BESOV-HANKEL NORMS IN TERMS OF THE CONTINUOUS
BESSEL WAVELET TRANSFORM

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Abstract. In this paper, we extend the concept of continuous Bessel wavelet transform in $L^p$-space and derived the Parseval’s as well as the inversion formulas. By using Bessel wavelet coefficients we characterized the Besov-Hankel space.

1. Introduction

In this paper, as usual $L^{p,\sigma}(\mathbb{R}^+ = (0, \infty))$ denotes the weighted $L^p$-space with norm

$$
\|f\|_{L^{p,\sigma}} = \|f\|_{p,\sigma} = \left( \int_0^{\infty} |f(x)|^p d\sigma(x) \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty),
$$

$$
\|f\|_{\infty,\sigma} = \text{ess sup}_{0 < x < \infty} |f(x)| < \infty.
$$

The Hankel transformation of the function $f \in L^{1,\sigma}(\mathbb{R}^+)$ is defined by

$$
\hat{f}(x) = \int_0^{\infty} j(xt) f(t) d\sigma(t), \quad 0 \leq x < \infty,
$$

where $\sigma(t) = \frac{t^{2\nu+1}}{2^{\nu+\frac{1}{2}} \Gamma(\nu+\frac{1}{2})}$, $j(.) = C_{\nu} x^{\nu - \frac{1}{2}} J_{\nu - \frac{1}{2}}$, $C_{\nu} = 2^{\nu + \frac{1}{2}} \Gamma(\nu + \frac{1}{2})$ and $J_{\nu - \frac{1}{2}}$ denote the Bessel function of first kind of order $\nu - \frac{1}{2}$.

2010 Mathematics Subject Classification. 33A40; 42C10; 44A05; 42C40.

Key words and phrases. Besov-Hankel space; Continuous Bessel wavelet transform; Hankel transform; Hankel convolution.

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If $f \in L_{1,\sigma}(\mathbb{R}^+)$, then the inverse of Hankel transformations is given by
\[
f(x) = \int_0^\infty j(xt) \hat{f}(t) d\sigma(t), \quad 0 < x < \infty.
\] (1.4)

Also, Parseval’s formula of the Hankel transformation for $f, g \in L_{1,\sigma} \cap L_{2,\sigma}$ is given by
\[
\int_0^\infty \hat{f}(x) \hat{g}(x) d\sigma(x) = \int_0^\infty f(u) g(u) d\sigma(u).
\] (1.5)

By denseness and continuity the Parseval’s formula can be extended to all $f, g \in L_{2,\sigma}(\mathbb{R}^+)$. Hence Hankel transform is isometry on $L_{2,\sigma}(\mathbb{R}^+)$. 

If $f, g \in L_{1,\sigma}(\mathbb{R}^+)$, then the convolution associated with the Hankel is defined as (see [10])
\[
(f \# g)(x) = \int_0^\infty f(x, y) g(y) d\sigma(y),
\] (1.6)

where the Hankel translation is given by
\[
f(x, y) = \tau_y f(x) =: \int_0^\infty f(z) D(x, y, z) d\sigma(z), \quad 0 < x, y < \infty,
\] (1.7)

and
\[
D(x, y, z) = \int_0^\infty j(xu) j(yu) j(zu) d\sigma(u)
\]
\[
= 2^{3\nu-\frac{3}{2}} \left[ \Gamma(\nu + \frac{1}{2}) \right]^2 \left( \Gamma(\nu) \pi^\frac{1}{2} \right)^{-1} (xyz)^{-2\nu-1} |\Delta(xyz)|^{2\nu-2}
\] (1.8)

where $\Delta$ denotes the area of a triangle. $D(x, y, z)$ is symmetric in $x, y, z$.

From (1.4) and (1.8), we have
\[
\int_0^\infty j(zu) D(x, y, z) d\sigma(z) = j(xu) j(yu), \quad 0 < x, y < \infty, \quad 0 \leq u < \infty,
\] (1.9)
and for \( u = 0 \), we get
\[
\int_0^\infty D(x, y, z) d\sigma(z) = 1, \tag{1.10}
\]
and
\[
(f \# g)(x) = \hat{f}(x)\hat{g}(x), \quad 0 \leq x < \infty. \tag{1.11}
\]

Now, we recall some properties of Hankel convolution (see [7, 2, 9, 6]) which are useful throughout the paper.

**Lemma 1.1.** Let \( f \in L^p,\sigma(\mathbb{R}^+) \), \( 1 \leq p < \infty \). Then we have
\[
||\tau_y f(x)||_{p,\sigma} \leq ||f||_{p,\sigma}. \tag{1.12}
\]

**Lemma 1.2.** Let \( f \in L^p,\sigma(\mathbb{R}^+) \) and \( g \in L^q,\sigma(\mathbb{R}^+) \), \( \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \). Then we have
\[
||f \# g||_{r,\sigma} \leq ||f||_{p,\sigma} ||g||_{p,\sigma}. \tag{1.13}
\]

Betancor and Rodriguez-Mesa defined the new function spaces called Besov-Hankel by using Hankel transform and its properties (see [9]).

**Definition 1.3. (Besov-Hankel Space):** Let measurable function \( \phi \) on \((0, \infty)\) belongs to \( BH^{p,q}_{\alpha,\sigma} \) if \( \phi \in L^p,\sigma(\mathbb{R}^+) \) and
\[
\int_0^\infty \left(h^{-\alpha} w_p(\phi)(h)\right)^q \frac{dh}{h} < \infty \quad \text{for} \quad \alpha > 0, 1 \leq p, q < \infty, \tag{1.14}
\]
where \( w_p(\phi)(h) = ||\tau_h \phi - \phi||_{p,\sigma}, \quad h \in (0, \infty) \).

1.1. **Bessel Wavelet.** Using the properties of Hankel transform Pathak and Dixit (see [2]) define the continuous Bessel wavelet for \( \psi \in L^p,\sigma(\mathbb{R}^+) \), \( 1 \leq p < \infty \), \( b \geq 0 \) and \( a > 0 \) as
\[
\psi_{b,a}(x) = D_a \tau_b \psi(x)
= a^{-2\nu - 1} \int_0^\infty \psi(z) D_a \left( \frac{b \cdot x}{a \cdot z} \right) d\sigma(z) \tag{1.15}
\]
where $D_a$ denote the dilation operator.

The continuous Bessel wavelet transform of $f \in L^{2,\sigma}(\mathbb{R}^+)$ with respect to a wavelet $\psi \in L^{2,\sigma}(\mathbb{R}^+)$ is defined as

$$
(B_\psi f)(b, a) = \int_0^\infty f(x)\widehat{\psi_{b,a}}(x) d\sigma(x)
= a^{-2\nu-1} \int_0^\infty \int_0^\infty f(x)\widehat{\psi(z)} D\left(\frac{b}{a}, \frac{x}{a}, z\right) d\sigma(z) d\sigma(x). \quad (1.16)
$$

Moreover, using (1.6), we have

$$
(B_\psi f)(b, a) = (f \# \psi_a)(b), \quad (1.17)
$$

where $\psi_a(t) = a^{-2\nu-1}\psi(t/a)$.

Betancor and Rodriguez-Mesa [9] first time introduce Besov-Hankel spaces and characterized by means of the Bochner-Riesz mean and the partial Hankel integrals. Perrier and Basdevant [5] established the characterization of Besov spaces by means of continuous wavelet transform. Motivated by these two works, we characterized Besov-Hankel spaces by the continuous Bessel wavelet transform.

Present paper is organized in following manner: section 1 is introductory, in which we recall some properties of Hankel transform, Besov-Hankel space and continuous Bessel wavelet transform. Section 2 is related to continuous the continuous Bessel wavelet transform in $L^{p,\sigma}(\mathbb{R}^+)$.

In this section we extend the concept of Bessel wavelet transform on $L^{p,\sigma}(\mathbb{R}^+)$.

**Theorem 2.1.** Suppose that the Bessel wavelet $\psi \in L^{p,\sigma}(\mathbb{R}^+)$ satisfies the admissibility condition

$$
A_\psi = \int_0^\infty \omega^{-2\nu-1}|\hat{\psi}(\omega)|^2 d\omega > 0,
$$
where \( \hat{\psi} \) denote the Hankel transform of \( \psi \) then continuous Bessel wavelet transform is a bounded linear operator

\[
L^{p,\sigma}(\mathbb{R}^+) \to L^{2,\sigma}(\mathbb{R}^+; \frac{d\sigma(a)}{a^{2\nu+1}}) \times L^{p,\sigma}(\mathbb{R}^+),
\]

moreover, for any \( f \in L^{p,\sigma}(\mathbb{R}^+) \), \( 1 < p < \infty \)

\[
\|f\|_{L^{p,\sigma}(\mathbb{R}^+)} = \left( \int_0^\infty \left( \int_0^\infty |B_\psi f(b,a)|^2 \frac{d\sigma(a)}{a^{2\nu+1}} \right)^\frac{p}{2} \sigma(b) \right)^{\frac{1}{p}}. \tag{2.1}
\]

Proof. Let \( S_p \) denote the space \( L^{2,\sigma}(\mathbb{R}^+; \frac{d\sigma(a)}{a^{2\nu+1}}) \times L^{p,\sigma}(\mathbb{R}^+) \) associated to the norm

\[
\|f\|_{S_p} = \left\{ \int_0^\infty \left( \int_0^\infty |f(b,a)|^2 \frac{d\sigma(a)}{a^{2\nu+1}} \right)^\frac{p}{2} \sigma(b) \right\}^{\frac{1}{p}}.
\]

If we take \( p = 2 \), then from Plancherel’s theorem:

\[
\|B_\psi f\|_{S_2} = \left\{ \int_0^\infty \left( \int_0^\infty |B_\psi f(b,a)|^2 \frac{d\sigma(a)}{a^{2\nu+1}} \right) \sigma(b) \right\}^{\frac{1}{2}} = \sqrt{A_\psi} \|f\|_{L_{2,\sigma}},
\]

where \( A_\psi = \int_0^\infty \omega^{-2\nu-1} |\hat{\psi}(\omega)|^2 d\omega > 0 \), if \( \psi \) is real. From singular integral theorem, the operators on \( L^{2,\sigma}(\mathbb{R}^+, \frac{d\sigma(a)}{a^{2\nu+1}}) \) holds inequality:

\[
\|B_\psi f\|_{S_p} \leq C_p \|f\|_{L^{p,\sigma}(\mathbb{R}^+)} \text{ for } 1 < p \leq 2,
\]

where the constant \( C_p \) depends only on \( p \) and \( \psi \)(see [12]). Due to duality the inequality is also valid for \( 1 < p < \infty \). It follows that

\[
\left\{ \int_0^\infty \left( \int_0^\infty |B_\psi f(b,a)|^2 \frac{d\sigma(a)}{a^{2\nu+1}} \right)^\frac{p}{2} \sigma(b) \right\}^{\frac{1}{p}} \leq C_p \|f\|_{L^{p,\sigma}(\mathbb{R}^+)}. \tag{2.2}
\]

Conversely suppose that \( f \in L^{2,\sigma}(\mathbb{R}^+) \cap L^{p,\sigma}(\mathbb{R}^+) \). Since continuous Bessel wavelet transform is isometry for every \( g \in L^{2,\sigma}(\mathbb{R}^+) \cap L^{p,\sigma}(\mathbb{R}^+) \), we can write

\[
\int_0^\infty \int_0^\infty B_\psi f(b,a)\overline{B_\psi g(b,a)}a^{-2\nu-1}d\sigma(a)d\sigma(b) = A_\psi \langle f, g \rangle
\]

\[
\frac{1}{A_\psi} \int_0^\infty \int_0^\infty B_\psi f(b,a)\overline{B_\psi g(b,a)}a^{-2\nu-1}d\sigma(a)d\sigma(b) = \int_0^\infty f(x)\overline{g(x)}d\sigma(x). \tag{2.3}
\]
Now,

\[
\left| \int_0^\infty f(x)g(x)d\sigma(x) \right| = \frac{1}{A_\psi} \left| \int_0^\infty \int_0^\infty B_\psi f(b, a)\overline{B_\psi g(b, a)} a^{-2\nu-1}d\sigma(a)d\sigma(b) \right|
\]

\[
\leq \frac{1}{A_\psi} \left( \int_0^\infty \left( \int_0^\infty |B_\psi f(b, a)|^2 a^{-2\nu-1}d\sigma(a) \right)^{\frac{p}{2}} d\sigma(b) \right)^{\frac{1}{p}} \times \left( \int_0^\infty \left( \int_0^\infty |B_\psi g(b, a)|^2 a^{-2\nu-1}d\sigma(a) \right)^{\frac{q}{2}} d\sigma(b) \right)^{\frac{1}{q}},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

From equation (2.2), we get

\[
\left\| f \right\|_{L^p(\mathbb{R}^+)} \leq A \left( \int_0^\infty \left( \int_0^\infty |B_\psi f(b, a)|^2 a^{-2\nu-1}d\sigma(a) \right)^{\frac{p}{2}} d\sigma(b) \right)^{\frac{1}{p}},
\]

By Density theorem

\[
\left\| f \right\|_{L^p(\mathbb{R}^+)} \leq A \left( \int_0^\infty \left( \int_0^\infty |B_\psi f(b, a)|^2 a^{-2\nu-1}d\sigma(a) \right)^{\frac{p}{2}} d\sigma(b) \right)^{\frac{1}{p}},
\]

where \( A = \frac{A_q}{A_\psi} \).

Theorem 2.2. (Parseval’s formula) Let us assume \( \phi_1 \in L^{p,\sigma}(\mathbb{R}^+), \phi_2 \in L^{q,\sigma}(\mathbb{R}^+) \) with \( 1 \leq p, q < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( \psi \) is a real wavelet then

\[
\frac{1}{A_\psi} \int_0^\infty \int_0^\infty B_\psi \phi_1(b, a)\overline{B_\psi \phi_2(b, a)} a^{-2\nu-1}d\sigma(a)d\sigma(b) = \int_0^\infty \phi_1(x)\overline{\phi_2(x)}d\sigma(x),
\]

(2.4)

where \( A_\psi = \int_0^\infty \omega^{-2\nu-1} |\hat{\psi}(\omega)|^2 d\omega > 0 \) and \( \hat{\psi} \) denote the Hankel transform.

Proof. Let us define bilinear transform \( T : L^{p,\sigma}(\mathbb{R}^+) \times L^{q,\sigma}(\mathbb{R}^+) \rightarrow \mathbb{R}^+ \) by

\[
T(\phi_1, \phi_2) = \langle B_\psi \phi_1(b, a), B_\psi \phi_2(b, a) \rangle \left( \frac{a^{-\nu-1}da}{\sigma(a)} \right)^{\frac{1}{p}} d\sigma(b).
\]
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Now, applying Holder’s inequality two times we obtain

\[ |T(\phi_1, \phi_2)| = \left| \langle B_\psi \phi_1(b, a), B_\psi \phi_2(b, a) \rangle \frac{d\sigma(a)}{a^{2\nu+1}} d\sigma(b) \right| \]
\[ \leq \int_0^\infty \left( \int_0^\infty |B_\psi \phi_1(b, a)|^2 d\sigma(a) \right)^{\frac{1}{2}} \left( \int_0^\infty |B_\psi \phi_2(b, a)|^2 d\sigma(b) \right)^{\frac{1}{2}} \]
\[ \leq \left( \int_0^\infty \left( \int_0^\infty |B_\psi \phi_1(b, a)|^2 \frac{d\sigma(a)}{a^{2\nu+1}} \right) \right)^{\frac{1}{p}} \times \left( \int_0^\infty \left( \int_0^\infty |B_\psi \phi_2(b, a)|^2 d\sigma(b) \right) \right)^{\frac{1}{q}} \]

using Theorem 2.1. we have

\[ |T(\phi_1, \phi_2)| \leq C \| \phi_1 \|_{L^p,\sigma(\mathbb{R}^+)} \| \phi_2 \|_{L^q,\sigma(\mathbb{R}^+)}. \] (2.5)

Moreover for all \( \phi_1 \in L^{2,\sigma}(\mathbb{R}^+) \cap L^{p,\sigma}(\mathbb{R}^+) \) and \( \phi_2 \in L^{2,\sigma}(\mathbb{R}^+) \cap L^{q,\sigma}(\mathbb{R}^+) \) we get

\[ T(\phi_1, \phi_2) = \langle B_\psi \phi_1(b, a), B_\psi \phi_2(b, a) \rangle \frac{d\sigma(a)}{a^{2\nu+1}} d\sigma(b) = A_\psi \langle \phi_1, \phi_2 \rangle. \] (2.6)

From equations (3.5), (3.6) and density of spaces \( L^{2,\sigma}(\mathbb{R}^+) \cap L^{p,\sigma}(\mathbb{R}^+) \) in \( L^{p,\sigma}(\mathbb{R}^+) \) gives the result. \( \square \)

2.1. An inversion formula.

**Theorem 2.3.** Let us consider \( \phi \in L^{p,\sigma}(\mathbb{R}^+) \) with \( 1 < p < \infty \) and \( \psi \) is a real wavelet. Then

\[ \phi(x) = \frac{1}{A_\psi} \int_0^\infty \int_0^\infty B_\psi \phi(b, a) \psi_{b,a}(x) \frac{d\sigma(a)}{a^{2\nu+1}} d\sigma(b). \] (2.7)

The equality holds in \( L^{p,\sigma}(\mathbb{R}^+) \) sense and the integral of right hand side have to be taken in the sense of distributions.

**Proof.** The proof followed from Theorem 2.2. \( \square \)

3. Characterization of Besov-Hankel Norms

In present section, By using the above results, we characterize the Besov-Hankel norms associated the Bessel wavelet transform.

**Theorem 3.1.** Let \( f \in B_{p,q,\sigma}^{\alpha,\sigma}(\mathbb{R}^+) \) (\( p, q > 1, \alpha \neq \mathbb{Z} \)) and analysing wavelet \( \psi \) has \([\alpha]+1\) cancellations and \( (2^{\alpha-|a|}) \psi \in L^{1,\sigma}(\mathbb{R}^+) \), then the wavelet coefficient of function \( f \) holds following conditions:
Suppose that $q < \infty$, \[ \int_0^\infty \left[ a^{-\alpha} \| B \psi(., a) \|_{L^p}^q \right] \frac{da}{a} < \infty \]

if $q = \infty$, \[ a \rightarrow a^{-\alpha} \| B \psi(., a) \|_{L^p} \in L^\infty(\mathbb{R}^+) \].

Moreover the function $a \rightarrow a^{-\alpha} \| B \psi(., a) \|_{L^p} \in L^q(\mathbb{R}^+, \frac{da}{a^{2\sigma+1}})$ and we have:

\[ \| a^{-\alpha} \| B \psi(., a) \|_{L^p} \|_{L^{q,\sigma}(\frac{da}{a^\nu})} \leq \| z^{\alpha-\alpha} \|_{L^1} \times \| h^{\alpha-\alpha} \psi_p(f, h) \|_{L^{q,\sigma}(\frac{da}{a^\nu})} \] \quad (3.1)

Proof. By the definition of continuous Bessel wavelet transform, we have

\[
B \psi(b, a) = \int_0^\infty f(x) \bar{\psi}_{b,a}(x) d\sigma(x)
\]

\[
= \int_0^\infty f(x) \left( \int_0^\infty a^{-2\nu-1} D \left( \frac{b}{a^{\nu}}, \frac{x}{a^{\nu}} \right) \bar{\psi}(z) d\sigma(z) \right) d\sigma(x)
\]

\[
= \int_0^\infty \bar{\psi}(z) \left( \int_0^\infty a^{-2\nu-1} D \left( \frac{b}{a^{\nu}}, \frac{x}{a^{\nu}} \right) f(x) d\sigma(x) \right) d\sigma(z)
\]

\[
= \left\{ \int_0^\infty (\tau_{az}f)(b) \bar{\psi}(z) d\sigma(z) - \int_0^\infty f(b) \bar{\psi}(z) d\sigma(z) \right\}
\]

\[
= \int_0^\infty \bar{\psi}(z) \left( (\tau_{az}f)(b) - f(b) \right) d\sigma(z).
\]

Taking $L^{p,\sigma}$—norm of the wavelet coefficient

\[
\| B \psi(b, a) \|_{L^{p,\sigma}} = \int_0^\infty \left\{ \int_0^\infty | \tau_{az}f(b) - f(b) \| d\sigma(z) \right\}^{\frac{1}{p'}} \| \bar{\psi}(z) \| d\sigma(z).
\]

Using Minkowski inequality of integrability for $p \neq \infty$

\[
\| B \psi(b, a) \|_{L^{p,\sigma}} \leq \int_0^\infty \left\{ \int_0^\infty | \tau_{az}f(b) - f(b) \| d\sigma(b) \right\}^{\frac{1}{p'}} \| \bar{\psi}(z) \| d\sigma(z). \] \quad (3.2)

Suppose that $q < \infty$ and integrating w.r.t. $a$, we get

\[
\int_0^\infty \left[ a^{-\alpha} \| B \psi(b, a) \|_{L^p}^q \right] \frac{da}{a} \leq \int_0^\infty \left[ a^{-\alpha} \int_0^\infty | \bar{\psi}(z) \| \omega_p(f, az) d\sigma(z) \right] \frac{da}{a}.
\]

Again using Minkowski integrability inequality

\[
\int_0^\infty \left[ a^{-\alpha} \| B \psi(b, a) \|_{L^p}^q \right] \frac{da}{a} \leq \left[ \int_0^\infty | \bar{\psi}(z) \| d\sigma(z) \left\{ \int_0^\infty \left( a^{-\alpha} \omega_p(f, az) \right) \frac{da}{a} \right\} \right]^q.
\]
Applying change of variable \( h = az \)

\[
\int_0^\infty z^{-\alpha} \psi(z) \, d\sigma(z) \left\{ \int_0^\infty (h^{-\alpha} \omega_{p,\sigma}(f,h))^q \frac{dh}{h} \right\}^{\frac{1}{q}} \leq \left\{ \int_0^\infty z^{-\alpha} \psi(z) \, d\sigma(z) \right\}^q \times \left\{ \int_0^\infty (h^{-\alpha} \omega_{p,\sigma}(f,h))^q \frac{dh}{h} \right\}^{\frac{1}{q}}.
\]

If \( q = \infty \) the hypothesis on \( f \) says that \( h^{-\alpha} \omega_{p,\sigma}(f,h) \in L_{\infty,\sigma}(\mathbb{R}^+) \), so

\[
\|B_\psi(b,a)\|_{L_{p,\sigma}} \leq a^\alpha \|h^{-\alpha} \omega_{p,\sigma}(f,h)\|_{L_{\infty,\sigma}(\mathbb{R}^+)} \int_0^\infty |z^{-\alpha} \psi(z)| \, d\sigma(z).
\]  

The theorem has been proved for \( 0 < \alpha < 1 \). For \( 1 < \alpha < 2 \), \( \alpha \) is not an integer, by hypothesis \( f' \) belongs to \( BH^p_{\alpha-1,\sigma} \). Since \( \psi \) has cancellation up to order 2 therefore \( (z - \int_{-\infty}^z \psi(t) \, d\sigma(t)) \) is an analysing wavelet and we can apply inequality (3.3) to the coefficient \( C(f', \int \psi, a, b) \)

\[
\int_0^\infty \left( \|C(f', \int \psi, a, b)\|_{L_{\infty,\sigma}(\mathbb{R}^+)} \right)^q \frac{da}{a} \leq \left\{ \int_0^\infty |z^{-\alpha-1} \psi(z)| \, d\sigma(z) \right\}^q \times \left\{ \int_0^\infty (h^{-\alpha-1} \omega_{p,\sigma}(f', h))^q \frac{dh}{h} \right\}.
\]

Now,

\[
\int_0^\infty (a^{-\alpha} \|B_\psi(b,a)\|_{L_{p,\sigma}})^q \frac{d\sigma(a)}{a^{2\nu}} \leq \left\{ \int_0^\infty |z^{-\alpha-1} \psi(z)| \, d\sigma(z) \right\}^q \times \left\{ \int_0^\infty (h^{-\alpha-1} \omega_{p,\sigma}(f', h))^q \frac{dh}{h} \right\} < \infty.
\]

This prove the result for \( 1 < \alpha < 2 \). Similarly for \( q = \infty \). Hence by recurrence on \([\alpha]\) proved the theorem. □

Next theorem is the converse of the above theorem. The Bessel wavelet coefficients at small value of \( a \) is sufficient to characterizes Besov-Hankel spaces.

**Theorem 3.2.** Suppose \( \alpha > 0 \), \( \alpha \) not an integer and a function \( \psi \) is a real \( C^{[\alpha]+1} \)-regular analysing wavelet with all derivatives rapidly decreasing. If \( f, f', f'', f''' \, \ldots, f^{[\alpha]} \in \)
and applying Minkowski’s inequality

\[ L^{p,\sigma}(\mathbb{R}^+) \text{ (} 1 < p < \infty \text{), and if } a^{-\alpha} \|(B\psi f)(a, \cdot)\|_{L^{p,\sigma}} \in L^{p,\sigma}(\mathbb{R}^+, \frac{da}{\alpha}), \text{ then } f \in B_{p,q}^{\nu,\alpha} \text{ and we have} \]

\[ \|h^{-(\alpha-\alpha)}w_{p,\sigma}(f^{(\alpha)}, h)\|_{L^{p,\sigma}} \leq \frac{1}{A_\psi} \left( \frac{2}{(\alpha - [\alpha])} \|\psi^{[\alpha]}\|_{L^{1,\sigma}} + \frac{1}{1 - (\alpha - [\alpha])} \|\psi^{[\alpha]+1}\|_{L^{1,\sigma}} \right) \times a^{-\alpha}\|(B\psi f)(a, \cdot)\|_{L^{p,\sigma}} \] (3.5)

**Proof.** Let \( f \in L^{p,\sigma}(\mathbb{R}^+) \). By inversion formula of Bessel wavelet transform

\[ f(x) = \frac{1}{A_\psi} \int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}} \int_0^\infty (B\psi f)(a, b) \psi_{a,b}(x) d\sigma(b) \] (3.6)

and

\[ \tau_h f(x) = \frac{1}{A_\psi} \int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}} \int_0^\infty (B\psi f)(a, b) \tau_h \psi_{a,b}(x) d\sigma(b). \] (3.7)

Then

\[ \tau_h f(x) - f(x) = \frac{1}{A_\psi} \int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}} \int_0^\infty (B\psi f)(a, b) \left\{ \tau_h \psi_{a,b}(x) - \psi_{a,b}(x) \right\} d\sigma(b) \]

\[ = \frac{1}{A_\psi} \int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}} \int_0^\infty (B\psi f)(a, b) \left\{ \tau_h \tau_h \psi_{a,b}(x) - \tau_h \psi_{a,b}(x) \right\} d\sigma(b) \]

\[ = \frac{1}{A_\psi} \int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}} \int_0^\infty (B\psi f)(a, b) a^{-2\nu-1} D(\frac{b}{a}, \frac{x}{a}, y) d\sigma(b) \]

\[ \times \int_0^\infty \left\{ \tau_h \psi(y) - \psi(y) \right\} d\sigma(y) \]

\[ = \frac{1}{A_\psi} \int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}} \tau_h (B\psi f)(a, \frac{x}{a}) \int_0^\infty \left\{ \tau_h \psi(y) - \psi(y) \right\} d\sigma(y) \]

Taking \( L^{p,\sigma} \)-norm on both side, we have

\[ w_{p,\sigma}(f, h) = \frac{1}{A_\psi} \left\{ \int_0^\infty \left| \int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}} \tau_h (B\psi f)(a, \frac{x}{a}) \int_0^\infty \left\{ \tau_h \psi(y) - \psi(y) \right\} d\sigma(y) \right|^p d\sigma(x) \right\} \]

applying Minkowski’s inequality

\[ \leq \frac{1}{A_\psi} \int_0^\infty \frac{d\sigma(t)}{t^{2\nu}} \| (B\psi f)(\frac{h}{t}, \cdot) \|_{L^{p,\sigma}} \int_0^\infty |\tau_h \psi(y) - \psi(y)| d\sigma(y) \]} (3.8)
Now, consider $0 < \alpha < 1$ and using Minkowski’s inequality

\[
\left\{ \int_0^\infty \frac{dh}{h} h^{-\alpha q} w_{p,\sigma}(f,h)^q \right\}^\frac{1}{q} \leq \frac{1}{A_\psi} \int_0^\infty \frac{d\sigma(t)}{t^{2\nu}} \int_0^\infty |\tau_{\frac{h}{t}} \psi(y) - \psi(y)|d\sigma(y) \\
\times \left\{ \int_0^\infty \frac{dh}{h} h^{-\alpha q} \| (B_\psi f)(\frac{h}{t},\cdot) \|_{L^p,\sigma}^q \right\}^\frac{1}{q}
\]

\[
= \frac{1}{A_\psi} \int_0^\infty \frac{d\sigma(t)}{t^{2\nu+\alpha}} \int_0^\infty |\tau_{\frac{h}{t}} \psi(y) - \psi(y)|d\sigma(y) \\
\times \left\{ \int_0^\infty \frac{da}{a} a^{-\alpha q} \| (B_\psi f)(a,\cdot) \|_{L^p,\sigma}^q \right\}^\frac{1}{q}
\]

\[
= \frac{C}{A_\psi} \int_0^\infty \frac{dt}{t^{1+\alpha}} \int_0^\infty |\tau_{\frac{h}{t}} \psi(y) - \psi(y)|d\sigma(y) \\
\times \left\{ \int_0^\infty \frac{da}{a} a^{-\alpha q} \| (B_\psi f)(a,\cdot) \|_{L^p,\sigma}^q \right\}^\frac{1}{q}.
\]

Using Lemma 1.1, we obtain

\[
\int_0^\infty |\tau_{\frac{h}{t}} \psi(y) - \psi(y)|d\sigma(y) \leq 2\|\psi\|_{L^1,\sigma}
\]  \hspace{1cm} (3.8)

and

\[
\int_0^\infty |\tau_{\frac{h}{t}} \psi(y) - \psi(y)|d\sigma(y) = \int_0^\infty \left| \int_0^t (\tau_z \psi(y))' dz \right|d\sigma(y) \\
\leq \int_0^t \int_0^\infty |(\tau_z \psi(y))' dz|d\sigma(y) \\
\leq \|\psi'\|_{L^1,\sigma} t.
\]  \hspace{1cm} (3.9)

Here, we observe that

\[
\int_0^\infty \frac{dt}{t^{1+\alpha}} \int_0^\infty |\tau_{\frac{h}{t}} \psi(y) - \psi(y)|d\sigma(y) = \int_0^1 \frac{dt}{t^{1+\alpha}} \int_0^\infty |\tau_{\frac{h}{t}} \psi(y) - \psi(y)|d\sigma(y) \\
+ \int_1^\infty \frac{dt}{t^{1+\alpha}} \int_0^\infty |\tau_{\frac{h}{t}} \psi(y) - \psi(y)|d\sigma(y) \\
\leq 2\|\psi\|_{L^1,\sigma} \int_1^\infty \frac{dt}{t^{1+\alpha}} + 2\|\psi'\|_{L^1,\sigma} \int_0^1 \frac{dt}{t^{1+\alpha}} \\
= \frac{2}{\alpha}\|\psi\|_{L^1,\sigma} + \frac{1}{1-\alpha}\|\psi'\|_{L^1,\sigma},
\]
which proved the result for $0 < \alpha < 1$.

If $1 < \alpha < 2$, by the hypothesis $f' \in L^p(\mathbb{R}^+) \text{ and } \psi$ is a $C^1$-regular function with $\psi'$ rapidly decreasing at infinity. From equations (3.6) and (3.7), we have the equality

$$\tau_h f'(x) - f'(x) = \frac{1}{A_\psi} \int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}} \left( \int_0^\infty (B_\psi f)(a, b) \{ \tau_h \psi'_{a,b}(x) - \psi'_{a,b}(x) \} \, d\sigma(b) \right) \, d\sigma(a).$$

Calculate in similar manner as above for $f'$ gives the following estimation

$$\left\{ \int_0^\infty \frac{d\sigma(a)}{h^{\alpha-1-q}w_{p,\sigma}(f, h)^q} \right\}^{\frac{1}{q}} \leq \frac{1}{A_\psi} \int_0^\infty \frac{dt}{t^{1+\alpha}} \int_0^\infty \left| \tau_h \psi'(y) - \psi(y) \right| d\sigma(y) \times \left\{ \int_0^\infty \frac{dh}{h} (a)^{-\alpha q} \| (B_\psi f)(a, \cdot) \|^q_{L^p,\sigma} \right\}^{\frac{1}{q}} \leq \frac{1}{A_\psi} \left( \frac{2}{(\alpha-1)} \| \psi' \|^q_{L^1,\sigma} + \frac{1}{1-(\alpha-1)} \| \psi'' \|^q_{L^1,\sigma} \right) \times \left\{ \int_0^\infty \frac{dh}{h} a^{-\alpha q} \| (B_\psi f)(a, \cdot) \|^q_{L^p,\sigma} \right\}^{\frac{1}{q}}$$

this proves that $f' \in BH^{p,q}_{\alpha-1,\sigma}$ the hypothesis $a \to a^{-\alpha} \| (B_\psi f)(a, \cdot) \|^q_{L^p,\sigma} \in L^q(\mathbb{R}^+)$ implies $a \to a^{-(\alpha-1)} \| (B_\psi f)(a, \cdot) \|^q_{L^p,\sigma} \in L^q(\mathbb{R}^+)$ then $f \in BH^{p,q}_{\alpha-1,\sigma}$. The theorem is established for $1 < \alpha < 2$, a recurrence on $[\alpha]$ gives the final result. □

**Corollary 3.3.** Let $f \in BH^{p,q}_{\alpha,\sigma}(\mathbb{R}^+) \ (p, q > 1, \alpha \neq \mathbb{Z})$, then

$$\| f \|_{BH^{p,q}_{\alpha,\sigma}} = \| f \|_{L^p,\sigma(\mathbb{R}^+)} + \| f \|_{BH^{p,q}_{\alpha,\sigma}}$$

where $\| f \|_{BH^{p,q}_{\alpha,\sigma}}$ is equal to

$$\| f \|_{BH^{p,q}_{\alpha,\sigma}} = \int_0^\infty \left( h^{-\alpha} w_p(\phi)(h) \right)^q \frac{dh}{h} \approx \int_0^\infty \left[ a^{-\alpha} \| B_\psi(\cdot, a) \|^q_{L^p,\sigma} \right] \frac{da}{a}.$$ 

**Remark 3.4.** Considerable work has already been done on the characterization of Besov-k-Hankel norms by means of k-Hankel wavelet transform and Mehler-Besov-Fock spaces by using Mehler-Fock wavelet transform etc. A further research in the context of different types of Besov space related to the different integral transform is needed.
Acknowledgement

The research of the second author is supported by University Grants Commission (UGC), grant number: F.No. 16-6(DEC. 2017)/2018(NET/CSIR), New Delhi, India.

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