LETTERPLACE AND CO-LETTERPLACE IDEALS OF POSETS

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Abstract. We study two types of monomial ideals associated to a poset $P$, the letterplace ideals $L(n, P)$ and co-letterplace ideals $L(P, n)$. By cutting down these ideals, or subideals of these, by regular sequences of variable differences, we obtain many ideals studied recently in the literature: multichain ideals and generalized Hibi type ideals, initial ideals of determinantal ideals, strongly stable ideals, $d$-partite $d$-uniform ideals, Ferrers ideals, and uniform face ideals.

Introduction

In [7] V.Ene, F.Mohammadi, and the third author introduced multichain ideals and generalized ideals of Hibi type associated to a partially ordered set $P$, and showed several fundamental results concerning these ideals.

Here we consider two basic classes of these ideals which we call letterplace and co-letterplace ideals of the poset $P$. We show that these ideals are rather fundamental and give a unified understanding of many ideals studied in monomial ideal theory in recent years. Classes of ideals which derive from these ideals are: multichain ideals and generalized Hibi ideals, initial ideals of determinantal ideals and of ladder determinantal ideals, strongly stable ideals and $d$-uniform $d$-partite hypergraph ideals, Ferrers ideals, edge ideals of cointerval $d$-hypergraphs, and uniform face ideals.

The $n$'th letterplace ideal of the poset $P$, written $L(n, P)$, is the monomial ideal generated by monomials

\[ x_{1,p_1}x_{2,p_2}\cdots x_{n,p_n} \]

where $p_1 \leq p_2 \leq \cdots \leq p_n$ is a weakly increasing sequence of elements from $P$. By [7 Thm. 2.4] this is a Cohen-Macaulay ideal.

Date: January 20, 2015.
2000 Mathematics Subject Classification. Primary: 13F20, 05E40; Secondary: 06A11.
Its Alexander dual, the \( n \)’th co-letterplace ideal \( L(P, n) \), is the monomial ideal generated by

\[
\prod_{p \in P} x_{i_p, p}
\]

where \( 1 \leq i_p \leq n \) and \( p < q \) implies \( i_p \leq i_q \). Such ideals have linear resolution.

Let \( S \) be the ambient polynomial ring of these ideals. A basic idea we introduce is to cut the associated rings \( S/L(n, P) \) and \( S/L(P, n) \) down by a sequence consisting of variable differences \( x_{i_p} - x_{i_q} \). The resulting quotient rings will also be defined by monomial ideals. The question is: When do such differences form a regular sequence? Significant sufficient criteria for such a sequence to be regular are given in Theorems 2.1, 2.2. This implies that the quotient ring has the same graded Betti numbers as the original ring.

Another key idea, to gain greater flexibility, is that we consider subideals \( L \subseteq L(P, n) \) generated by a subset of the monomials in (1). For notable classes of such subsets, the ideal \( L \) will also have linear resolution. Theorem 5.10 gives a sufficient criterion for a sequence of variable differences to be a regular sequence for the associated ring \( S/L \).

The resulting quotient ideals, when we divide \( S/L \) and \( S/L(n, P) \) out by various regular sequences, give the wealth of ideals mentioned in the second paragraph above.

The organization of the paper is as follows. In Section 1 we define ideals \( L(Q, P) \) associated to pairs of posets \( Q \) and \( P \). In particular for the totally ordered poset \([n]\) on \( n \) elements, we introduce the letter-place ideals \( L([n], P) \) and co-letterplace ideals \( L(P, [n]) \). We investigate how they behave under Alexander duality. In Section 2 we study when a sequence of variable differences is regular for letter-place and co-letterplace ideals. Section 3 gives classes of ideals, including multichain ideals and initial ideals of determinantal ideals, which are quotients of letterplace ideals by a regular sequence. Section 4 describes in more detail the generators and facets of various letterplace and co-letterplace ideals. Section 5 considers poset ideals \( J \) in \( \text{Hom}(P, [n]) \) and the associated co-letterplace ideal \( L(J) \). We show it has linear resolution, and compute its Alexander dual. Section 6 gives classes of ideals which are quotients of co-letterplace ideals by a regular sequence. This includes strongly stable ideals, \( d \)-uniform \( d \)-partite hypergraph ideals, Ferrers ideals, and uniform face ideals. The last sections 7 and 8 contain proofs of basic results in this paper on when sequences of variable differences
are regular, and how Alexander duality behaves when cutting down by such a regular sequence.

1. LETTERPLACE IDEALS AND THEIR ALExANDER DUALS

If $P$ is a partially ordered set (poset), a poset ideal $J \subseteq P$ is a subset of $P$ such that $p \in P$ and $q \in J$ with $p \leq q$, implies $p \in J$. The term order ideal is also much used in the literature for this notion. If $S$ is a subset of $P$, the poset ideal generated by $S$ is the set of all elements $p \in P$ such that $p \leq s$ for some $s \in S$.

1.1. Isotone maps. Let $P$ and $Q$ be two partially ordered sets. A map $\phi : Q \rightarrow P$ is isotone or order preserving, if $q \leq q'$ implies $\phi(q) \leq \phi(q')$. The set of isotone maps is denoted Hom($Q, P$). It is actually again a partially ordered set with $\phi \leq \psi$ if $\phi(q) \leq \psi(q)$ for all $q \in Q$. The following will be useful.

Lemma 1.1. If $P$ is a finite partially ordered set with a unique maximal or minimal element, then an isotone map $\phi : P \rightarrow P$ has a fix point.

Proof. We show this in case $P$ has a unique minimal element $p = p_0$. Then $p_1 = \phi(p_0)$ is $\geq p_0$. If $p_1 > p_0$, let $p_2 = \phi(p_1) \geq \phi(p_0) = p_1$. If $p_2 > p_1$ we continue. Since $P$ is finite, at some stage $p_n = p_{n-1}$ and since $p_n = \phi(p_{n-1})$, the element $p_{n-1}$ is a fix point. □

1.2. Alexander duality. Let $k$ be a field. If $R$ is a set, denote by $k[x_R]$ the polynomial ring in the variables $x_r$ where $r$ ranges over $R$. If $A$ is a subset of $R$ denote by $m_A$ the monomial $\Pi_{a \in A}x_a$.

Let $I$ be a squarefree ideal in a polynomial ring $k[x_R]$, i.e. its generators are monomials of the type $m_A$. It corresponds to a simplicial complex $\Delta$ on the vertex set $R$, consisting of all $S \subseteq R$, called faces of $\Delta$, such that $m_S \not\in I$.

The Alexander dual $J$ of $I$, written $J = I^A$, may be defined in different ways. Three definitions are the following.

1. The Alexander dual $J$ is the monomial ideal whose monomials are those in $k[x_R]$ with nontrivial common divisor with every monomial in $I$.

2. The Alexander dual $J$ is the ideal generated by all monomials $m_S$ where the $S$ are complements in $R$ of faces of $\Delta$.

3. If $I = \bigcap_{i=1}^r p_i$ is a decomposition into prime monomial ideals $p_i$ where $p_i$ is generated by the variables $x_a$ as a ranges over the subset $A_i$ of $R$, then $J$ is the ideal generated by the monomials $m_{A_i}$, $i = 1, \ldots, r$. (If the decomposition is a minimal primary decomposition, the $m_{A_i}$ is a minimal generating set of $J$.)
1.3. Ideals from Hom-posets. To an isotone map \( \phi : Q \to P \) we associate its graph \( \Gamma \phi \subseteq Q \times P \) where \( \Gamma \phi = \{(q, \phi(q)) \mid q \in Q\} \). As \( \phi \) ranges over Hom\((Q, P)\), the monomials \( m_{\Gamma \phi} \) generate a monomial ideal in \( \mathbb{k}[x_{Q \times P}] \) which we denote by \( L(Q, P) \). More generally, if \( S \) is a subset of Hom\((Q, P)\) we get ideals \( L(S) \) generated by \( m_{\Gamma \phi} \) where \( \phi \in S \).

If \( R \) is a subset of the product \( Q \times P \), we denote by \( R^\tau \) the subset of \( P \times Q \) we get by switching coordinates. As \( L(Q, P) \) is an ideal in \( \mathbb{k}[x_{Q \times P}] \), we may also consider it as an ideal in \( \mathbb{k}[x_{P \times Q}] \). In cases where we need to be precise about this, we write it then as \( L(Q, P)^\tau \).

If \( Q \) is the totally ordered poset on \( n \) elements \( Q = [n] = \{1 < 2 < \cdots < n\} \), we call \( L([n], P) \), written simply \( L(n, P) \), the \( n \)th letterplace ideal of \( P \). It is generated by the monomials

\[
x_{1,p_1}x_{2,p_2}\cdots x_{n,p_n} \text{ with } p_1 \leq p_2 \leq \cdots \leq p_n.
\]

This is precisely the same ideal as the multichain ideal \( I_{n,n}(P) \) defined in [7] (but with indices switched). The ideal \( L(P, [n]) \), written simply \( L(P,n) \), is the \( n \)th co-letterplace ideal of \( P \). In [7] it is denoted \( H_n(P) \) and is called a generalized Hibi type ideal.

The following is Theorem 1.1(a) in [7], suitably reformulated. Since it is a very basic fact, we include a proof of it.

**Proposition 1.2.** The ideals \( L(n, P) \) and \( L(P,n)^\tau \) are Alexander dual in \( \mathbb{k}[x_{[n] \times P}] \).

**Proof.** Let \( L(n, P)^A \) be the Alexander dual of \( L(n, P) \). First we show \( L(P,n) \subseteq L(n, P)^A \). This is equivalent to: For any \( \phi \in \text{Hom}([n], P) \) and any \( \psi \in \text{Hom}(P, [n]) \), the graphs \( \Gamma \phi \) and \( \Gamma \psi^\tau \) intersect in \( [n] \times P \).

Let \( i \) be a fix point for \( \psi \circ \phi \). Then \( i \xrightarrow{\phi} p \xrightarrow{\psi} i \) and so \((i, p)\) is in both \( \Gamma \phi \) and \( \Gamma \psi^\tau \).

Secondly, given a squarefree monomial \( m \) in \( L(n, P)^A \) we show that it is divisible by a monomial in \( L(P,n) \). This will show that \( L(n, P)^A \subseteq L(P,n) \) and force equality here. So let the monomial \( m \) correspond to the subset \( F \) of \( P \times [n] \). It intersects all graphs \( \Gamma \phi^\tau \) where \( \phi \in \text{Hom}([n], P) \). We must show it contains a graph \( \Gamma \psi \) where \( \psi \in \text{Hom}(P, [n]) \).

Given \( F \), let \( \mathcal{J}_n = P \) and let \( \mathcal{J}_{n-1} \) be the poset ideal of \( P \) generated by all \( p \in \mathcal{J}_n = P \) such that \((p,n) \notin F \). Inductively let \( \mathcal{J}_{i-1} \) be the poset ideal in \( \mathcal{J}_i \) generated by all \( p \) in \( \mathcal{J}_i \) with \((p,i) \) not in \( F \).

**Claim 1.** \( \mathcal{J}_0 = \emptyset \).

**Proof.** Otherwise let \( p \in \mathcal{J}_0 \). Then there is \( p \leq p_1 \) with \( p_1 \in \mathcal{J}_1 \) and \((p_1,1) \notin F \). Since \( p_1 \in \mathcal{J}_1 \) there is \( p_1 \leq p_2 \) with \( p_2 \in \mathcal{J}_2 \) such that \((p_2,2) \notin F \). We may continue this and get a chain \( p_1 \leq p_2 \leq \cdots \leq p_n \).
with \((p_i, i)\) not in \(F\). But this contradicts \(F\) intersecting all graphs \(\Gamma\phi\) where \(\phi \in \text{Hom}([n], P)\).

We thus get a filtration of poset ideals
\[
\emptyset = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_{n-1} \subseteq \mathcal{J}_n = P.
\]
This filtration corresponds to an isotone map \(\psi : P \to [n]\).

**Claim 2.** \(\Gamma\psi\) is a subset of \(F\).

**Proof.** Let \((p, i) \in \Gamma\psi\). Then \(p \in \mathcal{J}_i \setminus \mathcal{J}_{i-1}\) and so \(p \not\in \mathcal{J}_{i-1}\). Thus \((p, i) \in F\). □

**Remark 1.3.** The case \(n = 2\) was shown in [12] where the ideal \(H_P\) generated by \(\prod_{p \in J} x_p \prod_{q \in P \setminus J} y_q\) as \(J\) varies over the poset ideals in \(P\), was shown to be Alexander dual to the ideal generated by \(x_p y_q\) where \(p \leq q\).

**Remark 1.4.** That \(L(m,n)\) and \(L(n,m)\) are Alexander dual is Proposition 4.5 of [9]. There the elements of these ideals are interpreted as paths in a \(m \times n\) matrix with generic linear forms \((x_{ij})\) and the generators of the ideals are the products of the variables in these paths.

1.4. **Alexander dual of** \(L(Q, P)\). In general \(L(Q, P)\) and \(L(P, Q)\) are not Alexander dual. This is easily checked if for instance \(Q\) and \(P\) are antichains of sizes \(\geq 2\). However we have the following.

**Proposition 1.5.** Suppose \(Q\) has a unique maximal or minimal element. The least degree of a generator of the Alexander dual \(L(Q, P)^A\) and of \(L(P, Q)\) are both \(d = |P|\) and the degree \(d\) parts of these ideals are equal. In particular, since \(L(P, Q)\) is generated in this degree \(d\), it is contained in \(L(Q, P)^A\).

Note that the above is equivalent to say that the minimal primes of \(L(Q, P)\) of height \(\leq |P|\) are precisely the
\[
p_\psi = (\{x_{\psi(p), p} \mid p \in P\}), \quad \text{where } \psi \in \text{Hom}(P, Q).
\]

**Proof.** We show that:
1. \(L(Q, P) \subset p_\psi\) for all \(\psi \in \text{Hom}(P, Q)\).
2. \(p_\psi\) is a minimal prime of \(L(Q, P)\).
3. Any minimal prime \(p\) of \(L(Q, P)\) is \(p = p_\psi\) for some \(\psi\).

This will prove the proposition.
1. Given \(\phi \in \text{Hom}(Q, P)\) and \(\psi \in \text{Hom}(P, Q)\). We have to show that \(m_\phi = \prod_{q \in Q} x_{q, \phi(q)} \in p_\psi\). By Lemma 1.1 \(\psi \circ \phi\) has a fix point \(q\), and
let \( p = \phi(q) \). Then \( \psi(p) = q \). Therefore, \( x_{q,p} \) is a factor of \( m_\phi \) and a generator of \( p_\psi \). This implies that \( m_\phi \in p_\psi \).

2. Next we show that \( p_\psi \) is a minimal prime ideal of \( L(Q, P) \). Suppose this is not the case. Then we may skip one of its generators, say \( x_{\psi(p),p} \), to obtain the prime ideal \( p \subset p_\psi \) with \( L(Q, P) \subset p \). Let \( \phi \in \text{Hom}(Q, P) \) be the constant isotone map with \( \phi(q) = p \) for all \( q \in Q \). Then \( m_\phi = \prod_{q \in Q} x_{q,p} \in L(Q, P) \). Since no factor of \( m_\phi \) is divisible by a generator of \( p \), it follows that \( L(Q, P) \) is not contained in \( p \), a contradiction.

3. Now let \( p \) be any minimal prime ideal of \( L(Q, P) \). Since \( L(Q, P) \subset p \) it follows as in the previous paragraph that for each \( p \in P \) there exists an element \( \psi(p) \in Q \) such that \( x_{\psi(p),p} \in p \). This show that height \( L(Q, P) = |P| \). Assume now that height \( p = |P| \). Then \( p = \langle \{x_{\psi(p),p} \mid p \in P\} \rangle \). It remains to be shown that \( \psi P \to Q \) is isotone. Suppose this is not the case. Then there exist \( p, p' \in P \) such that \( p < p' \) and \( \psi(p) \not\leq \psi(p') \). Let \( \phi : Q \to P \) the map with \( \phi(q) = p \) if \( q \leq \psi(p') \) and \( \phi(q) = p' \) if \( q \not\leq \psi(p') \). Then \( \phi \) is isotone, and it follow that \( m_\phi = \prod_{q \leq \psi(p')} x_{q,p} \prod_{q \not\leq \psi(p')} x_{q,p'} \) does not belong to \( p \), a contradiction. \( \square \)

2. Quotients of letterplace ideals

A chain \( c \) in the product of two posets \( Q \times P \) is said to be left strict if for two elements in the chain, \( (q,p) < (q',p') \) implies \( q < q' \). Analogously we define right strict. The chain is bistriect if it is both left and right strict.

An isotone map of posets \( \phi : Q \times P \to R \) is said to have left strict chain fibers if all its fibers \( \phi^{-1}(r) \) are left strict chains in \( Q \times P^{op} \). Here \( P^{op} \) is the opposite poset of \( P \), i.e. \( p \leq^{op} p' \in P^{op} \) iff \( p' \leq p \in P \).

The map \( \phi \) gives a map of linear spaces \( \phi_1 : \langle x_{Q \times P} \rangle \to \langle x_R \rangle \) (the brackets here mean the \( k \)-vector space spanned by the set of variables). The map \( \phi_1 \) induces a map of polynomial rings \( \hat{\phi} : k[x_{Q \times P}] \to k[x_R] \). In the following \( B \) denotes a basis for the kernel of the map of degree one forms \( \phi_1 \), consisting of differences \( x_{q,p} - x_{q',p'} \) with \( \phi(q,p) = \phi(q',p') \).

**Theorem 2.1.** Given an isotone map \( \phi : [n] \times P \to R \) which has left strict chain fibers. Then the basis \( B \) is a regular sequence of the ring \( k[x_{[n] \times P}] / L(n, P) \).

**Theorem 2.2.** Given an isotone map \( \psi : P \times [n] \to R \) which has left strict chain fibers. Then the basis \( B \) is a regular sequence of the ring \( k[x_{P \times [n]}] / L(P, n) \).

We shall prove these in Section 8. For now we note that they require distinct proofs, with the proof of Theorem 2.2 the most delicate.
In the setting of Theorem 2.1 we let $L^\phi(n, P)$ be the ideal generated by the image of the $n$’th letterplace ideal $L(n, P)$ in $k[x_R]$, and in the setting of Theorem 2.2 we let $L^\psi(P, n)$ be the ideal generated by the image of the $n$’th co-letterplace ideal $L(P, n)$ in $k[x_R]$. Note that $L^\phi(n, P)$ is a squarefree ideal iff in the above the fibers $\phi^{-1}(r)$ are distict chains in $[n] \times P^{op}$, and similarly $L^\psi(P, n)$ is a squarefree ideal iff the fibers $\psi^{-1}(r)$ are distict chains in $P \times [n]^{op}$.

We get the following consequence of the above Theorems 2.1 and 2.2.

**Corollary 2.3.** The quotient rings $k[x_{[n]\times P}]/L(n, P)$ and $k[x_R]/L^\phi(n, P)$ have the same graded Betti numbers. Similarly for $k[x_{P\times [n]}]/L(P, n)$ and $k[x_R]/L^\psi(P, n)$.

**Proof.** We prove the first statement. Let $L^\im\phi(n, P)$ be the image of $L(n, P)$ in $k[x_{\im\phi}]$, and $S = R\setminus \im\phi$. Thus $k[x_{\im\phi}]/L^\im\phi(n, P)$ is a quotient of $k[x_{[n]\times P}]/L(n, P)$ by a regular sequence, and $k[x_R]/L^\phi(n, P)$ is $k[x_{\im\phi}]/L^\im\phi(n, P) \otimes k[x_S]$. □

For the poset $P$ consider the multichain ideal $I(n, P)$ in $k[x_P]$ generated by monomials $x_{p_1} x_{p_2} \cdots x_{p_n}$ where $p_1 \leq p_2 \leq \cdots \leq p_n$ is a multichain of length $n$ in $P$. The quotient $k[x_P]/I(n, P)$ is clearly artinian since $x_p^n$ is in $I(n, P)$ for every $p \in P$.

**Corollary 2.4.** The ring $k[x_P]/I(n, P)$ is an artinian reduction of $k[x_{[n]\times P}]/L(n, P)$ by a regular sequence. In particular $L(n, P)$ is a Cohen-Macaulay ideal. It is Gorenstein iff $P$ is an antichain.

**Proof.** The first part is because the map $[n] \times P \to P$ fulfills the criteria of Theorem 2.1 above. An artinian ideal is Gorenstein iff it is a complete intersection. Since all $x_p^n$ are in $I(n, P)$, this holds iff there are no more generators of $I(n, P)$, which means precisely that $P$ is an antichain. □

This recovers part of Theorem 2.4 of [7] showing that $L(n, P)$ is Cohen-Macaulay. The Gorenstein case above is Corollary 2.5 of [7]. Recall that a squarefree monomial ideal is bi-Cohen-Macaulay, [9], iff both the ideal and its Alexander dual are Cohen-Macaulay ideals.

**Corollary 2.5.** $L(n, P)$ is bi-Cohen-Macaulay iff $P$ is totally ordered.

**Proof.** Since $L(n, P)$ is Cohen-Macaulay, it is bi-Cohen-Macaulay iff it has a linear resolution, [5]. Equivalently $I(n, P)$ in $k[x_P]$ has a linear resolution. But since $I(n, P)$ gives an artinian quotient ring, this is equivalent to $I(n, P)$ being the $n$’th power of the maximal ideal. In this case every monomial $x_p^{n-1} x_q$ is in $I(n, P)$ and so every pair $p, q$ in $P$ is comparable. Thus $P$ is totally ordered. Conversely, if $P$ is
totally ordered, then clearly $I(n, P)$ is the $n$’th power of the maximal ideal.

Definition 2.6. Let $R' \to R$ be a surjective map of sets with $R'$ of cardinality one more than $R$, and let $r_1 \neq r_2$ in $R'$ map to the same element in $R$. Let $I$ be a monomial ideal in $k[x_R]$. A monomial ideal $J$ in $k[x_{R'}]$ is a separation of $I$ if i) $I$ is the image of $J$ by the natural map $k[x_{R'}] \to k[x_R]$, ii) $x_{r_1}$ occurs in some minimal generator of $J$ and similarly for $x_{r_2}$, and iii) $x_{r_1} - x_{r_2}$ is a regular element of $k[x_{R'}]/J$.

The ideal $I$ is separable if it has some separation $J$. Otherwise it is unseparable. If $J$ is obtained from $I$ by a succession of separations, we also call $J$ a separation of $I$. We say that $I$ is a regular quotient by variable differences of $J$, or simply a regular quotient of $J$. If $J$ is unseparable, then $J$ is a separated model for $I$.

This notion also occurs in [8] where unseparable monomial ideals are called maximal.

Lemma 2.7. Let $I$ be an ideal generated by a subset of the generators of $L(Q, P)$. Then $I$ is unseparable.

Proof. Let $R' \to Q \times P$ be a surjective map with $R'$ of cardinality one more than $Q \times P$. Suppose there is a monomial ideal $J$ in $k[x_{R'}]$ which is a separation of $I$. Let $a$ and $b$ in $R'$ both map to $(q, p)$. For any other element of $R'$, we identify it with its image in $Q \times P$. Suppose $m = x_{aq}m_0$ in $J$ maps to a generator $x_{aq}m_0$ of $L(Q, P)$, and $m' = x_{bm}m_1$ maps to another generator of $L(Q, P)$. Then $m_0$ does not contain a variable $x_{aq}$ with first index $q$, and similarly for $m_1$. Note that the least common multiple $m_{01}$ of $m_0$ and $m_1$ does not contain a variable with first index $q$. Hence $m_{01}$ is not in $L(Q, P)$ and so $m_{01}$ is not in $J$. But $(x_b - x_a)m_{01}$ is in $J$ since $x_bm_{01}$ and $x_am_{01}$ are in $J$. By the regularity of $x_b - x_a$ this implies $m_{01}$ in $J$, a contradiction.

As we shall see, many naturally occurring monomial ideals are separable and have separated models which are letterplace ideals $L(n, P)$ or are generated by a subset of the generators of co-letterplace ideals $L(P, n)$.

Remark 2.8. In [8] Section 2] the first author shows that the separated models of the squarefree power $(x_1, \ldots, x_n)^{n-1}$ are in bijection with trees on $n$ vertices.

We now consider the Alexander dual of $L^\phi(n, P)$.
Theorem 2.9. Let $\phi : [n] \times P \rightarrow R$ be an isotone map such that the fibers $\phi^{-1}(r)$ are bistrixt chains in $[n] \times P^{op}$. Then the ideals $L^\phi(n, P)$ and $L^{\phi^\tau}(P, n)$ are Alexander dual.

We prove this in Section 7.

Remark 2.10. The Alexander dual of the squarefree power in Remark 2.8 is the squarefree power $(x_1, \ldots, x_n)^{2}_{sq}$. Separations of this ideal are studied by H.Lohne, [15]. In particular he describes how the separated models are also in bijection with trees on $n$ vertices.

3. Examples of regular quotients of letterplace ideals

The ideals which originally inspired this paper are the multichain ideals of [7].

3.1. Multichain ideals. Let $P_m$ be $P \times [m]$ where $m \geq 1$. Consider the surjective map

$$[s] \times P_m \rightarrow P \times [m + s - 1]$$

$$(i, p, a) \mapsto (p, a + i - 1).$$

This map has left strict chain fibers. The image of $L(s, P_m)$ in $k[x_{P \times [m+s-1]}]$ is exactly the multichain ideal $I_{m+s-1,s}(P)$ of [7]. This is the ideal generated by monomials

$$x_{p_1,i_1}x_{p_2,i_2} \cdots x_{p_s,i_s}$$

where

$$p_1 \leq \cdots \leq p_s, \quad 1 \leq i_1 < \cdots < i_s \leq m + s - 1.$$ 

It is obtained from the $s$’th letterplace ideal $L(s, P_m) = L(s, P \times [m])$ by cutting down by a regular sequence. Thus we recover the fact, [7, Thm. 2.4], that these ideals are Cohen-Macaulay.

The Alexander dual of $L(s, P_m)$ is $L(P_m, s)$. An element $r$ of Hom($P \times [m], [s]$) may be represented by sequences

$$1 \leq r_{p1} \leq \cdots \leq r_{pm} \leq s$$

such that for each $p \leq q$ we have $r_{pq} \leq r_{aq}$. The element $r$ gives the monomial generator in $L(P_m, s)$

$$m_r = \prod_{p \in P} \prod_{i=1}^{m} x_{p,i,r_{pi}}.$$
By Theorem 2.9, the Alexander dual of the multichain ideal $I_{m+s-1,s}(P)$ is then generated by
\[
\prod_{p \in P} \prod_{i=1}^{m} x_{p,t_i} \bigcap_{1 \leq t_1 < t_2 < \cdots < t_m \leq m+s-1}
\]
(where $t_{pi} = r_{pi} + i - 1$) such that $p < q$ implies $t_{pj} \leq t_{qj}$. These are exactly the generators of the squarefree power ideal $L(P,s+m-1)_{m}$. This recovers Theorem 1.1(b) in [7].

3.2. Initial ideals of determinantal ideals: two minors. We now let $P = [n]$ and $s = 2$. Let $e, f \geq 0$. There are isotone maps
\[
[2] \times [n] \times [m] = [2] \times P_m \xrightarrow{\phi_{e,f}} [n + e] \times [m + f]
\]
\[
(1, a, b) \mapsto (a, b)
\]
\[
(2, a, b) \mapsto (a + e, b + f)
\]

These maps have left strict chain fibers and we get the ideal $L^{\phi_{e,f}}(2, P_m)$.

- When $(e, f) = (0, 1)$ we are in the situation of the previous Subsection 3.1 and we get the multichain ideal $I_{m+1,2}([n])$.
- When $(e, f) = (1, 0)$ we get the multichain ideal $I_{n+1,2}([m])$.
- When $(e, f) = (1, 1)$ we get the ideal in $k[x_{[n+1] \times [m+1]}]$ generated by monomials $x_{i,j}x_{i',j'}$ where $i < i'$ and $j < j'$. This is precisely the initial ideal $I$ of the ideal of two-minors of a generic $(n + 1) \times (m + 1)$ matrix of linear forms $(x_{i,j})$ with respect to a suitable monomial order with respect to a diagonal term order, [20].

In particular all of $I_{m+1,2}([n]), I_{n+1,2}([m])$ and $I$ have the same graded Betti numbers and the same $h$-vector, the same as $L(2, [n] \times [m])$.

Particularly noteworthy is the following: The ideal of two-minors of the generic $(n + 1) \times (m + 1)$ matrix is the homogeneous ideal of the Segre product of $\mathbb{P}^m \times \mathbb{P}^n$ in $\mathbb{P}^{nm+n+m}$. Since this Segre embedding is linearly normal, it is not the projection of a variety fully embedded in any higher dimensional projective space. Thus we cannot “lift” the ideal of two minors to an ideal in a polynomial ring with more variables than $(n + 1)(m + 1)$. However its initial ideal may be separated to the monomial ideal $L(2, [n] \times [m])$ with $2nm$ variables.

Varying $e$ and $f$, we get a whole family of ideals $L^{\phi_{e,f}}(2, [n] \times [m])$ with the same Betti numbers as the initial ideal of the ideal of two-minors. When $e = 0 = f$ we get an artinian reduction, not of the initial ideal of the ideal of two-minors, but of its separated model $L(2, [n] \times [m])$. When $e \geq n + 1$ and $f \geq m + 1$, the map $\phi_{e,f}$ is injective and $L^{\phi_{e,f}}(2, [n] \times [m])$
is isomorphic to the ideal generated by $L(2, [n] \times [m])$ in a polynomial ring with more variables.

3.3. Initial ideals of determinantal ideals: higher minors. We may generalize to arbitrary $s$ and two weakly increasing sequences

$$e = (e_1 = 0, e_2, \ldots, e_s), \quad f = (f_1 = 0, f_2, \ldots, f_s)$$

We get isotone maps

$$[s] \times [n] \times [m] \longmapsto [n + e_s] \times [m + f_s]$$

$$(i, a, b) \mapsto (a + e_i, b + f_i)$$

- When $e = (0, \ldots, 0)$ and $f = (0, 1, \ldots, s - 1)$ we get the multi-chain ideal $I_{m + s - 1, s}([n])$.
- When $e = (0, 1, \ldots, s - 1)$ and $f = (0, \ldots, 0)$ we get the multi-chain ideal $I_{n + s - 1, s}([n])$.
- When $e = (0, 1, \ldots, s - 1)$ and $f = (0, 1, \ldots, s - 1)$ we get the ideal $I$ generated by monomials

$$x_{i_1,j_1}x_{i_2,j_2} \cdots x_{i_s,j_s}$$

where $i_1 < \cdots < i_s$ and $j_1 < \cdots < j_s$. This is the initial ideal $I$ of the ideal of $s$-minors of a general $(n + s - 1) \times (m + s - 1)$ matrix $(x_{i,j})$ with respect to a diagonal term order, 20.

We thus see that this initial ideal $I$ has a lifting to $L(s, [n] \times [m])$ with $snm$ variables, in contrast to the $(n + s - 1)(m + s - 1)$ variables which are involved in the ideal of $s$-minors. We get maximal minors when, say $m = 1$. Then the initial ideal $I$ involves $sn$ variables. So in this case the initial ideal $I$ involves the same number of variables as $L(s, [n])$, i.e. the generators of these two ideals are in one to one correspondence by a bijection of variables.

3.4. Initial ideal of the ideal of two-minors of a symmetric matrix. Let $P = \text{Hom}([2], [n])$. The elements here may be identified with pairs $(i_1, i_2)$ where $1 \leq i_1 \leq i_2 \leq n$. There is an isotone map

$$\phi : [2] \times \text{Hom}([2], [n]) \to \text{Hom}([2], [n + 1])$$

$$(1, i_1, i_1) \mapsto (i_1, i_2)$$

$$(2, i_1, i_2) \mapsto (i_1 + 1, i_2 + 1).$$

This map has left strict chain fibers, and we get a regular quotient ideal $L^\phi(2, \text{Hom}([2], [n]))$, generated by $x_{i_1,i_2}x_{j_1,j_2}$ where $i_1 < j_1$ and $i_2 < j_2$ (and $i_1 \leq i_2$ and $j_1 \leq j_2$). This is the initial ideal of the ideal generated by 2-minors of a symmetric matrix of size $n + 1$, see [1, Sec.5].
3.5. **Ladder determinantal ideals.** Given a poset ideal $\mathcal{J}$ in $[m] \times [n]$. This gives the letterplace ideal $L(2, \mathcal{J})$. There is a map

$$\phi : [2] \times \mathcal{J} \rightarrow [m + 1] \times [n + 1]$$

$$(1, a, b) \mapsto (a, b)$$

$$(2, a, b) \mapsto (a + 1, b + 1)$$

The poset ideal $\mathcal{J}$ is sometimes also called a one-sided ladder in $[m] \times [n]$. The ideal $L^\phi(2, \mathcal{J})$ is the initial ideal of the ladder determinantal ideal associated to $\mathcal{J}$, [19, Cor.3.4]. Hence we recover the fact that these are Cohen-Macaulay, [14, Thm.4.9].

Consider the special case where $\mathcal{J}$ in $[n] \times [n]$ consists of all $(a, b)$ with $a + b \leq n + 1$. Then $L^\phi(2, \mathcal{J})$ is the initial ideal of the Grassmannian $G(2, n+3)$, [6, Ch.6]. It is also the initial ideal of the ideal of 4-Pfaffians of a skew-symmetric matrix of rank $n + 3$, [14, Sec.5].

### 4. Description of facets and ideals

As we have seen $\text{Hom}(Q, P)$ is itself a poset. The product $P \times Q$ makes the category of posets $\text{Poset}$ into a symmetric monoidal category, and with this internal $\text{Hom}$, it is a symmetric monoidal closed category [16, VII.7], i.e. there is an adjunction of functors

$$\begin{array}{ccc}
\text{Poset} & \to & \text{Poset} \\
\times P & \rightleftharpoons & \text{Hom}(P, -)
\end{array}$$

so that

$$\text{Hom}(Q \times P, R) \cong \text{Hom}(Q, \text{Hom}(P, R)).$$

This is an isomorphism of posets. Note that the distributive lattice $D(P)$ associated to $P$, consisting of the poset ideals in $P$, identifies with $\text{Hom}(P, [2])$. In particular $[n + 1]$ identifies as $\text{Hom}([n], [2])$. The adjunction above gives isomorphisms between the following posets.

1. $\text{Hom}([m], \text{Hom}(P, [n + 1]))$
2. $\text{Hom}([m] \times P, [n + 1]) = \text{Hom}([m] \times P, \text{Hom}([n], [2]))$
3. $\text{Hom}([m] \times P \times [n], [2])$
4. $\text{Hom}([n] \times P, \text{Hom}([m], [2])) = \text{Hom}([n] \times P, [m + 1])$
5. $\text{Hom}([n], \text{Hom}(P, [m + 1]))$

These Hom-posets normally give distinct letterplace or co-letterplace ideals associated to the same underlying (abstract) poset. There are natural bijections between the generators. The degrees of the generators are normally distinct, and so they have different resolutions.
Letting \( P \) be the one element poset, we get from 2., 3., and 4. above isomorphisms

\[(2) \quad \text{Hom}([m], [n + 1]) \cong \text{Hom}([m] \times [n], [2]) \cong \text{Hom}([n], [m + 1]).\]

An element \( \phi \) in \( \text{Hom}([m], [n + 1]) \) identifies as a partition \( \lambda_1 \geq \cdots \geq \lambda_m \geq 0 \) with \( m \) parts of sizes \( \leq n \) by \( \phi(i) = \lambda_{m+1-i} + 1 \). The left and right side of the isomorphisms above give the correspondence between a partition and its dual.

Letting \( m = 1 \) we get by 2., 3., and 5. isomorphisms:

\[\text{Hom}(P, [n + 1]) \cong \text{Hom}(P \times [n], [2]) \cong \text{Hom}(n, D(P))\]

and so we have ideals

\[L(P, n + 1), \quad L(P \times [n], [2]), \quad L(n, D(P))\]

whose generators are naturally in bijection with each other, in particular with elements of \( \text{Hom}([n], D(P)) \), which are chains of poset ideals in \( D(P) \):

\[(3) \quad \emptyset = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n \subseteq \mathcal{J}_{n+1} = P.\]

The facets of the simplicial complexes associated to their Alexander duals

\[L(n + 1, P), \quad L(2, P \times [n]), \quad L(D(P), n),\]

are then in bijection with elements of \( \text{Hom}([n], D(P)) \).

For a subset \( A \) of a set \( R \), let \( A^c \) denote its complement \( R \setminus A \).

1. The facets of the simplicial complex associated to \( L(n + 1, P) \) identifies as the complements \( (\Gamma \phi)^c \) of graphs of \( \phi : P \to [n + 1] \). This is because these facets correspond to the complements of the set of variables in the generators in the Alexander dual \( L(P, n + 1) \) of \( L(n + 1, P) \).

For isotone maps \( \alpha : [n+1] \times P \to R \) having bistrict chain fibers, the associated simplicial complex of the ideal \( L^\alpha(n + 1, P) \), has also facets in one-to-one correspondence with \( \phi : P \to [n + 1], \) or equivalently \( \phi' : [n] \to D(P) \), but the precise description varies according to \( \alpha \).

2. The facets of the simplicial complex associated to \( L(2, P \times [n]) \) identifies as the complements \( (\Gamma \phi)^c \) of the graphs of \( \phi : P \times [n] \to [2] \). Alternatively the facets identifies as the graphs \( \Gamma \phi' \) of \( \phi' : P \times [n] \to [2]^\circ \).

3. Let

\[\alpha : [2] \times P \times [n] \to P \times [n + 1], \quad (a, p, i) \mapsto (p, a + i - 1).\]
The ideal $L^a(2, P \times [n])$ is the multichain ideal $I_{n+1, 2}(P)$. The generators of this ideal are $x_{p,i}x_{q,j}$ where $p \leq q$ and $i < j$. The facets of the simplicial complex associated to this ideal are the graphs $\Gamma \phi$ of $\phi : P \to [n+1]^{op}$.

5. Co-letterplace ideals of poset ideals

5.1. **The ideal** $L(\mathcal{J})$. Since $\text{Hom}(Q, P)$ is itself a partially ordered set, we can consider poset ideals $\mathcal{J} \subseteq \text{Hom}(Q, P)$ and form the subideal $L(\mathcal{J})$ of $L(Q, P)$ generated by the monomials $m_{\Gamma \phi}$ where $\phi \in \mathcal{J}$. We say $L(\mathcal{J})$ is the **co-letterplace ideal of the poset ideal** $\mathcal{J}$. Often we simply call it a co-letterplace ideal.

**Proposition 5.1.** Let $\mathcal{J}$ be a poset ideal in $\text{Hom}(P, [n])$. Then $L(\mathcal{J})$ has linear quotients, and so it has linear resolution.

**Proof.** We extend the partial order $\leq$ on $\mathcal{J}$ to a total order, denoted $\leq'$, and set $m_{\Gamma \psi} \geq m_{\Gamma \phi}$ if and only if $\psi \leq' \phi$. We claim that $L(\mathcal{J})$ has linear quotients with respect to the given total order of the monomial generators of $L(\mathcal{J})$. Indeed, let $m_{\Gamma \psi} > m_{\Gamma \phi}$ where $\psi \in \mathcal{J}$. Then $\psi <^t \phi$, and hence there exists $p \in P$ such that $\psi(p) < \phi(p)$. We choose a $p \in P$ which is minimal with this property. Therefore, if $q < p$, then $\phi(q) < \psi(q) \leq \psi(p) < \phi(p)$. We set

$$\psi'(r) = \begin{cases} 
\psi(r), & \text{if } r = p, \\
\phi(r), & \text{otherwise}.
\end{cases}$$

Then $\psi' \in \text{Hom}(P, [n])$ and $\psi' < \phi$ for the original order. It follows that $\psi' \in \mathcal{J}$, and $m_{\Gamma \psi'} > m_{\Gamma \phi}$. Since $(m_{\Gamma \psi'}) : m_{\Gamma \phi} = (x_{p, \psi(p)})$ and since $x_{p, \psi(p)}$ divides $m_{\Gamma \psi'}$, the desired conclusion follows. $\square$

**Remark 5.2.** One may fix a maximal element $p \in P$. The statement above still holds if $\mathcal{J}$ in $\text{Hom}(P, [n])$ is a poset ideal for the weaker partial order on $\text{Hom}(P, [n])$ where $\phi \leq^w \psi$ if $\phi(q) \leq \psi(q)$ for $q \neq p$ and $\phi(p) = \psi(p)$. Then one deduces either $\psi' \leq^w \phi$ or $\psi' \leq^w \psi$. In either case this gives $\psi' \in \mathcal{J}$.

For an isotone map $\phi : P \to [n]$, we define the set

$$\Lambda \phi = \{(p, i) \mid \phi(q) \leq i < \phi(p) \text{ for all } q < p\}.$$ 

It will in the next subsection play a role somewhat analogous to the graph $\Gamma \phi$. For $\phi \in \mathcal{J}$ we let $J_\phi$ be the ideal generated by all $m_{\Gamma \psi}$ with $m_{\Gamma \psi} > m_{\Gamma \phi}$, where we use the total order in the proof of Proposition 5.1 above. In analogy to [7 Lemma 3.1] one obtains:

**Corollary 5.3.** Let $\phi \in \mathcal{J}$. Then $J_\phi : m_{\Gamma \phi}$ is $\{x_{p, i} \mid (p, i) \in \Lambda \phi\}$. 
Proof. The inclusion $\subseteq$ has been shown in the proof of Proposition 5.1. Conversely, let $x_{p,i}$ be an element of the right hand set. We set

$$\psi(r) = \begin{cases} i, & \text{if } r = p, \\ \phi(r), & \text{otherwise.} \end{cases}$$

Then $m_{\Gamma\psi} \in J_\phi$ and $(m_{\Gamma\psi}) : m_{\Gamma\phi} = (x_{p,i})$. This proves the other inclusion. \hfill \Box

Corollary 5.4. The projective dimension of $L(J)$ is the maximum of the cardinalities $|\Lambda\phi|$ for $\phi \in J$.

Proof. This follows by the above Corollary 5.3 and Lemma 1.5 of [13]. \hfill \Box

Remark 5.5. By [7, Cor.3.3] the projective dimension of $L(P,n)$ is $(n-1)s$ where $s$ is the size of a maximal antichain in $P$. It is not difficult to work this out as a consequence of the above when $J = \text{Hom}(P,[n])$.

An explicit form of the minimal free resolution of $L(P,n)$ is given in [7, Thm. 3.6].

5.2. Alexander dual of $L(J)$. We describe the Alexander dual of $L(J)$ when $J$ is a poset ideal in $\text{Hom}(P,[n])$. Since $L(J)$ has linear resolution, the Alexander dual $L(J)^A$ is a Cohen-Macaulay ideal containing $L(n,P)$, by [3]. Recall the set $\Lambda\phi$ defined above, associated to a map $\phi \in \text{Hom}(P,[n])$.

Lemma 5.6. Let $J$ be a poset ideal in $\text{Hom}(P,[n])$. Let $\phi \in J$ and $\psi$ be in the complement $J^c$. Then $\Lambda\psi \cap \Gamma\phi$ is nonempty.

Proof. There is some $pinP$ with $\psi(p) > \phi(p)$. Choose $p$ to be minimal with this property, and let $i = \phi(p)$. If $(p,i)$ is not in $\Lambda\psi$, there must be $q < p$ with $\psi(q) > i = \phi(p) \geq \phi(q)$. But this contradicts $p$ being minimal. Hence $(p,i) = (p,\phi(p))$ is both in $\Gamma\phi$ and $\Lambda\psi$. \hfill \Box

Lemma 5.7. Let $S$ be a subset of $P \times [n]$ which is disjoint from $\Gamma\phi$ for some $\phi$ in $\text{Hom}(P,[n])$. If $\phi$ is a minimal such w.r.t. the partial order on $\text{Hom}(P,[n])$, then $S \supseteq \Lambda\phi$.

Proof. Suppose $(p,i) \in \Lambda\phi$ and $(p,i)$ is not in $S$. Define $\phi' : P \to [n]$ by

$$\phi'(q) = \begin{cases} \phi(q), & q \neq p \\ i, & q = p \end{cases}$$

By definition of $\Lambda\phi$ we see that $\phi'$ is an isotone map, and $\phi' < \phi$. But since $S$ is disjoint from $\Gamma\phi$, we see that it is also disjoint from $\Gamma\phi'$. This contradicts $\phi$ being minimal. Hence every $(p,i) \in \Lambda\phi$ is also in $S$. \hfill \Box
For a subset $S$ of $\text{Hom}(P, [n])$ define $K(S) \subseteq \mathbb{k}[x_{[n] \times P}]$ to be the ideal generated by the monomials $m_{\Lambda \phi \tau}$ where $\phi \in S$.

**Theorem 5.8.** The Alexander dual of $L(J)$ is $L(n, P) + K(J^c)$.

**Proof.** We show the following.
1. The right ideal is contained in the Alexander dual of the left ideal: Every monomial in $L(n, P) + K(J^c)$ has non-trivial common divisor with every monomial in $L(J)$.
2. The Alexander dual of the left ideal is contained in the right ideal: If $S \subseteq [n] \times P$ intersects every $\Gamma \phi \tau$ where $\phi \in J$, the monomial $m_S$ is in $L(n, P) + K(J^c)$.

1a. Let $\psi \in \text{Hom}([n], P)$. Since $L(n, P)$ and $L(P, n)$ are Alexander dual, $\Gamma \psi \cap \Gamma \phi \tau$ is non-empty for every $\phi \in \text{Hom}(P, [n])$ and so in particular for every $\phi \in J$.
1b. If $\psi \in J^c$ then $\Lambda \psi \cap \Gamma \phi$ is nonempty for every $\phi \in J$ by Lemma 5.6.

Suppose now $S$ intersects every $\Gamma \phi \tau$ where $\phi$ is in $J$.
2a. If $S$ intersects every $\Gamma \phi \tau$ where $\phi$ is in $\text{Hom}(P, [n])$, then since $L(n, P)$ is the Alexander dual of $L(P, n)$, the monomial $m_S$ is in $L(n, P)$.
2b. If $S$ does not intersect $\Gamma \phi \tau$ where $\phi \in J^c$, then by Lemma 5.7 for a minimal such $\phi$ we will have $S \supseteq \Lambda \phi \tau$. Since $S$ intersects $\Gamma \phi \tau$ for all $\phi \in J$, a minimal such $\phi$ is in $J^c$. Thus $m_S$ is divided by $m_{\Lambda \phi \tau}$ in $K(J^c)$.

**Remark 5.9.** For a more concrete example, see the end of Subsection 6.4

5.3. **Quotients of** $L(J)$. We now consider co-letterplace ideals of poset ideals when we cut down by a regular sequence of variable differences. The following generalizes Theorem 2.2 and we prove it in Section 8

**Theorem 5.10.** Given an isotone map $\psi : P \times [n] \to R$ with left strict chain fibers. Let $J$ be a poset ideal in $\text{Hom}(P, n)$. Then the basis $B$ (as defined before Theorem 2.7) is a regular sequence for the ring $\mathbb{k}[x_{P \times [n]}]/L(J)$.

6. **Examples of regular quotients of co-letterplace ideals**

We give several examples of quotients of co-letterplace ideals which have been studied in the literature in recent years.
6.1. **Strongly stable ideals:** *Poset ideals in* $\text{Hom}([d],[n])$. Elements of $\text{Hom}([d],[n])$ are in one to one correspondence with monomials in $k[x_1,\ldots,x_n]$ of degree $d$: A map $\phi$ gives the monomial $\Pi_{i=1}^d x_{\phi(i)}$. By this association, the poset ideals in $\text{Hom}([d],[n])$ are in one to one correspondence with strongly stable ideals in $k[x_1,\ldots,x_n]$ generated in degree $d$.

Consider the projections $[d] \times [n] \to^p [n]$. The following is a consequence of Proposition 5.1 and Theorem 5.10.

**Corollary 6.1.** Let $J$ be a poset ideal of $\text{Hom}([d],[n])$. Then $L(J)$ has linear resolution. The quotient map

$$k[x_{[d] \times [n]}/L(J) \to^p \to k[x_{[n]}]/L^p(J)$$

is a quotient map by a regular sequence, and $L^p(J)$ is the strongly stable ideal in $k[x_1,\ldots,x_n]$ associated to $J$.

The ideals $L(J)$ are extensively studied by Nagel and Reiner in [18]. Poset ideals $J$ of $\text{Hom}([d],[n])$ are there called strongly stable $d$-uniform hypergraphs, [18, Def. 3.3]. If $M$ is the hypergraph corresponding to $J$, the ideal $L(M)$ is the ideal $I(F(M))$ of the $d$-partite $d$-uniform hypergraph $F(M)$ of [18, Def. 3.4, Ex. 3.5].

Furthermore the ideal $L^p(J)$ is the ideal $I(M)$ of [18, Ex. 3.5]. The squarefree ideal $I(K)$ of [18, Ex. 3.5] is the ideal $L^0(J)$ obtained from the map:

$$\phi : [d] \times [n] \to [d+n-1]$$

$$(a,b) \mapsto a+b-1$$

Corollary 6.1 above is a part of [18, Thm. 3.13].

Given a sequence $0 = a_0 \leq a_1 \leq \cdots \leq a_{d-1}$, we get an isotone map

$$\alpha : [d] \times [n] \to [n+a_{d-1}]$$

$$(i,j) \mapsto j + a_{i-1}$$

having left strict chain fibers. The ideal $L^0(J)$ is the ideal coming from the strongly stable ideal associated to $J$ by the stable operator of S.Murai [17, p.707]. When $a_{i-1} < a_i$ they are called alternative squarefree operators in [21, Sec. 4].

**Remark 6.2.** In [10] Francisco, Mermin and Schweig consider a poset $Q$ with underlying set $\{1,2,\ldots,n\}$ where $Q$ is a weakening of the natural total order, and study $Q$-Borel ideals. This is not quite within our setting, but adds extra structure: Isotone maps $\phi : [d] \to [n]$ uses the total order on $[n]$ but when studying poset ideals $J$ the weaker poset structure $Q$ is used on the codomain.
Let \( n \) be the poset which is the disjoint union of the one element posets \( \{1\}, \ldots, \{n\} \), so any two distinct elements are incomparable. This is the antichain on \( n \) elements.

6.2. Ferrers ideals: Poset ideals in \( \text{Hom}(2, [n]) \). By (2) partitions \( \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \) where \( \lambda_n \leq n \) correspond to elements of:

\[
\text{Hom}([n], [n+1]) \cong \text{Hom}([n] \times [n], [2]).
\]

Thus \( \lambda \) gives a poset ideal \( J \) in \([n] \times [n] = \text{Hom}(2, [n])\). The Ferrers ideal \( I_\lambda \) of [3, Sec. 2] is the ideal \( L(J) \) in \( k[x_{2 \times [n]}] \). In particular we recover the result from [3, Cor. 3.8] that it has linear resolution.

More generally, the poset ideals \( J \) of \( \text{Hom}(d, [n]) \) correspond to the \( d \)-partite \( d \)-uniform Ferrers hypergraphs \( F \) in [18, Def. 3.6]. That \( L(J) \) has linear resolution is [18, Thm. 3.13].

6.3. Edge ideals of cointerval \( d \)-hypergraphs. Let \( \text{Hom}_s(Q, P) \) be strict isotone maps \( \phi \), i.e. \( q < q' \) implies \( \phi(q) < \phi(q') \). There is an isomorphism of posets

\[
\text{Hom}([d], [n]) \cong \text{Hom}_s([d], [n+d-1]),
\]

by sending \( \phi \) to \( \phi_s \) given by \( \phi_s(j) = \phi(j) + j - 1 \).

Consider the weaker partial order on \( \preceq \) on \( \text{Hom}([d], [n]) \) where \( \phi \preceq \psi \) if \( \phi(i) \leq \psi(i) \) for \( i < d \) and \( \phi(d) = \psi(d) \). Via the isomorphism (4) this gives a partial order \( \preceq_s \) on \( \text{Hom}_s([d], [n+d-1]) \). The poset ideals for the partial order \( \preceq_s \) correspond to the cointerval \( d \)-hypergraphs of [4, Def. 4.1] on the set \( \{1, 2, \ldots, n+d-1\} \). Let \( J \) be such a poset ideal for \( \preceq_s \). It corresponds to a poset ideal \( J \) in \( \text{Hom}([d], [n]) \) for \( \preceq \). Let

\[
\phi : [d] \times [n] \to [d+n-1] \quad (a, b) \mapsto a + b - 1
\]

The ideal \( L^\phi(J) \) is the edge ideal of the cointerval hypergraph corresponding to \( J \), see [4, Def. 2.1]. By remarks 5.2 and 8.2, theorems 5.11 and 5.1 still holds for the weaker partial order \( \preceq \). Hence we recover the fact from [4, Cor. 4.7] that edge ideals of cointerval hypergraphs have linear resolution. In the case \( d = 2 \) these ideals are studied also in [3, Sec. 4] and [18, Sec. 2]. These are obtained by cutting down by a regular sequence of differences of variables from a skew Ferrers ideals \( I_{\lambda-\mu} \). The skewness implies the ideal comes from a poset ideal of \( \text{Hom}([2], [n]) \) rather than \( \text{Hom}(2, [n]) \). Due to this we get the map (5) which has left strict chain fibers, and so the ideal \( I_{\lambda-\mu} \), of [3, Sec. 4].
6.4. **Uniform face ideals:** Poset ideals in $\text{Hom}(n, [2])$. The uniform face ideal of a simplicial complex $\Delta$, introduced recently by D.Cook [2], see also [11], is the ideal generated by the monomials $\prod_{i \in F} x_i \cdot \prod_{i \notin F} y_i$ as $F$ varies among the faces of $\Delta$. The Boolean poset on $n$ elements is the distributive lattice $D(n) = \text{Hom}(n, [2])$. A simplicial complex $\Delta$ on the set $\{1, 2, \ldots, n\}$ corresponds to a poset ideal $J$ of $\text{Hom}(n, [2])$, and the uniform face ideal of $\Delta$ identifies as the subideal $L(J)$ of $L(n, [2])$.

More generally Cook considers a set of vertices which is a disjoint union of $k$ ordered sets $C_1 \cup \cdots \cup C_k$, each $C_i$ considered a colour class. He then considers simplicial complexes $\Delta$ which are nested with respect to these orders $\text{Hom}(U^k_{i=1}[c_i])$ $\text{Hom}(U^k_{i=1}[c_i])$. Let $c_i$ be the cardinality of $C_i$ and consider the poset which is the disjoint union $\bigcup_{i=1}^k [c_i]$. Then such a $\Delta$ corresponds precisely to a poset ideal $J$ of $\text{Hom}(U^k_{i=1}[c_i])$. In fact $J$ is isomorphic to the index poset $P(\Delta, C)$ of [2, Def. 6.1]. The uniform face ideal is obtained as follows: There are projection maps $p_i : [c_i] \times [2] \to [2]$ and so

$$
\bigcup_{i=1}^k p_i : (\bigcup_{i=1}^k [c_i]) \times [2] \to \bigcup_{i=1}^k [2].
$$

This map has left strict chain fibers and the ideal $L^{\bigcup_{i=1}^k \bigcup_{i=1}^k [c_i]}$ is exactly the uniform face ideal $I(\Delta, C)$. In [2] Thm. 6.8 it is stated that this ideal has linear resolution.

Returning again to the first case of the ideal $L(J)$ in $L(n, [2])$, its Alexander dual is by Theorem 5.8

$$
L(J)^A = L([2], \nu) + K(J^e).
$$

Here $L([2], \nu)$ is the complete intersection of $x_{1j}x_{2j}$ for $j = 1, \ldots, n$, while $K(J^e)$ is generated by $\prod_{j \in G} x_{1j}$ where $G$ is a nonface of $\Delta$. Thus $K(J^e)$ is the associated ideal $I_\Delta \subseteq \mathbb{k}[x_{11}, \ldots, x_{1n}]$. This is [11, Thm. 1.1]: $L(J)^A$ is the Stanley-Reisner ideal $I_\Delta$ with whiskers $x_{1j}x_{2j}$.

7. **Proof concerning Alexander duality**

In this section we prove Theorem 2.9 concerning the compatibility between Alexander duality and cutting down by a regular sequence. The following lemma holds for squarefree ideals. Surprisingly it does not hold for monomial ideals in general, for instance for $(x^n_0, x^n_1) \subseteq k[x_0, x_1]$. 

Lemma 7.1. Let \( I \subseteq S \) be a squarefree monomial ideal and let \( f \in S \) such that \( x_1 f = x_0 f \) considered in \( S/I \). Then for every monomial \( m \) in \( f \) we have \( x_1 m = 0 = x_0 m \) in \( S/I \).

Proof. Write \( f = x_0^a f_a + \cdots + x_0 f_1 + f_0 \) where each \( f_i \) does not contain \( x_0 \). The terms in \( (x_1 - x_0) f = 0 \) of degree \( a + 1 \) in \( x_0 \), are in \( x_0^{a+1} f_a \), and so this is zero. Since \( S/I \) is squarefree, \( x_0 f_a \) is zero, and so \( f = x_0^{a-1} f_{a-1} + \cdots \). We may continue and get \( f = f_0 \). But then again in \( (x_1 - x_0) f = 0 \) the terms with \( x_0 \) degree 1 is \( x_0 f_0 \) and so this is zero. The upshot is that \( x_0 f = 0 = x_1 f \). But then each of the multigraded terms of these must be zero, and this gives the conclusion. \[\square\]

Let \( S \) be the polynomial ring \( k[x_0, x_1, x_2, \ldots, x_n] \) and \( I \subseteq S \) a squarefree monomial ideal with Alexander dual \( J \subseteq S \). Let \( S_1 = k[x, x_2, \ldots, x_n] \) and \( S \rightarrow S_1 \) be the map given by \( x_i \mapsto x_i \) for \( i \geq 2 \) and \( x_0, x_1 \mapsto x \).

Let \( I_1 \) be the ideal of \( S_1 \) which is the image of \( I \), so the quotient ring of \( S/I \) by the element \( x_1 - x_0 \) is the ring \( S_1/I_1 \). Similarly we define \( J_1 \).

Proposition 7.2. a) If \( x_1 - x_0 \) is a regular element of \( S/I \), then \( J_1 \) is squarefree.

b) If \( I_1 \) is squarefree then \( x_1 - x_0 \) is a regular element on \( S/J \).

c) If both \( x_1 - x_0 \) is a regular element on \( S/I \) and \( I_1 \) is squarefree, then \( J_1 \) is the Alexander dual of \( I_1 \).

Proof. The Alexander dual \( J \) of \( I \) consists of all monomials in \( S \) with non-trivial common factor (ntcf.) with all monomials in \( I \).

a) Let \( F \) be a facet of the simplicial complex of \( I \). Let \( m_F = \prod_{i \in F} x_i \). Suppose \( F \) does not contain any of the vertices 0 and 1. Then \( x_1 m_F = 0 = x_0 m_F \) in \( S/I \) (since \( F \) is a facet). Since \( x_1 - x_0 \) is regular we get \( m_F = 0 \) in \( S/I \), a contradiction. Thus every facet \( F \) contains either 0 or 1. The generators of \( J \) are \( \prod_{i \in [n] \setminus F} x_i \), and so no such monomial contains \( x_0 x_1 \) and therefore \( J_1 \) will be squarefree.

b) Suppose \( (x_1 - x_0)f = 0 \) in \( S/J \). By the above for the monomials \( m \) in \( f \), we have \( x_1 m = 0 = x_0 m \) in \( S/J \). We may assume \( m \) is squarefree. So \( x_0 m \) has ntcf. with all monomials in \( I \) and the same goes for \( x_1 m \). But then \( m \) has ntcf. with all monomials in \( I \), since if \( m \) does not have ntcf. with the minimal monomial generator \( n \) in \( I \), then \( n = x_0 x_1 n' \). Hence it follows that the image of \( n \) in \( I_1 \) would not be squarefree, contrary to the assumption.

c) A monomial \( m \) in \( J \) has ntcf. with all monomials in \( I \). Then its image \( \overline{m} \) in \( S_1 \) has ntcf. with all monomials in \( I_1 \), and so \( J_1 \) is contained in the Alexander dual of \( I_1 \).
Assume now \( \overline{m} \in S_1 \) has ntcf. with all monomials in \( I_1 \). If \( \overline{m} \) does not contain \( x \) then \( m \) has ntcf. with every monomial in \( I \), and so \( \overline{m} \in J_1 \).

Otherwise \( \overline{m} = xm' \) and so \( \overline{m} \in J_1 \). We will show that either \( x_0 m' \) or \( x_1 m' \) is in \( J \). If not, then \( x_0 m' \) has no common factor with some monomial \( x_1 n_1 \) in \( I \), and \( x_1 m' \) has no common factor with some monomial \( x_0 n_0 \) in \( I \). Let \( n \) be the least common multiple of \( n_0 \) and \( n_1 \). Then \( x_0 n \) and \( x_1 n \) are both in \( I \) and so by the regularity assumption \( n \in I \). But \( n \) has no common factor with \( x_0 m' \) and \( x_1 m' \), and so \( n \in I_1 \). This is a contradiction. Hence either \( x_0 m \) or \( x_1 m \) is in \( J \) and so \( m \) is in \( J \).

We are ready to round off this section:

Proof of Theorem 2.9. Both ideals are squarefree and are obtained from the Alexander dual ideals \( L(n,P) \) and \( L(P,n) \) by cutting down by a regular sequence.

8. Proof that the poset maps induce regular sequences.

To prove Theorems 2.1, 2.2 and 5.10 we will use an induction argument. Let \([n] \times P \xrightarrow{\phi} R \) be an isotone map. Let \( r \in R \) have inverse image by \( \phi \) of cardinality \( \geq 2 \). Choose a partition into nonempty subsets \( \phi^{-1}(r) = R_1 \cup R_2 \) such that \((i,p) \in R_1 \) and \((j,q) \in R_2 \) implies \( i < j \). Let \( R' \) be \( R \setminus \{r\} \cup \{r_1, r_2\} \). We get the map

\[
[n] \times P \xrightarrow{\phi'} R' \to R
\]

factoring \( \phi \), where the elements of \( R_i \) map to \( r_i \). Let \( p', q' \) be distinct elements of \( R' \). For an element \( p' \) of \( R' \), denote by \( \overline{p'} \) its image in \( R \).

We define a partial order on \( R' \) by the following two types of strict inequalities:

- \( r_1 < r_2 \), and
- \( p' < q' \) if \( \overline{p'} < \overline{q'} \)

Lemma 8.1. This ordering is a partial order on \( R' \).

Proof. Transitivity: Suppose \( p' \leq q' \) and \( q' \leq r' \). Then \( \overline{p'} \leq \overline{q'} \) and \( \overline{q'} \leq \overline{r'} \) and so \( \overline{p'} \leq \overline{r'} \). If either \( \overline{p'} \) or \( \overline{r'} \) is distinct from \( r \) we conclude that \( p' \leq r' \). If both of them are equal to \( r \), then \( \overline{q'} = r \) also. Then either \( p' = q' = r \) or \( p' = r_1 \) and \( r' = r_2 \), and so \( p' \leq r' \).

Reflexivity: Suppose \( p' \leq q' \) and \( q' \leq p' \). Then \( \overline{p'} = \overline{q'} \). If this is not \( r \) we get \( p' = q' \). If it equals \( r \), then since we do not have \( r_2 \leq r_1 \), we must have again have \( p' = q' \).
Proof of Theorem 2.1. We show this by induction on the cardinality of $\text{im } \phi$. Assume that we have a factorization (3), such that

$$k[x_n]/L^\phi(n, P)$$

is obtained by cutting down from $k[x_{[n]\times P}]/L(n, P)$ by a regular sequence of variable differences.

For $(a, p)$ in $[n] \times P$ denote its image in $R'$ by $(\overline{a}, \overline{p})$ and its image in $R$ by $(\overline{a}, \overline{p})$. Let $(a, p)$ map to $r_1 \in R'$ and $(b, q)$ map to $r_2 \in R'$. We will show that $x_{r_1} - x_{r_2}$ is a regular element in the quotient ring (7). So let $f$ be a polynomial of this quotient ring such that $f(x_{r_1} - x_{r_2}) = 0$. Then by Lemma 7.1, for any monomial $m$ in $f$ we have $mx_{r_1} = 0 = mx_{r_2}$ in the quotient ring (7). We assume $m$ is nonzero in the quotient ring (7).

There is a monomial $x_{r_1}^{p_1}x_{r_2}^{p_2} \ldots x_{r_n}^{p_n}$ in $L^\phi(n, P)$ dividing $mx_{\overline{a}, \overline{p}}$ considered as monomials in $k[x_R]$. Then we must have $\overline{a}, \overline{p} = \overline{s}, \overline{p}$ for some $s$. Furthermore there is $x_{1,q_1}^{p_1}x_{1,q_2}^{p_2} \ldots x_{1,q_c}^{p_n}$ dividing $mx_{\overline{b}, \overline{q}}$ in $k[x_R]$, and so $\overline{b}, \overline{q} = \overline{t}, \overline{q}$ for some $t$.

In $R$ we now get

$$\overline{s}, \overline{p} = \overline{r_1} = \overline{b}, \overline{q} = \overline{r_2},$$

so $s = t$ would imply $q_t = p_s$ since $\phi$ has left strict chain fibers. But then

$$r_1 = \overline{a}, \overline{p} = s, \overline{p} = \overline{r_1} = \overline{b}, \overline{q} = r_2$$

which is not so. Assume, say $s < t$. Then $p_s \geq q_t$ since $\phi$ has left strict chain fibers, and so

$$p_t \geq p_s \geq q_t \geq q_s.$$  

1. Suppose $p_s > q_t$. Consider $x_{1,q_1}^{p_1} \ldots x_{t-1,q_t-1}^{p_{t-1}}$. This will divide $m$ since $x_{1,q_1}x_{1,q_2} \ldots x_{1,q_c}$ divides $mx_{\overline{a}, \overline{p}}$. Similarly $x_{s+1,p_{s+1}}^{p_{s+1}} \ldots x_{r,n}^{p_n}$ divides $m$. Chose $s \leq r \leq t$. Then $p_r \geq p_s > q_t \geq q_t$ and so $r, q_r < r, p_r$. Then $x_{1,q_1} \ldots x_{t-1,q_t-1}$ and $x_{r,p_r} \ldots x_{n,p_n}$ do not have a common factor since

$$\overline{i}, \overline{q} \leq \overline{r}, \overline{p} < \overline{r}, \overline{p} \leq \overline{j}, \overline{p},$$

for $i \leq r \leq j$. Hence the product of these monomials will divide $m$ and so $m = 0$ in the quotient ring $k[x_R]/L^\phi(n, P)$.

2. Assume $p_s = q_t$ and $q_t > q_s$. Then $s, \overline{p} > \overline{s}, \overline{q}$ since $\phi$ has left strict chain fibers. The monomials $x_{1,q_1}^{p_1} \ldots x_{1,q_c}$ and $x_{s+1,p_{s+1}}^{p_{s+1}} \ldots x_{n,p_n}$ then do not have any common factor, and the product divides $m$, showing that $m = 0$ in the quotient ring $k[x_R]/L^\phi(n, P)$.

If $p_t > p_s$ we may argue similarly.
3. Assume now that \( p_t = p_s = q_t = q_s \), and denote this element as \( p \). Note that for \( s \leq i \leq t \) we then have \( p_s \leq p_i \leq p_t \), so \( p_i = p \), and the same argument shows that \( q_i = p \) for \( i \) in this range.

Since \( \overline{s, p} = \overline{s, p} = \overline{a, p} \neq \overline{b, q} = \overline{t, q} = \overline{t, p} \)
there is \( s \leq r \leq t \) such that
\[
\overline{s, p} = \cdots = \overline{r, p} < \overline{r + 1, p} \leq \cdots \leq \overline{t, p}.
\]
This is the same sequence as
\[
\overline{s, q} = \cdots = \overline{r, q} < \overline{r + 1, q} \leq \cdots \leq \overline{t, q}.
\]
Then \( x_{q_1} \cdots x_{q_r} \) and \( x_{r+1} \cdots x_{t, p} \) divide \( m \) and do not have a common factor, and so \( m = 0 \) in the quotient ring \( \mathbb{k}[x_R]/L^\phi(n, P) \).

**Proof of Theorems 2.2 and 5.10**

By induction on the cardinality of \( \text{im} \phi \).

We assume we have a factorization
\[
P \times [n] \xrightarrow{\phi'} R' \rightarrow R
\]
and which is analogous to (8), such that
\[
\mathbb{k}[x_{R'}]/L^\phi(J)
\]
is obtained by cutting down from \( \mathbb{k}[x_{P \times [n]}]/L(J) \) by a regular sequence of variable differences.

Let \((p_0, a)\) map to \( r_1 \in R'\) and \((q_0, b)\) map to \( r_2 \in R'\). We will show that \( x_{r_1} - x_{r_2} \) is a regular element in the quotient ring (8).

So let \( f \) be a polynomial of this quotient ring such that \( f(x_{r_1} - x_{r_2}) = 0 \). Then by Lemma 7.1 for any monomial \( m \) in \( f \) we have \( mx_{r_1} = mx_{r_2} \) in the quotient ring \( \mathbb{k}[x_{R'}]/L^\phi(J) \). We assume \( m \) is nonzero in this quotient ring.

There is \( i \in J \subseteq \text{Hom}(P, n) \) such that the monomial \( m^i = \prod_{p \in P} x_{p, i_p} \) in \( L^\phi(J) \) divides \( mx_{p_0, a} \), and similarly a \( j \in J \) such that the monomial \( m^j = \prod_{p \in P} x_{p, j_p} \) divides \( mx_{q_0, b} \). Hence there are \( s \) and \( t \) in \( P \) such that \( s, i_s = \overline{p_0, a} \) and \( t, j_t = \overline{q_0, b} \) in \( R \) we then get:
\[
\overline{s, i_s} = \overline{p_0, a} = \overline{q_0, b} = \overline{t, j_t},
\]
so \( s = t \) would imply \( i_t = j_t \) since \( \phi \) has left strict chain fibers. But then
\[
r_1 = \overline{p_0, a} = \overline{s, i_s} = \overline{t, j_t} = \overline{q_0, b} = r_2
\]
which is not so. Assume then, say \( s < t \). Then \( i_s \geq j_t \) since \( \phi \) has left strict chain fibers, and so
\[
i_t \geq i_s \geq j_t \geq j_s.
\]

Now form the monomials
• $m_{i>s} = \prod_{p>s} x_{p,i_p}$.
• $m_{i>j} = \prod_{i_p > j_p, \text{not } (p>s)} x_{p,i_p}$.
• $m_{i<j} = \prod_{i_p < j_p, \text{not } (p>s)} x_{p,i_p}$.
• $m_{i=j} = \prod_{i_p = j_p, \text{not } (p>s)} x_{p,i_p}$.

Similarly we define $m^*_i$ for the various subscripts $*$. Then

$m^i = m^i_{i=j} \cdot m^i_{i>j} \cdot m^i_{i<j} \cdot m^i_{i>s}$

divides $x_{s,i_s} m$, and

$m^j = m^j_{i=j} \cdot m^j_{i>j} \cdot m^j_{i<j} \cdot m^j_{i>s}$

divides $x_{t,j_t} m$.

There is now a map $\ell : P \to [n]$ defined by

$$\ell(p) = \begin{cases} i_p & \text{for } p > s \\ \min(i_p, j_p) & \text{for not } (p > s) \end{cases}$$

This is an isotone map as is easily checked. Its associated monomial is

$m^\ell = m^i_{i=j} \cdot m^j_{i>j} \cdot m^i_{i<j} \cdot m^i_{i>s}$.

We will show that this divides $m$. Since the isotone map $\ell$ is $\leq$ the isotone map $i$, this will prove the theorem.

Claim 3. $m^j_{i>j}$ is relatively prime to $m^i_{i<j}$ and $m^i_{i>s}$.

Proof. Let $x_{p,j_p}$ be in $m^j_{i>j}$.

1. Suppose it equals the variable $x_{q,i_q}$ in $m^i_{i<j}$. Then $p$ and $q$ are comparable since $\phi$ has left strict chain fibers. If $p < q$ then $j_p \geq i_q \geq i_p$, contradicting $i_p > j_p$. If $q < p$ then $i_q \geq j_p \geq j_q$ contradicting $i_q < j_q$.

2. Suppose $x_{p,j_p}$ equals $x_{q,i_q}$ in $m^i_{i>s}$. Then $p$ and $q$ are comparable and so $p < q$ since $q > s$ and we do not have $p > s$. Then $j_p \geq i_q \geq i_p$ contradicting $i_p > j_p$. □

Let $abc = m^i_{i<j} \cdot m^i_{i<j} \cdot m^i_{i>s}$ which divides $m x_{s,i_s}$ and $ab' = m^j_{i=j} \cdot m^j_{i>j}$ which divides $m$ since $x_{t,j_t}$ is a factor of $m^j_{i>s}$ since $t > s$. Now if the product of monomials $abc$ divides the monomial $n$ and $ab'$ also divides $n$, and $b'$ is relatively prime to $bc$, then the least common multiple $ab'c$ divides $n$. We thus see that the monomial associated to the isotone map $\ell$

$m^\ell = m^i_{i=j} \cdot m^j_{i>j} \cdot m^i_{i<j} \cdot m^i_{i>s}$.
divides $m x_{s,i}$. We need now only show that the variable $x_{s,i}$ occurs to a power in the above product for $m^f$ less than or equal to that of its power in $m$.

Claim 4. $x_{s,i}$ is not a factor of $m^j_{i > j}$ or $m^j_{i < j}$.

Proof. 1. Suppose $s, i = p, i_p$ where $i_p < j_p$ and not $p > s$. Since $p$ and $s$ are comparable (they are both in a fiber of $\phi$), we have $p \leq s$. Since $\phi$ is isotope $i_p \leq i_s$ and since $\phi$ has left strict chain fibers $i_p \leq i_s$. Hence $i_p = i_s$. By (9) $j_s \leq i_s$ and so $j_p \leq j_s \leq i_s = i_p$. This contradicts $i_p < j_p$.

2. Suppose $s, i = p, j_p$ where $j_p < i_p$ and not $p > s$. Then again $p \leq s$ and $i_p \leq i_s \leq j_p$, giving a contradiction. □

If now $i_s > j_s$ then $x_{s,i}$ is a factor in $m^j_{i > j}$ but by the above, not in $m^j_{i > j}$. Since $m^f$ is obtained from $m^i$ by replacing $m^i_{i > j}$ with $m^j_{i > j}$, we see that $m^f$ contains a lower power of $x_{s,i}$ than $m^i$ and so $m^f$ divides $m$.

Claim 5. Suppose $i_s = j_s$. Then the power of $x_{s,i}$ in $m^j_{i > s}$ is less than or equal to its power in $m^j_{i > s}$.

Proof. Suppose $s, i = p, i_p$ where $p > s$. We will show that then $i_p = j_p$. This will prove the claim.

The above implies $p, i_p = s, i_s = t, j_t$, so either $s < p < t$ or $s < t \leq p$. If the latter holds, then since $\phi$ has left strict chain fibers, $i_s \geq j_t \geq i_p$ and also $i_s \leq i_p$ by isotonicity, and so $i_s = i_p = j_t$. Thus $s, i_s \leq t, j_t \leq p, i_p$ and since the extremes are equal, all three are equal contradicting the assumption that the two first are unequal.

Hence $s < p < t$. By assumption on the fibre of $\phi$ we have $i_s \geq i_p$ and by isotonicity $i_s \leq i_p$ and so $i_s = i_p$. Also by (9) and isotonicity

$$i_s \geq j_t \geq j_p \geq j_s.$$

Since $i_s = j_s$ we get equalities everywhere and so $i_p = j_p$, as we wanted to prove. □

In case $i_s > j_s$ we have shown that $m^f$ divides $m$. So suppose $i_s = j_s$. By the above two claims, the $x_{s,i}$ in $m^f$ occurs only in $m^i_{i = j} \cdot m^i_{i > s}$ and to a power less than or equal to that in $m^i_{i = j} \cdot m^j_{i > s}$. But since $s, i_s \neq t, j_t$ the power of $s, i_s$ in $m^j_{i > s}$ is less than or equal to its power in $m$. Hence the power of $x_{s,i}$ in $m^f$ is less or equal to its power in $m$ and $m^f$ divides $m$. □
Remark 8.2. Suppose $P$ has a unique maximal element $p$. The above proof still holds if $J$ in $\text{Hom}(P, [n])$ is a poset ideal for the weaker partial order $\leq^w$ on $\text{Hom}(P, [n])$ where the isotone maps $\phi \leq^w \psi$ if $\phi(q) \leq \psi(q)$ for $q < p$, and $\phi(p) = \psi(p)$.

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