamma < 2, \\
(1 + t)^{-1} \frac{1}{\tau} \frac{1}{p} \log(2 + t), & 1 \leq t \leq T, \gamma = 2
\end{array} \right.
\]

\[
+ CE_0^{(s,p)} (1 + L(T)) \left\{ \begin{array}{ll}
(1 + t)^{-1} \frac{1}{\tau} \frac{1}{p} \frac{1}{\tau} \frac{1}{p}, & 1 \leq t \leq T, 1 < \gamma < 2, \\
(1 + t)^{-1} \frac{1}{\tau} \frac{1}{p} \frac{1}{\tau} \frac{1}{p} \frac{1}{\tau} \frac{1}{p} \frac{1}{\gamma}, & 1 \leq t \leq T, \gamma = 2
\end{array} \right.
\]

(3.34)

Thus, combining (3.4), (3.28) through (3.32) and (3.34), we obtain

\[
\| \partial_x^k \phi(\cdot, t) \|_{L^p} \leq CE_0^{(s,p)} (1 + t)^{-1} \frac{1}{\tau} \frac{1}{p} + CE_0^{(s,p)} (1 + t)^{-1} \frac{1}{\tau} \frac{1}{p} \log(2 + t)
\]

\[
+ C_0 \left\{ \begin{array}{ll}
(1 + t)^{-1} \frac{1}{\tau} \frac{1}{p}, & 1 \leq t \leq T, 1 < \gamma < 2, \\
(1 + t)^{-1} \frac{1}{\tau} \frac{1}{p} \log(2 + t), & 1 \leq t \leq T, \gamma = 2
\end{array} \right.
\]

\[
+ CE_0^{(s,p)} (1 + L(T)) \left\{ \begin{array}{ll}
(1 + t)^{-1} \frac{1}{\tau} \frac{1}{p} \frac{1}{\tau} \frac{1}{p}, & 1 \leq t \leq T, 1 < \gamma < 2, \\
(1 + t)^{-1} \frac{1}{\tau} \frac{1}{p} \frac{1}{\tau} \frac{1}{p} \frac{1}{\gamma}, & 1 \leq t \leq T, \gamma = 2
\end{array} \right.
\]

(3.35)

For \(0 \leq t \leq 1\), in the same way to get (3.14) and (2.34), we easily see

\[
\| \partial_x^k \phi(\cdot, t) \|_{L^p} \leq \| \partial_x^k u(\cdot, t) \|_{L^p} + \| \partial_x^k \chi(\cdot, t) \|_{L^p} \leq CE_0^{(s,p)}, \quad 0 \leq t \leq 1.
\]

(3.36)

Summing up (3.35) and (3.36), it follows that

\[
L(T) \leq CE_0^{(s,p)} + C_0 + C_1 E_0^{(s,p)} L(T),
\]

where \(C_1\) is a positive constant. Therefore, we arrive at the desired estimate

\[
L(T) \leq 2CE_0^{(s,p)} + 2C_0
\]

if \(E_0^{(s,p)}\) is so small that \(C_1 E_0^{(s,p)} \leq \frac{1}{2}\). This completes the proof. \(\square\)

**End of the proof of Theorem 1.1.** Since we have already shown (1.16) and (1.17) with \(k = 0, 1\) (Corollary 3.3 and Proposition 3.4), we only need to prove (1.17) with \(k = 1, 2\). First, differentiating (3.1) with respect to \(t\), then we have

\[
\partial_t u(t) = \partial_t G(t) * (u_0 + u_1) + \partial_t^2 G(t) * u_0 - \int_0^t \partial_t G(t - \tau) * \partial_x (g(u)(\tau)) d\tau - G(0) * \partial_x (g(u))(t),
\]

where \(g(u) = \frac{3}{2} u^2 + \frac{5}{2} u^3\). Moreover, because (2.2) and (2.3), we see that \(G(0) * \rho = 0\) for any \(\rho\). Therefore, we arrive at

\[
\partial_t u(t) = \partial_t G(t) * (u_0 + u_1) + \partial_t^2 G(t) * u_0 - \int_0^t \partial_t G(t - \tau) * \partial_x (g(u)(\tau)) d\tau.
\]

(3.37)
Similarly, from (2.5), we obtain
\[ \partial_t \chi(t) = \partial_t G_0(t) * \chi_0 - \frac{b}{2} \int_0^t \partial_t G_0(t - \tau) * \partial_x (\chi^2)(\tau) d\tau - \frac{b}{2} \partial_x (\chi^2)(t), \] (3.38)
where \( \chi_0(x) = \chi(x,0) \). Thus, combining (3.37) and (3.38), it follows that
\[ \partial_t (u(t) - \chi(t)) = \partial_t (G - G_0)(t) * (u_0 + u_1) + \partial_t G_0(t) * (u_0 + u_1 - \chi_0) \]
\[ + \partial_t^2 G(t) * u_0 - \int_0^t \partial_t G(t - \tau) * \partial_x \left( g(u) - \frac{b}{2} \chi^2 \right)(\tau) d\tau \]
\[ - \frac{b}{2} \int_0^t \partial_t (G - G_0)(t - \tau) * \partial_x (\chi^2)(\tau) d\tau + \frac{b}{2} \partial_x (\chi^2)(t) \]
\[ \equiv J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \] (3.39)

Now, we shall evaluate the all terms of the right hand side of (3.39). Here and after in this
proof, we set \( \gamma \equiv \min \{ \alpha, \beta \} \). First for \( J_1 \), it follows from (2.7) that
\[ \| \partial_x^2 J_1(\cdot,t) \|_{L^p} \leq C E_{0}^{(s,p)}(1 + t)^{-\frac{1}{2} + \frac{1}{4} + \frac{3}{4} + \gamma} \]
\[ \leq C E_{0}^{(s,p)}(1 + t)^{-\frac{1}{2} + \frac{1}{4} + \gamma}, \quad t \geq 1. \] (3.40)

Also, since \( \int_{\mathbb{R}} (u_0 + u_1 - \chi_0) dx = 0 \), we have from (2.15)
\[ \| \partial_x^2 J_2(\cdot,t) \|_{L^p} \leq C \begin{cases} (1 + t)^{-\frac{1}{2} + \frac{1}{4} + \gamma}, & t \geq 1, 1 < \gamma < 2, \\ (1 + t)^{-\frac{1}{2} + \frac{1}{4} + \gamma} \log(2 + t), & t \geq 1, \gamma = 2. \end{cases} \] (3.41)

Similarly, from (2.5), we obtain
\[ \| \partial_x^2 J_3(\cdot,t) \|_{L^p} \leq C E_{0}^{(s,p)}(1 + t)^{-\frac{1}{2} + \frac{1}{4} + \gamma}, \quad t \geq 0. \] (3.42)

Next, we evaluate \( J_4 \). From (2.9), (2.10), (2.12), (3.25) and (3.26), we have
\[ \left\| \left( g(u) - \frac{b}{2} \chi^2 \right)(\cdot,t) \right\|_{L^1} \leq C \begin{cases} (1 + t)^{-\frac{1}{2}}, & t \geq 0, 1 < \gamma < 2, \\ (1 + t)^{-1 + \epsilon}, & t \geq 0, \gamma = 2, \end{cases} \] (3.43)

\[ \left\| \partial_x^2 \left( g(u) - \frac{b}{2} \chi^2 \right)(\cdot,t) \right\|_{L^p} \leq C \begin{cases} (1 + t)^{-\frac{1}{2} + \frac{1}{4} + \gamma}, & t \geq 0, 1 < \gamma < 2, \\ (1 + t)^{-\frac{1}{2} + \frac{1}{4} + \gamma}, & t \geq 0, \gamma = 2, \end{cases} \] (3.44)
for \(0 \leq l \leq s\). Therefore, we can compute that from (2.5), (3.43) and (3.44) that

\[
\| \partial_x^l J_4(\cdot, t) \|_{L^p} \leq \int_0^{t/2} \left\| \partial_t \partial_x^{l+1} G(t - \tau) * \left( g(u) - \frac{b}{2} \chi^2 \right)(\tau) \right\|_{L^p} d\tau \\
+ C \int_{t/2}^t \left\| \partial_t G(t - \tau) * \partial_x^{l+1} \left( g(u) - \frac{b}{2} \chi^2 \right)(\tau) \right\|_{L^p} d\tau \\
\leq C \int_0^{t/2} (1 + t - \tau)^{-\frac{3}{2} + \frac{1}{2p} - \frac{1}{4} (1 + \tau)^{-\frac{1}{2}} d\tau, \quad 1 < \gamma < 2, \\
+ C \int_{t/2}^t (1 + t - \tau)^{-\frac{3}{2} + \frac{1}{2p} - \frac{1}{4} (1 + \tau)^{-1 + \varepsilon} d\tau, \quad \gamma = 2
\]

On the other hand, by using (2.7) and (2.12), it follows that

\[
\| \partial_x^l J_5(\cdot, t) \|_{L^p} \leq C \int_0^{t/2} \left\| \partial_t \partial_x^{l+1} (G - G_0)(t - \tau) * (\chi^2)(\tau) \right\|_{L^p} d\tau \\
+ C \int_{t/2}^t \left\| \partial_t (G - G_0)(t - \tau) * \partial_x^{l+1} (\chi^2)(\tau) \right\|_{L^p} d\tau \\
\leq C \int_0^{t/2} (t - \tau)^{-\frac{3}{2} + \frac{1}{2p} - \frac{1}{4}} (1 + t - \tau)^{-\frac{1}{2}} (\| \chi^2(\cdot, \tau) \|_{L^1} + \| \chi^2(\cdot, \tau) \|_{W^{1+1, p}}) d\tau \\
+ C \int_{t/2}^t (t - \tau)^{-\frac{3}{2} + \frac{1}{2p} - \frac{1}{4} \| \partial_x^{l+1} (\chi^2)(\cdot, \tau) \|_{L^p} d\tau \\
\leq CE_0^{(s,p)} \int_0^{t/2} (t - \tau)^{-\frac{3}{2} + \frac{1}{2p} - \frac{1}{4} (1 + \tau)^{-\frac{1}{2}} (1 + t - \tau)^{-\frac{1}{2}} d\tau \\
+ CE_0^{(s,p)} \int_{t/2}^t (t - \tau)^{-\frac{3}{2} + \frac{1}{2p} - \frac{1}{4} (1 + \tau)^{-1 + \varepsilon} (1 + t - \tau)^{-\frac{1}{2}} d\tau \\
\leq CE_0^{(s,p)} (1 + t)^{-\frac{3}{2} + \frac{1}{2p} - \frac{1}{4} \log(2 + t), \quad t \geq 0.
\]

Finally, we easily see form (2.12)

\[
\| \partial_x^l J_6(\cdot, t) \|_{L^p} \leq CE_0^{(s,p)} (1 + t)^{-\frac{3}{2} + \frac{1}{2p} - \frac{1}{4}, \quad t \geq 0.
\]

Therefore, combining (3.39) through (3.42) and (3.45) through (3.47), we arrive at

\[
\| \partial_t \partial_x^l (u(\cdot, t) - \chi(\cdot, t)) \|_{L^p} \leq C \begin{cases} 
(1 + t)^{-\frac{3}{2} + \frac{1}{2p} - \frac{1}{4}, & t \geq 1, 1 < \gamma < 2, \\
(1 + t)^{-1 + \varepsilon}, & t \geq 1, \gamma = 2
\end{cases}
\]

for \(0 \leq l \leq s - 1\). For \(0 \leq l \leq 1\), we obtain from (2.10) and (2.12)

\[
\| \partial_t \partial_x^k (u(\cdot, t) - \chi(\cdot, t)) \|_{L^p} \leq \| \partial_t^k \partial_x^l u(\cdot, t) \|_{L^p} + \| \partial_t^k \partial_x^l \chi(\cdot, t) \|_{L^p} \leq CE_0^{(s,p)} \leq C,
\]

where \(k = 0, 1, 2\) and \(l \geq 0\) with \(0 \leq k + l \leq s\). Summing up (3.48) and (3.49), we get (1.17) with \(k = 1\) and \(0 \leq l \leq s - 1\).
Next, we shall show (1.17) with $k = 2$ and $0 \leq l \leq s - 2$. By using the integration by parts, in the same way to get (3.37), we obtain

$$
\partial_t^2 u(t) = \partial_t^2 G(t) * (u_0 + u_1) + \partial_t^2 G(t) * u_0 - \int_0^{t/2} \partial_t^2 \partial_x G(t - \tau) * (g(u)(\tau))d\tau - \int_{t/2}^t \partial_t G(t - \tau) * \partial_t \partial_x (g(u))((\tau))d\tau - \partial_t \partial_x G \left( \frac{t}{2} \right) * (g(u)) \left( \frac{t}{2} \right),
$$

(3.50)

$$
\partial_t^2 \chi(t) = \partial_t^2 G_0(t) * \chi_0 - \frac{b}{2} \int_0^{t/2} \partial_t^2 \partial_x G_0(t - \tau) * (\chi^2(\tau))d\tau - \frac{b}{2} \partial_t \partial_x (\chi^2)(t)
$$

$$
- \frac{b}{2} \int_{t/2}^t \partial_t G_0(t - \tau) * \partial_t \partial_x (\chi^2)(\tau)d\tau - \frac{b}{2} \partial_t \partial_x G_0 \left( \frac{t}{2} \right) * (\chi^2) \left( \frac{t}{2} \right).
$$

(3.51)

Thus, from (3.50) and (3.51), we have

$$
\partial_t^2 (u(t) - \chi(t)) = \partial_t^2 (G - G_0)(t) * (u_0 + u_1) + \partial_t^2 G_0(t) * (u_0 + u_1 - \chi_0) + \partial_t^2 G(t) * u_0
$$

$$
- \int_0^{t/2} \partial_t^2 \partial_x G(t - \tau) * (g(u) - \frac{b}{2} \chi^2) (\tau)d\tau - \int_{t/2}^t \partial_t G(t - \tau) * \partial_t \partial_x (g(u) - \frac{b}{2} \chi^2) (\tau)d\tau
$$

$$
- \frac{b}{2} \int_0^{t/2} \partial_t^2 \partial_x G(G - G_0)(t - \tau) * (\chi^2)(\tau)d\tau - \frac{b}{2} \int_{t/2}^t \partial_t (G - G_0)(t - \tau) * \partial_t \partial_x (\chi^2)(\tau)d\tau
$$

$$
+ \frac{b}{2} \partial_t \partial_x (\chi^2) - \partial_t \partial_x G \left( \frac{t}{2} \right) * \left( g(u) - \frac{b}{2} \chi^2 \right) \left( \frac{t}{2} \right) - \frac{b}{2} \partial_t \partial_x (G - G_0) \left( \frac{t}{2} \right) * (\chi^2) \left( \frac{t}{2} \right)
$$

$$
= \bar{J}_1 + \bar{J}_2 + \bar{J}_3 + \bar{J}_4 + \bar{J}_5 + \bar{J}_6 + \bar{J}_7 + \bar{J}_8 + \bar{J}_9 + \bar{J}_{10}.
$$

(3.52)

Similarly as (3.44), we note that

$$
\left\| \partial_t \partial_x^2 \left( g(u) - \frac{b}{2} \chi^2 \right) (\cdot) \right\|_{L^p} \leq C \begin{cases} 
(1 + t)^{-\frac{2}{p} + \frac{1}{\gamma} - \frac{1}{4} - \frac{1}{\gamma}}, & t \geq 0, 1 < \gamma < 2, \\
(1 + t)^{-2 - \frac{1}{p} + \frac{1}{\gamma} + \varepsilon}, & t \geq 0, \gamma = 2
\end{cases}
$$

(3.53)

for $0 \leq l \leq s - 1$. By using the same argument given in the above paragraph, we have the following estimates:

$$
\|\partial_t^{l} \bar{J}_1(\cdot, t)\|_{L^p} + \|\partial_t^{l} \bar{J}_3(\cdot, t)\|_{L^p} + \|\partial_t^{l} \bar{J}_5(\cdot, t)\|_{L^p} + \|\partial_t^{l} \bar{J}_5(\cdot, t)\|_{L^p} \leq C(1 + t)^{-2 + \frac{1}{p} - \frac{1}{2}}, t \geq 1,
$$

$$
\|\partial_t^{l} \bar{J}_2(\cdot, t)\|_{L^p} \leq C \begin{cases} 
(1 + t)^{-\frac{2}{p} + \frac{1}{\gamma} - \frac{1}{4} - \frac{1}{\gamma}}, & t \geq 1, 1 < \gamma < 2, \\
(1 + t)^{-2 + \frac{1}{p} - \frac{1}{4} + \varepsilon} \log(2 + t), & t \geq 1, \gamma = 2,
\end{cases}
$$

$$
\|\partial_t^{l} \bar{J}_4(\cdot, t)\|_{L^p} + \|\partial_t^{l} \bar{J}_5(\cdot, t)\|_{L^p} \leq C \begin{cases} 
(1 + t)^{-\frac{2}{p} + \frac{1}{\gamma} - \frac{1}{4} - \frac{1}{\gamma}}, & t \geq 0, 1 < \gamma < 2, \\
(1 + t)^{-2 + \frac{1}{p} - \frac{1}{4} + \varepsilon}, & t \geq 0, \gamma = 2
\end{cases}
$$

and

$$
\|\partial_t^{l} \bar{J}_7(\cdot, t)\|_{L^p} \leq C(1 + t)^{-2 + \frac{1}{p} - \frac{1}{2} \log(2 + t)}, t \geq 1.
$$

On the other hand, for $\bar{J}_9$, from (2.5) and (3.43), we get

$$
\|\partial_t^{l} \bar{J}_9(\cdot, t)\|_{L^p} \leq C(1 + t)^{-\frac{2}{p} + \frac{1}{2} - \frac{1}{2}} \left\| \left( g(u) - \frac{b}{2} \chi^2 \right) \left( \cdot \right) \right\|_{L^1}
$$

$$
\leq C \begin{cases} 
(1 + t)^{-\frac{2}{p} + \frac{1}{2} - \frac{1}{2}}, & t \geq 0, 1 < \gamma < 2, \\
(1 + t)^{-\frac{2}{p} + \frac{1}{2} + \epsilon}, & t \geq 0, \gamma = 2.
\end{cases}
$$
For $J_{10}$, it follows that from (2.7) and (2.12)
\[
\|\partial_t^j J_{10}(\cdot, t)\|_{L^p} \leq C t^{-\frac{j}{2} + \frac{1}{p} - \frac{1}{2}} (1 + t)^{-\frac{1}{2}} \left( \|\chi^2 \left( \cdot, \frac{t}{2} \right)\|_{L^1} + \|\chi^2 \left( \cdot, \frac{t}{2} \right)\|_{W^{l+1,p}} \right)
\]
\[
\leq C(1 + t)^{-\frac{j}{2} + \frac{1}{p} - \frac{1}{2}}, \quad t \geq 1.
\]
Therefore, combining all the above estimates, we can derive
\[
\|\partial_t^2 \partial_x^j (u(\cdot, t) - \chi(\cdot, t))\|_{L^p} \leq C \begin{cases} 
(1 + t)^{-\frac{j}{2} + \frac{1}{p} - \frac{1}{2}} & \text{for } 0 < t \leq 1, 1 < \gamma < 2, \\
(1 + t)^{-2 + \frac{1}{p}} & \text{for } 0 < t \leq 1, \gamma = 2
\end{cases}
\] (3.54)
for $0 \leq l \leq s - 2$. Thus, summing up (3.54) and (3.49), we can prove (1.17) with $k = 2$. This completes the proof. \hfill \Box

Finally in this section, we give the additional decay estimate for $u - \chi$. From the original equations (1.2) and (1.7), we see that
\[
(\partial_t + a\partial_x)(u - \chi) = (-\partial_t^2 + \partial_x^2)(u - \chi) - \frac{b}{2} \partial_x(u^2 - \chi^2) - \frac{c}{3!} \partial_x(u^3) - (\partial_t - a\partial_x)(\partial_t + a\partial_x)\chi.
\]
By virtue of this relation, we have the following estimate:

**Corollary 3.5.** Assume the same conditions on $u_0$ and $u_1$ in Theorem 1.1 are valid. Then, for any $\varepsilon > 0$, the estimate
\[
\|\partial_x^l (\partial_t + a\partial_x)((u - \chi)(\cdot, t))\|_{L^p} \leq C \begin{cases} 
(1 + t)^{-\frac{\min\{\alpha, \beta\}}{2} + \frac{1}{p} - \frac{1}{2}} & \text{for } 0 < t \leq 1, \min\{\alpha, \beta\} < 2, \\
(1 + t)^{-2 + \frac{1}{p} + \varepsilon} & \text{for } 0 < t \leq 1, \min\{\alpha, \beta\} = 2
\end{cases}
\] (3.55)
holds for $0 \leq l \leq s - 2$.

We will use this estimate in the proof of Theorem 1.2.

**4 Proof of Theorem 1.2 for $1 < \min\{\alpha, \beta\} < 2$**

In this section, we shall prove Theorem 1.2 in the case of $1 < \min\{\alpha, \beta\} < 2$. Namely, our purpose of this section is to derive (1.21) and (1.23). First, we set
\[
\psi(x, t) \equiv u(x, t) + u_1(x, t) - \chi(x, t), \quad \psi_0(x) \equiv u_0(x) + u_1(x) - \chi_0(x),
\] (4.1)
where $\chi_0$ is defined by $\chi_0(x) = \chi(x, 0)$. Then we have the following initial value problem:
\[
\psi_t + a\psi_x + (b\chi)\psi_x - \mu\psi_{xx} = a\partial_x(\partial_t + a\partial_x)(u - \chi) + a\partial_x(\partial_t + a\partial_x)\chi
\]
\[
- \frac{b}{2} \partial_x((u - \chi)^2) - \frac{c}{3!} \partial_x(u^3) + \partial_x(b\chi u_t - \mu u_{tx}), \quad x \in \mathbb{R}, \quad t > 0,
\]
\[
\psi(x, 0) = \psi_0(x), \quad x \in \mathbb{R}.
\] (4.2)

Therefore, from Lemma 2.6, we obtain
\[
\psi(x, t) = U[\psi_0](x, t, 0)
\]
\[
+ a \int_0^t U[\partial_x(\partial_t + a\partial_x)(u - \chi)(\tau)](x, t, \tau)d\tau + a \int_0^t U[\partial_x(\partial_t + a\partial_x)\chi(\tau)](x, t, \tau)d\tau
\]
\[
- \frac{b}{2} \int_0^t U[\partial_x((u - \chi)^2)(\tau)](x, t, \tau)d\tau - \frac{c}{3!} \int_0^t U[\partial_x(u^3)(\tau)](x, t, \tau)d\tau
\]
\[
+ b \int_0^t U[\partial_x(u_1\chi)(\tau)](x, t, \tau)d\tau - \mu \int_0^t U[\partial_x(u_{tx})(\tau)](x, t, \tau)d\tau.
\] (4.3)

For the first term of the right hand side in the above equation (4.3), we have the following asymptotic formula. (The proof is given by almost the same argument as in Proposition 4.1 in [2]):
Proposition 4.1. Assume the same conditions on \( u_0 \) and \( u_1 \) in Theorem 1.2 are valid. Then we have
\[
\lim_{t \to \infty} (1 + t)^{\min\{\alpha, \beta\}} \frac{\|U[\psi_0](\cdot, t, 0) - Z(\cdot, t)\|_{L^\infty}}{\log(1 + t)} = 0, \quad 1 < \min\{\alpha, \beta\} < 2,
\]
\[
\lim_{t \to \infty} \frac{(1 + t)^{\min\{\alpha, \beta\}} \|U[\psi_0](\cdot, t, 0) - Z(\cdot, t)\|_{L^\infty}}{\log(1 + t)} = 0, \quad \min\{\alpha, \beta\} = 2,
\]
where \( Z(x, t) \) is defined by (1.18).

Proof. From the definition of \( U \) given by (2.26) and \( \eta_0(x) = \eta(x, 0) \), we have
\[
U[\psi_0](x, t, 0) = \int_{\mathbb{R}} \partial_x(G(x - y, t)\eta(x, t))\eta_0(y)^{-1}\left(\int_{-\infty}^{y} (u_0(\xi) + u_1(\xi) - \chi_0(\xi))d\xi\right)dy = \int_{\mathbb{R}} \partial_x(G(x - y, t)\eta(x, t))z_0(y)dy,
\]
where \( z_0(y) \) is defined by (1.20). First, we shall check the following estimate:
\[
|z_0(x)| \leq C(1 + |x|)^{-\min\{\alpha, \beta\}-1}, \quad x \in \mathbb{R}.
\]
(4.7)

If \( x < 0 \), from (2.22), (1.3), (1.4) and (1.5), we have
\[
|z_0(x)| \leq C \int_{-\infty}^{x} \left(|u_0(y)| + |u_1(y)| + |\chi_0(y)|\right)dy
\leq C \int_{-\infty}^{x} (1 + |y|)^{-\min\{\alpha, \beta\}}dy + C \int_{-\infty}^{x} (1 + |y|)^{-N}dy \quad (\forall N \geq 0)
\leq C \int_{-\infty}^{x} (1 - y)^{-\min\{\alpha, \beta\}}dy \leq C(1 - x)^{-\min\{\alpha, \beta\}-1} = C(1 + |x|)^{-\min\{\alpha, \beta\}-1}.
\]

On the other hand, since \( \int_{\mathbb{R}} (u_0(x) + u_1(x))dx = \int_{\mathbb{R}} \chi_0(x)dx = M \), if \( x > 0 \), similarly we have
\[
|z_0(x)| \leq C \int_{-\infty}^{x} \left(u_0(y) + u_1(y) - \chi_0(y)\right)dy
\leq C \int_{-\infty}^{x} (1 + |y|)^{-\min\{\alpha, \beta\}}dy + \int_{x}^{\infty} \chi_0(y)dy
\leq C \int_{x}^{\infty} (1 + |y|)^{-\min\{\alpha, \beta\}}dy + C \int_{x}^{\infty} (1 + |y|)^{-N}dy \quad (\forall N \geq 0)
\leq C \int_{x}^{\infty} (1 + y)^{-\min\{\alpha, \beta\}}dy \leq C(1 + x)^{-\min\{\alpha, \beta\}-1} = C(1 + |x|)^{-\min\{\alpha, \beta\}-1}.
\]

Therefore we get (4.7). Thus, we obtain the boundedness of \( (1 + |y|)^{-\min\{\alpha, \beta\}-1}z_0(y) \). Moreover, from the assumption on \( z_0(y) \), for any \( \varepsilon > 0 \) there is a constant \( R = R(\varepsilon) > 0 \) such that
\[
|z_0(y) - c_{\alpha, \beta}^+(1 + |y|)^{-\min\{\alpha, \beta\}-1}| \leq \varepsilon(1 + |y|)^{-\min\{\alpha, \beta\}-1}, \quad y \geq R,
\]
\[
|z_0(y) - c_{\alpha, \beta}^-(1 + |y|)^{-\min\{\alpha, \beta\}-1}| \leq \varepsilon(1 + |y|)^{-\min\{\alpha, \beta\}-1}, \quad y \leq -R.
\]

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From (1.18) and (4.6), we have the following estimate

\[
|U[\psi_0](x, t, 0) - Z(x, t)| \leq \int \left| \partial_x(G_0(x - y, t)\eta(x, t)) \right| |z_0(y) - c_{\alpha, \beta}(y)(1 + |y|)^{-(\min\{\alpha, \beta\} - 1)}|dy
\]

\[
\leq \int_{|y| \leq R} \left| \partial_x(G_0(x - y, t)\eta(x, t)) \right| |z_0(y) - c_{\alpha, \beta}(y)(1 + |y|)^{-(\min\{\alpha, \beta\} - 1)}|dy
\]

\[+ \varepsilon \int_{|y| \geq R} \left| \partial_x(G_0(x - y, t)\eta(x, t)) \right| (1 + |y|)^{-(\min\{\alpha, \beta\} - 1)}dy \]

\[
\leq C \sum_{n=0}^{1} \| \partial_x^{1-n}\eta(\cdot, t) \|_{L^\infty} \| \partial_x^n G_0(\cdot, t) \|_{L^\infty} \int_{|y| \leq R} |z_0(y) - c_{\alpha, \beta}(y)(1 + |y|)^{-(\min\{\alpha, \beta\} - 1)}|dy
\]

\[+ \varepsilon C \sum_{n=0}^{1} \| \partial_x^{1-n}\eta(\cdot, t) \|_{L^\infty} \int_{R} \left| \partial_x^n G_0(x - y, t) \right| (1 + |y|)^{-(\min\{\alpha, \beta\} - 1)}dy.
\]

For the integral in the last term of the right hand side of (4.8), we can estimate it as follows

\[
\int_{R} \left| \partial_x^n G_0(x - y, t) \right| (1 + |y|)^{-(\min\{\alpha, \beta\} - 1)}dy
\]

\[
= \left( \int_{|y| \geq \sqrt{1 + t}} + \int_{|y| \leq \sqrt{1 + t}} \right) \left| \partial_x^n G_0(x - y, t) \right| (1 + |y|)^{-(\min\{\alpha, \beta\} - 1)}dy
\]

\[\leq \sup_{|y| \geq \sqrt{1 + t}} (1 + |y|)^{-(\min\{\alpha, \beta\} - 1)} \int_{|y| \geq \sqrt{1 + t}} \left| \partial_x^n G_0(x - y, t) \right|dy
\]

\[\leq \sup_{|y| \geq \sqrt{1 + t}} (1 + |y|)^{-(\min\{\alpha, \beta\} - 1)} \int_{|y| \leq \sqrt{1 + t}} (1 + |y|)^{-(\min\{\alpha, \beta\} - 1)}dy
\]

\[\leq (1 + t)^{-(\min\{\alpha, \beta\} - 1)} \| \partial_x^n G_0(\cdot, t) \|_{L^1} + \| \partial_x^n G_0(\cdot, t) \|_{L^\infty} \int_{|y| \leq \sqrt{1 + t}} (1 + |y|)^{-(\min\{\alpha, \beta\} - 1)}dy
\]

\[
\leq C(1 + t)^{-(\min\{\alpha, \beta\} - 1)} + C \frac{t^\frac{1}{2}}{(1 + t)^{\frac{1}{2}}}
\]

\[
\leq C(1 + t)^{-(\min\{\alpha, \beta\} - 1)} + C \frac{t^\frac{1}{2}}{(1 + t)^{\frac{1}{2}}}
\]

\[
\leq C \left\{ (1 + t)^{-(\min\{\alpha, \beta\} - 1)} + \frac{t^\frac{1}{2}}{(1 + t)^{\frac{1}{2}}} : 1 \leq \min\{\alpha, \beta\} < 2, \right. \]

\[\left. (1 + t)^{-\frac{1 + \alpha}{2}} \log(1 + t), \quad t \geq 1, \min\{\alpha, \beta\} = 2 \right. \]

Therefore, by using (4.8), Lemma 2.5, Lemma 2.4 and (4.9), we get

\[
\|U[\psi_0](\cdot, t, 0) - Z(\cdot, t)\|_{L^\infty} \leq C(1 + t)^{-\frac{1 + \alpha}{2}} \log(1 + t), \quad t \geq 1, \min\{\alpha, \beta\} = 2.
\]

Thus, we obtain

\[
\limsup_{t \to \infty} (1 + t)^{-\frac{1 + \alpha}{2}} \|U[\psi_0](\cdot, t, 0) - Z(\cdot, t)\|_{L^\infty} \leq C, \quad 1 < \min\{\alpha, \beta\} < 2,
\]

\[
\limsup_{t \to \infty} (1 + t)^{-\frac{1 + \alpha}{2}} \log(1 + t), \quad t \geq 1, \min\{\alpha, \beta\} = 2.
\]

Therefore, we get (4.4) and (4.5), because \(\varepsilon > 0\) can be chosen arbitrarily small.
End of the proof of Theorem 1.2 for $1 < \min\{\alpha, \beta\} < 2$. We shall prove (1.21) and (1.23). By using (1.18) and (4.3), we have

$$u(x, t) - \chi(x, t) - Z(x, t) = U[\psi_0](x, t, 0) - Z(x, t) = u_t(x, t)$$

$$+ \int_0^t U[\partial_x(\partial_t + a\partial_x)(u - \chi)(\tau)](x, t, \tau)d\tau + \int_0^t U[\partial_x(\partial_t + a\partial_x)\chi(\tau)](x, t, \tau)d\tau$$

$$- \frac{b}{2} \int_0^t U[\partial_x((u - \chi)^2)(\tau)](x, t, \tau)d\tau - \frac{c}{3!} \int_0^t U[\partial_x(u^3)(\tau)](x, t, \tau)d\tau$$

$$+ b \int_0^t U[\partial_x(u\chi)(\tau)](x, t, \tau)d\tau - \mu \int_0^t U[\partial_x(u_{\chi\chi})(\tau)](x, t, \tau)d\tau$$

$$\equiv U[\psi_0](x, t, 0) - Z(x, t) - u_t(x, t) + J_1 + J_2 + J_3 + J_4 + J_5 + J_6.$$
For $J_4$, in the same way to get (4.11), from Lemma 2.7 and (2.9), we get

\[ \|J_4(\cdot, t)\|_{L^\infty} \leq C \sum_{n=0}^{1} (1 + t)^{-\frac{1}{2} + \frac{n}{6}} \times \left( \int_{0}^{t/2} (t - \tau)^{-\frac{1}{2} - \frac{n}{6}} \|u^3(\cdot, \tau)\|_{L^1} d\tau + \int_{t/2}^{t} (t - \tau)^{-\frac{1}{2}} \|u^3(\cdot, \tau)\|_{L^\infty} d\tau \right) \]

\[ \leq C \sum_{n=0}^{1} (1 + t)^{-\frac{1}{2} + \frac{n}{6}} \times \left( \int_{0}^{t/2} (t - \tau)^{-\frac{1}{2} - \frac{n}{6}} (1 + \tau)^{-1} d\tau + \int_{t/2}^{t} (t - \tau)^{-\frac{1}{2} + \frac{n}{6}} (1 + \tau)^{-1} d\tau \right) \]

\[ \leq C(1 + t)^{-1} \log(1 + t), \quad t \geq 1. \]  

(4.14)

We note that $J_4$ does not appear if $c = 0$. Also, from Lemma 2.7, Gagliardo-Nirenberg inequality, (2.10) and (2.12), we have the estimate for $J_5$ as follows.

\[ \|J_5(\cdot, t)\|_{L^\infty} \leq C \sum_{n=0}^{1} (1 + t)^{-\frac{1}{2} + \frac{n}{6}} \left( \int_{0}^{t/2} (t - \tau)^{-\frac{1}{2} - \frac{n}{6}} \|u_t(\cdot, \tau)\|_{L^\infty} \|\chi(\cdot, \tau)\|_{L^1} d\tau \right) \]

\[ + \int_{t/2}^{t} (t - \tau)^{-\frac{1}{2}} \|u_t(\cdot, \tau)\|_{L^\infty} \|\chi(\cdot, \tau)\|_{L^\infty} d\tau \]

\[ \leq C \sum_{n=0}^{1} (1 + t)^{-\frac{1}{2} + \frac{n}{6}} \left( \int_{0}^{t/2} (t - \tau)^{-\frac{1}{2} - \frac{n}{6}} (1 + \tau)^{-1} d\tau + \int_{t/2}^{t} (t - \tau)^{-\frac{1}{2} + \frac{n}{6}} (1 + \tau)^{-1} d\tau \right) \]

\[ \leq C(1 + t)^{-1} \log(1 + t), \quad t \geq 1. \]  

(4.15)

Finally, we evaluate $J_6$. Similarly as before, it follows that

\[ \|J_6(\cdot, t)\|_{L^\infty} \leq C \sum_{n=0}^{1} (1 + t)^{-\frac{1}{2} + \frac{n}{6}} \times \left( \int_{0}^{t/2} (t - \tau)^{-\frac{1}{2} - \frac{n}{6}} \|u_{tx}(\cdot, \tau)\|_{L^1} d\tau + \int_{t/2}^{t} (t - \tau)^{-\frac{1}{2}} \|u_{tx}(\cdot, \tau)\|_{L^2} d\tau \right) \]

\[ \leq C \sum_{n=0}^{1} (1 + t)^{-\frac{1}{2} + \frac{n}{6}} \left( \int_{0}^{t/2} (t - \tau)^{-\frac{1}{2} - \frac{n}{6}} (1 + \tau)^{-1} d\tau + \int_{t/2}^{t} (t - \tau)^{-\frac{1}{2} + \frac{n}{6}} (1 + \tau)^{-1} d\tau \right) \]

\[ \leq C(1 + t)^{-1} \log(1 + t), \quad t \geq 1. \]  

(4.16)

By using (4.10), (2.10), Gagliardo-Nirenberg inequality and (4.11) through (4.16), we have

\[ \|u(\cdot, t) - \chi(\cdot, t) - Z(\cdot, t)\|_{L^\infty} \leq \|U[\psi_0](\cdot, t, 0) - Z(\cdot, t)\|_{L^\infty} + C(1 + t)^{-1} + C(1 + t)^{-1} \log(1 + t) \]

\[ + C \begin{cases} (1 + t)^{-1}, & t \geq 1, \quad 3/2 < \min\{\alpha, \beta\} < 2, \\ (1 + t)^{-1} \log(1 + t), & t \geq 1, \quad \min\{\alpha, \beta\} = 3/2, \\ (1 + t)^{-\min\{\alpha, \beta\}+1/2}, & t \geq 1, \quad 1 < \min\{\alpha, \beta\} < 3/2. \end{cases} \]

Therefore, from (4.4), we finally obtain

\[ \limsup_{t \to \infty} (1 + t)^{-\frac{1}{2}} \|u(\cdot, t) - \chi(\cdot, t) - Z(\cdot, t)\|_{L^\infty} = 0. \]

Thus we completes the proof of (1.21).
In the rest of this proof, we shall prove (1.23). First, in the same way to get (4.8), by using Lemma 2.5 and (4.9), we can derive the upper bound estimate of (1.23) as follows.

\[
|Z(x, t)| \leq C \max\{|c_{+,\alpha,\beta}^+|, |c_{-,\alpha,\beta}^-|\} \sum_{n=0}^{1} \|\partial_x^{n-1} y|^{\gamma}(\cdot, t)\|_{L^\infty} \int_{\mathbb{R}} |\partial_y^n G_0(x - y, t)|(1 + |y|)^{-\min(\alpha, \beta) - 1} dy
\]

\[
\leq C \max\{|c_{+,\alpha,\beta}^+|, |c_{-,\alpha,\beta}^-|\} \left\{ (1 + t)^{-\frac{\min(\alpha, \beta) - 1}{2}}, \quad t \geq 1, \quad 1 < \min(\alpha, \beta) < 2, \\
(1 + t)^{-\frac{\min(\alpha, \beta)}{2}} \log(1 + t), \quad t \geq 1, \quad \min(\alpha, \beta) = 2 \right\}
\]

\[
\leq C \max\{|c_{+,\alpha,\beta}^+|, |c_{-,\alpha,\beta}^-|\} \left\{ (1 + t)^{-\frac{\min(\alpha, \beta)}{2}}, \quad t \geq 1, \quad 1 < \min(\alpha, \beta) < 2, \\
(1 + t)^{-1} \log(1 + t), \quad t \geq 1, \quad \min(\alpha, \beta) = 2 \right\}.
\]

(4.17)

Next, we shall prove lower bound estimate of (1.23). For the simplicity, we set \(\gamma \equiv \min(\alpha, \beta)\). First, we take \(x = at\) in (1.18), then we have from (1.4) and (1.18)

\[
\begin{align*}
Z(at, t) &= \int_{\mathbb{R}} c_{\alpha,\beta}(y) \left( (\partial_y G_0)(at - y, t) y(at, t) + G_0(at - y, t) \frac{b}{2 \mu} \chi(at, t) y(at, t) \right) (1 + |y|)^{-(\gamma - 1)} dy \\
&= c_{+,\alpha,\beta}^+ y(at, t) \int_{0}^{\infty} (\partial_y G_0)(at - y, t) (1 + y)^{-(\gamma - 1)} dy \\
&\quad + c_{-,\alpha,\beta}^- y(at, t) \int_{-\infty}^{0} (\partial_y G_0)(at - y, t) (1 - y)^{-(\gamma - 1)} dy \\
&\quad + \frac{b c_{+,\alpha,\beta}^+}{2 \mu} \chi(at, t) \int_{0}^{\infty} G_0(at - y, t) (1 + y)^{-(\gamma - 1)} dy \\
&\quad + \frac{b c_{-,\alpha,\beta}^-}{2 \mu} \chi(at, t) \int_{0}^{\infty} G_0(at - y, t) (1 - y)^{-(\gamma - 1)} dy \\
&= \frac{(c_{+,\alpha,\beta}^+ - c_{-,\alpha,\beta}^-)}{4 \sqrt{\pi} \mu^{3/2}} y(at, t) t^{-\frac{\gamma}{2}} \int_{0}^{\infty} e^{-\frac{y^2}{4 \pi \mu}} y(1 + y)^{-(\gamma - 1)} dy \\
&\quad + \frac{b(c_{+,\alpha,\beta}^+ + c_{-,\alpha,\beta}^-)}{4 \sqrt{\pi} \mu^{3/2}} \chi(at, t) t^{-\frac{\gamma}{2}} \int_{0}^{\infty} e^{-\frac{y^2}{4 \pi \mu}} (1 + y)^{-(\gamma - 1)} dy
\end{align*}
\]

\[
\equiv L_1(t) + L_2(t).
\]

(4.18)

From the mean value theorem, there exists \(\theta_j \in (0, 1)\) such that

\[
(1 + y)^{-(\gamma - j)} - y^{-(\gamma - j)} = -(\gamma - j)(y + \theta_j)^{-(\gamma - j + 1)}.
\]

(4.19)

Therefore, since

\[
\int_{0}^{\infty} e^{-\frac{y^2}{4 \pi \mu}} y^{j-\gamma} dy = 2^{j-\gamma}(\mu t)^{j-\gamma} \Gamma\left(\frac{j + 1 - \gamma}{2}\right), \quad j \geq 1,
\]

(4.20)

it follows that

\[
L_1(t) = \frac{(c_{+,\alpha,\beta}^+ - c_{-,\alpha,\beta}^-)}{4 \sqrt{\pi} \mu^{3/2}} y(at, t) t^{-\frac{\gamma}{2}} \\
\times \left( \int_{0}^{\infty} e^{-\frac{y^2}{4 \pi \mu}} y^{2-j-\gamma} dy + \int_{0}^{\infty} e^{-\frac{y^2}{4 \pi \mu}} y((1 + y)^{-(\gamma - 1)} - y^{-(\gamma - 1)}) dy \right)
\]

\[
\frac{(c_{+,\alpha,\beta}^+ - c_{-,\alpha,\beta}^-)}{4 \sqrt{\pi} \mu^{3/2}} y(at, t) t^{-\frac{\gamma}{2}} \\
\times \left( 2^{j-\gamma}(\mu t)^{j-\gamma} \Gamma\left(\frac{j + 1 - \gamma}{2}\right) - (\gamma - 1) \int_{0}^{\infty} e^{-\frac{y^2}{4 \pi \mu}} y(y + \theta_1)^{-(\gamma - 1)} dy \right),
\]

(4.21)
where $\Gamma(s)$ is the Gamma function for $s > 0$. On the other hand, by making the integration by parts, we have from (4.19) and (4.20)

\[
L_2(t) = \frac{b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)}{4\sqrt{\pi\mu^{3/2}}} \eta(at, t)\chi(at, t)t^{-\frac{1}{2}}
\]

\[
\times \left(\frac{1}{\gamma - 2} + \frac{1}{2\mu(2 - \gamma)} \right) t^{-1} \int_0^\infty e^{-\frac{y^2}{2\mu t} - (1 + y)^{-\gamma - 2}} dy
\]

\[
= \frac{b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)}{4\sqrt{\pi\mu^{3/2}}} \eta(at, t)\chi(at, t)t^{-\frac{1}{2}} \left(\frac{1}{\gamma - 2} + \frac{1}{2\mu(2 - \gamma)} \right) t^{-1} \int_0^\infty e^{-\frac{y^2}{2\mu t} - (y + \theta_2)^{-(\gamma - 1)}} dy
\]

Therefore, from (4.18), (4.21), (4.22) and (2.21), we obtain

\[
\|Z(\cdot, t)\|_{L^\infty} \geq |Z(at, t)| = |L_1(t) + L_2(t)|
\]

\[
= \frac{|\eta(at, t)|}{4\sqrt{\pi\mu^{3/2}}} 2^{\gamma - 1} \mu^{\frac{\gamma - 1}{2}} (\frac{c_{\alpha,\beta}^+ - c_{\alpha,\beta}^-}{\gamma - 2}) \Gamma \left(\frac{3 - \gamma}{2} \right) t^{-\frac{1}{2}}
\]

\[
- (c_{\alpha,\beta}^+ - c_{\alpha,\beta}^-)(\gamma - 1)t^{-\frac{1}{2}} \int_0^\infty e^{-\frac{y^2}{2\mu t}} y(y + \theta_1)^{-(\gamma - 2)} dy - \frac{b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)}{2 - \gamma} \chi(at, t)t^{-\frac{1}{2}}
\]

\[
+ \frac{2^{\gamma - 1}}{2 - \gamma} b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-) \Gamma \left(2 - \frac{\gamma}{2} \right) \chi(at, t)t^{\frac{\gamma - 3}{2}}
\]

\[
\geq \min\{1, 2^{\gamma - 1}\} \frac{\mu^{\frac{\gamma - 1}{2}}}{4\sqrt{\pi\mu^{3/2}}} |M_1(t) + M_2(t) + M_3(t) + M_4(t) + M_5(t)|.
\]

First, we focus on the term $M_1(t)$ and $M_3(t)$. From the mean value theorem, there exists $\tilde{\theta} \in (0, 1)$ such that

\[
\chi^* \left(\frac{-a}{\sqrt{1 + t}}\right) = \chi^*(0) - \frac{a}{\sqrt{1 + t}} \chi'^* \left(\frac{-a\tilde{\theta}}{\sqrt{1 + t}}\right).
\]

Therefore, we get

\[
\chi(at, t) = (1 + t)^{-\frac{1}{2}} \chi^*(0) - a(1 + t)^{-1} \chi'^* \left(\frac{-a\tilde{\theta}}{\sqrt{1 + t}}\right).
\]

Thus, using (4.24), we have

\[
|M_1(t) + M_3(t)|
\]

\[
= 2^{\gamma - 1} \mu^{\frac{\gamma - 1}{2}} \left|\sqrt{\mu}(c_{\alpha,\beta}^+ - c_{\alpha,\beta}^-) \Gamma \left(\frac{3 - \gamma}{2} \right) t^{-\frac{3}{2}} + \frac{b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)}{2 - \gamma} \Gamma \left(2 - \frac{\gamma}{2} \right) \chi(at, t)t^{\frac{\gamma - 3}{2}}\right|
\]

\[
= 2^{\gamma - 1} \mu^{\frac{\gamma - 1}{2}} \left|\sqrt{\mu}(c_{\alpha,\beta}^+ - c_{\alpha,\beta}^-) \Gamma \left(\frac{3 - \gamma}{2} \right) t^{-\frac{3}{2}} + \frac{b\chi^*(0)(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)}{2 - \gamma} \Gamma \left(2 - \frac{\gamma}{2} \right) \chi(t)^{(1 + t)^{-\frac{1}{2}} t^{\frac{\gamma - 3}{2}}}\right|
\]

\[
- \frac{ab(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)}{2 - \gamma} \Gamma \left(2 - \frac{\gamma}{2} \right) (1 + t)^{-1} t^{\frac{\gamma - 3}{2}} \chi'^* \left(\frac{-a\tilde{\theta}}{\sqrt{1 + t}}\right)
\]

\[
= 2^{\gamma - 1} \mu^{\frac{\gamma - 1}{2}} \left|\sqrt{\mu}(c_{\alpha,\beta}^+ - c_{\alpha,\beta}^-) \Gamma \left(\frac{3 - \gamma}{2} \right) t^{-\frac{3}{2}} + \frac{b\chi^*(0)(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)}{2 - \gamma} \Gamma \left(2 - \frac{\gamma}{2} \right) t^{-\frac{2}{2}}\right|
\]

\[
+ \frac{b\chi^*(0)(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)}{2 - \gamma} \Gamma \left(2 - \frac{\gamma}{2} \right) (1 + t)^{-\frac{1}{2}} t^{\frac{\gamma - 2}{2}} \chi'^* \left(\frac{-a\tilde{\theta}}{\sqrt{1 + t}}\right)
\]

\[
- \frac{ab(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)}{2 - \gamma} \Gamma \left(2 - \frac{\gamma}{2} \right) (1 + t)^{-1} t^{\frac{\gamma - 3}{2}} \chi'^* \left(\frac{-a\tilde{\theta}}{\sqrt{1 + t}}\right).
\]
From (1.4) and (1.7), it is easy to see that
\[ \chi_*'(0) + \frac{x}{2\mu} \chi_* = \frac{b}{2\mu} \chi_*'. \]
Then, using (2.11), we obtain
\[ |\chi_*'| \leq C(|x| + |\chi_*|)|\chi_*| \leq C|\mu' e^{-\frac{x^2}{4\mu\nu^2}}, \]
and hence
\[ |\chi_*'(\frac{-a\theta}{\sqrt{1 + t}})| \leq C|\mu' e^{-\frac{x^2}{4\mu\nu^2}}| \leq C =: C_0. \]
Therefore, from (4.25), we get
\[ |M_1(t) + M_4(t)| \geq 2^{2-\gamma} \mu^{1-2} \left( |\tilde{\nu}_0|(1 + t)^{-\frac{\gamma}{2}} - \frac{|b\chi_*'(0)(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)|}{2 - \gamma} \Gamma \left( 2 - \left( \frac{\gamma}{2} \right)^2 \right) t^{-1 - \frac{\gamma}{2}} \right. \]
\[ - \left. \frac{C_0|b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)|}{2 - \gamma} \Gamma \left( 2 - \left( \frac{\gamma}{2} \right)^2 \right) t^{-1 - \frac{\gamma}{2}} \right) \leq |(\text{4.26})|, \]
where \( \tilde{\nu}_0 \) is defined by (1.25). Next, for \( M_2(t) \), we have from (4.20)
\[ |M_2(t)| \leq (\gamma - 1)|c_{\alpha,\beta}^+ - c_{\alpha,\beta}^-| t^{\frac{\gamma}{2}} \int_0^\infty e^{-\frac{x^2}{4\mu}} y^{1-\gamma} dy \]
\[ = \frac{\mu^{1-\frac{\gamma}{2}}(\gamma - 1)|c_{\alpha,\beta}^+ - c_{\alpha,\beta}^-|}{2\gamma - 1} \Gamma \left( 1 - \frac{\gamma}{2} \right) t^{-\frac{1 + \gamma}{2}}. \]
It is easy to see that
\[ |M_3(t)| \leq \frac{|b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)|}{2 - \gamma} \|\chi(\cdot), t\|_{L^\infty} t^{-\frac{\gamma}{2}} \leq \frac{C_1|b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)|}{2 - \gamma} t^{-1}. \]
Finally, for \( M_5(t) \), we obtain from (4.20)
\[ |M_5(t)| \leq \frac{|b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)|}{2\mu} \|\chi(\cdot), t\|_{L^\infty} t^{-\frac{\gamma}{2}} \int_0^\infty e^{-\frac{x^2}{4\mu}} y^{2-\gamma} dy \]
\[ \leq \frac{C_2|b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)|}{(2\sqrt{\mu})^{\frac{1}{2}}} \Gamma \left( \frac{3 - \gamma}{2} \right) t^{-\frac{1 + \gamma}{2}}. \]
Therefore, combining (4.23), (4.26) through (4.29), we have
\[ \|Z(\cdot), t\|_{L^\infty} \geq \frac{\min \{1, e^{\frac{x^2}{4\mu\nu^2}}\}}{4\sqrt{\pi\mu^3/2}} \{ |M_1(t) + M_4(t)| - |M_2(t)| - |M_3(t)| - |M_5(t)| \}
\[ \geq \frac{\min \{1, e^{\frac{x^2}{4\mu\nu^2}}\}}{4\sqrt{\pi\mu^3/2}} \left[ 2^{2-\gamma} \mu^{1-2} \left( |\tilde{\nu}_0|(1 + t)^{-\frac{\gamma}{2}} - \frac{|b\chi_*'(0)(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)|}{2 - \gamma} \Gamma \left( 2 - \left( \frac{\gamma}{2} \right)^2 \right) t^{-1 - \frac{\gamma}{2}} \right. \right. \]
\[ - \left. \frac{C_0|b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)|}{2 - \gamma} \Gamma \left( 2 - \left( \frac{\gamma}{2} \right)^2 \right) t^{-1 - \frac{\gamma}{2}} \right) \leq \frac{C_1|b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)|}{2 - \gamma} t^{-1} - \frac{C_2|b(c_{\alpha,\beta}^+ + c_{\alpha,\beta}^-)|}{(2\sqrt{\mu})^{\frac{1}{2}}} \Gamma \left( \frac{3 - \gamma}{2} \right) t^{-\frac{1 + \gamma}{2}} \right] \}
\]
Hence, there is a positive constant \( \nu_0 \) such that (1.23) holds. This completes the proof of Theorem 1.2 for \( 1 < \min \{\alpha, \beta\} < 2 \).
5 Proof of Theorem 1.2 for \( \min\{\alpha, \beta\} = 2 \)

Finally in this section, we shall completes the proof of Theorem 1.2. Namely, we prove (1.22) and (1.24). First, let us recall the following fact derived in [7]. We consider

\[
v_t + av_x + (b\chi v)_x - \mu v_{xx} = -\kappa \partial_x(\chi^3), \quad x \in \mathbb{R}, \ t > 0, \quad v(x, 0) = 0, \quad x \in \mathbb{R}. \tag{5.1}
\]

The leading term of the solution \( v(x, t) \) to (5.1) is given by \( V(x, t) \) defined by (1.9). More precisely, the following asymptotic formula can be shown (for the proof, see Proposition 4.3 in [7]).

**Proposition 5.1.** Assume that \( |M| \leq 1 \). Then the estimate

\[
\|v(\cdot, t) - V(\cdot, t)\|_{L^\infty} \leq C|M|(1 + t)^{-1}, \ t \geq 1
\]

holds. Here \( v(x, t) \) is the solution to (5.1) and \( V(x, t) \) is defined by (1.9).

This formula helps us to complete the proof of Theorem 1.2 with \( \min\{\alpha, \beta\} = 2 \).

**End of the proof of Theorem 1.2 for \( \min\{\alpha, \beta\} = 2 \).** First, we shall prove (1.22). We set

\[
\phi(x, t) \equiv u(x, t) + u_t(x, t) - \chi(x, t) - v(x, t).
\]

Then, from (1.2), (1.7) and (5.1), we have the following initial value problem:

\[
\phi_t + a \phi_x + (b\chi \phi)_x - \mu \phi_{xx} = \partial_x N_0(\chi) + \partial_x N_1(u, \chi), \quad x \in \mathbb{R}, \ t > 0, \quad \phi(x, 0) = \psi_0(x) = u_0(x) + u_1(x) - \chi_0(x), \quad x \in \mathbb{R},
\]

where

\[
N_1(\chi) \equiv 2a\mu \chi x - 2ab\chi x + \frac{ab^2}{4\mu} \chi^2,
\]

\[
N_2(u, \chi) \equiv a(\partial_t + a\partial_x)(u - \chi) - \mu \partial_x(\partial_t(\partial_x - \chi)) - \mu \partial_t(\partial_x(\partial_x - \chi)) + b\chi(\partial_t + a\partial_x)\chi - \frac{b}{2}(u - \chi)^2 - \frac{c}{3!}(u - \chi)^3 - \frac{c}{2}u\chi(u - \chi).
\]

Therefore, from Lemma 2.6, we obtain

\[
\phi(x, t) = U[\psi_0](x, t, 0) + \int_0^t U[\partial_x N_1(\chi)(\tau)](x, t, \tau) d\tau + \int_0^t U[\partial_x N_2(u, \chi)(\tau)](x, t, \tau) d\tau.
\]

Thus, we have

\[
u(x, t) - \chi(x, t) - Z(x, t) - V(x, t)
\]

\[
= U[\psi_0](x, t, 0) - Z(x, t) - u_t(x, t) + v(x, t) - V(x, t)
\]

\[
+ \int_0^t U[\partial_x N_1(\chi)(\tau)](x, t, \tau) d\tau + \int_0^t U[\partial_x N_2(u, \chi)(\tau)](x, t, \tau) d\tau
\]

\[
\equiv U[\psi_0](x, t, 0) - Z(x, t) - u_t(x, t) + v(x, t) - V(x, t) + K_1 + K_2,
\]

where \( Z(x, t) \) and \( V(x, t) \) are defined by (1.18) and (1.9), respectively.

Now, we only need to evaluate the last two terms in the right hand side of (5.6). First, we evaluate \( K_1 \). To estimate it, we introduce the useful property of \( N_1(\chi) \). Actually, if we set \( N_0(\chi) \equiv 2\mu \chi_x - \frac{b}{2}\chi^2 \), from \( N_1(\chi) = a(\partial_x N_0(\chi) - \frac{b}{2\mu} \chi N_0(\chi)) \), we get \( N_1(\chi) = \eta \partial_x(\eta^{-1} N_0(\chi)) \).
Therefore, from the definition of $K_1$ and (2.26), and by making the integration by parts, we have

$$K_1(x, t) = \int_0^t \int_\mathbb{R} \partial_x (G_0(x - y, t - \tau) \eta(x, t)) \partial_y ((\eta(y, \tau))^{-1} N_0(y, \tau)) dy d\tau$$

$$= \sum_{n=0}^l \int_0^t \int_\mathbb{R} \partial_x^{n-1} \eta(x, t) \left( \int_0^{t/2} + \int_{t/2}^t \right) \partial_x^n G_0(x - y, t - \tau) \partial_y ((\eta(y, \tau))^{-1} N_0(y, \tau)) dy d\tau$$

$$= \sum_{n=0}^l \int_0^t \int_\mathbb{R} \partial_x^{n-1} \eta(x, t) \left( \int_0^{t/2} \partial_x^n G_0(x - y, t - \tau) (\eta(y, \tau))^{-1} N_0(y, \tau) dy d\tau \right.$$  

$$+ \int_{t/2}^t \partial_x^n G_0(x - y, t - \tau) \partial_y ((\eta(y, \tau))^{-1} N_0(y, \tau)) dy d\tau) \right).$$  

(5.7)

Also from Lemma 2.3 and Lemma 2.5, for any non-negative integer $l$ and $1 \leq q \leq \infty$, it is easy to see that

$$\| \partial_x^l (\eta^{-1} N_0(\chi)) \|_{L^q} \leq C \sum_{j=0}^l (1 + t)^{-\frac{1}{2}(l-j)} \| \partial_x^j N_0(\cdot, t) \|_{L^q} \leq C (1 + t)^{-1 + \frac{1}{2} \cdot - \frac{1}{2}}.$$  

(5.8)

Hence, from (5.7), Young’s inequality, Lemma 2.5, (2.13) and (5.8), we have

$$\| K_1(\cdot, t) \|_{L^\infty} \leq C \sum_{n=0}^l \| \partial_x^{n-1} \eta(\cdot, t) \|_{L^\infty} \left( \int_0^{t/2} \| \partial_x^n G_0(\cdot, t - \tau) \|_{L^\infty} \| (\eta^{-1} N_0(\chi))(\cdot, \tau) \|_{L^q} d\tau \right.$$  

$$+ \int_{t/2}^t \| \partial_x^n G_0(\cdot, t - \tau) \|_{L^\infty} \| \partial_x (\eta^{-1} N_0(\chi))(\cdot, \tau) \|_{L^q} d\tau) \right)$$

$$\leq C \sum_{n=0}^l (1 + t)^{-\frac{1}{2} + \frac{1}{2}} \left( \int_0^{t/2} (t - \tau)^{-\frac{1}{2} - \frac{1}{2}} (1 + \tau)^{-\frac{1}{2}} d\tau + \int_{t/2}^t (t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{1}{2}} d\tau \right)$$

$$\leq C (1 + t)^{-1}, \quad t \geq 1.$$  

(5.9)

Next, we estimate $K_2$. Before do that, for $0 < \varepsilon < \frac{1}{2}$, we prepare the following estimates:

$$\| N_2(\cdot, t) \|_{L^1} \leq C (1 + t)^{-\frac{1}{2} + \varepsilon},$$  

(5.10)

$$\| N_2(\cdot, t) \|_{L^2} \leq C (1 + t)^{-\frac{1}{2} + \varepsilon}.$$  

(5.11)

We shall prove only (5.10), since we can prove (5.11) in the same way. From (3.55), (1.17), (2.12), (1.16) and (2.9), we have

$$\| N_2(\cdot, t) \|_{L^1} \leq C \| (\partial_t + a \partial_x)(u - \chi)(\cdot, t) \|_{L^1} + C \| \partial_t \partial_x (u - \chi)(\cdot, t) \|_{L^1},$$

$$+ C \| \chi(\cdot, t) \|_{L^\infty} \| \partial_x (u - \chi)(\cdot, t) \|_{L^1} + C \| \partial_t (u - \chi)(\cdot, t) \|_{L^1},$$

$$+ C \| (u - \chi)^3(\cdot, t) \|_{L^1} + C \| u(\cdot, t) \|_{L^1} \| (u - \chi)(\cdot, t) \|_{L^1}.$$  

$$\leq C (1 + t)^{-\frac{1}{2} + \varepsilon}.$$  

Therefore, by using Lemma 2.7, (5.10) and (5.11), we obtain

$$\| K_2(\cdot, t) \|_{L^\infty} \leq C \sum_{n=0}^l (1 + t)^{-\frac{1}{2} + \frac{1}{2}} \left( \int_0^{t/2} (t - \tau)^{-\frac{1}{2} - \frac{1}{2}} \| N_2(\cdot, \tau) \|_{L^1} d\tau + \int_{t/2}^t (t - \tau)^{-\frac{1}{2} - \frac{1}{2}} \| N_2(\cdot, \tau) \|_{L^2} d\tau \right)$$

$$\leq C \sum_{n=0}^l (1 + t)^{-\frac{1}{2} + \frac{1}{2}} \left( \int_0^{t/2} (t - \tau)^{-\frac{1}{2} - \frac{1}{2}} (1 + \tau)^{-\frac{1}{2} + \varepsilon} d\tau + \int_{t/2}^t (t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{1}{2} + \varepsilon} d\tau \right)$$

$$\leq C (1 + t)^{-1}, \quad t \geq 1.$$  

(5.12)
Thus, from (5.6), (2.10), Gagliardo-Nirenberg inequality, (5.2), (5.9) and (5.12), we obtain

\[ \|u(\cdot,t) - \chi(\cdot,t) - Z(\cdot,t) - V(\cdot,t)\|_{L^\infty} \leq \|U[\psi_0](\cdot,t,0) - Z(\cdot,t)\|_{L^\infty} + C(1 + t)^{-1}, \quad t \geq 1. \]

Therefore, from (4.5), we finally arrive at

\[ \limsup_{t \to \infty} \frac{(1 + t)}{\log(1 + t)} \|u(\cdot,t) - \chi(\cdot,t) - Z(\cdot,t) - V(\cdot,t)\|_{L^\infty} = 0. \]

This completes the proof of (1.22).

Finally, we shall derive the estimate of \( Z(x,t) + V(x,t) \), that is (1.24). Since

\[ \|V(\cdot,t)\|_{L^\infty} = |\kappa d|\|V_*(\cdot)\|_{L^\infty} (1 + t)^{-1} \log(1 + t), \]

and using (4.17), we can derive the upper bound estimate of (1.24). Then, we shall only prove the lower bound estimate. First, we take \( x = a t \) in (1.9). Since

\[ V_*(x) = \frac{1}{\sqrt{4\pi \mu}} \left( b \frac{b}{2\mu} \chi(x) \eta_\alpha(x) - \frac{x}{2\mu} \eta_\alpha(x) \right) = \frac{1}{4\sqrt{\pi \mu}^{3/2}} (b \chi(x) - x) \eta_\alpha(x), \]

from (1.10), (1.4) and (1.19), it follows that

\[ V(at,t) = -\kappa d V_* \left( \frac{-a}{\sqrt{1 + t}} \right) (1 + t)^{-1} \log(1 + t) \]

\[ = -\frac{\kappa d}{4\sqrt{\pi \mu}^{3/2}} \left( b \sqrt{1 + t} \chi(at,t) + \frac{a}{\sqrt{1 + t}} \right) \eta(\alpha, t)(1 + t)^{-1} \log(1 + t) \]

\[ = -\frac{\kappa d}{4\sqrt{\pi \mu}^{3/2}} \eta(\alpha, t) \left( b \chi(at,t)(1 + t) - \frac{\eta(\alpha, t)}{t} \log(1 + t) + a(1 + t) - \frac{\eta(\alpha, t)}{t} \log(1 + t) \right). \]

Combining (4.18) and (5.13), we have from the triangle inequality that

\[ \|Z(\cdot,t) + V(\cdot,t)\|_{L^\infty} \]

\[ \geq |Z(at,t) + V(at,t)| \]

\[ = \left| (c_{a,\beta}^+ - c_{a,\beta}^-) \eta(\alpha, t)t^{-\frac{1}{2}} \int_0^{\infty} e^{-\frac{y^2}{4\mu}} (1 + y)^{-1} dy \right| \]

\[ + \frac{b(c_{a,\beta}^+ + c_{a,\beta}^-)}{4\sqrt{\pi \mu}^{3/2}} \eta(\alpha, t) \chi(at,t)t^{-\frac{1}{2}} \int_0^{\infty} e^{-\frac{y^2}{4\mu}} (1 + y)^{-1} dy \]

\[ - \frac{\kappa d}{4\sqrt{\pi \mu}^{3/2}} \eta(\alpha, t) \left( b \chi(at,t)(1 + t) - \frac{\eta(\alpha, t)}{t} \log(1 + t) + a(1 + t) - \frac{\eta(\alpha, t)}{t} \log(1 + t) \right) \]

\[ \geq \left| \frac{b}{4\sqrt{\pi \mu}^{3/2}} \left( \chi(at,t) \right) \left( c_{a,\beta}^+ + c_{a,\beta}^- \right) t^{-\frac{1}{2}} \int_0^{\infty} e^{-\frac{y^2}{4\mu}} (1 + y)^{-1} dy \right| \]

\[ - \kappa d(1 + t)^{-\frac{1}{2}} \log(1 + t) \left| \left( c_{a,\beta}^+ - c_{a,\beta}^- \right) \eta \right| \right| t^{-\frac{1}{2}} \int_0^{\infty} e^{-\frac{y^2}{4\mu}} dy \]

\[ - \frac{|\kappa d| \max\{1, e^{\frac{bM}{\mu}}\}}{4\sqrt{\pi \mu}^{3/2}} (1 + t)^{-\frac{1}{2}} \log(1 + t) \]

\[ \equiv \left| \frac{b}{4\sqrt{\pi \mu}^{3/2}} \left( \chi(at,t) \right) W(t) - \frac{|c_{a,\beta}^+ - c_{a,\beta}^-| \max\{1, e^{\frac{bM}{\mu}}\}}{4\mu} \right| t^{-\frac{1}{2}} \int_0^{\infty} e^{-\frac{y^2}{4\mu}} dy \]

\[ - \frac{|\kappa d| \max\{1, e^{\frac{bM}{\mu}}\}}{4\sqrt{\pi \mu}^{3/2}} (1 + t)^{-\frac{1}{2}} \log(1 + t). \]

Now, we evaluate \( W(t) \) from below. Splitting the \( y \)-integral and using the triangle inequality, we
obtain

\[
W(t) = \left| (c_{α,β}^+ + c_{α,β}^-)t^{-\frac{1}{2}} \left( \int_0^{\sqrt{1+t-1}} + \int_{\sqrt{1+t-1}}^{\infty} \right) e^{-\frac{y^2}{4\mu t}} (1 + y)^{-1} dy - \kappa d(1+t)^{-\frac{1}{2}} \log(1+t) \right|
\]
\[
\geq \left| (c_{α,β}^+ + c_{α,β}^-)t^{-\frac{1}{2}} \int_0^{\sqrt{1+t-1}} e^{-\frac{y^2}{4\mu t}} (1 + y)^{-1} dy - \kappa d(1+t)^{-\frac{1}{2}} \log(1+t) \right|
\]
\[
- \left| (c_{α,β}^+ + c_{α,β}^-)t^{-\frac{1}{2}} \int_{\sqrt{1+t-1}}^{\infty} e^{-\frac{y^2}{4\mu t}} (1 + y)^{-1} dy \right|
\]
\[
\equiv W_1(t) - W_2(t).
\]

(5.15)

From the mean value theorem, there exists \( \theta \in (0, 1) \) such that

\[ e^{-\frac{\theta^2}{4\mu t}} = 1 - \frac{y^2}{4\mu t} e^{-\frac{\theta^2}{4\mu t}}. \]

Therefore, we have

\[
W_1(t) = \left| (c_{α,β}^+ + c_{α,β}^-)t^{-\frac{1}{2}} \int_0^{\sqrt{1+t-1}} \left( 1 - \frac{y^2}{4\mu t} e^{-\frac{\theta^2}{4\mu t}} \right) (1 + y)^{-1} dy - \kappa d(1+t)^{-\frac{1}{2}} \log(1+t) \right|
\]
\[
\geq \left| (c_{α,β}^+ + c_{α,β}^-)t^{-\frac{1}{2}} \int_0^{\sqrt{1+t-1}} (1 + y)^{-1} dy - \kappa d(1+t)^{-\frac{1}{2}} \log(1+t) \right|
\]
\[
- \frac{c_{α,β}^+ + c_{α,β}^-}{4\mu} \int_0^{\sqrt{1+t-1}} e^{-\frac{\theta^2}{4\mu t} y^2} (1 + y)^{-1} dy \]
\[
\equiv W_{1,1}(t) - W_{1,2}(t).
\]

(5.16)

For \( W_{1,1}(t) \), we obtain

\[
W_{1,1}(t) = \left| \frac{c_{α,β}^+ + c_{α,β}^-}{2} t^{-\frac{1}{2}} \log(1+t) - \kappa d(1+t)^{-\frac{1}{2}} \log(1+t) \right|
\]
\[
= \left| \frac{c_{α,β}^+ + c_{α,β}^-}{2} t^{-\frac{1}{2}} \log(1+t) - \frac{\kappa d}{2} (1+t)^{-\frac{1}{2}} \log(1+t) \right|
\]
\[
+ \left( \frac{c_{α,β}^+ + c_{α,β}^-}{2} - \kappa d \right) (1+t)^{-\frac{1}{2}} \log(1+t) \right|
\]
\[
\geq |\tilde{y}_1|(1+t)^{-\frac{1}{2}} \log(1+t) - \left| \frac{c_{α,β}^+ + c_{α,β}^-}{2} \right| \log(1+t) (t^{-\frac{1}{2}} - (1+t)^{-\frac{1}{2}})
\]
\[
\geq |\tilde{y}_1|(1+t)^{-\frac{1}{2}} \log(1+t) - \left| \frac{c_{α,β}^+ + c_{α,β}^-}{2} \right| t^{-\frac{1}{2}} \log(1+t),
\]

(5.17)

where \( \tilde{y}_1 \) is defined by (1.25). On the other hand, for \( W_{1,2}(t) \), we get

\[
W_{1,2}(t) \leq \frac{|c_{α,β}^+ + c_{α,β}^-|}{4\mu} t^{-\frac{1}{2}} \int_0^{\sqrt{1+t-1}} y dy = \frac{|c_{α,β}^+ + c_{α,β}^-|}{8\mu} t^{-\frac{1}{2}} (\sqrt{1+t-1})^2 \leq \frac{|c_{α,β}^+ + c_{α,β}^-|}{8\mu} t^{-\frac{1}{2}}.
\]

(5.18)

Analogously, for \( W_{2}(t) \), it is easy to see that

\[
W_{2}(t) \leq |c_{α,β}^+ + c_{α,β}^-| t^{-\frac{1}{2}} \left( \sup_{y \geq \sqrt{1+t-1}} (1 + |y|)^{-1} \right) \int_{y \geq \sqrt{1+t-1}} e^{-\frac{y^2}{4\mu t}} dy
\]
\[
\leq \sqrt{\mu t} |c_{α,β}^+ + c_{α,β}^-|(1+t)^{-\frac{1}{2}}.
\]

(5.19)
Finally, combining (5.14) through (5.19), (2.12) and (4.24), we obtain
\[ \| Z(\cdot, t) + V(\cdot, t) \|_{L^\infty} \]
\[ \geq \frac{|b| \min\{1, e^{\frac{|ab|}{\mu}}\}}{4\sqrt{\pi} \mu^{3/2}} |\chi(at, t)||\tilde{\nu}_1|(1 + t)^{-\frac{1}{2}} \log(1 + t) \]
\[ = \frac{|b| \min\{1, e^{\frac{|ab|}{\mu}}\}|c^+_{\alpha, \beta} + c^-_{\alpha, \beta}|}{8\sqrt{\pi} \mu^{3/2}} t^{-\frac{1}{2}} \log(1 + t) \| \chi(\cdot, t) \|_{L^\infty} \]
\[ = \frac{|b| \min\{1, e^{\frac{|ab|}{\mu}}\}|c^+_{\alpha, \beta} + c^-_{\alpha, \beta}|}{4\sqrt{\pi} \mu^{3/2}} \left( \sqrt{\frac{1}{8\mu}} \right) t^{-\frac{1}{2}} \| \chi(\cdot, t) \|_{L^\infty} \]
\[ = \frac{|c^+_{\alpha, \beta} - c^-_{\alpha, \beta}|}{4\mu} max\{1, e^{\frac{|ab|}{\mu}}\} t^{-1} - \frac{|abd| \max\{1, e^{\frac{|ab|}{\mu}}\}}{4\sqrt{\pi} \mu^{3/2}} (1 + t)^{-\frac{1}{2}} \log(1 + t) \]
\[ \geq \frac{|b\chi_x(0)| \min\{1, e^{\frac{|ab|}{\mu}}\}}{4\sqrt{\pi} \mu^{3/2}} |\tilde{\nu}_1|(1 + t)^{-1} \log(1 + t) - C_0 \frac{|ab| \min\{1, e^{\frac{|ab|}{\mu}}\}}{4\sqrt{\pi} \mu^{3/2}} |\tilde{\nu}_1|(1 + t)^{-\frac{1}{2}} \log(1 + t) \]
\[ = \frac{|b| \min\{1, e^{\frac{|ab|}{\mu}}\}|c^+_{\alpha, \beta} + c^-_{\alpha, \beta}|}{8\sqrt{\pi} \mu^{3/2}} t^{-2} \log(1 + t) \]
\[ = \frac{|b| \min\{1, e^{\frac{|ab|}{\mu}}\}|c^+_{\alpha, \beta} + c^-_{\alpha, \beta}|}{4\sqrt{\pi} \mu^{3/2}} \left( \sqrt{\frac{1}{8\mu}} \right) t^{-1} \]
\[ = \frac{|c^+_{\alpha, \beta} - c^-_{\alpha, \beta}|}{4\mu} max\{1, e^{\frac{|ab|}{\mu}}\} t^{-1} - \frac{|abd| \max\{1, e^{\frac{|ab|}{\mu}}\}}{4\sqrt{\pi} \mu^{3/2}} (1 + t)^{-\frac{1}{2}} \log(1 + t). \]

As a conclusion, there is a positive constant $\nu_1$ such that (1.24) holds. This completes the proof of Theorem 1.2 for $\min\{\alpha, \beta\} = 2$. \hfill \Box

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References
[1] I. Fukuda, Asymptotic behavior of solutions to the generalized KdV-Burgers equation, to appear in Osaka J. Math.
[2] I. Fukuda, Asymptotic behavior of solutions to the generalized KdV-Burgers equation with slowly decaying data, preprint, arXiv:1806.07323v2.
[3] N. Hayashi, E.I. Kaikina and P.I. Naumkin, Large time asymptotics for the BBM-Burgers equation, Ann. Henri Poincaré 8 (2007) 485-511.
[4] N. Hayashi and P.I. Naumkin, Asymptotics for the Korteweg-de Vries-Burgers equation, Acta Math. Sin. (Engl. Ser.), 22 (2006) 1441-1456.
[5] S. Jin and Z. Xin, The relaxation schemes for systems of conservation laws in arbitrary space dimensions, Commun. Pure Appl. Math. 48 (1995) 235-276.
[6] E.I. Kaikina and H.F. Ruiz-Paredes, Second term of asymptotics for KdVB equation with large initial data, Osaka J. Math. 42 (2005) 407-420.

[7] M. Kato, Large time behavior of solutions to the generalized Burgers equations, Osaka J. Math. 44 (2007) 923-943.

[8] M. Kato and Y. Ueda, Asymptotic profile of solutions for the damped wave equation with a nonlinear convection term, Math. Meth. Appl. Sci. 40 (2017) 7760-7779.

[9] T. Kawakami and Y. Ueda, Asymptotic profiles to the solutions for a nonlinear damped wave equation, Differential and Integral equations 26 No. 7-8 (2013) 781-814.

[10] S. Kawashima, Large-time behavior of solutions to hyperbolic-parabolic systems of conservation laws and applications, Proc. Roy. Soc. Edinburgh Sect. A 106 (1987) 169-194.

[11] Y. Kitagawa, Asymptotic behavior of solutions to the initial value problem for one dimensional single viscous conservation law, Osaka University, Master’s thesis, 2007 (in Japanese).

[12] A. Matsumura and K. Nishihara, Global solutions of nonlinear differential equations—Mathematical analysis for compressible viscous fluids, Nippon-Hyoron-Sha, Tokyo, 2004 (in Japanese).

[13] T. Narazaki and K. Nishihara, Asymptotic behavior of solutions for the damped wave equation with slowly decaying data, J. Math. Anal. Appl. 338 (2008) 803-819.

[14] R. Orive and E. Zuazua, Long-time behavior of solutions to a nonlinear hyperbolic relaxation system, J. Differential Equations 228 (2006) 17-38.

[15] R. Racke and Y. Ueda, Nonlinear thermoelastic plate equations—global existence and decay rates for the Cauchy problems, J. Differential Equations 263 (2017) 8138-8177.

[16] Y. Ueda and S. Kawashima, Large time behavior of solutions to a semilinear hyperbolic system with relaxation, J. Hyperbolic Differ. Equ. 4 (2007) 147-179.

[17] G. B. Whitham, Linear and nonlinear waves, Wiley, New York, 1974.

Ikki Fukuda
Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan
E-mail: i.fukuda@math.sci.hokudai.ac.jp