Large $N$ Matrix Field Theories

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Abstract: We discuss aspects of recent novel approaches towards understanding the large $N$ limit of matrix field theories with local or global non-abelian symmetry.

The large $N$ limit of field theories whose local degrees of freedom are $N \times N$ matrices continues to intrigue physicists since its discovery by 't Hooft 25 years ago \[1\]. This is striking given the fact that no field theory with propagating matrix degrees of freedom has up to now been exactly solved by large $N$ techniques. Certainly, one reason for this continuing interest is the significant evidence that this limit indeed entails important simplifications. Furthermore, in the most interesting cases, physics appears to be not too different as compared to the finite $N$ case. Lastly, a new aspect of “large $N$” has appeared over the years: $N = \infty$ may not only be a reasonable and tractable approximation to some theory, but may also define the theory. Indeed, we know this to be true in some simple cases such as non-critical string theory, and now there are serious proposals for the case of “real” (i.e. critical, supersymmetric) string theory and 11-dimensional M-theory.

One of the above mentioned simplifications that seems to occur can be roughly summarized by the correspondence

$$ (N = \infty) \quad \simeq \quad \text{suitable zero-dimensional matrix model} \quad (N = \infty) \quad (1) $$

That is, it appears that every matrix field theory can be replaced by a suitable matrix integral such that at least some of the physical observables on both sides become identical at $N = \infty$. A first instance of this was discovered by Eguchi and Kawai \[2\], but we now know quite a few further examples, and a much more general principle, as loosely stated in the equivalence (1), appears to be at work.

In part 1 we will discuss a naive (presumably too naive) version of the correspondence (1) for Yang-Mills field theory, where we point out that a surprisingly simple reduced matrix model, contrary to initial expectations, proves to be well-defined for large $N$. Our main point here is that the existence of the proposed Yang-Mills matrix integrals has been overlooked in the past, and that they are, apart from other important applications,
an ideal laboratory for testing new large $N$ gauge theory techniques. Part 2 deals with
known exact equivalences \([\mathbb{I}]\) for interacting $D$-dimensional lattice field theories with
global $U(N)$ symmetry, and we outline, taking as a specific example a $D = 2$ hermitian
model, a very general procedure for bootstrapping the $N = \infty$ solution.

1 Local $U(N)$: Yang-Mills integrals

*In this part we discuss some aspects of results obtained in collaboration with W. Krauth and H. Nicolai and published in [3],[4],[5]. Consider $D$-dimensional pure $SU(N)$ Yang-Mills field theory and, inspired by the principle \([\mathbb{I}]\), reduce it by brute force to zero dimensions. The continuum path integral, involving traceless hermitian gauge connections $X_\mu$, becomes an ordinary matrix integral:

$$Z_{D,N} = \int \prod_{A=1}^{N^2-1} \prod_{\mu=1}^{D} \frac{dX_A^\mu}{\sqrt{2\pi}} \exp \left[ \frac{1}{2} \text{Tr} [X_\mu, X_\nu] [X_\mu, X_\nu] \right].$$

(2)

Note that gauge fixing is no longer required here, since the overcounting of gauge-equivalent
configurations involves merely a factor of the compact, finite volume of the gauge group;
space time has become a point (or more precisely, an infinitesimal torus, since the “point”
still keeps a sense of the $D$ directions.). Now, as was explained in \([3]\), the integral eq.(2)
still “knows” something about $D$-dimensional space-time. Indeed, shifting

$$X_\mu \rightarrow P_\mu + X_\mu$$

(3)

by diagonal matrices $P_\mu = \text{diag}(p_1^\mu, \ldots, p_N^\mu)$ we formally recover Feynman rules which look
like the ordinary ones except that the momentum integrations are replaced by sums over
discretized momenta $p_i^\mu - p_j^\mu$. As $N \rightarrow \infty$ one might hope that the sums turn back into
loop integrals, motivating the correspondence \([\mathbb{I}]\). Now in \([3]\) a somewhat complicated
quenching and gauge fixing procedure was introduced in order to ensure the recovery of
the field theory. Indeed it would seem at first sight that the integral eq.(2) is meaningless
without the procedure of \([3]\) since there are unconstrained flat directions in integration space, due to mutually commuting matrices. However, the Monte Carlo results of \([4]\) suggest

**Proposition 1a:** The Yang-Mills integrals $Z_{D,N}$ exist iff $N > \frac{D}{D-2}$.

It would be quite important to find methods enabling one to rigorously prove this statement, or even calculate the partition sums $Z_{D,N}$. Some important analytic evidence comes from the perturbative calculations of \([4]\). For $SU(2)$, a proof of the proposition, as well as an analytic expression for $Z_{D,2}$, is known.

The matrix integrals eq.(2) have beautiful supersymmetric extensions in dimensions $D = 4, 6, 10$. These read

$$Z^N_{D,N} := \int \prod_{A=1}^{N^2-1} \left( \prod_{\mu=1}^{D} \frac{dX_A^\mu}{\sqrt{2\pi}} \right) \left( \prod_{\alpha=1}^{N} d\Psi_{A}^{\alpha} \right) \exp \left[ \frac{1}{2} \text{Tr} [X_\mu, X_\nu] [X_\mu, X_\nu] + \text{Tr} \Psi_{\alpha}^{\mu} \Gamma^{\mu}_{\alpha\beta} X_\mu \Psi_{\beta} \right].$$

(4)

where we have supersymmetrically added $N = 2(D-2)$ hermitian fermionic matrices
$\Psi_{\alpha}$ to the models. The $D = 10$ model corresponds to the dimensional reduction of
the maximally supersymmetric conformal $D = 4, N = 4$ Yang-Mills field theory to zero
dimensions. It is also the crucial ingredient in the IKKT model for IIB superstrings \[8\], which however, instead of taking the large \(N\) limit, sums \(Z_{10,N}^{16}\) over all values of \(N\).

Following the \(SU(2)\) calculations of \[9\], the perturbative arguments of \[10\], the arguments of \[11\], the calculations of \[12\], and our Monte Carlo work, we are led to

**Proposition 1b:** The susy Yang-Mills integrals \(Z_{4,N}^{4}, Z_{6,N}^{8}, Z_{10,N}^{16}\) exist iff \(N \geq 2\).

The analytic results of these integrals are believed to be known, and a rigorous mathematical proof would be welcome.

It is interesting to understand the similarities and differences of these little studied “new” matrix models eqs.(2),(4), whose existence has been missed until recently, in relation to the conventional “old” matrix models of Wigner type. A crucial quantity in the old matrix models is the distribution of eigenvalues of the random matrices. An interesting novel feature of the new matrix models is the fact that, at finite \(N\), only a finite number of one-matrix moments exist. The numerical results agree with perturbative power-counting arguments, and one is led, for the bosonic models eq.(2), to

**Proposition 2a:** \(\frac{1}{N}\text{Tr} X_1^{2k} < \infty\) iff \(k < N(D-2) - \frac{3}{2}D + 2\),

while in the supersymmetric cases \(D = 4, 6, 10\) eq.(4) one has

**Proposition 2b:** \(\frac{1}{N}\text{Tr} X_1^{2k} < \infty\) iff \(k < D - 3\).

Once again, except for \(SU(2)\), rigorous proofs of these conjectures are missing. These findings indicate that in the new matrix models the density of eigenvalues falls off much slower (powerlike) than in the old ones (exponential). As \(N \to \infty\) the bosonic densities behave once again rather conservatively (infinitely many moments exist), while for the susy densities the behavior indicated in proposition 2b is independent of \(N\).

A much more difficult question is whether these models might lead to a “self-quenching” effect where a background \(P_\mu\) (in eq.(3)), bearing some resemblance to real Yang-Mills theory, is dynamically generated as \(N \to \infty\).

The above Yang-Mills integrals have many applications even at finite \(N\) (for a recent unexpected one see \[13\]); however, here we would like to stress that they constitute an ideal laboratory for developing new large \(N\) techniques aimed at making progress with ‘t Hooft’s large \(N\) QCD \[1\].

## 2 Global \(U(N)\): Master partitions

The problem of finding the \(N = \infty\) solution to matrix field theories has not even been solved in the presumably simpler case of models with a global \(U(N)\) symmetry. The main obstacle has been that no systematic procedure was known to reduce the local number of degrees of freedom from \(\mathcal{O}(N^2)\) to \(\mathcal{O}(N)\). In \[13\] we outlined a general approach for achieving such a reduction for any field theory with a global matrix symmetry. Let us sketch the idea in the specific example of an interacting \(D = 2\) hermitian scalar field theory. It is convenient to put the theory on a lattice:

\[
\mathcal{Z} = \int \prod_x \mathcal{D}M(x) \ e^{-S},
\]
\[ S = N \text{Tr} \sum_x \left[ \frac{1}{2} M(x)^2 + \frac{g}{4} M(x)^4 - \frac{\beta}{2} \sum_{\mu=1,2} [M(x)M(x + \hat{\mu}) + M(x)M(x - \hat{\mu})] \right], \] (5)

where the field variables are \( N \times N \) hermitian matrices \( M(x) \) defined on the square lattice sites \( x \) and \( \hat{\mu} \) denotes the unit vector in the \( \mu \)-direction. The measure is the usual flat measure on hermitian matrices. The first step consists in applying the reduction principle \( (\ref{red}) \). Naively reducing the system as in the previous section down to a single point results in an ordinary one-matrix model where the information on the 2D lattice is lost. A more careful reduction has to hide the propagation on the lattice in group space; here we will use the beautiful procedure of “twisting”, see \( \text{(14)} \) and references therein. Using the \( N \times N \) Weyl-t Hooft matrices

\[ P = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 1 \\ \cdots & \cdots & \ddots & \ddots & \cdots \\ 1 & \cdots & \cdots & 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & \omega & \cdots & \cdots & \omega^{N-2} \\ \omega & \cdots & \ddots & \ddots & \cdots \\ \cdots & \ddots & \ddots & \ddots & \cdots \\ \cdots & \ddots & \ddots & \ddots & \cdots \\ \omega^{N-2} & \cdots & \cdots & \cdots & \omega \\ \omega^{N-1} \end{pmatrix}, \] (6)

where \( \omega = \exp{\frac{2\pi i}{N}} \) and \( PQ = \omegaQP \), one can show by a Fourier transform in matrix index space that the one-matrix integral

\[ Z = \int \mathcal{D} M \exp N \text{Tr} \left[ -\frac{1}{2} M^2 - \frac{g}{4} M^4 + \beta (M P M P^\dagger + M Q M Q^\dagger) \right], \] (7)

has the same vacuum energy as the path integral eq.(\ref{red}).

As a second step we need to reduce the number of variables from \( N^2 \) to \( N \). The brute force approach would be to diagonalize the matrix \( M \) and perform the integration over the unitary diagonalizing matrix. One would then obtain an effective action for the \( N \) eigenvalues of \( M \). However, calculations at small \( N \) show that this effective action is extremely complicated in the case at hand. On the other hand, if we change variables from the eigenvalues to \textit{partitions}, corresponding to a Fourier transform in group space, something very interesting happens. The \( N \) variables dual to the \( N \) eigenvalues are the Young weights \( h_i = N - i + m_i, i = 1, \ldots, N \), where the \( m_i \) are the lengths of the \( i \)-th row in the Young diagram corresponding to the partition. Denoting the partitions through \( h = (h_1, \ldots, h_N) \), the dual representation of the integral eq.\((\ref{red})\) is found to be

\[ Z = \sum_h \mathcal{I}_h \mathcal{L}_h \beta^{|h|}, \] (8)

where instead of an integration over the \( N \times N \) matrix \( M \) we now have a sum over all partitions \( h \) of the non-negative integer \( |h| = 0, 1, 2, \ldots \). Here \( \mathcal{I}_h \) contains all the information on the interaction, and essentially requires the general correlation function of the \( U(N) \)-invariant one-matrix integral

\[ \mathcal{I}_h = N^{|h|} \prod_{i=1}^N \frac{(N-i)!}{h_i!} \int \mathcal{D} M \exp N \text{Tr} \left[ -\frac{1}{2} M^2 - \frac{g}{4} M^4 \right] \chi_h(M), \] (9)

which are known. Here \( \chi_h(M) \) are the Schur functions on \( h \) which are nothing but a complete set of class functions (non-abelian Fourier modes) on the group. The information on the lattice is contained in the \textit{lattice polynomials}

\[ \mathcal{L}_h = \exp \frac{1}{N} \text{Tr} \left( \partial P \partial P^\dagger + \partial Q \partial Q^\dagger \right) \chi_h(J) \bigg|_{J=0}. \] (10)
Here $\partial$ denotes the $N \times N$ matrix differential operator whose matrix elements are $\partial_{ji} = \frac{\partial}{\partial J_{ij}}$. The $L_h$ are easily shown to be polynomials in the variable $\frac{1}{N}$ of maximal degree $\frac{1}{2}|h| - 1$.

Now the result of this harmonic analysis is that the terms to be summed over partitions in eq.(8) *factorize* into a piece $I_h$ containing the information on the local interaction and and the piece $L_h$ containing the information on the space-time structure. Since there are only $N$ variables $h_i$ we expect the sum eq.(8) to be dominated at $N = \infty$ by a saddle point, i.e. an effective master partition. In a third and final step we will need to write the full system of bootstrap equations for the saddle point. This will require a deeper analysis of the lattice polynomials. But it should be clear that the problem of solving the large $N$ lattice field theory has been reformulated in a rather non-trivial way: In fact, the interacting theory (i.e. $g \neq 0$ in eq.(3)) is no harder to solve in this dual space of Young weights than the free theory ($g = 0$).

**Acknowledgements**

I thank W. Krauth and H. Nicolai for fruitful collaboration, and J. Hoppe, V. A. Kazakov, I. K. Kostov, and J. Plefka for useful discussions. This work was supported in part by the EU under Contract FMRX-CT96-0012.

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