A short proof of Shelah’s eventual categoricity conjecture for AEC’s with amalgamation, under $GCH$

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Abstract

We provide a short proof of Shelah’s eventual categoricity conjecture for abstract elementary classes (AEC’s) with amalgamation, assuming the Generalized Continuum Hypothesis ($GCH$). The proof builds on an earlier topos-theoretic argument which was syntactic in nature and recurred to $\kappa$-classifying toposes. We carry out here the same proof idea but from the semantic perspective, making use of a connection between $\kappa$-classifying toposes on one hand and the Scott adjunction on the other hand, this latter developed independently. Along the way we prove a downward categoricity transfer for axiomatizable AEC’s with amalgamation and no maximal models.\footnote{This research has been supported through the grant 19-00902S from the Grant Agency of the Czech Republic. I would like to thank Jiří Rosický and Ivan Di Liberti for reading and commenting a first version.}

1 Introduction

Shelah’s eventual categoricity conjecture asserts that for any AEC there is a cardinal $\kappa$ such that if the AEC is categorical in some $\lambda > \kappa$, it is categorical in all $\lambda > \kappa$. This general conjecture was stated in [She09], while the version for the particular case of sentences in $L_{\omega_1, \omega}$ was conjectured circa 1977. Both conjectures are still open, though several approximations are known (see, e.g., [Vas18]). When the AEC has amalgamation and the Generalized Continuum Hypothesis ($GCH$) holds, the conjecture was proven to be true in [Esp19] through a topos-theoretic argument. We will here run the same proof, but taking a look at the semantic content of the $\kappa$-classifying toposes there employed. This is possible through an unexpected connection between these toposes and the ones arising from the Scott adjunction, this latter discovered by Simon Henry and Ivan Di Liberti (see [Hen] and Di Liberti’s PhD thesis [DL]). The use of the semantic description of the $\kappa$-classifying toposes allows us to simplify the proof considerably, obtaining in fact a downward categoricity transfer which provides information about the categoricity spectrum of AEC’s with amalgamation and no maximal models. In a future work we will study under what conditions an upward categoricity transfer can be established, thereby obtaining generalizations of Morley-Shelah categoricity theorem.

2 The theory $\lambda$-classified by the $\lambda$-Scott topos

The theorem that allows the topos-theoretic machinery to work is the following:

\textbf{Theorem 2.1.} Assume $\kappa^{<\kappa} = \kappa$. Then any $\kappa$-separable $\kappa$-topos has enough $\kappa$-points.

For a proof, see [Esp17]. Using this theorem, we can now find an important link between categoricity and atomic toposes:
Theorem 2.2. Assume $\kappa^{<\kappa} = \kappa$. A $\kappa$-separable $\kappa$-topos has a unique $\kappa$-point (up to isomorphism) of size at most $\kappa$ if and only if the topos is two-valued and Boolean.

Proof. The proof is a direct generalization of the case $\kappa = \omega$, proven by Barr and Makkai in [BM87], using Theorem 2.1 to generalize the analogous statement for separable toposes (which is the particular case $\kappa = \omega$).

Recall from [Hen] and [DL] that there is an adjunction between $\kappa$-exact localizations of presheaf toposes and accessible categories with $\kappa$-directed colimits. Given such an accessible category $A$, its $\kappa$-Scott topos $S_\kappa(A)$ is the topos of functors to $\text{Set}$ preserving $\kappa$-directed colimits. This assignment is functorial and gives one part of the adjunction, the other being the functor $pt$ sending a topos to its category of $\kappa$-points.

Any AEC $K$ is a $\lambda$-accessible category with directed colimits, and thus we can consider the corresponding Scott toposes. An AEC $(K, \prec)$ categorical in some $\lambda$ with $\text{cof}(\lambda) > LS(K)$ can be axiomatized, above the categoricity cardinal and under $GCH$, in $\mathcal{E}_{\lambda,LS(K)^+}$ (cf. Remark 4.2 in [Esp9]). We have now:

Lemma 2.3. Let $(K, \prec)$ be an AEC with the amalgamation property. Then the inclusion $f : \mathcal{K}_{\geq \kappa^+} \hookrightarrow \mathcal{K}_{\geq \kappa}$ induces an embedding of the corresponding $\kappa^+$-Scott toposes.

Proof. Consider the full and faithful inclusion $f : \mathcal{K}_{\geq \kappa^+} \hookrightarrow \mathcal{K}_{\geq \kappa}$, both accessible categories with $\kappa^+$-directed colimits. The $\kappa^+$-Scott functor takes this inclusion into a geometric morphism whose inverse image $F^*$ is given by corestricting the transpose $f^* : \text{Set}^{K^\kappa} \longrightarrow \text{Set}^{K_{\kappa^+}}$ along the inclusion $i : S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) \hookrightarrow \text{Set}^{K_{\kappa^+}}$, i.e., it satisfies $i F^* = f^*$ (cf. [DL]). The direct image $F_*$ of the geometric morphism is in turn given by the composite $f_* i : S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) \hookrightarrow \text{Set}^{K_{\kappa^+}} \longrightarrow \text{Set}^{K_{\kappa^+}} \cong S_{\kappa}(\mathcal{K}_{\geq \kappa})$, where $f_* = (f^* \circ Y)^*$ is the right adjoint of the left Kan extension, along Yoneda embedding $Y : \mathcal{K}_{\geq \kappa^+} \hookrightarrow \text{Set}^{K_{\kappa^+}}$, of the composite of $f^*$ and $Y$.

To prove that $F_*$ is an embedding we need to show that the counit is an isomorphism, for which in turn it is enough to prove that $f^* f_*$ is an isomorphism. At the level of objects, $f^* f_*$ is computed as follows:

$$f^* f_*(G)(a) = f^*(f^* \circ Y)^*(G)(a) = (f^* \circ Y)^*(G)(f(a)) \cong [f^*(Y(f(a))), G] \tag{1}$$

If $f(a) \cong \lim_i b_i$ is the canonical expression of a model as a $\kappa^+$-filtered colimit of models of size $\kappa$, we have $[f^*(Y(f(a))), G] \cong \lim_i [f^*(Y(b_i)), G]$. Using now that $f$ is full and faithful we get that $f^*(Y(b_i)) = [b_i, -]_{\mathcal{K}_{\geq \kappa^+}}$; indeed, again at the level of objects, we have:

$$f^*(Y(b_i))(a') = Y(b_i)(f(a')) = [b_i, f(a')] = [b_i, -] \circ f(a').$$

Replacing this value in (1) we get $f^* f_*(G)(a) \cong \lim_i [\lim_i [a_j', -], G]$. We want to show that this coincides with $G(a)$. Let $[b_i, -]_{\mathcal{K}_{\geq \kappa^+}} \cong \lim_j [a_j', -]$ be the canonical expression of a functor as a colimit of representables. Replacing this value we get:

$$f^* f_*(G)(a) \cong \lim_i \lim_j [a_j', -], G] \cong \lim_j G(a_j').$$

Now the limit $\lim_j G(a_j')$ is the set of compatible families of elements $x_{(a_j', f)} \in G(a_j')$, when $(a_j', f)$ runs through the category of elements associated to $[b_i, -]_{\mathcal{K}_{\geq \kappa^+}}$, whose objects are all the arrows $f : b_i \longrightarrow a_j$ with $a_j \in \mathcal{K}_{\geq \kappa^+}$. The filtered colimit of such compatible
families is simply computed by taking the classes according to the equivalence relation which identifies two compatible families on categories of elements associated to $b_i, b'_i$ if their restriction to a common subcategory of elements, associated to some $b_j$ containing both $b_i$ and $b'_i$, coincide. There is a map $s$ from this filtered colimit to the set $S$ of compatible families on the category of elements associated to $a$ (which is the filtered colimit of the $b_j$) given by restriction. But this latter category contains $a$ itself, and thus compatible families are necessarily in bijection with the elements of $G(a)$, so we just need to prove that $s$ is a bijection.

Given a compatible family $X$ on the category associated to $a$, corresponding to an element $e$, consider first the case in which $G = ev_e|_{\mathcal{K}_{\geq\kappa}^+}$ for some $\phi \in \mathcal{L}_{\kappa^{++},\kappa^+}$. Since $\phi(x) = \bigvee_{i<\kappa} \exists y \bigwedge_{j<i} \psi_{j}^i(x,y)$, there is a tuple $c$ in $a$ and a $i < \kappa$ for which $a \models \bigwedge_{j<i} \psi_{j}^i(e,c)$. Let $b$ be a submodel of size $\kappa$ containing $e$ and $c$. Then $b \models \phi(e)$, i.e., $e \in ev_e(b)$. This implies that, defining $x \in ev_e(a)$ as the image $h(e)$ for each $h : b \to a$, we get a compatible family on the category associated to $b$, whose image by $s$ is the compatible family $X$. Hence, $s$ is surjective for this choice of $G$. Moreover, for a model $a' \notin S$ in the category of elements corresponding to some $b'$, our assumptions imply that there is an embedding $h : a' \to b'$ of $a'$ into some $m \in S$, whence there is a unique possible choice of $e' \in G(a')$, as $G(h)$ is injective (since $h$ is a monomorphism). This implies that any two compatible families which by $s$ are mapped to $X$ must actually coincide when restricted to the category associated to some $b_j$. Thus, $s$ is also injective for this choice of $G$, i.e., $s$ is indeed a bijection.

Finally, each $G$ in $\text{Set}^{\mathcal{K}_{\kappa^+}}$ is of the form:

$$G \cong \lim_{\leftarrow i} [a_i, -] \cong \lim_{\leftarrow i} \lim_{\leftarrow j} [\phi_{i,j}^*, -], -] \cong \lim_{\leftarrow i} \lim_{\leftarrow j} ev_{\phi_{i,j}^*}.$$ 

Since by what we have proven so far the counit $\eta_{ev_{\phi_{i,j}^*}}$ is an isomorphism, so must be each $\eta_{[a_i, -]}$ (since the $ev_{\phi_{i,j}^*}$ are codense in the $[a_i, -]$), which in turn implies that so must be $\eta_G$ for every functor $G$ (since the $[a_i, -]$ are dense in $\text{Set}^{\mathcal{K}_{\kappa^+}}$).

Let $T$ be the theory axiomatizing an AEC with the hypothesis of Lemma 2.3, and define $T' := T \cup \{\text{there are $\kappa^+$ distinct elements}\}$, the $\kappa^+$-geometric theory which axiomatizes $\mathcal{K}_{\geq\kappa^+}$. We have the following:

**Theorem 2.4.** Assume $GCH$ and amalgamation. The $\kappa^+$-classifying topos of $T'$ is precisely the $\kappa^+$-Scott topos $S_{\kappa^+}(\mathcal{K}_{\geq\kappa^+})$. Moreover, the canonical embedding of the syntactic category $\mathcal{C}_{T'} \hookrightarrow S_{\kappa^+}(\mathcal{K}_{\geq\kappa^+})$ is given by the evaluation functor, which on objects acts by sending $(x, \phi)$ to the functor $\{M \mapsto [\phi]^M\}$.

**Proof.** By Theorem 4.1 in [Esp19], we know that the $\kappa^+$-classifying topos of the theory $T'' := T \cup \{\text{there are $\kappa$ distinct elements}\}$ is precisely the $\kappa^+$-Scott topos $S_{\kappa^+}(\mathcal{K}_{\geq\kappa}) \cong \text{Set}^{\mathcal{K}_{\kappa}}$. By Lemma 2.3, there is an embedding $i : S_{\kappa^+}(\mathcal{K}_{\geq\kappa^+}) \hookrightarrow \text{Set}^{\mathcal{K}_{\kappa}}$. On the other hand the functor $ev : \mathcal{C}_{T'} \to S_{\kappa^+}(\mathcal{K}_{\geq\kappa^+})$, given by evaluation $\phi \mapsto ev_{\phi}$, induces the counit $\eta_{\text{Set}^{T'}}$ whose inverse image is $\eta_{\text{Set}^{\mathcal{K}_{\kappa}}}$, such that $(\eta_{\text{Set}^{T'}})^*$, followed by the embedding $\text{Set}^{T'} \hookrightarrow \text{Set}^{\mathcal{T''}}$ gives $i$. This implies that $(\eta_{\text{Set}^{T'}})^*$ is an embedding. Since $\text{Set}^{T'}$ has enough $\kappa^+$-points, $\eta_{\text{Set}^{T'}}$ is in addition conservative, whence $\eta_{\text{Set}^{T'}}$ is an equivalence.

**Corollary 2.5.** Assume $GCH$ and amalgamation, and let $\kappa \geq LS(\mathcal{K})^+$ be regular. Then $\mathcal{K}$ is $\kappa$-categorical if and only if $S_\kappa(\mathcal{K}_{\geq\kappa})$ is two-valued and Boolean. Moreover, if $\kappa$ is
singular the result still holds provided we define the Scott topos $S_\mu(K_{\geq \mu})$ as the colimit of the Scott toposes $S_\mu(K_{\geq \mu})$ for regular $\mu < \kappa$.

Proof. For a successor cardinal $\kappa$ with $\kappa^{<\kappa} = \kappa$, the result follows directly from Theorem 2.2 and Theorem 2.4. For $\kappa$ limit, we can use the proof of the “if” implication appearing in [Esp19]; for the “only if” part, note that by the duality theory of [For12] between geometric theories with enough models and topological groupoids, each Scott topos $S_\mu(K_{\geq \mu})$ is the topos of equivariant sheaves on a topological groupoid built using models of size at most $\kappa$, whence the colimit $S_\kappa(K_{\geq \kappa})$ will be a topos of equivariant sheaves on the limit topological groupoid, which is a topological group, thus two-valued and Boolean (we use a result of [DL] stating that $\mu$-Scott toposes have enough $\mu$-points).

3 Eventual categoricity

We are now going to prove the following:

Theorem 3.1. Assume GCH. Suppose the AEC $(K, \preceq)$ is axiomatizable, has amalgamation and no maximal models and is categorical in some $\lambda > \text{LS}(K)$. Then $K$ is also categorical at any $\delta$ with $\text{LS}(K)^+ \leq \delta \leq \lambda$.

Proof. Take first $\delta = \kappa^+$. By Corollary 2.5, it is enough to show that $S_{\kappa^+}(K_{\geq \kappa^+})$ is two-valued and Boolean. Consider the inverse image functor $f^*: S_{\kappa^+}(K_{\geq \kappa^+}) \rightarrow S_\lambda(K_{\geq \lambda})$ given by restriction.

Note first that $f_\ast$ is a surjection. Indeed, by Theorem 2.4 it follows that evaluations generate and that a subfunctor of an evaluation is a join of evaluations. Hence, it is enough to prove that given two $\kappa^+$-geometric formulas $\phi \leq \psi$ whose extensions coincide in all models in $K_{\geq \lambda}$, it is the case that $\phi(x) \equiv \psi(x)$. Suppose this latter statement does not hold. By the completeness theorem for $L_{\kappa^+,\kappa^+}$ (cf. [Esp17]) there is a model $M$ and a tuple $a$ such that $M \models \psi(a)$ but $M \not\models \phi(a)$. By the assumption of no maximal models, there is an embedding $j: M \rightarrow N$ into the model of cardinality $\lambda$, and since $N \models \psi(j(a))$, by hypothesis we also have $N \models \phi(j(a))$. However in the presheaf topos $\text{Set}^{K_{\geq \kappa^+}}$ the functor $ev_a$ is complemented by some functor $F$, and so $a \in F(M)$ but $j(a) \notin F(N)$, which is absurd; so indeed $\phi$ and $\psi$ are equivalent.

We now claim that $f_\ast$ must be an open surjection, since we can easily see, using the definition of universal quantification $\forall_h$ in $S_{\kappa^+}(K_{\geq \kappa^+})$ (i.e., right adjoint to the pullback functor $h^{-1}$) and the fact that $f^*$ is just the restriction, that universal quantification is actually preserved if $\lambda$ is a successor, and by the result of Moerdijk in [Moe86] also if $\lambda$ is limit. As a consequence, since $S_\lambda(K_{\geq \lambda})$ is two-valued and Boolean by Corollary 2.5, it follows that $S_{\kappa^+}(K_{\geq \kappa^+})$ is also two-valued and Boolean, whence atomic and connected.

By [Joh02], C 3.5.6, it must have an unique model of size $\kappa^+$ up to isomorphism.

The case in which $\delta$ is a limit cardinal is easily handled by an inductive argument knowing that the AEC will be categorical at all $\gamma_i$ for a cofinal sequence of successors $\gamma_i < \delta$. Indeed, since the AEC is $\gamma_i$-categorical, $S_{\gamma_i}(K_{\geq \gamma_i})$ will be atomic and connected. We can then consider the (2-)colimit of the following chain:

$$S_\gamma(K_{\geq \gamma_0}) \rightarrow S_\gamma(K_{\geq \gamma_1}) \rightarrow \ldots \rightarrow S_\delta(K_{\geq \delta})$$

Each topos in the colimit diagram is atomic and connected and has a point of size $\delta$ (since $S_\delta(K_{\geq \delta})$ has a a surjection from $\text{Set}$, namely the model $M_\delta = \lim M_i$), hence
it is equivalent (see [Moe88]) to the topos of equivariant sheaves over the topological
group of automorphisms of its model, with the topology of pointwise convergence. By the
duality theory of [For12] between geometric theories with enough models and groupoids of
models, the morphisms between the toposes correspond to maps between the corresponding
topological groups, whence the colimit topos will be the topos of equivariant sheaves on
the limit topological group, in particular, it will also be two-valued and Boolean.

Corollary 3.2. (Shelah’s eventual categoricity conjecture for AEC’s) Assume GCH and
amalgamation. There exists a cardinal µ such that if K is categorical in some λ ≥ µ, it is
categorical in all λ ≥ µ.

Proof. Let µ₀ be the first cardinal above the Hanf number of K for categoricity. If the
AEC is categorical in some λ₀ ≥ µ₀, then it is categorical in unboundedly many cardinals,
whence in one of arbitrarily large cofinality; thus the AEC is axiomatizable. By Theorem
3.1, it will be categorical in all λ ≥ λ₀ (the hypothesis apply since above λ₀ there are
no maximal models). Since there is only a set of AEC’s with given Löwenheim-Skolem
number, there is a µ that depends only on this number such that categoricity in some
λ ≥ µ implies categoricity in every λ ≥ µ.

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