An Exact Integral-to-Sum Relation for Products of Bessel Functions

Oliver H.E. Philcox\textsuperscript{1,2,*} and Zachary Slepian\textsuperscript{3,4}

\textsuperscript{1}Department of Astrophysical Sciences, Princeton University, Princeton, NJ 08540, USA
\textsuperscript{2}School of Natural Sciences, Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540, USA
\textsuperscript{3}Department of Astronomy, University of Florida, 211 Bryant Space Science Center, Gainesville, FL 32611, USA
\textsuperscript{4}Physics Division, Lawrence Berkeley National Laboratory, 1 Cyclotron Road, Berkeley, CA 94720, USA
\textsuperscript{*}Electronic Address: ohep2@cantab.ac.uk

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Abstract

A useful identity relating the infinite sum of two Bessel functions to their infinite integral was discovered in Dominici et al. (2012). Here, we extend this result to products of \( N \) Bessel functions, and show it can be straightforwardly proven using the Abel-Plana theorem, or the Poisson summation formula. For \( N = 2 \), the proof is much simpler than that of Dominici et al., and significantly enlarges the range of validity.

1 Introduction

Integrals of Bessel functions appear in many guises across the physical sciences, populating fields as diverse as atomic physics, classical mechanics and cosmology [e.g., 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Of particular importance is their appearance in the basis decomposition of spherically- and circularly-symmetric functions. As a concrete example, consider the inverse Fourier transform of an isotropic function \( \tilde{f}(k) \equiv \tilde{f}(k) \) in three dimensions, defined by

\[
f(r) = \int \frac{d^3 k}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{r}} \tilde{f}(k).
\]  

(1)

Via the well-known plane-wave identity [e.g., 19, Eq. 16.63], the exponential may be expanded as a sum of spherical Bessel functions of the first kind, \( j_\ell(kr) \), and the angular part of the integral (1) can be performed analytically, leading to

\[
f(r) = f(r) = \int_0^\infty \frac{k^2 dk}{2\pi^2} j_0(kr) \tilde{f}(k).
\]

(2)

If instead one wishes to evaluate the function \( f \) at the difference of two positions, \( r - r' \), the same identity can be used to show

\[
f(r - r') = \sum_{\ell = 0}^\infty (2\ell + 1) \mathcal{L}_\ell(\hat{r} \cdot \hat{r}') \int_0^\infty \frac{k^2 dk}{2\pi^2} j_\ell(kr) j_\ell(kr') \tilde{f}(k),
\]

(3)
where $L_{\ell}(\hat{r} \cdot \hat{r}')$ is a Legendre polynomial of order $\ell$. This now involves the infinite integral of two spherical Bessel functions, which form the coefficients of a Legendre series in the separation angle $\hat{r} \cdot \hat{r}'$ [e.g., 20]. Whilst not strictly the subject of this work, the above examples serve to illustrate the ubiquity of (spherical) Bessel function integrals.

Given this, much work has been devoted to numerical computation of Bessel function integrals. Of particular note is the FFTLog algorithm [21] in which one expands a general function (subject to a set of regularity conditions) as a complex power law, allowing analytic computations of its Bessel-weighted infinite integral. This is both an accurate and fast procedure, and has found great use in the field of cosmology [e.g., 22].

Of interest to this work are the results of [23], which demonstrated the following relation between the infinite integral of two Bessel functions and their sum:

$$
\int_0^\infty dt J_\nu(at)J_\nu(bt) = \sum_{m=0}^\infty \varepsilon_m J_\nu(am)J_\nu(bm)
$$

where $J_\nu(x)$ is a Bessel function of the first kind, $\nu$ is a positive half-integer, $a, b \in [0, \pi]$ and

$$
\varepsilon_m = \begin{cases} 
1/2 & m = 0 \\
1 & m \geq 1 
\end{cases}
$$

The integral (4) may be additionally expressed in terms of spherical Bessel functions of the first kind, giving

$$
\int_0^\infty dt j_\ell(at)j_\ell(bt) = \sum_{m=0}^\infty \varepsilon_m j_\ell(am)j_\ell(bm)
$$

for non-negative integer $\ell$. (4) has been used in a variety of contexts [24, 25, 17, 26, 27], and was proven in [23] as a special case of the more general relation

$$
\int_0^\infty dt \frac{J_\mu(at)J_\nu(bt)}{t^{\mu+\nu-2k}} = \sum_{m=0}^\infty \varepsilon_m \frac{J_\mu(am)J_\nu(bm)}{m^{\mu+\nu-2k}}
$$

for $0 < b < a < \pi$, Re($\mu$) > $2k - 1/2$, Re($\nu$) > $-1/2$, and $k \in \mathbb{N}_0$ (where $\mathbb{N}_0$ is the set of all natural numbers and zero). To prove this, the authors of [23] use the known result for this infinite integral in terms of Gauss’ hypergeometric function (e.g. [28]), then manipulate this solution. Below, we show that a simpler proof is possible via the Abel-Plana theorem.

## 2 Main Result

The principal new result of this work is the following:

$$
\int_0^\infty dt t^{2k} \prod_{j=1}^N [t^{-\nu_j} J_{\nu_j}(a_j t)] = \sum_{m=0}^\infty \varepsilon_m m^{2k} \prod_{j=1}^N [m^{-\nu_j} J_{\nu_j}(a_j m)]
$$

i.e., that the integral of $N$ Bessel functions can be written as an infinite sum for arbitrary $N > 0$. This uses the definition (5) and is valid for integer $k$, real $\{a_j\}$ and complex $\{\nu_j\}$ subject to the conditions
We now introduce the Abel-Plana theorem \([29, 30]\), which will be used to prove our main result (8).

### Abel-Plana Theorem

For \(k \in \mathbb{N}_0 \equiv \{0, 1, 2, \ldots\}\)

- \(\sum_{j=1}^{N} |a_j| \leq 2\pi\)
- \(\sum_{j=1}^{N} \text{Re}(\nu_j) > 2k - N/2\)
- \(\sum_{j=1}^{N} \text{Re}(\nu_j) > 2k - N/2 + 1\) if:
  - \(\sum_{j=1}^{N} |a_j| = 2\pi\) and/or:
  - There exists some vector \(\{s_j\} = \{\pm 1, \pm 1, \ldots\}\) such that \(\sum_{j=1}^{N} s_j a_j = 0\).

We additionally assume \(a_j \neq 0 \forall j\), to avoid trivial results. The range of validity is somewhat increased if some of \(\{\nu_j\}\) are negative integers, in particular those with \(j \in J\), for some non-empty set \(J\). In this case, (8) applies also for integer \(k \geq -\sum_{j' \in J} |\nu_{j'}|\).

In practice, the condition that \(\sum_{j=1}^{N} |a_j| \leq 2\pi\) is not a limitation. Assuming \(\sum_{j=1}^{N} |a_j| = 2\pi A\) with \(A \geq 1\), we may rescale \(t \to \tilde{t} \equiv At\), \(a_j \to \tilde{a}_j \equiv a_j/A\), giving

\[
\int_0^\infty dt \int_{-\infty}^{\infty} \prod_{j=1}^{N} \left( e^{-\nu_j} J_{\nu_j}(a_j t) \right) = A \sum_{\nu_j=1}^{\nu_j=2} \int_0^\infty dt \int_{-\infty}^{\infty} \prod_{j=1}^{N} \left[ e^{-\nu_j} J_{\nu_j}(\tilde{a}_j \tilde{t}) \right] \]

noting that \(\sum_{j=1}^{N} |\tilde{a}_j| = 2\pi A^{1-N} \leq 2\pi\).

In the case \(N = 2\), labelling \(\{a_j\} = \{a, b\}\), \(\{\nu_j\} = \{\mu, \nu\}\), for real \(a, b\), (8) can be written

\[
\int_0^\infty dt \int_{-\infty}^{\infty} J_{\mu}(at) J_{\nu}(bt) = \sum_{m=0}^{\infty} \epsilon_m \frac{J_{\mu}(am) J_{\nu}(bm)}{m^{\mu+\nu-2k}}
\]

for \(k \in \mathbb{N}_0\) (assuming neither \(\mu\) nor \(\nu\) are negative integers), \(\text{Re}(\mu + \nu) > 2k - 1\), and \(|a| + |b| \leq 2\pi\). Additionally we require \(\text{Re}(\mu + \nu) > 2k\) if \(|a| + |b| = 2\pi\) or \(a = \pm b\). This matches the result of [23], but with a larger domain of validity, for example including \(a \geq \pi\).

### 3 Abel-Plana Theorem

We now introduce the Abel-Plana theorem [29, 30], which will be used to prove our main result (8) in §5. In the notation of [31], the theorem states that

\[
\sum_{k=0}^{\infty} f(k) = \int_0^\infty dx f(x) + \frac{1}{2} f(0) + i \int_0^\infty dy \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}
\]

where the function \(f : \mathbb{C} \to \mathbb{C}\) obeys the following conditions:

1. \(f(z)\) is analytic in the closed half-plane \(U = \{z \in \mathbb{C} : \text{Re}(z) \geq 0\}\).
2. \(\lim_{y \to \infty} |f(x \pm iy)| e^{2\pi y} = 0\) uniformly in \(x\) on every finite interval.
3. \(\int_0^\infty dy |f(x + iy) - f(x - iy)| e^{2\pi y}\) exists for every \(x \geq 0\) and tends to zero as \(x \to \infty\).
The additional conditions

4. $\int_0^\infty dx f(x)$ exists,
5. $\lim_{n \to \infty} f(n) = 0,$

are often imposed, though [31] consider the second to be superfluous. The theorem itself is straightforwardly proved from the argument principle and Cauchy’s integral theorem; in essence, one considers the integral of $f(z)/[e^{-2\pi iz} - 1]$, which has poles at integer $z$. See [32] and [33] for additional discussion of this theorem and its applications.

## 4 Validity Conditions

Consider the function $f : \mathbb{C} \to \mathbb{C}$:

$$f(z) = z^{-\lambda} \prod_{j=1}^{N} J_{\nu_j}(a_j z) \tag{12}$$

where $J_\mu(z)$ is a Bessel function of the first kind and $\mu_1, \mu_2, ..., \mu_N, a_1, a_2, ..., a_N$ and $\lambda$ are complex parameters. In the below, we demonstrate that $f(z)$ satisfies the Abel-Plana validity criteria given in §3, subject to certain restrictions on each parameter. We will ignore the trivial cases in which at least one element of $\{a_j\}$ is zero.

### 4.1 Condition 1

$x^{-\lambda}$ is analytic in the closed half-plane $U = \{z \in \mathbb{C} : \text{Re}(z) \geq 0\}$.

Since the Bessel functions are holomorphic on $\mathbb{C}$ except for a branch cut on the negative real axis, and $x^{-\lambda}$ is holomorphic in $\mathbb{C}\setminus\{0\}$, it follows that $f(z)$ is analytic in $U\setminus\{0\}$. Near $z = 0$, we consider the asymptotic form for Bessel functions with $0 < |z| \ll \sqrt{\nu + 1}$:

$$J_\nu(z) \approx \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu \quad (\nu \not\in \mathbb{Z}^-) \tag{13}$$

[34, Eq. 9.1.7], where $\mathbb{Z}^-$ is the set of negative integers, and $\Gamma(t)$ is the Gamma function. Thus, $f(z)$ has the asymptotic limit

$$f(z) \approx \prod_{j=1}^{N} \left[\frac{1}{\Gamma(\nu_j + 1)} \left(\frac{a_j}{2}\right)^{\nu_j}\right] \times z^{\sum_{j} \nu_j - \lambda} \tag{14}$$

assuming no element of $\{\nu_j\}$ is a negative integer. The limit $z \to 0$ (i.e. $f(0)$) thus exists if $\sum_j \text{Re}(\nu_j) - \text{Re}(\lambda) > 0$. The limit additionally exists if $\sum_j \nu_j - \lambda = 0$.

For negative integer order $\nu$, the Bessel function is instead approximated by

$$J_\nu(z) \approx \frac{(-1)^\nu}{|\nu|!} \left(\frac{z}{2}\right)^{|\nu|} \quad (\nu \in \mathbb{Z}^-), \tag{15}$$
thus \( \lim_{z \to 0} J_\nu(z) = 0 \) for all negative integer \( \nu \). Given this, the most general constraint on the existence of \( f(0) \) is that
\[
\sum_{j' \in J} |\nu_j'| + \sum_{j \notin J} \text{Re}(\nu_j) - \text{Re}(\lambda) > 0 \tag{16}
\]
or
\[
\sum_{j' \in J} |\nu_j'| + \sum_{j \notin J} \nu_j - \lambda = 0,
\]
where \( J \) is the set of indices corresponding to negative integer \( \nu_j \), i.e. \( J = \{ j : \nu_j \in \mathbb{Z}^- \} \). Assuming (16) holds, \( f(z) \) is analytic on \( U \) and its boundary, thus the condition is satisfied.

### 4.2 Condition 2

\[
\lim_{y \to \infty} |f(x \pm iy)|e^{-2\pi y} = 0 \quad \text{uniformly in } x \text{ on every finite interval.}
\]

At large \(|z|\), Bessel functions of the first kind have the following asymptotic form:
\[
J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \left[ \cos \left( z - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) + e^{\text{Im}(z)}O\left(|z|^{-1}\right) \right] \tag{17}
\]
[34, Eq. 9.2.1], assuming \(|\arg(z)| < \pi/2 \) (which is valid for all \( z \) within the closed half-plane \( U \)). To evaluate the large-\( y \) limit of \(|f(x \pm iy)|\), we require the term
\[
|J_\nu(az)| \approx \sqrt{\frac{1}{2\pi |a|}} e^{i(az - \nu \pi/2 - \pi/4)} - e^{-i(az - \nu \pi/2 - \pi/4)}, \tag{18}
\]
ignoring the subdominant \( O\left(|z|^{-1}\right) e^{\text{Im}(z)} \) term (appropriate given that \(|\cos z| \) is \( O(e^{\text{Im}(z)}) \) for large \(|\text{Im}(z)|\)). Writing \( z = x \pm iy \), the dominant term is the one involving \( y \) with a positive coefficient;
\[
|J_\nu(a(x \pm iy))| \approx \sqrt{\frac{1}{2\pi |a||y|}} \exp\left( \text{sgn}[\text{Re}(a)] \left[ \text{Re}(a)y \pm \frac{\pi}{2} \text{Im}(\nu) \mp \text{Im}(a)x \right] \right), \tag{19}
\]
noting that \(|e^{i\phi}| = 1 \) for all real \( \phi \). Utilizing definition (12), we find the following asymptotic limit of \( |f(x \pm iy)| \) for large \( y > 0 \):
\[
|f(x \pm iy)| \approx y^{\frac{N}{2} + \text{Re}(\lambda)} \exp\left[ \frac{\pi}{2} \text{Im}(\lambda) \right] \times \prod_{j=1}^{N} \left( 2\pi |a_j| \right)^{-1/2} \exp\left( \text{sgn}[\text{Re}(a_j)] \left[ \text{Re}(a_j)y \pm \frac{\pi}{2} \text{Im}(\nu_j) \mp \text{Im}(a_j)x \right] \right), \tag{20}
\]
noting that \(|z^\lambda| = |z|^{\text{Re}(\lambda) \text{arg}(z)} | \text{arg}(z) \approx \pm \pi/2 \).
We may now test the condition:

\[
\lim_{y \to \infty} |f(x \pm iy)| e^{-2\pi y} = \exp \left[ \pm \frac{\pi}{2} \text{Im}(\lambda) \prod_{j=1}^{N} (2\pi|a_j|)^{-1/2} \exp \left[ \pm \frac{\pi}{2} \text{sgn}(\text{Re}(a_j)) \text{Im}(\nu_j) \right] \right] \\
\times \exp \left[ \mp x \sum_{j=1}^{N} \text{sgn}(\text{Re}(a_j)) \text{Im}(a_j) \right] \\
\times \lim_{y \to \infty} \left[ y^{-N/2+\text{Re}(\lambda)} \exp \left( y \left[ \sum_{j=1}^{N} |\text{Re}(a_j)| - 2\pi \right] \right) \right].
\]

The limit is zero if (i) \(\sum_{j=1}^{N} |\text{Re}(a_j)| < 2\pi\) or (ii) \(\sum_{j=1}^{N} |\text{Re}(a_j)| = 2\pi\) and \(\text{Re}(\lambda) > -N/2\). Furthermore, if \(\sum_{j=1}^{N} \text{sgn}(\text{Re}(a_j)) \text{Im}(a_j) = 0\), the limit is uniformly approached in \(x\), as required. Henceforth, we will assume the stronger condition that all \(a_j\) are real; this is later required in §5.

4.3 Condition 3

\[\int_{0}^{\infty} dy |f(x + iy) - f(x - iy)| e^{-2\pi y} \text{ exists for every } x \geq 0 \text{ and tends to zero as } x \to \infty.\]

For this, we first note that the quantity of interest is bounded on both sides:

\[0 < \int_{0}^{\infty} dy |f(x + iy) - f(x - iy)| e^{-2\pi y} \leq \int_{0}^{\infty} dy |f(x + iy)| e^{-2\pi y} + \int_{0}^{\infty} dy |f(x - iy)| e^{-2\pi y} \tag{22}\]

using the triangle inequality and noting that \(|f(x + iy) - f(x - iy)| \geq 0\) and \(e^{-2\pi y} > 0\) for all \(y \geq 0\).

From condition 1 (§4.1), \(f(x \pm iy)\) is holomorphic for all \(x \geq 0\), and thus Riemann-integrable across any finite interval \(y \in [u, v]\) with \(u, v \geq 0\). To show that the infinite integral exists, we must consider its behavior at infinity. For sufficiently large \(y\), \(|f(x \pm iy)|\) can be replaced by (21), such that

\[|f(x \pm iy)| e^{-2\pi y} \approx K_0 y^{-\text{Re}(\lambda)} e^{-\left[2\pi - \sum_{j=1}^{N} |a_j|\right] y} \tag{23}\]

for some \(K_0 > 0\), assuming that \(\{a_j\}\) are real. If \(\sum_{j=1}^{N} |a_j| < 2\pi\), then there exist real numbers \(c_1, K_1\) such that

\[|f(x \pm iy)| e^{-2\pi y} < K_1 e^{-c_1 y} \tag{24}\]

for all \(y > M_0\) with \(M_0 \gg 0\), \(0 < c_1 < 2\pi - \sum_{j=1}^{N} |a_j|\) and \(K_1 > 0\). This implies

\[\int_{M}^{\infty} dy |f(x \pm iy)| e^{-2\pi y} < K_1 \int_{M}^{\infty} dy e^{-c_1 y} = K_1 / c_1 e^{-c_1 M} < \infty. \tag{25}\]

If instead we assume \(\sum_{j=1}^{N} |a_j| = 2\pi\) and \(\text{Re}(\lambda) > -N/2\), there must exist real numbers \(c_2, K_2\) such that

\[|f(x \pm iy)| e^{-2\pi y} < K_2 e^{-c_2} \tag{26}\]
for all \( y > M \gg 0 \), with \( 0 < c_2 < \text{Re}(\lambda) + N/2 \). In this case,

\[
\int_M^\infty dy |f(x \pm iy)|e^{-2\pi y} < K_2 \int_M^\infty dy y^{-c_2} = \frac{K_2}{1 - c_2} M^{1-c_2} < \infty
\]

(27)

if \( 1 - c_2 < 0 \), and thus \( \text{Re}(\lambda) > 1 - N/2 \), which is a slightly stronger bound than before.

If such restrictions are satisfied, the integral of \(|f(x \pm iy)|e^{-2\pi y}\) over the whole range \( y \in [0, \infty] \) must be finite, and thus, by (22), \( \int_0^\infty dy |f(x + iy) - f(x - iy)|e^{-2\pi y} \) must exist. Furthermore, (20) demonstrates that \(|f(x \pm iy)|e^{-2\pi y}\) is suppressed by \( x^{-(N/2+\text{Re}(\lambda))}\) at large \( x \); thus, if \( \text{Re}(\lambda) > -N/2 \), the integral tends to zero as \( x \to \infty \) and the condition is satisfied.

4.4 Condition 4

\( \int_0^\infty dx f(x) \) exists.

First we note that

\[
\left| \int_0^\infty dx f(x) \right| \leq \int_0^\infty dx |f(x)|;
\]

(28)

thus proving the latter to exist is a sufficient (but not necessary) condition for the existence of \( \int_0^\infty dx f(x) \). Proof of this proceeds analogously to condition 3 (§4.3), where we first note that, assuming condition 1 (§4.1) be satisfied, \( f(x) \) is Riemann-integrable on any finite interval. Secondly, using the asymptotic form of \( |f(x)| \) from the first line of (20), we note that there must exist constants \( K_3, c_3 \) such that

\[
|f(x)| < K_3 x^{-c_3}
\]

(29)

for all \( x > M' \gg 0 \), where \( 0 < c_3 < \text{Re}(\lambda) + N/2 \). Using this

\[
\int_{M'}^\infty dx |f(x)| < K_3 \int_{M'}^\infty dx x^{-c_3} = \frac{K_3}{1 - c_3} (M')^{1-c_3} < \infty
\]

(30)

if \( 1 - c_3 < 0 \) and thus \( \text{Re}(\lambda) > 1 - N/2 \).

In practice, a stronger bound is in fact possible in most cases, since the integrand is oscillatory. To show this, we again consider \( x > M' \) for large positive \( M' \), with the asymptotic form

\[
f(x) \approx x^{-\lambda - N/2} \sum_{j=1}^{N} \left(2\pi a_j\right)^{-1/2} \cos \left(a_j x - \nu_j \frac{\pi}{2} - \frac{\pi}{4}\right)
\]

(31)

from (17), with \( \text{Im}(z) = 0 \). Next, note that the product of two cosines can be rewritten as a sum via the standard relation

\[
\cos \theta \cos \phi = \frac{1}{2} \left[ \cos(\theta + \phi) + \cos(\theta - \phi) \right]
\]

(32)

for arbitrary complex \( \theta, \phi \). In this way, we may reduce the product of \( N \) cosine functions into a linear sum of \( 2^N \) terms of the form \( \cos(Ax + B) \). In particular the \( A \) coefficients are of the form \( A = \sum_{j=1}^{N} s_j a_j \) where \( s_j = \{ \pm 1, \pm 1, \ldots \} \). Each term is thus an oscillatory function of \( x \), provided that one cannot find an vector \( \{ s_j \} \) satisfying \( \sum_j s_j a_j = 0 \). If one \textit{can} be found, then we obtain a non-oscillatory term. Physically, this corresponds to a case when one of the ‘beat-frequencies’ is
equal to zero. In this case, \( \int_0^\infty dx \, f(x) \) contains a logarithmic divergence, and the condition is not satisfied.

Taking the above approach, we can write schematically

\[
f(x) \approx K_4 x^{-\lambda - N/2} \sum_{i=1}^{2N} \cos (A_i x + B_i)
\]

for some \( K_4 \). Assuming each \( A_i \neq 0 \), each integral may be divided up into regions where the integrand is negative and positive:

\[
\int_{M^*}^\infty dx \, f(x) \approx K_4 \sum_{i=1}^{2N} \int_{M^*}^\infty dx \, x^{-\lambda - N/2} \cos (A_i x + B_i)
\]

where \( M^* \) is the first increasing zero of \( \cos(A_i x + B_i) \) with \( M^* > M' \). Provided \( \text{Re}(\lambda) > -N/2 \), the magnitude of the integrand decreases monotonically with \( p \) (and to zero as \( p \to \infty \)) thus the alternating series, and hence the integral, converges by the Leibniz criterion. Using this approach, we find that \( \int_0^\infty dx \, f(x) \) exists for all \( \text{Re}(\lambda) > -N/2 \), provided that we exclude \( A_i = 0 \), i.e. provided that there does not exist \( \{s_j\} = \{\pm, \pm, \ldots\} \) such that \( \sum_{j=1}^{N} s_j \alpha_j = 0 \). If such a vector can be found, we instead require \( \text{Re}(\lambda) > 1 - N/2 \) to avoid logarithmic divergences in the integral.

Since the above requirements are somewhat non-trivial, it is useful to consider the simpler case \( N = 2 \), which has \( \{a_j\} = \{a, b\} \), \( \{\nu_j\} = \{\mu, \nu\} \) and was discussed in [23]. Here, \( \int_0^\infty dx \, f(x) \) is a known integral that can be written in terms of hypergeometric functions [e.g. Eq. 6.574, 28]. That solution requires \( \text{Re}(\lambda) > -1 \) and real \( a > 0 \), \( b > 0 \), with the additional constraint \( \text{Re}(\lambda) > 0 \) if \( a = b \). Given that \( J_\nu(-ax) = (-1)^\nu J_\nu(ax) \) for real \( ax \) (as here), the regime of validity may be extended also to \( a < 0 \) and \( b < 0 \), and is subject to the same limits as our approach.

### 4.5 Condition 5

\[
\lim_{n \to \infty} f(n) = 0.
\]

From (31), the behavior of \( f(n) \) at large (integer) \( n \) is that of a cosine bounded by the envelope \( n^{-\text{Re}(\lambda)+N/2} \). For real \( \{a_j\} \), the magnitude of each cosine is bounded by a constant (equal to unity if \( \nu_j \) is real), thus \( |f(n)| \leq K_5 n^{-\text{Re}(\lambda)+N/2} \) for some constant \( K_5 > 0 \). This implies

\[
0 \leq \lim_{n \to \infty} |f(n)| \leq K_5 \lim_{n \to \infty} n^{-\text{Re}(\lambda)+N/2},
\]

Assuming \( \text{Re}(\lambda) > -N/2 \), the limit is zero, satisfying the condition.
5 Proof of the Integral-to-Sum Relation

We now insert our definition of $f(z)$ into the Abel-Plana theorem (11). This gives

$$
\sum_{m=0}^{\infty} m^{-\lambda} \prod_{j=1}^{N} J_{\nu_{j}}(a_{j}m) = \int_{0}^{\infty} dt \ t^{-\lambda} \prod_{j=1}^{N} J_{\nu_{j}}(a_{j}t) + \frac{1}{2} \lim_{s \to 0} s^{-\lambda} \prod_{j=1}^{N} J_{\nu_{j}}(a_{j}s)
$$

(36)

$$
+ i \int_{0}^{\infty} dy \ \frac{dy}{e^{2\pi y} - 1} \left[ (iy)^{-\lambda} \prod_{j=1}^{N} J_{\nu_{j}}(ia_{j}y) - (-iy)^{-\lambda} \prod_{j=1}^{N} J_{\nu_{j}}(-ia_{j}y) \right].
$$

By introducing the $\varepsilon_{m}$ factor of (5), we may shift the second term on the RHS of the first line to the LHS. Secondly, we note that $J_{\mu}(it) \equiv i^{\mu} I_{\mu}(t)$, where $I_{\mu}$ is a modified Bessel function of the first kind. This leads to

$$
\sum_{m=0}^{\infty} \varepsilon_{m} m^{-\lambda} \prod_{j=1}^{N} J_{\nu_{j}}(a_{j}m) = \int_{0}^{\infty} dt \ t^{-\lambda} \prod_{j=1}^{N} J_{\nu_{j}}(a_{j}t)
$$

(37)

$$
+ i \int_{0}^{\infty} dy \ \frac{dy}{e^{2\pi y} - 1} \left[ \prod_{j=1}^{N} I_{\nu_{j}}(a_{j}y) - (-1)^{-\lambda} \prod_{j=1}^{N} I_{\nu_{j}}(-a_{j}y) \right].
$$

For real arguments $t$ (as here, since we assume all $a_{j}$ to be real), $I_{\nu}(t) \equiv (t)^{\nu} I_{\nu}(t) \equiv (-1)^{\nu} I_{\nu}(t)$, thus

$$
\sum_{m=0}^{\infty} \varepsilon_{m} m^{-\lambda} \prod_{j=1}^{N} J_{\nu_{j}}(a_{j}m) = \int_{0}^{\infty} dt \ t^{-\lambda} \prod_{j=1}^{N} J_{\nu_{j}}(a_{j}t)
$$

(38)

$$
+ i \int_{0}^{\infty} dy \ \frac{dy}{e^{2\pi y} - 1} \left[ 1 - (-1)^{-\lambda + \sum_{j=1}^{N} \nu_{j}} \right] \prod_{j=1}^{N} I_{\nu_{j}}(a_{j}y).
$$

If $\sum_{j=1}^{N} \nu_{j} - \lambda = 2k$ for $k \in \mathbb{Z}$, the second line vanishes identically and we obtain the desired result (8).

We finally consider the restrictions on $\lambda$, $\{a_{j}\}$ and $\{\nu_{j}\}$ arising from the Abel-Plana validity conditions of §3. If none of $\nu_{j}$ are negative integers, condition 1 (§4.1) necessitates $\mathrm{Re} \left( \sum_{j=1}^{N} \nu_{j} - \lambda \right) > 0 \quad \text{unless} \quad \sum_{j=1}^{N} \nu_{j} - \lambda = 0$; here, $\sum_{j=1}^{N} \nu_{j} - \lambda = 2k$, thus the condition is satisfied for all $k \in \mathbb{N}_{0}$. If some of $\{\nu_{j}\}$ are negative integers, in those with $j$ in some set $J$, we instead find a slightly more lenient condition

$$
\sum_{j \in J} (|\nu_{j}| + 2k) \geq 0 \quad \Rightarrow \quad k \geq - \sum_{j \in J} |\nu_{j}|.
$$

(39)

From condition 3 (§4.3), $\mathrm{Re}(\lambda) > -N/2$ and thus $\sum_{j=1}^{N} \mathrm{Re}(\nu_{j}) - 2k > -N/2$. Furthermore, we require $\sum_{j=1}^{N} |a_{j}| < 2\pi$ unless $\sum_{j=1}^{N} |a_{j}| = 2\pi$ and $\mathrm{Re}(\lambda) > 1 - N/2$: this implies that $\sum_{j=1}^{N} \mathrm{Re}(\nu_{j}) - 2k > 1 - N/2$. Finally, from condition 4 (§4.4), $\sum_{j=1}^{N} \mathrm{Re}(\nu_{j}) - 2k > 1 - N/2$ is also required if the exists a vector $\{s_{j}\} = \pm 1, \pm 1, \ldots$ such that $\sum_{j=1}^{N} s_{j}a_{j} = 0$. These assumptions additionally satisfy the remaining conditions, thus we arrive at the result of §2.
6 Alternate Proof via the Poisson Summation Formula

Below, we sketch an alternative proof of (8), using the Poisson summation formula.\(^1\) For this, first consider a general function \(g : \mathbb{R} \rightarrow \mathbb{C}\). Denoting the Fourier transform of \(g\) by \(\hat{g}\), the Poisson summation formula links the infinite sum of \(g\) and \(\hat{g}\):

\[
h \sum_{n=-\infty}^{\infty} g(nh) = \sum_{k=-\infty}^{\infty} \hat{g}(kh) = \sum_{k=-\infty}^{\infty} \hat{g}(k/h) \tag{40}
\]

for arbitrary \(h \in \mathbb{R}\) [e.g., 35], assuming that \(\hat{g}\) exists. If \(g\) is band-limited, such that \(\hat{g}(p)\) has support only for \(p \in (-h, h)\), then the RHS reduces to \(\hat{g}(0)\). Inserting the integral definition of \(\hat{g}(0)\), this gives

\[
h \sum_{n=-\infty}^{\infty} g(nh) = \hat{g}(0) = \int_{-\infty}^{\infty} dt \, g(t) \tag{41}
\]

To apply this result in our context, we must first ascertain whether \(f(t)\) (12) is band-limited. For this purpose, we first note that the Bessel functions may be written in integral form as

\[
J_{\nu}(t) = \frac{(t/2)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^\pi d\theta \, \sin^\nu(\theta/2) \cos(\theta/2) \tag{42}
\]

for \(\text{Re}(\nu) > -1/2\) [36, §10.9.4]. Since the RHS is a weighted average of cosines in \(t\) with frequencies in the range \([-1/(2\pi), 1/(2\pi)]\), it follows that the Fourier transform of \(J_{\nu}(t)\) has support only over this range. The product of \(N\) Bessel functions with scaling parameters \(\{a_j\}\) can thus be written in a form involving the product of \(N\) cosines in \(t\), each of which have frequencies in the range \([-a_j/(2\pi), a_j/(2\pi)]\). Since \(t^{-\lambda}\) is a polynomial, and thus of zero bandwidth, the function \(f(z)\) (12) contains only frequencies in the range \([-a_j/(2\pi), a_j/(2\pi)]\) and will be band-limited by \([-1, 1]\), provided that \(\sum_j a_j < 2\pi\) (for real \(a_j\)).\(^2\) Applying the Poisson summation formula with \(h = 1\) leads to

\[
\sum_{m=-\infty}^{\infty} f(m) = \hat{f}(0) = \int_{-\infty}^{\infty} dt \, f(t) \tag{43}
\]

We can remove the \(m < 0\) and \(t < 0\) terms by relabelling:

\[
\frac{1}{2} \sum_{m=0}^{\infty} \varepsilon_m \left[ f(m) + f(-m) \right] = \frac{1}{2} \hat{f}(0) = \frac{1}{2} \int_{0}^{\infty} dt \, [f(t) + f(-t)], \tag{44}
\]

introducing the \(\varepsilon_m\) coefficient (5) to capture the special case \(m = 0\). Finally, we insert the definition of \(f\), giving

\[
\sum_{m=0}^{\infty} \varepsilon_m m^{-\lambda} \left[ 1 + (-1)^{-\lambda} \sum_{j=1}^{N} \nu_j \right] \prod_{i=1}^{N} [J_{\nu}(a_j m)] = \int_{0}^{\infty} dt \, t^{-\lambda} \left[ 1 + (-1)^{-\lambda} \sum_{j=1}^{N} \nu_j \right] \prod_{j=1}^{N} [J_{\nu}(a_j t)] , \tag{45}
\]

where we have noted that \(J_{\nu}(-at) = (-1)^{\nu} J_{\nu}(at)\) [36, §10.11.1]. Setting \(\lambda = \sum_j \nu_j - 2k\) for \(k \in \mathbb{N}_0\) as before, we obtain the desired result (8). The remaining conditions in §4 ensure that (a) the Poisson summation formula is valid, and (b) the integral and sum in (45) exist.

\(^1\)We thank Jeremy Goodman for suggesting this approach.

\(^2\)This is true also for \(\sum_j a_j = 2\pi\) if there are no beat-frequencies, as in §4.4.
7 Summation Convergence

We briefly comment on the convergence of the infinite sum appearing in (8). This is important for assessing the utility of the result as a method to evaluate Bessel function integrals. For this purpose, we consider truncating \( \sum_{m=0}^{\infty} f(m) \) at \( m = M \) for large \( M \). By the integral test for convergence, \( \sum_{m=M}^{\infty} f(m) \) converges absolutely iff \( \int_{M}^{\infty} dx \, |f(x)| \) converges. Condition 5 (§4.5) implies that

\[
\int_{M}^{\infty} dx \, |f(x)| \leq K \int_{M}^{\infty} dx \, x^{-[\text{Re}(\lambda)+N/2]}, \quad (46)
\]

The integral exists for all \( \text{Re}(\lambda) + N/2 > 1 \), i.e. the sum is absolutely convergent for \( \sum_{j=0}^{n} \text{Re}(\nu_j) > 2k - N/2 + 1 \). Furthermore, the integral test also states that

\[
\sum_{m=M}^{\infty} |f(m)| \leq f(M) + \int_{M}^{\infty} dx \, f(x) = KM^{-[\text{Re}(\lambda)+N/2]} \left[ 1 + \frac{M}{1 - [\text{Re}(\lambda)+N/2]} \right] \quad (47)
\]

bounding the truncation error.

Considering the case \( 0 < \text{Re}(\lambda) + N/2 < 1 \), the sum \( \sum_{m=0}^{\infty} f(m) \) is instead conditionally convergent. This can be shown in a manner analogous to the derivation of condition 4 (§4.4), first separating out the various Fourier frequencies (cf. 33):

\[
\sum_{m=M}^{\infty} f(m) = K' \sum_{m=M}^{\infty} m^{-\lambda-N/2} \sum_{i=1}^{2^N} \cos(A_i m + B_i), \quad (48)
\]

for some \( K' \) and sufficiently large \( M \), where the validity conditions of §4 ensure that \( A_i \neq 0 \). Considering a single frequency \( A_i \) (i.e. one of the \( 2^N \) terms in the summation over \( i \)), the summation over \( m \) may be recast as an alternating series:

\[
K' \sum_{m=M_*}^{\infty} m^{-\lambda-N/2} \cos(A_i m + B_i) = \sum_{p=0}^{\infty} (-1)^p \sum_{m=M_*+[p\pi/A_i]}^{M_*+[p+\pi/A_i]} m^{-\lambda-N/2} \cos(A_i m + B_i) \quad (49)
\]

where we start at \( M_* > M \), the first increasing zero of \( \cos(A_i m + B_i) \) beyond \( M \). Note that the real part of the summand is explicitly positive. Each term in the \( p \) summation is positive and decreasing, and, assuming \( \text{Re}(\lambda) > -N/2 \), (49) converges by the alternating series test. Truncating at \( M_* \), the error in the term with frequency \( A_i \) cannot be greater than the first excluded \( p \) term, i.e.

\[
\left| \sum_{m=M_*}^{\infty} m^{-\lambda-N/2} \cos(A_i m + B_i) \right| \leq \sum_{m=M_*}^{M_*+[\pi/A_i]} \left| m^{-\lambda-N/2} \cos(A_i m + B_i) \right| \leq K'' M_*^{-[\text{Re}(\lambda)+N/2]} \quad (50)
\]

for some \( K'' > 0 \). This applies to all \( 2^N \) choices of frequency \( A_i \); combining, we find

\[
\left| \sum_{m=M}^{\infty} f(m) \right| = \mathcal{O} \left( M^{-[\text{Re}(\lambda)+N/2]} \right), \quad (51)
\]

giving the relevant truncation error.
Figure 1: Comparison of Bessel function integrals computed via numerical quadrature (light solid lines) and the discrete summations of this work using ten terms (8, dark dashed lines). Results are shown for products of two, three and four Bessel functions, as detailed in the captions. The \( a_j \) coefficients of \( N-1 \) Bessel function arguments are fixed to \( a_i = \frac{\pi}{16} \) (red), \( 3\frac{\pi}{16} \) (blue) or \( 5\frac{\pi}{16} \) (green), but the \( N \)-th, denoted by \( b \), is varied. The coefficients satisfy the validity criteria of §2 for all \( b \) to the left of the dotted vertical lines, thus we expect the sums and integrals to agree in the limit of infinite sampling points. Note that case (b) has \( \sum_j \Re(\nu_j) > 2k - N/2 + 1 \), and is thus not expected to converge at \( \sum_j a_j = 2\pi \); further, it is only conditionally convergent. Furthermore, case (c) has \( k = -1 \); possible since one of the Bessel functions has a negative integer argument. We find good convergence in all cases; this can be improved further by including more points in the discrete summation or increasing the domain limit (set here to \( t \in [0,10] \)).

8 Practical Demonstration & Conclusions

Fig. 1 presents a demonstration of our main result (8). For this purpose, we consider three representative choices of hyperparameters \( \{a_j\}, \{\nu_j\}, N \) and \( k \), which satisfy the conditions of §2. In each case, we compute both the LHS and RHS of (8) separately, using numerical quadrature to perform the integral (truncating at \( t_{\text{max}} = 10 \)) and evaluate the sum including the first 10 terms. In the regime \( \sum_{j=1}^N |a_j| < 2\pi \), the results are found to be in good agreement; discrepancies can be reduced
by using a greater number of terms or a larger \( s_{\text{max}} \). This is additionally true for the special case of negative integer \( \nu_j \) (since Fig. 1c converges with \( k = -1 \)), \( \sum_j \text{Re}(\nu_j) > 2k - N/2 + 1 \), which is only conditionally convergent. For \( \sum |a_j| > 2\pi \), the summation does not converge to the integral result, as expected. In practice, this can be avoided via (10) which extends the validity to all \( \sum_{j=1}^{N} |a_j| \).

In conclusion, we find that, subject to a number of assumptions, we can derive a useful formula relating the sums and integrals of products of \( N \) Bessel functions of both real and complex order, allowing straightforward evaluation of highly oscillatory integrals. The result is straightforward to prove using the Abel-Plana theorem, and agrees with previous work for \( N = 2 \), but with a significantly simpler proof. Whilst we consider only spherical Bessel functions in this work, we expect similar results to apply to other functions, provided they obey mild growth conditions and the relation \( f(iy) = f(-iy) \) for real \( y \).

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