Short-wavelength soliton in ultrarelativistic electron-positron-ion plasmas

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We derive a nonlinear equation governing dynamics of short-wavelength longitudinal waves in ultrarelativistic electron-positron-ion plasmas. In contrast to the recent work by Lashkin [Phys. Plasmas 27, 102302 (2020)], where a similar equation was suggested in the framework of the Wigner function approach for a nonrelativistic electron-ion degenerate plasma, in our case which is based on the Vlasov kinetic equation all three species of particles (electrons, positrons and ions) should be present. The nonlinearity arises only in the presence of a population of ions. By numerical simulations we demonstrate that collisions between even four solitons are fully elastic.

Ultrarelativistic plasmas exist in various astrophysical objects such as supernova remnants, pulsars, active galactic nuclei etc. \cite{1,2}, and also provide important insights about the early stage in the evolution of Universe \cite{3,4}. In laboratory conditions such plasmas can be produced in high-intensity laser fields \cite{5,6}. Such plasmas are always arises if the thermal energy of the particles exceeds twice rest mass energy of electrons $\sim 1.2$ MeV \cite{7}. Most astrophysical and laboratory ultrarelativistic plasmas consist of electrons, positrons and a minority population of ions (typically, the latter are nonrelativistic). Nonlinear waves and solitons in relativistic plasmas have been studied extensively for the past four decades (see for example a review \cite{8}). Specifically in a nondegenerate ultrarelativistic plasma, nonlinear evolution equations and their soliton solutions have been considered in a number of works. As was shown in Refs. \cite{9,10}, dynamics of ultrarelativistic Langmuir waves in electron-positron plasma in the framework of the fluid model is governed by the nonlinear Schrodinger (NLS) equation and the corresponding solution is the ultrarelativistic Langmuir soliton, and, in particular, its relevance to pulsar radiation was discussed. Later on, the NLS equation and ultrarelativistic Langmuir soliton in electron-positron plasma were obtained from the kinetic approach based on the Vlasov equation with ultrarelativistic Maxwellian distribution function \cite{11}. Ultrarelativistic Alf\’{v}én solitons propagating parallel and oblique to the external magnetic field correspond to the Korteweg-de Vries (KdV) equation and were studied in Refs. \cite{12} and \cite{13} respectively. The considered models are valid only in the short-wavelength case $k \ll 1$, where $k$ is suitably normalized dimensionless wavenumber. Then, the appearance of solitons is due to the balance of weak dispersion $k^2 \ll 1$ for the NLS equation, and $k^3 \ll 1$ for the KdV equation, and weak nonlinearity (cubic and quadratic respectively). Recently, ion-acoustic solitons in a plasma with ultrarelativistic electrons and positrons were studied in detail in Ref. \cite{14} using the fluid model and the Sagdeev potential formalism and a comparison between this approach (corresponding to strong nonlinearity), and the reductive perturbation approach (weak nonlinearity and weak dispersion $k^3 \ll 1$) leading to the KdV equation was made. The theory of nonlinear waves in a nondegenerate ultrarelativistic plasma in the short-wavelength limit $k \gg 1$, where the linear dispersion has an exponential character $\omega \sim k \exp(-k^2)$ (known in physics as the so-called dispersion of ”zero sound” \cite{15}), is fully absent. Recently, a novel nonlinear evolution equation with such dispersion and quadratic nonlinearity was derived by using kinetic equation for the Wigner function in Ref. \cite{16} for short-wavelength longitudinal waves in a nonrelativistic fully degenerate electron-ion quantum plasma. It was shown \cite{16} that despite the specific nature of the dispersion which has no counterpart in classic nonrelativistic plasmas, balance between the weak dispersion $k \gg 1$ and weak quadratic nonlinearity lead to the formation of solitons and the collisions between three solitons are elastic. The goal of the present Brief Communication is to derive a similar evolution equation describing short-wavelength nonlinear waves in a classic nondegenerate electron-positron-ion plasma with ultrarelativistic Maxwellian distribution function for electrons and positrons, and nonrelativistic one for ions. Unlike the work \cite{16}, all three species of particles (electrons, positrons and ions) should be present. Electrons and positrons account for the linear dispersion, while the ions do for the nonlinearity since the electron and positron nonlinear contributions are canceled each other. In addition, we numerically demonstrate that collisions between even four solitons are elastic.

Dielectric functions and dispersion relations of ultrarelativistic plasmas was first obtained by Silin in Ref. \cite{17}. Expression for the longitudinal $\varepsilon_L$ dielectric permittivity of an unmagnetized isotropic ultrarelativistic plasma is \cite{18}

$$
\varepsilon_L(\omega, k) = 1 + \frac{3 \omega_p^2}{k^2 c^2 \omega} \left[ 1 + \frac{\omega}{2k \epsilon} \ln \frac{\omega - kc}{\omega + kc} \right] + i \frac{\epsilon}{2k \omega} \theta(k^2 c^2 - \omega^2),
$$

where $\theta(x)$ is the Heaviside step function, $\omega$ and $k$ are the frequency and wave vector respectively, $\varepsilon = |k|$, $c$ is the speed of light, and $\omega_p^2 = \omega_{pe}^2 + \omega_{pp}^2$, where $\omega_{pe}$ is

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the plasma frequency of the particles of species \( \alpha = e, p \) (electrons, positrons) determined by

\[
\omega_{p\alpha} = \sqrt{\frac{4\pi e^2 c^2 n_{0\alpha}}{m_{\alpha}}}. \tag{2}
\]

Here, \( e \) is the elementary charge, \( n_{0\alpha} \) and \( T_\alpha \) are the equilibrium plasma density and the particle temperature of species \( \alpha \) respectively. Analytical expressions for the wave dispersion can be obtained from the dispersion equation \( \varepsilon_L(\omega, k) = 0 \) in the two limiting cases \( \ref{17} - \ref{19} \). In the long-wave limit \( \omega_p \gg kc \), the dispersion relation for longitudinal waves is

\[
\omega_k = \omega_p \left( 1 + \frac{3k^2c^2}{10\omega_p^2} \right). \tag{3}
\]

In the opposite case, i.e., in the short-wave limit \( \omega_p \ll kc \), the dispersion relation has the form

\[
\omega_k = kc \left[ 1 + 2 \exp \left( -\frac{2k^2c^2}{3\omega_p^2} - 2 \right) \right]. \tag{4}
\]

As is seen from Eq. \( \ref{1} \), the Landau damping is absent in both cases since \( \omega/k > c \). Equation \( \ref{3} \) is similar to the dispersion of Langmuir waves in classical nonrelativistic electron-ion plasmas. In particular, balance between the dispersion Eq. \( \ref{3} \) and cubic nonlinearity results in ultrarelativistic Langmuir soliton \( \ref{9} - \ref{11} \). The dispersion \( \ref{3} \) has no counterpart in nonrelativistic classical plasmas. However, the dispersion relation Eq. \( \ref{4} \) is the same as in a nonrelativistic degenerate quantum electron-ion plasma \( \ref{18} - \ref{20} \) in the limits \( T_e \to 0 \), \( \hbar k/m \to 0 \), and \( \omega_p \ll kv_F \) with the replacement \( c \to v_F \), where \( v_F = h(3\pi^2 n_0)^{1/3}/m \) is the electron Fermi speed and \( \omega_p \to \sqrt{4\pi e^2 n_0/m_c} \) is the electron Langmuir frequency, where \( m_c \) is the electron mass and \( \hbar \) is the Planck constant divided by \( 2\pi \). For such a nonrelativistic degenerate quantum plasma, a nonlinear evolution equation with this type of dispersion was obtained in Ref. \( \ref{16} \). Here we derive a similar nonlinear equation for an ultrarelativistic nondegenerate electron-positron-ion plasma. In contrast to Ref. \( \ref{16} \), on the one hand, we completely neglect quantum effects, so that all plasma particles are nondegenerate and the corresponding condition has the form \( T_\alpha \gg \hbar^2 n_{0\alpha}^{3/2}/m_\alpha \). On the other hand, the temperatures of electrons and positrons are so high that \( T_{e,p} \gg m_c c^2 \), while ions are assumed to be nonrelativistic \( T_i \ll m_c c^2 \), where \( m_i \) is the ion mass.

Obtaining of a nonlinear equation with the dispersion Eq. \( \ref{4} \) requires an essentially kinetic description (in Ref. \( \ref{16} \), the kinetic equation for the Wigner function was used). In the kinetic theory, the response of a plasma to longitudinal (i.e., electrostatic) wave fields is described by the linear response and a hierarchy of nonlinear susceptibilities \( \ref{21} - \ref{25} \). For an isotropic unmagnetized relativistic plasma, general expressions for the quadratic and cubic nonlinear response tensors were obtained in Ref. \( \ref{13} \). We consider a plasma without an external magnetic field. Laboratory ultrarelativistic plasmas created by extremely high laser fields are unmagnetized. As for the ultrarelativistic plasma of astrophysical objects, for example, pulsars, we note that ultrarelativistic electrons and positrons forming the pulsar plasma are forced to have a one-dimensional motion along the extremely strong pulsar magnetic field, due to fast radiative losses of perpendicular momentum. However, electron-positron plasmas in early universe and inside the gamma-ray burst fireball at the initial phase of its expansion are likely to be unmagnetized \( \ref{3} - \ref{7} \). With this in mind, for generality we consider the three-dimensional case for the moment. Further, when considering the one-dimensional case, in the subsequent equations \( k \) means the wave number along the external magnetic field. Throughout this paper we use the notation

\[
\sum_{q=q_1+q_2} \cdots \to \int \cdots \delta(q - q_1 - q_2) \frac{dq_1}{(2\pi)^4} \frac{dq_2}{(2\pi)^4}, \tag{5}
\]

where \( q = (k, \omega) \). Neglecting the collision integral, the kinetic Vlasov equation in the momentum space can be written as

\[
(\omega - k \cdot \mathbf{v}) f_q(p) + e_\alpha \varphi_q k \frac{\partial f_q^{(0)}(p)}{\partial p} + \varphi_p \sum_{q=q_1+q_2} p \cdot k_1 \frac{\partial f_q^{(0)}(p)}{\partial p} = 0, \tag{6}
\]

where \( f_q(p) \) is the deviation of the distribution function of each species, including ions \( \alpha = i \), from the equilibrium one \( f_q^{(0)}(p) \), \( e_\alpha \) is the charge of species, and \( \varphi \) is the electrostatic potential. The equilibrium distribution function for particles of species \( \alpha \) for isotropic unmagnetized relativistic plasma is the Jüttner distribution function \( \ref{18} - \ref{26} \)

\[
f_q^{(0)}(p) = \frac{n_{0\alpha}}{4\pi m_{\alpha}^2 c^3} \frac{\rho_\alpha}{K_2(\rho_\alpha)} \exp(-\rho_\alpha \gamma_\alpha), \tag{7}
\]

where \( \gamma_\alpha = \sqrt{1 + p^2/(m_\alpha c)^2} \) is the usual Lorentz factor, \( K_2 \) is the second kind modified Bessel function of the second order, and \( \rho_\alpha = m_\alpha c^2/T_\alpha \). In the ultrarelativistic limit \( \rho_\alpha \ll 1 \) for electrons and positrons (\( \alpha = e, p \)) one has

\[
f_q^{(0)}(p) = \frac{n_{0\alpha} c^3}{8\pi T_\alpha^{3/2}} \exp(-cp/T_\alpha), \tag{8}
\]

while for nonrelativistic \( (\rho_\alpha \gg 1) \) ions, the usual Maxwellian distribution follows from Eq. \( \ref{7} \). For ultrarelativistic electrons and positrons we have \( \mathbf{v} = p c/p \) and for nonrelativistic ions \( \mathbf{v} = p/m_i \). The distribution function \( f_q^{(0)}(p) \) is normalized to the equilibrium plasma density of each species.

\[
\int f_q^{(0)}(p) dp = n_{0\alpha}.
\]

We present the function \( f_q(p) \) as a series in powers of the field strength

\[
f_q(p) = \sum_{n=1}^{\infty} f_q^{(n)}(p). \tag{9}
\]
In the linear approximation, from Eqs. (6) and (9) one can obtain
\[ f_q^{(1)} = -\frac{e_\alpha \varphi_q}{(\omega - k \cdot v)} \frac{\partial f_{q_{\alpha}}^{(0)}}{\partial p}, \] (10)
and then we have the recurrence relation
\[ f_q^{(n)} = -\frac{e_\alpha}{(\omega - k \cdot v)} \sum_{q_1 + q_2} \varphi_{q_1} k_1 \cdot \frac{\partial f_{q_{\alpha}^{n-1}}}{\partial p}. \] (11)
Retaining terms in Eq. (9) up to second order in the wave fields and substituting \( f_q \) into the Poisson equation
\[ k^2 \varphi_q = 4\pi \sum_{\alpha} e_\alpha \int f_q(p) d^3p, \] (12)
where \( \sum \) stands for summation over the different species, we get
\[ \varepsilon_q \varphi_q = \sum_{q_1 + q_2} V_{q_1, q_2} \varphi_{q_1} \varphi_{q_2}, \] (13)
where
\[ \varepsilon_q = 1 + \sum_{\alpha} 4\pi e_\alpha k^2 \int \frac{k}{(\omega - k \cdot v)} \frac{\partial f_{q_{\alpha}^{(0)}}}{\partial p} d^3p \] (14)
is the linear dielectric response function, and neglecting ions one has Eq. (1) where \( f_{e_{\alpha}^{(0)}} \) is determined by Eq. (8).

The interaction matrix element \( V_{q_1, q_2} \) is determined by
\[ V_{q_1, q_2} = \sum_{\alpha} \frac{2\pi e_\alpha^3}{k^2} \int \frac{k_1 \cdot k_2}{(\omega_1 + \omega_2) - (k_1 + k_2) \cdot v} \times \frac{1}{\omega_2 - k_2 \cdot v} \frac{\partial f_{q_{\alpha}^{(0)}}}{\partial p} d^3p + (\omega_1, k_1 \rightarrow \omega_2, k_2). \] (15)

Note that the expression Eq. (15) for the interaction matrix element \( V_{q_1, q_2} \) is written in a symmetrized form. Singularities in the denominators in Eqs. (14) and (15) are avoided, as usual, using Landau’s rule by replacing \( \omega \rightarrow \omega + i\delta \). Linear Landau damping in ultrarelativistic plasma is absent. In this paper we neglect the nonlinear Landau damping (damping of the virtual beat wave) and only the principal value of the corresponding integral is understood, although the corresponding damping term can be easily obtained in the same way as the nonlinear Landau damping is obtained in the kinetic derivation of the NLS equation for Langmuir waves in classic plasmas. It is seen that the electron and positron terms in Eq. (15) have the opposite signs so that the contributions from electrons and positrons may cancel each other (complete mutual cancelation occurs if \( \omega_{pe} = \omega_{pp} \)). For a pure electron-positron plasma in the thermal equilibrium \( T_{e, p} = T \), quadratic nonlinearity vanishes identically. The situation changes drastically if ions are present. We write the interaction matrix element Eq. (15) as \( V_{q_1, q_2} = V_{q_1, q_2}^{(i)} + V_{q_1, q_2}^{(e, p)} \), where the first term corresponds to the ion contribution, and the second to electrons and positrons. For nonrelativistic ions, after two partial integrations in Eq. (15) one can write
\[ V_{q_1, q_2}^{(i)} = \frac{2\pi e_\alpha^3}{k^2} \int \frac{2(k \cdot k_1)(k \cdot k_2)}{((\omega - k \cdot v)^3(\omega_1 - k \cdot v)^2(\omega_2 - k \cdot v)^2)} f_{q_{\alpha}^{(0)}} d^3p + (\omega_1, k_1 \rightarrow \omega_2, k_2), \] (16)
where \( \omega = \omega_1 + \omega_2 \) and \( k = k_1 + k_2 \). For ultrarelativistic electrons and positrons with \( v = \frac{pc}{p} \) in an isotropic plasma we rewrite Eq. (15) as
\[ V_{q_1, q_2}^{(e, p)} = \sum_{\alpha = e, p} \frac{2\pi e_\alpha^3}{k^2} \int \frac{k_1 k_2 \cos^2 \theta}{(\omega_1 + \omega_2) - (k_1 + k_2) c \cos \theta} \times \frac{1}{\omega_2 - k_2 \cdot c \cos \theta} \frac{\partial f_{q_{\alpha}^{(0)}}}{\partial p} d^3p + (\omega_1, k_1 \rightarrow \omega_2, k_2). \] (17)
where \( \theta \) is an angle between \( k \) and \( v \), and \( d^3p = 2\pi p^2 \sin \theta dp d\theta \). When calculating the nonlinear term in Eqs. (10) and (17), we neglect the dispersion corrections (thermal corrections) to nonlinearity which correspond to \( k_2 \), in the denominators and take into account only the leading term. In the following, we consider the one-dimensional case \( k_2 = k, k_2 = k_2 = 0 \) and then from Eq. (10) we find for ions
\[ V_{q_1, q_2}^{(i)} = \frac{e^2}{2m_1} \frac{\omega^2}{k^2} \left\{ k_1 k_2^2 \begin{vmatrix} 2k_1 k_2 \omega_1 \omega_2 + k_1 k_2^2 \omega_2^2 \\ \omega_1^2 \omega_2 + \omega_2^3 \omega_2 \end{vmatrix} + (\omega_1, k_1 \rightarrow \omega_2, k_2) \right\}. \] (18)
For electrons and positrons from Eq. (17) one has
\[ V_{q_1, q_2}^{(e, p)} = -\frac{2\pi e_\alpha^3 n_0 c^2}{k^2 T^2} \left\{ k_1 k_2 \omega_{\omega_2} + (\omega_1, k_1 \rightarrow \omega_2, k_2) \right\}. \] (19)
Comparing Eqs. (18) and (19) one can see that the ion contribution in the nonlinearity is negligible. Essentially, that the wave dispersion in Eq. (1) has an acoustic type and in the leading term satisfies the three-wave resonance condition
\[ \omega_k = \omega_{k_1} + \omega_{k_2}, k = k_1 + k_2. \] (20)
In particular, this means that this condition, together with a quadratic nonlinearity, ensures the validity of the successive approximation in Eq. (9), and this is equivalent to the multi-time-scale perturbation expansion, i.e. the secular terms are removed. Thus, from Eqs. (4), (19) and (20) we have
\[ V_{q_1, q_2} = -\frac{2\pi e_\alpha^3 n_0 c^2}{k^2 T^2}. \] (21)
Expanding \( \varepsilon(\omega, k) \) in Eq. (1) near the eigenmode \( \omega_k \) determined by Eq. (4) yields
\[ \varepsilon_q = \varepsilon(\omega_k) + \varepsilon'(\omega_k)(\omega - \omega_k). \] (22)
where \( \varepsilon'(\omega_k) = \partial \varepsilon(\omega) / \partial \omega \big|_{\omega = \omega_k} \) and from Eq. (11) one can obtain

\[
\varepsilon'(\omega_k) = \frac{3 \omega_p^2}{4 k^3 c^2} \exp \left( \frac{2 k^2 c^2}{3 \omega_p^2} + 2 \right).
\] (23)

After inserting Eq. (22) into Eq. (13) we find

\[
(\omega - \omega_k)\varphi_q = \frac{1}{\varepsilon'(\omega_k)} \sum_{q=q_1+q_2} V_{q_1,q_2}\varphi_{q_1}\varphi_{q_2}.
\] (24)

Then, substituting Eqs. (21) and (22) into Eq. (24), and introducing the slow time scale \( \Omega = \omega - kc \) which balances the dispersion in Eq. (14) (compare, for example, kinetic derivation of the KdV equation in Ref. 29), we finally get

\[
\left[ \Omega - 2kc \exp \left( -\frac{2 k^2 c^2}{3 \omega_p^2} - 2 \right) \right] \varphi_k
= -\frac{2e\eta_{0i}}{T(n_0c + n_0p)} k \exp \left( -\frac{2 k^2 c^2}{3 \omega_p^2} - 2 \right) \sum_{q=q_1+q_2} \varphi_{q_1}\varphi_{q_2}.
\] (25)

After rescaling

\[
k \to \frac{c}{\omega_p} k, \quad \Omega \to \frac{c}{2\omega_p} \Omega, \quad \Phi \to -\frac{e\eta_{0i}}{T(n_0c + n_0p)} \varphi, \quad (26)
\]
equation (25) can be written in the dimensionless form

\[
\left[ \Omega - k \exp \left( -\frac{2 k^2}{3} \right) \right] \Phi_q = k \exp \left( -\frac{2 k^2}{3} \right) \sum_{q=q_1+q_2} \Phi_{q_1}\Phi_{q_2}.
\] (27)

By introducing the operator \( \hat{L} \) acting in the physical space as

\[
\hat{L}f(x) = \int ik \exp(-2k^2/3)e^{-ikx} \hat{f}(k) dk,
\] (28)

where \( f(x) \) is an arbitrary function and \( \hat{f}(k) \) is its Fourier transform, and using the convolution identity

\[
(f \ast g)_k = \int \hat{f}_k \hat{g}_k \delta(k-k_1-k_2) dk_1 dk_2.
\] (29)

one can write Eq. (27) in the physical space as

\[
\partial_t \Phi + \hat{L} \Phi + \hat{L} \Phi^2 = 0,
\] (30)

so that the nonlinearity has a nonlocal character. Note also that in the considered short-wavelength case \( k > 1 \), Eq. (27) can not be simplified by any expansion in \( k \). Equation (30) outwardly coincides with the equation obtained earlier in Ref. 16 for the case of a nonrelativistic degenerate quantum plasma using the kinetic equation for the Wigner function, but the physical meaning of its coefficients in dimensional form is completely different.

Stationary traveling soliton solutions of Eq. (30) of the form \( \Phi(x,t) = \Phi(x - vt) \), where \( v \) is the velocity of propagation in the \( x \) direction was obtained numerically in Ref. 16. Soliton solutions exist provided by the condition \( v > 1 \). In physical variables this means that the soliton velocity should satisfy the condition \( v > 2 \exp(-2) c \sim 0.27c \). In reality, the soliton velocity can not be superluminal as for any physical object, so that in the dimensionless variables the restriction above is \( v \sim 3.7 \). Note that for the group velocity \( v_{gr} = \partial \omega / \partial k \) of linear waves with dispersion Eq. (4) we have \( v_{gr} < c \). The time evolution of the solitons under their collisions was studied in Ref. 16, where it was shown that collisions between two and three solitons of Eq. (30) are fully elastic. In the present paper, we numerically solve the nonlinear equation Eq. (30) with the initial conditions given by a superposition of \( N = 4 \) soliton solutions

\[
\Phi(x,t) = \sum_{i=1}^{N} \Phi_i(x - x_i, t)
\] (31)

at the time \( t = 0 \), where \( \Phi_i \) correspond numerically found (up to machine accuracy) soliton solutions with essentially different velocities \( v_i \). An example of the elastic collision between four solitons with the velocities \( v_1 = 3.5 \), \( v_2 = 2.5 \), \( v_3 = 2 \) and \( v_4 = 1.3 \) is shown in Fig. 1. In particular, it can be seen that at the times \( t = 30 \) and \( t = 50 \) in the Fig. 1 the three solitons undergo strong distortion simultaneously so that two distant solitons feel each other through an intermediate soliton – this is a typical many-soliton effect (at the time \( t = 30 \) even all four solitons feel each other). Then, the solitons fully reconstruct their initial form without any emitting wakes of radiation \( (t = 100) \), resulting only in phase shifts. The overall picture closely resembles the elastic soliton collisions in the integrable models 30. The elastic collisions between solitons might suggest that equation (30) has exact \( N \)-soliton solutions and is completely integrable just like for KdV equation and others 30–32 and can be solved by the inverse scattering transform (IST) method, but, as was pointed out in Ref. 16, this is most likely not the case. The fact is that in the IST approach there exists a relationship between some function \( \hat{\omega}(\lambda) \), where \( \lambda \) is the spectral parameter and \( \omega(k) \) is the dispersion relation of the corresponding linearized equation 31. In all known cases \( \hat{\omega}(\lambda) \) is the rational function of \( \lambda \) though the associated spectral problem may involve meromorphic functions (like the elliptic Jacobi functions, as in the case of the Landau-Lifshitz equation 32) of the spectral parameter \( \lambda \). However, in addition we would like to make the following remark. In the Hirota bilinearization method, equations admitting \( N \)-soliton solutions are written in the so-called bilinearization form 33. In this case, the corresponding function of the so-called Hirota operators \( D_x \) and \( D_y \) is a polynomial (this reflects the character of linear dispersion and takes place for all known equations considered in the Hirota method) or an exponential (to our knowledge, such equations have not been considered) 34. An interesting question arises whether Eq. (30) can be written in the
FIG. 1: A typical example of an elastic collision between four solitons, with the velocities $v_1 = 3.5$, $v_2 = 2.5$, $v_3 = 2$ and $v_4 = 1.3$. The corresponding initial locations at the moment $t = 0$ are $x_1 = -185$, $x_2 = -160$, $x_3 = -130$ and $x_4 = -100$.

bilinearization form such as (in the Hirota notations), for example, $D_x[D_t + D_x \exp(-D_x^2)]F \cdot F = 0$ or something like this (for the KdV equation, the bilinearization form is $D_x(D_t + D_x^3)F \cdot F = 0$)?

Note also that, for illustrative purposes, we also considered Eq. (30) with the replacement of the nonlocal nonlinearity by the usual local nonlinearity $\Phi \partial_x \Phi$ of the KdV equation. It turned out that soliton solutions exist in a rather narrow range of soliton velocities and amplitudes (in particular, there are velocity limitations both from below and from above). Moreover, collisions of two solitons are not elastic in that model.

In summary, we have derived the nonlinear evolution equation governing dynamics of the short-wavelength longitudinal waves in the ultrarelativistic plasma. In contrast to the work [16], where a similar equation was derived for a nonrelativistic fully degenerate quantum electron-ion plasma, in our case all three species of particles (electrons, positrons and ions) must be present. Electrons and positrons account for the linear dispersion, while the ions do for the nonlinearity. We have demonstrated that the collisions between even four solitons are fully elastic resulting only in phase shifts.

**DATA AVAILABILITY**

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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