On quantum teleportation with beam-splitter-generated entanglement

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Following the lead of Cochrane, Milburn, and Munro [Phys. Rev. A 62, 062307 (2000)], we investigate theoretically quantum teleportation by means of the number-sum and phase-difference variables. We study Fock-state entanglement generated by a beam splitter and show that two-mode Fock-state inputs can be entangled by a beam splitter into close approximations of maximally entangled eigenstates of the phase difference and the photon-number sum (Einstein-Podolsky-Rosen — EPR — states). Such states could be experimentally feasible with on-demand single-photon sources. We show that the teleportation fidelity can reach near unity when such “quasi-EPR” states are used as the quantum channel.

I. INTRODUCTION

Quantum teleportation [1,2], “the disembodied transport of an unknown quantum state from one place to another” [3], is a cornerstone of quantum information. It is of primordial importance for communication between quantum computers, and the realization of quantum gates [4] and quantum error correction [5]. Quantum teleportation is based on maximally entangled states, a purely quantum mechanical feature, initially little noticed outside research on the fundamental issues of quantum theory such as the Einstein-Podolsky-Rosen (EPR) paradox [6]. Producing, preserving, and detecting high quality entanglement is an experimental challenge in making reliable quantum teleporters. Initial experiments, based on discrete and finite Hilbert spaces, have been successful in proving the principle but hindered by low efficiency in the production and detection of entangled photons [7–9]. The use of continuous quantum variables for teleportation [10–13] offers more straightforward detection methods and also the interesting access to an infinite Hilbert space, much richer in possibilities for encoding quantum information. The first continuous-variable teleportation experiment [10,11] used quadrature-squeezed electromagnetic fields and beam-splitter entanglement, and was based on the experimental realization [14,15] of the EPR paradox using continuous quantum optical variables [16,17].

Another set of interesting, in part continuous, variables is constituted by the photon number and the phase, which are canonically conjugate in the same sense as energy and time [18]. The use of these variables has been proposed to realize the EPR paradox [19] and the corresponding maximally entangled states are therefore usable for teleportation by means of the detection of the photon number difference and phase sum [20], or, more practically, of the photon number sum and phase difference [21]. In the latter reference, by Cochrane, Milburn, and Munro (CMM), quantum teleportation using phase-difference eigenstates as the EPR entanglement resource is proposed and studied in detail. At the difference of CMM, we follow, in this article, the definition of Luis and Sánchez-Soto [14] for phase-difference eigenstates, or, rather, relative-phase eigenstates, and also consider the most general ones [22]. Because how to create such a quantum state in the laboratory is not yet known for more than 2 photons, CMM also explored the use of approximate — but much simpler and possibly experimentally realizable using on-demand single-photon sources — EPR states such as created when a twin Fock state such as $|\psi\rangle = |n\rangle_a |m\rangle_b$ passes through a lossless balanced beam splitter. In their work, CMM found that, in this case, the maximum teleportation fidelity of an arbitrary coherent state is 50%, due to the fact that half the quantum amplitudes of this entangled state are null. The gist of this paper is to show that this is not the best one can hope for beam-splitter-generated entanglement, and we show that barely more complicated states can in fact yield near-unity teleportation fidelities.

The outline of this paper is as follows. In Section II we derive and evaluate close-to-maximally entangled, or quasi-EPR, states that can be created by sending two-mode Fock states through a lossless beam splitter, balanced or not. In Section III we use these quasi-EPR states as the entanglement resource in the number-phase teleportation scheme, and analyze the fidelity.

II. GENERATION OF APPROXIMATE EPR PHASE-DIFFERENCE EIGENSTATES BY A BEAM SPLITTER

A. The Schwinger representation

We begin by recalling the definition of the Schwinger representation [23], widely used in quantum optics, of a nondegenerate two-mode field in terms of a fictitious spin. This spin is defined as

$$J = \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a^\dagger b + b^\dagger a \\ -i(a^\dagger b - b^\dagger a) \\ a^\dagger a - b^\dagger b \end{pmatrix}$$  (1)
\[ J^2 = \frac{a^\dagger a + b^\dagger b}{2} \left( \frac{a^\dagger a + b^\dagger b}{2} + 1 \right), \]  

where \( a \) and \( b \) are the photon annihilation operators of each mode. The physical meaning of these operators is the following: \( J^2 \) represents the total photon number, \( J_z \) the photon number difference between the two modes, and \( J_{x,y} \) are the phase-difference, or interference, quadratures. It stems from this that \( e^{i\theta J_z} \) is the relative phase shift operator and \( e^{i\theta J_z} \) is the rotation carried out by a beam splitter (homo-/heterodyne measurement). The eigenstates of the fictitious spin are the two-mode Fock states

\[ |jm\rangle_z = |n_a\rangle_a |n_b\rangle_b, \tag{3} \]

and the respective eigenvalues of \( J^2 \) and \( J_z \) are given by

\[ j = \frac{n_a + n_b}{2} = \frac{N}{2}, \tag{4} \]

\[ m = \frac{n_a - n_b}{2}. \tag{5} \]

The Schwinger representation thus makes use of the homomorphism from \( SU(2) \) onto \( SO(3) \), which allows one to represent any unitary operation on the two-mode field by a rotation. The general \( SU(2) \) transformation

\[ \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \cos \frac{\beta}{2} e^{i(\alpha + \gamma)} & \sin \frac{\beta}{2} e^{i(\alpha - \gamma)} \\ -\sin \frac{\beta}{2} e^{-i(\alpha - \gamma)} & \cos \frac{\beta}{2} e^{-i(\alpha + \gamma)} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, \tag{6} \]

where \( \alpha, \beta, \) and \( \gamma \) are the Euler angles, corresponds to the rotation operator \( e^{-i\alpha J_z} e^{-i\beta J_z} e^{-i\gamma J_z} \) ( \( h = 1 \)). A lossless beam splitter corresponds to the values

\[ \alpha = -\gamma = \pi/2, \quad \beta = \pm 2 \arccos \sqrt{R}, \tag{7} \]

where \( R \) is the reflectivity of the beam splitter (the transmittivity \( T \) is such that \( R + T = 1 \)).

### B. Ideal number-phase EPR states

By definition, a maximally entangled two-mode state, or EPR state, is a two-mode quantum state

\[ |EPR\rangle = \sum_{k,l} s_{kl} |k\rangle_a |l\rangle_b \tag{8} \]

such that any reduced (single-mode) density matrix of this state \( \text{Tr}_{a,b}(|EPR\rangle \langle EPR|) \) is proportional to the identity matrix, which yields

\[ \sum_k s_{kl} s_{kl}^* = \delta_{ll'}, \sum_l s_{kl} s_{k'l}^* = \delta_{kk'}. \tag{9} \]

An example of EPR state is the eigenstate, introduced by Luis and Sánchez-Soto [19], of the operators of the photon-number sum and phase difference of two modes \( a \) and \( b \). Heeding the point made by Trifonov et al. [20], we will call this state a relative-phase eigenstate rather than a phase-difference eigenstate, thus recalling that the formal definition of a two-mode quantum phase difference operator does not coincide with the (problematic) definition of two single-mode quantum phase operators [23].

The relative-phase eigenstate is

\[ |\phi_r^{(N)}\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^N e^{i\phi_r^{(N)} n} |n\rangle_a |N-n\rangle_b, \tag{10} \]

where \( \phi_r^{(N)} = \phi_0 + 2\pi r/(N+1) \) is the phase difference, \( N \) is the total photon number, \( \phi_0 \) an arbitrary phase origin, and \( r \in [0, N] \). The phase difference is adequately defined with resolution \( 1/N \), i.e. at the Heisenberg limit. In the Schwinger representation, Eq. (10) becomes

\[ |\phi_r^{(2)}\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^j e^{i\phi_r^{(2)} m} |jm\rangle_z. \tag{11} \]

It is worth noting that the state

\[ |{\theta}^{(N)}\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^N e^{i\theta_n^{(N)} n} |n\rangle_a |N-n\rangle_b \tag{12} \]

can always be considered a relative-phase eigenstate, however involved or even arbitrary the real set \( \{\theta_n^{(N)}\}_n \) may be with respect to \( n \) [24]. The whole basis can always be constructed by applying the \( N+1 \) phase shift operators \( e^{i\theta_n^{(N)} J_z} \) to \( |{\theta}^{(N)}\rangle \). However, successful quantum teleportation will demand full initial knowledge of \( \{\theta_n^{(N)}\}_n \), which is the relative phase of the entanglement between Alice and Bob [24], and we will show later that the number-phase teleportation protocol becomes complicated if \( \theta_n^{(N)} \) is not linear in \( n \).

Finally, let us recall that maximal entanglement is only really attained when \( N \rightarrow \infty \).

### C. Generation of EPR and quasi-EPR states

The experimental realization of relative-phase eigenstates is an arduous problem. Recently, Trifonov et al. reported the experimental realization of a relative-phase eigenstate [22] for \( N = 2 \) [20]. Their astute method uses a nonbalanced beamsplitter to create a two-mode state, all of whose amplitudes have equal modulus. This method is not general in the sense that it cannot work perfectly for \( N > 2 \), as we show below. However, the use of a beam splitter to generate EPR or quasi-EPR states stems from quite general arguments indeed.

The studies of Heisenberg-limited interferometry [25] [26] [22] have led to a thorough understanding of the subtle quantum optical properties of the beam splitter [29] [26]. In particular, a balanced beam splitter
swaps the amplitudes and phase properties of the impinging two-mode wave, and can also be used to entangle nonclassical optical fields \[ |\rangle \]. As is readily seen in the Schwinger representation, a balanced beam splitter corresponds to a \(|\pi/2\rangle\) rotation around \(X\) and therefore transforms a state from axis \(Z\) (intensity difference) to axis \(Y\) (phase difference).

In the case of an EPR state such as Eq. (11), the phase difference is squeezed and the intensity difference is anti-squeezed. Experimentally, this is achievable by sending an intensity-difference-squeezed state through a beam splitter. To illustrate this, let us examine the beam-splitter output of the relative phase state (11):

\[
|\phi_0^{(2j)}(\beta)\rangle = e^{-i\beta J_x} |\phi_0^{(2j)}\rangle
\]

\[
= \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} i^{-m'} f_m(\beta) |j m'\rangle_z,
\]

where \( d_{m',m}^{j}(\beta) = \langle jm'|e^{-i\beta J_x}|jm\rangle \) is a rotation matrix element taken real by convention and proportional to a Jacobi polynomial \( P\), and where \( f_m(\beta) = \sum_{m=-j}^{j} e^{im\phi_0^{(2j)}(\pi/2)} d^{j}_{m',m}(\beta) \). This state is displayed for \( \phi_0 = 0 \) in Fig. 1. As expected, the result is a narrow state in photon number (i.e. \( f_m(\beta) \to 0 \) very fast as \( |m| \to j \)).

It is straightforward to see that sending \(|\phi_0^{(2j)}(\pi/2)\rangle\) through another, balanced \((\beta = \pm\pi/2)\), beam splitter will reconstruct the initial relative phase eigenstate \(|\phi_0^{(2j)}\rangle\). (One can see that \( \beta \to \pi/2 \) is necessary for maximal entanglement, since the spread of the state must cover all values of the projections of the spin.) Now, since \(|\phi_0^{(2j)}(\pi/2)\rangle\) contains but a few nonzero amplitudes, a reasonable method for generating close approximations of EPR states with a balanced beam splitter is to consider “quantum filtered” input states, which are derived from \(|\phi_0^{(2j)}(\pi/2)\rangle\) by keeping only the lowest values of \( m \) (this could be viewed as a sort of quantum Abbe experiment, with a low-\(m\)-pass filter). These states are, for \( N = 2j \) even,

\[
|j 0\rangle_z,
\]

\[
(f_{j} |j 0\rangle_z + f'_{j} |j 1\rangle_z + f''_{j} (j - 1)\rangle_z) / C_1,
\]

and so on, with \( C_\mu = (\sum_{m=\mu}^{2j} |f_{m}^{(2j)}|)^{1/2} \), and are

\[
(|j + \frac{1}{2}\rangle_z + |j - \frac{1}{2}\rangle_z) / \sqrt{2},
\]

\[
(f_{j} |j + \frac{1}{2}\rangle_z + f'_{j} |j - \frac{1}{2}\rangle_z + f''_{j} (j + \frac{1}{2})\rangle_z + f'''_{j} (j - \frac{1}{2})\rangle_z) / C_2,
\]

and so on, for \( N = 2j \) odd. Sending these states through a beam splitter gives output states closely resembling EPR states, as we will now see. We call these output states quasi-EPR states.

We start with the simplest one (15), which has already been considered by CMM. We define the general state rotated by a beam splitter as

\[
|j m(\beta)\rangle = e^{-i\beta J_x} |j m\rangle_z
\]

\[
= \sum_{m'=-j}^{j} e^{-im'm'} d_{m',m}^{j}(\beta) |j m'\rangle_z.
\]

Figure 2 displays the modulus of the quantum amplitudes of \(|j 0(\beta)\rangle\), versus \(\beta\) and \(m\).

One can see that \( \beta \to \pi/2 \) is still necessary for maximal entanglement. The very value \( \beta = \pi/2 \) leads to a problem, however, because every other amplitude of the state is zero (20). This was recalled by CMM when they investigated the use of this state as a teleportation channel and found that, because of this, teleportation fidelity was bounded by 50% (to the notable exception of some Schrödinger cat states). This situation, however, is changed if one considers an ever-so-slightly imbalanced beam splitter: Fig. 3 displays the modulus and phase of the coefficients of \(|10 0(90^o)\rangle\) and \(|10 0(85.5^o)\rangle\).

Clearly, \(|10 0(85.5^o)\rangle\) is closer to an EPR state than \(|10 0(90^o)\rangle\): it has the same spread but much more even amplitudes and no zeroes at all. The phase distribution is not constant but this just means that it is a general relative phase state of the form of Eq. (14), which is still a legitimate EPR state. In fact, \(|10 0(85.5^o)\rangle\) is the best quasi-EPR state for all \(\beta\). In general, we find that the angle \(\beta_Q\) that gives the best quasi-EPR state is given by the following phenomenological formula:

\[
\beta_Q = \frac{\pi}{2} \left( 1 - \frac{1}{N} \right).
\]

To test Eq. (21), we have plotted on Fig. 4 the state \(|j 0(\beta_Q)\rangle\) for \(j = 100, 1000, \text{ and } 10000\).

Note that all digits in the angle in Fig. 4 are significant, which points to an interesting situation. Let us assume that on-demand single-photon sources become a reality (not an unreasonable assumption), which would allow the production of \(|j 0\rangle_z\) in the laboratory. Equation (21) nevertheless poses a serious experimental constraint on the tolerance of the beam splitter reflectivity \(R\), because the required precision on \(\beta\), i.e. on \(R\), increases with \(N\) if one wants to resolve \(|j 0(\beta_Q)\rangle\) from \(|j 0(\pi/2)\rangle\) and its numerous inconvenient zeroes. Roughly, \(\Delta R \sim \Delta \beta \sim 1/N\).

Since a beam splitter using state-of-the-art optical coatings and polishing cannot give more than the (already unrealistic) \(\Delta R \sim 10^{-6}\), hence \(N\) cannot exceed \(10^6\) photons.

In fact, by taking a closer look at Fig. 4, one can see that the amplitudes present 1/N-period oscillations versus \(\beta\). These oscillations are of significant contrast, with the state amplitudes often reaching zero. This poses a problem for experimentally defining \(|j 0(\beta)\rangle\) as \(j\) increases.

This problem disappears, however, as soon as one uses a more elaborate input state, such as (17). Such an input state could be obtained using stimulated emission from a single atom, starting from a \(|j 0\rangle_z\) state and having the two beams shine simultaneously and noncollinearly on
the excited atom. One will also want to have fast nonradiative decay from the ground state of the transition so as to prevent subsequent absorption. Figure 6 displays the state \(|j N\alpha(\beta)| + |j - N\alpha(\beta)|/\sqrt{2}\).

Even though the \(1/N\) oscillations are still present, they are significantly attenuated as there are no zero amplitudes, even at \(\beta = \pi/2\). One can therefore use \(\beta = \pi/2\) in this case, which presents the nonnegligible advantage of yielding a constant phase distribution for this state \(|j N\alpha(\pi/2) + |j - N\alpha(\pi/2)|\) (unlike in Fig. 5). This is of importance for the teleportation protocol, as we will see in the next Section. Finally, it is straightforward and unsurprising to show that the more elaborate (less low-\(m\)-pass filtered) reconstructions (14) and (18) give even better results: more even amplitudes at still constant phase. We will, however, restrain our investigations to the two results: more even amplitudes at still constant phase.

### III. NUMBER-PHASE QUANTUM TELEPORTATION

In this section we briefly recall the definition of number-phase teleportation 13 and extend it to general relative-phase states. We then consider the use of the quasi-EPR states derived above.

#### A. Ideal entanglement resource

Quantum teleportation relies on a maximally entangled state shared by the sender, Alice, and the receiver, Bob. The entanglement concerns two physical systems, \(A\) and \(B\) respectively. Alice is in possession of \(A\) and also of system \(T\), whose “target” state is the quantum information she needs to transmit. The teleportation process consists, for Alice, in making a joint measurement on \(T\) and \(A\) such that both are projected onto a maximally entangled state. This prevents Alice from obtaining any quantum information about the target state, which as such is destroyed in the process. In turn, said target state is transferred, by “entanglement transitivity” from \(T\) to \(B\), i.e. to Bob, who may then reconstruct the exact target state on \(B\) using the classically transmitted results of Alice’s measurements (which contain no quantum information whatsoever). The conceptually difficult part is to figure out what measurements should be used by Alice to maximally entangle her two systems \(A\) and \(T\). This question was answered by Vaidman in connection with the EPR paradox 3.

In the case of number-phase teleportation, use is made of the commuting operators number sum \(\hat{N}_T + \hat{N}_A\) and (Hermitian) relative phase \(\hat{\phi}_{TA}\) whose measurements project the joint \(T-A\) state onto a joint eigenstate of the total number and the relative phase such as Eq. (14). If the same type of entangled state is shared between Alice and Bob, perfect teleportation can in principle be achieved. Let us consider the general case where the initial total state is

\[
\left|\psi\right>_T \otimes \left|\phi^{(N)}_r\right>_AB \equiv \frac{1}{\sqrt{N+1}} \sum_{m=0}^N c_m \left|m\right>_T \sum_{n=0}^N e^{i\pi m} \left|n\right>_A |N-n\rangle_B.
\]

We assume Alice’s measurements yield the eigenvalues \(\left<\hat{N}_T + \hat{N}_A\right> = q\) and \(\left<\hat{\phi}_{TA}\right> = \phi(q)\). The joint TA state is thus left in \(\left|\phi^{(q)}_{TA}\right>,\) and the total state after Alice’s measurement is

\[
\left|\psi_M\right> = e^{i\pi \phi(q)} \left|\phi^{(q)}_{s}\right>_TA \otimes \left|\psi_B\right>_B
\]

\[
\left|\psi_B\right>_B = C(q) \sum_{k_0} e^{-ik\pi(q) + \phi(q)} c_k |k + N - q\rangle_B,
\]

where \(k_0 = \text{Max}[0, q - N]\) and \(C(q) = (\sum_{k_0} |c_k|^2)^{-1/2}\). To exactly recover the target state, Bob must then perform, on \(B\), a photon number shift 13 of \(q - N\) and a phase shift of \(\phi^{(N)}_r + \phi^{(q)}_s\) i.e.

\[
\left|\psi_{out}\right>_B = e^{i\pi \phi(q)} e^{\phi(q)} J_z \mathcal{P}_{q-N} \left|\psi_B\right>_B.
\]

(In the particular case where \(q = N\), \(\phi^{(q)}_s = -\phi^{(N)}_r\), Bob does not have to do anything.) One can see from this that the phase distribution of the initial entanglement resource has to be corrected for, along with Alice’s measurement result. If this distribution is unknown or too complicated to correct, teleportation will fail. This correction can also be made by Alice, by simply shifting her phase operator, i.e. using

\[
e^{-i\phi^{(N)}_r J_z} \hat{\phi}_{TA} e^{i\phi^{(N)}_r J_z}
\]

instead of \(\hat{\phi}_{TA}\). [\(J_z = (\hat{N}_T - \hat{N}_A)/2\) here.]

Note that this requirement that the entanglement phase be perfectly known is a very general one and not at all specific to our particular choice of the optical phase variable for the teleportation protocol. This was pointed out by van Enk in Ref. 24.

In light of what precedes, it is interesting to investigate the use of the general relative phase state \(\{|\theta^{(N)}\rangle\}\) given by Eq. (12) (which still is a perfect EPR state). Indeed, if the phase distribution is more complicated than the mere relative phase offsets of Eq. (14), Bob will be faced with problems reconstructing the target state. If the initial entanglement is given by Eq. (12)
treat the cases of
\(|\psi\rangle_T \otimes |\{\theta^{(N)}\}\rangle_{AB}\),
and the phase difference operator by Eq. (22), then the post-measurement total state is
\[ |\psi_M\rangle = e^{i\theta}\phi^{(N)}_n |\phi^{(N)}_n\rangle_{TA} \otimes |\psi_B\rangle_B \]
(29)
\[ |\psi_B\rangle_B = C(q) \sum_{k=0}^{q} e^{ik\theta^{(N)}_n} c_k |k + N - q\rangle_B, \]
(30)
which shows that Bob needs more than a phase shift to properly reconstruct \(|\psi\rangle\), even if \(|\{\theta^{(N)}\}_n\) is fully known.

What is needed is the unitary transformation \(U_{\theta^{(N)}}\) that transforms \(|\{\theta^{(N)}\}\rangle\) into a “flat-phase” state for which \(\theta^{(N)}_n = \phi^{(N)}_n = \text{cst}, \forall n\). This transformation may be applied by Bob, as \(U_{\theta^{(N)}}|\psi_B\rangle_B = |\psi_{out}\rangle_B\), or by Alice, by measuring \(U_{\theta^{(N)}} \hat{\sigma}_{TA} U_{\theta^{(N)}}^\dagger\).

It is however somewhat puzzling that this additional step is needed if \(|\{\theta^{(N)}\}\rangle\) may indeed be considered as a legitimate relative phase eigenstate, since it is used in the corresponding relative phase measurement. There seems therefore to be a need for an absolute phase reference in number-phase teleportation, if Alice uses a relative phase operator. In other words yet, even though the whole relative-phase eigenbasis — and hence the operator — may be generally defined based upon any general state \(|\{\theta^{(N)}\}\rangle\) with arbitrary phase distribution \(|\{\theta^{(N)}\}_n\), it does in fact matter for teleportation that \(U_{\theta^{(N)}}\) correspond to a feasible physical measurement, thereby limiting the generality of relative phase states usable for teleportation. If \(U_{\theta^{(N)}}\) is a phase shift, then \(|\theta^{(N)}\rangle\) can only be linear in \(n\) at most.

One example of such a complicated situation is the quasi-EPR state \(|j0\rangle\), which has the phase distribution depicted in Fig. 3.

### B. Teleportation of a coherent state with a quasi-EPR resource

We now turn to the use of quasi-EPR states as the entanglement resource, and show how an arbitrary coherent state can be successfully teleported. Our evaluation of teleportation performance is based on the pure-state fidelity
\[ F = |\langle \psi_{out} | \psi \rangle|^2. \]
(31)
The entanglement resource is now a quasi-EPR state, i.e. of the general form (8), but where Eqs. (8) are not verified any more. The entangled state is of the form (10), but with nonequal amplitude moduli: we write
\[ |QEPR\rangle_{AB} = \sum_{n=0}^{N} s^{N}_n |n\rangle_A |N-n\rangle_B, \]
(32)
where \(\sum_{n=0}^{N} |s^{N}_n|^2 = 1\). As announced before, we only treat the cases of \(|j0\rangle\) and \(|j\frac{1}{2}\rangle\) where \(s^{N}_n\) are found using Eqs. (4, 5, 19), and their amplitudes are, respectively,
\[ s^{N}_n = i^{-n+n+\frac{s^{N}_n}{2}} \sum_{n=0}^{N} \frac{1}{\sqrt{2}} \left[ d^{N}_n e^{-\frac{i\pi}{4}} - id^{N}_n e^{\frac{i\pi}{4}} \right]. \]
(34)
The post-measurement total state is
\[ |\psi_M\rangle = |\phi^{(N)}_n\rangle_{TA} \otimes |\psi_B\rangle_B \]
(35)
\[ |\psi_B\rangle_B = C(q) \sum_{k=0}^{q} e^{-ik\theta^{(N)}_n} c_k s^{N}_{q-k} |k + N - q\rangle_B, \]
(36)
where \(C(q) = (\sum_{q=0}^{q} |c_k|^2 s^{N}_{q-k})^{-1/2}\). This yields
\[ |\psi_{out}\rangle_B = C(q) \sum_{k=0}^{q} c_k s^{N}_{q-k} |k\rangle_B, \]
(37)
and the fidelity
\[ F(q) = \left| \frac{\sum_{k=0}^{q} |c_k|^2 s^{N}_{q-k}}{\sum_{k=0}^{q} |c_k|^2 s^{N}_{q-k}} \right|^2. \]
(38)

Before we plot \(F(q)\) for the two quasi-EPR states, we must address the dependence of the fidelity on the measured value \(q\) of the number sum: this implies conditional teleportation even for an ideal relative-phase state \((s^{N}_n = \text{cst}, \forall N, n)\), which should not be.

Equation (38) has an upper bound, which can be found by using the Cauchy-Schwartz inequality:
\[ \sum_{k=0}^{q} |c_k|^2 s^{N}_{q-k} \leq \sum_{k=0}^{q} |c_k|^2 \sum_{k=0}^{q} |c_k|^2 |s^{N}_{q-k}|^2. \]
(39)
The fidelity is thus bound by
\[ F(q) \leq \sum_{k=0}^{q} |c_k|^2, \]
(40)
which corresponds to the fidelity of the ideal EPR resource. The Cauchy-Schwartz inequality thus has a precise physical meaning in this case. Note, however, that this limit (40) can be \(\ll 1\) unless \(k_0 = 0\) (i.e. \(q \leq N\) and \(q\) is very large compared to the spread of the target state (i.e. \(k_{\text{max}}\) such as \(c_k \neq 0\)). This can only be achieved if \(N \rightarrow \infty\), which is the rigorous condition for which a relative-phase eigenstate is truly maximally entangled. When this is fulfilled, the probability that \(q < k_{\text{max}}\) becomes negligible and the teleportation becomes unconditional \(F(q) \leq 1\).
Unfortunately, $N \rightarrow \infty$ cannot be satisfactorily approximated in numerical simulations with $N \sim 10$ to 100, therefore the fidelity displayed in our figures has high and low regions, depending on the value of $q$. It is simple to see that, for a coherent target state $|\alpha\rangle$ ($\alpha$ real), the high-fidelity region is given by $q \in [k_{\text{max}} - N - k_{\text{min}}]$ where $N > k_{\text{max}}$ i.e. $q \in [\alpha^2 + \alpha, N - \alpha^2 + \alpha]$

Figure 3 displays the fidelity versus $q$ and the beam splitter angle $\beta$ for the teleportation of a coherent state $\alpha = 3$ with the quasi-EPR resource $(|j/2(\beta)\rangle + |j - 1/2(\beta)\rangle)/\sqrt{2}$.

Once again, the limited spread in $q$ of the high fidelity region is only due to the limitations of our computation. What is essential is that $F > 80\%$ for $\beta = \pi/2$, thereby confirming our analysis of quasi-EPR states.

IV. CONCLUSION

We hope to have convincingly demonstrated that efficient quantum teleportation is indeed possible with only approximately entangled states (close enough to ideal, nevertheless). We chose the simplest quasi-EPR states, which could be good candidates for experimentation with on-demand single-photon sources. Finally, we investigated the use of generalized relative-phase states for quantum teleportation and showed that these states, although usable, lead to serious complications if the basis “generator” $\{|\theta\rangle\}$ has a more complicated phase distribution than a simple phase offset.

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FIG. 1. Modulus (top) and phase (bottom) of the quantum amplitudes of $|j 0 (\beta Q)\rangle$ (left) and $|10 0 (\beta Q)\rangle$ (right). The phase is $\pi/2$ for all nonzero components at $90^\circ$.

FIG. 2. Modulus of the quantum amplitudes of beam-splitter output $|\phi_j^{(N)} (\beta)\rangle$ versus beam-splitter angle $\beta$. $N = 21$ photons.

FIG. 3. Modulus (top) and phase (bottom) of the quantum amplitudes of $|10 0 (90^\circ)\rangle$ (left) and $|10 0 (85.5^\circ)\rangle$ (right). The phase is $\pi/2$ for all nonzero components at $90^\circ$.

FIG. 4. Modulus of the quantum amplitudes of $|j 0 (\beta Q)\rangle$ for $N = 200, 2000,$ and $20000$ photons (top to bottom). All states have almost identical appearance. See also $|10 0 (\beta Q)\rangle$ in Fig. 3.

FIG. 5. Modulus of the quantum amplitudes of beam-splitter output $(|j 1/2 (\beta)\rangle + |j - 1/2 (\beta)\rangle)/\sqrt{2}$ versus beam-splitter angle $\beta$. $N = 21$ photons.

FIG. 6. Teleportation fidelity using the quasi-EPR resource $(|j 0 (\beta Q)\rangle$, versus beam-splitter angle $\beta$ and number-sum measurement result $q$. $N = 21$ photons.

FIG. 7. Teleportation fidelity using the quasi-EPR resource $(|j 0 (\beta Q)\rangle$, versus beam-splitter angle $\beta$ for number-sum measurement result $q = 19$. $N = 20$ photons.
N.K. Tran and O. Pfister. Figure 1
N.K. Tran and O. Pfister. Figure 2
N.K. Tran and O. Pfister. Figure 3
Quantum amplitudes

N.K. Tran and O. Pfister. Figure 4
N.K. Tran and O. Pfister. Figure 5
N.K. Tran and O. Pfister. Figure 6
N.K. Tran and O. Pfister. Figure 7.