Rotating Einstein-Maxwell-Dilaton
Black Holes in D Dimensions

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Abstract

We construct exact charged rotating black holes in Einstein-Maxwell-dilaton theory in $D$ spacetime dimensions, $D \geq 5$, by embedding the $D$ dimensional Myers-Perry solutions in $D + 1$ dimensions, and performing a boost with a subsequent Kaluza-Klein reduction. Like the Myers-Perry solutions, these black holes generically possess $N = [(D - 1)/2]$ independent angular momenta. We present the global and horizon properties of these black holes, and discuss their domains of existence.
1 Introduction

In 4 dimensions the Kerr solution presents the unique family of stationary asymptotically flat vacuum black holes. Generalization of these rotating black holes to $D > 4$ dimensions leads to the Myers-Perry black holes \[1\]. These are characterized by their mass and their $N = [(D - 1)/2]$ independent angular momenta, associated with $N$ orthogonal planes of rotation (where $N = [(D - 1)/2]$ denotes the integer part of $(D - 1)/2$). The presence of rotating black rings in 5 dimensions \[2\] however shows, that the uniqueness theorems do not generalize to higher dimensions.

Coupling the electromagnetic field to gravity yields the unique family of Kerr-Newman black holes in 4 dimensions. Charged black rings \[3\] have been constructed in Einstein-Maxwell (EM) theory in 5 dimensions, but no exact solutions of rotating EM black holes with horizons of spherical topology are known in $D > 4$ dimensions. So far such black hole solutions have been obtained only numerically \[4\].

In contrast to pure EM theory, exact solutions of higher dimensional charged rotating black holes are known in theories with more symmetries. The presence of a Chern-Simons (CS) term, for instance, leads to a class of odd-dimensional Einstein-Maxwell-Chern-Simons (EMCS) theories, comprising the bosonic sector of minimal $D = 5$ supergravity, whose stationary black hole solutions \[5, 6\] possess surprising properties \[7, 8\]. In particular, in EMCS theory, even black holes with horizons of spherical topology are no longer uniquely characterized by their global charges (beyond a critical value of the CS coupling constant) \[8\].

The inclusion of additional fields, as required by supersymmetry or string theory, leads to further exact solutions of higher dimensional black holes \[9, 10\], since then certain constructive methods are available, such as Hassan-Sen transformations \[11\].

Here we construct exact solutions of rotating black holes in $D$ dimensions, by embedding the $D$-dimensional Myers-Perry solutions in $D + 1$ dimensions, and performing a boost with respect to the time and the additional coordinate, followed by a Kaluza-Klein reduction to $D$ dimensions \[12, 13, 14\]. This procedure leads to Einstein-Maxwell-dilaton (EMD) black holes in $D$ dimensions, for particular values of the dilaton coupling constant. The solutions are completely general, with dependence on the mass $M$, the angular momenta $J$, and the charge $Q$. Like their EM counterparts \[7\] they satisfy a Smarr formula and the first law of black hole mechanics. The corresponding EMD solutions in 4 dimensions have been known since long \[13\], including their dyonic generalizations \[14\].

After recalling the Myers-Perry black holes in section 2, we present the rotating EMD black holes in section 3, we obtain their global and horizon properties in section 4, and we discuss their domains of existence in section 5.
2 Myers-Perry black holes

The Myers-Perry (MP) solutions of $D$-dimensional gravity can be presented in a unified way for even and odd dimensional spacetimes \[15\]. For that purpose let us introduce the notation

\[ N \equiv \left[ \frac{D-1}{2} \right] , \quad \varepsilon \equiv \frac{1}{2} (1 + (-1)^D) , \]

where $D$ is the dimension of the spacetime and $\left[ \cdot \right]$ denotes the integer part. So for even $D$ we have $N = (D - 2)/2$ and $\varepsilon = 1$, whereas for odd $D$, $N = (D - 1)/2$ and $\varepsilon = 0$. In what follows we assume $D \geq 4$ and choose units such that $16\pi G_D = 1$.

The general MP solutions then read

\[
\begin{align*}
\sum_{i=1}^{N} \left( r^2 + a^2_i \right) \left( d\mu^2_i + \mu^2_i d\phi^2_i \right) + \\
\frac{m}{F} \left( dt - \sum_{i=1}^{N} a_i \mu^2_i d\phi_i \right)^2 + \varepsilon r^2 d\nu^2 ,
\end{align*}
\]

where

\[ F \equiv 1 - \sum_{i=1}^{N} \frac{a^2_i \mu^2_i}{r^2 + a^2_i} , \quad \Pi = \prod_{i=1}^{N} (r^2 + a^2_i) . \]

Note, that the coordinate $\nu$ enters only in even dimensions. The $\mu_i$ (and $\nu$, for even-dimensional cases) coordinates are not independent but have to obey the constraint

\[ \sum_{i=1}^{N} \mu^2_i + \varepsilon \nu^2 = 1 . \]

3 Kaluza-Klein black holes

Let us now generate Kaluza-Klein black holes using the MP solutions as seeds. In order to do so, we first embed the $D$-dimensional MP metric, Eq. (2), into a $(D+1)$ spacetime with extra coordinate $U$,

\[
ds_{D+1}^2 = dU^2 + ds_{D,MP}^2 ,
\]

and then perform a boost in the $t - U$ plane with the $2 \times 2$ matrix

\[ L = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} . \]

The resulting boosted $(D+1)$-dimensional metric is a solution of the $(D+1)$-dimensional vacuum Einstein equations.
To obtain $D$-dimensional EMD black holes, the boosted metric has to be compared to the Kaluza-Klein parametrization of a $(D + 1)$-dimensional metric, namely,

$$ds_{D+1}^2 = e^{2\Phi} g_{\rho\sigma} dx^\rho dx^\sigma + e^{-2(D-2)\Phi} (dU + A_\rho dx^\rho)^2,$$

where

$$l = \frac{1}{\sqrt{2(D-1)(D-2)}},$$

and $g_{\rho\sigma}$, $A_\rho$, and $\Phi$, obtained from this comparison, are then identified with the $D$-dimensional metric, the $D$-dimensional Maxwell potential, and the dilaton function, respectively, which satisfy the field equations of the $D$-dimensional Einstein-Maxwell-dilaton theory with action

$$S = \int d^Dx \sqrt{-g} \left( R - \frac{1}{2} \Phi_{,\rho} \Phi^{,\rho} - \frac{1}{4} e^{-2h\Phi} F_{\rho\sigma} F^{\rho\sigma} \right),$$

for the particular Kaluza-Klein value of the dilaton coupling constant $h$,

$$h = \frac{D-1}{\sqrt{2(D-1)(D-2)}}.$$

These field equations consist of the Einstein equations

$$G_{\rho\sigma} = \frac{1}{2} T_{\rho\sigma},$$

with stress-energy tensor

$$T_{\rho\sigma} = \partial_\rho \Phi \partial_\sigma \Phi - \frac{1}{2} g_{\rho\sigma} \partial_\tau \Phi \partial^\tau \Phi + e^{-2h\Phi} \left( F_{\rho\tau} F^{\sigma}_{\tau} - \frac{1}{4} g_{\rho\sigma} F^{\tau\beta} F_{\tau\beta} \right),$$

the Maxwell equations

$$\nabla_\rho \left( e^{-2h\Phi} F^{\rho\sigma} \right) = 0,$$

and the dilaton equation

$$\nabla^2 \Phi = -\frac{h}{2} e^{-2h\Phi} F_{\rho\sigma} F^{\rho\sigma}.$$

The solution to Eqs. (11)-(14) then has the $D$-dimensional metric

$$ds_D^2 = g_{\rho\sigma} dx^\rho dx^\sigma =$$

$$\left( 1 + \frac{mr^{2-\varepsilon}}{\Pi F} \sinh^2 \alpha \right)^{\frac{1}{2}} \left\{ -dt^2 + \frac{\Pi F}{\Pi - mr^{2-\varepsilon}} dr^2 + \right.$$  

$$\sum_{i=1}^{N} (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\varphi_i^2) + \varepsilon r^2 d\nu^2 +$$

$$\left. \frac{mr^{2-\varepsilon}}{\Pi F + mr^{2-\varepsilon} \sinh^2 \alpha} \left( \cosh \alpha dt - \sum_{i=1}^{N} a_i \mu_i^2 d\varphi_i \right)^2 \right\},$$

(15)
where $F$ and $\Pi$ are given in Eq. (3) and the constraint Eq. (4) holds. The gauge potential is given by

$$A_\rho dx^\rho = \frac{mr^{2-\varepsilon} \sinh \alpha}{\Pi F + mr^{2-\varepsilon} \sinh^2 \alpha} \left( \cosh \alpha dt - \sum_{i=1}^{N} a_i \mu_i^2 d\varphi_i \right),$$

(16)

and the dilaton function reads

$$\Phi = -\frac{1}{2(D-2)\iota} \log \left( 1 + \frac{mr^{2-\varepsilon}}{\Pi F} \sinh^2 \alpha \right),$$

(17)

with $\iota$ given in Eq. (8).

### 4 Black Hole Properties

The mass $M$, the angular momenta $J_i$, the electric charge $Q$, the magnetic moments $M_i$, and the dilaton charge $\Sigma$ can be read off the asymptotic behavior of the metric, the gauge potential, and the dilaton function

$$g_{tt} = -1 + \frac{M}{(D-3)A(S^{D-2})} \frac{1}{r^{D-3}} + \ldots,$$

(18)

$$g_{t\varphi_i} = -\frac{J_i}{2A(S^{D-2})} \frac{\mu_i^2}{r^{D-3}} + \ldots,$$

(19)

$$A_t = \frac{Q}{(D-3)A(S^{D-2})} \frac{1}{r^{D-3}} + \ldots,$$

(20)

$$A_{\varphi_i} = -\frac{M_i}{(D-3)A(S^{D-2})} \frac{\mu_i^2}{r^{D-3}} + \ldots,$$

(21)

$$\Phi = \frac{\Sigma}{(D-3)A(S^{D-2})} \frac{1}{r^{D-3}} + \ldots,$$

(22)

where $A(S^{D-2})$ is the area of the unit $(D-2)$-sphere.

Comparing these expansions to the asymptotic behavior of the solution, Eqs. (15)-(17), we obtain

$$M = m \left( 1 + (D-3) \cosh^2 \alpha \right) A(S^{D-2}),$$

(23)

$$J_i = 2ma_i \cosh \alpha A(S^{D-2}), \quad i = 1, \ldots, N,$$

(24)

$$Q = (D-3)m \sinh \alpha \cosh \alpha A(S^{D-2}),$$

(25)

$$M_i = (D-3)m a_i \sinh \alpha A(S^{D-2}), \quad i = 1, \ldots, N,$$

(26)
\[
\Sigma = -\frac{(D-3)m \sinh^2 \alpha}{2(D-2)t} A(S^{D-2}) . \tag{27}
\]

By combining these global charges one can derive the following quadratic relation

\[
\frac{Q^2}{M - \frac{2(D-2)t}{D-3} \Sigma} = -2(D-3)t \Sigma , \tag{28}
\]

which determines the dilaton charge in terms of the mass and the electric charge. Note that the angular momenta do not enter this relation.

The gyromagnetic ratios \(g_i\) are given by

\[
g_i = \frac{2M_i M}{QJ_i} = (D-3) + \frac{1}{\cosh^2 \alpha} = g , \quad i = 1, \ldots, N . \tag{29}
\]

The gyromagnetic ratio \(g\) depends only on the charge-to-mass ratio, \(q = Q/M\), and ranges between \(g = D-2\) for \(q = 0\) \cite{14} and \(g = D-3\) for \(|q| = 1\) \cite{13}.

Let us now turn to the horizon properties of these black hole solutions. Their event horizon is characterized as the largest non-negative root \(r = r_H\) of the function

\[
\Delta \equiv \Pi - m r^{2-\epsilon} , \tag{30}
\]

i.e.,

\[
\Delta|_H = \left( \Pi - m r^{2-\epsilon}\right)|_{r=r_H} = 0 , \tag{31}
\]

where \(\Pi\) is given by Eq. (3). In the general case, Eq. (31) cannot be solved for \(r_H\), but it can be easily solved for \(m\), which allows to express certain horizon quantities in terms of the horizon radius \(r_H\).

The (constant) horizon angular velocities \(\Omega_i\) can be defined by imposing the Killing vector field

\[
\chi = \partial_t + \sum_{i=1}^N \Omega_i \partial_{\varphi_i} \tag{32}
\]

to be null on and orthogonal to the horizon, yielding

\[
\Omega_i = \frac{a_i}{(r_H^2 + a_i^2) \cosh \alpha} , \quad i = 1, \ldots, N . \tag{33}
\]

The area of the horizon \(A_H\) of these black holes is given by \cite{17}

\[
A_H = \frac{\cosh \alpha}{r_H^{1-\epsilon}} A(S^{D-2}) \prod_{i=1}^N (r_H^2 + a_i^2) , \tag{34}
\]

and the surface gravity \(\kappa_{sg}\), defined by

\[
\kappa_{sg}^2 = -\frac{1}{2} \left( \nabla_{\mu} \chi_{\nu} (\nabla^\mu \chi^\nu) \right)|_{r=r_H} , \tag{35}
\]
takes the form
\[ \kappa_{sg} = \left. \frac{\Delta_r}{2m r_H^2 - \varepsilon} \right|_{r=r_H}, \]  
with \( \Delta \) given by Eq. (30).

Introducing further the horizon electrostatic potential \( \Psi_{el,H} \),
\[ \Psi_{el,H} \equiv \chi^\rho A_\rho \big|_{r=r_H} = \frac{\sinh \alpha}{\cosh \alpha}, \]  
and taking into account the quantities previously defined, it is straightforward to see that these black holes satisfy the Smarr mass formula
\[ M = 2 \frac{D-2}{D-3} \kappa_{sg} A_H + \frac{D-2}{D-3} \sum_{i=1}^{N} \Omega_i J_i + \Psi_{el,H} Q. \]  

Since Eqs. (10), (25), (27), and (37) yield the relation
\[ \frac{\Sigma}{h} = -\Psi_{el,H} Q, \]  
Eq. (38) leads to the modified Smarr-like formula
\[ M = 2 \frac{D-2}{D-3} \kappa_{sg} A_H + \frac{D-2}{D-3} \sum_{i=1}^{N} \Omega_i J_i + 2\Psi_{el,H} Q + \frac{\Sigma}{h}, \]  
also valid for non-Abelian black holes (in \( D = 4 \)) [18].

5 Domain of existence

In \( D = 4 \) dimensions the Kerr black holes satisfy the relation \( M^2 \geq |16\pi J_1| \), while for the Kerr-Newman black holes of EM theory \( M^2 \geq 4Q^2 + (16\pi J_1)^2/M^2 \) holds, ensuring cosmic censorship. The bounds are saturated for extremal solutions, which thus enclose the domain of existence of EM black hole solutions, exhibited in Fig. 1.

The 4-dimensional EMD black holes satisfy the relation \( M^2 \geq (2Q)^2 + (16\pi a_1)^2 - (2\Sigma)^2 \) [13] [14], which is precisely the condition for the original Kerr solution to have horizons. The domain of existence of these EMD black holes is also exhibited in Fig. 1 and contains the EM domain. Unlike the EM case, the extremal EMD solutions do not form a smooth boundary, but form a cusp at \( J = 0 \), and the associated static extremal solutions possess vanishing horizon area. In contrast, rotating extremal solutions possess finite horizon area [13] [19].

Considering the domain of existence of \( D = 5 \) MP black holes, a similar feature is observed. \( D = 5 \) black holes possess two independent angular momenta. The extremal
$D = 5$ MP solutions then form a square with respect to the scaled angular momenta $j_i = J_i / M^{3/2}$, $i = 1, 2$, as seen in Fig. 1. At the vertices of the square one of the two angular momenta vanishes, and the associated extremal single angular momentum solutions possess vanishing horizon area. Generic extremal solutions (with two non-vanishing angular momenta), in contrast, possess finite horizon area.

The domain of existence of the $D = 5$ EMD black holes is exhibited in Fig. 2. We observe, that extremal $D = 5$ EMD solutions retain the main features of extremal $D = 5$ MP solutions. Thus for a given scaled charge $q = Q / M$, the extremal EMD solutions form a square, and have vanishing horizon area at the vertices. With increasing $|q|$ the square shrinks and reaches zero size for $|q| = 1$.

When moving to $D = 6$ dimensions, the domain of existence of MP solutions changes distinctly. The vertices present in the $D = 5$ domain then move to infinity, while the boundary lines between the vertices are no longer straight but are given by

$$1 = \frac{256\sqrt{6}}{27\pi^2} \left[ (j_1^2 + j_2^2)^2 + 12j_1^2j_2^2 + 2(j_1^2 + j_2^2) \right] \sqrt{(j_1^2 + j_2^2)^2 + 12j_1^2j_2^2} - (j_1^2 + j_2^2).$$

Thus there are no extremal solutions for single angular momentum black holes. In contrast, generic black holes with two non-vanishing angular momenta possess extremal limits, as illustrated in Fig. 1.

The domain of existence of the $D = 6$ EMD black holes is exhibited in Fig. 2. Again,
the extremal $D = 6$ EMD solutions retain the main features of extremal $D = 6$ MP solutions. Thus for a given scaled charge $q = Q/M$, the extremal EMD solutions form hyperbola-like curves, diverging on the $j_1$ and $j_2$ axes. With increasing $|q|$ the curves approach the $j_1$ and $j_2$ axes, reaching them in the limit $|q| = 1$. The solutions on these axes have vanishing area.

This $D = 6$ pattern is now retained and generalized when going to higher dimensions. In $D > 6$ dimensions generic black holes with $N = \left(D - 1\right)/2$ non-vanishing angular momenta are delimited by extremal solutions. The domain of existence is unbounded though, since the $j$-axes are part of it. The inequality $|q| \leq 1$ remains valid for any dimension $D > 6$. As $|q|$ increases, the $N$-dimensional volume of black hole solutions for constant $q$ shrinks in size. In the limit $|q| = 1$, the resulting surface of extremal EMD solutions contains the $j$-axes, as illustrated for $D = 7$ in Fig. 3.

6 Conclusions

Based on the Kaluza-Klein action in $D$ dimensions, obtained by reducing the $D + 1$ dimensional vacuum Einstein action, we have constructed exact rotating black hole solutions by embedding the $D$ dimensional MP solutions in $D + 1$ dimensions and boosting in the extra direction.

The resulting black hole solutions are asymptotically flat, and possess a regular horizon of spherical topology. They are characterized by their global charges: their mass, their $N = \left(D - 1\right)/2$ angular momenta, and their electric charge. Their dilaton charge is not independent, but determined by their mass and electric charge, Eq. (28).
Figure 3: Left: Domain of existence of MP black holes in $D = 7$ in terms of scaled angular momenta $j_i = J_i / M^{(D-2)/(D-3)}$, $i = 1, 2, 3$. Right: Extremal EMD black holes in $D = 7$ for the extremal value of the scaled charge $|q| = 1$.

Their gyromagnetic ratio covers the range $D - 3 \leq g \leq D - 2$.

Combining their global charges and their horizon properties, these rotating black holes are seen to satisfy the first law and a Smarr formula Eq. (38), where the electric charge term can be replaced by a dilaton charge term, Eq. (40).

The domain of existence of these black holes is obtained by considering the set of extremal black holes. In 5 dimensions there exist always extremal solutions, but when one of the two angular momenta vanishes, these extremal solutions possess vanishing horizon area. In higher dimensions, for $|q| < 1$ extremal solutions do not exist, when one (for even $D$) or two (for odd $D$) of the angular momenta vanish. In the limit $|q| = 1$ the $j$-axes are contained in the surface of extremal EMD solutions.

Here these higher dimensional rotating EMD black hole solutions were obtained only for particular values of the dilaton coupling constant. It remains a challenge to generalize these solutions to arbitrary values of the dilaton coupling constant, including the pure Einstein-Maxwell case [20].

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