Some qualitative properties for the Kirchhoff total variation flow

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ABSTRACT
The first goal of this paper is to establish a result on the existence and uniqueness of solution to an initial-boundary value problem for parabolic equations of Kirchhoff type involving the 1-Laplace operator. The second goal is to discuss some qualitative properties, such as asymptotic behaviour and the extinction of solutions for the considered problem.

1. Introduction and the main results
In recent years, Andreu et al. [1,2] studied the following for the total variation flow problem

$$\begin{align*}
\begin{cases}
\frac{\partial u_t}{\partial t} - \text{div} \left( \frac{Du}{|Du|} \right) = 0 & \text{in } \Omega \times (0, +\infty), \\
u = 0 & \text{on } \partial \Omega \times (0, +\infty), \\
u(x, 0) = u_0(x) & \text{in } \Omega.
\end{cases}
\end{align*}$$

(P)

Under some assumptions on the initial datum $u_0$, they used the nonlinear semigroup theory to prove the existence and uniqueness of solution, as well as they also described the behaviour of solutions to the above problem near the extinction time.

The objective of this paper is to discuss the existence and asymptotic behaviour of solutions near the extinction time to the following class of Kirchhoff type problem involving the 1-Laplace operator

$$\begin{align*}
\begin{cases}
\frac{\partial u_t}{\partial t} - m \left( \int_\Omega |Du| + \int_{\partial \Omega} |u| \, dH^{N-1} \right) \Delta_1 u = 0 & \text{in } \Omega \times (0, +\infty), \\
u = 0 & \text{on } \partial \Omega \times (0, +\infty), \\
u(x, 0) = u_0(x) & \text{in } \Omega,
\end{cases}
\end{align*}$$

(1)

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N \geq 1$, and $\Delta_1 u = \text{div} \left( \frac{Du}{|Du|} \right)$ denotes the 1-Laplace operator. In the mathematical literature, equations in (P)–(1) are also known as very singular diffusion equations, see, for example, Giga et al. [3] and their references. These kinds of problems attracted large attention in the literature due to their applications in image processing, faceted crystal growth, continuum mechanics, for details, see Refs. [3, 4]. In geometry, as it is well known the
unit normal of the level set \( \{ u = k \} \) is given formally by \( \eta(x) = \frac{Du}{|Du|} \), then the mean curvature of this surface is formally given by
\[
H(u) = \text{div}(\eta)(x) = \text{div} \left( \frac{Du}{|Du|} \right),
\]
which turns out that the solution of a parabolic 1-Laplacian problem can be also seen as a solution to the evolution mean curvature flow for the level sets \( \{ u = k \} \), see Ecker [5] for more details.

The first difficulty we encounter when studying problem (P) is how to deal with the singularity of the 1-Laplace operator at \( |Du| = 0 \). Another difficulty here is to give sense to the boundary condition \( u = 0 \) on \( \partial\Omega \) which in general does not necessarily hold in the sense of trace. In order to handle such difficulties, Andreu et al. [1, 6] proposed a definition of weak solution based on the so-called Anzellotti pairing \( (z, Du) \) of \( L^\infty \)-divergence-measure vector field \( z \) and gradient of a BV function \( u \). In their definition, the bounded vector field \( z \) is considered as a substitute of \( \frac{Du}{|Du|} \), while the boundary condition \( u = 0 \) on \( \partial\Omega \) is taken in a very weak sense (see Definition 2.1). Since then the concept of solutions becomes known in the literature, there are two approaches used quite frequently to show the existence of solutions to total variation flow problems. The first one is based on the nonlinear semigroup theory, in particular on techniques of completely accretive operators and the Crandall-Liggett semigroup generation theorem, and when this approach has been used, different assumptions on the initial datum \( u_0 \) are assumed to establish the existence of solutions, see, for example, Refs. [1, 3, 4, 6–8]. In Ref. [9], Andreu et al. extended the work presented in Ref. [1] for (P) to the following total variation followed by nonlinear boundary conditions
\[
\left\{ \begin{array}{ll}
\frac{\partial u}{\partial t} - \text{div} \left( \frac{Du}{|Du|} \right) = 0 & \text{in } \Omega \times (0, +\infty), \\
-\frac{\partial u}{\partial v} \in \beta(u) & \text{on } \partial\Omega \times (0, +\infty), \\
u(x,0) = u_0(x) & \text{in } \Omega,
\end{array} \right. \quad \text{(NP)}
\]
where \( \Omega \) is an open bounded domain in \( \mathbb{R}^N \) with a \( C^1 \) boundary, \( \partial/\partial v \) is the Neumann boundary operator associated to \( \frac{Du}{|Du|} \), i.e.
\[
\frac{\partial u}{\partial v} := \left( \frac{Du}{|Du|}, v \right),
\]
with \( v \) being the unit outward normal vector on \( \partial\Omega \) and \( \beta \) is a maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \) with \( 0 \in \beta(0) \). By using the nonlinear semigroup theory and the Anzellotti pairing, the authors proved the following interesting results:

- If \( u_0 \in L^2(\Omega) \), then there exists a unique strong solution of (NP),
- If \( u_0 \in L^1(\Omega) \), then there exists a unique entropy solution of (NP).

Furthermore, they provided some explicit solutions to (NP). Mazón et al. [10] considered the following problem with dynamical boundary condition
\[
\left\{ \begin{array}{ll}
-\text{div} \left( \frac{Du}{|Du|} \right) = 0 & \text{in } \Omega \times (0, +\infty), \\
u_t + \left[ \frac{Du}{|Du|}, v \right] = g, & \text{on } \partial\Omega \times (0, +\infty), \\
u(x,0) = u_0(x) & \text{in } \Omega,
\end{array} \right. \quad \text{(PD)}
\]
and proved the existence and uniqueness of a semigroup solution of (PD) provided that \( u_0 \in L^2(\Omega) \) and \( g \in L^2(0,T;L^2(\partial\Omega)) \).
The second strategy is based on taking the limit as $p \to 1^+$ of solutions to the following problem

$$
\begin{aligned}
 &u_t - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) = 0 \quad \text{in } \Omega \times (0, +\infty), \\
 &u = 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
 &u(x,0) = u_0(x) \quad \text{in } \Omega.
\end{aligned}
$$

(P)

This approach has been used by many authors in recent years. Among them, we refer the reader to the work by Leon and Webler in Ref. [11], where the authors studied global existence and uniqueness of solutions for an inhomogeneous problem related to (P). Namely, they considered the following problem

$$
\begin{aligned}
 &u_t - \text{div} \left( \frac{Du}{|Du|} \right) = f(x,t) \quad \text{in } \Omega \times (0, +\infty), \\
 &u = 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
 &u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{aligned}
$$

(2)

and proved global existence and uniqueness of solution for (2) via a parabolic $p$-Laplacian problem and then taking the limit $p \to 1^+$ where $u_0 \in L^2(\Omega)$ and $f \in L^1_{\text{loc}}(0, +\infty; L^2(\Omega))$. In Ref. [12] Gianazza and Klaus showed that the solution of (P) converges to a solution of the total variation flow problem (P). Subsequently, in Ref. [13], Alves and Boudjeriou considered the following problem

$$
\begin{aligned}
 &u_t - \text{div} \left( \frac{Du}{|Du|} \right) = f(u) \quad \text{in } \Omega \times (0, +\infty), \\
 &u = 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
 &u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{aligned}
$$

(3)

where $f(u)$ behaving like these functions $f(u) = |u|^{q-2} u e^{\alpha |u|^2}$ for $q > 1, \alpha > 0$ and $f(u) = |u|^{q-2} u + |u|^{s-2} u$ with $q, s \in (1, N/(N - 1))$, and established the existence of global solutions by taking the limit as $p \to 1^+$ of solutions to a parabolic $p$-Laplace problem related to (3). It is worth mentioning that the approximation of proper solutions to the 1-Laplacian in terms of solutions to the $p$-Laplacian as $p \to 1^+$ has been also used to treat this stationary problem

$$
\begin{aligned}
 &-\text{div} \left( \frac{Du}{|Du|} \right) = f(u) \quad \text{in } \Omega, \\
 &u = 0 \quad \text{on } \partial \Omega,
\end{aligned}
$$

where $\Omega \subseteq \mathbb{R}^N$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies some conditions. For example, we refer the reader to Alves [14], Juutinen [15], Mercaldo et al. [16, 17], Molino Salas and Segura de León [18], Kawohl and Schuricht [19] and the references therein.

In conclusion, we point out that Bögelein et al. in Refs. [20, 21] developed a new approach based on a parabolic variational inequality to solve some classes of total variation flow. In this mentioned approach, the Anzellotti-pairing plays no role and the theory of nonlinear semigroup is ignored. Very recently, Kinnunen and Scheven in Ref. [22] showed that weak solutions to the total variation flow based on the Anzellotti pairing and the approach due to Bögelein et al. are in fact equivalent under natural assumptions.

When one compares problem (1) with the previous ones, there are important differences, for example, the differential operator in (1) is not homogeneous of zero degree and so many techniques used in Ref. [2] are not applicable here. It is important to stress out that this is the first time the Kirchhoff total variation flow is studied.
Throughout of this paper, we shall assume that \( m : \mathbb{R}_+ \to \mathbb{R}_+ \) is an increasing continuous function that satisfies:
\[
m(r) \geq m(0) > 0 \quad \text{for every } r \geq 0.
\]

For simplicity, the example of \( m \) that we shall keep in mind is \( m(s) = (s + 1)^p \) for any \( s \geq 0 \) and \( p \geq 1 \). However, in the next sections, we shall provide other examples of the function \( m \) where the condition \((m_1)\) holds.

**Outline of the paper.** In Section 2, we give the necessary definitions and some preliminary results on functions of bounded variation. In the others sections, we shall state and prove our results.

### 2. Preliminaries

In this section and throughout of this paper, we shall use the following notations:

- \( \Omega \subset \mathbb{R}^N \) is an open bounded set with Lipschitz boundary,
- \( \mathcal{H}^{N-1} \) is the \((N-1)\)-dimensional Hausdorff measure,
- \( |\Omega| \) stands for the \(N\)-dimensional Lebesgue measure,
- \( v(x) \) is the outer vector normal defined for \( \mathcal{H}^{N-1} \) almost everywhere \( x \in \partial \Omega \),
- \( D(\Omega) \) or \( C_0^\infty(\Omega) \) is the space of infinitely differentiable functions with compact support in \( \Omega \),
- The positive part of \( w \) will be denoted by \( w^+ = \max\{w, 0\} \),
- These notations \( u_t(t) \) or \( u'(t) \) will be used to denote the weak derivative of \( u \) with respect to time \( t \),
- The short hand notation \( u(t) = u(\cdot, t) \) will be used.
- By \( L^1_{w}(0, T; BV(\Omega)) \) we denote the space of functions \( u : [0, T] \to BV(\Omega) \) such that \( u \in L^1(\Omega \times (0, T)) \), the maps \( t \mapsto \langle Du(t), \varphi \rangle \) is measurable for every \( \varphi \in C_0^\infty(\Omega, \mathbb{R}^N) \) and such that \( \int_0^T \int_{\Omega} |Du(t)| \, dt < \infty \).

#### 2.1. Functions of bounded variation

Let \( \mathcal{M}(\Omega, \mathbb{R}^N) \) be the set of vector Radon measure. We define the space of functions of bounded variation \( BV(\Omega) \) by
\[
BV(\Omega) = \left\{ u \in L^1(\Omega) : Du \in \mathcal{M}(\Omega, \mathbb{R}^N) \right\}.
\]

From Ref. [23], we have that \( u \in BV(\Omega) \) is equivalent to \( u \in L^1(\Omega) \) and
\[
\int_{\Omega} |Du| := \sup \left\{ \int_{\Omega} u \, \text{div} \varphi \, dx : \varphi \in C_0^\infty(\Omega, \mathbb{R}^N), |\varphi(x)| \leq 1 \forall x \in \Omega \right\} < +\infty, \tag{4}
\]
where \( |Du| \) is the total variation of the vectorial Radon measure. The set \( BV(\Omega) \) is a Banach space when equipped with the norm
\[
\|u\|_{BV(\Omega)} := \int_{\Omega} |Du| + \|u\|_{L^1(\Omega)},
\]
which is non reflexive and non-separable. For more details on functions of bounded variation, see for example, Refs. [24–26].

We recall that \( \{u_n\} \) in \( BV(\Omega) \) converges strictly (or intermediate convergence) to \( u \in BV(\Omega) \) if

- \( u_n \to u \) strongly in \( L^1(\Omega) \), and
- \( \int_{\Omega} |Du_n| \to \int_{\Omega} |Du| \) as \( n \to \infty \).

In view of [23, Theorem 10.1.2], we note that with respect to the strict convergent, \( C^\infty(\overline{\Omega}) \) is dense in \( BV(\Omega) \). On the other hand, through a bounded (onto) operator \( BV(\Omega) \hookrightarrow L^1(\partial \Omega) \), the notion of
A trace on the boundary can be extended to functions $u \in BV(\Omega)$ (see, e.g. [24, Theorem 3.87]). As a consequence, an equivalent norm on $BV(\Omega)$ can be defined by

$$
\|u\| := \int_\Omega |Du| + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1}.
$$

Moreover, in view of [24, Corollary 3.49], the following continuous embeddings hold

$$
BV(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for every } 1 \leq q \leq 1^* = \frac{N}{N-1},
$$

which are compact for $1 \leq q < 1^*$. Hereinafter, we recall several important results from Ref. [27] which will be used throughout the paper. Let $1 \leq r \leq \infty$, we define the function space $X_r(\Omega)$ by

$$
X_r(\Omega) := \{ z \in L^\infty(\Omega; \mathbb{R}^N) : \text{div} z \in L^r(\Omega) \}.
$$

Following Ref. [27], let $1 \leq r \leq N$, $z \in X_r(\Omega)$, and $w \in BV(\Omega) \cap L^{r'}(\Omega)$ where $r'$ is the conjugate of $r$. Then we define a functional $(z, Dw) : C^\infty_0(\Omega) \to \mathbb{R}$ by formula

$$
\langle (z, Dw), \varphi \rangle = -\int_\Omega w \varphi \text{div}(z) \, dx - \int_\Omega wz \nabla \varphi \, dx, \quad \forall \varphi \in C^\infty_0(\Omega).
$$

Then, by Anzellotti [27, Theorem 1.5] $(z, Dw)$ is a Radon measure in $\Omega$,

$$
\int_\Omega (z, Dw) = \int_\Omega z \nabla w \, dx,
$$

for all $w \in W^{1,1}(\Omega)$ and

$$
\left| \int_\Omega (z, Dw) \right| \leq \int_B |(z, Dw)| \leq \|z\|_\infty \int_B |Dw|,
$$

for every Borel $B$ set with $B \subseteq \Omega$. Moreover, besides the $BV-$norm, the functional given by

$$
w \mapsto \int_\Omega \varphi |Dw|,
$$

is lower semicontinuous with respect to the $L^1-$convergence for any nonnegative smooth function $\varphi$ (see, e.g. Ref. [24]).

Let $1 \leq r \leq N$. Following [27], a weak trace on $\partial \Omega$ of normal component of $z \in X_r(\Omega)$ is defined as the application $[z, v] : \partial \Omega \to \mathbb{R}$, such that $[z, v] \in L^\infty(\partial \Omega)$ and $|[z, v]|_\infty \leq \|z\|_\infty$. We note that, the latter definition coincides with the classical one, that is

$$
[z(x), v(x)] = z(x) v(x), \quad \text{for all } x \in \partial \Omega \text{ if } z \in C^1(\overline{\Omega}, \mathbb{R}^N),
$$

Next, we recall the Green formula involving the measure $(z, Dw)$ and the weak trace $[z, v]$ which was given in [27, Theorem 1.9], namely:

$$
\int_\Omega (z, Dw) + \int_\Omega w \, \text{div} z \, dx = \int_{\partial \Omega} w [z, v] \, d\mathcal{H}^{N-1},
$$

for $z \in X_r(\Omega)$ and $w \in BV(\Omega) \cap L^{r'}(\Omega)$.

Now, based on the above preliminaries, for the reader’s convenience, we give the definition of solutions to (1) based on the Anzellotti pairing.
Definition 2.1: Let $u_0 \in L^2(\Omega) \cap BV(\Omega)$. A function $u \in C([0, T]; L^2(\Omega))$ will be called a weak solution of (1) if $u_t \in L^2(0, T; L^2(\Omega))$, $u \in L^1(0, T; BV(\Omega))$, and $u(0) = u_0$. Moreover, there exists $z(t) \in X_1(\Omega)$, $\|z(t)\|_{\infty} \leq 1$ a.e. $t \in [0, T]$ satisfying

\begin{itemize}
  \item (1) $u'(t) - m(\int_\Omega |Du(t)| + \int_{\partial \Omega} |u(t)| \, d\mathcal{H}^{N-1}) \, \text{div}(z(t)) = 0$, in $\mathcal{D}'(\Omega)$ a.e. $t \in [0, T]$,
  \item (2) $\int_\Omega (z(t), Du(t)) = \int_\Omega |Du(t)|$, 
  \item (3) $[z(t), v] \in \text{sign}(-u(t)) \, \mathcal{H}^{N-1}$ a.e on $\partial \Omega$.
\end{itemize}

We conclude this section with the following definition of the extinction in finite time:

Definition 2.2: We say that Problem (1) has the extinction in finite time property if there exists a number $T_0 > 0$ such that $u(\cdot, t) > 0$ in a time interval $0 < t < T_0$ and $u(\cdot, t) = 0$ for any $t \geq T_0$.

3. existence and uniqueness of solution to (1)

This section is devoted to the proof of the following theorem:

Theorem 3.1: Let $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$ and assume that $(m_1)$ holds. Then, Problem (1) has a unique solution $u$ in the sense of Definition 2.1, which satisfies

$$M \left( \int_\Omega |Du(t)| + \int_{\partial \Omega} |u(t)| \, d\mathcal{H}^{N-1} \right) \leq M \left( \int_\Omega |Du_0| + \int_{\partial \Omega} |u_0| \, d\mathcal{H}^{N-1} \right) \quad \forall t \geq 0, \quad (12)$$

where $M(\sigma) = \int_0^\sigma m(s) \, ds$. Moreover, assuming that there is $r > 0$ such that $B_r(0) \subset \subset \Omega$. Then the following conclusion hold:

- If $u_0(x) = k \chi_{B_r(0)}(x)$ and $m(\sigma) = (\sigma + 1)^p$ for $\sigma \geq 0$ with $p > 1$, $k > 0$. Then the explicit solution of (1) is given by

$$u(x, t) = \frac{\left( \left[ N(p - 1) \gamma_N r^{N-2} + (\gamma_N kr^{N-1} + 1)^{1-p} \right]^{\frac{1}{1-p}} - 1 \right)^+}{\gamma_N r^{N-1}} \chi_{B_r(0)}(x) \chi_{[0, T]}(t). \quad (13)$$

where $T = \frac{1 - (\gamma_N kr^{N-1} + 1)^{1-p}}{N(p-1)\gamma_N r^{N-2}}$, $\gamma_N = \frac{N^{N/2}}{\Gamma(\frac{N}{2}+1)}$.

- If $u_0(x) = k \chi_{B_r(0)}(x)$ and $m(\sigma) = 1 + \sigma$ for $\sigma \geq 0$, $k > 0$. Then the explicit solution of (1) is given by

$$u(x, t) = \frac{\left( e^{-N\gamma_N r^{N-2} t}(\gamma_N r^{N-1} k + 1)^+ - 1 \right)^+}{\gamma_N r^{N-1}} \chi_{B_r(0)}(x) \chi_{[0, T]}(t), \quad (14)$$

where $T = \frac{\log((\gamma_N kr^{N-1} + 1)^+)}{N\gamma_N r^{N-2}}$.

Next, we provide some examples on the function $m$ in order to illustrate Theorem 3.1.

Example 3.1: The first part in Theorem 3.1 can be illustrated with many increasing continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ satisfying the condition $(m_1)$. Among them, we give $m(r) = \exp(r)$, $m(r) = \log(2 + r)$, and $m(r) = \max\{r + 1)^p; (r + 1)^q\}$ for any $r \geq 0, p, q \geq 1$. 
Remark 3.1: In view of [1, Lemma 1], we observe that solving Problem (1) in the sense of Definition 2.1 is equivalent to solving the following Cauchy problem

\[
\begin{cases}
  u_t(t) + m \left( \int_{\Omega} |D u| + \int_{\partial \Omega} |u| \, d\mathcal{H}^{N-1} \right) A(u(t)) \equiv 0 \quad \text{in } t > 0, \\
  u(0) = u_0,
\end{cases}
\tag{15}
\]

where \( A(u) = \partial \Phi(u) \) and

\[
\Phi(u) = \begin{cases}
  \int_{\Omega} |D u| + \int_{\partial \Omega} |u| \, d\mathcal{H}^{N-1}, & \text{if } u \in \text{BV}(\Omega) \cap H, \\
  +\infty, & \text{if } u \in H \setminus \text{BV}(\Omega) \cap H.
\end{cases}
\]

Here and throughout the paper \( H = L^2(\Omega) \). We note that, by using the definition of the subdifferential of \( \Phi \) the Cauchy problem (15) can be rewritten as a variational inequality, which was studied in Refs. [28–30] and the references therein.

Remark 3.2: From (13)–(14), we observe that the solution of (1) is discontinuous and has the minimal required spatial regularity \( u(x, t) \in \text{BV}(\Omega) \setminus W^{1,1}(\Omega) \). We point out that the regularity of solutions to problem (P) still an open question.

Remark 3.3: In Refs. [2, 19] by using the method of separating variables, it has been observed that every solution for a parabolic 1-Laplacian problem must decay to zero in finite time \( T \). In view of (13)–(14), we observe that the same conclusion still holds true for the unique solution of (1).

The proof of the first part of Theorem 3.1 will be based on the existence of solutions for (P) and the existence of solutions for an ordinary differential equation. This argument has been introduced by Chipot and Lovat [31] to show the existence and uniqueness of solution for this nonlocal problem involving the Laplace operator

\[
\begin{cases}
  u_t - a \left( \int_{\Omega} u(x, t) \, dx \right) \Delta u = 0 \quad \text{in } \Omega \times (0, +\infty), \\
  u = 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
  u(x, 0) = u_0(x) \quad \text{in } \Omega,
\end{cases}
\]

where \( a \) is assumed to be continuous and \( a(s) > 0 \) for all \( s > 0 \). Later on, this approach have been used by Gobbino [32] to extend Chipot and Lovat result to an abstract setting. Namely, he discussed the existence and uniqueness of solution for this initial value problem

\[
\begin{cases}
  u_t + m \left( \|A^{1/2} u\|^2 \right) A u = 0 \quad \text{in } t > 0, \\
  u(0) = u_0 \in H,
\end{cases}
\tag{16}
\]

where \( A \) is a linear operator which is non-negative and self-adjoint on \( H \) with a domain \( D(A) \). The function \( m \) is continuous and behaving like \( m(\sigma) = a + b\sigma, \ a \geq 0, \ b > 0 \). In the previously mentioned work, the author proved the existence of at least one solution for (16) when \( m(\|A^{1/2} u_0\|^2) \neq 0 \). In this context, we refer the reader also to Refs. [33–36], where the idea of transforming nonlinear evolution equations into nonlinear ordinary differential equations has been employed to obtain the exact solutions.

The construction of explicit solutions to (1) is inspired by a work due to Andreu et al. Ref. [2], where the authors studied the parabolic 1-Laplacian problem when \( m \equiv 1 \).
Proof of Theorem 3.0.1.: We shall show that the following function solves (1) in the sense of Definition 2.1:

\[ u(t) = v(\alpha(t)), \]

where \( v \) is a solution for the following total variation flow problem

\[
\begin{aligned}
\frac{\partial v}{\partial t} - \text{div} \left( \frac{Dv}{|Dv|} \right) &= 0 \quad \text{in } \Omega \times (0, +\infty), \\
v &= 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
v(x, 0) &= u_0(x) \quad \text{in } \Omega,
\end{aligned}
\]  

(18)

and \( \alpha \) is a solution of the problem

\[
\begin{aligned}
\alpha &\in C([0, +\infty]) \cap C^1([0, +\infty]), \\
\alpha' &= \varphi(\alpha(t)) \quad t > 0, \\
\alpha(t) &> 0, \quad t > 0, \\
\alpha(0) &= 0,
\end{aligned}
\]

(19)

where \( \varphi(t) = m(\int_\Omega |Dv| + \int_{\partial\Omega} |v|) \, d\mathcal{H}^{N-1}) \).

It has been shown in Ref. [1] that problem (18) admits a unique solution in the sense of Definition 2.1, which solves equivalently the following Cauchy problem:

\[
\begin{aligned}
\frac{d}{dt} \Phi(v(t)) &= -\|v_t(t)\|_2^2, \quad \text{a.e. } t \in [0, +\infty), \\
v(0) &= u_0,
\end{aligned}
\]

(20)

where \( A(v) = \partial \Phi(v) \) and \( \Phi(v) \) is introduced in Remark 3.1.

Claim 3.1: \( t \mapsto \int_\Omega |Dv| + \int_{\partial\Omega} |v| \, d\mathcal{H}^{N-1} \) is a continuous map.

Indeed, in view of [37, Lemma p. 73] and (20), we have

\[
\frac{d}{dt} \Phi(v(t)) = -\|v_t(t)\|_2^2, \quad \text{a.e. } t \in [0, +\infty).
\]

(21)

Next fix \( t \) in \( [0, +\infty) \) and let \( t_k \to t \). From the last observation, we have

\[
\left| \int_\Omega |Dv(t_k)| + \int_{\partial\Omega} |v(t_k)| \, d\mathcal{H}^{N-1} - \int_\Omega |Dv(t)| + \int_{\partial\Omega} |v(t)| \, d\mathcal{H}^{N-1} \right| \\
= \left| \int_0^{t_k} \frac{d}{ds} \Phi(v(s)) \, ds \right| \leq \int_0^{t_k} \|v_2(s)\|_2^2 \, ds.
\]

(22)

According to (21), it is easy to observe that \( v_t \in L^2(0, +\infty; L^2(\Omega)) \). Hence, by (22) and the dominated convergence theorem, we conclude that Claim 3.1 holds. Thus, this shows \( \varphi \in C^0([0, +\infty)) \). Using \( (m_1) \) and the theory of ordinary differential equations, one can prove that problem (19) has a unique solution given by

\[
\alpha(t) = \psi^{-1}(t) \quad \text{where } \psi(r) = \int_0^r \frac{1}{\varphi(s)} \, ds,
\]

for details, see [32, Lemma 2.1]. From Remark 3.1, we need only to show that (17) is the unique solution to (15). Indeed, let \( \alpha \) be the unique solution of (19) and \( v \) be the unique solution of (20), let
The computations to construct (27) was carried out in [2, Lemma 1], so here we omit it. Integrating over \( \Omega \) the equation (24) yields the unique solution for the initial value problem

\[
\begin{cases}
  u'(t) + b(t)A(u(t)) \geq 0 & \text{in } t > 0, \\
  u(0) = u_0.
\end{cases}
\]

This implies that \( u(t) = \nu(\alpha(t)) \) where

\[ \alpha(t) = \int_0^t b(s) \, ds. \]

Now, it is easy to show that \( \alpha \) is a solution to (19). Finally, we note that the inequality (12) follows directly from (23).

In the last part of this section, we compute an explicit solution to Problem (1) where \( m(s) = (1 + s)^p \) for any \( p \geq 1 \). First let us assume that \( p > 1 \). We shall look for a solution of (1) of the form \( u(x, t) = \alpha(t) \chi_{B_r(0)}(x) \) on some interval \( (0, T) \). Therefore, we look for some \( z(t) \in X_1(\Omega) \) with \( \|z(t)\|_\infty \leq 1 \), such that

\[
\begin{align*}
  u'(t) - m \left( \int_\Omega |Du(t)| + \int_{\partial \Omega} |u(t)| \, d\mathcal{H}^{N-1} \right) \text{div}(z(t)) &= 0 & \text{in } \mathcal{D}'(\Omega), \ a.e \ t \in [0, T], \\
  \int_\Omega (z(t), Du(t)) &= \int_\Omega |Du(t)|, \\
  [z(t), v] &\in \text{sign}(-u(t)) \mathcal{H}^{N-1} \text{-a.e.}
\end{align*}
\]

The candidate to \( z(t) \) is the vector field

\[
z(t)(x) := \begin{cases} \
-x & \text{if } x \in B_r(0), \ 0 \leq t \leq T, \\
-\frac{r}{N-1}x & \text{if } x \in \Omega \setminus B_r(0), \ 0 \leq t \leq T, \\
\frac{|x|^N}{|x|^N} & \text{if } x \in \Omega, \ t > T,
\end{cases}
\]

The computations to construct (27) was carried out in [2, Lemma 1], so here we omit it. Integrating equation (24) over \( \Omega \) yields

\[ \alpha'(t)|B_r(0)| = -\left[ \alpha(t) \gamma_N r^{N-1} + 1 \right]^{p} N \gamma_N r^{N-1}, \]
where \( \gamma_N = \frac{N!^{N/2}}{\Gamma(N/2 + 1)} \). Therefore

\[
\alpha(t) = \frac{\left[ N(p - 1)\gamma_N r^{N-2} t + (\gamma_Nkr^{N-1} + 1)^{1-p} \right]^{\frac{1}{1-p}} - 1}{\gamma_N r^{N-1}}.
\]

In what follows, we check that

\[
u(x,t) = \frac{\left[ N(p - 1)\gamma_N r^{N-2} t + (\gamma_Nkr^{N-1} + 1)^{1-p} \right]^{\frac{1}{1-p}} - 1}{\gamma_N r^{N-1}} \chi_{B_r(0)}(x) \chi_{[0,T]}(t),
\]

where \( T = \frac{1 - (\gamma_Nkr^{N-1} + 1)^{1-p}}{N(p-1)\gamma_N r^{N-2}} \), satisfies (24)–(26). Since \( u(t,x) = 0 \) on \( \partial \Omega \), it easy to check that (26) holds. On the other hand, if \( \varphi \in D(\Omega) \) and \( 0 \leq t \leq T \), we have

\[
h(t) \int_{\Omega} \text{div}(z(t)) \varphi \, dx = -h(t) \int_{\Omega} \varphi \, dx = -h(t) \int_{\Omega} z(t) \nabla \varphi \, dx = -h(t) \int_{\partial \Omega, B_r(0)} z(t) \nabla \varphi \, d\mathcal{H}^{N-1} - h(t) \int_{\Omega \setminus B_r(0)} \text{div} \left( \frac{r^{N-1} x}{|x|^N} \right) \varphi \, dx
\]

where \( h(t) = \frac{[N(p - 1)\gamma_N r^{N-2} t + (\gamma_Nkr^{N-1} + 1)^{1-p} \right]^{\frac{p}{1-p}}}{\gamma_N r^{N-1}} \). Hence

\[
m \left( \int_{\Omega} |Du(t)| + \int_{\partial \Omega} |u(t)| \, d\mathcal{H}^{N-1} \right) \int_{\Omega} \text{div}(z(t)) \varphi \, dx = \int_{\Omega} u'(t) \varphi \, dx, \quad \text{a.e. } t \in [0,T],
\]

and consequently (24) holds. Finally, by Green’s formula (11), we have

\[
\int_{\Omega} (z(t), Du(t)) = -\int_{\Omega} \text{div}(z(t)) u(t) \, dx + \int_{\partial \Omega} [z(t), v] u(t) \, d\mathcal{H}^{N-1}
\]

\[
= -\frac{h(t)}{r} \int_{B_r(0)} \frac{|x|^2}{r^2} \, dx = -h(t) \int_{\partial B_r(0)} \frac{|x|^2}{r^2} \, d\mathcal{H}^{N-1}
\]

\[
= \frac{N}{r} \int_{B_r(0)} \left[ \frac{[N(p - 1)\gamma_N r^{N-2} t + (\gamma_Nkr^{N-1} + 1)^{1-p} \right]^{\frac{1}{1-p}} - 1}{\gamma_N r^{N-1}} \right] \, dx
\]

\[
= \left( \frac{[N(p - 1)\gamma_N r^{N-2} t + (\gamma_Nkr^{N-1} + 1)^{1-p} \right]^{\frac{1}{1-p}} - 1}{\gamma_N r^{N-1}} \right) \frac{N}{r} |B_r(0)|
\]

\[
= \left( \frac{[N(p - 1)\gamma_N r^{N-2} t + (\gamma_Nkr^{N-1} + 1)^{1-p} \right]^{\frac{1}{1-p}} - 1}{\gamma_N r^{N-1}} \right) \mathcal{H}^{N-1}(\partial B_r(0)) = \int_{\Omega} |Du(t)|.
\]

Therefore (25) holds, and consequently \( u(x,t) \) is a solution of (1) with initial datum \( u_0 \). In a similar fashion one can show that (14) satisfies (24)–(26). Hence the proof is now complete.
4. The extinction in finite time for (1)

In the next theorem, we show the unique solution of (1) decays to zero in finite time $T^*(u_0)$. Moreover, we provide an upper bound for the time extinction $T^*(u_0)$.

**Theorem 4.1:** Let $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$ and let $u(x, t)$ be a unique solution of Problem (1). Assume that $(m_1)$ holds and let $d(\Omega)$ be the smallest radius of a ball containing $\Omega$. If $T^*(u_0) = \inf\{t > 0 : u(t) = 0\}$, then

$$T^*(u_0) \leq \frac{d(\Omega)\|u_0\|_\infty}{Nm(0)}. \quad (28)$$

Moreover, there holds

$$\|u(t)\|_\infty \leq \|u_0\|_\infty, \quad \text{for any } t \geq 0. \quad (29)$$

This theorem can be illustrated in the following example:

**Example 4.1:** In a similar fashion to Example 3.1, we can illustrate the latter theorem with many increasing continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ satisfying the condition $(m_1)$, such as $m(r) = \exp(r)$, $m(r) = \log(2 + r)$, and $m(r) = \max\{(r + 1)^p; (r + 1)^q\}$ for any $r \geq 0, p, q \geq 1$.

Extinction of solutions for evolutionary equations has been widely studied in literature. As far as we know, the finite-time extinction for the total variation flow $(P)$ was given first by Andreu et al. in Ref. [2]. In that paper, the authors proved a comparison principle for total variation flow $(P)$ and obtained that

$$T^*(u_0) \leq \frac{d(\Omega)\|u_0\|_\infty}{N}. \quad (28)$$

Our approach to prove Theorem 4.1 is also based on a comparison principle established in this paper. Recently, Giga and Kohn in Ref. [3] rather than using a comparison principle they used an energy estimate with a suitable Sobolev-type inequality to prove the finite-time extinction of solutions to $(P)$ under different boundary conditions (Periodic BC, Neumann BC, Dirichlet BC). Namely, they showed that for every $u_0 \in L^2_{av}(T\mathbb{N})$, the extinction time satisfies

$$T^*(u_0) \leq S_N\|u_0\|_{L^N},$$

where $L^2_{av}(T\mathbb{N}) = \{v \in L^2(\mathbb{T}^N) : \int_{T\mathbb{N}} v \, dx = 0\}$ and $S_N$ is the best constant in the Sobolev inequality. Bonforte and Figalli [38] studied the explicit dynamic and sharp asymptotic behaviour for the following Cauchy problem of total variation flow in one dimension

$$\begin{cases}
  u_t - \left( \frac{u_x}{|u_x|} \right)_x = 0 & \text{in } \mathbb{R} \times (0, +\infty), \\
  u(x, 0) = u_0(x) & \text{in } \mathbb{R}.
\end{cases}$$

Moreover, for any given nonnegative compactly supported initial datum $u_0 \in L^1(\mathbb{R})$, the authors proved that

$$T^*(u_0) = \frac{1}{2} \int_{\mathbb{R}} u_0(x) \, dx.$$  

**Proof of Theorem 4.0.1:** As we have pointed out before, the proof is based on a comparison principle that we state in what follows. 

$\blacksquare$
Lemma 4.1: Let $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$ and let $u_1(x, t)$ be the unique solution of problem (1). Assume that $(m_1)$ holds and let $d(\Omega)$ be the smallest radius of a ball containing $\Omega$. Let $u_2(x, t) = \alpha(t)\chi_\Omega(x)$, satisfying

$$|\alpha'(t)| \leq \frac{m(0)N}{d(\Omega)}. \quad (30)$$

Then the following conclusion holds:

1. If $\alpha(t) \geq 0$ and $u_0 \leq \alpha(0)$, we have
   $$u_1(t) \leq u_2(t) \quad \text{a.e. on } \Omega,$$

2. If $\alpha(t) \leq 0$ and $u_0 \geq \alpha(0)$, we have
   $$u_1(t) \geq u_2(t) \quad \text{a.e. on } \Omega.$$

Proof: Since $\Omega$ is bounded, without loss of generality, we may assume that $\Omega \subseteq B(0, d(\Omega))$. According to Theorem 3.1, there exists $z_1(t) \in X_1(\Omega), \|z_1(t)\|_{\infty} \leq 1$, satisfying

$$u_1'(t) - m(\|u_1\|) \text{div}(z_1(t)) = 0, \text{ in } D'(\Omega) \text{ a.e } t \in [0, T] \quad (31)$$

$$\int_\Omega (z_1(t), Du_1(t)) = \int_\Omega |Du_1(t)|, \quad (32)$$

$$[z_1(t), v] \in \text{sign}(-u_1(t)) \quad \mathcal{H}^{N-1}\text{-a.e on } \partial\Omega. \quad (33)$$

In view of (30), one can show easily that $u_2(x, t) = \alpha(t)\chi_\Omega(x)$ satisfies (31) with the chosen vector field $z_2(t)(x) = \frac{\alpha'(t)x}{N m(0)}$. Thus, by the Green’s formula (11), we get

$$\frac{1}{2} \frac{d}{dt} \int_\Omega [(u_1(t) - u_2(t))^+]^2 \, dx = -m(\|u_1\|) \int_\Omega (z_1(t), D[(u_1(t) - u_2(t))^+])$$

$$+ m(0) \int_\Omega (z_2(t), D[(u_1(t) - u_2(t))^+])$$

$$+ m(\|u_1\|) \int_{\partial\Omega} [z_1(t), v](u_1(t) - u_2(t))^+ d\mathcal{H}^{N-1}$$

$$- m(0) \int_{\partial\Omega} [z_2(t), v](u_1(t) - u_2(t))^+ d\mathcal{H}^{N-1}.$$

If $R_t(r) = (r - \alpha(t))^+$, then by using similar calculations as in the proof of [1, Theorem 4], we arrive at

$$\int_\Omega (z_1(t), DR_t(u_1(t))) = \int_\Omega |DR_t(u_1(t))|. \quad (34)$$

Moreover, by (9) we deduce that

$$\left| \int_\Omega (z_2(t), R_t(u_1(t))) \right| \leq \|z_2(t)\|_{\infty} \int_\Omega |DR_t(u_1(t))| \leq \int_\Omega |DR_t(u_1(t))|. \quad (35)$$

Combining (34) and (35), and using the fact that $m$ is an increasing function, we obtain

$$- m(\|u_1\|) \int_\Omega (z_1(t), D[(u_1(t) - u_2(t))^+]) + m(0) \int_\Omega (z_2(t), D[(u_1(t) - u_2(t))^+])$$
Theorem 4.2: Let $u \in BV(\Omega) \cap L^{\infty}(\Omega)$ and denote by $u(t)$ the solution of (1) with initial data $u_0$. Assume that there is $r > 0$ such that $B_r(0) \subset \subset \Omega$. Then the following conclusion hold:

- If $m(\sigma) = (\sigma + 1)^p$ for $\sigma \geq 0$ and $p > 1$, then
  
  $$
  \|u(t)\|_{\infty} \geq \left( \frac{\left( N(p - 1)\gamma_N r^{N-2}t + (\gamma_N kr^{N-1} + 1)^{1-p} \right)^{\frac{1}{p}} - 1}{\gamma_N r^{N-1}} \right)^{+} \quad \text{for } 0 \leq t \leq T^*_1(u_0),
  $$

  where $T^*_1(u_0) = \frac{1}{N(p-1)\gamma_N r^{N-1} + 1} - \frac{1}{(\gamma_N + 1)^{1-p}}$, $\gamma_N = \frac{N\pi^{N/2}}{\Gamma\left(\frac{N}{2} + 1\right)}$. Moreover, if $\text{supp}(u_0) \subset B_r(0) \subset \subset \Omega$, then $\text{supp}(u(t)) \subset B_r(0)$ for all $t \geq 0$, and $u(t) = 0$ for all $t \geq \frac{1-\|u_0\|_{\infty}kr^{N-1} + 1}{N(p-1)\gamma_N r^{N-1} + 1}$. 

In light of $|z_2(t), v| \leq 1$, $z_1(t), v \in \text{sign}(-u_1(t))$ and $u_2(t) \geq 0$, we derive

$$
\begin{align*}
  &m(\|u_1\|) \int_{\partial \Omega} |z_1(t), v|(u_1(t) - u_2(t))^+ \, d\mathcal{H}^{N-1} - \int_{\partial \Omega} [z_2(t), v](u_1(t) - u_2(t))^+ \, d\mathcal{H}^{N-1} \\
  &\leq (m(0) - m(\|u_1\|)) \int_{\partial \Omega \cap \{u_1 > u_2\}} (u_1(t) - u_2(t)) \, d\mathcal{H}^{N-1} \leq 0.
\end{align*}
$$

Gathering (36) and (37) yields

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} [(u_1(t) - u_2(t))^+]^2 \, dx \leq 0.
$$

Hence the condition $u_1(0) \leq u_2(0)$ ensures $u_1 \leq u_2$. The proof of (2) is quite similar to (1), so here we omit it. \hfill \blacksquare

Taking

$$
\alpha(t) := \frac{Nm(0)}{d(\Omega)} \left( \frac{d(\Omega)\|u_0\|_{\infty}}{\alpha} - t \right)^+, \n$$

yields

$$
|\alpha'(t)| = \frac{Nm(0)}{d(\Omega)} \quad \text{and} \quad \alpha(0) = \|u_0\|_{\infty}.
$$

Invoking Lemma 4.1, we conclude that

$$
-\alpha(t) \leq u(t) \leq \alpha(t),
$$

and from which we obtain (29) holds. Hence the proof now is complete.

The first part of the next theorem demonstrates some lower and upper bounds for the solution of the Kirchhoff total variation flow. While the second part shows that there is no propagation of the support of the initial datum. We point out that it has been proved that if $p > 2$, then there is finite speed propagation property of solutions to $(P_p)$, that is, if $\text{supp}(u_0) \subset B_r(0) \subset \subset \Omega$, then the solution of $(P_p)$ satisfies that $\text{supp}(u(t))$ is a compact set for all $t > 0$; for more details, see Refs. [39, 40].

Theorem 4.2: Let $u_0 \in BV(\Omega) \cap L^{\infty}(\Omega)$ and denote by $u(t)$ the solution of (1) with initial data $u_0$. Assume that there is $r > 0$ such that $B_r(0) \subset \subset \Omega$. Then the following conclusion hold:

- If $m(\sigma) = (\sigma + 1)^p$ for $\sigma \geq 0$ and $p > 1$, then
  
  $$
  \|u(t)\|_{\infty} \geq \left( \frac{\left( N(p - 1)\gamma_N r^{N-2}t + (\gamma_N kr^{N-1} + 1)^{1-p} \right)^{\frac{1}{p}} - 1}{\gamma_N r^{N-1}} \right)^{+} \quad \text{for } 0 \leq t \leq T^*_1(u_0),
  $$

  where $T^*_1(u_0) = \frac{1}{N(p-1)\gamma_N r^{N-1} + 1} - \frac{1}{(\gamma_N + 1)^{1-p}}$, $\gamma_N = \frac{N\pi^{N/2}}{\Gamma\left(\frac{N}{2} + 1\right)}$. Moreover, if $\text{supp}(u_0) \subset B_r(0) \subset \subset \Omega$, then $\text{supp}(u(t)) \subset B_r(0)$ for all $t \geq 0$, and $u(t) = 0$ for all $t \geq \frac{1-\|u_0\|_{\infty}kr^{N-1} + 1}{N(p-1)\gamma_N r^{N-1} + 1}$.
Now let us consider the following functions:

\[ \|u(t)\|_\infty \geq \frac{\left(e^{-\gamma_N r^{N-2}t} (\gamma_N r^{N-1} k + 1) - 1 \right)^+}{\gamma_N r^{N-1}} \quad \text{for } 0 \leq t \leq T^*_2(u_0), \]

where \( T^*_2(u_0) = \frac{\log(\gamma_N r^{N-1}k+1)}{N \gamma_N r^{N-2}} \). Furthermore, if \( \text{supp}(u_0) \subset B_r(0) \subset \Omega \), then \( \text{supp}(u(t)) \subset B_r(0) \) for all \( t \geq 0 \), and \( u(t) = 0 \) for all \( t \geq \frac{\log(\gamma_N r^{N-1}\|u_0\|_\infty+1)}{N \gamma_N r^{N-2}} \).

**Proof:** Here we prove only the inequality in (38). The proof of (39) can be done similarly, so here we omit it. Since \( \Omega \) is a bounded domain, then without lost of generality, we may assume that \( \Omega \subset B_r(0) \), for some \( r > 0 \). By Theorem 3.1, we know that

\[ v(t, x) = \frac{\left( [N(p-1)\gamma_N r^{N-2}t + (\gamma_N kr^{N-1} + 1)^{1-p}]^{\frac{1}{1-p}} - 1 \right)^+}{\gamma_N r^{N-1}} \chi_{B_r(0)} \]

is a solution of problem (1) with the chosen vector field (27) and initial datum \( u_0 = k\chi_{B_r(0)} \). In order to prove (38), we argue by contradiction by assuming that, there exists \( t_0 \in (0, T^*_1(u_0)) \) such that

\[ \|u(t_0)\|_\infty < \|v(t_0)\|_\infty, \]

which implies that there exists \( \epsilon > 0 \) such that

\[ \|u(t_0)\|_\infty < \frac{\left( [N(p-1)\gamma_N r^{N-2}t_0 + (\gamma_N kr^{N-1} + 1)^{1-p}]^{\frac{1}{1-p}} - 1 \right)^+}{\gamma_N r^{N-1}} - \epsilon = k_1. \]

Now let us consider the following functions:

\[ v_1(x, t) := \frac{\left( [N(p-1)\gamma_N r^{N-2}t + (\gamma_N kr^{N-1} + 1)^{1-p}]^{\frac{1}{1-p}} - 1 \right)^+}{\gamma_N r^{N-1}} \chi_{B_r(0)}, \]

and

\[ v_2(x, t) := -\frac{\left( [N(p-1)\gamma_N r^{N-2}t + (\gamma_N kr^{N-1} + 1)^{1-p}]^{\frac{1}{1-p}} - 1 \right)^+}{\gamma_N r^{N-1}} \chi_{B_r(0)}. \]

In view of (40), clearly we have that \( v_2(0) \leq u(t_0) \leq v_1(0) \). Moreover, proceeding similarly as in the proof of Theorem 3.1, one can show that \( v_1 \) and \( v_2 \) are solutions to (1) with the candidate vector field (27) and initial datum \( u_0 = k_1 \chi_{B_r(0)} \) and \( u_0 = -k_1 \chi_{B_r(0)} \), respectively. Thus, by using the comparison principle in Lemma 4.1, we deduce \( v_2(t) \leq u(t_0 + t) \leq v_1(t) \). Hence, it follows that

\[ T^*_1(u_0) - t_0 = T^*_1(u(t_0)) \leq \frac{1 - (\gamma_N kr^{N-1} + 1)^{1-p}}{N(p-1)\gamma_N r^{N-2}} \]

\[ = \frac{1 - \left( [N(p-1)\gamma_N r^{N-2}t_0 + (\gamma_N kr^{N-1} + 1)^{1-p}]^{\frac{1}{1-p}} - \epsilon \gamma_N r^{N-1} \right)^{1-p}}{N(p-1)\gamma_N r^{N-2}} \]

\[ < T^*_1(u_0) - t_0. \]
But this is a contradiction. This completes the proof of the first statement. Now we turn to show the second statement of Theorem 4.2. Let \( \zeta = \|u_0\|_\infty \), according to Theorem 3.1 one can show that the following functions

\[
v_1(x, t) := \left(\frac{\left[ N(p - 1)pN - 2t + (pN\zeta^{-N} + 1)^{1-p}\right]^{\frac{1}{1-p}}}{\gamma Np - 1}\right) \chi_{B_r(0)},
\]

and

\[
v_2(x, t) := -\left(\frac{\left[ N(p - 1)pN - 2t + (pN\zeta^{-N} + 1)^{1-p}\right]^{\frac{1}{1-p}}}{\gamma Np - 1}\right) \chi_{B_r(0)}.
\]

are solutions of (1) with initial datum \( \zeta \chi_{B_r(0)} \) and \( -\zeta \chi_{B_r(0)} \) respectively. Thus, by the comparison principle seen in Lemma 4.1, we have \( v_2(x, t) \leq u(x, t) \leq v_1(x, t) \) for all \( t \geq 0 \) and \( x \in \Omega \). Therefore, \( \text{supp}(u(t)) \subset B_r(0) \) for all \( t \geq 0 \). Hence, this ends the proof.

In the last theorem of this paper, we establish lower and upper bounds on the rate of decay of \( \|u(t)\|_N \) and \( \|u(t)\|_\infty \), respectively.

**Theorem 4.3:** Let \( u_0 \in BV(\Omega) \cap L^\infty(\Omega) \) and \( u(x, t) \) is the unique solution of Problem (1). Assume that there exists \( \mu \in (0, 1) \) such that

\[
M(\sigma) \geq \mu m(\sigma)\sigma \quad \text{for every } \sigma \geq 0.
\]

Then we have

- There exists a constant \( \eta > 0 \) independent of the initial datum, such that

\[
\|u(t)\|_N \geq \eta \min\{m_0, 1\} N(T^*(u_0) - t) \quad \text{for } 0 \leq t \leq T^*(u_0).
\]

- Given \( 0 < \tau < T^*(u_0) \), we have

\[
\|u(t)\|_\infty \leq \frac{|u_0| m\left(\frac{1}{\mu m_0} M\left(\int_{\Omega} |Du_0| + \int_{\partial\Omega} |u_0| d\mathcal{H}^{N-1}\right)\right)}{\tau} (T^*(u_0) - t) \quad \text{for } \tau \leq t \leq T^*(u_0).
\]

**Example 4.2:** The previous theorem can be illustrated by some increasing continuous functions from \( \mathbb{R} \) into \( \mathbb{R} \) satisfying the conditions \( (m_1) \) and \( (45) \), such as \( m(r) = (r + 1)^p \) and \( m(r) = \log(2 + r) \) for any \( r \geq 0, p \geq 1 \).

**5. Proof of Theorem 4.1.2**

This section is devoted to the proof of Theorem 4.1.2. First, we prepare the following lemma, which will be employed in the proof.

**Lemma 5.1:** Let \( u(t) \) be a solution of the Dirichlet problem (1). Suppose that \( (45) \) holds. Then there holds

\[
|u'(t)| \leq \frac{|u_0| m\left(\frac{1}{\mu m_0} M\left(\int_{\Omega} |Du_0| + \int_{\partial\Omega} |u_0| d\mathcal{H}^{N-1}|u_0|\right)\right)}{t},
\]

for almost all \( t > 0 \).
Proof: From (45), (12) and \((m_1)\) we can easily obtain that
\[
m m\left(\int_\Omega |Du(t)| + \int_{\partial\Omega} |u(t)|\,d\mathcal{H}^{N-1}\right) \leq m\left(\frac{1}{\mu m_0} M\left(\int_\Omega |Du(t)| + \int_{\partial\Omega} |u(t)|\,d\mathcal{H}^{N-1}\right)\right)
\]
\[
(49)
\]
Remembering that the solution \((1)\) in the form \(u(t) = v(\alpha(t))\) where \(v\) is a solution of \((20)\) and by Andreu et al. \([2, \text{Lemma 2}]\) we have that the solution \(v\) of \((18)\) satisfies
\[
|v'(t)| \leq \frac{|u_0|}{t} \quad \text{for almost all } t > 0.
\]
(50)

Gathering (49), (50), and using [26, Theorem 2.2.2] yield
\[
|u'(t)| = \alpha'(t)|v'(t)| = m \left(\int_\Omega |Du(t)| + \int_{\partial\Omega} |u(t)|\,d\mathcal{H}^{N-1}\right) |v'(\alpha(t))| \leq \frac{|u_0| m\left(\frac{1}{\mu m_0} M\left(\int_\Omega |Du(t)| + \int_{\partial\Omega} |u(t)|\,d\mathcal{H}^{N-1}\right)\right)}{t},
\]
for almost all \(t > 0\).

\[
\text{Proof of Theorem 4.1.2:} \quad \text{Now let us turn to prove Theorem 4.3. By Theorem 3.1 there exists } z(t) \in X_1(\Omega), \|z(t)\|_\infty \leq 1, \text{ satisfying that}
\]
\[
u'(t) - m \left(\int_\Omega |Du(t)| + \int_{\partial\Omega} |u(t)|\,d\mathcal{H}^{N-1}\right) \text{div}(z(t)) = 0, \quad \text{in } D'(\Omega) \text{ a.e } t \in [0, T]
\]
\[
(51)
\]
\[
\int_\Omega (z(t), Du(t)) = \int_\Omega |Du(t)|,
\]
\[
(52)
\]
\[
[z(t), v] \in \text{sign}(-u(t)) \quad \mathcal{H}^{N-1}\text{-a.e on } \partial\Omega.
\]
\[
(53)
\]

Multiplying (51) by \(w \in BV(\Omega) \cap L^2(\Omega)\) (this is possible since \(\text{div}(z(t)) \in L^2(\Omega)\)), afterwards integrating over \(\Omega\) and using the Green’s formula (11), we get
\[
\int_\Omega u'(t)w + m \left(\int_\Omega |Du(t)| + \int_{\partial\Omega} |u(t)|\,d\mathcal{H}^{N-1}\right) \int_\Omega (z, Dw) = \int_{\partial\Omega} [z(t), v] w \,d\mathcal{H}^{N-1},
\]
\[
(54)
\]
for every \(w \in L^2(\Omega) \cap BV(\Omega)\). Let \(q \geq 1, \text{ and } \varphi(r) = |r|^{q-1}r. \text{ In view of (29) and [24, Theorem 3.99],}
\]
we have that \(\varphi(u) \in BV(\Omega) \cap L^2(\Omega).\) Then, taking \(w = \varphi(u)\) as a test function in (54), it follows that
\[
\int_\Omega u'(t)\varphi(u) + m \left(\int_\Omega |Du(t)| + \int_{\partial\Omega} |u(t)|\,d\mathcal{H}^{N-1}\right) \int_\Omega (z, D\varphi(u)) = \int_{\partial\Omega} [z(t), v] \varphi(u) \,d\mathcal{H}^{N-1},
\]
\[
(55)
\]
Now, by Anzellotti [27, Proposition 2.8] and having in mind (52), we obtain
\[
\int_\Omega (z, D\varphi(u)) = \int_\Omega \theta(z(t), D\varphi(u(t)), x)|D\varphi(u(t))| = \int_\Omega |D\varphi(u(t))|.
\]
Moreover, by (53)
\[
[z(t), v] \varphi(u(t)) = -|u(t)|^q \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega.
\]
Consequently, we get
\[
\frac{1}{q+1} \frac{d}{dt} \int_\Omega |u(t)|^{q+1} \,dx + m_0 \int_\Omega |D\varphi(u(t))| + \int_{\partial\Omega} |u|^q \,d\mathcal{H}^{N-1} \leq 0.
\]
\[
(56)
\]
By the continuous embedding (6) there exists $\eta > 0$ such that

$$
\frac{1}{q + 1} \frac{d}{dt} \int_{\Omega} |u(t)|^{q+1} \, dx + \eta \min\{m_0, 1\} \|u^q\|_{N^{-1}} \leq 0.
$$

Then, taking $q = N - 1$ yields

$$
\frac{d}{dt} \int_{\Omega} |u(t)|^N \, dx + \eta \min\{m_0, 1\} N \left( \int_{\Omega} |u(t)|^N \, dx \right)^{N-1/N} \leq 0. \tag{57}
$$

Hence

$$
\frac{d}{dt} \left( \int_{\Omega} |u(t)|^N \right)^{1/N} + \eta \min\{m_0, 1\} N \leq 0. \tag{58}
$$

Since $u(T^*(u_0)) = 0$, then by integrating (58) from $t$ to $T^*(u_0)$, we obtain (46). This shows the first statement. For the second statement, by Lemma 5.1 and $u(T^*(u_0)) = 0$, for every $t \geq \tau$ we have

$$
\frac{1}{T^*(u_0) - t} \left| u(x, T^*(u_0)) - u(x, t) \right| = \frac{1}{T^*(u_0) - t} \left| \int_t^{T^*(u_0)} u'(s) \, ds \right|
$$

$$
\leq \frac{1}{T^*(u_0) - t} \int_t^{T^*(u_0)} |u_0| m \left( \frac{1}{\mu m_0} M \left( \int_{\Omega} |Du_0| + \int_{\partial \Omega} |u_0| \, d\mathcal{H}^{N-1} \right) \right) \, ds
$$

$$
\leq \frac{|u_0| m \left( \frac{1}{\mu m_0} M \left( \int_{\Omega} |Du_0| + \int_{\partial \Omega} |u_0| \, d\mathcal{H}^{N-1} \right) \right)}{\tau}.
$$

Hence this ends the proof of Theorem 4.3.

\[ \blacksquare \]

6. Conclusion

We would like to conclude our paper by mentioning some problems related to our results, which we consider to be very interesting:

- We are wondering if the result stated in Theorem 3.1 still holds in the case when $m(0) = 0$.
- Let $u_0, u_1 : \Omega \to \mathbb{R}$ be smooth functions. Is there exist a function $u$ in the sense of Definition 2.1 obeying the equations

$$
\begin{align*}
\frac{d^2 u}{dt^2} - \text{div} \left( \frac{Du}{|Du|} \right) &= 0 \quad \text{in } \Omega \times (0, +\infty), \\
u &= 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega.
\end{align*}
$$

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References

[1] Andreu F, Ballester C, Caselles V, et al. The Dirichlet problem for the total variation flow. J Funct Anal. 2001;180:347–403.
[2] Andreu F, Caselles V, Díaz JL, et al. Some qualitative properties for the total variation flow. J Funct Anal. 2002;188:516–547.
[3] Giga M-H, Giga Y, Kobayashi R. Very singular diffusion equations. In: Taniguchi Conference on Mathematics Nara ’98. 2001, p. 93–125. (Adv. stud. pure math.; 31).
[4] Andreu F, Caselles V, Mazón JM. Parabolic quasilinear equations minimizing linear growth functionals. Basel: Birkhäuser Verlag; 2004. (Progress in nonlinear differential equations and their applications).
[5] Ecker K. Regularity theory for mean curvature flow. Boston (MA): Birkhäuser Boston Inc; 2004. (Progress in nonlinear differential equations and their applications).
[6] Andreu F, Ballester C, Caselles V, et al. Minimizing total variation flow. Differ Integral Equ. 2001;14:321–360.
[7] Hauer D, Mazón JM. Regularizing effects of homogeneous evolution equations: the case of homogeneity order zero. J Evol Equ. 2019;19:1–32.
[8] Mazón JM, Rossi JD, Toledo J. Fractional p-Laplacian evolution equations. J Math Pures Appl. 2016;105(6):810–844.
[9] Andreu F, Mazón JM, Moll JS. The total variation ow with nonlinear boundary conditions. Asymptot Anal. 2005;43(1–2):9–46.
[10] Mazón JM, Rossi JD, Segura de León S. Two different boundary value problems for the 1-Laplacian equation. Preprint.
[11] Segura de León S, Webler C. Global existence and uniqueness for the inhomogeneous 1-Laplace evolution equation. NoDEA Nonlinear Differ Equ Appl. 2015;22:1213–1246.
[12] Gianazza U, Klaus C. p-parabolic approximation of total variation flow solutions. Indiana Univ Math J. 2019;68(5):1519–1550.
[13] Alves CO, Boudjeriou T. Existence of solution for a class of heat equation involving the 1-Laplacian operator. JMAA. 2022;516:Article ID 126509. DOI: 10.1016/j.jmaa.2022.126509
[14] Alves CO. A Berestycki-Lions type result for a class of problems involving the 1-Laplacian operator. Comm Contemp Math. DOI: 10.1142/S021919972150022X
[15] Juutinen P. p-harmonic approximation of functions of least gradient. Indiana Univ Math J. 2005;54:1015–1029.
[16] Mercaldo A, Rossi JD, Segura de León S, et al. Behaviour of p-Laplacian problems with Neumann boundary conditions when p goes to 1. Commun Pure Appl Anal. 2013;12:253–267.
[17] Mercaldo A, Segura de León S, Trombetti C. On the behaviour of the solutions to p-Laplacian equation as p goes to 1. Publ Mat. 2008;52:377–411.
[18] Molino Salas A, Segura de León S. Elliptic equations involving the 1-Laplacian and a subcritical source term. Nonlinear Anal. 2018;168:50–66.
[19] Kawohl B, Schuricht F. Dirichlet problems for the 1−Laplace operator, including the eigenvalue problem. Commun Contemp Math. 2007;9(4):525–543.
[20] Bögelein V, Duzaar F, Marcellini P. A time dependent variational approach to image restoration. SIAM J Imaging Sci. 2015;8:968–1006.
[21] Bögelein V, Duzaar F, Scheven C. The total variation flow with time dependent boundary values. Calc Var Partial Differ Equ. 2016;55:108. DOI: 10.1007/s00526-016-1041-4
[22] Kinnunen J, Scheven C. On the definition of solution to the total variation flow. arXiv:2106.05711 [math.AP].
[23] Attouch H, Buttazzo G, Michaille G. Variational analysis in Sobolev and BV spaces: applications to PDEs and optimization. Philadelphia (PA): MPS-SIAM; 2006.
[24] Ambrosio L, Fusco N, Pallara D. Functions of bounded variation and free discontinuity problems. New York: The Clarendon Press, Oxford University Press; 2000. xviii+434 pp. (Oxford mathematical monographs).
[25] Evans LC, Gariepy RF. Measure theory and fine properties of functions. Boca Raton (FL): CRC Press; 1992. (Studies in advanced math.)
[26] Ziemer WP. Weakly differentiable functions. Berlin: Springer-Verlag; 1989. (GTM; 120).
[27] Anzellotti G. Pairings between measures and bounded functions and compensated compactness. Ann Mat Pura Appl. 1983;135(1):293–318.
[28] Migórski S, Khan AA, Zeng S. Inverse problems for nonlinear quasi-hemivariational inequalities with application to mixed boundary value problems. Inverse Probl. 2020;36:Article ID 024006.
[29] Zeng S, Liu Z, Migorski S. A class of fractional differential hemivariational inequalities with application to contact problem. Z Angew Math Phys. 2018;69:36, pages 23.
[30] Zeng S, Bai Y, Gasinski L, et al. Existence results for double phase implicit obstacle problems involving multivalued operators. Calc Var PDEs. 2020;59:pages 18.
[31] Chipot M, Lovat B. Some remarks on non local elliptic and parabolic problems. Nonlinear Anal. 1997;30(7):4619–4627.
[32] Gobbino M. Quasilinear degenerate parabolic equations of Kirchhoff type. Math Methods Appl Sci. 1999;22(5):375–388.

[33] Alam MN, Tunc C. An analytical method for solving exact solutions of the nonlinear Bogoyavlenskii equation and the nonlinear diffusive predator–prey system. Alex Eng J. 2016;11(1):152–161.

[34] Alam MN, Tunc C. New solitary wave structures to the (2 + 1)-dimensional KD and KP equations with spatio-temporal dispersion. J King Saud Univ Sci. 2020;32:3400–3409.

[35] Alam MN, Tunc C. The new solitary wave structures for the (2 + 1)-dimensional time-fractional Schrödinger equation and the space-time nonlinear conformable fractional Bogoyavlenskii equations. Alex Eng J. 2020;59:2221–2232.

[36] Islam S, Alam MN, Fayz-Al-Asad M, et al. An analytical technique for solving new computational of the modified Zakharov-Kuznetsov equation arising in electrical engineering. J Appl Comput Mech. 2021;7(2):715–726.

[37] Brézis H. Opérateurs maximaux monotones et semi-groupes des contractions dans les espaces de Hilbert. Amsterdam/London/New York: North-Holland/American Elsevier; 1971.

[38] Bonforte M, Figalli A. Total variation flow and sign fast diffusion in one dimension. J Differ Equ. 2012;252:4455–4480.

[39] Díaz JI, Herrero MA. Propriétés de support compact pour certaines équations elliptiques et parabolique non linéaires. C R Acad Sci Paris. 1978;286:815–817.

[40] Herrero M, Vazquez JL. On the propagation properties of a nonlinear degenerate parabolic equation. Comm Partial Differ Equ. 1982;7:1381–1402.