Shear Flow-Interchange Instability in Nightside Magnetotail as Proposed Cause of Auroral Beads as a Signature of Substorm Onset

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Abstract A geometric wedge model of the near-Earth nightside plasma sheet is used to derive a wave equation for low-frequency shear flow-interchange waves which transmit \( E \times B \) sheared zonal flows along magnetic flux tubes toward the ionosphere. Discrepancies with the wave equation result used in Kalmoni et al. (2015, https://doi.org/10.1002/2015JA021470) for shear flow-ballooning instability are discussed. The shear flow-interchange instability appears to be responsible for substorm onset. The wedge wave equation is used to compute rough expressions for dispersion relations and local growth rates in the midnight region of the nightside magnetotail where the instability develops, forming the auroral beads characteristic of geomagnetic substorm onset. Stability analysis for the shear flow-interchange modes demonstrates that nonlinear analysis is necessary for quantitatively accurate results and determines the spatial scale on which the instability varies.

1. Introduction

When the interplanetary magnetic field originating at the Sun contains a southward magnetic field component, the solar wind causes magnetic reconnection on the dayside of the Earth, followed by nightside reconnection in the magnetotail. The magnetic reconnection on the dayside changes the field configuration. Initially, there is a field line with both ends attached to the Sun and a field line with both ends attached to the Earth. Reconnection then produces new field lines, one attached to the Earth’s geomagnetic North Pole and extending into space and the other attached to the Earth’s geomagnetic South Pole and extending into space, creating open flux tubes. The flow of plasma from the solar wind then produces an electric field which causes convection of the magnetic field lines toward the nightside of the Earth due to flux freezing. The open flux tubes which connect to the polar regions of the Earth thereby provide the duskward directed electric field which drives the noon-midnight currents that carry the ionospheric ends of the magnetic field lines along with them. Reconnection on the nightside again closes magnetic field lines which connect to the Earth’s geomagnetic North and South Poles, forming stretched closed flux tubes on the nightside of the Earth (Dungey, 1961; Kivelson & Russell, 1995). A helpful diagram of the process is shown below in Figure 1.

Then a sequence of events referred to as a magnetic substorm occurs. The solar wind plasma deposits energy in the magnetotail during the growth phase of a magnetic substorm as the nightside magnetic field lines stretch tailward. This triggers a current disruption in the equatorial current sheet and initiates the expansion phase of the magnetic substorm, when the magnetic field lines snap back in response to the destabilization in the current sheet, and plasma is accelerated toward the polar regions of the Earth’s atmosphere. This sequence of events leads to the formation of aurorae in the \( E \) layer of the Earth’s ionosphere. Substorms occur about five times per day on average, and last for about 2–3 hr, but substorm onset occurs within a roughly 2 min time span (Angelopoulos et al., 2008; Angelopoulos et al., 2008; Coppi et al., 1966; Forsyth et al., 2014; Kivelson & Russell, 1995; McPherron et al., 2011; Sergeev et al., 2011; Wolf, 1995; Zeeuw et al., 2004; Zou et al., 2010).

At substorm onset (the initiation of the expansion phase), the most equatorward auroral arc suddenly brightens, followed by breakup of the arc and poleward expansion (Akasofu, 1964; Donovan et al., 2008). In the minutes leading to the breakup, small periodic fluctuations in the aurora aligned with magnetic longitude form (Nishimura et al., 2014), seen below in Figure 2. These fluctuations have come to be called “auroral
beads.” Henceforth, “longitudinal” will be used to refer to magnetic longitude. Auroral beads have been found to be likely pervasive in onset arcs, and the exponential growth of the beads indicates that a plasma instability in the magnetosphere is responsible for substorm onset (Gallardo-Lacourt et al., 2014; Kalmoni et al., 2017).

All-sky imagers (ASIs), which are a part of the NASA THEMIS mission to uncover the sequence of events which occur in the first few minutes of substorm onset, are distributed across North America, as seen in Figure 3. They have a 1 km spatial resolution and 3 s cadence image-capturing capacity and respond predominantly to 557.7 nm emissions. This spatiotemporal resolution is sufficient to capture the pertinent data for analyzing auroral bead structures for the green emissions corresponding to aurora at an altitude of approximately 110 km, namely, the $E$ layer (Burch & Angelopoulos, 2008; Mende et al., 2008).

Motoba et al. (2012) used ASI data from auroral beads in the Northern and Southern Hemispheres and proposed a common magnetospheric driver. Ultralow-frequency waves occurring within minutes of substorm onset are observed in the magnetosphere at frequencies similar to those of the auroral beads, and a single event was analyzed by Rae et al. (2010) to demonstrate that the beading is characteristic of a near-Earth

**Figure 1.** Solar wind-driven reconnection events for magnetic field lines. Field line numbers show the sequence of magnetic field line configurations. The first reconnection event occurs on the dayside of the Earth, where the southward Interplanetary Magnetic Field $1'$ connects with the northward Geomagnetic Field Line 1. The field lines are convected antisunward in Configurations 2 and 2' through 5 and 5', reconnecting again as 6 and 6'. Then the substorm expansion phase is initiated, and field lines connected to the geomagnetic North and South Poles dipolarize. Field Line 7', now closed as a roughly teardrop-shaped plasmoid, then continues tailward into interplanetary space. The geomagnetic field line will then swing back around from midnight to noon to become Field Line 9. The inset below shows the positions of the ionospheric anchors of the numbered field lines in the northern high-latitude ionosphere and the corresponding plasma currents of the Earth's polar caps. The currents flow from noon to midnight, convecting the feet of the field lines nightward and then closing back at noon via a lower-latitude field return flow. This figure is reproduced from Kivelson and Russell (1995).
magnetospheric instability triggering a current disruption in the central plasma sheet. Of the examined instabilities, cross-field current instability and shear flow-ballooning instability were the only two consistent with the analytical results. Kalmoni et al. (2015) used the ASI data for 17 substorm events over a 12-hr time span throughout the auroral oval (premidnight sector) across Canada and Alaska to perform an optical-statistical analysis that yielded maximum growth rates for the beads as a function of longitudinal
Figure 4. Exponential growth rate determination performed by Kalmoni et al. (2015). The log of the power spectral density plotted against time for a single wavenumber \(k_y = 0.9 \times 10^4 \text{ m}^{-1}\). This shows the duration of growth and growth rate from the linear fit. More details governing the fitting are given in (Kalmoni et al., 2015), from which this diagram is reproduced.

wavenumber, which were compared with theoretical calculations for growth rate dependence on wavenumber for various instabilities. Ultimately, the two mechanisms which remained unfuted were the shear flow-ballooning instability and the cross-field current instability.

The statistical analysis involved first spatially Fourier transforming longitudinal keograms to obtain the power spectral density. The longitudinal wavenumbers \(k_y, E\) measured in the ionosphere lay within the interval \(k_y, E \in [0.5 \times 10^{-4} \text{ m}^{-1}, 1.5 \times 10^{-4} \text{ m}^{-1}]\) during initial beading. The logarithm of the power spectral density was then plotted against time to determine the intervals of exponential growth for each wavenumber during onset. This is shown for one wavenumber in Figure 4. Since the exponential growth of each mode had a unique well-defined growth-rate during the interval until the breakup, only one instability is operating to produce the growth for each event. The growth rates were then examined as a function of wavenumber for determination of the most unstable waves. The maximum growth rates lie in the range \([0.03 \text{ Hz}, 0.3 \text{ Hz}]\) with median growth rate \(\gamma \sim 0.05 \text{ Hz}\). Note that wave propagation direction (eastward vs. westward) differed for the individual substorm events, but growth rates are independent of propagation direction (Nishimura et al., 2016).

Subsequently, Kalmoni et al. (2015) used the T96 model (Tsyganenko, 1995; Tsyganenko, 1996a; Tsyganenko, 1996b) to map the wavenumbers back to the equatorial magnetosphere to obtain the corresponding magnetospheric wavenumbers \(k_y \in [2.5 \times 10^{-6} \text{ m}^{-1}, 3.75 \times 10^{-6} \text{ m}^{-1}]\), or wavelength interval \(\lambda_\parallel \in [1,700 \text{ km}, 2,500 \text{ km}]\). The T96 model underestimates field-line stretching (and spatial scales) during the substorm growth phase. Equilibrium magnetic field mapping is thus unreliable at substorm times. This implies that the T96 model will determine the location of the instability to be closer to Earth in the equatorial plane. This is discussed in detail in Pulkkinen et al. (1991). Kalmoni et al. (2015) claims that the events can still be compared, even though the spatial scales are underestimated. Using the T96 model, Kalmoni et al. (2015) determined that the arcs map to the equatorial plane mostly in the range of 9–12 \(R_E\), with field strengths less than 20 nT. The relevant optical analysis plots are shown below in Figure 5.

Of the two instabilities which were not ruled out by the Kalmoni et al. (2015) analysis, the shear flow-ballooning instability provided the best explanation of the observed beading results, corroborating previous findings along these lines (Friedrich et al., 2000). This instability was characterized in the form used by Kalmoni et al. (2015) in Voronkov et al. (1997). This instability is a hybrid of the Kelvin-Helmholtz and Rayleigh-Taylor instabilities with larger growth rates operating on shorter growth time scales than pure
Figure 5. Optical analysis performed by (Kalmoni et al., 2015) for Gillam substorm on 2 October 2011. (a) North-south (geomagnetic) keogram to show auroral brightening and poleward propagation. (b) East-west (geomagnetic) keogram showing time evolution of periodic structures during substorm onset. (c) Power spectral density as a function of $k_y, E$. (d) Periods of exponential growth for each $k_y, E$, with growth rate denoted by color. Substorm onset interval is given by the vertical lines. Only wavenumbers that grow for over 30 s and start within one standard deviation of the median start time were used in the analysis. (e) Growth rate as a function of wavenumbers, with ionospheric wavenumbers $k_y, E$ below and magnetospheric wavenumbers $k_y$ above. This figure is reproduced from Kalmoni et al. (2015).
Kelvin-Helmholtz instability. The former are driven by shear flows and the latter by earthward pressure gradients. An extensive linear analysis of such hybrid instabilities and their relation to substorms has been conducted by Yamamoto (2008), Yamamoto (2009). In particular, it was found that the hybrid waves are driven by shear flows in the presence of an earthward particle energy density gradient. The auroral arc is tied to the boundary between the stretched field lines and the dipolarized field lines at the inner edge of the near-Earth plasma sheet. This is where pressure gradients are most relevant. The spatial scale of the shear flow-ballooning instability varies inversely as the size of the shear flow region. Kalmoni et al. (2015) determined that for this instability, the growth rates peak at 0.2 Hz in the wavenumber regions specified above.

After setting up a simple geometric wedge model following Wolf et al. (2018) for which perturbations will entail earthward flowing waves which carry the effects of the magnetospheric disturbance back to the ionosphere, we derive a wave equation for the plasma wedge which differs from that of both his original paper, (Wolf et al., 2018), and that from which Kalmoni et al. (2015) extracted the equation governing the shear flow-ballooning instability, namely, Voronkov et al. (1997). Voronkov et al. (1997) treated the coupling of shear flow and pressure gradient instabilities but incorrectly perturbed the momentum equation (see Appendix A). Wolf et al. (2018), on the other hand, did not treat the shear flow effect, thus obtaining low-frequency buoyancy waves which are not coupled to shear flow (see section 5). Either of these alterations shifts the growth rates and, more importantly, fails to capture some of the essential qualitative features of the instability mechanism.

We will begin with the linearization of the MHD field equations from which all equations under consideration can be derived by the addition of various constraints and assumptions. The continuity equation is an unnecessary constraint if one utilizes the flux tube volume given in terms of the magnetic field strength. Upon combining these equations to obtain an ordinary radial differential equation for the radial component of the velocity (sections 3.2–4), we take several limits to obtain a reduced low-frequency shear flow-interchange wave equation (section 5). Throughout, we pause to mention the equation obtained by Wolf et al. (2018) for the buoyancy waves in the absence of velocity shear, in general and under the same limits. Appendix A contains a discussion of the way in which Voronkov et al. (1997) obtained the shear flow-ballooning wave equation by a particular misuse of the momentum equation (section 3.1) in his linearization procedure.

The primary result is the shear flow-interchange wave equation and what it entails. The shear flow-interchange instability is the relevant instability which occurs in the magnetotail prior to substorm onset. The shear flow-ballooning instability equation is incorrect (resulting from a combination of inconsistencies and inappropriate assumptions in the perturbation of the momentum equation) and therefore eliminated as a proposal, and the interchange wave equation for buoyancy waves absent shear flow lacks the generality of the full shear flow-interchange wave equation. Shear flow-interchange instability should replace the Voronkov et al. (1997) shear flow-ballooning instability proposals to explain magnetospheric phenomena in the appropriate limits. More is said about this in section 5, as it is a somewhat subtle issue. What was a destabilizing ballooning term is really seen to be replaced by a stabilizing interchange term in the region of interest. The shear flow couples to the interchange instability in a way which reduces the growth rates relative to shear flow-ballooning instability, and a fortiori to pure shear flow (Kelvin-Helmholtz) instability. Local WKB stability analysis is performed in the regions under consideration, and spatial scale of the instability is determined.

In summary, it appears that a shear flow-interchange instability in the midnight region of the nightside magnetopause is the most plausible link in the causal chain of events which initiate substorm onset via earthward traveling shear flow-buoyancy waves, and results in structures in the aurorae in the E layer of the ionosphere. After perturbations drive an instability, the linear equations and dispersion relations become invalid. Nonlinear analysis will be developed in future work on the instability.

2. Wedge Model for Local Nightside Geomagnetic Tail Plasma

First, we set up a cylindrical coordinate system in the near-Earth nightside plasma sheet, seen to the right in Figure 6. The center of the magnetosphere is taken to be the origin of cylindrical coordinates \((r, \phi, y)\). The \(y\) axis is that of standard Solar Magnetic (SM) or Geocentric Solar Magnetic (GSM) coordinates, perpendicular to the magnetic dipole and the Earth-Sun line. Distance from the \(y\) axis is given by the \(r\) coordinate, which
Figure 6. Near-Earth nightside magnetosphere and wedge model. (a) Nightside region of magnetosphere showing equilibrium magnetic field lines shortly prior to substorm onset. (b) Diagram of the coordinate system for the wedge-shaped region of the magnetosphere under consideration. Field lines are approximated as concentric arcs. The $y$ axis is that of standard SM or GSM coordinates, perpendicular to the magnetic dipole and the Earth-Sun line. Distance from the $y$ axis is given by the $r$ coordinate, which specifies the distance of the tubes of magnetic flux from the center of the magnetosphere. Red boxes surround coincident regions in nightside magnetosphere and wedge model. Part (b) is reproduced from (Wolf et al., 2018).

specifies the distance of the tubes of magnetic flux from the center of the magnetosphere, and hence the local magnetic curvature. The transformation to the coordinates of Voronkov et al. (1997) is simple in the local plasma sheet region but globally aligns more naturally with the magnetospheric structures of interest.

Following Wolf et al. (2018), the simplest geometry has been chosen which still allows magnetic tension to support magnetic buoyancy oscillations to drive earthward flow, so that analytical solution for the eigenvalues is possible. Our more general equation includes coupling to shear flow velocity without sacrifice to this point. The two places at the upper and lower $\phi$-boundary (at $\pm \Delta \phi/2$) of the wedge represent the northern and southern ionospheres, taken by approximation to have no conductance owing to the absence of a field-aligned current in the model. The results of this oversimplification on thin filament oscillations for buoyancy waves in the absence of velocity shear are investigated in Wolf et al. (2012a), Wolf et al. (2012b), Wolf et al. (2018). The primary effect is to reduce the resistive damping of modes, which is not insignificant for realistic conductances on field lines in the inner magnetosphere. It is worth noting that damping becomes less significant for longer plasma sheet field lines with higher mass density owing to inertial effects.

We will for the background parameters use a model which has more appropriately stretched field lines for the substorm growth phase than the T96 model (Tsyganenko, 1995; Tsyganenko, 1996a; Tsyganenko, 1996b). Magnetic field lines get stretched substantially tailward during the growth phase prior to substorm onset (Pulkkinen et al., 1991; Yang et al., 2011). Plots of field line stretching during the growth phase are shown in Figure 7 below from three different models.
Figure 7. Plots of substorm growth phase current intensities and field line mappings. (top) On the left, integrated current intensities (total, ring, and tail currents) over the current sheet thickness in the ISEE-1 meridian (a) in the basic T89 model, and (b) in Pulkkinen modified model with maximal parameter values. On the right, field line projections to the ISEE-1 meridian (146°) computed using (c) the basic T89 model and (d) the Pulkkinen model. This figure is reproduced from Pulkkinen et al. (1991). (bottom) Plasma β and magnetic field line mappings (every degree from 62° to 66° latitude) in the midnight meridian plane at T = 0, 30, and 55 min during the growth phase from the Rice Convection Model. This figure is reproduced from Yang et al. (2011).
It is worth mentioning in passing that the lack of resistivity and the fixed field lines also exclude the necessary conditions for a characterization of any tearing modes which may be triggered by the shear flow-interchange instability in the region where the field lines are highly stretched. Since this instability will operate on longer timescales, it is left untreated, and as an independent hypothetical link in the sequence of events which constitute the substorm.

The system is taken to be at rest in equilibrium. Background equilibrium quantities are labeled with “0” subscripts, and \( \delta \) signifies perturbations. Magnetic field lines are approximated by concentric circles, and density, pressure, and magnetic field strength vary radially. We consider small perturbations which do not induce motion in the \( \phi \)-direction, so that \( k_\phi = k_\parallel = 0 \). Note that this implies that the field lines remain unperturbed. The pressure dynamics are modeled as adiabatic, entropy \( K := PV / \Gamma \) constant, with adiabatic gas constant \( \Gamma = (f + 2) / f \), where \( f \) is the number of degrees of freedom, and \( V(r) = r \Delta \phi / B(r) \) for the flux tube volume. Equilibrium force balance is given by

\[
-\frac{\partial}{\partial r} \left( P_0 + \frac{B_0^2}{2 \mu_0} \right) + \frac{B_0^2}{\mu_0 r} = 0.
\]

(1)

Note that the flux tube has a curvature toward the center of the Earth in the equatorial region, and the flux tube radius \( r \) is just the local radius of magnetic curvature.

Background parameters take the following form in this model:

\[
\vec{B}_0 = B_0(r) \hat{\phi}
\]

(2)

\[
\vec{v}_0 = v_0(r) \hat{y}
\]

(3)

\[
\rho_0 = \rho_0(r)
\]

(4)

\[
P_0 = P_0(r).
\]

(5)

So we assume that all equilibrium quantities are static and depend only on radius. The velocity has the form of an axially directed \( \vec{E} \times \vec{B} \) shear flow. More details about this model, such as the specific radial profiles of the background parameters and some of the perturbations which result under the assumption of no shear velocity (only buoyancy waves) can be found in Wolf et al. (2018).

It will be helpful now to introduce the concept of flux tube volume. Since flux tubes are the operative theoretical construct in this model, we will have an additional condition connecting the density to the magnetic field, through the flux tube volume. The flux tube volume is given by \( V(r) = \int ds / B \), where the performance of the integral in the case of the wedge model’s flux tubes yields \( V(r) = \frac{r \Delta \phi}{B(r)} \).

3. Linear Dynamics of Geomagnetic Tail Plasma

We begin with the MHD field equations:

\[
\frac{\partial}{\partial t} \left( \frac{P}{\rho^2} \right) + \vec{v} \cdot \nabla \left( \frac{P}{\rho^2} \right) = 0
\]

(6)

\[
\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) = 0
\]

(7)

\[
\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla \left( P + \frac{B^2}{2 \mu_0} \right) + \frac{\vec{B}}{\mu_0} \cdot \nabla \vec{B}.
\]

(8)

A flowchart (Figure 8) of the inferential pathways and corresponding assumptions for all equations to be analyzed is included below for reference. It should facilitate a global view of the interrelations.

3.1. Linearized MHD Equations for Wedge Flux Tube

Now, we apply the system of equations to the wedge formalism. Let us survey each equation and discuss.

We can from the start bypass the continuity equation which Voronkov et al. (1997) uses as an additional constraint by using the adiabaticity of a flux tube, with flux tube volume defined above:

\[
V := \frac{r \Delta \phi}{B}
\]

(9)
Figure 8. Inferential pathway flowchart. Nodes signify important equations or systems of equations, and arrows signify physical assumptions made in the transition between nodes. The assumptions for a particular transition are specified to the right of the given arrow. The equation which most plausibly supplies an explanation for the instabilities which occur in the magnetotail before substorm onset is the reduced low-frequency shear flow-interchange wave equation.

in the adiabatic pressure dynamics equation. Note that the $\Delta \phi$ is a constant. It is fitting to define the entropy:

$$K := PV^T$$

(10)

for future use in eliminating pressure and more intuitively representing interchange dynamics of the flux tubes under consideration. It should be noted that the flux tube volume should not be treated as inverse density and was defined with respect to adiabatic flux tube dynamics. The adiabatic pressure dynamics using the convective time derivative:

$$\frac{D}{Dt} (PV^T) = 0,$$

(11)

thus take the following form for a flux tube in the wedge formalism:

$$\frac{D}{Dt} \left( \rho \left( \frac{r}{B} \right)^r \right) = 0.$$

(12)
Substituting the total fields and linearizing, we obtain the equation which governs the pressure fluctuation dynamics:

\[
\left( \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial y} \right) \left( \frac{\delta P}{P_0} - \Gamma \frac{\delta B}{B_0} \right) + \frac{K'_0}{K_0} \delta v_r = 0.
\]  (13)

Henceforth, partial radial derivatives will be indicated by primes when convenient.

The magnetic field dynamics in the flux tube are governed by Faraday’s law with

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}),
\]  (14)

Substituting the total fields and linearizing, we obtain

\[
\frac{\partial \delta B_r}{\partial t} = -v_0 \frac{\partial \delta B_r}{\partial y},
\]  (15)

\[
\frac{\partial \delta B_\phi}{\partial t} = -v_0 \frac{\partial \delta B_\phi}{\partial y} - B_0 \frac{\partial \delta v_r}{\partial y} - B_0 B'_0 \delta v_r,
\]  (16)

\[
\frac{\partial \delta B_y}{\partial t} = \frac{v_0}{r} \delta B_r + v'_0 \delta B_r + v_0 \delta B'_r,
\]  (17)

Now, we examine the momentum equation, which governs the plasma dynamics:

\[
\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla \left( P + \frac{B^2}{2 \mu_0} \right) + \frac{\vec{B} \cdot \nabla \vec{B}}{\mu_0}.
\]  (18)

Substituting the total fields and linearizing, we obtain

\[
\rho_0 \frac{\partial \delta v_r}{\partial t} + \rho_0 v_0 \frac{\partial \delta v_r}{\partial y} = -\frac{\partial}{\partial r} \left( \delta P + \frac{B_0}{\mu_0} \delta B_\phi \right) - \frac{2B_0}{\mu_0} \delta B_\phi,
\]  (19)

\[
\rho_0 \frac{\partial \delta v_\phi}{\partial t} + \rho_0 v_0 \frac{\partial \delta v_\phi}{\partial y} = \frac{B_0}{\mu_0} \delta B_r + \frac{B'_0}{\mu_0} \delta B_r,
\]  (20)

\[
\rho_0 \frac{\partial \delta v_y}{\partial t} + \rho_0 v_0 \frac{\partial \delta v_y}{\partial y} + \rho_0 v'_0 \delta v_r = -\frac{\partial \delta P}{\partial y} = \frac{B_0}{\mu_0} \frac{\partial \delta B_\phi}{\partial y},
\]  (21)

which governs the plasma acceleration.

In the wedge formalism, the continuity equation is unnecessary, as the adiabaticity of the flux tube volume supplies the constraint which continuity would have supplied for density. Density is not perturbed in this model.

4. Magnetospheric Wave Equation for Plasma Wedge

Now, we assume all perturbations take the form of axially propagating waves \( e^{ik_y y - i\omega t} \) in the plasma sheet, denoting the Doppler-shifted frequency:

\[
\omega(r) : = \omega - k_y v_0(r),
\]  (22)

as these are the waves which will map back to the ionosphere to cause the longitudinally directed auroral beads. Note that this converts primes into total rather than partial radial derivatives.

Substituting this form into our self-consistent set of dynamical equations (13)–(21), we obtain
\[ i \tilde{\omega} \left( \frac{\delta P}{P_0} - \Gamma \frac{\delta B}{B_0} \right) = \frac{K'_0}{K_0} \delta v_r \]  

(23)

\[ -i \tilde{\omega} \delta v_r = 0 \]  

(24)

\[ -i \tilde{\omega} \delta B_\phi = -ik_r B_0 \delta v_r - B'_0 \delta v_r - B_0 \delta v'_r \]  

(25)

\[ -i \omega \delta B_r = \frac{v_0}{r} \delta B_r + v'_0 \delta B_r + v_0 B'_r \]  

(26)

\[ -i \tilde{\omega} \mu_0 \delta P = -\frac{\partial}{\partial r} \left( \delta P + \frac{B_0}{\mu_0} \delta B_\phi \right) - 2 \frac{B_0}{\mu_0} \delta B_\phi \]  

(27)

\[ -i \tilde{\omega} \mu_0 \delta v_\phi = \frac{B_0}{\mu_0} \delta B_\phi + \frac{B'_0}{\mu_0} \delta B_r \]  

(28)

\[ -i \tilde{\omega} \mu_0 \delta v_r + \rho_0 \left( \frac{v_0}{\mu_0} \delta P - ik_r B_0 \delta B_\phi \right) = 0 \]  

(29)

We can now see that (24), (26), and (28) imply the following perturbation components:

\[ \delta \vec{B} = \delta B_\phi \hat{\phi} \]  

(30)

\[ \delta \vec{v} = \delta v_r \hat{r} + \delta v_\phi \hat{\phi}. \]  

(31)

So three of the equations are now implicitly taken into account, and from the remaining of equations, (23)–(29), we obtain the following system:

\[ i \tilde{\omega} \left( \frac{\delta P}{P_0} - \Gamma \frac{\delta B}{B_0} \right) = \frac{K'_0}{K_0} \delta v_r \]  

(32)

\[ i \tilde{\omega} \frac{\delta B_\phi}{B_0} = ik_r \delta v_r + \frac{v'_0}{B_0} \delta v_r \]  

(33)

\[ i \tilde{\omega} \mu_0 \delta v_r = \frac{\partial}{\partial r} \left( \delta P + \frac{B_0}{\mu_0} \delta B_\phi \right) + 2 \frac{B_0}{\mu_0} \delta B_\phi \]  

(34)

\[ i \tilde{\omega} \mu_0 \delta v_r - \rho_0 \left( \frac{v_0}{\mu_0} \delta P - ik_r B_0 \delta B_\phi \right) = ik_r \delta P + ik_r \frac{B_0}{\mu_0} \delta B_\phi. \]  

(35)

From now on, it will be convenient to make frequent use of the Alfvén speed, sound speed, and fast mode wave speeds given by \( c_A^2 \) := \( \frac{B_0^2}{\mu_0 \rho_0} \), \( c_s^2 \) := \( \Gamma \frac{P_0}{\rho_0} \), and \( c_f^2 \) := \( c_A^2 + c_s^2 \), respectively. Now, for convenience, the equilibrium force balance equation can be recast as a condition to eliminate \( B_0 \) in lieu of \( K_0 \):

\[ \frac{B'_0}{B_0} = \frac{1}{c_f^2} \left( -\frac{c_A^2 K'_0}{\Gamma K_0} + \frac{c_s^2 - c_A^2}{r} \right). \]  

(36)

which will make manifest the interchange instability and its stability conditions.

Eliminating \( \delta v_r \), \( \delta P \), and \( \delta B_\phi \), we obtain the differential equation for the radial velocity fluctuations \( \delta v_r \), written in a form which most resembles that of Wolf et al. (2018):
Indeed, in this form, it is easy to see that dropping velocity shear terms yields precisely the equation in Wolf et al. (2018):

\[
\omega^2 \delta v_r = \frac{\omega}{\rho_0} \frac{d}{dr} \left( \frac{\delta v_r \rho_0 \omega}{r} \left( \frac{c_i^2 - c_j^2}{\omega^2 - k_j^2 c_j^2} \right) - \frac{d \delta v_r}{dr} \frac{c_j^2 \rho_0 \omega}{\omega^2 - k_j^2 c_j^2} - \frac{\delta v_r \rho_0 \omega}{r} \frac{k_j \omega \delta v_r'}{r \omega^2 - k_j^2 c_j^2} \right) \]

\[
+ \frac{2 c_i^2}{r} \frac{\delta v_r}{\Gamma c_j} \left[ \frac{K'_0}{c_j} \frac{c_i^2}{K_0} - \frac{\omega^2 \Gamma}{r} \left( \frac{c_i^2 - c_j^2}{\omega^2 - k_j^2 c_j^2} \right) \right] - \frac{d \delta v_r}{dr} \frac{\omega^2}{\omega^2 - k_j^2 c_j^2} - \frac{\delta v_r \omega v'}{r \omega^2 - k_j^2 c_j^2} \right) \].

(37)

Note that the frequencies are no longer Doppler-shifted (there is no shear velocity to supply the shift!). The objective of Wolf et al. (2018) was to study buoyancy waves in the magnetosphere, and velocity shear terms were thus neglected in order to facilitate a clearer understanding of the interchange-induced buoyancy waves, with buoyancy force arising from magnetic tension rather than gravity. This equation still describes both fast mode longitudinal and buoyancy waves in the plasma wedge, but the former are easily eliminated, which we will demonstrate in what follows.

With all derivatives performed, and all terms expanded, the magnetospheric wave equation takes the following form:

\[
\delta v_r'' + B(r) \delta v_r' + C(r) \delta v_r = 0,
\]

(39)

with coefficients

\[
B(r) := \frac{1}{r} + \frac{\omega^2}{\omega^2 - k_j^2 c_j^2} \frac{P_0'}{c_j} + \frac{\omega^2}{\omega^2 - k_j^2 c_j^2} \frac{2 c_i^2}{c_j} \frac{B_0'}{B_0} - \frac{k_j^2 c_j^2}{\omega^2 - k_j^2 c_j^2} \rho_0 + \frac{2}{\omega} \frac{\omega^2}{\omega^2 - k_j^2 c_j^2}
\]

\[
C(r) := \frac{k_j \omega}{\omega} + 2 \frac{\omega}{\omega^2} - \frac{2 c_i^2}{c_j} \frac{B_0'}{B_0} - \frac{2 c_i^2}{c_j} \frac{B_0'}{B_0} - \frac{2 c_i^2}{c_j} \frac{B_0'}{B_0} - \frac{2 c_i^2}{c_j} \frac{B_0'}{B_0} - \frac{2 c_i^2}{c_j} \frac{B_0'}{B_0} - \frac{2 c_i^2}{c_j} \frac{B_0'}{B_0}
\]

\[
+ \frac{k_j \omega}{\omega} \frac{\omega^2}{\omega^2 - k_j^2 c_j^2} \frac{B_0'}{B_0} + \frac{k_j \omega}{\omega} \frac{\omega^2}{\omega^2 - k_j^2 c_j^2} \frac{B_0'}{B_0} + \frac{k_j \omega}{\omega} \frac{\omega^2}{\omega^2 - k_j^2 c_j^2} \frac{B_0'}{B_0} + \frac{k_j \omega}{\omega} \frac{\omega^2}{\omega^2 - k_j^2 c_j^2} \frac{B_0'}{B_0}
\]

\[
+ \frac{1}{r} + \frac{\omega^2}{\omega^2 - k_j^2 c_j^2} \frac{1}{r} \frac{1}{r} - \frac{1}{r} \frac{1}{r} - \frac{1}{r} \frac{1}{r} - \frac{1}{r} \frac{1}{r} - \frac{1}{r} \frac{1}{r}
\]

\[
+ \frac{1}{r} + \frac{\omega^2}{\omega^2 - k_j^2 c_j^2} \frac{1}{r} \frac{1}{r} - \frac{1}{r} \frac{1}{r} - \frac{1}{r} \frac{1}{r} - \frac{1}{r} \frac{1}{r}
\]

(41)

5. Reduced Low-Frequency Wedge Wave Equation

We now eliminate fast modes, assuming \( \omega^2 \ll k_j^2 c_j^2 \), retaining only those modes which play a substantial role in substorm onset. Upon doing so, we obtain

\[
\delta v_r'' + \left( \frac{1}{r} + \frac{\rho_0}{\omega} - \frac{2 k_j \omega}{\omega^2} \frac{\omega^2}{k_j^2 c_j^2} \right) \delta v_r' + C(r) \delta v_r = 0
\]

(42)
As written by Voronkov et al. (1997), this has the form

Thus, the first term in parentheses yields immediately the buoyancy frequency for waves in a wedge:

\[
\delta v'' + \left( \frac{\rho_0'}{\rho_0} - \frac{2k\omega v_0' 2k^2 c_s^2}{\tilde{\omega}^2} \right) \delta v' + \left( \frac{k\omega v_0'}{\omega} + \frac{k\omega v_0'}{\omega} \frac{\rho_0'}{\rho_0} - 2 \left( \frac{\omega}{c_s} \right)^2 + \frac{k^2 c_s^2}{\omega^2} \frac{2}{\Gamma r} \frac{c_s^2 c_s K_0'}{c_s^2 K_0} - k_2^2 \right) \delta v_r = 0,
\]

(44)

with the $E \times B$ shear flow velocity $v_0(r)$ and local magnetic curvature determining the dynamic stability conditions. This equation for low-frequency waves in the wedge captures the most general dynamical phenomena relevant to the causal chain of events which we aim to describe. The first term in the first derivative coefficient is an inertial damping term. The first term in the zeroth derivative coefficient is the shear flow instability term, and the fourth term is the interchange instability term that supplies the buoyancy frequency.

Though it does not pertain to the more general analysis at hand, it should be mentioned that these limits, taken in the appropriate order, agree with those in Wolf et al. (2018), barring what appear to be minor typographical errors (as seen by a unit check) on his part (equations (19) and (20) of Wolf et al. (2018)).

The buoyancy frequency, which was thoroughly discussed in Wolf et al. (2018), is given by the next-to-last term in our equation. Let us perform this check. Dropping the shear velocity terms, we obtain

\[
\delta v'' + \left( \frac{\rho_0'}{\rho_0} - \frac{2k\omega v_0' 2k^2 c_s^2}{\tilde{\omega}^2} \right) \delta v' + \left( \frac{1}{\omega^2} \frac{2}{\Gamma r} \frac{c_s^2 c_s K_0'}{c_s^2 K_0} - 1 \right) k_2^2 \delta v_r = 0
\]

(45)

Thus, the first term in parentheses yields immediately the buoyancy frequency for waves in a wedge:

\[
\omega_b^2(r) = \frac{2}{\Gamma r} \frac{c_s^2 c_s K_0'}{c_s^2 K_0}
\]

(46)

It is demonstrated in Wolf et al. (2018) that this is just the oscillation frequency of thin magnetic filaments in the wedge. The speed $c_s c_a/c_s$ is just that of the slow mode buoyancy waves which result from pure interchange oscillations. Notice also that the frequencies are no longer Doppler-shifted, as there is no fluid velocity to supply the shift.

Recast in the above notation, the Voronkov et al. (1997) result (obtained in a methodologically similar way corresponding (roughly) to limits taken in section 5, though with the additional assumption that density gradient scales are large) utilized by Kalmoni et al. (2015) is

\[
\delta v'' + \left( \frac{v_0'}{k_2 \omega} - \frac{8}{\omega^2} \frac{\rho_0'}{\rho_0} - \frac{1}{\omega^2} \frac{2}{c_s^2} - 1 \right) k_2^2 \delta v_r = 0.
\]

(47)

As written by Voronkov et al. (1997), this has the form
We perform this formal simplification as follows. Let

\begin{equation}
\delta v''_r + \left( \frac{\nu''_r}{k_\nu} + \frac{W}{\omega^2} - 1 \right) k_y^2 \delta v_r = 0.
\end{equation}

where

\begin{equation}
W := -\frac{g_0^2}{\rho_0} - \frac{g^2}{c_f^2},
\end{equation}

with effective acceleration defined above. The term W obtained by Voronkov et al. (1997) was taken to be an analog of the buoyancy frequency, \(\omega_b^2\). Near the inner edge of the plasma sheet, at approximately 9–12 \(R_E\), the analysis of Kalmoni et al. (2015) indicates that these terms are destabilizing, whereas the above derivation of (44) produces corresponding terms which are positive definite, and hence stabilizing. The model setup for both Voronkov et al. (1997) and Wolf et al. (2018) disallows unstable modes to perturb the background equilibrium magnetic field, which is why interchange terms should result rather than ballooning terms.

These discussions are reproduced in the respective appendices so that they appear both for comparison with equilibrium magnetic field, which is why interchange terms should result rather than ballooning terms.

### 6. Examination of Stability Conditions

Stability analysis for the shear flow-interchange modes resulting from equation (44) is performed locally using WKB approximation. Since the length scales for several background gradients in the region have been assumed small in comparison to the shear flow width, we can examine the propagation and growth of a wave packet in the region centered on some wavenumber, with a width much smaller than the gradient length scales. Assuming that the growth rate is smaller than the frequency, the wave packet can propagate for some time governed by the linearized equations. Upon undergoing about 10 e-foldings, the linearized treatment must be replaced by the full nonlinear analysis. The subcase interchange instability (treated in more detail in Wolf et al. (2018)) is discussed in Appendix B for comparison. While we do not discuss stability conditions for the Voronkov et al. (1997), it is worth noting that what were ballooning terms under that treatment are seen to be interchange terms. The phenomenological import of this is that the former are localized, whereas the latter are globally distributed along the magnetic field lines.

#### 6.1. Shear Flow-Interchange Instability

In preparation for the obtainment of a dispersion relation, let us recast the differential equation in a simpler form to see the wavelike properties more clearly by extracting the radius-dependent prefactor as follows. For convenience, the equation for shear flow-interchange modes again is

\begin{equation}
\delta v''_r + \left( \frac{\rho_0^2}{\rho_0^2 - 2k_x v'_r - \omega^2} \right) k_y^2 \delta v'_r + \left( \frac{k_x v''_r}{\omega} + \frac{k_x v'_r}{\omega} - 2 \left( \frac{v'_r}{c_f} \right)^2 + \frac{k_x^2 c_f^2}{\omega^2} \frac{2}{\Gamma r} \frac{2 c_f^2}{c_f^2} k_y^2 \right) \delta v_r = 0.
\end{equation}

We perform this formal simplification as follows. Let

\begin{equation}
\delta v_r := \alpha(r) \delta u_r,
\end{equation}

so that

\begin{equation}
\delta v'_r = \alpha' \delta u'_r + \alpha \delta u''_r,
\end{equation}

\begin{equation}
\delta v''_r = \alpha'' \delta u_r + 2 \alpha' \delta u'_r + \alpha \delta u''_r.
\end{equation}

Now, we choose an \(\alpha\) such that the prefactor to \(\delta u'_r\) is 0 in the new variables:

\begin{equation}
\alpha(r) = \frac{\text{Const.}}{\sqrt{\rho_0}} \exp \left( \int \frac{\delta u''_r}{k_x c_f^2} \right),
\end{equation}

whereby

\begin{equation}
\frac{\alpha'}{\alpha} = \frac{k_x v'_r}{\omega} \frac{\omega^2}{k_x^2 c_f^2} - \frac{1}{2} \frac{\rho_0}{\rho_0^2}.
\end{equation}
so we obtain the following form:

$$
\delta u''_r + \left( \frac{1}{4} \left( \frac{\rho'_0}{\rho_0} \right)^2 - \frac{1}{2} \frac{\rho''_0}{\rho_0} + \frac{k_r v'_r}{c_r} - \frac{k_r v'_0}{c_r} \rho'_0 \rho_0 \right) - 3 \left( \frac{v'_0}{c_r} \right)^2 + \frac{k_r^2 c_r^2 k_0^0}{\Gamma c_r c_r c_r} - k_r^2 \delta u_r = 0.
$$

which manifestly has wave solutions (insofar as WKB approximation validates assuming the coefficients can be treated as roughly constant over the width of the wave packet):

$$
\delta u_r(r) \sim e^\Gamma r.
$$

Recall that it is the power spectral density which is obtained by Fourier transforming the ASI intensity keograms to obtain the instability growth rates. But the power spectral density is just the energy density being transported at the fluid velocity. It is the fluid velocity which will contain the growth rate, and radial and axial velocities have the same growth rate. This is just a reminder that we can straightforwardly proceed by identifying the radial fluid velocity growth rate obtained here with the empirically obtained power spectral density growth rates.

Upon converting our magnetospheric wedge wave equation into a dispersion relation for shear flow-interchange modes, with radial oscillations having wavenumber $k_r \sim 1/\delta$, keeping in mind that the radius needs to be Doppler-shifted, we obtain

$$
\left( \frac{1}{4} \rho'_0 \frac{\rho'_0}{\rho_0} - \frac{1}{2} \frac{\rho''_0}{\rho_0} + \frac{k_r v'_r}{c_r} - \frac{k_r v'_0}{c_r} \rho'_0 \rho_0 \right) - 3 \left( \frac{v'_0}{c_r} \right)^2 + \frac{k_r^2 c_r^2 k_0^0}{\Gamma c_r c_r c_r} - k_r^2 \delta u_r = 0.
$$

which manifestly has wave solutions (insofar as WKB approximation validates assuming the coefficients can be treated as roughly constant over the width of the wave packet):

$$
\delta u_r(r) \sim e^\Gamma r.
$$

Now, where growth and oscillations occur simultaneously, we should examine the situation where the growth rate is much smaller than the frequency, so that we can consider a propagating wave packet which exponentially grows as it travels until it reaches the point where nonlinearities must be considered. The growth rate is much larger than the frequency, the instability will grow too quickly for the packet to propagate before nonlinear analysis becomes necessary. Marginal stability analysis can now be used to determine the radius and radial wavenumber for the most unstable modes (by finding growth rate extrema). The wave packet will initially be centered at the radial wavenumber and begin its propagation at the radial extremum. The axial wavenumber remains fixed during propagation, since the frequency has no y-dependence. The instability extrema will occur near radial wavenumbers:

$$
k_r^2(r) = \pm \sqrt{\frac{1}{4} \left( \frac{\rho'_0}{\rho_0} \right)^2 - \frac{1}{2} \frac{\rho''_0}{\rho_0} - 3 \left( \frac{v'_0}{c_r} \right)^2 - \frac{1}{2} \frac{k_r^2 c_r^2 k_0^0}{\Gamma c_r c_r c_r} - k_r^2}.
$$
Without background profiles, the general expression for the radius at which growth rate is optimized is impossible to analytically express. The general constraint condition on background profiles which must hold at that radius is unilluminating. But we can examine the extremization condition on radius at the radial wavenumber extremum:

\[
\frac{2}{r} \frac{c_i^2}{\Gamma_i} \frac{K_0'}{K_0} = \frac{V}{v_0} \left( \frac{v_0''}{v_0} - \frac{v_0'}{v_0} \frac{\rho_0'}{\rho_0} \right).
\] (63)

where \( V \) is a radially constant velocity. We can write this as an explicit definition of the new velocity:

\[
V := \frac{2}{r} \frac{c_i^2}{\Gamma_i} \frac{K_0'}{K_0} \frac{v_0''}{v_0} - \frac{v_0'}{v_0} \frac{\rho_0'}{\rho_0} v_0. \] (64)

This means that our peak wavenumber can be recast in the following convenient form:

\[
k_r^* (r^*) = \pm \sqrt{\frac{1}{4} \left( \frac{\rho_0'}{\rho_0} \right)^2 - \frac{1}{2} \frac{\rho_0'}{\rho_0} - 3 \left( \frac{v_0''}{v_0} \right)^2 - \frac{1}{2} \frac{v_0'}{v_0} + \frac{1}{2} \frac{v_0'}{v_0} \frac{\rho_0'}{\rho_0} - k_y^2}. \] (65)

For the maximally sensitive mode, the growth rate and frequency are given by

\[
\omega_r = k_y (v_0 - V) \] (66)

\[
\gamma = \mp k_y V. \] (67)

Notice that this implies that the Doppler-shifted real part becomes precisely

\[
\tilde{\omega}_r = -k_y V. \] (68)

It is clear at this point that \( V \) is just the Doppler-shifted phase velocity of the propagating wave packet. Note in addition that the growth rate sign is always positive, and the choice of explicit sign cancels the implicit sign of \( k_r \) so as to maintain independence of propagation direction. Thus extremized, our growth-to-frequency ratio is seen to be given by

\[
\frac{\gamma}{\omega_r} = \pm \frac{V}{V - v_0}. \] (69)

which will be small when \( v_0 \gg V \). It is under these circumstances that we will have a growing propagating wave packet. The group velocity and time variation of peak wavenumber are generally given by

\[
\dot{r} = \frac{\partial \omega_r}{\partial k_r} \] (70)

\[
\dot{k}_y = \frac{\partial \omega_r}{\partial y} \] (71)

\[
\dot{k}_r = \frac{\partial \omega_r}{\partial r}. \] (72)

For the maximally unstable mode, the packet will evolve according to the following:

\[
\dot{r} = -\frac{4k_y k_r^* V}{\frac{v_0'}{v_0} - \frac{\rho_0'}{\rho_0} \frac{\rho_0'}{\rho_0} k_r}. \] (74)
The growth factor can thus be determined for this mode:

$$\int \gamma dt = \int \frac{\gamma}{F} dr = \frac{1}{4} \int \frac{\frac{\nu'_0}{V} - \frac{\nu'_0}{\rho_0}}{\sqrt{\frac{1}{4} \left( \frac{\nu'_0}{\rho_0} \right)^2 - \frac{1}{2} \frac{\rho'_0}{\rho_0} - 3 \left( \frac{\nu'_0}{c_f} \right)^2 - \frac{1}{2} \frac{\nu'_0}{V} + \frac{1}{2} \frac{\nu'_0}{\rho_0} - k_y^2}} dr. \quad (78)$$

This is the real part of the most unstable mode’s exponent during its linear evolution. When it becomes $\sim 10$, the linear approximations become invalid and nonlinear analysis becomes necessary.

### 6.2. WKB Applicability Conditions

Let us again examine the wavenumber for the most unstable mode to see if it satisfies the full applicability conditions for WKB:

$$k_y^*(r^*) = \sqrt{\frac{1}{4} \left( \frac{\rho'_0}{\rho_0} \right)^2 - \frac{1}{2} \frac{\rho'_0}{\rho_0} - 3 \left( \frac{\nu'_0}{c_f} \right)^2 - \frac{1}{2} \frac{\nu'_0}{V} + \frac{1}{2} \frac{\nu'_0}{\rho_0} - k_y^2}. \quad (79)$$

In order to apply WKB without sacrifice to quantitative accuracy, we had to be warranted in assuming that once we recast our equation in order to obtain wave solutions, we could treat $C(r)$ as constant over length scales on which the wave packet varied. That is in addition to the assumed $k_y^{-1} \ll r$ and $k_y^{-1} \ll L$, where $L$ was the length scale of magnetic field and pressure gradients, which is implicit in the reduced wedge wave equation.

We can see that the first term under the square root, however, that the wave packet will vary on the scale of the density gradient. That is, we will always have $k_y \sim \rho'_0/\rho_0$. On the one hand, this unfortunately entails that errors in the WKB analysis will be of order unity, but it also immediately yields the scale on which variations in the wave packet occur! The WKB analysis will only allow for a qualitative description of the time evolution of the instability in the linear regime, but the spatial grid size for a full nonlinear analysis is now determined. Once the location of the most unstable mode in the geotail is determined, the initial instability will vary on length scales approximately equal to the width of density gradient length scales in the region.

### 7. Conclusions

In summary, the shear flow-interchange wave equation best accommodates the circumstances under which an instability in the magnetotail initiates transmission of $B \times \nabla B$ sheared zonal flows along magnetic flux tubes toward the $E$ layer of the Earth’s ionosphere. WKB analysis yields a qualitative description of the most unstable propagating wave packet. A propagating wave packet undergoes growth as it travels, and is analyzed to the fullest extent that the linear analysis will allow, yielding results which can be supplemented by background parameter models to yield growth rates and dispersion properties for comparison with the auroral bead patterns mapped back along magnetic field lines to the magnetotail. For a quantitatively accurate picture, however, the full nonlinear analysis needs to be done and the reduced wedge wave equation needs solved numerically. The applicability conditions yield the spatial scale for variations in the instability for the nonlinear analysis.

The wedge model has several oversimplifications worth mentioning. One discussed above is the absence of a conductance (field-aligned current) on the angular boundaries of the wedge. Gravity and a background centrifugal acceleration were taken to be negligible. Frozen-in flux (via flux tube densities and obedience to Ohm’s law) and adiabaticity were also both assumed. The model is also symmetric about the equato-
rial plane. As a fluid model, it neglects kinetic effects, so that relevant physical features with spatial extent smaller than the ion Larmor radius are not taken into account. There may be some limiting procedures and relations between the instabilities arising from the MHD model in this paper and the results discussed in Kalmoni et al. (2018), which claims that auroral beads are likely the signature of kinetic shear Alfvén waves driven unstable in the magnetotail prior to substorm onset.

The wedge model and resulting shear flow-interchange instability analysis can be used to validate background parameter models and potentially allow for real-time prediction of substorm onset. Alternatively, combination of data yielding the background parameters as a function of radius in the region can be used to determine where precisely within the near-Earth magnetotail plasma sheet the instability is most likely to occur, and how an initial instability of local density gradient spatial scale size will propagate and disperse. Further work will involve modeling the full nonlinear equations using Tri Alpha Energy integrated with background parameters from models such as the Rice Convection Model (Harel et al., 1981; Sazykin & Wolf, 1981; Toffoletto et al., 2003; Wolf, 1983; Wolf et al., 2007; Wu et al., 2009) to model the time evolution of pressure, density, magnetic field strength, and fluid velocity in the magnetotail during magnetic substorms. The fully nonlinear wedge model will then be integrated with the WINMDI model (Horton et al., 1996; Horton et al., 2003; Horton et al., 2005; Horton & Doxas, 1998; Spencer et al., 2006) to attempt the real-time prediction of substorm onset and evolution.

Appendix A: Voronkov Treatment of Momentum Equation

Here I will briefly discuss the way in which the derivation of Voronkov et al. (1997) results from inconsistencies due to a misleading grouping of terms in Voronkov’s treatment of the momentum equation.

Equilibrium force balance can be written in general as

\[-\frac{\partial}{\partial r} \left( P_0 + \frac{B_0^2}{\mu_0} \right) + \frac{B_0^2}{\mu_0 r} = \frac{\rho_0 v_r^2}{r}. \tag{A1}\]

Voronkov recasts this in the following way:

\[\rho_g \dot{g} = \frac{\partial}{\partial r} \left( P_0 + \frac{B_0^2}{\mu_0} \right), \tag{A2}\]

where

\[\rho_g \dot{g} := \frac{B_0^2}{\mu_0 r} - \frac{\rho_0 v_r^2}{r}. \tag{A3}\]

Notice that \( g \) contains both a force term and an acceleration term.

Subsequently, the momentum equation gets perturbed with a \( \rho g \) term acting as a source, rather than \( B \cdot \nabla B / \mu_0 \). This is not a gravitational term, but a term with the \( g \) defined implicitly as above. There is no obvious reason that \( g \) in this form should be treated either as a background constant (it has radial dependence) or unperturbed. As far as the perturbation goes, the magnetic field term gets perturbed everywhere except within the \( g \) term. Also, equilibrium quantities are said to be in agreement with ours, contradicting the previous assumption of an azimuthal component of the background velocity. It is unclear how to fully characterize the discrepancy, but it is clear that the perturbation of the momentum equation is performed incorrectly.

Recast in the above notation, the Voronkov et al. (1997) result (obtained in a methodologically similar way corresponding [roughly] to limits taken in section 5, though with the additional assumption that density gradient scales are large) utilized by Kalmoni et al. (2015) is

\[\delta v_r'' + \left( \frac{v_r'}{k_f \omega_0} - \frac{\dot{g}}{\omega_0^2 \rho_0} - \frac{1}{\omega_0^2 c_s^2} - 1 \right) k_f^2 \delta v_r = 0. \tag{A4}\]

As written by Voronkov et al. (1997), this has the form
\[ \delta v'' + \left( \frac{v''}{k_\rho} + \frac{W}{\omega^2} - 1 \right) k^2 \delta v_r = 0, \]  
\text{(A5)}

where
\[ W := -\frac{g \rho'_0}{\rho_0} - \frac{g^2}{c_s^2}, \]  
\text{(A6)}

with effective acceleration defined above. The term \( W \) obtained by Voronkov et al. (1997) was taken to be an analog of the buoyancy frequency, \( \omega_b^2 \). Near the inner edge of the plasma sheet, at approximately 9–12 \( R_E \), the analysis of Kalmoni et al. (2015) indicates that these terms are destabilizing, whereas the above analysis reveals these terms to be stabilizing. This is due to differences between ballooning and interchange, where the former are often treated as localized and the latter is globally distributed along the magnetic field lines. The interchange instability results, rather than ballooning, since the unstable mode does not perturb the background equilibrium magnetic field.

Interchange instability is just a special case of the ballooning instability in which the unstable mode does not perturb the magnetic field lines. It is possible (though not necessary) that relaxing constraints on the wedge model will provide a real source of ballooning modes which would replace the interchange modes. But in order to obtain actual ballooning modes, the magnetic field lines must be free to move, which they are unable to do under the Voronkov et al. (1997) analysis. This should have been a phenomenological hint to the incorrectness of the result. The main point to stress is that it is the equation which is incorrect, but that a more correct treatment may indeed replace the shear flow-interchange instability with a shear flow-ballooning instability of some sort.

**Appendix B: Interchange Instability Analysis**

It should be mentioned that the limits from section 5, taken in the appropriate order, agree with those in Wolf et al. (2018), barring what appear to be minor typographical errors (as seen by a unit check) on his part. The buoyancy frequency, which was thoroughly discussed in Wolf et al. (2018), is given by the next-to-last term in our equation. Let us perform this check. Dropping the shear velocity terms in (44), we obtain
\[ \delta v'' + \frac{\rho_0'}{\rho_0} \delta v' + \left( \frac{1}{\omega^2} \frac{2}{\Gamma r} \frac{c_A c_s^2 K_0'}{c_f^2} - 1 \right) k^2 \delta v_r = 0 \]  
\text{(B1)}

Thus, the first term in parentheses yields immediately the buoyancy frequency for waves in a wedge:
\[ \omega_b^2(r) = \frac{2}{\Gamma r} \frac{c_A c_s^2 K_0'}{c_f^2} \]  
\text{(B2)}

where \( K \) is the entropy, defined
\[ K(r) := PV' = \frac{P}{\rho} \]  
\text{(B3)}

It is demonstrated in Wolf et al. (2018) that this is just the oscillation frequency of thin magnetic filaments in the wedge. The speed \( c_A c_s/c_f \) is just that of the slow mode buoyancy waves which result from pure interchange oscillations. Notice also that the frequencies are no longer Doppler-shifted, as there is no fluid velocity to supply the shift.

The interchange dispersion relation from Wolf et al. (2018), absent Doppler-shifts, is given by
\[ \omega^2 = \frac{k_z^2}{k_f^2 + k_z^2} \frac{2}{\Gamma r} \frac{c_A c_s^2 K_0'}{c_f^2} \]  
\text{(B4)}

By examination, we can see that there is no oscillatory growth. There is either propagation in the absence of growth or growth in the absence of propagation, and the condition for growth is just that
This situation is analyzed in much detail in Wolf et al. (2018).

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