Hitting Probability and the Hausdorff Measure of the Level sets for Spherical Gaussian Fields

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Abstract

Consider an isotropic spherical Gaussian random field $T$ with values in $\mathbb{R}^d$. We investigate two problems: (i) When is the level set $T^{-1}(t)$ nonempty with positive probability for any $t \in \mathbb{R}^d$? (ii) If the level set is nonempty, what is its Hausdorff measure? These two questions are not only very important in potential theory for random fields, but also fundamental in geometric measure theory. We give a complete answer to the questions under some very mild conditions.

Key words: Spherical Gaussian Fields, Level Sets, Hausdorff Measure, Local Times, Capacity.

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1 Introduction and Statement of Main Results

The investigation of geometric properties of level sets of random fields is an interesting topic in modern probability and in several applications. Depending on whether the sample functions of the random field are smooth (e.g. continuously differentiable) or not, the properties of the level sets have to be studied using different geometric or topological tools. We refer to Adler [1], Adler and Taylor [2, 3], Adler et al. [4], and Azaïs and Wschebor [5] for systematic accounts on geometry of random fields and applications.

One of our interest is to find the connection between the existence of nonempty level sets and capacity, or we say the hitting probability $\mathbb{P}\{T^{-1}(t) \neq \emptyset\}$ for any $t \in \mathbb{R}^d$ in our case. The problem has been studied widely for $d = 1$ with 1-dimensional random processes and we cite a few such as [8] and [16], etc. For $d > 1$ and multi-dimensional random fields on Euclidean space, it has been only known for a few, see for instance [6], [17] and [18]. Unfortunately, for $d > 1$ and multi-dimensional random fields on non-Euclidean space, there is no known
result on this problem. We show that when some conditions hold, the hitting probability is indeed positive.

Whence proved the existence of nonempty level sets under some circumstances, a naturally question is, what is the volume of this level set? When the sample functions satisfy certain smoothness/differentiability conditions, the expected value of the volume of these level sets can be computed explicitly in a wide variety of circumstances, by exploiting techniques based on the Kac-Rice formula and its generalizations (see [2], Chapters 11-12). For instance, the length of the level curve for stationary Gaussian random fields on the two-dimensional plane was investigated by Kratz and León [19, 20, 21].

The applicability of the Kac-Rice approach is restricted to cases where differentiability conditions are ensured. When the sample functions of the random field are nowhere differentiable, the level sets are often fractals and their geometric properties are closely related to the analytic properties of the local times of the random field. See Geman and Horowitz [15], Xiao [31, 32], and the references therein for more information. In this paper, we extend the methods for studying the Hausdorff measure of level sets for Gaussian fields to the spherical setting. The new main ingredient is the strong local nondeterminism and upper bound of higher order moment of local time established recently in [22] and [23], respectively.

More precisely, we shall focus on the level sets of an isotropic Gaussian random field $T = \{T(x), x \in \mathbb{S}^2\}$ with values in $\mathbb{R}^d$ defined on some probability space $(\Omega, \mathcal{F}, P)$ by

$$T(x) = (T_1(x), \ldots, T_d(x)), \quad x \in \mathbb{S}^2,$$

where $T_1, \ldots, T_d$ are independent copies of $T_0 = \{T_0(x), x \in \mathbb{S}^2\}$. We assume $T_0$ to be a zero-mean, mean square continuous and isotropic random fields on the sphere, for which the following spectral representation holds (24, Chapter 5) : for $x \in \mathbb{S}^2$,

$$T_0(x) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x)$$

(2)

where $\{a_{\ell m}\}_{\ell,m}$, $\ell = 0, 1, 2, \ldots$, $m = -\ell, \ldots, \ell$ is a triangular array of zero-mean, orthogonal, complex-valued random variables with variance $E|a_{\ell m}|^2 = C_\ell$, the angular power spectrum of the random field. For $m < 0$ we have $a_{\ell m} = (-1)^m a_{\ell,-m}$, whereas $a_{00}$ is real with the same mean and variance. (2) holds both in $L^2(\Omega)$ at every fixed $x$, and in $L^2(\Omega \times \mathbb{S}^2)$, i.e.

$$\lim_{L \to \infty} E \left[ T(x) - \sum_{\ell} \sum_{m=-\ell}^{\ell} a_{\ell m}(\omega) Y_{\ell m}(x) \right]^2 = 0,$$

(3)

and

$$\lim_{L \to \infty} E \left[ \int_{\mathbb{S}^2} \left\{ T_0(x) - \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x) \right\}^2 \, d\nu(x) \right] = 0,$$

(4)
Here $\nu$ denotes the canonical Lebesgue measure on the unit sphere with $d\nu(x) = \sin \vartheta d\vartheta d\varphi$ in the spherical coordinates $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi)$. The functions $\{Y_{\ell m}(x)\}$ are the so-called spherical harmonics, i.e. the eigenvectors of the Laplacian operator on the sphere. It is a well-known result that the spherical harmonics provide a complete orthonormal systems for $L^2(S^2)$, see again [24].

A celebrated theorem of Schoenberg [27] provides the following expansion for the covariance function:

$$E[T_0(x)T_0(y)] = \sum_{\ell=0}^{+\infty} C_{\ell} \frac{2\ell + 1}{4\pi} P_\ell(\langle x, y \rangle),$$

where for $\ell = 0, 1, 2, \ldots$, $P_\ell : [-1, 1] \to \mathbb{R}$ denotes the Legendre polynomial, which satisfy the normalization condition $P_\ell(1) = 1$. Thus, without loss of generality, we assume for every $x \in S^2$,

$$E\left[T_0(x)^2\right] = \sum_{\ell=0}^{+\infty} C_{\ell} \frac{2\ell + 1}{4\pi} := 1.$$

We introduce now the same, mild regularity conditions on the angular power spectrum $C_\ell$ of the random field $T_0(x)$ as in [22].

**Condition (A)** The random field $T_0(x)$ is Gaussian and isotropic with angular power spectrum such that, for all $\ell \geq 1$, there exist constants $\alpha > 2$, $K_0 > 1$, such that

$$C_\ell = \ell^{-\alpha}G(\ell) > 0,$$

where

$$K_0^{-1} \leq G(\ell) \leq K_0 \text{ for all } \ell \in \mathbb{N}^+.$$

Condition (A) entails a weak smoothness requirement on the tail behavior of the angular power spectrum, which is trivially satisfied by some cosmologically relevant models (where the angular power spectrum usually behaves as an inverse polynomial, see [24], pp.243-244).

Now denote by

$$T^{-1}(t) = \{x \in S^2 : T(x) = t\}$$

the level set at any $t \in \mathbb{R}^d$ and $\phi-m$ the Hausdorff measure associated to the function $\phi$. We state the following result for the critical condition on the existence of nonempty level sets as well as determining the exact Hausdorff measure function for the level set $T^{-1}(t)$. In particular, it implies that the Hausdorff dimension of $T^{-1}(t)$ equals $2 - (\alpha - 2)d/2$ in view of Theorem 1.2 in [23].

**Theorem 1.1** Let $T = \{T(x), x \in S^2\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined in [11]. Assume that the associated isotropic random field $T_0$ satisfies Condition (A) with $2 < \alpha < 4$. Then

$$P\{T^{-1}(0) \neq \emptyset\} > 0 \iff 4 - (\alpha - 2)d > 0;$$

(7)
Moreover, there exists a constant \( K_1 > 1 \) such that for every \( t \in \mathbb{R}^d \), the Hausdorff measure of the level set \( T^{-1}(t) \) associated with \( \phi \) satisfies that
\[
K_1^{-1} L\left(t, S^2 \right) \leq \phi - m\left(T^{-1}(t) \right) \leq K_1 L\left(t, S^2 \right), \text{ a.s. } \tag{8}
\]
where \( L\left(t, S^2 \right) \) is the local time defined in \( \text{(13)} \) Section 2, and the function \( \phi \) is defined by
\[
\phi (r) = \frac{r^2}{\left[ \rho_\alpha (r/ \sqrt{\log \log r}) \right]^\alpha} \tag{9}
\]
with \( \rho_\alpha (r) = r^{\frac{\alpha}{2} - 1} \) for \( r \geq 0 \).

In general, it has been known that Hausdorff measure of level sets can be controlled lower bounded by relative local time, see for instance Xiao \([31, 32]\) for Gaussian fields indexed in Euclidean space. Our theorem above significantly improves their results by proving that the upper bound by the local time also holds.

The rest of this paper is as follows: we collect some technical lemmas and their proofs in Section 2. The critical conditions on the existence of nonempty level set is presented in Section 3. Section 1 deals with the Hausdorff measure for the level sets. The lower bound for the Hausdorff measure is derived by applying our result on the local times and an upper density theorem of Roger and Taylor [25]. The proof of the upper bound is more difficult and we extend the covering argument in Talagrand [28, 29] and Xiao [31], and their strengthened version of Barak and Mountford [7] to the spherical setting. This part is technical and is presented in Section 3. As an illustration of Theorem 1.1 we give two examples in Section 4 as well.

We denote by \( |u|_\infty = \sup_{i=1,...,d} |u_i| \), the supremum of \( |u_i|, i = 1,...,d \), for \( u \in \mathbb{R}^d \), and use \( K \) to denote a constant whose value may change in each appearance, and \( K_{i,j} \) to denote the \( j \)th more specific positive finite constant in Section \( i \).

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2 Preliminaries

In this section, we collect a few technical results which will be instrumental for most of the proofs to follow. Recalling formula (4), for any two points \( x, y \in S^2 \), the variogram of \( T_0 \) is defined by
\[
\sigma^2(x, y) = \mathbb{E} \left\{ [T_0(x) - T_0(y)]^2 \right\} = 2 - 2\mathbb{E} [T_0(x)T_0(y)] = \sigma^2 (d_{S^2}(x, y)) \tag{10}
\]
Here we have made a little abuse of the notation $\sigma^2$. The following lemma from \cite{22} characterizes the variogram and the property of strong local nondeterminism of $T_0$.

**Lemma 2.1** Under Condition (A) with $2 < \alpha < 4$, there exist positive constants $K_{2,1} \geq 1, 0 < \delta_0 < 1$ depending only on $\alpha$ and $K_0$, such that for any $x, y \in S^2$, if $d_{S^2}(x, y) < \delta_0$, we have

$$K_{2,1}^{-1}\rho_\alpha^2(d_{S^2}(x, y)) \leq \sigma^2(d_{S^2}(x, y)) \leq K_{2,1}\rho_\alpha^2(d_{S^2}(x, y)).$$  \hspace{1cm} (11)

Moreover, there exists a constant $K_{2,2} > 0$ depending on $\alpha$ and $K_0$ only, such that for all integers $n \geq 1$ and all $x, x_1, ..., x_n \in S^2$, we have

$$\text{Var}(T_0(x)|T_0(x_1), ..., T_0(x_n)) \geq K_{2,2} \min_{1 \leq k \leq n} \rho_\alpha^2(d_{S^2}(x, x_k)).$$  \hspace{1cm} (12)

For any fixed point $x_0 \in S^2$, define the spherical random field $Y_{x_0}(x) = T_0(x) - T_0(x_0), x \in S^2$. An immediate consequence of Lemma 2.1 is as follows:

**Corollary 2.2** Under Condition (A) with $2 < \alpha < 4$, there exists a constant $K_{2,2}' > 0$, such that for all integers $n \geq 1$ and all $x, x_1, ..., x_n \in S^2$,

$$\text{Var}(Y_{x_0}(x)|Y_{x_0}(x_1), ..., Y_{x_0}(x_n)) \geq K_{2,2}' \min_{0 \leq k \leq n} \{\rho_\alpha^2(d_{S^2}(x, x_k))\}.$$  \hspace{1cm} (13)

Now let us introduce the local times of a random field on the unit sphere. Recall first that, for any Borel sets $B \subset \mathbb{R}^d$, $D \subset S^2$ and sample $\omega \in \Omega$, the occupation measure of a spherical random field $T$ in $B$ is defined by (c.f. \cite{15})

$$\mu_D(B, \omega) := \int_D 1_B(T(x, \omega)) d\nu(x) = \nu\{x \in D : T(x, \omega) \in B\},$$

with $1_B(\cdot)$ the index function. We recall also that local times of $T$ exist on $\mathbb{R}^d$ provided that the measure $\mu_D(dt)$ is a.s. absolutely continuous w.r.t the Lebesgue measure on $\mathbb{R}^d$. More precisely, when $\mu$ is absolutely continuous, there exists a function $L(t, D) = L(t, D, \omega) \geq 0$ which is measurable on $S^2$ and such that for almost all $\omega \in \Omega$ and each measurable set $B \subset \mathbb{R}^d$, \hspace{1cm} (13)

$$\mu_D(B, \omega) = \int_B L(t, D, \omega)dt.$$

We call $L(t, D)$ the local time of $T$ on $\mathbb{R}^d$. See \cite{22, 32} for more information about local times of random fields on the Euclidean spaces.

The following results are essential to our discussion about the inequalities (8), and have been established in \cite{23}.
Lemma 2.3 Under the conditions of Theorem 1.1 with \((α - 2)d < 4\), \(T\) a.s. has a jointly continuous local time \(L(t, D)\) w.r.t. \((t, x, r)\) for any open disk \(D = D(x, r) \subseteq \mathbb{S}^2\) and \(t \in \mathbb{R}^d\), which can be represented in the \(L^2(\mathbb{R}^d)\)-sense as

\[
L(t, D) = \frac{1}{2\pi} \int_{\mathbb{R}^d} e^{-i\xi^T t} \int_D e^{i\xi^T x} du(x) d\xi; \tag{14}
\]

Moreover, there exists a positive constant \(K_{2,3}\) depending on \(α, d, K_0\) and \(γ\), such that for any \(s, t \in \mathbb{R}^d\), all even integers \(n \geq 2\), \(0 < γ < 1\), we have

\[
\mathbb{E} \{ |L(t, D) - L(s, D)|^n \} \leq K_{2,3}^n(n!)^{2-γ/2} \|t - s\|^{nγ} ν(D)^{(n-1)γ/2+1},
\]

with \(η = \frac{d}{2} - (α - 2) γ > 0\).

Lemma 2.4 Under condition (A), there exists a positive constant \(K_{2,4}\) depending on \(K_0\) and \(α\), such that for any \(θ \in (0, δ_0)\), and positive integers \(L < \frac{K_{2,4}}{θ}\), we have

\[
\sum_{ℓ=1}^{L} \frac{2ℓ + 1}{4π} C_ℓ \left\{ 1 - P_ℓ(\cos θ) \right\} \leq K_{2,4} L^{1-α} θ^2,
\]

and, for any integer \(U > 1\),

\[
\sum_{ℓ=U}^{∞} \frac{2ℓ + 1}{4π} C_ℓ \left\{ 1 - P_ℓ(\cos θ) \right\} \leq K_{2,4} U^{2-α}.
\]

Note that here \(δ_0\) is the constant defined in Lemma 2.1.

In order to obtain the exact Hausdorff measure of the level sets in Theorem 1.1, we will also exploit the following two lemmas from Talagrand [28]. Let \(\{f(x), x \in M\}\) be a centered Gaussian field indexed by \(M\) and let \(d_f(x, y) = \sqrt{\mathbb{E}[(f(x) - f(y))^2]}\) be its canonical metric on \(M\). For a \(d_f\)-compact manifold \(M\), denote by \(N_{d_f}(M, ε)\) the smallest number of balls of radius \(ε\) in metric \(d_f\) that are needed to cover \(M\).

Lemma 2.5 If \(N_{d_f}(M, ε) \leq \Psi(ε)\) for all \(ε > 0\) and the function \(Ψ\) satisfies

\[
\frac{1}{K_{2,5}} Ψ(ε) \leq Ψ\left(\frac{ε}{2}\right) \leq K_{2,5} Ψ(ε), \quad ∀ ε > 0,
\]

where \(K_{2,5}\) is a finite constant. Then

\[
\mathbb{P} \left\{ \sup_{s, t \in M} |f(s) - f(t)| \leq u \right\} \geq \exp \left( -K_{2,6} Ψ(u) \right),
\]

where \(K_{2,6}\) is a constant depending only on \(K_{2,5}\).

Lemma 2.6 Let \(\{f(x), x \in M\}\) be a centered Gaussian field a.s. bounded on a \(d_f\)-compact set \(M\). There exists a universal constant \(K_{2,7}\) such that for any \(u > 0\), we have

\[
\mathbb{P} \left\{ \sup_{x \in M} f(x) \geq K_{2,7} \left( u + \int_0^d \sqrt{\log N_{d_f}(M, ε)} dε \right) \right\} \leq \exp \left( -\frac{u^2}{d} \right),
\]

where \(d = \sup \{d_f(x, y) : x, y \in M\}\) is the diameter of \(M\) in the metric \(d_f\).
Based on Lemmas 2.1 and 2.6 above, we obtain the following result:

**Lemma 2.7** Under the condition (A) with $2 < \alpha < 4$, there exist positive constants $K_{2,8}$ and $K_{2,9}$ depending only on $\alpha$ and $K_0$, such that for any $z \in S^2$ and $0 < r < \delta_0$, we have for any $u > K_{2,8}^{a-2}/2$,

$$
P\left\{ \sup_{x, y \in D(z, r)} |T_0(x) - T_0(y)| \geq u \right\} \geq \exp\left( -\frac{u^2}{K_{2,9} |\rho_2(2r)|^2} \right).
$$  \hspace{1cm} (15)

**Proof.** Recall (11) in Lemma 2.1, we have

$$
\sqrt{K_{2,1}^{-1} \rho_2(x, y)} \leq d_{T_0}(x, y) \leq \sqrt{K_{2,1} \rho_2(x, y)}.
$$

It follows immediately that, for any $D(z, r) \subset S^2$, and any $\epsilon \in (0, r)$,

$$
N_{d\epsilon}(D(z, r), \epsilon) \leq \frac{2 \pi r^2}{\pi (\epsilon/\sqrt{K_{2,1}})^{4/(\alpha-2)}} \leq 2(K_{2,1})^{2/(\alpha-2)} \frac{r^2}{\epsilon^{4/(\alpha-2)}},
$$

and

$$
\bar{d} = \sup \{ d_{T_0}(x, y) : x, y \in D(z, r) \} \leq \sqrt{K_{2,1} \rho_2(2r)},
$$

whence

$$
\int_0^{\bar{d}} \sqrt{K_{2,1} \rho_2(2r)} \int_0^{\sqrt{K_{2,1} \rho_2(2r)}} \sqrt{\log \left\{ 2(dK_{2,1})^{2/(\alpha-2)} \frac{r^2}{\epsilon^{4/(\alpha-2)}} \right\}} \, d\epsilon
$$

$$
\leq 2 \sqrt{K_{2,1} \rho_2(2r)} \int_0^{\alpha/2-1} \frac{d}{d\epsilon} \left( \int_1^{+\infty} u d(-e^{-u^2}) \right) \leq C_{2,1} r^{\alpha/2-1},
$$

where $(\cdot \vee \cdot)$ denotes as usual the maximum function. Taking $K_{2,7} = 2K_{2,6}C_{2,1}$, then by exploiting Lemma 2.6 we derive (15). This proves Lemma 2.7. \hfill \square

Finally, we recall briefly the definition of Hausdorff measure on the sphere $S^2$ and an upper density theorem due to [26], which will be used for proving the lower bound in Theorem 1.1. We refer to Falconer [14] for more information on geometry of fractals, and to [32] for its applications in studying sample path properties of Gaussian random fields.

For some $\epsilon > 0$, let $\Phi$ be the class of functions $\phi_1 : (0, \epsilon) \to (0, 1)$, which are right-continuous, monotonically increasing, with $\phi_1(0^+) = 0$ and such that there exists a finite constant $K > 0$ for which

$$
\phi_1(2r) \leq K, \text{ for } 0 < r < \frac{1}{2} \epsilon.
$$

The $\phi_1$-Hausdorff measure of $E \subseteq S^2$ is then defined as usual by

$$
\phi_1-m(E) = \liminf_{\epsilon \to 0} \left\{ \sum_{i=1}^{\infty} \phi_1(2r_i) : E \subseteq \cup_{i=1}^{\infty} D(x_i, r_i), r_i < \epsilon \right\}.
$$
Recall that $D(x, r)$ denotes the open disk of radius $r$ centered at $x \in \mathbb{S}^2$. Likewise, the Hausdorff dimension of $E$ is defined by

$$\dim E = \inf \{ \beta > 0 : s^\beta \cdot m(E) = 0 \} = \sup \{ \beta > 0 : s^\beta \cdot m(E) = \infty \}.$$

The following lemma is derived from the results in [26], and it allows to obtain lower bounds for $\phi_1 \cdot m(E)$. Recall first that for any Borel measure $\mu$ on $\mathbb{S}^2$ and $\phi_1 \in \Phi$, the upper $\phi_1$-density of $\mu$ at $x \in \mathbb{S}^2$ is defined by

$$D_{\phi_1} \mu(x) := \limsup_{r \to 0} \frac{\mu(D(x, r))}{\phi_1(2r)}.$$

**Lemma 2.8** For a given $\phi_1 \in \Phi$, there exists a positive constant $K_{2,10}$ such that for any Borel measure $\mu$ on $\mathbb{S}^2$ and every Borel set $E \subseteq \mathbb{S}^2$, we have

$$\phi_1 \cdot m(E) \geq K_{2,10} \mu(E) \inf_{x \in E} \left\{ \frac{1}{D_{\phi_1} \mu(x)} \right\}^{-1}.$$

### 3 Existence of Nonempty Level Sets and Capacity

In this section we will provide two more equivalent conditions for the existence of nonempty level sets. As an immediate consequence, we give a proof for (7) in Theorem 1.1.

Let us first introduce the capacity of the random field $T$. For any $t_2 \in \mathbb{R}^2$, the joint density of $T_0(x)$ and $T_0(y)$ can be represented as follows:

$$p(d_{d^2}(x,y); t_2) = \frac{1}{2\pi\sigma(d_{d^2}(x,y)) \sqrt{1 - \sigma^2(d_{d^2}(x,y))/4}} \exp \left\{ -\frac{1}{2} t_2^T \Sigma^{-1} t_2 \right\}, \quad (16)$$

where $\Sigma^{-1}$ is the inverse of positive definite covariance matrix $\Sigma(d_{d^2}(x,y))$ of $T_0(x)$ and $T_0(y)$. Hence, the joint density of $\Phi(d_{d^2}(x,y))$ of $T(x)$ and $T(y)$ at $(a, a)$ for any $a = (a_1, ..., a_d)^T \in \mathbb{R}^d$ can be represented as

$$\Phi(d_{d^2}(x,y); a) = \prod_{j=1}^{d} p(d_{d^2}(x,y); a_j, 1). \quad (17)$$

due to the independence of $T_1, ..., T_d$. Obviously, for any $a \in \mathbb{R}^d$,

$$\Phi(d_{d^2}(x,y); a) \leq \Phi(d_{d^2}(x,y)). \quad (18)$$

where $\Phi(\cdot) = \Phi(\cdot; 0)$. Now the $\Phi$-capacity of a Borel set $E \subseteq \mathbb{S}^2$ is then defined as

$$C_{\Phi}(E) = \left\{ \inf_{\mu \in \mathcal{P}(E)} \int \int \Phi(d_{d^2}(x,y)) \mu(dx) \mu(dy) \right\}^{-1},$$

8
where $\mathcal{P}(E)$ is the collection of all probability measures on $E$. Moreover, for any $\mu \in \mathcal{P}(E)$, we define the $\Phi$-energy of $\mu$ by

$$\varepsilon_\Phi(\mu) = \int \int_{S^2 \times S^2} \Phi(d_{S^2}(x, y)) \mu(dx) \mu(dy).$$

(19)

It is readily seen that

$$C_\Phi(E) = \left\{ \inf_{\mu \in \mathcal{P}(E)} \varepsilon_\Phi(\mu) \right\}^{-1}.$$

See [11, 18] for more informations about $\Phi$-capacity and $\Phi$-energy for probability measures on Borel sets in Euclidean spaces.

Let $N$ be the north pole of the unit sphere, then we have the following result.

**Proposition 3.1** Under the conditions of Theorem 1.1, the following are equivalent:

(i) $C_\Phi(D(N, r)) > 0$ for some $r > 0$;

(ii) for all $a \in \mathbb{R}^d$ and $x \in S^2$, $\mathbb{P}\{T^{-1}(a) \cap D(x, r) \neq \emptyset\} > 0$ for some $r > 0$;

(iii) $\Phi(d_{S^2}(N, \cdot)) \in L^1(D(N, r))$, for some $r > 0$.

**Proof.** We employ an argument that is similar, in spirit, to that used by Khoshnevisan and Xiao [18] in the proof of its Theorem 2.9. The steps of our proof is taken in the order (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

To prove (i) $\implies$ (ii), it is sufficient to prove

$$\mathbb{P}\{a \in T(D(x, r))\} > 0.$$ 

The idea is to find a random measure $\mathcal{H}$ such that

$$\mathbb{P}\{h(T^{-1}(a) \cap D(x, r)) > 0\} > 0.$$ 

More precisely, it satisfies

$$\mathbb{P}\{h(D(x, r)) > 0\} > 0,$$

(20)

as well as $h(D_T(\delta)) = 0$, a.s. for any $\delta > 0$ with

$$D_T(\delta) = \{y \in D(x, r) : |T(y) - a|_\infty > \delta\}.$$

(21)

Note that, under condition (i), there exists a probability measure $\mu \in \mathcal{P}(S^2)$ such that the $\Phi$-energy of $\mu$ in [19] is finite. Now for any $\varepsilon > 0$, $a \in \mathbb{R}^d$, we define a random measure

$$h_\varepsilon(E) = (2\varepsilon)^{-d} \int_E 1 \{|T(y) - a|_\infty \leq \varepsilon\} \mu(dy).$$

(22)
where $E$ is any Borel set on $\mathbb{S}^2$. Then it is readily seen that
\[
(2\varepsilon)^d \mathbb{E} [h_{\varepsilon}(E)] = \int_E \mathbb{P} \left\{ |\mathbf{T}(y) - \mathbf{a}|_\infty \leq \varepsilon \right\} \mu(dy)
\]
\[
= \int_E \left[ \prod_{j=1}^{d} \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} (t_j + a_j)^2 \right\} dt_j \right] \mu(dx).
\]
By Fatou’s Lemma, we obtain
\[
\lim_{\varepsilon \to 0} \mathbb{E} [h_{\varepsilon}(E)] = (2\pi)^{-d/2} \exp \left\{ -\frac{1}{2} \|\mathbf{a}\|^2 \right\} \mu(E), \tag{23}
\]
where $\|\cdot\|$ denotes the Euclidean distance in $\mathbb{R}^d$. Moreover,
\[
(2\varepsilon)^2d \mathbb{E} \left\{ [h_{\varepsilon}(E)]^2 \right\} \nonumber
\]
\[
= \int \int \mathbb{P} \left\{ |\mathbf{T}(y) - \mathbf{a}|_\infty \leq \varepsilon, |\mathbf{T}(z) - \mathbf{a}|_\infty \leq \varepsilon \right\} \mu(dy) \mu(dz)
\]
\[
= \int \int \left[ \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} p(d_{S^2}(y,z); t_2 + a_j \mathbf{1})\, dt_2 \right]^d \mu(dy) \mu(dz).
\]
Recall (16), we have for any $t_2 \in [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$,
\[
\lim_{\varepsilon \to 0} p(d_{S^2}(y,z); t_2 + a_j \mathbf{1}) = p(d_{S^2}(y,z); a_j \mathbf{1}). \tag{24}
\]
Thus, by Fatou’s Lemma again and the fact of (i), we have
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left\{ [h_{\varepsilon}(E)]^2 \right\} \nonumber
\]
\[
= \int \int \Phi \left( d_{S^2}(y,z) ; \mathbf{a} \right) \mu(dy) \mu(dz)
\]
\[
\leq \int \int \Phi \left( d_{S^2}(y,z) \right) \mu(dy) \mu(dz) < \infty. \tag{25}
\]
in view of the inequality (18). Now let $E = D(x,r)$, it is readily seen that for $h = \lim_{\varepsilon \to 0} h_{\varepsilon}$ in the weakly convergent sense, we have
\[
\mathbb{E} \left\{ h(D(x,r)) \right\} = (2\pi)^{-d/2} \mu(D(x,r)), \quad \mathbb{E} \left\{ [h(D(x,r))]^2 \right\} < \infty,
\]
in view of (20) and (24). Hence, by Cauchy-Schwarz inequality, we obtain
\[
\mathbb{P} \{ h(D(x,r)) > 0 \} \geq |\mathbb{E} [h(D(x,r))]|^2 \left\{ \mathbb{E} \left[ [h(D(x,r))]^2 \right] \right\}^{-1} > 0.
\]
It remains to prove that the random measure $h$ is supported on $T^{-1}(\mathbf{a})$, which can be obtained by the fact that for each $\delta > 0$,
\[
\mathbb{E} [h(D_T(\delta))] = \lim_{\varepsilon \to 0} \mathbb{E} [h_{\varepsilon}(D_T(\delta))] = 0,
\]
in view of the definitions $D_T(\delta)$ and $h_{\varepsilon}$ in (21) and (22).

The implication of $(iii) \implies (i)$ is obvious and we leave the proof of $(ii) \implies (iii)$ in the following Lemma. Thus the proof is completed. □
**Lemma 3.2** In Proposition 3.1, (ii) \( \implies \) (iii).

**Proof.** Recall (10) and note that for any \( x, y \in D(N, r) \),

\[
\mathbb{E}(T_0(x)|T_0(y)) = \frac{\mathbb{E}(T_0(x)T_0(y))}{\mathbb{E}(T_0(y)^2)} T_0(y) = C(x, y)T_0(y),
\]

where

\[ C(x, y) = \left( 1 - \frac{1}{2} \sigma^2(d_{S^2}(x, y)) \right). \]

Obviously, we have

\[
\left( 1 - \frac{1}{2} \sigma^2(2r) \right) \leq C(x, y) \leq 1.
\]

Now let \( Q_0(x, y) = T_0(x) - C(x, y)T_0(y) \), then it is readily seen that \( Q_0(x, y) \) is again Gaussian with \( \mathbb{E}[Q_0(x, y)] = 0 \), and

\[
\text{Var}(Q_0(x, y)) = 1 - C(x, y)^2 = [2\pi p(d_{S^2}(x, y); 0)]^{-2}
\]

in view of (16). Moreover, we have the following decomposition for \( T(x) \):

\[
T(x) = Q(x, y) + C(x, y)T(y),
\]

with \( Q(x, y) \) and \( T(y) \) independent. Now we are ready to prove (ii) \( \implies \) (iii).

For any \( \varepsilon \in (0, 1) \), define a random measure

\[
h_\varepsilon(B) = (2\varepsilon)^{-d} \int_B 1 \left\{ |T(x)|_\infty \leq \varepsilon \right\} d\nu(x).
\]

for any Borel set \( B \subseteq S^2 \), and consider the following conditional expectation

\[
M(y, \varepsilon) = \mathbb{E}\left[ h_\varepsilon(D(x, r))|T(y) \right], \quad y \in D(x, r).
\]

Then by the independence of \( Q(x, y) \) and \( T(y) \), we have

\[
M(y, \varepsilon) \geq (2\varepsilon)^{-d} \int_{D(N, r)} \mathbb{P}\left\{ |Q(x, y)|_\infty \leq \frac{\varepsilon}{2} \right\} d\nu(x)1 \left\{ |T(y)|_\infty \leq \frac{\varepsilon}{2} \right\}.
\]

Now, by first squaring both sides of the inequality above, then doing the expectation after taking the supremum over \( D(N, r) \), we obtain

\[
\mathbb{E}\left\{ \sup_{y \in D(N, r)} M^2(y, \varepsilon) \right\} \geq (2\varepsilon)^{-2d} \inf_{y \in D(N, r)} \left[ \int_{D(N, r)} \mathbb{P}\left\{ |Q(x, y)|_\infty \leq \frac{\varepsilon}{2} \right\} d\nu(x) \right]^2.
\]

(27)
Note that, by Jensen’s inequality, we have
\[
\mathbb{E} \left[ \sup_{y \in D(N,r)} M^2(y, \varepsilon) \right] \leq \mathbb{E} \left\{ \sup_{y \in D(N,r)} \mathbb{E} \left[ h_\varepsilon^0 \left( D(x, r) \right) \right]^2 \mathbb{E} \left[ T(y) \right] \right\}. \tag{28}
\]
Moreover, recall Lemma 2.1 and Var \((T_0(x)) = 1\) for any \(x \in S^2\), we have for any three points \(x_1, x_2, x_3 \in D(N, r)\) with some \(r \in (0, \delta_0)\),
\[
det \left[ \text{Cov} \left( T_0(x_1)T_0(x_2)|T_0(x_3) \right) \right] = \text{Var}(T_0(x_2))\text{Cov} \left( T_0(x_1)|T_0(x_2), T_0(x_3) \right)
\geq C_{3,1} \left[ p \left( \min_{i=2,3} d_{\delta_2}(x_1, x_i) ; 0 \right) \right]^{-2},
\]
where the constant \(C_{3,1}\) is positive and depends on \(\delta_0, K_{2,1}\) and \(K_{2,2}\). Hence,
\[
\mathbb{E} \left\{ \sup_{y \in D(N,r)} (2\varepsilon)^{-d} \int_{D(N,r)} \int_{D(N,r)} \text{d}\nu(x_1)\text{d}\nu(x_2) \right. \\
\cdot \mathbb{P} \left\{ |T(x_1)|_\infty \leq \varepsilon, |T(x_2)|_\infty \leq \varepsilon |T(y)| \right\} \\
\leq \mathbb{E} \sup_{x_3 \in D(N,r)} \int_{D(N,r)} \int_{D(N,r)} \text{d}\nu(x_1)\text{d}\nu(x_2) \\
\cdot (2\pi)^{-d} \{ \det \left[ \text{Cov} \left( T_0(x_1), T_0(x_2)|T_0(x_3) \right) \right] \}^{-d/2} \\
\leq C_{3,2} \sup_{x_3 \in D(N,r)} \int_{D(N,r)} \int_{D(N,r)} \Phi \left( d_{\delta_2}(x_1, x_2) \right) \text{d}\nu(x_1)\text{d}\nu(x_2) \\
\leq C_{3,2} \int_{D(N,r)} \int_{D(N,r)} \Phi \left( d_{\delta_2}(x_1, x_2) \right) \text{d}\nu(x_1)\text{d}\nu(x_2) \\
\left. + C_{3,2} \nu \{ D(N,r) \} \sup_{x_3 \in D(N,r)} \int_{D(N,r)} \Phi \left( d_{\delta_2}(x, x_3) \right) \text{d}\nu(x), \right.
\]
which gives
\[
\mathbb{E} \left[ \sup_{y \in D(N,r)} M^2(y, \varepsilon) \right] \leq 2C_{3,2} \nu \{ D(N,r) \} \int_{D(N,r)} \Phi \left( d_{\delta_2}(x, N) \right) \text{d}\nu(x). \tag{29}
\]
due to the fact that, for any \(y \in D(N,r)\)
\[
\int_{D(N,r)} \Phi \left( d_{\delta_2}(x, y) \right) \text{d}\nu(x) \leq \int_{D(N,r)} \Phi \left( d_{\delta_2}(x, N) \right) \text{d}\nu(x)
\]
Here the constant \(C_{3,2}\) is positive and depends on \(C_{3,1}\). In the meantime, by Fatou’s Lemma and the upper bound in \([24]\), we have
\[
\lim_{\varepsilon \to 0} (2\varepsilon)^{-d} \int_{D(N,r)} \mathbb{P} \left\{ \left| Z(x, y) \right|_\infty \leq \frac{\varepsilon}{2} \right\} \text{d}\nu(x) \\
= (2\pi)^{d/2} \int_{D(N,r)} \Phi \left( d_{\delta_2}(x, y) \right) \text{d}\nu(x) \\
\geq \frac{1}{3} (2\pi)^{d/2} \int_{D(N,r)} \Phi \left( d_{\delta_2}(x, N) \right) \text{d}\nu(x) \tag{30}
\]
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in view of the representation \((26)\) and the fact that, for any \(y \in D(N, r)\)

\[
\int_{D(N, r)} \Phi(d_{\mathbb{S}^2}(x, y)) \, d\nu(x) \geq \frac{1}{3} \int_{D(N, r)} \Phi(d_{\mathbb{S}^2}(x, N)) \, d\nu(x)
\]

Finally, combining inequalities \((27), (29)\) and \((30)\), together with

\[
P\left(\|T(y)\| \leq \frac{\varepsilon}{2} \right. \text{ for some } y \in D(N, r)\right)
\geq \frac{1}{2}
P\left(T(y) = 0 \text{ for some } y \in D(N, r)\right) > 0,
\]

we obtain

\[
C_{3,3} \left[ \int_{D(N, r)} \Phi(d_{\mathbb{S}^2}(x, N)) \, d\nu(x) \right]^{-1} \geq P\left(T^{-1}(0) \cap D(N, r) \neq \emptyset\right) > 0,
\]

where the constant \(C_{3,3}\) is positive and depends on \(C_{3,1}\) and \(C_{3,2}\) and the proof of Lemma is then completed. \(\blacksquare\)

As an immediate consequence, we prove the follows:

\textbf{Proof of (7) in Theorem 1.1} Recalling the formula \((10)\) and \((17)\) for the definition \(\Phi = \Phi_0\) and the estimations \((11)\) for variogram of \(T_0\) in Lemma 2.1, we have

\[
\int_{D(N, r)} \Phi(d_{\mathbb{S}^2}(x, N)) \, d\nu(x) \approx \int_0^r \int_0^{2\pi} \theta^{d(1-\alpha/2)} \sin \theta \, d\phi d\theta,
\]

where by \(A \approx B\) we mean that \(C_{3,4}B \leq A \leq C_{3,5}B\) for some positive constants \(C_{3,4}, C_{3,5}\) depending on \(K_{2,1}\). The right side of the equivalence above is finite if and only \(1 + d(1 - \alpha/2) > -1\), thus the equivalence \((7)\) is proved in view of Proposition 3.1. \(\blacksquare\)

### 4 Hausdorff Measure of the Level Sets

In this section, we give a proof of \((8)\) in Theorem 1.1 which is divided into proving lower and upper bounds separately. For the lower bound we will make the local time \(L(t, \cdot)\) as a natural measure on \(T^{-1}(t)\) and make use of Lemma 2.8.

The proof of upper bound is more involved. We will extend the covering argument by Baraka and Mountford \([7]\) to the spherical random fields. Their method strengthened the covering argument by Xiao \([31]\) for the level sets, see also Talagrand \([28, 29]\).

#### 4.1 Lower bound for the Hausdorff measure

We first establish the lower bound for the exact Hausdorff measure of the level set \(T^{-1}(t)\) by applying Lemma 2.8.

A basic tool to establish it is the proposition below:
Proposition 4.1 Under the conditions of Theorem [14], there exists a positive constant $K_{4,1}$, such that for all $x \in \mathbb{S}^2$ and $t \in \mathbb{R}^d$, with probability one,

$$\limsup_{r \to 0} \frac{L(t, D(x, r))}{\phi(r)} \leq K_{4,1},$$

where $\phi(\cdot)$ is defined in [4].

Proof. Denote by $f_m(x) = L(t, D(x, 2^{-m}))$, the local time of $T$ at $t \in \mathbb{R}^d$ in $D(x, 2^{-m}) \subseteq \mathbb{S}^2$, for every integer $m \geq 1$, and recall the representation ([14], we have

$$\mathbb{E} \left[ \int_{\mathbb{S}^2} |f_m(x)|^n L(t, d\nu(x)) \right] = (2\pi)^{-(n+1)} \times \int_{\mathbb{S}^2} \int_{D(x, 2^{-m})} \int_{\mathbb{R}^{d(n+1)}} \exp \left\{ \frac{n+1}{\alpha} \sum_{j=1}^{n+1} \xi_j^T (T(x_j) - t) \right\} d\xi d\nu(x),$$

with $d\nu(x) = d\nu(x_1) \cdots d\nu(x_{n+1})$ and $d\xi = d\xi_1 \cdots d\xi_{n+1}$, where $\xi_1, \cdots, \xi_n \in \mathbb{R}^d$. Using the similar argument as in [23], we obtain that the right-hand side of the equality above is bounded by

$$\int_{\mathbb{S}^2} \int_{D(x, 2^{-m})} \frac{d\nu(x)}{|\det Cov(T(x_1), \ldots, T(x_{n+1}))|^{1/2}} \leq (C_{4,1})^{n+1} (n!)^{\frac{d}{2}} (2^{-m})^{n(4-(\alpha-2)d)},$$

where $C_{4,1}$ is a positive constant and depends on $\alpha$, $d$ and $K_{2,2}$. Now, for each $m \geq 1$, consider the random set

$$D_m(\omega) = \{ x \in \mathbb{S}^2 : f_m(x) \geq K_{4,1} \phi(2^{-m}) \},$$

Let $K_{4,1} > C_{4,1}e^2$ and taking $n = \lfloor \log m \rfloor$, then by applying (31) and Stirling’s formula, we have

$$\mathbb{E} \left[ L(t, D_m) \right] = \mathbb{E} \left[ \int_{D_m} L(t, d\nu(x)) \right] \leq \frac{\mathbb{E} \left[ \int_{\mathbb{S}^2} |f_m(x)|^n L(t, d\nu(x)) \right]}{(K_{4,1} \phi(2^{-m}))^n} \leq (C_{4,1})^{n+1} (n!)^{d(\alpha-2)/4} [K_{4,1} \phi(2^{-m})]^{d(\alpha-2)/4} \leq m^{-2},$$

which implies that

$$\mathbb{E} \left[ \sum_{m=1}^{\infty} L(t, D_m) \right] < \infty.$$

Therefore, by Borel-Cantelli Lemma, we have for almost all $x \in \mathbb{S}^2$, with probability 1,

$$\limsup_{m \to \infty} \frac{L(t, D(x, 2^{-m}))}{\phi(2^{-m})} \leq K_{4,1}.$$  

(32)

Finally, for any $r > 0$ small enough, there always exists an integer $m$ such that $2^{-m} < r < 2^{-(m+1)}$ and (32) is applicable. Since the function $\phi(r)$ is increasing near $r = 0$, the result in Proposition 4.1 follows from (32) and a monotonicity argument.  

Theorem 4.2 Under the conditions of Theorem 1.1, there exists a positive constant $K_{4.2}$, such that for every $t \in \mathbb{R}^d$, with probability one,

$$\phi - m (T^{-1}(t)) \geq K_{4.2} L (t, S^2).$$

Proof. Let

$$D_0 = \left\{ x \in S^2 : \limsup_{r \to 0} \frac{L (t, D(x, r))}{\phi(r)} > K_{4.1} \right\},$$

it is readily seen that $D_0$ is a Borel set and $L (t, D_0) = 0$ almost surely, in view of Proposition 4.1. Therefore, we have almost surely

$$\phi - m (T^{-1}(t)) \geq \phi - m (T^{-1}(t) \cap (S^2 \setminus D_0)) \geq \frac{K_{2.10}}{K_{4.1}} L (t, S^2 \setminus D_0),$$

in view of Lemma 2.8. Let $K_{4.2} = \frac{K_{2.10}}{K_{4.1}}$, then the result in Theorem 4.2 is obtained by the fact that $L (t, S^2) = L (t, S^2 \setminus D_0)$ almost surely. \[\Box\]

### 4.2 Upper Bound for the Hausdorff Measure

Now we start working toward the upper bound for the exact Hausdorff measure of the level set $T^{-1}(t)$.

One important ingredient for establishing the upper bound for the exact Hausdorff measure of the level sets of $T$ is the following Proposition 4.3; the statement is similar to Proposition 4.1 of [28], but indeed much stronger. For any $x \in S^2$ and any $r_0 \in (0, \delta_0)$, let us consider the event $\Omega (x, r_0)$ defined by

$$\exists r \in (r_0^2, r_0) \text{ such that } \sup_{d(x,y) < r} \|T(x) - T(y)\| \leq 2Cw(r)$$

and $L (T(x), D(x, r)) > \frac{\pi}{C}\phi(r),$ \[\text{(33)}\]

where $w(r) = \rho_\alpha \left( r / \sqrt{\log \log r} \right)$, and $L (T(x), D(x, r))$ is the local time at $T(x)$ defined in the formula (13) in section 2. We have the following result:

Proposition 4.3 Under the conditions of Theorem 1.1, there exists a positive constant $K_{4.3}$ depending on $\alpha$, $d$ and $K_0$, such that for any $r_0 \in (0, \delta_0)$ and $x \in S^2$, we have

$$\mathbb{P} \{ \Omega_{r_0} (x, r_0, K_{4.3}) \} > 1 - \frac{1}{|\log r_0|^2}.$$  

The proof of Proposition 4.3 is quite involved and will be presented in Section 5. We now apply this proposition to prove the following upper bound.

Theorem 4.4 Under the conditions of Theorem 1.1, there exists a constant $K_{4.4} > 0$ depending on $\alpha$, $d$ and $K_0$ such that for every $t \in \mathbb{R}^d$, with probability one,

$$\phi - m (T^{-1}(t)) \leq K_{4.4} L (t, S^2).$$
Proof. Recall the event \( \Omega(x, r_0, C) \) defined in [33], for all integers \( p \geq 1 \), let
\[
R_p = \left\{ x \in \mathbb{S}^2 : \exists r \in \left[ 2^{-2^p}, 2^{-2^p-1} \right], \text{ such that } L(T(x), D(x, r)) > \frac{r}{\pi^{4/3}} \phi(r) \right\}
\]
and
\[
\Omega_{p,1} = \left\{ \omega : \nu(R_p) \geq \nu(\mathbb{S}^2)(1 - \frac{1}{2^{p-1}}) \right\}.
\]
In words, \( \Omega_{p,1} \) is the event that "a large portion of \( \mathbb{S}^2 \) consists of points at which \( T \) has the smallest oscillation". Recall that \( \nu(\mathbb{S}^2) = 4\pi \), thus the complement of \( \Omega_{p,1} \) is
\[
\Omega_{p,1}^c = \left\{ \nu(R_p) < 4\pi \left( 1 - \frac{1}{2^{p-1}} \right) \right\} = \left\{ \nu(R_p^c) > 4\pi \cdot \frac{1}{2^{p-1}} \right\}.
\]
By Markov’s inequality and Fubini’s theorem, we have
\[
\mathbb{P} \{ \Omega_{p,1}^c \} \leq \frac{2^{p-1}}{4\pi} \mathbb{E}(\nu(R_p^c)) = \frac{2^{p-1}}{4\pi} \int_{\mathbb{S}^2} \mathbb{P} \{ y \in R_p^c \} \, d\nu(y)
\]
\[
\leq \frac{2^{p-1}}{4\pi} \int_{\mathbb{S}^2} \frac{1}{2^{2(p-1)}} \, d\nu(y) \leq \frac{1}{2^{p-1}},
\]
where in the second inequality we have used the Proposition [33]. Therefore,
\[
\sum_{p=1}^{\infty} \mathbb{P} \{ \Omega_{p,1}^c \} < \infty. \tag{34}
\]
Before going to the next stage, we construct Voronoi cells on \( \mathbb{S}^2 \) for every integer \( k \geq 1 \) (see [24] as a reference). For \( k = 1 \), let \( \Xi_1 = \{ x_{1,1}, x_{1,2} \} \) where \( x_{1,1} \) and \( x_{1,2} \) are north and south poles respectively, and
\[
\mathcal{V}_1(x_{1,i}) = \{ y \in \mathbb{S}^2 : d(y, x_{1,i}) \leq d(y, x_{1,j}), \ j \neq i, j = 1, 2 \}
\]
for \( i = 1, 2 \). For \( k \geq 2 \), suppose \( \Xi_{k-1} \) is the centers of Voronoi cells of level \( k - 1 \) which has been chosen. Now choose a set of points \( \Xi_{k,i} = \{ x_{k,i_1}, \ldots, x_{k,i_l} \} \) in \( \mathcal{V}_{k-1}(x_{k-1,i}) \) for each \( i = 1, \ldots, N_{k-1} \) with \( N_{k-1} \) the cardinality of \( \Xi_{k-1} \), such that for any \( x_{k,i_j} \neq x_{k,i_j}', 2^{-k} \leq d(x_{k,i_j}, x_{k,i_j}') \leq 2^{1-k} \). The Voronoi cells of order \( k \) is then defined as follows:
\[
\mathcal{V}_k(x_{k,i}) = \{ y \in \mathcal{V}_{k-1}(x_{k-1,i}) : d(y, x_{k,i}) \leq d(y, x_{k,i_j}), \text{ for any } i_j \neq i_i \},
\]
Label the points in \( \Xi_{k,1}, \ldots, \Xi_{k,N_k} \) by \( x_{k,1}, \ldots, x_{k,N_k} \) with \( N_k \) the cardinality of the union set \( \cup_i \Xi_{k,i} \), and denote by \( \Xi_k = \{ x_{k,1}, \ldots, x_{k,N_k} \} \). The procedure of constructing Voronoi cells of order \( k \) and higher can be iterated. Obviously, these cells are nested for different \( k \)'s, and non-overlapping with \( \mathbb{S}^2 = \cup_i \mathcal{V}_k(x_{k,i}) \) for every \( k \). The cell \( \mathcal{V}_k(x_{k,i}) \) is called "good" if
\[
\sup_{x,y \in \mathcal{V}_k(x_{k,i})} \| T(x) - T(y) \| \leq 2K_{4,3}w(2^{-k}) \tag{35}
\]
\[\text{(35)}\]
as well as
\[ L(T(x_{k,i}), V_k(x_{k,i})) > \frac{\pi}{K_{4,3}} \phi(2^{-k}). \]

By Proposition 4.3, we can find a family \( H_{1,p} \) of good cells for some order \( k \in [2^{p-1}, 2^p] \) that covers \( R_p \); we denote by \( H_{2,p} \) the family of cells of order \( p \) that are not contained in any cell of \( H_{1,p} \), that is \( H_{2,p} \subset R^c_p \). Therefore, when \( \Omega_{p,1} \) occurs, the number of cells in \( H_{2,p} \) is at most \( M_p \) with
\[ 2^{2p+1} M_p \leq \frac{4\pi}{2^{p-1}}, \]
(36)

Let \( \varepsilon = C_{4,2}(2^{-2p})^{(\alpha/2-1)d/2p/2} \) with some positive constant \( C_{4,2} \) to be determined. Define \( \Omega_{p,2} \) to be the event that under the condition that
\[ \sup_{y, y' \in V_p(x_{p,i})} \| T(y) - T(y') \| \leq C_{4,2}(2^{-2p}2^{p/2})^{\alpha/2-1}, \]

it also follows that
\[ L(T(x_{p,i}), D(x_{p,i}, 2^{-2p})) \geq \frac{\pi(2^{-2p})^2}{\varepsilon} \]
holds for each cell of order \( p \) in \( S^{2n} \). Applying Lemma 2.3.1, we obtain that for some constants \( C_{4,2} > K_{2,7}\sqrt{K_{2,1}} \), the probability \( P \{ \Omega_{p,2}^c \} \) is bounded by
\[ \sum_{x_{p,i} \in H_{2,p} \cap \Xi_p} P \left\{ L(T(x_{p,j}), D(x_{p,j}, 2^{-2p})) < \nu(D(x_{p,j}, 2^{-2p})) \right\} \leq \sum_{x_{p,j} \in H_{2,p} \cap \Xi_p} P \left\{ \mu \left( B(T(x_{p,j}), \frac{\varepsilon}{2}), D \right) < \nu(D(x, 2^{-2p})) \right\} \leq \sum_{x_{p,j} \in H_{2,p} \cap \Xi_p} P \left\{ \sup_{y, y' \in D(x_{p,j}, 2^{-2p})} \| T(y) - T(y') \| > C_{4,2}(2^{-2p})^{\alpha/2-1}2^{p/2} \right\} \leq M_p \exp \{-2^n\} \leq 32\pi \exp \{-2^n(1 - \ln 2) - p \ln 2\}. \]

Hence we have
\[ \sum_{p=1}^{\infty} P \{ \Omega_{p,2}^c \} < \infty. \]
(37)

Finally, let \( \Omega_{p,3} \) be the event: for any \( x \in S^2 \) and \( t \in \mathbb{R}^d \),
\[ \sup_{\| s-t \| \leq 2K_{4,3}(2^{-2p})} \left| L(s, D(x, 2^{-2p})) - L(t, D(x, 2^{-2p})) \right| \leq \frac{\pi}{2K_{4,3}} \phi(2^{-2n}). \]
(38)

By Lemma 2.3, we have for any even integer \( n \geq 2 \)
\[ P \{ \Omega_{p,3}^c \} \leq \frac{(K_{2,3})^n}{2K_{4,3}w(2^{-2p})^{n\gamma}} \frac{2K_{4,3}w(2^{-2p})^{n\gamma} \phi(2^{-2p})^{(n-1)\eta+2}}{2\pi K_{4,3}} \]
\[ \leq C_{4,3}(2^{-2p})^{2-\eta-n\gamma(\alpha/2-1)} \frac{p^n(\alpha/2-1)(d+\gamma)}{p^n(\alpha/2-1)(d+\gamma)}. \]
where the constant \( C_{4,3} \) is positive and depends on \( n, K_{2,3} \) and \( K_{4,3} \). Let \( n \) be large enough so that \( n\gamma (\alpha/2 - 1) \geq 2 - \eta \), then we obtain

\[
\sum_{p=1}^{\infty} \mathbb{P}\{ (\Omega_{p,3})^c \} < \infty. \tag{39}
\]

Now set \( \mathcal{H}_p = \mathcal{H}_{1,p} \cup \mathcal{H}_{2,p} \). This family is well-defined for all \( p \geq 1 \), and it is a non-overlapping covering of \( \mathbb{S}^2 \). Set

\[
\begin{align*}
    r_A &= 2K_{4,3}w(2^{-2^p}), \quad \text{if } A \in \mathcal{H}_{1,p} \text{ and } A \text{ is of order } k \in [p, 2p], \\
    r_A &= C_{4,2}(2^{-2^p})(\alpha/2 - 1)d2^{-p/2}, \quad \text{if } A \in \mathcal{H}_{2,p}.
\end{align*}
\]

For each cell \( A \in \mathcal{H}_p \), denote by \( |A| \) the diameter of \( A \). We pick its center point \( x_A \), and define \( \Omega_A \) the set of events such that

\[
\left\{ \omega : \| T(x_A) - t \| \leq r_A, \, L(t, A) \geq \begin{cases} 
\frac{\pi}{2K_{4,3}} \phi \left( \frac{1}{2} |A| \right), & \text{for } A \in \mathcal{H}_{1,p} \\
\frac{\pi}{C_{4,2}K_{4,3}} \phi \left( \frac{1}{2} |A| \right), & \text{for } A \in \mathcal{H}_{2,p}
\end{cases} \right\}. \tag{35}
\]

Also, denote by \( \mathcal{F}_p \) the subfamily of \( \mathcal{H}_p \) such that

\[
\mathcal{F}_p = \{ A \in \mathcal{H}_p : \Omega_A \text{ occurs} \}.
\]

Letting \( \Omega_p = \Omega_{p,1} \cap \Omega_{p,2} \cap \Omega_{p,3} \), we have

\[
\sum_{p=1}^{\infty} \mathbb{P}\{ \Omega_p^c \} < \infty, \tag{40}
\]

in view of (34), (37) and (39). We claim that

For \( p \) large enough, on \( \Omega_p, \mathcal{F}_p \) covers \( T^{-1}(t) \). \tag{41}

To see this, consider that for any \( x \in T^{-1}(t) \), if \( x \) belongs to some cell \( A \) of order \( k \) in \( \mathcal{H}_{1,p} \), that is \( 2^{p-1} \leq k \leq 2^p \), then recall (34), we have that, for \( p \) large enough,

\[
\| T(x_A) - t \| = \| T(x_A) - T(x) \| \leq r_A,
\]

and

\[
L(t, A) \geq L(T(x_A), A) - |L(t, A) - L(T(x_A), A)| \geq \frac{\pi}{K_{4,3}} \phi \left( \frac{1}{2} |A| \right) - \frac{\pi}{2K_{4,3}} \phi \left( \frac{1}{2} |A| \right) \geq \frac{\pi}{2K_{4,3}} \phi \left( \frac{1}{2} |A| \right),
\]

in view of (38). Otherwise, \( x \) belongs to some cell \( A \) of order \( 2^p \) in \( \mathcal{H}_{2,p} \), and the inequality above is then readily seen in view of the definition of \( \Omega_{p,2} \). Therefore \( A \in \mathcal{F}_p \), and (41) is proved.
Now we see that on $\Omega_p$,
\[ \sum_{A \in H_{1,p}} \phi(|A|) \leq \pi^{-1} K_{4,3} 2^{3-d(\alpha/2-1)} \sum_{A \in H_{1,p}} L(t, A), \]
and
\[ \sum_{A \in H_{2,p}} \phi(|A|) \leq \pi^{-1} C_{4,2} 2^{3-d(\alpha/2-1)+p/2} 2^{d(\alpha-2)/4} \sum_{A \in H_{2,p}} L(t, A). \]

In the meantime, recalling Proposition 4.1 and the inequality (36), we have
\[ 2^{p/2} p^{d(\alpha-2)/4} \sum_{A \in H_{1,p}} L(t, A) \leq 4\pi K_{4,1} 2^{p/2} p^{d(\alpha-2)/4} 2^{2p+1 - \frac{1}{p-1}} \phi(2^{-2p}) \]
\[ \leq 4\pi K_{4,1} \frac{p^{d(\alpha-2)/4}}{2^{p/2-1}} \left[ 2^{2p+1} \phi(2^{-2p}) \right] \leq \frac{1}{1 - 2^{1-p}} \sum_{A \in H_{1,p}} \phi(|A|), \]
with $p$ large enough such that $4\pi K_{4,1} 2^{p(\alpha-2)/4} < 1$. Thus
\[ \sum_{A \in H_p} \phi(|A|) \leq K_{4,4} \sum_{A \in H_{1,p}} L(t, A) \leq K_{4,4} \sum_{A \in H_p} L(t, A), \]
with $K_{4,4} > 0$ depending on $\alpha, d$, $K_{4,3}$ and $C_{4,2}$. Hence by the Borel-Cantelli Lemma, almost surely we have
\[ \phi-m \left( T^{-1}(t) \right) \leq \liminf_{p \to \infty} \sum_{A \in H_p} \phi(|A|) \leq K_{4,4} L \left( t, S^2 \right), \]
in view of (10) and (11), which completes the proof. ■

**Proof of 8 in Theorem 1.1** The proof follows immediately by combining Theorems 4.2 and 4.4. ■

### 4.3 Two Examples

In this subsection, we illustrate Theorem 1.1 by two examples.

**Example 4.5** Let $H > 0$ with $H \neq 1$. Consider the centered isotropic Gaussian random field $W_H^0 = \{W_H^0(x), x \in S^2\}$ with covariance
\[ K_H \left( d_{S^2}(x, y) \right) = \text{Cov} \left( W_H^0(x), W_H^0(y) \right) = \int_{\mathbb{R}^+} \psi(\theta) \left( d_{S^2}(x, y), r \right) r^{2H-3} dr, \]
where
\[ \psi(\theta) = \begin{cases} \psi & \text{if } 0 < H < 1 \\ \psi - 4\pi & \text{if } H > 1 \end{cases} \]
with
\[ \psi (dS^2(x,y), r) = \int_{S^2} 1 \{dS^2(x,z) < r\} 1 \{dS^2(y,z) < r\} dv(z). \]

Notice that if \( dS^2(x,y) \geq 2r \), then \( \psi (dS^2(x,y), r) = 0 \). So we will just consider the case \( dS^2(x,y) \in (0, 2r) \). In this circumstance, careful calculations show that, the covariance \( K_H (dS^2(x,y)) \) can be represented as follows:
\[
K_H (dS^2(x,y)) = \sum_{\ell=1}^{+\infty} \frac{2\ell + 1}{4\pi} (C_\ell + d_\ell) P_\ell ((x,y)),
\]

where
\[ C_{4,4}^{-1} \ell^{-(2H+2)} \leq C_\ell \leq C_{4,4} \ell^{-(2H+2)}. \]
and \( d_\ell = 0 \) with
\[ C_{4,5}^{-1} (2\ell + 1)^{-2} (2\pi)^{-2\ell-1} \leq d_{2\ell+1} \leq C_{4,5} (2\ell + 1)^{-2} (2\pi)^{-2\ell-1}, \]
for \( \ell = 1, 2, \ldots \).

It is readily seen that the spherical random field \( W_H^0 \) satisfies Condition(A) with angular power spectrum index \( \alpha = 2H+2 \) if \( 0 < H < \frac{1}{2} \), and \( \alpha \geq 4 \) if \( H \geq \frac{1}{2} \) and \( H \neq 1 \). Therefore, the level sets of the random field \( W_H = (W_H^1, \ldots, W_H^d) \) with \( W_H^1, \ldots, W_H^d \) independent copies of \( W_H^0 \), is nonempty a.s. if and only if \( 0 < H < 1 \) and \( Hd < 2 \). Moreover, its exact Hausdorff measure of the level set \( W_H^{-1}(t) \) at any value \( t \in \mathbb{R}^d \) can be controlled upper and lower by its local time \( L(t, S^2) \), a.s. with Hausdorff dimension \( 2 - Hd \) for \( H \neq \frac{1}{2} \) in view of Theorem 1.2 in this paper and Theorem 1.2 in [23]. See [13] for more information on the statistical properties of the random field \( W_H^0 \).

**Example 4.6** Consider the zero-mean, isotropic Gaussian random field \( T_0 = \{T_0(x), x \in S^2\} \) with covariance
\[ C(x,y) = 1 - \frac{2}{\pi} dS^2(x,y) \]
for any \( x, y \in S^2 \) (c.f. [9], Example 3.3). By careful calculations, we can prove that,
\[
C(x,y) = C_1 P_1 ((x,y)) + \sum_{\ell=1}^{+\infty} (4\ell + 3) C_\ell P_{2\ell+1} ((x,y)) ,
\]
where for each \( \ell > 1 \), the angular power spectrum
\[
C_\ell = \sum_{n=\ell}^{\infty} \frac{1}{(2n + 2\ell + 3)!} \frac{[(2n-1)!!]^2}{(2n-2\ell)!!} ,
\]
and moreover,
\[
\frac{1}{8\sqrt{\pi}} \ell^{-1} \leq C_\ell \leq \frac{1}{4} \ell^{-1}.
\]

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That is, its angular power spectrum index \( \alpha = 1 \), which is less than 2. Thus, the condition \( \alpha \) does not hold, which leads to that Theorem 1.1 cannot be applied to this spherical random field. Actually, even the representation (2) does not hold in the sense of (3) and (4), as the right-hand side of (3) and (4) do not converge.

5 Proof of Proposition 4.3

For any two integers \( 1 \leq L < U \leq \infty \), we introduce the band-limited random field \( T_{L,U} = (T_{1}^{L,U}, \ldots, T_{d}^{L,U}) \) on \( S^2 \), where \( T_{k}^{L,U}, k = 1, \ldots, d \), are independent copies of

\[
T_{0}^{L,U}(x) = \sum_{\ell=L}^{U} \sum_{m=-\ell}^{\ell} a_{l\ell} Y_{\ell m}(x), \quad x \in S^2.
\]  

Observe that \( T_{0}^{L,U}(x) \) and \( T_{k}^{L',U'}(x) \) are independent for \( L < U < L' < U' \) in view of the orthogonality properties of the Fourier components of the field \( T_{0}(x) \) and the assumption of Gaussianity. Let \( D(z, r) \) be an open disk on \( S^2 \), we start by proving the following lemma.

**Lemma 5.1** Under the conditions of Theorem 1.1, there exists a positive constant \( K_{5,1} \) depending on \( \alpha, d \) and \( K_{0} \), such that for any \( 0 < r < \delta_0 \) and \( \varepsilon > 0 \), we have

\[
P \left\{ \sup_{x, y \in D(z, r)} \| T_{L,U}(x) - T_{L,U}(y) \| \leq \varepsilon \right\} \geq \exp \left( -K_{5,1} \frac{r^2}{\varepsilon^{4/(\alpha-2)}} \right),
\]

**Proof.** The canonical metric \( d_{T_{L,U}} \) on \( S^2 \) is defined by

\[
d_{T_{L,U}}(x, y) = \sqrt{\mathbb{E} \| T_{L,U}(x) - T_{L,U}(y) \|^2} = \sqrt{\mathbb{E} \left| T_{0}^{L,U}(x) - T_{0}^{L,U}(y) \right|^2}.
\]

By the lower bound in the inequalities (11) and the fact that \( d_{T_{L,U}}(x, y) \leq d_T(x, y) \) for any \( x, y \in S^2 \) with \( d_S(x, y) < \delta \), we have

\[
d_{T_{L,U}}(x, y) \leq \sqrt{d_{2,1}}(x, y).
\]

In the meantime, recall the metric entropy, it follows immediately that

\[
N_{d_{T_{L,U}}}(D(z, r), \varepsilon) \leq C_{5,1} \frac{r^2}{\varepsilon^{4/(\alpha-2)}}.
\]

where \( C_{5,1} \) is a positive constant depending on \( \alpha, d \) and \( K_{2,1} \). Therefore, the approximation (43) is derived by exploiting Lemma 2.5.

Let \( T^\Delta \) be the random field defined by

\[
T^\Delta = T - T_{L,U}.
\]
Meanwhile, for any \( r \in (0, \delta_0) \), take \( L = [B^{-\beta}r^{-1}] \) and \( U = [B^{1-\beta}r^{-1}] \), where the constants \( B, \beta \) satisfy \( \beta \in (0, \alpha/2 - 1] \) and \( \max \left\{ 1, (4K_{2,4})^{1/\alpha} \right\} < B < r^{-1} \) with \( K_{2,4} \) the constant in Lemma 2.4. Here \([\cdot] \) denotes integer part as usual. Then we have the following estimate on the tail probability of maximal oscillation of \( T^\Delta \).

**Lemma 5.2** Under the conditions of Theorem 1.1, there exists a positive constant \( K_{5,2} \) depending on \( \alpha, d \) and \( K_0 \), such that for any \( 0 < r < \delta_0, 0 < \beta \leq \alpha/2 - 1 \) and

\[
 u > K_{5,2} B^{-\beta/2} \sqrt{\log B} r^{\alpha - 2},
\]

we have

\[
 \Pr \left\{ \sup_{x,y \in D(z,r)} \|T^\Delta(x) - T^\Delta(y)\| \geq u \right\} \leq \exp \left( -\frac{1}{K_{5,2}} \frac{B^{\beta(4-\alpha)}u^2}{r^{\alpha - 2}} \right).
\]

**Proof.** Like in many other arguments in this paper, we start by considering a suitable Gaussian metric \( d_{T^\Delta} \) defined on \( D(z,r) \subset S^2 \) by

\[
d_{T^\Delta}(x,y) := \left[ \mathbb{E} \|T^\Delta(x) - T^\Delta(y)\|^2 \right]^{1/2}.
\]

Once again, we have \( d_{T^\Delta}(x,y) \leq \sqrt{d_{K_{2,4} \rho_\alpha}(x,y)} \). A simple metric entropy argument yields

\[
 N_{d_{T^\Delta}}(D(z,r), \varepsilon) \leq C_{5,1} \varepsilon^{4/\alpha - 2}.
\]

More precisely, recall

\[
d_{T^\Delta}^2(x,y) = d \left( \sum_{\ell=0}^{L-1} + \sum_{\ell=U+1}^\infty \frac{2\ell + 1}{4\pi} C_{\ell} \{1 - P_{\ell}(\cos \theta)\} \right),
\]

where \( \theta = d_{S^2}(x,y) \). Let \( 0 < \theta < r \), by Lemma 2.4 we obtain

\[
d_{T^\Delta}^2(x,y) \leq d_{K_{2,4}} \left( L^{4-\alpha} \theta^2 + U^{2-\alpha} \right) \leq d_{K_{2,4}} \left( B^{-\beta(4-\alpha)} + B^{-(1-\beta)(\alpha-2)} \right) r^{\alpha - 2} \leq d_{K_{2,4}} B^{-\beta(4-\alpha)} r^{\alpha - 2} := |f(B)|^2 r^{\alpha - 2},
\]

where

\[
f(B) = \sqrt{d_{K_{2,4}} B^{-\frac{\alpha}{2}(4-\alpha)}}.
\]

Hence, if we let

\[
 \overline{d} := \sup \{ d_{T^\Delta}(x,y) : x, y \in D(z,r) \},
\]

\[
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\]

\[
 \Pr \left\{ \sup_{x,y \in D(z,r)} \|T^\Delta(x) - T^\Delta(y)\| \geq u \right\} \leq \exp \left( -\frac{1}{K_{5,2}} \frac{B^{\beta(4-\alpha)}u^2}{r^{\alpha - 2}} \right).
\]
which obviously satisfies \( \bar{d} \leq f(B)r^{\bar{d}-1} \), then

\[
\int_0^1 \sqrt{\log N_{d+\Delta} (D(z, r), \varepsilon)} \, d\varepsilon \leq \int_0^1 \sqrt{\log C_{5.1} r^{\frac{r^2}{e^{3/\alpha-2}}}} \, d\varepsilon \\
\leq \frac{2}{\sqrt{\alpha - 2}} \sqrt{dK_{2.1}} r^{\frac{\bar{d}-1}{\alpha - 2}} \int_0^{+\infty} u \exp \left( -e^{-u^2} \right) \\
\leq \frac{4f(B)}{\sqrt{\alpha - 2}} \sqrt{dK_{2.1}} f(B) r^{\frac{\bar{d}-1}{\alpha - 2}}.
\]

By Lemma 2.6, we derive that, for any \( u \geq C_{5.2}B^{-\beta(2-\alpha/2)} \sqrt{\log B} r^{\alpha/2-1} \) with some constant \( C_{5.2} \) depending on \( \alpha \) and \( K_{2.7} \), it holds that

\[
P \left\{ \sup_{x, y \in D(z, r)} \| T^\Delta (x) - T^\Delta (y) \| \geq u \right\} \\
\leq P \left\{ \sup_{x, y \in D(z, r)} \| T^\Delta (x) - T^\Delta (y) \| \geq K_{2.7} \left( \frac{u}{2K_{2.7}} + \int_0^1 \sqrt{\log N_{d+\Delta} (D(z, r), \varepsilon)} \, d\varepsilon \right) \right\} \\
\leq \exp \left( - \frac{u^2}{4K_{2.7}^2 \| f(B) \|^2 r^{\alpha-2}} \right).
\]

The proof is then completed. \( \square \)

Now recalling the representation of local time (14), we introduce the following random field for any \( D = D(x, r) \subseteq S^2, x \in S^2 \)

\[
L^{L, U} (T(x), D) = \frac{1}{2\pi} \int_{D \times D} \exp \{ i\xi^T (T^{L, U}(y) - T^{L, U}(x)) \} \, d\nu(y) d\xi.
\]

Obviously, \( L^{L, U} (T(x), D) \) and \( L^{L', U'} (T(x), D) \) are independent for \( L < U < L' < U' \). We can now prove that

Lemma 5.3 Under the conditions of Theorem 1.1 there exist positive constants \( K_{5.3} \) and \( B_0 \) depending on \( K_{1.0}, d \) and \( \alpha \), such that for any \( x \in S^2 \) fixed, \( r \in (0, \delta_0) \) and \( B > B_0 \), we have for any \( A > 0 \),

\[
P \left\{ \| L (T(x), D) - L^{L, U} (T(x), D) \| \geq A^{-\phi(r)} \right\} \leq K_{5.3} A^{2.5} B^{-\kappa \beta(4-\alpha)} \left( \log |\log r| \right)^{(\alpha-2)d},
\]

where \( \kappa = \min \left\{ 2, \frac{4-d(\alpha-2)}{(\alpha-2)} \right\} \) and \( \phi(\cdot) \) is the function defined in (4).

Proof. For every \( x \in S^2 \) fixed, we introduce random fields \( Z_l = (Z_{l,1}, \ldots, Z_{l,d}) \), \( l = a, b, \Delta \), with \( Z_{a,k}, Z_{b,k}, Z_{\Delta,k}, k = 1, \ldots, d \), independent copies of \( Z_{a,0}, Z_{b,0}, \) and \( Z_{\Delta,0} \) where the random fields \( Z_{l,0} = \{ Z_{l,0}(y), y \in S^2 \}, l = a, b, \Delta, \) are defined as follows:

\[
Z_{a,0}(y) = T_{0}(y) - T_{0}(x), \\
Z_{b,0}(y) = T_{0}^{L, U}(y) - T_{0}^{L, U}(x), \\
Z_{\Delta,0}(y) = Z_{a,0}(y) - Z_{b,0}(x).
\]
Recall the representations (14) and (46), we have
\[
E \left[ L(\mathbf{T}(x), D) - L^{L,U}(\mathbf{T}(x), D) \right]^2 = \frac{1}{(2\pi)^2} \int_{D^2} \mathcal{II}(y_1, y_2) d\nu(y_1) d\nu(y_2), \quad (47)
\]
where
\[
\mathcal{II}(y_1, y_2) = \int_{\mathbb{R}^{2d}} \prod_{j=1,2} \left( E \exp \left\{ i \sum_{k=1}^d \xi_{k,j} Z_{a,k}(y_j) \right\} - E \exp \left\{ i \sum_{k=1}^d \xi_{k,j} Z_{b,k}(y_j) \right\} \right) d\xi.
\]
Now let us focus on the term \(\mathcal{II}(y_1, y_2)\). For any \(y_1, y_2 \in S^2\), denote by \(\sigma^2_{l,i} =: \text{Var}(Z_{l,0}(y_i))\), \(l = a, b, \Delta, \ i = 1, 2\), and \(\gamma_l =: \text{Cov}(Z_{l,0}(y_1), Z_{l,0}(y_2))\), \(l = a, b, \Delta\). Let \(\Upsilon_l\) be covariance matrices of \((Z_{l,0}(y_1), Z_{l,0}(y_2))\), that is
\[
\Upsilon_l =: \begin{bmatrix} \sigma^2_{l,1} & \gamma_l \\ \gamma_l & \sigma^2_{l,2} \end{bmatrix}
\]
for \(l = a, b, \Delta\). In the meantime, denote by
\[
\tilde{\Upsilon}_l =: \begin{bmatrix} \sigma^2_{l,1} & \gamma_l \\ \gamma_l & \sigma^2_{l,2} \end{bmatrix}, \ l, l' = a, b, \text{ and } l \neq l',
\]
then a standard calculation yields that
\[
\mathcal{II}(y_1, y_2) = \sum_{l=a,b} \frac{1}{(\det \Upsilon_l)^{d/2}} - \sum_{l=a,b} \frac{1}{(\det \tilde{\Upsilon}_l)^{d/2}}
\]
Notice that \(\Upsilon_a = \Upsilon_b + \Upsilon_\Delta, \ \tilde{\Upsilon}_a = \tilde{\Upsilon}_b + \text{diag}(\sigma^2_{\Delta,1}, 0)\) and \(\tilde{\Upsilon}_b = \tilde{\Upsilon}_b + \text{diag}(0, \sigma^2_{\Delta,2})\).
Moreover, careful calculations yield
\[
\det \Upsilon_a \geq \det \Upsilon_b + \det \Upsilon_\Delta + |\sigma_{b,1}\sigma_{\Delta,2} - \sigma_{\Delta,1}\sigma_{b,2}|^2 \geq \det \Upsilon_b
\]
\[
\det \tilde{\Upsilon}_a = \det \Upsilon_b + \sigma^2_{\Delta,1}\sigma^2_{b,2}, \text{ and } \det \tilde{\Upsilon}_b = \det \Upsilon_b + \sigma^2_{b,1}\sigma^2_{\Delta,2}.
\]
Therefore,
\[
\mathcal{II}(y_1, y_2) \leq \frac{2}{(\det \Upsilon_b)^{d/2}} - \frac{1}{(\det \Upsilon_b + \sigma^2_{\Delta,1}\sigma^2_{b,2})^{d/2}} - \frac{1}{(\det \Upsilon_b + \sigma^2_{b,1}\sigma^2_{\Delta,2})^{d/2}}. \quad (48)
\]
Now let \(\theta_i = dS^2(y_i, x), i = 1, 2\), then
\[
\sigma^2_{\Delta,i} = \left( \sum_{l=1}^{L-1} + \sum_{l=U+1}^{\infty} \right) \frac{2\ell + 1}{4\pi} C_\ell \left\{ 1 - P_\ell(\cos \theta_i) \right\},
\]
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whence by exploiting Lemmas 2.4 we obtain that
\[
\sigma_{\Delta,i}^2 \leq K_{2,4} \left( L^{4-\alpha} \beta_i^2 + U^{2-\alpha} \right) \leq K_{2,4} \left( B^{-\beta(4-\alpha)} + B^{-(1-\beta)(\alpha-2)} \right) \theta_i^{\alpha-2} \leq 2d^{-1} |f(B)|^2 \theta_i^{\alpha-2}. \tag{49}
\]
in view of (44). Let \( B > B_0 \) such that \(|f(B_0)|^2 < \frac{d}{4} K_{2,1}^{-1} \) as well as \( L = L(B_0, r), U = U(B_0, r) \) large enough so that \(|\theta_b| \leq \frac{1}{2} |\theta_a|\), then
\[
\sigma_{b,i}^2 = \sigma_{\Delta,i}^2 - \sigma_{\Delta,2}^2 \geq \frac{1}{2} \sigma_{\Delta,i}^2 \geq \frac{1}{2} K_{2,1}^{-1} \theta_i^{\alpha-2}, \quad i = 1, 2, \tag{50}
\]
which leads to
\[
\text{det } \mathbf{Y}_b = \sigma_{b,1}^2 \sigma_{b,2}^2 - \theta_b)^2 \geq \frac{1}{4} \left[ \sigma_{a,1}^2 \sigma_{a,2}^2 - \theta_a)^2 \right] = \frac{1}{4} \text{det } \mathbf{Y}_a. \tag{51}
\]
Thus, combining inequalities (48), (49), (50) and (51), we obtain
\[
II(y_1, y_2) \leq \frac{2}{(\text{det } \mathbf{Y}_b)^{d/2}} - \left( \frac{2}{\text{det } \mathbf{Y}_b + 2d^{-1} |f(B)|^2 \theta_1^{\alpha-2} \theta_2^{\alpha-2}} \right)^{d/2} \leq \frac{4d^{-1} C_{5,3} |f(B)|^2 \theta_1^{\alpha-2} \theta_2^{\alpha-2}}{\text{det } \mathbf{Y}_b \left( \text{det } \mathbf{Y}_b + 2d^{-1} |f(B)|^2 \theta_1^{\alpha-2} \theta_2^{\alpha-2} \right)^{d/2}} \leq \frac{2d^{4} C_{5,3} |f(B)|^2 \theta_1^{\alpha-2} \theta_2^{\alpha-2}}{\text{det } \mathbf{Y}_a \left( \text{det } \mathbf{Y}_a + 8d^{-1} |f(B)|^2 \theta_1^{\alpha-2} \theta_2^{\alpha-2} \right)^{d/2}} \tag{52}
\]
where the last but one inequality used the fact that \((1 + x)^{d/2} \leq 1 + C_{5,3} x \) for any \( x \in (0, \delta_0) \) with \( C_{5,3} = \frac{d}{4} \max \{ (1 + \delta_0)^{d/2-1}, 1 \} \). Recall (12) in Lemma 2.1 and Corollary 2.2 we have
\[
\text{det } \mathbf{Y}_a = \text{Var} \left( Z_{a,0}(y_1) \right) \text{Var} \left( Z_{a,0}(y_2) \right) \left| Z_a(y_1) \right| \left| Z_a(y_2) \right| \geq K_{2,1}^{-1} K_{2,2} \left[ \rho_a(\theta_1) \right]^2 \min \left( \rho_a^2(\theta_2), \rho_a^2(d_{\theta^2}(y_2, y_1)) \right). \tag{53}
\]
Replacing \( \text{det } \mathbf{Y}_a \) in the inequality (52) with the term on the right side of inequality (53) above, we obtain
\[
II(y_1, y_2) \leq C_{5,4} \frac{|f(B)|^2}{\theta_1^{\alpha-2}} \cdot A(x_2) \tag{54}
\]
where \( C_{5,4} \) is a positive constant depending on \( \alpha, d, C_{5,3}, K_{2,1}, K_{2,2}, \) and
\[
A(x_2) = \theta_2^{\alpha-2} \left[ \min \left( \theta_2^{\alpha-2}, d_{\theta^2}^{\alpha-2}(x_2, x_1) \right) \right]^{-1} \cdot \left\{ \min \left( \theta_2^{\alpha-2}, d_{\theta^2}^{\alpha-2}(x_2, x_1) \right) + 8d^{-1} |f(B)|^2 \theta_2^{\alpha-2} \right\}^{-d/2}.
\]

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Similar to the argument in the proof of Lemma 3.2 in \[23\], we define the following two sets that are disjoint except on the boundary,

\[
\Gamma_\tau = \{ y \in D : d_{g^2}(y, \tau) = \min \{ d_{g^2}(y, \tau'), \tau' = x, y_1 \} \}
\]

for \( \tau = x, y_1 \). Recall \( \theta_i = d_{g^2}(y_i, x) \), \( i = 1, 2 \), then it follows that

\[
\int_{\Gamma_x} A(x_2) d\nu(x_2) \leq \int_0^{2\pi} \int_0^{\tau(\phi)} \theta_2^{-(\alpha-2)d/2} \sin \theta_2 d\theta_2 d\phi \leq \frac{2}{4 - (\alpha-2)d} \int_0^{2\pi} [r(\phi)]^{\frac{1}{4}[-(\alpha-2)d]} d\phi \leq C_{5,5} r^{\frac{1}{2}[-(\alpha-2)d]}
\]

where \( C_{5,5} = \frac{4\pi}{4 - (\alpha-2)d} \), and hence,

\[
|f(B)|^2 \int_D \int_{\Gamma_x} \frac{A(y_2)}{\theta_1^{d(\alpha-2)/2}} d\nu(y_2) d\nu(y_1) \leq 2\pi C_{5,5} |f(B)|^2 r^{\frac{1}{4}[-(\alpha-2)d]} \int_0^{\tau} \theta_1^{\frac{1}{2}[-(\alpha-2)]} \sin \theta_1 d\theta_1 \leq C_{5,5} |f(B)|^2 r^{[-(\alpha-2)d]}.
\]

(55)

For \( \Gamma_{y_1} \), we split it into two domains,

\[
\Gamma_{y_1}^1 = \left\{ y \in \Gamma_{y_1} : d_{g^2}(y, y_1) \leq \left( 8d^{-1} |f(B)|^2 \right)^{1/(\alpha-2)} \theta_2 \right\},
\]

and

\[
\Gamma_{y_1}^2 = \left\{ y \in \Gamma_{y_1} : d_{g^2}(y, x_1) > \left( 8d^{-1} |f(B)|^2 \right)^{1/(\alpha-2)} \theta_2 \right\}.
\]

Denote by \( \theta_{12} = d_{g^2}(y_2, y_1) \) and let \( |f(B)|^2 \leq d^{-\alpha-1} \) as well as \( B > B_0 \), then by the triangle inequality

\[
\theta_1 - \theta_{12} \leq \theta_2 \leq \theta_1 + \theta_{12},
\]

we have \( \frac{1}{2} \theta_1 \leq \theta_2 \leq 2 \theta_1 \) for \( y_2 \in \Gamma_{y_1}^1 \). Hence,

\[
\int_{\Gamma_{y_1}^1} A(y_2) d\nu(y_2) \leq \frac{2\pi (2\theta_1)^{\alpha-2}}{\left\{ 8d^{-1} |f(B)|^2 \left( \theta_1/2 \right)^{\alpha-2} \right\}^{d/2}} \int_0^{2(8d^{-1} |f(B)|^2)^{1/(\alpha-2)} \theta_1} \theta_1^{2-\alpha} \sin \theta_1 d\theta_1 d\theta_1 \leq C_{5,6} |f(B)|^{-\frac{4-d(\alpha-2)}{\alpha-2}} \theta_1^{\frac{1}{2}[-(\alpha-2)d]},
\]

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with $C_{5,6}$ a positive constant depending on $\alpha, d$, which leads to

\[
|f(B)|^2 \int_D \int_{\mathbb{R}^2_+} \frac{A(y_2)}{\theta_1^{d(\alpha-2)/2}} d\nu(y_2) d\nu(y_1) \\
\leq C_{5,6} |f(B)|^{\frac{4-d(\alpha-2)}{(\alpha-2)}} \int_0^\tau \theta_1^{d(\alpha-2)} \sin \theta_1 d\theta_1 \\
\leq \frac{C_{5,6}}{4-d(\alpha-2)} |f(B)|^{\frac{4-d(\alpha-2)}{(\alpha-2)}} r^{4-d(\alpha-2)}. \tag{56}
\]

Meanwhile,

\[
\int_{\mathbb{R}^2_+} A(y_2) d\nu(y_2) \\
\leq r^{\alpha-2} \int_0^{2\pi} \int_0^{r(\phi)} \theta_1^{\alpha-2}(d/2+1) \sin \theta_1 d\theta_1 d\phi \\
\leq r^{\alpha-2} \int_0^{2\pi} \left[ r(\phi) \right]^{2-(\alpha-2)(d/2+1)} - C_{5,7} |f(B)|^{\frac{4-d(\alpha-2)(d/2+1)}{(\alpha-2)}} d\phi \\
\leq 2\pi \left( r^{\alpha-2} - C_{5,7} r^{\alpha-2} |f(B)|^{\frac{4-d(\alpha-2)}{(\alpha-2)}} \right),
\]

with $C_{5,7}$ a positive constant depending on $\alpha, d$, and hence

\[
|f(B)|^2 \int_D \int_{\mathbb{R}^2_+} \frac{A(y_2)}{\theta_1^{d(\alpha-2)/2}} d\nu(y_2) d\nu(y_1) \\
\leq 4\pi^2 |f(B)|^2 r^{\frac{2}{4-d(\alpha-2)}} \int_0^\tau \theta_1^{d(\alpha-2)} \sin \theta_1 d\theta_1 \\
\leq 4\pi^2 |f(B)|^2 r^{4-d(\alpha-2)}. \tag{57}
\]

Recall $f(B) = \sqrt{dK_{2,4}B^{-\beta(2-\alpha/2)}}$ and let $\kappa = \min \left\{ 2, \frac{4-d(\alpha-2)}{(\alpha-2)} \right\}$, then combining the inequalities (55), (56) and (57) together with (47) and (54), we have

\[
E \left[ L(T(x), D) - L^{L, \alpha}(T(x), D) \right]^2 \leq K_{5,3} B^{-\kappa \beta(2-\alpha)} r^{4-d(\alpha-2)},
\]

where $K_{5,3} > 0$ depends on $C_{5,4}, C_{5,6}, \alpha$ and $d$. The proof is then completed in view of the Chebyshev’s inequality. \hfill \blacksquare

**Proof of Proposition 4.3.** To establish the result, we only need to carefully construct a sequence $\{r_k\}_{k=1}^{k_0} \subset \left[ r_0^2, r_0 \right]$, such that there exists a positive constant $C_{5,8}$ such that

\[
P \left\{ \text{for every } r_k \in \left[ r_0^2, r_0 \right], \text{ } L \left( T(x), D(x, r_k) \right) \leq \frac{\phi(r_k)}{C_{5,8}} \right\} \\
\cup \sup_{d(x,y)=r_k} \|T(x) - T(y)\| > 2C_{5,8} w(r_k) \right\} \leq \exp \left\{ - (|\log r_0|)^{\frac{1}{2}} \right\}. \tag{58}
\]
Let $r_k = r_0 B^{-(k-1)}$, $k = 1, \ldots, k_0$, for some constants $k_0$ and $B > 1$ whose values will be determined later. Moreover, $k_0$ satisfies $k_0 \leq \frac{\log r_0}{\log B}$, so that $r_0^2 \leq r_k \leq r_0$ for all $1 \leq k \leq k_0$. Now let $L_k = \left[ \frac{B^k}{r_0} \right] + 1$, $U_k = \left[ \frac{B^{k+1}}{r_0} \right]$, where $\lfloor \cdot \rfloor$ denotes the integer part as before. Then by Lemma 5.1, we have

$$
\Pr \left\{ \sup_{d_2(x,y) \leq r_k} \| T^{L_k, U_k}(x) - T^{L_k, U_k}(y) \| \leq C_{5,8} w(r_k) \right\}
\geq \exp \left( -K_{5,1}(C_{5,8})^{-4/(\alpha-2)} \log |\log r_k| \right) \geq (|\log r_k|)^{-\frac{1}{4}},
$$

where we have chosen $C_{5,8}$ large enough such that $K_{5,1}(C_{5,8})^{-4/(\alpha-2)} \leq \frac{1}{4}$.

On the other hand, recall $\mathbf{T}^{\Delta_k} = \mathbf{T} - T^{L_k, U_k}$, and $2 - \alpha/2 > \alpha/2 - 1$ for $2 < \alpha < 4$, so we can let $B$ large enough such that

$$
\frac{B^{3(2-\alpha)/2}}{\log B} \geq (\log |\log r_0|)\frac{\alpha}{4} - 1,
$$

which leads to that, for any $k \in \{1, \ldots, k_0\}$,

$$
\rho_\alpha \left( \frac{r_k}{\log |\log r_k|} \right) \geq B^{-3(2-\alpha)/2} \log B r_k^{\alpha/2 - 1},
$$

and

$$
\Pr \left\{ \sup_{d_2(x,y) \leq r_k} \| T^{L_k, U_k}(x) - T^{L_k, U_k}(y) \| > C_{5,8} w(r_k) \right\}
\leq \exp \left( -C_{5,9} \frac{B^{3(2-\alpha)} \log |\log r_k|^\alpha}{\log |\log r_k|^\alpha} \right),
$$

where $C_{5,9} = C_{5,8}^2 / K_{5,2}$ in view of Lemma 5.2.

Meanwhile, for any $\varepsilon > 0$, let $B = B \left( \mathbf{T}^{L_k, U_k}(x), \frac{\varepsilon}{2} \right)$ be an open cube on $\mathbb{R}^d$, which is centered at $\mathbf{T}^{L_k, U_k}(x)$ with length $\varepsilon$. The occupation measure of $T^{L_k, U_k}$ in $B$ can be defined as

$$
\mu^{L_k, U_k}(I, D) = \int_{D(x,r)} 1_B \left( \mathbf{T}^{L_k, U_k}(y) \right) d\nu(y) = \int_B L^{L_k, U_k}(t, D(x,r_k)) dt,
$$

where $L^{L_k, U_k}$ is defined in (56). Now let $\varepsilon = C_{5,8} w(r)$, then by exploiting the continuity of $L(t, D)$ and Lemma 5.1, we have

$$
\Pr \left\{ L^{L_k, U_k}(\mathbf{T}(x), D) < \frac{\nu_\varepsilon(D(x,r_k))}{C_{5,8} w(r_k)} \right\}
\leq \Pr \left\{ \mu^{L_k, U_k}(I, D) < \frac{\nu_\varepsilon(D(x,r_k))}{C_{5,8} w(r_k)} \right\}
\leq \Pr \left\{ \sup_{y \in D(x,r_k)} \| T^{L_k, U_k}(y) - T^{L_k, U_k}(x) \| > C_{5,8} w(r_k) \right\}
\leq 1 - \exp \left( -K_{5,1} \frac{r_k^2}{(C_{5,8} w(r_k))^{4/(\alpha-2)}} \right)
= 1 - \exp \left( -K_{5,1} (C_{5,8})^{-4/(\alpha-2)} \log |\log r_k| \right). \quad (59)
$$

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Let the positive constant $C_{5.8}$ be large enough so that $(C_{5.8})^{4/(\alpha-2)/K_{5.1}} \geq 4$, then
\[ P \left\{ L_{k, U_k} (T(x), D(x, r_k)) \leq \frac{\pi}{C_{5.8}} \phi(r_k) \right\} \leq 1 - |\log r_k|^{-\frac{1}{2}}. \]
in view of the inequality (59). Now define $F_{k,i}, i = 1, ..., 4$, to be the events such that
\[ F_{k,1} = \left\{ \omega : \sup_{d_{r_k}(x, y) \leq r_k} \| T_{L_k, U_k}(y) - T_{L_k, U_k}(x) \| > C_{5.8} w(r_k) \right\}, \]
\[ F_{k,2} = \left\{ \omega : \sup_{d_{r_k}(x, y) \leq r_k} |T_{\Delta_k}(y) - T_{\Delta_k}(x)| > C_{5.8} w(r_k) \right\}, \]
\[ F_{k,3} = \left\{ \omega : \sup_{d_{r_k}(x, y) \leq r_k} |L_{L_k, U_k}(T(x), D(x, r_k)) - L_{L_k, U_k}(T(x), D(x, r_k))| \leq \frac{\pi}{C_{5.8}} \phi(r_k) \right\}, \]
\[ F_{k,4} = \left\{ \omega : \sup_{d_{r_k}(x, y) \leq r_k} |\log |T_{\Delta_k}(y) - T_{\Delta_k}(x)|| > \frac{\pi}{C_{5.8}} \phi(r_k) \right\}. \]
Obviously $F_{k,3} \subset F_{k,1}$ according to the discussion above in (59). Therefore, we obtain that
\[ \text{LHS} \leq P \left\{ \text{for every } k \in \{1, ..., k_0\} \cup F_{k,1} \cup F_{k,2} \cup F_{k,3} \cup F_{k,4} \right\} \]
\[ \leq \prod_{k=0}^{k_0} P \left\{ F_{k,1} \text{ or } F_{k,3} \text{ occurs} \right\} + \sum_{k=0}^{k_0} P \left\{ F_{k,2} \text{ occurs} \right\} + \sum_{k=0}^{k_0} P \left\{ F_{k,4} \text{ occurs} \right\} \]
\[ \leq \left( 1 - |\log r_0|^{-\frac{1}{2}} \right)^{k_0} + k_0 \exp \left\{ -C_{5.9} \frac{B^{\beta(4-\alpha)}}{(|\log \log r_0|)^{\alpha/2-1}} \right\} \]
\[ + \pi^{-2} K_{5.3} C_{5.8}^2 k_0 \frac{B^{-\kappa \beta(4-\alpha)}}{(|\log \log r_0|)^{\alpha/2-2}}. \]
Recall that $k_0 \leq |\log r_0| / \log B$. Let $B$ be large enough such that
\[ \min \left\{ K_{5.3} C_{5.8}^2 / \pi^2, C_{5.9} \right\} B^{\min \{\kappa,1\} \beta(4-\alpha)} \geq |\log r_0|^3 \]
as well as $B > B_0$, then it is readily seen that for $r_0$ small enough,
\[ \pi^{-2} K_{5.3} C_{5.8}^2 k_0 \frac{B^{-\kappa \beta(4-\alpha)}}{(|\log \log r_0|)^{\alpha/2-2}} \leq \frac{|\log r_0|}{|\log \log r_0|} \frac{(|\log r_0|)^{-3}}{(|\log \log r_0|)^{\alpha/2-2}} \]
\[ \leq \frac{1}{3} |\log r_0|^{-2}, \]
\[ k_0 \exp \left\{ -C_{5.9} \frac{B^{\beta(4-\alpha)}}{(|\log \log r_0|)^{\alpha/2-1}} \right\} \leq \frac{1}{3} |\log r_0|^{-2} \]
and
\[ \left( 1 - |\log r_0|^{-\frac{1}{2}} \right)^{k_0} \leq \exp \left\{ -k_0 |\log r_0|^{-\frac{1}{2}} \right\} \leq \frac{1}{3} |\log r_0|^{-2}. \]

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Therefore, $\Box \leq |\log r_0|^{-2},$

which leads to the conclusion of Proposition $\Box$. 

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