On the generality of a cosmological speculation based on Heisenberg’s Principle

Benito Hernández-Bermejo

Departamento de Física Fundamental, Universidad Nacional de Educación a Distancia. Senda del Rey S/N, 28040 Madrid, Spain.
E-mail: benito@fisfun.uned.es

Abstract

In a previous work [1], it was speculated that the lack of homogeneity of the large-scale structure of the universe may be due to quantum fluctuations of space in the early universe. In [1], this was argued for a Friedmann-type universe for which both the curvature and the cosmological constant were zero. Here it is shown that the same considerations are valid for arbitrary values of the curvature and Λ.

Keywords: large-scale structure, uncertainty principle.
One of the most compelling and fundamental problems of modern cosmology arose in the last decade, when a systematic redshift survey performed by Geller and Huchra [2] revealed that the large-scale structure of the universe is not homogeneous. Observations show that this inhomogeneity has two important features: First, the lack of homogeneity comprises very diverse scales, which range from the size of galaxies to very large structures of the order of hundreds of Mpc: As can be shown [3], the observed structure can only arise if the original fluctuations in the early universe are of small amplitude but embrace very different scales of length. Second: The unexpected large size of the largest irregularities. In fact, the extension of these structures is an open problem, since its determination is limited by the extent of the survey. Estimations suggest that the typical size of these formations could be of the order of 100 Mpc, that is, \( \sim 5\% \) of the extent of the observed universe, approximately.

Several explanations have been proposed in order to account for such properties (see [4–8]). However, none of the standard possibilities seems to account completely for the two important features indicated above.

In a recent work [1], I speculated that this problem could be overcome if we assume that the fluctuations which originated the observed distribution of matter were produced by position-momentum Heisenberg’s principle. The reader is referred to the original reference for the details, which I shall not recall here for the sake of brevity. The main conclusions, however, were that the irregularities thus produced should cover a very large (in fact infinite) range of scales, the fluctuations being weak at every scale. These two properties are in accordance with the above-mentioned requirements of the present-day cosmological models.

For simplicity, it can be assumed that the early universe may be well described by the Friedmann equation

\[
\frac{\dot{a}^2}{a^2c^2} + \frac{K}{a^2} - \Lambda = \frac{8\pi G \rho}{3c^4},
\]

(1)

where \( a \) is an arbitrary reference length, which changes as the universe
evolves, $\rho$ is the mass-energy density, $K$ is the curvature of the universe and $\Lambda$ is Einstein’s cosmological constant. Equation (1) is expressed in SI units.

In particular, when both $K$ and $\Lambda$ were taken as zero, (1) becomes

$$\frac{\dot{a}^2}{a^2 c^2} = \frac{8\pi G \rho}{3c^4}$$

(2)

This is a compromise among all possible models, since we can neglect $\Lambda$ in a first approach ($\Lambda \ll 1$), and also because the problem of the total mass of the universe remains unsolved and consequently we do not know whether it is open or closed. In an early universe composed of energy and a small amount of matter, which seems to be the most likely scenario [9, §1.5], we saw that $\rho \propto a^{\epsilon-4}$ can be taken, where $\epsilon > 0$, $\epsilon \ll 1$. In this case the universe expands following the dependence

$$a(t) \propto t^{\delta + 1/2},$$

(3)

where

$$\delta = \frac{\epsilon}{8 - 2\epsilon}$$

(4)

is also small and positive. It was also shown that, with full generality,

$$\Delta p \propto \dot{a},$$

(5)

where $\Delta p$ is the uncertainty associated to the expansion process. From this, together with equations (3) and

$$\Delta x \simeq \frac{\hbar}{\Delta p},$$

(6)

it is not difficult to obtain the important result:

$$\frac{\Delta x}{a} \propto t^{-2\delta}.$$
(7) diverges as $t \to 0$, which shows that the phenomenon embraces very
diverse (in fact infinite) scales of length. Making use of (6) and (5), this
divergence can be generally expressed as:

$$\lim_{t \to 0} a(t) \dot{a}(t) = 0.$$  \hspace{1cm} (8)

This criterion constitutes a sufficient condition for the process to be present,
and will often be employed in future calculations. It is also clear that these
fluctuations of space might have affected the matter distribution. In this
sense, it was shown that the perturbations are weak at every scale. All
these features are in full accordance [1] with the requirements of the cosmic
structure formation theories [3] for the generation of the observed large-scale
structure of the universe [2].

The logical step after the foregoing conclusions is the analysis of how these
results are influenced in the case of a nonzero value of the curvature $K$ and
the cosmological constant $\Lambda$. We shall proceed to do this in the present
work.

**The case $|K| \ll 1, \Lambda = 0$**

As pointed out by Hawking [10, ch. 8], the universe started its expansion
at a rate which is exceedingly close to the critical one, namely, that for which
the curvature is 0. It is remarkable the fact that, even today, ten billion years
after the Big Bang, the evolution of the universe does not allow, with the
present-day technology, an unambiguous determination of the sign of $K$.
Consequently, it seems justified to consider in detail the situation for which
$|K| \ll 1$, before we deal with the general problem.

We start by considering a universe purely composed of energy ($\rho \propto a^{-4}$).
After that, we shall study the more realistic case $\rho \propto a^{\epsilon-4}$ as a perturbation.
This is possible since $\epsilon$ is small and positive, as noted before. Thus, if we
write $\rho = \rho_0 a^{-4}$ the Friedmann equation (1) is:

$$\dot{a}^2 + K c^2 = \frac{8 \pi G \rho_0}{3 c^2} a^{-2}.$$  \hspace{1cm} (9)
If $K = 0$, the solution is $a(t) = \sqrt{2\alpha t^{1/2}}$, where $\alpha = (8\pi G p_0/3c^2)^{1/2}$. Since we are assuming $|K| \ll 1$, we can consider $a(t, K)$ as a function of the two variables $t$ and $K$ and expand as a Maclaurin series in $K$:

$$a(t, K) = a(t, K = 0) + K \frac{\partial a(t, K)}{\partial K} \bigg|_{K=0} + o(K^2) \simeq \sqrt{2\alpha t^{1/2} + Kf(t)} .$$

(10)

If we now substitute this expression into the Friedmann equation (9) and neglect all terms quadratic in $K$ we are led to a first order differential equation for $f$:

$$t\dot{f} + \frac{1}{2} f = -\frac{c^2}{\sqrt{2\alpha}} t^{3/2} .$$

(11)

This is a Cauchy equation [11]. Its general solution is:

$$f = \xi t^{-1/2} - \frac{c^2}{\sqrt{8\alpha}} t^{3/2} ,$$

with $\xi$ a real constant of integration. From (10) we have:

$$a(t, K) = \sqrt{2\alpha t^{1/2}} + K\xi t^{-1/2} - \frac{Kc^2}{\sqrt{8\alpha}} t^{3/2} + o(K^2) .$$

(13)

However, the physics of the problem shows that it must be $\xi = 0$, since $a$ cannot diverge in the limit $t \to 0$. Then we finally obtain:

$$a(t, K) = \sqrt{2\alpha t^{1/2}} - \frac{Kc^2}{\sqrt{8\alpha}} t^{3/2} + o(K^2) .$$

(14)

We can see that the first term is the dominant one as $t \to 0$. It is also noticeable that if $K > 0$ (closed universe) then $a(t, K)$ increases more slowly than in the case $K = 0$ (and vice versa if $K < 0$), as expected.

We can now contemplate the case $\epsilon \neq 0$ as a perturbation of the previous one. From equations (3) and (4) we see that the first term changes its dependence from $t^{1/2}$ to $t^{\delta+1/2}$. Consequently, now (14) must become:

$$a(t, K) = k_1(\delta) t^{\delta+1/2} + k_2(\delta) t^{g(\delta)+3/2} + o(K^2) ,$$

(15)

where $k_1(0) = \sqrt{2\alpha}$, $k_2(0) = -Kc^2/\sqrt{8\alpha}$ and $g(0) = 0$. Since $\delta$ is small and positive we can approximate $g(\delta) = \delta g'(0) + o(\delta^2) \simeq \beta\delta$. Then:

$$a(t, K) \simeq k_1(\delta) t^{\delta+1/2} + k_2(\delta) t^{3\delta+3/2} .$$

(16)
The perturbation is, as we see, controlled by the small parameter $\epsilon$ (or $\delta$ equivalently). Since $\epsilon \ll 1$, it is to be expected that the term $k_1(\delta)t^{\delta+1/2}$ of (16) is still the dominant one as $t \to 0$. This implies that:

$$\frac{1}{2} + \delta < \frac{3}{2} + \beta\delta \implies \beta\delta > \delta - 1 .$$  \hfill (17)

Now we can apply criterion (8) to solution (16):

$$\lim_{t \to 0} a(t, K) \dot{a}(t, K) = \lim_{t \to 0} [k_1 k_3 t^{2\delta} + (k_1 k_4 + k_2 k_3) t^{1+\delta+\beta\delta} + k_2 k_4 t^{2+2\beta\delta}] , \quad (18)$$

where $k_3 = (\delta + 1/2)k_1$, $k_4 = (\beta\delta + 3/2)k_2$ and $k_1$, $k_2$ are defined as in equation (15). If we take into account equation (17) we observe that: i) $1 + \delta + \beta\delta > 2\delta > 0$, and ii) $2 + 2\beta\delta > 2\delta > 0$. Consequently:

$$\lim_{t \to 0} a(t, K) \dot{a}(t, K) = 0 .$$  \hfill (19)

Thus the phenomenon is present for $\rho \propto a^{\epsilon-4}$ and $|K| \ll 1$, independently of the sign of $K$, as we wanted to show.

**The general problem: Arbitrary $K$ and $\Lambda$**

In this section we shall first demonstrate that criterion (8) is indeed satisfied for an arbitrary value of the curvature $K$. Although this result is far more general than that in the previous section, the proof is rather mathematical and the underlying physics is not as apparent as in the foregoing development. This is why both perspectives may be taken as complementary.

We start by considering the Friedmann equation (1) with density $\rho = \rho_0 a^{\epsilon-4}$, as usual:

$$\dot{a}^2 + Kc^2 = \alpha^2 a^{\epsilon-2} ,$$  \hfill (20)

where $\alpha = (8\pi G \rho_0/3c^2)^{1/2}$ as before. Now we multiply both sides by $a^2$ and regroup:

$$a^2 \dot{a}^2 = \alpha^2 a^\epsilon - Kc^2 a^2 .$$  \hfill (21)
We can make use of (21) and substitute in criterion (8):

\[
\lim_{t \to 0} a^2(t) \dot{a}^2(t) = \lim_{t \to 0} \left[ \alpha^2 a' - Kc^2 a^2 \right].
\]

(22)

But it is obvious that:

\[
\lim_{t \to 0} a(t) = 0.
\]

(23)

Since \( \epsilon > 0 \) the result is:

\[
\lim_{t \to 0} a^2(t) \dot{a}^2(t) = 0,
\]

(24)

and the criterion is therefore satisfied. It is interesting to observe that this property is, in fact, a direct consequence of the structure of the Friedmann equation, our only assumption concerning the exponent of the density. Indeed, repeating the previous calculations for a density of the form \( \rho = \rho_0 a^p \), with \( p \) a parameter, we conclude that criterion (8) is satisfied whenever \( p > -4 \). In particular, fluctuations due to the uncertainty principle need not be present in the unrealistic case \( p = -4 \) of a universe only composed of energy. Accordingly, it can be said that the radiation scenario is unstable since any perturbation due to the presence of matter ‘switches’ the process on. This was exactly the result found in [1] for the case \( K = 0 \), which is thus generalized for all curvatures.

We can now be concerned with the most general case [9,12] in which both the curvature \( K \) and \( \Lambda \) can take arbitrary values:

\[
\dot{a}^2 a^2 + K a^2 = \frac{8\pi G \rho}{3c^4},
\]

(25)

I shall employ again the most realistic expression for the density:

\[
\rho = \rho_0 a^{\epsilon - 4}, \quad \text{with } \epsilon > 0, \; \epsilon \ll 1
\]

(26)

As before, I shall make use of the criterion (8) to give a simple proof of the result we are interested in. For this, only the Friedmann equation (25) and the expression for the density (26) have to be used to write:

\[
a^2 \dot{a}^2 = \alpha^2 a' - (Kc^2) a^2 + (\Lambda c^2) a^4,
\]

(27)
where $\alpha$ is again a constant. It is then found that:

$$\lim_{t \to 0} a^2 \dot{a}^2 = \lim_{t \to 0} (\alpha^2 a^\epsilon - (Kc^2)a^2 + (\Lambda c^2)a^4) = 0,$$

(28)
since $\epsilon > 0$ and obviously $\lim_{t \to 0} a(t) = 0$. Criterion (8) is therefore satisfied. This generalizes the result found in [1] for the case $K = 0, \Lambda = 0$.

Conclusion

This work, which complements [1], shows how the uncertainty principle could provide a mechanism for the generation of fluctuations of very diverse scales in the early universe. These fluctuations comply to the requirements of the up-to-date cosmological models for the generation of the observed large-scale structure. Curiously, the required properties for the phenomenon to be present seem to be, to great extent, intrinsic to the Friedmann equation itself and not dependent on important features of the model such as the curvature or the cosmological constant. Consequently, these semiclassical considerations may provide an interesting starting point for a possible elucidation of the problem of the large-scale structure of the universe.

Acknowledgements

I would like to thank Professor Máximo Barón for his careful and constructive revision of several versions of this work.
References

1 Hernández-Bermejo, B. (1996) Heisenberg’s Principle: A cosmological speculation. Speculat. Sci. Technol., 19, 253–7.

2 Geller, M. J. and Huchra, J. P. (1989) Mapping the Universe. Science, 246, 897–903.

3 Bertschinger, E. (1994) Cosmic structure formation. Physica D, 77, 354–379.

4 Kaiser, N. (1984) On the spatial correlations of Abell clusters. Astrophys. J., 284(1), L9–L12.

5 Centrella, J. M., Gallagher, J. S., Melott, A. S. and Bushouse, H. A. (1988) A case-study of large-scale structure in a hot model universe. Astrophys. J., 333(1), 24–53.

6 Ikeuchi, S. (1981) Theory of galaxy formation triggered by quasar explosions. Publ. Astr. Soc. Jpn., 33(2), 211–222.

7 Shandarin, S. F. (1994) Nonlinear dynamics of the large-scale structure in the universe. Physica D, 77, 342–353.

8 Brown, W. K. (1994) A thick, rotating universe. Speculat. Sci. Technol., 17(3), 186–190.

9 Davies, P. (1984) The Accidental Universe. Cambridge: Cambridge University Press.

10 Hawking, S. W. (1990) A Brief History of Time: from the Big Bang to Black Holes. New York: Bantam Books.

11 Ayres, F. (1962) Differential Equations. New York: McGraw-Hill.

12 Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973) Gravitation. San Francisco: Freeman & Co.