MOD 2 MORAVA K-THEORY FOR FROBENIUS COMPLEMENTS OF EXPONENT DIVIDING $2^n \cdot 9$

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Abstract. We determine the cohomology rings $K(s)^*(BG)$ at 2 for all finite Frobenius complements $G$ of exponent dividing $2^n \cdot 9$.

Let $V$ be an abelian group, and let $G$ be a group of automorphisms of $V$. If $G$ has exponent $2^n \cdot 3^k$ for $0 \leq n$ and $0 \leq k \leq 2$ and $G$ acts freely on $V$, then $G$ is finite (see [6] Theorem 1.1). Every finite group that acts freely on an abelian group is isomorphic to a Frobenius complement in some finite Frobenius group (see [6] Lemma 2.6). By the classification of finite Frobenius complements (see [7]) the quotient of $G$ by its maximal normal 3-subgroup $H$ is isomorphic to a cyclic 2-group $C$, a generalized quaternion group $Q$, the binary tetrahedral group $2^2 T$ of order 24 (or SL(2,3)), or the binary octahedral group $2^2 O$ of order 48. Then Atiyah-Hirzebruch-Serre spectral sequence for $H \triangleleft G$ implies that at 2 the ring $K(s)^*(BG)$ is isomorphic to $K(s)^*(BK)$, for $K = G/H$ is either $C$, $Q$, $2^2 T$, $2^2 O$. For the cyclic group $C = Z/2^k$, $K(s)^*(BZ/2^k) = F_2[v_s, v_s^{-1}][u]/(u^{2^{k+1}})$. For the generalized quaternion group $Q_{2^{m+2}}$ we have Theorem 1.1 of [4]. We deduce Morava K-theory rings at 2 for the groups $2^2 T$ and $2^2 O$ as certain subgroups in $K(s)^*(BQ_8)$ and $K(s)^*(BQ_{16})$ respectively (Proposition 1.1 and Proposition 1.2).

In [3] we proved the following formula for the first Chern class of the transferred line complex bundle: Let $X \to Y$ be the regular two covering defined by free action of $Z/2$ on $X$ and let $\theta \to Y$ be the associated line complex bundle; Let $\xi \to X$ be a complex line bundle and let $\zeta \to Y$ be the plane bundle, transferred from $\xi$ by Atiyah transfer [2]. Then for $Tr^* : K(s)^*(X) \to K(s)^*(Y)$, the transfer homomorphism [1] for our covering $X \to Y$, one has

\[ Tr^*(c_1(\xi)) = c_1(\theta) + c_1(\zeta) + v_s \sum_{i=1}^{s-1} c_1(\theta)2^{s-2-2^i} c_2(\zeta)2^{s-2^i-1}. \]

We show that formula (0.1) plays major role in the ring structure $K(s)^*(BG)$ at 2 for aforementioned groups and gives another derivations for some related rank one Lie groups.

Much of our note is written in terms of Theorem 1.1 of [4]. Let

\[ G = \langle a, b | a^{2m+1} = 1, \; b^2 = a^e, bab^{-1} = a^r \rangle, \quad m \geq 1 \]

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and either $e = 0$, $r = -1$ (the dihedral group $D_{2m+2}$ of order $2^{m+2}$), $e = 2m$, $r = -1$ (the generalized quaternion group $Q_{2m+2}$) or $m \geq 2$, $e = 0$, $r = 2^m - 1$ (the semidihedral group $SD_{2m+2}$).

Spectral sequence consideration (see [3]) imply that $K(s)(BG)$ is generated by following Chern classes $|c| = |x| = 2$, $|c_2| = 4$:

$$c = c_1(\eta_1), \quad \eta_1 : G/\langle a \rangle \cong \mathbb{Z}/2 \rightarrow \mathbb{C}^*, \quad b \mapsto -1;$$

$$x = c_1(\eta_2), \quad \eta_2 : G/\langle a^2, b \rangle \cong \mathbb{Z}/2 \rightarrow \mathbb{C}^*, \quad a \mapsto -1;$$

and $c_2 = c_2(\xi_{\pi_1})$, where $\xi_{\pi_1} \rightarrow B(\eta)$ is the plane bundle transferred from the canonical line bundle $\xi \rightarrow B(\eta)$, for the double covering $\pi_1 : B(\eta) \rightarrow B(\eta)$ corresponding to $\eta_1$.

The ring structure is the result of the formula for transferred first Chern class [4]. See [4].

Let $N$ be the normalizer of $U(1)$ in $S^3$. The normalizes of the maximal torus in $SO(3)$ is $O(2) = U(1) \times \mathbb{Z}/2$ and $\mathbb{Z}/2$ acts on $K(s)^*BU(1) = K(s)^*[\langle u \rangle]$ by $[-1]_F(u)$ as above.

Since $BU(1) \rightarrow [colim_m B\mathbb{Z}/(p^s)]$, we have $K(s)^*(BO(2)) = K(s)^*(\lim_m (BD_{2m+2})) = K(s)^*(\lim_m (BSD_{2m+2}))$ and $K(s)^*(BN) = K(s)^*(\lim_m (BQ_{2m+2}))$.

Thus Theorem 1.1 of [4] implies

Corollary 0.1. $K(s)^*(BO(2)) = K(s)^*[c, c_2]/(c^{2^r} + v_s c \sum_{i=1}^s c^{2^{r-2^i}} c_2^{2^i-1})$, where $c = c_1(\det \eta)$ and $c_2 = c_2(\eta)$ are the Chern classes of the bundle $\eta \rightarrow BO(2)$, the complexification of canonical $O(2)$ bundle.

Corollary 0.2. $K(s)^*(BN) = K^*(s)[c, c_2]/(c^{2^r} + v_s c \sum_{i=1}^s c^{2^{r-2^i}} c_2^{2^i-1})$, where $c = c_1(\nu)$ is the Chern class of $\nu$ the pullback bundle of the canonical real line bundle by $N \rightarrow N/U(1) = \mathbb{Z}/2$ and $c_2 = c_2(p^*(\xi))$ is the Euler class of the pullback bundle of the canonical quaternionic line bundle by the inclusion $N \subset S^3$.

Then $RP^2 \rightarrow BO(2) \rightarrow BO(3)$ is the projective bundle of the canonical $SO(3)$ bundle. Hence the pullback of the complexification of this canonical $SO(3)$ bundle splits over $BO(2)$ as $\eta \oplus \det \eta$. Note that $c_1(\det \eta) = c_1(\eta) + v_s c_2(\eta c^{-1})$ modulo transfer for the covering $BU(1) \rightarrow BO(2)$. Thus $K(s)^*(BSO(3))$ is subring in $K(s)^*(BO(2))$ generated by $v = c^2 + v_s c_2^2 + c_2$ and $w = c c_2$. This implies

Corollary 0.3. $K(s)^*(BSO(3)) = K(s)^*[v, w][f_s(v, w), g_s(v, w)]$, where $|v| = 4$, $|w| = 6$, and $f_s = f_s(v, w)$, $g_s = g_s(v, w)$ are determined by $f_2 = vw$, $g_2 = w^2$ and for $s > 2$

$$f_s = \begin{cases} f_{s-1}^2 + w v^{2^{s-1} - 1} & \text{if } s \text{ even}, \\ f_{s-1}^2 v + w v^{2^{s-1} - 1} & \text{if } s \text{ odd}, \end{cases}$$

$$g_s = \begin{cases} g_{s-1}^2 + w v^{2^{s-1} - 1} & \text{if } s \text{ odd}, \\ g_{s-1}^2 v + w v^{2^{s-1} - 1} & \text{if } s \text{ even}. \end{cases}$$

Our main result is the following.
Let $\mathcal{G}$ be a group acting freely on an abelian group. Let $\mathcal{G}$ be of exponent dividing $2^n - 9$ (hence $\mathcal{G}$ is necessarily finite, as above) and let $\mathcal{H} < \mathcal{G}$ be the maximal normal 3-subgroup.

**Theorem 0.4.** As a ring $K(s)^*(B\mathcal{G})$ has one of the following forms

(i) If $\mathcal{G}/\mathcal{H} = Q_8$, then $K(s)^*(B\mathcal{G}) = K(s)^*[c, x, c_2]/R$ and the relations $R$ are determined by

$$c^{2^r} = x^{2^r} = 0, \quad v_s c^{2^r} = v_s \sum_{i=1}^{s-1} c^{2^{r-2^i+1}} c_i + c^2, \quad v_s c^{2^r} x = c^2 + cx + x^2,$$

$$v_s c^{2^r} x = v_s \sum_{i=1}^{s-1} x^{2^{r-2^i+1} + c_i} + x^2.$$

(ii) If $\mathcal{G}/\mathcal{H} = Q_{2m+2}$, $m > 1$, then $K(s)^*(B\mathcal{G}) = K(s)^*[c, x, c_2]/R$, and the relations $R$ are determined by

$$c^{2^r} = x^{2^r} = 0, \quad v_s c^{2^r} = v_s \sum_{i=1}^{s-1} c^{2^{r-2^i+1}} c_i + c^2, \quad v_s c^{2^r} x = c^2 + cx + x^2,$$

$$v_s c^{2^r} x = v_s \sum_{i=1}^{s-1} x^{2^{r-2^i+1} + c_i} + \sum_{i=1}^{m_s} v_s 1 + v_s^2 c^2 + v_s^{2^m} c^{2^{m+1}} c^{2^{m+1}} c^{2^{m+1}} = cx + x^2,$$

where $\kappa(m) = \frac{2^{m+1}}{2^{m+1}-1}$.

(iii) If $\mathcal{G}/\mathcal{H} = \mathbb{Z}/2^T$, then $K(s)^*(B\mathcal{G}) = K(s)^*[c_2]/c_2^{(2^{r+1})2^{r-1}}$.

(iv) If $\mathcal{G}/\mathcal{H} = \mathbb{Z}/2^T$, then

$$K(s)^*(B\mathcal{G}) = K(s)^*[c, c_2]/c_2^{2^{r+1}} + v_s c_2 \sum_{i=1}^{s-1} c^{2^{r-2^i+1} c_i + 1}.$$  

(v) If $\mathcal{G}/\mathcal{H} = \mathbb{Z}/2^k$, then $K(s)^*(B\mathcal{G}) = K(s)^*[c]/c^2 x$.

Here in all cases $|c| = |x| = 2$.

The statement (v) is clear. (i) and (ii) follow from Theorem 1.1 of [4] for $Q_8$ and $Q_{2m+2}$ respectively. What remains is to consider the cases of binary tetrahedral and binary octahedral groups.

1. **Binary Polyhedral groups**

As it is known any finite subgroup of $SO(3)$ is either a cyclic group, a dihedral group or one of the groups of a Platonic solid: tetrahedral group $T \cong A_4$, cube/octahedral group $O \cong S_4$, or icosahedral group $I \cong A_5$. We consider the preimages of the latter groups under the covering homomorphism $S^3 \to SO(3)$.

**Binary tetrahedral group.** Binary tetrahedral group $2T$ as the group of 24 units in the ring of Hurwitz integers $2T$ is given by $\{ \pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k) \}$.

This group can be written as a semidirect product $2T = Q_8 \rtimes \mathbb{Z}/3$, where $Q_8$ is the quaternion group consisting of the 8 Lipschitz units $\pm 1, \pm i, \pm j, \pm k$ and $\mathbb{Z}/3$ is the cyclic group generated by $-\frac{1}{2}(1 + i + j + k)$. The cyclic group acts on the normal subgroup $Q_8$ by conjugation. So that the generator of $\mathbb{Z}/3$ cyclically rotates $i, j, k$.

Consider now Morava $K$-theory at 2. Then relations of Theorem 1.1 of [4] for $K(s)^*(BQ_8)$ imply that its subring of invariants under $\mathbb{Z}/3$ action is generated by $c_2$: the generator of $\mathbb{Z}/3$ cyclically rotates $c, x$ and $c + x + v_s c^{2^{r-1}} x^{2^{r-1}}$. If ignoring the powers of $v_s$, then the first and second elementary symmetric functions in these three symbols are equal to $c^{2^{r-1}}$ and $c^2$ respectively and the third is zero. It follows that $K(s)^*(B2T) \cong [K(s)^*(BQ_8)]^{\mathbb{Z}/3}$. 


Proposition 1.1. $K(s)^*(B2T) \cong K(s)^*[c_2]/c_2^{(2^r+1)2^r-1}$, where $|c_2| = 4$.

Binary octahedral group $2O$. This group is given as the union of the 24 Hurwitz units $\{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$ with all 24 quaternions obtained from \(\frac{1}{\sqrt{2}}(\pm 1 \pm i \pm j \pm k)\) by permutation of coordinates.

The generalized quaternion group $Q_{16}$ forms a subgroup of $2O$ and its conjugacy classes has 3 members. Therefore by the transfer argument $B2O$ is a stable wedge summand of $BQ_{16}$ after localized at 2, meaning $K(s)^*(B2O)$ is the subring in $K(s)^*(BQ_{16})$ at 2. We show that this is the subring generated by two symbols $c$ and $c_2$ of Theorem 1.1 of [4]. Namely one has

Proposition 1.2. $K(s)^*(B2O)$ is isomorphic to

$$K(s)^*[c, c_2]/(c^{2^r}, c^{2^r}, \sum_{i=1}^{s} c^{2^r-2i^2} c_2^{(2^r+1)2^r-1})$$

where $|c| = 2$, $|c_2| = 4$.

Binary icosahedral group. $2I$ is given as the union of the 24 Hutwitz units $\{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$ with all 96 quaternions obtained from \(\frac{1}{2}(0 \pm 1 \pm i \mp \nu^{-1} j \pm \nu k)\) by even permutation of coordinates. Here $\nu = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio. This group is isomorphic to $SL_2(5)$-the group of all $2 \times 2$ matrices over $\mathbb{F}_5$ with unit determinant.

Among other subgroups the relevant subgroup is the binary tetrahedral group formed by Hurwitz units. Then coset $2I/2O$ has 5 members hence by the transfer argument again $B2I$ splits of $B2O$ after localized at 2. Thus we obtain

$$K(s)^* B(2I) \cong K(s)^* B(2T).$$

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