A Note on Banach Principle for \( JW \)-algebras

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Abstract: - In the sequel we establish the Banach Principle for semifinite \( JW \)-algebras without direct summand of type \( 1_2 \), which extends the recent results of Chilin and Litvinov on the Banach Principle for semifinite von Neumann algebras to the case of \( JW \)-algebras.

Key-Words: - von Neumann algebras, Jordan operator algebras, \( JW \)-algebras, Banach Principle, \( * \)-algebra of \( \tau \)-measurable operators affiliated to a semifinite von Neumann algebra, Jordan algebra of \( \tau \)-measurable operators affiliated to a semifinite \( JW \)-algebra.

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1 Introduction

Let \((\Omega, \Sigma, \mu)\) be a probability space. Denote by \( \mathcal{L} = \mathcal{L}(\Omega, \mu) \) the set of all (classes of) complex-valued measurable functions on \( \Omega \). Let \( \tau_\mu \) be the measure topology on \( \mathcal{L} \). The classical Banach Principle (see for example [13]) can be stated as follows:

Classical Banach Principle. Let \((X, \| \cdot \|)\) be a Banach space, and let \( a_n : (X, \| \cdot \|) \to (\mathcal{L}, \tau_\mu) \) be a sequence of continuous linear maps. Consider the following properties:

(I) \ the sequence \( \{a_n(x)\} \) converges almost everywhere (a.e.) for every \( x \in X \);

(II) \( \hat{a}(x)(\omega) = \sup_n |a_n(x)(\omega)| < \infty \) a.e. for every \( x \in X \);

(III) (II) holds, and the maximal operator \( \hat{a} : (X, \| \cdot \|) \to (\mathcal{L}, \tau_\mu) \) is continuous at \( 0 \);

(IV) the set \( \{x \in X : \{a_n(x)\} \text{ converges a.e.} \} \) is closed in \( X \).

Then the implications \( (I) \Rightarrow (II) \Rightarrow (III) \Rightarrow (IV) \) are always true. If in addition, there exists a dense subset \( D \subset X \), such that the sequence \( \{a_n(x)\} \) converges a.e. for every \( x \in D \), then all four conditions \( (I) - (IV) \) above are equivalent.

The Banach Principle above was often applied in the case \( X = (L^p, \| \cdot \|_p) \), where \( 1 \leq p < \infty \). However, in the case \( p = \infty \) the uniform topology on \( L^\infty \) appears to be too strong for the classical Banach Principle to be effective in \( L^\infty \). For example, one can notice that continuous functions are not uniformly dense in \( L^\infty \).

Bellow and Jones [9], using the fact that the unit ball \( L^\infty_1 = \{x \in L^\infty : \|x\|_\infty \leq 1\} \) is complete in \( \tau_\mu \), suggested to consider the measure topology on \( L^\infty \) by replacing \( (X, \| \cdot \|) \) by \( (L^\infty_1, \tau_\mu) \). Since \( L^\infty_1 \) is not a linear space, geometrical complications occur, which, however, were successfully resolved in [9].

Non-commutative versions of Banach Principle for measurable operators affiliated to a semifinite von Neumann algebra were established in [14] and [18]. These results were extended to the case of semifinite JBW-algebras in [16] and [17], following ideas introduced in [1], [2], [4], [5], [10]. A non-commutative version of the Banach Principle for \( L^\infty \) was proposed by Chilin and Litvinov in [12].

The present notes are devoted to a presentation of an
2 Preliminaries

Let $M$ be a semifinite von Neumann algebra of bounded operators acting on a complex Hilbert space $H$ ([11]), and let $B(H)$ be the algebra of all bounded operators on $H$. A densely defined closed operator $x$ on $H$ is called affiliated to $M$ if $y'z = zy'$, with $z \in M$ ([19], [21]). Denote by $P(M)$ the complete lattice of projections in $M$. Let $\tau$ be a faithful normal semifinite trace on $A$. Let $\tau$ be the set of all $\tau$-measurable operators affiliated to $\tau$-measurable operators affiliated to $M$. Denote by $e^1 = 1 - e$ the orthogonal complemented projection for the projection $e \in P(M)$. An operator $x$ affiliated to $M$ is called $\tau$-measurable if, for arbitrary $\epsilon > 0$ and $\delta > 0$, where $\| \cdot \|$ stands for the operator norm on $B(H)$. The topology $t_\tau$ defined on $L(M, \tau)$ by the family $\{ V(\epsilon, \delta) : \epsilon > 0, \delta > 0 \}$ of neighborhoods of zero is called the measure topology ([19], [21]).

**Theorem 1.** $(L(M, \tau), t_\tau)$ is a complete metrizable topological $*$-algebra.

**Proof.** See [19], [21] for details.

**Proposition 1.** For any $d > 0$, the sets $M_d = \{ x \in M : \| x \| \leq d \}$, $M_d^* = \{ x \in M_d : x = x^* \}$ are $t_\tau$-complete.

**Proof.** See [12] for details.

A sequence $\{ y_n \} \subset L(A, \tau)$ is said to converge almost uniformly (a.u.) to $y \in L(M, \tau)$ if $\forall \epsilon > 0$, $\exists \epsilon > 0$, $\epsilon > 0$, $\exists e \in P(M)$ with $\tau(e^+) < \epsilon$ such that $\| (y - y_n)e \| \to 0$.

**Proposition 2.** For $\{ y_n \} \subset L(M, \tau)$ the conditions

(i) $\{ y_n \}$ converges a.u. in $L(M, \tau)$;

(ii) $\forall \epsilon > 0$, $\exists e \in P(M)$ with $\tau(e^+) < \epsilon$ such that $\| (y_m - y_n)e \| \to 0$ as $m, n \to \infty$;

are equivalent.

**Proof.** See [12] for details.

The following theorem is a non-commutative version of Riesz theorem ([13]).

**Theorem 2.** If $\{ y_n \} \subset L(M, \tau)$ and $y = t_\tau - \lim_{n \to \infty} y_n$, then $y = a.u. - \lim_{k \to \infty} y_k$ for some subsequence $\{ y_n \} \subset \{ y_n \}$.

**Proof.** See [21] and [14] for details.

Let $A$ be a semifinite JW-subalgebra of $B(H)_{sa}$ without a direct summand of type $I_2$ (see [15] and [20] for definitions), $P(A)$ be the complete lattice of projections in $A$, and $\tau$ be a faithful normal semifinite trace on $A$. Let $M = M(A)$ be the von Neumann enveloping algebra of the Jordan algebra $A$. Then $\tau$ can be uniquely extended to a faithful normal semifinite trace on $M$, for which we will use the same symbol $\tau$ (see [3], [6] and [8]). An operator $x$ affiliated to $A$ is called $\tau$-measurable if $\forall \epsilon > 0$, $\exists e \in P(A)$ with $\tau(e^+) < \epsilon$ such that $eH$ belongs to the domain of the operator $x$. Let $L(A, \tau)$ be the set of all $\tau$-measurable operators affiliated to $A$.

**Proposition 3.** An operator $x \in L(M, \tau)_{sa}$ is affiliated to $A$ iff $x \in L(A, \tau)$.

**Proof.** Follows from arguments in [20].

**Theorem 3.** $(L(A, \tau), t_\tau)$ is a complete topological Jordan subalgebra of $(L(M, \tau), t_\tau)_{sa}$.

**Proof.** A direct consequence of Theorem 1 and arguments in [20].

A sequence $\{ y_n \} \subset L(A, \tau)$ is said to converge bilaterally with square almost uniformly (b.s.a.u.) to $y \in L(M, \tau)$ if $\forall \epsilon > 0$, $\exists e \in P(A)$ with $\tau(e^+) < \epsilon$ such that $\| e(y - y_n)^2e \| \to 0$.

**Proposition 4.** For $\{ y_n \} \subset L(A, \tau) \subset L(M, \tau)$ the conditions:

(i) $\{ y_n \}$ converges a.u. in $L(M, \tau)$;

(ii) $\forall \epsilon > 0$, $\exists e \in P(M)$ with $\tau(e^+) < \epsilon$ such that $\| (y_m - y_n)e \| \to 0$ as $m, n \to \infty$;

(iii) $\{ y_n \}$ converges b.s.a.u. in $L(A, \tau)$;

(iv) $\forall \epsilon > 0$, $\exists e \in P(A)$ with $\tau(e^+) < \epsilon$ such that $\| e(y_m - y_n)^2e \| \to 0$ as $m, n \to \infty$;

are equivalent.

**Proof.** From $\| e(y_m - y_n)^2e \| = e(y_m - y_n)(y_m - y_n)e = e(y_m - y_n)^2e \to 0$ as $m, n \to \infty$. 

□
\begin{align*}
-\| (y_m - y_n) e^* (y_m - y_n) e \| & \leq \| (y_m - y_n) e^* \| \cdot \| (y_m - y_n) e \| = \| (y_m - y_n) e \|^2, \\
\text{so we can see that b.s.a.u. fundamentalness of a sequence in a reversible JW-algebra ([15], [8]) is equivalent to a.u. fundamentalness of the same sequence in its von Neumann enveloping algebra } M = M(A). \\
\text{Thus the statement follows from Proposition 2 above.} \quad \square
\end{align*}

The Riesz theorem 2 above will take the following form.

**Theorem 4.** If \( \{ y_n \} \subset L(A, \tau) \) and \( y = t_x - \lim_{n \to \infty} y_n \), then \( y = \text{b.s.a.u.} - \lim_{n \to \infty} y_n \) for some subsequence \( \{ y_{n_k} \} \subset \{ y_n \} \).

**Proof.** Directly follows from Proposition 4 and Theorem 2 above. \( \square \)

### 3 Bilateral with square uniform equicontinuity for sequences of maps into \( L(A, \tau) \)

Let \( E \) be an arbitrary set. If \( a_n : E \to L(A, \tau) \), \( x \in E \), and \( b \in A \) such that \( \{ b(a_n(x))^2 b \} \subset A \).

Denote \( S(\{ a_n^2 \}, x, b) = \sup_n \| b(a_n(x))^2 b \| \).

The following Lemma is valid.

**Lemma 1.** Let \( (X, +) \) be a semigroup, and \( a_n : X \to L(A, \tau) \) be a sequence of additive maps. Assume that \( x \in X \) is such that \( \forall \varepsilon > 0 \), \( \exists \{ x_k \} \subset X \), and \( p \in P(A) \) with \( \tau(p^*) < \varepsilon \), such that:

(i) \( \{ a_n(x_k + x_n) \} \) converges b.s.a.u. as \( n \to \infty \), for every \( k \in N \);

(ii) \( S(\{ a_n^2 \}, x_k, p) \to 0 \), as \( k \to \infty \).

Then the sequence \( \{ a_n(x) \} \) converges b.s.a.u. in \( L(A, \tau) \).

**Proof.** Follows from [12] and Proposition 4. \( \square \)

Let \( (X, t) \) be a topological space, and \( a_n : X \to L(A, \tau) \) and \( x_0 \in X \) be such that \( a_n(x_0) = 0 \) for \( n \in N \). A sequence \( \{ a_n \} \) is called bilaterally with square equicontinuous at \( x_0 \) if \( \forall \varepsilon, \delta > 0 \), \( \exists \) a neighborhood \( U \) of \( x_0 \) in \( (X, t) \) such that \( a_n U \subset V(\varepsilon, \delta) \cap L(A, \tau) \), \( n \in N \), i.e. \( \forall x \in U \) and \( \forall n \in N \) one can find a projection \( e = e(x, n) \in P(A) \) with \( \tau(e^*) < \varepsilon \), satisfying \( \| e(a_n(x))^2 e \| < \delta \).

Let now \( x_0 \in E \subset X \). A sequence \( \{ a_n \} \) is called bilaterally with square uniformly equicontinuous at \( x_0 \) on \( E \) if \( \forall \varepsilon, \delta > 0 \), \( \exists \) a neighborhood \( U \) of \( x_0 \) in \( (X, t) \) such that \( \forall x \in E \cap U \), \( \exists e = e(x) \in P(A) \) with \( \tau(e^*) < \varepsilon \), satisfying \( S(\{ a_n^2 \}, x, e) < \delta \).

**Proposition 5.** Let the sequence \( \{ a_n \} \) and \( x_0 \in E \subset X \) be as above. Then,

(i) \( \{ a_n \} \) is equicontinuous at \( x_0 \) on \( E \) into \( L(M, \tau) \) iff it is bilaterally with square equicontinuous at \( x_0 \) on \( E \) into \( L(A, \tau) \);

(ii) \( \{ a_n \} \) is uniformly equicontinuous ([12]) at \( x_0 \) on \( E \) into \( L(M, \tau) \) iff it is bilaterally with square uniformly equicontinuous at \( x_0 \) on \( E \) into \( L(A, \tau) \).

**Proof.** Directly follows from Proposition 4 and arguments in [12]. \( \square \)

Theorem 1 and theorem 3 established that that \( (L(M, \tau), t_e) \) is a complete metrizable topological *-algebra, and \( (L(A, \tau), t_e) \) is a complete metrizable topological Jordan subalgebra of \( (L(M, \tau), t_e)|_{S_t} \). In [12] it has been established that for any \( d > 0 \), the sets

\( M_d = \{ x \in M : \| x \| \leq d \} \), and

\( M_d^b = \{ x \in M_d : x = x^* \} \) are \( t_e \)-complete. It is easy to see that the set \( A_d = M_d^b \cap A \) is \( t_e \)-complete too.

**Lemma 2.** Let \( d > 0 \). If \( a_n : A \to L(A, \tau) \) be a sequence of additive maps. Then it is bilaterally with square uniformly equicontinuous at 0 on \( A_d \) iff it is uniformly equicontinuous at 0 on \( M_d \) (where in the second condition we mean that all maps are extended by linearity to the sequence of additive maps \( M \to L(M, \tau) \)).

**Proof.** Directly follows from Proposition 5, and arguments in [12] and [8]. \( \square \)

**Lemma 3.** Let a sequence \( a_n : A \to L(A, \tau) \) of additive maps be bilaterally with square uniformly equicontinuous at 0 on \( A_d \) for some \( 0 < d \in \mathbb{R} \).

Then \( \{ a_n \} \) is as well bilaterally with square
uniformly equicontinuous at 0 on $A$, for every $0 < s \in \mathbb{R}$.

**Proof.** Directly follows from Lemma 2 and arguments in [12] and [8].

4 Main results

Let $0 \in E \subset A$. For a sequence $a_n : (A, t) \to L(A, \tau)$, consider the following conditions:

- Bilateral with square almost uniform convergence of $\{a_n(x)\}$ for every $x \in E$ (BSCNV $(E)$);
- Bilateral with square uniform equicontinuity at 0 on $E$ (BSCNT $(E)$);
- Closedness in $(E, t_s)$ of the set $C(E) = \{x \in E : \{a_n(x)\} \text{ converges b.s.a.u.} \}$ (BSCLS $(E)$).

In this section we will discuss relationships among the conditions (BSCNV $(A_1)$), (BSCNT $(A_1)$), and (BSCLS $(A_1)$).

**Theorem 5.** Let $a_n : A \to L(A, \tau)$ be a (BSCNV $(A_1)$) sequence of positive $t_s$-continuous linear maps with $a_n(1) \leq 1$, $n \in \mathbb{N}$. Then the sequence $\{a_n\}$ is also (BSCLS $(A_1)$).

**Proof.** Directly follows from arguments in [12] and the previous section.

**Theorem 6.** A (BSCNT $(A_1)$) sequence of additive maps $a_n : A \to L(A, \tau)$ is as well (BSCLS $(A_1)$).

**Proof.** Directly follows from arguments in [12] and the results of the previous section.

**Theorem 7.** Let $a_n : A \to L(A, \tau)$ be a sequence of positive $t_s$-continuous linear maps such that $a_n(1) \leq 1$, $n \in \mathbb{N}$. If a sequence $\{a_n\}$ is (BSCNV $(D)$) with $D$ being $t$-dense in $A$, the conditions (BSCNV $(A_1)$), (BSCNV $(A_1)$), and (BSCLS $(A_1)$) are equivalent.

**Proof.** Directly follows from [12] and the results of the previous section.

5 Conclusion

Results of the present notes extend the results of [12] to the case of JW-algebras without direct summand of type I$_2$. In a new manuscript under preparation we extend these results to the case of bilateral almost uniform convergence on semifinite von Neumann algebras and semifinite JBW-algebras without direct summand of type I$_2$. 

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