Harmonic Bergman spaces and Bergman projectors on homogeneous trees

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Abstract

In this paper we investigate some properties of the harmonic Bergman spaces \( A^p(\sigma) \) on a \( q \)-homogeneous tree, where \( q \geq 2 \), \( 1 \leq p < \infty \), and \( \sigma \) is a suitable reference measure on the tree. Such spaces were introduced by J. Cohen, F. Colonna, M. Picardello and D. Singman. When \( p = 2 \) these are reproducing kernel spaces and we compute explicitly their reproducing kernel. We then study the boundedness properties of the Bergman projector on \( L^p(\sigma) \).

Introduction

The aim of this work is the study of the harmonic Bergman spaces on homogeneous trees introduced in [3]. We show that, given a reference measure \( \sigma \) on the tree, the Bergman spaces \( A^p(\sigma) \), \( 1 \leq p < \infty \), enjoy some properties known for their holomorphic analogous on the hyperbolic disk. The space \( A^2(\sigma) \) is a reproducing kernel Hilbert space (RKHS), and then we provide an explicit formula for the kernel. Furthermore, we prove that, for a prototypical class of measures, the extension of the associated Bergman projector is bounded on \( L^p(\sigma) \) if and only if \( p > 1 \).

The notion of harmonic function is stated for functions defined on homogeneous trees using the mean value property. That is, a function on \( X \) is said to be harmonic if the mean of its values on the neighbors of a vertex coincides with the value at the vertex, for every vertex. J. Cohen, F. Colonna, M. Picardello, and D. Singman introduce harmonic Bergman spaces on homogeneous trees in [3]. They consider a family of reference measures which consist of finite measures absolutely continuous with respect to the counting measure and whose Radon-Nikodym derivative \( \sigma \) is a radial positive decreasing function on \( X \). Thus, for every \( 1 \leq p < \infty \), they define the Bergman space \( A^p(\sigma) \) as the closed subspace of \( L^p(\sigma) \) consisting of harmonic functions. The requirement for the measure \( \sigma \) to be finite is necessary in order to avoid the case in which the Bergman space consists of the null function alone.

In the context of the hyperbolic disk, when \( p = 2 \), the weighted Bergman spaces are RKHS, and the holomorphic Bergman kernel is known as well as the properties of the associated projector. Indeed, the extension of the holomorphic Bergman

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projector to the weighted $L^p$-spaces is bounded if and only if $p > 1$, see [7], [8] and [9]. Furthermore, it is of weak type $(1, 1)$, see [5] [11]. In our work, first of all, we show that $A^2(\sigma)$ is a reproducing kernel Hilbert space for every reference measure $\sigma$ and we provide an explicit formula for the kernel $K_\sigma$ in Theorem [14]. Since $A^2(\sigma)$ is closed in $L^2(\sigma)$, there exists an orthogonal projection $P_\sigma : L^2(\sigma) \to A^2(\sigma)$. We prove that the extension of $P_\sigma$ to an operator from $L^p(\sigma)$ to $A^p(\sigma)$ is bounded if and only if $p > 1$, for a particular class of reference measures.

The measures we focus on are called exponentially decreasing radial measures since they satisfy the requirement of $\sigma$ to be decreasing exponentially w.r.t. the distance from $o$, a natural counterpart of the measures considered on the hyperbolic disk in the definition of the weighted holomorphic Bergman spaces. This restriction allows us to consider Toeplitz-type operators and to prove for them more general boundedness results. The fact that the extension of the projector to the weighted $L^p$-spaces is bounded if and only if $p > 1$ follows from the fact that the projector coincides with a particular Toeplitz-type operator.

A natural question to investigate is whether the same holds for a general reference measure $\sigma$. In [3], the authors introduce the optimal measures, a subclass of the reference measures which, roughly speaking, decrease fast as the distance from the origin increases. The exponentially decreasing radial measure are optimal. The boundedness of the Bergman projector for optimal measures is still work in progress. We know that the boundedness of the operator on $L^1(\sigma)$ is false for every optimal measure. On the other hand, we are aware that for some optimal measures the boundedness of the projector on $L^p(\sigma)$ holds for every $p > 1$. Another natural question we are working on is the weak type $(1, 1)$ boundedness of $P_\sigma$.

The paper is organized as follows. In the first section we define the harmonic Bergman spaces and, for every reference measure, we provide an orthonormal basis of the Hilbert space $A^2(\sigma)$. The basis plays a fundamental role in Section 2 in the proof of the two formulae for the kernel of the RKHS $A^2(\sigma)$: the first is a recursive formula, while the second is the explicit formula of the kernel given in Theorem [14]. In Section 3 we focus on a specific class of reference measures and state two results characterizing the boundedness of the extension of the Toepliz-type operators related to the kernel of $A^2(\sigma)$ to the weighted $L^p$ spaces (see Theorem [13] and [14]). As a consequence, in Theorem [15] we show that the extension of the harmonic Bergman projector to the weighted $L^p$ space is bounded if and only if $p > 1$.

In the following, we shall use the symbol $\approx (\leq, \geq)$ between two quantities when the left hand side is equal (less than or equal, or greater than or equal, respectively) to the right hand side up to the multiplication for a (fixed) positive constant. Furthermore we assume the following convention on sums: the sum is null whenever the starting index is greater than the final index. If $Y \subseteq X$, we denote by $1_Y$ the characteristic function of $Y$, that is the function on $X$ which is 1 on $Y$ and 0 outside.

1 Harmonic Bergman spaces

1.1 Preliminaries on homogeneous trees

We present some preliminary notions and results on homogeneous trees, for a deeper analysis we refer to [2] [6].
A graph is a pair \((X, \mathcal{E})\), where \(X\) is the set of vertices and \(\mathcal{E}\) is the family of unoriented edges, where an edge is a two-element subset of \(X\). If two vertices \(u, v\) in \(X\) are joined by an edge, they are called adjacent and this is denoted by \(u \sim v\). A tree is an undirected, connected, loop-free graph. A \(q\)-homogeneous tree is a tree in which each vertex has exactly \(q + 1\) adjacent vertices. With slight abuse, we refer to the set of vertices \(X\) as the tree itself. We fix an origin \(o \in X\). Homogeneous trees are endowed with the canonical distance determined by the (minimal) number of contiguous edges between two vertices.

From now on we consider a \(q\)-homogeneous tree \(X\) with \(q \geq 2\). Given \(u, v \in X\), with \(u \neq v\), we denote by \([u, v]\) the unique ordered \(t\)-uple \((x_0 = u, x_1, \ldots, x_{t-1} = v) \in X^t\), where \(\{x_i, x_{i+1}\} \in \mathcal{E}\) and all the \(x_i\) are distinct. We denote by \([u, v]\) the path starting at \(u\) and ending at \(v\). With slight abuse of notation, if \([u, v] = (x_0, \ldots, x_{t-1})\) we write \(x_i \in [u, v]\), \(i \in \{0, \ldots, t-1\}\). In particular, if \(u\) and \(v\) are adjacent, both \([u, v], [v, u] \in X^2\) are oriented, unlike the edge \(\{u, v\} \in \mathcal{E}\) which is not. A homogeneous tree \(X\) carries a natural distance \(d: X \times X \to \mathbb{N}\), where for every \(u, v \in X\) the distance \(d(u, v)\) is the number of 2-chains in the path \([u, v]\). If \(v \in X\), then we denote by \(S(v, n)\) and \(B(v, n)\) the sphere and the ball centered at \(v\) with radius \(n \in \mathbb{N}\), respectively, i.e.

\[
S(v, n) = \{ x \in X : d(v, x) = n \} \quad \text{and} \quad B(v, n) = \{ x \in X : d(v, x) \leq n \}.
\]

It is straightforward to check that

\[
|S(v, n)| = \begin{cases} 
1, & n = 0; \\
(q + 1)q^{n-1}, & n \neq 0,
\end{cases} \quad \text{and} \quad |B(v, n)| = \frac{q^{n+1} + q^n - 2}{q - 1}.
\]

From now on we fix an origin \(o \in X\), and call norm of a vertex \(v\) in \(X\) its distance from \(o\), i.e. \(|v| = d(o, v)\). If \(v \neq o\), then we define the sector of \(v\) as the subset

\[
T_v := \{ x \in X : [o, v] \subseteq [o, x] \},
\]

and we adopt the convention \(T_o = X\). Moreover, we call sons of \(v\) the elements of \(s(v) = \{ u \in X : u \sim v, |u| = |v| + 1 \}\). It is easy to observe that

\[
|s(v)| = \begin{cases} 
q + 1, & \text{if } v \neq o; \\
q, & \text{if } v = o.
\end{cases}
\]

Hence if \(v \neq o\) we call predecessor of \(v\) and denote by \(p(v)\) the neighbor of \(v\) which is not a son of \(v\); it follows that \(|p(v)| = |v| - 1\). The vertex \(o\) is the only one having no predecessors, and \(s(o) = S(o, 1)\). Furthermore, in what follows, we consider the predecessor as a function \(p: X \setminus \{o\} \to X\) so that \(p^\ell: X \setminus B(o, \ell - 1) \to X\) denotes the \(\ell\)-th predecessor function.

### 1.2 Harmonic functions and harmonic Bergman spaces

**Definition 1.** Let \(f\) be a complex valued function on \(X\). The combinatorial Laplacian is defined on \(f\) by

\[
Lf(v) := \frac{1}{q + 1} \sum_{u \sim v} f(u) - f(v), \quad v \in X.
\]

Let \(Y \subseteq X\). We say that \(f\) is harmonic on \(Y\) if \(Lf = 0\) on \(Y\). We shall call a function harmonic if it is harmonic on \(X\).
Equivalently, $f$ is harmonic on $Y$ if
\begin{equation}
  f(v) = \frac{1}{q+1} \sum_{u \sim v} f(u), \quad v \in Y. 
\end{equation}

The harmonicity property (1) implies a certain rigidity for the function. In particular, the value of a harmonic function at a vertex $y \in X$ “propagates” to every layer of the sector $T_y$, as showed in the following proposition, which is a modified version of Lemma 4.1. In that lemma, the authors show that a function which is harmonic and radial on a sector $T_y$, $y \in X \setminus \{o\}$, is completely described by its values at $y$ and $p(y)$. We consider a harmonic function on the sector $T_y$ without the radial condition and we formulate the conclusion for the average on $S(o,n)$, $n \geq |y|$, instead of for each vertex of the sector.

**Proposition 2.** Let $y \in X \setminus \{o\}$. If $f : X \to \mathbb{C}$ is harmonic on $T_y$, then for every $n \in \mathbb{N}$, $n \geq |y|$, we have
\begin{equation}
  \sum_{|x|=n, x \in T_y} f(x) = \left( \frac{n-|y|}{q} \right) f(y) - \left( \frac{n-|y|-1}{q} \right) f(p(y)).
\end{equation}

Furthermore, if a function $f : X \to \mathbb{C}$ is radial on $T_y$ and satisfies (2) for every $n \geq |y|$, then $f$ is harmonic on $T_y$.

From Proposition 2 we deduce a generalization of Formula (1).

**Corollary 3.** Let $f$ be a harmonic function on $X$. Then the following mean value property holds true: for every $n \in \mathbb{N} \setminus \{0\}$
\begin{equation}
  f(o) = \frac{1}{|S(o,n)|} \sum_{|x|=n} f(x).
\end{equation}

We introduce a technique which permits to extend a function which is harmonic on a ball centered in $o$ which will be useful in what follows. Let $n \in \mathbb{N}$, $n \geq 1$, and $g$ be a function on $X$ which is harmonic on $B(o,n)$. It is easy to see that there are infinite ways to extend $g$ to a harmonic function on all of $X$ and which coincides with $g$ on $B(o,n+1)$. We define $g^H_n$ on $X$ to be radial when restricted on $T_y$ for every $y \in S(o,n+1)$ and harmonic on $X$.

Let $x \in X \setminus B(o,n)$. There exists a unique $y \in S(o,n+1)$ such that $x \in T_y$, and $y = p^{|x|-n-1}(x)$ (where $p^0 = \text{id}_X$). Since the function $g^H_n$ is supposed to be radial and harmonic on $T_y$, by Proposition 2 we have that
\begin{align*}
g^H_n(x) &= \frac{1}{|S(o,|x|) \cap T_y|} \sum_{|z|=|x|, z \in T_y} g^H(z) \\
&= q^{|x|-|x|} \left[ \left( \sum_{j=0}^{|x|-|y|} q^j \right) g(y) - \left( \sum_{j=0}^{|x|-|y|-1} q^j \right) g(p(y)) \right] \\
&= q^{n+1-|x|} \left[ \left( \sum_{j=0}^{|x|-n-1} q^j \right) g(p^{|x|-n-1}(x)) - \left( \sum_{j=0}^{|x|-n-2} q^j \right) g(p^{|x|-n}(x)) \right] \\
&= \left( \sum_{j=0}^{|x|-n-1} q^{-j} \right) g(p^{|x|-n-1}(x)) - \left( \sum_{j=1}^{|x|-n} q^{-j} \right) g(p^{|x|-n}(x)).
\end{align*}
For simplicity we introduce the notation

\[ a_n = \sum_{j=0}^{n} q^{-j} = \frac{q - q^{-n}}{q - 1}, \quad n \in \mathbb{N}, \]

and we put \( a_{-1} = 0 \). Hence

\[ g_n^H(x) = \begin{cases} g(x), & |x| \leq n; \\ a_{|x|-n-1} g(p^{|x|-n-1}(x)) - (a_{|x|-n-1} - 1) g(p^{|x|-n}(x)), & |x| > n, \end{cases} \]

where \( p^0 = \text{id}_X \). The function \( g_n^H \) defined above is harmonic on \( X \) by Proposition 2 and by using the fact that

\[ X = B(o, n) \cup \bigcup_{y \in S(o, n+1)} T_y. \]

Observe that we do not loose that \( g_n^H \) is harmonic on \( B(o, n) \) because, since \( a_0 = 1 \) and \( a_{-1} = 0 \), \( g_n^H = g \) on \( B(o, n + 1) \), and not only on \( B(o, n) \). Furthermore, the extension \( g_n^H \) is radial on every sector “starting” from a point in \( S(o, n + 1) \) by construction.

![Figure 1: The function \( g \) is harmonic on \( B(o, 2) \), that is the set of vertices in the blue area. The function \( g_2^H \) is obtained by extending the values of \( g \) in \( S(o, 3) \) (the green area) along sectors in such a way that \( g_2^H \) is harmonic and constant on the vertices lying on the same red arc, that is on the “layers” of the sectors.](image)

### 1.3 Harmonic Bergman spaces

Homogeneous trees are classically endowed with the counting measure. The main advantage of such measure is the invariance under the group of isometries of the tree. Let \( p \geq 1 \). The only harmonic function that is \( p \)-summable w.r.t. the counting measure is the null function, as we show in the following statement.
Proposition 4. If $f$ is a harmonic function in $L^p(X)$, then $f$ is the null function.

Proof. Suppose that $f$ is harmonic. We have that

$$
\sum_{x \in X} |f(x)|^p = \sum_{n=0}^{+\infty} \sum_{|x|=n} |f(x)|^p
\leq \frac{(q+1)^{p-1}}{(q+1)^p} \sum_{n=0}^{+\infty} \sum_{|y|=n} |f(y)|^p
= \frac{1}{q+1} (q+1)^p \|f\|_{L^p(X)}^p = \|f\|_{L^p(X)}^p < +\infty,
$$

since every vertex is neighbor of exactly $q+1$ other vertices. Hence the unique inequality in the computation above is an equality, so that

$$(q+1)^{p-1} \sum_{y \sim x} |f(y)|^p = \left| \sum_{y \sim x} f(y) \right|^p = (q+1)^p |f(x)|^p,
$$

which means that $|f|^p$ is harmonic, too. If $f$ is not the null function, then there exists $v \in X$ such that $f(v) \neq 0$. Hence by Corollary 3, we have

$$
\sum_{x \in X} |f(x)|^p = \sum_{n=0}^{+\infty} \sum_{d(v,x)=n} |f(x)|^p = |f(v)|^p \sum_{n=0}^{+\infty} |S(v,n)| = +\infty,
$$

which is a contradiction. Hence $f = 0$. \qed

If we want to work with Bergman spaces of harmonic functions, the previous proposition leads to consider finite measures on $X$. In [3], the authors have introduced harmonic Bergman spaces w.r.t the following class of measures on a $q$-homogeneous tree $X$ (see also [4]).

Definition 5. A reference measure on $X$ is a finite measure that is absolutely continuous w.r.t. the counting measure and whose Radon-Nikodym derivative $\sigma$ is a radial positive decreasing function on $X$. With slight abuse of notation we denote by $\sigma$ the reference measure, too. Given a reference measure $\sigma$ on $X$ for every $p \in [1, \infty)$ the Bergman space $A^p(\sigma)$ is the space of harmonic functions on $X$ such that

$$
\|f\|_{A^p(\sigma)}^p := \sum_{x \in X} |f(x)|^p \sigma(x) < +\infty.
$$

If $\sigma$ is a reference measure on $X$, and if we denote by $\sigma_n$ the value of $\sigma$ on the sphere $S(0,n)$, then the total mass of $\sigma$ is

$$
\sigma(X) = \sigma_0 + \frac{q+1}{q} \sum_{n=1}^{+\infty} \sigma_n q^n < +\infty.
$$

From now on, for a reference measure $\sigma$, we set $B_\sigma := \sigma(X) < +\infty$. 


Example 6. Let \( \alpha > 1 \). Interesting examples of reference measures are the class of exponentially decreasing radial measures, consisting of the measures having density

\[
\mu_\alpha(x) = q^{-\alpha|x|}, \quad x \in X.
\]

Indeed, \( \mu_\alpha \) is radial, positive and decreasing. Furthermore,

\[
B_{\mu_\alpha} = 1 + \frac{q + 1}{q} \sum_{n=1}^{+\infty} q^{(1-\alpha)n} = 1 + \frac{q + 1}{q} \frac{q^{1-\alpha}}{1 - q^{1-\alpha}} = \frac{q^{\alpha} + 1}{q^{\alpha} - q} < +\infty.
\]

In [3], the authors introduce the definition of optimal measure, that is a reference measure \( \sigma \) for which

\[
\sup_{n \in \mathbb{N}} \frac{1}{\sigma_n q^n} \sum_{m=n}^{+\infty} \sigma_m q^m < +\infty.
\]

It is easy to see that the exponentially decreasing radial measures are optimal.

Given a reference measure \( \sigma \), we introduce a decreasing sequence \( (b_m)_{m \in \mathbb{N}} \) which collects some important information on \( \sigma \). For every \( n \in \mathbb{N} \), we define

\[
b_n = b_n(\sigma) = \sum_{m=n+1}^{+\infty} \left[ \left( \sum_{k=0}^{m-n-1} q^k \right) \left( \sum_{j=0}^{m-n-1} q^{-j} \right) \right]. \tag{4}
\]

The sums are finite because \( \sigma \) is a finite measure on \( X \). In the following, we shall use the notation \( b_n \) instead of \( b_n(\sigma) \) whenever the measure is clear from the context.

The next lemma is a technical result that is very useful in what follows.

Lemma 7. Let \( n \in \mathbb{N} \) and \( g \) be a function on \( X \) which is harmonic on \( B(o,n) \) and vanishes on \( S(o,n) \). Then there exists a constant \( b'_n > 0 \) such that for every \( f \in \mathcal{A}^2(\sigma) \)

\[
\langle f, g_n^H \rangle_{\mathcal{A}^2(\sigma)} = \langle f|_{B(o,n)}, g|_{B(o,n)} \rangle_{\mathcal{A}^2(\sigma)} + \sum_{|y|=n+1} \left( b_n f(y) - b'_n f(p(y)) \right) g(y),
\]

where \( b_n \) is defined in (4).

Remark 8. The constant \( b'_n \) has a structure similar to that of \( b_n \), as can be seen in the proof below, but we are not interested in it.

Proof. Take \( f \in \mathcal{A}^2(\sigma) \). We have that

\[
\langle f, g_n^H \rangle_{\mathcal{A}^2(\sigma)} = \sum_{x \in B(o,n)} f(x)g(x)\sigma(x) + \sum_{m=n+1}^{+\infty} \sigma_m \sum_{|x|=m} f(x)g_n^H(x).
\]

Observe that from the definition of \( g_n^H \) and Proposition 2

\[
\sum_{|x|=m} f(x)g_n^H(x) = \sum_{|y|=n+1} \sum_{|x|=m} f(x)g_n^H(x) \\
= \sum_{|y|=n+1} \left( \sum_{j=0}^{m-n-1} q^{-j} \right) \left( \sum_{k=0}^{m-n-1} q^k \right) f(y) - \left( \sum_{k=0}^{m-n-2} q^k \right) f(p(y)) \left( \sum_{j=0}^{m-n-1} q^{-j} \right) \left( \sum_{k=0}^{m-n-2} q^k \right) g(y) \\
= \sum_{|y|=n+1} \left[ f(y)q^{m-n-1} \sum_{j=0}^{m-n-1} q^{-j} \right] \left[ f(p(y)) \sum_{j=0}^{m-n-1} q^{-j} \right] \left( \sum_{k=0}^{m-n-2} q^k \right) \left( \sum_{k=0}^{m-n-2} q^k \right) g(y).
\]
orthonormal basis for the space $W$.

We now focus on the case $A$. A canonical orthonormal basis of $A$ is independent of the choice of the reference measure $\sigma$ and that the harmonic extension that we have introduced is radial on sectors.

### 1.4 A canonical orthonormal basis of $A^2(\sigma)$

We now focus on the case $p = 2$. The goal of this section is the construction of an orthonormal basis for the space $A^2(\sigma)$.

Let us consider the linear spaces

$$W_v := \{ \varphi: s(v) \to \mathbb{C} : \sum_{z \in s(v)} \varphi(z) = 0 \} \simeq \left\{ \begin{array}{ll} \mathbb{C}^q, & v = o, \\
\mathbb{C}^{q-1}, & v \in X \setminus \{o\}. \end{array} \right.$$  

For convenience we introduce the following intervals of integer numbers: for every $v \in X$ we set $I_v = \{1, \ldots, |s(v)|\}$. We fix an orthonormal basis $\{e_{v,j}\}_{j \in I_v}$ of $W_v$ w.r.t. to the scalar product

$$\langle \varphi, \psi \rangle_{W_v} = \sum_{y \in s(v)} \varphi(y) \overline{\psi(y)}.$$  

Let $v \in X$ and $j \in I_v$. We consider the extension by zero to all of $X$ of $e_{v,j}$, namely,

$$E_{v,j}(x) = \begin{cases} e_{v,j}(x), & x \in s(v); \\
0, & x \notin s(v). \end{cases}$$  

It is easy to see that $E_{v,j}$ is harmonic on $X$ and vanishes on $B(o,|v|)$. We denote the harmonic extension of $E_{v,j}$ by $f_{v,j} = (E_{v,j})^{H}_{|v|}$, namely

$$f_{v,j}(x) = \begin{cases} 0, & \text{if } x \notin T_v \setminus \{v\}, \\
ap_{|x| - |v| - 1} E_{v,j}(p^{|x| - |v| - 1}(x)), & \text{otherwise}. \end{cases}$$  

Hence $f_{v,j}$ is harmonic for every $v \in X$ and $j \in I_v$. Furthermore $f_{v,j}$ is bounded, since $|f_{v,j}| \leq (1 - q^{-1})^{-1}\|e_{v,j}\|_{W_v,\infty}$, and then $f_{v,j} \in A^2(\sigma)$ for every reference measure $\sigma$. Observe that $f_{v,j}(v) = E_{v,j}(v) = 0$ for every $v \in X$ and $j \in I_v$. More precisely, if $v \neq o$, then

$$\text{supp} f_{v,j} \subseteq T_v \setminus \{v\}. \quad (5)$$  

Finally, we write $f_0 = 1_X$.

Notice that the family

$$\mathcal{F} = \{f_0\} \cup \{f_{v,j} : v \in X, j \in I_v\} \subseteq A^2(\sigma) \quad (6)$$  

is independent of the choice of the reference measure $\sigma$. Below we prove that $\mathcal{F}$ is an orthogonal system in every $A^2(\sigma)$. In the proofs we use the fact that $(e_{v,j})_{j \in I_v}$ are orthonormal and that the harmonic extension that we have introduced is radial on sectors.
Proposition 9. The family $\mathcal{F}$ is a complete orthogonal system in $A^2(\sigma)$ for every reference measure $\sigma$.

Proof. Fix a reference measure $\sigma$. The fact that $f_0$ is orthogonal to every function of the family follows from the harmonicity of $f_{v,j}$ and Corollary 3. Indeed

$$\langle f_{v,j}, f_0 \rangle_{A^2(\sigma)} = \sum_{x \in X} f_{v,j}(x)\sigma(x) = \sum_{n=0}^{+\infty} \sigma_n \sum_{|x|=n} f_{v,j}(x) = 0.$$ 

Let us consider $v, w \in X$ with $v \neq w$. Without loss of generality we may consider two situations: either $T_v \cap T_w = \emptyset$ or $T_v \subset T_w$. In the first case $f_{v,j} \perp f_{w,k}$ for every $j \in I_v$ and $k \in I_w$, because their supports are disjoint. If $T_v \subset T_w$, then we can suppose that $|w| \leq |v|$. Since $f_{v,j}|_{B(o|w|+1)} = 0$, from Lemma 7 we have

$$\langle f_{v,j}, f_{w,k} \rangle_{A^2(\sigma)} = \sum_{|y|=|w|+1} (b_{|w|}f_{v,j}(y) - B'_{|w|}f_{v,j}(p(y)))E_{w,k}(y) = 0.$$ 

It remains to prove orthogonality in the case $v = w$. Let $j, k \in I_v$ be such that $j \neq k$. We know that $f_{v,k}|_{B(o|v|)} = 0$, so that by Lemma 2

$$\langle f_{v,j}, f_{v,k} \rangle_{A^2(\sigma)} = b_{|v|} \sum_{|y|=|v|+1} E_{v,j}(y)E_{v,k}(y) = b_{|v|} \sum_{y \in s(v)} e_{v,j}(y)e_{v,k}(y) = 0,$$

where we used the fact that $\text{supp}(E_{v,k}) \subseteq s(v)$ and the orthogonality of $e_{v,j}$ and $e_{v,k}$ in $W_v$.

We show now that $\mathcal{F}$ is complete. Take $g \in A^2(\sigma)$ such that $\langle g, f \rangle_{A^2(\sigma)} = 0$ for every $f \in \mathcal{F}$. We will show that $g$ is the null function in $A^2(\sigma)$. In particular we prove by induction that $g \equiv 0$ on every $B(o, m), m \in \mathbb{N}$.

We start by observing that $\langle g, f_0 \rangle_{A^2(\sigma)} = 0$ implies $g(o) = 0$. Indeed by 3

$$0 = \langle g, f_0 \rangle_{A^2(\sigma)} = \sum_{n=0}^{+\infty} \sigma_n \sum_{|x|=n} g(x) = \left(1 + \frac{q+1}{q} \sum_{n=1}^{+\infty} q^n \sigma_n\right)g(o) = B_\sigma g(o).$$

We assume now $g = 0$ on $B(o, m)$ for some $m \in \mathbb{N}$. Let $v \in S(o, m)$. Observe that since $g$ is harmonic and $g(v) = 0$, we have $g|_{s(v)} \in W_v$. Hence for every $j \in I_v$

$$0 = \langle g, f_{v,j} \rangle_{A^2(\sigma)} = b_{|v|} \sum_{y \in s(v)} e_{v,j}(y)g(y)$$

and this implies that $g(y) = 0$ for every $y \in s(v)$ and so for every $y \in S(o, m+1)$, that is $g$ vanishes on $B(o, m+1)$. The statement follows by induction.

We have proved that, for every reference measure $\sigma$, $\mathcal{F}$ is a complete orthogonal system in $A^2(\sigma)$. We now fix a measure $\sigma$ and compute the norm of the functions of the family $\mathcal{F}$ in $A^2(\sigma)$. It is immediate to see that $\|f_0\|_{A^2(\sigma)} = B_\sigma$. Let $v \in X$ and $j \in I_v$. By 3, we have

$$\|f_{v,j}\|_{A^2(\sigma)}^2 = \langle f_{v,j}, f_{v,j} \rangle_{A^2(\sigma)} = b_{|v|} \sum_{y \in s(v)} e_{v,j}(y)e_{v,j}(y) = b_{|v|}.$$ 

Hence the norm of $f_{v,j}$ does not depend on $j$ and coincides with the constant in 3. Hence

$$\mathcal{F}_\sigma = \{B_\sigma^{-\frac{1}{2}}f_0\} \cup \{b_{\frac{1}{|v|}}f_{v,j} : v \in X, j \in I_v\}$$

is an orthonormal basis of $A^2(\sigma)$. 

2 The reproducing kernel of $A_2^2(\sigma)$

In this section we show that the Bergman spaces $A_2^2(\sigma)$ are reproducing kernel Hilbert spaces. Below we present a recursive formula for the kernel and then we derive a formula in closed form. Observe that the main ingredient used in the proofs are the harmonic extension and the orthonormal basis defined in the previous section together with the fact that $W_v$ are reproducing kernel Hilbert spaces, too.

Let $z \in X$. We consider the evaluation functional $\Phi_z : A_2^2(\sigma) \to \mathbb{C}$ defined by $\Phi_z g = g(z)$. Observe that $\Phi_z$ is a bounded operator, indeed by the Cauchy-Schwarz inequality

$$|g(z)| = \frac{1}{q + 1} \left| \sum_{x \sim z} g(x) \right| \leq \frac{1}{q + 1} \sum_{x \sim z} |g(x)| \leq \frac{1}{q + 1} \|g\|_{A_2^2(\sigma)} \frac{\|1_{S(z,1)}\|_{L^2(\sigma)}}{|z|},$$

where $1_{S(z,1)}$ is the characteristic function of the sphere $S(z,1)$. Since for every $x \in S(z,1)$, $\sigma(x) \geq \sigma_{|z|+1}$, we have that

$$\left\| \frac{1_{S(z,1)}}{\sigma} \right\|_{L^2(\sigma)}^2 = \sum_{d(z,x)=1} \frac{1}{\sigma(x)^2} \leq \sum_{d(z,x)=1} \frac{1}{\sigma_{|z|+1}^2} = \frac{q + 1}{\sigma_{|z|+1}^2}.$$ 

Hence

$$|g(z)| \leq (q + 1)^{-\frac{1}{2}} \sigma_{|z|+1}^{-1} \|g\|_{A_2^2(\sigma)}.$$

Thus $A_2^2(\sigma)$ is a reproducing kernel Hilbert space (RKHS), that is for every $z \in X$ there exists $K_z \in A_2^2(\sigma)$ such that

$$\langle g, K_z \rangle_{A_2^2(\sigma)} = g(z), \quad g \in A_2^2(\sigma).$$

Since $F_\sigma$ defined in (10) is an orthonormal basis of $A_2^2$, for every $z \in X$ we can write

$$K_z = \sum_{f \in F_\sigma} \langle f, K_z \rangle_{A_2^2} f = \sum_{f \in F_\sigma} f(z) f = \frac{1}{B_\sigma} + \sum_{v \in X} \sum_{j \in I_v} \frac{f_{v,j}(z) f_{v,j}}{b_{|v|}}.$$ (11)

We recall that by (12), for every $z \in X$

$$\{v \in X : f_{v,j}(z) \neq 0 \text{ for some } j \in I_v \} \subseteq B(o, |z| - 1).$$

Hence for every $z \in X$ the sum in (11) is finite and the decomposition of $K_z$ holds true pointwise.

Our goal is to compute $K_z$. To this end, we introduce the auxiliary function $\Gamma : X \times X \times X \to \mathbb{R}$ which is a parametrization of the family of reproducing kernels for the spaces $\{W_v\}_{v \in X}$. For every $(v, z, x) \in X \times X \times X$ we set

$$\Gamma(v, z, x) = \begin{cases} 0, & \text{if } \{z, x\} \not\subseteq T_v \setminus \{v\}; \\ \frac{|s(v)| - 1}{|s(v)|}, & \text{if } \{z, x\} \subseteq T_y \text{ for some } y \in s(v); \\ -\frac{1}{|s(v)|}, & \text{otherwise.} \end{cases}$$

Observe that $\Gamma$ is symmetric in the second and third variables. Furthermore, $\Gamma(v, z, \cdot)$ is the null function if $z \not\in T_v \setminus \{v\}$ and whenever $z \in T_v \setminus \{v\}$ we have supp$(\Gamma(v, z, \cdot)) = T_v \setminus \{v\}$. Moreover, the values of $\Gamma(v, z, \cdot)$ on $T_v \setminus \{v\}$ are completely determined by the values on $s(v)$, as the value of $\Gamma(v, z, \cdot)$ at $x \in T_v \setminus \{v\}$ is equal to the value at $p^{\frac{1}{2}}|v|^{-1}(x) \in s(v)$.
We show next that $\Gamma(v, z, \cdot)$ is the reproducing kernel of $W_v$, namely that for $z \in s(v)$ we have
\[ \varphi(z) = \langle \varphi, \Gamma(v, z, \cdot) \rangle_{W_v}, \quad \varphi \in W_v. \]
First of all $\Gamma(v, z, \cdot) \in W_v$ because
\[ \sum_{y \in s(v)} \Gamma(v, z, y) = -(|s(v)| - 1) \frac{1}{|s(v)|} + \frac{|s(v)| - 1}{|s(v)|} = 0. \]
Furthermore,
\[ \langle \varphi, \Gamma(v, z, \cdot) \rangle_{W_v} = \frac{|s(v)| - 1}{|s(v)|} \varphi(z) - \frac{1}{|s(v)|} \sum_{y \neq z, y \in s(v)} \varphi(y) \]
\[ = \frac{|s(v)| - 1}{|s(v)|} \varphi(z) + \frac{1}{|s(v)|} \varphi(z) = \varphi(z), \]
because $\varphi \in W_v$.
It is easy to see that $\Gamma(v, z, \cdot)$ is harmonic on $B(o, |v|)$ so that we can consider the harmonic extension $(\Gamma(v, z, \cdot))_{[v]}^H$, which is bounded by construction. Indeed from the definition of harmonic extension we have for every $x \in T_v \setminus \{v\}$
\[ (\Gamma(v, z, \cdot))_{[v]}^H(x) = \left( \sum_{j=0}^{[x]-[v]-1} q^{-j} \right) \Gamma(v, z, p^{[x]-[v]-1}(x)) = a_{[x]-[v]-1} \Gamma(v, z, x), \quad (12) \]
and it vanishes elsewhere. We recall that if $z \notin T_v$, then $\Gamma(v, z, \cdot) = (\Gamma(v, z, \cdot))_{[v]}^H$ is the null function.

**Proposition 10.** Let $z \in X$ and $[o, z] = \{v_t\}_{t=0}^{|z|}$. The kernel $K_z$ is
\[ K_z = \begin{cases} \frac{1}{B_o}, & \text{if } z = o, \\ K_o + \frac{1}{b_0} (\Gamma(o, z, \cdot))_{[v]}^H, & \text{if } |z| = 1, \\ -\frac{1}{q} K_{v_m-2} + \frac{q+1}{q} K_{v_{m-1}} + \frac{1}{b_{m-1}} (\Gamma(v_{m-1}, z, \cdot))_{[v]}^H, & \text{if } |z| = m > 1. \end{cases} \]
Proving. Since the measure $\sigma$ is finite and the constant functions are harmonic, $K_o = \frac{1}{p_o} \in A^2(\sigma)$. The reproducing property follows from (7).

Now we observe that for every $v, z \in X$ such that $z \in T_v$ and $g \in A^2(\sigma)$ we have that

$$\langle g, (\Gamma(v, z, \cdot)) \rangle_{A^2(\sigma)} = b_{|v|} \sum_{y \in \delta(v)} g(y) \Gamma(v, z, y).$$  \hspace{1cm} (13)

Indeed, by Lemma (7) and $\text{supp} \Gamma(v, z, \cdot) = T_v \setminus \{v\}$, we have

$$\langle g, (\Gamma(v, z, \cdot)) \rangle_{A^2(\sigma)}^{H} = \langle g, B(o, |v|), \Gamma(v, z, \cdot) \rangle_{A^2(\sigma)} + \sum_{|y| = |v| + 1} (b_{|v|} g(y) - b'_v g(p(y))) \Gamma(v, z, y) \hspace{1cm} (13.1)$$

where we used $\Gamma(v, z, \cdot) \delta(v) \in W_v$.

We prove the case $|z| = 1$. The function $K_z \in A^2(\sigma)$ because it is sum of functions in $A^2(\sigma)$. We prove the reproducing property. For $g \in A^2(\sigma)$, by the reproducing formula of $K_o$ and (13) with $v = o$, whenever $|z| = 1$

$$\langle g, K_z \rangle_{A^2(\sigma)} = g(o) + \frac{1}{b_0} \langle g, (\Gamma(o, z, \cdot)) \rangle_{A^2(\sigma)}^{H} \hspace{1cm} (13.2)$$

where we used that $g$ is harmonic at $o$. This proves the case $|z| = 1$.

It remains to prove the case $|z| = m > 1$. We have $K_z \in A^2(\sigma)$ since it is the sum of bounded and harmonic functions. For $g \in A^2(\sigma)$ by induction on $m$ and (13) with $v = v_{m-1}$ we have

$$\langle g, K_z \rangle_{A^2(\sigma)} = \frac{1}{q} g(v_{m-2}) + \frac{q + 1}{q} g(v_{m-1}) + \frac{1}{b_{m-1}} \langle g, (\Gamma(v_{m-1}, z, \cdot)) \rangle_{A^2(\sigma)}^{H} \hspace{1cm} (13.3)$$

where we used the fact that $g$ is harmonic at $v_{m-1}$. \hfill \square

In Proposition (13) the kernel $K_z$ is expressed through a two-step recursive formula. We aim to find an explicit formula for $K_z$. 

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Theorem 11. For every \((z, x) \in X \times X\)
\[
K(z, x) = \frac{1}{B_\sigma} + \frac{q^2}{(q - 1)^2} \sum_{v \in X} \frac{1}{b_{|v|}} \Gamma(v, z, x)(1 - q^{|v| - |z|})(1 - q^{|v| - |x|}).
\] (14)

Proof. Let \(z \in X\) and \([o, z] = \{v_t\}_{t=0}^{|z|}\). We start by proving that
\[
K_z = \frac{1}{B_\sigma} + \sum_{t=0}^{|z|} \left( \sum_{j=0}^{|z| - t} q^{-j} \right) \frac{1}{b_t} \Gamma(v_t, v_{t+1}, \cdot)^H.
\] (15)

The case \(z = o\) follows trivially from Proposition 10 and the convention on sums stated in the Introduction. We prove (15) by induction on \(m = |z| \geq 1\). The case \(m = 1\) directly follows from Proposition 10 too. Let \(m \in \mathbb{N}\), \(m > 1\) and \(z \in X\), with \(|z| = m\). Suppose that (15) holds for every vertex in \(B(o, m - 1)\). Hence by Proposition 10 we have
\[
K_z = -\frac{1}{q} K_{v_{m-2}} + \frac{q + 1}{q} K_{v_{m-1}} + \frac{1}{b_{m-1}} \Gamma(v_{m-1}, z, \cdot)^H
\]
\[
= -\frac{1}{q} \left[ \frac{1}{B_\sigma} + \sum_{t=0}^{m-3} \left( \sum_{j=0}^{m-t-3} q^{-j} \right) \frac{1}{b_t} \Gamma(v_t, v_{t+1}, \cdot)^H \right]
\]
\[
+ \frac{q + 1}{q} \left[ \frac{1}{B_\sigma} + \sum_{t=0}^{m-2} \left( \sum_{j=0}^{m-t-2} q^{-j} \right) \frac{1}{b_t} \Gamma(v_t, v_{t+1}, \cdot)^H \right]
\]
\[
+ \frac{1}{b_{m-1}} \Gamma(v_{m-1}, z, \cdot)^H
\]
\[
= \frac{1}{B_\sigma} + \sum_{t=0}^{m-2} \left( \frac{q + 1}{q} \left( \sum_{j=0}^{m-t-2} q^{-j} \right) - \frac{1}{q} \left( \sum_{j=0}^{m-t-3} q^{-j} \right) \right) \frac{1}{b_t} \Gamma(v_t, v_{t+1}, \cdot)^H
\]
\[
+ \frac{1}{b_{m-1}} \Gamma(v_{m-1}, z, \cdot)^H
\]
\[
= \frac{1}{B_\sigma} + \sum_{t=0}^{m-1} \left( \sum_{j=0}^{m-t-1} q^{-j} \right) \frac{1}{b_t} \Gamma(v_t, v_{t+1}, \cdot)^H.
\]

Hence we proved (15) by induction. Since \(\text{supp}(\Gamma(v_t, v_{t+1}, \cdot)^H) = T_{v_t} \setminus \{v_t\}\), we have that the \(t\)-th term of the sum in (15) does not vanish if and only if \(x \in T_{v_t} \setminus \{v_t\}\) and hence by (12), we have
\[
K(z, x) = K_z(x) = \frac{1}{B_\sigma} + \sum_{v \in X} \frac{1}{b_{|v|}} \left( \sum_{j=0}^{|z| - |v| - t} q^{-j} \right) \left( \sum_{j=0}^{|z| - |v|} q^{-j} \right) \Gamma(v, z, x)
\]
\[
= \frac{1}{B_\sigma} + \frac{q^2}{(q - 1)^2} \sum_{v \in X} \frac{1}{b_{|v|}} (1 - q^{|v| - |z|})(1 - q^{|v| - |x|}) \Gamma(v, z, x).
\]

\[\blacksquare\]

Remark 12. The confluent of two vertices \(z, x \in X\) is the common vertex of \([o, x]\) and \([o, z]\) farthest from \(o\), denoted by \(z \lor x\). It is possible to see that the value of the kernel \(K\) at \((z, x) \in X \times X\) depends only on the values of \(|x|, |z|\) and \(|z \lor x|\). Furthermore, from (14) it is clear that \(K\) is symmetric, that is \(K(z, x) = K(x, z)\).
3 Bergman projectors

3.1 Boundedness of the Bergman projector

In this section we study the boundedness properties of the extension of the Bergman projector to $L^p$ spaces.

We restrict our attention to the family of the exponentially decreasing radial measures $\mu_\alpha$, $\alpha > 1$ defined in Example 6. For this class of measures we are able to prove that the extension of the Bergman projector to $L^p(X)$ is bounded if and only if $p > 1$ (see Theorem 18).

We shall use the notation $L^p_\alpha$ and $A^p_\alpha$ for the Lebesgue and Bergman spaces w.r.t. $\mu_\alpha$, respectively. Furthermore, we denote by $K_\alpha : X \times X \to \mathbb{R}$ the reproducing kernel of $A^2_\alpha$. It will be useful to keep track of the weight in the constants introduced in (4), so we denote them by $b_{\alpha,n}$. In particular observe that in this case there is a relation between the constants: for every $n \in \mathbb{N}$

$$b_{\alpha,n} = \sum_{m=n+1}^{+\infty} \left[ q^{-am} \left( \sum_{k=0}^{m-n-1} q^k \right) \left( \sum_{j=0}^{m-n-1} q^{-j} \right) \right]$$

$$= \sum_{\ell=1}^{+\infty} \left[ q^{-a(\ell+n)} \left( \sum_{k=0}^{\ell-1} q^k \right) \left( \sum_{j=0}^{\ell-1} q^{-j} \right) \right] = q^{-an}b_{\alpha,0}. \quad (16)$$

Furthermore we set $B_\alpha = \mu_\alpha(X)$.

In analogy with Toeplitz-type operators studied by Zhu in [9], we introduce two families of operators. For any real parameters $a, b$ and for $c > 1$, we define the following integral operators

$$S_{a,b,c} f(z) = q^{-a|z|} \sum_{x \in X} |K_c(z, x)| f(x) q^{-b|x|},$$

and

$$T_{a,b,c} f(z) = q^{-a|z|} \sum_{x \in X} K_c(z, x) f(x) q^{-b|x|}.$$

We are now in a position to state two results, which will imply as a corollary the boundedness properties of the Bergman projectors. Theorem 13 is devoted to the study of the boundedness of the operators $S_{a,b,c}$ and $T_{a,b,c}$ on weighted $L^p$-spaces for $p > 1$; the case $p = 1$ needs different arguments and for this reason is treated apart in Theorem 14. The proofs of both theorems are postponed to Subsection 3.2.

**Theorem 13.** Let $\alpha \in \mathbb{R}$, $c > 1$ and $1 < p < \infty$. The following conditions are equivalent:

(i) the operator $S = S_{a,b,c}$ is bounded on $L^p_\alpha$;

(ii) the operator $T = T_{a,b,c}$ is bounded on $L^p_\alpha$;

(iii) the parameters satisfy

$$c \leq a + b, \quad -pa < \alpha - 1 < p(b - 1).$$

**Theorem 14.** Let $\alpha \in \mathbb{R}$ and $c > 1$. The following conditions are equivalent:

(i) the operator $S = S_{a,b,c}$ is bounded on $L^1_\alpha$;

(ii) the operator $T = T_{a,b,c}$ is bounded on $L^1_\alpha$.
(ii) the operator $T = T_{a,b,c}$ is bounded on $L^1_\alpha$;

(iii) the parameters either satisfy
$$c = a + b, \quad -a < \alpha - 1 < b - 1,$$

or satisfy
$$c < a + b, \quad -a < \alpha - 1 \leq b - 1.$$

We state a corollary which is simply a reformulation of the previous theorems when $c = a + b$.

**Corollary 15.** Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$. If $a, b \in \mathbb{R}$ are such that $a + b > 1$, then the following conditions are equivalent:

(i) the operator $S = S_{a,b,a+b}$ is bounded on $L^p_\alpha$;

(ii) the operator $T = T_{a,b,a+b}$ is bounded on $L^p_\alpha$;

(iii) the parameters satisfy
$$-pa < \alpha - 1 < p(b - 1).$$

Let $\beta > 1$. Since $A^2_{\beta} \subseteq L^2_{\beta}$ is a closed subspace of a Hilbert space, there exists an orthogonal projection $P_\beta : L^2_{\beta} \to A^2_{\beta}$. Observe that by the reproducing property of $K_{\beta,z} = K_{\beta}(z, \cdot)$, $z \in X$, we can write the projection $P_\beta f$ of $f \in L^2_{\beta}$ as follows
$$P_\beta f(z) = \langle P_\beta f, K_{\beta,z} \rangle_{A^2_{\beta}} = \langle f, P_\beta K_{\beta,z} \rangle_{L^2_{\beta}} = \langle f, K_{\beta,z} \rangle_{L^2_{\beta}},$$

where we used the orthogonality of $P_\beta$. Hence we can rewrite $P_\beta$ as the integral operator on $L^2_{\beta}$ induced by the reproducing kernel $K_\beta$, that is
$$P_\beta f(z) = \sum_{x \in X} K_\beta(z,x)f(x)q^{-\beta|x|}, \quad f \in L^2_{\beta}, z \in X. \quad (17)$$

Now we prove a preliminary result which is useful both in the proofs of the next subsection and to show that $P_\alpha$ is not bounded on $L^1_\alpha$.

**Lemma 16.** Let $\alpha > 1$. Then:
$$\sum_{x \in X} |K_\alpha(x,z)||q^{-\alpha|x|}| \geq |x|, \quad x \in X.$$

**Proof.** For every $x \in X \setminus \{0\}$, we put $\{v_t\}_{t=0}^{|x|} = \{a, x\}$. Then, by (14)
$$\sum_{x \in X} |K_\alpha(x,z)||q^{-\alpha|x|}| = \sum_{t=1}^{|x|} |K_\alpha(x,v_t)||q^{-\alpha t}$$
$$= \sum_{t=1}^{|x|} \left( \frac{1}{B_\alpha} + \frac{q^2}{(q+1)^2} \sum_{v \in X} \frac{1}{b_\alpha |v|} \Gamma(v, v_t, x)(1 - q^{|v|-t})(1 - q^{|v|-|x|}) \right) q^{-\alpha t}$$
$$\geq b_\alpha^{-1} \sum_{t=1}^{|x|} \sum_{v \in X} q^{\alpha(|v|-t)} \Gamma(v, v_t, x)(1 - q^{|v|-t})(1 - q^{|v|-|x|})$$
$$= \sum_{t=1}^{|x|} \sum_{\ell=0}^{|x|} q^{\alpha(t-\ell)} \Gamma(v_t, v_t, x)(1 - q^{t-\ell})(1 - q^{t-|x|})$$
$$\geq \sum_{t=1}^{|x|} \sum_{\ell=0}^{|x|} q^{\alpha(t-\ell)} \approx \sum_{t=1}^{|x|} q^{-\alpha t} q^{\alpha t} = |x|,$
where we used the fact that supp(\(\Gamma(\cdot, v, x)\)) = \([o, v_{t-1}] = [v_0, v_{t-1}]\) and the function is greater than or equal to \(\frac{2}{q}\) there.

Corollary 17. For every \(\alpha > 1\). The projector \(P_\alpha\) is unbounded on \(L^1_\alpha\).

Proof. For every \(n \in \mathbb{N}\), we fix a vertex \(v_n\) in \(S(o, n)\), and define

\[
f_n(x) = \mathbb{1}_{\{v_n\}}(x)q^{\alpha|x|}, \quad x \in X.
\]

Clearly, \(\|f_n\|_{L^1_\alpha} = 1\). On the other hand, we have \(P_\alpha f_n(z) = K_\alpha(z, v_n)\), and then by Lemma 16

\[
\|P_\alpha f_n\|_{L^1_\alpha} = \sum_{x \in X} |K_\alpha(z, v_n)|q^{-\alpha|z|} \geq |v_n| = n,
\]

which tends to \(+\infty\) as \(n \to +\infty\).

The case \(p = 1\) is actually the only value of \(1 \leq p < \infty\) for which \(P_\alpha\) is not bounded on \(L^p_\alpha\). This follows from Corollary 15.

Theorem 18. Let \(1 \leq p < \infty\), \(\alpha, \beta > 1\). The operator \(P_\beta\) is bounded from \(L^p_\alpha\) to \(\mathcal{A}_\alpha^p\) if and only if

\[
p(\beta - 1) > \alpha - 1.
\]

In particular, \(P_\alpha\) is bounded from \(L^p_\alpha\) to \(\mathcal{A}_\alpha^p\) if and only if \(p > 1\).

Proof. It is sufficient to observe that from (17), \(P_\beta = T_{0,\beta,\beta}\). Hence, from Corollary 15, the boundedness of \(P_\beta\) on \(L^p_\alpha\) is equivalent to \(p(\beta - 1) > \alpha - 1(> 0)\).

As a direct application of Theorem 18 we deduce the following result on the dual of Bergman spaces.

Corollary 19. Let \(1 < p < \infty\) and \(\alpha > 1\). Then

\[
(\mathcal{A}_\alpha^p)^* = \mathcal{A}_\alpha^{p'},
\]

with equivalent norms under the pairing

\[
\langle f, g \rangle_{\mathcal{A}_\alpha^p \times \mathcal{A}_\alpha^{p'}} = \sum_{z \in X} f(z)g(z)q^{-\alpha|z|} \quad f \in \mathcal{A}_\alpha^p, g \in \mathcal{A}_\alpha^{p'}.
\] (18)

Proof. Let \(g \in \mathcal{A}_\alpha^{p'}\). By Hölder inequality we have that

\[
|\langle f, g \rangle_{\mathcal{A}_\alpha^p \times \mathcal{A}_\alpha^{p'}}| \leq \|g\|_{\mathcal{A}_\alpha^{p'}}\|f\|_{\mathcal{A}_\alpha^p},
\]

for every \(f \in \mathcal{A}_\alpha^p\) and then \(g\) defines an operator in \((\mathcal{A}_\alpha^p)^*\). Conversely, for \(\Phi \in (\mathcal{A}_\alpha^p)^*\), then by the Hahn-Banach theorem, there exists \(\Phi \in (L^p_\alpha)^*\) such that \(\Phi|_{\mathcal{A}_\alpha^p} = \Phi\) and \(\|\Phi\|_{(L^p_\alpha)^*} \geq \|\Phi\|_{(L^p_\alpha)^*}\). Then by the duality of \(L^p\) spaces there exists \(h \in L^{p'}\) such that

\[
\Phi(f) = \Phi(h) = \langle f, h \rangle_{L^p_\alpha \times L^{p'}};
\]

for every \(f \in \mathcal{A}_\alpha^p\). By the orthogonality of \(P_\alpha\) and Theorem 18

\[
\Phi(f) = \langle P_\alpha f, P_\alpha h \rangle_{\mathcal{A}_\alpha^p \times \mathcal{A}_\alpha^{p'}} = \langle f, P_\alpha h \rangle_{\mathcal{A}_\alpha^p \times \mathcal{A}_\alpha^{p'}}.
\]

Hence \(\Phi\) corresponds to \(P_\alpha h \in \mathcal{A}_\alpha^2\) under the pairing (18).
3.2 Proofs of Theorems 13 and 14

This subsection is devoted to the proofs of Theorems 13 and 14, splitting up the proofs in various results. In both statements it is straightforward to see that (i) implies (ii). For the rest of the section $\alpha, a, b, c$ denote real parameters with $c > 1$.

3.2.1 Proof that (ii) implies (iii)

In this subsection we suppose that the operator $T_{a,b,c}$ is bounded on $L^p_\alpha$ and we deduce necessary conditions on the parameters $a, b, c, \alpha$ in various lemmas.

Lemma 20. Let $1 \leq p < \infty$. If $T_{a,b,c}f \in L^p_\alpha$ for every $f \in L^p_\alpha$, then $-pa < \alpha - 1$.

Proof. Consider $f(x) = q^{-R|x|}$ with $R \in \mathbb{R}$ such that

$$R > \max \left\{ \frac{1 - \alpha}{p}, 1 - b \right\}.$$

Since $R > \frac{1 - \alpha}{p}$ we have that $f \in L^p_\alpha$ and for every $z \in X$

$$T_{a,b,c}f(z) = q^{-a|z|} \sum_{x \in X} K_c(z, x)q^{-(b+R)|x|}$$

$$= q^{-a|z|} \sum_{n=0}^{+\infty} q^{-(b+R)n} \sum_{|x| = n} K_c(z, x)$$

$$= q^{-a|z|} \sum_{n=0}^{+\infty} q^{-(b+R)n} |S(o, n)| K_c(z, o)$$

by Corollary 3 applied to the harmonic function $K_c(z, \cdot)$. Hence, since $R > 1 - b$

$$T_{a,b,c}f(z) = q^{-a|z|} \frac{1}{B_c} \left[ 1 + \frac{q + 1}{q} \sum_{n=1}^{+\infty} q^{-(b-R+1)n} \right] = \frac{B_{b+R}}{B_c} q^{-a|z|}, \quad z \in X.$$

Now observe that $T_{a,b,c}f \in L^p_\alpha$ implies

$$\sum_{z \in X} q^{-(ap+\alpha)|z|} = 1 + \frac{q + 1}{q} \sum_{n=1}^{+\infty} q^{(1-\alpha)n} < +\infty,$$

which holds if and only if $-pa < \alpha - 1$, as required.

From now on we put

$$\|e_{v,j}\|_p = \left( \sum_{y \in s(v)} |e_{v,j}(y)|^p \right)^{1/p}, \quad v \in X, j \in I_v, 1 \leq p < \infty.$$

Lemma 21. Let $1 \leq p < \infty$. If $T_{a,b,c}$ is bounded on $L^p_\alpha$, then $a + b \geq c$.

Proof. Fix $R \in \mathbb{R}$ such that

$$R > \max \left\{ \frac{1 - \alpha}{p}, c - b \right\}.$$
For every \( v \in X \setminus \{o\} \) and \( j \in I_v \), we define \( g_{v,j}(x) = f_{v,j}(x)q^{-R|x|} \), where \( f_{v,j} \in \mathcal{F} \) are defined in (5). By \( R > \frac{1-\alpha}{p} \), we have that \( g_{v,j} \in L^p_\alpha \), then

\[
T_{a,b,c}g_{v,j}(z) = q^{-a|z|} \sum_{x \in X} K_c(z,x) f_{v,j}(x) q^{-(b+R)|x|}
\]

\[
= q^{-a|z|} \langle f_{v,j}, K_c, z \rangle_{L^2_{b+R}}
\]

since \( R > c - b \) implies \( K_{c,z} \in L^2_c \subseteq L^2_{b+R} \). Now we use the decomposition (11) of \( K_{c,z} \) on the orthonormal basis of \( A^2_c \) and obtain

\[
\langle K_{c,z}, f_{v,j} \rangle_{L^2_{b+R}} = \left\langle \frac{1}{B_c} + \sum_{u \in X} \sum_{k \in I_u} \frac{f_{u,k}(z) f_{u,k}}{b_{c,|v|}}, f_{v,j} \right\rangle_{L^2_{b+R}}
\]

\[
= \frac{f_{v,j}(z)}{b_{c,|v|}} \langle f_{v,j}, f_{v,j} \rangle_{L^2_{b+R}}
\]

\[
= \frac{b_{b+R,|v|}}{b_{c,|v|}} f_{v,j}(z),
\]

where we use the orthogonality of \( \mathcal{F} \) and (3). The norm of \( T_{a,b,c}g_{v,j} \) in \( L^p_\alpha \) is

\[
||T_{a,b,c}g_{v,j}||^p_{L^p_\alpha} = \left( \frac{b_{b+R,|v|}}{b_{c,|v|}} \right)^p \sum_{x \in X} |f_{v,j}(z)|^p q^{-(a+\alpha)|z|}
\]

\[
= \left( \frac{b_{b+R,|v|}}{b_{c,|v|}} \right)^p \sum_{n=0}^{+\infty} q^{-(a+\alpha)n} \sum_{|z| = n} |f_{v,j}(z)|^p.
\]

Since \( \text{supp}(f_{v,j}) \subseteq T_v \setminus \{v\} \), the integral of \( |f_{v,j}|^p \) on the sphere \( S(o, n) \) vanishes for every \( n \leq |v| \). If \( n > |v| \), then \( p^{|z|-n}|z| \) is the unique vertex in \( s(v) \) such that \( z \) lies in its sector. Hence

\[
\sum_{|z| = n} |f_{v,j}(z)|^p = \sum_{|z| = n} |e_{v,j}(p^{|z|-n}(z))|^p a_{n, |p^{z|-n}(z)|}^p
\]

\[
= a_{n-|v|-1}^p \sum_{|z| = n} |e_{v,j}(p^{z|-n}(z))|^p
\]

\[
= a_{n-|v|-1}^p q^{n-|v|-1} \sum_{y \in s(v)} |e_{v,j}(y)|^p
\]

\[
= a_{n-|v|-1}^p q^{n-|v|-1} \|e_{v,j}\|^p_{p}.
\]

For simplicity, for every \( s \in \mathbb{R} \) and \( 1 < p < \infty \), we put

\[
C(s, p) := \sum_{m=1}^{+\infty} q^{(1-s)m-1} a_{m-1}^p.
\]
Observe that $C(s, p)$ converges whenever $s < 1$. Hence we have

$$\|T_{a,b,c}g_{v,j}\|_{L^p}^p = \left(\frac{b_{b+R_1|v|}}{b_{c_1|v|}}\right)^p \sum_{n=|v|+1}^{+\infty} q^{-|ap+\alpha|n}a_{n-|v|-1}^p q^{n-|v|-1} \|e_{v,j}\|_{L^p}^p$$

$$= \|e_{v,j}\|_{L^p}^p \left(\frac{b_{b+R_1|v|}}{b_{c_1|v|}}\right)^p \sum_{m=1}^{+\infty} q^{-|ap+\alpha|(m+|v|)}a_{m-1}^p q^{m-1}$$

$$= \|e_{v,j}\|_{L^p}^p \left(\frac{b_{b+R_1|v|}}{b_{c_1|v|}}\right)^p q^{-|ap+\alpha||v|}a_{1}^p \sum_{m=1}^{+\infty} q^{(1-|ap+\alpha|)m-1}a_{m-1}^p$$

$$= \|e_{v,j}\|_{L^p}^p C(ap + \alpha, p) \left(\frac{b_{b+R_1|v|}}{b_{c_1|v|}}\right)^p q^{-|ap+\alpha||v|},$$

where $C(ap + \alpha, p)$ converges from $ap + \alpha > 1$, through Lemma 20. On the other hand,

$$\|g_{v,j}\|_{L^p}^p = \sum_{x \in X} |f_{v,j}(x)|^p q^{-(Rp+\alpha)|x|}$$

$$= \sum_{n=0}^{+\infty} q^{-(Rp+\alpha)n} \sum_{|x|=n} |f_{v,j}(x)|^p$$

$$= \sum_{n=|v|+1}^{+\infty} q^{-(Rp+\alpha)n}a_{n-|v|-1}^p q^{n-|v|-1} \|e_{v,j}\|_{L^p}^p$$

$$= \|e_{v,j}\|_{L^p}^p C(Rp + \alpha, p)q^{-|ap+\alpha||v|},$$

with $C(Rp + \alpha, p) \to 1$ when $R \to +\infty$. From the boundedness of $T_{a,b,c}$ we have that the following rapport is bounded as a function of $v \in X\setminus\{a\}$:

$$\frac{\|T_{a,b,c}g_{v,j}\|_{L^p}^p}{\|g_{v,j}\|_{L^p}^p} \approx \left(\frac{b_{b+R_1|v|}}{b_{c_1|v|}}\right)^p q^{-|ap+\alpha-Rp-\alpha||v|} \approx q^{-p(R+b-c)||v||p|-(ap-Rp)||v|},$$

from (10). Hence we have that $c \leq a + b$. 

**Lemma 22.** Let $1 < p < \infty$. If $T_{a,b,c}$ is bounded on $L^p_\alpha$, then $\alpha - 1 < p(b - 1)$.

**Proof.** From Theorem 1.9 in [9], the boundedness of $T_{a,b,c}$ on $L^p_\alpha$ is equivalent to the boundedness of the adjoint operator $T^*_{a,b,c}$ on $L^{p'}_\alpha$. It is easy to see that

$$T^*_{a,b,c}g(x) = q^{-(b-\alpha)|x|} \sum_{z \in X} K_c(x, z)g(z)q^{-(a+\alpha)|z|} = T_{b-\alpha,a+a,c}g(x) \quad g \in L^{p'}_\alpha.$$

Hence, the fact that $T^*_{a,b,c}$ is bounded on $L^{p'}_\alpha$ implies, through Lemma 20 that $-p'(b-\alpha) < \alpha - 1$, that is $\alpha - 1 < p(b - 1)$. 

Lemmas 20, 21, 22 show that (ii) implies (iii) in Theorem 16. Now we focus on the same implication in the case $p = 1$. 

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Lemma 23. If $T_{a,b,c}$ is bounded on $L^1_{\alpha}$, then

\[
\alpha < b, \quad \text{when } c = a + b; \\
\alpha \leq b, \quad \text{when } c < a + b.
\]

Proof. From Lemma 21 if $T_{a,b,c}$ is bounded on $L^1_{\alpha}$, then $c \leq a + b$. From Theorem 1.9 in [9], the boundedness of $T_{a,b,c}$ on $L^1_{\alpha}$ implies the boundedness of the adjoint operator $T^*_{a,b,c}$ on $L^\infty_{\alpha}$ defined by

\[
T^*_{a,b,c}g(x) = q^{-(b-\alpha)|x|} \sum_{z \in X} K_c(x,z)g(z)q^{-(a+\alpha)|z|}, \quad g \in L^\infty_{\alpha}.
\]

In particular, for $1_X \in L^\infty_{\alpha}$, we have

\[
T^*_{a,b,c}1_X(x) = q^{-(b-\alpha)|x|} \sum_{z \in X} K_c(x,z)q^{-(a+\alpha)|z|}
= q^{-(b-\alpha)|x|} \frac{1}{B_c} \sum_{n=0}^{+\infty} |S(o,n)|q^{-(a+\alpha)n} = \frac{B_{a+\alpha}}{B_c} q^{-(b-\alpha)|x|},
\]

which belongs to $L^\infty_{\alpha}$ if and only if $\alpha \leq b$.

Suppose now that $a + b = c$. We know that $\alpha \leq b$ and we want to prove that $\alpha < b$. Assume by contradiction that $\alpha = b$. Theorem 3.6 in [9] states that $T^*_{a,b,c}$ is bounded on $L\infty_{\beta}$, where

\[
T^*g(x) = \sum_{z \in X} K_c(x,z)g(z)q^{-|z|}, \quad g \in L^\infty_{\beta}.
\]

The boundedness of $T^*_{a,b,c}$ on $L^\infty_{\beta}$ implies that

\[
\sup_{x \in X} \sum_{z \in X} |K_c(x,z)|q^{-|z|} < +\infty. \tag{19}
\]

Which is a contradiction by Lemma 16. Hence $T_{a,b,c}$ is unbounded.

Lemmas 20 21 23 show that (ii) implies (iii) in Theorem 14.

3.2.2 Proof that (iii) implies (i)

We start by stating a technical lemma, which will be useful in both Proposition 25 and Proposition 26 that are devoted to prove that (iii) implies (i) in the case $p > 1$ and $p = 1$, respectively.

Lemma 24. Let $\beta, \gamma > 1$. Then there exist $C_1, C_2 > 0$ depending only on $\beta$ and $\gamma$ such that

\[
\sum_{x \in X} |K_\gamma(z,x)|q^{-\beta|x|} \leq \begin{cases} C_1(1 + q^{-(\beta-\gamma)|z|}), & \text{if } \gamma \neq \beta, \\ C_2(1 + |z|), & \text{if } \gamma = \beta. \end{cases}
\]

Proof. We start by observing that the orthogonal bases $\{e_{v,j}\}_{j \in I_v}$ of $W_v$, $v \in X$, involved in the construction of functions in $F$ are such that their 1-norms in $F$ are bounded from above, namely

\[
\|e_{v,j}\|_1 \leq \sqrt{|s(v)|} \|e_{v,j}\|_2 = \sqrt{|s(v)|} \leq \sqrt{q+1}, \quad v \in X, j \in I_v.
\]
Hence
\[
\sum_{x \in X} |K_{\gamma}(z, x)|q^{-\beta|x|} = \sum_{x \in X} \frac{1}{B_\gamma} \sum_{v \in X} \sum_{j \in I_v} f_{v, j}(x) f_{v, j}(x) \bigg| \frac{b_{\gamma, |v|}}{b_{\beta, |v|}} \bigg| q^{-\beta|x|} \\
\leq \frac{B_\beta}{B_\gamma} \sum_{v \in X} \sum_{j \in I_v} \sum_{x \in X} f_{v, j}(z) f_{v, j}(x) \bigg| \frac{b_{\gamma, |v|}}{b_{\beta, |v|}} \bigg| q^{-\beta|x|} \\
= \frac{B_\beta}{B_\gamma} \sum_{v \in X} \frac{1}{b_{\gamma, |v|}} \sum_{j \in I_v} f_{v, j}(z) \sum_{x \in X} |f_{v, j}(x)| q^{-\beta|x|} \\
\leq \frac{B_\beta}{B_\gamma} \sum_{v \in X} \frac{1}{b_{\gamma, |v|}} \sum_{j \in I_v} f_{v, j}(z) \|B_\beta \|_{L_2^2} \\
= \frac{B_\beta}{B_\gamma} + \frac{B_\beta^{-\frac{1}{2}}}{b_{\gamma, |v|}} \sum_{v \in X} \sum_{j \in I_v} |f_{v, j}(z)|,
\]
where we use the fact that the measure \(\mu_\beta\) is finite on \(X\) and thus \(\|\cdot\|_{L_2^2} \leq B_\beta^{-\frac{1}{2}} \|\cdot\|_{L_2^2}\), by Cauchy-Schwarz inequality. Now observe that from (5), we have that \(f_{v, j}(z) = 0\) if \(z \notin T_v \setminus \{v\}\). Hence, if we denote by \(\{v_t\}_{t=0}^{\infty}\) the path \([a, z]\), then
\[
\sum_{j \in I_v} |f_{v, j}(z)| = \begin{cases} a_{|z| - \ell - 1} \sum_{j \in I_v} |e_{v, j}(v_{\ell+1})|, & \text{if } v = v_{\ell}, 0 \leq \ell < |z|; \\
0, & \text{otherwise.}
\end{cases}
\tag{20}
\]
Therefore, since \(a_n \leq \frac{\delta}{\beta - 1}\) and by using (16), we have
\[
\sum_{x \in X} |K_{\gamma}(z, x)|q^{-\beta|x|} \leq \frac{B_\beta}{B_\gamma} + \frac{B_\beta^{-\frac{1}{2}}}{b_{\gamma, |v|}} \sum_{\ell=0}^{\lfloor \frac{|z| - 1}{|v|} \rfloor} \sum_{j \in I_v} |e_{v, j}(v_{\ell+1})| \\
\leq \frac{B_\beta}{B_\gamma} + \frac{B_\beta^{-\frac{1}{2}}}{b_{\gamma, |v|}} \sum_{\ell=0}^{\lfloor \frac{|z| - 1}{|v|} \rfloor} q^{-\beta|z|} \sup_{j \in I_v} |e_{v, j}|_1 \\
\leq \begin{cases} C_1 (1 + q^{-\beta|z|}), & \text{if } \gamma \neq \beta, \\
C_2 (1 + |z|), & \text{if } \gamma = \beta.
\end{cases}
\]
\[\square\]

**Proposition 25.** Let \(1 < p < \infty\). If \(a + b > c > 1\) and \(-pa < \alpha - 1 < p(b - 1)\), then \(S_{a, b, c}\) is bounded on \(L_p^\alpha\).

**Proof.** We set
\[
H(z, x) = |K_{\alpha}(z, x)|q^{-a|x|}q^{-(b-a)|x|},
\]
then we write the operator \(S_{a, b, c}\) as
\[
S_{a, b, c} f(z) = \sum_{x \in X} H(z, x) f(x) q^{-a|x|}.
\]
Our purpose is to apply Schur’s test (see Theorem 3.6 in [9]) to the integral operator with positive kernel \(H : X \times X \rightarrow [0, \infty)\). To do so, we have to show that there exists a positive function \(h\) on \(X\) such that
\[
\sum_{z \in X} H(z, x) h(z) p q^{-a|x|} \leq h(x) p, \quad \sum_{x \in X} H(z, x) h(x) p q^{-a|x|} \leq h(z) p. \tag{21}
\]
Observe that the two inequalities assumed for $\alpha$ are equivalent to
\[
\frac{-a + \alpha - 1}{p} < \frac{a}{p'}, \quad \frac{b - 1}{p'} < \frac{b - \alpha}{p}.
\]
Hence, since $a + b > 1$, it is possible to choose an element
\[
\gamma \in \left( -\frac{b - 1}{p'}, \frac{a}{p'} \right) \cap \left( -\frac{a + \alpha - 1}{p}, -\frac{b - \alpha}{p} \right) \neq \emptyset.
\]
(22)

We want to show that $h(x) = q^{-\gamma|x|}$ satisfies (21). Let $z \in X$. We suppose $\gamma \neq \frac{c-b}{p}$.
We can apply Lemma 24 since $b + \gamma p' > 1$ by (22), obtaining
\[
\sum_{x \in X} H(z,x)h(x)p'q^{-\alpha|x|} = q^{-\alpha|z|} \sum_{x \in X} |K_c(z,x)|q^{-(b+\gamma p')|x|}
\leq q^{-\alpha|z|}(1 + q^{-(b+\gamma p'-c)|z|})
\leq q^{-\gamma p'|z|} = h(z)p',
\]
where we used $a + b - c \geq 0$ and $a > \gamma p'$. Similarly, when $\gamma = \frac{c-b}{p}$ we can apply again Lemma 24 and conclude by using $a > \gamma p'$. On the other hand, we have that
if $\gamma \neq -\frac{c-a}{p}$, by $a + \gamma p + \alpha > 0$ and by Lemma 24
\[
\sum_{z \in X} H(z,x)h(z)p|z| = q^{-(b-a)|x|} \sum_{z \in X} |K_c(z,x)|q^{-(a+\gamma p + \alpha)|z|}
\leq q^{-(b-a)|z|}(1 + q^{-(a+\gamma p + \alpha-c)|z|})
\leq q^{-\gamma p'|z|} = h(z)p',
\]
since $a + b \geq c$ and, by (22), $b - \alpha > \gamma p$. Similarly when $\gamma = -\frac{c-a}{p}$.

In conclusion, (24) holds and by Schur’s test the operator $S_{a,b,c}$ is bounded on $L^p_{\alpha}(X)$. \qed

Notice that Proposition 26 shows that (iii) implies (i) in Theorem 13.

**Proposition 26.** If $a + b \geq c$ and
\[
-a < a - 1 < b - 1, \quad \text{when } c = a + b;
-a < a - 1 \leq b - 1, \quad \text{when } c < a + b,
\]
then $S_{a,b,c}$ is bounded on $L^1_{\alpha}$.

**Proof.** Let $f \in L^1_{\alpha}$. We suppose $c \neq a + \alpha$ and we observe that, since $a + \alpha > 1$, by Lemma 24
\[
\|S_{a,b,c}f\|_{L^1_{\alpha}} = \sum_{z \in X} |K_c(z,x)|f(x)q^{-(a+\alpha)|x|}
\leq \sum_{z \in X} |f(x)|q^{-(b-a)|x|} \sum_{z \in X} |K_c(z,x)|q^{-(a+\alpha)|z|}
\leq \sum_{z \in X} |f(x)|q^{-(b-a)|x|}(1 + q^{-(a+\alpha-c)|x|})
\leq \sum_{x \in X} |f(x)|q^{-\alpha|x|} = \|f\|_{L^1_{\alpha}},
\]
where we used the fact that $a + b - c \geq 0$ and $b \geq \alpha$. The case $c = a + \alpha$ follows similarly using again Lemma 24 and $b > \alpha$. Hence, $S_{a,b,c}$ is bounded on $L^1_{\alpha}$ \qed

Proposition 26 shows that (iii) implies (i) in Theorem 13.
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