Equivalence transformations of Euler-Bernoulli equation

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Abstract

We give a determination of the equivalence group of Euler-Bernoulli equation and of one of its generalizations, and thus derive some symmetry properties of this equation.

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1. Introduction

Consider a collection $\mathcal{F}$ of differential equations of the form

$$\Delta(x, y_{(n)}; \mathcal{A}) = 0,$$  \hspace{1cm} (1)

where $x = (x^1, \ldots, x^p)$ is the set of independent variables and $y_{(n)}$ denotes $y$ and all its derivatives up to the order $n$, while the parameter $A$ denotes collectively the set of all arbitrary functions specifying the family element in $\mathcal{F}$. On the other hand, let $G$ be the Lie pseudo-group of point transformations of the form

$$x = \varphi(z, w), \quad y = \psi(z, w),$$  \hspace{1cm} (2)

where $z = (z^1, \ldots, z^p)$ is the new set of independent variables, and $w = w(z)$ is the new dependent variable. As Tresse explained in [1, P. 11], the elements of $G$ depend in general on arbitrary functions and not on arbitrary constants, and

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$G$ is therefore infinite-dimensional. We say that $G$ is the group of equivalence transformations of (1) if it is the largest Lie pseudo-group of point transformations that map (1) into an equation of the same form, that is, if in terms of the same function $\Delta$ appearing in (1), the transformed equation has an expression of the form

$$\Delta(z, w_{(n)}; B) = 0,$$

where $B$ denotes collectively the new set of arbitrary functions of the equation. Equivalence groups play an important role in, amongst others, the classification of differential equations by means of their invariant functions, and they are also a valuable tool for the identification of these invariant functions [2, 3, 4, 5].

In this paper we obtain the equivalence transformations of Euler-Bernoulli equation (EB). The difficulty with this equation is due to the high order of derivatives that it involves, especially in the two-dimensional case, and this makes the standard method of finding the equivalence transformations by direct analysis intractable. We therefore recourse to the transformation of differential operators that we discuss in the next section. Next, we proceed to the determination of the equivalence group of (EB) and of one of its generalizations, and derive some symmetry properties of this equation. This includes the full symmetry group of the equation, which was obtained only partially in [6] in terms of some undetermined set of functions.

2. Transformation of differential operators

An expression of differential operators in terms of new variables yields an effective algorithm for implementing the transformation of a differential equation under a given change of variables, and this can also greatly simplify calculations, based on the properties of these differential operators.

Suppose for instance that the general change of variables [2] that maps (1)
into (3) is given more explicitly in the form

\[ x^i = \varphi_i(z, w), \quad z = (z^1, \ldots, z^p), \quad w = w(z) \]  
\[ u = \psi(z, w) \equiv T(z). \] (4a) (4b)

Then the last equality gives

\[ \frac{\partial T(z)}{\partial z^i} = \sum_j u_{x^j} \frac{\partial \varphi_j}{\partial z^i}, \quad \text{for} \quad i = 1, \ldots, p, \] (5)

and for invertible transformations (4), the linear system (5) yields solutions of the form

\[ u_{x^j} = \sum_{i=1}^{p} \Psi^j_i \frac{\partial T(z)}{\partial z^i}, \quad \text{for} \quad j = 1, \ldots, p, \]

for some functions \( \Psi^j_i = \Psi^j_i(z) \). In other words, an expression of differential operators in terms of the new independent variables is given by

\[ \partial_{x^j} = \Psi_j, \quad \text{where} \quad \Psi_j = \sum_i \Psi^j_i \partial_{z^i}, \] (6)

We shall make use of some properties of the linear differential operators \( \Psi_j \) to simplify calculations, especially when dealing with high-order derivatives.

3. Determination of the equivalence group

We now move on to consider in this section the problem of finding the equivalence group for the one-dimensional Euler-Bernoulli equation, which is a model for calculating the load-carrying and deflection characteristics of beams. Since the late 19th century, following the successful demonstration of this theory in the constructions of the Eiffel Tower and the Ferris wheel, Euler-Bernoulli equation (also known as Engineer’s beam theory) became the cornerstone of engineering.

This equation can be put in the form

\[ \frac{\partial^2}{\partial x^2} \left( f(x) \frac{\partial^2 u}{\partial x^2} \right) + m(x) \frac{\partial^2 u}{\partial t^2} = 0, \quad t > 0, \quad 0 < x < \mathcal{D}, \] (7)
where \( f(x) > 0 \) is the flexural rigidity, \( m(x) > 0 \) is the lineal mass density and \( u(t,x) \) is the transversal displacement at time \( t \) and position \( x \) from one end of the beam taken as origin.

In the relatively recent research literature, Euler-Bernoulli equation, which is often simply referred to as Euler-Bernoulli beam or beam equation, has been frequently discussed. In particular, Gotllieb [7] has investigated the iso-spectral properties of this equation and its non-homogeneous variants in connection with the unit beam (i.e. \( f = 1 \) and \( m = 1 \)). More recently, Wafo Soh [6] considered the equivalence problem for Euler-Bernoulli beam from the Lie symmetry approach, while Morozov and Wafo Soh [8] investigated the same problem using Cartan’s equivalence method.

It is clear that in any change of variables that preserves the form of (7), the transformation of \( x \) should depend on a single variable (due to the form of \( f \) and \( m \)), while the linearity of the equation forces the transformation of \( u \) to be linear, and that for the independent variables not to involve the dependent variable. We are therefore led to look for equivalence transformations of (7) in the form

\[
t = R(y,z), \quad x = S(z), \quad u = L(y,z)w + J(y,z)
\]

Under (8), the coefficient \( \gamma_1 \) of \( w_{yyyy} \) in the transformed equation takes the form

\[
\gamma_1 = \frac{(fR_y^4L)/(R_y^4S_y^4)}
\]

and its vanishing implies that \( R = R(y) \). Inserting this new expression for \( R \) into (8), the vanishing of the coefficient of \( w_y \) in the new transformed equation yields the condition

\[
2R_y L_y - LR_{yy} = 0,
\]

and thus \( L = h(z)\sqrt{R_y} \), for some function \( h \). If we now transform (7) using this additional information and write down the coefficient \( \gamma_2 \) of \( w_z \) as a polynomial in \( f(S) \) and its derivatives, then the coefficient \( \delta_1 \) of \( f_S S \) in this expansion is

\[
\delta_1 = S_y^4(2h_z S_z - h S_{zz}),
\]
and its vanishing gives \( h = k_1 \sqrt{S_z} \), where \( k_1 \) is a constant, so that \( u = k_1 (R_y S_z)^{1/2} w + J \). With this expression for \( h \), the vanishing of the coefficient of \( f_S \) in \( \gamma_2 \) leads to the condition

\[-3 S_{zz}^2 + 2 S_z S_{zzz} = 0,\]

and this in turn shows that

\[x \equiv S = (k_2 z + k_3)/(k_4 z + 1), \quad k_2 - k_3 k_4 \neq 0,\]

for some constants \( k_2, \ldots, k_4 \). With all the information obtained up to this point on \( R, S, \) and \( L \), the coefficient \( \gamma_3 \) of \( w \) now takes the form

\[\gamma_3 = -\frac{4(k_2 - k_3 k_4)^8 m (3R_{yy}^2 - 2R_y R_{yyy})}{(1 + k_4 z)^{16}},\]

showing that \( R \) must satisfy an equation similar to that for \( S \), namely

\[3R_{yy}^2 - 2R_y R_{yyy} = 0.\]

Consequently, we must have

\[t \equiv R = (k_5 y + k_6)/(k_7 y + 1), \quad k_5 - k_6 k_7 \neq 0,\]  \hspace{1cm} (9)

for some arbitrary constants \( k_5, \ldots, k_7 \). With this new information and all others obtained thus far, (8) now takes the form

\[t = \frac{k_5 y + k_6}{k_7 y + 1}, \quad x = \frac{k_2 z + k_3}{k_4 z + 1} \hspace{1cm} (10a)\]

\[u = k_1 \frac{(k_2 - k_3 k_4)^{1/2} (k_5 - k_6 k_7)^{1/2}}{(1 + k_7 y)(1 + k_4 z)} w + J(y, z). \hspace{1cm} (10b)\]

Under (10), the transformation of Euler-Bernoulli beam retains a constant component, that is, a component (or sum of terms) free from \( w \) and its derivatives, and which appears to contain the only terms involving \( J \) and its derivatives. In fact this component is simply the transformation of \( u = J \) under the change of independent variables (10a), and it is readily found that the vanishing of this constant component amounts to setting

\[J(y, z) = \frac{k_8 - k_10 k_7^2 y + k_4 (-k_9 + k_11 k_7^2 y) + k_4^2 (-k_9 z + k_11 k_7^2 y z)}{k_4 k_7 (1 + k_7 y)(1 + k_4 z)} \hspace{1cm} (11)\]
where \( k_8, \ldots, k_{11} \) are some additional constants of integration.

If we now perform a transformation of (7) under the change of variables (10) with \( J \) as in (11), the transformed equation takes on the form

\[
\frac{\partial^2}{\partial z^2} \left( F(y, z) \frac{\partial^2 w}{\partial z^2} \right) + M(y, z) \frac{\partial^2 w}{\partial y^2} = 0, \tag{12a}
\]

where

\[
F(y, z) = \frac{(1 + k_4 z)^5}{(1 + k_7 y)(1 + k_4 z)} f(S), \tag{12b}
\]

\[
M(y, z) = \frac{(1 + k_7 y)^3(1 + k_4 z)^3(k_2 - k_3 k_4)^4}{(k_5 - k_6 k_7)^3(1 + k_4 z)^7} m(S). \tag{12c}
\]

Therefore, the change of variables given by (10) and (11) will represent an equivalence transformation of (7) if and only if the functions \( F \) and \( M \) in (12) do not depend on \( y \), and this is possible if and only if \( k_7 = 0 \). However, as in (11) the value \( k_7 = 0 \) is not allowed, we must recompute the appropriate expression for \( J \) when \( k_7 = 0 \). So, letting \( k_7 = 0 \) in (9) and finding the corresponding value for \( J \) gives

\[
J = \frac{-k_0 - k_9 y + k_4(k_8 + k_{10} y) + k_4^2(k_8 + k_{10} y) z}{k_4(1 + k_4 z)}, \tag{13}
\]

for some new constants \( k_0 \), and \( k_8, \ldots, k_{10} \), and the transformed equation now takes on the required form

\[
\frac{\partial^2}{\partial z^2} \left( F(z) \frac{\partial^2 w}{\partial z^2} \right) + M(z) \frac{\partial^2 w}{\partial y^2} = 0, \tag{14a}
\]

where

\[
F(z) = (1 + k_4 z)^5 \left[ \frac{(k_2 - k_3 k_4)k_5}{(1 + k_4 z)^2} \right]^{1/2} f(S), \tag{14b}
\]

\[
M(z) = \frac{(k_2 - k_3 k_4)^5}{k_5^2(1 + k_4 z)^3} \left[ \frac{(k_2 - k_3 k_4)k_5}{(1 + k_4 z)^2} \right]^{1/2} m(S). \tag{14c}
\]

We have thus obtained the following result.
Theorem 1. The group $G$ of equivalence transformations of Euler-Bernoulli equation \((7)\) is given by the equivalence transformations

\[
\begin{align*}
t &= k_5 y + k_6, & x &= \frac{k_2 z + k_3}{k_4 z + 1} \\
u &= k_1 \left( \frac{(k_2 - k_3 k_4) k_5}{(1 + k_4 z)^2} \right)^{1/2} w + J,
\end{align*}
\]

where

\[
J = \frac{-k_0 - k_9 y + k_4 (k_8 + k_10 y) + k_4^2 (k_8 + k_10 y) z}{k_4 (1 + k_4 z)},
\]

is the function obtained in \((13)\), and the transformed equation is given by \((14)\).

We also note that in the successive transformations of \((7)\) which led to \((12)\), the arguments of the arbitrary function $f$ and $m$ were treated as dummy, by being ignored. Thus, in view of \((12)\), we readily obtain the following result for an extended version of Euler-Bernoulli Equation.

Lemma 1. For the generalized version of Euler-Bernoulli equation of the form

\[
\frac{\partial^2}{\partial x^2} \left( f(t, x) \frac{\partial^2 u}{\partial x^2} \right) + m(t, x) \frac{\partial^2 u}{\partial t^2} = 0,
\]

in which $f$ and $m$ are assumed to be functions of both $t$ and $x$, an equivalence subgroup is given by the equivalence transformations \((10)\) and \((11)\), and the transformed equation takes on the form \((12)\).

We may not however assert as yet that \((10)\) and \((11)\) define the equivalence group of \((16)\), because due to the form of $f$ and $m$ in \((7)\), these transformations were obtained under the assumption that $x = S(z)$, depends only on a single variable. But this restriction in no longer a priori permitted in the case of \((16)\) where $f$ and $m$ are functions of both $t$ and $x$.

The equivalence group of \((16)\) should thus be looked for in the form

\[
t = R(y, z), \quad x = S(y, z), \quad \text{and} \quad u = L(y, z) w + J(x, y),
\]

where as usual, the functions $R, S, L$ and $J$ are to be found. It first follows from the lemma and the invertibility of \((17)\), that

\[
R_y S_z L \neq 0, \quad \varepsilon L \neq 0,
\]

where $\varepsilon$ is the Jacobian of the transformation.
\[ \Xi = R_y S_z - R_z S_y. \]  

(18b)

It is also clear that in the transformed equation, terms of the highest order may only come from the transformation of \( u_{xxxx} \). Moreover, to find such terms, we may assume that \( u = L w \) in (17), and since \( L \neq 0 \), by Leibnitz rule we may assume for such purpose that \( u = w(y, z) \). Since the term \((R^4/\Xi)w_{yyyy}\) appears in the polynomial expansion of the transformation of \( u_{xxxx} \), the vanishing of this term yields \( R = R(y) \).

On the other hand, it follows from the formulas (16) that

\[
\partial_t = \frac{1}{\Xi}(S_z \partial_y - S_y \partial_z), \quad \partial_x = \frac{1}{\Xi}(-R_z \partial_y + R_y \partial_z),
\]

(19)

and for \( R = R(y) \), this reduces to

\[
\partial_t = \frac{1}{R_y} \partial_y - \frac{S_y}{R_y S_z} \partial_z, \quad \partial_x = \frac{1}{S_z} \partial_z.
\]

(20)

Consequently, terms involving derivatives of \( w \) w.r.t. \( y \) in the transformed equation can only come from the transformations of derivatives of \( u \) w.r.t. \( t \), i.e. from the transformation of \( u_{tt} \). So, transforming \( u_{tt} \) by setting \( u = L(y, z)w \) in (17) shows that the term in \( w_{yz} \) has coefficient

\[-(2S_y L)/(R^2_y S_z).
\]

The vanishing of this term gives \( S = S(z) \), and hence the transformations of (16) must also be sought in the form

\[
t = R(y), \quad x = S(z), \quad u = L(y, z)u + J(y, z),
\]

(21)

just as in the case of (7). Given that in the transformations of (7) starting with a change of variables of the form (21), and leading to (10), (11), and (12), the arguments of the arbitrary functions \( f \) and \( m \) are ignored, and because (7) and (16) differ only by these arguments, we may conclude that the equivalence transformations of (16) must also be of the form (10)- (11). Since the resulting
transformations of (16) which are also given by (12) are of the required form, we have thus obtained the following result.

**Theorem 2.** The equivalence transformations

\[
\begin{align*}
t &= \frac{k_5 y + k_6}{k_7 z + 1}, \quad x = \frac{k_2 z + k_3}{k_4 z + 1} \\
u &= k_1 \frac{(k_2 - k_3 k_4)^{1/2}(k_5 - k_6 k_7)^{1/2}}{(1 + k_7 y)(1 + k_4 z)} w + J(y, z)
\end{align*}
\]

where

\[
J(y, z) = \frac{k_8 - k_{10} k_7^2 y + k_4 \left(-k_9 + k_{11} k_7^2 y\right) + k_4^2 \left(-k_9 z + k_{11} k_7^2 yz\right)}{k_4 k_7 (1 + k_7 y)(1 + k_4 z)},
\]

and which are given by (10) and (11), define the equivalence group of the extended form (10) of (7).

4. Symmetry properties

It is clear that the equivalence group of a differential equation contains according to our definition the largest symmetry group \(G_e\) of the equation that is free from the arbitrary functions of the equation. However, such a symmetry group is only a subgroup of the so-called principal Lie group, which is the largest symmetry group that holds for all arbitrary functions, but which may depend on these arbitrary functions.

Symmetries of Euler-Bernoulli equation were calculated in [6], but were obtained only in a conditional form, and under the assumption that the functions \(f\) and \(m\) satisfy certain unsolved complicated equations.

It appears from the form of (14) that (15) will represent a symmetry transformation of (7) only if the transformation of \(x\) is the identity transformation, \(x = z\). This means that we have to assume \(k_4 = 0\) in (14), and because the corresponding expression for the denominator of \(J\) vanishes for \(k_4 = 0\), we have to recompute the appropriate value of \(J\). To do so, we transform (7) using (10), with \(k_7 = k_4 = k_3 = 0\), and \(k_2 = 1\). It then follows that (7) will be invariant under these transformations if and only if \(k_5 = 1\), and

\[
J = yp_2 + p_4 + z (yp_1 + p_3).
\]
We thus have the following result.

**Theorem 3.** The largest symmetry group of Euler-Bernoulli equation that is independent of its arbitrary functions is a six-parameter group $G_e$ given by the transformations

$$t = y + p_6, \quad x = z, \quad \text{and} \quad u = p_5 w + J,$$

(23)

where $J$ is given by (22), and with corresponding symmetry generator

$$v = p_6 \partial_t + (p_4 + p_2 t + p_3 x + p_1 tx + p_5 u) \partial_u,$$

(24)

where $p_1, \ldots, p_6$ are the group parameters.

We note that the function $J$ in (22) is the fundamental solution of Euler-Bernoulli equation that does not depend on its arbitrary functions. Owing to the linearity of (7), if we replace the function $J$ in (23) by any other solution of (7), the resulting transformation remains a symmetry [9, 10], but which may however depend on the arbitrary functions of the equation, and this is contrary to our assumptions, and for practical considerations such solutions are not always available.

On the other hand, due to the superposition principle of linear equations, (23) with $J$ replaced by an arbitrary solution $S$ of (7), remains a symmetry of Euler-Bernoulli equation, and one readily verifies that this is the most general symmetry group of this equation.

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