Properties of the nuclear medium

M Baldo and G F Burgio

Instituto Nazionale di Fisica Nucleare, Sez. di Catania, Via S Sofia 64 95123 Catania, Italy
E-mail: marcello.baldo@ct.infn.it and fiorella.burgio@ct.infn.it

Received 3 December 2010, in final form 4 July 2011
Published 9 January 2012
Online at stacks.iop.org/RoPP/75/026301

Abstract
We review our knowledge on the properties of the nuclear medium that have been studied, over many years, on the basis of many-body theory, laboratory experiments and astrophysical observations. Throughout the presentation particular emphasis is placed on the possible relationship and links between the nuclear medium and the structure of nuclei, including the limitations of such an approach. First we consider the realm of phenomenological laboratory data and astrophysical observations and the hints they can give on the characteristics that the nuclear medium should possess. The analysis is based on phenomenological models, that however have a strong basis on physical intuition and an impressive success. More microscopic models are also considered, and it is shown that they are able to give invaluable information on the nuclear medium, in particular on its equation of state. The interplay between laboratory experiments and astrophysical observations is particularly stressed, and it is shown how their complementarity enormously enriches our insights into the structure of the nuclear medium.

We then introduce the nucleon–nucleon interaction and the microscopic many-body theory of nuclear matter, with a critical discussion about the different approaches and their results. The Landau–Fermi liquid theory is introduced and briefly discussed, and it is shown how fruitful it can be in discussing the macroscopic and low-energy properties of the nuclear medium. As an illustrative example, we discuss neutron matter at very low density, and it is shown how it can be treated within the many-body theory. The general bulk properties of the nuclear medium are reviewed to indicate at which stage of our knowledge we stand, taking into account the most recent developments both in theory and experiments. A section is dedicated to the pairing problem. The connection with nuclear structure is then discussed, on the basis of the energy density functional method. The possibility of linking the physics of exotic nuclei and the astrophysics of neutron stars is particularly stressed. Finally, we discuss the thermal properties of the nuclear medium, in particular the liquid–gas phase transition and its connection with the phenomenology on heavy ion reactions and the cooling evolution of neutron stars. The presentation has been taken for non-specialists and possibly for non-nuclear physicists.

(Some figures may appear in colour only in the online journal)

This article was invited by P-H Heenen.

Contents
1. Introduction 2
2. Free Fermi gas of nucleons 2
2.1. The equation of state 2
2.2. The incompressibility 3
2.3. Momentum distribution 3
2.4. The symmetry energy 3
2.5. The single particle density of states 4
2.6. Other microscopic physical quantities 4
3. Basic phenomenology 4
3.1. Mass formula and saturation 4
3.2. Giant resonances in nuclei 6
3.3. Heavy ions 7
3.4. Astrophysics 8
4. From the nuclear interaction to the correlated nuclear medium 10
4.1. Sketch of the nucleon–nucleon interaction 11
4.2. Theoretical many-body methods 13
1. Introduction

Nuclear physics has so many facets that it looks impossible to find a common theoretical picture that is able to unify under a common view, at least to a certain extent, the whole realm of phenomena where nucleonic systems play a role. Indeed, the structure of nuclei, their excitations, nuclear collisions, the structure of neutron stars, supernovae explosion, very many astrophysical phenomena and processes, are all directly connected to that area of physics that can be called ‘nuclear physics’. The possible unification can come from the fundamental theory of strong and weak interactions, quantum chromodynamics (QCD), and the so-called Standard Model. However, in addition to the difficulty in solving QCD for multi-baryonic systems with the necessary accuracy, this would be hardly useful for the understanding of the simple physical basis of the rich structure that nuclear systems display in different contexts.

From a semi-classical or macroscopic point of view, all nuclear systems can be considered as pieces of a quite particular matter, the nuclear medium. The hypothetical uncharged infinite and homogeneous system formed by the nuclear medium is usually called nuclear matter. Actually, as we will discuss, in first approximation, supernovae and neutron stars contain macroscopic portions of nuclear matter. From this point of view, nuclei are considered as droplets of nuclear matter, and indeed this is the basis of the liquid drop model (LDM) of nuclei. The macroscopic view cannot of course exhaust all the numerous aspects of nuclear structure, where microscopic many-body effects are essential. It is however physically meaningful to ask for the properties of the nuclear medium, since this is a state of matter of fundamental relevance.

In this brief review paper we will present the status of our knowledge on the nuclear medium as can be extracted phenomenologically and established theoretically. On the other hand, we will discuss, on the basis of the works performed in the last few years, the possibility of using the properties of the nuclear medium, noticeably its equation of state (EoS), to guide the nuclear structure theory of normal and exotic nuclei. Along the same lines it can be of great physical insight to try to establish, to the extent that this is possible, a link between the macroscopic view and the general properties of finite nuclei.

The style of the review is intended for non-specialists. We introduce each subject by reporting standard results, leaving formal arguments to textbooks or original papers, before going to more advanced developments and the discussion about ongoing research works. The presentation is of course guided by the personal views of the authors as well as by their limitations.

2. Free Fermi gas of nucleons

Before going to the microscopic many-body theory of nuclear matter, we recall the elementary properties of a free Fermi gas of nucleons. This will serve as a starting point when the nuclear interaction will be introduced and at the same time as a reference for comparison with the realistic treatments and results.

2.1. The equation of state

If we assume that no interaction takes place between \( N \) nucleons inside a box of large volume \( V \), we have the simplest model of nuclear matter, the free fermion gas. In this case the energy per particle \( e \) is given by

\[
e = \frac{3}{5} E_F = \frac{3}{2 m} \left( \frac{3 \pi^2}{2} \right)^{\frac{2}{3}} \rho^{\frac{5}{3}}
\]

where \( E_F = \hbar^2 k_F^2 / 2m \) is the Fermi energy. Equation (1) relates the energy per particle \( e \) to the density \( \rho \), and therefore it is the EoS (the simplest one) for a free symmetric nucleon gas at zero temperature. If one measures the energy in MeV, the length in femtometers (fm; also called ‘fermi’), and adopts for the nucleon mass \( m c^2 = 938.9 \) MeV, an average value between neutron and proton masses, for simplicity, then \( \hbar^2 / 2m = 20.74 \), and

\[
e = 75.03 \rho^{\frac{5}{3}} \text{ MeV}.
\]
This well-known result indicates that the energy of a free fermion gas increases monotonically with the density. If nuclear matter must be stable in mechanical equilibrium, then, at a density \( \rho = \rho_0 \approx 0.16 \text{ fm}^{-3} \), the so-called saturation density, a net attractive potential energy must be present around this density. This attraction, coming from the nucleon–nucleon (NN) interaction, must produce a minimum in the EoS, namely in the curve \( e = e(\rho) \), at \( \rho = \rho_0 \). This requirement originates from the phenomenological observation that the central density of the medium and heavy nuclei (as extracted, e.g., from electron scattering data) is pretty constant throughout the nuclear mass table and close to the above-mentioned value of \( \rho_0 \). This is interpreted as the mechanical equilibrium density of nuclear matter and it is the starting point for the development of the empirical mass formula in its different versions.

2.2. The incompressibility

One of the characteristics of a free Fermi gas is the ‘Pauli pressure’ \( p \), namely the pressure due to the exclusion principle, a typical quantal effect. From the pressure, the incompressibility \( K_0 \) can be derived according to the usual definition

\[
K_0 = -\frac{V}{\rho} \left( \frac{dp}{dV} \right) = \rho \left( \frac{dp}{d\rho} \right) = \frac{2}{3} \frac{\hbar^2}{2m} \left( \frac{3\pi^2}{2} \right)^\frac{2}{3} \rho \approx 83.36 \rho^{\frac{5}{3}}.
\]

Anyhow, the connection between the frequency of the compressional monopole mode and incompressibility is less obvious than at first sight [3].

2.3. Momentum distribution

The ground state of the free fermion gas is characterized by a momentum distribution as indicated in figure 1(a). This picture is expected to be modified by the NN interaction, as shown in figure 1(b). Here the discontinuity at the Fermi energy is assumed to persist despite the NN correlations. The Fermi liquids that have this property are called ‘normal’ Fermi liquids. The deviation of the discontinuity from one is a measure of the strength of the correlations. The persistence of the discontinuity at \( k_F \) is the basis of the Landau theory of Fermi liquid and of the concept of the quasi-particle [4], to be discussed in section 5. If nuclear matter is a superfluid, as it appears to be in a range of densities, the discontinuity disappears. Apart from the possible onset of superfluidity, which affects the gross properties of the EoS only weakly, nuclear matter appears to be a normal Fermi liquid. Superfluidity of course dramatically changes the transport properties of nuclear matter.

2.4. The symmetry energy

If the proton number \( N_p \) is different from the neutron number \( N_n \), with \( N = N_n + N_p \), then the neutron and proton Fermi momenta are different, since the neutron and proton densities are different. Accordingly, the EoS of equation (1) has to be generalized. Defining

\[
\beta = \frac{N_n - N_p}{N_n + N_p} = \frac{\rho_n - \rho_p}{\rho}
\]

as the ‘asymmetry’ parameter, one easily gets

\[
E = E_n + E_p = N_n \frac{3}{5} \frac{e_p}{E_F} + N_p \frac{3}{5} \frac{e_n}{E_F}
\]

\[
e = \frac{E}{N} = \frac{3}{10} \frac{\hbar^2}{2m} \left( \frac{3\pi^2}{2} \right)^\frac{2}{3} \rho \left[ (1 + \beta)^\frac{2}{3} + (1 - \beta)^\frac{2}{3} \right]
\]

\[
\approx e(\beta = 0) + a_{sy} \beta^2 + \ldots
\]

\[
a_{sy} = \frac{1}{3} \frac{E_F}{\rho}
\]

Thus, for a fixed value of the total density \( \rho \), the energy per particle \( e \) has a minimum at \( \beta = 0 \). The coefficient \( a_{sy} \) is called the symmetry energy. At \( \rho \approx \rho_0 \), one finds \( a_{sy} \approx 12 \text{ MeV} \).

From the systematics on the asymmetry dependence of the binding energy of medium–heavy nuclei, \( a_{sy} \) turns out to be more than twice as large as this value. Here the interaction must play a major role. The density dependence of \( a_{sy} \) is one of the most relevant issues not only in nuclear astrophysics, but also in nuclear structure.

2.5. The single particle density of states

For many physical phenomena the single particle density of states at the Fermi energy is a relevant quantity. For the free gas model one can readily get an explicit expression

\[
D(E_F) = \sum_k \delta \left( E_F - \frac{\hbar^2 k^2}{2m} \right)
\]

\[
= \frac{V \cdot g}{(2\pi)^3} \int d^3 k \delta \left( E_F - \frac{\hbar^2 k^2}{2m} \right)
\]

\[
= \frac{N g}{\rho (2\pi)^3} \frac{4\pi m \hbar^2}{\hbar^2 k_F} = \frac{3N}{2E_F} \approx \frac{N}{23} \text{ MeV}^{-1}
\]

where the last equality holds at \( \rho \approx \rho_0 \) and \( g \) is the degeneracy. This elementary result is expected to be modified by the presence of the interaction [5]. The effect of the NN correlations on \( D(E_F) \) can be introduced by substituting the free nucleon mass with the so-called nucleon ‘effective mass’, which will be shortly discussed in the section on Landau theory. Another related useful quantity is the single particle level properties of nuclear matter.
density per unit volume, which for symmetric matter can be written
\[ d(E_F) = D(E_F)/V = \frac{2m}{\pi^2\hbar^2} k_F \] (7)
It depends only on the nuclear matter density.

2.6. Other microscopic physical quantities

In order to characterize the properties of the nuclear medium other quantities are necessary. First of all the nuclear surface properties are characterized by the values of surface thickness and surface tension. Bulk and shear viscosities are essential to describe the macroscopic dynamics of neutron stars. They are dominated by the NN interaction, and therefore they will be discussed after the correlations among nucleons are introduced and discussed. In finite nuclei viscosity must be treated in a different scheme from that in nuclear matter, since the presence of the nuclear surface plays a major role. This issue will also be discussed in the section on viscosity.

3. Basic phenomenology

3.1. Mass formula and saturation

Some of the basic phenomenological data on the nuclear medium come from the semi-empirical mass formula ([6, 7]). The aim of the mass formula is to express the total binding number \( B(A, Z) \) of a nucleus as a smooth function of the mass number \( A \) and the atomic number \( Z \). Several versions of the formula exist. In any case, the physical basis is the LDM or the so-called droplet model (DM). In these models, the nucleus is described as a drop of a quantal liquid, the nuclear medium, whose properties are derived as for a classical liquid, with the addition of some quantal corrections, typical of the nuclear systems. A set of parameters are introduced, some macroscopic in character, some other more connected to a Fermi liquid behavior. The refined versions of the purely phenomenological mass formula contain several terms and can be written as
\[ B(A, Z) = a_V A + a_S A^{\frac{3}{2}} + (a_I + a_{IS}/A^\frac{1}{2}) \frac{(N-Z)^2}{A} \]
\[ + a_c \frac{Z^2}{A^\frac{1}{2}} - \delta_P + E_D \] (8)
which, in the written order, contains the bulk contribution (parameter \( a_V \)), the surface correction (\( a_S \)), the bulk and surface symmetry energies (\( a_I \) and \( a_{IS} \) respectively), the Coulomb energy (\( a_c \)), the pairing energy (\( \delta_P \), to be discussed in detail in section 7.4, and the deformation energy (\( E_D \)). The overall trend of the empirical binding energy of nuclei and the way it can be reproduced by this simple formula, by adjusting the set of parameters \( a \), are discussed in basic books [8], where the meaning and possible forms of the different terms are discussed in more detail. The values of the parameters depend slightly on the particular form used for \( \delta_P \) and \( E_D \) [9–11].

The value of the bulk energy \( a_V \) in all cases is very close to \(-16 \text{ MeV}\). This formula provides an excellent fit to the smooth part of the binding energy of nuclei throughout the nuclear mass table with a few parameters. This fact supports the interpretation of each term as schematically indicated above. A partial justification of the mass formula can be obtained within the semi-classical scheme of approximation. In fact, it is possible to show [8] that the smooth function \( B(A, Z) \) can be considered the first term of the expansion in \( h \) of a mean field estimate of the nuclear binding energy. The deviations, which are actually very small in percentage, are therefore interpreted as ‘shell corrections’ [5], i.e. corrections coming from the quantal effects related to the finite size of nuclei. Systematic methods to estimate these effects have been devised by many authors, in particular by Strutinsky [12]. They will be discussed in section 8.1.

The very fact that in the fitting procedure a constant term \( a_V \) can be identified well as one of the relevant terms indicates that this term can be indeed interpreted most naturally as the bulk part of the binding energy, namely the energy per particle of the infinite symmetric nuclear matter. This can be also seen if one extrapolates the formula for \( A \to \infty \), provided \( N = Z \) and the Coulomb energy is neglected. Then, in this case, only the first term survives. Similarly, the coefficient \( a_I \) can be identified with the nuclear matter symmetry energy per particle. However, it has been argued recently [13, 14] that these parameters could reach the asymptotic values only at an exceedingly large value of the mass number \( A \), and therefore along the nuclear mass table they still contain a smooth dependence on the mass and atomic number. In any case, even the simple liquid drop formula (8) is able to accurately fit [5, 16] the smooth part of the nuclear binding along the stability valley.

The DM [15] extends the LDM by including additional contributions, in particular terms that introduce the finite width of the nuclear surface, curvature terms, a term taking into account the possibility of a slight compression of the nuclear medium in the nucleus, the displacement of the proton surface from the neutron one and the deformation. The model assumes that the total energy of the nucleus can be written as the integral over the volume of a local energy density, which is a function of the proton and neutron local density. In principle, the total energy is then a functional of the proton and neutron density profile. In the simplest form of the DM model the functional is extended to include the asymmetry, the deviation from saturation density in the bulk, the surface width and the curvature. The energy is minimized, keeping the coefficients of the expansion as the parameters of the model. Let us consider the energy per particle \( e(\rho, I) \) in nuclear matter as a function of density and asymmetry \( I = (N-Z)/A \). In general, in the vicinity of the saturation point for symmetric matter \( I = 0, \rho = \rho_0 \) this energy is usually expanded as
\[ e(\rho, I) = e(\rho, 0) + a_{sym}(\rho) I^2 \]
\[ = e(\rho_0) + \frac{1}{18} K \epsilon^2 + \left[ J(\rho_0) - \frac{L \epsilon}{3} + \frac{K_{sym} \epsilon^2}{8} \right] I^2 \] (9)
where \( \epsilon = (\rho - \rho_0)/\rho_0 \), \( K \) is the incompressibility at the saturation point, \( J(\rho_0) \) is the symmetry energy coefficient at saturation and the parameters \( L \) and \( K_{sym} \) in the square bracket characterize the density dependence of the symmetry energy around the saturation point. The particular numerical factors are used for convenience. In the DM the last term in square
and proton densities): of calculations in semi-infinite nuclear matter with over 100 place constraints on the parameters on the neutron skin in nuclei from anti-protonic atoms can be written \[ 15\], neglecting again quartic and curvature term proportional to the surface asymmetry coefficient is not possible, since the result can be written \[ 15\], neglecting again quartic and curvature term proportional to \( a_{\text{IS}} \) is given by the expression inside the round brackets. Here the quantity \( Q \) is a particular combination of the surface and bulk parameters:

\[
Q = H \left( 1 - \frac{2}{3} \frac{P}{J} \right).
\]

One can note that a clean separation between bulk and surface asymmetry coefficient is not possible, since the term proportional to \( J^2 A^{2/3} \) contains both surface and bulk parameters, and therefore it cannot be straightforwardly identified with the surface symmetry energy. This expression also indicates that \( a_{\text{IS}} \) is correlated with the density dependence of the bulk symmetry energy (the parameter \( L \)), on the one hand, and with the neutron skin thickness (parameters \( H, P \)), on the other.

More recently \[ 13, 14\] the issue of the surface symmetry energy was extensively analyzed within a variant of the DM, where a direct link of the surface symmetry energy was proposed with the density dependence of the symmetry energy in nuclear matter, i.e. the function \( a_{\text{sym}} \) in equation (9). The subject was also approached \[ 14\] with an extensive analysis of calculations in semi-infinite nuclear matter with over 100 Skyrme forces. In \[ 17\] it was shown that measurements on the neutron skin in nuclei from anti-protonic atoms can place constraints on the parameters \( L \) and \( K_{\text{sym}} \). The DM can be developed on a more microscopic level within the Thomas–Fermi (TF) approximation for the energy density functional (EDF) \[ 15\]. In this case the parameters to be fitted to the nuclear binding are the ones which characterize the TF functional, and are again directly related to the properties of nuclear matter. An extended version of the DM \[ 18, 19\] also includes deformation parameters. If shell and microscopic pairing corrections are systematically included, with a number of parameters close to 40, it was possible \[ 18, 19\] to fit binding energies of nuclei throughout the mass table \( (A \geq 16) \) with astonishing precision. The overall error for the total binding energy is 0.669 MeV, but it is reduced to 0.448 MeV if only medium–heavy nuclei \( (A > 65) \) are considered, in line with the macroscopic character of the model.

The appealing physical feature of these models is the direct relationship between each parameter and a definite property of the nuclear medium. In principle, a phenomenological analysis based on these models can provide basic physical quantities which characterize the nuclear medium, both in its homogeneous macroscopic phase and in finite nuclei, such as the symmetry energy, the surface tension, the compressibility and so on.

A similar but more microscopic method, to determine the gross (macroscopic) properties of finite nuclei and of the nuclear medium, has been developed in recent years \[ 20–23\]. In this scheme the procedure is in some sense reversed. The binding energies of nuclei are fitted along the mass table with effective forces (Skyrme, Gogny or relativistic mean field), which can be taken from the wide literature on the subject. Then the shell effects are subtracted and the remaining smooth part is used to extract the macroscopic physical parameters that characterize the LDM or the DM. In order to do that, in the smooth trend of the energy one extracts the terms proportional to \( A^{1/3} \) and to \( A^{2/3} \) as well as to \( J^2 \). To better identify the trend as a function of \( A \) of the terms that tend to a constant at increasing \( A \) (bulk energy, compressibility) calculations are extrapolated to huge values of \( A \) (up to \( 10^9 \)). The systematics on an ample set of Skyrme forces show fluctuations of the LDM parameters from one force to another, especially the surface energy and in general the quantities that depend on asymmetry. The isoscalar quantities, such as bulk energy and compressibility, appear more stable. Systematic discrepancies are also observed between Skyrme forces and relativistic mean field. A quantitative analysis, as well as a comparison with other schemes, particularly of \[ 14\], can be found in \[ 21\].

Of course the mass formula contains information only at the saturation density \( \rho_0 \), and therefore the knowledge of the complete EoS goes well beyond the content of equation (8). As mentioned briefly in the introduction, information on the EoS at \( \rho \neq \rho_0 \), finite temperature and large asymmetry are expected to come from heavy ion collision experiments and astrophysical observations.

Finally, one has to mention that the notion itself of saturation also comes from the observation that the central density of medium–heavy nuclei is pretty constant throughout the nuclear chart. This fundamental phenomenological result has been obtained mainly from elastic electron scattering, which provides the whole charge distribution in nuclei. The total density is then obtained by assuming that the neutron density scales as \( N/Z \) with respect to the proton density. For a recent analysis see \[ 25, 26\]. The value of the density is around 0.16 fm\(^{-3}\) and it is interpreted as the density at which symmetric nuclear matter displays its minimal energy (saturation point). Up to now asymmetry effects are within the
overall phenomenological uncertainty on the saturation point. The charge distribution obtained from electron scattering, in particular in the surface region, can be reproduced with great accuracy by most of the mean field calculations [25, 26] based on Skyrme forces. These effective forces seem to be able to incorporate all the correlations that affect the charge profile of nuclei.

3.2. Giant resonances in nuclei

If nuclei are viewed as droplets of nuclear matter, it is natural to consider the possibility of their excitations. The quantization of these modes corresponds to collective excitations of the nucleus as a whole. This is the physical basis of the Bohr–Mottelson model [5] for the nuclear modes of excitation. These collective modes have found clear and extensive experimental evidence [5, 8]. The vibrational excitations are classified according to the multipolarity of the surface oscillations and their isospin character, i.e. if neutrons and protons oscillate with the same or opposite phase (for simplicity we neglect spin flip). They are universally called 'giant resonances', since they usually carry a large fraction of the total strength of the corresponding spectral function.

The simplest oscillation is the isoscalar monopole vibration, corresponding to a compressional mode of the nucleus. The question that arises naturally is then whether it is possible to extract from the study of the monopole excitation, in particular from its energy, the compression modulus of nuclear matter at saturation. This possibility has been explored extensively for many years [3, 8]. From a purely macroscopic point of view there are essentially two difficulties along this line: (a) to calculate the excitation energy, as for an harmonic oscillator, not only the incompressibility is needed, giving the restoring force, but also the dynamical mass that should be used, and (b) the surface tension of the nucleus should play some role, but there is not any obvious relationship between surface tension and incompressibility. As for point (a), microscopically there is a general method to estimate the collective mass of an excitation, the so-called 'cranking mass' [5, 8]. At the macroscopic level it can be estimated assuming a particular velocity flow, in particular the scaling hypothesis implies a radial velocity proportional to the radius, in which case the inertial parameter has an analytic expression and it is proportional to the square of the radius of the nucleus [3, 27]. It is difficult to handle point (b) with a satisfactory accuracy at the macroscopic level, and it is necessary to introduce some microscopic elements in the theory. The most successful semi-microscopic method is the Skyrme functional method, to be discussed with some detail in section 8.2. According to this well-known method, an effective NN interaction is introduced and the energy of the nucleus is assumed to be equal to the Hartree or Hartree–Fock energy calculated with such a force, i.e. minimizing the mean field energy functional calculated with the force. The effective force is semi-phenomenological in character and therefore it contains a few parameters. With the same force it is also possible to calculate the nuclear matter EoS, and to tune the parameters in order to obtain the correct saturation point and a given incompressibility modulus.

In this way it is possible to check whether a correlation exists between the energy of the isoscalar monopole vibration and the nuclear matter incompressibility. All the parameters are fitted in any case to reproduce the binding energy of a large set of nuclei, as well as other phenomenological data. For an extensive application of this method, see, e.g., [28, 29]. One finds indeed that a correlation exists between incompressibility and the position of the monopole giant resonance, so that, in principle, it is possible to extract from the experimental data the value of the incompressibility in nuclear matter. For the method to be reliable, the result should be essentially independent of the particular Skyrme force used in the calculations. Unfortunately this is not the case. It has been shown more recently [37] that the relationship between the centroid of the monopole excitation and the value of the incompressibility is not unique, but it also depends on other details of the force, mainly it is correlated with the density dependence of the energy density and symmetry energy of the force [37, 38]. This also explains, at least partially, the reason why the incompressibility extracted from the relativistic mean field functional tends to be systematically higher than the one extracted for the non-relativistic Skyrme functional. At present, the constraints on the value of the nuclear matter incompressibility from the monopole excitation are not so tight. It is fair to say that it can be approximately constrained between 210 and 250 MeV. A more refined value can be expected to come out in the near future from additional analysis of phenomenological data.

The prototype of giant resonance is surely the dipole mode, where neutron and proton oscillate against each other. The restoring force in this case is the symmetry energy. Both volume and surface contribution can be present. In fact, a recent analysis [17, 39] on the correlation between dipole resonance energy and symmetry energy indicates that such a correlation can be obtained if the symmetry energy is taken at 0.1 fm$^{-3}$, about 2/3 the saturation density. In any case it is difficult to get a strong constraint on the nuclear matter symmetry energy at saturation from the giant dipole resonance.

The isoscalar quadrupole mode is more connected with the surface tension in nuclei, since, in first approximation, the mode occurs at a constant volume. However, this correlation has not been explored, probably because in this case it is more difficult to estimate the collective mass term.

The other giant resonances do not involve only one or a few characteristics of the effective forces, and therefore they can hardly be used to study the definite physical properties of the nuclear medium.

Finally, the study of the giant resonance damping has to be mentioned, which is measured by their width. From a macroscopic point of view such a damping should be connected to some sort of viscosity of the nuclear medium. Unfortunately, the physical situation is much more complicated. First of all, one should take into account that we are dealing with a quantal liquid, as discussed in section 5. Therefore, giant resonances should actually be considered as zero sound modes, and, in principle, no hydrodynamical picture should be adopted. Furthermore, the presence of the nuclear surface introduces a different type of damping, the so-called one-body dissipation.
The applicability of such a damping mechanism requires the onset of a certain degree of chaos in the single particle dynamics [44], and therefore it seems ill-suited to giant resonance of low multipolarity. Probably the octupole vibration can be partly affected by such a type of dissipation. Finally, the damping can be produced by the emission of nucleons, the so-called decay damping. For all these reasons the extraction from the width of any sort of viscosity is strongly hampered, and the study of the giant resonance width must rely completely on nuclear structure analysis [45, 46]. Shear and bulk viscosities in nuclear matter, as present in neutron stars, must be predicted only on a purely theoretical basis when microscopic models of astrophysical phenomena are developed.

3.3. Heavy ions

In a period that includes at least the last two decades, intensive studies of heavy ion reactions at energies ranging from a few tens to several hundred MeV per nucleon (hereafter indicated as MeV A\(^{-1}\)) have been performed in different laboratories throughout the world. One of the main goals, probably the principal one, has been the extraction from the data on suitable observable quantities of the gross properties of the nuclear EoS. An enormous body of literature exists on the subject, and observable quantities of the gross properties of the nuclear EoS and its viscosity. An enormous body of literature exists on the subject, and therefore we will focus on a few items that, according to our personal view, are connected with established and insightful results.

3.3.1. Flows and differential flows. It can be expected that in heavy ion collisions at large enough energy nuclear matter is compressed and that, at the same time, the two partners of the collisions produce flows of matter. In principle, the dynamics of the collisions should be connected with the nuclear medium EoS and its viscosity.

However at low enough energy the cross section is dominated by deep inelastic processes, where the target and projectile keep their identity during the collision, stick together for a while and separate again. This reaction mechanism persists up to about 10 MeV A\(^{-1}\). At increasing energy the so-called ‘multifragmentation’ regime is encountered, where after the collision numerous nucleons and fragments of different sizes are emitted. Usually, in non-central collisions, one distinguishes target-like and projectile-like fragments, the different sizes are emitted. Usually, in non-central collisions, one distinguishes target-like and projectile-like fragments, the so-called spectators, and the participant region, where matter is partly stopped and tends to form a partly equilibrated zone. In semi-classical simulations of heavy ion collisions two main ingredients are introduced, the single particle mean field \(U\) and the in-medium NN scattering cross section \(\sigma_{\text{NN}}\). A Boltzmann-like kinetic equation is assumed for the nucleons:

\[
\frac{df}{dt} + \nabla_{p} \epsilon \cdot \nabla_{f} - \nabla_{\epsilon} \cdot \nabla_{p} f = I \left( \frac{\sigma_{\text{NN}}}{2\pi} \right) \quad (14)
\]

where \(n = n(r, p, t)\) is the single particle density distribution in phase space, \(\epsilon = p^{2} / 2M + U(r, p, t)\) the local single particle energy and \(I\) the two-body collision integral that describes the loss and gain of particles, at a given phase space point, due to scattering of nucleons in the medium. The single particle potential \(U\) can be written as

\[
U(r, p, t) = \int \! d^3r' \! d^3p' v_{\text{eff}}(r, p; r', p') n(r', p', t) \quad (15)
\]

where \(v_{\text{eff}}\) is the effective NN interaction in the medium. In general \(U\) is identified with the single particle potential in nuclear matter at the local density, e.g. the Brueckner potential.

In practice it is not possible to obtain directly from the data indications on the EoS and the scattering cross section in the medium. Even if \(4\pi\) detectors, with which is possible an (almost) complete reconstruction of the collision dynamics, have been developed, the interpretation of the data is not unique. The usual procedure is to assume a set of possibility for the potential \(U\) and, more rarely, for \(d\sigma / d\Omega\), and to find which ones in the considered sets fit the data better. Once \(U\) is chosen, the EoS can be calculated, since the single particle potential fixes the interaction energy per particle. Analogously, the scattering cross section determines the transport properties of the nuclear medium. However, very often the NN scattering processes have only the effect of driving the system toward equilibrium, or quasi-equilibrium, at least in the participant zone, so that the information on the cross section is often too indirect to be accessible. In any case it is true that the results of the simulations depend, in general, on both quantities.

One of the quantities that is more often analyzed is the so-called transverse momentum, also in its differential form. If the reaction plane is the \((x, y)\) plane and the initial direction of the two colliding nuclei is along the \(y\)-axis, one calculates the average momentum \(p^{y}\) along the \(x\)-axis of the nucleons as a function of their velocity \(y\) (or ‘rapidity’) along the \(y\)-axis:

\[
F(y) = \langle p^{y} \rangle_{y}; \quad F^{y}(y) = d\langle p^{y} \rangle_{y} / dy. \quad (16)
\]

At high enough energy the flow is strongly affected by the matter compression during the collision and dominated by the corresponding pressure. Then the initial flow undergoes a strong repulsion from the interaction zone, which means that \(F(y)\) turns from negative to positive values as \(y\) changes sign, with a well-defined slope. In fact the negative and positive values of \(y\) label the target-like and projectile-like fragments or nucleons, at least for \(y\) around zero (‘mid-rapidity’ region). It was hoped that the slope \(F'(0)\) could sharply characterize the nuclear matter EoS. Unfortunately in the simulations only a weak dependence on the EoS stiffness is observed, somehow obscured by the numerical uncertainty of the simulations themselves [47]. This is probably due to the competing effect of the NN collisions incorporated in the collision integral \(I\). For the same reason, it is difficult to extract any solid information on the in-medium cross section. To further complicate the situation, different simulation methods, such as the quantum molecular dynamics [48–51], give slightly different results under the same physical conditions, which increases the uncertainty on the EoS that can be extracted.

Despite all these difficulties, some gross constraints on the nuclear EoS can be extracted. In [52] this effort was summarized by plotting the region where any reasonable EoS should pass through in the pressure versus density plane. The
they cannot be directly re-absorbed by nucleons. However, this does not necessarily mean that the interaction can be completely neglected. In particular, let us consider the lightest strange particle, the K-meson (kaon). In this case one has to distinguish between K− and K+, since the negative kaon forms resonances with nucleons even at a low energy and therefore their interaction with the nuclear medium cannot be neglected. Unfortunately the interaction potential felt by K− in the medium is not so well known theoretically, and this introduces an uncertainty in the analysis of the experimental data. The situation for K+ is different, since no resonance with nucleons is present and the interaction can be treated almost perturbatively, and actually the uncertainty is much reduced. The main mechanism of K+ production is through the excitations of a nucleon to a Δ, which in turn decays in a Λ and a K+.

\[ NN \rightarrow N\Delta \rightarrow N\Lambda K^+ . \]  

It is then clear that the simulations must include nucleon excitations and must be relativistic. The uncertainty is mainly due to the not so well-known potential of the Δ in the nuclear medium. Fortunately this does not affect the final results substantially, which appear to be under control also numerically and almost independent of the simulation method. An excellent and extensive review of the subject, both at experimental and theoretical level, can be found in [47, 55]. Here we restrict ourselves to some of the conclusions that can be drawn from this line of research, that has been developed over several years.

The optimal energy for this type of investigation is close to or even below the two-body threshold, since then the only way to produce the kaons is by compression of the matter. Since at the threshold the production rate increases steeply, there is a strong sensitivity to the value of the maximum density reached during the collision, and this is an ideal situation for studying the EoS and its incompressibility. The comparison of the simulations with the experimental data on K+ production, noticeably the ones from the KaoS [56] and FOPI [57] collaborations, points in the direction of a soft EoS. More precisely, the interval of compatible incompressibility is narrower than those obtained from the analysis of flows:

\[ 180 \leq K \leq 250 \text{ MeV}. \]  

These values are compatible with those obtained from the monopole oscillations, see section 3. However, it has to be kept in mind that, in the simulations, kaon production occurs at a density about 2–3 times larger than saturation, \( \rho \geq 2\,\rho_0 \), and therefore the two sets cannot be fairly compared. In any case, a stiff EoS above saturation seems to be excluded from this analysis, as is apparent from figure 3, taken from [55].

### 3.4. Astrophysics

The nuclear medium is directly and massively involved in core-collapse supernova explosions, where nuclear matter is compressed at supra-saturation density and triggers a shock wave that is the main agent of the outflow. The properties of nuclear matter completely determine the structure of neutron stars and the phenomena that occur in their interior or at the surface. Indirectly, the peculiarities of the nuclear medium are relevant for many other processes, such as nucleosynthesis.
3.4.1. Supernovae. One of the major puzzles in astrophysics is the mechanism that drives the explosion of core-collapse supernovae. In the complex simulations of the after-bounce stage of the supernovae the shock wave is stalling due to the energy loss, mainly produced by the disintegration of the nuclei in the envelope. It is common wisdom that only the revival of the shock by the blast of neutrinos, initially trapped, can produce the final explosion [59]. For a long time it was believed that a direct link should exist between the possibility of explosion and the value of the incompressibility in nuclear matter. Many years of effort on supernova event simulations have disproved this belief and the difficulty of getting a real explosion by computer simulation has indicated that only a detailed treatment of the different aspects of the phenomenon could lead to a definite solution of the puzzle [60]. It seems now [61, 62] that the key ingredient is the three-dimensional character of the process, that makes the structure of the turbulent flow of matter more tiny and renders (with respect to a two-dimensional situation) the energy deposition of neutrinos on the matter close to the shock more efficient. The possibility of the explosion is therefore not determined by the details of the nuclear EoS. However, many quantitative aspects of the explosion do depend on the EoS, such as the total energy release, neutrino luminosity and the timing of the whole phenomenon as determined by the neutrino mean free path (for which nucleon correlations in nuclear matter are essential). Unfortunately we are not yet at a stage where detailed quantitative predictions and comparisons can be made in relation to phenomenology because of the enormous complexity of supernovae explosions.

It has to be kept in mind that the nuclear medium present in supernovae, as well as in neutron stars, is very asymmetric and in different physical conditions than in heavy ion collisions. In particular the time scale is different by several orders of magnitude, so that many processes that the nuclear medium can undergo in supernovae cannot occur in heavy ion reactions because the time is very short.

3.4.2. Neutron stars. The compact remnant of supernovae explosions, if the collapse does not end in a black hole, is a neutron star (NS), an extremely dense object, which in the standard view is composed of nuclear matter, electrons and muons. Leptons are necessarily present because initially the star is of course neutral. The structure of the NS is determined by the properties of the nuclear medium in an extremely wide range of densities, from a few times the saturation one at the center, down to values several orders of magnitude smaller close to the surface. An old enough NS is virtually at zero temperature and its exterior part is actually formed by a Coulomb crystal of nuclei [63, 212]. Below this outer crust, an inner crust is present, where nuclei are surrounded by a gas of dripping neutrons. In this region the EoS of pure neutron matter at a low density is relevant. At the same time nuclei are very exotic, challenging our knowledge of the nuclear medium at large asymmetry. Below the crust asymmetric homogeneous nuclear matter fills the whole space. Again the asymmetry is very large, and a direct link with nuclear structure and heavy ion reactions in terrestrial laboratories is not possible. The challenge for microscopic many-body theory is to establish this link and try to test it by comparison with phenomenology. In this section we discuss the NS maximum mass and a few related issues, leaving other issues to sections 6–9. A NS is bound by gravity, and it is kept in hydrostatic equilibrium only by the pressure produced by the compressed nuclear matter. It is then apparent that the nuclear matter EoS is the main medium property that is relevant in this case, as can be seen in the celebrated Tolman–Oppenheimer–Volkoff [65, 66] equations, valid for spherically symmetric NS:

$$dP = -G \frac{\epsilon \rho}{r^2} \left(1 + \frac{\rho}{\epsilon}\right) \left(1 + \frac{4\pi P r^3}{m}\right) \left(1 - \frac{2 Gm}{r}\right)^{-1}$$ \hspace{1cm} (19)

$$dm = 4\pi r^2 \epsilon$$ \hspace{1cm} (20)

where $G$ is the gravitational constant, $P$ the pressure, $\epsilon$ the energy density and $r$ the (relativistic) radius coordinate. To close the equations we need the relation between pressure and density, $P = P(\epsilon)$, i.e. just the EoS. In the Newtonian limit the energy density is just the mass density and in each parenthesis the second term is neglected, and we get the equations of hydrostatic equilibrium in non-relativistic mechanics. The use of general relativity (GR) is demanded by the strong gravitational field. Integrating these equations one gets the mass and radius of the star for each central density. Typical values are 1–2 solar masses ($M_\odot$) and about 10 km, respectively. This indicates the extremely high density of the object. It turns out that the mass of the NS has a maximum value as a function of radius (or central density), above which the star is unstable against collapse to a black hole. The value of the maximum mass depends on the nuclear EoS, so that the observation of a mass higher than the maximum one allowed by a given EoS simply rules out that EoS. Up to now the best microscopic EoS are compatible with the largest observed masses, that are close to 1.7 solar mass [67]. It would of course be desirable to have some phenomenological data also on the radius of NS. Unfortunately this is quite difficult, but some tentative analyses look promising [68]. In particular, a recent
particular the EoS of [70] looks too repulsive at a high density other microscopic EoS do not show the same agreement, in with the extracted observational constraints. It turns out that also reported. The theoretical EoS appears to be compatible (BHF) calculations, to be discussed in the next section, is also reported. The theoretical EoS appears to be compatible with the extracted observational constraints. It turns out that other microscopic EoS do not show the same agreement, in particular the EoS of [70] looks too repulsive at a high density [69]. These boundaries obtained from astrophysical data are complementary to those obtained from heavy ion reactions, see figure 2 in the previous subsection. In fact, in heavy ion collisions the tested matter is essentially symmetric, while in NS the matter is highly asymmetric. Considered together, the two types of constraints probe the density dependence of the symmetry energy.

Unfortunately the theoretical situation for the EoS in NS and for the maximum mass is actually much more complicated. In fact, in NS weak processes have time to develop and, if energetically convenient, they can produce strange particles such as hyperons, and then change the composition of the nuclear medium. This is clearly at odds with what can happen in heavy ion reactions, where the collision time is short and the multiplicity of strange particles is so small that a bulk strange matter cannot be formed. In contrast in NS, at least above a certain density, the difference of the neutron and proton chemical potentials is so high to overcome the mass difference between hyperons and neutrons. This is indeed the case, according to microscopic calculations [71]. Above 2–3 times saturation density Σ− or Λ hyperons appear. This softens the EoS so much that the maximum mass becomes smaller than the most established NS mass [67, 72]. This result seems to be quite robust and not dependent on the not so well-known hyperon–nucleon or hyperon–hyperon interaction [73]. The only way out seems to be, up to now, a possible phase transition to quark matter. Indeed, calculations on the basis of simple models [74–78] can result in a maximum mass that is (marginally) compatible with the observed largest mass. Of course, it could also be that the quark matter EoS is stiffer than assumed in simple models [79–82], but in any case it seems that we are close to testing our knowledge on the QCD deconfined phase at high densities. All this makes it clear that the NS physics connected with the central high-density core is quite different from that in heavy ions, where in ultra-relativistic collisions at the Large Hadron Collider (LHC) the deconfined QCD phase is tested at zero baryon density and high temperature. However, it is a basic challenge to the theory to be able to connect the transition to quark matter in these two extreme different physical situations. Advances both in phenomenological observations and theoretical methods are needed.

The maximum mass problem can clearly lead far from the physics of the ‘nuclear medium’, at least as it is considered in traditional nuclear physics. However the distinction between traditional nuclear physics and QCD physics is partly artificial, and they should be considered as two complementary aspects of the same physical realm.

Finally, one has to observe that an observation of a maximum mass of 2 solar masses or higher would be a real breakthrough in our knowledge on high-density nuclear medium, since it would question the simple models of quark matter. Recent observations [83] on the pulsar of the binary system PSR J1614-2230 seem to indicate such a possibility, and as anticipated in [74], it would imply the necessity of a repulsive interaction in quark matter [84].

4. From the nuclear interaction to the correlated nuclear medium

The properties of the nuclear medium are determined or strongly affected microscopically by the features of the NN interaction. In particular, one of the main characteristics of the NN interaction is the presence of a hard repulsive core, whose relevance can be hardly overlooked. Furthermore, any realistic NN potential must include a complex structure of operators involving spin, isospin and orbital angular momentum. This non-trivial structure is one of the main reasons that renders the microscopic many-body theory of nuclear matter and nuclei so hard to handle. Another characteristic of the NN interaction is the presence of a quasi-bound state (1S0 channel) and a bound state (3S1−3D1 channel) in the s-wave. This peculiar
4.1. Sketch of the nucleon–nucleon interaction

The NN interaction was intensively studied when nuclear physics started developing. Over the years the phenomenological analysis has been more and more refined. Presently, the phase shifts in different two-body channels are known with high precision up to an energy of about 300 MeV in the laboratory, even if discrepancies between the results of different groups still persist [85]. For future reference we recall very briefly the connection between the two-body interaction and the cross section, the quantity which is actually measured. If one assumes that the interaction can be described by a static non-relativistic potential \( v \), the scattering process at the energy \( E \) can be described by the \( T \)-matrix. It can be calculated solving the integral equation

\[
T(E) = v + v \frac{P}{E - H_0} T(E)
\]

(21)

where \( H_0 \) is the free kinetic energy Hamiltonian and \( P \) indicates the principal value for the integral, which fixes the stationary wave boundary conditions. The phase shifts are connected with the diagonal matrix elements in the usual way. The NN interaction, as we will see, contains not only a central interaction part but also more complex operators, and one has both single and coupled channels \( \alpha \), characterized by the total angular momentum \( j \), total spin \( S \), total isospin \( T \) and the orbital angular momenta \( l, l' \) (\( l = l' \) for single channel). Fitting the data on the cross sections at different energies, the phase shifts \( \delta_\alpha \) for each channel can be extracted. For the NN interaction a particular form is assumed, as suggested by the meson–nucleon theory of strong interaction, which contains several parameters that are fitted to reproduce the phase shifts. In this way one can fix the NN potential, which however partly remains model dependent; for details, see e.g. [86]. Here we sketch the main ideas of the meson theory of the NN interaction. For simplicity we use a non-relativistic treatment, as a first schematic introduction to the theory of nuclear forces. Of course, the correct framework is the relativistic field theory of the meson–nucleon system. Let us consider the simplest possible case, i.e. the coupling of nucleons with a spinless neutral meson. If we indicate by \( b^{|q} (\hat{b}_q) \) the creation (annihilation) operator of a meson with momentum \( q \), the simplest coupling term is the scalar one

\[
H_c = G_s \int d^3x \psi(x) \psi(x) \phi(x)
\]

(22)

which describes the processes of emission and absorption of a meson. The two processes are included together, with the same weight, as required by the hermiticity of the interaction \( H_c \) and of the scalar field \( \phi(x) \). Momentum conservation has been explicitly worked out and the constant \( G_s \) is the meson–nucleon coupling constant. In equation (22) \( \omega(q) \) is the meson energy and the factor in front \( 1/\sqrt{2 \omega(q)} \) comes from the usual quantization of the boson (meson) field in a set of harmonic oscillators [87]. The creation and annihilation meson operators satisfy the usual boson commutation relations

\[
[b_q, b^{|q} ] = - [b^{|q} b_q] = 0 \quad b_q b^{|q} - b^{|q} b_q = \delta_q (k - k')
\]

(23)

In equation (22) the product \( a^ll'l' \) includes a scalar product in the spin component, \( a^ll'l' = \sum_\sigma a^ll'l' \). The total Hamiltonian will also include, besides the free nucleon Hamiltonian, the free meson part

\[
h_0 = \sum_q \omega(q) b_q b^{|q}
\]

(24)

and we can take the relativistic expression \( \omega(q) = \sqrt{(mc^2)^2 + q^2 c^2} \) since the meson mass \( m \) is usually much smaller than the nucleon mass, and therefore its kinematics is surely relativistic. We will indicate by \( H_0 \) the non-interacting part of the Hamiltonian. Perturbation theory in \( H_0 \) of different physical quantities can easily be developed and the different terms can be represented by diagrams. For our purposes only the lowest order has to be considered. In the framework of the meson–nucleon theory, the effective NN interaction can be identified with the irreducible part of the two-nucleon scattering matrix \( T^{(2)} \). By ‘irreducible’ here we mean the set of (connected) diagrams which cannot be separated into two distinct parts by cutting two nucleon lines at any given level along the diagram. The general perturbation theory for \( T^{(2)} \) can be obtained from the usual expansion for the scattering matrix

\[
T^{(2)}(E) = H_0 + H_c \frac{1}{E - H_0} T^{(2)}
\]

(25)

\[
= H_0 + H_c \frac{1}{E - H_0} H_c + H_c \left( \frac{1}{E - H_0} H_c \right)^2 \ldots
\]

In this expansion we have to select the processes which correspond to the scattering of two nucleons and are irreducible. The lowest order which can contribute is the second order, since to first order the coupling term \( H_c \) cannot describe only emission or absorption of a meson. Let us denote by \(|kk'\rangle \) the free (anti-symmetrized) state \( a^ll'l' \).
of two nucleons with momenta \( k \) and \( k' \). The amplitude for the scattering from the state \( |k_0 k'0\rangle \) to the state \( |k_1 k_1'\rangle \) can be extracted from the second-order term of \( T^{(2)} \):

\[
\langle k_1 k_1'|T^{(2)}|k_0 k_0\rangle \approx \langle k_1 k_1'|H_c \frac{1}{E - H_0} H_c |k_0 k_0\rangle. \tag{26}
\]

If we insert the expression of equation (22) for \( H_c \), since by definition \( H_0 \) is diagonal in the free state representation, we can use Wick’s theorem for the vacuum state in a straightforward way. For the meson operators this is trivial (they commute with the nucleon operators). The four contractions which can contribute can be depicted as in figure 5 which give four distinct contributions. The corresponding analytical expressions for the two-body scattering matrix is given by

\[
\langle k_1 k_1'|T^{(2)}|k_0 k_0\rangle = \sum_q \frac{\hbar}{v} G^2 \frac{\delta K}{2\omega q} \left( \delta_K(q - k_0 + k_1) - \delta_K(q - k_0 - k_1) \right)
\]

\[
\times \left\{ \frac{1}{E_0 - E + E_{k_0 - q} - E_{k_1} + \omega_q} + \frac{1}{E_0 - E + E_{k_0 - q} - E_{k_1} + \omega_q} \right\}
\]

\[
\times \left[ (\delta_K(q - k_0) + k_1) - \delta_K(q - k_0 - k_1)) \right]
\]

where \( E_0 = E_{k_0} + E_{k_1} \) is the initial energy. If we interpret this matrix element as the matrix element of a two-body potential \( v \) between nucleons, this potential is clearly non-local and energy dependent. It is convenient to introduce the relative and total momenta of the initial and final two nucleon states:

\[
Q = \frac{1}{2}(k_0' - k_0) \quad P = k_0' + k_0
\]

\[
Q' = \frac{1}{2}(k_1' - k_1) \quad P' = k_1' + k_1. \tag{28}
\]

Putting \( E = E_0 \), the matrix element of \( v \) can be written as

\[
\langle k_1 k_1'|v|k_0 k_0\rangle = \frac{\hbar}{2V} G^2 \delta_K (P - P') \left( \frac{1}{\omega q - \omega q'} \right)
\]

\[
\times \left\{ \frac{1}{E_{Q_+} - E_{Q_+} - \omega q + \omega q - \omega q' + 1} + \frac{1}{E_{Q_+} - E_{Q_+} - \omega q + \omega q - \omega q' + 1} \right\}
\]

\[
+ \frac{1}{\omega q + \omega q'} \left( \frac{1}{E_{Q_+} - E_{Q_+} + \omega q + \omega q'} + \frac{1}{E_{Q_+} - E_{Q_+} + \omega q + \omega q'} \right). \tag{29}
\]

where \( Q_{\pm} = \pm Q + P/2 \). In the limit of large nucleon mass, the terms corresponding to the nucleon recoil can be neglected, which is equivalent to putting \( E_k \approx M \) everywhere in the expression. In this approximation a very simple form is obtained:

\[
\langle k_1 k_1'|v|k_0 k_0\rangle = \frac{\hbar}{V} G^2 \delta_K (P - P') \left( \frac{1}{\omega q - \omega q'} \right)
\]

\[
= \frac{\hbar}{V} G^2 \delta_K (P - P') \left( \frac{1}{(Q - Q') + \omega q + \omega q'} \right) + \left( Q - Q' \right)^2 + (m c^2)^2 \tag{30}
\]

where the explicit form for \( \omega(q) \) has been used. The expression is Galilei invaraint, since it depends only on the relative momenta \( Q \) and \( Q' \), and is energy independent. The expression of equation (30) can be interpreted as the direct and exchange matrix elements of a local potential. In agreement with the scalar nature of the exchanged meson, the interaction is independent of the spins of the nucleons. The form of such a potential in the coordinate representation is the celebrated Yukawa potential:

\[
v(r) = -G^2 \frac{\hbar}{4\pi h} \left( \frac{1}{q^2 + (m c^2)^2} \right) \frac{e^{-mr}}{\mu r}
\]

The range \( a = 1/\mu = \hbar / mc \) of this potential is the Compton wavelength of the meson. This means that heavier mesons produce shorter potential range. For the validity of the static approximation (i.e. large nucleon mass) it is essential that the ratio between meson and nucleon masses is small. Unfortunately not all the possible mesons which can be considered involved in the NN interaction processes have a small mass compared with the nucleon one. The potentials derived from heavier meson exchange processes therefore have to be considered as effective ones, and the corresponding parameters as effective ones. The latter can therefore differ from the phenomenological ones extracted from meson–nucleon scattering. Equation (22) is schematic, since mesons and nucleons are not point-like particles, and therefore a more refined treatment must introduce vertex corrections in the interaction processes. Usually these corrections are described by phenomenological vertex form factors which multiply the expressions of the type of equation (30) for the NN potentials. The corresponding form in coordinate representation is modified accordingly. From the above results it turns out that the local potential mediated by a scalar meson is attractive. This is the case of the so-called \( \sigma \) meson, which is commonly believed to be responsible for the intermediate range attraction characteristic of the two-nucleon interaction. The lightest known (strongly interacting) meson, the \( \pi \) meson, is known to be a pseudoscalar meson, i.e. a meson with negative internal parity, which is therefore described by a field which changes sign under the parity operation. For the \( \pi \) meson the scalar coupling of equation (22) cannot be used, since the Hamiltonian of strong interaction must be parity invariant. In the non-relativistic limit the only possibility in this case is a pseudovector coupling. Furthermore, the \( \pi \) meson
for the long range attractive part of the NN interaction. Another important case is the exchange of a vector meson, namely a meson of spin one. This case is more complex and requires a fully relativistic treatment. It turns out that a vector meson mainly produces a repulsive interaction. Therefore, at least part of the repulsive core, characteristic of the NN interaction, can be described by the exchange of spin-one mesons, like the $\omega$ meson. However, at distances smaller than the typical core size (~0.4 fm) the structure of the nucleons, as described by QCD, starts to play a role and the meson picture cannot be maintained any more. The meson theory in this range can be regarded as an effective model for more complex processes and the corresponding coupling constants and cutoff have to be considered as parameters to be adjusted to fit the experimental data on NN scattering.

In all these considerations, one assumes that only one meson is exchanged at a time, so that the NN interaction is fully determined by the set of known mesons and by their couplings with nucleons. This is the so-called one boson exchange potential (OBEP). It turns out, however, that the intermediate range attraction cannot be obtained in this way. As already mentioned, it is customary then to introduce a fictitious scalar meson, the $\sigma$ meson, with suitable mass and coupling to reproduce the phenomenological intermediate range attraction. It is usually believed that the hypothetical $\sigma$ meson simulates the simultaneous exchange of two pions both correlated and uncorrelated. The phenomenology on pion–pion scattering gives only a broad structure in the s-wave channel, and therefore a fully satisfactory theoretical basis for the introduction of the $\sigma$ meson is still lacking. For a historical account of the OBEP theory, see [85].

4.2. Theoretical many-body methods

Once the interaction between two nucleons is established, one can try to solve the many-body problem for nuclear matter. However, it is not obvious that the nuclear Hamiltonian includes only two-body forces. Since we know that the nucleon is not an elementary particle, we can expect that the interaction in a system of nucleons is not fully additive, namely that it is not simply the sum of the interactions between pairs of nucleons, but also three or more nucleon forces must be considered. This important issue is discussed later. For the moment we restrict the treatment to the case of two-body forces, which are expected anyhow to be dominant around saturation or slightly above.

4.2.1. The Brueckner–Bethe–Goldstone expansion. The Brueckner–Bethe–Goldstone (BBG) many-body theory is based on the re-summation of the perturbation expansion of the ground state energy. The original bare NN interaction is systematically replaced by an effective interaction that describes the in-medium scattering processes. The in-vacuum $T$-matrix of the general equation (21) is replaced by the so-called $G$-matrix, that takes into account the effect of the Pauli principle on the scattered particles and the in-medium potential $U(k)$ felt by each nucleon. The corresponding

\[ H_c = G_{\mu\nu}(\mathbf{k}) \sum_{\ell q} \left[ \frac{\hbar}{2 \omega_0(q)} (a_{\ell \mu}^\dagger \mathbf{q} \cdot \mathbf{\tau} a_{\ell \nu} b^\dagger_q q) + a_{\ell \mu}^\dagger (\mathbf{v} \cdot \mathbf{\tau}) a_{\ell \nu} b_q \right] \]

where now the $b$ and $b^\dagger$ operators refer to the $\pi$ meson. The quantities $\sigma \equiv \sigma_x, \sigma_y, \sigma_z$ are the usual Pauli matrices which act on the spin variables of the nucleon creation and annihilation operators. The particular form ensures rotational invariance. Since the Pauli matrices form a pseudovector, the expression for $H_c$ is indeed a scalar. The matrices $\mathbf{\tau} \equiv \tau_x, \tau_y, \tau_z$ are the Pauli matrices in isospin spaces and as such they act on the isospin variables of the nucleon operators. The expression includes a scalar product of these three Pauli matrices, which form a three-vector, with the isospin variables of the meson operator, namely $\mathbf{\tau} \cdot \mathbf{b} \equiv \sum_i \tau_i b_i$ (and analogously for $b^\dagger$), where $i$ labels the three possible isospin (charge) states of the $\pi$ meson. The scalar product ensures that $H_c$ is scalar in isospin space, and this is dictated by the charge independence of the nuclear forces, which is phenomenologically observed to a very high degree of accuracy. Following the same procedure as in the case of a scalar meson, one gets the following expression for the direct matrix element of the interaction in the static limit, with $k = Q - Q'$:

\[ \langle Q' P' | v | Q P \rangle = -G_{\mu\nu}(\mathbf{k}) \frac{\hbar}{\sqrt{2}} \delta_k(P - P') \times \frac{\sigma_1 \cdot k (\sigma_2 \cdot k)}{k^2 c^2 + (mc^2)^2} (\tau_1 \cdot \tau_2) \]

(31)

where the matrix elements between spin and isospin states of the corresponding Pauli matrices have to be taken, i.e. the expression still has to be considered an operator in spin–isospin space. It is customary to introduce the tensor operator

\[ S_{12} = 3 (\sigma_1 \cdot k)(\sigma_2 \cdot k) - \sigma_1 \cdot \sigma_2 k^2 \]

and the expression can be written as

\[ \langle Q' P' | v | Q P \rangle = -\frac{1}{12} G_{\mu\nu}(\mathbf{k}) \frac{\hbar}{\sqrt{2}} \delta_k(P - P') \frac{S_{12}}{k^2 c^2 + (mc^2)^2} - \sigma_1 \cdot \sigma_2 \left[ \frac{m^2 c^2}{k^2 c^2 + (mc^2)^2} + \sigma_1 \cdot \sigma_2 \right] \tau_1 \cdot \tau_2 \]

(32)

A pseudoscalar meson gives rise to a tensor–isospin interaction plus a spin–isospin interaction. The last term is a contact interaction (a delta function in coordinate space). A more complete treatment should also include the vertex form factors in this case. It has to be note that the coupling constant has a different definition here from that in the relativistic treatment. More complex couplings are possible, and they naturally arise in a relativistic treatment, which is the framework in which the theory of nuclear forces has ultimately to be formulated. The interaction is attractive or repulsive according to the quantum numbers of the interacting nucleons, namely on the two-body channel (including isospin). It turns out that the tensor part is attractive in the s-wave channels. The $\pi$ meson is responsible
integral equation for the $G$-matrix can be written as
\[ \langle k_1 k_2 | G(\omega) | k_3 k_4 \rangle = \langle k_1 k_2 | v | k_3 k_4 \rangle + \sum_{k'_3 k'_4} \langle k_1 k_2 | v | k'_3 k'_4 \rangle \times \frac{(1 - \Theta_T(k'_3))(1 - \Theta_T(k'_4))}{\omega - e_{k'_3} + e_{k'_4}} \langle k'_3 k'_4 | G(\omega) | k_3 k_4 \rangle \] (33)
where the two factors $1 - \Theta_T(k)$ force the intermediate momenta to be above the Fermi momentum (‘particle states’), the single particle energy $e_k = \hbar^2 k^2/2m + U(k)$ and the summation includes spin–isospin variables. The $G$-matrix does not have the hard core of the original bare NN interaction and is defined even for bare interaction with an infinite hard core. In this way the perturbation expansion is more manageable. The introduction and choice of the single particle potential are essential to make the re-summed expansion convergent. In order to incorporate as much higher order correlations as possible the single particle potential is calculated self-consistently with the $G$-matrix itself:
\[ U(k) = \sum_{k' < k_T} \langle kk' | G(e_{k'} + e_k) | kk' \rangle. \] (34)

An account of the diagrammatic method, the degree of convergence of the BBG expansion and a summary of the results can be found in [53, 91]. Here we restrict ourselves to indicating the expression of the correlation energy at the so-called Brueckner level (‘two hole-line’ approximation)
\[ \Delta E_2 = \frac{1}{2} \sum \langle k_1 k_2 | G(e_{k_1} + e_{k_2}) | k_3 k_4 \rangle (35) \]
where $\langle k_1 k_2 | A = | k_3 k_4 \rangle$. At this level of approximation, which mainly includes two-body correlations, one can find that the corresponding ground state wave function $\Psi$ can be written consistently as
\[ |\Psi\rangle = e^{\hat{S}_2} |\Phi\rangle \] (36)
where $\Phi$ is the unperturbed free particle ground state and $\hat{S}_2$ is the two-particle correlator:
\[ \hat{S}_2 = \sum_{k_1 k_2 k_3 k_4} \frac{1}{4} \langle k_1 k_2 | S_0 | k_1 k_2 \rangle a^\dagger(k_1) a^\dagger(k_2) a(k_2) a(k_1) \] (37)
where the $k$s are hole momenta, i.e. inside the Fermi sphere, and the $k'$s are particle momenta, i.e. outside the Fermi sphere. The function $\hat{S}_2$ is the so-called ‘defect function’. It can be written in terms of the $G$-matrix and it is just the difference between the in-medium interacting and non-interacting two-body wave functions [53, 91]. A recent systematic study of the dependence of the resulting EoS on the NN interaction can be found in [94].

One of the well-known results of all these studies, that lasted for about half a century, is the need of TBF in order to get the correct saturation point in symmetric nuclear matter. Once the TBF are introduced, the resulting EoS, for symmetric matter and pure neutron matter, is reported in figure 6 for the two-body interaction $U_{138}$ (squares). The TBF produce a shift in energy of about $+1$ MeV in energy and of about $-0.01$ fm$^{-3}$ in density. This adjustment is obtained by tuning the two parameters contained in the TBF, as in [92–94] and was performed to get an optimal saturation point (the minimum). For comparison, the variational EoS of [70] is also reported, which will be discussed in the next section. The connection between two-body and three-body forces within the meson–nucleon theory of nuclear interaction is discussed and worked out in [95–97]. The possible interplay between two-body interaction and TBF for the resulting EoS is discussed in [98].

Special attention must be paid to the sub-saturation region of nuclear EoS. At very low-density homogeneous matter must behave as a free Fermi gas, and therefore, as discussed above, the energy per particle must increase with density. To join the region just below saturation, where the energy is a decreasing function of density, a point where the compressibility vanishes must exist. Below this point (lower density) the compressibility is negative, while above it is positive. If the compressibility is negative nuclear matter is unstable toward cluster formation, that is light nuclei are formed and coexist with the bulk. Of course this transition must be present in any calculated EoS, and it is not a peculiarity of the BBG method. The density of the transition point is estimated to be around $2/3$ of the saturation density (symmetric nuclear matter). Cluster formation in nuclear matter has been studied in [99–102] and its effect on the nuclear EoS was studied in [103]. At the same time, each cluster corresponding to a definite light nucleus is modified by the medium with respect to the free space. In the analysis of the experimental data this cluster formation of nuclear matter at low density must be carefully considered.

4.2.2. The variational method. In the variational method one assumes that the ground state wave function $\Psi$ can be written in a form as in equation (36), i.e.
\[ \Psi(r_1, r_2, ...) = \Pi_{i<j} f(r_{ij}) \Phi(r_1, r_2, ...) \] (38)
where $\Phi$ is the unperturbed ground state wave function, properly anti-symmetrized, and the product runs over all
possible distinct pairs of particles. The correlation factor is determined here by the variational principle, i.e. by assuming that the mean value of the Hamiltonian gets a minimum (or in general stationary point):

\[
\frac{\delta \langle \Psi | H | \Psi \rangle}{\delta f} \langle \Psi | \Psi \rangle = 0. \quad (39)
\]

In principle, this is a functional equation for the correlation function \( f \), which however can be written explicitly in a closed form only if additional suitable approximations are introduced. The function \( f(r_{ij}) \) is assumed to converge to 1 at large distances and to go rapidly to zero as \( r_{ij} \to 0 \), to take into account the repulsive hard core of the NN interaction. Furthermore, at distances just above the core radius a possible increase in the correlation function beyond the value 1 is possible.

For nuclear matter it is necessary to introduce a channel-dependent correlation factor, which is equivalent to assuming that \( f \) is actually a two-body operator \( \hat{F}_{ij} \). One then assumes that \( \hat{F} \) can be expanded in the same spin–isospin, spin–orbit and tensor operators appearing in the NN interaction. Momentum-dependent operators, like spin–orbit, are usually treated separately. The product in equation (38) must be then symmetrized since the different terms do not commute anymore.

If the two-body NN interaction is local and central, its mean value is directly related to the pair distribution function \( g(r) \):

\[
\langle V \rangle = \frac{1}{2} \rho \int d^3r v(r) g(r) \quad (40)
\]

where

\[
g(r_1 - r_2) = \frac{\int \prod_{i=1}^2 d^3r_i |\Psi(r_1, r_2, \ldots)|^2}{\int \prod_i d^3r_i |\Psi(r_1, r_2, \ldots)|^2}. \quad (41)
\]

The main job of the variational method is to relate the pair distribution function to the correlation factors \( F \). Again, in nuclear matter the pair distribution function must also be considered channel dependent and the relation with the correlation factor becomes more complex. In general this relation cannot be worked out exactly, and one has to rely on some suitable expansion. Furthermore, a three-body or a higher correlation function must in general be introduced, which will depend on three or more particle coordinates and describe higher order correlations in the medium. Many excellent review papers exist in the literature on the variational method and its extensive use for the determination of nuclear matter EoS [105, 106]. The best known and most used variational nuclear matter EoS is the one of [70], and it is reported in figure 6. A detailed discussion of the connection between the variational method and BBG expansion can be found in [53].

### 4.2.3. The relativistic approach

One of the deficiencies of the Hamiltonian considered in the previous sections is the use of the non-relativistic limit. The relativistic framework is of course the framework on which the nuclear EoS should be ultimately based. The best relativistic treatment developed so far is the Dirac–Brueckner approach. Excellent review papers on the method can be found in the literature [107] and in textbooks [108]. Here we restrict the presentation to the main basic elements of the theory.

In the relativistic context the only NN potentials which have been developed are the ones of OBE (one boson exchange) type. The starting point is the Lagrangian for the nucleon–meson coupling:

\[
\mathcal{L}_{\text{pN}} = -\frac{f_{\pi N}}{m_\pi} \psi \gamma^\mu \gamma^5 v \psi \partial_\mu \phi^{(pN)} - i \frac{f_\gamma}{4M} \psi \sigma^\mu\nu \partial_\mu \phi^{(pN)} \partial_\nu \phi^{(vP)} \quad (42)
\]

\[
\mathcal{L}_{\phi} = +g_s \bar{\psi} \gamma^\mu \gamma^5 \phi^{(s)} - \frac{f_c}{4M} \bar{\psi} \sigma^\mu\nu \partial_\mu \phi^{(s)} \partial_\nu \phi^{(s)} \quad (43)
\]

with \( \psi \) the nucleon and \( \phi^{(s)} \) the meson fields, where \( \alpha \) indicates the type of meson and \( \mu \) the Lorentz component in the case of vector mesons. For isospin 1 mesons, \( \phi^{(s)} \) is to be replaced by \( \tau \cdot \phi^{(s)} \), with \( t^l = (l = 1, 2, 3) \) the usual Pauli matrices. The labels \( pN, pV, s \) and \( v \) denote pseudoscalar, pseudovector, scalar and vector coupling/field, respectively.

The OBEP is defined as a sum of one-particle-exchange amplitudes of certain bosons with given mass and coupling. The main difference with respect to the non-relativistic case is the introduction of the Dirac-spinor amplitudes. The six non-strange bosons with masses below 1 GeV/c\(^2\) are used. Thus,

\[
V_{\text{OBEP}} = \sum_{\alpha=\pi,\eta,\rho,\sigma,\delta,\omega} V_{\alpha} \quad (45)
\]

with \( \pi \) and \( \eta \) pseudoscalar, \( \sigma \) and \( \delta \) scalar, and \( \rho \) and \( \omega \) vector particles. The contributions from the iso-vector bosons \( \pi, \sigma \) and \( \rho \) contain a factor \( \tau_1 \cdot \tau_2 \). In the so-called static limit, i.e. treating the nucleons as infinitely heavy (their energy equals the mass), the usual denominator of the interaction amplitude in momentum space, coming from the meson propagator, is exactly the same as in the non-relativistic case (since in both cases meson kinematics is relativistic). This has to be compared with the complete expression of the matrix element between nucleonic (positive energy) states, reported in [86] for the so-called CD Bonn potential. We will not discuss these implementations since they involve too many technical details.

The fully relativistic analogue of the two-body scattering matrix is the covariant Bethe–Salpeter (BS) equation. In place of the NN non-relativistic potential the sum \( V \) of all connected two-particle irreducible diagrams has to be used, together with the relativistic single particle propagators. Explicitly, the BS equation for the covariant scattering matrix \( T \) in an arbitrary frame can be written as

\[
T(q', q | P) = V(q', q | P) + \int d^4k \mathcal{V}(q', k | P) \mathcal{G}(k | P) T(k, q | P) \quad (46)
\]

with

\[
\mathcal{G}(k | P) = \frac{i}{(2\pi)^2} \frac{1}{\left( \frac{1}{2} P + k - M + i\epsilon \right)^{(1)}} \times \frac{1}{\left( \frac{1}{2} P - k - M + i\epsilon \right)^{(2)}} \quad (47)
\]
where \( q, k \) and \( q' \) are the initial, intermediate, and final relative four-momenta, respectively (with e.g. \( k = (k_0, k) \)), and \( P = (P_0, P) \) is the total four-momentum; \( k = \gamma^\mu k_\mu \). The superscripts refer to particles (1) and (2). Of course all quantities are appropriate matrices in spin (or helicity) and isospin indices. The use of the OBE potential as the kernel \( V \) is equivalent to the so-called ladder approximation, where one meson exchanges occur in disjoint time intervals with respect to each other, i.e. at any time only one meson is present. Unfortunately, even in the ladder approximation the BS equation is difficult to solve since it is in general non-local in time, or equivalently energy dependent, which means that the integral equation is four-dimensional. It is not even certain in general whether it admits solutions. It is then customary to reduce the four-dimensional integral equation to a three-dimensional one by approximating properly the energy dependence of the kernel. In most methods the energy exchange \( k_0 \) is fixed to zero and the resulting reduced BS equation is similar to its non-relativistic counterpart. In the Thompson reduction scheme this equation for matrix elements between positive-energy spinors (c.m. frame) reads

\[
T(q', q) = V(q', q) + \int \frac{d^3k}{(2\pi)^3} V(q', k) \frac{M^2}{E_k} \times \frac{1}{2E_q - 2E_k + ie} T(k; q|P)
\]

where both \( V(q', q) \) and \( T \) have to be considered as matrices acting on the two-particle helicity (or spin) space, and \( E_k = \sqrt{k^2 + M^2} \) is the relativistic particle energy. In the alternative Blankenbecler–Sugar [86] reduction scheme some different relativistic kinematical factors appear in the kernel. This shows that the reduction is not unique. The partial wave expansion of the \( T \)-matrix can then be performed starting from the helicity representation. The corresponding amplitudes include single as well as coupled channels, with the same classification in quantum numbers \( JLS \) as in the non-relativistic case and therefore their connection with phase shifts is the same. In the intermediate states of momentum \( k \) only the positive energy states are usually considered (by the proper Dirac projection operator). As in the case of the OBE potential, the main difference with respect to the non-relativistic case is the use of the Dirac spinors.

The DBHF method can be developed in analogy with the non-relativistic case. The two-body correlations are described by introducing the in-medium relativistic \( G \)-matrix. The DBHF scheme can be formulated as a self-consistent problem between the single particle self-energy \( \Sigma \) and the \( G \)-matrix. Schematically, the equations can be written as

\[
G = V + i \int V Q g g G
\]

\[
\Sigma = -i \int F [\text{Tr}[g G] - g G]
\]

where \( Q \) is the Pauli operator which projects the intermediate two particle momenta outside the Fermi sphere, as in the BHF \( G \)-matrix equation, and \( g \) is the single particle Green’s function. Self-consistency is entailed by the Dyson equation

\[
g = g_0 + g_0 \Sigma g
\]

where \( g_0 \) is the (relativistic) single particle Green’s function for a free gas of nucleons. The self-energy is a matrix in spinor indices, and therefore in general it can be expanded in the covariant form

\[
\Sigma(k, k_F) = \Sigma_s(k, k_F) - \gamma_0 \Sigma_0(k, k_F) + \gamma \cdot k \Sigma_e
\]

where \( \gamma_0, \gamma_e \) are the Dirac gamma matrices and the coefficients of the expansion are scalar functions, which in general depend on the modulus \( |k| \) of the three-momentum and on the energy \( k_0 \). Of course they also depend on the density, i.e. on the Fermi momentum \( k_F \). The free single particle eigenstates, which determine the spectral representation of the free Green’s function, are solutions of the Dirac equation

\[
[\gamma_\mu k^\mu - M]u(k) = 0
\]

where \( u \) is the Dirac spinor at four-momentum \( k \). For the full single particle Green’s function \( g \) the corresponding eigenstates satisfy

\[
[\gamma_\mu k^\mu - M + \Sigma]u(k)^* = 0.
\]

Inserting the above general expression for \( \Sigma \), after a little manipulation, one gets

\[
[\gamma_\mu k^\mu - M^*]u(k)^* = 0
\]

with

\[
k^0 = k^0_0 + \Sigma_0 \quad k^I = k^I_0 \quad M^* = \frac{M + \Sigma_s}{1 + \Sigma_0}.
\]

This is the Dirac equation for a single particle in the medium, and the corresponding solution is the spinor

\[
u^*(k, s) = \sqrt{\frac{E_k^* + M^*}{2M^*}} \left( \frac{\sigma \cdot k}{\sqrt{k^2 + M^2}} \right) \chi_s \quad E_k^* = \sqrt{k^2 + M^2}.
\]

In line with the Brueckner scheme, within the BBG expansion, in the self-energy of equation (50) only the contribution of the single particle Green’s function pole is considered (with strength equal to one). Furthermore, negative energy states are neglected and one gets the usual self-consistent condition between self-energy and scattering \( G \)-matrix. The functions to be determined in this case are the three scalar functions appearing in equation (51). However, to simplify the calculations these functions are often replaced by their value at the Fermi momentum.

In any case, the medium effect on the spinor of equation (53) is to replace the vacuum value of the nucleon mass and three-momentum with the in-medium values of equation (52). This means that the in-medium Dirac spinor

\[
\chi_{s_i} = \sqrt{E_k^2 + M^2}.
\]
is ‘rotated’ with respect to the corresponding one in vacuum, and a positive (particle) energy state in the medium has some non-zero component on the negative (particle) energy state in vacuum. In terms of vacuum single nucleon states, the nuclear medium produces automatically anti-nucleon states which contribute to the self-energy and to the total energy of the system. It has been shown in [109] that this relativistic effect is equivalent to the introduction of well-defined TBF at the non-relativistic level. These TBF turn out to be repulsive and consequently produce a saturating effect. The DBHF indeed gives in general a better SP than BHF. Of course one can wonder why these particular TBF should be selected, but anyhow a definite link between DBHF and BHF + TBF is, in this way, established. Indeed, including in BHF only these particular TBF one gets results close to DBHF calculations, see e.g. [94].

Despite the fact that DBHF is similar to the non-relativistic BHF, some features of this method are still controversial. The results depend strongly on the method used to determine the covariant structure of the in-medium $G$-matrix, which is not unique since only the positive energy states must be included. It has to be stressed that, in general, the self-energy is better calculated in the matter reference frame, while the $G$-matrix is more naturally calculated at the center of mass of the two interacting nucleons. This implies that the $G$-matrix has to be Lorentz transformed from one reference frame to the other, and its covariant structure is then crucial. Formally, the most accurate method appears to be the subtraction scheme of [110]. Generally speaking, the EoS calculated within the DBHF method turn out to be stiffer above saturation than those calculated from the BHF + TBF method.

4.2.4. The $V_{\text{low}}$ approach. We have seen that in the conventional approaches the NN interaction is characterized by a strong repulsive core, which is the main origin of difficulty in the microscopic many-body treatment of nuclear matter and nuclei starting from the bare NN interaction. The main effect of the hard core in the NN interaction is to produce scattering to high momenta of the interacting particles. The $G$-matrices of the BBG expansion, or the correlation functions in the variational method, are much softer, i.e. they present high-momentum components much smaller than the original bare NN interaction. It is possible to soften the hard core of the NN interaction from the start by integrating out all the momenta larger than a certain cutoff $\Lambda$ and ‘renormalizing’ the interaction to an effective interaction $V_{\text{low}}$ in such a way that it is equivalent to the original interaction for momenta $q < \Lambda$. By construction $V_{\text{low}}$ must give the same half of the energy shell scattering $T$-matrix $(q'|T(E_q)|q)$ as the original interaction, where $E_q$ is the energy of the initial state at relative momentum $q$. The main purpose of the procedure is to separate the low-energy scale, proper of nuclear physics problems, from the high-momenta components. The structure of the interaction below the cutoff should be mainly independent of the detailed structure of the interaction at high momenta, that is substantially not fixed by phenomenology on NN processes. It turns out that this goal can indeed be achieved, starting from realistic NN interaction with no cutoff. The procedure is similar to the renormalization in quantum electrodynamics (QED), where the results on physical processes are independent of the cutoff that determines the energy scale of the theory, leaving away the unknown physics at higher scales, and is obtained by fixing basic physical quantities such as the electron mass and charge. We will now try to summarize the results on the nuclear interaction along these lines. The problem of projecting out a subspace $Q$ and constructing an equivalent Hamiltonian acting only in the complementary space $P$ has a long history in nuclear physics and in other fields as well. Following Bloch and Horowitz [111, 112] and Feshbach [113] the equivalent effective Hamiltonian acting on the $P$ subspace can be written as

$$H_P = PH_P + PH_Q \frac{1}{E - QHQ} QHQ$$

(54)

where $P$ and $Q$ are the projection operators into the corresponding subspaces. In the present case of the two-body problem

$$P = \int_{k < \Lambda} d^3k |k > < k| \quad Q = 1 - P. \quad (55)$$

In equation (54) $H$ is the original Hamiltonian and $E$ is the exact total energy. If the eigenvalue problem has to be solved in the $P$ space then the calculation of the eigenvalue energy requires a self-consistent procedure. In a scattering problem this means that the effective Hamiltonian $H_P$ in $P$ space is energy dependent. To eliminate this energy dependence of the effective Hamiltonian several methods have been developed, the perturbative expansion of [111, 112], the diagrammatic expansion (‘folded diagrams’) of Kuo [114], the Lee–Suzuki diagonalization method [115, 116], the renormalization group (RG) [117], the similarity renormalization group (SRG) [118–120]. All these methods are in principle equivalent [121]. Each one has some convenience, according to the problem in hand. The main purpose of introducing a cutoff is to separate the small momenta region, where low-energy nuclear phenomena occur, from the high-momenta region, which is unresolved and not accessible experimentally. This is a typical procedure of effective theory, where the high-energy scale is disentangled, if possible, from the low-energy scale, where an effective interaction is introduced. It is outside the scope of this report to describe these methods and we refer to recent reviews and papers where they are extensively discussed and applied [122, 123]. Here we limit the presentation to the RG method, where the procedure of introducing the cutoff and its influence on the low-momenta effective interaction is quite transparent. The construction of $V_{\text{low}}$ requires that the original half-off shell NN scattering $T$-matrix is exactly preserved up to the cutoff. If $V$ is the original NN interaction (with hard core) in a given two-body channel, the $T$-matrix satisfies the Lippman–Schwinger (LS) equation (21), which can be more explicitly written as

$$T(k', k, e_k) = V(k', k) + \frac{2}{\pi} \int_0^{\infty} \frac{V(k', q)T(q, k, e_k)}{e_k - e_q} q^2 dq \quad (56)$$
where $k, k'$ are the initial and final relative momenta, $e_k$ the relative kinetic energy and $P$ denotes principal part of the integral. The interaction $V_{\text{low}}$ must satisfy the condition that the same $T$-matrix restricted to momenta $k, k' < \Lambda$ is preserved:

$$T(k', k, e_k) = V_{\text{low}}(k', k) + \frac{2}{\pi} P \int_0^\Lambda \frac{V_{\text{low}}(k', q)T(q, k, e_k)}{e_k - e_q} q^2 dq.$$  \hspace{1cm} (57)

By construction $V_{\text{low}}(k', k)$ is defined only for $k, k' < \Lambda$, where the scattering process can be exactly described. The $V_{\text{low}}$ interaction is dependent on the cutoff $\Lambda$, since it is an effective interaction, but the $T$-matrix is not, within the considered momentum range. The simplest way to calculate $V_{\text{low}}$ is then to impose the condition

$$\frac{d}{d\Lambda} T(q, k, e_k) = 0 \quad q, k < \Lambda$$  \hspace{1cm} (58)

where the dependence on $\Lambda$ is implicit through the potential $V_{\text{low}}$, and therefore equation (58) is an implicit differential equation for $V_{\text{low}}$. This equation can indeed be written as an explicit differential equation [121], to be treated numerically:

$$\frac{d}{d\Lambda} V_{\text{low}}(k', k, e_k) = \frac{2}{\pi} \frac{V_{\text{low}}(k', \Lambda, e_\Lambda)T(\Lambda, k, e_k)}{1 - (k/\Lambda)^2}.$$  \hspace{1cm} (59)

It has to be stressed that this equation is exact. It describes the RG ‘flow’ of each matrix element of $V_{\text{low}}$ as $\Lambda$ is varied. Equation (59) is typical of the RG method in general, as in theory of phase transitions and in field theory, where the differential form of the RG allows the inclusion of resummation of large sets of diagrams.

All possible $V_{\text{low}}$ interactions are of course much softer than the original NN interaction, since no high-momentum components are present. The short range repulsion is replaced by the non-local structure of the interaction. It has to be kept in mind that any $V_{\text{low}}$ is a legitimate realistic NN interaction. In fact, due to the mentioned equivalence, they fit exactly the same phase shifts up to an energy corresponding to the cutoff. The latter can be taken above 300 MeV in the laboratory, corresponding to relative momentum $q \approx 2.1$ fm$^{-1}$, which is the largest energy where the data are established. It turns out that the data and the NN interaction below this cutoff are not sensitive to the details of the hard core behavior. Indeed, all the $V_{\text{low}}$, at least their diagonal matrix elements, are almost identical up to the cutoff, provided it is not taken too large above 2.1 fm$^{-1}$. This point is illustrated in figure 7, taken from [124], for the $^3S_1$ channel. One can notice the remarkable agreement among the different $V_{\text{low}}(k, k)$ as the cutoff is decreased down to the value corresponding to the highest energy considered for the fit of the phase shifts for all the original interactions. This is not completely surprising, since all realistic NN interactions fit the same phase shifts up to about 2.1 fm$^{-1}$, but it clearly indicates that the hard core of the NN interaction is actually not resolved and its particular structure is not relevant. In other words, it demonstrates the decoupling between high and low momenta. The cutoff can be further lowered below this value, still keeping the same phase shifts up to the cutoff. In general the evolved interaction is not strictly Hermitian, but it can be put in the Hermitian form by a suitable similarity transformation [124].

The renormalization of the interaction when the cutoff is lowered should be accompanied by a similar renormalization of the operators corresponding to the physical observables. This can be a non-trivial task. However, it is important to notice that all observables that are sensitive only to the low-momenta components will not evolve appreciably. As an example one can mention the different types of multipole moments of the system or of a given transition.

The fact that $V_{\text{low}}$ is soft makes it much more manageable than a hard core interaction, in particular it can be used in

Figure 7. RG evolution of the diagonal matrix elements of $V_{\text{low}}$ as the cutoff is decreased, for different NN realistic bare interactions, in the $^3S_1$ channel. Reprinted from [124], copyright 2003, with permission from Elsevier.
perturbation expansions and in nuclear structure calculations in a more efficient way. Again, at a purely phenomenological level, a non-local interaction at a short distance is perfectly legitimate as a hard core, since the behavior of the potential at a short distance is not experimentally accessible. In principle, it is only a question of representation. The application of $V_{\text{low}}$ to nuclear matter is of fundamental relevance. Figure 8, from [125], summarizes the main results. At the Hartree–Fock level for $V_{\text{low}}$ one finds that symmetric nuclear matter does not saturate, i.e. no minimum of the energy as a function of density seems to be present. TBF are needed to obtain a reasonable saturation point. Figure 8 reports the results when TBF from chiral perturbation theory (to be introduced in the next section) are included. The figure also shows the sensitivity to the cutoff $\Lambda$ used in constructing $V_{\text{low}}$. The conclusion from this analysis is that with low-momentum interactions saturation in nuclear matter is driven by TBF. Although it is not surprising that the role of TBF is dependent on the two-body force employed, the comparison with the Brueckner $G$-matrix method cannot be avoided. In fact, the $V_{\text{low}}(k, k)$ interaction can be obtained from the traditional Argonne $v_{18}$ NN interaction, as in [125], and the procedure of projecting out the high-momentum components has some similarity with the construction of the $G$-matrix. It turns out that the Hartree–Fock approximation with the $G$-matrix calculated with the same interaction $v_{18}$ produces saturation in nuclear matter [187]. Also in this case TBF are necessary to get a good saturation point, a general conclusion valid for all the traditional NN forces. However, it is natural to ask what the reasons are for the discrepancy. This is a basic question, since it is desirable to have a physical picture of the saturation mechanism. The main difference between the $G$-matrix and $V_{\text{low}}$ is apparent. While $V_{\text{low}}$ is constructed in vacuum, the $G$-matrix is constructed in-medium, and as such it is dependent on the total momentum and energy (‘entry energy’). Furthermore, it turns out that the $G$-matrix still keeps some coupling between the low- and large-momentum components, which is systematically reduced [122] by the $V_{\text{low}}$ construction. An important consequence of the latter property of $V_{\text{low}}$ is that it can be treated perturbatively. This can be seen in the right panel of figure 8, where the inclusion of the second-order contribution in $V_{\text{low}}$ is of a few MeV at saturation, in contrast to the first-order contribution of about 40 MeV. As we have seen in section 4.2.1, the $G$-matrix approach needs to rely on non-perturbative methods. Although the three hole-line contributions is much smaller than the two hole-line one [187], with the $V_{\text{low}}$ interaction it is much easier to handle the many-body problem. The dependence on the value of the cutoff as displayed in figure 8, in particular above 2 fm$^{-1}$, is an indication that the repulsive core still plays a role in nuclear matter, in contrast with the free two-nucleon scattering in free space. It is also in agreement with the fact that within the BHF or variational methods the use of TBF fitted to few-body systems gives too much repulsion in nuclear matter [70, 126].

A closer connection to the $G$-matrix is obtained if $V_{\text{low}}$ is constructed directly in the medium. In this case the SRG method turns out to be more convenient [122]. In this method a unitary transformation is performed on the Hamiltonian $H$ that leaves the kinetic energy invariant:

$$H_s = U_s H U_s^\dagger$$

(60)

where $s$ is a running parameter. In the two-body case in vacuum, the interaction is evolved toward a diagonal form in momentum space and is strongly softened. This can be achieved by a proper choice of the family of unitary transformations, which can be identified by the generator of the transformations in differential form. According to [122] the differential equation for $H_s$ can be written as

$$\frac{dH_s}{ds} = [[G_s, H_s], H_s]$$

(61)

where $G_s$ is the generator. One of the most used choices is the kinetic energy, in which case the off-diagonal matrix elements of the evolved interaction are exponentially suppressed at increasing values of $s$ [122]. Other choices for $G_s$ are possible. Since the evolution of the interaction is through unitary transformations, all the matrix elements are preserved, provided the operators corresponding to the
physical observables are also evolved. This can be a quite complex procedure. Also in this case observables that are sensitive only to the low momenta of the wave function will not evolve appreciably.

When both RG and SRG methods are applied to an A-body system, they automatically generate many-body forces (up to A-body forces). This is because the evolution projects out some degrees of freedom (the high-momentum components), which ‘induces’ many-body forces just the same way when non-nucleonic degrees of freedom are integrated out. In principle, these induced many-body forces are of different origin from e.g. the standard TBF generated by the virtual excitation of the Δ. However, it is not possible to distinguish between them, and many-body forces are an unavoidable ingredient of the many-body problem. In any case, it is necessary to establish that higher order forces are much less important than the lower order ones, otherwise the method would be not manageable. It has been checked by specific numerical examples that the contributions of the n-body forces decrease rapidly for increasing n [122].

If the SRG is applied in-medium, the evolution also generates many-body forces, which will be dependent on the system. A simplification can be obtained within the second quantization formalism and introducing the normal ordering of the creation and annihilation operators with respect to the (unperturbed) ground state. In this case one can consider on the right-hand side of equation (61) only the contribution of the two-body normal order terms. In this way one gets only an evolved density-dependent two-body forces and the higher order forces are embodied in this density dependence. This is similar to the procedure of averaging a TBF over the coordinate of one of the particles, which automatically produces a density-dependent two-body force. Although the interaction evolution is not exactly unitary, it appears to be a quite accurate and promising approximation [122].

In conclusion of this section one can say that the $V_{\text{low}}$ approach has opened a new route to the microscopic many-body theory of the nuclear medium.

4.2.5. **Trying a link to QCD: the chiral symmetry approach.**

One of the main aims of nuclear physics is to connect the low-energy nuclear physics phenomena with the underlying more fundamental theory of strong interaction, i.e. QCD and the Standard Model, based on quark and gluon degrees of freedom, together with the Weinberg–Salam–Glashow theory of weak interaction. This is quite difficult because all hadron sectors are in the non-perturbative regime, due to confinement. A possible strategy is the systematic use of the symmetries embodied in the hadronic QCD structure. The main symmetry that remains visible in the confined matter is the chiral symmetry, the symmetry that QCD possesses if the bare quark masses are put equal to zero. The symmetry is spontaneously broken in the confined phase, i.e. in hadronic matter, but, according to the general theorem by Goldstone, a zero-mass boson should be present. This is indeed the $\pi$ meson, which in the limit of zero quark mass, should also have zero mass. This is the main signature of the underlying chiral symmetry. For non-strange matter, only $u$ and $d$ quarks are relevant, and they are indeed expected to have a mass of a few MeV. This small explicit breaking of chiral symmetry results in the physical mass of the pion, that is the lightest meson, even if it is not small at the energy scale of many nuclear phenomena. Along this line Weinberg [127–129] proposed a scheme for the construction of the multi-nucleon interactions within the general framework of effective field theory (EFT), constrained by the underlying chiral symmetry of QCD. In EFT the high-energy scale is chosen to be a typical QCD scale, e.g. the nucleon mass $M$. The amplitude $A$ for a typical nuclear phenomenon is then expanded in powers of the ratios $Q/M$, where $Q$ are the momenta appearing in the corresponding Feynman diagram. Since $Q/M$ is assumed to be a small quantity, the diagrams can be ordered at increasing value of the power $v$ in $Q/M$, which should correspond to a rapidly converging power series, with coefficients that are smooth functions of $Q/M$. Schematically

$$A = \sum_a C_a(Q/M) \left( \frac{Q}{M} \right)^a.$$  (62)

Chiral symmetry characterizes the underlying QCD Lagrangian in terms of quark and gluons, and must be imposed on the effective nuclear Lagrangian in terms of pion and nucleon fields. Other meson degrees of freedom must be integrated out, since their masses are comparable to the QCD scale. Chiral symmetry in the nuclear Lagrangian is realized by a non-linear representation of the chiral group [130]. Since chiral symmetry is actually broken, there will also be terms proportional to a power of the ratio of the pion mass to the large scale $m/\pi$, that is also small. The effective Lagrangian can also be expanded into a series of terms with increasing value of the power in $Q/M$, whose structure is dictated by chiral symmetry. We sketch here how to construct chiral invariant quantities using pion and nucleon fields, leaving the foundations and details to the original papers [127–130] and to more specialized review papers [131–133]. The chiral group has the structure of the direct product $SU(2) \times SU(2)$. The two $SU(2)$ groups can be chosen as the usual isospin group $SU(2)_V$ and the axial isotopic spin group $SU(2)_{A}$, with generators equal to $t$ and $x = \gamma t$, respectively, where $t$ is the isotopic spin, appropriate to a given field. The main point is to find the transformation of the pion and nucleon fields under chiral transformations. The isotopic group corresponds to an unbroken symmetry and we can choose the usual transformation law for a generic field $\phi$:

$$[t^{(i)}_j, \phi_k] = i f^{(i)}_{jk} \phi_k$$  (63)

where $t^{(i)}_j$ is the matrix corresponding to the representation of the isospin, e.g. the representation $T = 1/2$ for the nucleon field $\psi$ and $T = 1$ for the pion field $\pi$. The axial group $SU(2)_{A}$ corresponds to a broken symmetry (finite pion mass), and a non-linear representation is introduced [130] (summation over repeated indices):

$$[x^{(i)}_j, \pi_j] = -i f^{(i)}_{ij} (\pi) \quad [x^{(i)}_j, \psi] = v_{ij} f^{(i)}_{ij} \psi$$  (64)

for the pion and nucleon, respectively. The two functions $f$ and $v$ are not arbitrary, they are constrained by the condition that the standard commutation relations of the $t^{(i)}_j$ and $x^{(i)}_j$ are
satisfied. Up to a redefinition of the fields, they can be chosen as [130]
\[
f_{ij} = \delta_{ij} \frac{1}{2\lambda} \left( 1 - \lambda^2 \right) + \lambda \epsilon_{ij} \pi_j \quad i\nu_j = \lambda \epsilon_{ijk} \pi_k
\] (65)

where \( \epsilon \) is the antisymmetric tensor of order 3 and \( \lambda \) an arbitrary constant. It can be fixed by comparing the expression of the weak axial-vector current with the standard expression, then \( \lambda = 1/F_\pi \), with \( F_\pi \) is the unrenormalized pion decay constant. Although this realization of the chiral symmetry is non-linear, isoscalar quantities like \( \bar{\psi} \gamma_\mu \psi \), or powers of it, are also chiral invariants. Derivative terms must also be introduced, but the non-linearity prevents us to consider just the derivative of the fields, since they do not behave under chiral transformations as isospin vectors. A ‘covariant derivative’ \( D_\mu \) is then constructed in such a way that both the pion and nucleon fields (or any other fields) transform like a vector (with the proper isospin), i.e.
\[
[x^0, D_\mu \phi_j] = -iv_{ik} \gamma_k \lambda D_\mu \phi_j
\] (66)

where \( \phi \) is either \( \pi \) or \( \psi \) and \( \lambda \) the isospin matrix representation for each field. This condition fixes the form of the covariant derivative. With the choice (65) the result is [130]
\[
D_\mu \pi = (1 + \lambda^2 \pi^2)^{-1} \partial_\mu \pi
\]
\[
D_\mu \psi = \partial_\mu \psi + 2i\lambda (1 + \lambda^2 \pi^2)^{-1} t \cdot (\pi \times \partial_\mu \pi) \psi.
\] (67)

Under isospin transformations the covariant derivatives keep the same form as the fields themselves. Since the covariant derivative is the same (except for the isospin value), chiral invariant quantities can be constructed from any isoscalar quantities. For the pion Lagrangian one can assume
\[
\mathcal{L}_\pi = D_\mu \pi D^\mu \pi.
\] (68)

It has to be noted that the Lagrangian also includes direct interactions among an arbitrary number of pions. The terms involving the minimal number of nucleons and pions with at most one derivative are
\[
\bar{\psi} \gamma_\mu \psi \left( \frac{1}{2} \epsilon \bar{\psi} \gamma^\mu (\bar{\psi} \gamma^\nu \psi) \right)
\]
\[
\bar{\psi} \gamma_\mu \gamma_\nu \psi \cdot D_\nu \pi \quad \bar{\psi} \gamma_\mu D_\nu \psi \cdots.
\] (69)

The interacting NN and nucleon–pion effective Lagrangian \( \mathcal{L} \) corresponding to these terms will be their linear combination. In the non-relativistic limit for the nucleons one gets [128]
\[
\mathcal{L} = \mathcal{L}_{\pi \pi} + \mathcal{L}_{\pi NN}
\]
\[
\mathcal{L}_{\pi \pi} = -\bar{\psi} (2D^{-1}_\pi F^{-2}_\pi t (\pi \times \partial_0 \pi)
\]
\[+ 2D^{-1}_\pi F^{-2}_\pi g_A t \cdot (\sigma \cdot \nabla) \pi) \psi
\]
\[
\mathcal{L}_{\pi NN} = -\frac{1}{2} C_\pi (\bar{\psi} \gamma_\mu \psi)^2 - \frac{1}{2} C_T (\bar{\psi} \gamma_\mu \sigma \psi \cdot (\bar{\psi} \sigma \psi)
\] (70)

where \( g_A \) is the (unrenormalized) weak axial coupling constant, \( t \) is the nucleon isotopic spin matrix and \( C_\pi \), \( C_T \) two coupling constants that cannot be fixed on the basis of chiral symmetry only, but they can be determined phenomenologically. The introduction of \( g_A \) is again suggested by the comparison with the expected form of the axial current. This is in agreement with the form of EFT where the pion is treated explicitly, while all the other meson degrees of freedom are incorporated into nucleonic contact interactions, eventually with a certain number of derivatives. The interaction vertices in equation (70) are proportional to the lowest power in the small ratio \( r = Q/M \). The Lagrangian in principle contains an infinite number of terms, ordered according to the power in \( Q/M \) they are proportional to.

Let us consider a general Feynman diagram contributing to a given amplitude. In addition to the external lines, which are supposed to carry small momenta, the internal lines can bring loop integrals that are ultraviolet diverging. They can be regularized by fixing a renormalization point, also chosen at low momentum. Then the loop integrals have an effective momentum cutoff at scale \( Q/M \), and the whole diagram is proportional to a given power of \( Q/M \). Alternatively, one can introduce a ‘regulator’, i.e. a function of momenta that smoothly cuts off the integrals at the large scale. One can devise a power counting scheme to fix (superficially) the power that characterizes the diagram. The value of \( v \) is determined by the number \( I_e \) of nucleon propagators, each one contributing with \( r^{-1} \), the number \( I_p \) of pion propagators, each one contributing with \( r^{-2} \), the number and type of vertices, where each derivative contributes with \( r \), and finally the phase space factors coming from the independent four-momentum integrations on the internal lines, each one contributing with \( r^4 \).

Then, following Weinberg [128], if a diagram has \( L \) loops and \( V \) vertices with \( d_i \) derivatives, its overall power \( v \) is given by
\[
v = 4L - I_n - 2I_p + \sum_i V_i d_i.
\] (71)

Taking into account the topological properties of a generic diagram, this relation can be written as
\[
v = 2 - \frac{1}{2} E_n + 2L + \sum_i V_i \left( d_i + \frac{1}{2} n_i - 2 \right)
\] (72)

where \( E_n \) is the number of external nucleon lines and for each one of the \( V \) vertices of type \( i \), with \( n_i \) number of nucleon lines and \( d_i \) derivatives. Each vertex contributes a power \( \Delta_i = d_i + \frac{1}{2} n_i - 2 \), that is non-negative by chirality. The lowest order diagrams with given \( E_n \) and \( L \) are then the ones with \( \Delta_i = 0 \). The vertices of equation (70) have indeed \( \Delta = 0 \), with \( v = 0 \) or \( v = 1 \).

The next step is to introduce the breaking of chiral symmetry due to the finite pion mass. This is a delicate point, since the symmetry breaking terms in the Lagrangian can have different forms. They must contain the pion mass as a factor since they must vanish for zero pion mass. The most common choice is a mass term of the form
\[
\mathcal{L}_m = -\frac{1}{2} m_\pi^2 D^{-1} \pi^2.
\] (73)

This introduces a series of pion vertices with increasing number of interacting pions and power of \( m_\pi \). Since \( m_\pi/M \) is assumed to be a small number, in the power counting of a diagram each \( m_\pi \) factor must count as one additional factor \( r \). In practice, each \( m_\pi \) factor can be considered on the same footing as
a derivative. As the expansion proceeds additional contact terms are introduced, with higher number of nucleon lines and higher derivatives. They can also be viewed as counter-terms to regularize the divergences coming from the loop integrals appearing as the order of the expansion increases. This procedure of renormalization is common in quantum field theory, like QED. In renormalizable field theory the number of counter-terms are finite and their strengths are fixed by demanding that some quantities have their physical values. In QED they are fixed by imposing the physical values of the electron charge and mass. Then the perturbation expansion terms are all finite. In the case of nuclear physics one demands that the phase shifts in some channels and specific energies are reproduced correctly. However in this case the theory is not renormalizable and at any order new counter-terms must be introduced, which increase the number of parameters. This regularized expansion, the chiral perturbation expansion (ChPE), can be used to construct NN interactions. The diagrams contributing to the interaction processes are again classified according to the power index \( v \) that they bring and at increasing \( v \) the relevance of the diagrams should be rapidly decreasing in the region of validity of the expansion (i.e. the ‘low’-momentum region). Figure 9 reports the structure of the diagrams contributing to the NN interaction at the leading order (LO) \( v = 0 \) (first row), and at the next to leading order (NLO) \( v = 2 \) (second row). They are constructed with the vertex of equation (70), except for the NN derivative coupling, indicated by a full dot in the first diagram of the second row. For trivial reasons (including parity) no \( v = 1 \) terms exist in this case. As the expansion proceeds the diagrams proliferate. At present the nucleonic interaction has been calculated up to next-next-leading order (N\(^3\)LO). More details, as well as the list of higher order diagrams up to N\(^3\)LO, can be found in topical review papers [131, 132].

A more formal derivation of ChPT from QCD can be developed if one starts from the QCD generating functional, with the explicit inclusion of the nucleon fields [134].

Some difficulties, surely not unexpected, arise in the perturbation expansion of the nucleonic interactions. If in the interaction between two nucleons any one of the processes considered above is iterated along the two-nucleon propagation in a ladder fashion, the free two-nucleon propagations that arise between two successive interaction processes bring a small energy denominator, of order \( Q^2 / M \) (non-relativistic limit), and the contribution of the diagram is enhanced by a factor \( M / Q \) with respect to the standard (superficial) power counting scheme described above. This is well known in the theory of scattering. To calculate the scattering matrix one has to, in general, iterate the interaction to all order, i.e. to solve the Lippman–Schwinger equation or the Schrödinger equation for the scattering wave function (except for extremely peripheral collisions where the Born approximation can be used). It is then natural to define the NN interaction as the sum of the irreducible diagrams as a block to be inserted in the Lippman–Schwinger integral equation, which will iterate it to all orders. By ‘irreducible’ clearly one means diagrams that cannot be separated into two parts by cutting two free nucleon lines. The procedure is the same as in the theory of NN interaction based on meson exchange processes, where however it is difficult to go beyond the one meson exchange processes.

A second difficulty arises if ChPT is used to construct a realistic NN interaction that fits the low-energy data, in particular the binding energy of deuteron and the virtual state of the \( ^1S_0 \) channel in the neutron–neutron interaction, which results in an anomalously large scattering length. This requires [128] not only extremely fine tuning of the parameters, but also unnatural large values of the parameters of the contact interactions, e.g. \( C_3 \) and \( C_T \) of equation (70). The values are ‘unnatural’ because they are expected to be of order \( 1 / M^2 \), since their dimension is indeed (mass)\(^2\). This difficulty is, however, common to any theory of nuclear forces, because there is no simple physical explanation (surely not from QCD) for the so low value of the deuteron binding energy (or position of the virtual state), which is much smaller than any strong interaction scale. It is the low value of these energies that is ‘unnatural’. Of course this does not prevent us from constructing equally well a realistic NN interaction. On the basis of ChPT many two-nucleon interactions have been constructed and fitted to the deuteron properties and the available phase shifts analysis up to 300 MeV in the laboratory. They can be as accurate [131, 135] as the conventional high precision interactions based on meson exchange models, with a comparable number of parameters.

One of the major positive features of the ChPT approach to nuclear forces is that it also automatically provides TBF (or higher) on the same footing. They arise first at N\(^2\)LO level, as depicted schematically in figure 10. As a consequence, because of the hierarchy intrinsic in the chiral expansion, TBF are expected to be smaller than two-body forces, at least within the range of validity of the expansion. This is in agreement with the phenomenological findings in more conventional approaches, when applied to few-body nuclear systems and nuclear matter. This hierarchy of the nuclear forces continues as the number of interacting nucleons increases, since then also the order of the expansion must increase. Indeed four-body forces appear only at N\(^3\)LO level, and so on. It has to be stressed that in the TBF the same couplings that fix the two-body forces have to be used, and in general only a
few additional parameters must be introduced as the order increases, mainly for the contact interactions. The TBF are then automatically consistent with the two-body forces, and so on for the higher order many-nucleon forces. This consistency can also be enforced in the meson–nucleon coupling models of the nuclear interaction [136–139], but in that case it is not clear how to establish a hierarchy of the forces.

Many-nucleon forces are of course relevant for many-nucleon systems, in particular nuclear matter. We will focus the presentation on the application of ChPT forces to nuclear matter. In this case another scale appears, the Fermi momentum $k_F/M$, which is of the same order as $m_\pi$ at saturation. In the chiral limit it is then natural to expand in $k_F/M$. Such an expansion can be obtained from the vacuum ChPT expansion by introducing at each nucleon line a ‘Fermi sea insertion’ [140]. This is equivalent to assigning to each fermion line in a diagram the difference between the free nucleon propagator and the hole nucleon propagator in the Fermi sea. In particular, for nuclear matter the correction thus obtained with respect to the vacuum diagrams gives a direct contribution to the EoS of nuclear matter, and this correction is clearly proportional to a power of $k_F/M$, according to the number of sea insertions. Also in this case a cutoff must be introduced to regularize the vacuum loop integrals that are still present in the diagram. With pion degrees of freedom and no contact interactions, saturation can be obtained already including only one- and two-pion exchange diagrams. Tuning the cutoff, saturation point and compressibility is in fair agreement with phenomenology [140]. Along the same lines more sophisticated expansions can be developed, including a power counting modified for finite density systems, where the small scale is fixed by both $k_F$ and $m_\pi/M$. The results [141] are in good agreement with the most advanced non-relativistic many-body calculations.

A different approach can be developed, where the many-nucleon interactions built in vacuum are directly used in nuclear matter calculations. In this case the ChPT is used in conjunction with the EFT scheme. An effective low-momentum interaction $V_N$ is constructed by, e.g., the RG method, starting from a ChPT interaction, properly regularized at high momenta. Figure 11, from [142], reports the EoS calculated from the $V_N$ obtained by evolving the N$^3$LO interaction of [135], which contains a regularization cutoff $\Lambda_N = 500$ MeV. The calculation is performed perturbatively up to third order, as indicated. TBF are included following the procedure of [143] of averaging over the quantum numbers of the third particle in the Fermi sea, which results in density-dependent two-body interaction. Second and third order turn out to be quite close, which indicates the possibility of treating nuclear matter with a perturbative approach, in agreement with similar studies based on interactions $V_N$ derived from realistic non-relativistic interactions (see the previous section). The cutoff for the TBF is fixed at $\Lambda_N = 2.0$ fm$^{-1}$, while the cutoff of the evolved two-nucleon interaction $\Lambda$ is varied independently within the indicated range. As in previous studies, nuclear matter does not saturate with two-body forces only, and the contribution of the TBF is essential. It is important to note that the parameters of the TBF have been fitted to three- and four-body nuclear systems. The dependence on the cutoff is believed to be due to the approximations that have been introduced, because exact calculations should be cut off independently (as it is true for the nuclear interactions in vacuum). These approximations can be traced in the treatment of TBF, higher chiral order terms for the nuclear forces and higher order contributions in the perturbation expansion. As in previous studies with increasing cutoff the binding decreases. At the larger cutoff $\Lambda = 2.8$ fm$^{-1}$, which approximately corresponds to the regulator cutoff of the original interaction, the TBF look too repulsive. This is in agreement with Brueckner G-matrix [126] or variational [70] calculations, where the phenomenological TBF obtained by fitting few-body systems must have a reduced repulsive component if the saturation point is to be reproduced.

One of the main aims of the EFT of nuclear interaction is to describe the low momentum processes by introducing a momentum regulator and a few contact counter-terms in such
a way that the physical results are independent of the regulator cutoff, at least for (asymptotically) large cutoff. The strength of each counter-term depends on the regulator form and on the cutoff value, but the results, e.g. the phase shifts, should be ultimately insensitive to them, once the cutoff is chosen large enough. In other words, since a regulator is physically equivalent to a soft core in coordinate space, the low energy physics (i.e. below the QCD scale) should be independent of the interaction at a short distance, and a clear-cut separation of the low-energy physics would emerge. This goal can be achieved for the s-wave [144], where a single counter-term is enough to regularize the scattering NN $T$-matrix to all orders, which is necessary to describe the shallow states (deuteron bound state and large scattering length) typical of the NN interaction. Also in higher partial waves the program can be pursued, but in the triplet states with attractive tensor force (from one pion exchange) this cannot be trivially achieved because in this case the tensor force produces spurious bound states as the cutoff is increased. This requires additional counter-terms which do not follow the superficial power counting of Weinberg, i.e. contact interactions which should appear at higher order (i.e. they contain higher derivatives) are actually introduced already at lower order. To be specific, the $^3P_0$ and $^3P_1$ channels at LO need contact counter-terms of second order (second derivative), which according to the superficial power counting should appear only at NLO level. Similarly the $^3D_2$ channel needs fourth-order counter-terms. Once this is done, i.e. the power counting is properly modified, the cutoff independence is achieved for all relevant channels. In general cutoff independence is obtained for $\Lambda > 5$ GeV, with some variations according to the channel. The problem is directly related to the treatment of attractive singular potential, since indeed the tensor force diverges as $1/r^3$ at small distance $r$. Details of the proposed regularization strategy can be found in [144].

The extension of this program to finite density systems, in particular to nuclear matter, has been presented in [145]. Following the same lines, at LO level the nuclear EoS turns out independent of the cutoff $\Lambda > 5$ GeV. The EoS is calculated within the Brueckner $G$-matrix formalism and the binding turns out too small. Higher hole-line contribution should then be included, but unfortunately the wound parameter turns out to be about 0.4 around saturation, too big to expect convergence of the corresponding hole expansion. Further works are needed to assess and clarify this ambitious program.

The progress that has been made along the ChPT theory of nuclear forces is impressive. The link with QCD has been strengthened and the developments at the formal level as well as in the phenomenological analysis seem to indicate that we cannot be so far from a microscopic foundation of the nuclear forces. This opens the possibility of linking to QCD the whole realm of low-energy nuclear physics and in particular the main properties of the nuclear medium, and it also provides a practical and systematic method to approach the nuclear many-body problem.

5. Nuclear matter as a Fermi liquid

The many-fermion systems are characterized by a sharp Fermi surface. If the interaction is not very peculiar, this is a general result known as Migdal's theorem [4]. The nuclear medium is not an exception, and many properties of both finite nuclei and nuclear matter are strongly affected by this feature. In particular, low-energy excitations and low-temperature thermal properties can involve only particles close to the Fermi surface. For the same reason transport phenomena are determined by the scattering processes that occur close to the Fermi surface.

Landau theory of (normal) Fermi liquids systematically exploits this feature to develop a semi-phenomenological treatment of most of the low-energy phenomena in homogeneous Fermi systems. Nuclear matter can be treated along the same lines, provided the isospin degrees of freedom are properly included. Migdal and collaborators [4, 146] have extended the approach to finite systems, in particular nuclei, where the so-called finite Fermi system theory (FFST) has been extensively applied. Excellent and pedagogical expositions of Landau theory can be found in [4, 147]. Here we limit ourselves to recalling the main concepts and some basic applications. At the basis of the theory is the introduction of quasi-particle states. The suggestion comes from the so-called adiabatic switching on of the interaction in a many-body system. The Gell-Mann and Low [148] theorem states that if one evolves a many-particle system, starting from an independent particle eigenstate, by switching on the interaction adiabatically, i.e. with an infinitely slow variation, then one obtains a state of the interacting system. Since for independent particles eigenstates are identified by the values of the occupation number $n(p)$ for each single particle state $p \equiv (\vec{p}, \sigma, \tau)$, so will be the corresponding state of the interacting system. The statement is correct if no phase transition or cluster formation occurs during the switching on of the interaction, and this is what we assume for the moment. It follows that the ground state of the interacting system will be characterized by a distribution of occupation numbers as the non-interacting one, i.e. a sharp Fermi distribution (zero temperature). If we consider the excitation of the non-interacting system that is obtained by adding a particle or forming a hole (i.e. subtracting a particle) at the single particle state $p$, these excitations will be called quasi-particle and quasi-hole in the interacting system. The corresponding variation in energy of the interacting many-particle system is called quasi-particle or quasi-hole energy, denoted by $\epsilon(p)$. In general, if we vary the distribution of occupation numbers in a smooth way, or we average the distribution within neighboring states, the general variation in energy $\delta E$ of the system to first order in the variation $\delta n(p)$ of the occupations can be written as

$$\delta E = \frac{1}{\Omega} \sum_p \epsilon(p) \delta n(p)$$

where $\Omega$ is the volume of the system. It is clear that the quasi-particles are fermions. However, unlike in the non-interacting system, they do interact, since the variation of the total energy will be a complex function of the variation $\delta n(p)$ and not just an additive linear function. To second order the energy
variation will be
\[ \delta E = \frac{1}{\Omega} \sum_p \epsilon(p) \delta n(p) + \frac{1}{2 \Omega^2} \sum_{p,p'} f(p, p') \delta n(p) \delta n(p') \]
(75)

where
\[ f(p, p') = \Omega^2 \frac{\delta^2 E}{\delta n(p) \delta n(p')} = \Omega \frac{\delta \epsilon(p)}{\delta n(p')} \]
(76)
is the quasi-particle interaction, since it describes the variation of a quasi-particle energy due to the presence of the other quasi-particles. More precisely, if the quasi-particle distribution is changed by the amount \( \delta n(p') \) for each \( p' \), the energy of the quasi-particle of momentum \( p \) changes by an amount given by the expression
\[ \delta \epsilon(p) = \sum_{p'} f(p, p') \delta n(p'). \]
(77)

However, the Gell-Mann and Low theorem is valid only if the perturbative expansion is convergent, since its demonstration is developed by considering each term of the perturbation expansion. This is not necessarily a valid assumption, and the quasi-particle states have a finite lifetime and they actually decay. This can be best seen in the Green’s function formalism, where the Landau theory can be more rigorously formulated [4, 149]. It can be seen that the quasi-particles have an infinite lifetime only exactly at the Fermi surface, while they have a decreasing lifetime as one moves away from the Fermi surface. This is a general property based only on a phase space argument. In a perturbative picture, a single quasi-particle can decay into two particles–one hole state (for momenta above the Fermi momentum), or in two holes–one particle state (below the Fermi momentum), and one can easily see that the possible phase space vanishes exactly at the Fermi momentum. This remains true if one considers more complicated decay states (multi-particles–multi holes). Once the quasi-particles are introduced, these considerations hold for the scattering processes between quasi-particles, which are then responsible for the decay. If the quasi-particles have no decay width at the Fermi surface, it is reasonable to expect that close enough to the Fermi surface the width will remain small, and there will be a region around the Fermi surface where the quasi-particles can be considered as stable, provided the phenomena that are considered have characteristic time scales shorter than the lifetime of the quasi-particles involved.

Again in a more formal language, a quasi-particle corresponds to the singular part of the single particle Green’s functions, i.e. a pole in the complex energy plane at a given momentum. The non-singular part should give a negligible contribution in many dynamical processes, since it is expected to be highly incoherent. However, the non-singular part has the effect of renormalizing the properties of the quasi-particles. In summary, it is the pole contribution that behaves like a particle, with properly renormalized physical parameters. In this sense a quasi-particle can be viewed as a particle dressed by the interaction with the other particles. The pole moves from one sheet of the complex energy plane to the other, when the momentum moves from below to above the Fermi surface. This means that the occupation number has a jump at the Fermi surface, a property anticipated at the beginning of this section. If we neglect the non-singular part, the quasi-particles’ distribution in the ground state has to be considered as an unperturbed Fermi distribution, i.e. with occupation numbers 1 and 0 (at zero temperature), so they are fermion particles.

In summary, the dynamical processes that involve excitations close to the Fermi surface can be described in terms of quasi-particle kinetics, whose dynamics can be treated as particles (fermions), but with renormalized properties. In the semi-classical regime, valid in the long wavelength limit, the kinetic equations for the quasi-particle distribution \( n(r, p, t) \) must follow equations (14) and (15), where the effective NN interaction, in the momentum representation, is just the Landau effective interaction \( f(p, p') \) of equation (76).

One has to be aware that the distribution function \( n(r, p, t) \) that appears in equations (14) and (15) is essentially the semi-classical limit of the quantal density matrix \( \langle \psi^\dagger(r', t') \psi(r, t) \rangle \) (to be precise, its Wigner transform) [150]. If we consider a perturbation with total momentum \( q \), one has to consider the Fourier transform of the distribution function, which is equivalent to the density matrix in momentum representation \( \langle \psi^\dagger(p+q,t) \psi(p,t) \rangle \). Therefore, the momenta \( p+q \) and \( p \) must form a particle–hole pair, i.e. they must lie on opposite sides of the Fermi surface. In fact, any perturbation of a Fermi liquid must imply the promotion of a particle from below to above the Fermi surface, and if the particles are removed from a position slightly different from the position where they are promoted the process involves a variation \( \delta n(r, p, t) \) of the distribution function at each point \( r \) which satisfies the kinetic equations (14) and (15). In the long wavelength limit, i.e. when \( |q| \) is much smaller than the Fermi momentum, the quasi-particle momentum \( p \) must lie close to the Fermi surface. Since the effective interaction is expected to be a smooth function, it can then be calculated for values of the momenta just on the Fermi surface, i.e. for \( |p| = |p'| = p_F \), where \( p_F \) is the Fermi momentum. For rotational invariance \( f \) must depend only on \( |p - p'| \). The dependence on the angle \( \theta \) between \( p \) and \( p' \) can be expanded in Legendre polynomials \( P_l(\cos \theta) \). Of course one should also include the spin and isospin dependence, and the explicit expression for \( f \) is usually written, following Landau, as
\[ f(p, p') = f + g \sigma \sigma' + f' \tau \tau' + g' \sigma \sigma' \tau \tau' \]
(78)
and each coefficient can now be expanded, e.g.
\[ f = \sum_l f_l P_l(\cos \theta) \]
(79)
etc. This defines four sets of parameters, \{\( f_l \}\}, \{\( g_l \}\}, \{\( f'_l \}\}, \{\( g'_l \}\}. They are basic quantities in the Landau theory. One should not confuse this expansion with the expansion in partial waves of the NN interaction (effective or not), but rather the different terms in \( l \) are connected with the range in
non-locality of the particle–hole interaction, or, equivalently, to its momentum dependence. In most applications these parameters are multiplied by the single particle density of state \( N \) at the Fermi surface, and one then introduces the dimensionless parameters \( F_i = Nf_i \) and similar. The other basic quantity of the Landau theory is the collision integral. It can be worked out \[147\] for two-body collisions that are assumed to dominate for not too high density. The two-body collision probability must contain two factors \( n(k) \) (Fermi functions) to weight the two occupied initial single particle states and two factors \( 1 - n(k) \) for the Pauli blocking of the two unoccupied final single particle states. Explicitly, the collision integral that has to be included at the second member of the kinetic equations \(14\) and \(15\) reads \[147\]

\[
I(p_1) = \frac{2\pi}{h^2} \sum_{p_2,p_3,p_4} |\langle p_3 p_4 | T | p_1 p_2 \rangle|^2 \\
\times \delta(p_1 + p_2 - p_3 - p_4) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \\
\times [n_3 n_4 (1 - n_1) (1 - n_2) - n_1 n_2 (1 - n_3) (1 - n_4)]
\]

(80)

where \( T \) is the scattering \( T \)-matrix (in the medium!), \( n_i = n(p_i) \) and \( \omega_i \) is the single particle energy of momentum \( p_i \). Recall that in the adopted notation \( p_i = (p_i, \sigma_i, \tau_i) \). This collision integral gives the probability that a particle of a given momentum \( p_1 \) scatters with a particle of any momentum \( p_2 \) to all possible final momenta \( p_3, p_4 \) (first term in the square brackets) and the probability that two particles of momenta \( p_3, p_4 \) scatters to the final momentum \( p_1 \) and any other momentum \( p_2 \) (second term). The two terms correspond to the loss and gain processes, for the state of momentum \( p_1 \), that can occur in the medium. We have not indicated explicitly that actually all the distribution functions appearing in the collision integral are calculated at a given space–time point. This is justified if the range of the interaction producing the collisions is much smaller than the average distance between particles. In this sense the theory is valid for low density, which in this context means that the average number of quasi-particles must not be too large. The collision integral is the key quantity that determines the different transport coefficients, because if the particle distribution is macroscopically perturbed by an external action, it is through collisions that the system reacts to bring back the distribution to the equilibrium one. The particles through collisions transport different physical quantities, such as momentum, energy and so on, and the frequency of collisions is the main feature that fixes the speed of this restoration and therefore the corresponding transport coefficients or related quantities.

On the other hand, the set of Landau parameters that characterize the interaction are more related to the mechanical properties of the medium or to the dynamical microscopic processes that can take place.

5.1. Effective mass

We have introduced the concept of quasi-particles, and we have anticipated that their physical parameters have to be renormalized. If the interaction \( f(p, p') \) depends explicitly on momentum, it changes the relationship between energy and momentum of the quasi-particle due to the dragging effect, see equation \(77\). The standard result is that the quasi-particle velocity \( v_k \) for the momentum \( k \) can be written as

\[
v_k = \frac{d\epsilon(k)}{dk} = \frac{k}{m^*}
\]

(81)

where the effective mass \( m^* \) is given by

\[
m^* = \frac{m}{1 + \frac{F_i}{3}}.
\]

(82)

Here \( m \) is the bare mass, and \( F_i \) is the dimensionless constant previously introduced. Only the term \( l = 1 \) of the expansion \(79\) contributes. The relation \(82\) is a consequence of Galilei invariance \[147\]. The concept of effective mass can be extended also to finite nuclei. It is extensively used in EDF schemes or Skyrme forces (see section 8.2) as a parameter, possibly density dependent. Physical quantities that are particularly sensitive to its value are the energy of different giant resonances, notably the monopole one, see section 3.2, and the single particle density of states. The latter is mainly proportional to the effective mass.

The canonical value of the effective mass at the saturation density that appears in most of the Skyrme forces is close to \(0.7m\). However, the density of state close to the Fermi energy extracted phenomenologically in finite nuclei seems to require a value close to \( m \). This discrepancy can be explained and understood if one introduces dispersive effects in the single particle spectrum \[151\]. In fact in the nuclear medium it is essential to distinguish between the so-called \( k \)-mass \( m_k \) and \( \omega \)-mass \( m_{\omega} \). If one considers the single particle self-energy \( M(k, \omega) \), the total effective mass can be written as \[152\]

\[
\frac{m_{\omega}}{m} = \left(1 - \frac{\partial M}{\partial \omega}\right) \\
\frac{m_k}{m} = \left(1 + \frac{m \frac{\partial M}{\partial k}}{k}ight)^{-1} \\
\frac{m^*}{m} = \left(\frac{m_{\omega}}{m}, \frac{m_k}{m}\right).
\]

(83)

The \( \omega \)-mass is due to the energy dependence of the self-energy. In the calculation of the ground state energy and wave function with Skyrme forces the mean field is directly related to the \( k \)-mass. In the density of states what is involved is the single particle dynamics, which must also include the \( \omega \)-mass, and therefore the total mass must be used. Extensive calculations \[151\] of the single particle levels confirm that indeed the total effective mass around the Fermi energy is close to the bare mass \( m \). Similar results are obtained in microscopic calculations of symmetric nuclear matter \[153\].

In a general treatment of nuclear structure based on the EDF method, to be discussed in section 8.2, the effective mass is a parameter to be fixed or fitted to the experimental mass table, and not necessarily the distinction between the \( \omega \)-mass and \( k \)-mass is apparent or displayed. Therefore, the effective mass in nuclear structure is neither a well-defined concept, nor can it be given a well-defined value. In other words, its value depends on the theoretical scheme and on the physical quantity that is considered.
5.2. Static and equilibrium properties

Since Landau theory introduces the interaction between quasi-particles, it can be used to calculate some static and equilibrium properties of an interacting Fermi liquid with respect to a free gas.

In particular the incompressibility of equation (3) is modified as follows by simple arguments. If the system is compressed, the Fermi energy \( \epsilon_F \) increases, but the quasi-particles filling the new available states interact among each other, so that, according to equation (77), the variation in the quasi-particle energy is

\[
\delta \epsilon(p) = f_0 \frac{1}{V} \sum_p \delta n(p) = f_0 \delta n
\]

(84)

where the spherical symmetry of the distribution has been used. Then the Fermi energy will get an additional variation, with respect to a free gas model, given by equation (84). Since \( \delta P = n \delta \epsilon \), this means that the incompressibility is multiplied by the factor \( (1 + F_0) \), where \( F_0 \) is the Landau dimensionless constant for \( l = 0 \). At the same time the mass should be substituted by the effective mass. Then, referring to equation (3), the incompressibility \( K \) is related to the free gas one \( K_0 \) by

\[
K = K_0 \left( \frac{m}{m^*} \right) (1 + F_0).
\]

(85)

Other bulk properties are the specific heat \( c_V \) and entropy \( s \). They require the formulation of the Landau theory at a finite temperature [147]. In the low-temperature limit the standard result is

\[
c_V = \frac{m^*}{m} c_V^0, \quad s = \frac{m^*}{m} s^0
\]

(86)

where \( c_V^0 \) and \( s^0 \) are the corresponding values for a free gas. The interaction changes just the density of states at the Fermi surface. These quantities are relevant for the thermodynamic evolution of neutron stars. For supernovae the temperature is much higher, typically a few tens of MeV. Then the Landau theory cannot be applied and other methods must be used, either phenomenological (e.g. Skyrme functionals) or microscopic, see section 9. The thermodynamical properties of nuclear matter are only partially known at such temperatures. A clarification of this subject will be of great value for the detailed description of supernovae evolution.

5.3. Transport coefficients and macroscopic dynamics

Transport coefficients are fundamental properties of a Fermi liquid in general and of the nuclear medium in particular. They are physical parameters that describe the dynamical behavior of the system at the macroscopic level and as such their main interest for the nuclear medium is mainly related to astrophysical phenomena in compact objects. Shear viscosity, that can be viewed as a momentum transport coefficient, is essential for understanding the damping of NS oscillations and supernovae evolution. Heat diffusion coefficient determines the early cooling evolution of neutron stars. The evaluation of these physical parameters can be done within the Landau theory by assuming a macroscopic deviation from equilibrium and solving the kinetic equation under the stationary condition: the (linear) relation between the quantity that describes the deviation from equilibrium and the corresponding flux of the quantity that is driven by the deviation. A temperature gradient drives a heat flux, and a fluid velocity gradient produces a transmission of momentum perpendicularly to the gradient.

Generally speaking, if local thermodynamical equilibrium is established in a Fermi system of particles, the collision integral vanishes, see equation (80), no flux can flow and the driving quantity is the deviation from local equilibrium \( \delta \epsilon \). However, in a system of interacting quasi-particles one has to consider the true quasi-particle energies, that depend in turn on the quasi-particle distribution itself, see equation (77). Therefore, the collision integral is expanded with respect to the deviation of the quasi-particle distribution from the particle local equilibrium distribution. The profile of this deviation is then determined to first order by solving the kinetic equation that includes the collision term. Under stationary conditions the kinetic equation becomes a linear integral equation for the quasi-particle local equilibrium deviation, where the kernel of the integral is determined by the collision integral under the specific physical situation. Along these lines various approximations to solve the integral equations have been developed, such as the works by Brooker and Sykes [154] and Jensen et al [155, 156], who supplied the exact analytical solutions for both the viscosity \( \eta \) and the thermal conductivity \( K \), in addition to the spin diffusion coefficient. Details of the derivation can be found in the original work or in [147]. The results involve in all cases the same angular integral over the probability \( W \) for elastic scattering of two quasi-particles at the Fermi surface:

\[
I_W = \int \frac{d \Omega}{4 \pi} \frac{W(\theta, \phi)}{\cos(\theta/2)}
\]

(87)

where \( W \) is proportional to the square of the matrix element of the scattering \( T \)-matrix, see equation (80). Because of conservation of momenta, \( W \) depends only on two angles, taken here according to the Abrikosov–Khalatnikov convention, see [147, 157]. The results for the thermal conductivity and the shear viscosity read

\[
K = \frac{1}{2 \pi^2} C_V v_F^2 \tau A_K
\]

\[
\eta = \frac{1}{5} P v_F^3 \tau A_\eta
\]

(88)

where \( v_F \) is the Fermi velocity, \( C_V \) the specific heat, \( A_K \), \( A_\eta \), are numerical factor that can be expressed in terms of numerical series, and \( \tau \) is related to \( I_W \) by

\[
\tau = \frac{8 \pi^4 \hbar^6}{(m^3 I_W T^2)}
\]

(89)

that has the meaning of a relaxation time. Actually the factors \( A_K \) and \( A_\eta \) depend also on the probability \( W \) through an additional integral, characteristic of each one of them. This must be carefully considered when comparing different models for \( W \).

Several estimates of the shear viscosity have been presented in the literature, based on different models for
Calculations have been developed, but the superfluid properties play a role in the cooling process. Only recently microscopic electron component. The specific heat of baryon matter can get, in heavy ion collisions has been rather limited. Collision dynamics appears too complex to allow any study on the relevance of the shear viscosity, not to speak of its possible value.

The thermal conductivity in NS is dominated by the electron component. The specific heat of baryon matter can play a role in the cooling process. Only recently microscopic calculations have been developed, but the superfluid properties of neutron matter must be included [161, 162]. Still much work has to be done in this field.

Finally, one should also consider the bulk viscosity of the nuclear medium. However, the main contribution to the bulk viscosity in the nuclear matter present in compact stars comes from the weak processes, and therefore it falls outside the scope of this review. In any case it has been extensively studied, with controversial results to be clarified [163, 164]. Furthermore, the possible hyperon component can be of decisive relevance [165]

5.4. Collective excitations

One of the fundamental results of the Landau theory of Fermi liquid, and nuclear matter in particular, is the possibility of collective microscopic excitations. Indeed, since the foundation of the theory, it has been shown by Landau that at low excitation energies the quasi-particle interactions can produce a concentration of the excitation strength at a particular value of the energy, where the response function displays a pole, i.e. a resonant behavior, that physically corresponds to a coherent motion of quasi-particles. Microscopically, the collective excitation is produced by the particle–hole interaction, as displayed in equation (78). If we limit the interaction to such a form, the possible excitations can be classified according to the values of the total spin $S$ and isospin $T$. The $S = T = 0$ excitations correspond to density oscillation of the system. In finite nuclei they correspond to the isoscalar monopole giant resonance. The function $f$ in equation (78) determines the position and strength of the excitation. If the interaction is zero range (contact interaction), then only the value of $f_0$, i.e. $f_l$ for $l = 0$, is different from zero. The value of $f_0$ is therefore a fundamental constant that characterizes the nuclear medium. The onset of this collective motion can be seen by linearizing the kinetic equation (14) with respect to the variation of the quasi-particle distribution function, since we are considering small oscillations of the system. We then put $n(p, r, t) = n_0(p) + \delta n(r, p, t)$, where $n_0$ is the ground state quasi-particle distribution, i.e. a sharp Fermi distribution, and neglect any quadratic term in $\delta n$. We also neglect the collision integral $I$, that is we assume that the collision frequency is small with respect to the oscillation frequency. This also means that the quasi-particles’ decay can be neglected. After Fourier transform of the equation at momentum $q$ and frequency $\omega$ and simple manipulations, one gets, in the long wavelength limit ($q \to 0$),

$$\omega - q \cdot v_p \delta n(q, p, \omega) + \frac{\partial n_0}{\partial p} (q \cdot v_p) \sum_{p'} f(p' - p) \delta n(q, p, \omega).$$

(90)

This is an eigenvalue equation for the frequency $\omega$ at momentum $q$. Actually the equation depends only on the ratio $\omega/q$, which is the propagation velocity of the oscillatory wave. The corresponding distribution distortion $\delta n$ can be obtained by noting that the derivative $\partial n_0/\partial \epsilon_p$ equals $-\delta(\epsilon - \epsilon_F)$, since $n_0$ is just a sharp Fermi distribution. Then also $\delta n$ is proportional to the delta function

$$\delta n(q, p, \omega) = \delta(\epsilon - \epsilon_F) \xi(q, p, \omega).$$

(91)

If the interaction has only the $l = 0$ term different from zero, then $\xi$ depends only on the modulus of $p$, and substituting equation (91) into equation (90) one gets an explicit eigenvalue equation for $\omega$. Putting $s = \omega/q v_F$, it reads

$$1 + \frac{1}{2} F_0 \int_{-1}^{1} d(\cos \theta) \frac{\cos \theta}{\cos \theta - s} = 0$$

(92)

where $F_0 = N f_0$. If $s > 1$, corresponding to a repulsive interaction $f_0 > 0$, the integral is non-singular and one gets a dispersion relation for $s$:

$$1 + \frac{1}{2} F_0 \left( 1 + \ln \frac{s - 1}{s + 1} \right) = 0.$$  

(93)

This solution is called ‘zero sound’. It is an excitation mode of purely quantal origin, typical of any fermion liquid with a repulsive interaction. As $F_0$ increases from 0 to large values much larger than 1, the solution $s_0$ of the dispersion relation varies from 1 to $\sqrt{F_0/3}$. As already anticipated, this mode has an analogous mode in finite nuclei in the isoscalar monopole giant resonance, which can then be considered as the zero sound in finite nuclei. In nuclear matter other types of zero sounds can exist, corresponding to different spin–isospin total quantum number. $T = 1$ and $S = 0$ correspond to the dipole giant resonance in nuclei, $T = 0$ and $S = 1$ to the spin magnetic mode, and $T = 1$ and $S = 1$ to the Gamow–Teller resonance. They have all been extensively studied in nuclei. If one keeps the correspondence with finite nuclei, it is possible to have phenomenological indications on the Landau parameters from the positions of the giant resonances in nuclei. It turns out, following this line, that the Landau parameter $F_0$ must be slightly negative, approximately $-0.4 < F_0 < 0$. The analogous Landau parameter $G_0 = N g_0$ should be much smaller than 1, due to the observed lack of collectivity of the spin mode in nuclei. The value of $F_0$ for the dipole mode should also be slightly negative, a few tenths, but the extrapolation from nuclear matter to nuclei is questionable, because in nuclei there is a substantial contribution of the surface to this mode.
Indeed the analysis of [13] on the mass dependence of the giant dipole resonance gives some indication that the mode smoothly develops from a surface mode, as in the Goldhaber–Teller [166] model, to a bulk one, as in the Steinwedel–Jensen [167] model, as the atomic number increases. The Landau parameter for the Gamow–Teller mode is much less known due to the substantial contribution of the Δ excitation, not included in the usual Landau treatment. The Landau parameters characterize the nuclear medium at a fundamental level, but they are only partially known.

Going back to the dispersion relation (93), let us consider the case \( s < 1 \) that should appear when \( F_0 \leq 0 \). The integral is then singular and one has to specify how to handle the singularity. Physically speaking, if \( s < 1 \) the possible eigenfrequency falls inside the unperturbed particle–hole continuum. In fact the unperturbed continuum, in the long wavelength limit, spans the excitation energy up to \( +qVF \). This means that the mode couples directly to the particle–hole continuum and can decay. This implies that one should look for a complex solution of the dispersion relation, whose imaginary part will provide an estimate of the damping decay time of the mode. This decay mechanism is called ‘Landau damping’ and is a phenomenon characteristic of Fermi liquids in general. It is not related to the collisions between quasi-particles, and therefore it has no connection to any sort of viscosity, being a purely quantal effect. It has to be stressed that when Landau damping is active, the excitation mode is strongly damped and in practice it disappears.

One could ask whether Landau damping occurs in finite nuclei. In general the main strength of the giant resonances falls in the single particle continuum, that is at an excitation energy where the particle can escape, and therefore where the unperturbed particle–hole spectrum is continuous. In principle, the Landau damping can therefore be present; however, at the same time the so-called spreading width is also present, that is coupling with more complex configurations, like two particles–two holes states. In nuclear matter the latter turns out to be very small with respect to the Landau damping, just due to phase-space restrictions. In finite nuclei Landau damping is not so strong as in nuclear matter, due to the relatively small density of states at the excitation energy where they are located. The width of giant resonances, as already mentioned, is therefore mainly a nuclear structure problem, not related to the gross properties of the nuclear medium.

The elementary excitations in the nuclear medium have relevance for the physics of NS. In the homogeneous region of the star these excitations affect the emission and propagation of neutrinos and the specific heat of matter. They are expected to be present also in the crust region [168]. For illustration we report in figure 12, taken from [169], the spectral functions of neutrons, protons and electrons at a nucleonic density equal to the saturation one. The spectral functions reported here are just the density–density response functions. They describe the density fluctuations produced by an external probe. A sharp delta-like peak corresponds to an excitation mode of the system. One can see that the neutron strength is quite spread, showing that the excitations are in the region of Landau damping. One distinguishes the sharp electron peak corresponding to the plasma mode. Neutron and proton excitations are mixed, but their interaction can be considered weak, because the proton strength looks rather localized and not so much affected by the Landau damping. In these calculations three possible effective interactions have been taken, corresponding to the Landau parameter \( F_0 \). Since matter is asymmetric, in this case we have three different \( F_0 \) parameters, corresponding to neutron–neutron, proton–proton and neutron–proton interactions. The first choice was taken from BHF calculations, the other two from particular Skyrme forces. The matter was assumed to be normal. Superfluidity can partly change this picture [170]. In any case the overall picture that comes from these analyses characterizes some of the fundamental properties of the nuclear medium.

Finally, we mention that in nuclear matter, when the characteristic frequencies of the motion become very small, the number of quasi-particle collisions per period of oscillations can become so large that local equilibrium can be reached and the hydrodynamical regime can then be established. Density oscillations can still be present and we have in this case the ‘first sound’ mode. This is virtually identical to the sound in classical fluid, like air, in the same dynamical limit. In the hydrodynamical regime macroscopic motion is determined just by the conservation laws and the EoS of the nuclear medium. Macroscopic physical parameters, such as those discussed in sections 5.2 and 5.3, then play the central role.

### 5.5. Theoretical challenges

We have seen that the fundamental parameters of the Landau theory of Fermi liquid, as applied to the nuclear medium, are only poorly known on the basis of laboratory experiments on nuclei. Astrophysical observations can also give information on their values, but the data are still scarce and difficult to interpret. They are associated mainly with observations on NS oscillation or supernova evolution. It would therefore be quite desirable to have theoretical evaluations of their values and behavior with density on the basis of sound microscopic many-body theories. We have already discussed the scattering matrix terms which appear in the collision integral and the present state of the art. There is a vast body of literature on the theoretical evaluation of the Landau parameters \( \{f_i, g_i, f_i', g_i'\} \) of the quasi-particle interaction. Their ab initio theoretical determination is extremely difficult because of the hard core character of the bare NN interaction. Among the different schemes that have been employed we can mention the self-consistent Babu–Brown equations [171, 172], the self-consistent Green’s functions expansion [173, 174] and more recently the RG method [175]. The main difficulty in all methods lies in the estimation of the role of the screening processes. The quasi-particles can exchange collective excitations, such as the ones discussed in section 5.4, where again the quasi-particle interaction appears. This entails a self-consistent procedure to calculate the interaction itself. However, the results are sensitive to the scheme and approximations employed in the procedure and conclusive results are not yet available. Furthermore, for asymmetric matter, such as the one present in NS, calculations are quite scarce.
6. Neutron matter at very low density: an exercise in many-body theory

The low-density region of pure neutron matter, as present in the inner crust of NS, is less trivial than one could expect at first sight since the neutron–neutron scattering length is extremely large, about $-18$ fm, due to the well-known virtual state in the $^{1}\Sigma_{0}$ channel, and therefore even at very low density one cannot assume the neutrons to be uncorrelated. These considerations have also stimulated great interest in the so-called unitary limit, i.e. the limit of infinite (negative) scattering length of a gas of fermions at a vanishingly small density. A series of works [176–178] have been presented in the literature based on various approximations, and a recent Monte Carlo calculation [179] on a related physical system has shown that the unitary limit can present a quite complex structure, involving both fermionic and bosonic effective degrees of freedom, which has still to be elucidated. Variational [180] and finite volume Green’s function Monte Carlo calculations [181] for neutron matter at relatively low densities have shown that the EoS, in a definite density range, can be written as the free gas EoS multiplied by a factor $\xi$, which turns out to be close to 0.5. This is actually what one could expect in the unitary limit regime, since no scale exists in this case, except for the Fermi momentum. Monte Carlo calculations [176–178] with schematic forces in a regime close to the unitary limit have found a factor $\xi \approx 0.44$. The connection between the variational results and the unitary limit has been studied in [182] by means of effective theory methods.

6.1. A single G-matrix problem

Since the scattering length $a$ and effective range $r_0$ in the $^{1}\Sigma_{0}$ channel of the neutron–neutron interaction differ by about a factor 6, there is no density interval where the unitary limit can be considered strictly valid. However, in the range $r_0 < d < |a|$, where $d$ is the average inter-particle distance, the physical situation should be the ‘closest’ possible to the unitary limit. This range corresponds to the Fermi momentum range $0.4 \text{ fm}^{-1} < k_F < 0.8 \text{ fm}^{-1}$, which corresponds to densities between about 1/50 and 1/5 of the saturation density. Let us choose as a realistic NN potential the Argonne $v_{18}$ interaction.
Figure 13. Plots of the free $K$-matrix ($T$), divided by 3 for convenience, in comparison with the in-medium $K$-matrix ($G$), at the indicated Fermi momentum, for different total momentum $P$, as a function of the relative momentum $k$. All the momenta are in fm$^{-1}$. The results are for the $^{1}S_{0}$ channel in pure neutron matter. The arrow indicates the maximum momentum needed in the calculation of the EoS; see [184] for details. Reprinted with permission from [184]. Copyright 2008 by the American Physical Society.

[183]. The first finding is that the TBF of the Urbana model, adjusted to reproduce the correct saturation point [92], give a contribution which is less than 0.01 MeV, and therefore we can neglect TBF to a good approximation. The second finding is that the single particle potential is very small in this density range, and its effect can be neglected. It affects the energy per particle less than 0.1 MeV.

It is enlightening to compare the in-medium $G$-matrix with the free $T$-matrix in the $^{1}S_{0}$ channel. The diagonal matrix elements are reported in figure 13, taken from [184], at selected values of the relative momentum $k$ and total momentum $P$ (in fm$^{-1}$) at the Fermi momentum $k_F = 0.4$ fm$^{-1}$. For the sake of comparison the free $T$-matrix has been divided by 3. Due to Galilean invariance, the free $T$-matrix is independent of $P$. Despite the fact that Fermi momentum is quite small, a drastic difference between the two scattering matrices is apparent, not only in shape but also in absolute value. The Pauli operator effect is enhanced in this particular channel since the virtual state is suppressed in the medium. This illustrates the dramatic difference that can exist between the in-medium effective interaction and the free bare interaction. The large enhancement at the Fermi momentum and for small total momentum $P$ is due to the pairing singularity, to be discussed in section 7.4. The BBG expansion relies on the basic idea that the contributions of the diagrams of the expansion decrease with increasing number of hole-lines which are included. Although the BBG scheme is essentially a low-density expansion, it has been found [186, 187] that the convergence is valid up to densities as high as a few times saturation density in symmetric nuclear matter and even better in neutron matter. It is then likely that at the low densities considered this convergence should be even faster. This is indeed confirmed by explicit numerical calculations [184], and indeed the third finding is that the three hole-line contribution is at most 0.15 MeV at the highest density considered and rapidly decreasing to vanishingly small values as the density decreases. Finally, the fourth finding is that higher partial waves give a negligible contribution both at the two hole-lines (Brueckner) and three hole-lines level. These four findings, all together, point out that the many-body problem of neutron matter at low densities is reduced to a single $G$-matrix problem, i.e. to the calculation of the $^{1}S_{0}$ $G$-matrix.

6.2. The ‘exact’ equation of state

The two EoS, one calculated within the full BBG expansion up to the three hole-lines contributions and the other calculated with the single $^{1}S_{0}$ $G$-matrix, are compared in figure 14, taken from [184]. They are mainly indistinguishable. The energy per particle is very close to 1/2 of the kinetic energy. It turns out that the $G$-matrix is fully determined by the scattering length and effective range. One can construct a rank-one separable interaction [184]:

$$\langle k' | v | k \rangle = \lambda \phi(k') \phi(k)$$ (94)
Figure 14. Neutron matter EoS calculated within the BBG method (label G), within the variational method of [180] (triangles), according to the estimate of [185] (dotted line) and with the separable representation of the \( G \)-matrix (label K sep). The dashed-dotted line is one half of the free gas EoS. The square represents the result of the quantum Monte Carlo calculation of [188].

with a simple form factor

\[
\phi(k) = \frac{1}{(k^2 + b^2)}
\]

where the parameters \( \lambda \) and \( \beta \) are determined by imposing that the scattering length and effective range are reproduced. Then the \( G \)-matrix can be obtained analytically and the corresponding EoS by simple numerical integration. The procedure is equivalent to an effective theory with smooth cutoff and its accuracy is shown in figure 14. The calculation can be extended to a very small density, as reported in figure 15, where one can see that the EoS approaches the one for a free gas, as it must be for \( k_F < 1/|a| \). In both figures 14 and 15 the squares indicate the results of the quantum Monte Carlo calculation of [188]. They agree fairly well with the BBG results up to the density where the Monte Carlo calculation can be performed.

What is missing in the BBG calculations is the pairing correlations. The contribution to the EoS of pairing is not expected to be relevant, but it is important to know the value of the gap for many phenomena in NS and as an indication for finite nuclei. This subject will be covered in sections 8.3 and 7.4.

7. Bulk properties of nuclear matter

In this section we try to illustrate our knowledge of some of the physical parameters that characterize the properties of the nuclear medium. Some of them have been discussed in previous sections for matter close to saturation density, so we concentrate mainly on their density dependence. A subsection is devoted to the possible superfluid phases, that have not been discussed up to now.

7.1. Density dependence of the symmetry energy

Hints about the symmetry energy below saturation density have been claimed to come from experimental data on heavy ion reactions at intermediate energy. The iso-scaling regularity [189, 190] appears to be present in many experiments on multi-fragmentation, from which an estimate of the density dependence of the symmetry energy can be deduced [191]. These experimental data have been analyzed in [192] on the basis of a quantum statistical approach which includes the formation of clusters (light nuclei up to \( A = 4 \)). As discussed in section 4.2.1 at densities lower than about 2/3 saturation density homogeneous symmetric nuclear matter is unstable toward cluster formation and this affects the EoS and the symmetry energy, defined as the difference between the pure neutron matter EoS and the symmetric matter EoS with the inclusion of clusters. In [192] it was shown that the low-density symmetry energy thus defined is in agreement with the one extracted from the data on iso-scaling, while the low-density symmetry energy for homogeneous matter (that is without clusters) is too small. In fact the formation of clusters leads to a lowering of symmetric matter EoS.

Another possibility has been advocated [193, 194] from the data on the so-called ‘isospin diffusion’ in heavy ion reaction at moderate energy.

Microscopic calculations show fair agreement with these ‘data’ and among each other up to saturation and slightly above. A comparison with results from different Skyrme forces shows that only a few of them are compatible with this general behavior. In [195] isospin diffusion data on tin isotopes have been analyzed within quantum molecular dynamic simulations, and constraints on the symmetry energy at sub-saturation density (essentially the parameter \( L \)) have been extracted and compared with the constraints coming from other methods. In view of the discussion above, the comparison with the low-density symmetry energy extracted from iso-scaling analysis is at least questionable.

From the theoretical point of view and for application to the EDF method it is still meaningful to consider the symmetry energy for homogeneous matter. In figure 16, taken from [196], the symmetry energy in the low-density region calculated in the
BHF scheme is shown in comparison with a few Skyrme forces that show fair agreement with the microscopic calculations. The most modern Skyrme forces are constructed in such a way to fit the microscopic results, and the agreement is therefore enforced. Above saturation the situation is less under control. On the one hand the microscopic calculations need to extend the use of TBF to density where they are not well known. On the other hand experimental data on heavy ion collisions can provide hints on the density dependence of the symmetry energy only very indirectly through extensive simulations. This approach is extensively reviewed in [197], where the prospects in this line of research are presented in detail, in particular in connection with the development of the facilities for exotic nuclei.

Astrophysical observations on the mass of NS are also indirectly testing the symmetry energy at high densities, because the value of the symmetry energy can change the value of the incompressibility of the very asymmetric matter at the center of compact stars. As already mentioned, the interplay of the data extracted from the heavy ion reaction at intermediate energy, which test the EoS of almost symmetric matter, and the analysis of the observational data on NS, where the matter is highly asymmetric, can be of great help to clarify this difficult but fundamental issue.

7.2. Incompressibility

The incompressibility near saturation has already been discussed. Below saturation density usually the EoS of symmetric matter and neutron matter are assumed to be simple low-order polynomial functions. This can be a delicate point, since the small details of the EoS behavior at low density, especially for symmetric matter, can be relevant for the construction of accurate EDFs to be used in the calculations of the mass table. Microscopic calculations appear in close agreement in this density region, see figure 6. At higher density, well above saturation, the microscopic theories face the problem, already mentioned, of the TBF, that give a very large contribution but are not well known. As a reference density above which this problem can be serious, one can take a value around 3–4 times saturation density. Also in this case the interplay between theory and experiments is essential to make progress on this issue.

7.3. Viscosity

The notion of viscosity has already been discussed within the Landau theory of Fermi liquid. Here the effective interaction that enters, one way or another, in the calculation of the EoS must be the basic quantity to be used also to calculate the scattering probability $W$ of section 5.3. For instance the $G$-matrix of the BHF scheme should be used as the basic quantity. Of course, at high densities we meet the same problems as for the EoS.

7.4. Superfluidity

It was argued long ago [198, 199] that to explain the observed glitch phenomenon in several pulsars it would be natural to assume that nuclear matter is a superfluid in the interior of NS. In fact, the sudden increase in the rotational frequency and the long time needed to recover the initial rotational frequency suggest the presence in the crust of an almost decoupled component with low viscosity. Since then a vast body of literature has developed on the subject, and different superfluid phases have been found theoretically under the physical conditions expected inside NS. It is well known that theory of superfluidity, or better superconductivity, was first formulated by Bardeen, Cooper and Schrieffer (BCS) [200]. In a more general formulation, the constitutive equation for the onset of a superfluid phase is the gap equation [4,149]

$$\Delta(k) = - \frac{1}{2} \sum_{k'} V(k, k') \frac{\Delta(k)}{E_k}. \quad (96)$$

Here the gap function $\Delta$ is related to the pairing correlation function, or ‘pair wave function’ by

$$\Delta(k) = \langle \psi^\dagger(k) \psi(-k) \rangle. \quad (97)$$

For simplicity we assume pairing in the s-wave and the particles that form the Cooper pair have opposite momentum and spin. Then the so-called quasi-particle energy is given by

$$E_k = \sqrt{\tilde{\epsilon}_k^2 + \Delta_k^2} \quad (98)$$

where $\tilde{\epsilon}_k = E_k - \mu$, $\mu$ being the chemical potential and $E_k$ the single particle spectrum in the absence of pairing interaction. In this case all quantities depend only on the modulus of $k, k'$. The kernel $V$ is the irreducible particle–particle interaction. In the many-body language, it is the sum of all diagrams that describe the interaction of two particles and that cannot be divided into two disconnected parts by cutting two particle
lines. For strong pairing correlation it also depends on the pairing gap $\Delta$. The exact equation for the pairing gap is more complex than equation (96). It can be formulated in terms of single particle Green’s functions and it can be found in [4,149]. We will discuss later the possible modifications, but equation (96) already contains some of the main features of the pairing problem for the nuclear medium. For neutron–neutron or proton–proton pairing the only s-wave channel is the $^1S_0$ two-body channel, and we will first discuss this case. Equation (96) is a non-linear equation for the gap function $\Delta(k)$, due to its presence in $E_1$ in the denominator. The onset of the superfluid phase of the nuclear medium is indicated by a non-zero solution for the gap. The zero solution $\Delta(k) = 0$ is always present, but it can be shown that the energy of the superfluid phase, if it exists, is always lower than the normal phase. This is of course a general feature. What characterizes the pairing correlation in the nuclear medium is the value of the gap, which is estimated to be a substantial fraction of the Fermi energy. Another feature is that the effective interaction is not concentrated around the Fermi surface, like in an ordinary superconductor or in superfluid $^3$He. Although a large contribution in the gap equation comes from the interaction at the Fermi momentum, momenta that are far away from the Fermi surface are also essential and the pairing gap function $\Delta(k)$ is momentum dependent. All this can be seen by considering the simplest approximation for the pairing interaction, the bare NN interaction. It turns out that the bare NN interaction is able to produce a substantial pairing gap at a low density, but, due to the hard core of the interaction, high-momentum components must be taken into account. The approach can be extended to other channels, and the possible pairing gaps at the Fermi surface in this approximation as a function of density in NS matter is depicted in figure 17, taken from [201]. In this evaluation of the different pairing gaps the single particle spectrum has been taken from BHF calculations for the NS matter. Accordingly, the concentration of protons has been calculated at the beta equilibrium and the nuclear matter EoS from the BHF theory. Other microscopic EoS would give similar results. This is mainly equivalent to the use of the effective mass. In this case the effective mass is smaller than the bare one. This reduces the pairing gap because it increases the kinetic energy and reduces the density of states. The effect is larger at a higher density, such as for the proton and the $^3PF_2$ neutron pairing, and it is almost negligible for the $^1S_0$ neutron pairing. The proton–proton pairing is shifted at a higher density with respect to the neutron–neutron one just because the fraction of protons varies smoothly from a few percent to about 15%, according to the adopted EoS. The p-wave neutron pairing is present only at a high density because the phase shifts of the $^3PF_2$ channel becomes appreciable only at higher a momentum, and hence at a higher Fermi energy.

This overall picture is very important for many phenomena that occur in NS, in particular for the cooling process and the glitch events. However, a quantitative description of the observational data requires an accurate prediction of all these pairing gaps. This turns out to be an extremely difficult task. First of all the pairing gap is quite a sensitive quantity and small variations of the interaction can substantially change its value. This can be illustrated in the weak coupling limit, where one takes a momentum independent interaction and a constant density of states. In this limit the pairing gap can be obtained from a simple integration and it is a constant. It depends exponentially and non-analytically on the interaction strength

$$\Delta = E_0 \exp(1/\nu k_F m)$$

(99)

where the factor $E_0$ is related to the cutoff, but it varies smoothly and has a logarithmic dependence on it. The non-analyticity is a manifestation that a phase transition takes place, for any small value of an attractive interaction at the Fermi surface.

Then the question arises as to whether we know accurately enough the bare NN interaction that the pairing gap thus calculated is essentially independent of the particular realistic NN interaction. This turns out to be true for the $^1S_0$ neutron and proton pairing gap, but it is invalid for the $^3PF_2$ channel at the higher density. This is because the phase shifts are known up to about relative momentum $k = 2\text{ fm}^{-1}$, above which any potential gives essentially extrapolated values. This can be dramatically seen in figure 18, where the pairing gap in the $^3PF_2$ channel, calculated in the BCS approximation, is reported for different interactions. Above $k_F = 2\text{ fm}^{-1}$ the diverging results are a manifestation of this uncertainty. Further uncertainty is introduced at a higher density, where TBF start to play a role. In addition to this basic uncertainty, not present for the lower density case, the BCS approximation must be implemented by including the many-body effects in the gap equation. Correlations affect the single particle motion. As already mentioned, the momentum dependence of the normal part of the self-energy introduces the single particle potential, which can be approximated by substituting the bare mass with the effective mass. In addition, the energy dependence introduces the so-called quasi-particle strength or $Z$-factor that gives the weight with which the quasi-particles can take part in a Cooper pair. More generally, if one assumes that the effective pairing interaction is energy independent, the gap equation can
Figure 18. Pairing gap in the $^3P_F_2$ channel as a function of density in neutron matter.

Due to the exponential dependence, the $Z$-factor can substantially reduce the pairing gap. The total effective mass at the Fermi momentum, when dispersive effects are included, turns out to be close to 1 in the density range where neutron matter is expected to be a superfluid. These effects have been studied by several authors, both using the more general equation (100) and the pole approximation [202–206] of equation (102). The reduction of the gap looks to be moderate, approximately 20–30%.

The estimation of the many-body effects on the pairing interaction $\nu$ turns out to be the most difficult task. The main problem seems to be the inclusion of the medium polarization effects on the effective interaction. This problem has been approached by a variety of techniques and approximations. The results are summarized in [201]. They all display a strong reduction in the pairing gap in the neutron $^1S_0$ channel, but they do not agree on the size of the reduction, that has a value between 0.5 and 0.2 or even smaller. The discrepancy is again related to the extreme sensitivity of the pairing gap to the value of the effective pairing interaction, which requires an accuracy difficult to reach in this many-body problem. More recently the RG method has been used [175]. The results look similar to the previous calculation in [173]. In the region of lowest density Monte Carlo calculations are also available [188, 207, 208]. Despite some discrepancies among the results, they seem to indicate a much smaller reduction of the pairing gap for this channel. Whether this is due to some limitation of the Monte Carlo calculations or to a drawback of the other microscopic many-body methods is a question that is awaiting an answer.

The $^1S_0$ proton pairing gap is more difficult to treat, because the much higher density of neutrons can have a dominant effect. The most recent calculation [209] indicates that the effect of the neutron component strongly enhances the effective pairing interaction through the tensor component of the NN interaction, but this effect is in competition with the strong reduction due to the effective mass and $Z$-factor. To illustrate how difficult it is to control all these competing effects, figure 19 reports the pairing gap when each one of these many-body contributions is introduced. Notice that without the neutron polarization effect the gap would be zero. The final gap turns out to be more concentrated at a lower density with a smaller strength. Further studies are surely needed to confirm these results.

The $^3P_F_2$ neutron pairing gap turns out to be much smaller and relevant only in a deeper region of an NS. Application [210] of the RG method to this channel brings the gap to a very small value, in the range of a few tens of keV, even compatible with a vanishing one.

Observational data on NS cooling can indirectly give indications about the values of all these pairing gaps. An analysis based on the assumption of pure nucleonic matter [211] constrains the gaps to values that are compatible with the theoretical predictions. In particular, the $^3P_F_2$ is seen to be necessarily quite small, of the same order as the theoretical results. However, recent astrophysical observations [212, 213] seem to indicate that in the core the superfluid pairing gap should be close to 100 keV or larger. The proton pairing gap is deduced to be indeed restricted to a relatively lower baryon...
density, mainly outside the inner core. For the neutron $^{1}S_{0}$ the constraints are less stringent, but anyhow compatible with most of the theoretical predictions.

The possible onset of exotic matter components, such as hyperons and quarks, noticeably complicates the analysis. These components can also be superfluids, but unfortunately their pairing gaps are even more uncertain than in the nucleonic case. From the observational data on glitches one can extract indications on the pinning energy of vortices, and therefore indirectly on the pairing gap. However, this phenomenon still needs a complete dynamical theory before any conclusion on both the pairing gap and the pinning energy can be deduced.

The possible link of these results with the pairing phenomenon in nuclei is discussed in section 8.3.

8. Connection with nuclear structure

One of the basic questions that was posed since the first developments in nuclear physics is to what extent the properties of the nuclear medium, in the bulk and at the surface, can be transferred to finite nuclei or to what extent they influence the general trends that are observed in nuclear structure studies. The simplest schemes where this link is exploited is the LDM and the DM, introduced in section 3.1. As explained, in this semi-classical model the assumed constant saturation energy per particle is corrected by the surface and Coulomb energy to explain the binding energy of finite nuclei. In more sophisticated variants of the models the semi-classical TF approximation is used to calculate the smooth trend of the nuclear binding (and possibly other quantities) along the mass table. The TF functional is fixed by a set of parameters to be fitted to data. Quantal characteristics like shell effects are not included. We will briefly describe some of the methods that have been used and developed to understand more deeply this possible link and, at the same time, to devise practical and general theoretical scheme for nuclear structure which go beyond the TF scheme.

8.1. The Thomas–Fermi approximation and implementations

Historically, the first method to relate the properties of finite quantal systems to the corresponding homogeneous system was the semi-classical TF scheme. It was devised to calculate the ground state properties of large atoms and molecules in the case of the independent particle limit. In fact, even in this limit, the quantal calculations can become quite complex. The original form of the scheme is equivalent to taking the zero-order term in the expansion in $h$ of the density matrix for the independent particle wave function, but it can be easily derived by assuming that the system is locally equivalent to a free Fermi gas at the local density and local potential. We recall here some features of the approximation that are useful for the development of the presentation. In its simplest version it can be formulated within the density functional method, i.e. assuming that the energy of the system can be written as a functional of the density $\rho(r)$:

$$E_{TF} = T_{F}(\langle |\rho| \rangle) + V_{C}(\langle |\rho| \rangle) + V_{pp}(\langle |\rho| \rangle)$$

(103)

where $T_{F}$ is the kinetic energy contribution, $V_{C}$ the external potential and $V_{pp}$ the particle-particle interaction. In the case of atoms, $V_{C}$ is the energy due to the central Coulomb potential and $V_{pp}$ is the electron-electron Coulomb interaction energy:

$$T_{F}(\langle |\rho| \rangle) = \frac{3}{5} \int d^{3}r E_{F}(\rho(r))\rho(r);$$

$$V_{C}(\langle |\rho| \rangle) = -Z \int d^{3}r' v(r)\rho(r')$$

$$V_{pp}(\langle |\rho| \rangle) = \int d^{3}r \int d^{3}r' v(|r - r'|)\rho(r)\rho(r')$$

(104)

with $Z$ the charge of the nucleus, $v = 4\pi e^{2}/r$ the Coulomb potential between two electrons (of charge $e$) and $E_{F}$ the Fermi energy for a free gas at the density $\rho(r)$. The Euler–Lagrange equation corresponding to the minimization of this functional with the constraint of a fixed number of particles is

$$E_{F}(\rho(r)) - Zv(r) + \int d^{3}r' v(|r - r'|)\rho(r') = \mu$$

(105)

where $\mu$ is the Lagrange multiplier, that has the meaning of chemical potential. Note that $\mu$ is a constant, independent of the position. The solution of this equation gives the density and then the energy of the ground state. A way of solving this equation is to apply the Laplacian differential operator and use the Poisson equation $\Delta V_{C} = -4\pi e^{2}\rho(r)$, where
\( V_c \) is the last term on the left-hand side of equation (105), that is the potential produced by all electrons at \( r \). This gives the familiar differential equation of the TF scheme in its simplest form [214]. We will not discuss the many refinements of the approximation that have been developed, to include, e.g., the exchange interaction, but rather we consider the TF approximation in the nuclear case. The physical situation is rather different. There is no central interaction, and the binding must come from the NN interaction. The latter is not long range, like the Coulomb potential, but it is short range, even zero range if we adopt a Skyrme effective interaction. Then equation (105) has no solution, except in the trivial case of a homogeneous system. To get any sensible result for a nucleus one has to go to the next order in the expansion in \( \hbar \) of the kinetic energy term in the functional. This introduces gradient terms [8]. To order \( \hbar \) only one term contributes, proportional to \((\nabla \rho)^2\). With the inclusion of this term, a surface can develop and the nuclear TF approximation can describe the density profile of a nucleus.

This discussion makes clear that in the nuclear case the kinetic term must be treated at a different order in the expansion in \( \hbar \) and at a different level of approximation. The full quantal treatment of the kinetic term must be considered to construct any accurate nuclear energy functional.

Despite that, the TF approximation, or more advanced expansions in \( \hbar \) can be useful. In fact the TF approximation and its implementations are expected to describe the smooth part, e.g., of the density of states, leaving outside of their remit the description of the shell effects, which are a purely quantal effect. Indeed the expansion in \( \hbar \) of many quantities is actually asymptotic, and the quantal effects depend non-analytically on \( \hbar \). This property of the \( \hbar \) expansions can be used to evaluate the shell effects in the so-called microscopic–macroscopic approach, introduced in section 3.1. The difference between the exact quantal calculations and the results of the expansion is taken as an estimate of the shell effects. Implicitly, it is assumed that shell effects are essentially the same in the independent particle model as in the fully correlated system. The smooth part, obtained from the liquid drop or droplet models, is expected to include in an effective way the correlation contribution. Along these lines, recently in [215] the Kirkwood \( \hbar \) expansion to fourth order has been used to estimate the shell effects by comparing the results with the quantal calculations. This method and similar ones based on \( \hbar \) expansions are methods alternative to the Strutinski smoothing method [12].

A step further along these lines is the construction of general effective EDFs that are devised to include all the correlations in an effective way and without any \( \hbar \) expansion. Although they are necessarily in part phenomenological, i.e. they contain a certain number of parameters, the aim is to relate their characteristics in terms of the properties of the nuclear medium. On the other hand, if they are treated on a purely phenomenological level, they turn out to be extremely accurate. These items will be discussed in the next section.

8.2. The density functional method

A fully quantal microscopic method to connect the nuclear medium properties and the structure of finite nuclei is to introduce an effective NN force, whose parameters are fitted, on the one hand, to reproduce the nuclear matter EoS, as derived phenomenologically and microscopically, and on the other hand to reproduce binding energy and radii of a representative set of nuclei. This is the scheme of the Skyrme forces. These effective forces have been developed along the years with increasing success and have been widely used in nuclear structure and spectroscopic studies. It is surely not possible to review the enormous body of literature on the subject, and we limit ourselves here to sketching the main features of the method. The simplest Skyrme force can be written, for a symmetric system, as

\[
V = t_0 \rho(r) \delta(r - r') + t_2 \rho(r)^2 \delta(r - r') \delta(r' - r'')
\]  

(106)

where \( \rho \) is the nucleon number density at the point \( r \) and the parameters \( t_0 \) (negative) and \( t_2 \) (positive) are adjusted to reproduce the saturation point of nuclear matter, where the force depends only on the constant total density. They correspond to effective two-body and three-body interactions, respectively. In addition to these parameters, an effective mass \( m^* \) is also usually introduced in the kinetic part, so that the total Hamiltonian can be written as

\[
H = T + V = T + \frac{\hbar^2}{2m^*} \nabla \delta(r - r') \nabla.
\]  

(107)

Then the force is used to calculate the ground state of nuclei in the Hartree–Fock (mean field) approximation, thus establishing an indirect link between nuclear matter and finite nuclei. Much more elaborate forces have been developed, in which density gradient terms and asymmetry-dependent terms are introduced. Then the number of parameters increases, but the precision of the fit on the overall mass table can be really impressive, see e.g. [28, 29]. More elaborate terms involving higher degrees of the density derivatives can be still introduced, and the accuracy of the fit can be astonishingly good [216, 217]. However, along this line, the connection with NN bare forces and nuclear matter EoS is gradually lost, since the additional terms have a form loosely connected with the NN interaction and are all vanishing in a uniform matter.

Another method that tries to keep more closely the connection with the properties of the bulk nuclear medium is based on the Kohn and Sham (KS) [218–224] approach, first devised for atomic, molecular and solid systems and developed also for nuclear system. Let us consider the microscopic bulk EoS, as reported in figure 6, extended to asymmetric nuclear matter. This can be taken as the bulk contribution to the EDFs. For numerical applications it can be written in a polynomial form. The functional must then be implemented mainly by three additional contributions. The first one is the Coulomb energy, that can be calculated with different degrees of sophistication, e.g. by including the exchange and short range parts. The second one must take into account the presence of the surface, that in nuclei is sharply localized, within a length of the order of 1 fm, and therefore cannot be
described only by the bulk part. The additional contribution has to be localized at the surface, and the simplest way to do so is to introduce density gradient terms or non-local short range convolution terms. The surface terms are connected to the surface tension of nuclear matter, because, in the macroscopic limit, they modify the surface energy of the system. Finally, it is mandatory to add a spin–orbit term, since it strongly affects the single particle level scheme and it is mandatory in order to reproduce the shell sequence (i.e. the ‘magic numbers’). The spin–orbit interaction is roughly proportional to the gradient of the single particle potential, and therefore it is also localized at the surface. The strength of this term is severely constrained by phenomenology, but it has still some degree of uncertainty. It is desirable to keep the number of surface and spin–orbit terms to a minimum, since they introduce additional phenomenological parameters. It is one of the aims of the EDF method to get these parameters from microscopic many-body theory, but in order to get a high precision fit to the wide set of nuclear binding and radii throughout the mass table, they must be finely tuned beyond the possibility in accuracy of any microscopic theory. The possibility remains to have a guidance to their values within a reasonable accuracy and to get a deeper understanding of their microscopic origin. This program has still to be developed.

Following the above considerations, the EDF can be written as

\[ E = E_0 + E_{s.o.} + E_{int}^{\infty} + E_{int}^{FR} + E_C. \]  

For the surface term, following \([228]\), one can take a simple finite range term

\[ E_{int}^{FR}[\rho_n, \rho_p] = \frac{1}{2} \sum_{i,t} \int \int d^3 r d^3 r' \rho_i(r) v_{i,t}(r - r') \rho_t(r') - \frac{1}{2} \sum_{i,t} \gamma_{i,t} \int d^3 r \rho_i(r) \rho_t(r) \]  

with \( t = \text{proton/neutron} \) and \( \gamma_{i,t} \) the volume integral of \( v_{i,t}(r) \). The subtraction in (109) is made in order not to contaminate the bulk part. The finite range terms have been used in, e.g. \([225, 227]\), generalizing the usual Skyrme functionals. In \([228, 229]\), for the finite range form factor \( v_{i,t}(r) \) a simple Gaussian ansatz, \( v_{i,t}(r) = V_{i,t} e^{-r^2/\sigma^2} \), was taken, so that a minimum of three open parameters was introduced: \( V_{p,p} = V_{n,n} = V_z, V_{n,p} = V_{p,n} = V_U \) and \( \sigma \).

In equation (108), \( E_{int}^{FR} \) and \( E_C \) are the spin–orbit and Coulomb parts, respectively. More details on their determination can be found in \([228, 229]\). The first piece \( E_0 \) in equation (108) corresponds to the uncorrelated part of the kinetic energy and within the KS method it is written as

\[ E_0 = \frac{h^2}{2m} \sum_{i,t,s} \int d^3 r |\nabla \psi_i(r, s, t)|^2. \]  

where \( s \) and \( t \) stand for spin and isospin indices. At each point \( r \) the bulk term equals the nuclear matter EoS at the corresponding local density (and asymmetry). Then, upon variation to minimize the EDF, one gets a closed set of A Hartree-like equations with an effective potential, the functional derivative of the interaction part with respect to the local density \( \rho(r) \). Since the latter depends on the density, and therefore on the \( \psi_{i,s} \), a self-consistent procedure is necessary.

The equations are exact if the exact EDF is known. The existence of the latter is proved by the Hohenberg–Kohn (HK) theorem \([230]\), which states that for a Fermi system, with a non-degenerate ground state, the total energy can be expressed as a functional of the density \( \rho(r) \) only. Such a functional reaches its variational minimum when evaluated with the exact ground state density. In practice of course a reliable approximation must be found for the otherwise unknown density functional, taking inspiration from physical considerations and microscopic input, as discussed above. It has to be stressed that in the KS formalism the exact ground state wave function is actually not known, the density being the basic quantity. It has also to be note that in the standard KS scheme, the kinetic energy term includes the bare nucleon mass, but variants with an effective mass are possible (to incorporate the correlated part of the kinetic energy).

In \([229]\) the parameters of the functional have been fixed by fitting a set of deformed nuclei, both normal and super-heavy (functional BCP). The choice of fitting first deformed nuclei is suggested by the consideration that these nuclei should be better described by mean field, while spherical nuclei need corrections due to zero-point motion in the ground state, e.g. RPA correlations in the ground state. The mean error in binding energy in the fit of deformed nuclei is about \( \sigma_E = 0.52 \text{ MeV} \), while the discrepancy for the spherical nuclei, where the functional was fixed and no fit was any more performed, is about \( \sigma_E = 1.2 \text{ MeV} \). The corresponding deviation for the root mean square radii is 0.03 fm.

The performance is comparable to the best Skyrme forces or the finite range Gogny force DIS \([231]\), but of lower quality than the one from HFB calculations of e.g. \([28, 29]\). It must be stressed again that the functional of equation (108) has been developed by introducing a minimal set of finite size terms, i.e. surface and spin–orbit terms, in addition to the bulk part fixed once for ever from microscopic nuclear matter EoS. In this way one can clearly separate the bulk and surface contribution to the binding and the radius in finite nuclei and establish a link between the properties of the nuclear medium (EoS and the bulk symmetry energy) and the structure of finite nuclei. It would be interesting to further analyze this link on the basis of an extended set of results.

An effective force can be tested for systems that do not exist in nature but that can be treated accurately by the many-body method. Neutron droplets have been studied in \([30]\) within an accurate Monte Carlo method. They should have a physical connection with the neutron halo of ‘nuclei’ that are present in the NS crust. These droplets are constructed by filling a potential pocket with a varying number of neutrons. A comparison with the prediction of the SLy4 and other Skyrme forces suggests that, in order to reproduce the microscopic
calculations, the iso-vector surface term in the force should be substantially more repulsive, while the iso-vector spin–orbit should be reduced. It is a challenge to all EDF to be able to provide a good fit not only of ordinary or exotic nuclei but also of these extreme systems, where only neutrons are present and the surface region displays a diffusivity about a factor 2 larger than in ordinary stable nuclei. In this way one could construct functionals that should also be accurate for very exotic nuclei and nuclear matter in the NS crust.

The developed chiral effective field interactions, which includes both two-body and three-body forces, can be used to construct EDF, along the same lines as for Skyrme forces. In this way [33, 239, 240] one can get an EDF that has its root in QCD. In order to get a similar EDF which is competitive with the best Skyrme functionals, it turns out necessary to adjust the density-dependent couplings of the contact terms [34]. Despite that, a bridge between QCD and nuclear structure is for the first time established.

Generally speaking the density functional method is a prototype of a mean field scheme. It tries to incorporate in an effective way the NN (or many-nucleons) correlations through the form and parameters of the force or of the functional. The quality of the fitting procedure to the binding energy of nuclei (or other quantities) can give an indication to what extent this is possible with a given functional. As already mentioned, one way to improve the fitting is through additional derivative terms of higher order. Another line of thought is to estimate, for a given functional, the effect of the correlations not included in the functional. In [35] a study was performed on the correlations associated with the quadrupole field, both static (deformation) and dynamical (e.g. virtual excitations of two particle–two hole states). They are estimated by angular momentum projection and the generator coordinate method, the former being the dominant one, up to 5.5 MeV (total). These correlations turn out to be relevant to improving the binding energy fitting throughout the mass table with respect to the mean field (from the Skyrme force Sly4 [36]). The amount of correlation effects shows some systematic trends, in particular dynamical correlations appear as soon as the nucleus is not a double-magic one, while the static one becomes relevant only for larger number of nucleons in the middle of the shell. However, in general correlation effects are expected to depend on the initial mean field (that is the Skyrme force adopted). A clean separation between mean field and correlations appears a little problematic. The correlations involve a genuine nuclear structure problem and they cannot be easily connected with the gross properties of the nuclear medium, but their study and analysis is of overwhelming relevance to improve our understanding of the nuclear structure throughout the mass table.

8.3. Pairing in nuclei

Soon after the formulation of BCS theory [200], it was shown [232] that pairing correlation is present in nuclei. Since then, pairing correlation has played a major role in the development of nuclear structure. Despite the enormous development in this field, the main physical question that remains unanswered is the origin of the attractive pairing interaction in nuclei in terms of the NN interaction. In most applications the pairing interaction is treated as a phenomenological force. This approach is quite successful, particularly with the Gogny force [231], but still the physical processes at the basis of these forces are not yet clear. We will focus here only on pairing between like-particles, because the neutron–proton pairing is restricted mainly to symmetric or almost symmetric nuclei, where isospin symmetry is valid. We have seen that the bare NN potential is able to produce in nuclear matter a substantial pairing gap mainly at sub-saturation density. It is not trivial to relate these results to pairing in nuclei, since the bulk density is at saturation and the nuclear surface is too sharp to justify a local density approximation. Furthermore, in nuclei surface modes can play a role in the physical processes responsible for the effective pairing interaction [233–235].

In addition to the many-body aspects of the problem, at least two other features of the nuclear pairing have to be mentioned. One is related to the fact that the pairing phenomenon occurs close to the Fermi surface, while the bare NN potential necessarily also involves scattering to high energy (or momentum) due to its strong hard core component, which is one of the main characteristics of the nuclear interaction. It looks therefore natural to develop a procedure which removes the high-energy states and ‘renormaize’ the interaction into a region close to the Fermi energy. This can be done in different ways, among which the most commonly used seems to be the RG method [175]. A second feature is the relevance of the single particle spectrum, not only because the density of states at the Fermi surface plays of course a major role, but also because the whole single particle spectrum has an influence on the effective pairing interaction.

In the last few years relevant progress has been made in the microscopic calculations of pairing gap in nuclei [233–240]. The main established result is that the bare NN interaction, renormalized by projecting out the high momenta, is a reasonable starting point that is able to produce a pairing gap which shows a discrepancy with respect to the experimental value not larger than a factor 2. In view of the great sensitivity of the gap to the effective interaction this result does not appear obvious. The effective pairing interaction constructed within this renormalization scheme [241] also explains qualitatively the surface relevance. Indeed, this interaction at the surface can exceed the value inside by one order of magnitude. The direct effect of the surface enhancement of the gap was presented in [242, 243] for semi-infinite nuclear matter and in [244] for a nuclear slab. In addition to the renormalization of the bare interaction of the high-momentum components, other physical effects should be included in a microscopic approach, like the ones related to the effective mass, or more generally, the single particle spectrum, and the many-body renormalization of the pairing interaction.

Let us first consider the problem of reducing the interaction to an effective one close to the Fermi energy. In general language this can be seen as a typical case that can be approached by an ‘effective theory’, where the low-energy phenomena are decoupled from the high-energy components. In this procedure the resulting low-energy effective interaction
Taking for the pairing interaction the bare NN interaction \( V(\mathbf{k},\mathbf{k}') \), it is possible to project out the momenta larger than a cutoff \( k_c \) by introducing the interaction \( V_{\text{eff}}(\mathbf{k},\mathbf{k}') \), which is restricted to momenta \( k < k_c \). It satisfies the integral equation

\[
V_{\text{eff}}(\mathbf{k},\mathbf{k}') = V(\mathbf{k},\mathbf{k}') - \sum_{k' > k_c} \frac{V(\mathbf{k},\mathbf{k'})V_{\text{eff}}(\mathbf{k}'',\mathbf{k}'')}{2E_k''},
\]

The gap equation, restricted to momenta \( k < k_c \) and with the original interaction \( V \) replaced by \( V_{\text{eff}} \), is exactly equivalent to the original gap equation. The relevance of this equation is that for a not too small cutoff the gap \( \Delta(k) \) can be neglected in \( E(k) \) to a very good approximation and the effective interaction \( V_{\text{eff}} \) depends only on the normal single particle spectrum above the cutoff \( k_c \). As such \( V_{\text{eff}} \) contains necessarily some dependence on the in-medium single particle spectrum at high momenta. In the RG approach, if the procedure of constructing the low-momenta interaction \( V_{\text{low}}(\mathbf{k},\mathbf{k}') \), discussed in section 4.2.4, is carried out in vacuum, it implicitly assumes a free spectrum at high momenta. In the medium it faces the same uncertainty.

This procedure of projecting out the high momenta can be extended to finite nuclei, and the same uncertainty persists. To illustrate this point in table 1, taken from [245], are reported the values of the pairing gap for \(^{120}\text{Sn} \). The diagonal pairing gap matrix elements for the levels around the Fermi energy and the corresponding average values are compared for different types of calculations. For the mean field the Sly4 Skyrme force has been used. In the procedure of projecting out the high momenta the effective mass has been put equal to the bare one above a certain cutoff \( \Lambda \). This can be done within the so-called local potential approximation (LPA) [246, 247], which has been proved to be a quite accurate approximation. In the first column the effective mass has been taken equal (and constant) to the original Sly4 value only inside the model space, i.e. within the states where the effective interaction is calculated after the projection of the high-momentum components, according to equation (112). The model space corresponds to single particle energies \( \epsilon_\lambda < 40 \text{ MeV} \). For the second column \( \Lambda = 3 \text{ fm}^{-1} \), for the third \( \Lambda = 4 \text{ fm}^{-1} \) and for the fourth \( \Lambda = 6.2 \text{ fm}^{-1} \). The original interaction is the Argonne \( v_{18} \) NN potential. In all cases the original density dependence of the effective mass was kept within the LPA scheme. The strong sensitivity of the pairing gap to the value of \( \Lambda \) indicates that the separation between low- and high-momenta components can be problematic.

### Table 1. The Argonne \( v_{18} \) potential.

| \( \lambda \) | Sly4 | Sly4-1 | Sly4-2 | Sly4-3 |
|--------------|------|--------|--------|--------|
| \( 3s_{1/2} \) | 1.23 | 1.10   | 0.83   | 0.56   |
| \( 2d_{3/2} \) | 1.32 | 1.18   | 0.89   | 0.61   |
| \( 2d_{5/2} \) | 1.34 | 1.20   | 0.92   | 0.63   |
| \( 1f_{7/2} \) | 1.48 | 1.31   | 0.96   | 0.64   |
| \( 2h_{11/2} \) | 1.27 | 1.13   | 0.85   | 0.57   |
| \( \Delta_{v} \) | 1.34 | 1.19   | 0.89   | 0.60   |

We have seen in section 7.4 that in nuclear matter one of the main open questions is the role of the medium in screening the effective pairing interaction. It turns out [233–235] that the same processes of exchange of collective excitations between quasi-particles produce an overall attractive interaction, usually indicated as ‘induced interaction’, and could contribute substantially to the strength of the effective pairing interaction. This striking difference could be due to the strong collectivity of the low lying surface modes in nuclei and to the corresponding small strength of the spin modes. The above described calculations that include only the bare NN interaction, on comparison with the experimental data, can put limits on the relevance of the induced interaction. Due to the mentioned uncertainties, it is not yet possible to draw any quantitative conclusions. In any case this is an active field of research and some answers, at least partial, could come from future works.

Finally it is important to mention one of the main issues that is involved when one tries to relate pairing in nuclear matter and finite nuclei. The size \( \xi \) of the Cooper pairs in nuclear matter can be estimated in the weak coupling limit

\[
\xi \approx \frac{\hbar v_p}{\sqrt{8\Delta_F}}.
\]

In nuclear matter for the neutron pairing gap obtained with the free NN interaction one finds that \( \xi > 5 \text{ fm} \). If one extrapolates this expression to finite nuclei and takes a typical value for the gap of 1–2 MeV, then the value of \( \xi \) can exceed the size of the nucleus. It looks that we cannot describe pairing correlation in nuclei with a simple spatial picture. Also in this case the role of surface is essential. It has been found that the pairing correlation length \( \xi \) depends on the center of mass of the pair, and it shrinks just at the surface of the nucleus. In figure 20, taken from [248], is reported the case of a slab of nuclear matter. Here the Cooper pair size \( \xi_x \), in the direction perpendicular to the slab is reported as a function of the center of mass position \( X \). The minimum of \( \xi_x \) falls exactly at the surface of the slab, where \( \xi_x \approx 2 \text{ fm} \), independently of the pairing interaction used. This is in line with the results presented in [249] for realistic cases, where the Cooper pair size was also found to be minimal at the surface and around 2 fm. References to previous works on this subject can be found in this paper. Even if this shrinking of the pair size at the surface could be a general effect, not necessarily connected with the pairing interaction but rather just with the available phase space, it looks as if pairing correlation has room mainly at the surface. This is in agreement with the already mentioned enhancement of the local pairing gap at the surface. Loosely speaking, one can imagine that the Cooper pairs are formed mainly around the surface, where the pairing correlation can develop.

#### 8.4. Neutron and proton radii

It is not easy to establish at a formal level a direct connection between the properties of nuclei and nuclear matter. A different
strategy then has been pursued, which can be considered a semi-empirical theoretical method. One considers a wide set of Skyrme forces with different characteristics and one looks for a possible correlation between a specific nuclear matter property and a physical parameter in nuclei as the force is varied within the given set. This approach was followed in [261], where it was shown that the difference between the neutron and proton root mean square radii (‘neutron skin’) of Pb is linearly correlated with the slope of the neutron matter EoS at a density of 0.1 fm$^{-3}$. This linear correlation plot is shown in figure 21, taken from [196]. The linear correlation was shown to also hold for other nuclei and also if one considers, instead of the neutron matter EoS, the slope of the symmetry energy as a function of density in symmetric nuclear matter at approximately the same density [17]. Even if the approach is not formally microscopic, it is quite fruitful because it links in a direct way the nuclear EoS to the structure of nuclei. In other words, measuring a physical quantity in nuclei would give direct information on the nuclear EoS. Unfortunately it is not easy to measure the neutron radius, since the hadron probes are strongly interacting, at odds with electrons that provide the charge radius. There is a large expectation on the parity violation electron scattering experiment PREX that is going on at JLAB [262], because these measurements will directly give the difference between neutron and proton radii in $^{208}$Pb. It has to be seen if the accuracy of the data will be enough to distinguish among different functionals. Partial justifications of some of these correlations have been presented in the literature [263], but further microscopic investigations are surely needed to clarify the field. As an illustration, figure 22 reports the results for the neutron skin value from one of the versions of the functional BCP throughout the mass table. The set of results has overall agreement with the phenomenological data. It would be desirable to identify the main properties of the forces or functionals that determine the value of the radii difference [17, 265] (or other physical quantity), but this goal has still to be reached. This would indeed be real progress in our understanding of the fundamental properties of the nuclear medium, from nuclear matter to finite nuclei.

8.5. Exotic nuclei

The physics of exotic nuclei, i.e. nuclei far away from the stability valley that are not present on the Earth, is a field of rapid development where a large effort is concentrated in many laboratories throughout the world. Long-term projects with large collaboration networks have been established in different countries and continents (e.g. EURISOL, FIRB) to produce radioactive beams. They will allow us to study systematically

Figure 20. Size of the Cooper pair in the direction perpendicular to a slab of nuclear matter. The surface of the slab is located at about $X = 6$ fm, very close to the position of the minimum. Reprinted with permission from [248]. Copyright 2009 by the American Physical Society.

Figure 21. Linear correlations between the neutron and proton radii and the slope (MeV fm$^3$) of the symmetry energy at the density $\rho = 0.1$ fm$^{-3}$. Squares from the top correspond to $NL1$ [250], $NL3$ [251], $G1$, $G2$ [252] and $Z271$ [253]. Triangles to Gogny forces $D280$, $D300$, $D250$, $D260$ $D1$ and $D1S$ [254]. Circles correspond to the Skyrme forces $SV$ [255], $SV$ [255], $SkM^*$ [256], $Sly4$, $Sly5$ [257], $T6$ [258], $SG11$ [259], $SI$ [260], $SI1$ [260], $SI11$ and $SV1$ [255]. The result for the BCP functional is labeled by a star. Reprinted from [196], copyright 2004, with permission from Elsevier.

Figure 22. Neutron skin calculated with the BCP functional throughout the mass table. The uppermost line and lowermost line are the phenomenological boundaries obtained in [264]. The middle line is just the average value of the phenomenological boundaries.
this point, figure 23 reports the results of the calculations of the equilibrium with the surrounding neutron gas. To illustrate the interaction (neutrons are ‘dripping’ from them), but they are in nuclei are therefore unstable even with respect to the strong surrounded by a neutron gas (besides the electron gas). These neutrons that surround the nuclei and prevent their beta decay due to Pauli blocking. In the inner crust a neutron gas also exists, but at not too high density it is possible to separate out, at least approximately, to the ‘blob’ (approximately estimated). They are evaluated with the same functional, and are stable with respect to strong interaction. One observes the formation of the neutron gas outside the ‘nucleus’ at the center of the Wigner–Seitz cell as the density increases. Of course these nuclei would be strongly unstable with respect to weak interaction. Beyond the reported maximum density, it is not possible any more to find finite nuclei corresponding to the ‘blob’ at the center of the Wigner–Seitz cell, because they are unstable even with respect to strong interaction. Then it is not any more possible to distinguish the ‘blob’ as a nucleus separated from the surrounding neutron gas.

The structure of all these Wigner–Seitz cells can be studied solely on the basis of a strong extrapolation of the theoretical methods, in particular of the various EDF, developed and checked along the available mass table. Unfortunately, despite the fact that all EDF must agree, within a certain accuracy, for the nuclei that can be produced in laboratory, their predictions on the nuclei of such a large asymmetry are often diverging. This indicates that we are still far from having under control the microscopic theory of the asymmetry dependence of the nuclear medium properties. This uncertainty is reflected in the uncertainty of the NS crust, and the discrepancy extends to the region of higher density, where it is not possible to separate any more a definite nucleus at the center of cell, until the transition to homogeneous matter occurs. As an illustration table 2 reports the values of the atomic numbers of nuclei in the inner crust calculated in the classical paper by Negele and Vautherin [267] and with a different functional [268].

The latter also includes the pairing correlations with three different strengths (P1, P2 and P3). The reason for these discrepancies remains to be clarified, but it has to be stressed that the position of the minimum in the energy as a function of the atomic number is quite delicate, because it can often happen that different local minima are competing among each other. In the table the values of the Wigner–Seitz cell radius are also reported, which turns out to be a less sensitive quantity. It must be pointed out that these discrepancies persist at lower densities, down to the drip point and slightly below, as systematically explored in [269] for a wide set of Skyrme forces and relativistic mean field functionals. To illustrate the difficulty and uncertainty in these calculations, figure 24 reports typical energy curves for the Wigner–Seitz cell as a function of the atomic number [268]. In addition to the apparent competition among different energy minima, the tiny variation of the binding energy has to be noticed. The accuracy needed to predict the absolute minimum is of the order of 5–10 keV per particle, which is at the limit of performance of

![Figure 23. Neutron density profile of the nuclei at the upper edge of the inner crust of an NS. The dotted lines indicate the corresponding neutron density profile of a nucleus with the same atomic and mass numbers of the ‘cluster’ of matter at the center of the Wigner–Seitz cell.](image-url)

Table 2. Atomic number of the matter inside the Wigner–Seitz cell throughout the inner crust and the corresponding cell radius. The calculations labeled P1, P2, P3 are for three different pairing strengths, reported in [268], in comparison with the results (NV) of [267].

| kF (fm⁻¹) | Z     | Rc(fm) |
|-----------|-------|--------|
|           | P1    | P2    | P3    | NV    | P1    | P2    | P3    |
| 0.6       | 58    | 56    | 56    | 56    | 37.51 | 36.85 | 36.92 |
| 0.7       | 52    | 46    | 46    | 50    | 32.02 | 30.31 | 30.27 |
| 0.8       | 42    | 40    | 40    | 50    | 26.90 | 25.97 | 25.97 |
| 0.9       | 24    | 20    | 20    | 20    | 20.26 | 18.34 | 18.39 |
| 1.0       | 20    | 20    | 20    | 20    | 16.69 | 16.56 | 16.56 |
| 1.1       | 20    | 20    | 20    | 20    | 14.99 | 15.05 | 15.05 |
| 1.2       | 20    | 20    | 20    | 20    | 13.68 | 13.73 | 13.74 |

8.6. The neutron star crust

We have seen that the crust of NS is the place in the Universe where the most asymmetric nuclei are present in a stable manner. The reason is of course the existence of the electron gas that surrounds the nuclei and prevents their beta decay due to Pauli blocking. In the inner crust a neutron gas also exists, but at not too high density it is possible to separate out, at least approximately, the nucleus at the center of the lattice cell. In this way one can picture the inner crust as a lattice of nuclei surrounded by a neutron gas (besides the electron gas). These nuclei are therefore unstable even with respect to the strong interaction (neutrons are ‘dripping’ from them), but they are in equilibrium with the surrounding neutron gas. To illustrate this point, figure 23 reports the results of the calculations in [266], where the neutron density profile in the Wigner–Seitz cell is compared, at a different density, with the corresponding profile of a nucleus with the same atomic number and a mass number equal to the number of nucleons inside the radius of the ‘blob’ (approximately estimated). They are evaluated with the same functional, and are stable with respect to strong interaction. One observes the formation of the neutron gas.
the best EDF, and surely the extrapolation to so asymmetric matter is not yet under control. In any case, since the structure of these nuclei cannot be studied in laboratory, one has to look for observational data on NS that are sensitive to their properties. The main physical parameter of the crust is the values of the atomic number of nuclei in the lattice as a function of density and the lattice spacing. Other quantities, such as the shear modulus or the incompressibility are functions of these parameters on the basis of the well-known properties of Coulomb lattices. In fact, even in the inner crust, the effect of the neutron gas on these quantities is negligible. The observation of NS oscillations during flares of x-ray or gamma ray emission in accreting processes or in magnetar quakes can possibly be used to study the structure of the crust. These observations are now numerous [270–273]. The frequency spectrum of the oscillations is deduced from Fourier analysis of the luminosity signal, which reveals the beatings superimposed on the rotational pulses. If in the oscillations the crust is decoupled from the core, the spectral analysis of these data can be used to constrain the shear modulus and, consequently, the density dependence of the symmetry energy [274]. These types of analyses are partly model dependent, but they open a window on the possibility of studying, even if in a very indirect way, the nuclear medium for very high asymmetry at and below saturation density.

9. Liquid–gas phase transition

In the last stage of supernovae collapse the EoS of asymmetric nuclear matter at finite temperature plays a major role in determining the final evolution. Few microscopic calculations of the nuclear EoS at finite temperature exist. The variational calculation by Friedman and Pandharipande [275] was one of the first few semi-microscopic investigations of the finite temperature EoS. The results appear fairly close to the predictions based on Skyrme force models: symmetric nuclear matter undergoes a liquid–gas phase transition, with a critical temperature $T_c = 18–20$ MeV. This is a fundamental property of the nuclear medium. Different types of Skyrme forces give different critical temperatures, but they all lie close to this range of values. Later, Brueckner-like calculations at finite temperature [276] confirmed these findings with very similar values of $T_c$. The full finite temperature formalism by Bloch and De Dominicis (BD) [277] was followed. In this section we sketch the main results and the method followed in the finite temperature case. We then discuss the connection with laboratory experiments and data.

9.1. Critical temperature

The starting point in the many-body theory of finite temperature EoS is the calculations of the grand-canonical potential $\Omega$. In the BD formalism, in line with the Brueckner scheme, a self-consistent single particle potential $U(k)$ is introduced and finally the grand-canonical potential is given by

\[
\Omega = \Omega' + \Delta\Omega
\]  

(114)

where

\[
\Omega' = -\frac{2V}{\pi^2} \int_0^\infty k^2 dk \left[ \frac{1}{\beta} \log(1 + e^{-\beta(k\mu)}) + U(k)n(k) \right]
\]  

(115)

is the grand-canonical potential for independent particles with Hamiltonian

\[
H_0 = \sum_k e_k \alpha_k^+ \alpha_k + \sum_k \left( \frac{\hbar^2 k^2}{2m} + U(k) \right) \alpha_k^+ \alpha_k
\]  

(116)

and $\mu$ is the chemical potential. The interaction part $\Delta\Omega$ of the grand-canonical potential is given by

\[
\Delta\Omega = \frac{1}{2} e^{2\beta \mu} \int_{-\infty}^{\infty} \frac{e^{-\beta\omega}}{2\pi} d\omega \text{Tr}_2 \left[ \arctan \left( \frac{\beta \omega}{\mu} \right) \right] .
\]  

(117)
In this approach one calculates the free energy of symmetric nuclear matter as a function of density at different temperatures. The full lines are fits to the calculated points. The curves decrease systematically as the temperature increases.

The trace in the previous equation $\text{Tr}_2$ is taken in the space of anti-symmetrized two-body states and the two-body scattering matrix $K$ is defined by

$$
(k_1 k_2 | K(\omega) | k_3 k_4) = (n_+ (k_1) n_+ (k_2) n_- (k_3) n_- (k_4))^\frac{1}{2} \times (k_1 k_2 | K(\omega) | k_3 k_4)
$$

(118)

where the scattering matrix $K$ satisfies the integral equation

$$
(k_1 k_2 | K(\omega) | k_3 k_4) = (k_1 k_2 | v | k_3 k_4 + \sum_{k_i k_i'} (k_1 k_2 | v | k_i k_i')) \times \frac{n_+ (k_i') n_- (k_i)}{\omega - \epsilon} (k_i' k_i' | K(\omega) | k_3 k_4).
$$

(119)

In these equations $n_+ (k) = 1 - n(k)$, with $n(k)$ the Fermi distribution function at a given temperature and for the single particle spectrum $\epsilon(k)$. Then equation (119) coincides with the Brueckner equation for the $G$-matrix in the zero temperature limit. It has to be noted that only the principal part has to be considered in the integration, thus making $K$ a real matrix. The appearance of the arctan in equation (117) looks peculiar, but it comes from a ladder summation similar to the one for the zero temperature $G$-matrix. More details on the derivation and on the numerical treatment of the equations can be found in [276]. In this approach one calculates the free energy $F = \Omega + \mu N$ and then the pressure from the thermodynamical relationship

$$
p = \rho^2 \frac{\partial f}{\partial \rho} \tag{120}
$$

where $\rho$ is the total number density and $f$ the free energy per particle $F/N$. A typical result is reported in figure 25, where the full lines are interpolations of the calculated points, suitable for differentiation. The resulting EoS at finite temperature, i.e. pressure versus density, is reported in figure 26. One recognizes the familiar van der Waals shape, which entails a liquid–gas phase transition, with a definite critical temperature, i.e. the temperature at which the minimum in the van der Waals isotherm disappears. This is clearly a fundamental property of the nuclear medium: it behaves macroscopically at a finite temperature in a way similar to a classical liquid. The critical temperature turns out to be around $T_c = 18–20$ MeV.

A difficulty in this approach is the lack of thermodynamical consistency. In fact the thermodynamical relation $p = -f' + \mu \rho$, usually referred to as the ‘Hughenoltz–Van Hove theorem’, is not satisfied. Here $f'$ is the free energy per unit volume $F/V$. In other words, the pressure calculated from equation (120) does not coincide with the pressure calculated from $p = -\Omega/V$. This point, which is not necessarily a too serious drawback, is discussed in the next section.

9.2. Theoretical uncertainties and challenges

We have seen that symmetric nuclear matter undergoes a liquid–gas phase transition. This is the result of calculations with microscopic calculations, but also with effective forces, e.g. Skyrme forces. The values of the critical temperature, however, depend on the theoretical scheme, and on the particular effective force adopted. In particular, it turns out that the critical temperature within the Dirac–Brueckner scheme is definitely smaller [278, 279] than in the non-relativistic scheme, about 10 MeV against 18–20 MeV. This cannot be ascribed to relativistic effects, since the critical density is well below, about 1/3, the saturation density. Probably this is due to a different behavior of the Dirac–Brueckner EoS at low density. This point still needs clarification.

As anticipated in the previous subsection, another uncertainty stems from the violation of the Hughenoltz–Van Hove (HVH) theorem within the extension to finite temperature of the non-relativistic Brueckner scheme, as implemented by Bloch and De Dominicis [277]. In the applications, the pressure is calculated from the derivative of the free energy per particle and the theorem is actually automatically satisfied. The difficulty is then that the chemical potential determined by fixing the density in the Fermi distribution is not strictly the one extracted from the derivative of the free energy per unit volume, as it should be. In any case the procedure looks the most reliable within the Brueckner scheme, since the HVH theorem, as a thermodynamic relationship, is satisfied.
A general approximation scheme that strictly satisfies the theorem has been devised by Baym [280]. It is based on the self-consistent Green’s function method, where the single particle self-energy is approximated by a functional of the Green’s function itself, which is then calculated in a self-consistent manner. Numerical calculations have indeed [281] shown that the HVH is satisfied. The results at the two-body correlation level, at least when only two-body forces are used, in some cases are similar to the Brueckner ones, in some others they differ appreciably, according to the forces used [281, 282]. The main difference with the Brueckner scheme is the introduction in the ladder summation also of the hole–hole propagation. The expansion scheme is therefore at odds with the hole-line expansion, and actually it is not clear how to proceed to improve the approximation or if convergence has been reached. Therefore, on the one hand we have the hole-line expansion at zero temperature that has some definite sign of convergence at the two hole-line (Brueckner) level, on the other hand at finite temperature we have a different truncation scheme to satisfy the HVH theorem, that however is not proved to be a satisfactory approximation and does not reduce to the Brueckner scheme at zero temperature. It would be quite desirable to have a scheme that is able to have both requirements satisfied, i.e. to have a good degree of convergence and the fulfillment of the HVH theorem (at zero and finite temperature). Although this sort of dilemma is a challenge that requires further studies, the gross properties of nuclear matter at finite temperature appear well established.

9.3. Isospin dependence

If the nuclear matter is asymmetric, the existence of two components, neutrons and protons, complicates quite a bit the phase transition picture. The spinodal region is still well defined, the bulk incompressibility at a given asymmetry is negative in specific portions of the various possible thermodynamical planes. The coexistence line presents a new feature, the ‘distillation’ phenomenon. The name is suggested by the analogy with the process used in the distillation of liquors. The chemical equilibrium between liquid and vapor requires the equality of the proton and neutron chemical potentials. They are however different if matter is asymmetric, and therefore, in view of the density dependence of the symmetry energy, the fraction of neutrons and protons are different in the liquid and in the vapor. In general, it turns out that the vapor is expected to be more neutron rich. The effect is not dramatic, but it has been claimed that in heavy ion reactions, where a phase transition similar to the liquid–gas phase transition is expected to occur, this effect should be seen [283]. The distillation also has the effect that the Maxwell construction for asymmetric matter is modified with respect to the symmetric case. The horizontal line that characterizes the construction in the pressure versus density plane, within the coexistence region, is replaced by a non-horizontal line, see figure 27, taken from [284]. In fact, if the fractions of vapor and liquid are changed, while they have different compositions, the overall asymmetry must be kept constant. This can be achieved only by changing the equilibrium pressure in order to shift properly the neutron and proton chemical potentials.

Figure 27. Isotherms of asymmetric nuclear matter at the indicated proton/nucleon ratio Z/A. Upper panel, neutron chemical potential. Lower panel, pressure. The points A and B indicate the endpoints of the coexistent region. Courtesy of C Ducoin. Reprinted from [284], copyright 2007, with permission from Elsevier.

Both the spinodal decomposition [285] and the distillation phenomenon [284] in asymmetric matter at a finite temperature have been studied within the Skyrme functional scheme. The line that marks the onset of the spinodal instability is now characterized not only by the values of the total density and temperature (at a given overall asymmetry) but also by the direction along which the instability can develop, i.e. the fractions of protons and neutrons. Following this direction, the curvature of the free energy in the plane of proton versus neutron chemical potentials can vary. The direction where the curvature is minimal should indicate the most unstable direction and therefore the most probable composition of the liquid clusters that are produced due to the instability. This is illustrated in figure 28, taken from [285].

This can have direct relevance in astrophysics for the formation process of the NS crust, where however it is necessary to introduce the electron component and the Coulomb interaction.

9.4. Phenomenology: the limiting temperature

We have seen that symmetric nuclear matter undergoes a liquid–gas phase transition. However, if this phase transition exists, it does not possess a direct correspondence in finite nuclei, due to the Coulomb interaction and finite size effects. In particular, the Coulomb force is long range and strong enough to modify the nature of the phase transition. However some authors [286, 287] have pointed out that the nuclear EoS can be
linked to the maximal temperature a nucleus can sustain before reaching mechanical instability. This ‘limiting temperature’ \( T_{\text{lim}} \) is mainly the maximal temperature at which a nucleus can be observed.

It has to be stressed that the reaction dynamics can prevent the formation of a true compound nucleus. The onset of incomplete fusion reactions can mask completely the possible presence of fusion or quasi-fusion processes. At higher energies, the heavy ion reaction can be fast enough that no (nearly) thermodynamical equilibrium can be reached, as demanded in a genuine standard fusion–evaporation reaction. However, combined theoretical and experimental analyses [288] indicate that a nearly equilibrium condition is reached in properly selected multi-fragmentation heavy ion reactions at intermediate energy. The main experimental observation is the presence of a plateau in the so-called ‘caloric curve’, i.e. in the plot of temperature versus total excitation energy [289–292]. This behavior was qualitatively predicted by the Copenhagen statistical model [293] of nuclear multi-fragmentation. The relation between multi-fragmentation processes and the nuclear EoS was extensively studied by several authors within the statistical approach to heavy ion reaction at intermediate energy [294–300].

In different experiments, various methods were used to extract from the data the values of the temperature of the source which produces the observed fragments, but a careful analysis of the data [288] seems to indicate a satisfactory consistency of the results. In [288, 301] an extensive set of experimental data was analyzed and it was shown that the temperature at which the plateau starts decreases with increasing mass of the residual nucleus which is supposed to undergo fragmentation. Both the values and the decreasing trend of this temperature turn out to be consistent with its interpretation as limiting temperature \( T_{\text{lim}} \). According to this interpretation, at increasing excitation energy the point where the temperature plot deviates from Fermi gas behavior and the starting point of the plateau mark the critical point for mechanical instability and the onset of the multi-fragmentation regime. The corresponding value of the critical temperature can be calculated within the droplet model, and indeed many estimates based on Skyrme forces are in fairly good agreement with the values extracted from phenomenology [287, 288]. Moreover, the relation between nuclear matter critical temperature \( T_c \) and \( T_{\text{lim}} \) appears to be quite stable and independent of the particular EoS and method used, which allows [301] to estimate \( T_c \) from the set of values of \( T_{\text{lim}} \).

In general, one can expect that \( T_{\text{lim}} \) is substantially smaller than the critical one, \( T_c \). In fact, both the Coulomb repulsion and the lowering of the surface tension with increasing temperature tend to destabilize the nucleus. Since the surface tension goes to zero at the critical temperature, \( T_{\text{lim}} \) is reached much before \( T_c \). These predictions were checked in the seminal paper of [286], as well as in further studies based on macroscopic Skyrme forces [287], for which a simple relationship was established between \( T_{\text{lim}} \) and \( T_c \). If microscopic EoS are used, then the relationship between \( T_{\text{lim}} \) and \( T_c \) is not so simple and the ratio \( T_{\text{lim}}/T_c \) depends on the details of the EoS [302]. In principle the comparison with the phenomenological data can discriminate among different EoS. Indeed, most of the microscopic EoS reproduce the empirical saturation point, but their behavior at finite temperature can be quite different. This is mainly because the critical temperature, and therefore the limiting temperature, is very sensitive to the details of the EoS. In fact \( T_c \) is determined by the behavior of the derivative of the pressure with respect to the density, which in turn is the second derivative of the free energy. If the pressure is extracted directly from the grand-canonical potential as a function of the chemical potential, still a strong sensitivity to the low-density and high-temperature properties of the EoS remains. Figure 29 reports the pressure as a function of the density in symmetric nuclear matter, where one can recognize the liquid branch (the branch starting from the lowest cusp) and the vapor branch (the almost vertical one that starts from the upper cusp) [302]. Their intersection gives the coexistence point, while the smooth branch joining the two cusps is the unstable part of the EoS, corresponding to the Maxwell construction. The upward shifted liquid branch (dashed line) is the branch of a finite nucleus that takes into account the Coulomb interaction and finite size effects. Since the vapor branch is assumed to be almost unchanged,
with the values extracted from experimental data. The line starting from the lower cusp is the vapor branch. The intersection point is the coexistence point at that temperature. The line joining the two cusps is the unstable branch. The dashed line corresponds to a finite nucleus. It is obtained within a simplified LDM. The figure corresponds to the case of a temperature equal to the limiting temperature.

the shifted branch gives the shifted coexist point between the nucleus and the vapor (assumed uncharged). If, as in the figure, the liquid branch touches the upper cusp, then the corresponding temperature is the limiting temperature, since at increasing temperature the cusp is lowered and no coexistence point exists anymore. Figure 30, taken from [302], reports the results of a systematic calculation of \( T_{\text{lim}} \) for different nuclei and for different EoS in comparison with the values extracted from experimental data [288, 301]. One notices the strong dependence on the EoS derived from different microscopic schemes, and at the same time the relative small sensitivity to the NN force within the same scheme (BHF). It seems that the non-relativistic EoS based on the BHF approximation is the favored one.

9.5. The astrophysical link

The outer part of cold neutron stars is formed by a solid crystal of nuclei, while in a supernova, during the after-bounce stage, the temperature is so high that nuclear matter is in the homogeneous fluid phase. During the cooling process medium–heavy nuclei are formed from the homogeneous matter, and this transition is quite similar to a liquid–vapor phase transition, where liquid droplets are formed in the mixed phase. The droplet formation is directly related to the so-called ‘spinodal’ instability, i.e. the region in the phase diagram where the incompressibility is negative. However, the presence of Coulomb interaction changes the nature of the transition [304]. It turns out that, despite the fact that it is of first order, the thermodynamical potential does not display singularity as a function of the thermodynamical variables. In any case, the transition is accompanied by the formation of nuclear clusters. In fact one can expect that at a sufficiently low temperature, inside the spinodal region, nuclear matter is composed of nuclei of different sizes, light fragments (tritons, helium-3 and alpha particles) and nucleons. This phase has been described on the basis of the LDM [305] and the relativistic TF scheme [306]. In the low-density limit a virial expansion has also been applied [307]. The detailed theoretical description of this phase is quite relevant mainly for supernova simulations. A microscopic theory of the corresponding EoS, at the same level of accuracy as for the homogeneous matter case, is still missing.

Finally, it has to be mentioned that at even lower temperatures the clusters should undergo a liquid–solid phase transition, with the formation of a Coulomb crystal. This transition is even less known and no consensus exists on its general properties like the solidification energy or latent heat.

A discussion on this subject is outside the scope of this review.

10. Conclusions

The excursus we tried to perform on the properties of the peculiar matter, that can be called the ‘nuclear medium’, surely does not do justice to the impressive progress that has taken place both at the phenomenological and theoretical levels. The continuous and vivid interest in the subject is due to the extremely wide realm of phenomena and physical systems, on the Earth and in the Universe, where the nuclear medium plays a central role.

The continuous interplay between theoretical developments on the one hand and laboratory experiments and astrophysical observations on the other hand is the main driving force that makes possible progress in this exciting field. As we tried to illustrate, our knowledge of the properties of the nuclear medium has widened and sharpened in many respects. Many physical parameters start to be known with a certain accuracy not only at saturation density but also at lower and higher densities, and, in general, the equation of state has been severely constrained. Taking the risk of being too schematic, let us try
to summarize tentatively our knowledge of those properties of the nuclear medium that we touched in this review.

1. **Equation of state.** The constraints obtained from the data on heavy ion collisions and from the analysis of astrophysical observations on neutron stars are complementary between each other, since the nuclear matter asymmetry that is involved is quite different in the two cases. However, it is a challenge for the theory to be able to describe the nuclear medium under both physical conditions. If one takes together the two sets of data, one gets a severe test for the different microscopic theories. Although these constraints have to be confirmed and further analyzed, one can expect that in the near future new data and new results will put under serious test all existing microscopic theories.

2. **Incompressibility.** The value of the symmetric matter incompressibility near saturation seems now to be constrained to an interval approximately between 210 and 250 MeV. Microscopic theories are in fair agreement with these values. More difficult is to put constraints on the incompressibility at higher density, since its value is determined by the details of the density dependence of the pressure. Astrophysical observations should test in principle the incompressibility of very asymmetric nuclear matter, but the uncertainty on the structure of neutron stars’ inner cores hinders the progress in this direction.

3. **Symmetry energy.** The symmetry energy at saturation can be considered constrained in a narrow interval, essentially 30 ± 1 MeV. All microscopic theories agree with these values. Indirect hints about the density dependence of the symmetry energy come from both heavy ion collision data and astrophysical observation on neutron star phenomenology. A great variety of experimental and phenomenological data have been and are currently being analyzed. They include binding energy of nuclei, heavy ion collisions and astrophysical observations on compact objects. The cross comparison of the results of data analysis are extremely useful to tighten the constraints on the density dependence of the symmetry energy. Caution must be taken to compare results from different observables, as was indicated in the discussion about the iso-scaling and isospin diffusion data analysis. The constraints on the density dependence of the symmetry energy, in particular the parameter $L$, are still not stringent, but it looks like future experiments will be able to refine them. Microscopic theories for homogeneous matter agree at sub-saturation density and up to 2–3 times saturation density, but disagreements appear at higher densities. Correlations between symmetry energy at sub-saturation density and nuclear structure properties have been found. These can be of great help in clarifying many open questions in the field.

4. **Transport coefficients.** Several macroscopic phenomena that occur in neutron stars are determined by transport properties of the nuclear medium. The prediction of the damping of some of the possible overall oscillations of neutron stars requires the calculation of the shear and bulk viscosities. The study of cooling evolution of neutron stars requires knowledge of the thermal properties of the nuclear medium in different physical conditions. In principle these transport coefficients require only standard techniques based on the Fermi liquid theory. What is needed presently is a consistent scheme that is able to describe both the mechanical properties, like the EoS, of the nuclear medium and its transport properties. Progress in this direction has been made and one can be confident that in the near future a coherent picture of the nuclear medium will be possible on the basis of microscopic many-body theory, at least for not too high densities.

5. **Low energy excitations.** The elementary excitations of the nuclear medium are relevant for its thermal properties and for many phenomena occurring in neutron stars, such as emission and propagation of neutrinos. They are also a guide to the low-energy excitations in nuclei. They can be studied within the Landau theory of Fermi liquids. The fundamental physical parameters that are needed are the so-called ‘Landau parameters’ that fix the effective nucleon–nucleon interaction in the vicinity of the Fermi surface. The microscopic determination of these parameters is a difficult theoretical problem, that has been approached with a variety of techniques. Both in symmetric and pure neutron matter their values are not firmly established. The phenomenology on nuclei excitations gives only indications on some of them, since the physical conditions are quite different and finite size effects, like the presence of a surface, are essential.

6. **Nuclei and finite size effects.** The connection between the nuclear medium properties and the structure of nuclei is of course not simple. However, for a long time, the energy density functional method has been developed to describe the ground state of nuclei and their elementary excitations. This method is the most suited to establish a link between nuclear structure and the nuclear medium properties. Indeed the main assumption that can be taken within this method is to divide the functional to be minimized into a bulk and a surface part and identify the bulk part with the EoS of nuclear matter. Of course, in addition to these two building blocks, the spin–orbit and Coulomb interaction must be included. This is a theory that is semi-microscopic in character, since all finite size effects are assumed to be included in an effective way in surface terms, to be determined phenomenologically. However, the physical interpretation of the different terms can provide hints as to the values of the parameters that fix the non-bulk parts of the functional. If the functionals are treated completely phenomenologically and the number and complexity of the surface terms (i.e. containing density gradients) are increased, the accuracy of these functionals can be excellent throughout the mass table. The semi-microscopic and the purely phenomenological approach are complementary, and substantial refinements and increases in accuracy are expected to occur in the future.

7. **Pairing and superfluidity.** The pairing correlation is of paramount relevance both in the physics of neutron stars and in nuclear structure. Substantial progress has been
made in the theoretical determination of the effective pairing interaction in neutron matter and in neutron star matter for the different superfluid channels. Due to the extreme sensitivity of the pairing gap to the strength of the effective interaction, the various pairing gaps are still uncertain, but their overall trends as a function of density can be considered established, at least for not too high density. In nuclei it is not yet established as to what extent the bare nucleon–neucleon interaction, taken as pairing interaction, is able to reproduce the experimental pairing gaps. However, a substantial fraction can surely come from the bare interaction, while the remaining part is a matter of debate.

(8) Transition to quark matter. Neutron stars could be the only objects in the Universe where a macroscopic portion of quark matter is present under stable conditions. Massive neutron stars are indeed expected to have such a high-density baryonic matter in their interior that a transition to the deconfined phase could be possible. This is of course not firmly established, but progress in this field seems to be rapid. The maximum mass of observed neutron stars, with a fairly good degree of confidence, has increased in the last few years, up to a value close to $2M_{\odot}$. This result, if confirmed, would put definite constraints not only on the nuclear medium EoS but also on the possible quark matter EoS. This shows the interplay between the ‘traditional’ nuclear physics and the theory of the QCD matter in the physics of neutron stars. Further exciting and fundamental progress is expected in the near future in this field.

This schematic list surely does not exhaust all the facets of the properties of the nuclear medium, but we hope to have given at least some idea of the state of the art in the field, of what is going on and of the trends that can be expected to develop in the future.

Acknowledgments

We would like to thank all our collaborators over the years, who appear as co-authors of the papers quoted in this review. To them we would like to express our gratitude and friendship. One of the authors (MB) would like to review. To them we would like to express our gratitude who appear as co-authors of the papers quoted in this review.

We would like to thank all our collaborators over the years, We acknowledge the support of COMPSTAR, a research and networking program of the European Science Foundation.

References

[1] Youngblood D, Clark H L and Lui Y W 1999 Phys. Rev. Lett. 82 691
[2] Myers W D and Swiatecki W J 1996 Nucl. Phys. A 601 141
[3] Blaizot J P 1980 Phys. Rep. 64 171
[4] Migdal A B 1967 Theory of Finite Fermi Systems and Application to Atomic Nuclei (New York: Wiley)
[5] Bohr A and Mottelson B R 1974 and 1975 Nuclear Structure vols 1 and 2 (New York: Benjamin)
[6] von Weisäcker C F 1935 Z. Phys. 96 431
[7] Bethe H A and Bacher R F 1936 Rev. Mod. Phys. 8 82
[8] Ring P and Schuck P 1980 The Nuclear Many-Body Problem (Berlin: Springer)
[9] Myers W D and Swiatecki W J 1966 Nucl. Phys. 81 1
[10] Myers W D 1970 Nucl. Phys. A 145 387
[11] Myers W D and Swiatecki W J 1969 Ann. Phys. 55 395
[12] Strutinsky V M 1967 Nucl. Phys. A 95 420
[13] Strutinsky V M 1968 Nucl. Phys. A 122 1
[14] Strutinsky V M 1974 Nucl. Phys. A 218 169
[15] Danielewicz P and Lee J 2001 Nucl. Phys. A 727 233
[16] Danielewicz P 2009 Nucl. Phys. A 818 66
[17] Myers W D and Swiatecki W J 1969 Ann. Phys. 55 395
[18] Roca-Maza X and Piekarewicz J 2008 Phys. Rev. C 78 025807
[19] Centelles M, Roca-Maza X, Vinas X and Warda M 2009 Phys. Rev. Lett. 102 122502
[20] Möller P, Nix J R, Myers W D and Swiatecki W J 1995 At. Data Nucl. Data Tables 59 185
[21] Möller P, Nix J R and Kratz K-L 1997 At. Data Nucl. Data Tables 66 131
[22] Reinhard P G, Bender M, Nazarewicz W and Vertse T 2006 Phys. Rev. C 73 014309
[23] Nikolov N, Schunck N, Nazarewicz W, Bender M and Pei J 2011 Phys. Rev. C 83 034305
[24] Kleban M, Nerlo-Pomorska B, Berger J-F, Dechargé J, Girod M and Hilaire S 2002 Phys. Rev. C 65 024309
[25] Nerlo-Pomorska B and Mazurek K 2002 Phys. Rev. C 66 064305
[26] Myers W D and Swiatecki W J 1996 Nucl. Phys. A 601 141
[27] Bertulani C A 2006 arXiv:nucl-th/0607024
[28] Roca-Maza X, Centelles M, Salvat F and Vinas X 2008 Phys. Rev. C 78 044322
[29] Centelles M, Patra S K, Roca-Maza X, Sharma B K, Stevenson P D and Vinas X 2010 J. Phys. G: Nucl. Part. Phys. 37 075107
[30] Chamue E, Goriely S and Pearson J M 2008 Nucl. Phys. A 812 72
[31] Chamue E, Goriely S and Pearson J M 2009 Phys. Rev. C 80 054307
[32] Gandolfi S, Carlson J and Piper C 2011 Phys. Rev. Lett. 106 012501
[33] Gebremariam B, Duguet T and Bogner S K 2010 Phys. Rev. C 82 014305
[34] Gebremariam B, Bogner S K and Duguet T 2010 Comput. Phys. Commun. 181 1167
[35] Gebremariam B, Bogner S K and Duguet T 2011 Nucl. Phys. A 852 17
[36] Stoitsov M, Kortelainen M, Bogner S K, Duguet T, Furnstahl R J, Gebremariam B and Schunck N 2010 Phys. Rev. C 82 054307
[37] Bender M, Bertisch G F and Heenen P-H 2006 Phys. Rev. C 73 054322
[38] Chabanat E, Bonche P, Haensel P, Meyer J and Schaeffer R 1998 Nucl. Phys. A 635 231
[39] Chabanat E, Bonche P, Haensel P, Meyer J and Schaeffer R 1998 Nucl. Phys. A 643 441
[40] Colo’ G, Van Giai N, Meyer J, Bonnacure and Bonche P 2004 Phys. Rev. C 70 024307
[41] Sharma M M 2009 Nucl. Phys. A 816 65
[42] Trippa L, Colo’ G and Vigezz E 2008 Phys. Rev. C 77 061304
[43] Blocki J, Boneh Y, Nix J R, Randrup J, Mobel M, Sierk A J and Swiatecki W J 1978 Ann. Phys. 113 338
[44] Blocki J, Shi J J and Swiatecki W J 1993 Nucl. Phys. A 554 387
[45] Jarzynsky C and Swiatecki W J 1995 Nucl. Phys. A 552 1
[46] Blocki J, Skalski J and Swiatecki W J 1995 Nucl. Phys. A 594 137
[47] Baldo M, Burgio G F, Rapisarda A and Schuck P 1998 Phys. Rev. C 58 2821
[283] Colonna M, Baran V, Di Toro M and Wolter H H 2008 Phys. Rev. C 78 064618
[284] Ducoin C, Chomaz P and Gulminelli F 2007 Nucl. Phys. A 781 407
[285] Ducoin C, Chomaz P and Gulminelli F 2006 Nucl. Phys. A 771 68
[286] Leviš S and Bonche P 1985 Nucl. Phys. A 437 426
[287] Song H Q and Su R K 1991 Phys. Rev. C 44 2505
[288] Natowitz J B, Wada R, Hagel K, Keutgen T, Murray M, Makeev A, Qin L, Smith P and Hamilton C 2002 Phys. Rev. C 65 034618
[289] Pochdzalla J et al 1995 Phys. Rev. Lett. 75 1040
[290] Wada R et al 1997 Phys. Rev. C 55 227
[291] Cibor J et al 2000 Phys. Lett. B 473 29
[292] Cussol D et al 1993 Nucl. Phys. A 561 298
[293] Bondorf J, Donangelo R, Mishustin I N and Schulz H 1985 Nucl. Phys. A 444 460
[294] Bondorf J, Donangelo R, Mishustin I N, Pethick C, Schulz H and Sneppen K 1985 Phys. Lett. B 150 57
Bondorf J, Donangelo R, Mishustin I N, Pethick C, Schulz H and Sneppen K 1985 Nucl. Phys. A 443 321
[295] Bondorf J, Donangelo R, Schulz H and Sneppen K 1985 Phys. Lett. B 162 30
[296] Botvina A S et al 1987 Nucl. Phys. A 475 663
[297] Gross D et al 1993 Prog. Part. Nucl. Phys. 30 155 and references therein
[298] Friedman W A 1988 Phys. Rev. Lett. 60 2125
[299] Csernai L P and Kapusta J I 1986 Phys. Rep. 131 223 and references therein
[300] Bondorf J P et al 1995 Phys. Rep. 257 133
[301] Natowitz J B, Hagel K, Ma Y, Murray M, Qin L, Wada R and Wang J 2002 Phys. Rev. Lett. 89 212701
[302] Baldo M, Ferreira L S and Nicotra O 2004 Phys. Rev. C 69 034321
[303] Fritsch S, Kaiser N and Weise W 2002 Phys. Lett. B 545 73
[304] Ducoin C, Hasnoau K H O, Chomaz P and Gulminelli F 2007 Phys. Rev. C 75 065805
[305] Lattimer J and Swesty F D 1991 Nucl. Phys. A 535 331
[306] Shen H 2002 Phys. Rev. C 65 035802
[307] Horowitz C J and Schwenk A 2006 Phys. Lett. B 638 153