REFLECTION GROUPS AND DISCRETE INTEGRABLE SYSTEMS

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ABSTRACT. We present a method of constructing discrete integrable systems with crystallographic reflection group (Weyl) symmetries, thus clarifying the relationship between different discrete integrable systems in terms of their symmetry groups. Discrete integrable systems are associated with space-filling polytopes arise from the geometric representation of the Weyl groups in the n-dimensional real Euclidean space $\mathbb{R}^n$. The “multi-dimensional consistency” property of the discrete integrable system is shown to be inherited from the combinatorial properties of the polytope; while the dynamics of the system is described by the affine translations of the polytopes on the weight lattices of the Weyl groups. The connections between some well-known discrete systems such as the multi-dimensional consistent systems of quad-equations [1] and discrete Painlevé equations [28] are obtained via the geometric constraints that relate the polytope of one symmetry group to that of another symmetry group, a procedure which we call geometric reduction.

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1. Introduction

To find reductions of discrete dynamical systems, regardless of dimension, is an aim which has stimulated a great deal of research. We present a constructive geometric method to answer this question based on associating each system with polytopes in n-dimensional lattices.

Focusing in particular on discrete integrable systems, where geometric representations of the Weyl groups are fundamental, the connections between different systems are identified with relations between different Weyl groups, in particular in terms of their reflection subgroups. In this way, we find geometric reductions from partial difference equations posed on an n-dimensional quadrilateral lattice (known

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Figure 1.1. Degeneration of type of symmetries. Of the 22 types listed, 19 (three kinds of discrete equations for the $E_8$ type, two kinds each for the types $E_7$ and $E_6$, and excluding the three affine root systems of type $A_0$ on the last column) correspond to the symmetries of discrete Painlevé equations [28].

As quad-equations) to Sakai’s formulation of the discrete Painlevé equations [28], which are ordinary difference equations.

While there have been several examples associating discrete integrable systems with regular polytopes of the Weyl groups in the literature, they have either not been space-filling or have been restricted to hypercubes. For example, Hirota’s d-KP equation is a six-point equation associated with the vertices of an octahedron [11, 22, 29, 10, 4]; the ABS classification of four-point partial difference equations is associated with quadrilaterals [11], while six-point equations are associated with the octahedron (octahedron-equations) [2]; and the quadrilateral Yang-Baxter maps have variables associated with the edges of a 3-cube [26, 6]. In contrast, our approach extends to infinite-dimensional affine Weyl groups, which give rise to space-filling polytopes, given by translations of their Voronoi cells. The vertices of the Voronoi cell are some set of weights of the Weyl group.

This consideration has the advantage that the dynamics of the associated discrete system can then be easily understood as a tessellation of Euclidean space by translations of the Voronoi cell on the weight lattice. Polytopes associated with the roots of the Weyl groups, called Delaunay cells, are in general not space-filling polytopes. Another reason for considering the weights of the Weyl groups is that the fundamental object in the theory of discrete Painlevé equations, namely the $\tau$ functions are associated with the weights of the Weyl group [24, 25, 18].

Systems of quad-equations in the literature have been mainly defined on the vertices of the $n$-dimensional hypercube ($n$-cube) [11], or constructed as a consistent system of the same equations [4]. In contrast, the systems we consider have a wide variety of symmetries, related via reductions to the types of the discrete Painlevé equations in Sakai’s classification: 19 types of the discrete Painlevé equations, which follows a degeneration pattern of the affine Weyl symmetry groups [28]: from the root system of type $E_8^{(1)}$ down to $A_1^{(1)}$ (see Figure 1.1).

This means that we have on our hands a corresponding class of quad-equations with immensely rich combinatorial/geometrical structures. This paper reviews some of our earlier works in this direction and explain the method in detail with two illustrating examples.
The plan of the paper is as follows. In Section 2, we give some preliminaries on the Weyl groups. In particular, the actions of the translational elements of the affine Weyl group is discussed in detail. We define the Voronoi cell of types $A$ and $B$ of the Weyl groups, which will be central to our construction of the discrete integrable systems in this exposition. In particular, we explain a relation between the Voronoi cells of types $A$ and $B$ through projection. In Section 3, we construct systems of discrete equations on the Voronoi cells of types $A$ and $B$; and obtain explicit relations between two systems by exploiting the geometric relation obtained in Section 2. Two examples of different combinatorial constructions are presented: (i) a system of quad-equations with $W(B_3)$ symmetry and its reduction to a $q$-discrete Painlevé type equation with $\tilde{W}(A_2)$ symmetry; (ii) a system of quad-equations with $W(B_2 + A_1)$ symmetry and its reduction to a $q$-discrete Painlevé type equation with $\tilde{W}(A_1 + A_1)'$ symmetry. In Section 4, we construct the rational surface associated with the $(A_2 + A_1)$-type root system, which define the space of initial values of a discrete Painlevé equation. We discuss the singularity structures of the discrete Painlevé equations via intersection theories of rational surfaces and the affine Weyl groups on the Picard lattice. Finally, in Section 5 we give some concluding remarks, comment on some of the implications of our result and some future directions.

2. Reflection groups and their associated polytopes

We give the necessary properties and facts of the Weyl groups, assuming the reader is familiar with the theory of the irreducible root systems and their affine extensions. We follow closely the terminology and notation of Humphreys [12]. In particular, we discuss the two pictures associated to the actions of the translational elements of the affine Weyl group via a geometrical representation using a $(n+1)$-dimensional real vector space $V^{(1)}$ and its dual space $V^{(1)*}$. The two pictures manifest as two complementing aspects of our construction of the discrete integrable systems as will be seen in the later sections of the paper.

2.1. Irreducible finite root systems and the corresponding Weyl groups.

Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a Cartan matrix of type $A_n, B_n, C_n, D_n, F_4, G_2, E_6, E_7$ or $E_8$ [8] Appendices I-X]. Let $V$ and $V^*$ be $n$-dimensional real vector spaces spanned by

$$\Delta = \{\alpha_1, ..., \alpha_n\} \quad \text{and} \quad \Delta^\vee = \{\alpha_1^\vee, ..., \alpha_n^\vee\}, \quad (2.1)$$

respectively, and define a bilinear pairing $\langle , \rangle : V \times V^* \rightarrow \mathbb{R}$ by

$$\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij} \quad (2.2)$$

for all $i, j \in \{1, ..., n\}$. Since $A$ is non singular, $V^*$ is isomorphic to the dual space of $V$. The elements of $\Delta$ and $\Delta^\vee$ are called the simple roots and simple coroots; and

$$Q = \mathbb{Z}\Delta \quad \text{and} \quad Q^\vee = \mathbb{Z}\Delta^\vee, \quad (2.3)$$

are the root lattice and coroot lattice. The height of the vector $\sum \lambda_i \alpha_i \in Q$ is defined to be $\sum A_i$.

For each $i \in \{1, ..., n\}$ define a linear transformation $s_{\alpha_i} : V \rightarrow V$ by the requirement that

$$s_{\alpha_i} \alpha_j = \alpha_j - (\alpha_j, \alpha_i^\vee)\alpha_i = \alpha_j - A_{ji}\alpha_i. \quad (2.4)$$
It is well known that the linear group \( W' \) generated by \( s_{\alpha_1}, \ldots, s_{\alpha_n} \) is finite. Furthermore, \( W' \) is isomorphic to the Weyl group of the Cartan matrix \( A \), defined to be the abstract group \( W = W(A) \) generated by \( s_1, \ldots, s_n \) subject to the defining relations 
\[
(s_is_j)^{m_{ij}} = 1,
\]
for all \( i, j \in \{1, \ldots, n\} \), where \( (m_{ij})_{1 \leq i, j \leq n} \) is the Coxeter matrix associated with the Cartan matrix \( A \) [8, no. 1.5, Chap. VI]. Note that

\[
A_{ij}A_{ji} = 4 \cos^2 \left( \frac{\pi}{m_{ij}} \right) \tag{2.5}
\]

for \( i \neq j \) and \( A_{ii} = 2 \). (In particular, \( m_{ii} = 1 \) for all \( i \), and \( m_{ij} \in \{2, 3, 4, 6\} \) for \( i \neq j \).

Let \( \eta : W \to W' \) be the isomorphism satisfying \( \eta s_i = s_{\alpha_i} \) for all \( i \in \{1, \ldots, n\} \), and define an action of \( W' \) on \( V \) via \( w.v = (\eta w)v \) for all \( w \in W \) and \( v \in V \).

Since \( W' \) is finite it preserves a Euclidean inner product \( (, ) : V \times V \to \mathbb{R} \). If \( 0 \neq \alpha \in V \), then the reflection in the hyperplane orthogonal to \( \alpha \) is the orthogonal transformation \( V \to V \) given by

\[
v \mapsto v - \frac{2(\alpha, v)}{(\alpha, \alpha)} \alpha,
\]
for all \( v \in V \). Since \( (\alpha_i, \alpha_j) = (s_{\alpha_i}, s_{\alpha_i}, s_{\alpha_i}, s_{\alpha_i}) = (-\alpha_i, \alpha_j - A_{ji} \alpha_i) \), it follows that

\[
A_{ji} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \tag{2.6}
\]

for all \( i, j \in \{1, \ldots, n\} \) and that \( s_{\alpha_i} \) is the reflection in the hyperplane \( H_i \):

\[
H_i = \{ v \in V \mid (v, \alpha_i^\vee) = 0 \} = \{ v \in V \mid (v, \alpha_i) = 0 \}. \tag{2.7}
\]

Note that Equation (2.6) implies that

\[
(\alpha_i, \alpha_i)A_{ji} = A_{ij}(\alpha_j, \alpha_j) \tag{2.8}
\]

for all \( i \) and \( j \). In general, we write \( s_{\alpha_i} \) for the reflection in the hyperplane orthogonal to \( \alpha \). A trivial calculation shows that if \( w : V \to V \) preserves \( (, ) \) then

\[
s_{\alpha \vee w} = ws_{\alpha}w^{-1} \tag{2.9}
\]

for all nonzero \( \alpha \in V \).

The root system of \( W \) is defined to be the subset \( \Phi \) of \( Q \) given by \( \Phi = W' \Delta \). If \( \alpha \in \Phi \), then \( \alpha = w_0 \alpha_i \) for some \( w \in W' \) and \( i \in \{1, \ldots, n\} \), hence \( s_\alpha = ws_{\alpha_i}w^{-1} \in W' \). For simplicity of notation, we henceforth use the isomorphism \( \eta \) to identify elements of \( W' \) with elements of \( W \), so that \( s_i = s_{\alpha_i} \) and \( s_\alpha \in W \) whenever \( \alpha \in \Phi \).

Note that \( W \) acts on \( V^* \) via the contragredient action:

\[
\langle w^{-1} f, h \rangle = \langle f, w.h \rangle, \quad f \in V, h \in V^*, w \in W. \tag{2.10}
\]

It follows that

\[
s_i \alpha_j^\vee = \alpha_j^\vee - A_{ij} \alpha_i^\vee, \tag{2.11}
\]

and the coroot system is defined by \( \Phi^\vee = W' \Delta^\vee \).
Define a linear isomorphism $\theta : V^* \rightarrow V$ by $\theta \alpha^\vee_i = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$ for all $i$. Then Equations (2.8) and (2.4) combine to give

$$s_i.(\theta \alpha^\vee_j) = s_i\left(\frac{2\alpha_j}{(\alpha_j, \alpha_j)} - \frac{2}{(\alpha_i, \alpha_i)} A_{ij}\alpha_i\right) = \theta \alpha^\vee_j - A_{ij}\theta \alpha^\vee_i = \theta(\alpha^\vee_j - A_{ij}\alpha^\vee_i) = \theta(s_i.\alpha^\vee_j),$$

where we have used Equation (2.11) in the last line. It follows that $\theta$ commutes with the $W$–actions on $V^*$ and $V$. By the definition of $\theta$ we have

$$\alpha^\vee_i = \frac{2\alpha_i}{(\alpha_i, \alpha_i)},$$

for all $\alpha \in \Phi$. Observe that if $\alpha = w.\alpha_j$ for some $w \in W$ and $j \in \{1, 2, ..., n\}$ then

$$\theta(w.\alpha_j^\vee) = w.\theta(\alpha_j^\vee) = 2\frac{w.\alpha_j}{(\alpha_j, \alpha_j)} - \frac{2}{(w.\alpha_j, w.\alpha_j)} = \frac{2\alpha}{(\alpha, \alpha)} = \theta(\alpha^\vee),$$

since $W$ preserves the inner product. Hence $\alpha^\vee = w.\alpha^\vee_j$ whenever $\alpha = w.\alpha_j$. Since we are considering only irreducible root systems, $\Phi$ contains a unique root $\tilde{\alpha}$, whose height is maximal, called the highest root. Define $c_1, c_2, ..., c_n \in \mathbb{R}$ by

$$\tilde{\alpha} = \sum_{i=1}^{n} c_i\alpha_i.$$  

(2.13)

The values of the $c_i$ for all finite Weyl groups can be found in [12] for example. Similarly in $\Phi^\vee$ we have the highest coroot,

$$\tilde{\alpha}^\vee = \sum_{i=1}^{n} m_i\alpha_i^\vee,$$  

(2.14)

where $\tilde{\alpha}_s$ is the highest short root in $\Phi$, $m_i = c_i$ for the simply laced root systems (types $A, D, E$), and $m_i = c_{n-i+1}$ otherwise. Note that for the simply laced root systems $\tilde{\alpha}_s = \tilde{\alpha}^\vee$.

Let $h_1, ..., h_n \in V^*$ satisfy

$$\langle \alpha_i, h_j \rangle = \delta_{ij}, \quad (1 \leq i, j \leq n),$$

(2.15)

so that $\{h_1, ..., h_n\}$ is the basis of $V^*$ dual to the basis $\{\alpha_1, ..., \alpha_n\}$ of $V$. The elements $h_1, ..., h_n$ are called the fundamental weights, and $P = \mathbb{Z}\{h_1, ..., h_n\}$ is called the weight lattice. Observe that from Equations (2.2) and (2.15) we have

$$\alpha_i^\vee = \sum_{i=1}^{n} A_{ij}h_j,$$  

(2.16)
Moreover,

\[ s_i h_j = h_j - (\alpha_i, h_j) \alpha_i^\vee = \begin{cases} h_j, & \text{for } i \neq j, \\ h_j - \alpha_j^\vee = h_j - \sum_{k=1}^n A_{jk} h_k, & \text{for } i = j. \end{cases} \quad (2.17) \]

Since in this paper we will be primarily interested in the Weyl group of types \( A \) and \( B \), we give their definitions below.

### 2.2. Combinatorial description of Weyl group of types \( A \) and \( B \)

The Weyl group of type \( A \), denoted \( W(A_{n-1}) \), is isomorphic to \( S_n \), the symmetric group on \{1, ..., n\}, with \( s_i \) acting as the transposition \((i, i+1)\) for each \( i \in \{1, ..., n-1\} \).

The Dynkin diagram \( \Gamma(A_{n-1}) \) (shown in Figure 2.1) describes the relation satisfied by \( s_1, ..., s_{n-1} \) as explained in [12, page 31], namely

\[ (s_i s_j)^2 = 1 \text{ if } |i - j| > 1, \quad \text{and} \quad (s_i s_{i+1})^3 = 1 \text{ if } |i - j| = 1, \quad (2.18) \]

for \( i, j \in \{1, ..., n-1\} \).

The Weyl group \( W(B_n) = \langle s_1, ..., s_{n-1}, s_n \rangle \) acts on the set of all subsets of \{1, ..., n\}, while the subgroup \( \langle s_1, ..., s_{n-1} \rangle \) acting via permutations of \{1, ..., n\} as in type \( A \) above, and with \( s_n \) acting via

\[ s_n : S \mapsto S \cup \{n\} \quad \text{for all } S \subseteq \{1, ..., n\}, \quad (2.19) \]

where \( \cup \) denotes the symmetric difference, \( I \cup J = (I \cup J) \setminus (I \cap J) \). The Dynkin diagram \( \Gamma(B_n) \) is shown in Figure 2.2 and the corresponding relations are:

\[ (s_{n-1} s_n)^4 = 1, \quad (s_i s_j)^2 = 1 \text{ if } |i - j| > 1, \quad (s_i s_{i+1})^3 = 1 \text{ if } |i - j| = 1, \quad (2.20) \]

for \( i, j \in \{1, ..., n-1\} \). Now we are ready to define the Voronoi cell of the Weyl groups.

### 2.3. Voronoi cell: space-filling Polytopes.

**Definition 2.1** ([9]). Let the fundamental simplex \( F \) be the convex hull of

\[ \left\{ 0, \frac{h_1}{m_1}, \frac{h_2}{m_2}, ..., \frac{h_n}{m_n} \right\}, \]

where \( h_i \) are the fundamental weights, \( m_i \) are the coefficients in the expression \( \alpha_i^\vee \) for the highest coroot. The Voronoi cell of the Weyl group \( W \), denoted by \( \text{Vor}(W) \) is the union of the images of the fundamental simplex under \( W \),

\[ \text{Vor}(W) = \bigcup_{w \in W} w.F, \quad w \in W. \quad (2.21) \]
Theorem 2.1 ([23] [9]). Vor(W) is a space-filling polytope - the whole space $\mathbb{R}^n$ can be tessellated by translated copies of Vor(W).

The construction of the Voronoi cells for all finite Weyl groups is given in [23] via an approach of the decorated Dynkin diagrams. Here we restrict our attention to types $A$ and $B$.

Proposition 2.1 ([23]). The polytope Vor($B_n$) is an n-cube.

Proof. The coefficients $m_i$ for the root system of type $B_n$ are $2, 2, ..., 2, 1$. The fundamental simplex $F$ is then the convex hull of $\left\{ 0, h_{n-1}, h_{n-2}, ..., h_1, h_0 \right\}$, and the polytope Vor($B_n$) is the convex hull of $\text{Orb}(h_n) = W(B_n).h_n$ - the set of weight vectors in the orbit of the fundamental weight $h_n$ under the actions of $W(B_n)$. In the geometric representation of $W(B_n)$ in $\mathbb{R}^n$, we have

$$h_n = \frac{1}{2}(\epsilon_1 + ... + \epsilon_n),$$

where $\epsilon_i$ (i=1, ..., n) are the unit vectors of $\mathbb{R}^n$.

The actions of the generators of $W(B_n)$ on this geometric representation are that $s_i$ permutes $\epsilon_i \leftrightarrow \epsilon_{i+1}$ for $i \in \{1, ..., n-1\}$, and $s_n$ changes the sign of $\epsilon_n$. Then weights in $\text{Orb}(h_n)$ are of the form

$$\frac{1}{2}(\pm \epsilon_1 + ... \pm \epsilon_n).$$

The $2^n$ weights in this orbit, are exactly the $2^n$ vertices of an n-cube with edge length 1, centered at the origin 0.

To simplify the notation, we denote the vectors in $\text{Orb}(h_n)$ by $X_S$, $S \subseteq \{1, ..., n\}$. The subset $S$ encodes the positive components of the vector. For example, $h_n = \frac{1}{2}(\epsilon_1 + \epsilon_2 + ... + \epsilon_n) = X_{(1,2, ..., n)} = X_{12...n}$, and $\frac{1}{2}(\epsilon_1 - \epsilon_2 - ... - \epsilon_n) = X_{\emptyset} = X_0$. Therefore, the set of vertices of Vor($B_n$) in this notation is given by

$$\text{Orb}(h_n) = \{X_S\}_{S \subseteq \{1, ..., n\}}.$$  (2.23)

□

Proposition 2.2 ([15]). The polytope Vor($A_{n-1}$) is obtained from Vor($B_n$) by an orthogonal projection $\phi : \mathbb{R}^n \mapsto \mathbb{R}^{n-1}$ along the fundamental weight of $W(B_n)$: $h_n = \frac{1}{2}(\epsilon_1 + ... + \epsilon_n)$, defined by

$$\phi.v = v - \frac{(v, h_n) h_n}{\|h_n\|^2},$$

where $(\ , \ )$ is the Euclidean inner product and

$$\{U_S\}_{S \subseteq \{1, ..., n\}} = \{\phi.X_S\}, \quad 1 \leq |S| \leq n-1,$$  (2.25)

are the $2^n - 2$ vertices of Vor($A_{n-1}$).

Proof. The coefficients $m_i$ for the root system of type $A_{n-1}$ are $1, 1, ..., 1, 1$. The fundamental simplex $F$ is then the convex hull of $\left\{ 0, h_1, h_2, ..., h_{n-1} \right\}$, where

$$h_k = (\epsilon_1 + ... + \epsilon_k) - \frac{k}{n} \sum_{i=1}^n \epsilon_i, \quad 1 \leq k \leq n-1,$$  (2.26)

are the fundamental weights of $W(A_{n-1})$. We denote the vectors in $\text{Orb}(h_k)$ by $\{U_S\}$ for $S \subseteq \{1, ..., n\}$ and $|S| = k$, where the subset $S$ in $U_S$ records the positive
satisfying the fundamental relations \( \Gamma \) of \( h \).

The generators of \( W \) which correspond to the Dynkin diagram \( h \) to

Example 2.1. \( \Gamma \) of \( V \) or \( r \) of all the fundamental weights: the components of the vector. The vertices of \( \text{Vor}(A_{n-1}) \), are the union of the orbits of all the fundamental weights:

\[
\{ U_S \}_{S \in \{1, \ldots, n\}} = \bigcup_{1 \leq k \leq n-1} r.h_k, \ r \in W(A_{n-1}). \tag{2.27}
\]

The projection \( \phi \) chooses the \((n - 1)\)-dimensional hyperplane in \( \mathbb{R}^n \) orthogonal to \( h_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n) \), which corresponds to the usual representation of \( A_{n-1} \) in \( \mathbb{R}^n \).

It can be checked directly that \( \{ \phi.X_S \} = \{ U_S \} \). Note that two vertices of \( \text{Vor}(B_n) \) coincide after the orthogonal projection: \( \phi.X_{12 \ldots n} = \phi.X_0 = 0 \), and are now at the center of \( \text{Vor}(A_{n-1}) \).

\[ \square \]

Remark 2.1. The projection of \( \text{Vor}(B_n) \) to \( \text{Vor}(A_{n-1}) \) on the group level corresponds to taking the parabolic subgroup \( W(A_{n-1}) \) of \( W(B_n) \). On the Dynkin diagram, it is associated with deleting the node \( s_n \) from \( \Gamma(B_n) \) (Figure 2.3.1) to obtain \( \Gamma(A_{n-1}) \) (Figure 2.7).

We give an example of Propositions 2.1 and 2.2 for the case \( n = 3 \).

Example 2.1. The fundamental weight \( h_3 \) of \( W(B_3) \) is \( \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3) \). The vertices of \( \text{Vor}(B_3) \) are the vectors in \( \text{Orb}(h_3) \). They are of the form: \( \frac{1}{2}(\pm \epsilon_1 + \pm \epsilon_2 \pm \epsilon_3) \). In the \( X_S \) notation they are:

\[
\{ X_S \}_{S \in \{1, \ldots, 3\}} = \{ X_0, X_1, X_2, X_3, X_{12}, X_{13}, X_{23}, X_{123} \}. \tag{2.28}
\]

They are the 8 vertices of a 3-cube.

The generators of \( W(B_3) = \langle s_1, s_2, s_3 \rangle \) act on the variables \( X_S \) as follows:

\[
\begin{align*}
s_1 : & \{ X_1 \leftrightarrow X_2, \ X_{13} \leftrightarrow X_{23} \}, \\
s_2 : & \{ X_2 \leftrightarrow X_3, \ X_{12} \leftrightarrow X_{13} \}, \\
s_3 : & \{ X_0 \leftrightarrow X_3, \ X_1 \leftrightarrow X_{13}, \ X_2 \leftrightarrow X_{23}, \ X_{12} \leftrightarrow X_{123} \},
\end{align*} \tag{2.29}
\]

satisfying the following relations

\[
\begin{align*}
s_1^2 = s_2^2 = s_3^2 = 1, \quad (s_1s_2)^3 = 1, \quad (s_1s_3)^2 = 1, \quad (s_2s_3)^4 = 1,
\end{align*} \tag{2.30}
\]

which correspond to the Dynkin diagram \( \Gamma(B_3) \) (see Figure 2.3.1).

By Equation (2.27), the vertices of \( \text{Vor}(A_2) \) are the weight vectors in the orbits of \( h_1 \) and \( h_2 \). The fundamental weights of \( W(A_2) \) are:

\[
\begin{align*}
h_1 &= U_1 = \frac{1}{3}(2\epsilon_1 - \epsilon_2 - \epsilon_3), \\
h_2 &= U_{12} = \frac{1}{3}(\epsilon_1 + \epsilon_2 - 2\epsilon_3).
\end{align*} \tag{2.31}
\]

The generators of \( W(A_2) = \langle s_1, s_2 \rangle \) act on the variables \( U_S \) as follows:

\[
\begin{align*}
s_1 : & \{ U_1 \leftrightarrow U_2, \ U_{13} \leftrightarrow U_{23} \}, \\
s_2 : & \{ U_2 \leftrightarrow U_3, \ U_{12} \leftrightarrow U_{13} \},
\end{align*} \tag{2.32}
\]

satisfying the fundamental relations

\[
\begin{align*}
s_1^2 = s_2^2 = 1, \quad (s_1s_2)^3 = 1,
\end{align*} \tag{2.33}
\]

which correspond to the Dynkin diagram \( \Gamma(A_2) \) (see Figure 2.3.2).
$\text{Vor}(A_2)$ is a hexagon with 6 vertices:

\[ \{ U_S \}_{S \in \{1,\ldots,3\}} = \{ U_1, U_2, U_3, U_{12}, U_{13}, U_{23} \}, \quad (2.34) \]

centred at $U_0 = h_0 = 0$.

It can be checked easily that the set of vertices given in (2.34) results from the set of vertices given in (2.28) by orthogonal projection $\phi$ along $X_{123} = h_3 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)$. Geometrically this corresponds to the projection of a 3-cube to a hexagon.

2.4. **Affine Weyl group and translations.** Given the Cartan matrix $A = (A_{ij})_{1 \leq i,j \leq n}$ as in Section 2.1 above, we now define

\[ A_{j0} = (\alpha_j, -\alpha^\vee), \quad A_{0j} = (-\tilde{\alpha}, \alpha_j^\vee), \quad (2.35) \]

and let $A_{00} = 2$. The $(n+1) \times (n+1)$ matrix $A^{(1)} = (A_{ij})_{0 \leq i,j \leq n}$ is called the extended Cartan matrix. Let $W(A^{(1)})$ be the abstract group generated by $s_0, s_1, \ldots, s_n$, subject to the defining relations $(s_i s_j)^{m_{ij}} = 1$ for all $i, j \in \{0,1,\ldots,n\}$, where the $m_{ij}$ are given by Equation (2.34). These $m_{ij}$ can be read off from the extended Dynkin diagrams $\Gamma^{(1)}$ see [12] p.34. The group $W(A^{(1)})$ is called the affine Weyl group of type $A$.

The corresponding extended vector space $V^{(1)}$ is spanned by the set of simple affine roots, $\Delta^{(1)} = \Delta \cup \{ \alpha_0 \}$. In addition, we write $s_0$ for the reflection on $V^{(1)}$ corresponding to $\alpha_0$. Thus

\[ s_i \alpha_j = \alpha_j - A_{ji} \alpha_i \quad \text{for all } i, j \in \{0,1,\ldots,n\}. \quad (2.36) \]

We define $\Phi^{(1)} = W(A^{(1)}) \Delta^{(1)}$, and call $\Phi^{(1)}$ the *the affine root system*. It can be shown that

\[ \Phi^{(1)} = \{ \alpha + k \delta | \alpha \in \Phi, k \in \mathbb{Z} \}, \quad (2.37) \]

where

\[ \delta = \alpha_0 + \tilde{\alpha} = \sum_{i=0}^{n} c_i \alpha_i \quad (2.38) \]

is called *null root*. We have let $c_0 = 1$, and $c_i$ ($i \in \{1,\ldots,n\}$) are defined earlier. From the fact that $A^{(1)}$ is singular we have

\[ (\delta, \alpha_j^\vee) = \left( \sum_{i=0}^{n} c_i \alpha_i, \alpha_j^\vee \right) = \sum_{i=0}^{n} c_i A_{ij} = 0, \quad \text{for } 0 \leq j \leq n, \quad (2.39) \]

and that all elements of $W$ act trivially on $\delta$. The extended space $V^{(1)}$ is spanned by the simple affine roots $\{ \alpha_0, \alpha_1, \ldots, \alpha_n \}$, or equally well, by the basis $\{ \alpha_1, \ldots, \alpha_n, \delta \}$. We extend the bilinear pairing in Equation (2.22) by $\langle , \rangle: V^{(1)} \times V^{(1)*} \rightarrow \mathbb{R}$ by

\[ \langle \alpha_i, h_j \rangle = \delta_{ij}, \quad \text{for } (1 \leq i, j \leq n), \quad \text{and } \langle \delta, h_k \rangle = 1. \quad (2.40) \]

Then $\{ h_1, \ldots, h_n, h_\delta \}$ is the basis of $V^{(1)*}$ dual to the basis $\{ \alpha_1, \ldots, \alpha_n, \delta \}$ of $V^{(1)}$.

There are two pictures of the actions of $W(A^{(1)})$. In one they act linearly on the simple affine roots $\{ \alpha_0, \alpha_1, \ldots, \alpha_n \}$ in $V^{(1)}$ by formula (2.36) for the reflections $s_i$ ($i \in \{0,1,\ldots,n\}$). In the second picture, the actions can be seen as affine transformations in a hyperplane of $V^{(1)*}$, which give rise to translational motions that are nonlinear. We give a detailed explanation below. Define a hyperplane $H$ in $V^{(1)*}$ by

\[ H = \{ h \in V^{(1)*} | \langle \delta, h \rangle = 1 \}. \quad (2.41) \]
Then generators $s_{\alpha-k\delta}$ associated with the affine roots $\alpha - k\delta \in \Phi^{(1)}$ act on $h \in H$ by the formula
\[
s_{\alpha-k\delta}.h = h - ((\alpha, h) - k)\alpha^\vee,
\] (2.42)
where $\alpha^\vee \in \Phi^\vee$, $k \in \mathbb{N}$. The generator $s_{\alpha-k\delta}$ describes reflection about the hyperplane $H_{\alpha-k\delta}$ defined by,
\[
H_{\alpha-k\delta} = \{ h \in H \mid (\alpha, h) = k \}, \quad \text{note that } H_{\alpha-k\delta} = H_{-\alpha+k\delta}.
\] (2.43)
Observe that $s_{\alpha-k\delta}$ defined in Equation (2.42) are reflections about reflection planes that do not pass through the origin for $k \neq 0$. The hyperplane $H$ is called an affine plane. In particular, we see that
\[
s_0.h = s_{\alpha_0}.h = s_{\delta-\alpha_0}.h = s_{\tilde{\alpha}}.h = h - ((\tilde{\alpha}, h) - 1)\tilde{\alpha}^\vee,
\] (2.44)
and it follows that
\[
t_{\tilde{\alpha}} = s_0s_{\alpha_0}.h = s_{\delta-\alpha_0}s_{\tilde{\alpha}}.h = h + \tilde{\alpha}^\vee,
\]
where we have denoted the element of $W(A^{(1)})$ associated with translation of $\tilde{\alpha}^\vee$ on $H$ as $t_{\tilde{\alpha}}$. It is known that the affine Weyl group decomposes into the semidirect product of translations in the coroot lattice and the finite Weyl group:
\[
W(A^{(1)}) = (t_{\alpha_1}, ..., t_{\alpha_n}) \rtimes W(A) = T_{Q^\vee} \rtimes W(A),
\] (2.45)
where $t_{\alpha_i} \in W(A^{(1)})$ for ($i = 1, ..., n$), $\alpha_i \in \Delta$. The group $W(A^{(1)})$ can be further extended by Dynkin diagram automorphisms to the extended affine Weyl group $\tilde{W}(A^{(1)})$, which decomposes into the semidirect product of the translations in the weight lattice and the finite Weyl group acting on the weight lattice:
\[
\tilde{W}(A^{(1)}) = (t_{h_1}, ..., t_{h_n}) \rtimes W(A) = T_P \rtimes W(A),
\] (2.46)
where $t_{h_i} \in \tilde{W}(A^{(1)})$ ($i = 1, ..., n$) are the translational elements associated to the fundamental weights. Note that we denote the Dynkin diagram of $\tilde{W}(A^{(1)})$ as $\tilde{\Gamma}^{(1)}(A)$.

Now we give the definition of the translational elements and their actions on $V^{(1)}$ and $H \subset V^{(1)*}$ respectively.

\subsection{Translational elements of the affine and extended affine Weyl groups.}

**Definition 2.2** ([17]). Let $\mu \in V$, so that $\langle \delta, \mu^\vee \rangle = 0$ and $\langle \mu, \mu^\vee \rangle \neq 0$. Translational element associated to $\mu$ in $W(A^{(1)})$ is given by
\[
t_\mu = s_{\delta-\mu}s_\mu,
\] (2.47)
and
\[
wt_\mu w^{-1} = t_{w.\mu}, \quad w \in W(A^{(1)}).
\] (2.48)

The action of $t_\mu$ on $H$ is given by
\[
t_\mu.h = h + \mu^\vee,
\] (2.49)
and its action on the simple affine roots $\alpha_i \in V^{(1)}$ ($i \in \{0, 1, ..., n\}$) is given by
\[
t_\mu.\alpha_i = \alpha_i - (\alpha_i, \mu^\vee)\delta = \alpha_i - \mu_i\delta,
\] (2.50)
We refer to the linear action of $t\mu$ that the shift motion on the simple affine roots:

$$\langle \alpha, \mu \rangle = \langle \sum_{i=0}^{n} c_i \alpha_i, \mu \rangle = \sum_{i=0}^{n} c_i \mu_i. \tag{2.51}$$

We refer to the linear action of $t\mu$ on $\alpha_i \in V^{(1)} (i \in \{0, 1, \ldots, n\})$ given by Equation (2.50) as shift motion on the simple affine roots.

In what follows we give an explicit example of the ideas explained above for the root system of type $A_2$.

**Example 2.2.** Affine and extended affine Weyl group of type $A_2$ are: $W(A_2^{(1)}) = \langle s_0, s_1, s_2 \rangle$ and $\widetilde{W}(A_2^{(1)}) = \langle s_0, s_1, s_2, \rho \rangle$, respectively, where $\rho$ is the Dynkin diagram automorphism:

$$\rho : (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_1, \alpha_2, \alpha_0). \tag{2.52}$$

The generators satisfy the following relations

$$s_j^2 = 1, \quad (s_j s_{j+1})^3 = 1, \quad (j \in \mathbb{Z}/3\mathbb{Z}), \quad \rho^3 = 1, \quad \rho s_j = s_{j+1} \rho, \tag{2.53}$$

which correspond to the Dynkin diagram in Figure 2.3.

Using the equations (2.38), (2.47) and (2.48) we have the translational elements:

$$t_{\tilde{\alpha}} = s_{\tilde{\delta}} s_{\tilde{\alpha}} = s_{0\alpha_0} s_{0\tilde{\alpha}}, \tag{2.54}$$

$$t_{\alpha_1} = s_{2\alpha_0} s_{0\tilde{\alpha}} s_{\tilde{\alpha} s_2}, \tag{2.55}$$

$$t_{\alpha_2} = s_{1\alpha_0} s_{0\tilde{\alpha}} s_{\tilde{\alpha} s_1}. \tag{2.56}$$

The actions of $t_{\tilde{\alpha}}$, $t_{\alpha_1}$ and $t_{\alpha_2}$ on $V^{(1)}$ given by Equation (2.50) imply the following shift motions on the affine simple roots:

$$t_{\tilde{\alpha}} : \langle \alpha_0, \alpha_1, \alpha_2 \rangle \rightarrow \langle \alpha_0 + 2\delta, \alpha_1 - \delta, \alpha_2 - \delta \rangle, \tag{2.57}$$

$$t_{\alpha_1} : \langle \alpha_0, \alpha_1, \alpha_2 \rangle \rightarrow \langle \alpha_0 + \delta, \alpha_1 - 2\delta, \alpha_2 + \delta \rangle, \tag{2.58}$$

$$t_{\alpha_2} : \langle \alpha_0, \alpha_1, \alpha_2 \rangle \rightarrow \langle \alpha_0 + \delta, \alpha_1 + \delta, \alpha_2 - 2\delta \rangle. \tag{2.59}$$

Observe that the above transformations satisfy the constraint (2.51). The translational actions on the affine plane $h \in H$ is given by:

$$t_\alpha h = h + \alpha^\vee, \quad \alpha \in \{\tilde{\alpha}, \alpha_1, \alpha_2\}. \tag{2.60}$$

To describe translations correspond to the fundamental weights $h_1, h_2$, we need the extended affine Weyl group $\widetilde{W}(A_2^{(1)}) = T_P \rtimes W(A_2) = \langle h_1, h_2 \rangle \rtimes W(A_2)$. They are given by:

$$t_{h_1} = \rho s_2 s_1, \quad \text{and} \quad t_{h_2} = \rho^{-1} s_1 s_2. \tag{2.61}$$
Their actions result in shift motions on the simple affine roots given by:

\[ t_{h_i} : (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0 + \delta, \alpha_1 - \delta, \alpha_2), \]

(2.62)

\[ t_{h_2} : (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0 + \delta, \alpha_1, \alpha_2 - \delta), \]

(2.63)

and translational motions in the affine hyperplane \( H \) are given by:

\[ t_{h_i}.h = h + h_i, \quad (i = 1, 2). \]

(2.64)

If we write \( t_3 = t_{h_1} \), and let \( \rho t_i = t_{i+1} \rho, (i = 1, 2, 3) \), then we have:

\[ t_1 = \rho s_2 s_1, \quad t_2 = \rho s_0 s_2, \quad t_3 = \rho s_1 s_0, \]

(2.65)

which correspond to the translations by: \( \frac{2}{3}(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3) \), \( \frac{1}{3}(-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3) \) and \( \frac{1}{3}(-\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3) \), respectively. They satisfy the relation

\[ t_1 t_2 t_3 = 1. \]

(2.66)

The actions of \( t_2 \) and \( t_3 \) on the simple affine roots are given by:

\[ t_2 : (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0, \alpha_1 + \delta, \alpha_2 - \delta), \]

(2.67)

\[ t_3 : (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0 - \delta, \alpha_1, \alpha_2 + \delta). \]

(2.68)

We summarise the ideas explained in this section using Figure 2.4. It shows the affine plane \( H \) on which \( \widehat{W}(A_2^{(1)}) \) act as affine transformations by reflections about the reflection lines \( H_{\alpha_{1+k\delta}} \), and rotation about the center of the fundamental simplex \( F \) by \( \rho \). The fundamental simplex \( F \) is bounded by three red reflection lines \( H_{\alpha_0}, H_{\alpha_1} \) and \( H_{\alpha_2} \) that correspond to the three simple reflections \( s_0, s_1 \) and \( s_2 \). It is the triangle with vertices \( U_0, U_1, U_{12} \), where \( U_0 = 0 \) is the origin \( 0 \) and \( U_1 = h_1, U_{12} = h_2 \) are the two fundamental weights. The hexagon with vertices \( U_1, U_2, U_3, U_{12}, U_{13}, U_{23} \) centered at \( U_0 \) is \( \text{Vor}(A_2) \). We have also shown the vertex \( \tilde{\alpha}_V = U_{112} = U_{12} + U_1 \). Any point on the \( A_2 \) weight lattice can be obtained from the three vertices of \( F \) (which we call the initial value set): \( \{U_0, U_1, U_{12}\} \) using the elements of \( \widehat{W}(A_2^{(1)}) \). The transformations on the the initial value set is given by

\[ s_1 : \{U_1 \leftrightarrow U_2\}, \]

(2.69)

\[ s_2 : \{U_{12} \leftrightarrow U_{13}\}, \]

\[ s_0 : \{U_0 \leftrightarrow U_{112}\}, \]

\[ \rho : \{U_0, U_1, U_{12}\} \mapsto \{U_1, U_{12}, U_0\}, \]

which satisfy the defining relations of \( \widehat{W}(A_2^{(1)}) \) given in Equation 2.63. We have also shown the translational actions of \( t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_1}, t_2 \) and \( t_3 \) on \( F \).

Having introduced the preliminaries of the Weyl groups we are now ready to discuss them in the context of discrete integrable systems.

3. Discrete integrable systems on Voronoi cells – a combinatorial construction

An important class of discrete integrable equations, known as quad-equations, were classified by Adler-Bobenko-Suris [2]. They considered multi-dimensionally consistent, affine linear equations, which relate values of a function on vertices of a quadrilateral. These equations are the discrete analogues of many well-known nonlinear integrable partial differential equations.
Our construction relies on reformulating systems of quad-equations as birational group representations of the Weyl groups. In particular, we associate the system of quad-equations with a polytope of the Weyl group, where quad-equations are associated with some quadrilateral sub-structures of the polytope. The consistency and symmetry of the system of equations are the consequences of the combinatorial structure and the symmetry of the polytope.

To formulate discrete dynamical systems as birational groups is an approach well-established in the studies of quadrirational Yang Baxter maps \[32\] and discrete Painlevé equations \[24, 28, 18\]. In the context of quad-equations, it was first used by Atkinson \[3\] in his work on Yang Baxter maps as birational representation of a sequence of Coxeter groups (associated with the connected T-shaped Coxeter-Dynkin diagram). Our construction is different from those of \[3\], in that we allow different quad-equations in a system. Consequently, we can construct systems with symmetries associated with the disconnected Dynkin diagrams.

In order to associate equations and variables with the polytopes discussed in the previous section, let us consider the function $x : \{X_S\} \to \mathbb{C}$, on the vertices of $\text{Vor}(B_n)$ ($n$-cube), we write

$$x_S = x(X_S), \quad (3.1)$$
where $S \subseteq \{1, \ldots, n\}$, so that the function $x$ is indexed by the weight vectors of the $B_n$ root system, and elements $w \in W(B_n)$ act on the variables $x_S$ by

$$w(x_S) = x(w.X_S) = x(X_{w.S}) = x_{w.S}.$$  \hspace{1cm} (3.2)

Similarly, consider the function $u : \{U_S\} \to \mathbb{C}$ on the vertices of Vor$(A_{n-1})$, we write

$$u_S = u(U_S),$$  \hspace{1cm} (3.3)

where $S \subseteq \{1, \ldots, n\}$, $1 \leq |S| \leq n-1$, and $u_0 = u(0)$. The function $u$ is indexed by the weight vectors of the $A_{n-1}$ root system, in particular the centre and the vertices of Vor$(A_{n-1})$. Elements of $w \in \tilde{W}(A_{n-1}^{(1)})$ act on the variables $u_S$ by

$$w(u_S) = u(w.U_S) = u(U_{w.S}) = u_{w.S}.$$  \hspace{1cm} (3.4)

In what follows we give as examples the construction of two systems of quad-equations with different combinatorial structures: one associated with the 3-cube ($W(B_3)$ symmetry) and the other with the asymmetric 3-cube ($W(B_2 + A_1)$ symmetry), and give their reductions to some discrete Painlevé type equations.

3.1. A system of quad-equations with $W(B_3)$ symmetry. We construct a system of six l-mKdV equations consistent on Vor$(B_3)$ (a 3-cube), hence showing that the system has $W(B_3)$ symmetry.

The l-mKdV equation is defined by:

$$Q(x_0, x_1, x_3, x_{13}; a_1, a_3) = x_1x_{13} - x_0 x_3 + \frac{a_1}{a_3}(x_0 x_1 - x_3 x_{13}) = 0,$$  \hspace{1cm} (3.5)

where the variables $\{x_0, x_1, x_3, x_{13}\}$ are assigned to the vertices of the quadrilateral $\{X_0, X_1, X_3, X_{13}\}$, and $a_i$ are parameters associated with the edges.

**Proposition 3.1 (I).** A system of six such quad-equations, associated to the two-dimensional faces (2-faces) of the 3-cube (see Figure 3.1.1),

$$Q(x_0, x_1, x_3, x_{13}; a_1, a_3) = 0, \hspace{1cm} (3.6a)$$
$$Q(x_0, x_1, x_2, x_{12}; a_1, a_2) = 0, \hspace{1cm} (3.6b)$$
$$Q(x_0, x_2, x_3, x_{23}; a_2, a_3) = 0, \hspace{1cm} (3.6c)$$
$$Q(x_3, x_{13}, x_{23}, x_{123}; a_1, a_2) = 0, \hspace{1cm} (3.6d)$$
$$Q(x_2, x_{12}, x_{23}, x_{123}; a_1, a_3) = 0, \hspace{1cm} (3.6e)$$
$$Q(x_2, x_{13}, x_{12}, x_{123}; a_2, a_3) = 0, \hspace{1cm} (3.6f)$$

is said to be consistent on the 3-cube if given four initial values $\{x_0, x_1, x_2, x_3\}$ (indicated by black nodes), if $x_{123}$ can be uniquely and consistently determined from the equations (3.6a) - (3.6f).

The natural action of $W(B_3)$ on the index set $S$ of $X_S$ (given by Equation (2.20)) is extended to that on the variables $x_S$ by Equation (3.3). The equations in (3.3) can be obtained from Equation (3.5) by applying the following elements $\{1, s_2, s_1, s_3s_2, s_2s_3s_2, s_1s_2s_3s_2\}$ of $W(B_3)$, respectively.

The quad-equation (3.5) being affine linear means that any one of the four variables can be expressed rationally in terms of the other three, given as initial values. For example we can write

$$x_{13} = \frac{x_0(-a_1 x_1 + a_3 x_3)}{a_3 x_1 - a_1 x_3},$$  \hspace{1cm} (3.7)
Similarly, we can express $x_{12}, x_{23}$ and $x_{123}$ all in terms of $\{x_0, x_1, x_2, x_3\}$.

Let $\mathbb{C}(a;x)$ be the field of rational functions in $a_j (j \in \{1, 2, 3\})$ and $x_i (i \in I = \{0, 1, 2, 3\})$. In order to obtain a representation of $W(B_3)$ on the initial value set $x_i (i \in I = \{0, 1, 2, 3\})$ we need some appropriate algebraic relations among the variables $x_S$ which are associated with the vertices of the 3-cube. To this end, we use the quad-equations associated with the 2-faces of the cube such as expression (3.7). The whole system (3.6) involving eight variables then can be reformulated as a rational system of initial value set $\{x_0, x_1, x_2, x_3\}$, induced from the actions of the symmetry group of the 3-cube, that is $W(B_3)$.

**Definition 3.1.** The actions on the variables $x_S$ and $a_i$ to be associated with the generators of $W(B_3)$ on the initial value set $\{x_0, x_1, x_2, x_3\}$ are defined as:

$$
\begin{align*}
  s_1 &: \{x_1 \leftrightarrow x_2, \quad a_1 \leftrightarrow a_2\}, \\
  s_2 &: \{x_2 \leftrightarrow x_3, \quad a_2 \leftrightarrow a_3\}, \\
  s_3 &: \{x_0 \leftrightarrow x_3, \quad x_1 \rightarrow \frac{x_0(-a_1x_1 + a_3x_3)}{a_3x_1 - a_1x_3}, \quad x_2 \rightarrow \frac{x_0(-a_2x_2 + a_3x_3)}{a_3x_2 - a_2x_3}, \\
  a_1 &\rightarrow -a_1, \quad a_2 \rightarrow -a_2\}.
\end{align*}
$$

**Proposition 3.2.** The associated actions of $s_1, s_2$ and $s_3$ defined as above satisfy the fundamental relations in (2.30) for the simple reflections of $W(B_3)$, corresponding to the Dynkin diagram in Figure 2.3.1. That is, the actions of these generators define a birational representation of $W(B_3)$ on the field $\mathbb{C}(a;x)$ of rational functions. Furthermore, system (3.8) is equivalent to the system of quad-equations (3.6).

**Proof.** It can be checked by direct computation that transformations in (3.8) satisfy the fundamental relations in (2.30). \(\square\)
In what follows, we give a reduction of system (3.6) to a discrete equation of Painlevé type using the relation between Vor($B_3$) and Vor($A_2$) given in Example 2.1.

3.2. Reduction to a $q$-discrete Painlevé equation with $W(A^{(1)}_2)$ symmetry.

**Proposition 3.3.** On system (3.6) apply the map $\phi$ associated with the orthogonal projection $\phi$ defined in (2.24) along $X_{123} = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)$, then

$$
\begin{align*}
\phi(x_S) &= u_S, \quad 1 \leq |S| \leq 2, \\
u_{123} &= \phi(x_{123}) = \phi(x_0) = u_0,
\end{align*}
$$

(3.9)

we have the "periodic condition" (indicated by the dotted line in Figure 3.1.1):

$$
t_1 t_2 t_3 (u_S) = u_{S\cup\{1,2,3\}} = u_S,
$$

(3.10)

which puts the following constraints the edge parameters $a_i$:

$$
t_j(a_i) = \begin{cases} 
    a_i & \text{for } j \neq i, \\
    qa_i & \text{for } j = i,
\end{cases}
$$

(3.11)

where $q \in \mathbb{C}^*$ and $t_j(u_S) = u_{S\cup\{j\}}$ (defined in (2.63)) is the one-step shift in direction $j$ on the $A_2$ weight lattice (see Figure 2.4). System (3.6) is reduced to

$$
\begin{align*}
u_{112} &= \frac{u_0(qa_1 u_1 + a_3 u_{12})}{a_3 u_1 + qa_1 u_{12}}, & (3.12a) \\
u_2 &= \frac{u_1(a_1 u_0 + a_2 u_{12})}{a_2 u_0 + a_1 u_{12}}, & (3.12b) \\
u_{13} &= \frac{u_{12}(a_1 u_1 + a_3 u_0)}{a_3 u_1 + a_1 u_0}. & (3.12c)
\end{align*}
$$

Proof. It can be easily checked that condition (3.9) on system (3.6) leaves constraints (3.10)–(3.11) on the variables and parameters, respectively. Three essential quad-equations (3.12a)–(3.12c) describe the reduced system. For $u_{112}$, we shift $t_1 t_2 = t_{12}$; Equations (3.12a) and (3.12c) come from Equations (3.6b) and (3.6d), respectively. The rest of the equations of the reduced system (3.6) can be obtained from these three equations by shifts under the constraints given in Equations (3.10)–(3.11).

The natural action of $\widetilde{W}(A^{(1)}_2)$ on the index set $S$ of $U_S$ (given by Equation (2.69)) is extended to that on the variables $u_S$ by Equation (3.4). The three equations in (3.12) are associated with the three quadrilaterals around the fundamental simplex $F$ (see Figure 3.1.2). These are used to obtain a representation of $\widetilde{W}(A^{(1)}_2)$ on the initial value set $\{u_0, u_1, u_{12}\}$, which are the vertices of $F$, indicated by the rounded boxes in Figure 3.1.4.

**Proposition 3.4.** The reduced system (3.12) can be formulated as a birational representation of $\widetilde{W}(A^{(1)}_2) = \langle s_0, s_1, s_2, \rho \rangle$ on the three initial values $\{u_0, u_1, u_{12}\}$ by defining the actions associated to the generators as follows.
elements of the relations (2.53) of the reduced System (3.12) has

\[ s_1 : \{u_1 \rightarrow u_2 = \frac{u_1(a_1u_0 + a_2u_{12})}{a_2u_0 + a_1u_{12}}, \quad a_1 \leftrightarrow a_2\}, \]
\[ s_2 : \{u_{12} \rightarrow u_{13} = \frac{u_{12}(a_2u_1 + a_3u_0)}{a_3u_1 + a_1u_0}, \quad a_2 \leftrightarrow a_3\}, \]
\[ s_0 : \{u_0 \rightarrow u_{112} = \frac{u_0(qa_1u_1 + a_3u_{12})}{a_3u_1 + qa_1u_{12}}, \quad a_3 \rightarrow qa_1, \quad a_1 \rightarrow a_3/q\}, \]
\[ \rho : \{u_0 \rightarrow u_1, \quad u_1 \rightarrow u_{12}, \quad u_{12} \rightarrow u_0, \quad a_1 \rightarrow a_2, \quad a_2 \rightarrow a_3, \quad a_3 \rightarrow qa_1\}. \]

The associated actions of \( s_1, s_2, s_0, \rho \) defined as above satisfy the fundamental relations (2.53) of \( \tilde{W}(A_2^{(1)}) \) (corresponding to the Dynkin diagram in Figure 2.3).

Remark 3.1. We note that \( \rho^3 : \{a_1 \rightarrow qa_1, a_2 \rightarrow qa_2, a_3 \rightarrow qa_3\} \), but since the edge parameters \( a_i \) of system (3.12) can be always arranged as pairs of ratios and \( \rho^3 : \{\frac{a_i}{a_j} \rightarrow \frac{a_i}{a_j}\} \) for all \( i, j \in \{1, 2, 3\} \), the defining relation \( \rho^3 = 1 \) in (2.53) is satisfied.

Proof. It can be verified directly that the transformations given in Equation (3.13) satisfy the relations in Equation (2.53). That is the reduced System (3.12) has \( \tilde{W}(A_2^{(1)}) \) symmetry. \( \square \)

Theorem 3.1. Let

\[ f = \frac{u_1}{u_{12}}, \quad g = \frac{u_{12}}{u_0}. \] (3.14)

System (3.12) is equivalent to

\[ g_1 = \frac{(1 + ft)}{fg(f + t)}, \quad f_1 = \frac{(1 + a_1t)}{fg_1(g_1 + at)}, \] (3.15)

where \( t = qa_1/a_3, \quad a = a_3/a_1 \), and \( t_1(f) = f_1, \quad t_1(g) = g_1 \). System (3.12) is a second-order nonlinear ordinary q-discrete equation of Painlevé type with \( \tilde{W}(A_2^{(1)}) \) symmetry.

Proof. We want to find \( f, g \) shifted in the \( t_1 \) direction, that is:

\[ t_1(f) = f_1 = u_{11}/u_{112}, \] (3.16a)
\[ t_1(g) = g_1 = u_{112}/u_1. \] (3.16b)

For \( u_{11} \), shift Equation (3.12a) by \( t_1 \), rearrange, we have

\[ u_{11} = \frac{a_2u_1u_{12} + qa_1u_{12}u_{112}}{qa_1u_1 + a_2u_{112}}. \] (3.17)

We use Equation (3.12a) to express \( u_{112} \) in terms of \( \{u_0, u_1, u_{12}\} \), and finally in terms of \( f \) and \( g \). We have system (3.15), where \( t \) is the independent variable and \( a \) is a parameter of the equation. \( \square \)
Equations (3.19b), (3.19c), (3.19e), and (3.19f) are of $H$ and $\alpha$

The corresponding Dynkin diagram is given in Figure 3.1.5.

satisfy the following relations:

The natural actions of $s_2$, $s_3$ and $w_0$ on $X_S$ are given by the following transformations:

$$s_2 : \{X_2 \leftrightarrow X_3, \quad X_{12} \leftrightarrow X_{13}\},$$

$$s_3 : \{X_0 \leftrightarrow X_3, \quad X_1 \leftrightarrow X_{13}, \quad X_2 \leftrightarrow X_{23}, \quad X_{12} \leftrightarrow X_{123}\},$$

$$w_0 : \{X_0 \to 1/X_1, \quad X_1 \to 1/X_0, \quad X_2 \to 1/X_2, \quad X_{12} \to 1/X_{12}, \quad X_{13} \to 1/X_{123}, \quad X_{123} \to 1/X_{123}\}. $$

System (3.15) is a sub-case of the $\varphi$-discrete Painlevé equation (1.43) with $\widehat{W((A_2 + A_1)^{(1)})}$ symmetry, which has been obtained from a similar type of reduction of a system of quad-equations [16]. In Section 4 we provide an algebro-geometric description of Equation (1.43).

3.3. A system of quad-equations with $W(B_2 + A_1)$ symmetry. In the previous section we have constructed system (3.10) of six l-mKdV equations on the vertices of a 3-cube with $W(B_3)$ symmetry. In [17], it was shown that different quad-equations can be defined consistently on a 3-cube, sometimes referred to as the system on an asymmetric 3-cube.

Remark 3.2. System (3.19) is equivalent to Equations (3.29-3.30) given in [17] with $\delta_2 = \delta_3 = 0$, $\delta_1 = \rho$, and $x_0 \leftrightarrow x_1, x_2 \leftrightarrow x_{12}, x_3 \leftrightarrow x_3, x_{13} \leftrightarrow x_{123}, \beta \to 1/\beta, \gamma \to 1/\gamma$. System (3.19) no longer have $W(B_3)$ symmetry of the 3-cube. In fact we will prove that it has $W(B_2 + A_1)$ symmetry.

An asymmetric cube (drawn by solid black lines in Figure 3.1.3) can be constructed by considering its set of vertices $X_S$ ($S \subseteq \{1, \ldots, 3\}$) associated with a set of edges of two adjacent cubes (drawn by dashed lines in blue). The symmetry group of such an asymmetric 3-cube is $W(B_2 + A_1) = \langle s_2, r_3, w_0 \rangle$, where the generators satisfy the following relations:

$$s_3^2 = s_2^2 = w_0^2 = 1, \quad (s_2 s_3)^4 = 1, \quad (s_2 w_0)^2 = (s_3 w_0)^2 = 1. \quad (3.20)$$

The corresponding Dynkin diagram is given in Figure 3.1.3.

The natural actions of $s_2$, $s_3$ and $w_0$ on $X_S$ are given by the following transformations:

Proposition 3.5 ([1]). Given a $H^4$ type:

$$Q(x_0, x_2, x_3, x_{23}; \beta, \gamma) = x_2 x_3 + x_0 x_2 - \frac{\gamma}{\beta}(x_3 x_{23} + x_0 x_2) = 0, \quad (3.18a)$$

and a $H^6$ type:

$$H(x_0, x_1, x_2, x_{12}; \rho, \beta) = \beta x_2 x_{12} + \beta x_0 x_1 + \rho x_0 x_2 = 0, \quad (3.18b)$$

quad-equations in the ABS classification [1], the following system of quad-equations:

$$\begin{align*}
\frac{x_3}{x_2} &= \frac{\gamma x_0 - \beta x_{23}}{\beta x_0 - \gamma x_{23}}, \quad (3.19a) \\
\frac{x_1}{x_2} &= \frac{x_{12} - \frac{\rho}{\beta}}{x_0 - \frac{1}{\beta}}, \quad (3.19b) \\
\frac{x_{12}}{x_3} &= \frac{x_{123} - \frac{\rho}{\beta}}{x_3 - \frac{1}{\beta}}, \quad (3.19c) \\
\frac{x_{12}}{x_{23}} &= \frac{x_{123} - \frac{\rho}{\gamma}}{x_{23} - \frac{1}{\gamma}}. \quad (3.19f)
\end{align*}$$

is consistent on a 3-cube. Equations (3.19a) and (3.19e) are of $H^4$ type, and Equations (3.19d), (3.19a), (3.19c), and (3.19f) are of $H^6$ type.

$$\begin{align*}
\delta_2 &= \delta_3 = 0, \quad \delta_1 &= \rho, \quad \text{and} \quad x_0 \leftrightarrow x_1, x_2 \leftrightarrow x_{12}, x_3 \leftrightarrow x_3, x_{13} \leftrightarrow x_{123}, \beta \to 1/\beta, \gamma \to 1/\gamma.
\end{align*}$$

System (3.19) no longer have $W(B_3)$ symmetry of the 3-cube. In fact we will prove that it has $W(B_2 + A_1)$ symmetry.

An asymmetric cube (drawn by solid black lines in Figure 3.1.3) can be constructed by considering its set of vertices $X_S$ ($S \subseteq \{1, \ldots, 3\}$) associated with a set of edges of two adjacent cubes (drawn by dashed lines in blue). The symmetry group of such an asymmetric 3-cube is $W(B_2 + A_1) = \langle s_2, r_3, w_0 \rangle$, where the generators satisfy the following relations:

$$s_3^2 = s_2^2 = w_0^2 = 1, \quad (s_2 s_3)^4 = 1, \quad (s_2 w_0)^2 = (s_3 w_0)^2 = 1. \quad (3.20)$$

The corresponding Dynkin diagram is given in Figure 3.1.3.

The natural actions of $s_2$, $s_3$ and $w_0$ on $X_S$ are given by the following transformations:

$$\begin{align*}
s_2 : \{X_2 \leftrightarrow X_3, \quad X_{12} \leftrightarrow X_{13}\}, \\
s_3 : \{X_0 \leftrightarrow X_3, \quad X_1 \leftrightarrow X_{13}, \quad X_2 \leftrightarrow X_{23}, \quad X_{12} \leftrightarrow X_{123}\}, \\
w_0 : \{X_0 \to 1/X_1, \quad X_1 \to 1/X_0, \quad X_2 \to 1/X_2, \quad X_{12} \to 1/X_{12}, \quad X_{13} \to 1/X_{13}, \quad X_{123} \to 1/X_{123}\}.
\end{align*}$$
The actions of $s_2$ and $s_3$ above are the same as those of $W(B_3)$ given in Equation (2.29). The reflection plane associated with the generator $w_0$ is shown as the striped plane in Figure 3.1.3. It can be easily checked that the transformations in (3.21) satisfy the fundamental relations (3.20) of $W(B_2 + A_1)$.

Define the function $x : X_S \to \mathbb{C}$ on the vertices of the asymmetric cube, we write

$$x_S = x(X_S),$$

(3.22)
and the elements \( w \in W(B_2 + A_1) \) act on the variables \( x_S \) by

\[
    w(x_S) = x(wX_S) = x(X_{w,S}) = x_{w,S}.
\]

(3.23)

As in the case of system (3.6) on the symmetric 3-cube, the asymmetric system (3.19) can be constructed from four initial values. Here we have taken the initial value set to be \( \{x_0, x_2, x_{12}, x_{23}\} \) (indicated by the black nodes in Figure 3.1.3 and use the equations in (3.19) to express \( x_1, x_3, x_{13}, x_{123} \) in terms of the initial values as follows. For \( x_1 \) and \( x_3 \) use Equations (3.19b) and (3.19a), respectively. For \( x_{13} \), we use what is called the “tetrahedron equation” (see the tetrahedron outlined by the densely dotted lines in Figure 3.1.3),

\[
    T_H(x_0, x_{12}, x_{13}, x_{23}; \rho, \beta, \gamma)
\]

\[
= \beta^2 \gamma (x_0 x_{12} + x_{13} x_{23}) - \beta \gamma^2 (x_0 x_{13} + x_{12} x_{23}) - (\beta^2 - \gamma^2) \rho x_0 x_{23}.
\]

(3.24)

Equation (3.24) is a consequence of the Equations in system (3.19). It can be obtained by first getting the expressions for \( x_0 \), \( x_2 \) and \( x_3 \) using Equations (3.19c), (3.19b) and (3.19a), respectively; then relation \( x_0 x_2 x_3 = 1 \) implies Equation (3.24). Finally, for \( x_{123} \) we use Equation (3.19d).

The natural action of \( W(B_2 + A_1) \) on the index set \( S \) of \( X_S \) given in Equation (3.21) is extended to that on the variables \( x_S \) by Equation (3.23). System (3.19) is reformulated as a birational representation of \( W(B_2 + A_1) \) on the initial value set \( \{x_0, x_2, x_{12}, x_{23}\} \) by the following proposition.

**Proposition 3.6.** The actions on the initial value set \( \{x_0, x_1, x_{12}, x_{23}\} \) to be associated with the generators of \( W(B_2 + A_1) \) are defined as follows,

\[
    s_2 : \begin{align*}
    &x_2 \to \frac{x_2(x_0 \gamma - \beta x_{23})}{x_0 \beta - \gamma x_{23}}, \\
    &x_{12} \to \frac{-x_0 \beta^2 \gamma x_{12} + \beta \gamma^2 x_{12} x_{23} - x_0 x_{23}(\beta^2 - \gamma^2) \rho}{\beta \gamma (x_0 \gamma + \beta x_{23})}, \\
    &x_{23} \to x_{23},
    \end{align*}
\]

\[
    s_3 : \begin{align*}
    &x_0 \to \frac{x_2(x_0 \gamma - \beta x_{23})}{x_0 \beta - \gamma x_{23}}, \\
    &x_2 \leftrightarrow x_{23}, \\
    &x_{12} \to -\frac{x_2 x_{12} + x_{23} \rho}{x_{23}}, \\
    &x_{23} \to x_{23},
    \end{align*}
\]

\[
    w_0 : \begin{align*}
    &x_0 \to -\frac{x_2 (\beta x_{12} + \rho x_0)}{x_{12} - 1 / x_2}, \\
    &x_2 \to 1 / x_{12}, \\
    &x_{23} \to -\frac{x_{23}}{x_{23} x_{12} + x_{23} x_0 \rho / \gamma},
    \end{align*}
\]

(3.25)

The transformations (3.25) satisfy the fundamental relations in (3.20) for \( W(B_2 + A_1) \).

**Proof.** It can be checked by direct computation that the transformations satisfy the fundamental relations (3.20) of \( W(B_2 + A_1) \). System (3.19) of quad-equations can be generated from the two basic equations (3.18a)–(3.18b) by the actions of the group elements of \( W(B_2 + A_1) \). Equations (3.19a) and (3.19d) can be obtained from Equation (3.18a) by applying the group elements \( \{1, w_0\} \), respectively; Equations (3.19b), (3.19c), (3.19a) and (3.19b) can be obtained from Equation (3.18b) by applying the group elements \( 1, s_3, s_2 \) and \( s_2 s_3 \), respectively.

Now we show that system (3.19) is related by reduction to a discrete Painlevé equations with \( \tilde{W}((A_1 + A_1')^{(1)}) \) symmetry in Sakai’s classification.
3.4. Reduction to a $q$-discrete Painlevé equation with $\tilde{W}((A_1 + A'_1)^{(1)})$ symmetry. On system (3.19) letting

$$x_{123} = x_0$$  \hspace{1cm} (3.26)

impose the following conditions on the variables and parameters,

$$t_1 t_2 t_3 (x_S) = x_{S \cup \{1,2,3\}} = x_S, \quad \rho_1 = q \rho, \quad \beta_2 = q \beta, \quad \gamma_3 = q \gamma, \quad q \in \mathbb{C}^*, \hspace{1cm} (3.27)$$

where $t_j(x_S) = x_{S \cup \{j\}}$ denotes the shift in direction $j$ on the $\tilde{W}((A_1 + A'_1)^{(1)})$ lattice.

In section 3.2 reduction condition (3.9) on system (3.6) of six same quad-
equations corresponds to the orthogonal projection of a symmetric 3-cube with $W(B_3)$ symmetry to a triangular lattice with $\tilde{W}(A_2^{(1)})$ symmetry (see Figures 3.1.1 and 3.1.2). Here for the asymmetric system (3.19), reduction condition (3.26) corresponds to the projection (indicated by the loosely dotted line in Figures 3.1.3) of an asymmetric 3-cube with $W(B_2 + A_1)$ symmetry to a rectangular lattice (see Figure 3.1.4) with $\tilde{W}((A_1 + A'_1)^{(1)})$ symmetry. The extended affine Weyl group $\tilde{W}((A_1 + A'_1)^{(1)})$ is generated by the two sets of generators $\{s_2, s_0\}$ and $\{w_1, w_0\}$ that commute (realised as two sets of orthogonal affine reflection lines): and $\pi$ the Dynkin diagram automorphism (realised as rotation by 180° about the main diagonal of the two adjacent quadrilaterals shaded in blue). In other words, the generators satisfy the following relations:

$$s_0^2 = s_2^2 = (s_0 s_2) \infty = 1, \quad w_0^2 = w_1^2 = (w_0 w_2) \infty = 1,$$

$$s_i w_j = w_j s_i = 1, \quad i = 0, 2, \quad j = 0, 1,$$

$$\pi^2 = 1, \quad \pi w_0 = w_1 \pi, \quad \pi s_2 = s_0 \pi. \hspace{1cm} (3.28)$$

We note that the relation $(ww') \infty = 1$ means that there is no positive integer $N$ such that $(ww')^N = 1$ for transformations $w$ and $w'$. The corresponding Dynkin diagram is given in Figure 3.1.0. The natural action of $w \in \tilde{W}((A_1 + A'_1)^{(1)})$ on $X_S$, points of the rectangular $\tilde{W}((A_1 + A'_1)^{(1)})$ lattice, is extended to that on the variables $x_S$ as defined in Equation (3.23).

The first thing to notice is that the number of initial values of the reduced system becomes three. We choose $\{x_0, x_2, x_{23}\}$, and express $x_{12}$ in terms of these using Equation (3.19) with the reduction condition $x_{123} = x_0$. In fact it is easy to see that any $x$ variable on the lattice in Figure 3.1.4 can be reach from $\{x_0, x_2, x_{23}\}$ (indicated by the rounded boxes) by using the reflections $s_2, s_0, w_1, w_0$ and rotation $\pi$. System (3.19) with constraint (3.27) can then be rewritten as a birational representation of $\tilde{W}((A_1 + A'_1)^{(1)}) = \{s_2, s_0, w_1, w_0, \pi\}$ on the initial value set $\{x_0, x_2, x_{23}\}$. 

\footnote{The ' on $A'_1$ denotes the fact that these are two root systems of type $A_1$ with different root (weight) lengths.}
The transformations given in Equations (3.29)–(3.30) satisfy the fundamental relations of elements and the actions on the parameters are given by Proposition 3.7. The actions to be associated with the generators on the initial value set \( \{x_0, x_2, x_{23}\} \) are given by,

\[
\begin{align*}
s_2 &: \{x_2 \rightarrow \frac{x_2(x_0\gamma - \beta x_{23})}{x_0\beta - \gamma x_{23}}\}, \\
s_0 &: \{x_0 \rightarrow \frac{x_0(\gamma^2 x_2^2 + q\beta x_0 x_{23} + q\beta_0 x_{23})}{\gamma(\beta x_2^2 + x_{23}(\gamma x_0 + \rho x_2))} \}, \\
w_0 &: \{x_0 \rightarrow 1/x_0, \quad x_2 \rightarrow \frac{\gamma x_2}{x_{23}(\gamma x_0 + \rho x_2)} \}, \\
w_1 &: \{x_0 \leftrightarrow 1/x_{23}, \quad x_2 \rightarrow 1/x_2\}, \\
\pi &: \{x_0 \rightarrow 1/x_2, \quad x_2 \rightarrow 1/x_0, \quad x_{23} \rightarrow -\frac{\gamma x_2}{x_{23}(\gamma x_0 + \rho x_2)} \},
\end{align*}
\]

and the actions on the parameters are given by

\[
\begin{align*}
s_2 &: \{\beta, \gamma\} \rightarrow \{\gamma, \beta\}, \\
s_0 &: \{\beta, \gamma\} \rightarrow \{\gamma/q, q\beta\}, \\
w_0 &: \{\beta, \gamma, q\} \rightarrow \{\gamma, \beta, 1/q\}, \\
w_1 &: \{\beta, \gamma, \rho, q\} \rightarrow \{\gamma, \beta, \rho/q, 1/q\}, \\
\pi &: \{\beta, q\} \rightarrow \{q\beta, 1/q\}.
\end{align*}
\]

The transformations given in Equations (3.29)–(3.30) satisfy the fundamental relations of \( \bar{W}(A_1 + A_1^{(1)}) \) in Equation (3.28). In particular, the translational elements \( t_i \) (i = 1, 2, 3) are defined by

\[
t_1 = w_0 w_1, \quad t_2 = \pi s_1 w_0, \quad t_3 = \pi s_0 w_0.
\]

We note that \( t_1 t_2 t_3 : \{\rho \rightarrow q\beta, \beta \rightarrow q\beta, \gamma \rightarrow q\gamma\} \). However, since the parameters in system (3.29) can always be arranged as pairs of ratios so that the condition on the translational elements, \( t_1 t_2 t_3 = 1 \), is satisfied. This in fact corresponds to our reduction condition (3.26).

Proof. The relations between the \( x \) variables on any quadrilateral of the \( \bar{W}(A_1 + A_1^{(1)}) \) lattice in Figure 3.1.4 can be expressed in terms of Equations (3.19a), (3.19b), (3.19c), and their shifted versions under the constraint (3.27). The action of generator \( s_0 \) is \( \{x_0 \rightarrow x_{122}, x_{23} \rightarrow x_{22}\} \). We can express: \( x_{122} \) in terms of the initial values by shifting Equation (3.19a) by \( t_1 t_2 \), \( x_{22} \) by shifting Equation (3.19b) by \( t_2 t_3 \), and finally \( x_{22} \) by shifting Equation (3.19c) by \( t_2 \). The action of generator \( w_0 \) is: \( \{x_0 \rightarrow 1/x_0, x_2 \rightarrow 1/x_{12}, x_{23} \rightarrow 1/x_1\} \), which again can be expressed in terms of the initial values. \( \square \)
The dynamics of system defined by transformations (3.29)–(3.30) are described by the translational elements of \( \tilde{W}((A_1 + A'_1)^{(1)}) \). We show in the following proposition that it is equivalent to a \( q \)-discrete Painlevé equation.

**Proposition 3.8.** Let

\[
    f = \frac{x_0}{x_2}, \quad g = \frac{x_2}{x_{23}},
\]

then the reduced system (3.12) with constraint (3.27) can be rewritten as

\[
    f_1 = -\frac{1}{f} (1 + \frac{\rho}{\beta g_1}), \quad g_1 = -\frac{1}{g} (1 + \frac{\rho}{\gamma f}),
\]

where \( t_1 = w_0w_1 \) describes right horizontal translation on the \( \tilde{W}((A_1 + A'_1)^{(1)}) \) lattice, and \( t_1(\rho) = \rho_1 = q\rho, t_1(f) = f_1, \) and \( t_1(g) = g_1 \). System (3.33) is a second-order nonlinear ordinary \( q \)-discrete equation of Painlevé type with \( \tilde{W}((A_1 + A'_1)^{(1)}) \) symmetry, first obtained in [21].

**Proof.** System (3.33) can be obtained by applying the action of \( t_1 = w_0w_1 \) on \( f \) and \( g \) using the transformations given in Equations (3.29)–(3.30). Alternatively they can be obtained by rewriting Equation (3.19) and Equation (3.19) with constraint (3.27).

**Remark 3.3.** System (3.19) is related to the discrete Painlevé equation with \( \tilde{W}((A_1 + A'_1)^{(1)}) \) symmetry in Lemma 4.2 of [10] by a gauge transformation.

## 4. An algebro-geometric approach

In this section, we explain in detail how to construct a representation of the extended affine Weyl group \( \tilde{W}(A^{(1)}) \) associated with the Cartan matrix \( A^{(1)} \) on the field of rational functions and thereby derive discrete Painlevé equations as representations of \( \tilde{W}(A^{(1)}) \) on the spaces of point configurations in projective space. In particular, we shown how discrete Painlevé equations arise as Cremona transformations from the module (4.2).

In Section 3, we interpreted the dynamics of discrete integrable systems as translations on the weight lattice by the actions of \( \tilde{W}(A^{(1)}) \) on a hyperplane \( H \subset V^{(1)*} \). In this section, we look at the linear actions (in the sense explained in Section 2.4) on the set of simple affine roots \( \{\alpha_0, ..., \alpha_n\} \) in \( V^{(1)} \). To emphasise the fact that transformations are associated with the linear actions of \( \tilde{W}(A^{(1)}) \) on \( V^{(1)} \) we use the following convention

\[
    f.w = w^{-1}.f, \quad \text{for} \quad f \in V^{(1)}.
\]

That is, we use the right action to indicate the fact that we are acting on \( V^{(1)} \), whereas left actions are associated with transformations on the affine hyperplane \( H \subset V^{(1)*} \).

The action of the translational elements on \( V^{(1)} \), which we named “shifted motion”, is given by Equations (2.30), satisfying condition (2.31). This shifted motion on the simple affine roots is associated with the transformations of parameters of the discrete Painlevé equations [28]. For more background on the formulation of the discrete Painlevé equations as birational representations of the Weyl groups, in particular on the discussions of the translational elements of the affine Weyl groups associated with the discrete Painlevé equations see [24, 18, 28]. Here we give the a \((A_2 + A_1)^{(1)}\)-type discrete Painlevé equation as an example.
Let us consider the following module over \( \mathbb{Z} \):
\[
\bigoplus_{i=1}^{2} \mathbb{Z} H_i \bigoplus_{i=1}^{8} \mathbb{Z} E_i,
\]  
(4.2)
where \( \{H_1, H_2, E_1, \ldots, E_8\} \) is a basis of the module, equipped with the symmetric bilinear form \( (, ) \) given by
\[
(H_i, H_j) = 1 - \delta_{ij}, \quad (H_i, E_j) = 0, \quad (E_i, E_j) = -\delta_{ij}.
\]  
(4.3)
We define the root system of type \((A_2 + A_1)^{(1)}\) by
\[
Q((A_2 + A_1)^{(1)}) = \mathbb{Z} \alpha_0 \bigoplus \mathbb{Z} \alpha_1 \bigoplus \mathbb{Z} \alpha_2 \bigoplus \mathbb{Z} \beta_0 \bigoplus \mathbb{Z} \beta_1,
\]  
(4.4)
where
\[
\alpha_0 = H_1 - E_1 - E_4, \quad \alpha_1 = H_2 - E_2 - E_5, \quad \alpha_2 = H_1 + H_2 - E_3 - E_6 - E_7 - E_8,
\]  
(4.5a)
\[
\beta_0 = H_1 + H_2 - E_2 - E_4 - E_6 - E_8, \quad \beta_1 = H_1 + H_2 - E_1 - E_3 - E_5 - E_7.
\]  
(4.5b)
Note that the corresponding Cartan matrices of \( \bigoplus_{i=0}^{2} \mathbb{Z} \alpha_i \) and \( \bigoplus_{i=0}^{1} \mathbb{Z} \beta_i \) are of the \( A_2^{(1)} \)- and \( A_1^{(1)} \)-types, respectively:
\[
(a_{ij})_{i,j=0}^{2} = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix}, \quad (b_{ij})_{i,j=0}^{1} = \begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix},
\]  
(4.6)
where
\[
a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \quad b_{ij} = \frac{2(\beta_i, \beta_j)}{(\beta_j, \beta_j)}.
\]  
(4.7)

4.1. Picard group. We give the algebro-geometric meaning to the module \((4.2)\). Let \((f, g)\) be inhomogeneous coordinates of \( \mathbb{P}^1 \times \mathbb{P}^1 \) and \( P_i \in \mathbb{P}^3 \times \mathbb{P}^3 \) be the following points:
\[
P_1 : (f, g) = (-a_0^{-1}, 0), \quad P_2 : (f, g) = (0, -a_1),
\]  
(4.8a)
\[
P_3 : (f, g) = (0, \infty), \quad P_7 : (f, g; fg) = (0, \infty; -c^2 a_0 a_1 a_2^2),
\]  
(4.8c)
\[
P_4 : (f, g) = (-a_0, \infty), \quad P_8 : (f, g; fg) = (\infty, 0; -c^2 a_0 a_1),
\]  
(4.8f)
where \(a_0, a_1, a_2, c \in \mathbb{C}^*\). Let \( X \) be the rational surface obtained by the blowing-up at the eight points, \( \epsilon : X \to \mathbb{P}^3 \times \mathbb{P}^3 \), and \( E_i = \epsilon^{-1}(P_i), \ i = 1, \ldots, 8, \) and \( H_j, j = 1, 2, \) be the linear equivalence classes of the total transform of the point of the \( i \)-th blow up and the coordinate lines \( f=\)constant and \( g=\)constant, respectively. Then, the module \((4.2)\) is called the Picard group of the rational surface \( X \) and denoted by \( \text{Pic}(X) \):
\[
\text{Pic}(X) = \bigoplus_{i=1}^{2} \mathbb{Z} H_i \bigoplus_{i=1}^{8} \mathbb{Z} E_i.
\]  
(4.9)
Moreover, the bilinear form \( (, ) \) corresponds to the intersection form.
The anti-canonical divisor of $X$, denoted by $-K_X$, corresponds to the null root $\delta$ of the affine root system defined by Equation (2.38) in Section 2.4. It is uniquely decomposed into the prime divisors (or the simple affine roots of Section 2.4):

$$\delta = -K_X = 2H_1 + 2H_2 - \sum_{i=1}^{8} E_i = \sum_{i=0}^{5} D_i,$$

(4.10)

where

$$D_0 = E_6 - E_8, \quad D_1 = H_2 - E_1 - E_6, \quad D_2 = H_1 - E_2 - E_3,$$

(4.11a)

$$D_3 = E_3 - E_7, \quad D_4 = H_2 - E_3 - E_4, \quad D_5 = H_1 - E_5 - E_6.$$

(4.11b)

The submodule of $\text{Pic}(X)$

$$Q(A^{(1)}_5) = \bigoplus_{i=0}^{5} ZD_i,$$

(4.12)

is the root system of type $A^{(1)}_5$ since its corresponding Cartan matrix is of type $A^{(1)}_5$:

$$(d_{ij})_{i,j=0}^5 = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & 0 & -1 & 2
\end{pmatrix},$$

(4.13)

where

$$d_{ij} = \frac{2(D_i|D_j)}{(D_j|D_j)}.$$  

(4.14)

The corresponding Dynkin diagram is given in Figure 4.2 (where we have used the simple roots instead of the simple reflections to denote the nodes). Note that $mH_1 + nH_2 - \sum_{i=1}^{8} \mu_iE_i$ corresponds to a curve of bi-degree $(m, n)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ ($(m, n)$-curve of $(f, g)$) passing through the base points $P_i$ with multiplicity $\mu_i$. Moreover, $E_i - E_j$ corresponds to an exceptional curve inserted into the base point $P_j$, which passes through the base point $P_i$. Therefore, Equation (4.10) means that the $(2, 2)$-curve passing through the eight base points $P_i$ is decomposed into the six curves on the surface $X$. This decomposition, thus the position of the base points, characterises the type of rational surface $X$. Therefore, we refer to the surface $X$ as $A^{(1)}_5$-surface. The most general case is $A^{(1)}_0$-surface and its degenerations are illustrated in Figure 4.1 first given by Sakai [28].

The root systems orthogonal to those associated with surfaces are also important. In the case of $Q(A^{(1)}_5)$, defined in Equation (4.12), the orthogonal root system is $Q((A_2 + A_1)^{(1)})$ defined in Equation (4.13). The cascade of the orthogonal root system corresponding to the degenerations of surfaces in Figure 4.1 was given earlier in Figure 4.1.

4.2. Cremona isometries. We consider the Cremona isometries for the $A^{(1)}_5$-surface $X$. A Cremona isometry is defined by an automorphism of $\text{Pic}(X)$ which preserves

(i): the intersection form on $\text{Pic}(X)$;

(ii): the canonical divisor $K_X$;

(iii): effectiveness of each effective divisor of $\text{Pic}(X)$.
The reflections for simple roots $\alpha_i$, $i = 0, 1, 2$, and $\beta_i$, $i = 0, 1$, and automorphisms of the Dynkin diagram corresponding to the divisors $D_i$, $i = 0, \ldots, 5$, are Cremona isometries and form the extended affine Weyl group of type $(A_2 + A_1)_{1}^{1}$ \cite{28}.

We define the right actions of the reflections $s_i$, $i = 0, 1, 2$, and $w_j$, $j = 0, 1$, respectively across the hyperplane orthogonal to the root $\alpha_i$, $i = 0, 1, 2$, and $\beta_j$, $j = 0, 1$, by the following:

$$v.s_i = v - 2(v|\alpha_i)\alpha_i, \quad v.w_j = v - \frac{2(v|\beta_j)}{\beta_j}\beta_j$$ \hspace{1cm} (4.15)

for all $v \in \text{Pic}(X)$. We also define the right actions of the diagram automorphisms $\text{Aut}(A_5^{1}) = \{\sigma\}$: by

$$(H_1, H_2, E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8).\sigma$$

$$= (H_2, H_1 + H_2 - E_3 - E_6, E_2, E_7, H_2 - E_3, E_5, E_8, H_2 - E_6, E_4, E_1).$$ \hspace{1cm} (4.16)

We can easily verify that the linear actions of

$$\tilde{W}((A_2 + A_1)_{1}^{1}) = \{s_0, s_1, s_2, w_0, w_1, \sigma\}$$

on $\text{Pic}(X)$ satisfy the fundamental relations of the extended affine Weyl group of type $(A_2 + A_1)_{1}^{1}$:

$$s_i^2 = (s_is_{i+1})^3 = 1, \quad w_j^2 = (w_jw_{j+1})^\infty = 1, \quad (s_iw_j)^2 = 1,$$ \hspace{1cm} (4.17a)

$$\sigma^6 = 1, \quad s_i\sigma = \sigma s_{i+1}, \quad w_j\sigma = \sigma w_{j+1}.$$ \hspace{1cm} (4.17b)
We consider the birational action of 

\[ \sigma, \quad r = \sigma^3, \]  

(4.18)

define the right actions of \( \pi \) and \( r \) on the simple roots of \( Q((A_2 + A_1)^{(1)}) \) are given by

\[
(a_0, a_1, a_2, \beta_0, \beta_1). \pi = (a_2, a_0, a_1, \beta_0, \beta_1),  
\quad \pi = (a_2, a_0, a_1, \beta_0, \beta_1), 
\]

(4.19a)

\[
(a_0, a_1, a_2, \beta_0, \beta_1). r = (a_0, a_1, a_2, \beta_1, \beta_0).  
\quad (4.19b)
\]

The corresponding Dynkin diagram is given in Figure 4.3.

The action of \( \tilde{W}((A_2 + A_1)^{(1)}) \) is lifted to the level of the birational action on the variables \( f, g \) and the parameters \( a_i, \, i = 0, 1, 2, \) and \( c \) as stated in the following lemma.

**Lemma 4.1.** Let

\[
g = a_0 a_1 a_2, \quad f_0 = f, \quad f_1 = g, \quad f_2 = \frac{q c^2}{f g}. \quad (4.20)
\]

The left actions of \( \tilde{W}((A_2 + A_1)^{(1)}) \) on parameters are given by

\[
s_i : (a_i, a_{i+1}, a_{i+2}, c) \mapsto (a_i^{-1}, a_i a_{i+1}, a_i a_{i+2}, c), \quad \pi : (a_0, a_1, a_2, c) \mapsto (a_1, a_2, a_0, c),  
\]

\[
w_0 : (a_0, a_1, a_2, c) \mapsto (a_0, a_1, a_2, c^{-1}), \quad w_1 : (a_0, a_1, a_2, c) \mapsto (a_0, a_1, a_2, q^{-2} c^{-1}), \quad r : (a_0, a_1, a_2, c) \mapsto (a_0, a_1, a_2, q^{-1} c^{-1}), 
\]

where \( i \in \mathbb{Z}/3\mathbb{Z}, \) while its actions on variables are given by

\[
s_i(f_{i-1}) = \frac{f_{i-1}(1 + a_i f_i)}{a_i + f_i}, \quad s_i(f_i) = f_i, \quad s_i(f_{i+1}) = \frac{f_{i+1}(a_i + f_i)}{1 + a_i f_i}, \quad \pi(f_i) = f_{i+1}, 
\]

\[
w_0(f_i) = \frac{a_i a_{i+1}(a_{i-1} a_i + a_{i-1} f_i + f_{i-1} f_i)}{f_{i-1}(a_i a_{i+1} + a_i f_{i+1} + f_i f_{i+1})}, 
\]

\[
w_1(f_i) = \frac{1 + a_i f_i + a_i a_{i+1} f_i f_{i+1}}{a_i a_{i+1} f_{i+1}(1 + a_{i-1} f_{i-1} + a_{i-1} a_i f_{i-1} f_i)}, \quad r(f_i) = \frac{1}{f_i}, 
\]

where \( i \in \mathbb{Z}/3\mathbb{Z}. \) Note that for a function \( F = F(a_0, a_1, a_2, c, f, g), \) we let an element \( w \in \tilde{W}((A_2 + A_1)^{(1)}) \) act as

\[
w.F = F(w a_0, w a_1, w a_2, w c, w f, w g), \quad (4.21)
\]

and the parameter \( q \) is invariant under the action of \( \tilde{W}((A_2 + A_1)^{(1)}) \).

**Proof.** We consider the birational action of \( s_0 \) and denote its action by

\[
\bar{f} = s_0(f), \quad \bar{g} = s_0(g), \quad \bar{a}_i = s_0(a_i), \quad i = 0, 1, 2, \quad \bar{c} = s_0(c). \quad (4.22)
\]

We claim

\[
\left( \bar{a}_0, \bar{a}_1, \bar{a}_2; \bar{f}, \bar{g} \right) = \left( a_0^{-1}, a_0 a_1, a_0 a_2; f, g(a_0 + f) \right). \quad (4.23)
\]

Figure 4.3. Dynkin diagram for \( Q((A_2 + A_1)^{(1)}) \)
We use the following notation. Let $\mathcal{X}$ be the rational surface obtained by the blowing-up at the following base points:

\begin{align}
\mathcal{P}_1 &: (f, g) = (-\pi_0^{-1}, 0), \\
\mathcal{P}_2 &: (f, g) = (0, -\pi_1), \\
\mathcal{P}_3 &: (f, g) = (0, \infty), \\
\mathcal{P}_4 &: (f, g) = (-\pi_0, \infty), \\
\mathcal{P}_5 &: (f, g) = (\infty, -\pi_1^{-1}), \\
\mathcal{P}_6 &: (f, g) = (\infty, 0), \\
\mathcal{P}_7 &: (f, g; f\overline{g}) = (0, \infty; -\pi_0^2 \overline{\alpha_0} \overline{\alpha_2}^2), \\
\mathcal{P}_8 &: (f, g; \overline{f}\overline{g}) = (\infty, 0; -\pi_0^2 \overline{\alpha_0} \overline{\alpha_1}),
\end{align}

and $\mathcal{E}_i, i = 1, \ldots, 8$, and $\mathcal{P}_j, j = 1, 2$, be the linear equivalence classes of the total transform of the point of the $i$-th blow up and the coordinate lines $\overline{f}$=constant and $\overline{g}$=constant, respectively. Definition (4.13) gives the following actions:

\begin{align}
H_1.s_0 &= \mathcal{H}_1, \\
H_2.s_0 &= \mathcal{H}_1 + \mathcal{H}_2 - \mathcal{E}_1 - \mathcal{E}_4, \\
E_i.s_0 &= \mathcal{E}_i, i \neq 1, 4,
\end{align}

and

\begin{align}
\mathcal{H}_1.s_0 &= \mathcal{H}_1, \\
\mathcal{H}_2.s_0 &= \mathcal{H}_1 + \mathcal{H}_2 - \mathcal{E}_1 - \mathcal{E}_4, \\
\mathcal{E}_i.s_0 &= \mathcal{E}_i, i \neq 1, 4.
\end{align}

Since the relations

\begin{align}
H_1.s_0 &= \mathcal{H}_1, \\
\mathcal{P}_2.s_0 &= \mathcal{H}_1 + \mathcal{H}_2 - \mathcal{E}_1 - \mathcal{E}_4,
\end{align}

respectively indicate that the $(1, 0)$-curve of $(f, g)$ is paired with the $(1, 0)$-curve of $(\overline{f}, \overline{g})$, and the $(0, 1)$-curve of $(\overline{f}, \overline{g})$ is paired with the $(1, 1)$-curve of $(f, g)$ passing through the base points $P_1$ and $P_\overline{2}$, we can set

\begin{align}
\overline{f}(A_1 f + A_2) + A_3 f + A_4 &= 0, \\
\overline{g}(B_1 g(f + a_0) + B_2 (f + a_0^{-1})) + B_3 g(f + a_0) + B_4 (f + a_0^{-1}) &= 0.
\end{align}

Since the relations

\begin{align}
\mathcal{E}_i &= E_i.s_0, \quad i = 2, 3, 5, 6,
\end{align}

describe

\begin{align}
(\overline{f}, \overline{g}) |_{P_i} &= \left( -\frac{A_3 f + A_4}{A_1 f + A_2} - \frac{B_3 g(f + a_0) + B_4 (f + a_0^{-1})}{B_1 g(f + a_0) + B_2 (f + a_0^{-1})} \right) |_{P_i}, \quad i = 2, 3, 5, 6,
\end{align}

respectively, we obtain

\begin{align}
A_1 = A_4 = B_1 = B_4 = 0, \quad B_2 = -a_0 B_3, \quad \pi_1 = a_0 a_1.
\end{align}

Moreover, from

\begin{align}
\mathcal{E}_i &= E_i.s_0, \quad i = 7, 8,
\end{align}

we get

\begin{align}
(\overline{f}, \overline{g}; \overline{f}\overline{g}) |_{P_i} &= \left( -\frac{A_3}{A_2} f, g \frac{a_0 + f}{1 + a_0 f} - \frac{A_3}{A_2} a_0 + f \overline{f}\overline{g} \right) |_{P_i}, \quad i = 7, 8,
\end{align}
respectively, which lead to
\[ \frac{A_3}{A_2} = -\frac{a_0\bar{a}_0c^2}{c^2}, \quad \bar{a}_2 = a_0a_2. \] (4.34)

Finally, from
\[ \bar{H}_1 - E_4 = E_1s_0, \quad \bar{H}_1 - E_1 = E_4s_0, \] (4.35)
we obtain
\[ J|_{\bar{\rho}_4} = \frac{a_0\bar{a}_0c^2}{c^2}f|_{\rho_1}, \quad J|_{\bar{\rho}_1} = \frac{a_0\bar{a}_0c^2}{c^2}f|_{\rho_4}, \] (4.36)
respectively, which give
\[ \bar{a}_0 = a_0^{-1}, \quad \bar{c} = c. \] (4.37)
Equations (4.28), (4.31), (4.34) and (4.37) provide the claim for \( s_0 (4.23) \). In a similar manner, we can prove the other birational actions. \( \square \)

We can easily verify that the birational actions of \( \tilde{W}(A_2 + A_1) \) given in Lemma (4.1) also satisfy the fundamental relations (4.17).

4.3. Discrete Painlevé equations. It is well known that the translation part of \( \tilde{W}(A_2 + A_1) \), which are translations in \( Q(A_2 + A_1) \), give discrete Painlevé equations \( [28] \). We introduce the translations \( T_i, i = 1, 2, 3, 4, \) defined by
\[ T_1 = \pi s_2s_1, \quad T_2 = \pi s_0s_2, \quad T_3 = \pi s_1s_0, \quad T_4 = rw_0. \] (4.38)
Note that \( T_i, i = 1, 2, 3, 4, \) commute with each other and \( T_1T_2T_3 = 1 \).

Remark 4.1. Observe that by convention (4.1) we see that \( T_1 = t_1^{-1} = s_1s_2^{-1} \), where \( \pi = \rho^{-1} \), and \( t_1 \), defined in Equation (2.69), acts from the left is the translational element associated with translation by the fundamental weight \( h_1 \) in \( V^{(1)} \).

These are translation in the root system \( Q(A_2 + A_1) \):
\[
\begin{align*}
(a_0, a_1, a_2, b_0, b_0).T_1 &= (a_2, a_0, a_1, b_0, b_0).s_2s_1 \\
&= (a_2, a_0 + a_2, a_1, b_0, b_0).s_1 \\
&= (a_0, a_1, a_2, b_0, b_0) + (-1, 0, 0, 0, 0) \delta, \\
(a_0, a_1, a_2, b_0, b_0).T_2 &= (a_0, a_1, a_2, b_0, b_0) + (0, 0, 1, 0, 0) \delta, \\
(a_0, a_1, a_2, b_0, b_0).T_3 &= (a_0, a_1, a_2, b_0, b_0) + (1, 0, -1, 0, 0) \delta, \\
(a_0, a_1, a_2, b_0, b_0).T_4 &= (a_0, a_1, a_2, b_0, b_0) + (0, 0, 0, 1, -1) \delta,
\end{align*}
\] (4.39)
where we have used
\[ \delta = -K_X = a_0 + a_1 + a_2 = b_0 + b_1, \] (4.40)
and also these actions on the parameters are the translational motions as follows:
\[
\begin{align*}
T_1.(a_0, a_1, a_2, c) &= \pi s_2.(a_0a_1, a_1^{-1}, a_2a_1, c) = \pi.(qa_2, q^{-1}a_0, a_1, c) \\
&= (qa_0q^{-1}a_1, a_2, c), \\
T_2.(a_0, a_1, a_2, c) &= (a_0, qa_1, q^{-1}a_2, c), \\
T_3.(a_0, a_1, a_2, c) &= (q^{-1}a_0, a_1, qa_2, c), \\
T_4.(a_0, a_1, a_2, c) &= (a_0, a_1, a_2, qc).
\end{align*}
\] (4.41)
Moreover, their actions on the variables are given by

\[ T_i(f_i) = \frac{q^2}{f_i f_{i-1}} 1 + a_{i-1} f_{i-1}, \quad T_i(f_{i-1}) = \frac{q^2}{f_{i-1} f_i} 1 + a_{i-1} a_{i+1} T_i(f_i), \]  \hspace{1cm} (4.42a)

\[ T_i(f_{i-2}) = \frac{q^2}{T_i(f_i) T_i(f_{i-1})} a_i a_{i+1} f_{i-1} + 1 \quad T_i(f_i) = \frac{1}{1 + a_i f_i} (a_{i-1} f_i + 1) \]  \hspace{1cm} (4.42b)

where \( i \in \mathbb{Z}/3\mathbb{Z} \). The action of \( T_1 \) on \( f_0 \) and \( f_1 \) leads to a system of first-order ordinary difference equations:

\[ G_{n+1} G_n = \frac{q^2}{F_n} \frac{1 + q^n a_0 F_n}{q^n a_0 + F_n}, \quad F_{n+1} F_n = \frac{q^2}{G_{n+1}} \frac{1 + q^n a_0 a_2 G_{n+1}}{q^n a_0 a_2 + G_{n+1}}, \]  \hspace{1cm} (4.43)

where

\[ F_n = T_1^n(f_0), \quad G_n = T_1^n(f_1), \]  \hspace{1cm} (4.44)

which is known as a \( q \)-discrete analogue of Painlevé III equation [21, 28]. In a similar manner, in each of the \( T_2 \) and \( T_3 \) directions, we also obtain \( q \)-\( P_{1III} \). In contrast, the action of \( T_4 \) on \( f_i, i = 0, 1, 2 \) and \( f_0 f_1 f_2 = q^2 \) lead to a \( q \)-discrete analogue of Painlevé IV equation [20]:

\[ F_{n+1} = a_0 a_1 G_n \frac{1 + a_0 a_2 a_1 (a_0 F_n + 1)}{1 + a_0 a_1 (a_0 G_n + 1)} \]  \hspace{1cm} (4.45a)

\[ G_{n+1} = a_1 a_2 H_n \frac{1 + a_0 a_1 (a_1 G_n + 1)}{1 + a_1 G_n (a_2 H_n + 1)} \]  \hspace{1cm} (4.45b)

\[ H_{n+1} = a_2 a_0 F_n \frac{1 + a_1 G_n (a_2 H_n + 1)}{1 + a_2 H_n (a_0 F_n + 1)} \]  \hspace{1cm} (4.45c)

\[ F_n G_n H_n = q^{2n+1} c^2, \]  \hspace{1cm} (4.45d)

where

\[ H_n = T_4^n(f_0), \quad G_n = T_4^n(f_1), \quad H_n = T_4^n(f_2). \]  \hspace{1cm} (4.46)

It is also known that discrete dynamical systems of Painlevé type can be obtained from elements of infinite order of \( \tilde{W}(A_2 + A_1)^{(1)} \) which are not necessarily translations in \( Q((A_2 + A_1)^{(1)}) \) [20, 19]. We introduce the half-translation \( R_1 = \pi^2 s_1 \) satisfying

\[ R_1^2 = T_1. \]  \hspace{1cm} (4.47)

Transformation \( R_1 \) is not a translational motion on \( Q((A_2 + A_1)^{(1)}) \) and parameter space:

\[ (a_0, \alpha_1, \alpha_2, \beta_0, \beta_1).R_1 = (a_0 + \alpha_2 - \delta, \alpha_1 + \alpha_2, \delta - \alpha_2, \beta_0, \beta_1), \]  \hspace{1cm} (4.48a)

\[ R_1.(a_0, a_1, a_2, c) = (a_2 a_0, q^{-1} a_2 a_1, a_0, c), \]  \hspace{1cm} (4.48b)

but by letting \( a_2 = q^{1/2} \) it becomes the translational motion in the parameter subspace:

\[ R_1.(a_0, a_1, c) = (q^{1/2} a_0, q^{-1/2} a_1, c). \]  \hspace{1cm} (4.49)

The action of \( R_1 \):

\[ R_1(f_1) = f_0, \quad R_1(f_0) = \frac{q^2}{f_1 f_0} \frac{1 + a_0 f_0}{a_0 + f_0}. \]  \hspace{1cm} (4.50)
with the condition $a_2 = q^{1/2}$, gives the single second-order ordinary difference equation:

$$F_{n+1}F_{n-1} = \frac{q^{c^2} 1 + q^{n/2} a_0 F_n}{F_n q^{n/2} a_0 + F_n},$$

(4.51)

where

$$F_n = R_1^n(f_0),$$

(4.52)

which is known as a $q$-discrete analogue of Painlevé II equation [27].

Remark 4.2. Let

$$\gamma_0 = -\alpha_0 + \alpha_1, \quad \gamma_1 = 2\alpha_0 + \alpha_2,$$

(4.53)

where $\delta = \gamma_0 + \gamma_1$. The submodule of $\bigoplus_{i=0}^2 \mathbb{Z} \alpha_i$:

$$\mathbb{Z} \gamma_0 \bigoplus \mathbb{Z} \gamma_1$$

is the root system of type $A_1^{(1)}$ since its corresponding Cartan matrix is of type $A_1^{(1)}$:

$$(c_{ij})_{i,j=0}^1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

(4.55)

where

$$c_{ij} = \frac{2(\gamma_i|\gamma_j)}{\langle \gamma_j | \gamma_j \rangle}$$

(4.56)

It is obvious that the root system

$$Q((A_1 + A_1')^{(1)}) = \mathbb{Z} \gamma_0 \bigoplus \mathbb{Z} \gamma_1 \bigoplus \mathbb{Z} \beta_0 \bigoplus \mathbb{Z} \beta_1$$

(4.57)

is also orthogonal to the root system $Q(A_5^{(1)})$. The transformation $R_1$ is not a translation in $Q((A_2 + A_1)^{(1)})$ (see Equation (4.48a)) but a translation in $Q((A_1 + A_1')^{(1)})$:

$$(\gamma_0, \gamma_1, \beta_0, \beta_1) R_1 = (\gamma_0, \gamma_1, \beta_0, \beta_1) + (1, -1, 0, 0) \delta.$$  

(4.58)

5. Conclusion

The purpose of this article is to give a detailed exposition about our approach of associating discrete integrable systems to space-filling polytopes with the Weyl group symmetries. The main outcome of this approach is that the connection between the different discrete integrable systems can be clarified via the Weyl groups. These connections are realised as reductions from higher dimensional system (quad-equations) to lower dimensional system (discrete Painlevé equations). We associate the reductions with the degeneration of the symmetry groups, which are realised as geometric constraints that give rise to deformation/degeneration of the polytopes of these symmetry groups, thus providing a simple way of obtaining and understanding the connections between the different discrete integrable systems. Moreover, we have shown in Section 3 and 4 how the properties of the affine Weyl groups manifest in the two aspects of the discrete Painlevé equations, giving complementary combinatorial and geometrical information about the systems.

Having established the connections between quad-equations and discrete Painlevé equations some immediate consequences follow. First, Lax pairs for discrete Painlevé equations can be derived using Lax pairs of the quad-equations [13, 14]. Second, higher dimensional generalisations of discrete Painlevé equations can be obtained by
generalising the combinatorics of the associated quad-equations and the reduction conditions. Such generalisation of system (3.15) was obtained in [15].

There are still many questions that remain open. For instance, all the discrete Painlevé equations in Sakai’s classification are of the simply laced types (A, D, E). However, recently a nonlinear discrete system of $F_4$ type has been found as a reduction of the Q4 equation in the ABS list [5]. There have not been many such examples, and their natures are not yet well understood.

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