Calculation of Lebesgue Integrals by Using Uniformly Distributed Sequences

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Abstract: We present the proof of a certain modified version of Kolmogorov’s strong law of large numbers for calculation of Lebesgue Integrals by using uniformly distributed sequences in (0, 1). We extend the result of C. Baxa and J. Schoenberg (cf.[8], Theorem 1, p. 271) to a maximal set of uniformly distributed (in (0, 1)) sequences Sf ⊂ (0, 1)∞ which strictly contains the set of sequences of the form ({αn})n∈N with irrational number α and for which ℓ∞1(Sf) = 1, where ℓ∞1 denotes the infinite power of the linear Lebesgue measure ℓ1 in (0, 1).

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1. Introduction

In this note we show that the technique for numerical calculation of some one-dimensional Lebesgue integrals is similar to the technique which was given by Hermann Weyl’s [1] celebrated theorem as follows.

Theorem 1.1. ([2], Theorem 1.1, p. 2) The sequence (xn)n∈N of real numbers is u.d. mod 1 if and only if for every real-valued continuous function f defined on the closed unit interval [0, 1] we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f\left(\{x_n\}\right) = \int_{0}^{1} f(x)dx,$$

(1.1)

where \(\{\cdot\}\) denotes the fractional part of the real number.

Main corollaries of this theorem successfully were used in Diophantine approximations and have applications to Monte-Carlo integration (see, for example, [2],[3],[4]). During the last decades the methods of the theory of uniform distribution modulo one have been intensively used for calculation of improper Riemann integrals(see, for example, [6],[8]).

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In this note we are going to consider some applications of Kolmogorov strong
law of large numbers which can be considered as a certain extension of the Her-
mann Weyl’s above mentioned theorem from the class of Riemann’s integrable
functions to the class of Lebesgue integrable functions. We present our proof of
this century theorem which differs from Kolmogorov’s original proof. Further,
by using this theorem we present a certain improvement of the following result
of C. Baxa and J. Schoiβengeier

Theorem 1.2. ([8], Theorem 1, p. 271) Let \( \alpha \) be an irrational number, \( \mathbb{Q} \)
be a set of all rational numbers and \( F \subseteq [0, 1] \cap \mathbb{Q} \) be finite. Let \( f : [0, 1] \to \mathbb{R} \) be
an integrable, continuous almost everywhere and locally bounded on \([0, 1]\backslash F\).
Assume further that for every \( \beta \in F \) there is some neighbourhood \( U \) of \( \beta \)
such that \( f \) is either bounded or monotone in \([0, \beta) \cap U\) and in \((\beta, 1] \cap U\) as well.
Then the following conditions are equivalent:

1) \( \lim_{n \to \infty} \frac{f(x_n)}{n} = 0 \);
2) \( \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) \) exists;
3) \( \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_{(0,1)} f(x) dx \);

are equivalent

More precisely, we will extend the result of Theorem 1.2 to a maximal set
\( S_f \subset (0, 1) \) of uniformly distributed (in \((0, 1)\)) sequences strictly containing
all sequences of the form \( \{\alpha n\}_{n \in \mathbb{N}} \) where \( \alpha \) is an irrational numbers and for
which \( \ell_1^\infty(S_f) = 1 \), where \( \ell_1^\infty \) denotes the infinite power of the linear Lebesgue
measure \( \ell_1 \) in \((0, 1)\).

The paper is organized as follows.
In Section 2 we consider some auxiliary notions and facts from the theory of
uniformly distributed sequences and probability theory. In Section 3 we present
our main results.

2. Some auxiliary facts from probability theory

Definition 2.1. A sequence \( s_1, s_2, s_3, \cdots \) of real numbers from the interval \([0, 1]\)
is said to be uniformly distributed in the interval \([0, 1]\) if for any subinterval \([c, d]\)
of the \([0, 1]\) we have

\[
\lim_{n \to \infty} \frac{\#(\{s_1, s_2, s_3, \cdots, s_n\} \cap [c, d])}{n} = d - c,
\]

where \( \# \) denotes the counting measure.

Example 2.1. ([2], Exercise 1.12, p. 16) The sequence of all multiples of an
irrational \( \alpha \)

\[ 0, \{\alpha\}, \{2\alpha\}, \{3\alpha\} \cdots \]
is uniformly distributed in \((0, 1)\), where \( \{\cdot\} \) denotes the fractional part of the
real number.
Lemma 2.1. ([2] Theorem 2.2, p.183) Let $S$ be a set of all elements of $[0,1]^\infty$ which are uniformly distributed in the interval $[0,1]$. Then $\ell_1^\infty(S) = 1$, where $\ell_1^\infty$ denotes the infinite power of the standard linear Lebesgue measure $\ell_1$ in $[0,1]$.

We need some auxiliary fact from mathematical analysis and probability theory.

Lemma 2.2. (Kolmogorov-Khinchin ([7], Theorem 1, p.371)) Let $(X, S, \mu)$ be a probability space and let $(\xi_n)_{n \in \mathbb{N}}$ be the sequence of independent random variables for which $\int_X \xi_n(x) d\mu(x) = 0$. If $\sum_{n=1}^{\infty} \int_X \xi_n^2(x) d\mu(x) < \infty$, then the series $\sum_{n=1}^{\infty} \xi_n$ converges with probability 1.

Lemma 2.3. (Toeplitz Lemma ([7], Lemma 1, p. 377)) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of non-negative numbers, $b_n = \sum_{i=1}^{n} a_i$, $b_n > 0$ for each $n \geq 1$ and $b_n \uparrow \infty$, when $n \to \infty$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\lim_{n \to \infty} x_n = x$. Then

$$\lim_{n \to \infty} \frac{1}{b_n} \sum_{j=1}^{n} a_j x_j = x.$$ 

In particular, if $a_n = 1$ for $n \in \mathbb{N}$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k = x.$$ 

Lemma 2.4. (Kroneker Lemma ([7], Lemma 2, p.378)) Let $(b_n)_{n \in \mathbb{N}}$ be an increasing sequence of positive numbers such that $b_n \uparrow \infty$, when $n \to \infty$, and let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that the series $\sum_{k \in \mathbb{N}} x_k$ converges. Then

$$\lim_{n \to \infty} \frac{1}{b_n} \sum_{j=1}^{n} b_j x_j = 0.$$ 

In particular, if $b_n = 0$, $x_n = \frac{\omega_n}{n}$ and the series $\sum_{n=1}^{\infty} \frac{\omega_n}{n}$ converges then

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \omega_k}{n} = 0.$$ 

Below we give the proof of a certain modification of the Kolmogorov Strong Law of Large Numbers ([7], Theorem 3, p.379).

Lemma 2.5. Let $(X, \mathcal{F}, \mu)$ be a probability space and let $L(X)$ be a class of all real-valued Lebesgue measurable functions on $X$. Let $\mu_\infty$ be an infinite power of the probability measure $\mu$. Then for $f \in L(X)$ we have

$$\mu_\infty \{(x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in X^\infty \land \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_X f(x) dx\} = 1.$$ 

Proof. Without loss of generality, we can assume that $f$ is non-negative. We put $\xi_k((x_i)_{i \in \mathbb{N}}) = f(x_k)$ for $k \in \mathbb{N}$ and $(x_i)_{i \in \mathbb{N}} \in X^\infty$. We put also

$$\eta_k((x_i)_{i \in \mathbb{N}}) =$$
\[
\frac{1}{k} \sum_{i=1}^{\infty} \int_{X} \eta_k((x_i)_{i \in N}) \chi_{\{\omega : \xi_k(\omega) < k\}}((x_i)_{i \in N}) d\mu_\infty((x_i)_{i \in N}) - \int_{X} \sum_{i=1}^{\infty} \chi_{\{\omega : \xi_k(\omega) < k\}}((x_i)_{i \in N}) d\mu_\infty((x_i)_{i \in N})
\]

for \((x_i)_{i \in N} \in X^\infty\).

Note that \((\eta_k)_{k \in N}\) is the sequence of independent random variable for which
\[
\int_{X^\infty} \eta_k d\mu_\infty = 0.
\]

We have
\[
\sum_{n=1}^{\infty} \int_{X} \eta_n^2((x_i)_{i \in N}) d\mu_\infty((x_i)_{i \in N}) = 
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \int_{X} \xi_n^2((x_i)_{i \in N}) \chi((y_i)_{i \in N} : \xi_n((y_i)_{i \in N}) < n) d\mu_\infty((x_i)_{i \in N}) - 
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \int_{X} \xi_n((x_i)_{i \in N}) \chi((y_i)_{i \in N} : \xi_n((y_i)_{i \in N}) < n) d\mu_\infty((x_i)_{i \in N})^2 = 
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \int_{X} f(x_n)^2 \chi((y_i)_{i \in N} : f(y_n) < n) d\mu_\infty((x_i)_{i \in N}) - 
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \int_{X} f(x_n) \chi((y_i)_{i \in N} : f(y_n) < n) d\mu_\infty((x_i)_{i \in N})^2 = 
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \int_{X} f(x)^2 \chi_{\{\omega : f(\omega) < n\}} d\mu(x) - \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{X} f(x) \chi_{\{\omega : f(\omega) < n\}} d\mu(x)^2 \leq 
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \int_{X} f(x)^2 \chi_{\{\omega : f(\omega) < n\}} d\mu(x) = 
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \left( \sum_{k=1}^{n} \int_{X} f^2(x) \chi_{\{\omega : k-1 \leq f(\omega) < k\}} d\mu(x) \right) = 
\]
\[
\sum_{k=1}^{\infty} \int_{X} f^2(x) \chi_{\{\omega : k-1 \leq f(\omega) < k\}} d\mu((x)) \sum_{n=k}^{\infty} \frac{1}{n^2} \leq 
\]
\[
2 \sum_{k=1}^{\infty} \frac{1}{k} \int_{X} f^2(x) \chi_{\{\omega : k-1 \leq f(\omega) < k\}} d\mu(x) \leq 
\]
\[
2 \sum_{k=1}^{\infty} \int_{X} f(x) \chi_{\{\omega : k-1 \leq f(\omega) < k\}} d\mu((x)) + 2 \int_{X} f(x) d\mu(x). 
\]

Since
\[
\sum_{n=1}^{\infty} \int_{X} \eta_n^2((x_i)_{i \in N}) d\mu((x_i)_{i \in N}) < +\infty,
\]

by using Lemma 2.2 we get
\[
\mu\{(x_i)_{i \in N} : \sum_{k=1}^{\infty} \frac{1}{k} \int_{X} f(x_k) \chi_{\{\omega : f(x_k) < k\}}((x_i)_{i \in N}) \}
\]
\[
\int_{X}^{\infty} \xi_k((x_i)_{i \in N}) \chi_{\{ (x_i)_{i \in N} : f(x_k) < k \}} d\mu_{\infty}((x_i)_{i \in N}) \]

is convergent = 1.

On the other hand, we have
\[
\sum_{n=1}^{\infty} \mu_{\infty}(\{ (x_i)_{i \in N} : \xi_1((x_i)_{i \in N}) \geq n \}) = \sum_{n=1}^{\infty} \sum_{k \geq n} \mu_{\infty}( (x_i)_{i \in N} : k \leq \xi_1((x_i)_{i \in N}) < k+1 ) =
\]
\[
\sum_{k=1}^{\infty} k \mu_{\infty}( (x_i)_{i \in N} : k \leq \xi_1((x_i)_{i \in N}) < k+1 ) = \sum_{k=0}^{\infty} \int_{X}^{\infty} [k \chi_{\{ (x_j)_{j \in N} : k \leq \xi_1((x_j)_{j \in N}) < k+1 \}}]
\]
\[
\sum_{k=0}^{\infty} \int_{X}^{\infty} [\xi_1((x_i)_{i \in N}) \chi_{\{ (x_j)_{j \in N} : k \leq \xi_1((x_j)_{j \in N}) < k+1 \}}] = \int_{X}^{\infty} \xi_1((x_i)_{i \in N}) d\mu_{\infty}((x_i)_{i \in N}) < +\infty.
\]

Since \((\xi_k)_{k \in N}\) is a sequence of equally distributed random variables on \(X\), we have
\[
\sum_{n=1}^{\infty} \mu_{\infty}(\{ (x_i)_{i \in N} : \xi_k((x_i)_{i \in N}) \geq n \}) \leq \int_{X}^{\infty} \xi_1((x_i)_{i \in N}) d\mu_{\infty}((x_i)_{i \in N}) < +\infty,
\]

which by the well-known Borel-Cantelli lemma implies that
\[
\mu_{\infty}(\{ (x_i)_{i \in N} : \xi_n((x_i)_{i \in N}) \geq n \} i.m.) = 0.
\]

The last relation means that
\[
\mu_{\infty}(\{ (x_i)_{i \in N} : (\exists N((x_i)_{i \in N}))(\forall n \geq N((x_i)_{i \in N}) \rightarrow \xi_n((x_i)_{i \in N}) < n) \} = 1.
\]

Thus, we have obtained the validity of the following condition
\[
\mu_{\infty}( (x_i)_{i \in N} : \sum_{k=1}^{\infty} \frac{1}{k} [f(x_k) \chi_{\{ (y_i)_{i \in N} : f(y_k) < k \}}((x_i)_{i \in N}) -
\]
\[
\int_{X}^{\infty} \xi_k((x_i)_{i \in N}) \chi_{\{ (y_i)_{i \in N} : f(y_k) < k \}}((x_i)_{i \in N}) d\mu_{\infty}((x_i)_{i \in N}) ] is convergent &
\]
\[
(\exists N((x_i)_{i \in N}))(\forall n \geq N((x_i)_{i \in N}) \rightarrow \xi_n((x_i)_{i \in N}) < n) \} = 1.
\]

By Lemma 2.4 we get that \(\mu_{\infty}(D) = 1\), where
\[
D = \{ (x_i)_{i \in N} : \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} [f(x_k) \chi_{\{ (y_i)_{i \in N} : f(y_k) < k \}}((x_i)_{i \in N}) -
\]
\[
\int_{X}^{\infty} \xi_k((x_i)_{i \in N}) \chi_{\{ (y_i)_{i \in N} : f(y_k) < k \}}((x_i)_{i \in N}) d\mu_{\infty}((x_i)_{i \in N}) ] = 0 &
\]
\[
(\exists N((x_i)_{i \in N}))(\forall n > N((x_i)_{i \in N}) \rightarrow \xi_n((x_i)_{i \in N}) < n) \}.\]
Now it is obvious that for \((x_i)_{i \in \mathbb{N}} \in D\), we have

\[
0 = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \left[ f(x_k) \chi_{\{(y_i)_{i \in \mathbb{N}} : f(y_k) < k\}}(\{x_i\}_{i \in \mathbb{N}}) - \int_{X} \xi_k((x_i)_{i \in \mathbb{N}}) \chi_{\{(y_i)_{i \in \mathbb{N}} : f(y_k) < k\}}(\{x_i\}_{i \in \mathbb{N}}) \, d\mu(\{x_i\}_{i \in \mathbb{N}}) \right] = \\
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \int_{X} \xi_k((x_i)_{i \in \mathbb{N}}) \chi_{\{(y_i)_{i \in \mathbb{N}} : f(y_k) < k\}}(\{x_i\}_{i \in \mathbb{N}}) \, d\mu(\{x_i\}_{i \in \mathbb{N}}) = \\
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \int_{X} f(x_k) \chi_{\{y : f(y) < k\}}(x) \, d\mu(x) = \\
\lim_{N \to \infty} \frac{1}{N} \left[ \sum_{k=1}^{N} \int_{X} f(x_k) \chi_{\{y : f(y) < k\}}(x) \, d\mu(x) \right].
\]

Since

\[
\lim_{k \to \infty} \int_{X} f(x) \chi_{\{y : f(y) < k\}} \, d\mu(x) = \int_{X} f(x) \, d\mu(x),
\]

by Lemma 2.3 we get

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \int_{X} f(x_k) \chi_{\{y : f(y) < k\}}(x) \, d\mu(x) = \int_{X} f(x) \chi_{\{y : f(y) < k\}} \, d\mu(x)
\]

which implies that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_{X} f(x) \, d\mu(x)
\]

for each \((x_i)_{i \in \mathbb{N}} \in D\).

This ends the proof of theorem.

\[ \square \]

**Remark 2.1.** Formulation of Lemma 2.4 (cf. [5], p.285) needs a certain specification. More precisely, it should be formulated for sequences \((x_k)_{k \in \mathbb{N}} \in S \cap D\), where \(S\) comes from Lemma 2.1 and, \(D\) comes from Lemma 2.5 when \((X, F, \mu) = (0, 1), B(0, 1), \ell_1\). Since \(\ell_1(S \cap D) = 1\), such reformulated Lemma 2.4 can be used for the proof of Corollary 4.2 (cf. p. 296).
3. Main Results

By using Lemmas 2.1 and 2.5, we get

**Theorem 3.1.** Let $f$ be a Lebesgue integrable real-valued function on $(0, 1)$. Then we have

\[
\ell_1^\infty([\{x_k\}_k : (x_k)_k \in \mathbb{N}] \& (x_k)_{k \in \mathbb{N}} \text{ is uniformly distributed in } (0, 1) & \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_{0}^{1} f(x) \, dx) = 1.
\]

**Remark 3.1.** Let $f : (0, 1) \to \mathbb{R}$ be a Lebesgue integrable function. By Theorem 3.1 we have $\ell_1^\infty(A_f) = 1$, where

\[
A_f = \{(x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in (0, 1)^\infty & \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_{(0,1)} f(x) \, dx\}.
\]

**Corollary 3.1.** Let $f : (0, 1) \to \mathbb{R}$ be Lebesgue integrable function. Then we have $\ell_1^\infty(B_f) = 1$, where

\[
B_f = \{(x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in (0, 1)^\infty & \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) \text{ exists}\}.
\]

**Proof.** Since $A_f \subseteq B_f$, by Remark 3.1 we get

\[
1 = \ell_1(A_f) \leq \ell_1(B_f) \leq \ell_1((0, 1)^\infty) = 1.
\]

**Corollary 3.2.** Let $f : (0, 1) \to \mathbb{R}$ be Lebesgue integrable function. Then we have $\ell_1^\infty(C_f) = 1$, where

\[
C_f = \{(x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in (0, 1)^\infty & \lim_{N \to \infty} \frac{f(x_N)}{N} = 0\}.
\]

**Proof.** Note that $A_f \subseteq C_f$. Indeed, let $(x_k)_{k \in \mathbb{N}} \in A_f$. Then we get

\[
\lim_{N \to \infty} \frac{f(x_N)}{N} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N-1} f(x_k) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N-1} f(x_k) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \lim_{N \to \infty} \frac{1}{N-1} \sum_{k=1}^{N-1} f(x_k) = 0.
\]
By Remark 3.1 we know that $\ell_1^\infty(A_f) = 1$ which implies $1 = \ell_1^\infty(A_f) \leq \ell_1^\infty(C_f) \leq \ell_1^\infty((0,1)\infty) = 1$.

\[\blacktriangle\]

**Remark 3.2.** Note that for each Lebesgue integrable function $f$ in $(0,1)$, the following inclusion $S \cap A_f \subseteq S \cap C_f$ holds true, but the converse inclusion is not always valid. Indeed, let $(x_k)_{k \in \mathbb{N}}$ be an arbitrary sequence of uniformly distributed numbers in $(0,1)$. Then the function $f : (0,1) \to \mathbb{R}$, defined by $f(x) = \chi_{(0,1)\setminus\{x_k\}_{k \in \mathbb{N}}}(x)$ for $x \in (0,1)$, is Lebesgue integrable, $(x_k)_{k \in \mathbb{N}} \in C_f \cap S$ but $(x_k)_{k \in \mathbb{N}} \notin A_f \cap S$ because

\[\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = 0 \neq 1 = \int_{(0,1)} f(x) dx.\]

**Theorem 3.2.** Let $f : (0,1) \to \mathbb{R}$ be Lebesgue integrable function. Then the set $S_f$ of all sequences $(x_k)_{k \in \mathbb{N}} \in (0,1)\infty$ for which the following conditions hold:

1) $\lim_{n \to \infty} \frac{f(x_n)}{n} = 0$;
2) $\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k)$ exists;
3) $\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_{(0,1)} f(x) dx$;
4) $(x_k)_{k \in \mathbb{N}}$ is uniformly distributed in $(0,1)$

are equivalent, has $\ell_1^\infty$ measure one.

**Proof.** By Lemma 2.1 we know that $\ell_1^\infty(S) = 1$. By Remark 3.1 we have $\ell_1^\infty(A_f) = 1$. Following Corollaries 3.1 and 3.2 we have $\ell_1^\infty(B_f) = 1$ and $\ell_1^\infty(C_f) = 1$, respectively. Since $S_f = A_f \cap B_f \cap C_f \cap S$, we get

$$\ell_1^\infty(D) = \ell_1^\infty(A_f \cap B_f \cap C_f \cap S) = 1.$$ 

The next corollary is a simple consequence of Theorem 3.1.

**Corollary 3.3.** Let $Q$ be a set of all rational numbers of $[0,1]$ and $F \subseteq [0,1] \cap Q$ be finite. Let $f : [0,1] \to \mathbb{R}$ be Lebesgue integrable, $\ell_1$-almost everywhere continuous and locally bounded on $[0,1] \setminus F$. Assume that for every $\beta \in F$ there is some neighbourhood $U_\beta$ of $\beta$ such that $f$ is either bounded or monotone in $[0,\beta) \cap U_\beta$ and in $(\beta,1] \cap U_\beta$ as well. Let $S, A_f, B_f, C_f$ come from Lemma 2.1, Remark 3.1, Corollary 3.1,Corollary 3.2, respectively. We set $S_f = (A_f \cap B_f \cap C_f \cap S) \cup ((0,1)^\infty \setminus A_f) \cap ((0,1)^\infty \setminus B_f) \cap ((0,1)^\infty \setminus C_f) \cap S$.

Then for $(x_k)_{k \in \mathbb{N}} \in S_f$ the following conditions are equivalent:

1) $\lim_{n \to \infty} \frac{f(x_n)}{n} = 0$;
2) $\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k)$ exists;
3) $\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_{(0,1)} f(x) dx$;
Remark 3.3. Note that $S_f$ is maximal subset of the set $S$ for which conditions 1)-3) of Corollary 3.3 are equivalent, provided that for each $(x_k)_{k \in \mathbb{N}} \in S_f$ the sentences 1)-3) are true or false simultaneously, and for each $(x_k)_{k \in \mathbb{N}} \in S \setminus S_f$ the sentences 1)-3) are not true or false simultaneously. This extends the main result of Baxa and Schoiβengeier [8] because, the sequence of the form $(\{n\alpha\})_{n \in \mathbb{N}}$ is in $S_f$ for each irrational number $\alpha$, and no every element of $S_f$ can be presented in the same form. For example,

$$((n + 1/2(1 - \chi_{\{k: k \geq 2\}}(n)))\pi^{\chi_{\{k: k \geq 2\}}(n)})_{n \in \mathbb{N}} \in D \setminus S^*,$$

where $\{\cdot\}$ denotes the fractional part of the real number and $\chi_{\{k: k \geq 2\}}$ denotes the indicator function of the set $\{k: k \geq 2\}$.

Similarly, setting

$$D_f = (A_f \cap B_f \cap C_f) \cap (0,1)^{\infty} \setminus (A_f) \cap ((0,1)^{\infty} \setminus B_f) \cap ((0,1)^{\infty} \setminus C_f)).$$

we get a maximal subset of $(0,1)^{\infty}$ for which conditions 1)-3) of Corollary 3.3 are equivalent, provided that for each $(x_k)_{k \in \mathbb{N}} \in D_f$ the sentences 1)-3) are true or true simultaneously, and for each $(x_k)_{k \in \mathbb{N}} \in (0,1)^{\infty} \setminus D_f$ the sentences 1)-3) are not true or false simultaneously.

References

[1] H. Weyl, Über ein Problem aus dem Gebiete der diophantischen Approximation, Math.-phys. K1. (1916), 234-244.

[2] L. Kuipers, H. Niederreiter, Uniform distribution of sequences, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney (1974).

[3] G. Hardy, J. Littlewood, Some problems of diophantine approximation, Acta Math. 37 (1) (1914), 193–239.

[4] G. Hardy, J. Littlewood, Some problems of diophantine approximation, Acta Math. 37 (1) (1914), 155–191.

[5] G. R. Pantsulaia, Infinite-dimensional Monte-Carlo integration. Monte Carlo Methods Appl. 21 (2015), no. 4, 283–299.

[6] I. M. Sobol, Computation of improper integrals by means of equidistributed sequences, (Russian) Dokl. Akad. Nauk SSSR. 210 (1973), 278–281.

[7] Shiryaev A.N., Probability (in Russian), Izd.Nauka, Moscow, 1980.

[8] C. Baxa, J. Schoiβengeier, Calculation of improper integrals using $(n\alpha)$-sequences, Dedicated to Edmund Hlawka on the occasion of his 85 th birthday. Monatsh. Math. 135(4) (2002), 265–277.

[9] S.M. Nikolski, Course of mathematical analysis (in Russian), no. 1, Moscow (1983).