Encoding and decoding algorithms for unlabeled trees

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Abstract. Trees considered in this article, in which at least two receivers emanate from each internal vertex, are found in the works by E. Schroder, R. Stanley, O.V. Kuzmin, etc. Encoding and decoding algorithms for unlabeled planar rooted trees with a given number of end vertices, root receivers, and a sequence of degrees of internal tree vertices during depth-first search are built. Encoding is done in non-decreasing tuples, by assigning labels to the internal vertices of the tree. The proposed algorithms made it possible to prove the existence of a one-to-one correspondence between the studied set of trees and the set of non-decreasing tuples. To find the cardinality of a set of tuples, we used an approach based on the generalized Pascal pyramid.

1. Introduction

Today, the importance of tree research can hardly be overestimated. They are essential in modern approaches to data analysis. Specifying a weight function on a set of edges or vertices of a tree allows you to build various probabilistic models and apply them in decision-making problems. Particularly in data analysis, trees are used in the construction of clustering algorithms, and the tree counting according to different criteria makes it possible to construct various clustering quality criteria. In programming, trees are used to construct algorithms for various graph traversals, which are used in the construction of logistic models, the analysis of social networks, etc. Trees are used as a model for describing data structures in the theory of information systems, in partitioning and classification problems, in coding theory for constructing optimal codes, in biological problems related to evolutionary trees, in genetics, etc.

The concept of a tree as a formal mathematical object goes back to Kirchhoff and von Staudt [1]. The ideas of connection of combinatorial sequences with trees and their coding are found in the works by of many mathematicians [2, 3, 4, 5]. There are many works that deal with tree counting. For example, when counting planar trees, two parameters are mainly taken into account - the degrees of vertices and the number of components of forest connectivity. For them, explicit formulas were obtained that generate functions and various interpretations [1, 6]. Rooted trees were studied in the works by of F. Harari and E. M. Palmer [7]. They found generating functions for counting different specific types of trees. I.P. Goulden and D.M. Jackson [8] considered the enumeration of labeled rooted trees using enumerative lemmas related to exponential generating functions. The results obtained were used to enumerate functions, permutations, cycles, etc. The method of branching stochastic processes, used in the study of
graphs that are trees with labeled vertices, is considered in the books by V.F. Kolchin [9, 10], for other types of trees - in the works by of Yu. L. Pavlov [11]. The monograph by G.I. Kalmykov [12] describes a method for classifying connected labeled graphs (tree classification) and a method for studying formal power series based on it. A number of authors [1, 13], when considering the problem of calculating lattice paths in a plane, under certain constraints, obtained a large number of combinatorial identities for trees of different heights, etc. These results allow obtaining new generating functions and enumerative interpretations [14]. Work [13] proposes an approach to counting trees based on the generalized Pascal pyramid. Work [13] suggests a method for classifying planar rooted trees according to various criteria: root order, number of internal vertices, height, which was used to introduce new combinatorial numbers and enumerative interpretations. In [1], the formula for the number of flat forests with given degrees of vertices was used to prove the Lagrange formula for the expansion of a function in a series.

In contrast to [1], in this paper we consider enumeration of the subset of planar unlabeled trees in terms of the number of end vertices, root receivers, and the sequence of orders of the internal tree vertices under depth-first search. This result assumes further generalization by introducing a weight function on the set of vertices of the tree and is an important step in the construction of an explicit formula for combinatorial polynomials inverse to composition polynomials, following the ideas presented in [15, 16, 17] for partition polynomials that are used to describe various probabilistic diagrams of processes in technology, natural science, medicine, etc. [18, 19, 20].

Section 2 contains the basic definitions. Section 3 discusses the issues of calculating the cardinality of a set of non-decreasing tuples. Section 4 proposes the algorithms for encoding and decoding the considered set of trees. Encoding is done in non-decreasing tuples, by assigning labels to the internal vertices of the tree. A theorem on the existence of a one-to-one correspondence between the studied set of trees and the set of non-decreasing tuples is proved.

2. Basic concepts

A tuple of length $n$ is an ordered set of non-negative integers $(i_1, ..., i_n)$ [14]. Let $a_1, ..., a_n \in N, a_1 \leq ... \leq a_n$. Let us call $I(a_1, ..., a_n) = \{(i_1, ..., i_n) | i_j \leq a_j, i_j \leq i_{j+1}, 1 \leq j \leq n\}$, the set $I$ is called the set of non-decreasing tuples. $X(a_1, ..., a_n) = |I(a_1, ..., a_n)|$.

Partition of a natural number $n$ is a set of natural numbers in the sum of $n$, where the order of the terms is unimportant. If $n = \sum_{i=1}^{n} i r_i$, then the sequence $(r_1, r_2, ..., r_n)$ is called partition type [21].

Composition of a natural number $n$ into $k$ natural terms is a set of natural numbers $(n_1, n_2, ..., n_k)$ in the sum of components $n$, where the order of the terms is important [13]. Further in this work, we agree to assume that the terms equal to 1 do not affect the order of the terms in the composition, and we will always write them as last ones in the composition.

The rooted tree can be defined recursively. The rooted tree $d$ is such a set of vertices that: one specially selected vertex is called the root of the tree $d$, the remaining vertices (excluding the root) are split into $m \geq 0$ non-intersecting nonempty sets, each of which is a tree [13]. Vertices that do not have receivers are called end vertices. The vertices that have receivers are called internal vertices. In this paper we consider planar trees [1], that is, subtrees at any vertex are linearly ordered.

The structure of a tree will, when it is convenient, be reflected in the form of a way of placing parentheses on the set of its end vertices [1].

Sometimes, when it is convenient, we will represent a tree $T$ in the form of a graph with no closed paths, then $T = < V, R >$, where $V$ is the set of vertices, $R$ is the set of edges. Since
the order of the internal vertices of the trees under consideration is important, we will order the set $V$, additionally stipulating this. Let $n \geq 2$, $2 \leq k \leq n$. We call $D(n)$ the set of unlabeled planar rooted trees with exactly $n$ end vertices that have at least two vertices emanating from each internal vertex (and root); $D(n, k)$ be the the set of rooted unlabeled rooted trees that have exactly $n$ end vertices and $k$ root receivers that have at least two vertices emanating from them. Work [15] introduced the notation for similar sets of labeled trees $D(n)$ and $D(n, k)$. Obviously, the number of internal and end vertices for the sets $D(n, k)$ and $D(n, k), D(n)$ and $D(n)$ coincide. Therefore, we will adhere to the notation introduced in [15]:

$v(n, k)$ is the number of vertices in the tree $d$, excluding the root, $w(n, k)$ is the number of internal vertices in the tree $d$, excluding the root.

3. Non-decreasing tuples

Consider the problem of finding $X(a_1, ..., a_n)$.

**Statement 1.** For any $k, n \in N$, such that $k \leq n$, the following formula holds:

$$X(k, n) = \frac{(2n - k + 1)k}{2}. \quad (1)$$

*Proof.* Let $(i_1, i_2)$ be a tuple of length 2, $a_1 = k$, $a_2 = n$. The number of tuples for which $i_1 = 1$ equals to $n$, the number of tuples for which $i_1 = 2$ equals to $n - 1$ and so on, the number of tuples for which $i_1 = k$ equals to $(n - k + 1)$. Then $X(k, n) = n + (n - 1) + ... + (n + 1 - k) = \frac{(2n - k + 1)k}{2}$. The statement is proved.

**Statement 2.** Let $n \in N, n \geq 2$. The following recurrence relation holds:

$$X(a_1, ..., a_n) = \sum_{i=0}^{a_1-1} X(a_2 - i, ..., a_n - i). \quad (2)$$

*Proof.* We split the whole set $I$ into subsets $I_1, ..., I_{a_1}$ so that $I_i \cap I_j$ for $i \neq j$ and $\bigcup I_i = I$. In the set $I_k$ we include all tuples for which $i_1 = k, 1 \leq k \leq a_1$. Then it is sufficient to prove that $|I_k| = X(a_2 - i, ..., a_n - i), 0 \leq i \leq a_1 - 1$. If the first component of the tuple is equal to 1, we obtain that $|I_1| = X(a_2, ..., a_n)$. Further, since each next term is not less than the previous one and the number of options for each next $i_j, 2 \leq i \leq n$ decreases by one as $i_1$ increases by 1, we have: $|I_2| = X(a_2 - 1, ..., a_n - 1), |I_{a_1}| = X(a_2 - (a_1 - 1), ..., a_n - (a_1 - 1))$. The statement is proved.

**Theorem 1.** Let $n \in N, n \geq 4$. The following recurrence relation holds:

$$X(a_1, ..., a_n) = \sum_{i=0}^{a_n-2} f_{n-1,i} X(a_{n-1} - i, a_n - i), \quad (3)$$

where $f_{3,i} = i + 1, 0 \leq i \leq a_1 - 1$, $f_{3,i} = a_1, 1 \leq i \leq a_2 - 1$, $f_{n,k} = f_{n-1,k} + f_{n,k-1}$ for $0 \leq k \leq a_{n-2} - 1$, $f_{n,k} = f_{n,a_{n-2} - 1}$ for $a_{n-2} \leq k \leq a_{n-1} - 1$.

*Proof.* We will successively apply formula (2) until all the terms on the right-hand side correspond to the cardinality of some sets of tuples of length 2. We apply formula (2) to each term on the right-hand side of formula (2) and we get:

$$X(a_1, ..., a_n) = X(a_3, ..., a_n) + 2X(a_3 - 1, ..., a_n - 1) + 3X(a_3 - 2, ..., a_n - 2) + ... + a_1X(a_3 - a_1 + 1, ..., a_n - a_1 + 1) + ... + a_1X(a_3 - a_2 + 1, ..., a_n - a_2 + 1). \quad (4)$$

Further, acting in a similar way, we apply (2) to each term in (4).

We set $f_{m,k}$ as the coefficient with $X(a_m - k, ..., a_n - k)$ in expansions, $3 \leq m \leq n - 1$. From
(4) it follows that \( f_{3,0} = 1, f_{3,1} = 2, f_{3,2} = 3, \ldots, f_{3,a_1-1} = a_1, \ldots, f_{3,a_2-1} = a_1 \). Let \( 3 < m \leq n-1 \), expansion (4) is equal to:

\[
X(a_1, \ldots, a_n) = \sum_{i=0}^{a_2-1} f_{3,i} X(a_3 - i, \ldots, a_n - i) = \sum_{i=0}^{a_3-1} f_{4,i} X(a_4 - i, \ldots, a_n - i) = \ldots = \sum_{i=0}^{a_{m-1}-1} f_{m,i} X(a_m - i, \ldots, a_n - i) = \ldots = \sum_{i=0}^{a_{n-2}-1} f_{n-1,i} X(a_{n-1} - i, a_n - i).
\]

Note that according to (2) we have

\[
X(a_l - i, \ldots, a_n - i) - X(a_l - (i+1), \ldots, a_n - (i+1)) = X(a_{l+1} - i, \ldots, a_n - i),
\]

where \( 2 \leq l \leq n-1, 0 \leq i \leq a_l - 1 \).

Consider the expansion of each term in the sum \( \sum_{i=0}^{a_{m-2}-1} f_{m-1,i} X(a_{m-1} - i, \ldots, a_n - i) \), we write out the terms according to formula (2):

\[
X(a_{m-1}, \ldots, a_n) = X(a_m, \ldots, a_n) + X(a_m - 1, \ldots, a_n - 1) + \ldots + X(a_m - a_m - 1 + 1, \ldots, a_n - a_m - 1 + 1),
\]

(6)

\[
X(a_{m-1} - 1, \ldots, a_n - 1) = X(a_m - 1, \ldots, a_n - 1) + X(a_m - 2, \ldots, a_n - 2) + \ldots + X(a_m - a_m - 1 + 1, \ldots, a_n - a_m - 1 + 1),
\]

(7)

\[
X(a_{m-1} - 2, \ldots, a_n - 2) = X(a_m - 2, \ldots, a_n - 2) + X(a_m - 3, \ldots, a_n - 3) + \ldots + X(a_m - a_m - 1 + 1, \ldots, a_n - a_m - 1 + 1),
\]

(8)

\[
X(a_{m-1} - a_m - 2 + 1, \ldots, a_n - a_m - 2 + 1) = X(a_m - a_m - 2 + 1, \ldots, a_n - a_m - 2 + 1) + \ldots + X(a_m - a_m - 1 + 1, \ldots, a_n - a_m - 1 + 1).
\]

(9)

Then we substitute them into the sum and find the coefficients for them. Let us prove by induction on \( k \) that the coefficients \( f_{m,k} \) in the expansions are defined as follows: \( f_{m,k} = f_{m-1,k} + f_{m,k-1} \) for \( 0 \leq k \leq a_m - 2 - 1 \), \( f_{m,k} = f_{m,a_m - 2 - 1} \) for \( a_m - 2 \leq k \leq a_m - 1 - 1 \), \( f_{m,0} = 1 \).

Find \( f_{m,0} \). \( 3 \leq m \leq n - 1 \). According to (5) for \( i = 0 \), (6)-(9) the term \( X(a_{m-1}, \ldots, a_n) \) occurs only in (6) in the expansion \( X(a_{m-1}, \ldots, a_n) \) and once, at that. Hence \( f_{m,0} = f_{m-1,0} \). Arguing similarly, we come to the conclusion that \( f_{m,0} = \ldots = f_{3,0} \), hence \( f_{m,0} = 1 \).

Let \( k = 1 \). From (6)-(9) it follows that the term \( X(a_m - 1, \ldots, a_n - 1) \) occurs only in the expansions \( X(a_{m-1}, \ldots, a_n) \) and \( X(a_{m-1} - 1, \ldots, a_n - 1) \), therefore \( f_{m,1} = f_{m-1,0} + f_{m-1,1} = f_{m,0} + f_{m-1,1} \).

Let the statement be true for any natural number not exceeding \( k \). Let us prove correctness for \( k + 1 \). Let \( 0 \leq k + 1 \leq a_m - 2 - 1 \), case \( a_m - 2 \leq k + 1 \leq a_m - 1 - 1 \) we consider separately. From (6)-(9) it follows that the term \( X(a_m - (k+1), \ldots, a_n - (k+1)) \) occurs in the expansions \( X(a_{m-1}, \ldots, a_n), \ldots, X(a_{m-1} - (k+1), \ldots, a_n - (k+1)) \). It means that \( f_{m,k+1} = f_{m-1,0} + \ldots + f_{m-1,k} + f_{m-1,k+1} \). Taking into consideration \( f_{m,0} = f_{m-1,0} = 1 \) and induction assumptions we have: \( f_{m,k+1} = f_{m,0} + f_{m-1,1} + \ldots + f_{m-1,k} + f_{m-1,k+1} = f_{m,1} + f_{m-1,2} + \ldots + f_{m-1,k} + f_{m-1,k+1} = f_{m,2} + f_{m-1,3} + \ldots + f_{m-1,k} + f_{m-1,k+1} = \ldots = f_{m,k} + f_{m-1,k+1} \).

Now let \( a_m - 2 \leq k + 1 \leq a_m - 1 \). In (9), the expansion \( X(a_{m-1} - a_m - 2 + 1, \ldots, a_n - a_m - 2 + 1) \) includes only those terms that are in each of the previous expansions (6)-(8). All of them are included into these expansions one at a time, so the coefficients for them are the same and are equal to the coefficient for \( X(a_m - a_m - 2 + 1, \ldots, a_n - a_m - 2 + 1) \). Thus \( f_{m,k+1} = f_{m,a_m - 2 - 1} \) for \( a_m - 2 \leq k + 1 \leq a_m - 1 - 1 \). The theorem is proved.
4. Trees

Since the order of the internal vertices is important in the trees under consideration, in order to fix it, we will adhere to the well-known method of traversing the tree according to rule 1.

**Rule 1:** we will traverse the tree according to depth-first order [1], i.e. in depth, starting from the root, traversing from left to right.

Let us call the tree type \( d \in \bar{D}(n, k) \) the sequence \((n_1, n_2, ..., n_{w(n,k)})\) of degrees of internal vertices of the tree, excluding the root when traversing the tree according to rule 1. Let us denote \( D(n, k, n_1, n_2, ..., n_{w(n,k)}) \) as the set of trees \( d \in \bar{D}(n, k) \) of type \((n_1, n_2, ..., n_{w(n,k)})\).

Let us describe the way of encoding trees by non-decreasing tuples.

**Algorithm 1 (tree encoding)**

**Algorithm input:** tree diagram \( d \in \bar{D}(n, k) \).

**Algorithm output:** tree code \( a_1, ..., a_{w(n,k)} \).

Traverse the tree according to rule 1. Let \( v_1, ..., v_{w(n,k)} \) be the sequence of traversing its internal vertices (excluding the root).

For all internal vertices \( v_i \), \( 1 \leq i \leq w(n,k) \) we perform:
- count \( c(v_i) \) — the number of end vertices traversed to the internal vertex \( v_i \),
- encode the vertex \( v_i \): \( a_i := c(v_i) + 1 \).

Note that Algorithm 1 implies that the tree code is a non-decreasing tuple of length \( w(n,k) \).

To describe the decoding algorithm, we define a tree in the form of a graph \(< V, R >\), where \( V \) is a set of vertices, \( R \) is a set of edges. The set \( V \) is considered ordered in accordance with the order of traversing the tree vertices according to rule 1.

**Algorithm 2 (tree decoding)**

**Algorithm input:** tree code \( a_1, ..., a_{w(n,k)} \), tree type \((n_1, ..., n_{w(n,k)})\).

**Algorithm output:** tree \( T \in \bar{D}(n, k) \), \( T = < V, R > \).

Let \( V = \{v_0\}, R \) be the empty set.

Build all \( k \) root receivers: add vertices \( v_1, ..., v_k \) in \( V \), edges \((v_0, v_1), ..., (v_0, v_k)\) — in \( R \).

For \( 1 \leq i \leq w(n,k) \) we perform:
- traversing the built tree according to rule 1, we count \( a_i \) end vertices,
- build \( n_i \) receivers at \( a_i \)-th vertex:
  - add elements \( v_{k+(n_1+...+n_{i-1})+1}, ..., v_{k+(n_1+...+n_i)+1} \) after \( a_i \)-th vertex in \( V \),
  - rename \( a_i \)-th vertex as \( w_{a_i} \),
  - add elements \( (w_{a_1}, v_{k+(n_1+...+n_{i-1})+1}), ..., (w_{a_k}, v_{k+(n_1+...+n_i)+1}) \) in \( R \).

We introduce the notation for the set of all possible values of the \( i \)-th label of trees from the set under consideration: \( A_i = \{a_i : d \in \bar{D}(n,k,n_1,n_2,...,n_{w(n,k)})\}, 1 \leq i \leq w(n,k) \).

Further, to identify the belonging of tree types (and, accordingly, compositions) to which the sets of labels of internal vertices belong, additional indices will be introduced.

Let \( 1 \leq i \leq n - k - 1, 1 \leq j \leq n - k - i \). Let us introduce the following notation: if \( i^n_j = (i^{n_1}_j, i^{n_2}_j, ..., i^{n_{n-k}}_j) \) is some composition, then \( i_A^1, ..., i_A^{n-k} \) are sets of labels of trees of type \( (i^{n_1}_1, i^{n_2}_2, ..., i^{n_{n-k}}_{n-k}) \), where \( i_A^t \) is the set of possible labels \( t \)-th vertex of the tree \( 1 \leq t \leq n - k \). Let us describe an algorithm for constructing codes for trees of the entire set \( \bar{D}(n,k) \), which allows us to build the entire set of trees (using codes according to Algorithm 2) and reduce the problem of calculating the cardinality of the set \( \bar{D}(n,k) \) to the problem of calculating the cardinality of a set of non-decreasing tuples.

**Algorithm 3 (encoding the set \( \bar{D}(n,k) \))**

**Algorithm input:** \( n, k \).

**Algorithm output:** all types of \( i^n_j, 0 \leq i \leq n - k, 0 \leq j \leq n - k - i \) trees from the set \( \bar{D}(n,k) \), sets of labels \( i_A^1, i_A^2, ..., i_A^{n-k} \).

Let \( ^0n_1 = ... = ^0n_{n-k} = 2, ^0A_p = \{1, ..., k+p-1\}, 1 \leq p \leq n-k \).
For $i$ from 1 to $(n - k - 1)$

$$i_{n_1}^0 = \ldots = i_{n_{k-i-2}}^0 = 2, \quad i_{n_{k-i}}^0 = i = 2, \quad i_{n_{k-i+1}}^0 = \ldots = i_{n-k}^0 = 1,$$

$$i_{A_{n,i}}^0 = \{1, 2, \ldots, k\}, \quad i_{A_{n,k+1}}^0 = \{1, 2, \ldots, k + 1\}, \quad \ldots, \quad i_{A_{n,k-i}}^0 = \{1, 2, \ldots, n - i - 1\}.$$

While $n_1 \neq i + 2$ we perform

For $j$ from 1 to $(n - k - i)$

$$i_{n_j}^1 := i_{n_{k-i-j-1}}^1 = \ldots = i_{n_{k-i-j-2}}^1 = i_{n_{k-i-j-1}}^1 = 1, \quad i_{n_{k-i-j-2}}^1 = \ldots = i_{n_{k-i-j-1}}^1 = 1.$$
Let us build the set \( 0^1 A^0, 2 \leq i \leq n-k \). If the \( i \)-th vertex occurs as the first root receiver, then its label is 1, the maximum value is \( k+i-1 \) and corresponds to the way of placing parentheses \((u_1, u_2, ..., (u_i(u_{i-1}), ...))\). Hence \( 0^1 A^0 = \{1, ..., k + i - 1\}\).

Now let us consider compositions

\[
(2, ..., 2, 3, 1), (2, ..., 2, 4, 1, 1), ..., (n-k, k, 1, 1, ...), (n-k+1, 1, 1, ...). \tag{10}
\]

In (10) all compositions \( (i n^0_A, i n^0_A, ..., i n^0_A) \), \( 1 \leq i \leq n-k-1 \) are obtained by the formulas:

\[
i n^0_A = \ldots = i n^0_A, i n^0_A = i + 1, i n^0_A = \ldots = i n^0_A = 1.
\]

According to [15], the number of internal vertices in a tree of type \((2, ..., 2, i + 1, ..., 1)\) is equal to \( n-k-i \). Let us describe a method for constructing sets of labels for trees of type \((2, 2, 2, 3, 1)\) from \((2, 2, 2, ..., 2)\). The number of internal vertices is \( n-k-1 \). Let us build the sets \( 0^1 A^1, 1 \leq i \leq n-k-1 \). The degrees of the first \( n-k-2 \) vertices in these types coincide, so, according to Algorithm 1, the sets of labels \( 0^1 A^1 = 0^1 A^0, 1 \leq i \leq n-k-2 \). The last \( n-k-1 \) vertex is of degree 3, but since it is the last, according to rule 2, its degree does not affect its value and the value of the other labels and \( 1^1 A^1 = 0^1 A^1 \Rightarrow 1 \).

Let \( i-3 A^1, 1 \leq j \leq n-k-i+3 \) be the sets of labels for trees of type \((2, ..., 2, i-1, 1, ..., 1)\), we construct sets of labels for trees of type \((2, ..., 2, i, 1, ..., 1)\). The degrees of the first \( n-k-i+1 \) vertices coincide, so according to the coding rule for the set of labels \( i-3 A^1 = i-3 A^0, 1 \leq j \leq n-k-i+1 \) also coincide. The last \( n-k+i-2 \) vertex is of degree \( i \), but since it is the last, according to the encoding rule, its degree does not affect the value of its label and \( i-2 A^0 = \ldots = i-3 A^0 \Rightarrow 1 \). Thus, it is proved that the sets of labels are given by the following formulas \( 0 \leq i \leq n-k-1 \):

\[
i A^1 = \{1, 2, ..., k\}, i A^2 = \{1, 2, ..., k + 1\}, ..., i A^0 = \{1, 2, ..., n-i-1\}.
\]

Now let the tree have \( i \) internal vertices. Initial compositions are considered compositions of the form \((2, 2, 2, i, 1, 1, 1)\). Let us build all the other compositions from them. We will add 1 from left to right to the next term until we get the composition \((i, 2, 2, ..., 2, 1, 1, 1)\). This construction corresponds to the recurrent formulas \( 1 \leq j \leq n-k-i \):

\[
i n^1_A := i n^0_A, i n^1_A := i n^0_A, i n^2_A := i n^0_A + 1,
\]

\[
i n^1_A := i n^0_A, i n^1_A := i n^0_A - 1, i n^2_A := i n^0_A - 2.
\]

Let us describe the way of building a set of labels from the initial composition for all subsequent ones. For \((2, 2, 2, i, 1, 1, 1)\) sets \( i-2 A^0, 1 \leq j \leq n-k-i+2 \) have already been built and \( i-2 A^0 = \{1, 2, ..., n-i+1\} \). Let us increase the \( (n-k-i+1) \)-th term by one and decrease the \( (n-k-i+2) \)-th term by one. Thus we get a new composition \((2, 2, 2, 3, 1, i-1, 1, ..., 1)\), the set of labels of which will remain the same except for \( i A^1 \), to which element \( (n-k-i+1) \) will be added yet. According to the coding rule, the described change will increase the maximum degree of the penultimate vertex by one when traversing according to rule 1, and all other degrees of vertices and, accordingly, their set of labels will remain unchanged. Then \( i-2 A^1 = i-2 A^0 \) and \( i-2 A^1 = i-2 A^0 U \{i A^0, i A^0, n-k-i+2 \} \).

Let for composition \((i n^1_A, i n^1_A, i n^1_A, i n^1_A)\) the sets of labels \( 0^1 A^1, 0 A^2, 0 A^3 \) are built. Let us increase \( (n-k-i-j) \)-th term by 1 and decrease \( (n-k-i-j+1) \)-th term by 1, thus we obtain a new composition \((0 n^1_A, 0 n^1_A, 0 n^1_A, 0 n^1_A)\), the set of labels of which will remain the same except for \( i A^1 \), which another element \( (n-k-i+1) \) will be added yet. Then
\[ i_A^j m = i A_{m-1}^j \quad \text{and} \quad i_A^j n_{k-i-j+1} = i A_{n-k-i-j+1}^{j-1} \bigcup \{|i_A^j n_{k-i-j+1}| + 1\}. \]

Thus, we have proved a way of building a set of labels for \( \hat{D}(n, k) \), which corresponds to Algorithm 3. The theorem is proved.

**Note.** A one-to-one correspondence between the sets \( \hat{D} \) and \( I \) takes place only for the given parameters \((n, k, n_1, ... n_{nk})\). So even within the same partition, but in different compositions, there are different trees with the same code.

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**References**

[1] Stanley R P 1999 *Enumerative Combinatorics, vol. 2* (New York, Cambridge: Cambridge Univ. Press)
[2] Cayley A 1889 A theorem on trees *Quart. J. Math.* 23 376-8
[3] Neville E H 1953 The codifying of tree structure *Proc. Camb. Phil. Soc.* 49 381-5
[4] Prüfer H 1918 Neuer Beweis eines Satzesüber Permutationen *Arch. Math. Phys.* 27 142-4
[5] Moon J W 1967 Various proofs of Cayley’s formula for counting trees (A Seminar on Graph Theory) ed F Harary (New York: Holt, Rinehart, and Winston) pp 70-8
[6] Stanley R P 1978 Exponential structures *Studies in Applied Math.* 59 73-82
[7] Harari F and Palmer E M 1973 *Graphical Enumeration* (New York, London: Academic Press)
[8] Goulden I P and Jackson D M 1983 *Combinatorial Enumeration* (New Jersey: John Wiley Sons, Inc.)
[9] Kolchin V F 2010 *Random Graphs* (Cambridge: Cambridge Univ. Press)
[10] Kolchin V F 1986 *Random Mappings* (New York: Optimization Software Publications Division, Inc.)
[11] Pavlov Y L 1993 Some properties of plane trees with a hanging root *Discrete Math. Appl.* 3 (1) 97–102
[12] Kalmykov O I 2003 Tree Classification of Labeled Graphs (Moscow: Fizmatlit)
[13] Kuzmin O V 2000 *Generalized Pascal Pyramids and their Applications* (Novosibirsk: Nauka Publ.)
[14] Kuzmin O V 1994 Recurrence relations and enumerative interpretations of some combinatorial numbers and polynomials *Discrete Math. Appl.* 4 (4) 329-39.
[15] Balagura A A and Kuzmin O V 2011 The enumerative properties of combinatorial partition polynomials *Diskretn. Anal. Issled. Oper.* 18 (1) 3–14
[16] Balagura A A and Kuzmin O V 2008 Enumerative interpretation of homogeneous Platonov polynomials *Surveys on Applied and Industrial Mathematics* 15 (4) 735-8
[17] Balagura A A and Kuzmin O V 2007 Generalized Pascal pyramids and their reciprocals *Discrete Math. Appl.* 17 (6) 619-28
[18] Kuzmin O V, Balagura A A, Kuzmina V V and Khudonogov I A 2019 Partially ordered sets and combinatorial objects of the pyramidal structure *Advances and Applications in Discrete Mathematics* 20 (2) 229-42
[19] Kuzmin O V, Khomenko A P and Artyunin A I 2018 Discrete model of static loads distribution management on lattice structures *Advances and Applications in Discrete Mathematics* 19 (3) 183-93
[20] Kuzmin O V and Atalyan A V 2019 Decision Trees in the Problems of Diagnostics and Forecasting (Applied Discrete Analysis Problems vol 5) ed O Kuzmin (Irkutsk: Irkutsk Univ. Press) pp 64-80
[21] Andrews G E 1976 *The Theory of Partitions* (London: Addison-Wesley Publishing Company)