GEODESIC DISTANCES ON DENSITY MATRICES

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Abstract. We find an upper bound for geodesic distances associated to monotone Riemannian metrics on positive definite matrices and density matrices.

1. Introduction

The notion and importance of Fisher information is well established in statistics and probability theory. As a measure of distinguishability of probability densities, the Fisher information was used by Rao to define a Riemannian metric on probability spaces. On the simplex of probability vectors $\mathcal{P}_n = \{p = (p_1, \ldots, p_n), \sum_i p_i = 1, p_i > 0, i = 1, \ldots, n\}$, this is the unique metric contracting under markovian mappings, by the Chentsov uniqueness theorem. On $\mathcal{P}_n$, the Fisher metric is

$$\lambda_p(x, y) = \sum_i p_i^{-1} x_i y_i \quad x, y \in T_p \mathcal{P}_n$$

The geometry of $\mathcal{P}_n$ with this metric is quite simple. By

(1) \quad $p \mapsto 2(\sqrt{p_1}, \ldots, \sqrt{p_n})$,

it is isometric with an open subset in the sphere of radius 2 in $\mathbb{R}^n$, [3]. The metric can be extended to the set $\mathcal{M}_n = \{p = (p_1, \ldots, p_n), p_i > 0\}$ of all finite (strictly positive) measures on the set $\{1, \ldots, n\}$. Using the isometry (1) and elementary geometry in $\mathbb{R}^n$, we may compute the geodesic distance for the Fisher metric in $\mathcal{P}_n$ and $\mathcal{M}_n$:

$$D(p, q) = 2 \arccos\left(\sum_i \sqrt{p_i} \sqrt{q_i}\right) \quad p, q \in \mathcal{P}_n$$

(the Bhattacharya distance) and

$$d(p, q) = 2 \left(\sum_i (\sqrt{p_i} - \sqrt{q_i})^2\right)^{1/2} \quad p, q \in \mathcal{M}_n$$
The last expression is related to the Hellinger distance $H(p, q)$ by $d(p, q) = \sqrt{2H(p, q)}$. The Hellinger distance belongs to the family of Cziszar’s $f$-divergences $D_f(p, q) = \int f(q/p)dp$

here $f$ is a convex function. As it was shown in [1], the metric given by the hessian of $f$-divergence is a constant multiple of the Fisher metric.

In the case of a quantum system, the situation becomes more complicated. In the simplest case, the states of the system are represented by density matrices. In analogy with manifolds of classical probability densities, a quantum version of the Fisher information metric must be decreasing under stochastic maps. Contrary to the classical case, this monotonicity condition does not specify the metric uniquely. In fact, it was shown by Petz that the monotone metrics can be labelled by operator-monotone functions.

As it was mentioned in [5], there is no general formula for geodesic path and distance for a general monotone metrics. Explicit expressions are known only in two particular cases, namely the Bures metric and the Wigner-Yanase metric. In the present paper, we find an upper bound for the geodesic distances for all monotone metrics. This is done in a simple way: Following Uhlmann [18, 19], we obtain the Bures geodesics from certain purifying lifts of curves of density matrices and then make use of a duality relation between the smallest (Bures) and the largest (RLD) of monotone metrics. It is also shown that this upper bound is related to a particular non-commutative version of the Hellinger distance.

2. The manifold and monotone metrics.

Let $M_n$ be the algebra of $n$ by $n$ complex matrices. The set of faithful positive linear functionals on $M_n$ is identified with the cone of positive definite matrices. This set, with the differentiable manifold structure inherited from $M_n$, will be denoted by $\mathcal{M}$. Let $\mathcal{D} \subset \mathcal{M}$ denote the submanifold of density matrices in $\mathcal{M}$, that is

$$\mathcal{D} = \{ \rho \in \mathcal{M} : \text{Tr } \rho = 1 \}$$

The tangent space to $\mathcal{M}$ at $\rho \in \mathcal{M}$ is $T_\rho \mathcal{M} = \{ x \in M_n : x = x^* \}$. If $\rho \in \mathcal{D}$, then the tangent space $T_\rho \mathcal{D}$ is the subspace of traceless matrices in $T_\rho \mathcal{M}$.

Let $\lambda$ be a Riemannian metric on $\mathcal{M}$. Then we will say that $\lambda$ is a monotone metric if

$$\lambda_{T(\rho)}(T(h), T(h)) \leq \lambda_\rho(h, h), \quad \rho \in \mathcal{M}, \ h \in T_\rho \mathcal{M}$$
for all completely positive trace preserving maps $T$. It is an important result of Petz [16] that a Riemannian metric is monotone if and only if it has the form
\[
\lambda_{\rho}(h, k) = \text{Tr} \, h J_{\rho}(k)
\]
where $J_{\rho}$ is given by the operator mean
\[
J_{\rho} = R_{\rho}^{-1} [f(L_{\rho}/R_{\rho})]^{-1}
\]
Here $L_{\rho}$ and $R_{\rho}$ are the left and the right multiplication operator and $f : (0, \infty) \to \mathbb{R}$ is an operator monotone function which is symmetric, that is, $f(t) = tf(t^{-1})$. It is immediate from [2] that under the normalization $f(1) = 1$, any monotone metric is equal to the Fisher metric on commutative submanifolds. Moreover, we have
\[
\frac{2t}{1+t} \leq f(t) \leq \frac{1+t}{2}
\]
for all symmetric normalized operator monotone functions [12]. Accordingly, there is a greatest and a smallest element in the set of monotone metrics.

The smallest monotone metric is obtained for $f(t) = (1 + t)/2$. It is called the Bures metric, because it is related to the Bures distance, see also Section 4. The operator
\[
J_{\rho}(h) = g, \quad \rho g + g \rho = 2h
\]
is the symmetric logarithmic derivative, see [9, 16, 17].

The greatest monotone metric corresponds to the function $f(t) = 2t/(1 + t)$. In this case $J_{\rho}$ is the right logarithmic derivative (RLD)
\[
J_{\rho}(h) = \frac{1}{2}(\rho^{-1}h + h\rho^{-1})
\]
see [9, 16, 17]. More examples of monotone metrics can be found in Section 5.

3. **Standard representation and monotone metrics.**

The standard representation of the algebra $M_n$ is obtained if $M_n$ is endowed with the Hilbert-Schmidt inner product
\[
\langle x, y \rangle = \text{Tr} \, x^* y
\]
Let us denote the resulting Hilbert space by $H$. Then $M_n$ is represented on $H$ by
\[
\phi : M_n \to \mathcal{B}(H), \quad a \mapsto L_a
\]
where $L_a$ is the left multiplication operator $L_a w = aw$, $w \in H$. Each element $\rho$ in $\mathcal{M}$ has a vector representative, or purification, $w$ in $H$, such that
\[
\text{Tr} \rho a = \langle w, L_a w \rangle \quad \forall a \in M_n
\]
Then $w \in H$ is a vector representative of $\rho \in \mathcal{M}$ if and only if $\rho = ww^*$. Let $\rho_t$, $t \in I$, be a smooth curve in $\mathcal{M}$. A curve $w_t$ in $H$, such that $w_t$ is a vector representative of $\rho_t$ for all $t \in I$ is called a lift of $\rho_t$. In this case, the tangent vectors are related by
\[
\dot{\rho}_t = \dot{w}_t w_t^* + w_t \dot{w}_t^*
\]
Let us denote the corresponding projection of the tangent spaces $T_{\rho_t} \mathcal{M}$ by $\Pi$. Let $w_0 w_0^* = \rho_0$. There are many lifts of $\rho_t$ through $w_0$. Among such lifts, there is a unique lift with minimal Hilbert space length
\[
l_H(w_t) = \int_I \sqrt{\langle \dot{w}_t, \dot{w}_t \rangle} dt.
\]
It will be called the horizontal lift.

The horizontal lift was introduced in [18, 19], where the geometric phase was extended to mixed states. It was shown that the above minimalization problem leads to the condition
\[
w_t^* \dot{w}_t = \dot{w}_t^* w_t
\]
for all $t$. The curves $w_t$ in $H$, satisfying this condition, are called horizontal curves. The tangent vectors to horizontal curves at $w \in H$ form a real vector subspace $H_w = \{gw, g = g^*\}$. Let $H_w$ be endowed with the inner product $\text{Re} \langle \cdot, \cdot \rangle$, then it is a real Hilbert space, called the horizontal subspace. For each $h \in T_{ww^*} \mathcal{M}$, there is a unique element $\hat{h}$ in $H_w$, satisfying $h = \Pi(\hat{h})$. It follows that the inner product in $H_w$ can be projected onto $T_{ww^*} \mathcal{M}$. As it turns out, this projection defines a Riemannian metric on $\mathcal{M}$, moreover
\[
4 \text{Re} \langle \hat{h}, \hat{k} \rangle = 2 \text{Tr} h(L_\rho + R_\rho)^{-1}(k), \quad h, k \in T_\rho \mathcal{M}
\]
is exactly the Bures metric.

The commutant of $\phi(M_n)$ is the algebra of right multiplication operators $R_a w = wa$, $a \in M_n$, on $H$. For each $\sigma \in \mathcal{M}$, there is an element $w \in H$, such that
\[
\langle w, R_a w \rangle = \text{Tr} \sigma a
\]
This element is given by $\sigma = w^* w$. For each curve $\sigma_t$ in $\mathcal{M}$, let us consider the curves $w_t$ in $H$ satisfying $w_t^* w_t = \sigma_t$. The tangent vectors
of such curves satisfy $\dot{\sigma} = \tilde{\Pi}(\dot{w}_t)$, where $\tilde{\Pi} : T_wH \to T_{w^*w}\mathcal{M}$ is given by

$$\tilde{\Pi}(x) = x^*w + w^*x$$

We may now proceed exactly as before, choosing for each $\sigma_t$ the shortest of these curves. It is quite clear that $w_t$ is the shortest curve if and only if $w_t^*$ is horizontal, equivalently, $\dot{w}_t \in \tilde{H}_w := \{w_t g, g = g^*\}$ for all $t$. Moreover, we have

$$x \in H_w \iff x^* \in \tilde{H}_w^*.$$  

If we now project the real Hilbert space structure from $\tilde{H}_w$ to $T_{w^*w}\mathcal{M}$, using the projection $\tilde{\Pi}$, we will, of course, get the Bures metric again. On the other hand, it is easy to see that for each $\rho = w^*w$ and $h \in T_{\rho}\mathcal{M}$, $\tilde{h} := \frac{1}{2}h(w^*)^{-1}$ is the unique element in $\tilde{H}_w$ satisfying $h = \Pi(\tilde{h})$. We may therefore define

$$\lambda_{\rho}(h, k) := 4\text{Re} \langle \tilde{h}, \tilde{k} \rangle = \frac{1}{2}\text{Tr} \rho^{-1}(hk + kh)$$

which is the RLD metric. This shows that there is a duality relation between the Bures metric and RLD, see also [14, 10].

4. The geodesic distances

Let $\lambda$ be a Riemannian metric on $\mathcal{M}$. A curve $\rho_t$, $t \in [0, 1]$ is a geodesic path in $\mathcal{M}$ if its length

$$l_\lambda(\rho_t) = \int_0^1 \sqrt{\lambda_{\rho_t}(\dot{\rho}_t, \dot{\rho}_t)} dt$$

is the minimum of lengths of all curves connecting $\rho_0$ and $\rho_1$. This length is then the geodesic distance of $\rho_0$ and $\rho_1$. Let us denote by $d_\lambda$ the geodesic distance for the metric $\lambda$ in $\mathcal{M}$ and by $D_\lambda$ the geodesic distance in $\mathcal{D}$.

For Bures metric, the geodesic paths and distances were obtained by Uhlmann [18, 19] as follows. Let $\rho_0$ and $\rho_1$ be two elements in $\mathcal{M}$ and let $\rho_t$ be a curve connecting them. If $w_t$ is the horizontal lift of $\rho_t$, then by [5]

$$l_{\text{Bures}}(\rho_t) = 2l_H(w_t),$$

hence minimizing the Bures length means minimizing the Hilbert space length of horizontal lifts of curves connecting $\rho_0$ and $\rho_1$. From the definition of horizontality, this minimum is attained at the line segment $w_t = tw_1 + (1 - t)w_0$, such that $\|w_0 - w_1\|$ is minimal over $w_0w_1^* = \rho_0$, $w_1w_1^* = \rho_1$. This happens if and only if $w_1$ and $w_0$ are parallel amplitudes, that is, these satisfy Uhlmann’s parallelity condition

$$w_1^*w_0 \geq 0$$
For each \( w_0 \) there is a unique \( w_1 \) parallel to \( w_0 \), given by
\[
w_1 = \rho_0^{-1/2}(\rho_0^{1/2} \rho_1 \rho_0^{1/2})^{1/2} \rho_0^{-1/2} w_0
\]
The geodesic path in \( \mathcal{M} \), connecting \( \rho_0 \) and \( \rho_1 \) is then
\[
\rho_t = (tw_1 + (1-t)w_0)(tw_1 + (1-t)w_0)^* 
\]
and the geodesic distance is
\[
d_{\text{Bures}}(\rho_0, \rho_1) = 2\|w_0 - w_1\| = 2 \sqrt{\text{Tr} \, \rho_0 + \text{Tr} \, \rho_1 - 2 \text{Tr} \,(\rho_0^{1/2} \rho_1 \rho_0^{1/2})^{1/2}}
\]
this is called the Bures distance.

Let now \( \rho_t \) be a curve in \( \mathcal{D} \), then all lifts of \( \rho_t \) are curves on the unit sphere \( S \) in \( H \). If \( w_0, w_1 \in S \), the shortest curve connecting them lies on the large circle in \( S \) through them. The length of such arcs for \( w_0 w_1^* = \rho_0 \) and \( w_1 w_1^* = \rho_1 \) is minimal if \( w_0 \) and \( w_1 \) are parallel amplitudes and, by definition, in this case the arc is also horizontal. Hence, the Bures geodesic in \( \mathcal{D} \) is
\[
\rho_t = \frac{(w_0 + (1-t)w_1)(tw_0 + (1-t)w_1)^*}{\|tw_0 + (1-t)w_1\|^2}
\]
for parallel amplitudes \( w_0 \) and \( w_1 \) and the Bures distance
\[
D_{\text{Bures}}(\rho_0, \rho_1) = 2 \text{arccos} \, \text{Tr} \, w_0 w_1^* = 2 \text{arccos} \, \text{Tr} \,(\rho_0^{1/2} \rho_1 \rho_0^{1/2})^{1/2}
\]
The duality of the Bures and RLD metrics leads to the following upper bound for the RLD geodesic distance.

**Proposition 4.1.** Let \( \rho_0, \rho_1 \in \mathcal{M} \), then
\[
d_{\text{RLD}}(\rho_0, \rho_1) \leq d_{\text{Bures}}(\rho_0, \rho_0^{-1/2}(\rho_0 \# \rho_1)^2 \rho_0^{-1/2})
\]
where
\[
\rho_0 \# \rho_1 = \rho_0^{-1/2}(\rho_0^{-1/2} \rho_1 \rho_0^{-1/2})^{1/2} \rho_0^{1/2}
\]
is the geometric mean. If \( \rho_0 \) and \( \rho_1 \) are in \( \mathcal{D} \), the same holds for geodesic distances \( D_{\text{RLD}} \) and \( D_{\text{Bures}} \).

**Proof.** Let \( w_0 = \rho_0^{1/2} \) and let \( w \in H \) be such that \( w_0 \) and \( w \) satisfy the parallelity condition \([5]\). Then the curve \( w_t = tw + (1-t)w_0 \) is the horizontal lift of the Bures geodesic connecting \( \rho_0 \) and \( ww^* \), in particular, \( \tilde{w}_t \in H_{w_t} \) for all \( t \). Then \( w_t^* \) is a lift of a curve \( \rho_t \) in \( \mathcal{M} \), connecting \( \rho_0 \) and \( w^* w \) and by \([3]\), \( \tilde{w}_t^* \in \tilde{H}_{w_t} \). Consequently, by \([7]\),
\[
d_{\text{RLD}}(\rho_0, w^* w) \leq l_{\text{RLD}}(\rho_t) = 2\|w^* w_0^*\| = 2\|w - w_0\| = d_{\text{Bures}}(\rho_0, ww^*)
\]
From the parallelity condition, \( w = qw_0 \) for some \( q = q^* > 0 \). Let us choose \( w \) such that
\[
\rho_1 = w^* w = \rho_0^{1/2} q^2 \rho_0^{1/2}
\]
then \( q = (\rho_0^{-1/2} \rho_1 \rho_0^{-1/2})^{1/2} \) and
\[
w w^* = \rho_0^{-1/2} (\rho_0 \# \rho_1)^2 \rho_0^{-1/2}
\]
The statement for distances in \( \mathcal{D} \) is proved exactly the same way. □

**Remark 4.1.** Let \( w_0, w \) and \( q \) be as in the proof of the previous Proposition, then we have
\[
\|w_0 - w\|^2 = \text{Tr} \rho_0 + \text{Tr} \rho_1 - 2 \text{Tr} \rho_0 q
\]
and
\[
d_{Bures}(\rho_0, \rho_0^{-1/2} (\rho_1 \# \rho_0)^2 \rho_0^{-1/2}) = 2 \sqrt{\text{Tr} \rho_0 + \text{Tr} \rho_1 - 2 \text{Tr} \rho_0 \# \rho_1}
\]
so that \( \rho_0 \) and \( \rho_1 \) can be exchanged.

**Remark 4.2.** Let \( \rho_t = w_t^* w_t \) and \( q \) be as in the proof of Proposition 4.1. Then, in general, \( \rho_t \) is not the RLD geodesic. Indeed, it can be easily computed that for the RLD metric, the geodesic equation reads
\[
\ddot{\rho}_t + \frac{1}{L_{\rho_t} + R_{\rho_t}} (\dot{\rho}_t^2) - \dot{\rho}_t \rho_t^{-1} \dot{\rho}_t = a(t) \dot{\rho}_t
\]
where \( a \) is a smooth function \( a : I \to \mathbb{R} \), see also [3]. We have
\[
\rho_t = w_t^* w_t = \rho_0^{1/2} (1 + t(q - 1))^2 \rho_0^{1/2}
\]
It can be shown by direct computation that the geodesic equation is satisfied if and only if
\[
q(\rho_0 q - q \rho_0) = (\rho_0 q - q \rho_0)q
\]
which, for self-adjoint operators, implies \( q \rho_0 = \rho_0 q \). It follows that the inequality in Proposition 4.1 is strict, unless \( \rho_0 \) and \( \rho_1 \) commute. In that case, the geodesic distances are the same for all monotone metrics.

In [17], a class of generalized relative entropies
\[
H_g(\rho_0, \rho_1) = \text{Tr} \rho_0 g(\rho_0^{-1/2} \rho_1 \rho_0^{-1})
\]
was introduced, here \( g \) is an operator convex function. This is a non-commutative version of the \( f \)-divergence. It was shown in [17] that the generalized entropy \( H_g \) leads to a constant multiple of the RLD metric for infinitesimally close elements in \( \mathcal{D} \).

It is easy to see that the right hand side of (9) is equal to \( \sqrt{2 H_{g_0}(\rho_0, \rho_1)} \), where
\[
g_0(t) = 2 + 2t - 4t^{1/2}
\]
Note that on commuting elements, \( H_{g_0} \) is equal to the Hellinger distance.

By maximality of the RLD metric, we get
Corollary 4.1. Let $\rho_0, \rho_1 \in \mathcal{M}$ and let $\lambda$ be a monotone metric. Then
\[
d_{\text{Bures}}(\rho_0, \rho_1) \leq d_\lambda(\rho_0, \rho_1) \leq \sqrt{2H_{g_0}(\rho_0, \rho_1)} < 2\sqrt{\Tr \rho_0 + \Tr \rho_1}
\]
If $\rho_0, \rho_1 \in \mathcal{D}$, then
\[
2 \arccos \Tr \left( \rho_0^{1/2} \rho_1^* \rho_0^{1/2} \right)^{1/2} \leq D_\lambda(\rho_0, \rho_1) \leq 2 \arccos \Tr \rho_0^\# \rho_1 < \pi
\]

5. The WYD metrics

The Wigner-Yanase-Dyson (WYD) metrics are defined by
\[
\lambda^\alpha_\rho(h, k) = \partial^2_{t\partial s} \Tr f_\alpha(\rho + th)f_{-\alpha}(\rho + sk)|_{s,t=0}
\]
where
\[
f_\alpha(x) = \begin{cases}
\frac{1}{1-\alpha}x^{\frac{1-\alpha}{2}} & \alpha \neq 1 \\
\log(x) & \alpha = 1
\end{cases}
\]
As it was shown in [8], these metrics are monotone for $\alpha \in [-3, 3]$. The family of WYD metrics is important in quantum information geometry, see [7, 11, 6]. As special cases, for $\alpha = \pm 1$, we get the well known Bogoljubov-Kubo-Mori metric and for $\alpha = \pm 3$ we get the RLD metric.

The smallest in this family is the Wigner-Yanase (WY) metric, obtained for $\alpha = 0$. The WY metric has the form
\[
\lambda_\rho(h, k) = 4\Tr h(\sqrt{L_\rho} + \sqrt{R_\rho})^{-2}(k)
\]
The corresponding geodesic path and distance was computed in [5], using a non-commutative version of the square root map (1) and a pull-back technique. We will show that these can be also easily obtained using a similar method as in the Bures case.

Let $\rho_t$ be a curve in $\mathcal{M}$. Among its lifts $w_tw^*_t = \rho_t$, we will again choose a horizontal one. In this case, the lift $w_t$ is horizontal if it is contained in the natural positive cone at $w_0$, that is, if $w_t = \rho_t^{1/2}u_0$ for all $t$. In this case, the horizontal subspace is $H^0_w = \{gu, g = g^*\}$, where $w = \rho^{1/2}u$ is the polar decomposition of $w$. Each tangent vector $h \in T_{w_0}\mathcal{M}$ has a unique horizontal lift $h^0 = gu \in H^0_w$, such that $h = \Pi(h^0) = g\rho^{1/2} + \rho^{1/2}g$. The induced metric
\[
\lambda_\rho(h, k) = 4\Re\langle h^0, k^0 \rangle = 4\Tr h(L_\rho^{1/2} + R_\rho^{1/2})^2(k)
\]
is the WY metric. Note that in this case $x \in H^0_w$ if and only if $x^* \in H^0_w$, so that the WY metric is self-dual, in the sense mentioned in Section 3. Let us also remark that it is possible to obtain all the monotone metrics in a similar manner, see [4, 10].

Let now $\rho_0$ and $\rho_1$ be in $\mathcal{M}$ and let $\rho_t$ be a curve connecting them. Again, the WY length of $\rho_t$ is twice the Hilbert space length of its
horizontal lift \( w_t = \rho_t^{1/2} u_0 \). Therefore, \( \rho_t \) is the geodesic path if \( w_t = t \rho_t^{1/2} u_0 + (1 - t) \rho_0^{1/2} u_0 \), that is
\[
\rho_t = (t \rho_t^{1/2} + (1 - t) \rho_0^{1/2})^2
\]
and the geodesic distance is
\[
d_W Y(\rho_0, \rho_1) = 2 \| \rho_0^{1/2} - \rho_1^{1/2} \| = 2 \sqrt{\text{Tr} \rho_0 + \text{Tr} \rho_1 - 2 \text{Tr} \rho_0^{1/2} \rho_1^{1/2}}
\]
Similarly, if \( \rho_0, \rho_1 \in D \), then \( \rho_t \) is a geodesic path if and only if \( w_t \) lies on the large circle connecting \( \rho_0^{1/2} u_0 \) and \( \rho_1^{1/2} u_0 \). Hence
\[
\rho_t = \frac{(t \rho_1^{1/2} + (1 - t) \rho_0^{1/2})^2}{\| t \rho_1^{1/2} + (1 - t) \rho_0^{1/2} \|^2}
\]
and
\[
D_W Y(\rho_0, \rho_1) = 2 \arccos \text{Tr} \rho_0^{1/2} \rho_1^{1/2}
\]
Let us denote by \( \Delta_{\sigma, \rho} = L_{\sigma} R_{\rho}^{-1} \) the relative modular operator. In [15], a class of quasi-entropies was introduced by
\[
S_g(\rho, \sigma) = \text{Tr} \rho^{1/2} g(\Delta_{\sigma, \rho})(\rho^{1/2})
\]
where \( g \) is an operator convex function. This is another quantum version of the \( f \)-divergences. It is easy to see that
\[
(11) \quad d_W Y(\rho_0, \rho_1) = \sqrt{2 S_{g_0}(\rho_0, \rho_1)},
\]
where \( g_0 \) is given by (10). It was proved in [13] that each monotone metric can be obtained as the hessian of \( S_g \) for a suitable operator convex function \( g \). The choice \( g = g_0 \) leads to the WY metric.

From the previous Section and the fact that the WY metric is the least element in the family of WYD metrics, we get

**Corollary 5.1.** Let \( \lambda \) be a WYD metric and \( \rho_0, \rho_1 \in M \). Then
\[
\sqrt{2 S_{g_0}(\rho_0, \rho_1)} = d_W Y(\rho_0, \rho_1) \leq d_\lambda(\rho_0, \rho_1) \leq \sqrt{2 H_{g_0}(\rho_0, \rho_1)}
\]
where \( g_0(t) = 2 + 2t - 4t^{1/2} \). If \( \rho_0, \rho_1 \in D \), then
\[
2 \arccos \text{Tr} \rho_0^{1/2} \rho_1^{1/2} \leq D_\lambda(\rho_0, \rho_1) \leq 2 \arccos \text{Tr} \rho_0 \# \rho_1
\]

6. Acknowledgements

The research was supported by the grant VEGA 1/0264/03.
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