FREQUENCY DEPENDENCE OF HÖLDER CONTINUITY FOR QUASIPERIODIC SCHRÖDINGER OPERATORS

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ABSTRACT. We prove estimates on the Hölder exponent of the density of states measure for discrete Schrödinger operators with potential of the form $V(n) = \lambda \ell([n+1] \beta - [n \beta])$, with $\lambda$ large enough, and conclude that for almost all values of $\beta$, the density of states measure is not Hölder continuous.

1. INTRODUCTION

The discovery of quasicrystals by Dan Shechtman [13] has elicited considerable interest (for instance [15], [12]) in the subject of aperiodic order from mathematicians and physicists. The Fibonacci Hamiltonian has been one of the canonical models for a quasicrystal. It is the one-dimensional discrete Schrödinger operator on $\ell^2(\mathbb{Z})$ specified by

$$ (H\psi)(n) = \psi(n-1) + V(n)\psi(n) + \psi(n+1). $$

The sequence $V$ is called the potential; for the Fibonacci Hamiltonian, $V(n) = \lambda \ell((n+1)\beta - [n\beta])$, where $\lambda > 0$ is called the coupling constant and $\beta = \frac{\sqrt{5} - 1}{2}$ the frequency.

The spectral properties of the Fibonacci Hamiltonian qualitatively agree with those of physical quasicrystals. For example,

1. The spectrum $\Sigma$ is a Cantor set of zero Lebesgue measure ([14]). Its point spectrum is empty ([14], [7]), so the spectral measure is purely singular continuous. As $\lambda \to \infty$, the Hausdorff dimension of the spectrum behaves like $1.831 \cdot -\frac{\log \beta}{\log \lambda}$ ([14]).

2. The spectral measure $\mu$ is uniformly $\alpha$-Hölder continuous for some $\alpha > 0$. This means that there is a $\delta > 0$ such that for all $x$ and $y$ with $|x - y| < \delta$, $\mu[x, y] < |x - y|^-\alpha$ ([7]). An asymptotically optimal estimate of $\alpha$ has not been established, but see [5].

3. The density of states measure $N$ is also $\alpha$-Hölder continuous ([4], [6]). The distribution function of the density of states measure is given by the formula ([8])

$$ N([x,y]) = \lim_{n \to \infty} \frac{\# \text{eigenvalues of } H_n \text{ in } [x,y]}{n}, $$

where $H_n$ is the restriction of $H$ to the $\ell^2$ sequences supported on $[1,n]$. As $\lambda \to \infty$, the optimal Hölder exponent behaves like $\frac{-3\log \beta}{2\log \lambda}$.

One wonders how these properties depend on the frequency. It is already known ([11]) that $\Sigma$ is a Cantor set of zero Lebesgue measure for all irrational values of $\beta$. For $\lambda$ large enough, [10] established estimates for the Hausdorff dimension of $\Sigma$. Let $[0,a_1,a_2,a_3,\ldots]$
be the continued fraction expansion of $\beta$ (see [9] for an introduction to continued fractions). Then when $\mathcal{M}(\beta) := \liminf_{k \to \infty} \psi^{-1}a_1 \ldots a_k$ is finite,

$$\max \left\{ \frac{\log 2}{10 \log 2 + 3 \log (4(\lambda - 8))}, \frac{\log \mathcal{M}(\beta) - \log 3}{\log \mathcal{M}(\beta) + \log (12(\lambda - 8))} \right\} \leq \dim_H(\Sigma)$$

$$\leq \frac{2 \log \mathcal{M}(\beta) + \log 3}{2 \log \mathcal{M}(\beta) + \log (\lambda - 8) - \log 3}.$$ 

When $M$ is infinite, the Hausdorff dimension of the spectrum is 1; notice that the upper bound depends on $\lambda$ in the same way for all $\beta$. In [11], Liu, Qu, and Wen derive an expression for the Hausdorff dimension of $\Sigma$ and show that for all $\beta$ and $\lambda > 24$, the Hausdorff dimension is Lipschitz continuous.

Using methods like those of [4], this article determines how the $\alpha$-continuity of the density of states measure depends on $\beta$, assuming throughout that $\mathcal{M}(\beta) = \limsup_{k \to \infty} \psi^{-1}a_1a_2 \ldots a_k < \infty$ and $\lambda > 24$ (recall that $\mathcal{M}$ is finite for almost all $\beta$).

When the continued fraction coefficients of $\beta$ are constant, the behavior is like the Fibonacci Hamiltonian:

**Theorem 1.1.** Suppose $\beta = [0; b, b, b, \ldots]$. Then for every

$$\begin{cases} 
2 \log \beta & b > 3 \\
-b \log (\lambda + 5) - 3 \log (b + 2) & b = 2, 3 \\
3 \log \beta & b = 1
\end{cases}$$

there is a $\delta > 0$ such that the density of states measure $N$ associated to the family of Schrödinger operators with frequency $\beta$ and coupling strength $\lambda$ obeys

$$|N(x) - N(y)| \leq |x - y|^\gamma$$

for all $x, y$ with $|x - y| < \delta$.

**Theorem 1.2.** If $\beta = [0; b, b, b, \ldots]$, then for every

$$\begin{cases} 
2 \log \beta & b > 2 \\
-b \log (\lambda - 8) - \log (b) + b \log 3 & b = 2 \\
3 \log \beta & b = 1
\end{cases}$$

and any $0 < \delta < 1$, there are $x$ and $y$ with $0 < |x - y| < \delta$ such that $|N(x) - N(y)| \geq |x - y|^\gamma$.

**Corollary 1.3.** For constant continued fraction coefficients, this identifies the optimal asymptotic behavior of $\gamma$ as $\lambda \to \infty$. If $\Gamma(\lambda, b)$ is the optimal Hölder exponent, $\gamma \leq \Gamma \leq \bar{\gamma}$, so that

$$\lim_{\lambda \to \infty} \Gamma(\lambda, b) \log \lambda = \frac{-2 \log \beta}{b}$$

when $b > 3$, and similarly for smaller values of $b$.

More generally, the qualitative behavior is determined by $\check{d}(\beta) = \limsup_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} a_i$ and $\check{d}$, the limit inferior.
Theorem 1.4. If $d$ is finite, then $N$ is $\alpha$-Hölder continuous for some $\alpha$. If $d$ is infinite, $N$ is not Hölder continuous. It is well known that $d = d = \infty$ almost everywhere. Thus, for Lebesgue almost all $\beta$, $N$ is not Hölder continuous.

2. Structure of $\Sigma$

In [10], the fine structure of $\Sigma$ is developed enough (along the lines of [12]) to estimate its Hausdorff dimension. This article uses many parts of the apparatus Liu and Wen develop, so we recapitulate the necessary results (without proof). The idea is to approximate $\Sigma$ by finite unions of closed intervals, growing in number and shrinking in size at controlled rates.

The central objects to approximating $\Sigma$ are the continued fraction approximations to $\beta$. Let $p_k/q_k$ be the $k$th convergent to $\beta$. For $k \geq 1$ and $x \in \mathbb{R}$ define the transfer matrix over $q_k$ sites by

$$M_k(E) = \prod_{n=q_k}^{1} \begin{bmatrix} E-V(n) & -1 \\ 1 & 0 \end{bmatrix}$$

and put

$$M_{-1}(E) = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix}, \ M_0(E) = \begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix}.$$ 

These matrices arise in the spectral theory of a discrete Schrödinger operator because they produce the sequences that satisfy the formal difference equation

$$H \psi = \psi.$$ 

Proposition 2.1. This summarizes work that first appeared in [12]. Let $x_{(k,p)} = \text{tr} M_{k-1} M_k^p$ and $\sigma_{(k,p)} = \{ E \in \mathbb{R} : |x_{(k,p)}(E)| \leq \theta \}$. Then:

1. $M_{k+1} = M_{k-1} M_k^0$, so that $x_{(k+2,0)} = x_{(k,0)}$.
2. For $\lambda > 1$, $\sigma_{(k,p)}$ is made of disjoint closed intervals, equal in number to the degree of $x_{(k,p)}$. These intervals are called bands.
3. $\sigma_{(k+2,0)} \cap \sigma_{(k+1,0)} \subset \sigma_{(k,0)} \cap \sigma_{(k+1,0)}$.
4. $\sigma(H) = \bigcap \{ \sigma_{(k,0)} \cap \sigma_{(k+1,0)} \}$.
5. If $k \in \mathbb{N}$ and $p \geq -1$, $\sigma_{(k,p+1)} \subset \sigma_{(k+1,1)} \cap \sigma_{(k,p)}$.
6. If $k \in \mathbb{N}$, $p \geq -1$, and $\lambda > 1$, $\sigma_{(k+1,0)} \cap \sigma_{(k,p+1)} \cap \sigma_{(k+1,0)} = \emptyset$.

We will approximate $\Sigma$ using a certain subset of the above bands, called the generating bands. For $k \in \mathbb{N}$, define:

1. A band of type $(k, I)$ is a band of $\sigma_{(k,1)}$ contained in a band of $\sigma_{(k,0)}$.
2. A band of type $(k, II)$ is a band of $\sigma_{(k+1,0)}$ contained in a band of $\sigma_{(k,-1)}$.
3. A band of type $(k, III)$ is a band of $\sigma_{(k+1,0)}$ contained in a band of $\sigma_{(k,0)}$.

For each value of $k$, call the collection of all bands of the above three kinds the spectral generating bands at level $k$, written $\mathcal{G}_k$. These bands are useful because the combinatorial structure of $\mathcal{G}_{k+1}$ is easy to compute if $\mathcal{G}_k$ is known.

Lemma 2.2. For $k \in \mathbb{N}$,

1. Each band of type $(k, I)$ contains a single generating band; it is a band of $\sigma_{(k+2,0)}$ of type $(k+1,II)$.
2. Each band of type $(k, II)$ contains $a_k + 1$ bands of $\sigma_{(k+1,1)}$ of type $(k+1, I)$, and $a_k$ bands of $\sigma_{(k+2,0)}$ of type $(k+1, III)$.
3. Each band of type $(k, III)$ contains $a_k$ bands of $\sigma_{(k+1,1)}$ of type $(k+1, I)$ and $a_k - 1$ bands of $\sigma_{(k+2,0)}$ of type $(k+1, III)$.

To make use of Lemma 2.2 we have to understand how $\mathcal{G}_k$ approximates $\Sigma$: 

FIGURE 1. An illustration of Lemma 2.2 for $a_i \equiv 3$.

Lemma 2.3. Every generating band of level $k$ is contained in some generating band of level $k-1$, and $\Sigma = \bigcap_{k=0}^{\infty} \bigcup_{B \in \mathcal{G}_k} B$.

Let us recast the content of Lemma 2.2 in a form that is more useful for calculations. Put

$$T_k = \begin{bmatrix} 0 & 1 & 0 \\ a_k + 1 & 0 & a_k \\ a_k & a_k - 1 & 0 \end{bmatrix}.$$ 

Then Lemma 2.2 says that for $i, j \in \{I, II, III\}$, a band of type $(k, i)$ spawns $T(i, j)$ bands of type $(k+1, j)$. Given a band $B \in \mathcal{G}_k$, we can associate to it a type index, which is
the sequence of the types (I, II, or III) of each of its forbears. That is, for $B \in \mathcal{G}_k$, put
\[
\tau(B) = i(0), i(1), \ldots, i(k)
\]
where $B$ is of type $(k, i(k))$ and $B$ is contained in a band of type $(l, i(l))$ for each $l < k$. Using the matrices $T_l$, the number of bands with a given type index $\tau$ is seen to be $\prod_{l=0}^{k-1} T(\tau(l), \tau(l+1))$. Define $\mathcal{A}$ to be the set of all one-sided infinite words $\tau$ on \{I, II, III\} such that each prefix of $\tau$ is the type index of at least one band. Such type sequences are called admissible.

Because of the self-similarity of $\Sigma$, estimates for the lengths of generating bands are amenable to a similar formalism. Put
\[
P_k = \begin{bmatrix}
0 & (3/(\lambda - 8))^{a_k - 1} & 0 \\
3/(a_k(\lambda - 8)) & 0 & 3/(a_k(\lambda - 8)) \\
3/(a_k(\lambda - 8)) & 0 & 3/(a_k(\lambda - 8))
\end{bmatrix},
\]
\[
Q_k = \begin{bmatrix}
0 & (1/(\lambda + 5))^{a_k - 1} & 0 \\
(\lambda + 5)^{-1}(ak+2)^{-3} & 0 & (\lambda + 5)^{-1}(ak+2)^{-3} \\
(\lambda + 5)^{-1}(ak+2)^{-3} & 0 & (\lambda + 5)^{-1}(ak+2)^{-3}
\end{bmatrix}.
\]

**Lemma 2.4.** Then if $B$ is a generating band and $\tau$ its type,
\[
4 \prod Q_k(\tau(l), \tau(l+1)) \leq |B| \leq 4 \prod P_k(\tau(l), \tau(l+1)).
\]

3. Band Length Estimates

In this section we consider the asymptotic scaling rate of bands in $\mathfrak{g}_k$. It is easy to treat the case $\beta = [0; b, b, b, \ldots]$ quantitatively, so we take it up first.

**Lemma 3.1.** If $\beta = [0; b, b, b, \ldots]$, then for all $k$ and $B \in \mathcal{G}_k$,
\[
\log \frac{|B|}{4} \geq \begin{cases}
-\frac{k}{2} (b - 1) \log(\lambda + 5) - 3 \left[ \frac{k}{2} \log(b + 2) - \left| \frac{k}{2} \right| \log(\lambda + 5) \right], & b > 3 \\
-k \log(\lambda + 5) - 3k \log(b + 2), & b = 2, 3 \\
\left[ \frac{2k}{3} \right] (-\log(\lambda + 5) - 3 \log(b + 2)), & b = 1.
\end{cases}
\]

Let $L(k)$ be the above bound (that is, $|B| \geq L(k)$), and write $L(b, \lambda)$ for $\liminf_{k \to \infty} \frac{1}{k} \log L(k)$.

**Proof.** Our task is to bound the lower bound in Lemma 2.4 from below as $B$ ranges over all of $\mathfrak{g}_k$. A look at the matrix $Q_k$ shows that if $b \geq 4$ the bound in question is minimized whenever $\tau(B)$ has the greatest admissible amount of Is in it. This can occur at most half the time (by Lemma 2.2), proving the formula in the first case.

On the other hand, when $b < 4$ the minimum is achieved by a band with the greatest possible amount of Is and Ids in its type index. When $b \neq 1$, this is possible for every entry of $\tau(B)$, giving the second case, and for $b = 1$ it is possible $2/3$ of the time.

\[\square\]

**Lemma 3.2.** If $\beta = [0; b, b, b, \ldots]$, then for each $k$, there is a band of $\mathcal{G}_k$ with
\[
\log \frac{|B|}{4} \leq \begin{cases}
-\frac{k}{2} (b - 1) (\log(\lambda - 8) - \log 3) - \left| \frac{k}{2} \right| (\log b + \log(\lambda - 8) - \log 3), & b > 2 \\
-k (\log(\lambda - 8) - \log(b) + \log 3), & b = 2 \\
\left[ \frac{2k}{3} \right] (\log(\lambda - 8) - \log(b) + \log 3), & b = 1.
\end{cases}
\]

Let $U(k)$ be the above bound and define $U(b, \lambda) = \limsup_{k \to \infty} \frac{1}{k} \log \frac{U(k)}{4}$. 

Lemma 4.1. Recalling that $\sigma_k$ is the average of the first $k$ coefficients of $\beta$. Since $\log x$ is bounded above and below by geometric sequences.

Proof. Suppose not, so that for some faster-than-geometric sequence $a_0$ such that for some faster-than-geometric sequence $d_k$ approaches zero faster than any geometric sequence. □

Lemma 3.3. If $d$ is infinite, the sequence of lengths of shortest bands of $\sigma_{k+1,0}$ decays to zero faster than any geometric sequence.

Proof. We argue roughly as in the proof of Lemma 3.2. Indeed, $d$ is infinite, the sequence of lengths of shortest bands of $\sigma_{k+1,0}$ decays to zero faster than any geometric sequence.

Proof. We argue roughly as in the proof of Lemma 3.2. Indeed, $d = \infty$ means there is a $c > 0$ such that in each $\sigma_{k+1,0}$ there is a band whose length is less than $\exp(-ckd_k)$ (where $d_k$ is the average of the first $k$ coefficients of $\beta$). Since $c$ is independent of $k$ and $d_k$ diverges, $\exp(-ckd_k)$ approaches zero faster than any geometric sequence. □

Lemma 4.1. Recalling that $M(\beta) < \infty$, the sequence $q_k$ of denominators of convergents to $\beta$ is bounded above and below by geometric sequences.

Proof. By definition, $q_{k+1} = a_{k+1}q_k + q_{k-1}$.

Since $q_k$ increases monotonically, it follows that $q_{k+1} \leq (a_{k+1} + 1)q_k$. Thus

$$\log q_{k+1} \leq \sum_{i=1}^{k} \log (a_{i+1} + 1).$$

Since $\log (x+1) \leq \log(x) + 1/x$ and $\sum_{i=1}^{k} 1/a_i$ grows no faster than $k$, $\log q_k$ is bounded above by an arithmetic sequence.

Again using monotonicity of $q_k$, we get $q_{k+1} \geq a_{k+1}q_k$. Repeating the above reasoning, we finish the proof. □

Proof of Theorem 1.1. For arbitrary $x_0 < y_0$ that are close enough, we want to estimate $N(y_0) - N(x_0)$ from above. Because there is one Dirichlet eigenvalue of $H_{q_k}$ associated to each band of $\sigma_{k+1,0}$ (8),

$$N(y_0) - N(x_0) = \lim_{n \to \infty} \frac{|\sigma(H_{q_k}) \cap [x_0,y_0]|}{n} = \lim_{k \to \infty} \frac{|\sigma_{k+1,0} \cap [x_0,y_0]|}{q_k},$$

where $|X|$ is the cardinality of $X$. This amounts to finding a bound on $|\sigma_{k+1,0} \cap [x_0,y_0]|$. With $L$ the bound on band lengths of Lemma 3.1, define $m$ by

$$L(m+1) \leq y_0 - x_0 < L(m).$$

Because $N$ is supported on $\Sigma$, the interval $[x_0,y_0]$ can be replaced with $[x_0,y_0] \cap \Sigma$. Every point of $\Sigma$ is contained in a generating band (Lemma 2.3), so it is not a loss to assume $[x_0,y_0]$ is contained in a generating band. Then, by the definition of $m$, there is a band
[x, y] ∈ G_m containing it (notice that y − x is comparable in size to y_0 − x_0, because of how m is defined). We now have

\[ N(y_0) - N(x_0) \leq \lim_{k \to \infty} \frac{\# \sigma_{k+1,0} \cap [x, y]}{q_k}. \]

This ratio scales like 1/q_m. Indeed, it is equal to 1/q_m at k = m. As k increases to infinity, Lemma 2.2 shows that every band of \( \sigma_{m+1,0} \) produces a roughly constant proportion of the bands that comprise \( \sigma_{k+1,0} \). This means the share of bands in \( \sigma_{k+1,0} \) produced by \([x, y]\) remains practically constant. So, pick \( C \) so that \( N(y_0) - N(x_0) \leq C/q_m \).

Define \( \gamma \) by

\[ \gamma = \frac{\log C + \log q_k}{\log L(k + 1)}, \]

so that \( C/q_m = L(m + 1)^{\gamma m} \), which is by the definition of \( m \) less than \( (y - x)^\gamma m \).

Take any

\[ 0 < \gamma \leq \liminf_{k \to \infty} \gamma = \frac{\log \beta}{L(b, \lambda)}, \]

and choose \( k_0 \) so that \( \gamma_k > \gamma \) for \( k > k_0 \). Put \( \delta = L(k_0) \). Then if \( a < b \) satisfy \( b - a < \delta \), \( N(b) - N(a) \leq (b - a)^\gamma m \), where \( m \) is the integer corresponding to \( [a, b] \). This is less than \( (b - a)^\gamma \).

\[ \square \]

**Proof of Theorem 1.2**. Recall \( U(k) \) from Lemma 3.2; \( \log U \) goes to \( -\infty \) roughly linearly.

Let a supposed Hölder exponent \( \hat{\gamma} \) be given, satisfying the hypotheses of the theorem. Given \( \delta \), pick \( k_0 \) so that \( U(k_0) < \delta \) and

\[ \gamma_m = \frac{-m \log \beta}{\log U(m)} \]

is less than \( \hat{\gamma} \) for all \( m \geq k_0 \). This is possible for all \( \gamma > \log \beta / U(b, \lambda) \). Now, choose \([x, y] \in \sigma_{k_0} \) so that \( y - x \geq U(k_0) \).

The exponent \( \gamma_m \) is constructed so that \( U(m)^{\gamma_m} = \beta^m \). We have already seen in the proof of [1.1] that

\[ N(y) - N(x) \leq \lim_{k \to \infty} \frac{\# \sigma_k \cap [x, y]}{q_k} \simeq 1/q_{k_0}. \]

By construction, this is greater than \( (y - x)^\gamma m > (y - x)^\gamma \).

\[ \square \]

**Proof of Theorem 1.4**. We first prove that \( N \) is Hölder continuous when \( d \) is finite. The proof of Theorem 1.1 may be followed until [1.1]. By Lemma 3.4 \( \log L(k) \) lies between two arithmetic sequences. And by Lemma 4.1, the same goes for \( \log q_k \). This implies \( \liminf_{k \to \infty} \gamma_k > 0 \), which is the necessary input to obtain Hölder continuity.

Now assume \( d = \infty \). We will follow the framework of the previous proof of Theorem 1.2 and see that the optimal Hölder exponent is zero. Let \( U(k) \) stand for the sequence of upper bounds on band length obtained in Lemma 3.3; \( \log U \) goes to \( -\infty \) faster than any linear function. Also, recall that, by Lemma 4.1, there is an \( R \) so that \( R^m \) grows faster than \( q_m \).

Let a supposed Hölder exponent \( \gamma > 0 \) be given. Given \( \delta \), pick \( k_0 \) so that \( U(k_0) < \delta \) and

\[ \gamma_m = \frac{-m \log R}{\log U(m)} \]

is less than \( \gamma \) for all \( m \geq k_0 \). This is possible for all \( \gamma > 0 \) since \( \lim_{m \to \infty} \gamma_m = 0 \). Now, choose \([x, y] \in \sigma_{k_0} \) so that \( y - x \geq U(k_0) \).
The exponent $\gamma_m$ is constructed so that $U(m)^{\gamma_m} = R^{-m}$. Again,  
\[ N(y) - N(x) = \lim_{k \to \infty} \frac{\# \sigma_k \cap [x, y]}{q_k} \simeq 1/q_{k_0}. \]
Because of the hypothesis on $\beta$, $1/q_{k_0} \geq R^{k_0} = U(k_0)^{\gamma_{k_0}}$. By construction, this is greater than $(y - x)^{\gamma_{k_0}} > (y - x)^\gamma$, proving that $N$ is not Hölder continuous.

\[ \square \]

ACKNOWLEDGEMENT

The author is grateful to his advisor, David Damanik, for many helpful discussions and comments.

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