Supersymmetry and the KdV equations
for Integrable Hierarchies
with a Half-integer Gradation

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Abstract
Supersymmetry is formulated for integrable models based on the $sl(2|1)$ loop algebra endowed with a principal gradation. The symmetry transformations which have half-integer grades generate supersymmetry. The $sl(2|1)$ loop algebra leads to $N=2$ supersymmetric mKdV and sinh-Gordon equations. The corresponding $N=1$ mKdV and sinh-Gordon equations are obtained via reduction induced by twisted automorphism. Our method allows for a description of a non-local symmetry structure of supersymmetric integrable models.

1 Introduction

We consider a class of models based on the superalgebras $osp(1|2)$ and $sl(2|1)$ with the principal gradation. We show that supersymmetry, when present, appears naturally as a part of the graded symmetry structure of integrable models based on superalgebras. Our formalism differs from approaches existing in the literature in that no superspace or superfield techniques are used. Our derivation relies solely on algebraic techniques. Supersymmetry originates from a superalgebra structure underlying the integrable models.

In models based on the $sl(2|1)$ algebra the supersymmetry is associated with the symmetry generators of the half-integer gradations. Our convention is such that the generator $E$ of a space gradient has a unit gradation. The supersymmetry transformations are realized as symmetry flows generated by elements of half-integer grading in a centralizer $K$ associated with $E$. An anticommutator of elements of $K$ that square to $E$ generates a gradient. Thus, a fundamental property of supersymmetry is automatically satisfied. The remaining integer generators in the centralizer $K$ give rise to non-local symmetries intertwining different
supersymmetry transformations. Various $N = 1$ and $N = 2$ supersymmetric mKdV and sinh-Gordon equations are derived from the $sl(2|1)$ algebra. In contrast, the centralizer of the model based on the $osp(1|2)$ algebra does not contain any elements besides the isospectral deformation generators. Thus, for this model, there are no symmetry flows apart from the isospectral times.

The Riemann–Hilbert problem naturally allows for an extension to negative times [1]. As an outcome, each positive integrable hierarchy is accompanied by a related negative hierarchy defined by the flow equations of negative times. As standard examples, the mKdV and AKNS hierarchies serve as positive hierarchies while the sine-Gordon and complex sine-Gordon hierarchies serve as their negative counterparts [2, 3]. Here, we extend the list of examples to the fermionic extensions of the mKdV and sinh-Gordon hierarchies with or without supersymmetry. We also show that the relativistic hierarchies, obtained by the reduction of the two-loop WZWN model [4], can be identified with the flow of a negative hierarchy of an extended Riemann–Hilbert problem.

The content of the paper is as follows: In Section 2, we extend the dressing–Riemann–Hilbert procedure to the negative times and the half-integer gradations. We relate the negative hierarchy to contain the relativistic equations obtained from the reduction of the WZNW model. In this Section, we also construct the pertinent zero-curvature equations and related conservation laws. In addition to the family of the conserved bosonic Hamiltonians we discover a new conserved local fermionic charge. Section 3 introduces the mKdV hierarchy, the sine-Gordon hierarchy and its fermionic extensions based on the $osp(1|2)$ algebra endowed with the principal gradation. In Section 4, we construct the $N = 2$ mKdV hierarchy and the negative $N = 2$ sinh-Gordon hierarchy for the $sl(2|1)$ algebra. The $N = 1$ mKdV hierarchy is obtained in Section 5 via reduction employing the twisted automorphism. The reduced subalgebra is isomorphic to the $osp(2|2)^{(2)}$ twisted supersymmetric loop algebra.

The supersymmetric formulation of the (modified) KdV and sinh-Gordon equations have been a subject of many papers in the last 10–15 years. Here, we list a few of the contributions. They are based on an algebraic approach.

The supersymmetric KdV and mKdV equations were introduced in [5], (see also [6]). Soon after, Mathieu and Labelle wrote down the equations for the $N = 2$ supersymmetric KdV equations [7] (see also [8]). Kersten and Sorin studied the bihamiltonian structure of $N = 2$ supersymmetric KdV model within the superfield formalism [9].

Das and Galvão have derived the supersymmetric KdV equation from the self-duality conditions imposed on four-dimensional Yang-Mills potential [10].

In [11], Delduc and Gallot put forward a classification scheme of the supersymmetric integrable models in terms of constant, odd integer grade generators of the loop algebra that square to a semisimple element $E$. In a subsequent development, Miramontes and Madsen constructed supersymmetric non-local flows and corresponding non local conservation laws for a variety of models [12].

In [13, 14], Inami and Kanno proposed a fermionic Lax associated to the $osp(2|2)^{(2)}$ algebra and used it to obtain the supersymmetric mKdV and sinh-Gordon equations as well as their conservation laws. Their approach made use of the superfield formalism. The $N = 2$ KdV and mKdV equations together with the generalized Miura transformation were formulated in [14] using the superspace notation and the Lie algebra $sl(2, 2)^{(1)}$. In this paper
we use $sl(2,1)$ superalgebra which has a smaller rank.

2 Our Approach

2.1 The Riemann-Hilbert problem and the Dressing Formalism

In an algebraic approach to integrable models the algebra of symmetries is identified with a centralizer of generators of the isospectral deformations [15]. Recall, that the centralizer $K_\xi$ of an element $\xi$ in the algebra $G$ is the set of all elements commuting with $\xi$. Due to Jacobi identity $K_\xi$ is a subalgebra.

We will be working with a loop algebra $\hat{G}$ endowed with the principal gradation defined by a grading operator $Q$ to be given below for each specific model. The gradation induces decomposition into graded subspaces $\hat{G} = \bigoplus_{n \in \mathbb{Z}} \hat{G}_n$ with $\hat{G}_n$ such that $[Q, \hat{G}_n] = n \hat{G}_n$.

Furthermore, we define a Gauss decomposition for a group element $g$ of the corresponding group $G$ which with respect to the given grading takes a form

$$ g = g_- g_+ = NBM, \quad g_- = N, \quad g_+ = BM, \quad (2.1) $$

where

$$ N = \exp \left( \hat{G}_< \right), \quad M = \exp \left( \hat{G}_> \right), $$

are matrix exponentials constructed from strictly negative $\hat{G}_< \subseteq \bigoplus_{n=1}^{\infty} \hat{G}_n$ and strictly positive $\hat{G}_> \subseteq \bigoplus_{n=1}^{\infty} \hat{G}_n$ graded subalgebras. $B$ is a group element of grade zero.

Another fundamental object in this setting is a semisimple element $E^{(n)}$ of grade $n \in \mathbb{Z}$ which induces the decomposition $\hat{G} = K \oplus M$ where $K$ is a kernel and $M$ is an image of the adjoint operation $\text{ad}(E^{(n)}) X = [E^{(n)}, X]$.

The integrable structure we describe here is derived from an extended Riemann-Hilbert factorization problem [1]:

$$ \exp \left( -\sum_{n=1}^{\infty} E^{(n)} t_n \right) g \exp \left( \sum_{n=1}^{\infty} E^{(-n)} t_{-n} \right) = \Theta^{-1}(t) \Pi(t) \quad (2.2) $$

where $g$ is a constant element in $G$ while the dressing matrix $\Theta = \exp \left( \sum_{i<0}^{\infty} \theta^{(i)} \right)$ is an exponential in $G_<$. We further assume that the positive dressing matrix $\Pi$ decomposes as $\Pi = BM$ where $B$ is non-singular matrix of grade zero while $M = \exp \left( \sum_{i>0} m^{(i)} \right)$ is an exponential power series expansion of elements in $G_>$.  

By taking a derivative of identity (2.2) with respect to $t_n$ and $t_{-n}$ we obtain relations

$$ \frac{\partial}{\partial t_n} \Theta(t) = \left( \Theta E^{(n)} \Theta^{-1} \right)_- \Theta(t), \quad \frac{\partial}{\partial t_n} \Pi(t) = - \left( \Theta E^{(n)} \Theta^{-1} \right)_+ \Pi(t) \quad (2.3) $$

$$ \frac{\partial}{\partial t_{-n}} \Theta(t) = - \left( \Pi E^{(-n)} \Pi^{-1} \right)_- \Theta(t), \quad \frac{\partial}{\partial t_{-n}} \Pi(t) = \left( \Pi E^{(-n)} \Pi^{-1} \right)_+ \Pi(t) \quad (2.4) $$

These equations define action of the isospectral flows $\partial/\partial t_n$ and the negative flows $\partial/\partial t_{-n}$ on the dressing matrices. In the above equations, $(\ldots)_+$ denotes projection on positive terms with grades $\geq 0$ in $\hat{G}$ and $(\ldots)_-$ denotes projection on $\hat{G}_-$.  

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We now propose extension of equations (2.3)-(2.4) to all positive grade elements $K_i$ in $\mathcal{K}$ by associating to each such $K_i$ a transformation $\delta_{K_i}$ according to:

\[
\begin{align*}
\delta_{K_i} \Theta &= (\Theta K_i \Theta^{-1})_\cdot \Theta \\
\delta_{K_i} \Pi &= - (\Theta K_i \Theta^{-1})_+ \Pi .
\end{align*}
\]

The map $K_i \rightarrow \delta_{K_i}$ is a homomorphism [15]:

\[
[\delta_{K_i}, \delta_{K_j}] \Theta = \delta_{[K_i,K_j]} \Theta .
\]

Applying, respectively $\partial^2 / \partial t_{\pm n} \partial t_{\pm k}$ and $\partial^2 / \partial t_{\pm k} \partial t_{\pm n}$ on both sides of eq.(2.2) produces identical results due to commutativity of $E^n$ with $E^k$. Commutativity of $K_i$ with $E^n$ ensures that the flows from (2.5) indeed define symmetry transformations which commute with the isospectral flows $\partial / \partial t_n$.

These results can be extended to also yield commutativity of $\partial / \partial t_n$ and $\delta_{K_i}$ with the negative flows $\partial / \partial t_{-n}$. For example taking a difference of

\[
\begin{align*}
\delta_{K_i} \frac{\partial}{\partial t_{-n}} \Theta &= - \delta_{K_i} \left( \Pi E^{(-n)} \Pi^{-1} \right)_- \Theta = \left[ \left( \Theta K_i \Theta^{-1} \right)_+, \Pi E^{(-n)} \Pi^{-1} \right]_\cdot \Theta \\
&\quad - \left( \Pi E^{(-n)} \Pi^{-1} \right)_- \left( \Theta K_i \Theta^{-1} \right)_- \Theta
\end{align*}
\]

with

\[
\frac{\partial}{\partial t_{-n}} \delta_{K_i} \Theta = \frac{\partial}{\partial t_{-n}} \left( \Theta K_i \Theta^{-1} \right)_- \Theta = - \left( \left[ \Pi E^{(-n)} \Pi^{-1}, \Theta K_i \Theta^{-1} \right] \right)_- \Theta
\]

we obtain

\[
\left( \delta_{K_i} \frac{\partial}{\partial t_{-n}} - \frac{\partial}{\partial t_{-n}} \delta_{K_i} \right) \Theta \Theta^{-1} = \left[ \left( \Theta K_i \Theta^{-1} \right)_+, \Pi E^{(-n)} \Pi^{-1} \right]_+ + \left[ \left( \Theta K_i \Theta^{-1} \right)_-, \left( \Pi E^{(-n)} \Pi^{-1} \right)_- \right]
\]

\[
\quad + \left[ \left( \Pi E^{(-n)} \Pi^{-1} \right)_-, \Theta K_i \Theta^{-1} \right]_+ = 0 .
\]

Here the result is an identity and follows without invoking the commutativity of the underlying algebra generators. It follows quite generally that the commutation relations involving flows generated by algebra elements of opposite grading vanish identically.

The structure of the Lax operator follows from the underlying Riemann-Hilbert problem through the following standard construction. Let us take $n = 1$ in eq. (2.3) with $E \equiv E^{(1)}$ and identify $t_1$ with the space variable $x$. Then:

\[
\begin{align*}
\partial_x (\Theta) &= (\Theta E \Theta^{-1})_- \Theta = \left[ \Theta E \Theta^{-1} - (\Theta E \Theta^{-1})_+ \right] \Theta \\
&= \Theta E - \left( E + [\theta^{(-1)}, E] \right) \Theta \\
&= \Theta E - (E + A_0) \Theta
\end{align*}
\]

where $A_0 = [\theta^{(-1)}, E]$ is clearly in $\mathcal{M}$ and of grade zero. This leads to the dressing expression

\[
\Theta^{-1} (\partial_x + E + A_0) \Theta = \partial_x + E
\]

(2.9)
for the Lax operator $L = \partial_x + E + A_0$. Similarly, for higher flows we obtain
\[
\Theta^{-1} \left( \frac{\partial}{\partial t_n} + E^{(n)} + \sum_{i=0}^{n-1} D_n^{(i)} \right) \Theta = \frac{\partial}{\partial t_n} + E^{(n)} \tag{2.10}
\]
where
\[
(\Theta E^{(n)} \Theta^{-1})_+ = E^{(n)} + \sum_{i=0}^{n-1} D_n^{(i)}
\]
These dressing relations give rise to the zero-curvature conditions
\[
\left[ \partial_x + E + A_0, \frac{\partial}{\partial t_n} + E^{(n)} + \sum_{i=0}^{n-1} D_n^{(i)} \right] = \Theta \left[ \partial_x + E, \frac{\partial}{\partial t_n} + E^{(n)} \right] \Theta^{-1} = 0 . \tag{2.11}
\]
The structure of the Lax operators changes when terms with the half-integer grades appear in $\hat{G} = \oplus_{n \in \mathbb{Z}} \hat{G}_{n/2}$. As a consequence of such terms being present in the exponent of the dressing matrix
\[
\Theta = \exp \left( \sum_{i<0} \theta^{(i)} \right) = \exp \left( \theta^{(-1/2)} + \theta^{(-1)} + \theta^{(-3/2)} + \ldots \right)
\]
the form of the Lax operator obtained by the dressing procedure is changed as follows.
\[
\frac{\partial}{\partial t_1} (\Theta) = (\Theta E \Theta^{-1})_- \Theta = [\Theta E \Theta^{-1} - (\Theta E \Theta^{-1})_+] \Theta
\]
\[
= \Theta E + \left( E + [\theta^{(-1)}, E] + [\theta^{(-1/2)}, E] + \frac{1}{2} [\theta^{(-1/2)}, [\theta^{(-1/2)}, E]] \right) \Theta \tag{2.12}
\]
\[
= \Theta E + (E + A_0 + A_{1/2} + k_0) \Theta
\]
here
\[
A_0 = [\theta^{(-1)}, E] + \frac{1}{2} [\theta^{(-1/2)}, [\theta^{(-1/2)}, E]] \left|_\mathcal{M} \in \mathcal{M} \tag{2.13}
\right.
\]
\[
A_{1/2} = [\theta^{(-1/2)}, E] \in \mathcal{M} \tag{2.14}
\]
\[
k_0 = \frac{1}{2} [\theta^{(-1/2)}, [\theta^{(-1/2)}, E]] \left|_\mathcal{K} \in \mathcal{K} \tag{2.15}\right.
\]
where $\left|_\mathcal{K}$ and $\left|_\mathcal{M}$ denote projections on the kernel $\mathcal{K}$ and image $\mathcal{M}$, respectively. This shows that, in case of the half-integer grading, the Lax operator should be defined as
\[
\mathcal{L} = \partial_x + E + A_0 + A_{1/2} + k_0 . \tag{2.16}
\]
The unconventional grade zero term $k_0$ which resides in $\mathcal{K}$ is here present due to the half-integer grading (encountered in case of $sl(2|1)$ with principal gradation).
The dressing matrix $\Theta$ factorizes as $\Theta = U S$ [15] with $U$ which is entirely in $\mathcal{M}$ and given by a local power series in components of $A_0$ and $A_{1/2}$. $U$ rotates the Lax operator $\mathcal{L}$ into:

$$U^{-1} \left( \partial_x + E + A_0 + A_{1/2} + k_0 \right) U = \partial_x + E + K^{(-)},$$

(2.17)

here the term $K^{(-)} = \sum_{i<0} k^{(i)}$ contains only terms of a negative grade in $\mathcal{K}$. Additional factor $S$ in $\Theta$ rotates $\partial_x + E + K^{(-)}$ into

$$S^{-1} \left( \partial_x + E + K^{(-)} \right) S = \partial_x + E,$$

(2.18)

by a non-local gauge transformation.

Consider now the case of a superalgebra with a kernel $\mathcal{K}$ which contains a constant grade one-half element $D^{(1/2)}$. According to (2.5) this term gives rise to the symmetry flow

$$\partial_{1/2} \Theta \equiv \delta_{D^{(1/2)}} \Theta = \left( \Theta D^{(1/2)} \Theta^{-1} \right)_- \Theta.$$

(2.19)

We will show that $\partial_{1/2}$-flow enters the zero-curvature equation:

$$\left[ \partial_x + E + A_0 + A_{1/2} + k_0 , \partial_{1/2} + D^{(0)} + D^{(1/2)} \right] = 0.$$  

(2.20)

First, we rewrite the right hand side of eq. (2.19) as

$$(\Theta D^{(1/2)} \Theta^{-1})_- \Theta = \left[ \Theta D^{(1/2)} \Theta^{-1} - (\Theta D^{(1/2)} \Theta^{-1})_+ \right] \Theta$$

$$= \Theta D^{(1/2)} - D^{(1/2)} \Theta - [\theta^{-1/2}, D^{(1/2)}] \Theta$$

Introducing

$$D^{(0)} = [\theta^{-1/2}, D^{(1/2)}]$$

we can rewrite the expression

$$\partial_{1/2} \Theta = \Theta D^{(1/2)} - D^{(1/2)} \Theta - D^{(0)} \Theta$$

as a dressing formula

$$\Theta \left( \partial_{1/2} + D^{(1/2)} \right) \Theta^{-1} = \partial_{1/2} + D^{(0)} + D^{(1/2)}.$$  

(2.21)

By dressing the obvious identity

$$\left[ \partial_x + E, \partial_{1/2} + D^{(1/2)} \right] = 0 \quad \rightarrow \quad \Theta \left[ \partial_x + E, \partial_{1/2} + D^{(1/2)} \right] \Theta^{-1} = 0$$

we indeed arrive at the zero-curvature relation (2.20).

Similarly, in presence of the half-integer grading the zero-curvature condition (2.11) gives way to

$$\left[ \partial_x + E + A_0 + A_{1/2} + k_0 , \frac{\partial}{\partial t_n} + E^{(n)} + \sum_{k=1}^{m} (D^{(k)} + D^{(k-1/2)}) \right] = 0$$

(2.22)

Let us now turn our attention to the negative flows described in eq. (2.4). Assuming decomposition $\Pi = \Pi_0 \Pi_+ = B M$ with $M = \exp \left( \sum_{i>0} m^{(i)} \right)$ we obtain

$$\frac{\partial}{\partial t} \Theta(t) = - \left( B M E^{(-n)} M^{-1} B^{-1} \right)_- \Theta(t) = -B \left( M E^{(-n)} M^{-1} \right)_- B^{-1} \Theta(t).$$

(2.23)
Setting \( n = 1 \) in equation (2.23) yields (for integer gradation)
\[
\frac{\partial}{\partial t_{-1}} \Theta = -BE^{-1}B^{-1} \Theta
\]  
(2.24)

In case of a system with the half-integer gradation the expression for \( M \) becomes
\[
M = e^{\hat{\Theta}_0} = \exp \left( m^{(1/2)} + m^{(1)} + m^{(3/2)} + \ldots \right)
\]
and leads to an additional term in (2.24)
\[
\frac{\partial}{\partial t_{-1}} \Theta = -BE^{-1}B^{-1} \Theta - \mathcal{J}_{-1/2}B^{-1} \Theta,
\]  
(2.25)

where
\[
\mathcal{J}_{-1/2} = \left[ E^{-1}, m^{(1/2)} \right]
\]  
(2.26)

We can rewrite relation (2.25) as a dressing transformation:
\[
\Theta \frac{\partial}{\partial t_{-1}} \Theta^{-1} = \frac{\partial}{\partial t_{-1}} + \mathcal{J}_{-1/2}B^{-1} + BE^{-1}B^{-1}
\]

Such analysis leads us to propose the following three Lax operators :
\[
\mathcal{D}_{+1} = \mathcal{L} = \partial_x + A_0 + k_0 + A_{1/2} + E
\]  
(2.27)
\[
\mathcal{D}_{+1/2} = \partial_{1/2} + D^{(0)} + D^{(1/2)}
\]  
(2.28)
\[
\mathcal{D}_{-1} = \partial_{-1} + \mathcal{J}_{-1/2}B^{-1} + BE^{-1}B^{-1}
\]  
(2.29)

where
\[
\partial_{1/2}, \quad \partial_{-1} = \frac{\partial}{\partial t_{-1}}, \quad \text{and} \quad \partial_x = \frac{\partial}{\partial t_1}
\]

are all commuting flows as follows from their definition in (2.3), (2.4), (2.5), (2.6). Applying the dressing procedure one easily sees that this commutativity implies the zero-curvature equations :
\[
[D_{+1/2}, D_{-1}] = 0
\]  
(2.30)
\[
[D_{+1/2}, D_{+1}] = 0
\]  
(2.31)
\[
[D_{+1}, D_{-1}] = 0
\]  
(2.32)

The brackets with \( D_{+1/2} \) act as compatibility equations which define supersymmetry transformations and ensure invariance of equations of motion ((2.32)), i.e.
\[
[\partial_x + E + A_0 + k_0 + A_{1/2}, \partial_{-1} + \mathcal{J}_{-1/2}B^{-1} + BE^{-1}B^{-1}] = 0
\]  
(2.33)

under the supersymmetry transformation. The argument goes as follows. For supersymmetry to preserve (2.32) (or (2.33)) it must hold that \( \partial_{1/2} [D_{+1}, D_{-1}] = 0 \). This can be proved by using (2.30) and (2.31) and applying the Jacobi identity to get
\[
\partial_{1/2} [D_{+1}, D_{-1}] = - \left[ [D^{(0)} + D^{(1/2)}, D_{+1}], D_{-1} \right] - [D_{+1}, [D^{(0)} + D^{(1/2)}, D_{-1}]]
\]  
(2.34)
The grade $-1$ component of the zero-curvature relation (2.33)
\[
\partial_x (BE^{-1}B^{-1}) + [A_0 + k_0, BE^{-1}B^{-1}] = 0
\] (2.35)
has a solution
\[
A_0 + k_0 = -\partial_x B B^{-1}.
\] (2.36)
The grade $-1$ component of equation (2.30)
\[
\left[ \partial_{1/2} + D(0) + D(1/2), \partial_{-1} + B_{-1/2}B^{-1} + BE^{-1}B^{-1} \right] = 0
\] is equal to
\[
\partial_{1/2} (BE^{-1}B^{-1}) + [D(0), BE^{-1}B^{-1}] = 0.
\]
The obvious solution is
\[
D(0) = -\partial_{1/2} B B^{-1}.
\] (2.37)
With solutions (2.36) and (2.37) we can rewrite $\mathcal{D}_{+1}$ and $\mathcal{D}_{+1/2}$ as
\[
\mathcal{D}_{+1} = \mathcal{L} = \partial_x - \partial_x B B^{-1} + A_{1/2} + E
\] (2.38)
\[
\mathcal{D}_{+1/2} = \partial_{1/2} - \partial_{1/2} B B^{-1} + D(1/2)
\] (2.39)
We now write explicitly all the zero-curvature equations in components. Equation (2.30) implies
\[
\partial_{1/2} (-1/2) = [E^{-1}, B^{-1}D(1/2)B] \] (2.40)
\[
\partial_{-1} (\partial_{1/2} B B^{-1}) = [B_{-1/2}B^{-1}, D(1/2)].
\] (2.41)
From (2.31) we derive
\[
[E, \partial_{1/2} B B^{-1}] = [A_{1/2}, D(1/2)]
\] (2.42)
\[
\partial_{1/2} A_{1/2} = [\partial_{1/2} B B^{-1}, A_{1/2}] - [\partial_x B B^{-1}, D(1/2)].
\] (2.43)
Finally, equation (2.32) yields
\[
\partial_x (-1/2) = [E^{-1}, B^{-1}A_{1/2}B]
\] (2.44)
\[
\partial_{-1} A_{1/2} = [E, B_{-1/2}B^{-1}]
\] (2.45)
\[
\partial_{-1} (\partial_x B B^{-1}) = [BE^{-1}B^{-1}, E] + [B_{-1/2}B^{-1}, A_{+1/2}].
\] (2.46)
By dressing the identity
\[
\left[ \partial_{1/2} + D(1/2), \frac{\partial}{\partial t_n} + E(n) \right] = 0 \Rightarrow \Theta \left[ \partial_{1/2} + D(1/2), \frac{\partial}{\partial t_n} + E(n) \right] \Theta^{-1} = 0
\]
we obtain the zero-curvature relation
\[
\left[ \partial_{1/2} + D(0) + D(1/2), \frac{\partial}{\partial t_n} + E(n) + \sum_{k=1}^{n} (D_{n}(k-1) + D_{n}(k-1/2)) \right] = 0
\] (2.47)
Denote for brevity
\[ D_n = \frac{\partial}{\partial t_n} + E^{(n)} + \sum_{k=1}^{n} (D_{n}^{(k-1)} + D_{n}^{(k-1/2)}) , \]
then the above zero curvature condition
\[ [D_{1/2}, D_n] = 0 \]
enforces that the higher flows defined by \[ [D_{+1}, D_n] = 0 \] remain invariant under the supersymmetry transformation. Indeed, the statement \( \partial_{1/2} [D_{+1}, D_n] = 0 \) follows from relations (2.31) and (2.47) by employing the Jacobi identity in the same way as it was done in (2.34).

### 2.2 A Connection to the Relativistic Hierarchies

The negative hierarchy equations (2.44), (2.45), (2.46) can also be formulated by a Hamiltonian reduction procedure from the so-called two-loop WZNW model [4]. The WZNW model is defined in terms of two currents \( J, \bar{J} \) of the loop algebra \( \hat{G} \):
\[ J = g^{-1} \partial g, \quad \bar{J} = \bar{\partial} g g^{1} \]
According to the Gauss decomposition (2.1) we find
\[ J = M^{-1} K M, \quad \bar{J} = N \bar{K} N^{-1} \]
where
\[ K = B^{-1} N^{-1} (\partial N) B + B^{-1} \partial B + (\partial M) M^{-1} \]
\[ \bar{K} = N^{-1} (\bar{\partial} N) + \bar{\partial} B B^{-1} + B \bar{\partial} M M^{-1} B^{-1} \]
The WZNW currents satisfy the free equations of motion
\[ \bar{\partial} (g^{-1} \partial g) = 0, \quad \partial (\bar{\partial} g g^{-1}) = 0, \]
which imply the following equations of motion
\[ \bar{\partial} K + [K, \bar{\partial} M M^{-1}] = 0 \]
\[ \partial \bar{K} - [\bar{K}, N^{-1} \partial N] = 0 \]
Impose now the following constraints:
\[ J = E^{(-1)} + J^{-1/2} + J_0 + J_+ \]
\[ \bar{J} = E^{(1)} + \bar{J}^{1/2} + \bar{J}_0 + \bar{J}_< \]
Note, that in the expressions for \( J, \bar{J} \) we put to zero an infinite number of currents with grades \(< -1 \) and \( > 1 \), respectively [4].

From expressions (2.48)-(2.49) and the constraints (2.52)-(2.53) we find that
\[ B^{-1} N^{-1} (\partial N) B \bigg|_{\text{constr}} = E^{(-1)} + J^{-1/2} \]
\[ B \bar{\partial} M M^{-1} B^{-1} \bigg|_{\text{constr}} = E^{(1)} + \bar{J}^{1/2} \]
Projecting equation of motion (2.50) on grades $-1/2$ and 0 and using the constraint (2.54) we obtain:

$$\bar{\partial}J_{-1/2} + [E^{(-1)}, B^{-1}j_{1/2}B] = 0$$  \hspace{1cm} (2.56)

$$\bar{\partial} (B^{-1}\partial B) + [E^{(-1)}, B^{-1}E^{(1)}B] + [j_{-1/2}, B^{-1}j_{1/2}B] = 0$$  \hspace{1cm} (2.57)

Projecting equation of motion (2.51) on grades $1/2$ and 0 and using the constraint (2.55) we obtain:

$$\partial j_{1/2} - [E^{(1)}, Bj_{-1/2}B^{-1}] = 0$$  \hspace{1cm} (2.58)

$$\partial (\bar{\partial}BB^{-1}) - [E^{(1)}, BE^{(-1)}B^{-1}] - [\bar{j}_{1/2}, Bj_{-1/2}B^{-1}] = 0$$  \hspace{1cm} (2.59)

Notice, that conjugating equation (2.57) with $B$ turns it into eq. (2.59). Thus both equations are equivalent.

Identifying

$$A_{1/2} = j_{1/2}, \quad A_0 = \bar{\partial}B B^{-1}$$

and

$$\partial_{-1} = \partial, \quad \partial_x = -\bar{\partial}$$

turns eqs. (2.56), (2.58) and (2.59) into the zero curvature equations (2.44), (2.45) and (2.46). Below, we show explicitly that this identification holds for osp(1|2) and sl(2|1).

2.3 Conservation laws

The main object here is a current density (see [15]):

$$\mathcal{J}_n = \text{Tr} \left( [Q, \Theta] E^{(n)} \Theta^{-1} \right)$$  \hspace{1cm} (2.62)

Here $Q$ is the grading operator and the trace Tr includes projection on the zero grade in addition to the usual matrix trace. Using that $\partial \Theta / \partial t_m = \Theta E^{(n)} - B_m \Theta$ as follows from (2.3) for $B_m = (\Theta E^{(n)} \Theta^{-1})_+$ we find that the quantities

$$\partial_m \mathcal{J}_n = -\text{Tr} \left( [Q, B_m] \Theta E^{(n)} \Theta^{-1} \right), \quad \partial_m = \frac{\partial}{\partial t_m}$$

are local since $\Theta E^{(n)} \Theta^{-1} = U E^{(n)} U^{-1}$. Also,

$$\partial_m \mathcal{J}_n - \partial_n \mathcal{J}_m = \text{Tr} \left( [Q, \Theta] [E^{(m)}, E^{(n)}] \Theta^{-1} \right) = 0$$

Define now

$$\mathcal{H}_n = -\partial_x \mathcal{J}_n = \text{Tr} \left( (E + \frac{1}{2} A_{1/2}) \Theta E^{(n)} \Theta^{-1} \right)$$

$$= \text{Tr} \left( E \Theta E^{(n)} \Theta^{-1} \right) + \frac{1}{2} \text{Tr} \left( A_{1/2} \Theta E^{(n)} \Theta^{-1} \right)$$

where in obtaining the last equality we used equation (2.12).
It follows automatically that $\partial_m \mathcal{H}_n - \partial_n \mathcal{H}_m = 0$. Moreover, each $\int \mathcal{H}_n$ defines a conserved charge with respect to every isospectral flow as seen from

$$\partial_m \int \mathcal{H}_n = - \int \partial_x \partial_m \mathcal{J}_n = - \partial_m \mathcal{J}_n|_{-\infty}^{\infty} = 0.$$ 

The conclusion followed from locality of $\partial_m \mathcal{J}_n$. The lowest conserved density is

$$\mathcal{H}_1 = \text{Tr} \left( \frac{1}{2} \text{Tr} \left( A_{1/2} U E U^{-1} \right) \right)$$

Consider now a quantity

$$\mathcal{F}_n = \text{Tr} \left( D^{(1/2)} \Theta E^{(n)} \Theta^{-1} \right)$$ (2.63)

We will now derive the associated conservation laws. We start with an expression

$$\partial_n \mathcal{F}_m = \frac{\partial}{\partial t_n} \text{Tr} \left( D^{(1/2)} \Theta E^{(m)} \Theta^{-1} \right) = \text{Tr} \left( D^{(1/2)} \left[ (\Theta E^{(n)} \Theta^{-1})_-, (\Theta E^{(m)} \Theta^{-1})_+ \right] \right)$$

from which it follows that

$$\partial_m \mathcal{F}_n - \partial_n \mathcal{F}_m = 0.$$ 

Set from now $m = 1$ in expression for $\partial_n \mathcal{F}_m$. It simplifies to:

$$\partial_n \mathcal{F}_1 = \text{Tr} \left( D^{(1/2)} \left[ (\Theta E^{(n)} \Theta^{-1})_-, (\Theta E \Theta^{-1})_+ \right] \right)$$

$$= \text{Tr} \left( D^{(1/2)} \left[ (\Theta E^{(n)} \Theta^{-1})_-, E - [E, \theta^{(-1/2)}] - [E, \theta^{(-1)}] \right. \right.$$

$$+ \left. \frac{1}{2} [[E, \theta^{(-1/2)}], \theta^{(-1/2)}] \right)$$

$$(2.64)$$

$$= \text{Tr} \left( D^{(1/2)} \left[ (\Theta E^{(n)} \Theta^{-1})_-, E + A_{1/2} + A_0 + k_0 \right] \right)$$

Adding the total derivative of a local quantity $- \text{Tr} \left( D^{(1/2)} \left( \Theta E^{(n)} \Theta^{-1} \right)_{-1/2} \right)$ to $\partial_n \mathcal{F}_1$ in (2.64) yields

$$\partial_n \mathcal{F}_1 - \partial_x \text{Tr} \left( D^{(1/2)} \left( \Theta E^{(n)} \Theta^{-1} \right)_{-1/2} \right) = \text{Tr} \left( D^{(1/2)} \left[ (\Theta E^{(n)} \Theta^{-1})_-, \mathcal{L} \right] \right)$$ (2.65)

where $\mathcal{L} = \partial_x + E + A_{1/2} + A_0 + k_0$ is the Lax operator. Since $\mathcal{L}$ is equal to $\mathcal{L} = \Theta(\partial_x + E)\Theta^{-1}$ it satisfies a commutation relation

$$\left[ \left( \Theta E^{(n)} \Theta^{-1} \right)_-, \mathcal{L} \right] = - \left[ \left( \Theta E^{(n)} \Theta^{-1} \right)_+, \mathcal{L} \right].$$

The left hand side contains grades between $\frac{1}{2}$ and $-\infty$. The right hand side contains positive ($\geq 0$) grades and hence both sides must for consistency contain only grades $0$ and $1/2$. Therefore the right hand side of equation (2.65) vanishes after applying the trace and we arrive at:

$$\partial_n \mathcal{F}_1 = \partial_x \text{Tr} \left( D^{(1/2)} \left( U E^{(n)} U^{-1} \right)_{-1/2} \right).$$

Consequently,

$$\partial_n \int \mathcal{F}_1 = 0, \quad n > 0$$ (2.66)

expresses a new conservation law for the fermionic density $\mathcal{F}_1$.
3 The mKdV Hierarchy and its Extension to the $osp(1|2)$ Algebra

In this section, we illustrate our construction for the case of the $sl(2)$ algebra with the principal gradation and show how to extend the formalism to the $osp(1|2)$ algebra and include generators with the half-integer grading.

Consider first the $sl(2)$ algebra:

$$[H, E_{\pm}] = \pm 2E_{\pm}, \quad [E_+, E_-] = H.$$ 

This structure generalizes straightforwardly to the loop algebra $\hat{G}$ spanned by the generators $E^{(n)}_{\pm}, H^{(n)}, n \in \mathbb{Z}$ (where $X^{(n)} = \lambda^n X$). We define a principal gradation on $\hat{G}$ through the grading operator:

$$Q = 2d + \frac{1}{2}H, \quad d = \lambda \frac{d}{d\lambda} \quad (3.1)$$

Grading of the generators $E^{(n)}_{\pm}, H^{(n)}$ with respect to this grading operator $Q$ is defined by the commutation relations

$$[Q, E^{(n)}_{\pm}] = (2n \pm 1)E^{(n)}_{\pm}, \quad [Q, H^{(n)}] = 2nH^{(n)} \quad (3.2)$$

Define

$$E^{(2n+1)} = E^{(n)}_{+} + E^{(n+1)}_{-} \quad (3.3)$$

The kernel and image of the adjoint operation $\text{ad}(E^{(2n+1)}) X = [E^{(2n+1)}, X]$ are

$$\mathcal{K} = \{E^{(2m+1)}\} = \{E^{(m)}_{+} + E^{(m+1)}_{-}\} \quad (3.4)$$

$$\mathcal{M} = \{E^{(m)}_{+} - E^{(m+1)}_{-}, H^{(m)}\}, \quad m \in \mathbb{Z}. \quad (3.5)$$

It is an important feature that $E^{(2n+1)}$ is semisimple and therefore induces relation $\hat{G} = \mathcal{K} \oplus \mathcal{M}$.

The semisimple element of grade one:

$$E \equiv E^{(1)} = E^{(0)}_{+} + E^{(1)}_{-} \quad (3.6)$$

plays a particularly fundamental role in this formalism due to its connection to the Lax operator [16]:

$$\mathcal{L} = \partial_x + E + A_0 = \partial_x + \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix} + \begin{bmatrix} v & 0 \\ 0 & -v \end{bmatrix}, \quad (3.7)$$

here $A_0$ is a grade zero element of $\mathcal{M}$.

For $n = 3$ in (2.11) we obtain:

$$\left[ \partial_x + E + A_0, \frac{d}{dt_3} + D_3^{(0)} + D_3^{(1)} + D_3^{(2)} + E^{(3)} \right] = 0. \quad (3.8)$$

Decomposing this relation according to grading one finds a solution

$$D_3^{(0)} + D_3^{(1)} + D_3^{(2)} = \frac{1}{4} \partial_x^2 A_0 - \frac{1}{2} v^3 H - \frac{1}{4} [\partial_x A_0, E] - \frac{1}{2} v^2 E + \lambda^2 A_0.$$
Projecting the zero-curvature equation (3.8) on its grade zero component yields \( \partial A_0/\partial t_3 - \partial_x D_3^{(0)} - \left[ D_3^{(0)}, A_0 \right] = 0 \) which is equivalent to the celebrated mKdV equation:

\[
4 \frac{\partial}{\partial t_3} v = \partial_x^3 v - 6v^2 \partial_x v . \tag{3.9}
\]

We will now show that a compatibility of the positive and negative flows expressed by a commutation relation

\[
[\mathcal{L}, \mathcal{D}_{-1}] = \left[ \partial_x + E - \partial_x B^{-1}, \partial_{-1} + BE^{-1}B^{-1} \right] = 0
\]

leads in this setting to the sinh-Gordon equation. The zero grade component of \([\mathcal{L}, \mathcal{D}_{-1}] = 0\) is

\[
\partial_{-1} \left( \partial_x B^{-1} \right) = \left[ BE^{-1}B^{-1}, E \right] \tag{3.10}
\]

Let the group element \( B \) be given by the exponential of the grade zero element of the algebra:

\[
B = e^{-\phi H} \tag{3.11}
\]

Plugging \( B \) into equation (3.10) results in the sinh-Gordon equation

\[
\partial_x \partial_{-1} \phi = e^{2\phi} - e^{-2\phi} . \tag{3.12}
\]

We now generalize this formalism to the \( osp(1|2) \) algebra. In the three-dimensional matrix representation its generators are

\[
H = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad E^+ = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad E^- = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
F^+ = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad F^- = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
1 & 0 & 0
\end{bmatrix} .
\]

The corresponding loop algebra with generators \( E_\pm^{(n)}, F_\pm^{(n)}, H^{(n)}, n \in \mathbb{Z} \) (where \( X^{(n)} = \lambda^n X \)) reads

\[
\left[ H^{(n)}, E_\pm^{(m)} \right] = \pm 2E_\pm^{(n+m)}, \quad \left[ H^{(n)}, F_\pm^{(m)} \right] = \pm F_\pm^{(n+m)},
\]

\[
\left[ E_+^{(n)}, E_-^{(m)} \right] = \left\{ F_+^{(n)}, F_-^{(m)} \right\} = H^{(n+m)}
\]

\[
\left[ E_+^{(n)}, F_-^{(m)} \right] = -F_+^{(n+m)}, \quad \left[ E_-^{(n)}, F_+^{(m)} \right] = -F_-^{(n+m)},
\]

\[
\left\{ F_+^{(n)}, F_-^{(m)} \right\} = 2E_+^{(n+m)}, \quad \left\{ F_-^{(n)}, F_+^{(m)} \right\} = -2E_-^{(n+m)} .
\]

The principal gradation is again defined through the grading operator (3.1). The novel feature is appearance of the half-integer grades as seen in

\[
\left[ Q, E_\pm^{(n)} \right] = (2n \pm 1) E_\pm^{(n)}, \quad \left[ Q, F_\pm^{(n)} \right] = \left( 2n \pm \frac{1}{2} \right) F_\pm^{(n)}, \quad \left[ Q, H^{(n)} \right] = 2nH^{(n)}
\]
Consider again the semisimple element: $E = E_+^{(0)} + E_-^{(1)} \equiv E^{(1)}$ of grade one. It defines the following kernel and image:

\begin{align}
\mathcal{K} &= \{E^{(2n+1)}\} = \{E_+^{(n)} + E_-^{(n+1)}\} \\
\mathcal{M} &= \{E_+^{(n)} - E_-^{(n+1)}, H^{(n)}, F^{(n)}_\pm\}
\end{align}

In the model under consideration the term $k_0$ in (2.15) vanishes and the Lax operator is:

$$L = \partial_x + E + A_{1/2} + A_0, \quad A_0 = v H^{(0)}, \quad A_{1/2} = \bar{\psi} F_+^{(0)}.$$  

The zero curvature equation

$$\left[ \partial_x + E + A_0 + A_{1/2}, \partial_3 + D_3^{(0)} + D_3^{(1/2)} + D_3^{(1)} + D_3^{(3/2)} + D_3^{(2)} + D_3^{(5/2)} + E^{(3)} \right] = 0$$

for the $t_3$-flow is solved by

\begin{align}
D_3^{(5/2)} &= D_M^{(5/2)} = \lambda A_2 = \bar{\psi} F_+^{(1)} \\
D_3^{(2)} &= D_M^{(2)} = \lambda A_0 = v H^{(1)} \\
D_3^{(3/2)} &= D_M^{(3/2)} = \partial_x \bar{\psi} F_+^{(1)} \\
D_3^{(1)} &= \frac{1}{2} \left( \bar{\psi} \partial_x \bar{\psi} + \partial_x v \right) (E_+^{(0)} - E_-^{(1)}) \\
D_3^{(1)} &= - \left( \frac{1}{2} v^2 + \bar{\psi} \partial_x \bar{\psi} \right) (E_+^{(0)} + E_-^{(1)}) \\
D_3^{(1/2)} &= D_M^{(1/2)} = \left(-v \partial_x \bar{\psi} - \frac{1}{2} \bar{\psi} \partial_x v + \partial_x^2 \bar{\psi} - \frac{1}{2} \bar{\psi} v^2 \right) F_+^{(0)} \\
D_3^{(0)} &= D_M^{(0)} = \frac{1}{4} \left( \partial_x^2 v - 2v^3 - 6v \bar{\psi} \partial_x \bar{\psi} + 3 \bar{\psi} \partial_x^2 \bar{\psi} \right) H^{(0)}.
\end{align}

In this way we derive the equations of motion:

\begin{align}
4 \partial_3 v &= \partial_x \left( \partial_x^2 v - 2v^3 - 6v \bar{\psi} \partial_x \bar{\psi} + 3 \bar{\psi} \partial_x^2 \bar{\psi} \right) \\
&= v''' - 6v^2 v' - 6v \bar{\psi} \bar{\psi}' - 6v \bar{\psi} \bar{\psi}'' + 3 \bar{\psi} \bar{\psi}' + 3 \bar{\psi} \bar{\psi}'' \\
4 \partial_3 \bar{\psi} &= 4 \bar{\psi}''' - 6v' \bar{\psi}' - 6v^2 \bar{\psi}' - 6v \bar{\psi}' - 3 \bar{\psi} \bar{\psi}'.
\end{align}

A generalized version of the Miura transformation of the form:

\begin{align}
u &= c \left( v' - v^2 - \bar{\psi} \bar{\psi}' \right) \\
\eta &= \bar{\psi}' - v \bar{\psi}
\end{align}

(where $c$ is a non-zero constant parameter) transforms the equations of motion (3.17)-(3.18) into the fermionic extension of the KdV equation [17]:

\begin{align}
4 \partial_3 u &= u''' + \frac{6}{c} uu' + 12c \eta'' \\
4 \partial_3 \eta &= 4 \eta''' + \frac{6}{c} uu' + \frac{3}{c} u' \eta
\end{align}
Taking $c = -1$ in (3.19) and performing substitutions

\[ t_3 \rightarrow t = t_3/4, \ x \rightarrow -x, \ \eta \rightarrow \frac{\eta}{2} \]

we arrive at

\[ \partial_t u = -u''' + 6uu' - 3\eta\eta'' \]  
\[ \partial_t \eta = -4\eta''' + 6u\eta' + 3u'\eta \]  

(3.23) \hspace{1cm} (3.24)

which agrees with the Kupershmidt-KdV equation [17, 18]

These equations are not invariant under supersymmetry transformation in agreement with the kernel $K$ (3.13) possessing only generators of the isospectral deformations.

### 3.1 Dressing and conserved quantities for osp (1|2) algebra

We look for the lowest grade terms in $U^{-1}(\partial_x E + A_0 + A_{1/2})U = \partial_x E + E + \sum_{j=-1/2}^{\infty} k_j$. On the level of grade $1/2$ we get :

\[ A_{1/2} + [E, u^{-1/2}] = 0 \rightarrow u^{-1/2} = \bar{\psi}F^{(0)}_\perp \]

On the level of grade 0 we find :

\[ A_0 + [E, u^{-1}] = 0 \rightarrow u^{-1} = -\frac{1}{2}u\left(E^{(0)}_\perp - E^{(-1)}_+\right) \]

Contributions on both sides to grade $-1/2$ are

\[ \partial_x u^{-1/2} + [E, u^{-1/2}] + \frac{1}{2} [A_0, u^{-1/2}] + \frac{1}{2} [A_{1/2}, u^{-1}] = k_{-1/2} \]  

(3.25)

Since the left hand side of (3.25) is entirely in $\mathcal{M}$ the term $k_{-1/2}$ on the right hand side vanishes. Putting the left hand side of (3.25) to zero gives

\[ u^{-1/2} = \left(-\frac{3}{4}\bar{\psi}\psi + \bar{\psi}'\right)F^{(-1)}_\perp \]

The expression for the first conserved Hamiltonian density gives now :

\[ \mathcal{H}_1 = \text{Tr} \left(EUEU^{-1}\right) + \frac{1}{2} \text{Tr} \left(A_{1/2}UUEU^{-1}\right) = -v^2 - \bar{\psi}\psi' \]

which satisfies $\partial(-v^2 - \bar{\psi}\psi')/\partial t_3 = \partial_x(\ldots)$ for the Kupershmidt-mKdV flow from (3.17)-(3.18).

In case of $osp(1|2)$ we define an analog of $F_n$ from (2.63) as

\[ F_n = \text{Tr} \left(F_A \Theta E^{(n)}\Theta^{-1}\right) \]  

(3.26)
The argument leading to (2.66) also applies to osp (1|2) and therefore we are interested in the conserved density:

\[ F_1 = \text{Tr} (F_+ \Theta E \Theta^{-1}) = - \text{Tr} (F_+ [E, u^{(-3/2)}]) + \frac{1}{2} \text{Tr} (F_+ [[E, u^{(-1)}], u^{(-1/2)}]) \]
\[ + \frac{1}{2} \text{Tr} (F_+ [[E, u^{(-1/2)}], u^{(-1)}]) = - \text{Tr} (F_+ [E, u^{(-1/2)}]) - \frac{1}{2} \text{Tr} (F_+ [A_0, u^{(-1/2)}]) \]
\[ - \frac{1}{2} \text{Tr} (F_+ [A_{1/2}, u^{(-1)}]) = - \text{Tr} (F_+ \partial_x u^{(-1/2)}) = - \partial_x \tilde{\psi} \]

which clearly satisfies \( \partial_n F_1 = \partial_x (\ldots) \) in agreement with a general argument.

### 3.2 The Negative osp(1|2) Hierarchy

Let \( B \) be given again by \( e^{-\phi H} \) as in (3.11). This yields \( A_0 = -\partial_x B B^{-1} = \partial_x \phi H \). On the grade \( \pm 1/2 \) level we set

\[ A_{1/2} = \tilde{\psi} F_+^{(0)} = \tilde{\psi} F_+^{(0)}, \quad J_{-1/2} = \psi F_-^{(0)} \]

We now find that terms appearing in (2.45), (2.44) and (2.46) become in this parametrization

\[ BE^{(-1)} B^{-1} = e^{-2\phi} E_+^{(-1)} + e^{2\phi} E_-^{(0)} \]
\[ B_{J_{-1/2}} B^{-1} = \psi e^\phi F_-^{(0)} \]
\[ B^{-1} A_{1/2} B = \tilde{\psi} e^\phi F_+^{(0)} \]

Equations (2.45), (2.44) and (2.46) correspond in this case to the following osp(1|2) generalization of the sinh-Gordon equations

\[ \partial_{-1} \tilde{\psi} + \psi e^\phi = 0 \]
\[ \partial_x \psi + \tilde{\psi} e^\phi = 0 \]
\[ \partial_{-1} \partial_x \phi - e^{2\phi} + e^{-2\phi} + \psi \tilde{\psi} e^\phi = 0 \]

### 4 The \( sl(2|1) \) Hierarchy; Unreduced \( N = 2 \) Case

#### 4.1 \( sl(2|1): N = 2 \) mKdV from the zero curvature relations

The starting point the Lax operator \( L = \partial_x + A_0 + k_0 + A_{1/2} + E \) from (2.27) with \( A_0, k_0 \) and \( A_{1/2} \) being parametrized as (see Appendix A for the matrix representation of the \( sl(2|1) \) algebra used here)

\[ A_0 = uM_1^{(0)} + vM_2^{(0)} = \begin{bmatrix} v & -u\lambda^{-1} & 0 \\ u\lambda & -v & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
\[ k_0 = \frac{1}{2} [\theta^{(-1/2)}, [\theta^{(-1/2)}, E]] = \tilde{\psi}_1 \tilde{\psi}_2 \left( K_2^{(0)} - K_1^{(0)} \right) \]
and
\[
A_4 = \bar{\psi}_1 G_1^{(\frac{1}{2})} + \bar{\psi}_2 G_2^{(\frac{1}{2})} = \begin{bmatrix}
0 & 0 & -\bar{\psi}_1 + \bar{\psi}_2 \\
0 & 0 & \lambda (\bar{\psi}_1 + \bar{\psi}_2) \\
\lambda (\bar{\psi}_1 + \bar{\psi}_2) & \bar{\psi}_1 + \bar{\psi}_2 & 0
\end{bmatrix}
\]
(4.3)

where \(\bar{\psi}_i, i = 1, 2\) are anticommuting variables, i.e.
\[
\bar{\psi}_1^2 = \bar{\psi}_2^2 = 0, \quad \bar{\psi}_1 \bar{\psi}_2 + \bar{\psi}_2 \bar{\psi}_1 = 0
\]

The detailed solution to the zero curvature equation (2.22) with \(n = 3\) is given in Appendix B. Solving this zero curvature equation leads to the following equations of motion
\[
\begin{align*}
4\partial_3 \bar{\psi}_1 &= \partial_x \left( \partial_x^2 \bar{\psi}_1 - 2(v^2 - u^2) \bar{\psi}_1 \right) + 3\bar{\psi}_2 (v \partial_x u - u \partial_x \bar{\psi}_1) - u \left( \bar{\psi}_1 \partial_x u - u \partial_x \bar{\psi}_1 \right) \\
4\partial_3 \bar{\psi}_2 &= \partial_x \left( \partial_x^2 \bar{\psi}_2 - 2(v^2 - u^2) \bar{\psi}_2 \right) + 3\bar{\psi}_1 (v \partial_x u - u \partial_x \bar{\psi}_2) - v \left( \bar{\psi}_2 \partial_x v - v \partial_x \bar{\psi}_2 \right) \\
4\partial_3 u &= \partial_x \left[ \partial_x^2 u - 2(v^2 - u^2)u - 3u \partial_x \bar{\psi}_1 \bar{\psi}_1 + 3u \partial_x \bar{\psi}_2 \bar{\psi}_2 - 3v \partial_x (\bar{\psi}_1 \bar{\psi}_2) \right] + 4u \left( \partial_x \bar{\psi}_1 \partial_x u + (v^2 - u^2) \bar{\psi}_1 \bar{\psi}_2 \right) \\
4\partial_3 v &= \partial_x \left[ \partial_x^2 v - 2(v^2 - u^2)v - 3v \partial_x \bar{\psi}_1 \bar{\psi}_1 + 3v \partial_x \bar{\psi}_2 \bar{\psi}_2 - 3u \partial_x (\bar{\psi}_1 \bar{\psi}_2) \right] + 4u \left( \partial_x \bar{\psi}_1 \partial_x u + (v^2 - u^2) \bar{\psi}_1 \bar{\psi}_2 \right)
\end{align*}
\]
(4.4), (4.5), (4.6) and (4.7)

These are the \(N = 2\) supersymmetric mKdV equations.

To find the supersymmetry transformations generated by a constant generator of supersymmetry of grade \(1/2\):
\[
D^{(1/2)} = \epsilon_1 F^{(1/2)}_1 + \epsilon_2 F^{(1/2)}_2
\]
(4.8)

we consider the zero-curvature equation (2.20) with
\[
D^{(0)} = M_1^{(0)} (\bar{\psi}_1 \epsilon_1 - \bar{\psi}_2 \epsilon_2) + M_2^{(0)} (\bar{\psi}_1 \epsilon_2 - \bar{\psi}_2 \epsilon_1) + X_0 (K_2^{(0)} - K_1^{(0)})
\]
(4.9)

where
\[
X_0 = \int \left[ u (\bar{\psi}_1 \epsilon_2 - \bar{\psi}_2 \epsilon_1) + v (\bar{\psi}_2 \epsilon_2 - \bar{\psi}_1 \epsilon_1) \right].
\]
(4.10)

The zero-curvature equation (2.20) leads then to the transformations of \(u, v\):
\[
\begin{align*}
\partial_{1/2} u &= - (\bar{\psi}_2 \epsilon_2 - \bar{\psi}_1 \epsilon_1) + 2v X_0 \\
\partial_{1/2} v &= (\bar{\psi}_1 \epsilon_2 - \bar{\psi}_2 \epsilon_1) + 2u X_0
\end{align*}
\]
(4.11), (4.12)

and of the fermionic fields \(\bar{\psi}_1, \bar{\psi}_2\):
\[
\begin{align*}
\partial_{1/2} \bar{\psi}_1 &= u \epsilon_1 - v \epsilon_2 - 2 \bar{\psi}_2 X_0 \\
\partial_{1/2} \bar{\psi}_2 &= u \epsilon_2 - v \epsilon_1 - 2 \bar{\psi}_1 X_0
\end{align*}
\]
(4.13), (4.14)

The \(N = 2\) mKdV equations (4.4), (4.5), (4.6) and (4.7) are invariant under the supersymmetry transformations (4.11), (4.12), (4.13) and (4.14) as a result of \(D_{1/2}\) commuting with the Lax operators as in (2.47) and the usual dressing procedure arguments.
Associating $\delta_{\pm}$ to $\epsilon_{\pm}(F_1^{(1/2)} \pm F_2^{(1/2)})$ according to (2.5) and using the homomorphism property (2.7) we find the $N=2$ supersymmetry structure

$$\left[\delta_+, \delta_-\right] \Theta = 2\epsilon_+ \epsilon_- \delta_E \Theta = 2\epsilon_+ \epsilon_- \partial_x \Theta, \quad \left[\delta_{\pm}, \delta_{\pm}\right] \Theta = 0.$$  

Note that the $N=2$ mKdV equations (4.4)-(4.7) reduce to the supersymmetric mKdV systems [5, 18, 19] under reductions:

1) Set $v = 0$ and $\bar{\psi}_2 = 0$. Then the flows for remaining variables $u, \bar{\psi} = \bar{\psi}_1$ are:

$$4\partial_3 u = u''' + 6u^2 u' + 3\bar{\psi} \left(u\bar{\psi}'\right)' \quad (4.15)$$

$$4\partial_3 \bar{\psi} = \bar{\psi}''' + 3u \left(u\bar{\psi}'\right)' \quad (4.16)$$

1) Set $u = 0$ and $\bar{\psi}_1 = 0$. Then the flows for remaining variables $v, \bar{\psi} = \bar{\psi}_2$ are:

$$4\partial_3 v = v''' - 6v^2 v' - 3\bar{\psi} \left(v\bar{\psi}'\right)' \quad (4.17)$$

$$4\partial_3 \bar{\psi} = \bar{\psi}''' - 3v \left(v\bar{\psi}'\right)' \quad (4.18)$$

4.2 Relativistic sl(2|1) Hierarchy with Supersymmetry, Unreduced Case

The starting point here is a zero-curvature equation (2.33) with $B$ such that it reproduces

$$A_0 + k_0 = uM_1^{(0)} + vM_2^{(0)} + \bar{\psi}_1 \bar{\psi}_2 (K_2^{(0)} - K_1^{(0)})$$

via relation

$$A_0 + k_0 = \bar{\partial} B B^{-1}.$$  

The following definition of $B$

$$B = e^{\sigma K_1^{(0)}} e^{\rho K_2^{(0)}} e^{\phi M_1^{(0)}} e^{R M_2^{(0)}}$$  

(4.19)

includes all generators in grade zero subalgebra.

For an arbitrary derivation $\delta$, the vector field $\delta B B^{-1}$ decomposes as

$$\delta B B^{-1} = M_1^{(0)} \left(\delta \phi \cosh(2\sigma) - \delta R \cos(2\phi) \sinh(2\sigma)\right) + M_2^{(0)} \left(\delta R \cos(2\phi) \cosh(2\sigma) - \delta \phi \sinh(2\sigma)\right)$$

$$+ K_2^{(0)} \delta \rho + K_1^{(0)} \left(\delta \sigma - \delta R \sin(2\phi)\right)$$  

(4.20)

Let

$$j_{-1/2} = \psi_1 G_1^{(-1/2)} + \psi_2 G_2^{(-1/2)}, \quad A_{1/2} = \bar{\psi}_1 G_1^{(1/2)} + \bar{\psi}_2 G_2^{(1/2)}.$$  

Calculating the $K$-component of (2.43) and using (2.42) to find the $M$-component of $\partial_{1/2} B B^{-1}$ we obtain a constraint

$$-\partial_x B B^{-1} \big|_K = \bar{\psi}_1 \bar{\psi}_2 \left(K_2^{(0)} - K_1^{(0)}\right)$$  

(4.21)

\[\text{Notice that substituting } j_{-1/2} \text{ and } A_{1/2} \text{ in the r.h.s of eqn. (2.46), it can be seen that it does not contain terms proportional to } (K_1 + K_2) \text{ and henceforth we can take consistently the constraints (4.21) and (4.22). Such type of constraints in the } K \text{ subspace were considered in ref. [20] in order to construct dyonic integrable models.} \]
or in components using equation (4.20) with \( \delta = \partial_x \)

\[
\partial_x \rho = \partial_x R \sin(2\phi) - \partial_x \sigma = -\bar{\psi}_1 \bar{\psi}_2 \tag{4.22}
\]

Using equation (4.20) we write:

\[
-\partial_x B B^{-1} = M_1^{(0)} u + M_2^{(0)} v + \bar{\psi}_1 \bar{\psi}_2 (K_2^{(0)} - K_1^{(0)}) \tag{4.23}
\]

where

\[
u = \partial_x R \cos(2\phi) \sinh(2\sigma) - \partial_x \phi \cosh(2\sigma) \tag{4.24}
\]

\[
v = \partial_x \phi \sinh(2\sigma) - \partial_x R \cos(2\phi) \cosh(2\sigma) \tag{4.25}
\]

We use eq. (2.42) together with the identity

\[
\partial_1^2 (\partial_x B B^{-1}) - \partial_x (\partial_1^2 B B^{-1}) = [\partial_1^2 B B^{-1}, \partial_x B B^{-1}] \tag{4.26}
\]

to obtain

\[
\partial_1^2 B B^{-1} = (\bar{\psi}_2 \epsilon_2 - \bar{\psi}_1 \epsilon_1) M_1 - (\bar{\psi}_1 \epsilon_2 - \bar{\psi}_2 \epsilon_1) M_2 - E^{(0)} X_0 \tag{4.27}
\]

Comparing equation (4.27) and (4.20) yields:

\[
\partial_1^2 \phi = \partial_1^2 (\bar{\psi}_1 \bar{\psi}_2) = \bar{\psi}_2 \epsilon_2 - \bar{\psi}_1 \epsilon_1 \tag{4.28}
\]

\[
\partial_1^2 R \cos(2\phi) = \partial_1^2 \phi \sinh(2\sigma) + (\bar{\psi}_2 \epsilon_1 - \bar{\psi}_1 \epsilon_2) \cosh(2\sigma) \tag{4.29}
\]

\[
\partial_1^2 \rho = \partial_1^2 \sigma - \partial_1^2 R \sin(2\phi) = -X_0 \tag{4.30}
\]

from which we derive:

\[
\partial_1^2 \phi = (\bar{\psi}_2 \epsilon_2 - \bar{\psi}_1 \epsilon_1) \cosh(2\sigma) + (\bar{\psi}_2 \epsilon_1 - \bar{\psi}_1 \epsilon_2) \sinh(2\sigma) \tag{4.31}
\]

\[
\partial_1^2 R \cos(2\phi) = (\bar{\psi}_2 \epsilon_2 - \bar{\psi}_1 \epsilon_1) \sinh(2\sigma) + (\bar{\psi}_2 \epsilon_1 - \bar{\psi}_1 \epsilon_2) \cosh(2\sigma) \tag{4.32}
\]

Note, that the transformations (4.11) (4.12) also follow from the transformations (4.31), (4.32) through definitions (4.24)-(4.25).

From equations (4.13) (4.14) we notice a simple identity:

\[
\partial_1^2 \left( \bar{\psi}_1 \bar{\psi}_2 \right) = \partial_x X_0
\]

from which it follows that

\[
\partial_1^2 \rho = -X_0
\]

which confirms consistency of constraints (4.22) and (4.30).

One can integrate the constraint (4.22) to derive expression for the auxiliary field \( \sigma \) in terms of dynamical fields \( \bar{\psi}_1, \bar{\psi}_2, R, \phi \) as follows

\[
\sigma = -\rho + \int \partial_x R \sin(2\phi) = \int \left[ \bar{\psi}_1 \bar{\psi}_2 + \partial_x R \sin(2\phi) \right] \tag{4.33}
\]

from which

\[
\partial_1^2 \sigma = -\partial_1^2 \rho + \partial_1 R \int \partial_x R \sin(2\phi) = X_0 + \partial_1 \int \partial_x R \sin(2\phi) \tag{4.34}
\]
On the other hand we find from (4.30) that
\[ \partial \nu \sigma = \partial \nu \rho + \partial \nu R \sin(2\phi) = -X_0 + \partial \nu R \sin(2\phi) \] (4.35)
The identity
\[ 2X_0 = \partial \nu R \sin(2\phi) - \partial \nu \int \partial x R \sin(2\phi) \] (4.36)
ensures compatibility of equation (4.34) with equation (4.35) and shows that \( \sigma \) is consistently defined via the integral in equation (4.33).

We collect below the equations of motion of the model written in components:
\[ \partial \nu \bar{\psi}_1 = 2 \left( B_{J-1/2}B^{-1} \right) G_2, \quad \partial \nu \bar{\psi}_2 = 2 \left( B_{J-1/2}B^{-1} \right) G_1 \] (4.37)
\[ \partial \psi_1 = 2 \left( B^{-1}A_{1/2}B \right) G_2, \quad \partial \psi_2 = 2 \left( B^{-1}A_{1/2}B \right) G_1 \] (4.38)
\[ \partial \nu \psi_1 = 2 \left( B^{-1}D^{(1/2)}B \right) G_2, \quad \partial \nu \psi_2 = 2 \left( B^{-1}D^{(1/2)}B \right) G_1 \] (4.39)
\[ \partial \nu u = -2 \cosh(2R) \sin(2\phi) - 2 \left[ (B_{J-1/2}B^{-1})F_2 \bar{\psi}_1 - (B_{J-1/2}B^{-1})F_1 \bar{\psi}_2 \right] \] (4.40)
\[ \partial \nu v = -2 \sinh(2R) - 2 \left[ (B_{J-1/2}B^{-1})F_1 \bar{\psi}_1 - (B_{J-1/2}B^{-1})F_2 \bar{\psi}_2 \right] \] (4.41)
where \( (BXB^{-1})_Y \) denotes a component of expression \( (BXB^{-1}) \) along the \( Y \in \mathcal{G} \) direction.

5 Reduced Model and Supersymmetric KdV Equation

5.1 Step Operators and Twisted Automorphism of \( \mathfrak{sl}(2|1) \)

First, we define an automorphism of the finite dimensional algebra through:
\[ \sigma(E_\alpha) = -E_\alpha \] (5.1)
\[ \sigma(H) = -H \] (5.2)
where \( E_\alpha \) are bosonic step operators
\[ E_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \]

Recall, that roots of \( \mathfrak{sl}(2|1) \) can be realized in terms of unit length vectors \( e_i, f, i = 1, 2 \) \( (e_i \cdot e_j = \delta_{ij}, f \cdot f = -1) \) as:
\[ \alpha = e_1 - e_2, \gamma = e_2 - f, \alpha + \gamma = e_1 - f. \]

The above automorphism is compatible with the bosonic commutation relations:
\[ [H^i, E_\alpha] = \alpha^i E_\alpha \rightarrow [\sigma(H^i), \sigma(E_\alpha)] = \alpha^i \sigma(E_\alpha) \]
\[ [E_\alpha, E_\beta] = \epsilon(\alpha, \beta)E_{\alpha+\beta} \rightarrow [\sigma(E_\alpha), \sigma(E_\beta)] = \epsilon(\alpha, \beta)\sigma(E_{\alpha+\beta}) \]
\[ [E_\alpha, E_{-\alpha}] = \alpha.H \rightarrow [\sigma(E_\alpha), \sigma(E_{-\alpha})] = \sigma(\alpha.H) \]

The fermionic commutation relations:
\[
\{F_\alpha, F_\beta\} = \epsilon_F(\alpha, \beta)E_{\alpha+\beta}
\]
are consistent with the choice:
\[
\sigma(F_\alpha) = iF_{-\alpha}, \quad i = \sqrt{-1} \tag{5.3}
\]
Here, the fermionic step operators are:
\[
F_\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_{\alpha+\gamma} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

We now propose an extension of the automorphism \(\sigma \rightarrow \hat{\sigma}\) to the loop algebra of a form
\[
\hat{\sigma}(E_\alpha^{(n)}) = -(-1)^{\#(E_\alpha^{(n)})}E_{-\alpha}^{(n+\eta_{E_\alpha})} \tag{5.4}
\]
\[
\hat{\sigma}(F_\gamma^{(n)}) = i(-1)^{\#(F_\gamma^{(n)})}F_{-\gamma}^{(n+\eta_{F_\gamma})} \tag{5.5}
\]
where \(\#(E_\alpha^{(n)})\) is the (principal) grade of \(E_\alpha^{(n)}\):
\[
[Q, E_\alpha^{(n)}] = \#(E_\alpha^{(n)}) E_\alpha^{(n)}.
\]
for the grading operator \(Q = \lambda \frac{d}{d\lambda} + \frac{1}{2} \alpha.H\) from (A.1). The term \((-1)^{\#(E_\alpha^{(n)})}\) corresponds to the mapping \(\lambda \rightarrow -\lambda\) used in the case of the automorphism for the homogeneous gradation \([21, 1]\). The numbers \(\eta_{E_\alpha}, \eta_{F_\gamma}\) are eigenvalues in
\[
[\alpha.H, E_\alpha] = \eta_{E_\alpha} E_\alpha, \quad [\alpha.H, F_\gamma] = \eta_{F_\gamma} F_\gamma.
\]

The bosonic generators of \(\mathcal{M}\) are of the form \(M_2^{(n)} = \alpha.H^{(n)}\) and \(M_1^{(n)} = E_\alpha^{(n-1)} - E_{-\alpha}^{(n+1)}\). From their behavior under the \(\hat{\sigma}\) automorphism
\[
\hat{\sigma}(M_2^{(n)}) = -(-1)^n M_2^{(n)}, \quad \hat{\sigma}(M_1^{(n)}) = (-1)^n M_1^{(n)}
\]
we see that their zero grade components are respectively odd and even
\[
\hat{\sigma}(M_2^{(0)}) = -M_2^{(0)}, \quad \hat{\sigma}(M_1^{(0)}) = M_1^{(0)}. \tag{5.6}
\]
The fermionic generators of \(\mathcal{M}\) are:
\[
G_1^{(n+\frac{1}{2})} = \left( F_{\alpha+\gamma}^{(n)} + F_\gamma^{(n+1)} \right) + \left( F_{-\alpha-\gamma}^{(n+1)} + F_{-\gamma}^{(n)} \right)
\]
\[
G_2^{(n+\frac{1}{2})} = \left( F_{\alpha+\gamma}^{(n)} + F_\gamma^{(n+1)} \right) - \left( F_{-\alpha-\gamma}^{(n+1)} + F_{-\gamma}^{(n)} \right)
\]
and transform as:
\[
\hat{\sigma}(G_2^{(n+\frac{1}{2})}) = (-1)^n G_2^{(n+\frac{1}{2})}, \quad \hat{\sigma}(G_1^{(n+\frac{1}{2})}) = -(-1)^n G_1^{(n+\frac{1}{2})}
\]
and so their $\frac{1}{2}$-grade components are respectively, even and odd under the $\hat{\sigma}$ automorphism

$$\hat{\sigma}(G_2^{(\frac{1}{2})}) = G_2^{(\frac{1}{2})}, \quad \hat{\sigma}(G_1^{(\frac{1}{2})}) = -G_1^{(\frac{1}{2})}$$

We also have

$$\hat{\sigma}(K_2^{(n)}) = -(-1)^n K_2^{(n)}, \quad \hat{\sigma}(K_1^{(n)}) = -(-1)^n K_1^{(n)}$$

for

$$K_1^{(n)} = -E_\alpha^{(n-1)} + E_{-\alpha}^{(n+1)}$$

and

$$K_2^{(n)} = \alpha.H^{(n)} + 2\gamma.H^{(n)}$$

Also,

$$\hat{\sigma}(F_2^{(n+\frac{1}{2})}) = (-1)^n F_2^{(n+\frac{1}{2})}, \quad \hat{\sigma}(F_1^{(n+\frac{1}{2})}) = -(-1)^n F_1^{(n+\frac{1}{2})}$$

for the fermionic generators of $\mathcal{K}$ :

$$F_1^{(n+\frac{1}{2})} = \left(F_{\alpha+\gamma}^{(n)} - F_\gamma^{(n+1)} + F_{-\alpha-\gamma}^{(n)} - F_{-\gamma}^{(n)} \right)$$

$$F_2^{(n+\frac{1}{2})} = -\left(F_{\alpha+\gamma}^{(n)} - F_\gamma^{(n+1)} + F_{-\alpha-\gamma}^{(n)} - F_{-\gamma}^{(n)} \right)$$

### 5.2 Reduced Model and its Symmetries

To reduce the model we take equivalence classes under the automorphism $\hat{\sigma}$. The even algebra elements define an invariant subalgebra under $\hat{\sigma}$ consisting of :

$$\mathcal{M}_{\text{Bose}} = \{ M_1^{(2n)} = -E_\alpha^{(2n-1)} + E_{-\alpha}^{(2n+1)}, M_2^{(2n+1)} = \alpha.H^{(2n+1)} \}$$

$$\mathcal{M}_{\text{Fermi}} = \{ G_1^{(2n+\frac{1}{2})} = \left(F_{\alpha+\gamma}^{(2n+1)} + F_\gamma^{(2n+2)} \right) + \left(F_{-\alpha-\gamma}^{(2n+1)} + F_{-\gamma}^{(2n+1)} \right), \quad G_2^{(2n+\frac{1}{2})} = \left(F_{\alpha+\gamma}^{(2n)} + F_\gamma^{(2n+1)} \right) - \left(F_{-\alpha-\gamma}^{(2n)} + F_{-\gamma}^{(2n)} \right) \}$$

The even elements in $\mathcal{K}$ are :

$$\mathcal{K}_{\text{Bose}} = \{ K_2^{(2n+1)} = \alpha.H^{(2n+1)} + 2\gamma.H^{(2n+1)}, K_1^{(2n+1)} = -\left(E_\alpha^{(2n)} + E_{-\alpha}^{(2n+2)} \right) \}$$

$$\mathcal{K}_{\text{Fermi}} = \{ F_1^{(2n+\frac{1}{2})} = -\left(F_{\alpha+\gamma}^{(2n+1)} - F_\gamma^{(2n+2)} \right) + \left(F_{-\alpha-\gamma}^{(2n+1)} - F_{-\gamma}^{(2n+1)} \right), \quad F_2^{(2n+\frac{1}{2})} = \left(F_{\alpha+\gamma}^{(2n)} - F_\gamma^{(2n+1)} \right) + \left(F_{-\alpha-\gamma}^{(2n)} - F_{-\gamma}^{(2n)} \right) \}$$

We build the reduced model by projecting on even quantities

$$X^{\text{red}} = \frac{1}{2} (X + \hat{\sigma}(X))$$

The even elements $X^{\text{red}}$ constitute a subalgebra isomorphic to the $osp(2|2)^{(2)}$ twisted supersymmetric loop algebra. It is interesting to note that the affine version of this subalgebra
appeared in [22] where it was used to construct the \( N = 1 \) supersymmetric conformal affine Toda model based on the superfield formalism. In [23] this construction was generalized to include \( N = 2 \) supersymmetric conformal affine Toda model formulated in the superfield formalism based on \( sl(2, 2)^{(1)} \) supersymmetric KM algebra.

Note that \( E \) is invariant under \( \hat{\sigma} \). Thus the reduced Lax operator is:

\[
\partial_x + E + A_0^r + A_{1/2}^r + k_0^r
\]

(5.7)

with \( A_0^r \) and \( A_{1/2}^r \) parametrized as

\[
A_0^r = uM_1, \quad A_{1/2}^r = \bar{\psi}G^{(1/2)}_2
\]

and \( k_0^r \) to be determined.

The third flow of \( u, \bar{\psi} \)

\[
4\partial_3 u = u''' + 6u^2u' - 3\bar{\psi}(u\bar{\psi}')'
\]

(5.8)

\[
4\partial_3 \bar{\psi} = \bar{\psi}''' + 3u(u\bar{\psi})'
\]

(5.9)

is obtained from (4.4)-(4.7) by setting \( v = 0, \bar{\psi}_1 = 0 \).

Equations (5.8)-(5.9) are the supersymmetric generalization of mKdV equation (and can therefore be called super mKdV equations). A super-Miura transformation:

\[
\varphi = -iu' + u^2 - \bar{\psi}\bar{\psi}'
\]

(5.10)

\[
\eta = -i\bar{\psi}' + u\bar{\psi}
\]

(5.11)

defines new fields \( \varphi, \eta \) which satisfy the the supersymmetric generalization of KdV equation

\[
4\partial_3 \varphi = \varphi''' + 6\varphi\varphi' - 3\eta\eta''
\]

(5.12)

\[
4\partial_3 \eta = \eta''' + 3(\eta\varphi)'
\]

(5.13)

5.3 Dressing of the \( \text{sl}(2, 1) \) reduced model

Dressing the reduced Lax operator (5.7) according to relation (2.17) and requiring that the \( \mathcal{M} \)-component of \( U^{-1}(\partial_x + E + A_0^r + A_{1/2}^r + k_0^r)U \) vanishes i.e.:

\[
U^{-1}(\partial_x + E + A_0^r + A_{1/2}^r + k_0^r)U\big|_\mathcal{M} = 0
\]

provides a method to derive expressions for \( u^{(-i)}, i = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots \). For the grade 0 in \( \mathcal{M} \) we find:

\[
A_0^r + [E, u^{(-1)}] = 0 \rightarrow u^{(-1)} = \frac{1}{4\lambda^2} [A_0^r, E] = \frac{1}{2} uM_2^{(-1)}
\]

For the grade \( \frac{1}{2} \) in \( \mathcal{M} \) it holds that

\[
A_{1/2}^r + [E, u^{(-1/2)}] = 0 \rightarrow u^{(-1/2)} = \frac{1}{4\lambda^2} [A_{1/2}^r, E] = -\frac{1}{2} \bar{\psi}G^{(-1/2)}_1.
\]
Calculating $k_0^r$ according to (2.15) we find that it vanishes on the reduced manifold:

$$k_0^r = \frac{1}{2} \left[ u^{(-1/2)}, [u^{(-1/2)}, E] \right] = 0.$$ 

Contribution to $\mathcal{M}$ of degree $-1/2$ is found from the dressing formula to be:

$$[E, u^{(-3/2)}] + \frac{1}{2} \left[ [A_{1/2}, u^{(-1/2)}], u^{(-1/2)} \right] + \partial_x u^{(-1/2)} = 0,$$

which yields

$$u^{(-3/2)} = \frac{1}{4\lambda^2} \left[ \partial_x u^{(-1/2)}, E \right] = \frac{1}{16\lambda^2} \left[ [\partial_x A_{1/2}, E], E \right] = \frac{1}{4\lambda^2} \partial_x A_{1/2} = \frac{1}{4} \partial_x \bar{\psi} G_2^{(-3/2)}.$$

Contribution to $\mathcal{M}$ of degree $-1$ is found in the reduced case to be:

$$[E, u^{(-2)}] + \partial_x u^{(-1)} = 0,$$

which gives

$$u^{(-2)} = \frac{1}{4\lambda^2} \left[ \partial_x u^{(-1)}, E \right] = \frac{1}{16\lambda^2} \left[ [\partial_x A_0, E], E \right] = \frac{1}{4\lambda^2} \partial_x A_0 = \frac{1}{4} \partial_x u M^{(-2)}.$$

We next introduce $S = \exp(s)$ which enters equation (2.18). Contribution on $\mathcal{K}$ of degree $-1/2$ which are generated by the $U$ transformation are

$$\frac{1}{2} \left[ A_{1/2}, u^{(-1)} \right] + \frac{1}{2} \left[ A_0, u^{(-1/2)} \right] = k_{-1/2} = -\partial_x s^{(-1/2)},$$

and can be gauged away according to (2.18) provided

$$s^{(-1/2)} = -\frac{1}{2} F_1^{(-1/2)} \int (\bar{\psi} u).$$

Similarly, from

$$k_{-1} = \frac{1}{2} \left[ A_0, u^{(-1)} \right] + \frac{1}{2} \left[ A_{1/2}, u^{(-3/2)} \right] = -\frac{1}{4} \bar{\psi} \psi' \left( K_2^{(-1)} - K_1^{(-1)} \right) - \frac{1}{2} u^2 K_1^{(-1)}$$

and $\partial s^{(-1)} + (1/2) \left[ k_{-1/2}, s^{(-1/2)} \right] + k_{-1} = 0$ we obtain

$$s^{(-1)} = \frac{1}{4\lambda^2} \bar{\psi} u \int (\bar{\psi} u) E + \frac{1}{2} \int (u^2 K_1^{(-1)}) + \frac{1}{4} \left( K_2^{(-1)} - K_1^{(-1)} \right) \int (\bar{\psi} \psi').$$

We also find

$$k_{-3/2} = \frac{3}{8} \left( u' \bar{\psi} - u \bar{\psi}' \right) F_2^{(-3/2)}.$$
5.4 Symmetries and Conservation Laws

According to the definition (2.5) of a symmetry transformation we find that

\[
\delta_X A_{1/2} = - \left[ E, (\Theta X \Theta^{-1})_{-1/2} \right] \tag{5.14}
\]
\[
\delta_X A_0 = - \left[ E, (\Theta X \Theta^{-1})_{-1} \right] - \left[ A_{1/2}, (\Theta X \Theta^{-1})_{-1/2} \right]_{K} \tag{5.15}
\]

for \( X \in K \).

We now set \( X = \epsilon F_2^{(1/2)} \) with a constant Grassmannian parameter \( \epsilon \) and calculate

\[
\left( \Theta \epsilon F_2^{(1/2)} \Theta^{-1} \right)_{-1/2} = \left( U S \epsilon F_2^{(1/2)} S^{-1} U^{-1} \right)_{-1/2}
\]
\[
= \left( U \left( \epsilon F_2^{(1/2)} + \left[ s^{(-1/2)}, \epsilon F_2^{(1/2)} \right] + \frac{1}{2} \left[ s^{(-1/2)}, \left[ s^{(-1/2)}, \epsilon F_2^{(1/2)} \right] \right] \right) U^{-1} \right)_{-1/2}
\]
\[
= \left[ u^{(-1)}, \epsilon F_2^{(1/2)} \right] + \left[ s^{(-1)}, \epsilon F_2^{(1/2)} \right]
\]

where we took into consideration that \( \left[ s^{(-1/2)}, \epsilon F_2^{(1/2)} \right] = 0 \) and \( \left[ u^{(-1/2)}, \left[ s^{(-1/2)}, \epsilon F_2^{(1/2)} \right] \right] = 0 \).

In this way we obtain:

\[
\left( \Theta \epsilon F_2^{(1/2)} \Theta^{-1} \right)_{1/2} = -\frac{\epsilon}{2} u \bar{G}_1^{(-1/2)} + \frac{\epsilon}{2} \int (\bar{\psi} \psi') F_1^{(-1/2)} - \frac{\epsilon}{2} \int (u^2) F_1^{(-1/2)}
\]

Calculating, in the similar way

\[
\left( \Theta \epsilon F_2^{(1/2)} \Theta^{-1} \right)_{-1} \mid_{M} = \frac{\epsilon}{2} \bar{\psi}' M_2^{(-1)} - \frac{\epsilon}{4} \left( \int (u^2) - \int (\bar{\psi} \psi') \right) \bar{\psi} M_2^{(-1)}
\]

we find that in (5.14)-(5.15) the non-local terms cancel and the expression for the supersymmetry transformation

\[
\delta_{\text{susy}} = \delta_{\epsilon F_2}
\]

becomes:

\[
\delta_{\text{susy}} u = \epsilon \bar{\psi}', \quad \delta_{\text{susy}} \bar{\psi} = \epsilon u \tag{5.16}
\]

These results also follow by imposing the zero-curvature condition (2.20) with \( D^{(1/2)} = \epsilon F_2 \).

Correspondingly, the supersymmetry transformation of fields \( \varphi, \eta \) appearing in the supersymmetric KdV equation (5.12)-(5.13) is

\[
\delta_{\text{susy}} \varphi = \epsilon \eta', \quad \delta_{\text{susy}} \eta = \epsilon \varphi
\]

as follows from the super-Miura transformations (5.10)-(5.11).
Note, that the element $K^{(1)}_2$ is an even element of the kernel and therefore generates the symmetry transformation of the reduced hierarchy. One can show that

$$
\left( \Theta K^{(1)}_2 \Theta^{-1} \right)_{-1/2} = \left( U S K^{(1)}_2 S^{-1} U^{-1} \right)_{-1/2} = \left( U \left( K^{(1)}_2 + s^{(-1/2)}, K^{(1)}_2 \right) + \left[ s^{(-3/2)}, K^{(1)}_2 \right] \right)
$$

$$
+ \frac{1}{2} \left[ s^{(-1/2)}, \left[ s^{(-1/2)}, K^{(1)}_2 \right] \right] U^{-1} = \left[ u^{(-3/2)}, K^{(1)}_2 \right] + \frac{1}{2} \left[ u^{(-1)}, \left[ u^{(-1/2)}, K^{(1)}_2 \right] \right] + \left[ u^{(-1)}, \left[ s^{(-1/2)}, K^{(1)}_2 \right] \right]
$$

and

$$
\left[ s^{(-1/2)}, K^{(1)}_2 \right] = -\frac{1}{2} \int (\bar{\psi} u) \left[ F^{(-1/2)}_1, K^{(1)}_2 \right] = \frac{1}{2} \int (\bar{\psi} u) F^{(1/2)}_2
$$

and

$$
\left[ u^{(-1)}, \left[ s^{(-1/2)}, K^{(1)}_2 \right] \right] = \frac{1}{4} u \int (\bar{\psi} u) \left[ M^{(-1)}_1, F^{(1/2)}_2 \right] = -\frac{1}{4} u \int (\bar{\psi} u) G^{(-1/2)}_1
$$

and so

$$
\left( \Theta K^{(1)}_2 \Theta^{-1} \right)_{-1/2} |_M = -\frac{1}{4} \left( u \int (\bar{\psi} u) + \bar{\psi}' \right) G^{(-1/2)}_1.
$$

Combining these results we get

$$
\delta_{K^{(1)}_2} A_{1/2} = -E \left[ \Theta K^{(1)}_2 \Theta^{-1} \right]_{-1/2} = -E \int (\bar{\psi} u) G^{(-1/2)}_1 = \frac{1}{2} \left( \bar{\psi}' + u \int (\bar{\psi} u) \right) G^{(1/2)}_2
$$

or

$$
\delta_{K^{(1)}_2} \bar{\psi} = \frac{1}{2} \left( \bar{\psi}' + u \int (\bar{\psi} u) \right).
$$

Similarly, we find

$$
\delta_{K^{(1)}_2} u = -\frac{1}{2} \bar{\psi}' \int (\bar{\psi} u)
$$

Using, that $[K_2, F_2] = F_1$ we obtain

$$
\left[ \delta_{K^{(1)}_2}, \delta_{eF^{(1/2)}_2} \right] \bar{\psi} = \delta_{eF^{(1)}_1} \bar{\psi} = -\epsilon \bar{\psi}' \int (\bar{\psi} u) - \frac{1}{2} \epsilon u' - \frac{1}{2} \epsilon u \int (u)^2 + \frac{1}{2} \epsilon u \int (\bar{\psi} \bar{\psi}')
$$
\[
\left[ \delta_{K_2}, \delta_{\epsilon F_2^{(1/2)}} \right] u = \delta_{\epsilon F_1^{(3/2)}} u = \frac{1}{2} \epsilon \bar{\psi}'' + \epsilon u' \int (\bar{\psi} u) + \frac{1}{2} \epsilon u^2 \bar{\psi} - \frac{1}{2} \epsilon \bar{\psi}' \int (u)^2 + \frac{1}{2} \epsilon \bar{\psi} \int (\bar{\psi} \psi')
\]
which agrees with transformation generated by \(F_1^{3/2}\). One verifies that \([\delta_{\epsilon_{F_1}}, \delta_{\epsilon_{F_2}}] = 0\) in agreement with \([F_1, F_2] = 0\). Hence repeated commutation relations with \(\delta_{K_2}\) create the following chain
\[
d_{\epsilon F_2^{(1/2)}} \delta_{\epsilon F_2^{(3/2)}} \delta_{\epsilon F_2^{(5/2)}} \delta_{\epsilon F_2^{(7/2)}} \ldots
\]
of supersymmetry transformations of increasing grading.

Next, we turn to the discussion of conservation laws with respect to the (bosonic) isospectral flows generated by \(E^{(n)}\). The main object here is a current density (2.62). The lowest conserved density is
\[
\mathcal{H}_1 = \text{Tr} \left( EU E U^{-1} \right) + \frac{1}{2} \text{Tr} \left( A_{1/2} U E U^{-1} \right) = u^2 - \bar{\psi} \psi'
\]
where the fermionic contribution comes from the term \(\frac{1}{2} \text{Tr} \left( A_{1/2} \Theta E^{(n)} \Theta^{-1} \right)\) present due to the half-integer gradation.

Another way to obtain \(\mathcal{H}_1\) is to take \(\text{Tr} \left( E k_{-1} \right) = -u^2 + \bar{\psi} \psi'\) which must be conserved as a projection of \(K^-\) on the \(E\)-direction. Note, that only \(E\) is in the center of \(\mathcal{K}\).

Applying the supersymmetry transformation on the dressing matrix we get:
\[
\delta_{\text{susy}} \Theta = \left( \Theta \epsilon F_2^{(1/2)} \Theta^{-1} \right) - \Theta = \epsilon \Theta F_2^{(1/2)} - \left( \Theta \epsilon F_2^{(1/2)} \Theta^{-1} \right) + \Theta
\]
\[
= \epsilon \Theta F_2^{(1/2)} - \epsilon F_2^{(1/2)} \Theta + \epsilon A_0 \Theta, \quad A_0 = \left[ F_2^{(1/2)}, \theta^{(-1/2)} \right] \quad (5.17)
\]
Applying the above supersymmetry transformation on the current density \(\mathcal{J}_n\) from (2.62) yields:
\[
\delta_{\text{susy}} \mathcal{J}_n = -\frac{1}{2} \epsilon \text{Tr} \left( F_2^{(1/2)} \Theta E^{(n)} \Theta^{-1} \right) \quad (5.18)
\]
This leads to a Grassmannian quantity
\[
\mathcal{F}_n = \text{Tr} \left( F_2^{(1/2)} \Theta E^{(n)} \Theta^{-1} \right) \quad (5.19)
\]
which is an \(sl(2|1)\) analog of \(\mathcal{F}_n\) from (2.63).

Explicit calculation gives for \(n = 1\)
\[
\mathcal{F}_1 = \bar{\psi} u.
\]
In agreement with the general formalism this is a conserved supercharge which satisfies \(\partial (\bar{\psi} u) / \partial t_3 = \partial x(\ldots)\) for the \(t_3\)-mKdV flow from equations (5.8)-(5.9).

### 5.5 Relativistic \(sl(2|1)\) Hierarchy in the Reduced Case and its Supersymmetry Transformations

In this case,
\[
J_{-1/2} = \bar{\psi} G_1^{(-1/2)}, \quad \bar{J}_{1/2} = \bar{\psi} G_2^{(1/2)}, \quad B = e^{\phi M_1^{(0)}}.
\]
This parametrization leads to the following expressions
\[
B^{-1}E^{(1)}B = K^{(1)}_2 - \sin(2\phi)M^{(1)}_2 + \cos(2\phi)K^{(1)}_1
\]  
\[
BE^{-1}B^{-1} = K^{(-1)}_2 + \sin(2\phi)M^{(-1)}_2 + \cos(2\phi)K^{(-1)}_1
\]  
\[
B^{-1}j_{1/2}B = \bar{\psi} \left( \cos(\phi)G^{(1/2)}_2 + \sin(\phi)F^{(1/2)}_2 \right)
\]  
\[
BJ_{-1/2}B^{-1} = \psi \left( \cos(\phi)G^{(-1/2)}_1 - \sin(\phi)F^{(-1/2)}_1 \right).
\]

Plugging these results into eqs. (2.56), (2.58), (2.57) or (2.59) yields the relativistic supersymmetric sine-Gordon equations:
\[
\partial_x \psi = 2\bar{\psi} \cos(\phi), \quad \partial_{-1} \bar{\psi} = 2\psi \cos(\phi)
\]  
\[
\partial_x \partial_{-1} \phi = 2 \sin(2\phi) + 2\psi \bar{\psi} \sin(\phi)
\]

We find from equation (2.42) with \(D^{(1/2)} = \epsilon F_2\)
\[
\partial_{1/2} \phi = \epsilon \bar{\psi},
\]

Equation (2.40) yields a supersymmetry transformation
\[
\partial_{1/2} \bar{\psi} = \epsilon \phi'.
\]

This transformations agree with the supersymmetry transformations (5.16) obtained directly from the dressing procedure upon identification \(u = \phi'.\)

Furthermore, equation (2.43) leads to
\[
\partial_{1/2} \psi = 2\epsilon \sin \phi
\]

The supersymmetry transformations (5.26)-(5.28) leave all the sine-Gordon equations (5.24)-(5.25) invariant.

6 Outlook

This paper formulates supersymmetry for the integrable models based on superalgebras in an entirely algebraic fashion. This framework provides a convenient setup for an application of the vertex operator construction to deriving the soliton solutions [24]. At first glance, it may appear surprising that the same vertex structure would yield the soliton solutions of both the supersymmetric sinh-Gordon and mKdV models. After all, the sinh-Gordon solitons are topological solitons while the mKdV solitons are hydrodynamic in nature. Therefore, these two soliton configurations are stable due to different reasons. Note, however, that the vertex construction would be given in terms of the gradient field of the mKdV equation: \(u = \partial_x \phi\). The field \(u\) is not affected by the kink-like behavior of \(\phi\) at \(\pm \infty\). Hence, although both soliton solutions would be derived from the common algebraic structure the models remain defined in terms of different fields and some of their main characteristics disappear under a change of basis (\(\phi \rightarrow u\)) performed when going from one model to another.

An extension of this paper will include other algebraic structures. Work is being completed on the formalism of the supersymmetric AKNS and Lund-Regge models defined for superalgebras with the homogeneous gradation. We also plan to extend our results to the higher rank algebras and describe supersymmetric counterparts of the Boussinesq model.
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A  \(sl(2|1)\) realization in principal gradation

The principal grading for the \(sl(2|1)\) is generated by the operator

\[
Q = \lambda \frac{d}{d\lambda} + \frac{1}{2} \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  \(\text{(A.1)}\)

Relation

\[
[Q, E^{(n)}] = nE^{(n)}
\]  \(\text{(A.2)}\)

for

\[
E^{(n)} = \lambda^{n-1} \begin{bmatrix}
\lambda & -1 & 0 \\
-\lambda^2 & \lambda & 0 \\
0 & 0 & 2\lambda
\end{bmatrix}
\]  \(\text{(A.3)}\)

shows that the semisimple element \(E = E^{(1)}\) of \(sl(2|1)\) has grade one. It induces a decomposition of a loop algebra \(Sl(2,1)\) in \(\mathcal{M} = \text{Im}(ad_E)\) and \(\mathcal{K} = \text{Im}(ad_E)\):

\[
Sl(2,1) = \mathcal{M} \oplus \mathcal{K}
\]

The odd generators of \(\mathcal{K}\) are:

\[
F_1^{(n+\frac{1}{2})} = \lambda^n \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & -\lambda \\
\lambda & -1 & 0
\end{bmatrix}, \quad F_2^{(n+\frac{1}{2})} = \lambda^n \begin{bmatrix}
0 & 0 & -1 \\
0 & 0 & \lambda \\
\lambda & -1 & 0
\end{bmatrix}
\]  \(\text{(A.4)}\)

The even generators of \(\mathcal{K}\) are:

\[
K_1^{(n)} = \lambda^n \begin{bmatrix}
0 & -\lambda^{-1} & 0 \\
-\lambda & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad K_2^{(n)} = \lambda^n \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]  \(\text{(A.5)}\)

The odd generators of \(\mathcal{M}\) are:

\[
G_1^{(n+\frac{1}{2})} = \lambda^n \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & \lambda \\
\lambda & 1 & 0
\end{bmatrix}, \quad G_2^{(n+\frac{1}{2})} = \lambda^n \begin{bmatrix}
0 & 0 & -1 \\
0 & 0 & -\lambda \\
\lambda & 1 & 0
\end{bmatrix}
\]  \(\text{(A.6)}\)

The even generators of \(\mathcal{M}\) are:

\[
M_1^{(n)} = \lambda^n \begin{bmatrix}
0 & -\lambda^{-1} & 0 \\
\lambda & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad M_2^{(n)} = \lambda^n \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]  \(\text{(A.7)}\)
Two odd generators $F_i^{(\frac{1}{2})}, i = 1, 2$ have grade $\frac{1}{2}$ according to

$$[Q, F_i^{(n+\frac{1}{2})}] = (n + \frac{1}{2})F_i^{(n+\frac{1}{2})}, \; i = 1, 2$$

The element $E$ is a square of $F_1^{(\frac{1}{2})}$ and $F_2^{(\frac{1}{2})}$

$$E = \left(F_1^{(\frac{1}{2})}\right)^2 = - \left(F_2^{(\frac{1}{2})}\right)^2$$

which has a significance for the supersymmetry structure of the model.

The algebra can be split into $[\mathcal{K}, \mathcal{K}] = \mathcal{K}$ part containing :

$$\left[K_1^{(n)}, K_2^{(m)}\right] = 0, \quad \{F_1^{(n+\frac{1}{2})}, F_2^{(m+\frac{1}{2})}\} = 0, \quad (A.8)$$

$$\left[F_1^{(n+\frac{1}{2})}, K_1^{(m)}\right] = F_2^{(n+m+\frac{1}{2})}, \quad \left[F_1^{(n+\frac{1}{2})}, K_2^{(m)}\right] = -F_2^{(n+m+\frac{1}{2})}, \quad (A.9)$$

$$\left[F_2^{(n+\frac{1}{2})}, K_1^{(m)}\right] = F_1^{(n+m+\frac{1}{2})}, \quad \left[F_2^{(n+\frac{1}{2})}, K_2^{(m)}\right] = -F_1^{(n+m+\frac{1}{2})}, \quad (A.10)$$

$$\{F_1^{(n+\frac{1}{2})}, F_1^{(m+\frac{1}{2})}\} = \{-F_2^{(n+\frac{1}{2})}, F_2^{(m+\frac{1}{2})}\} = 2E^{(n+m+1)} \quad (A.11)$$

the $[\mathcal{K}, \mathcal{M}] = \mathcal{M}$ part containing :

$$\{F_2^{(n+\frac{1}{2})}, G_1^{(m+\frac{1}{2})}\} = -\{F_1^{(n+\frac{1}{2})}, G_2^{(m+\frac{1}{2})}\} = 2M_1^{(n+m+1)} \quad (A.12)$$

$$\{F_1^{(n+\frac{1}{2})}, G_1^{(m+\frac{1}{2})}\} = -\{F_2^{(n+\frac{1}{2})}, G_2^{(m+\frac{1}{2})}\} = 2M_2^{(n+m+1)} \quad (A.13)$$

$$\left[M_1^{(n)}, F_1^{(m+\frac{1}{2})}\right] = G_1^{(n+m+\frac{1}{2})}, \quad \left[M_1^{(n)}, F_2^{(m+\frac{1}{2})}\right] = G_2^{(n+m+\frac{1}{2})} \quad (A.14)$$

$$\left[M_2^{(n)}, F_1^{(m+\frac{1}{2})}\right] = -G_2^{(n+m+\frac{1}{2})}, \quad \left[M_2^{(n)}, F_2^{(m+\frac{1}{2})}\right] = -G_1^{(n+m+\frac{1}{2})} \quad (A.15)$$

$$\left[M_1^{(n)}, K_1^{(m)}\right] = 2M_2^{(n+m)}, \quad \left[M_1^{(n)}, K_2^{(m)}\right] = 0 \quad (A.16)$$

$$\left[M_2^{(n)}, K_1^{(m)}\right] = 2M_1^{(n+m)}, \quad \left[M_2^{(n)}, K_2^{(m)}\right] = 0 \quad (A.17)$$

$$\left[G_1^{(n+\frac{1}{2})}, K_1^{(m)}\right] = -G_2^{(n+m+\frac{1}{2})}, \quad \left[G_1^{(n+\frac{1}{2})}, K_2^{(m)}\right] = -G_2^{(n+m+\frac{1}{2})} \quad (A.18)$$

$$\left[G_2^{(n+\frac{1}{2})}, K_1^{(m)}\right] = -G_1^{(n+m+\frac{1}{2})}, \quad \left[G_2^{(n+\frac{1}{2})}, K_2^{(m)}\right] = -G_1^{(n+m+\frac{1}{2})} \quad (A.19)$$

and the part $[\mathcal{M}, \mathcal{M}] = \mathcal{K}$ characteristic for the symmetric space we have here

$$\{G_1^{(n+\frac{1}{2})}, G_2^{(m+\frac{1}{2})}\} = 0 \quad (A.20)$$

$$\{G_1^{(n+\frac{1}{2})}, G_1^{(m+\frac{1}{2})}\} = -\{G_2^{(n+\frac{1}{2})}, G_2^{(m+\frac{1}{2})}\} = 2(K_2^{(n+m+1)} - K_1^{(n+m+1)}) \quad (A.21)$$

$$\left[M_1^{(n)}, G_1^{(m+\frac{1}{2})}\right] = -F_1^{(n+m+\frac{1}{2})}, \quad \left[M_1^{(n)}, G_2^{(m+\frac{1}{2})}\right] = -F_2^{(n+m+\frac{1}{2})} \quad (A.22)$$
Consider the zero-curvature equation (2.22) with

\[ [M_2^{(n)}, G_1^{(m+\frac{1}{2})}] = -F_2^{(n+m+\frac{1}{2})}, \quad [M_2^{(n)}, G_2^{(m+\frac{1}{2})}] = -F_1^{(n+m+\frac{1}{2})} \]  

(A.23)

\[ [M_1^{(n)}, M_2^{(m)}] = -2K_1^{(n+m)} \]  

(A.24)

Some relations involving

\[ E^{(n)} = K_1^{(n)} + K_2^{(n)} \]

are

\[ [M_1^{(n)}, E^{(m)}] = 2M_2^{(n+m)}, \quad [M_2^{(n)}, E^{(m)}] = 2M_1^{(n+m)} \]  

(A.25)

\[ [G_1^{(n+\frac{1}{2})}, E^{(m)}] = -2G_2^{(n+m+\frac{1}{2})}, \quad [G_2^{(n+\frac{1}{2})}, E^{(m)}] = -2G_1^{(n+m+\frac{1}{2})} \]  

(A.26)

It follows from them that

\[ [[X_M^{(n)}, E^{(m)}], E^{(p)}] = 4X_M^{(n+m+p)} \]  

(A.27)

for any \( X_M \in \mathcal{M} \).

Clearly, \( \tilde{G}/\mathcal{K} \) is a symmetric space.

B  Zero-Curvature Equations for \( sl(2|1) \)

Consider the zero-curvature equation (2.22) with \( n = 3 \) :

\[ \left[ \partial_x + E + A_0 + A_{1/2} + k_0, \partial_3 + D_3^{(0)} + D_3^{(1/2)} + D_3^{(1)} + D_3^{(3/2)} + D_3^{(2)} + D_3^{(5/2)} + E^{(3)} \right] = 0 \]  

(B.1)

with \( A_0, k_0 \) and \( A_{1/2} \) parametrized as \( A_0 = u_0 M_1^{(0)} + u_0 M_2^{(0)}, k_0 = -\bar{\psi}_1 \bar{\psi}_2 (K_1^{(0)} - K_2^{(0)}) \) and

\[ A_{1/2} = \bar{\psi}_1 G_1^{(1/2)} + \bar{\psi}_2 G_2^{(1/2)}. \]

Below, for brevity we will suppress the subscript 3 in expressions for terms \( D_3^{(i)} \).

The solution to the zero-curvature equation is obtained by decomposing the zero-curvature equation according to grading into its \( \mathcal{M} = Im(ad_E) \) and \( \mathcal{K} = Ker(ad_E) \) components as follows.

Grade 7/2 :

\[ 0 = [E, D^{(7/2)}] + [A_{1/2}, E^{(3)}] \]  

(B.2)

Grade 3 :

\[ 0 = \partial_x D_K^{(3)} + [A_{1/2}, D_M^{(5/2)}] \]  

(B.3)

\[ 0 = [E, D^{(2)}] + [A_{1/2}, D_K^{(5/2)}] + [A_0, E^{(3)}] \]  

(B.4)

Grade 5/2 :

\[ 0 = \partial_x D_K^{(5/2)} + [A_{1/2}, D_M^{(2)}] + [A_0, D_M^{(5/2)}] + [k_0, D_K^{(5/2)}] \]  

(B.5)

\[ 0 = \partial_x D_M^{(5/2)} + [E, D^{(3/2)}] + [A_{1/2}, D_K^{(2)}] + [A_0, D_K^{(5/2)}] + [k_0, D_M^{(5/2)}] \]  

(B.6)
Grade 2 : 

\[ 0 = \partial_x D_K^{(2)} + \left[ A_{1/2}, D_M^{(3/2)} \right] + \left[ A_0, D_M^{(2)} \right] \] (B.7) 
\[ 0 = \partial_x D_M^{(2)} + \left[ E, D^{(1)} \right] + \left[ A_{1/2}, D_K^{(3/2)} \right] + \left[ A_0, D_K^{(2)} \right] + \left[ k_0, D_M^{(2)} \right] \] (B.8) 

Grade 3/2 : 

\[ 0 = \partial_x D_K^{(3/2)} + \left[ A_{1/2}, D_M^{(1)} \right] + \left[ A_0, D_M^{(3/2)} \right] + \left[ k_0, D_K^{(3/2)} \right] \] (B.9) 
\[ 0 = \partial_x D_M^{(3/2)} + \left[ E, D^{(1/2)} \right] + \left[ A_{1/2}, D_K^{(1)} \right] + \left[ A_0, D_K^{(3/2)} \right] + \left[ k_0, D_M^{(3/2)} \right] \] (B.10) 

Grade 1 : 

\[ 0 = \partial_x D_K^{(1)} + \left[ A_{1/2}, D_M^{(1/2)} \right] + \left[ A_0, D_M^{(1)} \right] \] (B.11) 
\[ 0 = \partial_x D_M^{(1)} + \left[ E, D^{(0)} \right] + \left[ A_{1/2}, D_K^{(1/2)} \right] + \left[ A_0, D_K^{(1)} \right] + \left[ k_0, D_M^{(1)} \right] \] (B.12) 

Grade 1/2 : 

\[ 0 = \partial_x D_K^{(1/2)} + \left[ A_{1/2}, D_M^{(0)} \right] + \left[ A_0, D_M^{(1/2)} \right] + \left[ k_0, D_K^{(1/2)} \right] \] (B.13) 
\[ 0 = -3 A_{1/2} + \partial_x D_M^{(1/2)} + \left[ A_{1/2}, D_K^{(0)} \right] + \left[ A_0, D_K^{(1/2)} \right] \] (B.14) 

Grade 0 : 

\[ 0 = \partial_x D_K^{(0)} - \partial_3 k_0 + \left[ A_0, D_M^{(0)} \right] \] (B.15) 
\[ 0 = -3 A_0 + \partial_x D_M^{(0)} + \left[ A_0, D_K^{(0)} \right] + \left[ k_0, D_M^{(0)} \right] \] (B.16)
We found the following solution for the $D$'s:

\[
D^{(5/2)} = D^{(5/2)}_M = \lambda^2 A_+,
\]

\[
D^{(2)}_M = \lambda^2 A_0, \quad D^{(2)}_K = -\lambda^2 \bar{\psi}_1 \psi_2 \left( K_1^{(0)} - K_2^{(0)} \right)
\]

\[
D^{(3/2)}_M = -\frac{1}{2} \lambda \left( \partial_x \bar{\psi}_2 G_1^{(1/2)} + \partial_x \bar{\psi}_1 G_2^{(1/2)} \right)
\]

\[
D^{(3/2)}_k = -\frac{1}{2} \lambda \left( v \bar{\psi}_1 + u \bar{\psi}_2 \right) F_1^{(1/2)} - \frac{1}{2} \lambda \left( v \bar{\psi}_2 + u \bar{\psi}_1 \right) F_2^{(1/2)}
\]

\[
D^{(1)}_M = \frac{1}{2} \lambda \left[ \partial_x v M_1^{(0)} + \partial_x u M_2^{(0)} - 2 \bar{\psi}_1 \bar{\psi}_2 A_0 \right]
\]

\[
D^{(1)}_K = -\frac{1}{2} \lambda \left( \partial_x \bar{\psi}_1 \bar{\psi}_2 - \partial_x \bar{\psi}_2 \bar{\psi}_2 + \left( v^2 - u^2 \right) \right) K_1^{(0)} + \frac{1}{2} \lambda \left( \partial_x \bar{\psi}_1 \bar{\psi}_2 - \partial_x \bar{\psi}_2 \bar{\psi}_2 \right) K_2^{(0)}
\]

\[
D^{(1/2)}_M = \left( \frac{1}{4} \partial_x^2 - \frac{1}{2} (v^2 - u^2) \right) A_+^2
\]

\[
D^{(1/2)}_K = -\frac{1}{4} \left( \partial_x u \bar{\psi}_1 - u \partial_x \bar{\psi}_1 + \partial_x v \bar{\psi}_2 - v \partial_x \bar{\psi}_2 \right) F_1^{(1/2)}
\]

\[
- \frac{1}{4} \left( \partial_x u \bar{\psi}_2 - u \partial_x \bar{\psi}_2 + \partial_x v \bar{\psi}_1 - v \partial_x \bar{\psi}_1 \right) F_2^{(1/2)}
\]

\[
D^{(0)}_M = \frac{1}{4} \left( \partial_x^2 u - 2(v^2 - u^2) u - 3u \partial_x \bar{\psi}_1 \psi_1 + 3u \partial_x \bar{\psi}_2 \psi_2 - 2 \bar{\psi}_1 \psi_2 \partial_x v - v \partial_x \left( \bar{\psi}_1 \psi_2 \right) \right) M_1^{(0)}
\]

\[
+ \frac{1}{4} \left( \partial_x^2 v - 2(v^2 - u^2) v - 3v \partial_x \bar{\psi}_1 \psi_1 + 3v \partial_x \bar{\psi}_2 \psi_2 - 2 \bar{\psi}_1 \psi_2 \partial_x u - u \partial_x \left( \bar{\psi}_1 \psi_2 \right) \right) M_2^{(0)}
\]

\[
D^{(0)}_K = \frac{1}{2} \left( u \partial_x v - v \partial_x u + \left( v^2 - u^2 \right) \bar{\psi}_1 \bar{\psi}_2 \right) K_1^{(0)}
\]

\[
- \frac{1}{4} \left( \partial_x^2 \bar{\psi}_1 \bar{\psi}_2 + \bar{\psi}_1 \partial_x^2 \bar{\psi}_2 - \partial_x \bar{\psi}_1 \partial_x \bar{\psi}_2 - 3(v^2 - u^2) \bar{\psi}_1 \bar{\psi}_2 \right) \left( K_1^{(0)} - K_2^{(0)} \right)
\]

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