Ray-Tracing studies in a perturbed atmosphere: I- The initial value problem.

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(Dated: March 28, 2001)

We report the development of a new ray-tracing simulation tool having the potential of the full characterization of a radio link through the accurate study of the propagation path of the signal from the transmitting to the receiving antennas across a perturbed atmosphere. The ray-tracing equations are solved, with controlled accuracy, in three dimensions (3D) and the propagation characteristics are obtained using various refractive index models. The launching of the rays, the atmospheric medium and its disturbances are characterized in 3D. The novelty in the approach stems from the use of special numerical techniques dealing with so called stiff differential equations without which no solution of the ray-tracing equations is possible. Starting with a given launching angle, the solution consists of the ray trajectory, the propagation time information at each point of the path, the beam spreading, the transmitted (resp. received) power taking account of the radiation pattern and orientation of the antennas and finally, the polarization state of the beam. Some previously known results are presented for comparative purposes and new results are presented as well as some of the capabilities of the software.

I. INTRODUCTION

Multipath propagation is believed to be the major cause of data transmission impairments in terrestrial line of sight microwave radio systems. Efficient antenna design requires the understanding of the propagation of individual rays across the channel and gauging the refractive index of the various atmospheric disturbances any given ray encounters during its propagation. Adopting a refractive index model for a given disturbance arising from spatial fluctuations in humidity, pressure or temperature (these fluctuations might be temporal as well, but we shall consider, for the time being, that the propagation time occurs on a time scale much smaller than the one associated with these fluctuations), we establish the ray propagation equations and solve them with several numerical techniques having a first, fourth and sixth order accuracy. The ray tracing equations are initially solved in two dimensions bypassing the effects of small and non-linear terms as explained in section 2. Later on, we switch to 3D in order to assess the effects the small and non-linear terms have on ray propagation. Several facts emerge from this approach:

- The small non-linear terms lead to a breakdown of standard integration techniques. The ray equations which constitute a system of 6 ordinary coupled non-linear differential equations become stiff. This means the integration step becomes so small (because of the presence of terms that differ by several orders of magnitude) making the integration process so slow that any progress in seeking a solution of the system is virtually stopped.
- The relation between the launching and arrival angles for a given disturbance are profoundly altered. What was previously believed to be a "good" or "bad" launching angle might have gotten its true attributes from reasons different from what is currently known.
- A very high sensitivity is observed around certain launching angles: a very small uncertainty in the launching angle can induce the ray to take a path radically different from what is normally expected.

This report is organized in the following way: In section 2, we establish the ray-tracing equations (RTE). In section 3 we describe some of the problems encountered during the solution of the RTE, namely those related to stiffness and present the algorithms to cure them (Appendix A contains a description and an example of a stiff system). In section 4 we compare our approach to previous ones and present some illustrative new cases in section 5. This section also describes the potential applications of the software and its capabilities. Section 6 discusses some possibilities for future developments. Appendix B shows how to avoid stiff differential equations in two dimensions and turn the RTE into a set of recursion relations.

II. RAY TRACING EQUATIONS

In terrestrial microwave radio systems, the range of frequencies used and in comparison the range of length scales present in the channel allow us to use a geometric (or ray) approach to electromagnetic propagation. The
fundamental equation of geometrical optics is the Eikonal equation:

$$(\mathbf{grad} S)^2 = n^2$$  \quad (1)$$

where $n$ is the local refractive index and $S$ is the local phase of the ray. Taking the gradient of both sides of the Eikonal equation gives the second order vector propagation equation:

$$\frac{d(\mathbf{2ndt})}{ds} = \mathbf{grad}n$$  \quad (2)$$

where $\mathbf{R}$ is the ray position and $ds$ is a differential displacement along the ray path, i.e. $ds = ||d\mathbf{R}||$, the norm of the vector $d\mathbf{R}$. This can be rewritten as a system of two first order equations:

$$\frac{d\mathbf{R}}{ds} = \mathbf{T}$$

$$\frac{d(n\mathbf{T})}{ds} = \mathbf{grad}n$$  \quad (3)$$

where $\mathbf{T}$ is a unit vector tangent to the ray path (The geometry is depicted in Fig.1). The advantage of solving a first order system rather than a single second order system is threefold:

- Stability problems are easier to handle.
- Validity of the solution is easy to monitor since one has to have for all times $||\mathbf{T}|| = 1$ providing a simple means to check the quality of the integration procedure.
- Accuracy of the solution is controlled within certain tolerance limits depending on the selected integration step.

This is discussed in detail in section 5. The refractive index function of the atmosphere is written as:

$$n = 1 + 10^{-6}N$$  \quad (4)$$

where $N$ depends on the frequency used, humidity conditions and height above the Earth ground. Several models exist for the range of frequencies and heights we are dealing with and are generally expressed in $N$ units. The following two models are of interest; the first for a normal atmosphere and the second for a disturbed one:

- Exponential model: $N = 315 \exp(-0.136h)$, with $h$ (height) in kms.
- Webster model: $N = 300 + kh + \frac{\Delta n}{\pi} \atan(12.63(k-h) \Delta h)$

where $k$ is the refractive index gradient with height $h$. The $\atan()$ term above is due to a disturbance located at a height $h_0$ having an extent $\Delta h$ and a refractive strength $\Delta n$. For a normal atmosphere ($\Delta n = 0$ in the Webster model) both models are linear in $h$ (after expanding the exponential to first order). Nevertheless, their dependence solely on height does not account for the 3D nature of the atmosphere and its disturbances. Some models like the recent one introduced by Costa[4] mimics a 3D atmospheric disturbance by multiplying the refractive index along the vertical with a Gaussian function along the horizontal perpendicular to the ray path plane. Going beyond these approaches, we introduce a full 3D profile:

$$N = p_x(x)p_y(y)p_h(h) + kh + N_0$$  \quad (5)$$

where $p_x, p_y$ and $p_h$ are the index profiles of the disturbance along the three directions in space $x$, $y$ and $h$. $N_0$ is an average normal atmosphere index and $k$ is the index gradient along the height. A profile function $p(X)$, along direction $X$ is typically taken as:

$$p(X) = (\Delta n_x/2)[\tanh((X-X_1)/\Delta X_1) - \tanh((X-X_2)/\Delta X_2)]$$  \quad (6)$$

where $X_1$ (resp. $X_2$) is the point where the hump starts growing (resp. decaying) and $\Delta X_1$ (resp. $\Delta X_2$) is a typical length scale for the growth (resp. decay). $\Delta n_x$ is the refractive strength of the disturbance. This model, though realistically representing a localized anisotropic disturbance in the atmosphere is based on a separable model of the refractive index function.

While our methodology can handle any arbitrary 3D model of the refractive index, any of these refractive models have to be modified in order to take account of the curvature of the Earth by the inclusion of a term \(1/2\) equal to $10^6 h/R_e$ where $R_e$ is the radius of the Earth.

### III. STIFF DIFFERENTIAL EQUATIONS ALGORITHMS

Using [4], the ray-tracing system [3] is rewritten as:

$$\frac{d\mathbf{R}}{ds} = \mathbf{T}$$

$$\frac{d\mathbf{T}}{ds} = [\mathbf{grad}N - \mathbf{T}(\mathbf{grad}N, \mathbf{T})]/(N + 10^6)$$  \quad (7)$$

Two important features appear in the RHS of the second equation in the system:

- The non-linear term in $\mathbf{T}$.
- The wide range of orders of magnitudes in the denominator.
These terms can be eliminated with the following procedure: Replace equation [7-b] by another equation defining the curvature of the ray path \( r \):

\[
\frac{dT}{ds} = \frac{U}{\rho}
\]  

(8)

where \( U \) is the normal to the trajectory. \( U \) is perpendicular to \( T \) and normalized: \(|U| = 1\). The unknown \( \rho \) can be determined by taking the scalar product of both sides of [7-b] with \( U \) and using [8]; one gets:

\[
1/\rho = U \cdot \nabla N / (N + 10^6)
\]  

(9)

Substituting [9] in [8] gives the following system:

\[
\frac{dR}{ds} = T
\]

\[
\frac{dT}{ds} = U(U \cdot \nabla N) / (N + 10^6)
\]  

(10)

In general, this system is not closed because it involves \( U \) besides \( R \) and \( T \). In two dimensions, one can close the system by invoking the orthogonality of \( U \) and \( T \) through:

\[
U = x \times T
\]  

(11)

where \( x \) is the unit vector along the x direction. With relation [11], system [10] is now closed and can be integrated by any standard explicit integration method (Predictor-corrector, Euler, Runge-Kutta, Richardson etc...). This will be illustrated in section 4. In general, \( N \) is a function of the position vector \( R \); when it depends only on the height, it is possible to further simplify the system and reduce it to a single scalar equation. In the case \( N \) depends only on height, \( \nabla N \) is along the vertical and if \( \psi \) is the angle \( T \) makes with the local horizontal, \( U \) being perpendicular to \( T \) will make the same angle with the vertical, [9] yields:

\[
\frac{1}{\rho} = |(dN/dh)\cos\psi| / (N + 10^6)
\]  

(12)

Livingston has derived an equation similar to [12]:

\[
\frac{1}{\rho} = -(1/n) (dn/dh)\cos\psi
\]  

(13)

Equation [13] is equivalent to [12] when the right sign is used. We have integrated system [10] in two dimensions and recovered typical results found in the literature, avoiding the difficulty arising from [7-b]. In the three dimensional case, one has to deal directly with system [7] with all terms retained, for, in general, the \( T \) vector does no longer have to be confined to the transmitter (TX) receiver (RX) plane. In this case, all standard explicit integration schemes break down. In other words, the norm of the vector \( T \) tangent to the ray path is no longer conserved. In order to fulfill the condition \(|R| = 1\), one has to take an integration step so small that the integration process is virtually stopped. This is called stiffness and an illustrative example is given in Appendix A.

Stiffness can be cured with the so called implicit integration schemes. In contrast to explicit integration schemes where a current system value depends only on the previous ones, implicit schemes couple present and past values of the system altogether. A price to pay is an increase in CPU time but the rewards are stability, accuracy and large integration steps. We have implemented two implicit schemes:

- Generalized Runge-Kutta (GRK) method of fourth order.
- Rosenbrock (ROW) method of sixth order.

In the first scheme, given a system of first order ordinary differential equations (ODE):

\[
dy/ds = f(y)
\]  

one builds the vectors from the system values at step \( n-1 \):

\[
k_i = \sigma f(y_{n-1} + \sum a_{ij} k_j) \quad \text{with: } i, j=1...m
\]  

(15)

and evaluates the next value \( n \) of the system with:

\[
y_n = y_{n-1} + \sum b_i k_i
\]  

(16)

\( \sigma \) is the integration step and the \( a_{ij} \) and \( b_i \) are coefficients depending on the scheme \( m \) of the integration order. In the Rosenbrock case, one adds to [15] the term \( \sigma (\partial f/\partial y) \sum d_{ij} k_i \), where the \( d_{ij} \)s are order dependent coefficients and \( (\partial f/\partial y) \) is the Jacobian of the system. The above equations are implicit since the unknown vectors \( k_i \) needed for integration step \( n \) appear on both sides of [15]. In the GRK method, only the vector function \( f \) is needed whereas in the ROW case both \( f \) and its first order derivative (Jacobian) are needed.

Both methods have been proven to perform very well up to stiffness parameters (ratio of the highest to the smallest eigenvalue of the Jacobian) as high as 10\(^7\). Incidentally, our stiffness parameter has been observed (while testing ROW algorithms) to be generally around 10\(^4\). We have used GRK of order 4 and ROW of order 6 because they have been extensively tested for a wide range of systems and are thoroughly documented.
IV. VALIDATION OF THE APPROACH AND COMPARISONS WITH PREVIOUS TREATMENTS

In order to validate our technique, we started with a comparison against analytically known solutions. Three models were tested, the sine-wave optical paths and the classical Luneburg lens (see, for instance, reference 7). In all three cases our results compared very accurately with the analytical ones. Then we went ahead and proceeded to solve in detail a case well documented in the literature and investigated by Webster [2] for various launching angles. This model is two dimensional (2D) and extensively referred to in the literature. We use the 2D version of the system of equations [10] which is non-linear (N is a non-linear function of R and a power of U appears in [10-b]).

The integration, started by taking values of R and T as the initial location and launching vectors, is done with a first-order Euler and fourth order Runge-Kutta methods. The TX-RX configuration and propagation conditions are the same as those given in Table 1 of Webster’s [2] paper. In Fig.2 we show the various ray paths between the TX and the RX for a series of launching angles (taken with respect to the horizontal) varying from -0.25 up to 0.5 degrees. The different launching angles, we use, are respectively, in degrees: -0.25, -0.20, -0.15, -0.10, -0.05, 0.0, 0.10, 0.20, 0.30, 0.40, 0.50. The refractive index profile used in the study is displayed in Fig.3.

While Fig.2 is based on a first order (Euler) integration method, some changes might occur if we rather use a fourth order Runge-Kutta method. In fact, the ray paths based on either scheme show no appreciable differences and compare well with the results found earlier by Webster in the same conditions. However, some discrepancies appear for positive launching angles and are probably due to the different levels of numerical accuracy between our treatment and Webster’s. Let us recall that in our case the numerical accuracy is monitored by checking the conservation of the norm of T. In these simulations, it is conserved with an error smaller than 10^{-7}. In order to compare our results to Webster’s directly, we derive, in the same fashion, recursion equations for the ray radial distance R (taken from the center of the Earth) and the angle ψ that T makes with the local horizontal. Referring to Appendix B and Fig.4, we can write the following relations:

\[ R_2 = R_1 + ds \sin(\psi_1) \]  \( (17) \)

\[ \psi_2 = \psi_1 + ds \frac{\cos(\psi_1)}{R_1} - \sin^{-1}\left(\frac{ds}{\rho_1}\right) \]  \( (18) \)

where the radius of curvature \( \rho_1 \) is given by [12] with \( \psi = \psi_1 \) and \( dN/dh \) is taken at the height \( R - R_e \) (\( R_e \) is the Earth radius). For a given step ds, one starts the set of iterations [17] and [18] with the launching radial distance \( R_1 \) and angle \( \psi_1 \). Using the same initial values as before we retrieve almost the same ray trajectories obtained in Fig.2. The validity of our results is monitored by the constancy of the modulus of T versus 1. Additionally, we compared our results (Euler and Runge-Kutta) to a very high accuracy integration technique based on the Butcher’s [8] algorithm (seven-stage sixth-order Runge-Kutta scheme). The sixth order results are virtually identical to the fourth order’s and Fig.5 depicts the ray trajectory obtained with the different levels of accuracy under the same atmospheric and launching conditions. Incidentally, the difference between fourth and sixth order trajectories in Fig.5 are on the order of a fraction of a millimeter. In spite of the above agreement, which is basically relative, one still has to gauge independently the accuracy of the results for a selected order and integration step. This is done with the following method: Pick an order \( p \) and an integration step \( \sigma \); integrate once with \( \sigma \) and twice with \( \sigma/2 \) in order to reach the same point; define a step ratio \( \kappa \) from the difference \( \Delta \) between the two results:

\[ \kappa = \frac{r^*\sqrt{2^p/(2^p-1)(\Delta/\epsilon)}}{p} \]  \( (19) \)

and monitor the value of \( \kappa \) for a given tolerance, during integration. Ideally, we should have \( \kappa \leq 2 \). In Fig.6, we display \( \kappa \) versus the integration step number for the first order (Euler, \( p=1 \)) case as well as the Runge-Kutta 4-th order (\( p=4 \)) and Butcher 6-th order (\( p=6 \)) for a tolerance of 1 millimeter. We use exactly the same condition as previously and a launching angle of 0.2 degrees. The figure shows clearly the superiority of 4-th and sixth order methods for the selected step when such a high accuracy is desired.

V. ILLUSTRATIVE RESULTS AND CAPABILITIES OF THE METHODOLOGY

We move on to the description of the 3D propagation case and show, with a simple example, how we evaluate the power from the antenna radiation pattern, the beam spreading and the state of polarization. We select a coordinate system such that the TX is somewhere on the z-axis whereas the y-axis is along the TX-RX line. The vertical plane is defined by the z axis and the TX-RX line. The beam spreading is evaluated by launching simultaneously several beams in the vertical and horizontal planes with angles differing by a small amount from those characterizing the main beam. The logarithm of the ratio of the surfaces swept by the different beams at the receiver location gives an estimate of the spreading loss. In order to account for the TX-RX antenna radiation pattern, we simply recall that the electric field radiated by a parabolic circular aperture antenna at a
point defined by its distance \( r \) from the main lobe origin and making an angle \( \theta \) with the lobe axis is given by:

\[
E(r, \theta) = j\beta E_0 a [\exp(-j\beta r)/r] J_1(\beta a \sin \theta)/\beta \sin \theta \tag{20}
\]

where \( a \) is the aperture radius, \( E_0 \) is a reference field, \( \beta = 2\pi/\lambda \) with \( \lambda \) the wavelength used, \( J_1 \) is the Bessel function of the first kind and \( j = \sqrt{-1} \). The antenna pattern is obtained after normalizing the value of \( |E(r, \theta)| \):

\[
f(\theta) = (2/\beta a)[J_1(\beta a \sin \theta)/\sin \theta] \tag{21}
\]

Alluding to our choice of axes, if the main lobe is pointing in a direction defined by the angles \( \beta, \gamma \) (in the vertical and horizontal plane respectively) and we have a ray along \( \beta', \gamma' \), the angle the ray makes with the main lobe axis is:

\[
\theta = \cos^{-1}(\cos \beta \sin \gamma \cos \beta' \sin \gamma' + \cos \beta \cos \gamma \cos \beta' \cos \gamma' + \sin \beta \sin \beta')(22)
\]

The power (in dB) is given by \( 20 \log_{10} f(\theta) \). The polarization state of a ray rotates, during propagation, by an angle calculated with the help of the following formula:

\[
\phi(A, B) = \int_A^B ds/\tau \tag{23}
\]

where \( A \) and \( B \) represent the two end points of the ray trajectory; \( \tau \), the local torsion of the ray is different from zero when the trajectory is not confined to a plane. Using the Frenet-Serret [1] formula:

\[
dB/ds = -\tau U \tag{24}
\]

Taking the dot product with \( U \) on both sides of equation [24] and replacing the value of \( \tau \) in [23], one gets:

\[
\phi(A, B) = -\int_A^B ds^2/(dB \cdot U) \tag{25}
\]

In order to evaluate the polarization rotation of the ray propagating from \( A \) to \( B \) with [25], a finite difference approximation \( B_n - B_{n-1} \) is used for the differential \( dB \), where the subscripts refer to the integration step. The final discrete formula for the polarization angle reads:

\[
\phi(A, B) = \sum_{n=1}^{N} ds^2/(B_{n-1} \cdot U_n) \tag{26}
\]

where \( N \) is the number of integration steps between \( A \) and \( B \). For illustration, we treat two 3D examples. In the first case, we take a refractive index model consisting of a refractive layer of finite length along the TX-RX line. The linear extent of the layer is taken respectively as 5, 10, 15, 20 and 25 kms. Fig.7 shows the dramatic effect the extent has on the ray path. Incidentally, the refractive index model along the height is taken as the same Webster model as before and the ray launching is made in the vertical plane. In the second case, we take a refractive index model given by a Webster profile along \( z \) and a profile \( p_0(y) \) given by [5]. Moreover we take an arbitrary 3D launching direction. The resulting 3D ray trajectory for the selected parameters listed in the corresponding caption is displayed in Fig.8.

VI. CONCLUSIONS AND FUTURE DEVELOPMENTS

We intend to use this technique to study the dynamics of microwave radio signals controlled by unstable atmospheric layers. The instabilities cause short error bursts lasting from many tens of micro-seconds to a few milliseconds [10]. Since, the error bursts have detrimental impact on communication networks [11], the future digital radio systems should be made immune to radio propagation degradations causing them. In order to develop defense strategies against the error bursts caused by atmospheric propagation instabilities, the physical characteristics of the instabilities have to be well understood. This 3D ray-tracing technique will be used to study the effects of dynamically changing atmospheric layers of limited size on microwave radio signals received simultaneously by a few parabolic antennas [12]. A propagation model simulating the recorded dynamics of received radio signals will, not only, help understanding the physical causes of the error bursts, but it will also be used in the computer optimization of antenna designs capable of minimizing the frequency of occurrence of the propagation caused error bursts. Highly accurate numerical techniques are required since small fluctuations of the atmospheric conditions are believed to be responsible for the flat phase fluctuations impairing the digital demodulation of the received microwave radio signals.

APPENDIX A

Let us consider the following first order system consisting of a pair of linear ordinary differential equations:

\[
dy_1/dx = \lambda_+ y_1 + \lambda_- y_2 \tag{27}
\]

\[
dy_2/dx = \lambda_- y_1 + \lambda_+ y_2 \tag{28}
\]

where \( \lambda_+ = (\lambda_1 + \lambda_2)/2, \lambda_- = (\lambda_1 - \lambda_2)/2 \) and \( x \geq 0 \).

The solution of the system is:
\begin{equation}
y_1 = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}
\end{equation}
\begin{equation}
y_2 = C_1 e^{\lambda_1 x} - C_2 e^{\lambda_2 x}
\end{equation}

where $C_1$ and $C_2$ are constants determined by the initial condition at $x=0$. In order to conform to our notation of Section 3, we define a column vector $y$ whose components are $y_1, y_2$ and write the system as:

\begin{equation}
dy/dt = f(y)
\end{equation}

The eigenvalues of the Jacobian of the system:

\begin{equation}
\frac{\partial f}{\partial y} = \begin{bmatrix} \lambda_+ \lambda_- \\ \lambda_- \lambda_+ \end{bmatrix}
\end{equation}

are solutions of:

\begin{equation}
det[\lambda I - \left(\frac{\partial f}{\partial y}\right)] = 0
\end{equation}

where $I$ is the $(2x2)$ unit matrix; they are nothing else than $\lambda_1$ and $\lambda_2$. If one picks $\lambda_1 = -1, \lambda_2 = -1000$, and chooses an explicit integration method, one finds the integration step should be smaller than $2/|\lambda_2|$, which is $0.002$. This is the origin of stiffness: even though the term $\exp(-1000 \times)$ contributes almost nothing to the solution for $x \geq 0$, its presence alone, virtually stops the integration process.

\section*{APPENDIX B}

The geometry of propagation is shown in Fig.4. At any point along the ray trajectory the tangent vector $T$ makes the angle $\psi$ with the local horizontal. When the ray propagates between two nearby locations, one may write:

\begin{equation}
R_2 = R_1 + ds \sin(\psi_1)
\end{equation}
\begin{equation}
\psi_2 = \psi_1 + ds \frac{\cos(\psi_1)}{R_1} - \sin^{-1}\left(\frac{ds}{\rho_1}\right)
\end{equation}

where $ds = ||R_2 - R_1||$. The radial distance $R_1$ (resp. $R_2$) is taken from the center of the Earth. The angle $\delta \theta$ between the two radial directions may be found by inspection:

\begin{equation}
R_1 \sin(\delta \theta) = ds \cos(\psi_1)
\end{equation}

which can be approximated by:

\begin{equation}
\delta \theta = \frac{ds \cos(\psi_1)}{R_1}
\end{equation}

In order to find the relation between the angles $\psi_1$ and $\psi_2$, we use the relation defining the derivative of $T, dT/ds = U/\rho$ in a discrete form:

\begin{equation}
T_2 - T_1 = ds \frac{U}{\rho_1}
\end{equation}

Taking the scalar product with $U_1$ on both sides of above, one gets:

\begin{equation}
T_2 \cdot U_1 = ds/\rho_1
\end{equation}

The inspection of Fig.4 provides the angle between $T_2$ and $U_1$:

\begin{equation}
(T_2, U_1) = \psi_2 - \psi_1 - \delta \theta + \pi/2
\end{equation}

Using the above result gives the relation sought:

\begin{equation}
\psi_2 = \psi_1 + \delta \theta - \sin^{-1}(ds/\rho_1)
\end{equation}
chronous Digital Hierarchy compatible digital radio systems” Presented to ICC 93 (Geneva).

[12] C. Tannous and J. Nigrin: Ray-tracing studies in a perturbed atmosphere: II- The boundary value problem (to be published).

**Figure Captions**

Fig. 1: Geometry of the system showing the coordinate system, the antennas in the vertical yz plane, a typical ray path and the local Frenet-Serret system $(\mathbf{T}, \mathbf{U}, \mathbf{B})$ attached to a point along the path.

Fig. 2: Euler first order 2D results. The rays are launched in the vertical plane and the angle they make with respect to the horizontal xy plane is respectively: -0.25, -0.20, -0.15, -0.10, -0.05, 0.0, 0.10, 0.20, 0.30, 0.40, 0.50 degrees. Equations [10] are used along with model [4-b] for a perturbed atmosphere $N = 300 \pm kh + \Delta n$ with the same parameters as those given in Table 1 of reference 2: $k = -39$, $\Delta n = -20$ (both in N units), $h_0=175$ meters, $\Delta h = 100$ meters, the transmitter height is 125 meters and the TX-RX separation is 60 kms.

Fig. 3: Webster model refractive index function (in N units) along the vertical showing an anomaly at a height of 175 meters and whose width is equal to 100 meters. The curvature of the Earth term $10^6h/R_e$ is present. The negative gradient of the layer refractive index is responsible for the multipath effects observed.

Fig. 4: Geometry of the ray trajectory used for establishing the recursion equations. The local tangent $\mathbf{T}$ vectors are shown making the angle $\psi$ with the local horizontal perpendicular to the ray vectors $\mathbf{R}$ drawn from the center of the Earth O. Two neighboring points along the ray paths are shown.

Fig. 5: Comparative study of the ray trajectories obtained from the recursion relations [17] and [18] (uppermost long dashed curve) and 1st order Euler (full line curve) on one hand, and the Runge-Kutta (4-th order) and Butcher (6-th order) on the other (short dashed curve). The launching angle in all cases is 0.2 degrees in the vertical plane and the model considered for the refracting layer is the same as Figure 2. The fourth and sixth order results are virtually identical.

Fig. 6: Comparative study of the behavior of the step ratio versus step number for the Euler (1st order), the Runge-Kutta (4-th order) and the Butcher (6-th order) methods when the step is fixed to its starting value. Ideally, this ratio should always be about 2. In the first order case, the bound is violated very rapidly (upper curve), whereas it is respected until almost the end of the trajectory in the 4-th (long dashed curve) and 6-th order (short dashed curve) cases. The tolerance is 1 mm and the step used is one hundredth the TX-RX distance.

Fig. 7: GRK (Implicit, 4-th order, 3D) results for the ray trajectories when the extent of the layer is a variable. Starting with a launching angle of 0.2 degrees in the vertical plane, the layer spans, initially, the entire hop of 60 kms (lowest curve). Moving upward from the next lower curve, the layer extent (along the TX-RX line) is from 5 to 25 kms, then 5 to 20 kms, 5 to 15 kms and finally 5 to 10 kms. In all cases, the refracting layer model is the same as in Figure 2.

Fig. 8: GRK (Implicit, 4-th order, 3D) results for the ray trajectory when the refractive index of the layer varies along two spatial directions (y and z) and round Earth profile considered. The normal atmosphere parameters are $N_0=300$ N units and the gradient $k=39$ N units/km. The 3D refractive index layer is described with a profile along y given by \[\tanh((y - y_1)/\Delta y) - \tanh((y - y_2)/\Delta y)/2\] with $y_1=0$ km, $y_2=60$ kms, $\Delta y = 100$ meters and a Webster profile along z given by $\Delta n \tan h[12.63(z - h_0)/\Delta h]/\pi$ with $h_0 = 175$ meters, $\Delta h = 100$ meters and $\Delta n = -20$ N units. The launching angles are 0.1, 0.2, 0.3, 0.4 and 0.5 degrees in the vertical plane with 0 and 0.001 degrees in the horizontal plane. The TX is at 125 meters along the z axis and the TX-RX antenna separation is 60 kms.
This figure "fig8.png" is available in "png" format from: http://arxiv.org/ps/physics/0104004v1