Riemann Solitons on Almost Co-Kähler Manifolds

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Abstract. The aim of the present paper is to characterize almost co-Kähler manifolds whose metrics are the Riemann solitons. At first we provide a necessary and sufficient condition for the metric of a 3-dimensional manifold to be Riemann soliton. Next it is proved that if the metric of an almost co-Kähler manifold is a Riemann soliton with the soliton vector field $\xi$, then the manifold is flat. It is also shown that if the metric of a $(\kappa, \mu)$-almost co-Kähler manifold with $\kappa < 0$ is a Riemann soliton, then the soliton is expanding and $\kappa, \mu, \lambda$ satisfies a relation. We also prove that there does not exist gradient almost Riemann solitons on $(\kappa, \mu)$-almost co-Kähler manifolds with $\kappa < 0$. Finally, the existence of a Riemann soliton on a three dimensional almost co-Kähler manifold is ensured by a proper example.

1. Introduction

Udrişte ([24], [25]) introduced the notion of Riemann flow. The Riemann flow is defined by

$$\frac{\partial}{\partial t} G(t) = -2R(g(t)), \quad (1)$$

where $G = \frac{1}{2}g \odot g$, $R$ is the Riemann curvature tensor of type $(0, 4)$ corresponding to the metric $g$ and $\odot$ denotes the Kulkarni-Nomizu product given by

$$(P \odot Q)(E, F, W, X) = P(E, X)Q(F, W) + P(F, W)Q(E, X) - P(E, W)Q(F, X) - P(F, X)Q(E, W).$$

In the same way as Ricci solitons, Riemann solitons were introduced by Hirica and Udrişte [16] which are the self-similar solution of Riemann flow. A Riemannian metric $g$ on a smooth manifold $M$ is said to be a Riemann soliton if there exists a smooth vector field $Z$ and a real constant $\lambda$ such that

$$2R + \lambda g \odot g + g \odot \mathcal{L}_Z g = 0, \quad (2)$$
where $\mathcal{L}_Z$ is the Lie derivative along the vector field $Z$. The vector field $Z$ is known as potential vector field. We denote a Riemann soliton by $(g, Z, \lambda)$. When $\lambda \in C^\infty(M)$, then $g$ is said to be an almost Riemann soliton. If $Z$ is a Killing vector field, then $M$ is a manifold of constant sectional curvature. Thus the Riemann soliton is the generalization of the space of constant curvature. The soliton will be called expanding, steady or shrinking according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. When the vector field $Z$ is a gradient of some smooth function $u$, then the Riemann soliton is called a gradient Riemann soliton and the equation (2) takes the form

$$R + \frac{\lambda}{2} g \circ g + g \circ \nabla^2 u = 0,$$

(3)

where $\nabla^2 u$ is the Hessian of the function $u$. If $\lambda$ is a smooth function in (3), then the metric $g$ is called a gradient almost Riemann soliton. Using Kulkarni-Nomizu product, the equation (2) can be written as

$$2R(E, F, W, X) + 2\lambda [g(E, X)g(F, W) - g(E, W)g(F, X)] + g(E, X)(\mathcal{L}_Z g)(F, W) + g(F, W)(\mathcal{L}_Z g)(E, X) - g(E, W)(\mathcal{L}_Z g)(F, X) - g(F, X)(\mathcal{L}_Z g)(E, W) = 0$$

(4)

for all vector fields $E, F, W, X$ on $M$. Contracting the equation (4), we lead

$$2S(F, W) + 2[(m - 1)\lambda + \text{div} Z]g(F, W) + (m - 2)(\mathcal{L}_Z g)(F, W) = 0,$$

(5)

where $S$ is the Ricci tensor, $m \geq 3$ is the dimension of the manifold $M$ and $\text{div}$ denotes the divergence operator. Contracting again the equation (5), we have

$$r + m(m - 1)\lambda + 2(m - 2) \text{div} Z = 0,$$

(6)

where $r$ is the scalar curvature. From the foregoing equation, we can easily see that $\text{div} Z$ is constant if and only if $r$ is constant.

In [16], Hiričă and Udriște studied Sasaki-Riemann soliton. They proved that, if the metric $g$ of a Sasaki manifold $M$ is a gradient Riemann soliton with potential function $u$ as harmonic or a Riemann soliton with potential vector field $Z$ is pointwise collinear to Reeb vector field $\xi$, then $M$ is a Sasaki-space form. In [14], Venkatesha et al. proved some interesting results on Riemann soliton within the framework of contact geometry. They also studied Riemann solitons and almost Riemann solitons on almost Kenmotsu manifolds (cf.[26]).

The present paper is organized as follows: After introduction, in Section 2 we recall the definition and basic properties of almost co-Kähler manifolds and $(\kappa, \mu)$-almost co-Kähler manifolds. In the next section, we characterize a three-dimensional manifold whose metric is the Riemann soliton. In Sections 4 and 5, we prove some lemmas and theorems on Riemann soliton in almost co-Kähler manifolds and $(\kappa, \mu)$-almost co-Kähler manifolds. In the Section 6, we consider gradient almost Riemann solitons on $(\kappa, \mu)$-almost co-Kähler manifolds. Finally, we construct an example to verify our results.

2. Almost co-Kähler manifolds

A smooth manifold $M^{2n+1}$ of dimension $(2n + 1)$ together with the triple $(\eta, \xi, \varphi)$, where $\eta$ is a 1-form, $\xi$ is a global vector field and $\varphi$ is a $(1, 1)$-tensor field, is said to be an almost contact manifold [2] if

$$\varphi^2 + \text{id} = \eta \otimes \xi, \quad \eta(\xi) = 1,$$

(7)

where $\text{id}$ is the identity automorphism. From (7) we can obtain $\varphi^2 \xi = 0$ and $\eta \circ \varphi = 0$. An almost contact structure $(\eta, \xi, \varphi)$ will be called normal if the almost complex structure $J$ on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J \left( E, \gamma \frac{d}{dt} \right) = \varphi E - \gamma \xi, \eta(\xi) \frac{d}{dt}$$
for all vector field $E$ on $M^{2n+1}$ and $\gamma \in C^\infty(M^{2n+1} \times \mathbb{R})$, is integrable. According to Blair [2], $[\varphi, \varphi] = -2d\eta \otimes \xi$ is the condition for normality of the almost contact structure $(\eta, \xi, \varphi)$ and conversely, where $[\varphi, \varphi]$ denotes the Nijenhuis tensor of $\varphi$ defined by

$$[\varphi, \varphi](E, F) = \varphi^2[E, F] + [\varphi E, \varphi F] - \varphi[\varphi E, F] - \varphi[E, \varphi F]$$

for all vector fields $E, F$ on $M^{2n+1}$. If a Riemannian metric $g$ on $M^{2n+1}$ satisfies

$$g(E, F) = g(\varphi E, \varphi F) + \eta(E)\eta(F)$$

for all vector fields $E, F$ on $M^{2n+1}$, then the manifold together with $(\eta, \xi, \varphi, g)$ is said to be an almost contact metric manifold and $g$ is called compatible metric with respect to the almost contact structure. The fundamental 2-form $\Phi$ on an almost contact metric manifold is defined by $\Phi(E, F) = g(\varphi E, F)$ for all vector fields $E, F$ on $M^{2n+1}$.

An almost contact metric manifold $M^{2n+1}$ is said to be an almost co-Kähler manifold if both $\eta$ and $\Phi$ are closed i.e., $d\eta = 0$ and $d\Phi = 0$, where $d$ denotes exterior derivative. In addition, if $M^{2n+1}$ is normal, then the manifold $M^{2n+1}$ is called co-Kähler manifold. An (almost) co-Kähler manifold is nothing but an (almost) cosymplectic manifold defined by Blair [3] and studied by several authors (see [1], [4]-[8], [11]-[13], [17, 18], [23], [27]-[32]).

On any almost co-Kähler manifold, we can define an $(1, 1)$-tensor field $h = \frac{1}{2} \mathcal{L}_\varphi \eta$. According to [19, 20] and [22], it is known that $h$ and $h'(= h \circ \varphi)$ are symmetric tensors and satisfy

$$h\xi = 0, \quad h' = -q\eta, \quad \text{tr } h = \text{tr } h' = 0,$$

$$\nabla \xi \varphi = 0, \quad \nabla \xi = h', \quad (10)$$

$$q\xi \varphi - \xi = 2h^2, \quad (11)$$

$$\nabla \xi h = -2q \xi \varphi - q\xi \ell, \quad (12)$$

$$S(\xi, \xi) + \text{tr } h^2 = 0, \quad (13)$$

where $\ell = R(., \xi)\xi$ is the Jacobi operator along the Reeb vector field, $\text{tr}$ denotes for trace and $\nabla$ is the Riemannian connection with respect to the metric $g$. Using the second equation of (10), we see that $(\xi, \xi \varphi)(E, F) = 2g(h' E, F)$ for all vector fields $E, F$ on $M^{2n+1}$. Thus, $\xi$ is a Killing vector field if and only if $h = 0$.

A $(\kappa, \mu)$-almost co-Kähler manifold $M^{2n+1}$, introduced by Endo [15], is an almost co-Kähler manifold whose structure vector field $\xi$ belongs to the $(\kappa, \mu)$-nullity distribution, i.e. the curvature tensor $R$ satisfies

$$R(E, F)\xi = \kappa(\eta(F)E - \eta(E)F) + \mu(\eta(F)hE - \eta(E)hF)$$

for all vector fields $E, F$ on $M^{2n+1}$ and $(\kappa, \mu) \in \mathbb{R}^2$. Taking $\xi$ instead of $F$ in (14), we have $\ell = -\kappa q^2 + \mu h$. Using this value of $\ell$ in (11), it follows that

$$h^2 = \kappa q^2. \quad (15)$$

From the above, it is easy to see that $\kappa \leq 0$ and $\kappa = 0$ if and only if $M^{2n+1}$ is a K-almost co-Kähler manifold. In particular, if $\mu = 0$ then the manifold is said to be $N(\kappa)$-almost co-Kähler manifold [9]. Any co-Kähler manifold satisfies (14) with $\kappa = \mu = 0$. Dacko and Olszak [10] defined almost co-Kähler $(\kappa, \mu, v)$-spaces. An almost co-Kähler manifold is said to be a $(\kappa, \mu, v)$-space if the curvature tensor $R$ satisfies

$$R(E, F)\xi = \kappa(\eta(F)E - \eta(E)F) + \mu(\eta(F)hE - \eta(E)hF)$$

$$- v(\eta(F)h'E - \eta(E)h'F)$$

for all vectors fields $E, F$ on $M^{2n+1}$.

In a $(\kappa, \mu)$-almost co-Kähler manifold the following relations hold [21] :

$$\nabla \xi h = \mu h', \quad (16)$$

$$\nabla \xi h^2 = 0, \quad (17)$$

$$\ell \varphi - \varphi \ell = 2\mu h'. \quad (18)$$

**Lemma 2.1.** [4] The Ricci operator $Q$ of a $(\kappa, \mu)$-almost co-Kähler manifold $M^{2n+1}$, $n \geq 1$, is given by

$$Q = \mu h + 2\kappa \eta \otimes \xi. \quad (19)$$
3. Riemann solitons on 3-dimensional manifolds

Suppose a metric \((g, Z, \lambda)\) of a 3-dimensional manifold \(M^3\) is a Riemann soliton. Then from the equation (5), we have

\[
2S(F, W) + (4\lambda + 2 \text{div} Z)g(F, W) + (\mathcal{L}_Z g)(F, W) = 0
\]

(20)

for all vector fields \(F, W\) on \(M^3\). Tracing the previous equation, we have

\[
\text{div} Z = -\frac{r + 6\lambda}{4}.
\]

(21)

Using this value of \(\text{div} Z\) in (20), we obtain

\[
2S(F, W) + \left(\lambda - \frac{r}{2}\right)g(F, W) + (\mathcal{L}_Z g)(F, W) = 0
\]

(22)

for all vector fields \(F, W\) on \(M^3\).

Conversely, suppose the equation (22) is satisfied. It is well known that the curvature tensor \(R\) of any 3-dimensional manifold is given by

\[
R(E, F)W = S(F, W)E - S(E, W)F + g(F, W)QE - g(E, W)QF
\]

\[-\frac{1}{2}[g(F, W)E - g(E, W)F].
\]

(23)

Taking inner product of (23) with \(X\), we have

\[
2R(E, F, W, X) = 2S(F, W)g(E, X) - 2S(E, W)g(F, X)
\]

\[+ 2g(F, W)S(E, X) - 2g(E, W)S(F, X)
\]

\[+ r[g(F, W)g(E, X) - g(E, W)g(F, X)].
\]

Using (22) in the foregoing equation, we obtain the equation (4). Hence \((g, Z, \lambda)\) is a Riemann soliton. Thus we obtain the following:

**Theorem 3.1.** Let \((M^3, g)\) be a three dimensional manifold. Then \((g, Z, \lambda)\) is a Riemann soliton if and only if

\[
2S + \left(\lambda - \frac{r}{2}\right)g + \mathcal{L}_Z g = 0.
\]

(24)

4. Riemann solitons on almost co-Kähler manifolds

Suppose the metric \(g\) of an almost co-Kähler manifold \(M\) is the Riemann soliton with the soliton vector field \(Z = f\xi\) and \(df \wedge \eta = 0\).

The condition \(df \wedge \eta = 0\) implies \(Ef = (\xi f)\eta(E)\) for all vector field \(E\) on \(M\). Taking covariant derivative of \(Z = f\xi\) along the vector field \(E\) and using the second equation of (10), we have

\[
\nabla_E Z = (\xi f)\eta(E)\xi + fh'E,
\]

which gives

\[
(\mathcal{L}_Z g)(E, F) = g(\nabla_E Z, F) + g(E, \nabla_F Z) = 2(\xi f)\eta(E)\eta(F) + 2f g(h'E, F)
\]

(25)

for all vector fields \(E, F\) on \(M\).
By virtue of (25) and (4), we get
\[2R(E, F, W, X) + 2\lambda [g(E, X)g(F, W) - g(E, W)g(F, X)]
+ g(E, X)[2(\xi f)\eta(\eta(F) + 2f g(h'E, W)]
+ g(F, W)[2(\xi f)\eta(\eta(X) + 2f g(h'E, X)]
- g(E, W)[2(\xi f)\eta(\eta(X) + 2f g(h'E, X)]
- g(F, X)[2(\xi f)\eta(\eta(W) + 2f g(h'E, W)] = 0
gives (26) for all vector fields \(E, F, W, X\) on \(M\). Taking \(\xi\) instead of \(W\) in (26), we obtain
\[R(E, F, \xi, X) + (\lambda + \xi f) [g(E, X)\eta(F) - g(F, X)\eta(E)]
+ f\{g(h'E, X)\eta(F) - g(h'E, X)\eta(E)\} = 0.
\]

Eliminating \(X\) in the previous equation, we infer
\[R(E, F)\xi = -(\lambda + \xi f)\eta(F)E - \eta(E)F + f\eta(F)\phi hE - \eta(E)\phi hF].
\]
This implies that \(M\) is a \((\kappa, \mu, \nu)\)-space with \(\kappa = -(\lambda + \xi f), \mu = 0\) and \(\nu = f\). Thus we can write the following:

**Lemma 4.1.** If the metric \((g, Z, \lambda)\) of an almost co-Kähler manifold \(M\) is a Riemann soliton with the soliton vector field \(Z = f\xi\) and \(df\wedge \eta = 0\), then \(M\) is a \((\kappa, \mu, \nu)\)-space with \(\kappa = -(\lambda + \xi f), \mu = 0\) and \(\nu = f\).

Putting \(f = 1\) in (27), we have
\[R(E, F)\xi = -\lambda \eta(F)E - \eta(E)F + \eta(F)\phi hE - \eta(E)\phi hF,
\]
which gives \(\ell = \lambda \phi q^2 + \phi h\). Using this value of \(\ell\) in (11), it follows that
\[h^2 = -\lambda \phi q^2.
\]

Taking covariant derivative of (29) and using first equation of (10), we lead
\[\nabla_\xi h^2 = 0.
\]

Using \(\ell = \lambda \phi q^2 + \phi h\) and (29) in (12), we get
\[\nabla_\xi h = h.
\]

Now
\[0 = (\nabla_\xi h^2)E = (\nabla_\xi h)hE + h(\nabla_\xi h)E = 2h^2 E = -2\lambda \phi q^2 E
\]
for all vector field \(E\) on \(M\), which gives \(\lambda = 0\). Since \(h\) is a symmetric tensor, from (29) we get \(h = 0\). Consequently, \((\xi g)(E, F) = 0\). From (4), we see that \(R(E, F)W = 0\) for all vector fields \(E, F, W\) on \(M\). From the above discussion, we can state the following:

**Theorem 4.2.** If the metric \(g\) of an almost co-Kähler manifold \(M\) is a Riemann soliton with the soliton vector field \(\xi\), then \(M\) is flat.

5. Riemann solitons on \((\kappa, \mu)\)-almost co-Kähler manifolds

Suppose the metric \((g, Z, \lambda)\) of a \((\kappa, \mu)\)-almost co-Kähler manifold \(M^{2n+1}\) of dimension \((2n + 1)\) is the Riemann soliton. The equation (5) can be written as
\[(\xi g)(F, W) = -\frac{2}{2n - 1}S(F, W) - \frac{2}{2n - 1} (2n\lambda + \text{div}Z)g(F, W)
\]
(32)
for all vector fields $F, W$ on $M^{2n+1}$. From (19) we have $r = 2n\kappa = \text{constant}$. Consequently, $\text{div} Z$ is a constant.

Taking covariant derivative of (32) along the arbitrary vector field $E$, we have

$$
(\nabla_E Z)(F, W) = -\frac{2}{2n-1}(\nabla_E S)(F, W).
$$

(33)

From Yano [33], we recall a well known formula

$$
(\xi Z\nabla F - \nabla F Z \xi - \nabla F \xi g)(F, W)
= -g((\xi Z\nabla)(E, F), W) - g((\xi Z\nabla)(E, W), F).
$$

Using the symmetry property of $\xi Z\nabla$ in the above formula, we have

$$
2g((\xi Z\nabla)(E, F), W) = (\nabla E Z)(g) + (\nabla F Z)(g)(W, E)
- (\nabla W Z)(g)(E, F).
$$

(34)

Using (19) and (33) in (34), we lead

$$
g((\xi Z\nabla)(E, F), W)
= -\frac{1}{2n-1} [\mu g((\nabla W h) h, F, E) - \mu g((\nabla F h) h, W, E)
- \mu g((\nabla h) h, W, E) - 4\mu \kappa \eta(W) g(h', F)].
$$

(35)

Taking $\xi$ instead of $F$ in (35) and using (10) and (16), we obtain

$$
g((\xi Z\nabla)(E, \xi), W)\rightleftharpoons \frac{1}{2n-1} [2\mu \kappa g(E, \nu W) - \mu^{2} g(h', W, E)],
$$

which gives

$$
(\xi Z\nabla)(E, \xi) = -\frac{1}{2n-1} (2\mu \kappa \varphi E + \mu^{2} h'E).
$$

(36)

Taking covariant derivative of (36) along the vector field $F$, we lead

$$
(\nabla F Z)(E, \xi) = -\frac{1}{2n-1} (2\mu \kappa (\nabla F \varphi)E + \mu^{2} (\nabla F h')E)
- (\xi Z\nabla)(E, h' F).
$$

(37)

Using the above equation in the formula [33]

$$
(\xi Z R)(E, F)W = (\nabla E Z\nabla)(F, W) - (\nabla F Z\nabla)(E, W),
$$

we get

$$
(\xi Z R)(E, \xi E) = \frac{2}{2n-1} (2\mu \kappa h E + \mu^{2} \kappa \varphi^{2} E) - \frac{\mu^{3}}{2n-1} h E,
$$

(38)

where we have used (10), (15) and (16).

On the other hand taking Lie-derivative of $R(E, \xi E) = \kappa [E - \eta (E) \xi] + \mu h E$ along $Z$, we have

$$
(\xi Z R)(E, \xi E) = -\kappa (\xi Z \eta) E \xi - \kappa \eta (E) E Z \xi + \mu (\xi Z h) E
- R(E, \xi Z) E \xi - R(E, \xi) E Z \xi.
$$

(39)

By virtue of (38) and (39) we obtain

$$
\frac{2}{2n-1} (2\mu \kappa E + \mu^{2} \kappa \varphi^{2} E) - \frac{\mu^{3}}{2n-1} h E
= -\kappa (\xi Z \eta) E \xi - \kappa \eta (E) E Z \xi + \mu (\xi Z h) E - R(E, \xi Z) E \xi - R(E, \xi) E Z \xi.
$$
Contracting the previous equation with respect to the orthonormal basis \( \{e_1, e_2, \cdots, e_n, \phi e_1, \phi e_2, \cdots, \phi e_n, \xi \} \), where \( h e_i = \sqrt{-\kappa} e_i \), we get
\[
S(\mathcal{L}Z \xi, \xi) = \frac{2n}{2n-1} \mu^2 \kappa.
\]
Utilizing (19) in the above equation, we lead
\[
g(\mathcal{L}Z \xi, \xi) = \frac{\mu^2}{2n-1}.
\] (40)
Putting \( F = W = \xi \) in (32) and using (40), we infer
\[
\text{div} Z = \mu^2 - 2n(\kappa + \lambda).
\] (41)
Contraction the equation (32), it follows that
\[
2 \text{div} Z = -\kappa - (2n + 1) \lambda.
\] (42)
By virtue of (41) and (42), we get
\[
\kappa = \frac{2\mu^2}{4n-1} - \frac{2n-1}{4n-1} \lambda.
\]
Thus we are in a position to state the following:

**Theorem 5.1.** If the metric \((g, Z, \lambda)\) of a \((\kappa, \mu)\)-almost co-Kähler manifold \(M^{2n+1}\) with \(\kappa < 0\) is a Riemann soliton, then \(\kappa, \mu\) and \(\lambda\) satisfy the relation
\[
\kappa = \frac{2\mu^2}{4n-1} - \frac{2n-1}{4n-1} \lambda.
\] (43)
Since \(\kappa < 0\), from (43) we easily see that \(\lambda > 0\). Hence, we can state that

**Corollary 5.2.** If the metric \((g, Z, \lambda)\) of a \((\kappa, \mu)\)-almost co-Kähler manifold \(M^{2n+1}\) with \(\kappa < 0\) is a Riemann soliton, then the soliton is expanding.

In particular, for \(\mu = 0\) we can state that the followings:

**Corollary 5.3.** If the metric \((g, Z, \lambda)\) of a \(N(\kappa)\)-almost co-Kähler manifold \(M^{2n+1}\) with \(\kappa < 0\) is a Riemann soliton, then \((4n-1)\kappa = -(2n-1)\lambda\).

**Corollary 5.4.** If the metric \((g, Z, \lambda)\) of a \(N(\kappa)\)-almost co-Kähler manifold \(M^{2n+1}\) with \(\kappa < 0\) is a Riemann soliton, then the soliton is expanding.

### 6. Gradient almost Riemann solitons on \((\kappa, \mu)\)-almost co-Kähler manifolds

In this section we consider a \((\kappa, \mu)\)-almost co-Kähler manifold \(M^{2n+1}\) whose metric \(g\) is a gradient almost Riemann soliton. We need the following lemma before proving the main results.

**Lemma 6.1.** (Lemma 3.8 of [14]) If the metric \(g\) of a Riemannian manifold \(M^{2n+1}\) is a gradient almost Riemann soliton, then for any vector fields \(E, F\) on \(M^{2n+1}\) the curvature tensor \(R\) satisfies
\[
R(E, F) Du = \frac{1}{2n-1} [(V_F Q) E - (V_E Q) F + F(2n\lambda + \Delta u) E - E(2n\lambda + \Delta u) F],
\] (44)
where \(D\) denotes the gradient operator and \(\Delta = \text{div} D\).
By virtue (10) and (19) the equation (44) can be written as

\[
R(E,F)Du = \frac{1}{2n-1}\left\{\mu(\nabla h)E - \mu(\nabla h)F
+ 2n\kappa \eta(E)h'F - 2n\kappa \eta(F)h'E
+ F(2n\lambda + \Delta u)E - E(2n\lambda + \Delta u)F\right]\}.
\] (45)

Balkan et al. [1] proved that in a \((\kappa, \mu)\)-almost co-Kähler manifold the tensor field \(h\) satisfies

\[
(\nabla h)E - (\nabla h)F = \kappa(\eta(F)\varphi E - \eta(E)\varphi F + 2g(\varphi E, F)\xi)
+ \mu(\eta(F)\varphi h E - \eta(E)\varphi h F).
\] (46)

Using (46) in (45), we obtain

\[
R(E,F)Du = \frac{1}{2n-1}\left\{\mu\kappa(\eta(F)\varphi E - \eta(E)\varphi F + 2g(\varphi E, F)\xi)
+ (\mu^2 - 2n\kappa)(\eta(F)h'F - \eta(E)h'F)
+ F(2n\lambda + \Delta u)E - E(2n\lambda + \Delta u)F\right\}.
\] (47)

Taking inner product of the foregoing equation with \(\xi\) and using (14), it follows that

\[
\kappa((Fu)\eta(E) - (Eu)\eta(F)) + \mu(g(\varphi E, Du)\eta(E) - g(\varphi E, Du)\eta(F))
= \frac{1}{2n-1}\left\{2\mu\kappa g(E, \varphi F) + F(2n\lambda + \Delta u)\eta(E) - E(2n\lambda + \Delta u)\eta(F)\right\}.
\] (48)

Replacing \(E\) and \(F\) by \(\varphi E\) and \(\varphi F\) respectively in (48), we infer

\[
0 = \frac{2\mu\kappa}{2n-1}g(E, \varphi F),
\]

which gives \(\mu = 0\), since \(\kappa < 0\).

Now letting \(E = \xi\) in (48) gives

\[
\kappa((Fu) - (\xi u)\eta(F)) = \frac{1}{2n-1}\left\{F(2n\lambda + \Delta u) - \xi(2n\lambda + \Delta u)\eta(F)\right\},
\]

that is,

\[
\frac{1}{2n-1}D(2n\lambda + \Delta u) = \kappa Du - \kappa \xi(u)\xi + \frac{1}{2n-1}\xi(2n\lambda + \Delta u)\xi.
\] (49)

On the other hand, by (9), contracting (47) with respect to \(E\) we find

\[
S(F, Du) = \frac{2n}{2n-1}F(2n\lambda + \Delta u).
\]

Recalling (19), we obtain

\[
\frac{2n}{2n-1}D(2n\lambda + \Delta u) = QDu = 2n\kappa \xi(u)\xi.
\]

Thus it follows from (49) that

\[
2\kappa \xi(u)\xi = \kappa Du + \frac{1}{2n-1}\xi(2n\lambda + \Delta u)\xi.
\]

From this we see \(Du = \xi(u)\xi\). Differentiating this along \(E\) and using the second term of (10), we have

\[
\nabla_E Du = E(\xi(u))\xi + \xi(u)h'E.
\] (50)
Since Equation (5) with $Z = Du$ may be expressed as
\[ V_E Du = -\frac{1}{2n-1} QE - \frac{1}{2n-1} (2n\lambda + \Delta u) E, \]
inserting (50) into the previous relation yields
\[ \frac{1}{2n-1} QE = -E(\xi(u))\xi - \xi(u)h'E - \frac{1}{2n-1} (2n\lambda + \Delta u) E. \]
Using (19) again in the above relation, we get
\[ \frac{2nK}{2n-1} \eta(E)\xi = -E(\xi(u))\xi - \xi(u)h'E - \frac{1}{2n-1} (2n\lambda + \Delta u) E. \]
Applying $\varphi$ to act on this equation and taking the inner product with $\varphi E$, we have
\[ -\xi(u)g(h'E, E) - \frac{1}{2n-1} (2n\lambda + \Delta u) g(\varphi E, \varphi E) = 0. \]
Because $trh' = 0$, the above formula shows $2n\lambda + \Delta u = 0$ and $\xi(u) = 0$. Since $Du = \xi(u)\xi$, $u$ is a constant. Further, it implies from (51) that $\kappa = 0$, which is contradictory with $\kappa < 0$.

**Theorem 6.2.** There does not exist gradient almost Riemann solitons on a $(\kappa, \mu)$-almost co-Kähler manifolds with $\kappa < 0$.

### 7. Example

In this section we construct an example of an almost co-Kähler manifold whose metric is a Riemann soliton.

Let $M^3 = \mathbb{R}^3(x, y, z)$, where $(x, y, z)$ are the standard coordinates of $\mathbb{R}^3$. Let $g$ be the Riemannian metric on $M^3$ defined by
\[ g = dx^2 + dy^2 + \frac{4(x^2 + y^2) + 1}{z^4} dz^2 - 4ye^{-z}dx dz - 4xe^{-z} dy dz. \]
Let $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, $e_3 = 2y\frac{\partial}{\partial z} + 2x\frac{\partial}{\partial y} + e^{2z}\frac{\partial}{\partial z}$. Then $\{e_1, e_2, e_3\}$ is an orthonormal basis of $(M^3, g)$. We have
\[ [e_1, e_2] = 0, \quad [e_1, e_3] = 2e_2, \quad [e_2, e_3] = 2e_1. \]

Let the 1-form $\eta$, the vector field $\xi$ and $(1, 1)$-tensor field $\varphi$ are defined by
\[ \eta = e^{-z} dz, \quad \xi = e_3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0. \]

The 2-form $\Phi$ is given by
\[ \Phi = -2dx \wedge dy - 4 ye^{-z} dy \wedge dz - 4xe^{-z} dz \wedge dx. \]
Since $d\eta = 0$ and $d\Phi = 0$, $M^3$ is an almost co-Kähler manifold.

The Riemannian connection $\nabla$ is given by
\[ \nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = -2e_3, \quad \nabla_{e_1} e_3 = 2e_2, \]
\[ \nabla_{e_2} e_1 = -2e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = 2e_1, \]
\[ \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \]

The components of the Riemannian curvature tensor $R$ are
\[ R(e_1, e_2)e_1 = -4e_2, \quad R(e_1, e_2)e_2 = 4e_1, \quad R(e_1, e_2)e_3 = 0, \]
Using the above expression of the curvature tensor $R$, it follows that

$$R(E,F)\xi = -4[\eta(F)E - \eta(E)F]$$

for all $E, F \in \chi(M^3)$. Hence $M^3$ is a $N(-4)$-almost co-Kähler manifold. The expression of the curvature tensor $R$ is

$$R(E,F)W = 4\{\eta(F)W - g(E,W)F\} - 8[\eta(F)\eta(W)E - \eta(E)\eta(W)F + g(F,W)\eta(E)\xi - g(E,W)\eta(F)\xi]$$

(52)

for all $E, F, W \in \chi(M^3)$. The components of the Ricci tensor $S$ are

$$S(e_1,e_1) = S(e_2,e_2) = 0, \quad S(e_3,e_3) = -8,$$

$$S(e_i,e_j) = 0, \quad \text{where } i, j = 1, 2, 3 \text{ and } i \neq j,$$

which gives $r = -8$. Also we have

$$S(E,F) = -8\eta(E)\eta(F)$$

(53)

for all $E, F \in \chi(M^3)$.

Let $Z = -8x_1 - 8y_2$ and $\lambda = 12$. By direct computations we obtain

$$(\xi_Z g)(E,F) = -16g(E,F) + 16\eta(E)\eta(F),$$

(54)

which gives $\text{div}Z = -16$. From (52), and (54) we see that the equation (4) is satisfied. Hence $g$ is a Riemann soliton. From (53) and (54), we lead

$$2S(E,F) + \left(\lambda - \frac{r}{2}\right)g(E,F) + (\xi_Z g)(E,F) = 0$$

for all $E, F \in \chi(M^3)$. Thus Theorem 3.1 is verified. Also Corollaries 5.3 and 5.4 are verified.

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