LINEAR FORMS AND QUADRATIC UNIFORMITY FOR FUNCTIONS ON $\mathbb{Z}_N$

W.T. GOWERS AND J. WOLF

Abstract. A very useful fact in additive combinatorics is that analytic expressions that can be used to count the number of structures of various kinds in subsets of Abelian groups are robust under quasirandom perturbations, and moreover that quasirandomness can often be measured by means of certain easily described norms, known as uniformity norms. However, determining which uniformity norms work for which structures turns out to be a surprisingly hard question. In [GW09a] and [GW09b, GW09c] we gave a complete answer to this question for groups of the form $G = \mathbb{F}_p^n$, provided $p$ is not too small. In $\mathbb{Z}_N$, substantial extra difficulties arise, of which the most important is that an “inverse theorem” even for the uniformity norm $\| \cdot \|_{U^3}$ requires a more sophisticated (local) formulation. When $N$ is prime, $\mathbb{Z}_N$ is not rich in subgroups, so one must use regular Bohr neighbourhoods instead. In this paper, we prove the first non-trivial case of the main conjecture from [GW09a].

Contents

1. Introduction
2. Bohr sets and their basic properties
3. Quadratic Fourier analysis on $\mathbb{Z}_N$
4. Generalized quadratic averages
5. The rank of a quadratic average
6. The structure of low-rank bilinear forms on Bohr sets
7. A more precise decomposition theorem
8. The structure of a function $QU$ when $Q$ has low rank
9. Some facts about ranks of quadratic and bilinear functions on Bohr sets
10. Computing with linear combinations of high-rank quadratic averages
11. Proof of the main result
References
1. Introduction

In additive combinatorics one is often interested in counting small structures in subsets of Abelian groups. For instance, one formulation of Szemerédi’s theorem is the assertion that if $\delta > 0$, $k$ is a positive integer and $N$ is large enough, then every subset $A$ of $\mathbb{Z}_N$ of density at least $\delta$ contains many arithmetic progressions of length $k$.

There is also an equivalent formulation of the theorem concerning functions, the formal statement of which is as follows.

**Theorem 1.1.** Let $\delta > 0$ and let $k$ be a positive integer. Then there is a constant $c = c(\delta, k) > 0$ such that for every $N$ and every function $f : \mathbb{Z}_N \to [0, 1]$ with $E_x f(x) \geq \delta$,

$$E_{x,d} f(x) f(x + d) \ldots f(x + (k - 1)d) \geq c.$$  

Here we use the symbol “$E$” to denote averages over $\mathbb{Z}_N$. For instance, $E_{x,d}$ is shorthand for $N^{-2} \sum_{x,d \in \mathbb{Z}_N}$. If we take $f$ to be the characteristic function of a set $A$ of density $\delta$, then the conclusion of Theorem 1.1 states that the number of arithmetic progressions of length $k$ in $A$, or rather the number of pairs $(x, d)$ such that $x, x + d, \ldots, x + (k - 1)d$ all lie in $A$, is at least $c(\delta, k)N^2$. (It is not necessary to assume that $N$ is sufficiently large, because for small $N$ the degenerate progressions where $d = 0$ are numerous enough for the theorem to be true. But $N$ has to be large for it to become a non-trivial statement.)

There are now several known ways of proving Szemerédi’s theorem. One of them, an analytic approach due to the first author [G01], relies heavily on the fact that the quantity $E_{x,d} f(x) f(x + d) \ldots f(x + (k - 1)d)$ is robust under perturbations of $f$ that are quasirandom in a suitable sense. More precisely, in [G01] a norm $\| \cdot \|_{U^k}$ was defined for each $k$, which has the property that if $f_1, \ldots, f_k$ are functions with $\|f_i\|_{\infty} \leq 1$ for every $i$, then

$$|E_{x,d} f_1(x) f_2(x + d) \ldots f_k(x + (k - 1)d)| \leq \min_i \|f_i\|_{U^{k-1}}.$$  

From this it is simple to deduce that if $f$ and $g$ are two functions from $\mathbb{Z}_N$ to $[0, 1]$, then

$$E_{x,d} f(x) f(x + d) \ldots f(x + (k - 1)d) - E_{x,d} g(x) g(x + d) \ldots g(x + (k - 1)d)$$

has magnitude at most $k\|f - g\|_{U^{k-1}}$. If we choose a function $h$ randomly by taking the values $h(x)$ to be bounded random variables of mean 0, then with high probability $\|h\|_{U^k}$ will be very small. Thus, the $U^k$ norms are measures of a certain kind of quasirandomness that is connected with cancellations in expressions such as $E_{x,d} h(x) h(x + d) \ldots h(x + (k - 1)d)$. 
The proof of inequality (1) is a relatively straightforward inductive argument that involves repeated application of the Cauchy-Schwarz inequality. Once one has this argument, it is natural to try to generalize it to other expressions such as $E_{x,y,z} f(x+y) f(x+z) f(y+z)$ or $E_{x,d} f(d) f(x) f(x+d) f(x+2d)$. In general, one can take a system of linear forms $L_1, \ldots, L_r$ in $k$ variables $x_1, \ldots, x_s$ (that is, for each $i$ we write $x_i$ for $(x_1, \ldots, x_s)$ and define $L_i(x) = \sum_{j=1}^s a_{ij} x_j$ for some integers $a_{i1}, \ldots, a_{is}$) and examine the quantity $E_x \prod_{i=1}^r f_i(L_i(x))$, or the more general quantity $E_x \prod_{i=1}^r f_i(L_i(x))$. We would then like to know for which uniformity norms $U^k$ it is true that these expressions are robust under small $U^k$ perturbations.

This question was first addressed by Green and Tao [GrT06], who were interested in proving asymptotic estimates for expressions such as these when $f$ is the characteristic function of the primes up to $n$ (or rather the closely related von Mangoldt function). They defined a notion of complexity for a system of linear forms. This is a positive integer $k$ with the property that if a system has complexity $k$, then the corresponding analytic expression will be robust under small $U^{k+1}$ perturbations. (As we shall see, there are good reasons for defining complexity in a way that leads to this difference of 1.) Roughly speaking, the property they identified picks out the minimal $k$ for which repeated use of the Cauchy-Schwarz inequality can be used to prove robustness under small $U^{k+1}$ perturbations.

However, it turns out that there are some systems of linear forms of complexity $k$ that are robust under small perturbations in $U^{j+1}$ for some $j < k$. (Since the $U^k$ norms increase as $k$ increases, the assumption that a function is small in $U^{j+1}$ is weaker than the assumption that it is small in $U^{k+1}$.) This phenomenon was first demonstrated in [GW09a], where we showed that if $G$ is the group $\mathbb{F}_p^n$, then there is a system of linear forms of complexity 2 such that the corresponding analytic expression is robust under small $U^2$ perturbations. (A similar phenomenon in ergodic theory was discovered independently by Leibman [L07].) Because Green and Tao’s definition, appropriately modified, appears to capture all systems of linear forms for which Cauchy-Schwarz-type arguments work (though we have not actually formulated and proved a statement along these lines), one must use additional tools. The particular tool we used was a new technique known as quadratic Fourier analysis, which we shall discuss in some detail in §3. A weak “local” form of quadratic Fourier analysis was introduced and used in [G01] to prove Szemerédi’s theorem (for progressions of length 4 – higher order Fourier analysis was needed for the general case). A more “global” version was developed by Green and Tao [GrT08] and will be essential to this paper.
In [GW09a] we made the following conjecture.

**Conjecture 1.2.** Let $L_1, \ldots, L_r$ be a system of linear forms in $x = (x_1, \ldots, x_s)$ and let $G$ be the group $\mathbb{Z}_N$ or $\mathbb{F}_p^n$ for sufficiently large $p$. Suppose also that the $k$th powers of the forms $L_i$ are linearly independent. Then $\mathbb{E}_{x} \prod_{i=1}^{r} f(L_i(x))$ is close to $\mathbb{E}_{x} \prod_{i=1}^{r} g(L_i(x))$ whenever $f$ and $g$ are bounded functions and $\|f - g\|_{U^k}$ is small.

It is not hard to prove the converse of this conjecture, so if it is true then it identifies precisely the minimal uniformity norm with respect to which the multilinear expression derived from the linear forms is continuous (where by “continuous” we mean continuous in a way that does not depend on the size of the group): it is given by the smallest $k$ such that the $k$th powers of the linear forms are linearly independent. In such a case, we shall say that the forms are $k$th-power independent. When $k = 2$ we shall say that they are square independent. We also formulated a more general conjecture that covers the case of $m$ different functions.

In [GW09c] we proved Conjecture 1.2 in the case where $G = \mathbb{F}_p^n$, using the very recent inverse theorem for the $U^k$ norm in that context, which was proved by Bergelson, Tao and Ziegler [BTZ09, TZ08]. This inverse theorem opens the way to cubic Fourier analysis, quartic Fourier analysis, and so on. Our result is the first application of this higher-order Fourier analysis. The first application of higher-order Fourier analysis on $\mathbb{Z}_N$, which has recently become a theorem (though so far only the $k = 4$ case is available [GrTZ09]), is to linear equations in the primes: Green and Tao have already obtained asymptotics for the numbers of solutions for all systems of finite complexity, conditional on the inverse conjecture for the $U^k$ norm in $\mathbb{Z}_N$, which they have now proved with Ziegler.

Quadratic Fourier analysis on $\mathbb{Z}_N$ has had other applications. For example, a modification of Theorem 7.5 was used by Candela [C08] to prove that if $A$ is a dense subset of $\{1, 2, \ldots, n\}$ then the set of all $d$ such that $A$ contains an arithmetic progression of length 3 and common difference $d$ must itself contain an arithmetic progression of length at least $(\log \log N)^c$.

In [GW09a] we proved the first non-trivial case of Conjecture 1.2 for $\mathbb{F}_p^n$, which is the case of square-independent systems of complexity 2. However, we obtained a bound of tower type, so from a quantitative point of view this result was not very satisfactory. In [GW09b], we improved this bound to one that was doubly exponential. The general inverse theorem for functions on $\mathbb{F}_p^n$ has so far been proved only as a purely qualitative statement, so we did not obtain any bounds at all for the other cases of Conjecture 1.2. In this paper, we shall prove Conjecture 1.2 for square independent systems of complexity 2 in $\mathbb{Z}_N$. In other
words, if \( f \) and \( g \) are bounded functions and \( L_1, \ldots, L_r \) is a square independent system of complexity 2, we are interested in how small the \( U^2 \) norm \( \| f - g \|_{U^2} \) has to be to guarantee that \( \mathbb{E}_x \prod_{i=1}^r f(L_i(x)) \) is within \( \epsilon \) of \( \mathbb{E}_x \prod_{i=1}^r g(L_i(x)) \). We go to considerable efforts to obtain a respectable bound, which in the end is a doubly exponential dependence on \( \epsilon \). If we had worked less hard then we would have had to settle for a tower-type bound. To obtain the good bound (relatively speaking) we shall use some of the ideas from [GW09b] as well as some new ideas to deal with problems that do not arise in \( \mathbb{F}_p^n \).

The big difference between \( \mathbb{F}_p^n \) and \( \mathbb{Z}_N \) is that \( \mathbb{F}_p^n \) has many subgroups that closely resemble \( \mathbb{F}_p^n \) itself. If \( N \) is prime, then \( \mathbb{Z}_N \) has no non-trivial subgroups at all, and it becomes necessary to consider subsets that are “approximately closed” under addition. These subsets are called regular Bohr sets, and we shall discuss them in the next section.

Here we remark that the notion of a Bohr set originated in the study of almost periodic functions and has played a very important role in additive combinatorics since Ruzsa’s pioneering proof [R94] of Freiman’s theorem [F73]. The additional hypothesis of regularity, which makes it possible to treat Bohr sets like subgroups, was introduced by Bourgain [B99] and has subsequently been used by several authors.

By proving our results first for \( \mathbb{F}_p^n \) and then adapting the arguments to the \( \mathbb{Z}_N \) context, we are following a general course urged by Green in [Gr07]. The reason for doing it is that it splits problems into two parts. The first part, which is in a sense more fundamental, is to get one’s result in a model context where certain distracting technicalities do not arise. Once one has done that, one has a global structure for the proof, and one can usually find a proof in \( \mathbb{Z}_N \) that has the same global structure as the proof in \( \mathbb{F}_p^n \).

That is the case for our result, so although we have made this paper self-contained, the reader will almost certainly prefer to begin by reading [GW09b]. However, the adaptation of our arguments to \( \mathbb{Z}_N \) is by no means a completely mechanical process. Some parts are, by now, fairly routine, but certain concepts that are quite useful for proving results in \( \mathbb{F}_p^n \) do not have obvious analogues in \( \mathbb{Z}_N \), and some lemmas that are almost trivial in \( \mathbb{F}_p^n \) become serious statements with non-obvious proofs in \( \mathbb{Z}_N \). We shall highlight the less obvious parts of the adaptation as they arise, since some of them may well find other uses.

Very recently indeed, Green and Tao [GrT10] have proved Conjecture 1.2 in full generality in \( \mathbb{Z}_N \), using the recent inverse theorem. Their method is completely different from ours, which was almost certainly necessary: it seems that they have found the right framework for studying the problem if one is content with arguments that do not give reasonable bounds. However, in order to obtain the quantitative statement that we prove
here, it seems to be necessary (at least given the technology as it is at present) to use different, more “old-fashioned” techniques. For the time being a proof of the full conjecture with good bounds looks out of reach: not the least of the difficulties would be obtaining a quantitative version of the inverse theorem.

2. Bohr sets and their basic properties

Let $K$ be a subset of $\mathbb{Z}_N$ and let $\rho > 0$. The Bohr set $B(K, \rho)$ is the set of all $x \in \mathbb{Z}_N$ such that $|\omega^{rx} - 1| \leq \delta$ for every $r \in K$. As we have just said, Bohr sets will play the role that subgroups played for functions defined on $\mathbb{F}_p^n$. However, they are not closed under addition, and this causes problems.

The way to deal with these problems is to use the fact that Bohr sets do have at least some closure properties. In particular, if $x \in B(K, \rho)$ and $y \in B(K, \sigma)$, then $x + y \in B(K, \rho + \sigma)$. To use this fact, one takes $\sigma$ small enough for $B(K, \rho + \sigma)$ to be approximately equal to $B(K, \rho)$.

However, such an approach can work only if the size of the set $B(K, \rho)$ depends sufficiently continuously on $\rho$, which is not always the case. This fact motivated an important definition due to Bourgain [B99]. Let $B = B(K, \rho)$ be a Bohr set. $B$ is said to be regular if, for every $\epsilon > 0$, the Bohr set $B(K, \rho(1+\epsilon))$ has cardinality at most $|B|(1+100|K|\epsilon)$ and the Bohr set $B(K, \rho(1-\epsilon))$ has cardinality at least $|B|(1-100|K|\epsilon)$. The precise form of this definition is what comes out of the following lemma (see for example [TV06]), which tells us that it is easy to find regular Bohr sets.

Lemma 2.1. Let $K$ be a subset of $\mathbb{Z}_N$ and let $\rho_0 > 0$. Then there exists $\rho$ such that $\rho \in [\rho_0, 2\rho_0]$ and the Bohr set $B(K, \rho)$ is regular.

It will be useful to have a concise notation that allows us to talk about pairs of Bohr sets that have the approximate closure property under addition.

Definition. Let $B$ be a regular Bohr set $B(K, \rho)$. Then we say that a subset $B' \subset B$ is $\epsilon$-central for $B$, and write $B' \prec_{\epsilon} B$, if $B' = B(K, \sigma)$ for some $\sigma \in [\epsilon \rho/400|K|, \epsilon \rho/200|K|]$ and $B'$ is also regular. Given a pair of Bohr sets $B' \prec_{\epsilon} B$, we define the closure of $B$ to be the set $B^+ = B(K, \rho + \sigma)$ and the interior to be the set $B^- = B(K, \rho - \sigma)$.

The definitions of closure and interior depend on the central set $B'$, so they cannot be used unless $B'$ has been specified. But this does not cause any problems.

Because we are dealing with quadratic rather than linear local Fourier analysis, we will sometimes have to repeat the closure and interior operations, which, unlike their topological
counterparts, are not idempotent (and therefore not strictly speaking closure and interior operations at all). Thus, we define $B^{++}$ to be $B(K, \rho + 2\sigma)$ and $B^{--}$ to be $B(K, \rho - 2\sigma)$.

Note that in many of the early lemmas we do not actually need the central Bohr set $B'$ to be regular. However, we often apply a sequence of such lemmas, so it is convenient to insist on regularity at all times.

There are many closely related ways of using the regularity condition on a Bohr set. The next lemma, which will be used later, is a typical one. It exploits the fact that regular Bohr sets have “small boundaries”.

**Lemma 2.2.** Let $K$ be a subset of $\mathbb{Z}_N$ and let $x_1, \ldots, x_m$ be a sequence of $m$ elements of $\mathbb{Z}_N$. Suppose that the Bohr sets $B$ and $B'$ satisfy $B' \prec_{\epsilon} B$. Then for all but at most $\epsilon m|B|$ values of $x$ the following statement is true: for every $i$, $B'+x$ is either contained in $B+x_i$ or disjoint from it.

**Proof.** If $x - x_i \in B^-$, then $B' + x - x_i$ is a subset of $B^- + B' \subset B$, and therefore $B' + x \subset B + x_i$. Similarly, but in the other direction, if $(B' + x) \cap (B + x_i) \neq \emptyset$, then $x - x_i \in B^+$. Therefore, the only way that $B' + x$ can fail to be either contained in $B + x_i$ or disjoint from it is if $x - x_i \in B^+ \setminus B^-$. However, by the definition of regularity, the cardinality of $B^+ \setminus B^-$ is at most $\epsilon|B|$. The lemma follows. \qed

Another very useful principle indeed is that if $B$ is a regular Bohr set, $B'$ is a central subset, and $f$ is a bounded function, then $\mathbb{E}_{x \in B} f(x)$ is approximately equal to $\mathbb{E}_{x \in B} \mathbb{E}_{y \in B'} f(x + y)$. Indeed, this is the most common way that regularity has been applied. We shall need some less standard (but not difficult) variants of this principle—for the convenience of the reader we give proofs of all the results of this kind that we need. We shall use the notation “$\approx_{\epsilon}$” to stand for the relation “differs by at most $\epsilon$ from”.

**Lemma 2.3.** Let $\epsilon > 0$. Let $B$ and $B'$ be Bohr sets satisfying $B' \prec_{\epsilon} B$. Then for every function $f : \mathbb{Z}_N \to \mathbb{C}$ such that $\|f\|_{\infty} \leq 1$ and for every function $g : \mathbb{Z}_N^2 \to \mathbb{C}$ such that $\|g\|_{\infty} \leq 1$ the following statements hold.

(i) $\mathbb{E}_{x \in B} f(x) \approx_{\epsilon} \mathbb{E}_{x \in B} \mathbb{E}_{y \in P} f(x + y)$ for every subset $P \subset B'$.

(ii) $\mathbb{E}_{x \in B} f(x) \approx_{\epsilon} \mathbb{E}_{x \in B^-} f(x)$.

(iii) $\mathbb{E}_{x \in B^-} f(x) \approx_{3\epsilon} \mathbb{E}_{x \in B^-} \mathbb{E}_{y \in B'} f(x + y)$.

(iv) $\mathbb{E}_{x,x' \in B} g(x,x') \approx_{4\epsilon} \mathbb{E}_{x,x' \in B} \mathbb{E}_{y \in B'} g(x + y, x' + y)$.

(v) $\mathbb{E}_{x,x' \in B} g(x,x') \approx_{8\epsilon} \mathbb{E}_{x,x' \in B} \mathbb{E}_{y \in B'} g(x + y, x' + y)$. 

To prove (ii), we begin by noting that 

\[ \frac{\epsilon}{2}. \]

By regularity, the right hand side is at most \( \epsilon \), by the regularity of \( B \). Since \( E_{x \in B} E_{y \in P} f(x + y) = E_{y \in P} E_{x \in B} f(x + y) \), part (i) follows from the triangle inequality.

To prove (ii), we begin by noting that

\[ |E_{x \in B} f(x) - |B|^{-1} \sum_{x \in B^-} f(x)| \leq |B|^{-1} |B \setminus B^-|. \]

By regularity, the right hand side is at most \( \epsilon/2 \). It is also easy to check that

\[ |E_{x \in B^-} f(x) - |B|^{-1} \sum_{x \in B^-} f(x)| \leq \epsilon/2. \]

It follows that \( |E_{x \in B} f(x) - E_{x \in B^-} f(x)| \leq \epsilon \). Applying (ii) to both sides of (i), we deduce (iii). The proof of (iv) is very similar to that of (i). For each \( y \in B' \) we have the inequality

\[ |E_{x,x' \in B} g(x + y, x' + y) - E_{x,x' \in B} g(x, x')| \leq |B|^{-2} |B^2 \setminus (B + y)^2|. \]

From the fact that \( |B \setminus (B + y)| \leq \epsilon |B| \) it follows that \( |B^2 \setminus (B + y)^2| \leq 4\epsilon |B|^2 \). This implies (iv), just as the analogous statement implied (i). Finally, if we apply (ii) twice to both sides of (iv) we obtain (v).

\[ \square \]

3. Quadratic Fourier analysis on \( \mathbb{Z}_N \)

Conventional Fourier analysis on an Abelian group \( G \) decomposes a function \( f : G \to \mathbb{C} \) into a linear combination of characters, which are homomorphisms from \( G \) to \( T = \{ z \in \mathbb{C} : |z| = 1 \} \). If we allow ourselves a phase shift—that is, if we multiply a character by \( e^{i\theta} \) for some \( \theta \)—then we obtain a function \( \gamma \) that may not be a group homomorphism, but it is still a (multiplicative) Freiman homomorphism, since it satisfies the identity \( \gamma(x + d) \gamma(x)^{-1} = \gamma(y + d) \gamma(y)^{-1} \) for every \( x, y \) and \( d \) in \( G \).

Quadratic Fourier analysis replaces Freiman homomorphisms by a natural quadratic analogue. We can restate the identity above as \( \gamma(x) \gamma(x + a)^{-1} \gamma(x + b)^{-1} \gamma(x + a + b) = 1 \) for every \( x, a \) and \( b \) in \( G \). If \( A \subseteq G \), then a function \( \gamma : A \to T \) is a (multiplicative) quadratic homomorphism if

\[ \gamma(x) \gamma(x + a)^{-1} \gamma(x + b)^{-1} \gamma(x + c)^{-1} \gamma(x + a + b) \gamma(x + a + c) \gamma(x + b + c) \gamma(x + a + b + c)^{-1} = 1 \]

for every \( x, a, b \) and \( c \) in \( G \). The word “quadratic” is used because if \( A = G = \mathbb{Z}_N \), then \( \gamma \) has to be of the form \( \gamma(x) = e^{2\pi i q(x)/N} \) for some quadratic function \( q : \mathbb{Z}_N \to \mathbb{Z}_N \), and
similar statements are true for several other groups. Because of this, we shall also refer to these functions as quadratic phase functions. For more general subsets \( A \) it is less easy to describe quadratic homomorphisms explicitly, but if \( A \) is a sufficiently structured set, such as a coset of a subgroup of \( \mathbb{F}_p^n \) (when \( p \) is not too small) or a Bohr set in \( \mathbb{Z}_N \), then for many purposes it is enough just to know that \( \gamma \) is a quadratic homomorphism, though in these cases one can also give explicit descriptions and it is sometimes important to do so.

The basic idea of quadratic Fourier analysis is that it is possible to decompose a function into a linear combination of a small number of quadratic phase functions defined on regular Bohr sets, plus an error that does not affect calculations. One can of course do the same with conventional Fourier analysis simply by taking only the characters with large coefficients: however, there are circumstances where the error does affect calculations in the linear case, but does not in the quadratic case.

A notable difference between linear and quadratic Fourier analysis is that there is not a unique way of decomposing a function into quadratic parts, for the simple reason that there are too many quadratic phase functions. Furthermore, there is not even a natural notion of the “best” decomposition. So instead one has to settle for decompositions that are somewhat arbitrary and try to control their properties. In order to get started, one needs an inverse theorem, which in our case is a statement to the effect that if \( \|f\|_{U^3} \) is not small (which is a way of saying that \( f \) is not already a “small error”) then \( f \) correlates with a quadratic phase function.

The following theorem to this effect was proved by Green and Tao [GrT 08].

**Theorem 3.1.** Let \( f : \mathbb{Z}_N \to \mathbb{C} \) be a function such that \( \|f\|_\infty \leq 1 \) and \( \|f\|_{U^3} \geq \delta \), and let \( C = 2^{24} \). Then there exists a regular Bohr set \( B = B(K, \rho) \) with \( |K| \leq (2/\delta)^C \) and \( \rho \geq (\delta/2)^C \) such that \( \mathbb{E}_y \|f\|_{U^3(B+y)} \geq (\delta/2)^C \).

Here, \( \|f\|_{U^3(B+y)} \) is defined to be the maximum correlation between \( f \) and any quadratic phase function \( \gamma \) defined on \( B+y \). More precisely, it is the maximum over all quadratic phase functions \( \gamma \) from \( B+y \) to \( \mathbb{T} \) of the quantity \( |\mathbb{E}_{x \in B+y} f(x) \gamma(x)^{-1}| \).

In their paper, Green and Tao remark that a slightly more precise theorem holds. The result as stated tells us that for each \( y \) we can find a quadratic phase function \( \omega^q_y \) defined on \( B+y \) such that the average of \( |\mathbb{E}_{x \in B+y} f(x) \omega^q_y(x)| \) is at least \( (\delta/2)^C \). However, it is actually possible to do this in such a way that the “quadratic parts” of the quadratic phase functions \( q_y \) are the same. That is, it can be done in such a way that each \( q_y(x) \) has the form \( q(x-y) + \phi_y(x-y) \) for some (additive) quadratic homomorphism \( q : B \to \mathbb{Z}_N \) (that is independent of \( y \)) and some Freiman homomorphism \( \phi_y : B \to \mathbb{Z}_N \).
This will be convenient to us later, so we make the following definition, which is a modification of a definition given in [GW09c] for the \( \mathbb{F}_p \) case.

**Definition.** Let \( B \) be a regular Bohr set and let \( q \) be a quadratic map from \( B \) to \( \mathbb{Z}_N \). A quadratic average with base \( (B,q) \) is a function of the form \( Q(x) = \mathbb{E}_{y \in x-B} \omega^{q_y(x)} \), where each function \( q_y \) is a quadratic map from \( B + y \) to \( \mathbb{Z}_N \) defined by a formula of the form \( q_y(x) = q(x - y) + \phi_y(x - y) \) for some Freiman homomorphism \( \phi_y : B \to \mathbb{Z}_N \).

An equivalent way of defining \( Q \), which may be clearer, is to start by defining for each \( y \in \mathbb{Z}_N \) the function \( \gamma_y \), which takes the value \( \omega^{q_y(x)} \) when \( x \in B + y \) and 0 otherwise. Then \( Q \) is \( |B|^{-1} \sum_y \gamma_y \). Thus, the value of \( Q \) at \( x \) is the average value of all the \( \gamma_y(x) \) such that \( x \) belongs to the support of \( \gamma_y \).

We can use the extra observation of Green and Tao to give a slightly more precise version of the inverse theorem.

**Theorem 3.2.** Let \( f : \mathbb{Z}_N \to \mathbb{C} \) be a function such that \( \|f\|_\infty \leq 1 \) and \( \|f\|_{U^3} \geq \delta \), and let \( C_0 = 2^{24} \). Then there exists a regular Bohr set \( B(K, \rho) \) with \( |K| \leq (2/\delta)^{C_0} \rho \) and \( \rho \geq (\delta/2)^{C_0} \), and a quadratic map \( q : B \to \mathbb{Z}_N \), such that \( \langle f, Q \rangle \geq (\delta/2)^{C_0}/2 \) for some quadratic average \( Q \) with base \( (B,q) \).

**Proof.** The results of Green and Tao tell us that we can find a regular Bohr set \( B = B(K, \rho) \), satisfying the above bounds, and a quadratic function \( q \), and for each \( y \) we can find a Freiman homomorphism \( \phi_y : B \to \mathbb{Z}_N \), such that, defining \( q_y(x) = q(x - y) + \phi_y(x - y) \) on \( B + y \), we have

\[
\mathbb{E}_y |\mathbb{E}_{x \in B+y} f(x) \omega^{-q_y(x)}| \geq (\delta/2)^{C_0}
\]

For each function \( q_y \) we can add a constant \( \lambda_y \) without affecting the left-hand side. If \( N \geq 3 \), as we are certainly assuming, then we can choose this constant so that

\[
\Re(\mathbb{E}_{x \in B+y} f(x) \omega^{-q_y(x)+\lambda_y}) \geq \frac{1}{2} |\mathbb{E}_{x \in B+y} f(x) \omega^{-q_y(x)}|
\]

Therefore, after suitably redefining the functions \( q_y \) and setting \( Q(x) = \mathbb{E}_{y \in x-B} \omega^{q_y(x)} \), we have

\[
|\langle f, Q \rangle| \geq \Re(\mathbb{E}_x \mathbb{E}_{y \in x-B} f(x) \omega^{-q_y(x)}) \geq \frac{1}{2} \mathbb{E}_y |\mathbb{E}_{x \in B+y} f(x) \omega^{-q_y(x)}|,
\]

which proves the theorem.

In the proof of the \( \mathbb{F}_p^n \) case, we defined quadratic averages in a similar way, but the role of Bohr sets was played by subgroups (or subspaces). This was simpler for several reasons. One reason was that translates of a subspace partition \( \mathbb{F}_p^n \), but this turns out not
to be a significant complication of the $\mathbb{Z}_N$ case. More problematic is that we made some use of the fact that subspaces of $\mathbb{F}_p^n$ have a codimension, and it is not obvious what one would mean by the “codimension” of a Bohr set. To answer this question, we focus on the two main properties of codimension that we used for the $\mathbb{F}_p^n$ case: that a subspace of codimension $d$ has density $p^{-d}$ and that the intersection of subspaces of codimension $d$ and $d'$ has codimension at most $d + d'$. The analogous facts about Bohr neighbourhoods are that the intersection of the neighbourhoods $B(K, \rho)$ and $B(L, \rho)$ equals the neighbourhood $B(K \cup L, \rho)$, and that the density of $B(K, \rho)$ is at least $\rho^{|K|}$. Thus, for fixed $\rho$ the cardinality of $K$ is a good analogue of the codimension.

At first, this seems odd, since the cardinality of $K$ is closely connected with the dimension of $B$. However, it can also be seen as the number of inequalities that a point in $B$ must satisfy, and these inequalities are analogous to the linear constraints that a point in a subspace must satisfy. Nevertheless, to avoid confusion we will not use the word “codimension” here. Instead, we shall define the complexity of the Bohr set $B(K, \rho)$ to be the pair $(|K|, \rho)$. Strictly speaking, this is not well-defined, since different pairs $(K, \rho)$ can define the same Bohr set. So a slightly stricter definition is as follows: the Bohr set $B$ has complexity at most $(d, \rho)$ if there exists a set $K$ of cardinality at most $d$ and a constant $\rho' \geq \rho$ such that $B = B(K, \rho')$. (We say “at most” because we regard a smaller $\rho$ as giving a higher complexity.) We say that a quadratic average $Q$ with base $(B, q)$ has complexity at most $(d, \rho)$ if $B$ has complexity at most $(d, \rho)$.

Now, as we did in the $\mathbb{F}_p^n$ case, we can use fairly abstract reasoning to deduce some decomposition results from the inverse theorem. First, we recall a result from [GW09b]. It is a straightforward consequence of the Hahn-Banach theorem and appears in [GW09b], with proof, as Corollary 2.4. It can be thought of as a general machine for converting inverse theorems into decomposition theorems.

**Proposition 3.3.** Let $k$ be a positive integer and for each $i \leq k$ let $\| \cdot \|_i$ be a norm defined on a subspace $V_i$ of $\mathbb{C}^n$. Suppose also that $V_1 + \cdots + V_k = \mathbb{C}^n$. Let $\alpha_1, \ldots, \alpha_k$ be positive real numbers, and suppose that it is not possible to write the function $f$ as a linear sum $f_1 + \cdots + f_k$ in such a way that $f_i \in V_i$ for each $i$ and $\alpha_1 \|f_1\|_1 + \cdots + \alpha_k \|f_k\|_k \leq 1$. Then there exists a function $\phi \in \mathbb{C}^n$ such that $\|f, \phi\| \geq 1$ and such that $\|\phi\|_i \leq \alpha_i$ for every $i$.

The final condition on $\phi$ means that $\|f, \phi\| \leq \alpha_i$ for every $i$ and every $g \in V_i$ with $\|g\|_i \leq 1$. 
We now apply Proposition 3.3 to obtain a theorem that tells us that an arbitrary function $f$ that is bounded in $L_2$ can be decomposed as a linear combination of quadratic averages plus a small error.

**Theorem 3.4.** Let $f : \mathbb{Z}_N \rightarrow \mathbb{C}$ be a function such that $\|f\|_2 \leq 1$. Let $C_0 = 2^{24}$. Then for every $\delta > 0$ and $\eta > 0$ there exist $C$, $d$ and $\rho$ such that $f$ has a decomposition of the form

$$f(x) = \sum_i \lambda_i Q_i(x) + g(x) + h(x),$$

where the functions $Q_i$ are quadratic averages of complexity at most $(d, \rho)$, and

$$\eta^{-1}\|g\|_1 + \delta^{-1}\|h\|_{U^3} + C^{-1}\sum_i |\lambda_i| \leq 1.$$

Moreover, we can take $C = 4(2/\eta\delta)^{C_0}$, $d = (2/\delta)^{C_0}$ and $\rho = (\delta/2)^{C_0}$.

**Proof.** For every quadratic average $Q$ on $\mathbb{Z}_N$ of complexity at most $(d, \rho)$, let $V(Q)$ be the one-dimensional subspace of $\mathbb{C}^{\mathbb{Z}_N}$ generated by $Q$, with the norm of $\lambda Q$ set to be $|\lambda|$. Let $\alpha(Q)$ be $C^{-1}$ for every $Q$. In addition, let us take the $L_1$ norm and $U^3$ norm defined on all of $\mathbb{C}^{\mathbb{Z}_N}$ and associate with them the constants $\eta$ and $\delta$, respectively.

Suppose that $f$ cannot be decomposed in the desired way. Applying Proposition 3.3 to the norms, subspaces and positive constants defined above, we obtain a function $\phi : \mathbb{Z}_N \rightarrow \mathbb{C}$ such that $\langle f, \phi \rangle \geq 1$, $\|\phi\|_\infty \leq \eta^{-1}$, $\|\phi\|_{U^3} \leq \delta^{-1}$ and $|\langle \phi, Q \rangle| \leq C^{-1}$ for every quadratic average $Q$ of complexity at most $(d, \rho)$.

Because $\|f\|_2 \leq 1$ and $\langle f, \phi \rangle \geq 1$, we find that $\langle \phi, \phi \rangle = \|\phi\|_2 \geq 1$. But then $\|\phi\|_{U^3} \|\phi\|_{U^3}^* \geq 1$, which implies that $\|\phi\|_{U^3} \geq \delta$. Applying Theorem 3.2 to $\eta \phi$, we obtain a quadratic average $Q$ of complexity at most $(d, \rho)$ such that $|\langle \phi, Q \rangle| \geq (\eta\delta/2)^{C_0}/2$, which is a contradiction since this inner product was supposed to be at most $C^{-1}$ for all such quadratic averages.

4. **Generalized quadratic averages**

Before we go any further, we must address a technical issue that did not arise for $\mathbb{F}_p^n$. There, it is a triviality that if $Q$ is a quadratic average with base $(V, q)$ and $Q'$ is a quadratic average with base $(V', q')$, then $Q \overline{Q'}$ is a quadratic average with base $(V \cap V', q - q')$. The analogous statement for $\mathbb{Z}_N$ is false, but an approximate version of it is true if we are prepared to generalize the notion of a quadratic average.

In fact, we shall begin by discussing an even more basic statement, which again does not quite hold for $\mathbb{Z}_N$, namely the statement that if $Q$ is a quadratic average with base $(V, q)$ and $V'$ is a subspace of $V$, then $Q$ is a quadratic average with base $(V', q)$.
In order to obtain an analogue of this statement for $\mathbb{Z}_N$, we first define a generalized quadratic average with base $(B, q)$ to be any average of quadratic averages with base $(B, q)$—that is, any function of the form $n^{-1}(Q_1 + \cdots + Q_n)$, where $n$ is a positive integer and each $Q_i$ is a quadratic average with base $(B, q)$.

**Lemma 4.1.** Let $\epsilon > 0$, let the Bohr sets $B$ and $B'$ satisfy $B' \prec_\epsilon B$. Let $q$ be a quadratic form on $B$ and let $Q$ be a generalized quadratic average with base $(B, q)$. Then there is a generalized quadratic average $Q'$ with base $(B', q)$ such that $\|Q - Q'\|_\infty \leq 4\epsilon$.

**Proof.** We begin by proving the result when $Q$ is a quadratic average, defined by the formula $Q(x) = \mathbb{E}_{y \in x - B} \omega^q_y(x)$. Let $A$ be the interior associated with the pair $B' \prec_\epsilon B$. From the proof of Lemma 4.3 (ii), we know that

$$\mathbb{E}_{y \in x - B} \omega^q_y(x) \approx \mathbb{E}_{y \in x - A} \omega^q_y(x),$$

so if we set $R(x)$ to equal $\mathbb{E}_{y \in x - A} \omega^q_y(x)$ then $\|R - Q\|_\infty \leq \epsilon$.

Next, we define $S(x)$ to be $\mathbb{E}_{y \in x - A} \mathbb{E}_{u \in B'} \omega^{q_y + u}(x)$. By Lemma 4.3 (iii), $\|S - R\|_\infty \leq 3\epsilon$. But $S(x)$ is equal to $\mathbb{E}_{y \in A} \mathbb{E}_{u \in B'} \omega^{q_y + u}(x)$. For each $y \in -A$ and $u \in B'$, $q_y + u$ is a local quadratic that is defined everywhere on $B + y + u$, and hence on $B' + u$ (since $B' - y \subset B' + A \subset B$). Therefore, since translating a local quadratic has the effect of adding a Freiman homomorphism, for each $y \in -A$ the function $Q_y(x) = \mathbb{E}_{u \in x - B'} \omega^{q_y + u}(x)$ is a quadratic average with base $(B', q)$. It follows that $S$ is a generalized quadratic average with base $(B', q)$. By the triangle inequality, $\|Q - S\|_\infty \leq 4\epsilon$.

The result for generalized quadratic averages follows easily: one simply applies the result just proved to each individual quadratic average and uses the triangle inequality. \qed

**Lemma 4.2.** Let $\epsilon > 0$. Let $B_1$ and $B_2$ be regular Bohr sets, let $q_1$ and $q_2$ be quadratic functions defined on them and let $Q_1$ and $Q_2$ be generalized quadratic averages with bases $(B_1, q_1)$ and $(B_2, q_2)$, respectively. Suppose that the Bohr sets $B$ and $B'$ satisfy the relations $B' \prec_\epsilon B \prec_\epsilon B_1 \cap B_2$. Then there exists a generalized quadratic average $Q'$ with base $(B', q_1 - q_2)$ such that $\|Q_1 \overline{Q_2} - Q'\|_\infty \leq 18\epsilon$.

**Proof.** The argument is similar to the proof of the previous lemma, but slightly more complicated. First of all, by that lemma we can uniformly approximate both $Q_1$ and $Q_2$ to within $4\epsilon$ by generalized quadratic averages $Q'_1$ and $Q'_2$ with bases $(B, q_1)$ and $(B, q_2)$, respectively. Suppose that $Q'_i = n_i^{-1}(Q_{i,1} + \cdots + Q_{i,n_i})$. If we can find for each pair $(r, s)$ a generalized quadratic average $Q_{rs}$ such that $\|Q_{1s} Q_{2s} - Q_{rs}\| \leq 10\epsilon$, then by taking the average over all $r$ and $s$ and applying the triangle inequality, we find that $\|Q'_1 \overline{Q'_2} - Q'\| \leq 18\epsilon$. \qed
10ε, where \( Q' \) is the average of the \( Q_{rs} \). Thus, it is enough to prove the result when \( Q'_1 \) and \( Q'_2 \) are non-generalized quadratic averages.

Let us do this and let \( Q'_1 \) and \( Q'_2 \) be given by the formulae \( Q'_1(x) = \mathbb{E}_{y \in x-B} \omega^{q_1,y}(x) \) and \( Q'_2(x) = \mathbb{E}_{y \in x-B} \omega^{q_2,y}(x) \), respectively.

Now let us imitate the previous proof. Let \( B^{-} \) be the interior associated with the pair \( B' \prec \epsilon B \). Then, as we did for \( Q \) in the previous lemma, we can uniformly approximate \( Q'_1 \) and \( Q'_2 \) to within \( \epsilon \) by quadratic averages \( R_1 \) and \( R_2 \) that are given by the formulae \( R_1(x) = \mathbb{E}_{y \in x-B} \omega^{q_1,y}(x) \) and \( R_2(x) = \mathbb{E}_{y \in x-B} \omega^{q_2,y}(x) \), respectively.

Let us examine the product \( R_1(x)R_2(x) \). It equals \( \mathbb{E}_{y,z \in x-B} \omega^{q_1,y}(x)q_2,z(x) \), which, by Lemma 2.3 (v), differs by at most 8\( \epsilon \) from

\[
\mathbb{E}_{y,z \in x-B} \mathbb{E}_{u \in -B'} \omega^{q_1,y+u(x)-q_2,z+u(x)} = \mathbb{E}_{y,z \in x-B} \mathbb{E}_{u \in -B'} \omega^{q_1,y+u(x)-q_2,z+u(x)}.
\]

But the right-hand side is the formula for a generalized quadratic average with base \((B',q_1-q_2)\), so we are done.

\( \square \)

5. The rank of a quadratic average

Recall that so far we have shown how to decompose a function defined on \( \mathbb{Z}_N \) into a linear combination of quadratic averages and an error that is small in a useful sense. As we did in [GW09b] in the proof for \( \mathbb{F}_p^n \), we shall now collect these quadratic averages into well-correlating clusters. However, before we do so, we must think about another concept that is very convenient when discussing quadratic forms on \( \mathbb{F}_p^n \) and that does not have an immediately obvious \( \mathbb{Z}_N \) analogue, namely the rank of a form.

Suppose that we have a quadratic form \( q \) defined on a subspace \( V \) of \( \mathbb{F}_p^n \). Then a simple calculation shows that \(|\mathbb{E}_{x \in V} \omega^{q(x)}| = p^{r/2}\), where \( r \) is the rank of the bilinear form \( \beta(u,v) = (q(u + v) - q(u) - q(v))/2 \) associated with \( q \). Indeed,

\[
|\mathbb{E}_{x \in V} \omega^{q(x)}|^2 = \mathbb{E}_{x,y \in V} \omega^{q(x)-q(y) = \mathbb{E}_{x,y \in V} \omega^{\beta(x+y,x-y)} = \mathbb{E}_{u,v \in V} \omega^{\beta(u,v)}.
\]

For each \( u \), the expectation over \( v \) is 0 unless \( \beta(u,v) = 0 \) for every \( v \), in which case it is 1. But the set of \( u \) such that \( \beta(u,v) \) vanishes is a subspace of \( V \) of codimension \( r \), so it has density \( p^{-r} \), which proves the result.

This calculation allowed us to argue as follows in [GW09b]. Given a quadratic form \( q \), we looked at its rank \( r \). If \( r \) was large, then \( q \) had a small average, whereas if \( r \) was small then \( q \) was constant on translates of a subspace of low codimension. This dichotomy played an important role in the proof, so we need to find an analogue for \( \mathbb{Z}_N \).
A close examination of the proof in [GW09b] shows that the main properties of rank that we used were that the rank of a sum of two quadratic forms is at most the sum of the ranks, and that if $\beta$ is a bilinear form of rank $r$ and $\phi$ and $\psi$ are linear functions, then $E_{x,y}\omega^{\beta(x,y)+\phi(x)+\psi(y)}$ has size at most $p^{-r}$.

The first of these properties looks very much like a fact of linear algebra, so it is tempting to try to develop an analogue of this linear algebra for Bohr sets. Unfortunately, although some analogues of linear algebra do exist, they are much less clean, and in any case they are completely inappropriate for our purposes, roughly speaking because the codimension of a subspace of $\mathbb{F}_{p}^n$ corresponds more to the dimension of a Bohr set in $\mathbb{Z}_N$. For example, if we are guided by linear algebra then we will be inclined to say that the function $\omega x^2$ has rank 1, but in fact we want to count it as having very high rank because the expectation $E_{x,y}\omega^{2xy}$ is tiny.

There is, however, a rather easy way to define an appropriate notion of rank in the $\mathbb{Z}_N$ context, which is to exploit the fact that there is a completely different alternative definition in $\mathbb{F}_{p}^n$. We observed above that $|E_x\omega^{q(x)}|$ is not just at most $p^{-r/2}$ but actually equal to $p^{-r/2}$. Therefore, we could, if we wanted, define the rank of $q$ to be $\log_p(\alpha^{-1})$, where $\alpha = |E_x\omega^{q(x)}|^2$. And this gives us a definition that can be carried over much more easily to functions defined on Bohr sets in $\mathbb{Z}_N$. (It can also be carried over much more easily to polynomial forms of higher degree and their associated multilinear forms. This was essential to us in [GW09c].)

We do of course pay a price for such a move. If we define rank in this way then it becomes true by definition that averages over quadratic phase functions of high rank are small. But we clearly cannot avoid doing any work: it is now not obvious that rank is subadditive or that a quadratic function of low rank has linear structure. In fact, neither of these statements is exactly true, but with some effort we will be able to prove usable approximations to them.

One final remark is that it turns out to be more convenient to focus on bilinear forms rather than quadratic forms. On a subspace of $\mathbb{F}_{p}^n$ the two are basically equivalent, but on a Bohr set $B$ in $\mathbb{Z}_N$ they no longer are, because $q(a+b) - q(a) - q(b)$ is not defined for every $a, b \in B$. We therefore have to look at a smaller structured set inside $B$.

Here, then, is the definition that we shall use. Some of the features of the definition may look a bit strange, but they are chosen to make later proofs run more smoothly.

**Definition.** Let $B$ be a Bohr set and let $q$ be a quadratic form on $B$. Let $B'$ be a Bohr set such that $2B' - 2B' \subset B$ and let $P$ be a subset of $B'$. The rank of the local quadratic form $q$ with respect to $P$ is defined as

$$\min \left\{ r : \min_{P'} E_{x,y}\omega^{q(x,y)+\phi(x)+\psi(y)} \leq p^{-r} \right\},$$

where $P'$ is a Bohr set of size $|P|^2$ and $\phi, \psi$ are linear functions.

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where $P'$ is a Bohr set of size $|P|^2$ and $\phi, \psi$ are linear functions.
Lemma 5.1. Let 0 < \epsilon < 1/20, let \( B \) and \( B' \) be Bohr sets satisfying \( B' \prec_\epsilon B \) and let \( P \subset B' \). Let \( q \) be a quadratic form on \( B \) and let \( Q \) be a generalized quadratic average with base \( (B, q) \). Suppose that the rank of \( Q \) relative to \( P \) is \( r \). Then \( |E_x Q(x)| \leq (11\epsilon + e^{-r})^{1/4} \).

**Proof.** Let \( h \) be a local quadratic phase function defined by a formula of the form \( h(x) = B(x-y)\omega^q(x-y) + \phi(x) \) for some Freiman homomorphism \( \phi \) (so in particular \( h \) is supported in the Bohr neighbourhood \( B + y \)). Let us estimate \( |E_{x \in B+y} h(x)| \), using Lemma 2.3 and the Cauchy-Schwarz inequality. Since \( \phi \) is an arbitrary Freiman homomorphism, it is enough to do this when \( y = 0 \), so let us assume that that is the case. Recall that \( \approx \) stands for the relation “differs by at most \( \epsilon \)” from.

By Lemma 2.3 (i) applied twice, we know that \( E_{x \in B} h(x) \approx 2\epsilon E_{x \in B} E_{a,b \in P} h(x + a + b) \). It is not hard to check that if \( \epsilon < 1/20 \) and \( \alpha \) and \( \beta \) are complex numbers such that \( |\alpha| \leq 1 \) and \( \alpha \approx 2\epsilon \beta \), then \( \alpha^4 \approx 10\epsilon \beta^4 \). Therefore, since \( ||h||_{\infty} \leq 1 \),

\[
|E_{x \in B} h(x)|^4 \approx 10\epsilon |E_{x \in B} E_{a,b \in P} h(x + a + b)|^4 \\
\leq E_{x \in B} |E_{a,b \in P} h(x + a + b)|^4.
\]
Now let us look at the inner expectation when $x \in B^-$. We have
\[
|E_{a,b \in P} h(x + a + b)|^4 \leq \left( E_{a \in P} |E_{b \in P} h(x + a + b)|^2 \right)^2
\]
\[
= (E_{b,b' \in P} E_{a \in P} h(x + a + b)h(x + a + b'))^2
\]
\[
\leq E_{b,b' \in P} |E_{a \in P} h(x + a + b)h(x + a + b')|^2
\]
\[
= E_{a,a' \in P} E_{b,b' \in P} h(x + a + b)h(x + a + b')h(x + a' + b)h(x + a' + b')
\]
\[
= E_{a,a' \in P} E_{b,b' \in P} \omega_{a+b-a'-b'}^{-q(a-a')-q(b-b')},
\]
which equals $e^{-r}$ by definition of the rank relative to $P$. The proportion of $x$ that belong to $B \setminus B^-$ is at most $\epsilon$, by regularity, and for these the inner expectation is at most 1.

This proves that $|E_{x \in B} h(x)|^4 \leq 11\epsilon + e^{-r}$, and hence that $|E_{x \in B} h(x)| \leq (11\epsilon + e^{-r})^{1/4}$.

Now let $Q$ be a quadratic average given by a formula of the kind $Q(x) = \sum_{y \in x - B} \omega_q(y)$. Then $E_x Q(x) = \sum_{y \in x - B} E_{y \in x} \omega_{q(y)}$. By the estimate just established, this has absolute value at most $(11\epsilon + e^{-r})^{1/4}$. Finally, this implies the same upper bound when $Q$ is a generalized quadratic average.

We remark that for the lemma just proved to be useful, one needs $\epsilon$ to be comparable to or smaller than $e^{-r}$. This may seem to be quite a strong requirement, given that we also need $B' \prec \epsilon B$. However, a recurring theme in this paper is that one can afford to take Bohr sets of small width: it is the dimension that one has to be careful about. So in fact the bound above is not too expensive for our later arguments to work.

We shall now prove a more general result. The proof we give is in two senses not optimal. The first is that we obtain a bound that is weaker than it needs to be, because we estimate an $\ell_4$ norm in terms of an $\ell_\infty$ norm. The second, more serious, is that we use Fourier analysis. The reason this is a defect is that it obscures the fact that the proof can be carried out in physical space and is therefore not hard to generalize. However, since in this paper we shall not be dealing with the cubic case for functions defined on $\mathbb{Z}_N$, this is not enough of a defect to outweigh the advantage that the proof we give is very simple and does not involve technicalities concerning regular Bohr sets.

**Lemma 5.2.** Let $B$ and $B'$ be Bohr sets satisfying $B' \prec \epsilon B$ and let $P$ be a subset of $B'$. Let $q$ be a quadratic form on $B$ and let $Q$ be a generalized quadratic average with base $(B,q)$. Suppose that the rank of $Q$ relative to $P$ is $r$. Then $\|Q\|_{L^2} \leq (11\epsilon + e^{-r})^{1/8}$. 
Proof. For every \( u \) the function \( Q(x)\omega^{ux} \) satisfies all the hypotheses of Lemma 5.1. Therefore, by that lemma, \( |\hat{Q}(u)| \leq (11\epsilon + e^{-r})^{1/4} \) for every \( u \). Since \( \|\hat{Q}\|_2 \leq \|Q\|_2 \leq 1 \), it follows that \( \|\hat{Q}\|_4 \leq (11\epsilon + e^{-r})^{1/2} \) and hence that \( \|Q\|_{U^2} \leq (11\epsilon + e^{-r})^{1/8} \). □

The next result expresses the idea that if two quadratic averages \( Q \) and \( Q' \) correlate well then they have a “low-rank difference” in the exponent. Very roughly speaking, this is because \( QQ' \) has large average, and is therefore a low-rank quadratic. Of course, this is not quite the correct argument, because \( Q \) and \( Q' \) are averages of quadratic phase functions defined on several different Bohr neighbourhoods. However, the basic idea is sound, as the next result shows.

**Corollary 5.3.** Let \( B \) and \( B' \) be two arbitrary Bohr sets, let \( q \) and \( q' \) be quadratic forms on \( B \) and \( B' \), and let \( Q \) and \( Q' \) be generalized quadratic averages with bases \((B, q)\) and \((B', q')\). Let \( B_1, B_2 \) and \( B_3 \) be Bohr sets satisfying the chain of relations \( B_3 \prec \epsilon B_2 \prec \epsilon B_1 \prec \epsilon B \cap B' \). Let \( P \) be a subset of \( B_3 \). Suppose that the rank of the function \( B_2(x)\omega^{q(x)-q'(x)} \) relative to \( P \) is at least \( r \). Then \( |\langle Q, Q' \rangle| \leq 18\epsilon + (11\epsilon + e^{-r})^{1/4} \).

Proof. By Lemma 4.2 there is a generalized quadratic average \( Q'' \) with base \((B_2, q-q')\) such that \( \|QQ' - Q''\|_\infty \leq 18\epsilon \). By Lemma 5.1 and our hypothesis, \( \|Q''(x)\| \leq (11\epsilon + e^{-r})^{1/4} \). Since \( \langle Q, Q' \rangle = \mathbb{E}_x(QQ')(x) \), the result follows. □

### 6. The Structure of Low-rank Bilinear Forms on Bohr Sets

Our next task is to understand the implications if the hypotheses of Corollary 5.3 do not hold. In the \( \mathbb{F}_p^n \) case, we argued that if \( Q \) and \( Q' \) are quadratic averages and \( \langle Q, Q' \rangle \) is not small, then \( QQ' \) has low rank, from which it follows that \( QQ' \) is constant on cosets of a low-codimensional subspace. From this we deduced that \( \|QQ'\|_{U^2} \) is not too large. In this paper, where \( Q \) and \( Q' \) are defined on a Bohr set \( B \), we shall argue that \( QQ' \) is approximately constant on translates of a small (but not too small) multidimensional arithmetic progression \( P \subset B \), and deduce that \( QQ' \) can be uniformly approximated by a function with smallish \( (U^2)^* \) norm.

A similar result to this was proved by Green and Tao in [GrT08] using a “local Bogolyubov lemma” that they developed specially for the purpose. Their argument can be used to show that \( QQ' \) is approximately constant on a Bohr subset \( B' \) of \( B \). However, the local Bogolyubov lemma is rather expensive, in that the dimension of \( B' \) is considerably larger than that of \( B \). This expense has to be iterated, and it turns out that if we were to use their result, then we would end up with a tower-type bound for our final estimate.
By contrast, the progression $P$ that we find has the same dimension as that of $B$ and the final estimate we obtain is doubly exponential. Unfortunately, our argument is rather uglier than that of Green and Tao since we rely on the fact that bilinear forms on multidimensional arithmetic progressions can be explicitly described, rather than just using the defining properties of bilinear forms on Bohr sets.

Although we eventually need a statement about Bohr sets, we shall begin by proving a dichotomy for bilinear phase functions defined on multidimensional progressions. As a prelude, here is a proof for the special case of one-dimensional progressions. It will be useful for the general case if we prove a non-symmetric result where one variable belongs to one progression and the other to another of a possibly different length. We shall also allow our bilinear forms to be non-homogeneous. That is, we shall consider functions of the form $b(x,y) = e(\alpha xy + \lambda x + \mu y + \nu)$ with $1 \leq x \leq m_1$ and $1 \leq y \leq m_2$. The aim will be to prove that such a function either has a small average (where this means smaller than a small positive constant $c$) or is approximately constant on a reasonably large subgrid of $[m_1] \times [m_2]$ (where this means a subgrid of size at least $c'm_1m_2$ for some not too small positive constant $c'$, but $c'$ is allowed to be smaller than $c$ and this elbow room will be quite helpful). The precise statement is as follows. As is standard, if $\theta$ is a real number then we write $|\theta|$ for the distance from $\theta$ to the nearest integer.

**Lemma 6.1.** Let $c > 0$ and let $m_1$ and $m_2$ be positive integers. Let $\beta(x,y)$ be the bilinear phase function $e(\alpha xy + \lambda x + \mu y + \nu)$, defined when $0 \leq x < m_1$ and $0 \leq y < m_2$. Suppose that $|\mathbb{E}_{x,y} b(x,y)| \geq 2c$. Then there exists a positive integer $q \leq 2c^{-1}$ and an integer $p$ such that $|\alpha - p/q| \leq 2c^{-2}/m_1m_2$. In particular, $|\alpha xy| \leq 2c$ whenever $x$ and $y$ are both multiples of $q$ and $x/m_1$ and $y/m_2$ are both at most $c'^{1/2}$.

**Proof.** Observe first that $\mathbb{E}_{x,y} b(x,y) \leq \mathbb{E}_y |\mathbb{E}_x b(x,y)| = \mathbb{E}_y |\mathbb{E}_x e(\alpha xy + \lambda x + \mu y + \nu)|$. (Here, as in the statement of the lemma, the expectations are over all $x$ and $y$ with $0 \leq x < m_1$ and $0 \leq y < m_2$.) Now for each $y$, the quantity $|\mathbb{E}_x e(\alpha xy + \lambda x + \mu y + \nu)|$ is at most $\min\{1, 1/m_1|\alpha y + \lambda|\}$, by the formula for summing a geometric progression. In particular, it is at most $c$ unless $|\alpha y + \lambda| \leq C/m_1$, where $C = 1/c$. So the only way that $\mathbb{E}_y |\mathbb{E}_x b(x,y)|$ can be at least $2c$ is if $|\alpha y + \lambda| \leq C/m_1$ for at least $cm_2$ values of $y$ (since otherwise we get less than $c + c$).

So now let us think about what is implied if $|\alpha y + \lambda| \leq C/m_1$ for at least $cm_2$ values of $y$. We shall show that $\alpha$ is within $C'/m_1m_2$ of a rational with small denominator, where with the benefit of hindsight we choose $C'$ to equal $2C/c = 2c^{-2}$. It is an easy and standard consequence of the pigeonhole principle that there is a rational $p/q$ with $q \leq m_1m_2/C'$
such that $|\alpha - p/q| \leq C'/m_1m_2q$. It follows that $|\alpha y - py/q| \leq C'/m_1q$ for every $y \leq m_2$.

Therefore, either $q \leq C'/C$ or $|\lambda + py/q| \leq 2C/m_1$ whenever $|\alpha y + \lambda| \leq C/m_1$.

So now let us bound the number of multiples $py/q$ of $p/q$ such that $\lambda + py/q$ can be within $2C/m_1$ of an integer, given that $p$ and $q$ are coprime. To do this we split into cases. If $q < m_1/4C$, then $1/q > 4C/m_1$, so two translates of multiples of $1/q$ that are distinct mod 1 cannot both be within $2C/m_1$ of an integer. But since $p$ and $q$ are coprime, any $q$ distinct multiples of $p/q$ are also distinct multiples of $1/q$ mod 1, so at most one of them is within $2C/m_1$ of an integer when you add $\lambda$ to it. So if $q < m_1/4C$, then $|\alpha y| \leq C/m_1$ for at most $q^{-1}m_2 + 1$ values of $y$. (The “+1” is there because $m_2$ doesn’t have to be a multiple of $q$.) If this is at least $cm_2$, then $q$ is certainly at most $2c^{-1}$.

If $q \geq m_1/4C$ then we argue differently. This time we argue that the number of multiples of $1/q$ that are distinct mod 1 and lie within $2C/m_1$ of an integer is at most $2Cq/m_1$.

Since $q \leq m_1m_2/C'$, this is at most $2Cm_2/C' = cm_2$. Therefore, the number of $y$ such that $|\alpha y| \leq C/m_1$ is also at most $cm_2$.

In conclusion, either $q \leq 2c^{-1}$ or $E_y|E_x b(x,y)| = E_y|E_x e(\alpha xy + \lambda x + \mu y + \nu)| \leq 2c$. In the first case, we have $|\alpha - p/q| \leq C'/m_1m_2q = 2c^{-2}/m_1m_2q$, which implies the first assertion. If $x$ and $y$ are multiples of $q$ then $|\alpha xy| \leq 2c^{-2}xy/m_1m_2$. Therefore, if in addition $xy \leq c^3m_1m_2$, then we have that $|\alpha xy| \leq 2c$. In particular, $|\alpha xy| \leq 2c$ if $x$ and $y$ are both multiples of $q$ and $x \leq c^{3/2}m_1$ and $y \leq c^{3/2}m_2$.

Let us now see how Lemma 6.1 generalizes to a similar statement for bilinear phase functions on $d$-dimensional arithmetic progressions. This turns out to follow fairly straightforwardly from the one-dimensional case.

So now $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ range over $d$-dimensional arithmetic progressions, and we are looking at a function of the form $b(x,y) = e(\sum_{i,j} \alpha_{ij} x_i y_j)$. We would like to show that either the average of $b(x,y)$ is small or every $\alpha_{ij}$ is extremely close to a rational with small denominator. In the latter case, we will be able to restrict to a subprogression of the same dimension that is not too much smaller on which $b$ is approximately constant.

**Corollary 6.2.** Let $c > 0$, let $m_1, \ldots, m_d$ be positive integers and let $P$ be the multidimensional progression $\prod_{i=1}^d \{0, 1, \ldots, m_i - 1\}$. Let $b$ be a bilinear phase function on $P$ given by the formula $b(x,y) = e(\sum_{i,j} \alpha_{ij} x_i y_j + \sum_i \lambda_i x_i + \sum_j \mu_j y_j + \nu)$. Then either $|E_{x,y} b(x,y)| \leq 2c$ or there exist positive integers $q_{rs} \leq 2c^{-1}$ and integers $p_{rs}$ such that $|\alpha_{rs} - p_{rs}/q_{rs}| \leq 2c^{-2}/m_r m_s$ for every $r,s \leq d$. In the second case, there are positive integers $q_1, \ldots, q_d \leq (2c^{-1})^{2d}$ such that $\|\sum_{rs} \alpha_{rs} x_r y_s\| \leq 2d^2c$ whenever $x$ and $y$ belong to
the subprogression $P'$ that consists of all $z \in P$ such that each $z_r$ is of the form $h_rq_r$ for some $h_r$ between 0 and $c^{3/2}m_r$.

**Proof.** Let us fix all coordinates of $x$ and $y$ apart from $x_r$ and $y_s$ and estimate the quantity $|E_{x_r}E_{y_s}b(x,y)|$. We can write this expression in the form $|E_{x_r,y_s}c(\alpha_{rs}x_ry_s + \lambda x_r + \mu y_s + \nu)|$, where $\lambda$, $\mu$ and $\nu$ depend on the other coordinates of $x$ and $y$. Therefore, by Lemma 6.1 either $|E_{x_r,y_s}b(x,y)| \leq 2c$ or there exists a positive integer $q_{rs} \leq 2c^{-1}$ and an integer $p_{rs}$ such that $|\alpha_{rs} - p_{rs}/q_{rs}| \leq 2c/m_rm_s$.

In the second case, $\|\alpha_{rs}x_ry_s\| \leq 2c$ whenever $x_r$ and $y_s$ are both multiples of $q_{rs}$ and $x_r/m_r$ and $y_s/m_s$ are both at most $c^{3/2}$. A quick examination of the proof of Lemma 6.1 shows that the choice of $q$ did not depend on $\lambda$, $\mu$ or $\nu$, but only on the rational approximations to $\alpha$. Therefore, by averaging over all possible values of the other coordinates of $x$ and $y$ we may conclude that either $|E_{x,y}b(x,y)| \leq 2c$ or there exists a positive integer $q_{rs} \leq 2c^{-1}$ and an integer $p_{rs}$ such that $|\alpha_{rs} - p_{rs}/q_{rs}| \leq 2c/m_rm_s$. (This is the same conclusion as that of the previous paragraph, but the assumption is different, since now we are averaging over all $x$ and $y$ rather than fixing all but one coordinate.) In the second case, $\|\alpha_{rs}x_ry_s\| \leq 2c$ whenever $x_r$ and $y_s$ are both multiples of $q_{rs}$ and $x_r/m_r$ and $y_s/m_s$ are both at most $c^{3/2}$.

Since this is true for every $r$ and $s$, either $|E_{x,y}b(x,y)| \leq 2c$ or there are $d^2$ positive integers $q_{rs} \leq 2c^{-1}$ such that $\|\alpha_{rs}x_ry_s\| \leq 2c$ whenever $x_r$ and $y_s$ are both multiples of $q_{rs}$ and $x_r/m_r$ and $y_s/m_s$ are both at most $c^{3/2}$. For each $r$, let $q_r$ be the product of all the $q_{rs}$ and all the $q_{sr}$. Then $q_r$ is at most $(2c^{-1})^{2d}$, and if $x_r$ is a multiple of $q_r$ and $y_s$ is a multiple of $q_s$ with $x_r/m_r$ and $y_s/m_s$ both at most $c^{3/2}$, then again $\|\alpha_{rs}x_ry_s\| \leq 2c$. But if that is true for every $r$ and every $s$, then $\|\sum_{r,s} \alpha_{rs}x_ry_s\| \leq 2d^2c$, which proves the result. \square

Our next target is to prove that quadratic averages either have small $U^2$ norms or are uniformly close to functions with moderately small $U^2$-dual norms. We begin with a lemma about linear phase functions on subsets of $\mathbb{Z}_N$. Before stating it, let us give a definition that generalizes our earlier concepts of interior, closure and boundary to arbitrary pairs of sets.

**Definition.** Given a pair $(A, B)$ of subsets of $\mathbb{Z}_N$, define the closure of $A$ (relative to $B$) to be $A + B$ and the interior to be $\{ x : x + B \subset A \}$. Denote these by $A^+$ and $A^-$, respectively. Define the boundary of $A$ to be $A^+ \setminus A^-$ and denote it by $\partial A$.

As before, when we use the notation $A^+$, $A^-$ and $\partial A$, it will always be clear from the contexts what the set $B$ is that we are implicitly talking about.
Lemma 6.3. Let $A$ be a subset of $\mathbb{Z}_N$, let $\phi : A \to \mathbb{Z}_N$ be a Freiman homomorphism, let $B$ be a subset of $A - A$ that contains 0, and let $\psi$ be the function $\psi(d) = \phi(x + d) - \phi(x)$ for some $x \in A \cap (A - d)$, which is well-defined everywhere on $B$. Let the densities of $A$ and $B$ be $\gamma$ and $\theta$. Let $f$ be the function defined by taking $f(x) = \gamma^{-1} \omega^{\phi(x)}$ when $x \in A$ and 0 otherwise, and let $g$ be defined by taking $g(d) = \theta^{-1} \omega^{\psi(d)}$ whenever $d \in B$, and 0 otherwise. Then $\|f - f \ast g\|_\infty \leq 2\gamma^{-1}$, and $f - f \ast g$ is supported inside the boundary $\partial A$.

Proof. First let us deal with the uniform bound for $f - f \ast g$. Since $\|f\|_\infty \leq \gamma^{-1}$, it is enough to prove that $\|f \ast g\|_\infty \leq \gamma^{-1}$. But this is clear because $f \ast g(x) = \mathbb{E}_{d \in B} f(x - d) \omega^{\psi(d)}$, which is an average of numbers with absolute value at most $\gamma^{-1}$. (This equality is the reason for normalizing $g$ with the constant $\theta^{-1}$.)

If $x \notin A^+ = A + B$, then $f(x - d) = 0$ for every $d \in B$, so $f \ast g(x) = 0$. Since $0 \in B$ and $f$ is supported in $A$, $f(x) = 0$ as well.

If $x \in A^-$, then

$$f \ast g(x) = \mathbb{E}_{d \in B} f(x - d) \omega^{\psi(d)} = \mathbb{E}_{d \in B} \omega^{\phi(x - d) + \psi(d)} = \mathbb{E}_{d \in B} \omega^{\phi(x)} = f(x).$$

This proves the lemma. \qed

We would like to think of $f \ast g$ as approximating $f$, so we shall apply Lemma 6.3 to a pair of sets $A$ and $B$ such that $\partial A$ is small. We have already seen such pairs in the context of regular Bohr neighbourhoods, but we now need to look at multidimensional arithmetic progressions as well.

Lemma 6.4. Let $P$ be a proper $d$-dimensional arithmetic progression consisting of all points $x_0 + \sum_{i=1}^d a_i x_i$ such that $0 \leq a_i < m_i$, let $\epsilon > 0$, and let $Q$ be the progression consisting of all points $\sum_{i=1}^d b_i x_i$ such that $0 \leq b_i < \epsilon m_i / d$. Let the density of $P$ be $\gamma$. Then the density of $P^+ \setminus P^-$ is at most $3\epsilon \gamma$ and the density of $Q$ is at least $(\epsilon / d)^d \gamma$.

Proof. The number of integers of the form $r + s$, where $r$ is an integer between 0 and $m - 1$ and $s$ is an integer such that $0 \leq s \leq \eta m$ is at most $(1 + \eta)m$, since we have equality when $\eta m$ is an integer, and if we increase $\eta m$ towards the next integer then we increase $(1 + \eta)m$ without increasing the number of elements of the set.

Now suppose that $r$ is an integer and that $r \geq \lfloor \eta m \rfloor$. Then $r - s \geq 0$ whenever $s$ is an integer and $s < \eta m$. The number of integers less than $m$ with this property is $m - \lfloor \eta m \rfloor \geq m(1 - \eta)$.
From these two calculations, we find that $P^+$ has density at most $(1 + \epsilon/d)^d \gamma$ and $P^-$ has density at least $(1 - \epsilon/d)^d \gamma$. The first result now follows from the simple estimates $(1 + \epsilon/d)^d \leq 1 + 2\epsilon$ and $(1 - \epsilon/d)^d \geq 1 - \epsilon$.

Also, the number of integers $r$ such that $0 \leq r < \eta m$ is $\lceil \eta m \rceil \geq \eta m$, so the density of $Q$ is at least $(\epsilon/d)^d$ times that of $P$, so we have the second assertion as well. \qed

Next, we show why approximating a function by a convolution of two functions helps us to control its $U^2$-dual norm.

**Lemma 6.5.** Let $A$ and $B$ be two sets and let $f$ and $g$ be two functions such that $|f|$ is bounded above by the characteristic measure of $A$ and $g$ is bounded above by the characteristic measure of $B$. Suppose that the density of $A$ is $\gamma$ and the density of $B$ is $\theta$. Then $\|f * g\|_{U^2} \leq \gamma^{-1/2} \theta^{-1/4}$.

**Proof.** Let $h$ be any other function, and define $g^*$ by $g^*(x) = \overline{g(-x)}$ for every $x$. Then

$$\langle f * g, h \rangle = \langle f, g^* * h \rangle \leq \|f\|_2 \|g^* * h\|_2 \leq \gamma^{-1/2} \|g\|_{U^2} \|h\|_{U^2} \leq \gamma^{-1/2} \theta^{-1/4} \|h\|_{U^2},$$

from which the result follows. Here we have used the fact that the characteristic measure of a set of density $\delta$ has $L_2$ norm at most $\delta^{-1/2}$ and $U^2$ norm at most $\delta^{-1/4}$. We have also made use of the inequality $\|u * v\|_2 \leq \|u\|_{U^2} \|v\|_{U^2}$, which can be thought of as a special case of Young’s inequality or as a special case of Lemma 3.8 of [G01], a Cauchy-Schwarz inequality for the uniformity norms. \qed

Putting the last three lemmas together, we deduce the following.

**Lemma 6.6.** Let $P$, $\gamma$, $\epsilon$ and $Q$ be as in Lemma 6.4. Let $\phi$ be a Freiman homomorphism defined on $P$, and let $f(x) = \gamma^{-1} \omega^{\phi(x)}$ if $x \in P$ and 0 otherwise. Then there exists a function $h$ such that $\|f - h\|_\infty \leq 2\gamma^{-1}$, $\|h\|_{U^2}^* \leq \gamma^{-3/4} (\epsilon/d)^{-d/4}$, and $f - h$ is supported in $P^+ \setminus P^-$, which has density at most $3 \epsilon \gamma$.

**Proof.** Let us apply Lemma 6.5 with $A = P$, $B = Q$, $f$ as given in this lemma, and $g$ as defined in the statement of Lemma 6.3. We shall prove that we can take $h$ to be the function $f * g$. Lemma 6.3 tells us that $\|f - f * g\|_\infty \leq 2\gamma^{-1}$ and that $f - f * g$ is supported in $P^+ \setminus P^-$. Lemma 6.4 tells us that $P^+ \setminus P^-$ has density at most $3 \epsilon \gamma$, and lemma 6.5 tells us that $\|f * g\|_{U^2} \leq \gamma^{-1/2} \theta^{-1/4}$, where $\theta$ is the density of $Q$. Lemma 6.4 tells us that $\theta$ is at least $(\epsilon/d)^d \gamma$, and this completes the proof. \qed
We are about to prove a slightly complicated technical lemma that will help us handle error terms without cluttering up proofs. Before we do so, here is a much simpler technical lemma that will help us to prove the complicated one without cluttering up its proof.

**Lemma 6.7.** Let \( \alpha, \beta, \rho \) and \( \sigma \) be positive constants. Let \( U \) and \( V \) be subsets of \( \mathbb{Z}_N \) of density \( \sigma \alpha \) and \( \beta \), respectively. For each \( y \in V \) let \( g_y \) be a function supported in \( y + U \) such that \( \|g_y\|_\infty \leq \rho \alpha^{-1} \). Then \( \|E_{y \in V} g_y\|_\infty \leq \rho \sigma \beta^{-1} \).

**Proof.** For each \( x \),
\[
|E_{y \in V} g_y(x)| \leq \rho \alpha^{-1} \mathbb{P}[x \in y + U | y \in V] \leq \rho \alpha^{-1} \sigma \alpha^{-1} = \rho \sigma \beta^{-1}.
\]
The lemma follows. \( \Box \)

**Lemma 6.8.** Let \((A, B)\) and \((C, D)\) be two pairs of subsets of \( \mathbb{Z}_N \) with \( C^+ \subset B \). Let the densities of \( A \) and \( C \) be \( \beta \) and \( \gamma \), respectively. Suppose also that \( \partial C \) has density at most \( \epsilon \gamma \). Let \( g \) be a function defined on \( \mathbb{Z}_N \) such that \( |g(x)| \leq \beta^{-1} \) for every \( x \in A \) and \( g(x) = 0 \) for every \( x \notin A \). For each \( y \in A \) let \( g_y \) be a function such that \( \|g_y\|_\infty \leq \gamma^{-1} \) and \( g_y \) is supported in \( y + C \). Suppose that \( E_{y \in A} g_y(x) = g(x) \) for every \( x \in A^- \). Now suppose that for each \( y \in A^- \) there is a function \( h_y \) such that \( |g_y(x) - h_y(x)| \leq \theta \gamma^{-1} \) for every \( x \in y + C^- \), \( |g_y(x) - h_y(x)| \leq \lambda \gamma^{-1} \) for every \( x \in y + \partial C \), and \( g_y(x) = h_y(x) = 0 \) whenever \( x \notin y + C^+ \). And for each \( y \in A \setminus A^- \), let \( h_y \) be identically zero. Let \( h(x) = E_{y \in A} h_y(x) \) for every \( x \in \mathbb{Z}_N \). Then \( |g(x) - h(x)| \leq (\theta + \lambda \epsilon) \beta^{-1} \) for every \( x \in A^- \), \( |g(x) - h(x)| \leq (4 + \lambda \epsilon) \beta^{-1} \) for every \( x \in \partial A \), and \( g(x) = h(x) = 0 \) for every \( x \notin A^+ \).

**Proof.** If \( x \in A^- \) then \( E_{y \in A} g_y(x) = g(x) \), by hypothesis. If \( x \in \partial A \), then Lemma 6.7 (with \( U = C \) and \( V = A \)) implies that \( |E_{y \in A} g_y(x)| \leq \beta^{-1} \), which implies that \( |g(x) - E_{y \in A} g_y(x)| \leq 2 \beta^{-1} \). And if \( x \notin A^+ \), then both \( g(x) \) and \( E_{y \in A} g_y(x) \) are zero.

Let us write \( u_y \) for the restriction of \( g_y - h_y \) to \( y + C^- \) and \( v_y \) for the restriction of \( g_y - h_y \) to \( y + \partial C \). Then \( g_y - h_y = u_y + v_y \) for every \( y \in A \). If \( y \in A^- \), then \( \|u_y\|_\infty \leq \theta \gamma^{-1} \) and \( \|v_y\|_\infty \leq \lambda \gamma^{-1} \). If \( y \in A \setminus A^- \) then \( \|u_y\|_\infty \) and \( \|v_y\| \) are both at most \( \gamma^{-1} \).

For every \( x \in A^- \), \( |E_{y \in A} u_y(x)| \leq \theta \beta^{-1} \) by Lemma 6.7 (with \( U = C^- \) and \( V = A \)). For every \( x \in \partial A \), \( |E_{y \in A} u_y(x)| \leq 2 \beta^{-1} \), again by Lemma 6.7. (In this case, we have the bound \( \|u_y\|_\infty \leq 2 \gamma^{-1} \).) And for every \( x \notin A^+ \), \( E_{y \in A} u_y(x) = 0 \).

If \( x \in A^+ \), then \( |E_{y \in A} v_y(x)| \leq \lambda \epsilon \beta^{-1} \), again by Lemma 6.7 (this time with \( U = \partial C \)). And if \( x \notin A^+ \), then \( E_{y \in A} v_y(x) = 0 \).
Adding these estimates together, we find that $|\mathbb{E}_{y \in A} g_y(x) - \mathbb{E}_{y \in A} h_y(x)|$ is at most $(\theta + \lambda \epsilon) \beta^{-1}$ if $x \in A^-$, at most $(2 + \lambda \epsilon) \beta^{-1}$ if $x \in \partial A$, and 0 if $x \notin A^+$. Finally, combining this with the estimates for $g - \mathbb{E}_{y \in A} g_y$ in the first paragraph, we obtain the result stated. \qed

In the next statement, it may not be clear why $\eta$ cannot be taken to be arbitrarily small. The reason is that the maximum possible density $\gamma$ decreases with $\eta$, so in fact the bound on $\|Q''\|_{U^2}$ increases as $\eta$ decreases.

**Corollary 6.9.** Let $0 < \epsilon \leq 1$ and let $0 < \eta \leq 1/20$. Let $B$ be a regular Bohr set of density $\beta$ and let $B'$ be a Bohr subset with $B' \prec \eta B$. Let $P$ be a $d$-dimensional arithmetic progression of density $\gamma$ such that $P + P$ lives inside $B'$, let $\ell$ be a quadratic form on $B$ and let $Q$ be a general quadratic average with base $(B, \ell)$. Then for every $\epsilon > 0$, either $\|Q\|_{U^2} \leq (11 \eta + \alpha)^{1/8}$ or there exists a function $Q''$ such that $\|Q - Q''\|_\infty \leq 4 \pi d^2 \alpha + 2 \epsilon + 7 \eta$ and $\|Q''\|_{U^2} \leq \gamma^{3/4}(\epsilon/d)^{-d/4}$, where $\gamma' \geq (\alpha/4)^{4d^2} \gamma$.

**Proof.** Suppose first that $Q$ has rank at most $\log(1/\alpha)$ relative to $P$. In this case, we are immediately done, since Lemma [5.2] tells us that $\|Q\|_{U^2} \leq (11 \eta + \alpha)^{1/8}$.

Now suppose that $Q$ has rank at least $\log(1/\alpha)$ relative to $P$. As usual, let us begin by assuming that $Q$ is a non-generalized quadratic average, so that it has a formula of the form $Q(x) = \mathbb{E}_{y \in x - B} \omega^{q_y(x)}$, where $q_y(x) = q(x - y) + \phi_y(x - y)$ for some Freiman homomorphism $\phi_y$ defined on $B$. For each $y$ let us define $f_y(x)$ to be $\beta^{-1} \omega^{q_y(x)}$ if $x \in y + B$ and 0 otherwise. Then, as we commented after defining quadratic averages, $Q$ is the average of all the functions $f_y$. The strategy of our proof will be to show that each function $f_y$ can be approximated by a function with small $U^2$-dual norm in such a way that the average of all the errors is uniformly small.

We shall begin by examining $f = f_0$, which is supported in $B$. By the definition of rank, we have the inequality

$$|\mathbb{E}_{a,a',b,b'} \in P \omega^{q(a + b - a' - b') - q(a - a') - q(b - b')}| \geq \gamma.$$

It follows that there exist $a'$ and $b'$ in $P$ such that

$$|\mathbb{E}_{a,b} \in P \omega^{q(a + b - a' - b') - q(a - a') - q(b - b')}| \geq \gamma.$$

Choose such an $a'$ and $b'$, and write $\beta(u, v)$ for $q(u + v) - q(u) - q(v)$. Then

$$q(a + b - a' - b') - q(a - a') - q(b - b') = \beta(a - a', b - b')$$

which we can expand into the homogeneous part $\beta(a, b)$ and the linear and constant terms $-\beta(a', b) - \beta(a, b') + \beta(a', b')$.}


Let us discuss further the relationship between $g$ and $\beta$. A quadratic homomorphism on $P$ must be given by a formula of the form $q(x) = \sum_{i,j} a_{ij} x_i x_j + \sum_i b_i x_i + c$ for a matrix $(a_{ij})$ that we may take to be symmetric (since we can replace it by $(a_{ij} + a_{ji})/2$). Then $\beta(u, v)$ works out to be $2 \sum_{i,j} a_{ij} u_i v_j$, and there are coefficients $b_i'$ and $c_j'$ and $d'$ such that $\beta(u - a', v - b') = 2 \sum_{i,j} a_{ij} u_i v_j + \sum_i b_i' u_i + \sum_j c_j' v_j + d'$. Moreover, $q(x) = \beta(x, x)/2$ for every $x$.

Since $|E_{a,b \in P} \omega^\beta(a-a', b-b')| \geq \alpha$, Corollary 6.2 (with $c = \alpha/2$) implies that there is a subprogression $P'$ of $P$ of density at least $(\alpha/4)^{2d^2}(\alpha/2)^{3d/2} \gamma$ and of dimension $d$ such that $|1 - \omega^\beta(a, b)| \leq 2\pi d^2 \alpha$ for every $a$ and $b$ in $P'$. If we restrict further, to pairs $(a, b)$ such that all their coordinates are even, then we obtain a progression $P''$ of density $\gamma' \geq 2^{-d}(\alpha/4)^{2d^2}(\alpha/2)^{3d/2} \gamma \geq (\alpha/4)^{d^2} \gamma$ and of dimension $d$ such that $|1 - \omega^\beta(a, b)/2| \leq 2\pi d^2 \alpha$ for every $a$ and $b$ in $P''$. Let us set $\theta$ to be $2\pi d^2 \alpha$. Then there is a Freiman homomorphism $\phi$ defined on $P''$ such that $|\omega^{\phi(a)} - \omega^{\phi(a')}| \leq \theta$ for every $a \in P''$.

We now apply Lemma 6.6 to the function $l$ defined by $l(x) = \gamma' - 1, \omega^{\phi(x)}$ when $x \in P''$ and $l(x) = 0$ otherwise. It gives us a subprogression $P_3 \subset P''$ and a function $h'$ such that $\|l - h'\|_\infty \leq 2\gamma' - 1, \|h'\|_\ast \leq \gamma' - 3/4(\epsilon/d)^{-d/4}$, and $l - h'$ is supported on $\partial P''$, which has density at most $3\epsilon \gamma'$. (Here the boundary is taken with respect to $P_3$, and $\epsilon$ denotes the proportion of $P''$ that we take to lie in $P_3$).

Let us define $f'(x)$ to be $\gamma' - 1, \omega^{\phi(x)}$ if $x \in P''$ and $0$ otherwise. The above calculations show that $|f'(x) - h'(x)|$ is at most $\theta \gamma' - 1$ when $x \in P''$, at most $(2 + \theta) \gamma' - 1$ when $x \in \partial P''$, and $0$ when $x \notin P''$. Moreover, the density of $\partial P''$ is at most $3\epsilon \gamma'$.

We are preparing to apply Lemma 6.8. Our pairs of sets are $(B, B')$ and $(P'', Q)$, which satisfy the hypothesis since $P'' = P'' + P_3 \subset P + P \subset B'$. Our function $g$ is defined by taking $g(x) = \beta^{-1} \omega^{\phi(x)}$ if $x \in B$ and $0$ otherwise. If $y \in B^-$ then we shall define $g_y(x)$ to be $\gamma' - 1, \omega^{\phi(x)}$ if $x \in y + P''$ and $0$ otherwise. (This is the normalized restriction of $\omega^{\phi(x)}$ to $y + P''$.) If $y \notin B^-$ we shall define $g_y$ to be identically zero. Then if $x \in B^-$, we have $E_{y \in B} g_y(x) = \gamma' - 1, \beta g(x)P[x \in y + P''|y \in B] = g(x)$, so the hypotheses about $g$ and the $g_y$ are satisfied.

The function $f'$ just discussed was equal to $g_0$. For each fixed $y \in B^-$ the function $q(x) - q(x - y)$ is a Freiman homomorphism on $y + P''$, so the argument used for $f_0$ can also be used to provide for us a function $h_y$ such that $\|h_y\|_\ast \leq \gamma' - 3/4(\epsilon/d)^{-d/4}$, and such that $|q_y(x) - h_y(x)|$ is at most $\theta \gamma' - 1$ when $x \in y + P''$, at most $(2 + \theta) \gamma' - 1$ when $x \in y + \partial P''$, and $0$ otherwise. Thus, in Lemma 6.8 we can take $\beta$ to be $\beta$, $\gamma$ to be $\gamma'$, $\theta$ to be $\theta$, $\epsilon$ to be $\epsilon$, and $\lambda$ to be $2 + \theta$. 
We then set \( h(x) = \mathbb{E}_{x \in B} h_y(x) \). By Lemma 6.8, \( |g(x) - h(x)| \) is at most \((\theta + 2\epsilon + \theta\epsilon)\beta^{-1}\) for every \( x \in B^- \), at most \((4 + 2\epsilon + \theta\epsilon)\beta^{-1}\) for every \( x \in \partial B \), and \( g(x) = h(x) = 0 \) for \( x \notin B^+ \). Moreover, since \( h \) is just the average of all the \( h_y \), the triangle inequality implies that \( \|h\|_{U^2}^{*} \leq \gamma^{-3/4}(\epsilon/d)^{-d/4} \).

Now \( g \) is the function \( f_0 \) defined at the beginning of the proof, where we defined \( f_y(x) \) to be \( \beta^{-1}\omega_{g_0}(x) \) if \( x \in y + B \) and 0 otherwise. Since \( g_y(x) - q(x - y) \) is a Freiman homomorphism on \( y + B \), the same argument gives us a function \( k_y \) such that \( |f_y(x) - k_y(x)| \) is at most \((\theta + 2\epsilon + \theta\epsilon)\beta^{-1}\) for every \( x \in y + B^- \), at most \((4 + 2\epsilon + \theta\epsilon)\beta^{-1}\) for every \( x \in \partial y + B \), and \( f_y(x) = k_y(x) = 0 \) for \( x \notin y + B^+ \). Also, \( \|k_y\|_{U^2}^{*} \leq \gamma^{-3/4}(\epsilon/d)^{-d/4} \).

We now apply Lemma 6.8 once again, but this time it is simpler because our set \( A \) will have empty boundary. Indeed, we take \( A \) and \( B \) to be \( \mathbb{Z}_N \), \( C \) to be \( B \), and \( D \) to be \( B' \). This time round we can take \( \beta \) to be 1, \( \gamma \) to be \( \beta \), \( \epsilon \) to be \( \eta \), \( \theta \) to be \( \theta + 2\epsilon + \theta\epsilon \), and \( \lambda \) to be \( 4 + 2\epsilon + \theta\epsilon \). Then \( Q(x) = \mathbb{E}_{y \in x - B} \omega_{g_0}(x) \) by definition, and this is equal to \( \mathbb{E}_{y \in \mathbb{Z}_N} f_y(x) \). Let \( Q''(x) = \mathbb{E}_{y \in \mathbb{Z}_N} k_y(x) \). Then Lemma 6.8 tells us that \( \|Q - Q''\|_{\infty} \leq \theta + 2\epsilon + \theta\epsilon + (4 + 2\epsilon + \theta\epsilon)\eta \). Moreover, \( \|Q''\|_{U^2}^{*} \leq \gamma^{-3/4}(\epsilon/d)^{-d/4} \), again by the triangle inequality. \( \square \)

We now combine Corollary 6.9 with a result of Ruzsa so that we can say something about bilinear phase functions defined on Bohr sets.

**Theorem 6.10.** Let \( Q \) be a generalized quadratic average of complexity \((d, \rho)\). Then for every \( \alpha \) with \( 0 < \alpha \leq 1/20 \), either \( \|Q\|_{U^2} \leq (12\alpha)^{1/8} \) or there exists a function \( Q'' \) such that \( \|Q - Q''\|_{\infty} \leq 16d^2\alpha \) and \( \|Q''\|_{U^2}^{*} \leq (4/\alpha)^{4d^2}(800d^2/\rho)^d \).

**Proof.** Suppose that \( Q \) has base \((B, q)\), where \( B = B(K, \rho) \) and \( K \) has cardinality \( d \). Let \( \eta = \alpha \) and let \( B' \prec_{\alpha} B \). Then \( B' = B(K, \sigma) \) for some \( \sigma \geq \alpha\rho/400d \). A theorem of Ruzsa [R94] (see also [N96]) tells us that \( B' \) contains a proper \( d \)-dimensional arithmetic progression of density at least \((\sigma/d)^d \). Therefore, there is a proper \( d \)-dimensional arithmetic progression \( P \) of density \( \gamma \geq (\sigma/2d)^d \) such that \( P + P \subset B' \). By Corollary 6.9 with \( \eta = \alpha \) and \( \epsilon = \alpha d^2 \) (if \( \epsilon > 1 \) then Corollary 6.9 is trivial so we do not need to worry about this), either \( \|Q\|_{U^2} \leq (12\alpha)^{1/8} \) or there exists a function \( Q'' \) such that \( \|Q - Q''\|_{\infty} \leq 16d^2\alpha \) and \( \|Q''\|_{U^2}^{*} \leq (\alpha/4)^{-3d^2}(\sigma/2d)^{-3d^4}(\alpha d)^{-d/4} \). A small back-of-envelope calculation shows that this is at most the bound stated for \( \|Q''\|_{U^2}^{*} \). \( \square \)

7. **A more precise decomposition theorem**

Theorem 3.4 stated that every function that is bounded above in \( L_2 \) can be decomposed into a linear combination of quadratic averages plus a sum of two error terms, one of which
is small in $U^3$ and one in $L_1$. The aim of this section is to prove a refinement of this statement. Once again, we shall show that a function $f$ with $\|f\|_2 \leq 1$ can be decomposed as a linear combination of quadratic averages plus a small error. However, we shall collect these quadratic averages into a small number of “clusters” in such a way that two quadratic averages that belong to the same cluster will have a low-rank difference. Then the results of the previous section will allow us to express each cluster as a product of just one quadratic average with a function with small $U^2$-dual norm. We proved an analogous theorem for $\mathbb{F}_p^n$: after the hard work of the previous section, the rest of the adaptation is relatively routine.

First let us combine Theorem 6.10 with Lemma 4.2 in order to describe what happens if two generalized quadratic averages have a significant correlation. The following lemma should be thought of as a companion to Corollary 5.3. The appearance of the generalized quadratic average $Q_0$ in the statement may look a bit strange: it is there for technical reasons that will be explained later.

**Lemma 7.1.** Let $B$ and $B'$ be two arbitrary Bohr sets and let the complexity of $B \cap B'$ be $(d, \rho)$. Let $q$ and $q'$ be quadratic forms on $B$ and $B'$. Let $Q$ and $Q'$ be generalized quadratic averages with bases $(B, q)$ and $(B', q')$ and suppose that $\langle Q, Q' \rangle \geq \zeta$. Let $Q_0$ be another generalized quadratic average with base $(B, q)$. Then there exists a function $Q'''$ such that $\|Q_0Q' - Q'''\|_\infty \leq \zeta/2 + d^2 \zeta^8/2^9$ and $\|Q'''\|_{U^2}^* \leq (2^{11}/\zeta^8)^{4d}(800d^2/\rho)^d$.

**Proof.** Let $\eta = \zeta/36$. Let $B_1$ and $B_2$ be regular Bohr sets such that $B_2 \prec \eta B_1 \prec \eta B \cap B'$. Then Lemma 4.2 tells us that there is a generalized quadratic average $Q''$ with base $(B_2, q - q')$ such that $\|QQ' - Q''\|_\infty \leq 18\eta = \zeta/2$.

Since $Q_0$ also has base $(B, q)$, the same argument gives us a generalized quadratic average $Q_0''$ with base $(B_2, q - q')$ such that $\|Q_0Q' - Q_0''\|_\infty \leq \zeta/2$.

Now $\mathbb{E}_x Q''(x) \geq \zeta - 18\eta = \zeta/2$, so $\|Q''\|_{U^2} \geq \zeta/2$. Therefore, if we set $\alpha$ to be $\zeta^8/2^9$, then Theorem 6.10 implies that there exists a function $Q'''$ such that $\|Q'' - Q''''\|_\infty \leq 16d^2\alpha$ and $\|Q''''\|_{U^2}^* \leq (4/\alpha)^{4d}(800d^2/\rho)^d$.

However, we wanted a similar statement for $Q_0''$ rather than $Q''$. This does not quite follow from Theorem 6.10 but it follows from the proof. A quick examination of Corollary 6.9 reveals that the alternatives in question depend just on the rank of $Q$ and not on $Q$ itself. (To be precise, if two quadratic forms have the same base and the same rank with respect to $P$, then there must be one half of the dichotomy that applies to both forms.) Therefore, we obtain the result stated. \qed
What will be crucial to us later, if we want a reasonable bound, is that the $U^2$-dual norm of $Q''$ in the above lemma depends polynomially on $\zeta$ for fixed $d$. It is for this reason that it would have been too expensive to use Green and Tao’s local Bogolyubov lemma to prove Theorem 6.10. That would have allowed us to prove an analogue of Corollary 6.2 for bilinear phase functions defined on Bohr sets. However, the subset we passed to would then have been a Bohr set whose dimension depended polynomially on $c$, whereas in fact we passed to a multidimensional progressions without any increase in dimension. That would have translated into an exponential dependence on $\zeta$ in Lemma 7.1.

Unfortunately, before we prove our more precise decomposition result we must deal with another technical difficulty that did not arise for quadratic averages on $F^*_p$, which is that $Q^{-1}$ does not in general equal $\overline{Q}$. In our previous paper it was convenient to write $Q_j$ as $Q_iQ_iQ_j$. In order to do something similar in the $Z_N$ case we shall first show that for every regular Bohr neighbourhood $B$ and every quadratic function $q : B \to Z_N$ we can find a smaller Bohr neighbourhood $B'$ and a quadratic average $Q$ with base $(B',q)$ such that $|Q(x)| = 1$ for almost every $x$. The statement of Lemma 7.1 is designed so that we will then be able to replace any given $Q_i$ by a quadratic average with this convenient property and with the same base.

We begin by proving, using a very standard argument, that $Z_N$ can be covered fairly efficiently by copies of $B$.

**Lemma 7.2.** Let $B = B(K,\rho)$ be a Bohr set and write $d$ for the size of $K$ and $\beta$ for the density of $B$. Then there is a set $\{B_1,\ldots,B_m\}$ of translates of $B$ such that $m \leq 5^d\beta^{-1}$ and every point in $Z_N$ belongs to at least one $B_i$.

**Proof.** A basic fact about Bohr sets is that the Bohr set $B'' = |B(K,\rho/2)|$ has density at least $5^{-d}\beta$. (See for example [GrT09b], Lemma 8.1.) Let $x_1,\ldots,x_m$ be a maximal collection of points with the property that the translates $x_i + B''$ are disjoint. Then $m \leq 5^d\beta^{-1}$. Also, the sets $x_i + B$ cover $Z_N$, since if $x \notin x_i + B$ for any $i$, then $x + B''$ and $x_i + B''$ are disjoint (or $x$ would belong to $x_i + B'' - B'' \subset x_i + B$).

The condition $B' \prec_{5^d} B$ that appears in the next corollary may look rather expensive with its exponential dependence on $d$, but the effect on our eventual bound is not particularly serious: the density of $B'$ is exponential in $d^2$ instead of $d$. When we come to apply the result, $d$ will be bounded above by $(2/\delta)^{C_0}$ for some absolute constant $C_0$, so this decrease in the density is comparable to the result of replacing $C_0$ by $2C_0$. 


Corollary 7.3. Let $\epsilon > 0$, let $B = B(K, \rho)$ be a regular Bohr set, and let $q$ be a quadratic function defined on $B$. Let $d = |K|$, let $B' \subset_{\epsilon/5} B$ and let $B''$ be a Bohr set such that $B'' - B'' \subset B'$. Then there is a quadratic average $Q$ with base $(B'', q)$ such that for all but at most $\epsilon N$ values of $x$ the restriction of $Q$ to $x + B''$ is a quadratic phase function. In particular, $|Q(x)| = 1$ for all but at most $\epsilon N$ values of $x$.

Proof. Let $B_1, \ldots, B_m$ be a sequence of translates of $B$ given by the previous lemma, with $B_i = x_i + B$. On each $B_i$, let $q_i$ be the function $q_i(x) = q(x - x_i)$. Now let us greedily make the sets $B_i$ disjoint, by letting $B'_i = B_i \setminus (B_1 \cup \cdots \cup B_{i-1})$ for each $i$.

We are trying to define a function of the form $Q(x) = \mathbb{E}_{y \in B''} \omega^{q_y(x)}$, so it remains to choose the functions $q_y$ appropriately. This we do by letting $q_y(x) = q_i(x)$ for the unique $i$ such that $y \in B'_i$. Since $q_i(x) = q(x - x_i) = q((x - y) - (x_i - y))$, this is of the form $q(x - y) + \phi_y(x - y)$ for some Freiman homomorphism $\phi_y : B' \to \mathbb{Z}_N$, as required.

Now each $q_i$ is a quadratic homomorphism on $B_i$, so the restriction of $Q$ to $x + B''$ will be a quadratic phase function if there exists $i$ such that $x + B'' - B'' \subset B'_i$. A sufficient condition for this is that, for every $i$, either $x - B' \subset B_i$ or $(x - B') \cap B_i = \emptyset$. But Lemma [2.2] implies that this is true for all but at most $5^{-d} \epsilon m |B| \leq \epsilon N$ values of $x$, as claimed. \qed

The property we have just obtained is a useful one, so let us give it a name. Note that the Bohr set $B$ from Corollary 7.3 is no longer explicitly mentioned, but its width and dimension appear (in disguised form) as the parameter $m$ below.

Definition. We say that a quadratic average $Q$ with base $(B'', q)$ is $(\epsilon, m)$-special if the following holds. There exist at most $m$ elements $x_1, \ldots, x_m \in \mathbb{Z}_N$ such that for all but at most $\epsilon N$ points $x \in \mathbb{Z}_N$ the restriction of $Q$ to $x + B''$ is equal to the restriction of $\omega^{q_1}$ to $x + B''$, where $q_i(x) = q(x - x_i)$.

We shall not need this definition in the rest of this section, but it will be used in the next section.

The following lemma (which has a simple proof) appears in [GW09a] as Corollary 2.11.

Lemma 7.4. Let $u_1, \ldots, u_n$ be a collection of vectors of norm at most 1 in a Hilbert space $H$, let $\lambda_1, \ldots, \lambda_n$ be scalars with $\sum_{i=1}^n |\lambda_i| \leq C$ and let $\delta > 0$. Then there are vectors $u_{i_1}, \ldots, u_{i_k}$ and a set $A \subset \{1, 2, \ldots, n\}$ such that $k \leq 2C^2/\delta^2$, and with the following properties: $\|\sum_{i \in A} \lambda_i u_i\|_2 \leq \delta$, and for every $i \notin A$ there exists $j$ such that $|\langle u_i, u_j \rangle| \geq \delta^2/2C$. 

We are now ready to state and prove the main result of this section. It is important for us to be able to vary the parameter \( \epsilon \) below independently of the quantity \( C \).

**Theorem 7.5.** Let \( f : \mathbb{Z}_N \to \mathbb{C} \) be a function such that \( \| f \|_2 \leq 1 \), and let \( \delta > 0 \). Let \( C_0 = 2^{2^4} \), \( d = (2/\delta)^{C_0} \), \( \rho = (\delta/2)^{C_0} \), and \( C = 4(2/\delta)^{C_0} \), and let \( \epsilon > 0 \) be at most \( (\delta/2)^{6C_0} \). Then \( f \) has a decomposition

\[
f(x) = \sum_{i=1}^k Q_i'(x)U_i(x) + g(x) + h(x),
\]

with the following properties: \( k \leq 2C/\delta^2 \), the \( Q_i' \) are quadratic averages on \( \mathbb{Z}_N \) with complexity at most \( (d, \epsilon \rho/800d5^d) \), \( \sum_{i=1}^k \| U_i \|_2 \leq (8/\epsilon^8)^{4d^2}(2^{20}d^35^d/\epsilon^4)C \), \( \sum_{i=1}^k \| U_i \|_\infty \leq 2C \), \( \| g \|_1 \leq 3\delta \) and \( \| h \|_{U^3} \leq \delta \). Moreover, the quadratic averages \( Q_i' \) are \((\epsilon, (5/\rho)^d)\)-special.

**Proof.** By Theorem 3.3, \( f \) can be decomposed into a sum \( \sum_i \lambda_i Q_i(x) + g'(x) + h(x) \), where each \( Q_i \) is a quadratic average of complexity at most \( (d, \rho) \), and \( \| g' \|_1 \leq \delta \), \( \| h \|_{U^3} \leq \delta \) and \( \sum_i |\lambda_i| \leq C \).

Suppose that \( Q_i \) has base \((B_i, q_i)\). Lemma 7.2 and Corollary 7.3 tell us that if \( B'_i \prec_{\epsilon/5^d} B_i \) then there is a quadratic average \( Q'_i \) with base \((B'_i, q_i)\) which is \((\epsilon, 5^d\beta^{-1})\)-special, where \( \beta \) is the density of the base of \( Q_i \). In particular, \( |Q'_i(x)| = 1 \) for all but at most \( \epsilon N \) values of \( x \). Furthermore, Lemma 4.1 gives us a generalized quadratic average \( Q''_i \) with base \((B'_i, q_i)\) such that \( \| Q_i - Q''_i \|_\infty \leq 6\epsilon/5^d \), which is at most \( 2\epsilon \). Note that the complexities of \( Q'_i \) and \( Q''_i \) are at most \((d, \epsilon \rho/800d5^d)\). (The additional factor of 2 stems from the requirement that \( B'' - B'' \subseteq B' \) in Corollary 7.3.)

Now we apply Lemma 7.4 to the linear combination \( \sum_i \lambda_i Q_i \). Without loss of generality, the functions that it gives us are \( Q_1, \ldots, Q_k \). Then Corollary 7.4 tells us that we can write \( \sum_i \lambda_i Q_i \) in the form \( \sum_{i=1}^k \sum_{j \in A_i} \lambda_j Q_j + g'' \), where \( k \leq 2C^2/\delta^2 \), \( \| g'' \|_2 \leq \delta \) and \( \| Q_i, Q_j \| \geq \delta^2/2C \) for every \( i \leq k \) and every \( j \in A_i \). In order to proceed, we must rewrite this decomposition in terms of the functions \( Q_i' \). That is, we wish to take the sum \( \sum_{j \in A_i} \lambda_j Q_j \) and replace it by \( Q'_i \sum_{j \in A_i} \lambda_j Q'_j \).

Since \( \| Q_i - Q''_i \|_\infty \leq 2\epsilon \leq \delta^2/4C \), we have that \( |\langle Q''_i, Q_j \rangle| \geq 2\epsilon \) for every \( j \in A_i \). Therefore, since \( Q'_i \) and \( Q''_i \) have the same base, Lemma 7.1 (with \( \zeta = 2\epsilon \) and \( \rho \) replaced by \( \epsilon \rho/800d5^d \)) tells us that each function \( Q'_i Q_j \) with \( j \in A_i \) can be written as \( F + G \), with \( \| G \|_\infty \leq \epsilon + 8d^2\epsilon^8 \) and \( \| F \|_2 \leq (8/\epsilon^8)^{4d^2}(2^{20}d^35^d/\epsilon^4)C \). Therefore, \( \sum_{j \in A_i} \lambda_j Q'_j \) can be written as \( F + G \) with \( \| G \|_\infty \leq (\epsilon + 8d^2\epsilon^8) \sum_{j \in A_i} |\lambda_j| \) and \( \| F \|_{U^2} \leq (8/\epsilon^8)^{4d^2}(2^{20}d^35^d/\epsilon^4)C \sum_{j \in A_i} |\lambda_j| \).

(In this proof the functions \( F \) and \( G \) may vary from line to line.) This implies also that \( \| F \|_\infty \leq 2 \sum_{j \in A_i} |\lambda_j| \) (since, as can easily be checked, \( \epsilon + 8d^2\epsilon^8 \leq 1 \).
Since \(|Q'(x)|^2 = 1\) for all but at most \(\epsilon N\) values of \(x\), we have the estimate \(\|1 - |Q'|^2\|_2 \leq \epsilon\). It follows that

\[
\left\| \sum_{j \in A_i} \lambda_j Q_j - Q'_i \sum_{j \in A_i} \lambda_j \overline{Q_j} Q_j \right\|_1 \leq \epsilon \sum_{j \in A_i} |\lambda_j|.
\]

Hence, \(\sum_{j \in A_i} \lambda_j Q_j\) can be written in the form \(Q'_i U_i + V_i\) with \(\|V_i\|_1 \leq (2\epsilon + 8d^2 \epsilon^2) \sum_{j \in A_i} |\lambda_j|\) and \(\|U_i\|_{U^2}^* \leq (8/\epsilon^8)^d (2^{20} d^{35} d^4 / \epsilon \rho)^d \sum_{j \in A_i} |\lambda_j|\). Our bound for \(\|F\|_{\infty}\) in the previous paragraph also gives us that \(\|U_i\|_{\infty} \leq 2 \sum_{j \in A_i} |\lambda_j|\).

Putting all this together, we find that \(\sum_{i=1}^k \sum_{j \in A_i} \lambda_j Q_j\) can be written as \(\sum_{i=1}^k U_i Q'_i + V\), with \(\sum_{i=1}^k \|U_i\|_{U^2} \leq (8/\epsilon^8)^d (2^{20} d^{35} d^4 / \epsilon \rho)^d C\), \(\sum_{i=1}^k \|U_i\|_{\infty} \leq 2C\), and \(\|V\|_1 \leq (2\epsilon + 8d^2 \epsilon^2)C \leq \delta\). We have therefore written \(f\) as \(\sum_{i=1}^k U_i Q'_i + (V + g'' + g') + h\). Since all of \(\|V\|_1\), \(\|g''\|_1\) and \(\|g'\|_1\) are at most \(\delta\), the theorem is proved. \(\square\)

8. The structure of a function \(QU\) when \(Q\) has low rank

The main result of the previous section gives us a decomposition of the form \(f = \sum_{i=1}^k Q'_i U_i + g + h\), where \(g\) and \(h\) are error terms, the functions \(U_i\) have bounded \(U^2\)-dual norms, and the \(Q'_i\) are quadratic averages. Moreover, the quadratic averages are \((\epsilon, m)\)-special, an important property which we shall make use of shortly.

The aim of this section is to find a “structured set” \(S\) such that the functions \(\omega^{g(x-x_i)} + \phi_i(x-x_i)\) are all approximately \(S\)-invariant, where this means that they do not vary much if you add an element of \(S\) to \(x\). We already have many of the tools to do this: the main task of this section will be to develop a little further some of the results of the last two sections. We shall soon say what a structured set is, but one can think of it as a set that resembles a lattice convex body in the way that a Bohr set or a multidimensional arithmetic progression does.

As in Section 6 we shall make use of the fact that a quadratic homomorphism defined on a multidimensional arithmetic progression can be explicitly described. We shall use elements from the proofs of some of the lemmas in that section.

It may seem as though the next lemma has basically already been proved in Section 6. In a sense, that is true, but we need to run the argument again in order to make very clear that the phase function \(f\) that appears in the statement below is independent of the translate of \(P'\). Later we shall see why that is so important.

**Lemma 8.1.** Let \(B\) be a regular Bohr set, let \(P\) be a \(d\)-dimensional arithmetic progression such that \(P + P \subset B\), let \(q\) be a quadratic homomorphism defined on \(B\), let \(Q(x) = \omega^{q(x)}\) for every \(x \in B\), and suppose that the rank of \(Q\) with respect to \(P\) is at most \(\log(1/\alpha)\). Let
Let \( \epsilon > 0 \) and let \( \theta = \alpha^2 \epsilon / 8d^2 \). Then there is a subprogression \( P' \subset P \) of dimension \( d \) and size at least \((\alpha/8)^2d^3|P|\), and a multiplicative Freiman homomorphism \( f \) from \( P \) to the unit circle in \( \mathbb{C} \), such that \(|Q(x)f(x) - Q(y)f(y)| \leq \epsilon \) whenever \( x - y \in P' \). Moreover, if \( Q' = Qg \) for some multiplicative homomorphism \( g \), then we can choose the same subprogression \( P' \) to work for \( Q' \).

**Proof.** First, recall from the proof of Corollary 6.9 that the restriction of \( q \) to \( P \) is given by a formula of the form \( q(x) = \sum a_{ij}x_ix_j + \sum b_i x_i + c \). Here, we are writing a typical point \( x \in P \) as \( u_0 + \sum x_i u_i \), where \( 0 \leq x_i < m_i \). Moreover, there are coefficients \( b'_i \) and \( c'_i \) such that, setting \( \beta(u,v) = 2 \sum a_{ij}u_iv_j + b'_i u_i + c'_i v_i \), we have \( |\mathbb{E}_{u,v \in P} e(\beta(u,v))| \geq \alpha \).

Corollary 6.2 (with \( \alpha = 2c \)) then gives us rational approximations \(|2a_{ij} - p_{ij}/q_{ij}| \leq 8\alpha^{-2}/m_im_j\), with \( q_{ij} \leq 4\alpha^{-1} \).

Now let \( x = u_0 + \sum x_i u_i \) and \( y = u_0 + \sum (x_i + w_i) u_i \) be two points in \( P \). Then

\[
q(y) - q(x) = \sum_{i,j} a_{ij} w_i w_j + \sum (b_i + \sum_j a_{ij}x_j) w_i + \sum (b_j + \sum_i a_{ij}x_i) w_j.
\]

As in the proof of Corollary 6.2, let \( q_i = \prod_j q_{ij} \times \prod_j q_{ji} \). Then \( q_i \leq (4\alpha^{-1})^{2d} \). Suppose now that each \( w_i \) is even and a multiple of \( q_i \) and that \( w_i \leq \theta m_i \). Then \(|a_{ij}w_ix_j|\) and \(|a_{ij}w_iw_j|\) are both at most \( 8\alpha^{-2}\theta \), since \( w_i \) is an even multiple of \( q_{ij} \), \(|2a_{ij} - p_{ij}/q_{ij}| \leq 8\alpha^{-2}/m_i m_j\), and \( w_j \) and \( x_j \) are both at most \( m_j \). Therefore, if \( \theta \leq \alpha^2 \epsilon / 8d^2 \), we find that

\[
\omega^{q(y) - q(x)} \approx e^{(2 \sum_i b_i w_i)}.
\]

Let us therefore define \( f(x) \) to be \( e(-2 \sum_i b_i x_i) \). Then

\[
|Q(y)f(y) - Q(x)f(x)| = |\omega^{q(y) - q(x)} e(-2 \sum_i b_i w_i) - 1| \leq \epsilon,
\]

which proves the first statement.

The second statement is trivial: if \( Q' = Qg \) then all we have to do is choose the same subprogression \( P' \) and replace \( f \) by \( fg^{-1} \).

It follows from this lemma that if \( X \) is a set on which \( f \) is approximately equal to 1, then \( Q \) is roughly constant on translates of \( X \cap P' \). We shall now prove that such sets have a structure that is similar to that of Bohr sets.

To do this, we shall make use of the notion of Bourgain systems. This is an abstract notion introduced by Green and Sanders [GrS07] that is designed to capture the properties one actually uses of Bohr sets in most applications. A Bourgain system of dimension \( d \) is a collection of sets \( X_\rho \), one for each \( \rho \in [0,4] \), satisfying the following properties.
• If \( \rho' \leq \rho \) then \( X_{\rho'} \subset X_{\rho} \).
• \( 0 \in X_{\rho} \).
• \( X_{\rho} = -X_{\rho} \).
• If \( \rho + \rho' \leq 4 \) then \( X_{\rho} + X_{\rho'} \subset X_{\rho + \rho'} \).
• If \( \rho \leq 1 \) then \( |X_{2\rho}| \leq 2^d|X_{\rho}| \).

An important fact about Bourgain systems is that there is an analogue of the notion of a regular Bohr set. The next lemma is Lemma 4.12 of [GrS07] (though we have stated it slightly differently).

**Lemma 8.2.** Let \( (X_{\rho}) \) be a Bourgain system of dimension \( d \) and let \( 0 < \tau \leq 1 \). Then there exists \( \rho \in [\tau/2, \tau] \) such that \( |X_{\rho(1+\kappa)}| \leq (1 + 10d\kappa)|X_{\rho}| \) and \( X_{\rho(1-\kappa)} \geq (1 - 10d\kappa)|X_{\rho}| \) whenever \( 0 \leq 10d\kappa \leq 1 \).

If \( \rho \) has this property, we shall call \( X_{\rho} \) a regular set in the system \( (X_{\rho}) \). (This terminology is not quite the same as that of Green and Sanders, but is close to the standard terminology for Bohr sets.)

As we did for Bohr sets, we define a notion of one set in a Bourgain system being “central” in another.

**Definition.** Let \( (X_{\rho}) \) be a Bourgain system and let \( 0 < \sigma < \rho \leq 1 \). We shall say that \( X_{\sigma} \) is \( \epsilon \)-central in \( X_{\rho} \) and write \( X_{\sigma} \prec_{\epsilon} X_{\rho} \) if both \( X_{\rho} \) and \( X_{\sigma} \) are regular sets, and \( \sigma \in [\epsilon\rho/400d, \epsilon\rho/200d] \).

Note that by Lemma 8.2, we know that if \( X_{\rho} \) is regular then there exists \( \sigma \) such that \( X_{\sigma} \prec_{\epsilon} X_{\rho} \). Lemma 4.4 of [GrS07] asserts that if \( (X_{\rho}) \) is a Bourgain system of dimension \( d \) and \( \eta \in [0, 1] \), then \( |X_{\eta\rho}| \geq (\eta/2)^d|X_{\rho}| \). Therefore, if \( X_{\sigma} \prec_{\epsilon} X_{\rho} \), we know that \( |X_{\sigma}| \geq (\epsilon/800d)^d|X_{\rho}| \). We also obtain a lower bound for the sizes of the sets in a dilated system \( (Y_{\rho}) = (X_{\eta\rho}) \), which will be useful to us later.

The next lemma we state without proof because the proof is almost identical to that of Lemma 2.3 (i).

**Lemma 8.3.** Let \( \epsilon > 0 \). Let \( (X_{\rho})_{0 \leq \rho \leq 4} \) be a Bourgain system and let \( 0 < \sigma < \rho \) be such that \( X_{\sigma} \prec_{\epsilon} X_{\rho} \). Let \( f \) be any function from \( \mathbb{Z}_N \) to \( \mathbb{C} \) such that \( ||f||_{\infty} \leq 1 \). Then

\[
\mathbb{E}_{x \in X_{\rho}} f(x) \approx_{\epsilon} \mathbb{E}_{x \in X_{\rho}} \mathbb{E}_{y \in X_{\sigma}} f(x + y).
\]

Obvious candidates for Bourgain systems are families of subgroups, Bohr sets and multidimensional arithmetic progressions. For example, given a Bohr set \( B = B(K, \sigma) \), the
set

\[ X_\rho = \{ x \in \mathbb{Z}_N : |1 - e(rx/N)| \leq \rho \sigma \text{ for all } r \in K \} \]

obviously satisfies the first four of the above properties, and it also satisfies the final one with \(2^d\) replaced by \(5^{|K|}\) (see for example Section 8 of [GrT08]). Therefore the sets \(X_\rho\) can be viewed as forming a Bourgain system of dimension \(d \leq 3|K|\). A similar statement holds for a family of multidimensional arithmetic progressions with the same basis but differing widths.

We shall not yet explain in detail why Bourgain systems are useful. Instead, we shall introduce the Bourgain system we wish to use, prove that it is a Bourgain system, and then when we need it to satisfy various properties we shall quote appropriate results that tell us that all Bourgain systems have those properties. The proofs are not too hard and can be found in [GrS07].

Lemma 8.4. Let \(m_1, \ldots, m_d\) be positive integers and let \(P\) be the set \(\prod_{i=1}^{d}[-m_i, m_i]\). Let \(f_1, \ldots, f_M\) be multiplicative Freiman homomorphisms from \(P\) to \(\mathbb{T}\) that take the value 1 at 0, and for each \(\rho \in [0, 4]\) let

\[ X_\rho = \{ x \in P : |1 - f_j(x)| \leq \rho \text{ for every } j \leq M \} \]

Then the sets \((X_\rho)\) form a \(2M\)-dimensional Bourgain system. Moreover, the relative density of \(X_\rho\) in \(P\) is at least \(3^{-d}(\rho/2\pi)^M\).

Proof. The first three properties hold trivially. The fourth is almost trivial: the simple calculation needed is that if \(f\) is a multiplicative homomorphism to \(\mathbb{T}\), \(|1 - f(x)| \leq \rho\), and \(|1 - f(y)| \leq \rho'\), then

\[ |1 - f(x + y)| \leq |1 - f(x)| + |f(x) - f(x + y)| = |1 - f(x)| + |1 - f(y)| \leq \rho + \rho'. \]

The only real work comes in proving the fifth property, which bounds the size of \(B_{2\rho}\) in terms of the size of \(B_\rho\). The argument here is essentially the same as it is for Bohr sets. Let us define a map \(\psi : P \to \mathbb{T}^d\) by \(\psi(x) = (f_1(x), \ldots, f_M(x))\). Then \(\psi(X_{2\rho}) \subset \mathbb{T}_{2\rho}^M\), where \(\mathbb{T}_{2\rho} = \{ z \in \mathbb{C} : |z| = 1, |1 - z| \leq 2\rho \}\).

We can cover \(\mathbb{T}_{2\rho}\) by four segments of the circle that have diameter at most \(\rho\). Let us use all \(4^M\) possible products of these sets to cover the set \(\mathbb{T}_{2\rho}^M\). If \(Z\) is one of these products and \(\psi(x)\) and \(\psi(y)\) both belong to \(Z\), then \(\psi(x - y) \in \mathbb{T}_{\rho}^M\), which implies that \(x - y \in X_\rho\), or equivalently that \(x \in y + X_\rho\). It follows that if we choose one \(y\) for each \(Z\) for which there exists \(y\) with \(\psi(y) \in Z\), then we have a system of at most \(4^M\) translates of \(X_\rho\) that cover \(X_{2\rho}\). Therefore, the sets \(X_\rho\) form a Bourgain system of dimension \(2M\), as claimed.
Now let us turn to the density estimate, which is proved in a similar way. For each 
$z = (z_1, \ldots, z_M) \subset \mathbb{T}^M$, let $\mathbb{T}^M_{\rho/2}(z)$ be the set of all $w = (w_1, \ldots, w_M) \in \mathbb{T}^M$ such that 
$|z_i - w_i| \leq \rho/2$ for every $i$. For any $z_i \in \mathbb{T}$, the arc of points $w_i \in \mathbb{T}$ such that $|z_i - w_i| \leq \rho/2$ has length at least $\rho$, so the density of $\mathbb{T}^M_{\rho/2}(z)$ is at least $(\rho/2\pi)^M$.

Let us write $P/2$ for the set $\prod_{i=1}^d [-m_i/2, m_i/2]$. Then $|P/2| \geq 3^{-d}|P|$ (because the worst case is when every $m_i$ is equal to 1). Hence, by averaging we can find $z \in \mathbb{T}^M$ such that $\psi(x) \in \mathbb{T}^M_{\rho/2}(z)$ for at least $3^{-d}(\rho/2\pi)^M|P|$ points $x \in P/2$. Let $x$ be any such point.

If $y$ is any other such point, then $y - x \in P$ and $\psi(x)$ and $\psi(y)$ both belong to $\mathbb{T}^M_{\rho/2}(z)$, which implies that $\psi(y - x) = \psi(y)\overline{\psi(x)} \in \mathbb{T}^M_{\rho}$, so $y - x \in X_\rho$. Hence $X_\rho$ must contain at least $3^{-d}(\rho/2\pi)^M|P|$ distinct points.

Let $P \subset B'$ be a proper generalized progression. We shall need a lemma to tell us that we can cover $\mathbb{Z}_N$ reasonably efficiently with translates of $P$. The proof is essentially the same as the proof of Lemma 7.2

**Lemma 8.5.** Let $P \subset \mathbb{Z}_N$ be a proper arithmetic progression of dimension $d$ and density $\gamma$. Then there is a system of at most $3^d \gamma^{-1}$ translates of $P$ that covers $\mathbb{Z}_N$.

**Proof.** Let $P = \{\sum_{i=1}^d a_i x_i : 0 \leq a_i < m_i\}$ and let $P' = \{\sum_{i=1}^d a_i x_i : 0 \leq a_i < m_i/2\}$. Then let $u_1, \ldots, u_M$ be a maximal set such that the sets $P' + u_i$ are disjoint. Note that $P' - P' = \{\sum_{i=1}^d a_i x_i : -m_i/2 < a_i < m_i/2\} \subset P - \sum_i [m_i/2] x_i$. Let $z = \sum_i [m_i/2] x_i$.

Then the sets $P + u_i - z$ form a cover, since for every $x$ there exists $u_i$ such that $(x + P) \cap (u_i + P') \neq \emptyset$, which implies that $x \in u_i + P' - P' \subset u_i + P - z$. Since $P'$ has cardinality at least $3^{-d}|P|$ and therefore density at least $3^{-d}\gamma$, the result is proved. \ □

For the next lemma we shall make use of the concept of "special" quadratic averages, which was defined just after the proof of Corollary 7.3

**Corollary 8.6.** Let $B$ be a Bohr set of dimension $d$ and let $q$ be a quadratic homomorphism defined on $B$. Let $m$ be a positive integer and let $B' \prec_{\epsilon/5^d} B$ be another Bohr set. Let $Q$ be an $(\epsilon, m)$-special quadratic average with base $(B', q)$; in other words, for all but at most $\epsilon N$ points $x \in \mathbb{Z}_N$ the restriction of $Q$ to $x + B'$ is equal to the restriction of $\omega^q$ to $x + B'$, where $q_i$ is one of at most $m$ translates of $q$. Let $P \subset \mathbb{Z}_N$ be a proper generalized arithmetic progression of dimension $d$ and density $\gamma$ such that $2P - 2P \subset B'$. Then there is a set $V$ of size at most $3^d \gamma^{-1}$ such that for at least $(1 - \epsilon)N$ values of $x$ there exists $i \leq m$ and $v \in V$ such that $x + P - P \subset v + 2P - P$ and the restriction of $Q$ to $v + 2P - P$ is equal to $\omega^q$.\□
Proof. By Lemma 8.5 there is a set $V$ of size at most $2^d \gamma^{-1}$ such that every $x$ is in $v + P$ for some $P$. If $x \in v + P$ then $u \in x - P$, so $v + P \subseteq x + P - P$. Since we assumed that $P$ was such that $2P - 2P \subseteq B'$, we find that $x + P - P \subseteq v + 2P - P \subseteq x + B'$. Since $Q$ is $(\epsilon, m)$-special, the proportion of $x$ such that the restriction of $Q$ to $x + B'$ is equal to $\omega^q_i$ for some $i$ is at least $1 - \epsilon$.

Therefore, as claimed, for at least this proportion of $x$, we have some $v \in V$ such that $x + P - P \subseteq v + 2P - P$ and the restriction of $Q$ to $v + 2P - P$ is equal to $\omega^q_i$ for some $i$. □

We now come to the main result of this section. The bound may look somewhat complicated, so let us draw attention to the one feature of it that is very important to us: that the dependence on $\alpha$ is of a power type rather than exponential. It is for this that we have put in the work of the last three sections rather than simply applying the local Bogolyubov lemma. (The fact that the power depends on $d$ is quite expensive, but it produces a doubly exponential bound rather than the tower-type bound that would have resulted if $\alpha$ had appeared in the exponent.)

**Lemma 8.7.** Let $Q$ be a quadratic average that satisfies all the assumptions of the previous lemma and suppose that the rank of $Q$ is at most $\log(1/\alpha)$ with respect to $P$. Let $\eta > 0$ and let $\theta = \alpha^2 \eta / 8d^2$. Then there is a subprogression $P' \subseteq P$ of relative density at least $(\alpha / 8)^{2d} \theta^d$ and a Bourgain system $(X_\rho)$ of dimension $2m$ such that each $X_\rho$ is a subset of $P'$, the relative density of $X_\rho$ is at least $3^{-d(\rho/2\pi)^m}$ inside $P'$, and for every $\rho$ and all but at most $\epsilon N$ values of $x$, $|Q(y) - Q(x)| \leq \eta + \rho$ for every $y \in x + X_\rho$.

Proof. Corollary 8.6 implies that for at least $(1 - \epsilon)N$ values of $x$ there is some $v \in V$ such that $x + P - P \subseteq v + 2P - P$ and the restriction of $Q$ to $v + 2P - P$ is $\omega^q_i$ for some translate $q_i$ of $q$. Lemma 8.1 then gives us a progression $P'$ of the density stated, and a multiplicative homomorphism $f_i$, such that $|Q(y) f_i(y) - Q(z) f_i(z)| \leq \eta$ whenever $y, z \in v + 2P - P$ and $y - z \in P'$. In particular, $|Q(x) f_i(x) - Q(y) f_i(y)| \leq \eta$ whenever $y \in x + P'$.

If in addition $|1 - f_i(y - x)| \leq \rho$ for each fixed $i$, then

$$|Q(x) - Q(y)| = |Q(x) f_i(x) - Q(y) f_i(x)| \leq |Q(x) f_i(x) - Q(y) f_i(y)| + |Q(y)||f_i(y) - f_i(x)|,$$

which, using the multiplicative property of $f_i$, equals

$$|Q(x) f_i(x) - Q(y) f_i(y)| + |f_i(y - x) - 1|$$

and can therefore be bounded above by $\eta + \rho$. 

****
By Lemma 8.8, the sets $X_\rho = \{ z \in P' : |1 - f_i(z)| \leq \rho \text{ for every } i \leq m \}$ form a Bourgain system of dimension $2m$, such that $X_\rho$ has relative density at least $3^{-d}(\rho/2\pi)^m$ inside $P'$.

We have just shown that one special quadratic average $Q$ is roughly invariant under convolution by sets $X_\rho$ that come from a certain Bourgain system. We now want to obtain a similar statement for a combination $\sum_{i=1}^k Q_i U_i$ of functions with small $U^2$ dual norm. The rough idea is to choose for each function $Q_i$ and each function $U_i$ a set from a Bourgain system with respect to which it is roughly translation invariant, and then to intersect all these sets. We shall use a lemma of Green and Sanders [GrS07] to prove that this intersection is reasonably large.

The next lemma is a standard application of Bogolyubov’s method.

**Lemma 8.8.** Let $f$ be a function from $\mathbb{Z}_N$ to $\mathbb{C}$ and suppose that $\|f\|_{U^2} \leq T$ and $\|f\|_\infty \leq C$. Let $K = \{ r \in \mathbb{Z}_N : |\hat{f}(r)| \geq \rho \}$ and let $B$ be the Bohr set $B(K, \rho)$. Then

$$\mathbb{E}_x |f(x + d) - f(x)|^2 \leq \rho^2 C^2 + 4T^{4/3} \rho^{2/3}$$

for every $d \in B$.

**Proof.** We apply the Fourier inversion formula and split the expectation into two parts in the usual manner:

$$\mathbb{E}_x |f(x + d) - f(x)|^2 = \mathbb{E}_x |\sum_r \hat{f}(r)(\omega^{rx+d} - \omega^{rx})|^2 \leq \sum_{r \in B} |\hat{f}(r)|^2 |\omega^d - 1|^2 + 4 \sum_{r \notin B} |\hat{f}(r)|^2,$$

which is bounded above by

$$\rho^2 \|f\|_2^2 + 4\|\hat{f}\|_{U^2}^{4/3} \rho^{2/3} \leq \rho^2 C^2 + 4T^{4/3} \rho^{2/3}$$

as claimed, using the fact that $\|\hat{f}\|_{U^2} = \|f\|_4/3$.

**Corollary 8.9.** Let $Q$ be a quadratic average and let $U$ be a function such that $\|U\|_\infty \leq C$. Let $X$ be a set such that for at least $(1 - \epsilon)N$ values of $x \in \mathbb{Z}_N$ we have $|Q(x + d) - Q(x)| \leq \eta$ for every $d \in X$. Let $B$ be a set such that $\mathbb{E}_x |U(x + d) - U(x)|^2 \leq \gamma$ for every $d \in B$. Then $\mathbb{E}_x |Q(x + d)U(x + d) - Q(x)U(x)|^2 \leq 2\eta^2 C^2 + 2\gamma + 4\epsilon C^2$ for every $d \in B \cap X$. Consequently, if $S$ is any subset of $B \cap X$ and $\sigma$ is the characteristic measure of $S$, then $\|QU - (QU)*\sigma\|_2 \leq 2\eta C + 2\gamma^{1/2} + 2\epsilon^{1/2} C$. 

Proof. Let $d \in B \cap X$ and let $x \in \mathbb{Z}_N$ be such that $|Q(x + d) - Q(x)| \leq \eta$ for every $d \in X$. Then

$$|Q(x + d)U(x + d) - Q(x)U(x)| \leq |Q(x + d) - Q(x)||U(x + d)| + |Q(x)||U(x + d) - U(x)|,$$

which, by assumption, is at most

$$\eta C + |U(x + d) - U(x)|.$$

It follows that for every such $x$ and every $d \in B \cap X$, we have

$$|Q(x + d)U(x + d) - Q(x)U(x)|^2 \leq 2\eta^2 C^2 + 2|U(x + d) - U(x)|^2.$$

The proportion of $x$ to which this applies is at least $1 - \epsilon$, by hypothesis. For all other $x$, we can at least say that $|Q(x + d)U(x + d) - Q(x)U(x)|^2 \leq 4||U||^2_\infty \leq 4C^2$. The first statement follows upon taking expectations.

Now

$$\|QU - (QU) \ast \sigma\|^2_2 = \mathbb{E}_x|\mathbb{E}_{d \in S} Q(x + d)U(x + d) - Q(x)U(x)|^2,$$

which by Cauchy-Schwarz and the first assertion is bounded above by

$$\mathbb{E}_{d \in S} \mathbb{E}_x|Q(x + d)U(x + d) - Q(x)U(x)|^2 \leq 2\eta^2 C^2 + 2\gamma + 4\epsilon C^2.$$

This proves the second statement. \hfill \qed

Recall that the aim of this section is to deal with a sum $\sum_i Q'_i U_i$ in which the functions $U_i$ have small $U^2$ dual norm and the quadratic averages $Q'_i$ have low rank. Putting together what we have proved so far enables us to find, for each $i$, a structured set $S_i$ with characteristic measure $\sigma_i$ such that $(Q'_i U_i) \ast \sigma_i$ is close to $Q'_i U_i$ in $L_2$. Thus, if we let $S = S_1 \cap \cdots \cap S_k$ then we have a measure $\sigma$ such that $(\sum_{i=1}^k Q'_i U_i) \ast \sigma$ is close to $\sum_{i=1}^k Q'_i U_i$ in $L_2$. As well as making these steps formal, we shall need to prove a lower bound for the size of $S_1 \cap \cdots \cap S_k$.

In order to do so, we generalize a lemma of Green and Sanders about intersections of sets from Bourgain systems. (It appears in a slightly different form in their paper [GrS07] as Lemma 4.10.)

Lemma 8.10. Let $(X_\rho)$ and $(Y_\rho)$ be two Bourgain systems in $\mathbb{Z}_N$ of dimensions $d$ and $d'$, and let the densities of each $X_\rho$ and $Y_\rho$ be $\mu_\rho$ and $\nu_\rho$, respectively. Then $(X_\rho \cap Y_\rho)$ is a Bourgain system of dimension at most $4(d + d')$ and $X_\rho \cap Y_\rho$ has density at least $2^{-3(d+d')} \mu_\rho \nu_\rho$ whenever $\rho \leq 1$. 

We will need to have a similar lemma for more than two Bourgain systems. We could imitate the proof of Green and Sanders for the case of two systems, but for simplicity let us just apply their result and obtain a slightly worse bound.

**Corollary 8.11.** For $i = 1, 2, \ldots, s$, let $(X^{(i)}_\rho)$ be a Bourgain system in $\mathbb{Z}_N$ of dimension $d_i$ and let $X^{(i)}_\rho$ have density $\mu^{(i)}_\rho$. Then the sets $X^{(1)}_\rho \cap \cdots \cap X^{(s)}_\rho$ form a Bourgain system of dimension at most $4s^2(d_1 + \cdots + d_s)$ and have density at least $2^{-4s^2(d_1 + \cdots + d_s)} \mu^{(1)}_\rho \cdots \mu^{(s)}_\rho$ whenever $\rho \leq 1$.

**Proof.** It is enough to prove the result when $\rho = 1$, since for smaller $\rho$ we can take a dilated system. This allows us to simplify our notation and write $\mu_i$ for $\mu^{(i)}_1$.

We begin by assuming that $s = 2^r$ for some positive integer $r$. Then we form a new collection of $2^{r-1}$ Bourgain systems by intersecting the old ones in pairs. For instance, one of the new systems is $(X^{(1)}_\rho \cap X^{(2)}_\rho)$, which has dimension at most $4(d_1 + d_2)$, and the density of $X^{(1)}_1 \cap X^{(2)}_1$ is at least $2^{-3(d_1 + d_2)} \mu_1 \mu_2$ by Lemma 8.10 above.

Now we pair off the new systems. The dimension of the first system that results will be at most $16(d_1 + d_2 + d_3 + d_4)$ and the density when $\rho = 1$ will be at least $2^{-15(d_1 + d_2 + d_3 + d_4)} \mu_1 \mu_2 \mu_3 \mu_4$.

In general, after $q$ stages we have a dimension of at most $4^q(d_1 + \cdots + d_{2^q})$ and a density when $\rho = 1$ of at least $2^{-(4^q-1)(d_1 + \cdots + d_{2^q})} \mu_1 \cdots \mu_{2^q}$, as can easily be checked by induction.

This proves the result when $s$ is a power of 2, with bounds of $s^2(d_1 + \cdots + d_s)$ and $2^{-(s^2-1)(d_1 + \cdots + d_s)} \mu_1 \cdots \mu_s$. For general $s$, one can simply take a few more Bourgain systems for which every set is equal to $\mathbb{Z}_N$ in order to make up their number to the next power of 2. \hfill $\square$

We are about to tackle one of the main results of this section, which will eventually allow us to eliminate the low-rank phases from the decomposition when the function $f$ to be decomposed has a sufficiently small $U^2$ norm. Very roughly, we shall find a structured set $S$ that is not too small such that when we convolve the low-rank part of the decomposition with the characteristic measure $\sigma$ of $S$, it remains approximately unchanged. Later, we shall also show that convolving $f$ and the rest of the decomposition of $f$ by $\sigma$ creates a function that is small. From this it follows that the low-rank part of the decomposition is small. This will give us the $\mathbb{Z}_N$ analogue of Theorem 5.7 in [GW09b]. (The proof has the same structure as well, but here the argument is substantially more complicated.)

The parameters in Proposition 8.12 below are chosen so that the proposition can be readily applied to the quadratic averages in the decomposition arising from Theorem 7.3.
An important feature of the precise statement is that the dimension of the Bourgain system \((S_{\rho'})\) it produces does not depend on the rank-related quantity \(\alpha\).

**Proposition 8.12.** Suppose that \(\alpha, \delta\) and \(\zeta\) are positive reals. Let \(C_0 = 2^{24}\), \(d = (2/\delta)C_0\), \(C = 4(2/\delta^2)C_0\) and \(\rho = (\delta/2)C_0\). Let \(k\) and \(m\) be integers bounded above by \(2C/\delta^2\) and \((5/\rho)^d\), respectively. Let \(\varepsilon > 0\) be at most \((\zeta/20kC)^2\) and let \(T = (8/\varepsilon^8)\cdot (20d^3 d^5 / \varepsilon \rho)^d C\). For each \(i = 1, 2, \ldots, k\), let \(Q'_i\) be a quadratic average with base \((B'_i, q_i)\) of complexity at most \((d, \varepsilon \rho / 800d^5\)). Moreover, suppose that each \(Q'_i\) is an \((\varepsilon, m)\)-special average, and that its rank with respect to some \(d'\)-dimensional progression \(P\) of density \(\gamma'\) satisfying \(2P - 2P \subseteq B'_i\) for each \(i\) is at most \(\log(1/\alpha)\). Suppose further that \(\sum_{i=1}^k \|U_i\|_\infty \leq 2C\) and that \(\sum_{i=1}^k \|U_i\|_{\ell^2} \leq T\). Then there exists a Bourgain system \((S'_{\rho})\) of dimension at most \(32k^3(m + 2^{33}k^6 T^4 C^2 / \zeta^6)\) such that each \(S'_{\rho}\) has density at least

\[
\gamma^k \left( \frac{\alpha^4 \zeta}{2^{15} k C d^2} \right)^{d^2 k} \left( \frac{\zeta^4 \rho'}{2^{27} k^4 C T^2} \right)^{64k^3(m + 2^{33}k^6 T^4 C^2 / \zeta^6)}
\]

such that

\[
\| \sum_{i=1}^k Q'_i U_i - \left( \sum_{i=1}^k Q'_i U_i \right) * \sigma_{\rho'} \|_{\ell^2} \leq \zeta
\]

for every \(\rho' \leq 1\), where \(\sigma_{\rho'}\) is the characteristic measure of \(S'_{\rho}\).

**Proof.** Let us begin by fixing some \(i \in \{1, 2, \ldots, k\}\). Let \(\eta = \zeta / (20kC)\), and set \(\theta = \alpha^2 \eta / 8d^2\). First we apply Lemma \(8.7\) to obtain a subprogression \(P'_i \subseteq P\) of relative density at least \((\alpha/8)^2 d^2 \theta^d\) and a Bourgain system \((X^{(i)}_{\rho'})\) of dimension at most \(2m\) such that each \(X^{(i)}_{\rho'}\) is a subset of \(P'_i\), the relative density of \(X^{(i)}_{\rho'}\) inside \(P'_i\) is at least \(3^{-d}(\rho' / 2\pi)^m\), and for every \(\rho'\) and for all but at most \(\varepsilon N\) values of \(x\), we have \(|Q'_i(x + y) - Q'_i(x)| \leq \eta + \rho'\) for all \(y \in X^{(i)}_{\rho'}\).

Set \(\xi = \zeta^3 / (2^{15} k^3 T^2)\), in which case we can check that \(\xi\) also satisfies \(4\xi C \leq \zeta / 5k\). Apply Lemma \(8.8\) with \(\rho = \xi\) and \(C\) replaced with \(2C\) to find a set \(K_i\) of cardinality at most \((2C / \xi)^2\) such that

\[
E_x |U_i(x + y) - U_i(x)|^2 \leq 4\xi^2 C^2 + 4T^4/3 \xi^{2/3}
\]

for every \(y\) in the Bohr set \(B(K_i, \xi)\), which has density at least \(\xi^{(2C / \xi)^2}\). From this we can create a Bourgain system \((A^{(i)}_{\rho'})\) of dimension at most \(3(2C / \xi)^2\) by setting \(A^{(i)}_{\rho'}\) to be the Bohr set \(B(K_i, \rho' \xi)\), in which case the above inequality holds whenever \(\rho' \leq 1\) and \(y \in A^{(i)}_{\rho'}\).

Note that for any value of \(\rho'\), the function \(U_i\) and the sets \(A^{(i)}_{\rho'}\) and \(X^{(i)}_{\rho'}\) satisfy the hypotheses of Corollary \(8.9\). More precisely, for any fixed \(\rho'\), Corollary \(8.9\) with \(X = X^{(i)}_{\rho'}\),
B = A^{(i)}_{\rho'}, \gamma = 4\xi^2C^2 + 4T^{4/3}\xi^{2/3}, \eta replaced by \eta + \rho' and C replaced with 2C tells us that if $S_i$ is any subset of $A^{(i)}_{\rho'} \cap X^{(i)}_{\rho'}$, and $\sigma_i$ is the characteristic measure of $S_i$, then
\[
\|Q_i'U_i - (Q_i'U_i) \ast \sigma_i\|_2 \leq 4(\eta + \rho')C + 2(4\xi^2C^2 + 4T^{4/3}\xi^{2/3})^{1/2} + 4\epsilon^{1/2}C.
\]

Our parameters $\eta$, $\xi$ and $\epsilon$ were chosen so that
\[
\|Q_i'U_i - (Q_i'U_i) \ast \sigma_i\|_2 \leq \zeta/k
\]
for each $i = 1, 2, \ldots, k$, provided that $\rho' \leq \zeta/(20kC)$. In particular, letting $S^{(i)}_{\rho'} = (A^{(i)}_{\rho'} \cap X^{(i)}_{\rho'}) \cap \cdots \cap (A^{(k)}_{\rho'} \cap X^{(k)}_{\rho'})$, and writing $\sigma_{\rho'}$ for the corresponding characteristic measure, we conclude that
\[
\|Q_i'U_i - Q_i'U_i \ast \sigma_{\rho'}\|_2 \leq \zeta/k
\]
for each $i = 1, 2, \ldots, k$, and hence that
\[
\left\| \sum_{i=1}^k Q_i'U_i - \left( \sum_{i=1}^k Q_i'U_i \right) \ast \sigma_{\rho'} \right\|_2 \leq \zeta.
\]

Unfortunately, since we are placing a restriction on the size of $\rho'$, the Bourgain system $(S^{(i)}_{\rho'})_{0 \leq \rho' \leq 4}$ is not quite the one we are looking for. However, we can easily get round this problem by rescaling: for each $\rho' \in [0, 4]$ let us define $S^{(i)}_{\rho'}$ to $S_{\rho'\zeta/80kC}$ and let us take the Bourgain system $(S^{(i)}_{\rho'})_{\rho' \in [0,4]}$.

It remains to verify the statements about the dimension and density of the sets $S^{(i)}_{\rho'}$. Recall that each set $X^{(i)}_{\rho'}$ had relative density $3^{-d} (\rho'/2\pi)^m$ with respect to $P'_i$. This subprogression $P'_i$ itself had relative density $(\alpha/8)^{2d^2} \theta^d$ with respect to $P$, and $P$ in turn was assumed to have density $\gamma'$ inside $\mathbb{Z}_N$. Therefore, the density of $X^{(i)}_{\rho'}$ inside $\mathbb{Z}_N$ is at least
\[
\gamma' = \gamma' \left( \frac{\alpha}{8} \right)^{2d^2} \left( \frac{\alpha^2\zeta}{480kCd^2} \right)^d \left( \frac{\rho'}{2\pi} \right)^m.
\]

The dimension of each $X^{(i)}_{\rho'}$ was simply $2m$, and the dimension of $A^{(i)}_{\rho'}$ at most $12(C/\xi)^2$. Hence by Lemma 8.1, we find that each $(A^{(i)}_{\rho'} \cap X^{(i)}_{\rho'})$ is a Bourgain system of dimension at most $4(2m + 12(C/\xi)^2)$, and the density of $A^{(i)}_{\rho'} \cap X^{(i)}_{\rho'}$ is at least $2^{-3(2m+12(C/\xi)^2)}\xi^{4(C/\xi)^2}\gamma_{\rho'}$. Finally, by Lemma 8.11 we establish that the Bourgain system $(S^{(i)}_{\rho'}) = ((A^{(i)}_{\rho'} \cap X^{(i)}_{\rho'}) \cap \cdots \cap (A^{(k)}_{\rho'} \cap X^{(k)}_{\rho'}))$ has dimension at most $16k^3(2m + 12(C/\xi)^2)$, and that $S^{(i)}_{\rho'}$ has density at least $2^{-(16k^3+3k)(2m+12(C/\xi)^2)}\xi^{4k(C/\xi)^2}\gamma_{\rho'}$, and therefore the dilated Bourgain system $(S^{(i)}_{\rho'})$ has dimension at most $16k^3(2m + 12(C/\xi)^2)$, and $S^{(i)}_{\rho'}$ has density at least $(\zeta/160kC)^{16k^3(2m+12(C/\xi)^2)}$ times the density of $S^{(i)}_{\rho'}$. 

Revisiting our choice of $\xi$, we find that $12(C/\xi)^2 \leq 2^{34}k^6T^4C^2/\xi^6$, and hence the dimension of the Bourgain system $(S'_{\rho'}\gamma)$ satisfies the desired bound. The density of $S'_{\rho'}$ is at least $(\xi/320kC)^32k^3(2m+23k^6T^4C^2/\xi^3)(\xi^3/2^{15}k^3T^2)^33k^2T^4C^2/\xi^6\rho_{\rho'}^k$, which can be simplified and bounded below by the quantity given in the statement of the proposition. 

Next, we need a technical lemma that we shall use repeatedly in the rest of the paper. It states that the rank of a quadratic average does not decrease too much when taken with respect to a slightly smaller set. This statement was proved for $\mathbb{F}_p^n$ using a simple algebraic argument in [GW09a]. As we have already discussed, arguments that depend on dimensions of subspaces do not have direct analogues in $\mathbb{Z}_N$, so instead we shall give an analytic proof. If $\beta$ is a bilinear form, let us define $\alpha_P(\beta)$ to be $E_{a',b',b''\in P} \omega^{\beta(a'-a',b-b')}$, and $r_P(\beta) = \log \alpha_P^{-1}$. Note that if $q$ is a quadratic function that is defined where it needs to be and $\beta(a,b) = q(a+b) - q(a) - q(b)$, then $r_P(\beta) = r_P(q)$, so all we are doing is attaching the rank of a quadratic function to the associated bilinear function as well. (By a “bilinear function” we mean a function that is a Freiman homomorphism in each variable separately.)

**Lemma 8.13.** Let $B'$ be a Bohr set, let $\beta$ be a bilinear function defined on $B' \times B'$ and let $P$ and $B''$ be subsets of $B'$ such that $2P - 2P \subseteq B'$ and $2B'' - 2B'' \subseteq B'$. Then

$$\alpha_P(\beta) \geq \left( \frac{|P \cap B''|}{|P|} \right)^4 \alpha_{P \cap B''}.$$ 

**Proof.** We shall repeatedly make use of the positivity property of the exponential sum that we used to define the rank of a bilinear form. We start by writing

$$\alpha_P(\beta) = E_{x,x',y,y'\in P} \omega^{\beta(x-x',y-y')} = E_{x\in P} E_{x'\in P} |E_{y\in P} \omega^{\beta(x-x',y)}|^2 = E_{x\in P} g_1(x),$$

where we have written $g_1(x) = E_{x'\in P} |E_{y\in P} \omega^{\beta(x-x',y)}|^2$. Note that $g_1$ maps into $[0, 1]$. Let $\rho = |P \cap B''|/|P|$. Then the positivity of $g_1$ implies that

$$\alpha_P(\beta) \geq \rho E_{x\in P \cap B''} g_1(x) = \rho E_{x'\in P} E_{x\in P \cap B''} |E_{y\in P} \omega^{\beta(x-x',y)}|^2 = \rho E_{x'\in P} g_2(x'),$$

where this time we have written $g_2(x') = E_{x\in P \cap B''} |E_{y\in P} \omega^{\beta(x-x',y)}|^2$. Again, $g_2$ is non-negative so that

$$\alpha_P(\beta) \geq \rho^2 E_{x'\in P \cap B''} g_2(x') = \rho^2 E_{x,x'\in P \cap B''} |E_{y\in P} \omega^{\beta(x-x',y)}|^2.$$ 

Interchanging summation, the latter expression equals

$$\rho^2 E_{y,y'\in P} |E_{x\in P \cap B''} \omega^{\beta(x-x',y-y')}|^2 = \rho^2 E_{y\in P} g_3(y) \geq \rho^3 E_{y\in P \cap B''} g_3(y),$$

where $g_3(y) = |E_{x\in P} \omega^{\beta(x-x',y)}|^2$.
with \( g_3(y) = \mathbb{E}_{y' \in P} |\mathbb{E}_{x \in P \cap B''} \omega^{\beta(x,y-y')}|^2 \), which is again non-negative. Applying the same argument one final time, we see that
\[
\rho^3 \mathbb{E}_{y' \in P} \mathbb{E}_{y \in P \cap B''} |\mathbb{E}_{x \in P \cap B''} \omega^{\beta(x,y-y')}|^2 = \rho^3 \mathbb{E}_{y' \in P} g_4(y') \geq \rho^4 \mathbb{E}_{y' \in P \cap B''} g_4(y'),
\]
where \( g_4(y') = \mathbb{E}_{y \in P \cap B''} |\mathbb{E}_{x \in P \cap B''} \omega^{\beta(x,y-y')}|^2 \) is non-negative. We have thus shown that
\[
\alpha_P(\beta) \geq \rho^4 \mathbb{E}_{x,x',y,y' \in P \cap B''} \omega^{\beta(x-x',y-y')} = \rho^4 \alpha_{P \cap B''}(\beta),
\]
which proves the result. \( \square \)

We shall also need the following lemma from [GW09b] that enables us to take a set of not too many quadratic functions and partition it into a “low-rank part” and a “high-rank part” in such a way that there is a large gap between the ranks in the two parts. We shall present the lemma in a slightly modified form and give the simple proof of the precise statement we need.

**Lemma 8.14.** Let \( R_0, b \geq 2 \) and \( t > 1 \) be constants. For each \( i = 1, 2, \ldots, k \), let \( Q_i \) be a quadratic average with base \((B'_i, q_i)\). Then for any \( P \subset \bigcap_{i=1}^k B'_i \) there is a partition of \( \{1, 2, \ldots, k\} \) into two sets \( L \) and \( H \), and a constant \( R \in [R_0, b^k(R_0 + t)] \), such that the rank of \( Q_i \) with respect to \( P \) is at most \( R \) for every \( i \in L \) and at least \( bR + t \) for every \( i \in H \).

**Proof.** Without loss of generality the \( Q_i \) are arranged in increasing order of rank with respect to \( P \). If there is no \( i \) such that \( Q_i \) has rank at least \( b^i(R_0 + t) \) with respect to \( P \), then let \( L = \{1, 2, \ldots, k\} \) and let \( R = b^k(R_0 + t) \) and we are done.

Otherwise, let \( i \) be minimal such that \( Q_i \) has rank at least \( b^iR_0 + (1 + b + \cdots + b^{i-1})t \). Set \( R = b^{i-1}R_0 + (1 + b + \cdots + b^{i-2})t \). Then for every \( j < i \) the rank of \( Q_j \) is at most \( R \), and for every \( j \geq i \) the rank of \( Q_j \) is at least \( bR + t \). Since \( R \leq b^kR_0 + (1 + b + \cdots + b^{k-1})t \leq b^k(R_0 + t) \), the lemma is proved. \( \square \)

**Lemma 8.15.** Let \( Q \) be a quadratic average with base \((B,q)\), let \( B_1 \prec \eta B \) and suppose that \( Q \) has rank \( r \) with respect to a subset \( P \subset B_1 \). Let \( Q' \) be another quadratic average, with base \((B',q')\), where \( B' \) has complexity at most \((d, \rho)\). Suppose that \( \epsilon \) and \( \alpha \) are positive constants such that
\[
16d^2\alpha + (11\eta + e^{-r})^{1/8}(4/\alpha)^{1/d}(800d^2/\rho)^d \leq 2\epsilon.
\]
Then if \( \langle Q, Q' \rangle \geq 2\epsilon \), it follows that \( \|Q'\|_{U^2} \leq (12\alpha)^{1/8} \).
Proof. The basic idea is that if \( \|Q'\|_{U^2} \) is not small, then by Theorem 6.10 we can approximate it by a quadratic average with smallish \( U^2 \) dual norm, which shows that \( Q' \) cannot after all correlate with \( Q \), which has small \( U^2 \) norm.

More precisely, Theorem 5.2 tells us that \( \|Q\|_{U^2} \leq (11\epsilon + e^{-r})^{1/8} \). Let \( \alpha > 0 \) and suppose that \( \|Q'\|_{U^2} > (12\alpha)^{1/8} \). Then Theorem 6.10 gives us a function \( Q'' \) such that \( \|Q' - Q''\|_{\infty} < 16\delta^2 \alpha \) and \( \|Q''\|_{U^2} < (4/\alpha)^{4d^2} (800d^2/\rho)^d \). It follows that

\[
\langle Q, Q' \rangle < \|Q\|_1 \|Q' - Q''\|_{\infty} + \|Q\|_{U^2} \|Q''\|_{U^2} \leq 16\delta^2 \alpha + (11\epsilon + e^{-r})^{1/8} (4/\alpha)^{4d^2} (800d^2/\rho)^d,
\]

which we are assuming to be at most \( 2\epsilon \). This proves the lemma.

\[\square\]

It turns out that we need to look some distance ahead in order to determine with respect to what sort of substructure we would like our quadratic averages to have large rank. So for the time being our choice of substructure will look rather arbitrary. For further justification the reader may wish to consult the proof of Proposition 10.2 a few pages further along.

It may help if we point out that the unpleasant bound for \( c \) in the theorem below is exponential in \( R_0 \) and doubly exponential in \( \delta \). This, rather than the precise form of the bound, is what mainly matters to us.

**Theorem 8.16.** Let \( C_0 = 2^{2^4} \), let \( \delta > 0 \) and let \( C = 4(2/\delta^2)^{C_0} \). Let \( f : \mathbb{Z}_N \to \mathbb{C} \) be a function such that \( \|f\|_2 \leq 1 \) and let \( R_0 \) be a positive real number. Let \( d = (2/\delta)^{C_0} \), \( \rho = (\delta/2)^{C_0} \), let \( \epsilon > 0 \) be bounded above by \( \delta^6/2^{12} C^5 \), let \( T = (8/\epsilon^8)^{4d^2} (2^{20} d^3 5^d/\epsilon \rho)^d C \) and let \( c > 0 \) be at most

\[
e^{-2^{15\delta} d^7 k^{6k} R_0} \left( \frac{\delta^2 \rho^2 \epsilon^2 \Phi}{2^{48k^4 5^d C^2}} \right)^{2^{26k^4 10k^4k^6}},
\]

where

\[
\Phi = \Phi(\delta, \epsilon) = \left( \frac{\delta^5 \epsilon \rho}{2^{34k^8 d^{15d} C^2 T^2}} \right)^{64k^2(m+d^2+2^{33k^6 T^4} C^2/\delta^6)}.
\]

Let \( f \) be any function such that \( \|f\|_{U^2} \leq c \). Then \( f \) has a decomposition of the form

\[
f(x) = \sum_{i=1}^k Q'_i(x) U_i(x) + g(x) + h(x),
\]

where \( k \leq 2C/\delta^2 \) and the \( Q'_i \) are quadratic averages on \( \mathbb{Z}_N \) with base \( (B'_i, q'_i) \) and of complexity at most \( (d, \epsilon \rho/800 d^5) \), such that \( \sum_{i=1}^k \|U_i\|_{U^2} \leq T, \sum_{i=1}^k \|U_i\|_{\infty} \leq 2C, \|g\|_1 \leq 10\delta \) and \( \|h\|_{U^3} \leq 2\delta \). Moreover, each quadratic average \( Q'_i \) is \( (\epsilon, m) \)-special for \( m \leq (5/\rho)^d \), and there exists a proper generalized arithmetic progression \( P \) inside \( B' = \bigcap_{i=1}^k B'_i \) of dimension \( d' \leq kd \) and density \( \gamma' \geq (\epsilon \rho/2^{12} d' 5^d)^{d'} \), such that each \( Q'_i \) has rank at least \( R_0 \) with respect to \( P \).
Proof. Because \( \epsilon \leq \delta^6/2^{12}C^5 \), it is also at most \((\delta/2)^{5C^6}\) and therefore satisfies the hypothesis of Theorem \[.3\] We deduce that \( f \) has a decomposition of the form

\[
f(x) = \sum_{i=1}^{k} Q_i'(x)U_i(x) + g'(x) + h'(x),
\]

with the following properties: \( k \leq 2C/\delta^2 \), the \( Q_i' \) are quadratic averages on \( \mathbb{Z}_N \) with base \((B'_i, q_i)\) and of complexity at most \((d, \epsilon \rho/800d5^d)\), \( \sum_{i=1}^{k} \|U_i\|_{*}^2 \leq T \), \( \sum_{i=1}^{k} \|U_i\|_{\infty} \leq 2C \), \( \|g'\|_1 \leq 3\delta \) and \( \|h'\|_{U^3} \leq \delta \). Moreover, each average \( Q_i' \) is \((\epsilon, m)\)-special for \( m \leq (5/\rho)^d \).

By a lemma of Ruzsa \[R94\] (see also \[N96\]) there is a proper generalized arithmetic progression \( P \subset B' \) with the properties claimed in the theorem. (The additional factor of 1/4 in the density of this progression arises from the requirement that \( 2P - 2P \subseteq B'_i \), which we need in order to be able to talk about the rank of the quadratic average with respect to \( P \).) Let us assume that the quadratic averages are arranged in increasing order of rank with respect to \( P \).

Applying Lemma \[8.14\] with \( b = 2^{13}d^5k^3 \) and

\[
t = 2^{11}d^3 \log \left( \frac{2^{3(k+15)}d^415^dC}{\delta \rho^2 \epsilon^2 \Phi} \right),
\]

we obtain positive integers \( R \in [R_0, b^k(R_0 + t)] \) and \( s \in \{0, 1, \ldots, k\} \) such that \( Q_i' \) has rank at most \( R \) when \( i \leq s \) and rank at least \( bR + t \) when \( i > s \). We collect together the low- and high-rank quadratic phases by setting \( f_L = \sum_{i=1}^{s} Q_i'U_i \) and \( f_H = \sum_{i=s+1}^{k} Q_i'U_i \).

Because \( \epsilon \leq \delta^6/2^{12}C^5 \) and \( k \leq 2C/\delta^2 \), we also have \( \epsilon \leq (\delta/20kC)^2 \), so it satisfies the hypothesis of Proposition \[8.12\] with \( \zeta = \delta \). Setting \( \log(1/\alpha) = R \) in Proposition \[8.12\] we obtain a Bourgain system \((S'_{\rho'})\) of dimension at most \( 32k^3(m + 2^{23}k^6T^4C^2/\delta^6) \) such that \( \|f_L - f_L * \sigma\|_2 \leq \delta \), where \( \sigma \) is the characteristic measure of \( S'_{\rho'} \). That proposition also gives us a lower bound for the density \( \gamma \) of \( S_{\rho'} \) of

\[
\left( \frac{\alpha^4 \delta \epsilon \rho}{2^{27}k^4d^415^dC} \right)^{d^2k^3} \left( \frac{\delta^4}{2^{27}k^4CT^2} \right)^{(64k^3(m + 2^{23}k^6T^4C^2/\delta^6)^2)} \geq e^{-4d^2k^3R} \cdot \Phi(\delta, \epsilon).
\]

Now let us reconsider our original decomposition \( f = f_L + f_H + g' + h' \). We shall convolve this equation with the measure \( \sigma \) on both sides. We shall show that all of \( f * \sigma \), \( f_H * \sigma \), \( g' * \sigma \) and \( h' * \sigma \) are small and we have already seen that \( f_L * \sigma \approx f_L \). From this it will follow that \( f_L \) is small enough to be absorbed into the error terms.

Let us deal with the easy parts first. Since \( \|\sigma\|_1 = 1 \) and the \( L_1 \) norm is translation invariant, the triangle inequality implies that \( \|g' * \sigma\|_1 \leq \|g'\|_1 \leq 3\delta \). Similarly, \( \|h' \sigma\|_{U^3} \leq \delta \), since the \( U^3 \) norm is also translation invariant.
Next, let us estimate $\|f \ast \sigma\|_1$. The Cauchy-Schwarz inequality (applied to the Fourier transform, though a direct argument is also possible) gives us that $\|f \ast \sigma\|_1 \leq \|f \ast \sigma\|_2 \leq \|f\|_{U^2} \|\sigma\|_{U^2}$. But we are assuming that $\|f\|_{U^2} \leq c$ and we know that $\|\sigma\|_{U^2} \leq \|\sigma\|_\infty = \gamma^{-1}$. Thus provided that $c \leq \delta \gamma$, we obtain the bound $\|f \ast \sigma\|_2 \leq \delta$. This gives us the upper bound that $c$ will be required to satisfy for the theorem to hold.

Our one remaining task is to show that $\|f_H \ast \sigma\|_1$ is small (when the parameters are appropriately chosen). This is significantly harder, and we shall need to use Lemma 8.15.

Recall first that each quadratic average $Q_j$ that appears in $f_H$ has base $(B'_i, q_i)$ and rank at least $bR + t$ with respect to the progression $P \subseteq B'$. We also recall from the proof of Theorem 7.5 that $Q'_j U_i(x)$ can be written as $\sum_{j \in A_i} \lambda_j Q_j(x) + V_i$, where $\|V_i\|_1 \leq (2\epsilon + 8d^2 \delta^8) \sum_{j \in A_i} |\lambda_j|$. The functions $Q_j$ are quadratic averages with base $(B_j, q_j)$ and complexity at most $(d, \rho)$. Let $f'_H(x) = \sum_{i > s} \sum_{j \in A_i} \lambda_j Q_j(x)$. We have

$$\|f_H \ast \sigma\|_1 \leq \|f'_H \ast \sigma\|_1 + \|(f_H - f'_H) \ast \sigma\|_1 \leq \|f'_H \ast \sigma\|_1 + \sum_{i > s} \sum_{j \in A_i} |\lambda_j|(2\epsilon + 8d^2 \delta^8).$$

The latter term was shown to be at most $\delta$ in the proof of Theorem 7.5. It follows that $\|f_H \ast \sigma\|_1 \leq \|f'_H \ast \sigma\|_1 + \delta$, and thus it suffices to estimate $\|f'_H \ast \sigma\|_1$. In fact, we shall obtain an upper bound for $\|f'_H \ast \sigma\|_2$.

By the Cauchy-Schwarz inequality as used on $\|f_L \ast \sigma\|_2$ earlier we have

$$\|f'_H \ast \sigma\|_2 \leq \|f'_H\|_{U^2} \|\sigma\|_{U^2} \leq \gamma^{-1} \sum_{i > s} \sum_{j \in A_i} |\lambda_j| \|Q_j\|_{U^2}$$

with $\sum_{i > s} \sum_{j \in A_i} |\lambda_j| \leq C$. In order to prove that $\|f'_H \ast \sigma\|_1 \leq \delta$ it will therefore be enough to show that each $Q_j$ has $U^2$ norm at most $\delta \gamma/C$. To do this, we extract further information from the proof of Theorem 7.5. It tells us that there is another quadratic average $Q''_i$ with the same base $(B''_i, q_i)$ as $Q'_i$ such that $\langle Q''_i, Q_j \rangle \geq 2\epsilon$. Since $Q''_i$ has the same high rank as $Q'_i$ and correlates with $Q_j$, we are in a position to apply Lemma 8.15.

To do this, we set $Q = Q''_i$, $B = B'_i$ and $q = q_i$. We shall let

$$\eta = \left(\frac{\epsilon}{4}\right)^8 \left(\frac{\rho}{800d^2}\right)^{8d} \left(\frac{\delta \gamma}{4C}\right)^{2^8d^2}$$

and we shall take $B'$ to be a Bohr subset $B''_i$ of $B'_i$ such that $B''_i \prec_\eta B'_i$. We then take $Q'$ to be $Q_j$, remarking that $Q_j$ has base $(B_j, q_j)$ for some $B_j$ of complexity at most $(d, \rho)$.

We shall take the set $P$ in Lemma 8.15 to be the set $P \cap B''_i$ here. We now need a lower bound for the rank of $Q = Q''_i$ with respect to $P \cap B''_i$, or equivalently an upper bound for the quantity $\alpha_{P \cap B''_i}(Q)$. Lemma 8.13 tells us that $\alpha_{P \cap B''_i}(Q) \leq \beta^{-4} \alpha_P \leq \beta^{-4} e^{-((bR+t) \eta)}$, where
where $\beta = |P \cap B''|/|P|$ is the relative density of $P \cap B''$ in $P$. By Lemma 8.10 we find that $\beta$ is at least $2^{-3(kd+3d)}$ times the density of $B''$ in $P$, so $\beta \geq (\eta \epsilon \rho/2^{3(k+10)}d^25^d)^d$. Therefore, we can take $e^{-r}$ in Lemma 8.15 to be $\beta^{-4}e^{-(bR+t)}$ with this value of $\beta$. It can now be checked (the checking, though painful, is routine) that if we take $\alpha = (\delta \gamma/C)^8/12$, then the conditions for Lemma 8.15 are satisfied. Therefore, by that lemma, $\|Q_j\|_{U^2} \leq \delta \gamma/C$.

This completes the proof that $\|f_{H^*\sigma}\|_1 \leq 2\delta$. We have therefore demonstrated that it is possible to write $f_L$ as a sum $g'' + h''$ with $\|g''\|_1 \leq 7\delta$ and $\|h''\|_{U^3} \leq \delta$. It follows that $f$ has a decomposition $f = f_H + g + h$ with $\|g\|_1 \leq 10\delta$ and $\|h\|_{U^3} \leq 2\delta$ as claimed. Finally, we remark that the rank $R$ was at most $b^k(R_0 + t)$, a condition which we insert into our bound for the uniformity parameter $c$ to obtain the theorem as stated.

\[\square\]

9. Some facts about ranks of quadratic and bilinear functions on Bohr sets

In $\mathbb{F}_p^n$, it was more or less self-evident that the rank of the sum of two quadratic forms was bounded above by the sum of the individual ranks. Such subadditivity, even in approximate form, is no longer a trivial statement for forms of higher degree such as those in [GW09c], and, as it turns out, for the locally defined quadratic forms that we are dealing with in this paper. Here we shall use regular sets from Bourgain systems to adapt the analytic proof of subadditivity for $\mathbb{F}_p^n$ given in [GW09c] to $\mathbb{Z}_N$. The reader may wish to consult the finite-fields argument in that paper before embarking on this section.

The following standard identity is the key ingredient in the proof of subadditivity.

**Lemma 9.1.** Let $B \subseteq \mathbb{Z}_N$ and let $\beta : B^2 \rightarrow \mathbb{Z}_N$ be a bilinear function and let $f(x, y) = \omega^{\beta(x, y)}$. Then

$$f(a - a', b - b') = f(x + a, y + b)f(x + a, y + b')f(x + a', y + b)f(x + a', y + b')$$

provided that all of $a - a', b - b', x + a, x + a', y + b, y + b'$ lie in $B$.

**Proof.** This follows immediately from the identity

$$\beta(a - a', b - b') = \beta(x + a, y + b) - \beta(x + a', y + b) - \beta(x + a, y + b') + \beta(x + a', y + b'),$$

which can easily be checked by hand. \[\square\]

**Lemma 9.2.** Let $B, B'$ be two sets from a Bourgain system and suppose that $B' \prec_\epsilon B$. Write $\pi$ for the characteristic measure of $B$. Then for every $s \in B'$ and every function
Let \( j : \mathbb{Z}_N \to \mathbb{C} \) with \( \| j \|_\infty \leq 1 \), we have
\[
\mathbb{E}_{u \in \mathbb{Z}_N} \pi * \pi(u)j(u + s) \approx \epsilon \mathbb{E}_{u \in \mathbb{Z}_N} \pi * \pi(u)j(u).
\]
In particular, it follows immediately that for any \( A \subseteq B' \),
\[
\mathbb{E}_{u \in \mathbb{Z}_N} \pi * \pi(u)j(u) \approx \epsilon \mathbb{E}_{u \in \mathbb{Z}_N} \mathbb{E}_{s \in A} \pi * \pi(u)j(u + s).
\]

**Proof.** We estimate the difference between the left- and right-hand side above by expanding out the convolution and using the triangle inequality.
\[
|\mathbb{E}_{u \in \mathbb{Z}_N} \pi * \pi(u)j(u + s) - \pi * \pi(u)j(u)| = |\mathbb{E}_{u,z \in \mathbb{Z}_N} \pi(z)\pi(u - z)(j(u + s) - j(u))|
\leq \mathbb{E}_{z \in \mathbb{Z}_N} \pi(z)|\mathbb{E}_{u \in \mathbb{Z}_N} \pi(u)(j_z(u + s) - j_z(u))|
= \mathbb{E}_{z \in \mathbb{Z}_N} \pi(z)|\mathbb{E}_{u \in B} j_z(u + s) - j_z(u)|,
\]
where \( j_z(u) = j(u + z) \) for all \( u \). The inner expectation is at most \( \epsilon \) for every \( s \in B' \) by Lemma 8.3. \( \square \)

We now apply Lemma 9.2 to derive an inequality reminiscent of the usual lemmas that say that a function behaves quasirandomly if its \( U^2 \) norm is small. However, our inequality concerns a “local” version of the \( U^2 \) norm. Given two sets \( B' \prec_\epsilon B \) from a Bourgain system and a function \( h : \mathbb{Z}_N \to \mathbb{C} \), we shall define \( \| h \|_{U^2(B+B',B')} \) by the formula
\[
\| h \|_{U^2(B+B',B')}^4 = \mathbb{E}_{x,y} \pi * \pi(x) \pi * \pi(y)
\mathbb{E}_{u',b,b' \in B'} h(x + a, y + b)h(x + a', y + b)h(x + a, y + b')h(x + a', y + b'),
\]
where \( \pi \) is the characteristic measure of \( B \).

**Lemma 9.3.** Let \( \epsilon > 0 \), let \( B' \prec_\epsilon B \) be a regular Bourgain pair and write \( \pi \) for the characteristic measure of \( B \). Then for any function \( h : (\mathbb{Z}_N)^2 \to \mathbb{C} \) with \( \| h \|_\infty \leq 1 \), we have the estimate
\[
|\mathbb{E}_{x,y \in \mathbb{Z}_N} \pi * \pi(x) \pi * \pi(y)h(x, y)|^4 \leq \| h \|_{U^2(B+B,B')}^4 + 6\epsilon.
\]

**Proof.** The Cauchy-Schwarz inequality implies that
\[
|\mathbb{E}_{x,y \in \mathbb{Z}_N} \pi * \pi(x) \pi * \pi(y)h(x, y)|^4 \leq |\mathbb{E}_{x \in \mathbb{Z}_N} \pi * \pi(x)|^2 |\mathbb{E}_{y \in \mathbb{Z}_N} \pi * \pi(y)|^2 |\mathbb{E}_{x,y} h(x, y)|^2.
\]
Lemma 9.2 tells us that
\[
\mathbb{E}_{y \in \mathbb{Z}_N} \pi * \pi(y) h(x, y) \approx \epsilon \mathbb{E}_{y \in \mathbb{Z}_N} \mathbb{E}_{b \in B'} \pi * \pi(y) h(x, y + b)
\]

\[
\mathbb{E}_{x \in \mathbb{Z}_N} \pi \mathbb{E}_{y \in \mathbb{Z}_N} \pi(y)h(x, y) \approx \epsilon \mathbb{E}_{x \in \mathbb{Z}_N} \mathbb{E}_{b \in B'} \pi h(x, y + b)
\]

\[
\mathbb{E}_{x \in \mathbb{Z}_N} \pi h(x) \approx \epsilon \mathbb{E}_{x \in \mathbb{Z}_N} \mathbb{E}_{b \in B'} h(x + b)
\]

\[
\mathbb{E}_{x \in \mathbb{Z}_N} h(x) \approx \epsilon \mathbb{E}_{x \in \mathbb{Z}_N} \mathbb{E}_{b \in B'} h(x + b)
\]
for every \(x\), from which it follows that
\[
|\mathbb{E}_{y \in \mathbb{Z}_N} \pi \ast \pi(y) h(x, y)|^2 \approx 2\epsilon |\mathbb{E}_{y \in \mathbb{Z}_N} \mathbb{E}_{b \in B'} \pi \ast \pi(y) h(x, y + b)|^2.
\]
From this it follows that
\[
|\mathbb{E}_{x \in \mathbb{Z}_N} \pi \ast \pi(x) |\mathbb{E}_{y \in \mathbb{Z}_N} \pi \ast \pi(y) h(x, y)|^2|^2 \leq |\mathbb{E}_{x \in \mathbb{Z}_N} \pi \ast \pi(x) |\mathbb{E}_{y \in \mathbb{Z}_N} \mathbb{E}_{b \in B'} \pi \ast \pi(y) h(x, y + b)|^2|^2 + 4\epsilon.
\]
(For these last two approximations we have used the fact that if \(a \approx \epsilon b\) and \(a\) and \(b\) both have modulus at most 1, then \(a^2 \approx 2\epsilon b^2\), which follows from the fact that \(a^2 - b^2 = (a + b)(a - b)\).) By the Cauchy-Schwarz inequality,
\[
|\mathbb{E}_{x \in \mathbb{Z}_N} \pi \ast \pi(x) |\mathbb{E}_{y \in \mathbb{Z}_N} \mathbb{E}_{b \in B'} \pi \ast \pi(y) h(x, y + b)|^2|^2 \\
\leq |\mathbb{E}_{x \in \mathbb{Z}_N} \pi \ast \pi(x) |\mathbb{E}_{y \in \mathbb{Z}_N} \pi \ast \pi(y) |\mathbb{E}_{b \in B'} h(x, y + b)|^2|^2 \\
= |\mathbb{E}_{y \in \mathbb{Z}_N} \pi \ast \pi(y) \mathbb{E}_{b, b' \in B'} |\mathbb{E}_{x \in \mathbb{Z}_N} \pi \ast \pi(x) h(x, y + b) h(x, y + b')|^2.
\]
Applying Lemma 9.2 in a similar way a second time, we see that this is at most
\[
|\mathbb{E}_{y \in \mathbb{Z}_N} \pi \ast \pi(y) \mathbb{E}_{b, b' \in B'} |\mathbb{E}_{x \in \mathbb{Z}_N} \pi \ast \pi(x) h(x + a, y + b) h(x + a, y + b')|^2 + 2\epsilon \\
\leq \mathbb{E}_{x, y \in \mathbb{Z}_N} \pi \ast \pi(x) \pi \ast \pi(y) \mathbb{E}_{a \in B'} |\mathbb{E}_{a \in B'} h(x + a, y + b) h(x + a, y + b')|^2 + 2\epsilon,
\]
which equals \(\|h\|_{U^2(B + B')}^4 + 2\epsilon\). This proves the lemma. \(\square\)

Finally, we need to exploit regularity once more to be able to shift our variables at a certain point in the proof. We isolate the lemma, which is very similar to Lemma 9.2, in order to keep the proof of the main result tidy.

**Lemma 9.4.** Let \(B\) and \(B'\) be sets from a Bourgain system with \(B' \prec \epsilon B\), and write \(\pi\) for the characteristic measure of \(B\). Write \(\sigma(x) = \pi \ast \pi(x)\), \(\rho\) for the density of \(B\) and let \(j : \mathbb{Z}_N \to \mathbb{C}\) be an arbitrary function with \(\|j\|_{\infty} \leq 1\). Then for any \(a \in B'\),
\[
\mathbb{E}_{x \in \mathbb{Z}_N} \sigma(x + a)^2 j(x) \approx_{2\epsilon / \rho} \mathbb{E}_{x \in \mathbb{Z}_N} \sigma(x)^2 j(x).
\]
Proof. As usual, we shall attempt to bound the difference between the two sides in absolute value.

\[
|\mathbb{E}_x (\sigma(x + a)^2 - \sigma(x)^2) j(x)| = |\mathbb{E}_x (\sigma(x + a) + \sigma(x)) (\sigma(x + a) - \sigma(x)) j(x)| \\
= |\mathbb{E}_x (\sigma(x + a) + \sigma(x)) j(x) \pi(v) (\pi(x + a - v) - \pi(x - v))| \\
\leq 2 \rho^{-1} \mathbb{E}_x |\pi(v) (\pi(x + a - v) - \pi(x - v))| \\
\leq 2 \rho^{-1} \mathbb{E}_x |\pi(v) \mathbb{E}_x |\pi(x + a - v) - \pi(x - v)|
\]

The expression \(|\pi(x + a - v) - \pi(x - v)|\) is non-zero if and only if \(x \in (v + B) \triangle (v - a + B)\), which by regularity assumptions is the case for at most \(\epsilon |B|\) values of \(x\). The non-zero value taken is \(\rho^{-1}\), and we conclude that \(\mathbb{E}_x |\pi(x + a - v) - \pi(x - v)| \leq \epsilon\). The lemma follows. \(\square\)

We are now fully prepared to prove subadditivity. We remind the reader that \(\alpha_P(\beta) = \mathbb{E}_{a,a',b,b'} \omega^{\beta(a-a',b-b')}\), and \(r_P(\beta) = \log \alpha_P^{-1}\) for any bilinear form \(\beta\) defined on a set that contains \(P - P\).

**Lemma 9.5.** Let \(\beta_1\) and \(\beta_2\) be bilinear forms defined on a Bohr set \(B\), and let \((B_\rho)\) be a Bourgain system of dimension \(d\) such that \(B_1\) has density \(\gamma < 1/2\) and \(2B_1 - 2B_1 \subseteq B\). Let \(\epsilon > 0\). Then

\[(\alpha_{B_1}(\beta_1) \alpha_{B_1}(\beta_2))^4 \leq \gamma^{-6}(800d/\epsilon)^{4d} \alpha_{B_1}(\beta_1 + \beta_2) + 9\epsilon \gamma^{-7}.
\]

**Proof.** Let \(B' \prec \epsilon B_1\), write \(\gamma'\) for the density of \(B'\), and note that \(\gamma' \geq (\epsilon/800d)^d \gamma\). We shall begin to prove the subadditivity statement by considering the expression

\[
\alpha_{B_1}(\beta_1) \alpha_{B_1}(\beta_2) = \mathbb{E}_{x,x',y,y' \in B_1} f(x - x', y - y') \mathbb{E}_{u,u',v,v' \in B_1} g(u - u', v - v') \\
= \mathbb{E}_{x,y \in \mathbb{Z}_N} \pi * \pi(x) \pi * \pi(y) f(x, y) \mathbb{E}_{u,v \in \mathbb{Z}_N} \pi * \pi(u) \pi * \pi(v) g(u, v),
\]

where \(\pi\) is the characteristic measure of \(B_1\). Shifting two of the variables, we obtain

\[
\mathbb{E}_{x,y \in \mathbb{Z}_N} \pi * \pi(x) \pi * \pi(y) f(x, y) \mathbb{E}_{u,v \in \mathbb{Z}_N} \pi * \pi(x + u) \pi * \pi(y + v) g(x + u, y + v).
\]

Writing \(\sigma = \pi * \pi\), we apply Hölder’s inequality (or the Cauchy-Schwarz inequality twice) to show that

\[
(\alpha_{B_1}(\beta_1) \alpha_{B_1}(\beta_2))^4 \leq \mathbb{E}_{u,v} |\mathbb{E}_{x,y} \sigma(x)\sigma(y) f(x,y)\sigma(x + u)\sigma(y + v) g(x + u, y + v)|^4 \\
= \mathbb{E}_{u,v} |\mathbb{E}_{x,y} \sigma(x)\sigma(y) h_{u,v}(x,y)|^4
\]
where we have set \( h_{u,v}(x, y) = f(x, y)g(x + u, y + v)\sigma(x + u)\sigma(y + v) \). For every fixed value of \( u \) and \( v \), we shall apply Lemma 9.3. From this, we deduce that

\[
(\alpha_B(\beta_1)\alpha_B(\beta_2))^4 \leq \mathbb{E}_{u,v}\mathbb{E}_{x,y}\sigma(x)\sigma(y)\mathbb{E}_{a,a',b,b'\in B'}h_{u,v}(x + a, y + b) \\
\hspace{1cm}(x + a', y + b') h_{u,v}(x + a', y + b') + 12\epsilon.
\]

(We have omitted the condition \( \epsilon < 1/3 \) from the statement of this lemma since if \( \epsilon \geq 1/3 \) then the lemma holds trivially.) Next, we expand out \( h_{u,v} \), which replaces the right-hand side by

\[
\mathbb{E}_{u,v}\mathbb{E}_{x,y}\sigma(x)\sigma(y)\mathbb{E}_{x,a',b,b'\in B'}f(x + a, y + b)\mathbb{E}_{x,x',y}f(x + a', y + b) f(x + a', y + b') \\
g(x + u + a, y + v + b)g(x + u + a', y + v + b)g(x + u + a', y + v + b')g(x + u + a', y + v + b') \\
\sigma(x + u + a)\sigma(x + u + a')^2\sigma(y + v + b)^2\sigma(y + v + b')^2 + 12\epsilon.
\]

Setting \( x' = x + u, y' = y + v \), we can rewrite this expression as

\[
\mathbb{E}_{x,y,x',y'}\sigma(x)\sigma(y)\mathbb{E}_{a,a',b,b'\in B'}f(x + a, y + b)\mathbb{E}_{x,x',y}f(x + a', y + b) f(x + a', y + b') \\
g(x' + a, y' + b)g(x' + a', y' + b')g(x' + a, y' + b')g(x' + a', y' + b') \\
\sigma(x' + a)\sigma(x' + a')^2\sigma(y' + b)^2\sigma(y' + b')^2 + 12\epsilon.
\]

Lemma 9.1 tells us that this expression is equal to

\[
\mathbb{E}_{x,y,x',y'}\sigma(x)\sigma(y)\mathbb{E}_{a,a',b,b'\in B'}\sigma(x' + a)\sigma(x' + a')^2\sigma(y' + b)^2\sigma(y' + b')^2 f(a - a', b - b')g(a - a', b - b') + 12\epsilon.
\]

Since \( \mathbb{E}_x\sigma(x) = 1 \), this in turn equals

\[
\mathbb{E}_{a,a',b,b'\in B'}f(a - a', b - b')g(a - a', b - b')\mathbb{E}_{x',y'}\sigma(x' + a)\sigma(x' + a')^2\sigma(y' + b)^2\sigma(y' + b')^2 + 12\epsilon.
\]

We would like to be able to evaluate the inner expectation independently of the choice of \( a, a', b, b' \). We cannot do this exactly, but Lemma 9.4 tells us that \( \sigma \) is approximately translation invariant, so we can do it if we introduce a small error. For instance, if we apply it to the first occurrence of the function \( \sigma^2 \) and let \( j(x') = \gamma^6\sigma(x' + a')\sigma(y' + b)^2\sigma(y' + b')^2 \), then \( \|j\|_\infty \leq 1 \), so we find that

\[
\mathbb{E}_{x',y'}\sigma(x' + a)\sigma(x' + a')^2\sigma(y' + b)^2\sigma(y' + b')^2 \leq \mathbb{E}_{x',y'}\sigma(x')^2\sigma(x' + a')^2\sigma(y' + b)^2\sigma(y' + b')^2 + 2\gamma^{-7}\epsilon.
\]

Applying the lemma three more times in this way, we find that

\[
(\alpha_{B_1}(\beta_1)\alpha_{B_1}(\beta_2))^4 \leq \mathbb{E}_{a,a',b,b'\in B'}f(a - a', b - b')g(a - a', b - b')\mathbb{E}_{x',y'}\sigma^4(x')\mathbb{E}_{y'}\sigma^4(y') + 8\gamma^{-7}\epsilon + 12\epsilon.
\]
But since $\mathbb{E}_{a,a',b,b' \in B'} f(a-a', b-b') g(a-a', b-b') = \alpha_{B'}(\beta_1 + \beta_2)$ and $\mathbb{E}_x \sigma^4(x) \leq \gamma^{-3}$, we have shown that

$$(\alpha_{B_1}(\beta_1) \alpha_{B_1}(\beta_2))^4 \leq \gamma^{-6} \alpha_{B'}(\beta_1 + \beta_2) + 8 \gamma^{-7} \epsilon + 12 \epsilon.$$ 

Unfortunately the exponential sum on the right-hand side is taken over $B'$, or we would be done. But we can remedy this situation by applying Lemma 8.13 which implies that $\alpha_{B'} \leq (\gamma/\gamma')^4 \alpha_{B_1}$. Therefore,

$$(\alpha_{B_1}(\beta_1) \alpha_{B_1}(\beta_2))^4 \leq \gamma^{-2} \gamma'^{-4} \alpha_{B_1}(\beta_1 + \beta_2) + 8 \gamma^{-7} \epsilon + 12 \epsilon.$$ 

The result follows from the lower bound for $\gamma'$ mentioned at the beginning of the proof. □

We need a slight generalization of Lemma 9.5 to be able to sum arbitrarily many bilinear forms. In fact, we shall not use Lemma 9.5 as stated to carry out the induction, but rather the main intermediate result in the proof above that related the rank of $\beta_1 + \beta_2$ with respect to $B'$ to the individual ranks with respect to $B_1$.

**Lemma 9.6.** Let $\epsilon > 0$. For $i = 1, 2, \ldots, m$, let $\beta_i$ be a bilinear form defined on a set $B$, and let $(B_\rho)$ be a Bourgain system of dimension $d$ such that $B_1$ has density $\gamma < 1/8$ and $2B_1 - 2B_1 \subseteq B$. Then

$$\prod_{i=1}^m \alpha_B(\beta_i) \leq \gamma^{-2m^2} (800d/\epsilon)^{d \log m/m^2} \alpha_B(\sum_{i=1}^m \beta_i)^{1/m^2} + 8 \gamma^{-2m^2} (\epsilon^{1/4}/\gamma^2)^{1/m^2}.$$ 

**Proof.** Let us start off by considering the case when $m = 2^s$. Let $B_{s+1} \prec_\epsilon B_s \prec_\epsilon \ldots \prec_\epsilon B_2 \prec_\epsilon B_1$ be a sequence of sets from the Bourgain system $(B_\rho)$. (Thus, the indices do not indicate values of $\rho$.) We shall prove that

$$\prod_{i=1}^{2^s} \alpha_{B_1}(\beta_i) \leq A^4 \alpha_{B_{s+1}}(\sum_{i=1}^{2^s} \beta_i)^{1/4^s} + 4a^{1/4^s} A^{4^s},$$ 

where $A = \gamma^{-3/2}$ and $a = 2\gamma^{-2} \epsilon^{1/4}$, and proceed by induction on $s$. The case where $s = 1$ is guaranteed by the proof of Lemma 9.5. Indeed, before we switched from $B'$ back to $B_1$, the inequality we had implied that

$$\alpha_{B_1}(\beta_1) \alpha_{B_1}(\beta_2) \leq A \alpha_{B'}(\beta_1 + \beta_2)^{1/4} + a,$$

on the assumption that $B' \prec_\epsilon B_1$. If we take $B' = B_2$, then this is in fact stronger than the case $s = 1$ of this lemma.
Suppose now that the statement is true for \( s \), and consider
\[
\prod_{i=1}^{2^s+1} \alpha_{B_1}(\beta_i) \leq (A^{4^s} \alpha_{B_{s+1}}(\sum_{i=1}^{2^s} \beta_i)\|1/4^s + 4a^{1/4^s} A^{4^s})(A^{4^s} \alpha_{B_{s+1}}(\sum_{i=2^s+1}^{2^s+1} \beta_i)\|1/4^s + 4a^{1/4^s} A^{4^s})
\]
\[
\leq A^{2^{4^s}}(\alpha_{B_{s+1}}(\sum_{i=1}^{2^s} \beta_i)\alpha_{B_{s+1}}(\sum_{i=2^s+1}^{2^s+1} \beta_i))\|1/4^s + 8a^{1/4^s} A^{2\cdot4^s} + 16a^{2/4^s} A^{2\cdot4^s},
\]
from which it follows by the strengthened version of the \( s = 1 \) case noted above that
\[
\prod_{i=1}^{2^s+1} \alpha_{B_1}(\beta_i) \leq A^{2^{4^s}}(A\alpha_{B_{s+2}}(\sum_{i=1}^{2^s+1} \beta_i)\|1/4^s + a)\|1/4^s + 8a^{1/4^s} A^{2\cdot4^s} + 16a^{2/4^s} A^{2\cdot4^s}
\]
\[
\leq A^{2^{4^s+1/4^s}}\alpha_{B_{s+2}}(\sum_{i=1}^{2^s+1} \beta_i)\|1/4^{s+1} + A^{2\cdot4^s} a^{1/4^s} + 8a^{1/4^s} A^{2\cdot4^s} + 16a^{2/4^s} A^{2\cdot4^s}.
\]
It is easily checked that this expression is bounded above by
\[
A^{4^{s+1}}\alpha_{B_{s+2}}(\sum_{i=1}^{2^s+1} \beta_i)\|1/4^{s+1} + 4a^{1/4^{s+1}} A^{4^{s+1}}
\]
as claimed, provided that \( \gamma < 1/8 \). This concludes the inductive step. To complete the proof, we apply Lemma 8.13 to obtain a statement about the rank with respect to \( B_1 \). It tells us that
\[
\prod_{i=1}^{2^s} \alpha_{B_1}(\beta_i) \leq A^{4^s} \alpha_{B_{s+1}}(\sum_{i=1}^{2^s} \beta_i)\|1/4^s + 4a^{1/4^s} A^{4^s} \leq A^{4^s}(|B_1|/|B_{s+1}|)\|1/4^s \alpha_{B_1}(\sum_{i=1}^{2^s} \beta_i)\|1/4^s + 4a^{1/4^s} A^{4^s},
\]
with \(|B_{s+1}| \geq (\epsilon/800d)^s |B_s| \geq ... \geq (\epsilon/800d)^{sd} |B_1| \). It follows that
\[
\prod_{i=1}^{2^s} \alpha_{B_1}(\beta_i) \leq A^{4^s} (800d/\epsilon)^{sd/4^s} \alpha_{B_1}(\sum_{i=1}^{2^s} \beta_i)\|1/4^s + 4a^{1/4^s} A^{4^s}.
\]
For general \( m \), note that we can add in bilinear forms that are identically zero without affecting the argument.

Next we state and prove a modified version of Lemma 6.3 and Corollary 6.4 from [GW09]. This is the first and only time we make use of the assumption that our system of linear forms is square independent.

**Lemma 9.7.** Let \( \epsilon > 0 \). Suppose that \( L_i(x) = \sum_{u=1}^{d} c_{iu} x_u \), \( i = 1, 2, \ldots, m \), is a square-independent system. Suppose that each of the (not necessarily distinct) bilinear forms \( \beta_i \), \( i = 1, 2, \ldots, m \) is defined on a Bohr set \( B \), and that \( (B_\rho) \) is a Bourgain system of dimension
functions with \( \| \) We have

\[
\text{Proof. For each } i = 1, 2, \ldots, m, \text{ let } M_i \text{ be the } (d \times d) \text{ matrix } (c_{iu}c_{iv})_{u,v}. \text{ Square independence implies that the matrices } M_i \text{ are linearly independent. It follows that the rank of the } d^2 \times m \text{ matrix whose } ((u, v), i) \text{ entry is } c_{iu}c_{iv} \text{ is } m. \text{ The rows of this matrix are the } (d \times d) \text{ matrices } M_1, \ldots, M_m. \text{ The columns are the vectors } C_{uv} = (c_{1u}c_{1v}, c_{2u}c_{2v}, \ldots, c_{mu}c_{mv}). \text{ Since row rank equals column rank, we can find } m \text{ linearly independent vectors } C_{uv}. \text{ We have just shown that there is a collection of } m \text{ forms } \eta_j = \sum_{i=1}^{m} B_{ij} \beta_i \text{ for an invertible matrix } B, \text{ so we can write } \beta_i = B_{ij}^{-1} \eta_j. \text{ But in this situation Lemma 9.6 tells us that }
\[
(\min_j \alpha_{B_1}(\eta_j))^m \leq \prod_j \alpha_{B_1}(\eta_j) \leq A\alpha_{B_1}(\beta_i)^{1/m^2} + a,
\]

where we have written \( A = \gamma^{-2m^2}(800d/e)^{d\log m/m^2} \) and \( a = 8\gamma^{-2m^2}(e^{1/4}/\gamma^2)^{1/m^2}. \) Therefore, there exists an index \( j \) such that \( \alpha_{B_1}(\eta_j) \leq A^{1/m} \alpha_{B_1}(\beta_i)^{1/m^3} + a^{1/m}. \) But \( \eta_j \) equals \( \beta_{uv} \) for some pair \( (u, v) \in [d]^2. \)

We continue by proving a lemma that says that high-rank bilinear phase functions defined on Bohr sets are quasirandom in the following sense: they do not correlate well with products of functions of one variable.

**Lemma 9.8.** Let \( \epsilon > 0 \) and let \( B \) and \( B' \) be part of a Bourgain system such that \( B' \prec_\epsilon B. \) Let \( \beta \) be a bilinear form defined on \( B^2, \) and suppose that \( P \subseteq B'. \) Let \( g \) and \( h \) be two functions with \( \|g\|_\infty \) and \( \|h\|_\infty \) at most 1. Then

\[
|\mathbb{E}_{x,y \in B} \omega^{\beta(x,y)} g(x) h(y)| \leq (\alpha_P(\beta) + 6\epsilon)^{1/4}.
\]

**Proof.** We have

\[
|\mathbb{E}_{x,y \in B} \omega^{\beta(x,y)} g(x) h(y)|^4 \leq (\mathbb{E}_{x \in B} |\mathbb{E}_{y \in B} \omega^{\beta(x,y)} h(y)|)^2,
\]

which, by Lemma 2.3 (ii) and the difference-of-squares argument used in the proof of Lemma 9.3, is to within \( 4\epsilon \) equal to

\[
(\mathbb{E}_{x \in B} |\mathbb{E}_{y \in B} - \mathbb{E}_{z \in P} \omega^{\beta(x,y+z)} h(y+z)|^2)^2.
\]
By the Cauchy-Schwarz inequality this is in turn bounded above by
\[
\left( \mathbb{E}_{x \in B} \mathbb{E}_{y \in B^-} |\mathbb{E}_{z \in P} \omega^\beta(x, y + z) \mu(y + z)|^2 \right)^2.
\]
Expanding out the inner square and applying the triangle inequality, we can bound this above by
\[
\left( \mathbb{E}_{y \in B^-} \mathbb{E}_{z, z' \in P} \mathbb{E}_{x \in B} \omega^\beta(x, z - z') \right)^2.
\]
The inner sum is to within \( \epsilon \) equal to \( \mathbb{E}_{x \in B^-} \mathbb{E}_{z, z' \in P} \mathbb{E}_{x \in B} \omega^\beta(x, z - z') \), so our next upper bound is
\[
\left( \mathbb{E}_{y \in B^-} \mathbb{E}_{z, z' \in P} \mathbb{E}_{x \in B} \omega^\beta(x, z - z') \right)^2 + 2\epsilon.
\]
Another application of Cauchy-Schwarz shows that this is at most
\[
\mathbb{E}_{x, y \in B^-} \mathbb{E}_{z, z' \in P} \mathbb{E}_{w \in P} \omega^\beta(w, z - z') \mathbb{E}_{x \in B} \mathbb{E}_{z, z' \in P} \mathbb{E}_{w \in P} \omega^\beta(w - w', z - z') + 2\epsilon.
\]
We recognize the first part of this expression as the definition of \( \alpha_P(\beta) \). This proves the result.

10. Computing with linear combinations of high-rank quadratic averages

We are now in a position to perform the computation over the structured parts of our decompositions, which will be a key ingredient in the proof of the main result of this paper. The next lemma is very straightforward and will help us keep the proof of the subsequent computation as tidy as possible.

Lemma 10.1. For each \( j = 1, 2, \ldots, r \), let \( g_j \) and \( g'_j \) be arbitrary functions on \( \mathbb{Z}_N^s \). Let \( G = \max_j \|g_j\|_\infty \), \( G' = \max_j \|g'_j\|_\infty \) and \( C = \max\{G, G'\} \). Then
\[
\mathbb{E}_{x \in A} \prod_{j=1}^r g_j(x) - \mathbb{E}_{x \in A} \prod_{j=1}^r g'_j(x)
\]
is bounded in absolute value by
\begin{enumerate}
  \item \( rC^r \max_j \|g_j - g'_j\|_2 \) if \( A = \mathbb{Z}_N^s \) or
  \item \( rC^r \max_j \|g_j - g'_j\|_\infty \) if \( A \subseteq \mathbb{Z}_N^s \).
\end{enumerate}

Proof. In both cases the bound stated follows from the observation that
\[
\prod_{j=1}^r g_j(x) - \prod_{j=1}^r g'_j(x) = \sum_{i<j} \prod_{j=1}^r g_i(x)(g_j(x) - g'_j(x)) \prod_{i>j} g_i(x).
\]
When \( A = \mathbb{Z}_N \), this actually implies a stronger upper bound of \( rC^r \max_j \|g_j - g'_j\|_1 \), though we shall only need the \( L_2 \) bound. For general \( A \) the above identity implies that
\[ |\prod_j g_j(x) - \prod_j g'_j(x)| \leq rC' \max_j \|g_j - g'_j\|_\infty \text{ for every } x, \text{ so it holds for the average over } x \text{ over any set } A. \] 

The next result has a long and complicated-looking proof. However, much of the complication is due to the need to keep track of ever more elaborate parameters as we apply the estimates of the preceding sections. So let us first give a qualitative discussion of the argument, to try to indicate what the underlying ideas are.

Recall that our ultimate aim is to obtain a small upper bound for the quantity

\[ \mathbb{E}_{x \in (\mathbb{Z}/N)gw} \prod_{i=1}^r f(L_i(x)) \]

when the linear forms \( L_i \) are square independent and \( \|f\|_{U^2} \) is sufficiently small. The basic idea behind the proof is to decompose \( f \) as a sum of the form \( \sum_i Q_i U_i + g + h \), where the \( Q_i \) are generalized quadratic averages, the \( U_i \) are functions with small \( U^2 \) dual norm, \( g \) has small \( L_1 \) norm and \( h \) has small \( U^3 \) norm, and then to substitute this expression in for \( f \) and do the computations.

If that were all there was to it, then this paper would be much shorter than it is. However, replacing the \( r \) occurrences of \( f \) by quadratic averages in the above expression does not give a small result unless those averages have high rank. So a major task was to show, using the hypothesis that \( \|f\|_{U^2} \) is small, that the decomposition could be made into high-rank averages. In the previous section, we proved that high-rank averages do indeed lead to small results.

There is one further difficulty, however. The most obvious thing to do at this stage would be to substitute \( \sum_i Q_i U_i + g + h \) for each occurrence of \( f \), with every \( Q_i \) of high rank. This would give us a big collection of terms to deal with. But not all of them would be small. For example, if we take \( g \) from every bracket, we obtain a term that has no reason to be small: the fact that \( \|g\|_1 \) is small is no guarantee that

\[ \mathbb{E}_{x \in (\mathbb{Z}/N)gw} \prod_{i=1}^r g(L_i(x)) \]

is small.

Instead, we do something slightly different. We first decompose just one copy of \( f \), obtaining an expression of the form

\[ \mathbb{E}_{x \in (\mathbb{Z}/N)gw} \prod_{i=1}^{r-1} f(L_i(x)) (\sum_i Q_i^{(r)}(L_r(x)) U_i^{(r)}(L_r(x)) + g_r(L_r(x)) + h_r(L_r(x))). \]
The effects of the $g_r$ and $h_r$ terms are now small: to deal with the $h_r$ term (which has small $U^3$ norm) we use a lemma of Green and Tao (Lemma 11.2 below), and to deal with the $g_r$ term we use the fact that it has small $L^1$ norm and the rest of the product is bounded. Thus, we can approximate the above expression by

$$E_{x \in \mathbb{Z}_N} \prod_{i=1}^{r-1} f(L_i(x)) f_r(L_r(x)),$$

where $f_r(x) = \sum_i Q_i^{(r)}(x) U_i^{(r)}(x)$. At this stage, we would like to repeat the process with the $(r-1)$st copy of $f$, but we have a much worse bound for $\|f_r\|_\infty$ than we had for $\|f\|_\infty$, so we have to choose a new decomposition $f = \sum_i Q_i^{(r-1)} U_i^{(r-1)} + g_{r-1} + h_{r-1}$ in such a way that $\|g_{r-1}\|_1 \|f_r\|_\infty$ is small (and not just $\|g_{r-1}\|_1$). And then we continue the process.

This explains why Proposition 10.2 below concerns $r$ different functions and $r$ different decompositions. Once we have these decompositions, then the above argument is a sketch proof that we can ignore all the error terms and just concentrate on the terms involving high-rank quadratic averages, which is what we do in the proposition. So our problem is now reduced to obtaining an upper bound for the size of terms of the form

$$E_{x \in \mathbb{Z}_N} \prod_{i=1}^{r} (Q_i U_i(L_i(x))),$$

when all the $Q_i$ have high rank and the $U_i$ have not too large $U^2$ dual norm, since the expression we are left wishing to estimate is a sum of a bounded number of terms of this form.

The next complication (or rather, apparent complication, since we have the tools to deal with it) is that the $Q_i$ will have different bases and the high ranks will be with respect to different sets. All we really have to do in order to deal with that kind of problem is intersect everything. We know that sets from Bourgain systems have intersections that are not too small, and will use that fact repeatedly.

The rough idea for dealing with a term of the above form is to find a set $D$ such that for every $i$ the functions $U_i(x)$ and $U_i(x+y)$ are close in $L_2$ for every $y \in D$. This we do by finding one such set for each $U_i$ and intersecting those sets. And for that we use the fact that $\|U_i\|_{U^2}^*$ is small for each $i$. Once we have done that, we use Lemma 8.13 to argue that our quadratic averages $Q_i$ still have high rank with respect to a generalized arithmetic progression sitting inside $D$. We then split the average we are trying to estimate into an average of averages taken over translates of $D$, which allows us to assume (after allowing for a small error) that the $U_i$ are constant. At this point we are doing a calculation that
Lemma 8.8 gives us, for each $j$

\[ \prod_{E}^{(2)} \]

Proof. We can split the expectation into individual terms of the form

\[
\mathbb{E}_{x_1 \in z_1 + D} \mathbb{E}_{x_2 \in z_2 + D} \ldots \mathbb{E}_{x_r \in z_r + D} \prod_{j=1}^{r} Q_j(L_j(x)).
\]

If we expand out terms such as $Q_j(L_j(x))$, then we obtain sums that involve bilinear functions, at which point we use Lemmas 9.7 and 9.8 to show that there is always a high-rank bilinear function involved, and therefore that the corresponding terms are small.

Now let us do the argument in detail.

**Proposition 10.2.** Let $\epsilon, \theta > 0$. For each $j = 1, 2, \ldots, r$, let $f_j = \sum_{i=1}^{k_j} Q_i^{(j)} U_i^{(j)}$ be a linear combination of $(\epsilon, m_j)$-special quadratic averages with bases $(B_i^{(j)}, q_i^{(j)})$ on $\mathbb{Z}_N$, each of complexity at most $(d_j, \epsilon_j \rho_j / 800 d_j 5^{d_j})$. Suppose further that each $Q_i^{(j)}$ is of rank $R_j$ with respect to some generalized arithmetic progression $P^{(j)} \subseteq B_i^{(j)} = \bigcap_{i=1}^{k_j} B_i^{(j)}$ of dimension $d'_j \leq k_j d_j$ and density $\gamma'_j$, and that $\sum_{i=1}^{k_j} \| U_i^{(j)} \|_{\infty} \leq 2 C_j$ and $\sum_{i=1}^{k_j} \| U_i^{(j)} \|_{U_2} \leq T_j$.

Set $C = \max_j C_j$, $T = \max_j T_j$, $R = \min_j R_j$, $d = \max_j d_j$, $k = \max_j k_j$, $\gamma' = \min_j \gamma'_j$ and $\rho = \min_j \rho_j$. Finally, suppose that $r (2kC)^r \epsilon \leq \theta$.

Let $L_1, \ldots, L_r$ be a square independent system of $r$ forms in $s$ variables, and set $M = \max_j \sum_{u=1}^{s} |a_{ju}|$. Then

\[
\left| \mathbb{E}_{x \in (\mathbb{Z}_N)^s} \prod_{j=1}^{r} f_j(L_j(x)) \right| \leq 5 \theta + \chi e^{-R/4r^3},
\]

where

\[
\chi = \chi(\epsilon, \theta) = \left( \frac{2^239\epsilon 0.5^{24} (3kC)^{50rT24M4}}{\gamma' \epsilon^2 \rho^2 6^{50}} \right)^{2^{32}r^2 d^4 (2kC)^{12r} \theta^{12}}.
\]

**Proof.** We can split the expectation into individual terms of the form

\[
\mathbb{E}_{x \in (\mathbb{Z}_N)^s} \prod_{j=1}^{r} (Q_i^{(j)} U_i^{(j)})(L_j(x))
\]

where each sequence $(i_1, \ldots, i_r)$ belongs to $[k_1] \times \cdots \times [k_r]$. Let us fix such a sequence, and for ease of notation let us write $Q_i^{(j)} U_i^{(j)}$ instead of $Q_i^{(j)} U_i^{(j)}$. We shall obtain a bound for (2) and then multiply it by $\prod_{j=1}^{r} f_j(L_j(x))$ to obtain a bound for $\left| \mathbb{E}_{x \in (\mathbb{Z}_N)^s} \prod_{j=1}^{r} f_j(L_j(x)) \right|$.

Since $\| U_j \|_{U_2} \leq \sum_{i=1}^{k_j} \| U_i^{(j)} \|_{U_2} \leq T_j \leq T$ and $\| U_j \|_{\infty} \leq \sum_{i=1}^{k_j} \| U_i^{(j)} \|_{\infty} \leq 2 C_j \leq 2 C$, Lemma 8.8 gives us, for each $j = 1, 2, \ldots, r$ and any $\xi > 0$, a Bohr set $E_j$ of complexity at most ($(2C/\xi)^2, \xi$) such that

\[
\mathbb{E}_{x} |U_j(x+y) - U_j(x)|^2 \leq 4 \xi^2 C^2 + 4 T^4 \xi^2 / 3
\]
for each \( y \in E_j \). Therefore, for each subset \( E \subseteq E_j \),
\[
\|U_j - U_j \ast \mu_E\|_2^2 \leq \mathbb{E}_{y \in E} \mathbb{E}_x |U_j(x + y) - U_j(x)|^2 \leq 4\xi^2 C^2 + 4T^{4/3} \gamma^{2/3},
\]
where \( \mu_E \) is the characteristic measure of \( E \). In particular, if we set \( \xi = (\theta/r(2kC)^r)^3/2^6T^2 \) (assuming, as usual, that \( T \) is much larger than \( C \)) and \( E = E_1 \cap \cdots \cap E_r \), then it is readily checked that
\[
\|U_j - U_j \ast \mu_E\|_2 \leq \theta/r(2kC)^r
\]
for all \( j = 1, 2, \ldots, r \). Using Lemma 10.1 (i), we can therefore replace the average (2) by the expression
\[
\mathbb{E}_{x \in \mathbb{Z}^N} \prod_{j=1}^r (Q_j'(U_j \ast \mu_E))(L_j(x))
\]
at the cost of an error of at most \( \theta/k^r \).

Now \( E \) is a Bohr set \( B(K, \xi) \) of dimension \( d_E \leq r(2C/\xi)^2 \leq 2^{14}r^7(2kC)^{6r}T^4/\theta^6 \) and density \( \gamma_E \geq \xi^{d_E} \geq (\theta^3/2^6)^3(2kC)^{3r}(T^2)^{214}r^7(2kC)^{6r}T^4/\theta^6 \). Moreover, \( U_j \ast \mu_E \) is roughly constant on translates of central subsets \( E' \). More precisely, in order for \( U_j \ast \mu_E \) to be constant to within \( \theta/(r(3kC)^r) \) on translates of \( E' = B(K, \xi') \), it is enough if \( E' \prec \theta/(r(3kC)^r) \) \( E \), by Lemma 2.3. Let us note for the record that in this case the dimension of \( E' \) is \( d_{E'} = d_E \) and the density is \( \gamma_{E'} \geq (\theta/r(3kC)^r)\gamma_E \), which is bounded below by
\[
\left(\theta^{10}/2^{30}r^{11}(3kC)^{10r}T^6M\right)^{214}r^7(2kC)^{6r}T^4/\theta^6.
\]

Suppose that the linear form \( L_i(x) \) is given by the formula \( \sum_{u=1}^s c_{iu}x_u \). Let \( E'' = B(K, \xi''/M) \), where \( M = \max_j \sum_{u=1}^s |c_{ju}| \). As a result, \( E'' \) has dimension \( d_{E''} = d_E \) and density \( \gamma_{E''} \geq M^{-d_{E''}}\gamma_{E''} \), which is at least \( (\theta^{10}/2^{30}r^{11}(3kC)^{10r}T^6M)^{214}r^7(2kC)^{6r}T^4/\theta^6 \). The reason for passing to this smaller Bohr set \( E'' \) is so that it will have the following property: if \( x_u \in E'' \) for every \( u \), then \( \sum_{u=1}^s c_{iu}x_u \in E' \).

Let \( B' = B_1' \cap \cdots \cap B_s' \). Then \( B' \) is a Bohr set of dimension \( d_{B'} \leq rd \) and density \( \gamma_{B'} \geq (\epsilon\rho/800d^5)^rd \). Let \( B'' \) be a narrowing of \( B' \) by the same factor \( 1/M \), so \( B'' \) is a Bohr set of dimension \( d_{B''} = d_{B'} \) and density \( \gamma_{B''} \geq (\epsilon\rho/800d^5M)^{rd} \). Finally, set \( D = E'' \cap B'' \), which is like a Bohr set but with “different widths in different directions”. Rather than go into the details of this, we merely observe that if \( E'' = B(K, \xi'') \) and \( B'' = B(L, \tau'') \), then we can define a Bourgain system \( (D_\mu) \) by setting \( D_\mu \) to be \( B(K, \mu\xi'') \cap B(L, \mu\tau'') \). By Lemma [8.10] and the remark following Lemma [8.3] (which says that the dimension of a Bohr set \( B(K, \rho) \) considered as part of a Bourgain system is at most \( 3|K| \)), this is a Bourgain system
LINEAR FORMS AND QUADRATIC UNIFORMITY FOR FUNCTIONS ON \( \mathbb{Z}_N \)

of dimension \( d_D \leq 12(d_{E''} + d_{B''}) \leq 2^{18} r^7 (2kC)^6 r^d / \theta^6 + 2^4 r d \) such that \( D = D_1 \) has density \( \gamma_D \geq 2^{-9(d_{E''} + d_{B''})} \gamma_{E''} \gamma_{B''} \geq (\epsilon \rho/2^{19} d S M)^r d (\theta^{10} / 2^{39} \rho^{11} (3kC)^{10r} T^6 M)^{214} r^7 (2kC)^6 r^4 / \theta^6 \) by Lemma \[8.10\].

We shall cover \( \mathbb{Z}_N \) with translates of \( D \), and compute the expectation

\[
\mathbb{E}_{x_1 \in z_1 + D} \mathbb{E}_{x_2 \in z_2 + D} \ldots \mathbb{E}_{x_s \in z_s + D} \prod_{j=1}^r (Q_j'(U_j * \mu_E))(L_j(x))
\]

for some fixed choice of \( z_1, \ldots, z_s \in \mathbb{Z}_N \). Now if each \( x_i \) is confined to a translate \( z_i + D \), then \( L_j(x) \) is contained in some particular translate \( y_j + E' \cap B' \), by our choice of \( E'' \) and \( B'' \). On this translate, \( U_j * \mu_E \) is constant to within \( \theta/(r(3kC)^r) \). More precisely, we can write \( U_j * \mu_E(x) = \lambda_{y_j} + \epsilon_j(x) \), where \( \|\epsilon_j\|_\infty \leq \theta/(r(3kC)^r) \) for all \( j = 1, 2, \ldots, r \). Taking into account the fact that \( \sum_{j=1}^k \|U_i^{(j)}\|_\infty \leq 2C_j \), we immediately note that \( |\lambda_{y_j}| \leq 3C_j \) for any \( j = 1, 2, \ldots, r \). It follows from Lemma \[10.1\](ii) that at the cost of an error of at most \( \theta/k^r \), we can focus on evaluating

\[
\left( \prod_{j=1}^r \lambda_{y_j} \right) \mathbb{E}_{x_1 \in z_1 + D} \mathbb{E}_{x_2 \in z_2 + D} \ldots \mathbb{E}_{x_s \in z_s + D} \prod_{j=1}^r Q_j'(L_j(x))
\]

instead of the earlier average \[4\]. We recall that each \( Q_j' \) was an \( (\epsilon, m_j) \)-special average with base \( B_j' \) and rank at least \( R_j \) with respect to \( P^{(j)} \subseteq B_j' \). In particular, for each \( j = 1, 2, \ldots, r \), since \( D \subseteq E' \cap B' \subseteq B_j' \), we find that for all but \( \epsilon N \) choices of \( y_j \in \mathbb{Z}_N \), the restriction of \( Q_j' \) to \( y_j + D \) is equal to the restriction of \( \omega_{q_j'} \) to \( y_j + D \), where \( q_j'(v) = q_j(v - v_j) \) for one of at most \( m_j \) fixed values \( v_j \in \mathbb{Z}_N \). Let us say that \( (y_1, \ldots, y_r) \) is \emph{good} if this is true for every \( j \leq r \).

Observe that as \( z_1, \ldots, z_s \) runs over \( \mathbb{Z}_N \), so does \( L_j(z_1, \ldots, z_s) \) for each \( j = 1, 2, \ldots, r \). Therefore a proportion of at least \( (1 - \sum_j \epsilon_j) \geq (1 - \epsilon r) \) of all choices of \( (z_1, \ldots, z_s) \in (\mathbb{Z}_N)^s \) gives rise to a good sequence \( (y_1, \ldots, y_r) \). If \( (y_1, \ldots, y_r) \) is good, then fix a value \( v_j \) for each \( j = 1, 2, \ldots, r \). Now since the \( \epsilon_j \) were required to satisfy \( r(2kC)^r \epsilon \leq \theta \), then incurring an error of at most \( \theta/k^r \), we can restrict our attention to

\[
\left( \prod_{j=1}^r \lambda_{y_j} \right) \mathbb{E}_{x_1 \in z_1 + D} \mathbb{E}_{x_2 \in z_2 + D} \ldots \mathbb{E}_{x_s \in z_s + D} \prod_{j=1}^r \omega_{q_j'}(L_j(x) - v_j)
\]

for some fixed choice of \( v_1, \ldots, v_r \). Recall that for each \( j = 1, 2, \ldots, r \), the linear form \( L_j(x) \) was given by the formula \( \sum_{u=1}^s c_{ju} x_u \). Writing \( \beta_j \) for the bilinear form associated with \( q_j \),
we have
\[ \sum_{j=1}^{r} q_j(L_j(x)) = \sum_{u,v=1}^{s} \sum_{j=1}^{r} c_{ju} c_{jv} \beta_j(x_u, x_v). \]

For each \( u \) and \( v \), let us write \( \beta_{uv} \) for the bilinear form \( \sum_{j=1}^{r} c_{ju} c_{jv} \beta_j \) as before.

Set \( P = P^{(1)} \cap \cdots \cap P^{(r)} \), which is part of a Bourgain system of dimension \( d_P \leq 4r^2 \sum_j d_j^i \leq 4r^3kd \) and has density \( \gamma_P \geq 2^{-d_P} \prod_j \gamma_j' \geq 2^{-4r^3kd}\gamma r^r \). We shall now consider the rank of each \( q_j \) with respect to \( P' = P \cap D' \), where \( D' \prec \theta^{4/6(3kC)^{4r}} D \). In order to do so, we need to determine the dimension and density of \( P' \), which is the main reason we have been carefully keeping track of our parameters since the start of the proof.

First note that \( D' \) is part of a Bourgain system of dimension \( d_{D'} = d_D \leq 2^{22r^8d(2kC)^{6r}T^4/\theta^6} \) as determined earlier and has density \( \gamma_{D'} \geq (\theta^4/2^{15}(3kC)^{4r}d_D)\gamma D \), which is bounded below by \( (\epsilon \theta^{20}/2^{95}r^{19}d^2(3kC)^{20r}T^{10}M^2)^{22r^8d(2kC)^{6r}T^4/\theta^6} \). Therefore \( P' \) is part of a Bourgain system of dimension
\[ d_{P'} \leq 4(d_P + d_{D'}) \leq 2^{26r^11kd^2(2kC)^{6r}T^4/\theta^6} \]
and has density
\[ \gamma_{P'} \geq 2^{-3(d_P + d_{D'})} \gamma_P \gamma_{D'} \geq \gamma r^r \left( \frac{\epsilon \theta^{20}}{2^{99}r^{19}d^2(3kC)^{20r}T^{10}M^2} \right)^{226r^11kd^2(2kC)^{6r}T^4/\theta^6} \]
by Lemma 8.10.

Finally, we use Lemma 8.13 to make the connection between the rank of our quadratic phases with respect to \( P \) and \( P' \). The lemma tells us that \( \alpha_{P'}(\beta_i) \leq (\gamma_P/\gamma_{P'}) \alpha_P(\beta_i) \) for each \( i = 1, 2, \ldots, r \).

Let \( \eta = \gamma_P^{8(1+r^4)}(\theta/4(3kC)^{r})^{16r^3} \). Lemma 9.4 with \( \epsilon = \eta \), \( B_1 = P' \) and \( m = r \) tells us that there exists a pair \( (u, v) \in [s]^2 \) such that the bilinear form \( \beta_{uv} \) defined above satisfies
\[ \alpha_{P'}(\beta_{uv}) \leq \gamma_{P'}^{2r}(800d_{P'}/\eta)^{d_{P'}^{log r/r^3}} \alpha_{P'}(\beta_i)^{1/r^3} + 4\gamma_{P'}^{-2r}(\eta^{1/4}/\gamma_{P'}^2)^{1/r^3}. \]

for any \( i = 1, 2, \ldots, r \). To conclude the proof, note that Lemma 9.8 implies that
\[ |\mathbb{E}_{x_1 + D \mathbb{E}_{x_2 + D} \cdots \mathbb{E}_{x_r + D} \prod_{j=1}^{r} \omega^d \beta_{uv}(x_u, x_v) + \sum_{i=1}^{d} \phi_{ui}(x_u) + \phi_1 | \leq \theta/(3kC)^r + \alpha_{P'}(\beta_{uv})^{1/4} \]
for any fixed linear forms \( \phi_{ui} \) and any constant \( \phi_1 \), which is at most
\[ \theta/(3kC)^r + \gamma_{P'}^{-r/2}(800d_{P'}/\eta)^{d_{P'}^{log r/4r^3}} \alpha_{P'}(\beta_i)^{1/4r^3} + 4\gamma_{P'}^{-r/2}(\eta^{1/4}/\gamma_{P'}^2)^{1/4r^3} \]
and therefore bounded above by
\[
\theta/(3kC)^r + \gamma_p^{-r/2}(800d_p/\eta)d_p\log r/4r^3 \left( \frac{\gamma_p}{\gamma_p'} \right)^{1/4r^3} \alpha_p(\beta_i)^{1/4r^3} + 2\gamma_p^{-r/2}(\eta^{1/4}/\gamma_p')^{1/4r^3}.
\]

Our choice of \(\eta\) implies that the third term is no larger than the first, and that the second term is at most
\[
\left( \frac{2^{239}r^{59}d_p^{6d(3kC)^{50\gamma}T^{24}M^4}}{\gamma c^2p^2\phi^{50}} \right)^{2^{233}r^{22}d^4(kC)_{12c}T^{9}/\theta^{12}} \alpha_p(\beta_i)^{1/4r^3}.
\]

Recalling that in (6) we had a pre-factor of \(\prod_{j=1}^r \lambda_j\) with each \(|\lambda_j| \leq 3C_j\) and in (3) a factor of \(k^r\), and that \(\alpha_p(\beta_i) \leq e^{-R}\) for every \(i = 1, 2, \ldots, r\), we obtain the final bound as stated. \(\square\)

11. Proof of the main result

Most of the work towards proving the main result was accomplished in the preceding section. Here we shall formally complete the proof of the following theorem.

**Theorem 11.1.** Let \(L_1, \ldots, L_r\) be a square independent system of linear forms in \(s\) variables of Cauchy-Schwarz complexity at most 2. For every \(\eta > 0\), there exists \(c > 0\) with the following property. Let \(f : \mathbb{Z}_N \rightarrow [-1, 1]\) be such that \(\|f\|_{U^2} \leq c\). Then
\[
\left| \mathbb{E}_{x \in \mathbb{Z}_N} \prod_{i=1}^r f(L_i(x)) \right| \leq \eta.
\]
Moreover, \(c\) can be taken to depend on \(\eta\) in a doubly exponential fashion.

As in [GW09a, GW09b], we need to recall a well-established result that will allow us to neglect the quadratically uniform part of the decomposition.

**Theorem 11.2.** Let \(f_1, \ldots, f_r\) be functions on \(\mathbb{Z}_N\), and let \(L_1, \ldots, L_r\) be a linear system of Cauchy-Schwarz complexity at most 2 consisting of \(r\) forms in \(s\) variables. Then
\[
\left| \mathbb{E}_{x \in \mathbb{Z}_N} \prod_{j=1}^r f_j(L_j(x)) \right| \leq \min_j \|f_j\|_{U^3} \prod_{i \neq j} \|f_i\|_{\infty}.
\]

**Proof of Theorem 11.1.** Let \(\eta > 0\), and let \(c > 0\) be chosen in terms of \(\eta\) later. Given \(f : \mathbb{Z}_N \rightarrow [-1, 1]\) with \(\|f\|_{U^2} \leq c\) we first apply Theorem 8.16 with \(\delta_1 = \eta/(24r)\) to obtain a decomposition
\[
f = f_1 + g_1 + h_1,
\]
where \( f_1 = \sum_j Q_j^{(1)} U_j^{(1)} \) with \( \sum_j \| U_j^{(1)} \|_\infty \leq 2C_1, \sum_j \| U_j^{(1)} \|_{L^2} \leq T_1, \| g_1 \|_1 \leq 10\delta_1 \) and \( \| h_1 \|_{U^3} \leq 2\delta_1 \). We have carefully ensured that each quadratic average \( Q_j^{(1)} \) has rank at least \( R_1 \) for some \( R_1 \) to be chosen later. Aiming to bound
\[
E_{x \in (\mathbb{Z}_N)^*} \prod_{j=1}^r f(L_j(x))
\]
above in absolute value by \( \eta \) for sufficiently uniform \( f \), we first replace the first instance of \( f \) in the product by \( g_1 + h_1 \). The product involving \( g_1 \) yields an error term of \( 10\delta_1 \) since all the remaining factors have \( L_\infty \) norm bounded by 1, while the product involving \( h_1 \) yields an error of \( 2\delta_1 \) by Theorem 11.2 above. Our choice of \( \delta_1 \) implies that the sum of these two errors is at most \( \eta/(2r) \).

Now we apply Theorem 8.16 again, this time with \( \delta_2 = \eta/(48rC_1) \), to obtain a decomposition
\[
f = f_2 + g_2 + h_2,
\]
where \( f_2 = \sum_j Q_j^{(2)} U_j^{(2)} \) with \( \sum_j \| U_j^{(2)} \|_\infty \leq 2C_2, \sum_j \| U_j^{(2)} \|_{L^2} \leq T_2, \| g_2 \|_1 \leq 10\delta_2 \) and \( \| h_2 \|_{U^3} \leq 2\delta_2 \). When replacing the first instance of \( f \) in the new product
\[
E_{x \in (\mathbb{Z}_N)^*} f_1(L_1(x)) \prod_{j=2}^r f(L_j(x))
\]
with \( g_2 + h_2 \), the product involving \( g_2 \) now contributes an error term of at most \( 20\delta_2 C_1 \) (since \( \| f_1 \|_\infty \leq 2C_1 \)). By Theorem 11.2 it follows that the contribution from the product involving \( h_2 \) is bounded above by \( 4\delta_2 C_1 \). Therefore the total error incurred is at most \( 24\delta_2 C_1 \), which is at most \( \eta/(2r) \) by our choice of \( \delta_2 \).

When we come to apply Theorem 8.16 to the \( k \)th instance of \( f \) in the original product, we need to do so with \( \delta_k \) satisfying \( 12 \cdot 2^{k-1} \delta_k C_1 \ldots C_{k-1} \leq \eta/(2r) \) for \( k = 2, \ldots, r \). This ensures that up to an error of \( \eta/2 \), it suffices to consider the product
\[
E_{x \in (\mathbb{Z}_N)^*} \prod_{j=1}^r f_j(L_j(x)),
\]
where each function \( f_j \) is quadratically structured. The key estimate, Proposition 10.2 with \( \theta = \eta/20 \), now implies that
\[
\left| E_{x \in (\mathbb{Z}_N)^*} \prod_{i=1}^r f_i(L_i(x)) \right| \leq \eta/4 + \chi e^{-R/4r^3},
\]
where
\[ \chi(\eta) = \left( \frac{2^{439}r^{50}d^{52}d(3kC)^{50r}T^{24}M^4}{\gamma'\epsilon^2\rho^2\eta^{30}} \right)^{2^{113}r^{22}k^2d^4(2kC)^{12r}T^8/\eta^{12}} \]
with \( C = \max_j C_j, T = \max_j T_j, R = \min_j R_j, d = \max_j d_j, k = \max_j k_j, \gamma' = \min_j \gamma'_j, \rho = \min_j \rho_j \) and \( \epsilon = \max_j \epsilon_j \). Choosing \( R_j \) large enough at each stage, we will be able to force \( \chi e^{-R/4r^3} \leq \eta/4 \).

The argument is essentially complete; it remains to check that the dependence we obtain is doubly exponential. First, note that every application of Theorem \[8.16\] returns \( \min_j \) with \( C, \delta, d, k \) as well as \( \rho_j \) and \( (\text{the upper bound on}) \epsilon_j \) as parameters that are polynomial in \( \delta_j \), and hence polynomial in \( \eta \). Only \( T_j \) is exponential in \( \eta \).

Also, in order to apply Proposition \[10.2\] we needed to assume that the parameters \( \epsilon_j \) satisfy \( r(2kC)^r \epsilon \leq \eta/20 \). This means that the density \( \gamma'_j \) of the progression \( P_j \) used in the \( j^{th} \) decomposition is at least \( (\eta \rho/2^j r(2kC)^r d^3 k^3 d^4) \), which does not affect the doubly exponential nature of \( \chi(\eta) \). Hence it is possible to choose \( R_j \) to be an exponential function of \( \eta \) at each stage. By Theorem \[8.16\] this is possible provided that \( \| f \|_{U^2} \leq c \), where \( c \) is bounded above by
\[ e^{-2^{15k}d^3k^6kR} \left( \frac{\delta^7 \rho^3 \epsilon^3}{2^{102k}k^{5}d^{85}d^{5}C^3T^2} \right)^{2^{65k}md^{12k}k^{15k}T^4C^2/\delta^6}, \]
where \( \delta = \min_j \delta_j \) and \( R \) was chosen to satisfy \( \chi e^{-R/4r^3} \leq \eta/4 \). More precisely, the average \( \mathbb{E}_x \prod_{j=1}^r f(L_j(x)) \) is less than \( \eta \) provided that \( c \) is at most
\[ \left( \frac{\eta^{54} \rho^3}{2^{460}k^{3}d^{85}d^{5}d(3kC)^{50r}T^{24}M^4} \right)^{2^{118k}k^{25}d^{11k}(2kC)^{12r}T^8/\eta^{12}} \left( \frac{\delta^7 \rho^3 \epsilon^3}{2^{102k}k^{5}d^{85}d^{5}C^3T^2} \right)^{2^{65k}md^{12k}k^{15k}T^4C^2/\delta^6}, \]

With \( m \) and \( r \) being fixed constants, \( M \) being a constant depending on the coefficients of the linear forms, \( C, \delta, d, k \) and \( \rho \) depending polynomially on \( \eta \) and \( T \) depending exponentially on \( \eta \), this bound on \( c \) is indeed doubly exponential in \( \eta \) as claimed.

\[ \square \]

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Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge CB3 0WB, UK.

E-mail address: W.T.Gowers@dpmms.cam.ac.uk

Rutgers, The State University of New Jersey, Department of Mathematics, 110 Friel-inghamyseen Rd., Piscataway, NJ 08854, U.S.A.

E-mail address: julia.wolf@cantab.net