INTEGRAL POINTS ON THE CONGRUENT NUMBER CURVE

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Abstract. We study integral points on the quadratic twists $E_D : y^2 = x^3 - D^2x$ of the congruent number curve. We give upper bounds on the number of integral points in each coset of $2E_D(Q)$ in $E_D(Q)$ and show that their total is $\ll (3.8)^{\text{rank}E_D(Q)}$. We further show that the average number of non-torsion integral points in this family is bounded above by 2. As an application we also deduce from our upper bounds that the system of simultaneous Pell equations $aX^2 - bY^2 = d, bY^2 - cZ^2 = d$ for pairwise coprime positive integers $a, b, c, d$, has at most $\ll (3.6)^{\omega(abcd)}$ integer solutions.

1. Introduction

For any squarefree positive integer $D$, we consider the elliptic curve $E_D : y^2 = x^3 - D^2x.$

We are interested in the set of integral points on the curve, defined as

$$E_D(Z) := \{(x, y) \in \mathbb{Z}^2 : y^2 = x^3 - D^2x\}.$$ 

Given an elliptic curve with Weierstrass equation $y^2 = x^3 + Ax + B$, Siegel [29] proved that there are only finitely many integral points, using techniques from the theory of Diophantine approximation. Baker [2, p.45] gave the first effective bound on the height of integral points: if an integral point $(x, y)$ exists, then

$$|x| \leq \exp((10^6 \max\{A, B\})^{10^6}).$$

Lang [22, p.140] conjectured that the number of integral points on a (quasi)minimal Weierstrass equation of an elliptic curve should be bounded only in terms of its rank. This was proven for elliptic curves with integral $j$-invariant [32, Theorem A] and for elliptic curves with bounded Szpiro ratio [20, Theorem 0.7]. The curves $E_D$ satisfy both of these properties, and more specifically the theorems show that there exists some constant $C$ that is computable but not made explicit, such that

$$\#E_D(Z) \ll C^{\text{rank}E_D(Q)}.$$

In [17], Gross and Silverman presented a bound with explicit constants, which in our case gives $C$ of order $10^9$ in [11]. The bound was improved in [13], which implies $C = 24$ in [11]. In [16, Theorem 2], the bound in [20, Theorem 0.7] was recovered by a different method and with some additional precision. From a more general theorem by Helfgott and Venkatesh [19, Corollary 3.11], we can deduce that

$$\#E_D(Z) \ll C^{\omega(D)}(\log D)^2(1.33)^{\text{rank}E_D(Q)},$$

where $C$ is some absolute constant and $\omega(D)$ denotes the number of distinct prime factors of the integer $D$. We obtain an upper bound in the form of [11] with a smaller and explicit base, specifically for the curves $E_D$.

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Theorem 1.1. We have
\[ \#E_D(\mathbb{Z}) \ll (3.8)^{r \text{rank } E_D(\mathbb{Q})}. \]

Therefore if we expect the rank to be uniformly bounded for all \( E_D(\mathbb{Q}) \) (as has been recently conjectured by various authors), then there would be a squarefree positive integer \( D \) such that \( \#E_D(\mathbb{Z}) \) attains its maximum.

We proceed by partitioning \( E_D(\mathbb{Z}) \) into cosets of \( 2E_D(\mathbb{Q}) \). For any \( R \in E_D(\mathbb{Q}) \), define
\[ Z_D(R) := E_D(\mathbb{Z}) \cap (R + 2E_D(\mathbb{Q})). \]
We obtain an upper bound on the size of each \( Z_D(R) \) in terms of the rank of \( E_D(\mathbb{Q}) \):

**Theorem 1.2.** If \( D \) is sufficiently large and \( R \in E_D(\mathbb{Q}) \) then
\[ \#Z_D(R) < 30 + (1.89)^{r + 19r^{1/3}}, \]
where \( r := \text{rank } E_D(\mathbb{Q}) \).

Since \( \#E_D(\mathbb{Q})/2E_D(\mathbb{Q}) = 2^{2+\text{rank } E_D(\mathbb{Q})} \), Theorem 1.1 is immediate from Theorem 1.2.

Fix \( \epsilon > 0 \). We partition \( Z_D(R) \) into the points with “small” \( x \)-coordinates,
\[ S_D(R) := \{ P \in Z_D(R) : x(P) \leq D^{2(1+r)} \} \]
and the points with “large” \( x \)-coordinates,
\[ L_D(R) := \{ P \in Z_D(R) : x(P) > D^{2(1+\epsilon)} \}, \]
which we will bound by very different techniques.

**Theorem 1.3** (Points with large \( x \)-coordinates). There exists some \( \epsilon > 0 \) such that the following holds for any sufficiently large \( D \) and \( R \in E_D(\mathbb{Q}) \).

1. \( \#L_D(R) \leq 30 \);
2. If the abc conjecture holds, then \( L_D(R) = \emptyset \).

We will complete the proof of Theorem 1.2 by showing that \( \#S_D(R) < (1.89)^{r + 19r^{1/3}} \). If \( x(R) \leq D \) then we can improve the bound to \( \#Z_D(R) \leq 4 \).

**Theorem 1.4** (Cosets with respect to points with very small \( x \)-coordinates).

1. \( Z_D(0) = \emptyset \);
2. \( Z_D((-D,0)) = \{ (-D,0) \} \) and \( Z_D((0,0)) = \{ (0,0) \} \);
3. \( Z_D((D,0)) \) contains \( (D,0) \) and no more than one other pair \( P, -P \in E_D(\mathbb{Z}) \), given by \( x(P) = (2v^2 - 1)D \), where \( v + u\sqrt{D} \) is the fundamental solution of the equation \( v^2 - Du^2 = 1 \);
4. If \( R \in E_D(\mathbb{Q}) \) and \( -D < x(R) < 0 \), then \( Z_D(R) \) contains at most one pair \( P, -P \in E_D(\mathbb{Z}) \), except for the sets
   \[ \{ (-98, \pm 12376), (-1058, \pm 21896) \} \text{ when } D = 1254, \]
   \[ \{ (-5184, \pm 398664), (-7056, \pm 233772) \} \text{ when } D = 7585. \]

The sets considered in Theorem 1.4 are contained in no non-trivial integral points, and the upper bounds obtained in (3), (4) are sharp. Indeed, on the curve \( E_9(\mathbb{Q}) \) of rank 1, the distinct cosets \( Z_6(R) \) of integral points are \( \{ (-6,0) \}, \{ (0,0) \} \), as well as
\[ \{ (-3, \pm 9) \}, \{ (-2, \pm 8) \}, \{ (6,0), (294, \pm 5040) \}, \{ (12, \pm 36) \}, \{ (18, \pm 72) \}. \]

Ordering the curves \( E_D \) with increasing \( D \), Heath-Brown [13, Theorem 1] showed that the moments of the 2-Selmer of \( E_D \) are bounded. Together with [13], this implies that the
average size of \( \mathcal{E}_D(\mathbb{Z}) \) is bounded. The boundedness of the average of \( \#\mathcal{E}_D(\mathbb{Z}) \) was also proved by Alpoge [11], but the upper bound was not explicitly evaluated.

Let \( D_N \) be the set of positive squarefree integers less than \( N \). Define \( \mathcal{T}_D \) to be the set of torsion points on \( \mathcal{E}_D(\mathbb{Q}) \). It is standard that \( \mathcal{T}_D = \{ O, (0,0), (\pm D,0) \} \) (see for example [27, Chapter I, Proposition 17]). Let \( s_{2k}(D) \) denote the \( \mathbb{Z}_2 \)-corank of the 2-power Selmer group of \( \mathcal{E}_D(\mathbb{Q}) \), and \( s_{2k}(D) \) denote the \( \mathbb{F}_2 \)-rank of the \( 2^k \)-Selmer rank of \( \mathcal{E}_D(\mathbb{Q}) \). Then \( s_{2k}(D) = \lim_{k \to \infty} s_{2k}(D) \). Each \( s_{2k}(D) \) and hence also \( s_{2\infty}(D) \) provides an upper bound on the rank of \( \mathcal{E}_D(\mathbb{Q}) \). Heath-Brown [18] notes that it can be derived from results of Cassels [12] and Birch and Stephens [6], that \( s_2(D) \) is even for \( D = 1, 2 \) or 3 mod 8, and odd for \( D = 5, 6 \) or 7 mod 8. An elementary proof of this parity condition was given by Monsky [18, Appendix]. Furthermore, the \( 2^{k+1} \)-Selmer group is computed from the kernel of the Cassels-Tate pairing on the \( 2^k \)-Selmer group. Since the Cassels-Tate pairing is always skew-symmetric [11], we have \( s_{2k}(D) = s_{2k+1}(D) \) mod 2 for all \( k \), so \( s_{2\infty}(D) \) and \( s_2(D) \) are of the same parity. Smith [35, Corollary 1.2] recently claimed that

\[
\{ D \in D_N : s_{2\infty}(D) \geq 2 \} = o(N).
\]

It then follows that for \( s \in \{ 0,1 \} \), we have

\[
\lim_{N \to \infty} \frac{1}{\#D_N} \# \{ D \in D_N : s_{2\infty}(D) = s \} = \frac{1}{2}.
\]

Since \( \text{rank} \mathcal{E}_D(\mathbb{Q}) \leq s_{2\infty}(D) \), asymptotically at most half of the curves are of rank 1, and density 0 of curves are of rank 2 or above. This allows us to focus on curves with rank 0 and 1, hence we can find a better upper bound on the average.

**Theorem 1.5.** We have

\[
\limsup_{N \to \infty} \frac{1}{\#D_N} \sum_{D \in D_N} (\#(\mathcal{E}_D(\mathbb{Z}) \setminus \mathcal{T}_D) \leq 2.
\]

If we further assume the abc conjecture, the upper bound can be improved to 1.

Note that non-torsion integral points come in pairs of \( (x, \pm y) \). The upper bound from Theorem 1.5 comes from the possible existence of a pair of small points in the range \( D^2/(\log D)^{12+\epsilon} < x < D^{2+\epsilon} \), and a pair of large points of size \( x > \exp(\exp(\frac{23}{11}\sqrt{\log D})) \) left from an application of Roth’s Theorem, which we are unable to eliminate on most curves of rank 1.

We expect the order of \( \sum_{D \in D_N} #(\mathcal{E}_D(\mathbb{Z}) \setminus \mathcal{T}_D) \) to be roughly \( N^{1/2} \). To obtain a lower bound, we attempt by counting a subset of integral points. Suppose \( u > v \) are squarefree positive coprime integers. Let \( w \) be the squarefree part of \( u^2 - v^2 \), so \( u, v, w \) are pairwise coprime. If \( D = ww \), then \( (u^2w, u^2v^2w^3\sqrt{u^2 - v^2}) \in \mathcal{E}_D(\mathbb{Z}) \), since \( w(u^2 - v^2) \) is a square by the definition of \( w \). If \( uv(u^2 - v^2) < N \), then \( D \in D_N \), so counting the number of squarefree coprime positive integers \( u, v \) in the range \( v < u < N^{1/4} \), gives a lower bound of \( \gg N^{1/2} \).

Now we give a heuristic on the maximum size of \( \sum_{D \in D_N} #(\mathcal{E}_D(\mathbb{Z}) \setminus \mathcal{T}_D) \). The larger points \( (x, y) \in \mathcal{E}_D(\mathbb{Z}) \) with \( x > D^{2+\epsilon} \) can be removed by assuming the abc conjecture as in Theorem 1.3, so let’s look at \( D \in D_N \) and \( |x| < D^{2+\epsilon} \). If \( x = -j, j - D \), or \( D + j \) for \( 1 \leq j \leq D/2 \), then \( x^3 - D^2x \approx jD^2 \). If \( \frac{3}{2}D < x < N^3 \) then \( x^3 - D^2x \approx x^3 \). Then we expect the number of pairs \( (D, x) \) such that \( x^3 - D^2x \) is a square to be approximately

\[
\sum_{\frac{1}{2}N \leq D < N} \left( \sum_{1 \leq j \leq D/2} \left( \frac{1}{jD^2} \right)^{1/2} + \sum_{D < x < D^3} \left( \frac{1}{x^3} \right)^{1/2} \right) \ll N^{1/2}.
\]
To prove Theorem 1.2, we bound \(\#S_D(R)\) and \(\#L_D(R)\) separately. We prove that \(\#S_D(R)\) is bounded above by

\[
\# \left\{ P \in R + 2E_D(Q) : \hat{h}(P) \leq 2(1 + \epsilon) \log D + O(1) \right\},
\]

where \(\hat{h}\) denotes the canonical height. Then viewing \(E_D(Q)\) as an \(r\)-dimensional Euclidean space, we apply sphere packing bounds to get an upper bound of \((1.89)^{r+19r^{1/3}}\) after fixing some appropriate \(\epsilon\).

On the other hand, we show that \(\#L_D(R)\) is bounded by some constant depending only on \(\epsilon\). Assume \(x(R) > D\) and \(R \notin T_D + 2E_D(Q)\), otherwise the result follows from Theorem 1.4. We first prove that points in \(L_D(R)\) obey a gap principle. Then, for points with larger heights in \(L_D(R)\), we apply Roth’s theorem in a way that is similar to a classical argument of Siegel’s Theorem, which also appeared in Alpoge’s work [1].

Let \(P = 2Q + R \in Z_D(R)\) and \(x(P)\) is large, \(P\) is close to the point at infinity. Let \(K\) be the minimal number field containing the \(x\)-coordinates of all points in \(\frac{1}{2}E_D(Q)\). If \(4S = R\) and \(2\hat{Q} = Q\), where \(\hat{Q}, S \in E_D(\overline{Q})\), then \(S\) and \(\hat{Q}\) are close together. Making this precise, we can show that \(x(\hat{Q})\) gives a \(K\)-approximation to \(x(S)\) with exponent close to 8. Roth’s theorem show that there are finitely many such \(\hat{Q}\). In [1], large integral points of the form \(P = 3Q + R\) were considered, where \(Q, R\) are rational points on a general elliptic curve. The main difference of our approach is that we apply Roth’s theorem over \(K\) instead of \(Q\). Given a class in \(E_D(Q)/nE_D(Q)\), the exponents of the \(Q\)-approximations obtained from the argument in [1] are close to \(\frac{1}{2}n^2\). If we had taken \(n = 2\), the exponent would be just under 2 which would not be large enough to apply Roth’s theorem. Applying the argument over \(K\) instead gives a large enough exponent.

2. Applications to other Diophantine equations

Given positive integers \(a, b, c\), Bennett [3, Theorem 1.2] proved that there exists at most one set of three consecutive integers of the form \(cZ^2, bY^2, aX^2\). In other words, the simultaneous equations

\[
aX^2 - bY^2 = 1, \quad bY^2 - cZ^2 = 1, \quad (X, Y, Z) \in \mathbb{Z}^3_{>0}
\]

possess at most one solution. We can ask a more general question replacing the 1 in the equations with an integer \(d\).

Theorem 2.1. Let \(a, b, c, d\) be pairwise coprime positive integers and set \(D = abcd\). Then for any sufficiently large \(D\), the system of equations

\[
aX^2 - bY^2 = d, \quad bY^2 - cZ^2 = d
\]

has at most \(15 + (1.89)^{r+19r^{1/3}} \leq 15 + (3.58)^{\omega(D)+12\omega(D)^{1/3}}\) solutions \((X, Y, Z) \in \mathbb{Z}^3_{>0}\), where \(r := \text{rank } E_D(Q)\).

A different upper bound on the number of solutions to the system [4] depending only on \(b\) and \(d\), but involving implicit constants, can be deduced from [16, p.41].

We prove Theorem 2.1 by relating the problem to our result in Theorem 1.2. If we take \(D = abcd\) and \(x = ac(bY)^2\), then \(x - D = ab(cZ)^2\) and \(x + D = bc(aX)^2\). Therefore \((ac(bY)^2, (abc)^2XYZ) \in E_D(Z)\). The image of such a point under the injective homomorphism

\[
\theta : E_D(Q)/2E_D(Q) \to Q/(Q^*)^2 \times Q/(Q^*)^2 \times Q/(Q^*)^2
\]
given at non-torsion points by

\[(x, y) \mapsto (x - D, x, x + D),\]
is \((ab, ac, bc)\). If \(P\) and \(R\) are both integral points on \(E_D\) that correspond to solutions to (3), then \(P - R \in 2E_D(\mathbb{Q})\). Moreover, \(x(P) > 0\) and \(b^2 \mid x(P)\). Theorem 2.1 is a corollary of Theorem 1.2 as \(\pm P\) corresponds to the same solution for (3).

More general forms of simultaneous Pell equations have been studied previously. For nonzero integers \(a_1, a_2, b_1, b_2, u, v\), let \(N(a_1, a_2, b_1, b_2, u, v)\) denote the number of solutions to the system of equations

\[
a_1X^2 - b_1Y^2 = u, \quad b_2Y^2 - a_2Z^2 = v
\]
in positive integers \(X, Y, Z\) such that \(\gcd(X, Y, Z, u, v) = 1\). Theorem 2.1 provides an upper bound to \(N(a, c, b, b, d, -d)\), where \(a, b, c, d\) are pairwise coprime positive integers. Transforming the equations (3) by \(X \mapsto aX\) and \(Z \mapsto cZ\), we get \(N(a, c, b, b, d, -d) = N(1, 1, a, bc, ad, -cd)\). Assuming \(a, b\) are distinct positive integers and \(-av \neq bu\), Bennett [3] Theorem 2.1] showed that

\[
N(1, 1, a, b, u, v) \leq 2^{\min\{\omega(u), \omega(v)\}} \log(|u| + |v|).
\]

This implies \(N(a, c, b, b, d, -d) \leq 2^{N(a, c, b, b, d, -d)}\). Bugeaud, Levesque and Waldschmidt [10] Théorème 2.2] gave the bound

\[
N(a_1, a_2, b_1, b_2, u, v) \leq 2 \pm 2^{3996(\omega(a_1a_2uv)+1)}.
\]

Translating to our case, this gives an upper bound \(N(a, c, b, b, d, -d) \leq 2 \pm 2^{3996(\omega(acd^2)+1)}\).

Theorem 2.1 can also provide an upper bound to a different Diophantine equation. In 1942, Ljunggren [24] showed that for a fixed integer \(d\), the equation

\[
X^4 - dY^2 = 1, \quad (X, Y) \in \mathbb{Z}_+^2
\]
has at most two solutions, through a study of units in certain quadratic and biquadratic fields. More recently, Bennett and Walsh [5] used the theory of linear forms in logarithms of algebraic numbers, to show that for squarefree positive integers \(b, d \geq 2\), the equation

\[
b^2X^4 - dY^2 = 1, \quad (X, Y) \in \mathbb{Z}_+^2
\]
has at most one solution. We prove the following as a corollary to Theorem 2.1:

**Theorem 2.2.** Let \(A, B, C\) be pairwise coprime positive squarefree integers. Then there are \(\ll 2^{(ABC)^2}\) integral solutions \((X, Y)\) to

\[
A^2X^4 - BY^2 = C^2.
\]

**Proof.** Let \(g := \gcd(X, C)\). Observe that

\[
Ag \left( \frac{X}{g} \right)^2 - \frac{C}{g} \left( \frac{X}{g} \right)^2 + \frac{C}{g} = B \left( \frac{Y}{g} \right)^2.
\]
The factors on the left hand side have common factor 1 or 2. Therefore we can write

\[
Ag \left( \frac{X}{g} \right)^2 - \frac{C}{g} = B_1Y_1^2 \text{ and } Ag \left( \frac{X}{g} \right)^2 + \frac{C}{g} = B_2Y_2^2,
\]
where \(B_1\) and \(B_2\) are positive integers such that \(B_1B_2 = B\) or \(4B\), and \(Y_1Y_2g = Y\). Now applying Theorem 2.1, the system of equations (4) has \(\ll 2^{(ABC)^2}\) solutions. There are \(2^{(C)}\) choices of \(g \mid C\) and \(\ll 2^{(B)}\) choices of pairs \((B_1, B_2)\). This proves Theorem 2.2. □
2.1. The abc conjecture. In Theorem 1.3 if we are allowed to assume the abc conjecture, we can show that there exists some $\epsilon$ (determined by the conjecture) such that the set $\mathcal{L}_D(R)$ is empty when $D$ is sufficiently large. The abc conjecture states that for every $\epsilon > 0$, for any pairwise coprime positive integers $a, b, c$, with $a + b = c$, we have

$$c \ll_\epsilon \prod_{p\mid abc} p^{1+\epsilon}.$$ 

Suppose that $(x, y) \in \mathcal{E}_D(\mathbb{Z})$ and $x > 0$. Let $g = \gcd(x, D)$. Dividing $y^2 = x^3 - D^2x$ by $xg^2$ and rearranging, we have

$$\left(\frac{D}{g}\right)^2 + \frac{y^2}{xg^2} = \left(\frac{x}{g}\right)^2.$$ 

Assuming the abc conjecture,

$$(5) \quad \left(\frac{x}{g}\right)^2 \ll_\epsilon \prod_{p\mid \left(\frac{D}{g}\right)^2 \left(\frac{y}{xg^2}\right)^2} p^{1+\epsilon}.$$ 

If $p \mid \left(\frac{D}{g}\right)^2 \left(\frac{y}{xg^2}\right)^2$, then $p \mid \left(\frac{D}{g}\right)^2 \left(\frac{y}{xg^2}\right)^2 = \frac{Dy^2}{g^4}$. By construction $g \mid D$ and $g^3 \mid y^2$. Since $g$ is squarefree, so $g^2 \mid y$. Therefore $p \mid \frac{Dy}{g^3}$. Putting this back to (5),

$$\left(\frac{x}{g}\right)^2 \ll_\epsilon \left(\frac{Dy}{g^2}\right)^{1+\epsilon} < \left(\frac{Dx^{3/2}}{g^2}\right)^{1+\epsilon}.$$ 

Then for $\epsilon < \frac{1}{15}$, since $g \leq D$,

$$x \ll_\epsilon \left(\frac{D^{2(1+\epsilon)}}{g^{4\epsilon}}\right)^{1-3\epsilon} \leq D^{2\left(\frac{1+\epsilon}{1-3\epsilon}\right)} < D^{2(1+5\epsilon)}.$$ 

This proves the last assertion in Theorem 1.3

3. Height estimates

Notice that if $(x, y) \in \mathcal{E}_D(\mathbb{Q})$, then either $x \geq D$ or $-D \leq x \leq 0$. For $\alpha \in \overline{\mathbb{Q}}$, define height functions $H(\alpha) := \prod_v \max\{1, |\alpha|_v\}$ and $h(\alpha) = \log H(\alpha) = \sum_v \log^+ |\alpha|_v$, where $v$ is taken over the set of places of $\mathbb{Q}(\alpha)$ and $\log^+$ is a function on the positive real numbers, defined as $\log^+ t = \max\{0, \log t\}$. For any point $P \in \mathcal{E}_D(\mathbb{Q})$, define $H(P) = H(x(P))$, denote the (Weil) height by $h(P) := h(x(P))$ and the canonical height by

$$\hat{h}(P) := \lim_{n \to \infty} h(nP) \quad \[1\]$$

Lemma 3.1. Let $P \in \mathcal{E}_D(\mathbb{Q})$ be a non-torsion point. Write $x(2P) = \frac{r}{s}$, where $r$ and $s$ are coprime integers and $s > 0$. Then $\gcd(r, D) = 1$.

**Proof.** Suppose $P \in \mathcal{E}_D(\mathbb{Q})$, then $\theta(2P) = (1, 1, 1)$, so write

$$x(2P) = \frac{r^2}{s^2}, \quad x(2P) - D = \frac{u^2}{v^2},$$

where $r, s, u, v \in \mathbb{Z}$ and $\gcd(r, s) = \gcd(u, v) = 1$. Combining gives

$$r^2v^2 - u^2s^2 = Dv^2s^2.$$ 

\[1\]This is sometimes defined with an extra factor of $\frac{1}{2}$ in literature.
We see that \( v = s \), since \( \gcd(r, s) = \gcd(u, v) = 1 \). Rewriting
\[
r^2 - u^2 = Ds^2.
\]
Since \( \gcd(r, s) = \gcd(u, v) = 1 \) and \( D \) is squarefree, \( \gcd(r, D) = 1 \).

We prove for points on \( E_D(\mathbb{Q}) \) that the Weil height and the canonical height are close together.

**Lemma 3.2.** Let \( P = (x, y) \in E_D(\mathbb{Q}) \setminus \{\mathcal{O}, (0, 0)\} \). Write \( x = \frac{u}{v} \), where \( r \) and \( s \) are coprime integers and \( s > 0 \). If \( x \geq D \), then

\[
- \log |\gcd(r, D)| - 2 \log 2 \leq \hat{h}(P) - h(P) \leq - \log |\gcd(r, D)| + \frac{2}{3} \log 2.
\]

If \(-D \leq x < 0\), then

\[
\log \left| \frac{D}{\gcd(r, D)} \right| - \log^+ |x| - 2 \log 2 \leq \hat{h}(P) - h(P) \leq \log \left| \frac{D}{\gcd(r, D)} \right| - \log^+ |x| + \frac{2}{3} \log 2.
\]

In particular,

\[
- 2 \log 2 \leq 4\hat{h}(P) - h(2P) = \hat{h}(2P) - h(2P) \leq \frac{2}{3} \log 2.
\]

**Proof.** Focusing on the \( h(2^n P) \) terms in the limit defining \( \hat{h}(P) \), we can express the canonical height as a telescoping series

\[
\hat{h}(P) = h(P) - \sum_{n=0}^{\infty} \frac{1}{4^n} \left( h(2^n P) - \frac{1}{4} h(2^{n+1} P) \right).
\]

Consider a point \( P \in E_D(\mathbb{Q}) \setminus \{\mathcal{O}, (0, 0)\} \). Write \( x(P) = \frac{u}{v} \), where \( r, s \) are coprime integers and \( s > 0 \). Then

\[
x(2P) = \frac{(r^2 + Ds^2)^2}{4rs(r - Ds)(r + Ds)}.
\]

If an odd prime \( p \) divides both \( (r^2 + Ds^2)^2 \) and \( 4rs(r - Ds)(r + Ds) \), then since \( \gcd(r, s) = 1 \), \( p \) divides both \( r \) and \( D \). If \( r^2 + Ds^2 \) is even, either \( r, D \) are all odd, or \( r, D \) are even and \( s \) is odd. The first case implies that \( r^2 + Ds^2 \equiv 2 \mod 8 \), so \( 4 \parallel (r^2 + Ds^2)^2 \). The second case note that \( D \) is squarefree so \( 2 \parallel D \). If \( 2 \parallel r \) we have \( 4 \cdot 2^4 \parallel (r^2 + Ds^2)^2 \), otherwise the \( 2^4 \parallel (r^2 + Ds^2)^2 \).

Therefore

\[
\gcd\left((r^2 + Ds^2)^2, 4rs(r - Ds)(r + Ds)\right) = (\gcd(r, D))^4 \text{ or } 4(\gcd(r, D))^4.
\]

Since \( x(2P) > D \), we have

\[
h(2P) = \log(r^2 + D^2 s^2) - \log \gcd\left((r^2 + Ds^2)^2, 4rs(r - Ds)(r + Ds)\right)
\]

\[
= 2 \log(r^2 + D^2 s^2) - 4 \log \gcd(r, D) - \mathbb{1}_{\{s \text{ odd}\}} \mathbb{1}_{\{\text{ord}_2 r = \text{ord}_2 D\}} 2 \log 2.
\]

We first prove (10). Suppose \( x(P) \geq D \). Then

\[
h(P) - \frac{1}{4} h(2P) = -\frac{1}{2} \log \left( 1 + \frac{D^2 s^2}{r^2} \right) - \log \gcd(r, D) + \mathbb{1}_{\{s \text{ odd}\}} \mathbb{1}_{\{\text{ord}_2 r = \text{ord}_2 D\}} 2 \log 2.
\]

Apply (11) to (9). Then

\[
0 < \log \left( 1 + \frac{D^2 s^2}{r^2} \right) < \log 2.
\]
Writing \( x(2^n P) = \frac{r(2^n P)}{s(2^n P)} \) in lowest term, we know from Lemma 3.1 that \( \gcd(r(2^n P), D) = 1 \) for any \( n \geq 1 \). The conditions \( 2 \nmid s \) and \( \ord_2 r = \ord_2 D \) can only hold simultaneously at most once in the sequence \( 2^n P \). For if \( s, r, D \) are all odd, then subsequent terms would have even \( s \). On the other hand, since the \( x \)-coordinates of double points must be squares, \( 2 \mid r \) can only happen in the first term. Noting that \( \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{4}{3} \), we get (9).

For (7), suppose instead \(-D < x(P) < 0\). Then from (10), we have

\[
\hat{h}(P) - \frac{1}{4} h(2P) = 1_{\{r > s\}} \log |x| - \frac{1}{2} \log \left(1 + \frac{r^2}{D^2 s^2}\right) - \log \left|\frac{D}{\gcd(r, D)}\right| + 1_{\{s \text{ odd}\}} 1_{\{\ord_2 r = \ord_2 D\}} 2 \log 2.
\]

Apply this to (9). Similar to the argument for (6), but here instead

\[
0 < \log \left(1 + \frac{r^2}{D^2 s^2}\right) < \log 2,
\]

we get (7).

Finally (8) follows from (11) and Lemma 3.1.

Estimates equivalent to (8) were obtained in [9, Section 2] by analysing the local height functions specifically for \( E_D \). The inequalities (6), (7) with larger constant terms can be obtained via a study of local heights by applying theorems for general elliptic curves [34, Theorem 4.1, Theorem 5.4], and [33, Theorem 5.2].

For general algebraic points on \( E_D \), we obtain the following estimate by applying [34, Equation(3)], noting that the discriminant of \( E_D \) is \( \Delta_D = (2D)^6 \) and \( j \)-invariant is 1728.

**Lemma 3.3.** Any \( P \in E_D(\mathbb{Q}) \) satisfies

\[
|\hat{h}(P) - h(P)| < \log D + 4.6.
\]

Since \( \hat{h}(2P) = h(2P) + O(1) \) by (8), we have

\[
\hat{h}(2P) = h(2P) - 2 \log 2 \geq \log D - 2 \log 2.
\]

Therefore for any \( P \in E_D(\mathbb{Q}) \setminus T_D \),

\[
\hat{h}(P) \geq \frac{1}{4} \log D - \frac{1}{2} \log 2.
\]

The equation (14) is a version of Lang’s conjecture, which says that the canonical height of a non-torsion point on an elliptic curve should satisfy

\[
\hat{h}(P) \gg \log |\Delta|,
\]

where \( \Delta \) is the discriminant of the elliptic curve. This conjecture was proven for elliptic curves with integral \( j \)-invariant [30], for elliptic curves which are twists [31], and for elliptic curves with bounded Szpiro ratio [20]. The curves \( E_D \) are in all three of these categories, as remarked in [9]. The bound (14) for curves \( E_D \) with the explicit constant factor \( \frac{1}{4} \) was first given in [9, (11)].

**4. Bounding small points via spherical codes**

In this section we prove the following lemma, which gives the upper bound of \( \#S_D(R) \) for Theorem 1.2.
Lemma 4.1. Suppose $R \in E_D(Q)$ with $x(R) > D$. Let $\epsilon < \frac{1}{650}$. Then for any sufficiently large $D$ we have

$$\#\{P \in R + 2E_D(Q) : \hat{h}(P) \leq 2(1 + \epsilon) \log D\} < (1.89)^{r+19r^{1/3}}.$$  

We first show how Lemma 4.1 implies the required upper bound of $\#S_D(R)$. This follows from the claim that  

$$S_D(R) \subseteq \{P \in R + 2E_D(Q) : \hat{h}(P) \leq 2(1 + \epsilon + o(1)) \log D\}.$$  

Suppose $P \in S_D(R)$, we need to show that $h(P) \leq 2(1 + \epsilon) \log D$ implies $\hat{h}(P) \leq 2(1 + \epsilon) \log D + O(1)$. If $x(P) \geq D$, from (6) we have $h(P) \leq h(P) + O(1) \leq 2(1 + \epsilon) \log D + O(1)$. If $-D \leq x(P) \leq 0$, from (7) we have $\hat{h}(P) \leq h(P) + \log D + O(1) \leq 2 \log D + O(1)$.

We now turn to the proof of Lemma 4.1. We know that the canonical height of equation (13). Viewing $R + 2E_D(Q)$ as a Euclidean space $\mathbb{R}^r$ of dimension $r$, we can bound the number of points by the maximum number of spheres of radius $\frac{1}{2}\sqrt{\log D} + O(1)$ with centres lying inside a sphere $S_R^{r-1}$ of radius $R = \sqrt{2(1 + \epsilon) \log D}$. This is similar to the covering argument used in [8] to obtain bounds on the number of rational points with bounded height on elliptic curves with full rational 2-torsion.

A spherical code in dimension $r$ with minimum angle $\theta$ is a set of points on the unit sphere $S_1^{r-1}$ in $\mathbb{R}^r$ with the property that no two points subtend an angle less than $\theta$ at the origin. Let $A(r, \theta)$ denote the greatest size of such a spherical code.

We can obtain an upper bound in terms of the function $A$ via a classical argument (see for example the proof of (2.1) in [14]). Project the sphere centres in $S_R^{r-1}$ onto the upper hemisphere of $S_R^r$ orthogonally to the hyperplane. The projections of the sphere centres are still at least distance $\sqrt{\log D + O(1)}$ apart, and thus separated by angles of at least $\theta$, where $\sin \frac{\pi}{2} = \frac{1}{2\sqrt{2(1 + \epsilon)}} - o(1)$. Therefore the number of small points is bounded above by $\leq A(r + 1, \theta)$.

4.1. For large dimensions. Kabatiansky and Levenshtein proved the following upper bound on $A(r, \theta)$.

Theorem 4.2 ([21] (52)). Let $r \geq 3$ and $\alpha = \frac{r - 3}{2}$, and let $t_k^\alpha$ be the largest root of

$$P_k^\alpha(t) = \frac{1}{2k} \sum_{i=0}^{k} \binom{k + \alpha}{i} \binom{k + \alpha}{k - i} (t + 1)^i (t - 1)^{k-i}.$$  

Take any $k$ such that $\cos \theta \leq t_k^\alpha$. Then

$$A(r, \theta) \leq \frac{4}{1 - t_{k+1}^\alpha} \binom{k + r - 2}{r}.$$  

From the proof of [21] Lemma 4], we know that

$$\tau_k \leq \frac{2\pi^{2/3}}{((k + \alpha)(k + \alpha + 1)\tau_k)^{1/3}} \leq t_k^\alpha \leq \tau_k,$$  

where

$$\tau_k = \sqrt{1 - \frac{\alpha^2 - 1}{(k + \alpha)(k + \alpha + 1)}}.$$
Therefore if we take $k$ such that
\begin{align*}
\cos \theta & \leq \tau_k - \frac{2\pi^{2/3}}{((k + \alpha)(k + \alpha + 1)\tau_k)^{1/3}},
\end{align*}
and since $t_{k+1}^\alpha \leq \tau_{k+1}$, we have
\begin{align*}
A(r, \theta) & \leq \frac{4}{1 - \tau_{k+1}} \left( \frac{k + r - 2}{r} \right).
\end{align*}
Take $\theta$ such that $\sin \frac{\theta}{2} = \frac{1}{2\sqrt{2(1+r)}} - o(1)$. Let
\begin{align*}
N & = \frac{1 - \sin \theta}{2\sin \theta} = \frac{2(1 + \epsilon)}{\sqrt{r} + 8\epsilon} - \frac{1}{2} + o(1),
\end{align*}
so that
\begin{align*}
\tau_k & \to \sqrt{1 - \frac{1}{(2N + 1)^2}}
\end{align*}
as $k \to \infty$ and $\frac{k}{r} \to N$.
Fix some
\begin{align*}
C^3 & > 16\pi^2 N(N + 1)(2N + 1)^5
\end{align*}
Take $k - 2 = \lfloor rN + Cr^{1/3} \rfloor$, so (15) is satisfied for large enough $r$. By Stirling’s formula, we have
\begin{align*}
\left( \frac{k + r - 2}{r} \right) & \leq \frac{e}{2\pi r^{1/2}} \frac{(k + r - 2)^{k + r - \frac{3}{2}}}{(k - 2)^{k - 2 + \frac{3}{2}}} \leq \frac{e}{2\pi r} \left( \frac{1 + \frac{1}{N}}{N} \right)^{Cr^{1/3} + \frac{3}{2}} \left( \frac{(1 + N)^{1 + N}}{NN} \right)^{r}
\end{align*}
for large enough $r$. Therefore for large enough $r$, we have the upper bound
\begin{align*}
A(r, \theta) & < \left( \frac{(1 + N)^{1 + N}}{NN} \right)^{r + \frac{\log(1 + N) - \log N}{(1 + N)\log(1 + N) - N\log N}} Cr^{1/3},
\end{align*}
taking some small $C$ in the range (16).
We can now prove Lemma 4.1 for $r \geq 2000$. Take $\epsilon < \frac{1}{2500}$ and $C = \frac{160}{25}$, so (16) is satisfied. Then we can rewrite the bound in (17) as $A(r, \theta) < (1.89)^{r + 10r^{1/3}}$.

4.2. For small dimensions. To prove Lemma 4.1 it remains to check the same bound holds for $r < 2000$. The two following bounds, obtained respectively by Rankin and Shannon, are weaker asymptotically when $r \to \infty$ but are better bounds for small $r$.

**Theorem 4.3** ([26] Theorem 2). If $0 < \theta < \frac{\pi}{4}$ and $\sin \beta = \sqrt{2} \sin \theta$, then
\begin{align*}
A(r, \theta) & \leq \frac{\sqrt{\pi} \Gamma \left( \frac{r - 1}{2} \right) \sin \beta \tan \beta}{2 \Gamma \left( \frac{r}{2} \right) \int_0^\beta \sin^{-2} x (\cos x - \cos \beta) dx}
\end{align*}
\begin{align*}
& \leq \frac{2\sqrt{\pi} \Gamma \left( \frac{r + 3}{2} \right) \cos \beta}{\Gamma \left( \frac{r}{2} \right) \sin^{-1} \beta(1 + \frac{3}{r + 3} \tan^2 \beta)} \sim \sqrt{\frac{1}{2}\pi r^3 \cos \theta}{\sin^{-1} \theta}. \sim \frac{1}{\sqrt{2} \sin \theta r^{-1}}.
\end{align*}

**Theorem 4.4** ([28] (21),(27)). Suppose $0 < \theta < \frac{\pi}{2}$. Then
\begin{align*}
A(r, \theta) & \leq \frac{\sqrt{\pi} \Gamma \left( \frac{r - 1}{2} \right)}{\Gamma \left( \frac{r}{2} \right) \int_0^\theta \sin^{-2} x dx} \leq \frac{2\sqrt{\pi} \Gamma \left( \frac{r - 1}{2} \right) \cos \theta}{\Gamma \left( \frac{r}{2} \right) \sin^{-1} \theta(1 + \frac{1}{r \tan^2 \theta})} \sim \sqrt{\frac{2\pi r}{\sin^{-1} \theta}}.
\end{align*}
Evaluating the bounds in Theorems 4.3 and 4.4 for $r < 2000$ proves Lemma 4.1 in those cases.
5. Repulsion between medium points

Suppose $P = (X, Y)$ and $R = (x, y)$ are integral points in the same coset of $2\mathcal{E}_D(\mathbb{Q})$. Assume $D^{2(1+\epsilon)} < x < X$ and $YY > 0$. Suppose $P = 2Q + R$ for some $Q \in \mathcal{E}_D(\mathbb{Q})$. Replacing $Q$ with one of $Q + (-0, 0), Q + (D, 0), Q + (-D, 0)$ if necessary, we can assume $x(Q) > (1 + \sqrt{2})D$.

**Lemma 5.1.** Suppose $P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in \mathcal{E}_D(\mathbb{Q})$ and $D < x_1 \leq x_2$. Let $B = \frac{x_2}{x_1}$ and $\mu = \frac{B^2}{x_1}$. Then

\[
(18) \quad \left(\frac{B + \mu}{\sqrt{B^2 - \mu + \sqrt{B(1 - \mu)}}} \right)^2 x_1 \leq x(P_1 + P_2) \leq \left(\frac{\sqrt{B^2 - \mu + \sqrt{B(1 - \mu)}}}{B - 1} \right)^2 x_1.
\]

Moreover

\[
(19) \quad \frac{1}{4} x(P_1) \leq x(P_1 + P_2) \leq \left(\frac{2B}{B - 1} \right)^2 x(P_1).
\]

**Proof.** Assuming $y_1y_2 > 0$,

\[
x(P_1 \pm P_2) = \left(\frac{\sqrt{B^2 - \mu + \sqrt{B(1 - \mu)}}}{B - 1} \right)^2 x_1 = \left(\frac{B + \mu}{\sqrt{B^2 - \mu + \sqrt{B(1 - \mu)}}} \right)^2 x_1.
\]

This shows that $x(P_1 + P_2)$ have to attain the lower bound in (18) when $y_1y_2 > 0$, and attain the upper bound in (18) when $y_1y_2 < 0$.

Noting that $B \geq 1$ and $\mu < 1$ by assumption, we deduce (19) from (18). $\square$

We now show that $x(Q + R)$ is properly bounded away from $D$. If $(1 + \sqrt{2})D < x(Q) < 4(1 + \sqrt{2})D$, since we assumed $x(R) > D^{2(1+\epsilon)}$, by the lower bound in (18) we have $x(Q + R) > \frac{3}{4}(1 + \sqrt{2})D$ for large enough $D$. If $x(Q) \geq 4(1 + \sqrt{2})D$, then $x(Q + R) \geq (1 + \sqrt{2})D$ by the lower bound in (19).

**Lemma 5.2.** Suppose $Q \in \mathcal{E}_D(\mathbb{Q}) \setminus \{O, (0, 0), (\pm D, 0)\}$ and $x(Q) \in \mathbb{R}$. Then

\[
x(2Q) \geq \frac{1}{4} |x(Q)|.
\]

Moreover, if in addition $x(Q) \geq \frac{1}{\delta}D$ for some $\delta > 1$, then

\[
\frac{1}{4} x(Q) \leq x(2Q) \ll_{\delta} x(Q).
\]

**Proof.** If $x(Q) > D$, this follows immediately from the formula

\[
x(2Q) = \frac{\left(1 + \left(\frac{D}{x(Q)}\right)^2\right)^2}{4 \left(1 - \left(\frac{D}{x(Q)}\right)^2\right)} x(Q).
\]

If $-D < x(Q) < 0$, then it is straightforward to check that $x(2Q) > D$, so $x(2Q) > |x(Q)|$. $\square$

Trivially $h(Q) \geq \log x(Q)$ and $h(Q + R) \geq \log x(Q + R)$. By Lemma 5.2 and the lower bounds on $x(Q)$ and $x(Q + R)$, we have $x(Q) \gg x(2Q) = x(P - R)$ and $x(Q + R) \gg x(2Q + 2R) = x(P + R)$. Also $x(P \pm R) \gg x$ by Lemma 5.1. Putting together we have
$h(Q), h(Q + R) \geq \log x + O(1)$. Now apply (6) to $P$ and $P - R$, then to $Q$ and $Q + R$, it follows that

$$\log X + \log x + O(1) \geq \hat{h}(P) + \hat{h}(R) = 2\hat{h}(Q) + 2\hat{h}(Q + R) \geq 4 \log x - 4 \log D + O(1).$$

Rearranging gives

(20) $$\log X \geq 3 \log x - 4 \log D + O(1).$$

6. Large integral points giving Diophantine approximations

Fix an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and write $| \cdot |$ for the corresponding absolute value. In this section we will prove the following lemma.

**Lemma 6.1.** Suppose $P \in \mathcal{E}_D(\mathbb{Z})$ such that $P = 4 \tilde{Q} + R$, for some $\tilde{Q} \in \frac{1}{2} \mathcal{E}_D(\mathbb{Q})$ and $R \in \mathcal{E}_D(\mathbb{Q}) \setminus 2 \mathcal{E}_D(\mathbb{Q})$. Assume that $x(R) > D$ and $h(P) > \max \{ \frac{1}{2} h(R), \frac{1}{2} \log D \}$, where $0 < \delta \leq \lambda < \frac{1}{1000}$. Take $S \in \{ \tilde{S} \in \mathcal{E}_D(\overline{\mathbb{Q}}) : 4 \tilde{S} = R \}$ such that $|x(\tilde{Q}) - x(S)|$ is minimum. Then

$$\frac{\log |x(\tilde{Q}) - x(S)|}{\hat{h}(P)} \leq -8 \cdot \frac{1 - 63 \lambda - 418 \delta}{(1 + \sqrt{\lambda})^2 + 16 \delta} + o(1).$$

Suppose $P, \tilde{Q}, R$ satisfies the assumptions of Lemma 6.1. Since $2 \tilde{Q} \in \mathcal{E}_D(\mathbb{Q})$, we have $x(\tilde{Q}) \in \mathbb{R}$. Then

(21) $$x(\tilde{Q}) \ll x(4 \tilde{Q}) = x(P - R) \ll x(R),$$

where the lower bound follows from applying Lemma 5.2 twice, and the upper bound is taken from (19). The upper bound does not depend on $\lambda$ as we have assumed that $\lambda$ is bounded from above.

6.1. **Height estimates.** Since $P = 4 \tilde{Q} + R$, we obtain from the triangle inequality

(22) $$4 \sqrt{\hat{h}(\tilde{Q})} - \sqrt{\hat{h}(R)} \leq \sqrt{\hat{h}(P)} \leq 4 \sqrt{\hat{h}(\tilde{Q})} + \sqrt{\hat{h}(R)}.$$

Apply the assumption $\log D < \delta h(P)$ to (6), we obtain

(23) $$(1 - \delta)h(P) - O(1) \leq \hat{h}(P) \leq h(P) + O(1).$$

Similarly using (6) and $h(R) < \lambda h(P)$, we have

(24) $$\hat{h}(R) \leq h(R) + O(1) \leq \lambda h(P) + O(1).$$

On the other hand, by (12) and the assumption $\log D < \delta h(P)$,

(25) $$h(\tilde{Q}) - \delta h(P) - O(1) \leq \hat{h}(\tilde{Q}) \leq h(\tilde{Q}) + \delta h(P) + O(1).$$

Applying (23) and (24) to (22), then squaring, we have

(26) $$(\sqrt{1 - \delta} - \sqrt{\lambda})^2 - 16 \delta - o(1) \leq \frac{16 h(\tilde{Q})}{h(P)} \leq (1 + \sqrt{\lambda})^2 + 16 \delta + o(1).$$
6.2. Approximation of algebraic numbers. If \( S \in \mathcal{E}_D(\overline{\mathbb{Q}}) \) such that \( 4S = R \), write
\[
x(R) = x(4S) = \frac{\phi_4(S)}{\psi_4(S)^2},
\]
where \( \psi_n \) is the \( n \)th-division polynomial of \( \mathcal{E}_D \) and \( \phi_n = x\psi_n^2 - \psi_{n+1}\psi_{n-1} \). Define
\[
(27) \quad f_R(T) := \prod_{S:4S=R} (T - x(S)) = \phi_4(T) - x(R)\psi_4(T)^2.
\]
Note that every elliptic curve has 16 4-torsion points over \( \overline{\mathbb{Q}} \). The two expressions in (27) are equivalent as they are both monic polynomials of degree 16 with roots \( \{x(S) \in \mathbb{Q} : 4S = R\} \). Put \( T = x(\tilde{Q}) \), then
\[
\prod_{4S=R} (x(\tilde{Q}) - x(S)) = \phi_4(\tilde{Q}) - x(R)\psi_4(\tilde{Q})^2.
\]
Substitute \( \phi_4(\tilde{Q}) = \psi_4(\tilde{Q})^2x(4\tilde{Q}) \), we get
\[
(28) \quad \prod_{4S=R} (x(\tilde{Q}) - x(S)) = \psi_4(\tilde{Q})^2(x(4\tilde{Q}) - x(R)).
\]
Now
\[
x(P) = x(4\tilde{Q} + R) = \left( \frac{y(4\tilde{Q}) - y(R)}{x(4\tilde{Q}) - x(R)} \right)^2 - x(4\tilde{Q}) - x(R)
\]
\[
= \frac{-y(4\tilde{Q})y(R) + x(4\tilde{Q})^2x(R) + x(4\tilde{Q})x(R)^2 - D^2(x(4\tilde{Q}) + x(R))}{(x(4\tilde{Q}) - x(R))^2}.
\]
Using (28), we have
\[
(29) \quad x(P) \left( \prod_{4S=R} (x(\tilde{Q}) - x(S)) \right)^2
\]
\[
= \psi_4(\tilde{Q})^4 \left( -y(4\tilde{Q})y(R) + x(4\tilde{Q})^2x(R) + x(4\tilde{Q})x(R)^2 - D^2(x(4\tilde{Q}) + x(R)) \right).
\]
If \( (x, y) \in \mathcal{E}_D(\overline{\mathbb{Q}}) \) and \( x \in \mathbb{R} \), then we must have \( y^2 \ll \max\{x, D\}^3 \), so
\[
\psi_4((x, y))^4 = (4y(x^6 - 5D^2x^4 - 5D^4x^2 + D^6))^4 \ll \max\{x, D\}^{14 + \frac{15}{2}},
\]
and we can bound (29) by
\[
\ll x(4\tilde{Q})x(R)^2\max\{x(\tilde{Q}), D\}^{30} \ll x(R)^{33} \ll x(P)^{33},
\]
where we have applied (21). Taking logs,
\[
(30) \quad \log \prod_{4S=R} |x(\tilde{Q}) - x(S)| \leq \frac{1}{2} + \frac{33}{2} \lambda + O \left( \frac{\delta}{\log D} \right).
\]
Let \( \alpha = x(S) \) be a root of \( f_R \). Apply [25, p.262 last line],
\[
|f'_R(\alpha)| \gg |\Delta(f_R)|^{1/2}\|f_R\|_1^{-14},
\]
where \( \Delta(\cdot) \) denotes the discriminant and \( \| \cdot \|_1 \) denotes the \( \ell_1 \)-norm. Write \( x(R) = \frac{z}{s} \), where \( \gcd(r, s) = 1 \). Since \( sf_R(T) \in \mathbb{Z}[T] \), so \( |\Delta(f_R)| \geq s^{-30} \). Also we can check that \( \|f_R\|_1 \ll D^{14}\max\{x(R), D^2\} \). Therefore noting that \( H(R) = r \geq Ds \),
\[
\prod_{\tilde{S} \neq S:4\tilde{S}=R} |x(\tilde{S}) - x(S)| = |f'_R(\alpha)| \gg |s|^{-15}(D^{14}\max\{x(R), D^2\})^{-14} \geq H(R)^{-15}D^{-209}.
\]
Take $S \in \{ \tilde{S} \in \mathcal{E}_D(\mathbb{Q}) : 4\tilde{S} = R \}$ such that $|x(\tilde{Q}) - x(S)|$ is minimum. By the triangle inequality

$$|x(S) - x(\tilde{S})| \leq |x(\tilde{Q}) - x(S)| + |x(\tilde{Q}) - x(\tilde{S})| \leq 2|x(\tilde{Q}) - x(\tilde{S})|.$$  

Taking products

$$\prod_{\tilde{S} \neq S : 4\tilde{S} = R} |x(\tilde{Q}) - x(S)| \gg \prod_{\tilde{S} \neq S : 4\tilde{S} = R} |x(S) - x(\tilde{S})| \gg H(R)^{-15} D^{-209}.$$  

Take logs

$$\log \prod_{\tilde{S} \neq S : 4\tilde{S} = R} |x(\tilde{Q}) - x(S)| \geq -15h(R) - 209 \log D + O(1).$$  

Put this back to (30),

$$\frac{\log |x(\tilde{Q}) - x(S)|}{h(P)} \leq -\frac{1}{2} + \frac{63}{2} \lambda + 209\delta + o(1).$$  

Applying the upper bound in (26) proves Lemma 6.1.

7. Roth’s Theorem

In this section we follow the proof of Roth’s Theorem in Chapter 6 of [7], specialising in the bivariate case.

Let $K \subseteq E$ be number fields such that $m := [E : K]$. Suppose $\alpha \in E$. Let $v_1$ be the infinite place of $E$ given by the embedding $v_E : E \hookrightarrow \mathbb{C}$, where $v : \mathbb{Q} \hookrightarrow \mathbb{C}$ is as fixed in Section 6 so we can write $| \cdot | = | \cdot |_{v_1}$. We call $\beta \in K$ a $K$-approximation to $\alpha$ with exponent $\kappa$, if

$$|\beta - \alpha| < H(\beta)^{-\kappa}.$$  

Approximations obey the following strong gap principle.

**Theorem 7.1** (strong gap principle [7 Theorem 6.5.4]). Let $\beta, \beta' \in K$ be distinct elements such that $|\alpha - \beta| < H(\beta)^{-\kappa}$, $|\alpha - \beta'| < H(\beta')^{-\kappa}$ and $h(\beta') \geq h(\beta)$. Then

$$h(\beta') \geq -2 \log 2 + (\kappa - 1)h(\beta).$$  

**Proof.** We have

$$\log |\beta - \beta'| = \log |(\alpha - \beta') - (\alpha - \beta)| \leq \max(\log |\alpha - \beta'|, \log |\alpha - \beta|) + \log 2$$  

$$\leq -\kappa \min(h(\beta'), h(\beta)) + \log 2 = -\kappa h(\beta) + \log 2.$$  

Also

$$\log |\beta - \beta'| \geq -h(\beta - \beta') \geq -h(\beta) - h(\beta') - \log 2.$$  

Combining the upper and lower bounds of $\log |\beta - \beta'|$ gives the required inequality.  

**Theorem 7.2.** Let $c < 1$, $M \geq 72$ and $L = \left(\frac{h(\alpha) + \log 2}{c - 1} + 4\right)M$. Assume

$$\kappa > \left(c - 4 \sqrt{\frac{m}{M}}\right)^{-1} \left(1 + \frac{c^{-2} + 1}{M}\right) \sqrt{2m}.$$  

Suppose $\beta_1, \beta_2 \in K$ are both approximations to $\alpha \in E$ with exponent $\kappa$. If $h(\beta_1) \geq L$, then $h(\beta_2) < Mh(\beta_1)$.

We prove Theorem 7.2 by contradiction. Suppose we can find $\beta_1, \beta_2$ under the assumptions in Theorem 7.2 and such that $h(\beta_1) \geq L$ and $h(\beta_2) \geq Mh(\beta_1)$. Let $\sigma := \sqrt{\frac{2}{M}}$ and $t := c \sqrt{\frac{2}{M}}$. 
7.1. The auxiliary polynomial. Take $N$ large. Choose

$$d_j = \left\lfloor \frac{N}{h(\beta_j)} \right\rfloor$$

for $j = 1, 2$.

Let $t < 1$, $\alpha = (\alpha, \alpha) \in E^2$ and $\beta = (\beta_1, \beta_2) \in K^2$. Let

$$V_2(t) := \text{vol}(\{(x_1, x_2) : x_1 + x_2 \leq t, \ 0 \leq x_1 \leq 1\}) = \frac{1}{2}t^2.$$

For a polynomial $F(x_1, x_2) = \sum_j a_j x^j \in \mathbb{Q}[x_1, x_2]$, define $|F|_v = \max_j |a_j|_v$, $H(F) := \prod_v |F|_v$ and $h(F) = \log H(F)$.

We apply the following lemma to construct an auxiliary polynomial.

**Lemma 7.3** ([7, Lemma 6.3.4]). Suppose $mV_2(t) < 1$. Then for all sufficiently large $d_1, d_2 \in \mathbb{Z}$, there exist $F \in K[x_1, x_2]$, $F \neq 0$, with partial degrees at most $d_1, d_2$ such that

$$\text{ind}(F; d, \alpha) := \min_{\mu} \left\{ \frac{\mu_1}{d_1} + \frac{\mu_2}{d_2} : \partial_\mu F(\alpha) \neq 0 \right\} \geq t;$$

and

$$h(F) \leq \frac{mV_2(t)}{1 - mV_2(t)} \sum_{j=1}^{2} (h(\alpha_j) + \log 2 + o(1))d_j,$$

as $d_j \to \infty$.

Since $\frac{1}{2}mt^2 < 1$, we have

$$\frac{mV_2(t)}{1 - mV_2(t)} = \frac{mt^2}{2 - mt^2} = \frac{1}{2m^{-1}t^{-2} - 1}.$$

Take $C_1 := \frac{h(\alpha) + \log 2}{\sigma^2 - 1}$, so $L = (C_1 + 4)M$. Then we can obtain a non-trivial polynomial $F \in K[x_1, x_2]$ with partial degrees at most $d_1, d_2$ such that

$$\text{(32)} \quad \text{ind}(F; d, \alpha) \geq t \quad \text{and} \quad h(F) < \frac{2C_1N}{L}.$$

7.2. Non-vanishing at the rational point. Next we apply Roth’s lemma to construct a suitable derivative of $F$ that does not vanish at $\beta$.

**Lemma 7.4** (Roth’s lemma ([7, Lemma 6.3.7])). Let $F \in \mathbb{Q}[x_1, x_2]$ with partial degrees at most $d_1, d_2$ and $F \neq 0$. Let $(\xi_1, \xi_2) \in \mathbb{Q}^2$ and $0 < \sigma^2 \leq \frac{1}{2}$. Suppose that $d_2 \leq \sigma^2 d_1$ and $\min_j d_j h(\xi_j) \geq \sigma^{-2}(h(F) + 8d_1)$. Then $\text{ind}(F; d, \xi) \leq 4\sigma$.

Since $L \geq 2\sigma^{-2}(C_1 + 4)$ and $M \geq 2\sigma^{-2}$, we can apply the lemma to get $\text{ind}(F; d, \beta) \leq 4\sigma$. Now we can take $\mu$ such that $\partial_\mu F(\beta) \neq 0$ and $\frac{\mu_1}{d_1} + \frac{\mu_2}{d_2} = \text{ind}(F; d, \beta)$. Let $G = \partial_\mu F$.

Since $\text{ind}(G; d, \alpha) \geq \text{ind}(F; d, \alpha) - \frac{\mu_1}{d_1} - \frac{\mu_2}{d_2}$ by [7, 6.3.2(c)], we deduce from (32) that

$$\text{(33)} \quad \text{ind}(G; d, \alpha) \geq t - 4\sigma, \quad G(\beta) \neq 0, \quad \text{and} \quad h(G) \leq \frac{4C_1N}{L}.$$
7.3. The upper bound. For places \( v \neq v_1 \), we have

\[
\log |G(\beta)|_v \leq \log |G|_v + \sum_{j=1}^{2} d_j (\log^+ |\beta_j|_v + \varepsilon_v o(1)),
\]

where \( o(1) \to 0 \) as \( d_j \to \infty \), and

\[
\varepsilon_v = \begin{cases} 
[K_v:Q_v] & \text{if } v \text{ is archimedean} \\
[K_v:Q] & \text{if } v \text{ is non-archimedean}
\end{cases}
\]

For the place \( v_1 \), expand \( G \) in Taylor series with center \( \alpha \)

\[
G(\beta) = \sum_k \partial_k G(\alpha) (\beta_1 - \alpha)^{k_1} (\beta_2 - \alpha)^{k_2}. 
\]

We have from (33), that

\[
\partial_k G(\alpha) = 0 \quad \text{if} \quad \frac{k_1}{d_1} + \frac{k_2}{d_2} < t - 4\sigma,
\]

and

\[
\log |\partial_k G(\alpha)|_{v_1} \leq \log |G|_{v_1} + \sum_{j=1}^{2} (d_j - k_j) \log^+ |\alpha|_{v_1} + \varepsilon_{v_1} (\log 2 + o(1)) d_j,
\]

so putting back to (35) and taking absolute values and logs,

\[
\log |G(\beta)|_{v_1} \leq \max_k \log \left| \partial_k G(\alpha) \prod_{j=1}^{2} (\beta_j - \alpha)^{k_j} \right|_{v_1} + \varepsilon_{v_1} \sum_{j=1}^{2} \log(d_j + 1)
\]

\[
\leq - \min_{\frac{k_1}{d_1} + \frac{k_2}{d_2} < t - 4\sigma} \left( \sum_{j=1}^{2} k_j \log^+ \frac{1}{|\beta_j - \alpha|_{v_1}} \right) + \log |G|_{v_1}
\]

\[
+ \sum_{j=1}^{2} \left( \log^+ |\beta_j|_{v_1} + \log^+ |\alpha|_{v_1} + \varepsilon_{v_1} (\log 2 + o(1)) \right) d_j.
\]

We now suppress the subscript \( v_1 \), as \(|·|\) is defined to be \(|·|_{v_1}\). Adding up the bounds (34) and (36) for all places \( v \), and noting that \( \sum_v \varepsilon_v = 1 \), we have

\[
\sum_v \log |G(\beta)|_v \leq - \min_{\frac{k_1}{d_1} + \frac{k_2}{d_2} \geq t - 4\sigma} \left( \sum_{j=1}^{2} k_j \log^+ \frac{1}{|\beta_j - \alpha|} \right) + \log |G|
\]

\[
+ \sum_{j=1}^{2} \left( \log^+ |\beta_j| + \log^+ |\alpha| + 2 \log 2 + o(1) \right) d_j
\]

\[
\leq - \min_{\frac{k_1}{d_1} + \frac{k_2}{d_2} \geq t - 4\sigma} \left( \sum_{j=1}^{2} k_j \log^+ \frac{1}{|\beta_j - \alpha|} \right) + \left( 2 + \frac{C_2}{L} \right) N + o(N),
\]

where \( C_2 = 4C_1 + 4 \log 2 + 2 \log^+ |\alpha| \).

From the assumption that \( \beta_1 \) and \( \beta_2 \) are both \( K \)-approximation to \( \alpha \) with exponent \( \kappa \), we see that

\[
\kappa \log^+ \frac{1}{|\beta_j - \alpha|},
\]
we have
\[
\sum_{j=1}^{2} k_j \log^+ \frac{1}{|\beta_j - \alpha|} \geq \kappa \sum_{j=1}^{2} (h(\beta_j)d_j) \frac{k_j}{d_j} \sim N\kappa \left(\frac{k_1}{d_1} + \frac{k_2}{d_2}\right).
\]
This gives us the upper bound
\[
\sum_v \log |G(\beta)|_v \leq -\kappa (t - 4\sigma) N + \left(2 + \frac{C_2}{L}\right) N + o(N).
\]

7.4. Obtaining the bound. Since \(G(\beta) \neq 0\), we have \(\sum_v \log |G(\beta)|_v = 0\). Put this into (37) and let \(N \to \infty\), we get
\[\kappa \left(\frac{t}{2} - 2\sigma\right) + 1 + \frac{C_2}{2L} \geq 0.\]
Since by assumption \(\sigma < \frac{1}{8}\), we have
\[\kappa \leq \left(\frac{t}{2} - 2\sigma\right)^{-1} \left(1 + \frac{C_2}{2L}\right),\]
which contradicts (31). The completes the proof of Theorem 7.2.

8. Bounding the number of points

In this section we prove the explicit upper bound of \(#\mathcal{L}_D(R)\) given in Theorem 1.3 when \(x(R) > D\) and \(R \notin \mathcal{T}_D + 2\mathcal{E}_D(\mathbb{Q})\). Take \(R\) to be the point with minimum canonical height in the coset \(R + 2\mathcal{E}_D(\mathbb{Q})\). Let \(\epsilon = 0.00153\), which satisfies the assumption in Lemma 4.1.

For each \(\tilde{Q} \in \frac{1}{2} \mathcal{E}_D(\mathbb{Q})\), define \(L_{\tilde{Q}} := \left(\frac{h(S) + \log 2}{c^2 - 1} + 4\right) M\) as in Theorem 7.2, where \(S\) is chosen in \(\frac{1}{4} R\) such that \(|x(S) - x(\tilde{Q})|\) is minimum, with absolute constants \(M\) and \(c\) to be specified later. We bound the number of medium points
\[A_1 := \left\{ P \in \mathcal{L}_D(R) : h(\tilde{Q}) < L_{\tilde{Q}} \text{ for some } \tilde{Q} \in \frac{1}{4}(P - R) \right\}\]
and large points
\[A_2 := \left\{ P \in \mathcal{L}_D(R) : h(\tilde{Q}) \geq L_{\tilde{Q}} \text{ for all } \tilde{Q} \in \frac{1}{4}(P - R) \right\}.\]

For each \(S \in \frac{1}{4} R\), define
\[B_2(S) := \left\{ \tilde{Q} \in \frac{1}{2} \mathcal{E}_D(\mathbb{Q}) : 4\tilde{Q} + R \in A_2, \ |x(S) - x(\tilde{Q})| \text{ minimum over } S \in \frac{1}{4} R \right\}.\]

8.1. Medium points. Let \(P_1, P_2, \ldots, P_s\) be points in \(A_1\) with strictly increasing height. From (20), we deduce that
\[h(P_{i+1}) \geq 3h(P_i) - 4 \log D + O(1)\]
for every \(i = 1, \ldots, s - 1\). Iterating this,
\[h(P_{i+1}) \geq 3^i h(P_1) - (1 + 3 + \cdots + 3^{i-1})(4 \log D + O(1)) = 3^i h(P_1) - 2(3^i - 1)(\log D + O(1)).\]
Therefore it follows from the assumptions \(x(P_1) > D^{2(1+\epsilon)}\) that
\[
h(P_s) > \left(1 - \frac{2 \log D + O(1)}{h(P_1)}\right) 3^{s-1} h(P_1) > \left(\frac{\epsilon}{1 + \epsilon} - o(1)\right) 3^{s-1} h(P_1).
\]
Take
\begin{equation}
\lambda > \frac{1 + \epsilon}{3^s - 1 \epsilon} \quad \text{and} \quad \delta > \frac{1}{2 \cdot 3^s - 1 \epsilon},
\end{equation}
so that \( \lambda h(P_s) > h(P_1) > h(R) \) and \( \delta h(P_s) > \frac{1}{2(1 + \epsilon)} h(P_1) > \log D \).

For each \( S \in \frac{1}{2} R \), since \( 16 h(S) = \hat{h}(R) \leq \hat{h}(P_1) \), we have by applying (12) to \( \hat{h}(S) \) and (4) to \( \hat{h}(P_1) \),
\[
h(S) < \frac{1}{16} h(P_1) + \log D - o(1).
\]
Writing \( P_s = 4\hat{Q}_s + R \) and using the lower bound in (26),
\[
\frac{1}{16} h(P_s) \left( (\sqrt{1 - \delta} - \sqrt{\lambda})^2 - 16 \delta - o(1) \right) \leq h(\hat{Q}_s) < L_{\hat{Q}_s} < \left( \frac{1}{16} h(P_1) + \log D + \log 2 + o(1) \right) \frac{M}{c^2 - 1} + 5 M.
\]
Now apply (38) and divide both sides by \( \frac{1}{16} h(P_1) \),
\[
\left( \frac{\epsilon}{1 + \epsilon} - o(1) \right) 3^{s-1} \left( (\sqrt{1 - \delta} - \sqrt{\lambda})^2 - 16 \delta + o(1) \right) < \left( 1 + \frac{8}{1 + \epsilon} + o(1) \right) \frac{M}{c^2 - 1}.
\]
Simplifying we have
\begin{equation}
3^{s-1} < \left( 1 + \frac{9}{\epsilon} \right) \frac{M}{(c^2 - 1) \left( (\sqrt{1 - \delta} - \sqrt{\lambda})^2 - 16 \delta \right) + o(1)}.
\end{equation}
Let \( s_0 \) be the maximum integer satisfying (40), then taking \( s = s_0 + 1 \) would be a contradiction as long as (39) is satisfied. Therefore \#(\mathcal{A}_1) \leq 2s_0 \), where the factor of 2 comes from the possible existence of \(-P_1, \ldots, -P_{s_0}\).

### 8.2. Large points

Now fix some \( S \in \frac{1}{2} R \) and consider the set \( \mathcal{B}_2(S) \). Let \( K \) be the minimal number field containing the \( x \)-coordinates of all points in \( \frac{1}{2} E_D(Q) \), and let \( E \) be the field \( K(x(S)) \).

Suppose \( \hat{Q} \in \mathcal{B}_2(S) \). Fix \( \lambda = 0.000137 \) and \( \delta = 0.0000684 \) so that (39) holds with \( s = 15 \), and take \( \kappa = 7.516 \), which satisfies
\[
\kappa < 8 \cdot \frac{1 - 63 \lambda - 418 \delta}{(1 + \sqrt{\lambda})^2 + 16 \delta} + o(1),
\]
then Lemma 6.1 implies that \( x(\hat{Q}) \) is a \( K \)-approximation to \( x(S) \) with exponent \( \kappa \). Now we can apply Theorem 7.2 with \( m = [E : K] \leq 4 \), \( M = 276.1 \) and \( c = 0.861 \), noting that (31) is satisfied. Take \( \beta_1 = x(\hat{Q}) \) such that \( h(\hat{Q}) \) is minimum over all \( \hat{Q} \in \mathcal{B}_2(S) \). Then Theorem 7.2 shows that all points in \( \mathcal{B}_2(S) \) must have height in the interval \( [h(\beta_1), M h(\beta_1)] \).

By Theorem 7.4 if \( t \) is the smallest integer such that
\[
(\kappa - 1)^t > M,
\]
then \#(\mathcal{B}_2(S)) \leq t. This is achieved by \( t = 3 \). There are 16 choices of \( S \), but since \( E_D(Q)[4] \subseteq \frac{1}{2} E_D(Q) \), if \( x(Q) \) is a \( K \)-approximation to \( x(S) \), then \( x(Q + T) \) is also a \( K \)-approximation to \( x(S + T) \) for any \( T \in E_D(Q)[4] \). Therefore \#(\mathcal{A}_2) \leq 3.

Returning to the medium points with our choice of constants, we have \#(\mathcal{A}_1) \leq 28. If \( P \in E_D(Z) \) then \(-P \in E_D(Z) \), so \#(\mathcal{A}_1 \cup \mathcal{A}_2) \leq 30.
We now prove the upper bounds in Theorem 1.4

9.1. Cosets with respect to a non-torsion point. Here we will treat the case in Theorem 1.4 (4), assuming \( R \notin T_D + 2\mathcal{E}_D(\mathbb{Q}) \). Suppose \( x(R) < 0 \). If \( P \in \mathcal{Z}_D(R) \), then \(-D < x(P) < 0 \) and so \( h(P) < \log D + \frac{3}{2} \log 2 \) by (7). Following the argument in Section 4, we obtain an upper bound of \( A(r+1, \theta) \) where \( \sin \frac{\theta}{2} = \frac{1}{2} \sqrt{\frac{\log D - 2 \log 2}{\log D + \frac{3}{2} \log 2}} = \frac{1}{2} - o(1) \).

For \( D \geq 97353 \), applying the following estimate by Rankin gives us an upper bound of 3. Since non-torsion integral points in comes in pairs of \( \pm P \), we can reduce the upper bound to 2 if \( R \notin T_D + 2\mathcal{E}_D(\mathbb{Q}) \).

Theorem 9.1 ([26] Lemma 2]). If \( \frac{\pi}{4} < \theta < \frac{\pi}{2} \), then
\[
A(r, \theta) \leq \frac{2 \sin^2 \theta}{2 \sin^2 \theta - 1}.
\]

Checking all the integral points in the range \(-D < x(P) < 0 \) on \( \mathcal{E}_D \) for each \( D < 97353 \), we see that the only exceptions are those listed in Theorem 1.4 (4).

9.2. Integral points in \( 2\mathcal{E}_D(\mathbb{Q}) + T_D \). We now prove cases (1), (2) and (3) in Theorem 1.4. We first show that if a rational point has a multiple which is an integral point, then the original point must also be integral.

Lemma 9.2. Suppose \( P \in \mathcal{E}_D(\mathbb{Q}) \). If \( mP \in \mathcal{E}_D(\mathbb{Z}) \) for some integer \( m \geq 2 \), then \( P \in \mathcal{E}_D(\mathbb{Z}) \).

Proof. Suppose \( P = mQ \), where \( Q \in \mathcal{E}_D(\mathbb{Q}) \). We have
\[
x(P) = \frac{\phi_m(Q)}{\psi_m(Q)^2},
\]
where \( \psi_m \) is the \( m \)th division polynomial, and \( \phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1} \) as usual. The polynomials \( \phi_m(x) \) and \( \psi_m(x)^2 \) have leading terms \( x^m \) and \( m^2x^{m^2-1} \) respectively. Putting \( x(Q) = \frac{u}{v} \) with \( \gcd(u, v) = 1 \), and clearing denominators we have
\[
x(Q) = \frac{u^{m^2} + vF(u, v)}{v(m^2u^{m^2-1} + vG(u, v))},
\]
for some polynomials \( F, G \in \mathbb{Z}[x, y] \). Therefore \( x(P) \in \mathbb{Z} \) implies \( v | u \), so \( v = 1 \) and \( Q \) is also integral.

We show that \( 2\mathcal{E}_D(\mathbb{Q}) \) contains no integral points.

Lemma 9.3. Suppose \( P \in \mathcal{E}_D(\mathbb{Q}) \) is non-torsion. Then \( 2P \notin \mathcal{E}_D(\mathbb{Z}) \).

Proof. Suppose \( P \in \mathcal{E}_D(\mathbb{Q}) \) and \( 2P \in \mathcal{E}_D(\mathbb{Z}) \), then \( P \) must be an integral point by Lemma 9.2. Write \( P = (x, y) \), so
\[
x(2P) = \left( \frac{x^2 + D^2}{2y} \right)^2.
\]
Suppose \( 2P \in \mathcal{E}_D(\mathbb{Z}) \). Then \( 4y^2 = 4x(x + D)(x - D) | (x^2 + D^2)^2 \). Therefore \( x \mid D \) and so \( x \) is squarefree. Write \( d = \frac{D}{x} \), and we have \( 4(d - 1)(d + 1) | x(d^2 + 1)^2 \). Since we assumed that \( P \) is not a torsion point, \( x \neq D \) and \( -D < x < 0 \). Suppose \( d \) is odd, then \( (d^2 + 1)^2 \equiv 4 \mod 8 \), and so \( 8 \mid (d - 1)(d + 1) \), so \( 8 \mid x \), but this contradicts with \( x \)}
being squarefree. Now suppose \( d \) is even, then \( (d^2 + 1)^2 \) is odd, so \( 4 \mid x \), which is also a contradiction. \( \square \)

We now look at points of the form \( 2P + (-D, 0) \) or \( 2P + (0, 0) \).

**Lemma 9.4.** Suppose \( P \in E_D(\mathbb{Q}) \). For each \( T \in \{(-D, 0), (0, 0)\} \), we have \( 2P + T \in E_D(\mathbb{Z}) \) if and only if \( P \) is a torsion point.

**Proof.** Notice that \( \theta(2P + (0, 0)) = (-D, -1, D) \) and \( \theta(2P + (-D, 0)) = (-2D, -D, 2) \). If \( 2P + (-D, 0) \in E_D(\mathbb{Z}) \), taking \( x(2P + (0, 0)) = -s^2 \), we see that the equation 
\[
s^2 + Dt^2 = D
\]
is solvable for \( s, t \in \mathbb{Z} \). Similarly if \( 2P + (0, 0) \in E_D(\mathbb{Z}) \), taking \( x(2P + (-D, 0)) = -Du^2 \), then
\[
Du^2 + 2v^2 = D
\]
is solvable for \( u, v \in \mathbb{Z} \). The only solutions to each of these equations over the integers are given by \( s = 0 \) and \( u^2 = 1 \). This implies that in both cases \( P \) is a torsion point. \( \square \)

The only possible non-torsion integral points in \( 2E_D(\mathbb{Q}) + T_D \) are in \((D, 0) + 2E_D(\mathbb{Q})\) and satisfies the property in the following theorem.

**Lemma 9.5.** Then there exists some \( P \in E_D(\mathbb{Q}) \) such that \( 2P + (D, 0) \in E_D(\mathbb{Z}) \) if and only if the system
\[
(41) \quad s^2 - 1 = 2Du^2, \quad s^2 + 1 = 2v^2
\]
is solvable for some \( s, u, v \in \mathbb{Z}_{>0} \). Furthermore, \( (11) \) has at most one solution for each \( D \). If a solution \( (s, u, v) \) exists, then \( x(2P + (D, 0)) = Ds^2 \),
\[
(42) \quad s^2 + 2uv\sqrt{D} = (v + u\sqrt{D})^2,
\]
and \( v + u\sqrt{D} \) is the fundamental solution to \( v^2 - Du^2 = 1 \).

**Proof.** Note that \( \theta(2P + (D, 0)) = (2, D, 2D) \). If \( 2P + (D, 0) \in E_D(\mathbb{Z}) \), then writing \( x(2P + (D, 0)) = Ds^2 \) finds us a solution \( (s, u, v) \) to the system \( (11) \). Conversely if \( (11) \) is solvable, it is easy to check that \( Ds^2 \) is the \( x \)-coordinate of an integral point on \( E_D \), and this point must be in the same coset of \( 2E_D(\mathbb{Q}) \) as \((D, 0)\) since \( \theta \) is injective.

If \( (11) \) is solvable, taking the difference of the two equations in \( (11) \), we get \( v^2 - Du^2 = 1 \). From \( (11) \), we see that \( (12) \) holds, and also \( s^2 - D(2uv)^2 = 1 \). Cohn showed that such equation has at most one solution unless \( D = 1785 \). More precisely, the main theorem in \( [15] \) implies that \( s^2 + 2uv\sqrt{D} \) is either \( a + b\sqrt{D} \) or \( (a + b\sqrt{D})^2 \) if \( a + b\sqrt{D} \) is the fundamental solution to \( v^2 - Du^2 = 1 \). This proves the final claim. \( \square \)

### 10. Average number of integral points

In this section we prove Theorem \([13, 15]\). From Theorem \([11]\)
\[
\#E_D(\mathbb{Z}) \ll 4^{\text{rank}E_D(\mathbb{Q})}.
\]

Heath-Brown \([18, \text{Theorem 1}]\) proved that\(^2\)
\[
\lim_{N \to \infty} \frac{1}{\#D_N} \sum_{D \in D_N} 2^{k \cdot \text{rank}E_D(\mathbb{Q})} \ll_k 1.
\]

\(^2\)To be precise, the theorem was only stated for odd \( D \), but it is possible to extend the proof to even \( D \).
Therefore
\[
\limsup_{N \to \infty} \frac{1}{\#D_N} \sum_{D \in D_N} (\#\mathcal{E}_D(\mathbb{Z}))^2 \ll 1.
\]

Suppose \( G_N \subseteq D_N \). By Cauchy–Schwarz inequality,
\[
\sum_{D \in G_N} \#\mathcal{E}_D(\mathbb{Z}) \leq \left( \sum_{D \in D_N} (\#\mathcal{E}_D(\mathbb{Z}))^2 \right)^{1/2} (\#G_N)^{1/2} \ll (\#D_N)^{1/2} (\#G_N)^{1/2}.
\]

Therefore the contribution from any subset \( G_N \) of \( D_N \) of size \( o(N) \) to the average of \( \#\mathcal{E}_D(\mathbb{Z}) \) over \( D_N \) tends to 0 as \( N \to \infty \).

Assuming \( (2) \) implies that we only need to consider the contribution from the curves \( \mathcal{E}_D \) with rank 0 or 1. A theorem by Le Boudec \([23, \text{Proposition 1}]\) shows that
\[
\sum_{D \geq 1} \# \left\{ P \in \mathcal{E}_D(\mathbb{Z}) : x(P) < N^2 (\log N)^\kappa \right\} \ll \frac{N}{(\log N)^{\kappa/2 - 6}},
\]
where we take \( \kappa > 12 \). Therefore we can also exclude all \( \mathcal{E}_D \) with any integral point \( H(P) < \frac{N^2}{(\log N)^\kappa} \) since there are \( o(N) \) of them.

If \( \text{rank} \mathcal{E}_D(\mathbb{Q}) = 0 \), then there are automatically no non-torsion integral points. In the following we consider \( \mathcal{E}_D \) such that \( \text{rank} \mathcal{E}_D(\mathbb{Q}) = 1 \) and any \( P \in \mathcal{E}_D(\mathbb{Z}) \setminus \mathcal{T}_D \) satisfy \( H(P) > \frac{D^2}{(\log D)^\kappa} \). This removes the need to consider the points arising from cases (1), (2), (4) in Theorem 1.4.

Our aim is to prove the following.

**Theorem 10.1.** Assume that \( \text{rank} \mathcal{E}_D(\mathbb{Q}) = 1 \) and that any \( P \in \mathcal{E}_D(\mathbb{Z}) \setminus \mathcal{T}_D \) satisfy \( H(P) > \frac{D^2}{(\log D)^\kappa} \). Then
\[
\#(\mathcal{E}_D(\mathbb{Z}) \setminus \mathcal{T}_D) \leq 4.
\]

We now demonstrate that the integral points that appear in Theorem 1.4 (3) are rare for \( D \in D_N \) and do not contribute to the average in Theorem 1.5. Recall that these points are classified in Lemma 9.5. Dirichlet class number formula for real quadratic number fields states that the class number of \( \mathbb{Q}(\sqrt{D}) \) equals
\[
\frac{\sqrt{D} L(1, \chi_D)}{\log \epsilon_D},
\]
where \( \chi_D \) is the Kronecker symbol \( \left( \frac{D}{\cdot} \right) \) and \( \epsilon_D \) is the fundamental unit of \( \mathbb{Q}(\sqrt{D}) \). Since the class number is at least 1, this gives an inequality
\[
\log \epsilon_D \leq \sqrt{D} L(1, \chi_D).
\]
It is well-known that \( L(1, \chi_D) \ll \log D \). Therefore together with \( (42) \), we have
\[
\log s < 2 \log \epsilon_D \ll \sqrt{D} \log D.
\]
On the other hand, since \( s^2 - 2v^2 = -1 \) and \( 1 + \sqrt{2} \) is the fundamental unit of \( \mathbb{Q}(\sqrt{2}) \), we the possible values of \( s \) is given by
\[
s = \frac{1}{2} \left( (1 + \sqrt{2})^k + (1 - \sqrt{2})^k \right),
\]
where \( k \) is any positive odd integer. For large values of \( k \), \( \left| (1 - \sqrt{2})^k \right| \) is bounded, so
\[
s \gg (1 + \sqrt{2})^k.
\]
Putting together the inequalities (43) and (44), we get

\[ k \ll \sqrt{D} \log D. \]

Therefore for \( D \in \mathcal{D}_N \), there are \( \ll \sqrt{N} \log N \) integral points of the form \( 2P + (D, 0) \), which does not contribute to the average in Theorem 1.5.

10.1. Odd multiples of a generator. It now remains to treat the points not covered by Theorem 10.2. Notice that if \( m \) is odd, \( mP + T = m(P + T) \) for any \( P \in \mathcal{E}_D(\mathbb{Q}) \) and \( T \in \mathcal{T}_D \), so any integral points not in \( 2\mathcal{E}_D(\mathbb{Q}) + \mathcal{T}_D \) are odd multiples of a generator of the free part of \( \mathcal{E}_D(\mathbb{Q}) \). By Lemma 10.3, if \( mP \in \mathcal{E}_D(\mathbb{Z}) \) then \( P \in \mathcal{E}_D(\mathbb{Z}) \).

If \( P \in \mathcal{E}_D(\mathbb{Z}) \) and \( H(P) > \frac{N^2}{(\log N)^2} \), then \( x(P + (0, 0)), x(P + (D, 0)), x(P + (D, 0)) \ll D \), therefore by assumption \( P + (0, 0), P + (D, 0), P + (D, 0) \notin \mathcal{E}_D(\mathbb{Z}) \). Therefore it is enough to consider odd multiples of one integral point that is also generator.

We show that small multiples of a reasonably sized rational point, as assumed in Theorem 10.1 which we wish to prove, cannot be integral.

**Theorem 10.2.** Let \( \kappa > 0 \) and \( C_1 < \sqrt{\frac{4}{3} \log 2} \). Suppose \( D \) is some sufficiently large squarefree integer, \( P \in \mathcal{E}_D(\mathbb{Q}) \) and assume \( x(P) > \frac{D^2}{(\log D)^2} \), then \( mP \notin \mathcal{E}_D(\mathbb{Z}) \) for all \( 1 < m \leq \exp(C_1 \log D) \).

We have shown that \( 2P \) cannot be integral, so assume \( m \geq 3 \). With the formulae

\[
\psi_{2m+1} = \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3,
\]

\[
\psi_{2m} = \frac{\psi_m}{2} \left( \psi_{m+2}\psi_{m-1} - \psi_{m-2}\psi_{m+1} \right),
\]

we prove the following by induction.

**Lemma 10.3.** Fix some \( C_2 > \frac{3}{2\log x} \). Let \( x > D \) such that \( (x, y) \in \mathcal{E}_D(\mathbb{Q}) \). Then for any positive integer \( m \) satisfying \( C_2(\log m)^2 < (2\log x - \log D) \), we have

\[
\psi_m(x) > \left( 1 - \exp \left( C_2(\log m)^2 \right) \left( \frac{D}{x} \right)^2 \right) mx^\frac{m^2 - 1}{2}.
\]

**Proof.** Write \( \psi_m(x) = (1 - E_m(D/x)^2)mx^\frac{m^2 - 1}{2} \). Assuming \( E_m(D/x)^2 < 1 \), we obtain from (43)

\[
E_{2m+1} < E_{m-1} + 3E_{m+1} + \frac{m^3(m+2)}{2m+1} (E_{m+1} + 3E_m),
\]

from (46)

\[
E_{m+2} < \frac{1}{2} + E_m + \frac{(m-1)^2(m+2)}{4}(E_{m-1} + 2E_{m+2}) + \frac{1}{2}(E_{m-2} + 2E_{m+1}).
\]

Assuming \( E_m < \exp(C_2(\log m)^2) \) for all \( m < N \), we obtain an upper bound for \( E_N < \exp(C_2(\log N)^2) \) from (47) and (48). Checking the base cases \( \psi_2 = 2x^{3/2}(1 - (D/x)^2)^{1/2} \) and \( \psi_3 = 3x^4(1 - 2(D/x)^2 - 1/3(D/x)^4) \) completes the induction.

Write uniquely \( x(mp) = \frac{x}{v_m} \), where \( \gcd(u, v_m) = 1 \) and \( v_m > 0 \). By [36, Lemma 11.4]

\[
\log v_m \leq \log |\psi_m(x)| \leq \log v_m + \frac{1}{8}m^2 \log |\Delta_D|,
\]

where \( \Delta_D = (2D)^6 \) is the discriminant of \( \mathcal{E}_D \).
Proof of Theorem 10.2. Let \( x := x(P) > \frac{D^2}{(\log D)^\kappa} \). Suppose \( mP \in \mathcal{E}_D(\mathbb{Z}) \), then (49) reduces to
\[
(50) \quad \psi_m(x) \leq (2D)^{\frac{3}{4}} m^2.
\]

Fix \( \epsilon > 0 \) such that
\[
\log m < \sqrt{\frac{1}{C_2}} (\log(1 - \epsilon) + 2 \log D - 2\kappa \log \log D),
\]
then by Lemma 10.3
\[
\psi_m(x) > \epsilon m x^{\frac{m^2-1}{2}} > \epsilon m x^{\left( \frac{D}{(\log D)^\kappa/2} \right)^{m^2-1}},
\]
which contradicts (50) for sufficiently large \( D \). \( \square \)

Now following Section 8, we have \#\( A_1 \) = 0 for the medium points using Theorem 10.2 and \#\( A_2 \) \leq 3 for the large points. Since non-torsion integral points come in pairs \( \pm P \), \#(\( A_1 \cup A_2 \)) \leq 2. Therefore the possible points contributing to the upper bound in Theorem 10.1 comes from the generator and its corresponding negative point, together with the pair of large points in \#\( A_2 \).

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