Kissing Numbers and the Centered Maximal Operator

J. M. Aldaz

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Abstract
We prove that in a metric measure space $X$, if for some $p \in (1, \infty)$ there are uniform bounds (independent of the measure) for the weak type $(p, p)$ of the centered maximal operator, then $X$ satisfies a certain geometric condition, the Besicovitch intersection property, which in turn implies the uniform weak type $(1, 1)$ of the centered operator. Thus, the following characterization is obtained: the centered maximal operator satisfies uniform weak type $(1, 1)$ bounds if and only if the space $X$ has the Besicovitch intersection property. In $\mathbb{R}^d$ with any norm, the constants coming from the Besicovitch intersection property are bounded above by the translative kissing numbers. The extensive literature on kissing numbers allows us to obtain, first, sharp estimates on the uniform bounds satisfied by the centered maximal operators defined by arbitrary norms on the plane, second, sharp estimates in every dimension when the $\ell_\infty$ norm is used, and third, improved estimates in all dimensions when considering euclidean balls, as well as the sharp constant in dimension 3. Additionally, we prove that the existence of uniform $L^1$ bounds for the averaging operators associated with arbitrary measures and radii, is equivalent to a weaker variant of the Besicovitch intersection property.

1 Introduction

It is well known that the centered maximal operator $M_\mu$ is of weak type $(1,1)$ for arbitrary, locally finite Borel measures $\mu$ on $\mathbb{R}^d$, with bounds exponential in $d$ but independent of the measure, because of the Besicovitch covering theorem.

Here we show, in the context of metric measure spaces $(X, d, \mu)$, that the full force of the theorem is not needed; in fact, the exact condition is given by the Besicovitch...
intersection property, which controls the maximal overlap $L(X, d)$ of families of balls such that each ball does not contain the center of any other ball in the collection. Since in metric spaces, in general neither centers nor radii of balls are unique, the preceding statement needs to be made more precise.

**Definition 1.1** A collection $C$ of balls in a metric space $(X, d)$ is a [Besicovitch family](#) if there is a choice function assigning a center and a radius to each ball in $C$, in such a way that for every pair of distinct balls $B(x, r), B(y, s) \in C$, $x \notin B(y, s)$ and $y \notin B(x, r)$. Denote by $BF(X, d)$ the collection of all Besicovitch families of $(X, d)$. The **Besicovitch constant** of $(X, d)$ is

$$L(X, d) := \sup \left\{ \sum_{B(x, r) \in C} 1_{B(x, r)}(y) : y \in X, \ C \in BF(X, d) \right\}.$$  

(1)

We say that $(X, d)$ has the **Besicovitch Intersection Property** if $L(X, d) < \infty$.

One of our main results is the following

**Theorem 1.2** The Besicovitch constant $L(X, d)$ is equal to $\sup_{\mu} \| M_\mu \|_{L^1 \rightarrow L^{1, \infty}}$, where the supremum is taken over all $\tau$-additive, locally finite Borel measures $\mu$ on $(X, d)$.

Thus, $M_\mu$ satisfies weak type $(1, 1)$ bounds that are uniform in $\mu$, if and only if $X$ has the Besicovitch intersection property, and the optimal constant is the same in both cases.

We mention that the preceding result applies to all Borel measures in separable metric spaces, since there $\tau$-additivity is automatic (cf. Definition 2.1).

Furthermore, if for some $1 < p < \infty$ the centered maximal operator $M_\mu$ satisfies uniform weak type $(p, p)$ bounds, then $X$ has the Besicovitch intersection property. So we obtain an extrapolation result, from uniform weak type $(p, p)$ to uniform weak type $(1, 1)$ (cf. Theorem 4.11). Of course, the other direction holds by interpolation.

Recall that spaces satisfying the conclusion of the Besicovitch covering theorem tend to be rather special, cf. [13, pp. 7-8]. The Besicovitch intersection property has clear advantages over stronger hypotheses of Besicovitch type: more spaces have it (for one example, see [19, Example 3.4]) and it is easier to handle technically (cf. [19]). For us, the main application here is that it leads to substantially better bounds than the previously known ones, when $(X, d) = (\mathbb{R}^d, \| \cdot \|)$ and $\| \cdot \|$ is a norm. The reason why one can use the Besicovitch intersection property instead of stronger hypotheses in order to obtain uniform bounds for the maximal operators, is that arbitrary measures can be replaced by finite sums of weighted Dirac deltas, by the “local discretization of measures” presented in Theorem 2.9.

Considerable efforts have been made to determine the boundedness properties of $M_\mu$ in many classes of spaces (as a very small sample, we mention [15,20,21,26,33]). When boundedness is known, it is often interesting to improve on the constants, finding the sharp ones if possible. Starting with the work of Stein, cf. [32], and Bourgain, cf. [6], the case of Lebesgue measure in $\mathbb{R}^d$ has been extensively studied, see [10] and the references contained therein. But the sharp constant for Lebesgue measure is known only in dimension 1, cf. [24].
We show (cf. Theorem 3.3) that $L(\mathbb{R}^d, \| \cdot \|)$, the Besicovitch constant of $(\mathbb{R}^d, \| \cdot \|)$, equals the maximum number of unit balls that can touch a central unit ball without touching each other (the so-called strict Hadwiger number) so in particular, $L(\mathbb{R}^d, \| \cdot \|)$ is bounded above by the translatable kissing number of $(\mathbb{R}^d, \| \cdot \|)$. This allows us to apply the extensive literature on kissing numbers to maximal function inequalities, thereby obtaining substantial improvements regarding the previously known bounds: on the plane, the uniform bounds are always either 4 or 5, depending on whether the unit ball of the given norm is a parallelogram or not. When $\| \cdot \| = \| \cdot \|_\infty$, we have that $L(\mathbb{R}^d, \| \cdot \|_\infty) = 2^d$ for every dimension $d$; furthermore, there is a locally finite Borel measure $\mu$ on $(\mathbb{R}^d, \| \cdot \|_\infty)$ for which $\| M_\mu \|_{L^1 \to L^1} = 2^d$, so this bound is attained. From the available information regarding kissing numbers for Euclidean balls, we obtain the sharp bound $L(\mathbb{R}^3, \| \cdot \|_2) = 12$; in dimension 4, $L(\mathbb{R}^4, \| \cdot \|_2) \leq 24$; for arbitrary $d$, we have the asymptotic estimates

$$(1 + o(1))1.1547^d \leq L(\mathbb{R}^d, \| \cdot \|_2) \leq 1.3205(1+o(1))^d.$$ 

We remark that for $d \gg 1$, these estimates are distinctly smaller than the bounds $2^d$ holding for cubes.

Motivated by a question of Prof. Przemysław Górka (personal communication), in the last section we show that averaging operators are of strong type $(1,1)$ for arbitrary, locally finite $\tau$-additive Borel measures $\mu$ on a metric space $X$, with bounds independent of $\mu$ and of $r$, if and only if $X$ satisfies a weaker version of the Besicovitch intersection property, called here the equal radius Besicovitch intersection property, cf. Definition 4.4 for the precise statement. It follows from the preceding results that if we have uniform weak type $(p, p)$ bounds for the centered maximal operator and some $p \in [1, \infty)$, then averaging operators are uniformly bounded on $L^1$.

Finally, we mention that via the uniform weak type of the maximal operator, the Besicovitch intersection property is sufficient to ensure, for all $\tau$-additive, locally finite Borel measures on arbitrary metric spaces, the validity of Lebesgue’s differentiation theorem. For the same class of measures in complete separable metric spaces, the necessary and sufficient condition was given by D. Preiss in terms of what might be called an asymptotic $\sigma$-Besicovitch intersection property (cf. [27] or [19, Theorem 6.2] for more details). This condition is strictly weaker than the Besicovitch intersection property.

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2 Definitions and General Results

We will use $B^o(x, r) := \{ y \in X : d(x, y) < r \}$ to denote metrically open balls, and $B^{cl}(x, r) := \{ y \in X : d(x, y) \leq r \}$ to refer to metrically closed balls; open and closed will always be understood in the metric (not the topological) sense. If we do not want to specify whether balls are open or closed, we write $B(x, r)$. But when we utilize $B(x, r)$, all balls are taken to be of the same kind, i.e., all open or all closed.
Definition 2.1 Let \((X, d)\) be a metric space. A Borel measure \(\mu\) on \(X\) is \(\tau\)-additive or \(\tau\)-smooth, if for every collection \(\{O_\alpha : \alpha \in \Lambda\}\) of open sets, we have
\[
\mu(\bigcup_\alpha O_\alpha) = \sup_{\mathcal{F}} \mu\left(\bigcup_{i=1}^n O_{\alpha_i}\right),
\]
where the supremum is taken over all finite subcollections \(\mathcal{F} = \{O_{\alpha_1}, \ldots, O_{\alpha_n}\}\) of \(\{O_\alpha : \alpha \in \Lambda\}\). If \(\mu\) assigns finite measure to bounded Borel sets, we say it is locally finite. Finally, we call \((X, d, \mu)\) a metric measure space if \(\mu\) is a \(\tau\)-additive, locally finite Borel measure on the metric space \((X, d)\).

Countable additivity tells us that countable unions of measurable sets can be approximated in measure by finite unions; \(\tau\)-additivity extends this property to arbitrary unions of open sets (which are open and hence Borel). Thus, the preceding definition includes all locally finite Borel measures on separable metric spaces and all Radon measures on arbitrary metric spaces, so it is more general than other commonly used definitions, cf. [14] for instance. From now on we always suppose that measures are locally finite, not identically zero, and that metric spaces have at least two points. The condition of local finiteness excludes some natural measures, such as the one on \(\mathbb{R}\) given by the density \(d\mu(x) = |x|^{-1}dx\); as a matter of fact, the weak type \((1,1)\) theory can be carried out without this assumption, since the average of an \(L^1\) function over a ball of infinite measure is zero; but the \(L^p\) theory fails for \(p > 1\), for it may happen that the maximal function is not well defined almost everywhere.

Definition 2.2 Let \((X, d, \mu)\) be a metric measure space and let \(g\) be a locally integrable function on \(X\). For any subset \(S \subset (0, \infty)\), the localized centered Hardy-Littlewood maximal operator \(M_S,\mu\) is given by
\[
M_{S,\mu}g(x) := \sup_{\{r \in S : 0 < \mu(B(x, r))\}} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |g|d\mu.
\]
Taking \(S = (0, \infty)\), we obtain the centered maximal operator \(M_\mu := M_{(0,\infty),\mu}\).

When the radii belong to an open set \(S\), by approximation it does not matter in the definition whether one takes the balls \(B(x, r)\) to be open or closed. We will utilize the same notation for the maximal operators, specifying which kind of balls we use whenever needed. Also, we often simplify notation by eliminating subscripts when the meaning is clear from the context. For instance, if only one measure \(\mu\) is being considered, we may write \(M\) instead of \(M_\mu\). We use \(\|M_\mu\|_{L^p \to L^p,\infty}\) to denote the weak type \((p, p)\) “norm” of \(M_\mu\), and \(\|M_\mu\|_{L^p \to L^p}\) to denote its operator norm on \(L^p\).

The Besicovitch intersection property appears in [19], where it is called the weak Besicovitch covering property. Our change in terminology is motivated by the fact that this property says nothing about sets to be covered; instead, given a Besicovitch family, it controls the cardinality of the intersections at any given point.

Call a Besicovitch family \(\mathcal{C}\) intersecting if \(\cap \mathcal{C} \neq \emptyset\). In [19] the presentation is local: the space \((X, d)\) has the Besicovitch intersection property with constant bounded by \(L\), if there exists an integer \(L \geq 1\) such that for every intersecting Besicovitch family \(\mathcal{C}\), the cardinality of \(\mathcal{C}\) is bounded by \(L\).
Remark 2.3 To see the equivalence of both formulations, just note that given any Besicovitch family $\mathcal{C}$ and any $z$ with $\sum_{B(x,r)\in \mathcal{C}} 1_{B(x,r)}(z) > 0$, the set $\{B(x,r) \in \mathcal{C} : z \in B(x,r)\}$ is an intersecting Besicovitch family.

Proposition 2.4 A metric space $(X, d)$ has the Besicovitch intersection property with constant $L(X, d)$ for collections of open balls, if and only if it has the Besicovitch intersection property for collections of closed balls, with the same constant.

Proof Denote by $L^0$ and $L^c$ the lowest constants for collections of open balls and for collections of closed balls, respectively. Suppose first that $L^0 < \infty$. Let $\mathcal{C}$ be an intersecting Besicovitch family of closed balls, and select any finite subcollection $\{B^c(x_1, r_1), \ldots, B^c(x_N, r_N)\}$. It is enough to prove that $N \leq L^0$. Let $t_i := \min\{d(x_j, x_i) : 1 \leq j \leq N, j \neq i\}$. Since $t_i > r_i$, it follows that $\{B^c(x_1, t_1), \ldots, B^c(x_N, t_N)\}$ is an intersecting Besicovitch family of open balls, so $N \leq L^0$.

Suppose next that $L^c < \infty$, and let $\mathcal{C}$ be an intersecting Besicovitch family of open balls. Select $y \in \cap \mathcal{C}$, and replace each ball $B^0(x, r) \in \mathcal{C}$ with the closed ball $B^c(x, d(x, y)) \subset B^0(x, r)$. The collection $\mathcal{C}'$ so obtained is an intersecting Besicovitch family of closed balls, so its cardinality is bounded by $L^c$.

Lemma 2.5 Let $(X, d)$ be a metric space and let $L > 0$. The following are equivalent:

1. $(X, d)$ has the Besicovitch weighted intersection property with constant bounded above by $L$.
2. For every finite sum of Dirac deltas $\mu := \sum_{i=1}^N c_i \delta_{x_i}$, the centered maximal operator satisfies $\|M_\mu\|_{L^1 \to L^1} \leq L$.

Proof First we show that (1) $\implies$ (2). Let $\mu = \sum_{i=1}^N c_i \delta_{x_i}$, where $0 < c_i < \infty$. Let $0 \leq f \in L^1(\mu)$ have norm $\|f\|_1 > 0$, and let $t > 0$ be such that $\mu\{M_\mu f > t\} > 0$. For each $x_i$ with $M_\mu f(x_i) > t$, select $r_i > 0$ such that $t \mu B(x_i, r_i) < \int_{B(x_i, r_i)} f d\mu$.

We reorder this finite collection of balls by non-increasing radii; to avoid more subscripts, we also relabel the chosen balls as $B(y_1, s_1), \ldots, B(y_J, s_J)$, so $s_i \geq s_{i+1}$ and $\{y_1, \ldots, y_J\}$ is just a permutation of $\{x_1, \ldots, x_N\}$. Then we apply the standard selection procedure: let $B(y_{i_1}, s_{i_1}) := B(y_1, s_1)$ be the ball with largest radius, let $B(y_{i_2}, s_{i_2})$ be the first ball in the list with $y_{i_2} \notin B(y_1, s_1)$, and suppose that $B(y_{i_1}, s_{i_1}), \ldots, B(y_{i_k}, s_{i_k})$ have been chosen, if all the centers $y_{i_j}$ have already been covered the process stops; otherwise, we let $B(y_{i_{k+1}}, s_{i_{k+1}})$ be the first ball in the list with $y_{i_{k+1}} \notin \bigcup_{j=1}^k B(y_{i_j}, s_{i_j})$.

In this way, we obtain a Besicovitch family $\mathcal{C}' = \{B(y_{i_j}, s_{i_j}), \ldots, B(y_{i_k}, s_{i_k})\}$ that covers the set $\{y_1, \ldots, y_J\}$. By the Besicovitch intersection property and the choice of $L$, $\sum_{B(y,s)\in \mathcal{C}'} 1_{B(y,s)} \leq L$, and since $\mu \{M_\mu f > t\} \setminus \{y_1, \ldots, y_J\} = 0$, we have...
\[ \mu\{M_\mu f > t\} \leq \mu(\cup C') \leq \sum_{B(y,s) \in C'} \mu B(y,s) \]

\[ < \sum_{B(y,s) \in C'} \frac{1}{t} \int 1_{B(y,s)} f \, d\mu \]

\[ = \frac{1}{t} \int \left( \sum_{B(y,s) \in C'} 1_{B(y,s)} \right) f \, d\mu \leq \frac{L}{t} \int f \, d\mu. \]

For 2) \(\implies\) 1), we prove that if \(C\) is an intersecting Besicovitch family in \((X, d)\) of cardinality \(> L\), then there exists a discrete measure \(\mu_c\) with finite support, for which \(\|M_{\mu_c}\|_{L^1 \to L^{1,\infty}} > L\). We may suppose that \(C = \{B(x_1, r_1), \ldots, B(x_{L+1}, r_{L+1})\}\) by throwing away some balls if needed. Let \(y \in \cap C\), and for \(0 < c \ll 1\), define \(\mu_c := c\delta_y + \sum_{i=1}^{L+1} \delta_{x_i}\). Set \(f_c = c^{-1}1_{\{y\}}\). Then \(\|f_c\|_1 = 1\) and for \(1 \leq i \leq L + 1\), \(M_{\mu_c} f_c(x_i) \geq 1/(1 + c)\). Taking \(y\) into account, we get

\[ \mu_c\{M_{\mu_c} f_c \geq 1/(1 + c)\} = L + 1 + c, \]

so for \(c\) small enough, \(\mu_c\{\{M_{\mu_c} f_c \geq 1/(1 + c)\}\} / (1 + c) > L. \)

The “local discretization of measures” Theorem 2.9 below, states that uniform bounds on weighted finite sums of Dirac deltas, extend to uniform bounds on arbitrary \((\tau\)-additive locally finite) Borel measures. Note that \(\|M_\mu\|_{L^1 \to L^{1,\infty}}\) is not assumed to be finite in either Lemma 2.7 or in Theorem 2.9.

Next we state three lemmas, some parts of which are well known in the absence of localization. The first one follows by a standard approximation argument, so the proof is omitted.

**Lemma 2.6** Let \((X, d, \mu)\) be a metric measure space. For \(0 \leq s < S \leq \infty\), the values of the localized centered maximal operator \(M_{(s,S),\mu}\) are independent of whether \(M_{(s,S),\mu}\) is defined using open or closed balls.

**Lemma 2.7** Let \((X, d)\) be a metric space. If there is a locally finite Borel measure \(\mu\) such that \(\|M_\mu\|_{L^1 \to L^{1,\infty}} > L\), then there exist an \(a > 0\), a \(T > 0\), a ball \(B^0(y, R)\), and a simple function \(f\) vanishing outside \(B^0(y, R + T)\), such that

\[ \mu\left( B^0(y, R) \cap \{M_\mu f > t\} \right) > \frac{L}{t} \int f \, d\mu. \]

**Proof** The argument proceeds by using several standard reductions to simpler cases. If \(\|M_\mu\|_{L^1 \to L^{1,\infty}} > L\), then we can select \(0 < h \in L^1(\mu), R > 0, t > 0,\) and \(y \in X\) such that

\[ \mu\left( B^0(y, R) \cap \{M_\mu h > t\} \right) > \frac{L}{t} \int h \, d\mu. \]

An additional approximation argument tells us that for some \(T \gg 1\), \(M_\mu\) can be replaced in the above inequality by its localized variant \(M_{(0,T),\mu}\).
Clearly, we only need to consider what happens inside $B^o(y, R + T)$ to determine the behavior of $M(0, T, \mu)$ in $B^o(y, R)$, so there is no loss in assuming that $h$ vanishes identically outside $B^o(y, R + T)$.

Next we show that $h$ can be suitably approximated by a simple function $f$, that is, of the form
$$f = \sum_{i=1}^{J} c_i 1_{S_i},$$
where the $S_i$ are disjoint Borel sets contained in $B^o(y, R + T)$, and the coefficients $c_i$ are strictly positive.

If $h$ is bounded then the result is clear, for given any $\varepsilon > 0$ we can always find a simple function $f = f 1_{B^o(y, R + T)}$ such that $0 \leq h \leq f$ and $\|f\|_1 < (1 + \varepsilon)\|h\|_1$. If $h$ is unbounded, we choose $H \gg 1$ so that the truncation $h \wedge H := \min\{h, H\}$ is sufficiently close to $h$ (and then we are back to the previous case) as follows. Set $E := B^o(y, R) \cap \{M_{\mu}h > t\}$ and note that for each $x \in E$ there exists an $r_x > 0$ such that
$$t < \frac{1}{\mu B(x, r_x)} \int_{B(x, r_x)} h \, d\mu.$$

By the Monotone Convergence Theorem, there is an $n_x \in \mathbb{N}$ such that
$$t < \frac{1}{\mu B(x, r_x)} \int_{B(x, r_x)} h \wedge n_x \, d\mu.$$

Let $E_n := E \cap \{M_{\mu}(h \wedge n) > t\}$. Then each $E_n$ is measurable, $E_n \subset E_{n+1}$, and $E = \bigcup_{n=1}^{\infty} E_n$, so
$$\lim_n \mu E_n = \mu E \geq \frac{L}{t} \int h \, d\mu.$$

Thus, there exists an $H$ such that
$$\mu E_H > \frac{L}{t} \int h \, d\mu \geq \frac{L}{t} \int h \wedge H \, d\mu.$$

From now on, it will be more convenient to use closed balls.

**Lemma 2.8** Let $(X, d, \mu)$ be a metric measure space. For $0 \leq s < S \leq \infty$, $0 \leq f \in L^1(\mu)$, $t > 0$ and $u \geq 0$, the set
$$O_{t,u} := \left\{ x \in X : \exists r \in (s, S) \text{ with } \frac{1}{\mu B^c(x, r)} \int_{B^c(x, r)} f \, d\mu > t \text{ and } \mu B^c(x, r) > u \right\}$$

is open.
Proof If $O_{t,u}$ is empty there is nothing to show, so suppose otherwise. Choose $x \in O_{t,u}$ and $r \in (s, S)$ such that

$$\frac{1}{\mu_{B^{cl}(x, r)}} \int_{B^{cl}(x, r)} f \, d\mu > t$$

and $\mu_{B^{cl}(x, r)} > u$.

Fix $\varepsilon > 0$ with

$$\frac{1}{\mu_{B^{cl}(x, r)}} \int_{B^{cl}(x, r)} f \, d\mu > (1 + \varepsilon)t.$$

Select $0 < \delta < r$ with $r + \delta < S$ and $\mu_{B^{cl}(x, r + \delta)} < (1 + \varepsilon)\mu_{B^{cl}(x, r)}$ (here we use that balls are metrically closed). Let $y \in B^o(x, \delta/2)$. Then

$$B^{cl}(x, r) \subset B^{cl}(y, r + \delta/2) \subset B^{cl}(x, r + \delta),$$

so $\mu_{B^{cl}(y, r + \delta/2)} > u$ and

$$(1 + \varepsilon)t < \frac{1}{\mu_{B^{cl}(x, r)}} \int_{B^{cl}(x, r)} f \, d\mu \leq \frac{1 + \varepsilon}{\mu_{B^{cl}(y, r + \delta/2)}} \int_{B^{cl}(y, r + \delta/2)} f \, d\mu.$$

□

Theorem 2.9 Let $(X, d)$ be a metric space and let $L > 0$. If there is a $\tau$-additive, locally finite Borel measure $\mu$ such that $\|M_{\mu}\|_{L^1 \rightarrow L^{1,\infty}} > L$, then there is a discrete, finite Borel measure $\nu$ with finite support in $X$, for which $\|M_{\nu}\|_{L^1 \rightarrow L^{1,\infty}} > L$.

Proof By the preceding lemmas, it is enough to show that given $\varepsilon > 0$, $R, T > 0$, $t > 0$, $y \in X$, and a simple function $0 \leq f \in L^1(\mu)$ vanishing outside $B^o(y, R + T)$, it is possible to select a finite discrete measure $\nu := \sum_{i=1}^m a_i \delta_{w_i}$, with $a_i > 0$, such that $\|f\|_{L^1(\nu)} = \|f\|_{L^1(\mu)}$ and

$$\mu \left( B^o(y, R) \cap \{M_{(0,T),\mu} f > t\} \right) \leq (1 + \varepsilon)\nu \left( B^o(y, R) \cap \{M_{(0,T),\nu} f > t/(1 + \varepsilon)\} \right),$$

where the localized maximal operators are defined using metrically closed balls.

For each $x \in O_t := B^o(y, R) \cap \{M_{(0,T),\mu} f > t\}$, select $0 < r_x < T$ such that

$$t\mu_{B^{cl}(x, r_x)} < \int_{B^{cl}(x, r_x)} f \, d\mu,$$

and choose $0 < \delta_x < T$ so that $\mu_{B^{cl}(x, r_x + \delta_x)} < (1 + \varepsilon)\mu_{B^{cl}(x, r_x)}$. It follows from Lemma 2.8 that $O_t$ is open; we select very small radii $0 < \delta_x < \min\{r_x, \delta_x\}/2$ so that for each $x$, we have $B^{cl}(x, \delta_x) \subset O_t$. Since
$O_t = \bigcup \{ B^\alpha(x, s_x) : x \in O_t \}$,

by $\tau$-additivity we can pick a finite collection of centers $x_1, \ldots, x_n$ in $O_t$ such that

$$\mu O_t < (1 + \varepsilon) \mu \cup_{i=1}^n B^\alpha(x_i, s_{x_i}).$$

Let $f = \sum_{i=1}^J c_i 1_{S_i}$, where the $S_i$ are disjoint Borel subsets of $B^\alpha(y, R + T)$, and the coefficients $c_i$ are strictly positive. The next step consists in defining a suitable finite subalgebra $A$ on $B^\alpha(y, R + T)$. We let $A$ be generated by the sets $S_i$ defining $f$, for $1 \leq i \leq J$, together with $B^\alpha(y, R)$, $B^\alpha(y, R + T)$, $O_t$, and the finite collection of balls $B^{cl}(x_i, u_i)$, where $u_i$ takes each of the three values $s_{x_i}$, $r_{x_i}$, and $r_{x_i} + \delta_{x_i}$, for $1 \leq i \leq n$.

Given $z \in B^\alpha(y, R + T)$, let $P_z := \cap \{ A \in A : z \in A \}$. The sets $P_z$ are the atoms of $A$, so they yield a finite partition $\{ P_1, \ldots, P_m \}$ of $B^\alpha(y, R + T)$ by non-empty measurable sets. Also, we may assume that for each $1 \leq i \leq m$, $\mu P_i > 0$, for otherwise we simply disregard a finite number of sets of measure zero. Since each $P_i$ cannot be split into smaller sets belonging to $A$, the value of any measure on $A$ is completely determined by its value on these atoms. Choose representatives $w_i \in P_i$, $1 \leq i \leq m$, and set $\nu \{ w_i \} = \mu P_i$. Then $\nu = \mu$ on $A$. Furthermore, $\nu$ is defined for all subsets of $X$, since it is a discrete measure.

By the $A$-measurability of $f$ and of the balls $B^{cl}(x_i, r_{x_i})$, $1 \leq i \leq n$, we have that $\nu B^{cl}(x_i, r_{x_i}) = \mu B^{cl}(x_i, r_{x_i})$ and

$$\int_{B^{cl}(w_i, r_i)} f d\nu = \int_{B^{cl}(w_i, r_i)} f d\mu.$$

We claim that for $\nu$-almost every point in $\cup_{i=1}^n B^{cl}(x_i, s_{x_i})$, the inequality $(1 + \varepsilon) M_{(0,T),<} f > t$ holds. This yields the result, since then

$$\nu \left( B^\alpha(y, R) \cap \{ M_{(0,T),<} f > t/(1 + \varepsilon) \} \right) \geq \nu \cup_{i=1}^n B^{cl}(x_i, s_{x_i})$$

$$= \mu \cup_{i=1}^n B^{cl}(x_i, s_{x_i}) > \frac{\mu O_t}{1 + \varepsilon}$$

$$> \frac{L}{(1 + \varepsilon)t} \int f d\mu = \frac{L}{(1 + \varepsilon)t} \int f d\nu.$$

To see why the claim is true, recall that the representatives $w_i$ constitute the support of $\nu$. Choose any $w_j \in \cup_{i=1}^n B^{cl}(x_i, s_{x_i})$. For some $1 \leq k \leq n$, we have $w_j \in B^{cl}(x_k, s_{x_k})$. Now

$$B^{cl}(x_k, r_{x_k}) \subset B^{cl}(w_j, s_{x_k} + r_{x_k}) \subset B^{cl}(x_k, r_{x_k} + \delta_{x_k}),$$

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The following is a restatement of our main result:

**Theorem 2.10** Let \((X, d)\) be a metric space and let \(L > 0\). The following are equivalent:

1. \((X, d)\) has the Besicovitch intersection property with constant bounded above by \(L\).
2. For every \(\tau\)-additive, locally finite Borel measure \(\mu\) on \(X\), the centered maximal operator associated with \(\mu\) satisfies \(\| M_\mu \|_{L^1 \to L^1, \infty} \leq L\).

**Proof** By Theorem 2.9, it is enough to prove the result for weighted finite sums of Dirac deltas. This is done in Lemma 2.5.

A modification of the proof of \((2) \implies (1)\) in Lemma 2.5, shows that for any \(p \in (1, \infty)\), the uniform weak type \((p, p)\) already implies the Besicovitch intersection property. Recall that the floor function \(\lfloor x \rfloor\) denotes the integer part of \(x\).

**Theorem 2.11** Let \((X, d)\) be a metric space. Each of the following statements implies the next:

1. There exist a \(p\) with \(1 < p < \infty\) and an integer \(N \geq 1\), such that for every discrete, finite Borel measure \(\mu\) with finite support in \(X\), the centered maximal operator associated with \(\mu\) satisfies \(\| M_\mu \|_{L^p \to L^p, \infty} \leq N\).
2. The space \((X, d)\) has the Besicovitch intersection property with constant at most \(\lfloor \frac{p}{p-1} \frac{1}{N^p - p} \rfloor + 1\).
3. For every \(\tau\)-additive, locally finite Borel measure \(\mu\) on \(X\), the centered maximal operator associated with \(\mu\) satisfies \(\| M_\mu \|_{L^1 \to L^1, \infty} \leq \lfloor \frac{p}{p-1} \frac{1}{N^p - p} \rfloor + 1\).

**Proof** Recall that \((2) \implies (3)\) has already been proved in Lemma 2.5, with a different expression for the constant. Regarding \((1) \implies (2)\), let \(q = p/(p-1)\) be the dual exponent of \(p\), let \(\mathcal{C} = \{ B(x_1, r_1), \ldots, B(x_J, r_J) \}\) be an intersecting Besicovitch family in \((X, d)\), and let \(y \in \cap \mathcal{C}\). Define, for \(c > 0\), the measure \(\mu_c := c\delta_y + \sum_{i=1}^J \delta_{x_i}\), and recall that for every \(\alpha > 0\),

\[
\mu_c(\{ M_\mu f \geq \alpha \}) \leq \left( \frac{\| M_{\mu_c} \|_{L^p \to L^p, \infty} \| f \|_{L^p}}{\alpha} \right)^p.
\]
Set \( f = \mathbf{1}_{[y]} \); then \( \| f \|_{L^p(\mu_c)} = c^{1/p} \). For \( 1 \leq i \leq J \), we have \( M_{\mu_c} f(x_i) \geq c/(1+c) \). Thus, \( \mu_c \{ M_{\mu_c} f \geq c/(1+c) \} = J + c \) (taking \( y \) into account) so with \( \alpha = c/(1+c) \), we have

\[
J + c \leq \left( \frac{\| M_{\mu_c} \|_{L^p \to L^\infty} (1+c)}{c^{1/q}} \right)^p.
\]

Maximizing \( g(c) = c^{1/q}/(1+c) \) we get \( c = p-1 \) and \( g(p-1) = (p-1)^{(p-1)/p}p^{p-1} \), so

\[
J + p - 1 \leq \| M_{\mu_c} \|_{L^p \to L^\infty}^p (p-1)^{1-p} \leq N^p p^p (p-1)^{1-p}.
\]

\( \Box \)

**Remark 2.12** The extrapolation result (1) \( \implies \) (3) tells us that for any \( 1 < p < \infty \), uniform weak type \((p, p)\) bounds of size \( N \) entail uniform weak type \((1, 1)\) bounds of size less than \( p^p (p-1)^{(1-p)}N^p \). Once we have these, by interpolation we get the following strong type \((p, p)\) bounds:

\[
\| M_\mu \|_{L^p \to L^p} \leq \frac{p^2 N}{(p-1)^{2-1/p}},
\]

(cf. [12, p. 42, Exercise 1.3.3 (a)]) so the factor \( p^2 (p-1)^{-2+1/p} \) bounds from above the ratio between the uniform strong and uniform weak \((p, p)\) bounds, for every \( p \in (1, \infty) \).

Note however that this bound might considerably overestimate the actual ratio, since in general interpolation will not yield the best possible constants, and occasionally it may yield bounds that are very far from optimality; for instance, it is known that for \( p = 2 \), for Lebesgue measure in \( \mathbb{R}^d \), and for balls defined by an arbitrary norm, optimal constants are uniformly bounded by 140 in every dimension (see [10, Theorem 5.2]). For cubes (balls with respect to the \( \ell_\infty \) norm), J. Bourgain has proved that dimension-independent bounds hold for every \( p > 1 \), cf. [7]. However, for cubes it is also known that the weak type \((1,1)\) constants diverge to infinity with the dimension (cf. [1]) and thus, for every \( p \in (1, \infty) \), so do the bounds obtained by interpolation.

### 3 Consequences for \( \mathbb{R}^d \)

Again we take balls to be closed. Recall that \( L(\mathbb{R}^d, \| \cdot \|) \) denotes the Besicovitch constant of \((\mathbb{R}^d, \| \cdot \|)\). The definition of strict Hadwiger number comes from [23, p. 123], but this notion had been used before.

**Definition 3.1** Let \( \| \cdot \| \) be any norm on \( \mathbb{R}^d \). The **Hadwiger number or translatative kissing number** \( H(d, \| \cdot \|) \), is the maximum number of translates of the closed unit ball \( B^c(0, 1) \) that can touch \( B^c(0, 1) \) without overlapping, i.e., all the translates have disjoint interiors. The **strict Hadwiger number** \( H^*(d, \| \cdot \|) \) is the maximum number

\( \mathcal{ Springer} \)
of translates of the closed unit ball $B^{cl}(0,1)$ that can touch $B^{cl}(0,1)$ without touching each other, that is, all the translates are disjoint. A spherical code is a finite set of unit vectors.

**Theorem 3.2** Let $\| \cdot \|$ be any norm on $\mathbb{R}^d$. The Besicovitch constant of $(\mathbb{R}^d, \| \cdot \|)$ equals its strict Hadwiger number, i.e., $L(\mathbb{R}^d, \| \cdot \|) = H^*(d, \| \cdot \|)$.

**Proof** To see that $L(\mathbb{R}^d, \| \cdot \|) \leq H^*(d, \| \cdot \|)$, let $C := \{ B^{cl}(x_1, r_1), \ldots, B^{cl}(x_n, r_n) \}$ be an intersecting Besicovitch family in $\mathbb{R}^d$ of maximal cardinality. Choose $y \in \cap C$, and let $r_y := \min(\|x_1 - y\|, \ldots, \|x_n - y\|)$. By a dilation and a translation, if needed, we can assume that $y = 0$ and $r_y = 1$. We claim that all the balls in $C' := \{ B^{cl}(2x_1/\|x_1\|, 1), \ldots, B^{cl}(2x_n/\|x_n\|, 1) \}$ are disjoint, and clearly they touch $B^{cl}(0,1)$, so $n \leq H^*(d, \| \cdot \|)$. To check the claim it is enough to verify that any two centers $2x_i/\|x_i\|$ and $2x_j/\|x_j\|$ are at distance $> 2$, or equivalently, that any two vectors in the spherical code $\{ x_1/\|x_1\|, \ldots, x_n/\|x_n\| \}$ are at distance $> 1$. So choose a pair of centers $x_i$ and $x_j$ of balls from $C$, with, say, $\|x_i\| \geq \|x_j\|$. Since $\|x_i - x_j\| > \|x_i\|$, using the lower bound for the angular distances from [22, Corollary 1.2], we get

$$\frac{\|x_i - x_j\|}{\|x_i\|} - \frac{\|x_j\|}{\|x_j\|} \geq \frac{\|x_i - x_j\| - \|x_i\| - \|x_j\|}{\min\{\|x_i\|, \|x_j\|\}} = \frac{\|x_i - x_j\| - \|x_i\| + \|x_j\|}{\|x_j\|} > 1.$$  

For the other direction, each set of unit vectors $S$ satisfying $\|x - y\| > 1$ for all $x, y \in S$ with $x \neq y$, defines an intersecting Besicovitch family $\{ B^{cl}(x, 1) : x \in S \}$, so $L(\mathbb{R}^d, \| \cdot \|) \geq H^*(d, \| \cdot \|)$.

In $\mathbb{R}^d$ there is “plenty of room,” so it is possible to construct a measure $\mu$ for which the supremum is attained.

**Theorem 3.3** Let $\| \cdot \|$ be any norm on $\mathbb{R}^d$. Then there exists a discrete measure $\mu$ such that $\|M_\mu\|_{L^1 \to L^{1, \infty}} = L(\mathbb{R}^d, \| \cdot \|)$.

**Proof** Given $\| \cdot \|$, by rescaling if needed we may assume that $\|e_1\| = 1$. Then we argue as in the proof of Lemma 2.5: choose a spherical code $\{ x_1, \ldots, x_N \}$ of cardinality $H^*(d, \| \cdot \|)$, with minimal separation strictly larger than 1. For $n \geq 1$, set $\mu_n := n^{-1} \delta_{3ne_1} + \sum_{i=1}^N \delta_{x_i+3ne_1}$, and let $\mu := \sum_{n=1}^\infty \mu_n$.

Note that the measure $\mu$ in the preceding result can be chosen to be finite, by assigning suitable weights to the measures $\mu_n$.

The best uniform bound in one dimension for the uncentered maximal operator is 2, cf. [8, Formula (6)]. Since $L(\mathbb{R}, | \cdot |) = 2$, the same uniform bound holds for both the centered and the uncentered operators in dimension 1. But already in dimension 2 (for squares and discs) the standard Gaussian measure provides an example where the uncentered maximal operator is not of weak type $(1, 1)$, cf. [31].

**Corollary 3.4** Given any norm $\| \cdot \|$ on the plane, if the unit ball is a parallelogram then $L(\mathbb{R}^2, \| \cdot \|) = 4$, while $L(\mathbb{R}^2, \| \cdot \|) = 5$ in every other case.
**Proof** This follows from the corresponding results for strict Hadwiger numbers, cf. [35, Proposition 23].

**Corollary 3.5** The sharp uniform bound $\sup_\mu \| M_\mu \|_{L^1 \to L^{1, \infty}}$ for the centered maximal operator on $(\mathbb{R}^d, \| \cdot \|_\infty)$ is $L(\mathbb{R}^d, \| \cdot \|_\infty) = 2^d$. Furthermore, the bound is attained.

**Proof** It is enough to check that $H^*(d, \| \cdot \|_\infty) = 2^d$, something that is both well known and easy to see. The inequality $H^*(d, \| \cdot \|_\infty) \geq 2^d$ follows by placing translates of the unit cube touching the central cube only at the vertices, and the other direction follows by noticing that any cube touching the central cube must touch some vertex. A more general result can be found in [37, Lemma 3.1].

Next we consider euclidean balls. In this context, the translatively kissing number is just the kissing number, $A(d, \theta)$ denotes the maximum number of unit vectors in $\mathbb{R}^d$ such that for any pair $x$, $y$ of them, $x \cdot y \leq \cos \theta$, and $A^o(d, \theta)$ is defined in the same way, but requiring the inequality to be strict, so $x \cdot y < \cos \theta$. Observe that $A(d, \pi/3) = H(d, \| \cdot \|_2)$, while $A^o(d, \pi/3) = H^*(d, \| \cdot \|_2)$.

**Corollary 3.6** The sharp uniform bound $\sup_\mu \| M_\mu \|_{L^1 \to L^{1, \infty}}$ for the centered maximal operator on $(\mathbb{R}^3, \| \cdot \|_2)$, is $L(\mathbb{R}^3, \| \cdot \|_2) = 12$. Asymptotically we have

$$(1 + o(1))\sqrt{\frac{3\pi}{8}} \log \frac{3}{2\sqrt{2}} d^{3/2} \left( \frac{2}{\sqrt{3}} \right)^d \leq L(\mathbb{R}^d, \| \cdot \|_2) \leq 2^{0.401(1+o(1))d}. \quad (7)$$

**Proof** For $d = 3$ it is well known that a spherical code of maximal cardinality (12 vectors) can be obtained from the vertices of a regular icosahedron inscribed in the unit sphere. Since the minimal separation between any two vertices of the icosahedron is strictly larger than 1, we have $12 = A(3, \pi/3) = A^o(3, \pi/3)$.

Regarding the asymptotic bounds, the left-hand side in formula (7) comes from [16, Theorem 1]; up to constants, it improves previously known bounds by a factor of $d$. Trivially $A^o(d, \pi/3) \geq A(d, \theta)$ for every $\theta > \pi/3$. Using the estimates in [16, Theorem 2] for $A(d, \theta)$, when $0 < \theta < \pi/2$, we conclude that the lower bounds given in [16, Theorem 1] by taking $\theta = \pi/3$ are also lower bounds for $A^o(d, \pi/3)$, since for $d$ fixed all the parameters in [16, Theorem 2] depend continuously on $\theta$. And the right-hand side of (7) follows directly from the upper bounds known for $A(d, \pi/3)$, cf. [17, Corollary 1, p. 20], or [9, Formulas (66) and (49)].

**Remark 3.7** For $d > 3$, the exact values of $A(d, \pi/3)$ presently known are $A(4, \pi/3) = 24$ (cf. [25]), $A(8, \pi/3) = 240$ and $A(24, \pi/3) = 196560$ (cf. [9, p.12, Table 1.1]) but additional upper and lower bounds can be found in the literature, cf. [5] for instance.

Ignoring the terms that are not exponential in $d$, the preceding corollary entails that $(1 + o(1))1.1547^d \leq L(\mathbb{R}^d, \| \cdot \|_2) \leq 1.3205^{(1+o(1))d}$, which are the bounds indicated in the introduction. Curiously, the uniform bounds satisfied by the centered operator associated with cubes are smaller than those associated with euclidean balls in dimensions 2, 3, and 4; in dimension 8 the situation is reversed, since $256 > 240$. 

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Finally, for $d \gg 1$, the bounds associated with cubes are much larger. Strict Hadwiger numbers for other norms have also been studied, cf. for instance [29,36,37].

Denote by $\mu_d(A) := \lambda^d(A \cap B(0, 1))$ the Lebesgue measure $\lambda^d$ restricted to the euclidean unit ball of $\mathbb{R}^d$; the measures $\mu_d$ provide a concrete family where the exponential factor in the left-hand side of (7) is present: by [2, Remark 2.7],

$$\|M_{\mu_d}\|_{L^1 \to L^1} \geq \frac{\sqrt{\pi(d + 1)}}{\sqrt{6}} \left(\frac{2}{\sqrt{3}}\right)^d.$$ 

Regarding the maximal number $\beta_d$ of disjoint collections appearing in the Besicovitch covering theorem in dimension $d$, it has been studied in [34] for euclidean balls and in [11] for balls associated with general norms. Specifically, in [34, Theorem] it is noted that $\beta_2 = 19$, $\beta_3 \leq 87$, and asymptotically, $\beta_d$ grows exponentially with $d$, to base at least $8/\sqrt{15}$ and at most 2.641. Thus, the bounds for the centered maximal operator using kissing numbers represent a substantial improvement over the $\beta_d$'s.

**Question 3.8** It would be desirable to have a better understanding of the relationship between $L(\mathbb{R}^d, \|\cdot\|_2) = A^\circ(d, \pi/3)$ and $A(d, \pi/3)$, a subject that hopefully will be of interest to specialists in spherical codes. For large angles this was solved in [28]. In particular, by [28, Theorem 1], $A^\circ(d, \pi/2) = d + 1$ and $A(d, \pi/2) = 2d$.

## 4 Boundedness of Averaging Operators on $L^1$

Recall that the complement of the support $(\text{supp } \mu)^c := \cup \{B^\circ(x, r) : x \in X, \mu B^\circ(x, r) = 0\}$ of a Borel measure $\mu$, is an open set, and hence measurable.

**Definition 4.1** Let $(X, d, \mu)$ be a metric measure space and let $g$ be a locally integrable function on $X$. For each fixed $r > 0$ and each $x \in \text{supp } \mu$, the averaging operator $A_{r, \mu}$ is defined as

$$A_{r, \mu}g(x) := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} g \, d\mu.$$ 

(8)

While maximal operators are well defined everywhere, by our convention averaging operators are defined only on the support of the measure under consideration.

**Definition 4.2** Let $(X, d)$ be a metric space and let $\mu$ be a locally finite Borel measure on $X$. If $\mu(X \setminus \text{supp } \mu) = 0$, we say that $\mu$ has full support.

By $\tau$-additivity, if $(X, d, \mu)$ is a metric measure space, then $\mu$ has full support, since the set $X \setminus \text{supp } \mu$ is a union of open balls of measure zero. Thus, averaging operators are a.e. defined on metric measure spaces.

To specify whether balls are open or closed, we use $A^\circ_{r, \mu}$ and $A^{cl}_{r, \mu}$ for the corresponding operators. When we consider only one measure $\mu$, we often write $A_r$ instead of $A_{r, \mu}$. 

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Definition 4.3 We call
\[ a_r(y) := \int_{\text{supp}(\mu)} \frac{1_{B(y,r)}(x)}{\mu(B(x,r))} \, d\mu(x) \] (9)
the conjugate function to the averaging operator \( A_r \).

Note that the conjugate function \( a_r \) is well defined a.e. (whenever \( y \) belongs to the support of \( \mu \)). According to [3, Theorem 3.3], \( A_r \) is bounded on \( L^1(\mu) \) if and only if \( a_r \in L^\infty(\mu) \), in which case \( \| A_r \|_{L^1(\mu)} = \| a_r \|_{L^\infty(\mu)} \). We will use \( a_{r_0}^o \) and \( a_{r_0}^c \) to specify whether balls are open or closed.

Definition 4.4 Denote by \( \mathcal{E}(X,d) \) the collection of all Besicovitch families \( \mathcal{C} \) of \((X,d)\) with the property that all balls in \( \mathcal{C} \) have equal radius, that is, for some choice of radius \( r_0 \) and some set \( A \subset X \), all balls in \( \mathcal{C} \) are of the form \( B(x,r_0) \), \( x \in A \). The equal radius Besicovitch constant of \((X,d)\) is
\[ E(X,d) := \sup \left\{ \sum_{B(x,r) \in \mathcal{C}} 1_{B(x,r)}(y) : y \in X, \mathcal{C} \in \mathcal{E}(X,d) \right\}. \] (10)

We say that \((X,d)\) has the equal radius Besicovitch Intersection Property with constant \( E(X,d) \) if \( E(X,d) < \infty \).

Example 4.5 It is well known that the Heisenberg groups \( \mathbb{H}^n \) with the Korány metric do not have the Besicovitch intersection property, cf. [18, pp. 17–18] or [30, Lemma 4.4]. However, they do have the equal radius Besicovitch intersection property, since they are geometrically doubling.

We mention why geometrically doubling entails \( E(X,d) < \infty \). Recall that a metric space is geometrically doubling if there exists a positive integer \( D \) such that every ball of radius \( r \) can be covered with no more than \( D \) balls of radius \( r/2 \). We call the smallest such \( D \) the doubling constant of the space. Now \( E(X,d) \leq D \). To see why, given an intersecting Besicovitch family \( \mathcal{C} \) with equal radius \( r \), choose any \( y \in \cap \mathcal{C} \), and note that the centers of all balls in \( \mathcal{C} \) form an \( r \)-net in \( B(y,r) \). Cover \( B(y,r) \) with \( \leq D \) balls of radius \( r/2 \). Since each such ball contains at most the center of one ball from \( \mathcal{C} \), the result follows.

While being geometrically doubling does not depend on whether we use open or closed balls in the definition, the value of the doubling constant may change. However, \( E(X,d) \) is the same for open and closed balls, as the next proposition indicates. Since the proof is a straightforward modification of the argument given for Proposition 2.4, we omit it.

Proposition 4.6 A metric space \((X,d)\) has the equal radius Besicovitch intersection property with constant \( E(X,d) \) for collections of open balls, if and only if it has the equal radius Besicovitch intersection property for collections of closed balls, with the same constant.
The situation regarding the existence of strong type $(1, 1)$ bounds for the averaging operators $A_{r, \mu}$, uniform in both $r$ and $\mu$, is analogous to the weak type case for the maximal operator, but with the equal radius Besicovitch intersection property replacing the Besicovitch intersection property.

**Theorem 4.7** Let $(X, d)$ be a metric space and let $E > 0$. The following are equivalent:

1. The space $(X, d)$ has the equal radius Besicovitch intersection property with constant bounded above by $E$.
2. For every $r > 0$ and every $\tau$-additive, locally finite Borel measure $\mu$ on $X$, we have $\|A_{r, \mu}\|_{L^1 \to L^1} \leq E$.
3. For every $r > 0$ and every finite weighted sum of Dirac deltas $\mu := \sum_{i=1}^N c_i \delta_{x_i}$, the averaging operator satisfies $\|A_{r, \mu}\|_{L^1(\mu) \to L^1(\mu)} \leq E$.

**Proof** Let us show that (1) $\implies$ (2). Disregarding a set of measure zero if needed, we suppose that $X = \text{supp} \mu$, so every ball has positive measure. Fix $y \in X$ and $r > 0$. First we consider the open balls case. Let $0 < s < r$, let

$$g(x) := \frac{1_{B^o(y, r)}(x)}{\mu B^o(x, r)}, \quad \text{and}$$

$$g_s(x) := \frac{1_{B^o(y, s)}(x)}{\mu B^o(x, r)}.$$

Since balls are open, $g_s \uparrow g$ everywhere as $s \uparrow r$, so we can use the monotone convergence theorem. It is thus enough to show that $\lim_{s \to r} \int_X g_s d\mu \leq E(X, d)$, to conclude that $\|a^o_s\|_{L^\infty(\mu)} \leq E(X, d)$. Then the result follows, since $\|A_r\|_{L^1(\mu) \to L^1(\mu)} = \|a_r\|_{L^\infty(\mu)}$ by [3, Theorem 3.3].

For the next step, we argue as in the proof of [3, Theorem 3.5], which dealt with the case where $X$ is geometrically doubling. We include the details for the readers convenience. Note first that $b_1 := \inf\{\mu B^o(x, r) : x \in B^o(y, s)\} > 0$. To see why, observe that for every $x \in B^o(y, s)$ and every $w \in B^o(y, r-s)$, $d(x, w) \leq d(x, y) + d(y, y) < s + r - s$, so $B^o(y, r-s) \subset B^o(x, r)$ and thus $0 < \mu B^o(y, r-s) \leq b_1$. Now take $0 < \varepsilon \ll 1$, and choose $u_1 \in B^o(y, s)$ so that $\mu B^o(u_1, r) < (1+\varepsilon)b_1$; let $b_2 := \inf\{\mu B^o(x, r) : x \in B^o(y, s) \setminus B^o(u_1, r)\}$, and select $u_2 \in B^o(y, s) \setminus B^o(u_1, r)$ so that $\mu B^o(u_2, r) < (1+\varepsilon)b_2$; repeat, with $b_{k+1} := \inf\{\mu B^o(x, r) : x \in B^o(y, s) \setminus \bigcup_{j=1}^k B^o(u_j, r)\}$, and $u_{k+1} \in B^o(y, s) \setminus \bigcup_{j=1}^k B^o(u_j, r)$ so that $\mu B^o(u_{k+1}, r) < (1+\varepsilon)b_{k+1}$. Since the balls $B^o(u_j, r)$ form a Besicovitch family and all contain $y$, there is an $m \leq E(X, d)$ such that $B^o(y, s) \setminus \bigcup_{j=1}^m B^o(u_j, r) = \emptyset$, and then the process stops.

Fix $x \in B^o(y, s)$, and let $i$ be the first index such that $x \in B^o(u_i, r)$. Then

$$\frac{1_{B^o(y, r)}(x)}{\mu B^o(x, r)} \leq (1+\varepsilon) \frac{1_{B^o(y, s) \cap B^o(u_i, r)}(x)}{\mu B^o(u_i, r)} \leq (1+\varepsilon) \sum_{j=1}^m \frac{1_{B^o(y, s) \cap B^o(u_j, r)}(x)}{\mu B^o(u_j, r)}.$$
so
\[
\int_X g_s \, d\mu = \int_X \frac{1_B^{\alpha}(y, x)}{\mu B^{\alpha}(x, r)} \, d\mu(x) \\
\leq \int_X (1 + \varepsilon) \sum_{j=1}^m \frac{1_B^{\alpha}(y, x) \cap B^{\alpha}(u_j, r)}{\mu B^{\alpha}(u_j, r)} \, d\mu(x) \\
\leq (1 + \varepsilon) \int_X \sum_{j=1}^m \frac{1_B^{\alpha}(u_j, r)}{\mu B^{\alpha}(u_j, r)} \, d\mu(x) \leq (1 + \varepsilon) E(X, d),
\]
and now \(\alpha^0(y) \leq E(X, d)\) follows by letting \(\varepsilon \downarrow 0\) and \(s \uparrow r\).

The closed balls case is proven using the result for open balls. Let \(0 \leq f \in L^1(\mu)\), \(w \in X\), \(R \geq 1\), and \(\varepsilon > 0\). By monotone convergence, taking \(R \uparrow \infty\), it is enough to show that
\[
\|1_B(w, R)^1_{\{A^{cl}_r f \leq R\}} A^{cl}_r f \|_{L^1} \leq \varepsilon + (1 + \varepsilon) E(X, d) \|f\|_{L^1}.
\]
For each \(x \in B(w, R)\), choose \(r_x > 0\) so that \(\mu B^{\alpha}(x, r + r_x) < (1 + \varepsilon) \mu B^{cl}(x, r)\), and let \(E_n := \{x \in B(w, R) : r_x > 1/n\}\). Then select \(N \gg 1\) satisfying \(\mu(B(w, R) \setminus E_N) < \varepsilon/R\). Now for all \(x \in E_N\), we have
\[
A^{cl}_r f(x) \leq \frac{1}{\mu B^{\alpha}(x, r)} \int_{B^{\alpha}(x, r+1/N)} f \, d\mu \\
\leq \frac{1 + \varepsilon}{\mu B^{\alpha}(x, r + 1/N)} \int_{B^{\alpha}(x, r+1/N)} f \, d\mu = (1 + \varepsilon) A^{\alpha}_{r+1/N} f(x),
\]
so
\[
\|1_B(w, R)^1_{\{A^{cl}_r f \leq R\}} A^{cl}_r f \|_{L^1} \leq \|1_B(w, R)^1_{\{A^{cl}_r f \leq R\}} A^{cl}_r f \|_{L^1} \leq \varepsilon + (1 + \varepsilon) E(X, d) \|f\|_{L^1}.
\]
Since (3) is a special case of (2), the only implication left is (3) \(\implies\) (1); we prove that if \(C\) is an intersecting Besicovitch family in \((X, d)\) of equal radius \(r\) and cardinality \(> E\), then there exists a discrete measure \(\mu_C\) with finite support, for which \(\|A_{r, \mu_C}\|_{L^1} > E\). We may suppose that \(C = \{B(x_1, r), \ldots, B(x_{E+1}, r)\}\). Let \(y \in \cap C\), and for \(0 < c < 1\), define \(\mu_c := c\delta_y + \sum_{i=1}^{L+1} \delta_{x_i}\). Set \(f_c = c^{-1}\text{1}_{\{y\}}\). Then \(\|f_c\|_1 = 1\), and for \(1 \leq i \leq E + 1\), we have \(A_{r, \mu_c} f_c(x_i) = 1/(1 + c)\), while \(A_{r, \mu_c} f_c(y) > 0\). Thus
\[
\|A_{r, \mu_c} f_c\|_1 > \frac{E + 1}{1 + c},
\]
and the result follows by taking \(c\) small enough. \(\square\)
Since $\|A_{r, \mu}\|_{L^\infty(\mu) \to L^\infty(\mu)} = 1$, by interpolation or by Jensen’s inequality (cf. [4, Theorem 2.10]) for all $1 < p < \infty$, we have $\|A_{r, \mu}\|_{L^p(\mu) \to L^p(\mu)} \leq E(X, d)^{1/p}$.

**Remark 4.8** In addition to having $\sup_{r, \mu} \|A_{r, \mu}\|_{L^1(\mu) \to L^1(\mu)} = E(X, d)$, using the same measures and functions it is easy to see that equality also holds for the weak type $(1, 1)$ bounds, that is, $\sup_{r, \mu} \|A_{r, \mu}\|_{L^1(\mu) \to L^{1, \infty}(\mu)} = E(X, d)$. In fact, since the function $f_c$ is a scalar multiple of an indicator function, this equality holds in the restricted weak type $(1, 1)$ case.

The preceding theorem entails that the uniform weak type $(1, 1)$ of the centered maximal operator is stronger than the uniform strong type $(1, 1)$ of the averaging operators.

**Corollary 4.9** Given any metric space $(X, d)$, we have

$$\sup_{r, \mu} \|A_{r, \mu}\|_{L^1 \to L^1} = E(X, d) \leq L(X, d) = \sup_{\mu} \|M_\mu\|_{L^1 \to L^{1, \infty}},$$

where the supremum on the left-hand side is taken over all $r > 0$ and all $\tau$-additive, locally finite Borel measures $\mu$ on $X$, and the supremum on the right, over all such $\mu$.

**Corollary 4.10** Let $1 \leq p < \infty$. If $(X, d)$ has the equal radius Besicovitch intersection property, and $\mu$ is a $\tau$-additive Borel measure on $X$, then for every $f \in L^p(\mu)$, we have $\lim_{r \to 0} A_r f = f$ in $L^p$.

This corollary follows in a standard fashion from the uniform boundedness of the averaging operators (cf. [3] for more details).

Analogously to the case of the centered maximal operator, given any $p \in (1, \infty)$, for averaging operators the uniform weak type $(p, p)$ implies the equal radius Besicovitch intersection property, and consequently, one can extrapolate from uniform weak type $(p, p)$ to uniform strong type $(1, 1)$.

**Theorem 4.11** Let $(X, d)$ be a metric space. Each of the following statements implies the next:

1. There exist a $p$ with $1 < p < \infty$ and an integer $N \geq 1$, such that for every discrete, finite Borel measure $\mu$ with finite support in $X$, and every $r > 0$, the averaging operators $A_{r, \mu}$ satisfy $\|A_{r, \mu}\|_{L^p \to L^{p, \infty}} \leq N$.
2. The space $(X, d)$ has the equal radius Besicovitch intersection property with constant $\lfloor p^p (p - 1)^{(1-p)N^p} \rfloor$.
3. For every $\tau$-additive, locally finite Borel measure $\mu$ on $X$ and every $r > 0$, the averaging operators $A_{r, \mu}$ satisfy $\|A_{r, \mu}\|_{L^1 \to L^1} \leq \lfloor p^p (p - 1)^{(1-p)N^p} \rfloor$.

**Proof** The implication (2) $\implies$ (3) is part of Theorem 4.7. Regarding 1) $\implies$ (2), we show that if $\mathcal{C}$ is an intersecting Besicovitch family in $(X, d)$, of cardinality strictly larger than $\lfloor p^p (p - 1)^{(1-p)N^p} \rfloor$ and equal radius $r$, then there exists a finite sum of weighted Dirac deltas $\mu$, for which $\|A_{r, \mu}\|_{L^p \to L^{p, \infty}} > N$.

Let $q = p/(p-1)$ be the dual exponent of $p$, and let $J := \lfloor p^p (p - 1)^{(1-p)N^p} \rfloor + 1$. We may suppose that $\mathcal{C} = \{B(x_1, r), \ldots, B(x_J, r)\}$. Let $y \in \cap \mathcal{C}$, and set, for $c > 0$,
\[ \mu_c := c \delta_y + \sum_{i=1}^{J} \delta_{x_i}. \]
Recall that for every \( \alpha > 0 \), by definition of the weak \((p, p)\)
constant \( \| A_{r, \mu_c} \|_{L^p \to L^p} \),
\[
\mu_c(\{ A_{r, \mu_c} f \geq \alpha \}) \leq \left( \frac{\| A_{r, \mu_c} \|_{L^p \to L^p} \| f \|_{L^p}}{\alpha} \right)^p. \tag{11}
\]

Set \( f = 1_{\{y\}} \); then \( \| f \|_{L^p(\mu_c)} = c^{1/p} \). For \( 1 \leq i \leq J \), we have \( A_{r, \mu_c} f(x_i) = c/(1+c) \).
Thus, \( \mu_c(\{ A_{r, \mu_c} f \geq c/(1 + c) \}) = J \), so with \( \alpha = c/(1 + c) \), we have
\[
J^{1/p} \leq \frac{\| A_{r, \mu_c} \|_{L^p \to L^p} \| (1 + c) }{c^{1/q}}. \tag{12}
\]

As before, maximizing \( g(c) = c^{1/q}/(1 + c) \) we get \( c = p - 1 \) and \( g(p - 1) = (p - 1)^{(p-1)/p} p^{-1} \), so
\[
N = \frac{(p - 1)^{(p-1)/p} (p^{p-1}(p - 1)^{(1-p)p})^{1/p}}{p} < \frac{(p - 1)^{(p-1)/p} J^{1/p}}{p} \tag{13}
\]

\( \Box \)

In what follows we take balls to be closed. Unlike the case of the Heisenberg groups (where we have \( E(X, d) < \infty \) and \( L(X, d) = \infty \) in Banach spaces the
equality \( E(X, d) = L(X, d) \) always holds.

**Theorem 4.12** If \((X, \| \cdot \|)\) is a Banach space, then \( E(X, \| \cdot \|) = L(X, \| \cdot \|) \).

**Proof** It suffices to show that \( E(X, \| \cdot \|) \geq L(X, \| \cdot \|) \). Both \( E(X, \| \cdot \|) \) and
\( L(X, \| \cdot \|) \) are defined as suprema, so it is enough to prove that given any finite,
intersecting Besicovitch family \( C := \{ B^{cl}(x_1, r_1), \ldots, B^{cl}(x_n, r_n) \} \), we can produce
an equal radius intersecting Besicovitch family of the same cardinality. Choose
\( y \in \cap C \); by a translation and a dilation, we may assume that \( y = 0 \) and \( r = 1 \).
By repeating the argument from the proof of Theorem 3.3, we see that \( C' := \{ B^{cl}(x_1/\|x_1\|, 1), \ldots, B^{cl}(x_n/\|x_n\|, 1) \} \) is the desired equal radius Besicovitch family.

As is the case with the maximal operator, in \( \mathbb{R}^d \) it is possible to construct a measure
\( \mu \) for which the supremum is attained, with \( r = 1 \). We omit the proof.

**Theorem 4.13** Let \( \| \cdot \| \) be any norm on \( \mathbb{R}^d \). Then there exists a discrete measure \( \mu \)
such that \( \| A^{cl}_{1, \mu} \|_{L^1(\mu) \to L^1(\mu)} = E(\mathbb{R}^d, \| \cdot \|) \).

The equality \( E(X, \| \cdot \|) = L(X, \| \cdot \|) \) allows one to transfer uniform bounds known
for the centered maximal operator to uniform bounds for the averaging operators.

**Remark 4.14** It is obvious that \( E(\mathbb{R}, \| \cdot \|) = 2 \), so \( \sup_{r, \mu} \| A_{r, \mu} \|_{L^1(\mu) \to L^1(\mu)} = 2 \) on \( \mathbb{R} \).
The special case \( \| A_{r, \mu} \|_{L^1(\mu) \to L^1(\mu)} \leq 2 \) for \( \mu \) the standard exponential distribution,
given by \( d\mu(t) = 1_{(0, \infty)}(t) e^{-t} dt \), had already been proven in [4, Theorem 4.2].
In higher dimensions, from Corollaries 3.4, 3.5, and 3.6, we obtain the following

**Corollary 4.15** Given any norm \( \| \cdot \| \) on the plane, if the unit ball is a parallelogram then \( \sup_{r, \mu} \| A_{r, \mu} \|_{L^1(\mu) \to L^1(\mu)} = 4 \), while \( \sup_{r, \mu} \| A_{r, \mu} \|_{L^1(\mu) \to L^1(\mu)} = 5 \) in every other case.

With balls defined using the \( \ell_\infty \) norm, the sharp uniform bound for \( \sup_{r, \mu} \| A_{r, \mu} \|_{L^1(\mu) \to L^1(\mu)} \) on \( (\mathbb{R}^d, \| \cdot \|_\infty) \) is \( 2^d \).

For the euclidean norm we have \( \sup_{r, \mu} \| A_{r, \mu} \|_{L^1(\mu) \to L^1(\mu)} = 12 \) in dimension 3.

Asymptotically, in dimension \( d \) the following bounds hold:

\[
(1 + o(1)) \sqrt{\frac{3\pi}{8}} \log \frac{3}{2\sqrt{2}} d^{3/2} \left( \frac{2}{\sqrt{3}} \right)^d \leq \sup_{r, \mu} \| A_{r, \mu} \|_{L^1(\mu) \to L^1(\mu)} \leq 2^{0.401(1+o(1))d}.
\]

(14)

**Remark 4.16** For \( \mathbb{R}^d \) with the euclidean norm and the standard gaussian measure \( \gamma \), it was shown in [4, Theorem 4.3] that \( \sup_{r>0} \| A_r \|_{L^1(\gamma) \to L^1(\gamma)} \leq (2 + \varepsilon)^d \), whenever \( \varepsilon > 0 \) and \( d \) is large enough. Note that the upper bounds from the preceding result (valid for all measures) represent a substantial improvement.

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