Strictly nef bundles

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Abstract. In this short note we will show that every homogeneous strictly
nef vector bundle on a complex flag variety is ample. We also consider the
question of when ampleness of vector bundles can be tested on curves.

1. Introduction

A line bundle $L$ on a projective variety $X$ is called strictly nef if $\text{deg}(L|_C) > 0$
for all integral curves $C \subset X$. A vector bundle $\mathcal{E}$ on $X$ is called strictly nef if
the corresponding hyperplane bundle $O_{\mathbb{P}E}(1)$ is strictly nef on $\mathbb{P}E$. There are old
examples of Mumford and Ramanujam (ex 10.6, 10.7 in [3]) which show that a
strictly nef (even effective) line bundle needn’t be ample. However, it is known
that strictly nef line bundles on homogeneous varieties are ample. For abelian
varieties, this is Prop 1.4 in [8], whereas for Flag varieties (being Fano), it follows
from the basepoint free theorem.

Given all this, it is natural to wonder what happens for strictly nef bundles of
higher rank on Homogeneous varieties. We will see in Section 1 that strictly nef bundles on a Flag Variety which are Homogeneous, are indeed ample. In Section 2, we will consider question 2.5 in [1] which asks whether for vector bundles on an
abelian variety, ampleness on curves implies ampleness.

2. Homogeneous bundles on Flag Varieties

Let $G$ be a semisimple complex Lie group, $B$ be a fixed Borel subgroup con-
taining a maximal torus $T$. Let $R$ be the set of roots of $G$ with respect to $T$ and let
$R_-$ denote the set of roots in the lie algebra of $B$, the so-called negative roots. Let
$P$ be a parabolic subgroup defined by a subset $I$ of the set of simple positive roots
$\{\alpha_1, ..., \alpha_l\}$. Let $\lambda_1, ..., \lambda_l$ be the fundamental weights, i.e., such that $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$. Then any arbitary weight $\lambda$ can be written as $\lambda = n_1 \lambda_1 + \ldots + n_l \lambda_l$ where $n_i \in \mathbb{Z}$
$\forall i$. If $n_i \geq 0 \forall i$, then we say $\lambda$ is dominant: $\lambda \in \Lambda^+$. There exists a partial order on
the weights: $\mu < \lambda$ if $\lambda - \mu$ is a positive (possibly 0) linear combination of dominant
weights. Weights maximal with respect to this partial order will be called maximal
weights.

Let $\mathcal{E} = G \times^P E_0$ be a homogeneous vector bundle on $X=G/P$ given by a $P$
module $E_0$. In what follows, $\Lambda(E_0)$ will denote the weights of $E_0$ and $\Lambda_{\text{max}}(E_0)$
will denote the subset of maximal weights. Here are a few facts we will be using:
(A) Corollary 11.5 on page 47 of [9] says: Let $E = G \times^P E_0$ be a homogeneous vector bundle on $X = G/P$ where $P$ is a parabolic subgroup of $G$. If $\Lambda_{\max}(E_0) \subset \Lambda^+$, then $E$ is generated by its global sections.

(B) We will also need the following well known fact: If $L(\lambda) \in \text{Pic } G/P$ is the homogeneous line bundle corresponding to the weight $\lambda$ (See beginning of page 18 of [9] for details on the correspondence.) and if $C(\alpha) \cong \mathbb{P}^1$ is the rational curve in $G/P$ corresponding to the roots $\pm \alpha$ (see the proof of Prop 4.4 in page 231 of [5] for details), then deg $L(\lambda)|_{C(\alpha)} = \langle \lambda, \alpha \rangle$. Here $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between roots and weights. (See loc. cit.)

**THEOREM 1.** If a homogeneous vector bundle $E$ on $G/P$ is strictly nef, then it is globally generated and hence ample.

**Proof.** Let $\Lambda(E_0) = \{\Lambda_1, \ldots, \Lambda_r\}$ counted with multiplicities, where $\Lambda_k = \sum_{i=1}^r n_i(k) \lambda_i$, then $E|_{C(\alpha_j)} = \mathcal{O}(n_j(1)) \oplus \cdots \oplus \mathcal{O}(n_j(r)) \forall j \notin I$. This is because $\Lambda(E_0|_{C(\alpha_j)}) = \{n_j(1) \lambda_1, \ldots, n_j(r) \lambda_j\}$ by (B) above. But since all line bundle quotients of a strict nef bundle restricted to a curve have positive degree (see Prop 2.1 in [7]), thus $n_j(k) > 0 \forall j \notin I, \forall k = 1, \ldots, r$. Now consider the projection $G/B \xrightarrow{\pi} G/P$. Since $\pi(C(\alpha_j)) = pt \forall j \in I$, thus $\langle \pi^*E|_{C(\alpha_j)} \rangle$ is trivial $\forall j \in I$ which means that $n_j(k) = 0 \forall j \in I, \forall k = 1, \ldots, r$ as above. Thus $n_j(k) \geq 0 \forall j$ and for all weights $\Lambda_k, k = 1, \ldots, r$ and thus $E$ is spanned by (A) above. Now by the canonical surjection $\pi^*E \rightarrow \mathcal{O}_{E^0}(1), \mathcal{O}_{E^0}(1)$ is generated by its global sections by (A) above and strict nef, hence ample.

\[ \square \]

**Remark 1:** The arguments in the above proof shows the more general fact that Homogeneous nef bundles on $G/P$ are globally generated.

**Remark 2:** The above theorem has also been proved independently around the same time by [1]. (See Theorem 3.1 in [1].)

3. Ampleness for bundles on curves

In this section, we consider question 2.5 of [1] which asks(in analogy with the line bundle case): Is a vector bundle on an abelian variety ample if its restriction to every curve is ample? We have the following lemma whose proof is an easy application of Hartshorne’s ampleness criterion for vector bundles on a smooth projective curve (Theorem 2.4 in [4]).

**Lemma 1.** Let $E$ be a vector bundle on a projective variety $X$. Then $E$ is ample when restricted to curves iff $\forall$ finite morphisms $f : C \rightarrow X$ where $C$ is a smooth projective curve, all vector bundle quotients of $f^*E$ are of positive degree.

**Proof.** $\iff$: If $C \subset X$ is a curve, consider its normalization $\tilde{C} \xrightarrow{\tilde{f}} C \subset X$. Then $f^*(E)$ is ample on $\tilde{C}$ by Hartshorne’s criterion. Hence $E|_C$ is also ample by Prop 6.1.8 (iii) in $[6]$. $\implies$: Let $f : C \rightarrow X$ be a finite morphism from a smooth projective curve and let $C' = f(C)$. Then $E|_{C'}$ is ample, thus $f^*(E|_{C'})$ is ample on $C$. By Hartshorne’s ampleness criterion on $C$, we are done. $\square$
Corollary 1. Let $X \xrightarrow{\pi} Y$ be a finite morphism of projective varieties, $E$ be a bundle on $Y$ that is ample when restricted to curves. Then $\pi^*(E)$ is also ample when restricted to curves.

Proof. Let $C \xrightarrow{f} X$ be a finite morphism, where $C$ is a smooth projective curve. Let $f^*\pi^*(E) \to Q \to 0$ be a quotient bundle. Now $\pi \circ f : C \to Y$ is finite and $\deg(Q) > 0$ by ampleness of $f^* \circ \pi^*(E)$. Thus $\pi^*(E)$ is ample by above lemma.

Remark 3: Let $X$ be an n-dimensional projective variety. $X$ admits a finite surjective morphism, say $f : X \to \mathbb{P}^n$. If $E$ is a vector bundle on $\mathbb{P}^n$ which is ample on curves, so is $f^*(E)$ by the above corollary. Moreover, by Prop 6.1.8 (iii) in [6], $E$ is ample iff $f^*(E)$ is. Thus, if there exists a vector bundle on $\mathbb{P}^n$ for some $n$, which is ample on curves, but not ample, that would negatively answer the question of [1].

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