Super Yang-Mills theory from nonlinear supersymmetry

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Abstract

The relation between a nonlinear supersymmetric (NLSUSY) theory and a SUSY Yang-Mills (SYM) theory is studied for $N = 3$ SUSY in two-dimensional space-time. We explicitly show the NL/L SUSY relation for the (pure) SYM theory by means of cancellations among Nambu-Goldstone fermion self-interaction terms.

PACS: 11.30.Pb, 12.60.Jv, 12.60.Rc, 12.10.-g

Keywords: supersymmetry, Nambu-Goldstone fermion, nonlinear/linear SUSY relation, composite unified theory

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Various linear supersymmetric (LSUSY) theories [1, 2, 3] are related to the Volkov-Akulov (VA) model [4] for nonlinear representation of supersymmetry (NLSUSY). The relation (abbreviated as NL/L SUSY relation) gives new insights into the low energy physics based on the nonlinear supersymmetric general relativity (NLSUSY GR) theory [5] in SGM scenario [5]-[9] towards the unified description of space-time, all forces and matter beyond the standard model (SM).

The linearization of NLSUSY [10]-[13] in flat space-time was shown in the various LSUSY (free) theories with the spontaneous SUSY breaking (SSB) [14] for \( N = 1 \) SUSY [10]-[15] and for extended SUSY, i.e. \( N = 2 \) [16] and \( N = 3 \) SUSY (in two-dimensional space-time \( (d = 2) \)) [17], and also shown in interacting \( N = 2 \) LSUSY (Yukawa interaction and SUSY QED) theories [18]-[21] in \( d = 2 \). (Note that for \( N = 2 \) SUSY in SGM scenario \( J^P = 1^- \) gauge field appears [16] in the linearization of NLSUSY. Therefore \( N = 2 \) SUSY is realistic minimal case.) By using the NL/L SUSY relation for the \( N = 2 \) SUSY QED theory \((d = 2)\) we have shown [22, 23] that LSUSY-supermultiplet fields are realized as the composite (massless) eigenstates of the spin-\( \frac{1}{2} \) massless Nambu-Goldstone (NG) fermions (superons) in the (fundamental) NLSUSY theory on the true vacuum (the minimum of the potential). As for the physical significanes of NL/L SUSY relation, we pointed out that since the SSB scale originated from the cosmological term in NLSUSY GR induces a fundamental mass scale on the true vacuum, it gives a natural explanation of the mysterious (observed) numerical relation between the (four dimensional) dark energy density of the universe and the neutrino (\( \nu \)) mass [9, 22, 23] in the asymptotically flat space-time, provided \( \nu \) is a composite of superons of these kinds.

Beyond the abovementioned works for the linearization of NLSUSY, the extension of NL/L SUSY relation to non-Abelian gauge theories is a crucial and interesting problem. In this letter we address the problem by focusing on \( N = 3 \) (pure) super Yang-Mills (SYM) theory in \( d = 2 \) for simplicity of calculations without the loss of generalities. SUSY invariant relations which connect LSUSY theories with the NL-SUSY model play an important role for establishing the NL/L SUSY relation, where basic fields in LSUSY supermultiplets are expressed as the composites of superons in a SUSY invariant way. We construct the SUSY invariant relations for the SYM theory and explicitly discuss the relation between the \( N = 3 \) NLSUSY action and a \( N = 3 \) (pure) SYM action. Each interaction terms in the \( d = 2, N = 3 \) SYM theory does not vanish in terms of superons, but the NL/L SUSY relation is realized by means of (nontrivial) cancellations among those terms as discussed later.
Let us first introduce the VA NLSUSY action for N SUSY \([4, 24]\),

\[
S_{\text{NLSUSY}} = -\frac{1}{2\kappa^2} \int d^2x \, |w|, 
\]

(1)

where \(\kappa\) is a constant whose dimension is \((\text{mass})^{-1}\) and \(|w|\) is the determinant describing the dynamics of (Majorana) superons \(\psi^i(x) (i, j, \cdots = 1, \cdots, N)\), which is written in \(d = 2\) \(^\dagger\) as

\[
|w| = \det(w^a_b) = \det(\delta^a_b + t^a_b) = 1 + t^a_a + \frac{1}{2i}(t^a_at^b_b - t^a_bt^b_a) 
\]

(2)

with \(t^a_b = -i\kappa^2 \bar{\psi}^i \gamma^a \partial_b \psi^i\). The NLSUSY action (1) is invariant under NLSUSY transformations for \(\psi^i\),

\[
\delta_\zeta \psi^i = \frac{1}{\kappa} \zeta^i - i\kappa \bar{\zeta}^j \gamma^a \partial_a \psi^i, 
\]

(3)

where \(\zeta^i\) are the constant (Majorana) spinor parameters. The NLSUSY transformations (3) satisfy a closed off-shell commutator algebra,

\[
[\delta_{\zeta_1}, \delta_{\zeta_2}] = \delta_P(\Xi^a), 
\]

(4)

where \(\delta_P(\Xi^a)\) means a translation with the parameters \(\Xi^a = 2i \bar{\zeta}_1^i \gamma^a \zeta_2^i\).

Next we shall exhibit a (pure) SYM action for \(N = 3\) SUSY in \(d = 2\) in terms of a gauge supermultiplet corresponding to the helicity states for the irreducible representation of \(SO(3)\) SP algebra,

\[
\left[ \mathbb{1}(+1), \mathbb{2} \left( \frac{1}{2} \right), \mathbb{2}(0), \mathbb{1} \left( -\frac{1}{2} \right) \right] + [\text{CPT conjugate}], 
\]

(5)

where \(\mathfrak{g}(\lambda)\) means the dimension \(\mathfrak{g}\) and the helicity \(\lambda\) of the irreducible representation. Component fields which belong to the adjoint representation of gauge group \(G\) are defined as

\[
V(x) = \{ v^a(x), \lambda^i(x), A^i(x), \chi(x), \phi(x), D^i(x) \} \quad (i, j, \cdots = 1, 2, 3), 
\]

(6)

\(^\dagger\)Minkowski space-time indices in \(d = 2\) are denoted by \(a, b, \cdots = 0, 1\). The Minkowski spacetime metric is \(\frac{1}{2} \{ \gamma^a, \gamma^b \} = \eta^{ab} = \text{diag}(+1, -)\) and \(\sigma^{ab} = \frac{1}{2} \{ \gamma^a, \gamma^b \} = i\epsilon^{ab} \gamma_5 \) \((\epsilon^{01} = 1 = -\epsilon_{01})\), where we use the \(\gamma\) matrices defined as \(\gamma^0 = \sigma^2, \gamma^1 = i\sigma^1, \gamma_5 = \gamma^0\gamma^1 = \sigma^3\) with \((\sigma^1, \sigma^2, \sigma^3)\) being Pauli matrices.
where we denote \( v^a \) for vector fields, \( \lambda^i \) and \( \chi \) for (Majorana) spinor fields, \( A^i \) for scalar fields, \( \phi \) for pseudo scalar fields and \( D^i \) for auxiliary scalar fields, respectively. These component fields are \( V(x) = V^I(x)T^I \) with generators \( T^I \) of \( G \) satisfying 
\[
[T^I, T^J] = if^{IJK}T^K.
\]

The \( N = 3 \) (pure) SYM action in terms of the component fields (6) is given by
\[
S_{\text{SYM}} = \int d^2x \, \text{tr} \left\{ -\frac{1}{4}(F_{ab})^2 + \frac{i}{2}\bar{\chi}D\chi + \frac{1}{2}(D_a A^i)^2 + \frac{i}{2}\bar{\chi}D\chi + \frac{1}{2}(D_a \phi)^2 + \frac{1}{2}(D_i)^2 
- ig\{\epsilon^{ijk}A^i\bar{\lambda}^j\lambda^k - [A^i, \bar{\lambda}^j]\chi + \phi(\bar{\lambda}^i\gamma_5\lambda^i + \bar{\chi}\gamma_5\chi)\}
+ \frac{1}{4}g^2([A^i, A^j]^2 + 2[A^i, \phi]^2) \right\},
\]
where \( g \) is the gauge coupling constant, \( D_a \) and \( F_{ab} \) are the covariant derivative and the non-Abelian gauge field strength defined as
\[
D_a\phi = \partial_a\phi - ig[v_a, \phi], \quad F_{ab} = \partial_a v_b - \partial_b v_a - ig[v_a, v_b].
\]

The SYM action (7) is invariant under the following \( N = 3 \) LSUSY transformations parametrized by \( \zeta^i \),
\[
\begin{align*}
\delta_\zeta v^a &= i\bar{\zeta}\gamma^a\lambda^i, \\
\delta_\zeta \lambda^i &= \epsilon^{ijk}(D^j - i\bar{\zeta}D\zeta^i)\zeta^k + \frac{1}{2}\epsilon^{ab}F_{ab}\gamma_5\zeta^i - i\gamma_5D\phi\zeta^i \quad + ig([A^i, A^j]\zeta^j + \epsilon^{ijk}[A^j, \phi]\gamma_5\zeta^k), \\
\delta_\zeta A^i &= \epsilon^{ijk}\bar{\zeta}\lambda^k - \bar{\zeta}\phi, \\
\delta_\zeta \chi &= (D^i + i\bar{\zeta}D\zeta^i)\zeta^i + ig(\epsilon^{ijk}A^i\omega^j\zeta^k - [A^i, \phi]\gamma_5\zeta^i), \\
\delta_\zeta \phi &= \bar{\zeta}\gamma_5\phi, \\
\delta_\zeta D^i &= -i\epsilon^{ijk}\bar{\zeta}\lambda^k - i\bar{\zeta}D\chi + ig([\bar{\zeta}\lambda^i, A^j] + \bar{\zeta}\lambda^i, A^j - \bar{\zeta}\lambda^i, A^j) \\
&- \epsilon^{ijk}\bar{\zeta}(\chi, A^k) + \epsilon^{ijk}\gamma_5[\lambda^k, \phi] + \bar{\zeta}(\chi, \phi),
\end{align*}
\]
which satisfy the ordinary off-shell commutator algebra,
\[
[\delta_\zeta, \delta_\zeta'] = \delta_P(\Xi^n) + \delta_G(\theta) + \delta_g(\theta),
\]
where \( \delta_G(\theta) \) means a commutator of the gauge transformation \( \delta_G(\theta)\phi = ig[\theta, \phi] \) with a generator \( \theta = -2(i\zeta_1\gamma^a\zeta_2 v_a - \epsilon^{ijk}\zeta_1\zeta_2^j A^k - \zeta_1\gamma_5\zeta_2^j \phi) \), while \( \delta_g(\theta) \) is the \( U(1) \) gauge transformation only for \( v^a \) with \( \theta \).
In order to see the relation between the NLSUSY action (1) and the SYM action (7), we construct the SUSY invariant relations in which the component fields (6) are expressed in terms of $\psi^i$ ($i, j, \cdots = 1, 2, 3$) as $V(x) = V(\psi(x))$. As discussed in the previous works in the superfield formulation [25], the hybrid transformations of $L$ and NLSUSY transformations of $V(x)$ and the adoption of the subsequent SUSY invariant constraints produce systematically the SUSY invariant relations. They satisfy the commutator algebra (4) as in the case of the Abelian gauge theories [16, 19] and the NLSUSY GR [5], i.e. the SUSY transformations are the square root of the translation ($GL(4, R)$ in NLSUSY GR). The explicit form of the SUSY invariant relations for the $d = 2$, $N = 3$ gauge supermultiplet in the leading orders of $\kappa$ is

$$v^aI = -\frac{i}{2} \kappa \epsilon^{ijk} \xi^{ij} \bar{\psi}^j \gamma^a \psi^k (1 - i \kappa^2 \bar{\psi}^j \partial \psi^j) + \frac{1}{4} \kappa^3 \epsilon^{ab} \epsilon^{ijk} \xi^{ij} \partial_a (\bar{\psi}^j \gamma_5 \psi^k \bar{\psi}^l \psi^l) + \mathcal{O}(\kappa^5),$$

$$\chi^I = \epsilon^{ijk} \xi^{ij} \bar{\psi}^j (1 - i \kappa^2 \bar{\psi}^j \partial \psi^j)$$

$$+ \frac{i}{2} \kappa \epsilon^{ijk} \partial_a \{\epsilon^{ijk} \gamma^a \bar{\psi}^j \psi^k + \epsilon^{ab} \epsilon^{ijk} (\gamma_b \psi^i \bar{\psi}^k \gamma_5 \psi^j - \gamma_5 \psi^i \bar{\psi}^k \gamma_b \psi^j)\} + \mathcal{O}(\kappa^4),$$

$$A^{ij} = \kappa \left(\frac{1}{2} \xi^{ij} \bar{\psi}^j \psi^j - \xi^{ij} \bar{\psi}^j \psi^j\right) (1 - i \kappa^2 \bar{\psi}^j \partial \psi^j) - \frac{i}{2} \kappa^2 \xi^{ij} \partial_a (\bar{\psi}^j \gamma^a \psi^j \bar{\psi}^k \psi^k) + \mathcal{O}(\kappa^5),$$

$$\chi^I = \xi^{ij} \psi^j (1 - i \kappa^2 \psi^j \partial \psi^j) + \frac{i}{2} \kappa \epsilon^{ijk} \partial_a (\gamma^a \bar{\psi}^j \bar{\psi}^j \psi^k) + \mathcal{O}(\kappa^4),$$

$$\phi^I = -\frac{1}{2} \kappa \epsilon^{ijk} \xi^{ij} \bar{\psi}^j \gamma_5 \psi^k (1 - i \kappa^2 \bar{\psi}^j \partial \psi^j) - \frac{i}{2} \kappa^3 \epsilon^{ab} \epsilon^{ijk} \xi^{ij} \partial_a (\bar{\psi}^j \gamma_b \psi^k \bar{\psi}^l \psi^l) + \mathcal{O}(\kappa^5),$$

$$D^{ij} = \frac{1}{\kappa} \xi^{ij} |w| - i \kappa \epsilon^{ijk} \partial_a \{\bar{\psi}^j \gamma^a \psi^j (1 - i \kappa^2 \bar{\psi}^j \partial \psi^j)\}$$

$$- \frac{1}{8} \kappa^3 \{\xi^{ij} \bar{\psi}^j \psi^j - 4 \xi^{ij} \bar{\psi}^j \psi^j\} \bar{\psi}^k \psi^k \} + \mathcal{O}(\kappa^5),$$

(11)

where $\xi^{ij}$ are arbitrary real constants related to the constant terms of the auxiliary fields $D^{ij} = D^{ij}(\psi)$. The relations (11) have the same form as those in Ref.[17] except the constants $\xi^{ij}$ with the indices of the gauge group $G$.

By substituting Eqs.(11) into the SYM action (7), we can show it reduces to the NLSUSY action (1) for $N = 3$ SUSY as

$$S_{\text{SYM}}(\psi) = - (\xi^{ij})^2 S_{\text{NLSUSY}} + [\text{surface terms}]$$

(12)

at least in the leading orders. Indeed, the kinetic terms (with the $(D^{ij})^2$-terms) in $S_{\text{SYM}}$ reduces to $S_{\text{NLSUSY}}$ as

$$S_{\text{kin,SYM}}(\psi) = \int d^2 x \text{ tr} \left\{-\frac{1}{4} (f_{ab})^2 + \frac{i}{2} \chi^i \partial \lambda^i + \frac{1}{2} (\partial_a A^i)^2 + \frac{i}{2} \chi^i \partial \chi + \frac{1}{2} (\partial_a \phi)^2 + \frac{1}{2} (D^i)^2 \right\}$$
\[-(\xi^I)^2 S_{\text{NLSUSY}} \] (at least up to \(O(\kappa^2)\)), where \(f_{ab} = \partial_a v_b - \partial_b v_a\). On the other hand, each interaction terms at \(O(g)\) in \(S_{\text{SYM}}\) give the following (non-vanishing) four and/or six superon (NG-fermion) self-interaction terms up to \(O(\kappa^3)\),

\[
\begin{align*}
(a) & \quad g_{K}^{ijk} f_{JKL} \xi^I \xi^J \xi^K \bar{\psi}^l \psi^m \bar{\psi}^n \phi^m \text{ (at } O(\kappa)\text{)}, \\
(b) & \quad ig_{3}^{ijk} f_{JKL} \xi^I \xi^J \xi^K \bar{\psi}^l \psi^m \bar{\psi}^n \phi^m \text{ (at } O(\kappa^3)\text{)}.
\end{align*}
\] (14)

However, they cancel with each other in \(S_{\text{SYM}}\), i.e. the interaction terms in \(S_{\text{SYM}}\) vanish as \(S_{\text{int}}(\psi) = \int d^2x \left\{ - f_{ab} v^a v^b + \partial_a A^i [v^a, A^i] + \partial_a \phi [v^a, \phi] + i \left( \bar{\lambda}^i \gamma^a [v_a, \lambda^i] + \bar{\lambda}^i \gamma^a [v_a, \chi^i] \right) + \epsilon^{ijk} A^i \bar{\lambda}^j \lambda^k - [A^i, \bar{\lambda}^i] \lambda^j + \phi (\bar{\lambda}^i \gamma_5 \lambda^i + \bar{\chi} \gamma_5 \chi) \right\} = 0 \) (15)

(at least up to \(O(\kappa^3)\)). Each interaction (potential) terms at \(O(g^2)\) in the SYM action (7) vanishes in terms of \(\psi^i\) due to \((\psi^i)^7 \equiv 0\) for \(d = 2, N = 3\) SUSY. From Eqs.(13) and (15) we conclude that the NLSUSY action (1) for \(N = 3\) SUSY is related to the \(N = 3\) SYM action (7) as

\[-(\xi^I)^2 S_{\text{NLSUSY}} = S_{\text{SYM}} + \text{[surface terms]} \] (16)

(From the previous works we anticipate that the relation (16) holds in all orders of \(\psi^i\), though yet to be confirmed.) For the case of \(d = 2, N = 2\) SUSY, each interaction terms at \(O(g)\) in a pure SYM action vanishes in terms of \(\psi^i\), which means that the relation (16) is trivial for the \(d = 2, N = 2\) pure SYM theory.

Let us summarize our results as follows. In this letter we explicitly discuss for \(N = 3\) SUSY in \(d = 2\) the relation between the NLSUSY action (1) and the SYM action (7). The SUSY invariant relations (11) are constructed by extending constant terms of the auxiliary fields to \(\xi^I (i, j, \cdots = 1, 2, 3\) and \(I, J, \cdots = 1, 2, \cdots, \text{dim}G\)) with the indices of the gauge group \(G\) in \(D_{II} = D_{II}(\psi)\). By substituting SUSY invariant relations into the SYM action we show the NL/L SUSY relation (16) for the \(d = 2, N = 3\) SYM theory, in which the interaction terms vanish as (15) by means of the cancellations among the superon (NG-fermion) self-interaction terms.
(14). We anticipate similar results for the (pure) SYM theory in $d = 4$. The further investigations for the SYM theory with matter supermultiplet (the SUSY QCD theory) in NL/L SUSY relation are interesting and crucial.
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