Some properties of the inverse error function

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Abstract. The inverse of the error function, \( \text{inverf}(x) \), has applications in diffusion problems, chemical potentials, ultrasound imaging, etc. We analyze the derivatives \( \frac{d^n}{dz^n} \text{inverf}(z) \big|_{z=0} \), as \( n \to \infty \) using nested derivatives and a discrete ray method. We obtain a very good approximation of \( \text{inverf}(x) \) through a high-order Taylor expansion around \( x = 0 \). We give numerical results showing the accuracy of our formulas.

1. Introduction

The error function \( \text{erf}(z) \), defined by

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) \, dt,
\]

occurs widely in almost every branch of applied mathematics and mathematical physics, e.g., probability and statistics [Wal50], data analysis [Her88], heat conduction [Jae46], etc. It plays a fundamental role in asymptotic expansions [Olv97] and exponential asymptotics [Ber89].

Its inverse, which we will denote by \( \text{inverf}(z) \),

\[
\text{inverf}(z) = \text{erf}^{-1}(z),
\]

appears in multiple areas of mathematics and the natural sciences. A few examples include concentration-dependent diffusion problems [Phi55, Sha73], solutions to Einstein’s scalar-field equations [LW95], chemical potentials [TM96], the distribution of lifetimes in coherent-noise models [WM99], diffusion rates in tree-ring chemistry [BKSH99] and 3D freehand ultrasound imaging [SJEMFAL+03].

Although some authors have studied the function \( \text{inverf}(z) \) (see [Dom03b] and references therein), little is known about its analytic properties, the major work having been done in developing algorithms for numerical calculations [Pet74]. Dan Lozier, remarked the need for new techniques in the computation of \( \text{inverf}(z) \) [Loz96].

In this paper, we analyze the asymptotic behavior of the derivatives \( \frac{d^n}{dz^n} \text{inverf}(z) \big|_{z=0} \) for large values of \( n \), using a discrete WKB method [CC96]. In Section 2 we present

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some properties of the derivatives of \( \text{inverf}(z) \) and review our previous work on nested derivatives. In Section 3 we study a family of polynomials \( P_n(x) \) associated with the derivatives of \( \text{inverf}(z) \), which were introduced by L. Carlitz in [Car63]. Theorem 3.3 contains our main result on the asymptotic analysis of \( P_n(x) \). In Section 4 we give asymptotic approximations for \( \frac{d^n}{dz^n} \text{inverf}(z) \) at \( z = 0 \) and some numerical results testing the accuracy of our formulas.

2. Derivatives

Let us denote the function \( \text{inverf}(z) \) by \( I(z) \) and its derivatives by

\[
d_n = \frac{d^n}{dz^n} \text{inverf}(z) \bigg|_{z=0}, \quad n = 0, 1, \ldots.
\]

Since \( \text{erf}(z) \) tends to \( \pm 1 \) as \( z \to \pm \infty \), it is clear that \( \text{inverf}(z) \) is defined in the interval \((-1, 1)\) and has singularities at the end points.

**Proposition 2.1.** The function \( I(z) \) satisfies the nonlinear differential equation

\[
I''(z) - 2 I'(z)^2 = 0
\]

with initial conditions

\[
I(0) = 0, \quad I'(0) = \frac{\sqrt{\pi}}{2}.
\]

**Proof.** It is clear that \( I(0) = 0 \), since \( \text{erf}(0) = 0 \). Using the chain rule, we have

\[
I'[\text{erf}(z)] = \frac{1}{\text{erf}'(z)} = \frac{\sqrt{\pi}}{2} \exp \left\{ I^2[\text{erf}(z)] \right\}
\]

and therefore

\[
I' = \frac{\sqrt{\pi}}{2} \exp (I^2).
\]

Setting \( z = 0 \) we get \( I'(0) = \frac{\sqrt{\pi}}{2} \) and taking the logarithmic derivative of \( 2.4 \) the result follows. \qed

To compute higher derivatives of \( I(z) \), we begin by establishing the following corollary.

**Corollary 2.2.** The function \( I(z) \) satisfies the nonlinear differential-integral equation

\[
I'(z) \int_0^z I(t) dt = -\frac{1}{2} + \frac{1}{\sqrt{\pi}} I'(z).
\]

**Proof.** Rewriting \( 2.2 \) as

\[
I = \frac{1}{2} \frac{I''}{(I')^2}
\]

and integrating, we get

\[
\int_0^z I(t) dt = \frac{1}{2} \left[ -\frac{1}{I'(z)} + \frac{1}{I'(0)} \right] = \frac{1}{2} \left[ -\frac{1}{I'(z)} + \frac{2}{\sqrt{\pi}} \right]
\]

and multiplying by \( I'(z) \) we obtain \( 2.3 \). \qed
Proposition 2.3. The derivatives of \( J(z) \) satisfy the nonlinear recurrence

\begin{equation}
(2.6) \quad d_{n+1} = \sqrt{\pi} \sum_{k=0}^{n-1} \binom{n}{k+1} d_k d_{n-k}, \quad n = 1, 2, \ldots
\end{equation}

with \( d_0 = 0 \) and \( d_1 = \frac{\sqrt{\pi}}{2} \).

**Proof.** Using

\( J(z) = \sum_{n=0}^{\infty} d_n \frac{z^n}{n!} \)

and \( d_1 = \frac{\sqrt{\pi}}{2} \) in (2.5), we have

\[
\left[ \frac{\sqrt{\pi}}{2} + \sum_{n=1}^{\infty} d_{n+1} \frac{z^n}{n!} \right] \left[ \sum_{n=1}^{\infty} d_{n-1} \frac{z^n}{n!} - \frac{1}{\sqrt{\pi}} \right] = -\frac{1}{2}
\]

or

\[
\frac{\sqrt{\pi}}{2} \sum_{n=1}^{\infty} d_{n-1} \frac{z^n}{n!} + \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \binom{n}{k+1} d_k d_{n-k} \frac{z^n}{n!} - \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} d_{n+1} \frac{z^n}{n!} = 0.
\]

Comparing powers of \( z^n \), we get

\[
\frac{\sqrt{\pi}}{2} d_{n-1} + \sum_{k=0}^{n-2} \binom{n}{k+1} d_k d_{n-k} - \frac{1}{\sqrt{\pi}} d_{n+1} = 0
\]

or

\[
\sum_{k=0}^{n-1} \binom{n}{k+1} d_k d_{n-k} - \frac{1}{\sqrt{\pi}} d_{n+1} = 0.
\]

\[\square\]

Although one could use (2.6) to compute the higher derivatives of \( \text{inverf}(z) \), the nonlinearity of the recurrence makes it hard to analyze the asymptotic behavior of \( d_n \) as \( n \to \infty \). Instead, we shall use an alternative technique that we developed in [Dom03a] and we called the method of "nested derivatives". The following theorem contains the main result presented in [Dom03a].

**Theorem 2.4.** Let

\[ H(x) = h^{-1}(x), \quad f(x) = \frac{1}{h'(x)}, \quad z_0 = h(x_0), \quad |f(x_0)| \in (0, \infty). \]

Then,

\[ H(z) = x_0 + f(x_0) \sum_{n=1}^{\infty} \mathfrak{D}^{n-1} f(x_0) \frac{(z - z_0)^n}{n!}, \]

where we define \( \mathfrak{D}^n f(x) \), the \( n \)th nested derivative of the function \( f(x) \), by \( \mathfrak{D}^0 f(x) = 1 \) and

\begin{equation}
(2.7) \quad \mathfrak{D}^{n+1} f(x) = \frac{d}{dx} \left[ f(x) \times \mathfrak{D}^n f(x) \right], \quad n = 0, 1, \ldots.
\end{equation}

The following proposition makes the computation of \( \mathfrak{D}^{n-1} f(x_0) \) easier in some cases.
Proposition 2.5. Let
\[ D^n[f](x) = \sum_{k=0}^{\infty} A_k^n (x-x_0)^k/k! , \quad f(x) = \sum_{k=0}^{\infty} B_k (x-x_0)^k/k! . \]
Then,
\[ A^{n+1}_k = (k+1) \sum_{j=0}^{k+1} A_{k+1-j}^n B_j . \]

**Proof.** From (2.8) we have
\[ f(x)D^n[f](x) = \sum_{k=0}^{\infty} \alpha^n_k (x-x_0)^k/k! , \]
with
\[ \alpha^n_k = \sum_{j=0}^{k} A^n_{k-j} B_j . \]
Using (2.8) and (2.10) in (2.7), we obtain
\[ \sum_{k=0}^{\infty} A^{n+1}_k (x-x_0)^k = \frac{d}{dx} \sum_{k=0}^{\infty} \alpha^n_k (x-x_0)^k = \sum_{k=0}^{\infty} (k+1) \alpha^{n+1}_k (x-x_0)^k \]
and the result follows from (2.11). \qed

To obtain a linear relation between successive nested derivatives, we start by establishing the following lemma.

**Lemma 2.6.** Let
\[ g_n(x) = \frac{D^n[f](x)}{f^n(x)} . \]
Then,
\[ g_{n+1}(x) = g'_n(x) + (n+1) \frac{f'(x)}{f^n(x)} g_n(x) , \quad n = 0, 1, \ldots . \]

**Proof.** Using (2.7) in (2.12), we have
\[ g_{n+1}(x) = \frac{D^{n+1}[f](x)}{f^{n+1}(x)} = \frac{\frac{d}{dx} [f(x) \times D^n[f](x)]}{f^{n+1}(x)} = \frac{\frac{d}{dx} [g_n(x) f^{n+1}(x)]}{f^{n+1}(x)} = \frac{g'_n(x) f^{n+1}(x) + g_n(x) (n+1) f^n(x) f'(x)}{f^{n+1}(x)} \]
and the result follows. \qed

**Corollary 2.7.** Let
\[ H(x) = h^{-1}(x) , \quad f(x) = \frac{1}{h'(x)} , \quad z_0 = h(x_0) , \quad |f(x_0)| \in (0, \infty) . \]
Then,
\[ \frac{d^n H}{dz^n}(z_0) = [f(x_0)]^n g_{n-1}(x_0) , \quad n = 1, 2, \ldots . \]
For the function \( h(x) = \text{erf}(z) \), we have

\[
(2.15) \quad f(x) = \frac{1}{h'(x)} = \frac{\sqrt{\pi}}{2} \exp \left( x^2 \right),
\]

and setting \( x_0 = 0 \) we obtain \( z_0 = \text{erf}(0) = 0 \). Using the Taylor series

\[
\frac{\sqrt{\pi}}{2} \exp \left( x^2 \right) = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}
\]

in \( (2.9) \), we get

\[
A_{n+1}^k = \frac{\sqrt{\pi}}{2} (k + 1) \sum_{j=0}^{\left\lfloor \frac{k+1}{2} \right\rfloor} \frac{A_{n+1-k}^j}{j!},
\]

with \( A_k^p \) defined in \( (2.8) \). Using \( (2.15) \) in \( (2.13) \), we have

\[
(2.16) \quad g_{n+1}(x) = g'_n(x) + 2(n + 1)xg_n(x), \quad n = 0, 1, \ldots,
\]

while \( (2.14) \) gives

\[
(2.17) \quad d_n = \left( \frac{\sqrt{\pi}}{2} \right)^n g_{n-1}(0), \quad n = 1, 2, \ldots.
\]

In the next section we shall find an asymptotic approximation for a family of polynomials closely related to \( g_n(x) \).

### 3. The polynomials \( P_n(x) \)

We define the polynomials \( P_n(x) \) by \( P_0(x) = 1 \) and

\[
P_n(x) = g_n \left( \frac{x}{\sqrt{2}} \right) 2^{-\frac{x}{2}}.
\]

\[
(3.1) \quad P_{n+1}(x) = P'_n(x) + (n + 1)xP_n(x),
\]

The first few \( P_n(x) \) are

\[
P_1(x) = x, \quad P_2(x) = 1 + 2x^2, \quad P_3(x) = 7x + 6x^3, \ldots.
\]

The following propositions describe some properties of \( P_n(x) \).

**Proposition 3.1.** Let

\[
P_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} C_n^k x^{n-2k},
\]

where \( \lfloor \cdot \rfloor \) denotes the integer part function. Then,

\[
C_n^0 = n!
\]

and

\[
C_n^k = n! \sum_{j_k = 0}^{k-1} \sum_{j_{k-1} = 0}^{j_k-1} \cdots \sum_{j_1 = 0}^{j_2-1} \prod_{i=1}^{k} \frac{j_i - 2i + 2}{j_i + 1}, \quad k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.
\]
Using (3.3) in (3.2) we have
\[
\sum_{0 \leq 2k \leq n+1} C^{n+1}_k x^{n+1-2k} = \sum_{0 \leq 2k \leq n} C^n_k (n-2k) x^{n-2k-1} + \sum_{0 \leq 2k \leq n} (n+1) C^n_k x^{n+1-2k} 
\]
\[
= \sum_{2 \leq 2k \leq n+2} C^{n-1}_k (n-2k+2) x^{n+1-2k} + \sum_{0 \leq 2k \leq n} (n+1) C^n_k x^{n+1-2k}. 
\]
Comparing coefficients in the equation above, we get

(3.6) \[ C^{n+1}_0 = C^n_0, \]

(3.7) \[ C^n_k = (n-2k+2) C^{n-1}_k + (n+1) C^n_k, \quad k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \]
and for \( n = 2m-1, \)
\[ C^{2m}_m = C^{2m-1}_{m-2}, \quad m = 1, 2, \ldots. \]
From (3.6) we immediately conclude that \( C^n_0 = n! \), while (3.7) gives

(3.8) \[ C^n_k = n! \sum_{j=0}^{n-1} \frac{j-2k+2}{(j+1)!} C^j_{k-1}, \quad n, k \geq 1. \]

Setting \( k = 1 \) in (3.8) and using (3.4), we have

(3.9) \[ C^n_1 = n! \sum_{j=0}^{n-1} \frac{j}{(j+1)!} C^j_0 = n! \sum_{j=0}^{n-1} \frac{j}{j+1}. \]

Similarly, setting \( k = 2 \) in (3.8) and using (3.4), we get

\[ C^n_2 = n! \sum_{j=0}^{n-1} \frac{j-2}{(j+1)!} \left[ \sum_{i=0}^{j-1} \frac{i}{j+1} \right] = n! \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} \frac{j-2}{j+1} i + 1 \]
and continuing this way we obtain (3.5). \( \square \)

**Proposition 3.2.** The zeros of the polynomials \( P_n(x) \) are purely imaginary for \( n \geq 1. \)

**Proof.** For \( n = 1 \) the result is obviously true. Assuming that it is true for \( n \) and that \( P_n(x) \) is written in the form

(3.10) \[ P_n(x) = n! \prod_{k=1}^{n} (z-z_k), \quad \text{Re}(z_k) = 0, \quad 1 \leq k \leq n, \]
we have two possibilities for \( z^* \), with \( P_{n+1}(z^*) = 0: \)

1. \( z^* = z_k, \) for some \( 1 \leq k \leq n. \)
   In this case, \( \text{Re}(z^*) = 0 \) and the proposition is proved.
2. \( z^* \neq z_k, \) for all \( 1 \leq k \leq n. \)
   From (3.2) and (3.10) we get
\[
\frac{P_{n+1}(x)}{P_n(x)} = \frac{d}{dx} \ln |P_n(x)| + (n+1)x = \sum_{k=1}^{n} \frac{1}{z-z_k} + (n+1)x. 
\]
Evaluating at \( z = z^* \), we obtain
\[
0 = \sum_{k=1}^{n} \frac{1}{z^*-z_k} + (n+1)z^*. 
\]
and taking Re(●), we have

\[ 0 = \text{Re} \left( \sum_{k=1}^{n} \frac{1}{z^* - z_k} + (n + 1)z^* \right) \]

\[ = \sum_{k=1}^{n} \frac{\text{Re} (z^* - z_k)}{|z^* - z_k|^2} + (n + 1) \text{Re}(z^*) = \text{Re}(z^*) \left[ \sum_{k=1}^{n} \frac{1}{|z^* - z_k|^2} + n + 1 \right] \]

which implies that \( \text{Re}(z^*) = 0 \).

3.1. Asymptotic analysis of \( P_n(x) \). We first consider solutions to (3.2) of the form

\[ (3.11) \quad P_n(x) = n!A^{(n+1)}(x), \]

with \( x > 0 \). Replacing (3.11) in (3.2) and simplifying the resulting expression, we obtain

\[ A^2(x) = A'(x) + xA(x), \]

with solution

\[ (3.12) \quad A(x) = \exp \left( \frac{-x^2}{2} \right) \left[ C - \sqrt{\frac{\pi}{2}} \text{erf} \left( \frac{x}{\sqrt{2}} \right) \right]^{-1}, \]

for some constant \( C \). Note that (3.11) is not an exact solution of (3.2), since it does not satisfy the initial condition \( P_0(x) = 1 \). To determine \( C \) in (3.12), we observe from (3.4) that

\[ (3.13) \quad P_n(x) \sim n!x^n, \quad x \to \infty. \]

As \( x \to \infty \), we get from (3.12)

\[ \ln[A(x)] \sim -\frac{x^2}{2} - \ln \left( C - \sqrt{\frac{\pi}{2}} \right) + \frac{\exp \left( \frac{-x^2}{2} \right)}{(C - \sqrt{\frac{\pi}{2}}) x}, \quad x \to \infty, \]

which is inconsistent with (3.13) unless \( C = \sqrt{\frac{\pi}{2}} \). In this case, we have

\[ (3.14) \quad A(x) \sim x + \frac{1}{x}, \quad x \to \infty, \]

matching (3.13). Thus,

\[ (3.15) \quad A(x) = \sqrt{\frac{2}{\pi}} \exp \left( -\frac{x^2}{2} \right) \left[ 1 - \text{erf} \left( \frac{x}{\sqrt{2}} \right) \right]^{-1}. \]

Since (3.11) and (3.14) give

\[ P_n(x) \sim n!x^{n+1}, \quad x \to \infty, \]

instead of (3.13), we need to consider

\[ (3.16) \quad P_n(x) = n!A^{(n+1)}(x)B(x,n). \]

Replacing (3.16) in (3.2) and simplifying, we get

\[ B(x, n + 1) = B(x,n) + \frac{1}{A(x)(n+1)} \frac{\partial B}{\partial x}(x,n). \]
Using the approximation
\[ B(x, n + 1) = B(x, n) + \frac{\partial B}{\partial n}(x, n) + \frac{1}{2} \frac{\partial^2 B}{\partial n^2}(x, n) + \cdots, \]
we obtain
\[ \frac{\partial B}{\partial n} = \frac{1}{A(x)(n+1)} \frac{\partial B}{\partial x}, \]
whose solution is
\[ (3.17) \quad B(x, n) = F \left[ \frac{n + 1}{1 - \text{erf} \left( \frac{x}{\sqrt{2}} \right)} \right], \]
for some function \( F(u) \). Matching (3.16) with (3.13) requires
\[ (3.18) \quad B(x, n) \sim \frac{1}{x}, \quad x \to \infty. \]
Since in the limit as \( x \to \infty \), with \( n \) fixed we have
\[ \ln \left[ \frac{n + 1}{1 - \text{erf} \left( \frac{x}{\sqrt{2}} \right)} \right] \sim \frac{x^2}{2}, \]
(3.17)-(3.18) imply
\[ F(u) = \frac{1}{\sqrt{2 \ln(u)}}. \]
Therefore, for \( x > 0 \),
\[ (3.19) \quad P_n(x) \sim n! \Phi(x, n), \quad n \to \infty, \]
with
\[ \Phi(x, n) = \left[ \sqrt{\frac{2}{\pi}} \frac{\exp \left( -\frac{x^2}{2} \right)}{1 - \text{erf} \left( \frac{x}{\sqrt{2}} \right)} \right]^{n+1} \left[ 2 \ln \left( \frac{n + 1}{1 - \text{erf} \left( \frac{x}{\sqrt{2}} \right)} \right) \right]^{-\frac{1}{2}}. \]
From (3.3) we know that the polynomials \( P_n(x) \) satisfy the reflection formula
\[ (3.20) \quad P_n(-x) = (-1)^n P_n(x). \]
Using (3.20), we can extend (3.19) to the whole real line and write
\[ (3.21) \quad P_n(x) \sim n! \left[ \Phi(x, n) + (-1)^n \Phi(-x, n) \right], \quad n \to \infty. \]
In Figure II we compare the values of \( P_{10}(x) \) with the asymptotic approximation (3.21).
We see that the approximation is very good, even for small values of \( n \). We summarize our results of this section in the following theorem.

**Theorem 3.3.** Let the polynomials \( P_n(x) \) be defined by
\[ P_{n+1}(x) = P_n'(x) + (n + 1) x P_n(x), \]
with \( P_0(x) = 1 \). Then, we have
\[ (3.22) \quad P_n(x) \sim P_n(x) \sim n! \left[ \Phi(x, n) + (-1)^n \Phi(-x, n) \right], \quad n \to \infty, \]
where

\begin{equation}
\Phi (x, n) = \left[ \sqrt{\frac{2}{\pi}} \exp \left( -\frac{x^2}{2} \right) \right]^{n+1} \left[ 2\ln \left( \frac{n+1}{1 - \text{erf} \left( \frac{x}{\sqrt{2}} \right)} \right) \right]^{-\frac{1}{2}}.
\end{equation}

### 4. Higher derivatives of \( \text{inverf} (z) \)

From (2.17) and (3.1), it follows that

\begin{equation}
d_n = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\pi}{2}} \right)^n P_{n-1}(0), \quad n = 1, 2, \ldots,
\end{equation}

where \( d_n \) was defined in (2.1). Using Theorem 3.3 in (4.1), we have

\[ d_n \sim \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\pi}{2}} \right)^n \Phi (0, n-1) \left[ 1 + (-1)^{n-1} \right], \]

as \( n \to \infty \). Using (3.23), we obtain

\begin{equation}
\frac{d_n}{n!} \sim \frac{1}{2n \sqrt{\ln(n)}} \left[ 1 + (-1)^{n-1} \right], \quad n \to \infty.
\end{equation}
Figure 2. A sketch of the exact (ooo) and asymptotic (solid curve) values of $\frac{d_{2k+1}}{(2k+1)!}$.

Setting $n = 2N + 1$ in (4.2), we have

$$(4.3) \quad \frac{d_{2N+1}}{(2N+1)!} \sim \frac{1}{(2N+1) \sqrt{\ln(2N+1)}}, \quad N \to \infty.$$  

4.1. Numerical results. In this section we demonstrate the accuracy of the approximation (4.2) and construct a high order Taylor series for inv erf ($x$). In Figure 2 we compare the logarithm of the exact values of $\frac{d_{2n+1}}{(2n+1)!}$ inv erf ($x$) $|_{x=0}$ and our asymptotic formula (4.2). We see that there is a very good agreement, even for moderate values of $n$.

Using (2.6), we compute the exact values

$$d_1 = \frac{1}{2} \pi^{\frac{1}{2}}, \quad d_3 = \frac{1}{4} \pi^{\frac{3}{2}}, \quad d_5 = \frac{7}{8} \pi^{\frac{5}{2}}, \quad d_7 = \frac{127}{16} \pi^{\frac{7}{2}}, \quad d_9 = \frac{4369}{32} \pi^{\frac{9}{2}}$$

and form the polynomial Taylor approximation

$$T_9(x) = \sum_{k=0}^{4} d_{2k+1} \frac{x^{2k+1}}{(2k+1)!}.$$
In Figure 3 we graph $\frac{T_9(x)}{\text{inverf}(x)}$ (solid curve), $\frac{T_9(x) + R_{10}(x)}{\text{inverf}(x)}$ (+++), and $\frac{T_9(x) + R_{20}(x)}{\text{inverf}(x)}$ (ooo).

The functions are virtually identical in most of the interval $(-1, 1)$ except for values close to $x = \pm 1$. We show the differences in detail in Figure 4. Clearly, the additional terms in $R_{20}(x)$ give a far better approximation for $x \approx 1$.

In the table below we compute the exact value of and optimal asymptotic approximation to $\text{inverf}(x)$ for some $x$:

| $x$  | $\text{inverf}(x)$ | $T_9(x) + R_N(x)$ | $N$  |
|------|---------------------|-------------------|------|
| 0.7  | .732869             | .732751           | 6    |
| 0.8  | .906194             | .905545           | 7    |
| 0.9  | 1.16309             | 1.16274           | 11   |
| 0.99 | 1.82139             | 1.82121           | 57   |
| 0.999| 2.32675             | 2.32676           | 423  |
| 0.9999| 2.75106             | 2.75105           | 3685 |
Clearly, (4.4) is still valid for $x \to 1$, but at the cost of having to compute many terms in the sum. In this region it is better to use the formula \cite{Dom03b}

$$\text{inverf}(x) \sim \sqrt{\frac{1}{2} \text{LW} \left[ \frac{2}{\pi (x-1)^2} \right]}, \quad x \to 1^-,$$

where $\text{LW}(\cdot)$ denotes the Lambert-W function \cite{CGH96}, which satisfies

$$\text{LW}(x) \exp[\text{LW}(x)] = x.$$
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