INVARIANT METRICS ON FINITE GROUPS

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Abstract. We study invariant and bi-invariant metrics on groups focusing on finite groups $G$. We show that non-equivalent (bi) invariant metrics on $G$ are in 1-1 correspondence with unitary symmetric (conjugate) partitions on $G$. To every metric group $(G, d)$ we associate to it the symmetry group and the weighted graph of distances. Using these objects we can classify all equivalence classes of invariant and bi-invariant metrics for small groups. We then study the number of non-equivalent invariant and bi-invariant metrics on $G$. We give an expression for the number of such metrics in terms of Bell numbers, with closed expressions for certain groups such as abelian, dihedral, quasidihedral and dicyclic groups. We then characterize all the groups (finite or not) in which every invariant metric is also bi-invariant. We give the number of non-equivalent invariant and bi-invariant metrics for all the groups of order up to 32.

1. Introduction

In this work we study invariant metrics on groups, focusing on the finite case. So, from now on $G$ will always denote a finite group (unless explicit mention of the contrary), $d$ a metric on $G$ and $w$ a weight function on $G$. We now recall these definitions.

A metric on a set $X$ is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that for every $x, y, z \in X$ satisfies the three conditions:

(a) $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$ (positiveness),
(b) $d(x, y) = d(y, x)$ (commutativity),
(c) $d(x, y) \leq d(x, z) + d(x, z)$ (triangle inequality).

One says that $(X, d)$ is a metric space. If $d$ only satisfies (a)-(b), $d$ is a semimetric and $(X, d)$ is a semimetric space. If $X = G$ is a group, then we will say that $(G, d)$ is a metric group, or a metric semigroup if condition (c) does not hold.

A weight on $G$ is a function $w : G \to \mathbb{R}_{\geq 0}$ such that for every $x, y \in G$ satisfies:

(a) $w(x) \geq 0$ (positiveness) and $w(x) = 0$ if and only if $x = e$,
(b) $w(x) = w(x^{-1})$ (symmetry),
(c) $w(xy) \leq w(x) + w(y)$ (triangle inequality).

In general, with minimum extra conditions, given a metric one can define a weight and conversely.

The metric $d$ is called integral if only takes integer values, that is $d : X \times X \to \mathbb{N}_0$ (equivalently, the associated weight function takes integer values). Since our main motivation to study metrics on groups are applications to combinatorics (coding theory, for instance), we will be mainly interested in integral metrics.

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The most common metrics used in coding theory are the Hamming distance for codes over fields and the Lee distance for codes over rings (see the seminal works of Lee [13], Nechaev [15] and Hammons et al [12]), which are integral invariant metrics. Bru\-al\-d\-i et al [3] introduced invariant metrics associated to a partially ordered set, the so called poset metrics. For instance, the Hamming metric is a poset metric for the antichain poset. The Rosen\-b\-lo\-om-Tsfas\-\-\-man metric or RT-metric [18] which is also widely used in coding theory is a particular case of poset metric [3]. Another important invariant metric is the one given by the homogeneous weight on rings [9] (when this weight indeed defines a metric). We note that poset metrics are also integral metrics and the homogeneous metrics can be rescaled to become integral.

We are interested in metrics defined over groups. A general family of such metrics is given by word metrics. A complete treatment of metrics on groups can be found in Chapter 10 of the Encyclopedia of distances [5]. Our approach to study metrics on groups will be basically through (unordered) partitions. In [2] the author considered metrics with ordered partitions, however metrics obtained from different orders of the same partition are all \(\Gamma\)-equivalent (see the comments after (3.1)) to each other. A relation between metrics and partitions over additive codes is given in [8], mainly for the study of duality and the MacWilliams identities.

Outline and results. We now summarize the main results in the paper. In Section 2 we study invariant metrics on an arbitrary group in terms of its symmetric partitions. In Section 3, for a finite metric group we study their symmetry group and the associated graph of distances. In Section 4 we give a detailed study of the invariant metrics for small groups. In Sections 5 and 6 we count the number of invariant and bi-invariant metrics on a finite group. In the final section we give the number of invariant and bi-invariant metrics for all the groups of order up to 32.

More precisely, in Section 2 we define right-invariant metrics on a group \(G\) and denote by \(\mathcal{M}(G)\) the set of all these metrics. The weight \(w\) associated to an invariant metric \(d\) induces a partition \(P(G, d)\) on \(G\) (see (2.4)). We denote by \(\mathcal{P}(G)\) the set of all unitary symmetric partitions on \(G\) (see the definition before (2.5)). Two metrics are \(\mathcal{P}\)-equivalent if they define the same unitary symmetric partitions on \(G\). In Proposition 2.5 we show that the set \(\mathcal{M}(G)/\sim_{\mathcal{P}}\) of non-equivalent invariant metrics on \(G\) are in 1-1 correspondence with \(\mathcal{P}(G)\). Also, the non-equivalent bi-invariant (left and right) metrics on \(G\) are in a 1-1 correspondence with unitary symmetric conjugate partitions of \(G\).

In the next section, given a metric group \((G, d)\), we study the symmetry group \(\Gamma(G, d)\) of permutations of \(G\) preserving the metric \(d\) and its relation with its distance graph. In general, it is difficult to determine the group \(\Gamma(G, d)\), however we can give some set contentions. In Proposition 3.2 we show that if \(G\) is abelian then we have \(\mathbb{D}(G) \leq \Gamma(G, d) \leq S_G\), where \(\mathbb{D}(G)\) is a generalization of the dihedral group (see Definition 3.1). For a general group, if \(d\) is bi-invariant, in Proposition 3.3 we get that \(G \rtimes \text{Aut}_{\text{ecpl}}(G) \leq \Gamma(G, d) \leq S_G\), where \(\text{Aut}_{\text{ecpl}}(G)\) is the subgroup of extended class preserving automorphisms of \(G\) (see the comments before (3.3)). Finally, in Theorem 3.5 we show that the symmetry group of \((G, d)\) can be put in terms of certain Cayley graphs associated to the partition. Namely, if \(P_0, P_1, \ldots, P_k\) is the unitary symmetric partition of \(G\) associated to \(d\), then \(\Gamma(G, s)\) is the intersection of all the Cayley graphs \(\text{Cay}(G, P_i)\).

In Section 4, using the results of Sections 2 and 3, we study the invariant and bi-invariant metrics in detail for small groups. For each group \(G\) of order up to 7 we give all their unitary symmetric (conjugate) partitions \(\mathcal{P}(G)\). For the equivalence class of metrics \(d\) associated to each partition in \(\mathcal{P}(G)\) we give the symmetry group \(\Gamma(G, d)\) and the graphs of distances \(G(G, d)\). Also, we indicate which of these metrics is a poset metric [3], a chain metric or an extended Lee metric—one both defined in [17]—or an homogeneous metric on rings [9].

In Section 5 we study the number of invariant metrics on a finite group \(G\). In Proposition 5.1 we give an expression for the number \(M(G)\) of non-equivalent invariant metrics on \(G\) in terms
of Bell numbers $B_k(G)$ where $k(G)$ is certain integer associated to $G$. This allows us to compute the number $M(\sigma)$ in closed form for abelian groups $G$ in Proposition 5.4 and for $G$ being dihedral groups $D_n$, dicyclic groups $Q_{4n}$ and quasidihedral groups $QD_n^\pm$ in Proposition 5.5. In Proposition 5.6 we give an expression for the number of invariant metrics for symmetric and alternating groups $S_n$, $A_n$. We then study the number of invariant metrics for some semidirect products $G \rtimes H$ in some particular cases (see (5.14)–(5.17)).

In Section 6 we study the number of invariant metrics on a group $G$. In Proposition 6.1 we compute the number $M^*(G)$ of bi-invariant metrics of a group $G$ in terms of the number of its conjugacy and real conjugacy classes. In Proposition 6.3 we give the precise number of bi-invariant metrics for dihedral, dicyclic and quasidihedral groups. The number of bi-invariant metrics of symmetric groups is considered in Example 6.4. In Example 6.5 we give the number of invariant and bi-invariant metrics of the special group $SL_2(F_q)$ of $2 \times 2$ matrices over a finite field. Finally, in Theorem 6.8 –one of the main results– we characterize all groups $G$ (finite or not) in which every invariant metric is also a bi-invariant one. We prove that either $G$ is abelian or $G = Q_8 \times H$ where $Q_8$ is the quaternion group and $H$ is an abelian elementary 2-group. In particular, if $G$ is finite it must be of the form $G = Q_8 \times \mathbb{Z}_2^k$ for some $k \in \mathbb{Z}_{\geq 0}$.

Finally, in Section 7, by using the results of Section 5 we count the number of invariant and bi-invariant metrics for the 144 groups of order up to 32 (see Theorem 7.1 and Tables 6–10).

2. Invariant Metrics and $\mathcal{P}$-Equivalence

From now on we assume that $(G, d)$ is a metric group. The metric $d$ is called right translation invariant or left translation invariant if for any $g, g', h \in G$ we respectively have

\[ d(gh, g' h) = d(g, g') \quad \text{or} \quad d(hg, hg') = d(g, g'). \]  

If $G$ is abelian both notions coincide and $d$ is called translation invariant (or bi-invariant).

Given a metric group $(G, d)$ we have the induced weight function $w : G \to \mathbb{R}_{\geq 0}$ defined by $w(x) = d(x, e)$ for any $x \in G$ where $e$ is the identity element of $G$. Conversely, if $(G, w)$ is a weight space, one can define a right (resp. left) translation invariant metric $d$ on $G$ by

\[ d(x, y) = w(xy^{-1}) \]  

(resp. $d(x, y) = w(y^{-1}x)$) for every $x, y \in G$ provided that $w(x^{-1}) = w(x)$ for every $x \in G$, which are denoted $d(x, y) = w(x - y)$ (resp. $d(x, y) = w(y - x)$) and $w(-x) = w(x)$ if $G$ is abelian.

It is equivalent to study right-invariant metrics or left-invariant metrics on non-abelian groups. To fix ideas and notations we will always assume that the metrics are right-invariant.

Let $(G, d)$ be a metric group and let $\mathcal{S}_G$ be the permutation group of $G$. Note that $G$ can be seen as a subgroup of $\mathcal{S}_G$ by identifying $g \in G$ with $\sigma_g \in \mathcal{S}_G$ where $\sigma_g(x) = xg$. We denote by $G_R$ the group $G$ seen as the group of right translations inside $\mathcal{S}_G$.

**Definition 2.1.** We say that $(G, d)$ is $\sigma$-invariant for $\sigma \in \mathcal{S}_G$, denoted $d^\sigma = d$, if

\[ d^\sigma(x, y) := d(\sigma(x), \sigma(y)) = d(x, y) \]  

for all $x, y \in G$. The symmetry group of $(G, d)$ is defined by $\Gamma(G, d) = \{ \sigma \in \mathcal{S}_G : d^\sigma = d \}$.

Clearly, to say that $(G, d)$ is $\sigma$-invariant for every $\sigma \in G_R$ is equivalent to say that $d$ is right translation invariant. From now on $(G, d)$ will denote a metric group where $d$ is a right translation invariant metric. We will use the following set

\[ M(G) = \{ \text{right-invariant metrics on } G \}. \]
Note that \( \mathcal{M}(G) \) is infinite; for example, given a metric \( d \) on \( G \) and \( \alpha > 0 \), we have that 
\[ d_{\alpha} = \alpha d \in \mathcal{M}(G), \]
where \( d_{\alpha} \) is defined by 
\[ d_{\alpha}(x, y) = \alpha d(x, y) \]
for all \( x, y \in G \). Similarly, given \( d_1, d_2 \in \mathcal{M}(G) \) then 
\[ d_1 + d_2 \in \mathcal{M}(G), \]
where 
\[ (d_1 + d_2)(x, y) = d_1(x, y) + d_2(x, y) \]
for all \( x, y \in G \). The addition of metrics is associative and hence \( \mathcal{M}(G) \) is a semigroup. It is worth noticing that in general the product of metrics \( (d_1 \cdot d_2)(x, y) = d_1(x, y)d_2(x, y) \) is not a metric, although it is a semimetric, that is a function that have all the properties of a metric but where the triangle inequality does not necessarily holds.

**Partitions and \( \mathcal{P} \)-equivalence of metrics.** Let \((G, d)\) be a metric group (recall that \( d \) is right translation invariant) with associated weight function \( w \). The **induced partition** of \((G, d)\), denoted \( \mathcal{P} = \mathcal{P}(G, d) \), is the partition of \( G \) determined by the equivalence relation 
\[(2.4) \quad g \sim h \iff w(g) = w(h) \iff d(g, e) = d(h, e)\]
for any \( g, h \in G \).

In general we will write \( \mathcal{P}(G, d) = \{P_i\}_{i \in \mathcal{I}} \) for a partition of \( G \), with \( 0 \in \mathcal{I} \), and thus we have 
\[ P_i = \{g \in G : w(g) = w_i\} \]
for \( i \in \mathcal{I} \) and some sequence \( \{w_i\}_{i \in \mathcal{I}} \) of different non-negative real numbers. We also write \( w(P_i) = w_i \) for \( i \in \mathcal{I} \). If the partition \( \mathcal{P}(G, d) \) is finite, in particular for finite groups, we will denote by \( P_0, \ldots, P_s \) the parts of \( \mathcal{P}(G, d) \), that is \( \mathcal{P} = \{P_0, \ldots, P_s\} \). Thus, if \( w_1, \ldots, w_s \) are the different nonzero weights of \( G \) (i.e. the different real values that \( w(g) \) can take for \( g \in G \)) we will say that \( G \) is an **s-weight** metric group.

We will say that \( \mathcal{P} \) is **unitary** if \( \{e\} \in \mathcal{P} \), in which case we assume that \( P_0 = \{e\} \), and that it is **symmetric** if \( P_i = P_i^{-1} \), i.e. \( g \in P_i \) if and only if \( g^{-1} \in P_i \), for all \( i \in \mathcal{I} \). We will use the following set
\[(2.5) \quad \mathcal{P}(G) = \{\text{unitary symmetric partitions of } G\}.\]

We now show the basic fact that any partition induced by a metric on a group is unitary and symmetric.

**Lemma 2.2.** If \((G, d)\) is a metric space then \( \mathcal{P} = \mathcal{P}(G, d) \) is unitary and symmetric.

**Proof.** Clearly we have that \( d(x, y) = 0 \) if and only if \( x = y \), then \( w(x) = 0 \) if and only if \( x = e \), where \( e \) is the identity of \( G \). Thus, \( \{e\} \in \mathcal{P} \), that is, \( \mathcal{P} \) is unitary. Now, we have that \( w(x) = d(x, e) = d(e, x) = w(x^{-1}) \) for all \( x \in G \) and hence \( P \) is also symmetric. \( \square \)

The lemma implies that we have a well defined map
\[(2.6) \quad \pi_G : \mathcal{M}(G) \longrightarrow \mathcal{P}(G), \quad d \mapsto \mathcal{P}(G, d).\]

**Definition 2.3.** We say that two metrics \( d_1, d_2 \) on \( G \) are **\( \mathcal{P} \)-equivalent**, and we denote it by \( d_1 \sim_\mathcal{P} d_2 \), if they induce the same unitary symmetric partition of \( G \). In symbols,
\[(2.7) \quad d_1 \sim_\mathcal{P} d_2 \iff \mathcal{P}(G, d_1) = \mathcal{P}(G, d_2).\]

We will denote by \( \mathcal{M}(G) / \sim_\mathcal{P} \) the space of \( \mathcal{P} \)-equivalence classes of metrics on \( G \).

Obviously, a metric \( d \) is \( \mathcal{P} \)-equivalent to any scalar multiple \( d_{\alpha} = \alpha d \) of \( d \) with \( \alpha \in \mathbb{R}_{>0} \). However, if \( d_1 \sim_\mathcal{P} d_1' \) and \( d_2 \sim_\mathcal{P} d_2' \) it is not true in general that \( d_1 + d_2 \sim_\mathcal{P} d_1' + d_2' \). For example, consider the following two metrics on \( G = \mathbb{Z}_4 \), the Lee metric \( d_{\text{Lee}} \) given by \( w_{\text{Lee}}(0) = 0 \), \( w_{\text{Lee}}(1) = w_{\text{Lee}}(3) = 1 \) and \( w_{\text{Lee}}(2) = 2 \), and the metric \( d \) induced by the weight function \( w(0) = 0 \), \( w(2) = 1 \) and \( w(1) = w(3) = 2 \). It is clear that \( d_{\text{Lee}} \sim_\mathcal{P} d \). On the other hand, \( d_{\text{Lee}} + d_{\text{Lee}} = 2d_{\text{Lee}} \sim_\mathcal{P} d_{\text{Lee}} \) and \( d_{\text{Lee}} + d = 3d_{\text{Ham}} \sim_\mathcal{P} d_{\text{Ham}} \). Since \( d_{\text{Lee}} \not\sim_\mathcal{P} d_{\text{Ham}} \) we have that
\[ d_{\text{Lee}} + d_{\text{Lee}} \not\sim_\mathcal{P} d_{\text{Lee}} + d. \]
This shows that it could be not so easy to attach a group structure to the space of equivalent metrics of \( G \). We pose the question: Which metric, topological or algebraic structure can be given to \( \mathcal{M}(G) / \sim_\mathcal{P} \)?
The map $\pi_G$ is clearly not injective. We now show that $\pi_G$ in (2.6) is surjective.

**Lemma 2.4.** If $\mathcal{P}$ is a unitary symmetric partition of $G$ with $|\mathcal{P}| \leq |\mathbb{R}|$, then there exists a right-invariant metric $d \in \mathcal{M}(G)$ such that $P(G, d) = \mathcal{P}$. Moreover, if $|\mathcal{P}|$ is finite, then $d$ can be taken to be integral.

**Proof.** Let $\mathcal{P} = \{P_i\}_{i \in I}$ with $P_0 = \{e\}$. Consider a sequence of real numbers $\{a_i\}_{i \in I} \subset [1, 2]$ all mutually different, and define the following weight function on $G$:

$$w(x) = \begin{cases} 0, & \text{if } x = e, \\ a_i, & \text{if } x \neq e, \ x \in P_i, \ i \in I \setminus \{0\}. \end{cases}$$

(2.8)

This defines the right-invariant metric $d$ given by $d(x, y) = w(xy^{-1})$. Since $\mathcal{P}$ is unitary, then $d(x, y) = 0$ if and only if $x = y$, and by symmetry of the partition we have that $d(x, y) = w(xy^{-1}) = w(yx^{-1}) = d(y, x)$. Also $d(x, y) \geq 0$, for all $x, y \in G$, and $d$ satisfies the triangle inequality since $d(x, y) \leq 2 \leq d(x, z) + d(z, y)$ for all $x, y, z \in G$ with $z \neq x, y$ (in the cases that $z = x$ or $z = y$ the inequality is trivial).

Finally, we have that $P_0(G, d) = \{e\} = P_0$ and $P_i(G, d) = \{x \in G : w(x) = a_i\} = P_i$ for any $i \in I, i \neq 0$, by construction. Moreover, if $I$ is finite, the numbers $a_i$ can be chosen to be rational, and hence multiplying them by a suitable integer one can obtain a new weight $w_N$ function having the same partition as before but taking only integer values. □

We have the following useful relation between invariant metrics and symmetric partitions.

**Proposition 2.5.** Let $(G, d)$ be a metric group. There is a 1–1 correspondence

$$\mathcal{M}(G)/\sim_{\mathcal{P}} \longleftrightarrow \mathcal{P}(G)$$

between $\mathcal{P}$-classes of right-invariant metrics of $G$ and unitary symmetric partitions $P$ of $G$ with $|P| \leq |\mathbb{R}|$.

**Proof.** Consider the map $\pi_G : \mathcal{M}(G)/\sim_{\mathcal{P}} \longrightarrow \mathcal{P}(G)$ given by $\pi_G([d]) = P(G, d)$. The map $\pi_G$ is well defined and surjective by Lemmas 2.2 and 2.4, and it is clearly injective by definition. □

We note that there is no canonical way to associate a metric to a given unitary symmetric partition of $G$. For example in the finite case, given such a partition $\mathcal{P} = \{P_0, P_1, \ldots, P_n\}$, using the weight defined in the proof of Lemma 2.4 one can obtain a metric, although this depends on the choice of the numbers $a_i$. For instance, we can take the numbers $a_i = 1 + \frac{1}{i}$ for $i = 1, \ldots, n$ and define the weight $w(x)$ on $G$ as in (2.8). Multiplying by $n$ we get an integral weight, but with a gap between $w_0 = 0$ and the minimum non-zero weight $w_1$.

**Bi-invariant metrics.** A bi-invariant metric is both a right-invariant and a left-invariant metric (see (2.1) and (2.2)). For abelian groups all these notions (R-invariance, L-invariance and bi-invariance) coincide.

As one can imagine, bi-invariant metrics are closely related with conjugacy classes. Let $(G, d)$ be a metric group with weight $w$. If $d$ is bi-invariant then $w$ is constant on conjugacy classes. That is, for every $x, y \in G$ we have

$$w(x) = d(x, e) = d(yx, y) = d(xy^{-1}, y) = d(yx^{-1}, e) = w(yxy^{-1})$$

where in the second and third equalities we have used left and right invariance, respectively. Conversely, if $d$ is right (or left) invariant and $w$ is constant on conjugacy classes then $d$ is bi-invariant. In fact, for every $x, y, z \in G$ we have

$$d(zx, zy) = w(zx(zy)^{-1}) = w(zxy^{-1}z^{-1}) = w(xy^{-1}) = d(x, y).$$
The partition $P_0, P_1, \ldots, P_s$ associated to a bi-invariant metric of $G$ is unitary symmetric and, by the previous comments, also conjugate, i.e. $gP_i g^{-1} = P_i$ for all $g \in G$ and $i = 0, \ldots, s$. We point out that the classes of equivalence of bi-invariant metrics are in 1–1 correspondence with the unitary symmetric conjugate partitions $P^c(G)$ of $G$, that is
\begin{equation}
\mathcal{M}(G) / \sim_{P^c} \leftrightarrow P^c(G)
\end{equation}
where $\sim_{P^c}$ denotes the equivalence relation on invariant metrics given by identifying those metrics having the same unitary symmetric conjugate partition of $G$. In fact, if $G$ has a unitary symmetric and conjugate partition $\mathcal{P}'$, then there is a bi-invariant metric $d$ on $G$ such that $P(G, d) = \mathcal{P}'$. In fact, since $\mathcal{P}'$ is unitary and symmetric then, by Lemma 2.4, there is a right-invariant metric $d$ such that $P(G, d) = \mathcal{P}'$. Since $\mathcal{P}'$ is conjugate, then $d$ is also left-invariant.

3. Symmetry groups and distance graphs

In this section we give a relation between the symmetry group and the distance graph of a metric group $(G, d)$.

The symmetry group of a metric group. From Definition 2.1, the symmetry group of a metric group $(G, d)$ is the set of all the permutations of $G$ preserving the metric $d$, that is
\begin{equation}
\Gamma(G, d) = \{ \sigma \in S_G : d(\sigma(x), \sigma(y)) = d(x, y) \}.
\end{equation}
We have the following equivalence relation in $\mathcal{M}(G)$, $d_1 \sim_\Gamma d_2$ if and only if $\Gamma(G, d_1) = \Gamma(G, d_2)$. In this case, we will say that $d_1$ and $d_2$ are $\Gamma$-equivalent. Note that $\mathcal{P}$-equivalence of metrics implies $\Gamma$-equivalence of metrics.

To study these symmetry groups we will need the following generalization of dihedral groups.

Definition 3.1. Let $G$ be an abelian group and $i : G \to G$ the inversion, given by $i(g) = g^{-1}$. We define the dihedral group of $G$ by $\mathbb{D}(G) = G \rtimes \langle i \rangle$.

If $G$ is the cyclic group of $n$ elements, then $\mathbb{D}(\mathbb{Z}_n) \simeq \mathbb{D}_n$, the classic dihedral group, for $n \geq 3$ while $\mathbb{D}(\mathbb{Z}_2) \simeq \mathbb{Z}_2$. That is, $\mathbb{D}(G)$ can be abelian or not. Also, $G \subseteq \mathbb{D}(G)$ but the equality may happen. Moreover, if $G$ is abelian and $G \neq \mathbb{Z}_2$ then $\mathbb{D}(G) = G \times \mathbb{Z}_2$.

By definition, $\mathbb{D}(G) \subseteq S_G$. However, notice that $\mathbb{D}(G) \leq \text{Hol}(G) \simeq N_{S_G}(G) \leq S_G$, where $\text{Hol}(G) = G \rtimes \text{Aut}(G)$ is the holomorph of $G$ and $N_{S_G}(G) = \{ \sigma \in S_G : \sigma G \sigma^{-1} = G \}$ is the normalizer of $G$ in $S_G$. Since $i \in \text{Aut}(G)$ we have that $\mathbb{D}(G)$ is a subgroup of $\text{Hol}(G)$, while the isomorphism $\text{Hol}(G) \simeq N_{S_G}(G)$ is obtained via the identification of $G$ with its right regular representation $G_R$. From now on, we will think of $\mathbb{D}(G)$ as a subgroup of $S_G$ where the inversion acts by permuting each element of $G$ with its inverse.

For abelian groups, the following is useful when trying to compute the symmetric group of a metric space $(G, d)$.

Proposition 3.2. Let $(G, d)$ be an abelian metric group. Then, the inversion automorphism on $G$ preserves distances, and hence we have that
\begin{equation}
\mathbb{D}(G) \leq \Gamma(G, d) \leq S_G.
\end{equation}

Proof. The metric $d$ is (right) traslation invariant, i.e. $G \leq \Gamma(G, d)$. For a group, the inversion is an automorphism if and only if the group is abelian. So, since $G$ is abelian, we only need to prove that it preserves distances. In fact, we have that
\begin{align*}
d(a, b) = w(ab^{-1}) = w(b^{-1}a) = d(b^{-1}, a^{-1}) = d(i(b), i(a)) = d(i(a), i(b)),
\end{align*}

hence $\mathbb{D}(G) \leq \Gamma(G, d)$, as we wanted to show. \qed
We now recall some classes of automorphisms. The group of inner automorphisms of $G$ is the one formed only by conjugations, that is $\text{Inn}(G) = \{ \varphi_g : g \in G \}$, where $\varphi_g(x) = g x g^{-1}$ for every $x \in G$. An automorphism $\sigma$ of a group $G$ is called a class-preserving automorphism if for every $g \in G$, there exists an element $h \in G$ such that $\sigma(g) = h g h^{-1}$. These automorphisms form the subgroup denoted as $\text{Aut}_{\text{ecp}}(G)$. Clearly every inner automorphism of $G$ is class-preserving. Similarly, an automorphism $\sigma$ of a group $G$ is called a class-inverting automorphism if, for any $g \in G$, there exists an element $h \in G$ such that $\sigma(g) = h g^{-1} h^{-1}$. The class-preserving and the class-inverting automorphisms form the subgroup of extended class-preserving automorphism group, denoted by $\text{Aut}_{\text{ecp}}(G)$. That is, the subgroup of all the automorphisms that sends every element either to an element in its conjugacy class or to an element in the conjugacy class of its inverse. Summing up, we have

(3.3) $\text{Inn}(G) \leq \text{Aut}_{\text{ecp}}(G) \leq \text{Aut}_{\text{ecp}}(G) \leq \text{Aut}(G)$.

For general metric groups we can give a similar result as the previous proposition, provided that the metric involved is bi-invariant.

**Proposition 3.3.** If $(G, d)$ is a bi-invariant metric group then

(3.4) $G \rtimes \text{Aut}_{\text{ecp}}(G) \leq \Gamma(G, d) \leq S_G$.

In particular if $G$ is abelian then $G \rtimes \text{Aut}_{\text{ecp}}(G) = D(G)$.

**Proof.** The metric $d$ is right translation invariant, i.e. $G \leq \Gamma(G, d)$. Let $\sigma \in \text{Aut}(G)$. If $\sigma$ is class-preserving then

$$d(a, b) = w(ab^{-1}) = w(\sigma(ab^{-1})) = w(\sigma(a)\sigma(b)^{-1}) = d(\sigma(a), \sigma(b)),$$

while if $\sigma \in \text{Aut}(G)$ is class-inverting then

$$d(a, b) = w(ab^{-1}) = w(ba^{-1}) = w(\sigma(ab^{-1})) = w(\sigma(a)\sigma(b)^{-1}) = d(\sigma(a), \sigma(b)),$$

hence $\text{Aut}_{\text{ecp}}(G) \leq \Gamma(G, d)$, which implies $G \rtimes \text{Aut}_{\text{ecp}}(G) \leq \Gamma(G, d)$ as we wanted to show.

If $G$ is abelian, the only class-preserving automorphism is the identity and the only class-inverting automorphism is the inversion. In this way, we have that $\text{Aut}_{\text{ecp}}(G) = \{ i \}$, and hence $G \rtimes \text{Aut}_{\text{ecp}}(G) = D(G)$, by Definition 3.1, as we wanted to show. \qed

It is known that

(3.5) $\text{Inn}(G) \simeq G/\mathcal{Z}(G)$

for any group $G$. Hence, we have that $\text{Inn}(S_n) = S_n$ for $n \geq 3$, $\text{Inn}(\mathbb{Z}_n) = \mathbb{D}_n$ for $n$ odd and $\text{Inn}(\mathbb{D}_n) = \mathbb{D}_n/\mathbb{Z}_2$ for $n$ even and also $\text{Inn}(\mathbb{Q}_4n) = \mathbb{Q}_{4n}/\mathbb{Z}_2$ for any $n \geq 2$.

**Example 3.4.** Let $G = \mathbb{D}_{2n}$ or $G = \mathbb{Q}_{4n}$ for any $n \geq 2$. By (3.3), (3.4) and (3.5) we have $G \rtimes G/\mathcal{Z}_2 \leq \Gamma(G, d)$ where $d$ is any bi-invariant metric on $G$. Thus, $8n^2 \leq |\Gamma(G, d)|$. If also the partition of $d$ has a part of the form $P_i = \{ g_i, g_i^{-1} \}$ with $g_i$ of order $2n$ (i.e. the graph of distances –see below– contains 2 disjoint cycles $C_{2n}$), as for instance in the case of the Lee metric, then $\Gamma(G, d) \leq \mathbb{D}_{2n} \rtimes \mathbb{Z}_2$ (see Corollary 3.6). Thus, $2^{3} n^2 \leq |\Gamma(G, d)| \leq 2^{5} n^2$, in this case. \end{example}
Next we will relate the symmetry group of \((G, d)\) with the automorphism group of the distance graph of \((G, d)\), which is the same automorphism of the graph \(\mathcal{G}(G, \mathcal{P})\). We need to recall Cayley graphs. Given a group \(G\) and a subset \(S\) of \(G\) one has a Cayley graph \(Cay(G, S)\) where \(G\) is the vertex set and two elements \(g, h\) form a (directed) edge from \(g\) to \(h\) if and only if \(g - h \in S\). If \(G\) is abelian we consider the Cayley graph undirected. If \(e \notin S\) the graph \(Cay(G, S)\) is loopless and if \(S\) is symmetric (\(S = S^{-1}\)) then it is a simple (undirected with no multiple edges) graph.

**Theorem 3.5.** Let \((G, d)\) be a metric group and let \(\mathcal{P} = \{P_0, P_1, \ldots, P_s\}\) be its unitary symmetric partition. Then, we have

\[
\Gamma(G, d) = \bigcap_{1 \leq i \leq s} \text{Aut}(Cay(G, P_i)).
\]

**Proof.** The automorphism group \(\text{Aut}(\mathcal{G}(G, d))\) of the distance graph \((G, d)\) is the group of all edge-preserving bijections \(\sigma\) of \(G\) also preserving the weights of the graph, i.e. if \(xy\) is an edge then \(\omega_d(\sigma(x)\sigma(y)) = \omega_d(xy)\). In this way, we have that

\[
\Gamma(G, d) = \text{Aut}(\mathcal{G}(G, d)).
\]

Now, as mentioned \(\text{Aut}(\mathcal{G}(G, d)) = \text{Aut}(\mathcal{G}(G, \mathcal{P}))\) and note that each part \(P_i\) of the partition induces a simple Cayley graph \(Cay(G, P_i)\). In fact, if \(0 < w_1 < \cdots < w_s\) are the different nonzero weights of \(d\) and \(P_i = \{g \in G : w(g) = w_i\}\) for \(1 \leq i \leq s\), then

\[
\{gh\} \in E \iff g - h \in P_i \iff d(g, h) = w_i.
\]

We can consider the graph \(Cay(G, P_i)\) naturally weighted by the distances of elements in \(G\). In this way, the distance graph of \((G, d)\) is the union of simple Cayley graphs, that is

\[
\mathcal{G}(G, d) = \bigcup_{1 \leq i \leq s} Cay(G, P_i).
\]

We recall that the union of graphs \(G_i = (V_i, E_i), i = 1, 2, \) is the graph \(G = (V_1 \cup V_2, E_1 \cup E_2)\). Then, by (3.8) we have that \(\text{Aut}(\mathcal{G}(G, d)) = \bigcap_{1 \leq i \leq s} \text{Aut}(Cay(G, P_i))\). In particular, if \(\sigma \in \text{Aut}(Cay(G, P_i))\), then \(\sigma\) is a permutation that preserves the weights of \(G_1 \cup G_2\). That is, it preserves the edges of \(G_1\) and \(G_2\), so the restriction to both of the graphs \(G_1, G_2\) is an automorphism of the corresponding graphs. On the other hand, a permutation that preserves the edges of \(G_1\) and \(G_2\) preserves the weights of \(G_1 \cup G_2\). The result is automatic from (3.7) and (3.9).

As usual, we denote by \(C_n\) the cycle graph of \(n\) vertices (\(n\)-cycle) and by \(mC_n\) the disjoint union of \(m\) copies of \(C_n\). We will also use the convention \(C_2 = K_2\). Given two groups \(G, H\), if \(H\) acts on \(\{1, 2, \ldots, n\}\) then the semidirect product \(G^n \rtimes H\) is called the wreath product of \(G\) and \(H\) and denoted by \(G \wr H\). The smallest non trivial wreath product is \(\mathbb{Z}_2 \wr \mathbb{Z}_2 \simeq \mathbb{D}_4\). We have the following consequence of Proposition 3.2 and Theorem 3.5.

**Corollary 3.6.** Consider a metric group \((G, d)\) of order \(n\) with associated unitary symmetric partition \(\mathcal{P} = \{P_0, P_1, \ldots, P_s\}\). If \(\text{Cay}(G, P_i) = mC_{\frac{n}{m}}\) for some \(m | n\) and \(i = 1, \ldots, s\), then

\[
\text{Aut}(\text{Cay}(G, P_i)) = \mathbb{D}_{\frac{n}{m}} \wr S_m
\]

and hence \(\Gamma(G, d) \leq \mathbb{D}_{\frac{n}{m}} \wr S_m\). In particular, if \(m = 1\) or \(m = n\) then \(\Gamma(G, d) = \mathbb{D}_n\) or \(\Gamma(G, d) = S_n\), respectively. Thus, \(\Gamma(G, d_{Ham}) \simeq S_n\) for any \(G\) and \(\Gamma(\mathbb{Z}_n, d_{Lee}) \simeq \mathbb{D}_n\) for \(n \in \mathbb{N}\).
Proof. By (3.2) and (3.6), we have that $\mathbb{D}_n \leq \Gamma(G, d) \leq \text{Aut}(\text{Cay}(G, P_i))$ for every $i = 1, \ldots, s$. Also, if $\text{Cay}(G, P_i) = C_n$ for some $i$ then $\text{Aut}(\text{Cay}(G, P_i)) = \mathbb{D}_n$. This immediately implies that $\Gamma(G, d) = \mathbb{D}_n$. Now, if $\text{Cay}((G, P_i)) = m C_{\frac{n}{m}}$ for some $i$ then we have (3.10) since we can permute the $m$ different $\frac{n}{m}$-cycles to each other and in each $\frac{n}{m}$-cycle we get the dihedral group $\mathbb{D}_m$ as permutation group. \hfill $\Box$

4. Invariant metrics for small groups

We now study the invariant and bi-invariant metrics and the associated partitions, symmetry groups and distance graphs of the smallest groups of order up to 7. More precisely, we find all the $P$-classes of metrics of $G \in \{\{0\}, \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_6, S_3, S_7\}$ by giving their unitary symmetric (conjugate) partitions and distance graphs. With the aid of these graphs, we will also compute the associated symmetry groups. Furthermore, we will indicate which of the metrics are poset metrics [3], homogeneous metrics [9], chain metrics or extended Lee metrics [17].

All the information on the metrics will be given in Tables 1–5. There, we give the weight functions $w_i$ associated to the metrics $d_i$ (omitting the trivial weights $w_i(0) = 0$). In the column labeled $s$ we count the number of non-trivial different weights of each given metric and in the second to last column we indicate some known metrics which fall in the same equivalence class of the given metric. In each row of the table, the weights $w_i$ correspond to the first metric listed in this penultimate column. In the last column we give the symmetry groups (as abstract groups).

Some general families of invariant metrics. We now recall the definitions of some particular classes of invariant metrics:

(a) Hamming metric: Let $G = G_1 \times \cdots \times G_n$ with $G_1, \ldots, G_n$ groups. The Hamming metric $d_{\text{Ham}}^n$ on $G$ is determined by the Hamming weight given by

$$w_{\text{Ham}}^n(x) = \#\{1 \leq i \leq n : x_i \neq e_{G_i}\}$$

where $x = (x_1, \ldots, x_n) \in G$ and $e_{G_i}$ denotes the identity element in $G_i$. For $n = 1$ we simply write $d_{\text{Ham}}$ and $w_{\text{Ham}}$.

(b) Poset metric [3]: Let $G = G_1 \times \cdots \times G_n$ with $G_1, \ldots, G_n$ groups, and consider the poset $P = \{[n], \leq\}$ where $\leq$ is a partial order on $[n] = \{1, 2, \ldots, n\}$. A subset $I \subset [n]$ is an ideal if given $a \in [n]$ and $b \in I$ with $a \leq b$ then $a \in I$. If $A \subset [n]$ is a subset, then $\langle A \rangle$ denotes the minimal ideal containing $A$. The weight function on $G$ associated to the poset $P$ is given by

$$w_P(g) = |\langle \text{supp}(g) \rangle| \quad \text{where} \quad \text{supp}(g) = \{i \in [n] : g_i \neq e\}$$

denotes the support of $g = (g_1, \ldots, g_n) \in G$. This induces the poset metric $d_P$ on $G$. For the $n$-antichain (trivial poset of $[n]$) we get the classic Hamming metric $d_{\text{Ham}}^n$ on $n$ coordinates.

(c) Chain metric [17]: Let $G$ be a group and $\mathcal{C}$ a chain $\langle e \rangle = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_n = G$ of subgroups of $G$. The chain metric $d_{\mathcal{C}}$ on $G$ associated to $\mathcal{C}$ is defined by

$$d_{\mathcal{C}}(x, y) = i \quad \Leftrightarrow \quad x - y \in H_i \setminus H_{i-1}$$

for $i = 0, \ldots, n$. Here we use the convention $H_{-1} = \emptyset$. If $\mathcal{C}$ is simply $\{e\} \subsetneq H \subsetneq G$ we denote the chain metric by $d_H$. In this case we have $w_H(e) = 0$ and

$$w_H(x) = \begin{cases} 1 & \text{if } x \in H \setminus \{e\}, \\ 2 & \text{if } x \in G \setminus H. \end{cases}$$

Also, for the trivial chain $\{e\} \subset G$ we get the Hamming metric $d_{\text{Ham}}$ on $G$.

(d) Lee and extended Lee metrics [17]: The well-known Lee metric $d_{\text{Lee}}$ on $\mathbb{Z}_m$ is given by

$$w_{\text{Lee}}(x) = \min\{x, m - x\}.$$
Now, for each \( n \mid m \), there is an extended Lee metric \( d_{n,\text{Lee}} \) on \( \mathbb{Z}_m \) with associated weight [17]:

\[
w_{n,\text{Lee}}(x) = \begin{cases} 
  x & \text{if } x \leq n, \\
  n & \text{if } n \leq x \leq m - n, \\
  n - x & \text{if } m - n \leq x \leq m - 1.
\end{cases}
\]

For even \( m \) and \( n = \frac{m}{2} \) we have \( d_{n,\text{Lee}} = d_{\text{Lee}} \).

(e) Homogeneous metric: There is a classic definition of a metric given by the homogeneous weight on the ring \( \mathbb{Z}_m \) [4]. Greferath and Schmidt [9] extended this to any finite ring \( R \) with identity \( 1 \neq 0 \) (see Definition 1.2). They showed that a weight function \( w \) on \( R \) (in a weaker sense) is an homogeneous weight on \( R \) if and only if there is a constant \( \lambda \geq 0 \) such that (see Theorem 1.3)

\[
w(x) = \lambda \left( 1 - \frac{\mu(x, Rx)}{|R : x|} \right)
\]

where \( R^x x = \{ y \in R : R y = Rx \} \) and \( \mu \) is the Möbius function on the poset of its principal left ideals \( \{ Rx : x \in R \} \) ordered by inclusion, that is \( \mu(Rx, Rx) = 1 \) and

\[
\mu(Rx, Rz) = - \sum_{Rx \leq Ry < Rz} \mu(Rx, Ry).
\]

This weight does not define a metric in general. When this is the case we denote the metric by \( d_{\text{hom}} \). Given an abelian group \( G \) one can consider all different ring structures on \( G \) and their corresponding homogeneous weights. It is known that for the group \( \mathbb{Z}_n \) of integers modulo \( n \) there is one ring structure for each \( k \mid n \) determined by \( 1 \cdot 1 = k \) (see [7]). However, only \( k = 1 \) gives rise to a ring with identity, which is necessarily the ring of integers modulo \( n \). We denote this metric by \( d_{\text{hom}} \).

**Invariant metrics for small groups.** For the trivial group \( \{0\} \) anything is trivial. Namely, \( \mathcal{P}_1(\{0\}) = \{\{0\}\} \), hence the metric is the Hamming one and \( \Gamma(\{0\}, d_{\text{Ham}}) = S_1 = \{0\} \). Next, we consider the smallest cyclic groups of prime order \( \mathbb{Z}_2, \mathbb{Z}_3 \) and \( \mathbb{Z}_5 \), which are also trivial.

**Example 4.1.** For the groups \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \) there is only one unitary symmetric partition: \( \mathcal{P}_1(\mathbb{Z}_2) = \{\{0\}, \{1\}\} \) and \( \mathcal{P}_1(\mathbb{Z}_3) = \{\{0\}, \{1, 2\}\} \). In both cases the metric is the Hamming metric and the symmetry groups are \( \Gamma(\mathbb{Z}_2, d_{\text{Ham}}) = S_2 \) and \( \Gamma(\mathbb{Z}_3, d_{\text{Ham}}) = S_3 \). For \( \mathbb{Z}_5 \) there are just two unitary symmetric partitions: \( \mathcal{P}_1(\mathbb{Z}_5) = \{\{0\}, \{1, 2, 3, 4\}\} \) and \( \mathcal{P}_2(\mathbb{Z}_5) = \{\{0\}, \{1, 4\}, \{2, 3\}\} \). The associated metrics are respectively the Hamming and the Lee metrics and hence the corresponding symmetry groups are \( \Gamma(\mathbb{Z}_5, d_{\text{Ham}}) = S_5 \) and \( \Gamma(\mathbb{Z}_5, d_{\text{Lee}}) = D_5 \).

In the next two examples we determine all invariant metrics for a group \( G \) of order 4.

**Example 4.2.** Consider the group \( \mathbb{Z}_4 = \{0, 1, 2, 3\} \). It is clear that \( \mathbb{Z}_4 \) has only the two unitary symmetric partitions: \( \mathcal{P}_1 = \{\{0\}, \{1, 2, 3\}\} \) and \( \mathcal{P}_2 = \{\{0\}, \{1, 3\}, \{2\}\} \), which correspond with the Hamming and the Lee metrics \( (d_1 = d_{\text{Ham}}, d_2 = d_{\text{Lee}}) \). The corresponding partition graphs and symmetry groups are given in Figures 1 and 2.

**Fig. 1.** \( \mathcal{G}_1 = \mathcal{G}(\mathbb{Z}_4, d_{\text{Ham}}) \)  
**Fig. 2.** \( \mathcal{G}_2 = \mathcal{G}(\mathbb{Z}_4, d_{\text{Lee}}) \)
Although we know by Corollary 3.6 that the symmetry group for \((\mathbb{Z}_4, d_{Lee})\) is \(\mathbb{D}_4\), we explain how it can be obtained using the graphs. By (3.6) we have
\[
\Gamma(\mathbb{Z}_4, d_{Lee}) = \text{Aut}(\text{Cay}(\mathbb{Z}_4, \{1, 3\})) \cap \text{Aut}(\text{Cay}(\mathbb{Z}_4, \{2\})).
\]
Also, \(\text{Aut}(\text{Cay}(\mathbb{Z}_4, \{1, 3\})) = \text{Aut}(C_4) \simeq \mathbb{D}_4\) and \(\text{Aut}(\text{Cay}(\mathbb{Z}_4, \{2\})) = \text{Aut}(2K_2) \simeq \mathbb{Z}_2 \wr \mathbb{Z}_2 \simeq \mathbb{D}_4\), by Corollary 3.6. The two automorphism groups of \(\text{Cay}(\mathbb{Z}_4, \{1, 3\})\) and \(\text{Cay}(\mathbb{Z}_4, \{2\})\) are not only isomorphic, they are actually equal by (3.2). Hence, we finally get \(\Gamma(\mathbb{Z}_4, d_{Lee}) = \mathbb{D}_4\).

From the graphs, we see that we have the following weight functions on \(\mathbb{Z}_4\):

| \(\mathbb{Z}_4\) | 1 | 2 | 3 | \(s\) | \(\text{chain hom.}\) | \(d_{Ham}\) | \(\Gamma\) |
|---|---|---|---|---|---|---|---|
| \(w_1\) | 1 | 1 | 1 | 1 | \(\checkmark\) | \(\times\) | \(d_{Lee} = d_{2, Lee}, d_{z_2}, d_{hom}\) | \(S_4\) |
| \(w_2\) | 1 | 2 | 1 | 2 | \(\checkmark\) | \(\checkmark\) | \(d_{Lee} = d_{2, Lee}, d_{z_2}, d_{hom}\) | \(\mathbb{D}_4\) |

We explain the metrics. Note that by (4.2) the chain metric \(d_{z_2}\) given by \(\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_4\) is equivalent to \(d_{Lee}\) but \(d_{Lee} \neq d_{z_2}\). Also, by (4.3) we have \(d_{2, Lee} = d_{Lee}\). The ring of integers modulo 4 has poset of principal ideals given by \(\{0\} \leq 2\mathbb{Z}_4 \leq \mathbb{Z}_4\) where \(\mu(\{0\}, \{0\}) = 1\), \(\mu(\{0\}, 2\mathbb{Z}_4) = -1\) and \(\mu(\{0\}, \mathbb{Z}_4) = 0\). Now, by (4.4), we have \(w_{hom}(1) = \lambda\), \(w_{hom}(2) = 2\lambda\), \(w_{hom}(3) = \lambda\). Finally, note that for \(\lambda = 1\) we have \(d_{Lee} = d_{hom}\).

**Example 4.3.** Consider the group \(\mathbb{Z}_2 \times \mathbb{Z}_2\). It is straightforward to check that \(\mathbb{Z}_2 \times \mathbb{Z}_2\) has the following five unitary symmetric partitions:

\[
\mathcal{P}_1 = \{(0,0), (0,1), (1,0), (1,1)\}, \quad \mathcal{P}_2 = \{(0,0), (0,1), (0,1), (1,1)\},
\]
\[
\mathcal{P}_3 = \{(0,0), (0,1), (1,0), (1,0)\}, \quad \mathcal{P}_4 = \{(0,0), (1,0), (1,1), (0,1)\},
\]
\[
\mathcal{P}_5 = \{(0,0), (1,0), (0,1), (1,1)\}.
\]

Their associated partition graphs are respectively given by Figures 3–7:

![Figure 3](image1.png) \(G_1 = G(\mathbb{Z}_2^2, d_1)\) \hspace{1cm} ![Figure 4](image2.png) \(G_2 = G(\mathbb{Z}_2^2, d_2)\) \hspace{1cm} ![Figure 5](image3.png) \(G_3 = G(\mathbb{Z}_2^2, d_3)\)

![Figure 6](image4.png) \(G_4 = G(\mathbb{Z}_2^2, d_4)\) \hspace{1cm} ![Figure 7](image5.png) \(G_5 = G(\mathbb{Z}_2^2, d_5)\)

We first compute the symmetry groups. It is obvious that \(\Gamma(\mathbb{Z}_2^2, d_1) \simeq S_4\). Also, proceeding as in the previous example, it is clear that the symmetry groups of \((\mathbb{Z}_2^2, d_2)\), \((\mathbb{Z}_2^2, d_3)\) and \((\mathbb{Z}_2^2, d_4)\)
are all isomorphic to $\mathbb{D}_4$. However, these groups are not equal. In fact, using the cycle notation of symmetric groups and the notations $a = (0,0)$, $b = (1,0)$, $c = (1,1)$ and $d = (0,1)$, if we consider the rotation of order four $\rho = (abcd)$ and the reflections $\tau_2 = (bd)$, $\tau_3 = (ab)(cd)$ and $\tau_4 = (ad)(bc)$, then we have $\Gamma(Z_2^2, d_2) = \langle \rho, \tau_2 \rangle$, $\Gamma(Z_2^2, d_3) = \langle \rho, \tau_3 \rangle$, and $\Gamma(Z_2^2, d_4) = \langle \rho, \tau_4 \rangle$. Finally, we compute the group associated to the Lee metric $d_5$. We have that

\[
\Gamma(Z_2^2, d_5) = \text{Aut}(Cay(Z_2^2, P_1)) \cap \text{Aut}(Cay(Z_2^2, P_2)) \cap \text{Aut}(Cay(Z_2^2, P_3)).
\]

Each $Cay(Z_2^2, P_i)$, $i = 1, 2, 3$, is the disjoint union of two copies of $K_2$. Hence,

$$\text{Aut}(Cay(Z_2^2, P_i)) = \text{Aut}(2K_2) = \mathbb{Z}_2 \ast \mathbb{Z}_2 \simeq \mathbb{D}_4$$

for $i = 1, 2, 3$. However, these automorphism groups are all different and we now look at the intersections in (4.5) more carefully. Note that

\[
\begin{align*}
\text{Aut}(Cay(Z_2^2, P_1)) &= \{id, (ab), (cd), (ab)(cd), (ac)(bd), (ad)(bc), (acbd), (adbc)\}, \\
\text{Aut}(Cay(Z_2^2, P_2)) &= \{id, (ac), (bd), (ab)(cd), (ad)(bc), (abcd), (adbc)\}, \\
\text{Aut}(Cay(Z_2^2, P_3)) &= \{id, (ad), (bc), (ab)(cd), (ad)(bc), (abdc), (acdb)\},
\end{align*}
\]

and hence we clearly have

$$\Gamma(Z_2^2, d_5) = \bigcap_{1 \leq i \leq 3} \text{Aut}(Cay(Z_2^2, P_i)) = \{id, (ab)(cd), (ac)(bd), (bc)(ad)\}.$$

Therefore, by (4.5) we finally obtain that $\Gamma(Z_2^2, d_5) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, as we wanted to see.

From the above graphs, we see that we have the following weight functions on $Z_2^2$:

| $Z_2^2$ | (1,0) | (1,1) | (0,1) | $s$ | poset | chain | hom. | metrics | $\Gamma$ |
|--------|--------|--------|--------|-----|-------|-------|------|---------|--------|
| $w_1$  | 1      | 1      | 1      | 1   | ✓     | ✓     | ✓    | $d_{Ham}, d_{hom}^i$ | $S_4$   |
| $w_2$  | 1      | 2      | 2      | 2   | ✓     | ✓     | ✓    | $d_{Ham}, d_{hom}^i, d_{hom}^R$ | $D_4$   |
| $w_3$  | 2      | 2      | 1      | 2   | ✓     | ✓     | ✓    | $d_{Ham}, d_{hom}^i, d_{hom}^R$ | $D_4$   |
| $w_4$  | 2      | 1      | 2      | 2   | ✓     | ✓     | ✓    | $d_{Ham}, d_{hom}^i, d_{hom}^R$ | $D_4$   |
| $w_5$  | 1      | 2      | 3      | 3   | ✓     | ✓     | ✓    | $d_{Ham}, d_{hom}^i, d_{hom}^R$ | $D_4$   |

The metric $d_1$ is the Hamming metric. We use the notation $d_{Lee}$ for $d_5$ since it is associated to the Lee (i.e. finest unitary symmetric) partition. We have that $d_2$ and $d_3$ are poset metrics with respect to the posets on the set of coordinates of $Z_2 \times Z_2$, namely $\{1,2\}$, determined by $1 \preceq 2$ and $2 \preceq 1$, respectively. On the other hand, the metric $d_4$ is the poset metric with respect to the antichain on $\{1,2\}$, that gives rise to the product Hamming metric $d_{Ham}$. The metrics $d_2, d_3$ and $d_4$ are chain metrics with respect to the chains $\{0\} \subset Z_2 \subset Z_2 \times Z_2$ given by the embeddings of $Z_2 \hookrightarrow Z_2 \times Z_2$ respectively defined by

$$i_1 : x \mapsto (x,0), \quad i_2 : x \mapsto (0,x), \quad \text{and} \quad i_1 \times i_2 : x \mapsto (x,x).$$

Finally, note that the Hamming metric $d_{Ham}$ is equivalent to the homogeneous metric $d_{hom}^i$ on the finite field of four elements $\mathbb{F}_4$, while the metrics $d_2, d_3$ and $d_4$ can be respectively obtained as homogeneous weights on the finite rings $R_1, R_2$ and $R_3$ given by the conditions $a \cdot a = a$, $b \cdot b = 0$ and $a \cdot b = b$ where $a = (1,0)$, $b = (0,1)$ for $R_1$, $a = (0,1)$, $b = (1,0)$ for $R_2$ and $a = (1,0)$, $b = (1,1)$ for $R_3$. $\diamond$
In the next two examples we study the invariant metrics for the groups of order 6.

Example 4.4. Consider now the group $\mathbb{Z}_6$. It is easy to check that

$$P_1 = \{\{0\}, \{1, 2, 3, 4, 5\}\}, \quad P_2 = \{\{0\}, \{1, 2, 4, 5\}, \{3\}\}, \quad P_3 = \{\{0\}, \{1, 3, 5\}, \{2, 4\}\},$$

$$P_4 = \{\{0\}, \{1, 5\}, \{2, 4, 3\}\}, \quad P_5 = \{\{0\}, \{1, 5\}, \{2, 4\}, \{3\}\},$$

are all the unitary symmetric partitions of $\mathbb{Z}_6$. From these partitions, in Figures 8–12 we give the corresponding partition graphs:

![Fig. 8. $G_1 = G(\mathbb{Z}_6, d_1)$](image1)

![Fig. 9. $G_2 = G(\mathbb{Z}_6, d_2)$](image2)

![Fig. 10. $G_3 = G(\mathbb{Z}_6, d_3)$](image3)

![Fig. 11. $G_4 = G(\mathbb{Z}_6, d_4)$](image4)

![Fig. 12. $G_5 = G(\mathbb{Z}_6, d_5)$](image5)

We know compute the symmetry groups. We will use the previous graphs together with Theorem 3.5 and Corollary 3.6. One must interpret the graphs as unions of the subgraphs given by each part of the partition (i.e. each color); for instance for the Lee metric $d_5$ we have the following decomposition of the graph of distances into simple Cayley graphs:

$$Cay(\mathbb{Z}_6, P_{Lee}) = Cay(\mathbb{Z}_6, \{1, 5\}) \cup Cay(\mathbb{Z}_6, \{2, 4\}) \cup Cay(\mathbb{Z}_6, \{3\})$$
Now, by (3.6) and (3.10) we have that
\[
\Gamma(Z_6, d_1) = \text{Aut}(K_6) \simeq S_6,
\]
\[
\Gamma(Z_6, d_2) = \text{Aut}(C_6 \cup 2C_3) \cap \text{Aut}(3K_2) = \text{Aut}(3K_2) \simeq S_2 \wr S_3,
\]
\[
\Gamma(Z_6, d_3) = \text{Aut}(C_6 \cup 3K_2) \cap \text{Aut}(2C_3) = \text{Aut}(2C_3) \simeq S_3 \wr S_2,
\]
\[
\Gamma(Z_6, d_4) = \text{Aut}(C_6) \cap \text{Aut}(C_6 \cup 3K_2) = \text{Aut}(C_6) \simeq D_6.
\]
\[
\Gamma(Z_6, d_5) = \text{Aut}(C_6) \cap \text{Aut}(2C_3) \cap \text{Aut}(3K_2) \simeq D_6 \cap (S_3 \wr S_2) \cap (S_2 \wr S_3) \simeq D_6.
\]

We now explain these isomorphisms. The first one is clear. For the second, third and fourth, note that the group of symmetries is the intersection of the automorphism group of a graph with the automorphism group of the complementary graph (see Figures 9–11), which are equal.

Hence, it is enough to compute the easiest case (second identities). For the last group we cannot use this trick, since we have 3 graphs. However, one can check that every automorphism of \(\text{Cay}(Z_6, \{1, 5\}) \simeq D_6\) is also an automorphism of \(\text{Cay}(Z_6, \{2, 4\}) \simeq S_3 \wr S_2\) and of \(\text{Cay}(Z_6, \{3\}) \simeq S_2 \wr S_3\), and hence we finally get \(\Gamma(Z_6, d_5) \simeq D_6\).

From the graphs, we see that we have the following weight functions on \(Z_6\):

| \(\mathbb{Z}_6^{w}\) | 1 | 2 | 3 | 4 | 5 | \(s\) | poset | chain | hom. | known metrics | \(\Gamma\) | order |
|-----------|---|---|---|---|---|-----|------|------|------|-------------|------|------|
| \(w_1\)   | 1 | 1 | 1 | 1 | 1 | 1   | ✓    | ✓    | ✓    | \(d_{\text{Ham}}\) | \(S_6\) | 720  |
| \(w_2\)   | 1 | 2 | 2 | 2 | 1 | 1   | ✓    | ✓    | ✓    | \(d_{1\leq 2}, d_{3}\) | \(S_2 \wr S_3\) | 48   |
| \(w_3\)   | 1 | 2 | 2 | 1 | 2 | 2   | ✓    | ✓    | ✓    | \(d_{2 \leq 1}, d_{3}\) | \(S_3 \wr S_2\) | 72   |
| \(w_4\)   | 1 | 2 | 2 | 2 | 1 | 2   | ✓    | X    | X    | \(d_{3, \text{Lee}}, d_{3, \text{Ham}}\) | \(D_6\) | 12   |
| \(w_5\)   | 1 | 2 | 3 | 2 | 1 | 3   | X    | X    | ✓    | \(d_{\text{Lee}} = d_{2, \text{Lee}}, d_{\text{hom}}\) | \(D_6\) | 12   |

The metrics \(d_1\) and \(d_5\) are Lee and Hamming respectively. For poset metrics we think \(Z_6\) as the product \(Z_2 \times Z_3\) (i.e. 2 coordinates) via the isomorphism \(1 \mapsto (1, 1)\), since otherwise we get the Hamming metric. We have that \(d_2, d_3, d_4\) are poset metrics with respect to the posets on the set of coordinates \(\{1, 2\}\), given by \(1 \preceq 2\) and \(2 \preceq 1\) and the trivial poset (2-antichain) respectively. The metrics \(d_3\) and \(d_4\) are chain metrics with respect to the chains \(\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_6\) and \(\{0\} \subset \mathbb{Z}_3 \subset \mathbb{Z}_6\) given by the embeddings \(x \mapsto 2x\) and \(x \mapsto 3x\), respectively. The ring of integers modulo 6 has the poset of principal ideals given by \(\{0\} \preceq 2\mathbb{Z}_6 \preceq \mathbb{Z}_6\) and \(\{0\} \preceq 3\mathbb{Z}_6 \preceq \mathbb{Z}_6\) where \(\mu(\{0\}, \{0\}) = 1\), \(\mu(\{0\}, 2\mathbb{Z}_6) = -1\), \(\mu(\{0\}, 3\mathbb{Z}_6) = -1\) and \(\mu(\{0\}, \mathbb{Z}_6) = 1\). By (4.4), we have \(w_{\text{hom}}(1) = w_{\text{hom}}(5) = \frac{1}{2} \lambda\), \(w_{\text{hom}}(2) = w_{\text{hom}}(4) = \frac{3}{2} \lambda\) and \(w_{\text{hom}}(3) = 2 \lambda\).

\[\Box\]

**Example 4.5.** Consider the group \(S_3 = \{id, (12), (13), (23), (123), (132)\}\). For simplicity we will write \(ij\) and \(ijk\) instead of \((ij)\) and \((ijk)\). By inspection one can see that \(S_3\) has 15 unitary symmetric partitions given by

- \(P_1 : \{12, 13, 23, 123, 132\}\),
- \(P_2 : \{12, 13, 23\}, \{123, 132\}\),
- \(P_3 : \{12, 13, 23, 123, 132\}\),
- \(P_4 : \{13\}, \{12, 23, 123, 132\}\),
- \(P_5 : \{23\}, \{12, 13, 123, 132\}\),
- \(P_6 : \{12, 13, 23, 123, 132\}\),
- \(P_7 : \{12, 23\}, \{13, 123, 132\}\),
- \(P_8 : \{13, 23\}, \{12, 123, 132\}\),
- \(P_9 : \{12\}, \{13, 23, 123, 132\}\),
- \(P_{10} : \{13\}, \{12, 23\}, \{123, 132\}\),
- \(P_{11} : \{23\}, \{12, 13\}, \{123, 132\}\),
- \(P_{12} : \{12\}, \{13\}, \{23, 123, 132\}\),
- \(P_{13} : \{12\}, \{23\}, \{13, 123, 132\}\),
- \(P_{14} : \{13\}, \{23\}, \{12, 123, 132\}\),
- \(P_{15} : \{12\}, \{13\}, \{23\}, \{123, 132\}\).

It is clear from this list that there are only two conjugate partitions (the first two ones) and hence \(S_3\) has only 2 bi-invariant metrics. The partition graphs of the metrics (except for the Hamming one) are given in Figures 13–18.
Applying automorphisms of $S_3$ one can obtain the isomorphic graphs corresponding to the rest of the metrics. Thus, $d_3, d_4, d_5$ are given by Figure 14, $d_6, d_7, d_8$ by Figure 15, $d_9, d_{10}, d_{11}$ by Figure 16 and $d_{12}, d_{13}, d_{14}$ by Figure 17.

Proceeding as in the previous examples we can obtain all the symmetry groups and metrics. We leave the details and summarize the information in the following table.

| #  | 12 | 13 | 23 | 123 | 132 | s | chain | metrics | $\Gamma(S_3, d)$ | order |
|----|----|----|----|-----|-----|---|-------|---------|------------|--------|
| 1  | 1  | 1  | 1  | 1   | 1   | 1 |✗     | $d_{Ham}$ | $S_6$      | 720    |
| 2  | 1  | 2  | 2  | 2   | 2   | 2 |✓     | $d_{(123)}$ | $S_3 \rtimes S_2$ | 72     |
| 3  | 1  | 2  | 2  | 2   | 2   | 2 |✓     | $d_{(12)}$  | $S_2 \wr S_3$   | 48     |
| 4  | 2  | 1  | 2  | 2   | 2   | 2 |✓     | $d_{(13)}$  | $S_2 \wr S_3$   | 48     |
| 5  | 2  | 2  | 1  | 2   | 2   | 2 |✓     | $d_{(23)}$  | $S_2 \wr S_3$   | 48     |
| 6  | 1  | 1  | 2  | 2   | 2   | 2 |✗     | $d_{6}$     | $D_6$       | 12     |
| 7  | 1  | 2  | 1  | 2   | 2   | 2 |✗     | $D_6'$      | $D_6'$      | 12     |
| 8  | 2  | 1  | 1  | 2   | 2   | 2 |✗     | $D_6'$      | $D_6'$      | 12     |
| 9  | 1  | 2  | 2  | 3   | 3   | 3 |✗     | $D_6'$      | $D_6'$      | 12     |
| 10 | 2  | 1  | 2  | 3   | 3   | 3 |✗     | $D_6'$      | $D_6'$      | 12     |
| 11 | 2  | 2  | 1  | 3   | 3   | 3 |✗     | $D_6'$      | $D_6'$      | 12     |
| 12 | 1  | 2  | 3  | 3   | 3   | 3 |✗     | $S_3$       | $S_3$       | 6      |
| 13 | 1  | 3  | 2  | 3   | 3   | 3 |✗     | $S_3$       | $S_3$       | 6      |
| 14 | 3  | 1  | 2  | 3   | 3   | 3 |✗     | $S_3$       | $S_3$       | 6      |
| 15 | 1  | 2  | 3  | 4   | 4   | 4 |✓     | $d_{Lee}$   | $S_3$       | 6      |

The metrics $d_4, d_6$ and $d_{10}$ are chain metrics with respect to the chains $Z_2 \subset S_3$ where $Z_2$ is respectively given by $\langle (23) \rangle$, $\langle (12) \rangle$ and $\langle (13) \rangle$; while $d_2$ is a chain metric with respect to $Z_3 \subset S_3$ with $Z_3 = \langle (123) \rangle$. 

\[\diamondsuit\]
Finally, we study the invariant metrics of the group $\mathbb{Z}_7$.

**Example 4.6.** The group $\mathbb{Z}_7$ has 5 unitary symmetric partitions given by

$$\mathcal{P}_1 = \{\{0\}, \{1, 2, 3, 4, 5, 6\}\}, \mathcal{P}_2 = \{\{0\}, \{1, 6\}, \{2, 3, 4, 5\}\},$$

$$\mathcal{P}_3 = \{\{0\}, \{2, 5\}, \{1, 3, 4, 6\}\}, \mathcal{P}_4 = \{\{0\}, \{3, 4\}, \{1, 2, 5, 6\}\}, \mathcal{P}_5 = \{\{0\}, \{1, 6\}, \{2, 5\}, \{3, 4\}\}.$$ 

Notice that in this case, partitions $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ are isomorphic since there are automorphisms of $\mathbb{Z}_7$ sending the parts of any partition to the parts of the other ones. For instance, one can pass from $\mathcal{P}_2$ to $\mathcal{P}_3$ by multiplying by 2 and from $\mathcal{P}_2$ to $\mathcal{P}_4$ by multiplying by 3. The partition graphs of these metrics are as follows:

$$\mathcal{G}_1 = \mathcal{G}(\mathbb{Z}_7, d_1) \quad \mathcal{G}_{i+1} = \mathcal{G}(\mathbb{Z}_7, d_{i+1}), \quad i = 1, 2, 3 \quad \mathcal{G}_5 = \mathcal{G}(\mathbb{Z}_7, d_5)$$

From the above graphs, we see that we have the following $s$-weight functions on $\mathbb{Z}_7$:

| $\mathbb{Z}_7$ | 1 | 2 | 3 | 4 | 5 | 6 | metrics | $\Gamma$ |
|---------------|---|---|---|---|---|---|---------|-------|
| $w_1$         | 1 | 1 | 1 | 1 | 1 | 1 | $d_{\text{Ham}}$, $d_{\text{hom}}^\pm$ | $\mathbb{S}_7$ |
| $w_2$         | 2 | 2 | 1 | 1 | 2 | 2 | $\mathbb{D}_7$ |
| $w_3$         | 2 | 1 | 2 | 1 | 2 | 2 | $\mathbb{D}_7$ |
| $w_4$         | 1 | 2 | 2 | 2 | 1 | 2 | $\mathbb{D}_7$ |
| $w_5$         | 1 | 2 | 3 | 3 | 2 | 1 | $d_{\text{Lee}}$ | $\mathbb{D}_7$ |

We know that $\Gamma(\mathbb{Z}_7, d_1) \simeq \mathbb{D}_7$ and $\Gamma(\mathbb{Z}_7, d_5) \simeq \mathbb{S}_7$. For the other groups, we have

$$\mathbb{D}_7 \subset \Gamma(\mathbb{Z}_7, d_i) = \text{Aut}(\text{Cay}(\mathbb{Z}_7, \{a, -a\})) \cap \text{Aut}(\text{Cay}(\mathbb{Z}_7, \{b, -b, c, -c\}))$$

with $a, b, c \in \{1, 2, 3\}$ for $i = 2, 3, 4$. Since $\text{Aut}(\text{Cay}(\mathbb{Z}_7, \{a, -a\})) = \text{Aut}(\mathbb{C}_7) \simeq \mathbb{D}_7$, we have that $\Gamma(\mathbb{Z}_7, d_i) \simeq \mathbb{D}_7$ for $i = 2, 3, 4$.

For groups or order $\geq 8$, the number of invariant metrics grows rapidly (as we will see in the next section) and the computation of the metrics and the symmetry groups gets more and more difficult. For this reason, we now study the symmetry groups for the non-abelian groups of order 8, $\mathbb{D}_4$ and $\mathbb{Q}_8$, only with the ‘Lee’ metrics, that is those associated to the finest unitary symmetric partition.

**Example 4.7.** Consider the quaternion and the dihedral groups $\mathbb{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with $i^2 = j^2 = k^2 = ijk = 1$ and $\mathbb{D}_4 = \{e, \rho, \rho^2, \rho^3, \tau, \tau \rho, \tau \rho^3, \tau \rho^2\}$ where $\rho^4 = \tau^2 = e$ and $\rho \tau \rho = \tau$. 
Their Lee partitions and conjugate Lee partitions are given by
\[ P_{\text{Lee}}(\mathbb{Q}_8) = \{\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}\} = P_{\text{Lee}}^c(\mathbb{Q}_8), \]
\[ P_{\text{Lee}}(\mathbb{D}_4) = \{\{e\}, \{\rho, \rho^3\}, \{\rho^2\}, \{\tau\}, \{\rho \tau\}, \{\rho^3 \tau\}\}, \]
\[ P_{\text{Lee}}^c(\mathbb{D}_4) = \{\{e\}, \{\rho, \rho^3\}, \{\rho^2\}, \{\tau\}, \{\rho \tau\}, \{\rho^3 \tau\}\}. \]

Note that the fact that the Lee partition and the conjugate Lee partition of \(\mathbb{Q}_8\) coincide implies that any invariant metric of \(\mathbb{Q}_8\) is also bi-invariant.

The partition graphs of \(\mathbb{D}_4\) and \(\mathbb{Q}_8\) for these metrics are respectively given by (black=1, blue=2, red=3, green-dashed=4, magenta-dotted=5, cyan=6):

\[ \mathcal{G}(\mathbb{Q}_8, d_{\text{Lee}}) \]
\[ \mathcal{G}(\mathbb{D}_4, d_{\text{Lee}}) \]
\[ \mathcal{G}(\mathbb{D}_4, d^c_{\text{Lee}}) \]

That is, \( \mathcal{G}(\mathbb{Q}_8, d^c_{\text{Lee}}) = \mathcal{G}(\mathbb{D}_4, d_{\text{Lee}}) = 4K_2 \cup 3(2C_4) \) and \( \mathcal{G}(\mathbb{D}_4, d_{\text{Lee}}) = 6(4K_2) \cup 2C_4 \). By Corollary 3.6, the groups \( \mathcal{G}(G, d_{\text{Lee}}) \) are intersections of groups isomorphic to \( \mathbb{Z}_2 \wr S_4 \) and \( \mathbb{D}_4 \wr \mathbb{Z}_2 \).

Hence, by Example 3.4 with \( n = 2 \), for \( G = \mathbb{D}_4, \mathbb{Q}_8 \) we have
\[ G \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \leq \Gamma(G, d_{\text{Lee}}) \leq \mathbb{D}_4 \wr \mathbb{Z}_2 \]
since \( G/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \). Thus, \( 2^5 \leq \#\Gamma(G, d_{\text{Lee}}) \leq 2^7 \) for \( G = \mathbb{D}_4, \mathbb{Q}_8 \).

We now compute the group \( \Gamma(G, d_{\text{Lee}})^c \) for \( G = \mathbb{D}_4, \mathbb{Q}_8 \). Notice that \( \Gamma(G, d_{\text{Lee}}) \leq \mathbb{S}_8 \). First we find the order. Using that \( \Gamma(G, d_{\text{Lee}})^c \) acts on \( G \) then we have
\[ |\Gamma(G, d_{\text{Lee}})^c| = |G||\Gamma_e|, \]

where \( \Gamma_e \) is the stabilizer of \( \Gamma(G, d_{\text{Lee}})^c \). From the graphs, one can see that \( \Gamma_e = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \). In fact, the group \( \Gamma_e \) for \( G = \mathbb{Q}_8 \) is determined by the 3 independent permutations sending \( i \mapsto -i, j \mapsto -j \) and \( k \mapsto -k \). Similarly, the group \( \Gamma_e \) for \( G = \mathbb{D}_4 \) is given by \( \tau \mapsto \rho^2 \tau, \rho \mapsto \rho^3 \) and \( \rho \tau \mapsto \rho^3 \tau \). Thus, we get that \( |\Gamma(G, d_{\text{Lee}})^c| = 2^6 \).

There are 267 groups of order 64, out of which 256 are non-abelian ones. Intuitively, this symmetry group should be a wreath product. But there are only 2 wreath products of order 64, \( \mathbb{Z}_2 \wr \mathbb{Z}_4 \) and \( \mathbb{Z}_2 \wr (\mathbb{Z}_2 \times \mathbb{Z}_2) \). Consider the group
\[ \mathbb{Z}_2 \wr (\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2^3 \times \varphi (\mathbb{Z}_2 \times \mathbb{Z}_2) \]
with \( \mathbb{Z}_2^3 = \langle \varphi_1, \varphi_i, \varphi_j, \varphi_k \rangle \) and \( \mathbb{Z}_2^2 = \langle \varphi_{ij}, \varphi_{ik} \rangle \) where the involution \( \varphi_a \) is determined by \( a \leftrightarrow -a \) for \( a \in \{1, i, j, k\} \), \( \varphi_{ij} \) is determined by \( i \leftrightarrow j, -i \leftrightarrow -j, 1 \leftrightarrow -1, k \leftrightarrow -k \) and \( \varphi_{ik} \) is determined by \( i \leftrightarrow k, -i \leftrightarrow -k, 1 \leftrightarrow -1, j \leftrightarrow -j \). We have that \( \mathbb{Z}_2^3 \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \) is a subgroup of \( \Gamma(G, d_{\text{Lee}})^c \), which has the same cardinality, and hence \( \Gamma(G, d_{\text{Lee}}) = \mathbb{Z}_2 \wr (\mathbb{Z}_2 \times \mathbb{Z}_2) \).
5. The number of invariant metrics on finite groups

How many equivalence \( P \)-classes of metrics are there on a finite group \( G \)? We will denote the number of non-equivalent invariant metrics of \( G \) by \( M(G) \), that is

\[
M(G) = \#(\widetilde{M}(G)).
\]

To answer this question we use Bell numbers. We recall that the \( n \)-th Bell number \( B_n \) counts the number of partitions of a set of \( n \) elements. The first Bell numbers are

\[
B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52, B_6 = 203, B_7 = 877, B_8 = 4.140.
\]

We now give the number of invariant metrics on a group \( G \). We denote by \( k_2(G) \) the number of elements of order \( \leq 2 \) of \( G \), that is

\[
k_2(G) = \#\{x \in G : x^2 = e\}.
\]

**Proposition 5.1.** If \( G \) is a finite group of order \( n \) then \( M(G) = B_{k(G)} \) where

\[
k(G) = \frac{1}{2} \{n + k_2(G)\} - 1.
\]

In particular, if \( n \) is odd then \( M(G) = B_{\frac{n-1}{2}} \).

**Proof.** In view of the correspondence (2.9), the number of \( P \)-classes of metrics of \( G \) is given by the number of unitary symmetric partitions of \( G \). The identity and the elements of order 2 in \( G \) are their own inverses. However, if the element \( g \in G \) has order \( > 2 \) then \( g \) and \( g^{-1} \) must be in the same part, since any partition is symmetric. Thus, it is clear that the number of symmetric partitions is given by the Bell number \( B_{k(G)} \) with

\[
k(G) = \frac{1}{2} \#\{g \in G : \text{ord}(g) > 2\} + \#\{g \in G : \text{ord}(g) = 2\}.
\]

Since \( n = k_2(G) + \#\{g \in G : \text{ord}(g) > 2\} \), we clearly get that \( k(G) = \frac{1}{2}\{n - k_2(G)\} + k_2(G) - 1 \), from where (5.2) follows.

Now, if \( G \) is a group of odd order then it has no elements of order 2, hence \( k_2(G) = 1 \) and

\[
k(G) = \frac{n-1}{2}.
\]

Notice that if \( G \) has even order then it has an odd number of elements of order 2, since we have the disjoint union \( G = \{e\} \cup \{\text{elements of order 2}\} \cup \{g, g^{-1}, h, h^{-1}, \ldots\} \). Hence, \( k(G) \) is a well defined positive integer.

For a group \( G \) with (not so) big cardinality \( n \) it could be difficult to compute \( k(G) \) by hand. In this case, we have the obvious (not so sharp) bound

\[
(5.4) \quad M(G) \leq B_{n-1}
\]

since \( 1 \leq k(G) \leq n - 1 \) by definition. That is, every group has at least one invariant metric, the Hamming metric.

**Example 5.2.** We now give the number of invariant metrics in some easy cases.

(i) If \( \mathbb{Z}_n \) denote the cyclic group of order \( n \) then \( M(\mathbb{Z}_n) = B_{\frac{n}{2}} \). In fact, we clearly have \( k(\mathbb{Z}_{2n}) = \frac{1}{2}(2n + 2) - 1 = n \), since there is only one element of order 2, and \( k(\mathbb{Z}_{2n+1}) = \frac{1}{2}((2n + 1) + 1) - 1 = n \), since there is no elements of order 2.

(ii) For any \( k \geq 1 \), since every nonzero element is of order 2, we have \( k(\mathbb{Z}_k^2) = k_2(\mathbb{Z}_k^2) \) and thus \( M(\mathbb{Z}_k^2) = B_{2k-1} \). Moreover, the equality in (5.4) holds if and only if \( G = \mathbb{Z}_2^k \) for some \( k \).

For an abelian group \( G \) we can give the precise number of invariant metrics. To this end we will need the following basic observation.
Lemma 5.3. If \( G, K \) are groups then
\[
(5.5) \quad k(G \times K) = \frac{1}{2} \{ |G||K| + k_2(G)k_2(K) \} - 1.
\]
In particular, \( k(G \times \mathbb{Z}_2^r) = 2^r(k(G) + 1) - 1 \) for every \( r \in \mathbb{N} \).

Proof. Just note that \( k_2(G \times K) = \# \{(x, y) \in G \times K : (x, y)^2 = (e, e) \} = k_2(G)k_2(K) \). The remaining assertion is clear since \( |\mathbb{Z}_2^r| = k_2(\mathbb{Z}_2^r) = 2^r \).

Using the previous proposition we will obtain closed expressions for the number of metrics of certain families of groups. For abelian groups we have the following result.

Proposition 5.4. Let \( G \) be an abelian group of order \( n \). Then, the number of invariant metrics on \( G \) is given by \( M(G) = B_{\frac{n}{2}+2s-1-1} \) where \( s \) is the number of cyclic factors of order some power of 2 in the canonical product decomposition of \( G \) into cyclic groups.

Proof. Since \( G \) is abelian it has a decomposition as product of cyclic groups
\[
G \cong \mathbb{Z}_{2m_1} \times \cdots \times \mathbb{Z}_{2m_s} \times \mathbb{Z}_{p_1^{e_1}} \times \cdots \times \mathbb{Z}_{p_t^{e_t}}
\]
where \( m_1, \ldots, m_s, r_1, \ldots, r_t, e_1, \ldots, e_t \in \mathbb{N} \) and \( p_1, \ldots, p_t \) are odd prime numbers (not necessarily distinct). Only the factors of even order have elements of order 2 (one per factor). Hence, using (5.5), we see that
\[
k_2(G) = k_2(\mathbb{Z}_{2m_1} \times \cdots \times \mathbb{Z}_{2m_s}) = k_2(\mathbb{Z}_{2^{m_1}}) \cdots k_2(\mathbb{Z}_{2^{m_s}}) = 2^s.
\]
We have \( k(G) = \frac{n+2s}{2} - 1 = \frac{n}{2} + 2^{s-1} - 1 \), by (5.2), and the result follows by Proposition 5.1.

We now give the number of invariant metrics for some families of non-abelian groups. We first consider dihedral, dicyclic and quasidihedral groups and later symmetric and alternating groups.

Proposition 5.5. The number of invariant metrics of the dihedral groups \( D_n \), dicyclic groups \( Q_{4n} \) and quasidihedral groups \( QD_n^\pm \) are given by:

(a) \( M(D_{2k+1}) = B_{3k+1} \) for \( k \geq 1 \) and \( M(D_{2k}) = B_{3k} \) for \( k \geq 2 \).

(b) \( M(Q_{4n}) = B_{2n} \) for \( n \geq 2 \).

(c) \( M(QD_n^+) = B_{2n+1} \) and \( M(QD_n^-) = B_{2n-1+1} \) for \( n \geq 4 \).

Proof. By Proposition 5.1, \( M(G) = B_{k(G)} \), so we will compute the numbers \( k(G) \) in each case.

(a) For \( n \geq 3 \), the dihedral group \( D_n \) is the group of symmetries of a regular \( n \)-gon and hence has order \( 2n \). One familiar presentation is
\[
(5.6) \quad D_n = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle.
\]
Notice that if \( n \) is odd then \( D_n \) has \( n \) elements of order 2 (the reflections) while if \( n \) is even \( D_n \) has \( n + 1 \) elements of order 2, the \( n \) reflections plus the half turn rotation. Thus, if \( n = 2k \) we have \( k(D_n) = \frac{2n+n+2}{2} - 1 = 3k \) and if \( n = 2k+1 \) we have \( k(D_n) = \frac{2n+n+1}{2} - 1 = 3k+1 \).

(b) The dicyclic groups \( Q_{4n} \) of order \( 4n \), also denoted \( \text{Dic}_n \), are defined for every \( n \geq 2 \) by
\[
(5.7) \quad Q_{4n} = \langle a, b : a^{2n} = e, b^2 = a^n, b^{-1}ab = a^{-1} \rangle.
\]
When \( n \) is a power of 2 these groups are the generalized quaternions \( Q_{2^m} \) for \( m \geq 4 \), in particular the quaternion group \( Q_8 \) for \( n = 2 \). Notice that \( Q_{4n} \) has only one element of order 2, namely \( a^n \). It is easy to see that every element in the group can be uniquely written in the form \( a^kb^j \), where \( 0 \leq k < 2n \) and \( j = 0, 1 \). It is clear that \( a^ka^k = e \) if and only if \( k = n \). Also, in general \( (a^kb)(a^jb) = a^{k-m+n} \), so we have \( a^kb^ka^j = a^n \neq e \). Thus, \( k(Q_{4n}) = \frac{1}{2}(2n+2) - 1 = 2n \).
(c) The quasidihedral groups $Q\mathbb{D}^\pm_n$ of order $2^n$ (also called semidihedral and denoted $S\mathbb{D}^\pm_n$) are defined for every $n \geq 4$ by
\begin{equation}
Q\mathbb{D}^\pm_n = \{x, y : x^{2^{n-1}} = y^2 = e, yxy = x^{2^{n-2} \pm 1}\}.
\end{equation}
The group $Q\mathbb{D}_n^-$ has $2^{n-2} + 1$ elements of order 2 while the group $Q\mathbb{D}_n^+$ has 3 elements of order 2. Therefore, $k(Q\mathbb{D}_n^-) = \frac{1}{2}(2^n + 2^{n-2} + 2) - 1 = 5 \cdot 2^{n-3}$ and $k(Q\mathbb{D}_n^+) = \frac{1}{2}(2^n + 4) - 1 = 2^{n-1} + 1$, and (c) holds.

Let us check the number of elements of order 2. First note that since $y$ has order 2, the elements of these groups are explicitly given by
\begin{equation}
Q\mathbb{D}_n^\pm = \{e, x, x^2, \ldots, x^{2^{n-2}}, \ldots, x^{2^{n-1}-1}, y, yx, yx^2, \ldots, yx^{2^{n-2}}, \ldots, yx^{2^{n-1}-1}\}
\end{equation}
Clearly, in both groups, $y$ is an element of order 2 and $x^{2^{n-2}}$ is the only element of order 2 in the cyclic subgroup $\langle x \rangle$. We now study the number of invariant metrics in the symmetric group $S_n$. We now study the number of invariant metrics for $Q\mathbb{D}^\pm_n$ and hence $2^n - 1$ and only if $k$ is even. 

We now study the number of invariant metrics in the symmetric group $S_n$ for $n \geq 3$ and the alternating subgroup $A_n$ for $n \geq 4$ (these groups for smaller values of $n$ are abelian). In this case, we will not have close expressions, but rather expressions involving the following numbers
\begin{equation}
s_n = \sum_{0 \leq k \leq \lfloor \frac{n}{2} \rfloor} \frac{1}{2^k k!(n-2k)!} \quad \text{and} \quad a_n = \sum_{0 \leq k \text{ even} \leq \lfloor \frac{n}{2} \rfloor} \frac{1}{2^k k!(n-2k)!}.
\end{equation}

**Proposition 5.6.** The number of invariant metrics for $S_n$ and $A_n$ are given by
\begin{equation}
M(S_n) = B_2\frac{1}{n!(s_n+1)} - 1 \quad \text{for } n \geq 3 \quad \text{and} \quad M(A_n) = B_2\frac{1}{n!(a_n+\frac{1}{2})} - 1 \quad \text{for } n \geq 4,
\end{equation}
where $s_n$ and $a_n$ are as in (5.10).

**Proof.** Any element in $S_n$ is a product of transpositions. The elements of order 2 are just the products of disjoint transpositions, which in this case commute with each other. We have to count all possible products of disjoint transpositions. For instance, the elements of order 2 in $S_6$ are of the form $(i_1i_2), (i_1i_2)(i_3i_4)$ or $(i_1i_2)(i_3i_4)(i_5i_6)$ with $i_1, i_2, i_3, i_4, i_5, i_6$ all different elements in $\{1, 2, 3, 4, 5, 6\}$; and there are \(\binom{6}{2} + \frac{1}{3!} \binom{6}{2} \binom{4}{2} + \frac{1}{6!} \binom{6}{2} \binom{4}{2} \binom{2}{2} \) of them. Thus, one can check that the number of elements of order $\leq 2$ in $S_n$ for $n \geq 3$ is given by
\begin{equation}
k_2(S_n) = 1 + \sum_{0 \leq k \leq n-1} \frac{1}{(k+1)!2^k} \binom{n-2}{2} \cdot \binom{n-4}{2} \cdot \binom{n-2k}{2}, \quad n = 2t, 2t + 1,
\end{equation}
which can be written more succinctly as $k_2(S_n) = n!s_n$. Hence, $M(S_n) = B_2\frac{1}{n!(s_n+1)}$. Similarly, $k_2(A_n) = 0$ for $n \geq 4$, hence $M(A_n) = B_2\frac{1}{n!(a_n+\frac{1}{2})}$. The result follows.
Semidirect products. Recall that if \( G \) and \( H \) are groups and \( \varphi : H \to \text{Aut}(G) \) is a homomorphism we have the semidirect product \( G \rtimes_\varphi H \) which is the set \( G \times H \) with the product given by \((g_1, h_1)(g_2, h_2) = (g_1\varphi_h(g_2), h_1h_2)\), where \( \varphi_h \) stands for the morphism \( \varphi(h) \). When \( \varphi \) is natural or the only one we omit it from the notation.

It is difficult to count the number of metrics on semidirect products \( G \rtimes_\varphi H \), unless we know the function \( \varphi \) explicitly. In general we can only estimate this number, although in some particular cases we can compute it explicitly. By (5.2) we have

\[
(5.12) \quad k(G \rtimes_\varphi H) = \frac{1}{2}(|G||H| + k_2(G \rtimes_\varphi H)) - 1.
\]

Now, since \((g, h)^2 = (g\varphi_h(g), h^2)\) we obtain

\[
(5.13) \quad k_2(G \rtimes_\varphi H) = k_2(G) + \sum_{h \in H, h^2 = e_H} \# \{ g \in G : g\varphi_h(g) = e_G \}.
\]

Separating the contribution of the identity element \( e_H \), since \( \varphi_{e_H} = id \), we have

\[
(5.14) \quad k_2(G \rtimes_\varphi H) = k_2(G) + \sum_{h \in H, h^2 = e_H} \# \{ g \in G : g\varphi_h(g) = e_G \}.
\]

Since \( 0 \leq \# \{ g \in G : \varphi_h(g) = g^{-1} \} \leq |G| \) we have that

\[
(5.15) \quad k_2(G) \leq k_2(G \rtimes_\varphi H) \leq k_2(G) + |G|(k_2(H) - 1).
\]

Applying these inequalities in (5.12), after some trivial computations we obtain

\[
(5.16) \quad \frac{1}{2}|G|(|H| - 1) + k(G) \leq k(G \rtimes_\varphi H) \leq k(H)|G| + k(G).
\]

Notice that both equalities hold if and only if \( \frac{1}{2}(|H| - 1) = k(H) \), that is if and only if \( k_2(H) = 1 \) which happens if and only if \( H \) is of odd order. Thus, if \( H \) is of odd order then we have

\[
(5.17) \quad k(G \rtimes H) = \frac{1}{2}(|G||H| - 1) + k(G).
\]

In particular, if \( G \) is also of odd order, then \( k(G \rtimes H) = \frac{1}{2}(|G||H|) = k(G \rtimes H) \).

We now consider the case of \( G \rtimes_\varphi \mathbb{Z}_2 \) where \( G \) is an abelian group. By (5.13), we have

\[
(5.18) \quad k_2(G \rtimes_\varphi \mathbb{Z}_2) = k_2(G) + \# \{ g \in G : g\varphi_1(g) = e_G \}.
\]

In the particular case of the generalized dihedral group \( \mathbb{D}(G) = G \rtimes \mathbb{Z}_2 \) of an abelian group \( G \), with \( G \neq \mathbb{Z}_2^2 \) since otherwise \( \mathbb{D}(G) = G \), where \( \varphi_1 \) is the inversion (see Definition 3.1) we have

\[
(5.19) \quad k_2(G \rtimes \mathbb{Z}_2) = k_2(G) + |G|.
\]

Thus,

\[
(5.20) \quad k(\mathbb{D}(G)) = \frac{1}{2}(3|G| + k_2(G)) - 1
\]

for any abelian group \( G \neq \mathbb{Z}_2^2 \). If \( G = \mathbb{Z}_n \) we get the dihedral group \( \mathbb{D}_n \) and (5.20) coincides with (a) in Proposition 5.5.

Notice also that if we take \( G = \mathbb{Z}_{2n-1} \) and consider the semidirect products \( G \rtimes_\varphi \mathbb{Z}_2 \) where the morphisms \( \varphi^\pm : \mathbb{Z}_2 \to \text{Aut}(\mathbb{Z}_{2n-1}) \) are given by \( x \mapsto x^{2n-2}\pm 1 \) we get the quasidihedral groups \( Q\mathbb{D}^\pm_n \) and that (5.20) in this case coincides with (c) in Proposition 5.5.

6. Bi-invariant metrics on groups

We now study the number of bi-invariant metrics on groups. Let \( G \) be a group, not necessarily finite. We will denote the number (if it is finite) of bi-invariant metrics on \( G \) by \( M^*(G) \). Clearly, we have

\[
(6.1) \quad M^*(G) \leq M(G).
\]
Let $C(G) = \{C_g : g \in G\}$ be the set of conjugacy classes of $G$ where
\[
C_g = \{hgh^{-1} : h \in G\}
\]
is the conjugacy class of an element $g \in G$. An element $g$ of $G$ is called real if it is conjugate to its inverse, i.e. if there is some $h \in G$ such that $hgh^{-1} = g^{-1}$. A conjugacy class $C_g$ is real if $C_g = C_{g^{-1}}$. We denote by $c(G) = \#C(G)$ the number of conjugacy classes of $G$ and by $c_2(G)$ the number of real conjugacy classes in $G$. From now on, when we write $c(G)$ or $c_2(G)$ we will assume that $G$ has a finite number of conjugacy classes. If $P_0, P_1, \ldots, P_s$ is a conjugate partition of $G$ then $g \in P_i$ implies that $C_g \subset P_i$.

We are now in a position to state the next result.

**Proposition 6.1.** If $G$ is a finite group, $M^*(G) = B_{k^*(G)}$ with
\[
k^*(G) = \frac{1}{2}(c(G) + c_2(G)) - 1.
\]

**Proof.** Consider the finest unitary symmetric conjugate partition $P = \{P_0, P_1, \ldots, P_s\}$ of $G$. Every unitary symmetric conjugate partition of $G$ is obtained by taking the unions of some parts of $P$. Hence, the number of unitary symmetric conjugate partitions of $G$ is given by the number of partitions of $\{1, 2, \ldots, s\}$. We proceed similarly as in the proof of Proposition 5.1. We have
\[
k^*(G) = \frac{1}{2}\#\{g \in G : C_g \neq C_{g^{-1}}\} + \#\{g \in G : C_g = C_{g^{-1}}\} - 1.
\]
Since $c(G) = c_2(G) + \#\{g \in G : C_g \neq C_{g^{-1}}\}$, we have that $k^*(G) = \frac{1}{2}(c(G) - c_2(G)) + c_2(G) - 1$, from where (6.3) follows. Thus, we finally obtain that $M^*(G) = B_{k^*(G)}$. □

We know that if $G$ is a finite abelian group then right-invariant, left-invariant and bi-invariant metrics are the same. We now check that $M^*(G) = M^*(G)$, i.e. equality holds in (6.1). If $G$ is abelian of order $n$, then $c(G) = n$ (since $C_g(G) = \{g\}$ for every $g \in G$) and $c_2(G) = 1 + k_2(G)$ since the only real elements are the identity and the involutions (order 2 elements). Thus, from (5.2) and (6.2) we have
\[
k^*(G) = \frac{n + c_2(G)}{2} - 1 = \frac{n - 1 + k_2(G)}{2} = k(G)
\]
as we wanted to show.

**Remark 6.2.** If $|G|$ is odd, then it has no real elements other than the identity. In fact, if $x \in G$ is real then $yxy^{-1} = x^{-1}$ for some $y \in G$. Hence $yx^ky^{-1} = x^{-k}$ for every $k$. Since $x$ has finite order we have $x^{-1} = x^k$ for some $k$ and thus we have $yxy^{-1} = x$. This implies that $y^2xy^{-2} = yx^{-1}y^{-1} = x$, so $x$ and $y^2$ commute, but since both $x, y$ have odd order then they must also commute. Thus, in this case we have that
\[
k^*(G) = \frac{1}{2}(c(G) - 1).
\]
For instance, the smallest non-abelian group of odd order is $G = \mathbb{Z}_7 \times \mathbb{Z}_3$, which has 5 conjugacy classes. Then $M(G) = B_{10} = 115.975$ and $M^*(G) = B_2 = 2$.

**Ambivalent groups.** If every element of $G$ is real, the group $G$ is called ambivalent. In this case $c(G) = c_2(G)$ and, hence
\[
k^*(G) = c(G) - 1
\]
for ambivalent groups $G$. By the previous remark, $G$ must have even order. Some families of groups, such as dihedral, generalized quaternions or symmetric, are known to be ambivalent. Now we give the analogous of Proposition 5.5 for bi-invariant metrics.
Proposition 6.3. The number of bi-invariant metrics of the dihedral groups $\mathbb{D}_n$, dicyclic groups $Q_{4n}$ and quasidihedral groups $Q\mathbb{D}_n^\pm$ are given by:

(a) $M^+(\mathbb{D}_{2k+1}) = B_{k+1}$ for $k \geq 1$ and $M(\mathbb{D}_{2k}) = B_{k+2}$ for $k \geq 2$.
(b) $M^+(Q_{4n}) = B_{n+2}$ for $n$ even and $M^+(Q_{4n}) = B_{n+1}$ for $n$ odd.
(c) $M^+(Q\mathbb{D}_n^\pm) = B_{3\cdot 2n-4+2}$ and $M(Q\mathbb{D}_n^\pm) = B_{3\cdot 2n-4+1}$ for $n \geq 4$.

Proof. (a) It is known that dihedral groups are ambivalent and that $c(\mathbb{D}_n) = \frac{n+3}{2}$ if $n$ is odd and $c(\mathbb{D}_n) = \frac{n+4}{2}$ if $n$ is even. In fact, the conjugacy classes are given by $\{e\}$, $\{a, a^{-1}\}$, $\{a^2, a^{-2}\}, \ldots, \{a^{n-2}, a^{n+2}\}$ and $\{b, ab, a^2b, \ldots, a^{n-1}b\}$ for $n$ odd and $\{e\}$, $\{a, a^{-1}\}$, $\{a^2, a^{-2}\}, \ldots, \{a^{n-2}, a^{n+2}\}$, $\{b, ab, a^2b, \ldots, a^{n-1}b\}$ and $\{ab, a^3b, a^5b, \ldots, a^{n-1}b\}$ for $n$ even. Hence, by (6.4) we have

\[
k^*(\mathbb{D}_n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n+2}{2} & \text{if } n \text{ is even.} \end{cases}
\]

Thus, $M^+(\mathbb{D}_n) = B_{n+2}$ for $n$ odd and $M^+(\mathbb{D}_n) = B_{n+4}$ for $n$ even.

(b) For every $n \geq 2$ we have the dicyclic groups $Q_{4n}$ of order $4n$ as defined in (5.7). We have that $c(Q_{4n}) = n + 3$. In fact, the conjugacy classes of $Q_{4n}$ are $C_e = \{e\}$, $C_{n} = \{a^n\}$, $C_{a^k} = \{a^k, a^{2n-k}\}$ for $k = 1, \ldots, n-1$, $C_0 = \{b, a^2b, a^4b, \ldots, a^{2n-2}b\}$ and $C_{ab} = \{ab, a^3b, a^5b, \ldots, a^{2n-1}b\}$. The $n+1$ classes $C_0$, $C_1$, $C_2$, $\ldots$, $C_n$ are clearly real. The class $C_0$ and $C_{ab}$ are real if and only if $n$ is even. Indeed, $(akb)(ak^n) = ak^{2m+n+1}$ hence $(akb)^{-1} = ak^{-n}b$. Therefore, if $n$ is even, $k-n$ has the same parity as $n$, and thus the inverse $(akb)^{-1}$ is in the same class as $akb$. In this way, $c_2(Q_{4n}) = n + 3$ if $n$ is even and $c_2(Q_{4n}) = n + 1$ if $n$ is odd. If $n$ is even, then $Q_{4n}$ is ambivalent and $k^*(Q_{4n}) = c(Q_{4n}) - 1$. If $n$ is odd then we have $k^*(Q_{4n}) = \frac{1}{2}(c(Q_{4n}) + c_2(Q_{4n})) - 1$. Finally, we have obtained that

\[
k^*(Q_{4n}) = \begin{cases} n+2 & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd,} \end{cases}
\]

for every $n \geq 1$.

(c) We study the conjugacy classes of the groups $Q\mathbb{D}_n^\pm$ simultaneously, using the presentations (5.8). By (5.9), the elements of $Q\mathbb{D}_n^\pm$ are of the form $x^k$ or $yxk^\pm$ with $k = 0, \ldots, 2n-1$. The conjugates of the elements $x^k$ are $x^k, x^kx^{-\ell} = x^k$ or $(yx^\ell)x^k(yx^\ell)^{-1} = yx^\ell x^{-\ell} y = x^k(2\ell \pm 1)$ for any $\ell$, where $\pm$ corresponds to the group $Q\mathbb{D}_n^\pm$. Hence, we have the conjugacy classes $C_e = \{e\}$ and $C_x = \{x^k, x^{(k+1)2n+1}\}$ for $k = 0, \ldots, 2n-1$. The conjugates of $yxk^\pm$ of the form

\[
x^\ell(yxk^\pm)x^{-\ell} = y(x^\ell y)x^{k-\ell} = yx^\ell x^{(2\ell \pm 1)2n+1} = (yxk^\pm) x^{(2\ell \pm 1)2n+1}.
\]

Thus, $C_{yxk} = \{yx^k, x^{(2\ell \pm 1)2n+1}, x^{(2n+1)(2\ell \pm 1)2n+1}\}$ for $k = 0, \ldots, 2n-1$. However, note that $C_{yxk} = C_y \cdot x^k$ for every $k$ and that $C_y = \{yx^k(2\ell \pm 1), yx^k(2n+2\ell \pm 1)\}_{\ell \geq 0}$. Summing up, the conjugacy classes of $Q\mathbb{D}_n^-$ (1st row) and $Q\mathbb{D}_n^+$ (2nd row) are given by

\[
C_e = \{e\}, \quad C_{k} = \{x^k, x^{k+1}\}, \quad C_{yxk} = \{yx^k(2\ell \pm 2), yx^k(2n+2\ell \pm 2)\}_{\ell \geq 0},
\]

\[
C_e = \{e\}, \quad C_{k} = \{x^k, x^{k+1}\}, \quad C_{yxk} = \{yx^{k+1}, yx^{k+2n-1}\}_{\ell \geq 0},
\]

for $k = 0, \ldots, 2n-1$ (there are some repetitions).

For instance, for $n = 4$ the conjugacy classes (real classes are underlined) of $Q\mathbb{D}_4^-$ are $\{e\}$, $\{x, x^3\}$, $\{x^2, x^6\}$, $\{x^4\}$, $\{x^5, x^7\}$, $\{y, yx^2, yx^4, yx^6\}$ and $\{yx, yx^3, yx^5, yx^7\}$ while the conjugacy
classes of $QD^+_4$ are $\{e\}, \{x, x^5\}, \{x^2\}, \{x^3, x^7\}, \{x^4\}, \{x^6\}, \{yx, yx^5\}, \{yx^2, yx^6\}$ and $\{yx^3, yx^7\}$. Thus $QD^-_4$ has 7 conjugacy classes (5 real) and $QD^+_4$ has 10 conjugacy classes (4 real). For $n = 5$ we have $\{e\}, \{x, x^7\}, \{x^2, x^{14}\}, \{x^3, x^5\}, \{x^4, x^{12}\}, \{x^6, x^{10}\}, \{x^8\}, \{x^9, x^{15}\}, \{x^{11}, x^{13}\}, \{xy, yx^2, yx^4, yx^8, yx^{10}, yx^{12}, yx^{14}\}$ and $\{yx, yx^3, yx^5, yx^7, yx^9, yx^{11}, yx^{13}, yx^{15}\}$ for $QD^-_5$ while $\{e\}, \{x, x^9\}, \{x^2\}, \{x^3, x^{11}\}, \{x^4\}, \{x^5, x^{13}\}, \{x^7, x^{15}\}, \{x^8\}, \{x^{10}\}, \{x^{12}\}, \{x^{14}\}, \{yx, yx^3\}, \{yx^2, yx^9\}, \{yx^2, yx^{10}\}, \{yx^3, yx^{11}\}, \{yx^4, yx^{12}\}, \{yx^7, yx^{15}\}$ for $QD^+_5$. Thus $QD^-_5$ has 11 conjugacy classes (7 real) and $QD^+_5$ has 20 conjugacy classes (8 real).

One can check that in general, $QD^-_n$ has $1 + 2^{n-2} + 2$ conjugacy classes given by $C_e$, all the $C_{x^i}$’s with $k = 1, \ldots, 2^{n-1} - 1$ (which are of the form $\{x^i, x^j\}$ with $8 \mid i + j$ except for $\{x^{2^{n-2}}\}$), $C_y$ and $C_{yx}$ out of which 1, $2^{n-3} + 2$ are real, namely $C_e$, those of the form $C_{x^{2k}}$ and $C_y$, $C_{yx}$. On the other hand, $QD^+_n$ has $(2^{n-2} + 2^{n-3}) + 2^{n-2}$ conjugacy classes given by $C_{x^k}$ with $k \geq 0$ and $C_y, C_{yx}, \ldots, C_{yx^{2^n-1}}$, out of which 4 are real, namely $C_e, C_{x^{2n-2}}, C_y$ and $C_{yx^{2n-3}}$.

Finally, we have that $c(QD^-_n) = 2^{n-2} + 3$ and $c_2(QD^-_n) = 2^{n-3} + 3$ and thus we get

$$k^*(QD^-_n) = \frac{1}{2}(2^{n-2} + 3 + 2^{n-3} + 3) - 1 = 3 \cdot 2^{n-4} + 2.$$ 

Also, $c(QD^+_n) = (2^{n-2} + 2^{n-3}) + 2^{n-2} = 5 \cdot 2^{n-3} + 4$ and $c_2(QD^-_n) = 4$, and hence we have $k^*(QD^-_n) = \frac{5 \cdot 2^{n-3} + 4}{2} - 1 = 5 \cdot 2^{n-4} + 1$, and the proof is complete. \hfill \Box

**Example 6.4.** The symmetric group $S_n$ is ambivalent for any $n$. It is a classic result that the number of conjugacy classes of $S_n$ is given by the number $p(n)$ of unordered integer partitions. Thus $k^*(S_n) = p(n) - 1$ and hence

$$M^*(S_n) = B_{p(n) - 1}.$$ 

Recall that $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5, p(5) = 7$ and $p(6) = 11$. So, for instance, we have $M^*(S_3) = B_2 = 2, M^*(S_4) = B_4 = 15, M^*(S_5) = B_6 = 203$ and $M^*(S_6) = B_{10} = 115.975$.

The conjugacy classes of the alternating groups are a bit more involved. It is known that

$$c(A_n) = 2f_n + \frac{p(n) - f_n}{2}$$

where $f_n$ is the number of self-conjugate integer partitions, that is the number of self-conjugate Ferrer diagrams of size $n$. The alternating groups $A_n$ are ambivalent only for $n = 1, 2, 5, 6, 10, 14$. Thus, $k^*(A_n) = 2f_n + \frac{p(n) - f_n}{2} - 1$ for $n = 1, 2, 5, 6, 10, 14$. \hfill \diamond

**Example 6.5.** Consider the special group $SL_2(F_q)$ of matrices $2 \times 2$ over a finite field $F_q$ of $q$ elements. That is $SL_2(F_q) = \{(a,b,c,d) : a, b, c, d \in F_q, ad - bc = 1\}$. The group $SL_2(F_q)$ has order $q^3 - q$. If $q$ is odd, this group has only one element of order 2, $-I$, and thus $k(SL_2(F_q)) = \frac{1}{2}(q^3 - q)$ and

$$M(SL_2(F_q)) = B_{\frac{1}{2}(q^3 - q)}.$$ 

For instance, $M(SL_2(F_3)) = B_{12} = 4.213.597$, $M(SL_2(F_5)) = B_{60}$ and $M(SL_2(F_{2^3})) = B_{360}$.

The number of conjugacy classes of $SL_2(F_q)$ is given by $q + 4$ for $q$ odd and by $q + 1$ if $q$ is even. It is known that $SL_2(F_q)$ is ambivalent if and only if $-1$ is a square in $F_q$, which in turn happens if and only if $q$ is even or if $q \equiv 1 \pmod{4}$. Suppose that $q$ is odd with $q \equiv 1 \pmod{4}$. Then, $k^*(SL_2(F_q)) = \frac{1}{2}(q + 4 - 1)$ by (6.4) and hence

$$M^*(SL_2(F_q)) = B_{\frac{q + 3}{2}}.$$ 

For instance, $M^*(SL_2(F_5)) = B_4 = 52$ and $M^*(SL_2(F_{2^3})) = B_6 = 203$. \hfill \diamond
Note that \( k^*(G) \) is the cardinal of the finest bi-invariant (unitary, symmetric and conjugate) partition \( \mathcal{P}^* \), and every bi-invariant partition is some union of parts of this partition. In particular, we have that

\[
k^*(G) \leq k(G) \leq n - 1,
\]

where \( n = |G| \), with both equalities holding only for \( G = \mathbb{Z}_k^2 \) for \( k \geq 1 \) (see Example 5.2 (ii)). One also have that \( k^*(G) \leq c(G) - 1 \) and every bi-invariant metric satisfies

\[
|\mathcal{P}^*| \leq k^*(G) \leq c(G) - 1.
\]

Hence, the number of bi-invariant metrics \( M^*(G) \) on \( G \) is bounded by the Bell number \( B_{c(G)} - 1 \).

We will need the following results.

**Lemma 6.6.** If \( H, K \) are groups then \( c(H \times K) = c(H)c(K), \ c_2(H \times K) = c_2(H)c_2(K) \) and

\[
(6.7) \quad k^*(H \times K) = \frac{1}{2}(c(H)c(K) + c_2(H)c_2(K)) - 1.
\]

As a consequence, the product of ambivalent groups is ambivalent.

**Proof.** Clearly, we have the identity \( C_{(h, k)} = C_h \times C_k \) of conjugacy classes for every \( (h, k) \in H \times K \) and hence \( c(H \times K) = c(H)c(K) \). Let us see that a real conjugacy class in the product \( H \times K \) is the product of real conjugacy classes in \( H \) and \( K \). The conjugacy class \( C_{(h, k)} \) is real if and only if \( C_{(h, k)} = C_{(h, k)}^{-1} = C_{(h^{-1}, k^{-1})} \). On the other hand, \( C_h \times C_k = C_{(h, k)} = C_{(k^{-1}, h^{-1})} = C_{k^{-1}} \times C_{h^{-1}} \). This implies that \( C_h = C_{h^{-1}} \) and \( C_k = C_{k^{-1}} \). The converse implication is analogous and thus we have that \( c_2(H \times K) = c_2(H)c_2(K) \). Equation (6.7) follows from (6.2) and these product formulas. \( \square \)

**Lemma 6.7.** Let \( G \) be a finite group. If \( A \) is any ambivalent finite group then

\[
(6.8) \quad k^*(G \times A) = k^*(A)(k^*(G) + 1) - 1.
\]

In particular, we have \( k^*(G \times \mathbb{Z}_2^r) = 2^r(k^*(G) + 1) - 1 \) for any \( r \in \mathbb{N}_0 \).

**Proof.** The result is a direct consequence of Lemma 6.6 and the definition (6.2) of \( k^*(G) \). \( \square \)

**Bi-invariance degree.** We know that for abelian groups, the number of invariant and bi-invariant metrics is the same. However, from our previous propositions and examples, this seems difficult for non-abelian groups to hold. In Example 4.7 we saw that the quaternions \( \mathbb{Q}_8 \) has the same number of invariant and bi-invariant metrics. Moreover, from Propositions 5.5 and 6.3 we see that this is the only dicyclic group \( \mathbb{Q}_{4n} \) having this property.

To measure the ratio of bi-invariant metrics on a finite group \( G \) out of the invariant ones we define the bi-invariance degree of \( G \) by

\[
(6.9) \quad b(G) = \frac{k^*(G)}{k(G)}.
\]

It is clear that \( 0 < b(G) \leq 1 \) and that \( b(G) = 1 \) if and only if every invariant metric is also bi-invariant. By (5.2) and (6.2) we have the expression

\[
b(G) = \frac{c(G) + c_2(G) - 2}{|G| + k_2(G) - 2}.
\]

By Propositions 5.5 and 6.3 we have

\[
b(\mathbb{D}_{2k+1}) = \frac{k+1}{2k+1}, \quad b(\mathbb{D}_{2k}) = \frac{k+2}{2k}, \quad b(\mathbb{Q}_{4n}) = \frac{n+2}{2n}, \quad b(Q\mathbb{D}_n^-) = \frac{3^2n^4+2}{5\cdot 2n^4+4}, \quad b(Q\mathbb{D}_n^+) = \frac{5\cdot 2n^4+4+1}{2n^4+1+1}.
\]

Therefore, asymptotically we have \( b(\mathbb{D}_n) \approx \frac{1}{4}, b(\mathbb{Q}_{4n}) \approx \frac{1}{2}, b(Q\mathbb{D}_n^-) \approx \frac{3}{10} \) and \( b(Q\mathbb{D}_n^+) \approx \frac{5}{3} \).

The following, one of the main results in the paper, characterizes all groups \( G \) (finite or not) in which the invariant metrics are also bi-invariant (i.e. with \( b(G) = 1 \) for the finite case).
Theorem 6.8. Every right (left) invariant metric on a group $G$ is also bi-invariant if and only if $G$ is abelian or $G = Q_8 \times H$ where $H$ is an elementary abelian 2-group. In particular, if $G$ is finite and non-abelian then $G = Q_8 \times \mathbb{Z}_2^k$ for some $k \in \mathbb{Z}_{\geq 0}$ and $M(G) = M^*(G) = B_{5,2^k-1}$.

Proof. If $G$ is abelian we know that any invariant metric is also bi-invariant. Let $G = Q_8 \times H$, where $H$ is an elementary abelian 2-group (i.e. every non trivial element has order 2). We want to show that every invariant metric $d$ on $G$ is also bi-invariant. That is, any unitary symmetric partition of $G$ is also conjugate. This happens if and only if the partition given by the conjugacy classes of $G$ is finer than the partition associated with the metric $d$, that is $C(G) \preceq P(G,d)$. The conjugacy classes of $G$ are of two forms, namely

$$C(G) = \{ \{x\} : x \in Z(G) \} \cup \{ \{x,x^{-1}\} : x \notin Z(G) \},$$

where $Z(G)$ denotes the center of $G$. In this way, we have that

$$C(G) \preceq P_{\text{Lee}}(G) \preceq P(G,d)$$

where $P_{\text{Lee}}(G)$ is the finest unitary symmetric partition of $G$, i.e. $P_{\text{Lee}}(G) = \{ \{x,x^{-1}\} : x \in G \}$.

We now prove the converse. Suppose that every invariant metric on $G$ is bi-invariant. For every $x \in G$ consider the metric given by the unitary symmetric partition

$$P_x = \{ \{e\}, \{x,x^{-1}\}, G \setminus \{e, x,x^{-1}\} \}.$$ 

Since $P_x$ is conjugate, by hypothesis, we have that $C_x \subset \{x,x^{-1}\}$.

Notice that every subgroup $N$ of $G$ is normal, i.e. $gNg^{-1} = N$ for every $g \in G$. This holds since if $x \in N$ then $C_x \subset \{x,x^{-1}\} \subset N$ and $C_x \subset N$ for every $x \in N$ clearly implies the normality of $N$. Groups in which all the subgroups are normal are characterized (see [1]), that is $G$ is either abelian or

$$G = Q_8 \times H \times K$$

where $H$ is an elementary abelian 2-group and $K$ is an abelian group in which every element has odd order. To finish the proof we show that $K$ is the trivial group. Take an element $x = (q,h,k) \in G$ with $\text{ord}(q) = 4$. On the one hand we have $C_{(q,h,k)} = \{(q,h,k),(-q,h,k)\}$. On the other hand,

$$C_{(q,h,k)} \subset \{(q,h,k),(q,h,k)^{-1}\} = \{(q,h,k),(-q,h,k)^{-1}\}.$$

Hence $k = k^{-1}$, which implies that $k = e$ since $k$ has odd order and hence $K = \{e\}$.

Finally, suppose that $G$ is finite. If $G$ is non-abelian then $G = Q_8 \times \mathbb{Z}_2^k$ for some $k \in \mathbb{N}$ and $G$ is ambivalent since $Q_8$ and $\mathbb{Z}_2^k$ are ambivalent by Lemma 6.6. Thus, by (6.4) we have

$$k^*(G) = c(G) - 1 = c(Q_8)c(\mathbb{Z}_2^k) - 1 = 5 \cdot 2^k - 1,$$

where we used (6.6) with $n = 2$ (one can also use Lemma 6.7). On the other hand, by Proposition 6.7 we obtain $k(G) = 2^k(k(Q_8) + 1) - 1 = 5 \cdot 2^k - 1$ where we used Proposition 5.5 (b) with $n = 2$. Thus, $k^*(G) = k(G)$ and hence $M^*(G) = M(G) = B_{5,2^k-1}$, as desired. \qed

To measure the commutativity of elements in non-abelian groups, Erdős and Turán defined in 1968 ([6]) the commutativity degree of a group $G$ as

$$d(G) = \frac{\# \{(x,y) \in G^2 : xy = yx\}}{|G|^2},$$

which is the probability that two elements of $G$ commute, and they proved that

(6.10) \hspace{1cm} d(G) = \frac{c(G)}{|G|}.

Later, in 1973, Gustafson [11] showed that if $G$ is non-abelian then $c(G) \leq \frac{5}{8}|G|$, hence $d(G) \leq \frac{5}{8}$. 

Note that for the groups $G = \mathbb{Q}_8 \times \mathbb{Z}_2^3$ it holds $c(G) = \frac{5}{8}|G|$. In this way, for the groups in the previous theorem we have $b(G) = 1$ and also that $d(G) = 1$ if $G$ is abelian or $d(G) = \frac{5}{8}$ if $G$ is non-abelian. In other words, if $d(G)$ is less that $\frac{5}{8}$ then $G$ has some invariant metric which is not bi-invariant, that is

$$d(G) < \frac{5}{8} \Rightarrow b(G) < 1.$$  

However, for example, if $d(G) = \frac{5}{8}$ we cannot assure that every invariant metric is bi-invariant. This is the case for the groups $\mathbb{D}_4 \times \mathbb{Z}_2^k$ (see Tables 6–10 for more examples). In fact, $c(\mathbb{D}_4 \times \mathbb{Z}_2^k) = c(\mathbb{D}_4)c(\mathbb{Z}_2^k) = 5 \cdot 2^k$ and $|\mathbb{D}_4 \times \mathbb{Z}_2^k| = 8 \cdot 2^k$, hence $d(\mathbb{D}_4 \times \mathbb{Z}_2^k) = \frac{5}{8}$. However, one checks that $b(\mathbb{D}_4 \times \mathbb{Z}_2^k) = \frac{5 \cdot 2^k - 1}{2^k} \sim \frac{5}{8} < 1$. Finally, we notice that all groups $G$ having $d(G) = \frac{5}{8}$ must be isoclinic to $\mathbb{Q}_8$ (see [14]).

7. The number of invariant metrics on groups of order up to 32

In this final section we compute the number of non-equivalent invariant and bi-invariant metrics for all the groups of order less than or equal to 32.

**Theorem 7.1.** Let $G$ be a finite group of order $n \leq 32$. The number of non-equivalent invariant and bi-invariant metrics of $G$ are given in Tables 6–10.

**Proof.** By Propositions 5.1 and 6.1, the number of invariant and bi-invariant metrics of $G$ are given respectively by $M(G) = B_{k(G)}$ and $M^*(G) = B_{k^*(G)}$ where $B_m$ is the $m$-th Bell number and $k(G)$ and $k^*(G)$ are given in (5.3) and (6.3).

We know some special cases in general. For instance, we know that $M(G) = B_{(n - 1)/2}$ if $n$ is odd (for any $G$) and that $M(G) = M^*(G)$ if $G$ is abelian. For $G$ abelian of even order $n = 2m$ it is enough to use Corollary 5.4 asserting that $M(G) = B_{m + 2 - 1 - 1}$ where $s$ is the number of factors of the form $2^r$ in the prime decomposition of $G$. For some families of non-abelian groups, we also know these numbers in general. Dihedral, dicyclic (including generalized quaternions) and quasidihedral groups are covered by Propositions 5.5 and 6.3 while symmetric, alternating and special linear groups are treated in Proposition 5.6 and Examples 6.4 and 6.5.

We will do a case by case study, proceeding by increasing order of the groups. The invariant and bi-invariant metrics for groups of order up to 7 were studied in Section 4 by inspection. With the mentioned results for abelian groups and dihedral or symmetric groups we check that the number of metrics are indeed correct. So we analyze the cases $n \geq 8$. We will do it in detail for $8 \leq n \leq 16$, i.e. the groups in Table 6. For the cases $17 \leq n \leq 32$ we will just give some comments, since it is clear how to proceed in each case.

**Order 8.** There are 5 groups of order 8, the abelian ones $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2^3$ and the non-abelian ones $\mathbb{D}_4$ and $\mathbb{Q}_8$. It is clear that $M(\mathbb{Z}_8) = B_3$, $M(\mathbb{Z}_4 \times \mathbb{Z}_2) = B_6$ and $M(\mathbb{Z}_2^3) = B_7$. For the dihedral group we have $M(\mathbb{D}_4) = B_6$ and $M^*(\mathbb{D}_4) = B_4$ and for the quaternion group $\mathbb{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ we have $M(\mathbb{Q}_8) = M^*(\mathbb{Q}_8) = B_4$.

**Order 9.** There are 2 groups of order 9, $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$, hence $M(G) = B_4$ in these two cases.

**Order 10.** There are two groups of order 10, the cyclic group $\mathbb{Z}_{10}$ and the dihedral group $\mathbb{D}_5$. Since $\mathbb{Z}_{10} = \mathbb{Z}_5 \times \mathbb{Z}_2$ we have that $M(\mathbb{Z}_{10}) = B_6$. Also, $M(\mathbb{D}_5) = B_7$ and $M^*(\mathbb{D}_5) = B_3$.

**Order 11.** There is only one group of order 11, $\mathbb{Z}_{11}$, hence $M(\mathbb{Z}_{11}) = B_5$.

**Order 12.** There are five groups of order 12, two abelian ones $\mathbb{Z}_{12} = \mathbb{Z}_4 \times \mathbb{Z}_3$ and $\mathbb{Z}_6 \times \mathbb{Z}_2 = \mathbb{Z}_3 \times \mathbb{Z}_2^2$, and three non-abelian ones $\mathbb{D}_6$, $\mathbb{Q}_{12}$ and $\mathbb{A}_4$. For the abelian groups we have $M(\mathbb{Z}_{12}) = B_6$ and $M(\mathbb{Z}_6 \times \mathbb{Z}_2) = B_7$. For the dihedral group we have $M(\mathbb{D}_6) = B_9$ and $M^*(\mathbb{D}_6) = B_5$ while for the dicyclic group $\mathbb{Q}_{12}$ we have that $M(\mathbb{Q}_{12}) = B_6$ and $M^*(\mathbb{Q}_{12}) = B_4$. For the alternating group
and

\begin{align*}
A_4 \text{ we have } M(A_4) = B_7 \text{ by Proposition 5.6. To compute the number of bi-invariant metrics on } A_4, \text{ notice that } A_4 \text{ has 4 conjugacy classes given by } \{id\}, \{(12)(34), (13)(24), (14)(23)\}, \{(123), (243), (142), (134)\} \text{ and } \{(132), (124), (143), (234)\}, \text{ the first two of which are real classes. Hence, } k^*(A_4) = \frac{k+2}{2} - 1 = 2 \text{ and thus } M^*(A_4) = B_2.
\end{align*}

Order 13. There is only one group of order 13, \(Z_{13}\), hence \(M(Z_{13}) = B_6\).

Order 14. There are two groups of order 14, the cyclic group \(Z_{14}\) and the dihedral group \(D_7\). Since \(Z_{14} = Z_7 \times Z_2\) we have that \(M(Z_{14}) = B_8\). Also, \(M(D_7) = B_{10}\) and \(M^*(D_7) = B_4\).

Order 15. There is only one group of order 15, \(Z_{15}\), hence \(M(Z_{15}) = B_7\).

Order 16. There are 14 groups of order 16; the 5 abelian groups \(Z_{16}, Z_8 \times Z_2, Z_4^2, Z_4 \times Z_2^2\) and \(Z_2^4\), and the 9 non-abelian groups \(D_8, Q_{16}, QD_{16}^+, QD_{16}^-, D_4 \times Z_2, Q_8 \times Z_2, Z_4 \times Z_4, Z_2^3 \times Z_4\), and \((Z_4 \times Z_2) \rtimes Z_2\). We have \(M(Z_{16}) = B_8\), \(M(Z_8 \times Z_2) = M(Z_2^4) = B_9\), \(M(Z_4 \times Z_2^2) = B_{11}\) and \(M(Z_2^4) = B_{15}\). For the dihedral groups we know that \(M(D_8) = B_{12}\) and \(M^*(D_8) = B_6\), for the quaternionic group we have \(M(Q_{16}) = B_8\) and \(M^*(Q_{16}) = B_6\) and for the semidihedral groups we have \(M(QD_4^+) = B_{20}\), \(M(QD_4^-) = B_9\) and \(M^*(QD_4^+) = B_5\), \(M^*(QD_4^-) = B_6\).

For the direct products \(G \times Z_2\) with \(G\) non-abelian of order 8, using Lemmas 5.3 and 6.7 we get that \(M(D_4 \times Z_2) = B_{13}\) and \(M^*(D_4 \times Z_2) = B_9\) and that \(M(Q_8 \times Z_2) = B_9\) and \(M^*(Q_8 \times Z_2) = B_9\).

It remains to study the three semidirect products \(Z_4 \rtimes \varphi Z_4, Z_2^2 \rtimes \varphi Z_4\) and \((Z_4 \times Z_2) \rtimes \varphi Z_2\). For the first semidirect product, \(\varphi : Z_4 \to \text{Aut}(Z_4)\) is determined by \(\varphi(1)\) and \(\varphi_1(a) = (-1)^a\). By \((5.13)\) we have that

\begin{equation}
k_2(Z_4 \rtimes \varphi Z_4) = k_2(Z_4) + \sum_{h \in Z_4, 2h = 0} \# \{g \in Z_4 : \varphi_h(g) = -g\} = 2 + 2 = 4,
\end{equation}

where \(\#\{g \in Z_4 : \varphi_2(g) = -g\} = 2\) since \(\varphi_2 = id\).

For the second semidirect product, \(\varphi : Z_4 \to \text{Aut}(Z_2^2) \simeq \text{GL}_2(Z_2)\) is determined by \(\varphi(1)\) which must be an element of order 2. Since all elements of order 2 in \(\text{GL}_2(Z_2)\) are conjugate, all possible automorphism give rise to the same semidirect product. Thus, we can take \(\varphi_1\) as multiplication by the matrix \(J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). By \((5.13)\) we have that \(\varphi_2 = id\) and

\begin{equation}
k_2(Z_2^2 \rtimes \varphi Z_4) = k_2(Z_2^2) + \# \{(x, y) \in Z_2^2 : 2(x, y) = (0, 0)\} = 4 + 4 = 8.
\end{equation}

For the third semidirect product, we use that \((Z_4 \times Z_2) \rtimes \varphi Z_2 \simeq Q_8 \rtimes \varphi Z_2\), where the morphism \(\varphi : Z_2 \to \text{Aut}(Q_8)\) is given by conjugation by \(i\), that is \(\varphi'_1(x) = -ix_i^{-1}\). By \((5.16)\) we have that

\begin{equation}
k_2(Q_8 \rtimes \varphi' Z_2) = k_2(Q_8) + \# \{x \in Q_8 : xix_i^{-1} = -1\} = 2 + 6 = 8,
\end{equation}

since \(xix(-i) = 1\) holds for every quaternion different from \(\pm i\).

Thus, we have that \(k(Z_4 \times Z_4) = \frac{16+4}{2} - 1 = 1\) and \(k(Z_2^2 \rtimes \varphi Z_4) = k((Z_4 \times Z_2) \rtimes \varphi Z_2) = \frac{16+8}{2} - 1\) by \((7.1)-(7.3)\), and hence \(M(Z_4 \times Z_4) = B_9\) and \(M(Z_2^2 \rtimes \varphi Z_4) = M((Z_4 \times Z_2) \rtimes \varphi Z_2) = B_{11}\).

For the group \(Z_4 \rtimes \varphi Z_4\) the conjugacy classes are

\[
\{e\}, \{a, a^3\}, \{x, a^2x\}, \{a^2\}, \{x^2\}, \{ax, a^3x\}, \{ax^2, a^3x^2\}, \{x^3, a^2x^3\}, \{a^2x^2\}, \{a^3, a^3x^3\}
\]

while the real conjugacy classes are \(\{e\}, \{a, a^3\}, \{a^2\}, \{x^2\}, \{ax^2, a^3x^2\}, \{a^2x^2\}\). Thus, we have \(k(Z_4 \rtimes \varphi Z_4) = \frac{16+6}{2} - 1\) and hence \(M^*(Z_4 \rtimes \varphi Z_4) = B_7\).
The groups

\[ \mathbb{Z}_2 \rtimes \mathbb{Z}_4 = \langle a, b, c : a^2 = b^2 = c^4 = 1, cac^{-1} = ab = ba, bc = cb \rangle, \]
\[ (\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle a, b, c : a^4 = c^2 = 1, b^2 = a^2, ab = ba, ac = ca, cbc = a^2b \rangle, \]

have character tables given by

\[
\begin{array}{c|cccccccccc}
\rho_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\rho_2 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
\rho_3 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
\rho_4 & 1 & -i & -i & 1 & 1 & -i & -i & 1 & 1 & 1 \\
\rho_5 & 1 & i & -i & 1 & 1 & i & -i & 1 & 1 & 1 \\
\rho_6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\rho_7 & 1 & i & -i & 1 & 1 & i & -i & 1 & 1 & 1 \\
\rho_8 & 1 & -i & -i & 1 & 1 & -i & -i & 1 & 1 & 1 \\
\rho_9 & 2 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & -2 & 0 \\
\rho_{10} & 2 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 2 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccccccccc}
\rho_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\rho_2 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
\rho_3 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
\rho_4 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
\rho_5 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
\rho_6 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
\rho_7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\rho_8 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\rho_9 & 2 & 0 & 0 & -2i & -2 & 0 & 0 & 0 & 2i & 0 \\
\rho_{10} & 2 & 0 & 0 & 2i & -2 & 0 & 0 & 0 & -2i & 0 \\
\end{array}
\]

In general, the number of characters of a group is the same as the number of conjugacy classes while the number of real characters coincides with the number of real conjugacy classes. Thus, we have \( k(\mathbb{Z}_2^3) = 1 + 1 + 8 - 1 = 7 \) and \( k((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) = 10 + 1 + 8 - 1 = 18 \) and hence \( M^*(\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4) = B_7 \) and \( M^*((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) = B_8 \).

For the remaining groups of order \( 17 \leq n \leq 32 \) (see Tables 7–10) the results follow in the same way as before from the results of the previous section. For the computation of the invariant metrics for the groups of even order, the groups that do not fall into the hypothesis of our previous results are the group \( \mathbb{Z}_5 \rtimes \mathbb{Z}_4 = \langle a, b : a^6 = b^4 = e, bab^{-1} = a^3 \rangle \) (or order 20), the group \( \mathbb{Z}_3 \times \mathbb{Z}_8 = \langle a, b : a^3 = b^8 = e, bab^{-1} = a^{-1} \rangle \) (or order 74) of order 24 and those groups of order 32 numbered \#116–121 and \#135–142 in Table 10. Using the presentations it is easy (though tedious) to check that \( k_2(\mathbb{Z}_5 \rtimes \mathbb{Z}_4) = 6 \) and \( k_2(\mathbb{Z}_3 \times \mathbb{Z}_8) = 2 \) from which one obtain the number of invariant metrics. For the rest of the groups one can also use their presentations with generators and relations (or some mathematical software to count the number of elements of order 2 of these groups). To compute the bi-invariant metrics of the non-abelian groups of order \( 17 \leq n \leq 32 \) which are not dihedral, semidihedral or dicyclic groups, one can use their character tables as before. For instance, one can use the webpage GroupNames ([10]). However, many of the direct products can be computed by reducing them to easier cases by using Lemma 6.7. □
Table 6. Invariant metrics on groups of order $1 \leq n \leq 16$

| #  | order | id   | $G$      | $M(G)$ | $M^*(G)$ | $b(G)$ | $d(G)$ | comment                  |
|----|-------|------|----------|--------|----------|--------|--------|--------------------------|
| 1  | 1 (1) | 1    | $\{e\}$ | $B_1$  | 1        | 1      | 1      | abelian                  |
| 2  | 2 (2) | 1    | $\mathbb{Z}_2$ | $B_1$  | 1        | 1      |        | abelian                  |
| 3  | 3 (1) | 1    | $\mathbb{Z}_3$ | $B_1$  | 1        |        |        | abelian                  |
| 4  | 4 (2) | 1    | $\mathbb{Z}_4$ | $B_2$  | 1        | 1      |        | abelian                  |
| 5  | 4 (2) | 2    | $\mathbb{Z}_2^2$ | $B_3$  | 1        | 1      |        | abelian                  |
| 6  | 5 (1) | 1    | $\mathbb{Z}_5$ | $B_2$  | 1        | 1      |        | abelian                  |
| 7  | 6 (2) | 2    | $\mathbb{Z}_6$ | $B_3$  | 1        |        |        | abelian                  |
| 8  | 6 (2) | 1    | $D_3$ | $B_4$  | $B_2$  | $\frac{1}{2}$ | $\frac{1}{4}$ | dihedral, $S_3$, $SL_2(\mathbb{F}_2)$ |
| 9  | 7 (1) | 1    | $\mathbb{Z}_7$ | $B_3$  | 1        |        |        | abelian                  |
| 10 | 8 (5) | 1    | $\mathbb{Z}_8$ | $B_4$  | 1        |        |        | abelian                  |
| 11 | 8 (5) | 2    | $\mathbb{Z}_4 \times \mathbb{Z}_2$ | $B_5$  | 1        |        |        | abelian                  |
| 12 | 8 (5) | 5    | $\mathbb{Z}_2^3$ | $B_7$  | 1        |        |        | abelian                  |
| 13 | 8 (5) | 3    | $D_4$ | $B_6$  | $B_4$  | $\frac{2}{3}$ |            | dihedral                  |
| 14 | 8 (5) | 4    | $Q_8$ | $B_4$  | $B_4$  | 1        |        | quaternionic              |
| 15 | 9 (2) | 1    | $\mathbb{Z}_9$ | $B_4$  | 1        |        |        | abelian                  |
| 16 | 9 (2) | 2    | $\mathbb{Z}_3^2$ | $B_4$  | 1        |        |        | abelian                  |
| 17 | 10 (2) | 2   | $\mathbb{Z}_{10}$ | $B_5$  | 1        |        |        | abelian                  |
| 18 | 10 (2) | 1  | $D_5$ | $B_7$  | $B_3$  | $\frac{1}{2}$ | $\frac{1}{4}$ | dihedral                  |
| 19 | 11 (1) | 1  | $\mathbb{Z}_{11}$ | $B_5$  | 1        |        |        | abelian                  |
| 20 | 12 (5) | 2  | $\mathbb{Z}_{12}$ | $B_6$  | 1        |        |        | abelian                  |
| 21 | 12 (5) | 5  | $\mathbb{Z}_4 \times \mathbb{Z}_2^2$ | $B_7$  | 1        |        |        | abelian                  |
| 22 | 12 (5) | 4  | $D_6$ | $B_9$  | $B_5$  | $\frac{5}{4}$ | $\frac{1}{4}$ | dihedral                  |
| 23 | 12 (5) | 1  | $Q_{12}$ | $B_6$  | $B_4$  | $\frac{2}{3}$ | $\frac{1}{3}$ | dicyclic                  |
| 24 | 13 (1) | 3  | $A_4$ | $B_7$  | $B_2$  | $\frac{2}{3}$ |            | alternating                |
| 25 | 13 (1) | 1  | $\mathbb{Z}_{13}$ | $B_6$  | 1        |        |        | abelian                  |
| 26 | 14 (2) | 2  | $\mathbb{Z}_{14}$ | $B_7$  | 1        |        |        | abelian                  |
| 27 | 14 (2) | 1  | $D_7$ | $B_{10}$ | $B_4$  | $\frac{2}{3}$ | $\frac{1}{3}$ | dihedral                  |
| 28 | 15 (1) | 1  | $\mathbb{Z}_{15}$ | $B_7$  | 1        |        |        | abelian                  |
| 29 | 16 (14) | 1 | $\mathbb{Z}_{16}$ | $B_8$  | 1        |        |        | abelian                  |
| 30 | 16 (14) | 5 | $\mathbb{Z}_8 \times \mathbb{Z}_2$ | $B_9$  | 1        |        |        | abelian                  |
| 31 | 16 (14) | 2 | $\mathbb{Z}_2^4$ | $B_9$  | 1        |        |        | abelian                  |
| 32 | 16 (14) | 10 | $\mathbb{Z}_4 \times \mathbb{Z}_2^2$ | $B_{11}$ | 1        |        |        | abelian                  |
| 33 | 16 (14) | 14 | $\mathbb{Z}_2^4$ | $B_{15}$ | 1        |        |        | abelian                  |
| 34 | 16 (14) | 7 | $D_8$ | $B_{12}$ | $B_6$  | $\frac{1}{3}$ | $\frac{1}{3}$ | dihedral                  |
| 35 | 16 (14) | 9 | $Q_{16}$ | $B_8$  | $B_6$  | $\frac{1}{3}$ | $\frac{1}{3}$ | quaternionic              |
| 36 | 16 (14) | 8 | $QD_{14}$ | $B_{10}$ | $B_5$  | $\frac{1}{3}$ | $\frac{1}{3}$ | quasidihedral             |
| 37 | 16 (14) | 6 | $QD_{14}^2$ | $B_9$  | $B_6$  | $\frac{1}{3}$ | $\frac{1}{3}$ | quasidihedral             |
| 38 | 16 (14) | 11 | $D_4 \times \mathbb{Z}_2$ | $B_{13}$ | $B_9$  | $\frac{1}{3}$ |            | product, gen. dihedral    |
| 39 | 16 (14) | 12 | $Q_8 \times \mathbb{Z}_2$ | $B_9$  | 1        |        |        | product                  |
| 40 | 16 (14) | 4 | $\mathbb{Z}_4 \times \mathbb{Z}_4$ | $B_7$  | 1        |        |        | semidirect product       |
| 41 | 16 (14) | 3 | $\mathbb{Z}_2^3 \times \mathbb{Z}_4$ | $B_{11}$ | $B_7$  | $\frac{1}{3}$ | $\frac{1}{3}$ | semidirect product       |
| 42 | 16 (14) | 13 | $(\mathbb{Z}_4 \times \mathbb{Z}_2) \times \mathbb{Z}_2$ | $B_{11}$ | $B_8$  | $\frac{1}{3}$ |            | semidirect product       |
Comments on Tables 6–10. In the second and third columns we list the GAP id of the groups. In the second column we also indicate between parenthesis the number of groups of the given order. Recall that \( b(G) = \frac{k^*(G)}{k(G)} \) and \( d(G) = \frac{c(G)}{|G|} \). For abelian groups \( G \) we omit the number of bi-invariant metrics, since \( M(G) = M^*(G) \), and \( d(G) = 1 \). In the last column we indicate the property of group that best describes it. In general there are some of them and we list the ones that can be used to obtain the number of metrics with our results. Finally, we mention that to find which groups of order \( 2n \) are generalized dihedral groups, we perform the semidirect product of non-cyclic abelian groups of order \( n \) different from \( \mathbb{Z}^k_2 \) and compare them with the non-abelian groups of order \( 2n \) in the tables. We get: \( \mathbb{D}(\mathbb{Z}_4 \times \mathbb{Z}_2) = \mathbb{D}_4 \times \mathbb{Z}_2, \mathbb{D}(\mathbb{Z}_3^2) = \mathbb{Z}_3^2 \times \mathbb{Z}_2, \mathbb{D}(\mathbb{Z}_3 \times \mathbb{Z}_2^2) = \mathbb{D}_6 \times \mathbb{Z}_2, \mathbb{D}(\mathbb{Z}_8 \times \mathbb{Z}_2) = \mathbb{D}_8 \times \mathbb{Z}_2, \mathbb{D}(\mathbb{Z}_4) = \mathbb{Z}_4 \times \mathbb{D}_4 \) and \( \mathbb{D}(\mathbb{Z}_4 \times \mathbb{Z}_2^2) = \mathbb{D}_4 \times \mathbb{Z}_2^2 \).

**Table 7. Invariant metrics on groups of order 17 ≤ \( n \) ≤ 24**

| #  | order     | id   | \( G \) | \( M(G) \) | \( M^*(G) \) | \( b(G) \) | \( d(G) \) | comment        |
|----|-----------|------|---------|-------------|--------------|------------|------------|----------------|
| 43 | 17 (1)    | 1    | \( \mathbb{Z}_{17} \) | \( B_8 \)   | 1            | abelian    |
| 44 | 18 (5)    | 2    | \( \mathbb{Z}_{18} \) | \( B_9 \)   | 1            | abelian    |
| 45 | 5         | 5    | \( \mathbb{Z}_5 \times \mathbb{Z}_2 \) | \( B_9 \)   | 1            | abelian    |
| 46 | 1         | 1    | \( \mathbb{D}_9 \) | \( B_{13} \) | \( B_5 \)  | dihedral   |
| 47 | 3         | 3    | \( \mathbb{D}_3 \times \mathbb{Z}_3 \) | \( B_{10} \) | \( B_5 \)  | product    |
| 48 | 4         | 4    | \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) | \( B_{13} \) | \( B_5 \)  | generalized dihedral |
| 49 | 19 (1)    | 1    | \( \mathbb{Z}_{19} \) | \( B_9 \)   | 1            | abelian    |
| 50 | 20 (5)    | 2    | \( \mathbb{Z}_{20} \) | \( B_{10} \) | 1            | abelian    |
| 51 | 5         | 5    | \( \mathbb{Z}_5 \times \mathbb{Z}_2^2 \) | \( B_{11} \) | 1            | abelian    |
| 52 | 4         | 4    | \( \mathbb{D}_{10} \) | \( B_{15} \) | \( B_7 \)  | dihedral   |
| 53 | 1         | 1    | \( \mathbb{Q}_{20} \) | \( B_{10} \) | \( B_6 \)  | cyclic     |
| 54 | 3         | 3    | \( \mathbb{Z}_5 \times \mathbb{Z}_4 \) | \( B_{12} \) | \( B_3 \)  | semidirect product |
| 55 | 21 (2)    | 2    | \( \mathbb{Z}_{21} \) | \( B_{10} \) | 1            | abelian    |
| 56 | 1         | 1    | \( \mathbb{Z}_7 \times \mathbb{Z}_3 \) | \( B_{10} \) | \( B_2 \)  | \( \frac{5}{7} \) |
| 57 | 22 (2)    | 2    | \( \mathbb{Z}_{22} \) | \( B_{11} \) | 1            | abelian    |
| 58 | 1         | 1    | \( \mathbb{D}_{11} \) | \( B_{16} \) | \( \frac{2}{7} \) | dihedral   |
| 59 | 23 (1)    | 1    | \( \mathbb{Z}_{23} \) | \( B_{11} \) | 1            | abelian    |
| 60 | 24 (15)   | 2    | \( \mathbb{Z}_{24} \) | \( B_{12} \) | 1            | abelian    |
| 61 | 9         | 9    | \( \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \) | \( B_{13} \) | 1            | abelian    |
| 62 | 15        | 15   | \( \mathbb{Z}_3 \times \mathbb{Z}_2^3 \) | \( B_{15} \) | 1            | abelian    |
| 63 | 6         | 6    | \( \mathbb{D}_{12} \) | \( B_{18} \) | \( B_8 \)  | dihedral   |
| 64 | 4         | 4    | \( \mathbb{Q}_{24} \) | \( B_{12} \) | \( B_8 \)  | quaternionic |
| 65 | 14        | 14   | \( \mathbb{D}_6 \times \mathbb{Z}_2 \) | \( B_{19} \) | \( B_{11} \)| product, gen. dihedral |
| 66 | 10        | 10   | \( \mathbb{D}_4 \times \mathbb{Z}_3 \) | \( B_{14} \) | \( B_9 \)  | \( \frac{11}{12} \) |
| 67 | 5         | 5    | \( \mathbb{D}_3 \times \mathbb{Z}_4 \) | \( B_{15} \) | \( B_8 \)  | product    |
| 68 | 7         | 7    | \( \mathbb{Q}_{12} \times \mathbb{Z}_2 \) | \( B_{13} \) | \( B_9 \)  | product    |
| 69 | 11        | 11   | \( \mathbb{Q}_8 \times \mathbb{Z}_3 \) | \( B_{12} \) | \( B_9 \)  | product    |
| 70 | 13        | 13   | \( \mathbb{A}_4 \times \mathbb{Z}_2 \) | \( B_{15} \) | \( B_5 \)  | product    |
| 71 | 12        | 12   | \( \mathbb{S}_4 \) | \( B_{16} \) | \( B_4 \)  | symmetric |
| 72 | 3         | 3    | \( \text{SL}_2(F_5) \) | \( B_{12} \) | \( B_4 \)  | special linear |
| 73 | 8         | 8    | \( \mathbb{Z}_6 \times \mathbb{Z}_2 \) | \( B_{16} \) | \( B_7 \)  | semidirect product |
| 74 | 1         | 1    | \( \mathbb{Z}_3 \times \mathbb{Z}_8 \) | \( B_{12} \) | \( B_7 \)  | semidirect product |
Table 8. Invariant metrics on groups of order $25 \leq n \leq 31$

| #  | order | id  | $G$     | $M(G)$ | $M^*(G)$ | $b(G)$ | $d(G)$ | comment          |
|----|-------|-----|---------|--------|----------|--------|--------|------------------|
| 75 | 25 (2) | 1   | $\mathbb{Z}_{25}$ | $B_{12}$ | 1        |        |        | abelian          |
| 76 | 2      | 2   | $\mathbb{Z}_2^5$ | $B_{12}$ | 1        |        |        | abelian          |
| 77 | 26 (2) | 2   | $\mathbb{Z}_{26}$ | $B_{13}$ | 1        | $\frac{7}{13}$ | $\frac{4}{13}$ | abelian          |
| 78 | 1      | 1   | $\mathbb{D}_{13}$ | $B_{19}$ | $B_7$    |        |        | dihedral         |
| 79 | 27 (5) | 1   | $\mathbb{Z}_{27}$ | $B_{13}$ | 1        |        |        | abelian          |
| 80 | 2      | 2   | $\mathbb{Z}_9 \times \mathbb{Z}_3$ | $B_{13}$ | 1        |        |        | abelian          |
| 81 | 5      | 4   | $\mathbb{Z}_3^4$ | $B_{13}$ | 1        |        |        | abelian          |
| 82 | 4      | 5   | $\mathbb{Z}_9 \times \mathbb{Z}_3$ | $B_{13}$ | $B_5$    | $\frac{5}{13}$ | $\frac{11}{27}$ | semidirect product |
| 83 | 3      | 6   | $\mathbb{Z}_3^4 \times \mathbb{Z}_3$ | $B_{13}$ | $B_5$    | $\frac{5}{13}$ | $\frac{11}{27}$ | semidirect product |
| 84 | 28 (4) | 2   | $\mathbb{Z}_{28}$ | $B_{14}$ | 1        |        |        | abelian          |
| 85 | 4      | 3   | $\mathbb{Z}_{14} \times \mathbb{Z}_2$ | $B_{15}$ | 1        |        |        | abelian          |
| 86 | 3      | 1   | $\mathbb{D}_{14}$ | $B_{21}$ | $B_9$    | $\frac{3}{7}$ | $\frac{5}{14}$ | dihedral         |
| 87 | 1      | 7   | $\mathbb{Q}_{28}$ | $B_{14}$ | $B_8$    | $\frac{5}{7}$ | $\frac{5}{7}$ | dicyclic         |
| 88 | 29 (1) | 1   | $\mathbb{Z}_{29}$ | $B_{14}$ | 1        |        |        | abelian          |
| 89 | 30 (4) | 4   | $\mathbb{Z}_{30}$ | $B_{15}$ | 1        |        |        | abelian          |
| 90 | 3      | 2   | $\mathbb{D}_{15}$ | $B_{22}$ | $B_8$    | $\frac{3}{11}$ | $\frac{3}{11}$ | dihedral         |
| 91 | 2      | 1   | $\mathbb{D}_{5} \times \mathbb{Z}_3$ | $B_{17}$ | $B_7$    | $\frac{7}{17}$ | $\frac{2}{5}$ | product          |
| 92 | 1      | 3   | $\mathbb{D}_{3} \times \mathbb{Z}_5$ | $B_{16}$ | $B_8$    | $\frac{1}{2}$ | $\frac{1}{2}$ | product          |
| 93 | 31 (1) | 1   | $\mathbb{Z}_{31}$ | $B_{15}$ | 1        |        |        | abelian          |

Table 9. Invariant metrics on groups of order $32$, I: abelian, dihedral, quasidihedral and dicyclic

| #  | id  | $G$     | $M(G)$ | $M^*(G)$ | $b(G)$ | $d(G)$ | comment          |
|----|-----|---------|--------|----------|--------|--------|------------------|
| 94 | 1   | $\mathbb{Z}_{32}$ | $B_{16}$ | 1        |        |        | abelian          |
| 95 | 16  | $\mathbb{Z}_{16} \times \mathbb{Z}_2$ | $B_{17}$ | 1        |        |        | abelian          |
| 96 | 3   | $\mathbb{Z}_8 \times \mathbb{Z}_4$ | $B_{17}$ | 1        |        |        | abelian          |
| 97 | 36  | $\mathbb{Z}_8 \times \mathbb{Z}_2^2$ | $B_{19}$ | 1        |        |        | abelian          |
| 98 | 21  | $\mathbb{Z}_4^2 \times \mathbb{Z}_2$ | $B_{19}$ | 1        |        |        | abelian          |
| 99 | 45  | $\mathbb{Z}_4 \times \mathbb{Z}_5^2$ | $B_{23}$ | 1        |        |        | abelian          |
| 100| 51  | $\mathbb{Z}_5^3$ | $B_{31}$ | 1        |        |        | abelian          |
| 101| 18  | $\mathbb{D}_{16}$ | $B_{24}$ | $B_{10}$ | $\frac{3}{17}$ | $\frac{11}{17}$ | dihedral         |
| 102| 20  | $\mathbb{Q}_{32}$ | $B_{16}$ | $B_{10}$ | $\frac{5}{32}$ | $\frac{11}{32}$ | quaternionic     |
| 103| 19  | $\mathbb{SD}_{5}$ | $B_{20}$ | $B_8$    | $\frac{4}{5}$ | $\frac{11}{32}$ | quasidihedral    |
| 104| 17  | $\mathbb{SD}_{5}$ | $B_{17}$ | $B_{11}$ | $\frac{11}{17}$ | $\frac{4}{5}$ | quasidihedral    |
Table 10. Invariant metrics on groups of order 32, II: products and extensions

| #  | id | G                      | $M(G)$ | $M^*(G)$ | $b(G)$ | $d(G)$ | comment                               |
|----|----|------------------------|--------|----------|--------|--------|----------------------------------------|
| 105| 39 | $D_8 \times Z_2$      | $B_{25}$ | $B_{13}$ | $\frac{15}{12}$ | $\frac{7}{16}$ | direct product, gen. dihedral           |
| 106| 25 | $D_4 \times Z_4$      | $B_{21}$ | $B_{14}$ | $\frac{7}{16}$ | $\frac{5}{16}$ | direct product                         |
| 107| 46 | $D_4 \times Z_2^2$    | $B_{27}$ | $B_{19}$ | $\frac{13}{16}$ | $\frac{7}{16}$ | direct product, gen. dihedral           |
| 108| 41 | $Q_{16} \times Z_2$   | $B_{17}$ | $B_{13}$ | $\frac{13}{16}$ | $\frac{7}{16}$ | direct product                         |
| 109| 26 | $Q_8 \times Z_4$      | $B_{17}$ | $B_{14}$ | $\frac{13}{16}$ | $\frac{7}{16}$ | direct product                         |
| 110| 47 | $Q_8 \times Z_2^2$    | $B_{19}$ | $B_{19}$ | $1$       | $\frac{7}{16}$ | direct product                         |
| 111| 40 | $SD_5^+ \times Z_2$   | $B_{21}$ | $B_{11}$ | $\frac{11}{16}$ | $\frac{7}{16}$ | direct product                         |
| 112| 37 | $SD_5^+ \times Z_2$   | $B_{19}$ | $B_{13}$ | $\frac{13}{16}$ | $\frac{7}{16}$ | direct product                         |
| 113| 23 | $(Z_4 \times Z_4) \times Z_2$ | $B_{19}$ | $B_{15}$ | $\frac{15}{16}$ | $\frac{7}{16}$ | direct product                         |
| 114| 22 | $(Z_2^2 \times Z_4) \times Z_2$ | $B_{23}$ | $B_{15}$ | $\frac{15}{16}$ | $\frac{7}{16}$ | direct product                         |
| 115| 48 | $(Z_4 \circ Z_4) \times Z_2$ | $B_{23}$ | $B_{17}$ | $\frac{17}{16}$ | $\frac{7}{16}$ | direct product                         |
| 116| 38 | $Z_8 \circ D_4$       | $B_{19}$ | $B_{13}$ | $\frac{13}{16}$ | $\frac{7}{16}$ | central product                        |
| 117| 42 | $Z_4 \circ D_8$       | $B_{21}$ | $B_{11}$ | $\frac{11}{16}$ | $\frac{7}{16}$ | central product                        |
| 118| 49 | $D_4 \circ D_4$       | $B_{25}$ | $B_{16}$ | $\frac{16}{17}$ | $\frac{7}{16}$ | central product                        |
| 119| 50 | $D_4 \circ Q_8$       | $B_{21}$ | $B_{13}$ | $\frac{13}{16}$ | $\frac{7}{16}$ | central product                        |
| 120| 11 | $Z_4 \wr Z_2$         | $B_{19}$ | $B_{19}$ | $1$       | $\frac{7}{16}$ | wreath product                         |
| 121| 27 | $Z_2^2 \wr Z_2$       | $B_{25}$ | $B_{13}$ | $\frac{13}{16}$ | $\frac{7}{16}$ | wreath product                         |
| 122| 24 | $Z_4^2 \rtimes_1 Z_2$ | $B_{19}$ | $B_{13}$ | $\frac{13}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 123| 33 | $Z_4^2 \rtimes_2 Z_2$ | $B_{33}$ | $B_{10}$ | $\frac{10}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 124| 43 | $Z_8 \rtimes Z_2^2$   | $B_{23}$ | $B_{10}$ | $\frac{10}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 125| 4  | $Z_8 \rtimes Z_4$     | $B_{17}$ | $B_{11}$ | $\frac{11}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 126| 6  | $Z_2^3 \rtimes Z_4$   | $B_{21}$ | $B_{8}$  | $\frac{8}{16}$  | $\frac{7}{16}$ | semidirect product                    |
| 127| 12 | $Z_4 \rtimes Z_8$     | $B_{17}$ | $B_{12}$ | $\frac{12}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 128| 5  | $Z_2^3 \rtimes Z_8$   | $B_{19}$ | $B_{12}$ | $\frac{12}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 129| 28 | $Z_4 \rtimes_1 D_4$   | $B_{23}$ | $B_{12}$ | $\frac{12}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 130| 34 | $Z_4 \rtimes_2 D_4$   | $B_{25}$ | $B_{13}$ | $\frac{13}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 131| 29 | $Z_2^2 \rtimes_1 Q_8$ | $B_{19}$ | $B_{12}$ | $\frac{12}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 132| 35 | $Z_4 \rtimes_1 Q_8$   | $B_{17}$ | $B_{13}$ | $\frac{13}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 133| 9  | $D_4 \rtimes Z_4$     | $B_{21}$ | $B_{10}$ | $\frac{10}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 134| 10 | $Q_8 \rtimes Z_4$     | $B_{17}$ | $B_{10}$ | $\frac{10}{16}$ | $\frac{7}{16}$ | semidirect product                    |
| 135| 15 | $Z_8 \cdot Z_4$       | $B_{17}$ | $B_{9}$  | $\frac{9}{16}$  | $\frac{7}{16}$ | non-split extension                   |
| 136| 44 | $Z_8 \cdot Z_2^2$     | $B_{19}$ | $B_{10}$ | $\frac{10}{16}$ | $\frac{7}{16}$ | non-split extension                   |
| 137| 32 | $Z_4^2 \cdot Z_2$     | $B_{17}$ | $B_{11}$ | $\frac{11}{16}$ | $\frac{7}{16}$ | non-split extension                   |
| 138| 2  | $Z_2 \cdot Z_4^2$     | $B_{19}$ | $B_{13}$ | $\frac{13}{16}$ | $\frac{7}{16}$ | non-split extension                   |
| 139| 7  | $Z_4 \cdot D_4$       | $B_{21}$ | $B_{8}$  | $\frac{8}{16}$  | $\frac{7}{16}$ | non-split extension                   |
| 140| 31 | $Z_4 \cdot D_4$       | $B_{21}$ | $B_{11}$ | $\frac{11}{16}$ | $\frac{7}{16}$ | non-split extension                   |
| 141| 8  | $Z_4 \cdot_10 D_4$    | $B_{17}$ | $B_{8}$  | $\frac{8}{16}$  | $\frac{7}{16}$ | non-split extension                   |
| 142| 13 | $Z_4 \cdot Q_8$       | $B_{17}$ | $B_{9}$  | $\frac{9}{16}$  | $\frac{7}{16}$ | non-split extension                   |
| 143| 30 | $Z_2^2 \cdot D_4$     | $B_{21}$ | $B_{11}$ | $\frac{11}{16}$ | $\frac{7}{16}$ | non-split extension                   |
| 144| 14 | $Z_2 \cdot Q_8$       | $B_{17}$ | $B_{11}$ | $\frac{11}{16}$ | $\frac{7}{16}$ | non-split extension                   |
Table 11. Bell numbers $B_1, \ldots, B_{32}$

| $B_1$ | $B_{17}$ | 82,864,869,804 |
| $B_2$ | $B_{18}$ | 682,076,806,159 |
| $B_3$ | $B_{19}$ | 5,832,742,205,057 |
| $B_4$ | $B_{20}$ | 51,724,158,235,372 |
| $B_5$ | $B_{21}$ | 474,869,816,156,751 |
| $B_6$ | $B_{22}$ | 4,506,715,738,447,323 |
| $B_7$ | $B_{23}$ | 44,152,005,855,084,346 |
| $B_8$ | $B_{24}$ | 445,958,869,294,805,289 |
| $B_9$ | $B_{25}$ | 4,638,590,332,229,999,353 |
| $B_{10}$ | $B_{26}$ | 49,631,246,523,618,756,274 |
| $B_{11}$ | $B_{27}$ | 545,717,936,059,989,389 |
| $B_{12}$ | $B_{28}$ | 6,160,539,404,599,934,652,455 |
| $B_{13}$ | $B_{29}$ | 71,339,801,938,860,275,191,172 |
| $B_{14}$ | $B_{30}$ | 846,749,014,511,809,332,500,147 |
| $B_{15}$ | $B_{31}$ | 10,293,358,946,226,376,485,035,653 |
| $B_{16}$ | $B_{32}$ | 128,064,670,049,908,713,825,644 |

Final remarks. With the results of Section 5, the number of invariant and bi-invariant metrics for several groups of low finite order $n$ can be computed. One needs an explicit presentation of the group $G$ to find the number of elements of order 2 and the table of characters of $G$ (for the information on the conjugacy classes). With the data of the groups given in the webpage GroupNames [10] one can obtain these numbers for almost all groups of order $n \leq 128$.

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