Black Hole Thermodynamics and Massive Gravity

Fabio Capela\textsuperscript{a} and Peter G. Tinyakov\textsuperscript{a}
\textsuperscript{a}Service de Physique Théorique, Université Libre de Bruxelles (ULB),
CP225 Boulevard du Triomphe, B-1050 Bruxelles, Belgium

E-mail: fregocap@ulb.ac.be, petr.tiniakov@ulb.ac.be

Abstract: We consider the generalized laws of thermodynamics in massive gravity. Making use of explicit black hole solutions, we devise black hole merger processes in which i) total entropy of the system decreases ii) the zero-temperature extremal black hole is created. Thus, both second and third laws of thermodynamics are violated. In both cases, the violation can be traced back to the presence of negative-mass black holes, which, in turn, is related to the violation of the null energy condition. The violation of the third law of thermodynamics implies, in particular, that a naked singularity may be created as a result of the evolution of a singularity-free state. This may signal a problem in the model, unless the creation of the negative-mass black holes from positive-mass states can be forbidden dynamically or the naked singularity may somehow be resolved in a full quantum theory.

Keywords: massive gravity, black hole thermodynamics, null energy condition

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1 Introduction

There has been recently an increasing interest in infrared modifications of gravity which could, in principle, explain the accelerated expansion of the Universe without introducing a "dark energy" component. However, before these models are taken seriously, their consistency has to be checked both from theoretical point of view and by comparison with the experimental data. One class of such models involves spontaneous breaking of Lorentz symmetry by the space-time dependent condensates of scalar fields coupled to gravity through derivative couplings \([1, 2, 3]\). It has been shown that such models may have a non-pathological behavior in the perturbative regime and may exhibit infrared modifications of the gravitational interactions. The models of this class are naturally called "massive gravity" since the graviton generically acquires a non-zero mass due to interactions with the scalar fields \([3, 4]\). An important feature of these models is that they are formulated in a non-perturbative way which makes it possible to study non-linear solutions such as those describing modified cosmological evolution \([5]\) or black holes \([6, 7]\) (for a review and further references see, e.g., Ref. \([8]\)).

In the conventional General Relativity (GR), the existence of solutions with horizons (e.g., black holes) raises questions of consistency of a more general kind, namely, the consistency with the general laws of thermodynamics. It has been argued \([9, 10, 11]\) that in GR such consistency can be obtained by assigning black holes a certain temperature and entropy. The temperature of a black hole characterizes its Hawking radiation, so that the black hole would be in a thermal equilibrium with the thermal bath at this temperature. The entropy of a black hole was argued to be proportional to its horizon area. In this way, the net entropy of the black hole and the outer region never decreases, generalizing the
second law of thermodynamics to processes that include black holes. These thermodynamical properties of black holes in GR are believed to be connected to fundamental principles of quantum physics, such as unitarity. The validity of the thermodynamical description of black holes should then give us some insight into quantum aspects of gravity. For example, in string theory for a certain class of extremal black holes the Bekenstein entropy has been reproduced by counting the microscopic states of the compact objects \[12, 13\].

A question that arises naturally is how the thermodynamic properties of the black holes are changed in the modified gravity models, in particular, in massive gravity, and whether these changes preserve consistency with the thermodynamic laws. In the context of the ghost condensate models \[2\] this question has been raised in Refs. \[14, 15\]. In Ref. \[14\] a gedanken experiment involving a black hole has been proposed which allows the transfer of heat from a cold body to a hot one, thus violating the second law of thermodynamics (for the discussion of the subtleties of the arguments see refs.\[16, 17\]). In Ref. \[15\] the violation of the second law of thermodynamics was related to the presence of negative energy states. On the contrary, in the context of the TeVeS models \[18\] it was conjectured \[19\] that the second law of thermodynamics holds if the effective graviton radiation temperature and the Hawking radiation temperature are equal.

The aim of this paper is to check the validity of the laws of thermodynamics in massive gravity by making use of the exact black hole solutions of Ref.\[7\] (solutions of this type were first found in bi-metric models in Ref. \[20\]). Unlike conventional black holes, these solutions depend on two parameters: the mass \(M\) and the “scalar charge” \(Q\). We will concentrate on solutions which can be interpreted as (modified) black holes, i.e., have a finite ADM mass (and, therefore, produce the Newtonian \(1/r\) potential at large distances) and have an event horizon. We will first find the expressions for the temperature and entropy of the modified black holes. This will be done by calculating the surface gravity and by making use of the Wald’s representation of entropy as a Noether charge. Having obtained the explicit expressions for the entropy and temperature in terms of the black hole mass and scalar charge, we then consider the process of merging of two black holes. As we will argue, in the limit when one of the black holes is much smaller than the other, the mass and the scalar charge of the resulting black hole is the sum of masses and charges of the constituents. Thus, we will be able to compare the entropies of the initial and final states and check the second law of thermodynamics. We will see that when the negative mass states are present, one can decrease the total entropy during the merger. Moreover, one can create a zero-temperature black hole — an extremal state which, by an infinitesimal change of initial parameters, can be converted into a state with naked singularity. Thus, the second and third laws of thermodynamics can be violated.

The paper is organized as follows. In Sect. 2 we review the static spherically symmetric solutions of massive gravity. In Sect. 3 we first compute the temperature and entropy of modified black holes in massive gravity. We then verify the second and the third laws of thermodynamics and show that both can be violated. Sect. 4 contains the summary of the results and their discussion.
2 Static Spherically Symmetric Solutions in Massive Gravity

The massive gravity model which is used in this paper is described by the following action [3]:

\[ S = \int dx^4 \sqrt{-g} \left[ -M_{pl}^2 R + \Lambda^4 F \right] \] (2.1)

where the first term is the usual Einstein-Hilbert action and the second one is a certain function (to be specified below) of the space-time derivatives of the four scalar fields \( \phi^0, \phi^i \) minimally coupled to gravity. This model should be viewed as a low-energy effective theory with the cutoff scale \( \Lambda \).

The model (2.1) generically admits a flat-space “vacuum” solution

\[ g_{\mu \nu} = \eta_{\mu \nu}; \quad \phi^0 = \Lambda^2 t; \quad \phi^i = \Lambda^2 x^i. \]

The vacuum possesses rotational symmetry provided the function \( F \) is invariant under the rotations of the fields \( \phi^i \) in the internal space. The Lorentz symmetry is, in general, broken. Requiring that the action (2.1) is invariant under the following symmetry,

\[ \phi^i \rightarrow \phi^i + \Xi^i (\phi^0), \] (2.2)

where \( \Xi^i \) are arbitrary functions of \( \phi^0 \), ensures that perturbations about this solution contain only two propagating degrees of freedom [3] which are two polarizations of a massive graviton with the mass of order \( m = \Lambda^2/M_{Pl} \), where \( M_{Pl} \) is the Planck mass. The model does not contradict the most obvious experimental constraints [5, 21, 22] for graviton masses as large as \( 10^{-20} \) eV.

The ansatz for the static spherically symmetric solution can be written in the following form [7]:

\[ ds^2 = \alpha(r) dt^2 - \beta(r) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

\[ \phi^0 = \Lambda^2 [t + h(r)], \]

\[ \phi^i = \phi(r) \frac{\Lambda^2 x^i}{r}, \] (2.3)

where the coordinate transformations \( r \rightarrow r' = r'(r) \) and \( t \rightarrow t' = t + \tau(r) \) have been used to eliminate some of the fields.

The analytical black hole solutions can be obtained if the function \( F \) has the form

\[ F = \frac{12}{\lambda X} + 6 \left( \frac{2}{\lambda} + 1 \right) w_1 - w_1^3 + 3w_1 w_2 - 2w_3 + 12, \]

where \( \lambda \) is a positive constant and

\[ X = g^{\mu \nu} \partial_\mu \phi^0 \partial_\nu \phi^0 / \Lambda^4, \quad w_n = \text{Tr}(W^{ij})^n, \]

\[ W^{ij} = \Lambda^{-4} \partial^\mu \phi^i \partial_\mu \phi^j - \frac{\partial^\mu \phi^i \partial_\mu \phi^0 \partial^\nu \phi^0 \partial_\nu \phi^j}{\Lambda^8 X}. \]

Note that the dependence of \( F \) on the scalar fields through two combinations \( X \) and \( W^{ij} \) ensures the symmetry (2.2).
With this function $F$, the black hole solution reads
\begin{align*}
\alpha(r) &= 1 - \frac{2MG}{r} - \frac{Q}{r^\lambda}, \\
\beta(r) &= \frac{1}{\alpha(r)}, \\
\phi(r) &= r.
\end{align*}

Here $M$ and $Q$ are two arbitrary integration constants. We will assume in what follows that $\lambda > 1$. In this case the asymptotics of the gravitational potential is Newtonian and is given by the parameter $M$ which determines the ADM mass of the solution.

The solution may possess an horizon which, in this case, is given by the largest root of the equation
\begin{equation}
\frac{2MG}{r_H} + \frac{Q}{r_H^\lambda} = 1
\end{equation}
Depending on the signs and relative values of the parameters $M$ and $Q$, there are three different cases when the horizon exists: $(M > 0, Q > 0)$, $(M > 0, Q < 0)$ and $(M < 0, Q > 0)$. The three corresponding solutions are shown in Fig. 1.

In the case $M > 0$ and $Q > 0$, the black hole has an attractive gravitational potential at all distances. The attraction is stronger than that of the usual Schwarzschild black hole of mass $M$; The horizon size of the modified black hole is also larger (left panel of Fig. 1).

When $M > 0$ and $Q < 0$, the horizon only exists when the condition
\begin{equation}
2MG \geq \lambda |Q|^{1/\lambda} \left( \frac{1}{\lambda - 1} \right)^{\frac{\lambda - 1}{\lambda}}
\end{equation}
is fulfilled. In this case, the Newton’s potential is also always attractive. However, the attraction is weaker than in the Schwarzschild case, and the horizon size is smaller.

Finally, in the case $M < 0$ and $Q > 0$ the Newton’s potential is repulsive at large distances and attractive near the horizon. This case is interesting since it doesn’t exist in GR. Although formally the black hole mass in GR can also be taken negative, this does not correspond to a physical solution because of the naked singularity. Another reason to disregard such solutions is the null energy condition which holds for the matter stress.

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**Figure 1.** Newtonian potential $2\Phi = g_{00} - 1$ in the three different cases: $M > 0$ and $Q > 0$, $M > 0$ and $Q < 0$, $M < 0$ and $Q > 0$. For comparison, the potential of the Schwarzschild solution of the mass $|M|$ is shown by the dashed line.
tensor \([23, 24, 25, 26]\). Neither of the arguments exist in the modified gravity case. At small distances the repulsion changes to the attraction, which creates the event horizon hiding the singularity (right panel of Fig. 1). Also, massive gravity doesn’t satisfy the null energy condition, allowing for negative mass states to be constructed, e.g., as in the ghost condensate model \([27]\).

3 The Thermodynamical Properties of the Black Hole Solutions

In order to analyse the validity of the second law of black hole thermodynamics we first compute the temperature and entropy of the modified black holes. We use the approach by Wald \([28, 29, 30, 31, 32]\) based on the Noether’s charge. As we will show, the entropy is given by the Bekenstein-Hawking formula, that is equal, in the appropriate units, to the one quarter of the horizon area.

3.1 The Black Hole Temperature

Due to quantum particle creation, black holes emit thermal radiation \([11]\). The temperature of this radiation is determined by the surface gravity \(\kappa\) of the black hole, that is, the acceleration experienced by a test body at the black hole horizon. The event horizon of a static spherically symmetric black hole corresponds to a killing horizon, i.e., a surface to which a killing vector field is normal (bifurcation surface). The surface gravity \(\kappa\) at any point of a killing horizon \(\mathcal{H}\) is defined by

\[
\xi^a \nabla_a \xi^b = \kappa \xi^b, \tag{3.1}
\]

where \(\xi^a\) is the killing field normal to \(\mathcal{H}\). For a static spherically symmetric metric of the form

\[
ds^2 = \alpha(r)dt^2 - \frac{1}{\alpha(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi), \tag{3.2}
\]

the surface gravity equals

\[
\kappa = \frac{\alpha'(r)}{2} \bigg|_{r=r_H}, \tag{3.3}
\]

where \(r_H\) is the horizon radius. The zeroth law of black hole mechanics asserts that \(\kappa\) is constant over the event horizon of a stationary black hole. Although in GR one needs to use the Einstein’s field equations to prove this statement, and thus its generalizations to other theories of gravity is questionable, the zeroth law trivially holds for spherically symmetric black holes.

Making use of the explicit solution (2.4) one finds

\[
T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_H} + \frac{1}{4\pi} (\lambda - 1) \frac{Q}{r_H^{\lambda+1}}
\]

\[
= \frac{1}{4\pi} \left( \frac{1 + (\lambda - 1)Q r_H^{-\lambda}}{r_H} \right). \tag{3.4}
\]

One recovers the temperature of the Schwarzschild black hole in the limit of zero scalar charge, in which case \(T_H = 1/4\pi r_H\) as expected. Moreover, since the temperature of the
black hole corresponds, from a mathematical point of view, to the tangent of the Newton’s potential at the event horizon, it is easy to conclude from Fig.1 that the existence of an event horizon implies $T_H \geq 0$.

Interestingly, the Hawking temperature behaves differently for positive and negative scalar charges. For positive $Q$, the temperature is larger than at $Q = 0$ and decreases with $r_H$ as in the case of a conventional Schwarzschild black hole. On the contrary, at $Q < 0$ the temperature is smaller than in the Schwarzschild case. Moreover, its dependence on $r_H$ is not monotonic, with a maximum reached for $r_H = r_{max} = [(\lambda^2 - 1)|Q|]^{1/\lambda}$ (see Fig. 2). When $r_H > r_{max}$, the specific heat is negative, as in the Schwarzschild case, corresponding to the fact that the black hole heats up as it radiates energy, while when $r_H < r_{max}$, the specific heat is positive, which means that the black hole cools down as it radiates, eventually reaching an equilibrium.

Another interesting observation is that the temperature reaches zero at a finite event horizon radius for $Q < 0$. This is closely related to the third law of black hole thermodynamics and will be discussed in the next section.

### 3.2 The Noether’s Charge as the Entropy

As has been proven in Refs. [28, 29, 30, 31, 32] by assuming a theory which admits black holes with a bifurcate horizon and a well-defined mass at infinity, the entropy of the black hole is given by

$$S_H = 2\pi \int_H Q[\xi],$$

(3.5)

where $Q[\xi]$ is the Noether charge related to the Killing field $\xi^a$ normal to the horizon $\mathcal{H}$. For a general theory with matter fields $\psi$ and the Lagrangian

$$\mathcal{L} = \mathcal{L} \left( g_{ab}, R_{abcd}, \psi, \nabla_{a_1} \psi, \nabla_{(a_1, \ldots, a_j)} \psi \right),$$

(3.6)
this quantity can be recast in the following form [29]:

\[ S_H = -2\pi \int_{\mathcal{H}} d^{D-2}x \sqrt{\sigma} \frac{\delta L}{\delta R_{abcd}} \hat{\epsilon}_{ab} \hat{\epsilon}_{cd}, \]  

(3.7)

where \( \sigma \) is the determinant of the metric on \( \mathcal{H} \) and \( \hat{\epsilon}_{ab} \) is the binormal vector to the bifurcation surface. The black hole entropy is then related to the Noether’s charge of diffeomorphism under the Killing vector field which produces the event horizon.

Eq. (3.7) can be directly applied to the case of massive gravity with the action (2.1). For metric tensors of the form (3.2), the relevant Killing vector is \( \partial_t \), while the binormal vector \( \hat{\epsilon}_{ab} \) has the following components: \( \hat{\epsilon}_{tr} = -\hat{\epsilon}_{rt} = 1 \) and the other components vanish. The explicit expression for the entropy is then

\[ S_H = -2\pi \int_{r=r_H} \left[ \left( \frac{\partial L}{\partial R_{abcd}} \right)^{(s)} \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} r^2 \sin \theta \right] d\theta d\phi \]

(3.8)

where the factor 4 is a consequence of the antisymmetry property of the Riemann tensor and the binormal vectors. The index \( (s) \) is to emphasize that the functional is evaluated on the solution (2.4). Substituting the Lagrangian (2.1) and making use of the fact that the function \( F \) does not dependent on the Riemann tensor (the metric and the Riemann tensor are treated as independent variables in eq (3.8)), we arrive at the Hawking-Bekenstein formula:

\[ S_H = -8\pi \int_{r=r_H} \left[ \left( \frac{\partial L}{\partial R_{rtt}} \right)^{(s)} r^2 \sin \theta \right] d\theta d\phi, \]

(3.9)

where \( A_H \) is the area of the black hole horizon in the Planck units \( G_N = 1 \).

As it has been shown in Ref. [28], this expression for the entropy is automatically consistent with the first law of thermodynamics. It can also be checked directly by making use of the expressions (3.9) and (2.5) at \( dQ = 0 \).

### 3.3 The Second and Third Laws of Thermodynamics

Let’s first examine the generalized second law of thermodynamics. To this end consider the change of entropy in the process of coalescence of two black holes characterized by the masses \( M_1 \) and \( m_2 \) and scalar charges \( Q_1 \) and \( q_2 \), respectively. Let both black holes be large enough so that the Hawking radiation can be neglected and their horizon radii are larger than the inverse cutoff scale, in which case the solutions are within the region of validity of the effective theory.

Before the coalescence, when the interaction of the two black holes can be neglected, their entropy is simply the sum of the entropies of the two isolated black holes as given by eqs. (3.9) and (2.5). In order to check the second law of thermodynamics we need to
determine the entropy of the final state. Let us argue that in the limit when one of the black holes is much larger than the other, and the scalar charges of the black holes are not parametrically larger than their masses,

$$|M_1| \gg |m_2|, \quad |Q_1|^{1/\lambda} \lesssim |M_1|, \quad |q_2|^{1/\lambda} \lesssim |m_2|,$$  \hfill (3.10)

the result of the coalescence is a black hole with the mass $M_1 + m_2$ and the scalar charge $Q_1 + q_2$. Then the entropy of the final state is given by eqs. (3.9) and (2.5), and the net change of entropy can be easily calculated.

The argument consists of two parts. First, let us show that in the above limit the total energy of gravity waves emitted in the process of coalescence is parametrically small as compared to the mass of the smallest black hole. Assume that the characteristic time scale of the coalescence is much smaller than the inverse graviton mass (this can always be arranged in view of the tiny value of the latter). Then the gravitational waves can be considered massless. The metric perturbation $h_{ij}$ in the case of the quadrupole radiation is estimated as follows:

$$h_{ij} \sim \frac{\dddot{Q}_{ij}}{M_{pl} r},$$ \hfill (3.11)

where $Q_{ij}$ is the second time derivative of the quadrupole moment and $r$ is the characteristic size of the system. The energy density in the gravitational waves is, therefore, of order

$$\rho \sim \omega^2 M_{pl}^2 h_{ij}^2 \sim \omega^2 \left( \frac{\dddot{Q}_{ij}}{M_{pl}^2 r^2} \right)^2,$$ \hfill (3.12)

where $\omega$ is the characteristic frequency of the emitted waves as seen by an asymptotic observer. Setting the size of the system to $r \sim R_H$ and the frequency to $\omega \sim 1/R_H$, $R_H$ being the horizon size of the large black hole, we have

$$\dddot{Q} \sim \omega^2 Q \sim \omega^2 m_2 R_H^2$$

and, therefore, the total energy emitted over the time period $\sim R_H$ is

$$E_{rad} \sim m_2 \left( \frac{m_2}{M_1} \right) \sim m_2 \left( \frac{r_H}{R_H} \right),$$ \hfill (3.13)

where $r_H$ is the horizon size of the small black hole and we have assumed that the presence of scalar charges does not change completely the horizon sizes of the two original black holes. We see from eq. (3.13) that in the limit $m_2 \ll M_1$ the gravitational radiation during the coalescence of the two black holes can be neglected.

We now turn to determining the mass and the scalar charge of the resulting black hole. This can be done by considering the asymptotic gravitational potential created by the coalescing black holes and extracting the coefficients in front of the $1/r$ and $1/r^\lambda$ terms. These coefficients determine the mass and the scalar charge of the resulting black hole, respectively.

To clarify the logic, consider first the coalescence of the two black holes of masses $M_1$ and $m_2$ in the conventional GR. In this case the asymptotic gravitational potential of each
black hole is determined by its mass and satisfies the linear equation. Thus, before the coalescence and to the leading order in $1/r$, the gravitational potential of the two black holes is given by the sum of the potentials of the individual black holes. The coefficient of $1/r$ is independent of the distance between the black holes (as long as this distance is much smaller than $r$) and is given by the sum of the black hole masses. Since after the coalescence this coefficient is determined by the mass $M$ of the resulting black hole and there is no substantial emission of the gravitational waves, we conclude that $M = M_1 + m_2$.

In the case of the modified black holes the situation is more complicated because the corresponding solution is non-linear at all distances. This introduces a correction to the simple addition of masses and charges,

$$M = M_1 + m_2 + \delta m; \quad Q = Q_1 + q_2 + \delta q.$$  \hspace{1cm}

However, one can choose the parameters $M_1$, $m_2$, $Q_1$ and $q_2$ in such a way that these corrections are negligible. Making use of an exactly treatable spherically symmetric case to estimate $\delta q$ and $\delta m$ (see Appendix A), one may argue that this can be achieved by requiring that

$$\left(\frac{R_H}{R_H}\right)^{1+1/\lambda} \gg \left(\frac{\mu}{M_{pl}}\right)^{3/\lambda}.$$  \hspace{1cm}

The last condition can always be satisfied for a sufficiently small value of the graviton mass $\mu$ without leaving the region of validity of the effective theory. In this case, the effects of non-linearity on the asymptotic potential can be neglected and the arguments presented above imply that the resulting black hole has, to a good accuracy, the mass $M = M_1 + m_2$ and the scalar charge $Q = Q_1 + q_2$.

Having determined the mass and the scalar charge of the final black hole, we can now calculate the change of the entropy in the process of coalescence,

$$\Delta S_{bh} = S_{final} - S_{initial} = \pi \left( R_H^2 - R_{H'}^2 - R_{H''}^2 \right) \approx \pi \left( R_H^2 - R_H^2 \right),$$  \hspace{1cm}

where $R_H$ is the radius of the event horizon of the final black hole.

In the limit (3.10) the entropy change can be calculated expanding in powers of $m_2/M_1$. To the leading order, the answer is linear in the mass of the small black hole,

$$\Delta S_{bh} \approx \frac{4\pi R_H m_2}{1 + (\lambda - 1)Q_1 R_{H}^{-\lambda}} = \frac{m_2}{T_H},$$  \hspace{1cm}

where $T_H$ is the temperature of the large black hole. Thus, the entropy may increase or decrease, depending on the sign of the mass of the small black hole. For negative masses, the generalized second law of thermodynamics is violated.

The decrease of the entropy when negative mass black holes are involved can be checked explicitly in the particular case of $\lambda = 2$. In this case eq. (2.5) for the horizon in terms of the mass and the scalar charge can be solved exactly,

$$R_H = M + \sqrt{M^2 + Q}.$$  \hspace{1cm} (3.17)
The existence of the horizon requires $M^2 + Q \geq 0$, which is always the case for a positive scalar charge. For negative $Q$, the condition (2.6) has to be imposed.

The change of the black hole entropy can be computed explicitly giving

$$
\Delta S_{bh} = 2\pi \left[ (M_1 + m_2) \sqrt{(M_1 + m_2)^2 + Q_1 + q_2} 
+ 2M_1 m_2 - M_1 \sqrt{M_1^2 + Q_1 - m_2 \sqrt{m_2^2 + q_2}} \right].
$$

(3.18)

One can see from this expression that the entropy always increases when both black holes have positive masses. However, when at least one of the masses is negative and $M_1 \gg |m_2|$, one arrives at the same conclusion as before: there is a decrease of entropy and the generalized second law of thermodynamics is violated.

Consider now the third law of thermodynamics. In application to black holes, it states that a black hole with a non-vanishing temperature cannot reach zero temperature in a finite sequence of operations [33]. In our case, the zero temperature corresponds to an “extremal” situation when the inequality in eq. (2.6) becomes an equality. Indeed, the black hole temperature can be expressed in terms of the surface gravity which, according to eq. (3.3), is proportional to the slope of $\alpha(\tau)$ at the horizon. It is then clear from Fig. 1 that the zero temperature can only be reached in the case $M > 0$, $Q < 0$ (represented on the middle panel) when the two roots of eq. (2.5) coincide. Thus, an extremal black hole satisfies the condition

$$
2M = \gamma |Q|^\beta
$$

(3.19)

with $\beta = 1/\lambda$ and

$$
\gamma = \lambda \left( \frac{1}{\lambda - 1} \right)^{\frac{\lambda - 1}{\lambda}} > 0.
$$

Note that both $\gamma$ and $\beta$ are positive real numbers. For simplicity, we will concentrate on the case $\lambda = 2$ from now on. In this case, $\beta = 1/2$ and $\gamma = 2$. The generalization to other cases is straightforward.

Consider again the coalescence of two black holes. Let the large black hole be nearly extremal, so that $Q_1 < 0$ and

$$
M_1 = (1 + \varepsilon)|Q_1|^{1/2},
$$

where $\varepsilon$ is a small positive number ($\varepsilon = 0$ corresponds to the extremal black hole). If the small black hole has a positive mass and a scalar charge satisfying the condition (2.6), it is easy to see that the black hole resulting from coalescence will also satisfy the condition (2.6) and thus will have a non-zero temperature. Indeed, the final state is characterized by the inequality

$$(M_1 + m_2)^2 + Q_1 + q_2 = (M_1^2 + Q_1) + (m_2^2 + q_2) + 2m_2 M_1 > 0.$$
still fulfilled for small enough values of $\varepsilon$. The scalar charge of the second black hole can be chosen as $q_2 = \delta |Q_1|$ with $\delta$ a small positive number. Then, the final state is a black hole with mass $M = (1 - \varepsilon) |Q_1|^{1/2}$ and a scalar charge $|Q| = (1 - \delta) |Q_1|$, which means that $M = |Q|^{1/2}$ if we choose $(1 - \varepsilon) = (1 - \delta)^{1/2}$. A non-extremal black hole has turned into an extremal one in a one-step process. Thus, when negative mass black holes are present, the third law of thermodynamics is not fulfilled.

The third law of black hole thermodynamics is closely related to the cosmic censorship conjecture \cite{34}, which states that all singularities resulting from a gravitational collapse are always hidden by the horizon. With an obvious modification, the above process that leads to an extreme black hole can be used to create a naked singularity. This possibility is again related to the existence of the negative-mass black holes and, ultimately, to the violation of the null energy condition in massive gravity.

4 Discussion

In this paper we have addressed the validity of the generalized laws of thermodynamics in massive gravity models. In these models the black hole solutions are modified by the presence of the scalar “hair”. The analog of the Schwarzschild black hole – the spherically symmetric solution – depends on two parameters, the mass and the “scalar charge” characterizing the hair strength. The presence of two free parameters makes the asymptotics of the gravitational field of the black hole essentially independent of its behavior near the horizon, allowing for negative-mass solutions without naked singularity.

Making use of the exact black hole solutions, we have constructed explicit examples of the black hole mergers which violate the generalized second and third laws of thermodynamics. The existence of such processes is in accord with the general theorems of the black hole thermodynamics which require the weak energy condition to hold. The latter condition, together with the null energy condition, is violated in massive gravity models. Indeed, the examples we have constructed involve the negative-mass black holes. Note that the violation of the null energy condition in massive gravity is related to the presence of the superluminal modes \cite{35} which are likely to be responsible for the existence of the black hole hair \cite{6}.

Although the negative mass black holes are classical solutions of massive gravity which exist on the same footing as the conventional positive-mass black holes, this does not guarantee that they can be created from positive-mass states in the course of the evolution. If this were the case, the violation of the second law would imply that one may devise a process involving black holes which would convert heat into work. Moreover, the violation of the third law would mean that the cosmic censorship conjecture is not true, so that one can create naked singularities starting from singularity-free states.

It is not inconceivable that the creation of the negative-mass states can be forbidden dynamically. In fact, it has been conjectured in the context of the ghost condensate model that an average null energy condition (ANEC) may prevent the entropy of the black hole from decreasing, in a coarse-grained sense \cite{17}. This means that the second law of thermodynamics may be violated locally, while in average (during long time periods) it holds.
The ANEC can be traced back to the presence of the conserved Noether’s charge related to the shift symmetry of the ghost condensate field. In our case, a similar internal shift symmetry is also present, so one may wonder whether this symmetry implies ANEC which may prevent the breakdown of thermodynamical laws.

The situation, however, is not exactly the same in massive gravity models. First, massive gravity possesses four scalar fields $\phi^0$ and $\phi^i$, the former being an analog of the ghost condensate field. Even though these fields decouple from the ordinary matter, they are coupled to each other, so the scalar sector is more complicated than in the ghost condensate model. For this reason we have not been able to directly generalize the argument based on ANEC, except for linear perturbations above the Minkowski background. This, however, is not sufficient to argue against the formation of the negative-mass black holes, so the question remains open.

Moreover, the original argument [17] is partially based on the fact that a negative Noether’s charge is strongly disfavored since it is leading to UV instabilities. However, the massive gravity is free from UV instabilities, at least for backgrounds which can be approximated as flat in the UV limit (see [5] for details). This implies that part of the argument in [17] doesn’t go through in our case.

The final observation is that ANEC cannot protect the third law of thermodynamics and the cosmic censorship conjecture. Indeed, unlike the entropy which may decrease locally but still increase in average, the naked singularity, once created, cannot be undone no matter what happens in other places. The breakdown of the cosmic censorship conjecture may therefore occur even if ANEC holds. The creation of naked singularities may signal the problem of the model, unless they are somehow regularized by an appropriate UV-completion.

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A Estimate of charge and mass correction

Since the equations of massive gravity do not linearize asymptotically because of the slow decay of the field $\phi^0$ [7], the state of two close black holes with parameters $M_1$, $Q_1$ and $m_2$, $q_2$ has asymptotic mass and scalar charge (i.e., the coefficients in front of $1/r$ and $1/r^\lambda$ in the Newtonian potential) equal to

$$M = M_1 + m_2 + \delta m, \quad Q = Q_1 + q_2 + \delta q.$$  

Our goal here is to estimate the corrections $\delta m$ and $\delta q$ and find the parameters for which these corrections can be neglected.
For the estimate we make use of the fact that for a spherically symmetric matter distribution the equations can be solved exactly. We model the large black hole by a sphere of constant density of the total mass $M_1$ and radius close to the horizon size $R \sim 2G_N M_1$, and a small black hole by a spherical shell of the same density and the mass $m_2$, covering the sphere $R$. Even though the geometry is different, we expect that our model configuration reproduces correctly the parametrical dependence of $\delta m$ and $\delta q$ on $R$, $M_1$ and $m_2$.

In a spherically symmetric case it is straightforward to obtain the scalar charge as of a sphere of a constant density as a function of the mass $M_1$ and radius $R$ [36]. Matching boundary conditions, we obtain the following relation for the scalar charge

$$Q_1 = C(\lambda) M_1 G_N \mu^2 R^{1+\lambda}, \quad (A.1)$$

where $C(\lambda)$ is a constant depending on $\lambda$ which is of order one for $\lambda$ varying between one and two, and $\mu$ is the graviton mass.

Making use of eq. (A.1), one may compute the change of the scalar charge when a thin shell around the constant density sphere is added. The result is

$$\delta q \sim m_2 G_N \mu^2 R^{1+\lambda}.$$

This estimate can be rewritten in terms of the parameters of the two black holes as follows:

$$\delta q \sim \mu^2 r_H R_H^{\lambda+1}.$$

Requiring $\delta q \ll q_2$ we arrive at the condition (3.14) for the worst case scenario, i.e. when $m_2 \sim \Lambda$. If the condition (3.14) is satisfied, it automatically implies that $\delta m \ll m_2$.

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