SINGULARITY FORMATION TO THE NONHOMOGENEOUS MAGNETO-MICROPOLAR FLUID EQUATIONS

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Abstract. We consider the Cauchy problem of nonhomogeneous magneto-micropolar fluid equations with zero density at infinity in the entire space $\mathbb{R}^2$. We show that for the initial density allowing vacuum, the strong solution exists globally if a weighted density is bounded from above. It should be noted that our blow-up criterion is independent of micro-rotational velocity and magnetic field.

1. Introduction. The three-dimensional (3D for short) nonhomogeneous magneto-micropolar fluid equations (see [10]) are given by

$$
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - (\mu_1 + \xi)\Delta u + \nabla P &= 2\xi \nabla \times w + b \cdot \nabla b, \\
b_t - \nu \Delta b + u \cdot \nabla b - b \cdot \nabla u &= 0, \\
(\rho w)_t + \text{div}(\rho u \otimes w) + 4\xi w - \mu_2 \Delta w - (\mu_2 + \lambda)\nabla \text{div} w &= 2\xi \nabla \times u, \\
\text{div} u &= \text{div} b = 0,
\end{align*}
$$

where $\rho$, $u$, $w$, $b$, and $P$ denote the density, velocity, micro-rotational velocity, magnetic field, and pressure of the fluid, respectively. The constants $\mu_1, \mu_2, \nu, \lambda, \xi$ stand for the viscosity coefficients of the fluid satisfying $\mu_1, \mu_2, \nu, \xi > 0$, $\mu_1 \geq 2\xi$, and $2\mu_2 + 3\lambda \geq 0$.

In the special case, when

$$
\begin{align*}
\rho &= \rho(x_1, x_2, t), \\
u &= (u^1(x_1, x_2, t), u^2(x_1, x_2, t), 0), \\
P &= P(x_1, x_2, t), \\
b &= (b^1(x_1, x_2, t), b^2(x_1, x_2, t), 0), \\
w &= (0, 0, w(x_1, x_2, t)),
\end{align*}
$$

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the system (1.1) reduces to the 2D nonhomogeneous magneto-micropolar fluid equations

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - (\mu_1 + \xi) \Delta u + \nabla P &= -2\xi \nabla \perp w + b \cdot \nabla b, \\
b_t - \nu \Delta b + u \cdot \nabla b - b \cdot \nabla u &= 0, \\
(\rho w)_t + \text{div}(\rho w u) + 4\xi w - \mu_2 \Delta w &= 2\xi \nabla \perp \cdot u, \\
\text{div} u &= \text{div} b = 0,
\end{aligned}
\]  

(1.2)

where \( u = (u^1, u^2) \) is a 2D vector with the corresponding scalar vorticity given by

\[ \nabla \perp \cdot u = \partial_1 u^2 - \partial_2 u^1, \]

while \( w \) represents a scalar function with

\[ \nabla \perp w = (-\partial_2 w, \partial_1 w). \]

The system (1.2) is supplemented with the initial condition

\[ (\rho, \rho u, b, \rho w)(x, 0) = (\rho_0, \rho_0 u_0, b_0, \rho_0 w_0)(x), \quad x \in \mathbb{R}^2, \]  

(1.3)

and the far field behavior

\[ (\rho, u, b, w)(x, t) \to (0, 0, 0, 0), \quad \text{as } |x| \to +\infty, \quad t > 0. \]  

(1.4)

Magneto-micropolar fluid equations describe the motion of electrically conducting micropolar fluids in the presence of a magnetic field. Due to the profound physical background and important mathematical significance, a great deal of attention has been focused on studying well-posedness of solutions to the magneto-micropolar fluid equations, both from a pure mathematical point of view and for concrete applications. For more background, we refer to [1,2] and references therein.

When \( \rho \) is constant, which means the fluid is homogeneous, the magneto-micropolar fluid equations have been extensively studied, such as existence and stability of solutions [3, 15], large time behavior of solutions [7, 17], blow-up criterion of solutions [20, 23], and so on. Recently, many authors investigated global existence and regularity of 2D homogeneous magneto-micropolar fluid equations with partial dissipation. Ma [12] showed the global existence and regularity of classical solutions to 2D magneto-micropolar fluid equations with mixed partial dissipation, magnetic diffusion, and angular viscosity. By making use of maximal regularity of heat operator and Littlewood-Paley decomposition theory, Shang and Zhao [16] studied the global regularity of classical solutions to 2D magneto-micropolar fluid equations with only micro-rotational velocity dissipation and magnetic diffusion. For other studies of homogeneous magneto-micropolar fluid equations, please refer to [14,19,22] and references therein.

When \( \rho \) is not constant, the system (1.1) is so-called nonhomogeneous magneto-micropolar fluid equations. For the initial density allowing vacuum states, imposing a compatibility condition on the initial data, Zhang and Zhu [24] showed the global existence of strong solution to the three-dimensional initial boundary value problem provided that some smallness condition holds. Using Desjardins’ interpolation inequality (see [5, Lemma 1]), Zhong [26] investigated the global existence and exponential decay of strong solution to the 2D initial boundary value problem with general large data and vacuum. Moreover, there is no need to impose the additional compatibility condition for the initial data via time-weighted techniques. Very recently, by fully exploiting the special structure of the system and weighted energy method, Zhong [27] showed the local existence of strong solutions to the Cauchy
problem of (1.2) in \( \mathbb{R}^2 \). In this paper, we will investigate the structure of possible singularities of strong solutions obtained in [27].

Before stating our main result, we first explain the notations and conventions used throughout this paper. For \( r > 0 \), set
\[
B_r := \{ x \in \mathbb{R}^2 \mid |x| < r \}, \quad \int_{B_r} \cdot dx := \int_{\mathbb{R}^2} \cdot dx.
\]

For \( 1 \leq p \leq \infty \) and integer \( k \geq 0 \), the standard Sobolev spaces are denoted by:
\[
L^p = L^p(\mathbb{R}^2), \quad W^{k,p} = W^{k,p}(\mathbb{R}^2), \quad H^k = H^k(\mathbb{R}^2), \quad D^{k,p} = \{ u \in L^1_{\text{loc}} | \nabla^k u \in L^p \}.
\]

Now, we wish to define precisely what we mean by strong solutions.

**Definition 1.1** (Strong solutions). \((\rho, u, P, b, w)\) is called a strong solution to (1.2)–(1.4) in \( \mathbb{R}^2 \times (0, T) \), if for some \( q > 2 \) and \( a > 1 \),
\[
\begin{aligned}
\rho &\geq 0, \quad \rho \in C([0, T]; L^1 \cap H^1 \cap W^{1,q}), \\
\rho \bar{x}^a &\in L^\infty(0, T; L^1 \cap H^1 \cap W^{1,q}), \\
\sqrt{\rho}u, \nabla u, \sqrt{\rho}u_t, \sqrt{\nabla}P, \sqrt{\nabla}u &\in L^\infty(0, T; L^2), \\
\sqrt{\rho}w, \nabla w, \sqrt{\nabla}w_t, \sqrt{\nabla}w &\in L^\infty(0, T; L^2), \\
b, b\bar{x}^\gamma, \nabla b, \sqrt{\nabla}b &\in L^\infty(0, T; L^2), \\
\nabla u, \nabla w &\in L^2(0, T; H^1) \cap L^{\frac{2+\gamma}{\gamma}}(0, T; W^{1,q}), \\
\nabla P &\in L^2(0, T; L^2) \cap L^{\frac{2+\gamma}{\gamma}}(0, T; L^q), \\
\nabla b &\in L^2(0, T; H^1), \quad \nabla b\bar{x}^\gamma &\in L^2(0, T; L^2), \\
\sqrt{\nabla}u_t, \sqrt{\nabla}b\bar{x}^\gamma, \sqrt{\nabla}u_t, \sqrt{\nabla}b_t, \sqrt{\rho}w_t &\in L^2(\mathbb{R}^2 \times (0, T)),
\end{aligned}
\]
and \((\rho, u, P, b, w)\) satisfies both (1.2) almost everywhere in \( \mathbb{R}^2 \times (0, T) \) and (1.3) almost everywhere in \( \mathbb{R}^2 \). Here
\[
\bar{x} := (3 + |x|^2)^{\frac{1}{2}} \log^{1+\eta_0}(3 + |x|^2)
\]
and \( \eta_0 \) is a positive number.

Without loss of generality, we assume that the initial density \( \rho_0 \) satisfies
\[
\int_{\mathbb{R}^2} \rho_0 dx = 1,
\]
which implies that there exists a positive constant \( N_0 \) such that
\[
\int_{B_{N_0}} \rho_0 dx \geq \frac{1}{2} \int_{\mathbb{R}^2} \rho_0 dx = \frac{1}{2}.
\]

Our main result can be stated as follows:

**Theorem 1.1.** In addition to (1.6) and (1.7), assume that the initial data \((\rho_0 \geq 0, u_0, b_0, w_0)\) satisfies for any given numbers \( a > 1 \) and \( q > 2 \),
\[
\begin{aligned}
\rho_0 \bar{x}^a &\in L^1 \cap H^1 \cap W^{1,q}, \quad \sqrt{\rho_0}u_0 \in L^2, \quad \nabla u_0 \in L^2, \\
b_0 \bar{x}^\gamma &\in L^2, \quad \nabla b_0 \in L^2, \quad \text{div} \, u_0 = \text{div} \, b_0 = 0,
\end{aligned}
\]
Let \((\rho, u, P, b, w)\) be a strong solution to the problem (1.2)–(1.4). If \( T^* < \infty \) is the maximal time of existence for that solution, then for any \( \delta > 0 \), we have
\[
\lim_{T \to T^*} \| \rho \bar{x}^a \|_{L^{\infty}(0, T; L^\infty(\mathbb{R}^2))} = \infty.
\]
Remark 1.1. The local existence and uniqueness of strong solutions with initial data as in Theorem 1.1 was established in [27]. Hence, the maximal time $T^*$ is well-defined.

Remark 1.2. The conclusion in Theorem 1.1 is somewhat surprising since the criterion (1.9) is independent of micro-rotational velocity and magnetic field. The result indicates that the mechanism of blowup of nonhomogeneous magneto-micropolar fluid equations is the same as the nonhomogeneous micropolar fluid equations [26, Theorem 1.1].

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Sections 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries. In this section, we will recall some known facts and elementary inequalities which will be used frequently later.

We begin with the following Gagliardo-Nirenberg inequality (see [13]).

\begin{align*}
\|f\|_{L^p(B_1)}^p & \leq C \|f\|_{L^2(B_1)}^2 \|\nabla f\|_{L^2(B_1)}^{p-2}, \\
\|g\|_{L^{\infty}(B_1)} & \leq C \|g\|_{L^{\frac{p}{2}}(B_1)}^{p-2} \|\nabla g\|_{L^{\infty}(B_1)}^{p-2}.
\end{align*}

The following weighted $L^m$ bounds for elements of the Hilbert space $\mathcal{D}^{1,2}(\mathbb{R}^2) := \{v \in H^1_{loc}(\mathbb{R}^2) | \nabla v \in L^2(\mathbb{R}^2)\}$ can be found in [9, Theorem B.1].

Lemma 2.2. For $m \in [2, \infty)$ and $\theta \in (1 + \frac{m}{2}, \infty)$, there exists a positive constant $C$ such that for all $v \in \mathcal{D}^{1,2}(\mathbb{R}^2)$,

\begin{equation}
\left( \int_{\mathbb{R}^2} \frac{|v|^m}{3 + |x|^2} (\log (3 + |x|^2))^{-\theta} dx \right)^{\frac{1}{m}} \leq C \|v\|_{L^2(B_1)} + C \|\nabla v\|_{L^2(\mathbb{R}^2)}. \tag{2.1}
\end{equation}

The combination of Lemma 2.2 and the Poincaré inequality yields the following useful results on weighted bounds, whose proof can be found in [6, Lemma 2.4].

Lemma 2.3. Let $\tilde{x}$ be as in (1.5). Assume that $\rho \in L^1(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)$ is a non-negative function such that

\begin{align*}
\|\rho\|_{L^1(B_{N_1})} & \geq M_1, \\
\|\rho\|_{L^1(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)} & \leq M_2,
\end{align*}

for positive constants $M_1, M_2,$ and $N_1 \geq 1$. Then for $\varepsilon > 0$ and $\eta > 0$, there is a positive constant $C$ depending only on $\varepsilon, \eta, M_1, M_2,$ and $N_1$, such that every $v \in \mathcal{D}^{1,2}(\mathbb{R}^2)$ satisfies

\begin{equation}
\|v\tilde{x}^{-\eta}\|_{L^{(2+\varepsilon)/\eta}(\mathbb{R}^2)} \leq C \|\sqrt{\rho} v\|_{L^2(\mathbb{R}^2)} + C \|\nabla v\|_{L^2(\mathbb{R}^2)}, \tag{2.2}
\end{equation}

with $\tilde{\eta} = \min\{1, \eta\}$.

Finally, we state a critical Sobolev inequality of logarithmic type, which is originally due to Brézis-Wainger [4]. The reader can refer to [8, Lemma 2.5] for the proof.
Lemma 2.4. Assume that \( f \in L^2(s, t; D^{1,2} \cap W^{1,q} (\mathbb{R}^2)) \) with some \( q > 2 \) and \( 0 \leq s < t \leq \infty \), then there is a constant \( C > 0 \) independent of \( s \) and \( t \) such that

\[
\| f \|_{L^2(s, t; L^\infty)}^2 \leq C + C \left( \| \nabla f \|_{L^2(s, t; L^2)}^2 + \| f \|_{L^2(s, t; L^8)}^2 \right) \log(3 + \| f \|_{L^2(s, t; W^{1,8})}^2). \tag{2.3}
\]

3. Proof of Theorem 1.1. Let \( (\rho, u, P, b, w) \) be a strong solution described in Theorem 1.1. Suppose that (1.9) were false, that is, there exists a constant \( M_0 > 0 \) such that

\[
\lim_{T \to T} \| \rho \|_{L^\infty(0, T; L^\infty)} \leq M_0 < \infty. \tag{3.1}
\]

We begin with the following standard energy estimate for \( 0 \leq t < \infty \), which gives

\[ \| f \|_{L^2(s, t; L^\infty)}^2 \leq C = 4 \left( \| f \|_{L^2(s, t; L^8)}^2 + \| f \|_{L^2(s, t; L^4)}^2 \right). \tag{3.2} \]

where (and in what follows) \( C \) denotes a generic positive constant depending only on \( \mu_1, \mu_2, \xi, \nu, \lambda, q, a, \gamma_0, M_0, N_0, T^*, \) and the initial data.

Proof. 1. Since \( \text{div} \, u = 0 \), it is easy to deduce from (1.2) that (see [9, Theorem 2.1]),

\[
\sup_{0 \leq t \leq T} \| \rho \|_{L^\infty(0, T; L^\infty)} \leq C. \tag{3.3}
\]

2. Multiplying (1.2)_2 by \( u \) and integrating by parts, we obtain from (1.2)_1 and (1.2)_5 that

\[
\frac{1}{2} \frac{d}{dt} \int \rho |u|^2 dx + (\mu_1 + \xi) \int |\nabla u|^2 dx = -2\xi \int \nabla^\perp \cdot u dx + \int b \cdot \nabla b \cdot u dx. \tag{3.4}
\]

Multiplying (1.2)_3 by \( b \) and integrating by parts, we find that

\[
\frac{1}{2} \frac{d}{dt} \int |b|^2 dx + \nu \int |\nabla b|^2 dx + \int b \cdot \nabla b \cdot u dx = 0. \tag{3.5}
\]

Testing (1.2)_4 by \( w \) and using (1.2)_1 and (1.2)_5, we get

\[
\frac{1}{2} \frac{d}{dt} \int \rho w^2 dx + \mu_2 \int |\nabla w|^2 dx + 4\xi \int |w|^2 dx = 2\xi \int \nabla^\perp \cdot u w dx. \tag{3.6}
\]

Combining (3.4)–(3.6) together, we have

\[
\frac{1}{2} \frac{d}{dt} \left( \| \sqrt{\rho} u \|_{L^2}^2 + \| b \|_{L^2}^2 + \| \sqrt{\rho} w \|_{L^2}^2 \right) + \left( (\mu_1 + \xi) \| \nabla u \|_{L^2}^2 + \nu \| \nabla b \|_{L^2}^2 + \mu_2 \| \nabla w \|_{L^2}^2 + 4\xi \| w \|_{L^2}^2 \right) \leq 0.
\]

which gives

\[
\frac{d}{dt} \left( \| \sqrt{\rho} u \|_{L^2}^2 + \| b \|_{L^2}^2 + \| \sqrt{\rho} w \|_{L^2}^2 \right) + (\mu_1 \| \nabla u \|_{L^2}^2 + \nu \| \nabla b \|_{L^2}^2 + \mu_2 \| \nabla w \|_{L^2}^2) \leq 0.
\]
Integrating the above inequality over \((0, T)\) leads to
\[
\sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\sqrt{\rho}w\|_{L^2}^2 \right)
+ \int_0^T \left( \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right) ds \leq C,
\]
which together with (3.3) yields (3.2) and completes the proof of Lemma 3.1.

Next, we will give some spatial weighted estimates on the density and the magnetic field.

**Lemma 3.2.** Under the condition (3.1), it holds that for any \(T \in [0, T^*)\),
\[
\sup_{0 \leq t \leq T} \left( \|\rho \bar{x}\|_{L^2} + \|\bar{b}\|_{L^2} + \int_0^T \|\nabla \bar{b}\|_{L^2}^2 dt \right) \leq C. \tag{3.7}
\]

**Proof.** 1. We obtain from [27, Lemma 3.2] that
\[
\inf_{0 \leq t \leq T} \int_{B_{N_0}} \rho dx \geq \frac{1}{4}. \tag{3.9}
\]

The combination of (3.9), (3.2), and (2.2) implies that for \(\varepsilon > 0, \eta > 0, v \in \dot{D}^{1,2}(\mathbb{R}^2)\), and \(\bar{\eta} = \min\{1, \eta\}\)
\[
\|v^{\bar{\eta}}\|_{L^{\frac{3\bar{\eta}}{2}}} \leq C(\varepsilon, \eta) \|\sqrt{\rho}v\|_{L^2}^2 + C(\varepsilon, \eta) \|\nabla v\|_{L^2}^2, \tag{3.10}
\]
which along with (3.2) and (3.9) yields
\[
\|\rho^\eta v\|_{L^{\frac{2+\eta}{\eta}}} \leq C\|\rho^\eta \bar{x}^{-\eta}\|_{L^1}^{\frac{3\eta}{2(2+\eta)}} \|v^{\bar{\eta}}\|_{L^{\frac{3\bar{\eta}}{2}}} + C\|\rho^\eta \bar{x}^{-\eta}\|_{L^1}^{\frac{3\eta}{2(2+\eta)}} \|\bar{v}\|_{L^{\frac{2+\eta}{\eta}}}
\leq C \left( \int \rho^{\frac{4(2+\eta)-\eta}{2\eta}} dx \right)^{\frac{3\eta}{2(2+\eta)}} \|v^{\bar{\eta}}\|_{L^{\frac{3\bar{\eta}}{2}}} + C\|\rho^\eta \bar{x}^{-\eta}\|_{L^1}^{\frac{3\eta}{2(2+\eta)}} \|\bar{v}\|_{L^{\frac{2+\eta}{\eta}}}
\leq C \|\sqrt{\rho}v\|_{L^2} + C\|\nabla v\|_{L^2}. \tag{3.11}
\]

In particular, this together with (3.2) and (3.10) yields
\[
\|\rho^\eta u\|_{L^{\frac{2+\eta}{\eta}}} + \|u^{\bar{\eta}}\|_{L^{\frac{2+\eta}{\eta}}} \leq C(1 + \|\nabla u\|_{L^2}), \tag{3.12}
\]
\[
\|\rho^\eta w\|_{L^{\frac{2+\eta}{\eta}}} + \|w^{\bar{\eta}}\|_{L^{\frac{2+\eta}{\eta}}} \leq C(1 + \|\nabla w\|_{L^2}). \tag{3.13}
\]

2. Multiplying (1.2) by \(u_t\) and integrating by parts, one has
\[
\frac{\mu_1 + \xi}{2} \int \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \int \|\rho u_t\|_{L^2}^2 dx
= \int b \cdot \nabla b \cdot u_t dx - \int \rho u \cdot \nabla u \cdot u_t dx - 2\xi \int \nabla^\perp w \cdot u_t dx
= \int b \cdot \nabla b \cdot u_t dx - \int \rho u \cdot \nabla u \cdot u_t dx + 2\xi \int \nabla^\perp \cdot u_t w dx
\]
Multiplying (1.2) by \( w_t \) and integrating by parts, we get
\[
\frac{1}{2} \frac{d}{dt} \int \left( (\mu_2 |w|^2 + 4\xi w^2) \right) dx + \int \rho w_t^2 dx = -\int \rho u \cdot \nabla w \cdot w_t dx + 2\xi \int \nabla^\perp \cdot u w_t dx,
\]
which combined with (3.14) leads to
\[
\frac{1}{2} \frac{d}{dt} \int \left( (\mu_1 + \xi)|\nabla u|^2 + \mu_2 |w|^2 + 4\xi w^2 - 4\xi \nabla^\perp \cdot u w \right) dx + \int \rho |u|^2 dx + \int \rho w_t^2 dx
\]
\[
= \int \mathbf{b} \cdot \nabla \mathbf{b} \cdot u_t dx - \int \rho u \cdot \nabla u \cdot u_t dx - \int \rho u \cdot \nabla w \cdot w_t dx
\]
\[
= -\frac{d}{dt} \int \mathbf{b} \cdot \nabla u \cdot b dx + \int \mathbf{b}_t \cdot \nabla u \cdot b dx + \int \mathbf{b} \cdot \nabla u \cdot b_t dx
\]
\[
- \int \rho u \cdot \nabla u \cdot u_t dx - \int \rho u \cdot \nabla w \cdot w_t dx.
\]
Thus, we have
\[
\frac{1}{2} B'(t) + \int \rho |u|^2 dx + \int \rho w_t^2 dx
\]
\[
= \int \mathbf{b}_t \cdot \nabla \mathbf{b} \cdot b dx + \int \mathbf{b} \cdot \nabla \mathbf{b} \cdot b_t dx - \int \rho u \cdot \nabla u \cdot u_t dx - \int \rho u \cdot \nabla w \cdot w_t dx
\]
\[
= \sum_{i=1}^{4} J_i,
\]
where
\[
B(t) := \int \left( (\mu_1 + \xi)|\nabla u|^2 + \mu_2 |w|^2 + 4\xi w^2 - 4\xi \nabla^\perp \cdot u w + \mathbf{b} \cdot \nabla u \cdot b \right) dx
\]
satisfies for some \( C_1 > 0 \),
\[
\frac{\mu_1}{2} \| \nabla u \|_{L^2}^2 + \mu_2 \| \nabla w \|_{L^2}^2 - C_1 \| \nabla b \|_{L^2}^2 - C
\]
\[
\leq B(t) \leq C \| \nabla u \|_{L^2}^2 + C \| \nabla w \|_{L^2}^2 + C \| w \|_{L^2}^2 + C \| \nabla b \|_{L^2}^2 + C,
\]
due to Hölder’s inequality, Gagliardo-Nirenberg inequality, and (3.2).

Multiplying (1.2) by \( 4|\mathbf{b}|^2 \mathbf{b} \) and integrating the resulting equation over \( \mathbb{R}^2 \), we derive from Gagliardo-Nirenberg inequality that
\[
\frac{d}{dt} \int |\mathbf{b}|^4 dx + 12\nu \int |\mathbf{b}|^2 |\nabla \mathbf{b}|^2 dx \leq C \int |\nabla \mathbf{u}| |\mathbf{b}|^4 dx
\]
\[
\leq C \| \nabla \mathbf{u} \|_{L^2} \| \mathbf{b} \|_{L^4}^4
\]
\[
\leq C \| \nabla \mathbf{u} \|_{L^2} \| \mathbf{b} \|_{L^2} \| \nabla \mathbf{b} \|_{L^2}^2
\]
\[
\leq \epsilon \| \nabla \mathbf{b} \|_{L^2}^2 + C \| \nabla \mathbf{u} \|_{L^2} \| \mathbf{b} \|_{L^4}^4,
\]
which along with Gronwall’s inequality and (3.2) yields

\[
\sup_{0 \leq t \leq T} \|b\|_{L^p} + \int_0^T \int |b|^2 |\nabla b|^2 dx dt \leq C
\]

(3.18)

after choosing \( \varepsilon \) suitably small. By virtue of (1.2), Hölder’s inequality, (3.18), (3.7), and Gagliardo-Nirenberg inequality, we derive that for any \( \varepsilon > 0 \),

\[
|J_1| + |J_2| \leq C \int |\nabla u| |b| (|\Delta b| + |\nabla b| + |b| |\nabla u|) dx
\]

\[
\leq C \|b\|_{L^4} \|\nabla u\|_{L^4} \|\Delta b\|_{L^2} + C \|b\|_{L^2}^2 \|u\|_{L^\infty} \|\nabla u\|_{L^4} \|\nabla b\|_{L^2} + C \|\nabla u\|_{L^4}^2
\]

\[
\leq \frac{\varepsilon}{4} \|\nabla^2 b\|^2_{L^2} + C \|u\|_{L^2}^2 \|\Delta u\|_{L^\infty} \|\nabla u\|_{L^4} \|\nabla b\|_{L^2} + C \|\nabla u\|_{L^4}^2
\]

(3.19)

We derive from Cauchy-Schwarz inequality and (3.1) that

\[
|J_3| \leq \frac{1}{2} \int \rho |u|^2 dx + \frac{1}{2} \int \rho |u|^2 |\nabla u|^2 dx
\]

\[
\leq \frac{1}{2} \int \rho |u|^2 dx + \frac{1}{2} \rho \|x\|^2 \|\nabla u\|^2_{L^\infty}
\]

\[
\leq \frac{1}{2} \int \rho |u|^2 dx + C \|u\|_{L^2}^2 \|\nabla u\|^2_{L^2}.
\]

(3.20)

Similarly, one has

\[
|J_4| \leq \frac{1}{2} \int \rho \omega^2 dx + C \|u\|_{L^\infty}^2 \|\nabla w\|^2_{L^2}.
\]

(3.21)

Hence, inserting (3.19)-(3.21) into (3.16) gives

\[
B'(t) + \|\sqrt{\rho} u\|^2_{L^2} + \|\sqrt{\rho} \omega\|^2_{L^2}
\]

\[
\leq C \left( 1 + \|u\|_{L^\infty}^2 \|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2} + \|\nabla w\|^2_{L^2} \right)
\]

\[
+ \nu \|\nabla^2 b\|^2_{L^2} + \varepsilon \|\nabla^2 u\|^2_{L^2}
\]

(3.22)

with \( \delta = \min\{\delta, \alpha\} \).

3. Notice that \( (\rho, u, P, w) \) satisfies the following Stokes system

\[
\begin{aligned}
-(\mu_1 + \xi) \Delta u + \nabla P &= -\rho u_t - \rho u \cdot \nabla u + b \cdot \nabla b - 2\xi \nabla^2 w, \quad x \in \mathbb{R}^2, \\
\text{div } u &= 0, \quad x \in \mathbb{R}^2, \\
u(x) &\to 0, \quad |x| \to \infty,
\end{aligned}
\]

(3.23)

applying regularity theory of Stokes system to (3.23) yields that for any \( p \in [2, \infty) \),

\[
\|\nabla^2 u\|_{L^p} + \|\nabla P\|_{L^p}
\]

\[
\leq C \|\rho u_t\|_{L^p} + C \|\rho u \cdot \nabla u\|_{L^p} + C \|b \cdot \nabla b\|_{L^p} + C \|\nabla^2 w\|_{L^p},
\]

(3.24)

which combined with (3.3), (3.12), (3.18), (3.1), and Gagliardo-Nirenberg inequality that

\[
\|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2}
\]

\[
\leq C \|\rho u_t\|_{L^2} + C \|\rho u \cdot \nabla u\|_{L^2} + C \|b \cdot \nabla b\|_{L^2} + C \|\nabla w\|_{L^2}
\]
\[ \leq C \| \rho \|_{L^\infty} \sqrt[4]{\rho} \| u \|_{L^2} + C \| \rho \|_{L^\infty} \sqrt[4]{\rho} \| u \|_{L^2} + C \| \nabla u \|_{L^2} \]
\[ + C \| b \|_{L^4} \| \nabla b \|_{L^4} + C \| \nabla w \|_{L^2} \]
\[ \leq C \| \sqrt[4]{\rho} u \|_{L^2} + C \| u \|_{L^2} \| \nabla u \|_{L^2} + C \| \nabla b \|_{L^2} \| \nabla^2 b \|_{L^2} + C \| \nabla w \|_{L^2} \]
\[ \leq C \| \sqrt[4]{\rho} u \|_{L^2} + C \| u \|_{L^2} \| \nabla u \|_{L^2} + C \| \nabla b \|_{L^2} \| \nabla^2 b \|_{L^2} + C \| \nabla w \|_{L^2} \]  
(3.25)

Multiplying (1.2) by \( \Delta b \) and integrating the resulting equality by parts over \( \mathbb{R}^2 \), it follows from Gagliardo-Nirenberg inequality, (2), and Young’s inequality that
\[ \frac{d}{dt} \int |\nabla b|^2 \, dx + 2\nu \int |\Delta b|^2 \, dx \]
\[ \leq C \int |\nabla u| |\nabla b|^2 \, dx + C \int |\nabla u| |\Delta b| \, dx \]
\[ \leq C \| \nabla u \|_{L^2} \| \nabla b \|_{L^4}^2 + C \| \nabla u \|_{L^4} \| \nabla b \|_{L^2} \| \Delta b \|_{L^2} \]
\[ \leq C \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \| \nabla b \|_{L^2} \| \nabla^2 b \|_{L^2} + C \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \| \nabla b \|_{L^2} \| \nabla^2 b \|_{L^2} \| \nabla b \|_{L^2} \]
\[ \leq C \| \nabla u \|_{L^2}^2 + C \| \nabla b \|_{L^2}^2 + \varepsilon \| \nabla^2 u \|_{L^2}^2 + \frac{\nu}{2(C_1 + 1)} \| \nabla^2 b \|_{L^2}^2. \]  
(3.26)

Adding (3.26) multiplied by \( C_1 + 1 \) to (3.22) and choosing \( \varepsilon \) small enough, we obtain from (3.25) that
\[ \frac{d}{dt} \left( B(t) + (C_1 + 1) \| \nabla b \|_{L^2} \right) \]
\[ \leq C \left( 1 + \| \nabla u \|_{L^2}^2 + \| \nabla b \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2 \right) \]
\[ \times \left( \| \nabla u \|_{L^2}^2 + \| \nabla b \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 \right), \]  
(3.27)

where we have used the standard \( L^2 \) estimate of elliptic system
\[ \| \nabla^2 b \|_{L^2} \leq C \| \Delta b \|_{L^2} \]

4. Let
\[ \Phi(t) := 3 + \sup_{0 \leq \tau \leq t} \left( \| \nabla u \|_{L^2}^2 + \| \nabla b \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 \right) \]
\[ + \int_0^t \left( \| \sqrt[4]{\rho} u \|_{L^2}^2 + \| \sqrt[4]{\rho} w \|_{L^2}^2 + \| \nabla^2 b \|_{L^2}^2 \right) \, d\tau. \]

Then we obtain from (3.22), (3.17), (2), Gronwall’s inequality, and Lemma 2.4 that for \( 0 \leq s \leq T < T^* \) and \( q > 2 \),
\[ \Phi(T) \leq C \Phi(s) \exp \left\{ C \int_s^T \left( \| u \|_{L^2}^2 \right)^2 \, d\tau \right\} \]
\[ \leq C \Phi(s) \exp \left\{ C \int_s^T \left( \| u \|_{L^2}^2 \right)^2 \, d\tau \right\} \]
\[ \leq C \exp \left\{ C \left( \left( \| \nabla (u \nabla \frac{1}{2}) \|_{L^2(s,T;L^2)} + \| u \nabla \frac{1}{2} \|_{L^2(s,T;L^q)} \right) \right) \right\} \times \Phi(s), \]  
(3.28)
Noting that
\[ |\nabla \tilde{x}| \leq (3 + 2\rho_0) \log^{1+\rho_0}(3 + |x|^2) \leq C \tilde{x}^{\frac{4}{1+\rho}}, \]
we then derive from (3.12) and \( a > 1 \) that
\[
\begin{align*}
\left\| \nabla (u \tilde{x}^\frac{a}{2}) \right\|_{L^2} &\leq C \left\| \nabla u \right\|_{L^2} + C \left\| \tilde{x}^{-\frac{a}{2} - 1} u \cdot \nabla \tilde{x} \right\|_{L^2} \\
&\leq C \left\| \nabla u \right\|_{L^2} + C \left\| \tilde{x}^{-1} \nabla \tilde{x} \right\|_{L^\infty} \left\| u \tilde{x}^{-\frac{a}{2}} \right\|_{L^2} \\
&\leq C \left\| \nabla u \right\|_{L^2} + C \left\| \tilde{x}^{-\frac{4+a}{1+\rho}} \right\|_{L^\infty} \left\| u \tilde{x}^{-\frac{a}{2}} \right\|_{L^2} \\
&\leq C \left\| \nabla u \right\|_{L^2} + C,
\end{align*}
\]
and
\[
\begin{align*}
\left\| u \tilde{x}^{-\frac{a}{2}} \right\|_{W^{1,q}} &\leq C \left\| u \tilde{x}^{-\frac{a}{2}} \right\|_{L^q} + C \left\| \nabla (u \tilde{x}^{-\frac{a}{2}}) \right\|_{L^q} \\
&\leq C + C \left\| \nabla u \right\|_{L^2} + C \left\| \nabla u \right\|_{L^q} + C \left\| \tilde{x}^{-\frac{a}{2} - 1} u \cdot \nabla \tilde{x} \right\|_{L^q} \\
&\leq C + C \left\| \nabla u \right\|_{L^2} + C \left\| \nabla u \right\|_{L^q} + C \left\| \tilde{x}^{-1} \nabla \tilde{x} \right\|_{L^\infty} \left\| u \tilde{x}^{-\frac{a}{2}} \right\|_{L^q} \\
&\leq C + C \left\| \nabla u \right\|_{L^2} + C \left\| \nabla u \right\|_{L^q} \left\| \nabla^2 u \right\|_{L^2}^{\frac{a}{2}} + C \left\| \tilde{x}^{-\frac{4+a}{1+\rho}} \right\|_{L^\infty} \left\| u \tilde{x}^{-\frac{a}{2}} \right\|_{L^q} \\
&\leq C + C \left\| \nabla u \right\|_{L^2} + C \left\| \nabla^2 u \right\|_{L^2}.
\end{align*}
\]
It follows from (3.24), (3.3), (3.12), (3.18), and Gagliardo-Nirenberg inequality that
\[
\begin{align*}
\left\| \nabla^2 u \right\|_{L^2} &+ \left\| \nabla P \right\|_{L^2} \\
&\leq C \left\| \rho u \right\|_{L^2} + C \left\| \rho u \cdot \nabla u \right\|_{L^2} + C \left\| b \cdot \nabla b \right\|_{L^2} + C \left\| \nabla w \right\|_{L^2} \\
&\leq C \left\| \rho \right\|_{L^\infty} \left\| \sqrt{\rho} u \right\|_{L^2} + C \left\| \rho u \right\|_{L^4} \left\| \nabla u \right\|_{L^4} + C \left\| b \right\|_{L^4} \left\| \nabla b \right\|_{L^4} + C \left\| \nabla w \right\|_{L^2} \\
&\leq C \left\| \sqrt{\rho} u \right\|_{L^2} + C \left(1 + \left\| \nabla u \right\|_{L^2} \right) \left\| \nabla u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \\
&\quad + C \left\| \nabla b \right\|_{L^2} \left\| \nabla^2 b \right\|_{L^2} + C \left\| \nabla w \right\|_{L^2} \\
&\leq C \left\| \sqrt{\rho} u \right\|_{L^2} + \frac{1}{2} \left\| \nabla^2 u \right\|_{L^2} + C \left\| \nabla u \right\|_{L^2}^2 \\
&\quad + C \left\| \nabla b \right\|_{L^2} + C \left\| \nabla^2 b \right\|_{L^2} + C \left\| \nabla w \right\|_{L^2}.
\end{align*}
\]
This together with (3.31) and Young’s inequality leads to
\[
\begin{align*}
\left\| u \tilde{x}^{-\frac{a}{2}} \right\|_{W^{1,q}} &\leq C + C \left\| \nabla u \right\|_{L^2}^{\frac{a}{2}} + C \left\| \sqrt{\rho} u \right\|_{L^2} + C \left\| \nabla^2 b \right\|_{L^2} + C \left\| \nabla b \right\|_{L^2} + C \left\| \nabla w \right\|_{L^2}.
\end{align*}
\]
Inserting (3.30) and (3.33) into (3.28) gives rise to for some \( \tilde{C} > 0 \),
\[
\Phi(T) \leq C \Phi(s)(C \Phi(T))^\frac{1}{\tilde{C}} \int_s^T (1 + \left\| \nabla u \right\|_{L^2}^2) d\tau.
\]
Recalling (3.2), one can choose \( s \) close enough to \( T^* \) such that
\[
\lim_{T \to T^*} \tilde{C} \int_s^T (1 + \left\| \nabla u \right\|_{L^2}^2) d\tau \leq \frac{1}{2},
\]
which along with (3.34) yields
\[
\Phi(T) \leq C \Phi^2(s) < \infty.
\]
The proof of Lemma 3.3 is finished.
The following result was essentially deduced in [11], we sketch it here for completeness.

**Lemma 3.4.** Under the condition (3.1), it holds that for any \( T \in [0, T^*) \),

\[
\sup_{0 \leq t \leq T} \| |b| \Delta \nabla b \|_{L^2}^2 + \int_0^T \| |b| \Delta b \|_{L^2}^2 \, dt \leq C. \tag{3.35}
\]

**Proof.** For \( a_1, a_2 \in \{-1, 0, 1\} \), denote

\[
\tilde{b}(a_1, a_2) = a_1 b^1 + a_2 b^2, \quad \tilde{u}(a_1, a_2) = a_1 u^1 + a_2 u^2, \tag{3.36}
\]

it is easy to infer from (1.2) that

\[
\tilde{b}_t - \nu \Delta \tilde{b} = b \cdot \nabla \tilde{u} - u \cdot \nabla \tilde{b}. \tag{3.37}
\]

Integrating (3.37) multiplied by \( 4\nu^{-1} \tilde{b} \Delta |\tilde{b}|^2 \) by parts over \( \mathbb{R}^2 \) leads to

\[
\nu^{-1} \left( \| \nabla |\tilde{b}|^2 \|_{L^2}^2 \right)_t + 2 \| \Delta |\tilde{b}|^2 \|_{L^2}^2
\]

\[
= 4 \int |\nabla \tilde{b}|^2 |\Delta \tilde{b}|^2 \, dx - 4\nu^{-1} \int b \cdot \nabla \tilde{u} \cdot \tilde{b} \Delta |\tilde{b}|^2 \, dx + 2\nu^{-1} \int u \cdot \nabla |\tilde{b}|^2 \Delta |\tilde{b}|^2 \, dx
\]

\[
\leq C \| \nabla u \|_{L^4}^4 + C \| \nabla b \|_{L^4}^4 + C \| b^2 \|_{L^4}^4 + \| \Delta |\tilde{b}|^2 \|_{L^2}^2
\]

\[
\leq C \| \nabla u \|_{L^2}^2 + C \| \nabla b \|_{L^2}^2 + C + \| \Delta |\tilde{b}|^2 \|_{L^2}^2
\]

\[
\leq C \| \sqrt{\rho} u \|_{L^2}^2 + C \| \nabla b \|_{L^2}^2 + C + \| \Delta |\tilde{b}|^2 \|_{L^2}^2, \tag{3.38}
\]

due to Gagliardo-Nirenberg inequality, (3.32), (3.8), and (3.2). Integrating (3.38) over \((0, T)\) and using (3.8) give that

\[
\sup_{0 \leq t \leq T} \| \nabla |\tilde{b}|^2 \|_{L^2}^2 + \int_0^T \| \Delta |\tilde{b}|^2 \|_{L^2}^2 \, dt \leq C. \tag{3.39}
\]

Noticing that

\[
\| |b| \Delta b \|_{L^2}^2 \leq C \| \nabla b \|_{L^4}^4 + \| \Delta |\tilde{b}(1, 0)|^2 \|_{L^2}^2 + \| \Delta |\tilde{b}(0, 1)|^2 \|_{L^2}^2
\]

\[
+ \| \Delta |\tilde{b}(1, 1)|^2 \|_{L^2}^2 + \| \Delta |\tilde{b}(0, -1)|^2 \|_{L^2}^2
\]

\[
\leq C \| \nabla b \|_{L^2}^2 + \| \Delta |\tilde{b}(1, 0)|^2 \|_{L^2}^2 + \| \Delta |\tilde{b}(0, 1)|^2 \|_{L^2}^2
\]

\[
+ \| \Delta |\tilde{b}(1, 1)|^2 \|_{L^2}^2 + \| \Delta |\tilde{b}(0, -1)|^2 \|_{L^2}^2, \tag{3.40}
\]

and

\[
\| |b| \nabla b \|_{L^2}^2 \leq G(t) \leq C \| |b| \nabla b \|_{L^2}^2 \tag{3.41}
\]

with

\[
G(t) := \| \nabla |\tilde{b}(1, 0)|^2 \|_{L^2}^2 + \| \nabla |\tilde{b}(0, 1)|^2 \|_{L^2}^2 + \| \nabla |\tilde{b}(1, 1)|^2 \|_{L^2}^2 + \| \nabla |\tilde{b}(0, -1)|^2 \|_{L^2}^2,
\]

thus we obtain the desired (3.35) from (3.39)–(3.41) and (3.8).

**Lemma 3.5.** Under the condition (3.1), it holds that for any \( T \in [0, T^*) \),

\[
\sup_{0 \leq t \leq T} \left( t \| \sqrt{\rho} u \|_{L^2}^2 + t \| \sqrt{\rho} w \|_{L^2}^2 + t \| b \|_{L^2}^2 \right)
\]

\[
+ \int_0^T \left( t \| \nabla u \|_{L^2}^2 + t \| \nabla w \|_{L^2}^2 + t \| b \|_{L^2}^2 \right) \, dt \leq C. \tag{3.42}
\]
Proof. 1. It follows from (3.12), (3.13), and (3.8) that for \( \varepsilon > 0, \eta > 0, \) and \( \tilde{\eta} = \min \{1, \eta\}, \)

\[
\|\rho\tilde{u}\|_{L^{2+\varepsilon}} + \|u\tilde{w}\|_{L^{2+\varepsilon}} + \|\rho\tilde{w}\|_{L^{2+\varepsilon}} + \|w\tilde{w}\|_{L^{2+\varepsilon}} \leq C,
\]

which combined with (3.24), (3.8), and (3.35) leads to

\[
\frac{d}{dt}\|\rho w_t\|_{L^2} \leq C\|\rho u_t\|_{L^2} + C\|\rho u \cdot \nabla u_{L^2} + C\|\nabla b\|_{L^2} + C\|\nabla w\|_{L^2}
\]

As a consequent, we get

\[
\|\nabla^2 w\|_{L^2} \leq C + C\|\sqrt{\rho} u_t\|_{L^2}.
\]

Employing \( L^2 \) theory of elliptic equations, we easily infer from (1.2)_4 and (3.43) that

\[
\|\nabla^2 w\|_{L^2} \leq C\|\rho w_t\|_{L^2} + C\|\rho u \cdot \nabla w_{L^2} + C\|\nabla u\|_{L^2}
\]

This implies that

\[
\|\nabla^2 w\|_{L^2} \leq C + C\|\sqrt{\rho} w_t\|_{L^2}.
\]

2. Differentiating (1.2)_2 and (1.2)_4 with respect to \( t \) and using (1.2)_1 give rise to

\[
\rho u_t + \rho u \cdot \nabla u_t - (\mu_1 + \xi)\Delta u_t + \nabla P_t
\]

\[
= (u \cdot \nabla \rho)(u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u - 2\xi \nabla u \cdot w_t + (b_t \cdot \nabla b + b \cdot \nabla b_t),
\]

\[
\rho w_{tt} + \rho w \cdot \nabla w_t + 4\xi w_t - \mu_2 \Delta w_t
\]

\[
= (u \cdot \nabla \rho)(w_t + u \cdot \nabla w) - \rho u_t \cdot \nabla w + 2\xi \nabla w \cdot u_t.
\]

Multiplying (3.46) by \( u_t \), (3.47) by \( w_t \), and integrating the resulting equality by parts over \( \mathbb{R}^2 \) and summing them, we obtain that

\[
\frac{1}{2} \frac{d}{dt} \int \rho|u_t|^2 + \rho w_t^2 \, dx + (\mu_1 + \xi) \int |\nabla u_t|^2 \, dx + \mu_2 \int |\nabla w_t|^2 \, dx + 4\xi \int w_t^2 \, dx
\]

\[
= \int ((u \cdot \nabla \rho)(u_t + u \cdot \nabla u) \cdot u_t - \rho u_t \cdot \nabla u \cdot u_t) \, dx
\]

\[
+ \int ((u \cdot \nabla \rho)(w_t + u \cdot \nabla w) \cdot w_t - \rho u_t \cdot \nabla w \cdot w_t) \, dx
\]

\[
- 2\xi \int (\nabla u \cdot w_t - \nabla w \cdot u_t) \, dx + \int b_t \cdot \nabla b \cdot u_t \, dx + \int b \cdot \nabla b_t \cdot u_t \, dx
\]

\[
= \sum_{i=1}^5 \tilde{I}_i.
\]
After integration by parts, we derive from (3.43), (3.11), (3.8), and Gagliardo-Nirenberg inequality that

\[
\tilde{I}_1 \leq C \int \rho |\mathbf{u}| (|\mathbf{u}| |\nabla \mathbf{u}| + |\mathbf{u}| |\nabla^2 \mathbf{u}| |\mathbf{u}|_t + |\mathbf{u}| |\nabla \mathbf{u}| |\nabla \mathbf{u}|_t + |\nabla \mathbf{u}|^2 |\mathbf{u}|_t) \, dx
\]
\[
\quad + \int \rho |\mathbf{u}|^2 |\nabla \mathbf{u}| \, dx
\]
\[
\leq C \|\sqrt{\rho} u\|_{L^6} \|\sqrt{\rho} u\|_{L^3} \|\nabla^2 \mathbf{u}\|_{L^6} \left( \|\nabla \mathbf{u}\|_{L^2} + \|\nabla \mathbf{u}\|^2_{L^2} \right)
\]
\[
+ C \|\rho^\frac{1}{2} u\|_{L^{2,12}} \|\sqrt{\rho} u\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^2} + \|\sqrt{\rho} u\|_{L^6} \|\nabla \mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^2}
\]
\[
\leq C \|\sqrt{\rho} u\|_{L^2} \left( \|\sqrt{\rho} u\|_{L^2} + \|\nabla \mathbf{u}\|^2_{L^2} \right) \left( 1 + \|\sqrt{\rho} u\|^2_{L^2} \right).
\]

(3.49)

Similarly, one has

\[
\tilde{I}_2 \leq C \int \rho |\mathbf{w}| (|\mathbf{w}| |\nabla \mathbf{w}| + |\mathbf{w}| |\nabla^2 \mathbf{w}| |\mathbf{w}|_t + |\mathbf{w}| |\nabla \mathbf{w}| |\nabla \mathbf{w}|_t + |\nabla \mathbf{w}| |\nabla \mathbf{w}| |\mathbf{w}|_t) \, dx
\]
\[
\quad + \int \rho |\mathbf{w}| |\nabla \mathbf{w}| \, dx
\]
\[
\leq C \|\sqrt{\rho} w\|_{L^6} \|\sqrt{\rho} w\|_{L^3} \|\nabla^2 \mathbf{w}\|_{L^6} \left( \|\nabla \mathbf{w}\|_{L^2} + \|\nabla \mathbf{w}\|^2_{L^2} \right)
\]
\[
+ C \|\rho^\frac{1}{2} w\|_{L^{2,12}} \|\sqrt{\rho} w\|_{L^2} \|\nabla^2 \mathbf{w}\|_{L^2} + \|\sqrt{\rho} w\|_{L^6} \|\nabla \mathbf{w}\|_{L^6} \|\nabla \mathbf{w}\|_{L^2}
\]
\[
\leq C \|\sqrt{\rho} w\|_{L^2} \left( \|\sqrt{\rho} w\|_{L^2} + \|\nabla \mathbf{w}\|^2_{L^2} \right) \left( 1 + \|\sqrt{\rho} w\|^2_{L^2} \right).
\]

(3.50)

We obtain from integration by parts and Cauchy-Schwarz inequality that

\[
\tilde{I}_3 = 4\xi \int \nabla^\perp \cdot \mathbf{u}_t \mathbf{w}_t \, dx \leq 4\xi \|\mathbf{w}_t\|^2_{L^2} + \xi \|\mathbf{u}_t\|^2_{L^2}.
\]

(3.51)

Integration by parts together with (1.2)5, (3.18), and Gagliardo-Nirenberg inequality indicates that

\[
\tilde{I}_4 + \tilde{I}_5 = - \int \mathbf{b}_t \cdot \nabla \mathbf{u}_t \cdot \mathbf{b}_t \, dx - \int \mathbf{b} \cdot \nabla \mathbf{u}_t \cdot \mathbf{b}_t \, dx
\]
\[
\leq \frac{\mu_1}{6} \|\mathbf{u}_t\|^2_{L^2} + C \|\mathbf{b}\|^2_{L^4} \|\mathbf{b}_t\|^2_{L^4}
\]
\[
\leq \frac{\mu_1}{6} \|\mathbf{u}_t\|^2_{L^2} + \frac{\mu_1 |\mathbf{v}|}{8(C_2 + 1)} \|\nabla \mathbf{b}_t\|^2_{L^2} + C \|\mathbf{b}_t\|^2_{L^2}.
\]

(3.52)
Substituting (3.49)–(3.52) into (3.48), we obtain after using (3.44) and (3.45) that
\[
\frac{d}{dt} \left( \|\sqrt{\rho} u_t\|^2_{L^2} + \|\sqrt{\rho} w_t\|^2_{L^2} \right) + \mu_1 \|\nabla u_t\|^2_{L^2} + \mu_2 \|\nabla w_t\|^2_{L^2} \\
\leq C \left( 1 + \|\sqrt{\rho} u_t\|^2_{L^2} + \|\sqrt{\rho} w_t\|^2_{L^2} \right)^2 + C\|b_t\|^2_{L^2} + \frac{\mu_1 \nu}{4(C_2 + 1)} \|\nabla b_t\|^2_{L^2}. \tag{3.53}
\]

3. Differentiating (1.2) with respect to \( t \) shows
\[
b_{tt} - b_t \cdot \nabla u - b \cdot \nabla u_t + u_t \cdot \nabla b + u \cdot \nabla b_t = \nu \Delta b_t. \tag{3.54}
\]
Multiplying (3.54) by \( b_t \) and integrating the resulting equality over \( \mathbb{R}^2 \) yield that
\[
\frac{1}{2} \frac{d}{dt} \int |b_t|^2 dx + \nu \int \|\nabla b_t\|^2 dx \\
= \int b_t \cdot \nabla u_t \cdot b_t dx - \int u_t \cdot \nabla b \cdot b_t dx + \int b_t \cdot \nabla u \cdot b_t dx - \int u \cdot \nabla b_t \cdot b_t dx \\
= \sum_{i=1}^4 S_i. \tag{3.55}
\]
We deduce from (3.18) and (3.10) that
\[
S_1 + S_2 \leq C \|\nabla u_t\|_{L^2} \|b_t\|_{L^4} \|b_t\|_{L^4} + C \|\nabla b_t\|_{L^2} \|u_t\|_{L^2} \|b_t\|_{L^2} \\
\leq C \|b_t\|^2_{L^4} + C \|\nabla u_t\|^2_{L^2} + \frac{\nu}{8} \|\nabla b_t\|^2_{L^2} + C \|u_t\|_{L^2} \|b_t\|_{L^2} \\
\leq \frac{\nu}{4} \|\nabla b_t\|^2_{L^2} + C \|b_t\|^2_{L^2} + C \|\nabla u_t\|^2_{L^2} + C \|\nabla b_t\|^2_{L^2}. \tag{3.56}
\]
Integration by parts combined with (1.2) and Gagliardo-Nirenberg inequality yields
\[
S_3 + S_4 = \int b_t \cdot \nabla u \cdot b_t dx \leq C \|b_t\|_{L^2} \|\nabla b_t\|_{L^2} \|\nabla u\|_{L^2} \\
\leq \frac{\nu}{4} \|\nabla b_t\|^2_{L^2} + C \|b_t\|^2_{L^2}. \tag{3.57}
\]
Inserting (3.56) and (3.57) into (3.55), one has
\[
\frac{d}{dt} \|b_t\|^2_{L^2} + \nu \|\nabla b_t\|^2_{L^2} \leq C \left( \|b_t\|^2_{L^2} + \|\sqrt{\rho} u_t\|^2_{L^2} \right) + C_2 \|\nabla u_t\|^2_{L^2}. \tag{3.58}
\]
From (3.53) multiplied by \( \mu_1^{-1}(C_2 + 1) \), (3.58), and (3.53), we get
\[
\frac{d}{dt} \left( \mu_1^{-1}(C_2 + 1) \|\sqrt{\rho} u_t\|^2_{L^2} + \|b_t\|^2_{L^2} + \|\sqrt{\rho} w_t\|^2_{L^2} \right) \\
+ \|\nabla u_t\|^2_{L^2} + \|\nabla b_t\|^2_{L^2} + \|\nabla w_t\|^2_{L^2} \\
\leq C \left( 1 + \|\sqrt{\rho} u_t\|^2_{L^2} + \|\sqrt{\rho} w_t\|^2_{L^2} \right) \left( 1 + \|b_t\|^2_{L^2} + \|\sqrt{\rho} u_t\|^2_{L^2} + \|\sqrt{\rho} w_t\|^2_{L^2} \right). \tag{3.59}
\]
Multiplying (3.59) by \( t \), we obtain (3.42) after using Gronwall’s inequality and (3.8). The proof of Lemma 3.5 is finished.

**Lemma 3.6.** Under the condition (3.1), it holds that for any \( T \in [0, T^*) \),
\[
\sup_{0 \leq t \leq T} \left( t\|\nabla^2 u\|^2_{L^2} + t\|\nabla^2 w\|^2_{L^2} + t\|\nabla^2 b\|^2_{L^2} + t\|\nabla b\|^2_{L^2} \right) \\
+ \int_0^T \left( \|\nabla^2 u\|_{L^4} + \|\nabla P\|_{L^4} + t\|\nabla u\|^2_{L^4} + t\|\nabla P\|_{L^4} + t\|\nabla^2 b\|^2_{L^2} \right) dt \\
\leq C. \tag{3.60}
\]
Proof. 1. We derive from (3.44) and (3.42) that

\[
\sup_{0 \leq t \leq T} t \| \nabla^2 u \|_{L^2} \leq C. \tag{3.61}
\]

Multiplying (3.45) by \( t \), one gets from (3.42) that

\[
\sup_{0 \leq t \leq T} t \| \nabla^2 w \|_{L^2} \leq C. \tag{3.62}
\]

2. Choosing \( p = q \) in (3.24), we deduce from (3.3), Gagliardo-Nirenberg inequality, (3.43), (3.8), (3.11), (3.44), and (3.45) that

\[
\| \nabla^2 u \|_{L^2} + t \| \nabla P \|_{L^2} \\
\leq C \left( \| \rho u \|_{L^2} + \| \rho u \cdot \nabla u \|_{L^2} + \| \mathbf{b} \cdot \nabla \mathbf{b} \|_{L^2} + \| \nabla w \|_{L^2} \right) \\
\leq C \left( \| \rho u \|_{L^2} + \| \rho u \cdot \nabla u \|_{L^2} \right) + C \left( 1 + \| \nabla^2 u \|_{L^2} + \| \nabla^2 b \|_{L^2} + \| \nabla^2 w \|_{L^2} \right)
\]

which together with (3.8), (3.42), and (3.61) implies that

\[
\int_0^T \left( \| \nabla^2 u \|_{L^2}^{\frac{q+1}{q+2}} + \| \nabla^2 w \|_{L^2} \right) dt \\
\leq C \int_0^T t \| \nabla u \|_{L^2}^{\frac{q-1}{q+2}} \left( \| \nabla u \|_{L^2} \right) \left( \| \nabla u \|_{L^2} \right)^{\frac{q-2(q+1)}{2(q+2)}} dt \\
+ C \int_0^T \| \nabla u \|_{L^2}^{\frac{q+1}{q+2}} dt \\
\leq C \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^{\frac{q-1}{q+2}} \int_0^T t \| \nabla u \|_{L^2}^{\frac{q+1}{q+2}} dt \\
+ C \int_0^T \left( 1 + \| \nabla u \|_{L^2}^{\frac{q+1}{q+2}} + \| \nabla^2 u \|_{L^2} + \| \nabla^2 w \|_{L^2} \right) dt \\
\leq C + C \int_0^T \left( t \| \nabla^2 u \|_{L^2} + t \| \nabla^2 w \|_{L^2} \right) dt \\
\leq C,
\]

and

\[
\int_0^T \left( t \| \nabla^2 u \|_{L^2} + t \| \nabla P \|_{L^2} \right) dt \\
\leq C \int_0^T \| \nabla u \|_{L^2} dt + C \int_0^T \left( t \| \nabla u \|_{L^2} \right)^{\frac{q+1}{q+2}} dt \\
+ C \int_0^T \left( 1 + \| \nabla^2 u \|_{L^2} + \| \nabla^2 w \|_{L^2} \right)^2 dt \\
\leq C \int_0^T \| \nabla u \|_{L^2} dt + C \int_0^T t \| \nabla u \|_{L^2} dt
\]
\[ + C \int_0^T \left( 1 + t \|\nabla^2 u\|_{L^2}^2 + t \|\nabla^2 b\|_{L^2}^2 + t \|\nabla^2 w\|_{L^2}^2 \right) dt \leq C. \]  

(3.65)

3. It deduces from (1.2)_3, (3.43), (3.8), (3.2), and Gagliardo-Nirenberg inequality that

\[
\|\nabla^2 b\|_{L^2}^2 \leq C \|b_t\|_{L^2}^2 + C \|u\| \|\nabla b\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \leq C \|b_t\|_{L^2}^2 + C \|u^{\alpha - \frac{\gamma}{2}}\|_{L^2}^2 \|\nabla b\|_{L^2}^2 \|\nabla b\|_{L^2}^2 + C \|b\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \leq C \|b_t\|_{L^2}^2 + C \|\nabla b\|_{L^2}^2 \|\nabla^2 b\|_{L^2}^2 \leq C \|b_t\|_{L^2}^2 + C \|\nabla b\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2 + C, \]

which gives that

\[
\|\nabla^2 b\|_{L^2}^2 \leq C \|b_t\|_{L^2}^2 + C \|\nabla b\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2}^2 + C. \]  

(3.66)

Multiplying (1.2)_3 by \(\Delta b \bar{x}^a\) and integrating by parts lead to

\[
\frac{1}{2} \frac{d}{dt} \int |\nabla b|^2 \bar{x}^a dx + \nu \int |\Delta b|^2 \bar{x}^a dx \leq C \int |\nabla b| |\nabla u| |\nabla b\|_{L^2}^2 dx + C \int |\nabla b|^2 |u| |\nabla \bar{x}^a| dx + C \int |\nabla b| |\Delta b| |\nabla \bar{x}^a| dx \]

\[
+ C \int |b| |\nabla u| |\Delta b| |\nabla \bar{x}^a| dx + C \int |\nabla u| |\nabla b|^2 \bar{x}^a dx =: \sum_{i=1}^5 J_i. \]  

(3.67)

Applying (3.7), (3.8), (3.43), (3.66), and Gagliardo-Nirenberg inequality, one gets by some direct calculations that

\[ J_1 \leq C \|b\|_{L^2} \|\nabla u\|_{L^4} \|\nabla b\|_{L^2} \]

\[ \leq C \|b\|_{L^2} \left( \|\nabla b\|_{L^2} + \|b\|_{L^2} \right) \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \|\nabla b\|_{L^2} \]

\[ \leq C \|\nabla b^2\|_{L^2}^2 + C \|\nabla b^2\|_{L^2}^2, \]

\[ J_2 \leq C \|\nabla b\|_{L^2}^2 \|u\|_{L^6} \|\nabla b\|_{L^2} \]

\[ \leq C \|\nabla b\|_{L^2}^2 + C \|\nabla b\|_{L^2}^2 \]

\[ \leq C \|\nabla b\|_{L^2}^2 + C \|b\|_{L^2} \|\nabla^2 u\|_{L^2}^2 \]

\[ \leq C \|\nabla b\|_{L^2}^2 + C \|\nabla b\|_{L^2}^2 \]

\[ \leq C \|\nabla^2 u\|_{L^2}^2 + C \|\nabla b\|_{L^2}^2, \]

\[ J_3 + J_4 \leq \frac{\nu}{8} \|\Delta b\|_{L^2}^2 + C \|\nabla b\|_{L^2}^2 \]

\[ \leq \frac{\nu}{8} \|\Delta b\|_{L^2}^2 + C \|\nabla b\|_{L^2}^2 \]

\[ + C \|b\|_{L^2} \left( \|\nabla b\|_{L^2} + \|b\|_{L^2} \right) \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \]

\[ \leq \frac{\nu}{8} \|\Delta b\|_{L^2}^2 + C \|\nabla b\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2}^2 + C, \]

\[ J_5 \leq C \|\nabla u\|_{L^2} \|\nabla b\|_{L^2} \]

\[ \leq C \|\nabla^2 u\|_{L^2} + C \|\nabla b\|_{L^2} \]

\[ \leq C \|\nabla^2 u\|_{L^2} + C \|\nabla b\|_{L^2} \]

\[ \leq C \left( 1 + \|\nabla^2 u\|_{L^2} \right) \|\nabla b\|_{L^2}. \]
Substituting the above estimates into (3.67) and noting the following fact
\[
\int |\nabla^2 b|^2 \bar{x}^a dx = \int |\Delta b|^2 \bar{x}^a dx - \int \partial_i \partial_i b \cdot \partial_k b \partial_k \bar{x}^a dx + \int \partial_i \partial_i b \cdot \partial_k b \partial_k \bar{x}^a dx
\]
we derive that
\[
\frac{d}{dt} \int |\nabla b|^2 \bar{x}^a dx + \frac{\nu}{2} \int |\nabla^2 b|^2 \bar{x}^a dx
\]
\[
\leq C \left( 1 + \|\nabla^2 u\|^2_{L^2} \right) \|\nabla b \bar{x}^\frac{3}{2}\|^2_{L^2} + C \left( 1 + \|\nabla^2 u\|^2_{L^2} + \|b_t\|^2_{L^2} \right).
\]
(3.68)
Thus, multiplying (3.68) by $t$, we infer from (3.61), (3.65), and Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \left( t \|\nabla b \bar{x}^\frac{3}{2}\|^2_{L^2} \right) + \int_0^T t \|\nabla^2 b\|^2_{L^2} dt \leq C.
\]
(3.69)
This along with (3.44), (3.42), and (3.61) implies that
\[
\sup_{0 \leq t \leq T} t \|\nabla^2 b\|^2_{L^2} \leq C.
\]
(3.70)
Then the desired (3.60) follows from (3.61), (3.62), (3.64), (3.65), (3.69), and (3.70).
This finishes the proof of Lemma 3.6.

**Lemma 3.7.** Under the condition (3.1), it holds that for any $T \in [0, T^*)$,
\[
\sup_{0 \leq t \leq T} \|\rho \bar{x}^a\|_{L^1 \cap H^1 \cap W^{1,q}} \leq C.
\]
(3.71)

**Proof.** 1. A straightforward calculation shows
\[
\bar{x}^{-1} |\nabla \bar{x}| \leq C (3 + |x|^2)^{-\frac{1}{2}},
\]
which combined with Sobolev’s inequality and (3.43) that for any $\sigma > 0$,
\[
\|u \bar{x}^{-\sigma}\|_{L^\infty} \leq C \|u \bar{x}^{-\sigma}\|_{L^3} + C \|\nabla (u \bar{x}^{-\sigma})\|_{L^3}
\]
\[
\leq C + C \|\nabla u\|_{L^1} \|\bar{x}^{-\sigma}\|_{L^\infty} + C \|u \bar{x}^{-\sigma}\|_{L^3} \|\bar{x}^{-1} \nabla \bar{x}\|_{L^\infty}
\]
\[
\leq C + C \|\nabla u\|_{L^3} \|\nabla^2 u\|_{L^2}^\frac{1}{2} + C \left( 1 + \|\nabla^2 u\|_{L^2} \right).
\]
(3.72)
2. One derives from (1.2) that $\rho \bar{x}^a$ satisfies
\[
\partial_t (\rho \bar{x}^a) + u \cdot \nabla (\rho \bar{x}^a) - a \rho \bar{x}^a \cdot \nabla \log \bar{x} = 0,
\]
(3.73)
which along with (3.72) and (3.8) gives that for any $r \in [2, q]$,
\[
\frac{d}{dt} \|\nabla (\rho \bar{x}^a)\|_{L^r}
\]
\[
\leq C \left( 1 + \|\nabla u\|_{L^\infty} + \|u \cdot \nabla \log \bar{x}\|_{L^\infty} \right) \|\nabla (\rho \bar{x}^a)\|_{L^r}
\]
\[
+ C \|\rho \bar{x}^a\|_{L^r} \left( \|\nabla u\|_{L^1} \|\nabla \log \bar{x}\|_{L^\infty} + \|u\|_{L^2} \log \bar{x}\|_{L^\infty} \right)
\]
\[
\leq C \left( 1 + \|\nabla^2 u\|_{L^2} + \|\nabla^2 u\|_{L^q} \right) \left( 1 + \|\nabla (\rho \bar{x}^a)\|_{L^r} + \|\nabla (\rho \bar{x}^a)\|_{L^q} \right),
\]
(3.74)
where we have used the following
\[
\|u \cdot \nabla \log \bar{x}\|_{L^\infty} + \|u\|_{L^2} \log \bar{x}\|_{L^\infty} \leq C \|u \bar{x}^{-\frac{3}{2}}\|_{L^\infty} \leq C \left( 1 + \|\nabla^2 u\|_{L^2} \right),
\]
owing to (3.29) and (3.72). Hence, we get the desired (3.71) from (3.74), Gronwall’s inequality, (3.60), (3.44), (3.8), and (3.9). This completes the proof of Lemma 3.7.

With Lemmas 3.1–3.7 at hand, we are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We argue by contradiction. Suppose that (1.9) were false, that is, (3.1) holds. Note that the general constant C in Lemmas 3.1–3.7 is independent of \( t < T^* \), that is, all the a priori estimates obtained in Lemmas 3.1–3.7 are uniformly bounded for any \( t < T^* \). Hence, the function 

\[
(\rho, u, b, w)(x, T^*) := \lim_{t \to T^*} (\rho, u, b, w)(x, t)
\]

satisfy the initial condition (1.8) at \( t = T^* \). Therefore, taking \( (\rho, u, b, w)(x, T^*) \) as the initial data, one can extend the local strong solution beyond \( T^* \), which contradicts the maximality of \( T^* \). Thus we finish the proof of Theorem 1.1.

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