The homomorphisms between scalar generalized Verma modules associated to maximal parabolic subalgebras

Hisayosi Matumoto
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Tokyo
153-8914, JAPAN
e-mail: hisayosi@ms.u-tokyo.ac.jp

Abstract

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. We classify the homomorphisms between $\mathfrak{g}$-modules induced from one-dimensional modules of maximal parabolic subalgebras.

§ 0. Introduction

In this article, we consider the existence problem of homomorphisms between generalized Verma modules, which are induced from one dimensional representations (such generalized Verma modules are called scalar, cf. [Boe 1985]). Our main result is the classification of the homomorphisms between scalar generalized Verma modules with respect to the maximal parabolic subalgebras.

A sufficient condition for the existence of the homomorphisms between Verma modules is given by [Verma 1968]. Bernstein, I. M. Gelfand, and S. I. Gelfand proved the condition of Verma is also a necessary condition. ([Bernstein-Gelfand-Gelfand 1975])

Later, Lepowsky studied the problem for the generalized Verma modules. In particular, Lepowsky ([Lepowsky 1975a]) solved the existence problem of nontrivial homomorphisms between scalar generalized Verma modules associated to the parabolic subalgebras which are the complexifications of the minimal parabolic subalgebras of real rank one simple Lie algebras (so-called the real rank one case).

Lepowsky also obtained a sufficient condition for the existence of the homomorphisms between scalar generalized Verma modules associated to the complexification of the minimal parabolic subalgebras of (not necessarily rank one) real semisimple Lie algebras. His condition is quite similar to that of Verma and he conjectured it is also a sufficient condition in the setting of complexified minimal parabolic algebras ([Lepowsky 1975b]).

Boe ([Boe 1985]) solved the existence problem in the case of parabolic subalgebras whose nilradical is commutative (so-called the Hermitian symmetric case).

The existence problem for maximal parabolic algebras is, in principle, reduced to the Kazhdan-Lusztig algorithm. Casian and Collingwood ([Casian-Collingwood 1987]) proposed a direct

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method of computing the Kazdhan-Lusztig data involving the generalized Verma modules. Applying the Kazdhan-Lusztig algorithm works very well in some cases. The structures of the (not necessarily scalar) generalized Verma modules in the real rank one case and in the Hermitian symmetric case are studied precisely ([Boe-Collingwood 1985], [Boe-Enright-Shelton 1988], [Collingwood-Irving-Shelton 1988]). Boe and Collingwood ([Boe-Collingwood 1990]) studied the case that all the (not necessarily scalar) generalized Verma modules are multiplicity free. The cases treated by Boe and Collingwood is more general than the real rank one case and the Hermitian symmetric case (so-called the multiplicity free case). In that case, they studied the structures of the (not necessarily scalar) generalized Verma modules precisely. In particular, they solved the existence problem of the non-trivial homomorphisms for regular integral infinitesimal characters in the multiplicity free case.

However, I surmise it is not easy to give the explicit answer to the existence problem by the Kazdhan-Lusztig algorithm in the general setting. Our central dogma is “Consider the most singular parameter, then everything turns to be easy.” Our approach to the problem consists of the following three main ingredients.

1. The translation principle

The translation principle has a long history. In [Vogan 1988], Vogan proposed an idea on translation principle in order to establish the irreducibility of a discrete series representation of a semisimple symmetric space in some case. His idea is extremely useful for the study of the existence problem. Depending Vogan’s idea, we formulated a version of translation principle in [Matumoto 1993] Proposition 2.2.3. In some cases, this enable us to reduce the existence of a non-trivial homomorphism to the most singular case in which the problem is often trivial.

2. Jantzen’s irreducibility criterion

Applying a version of the translation principle, we can often reduce the nonexistence of a non-trivial homomorphism to the irreducibility of a particular generalized Verma module. In [Jantzen 1977], Jantzen gave a sufficient and necessary condition for the irreducibility of a generated Verma module. His result is extremely useful for our purpose and we can establish the nonexistence of nontrivial homomorphisms in many cases.

3. The Kazdhan-Lusztig theory ([Kazdhan-Lusztig 1979], [Brylinski-Kashiwara 1981], [Beilinson-Bernstein 1993])

Although we do not compute Kazdhan-Lusztig polynomials, the existence of the Kazdhan-Lusztig algorithm plays an important role in our approach. The point is that the definition of the Kazdhan-Lusztig polynomials only depends on Coxeter systems. In some cases, this enable us to reduce the problem to that of a different maximal parabolic subalgebra of a different simple Lie algebra, which is easier than the original problem.

In the most of the cases we can solve the existence problem by the above three ideas. However, in some cases we need extra arguments.

This article consists of five sections.

We fix notations and introduce some fundamental material in §1.

In §2, we introduce sufficient conditions for the existence problem.

In §3, we treat the case of the classical algebras. The type A case is in the Hermitian symmetric case. So, we only consider the case of the type B,C, and D. The main theorems are Theorem 3.2.1, 3.2.2, and 3.2.3.

In §4, we treated the case of the exceptional algebras.

In §5, using a comparison result, we explain how to construct a homomorphism between scalar generalized Verma modules associated to a general parabolic subalgebra from a homomorphism
between scalar generalized Verma modules associated to a maximal parabolic subalgebra. We call such a homomorphism an elementary homomorphism.

I would like to propose:

**Working Hypothesis** An arbitrary nontrivial homomorphism between scalar generalized Verma modules is a composition of elementary homomorphisms.

The working hypothesis in the case of the Verma modules is nothing but the result of Bernstein-Gelfand-Gelfand. The first statement of the Lepowsky conjecture ([Lepowsky 1975b] Conjecture 6.13) means that, in the case of complexified minimal parabolic subalgebras of real semisimple Lie algebras, the above working hypothesis is affirmative.

The result of this article solves the existence of edge-of-wedge type embeddings in the case of the maximal parabolic subgroups of complex reductive groups ([Matumoto 2003]).

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§ 1. Notations and Preliminaries

**1.1 General notations**

In this article, we use the following notations and conventions.

As usual we denote the complex number field, the real number field, the ring of (rational) integers, and the set of non-negative integers by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively. $\frac{1}{2}\mathbb{N}$ means the set $\{\frac{n}{2} | n \in \mathbb{N}\}$, and $\frac{1}{2} + \mathbb{N}$ means the set $\{\frac{1}{2} + n | n \in \mathbb{N}\}$. We denote by $\emptyset$ the empty set. For any (non-commutative) $\mathbb{C}$-algebra $R$, “ideal” means “2-sided ideal”, “$R$-module” means “left $R$-module”, and sometimes we denote by 0 (resp. 1) the trivial $R$-module $\{0\}$ (resp. $\mathbb{C}$). Often, we identify a (small) category and the set of its objects. Hereafter “dim” means the dimension as a complex vector space, and “$\otimes$” (resp. Hom) means the tensor product over $\mathbb{C}$ (resp. the space of $\mathbb{C}$-linear mappings), unless we specify. For a complex vector space $V$, we denote by $V^*$ the dual vector space. For $a, b \in \mathbb{C}$, “$a \leq b$” means that $a, b \in \mathbb{R}$ and $a \leq b$. We denote by $A - B$ the set theoretical difference. $\text{card}A$ means the cardinality of a set $A$.

**1.2 Notations for reductive Lie algebras**

Let $\mathfrak{g}$ be a complex reductive Lie algebra, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote by $\Delta$ the root system with respect to $(\mathfrak{g}, \mathfrak{h})$. We fix some positive root system $\Delta^+$ and let $\Pi$ be the set of simple roots. Let $W$ be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ and let $\langle , \rangle$ be a non-degenerate invariant bilinear form on $\mathfrak{g}$. For $w \in W$, we denote by $\ell(w)$ the length of $w$ as usual. We also denote the inner product on $\mathfrak{h}^*$ which is induced from the above form by the same symbols $\langle , \rangle$. For $\alpha \in \Delta$, we denote by $s_\alpha$ the reflection in $W$ with respect to $\alpha$. We denote by $w_0$ the longest element of $W$. For $\alpha \in \Delta$, we define the coroot $\check{\alpha}$ by $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$, as usual. We call $\lambda \in \mathfrak{h}^*$ is dominant (resp. anti-dominant), if $\langle \lambda, \check{\alpha} \rangle$ is not a negative (resp. positive) integer, for each $\alpha \in \Delta^+$. We call $\lambda \in \mathfrak{h}^*$ regular, if $\langle \lambda, \alpha \rangle \neq 0$, for each $\alpha \in \Delta$. We denote by $\mathcal{P}$ the integral weight lattice, namely $\mathcal{P} = \{\lambda \in \mathfrak{h}^* \mid \langle \alpha, \check{\alpha} \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$.
If \( \lambda \in \mathfrak{h}^* \) is contained in \( \mathfrak{p} \), we call \( \lambda \) an integral weight. We define \( \rho \in \mathfrak{p} \) by \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \).

Put \( \mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid \forall H \in \mathfrak{h} \ [H, X] = \alpha(H)X \} \), \( u = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \), \( b = \mathfrak{h} + u \). Then \( b \) is a Borel subalgebra of \( \mathfrak{g} \). We denote by \( \mathcal{Q} \) the root lattice, namely \( \mathbb{Z} \)-linear span of \( \Delta \). We also denote by \( \mathcal{Q}^+ \) the linear combination of \( \Pi \) with non-negative integral coefficients. For \( \lambda \in \mathfrak{h}^* \), we denote by \( W_\lambda \) the integral Weyl group. Namely,

\[
W_\lambda = \{ w \in W \mid w\lambda - \lambda \in \mathcal{Q} \}.
\]

We denote by \( \Delta_\lambda \) the set of integral roots.

\[
\Delta_\lambda = \{ \alpha \in \Delta \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z} \}.
\]

It is well-known that \( W_\lambda \) is the Weyl group for \( \Delta_\lambda \). We put \( \Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda \). This is a positive system of \( \Delta_\lambda \). We denote by \( \Pi_\lambda \) the set of simple roots for \( \Delta_\lambda^+ \) and denote by \( \Phi_\lambda \) the set of reflection corresponding to the elements in \( \Pi_\lambda \). So, \( (W_\lambda, \Phi_\lambda) \) is a Coxeter system. We denote by \( \mathcal{Q}_\lambda \) the integral root lattice, namely \( \mathcal{Q}_\lambda = \mathbb{Z}\Delta_\lambda^+ \) and put \( \mathcal{Q}_\lambda^+ = \mathbb{N}\Pi_\lambda \).

Next, we fix notations for a parabolic subalgebra (which contains \( \mathfrak{b} \)). Hereafter, through this article we fix an arbitrary subset \( \Theta \) of \( \Pi \). Let \( \Theta \) be the set of the elements of \( \Delta \) which are written by linear combinations of elements of \( \Theta \) over \( \mathbb{Z} \). Put \( \mathfrak{a}_\Theta = \{ H \in \mathfrak{h} \mid \forall \alpha \in \Theta \ \alpha(H) = 0 \} \), \( \mathfrak{l}_\Theta = \mathfrak{h} + \sum_{\alpha \in \Theta} \mathfrak{g}_\alpha \), \( \mathfrak{n}_\Theta = \sum_{\alpha \in \Delta^+ \setminus \Theta} \mathfrak{g}_\alpha \), \( \mathfrak{p}_\Theta = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta \). Then \( \mathfrak{p}_\Theta \) is a parabolic subalgebra of \( \mathfrak{g} \) which contains \( \mathfrak{b} \). Conversely, for an arbitrary parabolic subalgebra \( \mathfrak{p} \supseteq \mathfrak{b} \), there exists some \( \Theta \subseteq \Pi \) such that \( \mathfrak{p} = \mathfrak{p}_\Theta \). We denote by \( W_\Theta \) the Weyl group for \( (\mathfrak{l}_\Theta, \mathfrak{h}) \). \( W_\Theta \) is identified with a subgroup of \( W \) generated by \( \{ s_\alpha \mid \alpha \in \Theta \} \). We denote by \( w_\Theta \) the longest element of \( \Theta \).

Using the invariant non-degenerate bilinear form \( \langle , \rangle \), we regard \( \mathfrak{a}_\Theta^* \) as a subspace of \( \mathfrak{h}^* \). It is known that there is a unique nilpotent (adjoint) orbit (say \( \mathcal{O}_{\mathfrak{p}_\Theta} \)) whose intersection with \( \mathfrak{n}_\Theta \) is Zarisky dense in \( \mathfrak{n}_\Theta \). \( \mathcal{O}_{\mathfrak{p}_\Theta} \) is called the Richardson orbit with respect to \( \mathfrak{p}_\Theta \). We denote by \( \mathcal{O}_{\mathfrak{p}_\Theta} \) in \( \mathfrak{g} \). Put \( \rho_\Theta = \frac{1}{2}(\rho - w_\Theta \rho) \) and \( \rho^\Theta = \frac{1}{2}(\rho + w_\Theta \rho) \). Then, \( \rho^\Theta \in \mathfrak{a}_\Theta^* \).

1.3 Generalized Verma modules

Define

\[
P^++_\Theta = \{ \lambda \in \mathfrak{h}^* \mid \forall \alpha \in \Theta \ \langle \lambda, \check{\alpha} \rangle \in \{ 1, 2, \ldots \} \}.
\]

\[
c^+P^+_\Theta = \{ \lambda \in \mathfrak{h}^* \mid \forall \alpha \in \Theta \ \langle \lambda, \check{\alpha} \rangle = 1 \}
\]

We easily have

\[
c^+P^+_\Theta = \{ \rho_\Theta + \mu \mid \mu \in \mathfrak{a}_\Theta^* \}.
\]

For \( \mu \in \mathfrak{h}^* \) such that \( \mu + \rho \in \mathfrak{P}^+_\Theta \), we denote by \( \sigma_{\mathfrak{p}_\Theta}(\mu) \) the irreducible finite-dimensional \( \mathfrak{p}_\Theta \)-representation whose highest weight is \( \mu \). Let \( E_{\mathfrak{p}_\Theta}(\mu) \) be the representation space of \( \sigma_{\mathfrak{p}_\Theta}(\mu) \). We define a left action of \( \mathfrak{n}_{\mathfrak{p}_\Theta} \) on \( E_{\mathfrak{p}_\Theta}(\mu) \) by \( X \cdot v = 0 \) for all \( X \in \mathfrak{n}_{\mathfrak{p}_\Theta} \) and \( v \in E_{\mathfrak{p}_\Theta}(\mu) \). So, we regard \( E_{\mathfrak{p}_\Theta}(\mu) \) as a \( U(\mathfrak{p}_\Theta) \)-module.

For \( \mu \in \mathfrak{P}^+_\Theta \), we define a generalized Verma module ([Lepowsky 1977]) as follows.

\[
M_{\mathfrak{p}_\Theta}(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} E_{\mathfrak{p}_\Theta}(\mu - \rho).
\]

For all \( \lambda \in \mathfrak{h}^* \), we write \( M(\lambda) = M_{\mathfrak{p}_\Theta}(\lambda) \). \( M(\lambda) \) is called a Verma module. For \( \mu \in \mathfrak{P}^+_\Theta \), \( M_{\mathfrak{p}_\Theta}(\mu) \) is a quotient module of \( M(\mu) \). Let \( L(\mu) \) be the unique highest weight \( U(\mathfrak{g}) \)-module with the
highest weight \( \mu - \rho \). Namely, \( L(\mu) \) is a unique irreducible quotient of \( M(\mu) \). For \( \mu \in \mathcal{P}_{\Theta}^{++} \), the canonical projection of \( M(\mu) \) to \( L(\mu) \) is factored by \( M_{\Theta}(\mu) \).

\[ \dim E_{\Theta}(\mu - \rho) = 1 \text{ if and only if } \mu \in \mathcal{P}_{\Theta}^{++} \Theta. \]

For a finitely generated \( U(g) \)-module \( V \), we denote by \( \text{Dim}(V) \) (resp. \( c(V) \)) the Gelfand-Kirillov dimension (resp. the multiplicity) of \( V \). (See [Vogan 1978]). We easily see \( \text{Dim}(M_{\Theta}(\mu)) = \dim n_{\Theta} \) and \( c(M_{\Theta}(\mu)) = \dim E_{\Theta}(\mu - \rho) \).

The following result is one of the fundamental results on the existence problem of homomorphisms between scalar generalized Verma modules.

**Theorem 1.3.1.** ([Lepowsky 1976])

Let \( \mu, \nu \in \mathcal{P}_{\Theta}^{++} \).

1. \( \dim \text{Hom}_{U(g)}(M_{\Theta}(\mu), M_{\Theta}(\nu)) \leq 1 \).
2. Any non-zero homomorphism of \( M_{\Theta}(\mu) \) to \( M_{\Theta}(\nu) \) is injective.

Hence, the existence problem of homomorphisms between scalar generalized Verma modules is reduce to the following problem.

**Problem** Let \( \mu, \nu \in \mathcal{P}_{\Theta}^{++} \). When is \( M_{\Theta}(\mu) \subseteq M_{\Theta}(\nu) \)?

### 1.4 Homomorphisms associated with Duflo involutions

Hereafter we assume \( \Theta \subseteq \Pi, \mu \in \mathcal{P}_{\Theta}^{++} \) and \( \mu \) is dominant and regular. Then, we easily have \( w_{\Theta}w_{0}\mu \in \mathcal{P}_{\Theta}^{++} \) and \( M_{\Theta}(w_{\Theta}w_{0}\mu) \) is irreducible. Here, we consider the following problem

**Problem 1.4.1.** When is \( M_{\Theta}(w_{\Theta}w_{0}\mu) \hookrightarrow M_{\Theta}(\mu) \)?

Concerning to Problem 1.4.1, a necessary and sufficient condition is known.

**Theorem 1.4.2.** ([Matumoto 1993]) Let \( \Theta \subseteq \Pi \) and \( \mu \in \mathcal{P}_{\Theta}^{++} \). If \( \mu \) is dominant and regular, then the following two conditions (1) and (2) are equivalent.

1. \( M_{\Theta}(w_{\Theta}w_{0}\mu) \hookrightarrow M_{\Theta}(\mu) \).
2. \( w_{\Theta}w_{0} \) is a Duflo involution for the Coxeter system \((W_{\mu}, \Phi_{\mu})\).

In particular, the answer of Problem 2.1.1 only depend on the Coxeter system \((W_{\mu}, \Phi_{\mu})\). (In fact, this fact is a conclusion of the Kazdhan-Lusztig conjecture.) We can find a complex reductive Lie algebra whose Weyl group (with a set of the simple reflections) are isomorphic to \((W_{\mu}, \Phi_{\mu})\). So, we can deduce Problem 1.4.1 to the following special case.

**Problem 1.4.3.** When is \( M_{\Theta}(w_{\Theta}w_{0}\rho) \hookrightarrow M_{\Theta}(\rho) \)?

We also remark that the following easy fact.

**Lemma 1.4.4.** Let \( \Theta \subseteq \Pi \) and \( \lambda \in \mathcal{P}_{\Theta}^{++} \) such that \( w_{\Theta}w_{0} = w_{0}w_{\Theta} \) and \( w_{\Theta}w_{0} \in W_{\lambda} \). We denote by \( w'_{0} \) be the longest element of \( W_{\lambda} \) with respect to \( \Pi_{\lambda} \). Then, \( w_{0} = w'_{0} \) and \( w_{\Theta}w'_{0} = w'_{0}w_{\Theta} \).
We denote by $Z(g)$ the center of $U(g)$. It is well-known that $Z(g)$ acts on $M(\lambda)$ by the Harish-Chandra homomorphism $\chi_{\lambda} : Z(g) \rightarrow \mathbb{C}$ for all $\lambda$. $\chi_{\lambda} = \chi_{\mu}$ if and only if there exists some $w \in W$ such that $\lambda = w\mu$. We denote by $Z_{\lambda}$ the kernel of $\chi_{\lambda}$ in $Z(g)$. Let $M$ be a $U(g)$-module and $\lambda \in \mathfrak{h}^*$. We say that $M$ has an infinitesimal character $\lambda$ if and only if $Z(g)$ acts on $M$ by $\chi_{\lambda}$. We say that $M$ has a generalized infinitesimal character $\lambda$ if and only if for any $v \in M$ there is some positive integer $n$ such that $Z_{\lambda}^nv = 0$. We say $M$ is locally $Z(g)$-finite, if and only if for any $v \in M$ we have $\dim Z(g)v < \infty$. We denote by $M_{Zf}$ (cf. [Bernstein-Gelfand 1980]) the category of $Z(g)$-finite $U(g)$-modules. We also denote by $\mathcal{M}[\lambda]$ the category of $U(g)$-modules with generalized infinitesimal character $\lambda$. Then, from the Chinese remainder theorem, we have a direct sum of abelian categories $M_{Zf} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathcal{M}[\lambda]$. We denote by $P_{\lambda}$ the projection functor from $M_{Zf}$ to $\mathcal{M}[\lambda]$. For $\mu \in \mathbb{P}$, we denote by $V_{\mu}$ the irreducible finite-dimensional $U(g)$-module with an extreme weight $\mu$. Let $\mu, \lambda \in \mathfrak{h}^*$ satisfy $\mu - \lambda \in \mathbb{P}$. Let $M$ be an object of $\mathcal{M}[\lambda]$. Then, from a result of Kostant we have that $M \otimes V_{\mu-\lambda}$ is an object of $M_{Zf}$. So, we can define translation functor $T^\mu_{\lambda}$ from $\mathcal{M}[\lambda]$ to $\mathcal{M}[\mu]$ as follows.

$$T^\mu_{\lambda}(M) = P_{\mu}(M \otimes V_{\mu-\lambda}).$$

The translation functors are exact.

We put

$$W(\Theta) = \{w \in W \mid w\Theta = \Theta\}.$$ 

Then, $W(\Theta)$ is a subgroup of $W$. Moreover, $w\rho_\Theta = \rho_\Theta$ and $w\Theta w = w\Theta$ hold for all $w \in W(\Theta)$ and $W(\Theta)$ preserves $\mathfrak{a}_\Theta^\theta$. In particular, $W(\Theta) \subseteq W_{\rho_\Theta}$.

We say that $\lambda \in \mathfrak{a}_\Theta^\theta$ is strongly $\Theta$-antidominant if and only if $\langle \lambda, \alpha \rangle \leqslant 0$ for all $\alpha \in \mathbb{Q}^+ \cap \mathbb{Q}_{\rho_\Theta+\lambda}$. (Cf. [Matumoto 1993])

Next, we consider the images of generalized Verma modules under certain translation functors.

**Lemma 1.5.1.** (Cf. [Matumoto 1993] Lemma 1.2.3) Assume that $\mu, \lambda \in \mathfrak{a}_\Theta^\theta$ are strongly $\Theta$-antidominant and that $\lambda - \mu$ is dominant and integral. Let $w \in W(\Theta)$. Then, we have

$$T^{-\rho_\Theta+\lambda}_{-\rho_\Theta+\mu}(M_\Theta(\rho_\Theta + w\mu)) = M_\Theta(\rho_\Theta + w\lambda).$$

### § 2. Sufficient conditions

For almost all $(g, \mathfrak{h}, \Delta^+, \Theta)$, the necessary and sufficient condition given in Theorem 1.4.2 is hard to check. So, we consider sufficient conditions, which we can check easily.

#### 2.1 A sufficient condition

We fix the notations for characters. (Cf. [Dixmier 1977] 7.5.1, [Knapp 2002] V. 6) Let $\mathbb{C}^{\mathfrak{h}^*}$ be the $\mathbb{C}$-vector space of all functions from $\mathfrak{h}^*$ to $\mathbb{C}$. For $f \in \mathbb{C}^{\mathfrak{h}^*}$, we define $\text{supp}(f) = \{\lambda \in \mathfrak{h}^* \mid f(\lambda) \neq 0\}$. For $\lambda \in \mathfrak{h}^*$, we define $e^\lambda$ the member of $\mathbb{C}^{\mathfrak{h}^*}$, that is 1 at $\lambda$ and 0 elsewhere. Let $\mathbb{C} < \mathfrak{h}^*$ be the set of all $f \in \mathbb{C}^{\mathfrak{h}^*}$ such that $\text{supp}(f)$ is contained in the union of a finite number of sets $\nu_i - Q^+$ with each $\nu_i$ in $\mathfrak{h}^*$. We introduce the structure of $\mathbb{C}$-algebra on $\mathbb{C}^{\mathfrak{h}^*}$ as in [Knapp 2002] (5.65).
Let $V$ be a $U(\mathfrak{g})$-module. For $\lambda \in \mathfrak{h}^*$, we define the weight space with respect to $\lambda$ as follows.

$$V_\lambda = \{v \in V \mid \forall H \in \mathfrak{h} \; Hv = \lambda(H)v\}$$

We say that $V$ has a character if $V$ is the direct sum of its weight spaces under $\mathfrak{h}$ and if $\dim V_\lambda \leq \infty$ for all $\lambda \in \mathfrak{h}^*$. In this case, the character is

$$[V] = \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda)e^\lambda.$$ 

For example, for $\lambda \in P_\Theta^+$, the following formula is well-known.

$$[M_\Theta(\lambda)] = D^{-1} \sum_{w \in W_\Theta} (-1)^{\ell(w)}e^{w\lambda}.$$ 

Here, we denote by $D$ the Weyl denominator, namely $D = e^\mu \prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})$. In particular, we have $[M(\lambda)] = D^{-1}e^\lambda$.

We put $B_{st} = \{[M(wp)] \mid w \in W\}$ and denote by $\mathcal{C}$ the subspace of $\mathbb{C}\mathfrak{h}^*$ spanned by $B_{st}$. We also put $B_{irr} = \{[L(wp)] \mid w \in W\}$. Then, $B_{st}$ and $B_{irr}$ are bases of $\mathcal{C}$. We identify the group algebra $\mathbb{C}[W]$ with $\mathcal{C}$ via the correspondence $W \ni w \mapsto [M(wp)] \in B_{st}$. We can introduce a $W$-module structure on $\mathcal{C}$ identifying $\mathcal{C}$ and the right regular representation on $\mathbb{C}[W]$. We consider the equivalence relations $\sim^L$ and $\sim^R$ on $W$ defined in [Kazhdan-Lusztig 1979]. The representation theoretic meanings of these equivalence relations are as follows.

**Theorem 2.1.1.** ([Joseph 1977], [Vogan 1980]) Let $x, y \in W$. Then we have

1. $x \sim^L y$ if and only if $\text{Ann}_{U(\mathfrak{g})}(L(xw_0\rho)) = \text{Ann}_{U(\mathfrak{g})}(L(yw_0\rho))$.
2. $x \sim^R y$ if and only if there exist some finite-dimensional $U(\mathfrak{g})$-modules $E_1$ and $E_2$ such that $L(xw_0\rho)$ and $L(yw_0\rho)$ are irreducible constituents of $E_1 \otimes L(yw_0\rho)$ and $E_2 \otimes L(xw_0\rho)$, respectively.

For $x \in W$, we denote by $V^R_x$ the $\mathbb{C}$-vector space with a basis $B_x = \{[L(yw_0\rho)] \mid y \sim^R x\}$. If we identify $V^R_x$ with a subquotient of $\mathcal{C}$ appropriately, we may regard $V^R_x$ a $W$-module. (For example, see [Barbasch-Vogan 1983]). $V^R_x$ is called a right cell representation.

We denote by $\mathcal{H}$ the space of $W$-harmonic polynomials on $\mathfrak{h}^*$, which can be regarded as a $W$-module in a usual manner (cf. [Vogan 1978]).

We quote:

**Theorem 2.1.2.** ([Vogan 1978], [Joseph 1980a, 1980b])

For $x \in W$, there is a $W$-homomorphism $\phi_x$ of $V^R_x$ to $\mathcal{H}$ satisfying the following conditions.

1. The image $\phi_x(V^R_x) \subseteq \mathcal{H}$ is an irreducible representation.
2. The image $\phi_x(V^R_x)$ is the special representation corresponding to the unique open dense nilpotent orbit in the associated variety of $\text{Ann}_{U(\mathfrak{g})}(L(xw_0\rho))$ via the Springer correspondence. In particular, $\phi_{w_0}(V^R_{w_0})$ is the special representation corresponding to the Richardson orbit $\mathcal{O}_{p_0}$.
3. For $y, z \in W$ such that $y \sim x \sim z, y \sim z$ if and only if $\phi_x([L(yw_0\rho)])$ and $\phi_x([L(zw_0\rho)])$ are proportional to each other.

Now, we state a sufficient condition.
Moreover, we have

Hereafter we fix

\[ M \sigma (w_0 w_0 \rho) \subseteq M (\rho). \]

**Proof.** We put \( I = \text{Ann}_{U (g)}(M \sigma (w_0 w_0 \rho)). \) \( I \) is primitive, since \( M \sigma (w_0 w_0 \rho) \) is irreducible. Since \( w_0 w_0 = w_0 w_0 \), we have \( w_0 w_0 \in W (S) \). So, from [Borho-Jantzen 1977] 4.10 Corollary, we have \( I = \text{Ann}_{U (g)}(M (\rho)). \) Since \( c(M \sigma (\rho)) = 1, M (\rho) \) has a unique irreducible constituent of maximal Gelfand-Kirillov dimension (say \( L (\sigma \rho) \)). Here, \( \sigma \) is an element of \( W \). (In fact we easy to prove \( L (\sigma \rho) \) is the unique irreducible submodule of \( M (\rho) \).) Assuming that \( \sigma \neq w_0 w_0 \), we shall deduce a contradiction.

When we regard a \( U (g) \)-module \( E \) as a \( U (p_\theta) \)-module, we write it by \( E |_{p_\theta} \). Since \( M \sigma (\mu) \otimes E \cong U (g) \otimes U (p_\theta) (E \otimes \mu - \rho) \otimes E |_{p_\theta} \) holds, we easily see that \( \sigma w_0 \not\sim w_\theta \).

From [Borho-Kraft 1976] 3.6, we see \( I = \text{Ann}_{U (g)}(L (\sigma \rho)). \) From Theorem 1.6.1, we have

\[ w_\theta \not\sim \sigma w_0. \]

Hence, from Theorem 1.6.2, \( \phi_{w_\theta} \) is not injective. This contradicts our assumption that \( V^{R}_{w_\theta} \) is irreducible. Q.E.D.

The multiplicity of a special representation \( \phi_x (V^{R}_{x}) \) in a right cell \( V^{R}_{x} \) is one. Moreover, any irreducible constituents in the right cell \( V^{R}_{x} \) belongs to the same family (see [Lusztig 1981] p78) as \( \phi_x (V^{R}_{x}). \) So, we have:

**Corollary 2.1.4.** Assume that \( \Theta \subseteq \Pi \) satisfy \( w_\Theta w_0 = w_0 w_\Theta \) and that the family of the special representation corresponding to the Richardson orbit of \( O_{p_\theta} \) does not contain any other element. Then, we have \( M \sigma (w_\Theta w_0 \rho) \subseteq M (\rho). \)

We denote by \( \not\sim \) the equivalence relation on \( W \) generated by \( \sim \) and \( \not\sim \). The following result is well-known and follows from Theorem 2.1.1.

**Corollary 2.1.5.** Let \( x, y \in W \) be such that \( x \not\sim y. \) Then \( \text{Dim}(L (x w_0 \rho)) = \text{Dim}(L (y w_0 \rho)). \)

From [Barbasch-Vogan 1983] Corollary 2.24 implies that \( x \not\sim y \) if and only if \( x w_0 \not\sim y w_0. \) Hence, we have:

**Lemma 2.1.6.** Let \( x, y \in W \) be such that \( x \not\sim y. \) Then \( \text{Dim}(L (x \rho)) = \text{Dim}(L (y \rho)). \)

### 2.2 Maximal parabolic subalgebras

Hereafter we fix \( \alpha \in \Pi. \) Put \( \Theta^\alpha = \Pi - \{ \alpha \}. \) \( a_{\Theta^\alpha} \) is one-dimensional and spanned by \( \rho_{\Theta^\alpha}. \)

Moreover, we have

\[ ^0 P^{++}_{\Theta^\alpha} = \{ \rho_{\Theta^\alpha} + t \rho_{\Theta^\alpha} \mid t \in \mathbb{C} \}. \]

We denote by \( \omega_\alpha \) the fundamental weight corresponding to \( \alpha. \) For any \( \beta \in \Theta^\alpha = \Pi - \{ \alpha \}, \) we have \( \langle \beta, \rho_{\Theta^\alpha} \rangle = 0. \) Hence there exists some \( d_\alpha \in \mathbb{R} \) such that \( d_\alpha \omega_\alpha = \rho_{\Theta^\alpha}. \) Since \( 2 \rho_{\Theta^\alpha} \) is integral, we have \( d_\alpha \in \mathbb{Z} \cap \mathbb{N}. \)

For simplicity, for \( t \in \mathbb{C}, \) we write \( M_{\Theta^\alpha} [t] \) for \( M_{\Theta^\alpha} (\rho_{\Theta^\alpha} + t \omega_\alpha). \) We have:

**Lemma 2.2.1.** Let \( s \) and \( t \) be distinct complex numbers such that \( M_{\Theta^\alpha} [s] \subseteq M_{\Theta^\alpha} [t]. \) Then, \( s = -t. \)
Proof. Since $M_{\Theta^0}[s]$ and $M_{\Theta^0}[t]$ have the same infinitesimal character, there exists some $w \in W$ such that $\rho_{\Theta^0} + tw_\alpha = w(\rho_{\Theta^0} + s_\omega_\alpha)$. Hence, $\langle \rho_{\Theta^0} + tw_\alpha, \rho_{\Theta^0} + tw_\alpha \rangle = \langle \rho_{\Theta^0} + s_\omega_\alpha, \rho_{\Theta^0} + s_\omega_\alpha \rangle$. From $\langle \omega_\alpha, \rho_{\Theta^0} \rangle = 0$, we have $t^2 = s^2$. So, $s = -t$. \qed

We easily have:

Lemma 2.2.2. (1) If $w_{\Theta^0}w_0 = w_0w_{\Theta^0}$, then $w_{\Theta^0}w_0(\rho_{\Theta^0} + tw_\alpha) = \rho_{\Theta^0} - tw_\alpha$ for all $t \in \mathbb{C}$.

(2) If $w_{\Theta^0}w_0 \neq w_0w_{\Theta^0}$, then $w_{\Theta^0}w_0(\rho_{\Theta^0} + tw_\alpha) = \rho_{\Theta^0} - tw_\alpha$ if and only if $t = d_\alpha$.

We put

$$c_\alpha = \min\{c \in \mathbb{R} \mid 2c_\omega_\alpha \in \mathbb{Q}^+\}.$$

Clearly $2c_\alpha$ is a positive integer.

Lemma 2.2.3. If $w_{\Theta^0}w_0 = w_0w_{\Theta^0}$, then either $c_\alpha = 1$ or $c_\alpha = \frac{1}{2}$.

Proof. We have only to show $c_\alpha \leq 1$. We may assume $g$ is simple. If $g$ is a simple Lie algebra of the type other than $A_n$, $D_{2n+1}$, and $E_6$, then the exponent of $Q/P$ is 1 or 2. So, in this case $2\omega_\alpha \in \mathbb{Q}^+$ for any $\alpha \in \Pi$. For the case of the type $A_n$, $D_{2n+1}$, or $E_6$, we can check $2\omega_\alpha \in \mathbb{Q}^+$ under the assumption $w_{\Theta^0}w_0 = w_0w_{\Theta^0}$ by the case-by-case analysis. \qed

Corollary 2.2.4. If $w_{\Theta^0}w_0 = w_0w_{\Theta^0}$ and if $M_{\Theta^0}[-t] \subseteq M_{\Theta^0}[t]$ holds, then $t \in \frac{1}{2}\mathbb{N}$. Moreover, if $c_\alpha = 1$, then $M_{\Theta^0}[-t] \hookrightarrow M_{\Theta^0}[t]$ implies $t \in \mathbb{N}$.

Definition 2.2.5. If $t \in \frac{1}{2}\mathbb{N}$ and $\rho_{\Theta^0} + tw_\alpha$ is not integral, we say $\rho_{\Theta^0} + tw_\alpha$ is half-integral.

We examine behavior of the translation functors in the setting of this subsection.

First, Lemma 1.5.1 and the exactness of the translation functor imply:

Lemma 2.2.6. Assume $w_{\Theta^0}w_0 = w_0w_{\Theta^0}$. Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$ be such that $t - n \geq 0$. Then, $M_{\Theta^0}[-t] \subseteq M_{\Theta^0}[t]$ implies $M_{\Theta^0}[-t + n] \subseteq M_{\Theta^0}[t - n]$.

From the translation principle, we also have:

Lemma 2.2.7. Assume $w_{\Theta^0}w_0 = w_0w_{\Theta^0}$. Let $t \in \mathbb{R}$ be such that $\rho_{\Theta^0} + tw_\alpha$ is dominant regular. Let $n \in \mathbb{N}$. Then, $M_{\Theta^0}[-t] \subseteq M_{\Theta^0}[t]$ implies $M_{\Theta^0}[-t - n] \subseteq M_{\Theta^0}[t + n]$.

In case $\rho_{\Theta^0} + tw_\alpha$ is not dominant regular, the corresponding statement to Lemma 1.7.6 is not necessarily correct. In fact, we need an extra assumption.

Let $G$ be a complex connected reductive Lie group, whose Lie algebra is $g$. Let $P_{\Theta}$ be the parabolic subgroup of $G$ corresponding to $p_{\Theta}$. We consider the generalised flag variety $X_\Theta = G/P_{\Theta}$. Since the holomorphic cotangent bundle $T^*X_\Theta$ has a natural symplectic structure, we can construct the moment map $m_{\Theta} : T^*X_\Theta \to g^*$. Using $\langle , \rangle$, we identify $g$ and $g^*$. Then, we regard the moment map as a surjective map of $T^*X_\Theta$ to the closure of the Richardson orbit $O_{p_{\Theta}}$.

We easily see the number $N_1(P_{\Theta})$ defined in [Hesselink 1978] 1.4 Step3 is the degree of the moment map $m_{\Theta} : T^*X_\Theta \to \overline{O_{p_{\Theta}}}$, So, $N_1(P_{\Theta}) = 1$ if and only if $m_{\Theta}$ is birational. For classical Lie algebras, $N_1(P_{\Theta})$ is obtained in [Hesselink 1978] 7.1 Theorem.

Lemma 2.2.8. ([Matumoto 1993] Proposition 2.2.3) Assume $w_{\Theta^0}w_0 = w_0w_{\Theta^0}$ and the moment map $m_{\Theta} : T^*X_{\Theta^0} \to \overline{O_{p_{\Theta^0}}}$ is birational. Let $n \in \mathbb{N}$. Then, $M_{\Theta^0}[-t] \subseteq M_{\Theta^0}[t]$ implies $M_{\Theta^0}[-t - n] \subseteq M_{\Theta^0}[t + n]$. 

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Since $M_{\Theta^0}[0] \subseteq M_{\Theta^0}[1]$, we have

**Corollary 2.2.9. ([Matumoto 1993] Corollary 2.2.4)**

Assume $w_0 \Theta^0 w_0 = w_0 \Theta^0 w_0$ and the moment map $m_{\Theta^0} : T^* X_{\Theta^0} \to \mathcal{O}_{\Theta^0}$ is birational. Then, we have $M_{\Theta^0}[-n] \subseteq M_{\Theta^0}[n]$ for all $n \in \mathbb{N}$.

Next, we introduce Jantzen’s criterion for the irreducibility of a generalized Verma module. For any $\lambda \in \mathfrak{h}^*$, we define an element of $\mathbb{C}[\mathfrak{h}^*]$ as follows.

$$\Upsilon_{\Theta^0}(\lambda) = D^{-1} \sum_{w \in W_{\Theta^0}} (-1)^{\ell(w)} e^{w\lambda}.$$  

Of course, for $\lambda \in \mathcal{P}_{\Theta^0}^+$, we have $\Upsilon_{\Theta^0}(\lambda) = [M_{\Theta^0}(\lambda)]$.

Put $\Delta_{\Theta^0} = \mathbb{Z}_{\Theta^0} \cap \Delta$ and $(\Delta_{\Theta^0})^+ = \Delta^+ - \Delta_{\Theta^0}$. We immediately have:

**Corollary 2.2.10.**

(1) If $\lambda \in \mathfrak{h}^*$ satisfies $\langle \beta, \lambda \rangle = 0$ for some $\beta \in \Delta_{\Theta^0}$, then $\Upsilon_{\Theta^0}(\lambda) = 0$.

(2) For $\lambda \in \mathfrak{h}^*$ and $w \in W_{\Theta^0}$, we have $\Upsilon_{\Theta^0}(\lambda) - (-1)^{\ell(w)} \Upsilon_{\Theta^0}(w\lambda) = 0$.

The following result is a special case of [Jantzen 1977] Satz 3.

**Theorem 2.2.11. (Jantzen)**

For $\lambda \in \mathcal{P}_{\Theta^0}^+$, the following (1) and (2) are equivalent.

(1) $M_{\Theta^0}(\lambda)$ is irreducible.

(2) We have

$$\sum_{\beta \in (\Delta_{\Theta^0})^+ \cap \langle \lambda, \beta^\vee \rangle \in \mathbb{N} - \{0\}} \Upsilon_{\Theta^0}(s_\beta \lambda) = 0.$$

**Remark** The above statement is slightly different from that in Jantzen’s paper. However, we consider maximal parabolic subalgebras. In this case, we can easily see the above condition (2) is equivalent to the condition of Jantzen.

§ 3. Classical Lie algebras

Throughout this section, $n$ means a positive integer such that $n \geq 2$ (resp. $n \geq 4$) whenever we consider the simple Lie algebra of the type $B_n$ or $C_n$ (resp. $D_n$).

3.1 The root systems

We retain the notations in §1 and §2.

(B_n type) We consider the root system $\Delta$ for $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. Then we can choose an orthonormal basis $e_1, \ldots, e_n$ of $\mathfrak{h}^*$ such that

$$\Delta = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{ \pm e_i \mid 1 \leq i \leq n \}.$$

We choose a positive system as follows.

$$\Delta^+ = \{ e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{ e_i \mid 1 \leq i \leq n \}.$$
If we put \( \alpha_i = e_i - e_{i+1} \) (1 \( \leq \) \( i < n \)) and \( \alpha_n = e_n \), then \( \Pi = \{ \alpha_1, ..., \alpha_n \} \).

For simplicity, we write \( \Theta^k \) for \( \Theta^{\alpha_k} \) for 1 \( \leq \) \( k \leq n \). Then \( \mathfrak{t}_{\Theta^k} \) is isomorphic to \( \mathfrak{gl}(k, \mathbb{C}) \times \mathfrak{so}(2(n-k) + 1, \mathbb{C}) \). Since \( w_0 \) is contained in the center of \( W \), we have \( w_0 w_{\Theta^k} = w_{\Theta^k} w_0 \) for any 1 \( \leq \) \( k \leq n \).

We write \( \omega_k, c_k, \) and \( d_k \) for \( \omega_{\alpha_k}, c_{\alpha_k}, \) and \( d_{\alpha_k}, \) respectively. Then, for 1 \( \leq \) \( k < n \), we have \( \omega_k = e_1 + \cdots + e_k, \; c_k = \frac{k}{2}, \) and \( d_k = n - \frac{k}{2} \). If \( k = n \), then \( \omega_n = \frac{1}{2}(e_1 + \cdots + e_n), \; c_n = 1, \) and \( d_n = n \).

Assume \( k \) is even or \( k = n \). Then, \( \rho_{\Theta^k} + tw_k \) is integral if and only if \( t \in \mathbb{Z} \).

Assume \( k \) is odd and 1 \( \leq \) \( k < n \). Then, \( \rho_{\Theta^k} + tw_k \) is integral if and only if \( t - \frac{1}{2} \in \mathbb{Z} \).

(C_n type)

We consider the root system \( \Delta \) for \( \mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) \). Then we can choose an orthonormal basis \( e_1, ..., e_n \) of \( \mathfrak{h}^* \) such that

\[
\Delta = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{ \pm 2e_i \mid 1 \leq i \leq n \}.
\]

We choose a positive system as follows.

\[
\Delta^+ = \{ e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{ 2e_i \mid 1 \leq i \leq n \}.
\]

If we put \( \alpha_i = e_i - e_{i+1} \) (1 \( \leq \) \( i < n \)) and \( \alpha_n = 2e_n \), then \( \Pi = \{ \alpha_1, ..., \alpha_n \} \).

For simplicity, we write \( \Theta^k \) for \( \Theta^{\alpha_k} \) for 1 \( \leq \) \( k \leq n \). Then \( \mathfrak{t}_{\Theta^k} \) is isomorphic to \( \mathfrak{gl}(k, \mathbb{C}) \times \mathfrak{sp}(n-k, \mathbb{C}) \). Since \( w_0 \) is contained in the center of \( W \), we have \( w_0 w_{\Theta^k} = w_{\Theta^k} w_0 \) for any 1 \( \leq \) \( k \leq n \).

We write \( \omega_k, c_k, \) and \( d_k \) for \( \omega_{\alpha_k}, c_{\alpha_k}, \) and \( d_{\alpha_k}, \) respectively. Then, for 1 \( \leq \) \( k \leq n \), we have \( \omega_k = e_1 + \cdots + e_k \) and \( d_k = n - \frac{k-1}{2} \). If \( k \) is even (resp. odd), then \( c_k = \frac{k}{2} \) (resp. \( c_k = 1 \)).

Assume \( k \) is odd. Then, \( \rho_{\Theta^k} + tw_k \) is integral if and only if \( t \in \mathbb{Z} \).

Assume \( k \) is even. Then, \( \rho_{\Theta^k} + tw_k \) is integral if and only if \( t - \frac{1}{2} \in \mathbb{Z} \).

(D_n type)

We consider the root system \( \Delta \) for \( \mathfrak{g} = \mathfrak{so}(2n, \mathbb{C}) \). Then we can choose an orthonormal basis \( e_1, ..., e_n \) of \( \mathfrak{h}^* \) such that

\[
\Delta = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \}.
\]

We choose a positive system as follows.

\[
\Delta^+ = \{ e_i \pm e_j \mid 1 \leq i < j \leq n \}.
\]

If we put \( \alpha_i = e_i - e_{i+1} \) (1 \( \leq \) \( i < n \)) and \( \alpha_n = e_{n-1} + e_n \), then \( \Pi = \{ \alpha_1, ..., \alpha_n \} \).

For simplicity, we write \( \Theta^k \) for \( \Theta^{\alpha_k} \) for 1 \( \leq \) \( k \leq n \). Since the case of \( k = n - 1 \) is essentially same as the case of \( k = n \), since \( \mathfrak{p}_{\Theta^{n-1}} \) and \( \mathfrak{p}_{\Theta^n} \) are conjugate under an automorphism of \( \mathfrak{g} \). So, when we consider the type D case, we omit the case of \( k = n - 1 \).

Then, \( \mathfrak{t}_{\Theta^k} \) is isomorphic to \( \mathfrak{gl}(k, \mathbb{C}) \times \mathfrak{so}(2(n-k), \mathbb{C}) \). We have \( w_0 w_{\Theta^k} = w_{\Theta^k} w_0 \) for any 1 \( \leq \) \( k < n \). However, \( w_0 w_{\Theta^n} = w_{\Theta^n} w_0 \) if and only if \( n \) is even.

We write \( \omega_k, c_k, \) and \( d_k \) for \( \omega_{\alpha_k}, c_{\alpha_k}, \) and \( d_{\alpha_k}, \) respectively. Then, for 1 \( \leq \) \( k < n - 1 \), we have \( \omega_k = e_1 + \cdots + e_k \) and \( d_k = n - \frac{k+1}{2} \). If \( k \) is even (resp. odd) and \( k \neq n \), then \( c_k = \frac{k}{2} \) (resp.

For \( k = n \), we have \( \omega_n = \frac{1}{2}(e_1 + \cdots + e_n), \; c_n = 1, \) and \( d_n = n \). If \( n \) is even (resp. odd), then \( c_n = 1 \) (resp. \( c_n = 2 \)).

Assume \( k \) is odd or \( k = n \). Then, \( \rho_{\Theta^k} + tw_k \) is integral if and only if \( t \in \mathbb{Z} \).

Assume \( k \) is even and 1 \( \leq \) \( k < n \). Then, \( \rho_{\Theta^k} + tw_k \) is integral if and only if \( t - \frac{1}{2} \in \mathbb{Z} \).
3.2 Statements of the main results

Here, we describe the existence of homomorphisms between scalar generalized Verma modules with respect to the maximal parabolic subalgebras of classical Lie algebras. For simple Lie algebras of type A, the answer is given in [Boe 1985]. So, we treat the case of the types of B, C, and D.

**Theorem 3.2.1.** (\(B_n\)-type)

Consider the case of \(\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C})\). Let \(k\) be a positive integer such that \(k \leq n\).

1. We consider the case that \(3k < 2n + 1\). Then, \(M_{\mathfrak{sk}}[-t] \subseteq M_{\mathfrak{sk}}[t]\) if and only if \(t \in \mathbb{N}\).
2. We consider the case that \(3k \geq 2n + 1\).
   
   (2a) Assume \(k\) is odd and \(k \neq n\). Then, \(M_{\mathfrak{sk}}[-t] \subseteq M_{\mathfrak{sk}}[t]\) if and only if \(t \in \frac{1}{2}\mathbb{N}\).
   
   (2b) Assume \(n\) is odd. Then, \(M_{\mathfrak{sk}}[-t] \subseteq M_{\mathfrak{sk}}[t]\) if and only if \(t \in \mathbb{N}\).
   
   (2c) Assume \(k\) is even. Then, \(M_{\mathfrak{sk}}[-t] \subseteq M_{\mathfrak{sk}}[t]\) if and only if \(t = 0\).

**Remark** The case of \(k = 1\) is due to [Lepowsky 1975a]. The cases of \(k = 1, 2, 3, n\) are in the multiplicity free case in [Boe-Collingwood 1990].

**Theorem 3.2.2.** (\(C_n\)-type)

Consider the case of \(\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})\). Let \(k\) be a positive integer such that \(k \leq n\).

1. We consider the case that \(3k < 2n\).
   
   (1a) Assume \(k\) is odd. Then, \(M_{\mathfrak{sk}}[-t] \subseteq M_{\mathfrak{sk}}[t]\) if and only if \(t = 0\).
   
   (1b) Assume \(k\) is even. Then, \(M_{\mathfrak{sk}}[-t] \subseteq M_{\mathfrak{sk}}[t]\) if and only if \(t \in \frac{1}{2}\mathbb{N}\).
2. We consider the case that \(3k > 2n\). Then, \(M_{\mathfrak{sk}}[-t] \subseteq M_{\mathfrak{sk}}[t]\) if and only if \(t \in \mathbb{N}\).

**Remark** The case of \(k = 2\) is due to [Lepowsky 1975a]. The case of \(k = n\) is due to [Boe 1985]. The cases of \(k = 1, 2, 3, n\) are in the multiplicity free case in [Boe-Collingwood 1990].

**Theorem 3.2.3.** (\(D_n\)-type)

Consider the case of \(\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})\). Let \(k\) be a positive integer such that \(k \leq n - 2\) or \(k = n\).

1. We consider the case that \(3k < 2n\). Then, \(M_{\mathfrak{sk}}[-t] \subseteq M_{\mathfrak{sk}}[t]\) if and only if \(t \in \mathbb{N}\).
2. We consider the case that \(3k \geq 2n\).
   
   (2a) Assume \(k\) is odd. Then, \(M_{\mathfrak{sk}}[-t] \subseteq M_{\mathfrak{sk}}[t]\) if and only if \(t = 0\).
   
   (2b) Assume \(k\) is even and \(k \neq n\). Then, \(M_{\mathfrak{sk}}[-t] \subseteq M_{\mathfrak{sk}}[t]\) if and only if \(t \in \frac{1}{2}\mathbb{N}\).
   
   (2c) Assume \(n\) is even. Then, \(M_{\mathfrak{sk}}[-t] \subseteq M_{\mathfrak{sk}}[t]\) if and only if \(t \in \mathbb{N}\).

**Remark** The case of \(k = 1\) is due to [Lepowsky 1975a]. The case of \(k = n\) is due to [Boe 1985]. The cases of \(k = 1, 2, n\) are in the multiplicity free case in [Boe-Collingwood 1990].

We give proofs of the above theorems in the subsequent sections.

3.3 Richardson orbits

We consider a partition \(\pi = (p_1, \ldots, p_k)\) of a positive integer \(m\) such that \(0 < p_1 \leq p_2 \leq \cdots \leq p_k\) and \(p_1 + p_2 + \cdots + p_k = m\). We put \(\pi[i] = \text{card}\{j \mid p_j = i\}\) for any positive integer \(i\). Let \(\{h_1, \ldots, h_r\} = \{i \in \mathbb{N} \mid i > 0, \pi[i] > 0\}\). Then, we write \(h_1^{\pi[h_1]} \cdots h_r^{\pi[h_r]}\) for \(\pi\). For example, \(5 \cdot 3^2 \cdot 1^5\) means \((1, 1, 1, 1, 3, 3, 3, 5)\).

**Type B_n** The nilpotent orbits in \(\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C})\) are parametrized by partitions \(\pi\) of \(2n + 1\) such that, for any even number \(2r\), \(\pi[2r]\) is even. (For example see [Carter 1985] p394) From [Collingwood-McGovern 1993] 7 and [Hesselink 1978] 7.1, we have:
Lemma 3.3.1. Let $g = \mathfrak{so}(2n+1, \mathbb{C})$.

(1) If $3k < 2n+1$, then the Richardson orbit $O_{p_{\Theta}k}$ corresponds to the partition $3^k \cdot 1^{2n+1-3k}$. In this case, the moment map $m_{\Theta k}$ is birational.

(2) If $3k \geq 2n+1$ and $k$ is odd, then the Richardson orbit $O_{p_{\Theta}k}$ corresponds to the partition $3^{2n+1-2k} \cdot 2^{3k-2n-1}$. In this case, the moment map $m_{\Theta k}$ is birational.

(3) If $3k \geq 2n+1$ and $k$ is even, then the Richardson orbit $O_{p_{\Theta}k}$ corresponds to the partition $3^{2n+1-2k} \cdot 2^{3k-2n-2} \cdot 1^2$.

From Corollary 2.2.9, and Lemma 3.3.1, we have:

Lemma 3.3.2. Let $g = \mathfrak{so}(2n+1, \mathbb{C})$. Assume that $3k \leq 2n+1$ or $k$ is odd. Then, we have $M_{\Theta k}[-t] \subseteq M_{\Theta k}[t]$ for all $t \in \mathbb{N}$.

Type $C_n$ The nilpotent orbits in $g = \mathfrak{sp}(n, \mathbb{C})$ are parametrized by partitions $\pi$ of $2n$ such that, for any odd number $2i+1$, $\pi[2i+1]$ is even. (For example see [Carter 1985] p394) From [Collingwood-McGovern 1993] 7 and [Carter 1985] Chapter 13, we have:

Lemma 3.3.3. Let $g = \mathfrak{sp}(2n, \mathbb{C})$.

(1) If $3k \leq 2n$ and $k$ is even, then the Richardson orbit $O_{p_{\Theta}k}$ corresponds to the partition $3^k \cdot 1^{2n-3k}$. In this case the moment map $m_{\Theta k}$ is birational.

(2) If $3k \leq 2n$ and $k$ is odd, then the Richardson orbit $O_{p_{\Theta}k}$ corresponds to the partition $3^{k-1} \cdot 2^2 \cdot 1^{3k-3k-1}$.

(3) If $3k > 2n$, then the Richardson orbit $O_{p_{\Theta}k}$ corresponds to the partition $3^{2n-2k} \cdot 2^{3k-2n}$. In this case, the moment map $m_{\Theta k}$ is birational.

From Corollary 2.2.9, and Lemma 3.3.3, we have:

Lemma 3.3.4. Let $g = \mathfrak{sp}(2n, \mathbb{C})$. Assume that $3k > 2n$ or $k$ is even. Then, we have $M_{\Theta k}[-t] \subseteq M_{\Theta k}[t]$ for all $t \in \mathbb{N}$.

Type $D_n$ We can associate a nilpotent orbit in $g = \mathfrak{so}(2n, \mathbb{C})$ with a partition $\pi = (p_1, \ldots, p_k)$ of $2n$ such that, for any even number $2i$, $\pi[2i]$ is even. If there is at least one odd number in $p_1, \ldots, p_k$, there is a unique nilpotent orbit associated with $(p_1, \ldots, p_k)$. The exceptional orbits are so-called very even nilpotent orbits. If $p_1, \ldots, p_k$ are all even, there are two nilpotent orbits associated with the partition $(p_1, \ldots, p_k)$. (For example see [Carter 1985] p395) From [Collingwood-McGovern 1993] 7 and [Hesselink 1978] 7.1, we have:

Lemma 3.3.5. Let $g = \mathfrak{so}(2n, \mathbb{C})$ and let $1 \leq k \leq n-2$ or $k = n$.

(1) If $3k \leq 2n$, then the Richardson orbit $O_{p_{\Theta}k}$ corresponds to the partition $3^k \cdot 1^{2n-3k}$. In this case, the moment map $m_{\Theta k}$ is birational.

(2) If $3k > 2n$ and $k$ is odd, then the Richardson orbit $O_{p_{\Theta}k}$ corresponds to the partition $3^{2n-2k} \cdot 2^{3k-2n-1} \cdot 1^2$.

(3) If $3k > 2n$ and $k$ is even, then the Richardson orbit $O_{p_{\Theta}k}$ corresponds to the partition $3^{2n-2k} \cdot 2^{3k-2n}$. In this case, the moment map $m_{\Theta k}$ is birational.

From Corollary 2.2.9, and Lemma 3.3.5, we have:

Lemma 3.3.6. Let $g = \mathfrak{so}(2n, \mathbb{C})$. Assume that $3k \leq 2n$ or $k$ is even. Then, we have $M_{\Theta k}[-t] \subseteq M_{\Theta k}[t]$ for all $t \in \mathbb{N}$.
So, from Lemma 2.2.6 and Lemma 2.2.7, we have

In this subsection, we prove the following result.

3.4 Existence results via comparison

Proof. If \( k \) and \( w = g \) then we show:

In this subsection, we treat the remaining case that nontrivial homomorphisms exist. Namely, \( \lambda_k = 3 \) can associate an irreducible representation of the Weyl group of the type \( \theta_k \). As in [Carter 1985] 13.3, we can associate the partition \( 3^{2n-2k} \cdot 2^{3k-2n} \) to a pair of partitions as follows.

This is the symbol associated with the special representation (say \( \pi_k \)) corresponding to the Richardson orbit \( O_{\rho_k} \) via the Springer correspondence. (Lusztig, Shoji)

As in [Carter 1985] 13.2, two irreducible representation with symbols:

\[
\begin{pmatrix}
\lambda_1, \ldots, \lambda_k \\
\mu_1, \ldots, \mu_k
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\mu_1, \ldots, \mu_k \\
\lambda_1, \ldots, \lambda_k
\end{pmatrix}
\]

3.5 Irreducibility of a right cell

In this subsection, we treat the remaining case that nontrivial homomorphisms exist. Namely, we show:

Lemma 3.5.1. Assume that \( g = \mathfrak{sp}(n, \mathbb{C}) \) and that \( k \) is an even positive integer such that \( 3k \leq 2n \). Then, we have \( M_{\rho_k}[-t] \subseteq M_{\rho_k}[t] \) for all \( t \in \frac{1}{2} + \mathbb{N} \).

Proof. We prove (1). We can prove (2) in a similar way.

Let \( k \) be an even positive integer such that \( 3k \leq 2n \). At first, we consider the case that \( g = \mathfrak{so}(2n+1, \mathbb{C}) \). In this case \( \rho \Theta_k^g = d_k \omega_k = (n - \frac{k}{2}) \omega_k \) from 3.1. Therefore, \( \rho = \rho \Theta_k + (n - \frac{k}{2}) \omega_k \) and \( w_0 \rho = \rho \Theta_k - (n - \frac{k}{2}) \omega_k \). Since \( n - \frac{k}{2} \in \mathbb{N} \), Lemma 3.3.2 implies \( M_{\rho_k}(w_0 \rho) \subseteq M_{\rho_k}(\rho) \).

From Theorem 1.4.2, \( w_0 \rho \) is a Duflo involution in the Weyl group for \( (g, h) \), where \( g = \mathfrak{so}(2n+1, \mathbb{C}) \). However, the Weyl group of the \( B_n \)-type and that of the \( C_n \)-type are isomorphic to each other as a Coxeter system. Since the notion of the Duflo involutions only depends on the structure of the Coxeter system, we see that as an element of the Weyl group for \( \mathfrak{sp}(n, \mathbb{C}) \), \( w_0 \rho \) is a Duflo involution. From Theorem 1.4.2 implies \( M_{\rho_k}(w_0 \rho) \subseteq M_{\rho_k}(\rho) \) holds for the case that \( g = \mathfrak{sp}(n, \mathbb{C}) \). Namely, \( M_{\rho_k}[-d_k] \subseteq M_{\rho_k}[d_k] \). In this case, \( d_k = n - \frac{n-1}{2} \in \frac{1}{2} + \mathbb{N} \). So, from Lemma 2.2.6 and Lemma 2.2.7, we have \( M_{\rho_k}[-t] \subseteq M_{\rho_k}[t] \) for all \( t \in \frac{1}{2} + \mathbb{N} \). Q.E.D.
are contained in the same family if and only if \( \{ \lambda_1, ..., \lambda_k, \mu_1, ..., \mu_k \} = \{ \lambda'_1, ..., \lambda'_k, \mu'_1, ..., \mu'_k \} \). So, we easily see the family of \( \pi_k \) consists of only one element. From Corollary 2.1.4, we have \( M_{\mathfrak{g}}(w_{\mathfrak{g}}) \subseteq M_{\mathfrak{h}}(\rho) \). In this case, \( d_k = n - \frac{k+1}{2} \in \frac{1}{2} + \mathbb{N} \). So, from Lemma 2.2.6 and Lemma 2.2.7, we have \( M_{\mathfrak{g}}[-t] \subseteq M_{\mathfrak{h}}[t] \) for all \( t \in \frac{1}{2} + \mathbb{N} \). Q.E.D.

Remark We can prove Lemma 3.4.1 in a similar way to Lemma 3.4.2.

3.6 Irreducibility of some generalized Verma modules

We have:

**Lemma 3.6.1.**

1. Assume that \( \mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C}) \) and that \( k \) is an even positive integer such that \( k < n \). Then, \( M_{\mathfrak{g}}(\frac{k}{2}) \) is irreducible.
2. Assume that \( \mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C}) \) and that \( n \) is even. Then, \( M_{\mathfrak{g}}(1) \) is irreducible.
3. Assume that \( \mathfrak{g} = \mathfrak{so}(2n, \mathbb{C}) \) and that \( k \) is an even positive integer such that \( 3k < 2n \). Then, \( M_{\mathfrak{g}}(\frac{k}{2}) \) is irreducible.
4. Assume that \( \mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C}) \) and that \( k \) is an odd positive integer such that \( 3k < 2n+1 \). Then, \( M_{\mathfrak{g}}(\frac{k}{2}) \) is irreducible.
5. Assume that \( \mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) \) and that \( k \) is an even positive integer such that \( 3k > 2n \). Then, \( M_{\mathfrak{g}}(\frac{k}{2}) \) is irreducible.
6. Assume that \( \mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) \). Then, \( M_{\mathfrak{g}}(1) \) is irreducible.

**Proof.** For \( 2 \leq r \leq k \), we put \( r^* = k + 2 - r \).

First, we prove (1). We put \( s = \frac{k}{2} \). Then, we see

\[
\rho_{\Theta^k} + \frac{1}{2} \omega_k = \sum_{i=1}^{2s} (s - i + 1) e_i + \sum_{j=2s+1}^{n} (n - j + \frac{1}{2}) e_j.
\]

We easily see

\[
\left\{ \beta \in (\Delta^{\Theta^k})^+ \mid \langle \rho_{\Theta^k} + \frac{1}{2} \omega_k, \beta \rangle \in \mathbb{N} \setminus \{0\} \right\} = \left\{ e_i \mid 1 \leq i \leq s \right\} \cup \left\{ e_i + e_j \mid 1 \leq i < j \leq 2s, 2s + 2 > i + j \right\}.
\]

For \( 2 \leq i \leq s \), \( \langle e_i - e_{2s-i+2}, s_{e_i}(\rho_{\Theta^k} + \frac{1}{2} \omega_k) \rangle = 0 \). Hence, Corollary 2.2.10 implies \( \Upsilon_{\Theta^k}(s_{e_i}(\rho_{\Theta^k} + \frac{1}{2} \omega_k)) = 0 \).

Next, we assume that \( 1 \leq i < j \leq 2s \) and that \( 2s + 2 > i + j \). Since \( i < j \), \( 2s + 2 > i + j \) implies \( i \leq s \). If \( 2 \leq i \), then we see \( j \neq i^* \) and \( \langle e_j - e_{i^*}, s_{e_i+e_j}(\rho_{\Theta^k} + \frac{1}{2} \omega_k) \rangle = 0 \). So, Corollary 2.2.10 implies \( \Upsilon_{\Theta^k}(s_{e_i+e_j}(\rho_{\Theta^k} + \frac{1}{2} \omega_k)) = 0 \). If \( i = 1 \) and \( j \neq s + 1 \), then \( j \neq j^* \neq 1 \) and \( \langle e_1 - e_{j^*}, s_{e_i+e_j}(\rho_{\Theta^k} + \frac{1}{2} \omega_k) \rangle = 0 \). So, Corollary 2.2.10 implies \( \Upsilon_{\Theta^k}(s_{e_i+e_j}(\rho_{\Theta^k} + \frac{1}{2} \omega_k)) = 0 \).

Hence, we have

\[
\sum_{\beta \in (\Delta^{\Theta^k})^+} \Upsilon_{\Theta^k}(s_{\beta}(\rho_{\Theta^k} + \frac{1}{2} \omega_k)) = \Upsilon_{\Theta^k}(s_{e_1}(\rho_{\Theta^k} + \frac{1}{2} \omega_k)) + \Upsilon_{\Theta^k}(s_{e_{s+1}}(\rho_{\Theta^k} + \frac{1}{2} \omega_k)).
\]
Moreover, we assume \( \beta \) is neither \( e_1 + e_{s+1} \) nor \( e_1 + e_{n-s} \). Then, we see there exists some \( \gamma \in \Delta_{\Theta^{k}} \) such that \( \langle s_\beta (\rho_{\Theta^{k}} + \frac{1}{2} \omega_k), \gamma^\vee \rangle = 0 \). From Corollary 2.2.10, we have
\[
\sum_{\beta \in (\Delta_{\Theta^{k}})^+ \atop \langle \rho_{\Theta^{k}} + \frac{1}{2} \omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\}} \Upsilon_{\Theta^{k}} (s_\beta (\rho_{\Theta^{k}} + \frac{1}{2} \omega_k)) = 0.
\]

From Jantzen’s criterion (Theorem 2.2.11), \( M_{\Theta^{k}}[\frac{1}{2}] \) is irreducible. So, we get (1).

(2) is proved in the same way as (1).

Next, we prove (3).

We put \( s = \frac{k}{2} \). Then, we see
\[
\rho_{\Theta^{k}} + \frac{1}{2} \omega_k = \sum_{i=1}^{2s} (s - i + 1) e_i + \sum_{j=2s+1}^{n} (n - j) e_j.
\]

We easily see
\[
\left\{ \beta \in (\Delta_{\Theta^{k}})^+ \atop \langle \rho_{\Theta^{k}} + \frac{1}{2} \omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\}
= \{ e_i + e_j \mid 1 \leq i < j \leq 2s, 2s + 2 > i + j \}
\quad \cup \{ e_i \pm e_j \mid 1 \leq i \leq 2s < j \leq n, (s - i + 1) \pm (n - j) > 0 \}.
\]

Since \( 3k < 2n \), we have \( 2s < n - s \). Let \( \beta \in (\Delta_{\Theta^{k}})^+ \) be such that \( \langle \rho_{\Theta^{k}} + \frac{1}{2} \omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\} \). Moreover, we assume \( \beta \) is neither \( e_1 + e_{s+1} \) nor \( e_1 + e_{n-s} \). Then, we can easily see there exists some \( \gamma \in \Delta_{\Theta^{k}} \) such that \( \langle s_\beta (\rho_{\Theta^{k}} + \frac{1}{2} \omega_k), \gamma^\vee \rangle = 0 \). From Corollary 2.2.10, we have
\[
\sum_{\beta \in (\Delta_{\Theta^{k}})^+ \atop \langle \rho_{\Theta^{k}} + \frac{1}{2} \omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\}} \Upsilon_{\Theta^{k}} (s_\beta (\rho_{\Theta^{k}} + \frac{1}{2} \omega_k)) = \Upsilon_{\Theta^{k}} (s_{e_1 + e_{s+1}} (\rho_{\Theta^{k}} + \frac{1}{2} \omega_k)) + \Upsilon_{\Theta^{k}} (s_{e_1 + e_{n-s}} (\rho_{\Theta^{k}} + \frac{1}{2} \omega_k)).
\]

However, \( s_{e_1 + e_{s+1}} (\rho_{\Theta^{k}} + \frac{1}{2} \omega_k) = s_{e_1 - e_{s+1}} s_{e_{n-s} - e_n} s_{e_{n-s} + e_n} s_{e_1 + e_{n-s}} (\rho_{\Theta^{k}} + \frac{1}{2} \omega_k) \) holds. So, Corollary 2.2.10 implies
\[
\sum_{\beta \in (\Delta_{\Theta^{k}})^+ \atop \langle \rho_{\Theta^{k}} + \frac{1}{2} \omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\}} \Upsilon_{\Theta^{k}} (s_\beta (\rho_{\Theta^{k}} + \frac{1}{2} \omega_k)) = 0.
\]

From Jantzen’s criterion (Theorem 2.2.11), \( M_{\Theta^{k}}[\frac{1}{2}] \) is irreducible. So, we get (3).

Next, we prove (4). We put \( s = \frac{k-1}{2} \). Then, we see
\[
\rho_{\Theta^{k}} + \frac{1}{2} \omega_k = \sum_{i=1}^{2s+1} \left( s - \frac{i + \frac{3}{2}}{2} \right) e_i + \sum_{j=2s+2}^{n} \left( n - j + \frac{1}{2} \right) e_j.
\]
We easily see
\[
\left\{ \beta \in \left( \Delta_{\Theta k} \right)^{+} \left| \langle \rho_{\Theta k} + \frac{1}{2} \omega_k, \beta \rangle \in \mathbb{N} - \{0\} \right. \right\} \\
= \{ e_i + e_j \mid 1 \leq i < j \leq 2s + 1, 2s + 3 > i + j \} \\
\cup \left\{ e_i \pm e_j \mid 1 \leq i \leq 2s < j \leq n, \left( s - i + \frac{3}{2} \right) \pm \left( n - j + \frac{1}{2} \right) > 0 \right\}.
\]

Since $3k < 2n + 1$, we have $s + \frac{1}{2} \leq n - 2s - \frac{3}{2}$. Let $\beta \in \left( \Delta_{\Theta k} \right)^{+}$ be such that $\langle \rho_{\Theta k} + \frac{1}{2} \omega_k, \beta \rangle \in \mathbb{N} - \{0\}$. Moreover, we assume $\beta$ is neither $e_1$ nor $e_{n-s}$. Then, we can easily see there exists some $\gamma \in \Delta_{\Theta k}$ such that $\langle s_\beta (\rho_{\Theta k} + \frac{1}{2} \omega_k), \gamma \rangle = 0$. From Corollary 2.2.10, we have
\[
\sum_{\beta \in \left( \Delta_{\Theta k} \right)^{+}} \left( \langle \rho_{\Theta k} + \frac{1}{2} \omega_k, \beta \rangle \right) \sum_{\langle \rho_{\Theta k} + \frac{1}{2} \omega_k, \beta \rangle \in \mathbb{N} - \{0\}} \Upsilon_{\Theta k}(s_\beta (\rho_{\Theta k} + \frac{1}{2} \omega_k)) = \Upsilon_{\Theta k}(s_{e_1} (\rho_{\Theta k} + \frac{1}{2} \omega_k)) + \Upsilon_{\Theta k}(s_{e_1 + e_{n-s}} (\rho_{\Theta k} + \frac{1}{2} \omega_k)).
\]

However, $s_{e_1} (\rho_{\Theta k} + \frac{1}{2} \omega_k) = s_{e_{n-s}} s_{e_1 + e_{n-s}} (\rho_{\Theta k} + \frac{1}{2} \omega_k)$ holds. So, Corollary 2.2.10 implies
\[
\sum_{\beta \in \left( \Delta_{\Theta k} \right)^{+}} \left( \langle \rho_{\Theta k} + \frac{1}{2} \omega_k, \beta \rangle \right) \sum_{\langle \rho_{\Theta k} + \frac{1}{2} \omega_k, \beta \rangle \in \mathbb{N} - \{0\}} \Upsilon_{\Theta k}(s_\beta (\rho_{\Theta k} + \frac{1}{2} \omega_k)) = 0.
\]

From Jantzen’s criterion (Theorem 2.2.11), $M_{\Theta k}^{\frac{1}{2}}$ is irreducible. So, we get (4).

(5) and (6) is due to [Gyoja 1994] p394. They are proved similarly. Q.E.D.

Lemma 3.6.1 and Lemma 2.2.6 imply:

**Lemma 3.6.2.** (1) Assume that $\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C})$ and that $k$ is an even positive integer such that $k < n$. Then, $M_{\Theta k}^{\frac{1}{2}} \nsubseteq M_{\Theta k}^{t + \frac{1}{2}}$ for all $t \in \mathbb{N}$.

(2) Assume that $\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C})$ and that $n$ is even. Then, $M_{\Theta k}^{\frac{1}{2}} \nsubseteq M_{\Theta k}^{t}$ for all $t \in \mathbb{N}$.

(3) Assume that $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and that $k$ is an even positive integer such that $3k < 2n$. Then, $M_{\Theta k}^{\frac{1}{2}} \nsubseteq M_{\Theta k}^{t + \frac{1}{2}}$ for all $t \in \mathbb{N}$.

(4) Assume that $\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C})$ and that $k$ is an odd positive integer such that $3k < 2n + 1$. Then, $M_{\Theta k}^{\frac{1}{2}} \nsubseteq M_{\Theta k}^{t + \frac{1}{2}}$ for all $t \in \mathbb{N}$.

(5) Assume that $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ and that $k$ is an even positive integer such that $3k > 2n$. Then, $M_{\Theta k}^{\frac{1}{2}} \nsubseteq M_{\Theta k}^{t + \frac{1}{2}}$ for all $t \in \mathbb{N}$.

(6) Assume that $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. Then, $M_{\Theta k}^{\frac{1}{2}} \nsubseteq M_{\Theta k}^{t}$ for all $t \in \mathbb{N}$.

### 3.7 Nonexistence results for the remaining cases

First, we assume that $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, $n$ is odd, and $k = n$. This case is treated in [Boe 1985]. In fact, his result contains:

**Lemma 3.7.1.** (Boe) Assume $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and $n$ is odd. Then, $M_{\Theta k}^{\frac{1}{2}} \nsubseteq M_{\Theta k}^{t}$ if and only if $t = 0$. 

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Hereafter we do not consider the above case. Therefore, from 2.1, we have $w_{\mathfrak{g}k}w_0 = w_0w_{\mathfrak{g}k}$.

The results in 3.3-3.6 and Corollary 2.2.4 imply that Theorem 3.2.1-3.2.3 is reduced to the following lemma.

**Lemma 3.7.2.**

1. Assume that $g = \mathfrak{so}(2n + 1, \mathbb{C})$ and that $k$ is an even positive integer such that $3k > 2n + 1$ and $k < n$. Then, $M_{\mathfrak{g}k}[-t - 1] \nsubseteq M_{\mathfrak{g}k}[t + 1]$ for all $t \in \mathbb{N}$.

2. Assume that $g = \mathfrak{sp}(n, \mathbb{C})$ and that $k$ is an odd positive integer such that $3k < 2n$ and $1 < k$. Then, $M_{\mathfrak{g}k}[-t - 1] \nsubseteq M_{\mathfrak{g}k}[t + 1]$ for all $t \in \mathbb{N}$.

3. Assume that $g = \mathfrak{so}(2n, \mathbb{C})$ and that $k$ is an odd positive integer such that $3k \geq 2n$. Then, $M_{\mathfrak{g}k}[-t - 1] \nsubseteq M_{\mathfrak{g}k}[t + 1]$ for all $t \in \mathbb{N}$.

**Proof.** From Lemma 2.2.6, we have only to show that $M_{\mathfrak{g}k}[-1] \nsubseteq M_{\mathfrak{g}k}[1]$.

We put $\Omega = W \cdot (\rho_{\mathfrak{g}k} + \omega_k) \cap \mathbb{P}_{\mathfrak{g}k}^{++}$. Then, in the settings of (1), (2), and (3) above, $\Omega$ consists of four elements (say $\lambda_1, \ldots, \lambda_4$). We can write $\lambda_1 = \rho_{\mathfrak{g}k} + \omega_k$ and $\lambda_4 = \rho_{\mathfrak{g}k} - \omega_k$. The remaining two elements are as follows.

If $g = \mathfrak{so}(2n + 1, \mathbb{C})$ and if $k = 2s$ is an even positive integer such that $3k > 2n + 1$ and $k < n$, then we may write:

$$
\lambda_2 = \left( s + \frac{1}{2} \right) e_1 + \sum_{i=2}^{2s} \left( s + \frac{1}{2} - i \right) e_i + \sum_{j=2s+1}^{n} \left( n + \frac{1}{2} - j \right) e_j,
$$

$$
\lambda_3 = \sum_{i=1}^{2s-1} \left( s + \frac{1}{2} - i \right) e_i + \sum_{j=2s+1} \left( n + \frac{1}{2} - j \right) e_j.
$$

If $g = \mathfrak{sp}(n, \mathbb{C})$ and if $k = 2s + 1$ is an odd positive integer such that $3k < 2n$ and $1 < k$, then we may write:

$$
\lambda_2 = (s + 1) e_1 + \sum_{i=2}^{2s+1} (s + 1 - i) e_i + \sum_{j=2s+2}^{n} (n + 1 - j) e_j,
$$

$$
\lambda_3 = \sum_{i=1}^{2s} (s + 1 - i) e_i + \sum_{j=2s+2} (n + 1 - j) e_j.
$$

If $g = \mathfrak{so}(2n, \mathbb{C})$ and if $k = 2s + 1$ is an odd positive integer such that $3k \geq 2n + 1$ and $k < n$, then we may write:

$$
\lambda_2 = (s + 1) e_1 + \sum_{i=2}^{2s+1} (s + 1 - i) e_i + \sum_{j=2s+2}^{n} (n - j) e_j,
$$

$$
\lambda_3 = \sum_{i=1}^{2s} (s + 1 - i) e_i + \sum_{j=2s+2} (n - j) e_j.
$$

**Claim 1** \hspace{1em} $M_{\mathfrak{g}k}(\lambda_3)$ is reducible.

**Proof of Claim 1** \hspace{1em} We apply Jantzen’s Criterion of irreducibility (Theorem 2.2.11).

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If $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and if $k = 2s$ is an even positive integer such that $3k > 2n + 1$ and $k < n$, then we have

$$\sum_{\beta \in (\Delta^{\Theta k})^+} \Upsilon_{\Theta k}(s_{\beta} \lambda_3) = \Upsilon_{\Theta k}(s_{e_1} \lambda_3) = -[M_{\Theta k}(\lambda_4)] \neq 0.$$ 

If $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ and if $k = 2s + 1$ is an odd positive integer such that $3k \leq 2n$ and $1 < k$, then we may have

$$\sum_{\beta \in (\Delta^{\Theta k})^+} \Upsilon_{\Theta k}(s_{\beta} \lambda_3) = \Upsilon_{\Theta k}(s_{e_1} \lambda_3) + \Upsilon_{\Theta k}(s_{e_1 + e_{s+1}} \lambda_3) + \Upsilon_{\Theta k}(s_{e_1 + e_{n-s+1}} \lambda_3)$$

$$= [M_{\Theta k}(\lambda_4)] - [M_{\Theta k}(\lambda_4)] - [M_{\Theta k}(\lambda_4)] = -[M_{\Theta k}(\lambda_4)] \neq 0.$$ 

If $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and if $k = 2s + 1$ is an odd positive integer such that $3k \geq 2n + 1$ and $k < n$, then we have

$$\sum_{\beta \in (\Delta^{\Theta k})^+} \Upsilon_{\Theta k}(s_{\beta} \lambda_3) = \Upsilon_{\Theta k}(s_{e_1 + e_{n-s+1}} \lambda_3) = -[M_{\Theta k}(\lambda_4)] \neq 0.$$ 

Therefore we have Claim 1. \qed

**Claim 2** There is a non-trivial $U(\mathfrak{g})$-homomorphism $\varphi : M_{\Theta k}(\lambda_2) \to M_{\Theta k}(\lambda_1)$.

**Proof of Claim 2** We remark that neither $\lambda_3 - \lambda_1$ nor $\lambda_3 - \lambda_2$ are contained in $\mathbb{Q}^+$. So, Claim 1 implies that $M_{\Theta k}(\lambda_4) \subseteq M_{\Theta k}(\lambda_3)$. On the other hand, Zuckerman’s duality theorem ([Boe-Collingwood 1985] 4.9, [Collingwood-Shelton 1990] Theorem 1.1, [Gyoja 2000]) implies

$$\dim \text{Hom}_{U(\mathfrak{g})}(M_{\Theta k}(\lambda_4), M_{\Theta k}(\lambda_3)) = \dim \text{Hom}_{U(\mathfrak{g})}(M_{\Theta k}(\lambda_2), M_{\Theta k}(\lambda_1)).$$

Hence there is a non-trivial $\varphi \in \text{Hom}_{U(\mathfrak{g})}(M_{\Theta k}(\lambda_2), (M_{\Theta k}(\lambda_1))$. \qed

We need:

**Claim 3** $\dim(L(\lambda_2)) = \dim \mathfrak{n}_{\Theta k}$.

Since the multiplicity (the Bernstein degree) of any scalar generalized Verma module is one, Claim 2 and Claim 3 imply $L(\lambda_2)$ is a unique irreducible constituent of $M_{\Theta k}(\lambda_1) = M_{\Theta k}[1]$ whose Gelfand-Kirillov dimension is $\dim \mathfrak{n}_{\Theta k}$. On the other hand, $\dim(M_{\Theta k}[-1]) = \dim \mathfrak{n}_{\Theta k}$ and $M_{\Theta k}[-1]$ is irreducible. Hence, $M_{\Theta k}[-1] \not\subseteq M_{\Theta k}[1]$ and we have Lemma 3.7.2. So, we have only to show Claim 3.

**Proof of Claim 3**

Let $\sigma \in W$ be the longest element (with respect to the length $\ell(\cdot)$) in

$$\{ w \in W \mid \lambda_2 \text{ is dominant with respect to } w \Delta^+ \}.$$ 

Then, $\sigma \rho$ and $\lambda_2$ are contained in the same closed Weyl chamber and we can regard $L(\lambda_2)$ as a limit of $L(\sigma \rho)$. Namely, $T_{\sigma \rho}^L(L(\sigma \rho)) \cong L(\lambda_2)$. Since $\dim(L(\sigma \rho)) = \dim(\lambda_2)$, Claim 3 is reduced to the following Claim 4. (Cf. Lemma 2.1.6.)
Claim 4 \( \sigma^{LR} w_{\Theta^k} w_0. \)

Proof of Claim 4 Claim 4 is obtained by the algorithm described in [Barbasch-Vogan 1982] p171-175.

First, we assume \( \mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) \) and that \( k \) is an odd positive integer such that \( 3k \leq 2n \) and \( 1 < k. \)

We put \( \delta = 10^{-23}. \) (In fact, \( \delta \) can be any real number such that \( 0 < \delta < \frac{1}{2} ). \) Put \( \lambda'_2 = \lambda_2 - \delta \omega_n \). We also put \( a_i = (\epsilon_i, \lambda'_2) \) for \( 1 \leq i \leq n. \) Namely, \( \lambda'_2 = \sum_{i=1}^{n} a_i \epsilon_i. \) Then, we easily see \( |a_1|, \ldots, |a_n| \) are distinct. Let \( b_1, \ldots, b_n \) be positive real numbers such that \( b_1 > b_2 > \cdots > b_n \) and \( \{b_1, \ldots, b_n\} = \{|a_1|, \ldots, |a_n|\}. \) So, there is a permutation \( \tau \in \mathfrak{S}_n \) such that \( |a_i| = b_\tau(i) \) for \( 1 \leq i \leq n. \) For \( 1 \leq i \leq n, \) we put \( c_i = \frac{a_i}{|a_i|} (n - \tau(i) + 1). \) Then, we have \( \sigma \rho = \sum_{i=1}^{n} c_i e_n. \)

Barbasch and Vogan attach \( \sigma \) to the sequence \( (c_1, \ldots, c_n, -c_n, \ldots, -c_1) \) ([Barbasch-Vogan 1982] p173). Applying the Robinson-Schensted algorithm to this sequence, we get a pair of Young tableaux. (These Young tableaux have the same shape.) Remark that the Robinson-Schensted algorithm in [Barbasch-Vogan 1982] is a little bit different from the usual one (see [Barbasch-Vogan 1982] p171). For our purpose, the important information is the shape of these Young tableaux. In order to obtain such a Young diagram, we need not compute \( c_1, \ldots, c_n. \) In fact, applying the Robinson-Schensted algorithm to the sequence \( (\epsilon_1, \ldots, \epsilon_n, -\epsilon_n, \ldots, -\epsilon_1) \) directly, we have the same Young diagram. In this case, the Young diagram corresponds to the partition of \( 2n, (2n - 2k, k + 1, k - 1). \) It corresponds to the symbol \( \binom{n - k - 1}{\frac{k}{2}} \binom{k + 1}{\frac{k}{2}} \).

On the other hand, \( w_{\Theta^k} w_0 \) corresponds to the partition of \( 2n, (2n - 2k, k, k). \) It corresponds to the symbol \( \binom{n - k}{\frac{k}{2}} \binom{k + 1}{\frac{k}{2}} \). Hence, from [Barbasch-Vogan 1982] Theorem 18, we have \( \sigma^{LR} w_{\Theta^k} w_0. \)

Next, we consider the case of \( \mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C}) \). We define \( \lambda'_2, a_i \) \((1 \leq i \leq n)\), \( b_i \) \((1 \leq i \leq n)\), \( \tau \in \mathfrak{S}_n \), and \( c_i \) \((1 \leq i \leq n)\) in the same way as the case of \( \mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) \). In the Barbasch-Vogan algorithm, \( W \) is regarded as the Weyl group of the type \( C_n \) rather than \( B_n \). So, \( \sigma \) is attached to \( (c_1, \ldots, c_n, -c_n, \ldots, -c_1) \). Again, we may apply the Robinson-Schensted algorithm to the sequence \( (\epsilon_1, \ldots, \epsilon_n, -\epsilon_n, \ldots, -\epsilon_1) \) directly and obtain a partition \( (k + 1, k - 1, 2n - 2k) \) of \( 2n. \) It corresponds to a symbol \( \binom{n - k}{\frac{k}{2}} \binom{k + 1}{\frac{k}{2}} \). On the other hand, \( w_{\Theta^k} w_0 \) corresponds to the partition of \( 2n, (k, k, 2n - 2k) \). It corresponds to a symbol \( \binom{n - k}{\frac{k}{2}} \binom{k + 1}{\frac{k}{2}} \). Hence, from [Barbasch-Vogan 1982] Theorem 18, we have \( \sigma^{LR} w_{\Theta^k} w_0. \)

Finally, we consider the case of \( \mathfrak{g} = \mathfrak{so}(2n, \mathbb{C}) \). In this case, we put \( \lambda'_2 = \lambda_2 - \delta \omega_n + \frac{\delta}{2} e_n. \) Here, \( \delta \) is a fixed real number such that \( 0 < \delta < \frac{1}{2} \). We define \( a_i \) \((1 \leq i \leq n)\), \( b_i \) \((1 \leq i \leq n)\), \( \tau \in \mathfrak{S}_n \), and \( c_i \) \((1 \leq i \leq n)\) in the same way as the case of \( \mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) \). \( \sigma \) is attached to \( (c_1, \ldots, c_n, -c_n, \ldots, -c_1) \). Again, we may apply the Robinson-Schensted algorithm to the sequence \( (\epsilon_1, \ldots, \epsilon_n, -\epsilon_n, \ldots, -\epsilon_1) \) directly and obtain a partition \( (k + 1, k - 1, 2n - 2k) \) of \( 2n. \) It corresponds to a symbol \( \binom{n - k}{\frac{k}{2}} \binom{k + 1}{\frac{k}{2}} \). On the other hand, \( w_{\Theta^k} w_0 \) corresponds to the partition of \( 2n, (k, k, 2n - 2k) \). It corresponds to a symbol \( \binom{n - k}{\frac{k}{2}} \binom{k + 1}{\frac{k}{2}} \). Hence, from [Barbasch-Vogan 1982] Theorem 18, we have \( \sigma^{LR} w_{\Theta^k} w_0. \) Q.E.D.
§ 4. Exceptional algebras

As in §3, we write $\Theta^k = \Pi - \{\alpha_k\}$, $\omega_k = \omega_{\alpha_k}$, $c_k = c_{\alpha_k}$, $d_k = d_{\alpha_k}$, etc.

4.1 Even parabolic subalgebras

Let $u \in \mathfrak{g}$ be a nilpotent element. From the Jacobson-Morozov theorem, there is an $\mathfrak{sl}_2$-triple $(v, h, u)$ in $\mathfrak{g}$. Namely, we have $[u, v] = h$, $[h, u] = 2u$, and $[h, v] = -2v$. It is well known that any eigenvalue of the operator $\text{ad}(h) \in \text{End}(\mathfrak{g})$ is an integer. $u$ is called an even nilpotent element if each eigenvalue of $\text{ad}(h)$ is an even number. Put $p_u = \sum_{i \geq 0} \{X \in \mathfrak{g} \mid [h, X] = iX\}$.

**Definition 4.1.1.** A parabolic subalgebra $\mathfrak{p}$ is called even, if there exists an even nilpotent element $u$ such that $\mathfrak{p} = p_u$.

For an even parabolic subalgebra $p_u$, the Richardson orbit $O_{p_u}$ contains $u$.

The following result is well-known. For example, it is an easy consequence of [Hesselink 1978] p218 and [Yamashita 1986] Lemma 3.5.

**Lemma 4.1.2.** Let $\mathfrak{p}_\Theta$ be an even parabolic subalgebra of $\mathfrak{g}$. Then, the moment map $m_\Theta : T^*X_\Theta \rightarrow \overline{O_{p_\Theta}}$ is birational.

4.2 $G_2$

In the regular integral case, homomorphisms between (not necessarily scalar) generalized Verma modules are classified for $G_2$ in [Boe-Collingwood 1990].

I imagine in the case of $G_2$ the classification of the homomorphisms between scalar generalized Verma modules is well-known, but I would like to state the result for the completeness.

Let $\mathfrak{g}$ be a simple Lie algebra of the type $G_2$. Then, we may write $\Pi = \{\alpha_1, \alpha_2\}$ such that $\langle \alpha_1, \alpha_1 \rangle > \langle \alpha_2, \alpha_2 \rangle$.

**Theorem 4.2.1.** Let $\mathfrak{g}$ be a simple Lie algebra of the type $G_2$ and let $k$ be 1 or 2. Then, $M_{\Theta^k} [-t] \subseteq M_{\Theta^k} [t]$ if and only if $t \in \mathbb{N}$.

**Proof.** We see $c_1 = c_2 = \frac{1}{2}$, $d_1 = \frac{3}{2}$, and $d_2 = \frac{5}{2}$. Thus, for $k = 1, 2$, $\rho_{\Theta^k} + t\omega_k$ is integral if and only if $t - \frac{1}{2} \in \mathbb{Z}$. For $k = 1, 2$, it is easy to see that $w_{\Theta^k}w_0 = w_0w_{\Theta^k}$ but they are not a Duflo involution of $W$. Moreover, we have:

**Lemma 4.2.2.** Let $k$ be 1 or 2. If $\rho_{\Theta^k} + t\omega_k$ is singular and integral, then $M_{\Theta^k} [t]$ is irreducible. In particular, $M_{\Theta^k} [\frac{1}{2}]$ is irreducible.

**Proof.** $\mathfrak{g}$ has only three special nilpotent orbits, namely the regular nilpotent orbit, the subregular nilpotent orbit, and $\{0\}$. (Cf. [Carter 1985]) Hence, the Gelfand-Kirillov dimension of any infinite-dimensional irreducible constituent of $M_{\Theta^k} [t]$ is $\dim \mathfrak{g}_{\Theta^k} = \dim (M_{\Theta^k} [t])$, if $\rho_{\Theta^k} + t\omega_k$ is integral. On the other hand, the multiplicity (the Bernstein degree) of $M_{\Theta^k} [t]$ is one. So, if $\rho_{\Theta^k} + t\omega_k$ is integral and singular, $M_{\Theta^k} [t]$ is irreducible. \[\square\]

We continue the proof of Theorem 4.2.1. From Lemma 4.2.2 and Lemma 2.2.6, we have $M_{\Theta^k} [-\frac{1}{2} - t] \not\subseteq M_{\Theta^k} [\frac{1}{2} + t]$ for all $t \in \mathbb{N}$.

For $t \in \mathbb{Z}$, $\rho_{\Theta^k} + t\omega_k$ is half-integral and its integral Weyl group is of type $A_2 \times A_2$. This case, we easily see $w_{\Theta^k}w_0$ is a Duflo involution of the integral Weyl group. Since $\rho_{\Theta^k} + t\omega_k$ is
Theorem 4.3.1.

Since \( P \) is dominant and regular if \( t \in \mathbb{N} - \{0\} \), Theorem 1.4.2 implies \( M_{G^k}[-1 - t] \subseteq M_{G^k}[1 + t] \) for all \( t \in \mathbb{N} \). Q.E.D.

4.3 \( F_4 \)

For the simple algebra of the type \( F_4 \), [Boe-Collingwood] treated the regular integral case. The half-integral case is somewhat easier, but I would like to mention the results for the completeness.

We consider the root system \( \Delta \) for a simple Lie algebra \( \mathfrak{g} \) of the type \( F_4 \). (For example, see [Knapp 2002] p691.) We can choose an orthonormal basis \( e_1, ..., e_4 \) of \( \mathfrak{h}^* \) such that

\[
\Delta = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq 4 \} \cup \{ \pm e_i \mid 1 \leq i \leq 4 \} \cup \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.
\]

We choose a positive system as follows.

\[
\Delta^+ = \{ e_i \pm e_j \mid 1 \leq i < j \leq 4 \} \cup \{ e_i \mid 1 \leq i \leq 4 \} \cup \left\{ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.
\]

Put \( \alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4) \), \( \alpha_2 = e_4 \), \( \alpha_3 = e_3 - e_4 \), and \( \alpha_4 = e_2 - e_3 \). Then, \( \Pi = \{ \alpha_1, ..., \alpha_4 \} \).

Since \( P = Q \) in this case, we have \( c_k = \frac{1}{2} \) for \( 1 \leq k \leq 4 \).

The result is:

**Theorem 4.3.1.** (1) is due to [Lepowsky 1975a].

1. \( M_{G^1}[-t] \subseteq M_{G^1}[t] \) if and only if \( t \in \frac{1}{2}\mathbb{N} \).
2. \( M_{G^2}[-t] \subseteq M_{G^2}[t] \) if and only if \( t \in \mathbb{N} \).
3. \( M_{G^3}[-t] \subseteq M_{G^3}[t] \) if and only if \( t \in \mathbb{N} \).
4. \( M_{G^4}[-t] \subseteq M_{G^4}[t] \) if and only if \( t \in \frac{1}{2}\mathbb{N} \).

**Proof.** (1) is proved in [Lepowsky 1975a] (Theorem 1.2). So, we consider the other cases.

Since \( c_k = \frac{1}{2} \) for \( k = 1, 2, 3, 4 \), \( M_{G^k}[-t] \subseteq M_{G^k}[t] \) implies \( t \in \frac{1}{2}\mathbb{N} \).

For each \( k = 1, 3, 4 \), \( p_{G^k} \) is an even parabolic subalgebra. (Cf. [Carter 1985] p401.) Hence, \( M_{G^k}[-t] \subseteq M_{G^k}[t] \) for \( t \in \mathbb{N} \) for \( k = 1, 3, 4 \).

We consider the case of \( k = 2 \). \( p_{G^2} + t\omega_2 \) is half-integral for \( t \in \mathbb{Z} \). The integral root system \( \Delta_{p_{G^2}} \) for \( p_{G^2} + t\omega_2 \) \( (t \in \mathbb{N}) \) is of the type \( A_1 \times B_3 \). \( \Theta^2 \) corresponds to the set of the long simple roots in the \( B_2 \) factor of \( \Delta_{p_{G^2}} \) and the simple root in the \( A_1 \)-factor. Hence, Theorem 3.2.1 (2a), Lemma 1.4.4, and Theorem 1.4.2 imply that \( w_{G^2}w_0 \) is a Duflo involution in the integral Weyl group. So, Theorem 1.4.2 implies \( M_{G^2}[-t] \subseteq M_{G^2}[t] \) for \( t \in \mathbb{N} \).

For \( k = 2, 3 \), we can prove that \( M_{G^k}[\frac{1}{2}] \) is irreducible using Jantzen’s criterion (Theorem 2.2.11). Hence, Lemma 2.2.6 implies that \( M_{G^k}[-\frac{1}{2} - t] \subseteq M_{G^k}[\frac{1}{2} + t] \) for \( t \in \mathbb{N} \) and \( k = 2, 3 \).

For \( k = 4 \), \( p_{G^4} + (\frac{1}{2} + t)\omega_4 \) is half-integral for \( t \in \mathbb{N} \). The integral root system \( \Delta_{p_{G^4}} \) for \( p_{G^4} + t\omega_2 \) \( (t \in \mathbb{N}) \) is of the type \( A_1 \times B_3 \). In this case \( w_{G^4} \) is the non-trivial element of the \( A_1 \)-factor of the integral Weyl group. So, it is a Duflo involution in the integral Weyl group and we have \( M_{G^4}[-\frac{1}{2} - t] \subseteq M_{G^4}[\frac{1}{2} + t] \) for \( t \in \mathbb{N} \). Q.E.D.
4.4 \textbf{E}_6

We consider the root system $\Delta$ for a simple Lie algebra $\mathfrak{g}$ of the type $E_6$. Put $\kappa = \frac{1}{2\sqrt{3}}$. We can choose an orthonormal basis $e_1, \ldots, e_6$ of $\mathfrak{h}^*$ such that

$$\Delta = \{e_i - e_j \mid 1 \leq i, j \leq 6, i \neq j\}$$

$$\cup \left\{ \pm \sum_{i=1}^{6} \left( \kappa + (-1)^{n(i)} \frac{1}{2} \right) e_i \mid n(i) = 1 \text{ or } n(i) = 0 \text{ for } 1 \leq i \leq 6, \sum_{i=1}^{6} n(i) = 3 \right\}$$

$$\cup \left\{ \pm 2\kappa \sum_{i=1}^{6} e_i \right\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i - e_j \mid 1 \leq i < j \leq 6\}$$

$$\cup \left\{ \sum_{i=1}^{6} \left( \kappa + (-1)^{n(i)} \frac{1}{2} \right) e_i \mid n(i) = 1 \text{ or } n(i) = 0 \text{ for } 1 \leq i \leq 6, \sum_{i=1}^{6} n(i) = 3 \right\}$$

$$\cup \left\{ 2\kappa \sum_{i=1}^{6} e_i \right\}.$$

Put $\alpha_i = e_i - e_{i+1}$ (1 $\leq i \leq 5$) and $\alpha_6 = \sum_{i=1}^{3} (\kappa - \frac{1}{2}) e_i + \sum_{i=4}^{6} (\kappa + \frac{1}{2}) e_i$. Then, $\Pi = \{\alpha_1, \ldots, \alpha_6\}$.

$$\begin{array}{cccccc}
1 & - & 2 & - & 3 & - 4 & - 5 \\
| & & | & | & |
6 & & & & & \\
\end{array}$$

In this case, $w_0$ is not contained in the center of the Weyl group. So, $w_\Theta w_0 = w_0 w_\Theta$ may fail for some $\Theta \subseteq \Pi$. In fact, $w_\Theta w_0 = w_0 w_\Theta^k$ holds for $k = 3, 6$, but it fails for $k = 1, 2, 4, 5$.

Since the Dynkin diagram of $E_6$ has a symmetry, the cases of $\Theta^4$ and $\Theta^5$ are similar to $\Theta^2$ and $\Theta^1$, respectively. So, we only consider $\Theta^1, \Theta^2, \Theta^3$, and $\Theta^6$.

We have:

\textbf{Theorem 4.4.1.} ((1) is due to [Boe 1985].)

(1) $M_{\Theta^1}[-t] \subseteq M_{\Theta^1}[t]$ if and only if $t = 0$.

(2) $M_{\Theta^2}[-t] \subseteq M_{\Theta^2}[t]$ if and only if $t = 0$.

(3) $M_{\Theta^3}[-t] \subseteq M_{\Theta^3}[t]$ if and only if $t \in \mathbb{N}$.

(4) $M_{\Theta^6}[-t] \subseteq M_{\Theta^6}[t]$ if and only if $t \in \mathbb{N}$.

\textbf{Proof.} (1) is proved in [Boe 1985]. So, we consider the other cases.

First, we prove (2). In this case, we have $\rho^{\Theta^2} = 3\omega_2$. We also have $\rho^{\Theta^2} \in \mathbb{Q}$ and $c_2 = \frac{3}{2}$.

Hence, $M_{\Theta^2}[-t] \subseteq M_{\Theta^2}[t]$ implies $t \in \frac{3}{2} \mathbb{N}$. If $t \in 3\mathbb{N}$, then $\rho_{\Theta^2} + t\omega_2$ is integral.

Assume that $t - \frac{3}{2} \in 3\mathbb{N}$. Then, the integral root system of $\rho_{\Theta^2} + t\omega_2$ is of the type $A_1 \times A_5$.

The set of simple roots is $\{\alpha_1, \alpha_6, \alpha_3, \alpha_4, \alpha_5, \beta\}$. Here, $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6 = \sum_{i=1}^{2} (\kappa + \frac{1}{2}) e_i + \sum_{i=3}^{5} (\kappa - \frac{1}{2}) e_i + (\kappa + \frac{1}{2}) e_6$. Among them, $\alpha_1$ is the simple root in the $A_1$-factor.
For $t \in \frac{3}{2}\mathbb{N}$, we denote by $W(t)$ the integral Weyl group of $\rho_{E_7} + t\omega_3$. Let $w'_0$ be the longest element of the integral Weyl group. We easily see that $w_{E_7}w'_0$ is not an involution. We also see $\rho_{E_7} + t\omega_3$ is dominant regular for $t \in \frac{3}{2}\mathbb{N} - \{0\}$. Since $w_{E_7}(\rho_{E_7} + t\omega_3) = -(\rho_{E_7} + t\omega_3)$, one of the following two conditions must hold.

(a) There is no $w \in W(t)$ such that $w(\rho_{E_7} + t\omega_3) = \rho_{E_7} - t\omega_3$.

(b) $w_{E_7}w'_0(\rho_{E_7} + t\omega_3) = \rho_{E_7} - t\omega_3$.

If (a) holds, then $M_{E_7}[t]$ has an infinitesimal character different from that of $M_{E_7}[-t]$. Hence, we have $M_{E_7}[-t] \not\subseteq M_{E_7}[t]$.

Assume (b) holds. Then, Theorem 1.4.2 implies that $M_{E_7}[-t] \not\subseteq M_{E_7}[t]$, since $w_{E_7}w'_0$ is not an involution.

Next, we prove (3) and (4). Let $k$ be either 3 or 6. Then we have $w_{E_k}w_0 = w_0w_{E_k}$ and $p_k$ is an even parabolic subalgebra (see [Carter 1985] p402). Hence, Lemma 4.1.2 and Corollary 2.2.9 imply $M_{E_k}[-t] \subseteq M_{E_k}[t]$ for all $t \in \mathbb{N}$. Since $c_k = \frac{1}{2}$ in these case, we have only to show $M_{E_k}[-\frac{1}{2} - t] \not\subseteq M_{E_k}[\frac{1}{2} + t]$ for all $t \in \mathbb{N}$. In this case, we can check the irreducibility of $M_{E_k}[\frac{1}{2}]$ via Jantzen’s criterion (Theorem 2.2.11). So, from Lemma 2.2.6, we have the desired result. We describe the computation briefly.

First, we consider the case of $k = 3$. In this case $d_3 = \frac{7}{2}$ and $\rho_{E_3} + \frac{1}{2}\omega_3$ is integral. We put $\gamma_1 = \sum_{i=1}^{6} 2\kappa e_i$ and $\gamma_2 = \sum_{i=1}^{6} (\kappa + \frac{1}{2}) e_i + \sum_{i=4}^{6} (\kappa - \frac{1}{2}) e_i$.

Put $\Xi = \{ \beta \in (\Delta_{E_3})^+ \mid \langle \rho_{E_3} + \frac{1}{2}\omega_3, \beta^\vee \rangle \in \mathbb{N} - \{0\} \}$. Then, $\gamma_1, \gamma_2 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \gamma_2\}$, we can find $\eta \in \Delta_{E_3}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{E_3}(s_{\beta}(\rho_{E_3} + \frac{1}{2}\omega_3)) = 0$. Moreover, we have $\Upsilon_{E_3}(s_{\gamma_1}(\rho_{E_3} + \frac{1}{2}\omega_3)) = -\Upsilon_{E_3}(\rho_{E_3} - \frac{1}{2}\omega_3)$ and $\Upsilon_{E_3}(s_{\gamma_2}(\rho_{E_3} + \frac{1}{2}\omega_3)) = \Upsilon_{E_3}(\rho_{E_3} - \frac{1}{2}\omega_3)$.

Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{E_3}(s_{\beta}(\rho_{E_3} + \frac{1}{2}\omega_3)) = 0.$$ 

This means that Jantzen’s criterion is satisfied.

Finally, we consider the case of $k = 6$. In this case $d_6 = \frac{11}{2}$ and $\rho_{E_6} + \frac{1}{2}\omega_6$ is integral. We put $\gamma_1 = \sum_{i=1}^{6} 2\kappa e_i$ and $\gamma_2 = \sum_{i=1}^{6} (\kappa + (-1)^{i-1}\frac{1}{2}) e_i$.

Put $\Xi = \{ \beta \in (\Delta_{E_6})^+ \mid \langle \rho_{E_6} + \frac{1}{2}\omega_6, \beta^\vee \rangle \in \mathbb{N} - \{0\} \}$. Then, $\gamma_1, \gamma_2 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \gamma_2\}$, we can find $\eta \in \Delta_{E_6}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{E_6}(s_{\beta}(\rho_{E_6} + \frac{1}{2}\omega_6)) = 0$. Moreover, we have $\Upsilon_{E_6}(s_{\gamma_1}(\rho_{E_6} + \frac{1}{2}\omega_6)) = \Upsilon_{E_6}(\rho_{E_6} - \frac{1}{2}\omega_6)$ and $\Upsilon_{E_6}(s_{\gamma_2}(\rho_{E_6} + \frac{1}{2}\omega_6)) = -\Upsilon_{E_6}(\rho_{E_6} - \frac{1}{2}\omega_6)$.

Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{E_6}(s_{\beta}(\rho_{E_6} + \frac{1}{2}\omega_6)) = 0.$$ 

This means that Jantzen’s criterion is satisfied. Q.E.D.

Remark In the case of $k = 6$, the non-existence of the homomorphism is proved in [Boe-Collingwood 1990] for the regular integral case.

4.5 $E_7$

Let $g$ be a simple Lie algebra of the type $E_7$. 

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We fix an orthonormal basis $e_1, ..., e_8$ in $\mathbb{R}^8$. We identify $\mathfrak{h}^*$ with $\{v \in \mathbb{R}^8 \mid \langle v, e_1 - e_2 \rangle = 0\}$ so that

$$
\Delta = \{\pm(e_1 + e_2)\} \cup \{\pm e_i \pm e_j \mid 3 \leq i < j \leq 8\}
$$

$$
\cup \left\{ \pm \frac{1}{2} \left( e_1 + e_2 + \sum_{i=3}^{8} (-1)^{\nu(i)} e_i \right) \mid \nu(i) \text{ is either 0 or 1 for } 3 \leq i \leq 8 \text{ and } \sum_{i=3}^{8} \nu(i) \text{ is even.} \right\}
$$

We choose a positive system as follows.

$$
\Delta^+ = \{(e_1 + e_2)\} \cup \{e_i \pm e_j \mid 3 \leq i < j \leq 8\}
$$

$$
\cup \left\{ \frac{1}{2} \left( e_1 + e_2 + \sum_{i=3}^{8} (-1)^{\nu(i)} e_i \right) \mid \nu(i) \text{ is either 0 or 1 for } 3 \leq i \leq 8 \text{ and } \sum_{i=3}^{8} \nu(i) \text{ is even.} \right\}
$$

Put $\alpha_i = e_{i+2} - e_{i+3}$ for $1 \leq i \leq 5$, $\alpha_6 = e_7 + e_8$, and $\alpha_7 = \frac{1}{2}(e_1 + e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$. Then, $\Pi = \{\alpha_1, ..., \alpha_7\}$ is the set of simple roots in $\Delta^+$.

$$
1 \quad -2 \quad -3 \quad -4 \quad -6 \quad -7 \quad +2
$$

We have:

**Theorem 4.5.1.** (The case of $k = 1$ is due to [Boe 1985].)

1. Assume $k \in \{1, 3, 5, 6, 7\}$. Then, $M_{\mathfrak{g}_k}[-t] \subseteq M_{\mathfrak{g}_k}[t]$ if and only if $t \in \mathbb{N}$.

2. Assume $k \in \{2, 4\}$. Then, $M_{\mathfrak{g}_k}[-t] \subseteq M_{\mathfrak{g}_k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.

**Proof.** For the simple Lie algebra of the type $E_7$, $w_0$ is contained in the center of the Weyl group. Moreover, all the maximal parabolic subalgebras are even. (See [Carter 1985] p403-404.) So, Lemma 4.1.2 and Corollary 2.2.9 imply that, for all $1 \leq k \leq 8$, $t \in \mathbb{N}$ implies $M_{\mathfrak{g}_k}[-t] \subseteq M_{\mathfrak{g}_k}[t]$. Hence, we have only to take care of the case of $t - \frac{1}{2} \in \mathbb{N}$.

The case of $k = 1$ is due to [Boe 1985]. In fact $M_{\mathfrak{g}_1}[-t] \nsubseteq M_{\mathfrak{g}_1}[t]$ for $t - \frac{1}{2} \notin \mathbb{N}$.

For the case $k = 2$, we have $d_2 = 7$ and $\omega_2 \in \mathbb{Q}$. The integral Weyl group for $\rho_{\mathfrak{g}_2} + t\omega_2$ ($t \in \frac{1}{2} + \mathbb{N}$) is of the type $A_1 \times D_6$. In fact, we have $\Pi_{\rho_{\mathfrak{g}_2} + t\omega_2} = \{e_3 + e_4\} \cup \Theta^2$. From Theorem 1.4.2 and Theorem 3.2.3, we have $w_{\mathfrak{g}_2} w_0$ is a Duflo involution of the integral Weyl group for $\rho_{\mathfrak{g}_2} + t\omega_2$. So, Theorem 1.4.2, Lemma 2.2.6 imply that $M_{\mathfrak{g}_2}[-t] \subseteq M_{\mathfrak{g}_2}[t]$ for $t - \frac{1}{2} \in \mathbb{N}$.

For the case $k = 3$, we have $d_3 = 5$ and $\omega_3 = \frac{1}{2}(e_1 + e_2) + e_3 + e_4 + e_5 \notin \mathbb{Q}$. So, we have $c_3 = 1$. Hence, $M_{\mathfrak{g}_3}[-t] \nsubseteq M_{\mathfrak{g}_3}[t]$ for $t - \frac{1}{2} \notin \mathbb{N}$.

For the case $k = 4$, we have $d_4 = 4$ and $\omega_4 \in \mathbb{Q}$. The integral Weyl group for $\rho_{\mathfrak{g}_4} + t\omega_4$ ($t \in \frac{1}{2} + \mathbb{N}$) is of the type $A_1 \times D_6$. In fact, we have $\Pi_{\rho_{\mathfrak{g}_4} + t\omega_4} = \{e_5 + e_6\} \cup \Theta^4$. From Theorem 1.4.2 and Theorem 3.2.3, we have $w_{\mathfrak{g}_4} w_0$ is a Duflo involution of the integral Weyl group for $\rho_{\mathfrak{g}_4} + t\omega_4$. So, Theorem 1.4.2, Lemma 2.2.6 imply that $M_{\mathfrak{g}_4}[-t] \subseteq M_{\mathfrak{g}_4}[t]$ for $t - \frac{1}{2} \in \mathbb{N}$.

For the case $k = 5$, we have $d_5 = 7$ and $\omega_5 = e_1 + e_2 + \frac{1}{2}(e_3 + e_4 + e_5 + e_6 + e_7 - e_8) \notin \mathbb{Q}$. So, we have $c_5 = 1$. Hence, $M_{\mathfrak{g}_5}[-t] \nsubseteq M_{\mathfrak{g}_5}[t]$ for $t - \frac{1}{2} \notin \mathbb{N}$.

For the cases $k = 6$ and $k = 7$, we have $d_6 = \frac{11}{2}$ and $d_7 = \frac{17}{2}$. In this case, we can show the irreducibility of $M_{\mathfrak{g}_k}[-\frac{1}{2}]$ ($k = 6, 7$) via Jantzen’s criterion (Theorem 2.2.11). So, from Lemma 2.2.6, for $k = 6, 7$, we have $M_{\mathfrak{g}_k}[-t] \nsubseteq M_{\mathfrak{g}_k}[t]$ for $t - \frac{1}{2} \notin \mathbb{N}$. We describe the computation briefly.
First, we consider the case of \( k = 6 \). We remark that \( P^{\ast \ast} \cap W(\rho_{\mathfrak{e}^6} + \frac{1}{2}\omega) \) consists of the 3 elements (say \( \lambda_1, \lambda_2, \lambda_3 \)). We put \( \lambda_1 = \rho_{\mathfrak{e}^6} + \frac{1}{2}\omega \) and \( \lambda_3 = \rho_{\mathfrak{e}^6} - \frac{1}{2}\omega \). The remaining element is \( \lambda_2 = \frac{1}{2}e_1 + \frac{1}{2}e_2 + 2e_3 - e_4 - e_5 - 2e_6 - 3e_7 - 3e_8 \). We put \( \gamma_1 = e_1 + e_2, \gamma_2 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 + e_3 + 2e_4 - 2e_6 - 3e_7 - 3e_8, \gamma_3 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 + e_3 + 3e_4 + 2e_5 - 2e_6 - e_8, \) and \( \gamma_4 = \frac{1}{2}e_1 + \frac{1}{2}e_2 + 3e_3 + e_4 - e_5 - 2e_7 - 3e_8 \). Put \( \Xi = \left\{ \beta \in \left( \Delta^{\ast \ast} \right)^{+} \middle| \left( \rho_{\mathfrak{e}^6} + \frac{1}{2}\omega, \beta^\vee \right) \in \mathbb{N} - \{0\} \right\} \). Then, \( \gamma_1, ..., \gamma_4 \in \Xi \). For \( \beta \in \Xi - \{ \gamma_1, ..., \gamma_4 \} \), we can find \( \eta \in \Delta_{\mathfrak{e}^6} \) such that \( \langle \beta, \eta \rangle = 0 \). Hence, we have \( \Upsilon_{\mathfrak{e}^6}(s_\beta(\rho_{\mathfrak{e}^6} + \frac{1}{2}\omega)) = 0 \). Moreover, we have \( \Upsilon_{\mathfrak{e}^6}(s_{\gamma_1}(\rho_{\mathfrak{e}^6})) = -\Upsilon_{\mathfrak{e}^6}(\lambda_3), \Upsilon_{\mathfrak{e}^6}(s_{\gamma_2}(\rho_{\mathfrak{e}^6} + \frac{1}{2}\omega)) = \Upsilon_{\mathfrak{e}^6}(\lambda_3), \Upsilon_{\mathfrak{e}^6}(s_{\gamma_3}(\rho_{\mathfrak{e}^6} + \frac{1}{2}\omega)) = -\Upsilon_{\mathfrak{e}^6}(\lambda_2), \) and \( \Upsilon_{\mathfrak{e}^6}(s_{\gamma_4}(\rho_{\mathfrak{e}^6} + \frac{1}{2}\omega)) = \Upsilon_{\mathfrak{e}^6}(\lambda_2) \). Hence, we have

\[
\sum_{\beta \in \Xi} \Upsilon_{\mathfrak{e}^6}(s_\beta(\rho_{\mathfrak{e}^6} + \frac{1}{2}\omega)) = 0.
\]

This means that Jantzen’s criterion is satisfied.

First, we consider the case of \( k = 7 \). We put \( \gamma_1 = e_1 + e_2 \) and \( \gamma_2 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 + e_5 - e_6 + e_7 + e_8) \).

Put \( \Xi = \left\{ \beta \in \left( \Delta^{\ast \ast} \right)^{+} \middle| \left( \rho_{\mathfrak{e}^6} + \frac{1}{2}\omega, \beta^\vee \right) \in \mathbb{N} - \{0\} \right\} \). Then, \( \gamma_1, \gamma_2 \in \Xi \). For \( \beta \in \Xi - \{ \gamma_1, \gamma_2 \} \), we can find \( \eta \in \Delta_{\mathfrak{e}^6} \) such that \( \langle \beta, \eta \rangle = 0 \). Hence, we have \( \Upsilon_{\mathfrak{e}^6}(s_\beta(\rho_{\mathfrak{e}^6} + \frac{1}{2}\omega)) = 0 \). Moreover, we have \( \Upsilon_{\mathfrak{e}^6}(s_{\gamma_1}(\rho_{\mathfrak{e}^6})) = \Upsilon_{\mathfrak{e}^6}(\lambda_3), \Upsilon_{\mathfrak{e}^6}(s_{\gamma_2}(\rho_{\mathfrak{e}^6} + \frac{1}{2}\omega)) = -\Upsilon_{\mathfrak{e}^6}(\lambda_3), \) and \( \Upsilon_{\mathfrak{e}^6}(s_{\gamma_4}(\rho_{\mathfrak{e}^6} + \frac{1}{2}\omega)) = \Upsilon_{\mathfrak{e}^6}(\lambda_2) \). Hence, we have

\[
\sum_{\beta \in \Xi} \Upsilon_{\mathfrak{e}^6}(s_\beta(\rho_{\mathfrak{e}^6} + \frac{1}{2}\omega)) = 0.
\]

This means that Jantzen’s criterion is satisfied.

Q.E.D.

Remark In the case of \( k = 7 \), the non-existence of the homomorphism is proved in [Boe-Collingwood 1990] for the regular integral case.

4.6 \( \mathfrak{e}^8 \)

We fix an orthonormal basis \( e_1, ..., e_8 \) in \( \mathfrak{h}^* \) such that

\[
\Delta = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq 8 \}
\cup \left\{ \pm \frac{1}{2} \left( \sum_{i=1}^{8} (-1)^{n(i)} e_i \right) \middle| n(i) \text{ is either 0 or 1 for } 3 \leq i \leq 8 \text{ and } \sum_{i=3}^{8} n(i) \text{ is odd} \right\}
\]

We choose a positive system as follows.

\[
\Delta^+ = \{ e_i \pm e_j \mid 1 \leq i < j \leq 8 \}
\cup \left\{ \frac{1}{2} \left( e_1 + \sum_{i=2}^{8} (-1)^{n(i)} e_i \right) \middle| n(i) \text{ is either 0 or 1 for } 2 \leq i \leq 8 \text{ and } \sum_{i=2}^{8} n(i) \text{ is odd} \right\}
\]
Put $\alpha_i = e_{i+1} - e_{i+2}$ for $1 \leq i \leq 5$, $\alpha_7 = e_7 + e_8$, and $\alpha_7 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$. Then, $\Pi = \{\alpha_1, ..., \alpha_8\}$ is the set of simple roots in $\Delta^+$.\\

$$
\begin{array}{cccccccc}
1 & -2 & -3 & -4 & -5 & -7 & -8 \\
\hline
\end{array}
$$

We have:

**Theorem 4.6.1.**

(1) Assume $k \in \{1, 2, 4, 6, 8\}$. Then, $M_{\Theta_k}[-t] \subseteq M_{\Theta_k}[t]$ if and only if $t \in \mathbb{N}$.

(2) Assume $k \in \{3, 5, 7\}$. Then, $M_{\Theta_k}[-t] \subseteq M_{\Theta_k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.

**Proof.** For any $k$, we have $w_{\Theta_k}w_0 = w_0w_{\Theta_k}$ since $w_0$ is contained in the center of $W$. Hence, Lemma 2.2.3 implies that $M_{\Theta_k}[-t] \subseteq M_{\Theta_k}[t]$, only if $t \in \frac{1}{2}\mathbb{N}$.

Next we consider the case of $t \in \mathbb{N}$. For $k \in \{1, 2, 3, 4, 6, 8\}$, $p_{\Theta_k}$ is even (cf. [Carter 1985] p405-406). In this case, Corollary 2.2.9 and Lemma 4.1.2 imply $M_{\Theta_k}[-t] \subseteq M_{\Theta_k}[t]$ for $t \in \mathbb{N}$. If $k = 5$, we have $d_5 = \frac{9}{2}$. So, $p_{\Theta_5}$ is not integral. A basis of integral root system for $p_{\Theta_5}$ is $\Theta^5 \cup \{e_5 + e_6\}$. We see that the integral root system is of the type $A_1 \times E_7$. So, the problem is reduced to the case of $k = 3$ in the type $E_7$. Hence, $M_{\Theta_5}[-t] \subseteq M_{\Theta_5}[t]$ for $t \in \mathbb{N}$ such that $p_{\Theta_5} + t\omega_5$ is dominant and regular. From Lemma 2.2.6, we have $M_{\Theta_5}[-t] \subseteq M_{\Theta_5}[t]$ for all $t \in \mathbb{N}$.

The case of $k = 7$ is similar to the case of $k = 5$. This time, a basis of the integral Weyl group of $p_{\Theta_7}$ is $\Theta^7 \cup \{\frac{1}{2}(e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7 + e_8)\}$ and the integral root system is of the type $A_1 \times E_7$. Hence, the problem is reduced to the case of $k = 5$ in the type $E_7$.

Next, we consider the case of $t \in \frac{1}{2} + \mathbb{N}$.

First, we consider the case of $k = 3$. In this case, $p_{\Theta_3} + t\omega_3$ is not integral for $t \in \frac{1}{2} + \mathbb{N}$. A basis of integral root system for $p_{\Theta_3} + \frac{1}{2}\omega_3$ is $\Theta^3 \cup \{e_3 + e_4\}$ and the integral root system is of the type $D_8$. This time, the problem is reduced to the case of $k = 3$ in the type $D_8$.

Next, we consider the case of $k = 5, 7$. In this case, $p_{\Theta_k} + t\omega_k$ is integral for $t \in \frac{1}{2} + \mathbb{N}$. The special representation corresponding to a Richardson orbit can be constructed as a MacDonald representation ([MacDonald 1972]). If $k = 5$ (resp. $k = 7$), then the number of positive roots in the Levi part of $p_{\Theta_k}$ is 14 (resp. 22). Hence, the special representation occurs as component of $S^{14}(\mathfrak{h}^*)$ (resp. $S^{22}(\mathfrak{h}^*)$) but not of $S^d(\mathfrak{h}^*)$ for $d > 14$ (resp. $d > 22$). Hence, from [Carter 1985] p417-418, we see the family of the special representation consists of a single element. Hence, Corollary 2.1.4, Lemma 2.2.6 and Lemma 2.2.7 imply that $M_{\Theta_k}[-t] \subseteq M_{\Theta_k}[t]$ for $t \in \frac{1}{2} + \mathbb{N}$.

If $k = 1, 2, 4, 6, 8$, we can prove that $M_{\Theta_k}[\frac{1}{2}]$ is irreducible via Jantzen’s criterion. So, we have $M_{\Theta_k}[-t] \not\subseteq M_{\Theta_k}[t]$ for $t \in \frac{1}{2} + \mathbb{N}$. We describe the computation briefly.

First, we consider the case of $k = 1$. In this case $d_1 = \frac{29}{2}$ and $p_{\Theta_1} + \frac{1}{2}\omega_1$ is integral. We put $\gamma_1 = e_1 + e_2$ and $\gamma_2 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 + e_5 - e_6 - e_7 - e_8)$.

Put $\Xi = \left\{ \beta \in \left(\Delta^\Theta_1\right)^+ : \langle \rho_{\Theta_1} + \frac{1}{2}\omega_1, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \gamma_2 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \gamma_2\}$, we can find $\eta \in \Delta_{\Theta_1}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{\Theta_1}(s_{\beta}(\rho_{\Theta_1} + \frac{1}{2}\omega_1)) = 0$. Moreover, we have $\Upsilon_{\Theta_1}(s_{\gamma_1}(\rho_{\Theta_1} + \frac{1}{2}\omega_1)) = \Upsilon_{\Theta_1}(\rho_{\Theta_1} - \frac{3}{2}\omega_1)$ and $\Upsilon_{\Theta_1}(s_{\gamma_2}(\rho_{\Theta_1} + \frac{1}{2}\omega_1)) = -\Upsilon_{\Theta_1}(\rho_{\Theta_1} - \frac{1}{2}\omega_1)$. Hence, we have

$$
\sum_{\beta \in \Xi} \Upsilon_{\Theta_1}(s_{\beta}(\rho_{\Theta_1} + \frac{1}{2}\omega_1)) = 0.
$$

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This means that Jantzen’s criterion is satisfied.

Next, we consider the case of $k = 2$. In this case $d_2 = \frac{19}{2}$ and $\rho_{\Theta^2} + \frac{1}{2}w_2$ is integral. We remark that $P^+_{\Theta^2} \cap W(\rho_{\Theta^2} + \frac{1}{2}w_2)$ consists of the 3 elements (say $\lambda_1, \lambda_2, \lambda_3$). We put $\lambda_1 = \rho_{\Theta^2} + \frac{1}{2}w_2$ and $\lambda_3 = \rho_{\Theta^2} - \frac{1}{2}w_2$. The remaining element is $\lambda_2 = 4e_1 - 3e_2 - 5e_3 + 4e_4 + 3e_5 + 2e_6 + e_7$. We put $\gamma_1 = e_1 + e_2$, $\gamma_2 = e_1 + e_3$, $\gamma_3 = e_1 - e_6$, and $\gamma_4 = \frac{1}{2}(e_1 + e_2 - e_3 + e_4 - e_5 - e_6 + e_7 + e_8)$. Put $\Xi = \left\{ \beta \in \left( \Delta^\Theta \right)^+ | \langle \rho_{\Theta^2} + \frac{1}{2}w_2, \beta' \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \ldots, \gamma_4 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \ldots, \gamma_4\}$, we can find $\eta \in \Delta_{\Theta^2}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $Y_{\Theta^2}(s_\eta(\rho_{\Theta^2} + \frac{1}{2}w_2)) = 0$. Moreover, we have $Y_{\Theta^2}(s_{\gamma_1}(\rho_{\Theta^2} + \frac{1}{2}w_2)) = -Y_{\Theta^2}(\lambda_3)$, $Y_{\Theta^2}(s_{\gamma_2}(\rho_{\Theta^2} + \frac{1}{2}w_2)) = Y_{\Theta^2}(\lambda_2)$, $Y_{\Theta^2}(s_{\gamma_3}(\rho_{\Theta^2} + \frac{1}{2}w_2)) = Y_{\Theta^2}(\lambda_3)$, and $Y_{\Theta^2}(s_{\gamma_4}(\rho_{\Theta^2} + \frac{1}{2}w_2)) = -Y_{\Theta^2}(\lambda_2)$. Hence, we have

$$\sum_{\beta \in \Xi} Y_{\Theta^2}(s_\beta(\rho_{\Theta^2} + \frac{1}{2}w_2)) = 0.$$  

This means that Jantzen’s criterion is satisfied.

Next, we consider the case of $k = 4$. In this case $d_4 = \frac{11}{2}$ and $\rho_{\Theta^4} + \frac{1}{2}w_4$ is integral. We remark that $P^+_{\Theta^4} \cap W(\rho_{\Theta^4} + \frac{1}{2}w_4)$ consists of the 7 elements $\lambda_1, \ldots, \lambda_7$. They are characterized as follows. $\langle \lambda_1, \omega_4 \rangle = 10$, $\langle \lambda_2, \omega_4 \rangle = 5$, $\langle \lambda_3, \omega_4 \rangle = 2$, $\langle \lambda_4, \omega_4 \rangle = 0$, $\langle \lambda_5, \omega_4 \rangle = -2$, $\langle \lambda_6, \omega_4 \rangle = -5$, and $\langle \lambda_7, \omega_4 \rangle = -10$. We have $\lambda_1 = \rho_{\Theta^4} + \frac{1}{2}w_4$ and $\lambda_7 = \rho_{\Theta^4} - \frac{1}{2}w_4$. We put $\gamma_1 = e_1 + e_2$, $\gamma_2 = e_1 + e_3$, $\gamma_3 = e_1 + e_4$, $\gamma_4 = e_1 + e_5$, $\gamma_5 = e_1 + e_6$, $\gamma_6 = e_1 - e_5$, $\gamma_7 = e_1 - e_7$, $\gamma_8 = e_1 - e_8$, and $\gamma_9 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 + e_6 + e_7 + e_8)$. Hence, we have

$$\sum_{\beta \in \Xi} Y_{\Theta^4}(s_\beta(\rho_{\Theta^4} + \frac{1}{2}w_4)) = 0.$$  

This means that Jantzen’s criterion is satisfied.

Next, we consider the case of $k = 8$. In this case $d_8 = \frac{23}{2}$ and $\rho_{\Theta^8} + \frac{1}{2}w_8$ is integral. We put $\gamma_1 = e_1 + e_7$ and $\gamma_2 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 + e_6 + e_7 - e_8)$. Put $\Xi = \left\{ \beta \in \left( \Delta^\Theta \right)^+ | \langle \rho_{\Theta^8} + \frac{1}{2}w_8, \beta' \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \gamma_2 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \gamma_2\}$, we can find $\eta \in \Delta_{\Theta^8}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $Y_{\Theta^8}(s_\eta(\rho_{\Theta^8} + \frac{1}{2}w_8)) = 0$. Moreover, we have $Y_{\Theta^8}(s_{\gamma_1}(\rho_{\Theta^8} + \frac{1}{2}w_8)) = Y_{\Theta^8}(\rho_{\Theta^8} - \frac{1}{2}w_8)$ and $Y_{\Theta^8}(s_{\gamma_2}(\rho_{\Theta^8} + \frac{1}{2}w_8)) = -Y_{\Theta^8}(\rho_{\Theta^8} - \frac{1}{2}w_8)$. Hence, we have

$$\sum_{\beta \in \Xi} Y_{\Theta^8}(s_\beta(\rho_{\Theta^8} + \frac{1}{2}w_8)) = 0.$$  

This means that Jantzen’s criterion is satisfied.
Finally, we consider the case $k = 6$. In this case, we choose the following basis of the root system in order to make computation easier. $\alpha_1 = e_7 - e_8$, $\alpha_2 = e_6 - e_7$, $\alpha_3 = e_5 - e_6$, $\alpha_4 = e_4 - e_5$, $\alpha_5 = e_3 - e_4$, $\alpha_6 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + e_7 + e_8)$, $\alpha_7 = e_2 - e_3$, and $\alpha_8 = e_1 - e_2$. In this case $d_6 = \frac{17}{2}$ and $\rho_{\Theta} + \frac{1}{2} \omega_6$ is integral. We put $\gamma_1 = e_1 + e_5$ and $\gamma_2 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 - e_8)$. Put $\Xi = \left\{ \beta \in \left( \Delta_{\Theta} \right)^+ \mid \left( \rho_{\Theta} + \frac{1}{2} \omega_6, \beta' \right) \in \mathbb{N} - \{0\} \right\}$.

Then, $\gamma_1, \gamma_2 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \gamma_2\}$, we can find $\eta \in \Delta_{\Theta}$ such that $\left( \beta, \eta \right) = 0$. Hence, we have $\Upsilon_{\Theta}(s_2(\rho_{\Theta} + \frac{1}{2} \omega_6)) = 0$. Moreover, we have $\Upsilon_{\Theta}(s_{\gamma_1}(\rho_{\Theta} + \frac{1}{2} \omega_6)) = \Upsilon_{\Theta}(\rho_{\Theta} - \frac{1}{2} \omega_6)$ and $\Upsilon_{\Theta}(s_{\gamma_2}(\rho_{\Theta} + \frac{1}{2} \omega_6)) = -\Upsilon_{\Theta}(\rho_{\Theta} - \frac{1}{2} \omega_6)$. Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{\Theta}(s_{\beta}(\rho_{\Theta} + \frac{1}{2} \omega_6)) = 0.$$ 

This means that Jantzen’s criterion is satisfied. Q.E.D.

**Remark** In the case of $k = 1$, the non-existence of the homomorphism is proved in [Boe-Collingwood 1990] for the regular integral case.

§ 5. Elementary homomorphisms

Here, we explain that we can construct homomorphisms in the setting of general parabolic subalgebras from the case of maximal parabolic subalgebras.

5.1 A comparison result

Here, we review some notion in [Matumoto 1993] §3. Hereafter, $\mathfrak{g}$ means a reductive Lie algebra over $\mathbb{C}$ and retain the notations in §1. We fix a subset $\Theta$ of $\Pi$. For $\alpha \in \Delta$, we put

$$\Delta(\alpha) = \{ \beta \in \Delta \mid \exists c \in \mathbb{R} \ \beta|_{\mathfrak{a}_\alpha} = c\alpha|_{\mathfrak{a}_\alpha} \},$$

$$\Delta^+(\alpha) = \Delta(\alpha) \cap \Delta^+, \quad U_\alpha = \mathbb{C}S + \mathbb{C}\alpha \subseteq \mathfrak{h}^*.$$

Then $(U_\alpha, \Delta(\alpha), \langle \ , \ \rangle)$ is a subroot system of $(\mathfrak{h}^*, \Delta, \langle \ , \ \rangle)$. The set of simple roots for $\Delta^+(\alpha)$ is denoted by $\Pi(\alpha)$. If $\alpha|_{\mathfrak{a}_\alpha} \neq 0$, then there exists a unique $\tilde{\alpha} \in \Delta^+$ such that $\Pi(\alpha) = \Theta \cup \{\tilde{\alpha}\}$. If $\alpha \in \Delta$ satisfies $\alpha|_{\mathfrak{a}_\alpha} \neq 0$ and $\alpha = \tilde{\alpha}$, then we call $\alpha$ $\Theta$-reduced. For $\alpha \in \Delta^+$, we denote by $W_{\Theta}(\alpha)$ the Weyl group of $(\mathfrak{h}^*, \Delta(\alpha))$. Clearly, $W_{\Theta} \subseteq W_{\Theta}(\alpha) \subseteq W$. We denote by $w^\alpha$ the longest element of $W_{\Theta}(\alpha)$. We call $\alpha \in \Delta$ $\Theta$-acceptable if $w^\alpha w_{\Theta} = w_{\Theta} w^\alpha$. We denote by $\Delta_{\Theta}^\alpha$ the set of $\Theta$-reduced $\Theta$-acceptable roots. Put $(\Delta_{\Theta}^\alpha)^+ = \Delta^+ \cap \Delta_{\Theta}^\alpha$. For $\alpha \in \Delta_{\Theta}^\alpha$, we define

$$\sigma_\alpha = w^\alpha w_{\Theta} = w_{\Theta} w^\alpha.$$

Clearly, $\sigma_\alpha^2 = 1$. For $\alpha \in \Delta$, we put

$$V_\alpha = \{ \lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \alpha \rangle = 0 \}.$$

We denote by $\omega_\alpha \in \mathfrak{a}_\Theta^* \subseteq \mathfrak{h}^*$ the fundamental weight for $\alpha$ with respect to the basis $\Pi(\alpha) = \Theta \cup \{\alpha\}$. Namely $\omega_\alpha$ satisfies that $\langle \omega_\alpha, \beta \rangle = 0$ for $\beta \in \Theta$, $\langle \beta, \tilde{\alpha} \rangle = 1$, and $\omega_\alpha|_{\mathfrak{g} - \mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Here, $\mathfrak{c}(\mathfrak{g}(\alpha))$ is the center of $\mathfrak{g}(\alpha)$. We see that there is some positive real number $a$ such that $\omega_\alpha = a\alpha|_{\mathfrak{a}_\alpha}$, since $\alpha|_{\mathfrak{h} \cap \mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Hence, we have $V_\alpha = \{ \lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \omega_\alpha \rangle = 0 \}$.

We can easily see:
Lemma 5.1.1. Let $\alpha \in \Delta^\Theta$. Then, we have

1. $\sigma_\alpha$ preserves $\mathfrak{a}_\Theta$.
2. $\sigma_\alpha \in W(\Theta)$. In particular, $\sigma_\alpha \rho_\Theta = \rho_\Theta$.
3. $\sigma_\alpha \omega_\alpha = -\omega_\alpha$.
4. $\sigma_\alpha |_{\mathfrak{a}_0}$ is the reflection with respect to $V_\alpha$.

For $\alpha \in (\Delta^\Theta)^+$, we define

$$\mathfrak{g}(\alpha) = \mathfrak{h} + \sum_{\beta \in \Delta(\alpha)} \mathfrak{g}_\beta, \quad \mathfrak{p}_\Theta(\alpha) = \mathfrak{g}(\alpha) \cap \mathfrak{p}_\Theta.$$  

Then, $\mathfrak{g}(\alpha)$ is a reductive Lie subalgebra of $\mathfrak{g}$ whose root system is $\Delta(\alpha)$ and $\mathfrak{p}_\Theta(\alpha)$ is a maximal parabolic subalgebra of $\mathfrak{g}(\alpha)$.

Put $\rho(\alpha) = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta$. For $\nu \in \mathfrak{a}_\Theta^*$, we denote by $C_\nu$ the one-dimensional $U(\mathfrak{p}_\Theta(\alpha))$-module corresponding to $\nu$. For $\nu \in \mathfrak{a}_\Theta^*$ we define a generalized Verma module for $\mathfrak{g}(\alpha)$ as follows.

$$M^g(\alpha)(\rho_\Theta + \nu) = U(\mathfrak{g}(\alpha)) \otimes_{U(\mathfrak{p}_\Theta(\alpha))} C_{\nu - \rho(\alpha)}.$$  

Then, we have:

Theorem 5.1.2. Let $\nu$ be an arbitrary element in $V_\alpha$ and let $c$ be either 1 or $\frac{1}{2}$. Assume that

$$M^g(\alpha)(\rho_\Theta - n c \omega_\alpha) \subseteq M^g(\alpha)(\rho_\Theta + n c \omega_\alpha)$$

for all $n \in \mathbb{N}$. Then, we have $M_\Theta(\rho_\Theta + \nu - n c \omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + n c \omega_\alpha)$ for all $n \in \mathbb{N}$. (We call the above homomorphism of $M_\Theta(\rho_\Theta + \nu - n c \omega_\alpha)$ into $M_\Theta(\rho_\Theta + \nu + n c \omega_\alpha)$ an elementary homomorphism.)

Proof. Assume that $M^g(\alpha)(\rho_\Theta - n c \omega_\alpha) \subseteq M^g(\alpha)(\rho_\Theta + n c \omega_\alpha)$ for all $n \in \mathbb{N}$. Remark that $\sigma_\alpha(\rho_\Theta + n c \omega_\alpha) = \rho_\Theta - n c \omega_\alpha$. From Theorem 1.4.2, this implies that $\sigma_\alpha$ is a Duflo involution of the integral Weyl group $W_{\rho_\Theta + n c \omega_\alpha}$ for a sufficiently large $n$.

Put $Q_{\alpha,n} = \{ \nu \in V_\alpha \mid \Delta(\alpha)_{\rho_\Theta + n c \omega_\alpha} = \Delta_{\rho_\Theta + \nu + n c \omega_\alpha} \}$. From Theorem 1.4.2, for sufficiently large $n \in \mathbb{N}$ and $\nu \in Q_{a,n}$, we have $M_\Theta(\rho_\Theta + \nu - n c \omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + n c \omega_\alpha)$, since we have $\sigma_\alpha(\rho_\Theta + \nu + n c \omega_\alpha) = \rho_\Theta + \nu - n c \omega_\alpha$. We easily see $\nu - n c \omega_\alpha$ is strongly $\Theta$-antidominant for all $n \in \mathbb{N}$. Applying Lemma 1.5.1 and the exactness of the translation functor, we can remove the extra assumption that $n$ is sufficiently large.

On the other hand $Q_{\alpha,n}$ is Zarisky dense in $V_\alpha$ (Cf. [Matumoto 1993] Lemma 3.2.2 (1)). Moreover, we can prove that $\{ \nu \in \mathfrak{a}_\Theta^* \mid M_\Theta(\rho_\Theta + \nu - \mu) \subseteq M_\Theta(\rho_\Theta + \nu) \}$ is Zarisky closed in $\mathfrak{a}_\Theta^*$ for each $\mu \in \mathfrak{a}_\Theta^*$ in the same way as [Lepowsky 1975b] Lemma 5.4. Hence, for each $\nu \in V_\alpha$ and each $n \in \mathbb{N}$, we have $M_\Theta(\rho_\Theta + \nu - n c \omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + n c \omega_\alpha)$. Q.E.D.

Remark Taking this opportunity, I would like to fix an error in [Matumoto 1993]. In page 269 line 18, the definitions of $\mathfrak{g}(\alpha, c)$ and $\mathfrak{p}_S(\alpha, c)$ are incorrect. $\mathfrak{g}(\alpha, c)$ should be an abstract reductive Lie algebra associated with the pair $(\mathfrak{h}, \Delta_{\alpha, c})$. $\mathfrak{p}_S(\alpha, c)$ should be the standard parabolic subalgebra corresponding to $\Theta$. Since $\Delta_{\alpha, c}$ need not be closed under the addition in $\Delta$, $\mathfrak{g}(\alpha, c)$ need not be a subalgebra of $\mathfrak{g}$.

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5.2 C\textsubscript{n} case

As an example, we describe elementary homomorphisms in the C\textsubscript{n} case. Let g = sp(n, \mathbb{C}). We use the notation in the root system in 3.1.

Let \kappa = (k_1, ..., k_s) be a finite sequence of positive integers such that k_1 + \cdots + k_s \leq n. We put k_i^* = k_1 + \cdots + k_i for 1 \leq i \leq s and k_0^* = 0. We define a subset \Theta^\kappa of \Pi as follows.

\[ \Theta^\kappa = \begin{cases} 
\Pi - \{e_{k_i^*} - e_{k_{i+1}^*} | 1 \leq i \leq s\} & \text{if } k_s^* < n, \\
\Pi - \{e_{k_i^*} - e_{k_{i+1}^*} | 1 \leq i \leq s - 1\} \cup \{2e_n\} & \text{if } k_s^* = n.
\end{cases} \]

Then the corresponding standard Levi subalgebra l_{\Theta^\kappa} is isomorphic to gl(k_1, \mathbb{C}) \oplus gl(k_2, \mathbb{C}) \oplus \cdots \oplus gl(k_s, \mathbb{C}) \oplus sp(n - k_s^*, \mathbb{C}). Here, we regard sp(0, \mathbb{C}) as a trivial Lie algebra \{0\}. Obviously any proper subset of \Pi is written as the form of \Theta^\kappa.

We put \( a_i = \sum_{j=1}^{k_i} e_{k_{j+1}^* - j} \) (1 \leq i \leq s). Then, \( a_1, ..., a_s \) form a basis of \( a_{\Theta^\kappa} \). We write \( M_{\Theta^\kappa}[t_1, ..., t_s] \) for \( M_{\Theta^\kappa}(t_1 a_1 + \cdots + t_s a_s) \) for \( t_1, ..., t_s \in \mathbb{C} \).

We easily have:

Lemma 5.2.1.

(1) If \( k_s^* < n \), then

\[ (\Delta^\Theta_\kappa)^+ = \{e_{k_i^*} - e_{k_{i+1}^*} | 1 \leq i < s, k_i = k_{i+1}\} \cup \{e_{k_i^*} - e_{k_{i+1}^*} | 1 \leq i \leq s\} \]

(2) If \( k_s^* = n \), then

\[ (\Delta^\Theta_\kappa)^+ = \{e_{k_i^*} - e_{k_{i+1}^*} | 1 \leq i < s, k_i = k_{i+1}\} \cup \{2e_{k_i^*} | 1 \leq i \leq s\} \]

Combining [Boe 1985] 4.4 Theorem, Theorem 3.2.2, and Theorem 5.1.2, we have

Proposition 5.2.2. (1) Let 1 \leq p < q \leq s be such that \( k_i = k_j \). If \( t_p - t_q \in \mathbb{N} \), we have

\[ M_{\Theta^\kappa} \left( \rho_{\Theta^\kappa} + \sum_{1 \leq i \leq s \atop i \neq p, q} t_i a_i + t_q a_p + t_p a_q \right) \subseteq M_{\Theta^\kappa} \left( \rho_{\Theta^\kappa} + \sum_{1 \leq i \leq s} t_i a_i \right). \]

(2) Let 1 \leq p \leq q be such that 3k_p > 2(k_p + n - k_s^*). If \( t_p \in \mathbb{N} \), we have

\[ M_{\Theta^\kappa} \left( \rho_{\Theta^\kappa} + \sum_{1 \leq i \leq s \atop i \neq p} t_i a_i - t_p a_p \right) \subseteq M_{\Theta^\kappa} \left( \rho_{\Theta^\kappa} + \sum_{1 \leq i \leq s} t_i a_i \right). \]

(3) Let 1 \leq p \leq q be such that 3k_p \leq 2(k_p + n - k_s^*) and \( k_p \) is even. If \( t_p \in \frac{1}{2} \mathbb{N} \), we have

\[ M_{\Theta^\kappa} \left( \rho_{\Theta^\kappa} + \sum_{1 \leq i \leq s \atop i \neq p} t_i a_i - t_p a_p \right) \subseteq M_{\Theta^\kappa} \left( \rho_{\Theta^\kappa} + \sum_{1 \leq i \leq s} t_i a_i \right). \]
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