Ruppeiner geometry for charged AdS black holes surrounded by quintessence with a cloud of strings background

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In this letter, we present a study of Ruppeiner’s thermodynamic geometry for charged AdS black holes with a cloud of strings and surrounded by quintessence with the motivation to probe the nature of interactions between black hole microstructures. We report that the scalar curvature of the Ruppeiner metric can assume both positive and negative values depending on certain parameters of the black hole indicating the presence of both repulsion and attraction dominated regions. The point at which the Ruppeiner curvature changes sign is determined by the electric charge and the coupling constants associated with quintessence and the cloud of strings. Such a crossing of curvature is also observed in the neutral case in this system.

I. INTRODUCTION

The fact that a black hole horizon can be assigned the notion of a temperature indicates towards a microscopic description of black holes which share the degrees of freedom of the Bekenstein-Hawking entropy [1–5]. These microscopic structures have associated with them, thermal degrees of freedom and respect an equipartition theorem (see for example [6]). On the other hand, in extended black hole thermodynamics, the black holes asymptotic to anti-de Sitter (AdS) spacetimes [7] have with them associated, the notion of a positive thermodynamic pressure [8–11] arising out of the negative cosmological constant of the AdS spacetime which is taken to be dynamical [8–19]. This allows one to associate a fluid-like description to these black holes at the thermodynamic level wherein, the critical behaviour can be mapped to that of non-ideal fluids [20–24]. With this picture in mind, it is useful to think of the microstructures of the black hole in parallel with molecules that constitute a fluid system in standard thermodynamics and at this phenomenological level, the study of thermodynamic geometry proves to be of outmost importance in its usefulness in probing the nature of interactions between the microstructures [25–41]. For example, it is now known that charged AdS black holes including those with a Gauss-Bonnet term in the Einstein-Hilbert action, are associated with both attraction and repulsion dominated regions [35–39] determined by their electric charge (and Gauss-Bonnet coupling, α_{GB} in d ≥ 6) whereas, their neutral counterparts are dominated by attraction [38–40]. Charged BTZ black holes (see [41] and references therein) on the other hand, are associated with purely repulsive behaviour. We consider, in this work charged AdS black holes coupled with a cloud of strings and surrounded by quintessence (see [42–43] and references therein). In the recent years, there has been a considerable amount of interest in quintessence [44], including black holes in its presence, with the first spherically symmetric general solution of the Einstein equations in (3 + 1)-dimensions satisfying linearity being given by Kiselev [45]. Usage of the terminology of quintessence in this model should be done carefully as emphasised in [45]. The interest in studying black hole systems with quintessence [47–48] is due to exciting developments through astronomical observations showing the accelerated expansion of the universe [49–50], with a possible explanation coming from the anti-gravity nature of dark energy. On the other hand, motivations for considering systems with cloud of strings also stem from the possibility of considering a universe with one-dimensional strings as building blocks, rather than point like particles. The first general solution of Einstein equations with a cloud of string background was studied long back by Letelier [51], with more recent generalizations to charged black holes [52–53]. Within extended black hole thermodynamics, with the availability of the equation of state, it is of great interest to ask, how are the microstructures sharing the degrees of freedom of the horizon interacting with one another for black holes coupled to a cloud of strings and surrounded with quintessence? We precisely address this problem here from purely phenomenological grounds.

Since the last few decades, Ruppeiner geometry [54] and its generalizations have been the subject of several exhaustive studies involving several classes of thermodynamic systems including quantum gases, magnetic systems [55–56] and most notably black holes. In black hole thermodynamics and especially for black holes in the extended phase space, Ruppeiner geometry has been of special interest due to several reasons. First, the singularities of curvature scalar of the metric signal critical behaviours. Even though this has been quite a motivation to study Ruppeiner geometry for black holes, more recently there has been a considerable amount of interest in probing the nature of black hole microstructures from Ruppeiner geometry. In standard thermodynamics,
it has been observed that the Ruppeiner metric is flat for non-interacting systems such as the ideal gas whereas, the Ricci scalar is negative for systems such as the van der Waals' fluid where attractive interactions are dominant. With this observation in mind, one can predict, at least phenomenologically the nature of dominant interactions between black hole microstructures from the sign of the curvature scalar of the Ruppeiner metric calculated for that particular black hole. The generic form of the Ruppeiner metric is defined as the negative Hessian of the entropy as,

$$ds_R^2 = -\frac{\partial^2 S}{\partial x^i \partial x^j} dx^i dx^j,$$

where \( x^i \) and \( x^j \) are independent thermodynamic fluctuation coordinates and the indices \( i, j \in \{1, 2, ..., n\} \). Even though the metric is initially defined as the Hessian of the entropy, one can calculate equivalent forms of the metric using other potentials such as the enthalpy, \( H = H(S, P) \) where the fluctuation coordinates are simply \( S \) and \( P \).

We shall, in this work, study Ruppeiner geometry on both \((T, V)\)- and \((S, P)\)-planes, i.e. using both enthalpy and Helmholtz free energy representations. The line element on the \((S, P)\)-plane can be calculated without much difficulty and reads [38, 39],

$$ds_R^2 = \frac{1}{C_P} dS^2 + \frac{2}{T} \left( \frac{\partial T}{\partial P} \right)_S dS dP - \frac{V}{TB_S} dP^2,$$

where \( B_S = -V(\partial P/\partial V)_S \) is the adiabatic bulk modulus which diverges for non-rotating black holes as considered in this paper where the entropy and volume are not independent. Similarly, the Ruppeiner line element on the \((T, V)\)-plane gives [38, 39],

$$ds_R^2 = \frac{1}{T} \left( \frac{\partial P}{\partial V} \right)_T dV^2 + \frac{2}{T} \left( \frac{\partial P}{\partial T} \right)_V dT dV + \frac{C_V}{T^2} dT^2,$$

where \( C_V \) is the specific heat at constant volume which is zero for non-rotating black holes. The last terms in eqns (2) and (3) therefore, drop out.

**Motivation and results:** The motivation of this work is to study the thermodynamic geometry for RN-AdS black holes with a cloud of strings and surrounded with quintessence so that one can probe the nature of interactions from the sign of the thermodynamic curvature. It is noted that these black holes can be associated with both positive and negative signs of the Ruppeiner curvature depending on certain parameters of the black hole. This indicates to the existence of both attraction and repulsion dominated regions. It therefore follows that there is a zero crossing of the Ruppeiner curvature where the black hole can be regarded as effectively non-interacting. Even though this feature is observed for charged black holes in general, the point at which the Ruppeiner metric is flat depends in this case on the coupling constants associated with quintessence and the string cloud in addition to the electric charge of the black hole.

**II. Quintessence Surrounding RN-AdS Black Holes With a Cloud of Strings**

We now consider charged AdS black holes with a cloud of strings and surrounded with quintessence in \((3 + 1)\)-dimensions. The metric is given as,

$$ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2,$$

where \( d\Omega^2 = \sum_{i=1}^{n-3} \sin^{2\alpha-2}(\rho_i) d\phi_i^2 \), is the line element on the 2-dimensional maximally symmetric Einstein space with a spherical topology. With the assumption that the cloud of strings and the quintessence are non-interacting, the lapse function \( f(r) \) of the so called Letelier metric [42, 43] asymptotic to AdS can be written down as,

$$f(r) = 1 - a - \frac{m}{r} + \frac{q^2}{r^2} + \frac{r^2}{\ell^2} - \frac{\alpha}{r^{\omega_q} + r^{\omega_q}},$$

where \( \ell \) is the radius of the AdS spacetime, \( m \) and \( q \) are simply constants respectively equal to the mass and charge of the black hole while \( \omega_q \) characterizes the quintessential state and \( a \) is an integration constant arising out of the cloud of strings that can effectively be treated as a coupling constant. The constant \( \alpha \) is related to the quintessence density \( \rho_q \) as,

$$\rho_q = -\frac{3\alpha \omega_q}{2\pi^3 (\omega_q + 1)},$$

Throughout the rest of the paper, we set following Kiselev [45] the quintessential state parameter to \( \omega_q = -2/3 \) so that the dominant energy condition [57, 58] is satisfied. To start out with the thermodynamics, we first identify that the pressure in this case is given by,

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi \ell^2},$$

whereas the volume and entropy are given by the standard relations,

$$S = \pi r_+^2, \quad V = \frac{4\pi r_+^3}{3},$$

where \( r_+ \) is the radius of the outer horizon satisfying \( f(r_+) = 0 \). The Hawking temperature of the black hole horizon is calculated to be,

$$T = \frac{1}{4\pi} \left( \frac{1 - a}{r_+} - \frac{q^2}{r_+^3} + 2\alpha + 8\pi Pr_+ \right),$$

where we have used eqn (7) to eliminate \( I \) in favour of \( P \). Solving eqn (5) for the pressure and identifying the specific volume to be \( v = 2r_+ \) gives the standard fluid-like equation of state,

$$P = \frac{T}{v} + \frac{\alpha}{2\pi v} - \frac{1 - a}{2\pi v^2} + \frac{q^2}{\pi v^4}.$$

It looks like that of a non-ideal fluid, i.e. one with non-trivial interactions between molecules. Each of the last three terms signify interactions between the microstructures. We take \( a < 1 \) so that the third term represents an attractive interaction of the van der Waals' type.
A. Thermodynamic geometry

We now study the thermodynamic geometry of these black holes. First, considering the \((T, V)\)-plane, the Ricci scalar of the Ruppeiner metric is calculated to be,
\[
R_{TV} = \frac{8\pi q^2 + 6^{2/3}(1 - a)\pi^{1/3}V^{2/3} + 3V\alpha}{3 \times 6^{2/3}\pi^{4/3}TV^{5/3}}.
\] (11)

It is simple to check that for \(a = \alpha = 0\), the curvature scalar reduces to that for the RN-AdS black hole on the \((S, P)\)-plane [39]. The Ricci scalar is plotted in figure-(1) as a function of \(V\) for fixed \(T\). Note that the curvature crosses a zero which indicates that the black hole microstructures are non-interacting at that point. The Ruppeiner curvature is positive for the black holes whose horizon radii are smaller than this point. This means that smaller black holes are dominated by repulsive interactions between the microstructures whereas for black holes larger than this bound, the curvature scalar is negative indicating dominance of attraction. Of special interest is the point which is the physical solution of the equation,
\[
\alpha r_+^3 - (1 - a)r_+^2 + 2q^2 = 0,
\] (12)
where the metric is flat. This point where the curvature has a zero crossing depends on the parameters \(a\), \(\alpha\) and \(q\) of the black hole. Considering now the \((S, P)\)-plane, the Ricci scalar can be expressed as,
\[
R_{SP} = -\frac{\sqrt{\pi}(2\pi q^2 + (-1 + a)S) + S^{3/2}\alpha}{\sqrt{\pi S(\pi q^2 + S(-1 + a - 8PS))} + 2S^{5/2}\alpha}.
\] (13)

It can be checked that the curvature scalars on both the planes are equivalent. The figure-(2) shows the variation of the Ricci scalar as a function of \(S\) for fixed \(P\). In this case too, a zero crossing of the curvature can be seen. It can be verified that this exactly corresponds to the point where the horizon radius is given by the real solution of eqn (12).

The interplay between the attraction and repulsion dominated regimes arises out of the presence of both attractive and repulsive interactions as can be seen from the equation of state [eqn (10)] corresponding to the black hole. It can be checked that for \(a \geq 1\), there is no bound in the horizon radius separating the attractive and repulsive regimes. This is because the case with \(a \geq 1\) corresponds to the case where there are no attraction terms in the equation of state. It can be verified that eqn (12) can alternatively be obtained as,
\[
\left(\frac{\partial P}{\partial v}\right)_{T=0} = 0.
\] (14)

We have set \(T = 0\) to remove the effect of the first term in eqn (10), which is purely of the kinetic type. Such kinetic effects activate only when there is a thermal background whereas, the remaining interaction terms are present even at zero temperature and these are the terms which compete with one another in determining the crossing point of the Ruppeiner curvature. The condition given by eqn (14) is reminiscent of a phase transition, except for setting \(T = 0\). We say that at this point, the black hole transitions from an attraction dominated regime to a repulsion dominated one and vice versa [39]. One should also be careful in using the term transition because this is not a true phase transition but only a transition between the attraction and repulsion dominated regions.

With the eqn (12) being quite complicated, one focus on simpler cases where the crossing point is more transparent. We consider first, the simplest case with \(q = 0\). This corresponds to the case of a Schwarzschild-AdS black hole with a cloud of strings and surrounded by quintessence. In this case one gets the crossing point to be simply,
\[
r_+ = \frac{1 - a}{\alpha}.
\] (15)

This variation \(r_+\) with \(\alpha\) and \(a\) is plotted in figure-(3). Note that the crossing point increases with decrease in the quintessence parameter \(\alpha\). Next, consider the case with \(\alpha = 0\). This is the case of the RN-AdS black hole with a cloud of strings in the absence of quintessence. It is easy to check that the curvature has a zero crossing at,
\[
r_+ = \frac{\sqrt{2}|q|}{\sqrt{1 - a}}.
\] (16)
This result looks similar to that for the RN-AdS black hole \cite{39} for which one simply sets $a = 0$. The variation of $r_+$ as a function of $q$ and $a$, indicating the fact that a point in the attraction dominated region can be put in the repulsion dominated region by suitably adjusting the black hole parameters. It also tells us that microstructures associated with electric charge and the string cloud are essentially of the repulsive kind. A similar crossing point for the case with $a = 0$ while other parameters nonzero can be obtained but is somewhat more complicated than the previous two results. For this case, the variation $r_+$ with $\alpha$ and $q$ is shown in figure-\ref{fig:5}. The crossing point increases with decrease in $\alpha$ as was also observed in the case with $q = 0$.

III. REMARKS

In this work, we have focussed on probing the nature of interactions among the microstructures for charged AdS black holes surrounded by quintessence and with a cloud of strings from purely phenomenological grounds. Our analysis, essentially answers the following question: If the black hole were a fluid, how would the molecules interact? It is now known that charged black holes in $d \geq 4$ are associated with both attraction and repulsion dominated regions. Such a result is phenomenologically explained by considering two distinct kinds of microstructures \cite{37,39} that a charged black hole should consist of. The first one, which is of the neutral type interacts only attractively while the second one is the electrically charged kind interacting repulsively. If such arguments be extended to this case, one has to introduce the notion of additional kinds of interacting microstructures arising out of the cloud of strings and also due to the effect of quintessence, both being repulsive in nature. It should also be remarked that in the case of the four dimensional charged black hole with a cloud of strings and quintessence, the string parameter $a \in [0,1]$ increases the zero crossing of the curvature scalar until the crossing point finally disappears at $a = 1$. Beyond this point, the microstructures are purely repulsive and there is no such attraction-repulsion interplay. This implies that even though it is well known that the standard RN-AdS black hole admits a nice critical behaviour which can get mapped to that of the van der Waals' system \cite{20}, the introduction of a cloud of strings in the background with $a = 1$ kills the critical behaviour altogether.

Another remarkable thing that comes out of our results is that even though the Schwarzschild-AdS black hole ($q = 0$) is associated with a negative sign of Ruppeiner curvature indicating attractive interactions \cite{35},...
FIG. 6: Ruppeiner curvature for the $q = 0$ case on the $(S,P)$-plane as a function of $S$ for fixed $P$.

the introduction of quintessence in the background, i.e. $\alpha \neq 0$ essentially introduces a repulsion dominated region as is shown in figure-6. In this case, the curvature has a zero crossing given by eqn (15). It therefore follows that for Schwarzschild-AdS black holes surrounded by quintessence, smaller black holes are attraction dominated whereas the larger ones are repulsion dominated. One would also expect a similar result to hold for neutral Gauss-Bonnet-AdS black holes. It would be interesting to see the effects of quintessence and a string cloud on the microstructures for black holes in higher derivative theories such as Gauss-Bonnet-AdS black holes or even more general black holes in Lovelock gravity.

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