Extrinsic eigenvalue estimates for Dirac operator✩

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Abstract

In this note, we prove lower and upper bounds for Dirac operators of submanifolds in certain ambient manifolds in terms of conformal and extrinsic quantities.

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1. Introduction

The eigenvalues of Dirac operators on spin manifolds are extensively studied. In 1980, Friedrich [9] first derived the lower bound of the first eigenvalues of a Dirac operator \( D \) in terms of the scalar curvature \( S_M \) and dimension \( m \) of the underlying manifold \( M^m \):

\[ \lambda^2(D) \geq \frac{m}{4(m-1)} S_M. \]

Since then, various kinds of estimates in terms of intrinsic geometric quantities have been proved (see e.g. [11, 13] and the references therein). A well known result of Hijazi [16] states that

\[ \lambda^2(D) \geq \frac{m}{4(m-1)} \lambda_1(L_M) \]

for \( m \geq 3 \), where \( L_M = -\frac{4(m-1)}{m-2} \Delta + S_M \) is the Yamabe operator of \( M \). If \( m = 2 \), Bär [2] proved that

\[ \lambda^2(D) \geq \frac{4\pi(1-g_M)}{\text{area}(M)}, \]

where \( g_M \) is the genus of \( M \).

On the other hand, the submanifold theory for Dirac operators was introduced by Bär in [3]. Let \( M^m \hookrightarrow \bar{M}^m+n \) be a closed oriented connected spin submanifold isometrically immersed in a Riemannian spin manifold \( \bar{M}^m+n \) with fixed spin structures. Milnor’s Lemma claims that there is a unique spin structure [23] on the normal bundle \( N \) of \( M \) in \( \bar{M} \). Denoted by \( \bar{\nabla}, \nabla \) and \( \nabla^\perp \) the Levi-Civita connections on \( \bar{M}, M, N \) respectively. Denoted by \( \bar{\nabla}^{\Sigma}, \nabla^{\Sigma_M} \) and \( \nabla^{\Sigma_N} \) the Levi-Civita connections on \( \Sigma \bar{M}, \Sigma M \) and \( \Sigma N \) respectively. For every \( X, Y \in TM \), define

\[
\begin{align*}
\bar{R}(X, Y) &= [\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]}, \\
R(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \\
R^+(X, Y) &= [\nabla^+_X, \nabla^+_Y] - \nabla^+_{[X,Y]}, \\
R^{\Sigma_M}(X, Y) &= [\nabla^{\Sigma_M}_X, \nabla^{\Sigma_M}_Y] - \nabla^{\Sigma_M}_{[X,Y]},
\end{align*}
\]

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\[ R^E(X, Y) := [\nabla_X^E, \nabla_Y^E] - \nabla_{[X,Y]}^E, \]
\[ R^N(X, Y) := [\nabla_X^N, \nabla_Y^N] - \nabla_{[X,Y]}^N. \]

Denoted by \( \gamma, \gamma^+ \) the Clifford multiplications on \( \Sigma \) \( \Sigma M \) and \( \Sigma N \) respectively. Denoted by \( D, D, D^\perp \) the Dirac operators on \( \Sigma M, \Sigma M \) and \( \Sigma N \) respectively. Let \( A^\mu \) be the shape operator of \( M \) in \( \bar{M} \) with respect to the normal vector field \( \mu \). \( B \) be the second fundamental form of \( M \) in \( \bar{M} \) and \( H \) be the normalized mean curvature vector of \( M \) in \( \bar{M} \). If \( M \) is a hypersurface of \( \bar{M} \), we denote \( A \) be the shape operator of \( M \) in \( \bar{M} \) with respect to the unit outward normal vector field. Finally, denote \( R(\iota) \) be the normalized trace of the ambient sectional curvature on the tangent space, i.e.,

\[ R(\iota) = \frac{1}{m(m-1)} \sum_{i,j=1}^m \bar{R}(e_i, e_j, e_i, e_j), \]

where \( \{e_i\} \) is a local orthonormal frame of \( TM \).

Bär [3] derived upper eigenvalue estimates for Dirac operators of closed hypersurfaces in real space forms. According to [3], we know that

\[ \Sigma M|_{\Sigma M} = \begin{cases} \Sigma M \otimes \Sigma N, & mn = 0 \mod 2 \\ (\Sigma M \otimes \Sigma N) \oplus (\Sigma M \otimes \Sigma N), & mn = 1 \mod 2. \end{cases} \]

By \( D^N \) we mean the Dirac operator on \( M \) twisted with the bundle \( \Sigma N \).

A spinor \( \psi \) on \( \bar{M} \) is called a Killing spinor with Killing constant \( \alpha \in \mathbb{C} \) if

\[ \nabla^\bar{M}_X \psi + \alpha \bar{\psi}(X) \psi = 0. \]

Bär [3] proved that if \( \bar{M} \) admits a nontrivial Killing spinor with constant \( \alpha \in \mathbb{R} \), then the first eigenvalue \( \lambda_1(D^\Sigma N) \) of \( D^\Sigma N \) (in the sense all other eigenvalue \( \lambda \) of \( D^\Sigma N \) satisfying \( |\lambda| \geq |\lambda_1| \)) satisfies the following estimate

\[ \lambda_1^2(D^\Sigma N) \leq m^2 |\alpha|^2 + \frac{m^2}{4 \text{vol}(M)} \int_M |H|^2. \]

If \( \alpha \in \sqrt{-1} \mathbb{R} \), he obtained the following estimate

\[ \left| \lambda_1(D^\Sigma N) \right| \leq m \left( |\alpha| + \frac{1}{2} \|H\|_{L^\infty(M)} \right). \]

Especially, if \( \bar{M} \) is the Euclidean space \( \mathbb{R}^{m+n} \), then

\[ \lambda_1^2(D^\Sigma N) \leq \frac{m^2}{4 \text{vol}(M)} \int_M |H|^2. \]

If \( \bar{M} \) is the unit sphere \( S^{m+n}(1) \), then

\[ \lambda_1^2(D^\Sigma N) \leq \frac{m^2}{4 \text{vol}(M)} \int_M (|H|^2 + 1). \]

Finally, if \( \bar{M} \) is the hyperbolic space \( \mathbb{H}^{m+n}(-1) \), then

\[ \left| \lambda_1(D^\Sigma N) \right| \leq \frac{m}{2} \left( 1 + \|H\|_{L^\infty(M)} \right). \]

When \( \bar{M} \) is the hyperbolic space \( \mathbb{H}^{m+1} \), the result has been improved by Ginoux (cf. [12]). It was proved that

\[ \left| \lambda_1(D^\Sigma N) \right| \leq \frac{m}{2} \left( \|H\|_{L^\infty(M)} - 1 \right). \]
Suppose \( n = \psi \) which is independent of \( \Sigma \). Raulot introduced in [16] a differential operator \( L_\phi \) acting on smooth functions on \( M \) by
\[
L_\phi f := -\Delta f - 2(\nabla \ln |\phi|, \nabla f) + \frac{m^2}{4} \left(|H|^2 + R(\iota)\right)f, \quad f \in C^\infty(M),
\]
where \( R(\iota) := \frac{1}{m(m-1)} \left( \tilde{S} - 2\tilde{Ric}(\nu, \nu) \right) \). \( \tilde{S}, \tilde{Ric} \) and \( \nu \) are the scalar curvature, the Ricci curvature of \( \tilde{M} \) and the unit outward norm vector field of \( M \) in \( \tilde{M} \) respectively. It was proved in [14] that if \( \tilde{M} \) admits a nontrivial twistor-spinor \( \phi \) with no zero on \( M \), then
\[
\lambda_1^2(D) \leq \lambda_1(L_\phi).
\]
Notice that if \( \phi \) is a Killing spinor, then
\[
L_\phi = -\Delta + \frac{m^2}{4} \left(|H|^2 + R(\iota)\right) = -\Delta + \frac{m}{4(m-1)} \left(S_M + |\tilde{A}|^2\right)
\]
which is independent of \( \phi \). Here \( \tilde{A} \) is the traceless part of \( A \).

For lower bounds estimates of submanifold Dirac operators, Hijazi and Zhang in [19, 20] proved that for \( D_H \psi = \lambda_H \psi, \psi \in \Gamma(\Sigma \tilde{M})_{|M} \), it holds:
\[
\lambda_H^2 \geq \frac{1}{4} \sup_a \inf_{M_\varphi} \frac{1}{1 + ma^2 - 2a} \left(S_M + R_{\perp,\varphi} - \frac{(m - 1)m^2 |H|^2}{(1 - ma)^2}\right),
\]
where \( a \) is some real function on \( M \), \( M_\varphi = \{ x \in M | \varphi(x) \neq 0 \} \), and
\[
R_{\perp,\varphi} = -\frac{1}{2} \left( \sum_{i,j,a,b} \tilde{R}_{ijab} e^i \cdot e^j \cdot e^a \cdot e^b \cdot \varphi, \varphi \right)_{|\psi|^2}.
\]
Under some extra conditions on the extrinsic curvature, they also obtained some lower bound in terms of the Yamabe constant, the curvature and volume of \( M \) (see [20] for details).

In this paper, we will prove lower and upper bound estimates for submanifold Dirac operators in terms of conformal and extrinsic quantities. Firstly, we have the following lower bound estimate:

**Theorem 1.1.** Let \( M^m \) be a closed oriented submanifold isometrically immersed in a Riemannian spin manifold \( \tilde{M}^{m+m} \). Suppose \( n = 1 \) or \( \tilde{M} \) is locally conformally flat. Then the eigenvalue \( \lambda \) of the Dirac operator \( D^\Sigma \) of the twisted bundle \( \Sigma M \otimes \Sigma N \) satisfies
\[
\lambda^2 \geq \left( \frac{4(1 - g_M)}{\text{area}(M)} - \frac{(n - 1) \int_M |\tilde{A}|^2}{2 \text{area}(M)} \right), \quad m = 2,
\]
\[
\lambda^2 \geq \left( \frac{4m - 1}{m - 2} \right) \lambda_1(L), \quad m > 2.
\]

Here \( \lambda_1(L) \) (if \( m > 2 \)) is the first eigenvalue of the operator \( L \) defined by
\[
L = -\frac{4(m - 1)}{m - 2} \Delta + S_M - (n - 1) |\tilde{A}|^2.
\]

Moreover, the equality implies that the Ricci curvature of \( M \) satisfies
\[
Ric = (n - 1) \sum_{a=1}^{n} (\tilde{A}^a)^2 + \frac{4(m - 1)\lambda^2}{m}. \quad g.
\]

**Remark 1.1.**
- When \( m = 2 \),
\[
\int_M |\tilde{A}|^2
\]
is invariant under the conformal change of the metric \( \tilde{g} \). The equality implies that \( g_M = 0 \) or \( g_M = 1 \) and \( \tilde{A} = 0 \), i.e., \( M \) is a 2-sphere or a totally umbilici 2-torus.
• If $m > 2$, the operator is conformally invariant in the following sense. If $\tilde{g}' = u^{4/(m-2)}\tilde{g}$ is a metric conformal to $\tilde{g}$, and $L'$ is similarly defined with respect to the metric $\tilde{g}'$, then

$$L'(u^{-1}f) = u^{-(m+2)/(m-2)}Lf.$$  

• If $m = n = 2$, then the first nonzero eigenvalue $\lambda$ of $D^{2\Sigma}$ satisfies

$$\lambda^2 \geq \frac{4\pi(1 - g_M) + 2\pi |\chi(N)|}{\text{area}(M)}.$$  

For a Dirac operator $D$, let $\lambda_i$ be the eigenvalues. We recall the conformal eigenvalue $\sigma_i(D)$ of $D$ (cf. [1]) given by

$$\sigma_i(D) = \inf_{[\tilde{g}] \in [g]} |\lambda_i(\tilde{g})| \text{vol}^{1/m}_{M_{\tilde{g}}}.$$  

Here $[g]$ stands for the conformal class of $g$. Similarly, for a second positive self adjoint elliptic operator $L$, we have the conformal eigenvalue $\lambda_i(L)$ of $L$ by

$$\sigma_i(L) = \inf_{[\tilde{g}] \in [g]} |\lambda_i(\tilde{g})| \text{vol}^{2/m}_{M_{\tilde{g}}}.$$  

Now Theorem 1.1 implies that

$$\sigma_1^2(D^{2\Sigma}) \geq \begin{cases} \frac{4\pi(1 - g_M) - \frac{n-1}{2} \int_M |\tilde{A}|^2}{m}, & m = 2, \\ \frac{m}{4(m-1)} \sigma_1(L), & m > 2. \end{cases}$$  

We say that $\psi$ is a twistor spinor on $\tilde{M}$ if

$$\nabla^\Sigma_M \psi + \frac{1}{m + n} \tilde{g}(X)D\psi = 0, \quad \forall X \in T\tilde{M}.$$  

By definition, we know that each Killing spinor is a twistor spinor. For the upper bound of the Dirac operator $D^{2\Sigma}$, we will prove the following

**Theorem 1.2.** Let $M, \tilde{M}$ be as in Theorem 1.1. Suppose $\tilde{M}$ admits a nontrivial twistor spinor, then there are at least $\mu$ conformal eigenvalues $\sigma_i$ of the Dirac operator $D^{2\Sigma}$ of the twisted bundle $\Sigma M \otimes \Sigma N$ such that

• If $m = 2$,

$$\sigma_1^2 \leq 4\pi(1 - g_M) + \frac{1}{2} \int_M |\tilde{A}|^2.$$  

• If $m \geq 3$,

$$\sigma_1^2 \leq \frac{m}{4(m-1)} \sigma_1 \left( L_M + |\tilde{A}|^2 \right) = \frac{m}{4(m-1)} \inf_{\phi \neq 0} \frac{\int_M \phi \left( L_M + |\tilde{A}|^2 \right) \phi}{\left( \int_M \phi^2 \text{vol}^{2m/(m-2)}_{(\text{area})^{(m-2)/m}} \right)}.$$  

Where $\mu = \dim \Sigma$ (twistor spinors on $\tilde{M}$) and $L_M = -\frac{4(m-1)}{m-2} \Delta + S_M$ is the Yamabe operator of $M$.

**2. Preliminaries**

We first compare the Dirac operator on $\tilde{M}$ with the one on $M$. We will use notations in [3]. We also refer the reader to [6, 15, 17, 18, 19, 20] and the references therein. Basic facts concerning Clifford algebras and spinor representations can be found in classical books [4, 23].
2.1. Algebra preliminaries

Let \( E \) be an oriented Euclidean vector space. If \( \dim E = m \) is even, then the the complex Clifford algebra of \( E \), denoted by \( \mathcal{C}(E) \), has precisely one irreducible module, the spinor module \( \Sigma E \) with dimension \( 2^{m/2} \). When restricted to the even subalgebra \( \mathcal{C}^0(E) \) the spinor module decomposes into even and odd half-spinors \( \Sigma E = \Sigma^+ E \oplus \Sigma^- E \) associated the eigenspaces of the complex volume element \( \omega_E = \sqrt{-1}^{m/2} \gamma_E(e_1 \ldots e_m) \). On \( \Sigma E \) it acts as \( \pm 1 \). Here \( \{e_i\} \) stand for a positively oriented orthonormal frame of \( E \) and \( \gamma_E : \mathcal{C}(E) \to \mathrm{End}(E) \) stands for the Clifford multiplication.

If \( m \) is odd there are exactly two irreducible modules, \( \Sigma^0 E \) and \( \Sigma^1 E \), again called spinor modules. In this case \( \dim \Sigma^0 E = \dim \Sigma^1 E = 2^{(m-1)/2} \). Also the two modules \( \Sigma^0 E \) and \( \Sigma^1 E \) can be distinguished by the action of the complex volume element \( \omega_E = \sqrt{-1}^{(m+1)/2} \gamma_E(e_1 \cdots e_m) \). On \( \Sigma E \) it acts as \( (-1)^j, j = 0, 1 \). There exists a vector space isomorphism \( \Phi : \Sigma^0 E \to \Sigma^1 E \) such that \( \Phi \circ \gamma_{E,0} = -\gamma_{E,1} \circ \Phi \), where \( \gamma_{E,j} : \mathcal{C}^j(E) \to \mathrm{End} \Sigma E \) stand for the Clifford multiplication, \( j = 0, 1 \).

Let \( E \) and \( F \) be two oriented Euclidean vector spaces. Let \( \dim E = m \) and \( \dim F = n \). We will construct the spinor module of \( E \oplus F \) from those of \( E \) and \( F \).

Case 1. \( m \) and \( n \) are both even.

Put \( \Sigma := \Sigma E \otimes \Sigma F \) and define
\[
\gamma : E \oplus F \to \mathrm{End} \Sigma,
\]
\[
\gamma(X \oplus Y)(\sigma \otimes \tau) = (\gamma_E(X)\sigma) \otimes \tau + (-1)^{\deg \sigma} \sigma \otimes (\gamma_F(Y)\tau).
\]

Here
\[
\deg \sigma = \begin{cases} 0, & \sigma \in \Sigma^+ E; \\ 1, & \sigma \in \Sigma^- E. \end{cases}
\]

In this case
\[
\Sigma^+ (E \oplus F) = (\Sigma^+ E \otimes \Sigma^+ F) \oplus (\Sigma^- E \otimes \Sigma^- F),
\]
\[
\Sigma^- (E \oplus F) = (\Sigma^+ E \otimes \Sigma^- F) \oplus (\Sigma^- E \otimes \Sigma^+ F).
\]

Case 2. \( m \) is even and \( n \) is odd.

Put \( \Sigma^j := \Sigma E \otimes \Sigma^j F \) for \( j = 0, 1 \). As similar to Case 1, we can define \( \gamma_j : E \oplus F \to \mathrm{End} \Sigma^j \) with obvious modification.

Case 3. \( m \) is odd and \( n \) is even.

This case is symmetric to the second one. Put \( \Sigma^j := \Sigma^j E \otimes \Sigma F \) and define
\[
\gamma_j : E \oplus F \to \mathrm{End} \Sigma^j,
\]
\[
\gamma_j(X \oplus Y)(\sigma \otimes \tau) = (-1)^{\deg \sigma} (\gamma_E(X)\sigma) \otimes \tau + \sigma \otimes (\gamma_F(Y)\tau).
\]

Case 4. \( m \) and \( n \) are both odd.

Set
\[
\Sigma^+ := \Sigma^0 E \otimes \Sigma^0 F,
\]
\[
\Sigma^- := \Sigma^0 E \otimes \Sigma^1 F,
\]
\[
\Sigma := \Sigma^+ \oplus \Sigma^-.
\]

Recall that there exists a vector space isomorphism \( \Phi : \Sigma^0 F \to \Sigma^1 F \) such that \( \Phi \circ \gamma_{F,0} = -\gamma_{F,1} \circ \Phi \). With respect to the splitting \( \Sigma = \Sigma^+ \oplus \Sigma^- \), we define
\[
\gamma : E \oplus F \to \mathrm{End} \Sigma,
\]
\[
\gamma(X \oplus Y) = \begin{pmatrix} 0 & \sqrt{-1} \gamma_{E,0}(X) \otimes \Phi^{-1} + \Phi \otimes \Phi \circ \gamma_{F,1}(Y) \\ -\sqrt{-1} \gamma_{E,0}(X) \otimes \Phi - \Phi \otimes \Phi^{-1} \circ \gamma_{F,0}(Y) & 0 \end{pmatrix}.
\]
2.2. Geometric preliminaries

With respect to the orthogonal splitting $TM|_M = TM \oplus N$, the Gauss formula says

$$\bar{\nabla}_X = \begin{pmatrix} \nabla_X & -B(X, \cdot)^* \\ B(X, \cdot) & \bar{\nabla}_X^\perp \end{pmatrix}.$$  

The following equations are well known, i.e., Gauss equations, Codazzi equations and Ricci equations (cf. [25]). For all $X, Y, Z \in TM, \mu \in N$,

$$\bar{\nabla}(X, Y)Z = R(X, Y)Z + \langle B(Y, Z), B(X, \cdot) \rangle - \langle B(Y, Z), B(X, \cdot) \rangle + \langle \nabla_X B(Y, Z) - (\nabla_Y B)(X, Z), \rangle - \langle \nabla_Y B(X, Z), \cdot \rangle.$$

$$\bar{\nabla}(X, Y)\mu = (\nabla_Y A)^\mu(X) - (\nabla_X A)^\mu(Y) + R^\mu(X, Y)\mu + \langle B(A^\mu(X), Y), \cdot \rangle - \langle B(A^\mu(Y), X), \cdot \rangle.$$

Using a standard formula (cf. [23]), we have

$$\nabla^{\Sigma M}_{\Sigma N} = \nabla^{\Sigma M}_X \otimes \text{Id} \otimes \nabla_X^{\Sigma N} + \frac{1}{2} \sum_{\alpha=1}^n \bar{\gamma}(\nabla^{\Sigma M}_X \cdot v_{\alpha}),$$

$$R^{\Sigma M}_{\Sigma N}(X, Y) = R^{\Sigma M}(X, Y) \otimes \text{Id} \otimes \nabla_X^{\Sigma N} + \frac{1}{4} \sum_{\alpha=1}^n \gamma([A^\alpha(X), A^\alpha(Y)]) \otimes \text{Id}$$

$$+ \frac{1}{4} \sum_{\alpha, \beta=1}^n \left( \left[ A^\alpha(X), A^\beta(Y) \right] - \left[ A^\alpha(Y), A^\beta(X) \right] \right) \text{Id} \otimes \gamma^{\perp}(v_{\alpha} \cdot v_{\beta}),$$

$$+ \frac{1}{2} \sum_{\alpha=1}^n \bar{\gamma}(\langle (\nabla_X A)^\alpha(Y) - (\nabla_Y A)^\alpha(X) \rangle \cdot v_{\alpha}).$$

Here $\{v_{\alpha}\}$ is a local orthonormal frame of the normal bundle $N$.

Define

$$\tilde{D} := \sum_{i=1}^{m} \bar{\gamma}(e_i)\nabla^{\Sigma M \otimes \Sigma N}_{e_i}.$$  

Then (cf. [3])

$$\tilde{D}^2 = \begin{pmatrix} (D^{\Sigma N})^2, & mn = 0 \mod 2 \\ (D^{\Sigma N} \oplus (-D^{\Sigma N}))^2, & mn = 1 \mod 2. \end{pmatrix}$$

Recall the Bochner formula (cf., [22, 23]),

$$\left( D^{\Sigma N} \right)^2 = \left( \nabla^{\Sigma M \otimes \Sigma N} \right)^* \nabla^{\Sigma M \otimes \Sigma N} + R^{\Sigma N},$$

where

$$R^{\Sigma N} = \frac{1}{2} \bar{\gamma}(e_i \cdot e_j)R^{\Sigma M \otimes \Sigma N}(e_i, e_j).$$

Recall the curvature decomposition of $\bar{\nabla}$. Denoted $P$ by the Schouten tensor which is defined by

$$P_{AB} := \frac{1}{n + m - 2} \left( Ric_{AB} - \frac{\delta}{2(n + m - 1)} \bar{g}_{AB} \right), \quad 1 \leq A, B \leq n + m,$$

the Weyl tensor $\bar{W}$ is given by

$$\bar{W}_{ABCD} := R_{ABCD} - \left( P_{AC} \bar{g}_{BD} + P_{BD} \bar{g}_{AC} - P_{AD} \bar{g}_{BC} - P_{BC} \bar{g}_{AD} \right).$$

Therefore, for every orthonormal 4-frame $\{e_A, e_B, e_C, e_D\}$, we have

$$\bar{W}_{ABCD} = R_{ABCD}.$$
Lemma 2.1.

\[ R^{\Sigma N} = \frac{m(m-1)}{4} \left( R(\ell) + |H|^2 \right) + \frac{1}{4} \sum_{i=1}^{m} \left( \sum_{a=1}^{n} \bar{\gamma}(A^a(e_i) \cdot \nu_a) \right)^2 - \frac{1}{8} \bar{\mathcal{W}}_{i,j} \bar{\gamma}(e_1 \cdot e_j \cdot \nu_a \cdot \nu_{\bar{g}}). \]  

(2.1)

Proof. A standard computation (cf. [23]) gives a formula

\[ R^{\Sigma N} = \frac{1}{8} \left( R(e_i, e_j) e_k e_l + \frac{1}{8} \left( R^2(e_i, e_j) \nu_a \cdot \nu_{\bar{g}} \right) \right) \bar{\gamma}(e_1 \cdot e_j \cdot \nu_a \cdot \nu_{\bar{g}}). \]  

(2.2)

The first term is

\[ \frac{S M}{4} = \frac{1}{4} \left( \sum_{i,j=1}^{m} \bar{R}(e_i, e_j, e_j, e_j) + m(m-1) |H|^2 - |\bar{A}|^2 \right). \]  

(2.3)

According to the Codazzi equation, we compute the second term as follows,

\[ \frac{1}{8} \left( R^2(e_i, e_j) \nu_a, \nu_{\bar{g}} \right) \bar{\gamma}(e_1 \cdot e_j \cdot \nu_a \cdot \nu_{\bar{g}}) \]

\[ = \frac{1}{8} \left( \left( R(e_i, e_j) \nu_a, \nu_{\bar{g}} \right) + \left( A^a(e_i), A^\beta(e_j) \right) - \left( A^a(e_i), A^\beta(e_j) \right) \right) \bar{\gamma}(e_1 \cdot e_j \cdot \nu_a \cdot \nu_{\bar{g}}) \]

\[ = \frac{1}{8} \left( \bar{W}(e_i, e_j) \nu_a, \nu_{\bar{g}} \right) \bar{\gamma}(e_1 \cdot e_j \cdot \nu_a \cdot \nu_{\bar{g}}) + \frac{1}{8} \left( \left( A^a(e_i), A^\beta(e_j) \right) - \left( A^a(e_i), A^\beta(e_j) \right) \right) \bar{\gamma}(e_1 \cdot e_j \cdot \nu_a \cdot \nu_{\bar{g}}) \]

\[ = \frac{1}{4} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{\gamma} \left( A^a(e_i) \cdot \nu_a \cdot A^\beta(e_j) \cdot \nu_{\bar{g}} \right) + |\bar{A}|^2 \bar{\gamma}(e_1 \cdot e_j \cdot \nu_a \cdot \nu_{\bar{g}}) \right) + \frac{1}{8} \left( \bar{W}(e_i, e_j) \nu_a, \nu_{\bar{g}} \right) \bar{\gamma}(e_1 \cdot e_j \cdot \nu_a \cdot \nu_{\bar{g}}), \]

where we used the fact

\[ \bar{W}_{i,j} = \bar{R}_{i,j}. \quad \forall i \neq j, \alpha \neq \beta. \]

Thus, the second term is

\[ \frac{1}{4} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{\gamma} \left( A^a(e_i) \cdot \nu_a \cdot A^\beta(e_j) \cdot \nu_{\bar{g}} \right) + |\bar{A}|^2 \right) - \frac{1}{8} \bar{W}_{i,j} \bar{\gamma}(e_1 \cdot e_j \cdot \nu_a \cdot \nu_{\bar{g}}). \]  

(2.4)

Now (2.1) follows from (2.2), (2.3) and (2.4). \qed

Remark 2.1. 1. If \( n = 1 \),

\[ R^{\Sigma N} = \frac{1}{4} S_M = \frac{m(m-1)}{4} \left( R(\ell) + |H|^2 \right) - \frac{1}{4} |\bar{A}|^2. \]

2. If \( m = 2, n = 2 \),

\[ R^{\Sigma N} \big|_{\Sigma^2} = \frac{1}{2} \kappa_M \pm \frac{1}{2} \kappa_N = \frac{1}{2} \left( R(e_1, e_2, e_1, e_2) + |H|^2 \right) - \frac{1}{4} |\bar{A}|^2 \pm \frac{1}{2} \kappa_N. \]

\[ - \frac{1}{4} \sum_{i=1}^{m} \left( \sum_{a=1}^{n} \bar{\gamma}(A^a(e_i) \cdot \nu_a) \right)^2 \bigg|_{\Sigma^2} = \frac{1}{4} |\bar{A}|^2 \pm \frac{1}{2} \left( \kappa_N - R(e_1, e_2, v_1, v_2) \right). \]

Here

\[ \kappa_N = \left( R^{\Sigma}(e_1, e_2) v_2, v_1 \right). \]

A direct consequence is

\[ \int_M |\bar{A}|^2 \geq 2 \left| 2 \pi \kappa(N) - \int_M \bar{R}(e_1, e_2, v_1, v_2) \right|. \]

Therefore,

\[ \chi(M) + \left| \kappa(N) - \frac{1}{2 \pi} \int_M \bar{R}(e_1, e_2, v_1, v_2) \right| \leq \frac{1}{2 \pi} \left( \int_M \bar{R}(e_1, e_2, e_1, e_2) + |H|^2 \right). \]

In particular, if \( \bar{M} \) is flat and \( M \) is minimal (cf. [21]), then

\[ \chi(M) + |\kappa(N)| \leq 0. \]
Hence, according to the Gauss equation, moreover, when restricted to the boundary, we have

\[ \chi(M) + \left| \chi(N) - \frac{1}{2\pi} \int_M \bar{R}(e_1, e_2, v_1, v_2) \right| \leq \frac{1}{2\pi} \left( \int_M \bar{R}(e_1, e_2, e_1, e_2) + |H|^2 \right). \]

\[ \square \]

### 2.3. Conformal transformation

Consider a conformal change \( \tilde{g}^\prime = e^{2\varphi} \tilde{g} \) of \( \tilde{M} \), then there is an isometric between \( \Sigma \tilde{M} \) and \( \Sigma \tilde{M}^\prime \), \( \psi \mapsto \psi^\prime \), with

\[ \nabla_{\tilde{\chi}} \psi^\prime = \left( \nabla_{\tilde{\chi}} \psi - \frac{1}{2} \tilde{g}(X \cdot \nabla u) \psi - \frac{1}{2} \tilde{X}(u) \psi \right)^\prime. \]

Moreover, when restricted to the boundary, we have

\[ \nabla_{\tilde{\chi}}^\prime \sigma^\prime = \left( \nabla_{\tilde{\chi}} \sigma - \frac{1}{2} \tilde{g}(X \cdot \nabla u) \sigma - \frac{1}{2} \tilde{X}(u) \sigma \right)^\prime, \quad (2.5) \]

\[ \nabla_{\tilde{\chi}^\prime}^\prime \lambda^\prime = \left( \nabla_{\tilde{\chi}^\prime} \lambda^\prime \right)^\prime. \quad (2.6) \]
Theorem 1.1

\[ \langle \nabla_X Y, v \rangle = \langle \nabla_X v, Y \rangle, \quad \forall X \in TM. \]

In particular,

\[ \omega^\perp_{\alpha \beta}(X) = \tilde{\omega}_{\alpha \beta}(X). \]

Here \( \omega^\perp \) and \( \tilde{\omega} \) are the connection 1-forms on the normal bundle \( N \) and the target manifold \( \bar{M} \) respectively. Since we have the transformation formula between connection 1-forms

\[ \tilde{\omega}^\perp_{\alpha \beta}(X) = \omega_{\alpha \beta}(X) + e_A(u) \langle X, e_B \rangle - e_B(u) \langle X, e_A \rangle. \]

Hence,

\[ \omega^\perp_{\alpha \beta}(X) = \tilde{\omega}_{\alpha \beta}(X) = \omega^\perp_{\alpha \beta}(X). \]

Now according to definition of the connection on \( N \), we get (cf. [23])

\[ \nabla^\perp_X \tau' = (\nabla_X \tau)'. \]

Now we can prove the following

Lemma 2.2. The Dirac operator on the twisted bundle \( \Sigma M \otimes \Sigma N \) is conformal invariant, i.e., for every \( \psi \in \Gamma(\Sigma M \otimes \Sigma N) \)

\[ D^{\Sigma N} \left( e^{-(m-1)u/2} \psi' \right) = e^{-(m+1)u/2} \left( D^{\Sigma N} \psi \right)'. \]

Proof. Without loss generality, set \( \psi = \sigma \otimes \tau \), then according to (2.5), we have

\[ D' \left( e^{-(m-1)u/2} \sigma \right) = e^{-(m+1)u/2} \left( D \sigma \right)' \]

Hence by using (2.6), we get

\[
\begin{align*}
D^{\Sigma N} \left( e^{-(m-1)u/2} \sigma \otimes \tau \right)' &= D^{\Sigma N} \left( e^{-(m-1)u/2} \sigma' \otimes \tau' \right) \\
&= D^{\Sigma N} \left( e^{-(m-1)u/2} \sigma' \right) \otimes \tau' + \gamma'(e') e^{-(m-1)u/2} \sigma' \otimes \nabla^+_\gamma \tau' \\
&= e^{-(m+1)u/2} \left( D \sigma \otimes \tau + \gamma(e) \sigma \otimes \nabla^+_\gamma \tau \right) \\
&= e^{-(m+1)u/2} \left( D^{\Sigma N} \left( \sigma \otimes \tau \right) \right)' .
\end{align*}
\]

\[ \square \]

3. Lower Bound Estimate

In this section, we will give a conformal lower bound of the first eigenvalue of the Dirac operator on the twisted bundle \( \Sigma M \otimes \Sigma N \), i.e., we will give a proof of Theorem 1.1.

Proof of Theorem 1.1. For every smooth function \( f \), we have the following weighted Bochner formula (cf. [7])

\[
\frac{m-1}{m} \int_M \exp(f) |D^{\Sigma N} \psi|^2 = \int_M \exp(f) \left( \frac{m-1}{2} \Delta f - \frac{(m-1)(m-2)}{4} |\nabla f|^2 + R^{\Sigma N}_\psi \right) |\psi|^2 + \int_M \exp((1-m)f) \left| p^{\Sigma N} \left( \exp \left( \frac{m}{2} f \right) \psi \right) \right|^2 , \tag{3.1}
\]

where

\[ R^{\Sigma N}_\psi |\psi|^2 = (R^{\Sigma N} \psi, \psi) . \]
and $P^{\Sigma N}$ is the twistor operator defined by

$$P^{\Sigma N}_X \psi := \nabla^{\Sigma M \Sigma N}_X + \frac{1}{m} \hat{\gamma}(X) D^{\Sigma N} \psi.$$ 

First, we estimate the curvature term $\mathcal{R}^{\Sigma N}_\psi$. According to the proof of Lemma 2.1, (2.4) implies that

$$\frac{1}{8} \left( R^+(e_i, e_j) v_{a, \alpha} \gamma(e_i \cdot e_j \cdot v_a \cdot v_\beta) \psi, \psi \right) = \frac{1}{4} \left( \sum_{i, a} |\tilde{\gamma}(A^a(e_i))\psi|^2 - \sum_{i, a} |\tilde{\gamma}(\tilde{A}^a(e_i) \cdot v_a)\psi|^2 \right)$$

$$= \frac{1}{4} \left( \sum_{i, a} |\tilde{\gamma}(\tilde{A}^a(e_i))\psi|^2 - \sum_{i, a} |\tilde{\gamma}(\tilde{A}^a(e_i) \cdot v_a)\psi|^2 \right)$$

$$= \frac{1}{4} \left( n \sum_{i, \beta} |\tilde{\gamma}(\tilde{A}^\beta(e_i))\psi - \frac{1}{n} \sum_{i, \alpha} |\tilde{\gamma}(\tilde{A}^\alpha(e_i) \cdot v_a)\psi|^2 \right) - (n - 1) |\tilde{A}|^2 |\psi|^2.$$ 

In particular,

$$\frac{1}{8} \left( R^+(e_i, e_j) v_{a, \alpha} \gamma(e_i \cdot e_j \cdot v_a \cdot v_\beta) \psi, \psi \right) \geq -\frac{1}{8} \tilde{W}_{i j a \beta} \gamma(e_i \cdot e_j \cdot v_a \cdot v_\beta) - \frac{n - 1}{4} |\tilde{A}|^2 |\psi|^2.$$ \hspace{1cm} (3.2)

Insert (3.2) into (2.2) to get

$$\mathcal{R}^{\Sigma N}_\psi \geq \frac{\sum_M - (n - 1) |\tilde{A}|^2}{4} - \frac{\tilde{W}_{i j a \beta} \gamma(e_i \cdot e_j \cdot v_a \cdot v_\beta) \psi, \psi}{8 |\psi|^2}.$$ \hspace{1cm} (3.3)

Suppose $\psi$ is an eigenspinor of $D^{\Sigma N}$ associated with $\lambda$, i.e.,

$$D^{\Sigma N} \psi = \lambda \psi.$$ 

Inserting (3.3) into (3.1), we obtain

$$\frac{m - 1}{m} \int_M e^f |\psi|^2 \geq \int_M e^f \left( \frac{m - 1}{2} \Delta f - \frac{(m - 1)(m - 2)}{4} |\nabla f|^2 + \frac{\sum_M - (n - 1) |\tilde{A}|^2}{4} \right) |\psi|^2.$$ \hspace{1cm} (3.4)

We consider two cases.

Case 1. $m = 2$.

In this case, we choose $f$ as a solution of the following PDE

$$\Delta f + \kappa_M - \frac{n - 1}{2} \tilde{A}^2 = \frac{4\pi(1 - g_M)}{\text{area}(M)} - \frac{(n - 1) \int_M |\tilde{A}|^2}{2 \text{area}(M)}, \quad \int_M f = 0,$$

on $M$. Therefore, according to (3.4), we get

$$\lambda^2 \geq \frac{4\pi(1 - g_M)}{\text{area}(M)} - \frac{(n - 1) \int_M |\tilde{A}|^2}{2 \text{area}(M)}.$$ 

Moreover, if $n = 2$, according to Remark 2.1

$$\mathcal{R}^{\Sigma N}_\psi |_{\Sigma^*} = \mathcal{R}^{\Sigma N}_\psi |_{\Sigma^*} = \frac{1}{2} \kappa_M \pm \frac{1}{2} \kappa_N.$$
A direct computation implies that if $D^E\psi = \lambda\psi$, then $D^E\psi^b = \lambda\psi^b$. Since $\lambda \neq 0$, we get $\psi^b \neq 0$ since $\psi$ is a nontrivial eigenspinor. Using a similar argument mentioned before, one can proved that

$$A^2 \geq \frac{4\pi(1 - g_M) \pm 2\pi\chi(N)}{\text{area}(M)}.$$ 

Therefore,

$$A^2 \geq \frac{4\pi(1 - g_M) + 2\pi\chi(N)}{\text{area}(M)}.$$ 

Here we used two formulae

$$\int_M \kappa_M = 2\pi\chi(M) = 4\pi(1 - g_M),$$

and

$$\int_M \kappa_N = 2\pi\chi(N).$$

Case 2. $m > 2$.

In this case, (3.4) implies that for every positive function $u$,

$$\frac{m - 1}{m} A^2 \int_M u^{4-m/(m-2)}|\psi|^2 \geq \int_M u^{-m/(m-2)} \left( \frac{m - 1}{m - 2} \Delta_M + \frac{S_M - (n - 1)|A|^2}{4} u \right) |\psi|^2. \quad (3.5)$$

Choose $u$ as an eigenfunction of the operator $L$, i.e.,

$$Lu = -\frac{4(m - 1)}{m - 2} \Delta_M u + \left( S_M - (n - 1)|A|^2 \right) u = \lambda_1(L)u.$$ 

Moreover, we can choose $u$ satisfying

$$\int_M u^2 = \text{vol}(M).$$ 

Then the inequality (3.5) implies that

$$A^2 \geq \frac{m}{4(m - 1)} \lambda_1(L).$$ 

Next, we will consider the limit case. If suppose

$$A^2 = \frac{4\pi(1 - g_M)}{\text{area}(M)} - \frac{(n - 1) \int_M |A|^2}{2 \text{area}(M)},$$ 

as $m = 2$ is the case and

$$A^2 = \frac{m}{4(m - 1)} \lambda_1(L),$$ 

as $m > 2$ is the case. Consider a new metric $\tilde{g}' = e^{-2f} \tilde{g}$, then $\tilde{\psi} = e^{(m-1)f/2} \psi' \ (f = \frac{2\log u}{2m} \text{ if } m > 2)$ satisfies

$$\nabla_{e^*}^E \tilde{\psi} + \frac{\lambda e^f}{m} \gamma(e') \tilde{\psi} = 0. \quad (3.6)$$

Consequently, $|\tilde{\psi}|_e \neq 0$ is a constant on $M$. Moreover, the equality in (3.2) gives

$$\tilde{\gamma}(\tilde{A}^\alpha(e) \cdot v_\alpha) \psi = \gamma(\tilde{A}^\alpha(e) \cdot v_\alpha) \psi, \quad \forall i, \alpha, \beta. \quad (3.7)$$

Form (3.6), we get

$$\sum_{i=1}^m \gamma'(e_i) R_{E^M \tilde{g}^2}(e_i, e') \tilde{\psi} = \frac{2(m - 1) \lambda e^f}{2m^2} \gamma'(e') \tilde{\psi} - \frac{\lambda e^f}{m} \gamma'(\nabla' f \cdot e') \tilde{\psi} - \lambda e^f e'(f) \tilde{\psi}.$$
Thus,
\[
\frac{(1 - m)}{m} \lambda e^t e'(f) \left| \dot{\phi}_t \right|^2 = 0.
\]
Therefore, \( f \) is a constant and \( f = 0 \) according to the normalizing condition. As a consequence,
\[
\sum_{i=1}^{m} \hat{\gamma}(e_i) R^{\Sigma M @ \Sigma N}(e_i, e_j) \psi = \frac{2(m - 1)J^2}{m^2} \hat{\gamma}(e_j) \psi.
\]
On the other hand, one can get (cf. [16, 23]),
\[
\sum_{i=1}^{m} \hat{\gamma}(e_i) R^{\Sigma M @ \Sigma N}(e_i, e_j) \psi = \frac{1}{4} \sum_{i,k,l=1}^{m} \left< R(e_i, e_j) e_k, e_l \right> \hat{\gamma}(e_i \cdot e_k \cdot e_l) \psi
\]
\[
+ \frac{1}{4} \sum_{i=1}^{m} \sum_{\alpha, \beta=1}^{n} \left< R^+(e_i, e_j) \gamma_{\alpha}, \gamma_{\beta} \right> \hat{\gamma}(e_i \cdot \gamma_{\alpha} \cdot \gamma_{\beta}) \psi
\]
\[
= \frac{1}{2} \hat{\gamma} \left( Ric(e_j) \right) \psi
\]
\[
- \frac{1}{4} \sum_{i=1}^{m} \sum_{a=1}^{n} \hat{\gamma} \left( \hat{B}(e_j, e_i) \cdot \hat{A}^\alpha(e_i) \cdot \gamma_{\alpha} + \hat{A}^\alpha(e_i) \cdot \gamma_{\alpha} \cdot \hat{B}(e_j, e_i) \right) \psi.
\]
According to (3.7), we get
\[
\sum_{i=1}^{m} \hat{\gamma}(e_i) R^{\Sigma M @ \Sigma N}(e_i, e_j) \psi = \frac{1}{2} \hat{\gamma} \left( Ric(e_j) \right) \psi + \frac{1}{2} \sum_{a=1}^{n} \hat{\gamma} \left( \left( \hat{A}^\alpha(e_j) \right)^2 \psi \right).
\]
Summarize these identities, we get
\[
\frac{1}{2} \hat{\gamma} \left( Ric(e_j) \right) \psi + \frac{1}{2} \sum_{a=1}^{n} \hat{\gamma} \left( \left( \hat{A}^\alpha(e_j) \right)^2 \psi \right) = \frac{2(m - 1)J^2}{m^2} \hat{\gamma}(e_j) \psi.
\]
Since \( \psi \) can not vanish anywhere on \( M \), then (3.8) implies that
\[
Ric = (n - 1) \sum_{a=1}^{n} \left( \hat{A}^\alpha \right)^2 + \frac{4(m - 1)J^2}{m^2} g.
\]
\( \square \)

4. Upper bound estimate

In this section, we want to bound the first conformal eigenvalue of the Dirac operator \( D_{\Sigma N} \) by extrinsic data provided \( M \) admits a twistor spinor \( \psi \), i.e.,
\[
P_{\Sigma N} \psi := \nabla^{\Sigma M}_X \psi + \frac{1}{m + n} \hat{\gamma}(X) D \psi = 0, \quad \forall X \in T M.
\]
When restricted to the boundary, we first prove the following Lemma.

**Lemma 4.1.** For every tangent vector field \( X \in \Gamma(TM) \), we have
\[
P_{\Sigma N}^X \psi = \tilde{P}_X \psi + \frac{1}{m} \sum_{i=1}^{m} \hat{\gamma}(e_i) \tilde{P}_{e_i} \psi - \frac{1}{2} \sum_{a=1}^{n} \hat{\gamma} \left( \hat{A}^\alpha(X) \cdot \gamma_{\alpha} \right) \psi.
\]
Proof. According to the definition of the connections given in the previous sections, we get
\[ \hat{D}\psi = \sum_{j=1}^{m} \tilde{g}(e_j)P_{e_j}\psi + \frac{m}{m+n} \hat{D}\psi + \frac{m}{2} \tilde{g}(H)\psi. \]
Thus (cf. [3]),
\[ P_{\Sigma}\psi := \nabla^{\Sigma}\nabla_{\Sigma}\psi + \frac{1}{m} \tilde{g}(X)D^{\Sigma}\psi = \nabla^{\Sigma}\nabla_{\Sigma}\psi + \frac{1}{m} \tilde{g}(X)\hat{D}\psi = \nabla^{\Sigma}\nabla_{\Sigma}\psi - \frac{1}{2} \sum_{a=1}^{n} \tilde{g}(\check{\alpha}}(X) \cdot \nu_{\alpha})\psi \]
\[ = P_X\psi + \frac{1}{m} \sum_{a=1}^{m} \tilde{g}(e_a)P_{e_a}\psi - \frac{1}{2} \sum_{a=1}^{n} \tilde{g}(\check{\alpha}}(X) \cdot \nu_{\alpha})\psi. \]

Now, we give a proof of Theorem 1.2

Proof of Theorem 1.2. Applying the weighted Bochner formula (3.1), (replacing \( \psi \) by \( f\psi \) and \( u \) by \( u \),
\[ \frac{m-1}{m} \int_{M} e^{\mu} |D^{\Sigma}(f\psi)|^2 = \int_{M} e^{\mu} \left( \frac{m-1}{2} \Delta u - \frac{(m-1)(m-2)}{4} |\nabla u|^2 + \mathcal{R}_{\Sigma}^{\Sigma} \right) |f\psi|^2 + \int_{M} e^{\mu} |\mathcal{P}\psi|^2. \]
In particular, taking \( e^{m/2} = 1 \), we get
\[ \frac{m-1}{m} \int_{M} e^{\mu} |D^{\Sigma}(e^{-m/2}\psi)|^2 = \int_{M} e^{\mu} \left( \frac{m-1}{2} \Delta u - \frac{(m-1)(m-2)}{4} |\nabla u|^2 + \mathcal{R}_{\Sigma}^{\Sigma} \right) |\psi|^2 \]
(4.1)
Now Lemma 2.1 gives
\[ \mathcal{R}_{\Sigma}^{\Sigma} |\psi|^2 = \frac{m(m-1)}{4} \left( R^{(2)} + |H|^2 \right) |\psi|^2 - \frac{1}{4} \sum_{a=1}^{n} \tilde{g}(\check{\alpha}}(e_a) \cdot \nu_{\alpha}) \psi \]
\[ - \frac{1}{8} W_{ij \alpha \beta} \left( \tilde{g}(e_i) \cdot e_j \cdot \nu_{\alpha} \cdot \nu_{\beta} \right) \psi, \]
and Lemma 4.1 gives
\[ |\mathcal{P}\psi|^2 = \frac{1}{4} \sum_{a=1}^{n} \tilde{g}(\check{\alpha}}(e_a) \cdot \nu_{\alpha}) \psi \]
if \( \psi \) is a twistor spinor of \( \Sigma M \). Therefore, (4.1) can be rewritten as follows:
\[ \frac{m-1}{m} \int_{M} e^{\mu} |D^{\Sigma}(e^{-m/2}\psi)|^2 \]
\[ = \int_{M} e^{\mu} \left( \frac{m-1}{2} \Delta u - \frac{(m-1)(m-2)}{4} |\nabla u|^2 + \frac{m(m-1)}{4} \left( R^{(2)} + |H|^2 \right) \right) |\psi|^2 \]
\[ - \int_{M} \frac{1}{8} e^{\mu} W_{ij \alpha \beta} \left( \tilde{g}(e_i) \cdot e_j \cdot \nu_{\alpha} \cdot \nu_{\beta} \right) \psi, \psi. \]
Since \( \psi \) is a nontrivial twistor spinor on \( \bar{M} \), we know that the zeros of \( \psi \) is isolated ([10]). In particular, \( \psi \) is nontrivial on \( M \). Considering a conformal change of the metric \( \bar{g}' = e^{-2u} \bar{g} \), we get

\[
\frac{m - 1}{m} \int_M |D^N(e^{-u/2} \psi')]^2_{\bar{g}'} = \frac{\int_M e^{(1-m)u} \left( \frac{m - 1}{2} \Delta u - \frac{(m-1)(m-2)}{4} |\nabla u|^2 + \frac{m(m-1)}{4} \left( R(\tau) + |H|^2 \right) \right) |\psi'|^2}{\int_M e^{-1+m}u |\phi|^2}.
\]

By assumption, \( n = 1 \) or \( \bar{M} \) is locally conformally flat, we obtain that the second term of the above equation is zero. We consider two cases

**U1** \( m = 2 \). We get

\[
\frac{\int_M |D^N(e^{-u/2} \psi')]^2_{\bar{g}'} = \frac{\int_M e^{-u} \left( \Delta u + \left( R(\tau) + |H|^2 \right) \right) |\psi'|^2}{\int_M e^{-2u} |\phi|^2}.
\]

We consider the following Liouville-type equations

\[
\Delta u_j + \kappa_j + \frac{1}{2} |\bar{A}|^2 + \epsilon_j = \mu_j e^{-2u_j}, \quad \int_M e^{-2u_j} = 1.
\]

Here \( \{\epsilon_j\} \) is some sequence consists of positive numbers such that \( \lim_{j \to \infty} \epsilon_j = 0 \) and \( \mu_j \) is constant for each \( j \). For the existence of \( \epsilon_j \), we refer the reader to Chen-Lin’s paper [5] for genus \( g_M \geq 1 \) and Djadli’s paper [8] for arbitrary genus. Then

\[
\lim_{j \to \infty} \mu_j = 4\pi(1 - g_M) + \frac{1}{2} \int_M |\bar{A}|^2.
\]

Thus the first conformal eigenvalue of \( D^N \) satisfies

\[
\lambda_1^2 = \inf \lambda^2 \text{area}(M) \leq \lim_{j \to \infty} \mu_j = 4\pi(1 - g_M) + \frac{1}{2} \int_M |\bar{A}|^2.
\]

**U2** \( m > 2 \).

In this case, let \( e^u = \phi^{2/(2-m)} \), where \( \phi \) is a positive function. Then a direct computation implies that

\[
\frac{\int_M |D^N(e^{-u/2} \psi')]^2_{\bar{g}'} = \frac{\int_M \left( -\frac{m}{m-2} \Delta \phi + \frac{m^2}{4} \left( |H|^2 + R(\tau) \right) \right) \phi^{m/(m-2)} |\psi'|^2}{\int_M \phi^{2(m+1)/(m-2)} |\phi|^2}.
\]

We consider the following nonlinear equations

\[
-\frac{4(m-1)}{m-2} \Delta \phi_j + m(m-1) \left( |H|^2 + R(\tau) \right) \phi_j = \tau_j \phi^{p_j-1},
\]

or equivalently

\[
\left( L_M + |\bar{A}|^2 \right) \phi_j = -\frac{4(m-1)}{m-2} \Delta \phi_j + \left( S_M + |\bar{A}|^2 \right) \phi_j = \tau_j \phi^{p_j-1},
\]

where \( L_M = \Delta + \left( m - 1 \right) \left( |H|^2 + R(\tau) \right) \), \( S_M = m(m-1) \left( |H|^2 + R(\tau) \right) \), and \( p_j \) is the optimal power for the \( j \)-th eigenvalue.

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where
\[
\tau_j = \inf_{\phi > 0} \frac{\int_M \phi \left( L_M + |\bar{A}|^2 \right) \phi}{\left( \int_M \phi^p \right)^{1/(2p)}},
\]
and \(2 < p_j < 2m/(m-2)\). It is obvious that \(\tau_j \geq 0\).

Choose \(\phi_j > 0\) satisfying
\[
\left( L_M + |\bar{A}|^2 \right) \phi_j = \tau_j \phi_j^{p_j - 1}, \quad \int_M \phi_j^{p_j} = 1.
\]

By using a similar argument to the Yamabe constant (cf. [24]), it can be shown that \(\tau_j \leq \sigma_1 \left( L_M + |\bar{A}|^2 \right)\) and
\[
\lim_{p_j \to 2m/(m-2)} \tau_j = \sigma_1 \left( L_M + |\bar{A}|^2 \right) = \inf_{\phi > 0} \frac{\int_M \phi \left( L_M + |\bar{A}|^2 \right) \phi}{\left( \int_M \phi^2 \right)^{m/(m-2)}}.
\]

Thus, we obtain
\[
\sigma_j^2 = \inf \lambda^2 \operatorname{vol}^{2/m} \leq \frac{m}{4(m-1)} \lim_{p_j \to 2m/(m-2)} \tau_j \geq \frac{m}{4(m-1)} \sigma_1 \left( L_M + |\bar{A}|^2 \right).
\]

\[\square\]

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