Quiver relations and associated symmetric polynomials

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Abstract

The idea is to identify certain path algebra elements with symmetric functions. We propose such a morphism by solving the quiver relations, which describe the Plucker-type embedding for quiver Grassmannians.

1 Introduction

Quiver is a directed graph with multiply edges and loops allowed. Let $Q$ = finite quiver with $k$ vertices and $l$ arrows. Quiver representation $M$ is a collection of vector spaces and linear maps, attached to each vertex and arrow of $Q$:

$$M = \{V_1, \ldots, V_k\} \cup \{M_{v,i,j}\}, \quad \dim(V_i) = d_i, \quad M_{v,i,j} = \text{Mat}(d_i \times d_j, k): V_i \to V_j$$

Quiver Grassmannian $Gr_e(Q, M)$ of dimension vector $\underline{e} = (e_1, \ldots, e_k)$ contains all $e$-dimensional subspaces of $M$, compatible with the underlying linear maps.

Fix an ordered basis $B = \{1, 2, \ldots, d_1, d_1 + 1, \ldots, d_l\}, I \subset B$ for all $V_i$s. Quiver relations (QR) are of the form:

$$\sum_{s',t'} (-1)^{c(s',t')} M_{v,s',t'} \Delta_{s'}^{s} \Delta_{t'}^{t} - \sum_{s'} (-1)^{c(s',t')} M_{v,s',t} \Delta_{s'}^{s} \Delta_{I}^{t} = 0, \quad I \subset B, \quad |I| = e$$

Define $Fset$ as a collection of all non-vanishing $\Delta_I^{s}$s in QR. Our aim is to generate polynomials which are sums of $GL$ characters with coefficients given by the embedding

$$Gr_{\underline{e}}(Q, M) \xrightarrow{\phi_1} Gr(e, d) \xrightarrow{\phi_2} \mathbb{P}^{e+d-1}, \quad e = \sum e_i, d = \sum d_i$$

Define 2 kind of invariants:

$$P(Q, s, M) = \sum_R \Delta_{R\chi_R}(\overline{t})$$

with $\Delta = \text{homogeneous coordinate on the image of embedding}$ and

$$\tilde{P}(Q, s, M) = \sum_{R'} \Delta_{R'\chi_R}(\overline{t})$$

with $\tilde{\Delta} \in Fset$ only (PR are not taken into part). Note that $P, P'$ are inhomogeneous in $R, R'$ (may contain Young diagrams of different size).

Proposition 1. In case of $\chi_{\lambda} = s_{\lambda}$ polynomials $P(Q, s, M)$ form a basis of $\text{Symm}$ algebra. This seems
to be obvious, but not the only one possible choice of $\chi$. Write $M$ as block matrix:

$$M = \begin{bmatrix}
    M_{1,1} & M_{1,2} & M_{1,3} & \ldots \\
    M_{2,1} & M_{2,2} & M_{2,3} \\
    M_{3,1} & M_{3,2} & M_{3,3} \\
    \vdots & \ddots & \ddots & \ddots
\end{bmatrix}$$

(6)

with each $M_{i,j}$ corresponding to $(i, j)$ linear map.

## 2 Computation of $P, \tilde{P}$. Quiver relations as vanishing minors

### 2.1 $Q = A_2$, $M = \{ \mathbb{C}, \mathbb{C}, \mu = \text{Mat}(2, 2, \mathbb{C}) \}, \varnothing = (1, 1)$

$I = \{2, 4\}$: One non-trivial equation, $\#(+, -) = (2, 2)$

$$F(v, 2, 3) = \mu_{v,1,4} \Delta_{\{1,4\}} \Delta_{\{2,3\}} + \mu_{v,2,4} \Delta_{\{2,4\}} \Delta_{\{3,2\}} - \mu_{v,1,3} \Delta_{\{1,4\}} \Delta_{\{2,4\}} - \mu_{v,2,3} \Delta_{\{2,4\}} = 0$$

(7)

Or:

$$F(v, 2, 3) = \begin{vmatrix}
    \mu_{v,1,4} & 0 & \Delta_{\{2,4\}} & \Delta_{\{2,3\}} \\
    \mu_{v,2,4} & \Delta_{\{1,4\}} & \frac{\mu_{v,2,4} \Delta_{\{2,3\}}}{\mu_{v,1,4} \Delta_{\{2,3\}}} & \Delta_{\{2,4\}} \\
    \mu_{v,1,3} & \Delta_{\{2,3\}} & \Delta_{\{2,3\}} & 0 \\
    \mu_{v,1,3} & \Delta_{\{2,3\}} & \Delta_{\{2,3\}} & \Delta_{\{2,3\}}
\end{vmatrix} = 0$$

(8)

Note that if we inverse the arrow, our equation will change:

$$F(v, 4, 1) = \mu'_{v,3,2} \Delta_{\{2,3\}} \Delta_{\{1,4\}} + \mu'_{v,4,2} \Delta_{\{2,4\}} \Delta_{\{1,4\}} - \mu'_{v,3,1} \Delta_{\{2,3\}} \Delta_{\{2,4\}} - \mu'_{v,4,1} \Delta_{\{2,4\}} = 0$$

(10)

where $\mu'$ is the inverse map. Now fix $\mu = \begin{pmatrix} p & 1 - p \\ 1 - p & p \end{pmatrix}$ — quantum permutation matrix from https://arxiv.org/abs/1109.4888. Combining this with the only Plücker identity

$$\Delta_{\{1,2\}} \Delta_{\{3,4\}} - \Delta_{\{1,3\}} \Delta_{\{2,4\}} + \Delta_{\{1,4\}} \Delta_{\{2,3\}} = 0$$

(11)

we obtain our polynomial invariant:

$$P(A_2, 4, \mu) = \frac{Q(h_1 - \frac{Q(-Wp - Q + pQ)h_2}{W - Wp + pQ} + W(h_1^2 - h_2h_0) - (-Wp - Q + pQ)W(h_1h_2 - h_3h_0)}{W - Wp + pQ}$$

(12)

where $Q, W$ are arbitrary complex numbers.

### 2.2 $M = \{ \mathbb{C}^2, \mathbb{C}^2, \mu = \text{Mat}(4, 4, \mathbb{C}) \}, \varnothing = (2, 2)$

$I = 3, 4, 7, 8$: We have 2x2 non-trivial quiver equations () with $\#(+, -) = (4, 5), (6, 3)$ One of them ()

$$F(v, 3, 5) = \begin{vmatrix}
    \mu_{v,1,7} + \mu_{v,2,7} + \mu_{v,2,8} & \Delta_{\{3,4,7,8\}} & \Delta_{\{3,4,5,7\}} + 2 \Delta_{\{3,4,7,8\}} \\
    \mu_{v,3,8} - \mu_{v,3,7} - \mu_{v,1,5} & \Delta_{\{1,4,7,8\}} + 2 \Delta_{\{2,4,7,8\}} & 0 \\
    -\mu_{v,1,8} + \mu_{v,2,5} + \frac{\mu_{v,3,5} \Delta_{\{3,4,7,8\}}}{\Delta_{\{3,4,7,8\}}} & \Delta_{\{3,4,7,8\}} + \Delta_{\{1,4,7,8\}} & 2 \Delta_{\{3,4,5,8\}} + \Delta_{\{3,4,5,7\}}
\end{vmatrix} = 0$$

(13)
\[ \mu_{v, 2.8} \Delta_{(2, 4, 7, 8)} \Delta_{(3, 4, 5, 7)} + \mu_{v, 3.8} \Delta_{(3, 4, 7, 8)} \Delta_{(3, 4, 5, 7)} - \mu_{v, 1.5} \Delta_{(1, 4, 7, 8)} \Delta_{(3, 4, 7, 8)} - \mu_{v, 2.5} \Delta_{(2, 4, 7, 8)} \Delta_{(3, 4, 7, 8)} - \mu_{v, 3.5} \Delta_{(3, 4, 7, 8)} ^2 \]

Can we rewrite them in a determinantal form?

\[ F(v, 4, 5) = \begin{vmatrix} \mu_{v, 4.7} & m_{21.2} & \Delta_{(3, 4, 5, 7)} \\ 0 & \Delta_{(3, 4, 7, 8)} & 0 \\ -\mu_{v, 4.8} & m_{23.2} & \Delta_{(3, 4, 5, 8)} \end{vmatrix} + \begin{vmatrix} \mu_{v, 2.5} & 0 & \Delta_{(3, 4, 5, 8)} \\ -\mu_{v, 1.7} & \Delta_{(2, 3, 7, 8)} & 0 \\ \mu_{v, 2.7} & \Delta_{(1, 3, 7, 8)} & \Delta_{(3, 4, 7, 8)} \end{vmatrix} + \begin{vmatrix} -\mu_{v, 1.8} & \Delta_{(2, 3, 7, 8)} & \Delta_{(3, 4, 7, 8)} \\ \mu_{v, 2.8} & \Delta_{(1, 3, 7, 8)} & 0 \\ -\mu_{v, 1.5} & -\frac{\mu_{v, 4.5} \Delta_{(3, 4, 7, 8)}}{\mu_{v, 2.8}} & \Delta_{(3, 4, 5, 7)} \end{vmatrix} = 0. \]

Polynomial \( P(A_2, 4, Id) \) define \( P(Q, n, M) = \) linear combination of GL characters with coefficients running through the Fset. **this turns out to be a special base of Symm.** This case gives

\[ Fset = \begin{pmatrix} \Delta_{(1, 4, 7, 8)} & \Delta_{(1, 3, 7, 8)} \\ \Delta_{(2, 4, 7, 8)} & \Delta_{(2, 3, 7, 8)} \\ \Delta_{(3, 4, 7, 8)} & \Delta_{(3, 4, 7, 8)} \end{pmatrix} \]

Now we can write down the explicit form of \( P(A_2, 4, Id): \)

\[ Fset = [\Delta_{(1, 3, 7, 8)}, \Delta_{(1, 4, 7, 8)}, \Delta_{(2, 3, 7, 8)}, \Delta_{(2, 4, 7, 8)}, \Delta_{(3, 4, 5, 7)}, \Delta_{(3, 4, 5, 8)}, \Delta_{(3, 4, 6, 7)}, \Delta_{(3, 4, 6, 8)}, \Delta_{(3, 4, 7, 8)}] \]

Assuming all free parameters = ±1, we get the following solution:

\[ Fset = [1, 1, 1, -1, -1, 1, -1, -1, 1] \]

Now choose \( \mu = \mu(p, q). \) Solving QR’s independently, we get:

\[ \Delta_{(1, 3, 7, 8)} = -\frac{1}{pq}(\Delta_{(3, 4, 5, 7)} - 2 \Delta_{(3, 4, 5, 7)} q - \Delta_{(1, 4, 7, 8)} p - \Delta_{(2, 4, 7, 8)} + p \Delta_{(2, 4, 7, 8)} + \Delta_{(1, 4, 7, 8)} p q + \Delta_{(2, 4, 7, 8)} q - p \Delta_{(2, 4, 7, 8)} q - \Delta_{(2, 3, 7, 8)} q + p \Delta_{(2, 3, 7, 8)} q) \]

\[ \Delta_{(3, 4, 5, 8)} = -\frac{1}{q}(\Delta_{(3, 4, 5, 7)} - \Delta_{(3, 4, 5, 7)} q - \Delta_{(1, 4, 7, 8)} p - \Delta_{(2, 4, 7, 8)} + p \Delta_{(2, 4, 7, 8)} \) \]

\[ \Delta_{(3, 4, 6, 7)} = -\frac{1}{p(2q - 1)}(2 p \Delta_{(2, 3, 7, 8)} q + 2 \Delta_{(3, 4, 5, 7)} p q - 2 p \Delta_{(2, 4, 7, 8)} q - p \Delta_{(3, 4, 5, 7)} + 2 p \Delta_{(2, 4, 7, 8)} - 2 \Delta_{(3, 4, 5, 7)} q - \Delta_{(2, 4, 7, 8)} q + \Delta_{(3, 4, 5, 7)} - \Delta_{(2, 4, 7, 8)} \)

\[ \Delta_{(3, 4, 6, 8)} = \frac{1}{pq(2q - 1)}(\Delta_{(3, 4, 6, 8)} = 2 p \Delta_{(2, 4, 7, 8)} - 2 \Delta_{(3, 4, 5, 7)} q^2 p - 2 \Delta_{(1, 4, 7, 8)} q^2 p + 2 p^2 \Delta_{(2, 4, 7, 8)} q + 2 \Delta_{(2, 4, 7, 8)} q^2 p - (21) \)
-2 \Delta_{(2,3,7,8)} q^2 p - p \Delta_{(3,4,5,7)} + \Delta_{(1,4,7,8)} p^2 - p^2 \Delta_{(2,4,7,8)} + 3 \Delta_{(3,4,5,7)} p q - 4 p \Delta_{(2,4,7,8)} q + \\
+ 2 \Delta_{(1,4,7,8)} p q + 2 p \Delta_{(2,3,7,8)} q + 2 \Delta_{(2,4,7,8)} q - \Delta_{(2,3,7,8)} q + \\
\Delta_{(3,4,5,7)} - \Delta_{(2,4,7,8)} + 2 \Delta_{(3,4,5,7)} q^2 - \\
- \Delta_{(2,4,7,8)} q^2 + \Delta_{(2,3,7,8)} q^2 - \Delta_{(1,4,7,8)} p - 3 \Delta_{(3,4,5,7)} q

Assuming \{\Delta_{(1,4,7,8)}, \Delta_{(2,3,7,8)}, \Delta_{(2,4,7,8)}, \Delta_{(3,4,5,7)}, \Delta_{(3,4,7,8)}\} = \{1\}, we can try to resolve all remaining (Plücker) relations. In this case

\[
Fset = \begin{bmatrix}
- \frac{2q - qp}{qp}, 1, 1, 1, 1, - \frac{2qp - 2q + p}{p(1 + 2q)}, - \frac{2q + 3qp - 2q^2 + 2q^2}{pq(1 + 2q)}, 1
\end{bmatrix}
\]  

(22)

Now we’re almost ready to write our target polynomial. The only thing left is to convert the indices \(i_1, i_2, \ldots, i_8\) to the corresponding Young diagram. In order to do this, expand the following determinant

\[
\Delta_{(1,3,7,8)} \simeq \begin{vmatrix}
  h_2 & h_4 & h_5 & h_6 \\
  h_1 & h_3 & h_4 & h_5 \\
  0 & h_2 & h_3 & h_4 \\
  0 & h_1 & h_2 & h_3
\end{vmatrix} = \delta_{1,1}, 
\Delta_{(1,4,7,8)} \simeq \begin{vmatrix}
  h_2 & h_3 & h_5 & h_6 \\
  h_1 & h_2 & h_4 & h_5 \\
  0 & h_1 & h_3 & h_4 \\
  0 & 0 & h_2 & h_3
\end{vmatrix} = \delta_{1,1}, 
\Delta_{(2,3,7,8)} \simeq \begin{vmatrix}
  h_2 & h_4 & h_5 & h_6 \\
  h_1 & h_3 & h_4 & h_5 \\
  0 & h_2 & h_3 & h_4 \\
  0 & 0 & h_1 & h_2
\end{vmatrix} = \delta_{1,1}.
\]  

(23)

2.3 \( Q = A_2, d_i = 6, e_i = 3, \mu = \sigma(p, q, 6) \)

Take 2 vertices \(v_1, v_2\) with \(V_1 = V_2 = \mathbb{C}^6, \)

\[
I = \begin{bmatrix} 4, 5, 6, 10, 11, 12 \end{bmatrix}
\]  

(24)

Recall that each \(\Delta\) is a \((6, 6)\) complex minor. The following minors are supposed to vanish:

\[
\{\Delta_{(1,4,5,6,10,11)}, \Delta_{(1,4,5,6,10,12)}, \Delta_{(1,4,5,6,11,12)}, \Delta_{(2,4,5,6,10,11)}, \Delta_{(2,4,5,6,10,12)}, \Delta_{(2,4,5,6,11,12)}\},
\]  

(25)

Total \#QR = 3^3 \cdot (3 \text{ for each } iin [4, 5, 6]: F(v, i, j)). For arbitrary \(\mu\) these relations look really huge!

The choice \(\mu = Id(6)\) kills most of the \(Fset\) entries, so we can write \(Fset\) as \((12, 12)\) sparse matrix with \{0, \pm1\} entries of the shape (empty space is filled with zeroes):}

\[
Fset = \text{Matrix}([\text{seq}(\text{seq}(Fset[1+i .. 12+i], i = 12*j), j = 0 .. 11)])
\]  

(26)
Finally, convert all survived \( Fset \) entries to corresponding Young diagrams to get out target polynomial:

\[
\tilde{P}(A_2, 6, \text{Id}) = s - s + s + s - s + s + \ldots,
\]

(27)

\[
\text{total } \#(0, +, -) = (120, 21, 3)
\]

2.4 \( Q = A_3, d_i = 2, \mu_{i=1,2} = Mat(2, 2, \mathbb{C}), \bar{c} = (1, 1, 1) \) \}

Now let’s add one more vertex:

\[
Q = A_3 : \quad \mu_1 \circ \mu_2 \circ \mu_i \quad (28)
\]

Choose \( I = \{2, 4, 6\} \), so now we have the embedding

\[
Gr_{(1,1,1)}(A_3, M) = Gr(3, 6) \rightarrow \mathbb{P}^{6+3-1} \quad (29)
\]

For \( v = 1 \) we have the following non-trivial quiver relations:

\[
F(1, 2, 3) = \mu_{1,1,4} \Delta_{(1,4,6)} \Delta_{(2,3,6)} + \mu_{1,2,4} \Delta_{(2,4,6)} \Delta_{(2,3,6)} - \mu_{1,1,3} \Delta_{(1,4,6)} \Delta_{(2,4,6)} - \mu_{1,2,3} \Delta_{(2,4,6)}^2 = 0
\]

(31)

Note that if \( \mu = \text{Id} \), the solution looks as follows:

\[
Fset = \begin{pmatrix}
\Delta_{(1,4,6)} & \Delta_{(2,3,6)} & \Delta_{(2,4,6)} \\
1 & \Delta_{(1,4,6)} & 1 \\
& \Delta_{(1,4,6)} & 1 \\
\end{pmatrix}, \quad (32)
\]

where all empty cells are zeroes.

2.5 \( Q = A_3, d_i = 4, \mu_1 = \sigma(p, q, 4), \mu_2 = \text{Id} \)

Initial data:

\[
B = \{1, 2, \ldots, 12\}, I = \{3, 4, 7, 8, 11, 12\} \quad (33)
\]

\[
\mu_1 = \begin{pmatrix}
p & 1 - p & 0 & 0 \\
1 - p & p & 0 & 0 \\
0 & 0 & q & 1 - q \\
0 & 0 & 1 - q & q \\
\end{pmatrix}, \quad \mu_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \quad (34)
\]

This case reads

\[
Fset' = \left\{ 1, \Delta_{(2,3,4,7,8,11)}, \Delta_{(2,3,4,7,8,12)}, \Delta_{(3,4,7,8,9,11)}, \Delta_{(3,4,7,8,10,11)}, - \frac{-q + 2qp}{q(-1 + 2p)}, - \frac{2p^2q - qp}{q(-1 + 2p)} \right\}
\]

(35)
3 Diagrammatic notation and exact formulas for invariants

\( \hat{P}(A_n, 2k, \sigma) : \)
\[
\hat{P}(A_2, 4, \sigma) = \left( \frac{(-2q - qp)}{qp} \right) \left( h_3^2 h_2^2 - h_3 h_2 h_4 - 2 h_3 h_4 h_2^2 + 2 h_4 h_2 h_1 - h_3 h_4 h_1^2 + h_5 h_2^2 - h_6 h_2^2 h_1 + h_6 h_3 h_1^2 \right) + \\
+ \frac{h_3^2 h_2^2 - h_3 h_2 h_1^2 - h_5 h_2^2 + 2 h_5 h_2 h_1^2 - 2 h_3 h_4 h_2 h_1 + 2 h_4 h_2 h_1^2 + h_3^2 h_2 h_1 - h_3 h_4 h_1^2 + (36)}{p (2q - 1) - 2q + p - 2 q^2 p + 2 q^2} \left( h_3 h_4 h_1^2 - h_5 h_2 h_1^2 \right)
\]

What is the asymptotic formula for \( \#_+ - \#_- \) in \( Fset(Id, m) \), when \( m \to \infty \)? The first few terms are shown on the figure: \( Fset \) matrix arrangement for \( A_4 \) quiver with \( d_i = 4, \mu_i = Id \):

![Figure 1: Convergence of \( \frac{\#_+ - \#_-}{\#_-} \) sequence, \( A_2 \) quiver, \( \mu = Id, \text{dim}(M) = i \)](image)

\[
Fset = \begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
x_1 & x_11 & x_12 & x_13 & x_14 & x_15 & x_16 & x_17 \\
1 & 1 & 1 & 1 & x_1 & x_1 - x_2 & x_1 - x_11 & x_18 & x_19 & x_2 & x_21 & x_22 & x_23 & x_24 & x_25 \\
1 & 1 & 1 & 1 & x_1 & x_2 & x_1 & x_11 & 1 & 1 & x_1 & x_2 & x_1 & x_11 & 1 & 1 \end{pmatrix}
\]
where $x_j = \Delta_{I(j)}$. Compare this to $A_2, d_i = 8, \mu_i = Id$ case:

\[
\begin{pmatrix}
  x_1 & -x_2 & x_3 & -x_4 \\
  x_5 & x_6 & x_7 & x_8 \\
  x_9 & x_{11} & x_{12} \\
  x_{13} & x_{14} & x_{15} & x_{16} \\
  1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

(38)

Totally positive $Fset$ arrangement for $A_4 \times$ mutations, $d_i = 6, \mu = Id$:

\[
\begin{pmatrix}
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

(39)
\(\frac{1}{2}(\hat{P}_+ - \hat{P}_- )\) for cyclic 6-vertex graph:
Lemma 1. Asymptotic formula for $\tilde{P}_{max}(Q_{univ}(n), n, Id)$:

- $m \times m$ “cubic ladder” decomposition $\simeq$ $\Delta$-summands in $\tilde{P}_I$
- each $\Delta$-summand consists of Young diagrams factorized by the “ladder-block” addition

The formula:

$$\tilde{P}_{max}(Q(m), n, Id) = \sum_{i=1}^{m} \sum_{j=1}^{n} s_{\lambda(i, j)} \Delta_{\lambda(i, *)},$$ \hspace{1cm} (42)

$\frac{1}{2}(\tilde{P}_+ - \tilde{P}_-)$ for cyclic 8-vertex graph:
5 Re-evaluation of $\tilde{P}$ for Coxeter quivers. “Vertex at $\infty$”

Let $Q$ be a quiver with $m$ vertices and all possible arrows (but only 2 opposite arrows for each pair of vertices is allowed, so the direction $v = \pm 1$). Choose $n, e \in \mathbb{N}$ and the integer-valued vectors

$$B = \{1, \ldots, m \cdot n\}, I = \{jn - i + 1\}_{i=1,\ldots,[n/2],j=1,\ldots,m} \subset B, |I| = e,$$

(44)

where $[x]$ is the integer part of $x$. Fix the representation $R(Q)$: vertices $\sim V_i(\mathbb{Z}^n)$, arrows $\sim$ transposition matrices $M_{v,i,j} = \sigma(\lambda_{i,j})^{\text{sign}(v)}$.

Each $\sigma(\lambda_{i,j})$ is a $(n \times n)$ transposition matrix, partitioned by $\lambda_{i,j}$ for the $(i,j)$-th arrow. We fix the particular choice $R = \Sigma_0$

$$(i,j) \rightarrow \lambda_{i,j} : (1,4) \rightarrow \begin{array}{c} \begin{array}{c} \begin{array}{c} 1 \end{array} \end{array} \\ \begin{array}{c} \begin{array}{c} 2 \end{array} \end{array} \end{array} (1,3) \rightarrow \begin{array}{c} \begin{array}{c} \begin{array}{c} 1 \end{array} \end{array} \\ \begin{array}{c} \begin{array}{c} 3 \end{array} \end{array} \end{array} (1,2) \rightarrow \begin{array}{c} \begin{array}{c} \begin{array}{c} 1 \end{array} \end{array} \\ \begin{array}{c} \begin{array}{c} 4 \end{array} \end{array} \end{array} (2,4) \rightarrow \begin{array}{c} \begin{array}{c} \begin{array}{c} 2 \end{array} \end{array} \\ \begin{array}{c} \begin{array}{c} 3 \end{array} \end{array} \end{array} (2,3) \rightarrow \begin{array}{c} \begin{array}{c} \begin{array}{c} 3 \end{array} \end{array} \\ \begin{array}{c} \begin{array}{c} 4 \end{array} \end{array} \end{array} \ldots$$

(45)

Assume also $\dim(\text{Gr}(Q,R)) = \left[ \frac{1}{2} n \right]$ for the corresponding quiver grassmannian $[2,3]$. $e_i = e_j \forall i,j$.

For instance, the case $(m,n) = (4,4)$ is given by

$$M_+ = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix} \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

(46)

and $M_-$ equals by-block inverse of $M_+$. Choose some path $x \in \mathcal{P}(Q)$ and solve the quiver relations with vanishing conditions $[4]$:

$$\sum_{s',t'} (-1)^{\epsilon(s',t')} M_{v,s',t'} \Delta_{s'} \Delta_t - \sum_{s'} (-1)^{\epsilon(s',t)} M_{v,s',t} \Delta_s' \Delta_t = 0,$$

$$\Delta_J = 0 \text{ for some non-proper } J's$$

Define symmetric polynomials

$$\bar{P}_R(x) = \sum_{i,j} \Delta_{\mu_i,s_{\mu_i,j}}(p_1,p_2,\ldots),$$

(48)

where $\mu_{i,j}$ is a partition derived from the Cauchy-Binet expansion for each $\Delta's$, $s_{\mu}$ is a Schur function.

**Prop 1.** The map $f : x \mapsto \bar{P}_R(x)$ is a homomorphism on a “proper” subset $S \subseteq \mathcal{P}$, s.t. for any $x, y \in S$

$$\bar{P}_R(x \cdot y) = \frac{1}{2} \sum_{\mu,\eta,\nu} \left( \text{coeff}(\bar{P}_R(x), \Delta_{\mu}) - \text{coeff}(\bar{P}_R(y), \Delta_{\eta}) \right) \Delta_{\nu} \simeq_{|\text{Res} (\Delta)|} \frac{1}{2} \left( \bar{P}_R(x) - \bar{P}_R(y) \right)$$

(49)

Note: this construction is very sensitive to the choice of $R$ (if we swap the transposition matrices, $S$ will change).

**Lemma 2.** If $x \in \mathcal{P}(Q)$ is of spiral type 1, then

$$\bar{P}_+(x) \equiv 0,$$

(50)
Figure 2: Examples of "good" (left) and "bad" (right) combinations of paths in $S \subseteq \mathcal{P}$ for $(m, n) = (10, 10)$. "Good" means the image of $f$ is not empty.

Figure 3: Examples of a non-trivial loops spiral type 1 (Coxeter odd type $\tilde{A}_n$) for $(m, n) = (6, 6), (8, 8), (10, 10), \ldots$.

where "±" means that we take a doubled path with $v = \pm 1$. This lemma can be improved if we take all irreducible subsets of the set of all arrows of $Q$, on which our polynomial vanishes. For instance, each minimal null-set for 4-vertex quiver is trivial, 6-vertex: $\{(1, 6), (3, 4), (1, 5), (3, 6)\}$ - is not minimal, but irreducible (recall that in this case orientation does not matter). 8-vertex: $\{(2, 7), (3, 6), (2, 6), (3, 5)\}$, 12-vertex: $\{(5, 8), (3, 10), (4, 9), (4, 8), (5, 7), (3, 9)\}$ and so on. Let’s write the corresponding symmetric polynomials explicitly:

$$
\hat{P}_i = \frac{\tilde{P}_+(x_i) - \tilde{P}_-(x_i)}{2} \bigg|_{\lambda_j, k \rightarrow \mathcal{Y}(\lambda_j, k)}, \quad \mathcal{Y}(l_i) = 0, \mathcal{Y}(\lambda > l_i) = [\lambda - l_i]
$$

(51)
\[ \hat{P}_4 = 0 \]

\[ \hat{P}_6 = \left( \begin{array}{c} s_{a_1} - s_{a_2} \\ s_{a_2} - s_{a_3} \end{array} \right) \Delta_{c_{a_1}} + \left( \begin{array}{c} s_{a_3} - s_{a_4} \\ s_{a_1} - s_{a_2} \end{array} \right) \Delta_{c_{a_2}} + \left( \begin{array}{c} s_{a_5} - s_{a_6} \\ s_{a_7} - s_{a_8} \end{array} \right) \Delta_{c_{a_3}} \]

\[ \hat{P}_8 = \left( \begin{array}{c} s_{a_1} - s_{a_2} \\ s_{a_2} - s_{a_3} \end{array} \right) \Delta_{c_{a_1}} + \left( \begin{array}{c} s_{a_3} - s_{a_4} \\ s_{a_5} - s_{a_6} \end{array} \right) \Delta_{c_{a_2}} + \left( \begin{array}{c} s_{a_7} - s_{a_8} \\ s_{a_9} - s_{a_{10}} \end{array} \right) \Delta_{c_{a_3}} + \left( \begin{array}{c} s_{a_{11}} - s_{a_{12}} \\ s_{a_{13}} - s_{a_{14}} \end{array} \right) \Delta_{c_{a_4}} \]

\[ \hat{P}_{10} = \left( \begin{array}{c} s_{a_1} - s_{a_2} \\ s_{a_2} - s_{a_3} \\ s_{a_3} - s_{a_4} \end{array} \right) \Delta_{c_{a_1}} + \left( \begin{array}{c} s_{a_5} - s_{a_6} \\ s_{a_7} - s_{a_8} \\ s_{a_9} - s_{a_{10}} \end{array} \right) \Delta_{c_{a_2}} + \left( \begin{array}{c} s_{a_{11}} - s_{a_{12}} \\ s_{a_{13}} - s_{a_{14}} \end{array} \right) \Delta_{c_{a_3}} \]

\[ \hat{P}_{12} = \left( \begin{array}{c} s_{a_1} - s_{a_2} \\ s_{a_2} - s_{a_3} \\ s_{a_3} - s_{a_4} \end{array} \right) \Delta_{c_{a_1}} + \left( \begin{array}{c} s_{a_5} - s_{a_6} \\ s_{a_7} - s_{a_8} \\ s_{a_9} - s_{a_{10}} \end{array} \right) \Delta_{c_{a_2}} + \left( \begin{array}{c} s_{a_{11}} - s_{a_{12}} \\ s_{a_{13}} - s_{a_{14}} \end{array} \right) \Delta_{c_{a_3}} \]

(52)
Let \((m, n) = (6, 6)\); consider the following two paths and their concatenation:

\[
\begin{align*}
\hat{P}(Q_1) &= \left( s_1 + s_3 + s_5 + s_{10} \right) \Delta_{b_1} + \left(-s_2 + s_4 + s_{10} - s_{15}\right) \Delta_{b_2} + \\
&\quad + \left(s_5 + s_6 + s_7 + s_9 \right) \Delta_{b_3} + \left(-s_4 + s_5 + s_8 + s_{10}\right) \Delta_{b_4} + \\
&\quad + \left(-s_1 + s_3 + s_4 + s_{10}\right) \Delta_{b_5} + \left(s_5 - s_6 + s_8 + s_{10}\right) \Delta_{b_6} + \\
&\quad + \left(s_1 + s_2 + s_4 + s_{10}\right) \Delta_{b_7} \quad \# 5 \text{ is the longest arc} \\
\hat{P}(Q_2) &= \left(s_1 + s_3 - s_4 + s_5 \right) \Delta_{b_1} + \left(-s_2 - s_4 + s_7 + s_{10}\right) \Delta_{b_2} + \\
&\quad + \left(-s_4 + s_6 + s_7 + s_{10}\right) \Delta_{b_3} + \left(s_5 - s_6 + s_8 + s_{10}\right) \Delta_{b_4} + \\
&\quad + \left(s_1 + s_2 + s_4 + s_{10}\right) \Delta_{b_5} + \left(-s_6 - s_8 + s_9 + s_{10}\right) \Delta_{b_6} + \\
&\quad + \left(s_1 + s_2 + s_4 + s_{10}\right) \Delta_{b_7} \quad \# 4 \text{ is the longest arc} \\
\hat{P}(Q_{1,2}) &= \left(-s_1 - s_3 + s_5 - s_6 + s_8 + s_{10}\right) \Delta_{b_1} + \left(s_2 + s_4 + s_7 - s_8 - s_{11} - s_{12}\right) \Delta_{b_2} + \\
&\quad + \left(s_5 - s_6 + s_7 + s_{10}\right) \Delta_{b_3} = \\
= &\frac{1}{2} \left( \text{coeff}(P(Q_1), \Delta_{b_1}) - \text{coeff}(P(Q_2), \Delta_{b_1}) \right) \Delta_{b_1} + \frac{1}{2} \left( \text{coeff}(P(Q_1), \Delta_{b_6}) - \text{coeff}(P(Q_2), \Delta_{b_6}) \right) \Delta_{b_6} + \\
&\quad + \frac{1}{2} \left( \text{coeff}(P(Q_1), \Delta_{b_3}) - \text{coeff}(P(Q_2), \Delta_{b_3}) \right) \Delta_{b_3} = \frac{1}{2} \left( P(Q_1) - \text{permut}(P(Q_2), \Delta_1) \right)
\end{align*}
\]

(This is a particular example related to the proposition 1).

\[
-I\hat{P}_{b}(\text{spiral}_2) = \begin{pmatrix}
s_1 & s_3 & s_5 & s_{10} \\
s_2 & s_4 & s_{10} & s_1 \\
s_4 & s_5 & s_6 & s_7 \\
s_5 & s_8 & s_7 & s_6 \\
s_9 & s_8 & s_7 & s_6 \\
s_{10} & s_9 & s_8 & s_7 \\
\end{pmatrix}
\]
The next goal is to build infinite series like this

\[ Z_R(S) = \frac{1}{2k} \sum_{i,j} \left( \hat{P}_R(x_i) - \hat{P}_R(x_j) \right), \]  

(55)

where \( S = \{x_1, x_2, \ldots, x_i, \ldots \} \subseteq \mathcal{P}(Q_{n \to \infty}) \) is a proper union of loops. Of course it can be rewritten in terms of Schur function expansion

\[ Z_R(S) = \sum_{\lambda} g_{\lambda}(\Delta) s_{\lambda}(p_1, p_2, \ldots), \]  

(56)

or, in our “ladder notation”, over all \( \Gamma \)-type weighted Young diagrams, using the \( \mathcal{Y} \) operator

\[ Z_R(S) = \sum_{\lambda_w} g_{\lambda_w}(\Delta) \mathcal{Y}_{\lambda_w}(p_1, p_2, \ldots), \quad w \in \mathbb{N}, \quad \lambda_w = \mathcal{Y}(\lambda) \]  

(57)

Take \( S' = \{x_1, x_2\} \subseteq \mathcal{P}(Q_{10}), (m,n) = (10,10) \). In this case the double-path polynomial is non-trivial (!)

\[ \hat{P}_\pm(S') = \left( \begin{array}{c} \Delta_8 + \Delta_9 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6 + \Delta_7 + \Delta_8 + \Delta_9 \end{array} \right) \]  

(58)

(Here we assumed \( \Delta_{16,7,8,9,10,16,17,18,19,20,\ldots,96,97,98,99,100} = 1 \) to simplify the formula). Its configuration is drawn on the figure 5 (right). Now let’s extend the picture for \( x_3 \): The problem is that in this case \((m,n) = (16,16)\) the image is empty, so we should add some isolated points to the graph 5 (left).
Let
\[ X = \{G_{i_1}^{(j_1)}, G_{i_2}^{(j_2)}, \ldots, G_{i_s}^{(j_s)}\}^k \in \mathcal{P} \]  
be an element of path algebra of \( Q_{\text{univ}}(k) \), built as concatenation normal loops \( G_{i_1}, G_{i_2}, \) each one shifted on the \( j \)-th position. For example, consider 3-union \( \{G_{4}^{(0)}, G_{6}^{(3)}, G_{8}^{(2)}\}^{10} \). QR has unique solution; furthermore, we take \( \Delta = 1 \) (denominator) to homogenise the resulting formula. Here is the associated quiver:

\[ Q(X) = \]

Then \( \tilde{P}_\pm(X) \) become homogeneous in \( \Delta \):

\[ \tilde{P}_\pm(X) = \left( s_{0} + s_{1} + s_{3} + s_{4} + s_{5} + s_{6} + s_{7} + s_{8} + s_{9} \right) \Delta + \]

\[ + \left( s_{0} + s_{1} + s_{3} + s_{4} + s_{5} + s_{6} + s_{7} + s_{8} + s_{9} \right) \Delta + \]

\[ + \left( s_{2} + s_{3} + s_{4} + s_{5} + s_{6} + s_{7} + s_{8} + s_{9} \right) \Delta + \]

\[ + \left( s_{0} + s_{1} + s_{3} + s_{4} + s_{5} + s_{6} + s_{7} + s_{8} + s_{9} \right) \Delta + \]

\[ + \left( s_{2} + s_{3} + s_{4} + s_{5} + s_{6} + s_{7} + s_{8} + s_{9} \right) \Delta + \]

\[ + \left( s_{0} + s_{1} + s_{3} + s_{4} + s_{5} + s_{6} + s_{7} + s_{8} + s_{9} \right) \Delta + \]

\[ \text{Remark: we can see only antisymmetric coefficients } \Delta \text{ here! To extend this case for } G_{10} \text{ one should exclude } G_{8}^{(2)}. \]
Figure 6: Non-trivial concatenation of loops $X = \{G^{(0)}_4, G^{(3)}_6, G^{(2)}_8\}_1^{10}$ and $X' = \{G^{(0)}_4, G^{(3)}_6, G^{(2)}_8, G^{(2)}_{10}\}_1^{10}$ (only the largest arcs are shown)

The resulting picture is that each loop can be scaled separately, such that increasing $k, s$ simultaneously does not change the consistency of QR. Our next aim is to investigate asymptotical properties, derived from $\{G^{(j_1)}_{i_1}, G^{(j_2)}_{i_2}, \ldots, G^{(j_s)}_{i_s}\}_k$ when $k, s \to \infty$. This is of further development.

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