ON THE PROBLEM OF ELECTROMAGNETIC-FIELD QUANTIZATION

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Abstract. We consider the radiation field operators in a cavity with varying dielectric medium in terms of solutions of Heisenberg’s equations of motion for the most general one-dimensional quadratic Hamiltonian. Explicit solutions of these equations are obtained and applications to the radiation field quantization, including randomly varying media, are briefly discussed.

1. Canonical Quantization

Radiation field quantization in the vacuum was introduced in original works of Born, Heisenberg and Jordan [10], [11] (see also books on quantum electrodynamics [1], [3], [6] and quantum optics [46], [63], [82], [84], [93]). A modern mathematical approach to quantization of mechanical systems is discussed in detail, for example, in [9], [20], [89], and/or [90] (see also [45] and the references therein). For a classical Hamiltonian system one replaces canonically conjugate coordinates and momenta by time-dependent operators $q_\lambda(t)$ and $p_\lambda(t)$ that satisfy the commutation rules
\[
[q_\lambda(t), q_\mu(t)] = [p_\lambda(t), p_\mu(t)] = 0, \quad [q_\lambda(t), p_\mu(t)] = i\hbar\delta_{\lambda\mu}.
\]
(1.1)
The time-evolution is determined by the Heisenberg equations of motion [35]:
\[
\frac{d}{dt}p_\lambda(t) = \frac{i}{\hbar} [p_\lambda(t), \mathcal{H}], \quad \frac{d}{dt}q_\lambda(t) = \frac{i}{\hbar} [q_\lambda(t), \mathcal{H}],
\]
(1.2)
with appropriate initial conditions.

Traditionally, the electromagnetic-field quantization is considered under the assumption that the field occupies an empty box [1], [3], [20], [27]. The quantization of the field in a uniform dielectric medium was discussed in [29], [37], [39], [40], [41], [47] (see also [20], [24], [88] and the references therein). Yet the problem of electromagnetic-field quantization in time-dependent nonuniform linear nondispersive media remains an active research topic up to now [2], [7], [12], [13], [20], [30], [36], [38], [55], [58], [64], [74], [76], [77], [88], [92].

In the present letter, we study the radiation field operators in a varying dispersive medium, which is mathematically described by the most general phenomenological quadratic Hamiltonian $\mathcal{H}$ in an abstract Hilbert space. (We concentrate on a single photon cavity mode, say $\nu$, with frequency $\omega_\nu = 1$ and use the units $c = \hbar = 1$. From now on, we shall usually omit the indices when dealing with the single mode under consideration.) In particular, our approach gives a natural description of squeezed photons that can be created as a result of parametric amplification of...
quantum fluctuations in the dynamic Casimir effect \cite{17, 18, 51, 67, 69, 83, 94} and/or by similar dynamical amplification mechanisms including the Unruh effect \cite{91} and Hawking radiation \cite{8, 33, 34}. It is also useful for quantum fields propagating in nonstationary external potentials \cite{7, 60, 92, 83}, and for photon quantization in randomly varying media.

2. Solution of Heisenberg Equations for Nonautonomous Quadratic Systems

Our main result is the following.

Theorem 1. The solution of the Heisenberg equations of motion (1.2) for the nonautonomous quadratic Hamiltonian

$$H = a(t)p^2 + b(t)x^2 + c(t)xp - id(t) - f(t)x - g(t)p$$

(a, b, c, d, f, and g are suitable real-valued functions of time only) has the form

$$p_\lambda(t) = \frac{\hat{b}(t) - \hat{b}^\dagger(t)}{i\sqrt{2}}, \quad q_\lambda(t) = \frac{\hat{b}(t) + \hat{b}^\dagger(t)}{\sqrt{2}}.$$  \hspace{1cm} (2.2)

Here, the time-dependent annihilation $\hat{b}(t)$ and creation $\hat{b}^\dagger(t)$ operators are given by the Ansatz

$$\hat{b}(t) = e^{-2i\gamma(t)} \left( \frac{\beta(t) x + \varepsilon(t) + i\frac{p - 2\alpha(t) x - \delta(t)}{\beta(t)}}{\sqrt{2}} \right),$$

$$\hat{b}^\dagger(t) = e^{2i\gamma(t)} \left( \frac{\beta(t) x + \varepsilon(t) - i\frac{p - 2\alpha(t) x - \delta(t)}{\beta(t)}}{\sqrt{2}} \right)$$  \hspace{1cm} (2.3)

in terms of solutions of the Ermakov-type system

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = a\beta^2,$$  \hspace{1cm} (2.4)

$$\frac{d\beta}{dt} + (c + 4a\alpha) \beta = 0,$$  \hspace{1cm} (2.5)

$$\frac{d\gamma}{dt} + a\beta^2 = 0$$  \hspace{1cm} (2.6)

and

$$\frac{d\delta}{dt} + (c + 4a\alpha) \delta - f - 2g\alpha = 2a\beta^3 \varepsilon,$$  \hspace{1cm} (2.7)

$$\frac{d\varepsilon}{dt} - (g - 2a\delta) \beta = 0,$$  \hspace{1cm} (2.8)

$$\frac{d\kappa}{dt} - g\delta + a\alpha^2 = a\beta^2 \varepsilon^2.$$  \hspace{1cm} (2.9)

The time-independent (self-adjoint) operators $x$ and $p$ obey the canonical commutation rule $[x, p] = i$ in an abstract (complex) Hilbert space which implies that the relation

$$\hat{b}(t)\hat{b}^\dagger(t) - \hat{b}^\dagger(t)\hat{b}(t) = 1$$  \hspace{1cm} (2.10)

holds at all times.
Proof. These results can be verified by a direct, but somewhat tedious, calculation when one expands the solution in generators of the Heisenberg–Weyl algebra, namely \{1, x, p\}, with undetermined time-dependent complex coefficients and simplifies the commutators. The substitutions (2.2)–(2.3) allow us to derive equations (2.4)–(2.8), say from the first Heisenberg equation (2.11) below. (A Mathematica based proof is available on the article’s website; see notebook HeisenbergOscillators.nb for an important program ingredient.) Equation (2.9), which determines the global phase of the corresponding Fock states in the Schrödinger representation, does not show up in this proof, but will appear later (see Lemma 2).

By back substitution, we see that the operators \( \hat{b}(t) \) and \( \hat{b}^\dagger(t) \) are solutions of the Heisenberg equations

\[
\frac{d}{dt} \hat{b}(t) = i \left[ \hat{b}(t), H \right], \quad \frac{d}{dt} \hat{b}^\dagger(t) = i \left[ \hat{b}^\dagger(t), H \right],
\]

subject to the initial conditions

\[
\hat{b}(0) = e^{-2i\gamma(0)} \sqrt{2} \left( \beta(0)x + \epsilon(0) + ip - 2\alpha(0)x - \delta(0) \right), \quad \hat{b}^\dagger(0) = e^{2i\gamma(0)} \sqrt{2} \left( \beta(0)x + \epsilon(0) - ip - 2\alpha(0)x - \delta(0) \right).
\]

To a certain extent, the creation and annihilation operators (2.3) allow us to incorporate the Schrödinger symmetry group of the harmonic oscillator, originally found in coordinate representation [70], [71], into a more abstract Heisenberg picture. (For the sake of simplicity, we have restricted ourselves to a single photon mode \( \nu \) with frequency \( \omega_\nu = 1 \); see [49] for a detailed investigation of the special case of uniform media.)

A concept of dynamical invariants for generalized harmonic oscillators, which is crucial for constructing the corresponding Fock states from our creation and annihilation operators, has been recently revisited in [15], [81], and [86] (see [21], [23], [56], [65], [66] and the references therein for classical accounts).

3. Solving The Ermakov-Type System

A general solution of (2.4)–(2.9) is given by Lemma 3 of [54] in a real form (see also [48] and [61]). In order to proceed to a more compact form, one needs to recall some notation. The substitution

\[
\alpha = \frac{1}{4a} \frac{\mu'}{\mu} - \frac{d}{2a}
\]

reduces the inhomogeneous equation (2.4) to the second order ordinary differential equation

\[
\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = c_0(2a)^2\beta^4\mu,
\]

which has the familiar time-varying coefficients

\[
\tau(t) = \frac{a'}{a} - 2c + 4d, \quad \sigma(t) = ab - cd + d^2 + \frac{d}{2} \left( \frac{a'}{a} - \frac{d'}{d} \right).
\]

(In (3.2) and in the rest of the paper, we use a formal ‘binary’ parameter \( c_0 = 0,1 \) for the sake of convenience.)

\[^2\text{See also Koutschan.nb [48].}\]
The time-dependent coefficients $\alpha_0$, $\beta_0$, $\gamma_0$, $\delta_0$, $\varepsilon_0$, $\kappa_0$, which satisfy the homogeneous (Riccati-type) system \((2.4)-(2.9)\), are given by (cf. \[14\], \[54\], \[86\])

$$\alpha_0(t) = \frac{1}{4a(t)\mu_0(t)} - \frac{d(t)}{2a(t)}, \quad (3.4)$$

$$\beta_0(t) = -\frac{\lambda(t)}{\mu_0(t)}, \quad \lambda(t) = \exp\left(-\int_0^t (c(s) - 2d(s)) \, ds\right), \quad (3.5)$$

$$\gamma_0(t) = \frac{1}{2\mu_1(0)\mu_0(t)} + \frac{d(0)}{2a(0)}, \quad (3.6)$$

and

$$\delta_0(t) = \frac{\lambda(t)}{\mu_0(t)} \int_0^t \left[ \left(f(s) - \frac{d(s)}{a(s)}g(s)\right)\mu_0(s) + \frac{g(s)}{2a(s)}\mu'_0(s) \right] \, ds \lambda(s), \quad (3.7)$$

$$\varepsilon_0(t) = -\frac{2a(t)\lambda(t)}{\mu'_0(t)}\delta_0(t) + 8 \int_0^t \frac{a(s)\sigma(s)\lambda(s)}{\mu'_0(s)} (\mu_0(s)\delta_0(s)) \, ds$$

$$+ 2 \int_0^t \frac{a(s)\lambda(s)}{\mu'_0(s)} \left(f(s) - \frac{d(s)}{a(s)}g(s)\right) \, ds, \quad (3.8)$$

$$\kappa_0(t) = \frac{a(t)\mu_0(t)}{\mu'_0(t)}\delta_0^2(t) - 4 \int_0^t \frac{a(s)\sigma(s)}{\mu'_0(s)} (\mu_0(s)\delta_0(s))^2 \, ds - 2 \int_0^t \frac{a(s)}{\mu'_0(s)} (\mu_0(s)\delta_0(s)) \left(f(s) - \frac{d(s)}{a(s)}g(s)\right) \, ds, \quad (3.9)$$

($$\delta_0(0) = -\varepsilon_0(0) = g(0)/(2a(0))$$ and $\kappa_0(0) = 0$), provided that $\mu_0$ and $\mu_1$ are the standard (real-valued) solutions of equation \[(3.2)\] when $c_0 = 0$ corresponding to the initial conditions $\mu_0(0) = 0$, $\mu'_0(0) = 2a(0) \neq 0$ and $\mu_1(0) \neq 0$, $\mu'_1(0) = 0$. (Proofs of these facts are outlined in \[14\] and \[16\]. The integrals are treated in the most general way, which may include stochastic calculus; see, for example, \[28\], \[73\] and the references therein.)

Here, we would like to present a new compact form of these solutions. Let us introduce the complex-valued function

$$z(t) = \left(2a(0) + \frac{d(0)}{a(0)}\right)\mu_0(t) + \frac{\mu_1(t)}{\mu_1(0)} + i\beta^2(0)\mu_0(t) \quad (3.10)$$

(a complex parametrization of Green’s function and linear invariants of generalized harmonic oscillators are also discussed in \[22\], \[23\] and \[31\]). Then

$$z(t) = c_1 E(t) + c_2 E^*(t), \quad (3.11)$$

where the complex-valued solutions are given by

$$E(t) = \frac{\mu_1(t)}{\mu_1(0)} + i\mu_0(t), \quad E^*(t) = \frac{\mu_1(t)}{\mu_1(0)} - i\mu_0(t), \quad (3.12)$$

and the corresponding complex-valued parameters are defined by

$$c_1 = \frac{1 + \beta^2(0)}{2} - i \left(\alpha(0) + \frac{d(0)}{2a(0)}\right), \quad c_2 = \frac{1 - \beta^2(0)}{2} + i \left(\alpha(0) + \frac{d(0)}{2a(0)}\right), \quad (3.13)$$

with

$$c_1 + c_2 = 1, \quad |c_1|^2 - |c_2|^2 = c_1 - c_2^* = \beta^2(0). \quad (3.14)$$
In addition,
\[ z(0) = c_1 + c_2 = 1, \quad z'(0) = 2ia(0)(c_1 - c_2). \] (3.15)

When written in terms of the complex function \( z \) in (3.10), the complex conjugate functions \( E \) and \( E^* \) defined in (3.12) become
\[
E = \frac{c_1^* z - c_2 z^*}{|c_1|^2 - |c_2|^2}, \quad E^* = \frac{c_1 z^* - c_2^* z}{|c_1|^2 - |c_2|^2}
\] (3.16)
and
\[
\mu_0 = \frac{z - z^*}{2i (c_1 - c_2)^2}, \quad \mu_1(0) = \frac{(c_1 - c_2^*) z + (c_1 - c_2) z^*}{2(c_1 - c_2)}.
\] (3.17)

One can readily verify that
\[
\alpha_0 = \frac{1}{4a} \frac{(z - z^*)'}{z - z^*} + \frac{d}{2a}, \quad \beta_0 = -2i\lambda \frac{c_1 - c_2^*}{z - z^*}, \\
\gamma_0 = \frac{(c_1^* - c_2^*) z + (c_1 - c_2) z^*}{2i(z - z^*)} + \frac{d(0)}{2a(0)}.
\] (3.18)
and equations (3.7)–(3.9) can also be rewritten in terms of the function \( z \) in view of (3.11). Finally, we introduce a second complex function,
\[
\zeta(t) = \varepsilon(0)\beta(0) + i(\delta(0) + \varepsilon_0(t)) = c_3 + i\varepsilon_0, \quad c_3 = \varepsilon(0)\beta(0) + i\delta(0),
\] (3.19)
and indicate the inverse relations between the essential, real and complex, parameters:
\[
\alpha(0) = \frac{c_1^* - c_1}{2i} - \frac{d(0)}{2a(0)}, \quad \beta^2(0) = c_1 - c_2^* = |c_1|^2 - |c_2|^2,
\] (3.20)
and
\[
\delta(0) = \frac{c_3 - c_3^*}{2i}, \quad \varepsilon(0) = \pm \frac{c_3 + c_3^*}{2\sqrt{|c_1|^2 - |c_2|^2}}.
\] (3.21)

Then the solution of the initial value problem for the Ermakov-type system can be expressed in terms of the complex function \( z \) in (3.10) as given in the lemma below.

**Lemma 1.** The system (2.1)–(2.9) is solved by
\[
\alpha = \alpha_0 + \lambda^2 \frac{c_1^* - c_2^*}{2i|z|^2} \frac{z + z^*}{z - z^*}, \quad \beta = \pm \lambda \frac{\sqrt{|c_1|^2 - |c_2|^2}}{|z|}, \quad \gamma = \gamma(0) - \frac{1}{2} \arg z,
\] (3.22)
\[
\delta = \delta_0 + \frac{\zeta z - \zeta^* z^*}{2i|z|^2}, \quad \varepsilon = \pm \frac{\zeta z + \zeta^* z^*}{2|z|\sqrt{|c_1|^2 - |c_2|^2}},
\] (3.23)
\[
\kappa = \kappa(0) + \frac{\zeta^2 z + \zeta^* z^*}{2|z|\sqrt{|c_1|^2 - |c_2|^2}} (z - z^*),
\] (3.24)
with \( z \) and \( \zeta \) as in (3.10) and (3.19), respectively. (The solution of the Ermakov-type equation (3.12) is given by \( \mu = \mu(0)|z| \).)

**Proof.** This amounts to a straightforward calculation using Lemma 3 in [54]. \( \square \)
As a consequence, one gets
\[
\frac{2i(\alpha - \alpha_0)}{\beta^2} = \frac{z + z^*}{z - z^*},
\]

(3.25)
\[
i(\alpha - \alpha_0) + \frac{\beta^2}{2} = \beta^2 \frac{z}{z - z^*}, \quad i(\alpha - \alpha_0) - \frac{\beta^2}{2} = \beta^2 \frac{z^*}{z - z^*},
\]

(3.26)
\[
\varepsilon + i\frac{\delta - \delta_0}{\beta} = \frac{\zeta z}{\beta(0)|z|}, \quad \varepsilon - i\frac{\delta - \delta_0}{\beta} = \frac{\zeta^* z^*}{\beta(0)|z|},
\]

(3.27)
\[
\varepsilon^2 + \left(\frac{\delta - \delta_0}{\beta}\right)^2 = \varepsilon^2(0) + \left(\frac{\delta(0) + \varepsilon_0}{\beta(0)}\right)^2,
\]

(3.28)
and
\[
\kappa = \kappa(0) + \frac{\delta - \delta_0}{2\beta} \varepsilon - \frac{\varepsilon_0 + \delta(0)}{2\beta(0)} \varepsilon(0).
\]

(3.29)

These “quasi-invariants” can be useful, for example, when making a comparison of calculations done by different approximation methods.

Examples of explicitly integrable quadratic systems are discussed in [15], [16], [22], [23], [56], [57], [61], [62], [65], and [95] (see also the references therein).

4. SINGLE MODE FOCK STATES FOR NONAUTONOMOUS QUADRATIC HAMILTONIANS

The time-dependent quadratic operator (see [81])
\[
\hat{E}(t) = \frac{1}{2} \left[ \frac{(p - 2\alpha x - \delta)^2}{\beta^2} + (\beta x + \varepsilon)^2 \right] = \frac{1}{2} \left[ \hat{a}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{a}(t) \right],
\]

(4.1)
with the defining property
\[
i\frac{d\hat{E}}{dt} + \hat{E}H - H\hat{E} = 0,
\]

(4.2)
extends the standard Hamiltonian (and/or number operator) for any given solution of the Ermakov-type system. (The initial data play a role of integrals of motion and/or quantum numbers for the creation and annihilation operators in the Heisenberg representation.) We use the substitution
\[
\hat{b}(t) = e^{-2i\gamma}\hat{a}(t) \quad \text{and} \quad \hat{b}^\dagger(t) = \hat{a}^\dagger(t)e^{2i\gamma} \quad \text{with} \quad [\hat{a}(t), \hat{a}^\dagger(t)] = 1.
\]
The oscillator-type spectrum,
\[
\hat{E}(t) |\Psi_n(t)\rangle = \left( n + \frac{1}{2} \right) |\Psi_n(t)\rangle,
\]

(4.3)
can be obtained in a standard fashion (with the aid of modified variable creation and annihilation operators; cf. [1], [9]):
\[
\hat{a}(t) |\Psi_n(t)\rangle = \sqrt{n} \ |\Psi_{n-1}(t)\rangle, \quad \hat{a}^\dagger(t) |\Psi_n(t)\rangle = \sqrt{n+1} \ |\Psi_{n+1}(t)\rangle.
\]

(4.4)
Here and in what follows, it is convenient to use the orthogonality relation \( \langle \Psi_m(t), \Psi_n(t) \rangle = \delta_{mn} \lambda^{-1} \) with \( \beta(0)\mu(0) = 1 \).

Now we can analyze abstract Fock states in the Schrödinger representation.
Lemma 2. Let $\hat{a}(t) |\Psi_0(t)\rangle = 0$. The dynamic Fock states given by

$$ |\psi_n(t)\rangle = e^{i(2n+1)\gamma + i\kappa} |\Psi_n(t)\rangle = \frac{e^{i(2n+1)\gamma + i\kappa}}{\sqrt{n!}} (\hat{a}^\dagger(t))^n |\Psi_0(t)\rangle $$

satisfy the time-dependent Schrödinger equation

$$ i\frac{d}{dt} |\psi_n(t)\rangle = H |\psi_n(t)\rangle $$

with the general quadratic Hamiltonian \((2.1)\) provided that equation \((2.9)\) for the global phase holds and \(\langle \Phi_n, d\Phi_n/dt \rangle = 0\) for \(\Phi_n = \lambda^{1/2} e^{-i(ax^2+\delta x)} \Psi_n\).

Proof. From \((4.2)\)–\((4.3)\), one gets, formally,

$$ \hat{E} \left( i\frac{d}{dt} |\psi_n(t)\rangle - H |\psi_n(t)\rangle \right) = \left( n + \frac{1}{2} \right) \left( i\frac{d}{dt} |\psi_n(t)\rangle - H |\psi_n(t)\rangle \right). $$

Therefore

$$ i\frac{d}{dt} |\psi_n(t)\rangle - H |\psi_n(t)\rangle = c_n |\psi_n(t)\rangle, $$

where \(\text{Im} \ c_n = 0\) in view of the normalization condition; see \([15], [56]\). (We assume also that the vacuum state is nondegenerate.)

Here,

$$ H = ap^2 + bx^2 + \frac{c}{2} (px + xp) + \frac{i}{2} (c - 2d) - fx - gp, $$

and the position and linear momentum operators are given by

$$ x = \frac{1}{\beta} \left[ \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger) - \varepsilon \right], $$

$$ p = \frac{\beta}{i\sqrt{2}} (\hat{a} - \hat{a}^\dagger) + \frac{\sqrt{2}\alpha}{\beta} (\hat{a} + \hat{a}^\dagger) + \frac{2\alpha\varepsilon}{\beta}. $$

In terms of creation and annihilation operators, the Hamiltonian takes the form

$$ H = \left[ \frac{a}{2} \left( \frac{4\alpha^2}{\beta^2} - \beta^2 \right) + \frac{b + 2c\alpha}{2\beta^2} - \frac{i}{2} (c + 4a\alpha) \right] (\hat{a})^2 $$

$$ + \left[ \frac{a}{2} \left( \frac{4\alpha^2}{\beta^2} - \beta^2 \right) + \frac{b + 2c\alpha}{2\beta^2} + \frac{i}{2} (c + 4a\alpha) \right] (\hat{a}^\dagger)^2 $$

$$ + \frac{1}{2} \left[ a \left( \beta^2 + \frac{4\alpha^2}{\beta^2} \right) + \frac{b + 2c\alpha}{\beta^2} \right] (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) + \frac{i}{2} (c - 2d) $$

$$ + \sqrt{2} \left[ \frac{4a^2 + c}{2\beta} \left( \delta - \frac{2\alpha\varepsilon}{\beta^2} \right) - \frac{\varepsilon}{\beta^2} (b + c\alpha) - \frac{f + 2g\alpha}{2\beta} $$

$$ + \frac{i}{2} (\beta (g - 2a\delta) + \varepsilon (c + 4a\alpha)) \right] \hat{a} $$

$$ + \sqrt{2} \left[ \frac{4a^2 + c}{2\beta} \left( \delta - \frac{2\alpha\varepsilon}{\beta^2} \right) - \frac{\varepsilon}{\beta^2} (b + c\alpha) - \frac{f + 2g\alpha}{2\beta} $$

$$ - \frac{i}{2} (\beta (g - 2a\delta) + \varepsilon (c + 4a\alpha)) \right] \hat{a}^\dagger. $$
Remark 1 for any given solution of the Ermakov-type system (2.4)–(2.9). This completes the proof.

Finally, in view of (4.8), (4.13) and (4.18), we obtain (after some simplification) that

\[ a \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right)^2 + \frac{\varepsilon}{\beta} \left( f + \frac{b\varepsilon}{\beta} \right) - \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right) \left( g + \frac{c\varepsilon}{\beta} \right), \]

where we have corrected a typo in [81]. As a result, by (4.4)–(4.3) we have

\[ \lambda \text{Re} \langle \Psi_n, H\Psi_n \rangle = \left( n + \frac{1}{2} \right) \left[ a \left( \beta^2 + \frac{4\alpha^2}{\beta^2} \right) + \frac{b + 2\alpha}{\beta} \right] \]
\[ + a \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right)^2 + \frac{\varepsilon}{\beta} \left( f + \frac{b\varepsilon}{\beta} \right) - \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right) \left( g + \frac{c\varepsilon}{\beta} \right) \]

(4.13)
in terms of solutions of the Ermakov-type system.

In order to complete the proof, one can repeat the evaluation of Berry’s phase [4] for generalized harmonic oscillators, given in [81] in coordinate representation, in a more abstract form. Indeed,

\[ \langle \psi_n, H\psi_n \rangle = \left( n + \frac{1}{2} \right) \left[ a \left( \beta^2 + \frac{4\alpha^2}{\beta^2} \right) + \frac{b + 2\alpha}{\beta} \right] + \lambda \langle \Psi_n, H\Psi_n \rangle. \]

(4.14)

In general,

\[ \Psi_n = e^{(1/2) f(c-2d) dt} e^{i(\alpha x^2 + \delta x)} \Phi_n, \quad \langle \Phi_n, \Phi_n \rangle = \delta_{mn}, \]

(4.15)
and

\[ \lambda \langle \Psi_n, d\Psi_n/dt \rangle = i \langle \Phi_n, \left( \frac{d\alpha}{dt} x^2 + \frac{d\delta}{dt} x \right) \Phi_n \rangle + \frac{1}{2} (c - 2d) + \langle \Phi_n, \frac{d\Phi_n}{dt} \rangle \]
\[ = i \frac{d\alpha}{dt} \lambda \langle \Psi_n, x^2 \Psi_n \rangle + i \frac{d\delta}{dt} \lambda \langle \Psi_n, x\Psi_n \rangle + \frac{1}{2} (c - 2d), \]

(4.16)
where \( \langle \Phi_n, d\Phi_n/dt \rangle = 0 \) by our hypothesis. Here,

\[ \lambda \langle \Psi_n, x\Psi_n \rangle = -\varepsilon \beta^{-1}, \quad \lambda \langle \Psi_n, x^2 \Psi_n \rangle = \beta^{-2} \left( \varepsilon^2 + n + \frac{1}{2} \right). \]

(4.17)
As a result,

\[ \lambda \text{Re} \left( i \langle \Psi_n, d\Psi_n/dt \rangle \right) = -\beta^{-2} \left( \varepsilon^2 + n + \frac{1}{2} \right) \frac{d\alpha}{dt} + \varepsilon \beta^{-1} \frac{d\delta}{dt}. \]

(4.18)
Finally, in view of (4.8), (4.13) and (4.18), we obtain (after some simplification) that

\[ c_n = \langle \Phi_n, i d\Phi_n/dt \rangle = 0 \]
for any given solution of the Ermakov-type system (2.4)–(2.9). This completes the proof. □

Remark 1. In coordinate representation, when \( \Phi_n \) is, essentially, the real-valued stationary orthonormal wave function for the simple harmonic oscillator with respect to the new variable \( \xi = \beta x + \varepsilon \) (see [52], [54], [72], and [81] for more details), the equation \( \langle \Phi_n, d\Phi_n/dt \rangle = 0 \) is valid due to the normalization condition \( \|\Phi_n\|^2 = 1 \). In general, one gets \( c_n = \langle \Phi_n, i d\Phi_n/dt \rangle \), and the previous connection is associated with a transport law for line bundles in the Hilbert space, namely, the change \( d\Phi_n \) is orthogonal to \( \Phi_n \) [85].

The last lemma can be reformulated in terms of an analog of Berry’s phase [4],

\[ \frac{d\theta_n}{dt} = \lambda \text{Re} \langle \psi_n, i d\psi_n/dt \rangle - \frac{d\varphi_n}{dt}, \]

(4.19)
where \( \varphi_n(t) = -(2n + 1)\gamma(t) \) is the dynamical phase [81] (see also [87] for an example).
Lemma 3. The Fock states, given by (4.5) in terms of solutions of the Ermakov-type system, satisfy the Schrödinger equation (4.6) if and only if the derivative of the Berry phase is evaluated by two equivalent expressions (40) and (48) in [81].

5. Expectation Values and Variances

By (4.10)–(4.11) and (4.4), one gets

\[
\langle \psi_n(t), x \psi_n(t) \rangle = -\frac{\lambda \varepsilon}{\beta}, \quad \bar{x} = \frac{\langle x \rangle}{\langle 1 \rangle} = \frac{\langle \psi_n, x \psi_n \rangle}{\langle \psi_n, \psi_n \rangle} = -\frac{\varepsilon(t)}{\beta(t)},
\]

(5.1)

\[
\langle \psi_n(t), p \psi_n(t) \rangle = \lambda \left( \delta - \frac{2\alpha \varepsilon}{\beta} \right), \quad \bar{p} = \frac{\langle p \rangle}{\langle 1 \rangle} = \frac{\langle \psi_n, p \psi_n \rangle}{\langle \psi_n, \psi_n \rangle} = \delta(t) - \frac{2\alpha(t) \varepsilon(t)}{\beta(t)}
\]

(5.2)

are given by

\[
\langle \Delta p \rangle^2 = \frac{\langle (\Delta p)^2 \rangle}{\langle 1 \rangle} = (\bar{p})^2 - (\bar{p})^2, \quad \langle \Delta x \rangle^2 = \frac{\langle (\Delta x)^2 \rangle}{\langle 1 \rangle} = (\bar{x})^2 - (\bar{x})^2,
\]

(5.3)

in terms of solutions of the Ermakov-type system. In particular,

\[
\langle \Delta p \rangle^2 \langle \Delta x \rangle^2 = \left( n + \frac{1}{2} \right) \left( \beta^2 + \frac{4\alpha^2}{\beta^2} \right), \quad \langle \Delta x \rangle^2 = \left( n + \frac{1}{2} \right) \beta^{-2}
\]

(5.4)

as required by the fundamental Heisenberg uncertainty relation [7], [15], [35]. The minimum-uncertainty squeezed states occur for \( n = 0 \) if \( \alpha(t_{\text{min}}) = 0 \).

6. Electromagnetic-Field Quantization in Varying Media

In the macroscopic approach, one can present the (noncommuting) vector field operators of the electric displacement \( D(r, t) \) and the magnetic induction \( B(r, t) \), which fully describe the properties of the quantized electromagnetic radiation inside a cavity filled with linear nonstationary dielectric material (with factorized electric permittivity and magnetic permeability tensors [20]), by the eigenfunction expansions (cf. [7], [25], [35], [42], [75], [82])

\[
D(r, t) = \sum_v \chi_v(t) p_v(t) D_v(r),
\]

\[
B(r, t) = \sum_v \omega_v(t) q_v(t) B_v(r),
\]

(6.1)

in the Heisenberg picture when the time evolution is introduced through the equations (2.2)–(2.3), which provides a more direct analogy between quantum and classical physics [7], [32]. For a discussion of properties of the stationary orthonormal eigenfunctions \( D_v(r) \) and \( B_v(r) \) defined by the geometry of the cavity and given boundary conditions, see [2], [7], [12], [13], [19], [20], [25], [29], [47], [58], [64], [74], [75], and [78].
In view of the phenomenological Maxwell equations, the single electromagnetic radiation mode $\nu$ in a cavity resonator is analogous to a parametric driven harmonic oscillator [13], [20], [27], [64], [74], [75]. This analogy between classical mechanics and electrodynamics allows one to determine functions $\chi_{\nu}(t)$ and $\varpi_{\nu}(t)$ from the electric permittivity, magnetic permeability, and conductivity of the (slowly-)varying medium in connection with the quadratic Hamiltonian (2.1) (see Appendix A for more details). After quantization of the field Hamiltonian, the time-dependent operators $p_{\nu}(t)$ and $q_{\nu}(t)$ are determined by Theorem 1, and the corresponding Fock states are constructed in Lemma 2. (The average fields obey the classical Maxwell equations [7].) Methods of stochastic calculus [73] can be used in the case of randomly varying media.

In Schrödinger’s picture, for the diagonal matrix elements of the field oscillators, we get

$$
\langle D(r, t) \rangle = D_{\nu}(r)\chi_{\nu}(t) \langle \psi_n(t), p\psi_n(t) \rangle,
$$

$$
\langle B(r, t) \rangle = B_{\nu}(r)\varpi_{\nu}(t) \langle \psi_n(t), x\psi_n(t) \rangle,
$$

with (5.1)–(5.2) for a single mode $\nu$, and the corresponding variances can be obtained with the help of (5.3). In the autonomous case (see [49]), the variances are given (up to a normalization) by equations (A.4)–(A.5) of [62].

7. Summary and Applications

In this letter, the radiation field operators in a cavity with varying dielectric medium are constructed in terms of explicit solutions of Heisenberg’s equations of motion. The phenomenological quadratic Hamiltonian under consideration corresponds to the most general (single mode) one-dimensional mathematical model of quantization in an abstract Hilbert space. Nonstandard solutions of these equations are obtained and applications to the radiation field quantization, including randomly varying media, are briefly discussed.

For most applications in (nonlinear) optics, the electromagnetic field can be treated classically. But, when quantum limits are approached and one is interested in the photon statistics of the field, a quantum description is required (see [24], [29], [36], [37] and the references therein). Explicit form of the Bogoliubov transformation (2.3) (for nonautonomous quadratic systems) is one of the starting points in this approach [7], [20], [74], [94]. Interaction of multiple modes and microscopic (lossy) medium models are also under consideration [39], [44], [47], [55], [58], [59], [77], [83], [88].

The problem of quantization of electromagnetic field in material media remains important in view of recent trends in the flourishing cavity QED [17], [19], [44], [69], [77] and for experiments in quantum optics in media [7], [29], [30], [55], [78], [92], which may result in a better understanding of the interaction of light with matter. Among other possible applications of the electromagnetic wave propagation in time-dependent media are the modulation of microwave power [68], wave propagation in ionized plasma [50], and magnetoelastic delay lines [79], [80] (see also [13] and the references therein).

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Appendix: Factorized Media

The phenomenological Maxwell equations in linear, passive, nondispersive, time-varying dielectric and magnetic media without sources, namely
\begin{align}
\text{curl } E &= -\frac{1}{c} \frac{\partial B}{\partial t}, \\
\text{div } D &= 0, \\
\text{curl } H &= \frac{1}{c} \frac{\partial D}{\partial t}, \\
\text{div } B &= 0,
\end{align}
(A.1)
\begin{align}
D &= \tilde{\varepsilon}(r, t) E, \\
B &= \tilde{\mu}(r, t) H,
\end{align}
(A.2)
with the help of the vector potential
\begin{align}
B &= \text{curl } A, \\
E &= -\frac{1}{c} \frac{\partial A}{\partial t},
\end{align}
(A.3)
and imposed gauge conditions
\begin{align}
\text{div } \left( \tilde{\varepsilon} \frac{\partial A}{\partial t} \right) &= 0, \\
\phi &= 0,
\end{align}
(A.4)
can be reduced to the single second-order generalized wave equation
\begin{align}
\text{curl } \left( \tilde{\mu}^{-1} \text{curl } A \right) + \frac{1}{c^2} \frac{\partial}{\partial t} \left( \tilde{\varepsilon} \frac{\partial A}{\partial t} \right) &= 0.
\end{align}
(A.5)
Here, we recall the case of factorized (real-valued) dielectric permittivity and magnetic permeability (tensors),
\begin{align}
\tilde{\varepsilon}(r, t) &= \xi(t) \varepsilon(r), \\
\tilde{\mu}(r, t) &= \eta(t) \mu(r),
\end{align}
(A.6)
considered in [20].

The solution of the classical problem for a given single mode, say \( \upsilon \), has the form \( A(r, t) = u(r) q(t) \) and
\begin{align}
B &= q \text{ curl } u, \\
D &= -\frac{\xi}{c} \frac{dq}{dt} \varepsilon u,
\end{align}
(A.7)
provided that
\begin{align}
\text{curl } \left( \frac{1}{\tilde{\mu}} \text{curl } u \right) &= \upsilon^2 \varepsilon u, \\
\text{div } (\varepsilon u) &= 0, \\
\frac{d^2 q}{dt^2} + \frac{\xi'}{\xi} \frac{dq}{dt} + \frac{c^2 \upsilon^2}{\xi \eta} q &= 0, \\
\upsilon &= \text{constant},
\end{align}
(A.8)
and certain required boundary conditions are satisfied on the boundary of the cavity (see [20], [25], [29], [17], [52] for more details).

Thus we can choose \( a = -\xi, \ b = -c^2 \upsilon^2 / (4 \xi^2 \eta) \) and \( c = d = f = g = 0 \) in Theorem 1 for the quantization of the mode of the electromagnetic field under consideration. (See also [2], [13], [64], [74], and [75].)
References

[1] A. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics*, Interscience Publishers, New York, 1965.

[2] X.-M. Bei and Z.-Z. Liu, *Quantum radiation in time-dependent dielectric media*, J. Phys. B: At. Mol. Opt. Phys. 44 (2011), 205501 (7pp).

[3] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Relativistic Quantum Theory*, Pergamon Press, Oxford, 1971.

[4] M. V. Berry, *Quantal phase factors accompanying adiabatic changes*, Proc. Roy. Soc. London, A392 (1984) #1802, 45–57.

[5] M. V. Berry and M. R. Dennis, *Quantum cores of optical phase singularities*, J. Opt. A: Pure Appl. Opt. 6 (2004), S178–S180.

[6] I. Białynicki-Birula and Z. Białynicki-Birula, *Quantum Electrodynamics*, Pergamon Press Ltd. and PWN–Polish Scientific Publishers, Oxford, New York, Toronto, Sydney, Warszawa, 1975.

[7] Z. Białynicka-Birula and I. Białynicki-Birula, *Space-time description of squeezing*, J. Opt. Soc. Am. B 4 (1987) #10, 1621–1626.

[8] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge, 1982.

[9] F. A. Berezin and M. A. Shubin, *The Schrödinger Equation*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1991.

[10] M. Born and P. Jordan, *Zur Quantenmechanik*, Z. f. Physik 34 (1925), 858–889.

[11] M. Born, W. Heisenberg, and P. Jordan, *Zur Quantenmechanik II*, Z. f. Physik 35 (1925), 557–615.

[12] J. R. Choi, *Dissipative blackbody radiation: radiation in a lossy cavity*, Int. J. Mod. Phys. B 12 (2004) #3, 317–324.

[13] J. R. Choi, *Interpreting quantum states of electromagnetic field in time-dependent linear media*, Phys. Rev. A 82 (2010), 055803 (4 pages).

[14] R. Cordero-Soto, R. M. López, E. Suazo, and S. K. Suslov, *Propagator of a charged particle with a spin in uniform magnetic and perpendicular electric fields*, Lett. Math. Phys. 84 (2008) #2–3, 159–178.

[15] R. Cordero-Soto, E. Suazo, and S. K. Suslov, *Quantum integrals of motion for variable quadratic Hamiltonians*, Ann. Phys. 325 (2010) #9, 1884–1912.

[16] R. Cordero-Soto and S. K. Suslov, *The degenerate parametric oscillator and Ince’s equation*, J. Phys. A: Math. Theor. 44 (2011) #1, 015101 (9 pages); see also arXiv:1006.3362v3 [math-ph] 2 Jul 2010.

[17] V. V. Dodonov, *Photon distribution in the dynamical Casimir effect with an account of dissipation*, Phys. Rev. A 80 (2009), 023814 (11 pages).

[18] V. V. Dodonov, *Current status of dynamical Casimir effect*, Physica Scripta 82 (2010) #3, 038105 (10 pp).

[19] A. V. Dodonov and V. V. Dodonov, *Resonance generation of photons from vacuum in cavities due to strong periodical changes of conductivity in a thin semiconductor boundary layer*, J. Opt. B: Quantum Semiclass. Opt. 7 (2005) S47–S58.

[20] V. V. Dodonov, A. B. Klimov, and D. E. Nikonov, *Quantum phenomena in nonstationary media*, Phys. Rev. A 47 (1993) #5, 4422–4429.

[21] V. V. Dodonov, I. A. Malkin, and V. I. Man’ko, *Integrals of motion, Green functions, and coherent states of dynamical systems*, Int. J. Theor. Phys. 14 (1975) #1, 37–54.

[22] V. V. Dodonov and V. I. Man’ko, *Coherent states and the resonance of a quantum damped oscillator*, Phys. Rev. A 20 (1979) #2, 550–560.

[23] V. V. Dodonov and V. I. Man’ko, *Invariants and correlated states of nonstationary quantum systems*, in: *Invariants and the Evolution of Nonstationary Quantum Systems*, Proceedings of Lebedev Physics Institute, vol. 183, pp. 71-181, Nauka, Moscow, 1987 [in Russian]; English translation published by Nova Science, Commack, New York, 1989, pp. 103-261.

[24] P. D. Drummond, *Electromagnetic quantization in dispersive inhomogeneous nonlinear dielectrics*, Phys. Rev. A 42 (1990) #11, 6845–6857.

[25] S. M. Dutra, *Cavity Quantum Electrodynamics: The Strange Theory of Light in a Box*, Hoboken, NJ, USA: Wiley, 2005.

[26] L. D. Faddeev, *Feynman integrals for singular Lagrangians*, Theoret. and Math. Phys. 1 (1969) #1, 1–13.

[27] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw–Hill, New York, 1965.
[28] J-P. Fouque, J. Garnier, G. Papanicolaou, and K. Solna, Wave Propagation and Time Reversal in Randomly Layered Media, Springer–Verlag, New York, 2007.
[29] R. J. Glauber and M. Lewenstein, Quantum optics of dielectric media, Phys. Rev. A 43 (1991) # 1, 467–491.
[30] T. Gruner and D.-G. Welsch, Quantum-optical input-output relations for dispersive and lossy multilayer dielectric plates, Phys. Rev. A 54 (1996) # 2, 1661–1677.
[31] G. Harari, Ya. Ben-Aryeh, and Ady Mann, Propagator for the general time-dependent harmonic oscillator with application to an ion trap, Phys. Rev. A 84 (2011) # 6, 062104 (4 pages).
[32] S. Haroche and J.-M. Raimond, Exploring the Quantum: Atoms, Cavities, and Photons, Oxford University Press, Oxford, 2006.
[33] S. W. Hawking, Black hole explosions?, Nature, London 248 (1974), 30–31.
[34] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. 43 (1975) #3, 199–220.
[35] W. Heisenberg, The Physical Principles of the Quantum Theory, University of Chicago Press, Chicago, 1930; Dover, New York, 1949.
[36] M. Hillery, An introduction to the quantum theory of nonlinear optics, Acta Physica Slovaca 59 (2009) # 1, 1–80.
[37] M. Hillery and L. D. Mlodinow, Quantization of electrodynamics in nonlinear dielectric media, Phys. Rev. A 30 (1984) # 4, 1860–1865.
[38] S. A. R. Horsley, Canonical quantization of the electromagnetic field interacting with a moving dielectric media, Phys. Rev. A 30 (2012), 023830 (12 pages).
[39] B. Huttner and S. M. Barnett, Quantization of the electromagnetic field in dielectrics, Phys. Rev. A 46 (1992) # 7, 4306–4322.
[40] J. M. Jauch and K. M. Watson, Phenomenological quantum-electrodynamics, Phys. Rev. 74 (1948) # 8, 950–957.
[41] J. M. Jauch and K. M. Watson, Phenomenological quantum electrodynamics. Part II. Interaction of the field with charges, Phys. Rev. 74 (1948) # 10, 1485–1493.
[42] E. T. Jaynes and F. W. Cummings, Comparison of quantum and semiclassical radiation theories with application to the beam maser, Proc. IEEE. 51 (1963) #1, 89–109.
[43] A. Kasman, Glimpses of Soliton Theory: The Algebra and Geometry of Nonlinear PDEs, American Mathematical Society, Student Mathematical Library, Vol. 54, New York, 2010.
[44] M. Khanbekyan, L. Knöll, D.-G. Welsch, A. A. Semenov, and W. Vogel, QED of lossy cavities: operator and quantum-state input-output relations, Phys. Rev. A 72 (2005), 053813 (16 pages).
[45] J. R. Klauder, Enhanced quantization: a primer, J. Phys. A: Math. Theor. 45 (2012), 285304 (8 pages).
[46] J. R. Klauder and E. C. G. Sudarshan, Fundamentals of Quantum Optics, W. A. Benjamin, Inc., New York, Amsterdam, 1968.
[47] L. Knöll, W. Vogel, and D. -G. Welsch, Action of passive, lossless optical systems in quantum optics, Phys. Rev. A 36 (1987) # 8, 3803–3818.
[48] C. Koutschan, http://hahn.la.asu.edu/~suslov/currens/index.htm; see Mathematica notebook: Koutschan.nb; see also http://www.risc.jku.at/people/ckoutsch/pekeris/
[49] S. I. Kryuchkov, S. K. Suslov, and J. M. Vega-Guzmán, The minimum-uncertainty squeezed states for atoms and photons in a cavity, arXiv:1201.0841v5 [quant-ph] 10 Nov 2012.
[50] S. Kozuki, Reflection of electromagnetic wave from a time-varying medium, Electron. Lett. 14 (1978) # 25, 826–828.
[51] P. Lähteenmäki, G. S. Paraoanu, J. Hassel, and P. J. Hakonen, Dynamical Casimir effect in a Josephson metamaterial, arXiv:1111.5008v2 [cond-mat.mes-hall] 1 Dec 2011.
[52] L. D. Landau and E. M. Lifshitz, Quantum Mechanics: Nonrelativistic Theory, Pergamon Press, Oxford, 1977.
[53] N. Lanfear and S. K. Suslov, The Cauchy problem for a forced harmonic oscillator, Revista Mexicana de Física, 55 (2009) #2, 195–215; see also arXiv:0707.1902v8 [math-ph] 27 Dec 2007.
[58] A. A. Lobashev and V. M. Mostepanenko, Quantum effects in nonlinear insulating materials in the presence of a nonstationary electromagnetic field, Theoret. and Math. Phys. 86 (1991) #3, 438–447.

[59] A. A. Lobashev and V. M. Mostepanenko, Quantum effects associated with parametric generation of light and the theory of squeezed states, Theoret. and Math. Phys. 88 (1991) #3, 303–309.

[60] A. A. Lobashev and V. M. Mostepanenko, Heisenberg representation of second-quantized fields in stationary external fields and nonlinear dielectric media, Theoret. and Math. Phys. 97 (1993) #3, 1393–1404.

[61] R. M. López, S. K. Suslov, and J. M. Vega-Guzmán, Reconstructing the Schrödinger groups, Physica Scripta, to appear; see also On the harmonic oscillator group, arXiv:1111.5509v2 [math-ph] 4 Dec 2011.

[62] R. M. López, S. K. Suslov, and J. M. Vega-Guzmán, On a hidden symmetry of harmonic oscillators, arXiv:1112.2586v2 [quant-ph] 3 Jan 2012.

[63] W. H. Louisell, Quantum Statistical Properties of Radiation, Wiley, New York, 1973.

[64] M. Maamache, N. Chaabi, and J. R. Choi, Geometric phase of quantized electromagnetic field in time-dependent linear media, Eur. Phys. Lett. 89 (2010), 40009 (6 pages); Erratum, 90 (2010), 59901 (1 page).

[65] I. A. Malkin and V. I. Man’ko, Dynamical Symmetries and Coherent States of Quantum System, Nauka, Moscow, 1979 [in Russian].

[66] I. A. Malkin, V. I. Man’ko, and D. A. Trifonov, Linear adiabatic invariants and coherent states, J. Math. Phys. 14 (1973) #5, 576–582.

[67] V. I. Man’ko, The Casimir effect and quantum vacuum generator, J. Sov. Laser Res. 12 (1991) #5, 383–385.

[68] F. R. Morgenthaler, Velocity modulation of electromagnetic waves, IRE Transactions on microwave theory and techniques 6 (1958), 167–172.

[69] W. Naylor, Towards particle creation in a microwave cylindrical cavity, Phys. Rev. A 86 (2012), 023842 (9 pages).

[70] U. Niederer, The maximal kinematical invariance group of the free Schrödinger equations, Helv. Phys. Acta 45 (1972), 802–810.

[71] U. Niederer, The maximal kinematical invariance group of the harmonic oscillator, Helv. Phys. Acta 46 (1973), 191–200.

[72] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer–Verlag, Berlin, New York, 1991.

[73] B. Øksendal, Stochastic Differential Equations, Springer–Verlag, Berlin, 2000.

[74] I. A. Pedrosa, Quantum electromagnetic waves in nonstationary linear media, Phys. Rev. A 83 (2011), 032108 (5 pages).

[75] I. A. Pedrosa, Quantum description of electromagnetic waves in time-dependent linear media, J. Phys.: Conf. Ser. 306 (2011), 012074 (8 pages).

[76] T. G. Philbin, Canonical quantization of macroscopic electromagnetism, New J. Phys. 12 (2010) 123008 (21 pages).

[77] C. Raabe, S. Schee, and D.-G. Welsch, Unified approach to QED in arbitrary linear media, Phys. Rev. A 75 (2007), 053813 (22 pages).

[78] A. M. C. Reyes and C. Eberlein, Completeness of evanescent modes in layered dielectrics, Phys. Rev. A 79 (2009), 043834 (7 pages).

[79] S. M. Rezende and F. R. Morgenthaler, Magnetoelastic waves in time-varying magnetic fields. II. Experiments, J. Appl. Phys. 40 (1969) #2, 524–536.

[80] S. M. Rezende and F. R. Morgenthaler, Magnetoelastic waves in time-varying magnetic fields. I. Theory, J. Appl. Phys. 40 (1969) #2, 537–545.

[81] B. Sanborn, S. K. Suslov, and L. Vinet, Dynamic invariants and the Berry phase for generalized driven harmonic oscillators, Journal of Russian Laser Research 32 (2011) #5, 486–494; see also arXiv:1108.5144v1 [math-ph] 25 Aug 2011.

[82] W. P. Schleich, Quantum Optics in Phase Space, Wiley–Vch Publishing Company, Berlin etc, 2001.

[83] R. Schützhold, G. Plunien, and G. Soff, Trembling cavities in the canonical approach, Phys. Rev. A 57 (1998) #4, 2311–2318.

[84] M. O. Scully and M. S. Zubairy, Quantum Optics, Cambridge University Press, Cambridge, 1997.

[85] B. Simon, Holonomy, the quantum adiabatic theorem, and Berry’s phase, Phys. Rev. Lett. 51 (1983) #24, 2167–2170.
[86] S. K. Suslov, *Dynamical invariants for variable quadratic Hamiltonians*, Physica Scripta **81** (2010) #5, 055006 (11 pp); see also arXiv:1002.0144v6 [math-ph] 11 Mar 2010.

[87] S. K. Suslov, *An analog of the Berry phase for simple harmonic oscillators*, Physica Scripta, to appear; see also arXiv:1112.2418v1 [quant-ph] 12 Dec 2011.

[88] L. G. Suttorp and M. Wubs, *Field quantization in inhomogeneous absorptive dielectrics*, Phys. Rev. A **70** (2004), 013816 (18 pages).

[89] L. A. Takhtajan, *Quantum Mechanics for Mathematicians*, Graduate Studies in Mathematics, Vol. 95, American Mathematical Society, Providence, Rhode Island, 2008.

[90] G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, Graduate Studies in Mathematics, Vol. 99, American Mathematical Society, Providence, Rhode Island, 2009.

[91] W. G. Unruh, *Notes on black-hole evaporation*, Phys. Rev. D **14** (1976) #4, 870–892.

[92] D. Vasylyev, W. Vogel, K. Henneberger, and F. Richter, *Propagation of quantized light through bounded dispersive and absorptive media*, Physica Scripta **T140** (2010), 014039 (3pp).

[93] D. F. Walls and G. J. Milburn, *Quantum Optics*, Springer, Berlin, Heidelberg, 2008.

[94] C. M. Wilson, G. Johansson, A. Pourkabirian, M. Simoen, J. R. Johansson, T. Duty, F. Nori, and P. Delsing, *Observation of the dynamical Casimir effect in a superconducting circuit*, Nature **479** (2011) November 17, 376–379.

[95] H. P. Yuen, *Two-photon coherent states of the radiation field*, Phys. Rev. A **13** (1976) #6, 2226–2243.

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