Topology of certain symplectic conifold transitions of $\mathbb{C}P^1$-bundles

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Abstract In this paper, we first extend Smith, Thomas and Yau’s examples of certain symplectic conifold transitions on trivial $\mathbb{C}P^1$-bundles over Kähler surfaces to all $\mathbb{C}P^1$-bundles over symplectic 4-manifolds. Then we determine the diffeomorphism types of all these symplectic conifold transitions. In particular, this implies that in the case of trivial $\mathbb{C}P^1$-bundles over projective complex surfaces, Smith, Thomas and Yau’s examples of symplectic conifold transitions are diffeomorphic to Kähler three-folds.

Keywords Symplectic conifold transitions · $\mathbb{C}P^1$-bundles · Kähler three-folds · Characteristic classes

Mathematics Subject Classification 57R17 · 57R22

1 Introduction

In this paper, all manifolds under consideration are closed, oriented and differentiable, unless otherwise stated. By a $\mathbb{C}P^1$-bundle, we always mean the projectivization $\mathbb{P}(E)$ of a rank two complex vector bundle $E$.

Symplectic conifold transitions introduced by Smith, Thomas and Yau [17] are symplectic surgeries on symplectic 6-manifolds which collapse embedded Lagrangian 3-spheres and replace them by symplectic 2-spheres. It is unknown that if symplectic conifold transitions of Kähler manifolds are still Kähler. This problem was raised in [17] and it was also shown in [16, Proposition 4.3] that this problem is related to a question of Donaldson [5, Question 4].

In trivial $\mathbb{C}P^1$-bundles over Kähler surfaces, Smith, Thomas and Yau [17] gave examples of symplectic conifold transitions along local embedded Lagrangian 3-spheres (see Definition 2.1). They pointed out that it should be possible for these examples to contain non-Kähler
manifolds. In fact, Corti and Smith [4] stated that such symplectic conifold transitions of the trivial $\mathbb{C}P^1$-bundle over certain projective complex surface were not deformation equivalent to any Kähler three-fold. However, Corti and Smith have now withdrawn their paper from the arXiv since a mistake was found in their proof.

In this paper, we will show that in the case of trivial $\mathbb{C}P^1$-bundles over projective complex surfaces, Smith, Thomas and Yau’s examples of symplectic conifold transitions are diffeomorphic to Kähler three-folds. Actually, we have a similar result for more general $\mathbb{C}P^1$-bundles (see Corollary 1.3). This result can be deduced from our main theorem, which determines the diffeomorphism types of symplectic conifold transitions of all $\mathbb{C}P^1$-bundles over symplectic 4-manifolds along local embedded Lagrangian 3-spheres, where the existence of such Lagrangian 3-spheres is given in Lemma 2.2. The main theorem is stated below.

For simplicity, denote $\overline{\mathbb{C}P^2}$ and $S^4$ by $N_k$, $k = 1, 2$, respectively, where $\overline{\mathbb{C}P^n}$ denotes the complex projective space $\mathbb{C}P^n$ with the opposite orientation. For $k = 1, 2$, let $\sigma^*_k \in H^2(N_k; \mathbb{Z})$ such that $\sigma^*_2 = 0$ and $\sigma^*_1$ is the dual class of the preferred generator $\sigma_1 \in H_2(\overline{\mathbb{C}P^2}; \mathbb{Z})$, i.e. $\langle \sigma^*_1, \sigma_1 \rangle = 1$. Denote $[M]$ for the fundamental class of a manifold $M$. Unless otherwise stated, we always choose symplectic forms on $\mathbb{C}P^1$-bundles over symplectic 4-manifolds to be the ones compatible with the fibrations [12, Theorem 6.3]. As there are exactly two distinct conifold transitions along an embedded Lagrangian 3-sphere up to diffeomorphism [17], we can state our main results as follows.

**Theorem 1.1** Let $\mathbb{P}(E)$ be the projectivization of a rank two complex vector bundle $E$ over a symplectic 4-manifold $N$. Then the two symplectic conifold transitions of $\mathbb{P}(E)$ along a local embedded Lagrangian 3-sphere are diffeomorphic to $K\ddot{a}hler$ three-folds. Moreover, the diffeomorphisms above can be chosen to preserve the homotopy classes of almost complex structures.

**Remark 1.2** For a 4-manifold $N$, every pair in $H^2(N; \mathbb{Z}) \times H^4(N; \mathbb{Z})$ can be realized as the Chern classes of a rank two complex vector bundles $E$ over $N$ and the isomorphism classes of the bundles $E_k$ in Theorem 1.1 can be completely determined by $c_1(E_k), c_2(E_k)$ [8, Theorem 1.4.20]. Moreover, it is not hard to prove the manifolds $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)\#\overline{\mathbb{C}P^3}$ are in different diffeomorphism classes by comparing their cohomology rings.

As it is well-known that the projectivization of a holomorphic vector bundle over a Kähler manifold admits a Kähler structure [18, Proposition 3.18], Theorem 1.1 can imply the corollary below.

**Corollary 1.3** Let $\mathbb{P}(E)$ be the projectivization of a rank two holomorphic vector bundle $E$ over a projective complex surface $N$, then the symplectic conifold transitions of $\mathbb{P}(E)$ along a local embedded Lagrangian 3-sphere are diffeomorphic to Kähler three-folds.
Remark 1.4 It is easy to extend Theorem 1.1 and Corollary 1.3 to the case of symplectic conifold transitions of $\mathbb{P}(E)$ along arbitrarily many disjoint local embedded Lagrangian 3-spheres.

We will finish the proof of Theorem 1.1 and Corollary 1.3 in Sect. 3.3. In the course of establishing Theorem 1.1, we also compute the topological invariants of $\mathbb{C}P^1$-bundles over simply-connected 4-manifolds in Example 3.1. According to [19] and [18], combining this computation with Theorem 1.1 will give diffeomorphism classification of symplectic conifold transitions of simply-connected $\mathbb{C}P^1$-bundles along local embedded Lagrangian 3-spheres.

2 Symplectic conifold transitions on $\mathbb{C}P^1$-bundles

We first recall the definition of conifold transitions [17]. Begin with a Lagrangian embedding $f: S^3 \to X$ in a symplectic 6-manifold $X$. By the Lagrangian neighborhood theorem [12, Theorem 3.33], the embedding $f$ can extend to a symplectic embedding $f': T^*_\epsilon S^3 \to X$ with $T^*_\epsilon S^3 \subset T^* S^3$ a neighborhood of the zero section of the cotangent bundle. Define a conifold transition along $f$ to be the smooth manifold

$$Y_k := X/f[S^3] \cup f'_{\psi_k} W_k^e$$

for $k = 1, 2$, where $W_k$ are two small resolutions of the complex singularity $W = \left\{ \sum z_j^2 = 0 \right\} \subset \mathbb{C}^4$ with exceptional set $\mathbb{C}P^1$ over $\{0\} \in W$ and either of $W_k$ is a complex vector bundle over $\mathbb{C}P^1$ with first Chern number $-2$; fixing coordinates on $T^* S^3$ as

$$T^* S^3 = \left\{(u, v) \in \mathbb{R}^4 \times \mathbb{R}^4 | u = 1, \langle u, v \rangle = 0 \right\},$$

the maps $\psi_k : (W_k \setminus \mathbb{C}P^1 \cong W \setminus \{0\}, \omega_0) \to (T^* S^3 \setminus \{v = 0\}, d(vdu))$ are symplectomorphisms defined in [17, (2.1)] with $\omega_0$ the restriction of the symplectic form $\frac{i}{2} \sum j d z_j \land d \overline{z}_j$ on $\mathbb{C}^4$; the submanifolds $W_k^e \subset W_k$ are neighborhoods of the exceptional set $\mathbb{C}P^1$ such that $W_k^e \cap \mathbb{C}P^1 = \psi_k^{-1}[T^*_\epsilon S^3 \setminus \{v = 0\}]$.

There are more choices in conifold transitions along a Lagrangian 3-sphere than along a Lagrangian embedding $S^3 \to X$, as changing the orientation of the Lagrangian 3-sphere $f[S^3]$ would induce a new Lagrangian embedding $S^3 \to X$ different from $f$. However, this change would just swap the diffeomorphism types of the conifold transitions, so there are exactly two distinct conifold transitions $Y_k, k = 1, 2$ along the Lagrangian 3-sphere $f[S^3]$ up to diffeomorphism [17]. It follows from [17, Theorem 2.9] that the two conifold transitions of a symplectic 6-manifold along a nullhomologous Lagrangian 3-sphere both admit distinguished symplectic structures. Hence to realize such symplectic conifold transitions on $\mathbb{C}P^1$-bundles, it suffices to find nullhomologous Lagrangian 3-spheres.

Inside the product $(\mathbb{C}^2 \times \mathbb{C}P^1, \omega_0 \times \omega_{\mathbb{C}P^1})$ of symplectic manifolds with $\omega_0 = \frac{i}{2} \sum j dz_j \land d \overline{z}_j$ on $\mathbb{C}^2$ and $\omega_{\mathbb{C}P^1}$ the Fubini–Study form on $\mathbb{C}P^1$, a well-known construction [1] of a nullhomologous Lagrangian 3-sphere is given by the composition of maps

$$f: S^3 \ni (i, h) \mapsto \mathbb{C}^2 \times \mathbb{C}P^1 \times id_{\mathbb{C}P^1} \rightarrow \mathbb{C}^2 \times \mathbb{C}P^1$$

(1.1)

where $i : S^3 \subset \mathbb{C}^2$ is the inclusion of the unit sphere, $h : S^3 \to \mathbb{C}P^1$ is the Hopf map and $\iota$ is the complex conjugation on $\mathbb{C}^2$. As the image $f[S^3]$ is entirely contained in $B^4(l) \times \mathbb{C}P^1$ with $B^4(l)$ a ball in $\mathbb{C}^2$ of radius $l > 1$, finding symplectic embeddings of $B^4(l) \times \mathbb{C}P^1$...
in \(\mathbb{C}P^1\)-bundles would induce nullhomologous Lagrangian 3-spheres in these bundles. This may lead to the following definition.

**Definition 2.1** Let \(\mathbb{P}(E)\) be a symplectic manifold which is a \(\mathbb{C}P^1\)-bundle over a 4-manifold \(N\). A Lagrangian 3-sphere in \(\mathbb{P}(E)\) is called *local* if it is the image of the composition of embeddings

\[
S^3 \xrightarrow{f} B^4(l) \times \mathbb{C}P^1 \xrightarrow{\eta} \mathbb{P}(E)
\]

for \(l > 1\) where the symplectic embedding \(\eta\) can induce a local trivialization of the bundle \(\pi : \mathbb{P}(E) \to N\), i.e. there is a differentiable embedding \(k : B^4(l) \to N\) such that

\[
\pi^{-1}[k(B^4(l))] = \eta[B^4(l) \times \mathbb{C}P^1] \xrightarrow{\eta^{-1}} B^4(l) \times \mathbb{C}P^1 \xrightarrow{k \times id_{\mathbb{C}P^1}} k[B^4(l)] \times \mathbb{C}P^1
\]

is a local trivialization of the \(\mathbb{C}P^1\)-bundle \(\mathbb{P}(E)\).

[4,17] have shown the existence of local Lagrangian 3-spheres in trivial \(\mathbb{C}P^1\)-bundles over Kähler surfaces. We generalize their result in the following Lemma by using Thurston’s construction of symplectic form [12, Theorem 6.3] and the construction of Kähler forms on \(\mathbb{P}(E)\) [18, Proposition 3.18].

**Lemma 2.2** Let \(\mathbb{P}(E)\) be the projectivization of a rank two complex vector bundle \(E\) over a symplectic 4-manifold \(N\). Then for any positive integer \(K\), the total space \(\mathbb{P}(E)\) admits a symplectic form compatible with the fibration such that there are \(K\) disjoint local Lagrangian 3-spheres in \(\mathbb{P}(E)\). Moreover, if \(N\) is Kähler and \(E\) admits a holomorphic structure, then for any positive integer \(K\), the total space \(\mathbb{P}(E)\) admits a Kähler form compatible with the fibration such that there are \(K\) disjoint local Lagrangian 3-spheres in \(\mathbb{P}(E)\).

**Proof** By Definition 2.1, it suffices to construct appropriate symplectic forms on \(\mathbb{P}(E)\) and find symplectic embeddings \(B^4(l) \times \mathbb{C}P^1 \to \mathbb{P}(E)\) for \(l > 1\) which can induce local trivializations. The key point is to note that there exists a system of projective-unity local trivializations \(\{(U_j, \phi_j)\}_{j=0}^m\) of the \(\mathbb{C}P^1\)-bundle \(\pi : \mathbb{P}(E) \to N\) and a partition of unity \(\rho_j : N \to [0, 1]\) subordinating to the open cover \(\{U_j\}_{j=0}^m\) of \(N\) such that each \(U_j\) is contractible and \(\rho_0 \equiv 1\) on a nonempty open subset \(V \subset U_0\). In fact, such \(U_j\) and \(\rho_0\) can be obtained by using [2, Corollary 5.2] and shrinking all open sets \(U_j\) except \(U_0\) if necessary.

For the symplectic case, apply Thurston’s construction. According to the proof of [12, Theorem 6.3], the local trivializations \(\{(U_j, \phi_j)\}_{j=0}^m\) and the partition of unity \(\rho_j : N \to [0, 1]\), together with the first Chern class of the dual bundle of the tautological line bundle over \(\mathbb{P}(E)\), can contribute to define a closed 2-form \(\tau\) on \(\mathbb{P}(E)\) such that the restriction of \(\tau\) on each fiber \(\mathbb{C}P^1\) is just \(\omega_{\mathbb{C}P^1}\), and the 2-form \(\tau + \lambda \pi^*\omega_N\) on \(\mathbb{P}(E)\) is symplectic for \(\lambda > 0\) sufficiently large where \(\omega_N\) denotes the symplectic form on \(N\). Since \(\rho_0 \equiv 1\) on \(V\), then the restriction of the form \(\tau\) to \(\pi^{-1}[V]\) is equal to the pullback \(\phi_0^*(0 \times \omega_{\mathbb{C}P^1})\) of the form \(0 \times \omega_{\mathbb{C}P^1}\) on \(V \times \mathbb{C}P^1\). By the Darboux neighborhood theorem, for any positive integer \(K\), there are \(K\) disjoint symplectic embeddings \(B^4(l) \to (V, \lambda \omega_N)\) with \(l > 1\) for \(\lambda\) sufficiently large. So in this case, we have \(K\) disjoint compositions of symplectic embeddings

\[
B^4(l) \times \mathbb{C}P^1 \to (V \times \mathbb{C}P^1, \lambda \omega_N \times \omega_{\mathbb{C}P^1}) \xrightarrow{\phi_0^{-1}} (\mathbb{P}(E), \tau + \lambda \pi^*\omega_N)
\]

which are the desired embeddings.

Now for the Kähler case, assume \(\omega_N\) is the Kähler form on \(N\) and \(E\) is holomorphic. Using the previous partition of unity \(\rho_J\) and the system of local trivializations \(\{(U_j, \varphi_j)\}_{j=0}^m\)
of $E$ associated to $\{(U_j, \phi_j)\}_{j=0}^m$, we can obtain a Hermitian metric $h$ on $E$ such that on the restriction $E|_V$ of $E$ to $V$, the metric $h$ is the pullback of the local Hermitian metric on $\mathbb{C}^2$ via the projection $E|_V \xrightarrow{\psi_0} V \times \mathbb{C}^2 \to \mathbb{C}^2$. [18, Proposition 3.18] shows that $h$ induces a Hermitian metric on the bundle $L^*$ and the Chern form $\omega_E$ associated to this metric can contribute to obtain a Kähler form $\omega_E + \lambda \tau^* \omega_N$ for $\lambda > 0$ sufficiently large. Replacing $\tau$ by $\omega_E$ in proof of the symplectic case can produce the desired symplectic embeddings. This completes the proof. □

3 Topology of symplectic conifold transitions of $\mathbb{C}P^1$-bundles

The aim of this section is to study the topology of symplectic conifold transitions of $\mathbb{C}P^1$-bundles along local Lagrangian 3-spheres and prove Theorem 1.1 and Corollary 1.3. For this purpose, we first recall in Sect. 3.1 the invariants of simply-connected 6-manifolds with torsion-free homology, and compute the invariants of $\mathbb{C}P^1$-bundles over simply-connected 4-manifolds; then determine in Sect. 3.2 the topology of conifold transitions of $\mathbb{C}P^1$ along $f[S^3]$, i.e. to establish Lemma 3.3, which is a key ingredient in the proof of Theorem 1.1.

3.1 Invariants of simply-connected 6-manifolds with torsion-free homology

By Wall [19] and Jupp [11], the third Betti number $b_3$, the integral cohomology ring $H^*$, the first Pontrjagin class $p_1$ and the second Whitney–Stiefel class $w_2$ form a system of invariants, which can distinguish all diffeomorphism classes of simply-connected 6-manifolds with torsion-free homology. As an example, we will compute these invariants for $\mathbb{C}P^1$-bundles over simply-connected 4-manifolds.

Example 3.1 Let $\pi : \mathbb{P}(E) \to N$ be the projectivization of a rank two complex vector bundle $E$ over a simply-connected 4-manifold. Then $\mathbb{P}(E)$ has a natural orientation which is compatible with that of the base and fibers. By the homotopy exact sequence and Gysin sequence, the 6-manifold $\mathbb{P}(E)$ is a simply-connected with $b_3 = 0$. The cohomology ring and the characteristic classes $w_2$, $p_1$ can be computed as follows.

(i) By the definition of Chern classes [2, Section 20], we have

$$H^*(\mathbb{P}(E)) \cong H^*(N)[a]/(a^2 + \pi^* c_1(E) \cdot a + \pi^* c_2(E))$$

where $a = c_1(L^*)$ with $L^*$ the dual bundle of the tautological line bundle $L = \{(l, v) \in \mathbb{P}(E) \times E \mid v \in l\}$ over $\mathbb{P}(E)$. Let $\{y_i\}$ be a basis of the free $\mathbb{Z}$-module $H^2(N)$ and then $\{a, \pi^* y_i\}$ forms a basis of $H^2(\mathbb{P}(E))$. By the relations $a^2 + \pi^* c_1(E) \cdot a + \pi^* c_2(E) = 0$ and $\{(N)^* \cup a, [\mathbb{P}(E)]\} = 1$ with $\{N)^* \in H^4(N)$ satisfying $\langle [N]^*, [N]\rangle = 1$, we can obtain

$$\langle a^3, [\mathbb{P}(E)]\rangle = \langle c_1(E)^2 - c_2(E), [N]\rangle$$

$$\langle a^2 \cup \pi^* y_i, [\mathbb{P}(E)]\rangle = -\langle c_1(E) y_i, [N]\rangle$$

$$\langle a \cup \pi^* y_i \cup \pi^* y_j, [\mathbb{P}(E)]\rangle = \langle y_i y_j, [N]\rangle$$

$$\langle \pi^* y_i \cup \pi^* y_j \cup \pi^* y_k, [\mathbb{P}(E)]\rangle = 0.$$  

(ii) Similar to the proof of [13, Theorem 14.10], as the tautological line bundle $L$ is a subbundle of the pullback $\pi^* E$ and a Hermitian metric on $E$ pulls back to a Hermitian
metric on $\pi^* E$, we have a splitting $\pi^* E = L \oplus L^\perp$ where $L^\perp$ is the orthogonal complement bundle of $L$ and hence

$$T\mathbb{P}(E) \cong \pi^* TN \oplus \text{Hom}_\mathbb{C}(L, L^\perp); \tag{3.1}$$

$$\text{Hom}_\mathbb{C}(L, L^\perp) \oplus \varepsilon_\mathbb{C}^1 \cong L^* \otimes \pi^* E$$

with $\varepsilon_\mathbb{C}^1$ the trivial complex line bundle. These isomorphisms, together with the relations $a^2 + \pi^* c_1(E) \cdot a + \pi^* c_2(E) = 0$, $p_1 = c_1^2 - 2c_2$ and $c_1(L_1 \otimes L_2) = 2c_1(L_1) + c_1(L_2)$; $c_2(L_1 \otimes L_2) = c_1(L_1)^2 + c_1(L_1)c_1(L_2) + c_2(L_2)$

with $L_i$ a complex vector bundle of rank $i = 1, 2$ [13, Problem 16-B], imply

$$w_2(T\mathbb{P}(E)) \equiv \pi^*(w_2(TN) + w_2(E));$$

$$p_1(T\mathbb{P}(E)) = \pi^*(p_1(TN) + c_1(E)^2 - 4c_2(E)).$$

Thus we have

$$\langle a^2 \cup w_2(T\mathbb{P}(E)), [\mathbb{P}(E)] \rangle = \langle w_2(E) \cup (w_2(E) + w_2(TN)), [N] \rangle;$$

$$\langle a \cup \pi^* y_i \cup w_2(T\mathbb{P}(E)), [\mathbb{P}(E)] \rangle = \langle y_i \cup (w_2(E) + w_2(TN)), [N] \rangle;$$

$$\langle \pi^* y_i \cup \pi^* y_j \cup w_2(T\mathbb{P}(E)), [\mathbb{P}(E)] \rangle = 0.$$  

$$\langle a \cup p_1(T\mathbb{P}(E)), [\mathbb{P}(E)] \rangle = 3\sigma(N) + \langle c_1(E)^2 - 4c_2(E), [N] \rangle$$

$$\langle \pi^* y_i \cup p_1(T\mathbb{P}(E)), [\mathbb{P}(E)] \rangle = 0.$$  

where $\sigma(N)$ is the signature of the 4-manifold $N$ [13, SIGNATURE THEOREM 19.4].

Remark 3.2 In Example 3.1, the isomorphisms (3.1) still hold in the case when the 4-manifold $N$ is not simply-connected, and it follows that

$$c_1(T\mathbb{P}(E)) = 2a + \pi^*(c_1(TN) + c_1(E))$$

which will be applied to the proof of Theorem 1.1 in Sect. 3.3.

3.2 Topology of conifold transitions of $B^4(l) \times \mathbb{C}P^1$ along $f[S^3]$  

It is easy to see that the definition of conifold transitions can extend to symplectic manifolds with boundaries. In this subsection we will prove Lemma 3.3, determining the topology of $Y_k$, $k = 1, 2$, which denote the two conifold transitions of $B^4(l) \times \mathbb{C}P^1$ along the Lagrangian embedding $f : S^3 \to B^4(l) \times \mathbb{C}P^1$ in (1.1).

As in Sect. 1, denote $\overline{\mathbb{C}P^2}$ and $S^4$ by $N_k$, $k = 1, 2$, respectively. Let $\sigma_k \in H_2(N_k)$ and $\sigma_k^* \in H^2(N_k)$ such that $\sigma_k$ is the dual class of the preferred generator $\sigma_1$ and $\sigma_2 = 0$, $\sigma_2^* = 0$. As $\partial Y_k = \partial B^4(l) \times \mathbb{C}P^1$, the lemma can be stated as follows.

Lemma 3.3 Let $id_\partial$ denote the identity map of $\partial Y_k = \partial B^4(l) \times \mathbb{C}P^1$. Then there are two diffeomorphisms

$$\phi_1 : B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_1 \to \mathbb{P}(E_1');$$

$$\phi_2 : B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_2 \to \mathbb{P}(E_2');$$

such that the restriction of $\phi_k$ on $B^4(l) \times \mathbb{C}P^1$ can induce a local trivialization of the bundle $\mathbb{P}(E_k')$ for $k = 1, 2$, where $E_k'$ is the rank two complex bundle over $N_k$ with $c_1(E_k') = -\sigma_k^*$ and $c_2(E_k') = -1.$

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To show this lemma, it needs to compute the topological invariants of \( M_k := B^4(l) \times \mathbb{CP}^1 \cup_{id_k} Y_k \). As Smith and Thomas [16, Proposition 4.2] have computed the intersection forms of the conifold transitions of \( \mathbb{CP}^2 \times \mathbb{CP}^1 \) along a local Lagrangian 3-sphere, we will extend their computation to the topological invariants of \( M_k \) in Lemma 3.4 and Example 3.5.

The following Lemma will be very useful for the computation of invariants of \( M_k \). Referring to the definition of conifold transitions recalled in Sect. 2, as we have inclusions of the exceptional set \( \mathbb{CP}^1 \subset W_k^e \) and the set \( O \times \mathbb{CP}^1 \subset B^4(l) \times \mathbb{CP}^1 \setminus f[S^3] \) with \( O \subset B^4(l) \) the origin, let \( C_k \) and \( P_k \) denote the images of the exceptional set \( \mathbb{CP}^1 \) and the set \( O \times \mathbb{CP}^1 \) under the natural inclusions \( W_k^e \to Y_k \to M_k \) and \( B^4(l) \times \mathbb{CP}^1 \setminus f[S^3] \to Y_k \to M_k \), respectively.

**Lemma 3.4** Let \( \sigma \in H_2(\mathbb{CP}^2) \) be the preferred generator. Then there are two differentiable embeddings \( r_k : \mathbb{CP}^2 \sharp N_k \to M_k, \ k = 1, 2 \) satisfying the following conditions:

(i) Under the homomorphism \( r_k^* : H_2(\mathbb{CP}^2 \sharp N_k) \cong H_2(\mathbb{CP}^2) \oplus H_2(N_k) \to H_2(M_k) \), the images of \( \sigma \) and \( \sigma_k \) are the homology classes \([P_k]\) and \( 1 - (-1)^k \cdot [C_k] \) in \( H_2(M_k) \), respectively.

(ii) The Euler class of the normal bundle of \( r_k \) is

\[
(-\sigma^*, -\sigma_k^*) \in H^2(\mathbb{CP}^2 \sharp N_k) \cong H^2(\mathbb{CP}^2) \oplus H^2(N_k),
\]

where \( \sigma^* \in H^2(\mathbb{CP}^2) \) is the dual cohomology class of \( \sigma \).

(iii) In \( M_k \), the intersection number of the submanifolds \( r_k[\mathbb{CP}^2 \sharp N_k] \) and \( C_k \) is \((-1)^k\).

To show this lemma, first recall some results in the proof of [17, Theorem 2.9] and [12, Theorem 3.33]. Let

\[
\Delta_\epsilon = \{(u, v) \in T^*_\epsilon S^3 | (v_1, v_2, v_3, v_4) = \lambda(-u_2, u_1, -u_4, u_3); \lambda \geq 0 \}
\]

and fix \( W_k, k = 1, 2 \) as \( W^\pm \) in [17], respectively. [17, Theorem 2.9] finds 4-dimensional submanifolds \( \tilde{S}_k \subset W_k^\epsilon, k = 1, 2 \) such that

1. \( \tilde{S}_1 \) is the complex line bundle over the exceptional set \( \mathbb{CP}^1 \subset W_1^\epsilon \) with Euler class \( -1 \) and \( \psi_1^{-1}[\Delta_\epsilon \setminus \{v = 0\}] = \tilde{S}_1 \setminus \mathbb{CP}^1 \);
2. \( \tilde{S}_2 \) is diffeomorphic to \( \mathbb{R}^4 \) and \( \psi_2^{-1}[\Delta_\epsilon \setminus \{v = 0\}] \) is equal to \( \tilde{S}_2 \) with a point removed.

3. The intersection number of \( \tilde{S}_k \) and the exceptional set \( \mathbb{CP}^1 \) in \( W_k^\epsilon \) is \((-1)^k\).

Considering the symplectic form \( d(vdu) \) on \( T^*S^3 \) and applying [12, Theorem 3.33] to the Lagrangian embedding \( f \), this defines an embedding \( \tilde{f} : T^*_\epsilon S^3 \to B^4(l) \times \mathbb{CP}^1 \) by \( \tilde{f}(u, v) = \exp_{f(u)}(-J_{f(u)} \circ df_u \circ \Phi_u(v)) \), where \( J \) is a compatible almost complex structure on \( (B^4(l) \times \mathbb{CP}^1, \omega_0 \times \omega_{\mathbb{CP}^1}) \) and \( \Phi_u : T^*_u S^3 \to T_u S^3 \) is an isomorphism determined by the relation \( \omega_0 \times \omega_{\mathbb{CP}^1}(df_u \circ \Phi_u(v), J_{f(u)} \circ df_u(v')) = v(v') \) for \( v' \in T_{q'} S^3 \).

**Proof of Lemma 3.4** As [12, Theorem 3.33] shows that \( \tilde{f} \) is isotopic to a symplectic embedding which represents a Lagrangian neighborhood of \( f \), thus \( Y_k \) is diffeomorphic to \( B^4(l) \times \mathbb{CP}^1 \setminus f[S^3] \cup_{\tilde{f}} W_k^\epsilon \). We claim that the restriction of \( \tilde{f} \) on \( \Delta_\epsilon \setminus \{v = 0\} \) is a diffeomorphism onto the relative complement of a closed neighborhood of

\[
O \times \mathbb{CP}^1 \subset R_0 = \{(w, \infty) | w \in B^4(l) \times \mathbb{CP}^1 | \|w\| < 1 \}.
\]
If it is true, then combining this claim with the conditions (1), (2), (3) above and the fact that \( R_0 \) is the open disc bundle over \( O \times \mathbb{C}P^1 \) with Euler class 1, it would imply that \( R_0 \cup_{\tau_{\psi^k}} \psi^{-1} \mathcal{N}_k \cong \mathbb{C}P^2 \mathbb{N}_k \) are well-defined differentiable submanifolds of \( \mathbb{B}^4(l) \times \mathbb{C}P^1 \setminus f(S^3) \cup_{\tau_{\psi^k}} \mathcal{N}_k \cong Y_k \subset M_k \) for \( k = 1, 2 \), respectively, which gives embeddings \( r_k : \mathbb{C}P^2 \mathbb{N}_k \hookrightarrow M_k \) satisfying (i) (iii). (ii) would follow from the fact that the restriction of the normal bundle of \( R_0 \subset \mathbb{B}^4(l) \times \mathbb{C}P^1 \) to \( O \times \mathbb{C}P^1 \) has Euler class \(-1\) and so does the restriction of the normal bundle of \( \mathcal{S}_1 \subset \mathcal{W}_1 \) to the exceptional set \( \mathbb{C}P^1 \).

Now it remains to show our claim. Under the identifications
\[
T \mathcal{S}^3 = T^* \mathcal{S}^3 = \{(u, v) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid |u| = 1, \langle u, v \rangle = 0\},
\]
\[
\mathbb{R}^4 = \mathbb{C}^2 : (r_1, r_2, r_3, r_4) \mapsto (r_1 + ir_2, r_3 + ir_4),
\]
it is easy to see that \( v(v') = o_0(v, Jv') = o_0(\overline{v}, J\overline{v'}) \) with \((\overline{v}, \overline{v'})\) the complex conjugate of \((v, v') \in T_u \mathcal{S}^3 \times T_u \mathcal{S}^3\). Thus for any \((u, v) \in \Delta_\epsilon \setminus \{v = 0\}\), we have
\[
v = \lambda iu = \lambda \sqrt{-1}u, \lambda > 0;
\]
\[
df(u) = (\overline{v}, [v]) = (\overline{v}, [0]) \in T_{(\pi, [u])} \mathbb{B}^4(l) \times \mathbb{C}P^1 = \mathbb{C}^2 \times \mathbb{C}^2/Cu
\]
and hence \( \Phi_u(v) = v \). These relations, together with the fact that the complex structure \( J_{f(u)} \) on \( T_{f(u)} \mathbb{B}^4(l) \times \mathbb{C}P^1 \) is induced by the multiplication by \( i = \sqrt{-1} \) on \( \mathbb{C}^2 \times \mathbb{C}^2/Cu \), imply

\[
\overline{f}(u, v) = \exp_{(\overline{u}, [u])}(-\lambda \overline{u}, [0]) = ((1 - \lambda)\overline{u}, [u]) \in R_0 \setminus O \times \mathbb{C}P^1.
\]

This completes the proof. \( \square \)

Using Lemma 3.4, we can compute the topological invariants of \( M_k = \mathbb{B}^4(l) \times \mathbb{C}P^1 \cup_{id_y} Y_k \) for \( k = 1, 2 \).

**Example 3.5** We first claim that \( M_k \) is a simply-connected 6-manifold with \( b_3 = 0 \) and \( H^2(M_k) \) has a basis consists of \( z_k \) and \( x_k \), where \( z_k \) is the Poincaré dual of the submanifold \( R_k = r_k[\mathbb{C}P^2 \mathbb{N}_k] \subset M_k \) and the definition of \( x_k \) is contained in the following proof of the claim. Since \( M_k \) is obtained by surgery along an embedding \( \mathcal{S}^3 \times \mathcal{B}^3 \hookrightarrow \mathcal{S}^2 \times \mathcal{S}^4 \) with \( C_k \) the resulting 2-sphere [3], then \( M_k \) is simply-connected and there is a cobordism \( W_k \) between \( \mathcal{S}^2 \times \mathcal{S}^4 \) and \( M_k \), assuming \( j_k : \mathcal{S}^2 \times \mathcal{S}^4 \hookrightarrow W_k \) and \( j_k' : M_k \hookrightarrow W_k \) are the inclusions. From the cohomology exact sequence of the pairs \((W_k, M_k)\) and \((W_k, \mathcal{S}^2 \times \mathcal{S}^4)\), it is easy to show that \( H_3(M_k) \cong H^3(M_k) \) is trivial. Furthermore, consider the exact sequence

\[
0 \rightarrow H^2(W_k) \xrightarrow{j_k^*} H^2(M_k) \xrightarrow{\delta} H^3(W_k, M_k), \tag{3.2}
\]
then \( \delta z_k \) is a generator of \( H^3(W_k, M_k) \) because the value of \( \delta z_k \) on the generator of \( H^2(W_k, M_k) \) is equal to \( \langle z_k, [C_k] \rangle \) by Lemma 3.4 (iii). This, together with the isomorphism \( H^2(W_k) \xrightarrow{j_k^*} H^2(\mathcal{S}^2 \times \mathcal{S}^4) \) and the exact sequence (3.2), implies that \( x_k := j_k^* j_k'^{-1} a \) and \( z_k \) form a basis of \( H^2(M_k) \), where \( a \in H^2(\mathcal{S}^2 \times \mathcal{S}^4) \) is the dual class of the preferred generator \( [\mathcal{S}^2] \) of \( H_2(\mathcal{S}^2 \times \mathcal{S}^4) \).

(i) The cohomology ring of \( M_k \): The relations \( j_k^*[P_k] = j_k*[\mathcal{S}^2] \) and \( \delta x_k = 0 \), together with Lemma 3.4 (i) and the fact that \( \langle x_k, [C_k] \rangle \) is equal to the value of \( \delta x_k \in H^3(W_k, M_k) \) on the generator of \( H_3(W_k, M_k) \), imply

\[
\langle r_k^* x_k, \sigma \rangle = \langle x_k, [P_k] \rangle = \langle a, [\mathcal{S}^2] \rangle = 1; \langle r_k^* x_k, \sigma_k \rangle = 0 \tag{3.3}
\]
for the basis $\sigma, \sigma_k \in H_2(\mathbb{C}P^2 \# N_k) \cong H_2(\mathbb{C}P^2) \oplus H_2(N_k)$. Let $e(\nu r_k)$ denote the Euler class of the normal bundle $\nu r_k$ of $r_k$, then it follows from the values (3.3) and Lemma 3.4 (ii) that

$$\langle z_k^3, [M_k] \rangle = \langle z_k^3, [R_k] \rangle = \langle e(\nu r_k)^2, [\mathbb{C}P^2 \# N_k] \rangle = \frac{1 + (-1)^k}{2};$$

$$\langle x_k^3 [M_k] \rangle = \langle x_k^3, [R_k] \rangle = \langle r_k^* x_k^2, [\mathbb{C}P^2 \# N_k] \rangle = 1;$$

$$\langle x_k^2 [M_k] \rangle = \langle x_k^2, [R_k] \rangle = \langle r_k^* x_k, [\mathbb{C}P^2 \# N_k] \rangle = -1;$$

$$\langle x_k^3, [M_k] \rangle = \langle j_k^* j_k^{-1} a^3, [M_k] \rangle = 0.$$

(ii) The first Pontrjagin class of $M_k$: The exact sequence

$$H_7(W_k, \partial W_k) \to H_6(S^2 \times S^4 \sqcup M_k) \to H_6(W_k),$$

together with the relations $\partial [W_k] = [M_k] - [S^2 \times S^4]$, $p_1(M_k) = j_k^* p_1(W_k)$ and $j_k^* p_1(W_k) = p_1(S^2 \times S^4) = 0$, imply

$$\langle p_1(M_k) x_k, [M_k] \rangle = \langle p_1(W_k) \cup j_k^{-1} a, j_k^* [M_k] - j_k^* [S^2 \times S^4] \rangle = 0.$$

From the relations $p_1(\nu r_k) = e(\nu r_k)^2$, $\langle p_1(\mathbb{C}P^2 \# N_k), [\mathbb{C}P^2 \# N_k] \rangle = 3 \cdot \frac{1 + (-1)^k}{2}$ and $z_k \cap [M_k] = r_k^* [\mathbb{C}P^2 \# N_k]$, together with Lemma 3.4 (ii) and the decomposition $r_k^* T M_k = T(\mathbb{C}P^2 \# N_k) \oplus \nu r_k$, we get

$$\langle p_1(M_k) x_k, [M_k] \rangle = \langle r_k^* p_1(M_k), [\mathbb{C}P^2 \# N_k] \rangle = 2 \times (1 + (-1)^k).$$

(iii) The second Whitney class of $M_k$: As the value $w_2(S^2 \times S^4) = 0$ and the isomorphism $j_k^*: H^2(W_k) \to H^2(S^2 \times S^4)$ imply $w_2(W_k) = 0$, thus

$$w_2(M_k) = j_k^* w_2(W_k) = 0.$$

Now we can prove the Lemma 3.3.

**Proof of Lemma 3.3** Denote $S^6$ and $\overline{\mathbb{C}P^3}$ by $Q_1$ and $Q_2$, respectively. By Wall and Jupp’s classification of simply-connected 6-manifolds with torsion-free homology [11, 19], comparing the invariants of $M_k$ and $\mathbb{P}(E_k')$ (see Example 3.5 and 3.1), we get two diffeomorphisms

$$\varphi_k : M_k \to \mathbb{P}(E_k') \# Q_k, k = 1, 2$$

such that $\varphi_k^* a_k = x_k + \frac{1 + (-1)^k}{2} \cdot z_k$ for $k = 1, 2$, $\varphi_1^* \pi_1^* (-\sigma_1^*) = z_1$ and $\varphi_2^* z' = z_2$, where

$$a_k \in H^2(\mathbb{P}(E_k') z Q_k) \cong H^2(\mathbb{P}(E_k')) \oplus H^2(Q_k)$$

denote the first Chern classes of the dual bundles of the tautological line bundles over $\mathbb{P}(E_k')$ for $k = 1, 2$, respectively, $\pi_1 : P(E') \to \overline{\mathbb{C}P^3}$ is the bundle projection, and

$$z' \in H^2(\mathbb{P}(E_2') \# \overline{\mathbb{C}P^3}) \cong H^2(\mathbb{P}(E_2')) \oplus H^2(\overline{\mathbb{C}P^3})$$

is the Poincaré dual of the submanifold $\mathbb{C}P^2 \subset \overline{\mathbb{C}P^3}$.

We claim that $\varphi_k|_k O \times \mathbb{C}P^1) = f_k[\mathbb{C}P^1]$ for the submanifold $O \times \mathbb{C}P^1 \subset B^4(l) \times \mathbb{C}P^1 \subset M_k$ and embeddings $f_k : \mathbb{C}P^1 \to \mathbb{P}(E_k') \# Q_k$ representing a fiber of $\mathbb{P}(E_k')$. As the relations
\[ \{ z_k, [O \times \mathbb{C}P^1] \} = 0 \text{ and } j'_k \cdot [P_k] = j_k[\mathbb{S}^2] = j'_k [O \times \mathbb{C}P^1] \] imply that \([O \times \mathbb{C}P^1]\) is the dual base of \(x_k + \frac{1+(-1)^k}{2} \cdot z_k = \phi_k^* a_k \) in the basis

\[
\begin{aligned}
\left\{ x_k + \frac{1+(-1)^k}{2} \cdot z_k, z_k \right\} &= \left\{ \phi_k^* a_1, \phi_k^* \pi_k^*(-\sigma_1^*) \right\} \text{ for } k = 1, \\
&= \left\{ \phi_k^* a_2, \phi_k^* \sigma_2^* \right\} \text{ for } k = 2,
\end{aligned}
\]

comparing this with the fact that \(f_k[\mathbb{C}P^1]\) is the dual base of \(a_k\) in the basis \(\{a_1, \pi_k^*(-\sigma_1^*)\}\) for \(k = 1\) and in the basis \(\{a_2, \sigma_2^*\}\) for \(k = 2\), respectively, shows the claim.

Since the claim above implies that \(\varphi_k|_{O \times \mathbb{C}P^1}\) is homotopic to \(f_k\), then by [9, THEOREM 1] and the isotopy extension theorem [10, Chapter 8, 1.3. Theorem], there is an isotopy \(F_t^k : \mathbb{P}(E'_k) \ni Q_k \to \mathbb{P}(E'_k) \ni Q_k, \ 0 \leq t \leq 1, \) such that \(F_0^k = id \) and \(F_1^k \circ \varphi_k|_{O \times \mathbb{C}P^1} = f_k\). Let \(f_k : B^4(l) \times \mathbb{C}P^1 \to \mathbb{P}(E'_k) \ni Q_k\) be an extension of \(f_k\) which can induce a local trivialisition of the bundle \(\mathbb{P}(E'_k)\), then \(F_t^k \circ \varphi_k|_{B^4(l) \times \mathbb{C}P^1}\) and \(f_k\) determine two closed tubular neighborhoods of \(f_k[\mathbb{C}P^1]\). By the ambient isotopy theorem for closed tubular neighborhoods [10, Chapter 4, Section 6, Exercises 9], there exists an isotopy \(H_t^k : \mathbb{P}(E'_k) \ni Q_k \to \mathbb{P}(E'_k) \ni Q_k, \ 0 \leq t \leq 1, \) such that \(H_0^k = id, \ H_t^k \circ F_t^k \circ \varphi_k|_{B^4(l) \times \mathbb{C}P^1} = f_k|_{B^4(l) \times \mathbb{C}P^1}\) and

\[
g := f_k^{-1} \circ H_t^k \circ F_t^k \circ \varphi_k|_{B^4(l) \times \mathbb{C}P^1} : B^4(l) \times \mathbb{C}P^1 \to B^4(l) \times \mathbb{C}P^1
\]

is a \(B^4(l)\)-bundle isomorphism. As the homotopy group \(\pi_2(\mathbb{O}(4))\) of the real orthogonal group \(\mathbb{O}(4)\) is trivial, this implies \(g|_{\partial B^4(l) \times \mathbb{C}P^1}\) is isotopic to the identity map of \(\partial B^4(l) \times \mathbb{C}P^1\) and then similar to the proof of [10, Chapter 8, 2.3], we can extend \(g\) to a self-diffeomorphism \(\phi\) of \(M_k = B^4(l) \times \mathbb{C}P^1 \cup_{id_y} Y_k\) which is identity outside a neighborhood of \(B^4(l) \times \mathbb{C}P^1\). Consequently, the restriction of \(\phi_k := H_t^k \circ F_t^k \circ \varphi_k \circ \phi^{-1}\) on \(B^4(l) \times \mathbb{C}P^1\) is equal to \(f_k\) and hence \(\phi_k, k = 1, 2,\) are the desired diffeomorphisms. \(\square\)

### 3.3 Topology of symplectic conifold transitions of \(\mathbb{C}P^1\)-bundles

The establishment of Lemma 3.3 make it possible to prove Theorem 1.1, which determines the diffeomorphism types of symplectic conifold transitions of \(\mathbb{C}P^1\)-bundles over 4-manifolds along local Lagrangian 3-spheres. In this section, we show this theorem and Corollary 1.3.

**Proof of Theorem 1.1** From [17, Theorem 2.9] and the definition of the two symplectic conifold transitions \(Z_k, k = 1, 2\) along a local Lagrangian embedding \(S^3 \xrightarrow{f} B^4(l) \times \mathbb{C}P^1 \xrightarrow{\eta} \mathbb{P}(E),\) we get the identification

\[
Z_k = \mathbb{P}(E) \cup_{\eta} M_k \setminus (\text{Interior } (\mathbb{C}P^1) \cup \mathbb{C}P^1)
\]

as almost complex manifolds with \(B^4(l) \times \mathbb{C}P^1\) seen as a subset of \(M_k = B^4(l) \times \mathbb{C}P^1 \cup_{id} Y_k\). Denote \(S^5\) and \(\mathbb{C}P^3\) by \(Q_1\) and \(Q_2\), respectively, and let \(E \cup_{c_2} E'_k\) denote the complex vector bundle over the one point union \(N \cup N_k\) obtained by identifying one fiber \(\mathbb{C}^2\) of the two bundles \(E\) and \(E'_k\), respectively. The identity map \(id\) of \(\mathbb{P}(E)\) and the diffeomorphisms \(\phi_k : M_k \to \mathbb{P}(E'_k) \ni Q_k\) in Lemma 3.3 contribute to define diffeomorphisms

\[
\Psi_k : Z_k \xrightarrow{\cong} \mathbb{P}(E) \ni Q_k, \ k = 1, 2
\]

where \(E_k\) is the pullback bundle of the bundle \(E \cup_{c_2} E'_k\) under the natural map \(N \ni N_k \to N \cup N_k\). It is very easy to get the Chern class of \(E_k\) from the isomorphism \(H^2(N \cup N_k) \cong H^2(N \ni N_k)\), the homomorphisms
and the values
\[ c_j(E \cup_C^2 E_k') = (c_j(E), c_j(E_k')) \in H^{2j}(N) \oplus H^{2j}(N_k) \cong H^{2j}(N \cup N_k) \]
for \( j = 1, 2 \).

To prove the diffeomorphisms \( \Psi_k \) preserve the homotopy classes of almost complex structure, it suffices to show it preserves \( c_1 \) [19, Theorem 9]. Consider the commutative diagram
\[
\begin{array}{ccc}
H^2(P(E) \notimes Q_k) & \xrightarrow{\Psi_k^*} & H^2(Z_k) \\
\uparrow \cong & & \uparrow \cong \\
H^2(P(E) \cup_{\eta \circ f_k} P(E_k') \notimes Q_k) & \xrightarrow{(id \cup \phi_k)^*} & H^2(P(E) \cup_{\eta} M_k) \\
\downarrow & & \downarrow \\
H^2(P(E)) \oplus H^2(P(E_k') \notimes Q_k) & \xrightarrow{id^* \circ \phi_k^*} & H^2(P(E)) \oplus H^2(M_k)
\end{array}
\]
where \( \eta \circ f_k : B^4(l) \times CP^1 \to P(E_k') \notimes Q_k \) is the restriction of \( \phi_k \) as in the proof of Lemma 3.3 and the vertical homomorphisms are induced by the natural inclusions. As the conifold transitions is an almost complex operation preserving the first Chern class [17] [4, Lemma 2], the formula of the first Chern class of a one point blowup [6, p. 608] [7] and Remark 3.2 imply that the images of \( c_1(TP(E_k) \notimes Q_k) \) and \( c_1(TZ_k) \) under the vertical composite homomorphisms are
\[
\left( 2a + \pi^*(c_1(TN) + c_1(E)), 2a_k - (1 + (-1)^k) \cdot z', \right) \),
\]
\[
\left( 2a + \pi^*(c_1(TN) + c_1(E)), 2x_k \right),
\]
respectively, with \( a_k, z' \) and \( x_k \) defined in the proof of Lemma 3.3 and Example 3.5. Since \( \phi_k^* = x_k + \frac{1 + (-1)^k}{2} \cdot z_k, \phi_k^* z' = z_2 \) by the proof of Lemma 3.3, then the horizontal homomorphism \( id^* \circ \phi_k^* \) maps the class (3.5)–(3.6) and hence \( c_1(TZ_k) = \Psi_k^* c_1(TP(E_k)) \) as the vertical homomorphisms in the diagram (3.4) are injective. This completes the proof.

Now we turn to show Corollary 1.3.

**Proof of Corollary 1.3** As the blowup of a Kähler manifold at a point is also Kähler [18, Proposition 3.24], this Corollary follows easily from Theorem 1.1 and the claim that both \( E_k \) over the projective complex surfaces \( N \cup N_k \) admit holomorphic structures. To prove the claim, it suffices to note Schwarzenberger [15, Theorem 9] showed that a complex vector bundle over a projective complex surface \( S \) admits a holomorphic structure if and only if the first Chern class of the bundle belongs to \( H^{1,1}(S) \). As \( c_1(E_2) = c_1(E) \) and \( c_1(E_1) \) is equal to \( c_1(E) \) plus the exceptional divisor \( -\sigma^1 \), so \( c_1(E_k) \in H^{1,1}(N \cup N_k) \) by the Lefschetz theorem on (1,1) classes [18, Theorem 11.30]. This completes the proof.

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