COHOMOLOGY OF MODULI OF REPRESENTATIONS OF MONOMIAL ALGEBRAS

MATTHEW WOOLF

Abstract. In this paper, we study moduli spaces of representations of certain quivers with relations. For quivers without relations and other categories of homological dimension one, a lot of information is known about the cohomology of their moduli spaces of objects. On the other hand, categories of higher homological dimension remain more mysterious from this point of view, with few general methods. In this paper, we will see how some of the methods used to study quivers can be extended to work for representations of any (noncommutative) monomial algebra with relations of length two. In particular, we will give an algorithm to calculate in many cases the classes of these moduli spaces in the Grothendieck ring of varieties.

1. Introduction

Moduli spaces of stable representations of quivers have a beautiful theory. Originally introduced by King [1994] their topological and geometric properties have been studied extensively. For us, we are most interested in their cohomology, which was calculated by Reineke [2003] and Harada and Wilkin [2011]. Why should we care about what happens when we allow relations? Apart from pure curiosity, there are three reasons this is an interesting question.

The first reason to study quivers comes from a derived category perspective. The derived category of representations of a quiver without relations can be equivalent to the derived category of representations of a different quiver with relations. If we’re interested in studying all moduli spaces of stable objects in this derived category, we need to understand moduli spaces of representations of a quiver with relations.

The second reason comes from algebraic geometry. Certain smooth projective varieties have full strong exceptional collections, which means the derived category of coherent sheaves is equivalent to the derived category of representations of a quiver with relations. Understanding moduli spaces of representations of the quiver then tells us about moduli spaces of sheaves on the variety.

The third reason also comes from algebraic geometry, though more at the level of an analogy. There is a close analogy between representations of a quiver and vector bundles on a curve, c.f., Rapoport [1997]. The cohomology of moduli spaces of stable vector bundles on a curve has been calculated using gauge theory by Atiyah and Bott [1983], and using arithmetic methods by Harder and Narasimhan [1974].

All four methods of calculating cohomology of moduli spaces of either representations of a quiver or vector bundles on a curve have a similar basic structure. At a

1There have been other calculations of these cohomology groups, but as they don’t seem to fall into the general pattern described below, we will ignore them.
very general level, they all copy the following strategy. First, calculate the cohomology of the moduli stack of all (not necessarily stable) objects under consideration (either representations or vector bundles). Next, stratify this stack by the numerical invariants (dimension vectors or Chern characters) of the subquotients which occur in the Harder-Narasimhan filtration. Next, inductively calculate the cohomology of each non-open stratum. Finally, subtract off these contributions to get the cohomology of the open stratum corresponding to semistable vector bundles.

The category of representations of a quiver and the category of coherent sheaves on a curve have one important fact in common: they both have homological dimension one. When calculating the cohomology of each stratum, it is necessary to determine the space of extensions of one object by another. The space of extensions of $B$ by $A$ is given by $\text{Ext}^1(B, A)$, but maps $\text{Hom}(B, A)$ give automorphisms of these extensions, so we can identify isomorphism classes of extensions with elements of the vector space $\text{Ext}^1(B, A)/\text{Hom}(B, A)$. But in a category of homological dimension one, this dimension is constant (being essentially the Euler characteristic). This means that given two moduli stacks of objects $\mathcal{M}$ and $\mathcal{N}$, the stack of objects which are an extension of something in $\mathcal{M}$ by something in $\mathcal{N}$ is a vector bundle over $\mathcal{M} \times \mathcal{N}$.

If we want to extend this type of argument to more general abelian categories, we instead get something mapping to $\mathcal{M} \times \mathcal{N}$, where the fibers are all vector spaces, but of possibly varying dimensions. This causes serious problems when we try and use point-counting or the Gysin sequence to relate the cohomology of the strata to the cohomology of the moduli stack of all objects.

In this paper, we will show in some toy examples with homological dimension not equal to one how to calculate cohomology of moduli spaces of semistable objects. Specifically, we will slightly modify the case of quivers by allowing certain fairly simple relations, and show that we can still calculate the cohomology of moduli spaces of semistable objects. In the case of quivers with no relations, our method is still slightly different from what has been done before. More specifically, we will show the following.

**Theorem.** Let $Q$ be a quiver, and $kQ$ its path algebra. Let $I$ be a two-sided ideal generated by paths of length two in $Q$. Let $\langle d_i \rangle$ be a dimension vector for $Q$, $\sigma$ a stability condition, and $M(Q, I, d, \sigma)$ the moduli space of stable representations of $(Q, I)$. Moreover, suppose that the $d_i$ are coprime and the stability condition is sufficiently generic (i.e., there are no strictly semistable representations). Then the class of $M(Q, I, d, \sigma)$ in the Grothendieck ring of varieties is a rational function in the class of $\mathbb{A}^1$ which can in principle be calculated explicitly.

As part of the proof, will also see the following facts about $M(Q, I, d, \sigma)$.

**Theorem.** Each irreducible component of $M(Q, I, d, \sigma)$ is unirational.

We note that results of Bardzell [1997] imply that we can construct examples of such $(Q, I)$ with arbitrarily large global dimension.

We will begin by reviewing some basic facts about stability of quiver representations and about the Grothendieck ring of varieties. We will then determine the class of certain relatively simple linear algebraic moduli spaces in the Grothendieck ring. We will then essentially adapt the strategy used in Reineke [2008] to study framed quiver moduli for our situation.
Acknowledgements

I would like to thank Arend Bayer, Tom Bridgeland, and Emanuele Macrì for their very helpful conversations about both the mathematical and geopolitical content of this work. I would also like to thank Dawei Chen, Izzet Coskun, Joe Harris, Alex Perry, and Eric Riedl for listening to me talk about this and providing their input. Finally, I’d like to thank Miles Reid and Markus Reineke for pointing out some relevant work I was not aware of.

2. Stability Conditions on an Abelian Category

We now review some basic material on stability conditions on Abelian categories from [Bridgeland 2007].

Definition. Let \( A \) be an Abelian category. A stability function will be a pair of homomorphisms \( \deg, \text{rk} : K(A) \to \mathbb{R} \) satisfying certain properties.

1. For any nonzero object \( E \) of \( A \), \( \text{rk}(E) \geq 0 \). If \( \text{rk}(E) = 0 \), then \( \deg(E) > 0 \).

   We can conclude that the slope of any nonzero \( E \),
   \[
   \mu(E) = \frac{\deg E}{\text{rk} E}
   \]

   is a well-defined element of \( \mathbb{R} \cup \{\infty\} \).

2. Any object \( E \) has a Harder-Narasimhan filtration

   \[
   0 = E_0 \subset E_1 \subset \cdots \subset E_n
   \]

   where
   \[
   \mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)
   \]

   and the \( E_i/E_{i-1} \) are semistable, i.e., given a nonzero proper subobject \( F \subset E_i/E_{i-1} \), we have \( \mu(F) \leq \mu(E_i/E_{i-1}) \). We note that this Harder-Narasimhan filtration will be unique.

Definition. An object \( E \) is stable if for any nonzero object \( F \subset E \), we have \( \mu(F) < \mu(E) \).

The automorphism group of a stable object is \( \mathbb{G}_m \). Any semistable object has a Jordan-Hölder filtration, i.e., a filtration where the subquotients are all stable of the same slope. The Jordan-Hölder filtration is not unique, e.g., take the direct sum of two different stable objects of the same slope, but the isomorphism classes of the subquotients are uniquely determined up to reordering.

Definition. Two semistable objects are said to be S-equivalent if the subquotients occurring in their Jordan-Hölder filtrations are the same.

3. Stability Conditions on the Category of Representations of a Quiver

We now recall the material on moduli spaces of stable representations of quivers from [King 1994]. We note that we describe the results using the more modern language of [Bridgeland 2007].

Definition. A quiver consists of a set of vertices \( V \) and a set of edges \( E \) together with two maps \( h, t : E \to V \).

We think of \( h \) and \( t \) as giving the head and tail of each edge.
Definition. A quiver is called finite if both $V$ and $E$ are finite sets.

Definition. A path in the quiver consists of an ordered collection of edges $e_i$ such that $t(e_i) = h(e_{i+1})$ for all $i$. A vertex is considered to be a path of length 0. A cycle is a path which begins and ends at the same vertex. An acyclic quiver is a quiver with no cycles.

Given a quiver, we can form its path algebra $kQ$. This algebra is generated by $V$ and $E$ with multiplication given by composition when it’s defined, and 0 otherwise. Equivalently, we can take the free vector space on the paths in $Q$ and define multiplication as before. We note that the path algebra of a finite acyclic quiver is finite-dimensional.

Definition. A representation of the quiver is defined to be a left module over the path algebra.

Note that this is equivalent to giving a vector space for each vertex and a map for each edge.

Definition. The dimension vector of a representation is the tuple of the dimensions of each of these vector spaces.

Given a two-sided ideal $I$ of $kQ$, we can form the quotient algebra, which is called the path algebra of the bound quiver, or quiver with relations. We can then talk about representations of the bound quiver.

Definition. A representation of the bound quiver $(Q, I)$ is a module over the quotient algebra $kQ/I$, or equivalently, a representation of the quiver such that the maps satisfy the relations determined by $I$.

We will be especially interested in the following class of bound quivers.

Definition. A monomial ideal in $kQ$ is a two-sided ideal generated by paths (rather than just linear combinations of paths). A monomial algebra is the quotient of $kQ$ by a monomial ideal.

Given any bound finite acyclic quiver, we can form $kQ/I - Mod$, the category of its finite-dimensional representations.

Lemma. The Grothendieck group $K(kQ - Mod) \cong ZV$. By choosing a rank and degree satisfying the positivity conditions for each vertex, we get a stability condition on $kQ/I - Mod$. Moreover, every stability condition arises this way.

Proof. We prove the first statement by induction on the dimension vector. Given a representation, there must be a vertex with nonzero dimension which does not map to any other such vertices. We can construct a representation of the bound quiver with a one-dimensional vector space at that vertex, and zero for all other vertices. The representation we started with surjects onto this representation, and the kernel has smaller dimension vector.

For the second statement, we just need to check the Harder-Narasimhan property, but this is easy since every object of $kQ/I$-mod has finite length – just pick the semistable quotient with smallest slope (we’ve seen that there is a semistable quotient), and continue this process with the kernel. This process must eventually terminate by the finite length condition.

The third statement is clear. □
Translating theorem 4.1 of [King 1994] into the above language, we get the following result.

**Theorem (King).** Let $Q$ be a finite acyclic quiver. Given any two-sided ideal of relations $I \subset kQ$, stability condition $\sigma$ on $kQ/I - \text{mod}$ and dimension vector $d$, there is a quasi-projective moduli space of stable representations of $(Q, I)$ of dimension vector $d$, and a projective moduli space of semi-stable representations.

4. Grothendieck Ring of Varieties

We recall some basic facts about the Grothendieck ring of varieties from [Bridgeland 2012].

We define the Grothendieck ring of varieties $K(\text{Var}_k)$ to be the quotient of the free Abelian group on varieties over $k$ by the scissor relations, namely if $X$ can be written as a union of $Y$ and $Z$ with $Y \cap Z = \emptyset$, then we write $[X] = [Y] + [Z]$. Multiplication is given by $[X] \cdot [Y] = [X \times Y]$.

One of the most important classes in the Grothendieck ring is the Lefschetz motive, $L = [\mathbb{A}^1]$. Many linear algebraic moduli spaces can be built out of the Lefschetz motive, so we make the following definition.

**Definition.** A variety is said to have a motivic cell decomposition if its class in the Grothendieck ring of varieties is a polynomial in $L$ with rational coefficients with degree bounded by the dimension of the variety.

The theory of mixed Hodge structures implies that the class of a smooth projective variety in the Grothendieck ring of varieties over $\mathbb{C}$ determines its Betti numbers, and even the Hodge numbers $h^{p,q} = \dim H^{p,q}(X, \mathbb{C})$. The class of a variety over a finite field determines the number of points, so the Weil conjectures imply that its $\ell$-adic Betti numbers are determined.

We now show that having a motivic cell decomposition is actually equivalent to an a priori weaker condition.

**Lemma.** If the class of a variety $X$ defined over $\mathbb{Z}$ is a rational function in $L$ with rational coefficients,

$$[X] = R(L) = \frac{P(L)}{Q(L)}$$

then $Q \mid P$, so in particular, $R$ is actually a polynomial. Furthermore, the degree of $R$ is bounded by the dimension of $X$.

**Proof.** By counting points over $\mathbb{F}_q$, we get a map $K(\text{Var}_k) \to \mathbb{Z}$ which sends $L$ to $q$. This means that $R(q)$ is an integer for any prime power $q$. But any rational function which takes integer values at integers infinitely many times must be an integer [http://mathoverflow.net/users/2384/gjergji zaimi](http://mathoverflow.net/users/2384/gjergji zaimi).

The bound on the degree of the polynomial follows for example from the Lang-Weil estimates for the number of rational points of a geometrically irreducible variety over a finite field [Lang and Weil, 1954].

We now show that having a motivic cell decomposition is actually equivalent to an a priori weaker condition.

**Proposition.** Let $X$ be a smooth projective variety of dimension $n$. Suppose that $[X] = \sum_{i=0}^n a_i L^i$. Then $h^{n-i, n-i} = h^{i,i} = a_i$, and $h^{i,j} = 0$ when $i \neq j$. 

**Proof.** By counting points over $\mathbb{F}_q$, we get a map $K(\text{Var}_k) \to \mathbb{Z}$ which sends $L$ to $q$. This means that $R(q)$ is an integer for any prime power $q$. But any rational function which takes integer values at integers infinitely many times must be an integer [http://mathoverflow.net/users/2384/gjergji zaimi](http://mathoverflow.net/users/2384/gjergji zaimi). The bound on the degree of the polynomial follows for example from the Lang-Weil estimates for the number of rational points of a geometrically irreducible variety over a finite field [Lang and Weil, 1954].

We now collect the following results about $K(\text{Var}_k)$, many of which are in [Bridgeland 2012].

**Proposition.** Let $X$ be a smooth projective variety of dimension $n$. Suppose that $[X] = \sum_{i=0}^n a_i L^i$. Then $h^{n-i, n-i} = h^{i,i} = a_i$, and $h^{i,j} = 0$ when $i \neq j$.
Lemma. If \( \pi : X \to Y \) is a Zariski-locally trivial fibration, and both \( Y \) and the fibers of \( \pi \) have motivic cell decompositions, then so does \( X \). More precisely, if \([F]\) is the class of a fiber, then \([X] = [F][Y]\).

Proof. Stratify \( Y \) such that \( \pi \) is trivial on each stratum \( S_i \). Then

\[
[X] = \sum [S_i][F] = [F] \sum [S_i] = [F][Y].
\]

\( \square \)

Lemma. Let \( \pi : X \to Y \) be a Zariski-locally trivial fibration. Suppose that \( X \) and the fibers of \( \pi \) have motivic cell decompositions. Then \( Y \) has a motivic cell decomposition.

Proof. The class of \( Y \) will be a rational function in \( L \), but by lemma 4, this implies that \( Y \) actually has a motivic cell decomposition. \( \square \)

Lemma. If \( X \) has a finite stratification such that each stratum has a motivic cell decomposition, then \( X \) has a motivic cell decomposition.

Lemma. If \( X \) can be covered by finitely many locally closed subvarieties \( V_i \) such that the intersection of any number of the \( V_i \) has a motivic cell decomposition, then \( X \) has a motivic cell decomposition.

Proof. This comes from an application of the inclusion-exclusion principle. \( \square \)

Lemma. Suppose we have a map \( \pi : X \to Y \) such that the fibers all admit a motivic cell decomposition, and there is a stratification of \( Y \) such that all the strata have motivic cell decomposition and the restriction of \( \pi \) to each stratum is a Zariski-locally trivial fibration, then \( X \) has a motivic cell decomposition.

Corollary. Flag varieties have a motivic cell decomposition.

Proof. We know that flag varieties can be realized as iterated Grassmannian bundles. By the above lemma, it suffices to show that Grassmannians have a motivic cell decomposition, but this follows from the Schubert decomposition [Griffiths and Harris, 1994]. \( \square \)

Lemma. \( \text{Gl}_n \) has a motivic cell decomposition.

Proof. Fix a point \( p \in \mathbb{P}^n \). We get a Zariski-locally trivial fibration \( \text{Gl}_n/P \to \mathbb{P}^n \). It suffices to show that \( P \) has a motivic cell decomposition. Without loss of generality, \( p = [1 : 0 : \cdots : 0] \). Then we can identify \( P \) with the space of matrices with first column

\[
\begin{bmatrix}
\lambda \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

with \( \lambda \neq 0 \), but this clearly has a motivic cell decomposition. \( \square \)
5. Decompositions of Linear Algebraic Moduli Spaces

In this section, we will show that various moduli spaces parametrizing linear algebraic objects such as maps or subspaces satisfying various conditions have an effective motivic cell decomposition.

**Lemma.** If $V$ and $W$ are two vector spaces of dimension $d$ and $e$ respectively, then the space of injective maps from $V$ to $W$ has an effective motivic cell decomposition.

**Proof.** Let $d = \dim V$. An injective map from $V$ to $W$ is the same as a $d$-dimensional subspace of $W$ and an isomorphism from $V$ to that subspace, so the space of injective maps is a Zariski-locally trivial $\text{Gl}_d$ fibration over a Grassmannian. □

Note that dualizing the above lemma shows that the space of surjective maps from $V$ to $W$ has an effective motivic cell decomposition.

This lemma also implies the a priori more general corollary.

**Corollary.** If $V$ and $W$ are vector spaces, then the space of maps from $V$ to $W$ of rank $k$ has an effective motivic cell decomposition.

**Proof.** This space can be identified with the product of the Grassmannian of $k$-dimensional quotients of $V$, the Grassmannian of $k$-dimensional subspaces of $W$, and the space of isomorphisms from the first space to the second space. □

**Lemma.** Fix a vector space $V$ and a collection of disjoint subspaces $W_j$. Then the space of subspaces $X \subset V$ of dimension $d$ which meet at least one of the $W_j$ has an effective motivic description.

**Proof.** For each $J \subset 2^n$ (the power set of the set of the first $n$ positive integers), we get a subset $W_J$ of $V$ by taking

$$\{v \in V : \forall I \in 2^n, v \in \sum_{i \in I} W_i \iff I \in J\}.$$ 

Using the inclusion-exclusion principle, we see that each of these subsets has an effective motivic cell decomposition. We note that each vector is in precisely one of the $W_J$.

To get a subspace disjoint from all the $W_j$, first pick a vector. For each of the $W_j$, either all of the vectors it contains or none of them will be allowable. Once we pick the first vector, we must pick a second vector.

To do this, define $W'_J$ by taking all the $W_j$ as well as the one-dimensional subspace we just defined, and repeat the previous construction. The effective motivic cell decompositions of the $W'_J$ will only depend on which of the $W_j$ our first vector lay in. Which vectors we can pick so the span with the first vector is disjoint from the $W_j$ will only depend on which $W'_J$ our second vector lies in. We can then continue our process for any subsequent vectors.

This shows that the space of injective maps from a fixed vector space to $V$ such that the image is disjoint from the $W_j$ has an effective motivic cell decomposition. We can then take the quotient by changes of basis of the vector space to get the desired result. □

We give an example to show how this idea works in practice. Consider a three-dimensional vector space $k^3$ and three non-coplanar one-dimensional subspaces $L_1$, $L_2$, and $L_3$. Let us try and find a two-dimensional vector space disjoint from the three lines.
The first vector we pick will either be in the span of two of the lines, or not. Let’s say first it’s in the span of two of the lines, say $L_1$ and $L_2$. The space of possible vectors is given by $L^2 - 2(L - 1) - 1$, since we’re considering vectors in the plane not on either of the lines. The second vector we pick cannot be on this plane, or the plane spanned by the first vector and $L_3$, so there are $L^3 - 2(L^2 - L) - L$ choices for the second vector.

In the second case, our first vector is not in the plane spanned by any two of the lines. In this case, we have $L^3 - 3(L - 1) - 1$ choices for the first vector and $L^3 - 3(L^2 - L) - L$ choices for the second vector.

**Definition.** An intersection dimension lattice for a vector space $V$ will consist of a finite set $I$ and a map $f : 2^I \to \{1, 2, \ldots, \dim V\}$.

Note that a collection of subspaces $W_i, i \in I$ of $V$ naturally give rise to an intersection dimension lattice, by setting for $K \subset I$, $f(K) = \dim \bigcap_{i \in K} W_i$.

**Proposition.** Fix a vector space $V$ and an intersection dimension lattice $(J, f)$ for $V$. Then the space of collections of subspaces of $V$ which give rise to that intersection dimension lattice has an effective motivic cell decomposition. We call this space the UK flag variety.

**Proof.** The proof of this is a generalization of the method used in the previous lemma. We can choose $W_I$, the intersection of all the $W_i$, freely. This gives us a Grassmannian. The space of possible such choices is a Grassmannian. We then pick $W_{I \setminus \{k\}}$ containing $W_I$ using the method of the previous lemma, i.e., adding one vector at a time by picking one vector at a time from a stratum of a recursively defined stratification. We then want to pick $W_{I \setminus \{1, 2\}}$ which must contain $W_{I \setminus \{1\}}$ and $W_{I \setminus \{2\}}$ but not $W_{I \setminus \{k\}}$ for $k > 2$. We continue this process for all $W_K$ with $K \subset I$. □

From the above construction of the effective motivic cell decomposition of the UK flag varieties, we see immediately the following fact.

**Corollary.** Each irreducible component of a UK flag variety is unirational.

It seems likely that with a little more effort, one can in fact prove that each irreducible component of the UK flag variety is rational. This would actually imply that certain moduli spaces of quiver representations were stably rational.

**Lemma.** Fix two vector spaces $A$ and $B$ and subspaces $V \subset A$ and $W \subset B$. The space of maps $f : A \to B$ with kernel $V$ and image $W$ has an effective motivic cell decomposition which only depends on the dimensions of the four spaces.

**Proof.** We note that we are essentially looking at the space of isomorphisms from $A/V$ to $W$. □

### 6. Decomposition of the Representation Space

Let $(Q, I)$ be a monomial algebra generated by paths of length two, and $w$ a dimension vector. By $R(Q, w)$, we will mean the representation space, i.e. the subvariety of

$$\prod_{e \in E} \Hom(h(e), t(e))$$
of tuples of maps satisfying the given relations. We will now show that $R(Q, w)$ has an effective motivic cell decomposition, or more precisely, that in a certain stratification, each stratum does.

**Theorem.** Fix a dimension vector $w$ and an intersection dimension lattice for each vertex of $Q$. The space of representations of $Q$ such that the kernels and intersections of the maps corresponding to the arrows of $Q$ give rise to this intersection dimension lattice has an effective motivic cell decomposition.

**Proof.** We can map this locus to the product of the UK flag varieties. We will show that this is a Zariski-local fibration, and that the fibers have an effective motivic cell decomposition.

First, we note that the relations for the quiver simply amount to requiring images of some maps to be contained in kernels of others, so whether the relations hold is determined entirely by the intersection dimension lattices. If the relations do not hold, then the space of representations is empty, so trivially has an effective motivic cell decomposition.

Suppose that the relations are satisfied. Given this, we can identify the space of these representations with the collections of isomorphisms $V_{h(e_i)}/K_i \to I_i$ (where $K_i$ is the kernel of the $i$th arrow and $I_i$ its image), which is clearly a Zariski-local fibration. □

Note that this argument already shows that each component of the moduli space of semistable representations is unirational. If each component of the UK flag variety is rational, then we similarly get rationality of each component of the representation space.

### 7. Decomposition of the Semistable Locus

To calculate the semistable locus of the space of representations, the idea will be to use stratify the entire representation variety by the dimension vectors which occur in the Harder-Narasimhan filtration and inductively show that each stratum corresponding to unstable representations has an effective motivic cell decomposition. Since the representation variety as a whole has an effective motivic cell decomposition, it will follow that the locus of semistable representations has an effective motivic cell decomposition.

**Definition.** Given a dimension vector $w$, a Harder-Narasimhan type will be a finite sequence of dimension vectors $w_i$ with $\sum w_i = w$ such that the slopes of the $w_i$ are decreasing.

Suppose we fix a Harder-Narasimhan type. We can consider the locus of representations which have a Harder-Narasimhan filtration of that type. We want the following result.

**Theorem.** Consider the space of representations of the bound quiver $(Q, I)$ with fixed Harder-Narasimhan type. Inside this, consider the locus where the kernels and images of all the arrows, together with all the subspaces from the Harder-Narasimhan stratification, have a fixed intersection dimension lattice. This locus has an effective motivic cell decomposition.

**Proof.** We can further refine this stratification by considering the intersection dimension lattice on each subquotient. This maps to the products of the spaces of
semistable representations with fixed intersection dimension lattices. By induction on the dimension vector, we know that each of those semistable loci has an effective motivic cell decomposition. We will show that the fibers all have the same effective motivic cell decomposition.

This follows from induction on the length of the Harder-Narasimhan filtration and the following result.

**Proposition.** Fix two representations of $(Q, I)$ such that for each of the representations, the kernels and images of all the maps have a given intersection dimension lattice at each vertex. Consider the space of representations of $(Q, I)$ on a fixed vector space which are an extension of the second representation by the first. Inside this, take the locus of representations where for each vertex of $Q$, the subspace corresponding to the subrepresentation and the kernels and images of the new representation have a fixed intersection dimension lattice. Then this locus has an effective motivic cell decomposition which depends only on the three intersection dimension lattices.

**Proof.** We note that whether the monomial relations for the representation are satisfied can be determined entirely from the intersection dimension lattice, so it suffices to show the space of extensions that have a given intersection dimension lattice has an effective motivic cell decomposition which only depends on the intersection dimension lattice. We can map the space of extensions to the corresponding UK flag variety, and it suffices to show that each fiber has the same effective motivic cell decomposition. For this, we note that we can essentially study this question one arrow at a time, since there’s no interaction between the arrows other than the monomial relations.

Let $f$ be an arrow in $Q$. Let $a : A \to A'$ be the corresponding map in the subrepresentation and $c : C \to C'$ the corresponding map in the quotient representation.
It suffices to show the above property for the space of \( \alpha : C \to A' \) such that the induced map \( b : B \to B' \) is compatible with the given UK flag.

What does it mean for a subspace \( V \subset B \) to be the kernel of \( b \)? Clearly, we must have \( V \cap A = \ker a \). Let \( \pi : B \to C \) be the natural map. We must clearly have \( \pi(V) \subset \ker c \). Let \( (x_1, x_2) \in V \). Then

\[
b(x_1, x_2) = (a(x_1) + \alpha(x_2), c(x_2))
\]

which is 0 if and only if \( \alpha(x_2) = -a(x_1) \). All elements of \( V \) with the same \( x_2 \) differ by an element of \( V \cap A \), so \( a \) takes the same value on each, so essentially we require \( \alpha \) is completely determined on \( \pi(V) \). Conversely, given \( x_2 \in \ker C \), if \( \alpha(x_2) = -a(x_1) \), then \( (x_1, x_2) \) is in the kernel of \( b \). We conclude that determining the kernel of \( b \) is equivalent to choosing a subspace \( W \) of \( \ker C \) such that

\[
\alpha^{-1}(\im a) \cap \ker c = W
\]

Once we have this subspace, the value of \( \alpha \) there is clearly determined. Furthermore, we can determine \( V \) as the space of

\[
\{(x_1, x_2) ; x_2 \in w, \alpha(x_2) + a(x_1) = 0\}.
\]

Since we’ve fixed \( V \), we know \( W = \pi(V) \) (whose dimension is determined by the intersection dimension lattice), and we know

\[
\alpha : W \to \im a,
\]

and that

\[
\alpha^{-1}(\im a) \cap \ker c = W.
\]

Dualizing the above argument, we see that since we’ve fixed the image of \( b \), we get a space \( U \) containing \( \im a \) (namely \( U = \im b \cap A' \)) such that

\[
\alpha(\ker c) + \im a = U,
\]

and we know the induced map

\[
\alpha : \ker c \to A'/U
\]

We note that the dimension of \( U \) is determined by the intersection dimension lattice.

We now pick a bilinear form on \( C \) and \( A' \) which we will use to freely identify quotients and subspaces. Putting everything together, we see that picking a suitable \( \alpha \) is equivalent to finding a subspace

\[
W \subset \ker c,
\]

a space

\[
U \supset \im a,
\]

a surjective map

\[
\ker c/W \to U/\im a,
\]

an injective map

\[
\ker c/W \to \im a,
\]

and an arbitrary map

\[
C/\ker C \to A'.
\]

The space of such choices has a motivic cell decomposition which depends only on the intersection dimension lattices.

\( \square \)
Since there are only finitely many possible Harder-Narasimhan types and intersection dimension lattices, we get the following consequence.

**Theorem.** The locus in $R(Q, I, w, \mu)$ of semistable representations has a motivic cell decomposition.

8. Motivic Cell Decomposition of the Moduli Space

We now want to understand the relationship between the semistable locus in the representation variety and the moduli space of semistable representations. We will make the assumption that all semistable representations are stable, which will hold for example if the dimension vector is coprime and the stability condition is sufficiently general.

By King [1994], the moduli space of stable representations is a GIT quotient of the representation variety by $\prod_{i \in \text{Vertex}(Q)} Gl_{w_i}$, and the stable locus is the locus of stable representations. Since the automorphism group of a stable representation is $G_m$, this is a principal $Gl_w/G_m$-bundle over the moduli space of stable representations [Mumford et al., 1994].

**Proposition.** When the dimension vector is coprime, this bundle is Zariski-locally trivial.

**Proof.** We know any principal $G$-bundle with $G \cong Gl_n$ or $G \cong Sl_n$ is Zariski-locally trivial, since it’s essentially the same as a vector bundle, which has a Zariski-local trivialization. The same will then be true for products of such groups. We will use standard facts about étale cohomology from Milne [1980]. We want to show that the $Gl_w/G$-bundle actually comes from a $Gl_w$-bundle. Consider the short exact sequence

$$1 \rightarrow G_m \rightarrow Gl_w \rightarrow Gl_w/G_m \rightarrow 1.$$  

We get an associated exact sequence of cohomology

$$H^1(Gl_w) \rightarrow H^1(Gl_w/G_m) \rightarrow H^2(G_m).$$

Recalling that $H^1$ classifies torsors, we want to show that the last map is 0. Looking at the short exact sequence

$$1 \rightarrow \mu_{a_i} \rightarrow Sl_{a_i} \times \prod_{k \neq i} Gl_{a_k} \rightarrow Sl_{a_i} \times \prod_{k \neq i} Gl_{a_k}/\mu_{a_i} \rightarrow 1$$

we see that the map $H^1(Gl_w/G_m) \rightarrow H^2(G_m)$ factors through $H^2(\mu_{a_k})$. Using the Kummer exact sequence, we can identify $H^2(\mu_{a_k})$ with the $a_k$-torsion of $H^2(G_m)$. If the $a_i$ are all coprime, then we deduce that we get the zero element of $H^2(G_m)$, as desired.

**Corollary.** The moduli space of stable representations has a motivic cell decomposition when the dimension vector is coprime and the stability condition is generic.

**Proof.** The class of the moduli space of stable representations will be the quotient of the class of the stable locus in the representation variety by the class $G$, but both have motivic cell decompositions.
9. Further Thoughts

We have now proved that the class of the moduli space of stable representations of a quiver in the Grothendieck ring of varieties is a polynomial in \( L \). A natural question to ask is whether we can explicitly calculate the coefficients of this polynomial. The method of proof we have used is in principle effective, but the observant reader will have noted that in all but the very simplest of examples, carrying out this procedure is a truly terrifying prospect.

However, we note that the knowledge that there is such a polynomial is enough to give us a much simpler algorithm. We know that degree of this polynomial is bounded by the dimension of the moduli space. It is easy to bound the dimension of the moduli space by the dimension of the moduli space of representations of the unbound quiver, which is easy to calculate in terms of the dimension vector. We can then count points of the moduli space over finite fields and use Lagrange interpolation to find the polynomial.

Apart from this, there are a number of natural ways to try to extend this work. The first would be to allow monomial algebras with relations of any length. Much of the work done in this paper carries through to that case, but there appear more complicated linear algebraic moduli spaces for which it is harder to produce a motivic cell decomposition. There are still a number special cases in which this is possible, but the general case is still just out of reach of the current author.

Even better, though, would be to get similar results for quivers with binomial relations. Even the case of binomial relations of length two would give an explicit method for calculating the cohomology of moduli spaces of stable sheaves on the projective plane. More generally, understanding this case better would give a great deal of information about many moduli spaces related to varieties with full exceptional collections.

One might hope that perhaps any moduli space of stable representations of a finite-dimensional algebra has a motivic cell decomposition, but this is impossible for the following reason. First, any Mori dream space is a fine moduli space of representations of a finite-dimensional algebra [Craw and Winn, 2013]. In particular, any Mori dream space would have a motivic cell decomposition. On the one hand, any smooth quintic threefold is a Mori dream space, but if it had a motivic cell decomposition, its only nonzero Hodge numbers would be \( h^{p,p} \), but this is certainly false for quintic threefolds.

References

M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. Philosophical Transactions of the Royal Society of London. Series A. Mathematical and Physical Sciences, 308(1505):523–615, 1983. ISSN 0080-4614. doi: 10.1098/rsta.1983.0017. URL http://www.ams.org/mathscinet-getitem?mr=702806

Michael J. Bardzell. The alternating syzygy behavior of monomial algebras. Journal of Algebra, 188(1):69–89, 1997. ISSN 0021-8693. doi: 10.1006/jabr.1996.6813. URL http://www.ams.org/mathscinet-getitem?mr=1432347

Tom Bridgeland. Stability conditions on triangulated categories. Annals of Mathematics. Second Series, 166(2):317–345, 2007. ISSN 0003-486X. doi: 10.4007/annals.2007.166.317. URL http://www.ams.org/mathscinet-getitem?mr=2373143
Tom Bridgeland. An introduction to motivic Hall algebras. *Advances in Mathematics*, 229(1):102–138, 2012. ISSN 0001-8708. doi: 10.1016/j.aim.2011.09.003. URL [http://www.ams.org/mathscinet-getitem?mr=2854172](http://www.ams.org/mathscinet-getitem?mr=2854172).

Alastair Craw and Dorothy Winn. Mori dream spaces as fine moduli of quiver representations. *Journal of Pure and Applied Algebra*, 217(1):172–189, 2013. ISSN 0022-4049. doi: 10.1016/j.jpaa.2012.06.014. URL [http://www.ams.org/mathscinet-getitem?mr=2965915](http://www.ams.org/mathscinet-getitem?mr=2965915).

Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. ISBN 0-471-05059-8. URL [http://www.ams.org/mathscinet-getitem?mr=1288523](http://www.ams.org/mathscinet-getitem?mr=1288523). Reprint of the 1978 original.

Megumi Harada and Graeme Wilkin. Morse theory of the moment map for representations of quivers. *Deductions*, 150:307–353, 2011. ISSN 0046-5755. doi: 10.1007/s10711-010-9508-5. URL [http://www.ams.org/mathscinet-getitem?mr=2753709](http://www.ams.org/mathscinet-getitem?mr=2753709).

G. Harder and M. S. Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Mathematische Annalen*, 212:215–248, 1974. ISSN 0025-5831. URL [http://www.ams.org/mathscinet-getitem?mr=0364254](http://www.ams.org/mathscinet-getitem?mr=0364254).

Gjergji Zaimi (http://mathoverflow.net/users/2384/gjergjizaimi). When does a rational function have infinitely many integer values for integer inputs? MathOverflow, 2010. URL [http://mathoverflow.net/q/37975](http://mathoverflow.net/q/37975).

A. D. King. Moduli of representations of finite-dimensional algebras. *The Quarterly Journal of Mathematics. Oxford. Second Series*, 45(180):515–530, 1994. ISSN 0033-5606. doi: 10.1093/qmath/45.4.515. URL [http://www.ams.org/mathscinet-getitem?mr=1315461](http://www.ams.org/mathscinet-getitem?mr=1315461).

Serge Lang and André Weil. Number of points of varieties in finite fields. *American Journal of Mathematics*, 76:819–827, 1954. ISSN 0002-9327. URL [http://www.ams.org/mathscinet-getitem?mr=0065218](http://www.ams.org/mathscinet-getitem?mr=0065218).

D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, third edition, 1994. ISBN 3-540-56963-4. URL [http://www.ams.org/mathscinet-getitem?mr=1304906](http://www.ams.org/mathscinet-getitem?mr=1304906).

M. Rapoport. Analogien zwischen den Modulräumen von Vektorbündeln und von Flaggen. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 99(4):164–180, 1997. ISSN 0012-0456. URL [http://www.ams.org/mathscinet-getitem?mr=1480327](http://www.ams.org/mathscinet-getitem?mr=1480327).

Markus Reineke. The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli. *Inventiones Mathematicae*, 152(2):349–368, 2003. ISSN 0020-9910. doi: 10.1007/s00222-002-0273-4. URL [http://www.ams.org/mathscinet-getitem?mr=1974891](http://www.ams.org/mathscinet-getitem?mr=1974891).

Markus Reineke. Framed quiver moduli, cohomology, and quantum groups. *Journal of Algebra*, 320(1):94–115, 2008. ISSN 0021-8693. doi: 10.1016/j.jalgebra.2008.01.025. URL [http://www.ams.org/mathscinet-getitem?mr=2417980](http://www.ams.org/mathscinet-getitem?mr=2417980).