Non-commutative space: boon or bane for quantum engines and refrigerators

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Various quantum systems are considered as the working substance for the analysis of quantum heat cycles and quantum refrigerators. The ongoing technological challenge is how efficiently can a heat engine convert thermal energy to mechanical work. The seminal work of Carnot has proposed a fundamental upper limit—the Carnot limit on the efficiency of the heat engine. However, the heat engines can be operated beyond the fundamental upper limit by exploiting non-equilibrium reservoirs. Here, the change in the space structure introduces the non-equilibrium effect. So, a question arises whether a change in the space structure can provide any boost for the quantum engines and refrigerators. The efficiency of the heat cycle and the coefficient of performance (COP) of the refrigerator cycles in the non-commutative space are analyzed here. The efficiency of the quantum heat engines gets a boost with the change in the space structure than the traditional quantum heat engine but the effectiveness of the non-commutative parameter is less for the efficiency compared to the COP of the refrigerator. There is a steep boost for the coefficient of performance of the refrigerator cycles for the non-commutative space harmonic oscillator compared to the harmonic oscillator.

I. INTRODUCTION

In the early stage of quantum field theory, it was Heisenberg suggestion to use the non-commutative structure for space-time coordinates for small length scale. Synder [1] proposed a formalism in this area to control the divergence, which had troubled theories like quantum electrodynamics. To define the non-commutative space, we surrogate the space-time coordinate \( x^\mu \) by their hermitian operators \( \hat{x}^\mu \) of non-commutative \( C^* \)- algebra of space-time [2]. An example of physics in the non-commutative space-time is Yang–Mills theory on non-commutative torus [3]. Out of various versions, the most commonly studied version consists of replacing the standard version of commutation relations by their non-commutative version such as \( [x^\mu , x^\nu ] = i\theta ^{\mu \nu } \), where \( x^\mu , x^\nu \) are the canonical coordinates and \( \theta ^{\mu \nu } \) is considered as a constant antisymmetric tensor. Other interesting structures exist in the literature which leads to instances like minimal length and generalized version of Heisenberg’s uncertainty relations. These are obtained when \( \theta ^{\mu \nu } \) is considered to be a function of the coordinates and momenta [4, 5].

Thermodynamics is a pre-eminent theory to evaluate the performance of the engines. It is one of the pillars of theoretical physics. Quantum thermodynamics deals with the exploration of the thermodynamic variables such as heat, temperature, entropy, energy and work for microscopic quantum systems and even for a single particle. The exploration of quantum thermodynamics involves the study of heat engines and refrigerators in microscopic regime [6–9]. Heat engines exist in two forms one, is discrete and the other is continuous in nature. In a discrete group, we have two-stroke and the four-stroke engines and in the continuous group, we have a turbine. Various working models for the realization of the quantum engines have been proposed in previous works [10–13]. The pioneering work of Carnot suggested that the efficiency of the heat engine to convert thermal energy to mechanical work has a limit. The modern-day challenge is to develop a more efficient heat engine to convert thermal energy to mechanical work with the different working mediums. Theoretical studies suggest that the limit to the efficiency of the engine, i.e., the Carnot limit, can be surpassed by exploiting the non-equilibrium reservoirs. Now the question is can any working model in quantum regime exceed the Carnot efficiency and can it boost the Coefficient of performance (COP) of the refrigerator?

In this paper, we have proposed an approach to surpass the Carnot efficiency of the thermal machine based on the non-commutative space structure. Progress in this direction but with different approaches is shown in the works [14–17]. For our analysis, we utilize the latter version of non-commutative space-time where \( \theta ^{\mu \nu } \) is considered to be a function of the coordinates and momenta. Our prime motivation is to develop engine and refrigerator in non-commutative space where the working substance will be the perturbed harmonic oscillator in this space. We employ this harmonic oscillator in the Stirling and Otto cycle which is the working principle for different engines and refrigerator. We analyze all the stages of the cycle to compute the efficiency of this model. The outcomes are astonishing when compared with the results of the usual spaces. We always observe higher efficiency in non-commutative space than the usual spaces. Along with that, the most interesting observation is that the efficiency is more for this space structure than the commutative phase space when we switch on the non-commutative parameter but it decreases with the increase in the parameter. Whereas, with the increase in the non-commutative parameter, the COP rises correspondingly. This guides us to the possibility of using non-commutative systems for the exploration of quantum information processing to obtain better results. One immediate question that arises is whether the defined non-commutative system is accessible physically. The obvious answer is yes and it is shown in previous works [18, 19]. The schematic analysis of the experimental model to access non-commutative space using optics is analyzed [19]. Using the same methodology, one can think of modeling the heat engine of non-commutative space.

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II. NON-COMMUTATIVE HARMONIC OSCILLATOR
(NHO)

We initiate our discussion with the basic notion of squeezed states. We can obtain the squeezed states by applying the Glaubers unitary displacement operator $D(\alpha)$ on the squeezed vacuum state \[20\]. The mathematical form is $|\alpha, \xi\rangle = D(\alpha)A(\xi)|0\rangle$, where $D(\alpha) = e^{(\alpha a - \alpha^* a^*)}$ and $A(\xi) = e^{(\xi a^* - \xi^* a)}$. Here $\alpha, \xi$ are the displacement and squeezing parameters, respectively, and $A(\xi)$ is the unitary squeezing operator. The ordering of the displacement and the squeezing operator are equivalent and it amounts to a change of parameter \[20\]. We can also construct the squeezed state $|\alpha, \xi\rangle$ using the ladder operator. It is obtained by performing the Holstein-Primakoff/Bogoliubov transformation on the squeezing operator \[20\]. It is defined as $(a + \xi a^\dagger)|\alpha, \xi\rangle = \alpha|\alpha, \xi\rangle$ where $a, \xi \in \mathbb{C}$. The operators $a, a^\dagger$ are the bosonic annihilation and creation operators, i.e., $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$, where $k(n)$ is a general function which leads to different generalized models.

The one-dimensional harmonic oscillator in the non-commutative space is defined as \[4, 21, 22\]. It is defined as $\gamma X = \gamma_0 + \gamma_2 p^2$, where $\gamma_0 = (1 + \gamma_0^2)$ and $\gamma_2 = \gamma_0^2/2$. The Hamiltonian (2) is non-Hermitian with respect to the inner product. The non-commutative Hamiltonian of the one-dimensional harmonic oscillator is derived from the standard one-dimensional harmonic oscillator which satisfies the condition $[X, P] = i\hbar(1 + \gamma_0^2)$.

III. QUANTUM HEAT ENGINES OF NON-COMMUTATIVE SPACE

The canonical partition function for the defined Hamiltonian \[1\] can be evaluated with the help of its corresponding eigenvalue to

$$Z = \sum_n e^{-\beta E_n}$$

subjected to the condition that $Re[\beta \omega \mu] > 0$. We are neglecting the higher order terms because they tend to zero for the higher order. Erfc is the complementary error function, it is defined as $erfc(x) = \Gamma(1/2, x^2)/\sqrt{\pi}$. We will now be able to evaluate all the thermodynamic variables in terms of the established partition function of the considered system for the analysis of the engine model.
A. Quantum stirling heat cycle

We will analyze the well-known Stirling cycle with non-commutating harmonic oscillator as the working substance. The four stages of the Stirling cycle (Fig. 1) is as follows:

(i) The first step of the cycle is the isothermal (A→B) process. In this process, the working substance will be kept in contact with a heat bath of temperature $T_h$. The system stays in thermal equilibrium with the heat bath throughout every instant of time. The energy spectrum $E_n$ and the internal energy $U$ are changed as a result of the slow change in the working substance, i.e., the changes that take place in Hamiltonian during the execution of this phase. So, heat is absorbed from the bath in this phase. The heat exchange during this phase of the cycle is

$$Q_{AB} = U_B - U_A + k_B T_h ln Z_B - k_B T_h ln Z_A,$$

where $k_B$ is the Boltzmann constant.

(ii) The second phase of the cycle is the isochoric (B→C) process. During this process, the system undergoes an isochoric heat exchange. The system is connected to a bath with lower temperature $T_c$, so heat is released. The heat exchange for this process is expressed as

$$Q_{BC} = U_C - U_B.$$

(iii) The third phase is an isothermal (C→D) process. During this phase of the cycle, the working substance is kept in contact with a bath of lower temperature $T_c$. Similar to phase one, the system is in thermal equilibrium with the bath. In this process, heat is released. The heat exchange during this stage of the cycle is

$$Q_{CD} = U_D - U_C + k_B T_c ln Z_D - k_B T_c ln Z_C.$$

(iv) The last stage of the cycle is the isochoric (D→A) process. The system is connected back to the bath with a higher temperature $T_h$. The heat exchange for the last stage of the cycle is expressed as

$$Q_{DA} = U_A - U_D.$$

For all the phases, the internal energy of the system can be evaluated using the partition function as $U = -\frac{\delta \ln Z}{\delta \beta}$. The different form of the partition function $(Z_A, Z_B, Z_C, Z_D)$ arises due to the changes that occur in the Hamiltonian of the system during the different phases of the cycle. The total work done is $W_{tot} = Q_{AB} + Q_{BC} + Q_{CD} + Q_{DA}$. The efficiency of the Stirling heat cycle is expressed as

$$\eta_{Stir} = 1 + \frac{Q_{BC} + Q_{CD}}{Q_{DA} + Q_{AB}}.$$

B. Stirling refrigerator cycle

If we reverse the cycle, we will have a Stirling refrigerator [30]. Following the same methodology, as done above, we can analyze all the four stages of the cycle.

(i) The first phase is the isothermal process. In this process, the system is paired to a cold bath at temperature $T_c$. This is just the reverse of the first phase of the heat cycle. The entropy of the system changes during this process. The heat absorbed is

$$Q_{AB} = T_c \Delta S.$$

(ii) The second stage is the isochoric process. During this phase of the cycle, the temperature of the system increases when connected to $T_h$ from $T_c$. The mean internal energy of the system changes during this phase of the working cycle. The heat gain for this phase is

$$Q_{BC} = U_C - U_B.$$
(iii) This phase is an *isothermal* process. During this stage of the cycle, the system is bridged with the hot reservoir with a temperature $T_h$. Heat is rejected from the system and is described as

$$Q_{CD} = T_h \Delta S.$$  \hfill (14)

(iv) The last stage is an *isochoric* process. The system is reverted to the cold reservoir $T_c$ from the hot reservoir $T_h$, which leads to a decrease in the internal energy of the system. The amount of heat released is

$$Q_{DA} = U_A - U_D.$$  \hfill (15)

The entropy of the system can be evaluated from the partition function of the system as $S = \ln Z + \beta U$. The internal energy can be evaluated as shown while we have analyzed the heat cycle. The COP of the Stirling refrigerator cycle is expressed as,

$$COP_{\text{Stir}} = \frac{Q_{AB} + Q_{BC}}{W_T},$$  \hfill (16)

where $W_T = Q_{AB} + Q_{BC} + Q_{CD} + Q_{DA}$ is the total work done on the system.

We can visualize the growth in the COP for the Stirling cycle for the Harmonic Oscillator (HO) and NHO. The COP of the Stirling refrigerator cycle with the NC parameter is shown in Fig. 3. In Fig. 2, the efficiency of the engine decreases with the increase in the NC parameter. So the non-commutative parameter is less effective when the NC parameter is high for the engine model that we have considered for our analysis. For the generic statement of the less effectiveness of NC parameter on the engine model, we have to explore other cycles which is an open area for exploration.

The maximum attainable efficiency of the heat engine by the standard harmonic oscillator has been plotted as a reference point for the analysis of the advantage due to the NC space. Now, due to the change in the space structure, the standard Hamiltonian changes to the non-commutative harmonic oscillator by applying the transformation from the commutative space to the non-commutative space. So, to compare the advantage of the change introduced by the non-commutative space, we have considered the standard harmonic oscillator as a reference point. For the Stirling cycle, we can visualize the advantage in Fig. 2 and Fig. 3.

C. Quantum Otto cycle

Now we will study the quantum Otto refrigerator cycle [31–33] with non-commutative space harmonic oscillator as the working substance. The four phases of the Otto refrigeration cycle (Fig. 4) are described as follows:

(i) The first phase of the cycle is an *isochoric* ($A \rightarrow B$) process. During this process, the system is coupled to a cold reservoir at a temperature $T_C$ while the Hamiltonian remains constant. The heat absorbed from the reservoir during this process is

$$Q_{\text{cold}} = \sum_n E_n^{\text{cold}} (p_n^{\text{hot}} - p_n^{\text{cold}}),$$  \hfill (17)

where $p_n^{\text{cold}} = \frac{\exp(-\beta E_n)}{Z} \mid \beta = \beta_{\text{cold}}, \omega = \omega'$, and $p_n^{\text{hot}} = \frac{\exp(-\beta E_n)}{Z} \mid \beta = \beta_{\text{hot}}, \omega = \omega'$. The occupation probabilities of the system in the nth eigenstate and $E_n^{\text{cold}} = E_n$ for $\omega = \omega'$.

(ii) The second stage of the cycle is an *adiabatic* ($B \rightarrow C$) process. During this phase of the cycle, the entropy of the system is conserved. Throughout the evolution of the system in this phase, the occupation distribution remains invariant.

(iii) The third stage of the cycle is an *isochoric* ($C \rightarrow D$) heating process. In this process, the system is connected to a hot reservoir at temperature $T_H$. The heat rejected to the hot reservoir in this phase is

$$Q_{\text{hot}} = \sum_n E_n^{\text{hot}} (p_n^{\text{cold}} - p_n^{\text{hot}}),$$  \hfill (18)

where $E_n^{\text{hot}} = E_n$ for $\omega = \omega$.

(iv) The last stage of the cycle is an *adiabatic* ($D \rightarrow A$) process. In this process, the system changes quasi-statically while the entropy of the system remains constant during the execution of this phase. The total work done on the cycle can be evaluated as, $W_{\text{total}} = Q_{\text{hot}} + Q_{\text{cold}}$. The COP for the Otto refrigerator is defined as the ratio of the amount of heat removed form the cold reservoir to the net amount of work done on the system under analysis. It is represented as

$$\Xi_{\text{Otto}} = \frac{Q_{\text{cold}}}{|W_{\text{total}}|}. $$  \hfill (19)
IV. DISCUSSION AND CONCLUSION

The non-commutative harmonic oscillator outperforms the harmonic oscillator in terms of the COP for the Stirling cycle and the Otto refrigerator. The contribution for this is provided by the non-commutative space parameter. So, we can infer that non-commutative is a boon for the refrigerators if considered for the growth of the COP. Whereas the NC parameter is less effective for a boost in the efficiency of the heat cycle for the considered model.

For the appropriate implementation of Otto refrigerator, it requires a slow implementation of the adiabatic processes so that we can maintain no further coherence generation on the eigenstates of the non-commutative space Hamiltonian. If there is any change, then the mean population will also change. To achieve thermal equilibrium with the reservoir, the system must spend a long time during the thermalization processes. The non-linearity that is generated in the Hamiltonian appears due to the non-commutative parameter which requires some energy cost for the implementation of the cycle. This model can result to a better resource in the applicable areas of quantum theory which needs further analysis. This model can be used for the analysis of the coupled working medium as shown in previous works. It can also be utilized for exploring the non-Markovian reservoirs in NC space. We have analyzed two thermodynamic cycles. One can analyze the different existing reversible cycles and irreversible cycles. It will be interesting to explore the effect of NC parameters in the irreversible cycles and on the quantum phase transition.

For the analysis of our work, we have used one of the existing models of the non-commutative space. To make the generic statement of the gain in COP for different cycles, one has to explore all the existing models in the non-commutative space. This is an open area to explore. One can also analyze the effect of the NC parameter on the different thermodynamic variables. We can analyze this model from uncertainty viewpoint to reduce the cost for the analysis of the

In the case of the Otto refrigerator, we have encountered a similar effect as in the case of Stirling cycle. We have detected the growth in the COP for the Otto cycle for NHO due to the non-commutating parameter shown in Fig. 5. But in the case of HO the COP remains constant. A steep rise in the coefficient of the NC parameter occurs for $\omega > \omega'$ and $\beta_h < \beta_c$ and this gives rise to this phenomenon. Following a similar pattern as done during the analysis of the Stirling cycle, we have considered the maximum attainable efficiency of the heat engine by the standard harmonic oscillator has been plotted as a reference point for the analysis of the advantage due to the NC space. For the Otto cycle, we can visualize the advantage in Fig. 5.

The immediate question that pokes our intuition is whether it is feasible with quantum technology we have? The answer to this is yes. One can analyze the non-commutative space effect using optical setup and measure the effect of the non-commutative parameter as shown in [19]. They have provided a schematic representation of the experimental setup for the following analysis. Following the same methodology, we can develop the setup for the analysis of different thermodynamic cycles. For experimental realization of the cycle in non-commutative space, one should keep in mind the basic ingredients that are required for the analysis. One of which is the availability of thermal heat bath for the different processes. The second one is about the measurement of the work performed during the different phases of the cycle, as in the case of the Otto cycle, the phases are two adiabatic processes. One of which expands the working medium and the other compresses the working medium. Third one is maintaining the thermal equilibrium during the thermalization processes. The experimental analysis of the engines and refrigerators of NC space is an open area to explore.
cycles. Along with that, it will be interesting to explore the experimental realization of these cycles in NC space.

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