CASIMIR INTERACTION BETWEEN CYLINDERS

FRANCISCO D. MAZZITELLI

Departamento de Física J.J. Giambiagi, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires - Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina.

We compute the Casimir interaction energy between two perfectly conducting, concentric cylinders, using the mode-by-mode summation technique. Then we compare it with the approximate results obtained using the proximity theorem and a semiclassical approximation based on classical periodic orbits. We show that the proximity theorem with a particular choice for the effective area coincides with the semiclassical approximation and reproduces the exact result far beyond its expected range of validity. We also compute the force between slightly eccentric cylinders and discuss the advantages of using a cylindrical geometry to measure the Casimir force.

1 Introduction

Many of the recent experiments that measured the Casimir force between conductors involved the interaction between a plane and a sphere, with only one exception where the force between parallel plates was measured. The plane-sphere configuration is more appealing from an experimental point of view, mainly because the alignment problem that complicates the measurement for the parallel plates is not present. However, from a theoretical point of view, the inconvenience is that there is no exact evaluation of the force between a sphere and a plane. The comparison between theory and experiment is done with the “proximity” approximation for the force.

It is then of interest to explore other geometries for which the Casimir force can be computed exactly. This would be useful both for testing the validity of the approximations and, eventually, for proposing new experiments. In what follows we will be concerned with the Casimir interaction between two long, perfectly conducting cylindrical shells of radii $a$ and $b$ and length $L \gg a, b$.

2 Exact Casimir energy

In order to define properly the Casimir energy we introduce a cutoff $\sigma$ as follows

$$E_C(\sigma) = \frac{\hbar}{2} \sum_{p=n,m,k} \left( e^{-\sigma \tilde{w}_p} \tilde{w}_p - e^{-\sigma \tilde{w}_p} \tilde{w}_p \right).$$

The exact Casimir energy $E_C$ is the limit of $E_C(\sigma)$ as $\sigma \to 0$. In our definition we take $w_p$ as the eigenfrequencies of the electromagnetic field satisfying perfect conductor boundary conditions at $r = a, r = b$ and $r = R$, and $\tilde{w}_p$ are
those corresponding to the boundary conditions at \( r = R_1, r = R_2 \) and \( r = R \), in the limit \( R > R_2 > R_1 \gg a > b \). The parameters \( R_1, R_2 \) and \( R \) define the reference vacuum.

In cylindrical coordinates, the eigenfunctions are of the form

\[
h_{nkz}(t, r, \theta, z) = e^{-i\omega_{nkz}t + ik_z z - in\theta}R_n(\lambda r),
\]

where the function \( R_n \) is a combination of Bessel functions satisfying the perfect conductor boundary conditions. These boundary conditions define the possible values of the constant \( \lambda \). The eigenfrequencies are

\[
\omega_{nkz} = c\sqrt{k_z^2 + \lambda^2}.
\]

The frequencies of the TE modes are defined by

\[
F_{TE}^n(z, a, b, R) \equiv J_n(za)\left(J_n(za)N_n(zb) - J_n(zb)N_n(za)\right) \\
\times \left[J_n(zb)N_n(zR) - J_n(zR)N_n(za)\right] = 0.
\]

The frequencies of the TM modes involve derivatives of the Bessel functions, and one can define an analogous function \( F_{TM}^n \). In the set of quantum numbers \( p = (n, m, k_z) \) appearing in Eq. (1), \( m \) denotes the different solutions \( \lambda_{nm} \) of both \( F_{TE}^n(z, a, b, R) = 0 \) and \( F_{TM}^n(z, a, b, R) = 0 \).

From Cauchy’s theorem it follows that

\[
\frac{1}{2\pi i} \int_C dz \frac{e^{-\sigma z} d}{dz} \ln f(z) = \sum_{i} x_i e^{-\sigma x_i},
\]

where \( f(z) \) is an analytic function within the closed contour \( C \), with simple zeros at \( x_1, x_2, \ldots \), within \( C \). We use this result to replace the sum over \( m \) in Eq. (1) by a contour integral in the complex plane. We find

\[
E_C(\sigma) = \frac{Lhc}{4\pi i} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sum_n \int_C dz \sqrt{k_z^2 + z^2} e^{-\sigma c\sqrt{k_z^2 + z^2}} \frac{d}{dz} \ln F_{EM}^n(z, a, b),
\]

where

\[
F_{EM}^n(z, a, b) = \lim_{R_1, R_2, R \to \infty} \frac{F_{TE}^n(z, a, b, R)F_{TM}^n(z, a, b, R)}{F_{TE}^n(z, R_1, R_2, R)F_{TM}^n(z, R_1, R_2, R)}.
\]

An adequate contour for the integration in the complex plane is a circular segment \( C_r \) of radius \( \Gamma \) and two straight line segments forming an angle \( \phi \) and \( \pi - \phi \) with respect to the imaginary axis. The nonzero angle \( \phi \) is needed to show that the contribution of \( C_r \) vanishes in the limit \( \Gamma \to \infty \) when \( \sigma > 0 \).

It proves to be convenient to compute the difference between the energy of the system of two concentric cylinders and the energy of two isolated cylinders of radii \( a \) and \( b \)

\[
E_{12}(\sigma) = E_C(\sigma) - E_1(\sigma, a) - E_1(\sigma, b).
\]

The divergences in \( E_C(\sigma) \) are cancelled out by those of \( E_1(\sigma, a) \) and \( E_1(\sigma, b) \). Therefore, to compute \( E_{12}(\sigma) \) we can set \( \phi = 0 \) and \( \sigma = 0 \). The contour
integral reduces to an integral on the imaginary axis. After some steps we find
\[ E_{12} = -\frac{\hbar c}{2\pi a^2} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sum_n \text{Im} \left\{ \int_0^\infty dy \sqrt{k_z^2 - y^2} \frac{d}{dy} \ln F_n(y, \alpha) \right\}, \quad (8) \]
where \( \alpha = b/a \) and
\[ F_n(y, \alpha) = \left[ 1 - \frac{I_n(y)K_n(\alpha y)}{I_n(\alpha y)K_n(y)} \right] \left[ 1 - \frac{I'_n(y)K'_n(\alpha y)}{I'_n(\alpha y)K'_n(y)} \right]. \quad (9) \]
Eqs. (8) and (9) allow a simple numerical evaluation of \( E_{12} \). The exact energy for the two concentric cylinders is the sum of \( E_{12} \) and the Casimir energies for single cylinders of radii \( a \) and \( b \)
\[ E_C = E_{12} - 0.01356 \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \hbar c. \quad (10) \]

3 Proximity and semiclassical approximations

The Casimir energy per unit area for parallel plates separated by a distance \( l \) is given by \( E_{pp}(l) = -\frac{\pi^2\hbar c}{120l^3} \). In the proximity approximation, the interaction energy between two conductors that form a curved gap of variable width \( z \) can be computed as
\[ E_I = \int_{\Sigma} E_{pp}(z) d\sigma, \quad (11) \]
where \( \Sigma \) is one of the two surfaces that define the gap. It is clear that in the above approximation the non-parallelism of the area elements is not taken into account. Moreover, the result is different if the other surface is chosen to perform the calculation. These corrections are expected to be small for low curvature and close surfaces.

For the concentric cylinders, the proximity approximation gives
\[ E_I(b - a) = -\frac{\pi^2\hbar c}{120} \frac{A_{eff}}{(b - a)^3}. \quad (12) \]
As mentioned before, there is an ambiguity in the choice of the effective area \( A_{eff} \). A calculation based on the inner cylinder gives \( A_{eff} = 2\pi aL \), while for a calculation based on the outer one gives \( A_{eff} = 2\pi bL \). The difference is of course harmless in the limit \( \alpha \to 1 \). In this limit, the exact result for \( E_{12} \) coincides with the proximity approximation. Indeed, using the uniform expansion of Bessel functions in Eqs. (8) and (9), one can show that to leading order in \( \alpha - 1 \), \( E_{12} \approx E_I(b - a) \).

For larger values of \( \alpha \) one expects the exact result to differ significantly from the proximity approximation. We have computed numerically the interaction energy \( E_{12} \), and compared it with the proximity approximation, using
different choices for the effective area. In particular, we parametrized the effective area as $A_{\text{eff}} = 2\pi La^p b^{1-p}$, and found that the value $p = 1/2$ provides the best fit for the numerical data. For this value of $p$, the effective area is the geometric mean of the areas of both cylinders. As shown in Fig. 1, the proximity approximation with this particular choice for the area reproduces the exact results far beyond its expected range of validity: the discrepancy for the pressure on the inner cylinder due to the presence of the outer one is less than 10% for $1 < \alpha < 4$. The choice of the geometric mean is crucial for this agreement.

Let us now discuss briefly the semiclassical approximation for the Casimir

![Figure 1. Dimensionless Casimir interaction energy (upper panel) and pressure (lower panel), as a function of $\alpha = b/a$. In both panels the dashed line corresponds to the exact result and the full line to the $p = 1/2$ proximity result.]
energy, which can be written as

\[ E_C = \int_0^{\infty} \frac{1}{2} E \rho_{osc}(E) \, dE , \]

(13)

where \( \rho_{osc}(E) \) is the difference between the spectral density of electromagnetic modes in the presence of conductors and the spectral density in vacuum. Periodic orbit theory relates \( \rho_{osc} \) of a given Hamiltonian to the periodic orbits in the corresponding classical system. To leading order in \( \hbar \),

\[ \rho_{osc}(E) = \frac{1}{\hbar^{\nu}} \sum_t A_t(E) \sin(S_t(E)/\hbar + \mu_t) , \]

(14)

where the sum runs over periodic orbits labeled by \( t \), and \( S_t \) is the classical action of the periodic orbit \( t \). For photons of energy \( E \), \( S_t(E) = EL_t/c \), with \( L_t \) the length of the periodic orbit. The phase \( \mu_t \) is the so-called Maslov index. The exponent \( \nu \) and the amplitudes \( A_t \) depend on the type of the periodic orbit.

The periodic orbits in the region between very long concentric cylinders are contained in planes perpendicular to the axis of the cylinders. There are two different classes of orbits. The type-I orbits do not touch the inner cylinder, and are polygons that may be uniquely labeled by two integers \( (v, w) \), where \( v \) is the number of bounces in the outer cylinder and \( w \) the winding number around the center. Type-II trajectories do touch both cylinders, and are also labeled by \( (v, w) \). We have have computed the Casimir interaction energy using this semiclassical approximation. The final result is dominated by the self-retracing type II periodic orbit \( (w = 0) \) and its repetitions. Moreover, it coincides with the proximity approximation if the effective area is taken as the geometric mean of the areas of both cylinders.

4 Eccentric cylinders: an experimental proposal

Up to now we considered concentric cylinders. Obviously, the force between them vanishes in this case. Let us now consider two slightly eccentric cylinders of radii \( a \) and \( b \). We will mainly focus on the particular case \( a \approx b \), since the Casimir interaction between cylinders is then stronger. The distance between the axis of the cylinders will be denoted by \( \epsilon \) (see Fig. 2). In order to evaluate the Casimir energy for this configuration, we will use the proximity approximation. This is partially justified by our previous results. The interaction energy between cylinders is

\[
E_I \simeq -\frac{\pi^2 \hbar c}{720} \int_0^{2\pi} \frac{dA_{eff}(\theta)}{r(\theta) - a}^3 ,
\]

(15)

where \( r(\theta) \) is the distance of a point of the external cylinder to the axis of the inner one and \( dA_{eff}(\theta) \) is the geometric mean of two small adjacent areas on both cylinders. From Fig.2 one can check that \( r(\theta) = \sqrt{b^2 - \epsilon^2 \cos^2 \theta} + \epsilon \sin \theta \)
Figure 2. Two eccentric cylinders. (a) An inner cylinder of radius $a$ and a hollow cylinder of radius $b$, with the origin of coordinates on the axis of the inner cylinder, and distance $\epsilon$ between the two axis. (b) The effective area for the application of the proximity approximation, as the geometric mean of $S_1$ and $S_2$: $dA_{\text{eff}}(\theta) = \sqrt{S_1 S_2}$

and $dA_{\text{eff}} = L \sqrt{ab + \epsilon a \sin \theta d\theta}$. Taking the derivative of the energy with respect to $\epsilon$ we obtain the force on the outer cylinder due to the inner one. Since we are considering $a \approx b$, we will always have $\epsilon/b \ll 1$. Thus to the lowest non trivial order we obtain

$$F_y = -\frac{\pi^2 \hbar c L a}{240 b^4} \int_0^{2\pi} \frac{d\theta \sin \theta}{\left[1 + \frac{\epsilon}{b} \sin \theta - \frac{a}{b}\right]^4} \approx F_0 \left(\tilde{\epsilon} + \frac{\tilde{\epsilon}^3}{7}\right)^{-1/2},$$

where $\tilde{\epsilon} = \epsilon/(b-a)$ and $F_0 = \pi^3 \hbar c L a / 60(b-a)^4$. It is worth noting that closest surfaces are attracted together, and that the force only vanishes when the cylinders are exactly concentric. The equilibrium position is unstable. In the particular case in which $\tilde{\epsilon} \ll 1$, the force is linear in the distance between the axis of the cylinders $F_y \approx \tilde{\epsilon} F_0$. This corresponds to an inverted harmonic oscillator, and explicitly shows the instability. In the opposite case, when $\tilde{\epsilon} \to 1$, the force scales like $d^{-7/2}$, where $d = b - a - \epsilon$ is the minimum distance between cylinders.

We now discuss a possible experimental arrangement that could be used to measure the Casimir force between cylinders. We consider the almost coaxial configuration $\tilde{\epsilon} \ll 1$. The external cylinder could be mounted on a resonator of effective mass $M$ and natural frequency $\omega_0$. The presence of the inner cylinder renormalizes the frequency of the resonator. Assuming a small frequency shift we obtain $\Delta \omega / \omega_0 = -\frac{F_0}{2(b-a) M \omega_0^2}$. For typical values of the parameters it is simple to reach a frequency shift of 0.1%. From an experimental point of view, this configuration has some advantages over the parallel plates configuration. On the one hand, the alignment procedure is easier for the cylinders than for the plates (see below). On the other hand, when there is no residual charge in the inner cylinder, the system remains neutral and screened by the external one from outer noises, interferences, and from residual charges in the outer cylinder. The expected gravitational force is...
obviously null. When the inner cylinder has a residual charge, there will be a small potential difference between the cylinders, and the coaxial configuration will be electrostatically unstable. To avoid it, one could start the experiment by putting both cylinders in contact. Ideally, the charge of the inner cylinder would flow to the outer one, minimizing the effect if residual electrostatic forces. Alternatively, one could use the electrostatic instability to improve the alignment: if a time dependent differential potential is applied between cylinders (as in the null experiments to test the exactness of the electrostatic inverse square law), parallelism would be optimized when the value of the force is minimum.

To summarize, we have shown that the interaction between cylindrical shells is both of theoretical and experimental interest for the study of Casimir forces. On the theoretical side, it is a simple but non trivial example for testing the validity of different approximations to the Casimir energy. On the experimental side, it is a promising geometry for measuring the Casimir force.

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