Completions of $\mu$-Algebras

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Abstract

A $\mu$-algebra is a model of a first order theory that is an extension of the theory of bounded lattices, that comes with pairs of terms $(f, \mu_x.f)$ where $\mu_x.f$ is axiomatized as the least prefixed point of $f$, whose axioms are equations or equational implications.

Standard $\mu$-algebras are complete meaning that their lattice reduct is a complete lattice. We prove that any non trivial quasivariety of $\mu$-algebras contains a $\mu$-algebra that has no embedding into a complete $\mu$-algebra.

We focus then on modal $\mu$-algebras, i.e. algebraic models of the propositional modal $\mu$-calculus. We prove that free modal $\mu$-algebras satisfy a condition – reminiscent of Whitman’s condition for free lattices – which allows us to prove that (i) modal operators are adjoints on free modal $\mu$-algebras, (ii) least prefixed points of $\Sigma_1$-operations satisfy the constructive relation $\mu_x.f = \bigvee_{n \geq 0} f^n(\bot)$. These properties imply the following statement: the MacNeille-Dedekind completion of a free modal $\mu$-algebra is a complete modal $\mu$-algebra and moreover the canonical embedding preserves all the operations in the class $\text{Comp}(\Sigma_1, \Pi_1)$ of the fixed point alternation hierarchy.

Introduction

When $L$ is a complete lattice, the least fixed point $\mu_x.f$ of a monotone function $f : L \longrightarrow L$ enjoys a remarkable property. We like to say that the least fixed point is constructive: the equality

$$\mu_x.f = \bigvee_{\alpha \in \text{Ord}} f^\alpha(\bot)$$

holds and provides a method to construct $\mu_x.f$ from the bottom of the lattice. The expressions $f^\alpha(\bot)$, indexed by ordinals, are commonly called the
approximants of $\mu_x.f$. They are defined by transfinite induction as expected: $f^0(\bot) = \bot$, $f^{\alpha+1} = f(f^\alpha(\bot))$, and $f^\alpha(\bot) = \bigvee_{\beta<\alpha} f^\beta(\bot)$ for a limit ordinal $\alpha$.

A careful reading of Tarski’s original fixpoint theorem [22] reveals that the completeness assumption is not needed for $f$ to have a fixed point. If $L$ is merely a poset a least prefixed point of a monotone $f$ is an element $\mu_x.f \in L$ satisfying

$$f(\mu_x.f) \leq \mu_x.f,$$

$$f(y) \leq y \quad \Rightarrow \quad \mu_x.f \leq y.$$  

Tarski’s theorem can be rephrased by saying that the least prefixed point, whenever it exists, is also a fixed point, hence it is the least fixed point. The two notions are similar and coincide on complete lattices. Properties (2) and (3) – the latter known as the Park induction rule [16, 6] – provide a natural axiomatization by equations and equational implications of least fixed points, provided that an order relation definable by equations is given. They have been used often to axiomatize concrete mathematical objects where implicit or explicit fixed points are at work: relational algebras with transitive closure [14], regular languages [11], powersets of Kripke frames [20, 10].

Many considerations induce to study classes of models of axioms (2) and (3). For example, model theory suggests that models of theories axiomatized by equational implications are preferable to models that are complete lattices: the former build up a quasivariety, colimits exist, free models exist, etc. The goal of this paper is to compare the models of theories where the least fixed points are defined by means of (2) and (3) – we shall refer to them as $\mu$-algebras – with a more restricted class of models, the standard, concrete, or complete models. These are the models whose underlying lattice is complete and where least fixed points are constructive. Mathematically, the comparison amounts to asking whether a model can be embedded into a complete one.

Despite the difference in the respective lengths, the paper is divided into two parts. In the first part we show that almost never $\mu$-algebras are completable. That is, within a non trivial fixed quasi-variety of $\mu$-algebras, we construct a $\mu$-algebra that has no embedding into a complete one. The second part of the paper is devoted to studying free modal $\mu$-algebras. Modal $\mu$-algebras are algebraic models of the modal $\mu$-calculus [10]; by the completeness theorem w.r.t. the class of Kripke frames [10, 25] we already know that free modal $\mu$-algebras are completable. We pursue an algebraic understanding of this fact, which eventually will provide us with some algebraic interpretation of the completeness theorem. Our analysis of free $\mu$-algebras, which never takes the completeness theorem as granted, can be synthesized as follows. We observe first a phenomenon that we classify as “definability of adjoints”. Using adjoints and their generalizations, $O_\Sigma$-adjoints, we argue that a restricted class of least fixed points are constructive on free modal $\mu$-algebras. This means that many relations like (1) hold on a free modal $\mu$-algebra even if this is presumably not complete. A detour through least solutions of systems of equations allow us to extend the class of constructive operations on free modal $\mu$-algebras to include all the $\Sigma_1$-operations. It is
easily argued that constructiveness is an essential property for an embedding into a complete $\mu$-algebra to exist. Indeed, the outcome of our analysis is the following result: the MacNeille-Dedekind completion of a free modal $\mu$-algebra is a complete modal $\mu$-algebra and the canonical embedding preserves all the operations in the class $\Sigma_1$ of the fixed point alternation hierarchy. The result is easily extended to the class $\text{Comp}(\Sigma_1, \Pi_1)$ of the alternation hierarchy.

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1 Notation, background

Cartesian structure

If $X$ is a finite set, then $L^X$ will denote the power of $L$ by $X$, i.e. $L^X = \prod_{x \in X} L$. Projection functions from a product to one of its factors will be denoted by $\text{pr}_x$ with the necessary sub/superscripts. For example if $x \in X$, then $\text{pr}^X_x : L^X \to L$ denotes the projection taking a vector $v \in L^X$ to $v(x)$.

If $f_x : L \to M$, $x \in X$, is a collection of functions, then shall use the notation $\langle f_x \rangle_{x \in X} : L \to M^X$ for the unique function $f : L \to M^X$ such that $f_x = \text{pr}_x \circ f$.  

3
Parametrized fixed points

Let \( f : P \times Q \to P \) be a monotone function. For \( q \in Q \), we use the notation \( f_q : P \to P \) for the monotone function sending \( p \) to \( f(p, q) \).

**Lemma 1.1.** Suppose that for each \( q \in Q \) the least prefixed point \( \mu_x.f_q \) of \( f_q : P \to P \) exists. Then the correspondence \( \mu_x.f : Q \to P \) sending \( q \) to \( \mu_x.f_q \) is monotone.

The notation \( f : P^x \times Q^y \to M \) will be used here to mean that \( f \) is considered as a function of two variables \( x \in P \) and \( y \in Q \). For example, if \( f : P^x \times P^y \to P \), then we shall write \( \mu_y.f \) or \( \mu_y.f(x, y) \) for unary function sending \( p \in P \) to the least prefixed point of the unary function \( y \mapsto f(p, y) \).

The Bekič property

We shall often make use of the Bekič property which usually is stated as an identity between existing (least pre-)fixed points. We shall also be concerned with existence of least prefixed points, hence we need a stronger form of the property which emphasizes this issue as well.

**Proposition 1.2.** Let \( P, Q \) be posets, let \( f : P^x \times Q^y \to P \) and \( g : P \times Q \to Q \) be monotone functions, and suppose that for each \( q \in Q \) the least prefixed point \( \mu_x.f_q \) of \( f_q : P \to P \) exists. Consider the monotone functions

\[
\langle f, g \rangle : P \times Q \to P \times Q,
\]

\[
\langle \mu_x.f \circ \text{pr}_Q, g \rangle : P \times Q \to P \times Q.
\]

A least prefixed point \( (\mu_1, \mu_2) \) of \( \langle f, g \rangle \) exists if and only if the least prefixed point \( (\mu_3, \mu_4) \) of \( \langle \mu_x.f \circ \text{pr}_Q, g \rangle \) exists, and if any of them exists then \( (\mu_1, \mu_2) = (\mu_3, \mu_4) \).

**Proposition 1.3.** Let \( P, Q \) be posets and \( f : P \to Q \), \( g : P \times Q \to Q \) be monotone functions. Consider the monotone functions

\[
\langle f \circ \text{pr}_Q, g \rangle : P \times Q \to P \times Q,
\]

\[
g \circ \langle f, \text{id}_Q \rangle : Q \to P \times Q \to Q.
\]

A least prefixed point \( (\mu_1, \mu_2) \in P \times Q \) of \( \langle f \circ \text{pr}_Q, g \rangle \) exists if and only if a least prefixed point \( \mu_3 \in Q \) of \( g \circ \langle f, \text{id}_Q \rangle \) exists, and they are determined by each other as follows:

\[
(\mu_1, \mu_2) = (f(\mu_3), \mu_3), \quad \mu_3 = \mu_2.
\]

A proof of the propositions appears in the extended version of [18] within the more general context of initial algebras of functors.
2 $\mu$-algebras are not completable

$\mu$-theories and $\mu$-algebras

Our first goal is to setup a generic logical framework within which to develop a theory of ordered algebras with least fixed point operators. Analogous frameworks [3, 1] can be coded within this framework.

Definition 2.1. A $\mu$-theory is a first order theory with the following properties:

- it is an extension of the theory of bounded lattices,
- it comes with fixed point pairs, that is, pairs of terms $(f, \mu_x.f)$ axiomatized by (2) and (3) so that (the interpretation of) $f$ is an order preserving operation in the variable $x$, and (the interpretation of) $\mu_x.f$ is a least prefixed point of $f$,
- its axioms are either equations or equational implications.

A $\mu$-algebra is model of a fixed $\mu$-theory. A $\mu$-algebra is complete if its lattice reduct is a complete lattice.

The notion of a morphism of $\mu$-algebras is standard from model theory: a function $g : A \longrightarrow B$ between the underlying sets of $\mu$-algebras $A$ and $B$ is a morphism if it preserves the interpretation of all the terms of the $\mu$-theory.

Let $f$ be a term of a $\mu$-theory and let $X$ be its set of free variables. For a $\mu$-algebra $A$, we shall overload the notation and write $f : A^X \longrightarrow A$ for the interpretation of $f$ on $A$. If $f$ is part of a fixed point pair $(f, \mu_x.f)$, so that $X$ is the disjoint union of $\{x\}$ and $Y$, and $v \in A^Y$, then we use the notation $f_v : A \longrightarrow A$ consistently with what exposed in Section 1 and say that $f_v : A \longrightarrow A$ is a fixed point polynomial. To simplify the notation, we shall also omit the subscript $v$ and say that $f : A \longrightarrow A$ is a fixed point polynomial.

A $\mu$-algebra with no complete extension

Our goal is to assess relations between $\mu$-algebras and complete $\mu$-algebras and to understand when a $\mu$-algebra embeds into a complete one. Contrarily to what happens for several algebraic structures related to logic (Boolean algebras, modal algebras K, Heyting algebras, quantales), we show next that this is not always possible for $\mu$-algebras.

Example 2.2. Choose a $\mu$-algebra $A$ and a fixed point polynomial $f : A \longrightarrow A$ for which the chain of finite approximants

$$\bot < f(\bot) < f^2(\bot) < \ldots < f^n(\bot) < \ldots$$
is infinite. Define the infinite sequences \( \phi_n \) by
\[
\phi_n = (\perp, \ldots, \perp, f(\perp), f^2(\perp), \ldots), \; n \geq 0,
\]
and consider them as elements of the product algebra \( A^\omega \). Since \( f \) is computed pointwise, observe that \( f(\phi_n) \) is equal to \( \phi_{n-1} \) for all but a finite number of coordinates.

Define the equivalence relation \( \sim \) on \( A^\omega \) by saying that two infinite sequences are equivalent if they coincide in all but a finite number of coordinates. The quotient \( A^\omega/\sim \) is a reduced product of \( A \) and all the equations and equational implications that hold in \( A \) hold in \( A^\omega/\sim \) as well, cf. [4, chapter 6]. In particular, \( A^\omega/\sim \) is a \( \mu \)-algebra in the same quasivariety as \( A \).

Denote by \( \bar{\phi}_n \) the equivalence class of \( \phi_n \) and recall that in \( A^\omega/\sim \) the least fixed point \( \mu_x.f \) is simply the equivalence class of the infinite sequence with constant value \( \mu_x.f \). The relations
\[
f(\bar{\phi}_n) \leq \bar{\phi}_{n-1}, \; n \geq 1 \quad \mu_x.f \nleq \bar{\phi}_0 \tag{4}
\]
hold in \( A^\omega/\sim \) and we claim that a configuration such as the one described by (4) is not compatible with \( A^\omega/\sim \) being complete. If \( \bigwedge_{n \geq 0} \bar{\phi}_n \) exists then
\[
f(\bigwedge_{n \geq 0} \bar{\phi}_n) \leq f(\bar{\phi}_{n+1}) \leq \bar{\phi}_n,
\]
for all \( n \geq 0 \), and therefore \( f(\bigwedge_{n \geq 0} \bar{\phi}_n) \leq \bigwedge_{n \geq 0} \bar{\phi}_n \). Then \( \mu_x.f \leq \bigwedge_{n \geq 0} \bar{\phi}_n \leq \bar{\phi}_0 \) gives a contradiction.

Finally observe that such a configuration is preserved by any extension of \( A^\omega/\sim \), and therefore this \( \mu \)-algebra has no complete extension.

It can be observed that in the \( \mu \)-algebra \( A^\omega/\sim \) the stronger relations \( f(\bar{\phi}_n) = \bar{\phi}_{n-1} \) hold and moreover \( \phi_{n+1} \leq \phi_n \). These stronger relations, which are not needed to prove that \( A^\omega/\sim \) has no completion, conceal the original idea behind the impossibility proof. This amounts to the construction of a chain of approximants indexed by natural numbers with the reverse order.

We say that a \( \mu \)-theory (or the quasivariety of the \( \mu \)-algebras) is non-trivial if we can find a \( \mu \)-algebra \( A \) and a fixed point polynomial \( f \) for which its finite approximants are all distinct. If this is not possible, then for each fixed point pair \((f, \mu_x.f)\) some equation of the form \( \mu_x.f = f^n(\perp) \) holds, showing that all the least fixed point are superfluous. We collect these observations in a Theorem.

**Theorem 2.3.** Any non-trivial quasivariety of \( \mu \)-algebras contains a \( \mu \)-algebra which does not admit an embedding into a complete \( \mu \)-algebra.

For simple \( \mu \)-theories, if a \( \mu \)-algebra has no configuration such as (4), then the principal filter embedding is a morphism of \( \mu \)-algebras.
Example 2.4. Consider a $\mu$-theory $T$ with just a unary function symbol $f$ and a constant $\mu_x.f$ in addition to the signature of bounded lattices. The axioms of $T$ are those of bounded lattices, additional equations, and the fixed point axioms (2) and (3) for the unique fixed point pair $(f(x), \mu_x.f)$.

Let $A$ be a $\mu$-algebra with no configuration such as (4) and let $F(A)$ be the standard algebra of filters of $A$ in the signature of $T$. Then all the equations of $T$ holds in $F(A)$ and all we need to observe is that the principal filter $\uparrow \mu_x.f$ is the least fixed point of the extension of $f$ to $F(A)$. Considering that the order in $F(A)$ is reverse inclusion, we need to verify that $\mu_x.f$ is below any element of an arbitrary filter $F$ such that $F \subseteq f(F)$. Recalling that

$$f(F) = \{ x \mid \exists y \in F f(y) \leq x \},$$

if $\phi_0 \in F$ then we can construct a sequence $\{ \phi_n \}_{n \geq 0}$ such that $f(\phi_{n+1}) \leq \phi_n$ for $n \geq 0$. Since $A$ lacks a configuration such as (4), we deduce $\mu_x.f \leq \phi_0$. □

Continuity of the algebra of ideals over a lattice is a major obstacle to exploit such a construction for completions of $\mu$-algebras. For example, the principal filter embedding becomes useless for $\mu$-theories where greatest fixed points are also an issue. Other conditions are needed to ensure that a $\mu$-algebra has an embedding into a complete $\mu$-algebra.

3 Completions for free modal $\mu$-algebras, overview

Recall that a free $\mu$-algebra embeds into a complete one if and only if the class of complete $\mu$-algebras generates the class of all $\mu$-algebras. If we adopt the perspective of algebraic logic, the statement that free $\mu$-algebras embed into complete ones amounts to a completeness theorem for the logic with respect to the semantics of all complete models.

It is often the case that free $\mu$-algebras embed into complete ones, for example free $\mu$-lattices [17] and free modal $\mu$-algebras, i.e. Lindenbaum algebras for the propositional modal $\mu$-calculus [10]. The rest of this paper will be concerned with studying free modal $\mu$-algebras. We present here their $\mu$-theory, i.e. the theory of modal $\mu$-algebras. The terms of the theory are generated according to the grammar:

$$t = p \mid x \mid \top \mid t_1 \land t_2 \mid \neg t \mid (\sigma)t \mid \mu_x.t,$$

where $\sigma$ ranges on a finite set of actions $\text{Act}$ and the fixed point generation rule applies only when the variable $x$ occurs under an even number of negations. The reader has surely recognized the framework of multimodal algebras, in addition to which we have least fixed points. Accordingly, the axioms of the theory are those of multimodal algebras $K$ as well as (2) and (3) for the fixed point.
pairs \((t, \mu_x.t)\). In the grammar we have distinguished a generator \(p\) from a variable \(x\). This will be useful when considering the interpretation of terms as operations on free modal \(\mu\)-algebras, where the generators become operations. This kind of term generation is standard from fixed point theory \([15]\), but it is also possible to code these terms as terms generated from an infinite signature using substitution only \([13]\). Finally, it can be shown that modal \(\mu\)-algebras form a variety of algebras \([19]\).

The completeness results for the propositional modal \(\mu\)-calculus \([10, 25]\) paired with the small Kripke model property \([21]\) imply that a free modal \(\mu\)-algebra has an embedding into an infinite product of finite modal \(\mu\)-algebras. This infinite product is of course a complete lattice. In the rest of the paper we shall prove a weaker embedding result concerning \(\Sigma_1\)-terms and \(\Sigma_1\)-operations. \(\Sigma_1\)-terms are defined by the grammar:

\[
t = x | p | \neg p | \top | t \land t | t \lor t | \langle \sigma \rangle t | [\sigma] t | \mu_x.t,
\]

(5)

and \(f : A^X \rightarrow A\) is a \(\Sigma_1\)-operation if it is the interpretation of a \(\Sigma_1\)-term. Observe that the fixed point formation rule is no longer constrained in the above grammar. By duality, the greatest fixed point \(\nu_x.f(x,y)\) of an operation \(f(x,y)\) is definable in the given signature: \(\nu_x.f(x,y) = \neg \mu_x.\neg f(\neg x,y)\). The class of \(\Pi_1\)-terms is then defined as above with the exception that least fixed point formation is replaced by greatest fixed point formation. The class of \(\text{Comp}(\Sigma_1, \Pi_1)\)-operations is obtained by composing in all the possible ways operations in the classes \(\Sigma_1\) and \(\Pi_1\). The reader is invited to consult \([1, \text{chapter 8}]\) for an exposition of the full fixed point alternation hierarchy. Our result can be stated as follows:

**Theorem 3.1.** Let \(\mathcal{F}\) be a free modal \(\mu\)-algebra. There exists a complete modal algebra \(\overline{\mathcal{F}}\) and an injective morphism of Boolean modal algebras \(i : \mathcal{F} \rightarrow \overline{\mathcal{F}}\) which preserves all the \(\text{Comp}(\Sigma_1, \Pi_1)\)-operations of the algebra \(\mathcal{F}\).

With respect to \([25]\), where algorithmic and game-theoretic ideas as well as tableaux manipulations are the main tools, we shall use purely algebraic and order theoretic tools. Under some respect, our work can also be understood as an effort to translate ideas from \([10, 25]\) into an algebraic and order theoretic framework.

We sketch in the rest of the section the strategy followed to prove Theorem 3.1. The algebra \(\overline{\mathcal{F}}\) is the MacNeille-Dedekind completion of \(\mathcal{F}\). For our goals, we recall that if \(L\) is a Boolean algebra, then \(\overline{L}\) is a Boolean algebra as well, see \([2, \text{Chapter V, Theorem 27}]\). Recall that an order preserving \(f : L \rightarrow M\) is a left adjoint if there exists \(g : M \rightarrow L\) (the right adjoint) such that \(f(x) \leq y\) if and only if \(x \leq g(y)\), for all \(x \in L\) and \(y \in M\). For our goals, we also need the following statement:

**Lemma 3.2.** Let \(L\) be a lattice and \(\overline{L}\) be its MacNeille-Dedekind completion. A left adjoint \(f : L \rightarrow L\) has an extension – necessarily unique – to a left adjoint \(f^\vee : \overline{L} \rightarrow \overline{L}\).
Using the notation of [8], if \( g \) is right adjoint to \( f \), then \( g^\wedge \) is right adjoint to \( f^\vee \). A first step towards our main result will be to prove:

**Claim 3.3.** The modal operators \( \langle \sigma \rangle \) of a free modal \( \mu \)-algebra are left adjoints.

Using Lemma 3.2 and Claim 3.3 we can state:

**Proposition 3.4.** The MacNeille-Dedekind completion \( \mathcal{F} \) of a free modal \( \mu \)-algebra is a multi-modal algebra \( K \) and the principal ideal embedding is a morphism of multi-modal algebras.\(^1\)

Since \( \mathcal{F} \) is a complete lattice, it is a complete modal \( \mu \)-algebra, and therefore we are also interested in preservation of fixed points. To this goal we shall use the following Lemma:

**Lemma 3.5.** Let \( A \) be a \( \mu \)-algebra, \( i : A \rightarrow \overline{A} \) its MacNeille-Dedekind completion, and \( f_v \) a fixed point polynomial. Suppose that

- \( f_v \) is preserved by \( i \), that is, \( i(f_v(x)) = f_{i(v)}(i(x)) \),
- \( \mu_x f_v \) is constructive: \( \mu_x f_v = \bigvee_{\alpha \in \text{Ord}} f_{i(v)}^\alpha(\bot) \).

Then the least fixed point \( \mu_x f_v \) is preserved: \( i(\mu_x f_v) = \mu_x f_{i(v)} \).

**Proof.** Observe first that

\[
 f_{i(v)}(i(\mu_x f_v)) = i(f_v(\mu_x f_v)) = i(\mu_x f_v) ,
\]

from which we deduce \( \mu_x f_{i(v)} \leq i(\mu_x f_v) \). For the converse we argue that approximants are preserved using continuity of the embedding of a lattice into its MacNeille-Dedekind completion. We have that \( f_{i(v)}^\beta(\bot) \) is preserved since \( i \) preserves the bottom, and \( f_{i(v)}^{\alpha+1}(\bot) \) is preserved since \( i \) preserves \( f \). For a limit ordinal \( \alpha \), suppose that \( i \) preserves \( f^{\beta}(\bot) \) for \( \beta < \alpha \). Then:

\[
 i\left( \bigvee_{\beta < \alpha} f_{i(v)}^{\beta}(\bot) \right) = \bigvee_{\beta < \alpha} i(f_{i(v)}^{\beta}(\bot)) = \bigvee_{\beta < \alpha} f_{i(v)}^{\beta}(\bot) ,
\]

since \( i \) preserves all existing joins. Consequently \( i(\mu_x f_v) = \bigvee_{\alpha \in \text{Ord}} f_{i(v)}^\alpha(\bot) \) which clearly is below \( \mu_x f_{i(v)} \).

We shall prove that all the \( \Sigma_1 \)-operations are preserved by showing that all these functions are constructive:

**Claim 3.6.** Every fixed point \( \Sigma_1 \)-polynomial \( f_v : \mathcal{F} \rightarrow \mathcal{F} \) over a free modal \( \mu \)-algebra satisfies the constructive relation

\[
 \mu_x f_v = \bigvee_{n \geq 0} f_v^n(\bot) .
\]

\(^1\)The same statement holds if we replace “free modal \( \mu \)-algebra” with “free multi-modal algebra \( K \)”.
Lemma 3.5 and Claim 3.6 imply that each $\Sigma_1$-operation on a free modal $\mu$-algebra is preserved.

A proper dualisation of the notions and results exposed so far can be used to prove that $\Pi_1$-operations are preserved by the embedding of a free modal $\mu$-algebra into its MacNeille-Dedekind completion. Consequently, all the operations in the class $\text{Comp}(\Sigma_1, \Pi_1)$ are preserved as well.

Finally, it should be observed that – using the completeness of the propositional modal $\mu$-calculus and the small model theorem [10, 25, 21] – it is possible to directly argue that every fixed point polynomial on a free modal $\mu$-algebra satisfies the relation (6). Hence the embedding of a free modal $\mu$-algebra into its MacNeille-Dedekind completion is indeed a morphism of modal $\mu$-algebras. On the other hand, it is implicit from [24] that the constructive relation (6) is at the core of the completeness problem for the modal $\mu$-calculus. There a proof-system is described which can easily be proved complete once it is known that the relations (6) hold in a free modal $\mu$-algebras.

We introduce the main property of free modal $\mu$-algebras in the next section. Using this property we shall immediately be able to prove Claim 3.3. We introduce next the notion of a $\mathcal{O}_f$-adjoint of finite type, using which we shall be able to prove constructiveness of several monotone endofunctions on products of free modal $\mu$-algebras, i.e. systems of equations on free modal $\mu$-algebras. We shall prove Claim 3.6 at the end of the paper, after devising a method to transport constructiveness from systems of equations to operations.

4 A property of free modal $\mu$-algebras

In this section we prove that free modal $\mu$-algebras enjoy a property similar to Whitman’s condition for free lattices, cf. [26, 7]. In proof theory this sort of property is often called a last rule and implies a cut-elimination theorem. We do not know yet if this property, stated in the next Theorem, characterizes free modal $\mu$-algebras. However this property is quite powerful and will eventually allow us to prove Claim 3.3 and Claim 3.6.

We briefly recall the universal property of a modal $\mu$-algebra $\mathcal{F}_P$ freely generated by a set $P$. Such a $\mu$-algebra comes with a function $j : P \rightarrow \mathcal{F}_P$ such that for each pair $(f, A)$ – where $A$ is a modal $\mu$-algebra and $f : P \rightarrow A$ – there exists a unique $\mu$-algebra morphism $\tilde{f} : \mathcal{F}_P \rightarrow A$ such that $f = \tilde{f} \circ j$. A generator in $\mathcal{F}_P$ is of the form $j(p)$ for some $p \in P$. It is easily argued that $j$ is injective (see the end of this section) and therefore we shall abuse notation and identify $P$ with its image $j(P)$.

**Theorem 4.1.** Let $\mathcal{F}$ be a free modal $\mu$-algebra and $\Lambda$ be a finite set of literals

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2It is easily argued that this property characterizes free modal algebras $\mathcal{K}$ among the finitely generated ones.
(generators or negated generators). The following implication holds in $\mathcal{F}$: if

$$
\bigwedge \Lambda \wedge \bigwedge_{\sigma \in \text{Act}} \left( \left[ \sigma \right] x_{\sigma} \wedge \bigwedge_{y \in Y_{\sigma}} (\sigma)y \right) \leq \bot,
$$

then either $p, \neg p \in \Lambda$ for some generator $p$, or $x_{\sigma} \wedge y \leq \bot$ for some $\sigma \in \text{Act}$ and $y \in Y_{\sigma}$.

We prove first that:

**Proposition 4.2.** Let $\mathcal{F}$ be a free modal algebra. The implication

$$
\bigwedge \Lambda \wedge \bigwedge_{\sigma \in \Sigma} \left( \bigvee [\sigma] Y_{\sigma} \wedge \bigwedge_{y \in Y_{\sigma}} (\sigma)y \right) \leq \bot
$$

implies

$$
\bigwedge \Lambda \leq \bot \text{ or } \exists \sigma \in \Sigma, y \in Y_{\sigma} \text{ s.t. } y \leq \bot
$$

holds in $\mathcal{F}$, where $\Lambda$ is a finite set of literals, $\Sigma \subseteq \text{Act}$, and, for each $\sigma \in \Sigma$, $Y_{\sigma}$ is a finite possibly empty set of elements of $\mathcal{F}$.

**Proof.** Let $\mathcal{A}$ be any modal algebra and suppose that for each $\sigma \in \Sigma$ we are given a set $Y_{\sigma}$ such that $y \not\leq \bot$ for each $y \in Y_{\sigma}$. For each $\sigma \in \Sigma$ and $y \in Y_{\sigma}$ let $\chi^y : \mathcal{A} \to 2$ be morphism of Boolean algebras such that $\chi^y(y) = \top$ (such a morphism exists by the prime filter theorem). Define

$$
\chi_\sigma(z) = \begin{cases} 
\bigvee_{y \in Y_{\sigma}} \chi^y(z), & \sigma \in \Sigma, \\
\bot, & \sigma \not\in \Sigma.
\end{cases}
$$

For $\sigma \in \Sigma$, observe that $\chi_\sigma(z) = \bot$ if $Y_{\sigma}$ is empty and otherwise that $\chi_\sigma(z) = \top$ if and only if $\chi^y(z) = \top$ for some $y \in Y_{\sigma}$.

We define a modal algebra structure on the product Boolean algebra $\mathcal{A} \times 2$. The modal operators $\langle \sigma \rangle$ are defined by:

$$
\langle \sigma \rangle(z, w) = (\langle \sigma \rangle z, \chi_\sigma(z)).
$$

Since the functions $\chi_\sigma$ preserve joins, these modal operators are normal (i.e. they preserve finite joins). Also, observe that the first projection $\text{pr}_1 : \mathcal{A} \times 2 \to \mathcal{A}$ is a morphism of modal algebras.

Suppose now that $\mathcal{A}$ is freely generated by a set $P$, $\mathcal{A} = \mathcal{F}_{\mathcal{P}}$, and let $\Lambda$ be a set of literals such that $\bigwedge \Lambda \not\leq \bot$. Since $p$ and $\neg p$ cannot belong both to $\Lambda$, we can choose a function $f : P \to \mathcal{A} \times 2$ with these properties: (i) $f(p) \in \{ (p, \bot), (p, \top) \}$ for each $p \in P$, (ii) $f(p) = (p, \top)$ if $p \in \Lambda$ and $f(p) = (p, \bot)$ if $\neg p \in \Lambda$.

Let $\hat{f} : \mathcal{F}_{\mathcal{P}} \to \mathcal{F}_{\mathcal{P}} \times 2$ be the extension of $f$ to a modal-algebra homomorphism, and observe that $\text{pr}_1 \circ \hat{f} = \text{id}_{\mathcal{F}_{\mathcal{P}}}$, since this relation holds on generators.
and that \( \tilde{f}(l) = (l, \top) \) for \( l \in \Lambda \). Suppose that
\[
\bigwedge_{\sigma \in \Sigma} \bigwedge_{\sigma \in \Sigma} \left( [\sigma] \bigvee_{Y' \sigma} (\sigma) y \right) \leq \bot.
\]
If we apply the morphism \( \tilde{f} \) to the above expression we obtain
\[
\left( \bigwedge_{\sigma \in \Sigma} \bigwedge_{\sigma \in \Sigma} \left( [\sigma] \bigvee_{Y' \sigma} (\sigma) y \right), a \wedge \bigwedge_{\sigma \in \Sigma} (b_\sigma \wedge c_\sigma) \right) \leq (\bot, \bot),
\]
where
\[
a = \bigwedge_{l \in \Lambda} \text{pr}_2(\tilde{f}(l)) = \bigwedge_{l \in \Lambda} \top = \top, \quad \text{since } \tilde{f}(l) = (l, \top)
\]
\[
b_\sigma = \neg \chi_\sigma(\neg \bigvee_{Y' \sigma}) = \top,
\]
− this relation is trivial if \( Y' \sigma \) is empty, and otherwise note that \( \chi_\sigma(\neg \bigvee_{Y' \sigma}) = \top \) iff \( \chi_\sigma^y(\neg \bigvee_{Y' \sigma}) = \top \) for some \( y \in Y' \sigma \), which cannot be because of \( \bot = \chi_\sigma^y(\bot) = \chi_\sigma^y(\bot \wedge \neg \bigvee_{Y' \sigma}) = \chi_\sigma^y(\bot) \wedge \chi_\sigma^y(\neg \bigvee_{Y' \sigma}) = \bot \) and finally
\[
c_\sigma = \bigwedge_{y \in Y' \sigma} \chi_\sigma(y) = \bigwedge_{y \in Y' \sigma} \top = \top.
\]
We obtain \( a \wedge \bigwedge_{\sigma \in \Sigma} b_\sigma \wedge c_\sigma = \top \) which contradicts \( a \wedge \bigwedge_{\sigma \in \Sigma} b_\sigma \wedge c_\sigma \leq \bot \).

We extend now the previous result from modal algebras to modal \( \mu \)-algebras.

**Proposition 4.3.** The implication (7) holds in a free modal \( \mu \)-algebra.

**Proof.** The proposition follows since if \( A \) is a modal \( \mu \)-algebra, then the modal algebra \( A \times 2 \) is also a modal \( \mu \)-algebra and the first projection is a morphism of modal \( \mu \)-algebras. This can be seen as follows: suppose that we have defined the interpretation of a term \( f \) in the algebra \( A \times 2 \) as an operation \( f : (A \times 2)^{(x) \cup Y} \longrightarrow A \times 2 \) so that the first projection preserves the interpretation. This is equivalent to saying that, for any fixed \( v \in (A \times 2)^{Y} \), the following diagram commutes:

\[
\begin{array}{ccc}
A \times 2 & \xrightarrow{f_v} & A \times 2 \\
\downarrow \text{pr} & & \downarrow \text{pr} \\
A & \xrightarrow{f_{\text{pr}(v)}} & A
\end{array}
\]

Then \( f_v = (\text{pr} \circ f_v, \psi) = (f_{\text{pr}(v)} \circ \text{pr}, \psi) \) for some \( \psi : A \times 2 \longrightarrow 2 \). Considering that for each fixed \( a \in A \) \( \mu_y \psi(a, y) \) exists -- 2 is a complete lattice -- we can
use the Bekić property to argue that the least fixed point of \( f_v \) exists and is equal to the pair \( (\mu_x.f_{\mathcal{P}(v)}), \mu_y.\psi(\mu_x.f_{\mathcal{P}(v)}, y) \) \). Therefore we interpret the term \( \mu_x.f \) in \( \mathcal{A} \times 2 \) as suggested above, so that the first projection \( \mathcal{P} \) preserves the interpretation of the term \( \mu_x.f \).

Since all the terms of the theory of modal \( \mu \)-algebras are generated either by substitution or by formation of fixed points from the terms of the theory of multi-modal algebras, we deduce that \( \mathcal{A} \times 2 \) is a modal \( \mu \)-algebra.

Lemma 4.4. On any modal algebra \( \mathcal{A} \) condition (7) is equivalent to

\[
\bigwedge \Lambda \land \bigwedge_{\sigma \in \text{Act}} ([\sigma]x_\sigma \land \bigwedge_{y \in Y_\sigma} \langle \sigma \rangle y) \leq \bot
\]

implies

\[
\bigwedge \Lambda \leq \bot \quad \text{or} \quad \exists \sigma \in \text{Act}, y \in Y_\sigma \text{ s.t. } x_\sigma \land y \leq \bot.
\]

Proof. Assume that (8) holds and that the antecedent of (7) holds for some \( \Sigma \subseteq \text{Act} \) and sets \( Y_\sigma \). In (8) let \( x_\sigma = \bigvee Y_\sigma \) if \( \sigma \in \Sigma \) and \( x_\sigma = \top \) and \( Y_\sigma = \emptyset \) if \( \sigma \notin \Sigma \). It immediately follows that either \( \bigwedge \Lambda \leq \bot \), or there exists \( \sigma \in \Sigma \) and some \( y \in Y_\sigma \) such that \( y \leq \bot \).

Conversely, assume that

\[
\bigwedge \Lambda \land \bigwedge_{\sigma \in \text{Act}} ([\sigma]x_\sigma \land \bigwedge_{y \in Y_\sigma} \langle \sigma \rangle y) \leq \bot
\]

and derive

\[
\bigwedge \Lambda \land \bigwedge_{\sigma \in \text{Act}} ([\sigma](\bigvee_{y \in Y_\sigma} y \land x_\sigma) \land \bigwedge_{y \in Y_\sigma} \langle \sigma \rangle (y \land x_\sigma)) =
\]

\[
\bigwedge \Lambda \land \bigwedge_{\sigma \in \text{Act}} ([\sigma](\bigvee_{y \in Y_\sigma} (y \land x_\sigma)) \land \bigwedge_{y \in Y_\sigma} \langle \sigma \rangle (y \land x_\sigma)) \leq \bot,
\]

using the fact that all operations involved are order preserving and distributivity. If we also assume that condition (7) holds, then it follows that \( \bigwedge \Lambda \leq \bot \) or \( x \land y_\sigma \leq \bot \) for some \( \sigma \in \text{Act} \) and \( y \in Y_\sigma \).

The last Proposition has almost lead us to a proof of Theorem 4.1. In order to complete the proof, we need to argue that if \( \bigwedge \Lambda \leq \bot \) in a free modal \( \mu \)-algebra, then \( p, \neg p \in \Lambda \) for some generator \( p \). However the latter property holds in a Boolean algebra \( \mathcal{B}_P \) freely generated by the set \( P \), so that it is enough to argue that the unique Boolean algebra homomorphism \( \kappa : \mathcal{B}_P \rightarrow \mathcal{F}_P \) extending the inclusion of generators \( j : P \rightarrow \mathcal{F}_P \) is an embedding. To this goal, observe that we can assume \( P \) to be finite so that the Boolean algebra \( \mathcal{B}_P \) is finite as well, hence it is complete. \( \mathcal{B}_P \) can also be given a trivial structure.
of a modal algebra (say $\langle \sigma \rangle x = x$) and therefore it is a modal $\mu$-algebra. Let $f : \mathcal{F}_P \rightarrow \mathcal{B}_P$ be the morphism of modal $\mu$-algebras such that $f \circ j(p) = p$ for $p \in P$, then $f \circ \kappa = \text{id}_{\mathcal{B}_P}$, since this relation holds on generators, and $\kappa$ is an embedding.

5 First consequences

In this section we present the first consequences of the property stated in Theorem 4.1. We shall prove Claim 3.3 stating that modal operators $\langle \sigma \rangle$ are left adjoints. Later we shall prove that a Kleene star modality, $\langle \sigma^* \rangle$ in PDL notation, is constructive. This means that this operation is a parametrized least prefixed point which is the supremum over the chain of finite approximants. A proof of this fact is included since it well exemplifies the theory that we shall develop in the next sections.

Modal operators are adjoints

Claim 3.3 can also be understood by saying that reverse or backward modalities are definable in free modal $\mu$-algebras. This property is analogous to Brzozowski derivatives being definable on free Kleene-algebras [12] and part of our contriibution consists in adapting the ideas presented there to the context of the propositional modal $\mu$-calculus.

Proposition 5.1 (i.e. Claim 3.3). On a free modal $\mu$-algebra each modal operator $\langle \sigma \rangle$ is a left adjoint.

Proof. Each element of a free modal $\mu$-algebra is a meet of elements of the form $\bigvee \Lambda \lor \bigvee_{\tau \in \text{Act}} (\langle \tau \rangle x_\tau \lor \bigvee_{y \in Y_\tau} [\tau]y)$ where $\Lambda$ is a set of literals. The previous statement holds since every term of the modal $\mu$-calculus is provably equivalent to a guarded term, see [10], i.e. to a term where negation appears only in front of generators and every bound fixed point variable is in the scope of some modal operator. Using fixed point equalities it is possible to unravel the term to extract its first modal level. The statement then follows by distributivity.

Therefore, we begin by defining the right adjoint for an element having this form: if

$$b = \bigvee \Lambda \lor \bigvee_{\tau \in \text{Act}} (\langle \tau \rangle x_\tau \lor \bigvee_{y \in Y_\tau} [\tau]y),$$

then we define

$$r_\sigma(b) = \begin{cases} \top, & \text{if } b = \top, \\ x_\sigma, & \text{otherwise.} \end{cases}$$
We argue now that $⟨\sigma⟩x ≤ b$ iff $x ≤ r_σ(b)$. Suppose that $⟨\sigma⟩x ≤ b$: if $b = \top$ then clearly $x ≤ \top = r_σ(x)$, and if $b ≠ \top$, then we deduce $x ≤ x_σ = r_σ(b)$. The latter statement is a consequence of Theorem 4.1 when properly dualized, taking into account that all the disjuncts other than $x ∧ ¬x_σ ≤ ⊥$ in the consequent of 4.1 imply $b = \top$. Conversely, the relation $⟨\sigma⟩r_σ(b) ≤ b$ clearly holds and implies that $x ≤ r_σ(b)$ implies $⟨\sigma⟩x ≤ b$. Note also that $r_σ(b)$ does not depend on the representation of $b$, as it is uniquely determined by the property $x ≤ r_σ(b)$ iff $⟨\sigma⟩x ≤ b$.

It is a standard step then to extend the right adjoint to all the elements of a free modal $μ$-algebra: if $x = \bigwedge_{j \in J} b_j$, then we define $r_σ(x) = \bigwedge_{j \in J} r_σ(b_j)$.\[ ∎\]

**The Kleene star is constructive**

An important property of $r_σ(z)$ – the right adjoint to $⟨\sigma⟩$ defined in the proof of Proposition 5.1 – is that it is computed out of the syntax of $z$. More precisely, $r_σ(z)$ is computed as a meet of terms belonging to the Fisher-Ladner closure, see [10], of a term representing $z$. The Fisher-Ladner closure has to be thought as the space of subterms of $z$, in particular it is finite. Consequently, the set $\{ r_σ^n(z) \mid n ≥ 0 \}$ is finite and $\bigwedge_{n ≥ 0} r_σ^n(z)$ exists in a free modal $μ$-algebra. We exemplify how to exploit this fact by proving that $μ_y.(x ∨ ⟨\sigma⟩y)$ is the supremum over the chain of its finite approximants.

We shall use the standard Propositional Dynamic Logic notation and let $⟨\sigma^*⟩x = μ_y.(x ∨ ⟨\sigma^*⟩y)$.

**Lemma 5.2.** The relation

$$⟨\sigma^*⟩a = \bigvee_{n ≥ 0} ⟨\sigma⟩^n a$$

holds in a free modal $μ$-algebra.

**Proof.** We only need to prove that if $⟨\sigma⟩^n a ≤ b$ for each $n ≥ 0$, then $⟨\sigma^*⟩a ≤ b$.

Assume that $⟨\sigma⟩^n a ≤ b$ for each $n ≥ 0$ and transpose these relations to obtain $a ≤ r_σ^n(b)$ for each $n ≥ 0$, hence $a ≤ \bigwedge_{n ≥ 0} r_σ^n(b)$. We claim that $\bigwedge_{n ≥ 0} r_σ^n(b)$ is a $⟨\sigma⟩$-prefixed point. Indeed:

$$⟨\sigma⟩ \bigwedge_{n ≥ 0} r_σ^n(b) ≤ \bigwedge_{n ≥ 0} ⟨\sigma⟩r_σ^n(b) = ⟨\sigma⟩b ∧ \bigwedge_{n ≥ 0} ⟨\sigma⟩r_σ^{n+1}(b) \leq ⟨\sigma⟩b ∧ \bigwedge_{n ≥ 0} r_σ^n(b) ≤ \bigwedge_{n ≥ 0} r_σ^n(b)$$

by the counit relation $⟨\sigma⟩r_σ x ≤ x$.\[ 15\]
Thus $\bigwedge_{n \geq 0} \tau_{\sigma}^n(b)$ is a $\langle \sigma \rangle$-prefixed point above $a$ and therefore $\langle \sigma^* \rangle a \leq \bigwedge_{n \geq 0} \tau_{\sigma}^n(b)$. Since $\bigwedge_{n \geq 0} \tau_{\sigma}^n(b) \leq b$ we deduce $\langle \sigma^* \rangle a \leq b$.

6 $O_f$-adjoints of finite type

The proof that the parametrized least prefixed point corresponding to the PDL star modality $\langle \sigma^* \rangle$ is the supremum over the chain of its finite approximants relies on the modality $\langle \sigma \rangle$ being a left adjoint. We cannot use this idea on the nose to prove constructiveness of other operations that are not left adjoints. For example, a necessity modal operation $[\sigma]$ is not a left adjoint on free modal $\mu$-algebras since it doesn’t preserve joins. To deal with the general case left adjoints are generalized as follows.

**Definition 6.1.** Let $L$ and $M$ be posets. An order preserving function $f : L \rightarrow M$ is a left $O_f$-adjoint if for each $m \in M$ the set

$$\{ x \mid f(x) \leq m \}$$

is a finitely generated lower set.

That is, $f$ is a $O_f$-adjoint iff the above set is a finite union of principal ideals, or equivalently iff for each $m \in M$ there exists a finite set $\mathcal{C}(f; m)$ such that for all $x \in L f(x) \leq m$ if and only if $x \leq c$ for some $c \in \mathcal{C}(f; m)$. We shall say that $\mathcal{C}(f; m)$ is the set of $f$-covers of $m$ or the covering set of $f$ and $m$.

It is easily seen that $f$ is a left adjoint if and only if $\{ x \mid f(x) \leq m \}$ is a principal ideal, thus every left adjoint is a left $O_f$-adjoint. Also, $f$ is a left $O_f$-adjoint if and only if

$$O_f(f) : O_f(L) \rightarrow O_f(M)$$

is a left adjoint; here $O_f(P)$ is the set of finitely generated lower sets of the poset $P$ and $O_f(f)$ is the obvious map induced by this functorial construction. The notion of $O_f$-adjoint presented here corresponds to that of a $\text{Pro}(D)$-adjoint [23] where $D$ is the class of all finite discrete categories. Similar but slightly different is the notion of a multiadjoint [5]. In the following, $O_f$-adjoint will abbreviate left $O_f$-adjoint.

We begin presenting an interesting order theoretic property of $O_f$-adjoints:

**Lemma 6.2.** A $O_f$-adjoint $f$ is continuous: if $I$ is a directed set and $\bigvee I$ exists, then $\bigvee_{i \in I} f(i)$ exists as well and is equal to $f(\bigvee I)$.

**Proof.** Suppose that for all $i \in I f(i) \leq m$. We can find $c_i \in \mathcal{C}(f; m)$ such that $i \leq c_i$. Since $I$ is directed and the $c_i$ are finite, we can find $i_0$ such that $i \leq c_{i_0}$ for all $i \in I$ and consequently $\bigvee I \leq c_{i_0}$. It follows that $f(\bigvee I) \leq f(c_{i_0}) \leq m$. \hfill $\square$
We can argue that being a $O_f$-adjoint is a stronger property than merely being continuous by considering the binary meet $\land : B \times B \rightarrow B$ on an infinite Boolean algebra $B$. The binary meet is continuous – since it is continuous in each variable – but it is not a $O_f$-adjoint. This can be seen by computing a candidate covering set $\mathcal{C}(\land; \bot)$. Since $x \land \neg x \leq \bot$, then we should be able to find $(\alpha_x, \beta_x) \in \mathcal{C}(f; \bot)$ such that $x \leq \alpha_x$, $\neg x \leq \beta_x$, and moreover $\alpha_x \land \beta_x \leq \bot$. It follows that $\alpha_x \leq \neg \beta_x \leq x$ and $\alpha_x = x$. Thus, for an infinite Boolean algebra the covering set $\mathcal{C}(\land; \bot)$ has to be infinite.

We list next some properties of $O_f$-adjoints:

**Proposition 6.3.**

1. An order preserving function $f : L \rightarrow M$ is a left adjoint if and only if it is a $O_f$-adjoint and preserves finite joins.

2. If a lattice $M$ is finitely meet-generated by a subset $B \subseteq M$, then $f : L \rightarrow M$ is a $O_f$-adjoint if and only if the covering set $\mathcal{C}(f; b)$ exists for each $b \in B$.

3. Identities are $O_f$-adjoints, and $O_f$-adjoints are closed under composition.

4. If the domain posets are meet semilattices, then the projections $\text{pr}_i : L_1 \times L_2 \rightarrow L_i$, $i = 1, 2$, are $O_f$-adjoints. Moreover $(f_1, f_2) : L \rightarrow M_1 \times M_2$ is a $O_f$-adjoint provided that $f_i : L \rightarrow M_i$, $i = 1, 2$, are $O_f$-adjoints.

5. Finite joins are $O_f$-adjoints.

6. Constant functions are $O_f$-adjoints. If $L$ is an Heyting algebra (or a Browerian semilattice), then $f(x) = k \land x : L \rightarrow L$ is a $O_f$-adjoint, where $k$ is a constant.

**Proof.**

1: If $r_f$ is right adjoint to $f$, then the lower set $\{ y \mid f(y) \leq m \}$ is generated by $r_f(m)$, thus $f$ is a $O_f$-adjoint. For the second statement, define the right adjoint $r_f(m)$ as $\bigvee \mathcal{C}(f; m)$.

2: Let $m = \bigwedge_{i \in I} b_i$. If $f(x) \leq m$, then $f(x) \leq b_i$ for all $i \in I$ and there exists $c_i \in \mathcal{C}(f; b_i)$ such that $x \leq c_i$; therefore $x \leq \bigwedge_{i \in I} c_i$. Conversely, if $x \leq \bigwedge_{i \in I} c_i$ with $c_i \in \mathcal{C}(f; b_i)$ for each $i \in I$, then $f(x) \leq b_i$, $i \in I$, and $f(x) \leq m$. That is, we can define

$$\mathcal{C}(f; \bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} \mathcal{C}(f; b_i).$$

This set is finite if $I$ is finite.

3: The identity is left adjoint to itself. For composition we can define:

$$\mathcal{C}(f \circ g; m) = \bigcup_{c \in \mathcal{C}(f; m)} \mathcal{C}(g; c).$$
4: Since we are assuming existence of \( \top \), projection functions are left adjoints. For pairing we define:
\[
C((f_1, f_2); (m_1, m_2)) = \{ c_1 \land c_2 \mid c_1 \in C(f_1; m_1), c_2 \in C(f_2; m_2) \}.
\]

5: The diagonal is right adjoint to \( \lor : L \times L \to L \).

6: Let \( f_k \) be the constant function taking every \( x \) to the constant value \( k \). We can define
\[
C(f_k; m) = \begin{cases} 
\emptyset & k \not\leq m \\
\top & \text{otherwise}
\end{cases}
\]
The operation \( k \to y \) is right adjoint to \( k \land x \).

\( \mathcal{O}_f \)-adjoints and fixed points

We analyze next \( \mathcal{O}_f \)-adjoints for which it makes sense to consider least fixed points, i.e. those of the form \( f : L^x \times M^y \to L \). For such an \( f \), we define a directed multi-graph \( G_x(f, L) \) as follows:

- its vertices are elements of \( L \),
- there is a transition \( l \xrightarrow{m'} l' \) iff \( (l', m') \in C(f; l) \).

We write \( G_x(f, l) \) for the full subgraph of \( G_x(f, L) \) of elements of \( L \) that are reachable from \( l \): \( l' \in L \) is a vertex of \( G_x(f, l) \) iff there exists a path from \( l \) to \( l' \) in \( G_x(f, l) \).

**Definition 6.4.** We say that the \( \mathcal{O}_f \)-adjoint \( f : L^x \times M^y \to L \) has finite type for the variable \( x \) if for each \( l \in L \) the graph \( G_x(f, l) \) is finite.

**Lemma 6.5.** Suppose that \( M \) is a meet semilattice, the \( \mathcal{O}_f \)-adjoint \( f : L^x \times M^y \to L \) has finite type, and \( \mu_x.f(x, y) \) exists for each \( y \in M \). Then the order preserving parametrized fixed point \( \mu_x.f : M^y \to L \) is again a \( \mathcal{O}_f \)-adjoint.

**Proof.** Recall that a path of length \( n \) in \( G_x(l, M) \) is a sequence of transitions \( l_i \xrightarrow{m_{i+1}} l_{i+1} \) with \( 0 \leq i < n \). Such a path is infinite if \( n = \omega \). The path is from \( l \) if \( l_0 = l \).

Remark that in an infinite path \( l_i \xrightarrow{m_{i+1}} l_{i+1}, i < \omega \), there exists only a finite number of \( m \)'s such that \( m = m_i \) for some \( i \). Hence the meet \( \bigwedge_{i \geq 1} m_i \) exists in \( M \). We define
\[
m \in C(\mu_x.f; l) \text{ iff } m = \bigwedge_{i \geq 1} m_i
\]
for some infinite path \( \{ l_i \xrightarrow{m_{i+1}} l_{i+1} \}_{i \geq 0} \) from \( l \).
Observe that this set is actually finite, as a consequence of \( G_x(f, l) \) being finite.

We begin verifying that \( \mu_x.f(m) \leq l \) if \( m \in \mathcal{C}(\mu_x.f;l) \). Observe that, by monotonicity, \( f(l_{i+1}, m) \leq f(l_{i+1}, m_{i+1}) \leq l_i \) for all \( i \geq 0 \), and more generally \( f^k_m(l_{i+k}) \leq l_i \) for all \( i, k \geq 0 \). Choose \( i < j \) such that \( l_i = l_j \) and let \( k = j - i \), then \( f^k_m(l_i) = f^k_m(l_j) \leq l_i \), hence \( \mu_x.f(x,m) = \mu_x.f^k_m(x) \leq l_i \). We deduce \( \mu_x.f(x,m) = f^i_m(\mu_x.f(x,m)) \leq f^i_m(l_i) \leq l_0 = l \).

Conversely, assume that \( \mu_x.f(x,y) \leq l_0 \); we can use the fixed point equation to deduce \( f(\mu_x.f(x,y), y) \leq l_0 \) which in turn implies \( (\mu_x.f(x,y), y) \leq (l_1, m_1) \) for some pair \((l_1, m_1) \in \mathcal{C}(f;l_0) \). By iterating the procedure, we can construct an infinite path \( \{ l_i \xrightarrow{m_{i+1}} l_{i+1} \}_{i \geq 0} \) from \( l \) such that for all \( i \geq 1 \) we have \( (\mu_x.f(x,y), y) \leq (l_i, m_i) \). We have therefore \( y \leq \bigwedge_{i \geq 1} m_i \in \mathcal{C}(\mu_x.f;l) \).

It is a natural step to prune covering sets \( \mathcal{C}(f;m) \) to extract the antichain of maximal elements. If this operation is performed on \( \mathcal{C}(\mu_x.f;l) \), we see that a maximal element is a meet indexed by some pan in \( G_x(f,l) \). By a pan, we mean a finite path that can be split into a simple path followed by a simple cycle.

**Lemma 6.6.** Under the conditions of the previous Lemma, the least prefixed point of \( f : L^x \times M^y \rightarrow L \) is constructive:

\[
\mu_x.f(x,y) = \bigvee_{n \geq 0} f^n_y(\bot).
\]

**Proof.** Assume \( l \) is such that \( f^n_y(\bot) \leq l \) for each \( n \geq 0 \). Let \( k \) be the number of vertices in the graph \( G_x(f,l) \) and observe that the relation \( f^n_y(\bot) \leq l \) implies that we can find a path of length \( k \) \( l_i \xrightarrow{m_{i+1}} l_{i+1} \) from \( l \) with the property that \( y \leq m_i \) for \( i = 1, \ldots, k \). By choosing \( i, j \) such that \( 0 \leq i < j \leq k \) and \( l_i = l_j \), construct an infinite path \( l_i \xrightarrow{m_{i+1}} l_{i+1} \) from \( l \) such that \( y \leq m_i \) for \( i \geq 1 \).

Thus \( \bigwedge m_i \in \mathcal{C}(\mu_x.f;l) \) and therefore \( \mu_x.f(x,y) \leq \mu_x.f(x, \bigwedge m_i) \leq l \).

\( \square \)

**\( O_f \)-adjoints on free modal \( \mu \)-algebras**

We continue by considering \( O_f \)-adjoints on free modal \( \mu \)-algebras. We have seen that meets provide a counter-example for \( O_f \)-adjointness. In [9] the authors suggest a sort of best approximation of meets as \( O_f \)-adjoints. They define the arrow term by:

\[
\vartriangleright X = [\sigma] \bigvee_{x \in X} X \wedge \bigwedge_{x \in X} (\sigma)x, \quad (9)
\]

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and, for a set of literals $\Lambda$, for a subset $\Sigma \subseteq \text{Act}$, and for disjoint sets of variables $\{X_\sigma\}_{\sigma \in \Sigma}$, they also define the special conjunction term by:

$$\bigwedge_{\Lambda, \Sigma} \{X_\sigma\} = \Lambda \wedge \bigwedge_{\sigma \in \Sigma} \overset{\sigma}{\sim} X_\sigma.$$

Let $X = \bigcup_{\sigma \in \Sigma} X_\sigma$ and $v \in \mathcal{F}^X$ be a vector of elements of a free modal $\mu$-algebra. We have seen in 4.4 that $\bigwedge_{\Lambda, \Sigma} v = \bot$ if either the literals in $\Lambda$ are inconsistent or $v(x) = \bot$ for some $\sigma \in \Sigma$ and $x \in X_\sigma$.

**Lemma 6.7.** Special conjunctions on free modal $\mu$-algebras are $O_f$-adjoints of finite type.

**Proof.** Recall from 5.1 that the free modal $\mu$-algebra is finitely meet-generated by elements of the form

$$b = \bigvee \Gamma \vee \bigvee_{\tau \in \text{Act}} (\langle \tau \rangle d_\tau \vee \bigvee_{e \in E_\tau} [\tau] e),$$

where $\Gamma$ is a set of literals. By Proposition 6.3.2, it is enough to define the covering sets $\mathcal{C}(\bigwedge_{\Lambda, \Sigma}; b)$ for such $b$’s. Observe that, if $b = \top$, then we can define $\mathcal{C}(f; \top) = \{ \top \}$ for any monotone $f$. Also, if $\bigwedge \Lambda \leq \bigvee \Gamma$, then we can define $\mathcal{C}(\bigwedge_{\Lambda, \Sigma}; b) = \{ \top \}$.

Hence, let $b$ be as in (11) and suppose that $b \neq \top$ and $\bigwedge \Lambda \not\leq \bigvee \Gamma$. Recalling that $X$ is the disjoint union of the $X_\sigma$, $\sigma \in \Sigma$, we define

$$\mathcal{C}(\bigwedge_{\Lambda, \Sigma}; b) = \{ c_{\sigma,y} \mid \sigma \in \Sigma, y \in X_\sigma \} \cup \{ c_{\sigma,e} \mid \sigma \in \Sigma, e \in E_\sigma \},$$

where the vectors $c_{\sigma,y}, c_{\sigma,e} \in \mathcal{F}^X$ are as follows:

$$c_{\sigma,y}(x) = \begin{cases} d_\sigma, & x = y, \\ \top, & \text{otherwise}, \end{cases}$$

and

$$c_{\sigma,e}(x) = \begin{cases} \top, & x \in X_\tau, \tau \neq \sigma \\ d_\sigma \vee e, & x \in X_\sigma. \end{cases}$$

Observe that

$$\bigwedge_{\Lambda, \Sigma} (c_{\sigma,y}) \leq \langle \sigma \rangle d_\sigma \leq b,$$

and

$$\bigwedge_{\Lambda, \Sigma} (c_{\sigma,e}) \leq \overset{\sigma}{\sim} \{ d_\sigma \vee e \} \leq [\sigma](d_\sigma \vee e) \leq \langle \sigma \rangle d_\sigma \vee [\sigma] e \leq b.$$
It follows that if \( v \leq c \in \mathbb{C}(\bigwedge_{\Lambda, \Sigma}; b) \), then \( \bigwedge_{\Lambda, \Sigma} v \leq b \).

Conversely, let \( v \in \mathcal{F}^X \), and suppose that \( \bigwedge_{\Lambda, \Sigma}(v) \leq b \). We apply Theorem 4.1 to this relation, whose explicit expression is

\[
\bigwedge_{\Lambda} \land \bigwedge_{\sigma \in \Sigma} \left( \bigvee_{x \in X_\sigma} v(x) \land \bigwedge_{x \in X_\sigma} (\sigma) v(x) \right) \leq \bigvee_{\Gamma} \lor \bigvee_{\tau \in \text{Act}} (\langle \tau \rangle d_\tau \lor \bigvee_{e \in E_\tau} [\tau] e).
\]

Since \( b \neq \top \) and \( \bigwedge \Lambda \not\leq \bigvee \Gamma \), one of the following two cases holds:

1. there exists \( \sigma \in \Sigma \) and \( x \in X_\sigma \) such that \( v(x) \leq d_\sigma \); in this case \( v \leq c_{\sigma, x} \),
2. there exists \( \sigma \in \Sigma \) and \( e \in E_\sigma \) such that \( v(x) \leq d_\sigma \lor e \) for each \( x \in X_\sigma \),
   in this case \( v \leq c_{\sigma, e} \).

To end the proof, we remark that covers of an element \( c \in \mathcal{F} \) are meets of subterms of a term representing \( c \), showing that special conjunctions have finite type.

It is now easy to argue that the necessity modal operation \( [\sigma] \) is an \( \mathcal{O}_f \)-adjoint on a free modal \( \mu \)-algebra. By Proposition 6.3, this is a consequence of \( [\sigma] \) belonging to the cone generated by joins and special conjunctions, since the relation \( [\sigma] x = \overset{\sigma}{\rightarrow} \{ x \} \lor \overset{\sigma}{\rightarrow} \emptyset \) holds on every modal algebra.

**Uniform families of \( \mathcal{O}_f \)-adjoint of finite type**

In the previous subsection we have studied properties of \( \mathcal{O}_f \)-adjoint and seen that having finite type is quite relevant for least fixed points. We develop next some tools by which it will be easier to compute the type of a \( \mathcal{O}_f \)-adjoint.

A function scheme is a triple \( (f, X, Y) \): intuitively \( f \) is a function symbol, \( X \) and \( Y \) are finite sets of variables, \( X \) being the arity of \( f \) and \( Y \) being its coarity. That is, \( f \) is meant to represent a function of the form \( f : L^X \longrightarrow L^Y \). For a function scheme \( (f, X, Y) \), an \( f \)-automaton is a pair \( \langle Q, \Delta_f \rangle \) where \( Q \) is a set of states and \( \Delta_f \subseteq Q^Y \times Q^X \). For a family \( \mathcal{F} \) of function schemes an \( \mathcal{F} \)-automaton is a tuple \( \langle Q, \{ \Delta_f \}_{f \in \mathcal{F}} \rangle \), where \( \langle Q, \Delta_f \rangle \) is an \( f \)-automaton for each \( f \in \mathcal{F} \). Let \( \langle Q, \{ \Delta_f \}_{f \in \mathcal{F}} \rangle \) be an \( \mathcal{F} \)-automaton and \( Q_0 \subseteq Q \): we let \( \langle Q, \{ \Delta_f \}_{f \in \mathcal{F}} \rangle, Q_0 = \langle P, \{ \Delta'_f \} \rangle \) be the least sub-\( \mathcal{F} \)-automaton of \( \langle Q, \{ \Delta_f \} \rangle \) such that \( Q_0 \subseteq P \) and \( v \in P^X \) and \( v \Delta_f w \) implies \( w \in P^X \) for each function scheme \( (f, X, Y) \in \mathcal{F} \). The relations \( \Delta'_f \) are the restriction of the \( \Delta_f \) to \( P \).

For a family \( \mathcal{F} \) of \( \mathcal{O}_f \)-adjoints of the form \( f : L^X \longrightarrow L^Y \), the \( \mathcal{F} \)-automaton \( \mathcal{A}_\mathcal{F} \) is defined as follows: its set of states is \( L \) and \( v \Delta f c \iff c \in \mathbb{C}(f; v) \).

**Definition 6.8.** A family \( \mathcal{F} \) of \( \mathcal{O}_f \)-adjoints is a **uniform family of finite type** if the underlying set of the \( \mathcal{F} \)-automaton \( \mathcal{A}_\mathcal{F}, Q_0 \) is finite whenever \( Q_0 \subseteq L \) is finite.
The obvious reason to introduce this notion is:

**Lemma 6.9.** Let $\mathcal{F}$ be a uniform family of finite type. If $f \in \mathcal{F}$ and $f : L^X \times L^Y \rightarrow L^X$, then $f$ has finite type for the variable $X$.

**Proof.** Let $v_0 \in L^X$ and $S$ be the set of states of $G_X(f, v_0)$. If $Q_0 = \{ l \mid v_0(x) = l \text{ for some } x \in X \}$, then we claim that $S \subseteq P^X$, where $P$ is the underlying set of $\overline{A}, Q_0$. Indeed, $v_0 \in P^X$ and if $v \in P^X$ and $v \xrightarrow{m'} v'$, then $v \Delta_f (v', m')$ so that $(v', m') \in P^X \times P^Y$ and $v' \in P^X$.

We investigate now closure properties of a family $\mathcal{F}$ w.r.t. finiteness.

**Lemma 6.10.** If $\mathcal{F}$ is a uniform family of finite type and $x \in X$, then $\mathcal{F} \cup \{ \text{pr}^X_x \}$ is a uniform family of finite type.

**Proof.** Recall that $\mathcal{C}(\text{pr}^X_x; b) = \{ c \}$ where $c(x) = b$ and $c(y) = \top$ for $y \neq x$. It is easily argued that the underlying set of $\overline{A}_{\mathcal{F} \cup \{ \text{pr}^X_x \}}, Q_0$ is contained in the underlying set of $\overline{A}_{\mathcal{F}, Q_0 \cup \{ \top \}}$.

**Lemma 6.11.** If $\mathcal{F}$ is a uniform family of finite type and $f, g \in \mathcal{F}$ with $f : L^X \rightarrow L^Y$ and $g : L^Y \rightarrow L^Z$, then $\mathcal{F} \cup \{ f \circ g \}$ is a uniform family of finite type.

**Proof.** Recall that $\mathcal{C}(g \circ f; b) = \bigcup_{c \in \mathcal{C}(g; b)} \mathcal{C}(g \circ f; c)$, from which it results that the underlying set of $\overline{A}_{\mathcal{F} \cup \{ f \circ g \}}, Q_0$ is the same as the underlying set of $\overline{A}_{\mathcal{F}, Q_0}$.

By the previous Lemmas, we can always assume that a uniform family of finite type $\mathcal{F}$ is closed under post-composition with projections, that is, if $\langle f_y \rangle_{y \in Y} : L^X \rightarrow L^Y$, then $f_y \in \mathcal{F}$ for each $y \in Y$.

**Lemma 6.12.** Let $\mathcal{F}$ be a uniform family of finite type which is closed under post-composition with projections, and let $f : L^X \rightarrow L$, $y \in Y$, be elements of $\mathcal{F}$. Then $\mathcal{F} \cup \{ \langle f_y \rangle_{y \in Y} \}$ is a uniform family of finite type.

**Proof.** Let $Q_0 \subseteq L$ be a finite subset of $L$, and let $\overline{A}_{\mathcal{F}, Q_0} = \langle Q, \{ \Delta_f \} \rangle$, so that, by assumption, $Q$ is finite.

Let $S$ be the meet-semilattice generated by $Q$ and let $P$ be the underlying set of $\overline{A}_{\mathcal{F} \cup \{ (f_y) \}}, Q_0$; we claim that $P \subseteq S$. Clearly, $Q_0 \subseteq Q \subseteq S$.

We show now that for each $f : L^X \rightarrow L^Y$ in $\mathcal{F} \cup \{ \langle f \rangle \}$, $w \in S^Y$ and $w \Delta_f c$ implies $c \in S^Y$.
We analyze first the case of a function of the form \( f : L^X \to L \). Let \( w \in S \) and suppose that \( w \Delta f c \), i.e. \( c \in \mathcal{C}(f; w) \). Since \( w \in S \), we can write \( w = \bigwedge w_i \) where \( w_i \in Q \). Hence \( c = \bigwedge c_i \) where \( c_i \in \mathcal{C}(f; w_i) \): we have, therefore, \( w_i \Delta f c_i \) and \( c_i \in Q^X \). Hence, \( c = \bigwedge c_i \) belongs to \( S^X \).

We analyze now the case of a function of the form \( f : L^X \to L^Y \), with \( Y \) not a singleton. Thus \( f = \langle f_y \rangle \) where each \( f_y : L^X \to L \) belongs to \( F \). Let \( w \in S^Y \), and suppose that \( w \Delta \langle f_y \rangle c \). This means that \( c \in \mathcal{C}(\langle f_y \rangle; w) \) so that \( c = \bigwedge c_y \) where \( c_y \in \mathcal{C}(f_y; w(y)) \) for each \( y \in Y \). We have already argued that \( c_y \in S^X \), hence \( c = \bigwedge c_y \in S^X \) as well.

7 Some constructive systems of equations

An order preserving \( F : A^X \times A^Y \to A^X \) can be thought to be a system of equations whose least solution is given by the least fixed point. The set \( X = \{ x_1, \ldots, x_n \} \) is the set of bound variables of the system and \( Y = \{ y_1, \ldots, y_m \} \) is the set of free variables, the sets \( X \) and \( Y \) being disjoint. If \( F = \langle F_x \rangle_{x \in X} \), then we represent such systems as expected:

\[
\begin{align*}
\vdots \\
x_i &= F_x(x_1, \ldots, x_n, y_1, \ldots, y_m) \\
\vdots
\end{align*}
\]

The Bekić property ensures that such a system of equations has a least solution in every modal \( \mu \)-algebra if each \( F_x \) is the interpretation of a term of the theory of modal \( \mu \)-algebras.

In this section we shall prove that, for many such \( F \) on a free modal \( \mu \)-algebra \( F \), the least prefixed point is the supremum over the chain of its finite approximants. The results of the previous sections allow us to easily derive this property for a restricted set of systems called here disjunctive-simple. Then, we freely use ideas and tools from [1, \S 9] to enlarge the class of systems that can be proved to be constructive. An improvement w.r.t. this monograph consists in adapting these tools in order to argue about existence of infinite suprema and approximants. Our last effort will be to prove that all the systems \( F = \langle F_x \rangle \) whose \( F_x \) are elementary operations of the theory of modal algebras enjoy this property.

**Definition 7.1.** We say that a term of the theory of modal \( \mu \)-algebras

- is *elementary* if it is among \( x, \top, x_1 \land x_2, \bot, x_1 \lor x_2, \sigma \to X_\sigma \).

With respect to two sets of variables \( X \) and \( Y \), we say that a term of the theory of modal \( \mu \)-algebras
• is simple if it is a distributive combination of terms of the form $\bigwedge Y' \wedge \bigwedge_{\emptyset, \Sigma} \{ D_\sigma \}$, where $Y' \subseteq Y$ and each $d \in D_\sigma$ is a distributive term on the variables in $X$,

• is disjunctive-simple if it is a join of terms of the form $\bigwedge Y' \wedge \bigwedge_{\emptyset, \Sigma} \{ D_\sigma \}$, where $Y' \subseteq Y$ and each $d \in D_\sigma$ is a join of a set of variables in $X$: $d = \bigvee X'$ with $X' \subseteq X$.

For a $\mu$-algebra $A$ we say that a map $F = \langle F_x \rangle_{x \in X} : A^X \times A^Y \longrightarrow A^X$ is elementary (resp. simple, resp. disjunctive-simple w.r.t. $X$ and $Y$) if each component $F_x : A^X \times A^Y \longrightarrow A$ is the interpretation of an elementary (resp. simple, resp. disjunctive-simple w.r.t. $X$ and $Y$) term.

Disjunctive-simple systems

**Proposition 7.2.** Let $F$ be a free modal $\mu$-algebra, $G : F^X \times F^Y \longrightarrow F^X$ be a disjunctive-simple map, and let $k \in F^Y$. Then $G_k : F^X \longrightarrow F^X$ is a $O_f$-adjoint of finite type.

**Proof.** Proposition 6.3 and Lemma 6.7 imply that for each $x \in X$ the $x$ component of $G_k$ – which we shall denote $G_x$ abusing notation – is a $O_f$-adjoint. Item 4 in Proposition 6.3 then imply that $G_k$ is a $O_f$-adjoint. Thus we are mainly concerned with arguing that $G_k$ has finite type, and in view of Lemma 6.9 and Lemma 6.12, it will be enough to show that the $G_x$ form a uniform family of finite type.

Each $G_x$ has the form $\bigvee_{i \in I_x} k_{x,i} \wedge \bigwedge_{\emptyset, \Sigma_{x,i}} \{ D_\sigma \}$, where, for each $i \in I_x$, $k_{x,i}$ is a constant element of the free modal $\mu$-algebra $F$, and for each $\sigma \in \Sigma_{x,i}$ and $d \in D_\sigma$ $d = \bigvee X'$. We use Lemma 6.11 and prove that the family

$$F = \bigcup \{ \bigvee Z \mid Z \text{ is finite} \} \cup \bigcup_{x \in X} \{ k_{x,i} \wedge z \mid i \in I_x \} \cup \bigwedge_{\emptyset, \Sigma_{x,i}} \{ X_\sigma \}$$

is uniform of finite type. Here $\bigvee Z : F^Z \longrightarrow F$ is the join operation of arity $Z$.

For each constant $k_{x,i}$, choose a term $t_{x,i}$ representing the element $\neg k_{x,i}$. Let now $Q_0$ be a finite subset of $F^X$ and, for each $q \in Q_0$, let $s_q$ be a term representing $q$.

Let $FL$ be the Fisher-Ladner closure of the terms $t_{x,i}$ and $s_q$, it is well known [10] that $FL$ is a finite set which, by its definition, comprises all the subterms of $t_{x,i}$ and $s_q$.

Let $\overline{FL} \subseteq F$ be the set of interpretations of terms in $FL$ in the $\mu$-algebra $F$, and let $D$ be the distributive lattice generated by $\overline{FL}$. It will be useful to think of $D$ as the meet closure of the join closure of $\overline{FL}$. We need to prove that $f \in F$, $d \in D$, and $c \in \mathcal{C}(f; d)$ with $c \in F^Z$, imply $c(z) \in D$ for each $z \in Z$. 24
Observe that if \( c \in \mathcal{C}(\vee; d) \), then \( c \) is the vector with \( d \) at each projection.

If \( f(z) = k_{x,i} \land z \), then \( \mathcal{C}(f; d) = \{ \neg k_{x,i} \lor d \} \subseteq D \).

Let \( f = \bigwedge_{\emptyset, \Sigma} \) and consider \( d \in D \): since a cover in \( \mathcal{C}(f; d) \) is a meet of covers in \( \mathcal{C}(f; d_i) \) where each \( d_i \) belongs to the join closure of \( FL \), we can assume that \( d \) is in the join closure of \( FL \), that is, \( d \) is the interpretation of a term of the form \( t_1 \lor \ldots \lor t_n \) with \( t_i \in FL \). Lemma 6.7 shows that a \( c \in \mathcal{C}(\bigwedge \emptyset, \Sigma; d) \) is a meet of elements \( c_j \), where each projection of a vector \( c_j \) is either \( \top \) or a join of subterms of \( t_1 \ldots t_n \), hence it belongs to \( D \).

Lemma 6.6 and the previous Lemma imply:

**Corollary 7.3.** The least prefixed point of a disjunctive-simple system \( G : \mathcal{F}^X \times \mathcal{F}^Y \rightarrow \mathcal{F}^X \) is constructive:

\[
\mu_x.G = \bigvee_{n \geq 0} G^n_v(\bot).
\]

for each \( v \in \mathcal{F}^Y \).

**From disjunctive-simple to simple systems**

Our next goal is to transfer constructiveness from a disjunctive-simple \( G \) to a simple \( F \). The main tool is the following Lemma:

**Lemma 7.4.** Consider a commuting diagram of posets with bottom

\[
\begin{array}{ccc}
L & \xrightarrow{f} & L \\
\downarrow i & & \downarrow i \\
M & \xleftarrow{g} & M
\end{array}
\]

where \( i \) is split by an order preserving \( \pi, \pi \circ i = \text{id}_L \). Let \( \alpha \) be a limit ordinal and suppose that (i) for \( \beta < \alpha, f^\beta(\bot) \) and \( g^\beta(\bot) \) exist and \( i(f^\beta(\bot)) = g^\beta(\bot) \), (ii) the approximant \( g^\alpha(\bot) \) exists. Then the approximant \( f^\alpha(\bot) \) exists as well and is equal to \( \pi(g^\alpha(\bot)) \). If moreover \( i \) is continuous, then \( i(f^\alpha(\bot)) = g^\alpha(\bot) \).

**Proof.** Let \( \alpha \) be an ordinal satisfying the hypothesis, we are going to argue that \( \pi(g^\alpha(\bot)) = \bigvee_{\beta < \alpha} f^\beta(\bot) \). Let us begin supposing that, for some \( l \in L \) and every \( \beta < \alpha, f^\beta(\bot) \leq l \). Apply \( i \) to these relations and deduce that \( g^\beta(\bot) \leq i(l) \) for \( \beta < \alpha \), hence \( g^\alpha(\bot) \leq i(l) \); apply \( \pi \) and deduce \( \pi(g^\alpha(\bot)) \leq l \). Conversely, apply \( \pi \) to \( i(f^\beta(\bot)) = g^\beta(\bot) \leq g^\alpha(\bot) \) to deduce \( f^\beta(\bot) \leq \pi(g^\alpha(\bot)) \) for \( \beta < \alpha \).
If moreover $i$ is continuous, then:

$$i(f^a(\bot)) = i(\bigvee_{\beta < \alpha} f^\beta(\bot)) = \bigvee_{\beta < \alpha} i(f^\beta(\bot)) = \bigvee_{\beta < \alpha} g^\beta(\bot) = g^a(\bot).$$

We shall make use of the Lemma as follows. For a finite set of variables $X$, let $\mathcal{P}_+(X)$ be the set of nonempty subsets of $X$. For each $S \in \mathcal{P}_+(X)$, the map $i_S : L^X \to L$, defined by

$$i_S(x) = \bigwedge_{j \in S} x_j,$$

is continuous. These maps, collected together, define a continuous map

$$i = \langle i_S \rangle_{S \in \mathcal{P}_+(X)} : L^X \to L^{\mathcal{P}_+(X)}$$

which moreover preserves the bottom element. For each $x \in X$ there is a projection onto the singleton set $\text{pr}_{\{x\}} : L^{\mathcal{P}_+(X)} \to L$. These projections, collected into a common projection $\text{pr} = \langle \text{pr}_{\{x\}} \rangle_{x \in X} : L^{\mathcal{P}_+(X)} \to L^X$, split $i$: $\text{pr} \circ i = \text{id}_{F^X}$. Thus we shall prove:

**Proposition 7.5.** For each simple $F : \mathcal{F}^X \times \mathcal{F}^Y \to \mathcal{F}^X$ there is a disjunctive-simple $G : \mathcal{F}^{\mathcal{P}_+(X)} \times \mathcal{F}^Y \to \mathcal{F}^{\mathcal{P}_+(X)}$ such that the diagram

$$\begin{array}{ccc}
\mathcal{F}^X \times \mathcal{F}^Y & \xrightarrow{F} & \mathcal{F}^X \\
\downarrow{i \times \text{id}_Y} & & \downarrow{i} \\
\mathcal{F}^{\mathcal{P}_+(X)} \times \mathcal{F}^Y & \xrightarrow{G} & \mathcal{F}^{\mathcal{P}_+(X)}
\end{array}$$

(12)

commutes.

Together with Corollary 7.3 and Lemma 7.4, the Proposition implies:

**Corollary 7.6.** Let $F : \mathcal{F}^X \times \mathcal{F}^Y \to \mathcal{F}^X$ be simple and $v \in \mathcal{F}^Y$. Then

$$\mu_X.F_v = \bigvee_{n \geq 0} F^n_v(\bot).$$

Diagram (12) commutes if for each nonempty subset $S \subseteq X$ we can find a disjunctive-simple $G_S : \mathcal{F}^{\mathcal{P}_+(X)} \to \mathcal{F}$ such that

$$\bigwedge_{j \in S} F_j(x) = G_S(\bigwedge_{j \in S_1} x_j, \ldots, \bigwedge_{j \in S_{2^n-1}} x_j),$$

(13)
where $S_1, \ldots, S_{2^n-1}$ is the list of nonempty subsets of $X$. We shall sketch the proof of Proposition 7.5, skipping on the details since its structure strictly follows [1, §9.4]. To be coherent with this monograph, we use $ar(t_1, \ldots, t_n)$ for the set of variables in $X$ appearing in the terms $t_i$, while $t[t_x/x]$ denotes a standard substitution applied to the term $t$.

**Lemma 7.7.** For every pair of disjunctive-simple terms $t_1, t_2$ there exists a disjunctive-simple term $t_3$ and $\phi : ar(t_3) \rightarrow P_+(ar(t_1, t_2))$ such that the equation

$$t_1 \land t_2 = t_3[\bigwedge \phi(x)/x]$$

holds in every modal algebra.

**Proof.** Let $t_1 = \bigvee_i \gamma_{1,i}$ and $t_2 = \bigvee_j \gamma_{j,j}$, where the $\gamma_{k,i}$ have the form $\bigwedge_{\Lambda \subseteq Y \in D_\sigma}$ with $\Lambda \subseteq Y$ and every $d \in D_\sigma$ a disjunction of variables from $X$. Clearly $t_1 \land t_2 = \bigvee_{i,j} \gamma_{1,i} \land \gamma_{2,j}$, thus it is enough to observe that

$$\bigwedge_{\Lambda_1, \Sigma_1} \{D_1, \Sigma_1\} \land \bigwedge_{\Lambda_2, \Sigma_2} \{D_2, \Sigma_2\} = \{\bigwedge_{\Lambda_3, \Sigma_3} \{D_3, \Sigma_3\}\} [\bigwedge \phi(x)/x]$$

for some some $\Lambda_3, \Sigma_3, D_3, \sigma$ and a $\phi$. To this goal, we observe that

$$(\sigma \downarrow) D_1, \sigma \land (\sigma \downarrow) D_2, \sigma = \begin{cases} \sigma \downarrow \emptyset, & \text{if } D_1, \sigma = D_2, \sigma = \emptyset, \\ \bot, & \text{if } D_i, \sigma = \emptyset \text{ for only one } i, \\ \sigma \downarrow \{d_1 \land \bigvee D_2, \sigma, \bigvee D_1, \sigma \land d_2 \mid d_1 \in D_1, \sigma, d_2 \in D_2, \sigma\}, & \text{otherwise.} \end{cases}$$

We give the explicit definition of $\Lambda_3, \Sigma_3, D_3, \sigma$ and $\phi$, under the simplifying assumption that only the last case occurs. We let $\Lambda_3 = \Lambda_1 \cup \Lambda_2$, $\Sigma_3 = \Sigma_1 \cup \Sigma_2$, $D_3, \sigma = D_1, \sigma$ for $\sigma \in \Sigma_1 \setminus \Sigma_2$, $D_3, \sigma = D_2, \sigma$ for $\sigma \in \Sigma_2 \setminus \Sigma_3$. For $\sigma \in \Sigma_1 \cap \Sigma_2$, we let

$$D_3, \sigma = \{\bigvee_{x \in X} w_{x,z} \mid X \in D_1, \sigma \}, \bigvee_{z \in Z} w_{z,y} \mid Y \in D_2, \sigma\}$$

where $w_{x,z}$ and $w_{z,y}$ are new variables and let $\phi(w_{x,z}) = \{x, z\}$, $\phi(w_{z,y}) = \{z, y\}$.

The proof of Proposition 7.5 is then achieved through the following steps:

1. It is shown that for every sequence of disjunctive-simple terms $t_1, \ldots, t_n$ there exists a disjunctive simple term $t_0$ and $\phi: ar(t_0) \rightarrow P_+(ar(t_1, \ldots, t_n))$
such that

\[ \bigwedge_{i=1,\ldots,n} t_i = t_0[\bigwedge \phi(x)/x] \]

is an equation of the theory of modal algebras.

2. It is shown that for each simple term \( s \) there exists a disjunctive-simple term \( d \) and a function \( \phi : ar(d) \longrightarrow \mathcal{P}_t(ar(s)) \) such that

\[ s = d[\bigwedge \phi(x)/x] \]

is an equation of the theory of modal algebras.

Collecting together these properties, we obtain a rephrasing of Proposition 7.5:

3. For every sequence of simple terms \( s_1, \ldots, s_n \) there exists a disjunctive simple term \( d \) and \( \phi : ar(d) \longrightarrow \mathcal{P}_t(ar(s_1, \ldots, s_n)) \) such that

\[ \bigwedge_{i=1,\ldots,n} s_i = d[\bigwedge \phi(x)/x] \]

is an equation of the theory of modal algebras.

**From simple to elementary systems**

Finally, we transfer constructiveness to elementary systems. To this goal, we say that two systems \( F : \mathcal{A}^X \times \mathcal{A}^Y \longrightarrow \mathcal{A}^X \) and \( G : \mathcal{A}^X \times \mathcal{A}^Y \longrightarrow \mathcal{A}^X \) are equivalent if for each \( v \in \mathcal{A}^Y \), the two chains of finite approximants \( \{ F^n_v(\bot) \}_{n \geq 0} \) and \( \{ G^n_v(\bot) \}_{n \geq 0} \) are cofinal into each other. This means that for each \( n \geq 0 \) there exists \( k \geq 0 \) such that \( F^n_v(\bot) \leq G^k_v(\bot) \), and vice-versa.

**Fact 7.8.** If \( F \) and \( G \) are equivalent systems then \( \bigvee_{n \geq 0} F^n_v(\bot) \) exists if and only if \( \bigvee_{n \geq 0} G^n_v(\bot) \) exists, and in both cases they are equal.

We introduce now the notion of a guarded system. An occurrence of a variable \( x \) in a term \( t \) is guarded if it is in the scope of a modal operator. A term is guarded (w.r.t. \( X \) and \( Y \)) if each occurrence of a variable \( x \in X \) in \( t \) is guarded. A system \( F : \mathcal{A}^X \times \mathcal{A}^Y \longrightarrow \mathcal{A}^X \) is guarded if each \( F_x \) is the interpretation of a guarded term (w.r.t. \( X \) and \( Y \)). The following Lemma is analogous to the well known fact that every formula of the modal \( \mu \)-calculus is equivalent to a guarded one [10]. The reader may wish to consult [1, §9.2.4] as well.

**Lemma 7.9.** For each elementary system \( F \) there exists a guarded system \( G \) which is equivalent to \( F \).
Proof. The following is a procedure – analogous to $\epsilon$-transitions elimination in automata theory – which eventually produces a system $G$, equivalent to a given system $F$, in which all the bound variables appear guarded.

For a system $F$, let us define the graph of $\epsilon$-transitions: its nodes are the bound variables of $F$, and we say $x_i \rightarrow x_j$ if $x_j$ is not guarded in $F_{x_i}$. Let us recall the notion of distance between two nodes of this graph: if the two nodes $x_i, x_j$ are connected by an $\epsilon$-path, then the distance between them is then minimum length of a path connecting them, and otherwise it is $\infty$.

The procedure alternates among two kind of steps: elimination of loops, and reduction of cycles.

We can eliminate $\epsilon$-loops from $F$: disjunctive normal forms ensure that if $x_i$ is not guarded by a modal operator in $F_{x_i}$, then

$$F_{x_i}(\ldots, x_i, \ldots) = (x_i \land f_i(\ldots, x_i, \ldots)) \lor g_i(\ldots, x_i, \ldots)$$

for some terms $f_i$ and $g_i$ in which $x_i$ are guarded. We can modify $F$ to eliminate all the loops by the following rewrite of systems:

$$\begin{cases}
\vdots \\
x_i = F_{x_i}(\ldots, x_i, \ldots) \\
\vdots
\end{cases} 
\rightsquigarrow 
\begin{cases}
\vdots \\
x_i = g_i(\ldots, x_i, \ldots) \\
\vdots
\end{cases}$$

Let $G$ be the system on the right, then it is easily checked that $F$ and $G$ have the same chains of approximants, hence they are equivalent.

If $F$ does not contain loops, we can operate the following rewrite in which, for every pair of bound variables $x_i$ and $x_j$, if $x_j$ occurs unguarded in $F_i$ then it is substituted with $F_j$:

$$\begin{cases}
\vdots \\
x_i = F_i(\ldots, x_j, \ldots) \\
\vdots
\end{cases} 
\rightsquigarrow 
\begin{cases}
\vdots \\
x_i = F_i(\ldots, F_j(\ldots), \ldots) \\
\vdots
\end{cases}$$

Observe that these rewrite reduce the distance between distinct connected nodes in the graph of $\epsilon$-transitions, thus the combined procedure terminates. Let $G$ be the system on the right, then it is easily checked that $F^n_v(\bot) \leq G^n_v(\bot) \leq F^{2n}_v(\bot)$ for each $v \in A^Y$, hence the two systems are equivalent.

The system $G$ obtained from the elementary $F$ by means of the procedure described above need not to be a simple system. On the other hand, all the $G_x$ are terms of the theory of modal algebras where all the variables in $X$ have modal depth at least 1, so that the system $G$ is quite similar to a simple one. This means that by adding new variables and cutting along substitutions we can “unravel” such a system $G$ to a simple system $H$. We only need to
justify these operations on systems. To this goal, we modify the previously
proposed equivalence of systems. Let \( X \subseteq Z \), \( F : A^X \times A^Y \rightarrow A^X \), and
\( G : A^Z \times A^Y \rightarrow A^Z \). We say that \( G \) determines \( F \) iff the chains \( \{ F^n(\bot) \}_{n \geq 0} \)
and \( \{ \text{pr}_X(G^n(\bot)) \}_{n \geq 0} \) are cofinal into each other.

**Fact 7.10.** Suppose that \( G \) determines \( F \), let \( A \) be a modal \( \mu \)-algebra and
\( v \in A^Y \). If \( \bigvee_{n \geq 0} G^n(\bot) \) exists in \( A^Z \), then \( \bigvee_{n \geq 0} F^n(\bot) \) exists in \( A^X \) as well
and is equal to \( \text{pr}_X(\bigvee_{n \geq 0} G^n(\bot)) \).

**Lemma 7.11.** For each elementary system \( F \) there exists a simple system \( H \)
which determines \( F \).

**Proof.** We only sketch the proof. We apply the following kind of rewrite rules
to the system \( G \) obtained from \( F \) by Lemma 7.9:

\[
\begin{align*}
\{ x_1 = g(f(x_1, x_2, y), x_1, x_2, y) \\
x_2 = h(x_1, x_2, y)
\} & \rightsquigarrow \\
\{ x_0 = f(x_1, x_2, y) \\
x_1 = g(f(x_1, x_2, y), x_1, x_2, y) \\
x_2 = h(x_1, x_2, y)
\}
\end{align*}
\]

Let us call \( G_0 \), \( G_1 \), and \( G_2 \) the three systems in the order. Clearly \( G_1 \) determines
\( G_0 \), while we have argued in the proof of Lemma 7.9 that \( G_2 \) is equivalent to
\( G_1 \). Hence \( G_2 \) determines \( G_0 \).

Iteration of this rewriting produces a simple system \( H \) determining the original
system \( F \).

**Proposition 7.12.** Each elementary system is constructive on a free modal
\( \mu \)-algebra.

**Proof.** In the previous subsection we have seen that simple systems are constructive
on free modal \( \mu \)-algebras. In this subsection we have argued that given an
elementary system there is a simple systems determining it, hence, on a free
modal \( \mu \)-algebra, every elementary system is constructive by Fact 7.10.

\[\square\]

## 8 \( \Sigma_1 \)-operations are constructive

The valid equations

\[
(\sigma)x \xrightarrow{\sigma} \{ x, \top \} \\
[\sigma]x \xrightarrow{\sigma} \{ x \} \lor \xrightarrow{\sigma} \emptyset
\]

show that all the operations of the theory of modal \( \mu \)-algebras are definable from
the Boolean algebra terms and the arrow terms (9). Accordingly we modify the
definition (5) of \( \Sigma_1 \)-terms as follows:

\[
t = x \mid \top \mid t \land \bot \mid t \lor t \mid \xrightarrow{\sigma} T \mid \mu x.t,
\]
where \( x \) is a variable and \( T \) is a set of previously defined terms. We remark that such a modification leaves invariant the class of \( \Sigma_1 \)-operations (i.e. interpretations of \( \Sigma_1 \)-terms).

The following concept is needed in the following:

**Definition 8.1.** We say that an order preserving map \( f : L^x \times M^y \to L \) is regular if it is continuous in each variable and constructive for the variable \( x \).

More generally, we shall say that an order preserving map \( f : L^X \to L \) is regular if it is continuous and constructive in each variable. Recall that being constructive means that the approximant \( f^\alpha_v(\bot) \) exists for each \( v \in M \) and each ordinal \( \alpha \). It is easily seen that for a continuous \( f \) existence of the approximant \( f^\omega_v(\bot) = \bigvee_{n \geq 0} f^n_v(\bot) \) suffices for existence of all approximants.

Hence, when arguing that a continuous order preserving function is regular, we shall only be concerned with existence of \( f^\omega_v(\bot) \). As an example, we have seen in the previous section that all the elementary \( G : \mathcal{F}^X \times \mathcal{F}^Y \to \mathcal{F}^X \) are constructive on a free modal \( \mu \)-algebra \( \mathcal{F} \). Since each \( G_x : \mathcal{F}^x \times \mathcal{F}^Y \to \mathcal{F} \) is also continuous, \( G \) is continuous as well. Hence such elementary \( G \) is regular on a free modal \( \mu \)-algebra. We want to transfer regularity, hence constructiveness, to \( \Sigma_1 \)-operations, for which we need to consider them as solutions of elementary systems:

**Lemma 8.2.** For each \( \Sigma_1 \)-operation \( f : A^Y \to A \) there exists an elementary system \( F : \mathcal{A}^X \times \mathcal{A}^Y \to \mathcal{A}^X \) and \( x \in X \) such that \( f = \text{pr}_x \circ \mu_X.F \).

**Proof.** The elementary systems are constructed by induction on the structure of \( \Sigma_1 \)-terms. For example, suppose \( t = t_1 \land t_2 \) and that \( G_i : \mathcal{A}^{X_i} \times \mathcal{A}^{Y_i} \to \mathcal{A}^{X_i} \) and \( x_i \) have the property stated in the Lemma w.r.t. \( t_i \), \( i = 1, 2 \). We let \( X = \{ x \} \cup X_1 \cup X_2 \), \( Y = Y_1 \cup Y_2 \), and \( F \) is the system:

\[
\begin{cases}
x = x_1 \land x_2 \\
x_1 = G_{1,x_1}(\ldots) \\
\vdots \\
x_2 = G_{2,x_2}(\ldots) \\
\vdots
\end{cases}
\]

Similar constructions work for \( t = t_1 \lor t_2 \) and \( t = \mu_y.t_1 \). Suppose therefore that \( t = \mu_y.t_1 \). Assume that \( G_1 : \mathcal{A}^{X_1} \times \mathcal{A}^{Y_1} \to \mathcal{A}^{X_1} \) and \( x_1 \) have the property stated in the Lemma w.r.t. \( t_1 \), then we let \( X = X_1 \cup \{ y \} \), \( Y = Y_1 \setminus \{ y \} \), and \( F \) is the system:

\[
\begin{cases}
y = x_1 \\
x_1 = G_{x_1}(\ldots) \\
\vdots
\end{cases}
\]

\( \square \)
To achieve the proof of Claim 3.6 we need one more result, which is a Bekič-like property for regular functions:

**Proposition 8.3.** Suppose that $F : L^x \times M^y \times N^z \rightarrow L$ is regular in $x$ and $G : L^x \times M^y \times N^z \rightarrow M$ is continuous. Then $(F,G) : L^x \times M^y \times N^z \rightarrow L \times M$ is regular in $(x,y)$ if and only if $G \circ (\mu_x.F, \text{id}_{M \times N}) : M^y \times N^z \rightarrow M$ is regular in $y$.

Using the Proposition above we can immediately state our goal:

**Theorem 8.4.** (cf. Claim 3.6.) Every $\Sigma_1$-operation $f : F^Z \rightarrow F$ is regular on a free modal $\mu$-algebra $F$, hence constructive.

**Proof.** By Lemma 8.2 $f = \text{pr}_x \circ \mu_x.F$ for some elementary $F : F^X \times F^Z \rightarrow F^X$ and some $x \in X$. Choose $y \in Z$ and observe that the system

$$(F, \text{pr}_x \cup Z_x) : F^X \times F^y \times F^{Z \setminus \{y\}} \rightarrow F^X \times F^y$$

is elementary hence regular. Since $\text{pr}_x \cup Z_x$ is continuous, we can use Proposition 8.3 with $G = \text{pr}_x \cup Z_x$ and deduce that $f = \text{pr}_x \circ \mu_x.F = \text{pr}_x \cup Z_x \circ (\mu_x.F, \text{id}_{F^Z}) : F^Z \rightarrow F^y$ is regular for $y \in Z$.

Our main goal in the rest of the paper will be to prove Proposition 8.3. The next Lemma, needed often later, also simplifies the statement of the Proposition.

**Lemma 8.5.** If $g : L^x \times M^y \rightarrow M$ is regular, then $\mu_y.g : L^x \rightarrow M$ is continuous.

**Proof.** Let $I$ be a directed set and suppose that $\bigvee I$ exists in $L$. We argue first that $g^{m+1}_I(\bot) = \bigvee_{i \in I} g^m(i,\bot)$ for all $m \geq 0$. The relation trivially holds for $m = 0$. Suppose it holds for $m$, then

$$g^{m+1}_I(\bot) = g(\bigvee_I g^m(\bot)) = \bigvee_{i \in I} g(i, \bigvee_i g^m(\bot)) = \bigvee_{i \in I} \bigvee_{j \in I} g(j, g^m_i(\bot)) = \bigvee_{i \in I} g^{m+1}_i(\bot).$$

Consequently, we obtain

$$(\mu_y.g)(\bigvee I) = \bigvee_{m \geq 0} g^{m+1}_I(\bot) = \bigvee_{m \geq 0} \bigvee_{i \in I} g^m_i(\bot) = \bigvee_{i \in I} (\mu_y.g)(i).$$
Proposition 8.6. Consider \( \langle f, g \rangle : L^x \times M^y \rightarrow L \times M \), where \( f : L^x \times M^y \rightarrow L \) is continuous and \( g : L^x \times M^y \rightarrow M \) is regular. Then \( \langle f, g \rangle \) is regular if and only if \( f \circ (\id_L, \mu_y, g) : L^x \rightarrow L \) is regular.

To prove the Proposition we shall fix a continuous \( f \) and a regular \( g \). We introduce an explicit notation for the approximants of \( \langle f, g \rangle \) and \( h(x) = f(x, \mu_y, g(x, y)) \):

\[
\begin{align*}
f_0 &= \bot \\
n_{n+1} &= f(n, g_n) \\
g_0 &= \bot \\
n_{n+1} &= g(n, g_n) ,
\end{align*}
\]

and

\[
\begin{align*}
h_0 &= \bot \\
n_{n+1} &= f(h_{n}, i_{n+1}) \\
i_0 &= \bot \\
n_{n+1} &= \mu_y(g(h_{n}, y)) .
\end{align*}
\]

Using this notation we shall prove:

Proposition 8.7. The sequences \( \{ f_n, g_n \}_{n \geq 0} \) and \( \{ h_n, i_n \}_{n \geq 0} \) have the same upper bounds.

Proof of Proposition 8.6. Suppose \( \langle f, g \rangle \) is regular. Since \( g \) is regular, \( \mu_y \cdot g \) is continuous, by Lemma 8.5, hence \( h(x) = f(x, \mu_y, g(x, y)) \) is continuous, since \( f \) is also continuous. Thus we only need to show that \( \bigvee_{n \geq 0} h^n(\bot) = \bigvee_{n \geq 0} h_n \) exists. Since \( \langle f, g \rangle \) is regular \( \bigvee_{n \geq 0} \langle f, g \rangle^n(\bot) \) exists, and by Proposition 8.7 \( \bigvee_{n \geq 0} \langle f, g \rangle^n(\bot) = \bigvee_{n \geq 0} (f_n, g_n) = \bigvee_{n \geq 0} (h_n, i_n) \). Finally \( \bigvee_{n \geq 0} h_n \) exists, since by continuity of projections it is equal to \( \pr_1(\bigvee_{n \geq 0} (h_n, i_n)) \).

Conversely, suppose that \( h(x) = f(x, \mu_y, g(x, y)) \) is regular. In particular we see that \( \bigvee_{n \geq 0} h_n \) exists, and by Lemma 8.5. \( \bigvee_{n \geq 0} i_n = \bigvee_{n \geq 0} \mu_y(g(h_n, y) = \mu_y(g(\bigvee_{n \geq 0} h_n, y)) \) exists as well. Clearly \( \langle f, g \rangle \) is continuous and \( \bigvee_{n \geq 0} \langle f, g \rangle^n(\bot) = \bigvee_{n \geq 0} (f_n, g_n) = \bigvee_{n \geq 0} (h_n, i_n) \) exists, by Proposition 8.7. \( \square \)

Proof of Proposition 8.7. For a set \( A \) we let \( M(A) \) be the set of upper bounds of \( A \), i.e. \( M(A) = \{ x \mid \forall a \in A \ a \leq x \} \).

Our first observation is that \( (f_n, g_n) \leq (h_n, i_n) \), for all \( n \geq 0 \). This is clearly true for \( n = 0 \), and if the relation holds for \( n \), then

\[
\begin{align*}
f_{n+1} &= f(f_n, g_n) \leq f(h_n, i_n) \leq f(h_n, i_{n+1}) = h_{n+1} , \\
g_{n+1} &= g(f_n, g_n) \leq g(h_n, i_n) \leq g(h_n, i_{n+1}) \\
&= g(h_n, \mu_y(g(h_n, y)) = \mu_y(g(h_n, y)) = i_{n+1} .
\end{align*}
\]

Therefore we have \( M(\{ (h_n, i_n) \mid n \geq 0 \}) \subseteq M(\{ (f_n, g_n) \mid n \geq 0 \}) \).

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To prove the converse inclusion, we introduce a third sequence indexed by words of natural numbers:

\[
\begin{align*}
l_e &= \perp \\
l_{wk} &= f(l_w, g_{l_w}^k(\perp)) \\
m_e &= \perp \\
m_{wk} &= g_{l_w}^k(\perp).
\end{align*}
\]

Claim. The sequence \( \{(f_n, g_n)\}_{n \geq 0} \) is cofinal into \( \{ (l_w, i_w) \}_{w \in N^*} \): for all \( w \in N^* \) there exists \( n \in N \) such that \( (l_w, m_w) \leq (f_n, g_n) \).

Proof of the Claim. Observe first that \( g_{f_n}^k(\perp) \leq g_{n+k} \), for all \( n, k \geq 0 \). This relation is trivial if \( k = 0 \), and supposing it holds for \( k \), then

\[
g_{f_n}^{k+1}(\perp) = g(f_n, g_{f_n}^k(\perp)) \leq g(f_{n+k}, g_{n+k}) = g_{n+k+1}.
\]

Clearly \( (l_e, m_e) \leq (f_0, g_0) \), and if \( (l_w, m_w) \leq (f_n, g_n) \), then

\[
m_{wk} = g_{l_w}^k(\perp) \leq g_{f_n}^k(\perp) \leq g_{n+k} \leq g_{n+k+1} \leq g_{n+k+1}.
\]

\[
l_{wk} = f(l_w, m_{wk}) \leq f(f_n, g_{n+k}) \leq f(f_{n+k}, g_{n+k}) = f_{n+k+1}.
\]

Consequently we have \( M(\{(f_n, g_n) | n \geq 0\}) \subseteq M(\{(l_w, g_w) | w \in N^*\}) \).

Claim. The two relations

\[
h_n = \bigvee_{w \in N^n} l_w \quad \quad i_n = \bigvee_{w \in N^n} m_w
\]

hold.

Proof of the Claim. Observe first that if \( w, u \in N^n \) and \( w \leq u \), then \( l_w \leq l_u \) and \( m_w \leq m_u \). This is easily verified by induction on \( n \). It follows that for \( n \) fixed, the sets \( \{ l_w \}_{w \in N^n} \) and \( \{ m_w \}_{w \in N^n} \) are directed.

The relations stated in the Claim trivially hold for \( n = 0 \). Suppose they hold for \( n \). Then

\[
i_{n+1} = \mu_{y, g(h_n, y)} = \mu_{y, g(l_w, y)} = \bigvee_{w \in N^n} \mu_{y, g(l_w, y)}
\]

\[
= \bigvee_{w \in N^n} \bigvee_{k \geq 0} g_{l_w}^k(\perp) = \bigvee_{w \in N^n} \bigvee_{k \geq 0} m_{wk} = \bigvee_{w \in N^{n+1}} m_u
\]

and

\[
h_{n+1} = f(h_n, \mu_{y, g(h_n, y)}) = \bigvee_{w \in N^n} f(l_w, \mu_{y, g(l_w, y)})
\]

\[
= \bigvee_{w \in N^n} \bigvee_{k \geq 0} g_{l_w}^k(\perp) = \bigvee_{w \in N^n} \bigvee_{k \geq 0} f(l_w, g_{l_w}^k(\perp)) = \bigvee_{w \in N^{n+1}} l_u.
\]
Consequently, $M(\{(h_n, i_n) | n \geq 0\}) = M(\{(l_w, m_w) | w \in \mathbb{N}^*\})$ and, by the previous results, $M(\{(h_n, i_n) | n \geq 0\}) = M(\{(f_n, g_n) | n \geq 0\})$. This terminates the proof of Proposition 8.7.

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