Charged dust solutions for the warp drive spacetime

Osvaldo L. Santos-Pereira 1 · Everton M. C. Abreu 2,3,4 · Marcelo B. Ribeiro 1,4,5

Received: 22 December 2020 / Accepted: 13 February 2021 / Published online: 22 February 2021
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC part of Springer Nature 2021

Abstract
The Alcubierre warp drive metric is a spacetime construction where a massive particle located inside a spacetime distortion, called warp bubble, travels at velocities arbitrarily higher than the velocity of light. This theoretically constructed spacetime geometry is a consequence of general relativity where global superluminal velocities, also known as warp speeds, are possible, whereas local speeds are limited to subluminal ones as required by special relativity. In this work we analyze the solutions of the Einstein equations having charged dust energy-momentum tensor as source for warp velocities. The Einstein equations with the cosmological constant are written and all solutions having energy-momentum tensor components for electromagnetic fields generated by charged dust are presented, as well as the respective energy conditions. The results show an interplay between the energy conditions and the electromagnetic field such that in some cases the former can be satisfied by both positive and negative matter density. In other cases the dominant and null energy conditions are violated. A result connecting the electric energy density with the cosmological constant is also presented, as well as the effects of the electromagnetic field on the bubble dynamics.

Keywords Warp drive · Charged dust · Electromagnetic tensor · Curved spacetime

Marcelo B. Ribeiro
mbr@if.ufrj.br
Osvaldo L. Santos-Pereira
olsp@if.ufrj.br
Everton M. C. Abreu
evertonabreu@ufrrj.br
1 Physics Institute, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil
2 Department of Physics, Universidade Federal Rural do Rio de Janeiro, Seropédica, Brazil
3 Department of Physics, Universidade Federal de Juiz de Fora, Juiz de Fora, Brazil
4 Applied Physics Graduate Program, Physics Institute, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil
5 Valongo Observatory, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil
Contents

1 Introduction ............................................. 2
2 Basic concepts ............................................ 4
   2.1 Warp drive metric ........................................ 4
   2.2 Electromagnetism in curved spacetime ......................... 5
3 The energy-momentum and Einstein tensor components ...................... 7
   3.1 Electromagnetic energy momentum tensor ....................... 7
   3.2 Energy density $T_{00}$ ....................................... 8
   3.3 Einstein tensor components ................................... 9
4 Energy conditions .......................................... 11
   4.1 Weak energy conditions ..................................... 11
   4.2 Dominant energy conditions .................................. 12
   4.3 Strong energy conditions .................................... 13
   4.4 Null energy conditions ..................................... 14
5 Another expression of the energy momentum tensor ........................ 15
   5.1 Choosing a specific observer ................................. 15
   5.2 Energy conditions for the specific electromagnetic field ............... 16
      5.2.1 Weak energy condition for the simplified choice of EEMT ............... 16
      5.2.2 Strong energy condition for the simplified choice of EEMT ............... 16
      5.2.3 Dominant energy condition for the simplified choice of EEMT ............. 16
      5.2.4 Null energy condition for the simplified choice of EEMT ................ 17
   5.3 Solving the Einstein equations ................................. 17
   5.4 Divergence of the specific electromagnetic field .................... 18
      5.4.1 Case $\partial_\beta = 0$ and $E_1 = 0$ ................................. 19
      5.4.2 Case $B_3 = 0$ and $B_2 = 0$ ................................. 20
6 Conclusions and final remarks .................................... 20
References ................................................ 21

1 Introduction

It has been known for some time that general relativity allows for particles to travel globally with superluminal velocities because special relativity limits the particles velocities to subluminal ones only locally. In other words, special relativity basically states that the light speed limit must be obeyed inside a local light cone. The warp drive metric proposed by Alcubierre [1] satisfies both conditions, since it advances a geometrical construction characterized by a spacetime distortion, called warp bubble, such that a particle inside this bubble travels with superluminal velocity in global terms, whereas locally its speed remains subluminal. More specifically, locally the warp bubble guarantees that the particle’s velocity is kept below the light speed, whereas outside the local light cone, created by the bubble distortion, the whole bubble structure travels with superluminal velocities, or warp speeds. The warp bubble is created in such a way that the spacetime in front of it is contracted while behind the bubble the spacetime is expanded. In its original proposition and first studies it was foresaw that the warp metric would violate the energy conditions, as well as supposedly requiring huge quantities of negative energy density.

After Ref. [1] several papers tried to comprehend the main aspects of the warp drive metric. For example, Ref. [2] discussed some quantum inequalities that should be valid as a result of the Alcubierre warp drive metric, concluding that large amounts of negative energy would be needed to convey particles with small masses across small
distances at warp speeds. Therefore, they figured that prohibitive huge quantities of negative energy density would be necessary to generate a warp bubble. Also dealing with quantum inequalities, Ref. [3] computed the limits required for the energy values and the bubble parameters necessary for the existence of the warp drive. The conclusion reached by this author was that the energy needed for a warp bubble is ten orders of magnitude greater than the total mass of all the visible universe, also negative.

Looking at the same problem of the physics of superluminal propulsion systems for interstellar travel, but from a different viewpoint, Krasnikov [4] analyzed the scenario of a massive particle making a round trip between two distant points in space at speeds faster than a photon. He questioned that this is not viable when reasonable conditions for global hyperbolic spacetimes are made. He analyzed in details some peculiar spacetime topologies, supposing that, for some of them, they need tachyons for superluminal trips to occur. He also assumed the need for a possible particular spacetime prearranged with some devices along the travel path, which would be set up and activated as needed in order to the superluminal travel to occur without tachyons. Such spacetime constructions was called as the Krasnikov tube by Ref. [5].

The metric proposed by Krasnikov was further generalized by Everett and Roman [5] by conceiving a tube along the path of the particle connecting Earth to a distant star. Inside this tube the spacetime is flat, however the lightcones are opened out in such a way as to allow the superluminal travel in one direction. One of the issues analyzed in Ref. [5] is that since the Krasnikov tube does not involve closed timelike curves we are able to construct a two way non-overlapping tube system such that it would work as a time machine. It was also demonstrated in Ref. [5] that a great quantity of negative energy density is necessary for the Krasnikov tube to function. These authors also used the generalized Krasnikov tube metric to compute an energy-momentum tensor (EMT) which would be positive in some particular regions.

Further studies concerning the metric proposed by Everett and Roman [5] were carried out by Lobo and Crawford [6,7]. They investigated in detail the metric and the respective EMT obtained from it and if it were possible for a superluminal travel to exist without violating the weak energy condition. Quantum inequalities used in Ref. [5] were also explored.

Van de Broeck [8] demonstrated that a minor modification of the Alcubierre geometry would reduce the total energy necessary for the creation of the warp bubble. By introducing a modification of the original warp drive metric the total negative mass-energy necessary to describe the spacetime distortion capable of warp speeds would be of the order of some solar masses. Natario [9] questioned if both the expansion and contraction of the space of the bubble is a matter of choice. He suggested a new version of the warp drive metric with zero expansion. Lobo and Visser [10,11] discussed that for the Alcubierre warp drive, and its version proposed in Ref. [9], the center of the bubble must be massless. They presented a linearized theory for both concepts and found out that even for low speeds the negative energy stored in the warp fields must be just a relevant part of the mass of the particle at the center of the warp bubble. White [12,13] depicted how a warp field interferometer could be implemented at the Advanced Propulsion Physics Laboratory. Lee and Cleaver [14,15] looked at how external radiation might affect the Alcubierre warp bubble, possibly making it energetically unsustainable, and how a proposed warp field interferometer could not
detect spacetime distortions. Mattingly et al. [16] discussed curvature invariants in the Natario warp drive.

One aspect that is often overlooked regarding the Alcubierre warp drive metric is that the original proposal did not come from solving the Einstein field equations (Alcubierre 2018, private communication), but as a geometrical construction theoretically capable of generating warp speeds. Indeed, the original proposal of the warp drive metric was not accompanied by a dynamical equation, which is the case when a metric is reached from solutions of the Einstein equations.

In a previous paper [17] we started from this realization and then endeavored to actually solve the Einstein equations using the simplest possible mass-energy distribution, incoherent matter or dust, as a starter in order to verify if this distribution were actually capable of creating a superluminal warp field, that is, a warp bubble. Although the results went back to vacuum, the solutions have indeed generated a dynamical equation for the warp metric regulating function $\beta$ (see below), which was found to obey in a particular case the Burgers equation for inviscid fluid with shockwaves in the form of plane waves (see also Ref. [18]).

In this work we intend to pursue a similar path, that is, to discuss solutions of the Einstein equations having the Alcubierre warp drive geometry and considering a non null cosmological constant. To do that we considered a charged dust capable of generating an electromagnetic field, wrote the EMT for both components, solved the equations and wrote the respective energy conditions. The results showed an interplay between the energy conditions and the electromagnetic field such that in some cases the former can be satisfied by both positive and negative matter density. In other cases the dominant and null energy conditions are violated. A result connecting the electric energy density with the cosmological constant is also presented, as well as the effects of the electromagnetic field on the bubble dynamics.

The plan of the paper is as follows. Section 2 presents both a brief review of the warp drive metric and electromagnetism in curved spacetimes necessary for a self contained paper. In Sect. 3 we calculated the electromagnetic EMT and the respective Einstein tensor components. In Sect. 4 we analyzed the energy conditions and Sect. 5 is dedicated to the investigation of conditions concerning the electric and magnetic fields under the warp drive metric. Section 6 presents our conclusions and final remarks.

2 Basic concepts

2.1 Warp drive metric

From special relativity it is well known that nothing moves faster than light, but in the scope of general relativity, the spacetime is not flat anymore, but dynamic. This allows the possibility of exotic solutions such as wormholes and warp drive, for example. This last one is a way of going from a point A to a point B in space in times arbitrarily smaller than the light would take to travel between those points.

In [1] it was described a possible way that a mass particle could travel from one point to another in spacetime measured by an external observer in a time interval smaller than the light would travel the same distance. In few words, consider a particle that
leaves an inertial reference frame $A$, which remains at rest, towards another inertial reference frame, also at rest, named $B$, at a distance $D$ from $A$. The particle is inside a bubble that can modify the spacetime in a way that the space behind the bubble is expanded whereas it is contracted in front of it. This dynamics allows the particle inside the bubble to travel the distance $D$ in a time less than $D/c$, where $c$ is the speed of light, as measured by external observers distant from the bubble, although it is still moving inside a light cone, which means that the particle does not travel faster than light locally.

The warp drive metric [1], using Cartesian coordinates $x^\mu = (t, x, y, z)$, is given by

$$ds^2 = -(1 - \beta^2)dt^2 - 2\beta(rs, t)dx
dt + dx^2 + dy^2 + dz^2, \quad (2.1)$$

where the term $\beta(rs, t)$ is the shift vector, a boost in the $x$-direction where the particle describes its trajectory inside the bubble. The shift vector is given by

$$\beta(rs, t) = v_s(t)f(rs), \quad (2.2)$$

where $v_s(t) = dx_s/dt$ is the warp bubble speed, $x_s$ is the coordinate for the center of the bubble, $f = f(rs)$ is the regulating function [1], which regulates the warp bubble shape. The parameter $rs$ is the radius of the bubble given by

$$rs = \sqrt{(x - x_s(t))^2 + y^2 + z^2}. \quad (2.3)$$

In this work we shall assume $\mu_0 = \epsilon_0 = 1$ and $G = c = 1$.

### 2.2 Electromagnetism in curved spacetime

In this section we will calculate the EMT for this warp drive metric. But first we will introduce the electromagnetic energy-momentum tensor (EMT) $T_{\alpha\beta}$ using, of course, the electromagnetic field strength tensor $F_{\alpha\beta}$.

It is well known that Maxwell electromagnetism is consistent with special relativity, the Lorentz force law, that the Maxwell equations are valid for any inertial reference system, and that they can be put in what is known as the covariant formulation, which means to describe electromagnetism in special relativity language in a manifestly invariant form under Lorentz transformations. This formalism is constructed in a flat spacetime with Minkowski metric in Cartesian coordinates given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \quad (2.4)$$

Due to the manifest covariance of the Maxwell equations in spacetime notation, if partial derivatives are replaced by covariant derivatives the extra terms cancels out and the equations remain the same, making it possible to substitute the Minkowski metric $\eta_{\mu\nu}$ for a curved spacetime metric $g_{\mu\nu}$ in a general curvilinear coordinate system.
The 4-gradient is given by the following expression
\[
\partial \mu = \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, -\nabla \right),
\] (2.5)
and we have that \( g_{\alpha\beta} \), and the 4-gradient in covariant form is \( \partial_\alpha = g_{\alpha\beta} \partial^\beta \).

The electric \( E \) and magnetic \( B \) fields are described through the electromagnetic 4-potential, which is a covariant 4-vector with the electric scalar potential \( \phi \) as its first component and the magnetic vector potential \( A \) as the other components,
\[
A^\alpha = (\phi, A). \tag{2.6}
\]

The electric and magnetic fields above can be written in tensor notation, such as
\[
B^a = \epsilon^{abc} \left( \partial_b A_c - \partial_c A_b \right), \tag{2.7}
\]
\[
E^a = \partial_0 A_a - \partial_a A_0. \tag{2.8}
\]

In matrix form, the field strength \( F^{\alpha\beta} \) can be written as,
\[
F^{\alpha\beta} = \begin{pmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & -B_3 & B_2 \\
E_2 & B_3 & 0 & -B_1 \\
E_3 & -B_2 & B_1 & 0
\end{pmatrix}, \tag{2.9}
\]
where the electric field \( E \) and the magnetic field \( B \) are in vector form
\[
E = (E_1, E_2, E_3), \tag{2.10}
\]
\[
B = (B_1, B_2, B_3). \tag{2.11}
\]

The covariant form of the electromagnetic tensor in general curved spacetime is given by the following expression,
\[
F_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} F^{\mu\nu}. \tag{2.12}
\]

The EMT for curved space time can be written as, where we use here that the signature is \((-+++\))
\[
T_{\alpha\beta} = \frac{1}{4\pi} \left( \frac{1}{4} g_{\alpha\beta} F_{\gamma\nu} F^{\gamma\nu} - g^{\gamma\nu} F_{\alpha\gamma} F_{\beta\nu} \right). \tag{2.13}
\]

If an extra term \((\mu u_\alpha u_\beta)\) is added to EEMT it would describe a charged matter with density \( \mu \) and proper velocity \( u_\alpha \) [19]. The EMT can be given by
\[ T^{\alpha\beta} = T^{(dust)} + T^{(elec)} \]
\[ = \frac{\mu}{4\pi} u^\alpha u^\beta + \frac{1}{4\pi} \left( \frac{1}{g^{\alpha\beta}} F_{\nu\sigma} F^{\nu\sigma} - F^{\alpha\nu} F_{\nu\beta} \right), \] (2.14)

which is the EEMT for the dust embedded in an electromagnetic field in a curved spacetime. It is very clear in the last equation what each part of the tensor means.

### 3 The energy-momentum and Einstein tensor components

From now on we will discuss the conditions for the energy density \( T_{00} \) from the EMT to be positive and how the radiant matter density \( \mu \) together with the electromagnetic components influence the warp drive and how higher orders of the warp drive shift \( O(\beta^2) \) influence the spacetime. We will discuss interesting expressions found from the computation of Einstein equations that provide us some insight on the plausibility of the warp drive concept.

#### 3.1 Electromagnetic energy momentum tensor

Considering the 4-velocity of the Eulerian observers \( u_\alpha = (-1, 0, 0, 0) \) [1], the dust matter radiation part \( \mu u^\alpha u^\beta \) from Eq. (2.14) has only one non zero component, i.e., \( T_{00}^{(dust)} = \mu \). The non zero and non redundant components of the EEMT are given by

\[ 4\pi T_{00} = \mu + \frac{1}{2} (B^2 + E^2) + (B_3 E_2 - B_2 E_3) \beta + \frac{\beta^2}{2} \left( B^2 - 2B_1^2 - E^2 - E_3^2 \right) \]
\[ + (B_2 E_3 - B_3 E_2) \beta^3 + \frac{\beta^4}{2} \left( E_2^2 + E_3^2 \right), \] (3.1)

\[ 4\pi T_{01} = B_2 E_3 - B_3 E_2 + \frac{\beta}{2} \left( 2B_1^2 - B^2 + E^2 \right) \]
\[ + (B_3 E_2 - B_2 E_3) \beta^2 - \frac{\beta^3}{2} \left( E_2^2 + E_3^2 \right), \] (3.2)

\[ 4\pi T_{02} = B_1 E_3 \beta^2 + B_1 B_2 \beta + B_3 E_1 - B_1 E_3, \] (3.3)

\[ 4\pi T_{03} = -B_1 E_2 \beta^2 + B_1 B_3 \beta - B_2 E_1 + B_1 E_2, \] (3.4)

\[ 4\pi T_{11} = \frac{\beta^2}{2} \left( E_2^2 + E_3^2 \right) + \frac{1}{2} \left( B^2 - 2B_1^2 + E^2 - 2E_1^2 \right) - (B_3 E_2 - B_2 E_3) \beta, \] (3.5)

\[ 4\pi T_{12} = -B_1 E_3 \beta - B_1 B_2 - E_1 E_2, \] (3.6)

\[ 4\pi T_{13} = B_1 E_2 \beta - B_1 B_3 - E_1 E_3, \] (3.7)

\[ 4\pi T_{22} = \frac{\beta^2}{2} \left( E_2^2 - E_3^2 \right) - (B_3 E_2 + B_2 E_3) \beta + \frac{1}{2} \left( B^2 - 2B_2^2 + E^2 - 2E_2^2 \right), \] (3.8)
\[ 4\pi T_{23} = E_2 E_3 \beta^2 + (B_2 E_2 - B_3 E_3) \beta - (B_2 B_3 + E_2 E_3), \]  
(3.9)

\[ 4\pi T_{33} = -\frac{1}{2} \left( E_2^2 - E_3^2 \right) \beta^2 + (B_3 E_2 + B_2 E_3) \beta + \frac{1}{2} \left( B^2 - 2B_3^2 + E^2 - 2E_3^2 \right). \]  
(3.10)

If we take \( \beta \to 0 \) in the warp drive metric in Eq. (2.14), it becomes asymptotically the Minkowski metric written in Eq. (2.4) and there is no warp drive. Using this result, the EEMT will become the one calculated for the Minkowski metric

\[
T_{\alpha\nu} = \begin{pmatrix}
4\pi \mu + \frac{1}{2} (E^2 + B^2) & -S_1 & -S_2 & -S_3 \\
-S_1 & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\
-S_2 & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\
-S_3 & -\sigma_{31} & -\sigma_{32} & -\sigma_{33}
\end{pmatrix},
\]  
(3.11)

where \( S_1, S_2 \) and \( S_3 \) are the vector components of the Poynting vector, \( S = E \times B/\mu_0 \), and \( \sigma_{ij} \) are the nine components of the Maxwell stress tensor defined by

\[
\sigma_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) \delta_{ij},
\]  
(3.12)

which will be rewritten as

\[
\sigma_{ij} = E_i E_j + B_i B_j - \frac{1}{2} (E^2 + B^2) \delta_{ij},
\]  
(3.13)

where \( i, j = 1, 2, 3 \). We can notice that considering a warp drive spacetime background, the EEMT components have extra \( \beta \)-correction terms in comparison with Minkowski background. In this scenario we can consider that the warp drive can generate electromagnetic fields or at least it reinforces an already existing one, as it can be seen from Eqs. (3.1) to (3.10). In particular, the component \( T_{00} \), which is the energy density, has fourth order \( \beta \)-corrections.

### 3.2 Energy density \( T_{00} \)

Analyzing the EEMT \( T_{00} \) term, which is given by Eq. (3.1), it shows that this term is a \( \beta \)-fourth order degree polynomial that has the form

\[
T_{00}(\beta) = \omega(0) + \omega(1) \beta + \omega(2) \beta^2 + \omega(3) \beta^3 + \omega(4) \beta^4,
\]  
(3.14)

where the coefficients \( \omega(k) \) are given by the expressions

\[
\omega(0) = \mu + \frac{1}{2} (B^2 + E^2),
\]  
(3.15)

\[
\omega(1) = B_3 E_2 - B_2 E_3,
\]  
(3.16)

\[
\omega(2) = \frac{1}{2} \left( B^2 - 2B_3^2 - E^2 - E_3^2 \right),
\]  
(3.17)
\( \omega(3) = B_2 E_3 - B_3 E_2 \), \hspace{1cm} (3.18) \\

\( \omega(4) = \frac{1}{2} \left( E_2^2 + E_3^2 \right) \). \hspace{1cm} (3.19)

for \( \beta = 0 \), the warp drive metric becomes the Minkowski metric and the only non zero component of the \( T_{00} \) term would be \( \omega_0 \) as expected from Eq. (3.11). Assuming a “weak” warp drive, that considers only shift vector \( O(\beta) \) terms, the condition for \( \beta \), where \( T_{00} \approx \omega(0) + \omega(1) \beta > 0 \) is

\[ \beta < \frac{\mu}{B_2 E_3 - B_3 E_2} + \frac{1}{2} \frac{B_2^2 + E_2^2}{B_2 E_3 - B_3 E_2} \], \hspace{1cm} (3.20)

which means that for this condition the shift vector has an upper bound limit which can be very large if \( B_2 E_3 \approx B_3 E_2 \). Now considering an usual warp drive, with higher orders of the shift vector \( O(\beta^2) \), the conditions for \( T_{00} > 0 \) requires that the terms \( \omega(2) \) and \( \omega(3) \) from Eqs. (3.17) and (3.18) are positive, and consequently we have the following conditions

\[ B_2^2 + B_3^2 > E_1^2 + E_2^2 + 2E_3^2 + B_1^2 \]. \hspace{1cm} (3.21) \\

\[ B_2 E_3 - B_3 E_2 > 0 \], \hspace{1cm} (3.22)

where we have neglected the term \( \omega(4) \) in Eq. (3.19), since it is always positive and it is multiplied by \( \beta^4 \). Both Eqs. (3.21) and (3.22) show connections between the components of the electromagnetic field in a way that the energy density \( T_{00} \) is positive for the warp drive spacetime in an electromagnetic background with a cosmological constant.

### 3.3 Einstein tensor components

Let us now calculate the Einstein tensor components added by the cosmological constant. Considering

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu}, \]

where \( g_{\mu\nu} \) is the warp drive metric tensor given in Eq. (2.1) and \( \Lambda \) is the cosmological constant. So, we have that

\[ G_{00} = \Lambda(1 - \beta^2) - \frac{1}{4}(1 + 3\beta^2) \left[ \left( \frac{\partial \beta}{\partial y} \right)^2 + \left( \frac{\partial \beta}{\partial z} \right)^2 \right] \]

\[ - \beta \left( \frac{\partial^2 \beta}{\partial y^2} + \frac{\partial^2 \beta}{\partial z^2} \right) \], \hspace{1cm} (3.24) \\

\[ G_{01} = \Lambda \beta + \frac{3}{4} \beta \left[ \left( \frac{\partial \beta}{\partial y} \right)^2 + \left( \frac{\partial \beta}{\partial z} \right)^2 \right] + \frac{1}{2} \left( \frac{\partial^2 \beta}{\partial y^2} + \frac{\partial^2 \beta}{\partial z^2} \right), \hspace{1cm} (3.25) \]
\[ G_{02} = -\frac{1}{2} \frac{\partial^2 \beta}{\partial x \partial y} - \frac{\beta}{2} \left( 2 \frac{\partial \beta}{\partial y} \frac{\partial \beta}{\partial x} + \beta \frac{\partial^2 \beta}{\partial^2 x} + \frac{\partial^2 \beta}{\partial t \partial y} \right), \quad (3.26) \]
\[ G_{03} = -\frac{1}{2} \frac{\partial^2 \beta}{\partial x \partial z} - \frac{\beta}{2} \left( 2 \frac{\partial \beta}{\partial z} \frac{\partial \beta}{\partial x} + \beta \frac{\partial^2 \beta}{\partial^2 x} + \frac{\partial^2 \beta}{\partial t \partial z} \right), \quad (3.27) \]
\[ G_{11} = \Lambda - \frac{3}{4} \left[ \left( \frac{\partial \beta}{\partial y} \right)^2 + \left( \frac{\partial \beta}{\partial z} \right)^2 \right], \quad (3.28) \]
\[ G_{12} = \frac{1}{2} \left( 2 \frac{\partial \beta}{\partial y} \frac{\partial \beta}{\partial x} + \beta \frac{\partial^2 \beta}{\partial x \partial y} + \frac{\partial^2 \beta}{\partial t \partial y} \right), \quad (3.29) \]
\[ G_{13} = \frac{1}{2} \left( 2 \frac{\partial \beta}{\partial z} \frac{\partial \beta}{\partial x} + \beta \frac{\partial^2 \beta}{\partial x \partial z} + \frac{\partial^2 \beta}{\partial t \partial z} \right), \quad (3.30) \]
\[ G_{23} = \frac{1}{2} \frac{\partial \beta}{\partial x} \frac{\partial \beta}{\partial y}, \quad (3.31) \]
\[ G_{22} = -\Lambda - \frac{\partial \beta}{\partial x} \left[ \frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial \beta}{\partial x} (\beta^2) \right] - \frac{1}{4} \left[ \left( \frac{\partial \beta}{\partial y} \right)^2 - \left( \frac{\partial \beta}{\partial z} \right)^2 \right]. \quad (3.32) \]
\[ G_{33} = -\Lambda - \frac{\partial \beta}{\partial x} \left[ \frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial \beta}{\partial x} (\beta^2) \right] + \frac{1}{4} \left[ \left( \frac{\partial \beta}{\partial y} \right)^2 - \left( \frac{\partial \beta}{\partial z} \right)^2 \right]. \quad (3.33) \]

After a long algebraic manipulation of these expressions and considering the Einstein equations \( G_{\mu \nu} = 8\pi T_{\mu \nu} \), we obtain the following set of partial differential equations

\[ \frac{4}{3} \Lambda = 8\pi \left[ T_{00} + 2\beta T_{01} + \left( \beta^2 - \frac{1}{3} \right) T_{11} \right], \quad (3.34) \]
\[ \frac{1}{2} \frac{\partial^2 \beta}{\partial y^2} + \frac{1}{2} \frac{\partial^2 \beta}{\partial z^2} = 8\pi (T_{01} + \beta T_{11}), \quad (3.35) \]
\[ -\frac{3}{4} \left( \frac{\partial \beta}{\partial y} \right)^2 - \frac{3}{4} \left( \frac{\partial \beta}{\partial z} \right)^2 - \Lambda = 8\pi T_{11}, \quad (3.36) \]
\[ \frac{1}{2} \left( \frac{\partial \beta}{\partial y} \right)^2 - \frac{1}{2} \left( \frac{\partial \beta}{\partial z} \right)^2 = 8\pi (T_{22} - T_{33}), \quad (3.37) \]
\[ -\frac{\partial \beta}{\partial x} \left[ \frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial \beta}{\partial x} (\beta^2) \right] - 2\Lambda = 8\pi (T_{33} + T_{22}), \quad (3.38) \]
\[ -\frac{1}{2} \frac{\partial^2 \beta}{\partial x \partial y} = 8\pi (T_{02} + \beta T_{12}), \quad (3.39) \]
\[ -\frac{1}{2} \frac{\partial^2 \beta}{\partial x \partial z} = 8\pi (T_{03} + \beta T_{13}), \quad (3.40) \]
\[ \frac{1}{2} \frac{\partial \beta}{\partial y} \frac{\partial \beta}{\partial z} = 8\pi T_{23}, \quad (3.41) \]
\[ -\frac{1}{4} \left( \frac{\partial \beta}{\partial y} \right)^2 - \frac{1}{4} \left( \frac{\partial \beta}{\partial z} \right)^2 + \Lambda = 8\pi \left( T_{00} + 2\beta T_{01} + \beta^2 T_{11} \right). \quad (3.42) \]
Note that the set of partial differential equations from Eq. (3.34) to Eq. (3.42) are quite cumbersome to solve if we consider the electromagnetic components that appear in the EEMT components $T_{\mu\nu}$, even if we consider them as constants components.

4 Energy conditions

In this section we will calculate the energy conditions for the EEMT and we will see that the inequalities written above will be satisfied. We will find connections between the components of the electromagnetic field, the radiant matter density $\mu$ and the shift vector $\beta$.

4.1 Weak energy conditions

For the weak energy condition, the EMT at each point of the spacetime must satisfy the condition

$$T_{\alpha\sigma} u^\alpha u^\sigma \geq 0,$$

where, for any timelike vector $u$ ($u_\alpha u^\alpha < 0$), and any zero vector $k$ ($k_\alpha k^\alpha = 0$), for an observer with unit tangent vector $v$ at a certain point of the spacetime, the local energy density measured by any observer is non-negative [20]. For the EEMT, the expression $T_{\alpha\sigma} u^\alpha u^\sigma$ is given by

$$T_{\alpha\sigma} u^\alpha u^\sigma = \frac{1}{2} \left( E_2^2 + E_3^2 \right) \beta^2 - \left( B_3 E_2 - B_2 E_3 \right) \beta + \frac{1}{2} \left( B^2 + E^2 \right) + \mu. \quad (4.2)$$

Notice that Eq. (4.2) is a quadratic function of $\beta$. So, to be solved the discriminant of this quadratic equation must be positive, namely

$$\left( B_3 E_2 - B_2 E_3 \right)^2 - 2 \left( E_2^2 + E_3^2 \right) \left[ \frac{1}{2} \left( B^2 + E^2 \right) + \mu \right] > 0,$$

and we have for the matter density $\mu$ that,

$$0 < \mu < \frac{(B_3 E_2 - B_2 E_3)^2}{2 \left( E_2^2 + E_3^2 \right)} - \frac{1}{2} \left( B^2 + E^2 \right), \quad (4.4)$$

which shows a condition between the electromagnetic components,

$$\left( B_3 E_2 - B_2 E_3 \right)^2 > \left( B^2 + E^2 \right) \left( E_2^2 - E_1 \right)$$

and that the matter density must have a positive inferior minimum value, since the r.h.s. of the inequality in Eq. (4.4) must be always positive. This result tells us that
the weak energy condition must be satisfied if we consider both positive and negative matter density. Solving exactly the inequality in Eq. (4.2) we have that

\[ \beta_- < \beta < \beta_+ \] (4.5)

where

\[ \beta_{\pm} = \frac{B_3 E_2 - B_2 E_3}{E^2 - E_1^2} \pm \sqrt{\left[ \frac{B_3 E_2 - B_2 E_3}{E^2 - E_1^2} \right]^2 - \frac{B^2 + E^2 + 2\mu}{E^2 - E_1^2}} \] (4.6)

and considering a “weak” warp drive where \( \beta^2 \approx 0 \) in Eq. (4.2), then for the weak energy condition to be valid it requires that

\[ \beta > \frac{1}{B_3 E_2 - B_2 E_3} \left[ \frac{1}{2} \left( B^2 + E^2 \right) + \mu \right], \] (4.7)

which tells us that in this case, the shift vector has a limiting minimum value condition. This is an interesting result since it implies that even for lower order of \( \beta \) the warp bubble speed is not limited by the weak energy condition. Notice that if \( B_3 E_2 \to B_2 E_3 \), then the right hand side of this last equation assumes unlimited values, which could be observed in fact in low strength electromagnetic fields and low density of radiant matter.

### 4.2 Dominant energy conditions

For every timelike vector \( u_\alpha \) the following inequalities must be satisfied

\[ T^{\alpha \beta} u_\alpha u_\beta \geq 0, \quad \text{and} \quad F^\alpha F_\alpha \leq 0, \] (4.8)

where \( F^\alpha = T^{\alpha \beta} u_\beta \) is a non-spacelike vector. We can realize that these conditions mean that for any observer, the local energy density appears to be non-negative and the local energy flow vector is non-spacelike. In any orthonormal basis the energy dominates the other components of the EMT,

\[ T^{00} \geq |T^{ab}| \quad \text{for each} \ a, b. \] (4.9)

Evaluating the first condition \( T^{\alpha \beta} u_\alpha u_\beta \geq 0 \) for the dominant energy condition gives us the same result as the weak energy condition seen in Eqs. (4.3) and (4.4), so this energy condition term can be satisfied by both positive and negative radiant matter density. Calculating the second condition for the dominant energy condition \( F^\alpha F_\alpha \), it will be given by a fourth degree polynomial on \( \beta \) given by the following expression,

\[ F^\alpha F_\alpha = \omega_4 \beta^4 + \omega_3 \beta^3 + \omega_2 \beta^2 + \omega_1 \beta + \omega_0, \] (4.10)
where $\omega(0), \ldots, \omega(4)$ are implicit functions of the spacetime coordinates $(t, x, y, z)$ and also explicit functions in terms of the electromagnetic field components. Notice that the subindexes of $\omega(k)$'s are not tensor indexes and the terms $\omega(k)\beta^k$ are not tensor contractions. The coefficients are given by

$$\omega(4) = \frac{1}{4} \left( E_2^2 + E_3^2 \right)^2, \quad (4.11)$$
$$\omega(3) = (E_2^2 + E_3^2)(E_2 B_3 - E_3 B_2), \quad (4.12)$$
$$\omega(2) = \frac{1}{2} (E_2^2 + E_3^2)(E^2 - B^2 - 2\mu) - (B_2 E_3 - B_3 E_2)^2, \quad (4.13)$$
$$\omega(1) = (E_2 B_3 - E_3 B_2)(B^2 - E^2 + 2\mu), \quad (4.14)$$
$$\omega(0) = -\frac{1}{4}(B^4 + E^2) - (E^2 + B^2)\mu - \mu^2$$
$$-\frac{1}{2}(B_1 E_1 + B_2 E_2 + B_3 E_3)^2 + \frac{1}{2}(B_1 E_2 - B_2 E_1)^2$$
$$+\frac{1}{2}(B_1 E_3 - B_3 E_1)^2 + \frac{1}{2}(B_2^2 E_3^2 - B_2 E_3)^2. \quad (4.15)$$

Since $F^\alpha F_\alpha$ is a fourth order polynomial on $\beta$ with complicated expressions for its components it is a rather challenging work to calculate all the roots for this polynomial and to find the general requirements for the dominant energy condition to be true. However, a simple and pragmatic way to impose the validity for this energy condition is to require that all the coefficients $\omega(k)$ to be a positive value.

4.3 Strong energy conditions

The strong energy condition is given by

$$\left( T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) u^\alpha u^\beta \geq 0, \quad (4.16)$$

for any timelike vector $u$. The expression from the strong energy condition in Eq. (4.16) for the EEMT and the warp drive metric is given by the following equation

$$\left( E_2^2 + E_3^2 \right) \beta^2 - 2 (B_3 E_2 - B_2 E_3) \beta + B^2 + E^2 + \mu \geq 0, \quad (4.17)$$

which is a quadratic $\beta$-inequality. For the strong energy condition to be satisfied it is necessary that the discriminant of the above quadratic equation be real, that is

$$(B_3 E_2 - B_2 E_3)^2 - \left( E_2^2 + E_3^2 \right) \left[ B^2 + E^2 + \mu \right] < 0. \quad (4.18)$$
which is very similar to the equation we found for the weak and first dominant energy conditions with the difference of a multiplication by a factor of 2

\[
\mu \leq \frac{(B_3 E_2 - B_2 E_3)^2}{(E_2^2 + E_3^2)} - (B^2 + E^2),
\] (4.19)

which has the same meaning of the matter density to satisfy the strong energy condition as in Eq. (4.4) for the weak energy condition. The matter density has an inferior bound and it may be negative. But the strong energy condition could still be satisfied as long as \( \mu \) is less than or equal to the right-hand side of the inequality in Eq. (4.19).

### 4.4 Null energy conditions

The null energy conditions are satisfied for the null vector \( k \) since the following inequality must be satisfied

\[
T_{\alpha\sigma} k^\alpha k^\sigma \geq 0, \quad \text{for any null vector } k^\alpha. \tag{4.20}
\]

Assuming the following vector \( k^\alpha = (a, b, 0, 0) \) where \( a \) and \( b \) can be determined by solving the equation \( k_{\alpha} k^{\alpha} = 0 \). Hence, we can find two possible results connecting \( a \) and \( b \),

\[
a = \frac{b}{\beta + 1} \quad \text{or} \quad a = \frac{b}{\beta - 1}, \tag{4.21}
\]

Considering this last result, the calculation of \( T_{\alpha\sigma} k^\alpha k^\sigma \) is given by

\[
T_{\alpha\sigma} k^\alpha k^\sigma = T_{00} k^0 k^0 + 2T_{01} k^0 k^1 + T_{11} k^1 k^1. \tag{4.22}
\]

Substituting Eq. (4.21) and that \( k^0 = a \) and \( k^1 = b \) into Eq. (4.22), the result of the null energy condition for the EEMT is a complicated function of \( \beta \) given by

\[
T_{\alpha\sigma} k^\alpha k^\sigma = \omega(4) \beta^4 + \omega(3) \beta^3 + \omega(2) \beta^2 + \omega(1) \beta + \omega(0), \tag{4.23}
\]

where the coefficients \( \omega(k) \) are also functions of \( \beta \) given by the expressions

\[
\omega(4) = \frac{a^2}{2}(E_2^2 + E_3^2), \tag{4.24}
\]

\[
\omega(3) = a^2(B_2 E_3 - B_3 E_2) - ab(E_2^2 + E_3^2), \tag{4.25}
\]

\[
\omega(2) = \frac{1}{2} \left[ a^2(B^2 - 2B_1^2 - E_1^2) + 4ab(B_3 E_2 - B_2 E_3) + (b^2 - 2a^2)(E_2^2 + E_3^2) \right], \tag{4.26}
\]

\[
\omega(1) = ab(B_1^2 - B_2^2 - B_3^2 + E^2) + (a^2 - b^2)(B_3 E_2 - B_2 E_3), \tag{4.27}
\]
\[ \omega_{(0)} = \frac{a^2}{2} (B^2 + E^2) + \frac{b^2}{2} (B^2 - 2B_1^2 + E^2 - 2E_1^2) + 2ab(B_2E_3 - B_3E_2) + a^2 \mu, \]

which are hard-working algebraic expressions concerning the null energy condition. The next step would be to solve them in a general way and to find the specific connections for the energy condition to be valid.

5 Another expression of the energy momentum tensor

In this section we will introduce a specific observer that simplifies the EMT components. We will use this approach as a laboratory relative to the warp drive in an electromagnetic field background before analyzing and solving the generalized Einstein equations.

5.1 Choosing a specific observer

Let us consider that the electric and magnetic fields are orthogonal to each other, like in an electromagnetic wave. In addition, they do not obey the wave equation, but they do satisfy the relation below,

\[ E_i B^i = 0. \] (5.1)

Now, let us assume that from the point of view of this observer the electric field has only one component, \( E_1 \), which is the only one non zero component and it points to the direction of the particle trajectory inside the warp bubble. Hence, from Eq. (5.1), it is straightforward that \( B_1 = 0 \), but we can still have the other magnetic components. Having said that, the non zero components of the EEMT can be computed such that,

\[ 4\pi T_{00} = \mu + \frac{1}{2} (B^2 + E_1^2) + \frac{\beta^2}{2} \left( B^2 - E_1^2 \right), \] (5.2)
\[ 4\pi T_{01} = \frac{\beta}{2} \left( E_1^2 - B^2 \right), \] (5.3)
\[ 4\pi T_{02} = B_3 E_1, \] (5.4)
\[ 4\pi T_{03} = -B_2 E_1, \] (5.5)
\[ 4\pi T_{11} = \frac{1}{2} \left( B^2 - E_1^2 \right), \] (5.6)
\[ 4\pi T_{22} = \frac{1}{2} \left( B_3^2 - B_2^2 + E_1^2 \right), \] (5.7)
\[ 4\pi T_{23} = -B_2 B_3, \] (5.8)
\[ 4\pi T_{33} = \frac{1}{2} \left( B_2^2 - B_3^2 + E_1^2 \right). \] (5.9)

Notice that \( B^2 = B_2^2 + B_3^2 \) and now that \( E = E_1 \).
5.2 Energy conditions for the specific electromagnetic field

Now we will use our specific choice of the electromagnetic field, namely, \( \mathbf{E} = (E_1, 0, 0) \) and \( \mathbf{B} = (0, B_2, B_3) \) in the energy conditions. We have demonstrated in the last section that both the weak and strong energy conditions could be satisfied in a general manner, but the dominant and null conditions require a hard-working calculation to show that they are valid.

5.2.1 Weak energy condition for the simplified choice of EEMT

Using the results obtained in the last section together with the specific choice of the electromagnetic field, we can find that the weak energy condition from Eq. (4.2) can be satisfied if

\[
T_{\alpha\sigma} u^\alpha u^\sigma = \frac{1}{2} (B^2 + E^2) + \mu \geq 0,
\]

which means that even with no radiant matter density, i.e., \( \mu = 0 \), the weak energy inequality is still non-negative. Besides, even with a negative matter density, it could still be positive for this energy condition to be satisfied if the electromagnetic field strength \( (B^2 + E^2)/2 \) is bigger than the matter density \( \mu \) as in Eq. (5.10).

5.2.2 Strong energy condition for the simplified choice of EEMT

From Eq. (4.17) the inequality simplifies to

\[
\left( T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) u^\alpha u^\beta = B^2 + E^2 + \mu \geq 0,
\]

and it is clear that, since \( B^2 + E^2 \) is obviously always positive, the strong energy condition is valid for the simplified choice of electromagnetic field since the matter density \( \mu \) is positive. In the case of a negative matter density, the modulo of electromagnetic energy density \( B^2 + E^2 \) must be larger than the negative matter density \( \mu \).

5.2.3 Dominant energy condition for the simplified choice of EEMT

For the dominant energy inequality, the first condition \( T^{\alpha\beta} u_\alpha u_\beta \geq 0 \), still gives us the same result as the one from the weak energy inequality. But now with this simplified choice for the electromagnetic field, the second requirement for the dominant energy condition is given by

\[
F^\alpha F_\alpha = -\mu^2 - (E^2 + B^2) \mu - \frac{1}{2} (E^2 - B^2)^2 - \frac{1}{2} E^2 B^2,
\]

which is a quadratic function of the matter density \( \mu \). The sign of this expression will depend on the value of the electromagnetic components, of course. But, for real solutions, we must have that \( E^2 B^2 \geq (E^2 - B^2)^2 / 2 \).
5.2.4 Null energy condition for the simplified choice of EEMT

Recovering the null energy condition from Eq. (4.23), we have that

$$T_{\alpha\sigma} k^\alpha k^\sigma = \frac{a^2}{2} (B^2 - E^2) \beta^2 + ab(E^2 - B^2) \beta + \frac{1}{2} (a^2 + b^2) B^2 + \frac{1}{2} (a^2 - b^2) E^2 + a^2 \mu, \quad (5.13)$$

Disregarding the fact that $a$ and $b$ are functions of $\beta$, we will use the assumption that Eq. (5.13) is a quadratic function of $\beta$ and we will impose that the discriminant of this equation must be positive or zero for real solutions. Hence, the null energy condition might be satisfied for any value of $\beta$ if

$$a^4 (E^2 - B^2) (\mu + B^2 + E^2) \geq 0. \quad (5.14)$$

which means that, considering a positive $\mu$, the main condition is $E^2 \geq B^2$.

Table 1 summarizes all the necessary requirements for the energy conditions to be valid simultaneously for the specific simplified choice of the electromagnetic field.

### 5.3 Solving the Einstein equations

For the simplified choice of the electromagnetic field, the set of partial differential Eqs. (3.34) to (3.42), that resulted from algebraic simplifications of the Einstein equations are

$$\frac{4}{3} \Lambda = 8\pi \left( \mu + \frac{1}{3} B^2 + \frac{2}{3} E_1^2 \right), \quad (5.15)$$

$$\frac{1}{2} \frac{\partial^2 \beta}{\partial y^2} + \frac{1}{2} \frac{\partial^2 \beta}{\partial z^2} = 0, \quad (5.16)$$

$$-\frac{3}{4} \left( \frac{\partial \beta}{\partial y} \right)^2 - \frac{3}{4} \left( \frac{\partial \beta}{\partial z} \right)^2 - \Lambda = 4\pi \left( B^2 - E_1^2 \right), \quad (5.17)$$

$$\frac{1}{2} \left( \frac{\partial \beta}{\partial y} \right)^2 - \frac{1}{2} \left( \frac{\partial \beta}{\partial z} \right)^2 = 8\pi \left( B_3^2 - B_2^2 \right), \quad (5.18)$$

| Energy condition | Results |
|------------------|---------|
| Weak             | $\frac{1}{2} (B^2 + E^2) + \mu \geq 0$ |
| Strong           | $B^2 + E^2 + \mu \geq 0$ |
| Dominant         | $E^2 B^2 \geq (E^2 - B^2)^2 / 2$ |
| Null             | $E^2 \geq B^2$, for $\mu > 0$. |
\[
- \frac{\partial}{\partial x} \left( \frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} (\beta^2) \right) - 2\Lambda = 8\pi E_1^2, \tag{5.19}
\]
\[
- \frac{1}{2} \frac{\partial^2 \beta}{\partial x \partial y} = 8\pi B_3 E_1, \tag{5.20}
\]
\[
- \frac{1}{2} \frac{\partial^2 \beta}{\partial x \partial z} = -8\pi B_3 E_1, \tag{5.21}
\]
\[
\frac{1}{2} \frac{\partial \beta}{\partial y} \frac{\partial \beta}{\partial z} = -8\pi B_2 B_3, \tag{5.22}
\]
\[
- \frac{1}{4} \left( \frac{\partial \beta}{\partial y} \right)^2 - \frac{1}{4} \left( \frac{\partial \beta}{\partial z} \right)^2 + \Lambda = 4\pi \left( 2\mu + E_1^2 + B^2 \right), \tag{5.23}
\]

5.4 Divergence of the specific electromagnetic field

In this section we will calculate the divergence of the EEMT considering that the components of the electromagnetic field are functions of the spacetime coordinates \( x^\mu \). The results are as follows,

\[
T^{0\alpha} : \alpha = - \frac{\partial (\beta B_2^2)}{\partial x} - \frac{1}{2} \frac{\partial (B^2 + E_1^2)}{\partial t} - \frac{\partial (B_2 E_1)}{\partial y} - \frac{\partial (B_3 E_1)}{\partial x} - \frac{\partial (\mu \beta)}{\partial t}, \tag{5.24}
\]
\[
T^{1\alpha} : \alpha = \frac{1}{2} \frac{\partial (B^2 - E_1^2)}{\partial x}, \tag{5.25}
\]
\[
T^{2\alpha} : \alpha = - \frac{\partial (\beta B_3 E_1)}{\partial x} - \frac{\partial (B_2 B_3)}{\partial z} - \frac{\partial (B_3 E_1)}{\partial t} + \frac{1}{2} \frac{\partial (B_3 E_1)}{\partial y} (B_3^2 - B_2^2 + E_1^2), \tag{5.26}
\]
\[
T^{3\alpha} : \alpha = \frac{\partial (\beta B_2 E_1)}{\partial x} + \frac{\partial (B_2 E_1)}{\partial t} - \frac{\partial (B_2 E_3)}{\partial y} + \frac{1}{2} \frac{\partial (B_2 E_1)}{\partial z} (B_2^2 - B_3^2 + E_1^2). \tag{5.27}
\]

Considering that \( E_1, B_1 \) and \( B_2 \) are constants, Eqs. (5.24) to (5.27) can be written such as

\[
T^{0\alpha} : \alpha = - B_2 \frac{\partial \beta}{\partial x} - \frac{\partial (\mu \beta)}{\partial x} - \frac{\partial \mu}{\partial t} \tag{5.28}
\]
\[
T^{1\alpha} : \alpha = 0 \tag{5.29}
\]
\[
T^{2\alpha} : \alpha = - B_3 E_1 \frac{\partial \beta}{\partial x} \tag{5.30}
\]
\[
T^{3\alpha} : \alpha = B_2 E_1 \frac{\partial \beta}{\partial x} \tag{5.31}
\]

Imposing that the EEMT must be conserved, i.e., the divergence must be zero, which shows that Eq. (5.28) is a continuity equation, but Eq. (5.30) implies that either \( B_3 = 0 \), or \( E_1 = 0 \) or \( \frac{\partial \beta}{\partial x} = 0 \), and Eq. (5.31) implies that either \( B_2 = 0 \), or \( E_1 = 0 \) or \( \frac{\partial \beta}{\partial x} = 0 \). Next we will analyze how each one of these cases can affect the set of Eqs. (5.15) to (5.23).

\[\square\] Springer
5.4.1 Case $\frac{\partial \beta}{\partial x} = 0$ and $E_1 = 0$

These conditions are satisfied simultaneously and it can be seen from Eq. (5.19). Moreover, Eqs. (5.20) and (5.21) are identically zero and the set of Einstein differential equations from Eq. (5.15) to (5.23) can be written such as

\[
\Lambda = 6\pi \mu + 2\pi B^2, \tag{5.32}
\]

\[
\frac{\partial^2 \beta}{\partial y^2} + \frac{\partial^2 \beta}{\partial z^2} = 0, \tag{5.33}
\]

\[
\left( \frac{\partial \beta}{\partial y} \right)^2 + \left( \frac{\partial \beta}{\partial z} \right)^2 = -\frac{4}{3} \Lambda - \frac{16\pi}{3} B^2, \tag{5.34}
\]

\[
\left( \frac{\partial \beta}{\partial y} \right)^2 - \left( \frac{\partial \beta}{\partial z} \right)^2 = 16\pi \left( B_3^2 - B_2^2 \right), \tag{5.35}
\]

\[
\frac{\partial \beta}{\partial y} \frac{\partial \beta}{\partial z} = -16\pi B_2 B_3, \tag{5.36}
\]

\[
\left( \frac{\partial \beta}{\partial y} \right)^2 + \left( \frac{\partial \beta}{\partial z} \right)^2 = -4\Lambda - 16\pi \left( 2\mu + B^2 \right). \tag{5.37}
\]

The above set of equations imply that the cosmological constant is null and the following relation between the electromagnetic field and the matter density

\[
\mu = -\frac{1}{3} B^2. \tag{5.38}
\]

So, the matter density will always be negative for this case, but the energy conditions will still be satisfied. The shift vector will not depend on the $x$ spacetime coordinate, and there is no Burgers equation, i.e., no shock wave, but $\beta$ is function of $(t, y, z)$ and it also satisfies the Laplace equation according to Eq. (5.33). Both Eqs. (5.34) and (5.37) are specific cases of the well known Eikonal equation, and they imply that the solution for the shift vector is not unique and may have a complex component. For this case the EMT in matrix form is given by,

\[
T_{\alpha\nu} = \begin{pmatrix}
-\frac{1}{2} \mu (1 + \beta^2) & -\frac{1}{2} \beta B^2 & 0 & 0 \\
-\frac{1}{2} \beta B^2 & -\frac{1}{2} B^2 & 0 & 0 \\
0 & 0 & \frac{1}{2} (B_3^2 - B_2^2) & -B_2 B_3 \\
0 & 0 & -B_2 B_3 & -\frac{1}{2} (B_3^2 - B_2^2)
\end{pmatrix}. \tag{5.39}
\]

It is clear that since the matter density is negative, i.e., $\mu < 0$, then the energy density for the energy momentum tensor is positive, namely, $T_{00} > 0$. 
5.4.2 Case $B_3 = 0$ and $B_2 = 0$

For this case it is straightforward to see that $B_2 = B_3 = 0$ implies that $\frac{\partial \beta}{\partial y} = 0$ and $\frac{\partial \beta}{\partial z} = 0$ from Eqs. (5.20) to (5.22). The set of equations for this case is the following

$$\begin{align*}
\Lambda &= 4\pi E_1^2, \\
\mu &= 0, \\
-\frac{\partial}{\partial x} \left( \frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x}(\beta^2) \right) &= 2\Lambda + 8\pi E_1^2.
\end{align*}$$

(5.40) (5.41) (5.42)

This result violates both null and dominant energy conditions, but it is an interesting theoretical result since both matter density $\mu$ and magnetic field $\mathbf{B}$ are null, but the cosmological constant is positive and proportional to the electric field energy as seen in Eq. (5.40).

$$T_{\mu\nu} = \begin{pmatrix}
-\frac{1}{2} E^2 (1 - \beta^2) & -\frac{1}{2} \beta E^2 & 0 & 0 \\
-\frac{1}{2} \beta E^2 & -\frac{1}{2} E^2 & 0 & 0 \\
0 & 0 & \frac{1}{2} E^2 & 0 \\
0 & 0 & 0 & \frac{1}{2} E^2
\end{pmatrix}.$$  

(5.43)

which is the final and symmetric form of the EEMT. Notice the presence of only the electric field and the shift function since $B = 0$.

6 Conclusions and final remarks

In this work we have investigated the solutions of the Einstein equations for the Alcubierre warp drive metric with the choice of dust and electromagnetic field energy-momentum tensor (EMT) as possible sources of global superluminal particle speeds, that is, warp velocities. The Einstein equations were analyzed and all results concerning the components of the electromagnetic field were discussed. The energy conditions were presented as functions of the electromagnetic field components, and we have established conditions on these components such that they obey, or not, the energy conditions the warp drive metric. The connections found between electromagnetic field and the superluminal effects of the warp drive resulting from the Einstein field equations, namely, being solutions of them, are new. We have also discussed the potential dynamic consequences of these results on the warp bubble dynamics.

We found that the energy conditions can be satisfied for positive and negative matter density, requirements which were summarized in a table. We also showed that the null divergence of the electromagnetic tensor results in some interesting cases, such as, the matter density becoming negative and proportional to the magnetic field energy density, the shift vector $\beta$ being a function of only $(t, y, z)$, and the absence of the Burgers equation as found in Ref. [17]. Nevertheless, a specific case of Eikonal equation has to be solved, leading to a wave equation, in order to find solutions for $\beta$.

In another case we found a violation of both the dominant and null energy conditions,
the matter density and the magnetic field are null and the cosmological constant is proportional to the electric energy density.

Acknowledgements  Everton M. C. Abreu thanks CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), Brazilian federal scientific supporting agency, for partial financial support, Grant Number 406894/2018-3.

References

1. Alcubierre, M.: The warp drive: hyper-fast travel within general relativity. Class. Quant. Gravit. arXiv: gr-qc/0009013)
2. Ford, L.H., Roman, T.A.: Quantum field theory constrains traversable wormhole geometries. Phys. Rev. D arXiv: gr-qc/9510071)
3. Pfennning, M.J., Ford, L.H.: The unphysical nature of Warp Drive. Class. Quant. Gravit. arXiv: gr-qc/9702026)
4. Krasnikov, S.V.: Hyperfast interstellar travel in general relativity. Phys. Rev. D arXiv: gr-qc/9511068)
5. Everett, A., Roman, T.A.: A superluminal subway: the Krasnikov tube. Phys. Rev. D arXiv: gr-qc/9702049)
6. Lobo, F.S.N., Crawford, P.: Weak energy condition violation and superluminal travel. Lect. Notes Phys. arXiv: gr-qc/0204038)
7. Lobo, F.S.N., Crawford, P.: Weak energy condition violation and superluminal travel. In: Fernández-Jambrina, L., González-Romero, L.M. (eds.) Current Trends in Relativistic Astrophysics. Springer, Berlin (2003). (arXiv: gr-qc/0204038)
8. Van Den Broeck, C.: A warp drive with more reasonable total energy. Class. Quant. Gravit. arXiv: gr-qc/9905084)
9. Natario, J.: Warp drive with zero expansion. Class. Quant. Gravit. arXiv: gr-qc/0110086)
10. Lobo, F.S.N., Visser, M.: Fundamental limitations on ‘warp drive’ spacetimes. Class. Quant. Gravit. 21, 5871 (2004). arXiv: gr-qc/0406083)
11. Lobo, F.S.N., Visser, M.: Linearized warp drive and the energy conditions (2004). arXiv: gr-qc/0412065)
12. White, H.G.: A discussion of space-time metric engineering. Gen. Relativ. Gravit. 35, 2025 (2003)
13. White, H.G.: Warp field mechanics 101. J. Br. Interplanet. Soc. 66, 242 (2011)
14. Lee, J., Cleaver, G.: Effects of external radiation on an Alcubierre warp bubble. Phys. Essays 29, 201 (2016)
15. Lee, J., Cleaver, G.: The inability of the White-Juday warp field interferometer to spectrally resolve spacetime distortions. Int. J. Mod. Phys. Adv. Theory Appl. 2, 35 (2017). (arXiv:1407.7772)
16. Mattingly, B., Kar, A., Gorban, M., Julius, W., Watson, C., Ali, M.D., Baas, A., Elmore, C., Lee, J., Shakerin, B., Davis, E., Cleaver, G.: Curvature invariants for the accelerating Natario warp drive. Particles 3, 642–659 (2020). (arXiv:2008.03366)
17. Santos-Pereira, O.L., Abreu, E.M.C., Ribeiro, M.B.: Dust content solutions for the Alcubierre warp drive spacetime. Eur. Phys. J. C arXiv: 2008.06560 [gr-qc])
18. Santos-Pereira, O.L., Abreu, E.M.C., Ribeiro, M.B.: Fluid dynamics in the warp drive spacetime geometry. Eur. Phys. J. C arXiv: 2101.11467 [gr-qc])
19. d’Inverno, R.: Introducing Einstein’s Relativity. Clarendon Press, Oxford (1992)
20. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Space-Time. Cambridge University Press, Cambridge (1973)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.