Supplementary material for ”Effect of small-world topology on wave propagation on networks of excitable elements”

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A  Excitability in the FitzHugh-Nagumo system

The local dynamics of Eqs. (1) are given by

\[
\begin{align*}
\dot{u} &= u - \frac{u^3}{3} - v, \\
\dot{v} &= \varepsilon(u - \beta). 
\end{align*}
\]  

(A.1a)  

(A.1b)

The periodic orbit which emerges from the Hopf bifurcations at $|\beta| = 1$ shows a canard explosion \[1\] immediately after the bifurcation point ($|\beta| \ll 1$), evolving to the large-amplitude limit cycle of a relaxation oscillator. In the regime where $|\beta| > 1$ and no limit cycle (stable or unstable) exists in the local dynamics, Eqs. (A.1) serve as a paradigmatic example for type-II excitability.

If we view the Eqs. (A.1) as a singularly perturbed system, we note that by Fenichel theory \[2\], the normally hyperbolic parts of the $u$-nullcline (which give the critical manifold of the system in the limit $\varepsilon \to 0$) are perturbed to attracting and repelling slow manifolds $S_{a\varepsilon}$, $S_{r\varepsilon}$, respectively. Any trajectory that passes near $S_{r\varepsilon}$ is repelled from it with $v$ staying almost constant (because $\varepsilon \ll 1$) until it hits $S_{a\varepsilon}$. A trajectory that follows the repelling slow manifold $S_{r\varepsilon}$ for a ‘considerable amount of time’ is called a canard trajectory \[1\]. In the case of the FitzHugh-Nagumo dynamics it is a common choice to take such a canard trajectory as the threshold trajectory, e.g., one uses the trajectory that goes through the point where the repelling and attracting parts of the critical manifold meet (which is given by the $u$-nullcline), see Fig. (1b).

B  Dispersion relation of traveling waves in the continuous system

With the parameter values being in the excitable regime (as are the parameters chosen here, $\beta = 1.1$, $\varepsilon = 0.04$), Eqs. (6) are known to support traveling-wave solutions. These solutions move at constant speed $c$ and in a comoving frame they do not change their shape. At a given $L$ there can be different traveling wave solutions traveling at different speeds, namely stable ‘fast waves’ and unstable ‘slow waves’. When varying $L$, these branches are connected, however, and together they form the dispersion relation ($L, c(L)$). For more details, we refer to \[3, 4, 5\].

We calculate the dispersion relation of traveling wave solutions of Eqs. (6) by transforming to the comoving frame $\xi = x - ct$ and doing numerical branch continuation using AUTO-07p \[6\]. The equations in the comoving frame (and written as a first-order system) are called the profile equations and read

\[
\begin{align*}
\dot{u} &= w, \\
\dot{v} &= -\frac{\varepsilon}{c}(u - \beta), \\
\dot{w} &= -cw - \left(u - \frac{u^3}{3} - v\right).
\end{align*}
\]  

(B.2a)  

(B.2b)  

(B.2c)

Due to the periodic boundary conditions, a traveling wave solution of Eqs. (6) appears as a periodic orbit in Eqs. (B.2). The domain size $L$ defines the period
Figure B.1: Dispersion relation for wave trains (or traveling pulses subject to periodic boundary conditions on a domain of size $L$). (a) Dispersion relation of Eq. (6) i.e., propagation velocity $c$ vs. $L$. (b) Same dispersion relation in the transformed coordinates $D = 1/L^2$ and $c/L$ which are more adequate for networks. (c)-(k) Leading part of the essential spectrum for selected points marked in panels (a),(b). The insets in (a), (b) show an enlarged view of the yellow rectangles, the insets in (c)-(k) show a blow-up of the spectrum near the origin. Those points that are in the spectrum for the system with periodic boundary conditions are marked by an asterisk. Parameters: $\beta = 1.1, \varepsilon = 0.04$
of this orbit and the propagation velocity \( c \) is an additional parameter that needs to be solved for during the continuation.

The spectrum of the linear stability analysis around such a traveling wave solution is very closely connected to the spectrum around a wave-train in an infinitely extended spatial domain subject to the same differential equation. The spectrum around this wave-train consists only of continuous components which together are called the essential spectrum \([7]\). By linearizing Eqs. (6) around a traveling wave solution using a Bloch-expansion ansatz and transforming to the comoving frame, the essential spectrum can be calculated by numerical continuation of the resulting boundary value problem using the Bloch wavenumber as continuation parameter. This boundary value problem reads

\[
\begin{align*}
\delta u' &= \delta w \\
\delta v' &= \frac{1}{c} (\lambda \delta v - \varepsilon \delta u) \\
\delta w' &= \lambda \delta u - [(1 - u_{tw}^2) \delta u - \delta v] - c \delta w \\
\delta u(L) &= e^{i2\pi\nu} \delta u(0) \\
\delta v(L) &= e^{i2\pi\nu} \delta v(0) \\
\delta w(L) &= e^{i2\pi\nu} \delta w(0),
\end{align*}
\]

where \( u_{tw} \) is (the \( u \) part of) the profile of a traveling wave solution to Eqs. (B.2) with spatial period \( L \) and propagation speed \( c \), \( \nu \) is the Bloch wavenumber in units of \( L/(2\pi) \), and \( \lambda \in \mathbb{C} \) is an element of the spectrum.

We have used this method which was proposed in \([7]\) to calculate the leading part of the essential spectrum of the linearization of Eq. (6) around the connected wave train. The essential spectrum is continuous and contains every point of the spectrum of a traveling wave solution of Eq. (6) subject to periodic boundary conditions. These points additionally fulfill the condition of an integer Bloch-wavenumber \( \nu \). The method we use to calculate the spectrum is elaborated in great detail in \([7]\) including hints on the implementation in AUTO.

We have calculated the dispersion relation for a traveling wave solution of Eqs. (6) (subject to periodic boundary conditions) as well as the essential spectrum of the related wave-train at selected solutions and those points of the essential spectrum that are in the spectrum of the traveling wave subject to periodic boundary conditions. The results are shown in Fig. B.1. For the dispersion relation of traveling wave solutions of Eqs. (6), we report the following: Starting with a stable traveling-wave solutions and large \( L \), the calculated spectrum tells us that a destabilization of the wave occurs at a critical value of \( L_{cr} \approx 30.756 \) by two complex conjugate eigenvalues with nonzero imaginary part. However, this point does not yet mark the lowest possible propagation speed \( c \). This is attained by the branch of unstable solutions after lowering \( L \) further to a value of \( L \approx 19.6 \). After that, the branch of unstable solutions continues in form of a fold bifurcation in direction of rising \( L \) with the speed converging against \( c \approx 0.493 \). On this branch, \( L \) can increase without bounds, and this branch is associated with the transition to an (unstable) solitary traveling pulse. Also on the stable branch, \( L \) can increase without bounds but the solutions do not lose stability and converge to a stable propagating solitary pulse.

After the destabilization at \( L_{cr} \), the spectrum changes in a complex way, leaving in the end an isolated eigenvalue as the only object in the right half-plane.
The dispersion relation (propagation velocity $c$ vs. domain size $L$) including the leading parts of the (essential) spectrum for selected points (c)-(k) is illustrated in Fig. B.1(a). In the pictures of the spectrum, the essential spectrum is marked by continuous lines and the values of the essential spectrum that occur for periodic boundary conditions are marked with asterisks. The latter are the relevant ones for our model. The order of occurrence of the points (c)-(k) is the following. (c) All eigenvalues are in the left half-plane (except for the one that is always present at zero corresponding to the Goldstone mode of translation invariance), (d) two complex conjugate eigenvalues crossing the imaginary axis at $L_{cr}$, (e) a second pair of complex conjugate eigenvalues crossing the imaginary axis, (f),(g) from the leading part of the essential spectrum, a circle comprising the Goldstone mode eigenvalue is forming and detaching, (h) the first two eigenvalues that have crossed the imaginary axis merge on the real axis and split in different directions, (i) one of the eigenvalues that has merged on the real axis crosses zero, (j) the second two eigenvalues that have crossed the imaginary axis cross the imaginary axis again in the opposite direction. (k) One eigenvalue is left in the right half-plane, the rest of the eigenvalues are in the left half-plane (again except for the Goldstone mode eigenvalue at zero). Thus after the first instability at $L_{cr}$, a scenario with several secondary instabilities emerges.

As discussed in the main text, the reaction-diffusion system Eq. (6) is a good approximation for the ring network Eq. (2) for large $N$ and large $D$. In Fig. B.1(b), we have displayed the dispersion relation of Fig. B.1(a) in transformed coordinates $c/L$ vs. $D = 1/L^2$. In these coordinates it is easier to compare this dispersion relation of a continuum system to that of the discrete ring system Eqs. (2). On the ring network, the above sequence of changes in the spectrum would translate to the following sequence of bifurcations: A torus bifurcation through which the wave loses its stability at (d), another torus bifurcation at (e), a saddle-node bifurcation at (i), a torus bifurcation at (j). The minimum domain size $L_{cr}$ for which a traveling-wave solution is stable translates to a maximum coupling strength according to Eq. (4). We denote this maximum coupling strength by $D_{\text{high}}$. 

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