Getting information via a quantum measurement: the role of decoherence

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Abstract In this work we investigate the relation between quantum measurements and decoherence, in order to formally express the necessity of the latter for obtaining an informative output from the former. To this aim, referring to the Von Neumann scheme for ideal quantum measurements, we first look for the minimal structure that the interaction between principal system and measurement apparatus must have for properly describing the process, beyond the quantum measurement limit, and then analyze the dynamical evolution induced by one such interaction. The analysis is developed by means of a recently introduced method for studying open quantum systems, namely the parametric representation with environmental coherent states, that allows us to determine a necessary condition that the quantum state of the apparatus must fulfill in order to give information on the observable being measured. We find that this condition strictly implies decoherence in the principal system, with respect to the eigenstates of the hermitian operator that represents the measured observable, thus establishing that there cannot be information flux from a quantum system towards a readable analyser unless decoherence occurs. The relevance of dynamical entanglement generation is highlighted, and consequences of the possible macroscopic structure of the measurement apparatus are also commented upon.

Keywords open quantum systems · decoherence · quantum measurement

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1 Introduction

The profound relation between quantum measurement process and decoherence is nowadays recognized as a key feature of quantum mechanics, not only from a foundational viewpoint but also when designing theoretical models or experimental setups aimed at capturing genuinely quantum behaviours of physical systems \[1,2,3,4\]. However, in the formal construction of such relation there are still unclear points, that enforce the introduction of otherwise unnecessary concepts or even of additional axioms. It is not too stretched to say that these unclear points tend to nest where the crossover towards a macroscopic measurement apparatus comes into play \[5,6,7,8\], and the quantum-to-classical transition consequently bursts into the description (see the last chapter of Ref.\[4\] for an up to date discussion of the subject, and an extensive bibliography). What makes particularly problematic the formal treatment of such transition, in the specific case of the measurement process, is the fact that it must exclusively concern the apparatus, without affecting the object of the measurement, dubbed principal system, whose quantum character is not at issue. Moreover, measuring is an inherently dynamical process, as it entails the definition of what is

- before (the principal system in the state about which we want to acquire information, and the apparatus initialized in some dumb configuration),
- during (the evolution ruled by the interaction that generates the information flow between principal system and apparatus),
- and after (the principal system in some final state and the apparatus in an informative and readable output configuration).

Collecting clues from the above reflections, we propose a description of the quantum measurement process as the dynamical evolution of an Open Quantum System (OQS) whose environment is the measurement apparatus. In fact, aim of this work is that of giving a formal content to the role played by decoherence in the way we effectively probe the quantum world. We set the problem into the framework of the OQS dynamics\[2,9\], where both principal system and environment are treated at a quantum level, and entanglement generation is the essential phenomenon allowing for information flow. We resort to a recently introduced method\[10\] for studying OQS, namely the Parametric Representation with Environmental Coherent States (PRECS), which is specifically tailored to follow the environmental quantum-to-classical crossover. Indeed one of the main feature of the PRECS is that of allowing an exact, and yet essentially asymmetric description of principal system and environment, with the former given in terms of parametrized pure states, and the latter strongly characterized by the use of generalized coherent states.

The structure of the paper is as follows: In Sec.\[2\] we introduce the Von Neumann scheme for ideal quantum measurements\[11\], and briefly review its problematic features\[3\]. In Sec.\[3\] we propose a description of the Von Neumann scheme as unitary dynamics of a composite quantum system. The resulting evolution is studied in Sec.\[3\] by the PRECS, which is briefly reviewed and commented upon in this same section. The crucial point of how information about the principal system becomes available through the apparatus is finally tackled in Sec.\[5\] where decoherence appears as a necessary phenomenon in order for the measurement process to produce an informative output. Results are commented and conclusions drawn in Sec.\[6\].
2 The Von Neumann scheme for Ideal Quantum Measurement

Let us briefly recall, following Chap.2 of Ref.[4], how Von Neumann proposed to formally describe an ideal quantum measurement [11]. The key players of the process are the system to be measured $\Gamma$ (principal system) and the measurement apparatus $\Xi$ (environment), with Hilbert space $\mathcal{H}_\Gamma$ and $\mathcal{H}_\Xi$, respectively. Both $\Gamma$ and $\Xi$ are taken as genuinely quantum.

Von Neumann assumes that before the measuring starts the composite system $\Psi \equiv \Gamma \cup \Xi$ be isolated, and therefore described by a pure state $|\Psi(0)\rangle \in \mathcal{H}_\Psi = \mathcal{H}_\Gamma \otimes \mathcal{H}_\Xi$, which is specifically chosen separable

$$|\psi(0)\rangle = |\Gamma\rangle \otimes |\Xi\rangle ;$$

(1)

the necessity of the hypothesis underlying Eq. (1) follows from the idea that object of a meaningful measurement must be a system in a well defined physical state, which means, in the quantum framework, in a pure state. Entanglement with any other system, at this level, is therefore excluded.

The existence of a “preferred basis”, $\{|p\rangle\}$ for $\mathcal{H}_\Gamma$, is then postulated, such that its elements are not disturbed by the measurement, i.e.

$$|p\rangle|\Xi\rangle \rightarrow |p\rangle|\Xi^p\rangle \quad \forall |p\rangle \in \{|p\rangle\} ,$$

(2)

where the symbol $\rightarrow$ indicates the transformation caused by the measuring, and the tensor product is, and will be hereafter, implicitly understood. The process (2) generates a correspondence between elements of the preferred basis and environmental states, which is at the heart of the information gain one aims at obtaining via the measuring itself.

Finally, Von Neumann assumes that what holds for each $|p\rangle$ separately keeps holding when a linear combination of them is taken, so that

$$|\Gamma\rangle|\Xi\rangle = \left( \sum_p c_p |p\rangle \right) |\Xi\rangle \rightarrow \sum_p c_p |p\rangle|\Xi^p\rangle ,$$

(3)

which summarizes the whole process. The main criticality of the scheme is recognized in the need of assuming not only the existence of a “preferred basis” but also its behaving according to Eq. (3). This seems quite a top-down assumption and that is why it is considered a problem, namely the problem of the preferred basis [4]. In fact, the Von Neumann scheme leaves us with another, perhaps even subtler, question: Eq. (3) suggests that superpositions of states $|p\rangle|\Xi^p\rangle$, i.e. relative to the composition of a quantum system with the macroscopic measurement device used for probing it, goes together with any observation, and should therefore be part of our everyday experience. Acknowledging that this is not actually the case implies recognition that a further problem exists, which usually goes under the name of the problem of non-observability of interference. This is related with the fact that the last step of the measurement process, i.e. that of extracting the relevant information from $\Xi$ is not actually considered in the Von Neumann scheme, which is why Eq. (3) is often considered, and referred to as, the arrival point of a “pre-measurement” process.
3 Pre-measurement process as unitary dynamics

Moving within the same framework described in Ref. [4] when referring to the "environment-induced superselection" (or "einselection" [3]), aim of this section is that of reformulating the Von Neumann scheme by explicitly expressing the \( \rightarrow \) symbol as a unitary evolution of \( \Psi \), so as to obtain the right hand side of Eq. (3) in the form

\[
\sum_p c_p |p\rangle|\Xi_p\rangle = U_\tau |\Psi(0)\rangle = |\Psi(\tau)\rangle,
\]

with \( U_\tau^\dagger U_\tau = 1 \rangle_{\mathcal{H}_\Xi}, \forall \tau \leq \tau \); here \( \tau \) indicates a time when the pre-measurement step can be considered ended. Given that unitary operators can be written as imaginary exponentials of hermitian ones, we take

\[
U_\tau = e^{-i \hbar t H_{\Gamma \Xi}},
\]

with \( H_{\Gamma \Xi} \) the total Hamiltonian for \( \Psi \). Although more general forms are in principle possible, most physical interactions can be written as

\[
H_{\Gamma \Xi} = \sum_{ij} g_{ij} O^\Gamma_i O^\Xi_j,
\]

where \( O^\Gamma_i \) and \( O^\Xi_i \) are operators acting on \( \mathcal{H}_\Gamma \) and \( \mathcal{H}_\Xi \), and local terms on \( \Gamma \) and \( \Xi \) are those with \( O^\Xi_i = 1 \rangle_{\mathcal{H}_\Xi} \) and \( O^\Gamma_i = 1 \rangle_{\mathcal{H}_\Gamma} \), respectively.

We now ask the question: what properties must \( H_{\Gamma \Xi} \) feature in order for \( U_\tau \) to describe a meaningful measurement process?

Consider the hermitian operator \( M \) corresponding to the observable we want to measure. Necessary condition for the measuring to have a cogent relation with the observable is that \( M \) does not change under \( U_\tau \) (we would not otherwise know what we are actually measuring), i.e. it must fulfill the commutativity criterion [3, 4]

\[
[M, H_{\Gamma \Xi}] = 0.
\]

Further consider the spectral decomposition

\[
M = \sum_\gamma \mu_\gamma |\gamma\rangle\langle\gamma| = \sum_{i=1}^n \mu_i P_i,
\]

where \( \mu_i \) are the \( n \) distinct eigenvalues of \( M \), \( K_i \) are the \( n \) \( M \)-invariant subspaces of \( \mathcal{H}_\Gamma \), and \( P_i \equiv \sum_{|\gamma\rangle \in K_i} |\gamma\rangle\langle\gamma| \) are projectors on different \( M \)-invariant subspaces \( K_i \). According to the logic underlying the Von Neumann scheme, as expressed by Eq. (2), it must be \( \langle \Xi' | \Xi' \rangle \neq 1 \) iff \( |p\rangle \) and \( |p'\rangle \) belong to different subspaces. This implies that \( i \) and \( i' \) at least one projector on each subspace \( K_i \) enters \( H_{\Gamma \Xi} \), and \( ii \) projectors on the same subspace are coupled with the same operator \( O^\Gamma_i \).

Taking into account that condition (7) must hold, the choice

\[
O^\Gamma_i = P_i, \quad i = 1, \ldots, n
\]

is recognized as appropriate for describing the measurement process as far as \( \sum_j g_{ij} O^\Xi_j \neq \sum_j g_{ij'} O^\Xi_j \) for \( i \neq i' \). Notice that different projectors commute, and simultaneous measurement of non commuting observables is hence excluded.
Writing the initial state on the basis \( \{ |\gamma\rangle \} \) of the \( M \)-eigenstates, from Eqs. (4-7) and (9) we finally obtain
\[
|\Psi(\tau)\rangle = e^{-\frac{i}{\hbar} \tau \sum_{ij} g_{ij} O_j^\Xi |\Xi\rangle} = \sum_{\gamma} c_{\gamma} |\gamma\rangle e^{-\frac{i}{\hbar} \tau H_\gamma^{\Xi}} |\Xi\rangle \ , \tag{10}
\]
where we have defined \( H_\gamma^{\Xi} \equiv \langle \gamma | H_{\Gamma^\Xi} |\gamma\rangle = \sum_j g_{\gamma j} O_j^{\Xi} \), with \( g_{\gamma j} = g_{ij} \forall \gamma \in \mathcal{K}_i \). Eq. (10) corresponds to the Von Neumann scheme, Eq. (4), with \( \{ |p\rangle \} = \{ |\gamma\rangle \} \), and
\[
|\Xi^{\gamma}\rangle \equiv |\Xi^{\gamma_\tau}\rangle = e^{-\frac{i}{\hbar} \tau H_\gamma^{\Xi}} |\Xi\rangle \ . \tag{11}
\]
Notice that from the above description the “preferred basis” naturally emerges as that of the eigenvectors of \( M \), univocally defined by the choice of the physical observable one wants to measure, which is a legitimate decision of the observer. This result generalizes what is assumed in the so called quantum measurement limit, where local terms, both for \( \Gamma \) and \( \Xi \), are disregarded.

Before ending this section, let us introduce the density operator for the principal system, \( \rho_\Gamma \equiv \text{Tr}_{H_\Xi} |\Psi\rangle \langle \Psi| \); its expression in terms of the eigenstates \( \{ |\gamma\rangle \} \), before and during the pre-measurement process is
\[
\rho_\Gamma(0) = \sum_{\gamma\gamma'} c_{\gamma} c_{\gamma'}^* |\gamma\rangle \langle \gamma' | \ , \tag{12}
\]
and, according to Eq. (10),
\[
\rho_\Gamma(t) = \sum_{\gamma} |c_{\gamma}|^2 |\gamma\rangle \langle \gamma| + \sum_{\gamma \neq \gamma'} c_{\gamma} c_{\gamma'}^* \langle \Xi^{\gamma_\tau}_t | \Xi^{\gamma_\tau}_t \rangle |\gamma\rangle \langle \gamma' | \ , \tag{13}
\]
showing that only the off-diagonal elements of the \( \rho_\Gamma \) matrix-representation on the \( M \)-eigenvectors basis evolve in time (which is why this type of evolution has been recently dubbed\[12\] “off-diagonal dynamics”). It is of absolute relevance, as it will further result in Sec. 4 that the evolution of \( \rho_\Gamma \) is exclusively ruled by the time dependence of the overlaps \( \langle \Xi^{\gamma_\tau}_t | \Xi^{\gamma_\tau}_t \rangle \).

4 Off-diagonal dynamics by the PRECS

Our next step is that of obtaining an expression for \( \rho_\Gamma(t) \) that allow us to go beyond the pre-measurement stage. To this aim we resort to the parametric representation with environmental coherent states (PRECS); the method has been recently introduced\[10\] as a tool for studying OQS with an environment that needs being considered quantum, but yet may have an extremely large Hilbert space. It is based on the construction of generalized coherent states\[13,14\] for the environment, or environmental coherent states (ECS), relative to the group, usually referred to as “dynamical group”, in terms of whose generators one can write all the operators \( O_j^{\Xi} \) in \( H_{\Gamma^\Xi} \). Without entering into the details of their construction and properties\[15\], we recall that ECS, hereafter indicated by \( |\Omega\rangle \), form an overcomplete set on \( \mathcal{H}_\Xi \) and are in one-to-one correspondence with points \( \Omega \) on a differentiable manifold \( \mathcal{M} \). Due to their overcompleteness, coherent states are not orthogonal; however, and this is just one of the many ECS properties that make
them the ideal tool for investigating the quantum to classical transition, their overlaps exponentially vanish as $\dim \mathcal{H}_\Xi$ grows, and the manifold $\mathcal{M}$ is demonstrated to be a proper phase-space in the classical limit\cite{16}.

The construction of ECS requires the (arbitrary) choice of a reference state $|R\rangle \in \mathcal{H}_\Xi$, whose representative point will define the origin of the reference frame on $\mathcal{M}$; the procedure entails the definition of an invariant (with respect to the dynamical group) measure $d\mu(\Omega)$ on $\mathcal{M}$, as well as a metric tensor $m$. ECS provide an identity resolution on $\mathcal{H}_\Xi$ in the form

$$1|\mathcal{H}_\Xi = \int_{\mathcal{M}} d\mu(\Omega)|\Omega\rangle\langle\Omega| .$$

(14)

Due to their being constructed in relation to the dynamical group, coherent states have peculiar dynamical properties, which are often summarized by the motto “once a coherent state, always a coherent state”\cite{13}. Referring to our specific setup, if the initial state of the apparatus is a coherent state ($|\Xi\rangle = |\Omega(0)\rangle$), in Eq. (1), from Eq. (11) it follows

$$|\Xi_{\gamma}\rangle = |\Omega_{\gamma}^{\gamma}(\tau)\rangle = e^{i\phi_{\gamma}}|\Omega_{\gamma}\rangle ,$$

(15)

with

- $|\Omega_{\gamma}\rangle$ the coherent state corresponding to the point $\Omega(\tau)$ on the trajectory on $\mathcal{M}$ defined by the solution of the classical-like equations of motion

$$im\frac{d\Omega}{dt} = \frac{\partial}{\partial \Omega^*} H_{\gamma}(\Omega) \quad \text{and c.c.} ,$$

(16)

with $H_{\gamma}(\Omega) = \langle \Omega|H|\Omega\rangle$, and

$$\phi_{\gamma} = \int_0^\tau dt \langle \Omega_{\gamma}^{\gamma}\rangle \left( i \frac{\partial}{\partial t} - H_{\gamma} \right) |\Omega_{\gamma}\rangle$$

(17)

Getting back to our composite system $\Psi$, once the environmental coherent states are constructed, its subsystems $\Gamma$ and $\Xi$ can be formally split by inserting $1|\mathcal{H}_\Xi$ as from Eq. (14) into any state $|\Psi\rangle$, including one written in the form (10). In particular, choosing the initial state of the measuring apparatus as the reference state for the ECS construction, $|R\rangle = |\Xi\rangle$, and exploiting the fact that $d\mu(\Omega)$ is group-invariant, we can write

$$|\Psi(t)\rangle = \int_{\mathcal{M}} d\mu(\Omega) \chi_{\xi}(\Omega)|\phi_{\xi}(\Omega)\rangle|\Omega\rangle ,$$

(18)

with

$$|\phi_{\xi}(\Omega)\rangle = \frac{1}{\chi_{\xi}(\Omega)} \sum_{\gamma} c_{\gamma} |\Omega|_{R_{\xi}^{\gamma}}\rangle |\gamma\rangle ,$$

(19)

$$\chi_{\xi}(\Omega) = \sqrt{\sum_{\gamma} |c_{\gamma}|^2 h_{\xi}^{\gamma}(\Omega)} ,$$

(20)

$$h_{\xi}^{\gamma}(\Omega) = |\langle \Omega|R_{\xi}^{\gamma}\rangle|^2 ,$$

(21)
where we have set \( \chi_t(\Omega) \) in \( R^+ \) by choosing its arbitrary phase equal to 0. Due to \( \langle \Psi(t)|\Psi(t) \rangle = 1 \) at any time, it is

\[
\int_{\mathcal{M}} d\mu(\Omega) \chi_t^2(\Omega) = 1 \quad \forall t .
\]  

The above Eqs. (18–21) define the parametric representation with environmental coherent states of \( |\Psi(t)\rangle \). It can be shown that the corresponding form for \( \rho^\Gamma(t) \) is

\[
\rho^\Gamma(t) = \int_{\mathcal{M}} d\mu(\Omega) \chi_t^2(\Omega) |\phi_t(\Omega)\rangle \langle \phi_t(\Omega)| ,
\]  

suggesting that \( \chi_t^2(\Omega) \) can be interpreted, consistently with Eq. (22), as the density distribution of ECS on \( M \). To this respect it is worth noticing that \( \chi_t^2(\Omega) = \langle \Omega | \rho^\Xi(t) | \Omega \rangle \) where \( \rho^\Xi(t) \equiv \text{Tr}_{\mathcal{H}_r} |\Psi\rangle \langle \Psi| \) is the reduced density operator for the environment.

5 Extracting information from the apparatus: emergence of decoherence

The description of the Von Neumann measurement scheme as unitary dynamics of a composite system by the PRECS has finally brought us to a formal expression for the right-hand side of Eq. (3),

\[
\sum_p \alpha_p |p\rangle |\Xi^p\rangle = \sum_\gamma c_\gamma |\gamma\rangle |R_\gamma^\tau\rangle = \int_{\mathcal{M}} d\mu(\Omega) |\Omega\rangle \chi^\tau(\Omega) |\phi(\Omega)\rangle ,
\]  

with \( \chi^\tau(\Omega) = \sum_\gamma c_\gamma |\gamma\rangle |h^\tau_\gamma(\Omega)\rangle \), and the initial state of the measuring apparatus chosen as reference state for constructing ECS. Looking at Eq. (24) it might seem that we ended up with having overturned the dependences with respect to the Von Neumann scheme: In fact, Eq. (24) shows that \( |\phi(\Omega)\rangle \) depends on the environmental parameter \( \Omega \), while the coherent state \( |\Omega\rangle \) of the measuring apparatus is not marked by the label “\( \gamma \)”. This is because the signature of the interaction with \( \Gamma \) is not in the ECS, that are defined independently of the \( G^\Gamma_i \) entering Eq. (6), but rather in their density distributions \( \chi^\tau(\Omega) \), which is where one should therefore look into, in order to extract information on \( \Gamma \) via \( \Xi \).

Let us now leave the pre-measurement process and consider the actual production of an outcome. Distinct coherent states, corresponding to distinct states of the measurement apparatus, will produce different outcomes, whose distribution will thus be associated with \( \chi^\tau(\Omega) \). On the other hand, in order for this setup to produce an outcome with some informational content, it is necessary that the \( \gamma \)–components entering \( \chi^\tau(\Omega) \), i.e. the terms \( |c_\gamma|^2 h^\tau_\gamma(\Omega) \), be sufficiently separated from each other to be distinguishable. Aiming at formally expressing this condition, let us consider the functions \( h^\tau_\gamma(\Omega) \) in Eq. (21). They are normalized distributions on \( \mathcal{M} \) whose support, defined as the region \( S^\tau_\gamma \in \mathcal{M} \) such that \( h^\tau_\gamma(\Omega) > 0 \forall \Omega \in S^\tau_\gamma \) (with \( \varepsilon \) a reasonably small number in \( R^+ \)), moves on such manifold with time. If, after some time, it is

\[
S^\gamma(t) \cap S^{\gamma'}(t) = \emptyset , \quad \forall \gamma, \gamma' \text{ s.t. } \mu_\gamma \neq \mu_{\gamma'} ,
\]  

\[5\]
then each distribution $h_\gamma^\pm$ can be individually located, and a one-to-one correspondence between the label $\gamma$ and the region $S_\gamma$ on $\mathcal{M}$ is established.

Condition (25) can hence be identified as that guaranteeing that an informative output can be extracted from the apparatus, which finally brings us to the question we aimed at answering: how and why this condition, that somehow regards $\Xi$ only, is related with the occurrence of decoherence in the principal system $\Gamma$? In order to take this last step forward, consider Eq. (13): If condition (25) holds, it is

$$\rho_\Gamma(t) \approx \sum_\gamma |c_\gamma|^2 \int_{S_\gamma^\pm} d\mu(\Omega) h_\gamma^\pm(\Omega) |\phi(\Omega, t)\rangle \langle \phi(\Omega, t)|$$

$$= \sum_\gamma |c_\gamma|^2 |\gamma\rangle \langle \gamma|,$$

(26)

that exactly express the vanishing of the off-diagonal elements of $\rho_\Gamma$ on the $\{|\gamma\rangle\}$ basis, i.e. the formal definition of decoherence for the principal system $\Gamma$. This makes finally evident that decoherence is not one of the many byproducts of the measurement process, but rather a necessary condition for the configuration of the apparatus to embody some usable information on $\Gamma$.

In order to better understand the construction that brought us to the above result, let us consider a simple example. Take $\Gamma$ as a quantum system with $\dim \mathcal{H}_\Gamma = 2$ (usually referred to as qubit), and $\Xi$ as a single-mode bosonic field. Be the hamiltonian

$$H_{ab} = \hbar v b^\dagger b + g\sqrt{\hbar} \sigma^z (b + b^\dagger)$$

(27)

with $[b, b^\dagger] = \hbar$, $[\sigma^a, \sigma^b] = i e^{\alpha \delta - \beta \delta} \sigma^d$, $\alpha(\beta, \delta) = x, y, z$, and $\sigma$ the Pauli operator. The above Hamiltonian is in the form (6) and the corresponding ECS, taking the reference state $|R\rangle$ such that $b|R\rangle = 0$, are the usual field coherent states, with $\mathcal{M}$ the complex plane, $d\mu = d\Omega d\Omega^*/(\pi \hbar)$ the invariant measure, and $m = \hbar^{-1}$ the (diagonal) metric tensor. Being $\dim \mathcal{H}_\Gamma = 2$, the label $\gamma$ can only take two values, hereafter indicated by $\pm$, and the initial state $\Gamma$ can be written as $(c_+|+\rangle + c_-|\mp\rangle)$, where $|\pm\rangle$ are the eigenstates of $\sigma^z$. Solutions of Eq. (16) are two circles $R_{\Omega}^\pm$ on the Re($\Omega$) – Im($\Omega$) plane, passing through $(0, 0)$, centered in $(\mp g/\nu, 0)$, and gone through clockwise. A snapshot of the ECS density distribution $\chi^2(\Omega)$ on the complex plane is shown in Fig. 1 together with part of the orbits $R_{\Omega}^\pm$; the two components $|c_\pm|^2 h_\gamma^\pm(\Omega)$ are already distinguishable, with the respective supports $S_\gamma^\pm$ quite well separated. Notice that, despite this image portrays the environment $\Xi$, it also mirrors the structure of the principal system state, $\rho_\Gamma$, due to the relation between condition (25) and Eq. (26). Although we have never mentioned it so far, it is worth noticing that the above relation between distinguishability of different $h_\gamma^\pm$ and diagonal form of $\rho_\Gamma(t)$ is established by the entanglement generation entailed by a non-trivial dynamical evolution of $\Psi$, such as that resulting from the interaction (6).

The idea that the ECS distribution $\chi^2(\Omega)$ be the “image” from which we can extract information on $\Gamma$ can be made more precise by introducing the differential entropy\(^1\) for $\chi^2(\Omega)$

$$\mathcal{E}(t) = - \int_{\mathcal{M}} d\mu(\Omega) \chi^2(\Omega) \log \chi^2(\Omega) =$$

(28)

\(^1\) In information theory, it is the Shannon Entropy generalization to continuous probability distributions.
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Fig. 1 Distribution $\chi_t^2(\Omega)$ at $t = \pi/30$ for the qubit-boson model, Eq. (27), with $g\sqrt{\hbar} = 2$ and $\nu = 1$. In the initial state of the qubit it is $|c_+|^2 = 1/4$ and $|c_-|^2 = 3/4$

$$=-\sum_{\gamma} \int_{\mathcal{M}} d\mu(\Omega) |c_{\gamma}|^2 h^\gamma_t(\Omega) \log \left( \sum_{\gamma} |c_{\gamma}|^2 h^\gamma_t(\Omega) \right).$$  (29)

Referring to our example Eq. (27), the explicit form of the distributions $h^\pm_t(\Omega)$ is

$$h^\pm_t(\Omega) = \frac{1}{\pi\hbar} |\langle \Omega | R^\pm_t \rangle |^2 = \frac{1}{\pi\hbar} e^{-\frac{1}{\hbar} |\Omega - R^\pm_t|^2},$$  (30)

corresponding to gaussians centered in $R^\pm_t$ with constant variance $\hbar$. As the orbits $R^\pm_t$ initially coincide, it exists an early stage of the process, no matter the coefficients $c_\pm$, during which $\chi_t^2(\Omega)$ keeps being an essentially unimodal distribution, centered in the origin of the complex plane, which implies $\mathcal{E} \sim \text{const}$, with no dependence on $c_\pm$, whatsoever. On the other hand, the trajectories $R^\pm_t$ dynamically separate from each other and, after a certain time, condition (25) starts holding, and the entropy

$$\mathcal{E} \sim -\sum_{\gamma=\pm} |c_{\gamma}|^2 \int_{S_\gamma} d\Omega d\Omega^* h^\gamma_t(\Omega) \log \left( |c_{\gamma}|^2 h^\gamma_t(\Omega) \right).$$  (31)

is seen to depend on the coefficients $|c_\pm|^2$, and to quantify the amount of information on the state $\Gamma$ that we can obtain adopting a measurement procedure based on the interaction (27). Fig.2 offers a visual rendering of the above result via the contour plot of $\chi_t^2(\Omega)$ on the complex plane, at different times: it is evident that the information content of the initial plot has nothing to do with the principal system $\Gamma$, while the later emergence of two distinct spots can be used to extract data on $c_+$ and $c_-$. 
6 Conclusions

The analysis we have presented formally shows that the reason why decoherence of the principal system is a necessary ingredient of a significant measurement process is that the information content of the apparatus would be otherwise null. To this respect it is important to recall that decoherence is defined as the dynamical process causing the vanishing of the off-diagonal elements of the system’s density matrix, with respect to a precise basis on its Hilbert space. When considering the measurement process, the relevant decoherence phenomenon is just that relative to the basis of eigenstates for the hermitian operator describing the observable to be measured. In fact, such basis explicitly comes into play when designing the interaction between measured system and measurement apparatus, which is ultimately responsible for the information flux between the twos. Indeed, what does not depend on the specific quantity to be measured, is the essential role of the dynamical entanglement generation, without which there would be no correlation.
between $\Gamma$ and $\Xi$ capable of leaving on the latter any trace of the quantum state of the former. This is an essential feature of quantum measurement, that accounts for the inability of approaches based on classical-like treatments of $\Xi$ to describe quantum measurements, as there cannot be entanglement between a quantum system to be observed and a classical apparatus that makes the measuring.

The necessary condition that both $\Gamma$ and $\Xi$ be quantum systems, on the other hand, raises another question worth being considered, namely whether one should expect coherence to be restored after a certain time or not. In fact, being $\Psi$ a quantum system, and its dynamics unitary, the evolution of both $\Gamma$ and $\Xi$ are superpositions of periodic motions, so that the time interval during which the apparatus is capable of conveying information is in principle finite. However, it can be shown\[15,17,10\] that as the dimension of the environmental Hilbert space grows, reflecting the fact that the apparatus is macroscopic, the environmental distributions $h^\gamma_t(\Omega)$ tend to Dirac $\delta$-functions and the recurrence time, i.e. the period of the unitary dynamics, diverges: as a consequence, decoherence occurs after an infinitesimally small time $\tau$, and coherence is never restored. To this respect, we underline that we have not considered the very last stage of the process, namely that where the Born’s rule and the ”wave-function collapse” come into play, and the unitarity of dynamics is lost. We believe the PRECS formalism can give original clues also with respect to these fundamental issues, but we postpone their possible analysis to future works.

7 Acknowledgements

This work has been done in the framework of the Convenzione operativa between the Institute for Complex Systems of the Italian National Research Council, and the Physics and Astronomy Department of the University of Florence.

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