We study gauge symmetry breaking by adiabatic approximation in the adiabatic self-consistent collective coordinate (ASCC) method. In the previous study, we found that the gauge symmetry of the equation of collective submanifold is (partially) broken by its decomposition into the three moving-frame equations depending on the order of $p$. In this study, we discuss the gauge symmetry breaking by the truncation of the adiabatic expansion. A particular emphasis is placed on the symmetry under the gauge transformations which are not point transformations. We also discuss a possible version of the ASCC method including the higher-order operators which can keep the gauge symmetry.

Subject Index

1. Introduction

The adiabatic self-consistent collective coordinate (ASCC) method [1] is a practical method for describing large-amplitude collective motion of atomic nuclei with superfluidity [2, 3]. It is an adiabatic approximation to the self-consistent collective coordinate (SCC) method [4, 5] and can be regarded as an advanced version of the adiabatic time-dependent Hartree-Fock-Bogoliubov (ATDHF) theory. The ASCC method overcomes the difficulties several versions of the ATDHF(B) theory encountered (see [3] for a review). It also provides with a non-perturbative scheme to solve the basic equations of the SCC method and is applicable to large-amplitude collective dynamics which is not accessible with the $(\eta, \eta^*)$ expansion method of the SCC method.

The “gauge” symmetry in the ASCC method was first pointed out by Hinohara et al [6]. They encountered a numerical instability in solving the basic equations of the ASCC method, the moving-frame HFB & QRPA equations. They found that the instability was caused by symmetry under some continuous transformation, under which the basic ASCC equations are invariant. Because the transformation changes the phase of the state vector, it is called the gauge symmetry. They proposed a prescription to fix the gauge and successfully applied it to multi-O(4) model [6] and to the shape coexistence/mixing in proton-rich Se and Kr isotopes with the pairing-plus-quadrupole model [7, 8].

After the successful application of the one-dimensional ASCC method, an approximated version of the two-dimensional ASCC method, the constrained HFB plus local QRPA method, was proposed [9] and applied to large-amplitude quadrupole collective dynamics [10–16]. (We mean by the $D$-dimensional ASCC method that the dimension of the collective
coordinate \( q \) is \( D \). However, little progress had been made in the understanding on the
gauge symmetry in the ASCC method.

Quite recently, we investigated the gauge symmetry in the ASCC method on the basis of
the Dirac-Bergmann theory of constrained systems, which brought about a new insight [17].
According to the theory of constrained systems initiated by Dirac and Bergmann [18–20],
the gauge symmetry is associated with constraints which are originated from the singularity
of the Lagrangian. In the ASCC method, the linear term of \( n \) in the collective Hamiltonian
plays the role of a constraint and leads to the gauge symmetry. In Ref. [17], we discussed
possible gauge transformations in the ASCC method from a general point of view based on
the Dirac-Bergmann theory of constrained systems. We found that the four examples or four
types of the gauge transformations play essential roles to discuss the gauge symmetry and
its breaking and to determine the form of the general gauge transformation under which the
ASCC equations are invariant. [The four examples are listed in Eqs. (2.22)-(2.35).]

The basic equations of the ASCC method, i.e., the moving-frame HFB & QRPA equations
and the canonical-variable conditions, are derived from the equation of collective submanifold
and the canonicity conditions, respectively, with use of the adiabatic expansion. Although
the equation of collective submanifold and the canonicity conditions are invariant under the
general gauge transformation, the gauge symmetry of the moving-frame QRPA equations
is partially broken. For example, we found in Ref. [17] that the gauge symmetry in the
moving-frame QRPA equations is broken by the decomposition of the equation of collective
submanifold into the three moving-frame equations in Example 3. On the other hand, in
Example 1, the gauge symmetry of the moving-frame QRPA equations is broken unless \( \dot{Q} \) and \( \dot{N} \) commute, but it has not been elucidated why the gauge symmetry is broken in
the moving-frame QRPA equations. Besides the moving-frame QRPA equations, the gauge
symmetry of the canonical-variable conditions also can be regarded to be broken by the
truncation of the adiabatic expansion, as we show in this paper.

As mentioned above, the ASCC method is an adiabatic approximation to the SCC method.
The term “adiabatic approximation” is often used for different meanings. In Ref. [21], it is
used for the approximate solution of the equation of collective submanifold by taking into
account up to the second order in the adiabatic expansion with respect to the collective
momenta \( p \). In this paper, we shall use this term in a broader sense to mean an approximate
solution of the equation of collective submanifold with use of the adiabatic expansion up to
a certain order with respect to \( p \). Precisely speaking, the “approximate solution” implies the
following two things. One is that the equation of collective submanifold is decomposed into a
certain number of moving-frame equations depending on the order of \( p \). (In the ASCC method
in this paper, we consider three moving-frame equations.) The other is that the adiabatic
expansion is truncated up to a certain order. These “decomposition” and “truncation” are
sources of the gauge symmetry breaking in the ASCC method.

In this paper, we investigate the gauge symmetry breaking by the adiabatic approximation
in the ASCC method and discuss a possible extension of the ASCC method to keep the
gauge symmetry by including the higher-order terms. In the ASCC method, we assume the
following form of the state vector

\[
|\phi(q,p,\varphi,n)\rangle = e^{-i\varphi\hat{N}}|\phi(q,p,n)\rangle = e^{-i\varphi\hat{N}}e^{i\hat{Q}(q,p,n)}|\phi(q)\rangle.
\]
Here, \( q \) is a collective coordinate and \( p \) is its conjugate momentum. \( \varphi \) is the gauge angle conjugate to the particle number \( n = N - N_0 \). \( \tilde{N} = \hat{N} - N_0 \) is the particle number operator measured from a reference value \( N_0 \). In the ASCC method we have considered so far, \( \hat{G} \) is expanded up to the first order with respect to \((p, n)\). We show that the gauge symmetry breaking of the moving-frame QRPA equations in Example 1 is due to the truncation of the expansion of \( \hat{G} \) to the first order.

The paper is organized as follows. In Sect. 2, we discuss the gauge symmetry of the canonicity conditions and that of the canonical-variable conditions. We show that the gauge symmetry of the canonical-variable conditions under the non-point gauge transformations is broken by the truncation of the adiabatic expansion. In Sect. 3, we expand \( \hat{G} \) up to the second order and show that the gauge symmetry breaking at the order of \( p \) is recovered if the second-order operators in \( \hat{G} \) are taken into account. We illustrate how the higher-order operators are transformed under the gauge transformations. In Sect. 4, a version of the ASCC method including the third-order operators is given. With the third-order operators, the moving-frame HFB & QRPA equations and the canonical-variable conditions up to the second order are gauge invariant under the non-point gauge transformations. The concluding remarks are given in Sect. 5. In Appendix A, the derivation of the basic equations for the second-order and third-order expansions of \( \hat{G} \) is given. In Appendix B, we present the gauge transformations of the third-order operators in Examples 2-4.

As in Ref. [17], we consider the one-dimensional ASCC method with a single component for simplicity in this paper. However, the extension to the multi-dimensional and multi-component cases is straightforward.

2. Canonicity conditions and gauge transformation

2.1. Canonicity and canonical-variable conditions

We consider gauge transformations of the canonicity conditions and those of the canonical-variable conditions, which are derived from the canonicity conditions with use of the adiabatic expansion. As in Ref. [17], we assume the state vector in the following form.

\[
|\phi(q, p, \varphi, n)\rangle = e^{-i\varphi\tilde{N}}|\phi(q, p, n)\rangle, \tag{2.1}
\]

\[
|\phi(q, p, n)\rangle = e^{i\vartheta(q)}|\phi(q)\rangle, \tag{2.2}
\]

\[
\hat{G}(q, p, n) = p\tilde{Q}(q) + n\tilde{\Theta}(q). \tag{2.3}
\]

The canonicity conditions are given by

\[
\langle \phi(q, p, \varphi, n) | \tilde{Q} | \phi(q, p, \varphi, n) \rangle = \langle \phi(q, p, \varphi, n) | i \partial_p | \phi(q, p, \varphi, n) \rangle = -\frac{\partial s}{\partial p}, \tag{2.4}
\]

\[
\langle \phi(q, p, \varphi, n) | \tilde{P} | \phi(q, p, \varphi, n) \rangle = \langle \phi(q, p, \varphi, n) | i \partial_q | \phi(q, p, \varphi, n) \rangle = p + \frac{\partial s}{\partial q}, \tag{2.5}
\]

\[
\langle \phi(q, p, \varphi, n) | \tilde{\Theta} | \phi(q, p, \varphi, n) \rangle = \langle \phi(q, p, \varphi, n) | i \partial_n | \phi(q, p, \varphi, n) \rangle = -\frac{\partial s}{\partial n}, \tag{2.6}
\]

\[
\langle \phi(q, p, \varphi, n) | \tilde{N} | \phi(q, p, \varphi, n) \rangle = \langle \phi(q, p, \varphi, n) | i \partial_\varphi | \phi(q, p, \varphi, n) \rangle = n + \frac{\partial s}{\partial \varphi}. \tag{2.7}
\]

Here, \( s \) is an arbitrary function of \((q, p, \varphi, n)\) and is set to \( s = 0 \) in the ASCC method. As discussed in Ref. [17], \( s \) is related to the generating function of a canonical transformation.
As shown in Ref. [1], \[ \hat{P}' = e^{-i\hat{G} e^{\hat{\varphi} \hat{N}}} P e^{-i \varphi \hat{N} e^{i\hat{G}}}, \quad \hat{Q}' = e^{-i\hat{G} e^{\hat{\varphi} \hat{N}}} Q e^{-i \varphi \hat{N} e^{i\hat{G}}}, \quad \hat{\Theta}' = e^{-i\hat{G} e^{\hat{\varphi} \hat{N}}} \hat{\Theta} e^{-i \varphi \hat{N} e^{i\hat{G}}}, \quad \hat{N}' = e^{-i\hat{G} e^{\hat{\varphi} \hat{N}}} \hat{N} e^{-i \varphi \hat{N} e^{i\hat{G}}} \]
are expanded as follows.

\begin{align*}
\hat{P}' &= i \partial_q - p \partial_q \hat{Q} - n \partial_q \hat{\Theta} + \cdots, \quad (2.8) \\
\hat{Q}' &= \hat{Q} + i \left[ \hat{Q}, p \hat{Q} + n \hat{\Theta} \right] + \cdots = \hat{Q} + n \frac{\hat{\varphi}}{2} [\hat{Q}, \hat{\Theta}] + \cdots, \quad (2.9) \\
\hat{\Theta}' &= \hat{\Theta} + i \left[ \hat{\Theta}, p \hat{Q} + n \hat{\Theta} \right] + \cdots = \hat{\Theta} + p \frac{\hat{\varphi}}{2} [\hat{\Theta}, \hat{Q}] + \cdots, \quad (2.10) \\
\hat{N}' &= \hat{N} + i p [\hat{N}, \hat{Q}] + i n [\hat{N}, \hat{\Theta}] + \cdots. \quad (2.11)
\end{align*}

By substituting (2.8)-(2.11) into the canonicity conditions (2.4)-(2.7) and setting \( s = 0 \), we obtain the zeroth- and first-order canonical-variable conditions.

Canonical-variable conditions

\begin{align*}
\langle \phi(q) \vert \hat{Q} \vert \phi(q) \rangle &= 0, \quad (2.12) \\
\langle \phi(q) \vert \hat{P} \vert \phi(q) \rangle &= 0, \quad (2.13) \\
\langle \phi(q) \vert \hat{\Theta} \vert \phi(q) \rangle &= 0, \quad (2.14) \\
\langle \phi(q) \vert \hat{N} \vert \phi(q) \rangle &= 0, \quad (2.15)
\end{align*}

\begin{align*}
\langle \phi(q) \vert [\hat{Q}, \hat{P}] \vert \phi(q) \rangle &= i, \quad (2.16) \\
\langle \phi(q) \vert [\hat{P}, \hat{\Theta}] \vert \phi(q) \rangle &= 0, \quad (2.17) \\
\frac{i}{2} \langle \phi(q) \vert [\hat{Q}, \hat{\Theta}] \vert \phi(q) \rangle &= 0, \quad (2.18) \\
\langle \phi(q) \vert [\hat{\Theta}, \hat{N}] \vert \phi(q) \rangle &= i, \quad (2.19) \\
\langle \phi(q) \vert [\hat{N}, \hat{Q}] \vert \phi(q) \rangle &= 0, \quad (2.20) \\
\langle \phi(q) \vert [\hat{N}, \hat{P}] \vert \phi(q) \rangle &= 0. \quad (2.21)
\end{align*}

The condition (2.21) is obtained by differentiating the condition (2.15) with respect to \( q \).

Here we have kept the factor \( \frac{1}{2} \) in Eq. (2.18), because it is necessary for discussion on the gauge symmetry later.

In Ref. [17], we found that the following four examples of the gauge transformations play essential roles for the discussion on the gauge symmetry in the ASCC method. Below we list the generators \( G \) of the gauge transformations and the transformations of collective variables and operators for each example. (The generator \( G \) should not be confused with \( \hat{G} \) in Eq. (2.2).) Hereinafter let \( \alpha \) be an infinitesimal. Although a more general linear gauge transformation is given in Ref. [17], it is not necessary for our purpose in this paper.

Example 1: \( G = \alpha p n \)

\begin{align*}
q &\rightarrow q + \alpha n, \quad (2.22) \\
\varphi &\rightarrow \varphi + \alpha p, \quad (2.23) \\
\hat{Q} &\rightarrow \hat{Q} + \alpha \hat{N}, \quad (2.24) \\
\hat{\Theta} &\rightarrow \hat{\Theta} + \alpha \hat{P}. \quad (2.25)
\end{align*}
Example 2: $G = \alpha n^2 / 2$

\[ \varphi \rightarrow \varphi + \alpha n, \quad (2.26) \]

\[ \hat{\Theta} \rightarrow \hat{\Theta} + \alpha \hat{N}. \quad (2.27) \]

Example 3: $G = \varepsilon_n = \alpha q n$

\[ p \rightarrow p - \alpha n, \quad (2.28) \]

\[ \varphi \rightarrow \varphi + \alpha q, \quad (2.29) \]

\[ \hat{P} \rightarrow \hat{P} - \alpha \hat{N}, \quad (2.30) \]

\[ \hat{\Theta} \rightarrow \hat{\Theta} + \alpha \hat{Q}. \quad (2.31) \]

Example 4: $G = \varepsilon_n = \alpha \varphi n$

\[ \varphi \rightarrow \varphi + \alpha \varphi = e^{\alpha \varphi}, \quad (2.32) \]

\[ n \rightarrow n - \alpha n = e^{-\alpha n}, \quad (2.33) \]

\[ \hat{\Theta} \rightarrow \hat{\Theta} + \alpha \hat{P} = e^{\alpha \hat{\Theta}}, \quad (2.34) \]

\[ \hat{N} \rightarrow \hat{N} - \alpha \hat{N} = e^{-\alpha \hat{N}}. \quad (2.35) \]

As one can see from the canonicity conditions (2.4)-(2.7), $(q, \varphi)$ are coordinates and $(p, n)$ are the conjugate momenta. Therefore, Examples 3 and 4 are point transformations. Example 1 is the first example of the gauge transformations found in [6], and the general gauge transformation including Examples 1-4 is discussed in Ref. [17].

Before moving to the discussion on the canonicity conditions, let us see the relation between the transformations of the c-numbers $(q, p, \varphi, n)$ and those of the corresponding operators. Let us take Example 1. In correspondence with (2.22)-(2.23), the differential operators are transformed as

\[ \partial_p \rightarrow \partial_p - \alpha \partial_\varphi, \quad (2.36) \]

\[ \partial_n \rightarrow \partial_n - \alpha \partial_q, \quad (2.37) \]

which leads to

\[ \hat{Q}|\phi(q, p, \varphi, n)\rangle \rightarrow (\hat{Q} + \alpha \hat{N})|\phi(q, p, \varphi, n)\rangle, \quad (2.38) \]

\[ \hat{\Theta}|\phi(q, p, \varphi, n)\rangle \rightarrow (\hat{\Theta} + \alpha \hat{P})|\phi(q, p, \varphi, n)\rangle. \quad (2.39) \]

By considering the leading order with respect to $(p, n)$, we obtain

\[ \hat{Q} \rightarrow \hat{Q} + \alpha \hat{N}, \quad (2.24) \]

\[ \hat{\Theta} \rightarrow \hat{\Theta} + \alpha \hat{P}. \quad (2.25) \]

When the transformation (2.24)-(2.25) is applied to the state vector, it implies the transformation of the argument of the state vector as follows.

\[ |\phi(q, p, \varphi, n)\rangle \rightarrow e^{-i\varepsilon \hat{N}} e^{i\varepsilon (Q(q) + \alpha \hat{N}) + i\beta (\Theta(q) + \alpha \hat{P}(q))}|\phi(q)\rangle \]

\[ = e^{-i(\varphi - \alpha p)\hat{N}} e^{i\varphi (q - \alpha n) + i\beta \Theta(q - \alpha n)}|\phi(q - \alpha n)\rangle, \]

\[ = |\phi(q - \alpha n, p, \varphi - \alpha p, n)\rangle, \quad (2.40) \]

where we ignored the second-order terms with respect to $(p, n)$. Note that the signs of $\alpha$ in the last expression are opposite to those in (2.22)-(2.23). Conversely, by transforming the
argument of the state vector as \( q \to q - \alpha n, \varphi \to \varphi - \alpha p \), one can obtain the transformation (2.24)-(2.25).

### 2.2. Gauge transformation of canonicity conditions

In this subsection, we discuss the gauge symmetry of the canonicity conditions. First, we shall consider the gauge symmetry of the canonical-variable conditions in Example 1. One can easily ascertain that the canonical-variable conditions (2.12)-(2.21) are invariant under the transformation of Example 1 (2.24)-(2.25). For example, the condition (2.18) is transformed as follows.

\[
\begin{align*}
\frac{i}{2} \langle \phi(q)|[\hat{\Theta}, \hat{Q}]|\phi(q) \rangle & \rightarrow \frac{i}{2} \langle \phi(q)|[\hat{\Theta} + \alpha \hat{P}, \hat{Q} + \alpha \hat{N}]|\phi(q) \rangle \\
& = \frac{i}{2} \left( \langle \phi(q)|[\hat{\Theta}, \hat{Q}]|\phi(q) \rangle + \alpha \langle \phi(q)|[\hat{P}, \hat{Q}]|\phi(q) \rangle \\
& \quad + \alpha \langle \phi(q)|[\hat{\Theta}, \hat{N}]|\phi(q) \rangle + \alpha^2 \langle \phi(q)|[\hat{P}, \hat{N}]|\phi(q) \rangle \right) \\
& = \frac{i}{2} \left( 0 - \alpha i + \alpha i + 0 \right) = 0,
\end{align*}
\]

that is, the operators after the transformation \( \hat{\Theta}' = \hat{\Theta} + \alpha \hat{P}, \hat{Q}' = \hat{Q} + \alpha \hat{N} \) also satisfy the weak canonical commutation relation

\[
\frac{i}{2} \langle \phi(q)|[\hat{\Theta}', \hat{Q}']|\phi(q) \rangle = 0.
\] (2.42)

In Refs. [6] and [17], this canonical-variable condition is said to be “gauge invariant” in this sense. However, as we shall see below, this “invariance” of the canonical-variable condition implies that the gauge symmetry is actually broken.

In the following discussions, the arbitrary function \( s \) in the canonicity conditions plays a key role. When the set of canonical variables with the arbitrary function \( (q^i, p_i, s = 0) \) is transformed to \( (q'^i, p'_i, S) \) by a time-independent canonical transformation, the following relation holds [22].

\[
p_i dq^i = p'_i dq'^i + dS.
\] (2.43)

Thus, \( S \) is the generating function of the time-independent canonical transformation. While \( S = const. \) in Examples 3 and 4, which are point transformations, in Examples 1 and 2, which are not point transformations, \( S = -G \) [17]. Below we consider the gauge transformation of the canonicity conditions in Examples 1 and 2. In point transformations, \( S \) does not contribute to the canonicity conditions after the gauge transformation, because \( S = const. \). On the other hand, in Examples 1 and 2, \( S = -G \) does contribute as they are not point transformations. We shall consider Examples 2 and 1.
Example 2: \( G = \alpha n^2/2 \)

\( G = \alpha n^2/2 \) generate \( \varphi \to \varphi + \alpha n \), which leads \( \partial_n \to \partial_n - \alpha \partial_\varphi \). \( S = -G \) contributes to the canonicity condition (2.6) and it is transformed as below.

\[
\langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_n | \phi(q, p, \varphi, n) \rangle = 0
\]

\[
\to \langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_n' | \phi(q, p, \varphi, n) \rangle = \langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_n + \alpha i \partial_\varphi | \phi(q, p, \varphi, n) \rangle
\]

\[
= \alpha n = - \frac{\partial S}{\partial n'}.
\]

(2.44)

One can see that the canonicity condition is satisfied after the gauge transformation. The gauge transformation can be also discussed by transforming the arguments of the state vector as below. (Recall the discussion on the relation between the transformations of the c-numbers and those of the corresponding operators in the last subsection.)

\[
\langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_n | \phi(q, p, \varphi, n) \rangle = 0
\]

\[
\to \langle \phi(q, p, \varphi - \alpha n, n) \mid \frac{1}{i} \partial_n | \phi(q, p, \varphi - \alpha n, n) \rangle
\]

\[
= \langle \phi(q, p, \varphi, n) \mid - \phi(q, p, \varphi, n) | \tilde{\partial}_\varphi \alpha n \mid \partial_n (| \phi(q, p, \varphi, n) \rangle - \alpha n \tilde{\partial}_\varphi | \phi(q, p, \varphi, n) \rangle)
\]

\[
= \langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_n | \phi(q, p, \varphi, n) \rangle - \alpha n \partial_\varphi \langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_n | \phi(q, p, \varphi, n) \rangle
\]

\[
\quad + \alpha \langle \phi(q, p, \varphi, n) | i \tilde{\partial}_\varphi | \phi(q, p, \varphi, n) \rangle
\]

\[
= \alpha n = - \frac{\partial S}{\partial n'}.
\]

(2.45)

We have used \( \langle \phi | \tilde{\partial}_\varphi \partial_n | \phi \rangle = \langle \phi | \partial_n \partial_\varphi | \phi \rangle \) and omitted higher-order infinitesimals.

Example 1: \( G = \alpha pn \)

\( G = \alpha pn \) generates the transformation \( (q, \varphi) \to (q', \varphi') = (q + \alpha n, \varphi + \alpha p) \). In correspondence, the differential operators are transformed as \( (\partial_p, \partial_n) \to (\partial_{p'}, \partial_{n'}) = (\partial_p - \alpha \partial_\varphi, \partial_n - \alpha \partial_q) \). Then, the canonicity conditions (2.6) is transformed as

\[
\langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_n | \phi(q, p, \varphi, n) \rangle = 0
\]

\[
\to \langle \phi(q, p, \varphi - \alpha n, n) \mid \frac{1}{i} \partial_n | \phi(q, p, \varphi - \alpha n, n) \rangle
\]

\[
= \langle \phi(q, p, \varphi, n) \mid - \phi(q, p, \varphi, n) | \tilde{\partial}_\varphi \alpha n \mid \partial_n (| \phi(q, p, \varphi, n) \rangle - \alpha n \tilde{\partial}_\varphi | \phi(q, p, \varphi, n) \rangle)
\]

\[
= \langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_n | \phi(q, p, \varphi, n) \rangle - \alpha n \partial_\varphi \langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_n | \phi(q, p, \varphi, n) \rangle
\]

\[
\quad + \alpha \langle \phi(q, p, \varphi, n) | i \tilde{\partial}_\varphi | \phi(q, p, \varphi, n) \rangle
\]

\[
= \alpha n = - \frac{\partial S}{\partial n'}.
\]

(2.46)

The canonicity condition (2.4) is transformed as

\[
\langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_p | \phi(q, p, \varphi, n) \rangle = 0
\]

\[
\to \langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_{p'} | \phi(q, p, \varphi, n) \rangle = \langle \phi(q, p, \varphi, n) \mid \frac{1}{i} \partial_p + \alpha i \partial_\varphi | \phi(q, p, \varphi, n) \rangle
\]

\[
= \alpha n = - \frac{\partial S}{\partial p'}.
\]

(2.47)

\( G = \alpha pn \) gives contributions of \( O(p) \) and of \( O(n) \) to the canonicity conditions (2.6) and (2.4), respectively.
2.3. Gauge symmetry breaking

The transformation of the canonicity condition (2.6) in Example 1 we have seen above,

\[ \langle \phi(q,p,\varphi,n) \mid \frac{1}{i} \partial_n \mid \phi(q,p,\varphi,n) \rangle = 0 \rightarrow \langle \phi(q,p,\varphi,n) \mid \frac{1}{i} \partial_{n'} \mid \phi(q,p,\varphi,n) \rangle = \alpha p, \]  

implies that the canonical-variable condition (2.18)

\[ \frac{i}{2} \langle \phi(q) \mid [\hat{Q},\hat{\Theta}] \mid \phi(q) \rangle = 0, \]  

which is derived from the \( O(p) \) term in the canonicity condition (2.6), should be changed by \( \alpha \) by the gauge transformation. However, this condition (2.18) remains 0 after the gauge transformation as we have seen in the beginning of Sect. 2.1. This implies that the gauge symmetry in the canonicity conditions is broken by the adiabatic approximation. As we shall see below, this is because we truncate the adiabatic expansion of \( \hat{G}(q,p,n) \) to the first order and adopt \( \hat{G} \) of the form \( \hat{G} = p\hat{Q} + n\hat{\Theta} \).

The canonical-variable condition (2.18) is also obtained from the \( O(n) \) term in the canonicity condition (2.4). As shown in Eq. (2.47), the canonicity condition (2.4) is changed by \( \alpha n \) by the gauge transformation and it is consistent with the discussion above.

Next we shall see the transformation of the canonicity condition (2.6) in Example 2.

\[ \langle \phi(q,p,\varphi,n) \mid \frac{1}{i} \partial_n \mid \phi(q,p,\varphi,n) \rangle = 0 \rightarrow \langle \phi(q,p,\varphi,n) \mid \frac{1}{i} \partial_{n'} \mid \phi(q,p,\varphi,n) \rangle = \langle \phi(q,p,\varphi,n) \mid \frac{1}{i} \partial_n + \alpha i \partial_\varphi \mid \phi(q,p,\varphi,n) \rangle \]

\[ = \alpha n = -\frac{\partial S}{\partial n}. \]  

This implies that the canonical-variable condition of \( O(n) \) is changed by \( +\alpha \). However, unless we take into account higher-order terms in the expansion of \( \hat{G} \), there appears no term of \( O(n) \) in the canonicity condition (2.6) and so it does not give a canonical-variable condition of \( O(n) \) [see (2.10) and (A20)].

3. Expansion of \( \hat{G} \) up to the second order

Above we have seen that the gauge symmetry of the first-order canonical-variable conditions is broken under the gauge transformations which are not point transformations. In this section, we show that the gauge symmetry is conserved if the second-order terms with respect to \( p \) and \( n \) in the generator \( \hat{G} \) are taken into account. We first present the basic definitions and equations in the case of the second-order expansion of \( \hat{G} \), and then discuss the gauge symmetry. For details of the derivation of the basic equations, see Appendix A.

3.1. Basic definitions and equations

So far, we have taken into account only the first-order terms in the expansion of \( \hat{G} \). Here we consider \( \hat{G} \) including the second-order terms as below.

\[ \hat{G}(q,p,n) = p\hat{Q}^{(1)}(q) + n\hat{\Theta}^{(1)}(q) + \frac{1}{2} p^2 \hat{Q}^{(2)}(q) + \frac{1}{2} n^2 \hat{\Theta}^{(2)}(q) + pn\hat{X} \]  

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\[
\hat{Q}^{(i)}(q) = \sum_{\alpha > \beta} Q^{(i)}_{\alpha\beta} a^\dagger_\alpha a_\beta + Q^{(i)*}_{\alpha\beta} a^\dagger_\beta a_\alpha, \quad (i = 1, 2),
\]
\[
\hat{\Theta}^{(i)}(q) = \sum_{\alpha > \beta} \Theta^{(i)}_{\alpha\beta} a^\dagger_\alpha a_\beta + \Theta^{(i)*}_{\alpha\beta} a^\dagger_\beta a_\alpha, \quad (i = 1, 2),
\]
\[
\hat{X}(q) = \sum_{\alpha > \beta} X_{\alpha\beta} a^\dagger_\alpha a_\beta + X^{*}_{\alpha\beta} a^\dagger_\beta a_\alpha.
\] (3.2)

These operators obey the time-reversal symmetry as follows.
\[
T\hat{Q}^{(1)}(q)T^{-1} = \hat{Q}^{(1)}(q),
\]
\[
T\hat{Q}^{(2)}(q)T^{-1} = -\hat{Q}^{(2)}(q),
\]
\[
T\hat{\Theta}^{(i)}(q)T^{-1} = -\hat{\Theta}^{(i)}(q), \quad (i = 1, 2),
\]
\[
T\hat{X}(q)T^{-1} = \hat{X}(q).
\] (3.3)

The first-order operators in the previous section are denoted by \(\hat{Q} = \hat{Q}^{(1)}\) and \(\hat{\Theta} = \hat{\Theta}^{(1)}\). If we expand the collective Hamiltonian up to the second order, we obtain
\[
\mathcal{H}(q, p, N) = V(q) + \frac{1}{2} B(q)p^2 + \frac{1}{2} D(q)n^2
\] (3.4)

with
\[
V(q) = \langle \phi(q) | \hat{H} | \phi(q) \rangle,
\]
\[
B(q) = \langle \phi(q) | [\hat{H}, i\hat{Q}^{(2)}] | \phi(q) \rangle - \langle \phi(q) | [[\hat{H}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] | \phi(q) \rangle,
\]
\[
\lambda(q) = \langle \phi(q) | [\hat{H}, i\hat{\Theta}^{(1)}] | \phi(q) \rangle,
\]
\[
D(q) = \langle \phi(q) | [\hat{H}, i\hat{\Theta}^{(2)}] | \phi(q) \rangle - \langle \phi(q) | [[\hat{H}, \hat{\Theta}^{(1)}], \hat{\Theta}^{(1)}] | \phi(q) \rangle.
\] (3.5) (3.6) (3.7) (3.8)

As discussed in Ref. [17], there appears no gauge symmetry if we employ the collective Hamiltonian (3.4). Therefore, we adopt the collective Hamiltonian up to \(O(n)\):
\[
\mathcal{H}(q, p, N) = V(q) + \frac{1}{2} B(q)p^2 + \lambda n.
\] (3.9)

The canonical-variable conditions are given as follows.

**Zeroth-order canonical-variable conditions**
\[
\langle \phi(q) | \hat{P} | \phi(q) \rangle = 0.
\]
\[
\langle \phi(q) | \hat{Q}^{(1)} | \phi(q) \rangle = 0.
\]
\[
\langle \phi(q) | \hat{\Theta}^{(1)} | \phi(q) \rangle = 0.
\]
\[
\langle \phi(q) | \hat{N} | \phi(q) \rangle = 0.
\]
\[
\langle \phi(q) | [\hat{N}, \hat{P}] | \phi(q) \rangle = 0.
\] (3.10) (3.11) (3.12) (3.13) (3.14)
First-order canonical-variable conditions

\[ \langle \phi(q) | [\hat{Q}^{(1)}, \hat{P}] | \phi(q) \rangle = i. \] (3.15)
\[ \langle \phi(q) | [\hat{\Theta}^{(1)}, \hat{N}] | \phi(q) \rangle = i. \] (3.16)
\[ \langle \phi(q) | [\hat{Q}^{(1)}, \hat{N}] | \phi(q) \rangle = 0. \] (3.17)
\[ \langle \phi(q) | [\hat{\Theta}^{(1)}, \hat{P}] | \phi(q) \rangle = 0. \] (3.18)
\[ \langle \phi(q) | \hat{X} + i \frac{1}{2}[\hat{Q}^{(1)}, \hat{\Theta}^{(1)}] | \phi(q) \rangle = 0. \] (3.19)
\[ \langle \phi(q) | \hat{X} + i \frac{1}{2}[\hat{\Theta}^{(1)}, \hat{Q}^{(1)}] | \phi(q) \rangle = 0. \] (3.20)
\[ \langle \phi(q) | \hat{Q}^{(2)} | \phi(q) \rangle = 0. \] (3.19)
\[ \langle \phi(q) | \hat{\Theta}^{(2)} | \phi(q) \rangle = 0. \] (3.20)

The conditions (3.19) and (3.20) can be rewritten as

\[ \langle \phi(q) | [\hat{Q}^{(1)}, \hat{\Theta}^{(1)}] | \phi(q) \rangle = 0, \] (3.23)
\[ \langle \phi(q) | \hat{X} | \phi(q) \rangle = 0. \] (3.24)

The moving-frame HFB & QRPA equations are as below.

**Moving-frame HFB equation**

\[ \delta \langle \phi(q) | \hat{H} - \lambda \hat{N} - \partial_q V \hat{Q}^{(1)} | \phi(q) \rangle = 0. \] (3.25)

**Moving-frame QRPA equations**

\[ \delta \langle \phi(q) | [\hat{H} - \lambda \hat{N} - \partial_q V \hat{Q}^{(1)}, \frac{1}{i} \hat{P}] - C(q) \hat{Q}^{(1)} - \partial_q \lambda \hat{N} - \frac{1}{2} B \partial_q V \left\{ [[\hat{H} - \lambda \hat{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - i[\hat{H} - \lambda \hat{N}, \hat{Q}^{(2)}] - \frac{i}{2} \partial_q V [\hat{Q}^{(1)}, \hat{Q}^{(2)}] \right\} \rangle = 0. \] (3.27)

The moving-frame HFB equation remains unchanged when we include the second-order operators \( \hat{Q}^{(2)}, \hat{\Theta}^{(2)} \) and \( \hat{X} \). In the moving-frame QRPA equations of \( O(p) \) and \( O(p^2) \), \( \hat{Q}^{(2)} \) is involved.

### 3.2. Gauge symmetry in the case of the second-order expansion

We shall investigate the gauge symmetry in the above case where we take into account the second-order operators. Although we are most interested in the non-point transformations, we discuss not only Examples 1 and 2 but also Examples 3 and 4, for completeness.

#### 3.2.1. Gauge symmetry in Example 1

We shall see how the operators are transformed under the gauge transformation in Example 1. It can be seen by transforming the arguments of the state vector as \( q \rightarrow q - \alpha n \) and \( \varphi \rightarrow \varphi - \alpha p \).
\[ e^{-i\varphi N} e^{i\hat{G}(q,p,n)} |\phi(q)\rangle \]
\[ \rightarrow e^{-i(\varphi - \alpha p) \hat{N}} e^{i\hat{G}(q-an,p,n)} |\phi(q - an)\rangle \]
\[ = e^{-i\varphi N} e^{i\alpha p \hat{N}} e^{i\alpha n \hat{P}} e^{i\hat{G}(p,n)} |\phi(q)\rangle \]
\[ = e^{-i\varphi N} \exp \left\{ i\alpha p \hat{N} + i\alpha n \hat{P} + i\hat{G} + \frac{1}{2} [i\alpha p \hat{N} + i\alpha n \hat{P}, i\hat{G}] + \cdots \right\} |\phi(q)\rangle \]
\[ = e^{-i\varphi N} \exp \left\{ ip(\hat{Q}^{(1)} + \alpha \hat{N}) + i\alpha (\hat{N}, \hat{P}) + \frac{i\alpha^2}{2} (\hat{Q}^{(2)} + i\alpha [\hat{N}, \hat{Q}^{(1)}]) \right. \]
\[ + \left. \frac{n^2}{2} (\hat{Q}^{(2)} + i\alpha [\hat{P}, \hat{Q}^{(1)}]) + i\alpha n [\hat{N}, \hat{Q}^{(1)}] \right\} |\phi(q)\rangle \] (3.28)

Here we have used the Baker-Campbell-Hausdorff formula [23]:
\[ e^X e^Y = \exp \left\{ X + Y + \frac{1}{2} [X, Y] + \frac{1}{12} ([X, [X, Y]] + [Y, [Y, X]] + \cdots) \right\}, \] (3.29)
and omitted the second-order infinitesimals. From Eq. (3.28), one can read the following transformation:
\[ \hat{Q}^{(1)} \rightarrow \hat{Q}^{(1)} + \alpha \hat{N}, \] (3.30)
\[ \hat{Q}^{(2)} \rightarrow \hat{Q}^{(2)} + i\alpha [\hat{N}, \hat{Q}^{(1)}], \] (3.32)
\[ X \rightarrow X + i\alpha [\hat{N}, \hat{Q}^{(1)}]. \] (3.34)

Let us see the gauge transformations of the canonical-variable conditions involving the second-order operators. The canonical-variable condition (3.21) which is derived from the \( O(p) \) term of the canonicity condition (2.4) is transformed as
\[ \langle \phi | \hat{Q}^{(2)} | \phi \rangle = 0 \rightarrow \langle \phi | \hat{Q}^{(2)} + i\alpha [\hat{N}, \hat{Q}^{(1)}] | \phi \rangle = 0, \] (3.35)
so it is gauge invariant. The canonical-variable condition (3.22) derived from the \( O(n) \) term of the canonicity condition (2.6) is transformed as
\[ \langle \phi | \hat{Q}^{(2)} | \phi \rangle = 0 \rightarrow \langle \phi | \hat{Q}^{(2)} + i\alpha [\hat{P}, \hat{Q}^{(1)}] | \phi \rangle = 0, \] (3.36)
so it is also gauge invariant. The canonical-variable condition (3.19) derived from the \( O(n) \) term of the canonicity condition (2.4) is transformed as
\[ \langle \phi(q) | \hat{X} + \frac{i}{2} (\hat{Q}^{(1)}, \hat{Q}^{(1)}) | \phi(q) \rangle = 0 \]
\[ \rightarrow \langle \phi(q) | \hat{X} + \frac{i\alpha}{2} ([\hat{N}, \hat{Q}^{(1)}] + [\hat{P}, \hat{Q}^{(1)}]) + \frac{i\alpha^2}{2} (\hat{Q}^{(1)} + \alpha \hat{N}, \hat{Q}^{(1)} + \alpha \hat{P}) | \phi(q) \rangle = \alpha. \] (3.37)
It is changed by \( \alpha \) as discussed in the previous section, and thus the gauge symmetry is conserved. It is easily seen that the canonical-variable condition (3.20), which is derived from the \( O(p) \) term of the canonicity condition (2.6), is also changed by \( \alpha \) and that the
gauge symmetry is not broken. Note that the new operators after the gauge transformation \( \hat{Q}^{(1)'} = \hat{Q}^{(1)} + \alpha \tilde{N} \) and \( \hat{\Theta}^{(1)'} = \hat{\Theta}^{(1)} + \alpha \hat{P} \) satisfy the weak canonical commutation relation, 
\[ \langle \phi |[\hat{Q}^{(1)'}, \hat{\Theta}^{(1)'}]| \phi \rangle = 0. \]

Next, we shall investigate the gauge symmetry of the moving-frame HFB & QRPA equations. In the case of the expansion of \( \hat{G} \) up to the first order, if \([\tilde{N}, \hat{Q}^{(1)}] = 0\), the moving-frame HFB & QRPA equations are invariant with the transformation of the Lagrange multiplier:
\[ \lambda(q) \rightarrow \lambda(q) - \alpha \partial_q V(q), \]  
\[ \partial_q \lambda(q) \rightarrow \partial_q \lambda(q) - \alpha C(q). \]  

However, the gauge symmetry of the moving-frame QRPA equations is actually broken because, as easily confirmed, the commutator \([\tilde{N}, \hat{Q}^{(1)}] \) is non-zero.

Because \( \hat{Q}^{(2)} \) appears only in the moving-frame QRPA equations, the moving-frame HFB equation remains gauge invariant if we take into account the second-order operators. The moving-frame QRPA equation of \( O(p) \) (3.26) is transformed as
\[ \delta \langle \phi(q) |[\hat{H} - \lambda \tilde{N} - \partial_q V \hat{Q}^{(1)}, \hat{Q}^{(1)}] - \frac{1}{i} B(q) \hat{P} - \frac{1}{i} \partial_q V \hat{Q}^{(2)} | \phi(q) \rangle = 0, \]
\[ \rightarrow \delta \langle \phi(q) |[\hat{H} - \lambda \tilde{N} - \partial_q V \hat{Q}^{(1)}, \hat{Q}^{(1)} + \alpha \tilde{N}] - \frac{1}{i} B(q) \hat{P} \]
\[ - \frac{1}{i} \partial_q V (\hat{Q}^{(2)} + i \alpha [\tilde{N}, \hat{Q}^{(1)}]) | \phi(q) \rangle = 0, \]
\[ \Leftrightarrow \delta \langle \phi(q) |[\hat{H} - \lambda \tilde{N} - \partial_q V \hat{Q}^{(1)}, \hat{Q}^{(1)}] - \frac{1}{i} B(q) \hat{P} - \frac{1}{i} \partial_q V \hat{Q}^{(2)} | \phi(q) \rangle = 0, \]  
and thus it is gauge invariant. Without \( Q^{(2)} \), the moving-frame QRPA equation of \( O(p) \) is not gauge invariant because \([\tilde{N}, \hat{Q}^{(1)}] \neq 0 \). With \( Q^{(2)} \) included, it is gauge invariant even if \([\tilde{N}, \hat{Q}^{(1)}] \neq 0 \). Thus one can see that the gauge symmetry breaking in the moving-frame QRPA equation of \( O(p) \) is because of the truncation of the adiabatic expansion of \( \hat{G} \) to the first order.

As shown below, the moving-frame QRPA equation of \( O(p^2) \) (3.27) is not gauge invariant. As in the case of the first-order expansion, the first three terms in Eq. (3.27)
\[ [\hat{H} - \lambda \tilde{N} - \partial_q V \hat{Q}^{(1)}, \frac{1}{i} \hat{P}] - C(q) \hat{Q}^{(1)} - \partial_q \lambda \tilde{N} \]
are gauge invariant. In the case of the first-order expansion, the fourth term
\[ -\frac{1}{2B} \partial_q V \left\{ [[[\hat{H} - \lambda \tilde{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}]] \right\} \]  
(3.42)
is not gauge invariant, and \([\tilde{N}, \hat{Q}^{(1)}] = 0 \) is required for the gauge symmetry. Therefore, we only have to check the gauge invariance of the part
\[ [[\hat{H} - \lambda \tilde{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - i[[\hat{H} - \lambda \tilde{N}, \hat{Q}^{(2)}] - \frac{i}{2} \partial_q V [\hat{Q}^{(1)}, \hat{Q}^{(2)}]. \]  
(3.43)
After a lengthy calculation, one finds that it transforms as
\[ [[\hat{H}_M, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - i[[\hat{H} - \lambda \tilde{N}, \hat{Q}^{(2)}] - \frac{i}{2} \partial_q V [\hat{Q}^{(1)}, \hat{Q}^{(2)}] \]
\[ \rightarrow [[\hat{H}_M, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - i[[\hat{H} - \lambda \tilde{N}, \hat{Q}^{(2)}] - \frac{i}{2} \partial_q V [\hat{Q}^{(1)}, \hat{Q}^{(2)}] \]
\[ + \frac{1}{2} \alpha \partial_q V [[\tilde{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - \frac{3}{2} i \alpha \partial_q V [\tilde{N}, \hat{Q}^{(2)}] + \frac{1}{2} \alpha^2 \partial_q V [\tilde{N}, [\tilde{N}, \hat{Q}^{(1)}]]. \]  
(3.44)
Thus the moving-frame QRPA equation of $O(p^2)$ is not gauge invariant because $[[\hat{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}]$ and $[\hat{N}, \hat{Q}^{(2)}]$ do not vanish. In fact, it is gauge invariant if we take into account the third-order operator $Q^{(3)}$, as we shall see in the next section. To sum up, with $\hat{Q}^{(2)}$ included, all the ASCC equations and conditions of $O(1)$ and $O(p)$ are invariant under the gauge transformation in Example 1. The moving-frame QRPA equation of $O(p^2)$ is not gauge invariant.

3.2.2. Gauge symmetry in Example 2. In Example 2, the generator $G = \frac{2}{3} n^2$ generates $\varphi \rightarrow \varphi + \alpha n$. By transforming the argument of the state vector as $\varphi \rightarrow \varphi - \alpha n$,

$$ e^{-i\varphi \hat{N}} e^{i\hat{G}} |\phi(q)\rangle $$

$$ \rightarrow e^{-i(\varphi - \alpha n)\hat{N}} e^{i\hat{G}} |\phi(q)\rangle = e^{-i\varphi \hat{N}} e^{i\alpha n \hat{N}} e^{i\hat{G}} |\phi(q)\rangle $$

$$ = e^{-i\varphi \hat{N}} \exp \left\{ ip\hat{Q}^{(1)} + in\Theta^{(1)} + \frac{i}{2} p^2 \hat{Q}^{(2)} + \frac{i}{2} n^2 \hat{X} + i\alpha n \hat{N} + \frac{1}{2} [i\alpha n \hat{N}, ip\hat{Q}^{(1)} + in\Theta^{(1)}] + \cdots \right\} |\phi(q)\rangle $$

$$ = e^{-i\varphi \hat{N}} \exp \left\{ ip\hat{Q}^{(1)} + in(\Theta^{(1)} + \alpha \hat{N}) + \frac{i}{2} p^2 \hat{Q}^{(2)} + \frac{i}{2} n^2 (\hat{X} + i\alpha [\hat{N}, \hat{Q}^{(1)}]) + \cdots \right\} |\phi(q)\rangle , \quad (3.45) $$

we obtain

$$ \hat{\Theta}^{(1)} \rightarrow \hat{\Theta}^{(1)} + \alpha \hat{N}, \quad (3.46) $$

$$ \hat{\Theta}^{(2)} \rightarrow \hat{\Theta}^{(2)} + i\alpha [\hat{N}, \hat{\Theta}^{(1)}], \quad (3.47) $$

$$ \hat{X} \rightarrow \hat{X} + \frac{i}{2} \alpha [\hat{N}, \hat{Q}^{(1)}] \quad (3.48) $$

Because $\hat{Q}^{(2)}$ is not transformed, the moving-frame HFB & QRPA equations are invariant as in the case of the first-order expansion of $\hat{G}$. The canonical-variable conditions involving the second-order operators are transformed as

$$ \langle \phi(q)|\hat{\Theta}^{(2)}|\phi(q)\rangle \rightarrow \langle \phi(q)|\hat{\Theta}^{(2)}|\phi(q)\rangle + i\alpha \langle \phi(q)|[\hat{N}, \hat{\Theta}^{(1)}]|\phi(q)\rangle = \alpha \quad (3.49) $$

$$ \langle \phi(q)|\hat{X}|\phi(q)\rangle \rightarrow \langle \phi(q)|\hat{X}|\phi(q)\rangle + \frac{i}{2} \alpha \langle \phi(q)|[\hat{N}, \hat{Q}^{(1)}]|\phi(q)\rangle = 0 \quad (3.50) $$

Here we employ the conditions (3.23) and (3.24) instead of (3.19) and (3.20). The other canonical-variable conditions are invariant under the gauge transformation. Among the canonical-variable conditions, most noteworthy is (3.49). $\hat{G} = \alpha n^2/2$ gives a contribution of $\alpha n$ to the gauge transformation of the canonicity condition (2.6). The transformation (3.49) correctly reflects the gauge transformation of the canonicity condition (2.6).
3.2.3. **Gauge symmetry in Example 3.** \( G = \alpha q n \) generates \((\varphi, p) \to (\varphi + \alpha q, p - \alpha n)\). By considering the transformation,

\[
e^{-i\varphi \hat{N}} e^{i\hat{G}} \left| \phi(q) \right> \to e^{-i(\varphi - \alpha q) \hat{N}} \exp \left\{ i \left[ (p + \alpha n) \hat{Q}^{(1)} + \frac{1}{2} (p + \alpha n)^2 \hat{Q}^{(2)} 
+ n \hat{\Theta}^{(1)} + \frac{1}{2} n^2 \hat{\Theta}^{(2)} + (p + \alpha n) n \hat{X} \right] \right\} |\phi(q)\rangle
\]

\[
= e^{-i\varphi \hat{N}} e^{i\alpha q \hat{N}} e^{-iq\hat{P}} \exp \left\{ i \left[ p \hat{Q}^{(1)} + n (\hat{\Theta}^{(1)} + \alpha \hat{Q}^{(1)}) + \frac{1}{2} p^2 \hat{Q}^{(2)} 
+ \frac{1}{2} n^2 (\hat{\Theta}^{(2)} + 2\alpha \hat{X} + \alpha^2 \hat{Q}^{(2)}) + pn (\hat{X} + \alpha \hat{Q}^{(2)}) \right] \right\} |q=0|\phi(0)\rangle
\]

\[
= e^{-i\varphi \hat{N}} e^{-iq(\hat{P} - \alpha \hat{N})} \exp \left\{ i \left[ p \hat{Q}^{(1)} + n (\hat{\Theta}^{(1)} + \alpha \hat{Q}^{(1)}) + \frac{1}{2} p^2 \hat{Q}^{(2)} 
+ \frac{1}{2} n^2 (\hat{\Theta}^{(2)} + 2\alpha \hat{X}) + pn (\hat{X} + \alpha \hat{Q}^{(2)}) \right] \right\} |q=0|\phi(0)\rangle,
\] (3.51)

we find

\[
\hat{P} \to \hat{P} - \alpha \hat{N},
\] (3.52)

\[
\hat{\Theta}^{(1)} \to \hat{\Theta}^{(1)} + \alpha \hat{Q}^{(1)},
\] (3.53)

\[
\hat{\Theta}^{(2)} \to \hat{\Theta}^{(2)} + 2\alpha \hat{X},
\] (3.54)

\[
\hat{X} \to \hat{X} + \alpha \hat{Q}^{(2)}.
\] (3.55)

The inclusion of the second-order operators does not affect the discussion in Ref. [17]. While the moving-frame HFB equation is gauge invariant, the moving-frame QRPA equation of \( O(p) \) is not. The moving-frame QRPA equation of \( O(p^2) \) is not gauge invariant because \([\hat{N}, \hat{Q}^{(1)}] \neq 0\). As easily seen, the canonical-variable conditions are all gauge invariant.

3.2.4. **Gauge symmetry in Example 4.** \( G = \alpha q n \) generates \( \varphi \to (1 + \alpha)\varphi = e^{\alpha}\varphi, \ n \to (1 - \alpha)n = e^{-\alpha}n \). By considering the transformation,

\[
e^{-i\varphi \hat{N}} e^{i\hat{G}} \left| \phi(q) \right> \phi(q)\rangle
\]

\[
= e^{-i(1 - \alpha) \varphi \hat{N}} \exp \left\{ i p \hat{Q}^{(1)} + i (1 + \alpha) n \hat{\Theta}^{(1)} + \frac{i}{2} p^2 \hat{Q}^{(2)} 
+ \frac{i}{2} (1 + \alpha)^2 n^2 \hat{\Theta}^{(2)} + p(1 + \alpha) n \hat{X} \right\} |\phi(q)\rangle
\]

\[
= e^{-i(1 - \alpha) \varphi \hat{N}} \exp \left\{ i p \hat{Q}^{(1)} + i n (1 + \alpha) \hat{\Theta}^{(1)} + \frac{i}{2} p^2 \hat{Q}^{(2)} 
+ \frac{i}{2} n^2 (1 + 2\alpha) \hat{\Theta}^{(2)} + pn (1 + \alpha) \hat{X} \right\} |\phi(q)\rangle,
\] (3.56)
we find

\[ \hat{\Theta}^{(1)} \rightarrow (1 + \alpha) \hat{\Theta}^{(1)} = e^{\alpha \Theta^{(1)}}, \]
\[ (3.57) \]
\[ \hat{\Theta}^{(2)} \rightarrow (1 + 2\alpha) \hat{\Theta}^{(2)} = e^{2\alpha \Theta^{(2)}}, \]
\[ (3.58) \]
\[ \hat{X} \rightarrow (1 + \alpha) \hat{X} = e^{\alpha \hat{X}}, \]
\[ (3.59) \]
\[ \hat{N} \rightarrow (1 - \alpha) \hat{N} = e^{-\alpha \hat{N}}. \]
\[ (3.60) \]

Also in this case, the inclusion of the second-order operators does not affect the discussion in Ref. [17]. It is clear that the moving-frame HFB & QRPA equations and the canonical-variable conditions are gauge invariant.

4. Expansion of \( \hat{G} \) up to the third order

In this section, we consider the expansion of \( \hat{G} \) up to the third order. (For the derivation of the basic equations in this section, see Appendix A.) \( \hat{G} \) is expanded as

\[
\hat{G}(q, p, n) = p\hat{Q}^{(1)}(q) + n\hat{\Theta}^{(1)}(q) + \frac{1}{2} p^2 \hat{Q}^{(2)}(q) + \frac{1}{2} n^2 \hat{\Theta}^{(2)}(q) + pn\hat{X} \\
+ \frac{1}{3!} p^3 \hat{Q}^{(3)}(q) + \frac{1}{3!} n^3 \hat{\Theta}^{(3)}(q) + \frac{1}{2} p^2 n \hat{O}^{(2,1)}(q) + \frac{1}{2} pn^2 \hat{O}^{(1,2)}(q). 
\]
\[ (4.1) \]

The time-reversal symmetry of these operators are as follows.

\[
\mathcal{T}\hat{Q}^{(i)}(q)\mathcal{T}^{-1} = (-1)^{(i-1)}\hat{Q}^{(i)}(q), \quad (i = 1, 2, 3), \\
\mathcal{T}\hat{\Theta}^{(i)}(q)\mathcal{T}^{-1} = -\hat{\Theta}^{(i)}(q), \quad (i = 1, 2, 3), \\
\mathcal{T}\hat{X}(q)\mathcal{T}^{-1} = \hat{X}(q), \\
\mathcal{T}\hat{O}^{(2,1)}(q)\mathcal{T}^{-1} = -\hat{O}^{(2,1)}(q), \\
\mathcal{T}\hat{O}^{(1,2)}(q)\mathcal{T}^{-1} = \hat{O}^{(1,2)}(q). 
\]

The third-order operators do not appear in the zeroth- and first-order canonical-variable conditions. They are involved in the second-order canonical-variable conditions:
The second-order canonical-variable conditions are gauge invariant if the expansion of \( \hat{G} \) up to the third order is taken into account. The conditions (4.4)-(4.7) can be rewritten as follows.

\[ \frac{1}{2} \langle \phi(q) | \hat{O}^{(3)} - \frac{i}{2} [\hat{O}^{(1)}, \hat{Q}^{(2)}] | \phi(q) \rangle = 0, \tag{4.2} \]
\[ \frac{1}{2} \langle \phi(q) | \hat{\Theta}^{(3)} - \frac{i}{2} [\hat{\Theta}^{(1)}, \hat{\Theta}^{(2)}] | \phi(q) \rangle = 0, \tag{4.3} \]
\[ \frac{1}{2} \langle \phi(q) | \hat{O}^{(2,1)} + \frac{i}{2} [\hat{\Theta}^{(1)}, \hat{Q}^{(2)}] - \frac{1}{3} [\hat{\Theta}^{(1)}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] + i[\hat{X}, \hat{Q}^{(1)}] | \phi(q) \rangle = 0, \tag{4.4} \]
\[ \frac{1}{2} \langle \phi(q) | \hat{O}^{(1,2)} + \frac{i}{2} [\hat{Q}^{(1)}, \hat{\Theta}^{(2)}] - \frac{1}{3} [\hat{Q}^{(1)}, \hat{\Theta}^{(1)}], \hat{\Theta}^{(1)}] + i[\hat{X}, \hat{\Theta}^{(1)}] | \phi(q) \rangle = 0, \tag{4.5} \]
\[ \langle \phi(q) | \hat{O}^{(2,1)} + \frac{i}{2} [\hat{Q}^{(2)}, \hat{\Theta}^{(1)}] + \frac{1}{6} [\hat{\Theta}^{(1)}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] | \phi(q) \rangle = 0, \tag{4.6} \]
\[ \langle \phi(q) | \hat{O}^{(1,2)} + \frac{i}{2} [\hat{\Theta}^{(2)}, \hat{Q}^{(1)}] + \frac{1}{6} [\hat{Q}^{(1)}, \hat{\Theta}^{(1)}], \hat{\Theta}^{(1)}] | \phi(q) \rangle = 0, \tag{4.7} \]
\[ \frac{1}{2} \langle \phi(q) | (i[\hat{P}, \hat{Q}^{(2)}] - [[\hat{P}, \hat{Q}^{(1)}], \hat{Q}^{(1)}]) | \phi(q) \rangle = 0, \tag{4.8} \]
\[ \frac{1}{2} \langle \phi(q) | (i[\hat{P}, \hat{\Theta}^{(2)}] - [[\hat{P}, \hat{\Theta}^{(1)}], \hat{\Theta}^{(1)}]) | \phi(q) \rangle = 0, \tag{4.9} \]
\[ \frac{1}{2} \langle \phi(q) | (i[\hat{N}, \hat{Q}^{(2)}] - [[\hat{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}]) | \phi(q) \rangle = 0, \tag{4.10} \]
\[ \frac{1}{2} \langle \phi(q) | (i[\hat{N}, \hat{\Theta}^{(2)}] - [[\hat{N}, \hat{\Theta}^{(1)}], \hat{\Theta}^{(1)}]) | \phi(q) \rangle = 0, \tag{4.11} \]
\[ \langle \phi(q) | i[\hat{P}, \hat{X}] - \frac{1}{2} ([\hat{P}, \hat{Q}^{(1)}], \hat{\Theta}^{(1)}] + [[\hat{P}, \hat{\Theta}^{(1)}], \hat{Q}^{(1)}]) | \phi(q) \rangle = 0, \tag{4.12} \]
\[ \langle \phi(q) | i[\hat{N}, \hat{X}] - \frac{1}{2} ([\hat{N}, \hat{Q}^{(1)}], \hat{\Theta}^{(1)}] + [[\hat{N}, \hat{\Theta}^{(1)}], \hat{Q}^{(1)}]) | \phi(q) \rangle = 0. \tag{4.13} \]
We shall consider the gauge transformation of operators in Example 1. By transforming the argument of the state vector as \( q \rightarrow q - \alpha n \) and \( \varphi \rightarrow \varphi - \alpha p \),

\[
e^{-i\varphi N} e^{iG(q,p,n)} |\phi(q)\rangle
\]

\[
\rightarrow e^{-i(\varphi - \alpha p)N} e^{iG(q-an,p,n)} |\phi(q - \alpha n)\rangle = e^{-i\varphi N} e^{i\alpha nP + i\alpha p N} e^{iG(q,p,n)} |\phi(q)\rangle
\]

\[
e^{-i\varphi N} \exp \left\{ i\hat{G} + i\alpha (p\tilde{N} + n\hat{P}) + \frac{1}{2} [i\alpha (p\tilde{N} + n\hat{P}), i\hat{G}] + \frac{1}{12} [i\hat{G}, [i\hat{G}, i\alpha (p\tilde{N} + n\hat{P})]] \right\} |\phi(q)\rangle
\]

\[
e^{-i\varphi N} \exp \left\{ ip(Q^{(1)} + \alpha \tilde{N}) + in(\hat{\Theta}^{(1)} + \alpha \hat{P})
\right. 
\]

\[
+ \frac{i}{2} n^2 (\hat{\Theta}^{(2)} + i\alpha \tilde{N})] + \frac{i}{2} [\tilde{N}, \tilde{\Theta}^{(1)}] + \frac{i}{2} [\tilde{P}, \tilde{\Theta}^{(1)}])
\]

\[
+ \frac{i}{2} n^2 (\hat{\Theta}^{(2)} + i\alpha \tilde{N})] + \frac{i}{2} [\tilde{N}, \tilde{\Theta}^{(1)}] + \frac{i}{2} [\hat{P}, \hat{\Theta}^{(1)}])
\]

\[
\left. + \frac{i}{2} [\hat{P}, \hat{\Theta}^{(1)}] \right\} |\phi(q)\rangle, \quad (4.18)
\]

we find

\[
\hat{Q}^{(1)} \rightarrow \hat{Q}^{(1)} + \alpha \tilde{N}, \quad (4.19)
\]

\[
\hat{\Theta}^{(1)} \rightarrow \hat{\Theta}^{(1)} + \alpha \hat{P}, \quad (4.20)
\]

\[
\hat{Q}^{(2)} \rightarrow \hat{Q}^{(2)} + i\alpha [\tilde{N}, \hat{Q}^{(1)}], \quad (4.21)
\]

\[
\hat{\Theta}^{(2)} \rightarrow \hat{\Theta}^{(2)} + i\alpha [\hat{P}, \hat{\Theta}^{(1)}], \quad (4.22)
\]

\[
\hat{X} \rightarrow \hat{X} + \frac{i}{2} [\hat{P}, \hat{\Theta}^{(1)}] + \frac{i}{2} [\tilde{P}, \tilde{\Theta}^{(1)}], \quad (4.23)
\]

\[
\hat{Q}^{(3)} \rightarrow \hat{Q}^{(3)} + \frac{3}{2} \alpha i [\tilde{N}, \hat{Q}^{(2)}] - \frac{1}{2} \alpha [\hat{Q}^{(1)}, [\hat{Q}^{(1)}, \tilde{N}]], \quad (4.24)
\]

\[
\hat{\Theta}^{(3)} \rightarrow \hat{\Theta}^{(3)} + \frac{3}{2} \alpha i [\hat{P}, \hat{\Theta}^{(2)}] - \frac{1}{2} \alpha [\hat{\Theta}^{(1)}, [\hat{\Theta}^{(1)}, \hat{P}]], \quad (4.25)
\]

\[
\hat{\hat{O}}^{(1,2)} \rightarrow \hat{\hat{O}}^{(1,2)} + \frac{i}{2} \alpha [\tilde{N}, \tilde{\Theta}^{(2)}] + i\alpha [\tilde{P}, \tilde{X}]
\]

\[
- \frac{1}{6} \alpha ([\hat{Q}^{(1)}, [\hat{\Theta}^{(1)}, \hat{P}]] + [\hat{\Theta}^{(1)}, [\hat{Q}^{(1)}, \hat{P}]] + [\hat{\Theta}^{(1)}, [\hat{\Theta}^{(1)}, \tilde{N}]]), \quad (4.26)
\]

\[
\hat{\hat{O}}^{(2,1)} \rightarrow \hat{\hat{O}}^{(2,1)} + \frac{i}{2} \alpha [\hat{P}, \hat{Q}^{(2)}] + i\alpha [\tilde{N}, \tilde{X}]
\]

\[
- \frac{1}{6} \alpha ([\hat{\Theta}^{(1)}, [\hat{Q}^{(1)}, \tilde{N}]] + [\hat{Q}^{(1)}, [\hat{\Theta}^{(1)}, \tilde{N}]] + [\hat{Q}^{(1)}, [\hat{Q}^{(1)}, \hat{P}]]). \quad (4.27)
\]
As one can ascertain easily, the second-order canonical-variable conditions (4.2)-(4.13) are invariant under this transformation. For example, in the case of the $O(p^2)$ canonical-variable condition (4.2),

\[
\langle \phi(q)|\hat{Q}^{(3)} - \frac{i}{2}[\hat{Q}^{(1)}, \hat{Q}^{(2)}]|\phi(q)\rangle = 0
\]

\[
\rightarrow \langle \phi(q)|\hat{Q}^{(3)} + \frac{3}{2} \alpha_i[\hat{N}, \hat{Q}^{(2)}] - \frac{1}{2} \alpha_i[\hat{Q}^{(1)}, [\hat{Q}^{(1)}, \hat{N}]] - \frac{i}{2}[\hat{Q}^{(1)} + \alpha \hat{N}, \hat{Q}^{(2)} + i \alpha [\hat{N}, \hat{Q}^{(1)}]]|\phi(q)\rangle
\]

\[
= \langle \phi(q)|\hat{Q}^{(3)} - \frac{i}{2}[\hat{Q}^{(1)}, \hat{Q}^{(2)}]|\phi(q)\rangle + \alpha \langle \phi(q)|i[\hat{N}, \hat{Q}^{(2)}] - [\hat{Q}^{(1)}, [\hat{Q}^{(1)}, \hat{N}]]|\phi(q)\rangle
\]

\[
= 0.
\]

(4.28)

It is clear that this canonical-variable condition is not gauge invariant unless $\hat{Q}^{(3)}$ is taken into account.

We shall move on to the gauge symmetry of the moving-frame QRPA equation of $O(p^2)$. The moving-frame HFB & QRPA equations are given as follows.

Moving-frame HFB equation

\[
\delta \langle \phi(q)|\hat{H} - \lambda \hat{N} - \partial_q V \hat{Q}^{(1)}|\phi(q)\rangle = 0.
\]

(4.29)

Moving-frame QRPA equations

\[
\delta \langle \phi(q)|[\hat{H} - \lambda \hat{N}, \hat{Q}^{(1)}] - \frac{1}{i} B(q) \hat{P} - \frac{1}{i} \partial_q V \hat{Q}^{(2)}|\phi(q)\rangle = 0.
\]

(4.30)

\[
\delta \langle \phi(q)|[\hat{H} - \lambda \hat{N} - \partial_q V \hat{Q}^{(1)}, \frac{1}{i} \hat{P}] - C(q) \hat{Q}^{(1)} - \partial_q \lambda \hat{N}
\]

\[
- \frac{1}{2B} \partial_q V \left\{[[\hat{H} - \lambda \hat{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - i[\hat{H} - \lambda \hat{N}, \hat{Q}^{(2)}] + \partial_q V (\hat{Q}^{(3)} - \frac{i}{2}[\hat{Q}^{(1)}, \hat{Q}^{(2)}])\right\}|\phi(q)\rangle = 0.
\]

(4.31)

As discussed in the previous section, the first three terms are gauge invariant in Eq. (4.31). We only have to check the gauge symmetry of the rest. With the use of the previous result (3.44), we easily obtain

\[
[[\hat{H} - \lambda \hat{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - i[\hat{H} - \lambda \hat{N}, \hat{Q}^{(2)}] + \partial_q V (\hat{Q}^{(3)} - \frac{i}{2}[\hat{Q}^{(1)}, \hat{Q}^{(2)}])
\]

\[
\rightarrow [[\hat{H} - \lambda \hat{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - i[\hat{H} - \lambda \hat{N}, \hat{Q}^{(2)}] - \frac{i}{2} \partial_q V (\hat{Q}^{(1)}, \hat{Q}^{(2)})
\]

\[
+ \frac{1}{2} \alpha \partial_q V ([\hat{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}) - \frac{3}{2} i \alpha \partial_q V [\hat{N}, \hat{Q}^{(2)}]
\]

\[
+ \partial_q V (\hat{Q}^{(3)} + \frac{3}{2} \alpha i [\hat{N}, \hat{Q}^{(2)}] - \frac{1}{2} \alpha [\hat{Q}^{(1)}, [\hat{Q}^{(1)}, \hat{N}]]])
\]

\[
= [[\hat{H} - \lambda \hat{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - i[\hat{H} - \lambda \hat{N}, \hat{Q}^{(2)}] - \frac{i}{2} \partial_q V (\hat{Q}^{(1)}, \hat{Q}^{(2)}) + \partial_q V \hat{Q}^{(3)}.
\]

(4.32)

Thus, the moving-frame QRPA equation of $O(p^2)$ is gauge invariant. As discussed in Ref. [17], it is no trivial that the moving-frame QRPA equation of $O(p^2)$ is gauge invariant if the third-order operator is included. At the level of the equation of collective submanifold, the
gauge symmetry of the equation of motion is conserved. Specifically, with the transformation of the Lagrange multiplier
\[ \lambda \rightarrow \lambda - \alpha \partial q V - \frac{1}{2} \alpha \partial q B p^2 \]
the equation of collective submanifold is gauge invariant. The change of the Lagrange multiplier \( \lambda \) under the gauge transformation (4.33) contains a term of \( O(p^2) \). Therefore, when the equation of collective submanifold is divided into the three equations depending on the order of \( p \) as shown in Eqs. (A22)-(A24), the equation of \( O(p^2) \) is not gauge invariant. The gauge invariance of the moving-frame QRPA equation of \( O(p^2) \) can be attributed by the fact that it is derived by using both of the \( O(1) \) and \( O(p^2) \) expansions of the collective submanifold.

5. Concluding remarks
In this paper, we have analyzed the gauge symmetry breaking by the adiabatic approximation in the ASCC method. A particular emphasis is put on the gauge symmetry breaking for the non-point gauge transformations. We have discussed the gauge symmetry breaking due to the adiabatic approximation to \( \hat{G} \) and a possible extension of the ASCC method including the higher-order operators. As we have seen in Ref. [17] and in this paper, there are two sources of the gauge symmetry breaking in the ASCC method. One is the decomposition of the equation of collective submanifold into the three equations, namely the moving-frame HFB & QRPA equations, depending on the order of \( p \). Example 3 is one example of the gauge symmetry breaking by the decomposition [17]. The other is the truncation of the adiabatic expansion we have discussed in details in this paper.

According to the generalized Thouless theorem [24], to describe states which are not orthogonal to the vacuum \( |\phi(q)\rangle \), it is sufficient to include only \( a^\dagger a^\dagger \) and \( aa \) terms (so-called A-terms) in \( \hat{G} \), and the \( a^\dagger a \) terms (so-called B-terms) are not necessary. However, when the expansion of \( \hat{G} \) is truncated to the first-order, \( [\hat{N}, \hat{Q}] = 0 \) is required for the gauge symmetry of the moving-frame QRPA equations, which implies that \( \hat{Q} \) should contain B-terms. In Refs. [6–8], Hinohara et al. required \( [\hat{N}, \hat{Q}] = 0 \) and successfully solve the moving-frame HFB & QRPA equations numerically. It may be one justification for the requirement of \( [\hat{N}, \hat{Q}] = 0 \) that one can keep the gauge symmetry which exists in the equation of collective submanifold before the adiabatic expansion.

In this paper, to conserve the gauge symmetry, we have introduced the higher-order operators \( \hat{Q}^{(i)} \) \((i > 1)\), instead of introducing the B-part of \( \hat{Q} \) and requiring \( [\hat{N}, \hat{Q}] = 0 \). It would be interesting to investigate the correspondence between the two approaches; one is the approach with the higher-order operators consisting of only A-terms and the other with only the first-order operator containing B-terms as well as the A-terms. This will be investigated in a future publication.

Acknowledgments
The author thanks K. Matsuyanagi and N. Hinohara for their fruitful discussions and comments.

A. Derivation of basic equations
We derive the basic equations in the cases where \( \hat{G} \) is expanded up to the second order and up to the third order. They are derived in a parallel way to Ref. [1].
We start with the expansion up to the second order,
\[
\hat{G}(q, p, n) = p\hat{Q}^{(1)}(q) + n\hat{\Theta}^{(1)}(q) + \frac{1}{2}p^2\hat{Q}^{(2)}(q) + \frac{1}{2}n^2\hat{\Theta}^{(2)}(q) + pn\hat{X}. \tag{A1}
\]

Using the Hadamard lemma [23],
\[
e^X Y e^{-X} = e^{\text{ad}_X Y} = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \cdots, \tag{A2}
\]
we expand the collective Hamiltonian up to the second order
\[
\mathcal{H}(q, p, n) = \langle \phi(q, p, \varphi, n) | \hat{\mathcal{H}} | \phi(q, p, \varphi, n) \rangle = \langle \phi(q) | e^{-i\hat{G}(q, p, n)} \hat{\mathcal{H}} e^{i\hat{G}(q, p, n)} | \phi(q) \rangle = V(q) + \frac{1}{2}B(q)p^2 + \lambda n + \frac{1}{2}D(q)n^2 \tag{A3}
\]
with
\[
V(q) = \langle \phi(q) | \hat{\mathcal{H}} | \phi(q) \rangle, \tag{A4}
\]
\[
B(q) = \langle \phi(q) | [\hat{\mathcal{H}}, i\hat{Q}^{(2)}] | \phi(q) \rangle - \langle \phi(q) | [\hat{\mathcal{H}}, \hat{Q}^{(1)}] | \phi(q) \rangle, \tag{A5}
\]
\[
\lambda(q) = \langle \phi(q) | [\hat{\mathcal{H}}, i\hat{\Theta}^{(1)}(q)] | \phi(q) \rangle, \tag{A6}
\]
\[
D(q) = \langle \phi(q) | [\hat{\mathcal{H}}, i\hat{\Theta}^{(2)}] | \phi(q) \rangle - \langle \phi(q) | [\hat{\mathcal{H}}, \hat{\Theta}^{(1)}(q)] | \phi(q) \rangle. \tag{A7}
\]
Here we have used
\[
\langle \phi(q) | [\hat{\mathcal{H}}, \hat{Q}^{(1)}] | \phi(q) \rangle = 0, \tag{A8}
\]
\[
\langle \phi(q) | i[\hat{\mathcal{H}}, \hat{X}] - \frac{1}{2}([\hat{\mathcal{H}}, \hat{\Theta}^{(1)}(q)], \hat{Q}^{(1)}(q)) + ([\hat{\mathcal{H}}, \hat{Q}^{(1)}(q)], \hat{\Theta}^{(1)}(q)) | \phi(q) \rangle = 0. \tag{A9}
\]
Because we are interested in the gauge symmetry, we adopt the collective Hamiltonian up to \(O(n)\),
\[
\mathcal{H}(q, p, n) = V(q) + \frac{1}{2}B(q)p^2 + \lambda(q)n. \tag{A10}
\]

The moving-frame HFB & QRPA equations are derived from
\underline{Eq. of collective submanifold:}
\[
\delta\langle \phi(q, p, \varphi, n) | \hat{\mathcal{H}} - \frac{\partial \mathcal{H}}{\partial p} \hat{P} - \frac{\partial \mathcal{H}}{\partial q} \hat{Q} - \frac{\partial \mathcal{H}}{\partial \varphi} \hat{\Theta} - \frac{\partial \mathcal{H}}{\partial n} \hat{N} | \phi(q, p, \varphi, n) \rangle = 0, \tag{A11}
\]
\[
\iff \delta\langle \phi(q, p, n) | \hat{\mathcal{H}} - \frac{\partial \mathcal{H}}{\partial p} \hat{P} - \frac{\partial \mathcal{H}}{\partial q} \hat{Q} - \frac{\partial \mathcal{H}}{\partial n} \hat{N} | \phi(q, p, n) \rangle = 0, \tag{A12}
\]
\[
\iff \delta\langle \phi(q) | e^{-i\hat{G}(q)} \hat{\mathcal{H}} e^{i\hat{G}(q)} - \frac{\partial \mathcal{H}}{\partial p} \hat{P}' - \frac{\partial \mathcal{H}}{\partial q} \hat{Q}' - \frac{\partial \mathcal{H}}{\partial n} \hat{N}' | \phi(q) \rangle = 0, \tag{A13}
\]
with \(\hat{P}' = e^{-i\hat{G}} \hat{P} e^{i\hat{G}}, \hat{Q}' = e^{-i\hat{G}} \hat{Q} e^{i\hat{G}}, \text{ and } \hat{N}' = e^{-i\hat{G}} \hat{N} e^{i\hat{G}}\). For the first equivalence, \(\partial \mathcal{H} / \partial \varphi = 0\) is used.

The canonical-variable conditions are derived from the canonicity conditions.
Canoncity conditions:

\[
\langle \phi(q, p, \varphi, n) | \hat{P} | \phi(q, p, \varphi, n) \rangle = \langle \phi(q) | \hat{P}' | \phi(q) \rangle = p, \tag{A14}
\]

\[
\langle \phi(q, p, \varphi, n) | \hat{Q} | \phi(q, p, \varphi, n) \rangle = \langle \phi(q) | \hat{Q}' | \phi(q) \rangle = 0, \tag{A15}
\]

\[
\langle \phi(q, p, \varphi, n) | \hat{N} | \phi(q, p, \varphi, n) \rangle = \langle \phi(q) | \hat{N}' | \phi(q) \rangle = n, \tag{A16}
\]

\[
\langle \phi(q, p, \varphi, n) | \hat{\Theta} | \phi(q, p, \varphi, n) \rangle = \langle \phi(q) | \hat{\Theta}' | \phi(q) \rangle = 0, \tag{A17}
\]

with \( \hat{\Theta}' = e^{-iG} \hat{\Theta} e^{iG} \). With the use of the general formula (A2), the unitary transformations of the generators are expanded as

\[
\hat{P}' = i\partial_q - p\partial_q \hat{Q}^{(1)} - n\partial_q \hat{X} - \frac{1}{2} p^2 \left( i[\partial_q \hat{Q}^{(1)}, \hat{Q}^{(1)}] + \partial_q \hat{Q}^{(2)} \right)
- \frac{1}{2} p^2 \left( i[\partial_q \hat{\Theta}^{(1)}, \hat{\Theta}^{(1)}] + \partial_q \hat{\Theta}^{(2)} \right) - \frac{i}{2} pm \left( [\partial_q \hat{Q}^{(1)}, \hat{\Theta}^{(1)}] + [\partial_q \hat{\Theta}^{(1)}, \hat{Q}^{(1)}] \right) + \cdots
= \hat{P} + ip[\hat{P}, \hat{Q}^{(1)}] + in[\hat{P}, \hat{\Theta}^{(1)}] + \frac{1}{2} p^2 \left( i[\hat{P}, \hat{Q}^{(2)}] - [\hat{P}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] \right)
+ \frac{1}{2} p^2 \left( i[\hat{P}, \hat{\Theta}^{(2)}] - [\hat{P}, \hat{\Theta}^{(1)}], \hat{\Theta}^{(1)}] \right)
+ pm \left( i[\hat{P}, \hat{X}] - \frac{1}{2} \left( [\hat{P}, \hat{Q}^{(1)}], \hat{\Theta}^{(1)}] + [\hat{P}, \hat{\Theta}^{(1)}], \hat{Q}^{(1)}] \right) \right) + \cdots, \tag{A18}
\]

\[
\hat{Q}' = \hat{Q}^{(1)} + p\hat{Q}^{(2)} + n(\hat{X} + \frac{i}{2}[\hat{Q}^{(1)}, \hat{\Theta}^{(1)}])
- \frac{i}{4} p^2 [\hat{Q}^{(1)}, \hat{Q}^{(2)}] + \frac{1}{2} p^2 \left( \frac{i}{2} [\hat{Q}^{(1)}, \hat{\Theta}^{(1)}] - \frac{1}{3} [\hat{Q}^{(1)}, \hat{\Theta}^{(1)}], \hat{Q}^{(1)}] + i[\hat{X}, \hat{\Theta}^{(1)}] \right)
+ \frac{pm}{2} \left( i[\hat{Q}^{(2)}, \hat{\Theta}^{(1)}] - \frac{1}{3} [\hat{Q}^{(1)}, \hat{\Theta}^{(1)}], \hat{Q}^{(1)}] \right) + \cdots, \tag{A19}
\]

\[
\hat{\Theta}' = \hat{\Theta}^{(1)} + n\hat{\Theta}^{(2)} + p(\hat{X} + \frac{i}{2}[\hat{Q}^{(1)}, \hat{\Theta}^{(1)}])
- \frac{i}{4} n^2 [\hat{\Theta}^{(1)}, \hat{\Theta}^{(2)}] + \frac{1}{2} p^2 \left( \frac{i}{2} [\hat{\Theta}^{(1)}, \hat{Q}^{(2)}] - \frac{1}{3} [\hat{\Theta}^{(1)}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] + i[\hat{X}, \hat{Q}^{(1)}] \right)
+ \frac{1}{2} pm \left( i[\hat{\Theta}^{(2)}, \hat{Q}^{(1)}] - \frac{1}{3} [\hat{\Theta}^{(1)}, \hat{Q}^{(1)}], \hat{\Theta}^{(1)}] \right) + \cdots, \tag{A20}
\]

\[
\hat{N}' = \hat{N} + ip[\hat{N}, \hat{Q}^{(1)}] + in[\hat{N}, \hat{\Theta}^{(1)}]
+ \frac{1}{2} p^2 \left( i[\hat{N}, \hat{Q}^{(2)}] - [\hat{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] \right) + \frac{1}{2} p^2 \left( i[\hat{N}, \hat{\Theta}^{(2)}] - [\hat{N}, \hat{\Theta}^{(1)}], \hat{\Theta}^{(1)}] \right)
+ pm \left( i[\hat{N}, \hat{X}] - \frac{1}{2} \left( [\hat{N}, \hat{Q}^{(1)}], \hat{\Theta}^{(1)}] + [\hat{N}, \hat{\Theta}^{(1)}], \hat{Q}^{(1)}] \right) \right) + \cdots. \tag{A21}
\]

By substituting (A18)- (A21) to (A14)-(A17), we obtain the zeroth- and first-order canonical-variable conditions (3.10)-(3.22). [One can also define the second-order canonical-variable conditions at this point. However, as shown in Eqs. (4.2)-(4.13), the second-order canonical-variable conditions contain contributions from the third-order operators, which are not taken into account now.]

By substituting (A18)- (A21) to the equation of collective submanifold, we obtain
The zeroth-order equation
\[ \delta\langle \phi(q)|\hat{H}_M|\phi(q)\rangle = 0, \] (A22)
The equation of the order of \( p \)
\[ \delta\langle \phi(q)|[\hat{H}_M, \hat{Q}^{(1)}] - \frac{i}{\hbar}B(q)\hat{P} - \frac{1}{4}\partial_q V\hat{Q}^{(2)}|\phi(q)\rangle = 0, \] (A23)
The equation of the order of \( p^2 \)
\[ \delta\langle \phi(q)|\frac{1}{2}[[\hat{H}_M, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - B(q)\Delta\hat{Q}^{(1)} - \frac{i}{2}\frac{\partial_q}{\partial_q V}\hat{Q}^{(2)}|\phi(q)\rangle = 0, \] (A24)

where
\[ \hat{H}_M = \hat{H} - \lambda\hat{N} - \partial_q \hat{Q}^{(1)}, \] (A25)
\[ \Delta\hat{Q}^{(1)} = \partial_q \hat{Q}^{(1)} + \Gamma(q)\hat{Q}^{(1)}, \] (A26)
\[ \Gamma(q) = -\frac{1}{2B(q)}\partial_q B(q). \] (A27)

We take the first derivative of the zeroth-order equation with respect to \( q \) and obtain
\[ \delta\langle \phi(q)|[\hat{H}_M, \frac{1}{i}\hat{P}] - C(q)\hat{Q}^{(1)} - \partial_q V\Delta\hat{Q}^{(1)} - \partial_q \lambda\hat{N}|\phi(q)\rangle = 0, \] (A28)
with \( C(q) = \partial_q^2 V - \Gamma(q)\partial_q V \). We eliminate \( \Delta\hat{Q}^{(1)} \) from Eq. (A24) with use of Eq. (A28), which leads to the moving-frame QRPA equation of \( O(p^2) \)
\[ \delta\langle \phi(q)|[\hat{H} - \lambda\hat{N} - \partial_q V\hat{Q}^{(1)}, \frac{1}{i}\hat{P}] - C(q)\hat{Q}^{(1)} - \partial_q \lambda\hat{N} - \frac{1}{2B} \left\{ [[\hat{H} - \lambda\hat{N}, \partial_q V\hat{Q}^{(1)}], \hat{Q}^{(1)}] - i\partial_q V[\hat{H} - \lambda\hat{N}, \hat{Q}^{(2)}] \right\} |\phi(q)\rangle = 0. \] (A29)

Equations (A22), (A23) and (A29) are the moving-frame HFB&QRPA equations in the case of the second-order expansion of \( \hat{G} \).

Then we move to the expansion of \( \hat{G} \) up to the third order
\[ \hat{G}(q, p, n) = p\hat{Q}^{(1)}(q) + n\hat{Q}^{(1)}(q) + \frac{1}{2}p^2\hat{Q}^{(2)}(q) + \frac{1}{2}n^2\hat{Q}^{(2)}(q) + pn\hat{X} + \frac{1}{3!}p^3\hat{Q}^{(3)}(q) + \frac{1}{3!}n^3\hat{Q}^{(3)}(q) + \frac{1}{2}p^2 n\hat{Q}^{(2,1)}(q) + \frac{1}{2}n^2\hat{Q}^{(2,1)}(q). \] (A30)

While the third-order operators do not contribute to \( \hat{P}' \), \( \hat{N}' \) and \( e^{-i\hat{G}}\hat{H}e^{i\hat{G}} \) up to the second order, they are involved in the second-order terms of \( \hat{Q}' \) and \( \hat{\Theta}' \) as follows.
\[ \hat{Q}' = \hat{Q}^{(1)} + p\hat{Q}^{(2)} + n(\hat{X} + \frac{i}{2}[\hat{Q}^{(1)}, \hat{\Theta}^{(1)}]) + \frac{1}{2}p^2 \left( \hat{Q}^{(3)} - \frac{i}{2}[\hat{Q}^{(1)}, \hat{Q}^{(2)}] \right) + \frac{1}{2}n^2 \left( \hat{Q}^{(1,2)} + \frac{i}{2}[\hat{Q}^{(1)}, \hat{\Theta}^{(2)}] - \frac{1}{3}[\hat{Q}^{(1)}, \hat{\Theta}^{(1)}], \hat{\Theta}^{(1)}] + i[\hat{X}, \Theta^{(1)}] \right) + pn \left( \hat{Q}^{(2,1)} + \frac{i}{2}[\hat{Q}^{(2)}, \hat{\Theta}^{(1)}] + \frac{1}{3}[\hat{\Theta}^{(1)}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] \right) \cdots, \] (A31)
In this example, order. In this Appendix, we briefly discuss the gauge symmetry in Examples 2-4.

In Sect. 4, we discuss the gauge symmetry in Example 1 when \( \hat{G} \). Gauge transformation in the case of the third-order expansion.

Equations (A22), (A23) and (A33) are the moving-frame HFB & QRPA equations in the \( O(1) \) and \( O(p) \) are unchanged after the inclusion of the third-order operators. The moving-frame QRPA equation of \( O(p^2) \) is derived similarly to the case of the second-order expansion and now given by

\[
\delta\langle \phi(q)|[\hat{H} - \lambda \hat{N} - \partial_q V \hat{Q}^{(1)} - \frac{1}{i} \hat{P}] - C(q)\hat{Q}^{(1)} - \partial_q \lambda \hat{N} - \frac{1}{2B} \partial_q V \left( [\hat{H} - \lambda \hat{N}, \hat{Q}^{(1)}] - i[\hat{H} - \lambda \hat{N}, \hat{Q}^{(2)}] + \partial_q V(\hat{Q}^{(3)} - \frac{i}{2}[\hat{Q}^{(1)}, \hat{Q}^{(2)}]) \right) |\phi(q)\rangle = 0. \tag{A33}
\]

Equations (A22), (A23) and (A33) are the moving-frame HFB & QRPA equations in the case of the third-order expansion.

B. Gauge transformation in the case of the third-order expansion

In Sect. 4, we discuss the gauge symmetry in Example 1 when \( \hat{G} \) is taken up to the third order. In this Appendix, we briefly discuss the gauge symmetry in Examples 2-4.

Example 2

In this example, \( G = \frac{g}{2} n^2 \) generates \( \varphi \rightarrow \varphi + \alpha n \). By transforming the state vector as

\[
e^{-i\varphi \hat{N}} e^{iG} |\phi(q)\rangle
\]

\[
e^{-i(\varphi - \alpha n) \hat{N}} e^{iG} |\phi(q)\rangle
\]

\[
e^{-i\varphi \hat{N}} \exp \left\{ i\hat{G} + ian\hat{N} + \frac{1}{2}[ian\hat{N}, i\hat{G}] + \frac{1}{12}([ian\hat{N}, [ian\hat{N}, i\hat{G}]] + [i\hat{G}, [ian\hat{N}, i\hat{G}]] + \cdots) |\phi(q)\rangle
\]

\[
e^{-i\varphi \hat{N}} \exp \left\{ ip\hat{Q}^{(1)} + in(\Theta^{(1)} + \alpha \hat{N}) + \frac{i}{2} p^2 \hat{Q}^{(2)}
+ \frac{i}{2} n^2(\Theta^{(2)} + i\alpha[\hat{N}, \Theta^{(1)}]) + pm(\hat{X} + \frac{i}{2} \alpha[\hat{N}, \hat{Q}^{(1)}])
+ \frac{i}{6} p^3 \Theta^{(3)} + \frac{i}{6} n^3 \Theta^{(3)} + \frac{i}{2} \alpha[\hat{N}, \hat{Q}^{(2)}] - \frac{1}{2} \alpha[\hat{Q}^{(1)}, [\hat{N}, \hat{N}]]
+ \frac{i}{6} p^2 n \left( \Theta^{(2,1)} + \frac{i}{2} \alpha[\hat{N}, \hat{Q}^{(2)}] - \frac{1}{6} \alpha[\hat{Q}^{(4)}, [\hat{Q}^{(1)}, \hat{N}]] \right)
+ \frac{i}{2} p^2 m \left( \Theta^{(1,2)} + \frac{1}{2} \alpha[\hat{N}, \hat{X}] - \frac{1}{6} \alpha \left( [\hat{Q}^{(1)}, [\hat{N}, \hat{X}]] + [\hat{N}, [\hat{Q}^{(1)}, \hat{N}]] \right) \right) \right\} |\phi(q)\rangle, \tag{B1}
\]
we find the following transformation.

\[
\begin{align*}
\hat{\Theta}^{(1)} & \rightarrow \hat{\Theta}^{(1)} + \alpha \tilde{N}, \\
\hat{\Theta}^{(2)} & \rightarrow \hat{\Theta}^{(2)} + i\alpha [\tilde{N}, \hat{\Theta}^{(1)}], \\
\hat{X} & \rightarrow \hat{X} + \frac{i}{2} \alpha [\tilde{N}, \hat{Q}^{(1)}], \\
\hat{\Theta}^{(3)} & \rightarrow \hat{\Theta}^{(3)} + \frac{3}{2} i\alpha [\tilde{N}, \hat{\Theta}^{(2)}] - \frac{1}{2} \alpha [\hat{\Theta}^{(1)}, [\hat{\Theta}^{(1)}, \tilde{N}]], \\
\tilde{O}^{(2,1)} & \rightarrow \tilde{O}^{(2,1)} + \frac{i}{2} \alpha [\tilde{N}, \tilde{Q}^{(2)}] - \frac{1}{6} \alpha [\tilde{Q}^{(1)}, [\tilde{Q}^{(1)}, \tilde{N}]], \\
\tilde{O}^{(1,2)} & \rightarrow \tilde{O}^{(1,2)} + i\alpha [\tilde{N}, \tilde{X}] - \frac{1}{6} \alpha \left( [\tilde{Q}^{(1)}, [\tilde{\Theta}^{(1)}, \tilde{N}]] + [\tilde{\Theta}^{(1)}, [\tilde{Q}^{(1)}, \tilde{N}]] \right).
\end{align*}
\]  

(B2)  

(B3)  

(B4)  

(B5)  

(B6)  

(B7)

It is clear that the moving-frame HFB equation and the moving-frame QRPA equations of \(O(p)\) and \(O(p^2)\) are invariant under the above transformation. The second-order canonical-variable conditions are also gauge invariant.

Example 3

\(G = \alpha q n\) generates \(\varphi \rightarrow \varphi + \alpha q\) and \(p \rightarrow p - \alpha n\). By considering the following transformation,

\[
e^{-i\varphi \tilde{N}} e^{i\tilde{G}} |\phi(q)\rangle
\]

\[
\rightarrow e^{-i(\varphi - \alpha q)\tilde{N}} \exp \left\{ i \left[ (p + \alpha n) \tilde{Q}^{(1)} + \frac{1}{2} (p + \alpha n)^2 \tilde{Q}^{(2)} + n\tilde{\Theta}^{(1)} + \frac{1}{2} n^2 \tilde{\Theta}^{(2)} \\
+ (p + \alpha n) n \tilde{X} + \frac{1}{6} (p + \alpha n)^3 \tilde{Q}^{(3)} + \frac{1}{2} (p + \alpha n)^2 n \tilde{O}^{(2,1)} \\
+ \frac{1}{2} (p + \alpha n) n^2 \tilde{\Theta}^{(1,2)} + \frac{1}{6} n^3 \tilde{\Theta}^{(3)} \right] \right\} |\phi(q)\rangle
\]

\[
= e^{-i\varphi \tilde{N}} e^{-i\varphi (\tilde{P} - \alpha \tilde{N})} \exp \left\{ i \left[ p \tilde{Q}^{(1)} + n (\tilde{\Theta}^{(1)} + \alpha \tilde{Q}^{(1)}) + \frac{1}{2} p^2 \tilde{Q}^{(2)} \\
+ \frac{1}{2} n^2 (\tilde{\Theta}^{(2)} + 2 \alpha \tilde{X}) + p n (\tilde{X} + \alpha \tilde{Q}^{(2)}) + \frac{1}{6} p^3 \tilde{Q}^{(3)} \\
+ \frac{1}{2} p^2 n (\tilde{O}^{(2,1)} + \alpha \tilde{Q}^{(3)}) + \frac{1}{2} p n^2 (\tilde{O}^{(1,2)} + 2 \alpha \tilde{O}^{(2,1)}) + \frac{1}{6} n^3 \left( \tilde{\Theta}^{(3)} + 3 \alpha \tilde{O}^{(1,2)} \right) \right] \right\} |\varphi = 0|\phi(0)\rangle,
\]

(B8)

we find

\[
\begin{align*}
\tilde{P} & \rightarrow \tilde{P} - \alpha \tilde{N}, \\
\tilde{\Theta}^{(1)} & \rightarrow \tilde{\Theta}^{(1)} + \alpha \tilde{Q}^{(1)}, \\
\tilde{\Theta}^{(2)} & \rightarrow \tilde{\Theta}^{(2)} + 2 \alpha \tilde{X}, \\
\tilde{\Theta}^{(3)} & \rightarrow \tilde{\Theta}^{(3)} + 3 \alpha \tilde{O}^{(1,2)}, \\
\tilde{O}^{(2,1)} & \rightarrow \tilde{O}^{(2,1)} + \alpha \tilde{Q}^{(3)}, \\
\tilde{O}^{(1,2)} & \rightarrow \tilde{O}^{(1,2)} + 2 \alpha \tilde{O}^{(2,1)}.
\end{align*}
\]  

(B9)  

(B10)  

(B11)  

(B12)  

(B13)  

(B14)  

(B15)
As in the case of the second-order expansion in Sect. 3, the moving-frame HFB equation is gauge invariant, while the moving-frame QRPA equation of $O(p)$ is not. The moving-frame QRPA equation of $O(p^2)$ is not gauge invariant because $[\tilde{N}, \hat{Q}^{(1)}] \neq 0$. The canonical-variable conditions up to the second order are gauge invariant.

**Example 4**

$G = \alpha \varphi n$ generates $\varphi \rightarrow (1 + \alpha) \varphi = e^{\alpha \varphi}$ and $n \rightarrow (1 - \alpha) n = e^{-\alpha} n$. Therefore, by considering

$$e^{-i\tilde{N}} e^{i\hat{G}} |\phi(q)| \phi(q)\rangle$$

$$\rightarrow e^{-i(1-\alpha)\varphi N} \exp \left[ i\hat{G}(q, p, (1+\alpha)n) \right] |\phi(q)\rangle,$$

we find

\begin{align*}
\hat{\Theta}^{(1)} &\rightarrow (1 + \alpha) \hat{\Theta}^{(1)} = e^{\alpha} \Theta^{(1)}, & (B16) \\
\hat{\Theta}^{(2)} &\rightarrow (1 + 2\alpha) \hat{\Theta}^{(2)} = e^{2\alpha} \Theta^{(2)}, & (B17) \\
\hat{X} &\rightarrow (1 + \alpha) \hat{X} = e^{\alpha} \hat{X}, & (B18) \\
\tilde{N} &\rightarrow (1 - \alpha) \tilde{N} = e^{-\alpha} \tilde{N}, & (B19) \\
\hat{\Theta}^{(3)} &\rightarrow (1 + 3\alpha) \hat{\Theta}^{(3)} = e^{3\alpha} \Theta^{(3)}, & (B20) \\
\hat{O}^{(2, 1)} &\rightarrow (1 + \alpha) \hat{O}^{(2, 1)} = e^{\alpha} O^{(2, 1)} , & (B21) \\
\hat{O}^{(1, 2)} &\rightarrow (1 + 2\alpha) \hat{O}^{(1, 2)} = e^{2\alpha} O^{(1, 2)} . & (B22) 
\end{align*}

The moving-frame HFB & QRPA equations are gauge invariant. It is also clear that the canonical-variable conditions are gauge invariant.

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