**d-SEMISTABLE CALABI–YAU THREEFOLDS OF TYPE III**

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**Abstract.** We develop some methods to construct normal crossing varieties whose dual complexes are two-dimensional, which are smoothable to Calabi–Yau threefolds. We calculate topological invariants of smoothed Calabi–Yau threefolds and show that several of them are new examples.

1. Introduction

A Calabi-Yau manifold is a compact Kähler manifold with trivial canonical class such that the intermediate cohomologies of its structure sheaf are all trivial \( h^i(M, \mathcal{O}_M) = 0 \) for \( 0 < i < \dim(M) \). Calabi–Yau threefolds have attracted much interest from both of mathematics and physics but the classification of Calabi–Yau threefolds is widely open. Even boundedness of their Hodge numbers is still unknown. Thus developing method of constructing Calabi–Yau threefolds and finding new examples are of interest.

If a normal crossing variety is the central fiber of a semistable degeneration of a Calabi–Yau manifolds, it can be regarded as a member in a deformation family of those Calabi–Yau manifolds. Semistable degenerations of \( K3 \) surfaces, which are Calabi–Yau twofolds, have been investigated by several authors ([9, 15]). V. Kulikov proved that any degeneration of \( K3 \) surfaces can be modified to be semistable one whose total space has trivial canonical divisor and he also classified the central fibers into three types. R. Friedman proved a smoothing theorem for a \( d \)-semistable normal crossing variety to \( K3 \) surfaces ([4]), which is a converse of V. Kulikov’s result. More concretely he found a sufficient condition for a normal crossing variety to be a central fiber of semistable degeneration of \( K3 \) surfaces. A smoothing theorem for the higher dimensional case has been introduced by Y. Kawamata and Y. Namikawa in [6]. A remarkable difference between two-dimensional cases of \( K3 \) surfaces and higher dimensional cases is that there are multiple deformation types for higher dimensional Calabi–Yau manifolds. So building a normal crossing variety smoothable to a Calabi–Yau manifold can be regarded as building a deformation type of Calabi–Yau manifolds.

Smoothing of normal crossing variety with two components to Calabi–Yau threefolds has been actively investigated. A. Tyurin studied Calabi–Yau threefolds via degeneration to normal crossing varieties, in particular 2-component varieties glued along a common anticanonical divisor in his

2010 Mathematics Subject Classification. 14J32, 14D05, 32G20.
posthumous paper [16]. R. Donagi, S. Katz and M. Wijnholt considered a holographic relation between F-theory on a degenerate Calabi-Yau and a dual theory on its boundary in [2]. A possible mirror construction of Calabi–Yau manifolds by smoothing has been suggested by C. Doran, A. Harder, A. Thompson ([3]) and materialized more in [12]. In this case the dual complexes of the normal crossing varieties are one-dimensional (line segments). In this note, we consider normal crossing varieties, smoothable to Calabi–Yau threefolds, whose dual complexes are two-dimensional. There are some difficulties in making such normal crossing varieties that do not appear in the case of normal crossing varieties with only two components. We explain this with examples and demonstrate how to circumvent the difficulties. We also develop a method for calculating their topological invariants such as Hodge numbers. It turns out that several of them have Hodge numbers different from Calabi–Yau threefolds from toric setting. It is also notable that the methods in this note naturally leads to the construction of multiple non-homeomorphic Calabi–Yau threefolds with the same Hodge numbers.

The structure of this note is as follows.

Section 2 is a background section for smoothing and degeneration of Calabi–Yau manifolds. We introduce basic definitions and the smoothing theorem of Kawamata-Namikawa, which is a main tool of the construction of Calabi–Yau manifolds in this note.

In Section 3, we concentrate on the case of normal crossing varieties of three components. Some formulas for Hodge numbers of their smoothing are developed.

Section 4 is devoted to the $d$-semitablility condition. Considering a concrete example, we demonstrate the difficulty in making $d$-semistable normal crossing varieties when they have three components and how to circumvent the difficulty.

In Section 5, we generalize and systemize the procedure in Section 4, stating exact conditions for the construction of $d$-semistable normal crossing varieties.

Applying our methods, we give several examples of Calabi–Yau threefolds of type III in Section 6. We consider six configurations which together produce more than fifty examples, including ones with new Hodge numbers. We also introduce a different kind of Calabi–Yau threefold of type III, which demonstrates that there are some room for generalization of our method.

2. Degeneration and smoothing of Calabi–Yau manifolds

A normal crossing variety is a reduced complex analytic space which is locally isomorphic to a normal crossing divisor on a smooth variety. It is said to be simple if all of its components are smooth varieties. In this note, we only consider simple ones. Let $X$ be a normal crossing variety with irreducible components $\{X_i| i \in I\}$. A stratum $S$ of $X$ is any irreducible component of an intersection $\bigcap_{i \in J} X_i$ for some $J \subset I$. The dual complex
\( D(X) \) of \( X \) is a simplicial complex whose vertices are labeled by the irreducible components of \( X \) and for every stratum of dimension \( r \) we attach a \((\dim X - r)\)-dimensional simplex.

A semistable degeneration is a proper flat holomorphic map \( \varphi : \mathcal{X} \to \Delta \) from a Kähler manifold \( \mathcal{X} \) onto the complex unit disk \( \Delta \) such that the fiber \( \mathcal{X}_t = \varphi^{-1}(t) \) is a smooth complex variety for \( t \neq 0 \) and the central fiber \( \mathcal{X}_0 \) is a simple normal crossing divisor of \( \mathcal{X} \). Following naming in \([4, 9]\), we give following definition:

**Definition 2.1.** A projective normal crossing variety of dimension \( n \) is called a \( d \)-semistable Calabi–Yau \( n \)-fold of type \( k+1 \) if it has trivial dualizing sheaf, it is the central fiber in a semistable degenerations of Calabi–Yau manifolds and its dual complex is \( k \)-dimensional.

It seems natural to include some mild singularities for Calabi–Yau manifolds of dimension higher than two, but in this note we stick to only smooth ones. In this definition, the usual Calabi–Yau \( n \)-fold is a \( d \)-semistable Calabi–Yau \( n \)-fold of type I. We also say that \( \mathcal{X}_0 \) is smoothable to \( \mathcal{X}_t(t \neq 0) \) with smooth total space \( \mathcal{X} \). In this note, we mainly consider \( d \)-semistable Calabi–Yau threefolds of type III that are composed of three components. We also briefly discuss examples of Calabi–Yau threefolds of type III that have four or more components. Our main tool is the smoothing theorem of Y. Kawamata and Y. Namikawa ([6]), which is stated below for the readers’ convenience.

**Theorem 2.2** (Y. Kawamata, Y. Namikawa). Let \( Y = \bigcup_i Y_i \) be a compact normal crossing variety of dimension \( n \) such that

1. It is Kähler and \( d \)-semistable.
2. Its dualizing sheaf is trivial: \( \omega_Y = \mathcal{O}_Y \).
3. \( H^{n-2}(Y_i, \mathcal{O}_{Y_i}) = 0 \) for any \( i \) and \( H^{n-1}(Y, \mathcal{O}_Y) = 0 \).

Then \( Y \) is smoothable to an \( n \)-fold \( M_Y \) with trivial canonical class and the total space of smoothing is smooth.

It was showed that the Hodge numbers of the Calabi–Yau manifolds can be calculated from the geometry of the normal crossing varieties ([10, 11]).

Let us take a very simple example.

**Example 2.3.** Let \( W_1, W_2 \) be copies of \( \mathbb{P}^3 \) and \( D \) be a smooth quartic surface in \( \mathbb{P}^3 \). Then \( W_1, W_2 \) contain copies of \( D \). Let \( W = W_1 \cup_D W_2 \), where ‘\( \cup_D \)’ means gluing along \( D \). Then the normal crossing variety \( W \) is projective and has trivial dualizing sheaf but it is not \( d \)-semistable. In [6], \( W_2 \) is blowed up along a smooth curve \( c \) in the linear system \([\mathcal{O}_D(8)]\) to become \( \tilde{W}_2 \). The proper transform \( \tilde{D} \) in \( \tilde{W}_2 \) of \( D \) is isomorphic to \( D \). So one can paste \( W_1 \) and \( \tilde{W}_2 \) along \( D \) and \( \tilde{D} \) to get a \( d \)-semistable normal crossing variety \( \tilde{W} \). It is smoothable to a Calabi–Yau threefold whose invariants are calculated in [11]. Note that \( \tilde{W} \) is a \( d \)-semistable Calabi–Yau threefold of type II.
3. The case of type III

Consider a normal crossing variety \( Y = Y_1 \cup Y_2 \cup Y_3 \). Let \( Y_{ij} = Y_i \cap Y_j, Y_{ijk} = Y_i \cap Y_j \cap Y_k \). If we treat the double locus \( Y_{ij} \) and the triple locus \( Y_{ijk} \) as subvarieties or divisors of \( Y_j \) and \( Y_{jk} \), we denote them by \( Y_{(ij)} \) and \( Y_{(ijk)} \) respectively. Let \( D_1 = Y_{23}, D_2 = Y_{31}, D_3 = Y_{12} \) and \( \tau = Y_{123} \). We want to smooth \( Y \) to a Calabi–Yau manifold, using Theorem 2.2. Throughout this note, we assume that \( Y \) satisfies the following conditions, which contains all conditions except the \( d \)-semistability in Theorem 2.2.

**Condition 3.1.**

1. \( n = \dim Y \geq 3 \)
2. \( H^a(Y_i, \mathcal{O}_{Y_i}) = 0 \) and \( H^a(D_i, \mathcal{O}_{D_i}) = 0 \) for each \( i = 1, 2, 3 \) and \( a = 1, 2 \). \( D_i \)'s and \( \tau \) are all connected.
3. There is an ample divisor \( H_i \) of \( Y_i \) such that \( H_i|_{Y_{ij}} \sim H_j|_{Y_{ji}} \) for every \( i, j \).
4. For each fixed \( j \), \(- \sum_{i \neq j} Y_{(ij)} \) is a canonical divisor of \( Y_j \).

Note that the pair \((Y_i, Y_{(ji)} \cup Y_{(ki)})\) is a log Calabi–Yau pair for \( \{i, j, k\} = \{1, 2, 3\} \) and \( Y_{(ji)} \cup Y_{(ki)} \) is a normal crossing of two rational surfaces if \( \dim Y = 3 \).

Now assume that \( Y \) is smoothable to a Calabi–Yau manifold \( M_Y \) with smooth total space and let us calculate topological invariants of \( M_Y \). In §7 of [11], we have introduced \( G^{2i}(Y, \mathbb{Z}) \) as a subgroup of \( \bigoplus_\alpha H^{2i}(Y_\alpha, \mathbb{Z}) \), that is, the image of the map

\[
H^{2i}(Y, \mathbb{Z}) \to \bigoplus_\alpha H^{2i}(Y_\alpha, \mathbb{Z}).
\]

We also located the Chern class of \( M_Y \) as an element of \( \bigoplus_i G^{2i}(Y, \mathbb{Z}) \). The group \( G^{2i}(Y, \mathbb{Z}) \) inherits the cup product from those of \( \bigoplus_\alpha H^{2i}(Y_\alpha, \mathbb{Z}) \) with the mixed terms set to be zero.

It is (Definition 7.1 in [11]):

\[
(3.1) \quad c(Y) = \sum_i \left( 1^{(i)} - \sum_{j \neq i} Y_{(ji)} \right) c(Y_i),
\]

where \( 1^{(i)} \) is the generator of \( H^0(Y_i, \mathbb{Z}) \). The cup product of \( c(Y) \) with \( \bigoplus_i G^{2i}(Y, \mathbb{Z}) \) gives some information about the cup product of \( c(M_Y) \) with \( \bigoplus_i H^{2i}(M_Y, \mathbb{Z}) \). Using this formula, one can calculate the topological Euler characteristic of \( M_Y \).

**Proposition 3.2.** If \( Y \) is smoothable to a Calabi–Yau manifold \( M_Y \) with smooth total space, then

\[
e(M_Y) = e(Y_1) + e(Y_2) + e(Y_3) - 2(e(D_1) + e(D_2) + e(D_3)) + 3e(\tau).
\]

**Proof.** From equation 3.1,

\[
e(M_Y) = c_3(Y) = \sum_i c_3(Y_i) - \sum_{i \neq j} Y_{(ji)} \cdot c_2(Y_i).
\]
By the adjunction formula, for \( \{i, j, k\} = \{1, 2, 3\} \),
\[
Y_{(ji)} \cdot c_2(Y_i) = c_2(Y_{ji}) + Y_{(ji)}|_{Y_j} \cdot c_1(Y_{ji}).
\]
Since \( Y \) is \( d \)-semistable, \( N(Y_{ij}) = 0 \), i.e. \( Y_{(ji)}|_{Y_j} + Y_{(ij)}|_{Y_i} = -Y_{(kiij)} \). So we have
\[
Y_{(ji)} \cdot c_2(Y_i) + Y_{(ij)} \cdot c_2(Y_j) = (c_2(Y_{ji}) + Y_{(ji)}|_{Y_j} \cdot c_1(Y_{ji})) + (c_2(Y_{ij}) + Y_{(ij)}|_{Y_i} \cdot c_1(Y_{ij})) = c_2(Y_{ji}) + c_2(Y_{ij}) + Y_{(ji)}|_{Y_j} + Y_{(ij)}|_{Y_i} \cdot c_1(Y_{ij}) = e(Y_{ji}) + e(Y_{ij}) + (-Y_{123}) \cdot c_1(Y_{ij}) = 2e(Y_{ij}) - Y^2_{(kiij)} - c_1(Y_{kiij}) = 2e(D_k) - Y^2_{(kiij)} - e(\tau).
\]
Hence
\[
e(M_Y) = \sum_i c_3(Y_i) - \sum_{i \neq j} Y_{(ji)} \cdot c_2(Y_i)
= \sum_i e(Y_i) - \sum_{i < j} (Y_{(ji)} \cdot c_2(Y_i) + Y_{(ij)} \cdot c_2(Y_j))
= \sum_i e(Y_i) - 2 \sum_k e(D_k) + 3e(\tau) - (Y^2_{(312)} + Y^2_{(123)} + Y^2_{(213)}).
\]
Note
\[
0 = N(Y_{12})|_{Y_{123}} + N(Y_{23})|_{Y_{123}} + N(Y_{13})|_{Y_{123}} = 3(Y^2_{(312)} + Y^2_{(123)} + Y^2_{(213)}).
\]
Therefore we have the formula. 

Next, we determine the Hodge number \( h^{1,1}(M_Y) \).

**Proposition 3.3.** There is an exact sequence
\[
0 \to H^2(Y, \mathbb{Z}) \xrightarrow{\eta} H^2(Y_1, \mathbb{Z}) \oplus H^2(Y_2, \mathbb{Z}) \oplus H^2(Y_3, \mathbb{Z}) \xrightarrow{\mu} H^2(D_1, \mathbb{Z}) \oplus H^2(D_2, \mathbb{Z}) \oplus H^2(D_3, \mathbb{Z}),
\]
where \( \eta \) is the restriction map and the map \( \mu \) is defined by
\[
\mu(H_1, H_2, H_3) = (H_2|_{D_1} - H_3|_{D_1}, H_3|_{D_2} - H_1|_{D_2}, H_1|_{D_3} - H_2|_{D_3}).
\]

**Proof.** Using the exponential sequence, there are isomorphisms
\[
\text{Pic}(Y) \simeq H^2(Y, \mathbb{Z}), \text{Pic}(Y_i) \simeq H^2(Y_i, \mathbb{Z}), \text{Pic}(D_i) \simeq H^2(D_i, \mathbb{Z}).
\]
Hence it is enough to show that the following sequence is exact:
\[
0 \to \text{Pic}(Y) \to \text{Pic}(Y_1) \oplus \text{Pic}(Y_2) \oplus \text{Pic}(Y_3) \to \text{Pic}(D_1) \oplus \text{Pic}(D_2) \oplus \text{Pic}(D_3).
\]
The exactness of the above sequence comes from Proposition 2.6 of [5].

\[\square\]

The subgroup \( G^2(Y, \mathbb{Z}) \) of
\[
H^2(Y_1, \mathbb{Z}) \oplus H^2(Y_2, \mathbb{Z}) \oplus H^2(Y_3, \mathbb{Z})
\]
is defined by \( G^2(Y, \mathbb{Z}) = \text{im}(\eta) \) (p. 704 of [11]) and now \( G^2(Y, \mathbb{Z}) = \ker \mu \) by Proposition 3.3. It inherits the cup products from those of \( H^2(Y_1, \mathbb{Z}) \oplus \)
$H^2(Y_2, \mathbb{Z}) \oplus H^2(Y_3, \mathbb{Z})$, where the mixed terms are defined to be zero. Let $NG^2(Y) = \langle e_1, e_2 \rangle$ be a subgroup of $H^2(Y_1, \mathbb{Z}) \oplus H^2(Y_2, \mathbb{Z}) \oplus H^2(Y_3, \mathbb{Z})$, where

$$e_1 = (-Y_{(21)} - Y_{(12)}, Y_{(13)}), e_2 = (Y_{(21)} - Y_{(32)}, Y_{(23)}).$$

Then it was showed in §5 of [11] that $NG^2(Y)$ is a subgroup of $G^2(Y)$ that is degenerated with respect to the cup product and $\text{rk}(NG^2(Y, \mathbb{Z})) = 2$. Furthermore, it is known that there is an injection (Proposition 5.4 in [11]):

$$\left( G^2(Y, \mathbb{Z})/NG^2(Y, \mathbb{Z}) \right)_f \rightarrow H^2(M_Y, \mathbb{Z})_f$$

with finite index, where $A_f$ for an Abelian group $A$ means its quotient by torsion part. This injection preserves the cup product, so one can use it to calculate the cup product on $H^2(M_Y, \mathbb{Z})$ (See [11] and §6 of [12] for more details). Since $G^2(Y, \mathbb{Z}) = \text{ker} \mu$, we can easily calculate $G^2(Y, \mathbb{Z})$.

**Corollary 3.4.** If $Y$ is smoothable to a Calabi–Yau manifold $M_Y$ with smooth total space, then

$$h^{1,1}(M_Y) = \dim(\ker(H^2(Y_1, \mathbb{Z}) \oplus H^2(Y_2, \mathbb{Z}) \oplus H^2(Y_3, \mathbb{Z})) \xrightarrow{\text{Hodge}} H^2(Y_1, \mathbb{Z}) \oplus H^2(Y_2, \mathbb{Z}) \oplus H^2(Y_3, \mathbb{Z})) - 2.$$ 

**Proof.** From Theorem 4.3 in [11], we note

$$h^{1,1}(M_Y) = h^2(M_Y) = h^2(Y) - 3 + 1 = h^2(Y) - 2.$$ 

Hence we are done by Proposition 3.3. $\square$

For a Calabi–Yau threefold $M_Y$, $h^{1,2}(M_Y) = h^{1,1}(M_Y) - \frac{1}{2} e(M_Y)$, So Proposition 3.2 and Corollary 3.4 determine all the Hodge numbers of $M_Y$.

4. $d$-SEMISTABILITY AND AN EXAMPLE

Consider a normal crossing variety $Y = Y_1 \cup Y_2 \cup Y_3$, satisfying Condition 3.1. By Theorem 2.2, one can show that $Y$ is smoothable to a Calabi–Yau manifold of dimension $n$ if it is $d$-semistable. The $d$-semistability (also called as ‘logarithmic structure’ in [6]) is the condition that the normal crossing variety $Y$ is the central fiber in semistable degeneration. Suppose that a normal crossing variety $X = X_1 \cup X_2 \cup X_3$ is the central fiber in a semistable degeneration $\varphi : X \rightarrow \Delta$. Note

$$X|_{X_j} = X_0|_{X_j} \sim X_t|_{X_j} = 0$$

on $X_j$, where $t \neq 0$. We have $(X_j|_{X_j})|_{X_{ij}} = (X_j|_{X_i})|_{X_{ij}}$ in Pic($X_{ij}$) since $X_{ij}$ is subvariety of both of $X_i, X_j$. For distinct $i, j, k$ in $\{1, 2, 3\}$,

$$0 = (X|_{X_j})|_{X_{ij}} = ((X_i + X_j + X_k)|_{X_j})|_{X_{ij}} = (X_i|_{X_j})|_{X_{ij}} + (X_j|_{X_j})|_{X_{ij}} + (X_k|_{X_j})|_{X_{ij}} = X_{(ij)}|_{X_{ij}} + (X_j|_{X_i})|_{X_{ij}} + X_{(kij)}$$

$$= X_{(ij)}|_{X_{ij}} + X_{(ji)}|_{X_{ij}} + X_{(kij)}$$
in Pic(X_{ij}).

So let

\[ N_Y(Y_{ij}) = Y_{(ij)}|Y_{ij} + Y_{(ji)}|Y_{ij} + Y_{(ki)}|Y_{ij} \in \text{Pic}(Y_{ij}). \]

Noting \( D_i = Y_{jk} \) for \( \{i, j, k\} = \{1, 2, 3\} \), let us call

\[(N_Y(D_1), N_Y(D_2), N_Y(D_3)) \in \text{Pic}(D_1) \oplus \text{Pic}(D_2) \oplus \text{Pic}(D_3)\]

the collective normal class of \( Y \). The triviality of the collective normal class of \( Y \) is called the ‘triple point formula’ in two dimensional case ([14]) and is a necessary condition for the \( d \)-semistability. On the other hand, if \( \text{Ext}^1_Y(\Omega^1_Y, O_Y)|_D \) is trivial, then \( Y \) is said to be \( d \)-semistable ([4]), where \( D = D_1 \cup D_2 \cup D_3 \). The triviality of the collective normal class is equivalent with the \( d \)-semistability in our case.

**Proposition 4.1.** The triviality of the collective normal class of \( Y \) implies the \( d \)-semistability of \( Y \). Hence if the collective normal class of \( Y \) is trivial, then \( Y \) is a Calabi–Yau manifold of type III.

**Proof.** Note that \( \text{Ext}^1_Y(\Omega^1_Y, O_Y)|_D \) is a line bundle on \( D \) and \( \text{Ext}^1_Y(\Omega^1_Y, O_Y)|_{D_i} = N_Y(D_i) \) ([4]). Suppose that \( Y \) has a trivial collective normal class, i.e. \( \text{Ext}^1_Y(\Omega^1_Y, O_Y)|_{D_i} = N_Y(D_i) = O_{D_i} \) for any \( i \neq j \). In order to show \( \text{Ext}^1_Y(\Omega^1_Y, O_Y)|_D =0 \), it is enough to show that the map

\[ \eta : \text{Pic}(D_1 \cup D_2 \cup D_3) \to \text{Pic}(D_1) \oplus \text{Pic}(D_2) \oplus \text{Pic}(D_3) \]

is injective, where \( \eta \) is the restriction map. Consider the exact sequence of sheaves of Abelian groups:

\[ 1 \to O^*_{D_1 \cup D_2} \to O^*_{D_1} \times O^*_{D_2} \to O^*_{D_1 \cap D_2} \to 1 \]

to deduce a long exact sequence

\[ 1 \to \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^* \xrightarrow{v} \mathbb{C}^* \to \text{Pic}(D_1 \cup D_2) \xrightarrow{\lambda} \text{Pic}(D_1) \oplus \text{Pic}(D_2) \]

because \( D_1 \cup D_2, D_2 \), and \( D_1 \cap D_2 = \tau \) are all connected. Since the map \( v \) is surjective, the map \( \lambda \) is injective. Consider again the exact sequence of sheaves of Abelian groups:

\[ 1 \to O^*_{D_1 \cup D_2 \cup D_3} \to O^*_{D_1 \cup D_2} \times O^*_{D_3} \to O^*_{(D_1 \cup D_2) \cap D_3} \to 1 \]

to get another long exact sequence

\[ 1 \to \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^* \xrightarrow{v'} \mathbb{C}^* \to \text{Pic}(D_1 \cup D_2 \cup D_3) \xrightarrow{\lambda'} \text{Pic}(D_1 \cup D_2) \oplus \text{Pic}(D_3) \]

because \( D_1 \cup D_2 \cup D_3, D_1 \cup D_2 \), and \( (D_1 \cup D_2) \cap D_3 = \tau \) are all connected. The map \( v' \) is surjective and so the map \( \lambda' \) is injective. The injectiveness of \( \lambda' \) and \( \lambda \) imply that of \( \eta \).

It is relatively easy to find a normal crossing variety that satisfies Condition 3.1 only but is not \( d \)-semistable. One usually needs to blow up along some suitable divisors of its double loci to make it \( d \)-semistable as in Example 2.3. If there are triple loci in the normal crossing varieties, the construction
get quite complicated. We devote the rest of this section to making $Y$ of the following example $d$-semistable.

**Example 4.2.** Consider a normal crossing $Y$ of two hyperplanes $Y_1, Y_2$ and one cubic threefold $Y_3$ in $\mathbb{P}^4$. We all know that $Y$ is smoothable to a quintic Calabi–Yau threefold but the total space is not smooth. We want to modify $Y$ so that we can apply Theorem 2.2. It has trivial dualizing sheaf but is not $d$-semistable — its collective normal class is a divisor class $((\mathcal{O}_{D_1}(5), \mathcal{O}_{D_2}(5), \mathcal{O}_{D_3}(5))$. Its dual complex is two-dimensional.

As in Example 2.3, we may need to choose smooth curves $C_i$ in the linear system $|\mathcal{O}_{D_i}(5)|$ on $D_i$ and do blow-ups. We assume for simplicity that $C_i$’s are all disjoint. Then blow up $Y_1, Y_2, Y_3$ along a smooth curve $C_3, C_1, C_2$ in the linear systems $|\mathcal{O}_{D_1}(5)|, |\mathcal{O}_{D_2}(5)|, |\mathcal{O}_{D_3}(5)|$ to get $\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3$ respectively. Let $\tilde{Y}_{ij}(j)$ be the proper transform of $Y_{ij}(j)$ in $\tilde{Y}_j$. When we try to paste them, we have matching problems. For example, we need to paste $\tilde{Y}_{(21)}$ in $\tilde{Y}_1$ with $\tilde{Y}_{(12)}$ in $\tilde{Y}_2$. Since the blow-up center $C_3$ lies on $Y_{(21)}$, $\tilde{Y}_{(21)}$ is isomorphic with $Y_{(21)}$ but $C_1$ intersects with $Y_{(12)}$ at 15 points. So $\tilde{Y}_{(12)}$ is the blow-up of $Y_{(12)}$ of those 15 points and accordingly $\tilde{Y}_{(12)}$ is not isomorphic to $\tilde{Y}_{(21)}$.

Hence we cannot paste them. This kind of problem does not occur in case of normal crossing varieties with only two components, where there is no triple locus. We will show that one can still paste the varieties after blow-ups if one choose the blow-up center carefully and choose the order of blow-ups in some suitable manner.

Choose smooth curves $C_i$’s that intersect with $\tau = Y_{123}$ transversely, satisfying the condition:

$$\begin{align*}
C_i \cap \tau &= C_j \cap \tau 
\end{align*}$$

for each $i, j$ (Figure 1). Note $C_i$ meets $\tau$ at 15 points but Figure 1 is simplified.

For $\{i, j\} = \{1, 2\}$, let $\pi_i : Y_i' \rightarrow Y_i$ be the blow-up along $C_j$ on $D_j$,

$$\begin{align*}
Y_{(31)}', Y_{(ji)}' &\text{ be the proper transform of } Y_{(31)}, Y_{(ji)} \text{ respectively and } E_j \text{ be the exceptional divisor over } C_j \text{ (Figure 2). Then } Y_{(31)}' \text{ is isomorphic to } Y_{(31)}.
\end{align*}$$

Note $Y_{(31)}$ is isomorphic with $Y_{(i3)}$. So $Y_{(31)}' (\subset Y_i')$ is isomorphic to $Y_{(i3)} (\subset Y_3)$. Note that $Y_{(ji)}' (\subset Y_j')$ is the blow-up of $Y_{(ji)}$ at the points $C_i \cap \tau$.

Since $C_1 \cap \tau = C_2 \cap \tau$, $Y_{(21)}'(\subset Y_1')$ and $Y_{(12)}'(\subset Y_2')$ are isomorphic. Let $C_3'$ be the proper transform of $C_3$ on $Y_{(21)}' (\subset Y_1')$ in the blow-up $Y_1' \rightarrow Y_1$. Since $C_1 \cap \tau = C_3 \cap \tau$ and $Y_{(21)}'(\subset Y_1')$ is the blow-up of $Y_{(21)}$ at the points $C_1 \cap \tau$, $C_3'$ does not meet with $\tau'$, where $\tau' = Y_{(21)}' \cap Y_{(31)}'$ and accordingly does not meet with $Y_{(31)}'$. In sum, the curve $C_3'$ is disjoint with $Y_{(31)}'$. Let $\pi'_1 : Y''_1 \rightarrow Y_1'$ be the blow-up along the curve $C_3'$ on $Y_{(21)}'$, $E_2'$ be the proper transform of $E_2$ and $E_3'$ be the exceptional divisor. Then the blow-up $Y''_1 \rightarrow Y_1'$ does not change $Y_{(31)}'$, i.e. the proper transform $Y_{(31)}''$ in $Y_1'$ of $Y_{(31)}'$ is isomorphic
with $Y'_{(31)}$ (Figure 3). Since the curve $C_3'$ lies on $Y'_{(21)}$, the proper transform $Y''_{(21)}$ in $Y''_1$ of $Y'_{(21)}$ is also isomorphic with $Y'_{(21)}$.

Now we can make a normal crossing variety by pasting $Y''_1, Y'_2, Y_3$. We note that $Y''_{(21)} \simeq Y'_{(12)}$, $Y'_{(32)} \simeq Y_{(23)}$ and $Y_{(13)} \simeq Y''_{(31)}$. Moreover those isomorphism induce isomorphisms

$$Y''_{(21)} \cap Y''_{(31)} \simeq Y'_{(12)} \cap Y'_{(32)} \simeq Y_{(13)} \cap Y''_{(23)}.$$
Hence we can make a normal crossing variety $\tilde{Y}$ such that there is a normalization

$$
\psi : Y''_1 \sqcup Y'_2 \sqcup Y_3 \to \tilde{Y}
$$

with $\psi(Y''_1) = \tilde{Y}_1$, $\psi(Y'_2) = \tilde{Y}_2$, $\psi(Y_3) = \tilde{Y}_3$ — this is already drawn in Figure 3. Let $\tilde{D}_i = \tilde{Y}_{jk}$ for $\{i, j, k\} = \{1, 2, 3\}$. It is not hard to see that $\tilde{Y}$ is $d$-semistable and projective (see also Theorem 5.3, Theorem 5.5).

By Theorem 2.2, $\tilde{Y}$ is smoothable to a Calabi–Yau threefold $M_{\tilde{Y}}$. By the formula in Proposition 3.2, the topological Euler number of $M_{\tilde{Y}}$ is

$$
e(M_{\tilde{Y}}) = \sum_i e(\tilde{Y}_i) - 2\sum_{i<j} e(\tilde{Y}_{ij}) + 3e(\tilde{Y}_{123}).
$$

Note

$$
e(\tilde{Y}_1) = e(Y''_1) = e(Y'_1) + e(C'_3) = e(Y'_1) + e(C_2) + e(C'_3) = e(Y'_1) + e(C_2) + e(C_3),
$$

$$
e(\tilde{Y}_2) = e(Y'_2) = e(Y_2) + e(C_1), \quad e(\tilde{Y}_3) = e(Y_3), \quad e(\tilde{Y}_{12}) = e(Y'_{12}) = e(Y_{12}) + e(C_1 \cap \tau),
$$

$$
e(\tilde{Y}_{23}) = e(Y_{23}), \quad e(\tilde{Y}_{13}) = e(Y_{13}), \quad e(\tilde{Y}_{123}) = e(Y_{123}).
$$

Hence

$$
e(M_{\tilde{Y}}) = \sum_i e(Y_i) - 2\sum_{i<j} e(Y_{ij}) + 3e(Y_{123}) + \sum_i e(C_i) - 2e(C_1 \cap \tau)
$$

$$
= (4 + 4 - 6) - 2(3 + 9 + 9) + 3 \cdot 0 + (-10 - 60 - 60) - 2 \cdot 15
$$

$$
= -200.
$$

On the other hand, one can show that $\text{rank} H^2(\tilde{Y}, \mathbb{Z}) = 3$ and

$$(\pi_1''(\pi_1^*(H_1)), \pi_2''(H_2), H_3)$$
belongs to $G^2(\tilde{Y},\mathbb{Z})$, where $H_i$ is the hyperplane section of $Y_i$. Let $\hat{H}$ be the image in $H^2(\tilde{Y},\mathbb{Z})_f$ of $(\pi_1^*(\pi_1^*(H_1)), \pi_2^*(H_2), H_3)$ by the map (3.2), Since
\[ \hat{H}^3 = \pi_1^*(\pi_1^*(H_1))^3 + \pi_2^*(H_2)^3 + H_3^3 = 1 + 1 + 3 = 5 \]
is non-zero, $\hat{H}$ is a non-zero element of $H^2(\tilde{Y},\mathbb{Z})_f$. On the other hand, using Corollary 3.4, we have $h^2(M_{\hat{Y}}) = 3 - 2 = 1$ (see also Lemma 5.6). Hence $\{\hat{H}\}$ is a basis for $H^2(M_{\hat{Y}},\mathbb{Q})$. Since the number $\hat{H}^3 = 5$ is positive and not a cube of an integer, $\hat{H}$ is the ample generator of $H^2(M_{\hat{Y}},\mathbb{Z})_f$. By Equation (3.1),
\[ c_2(\tilde{Y}) = \sum_i c_2(\tilde{Y}_i) - \sum_{i \neq j} \tilde{Y}_{(ji)} \cdot c_1(\tilde{Y}_i) = \sum_i c_2(\tilde{Y}_i) - \sum_i c_1(\tilde{Y}_i)^2, \]
where we used $c_1(\tilde{Y}_i) = \sum_{j \neq i} \tilde{Y}_{(ji)}$. Hence we have
\[ \hat{H} \cdot c_2(M_{\hat{Y}}) = \pi_1^*(H_1) \cdot (c_2(\tilde{Y}_1) - c_1(\tilde{Y}_1)^2) + \pi_2^*(H_2) \cdot (c_2(\tilde{Y}_2) - c_1(\tilde{Y}_2)^2) + \pi_3^*(H_3) \cdot (c_2(\tilde{Y}_3) - c_1(\tilde{Y}_3)^2) \]
\[ = (26 - (-4)) + (21 - 1) + (12 - 12) = 50. \]
So $M_{\hat{Y}}$ is a Calabi–Yau threefold of Picard number one with invariants:
\[ h^{1,1}(M_{\hat{Y}}) = 1, h^{1,2}(M_{\hat{Y}}) = 101, \hat{H}^3 = 5, \hat{H} \cdot c_2(M_{\hat{Y}}) = 50, \]
which are invariants of a quintic Calabi–Yau threefold. This is expected since the smoothing is a quintic Calabi–Yau threefold in $\mathbb{P}^4$.

5. PRODUCING $d$-SEMISTABLE MODELS

We apply the procedure in the previous section to more general normal crossing variety $Y$, satisfying Condition 3.1. We also want to consider the case that the blow-up curves $C_i$‘s have multiple components since it gives us much more examples.

Before it, we need a notion of sequential blow-ups. Let $Z$ be a smooth variety and $S$ be its smooth subvariety. For a sequence of smooth divisors $c_1, c_2, \ldots, c_k$ on $S$. We define the sequential blow-up $Z' \to Z$ along $c_1, c_2, \ldots, c_k$ on $S$ as follows: Let $Z(1) \to Z$ be the blow-up along $c_1$ and $S(1)$ be the proper transform of $S$. Since the blow-up center $c_1$ lies on $S$, $S(1)$ is isomorphic to $S$. So $S(1)$ contains copies of $c_1, c_2, \ldots, c_k$. We denote them by $c_1^{(1)}, c_2^{(1)}, \ldots, c_k^{(1)}$. We construct $Z(2), Z(3), \ldots, Z(k)$ inductively as follows. Let $Z(l+1) \to Z(l)$ be the blow-up along $c_1^{(l)}$ and $S(l+1)$ be the proper transform of $S(l)$. Since the blow-up center $c_1^{(l+1)}$ lies on $S(l)$, $S(l+1)$ is isomorphic to $S(l)$. So $S(l+1)$ contains copies of $c_1^{(l)}, c_2^{(l)}, \ldots, c_k^{(l)}$. We denote them by $c_1^{(l+1)}, c_2^{(l+1)}, \ldots, c_k^{(l+1)}$. Let $Z' = Z(k)$, then the sequential blow-up is the composite $Z' \to Z$ of the above blow-ups. Let $S'$ be the proper transform of $S$, then $S'$ is isomorphic to $S$. Let $c'_i = c_i^{(k)}$, then $c'_i$ is isomorphic to
$c_i$ for each $i$. Let $E^{(l)}$ be the exceptional divisor of the blow-up $Z^{(l)} \to Z^{(l-1)}$ over $c^{l-1}_l$. We denote the proper transform of $E^{(l)}$ with respect to the map $Z' \to Z^{(l)}$ by $E_l$ and call it the exceptional divisor over the center $c_l$ of the sequential blow-up $Z' \to Z$. Note the sequential blow-up depends on the order of blow-ups unless $S$ is a curve and so $c_i$’s are points. We note some useful facts about sequential blow-ups.

**Lemma 5.1.** Let $S$ have codimension one in $Z$ and $\pi : Z' \to Z$ be the sequential blow-up along smooth divisors $c_1, c_2, \ldots, c_k$ of $S$. Then

1. $S'|_{S'} \sim (\pi|_{S'})^*(S|_S - c_1 - c_2 - \cdots - c_k)$.

2. Let $H$ be an ample divisor on $Z$, then there is some positive number $M$ such that, for any fixed positive integer $m$ with $m \geq M$, the divisor

$$H' := n\pi^*H - E_k - mE_{k-1} - m^2E_{k-2} - \cdots - m^{k-1}E_1$$

is ample on $Z'$ for sufficiently large $n$.

3. Furthermore suppose that a normal crossing divisor $S+D_1 + \cdots + D_m$ is an anticanonical divisor of $Z$ such that any of $D_1, \cdots, D_m$ do not contain any of $c_1, c_2, \ldots, c_k$. Then $S' + D'_1 + \cdots + D'_m$ is an anticanonical divisor of $Z'$, where $D'_l$ is the proper transform of $D_l$.

**Proof.** Let $\pi^{(l)} : Z^{(l)} \to Z^{(l-1)}$ be the blow-up along $c^{l-1}_l$. Then

$$S^{(l)}|_{S^{(l)}} \sim (\pi^{(l)}|_{S^{(l)}})^*(S^{(l-1)}|_{S^{(l-1)}}) - c^{(l)}_l$$

$$= (\pi^{(l)}|_{S^{(l)}})^*(S^{(l-1)}|_{S^{(l-1)}}) - c^{(l-1)}_l).$$

Let $\pi^{[l]} = \pi^{(l)} \circ \pi^{(l-1)} \circ \cdots \circ \pi^{(1)}$, then one can inductively show

$$S^{(l)}|_{S^{(l)}} \sim (\pi^{[l]}|_{S^{(l)}})^*(S|_S - c_1 - c_2 - \cdots - c_i).$$

By letting $l = k$, we have the first claim.

Consider fibers over points in $c^{l-1}_l$ in the blow-up $\pi^{(l)} : Z^{(l)} \to Z^{(l-1)}$ and let $N_l$ be the set of proper transforms in $Z'$ of those fibers. Then the relative cone $\text{NE}(Z'/Z)$ of effective curves with respect to $\pi : Z' \to Z$ is generated by curves in $\bigcup_l N_l$. For any $L \in N_l$, $E_{l'} \cdot L = -1$ for $l' = l$, $E_{l'} \cdot L = 0$ for $l' < l$ and $E_{l'} \cdot L \geq 0$ for $l' > l$. For a divisor

$$D = -a_1E_1 - a_2E_2 - \cdots - a_kE_k,$$

if $D \cdot L > 0$ for any $L \in \bigcup_l N_l$, then $D$ is relatively ample with respect to the map $Z' \to Z$. Note that the set $\{E_{l'} \cdot L | L \in N_l, l' < l\}$ has the maximum $\beta$. Let $M = \beta + 2$. For a divisor

$$D' = -E_k - mE_{k-1} - m^2E_{k-2} - \cdots - m^{k-1}E_1$$
with $m > 1$ and $L \in N_l$,
\[
D' \cdot L = -E_k \cdot L - mE_{k-1} \cdot L - m^2E_{k-2} \cdot L - \cdots - m^{k-l}E_l \cdot L \\
= -E_k \cdot L - mE_{k-1} \cdot L - m^2E_{k-2} \cdot L - \cdots + m^{k-l-1}E_{l-1} \cdot L - m^{k-l} \\
\geq -\beta - m\beta - m^2\beta - \cdots - m^{k-l-1}\beta + m^{k-l} \\
\geq m^{k-l} \left( 1 - \frac{\beta}{m-1} \right).
\]

So if $m \geq M$, then $D' \cdot L > 0$ and accordingly $D'$ is relatively ample with respect to the map $Z' \to Z$. Hence we have the second claim.

Let $D_i^{(l)}$ be the proper transform in $Z^{(l)}$ of $D_i$ by the map $Z^{(l)} \to Z$. One can also inductively show
\[-K_{Z^{(l)}} \sim S^{(l)} + D_1^{(l)} + \cdots + D_m^{(l)}\]
and by letting $l = k$ we have the third claim.

From now on, assume that the normal crossing variety $Y$ is three dimensional (Of course, we always assume that $Y$ satisfies Condition 3.1). Then its triple loci $\tau$ is a curve.

Let $C_i = c_{i1} + c_{i2} + \cdots + c_{ai}$ be a divisor on $D_i$ for each $i$ that is a sum of distinct irreducible smooth curves $c_{ji}$'s on $D_i$. We require that $C_i$'s satisfy:

**Condition 5.2.** Each $C_i$ intersects with $\tau$ transversely and
\[C_1 \cap \tau = C_2 \cap \tau = C_3 \cap \tau.\]

If each $C_i$ is linearly equivalent to the divisor class $N_Y(D_i)$, defined in (4.1), we call $\{C_1, C_2, C_3\}$ a collective normal divisor of $Y$.

For $\{i,j\} = \{1,2\}$, let $\pi_{ij} : Y'_{ij} \to Y_i$ be the sequential blow-up along $c_{1j}, c_{2j}, \cdots, c_{aj}$ on $D_j$, $Y_{(3i)}$ be the proper transform of $Y_{(3i)}$ and $E_{ij}$ be the exceptional divisor over $c_{ij}$. Then $Y_{(3i)}'$ is isomorphic to $Y_{(3i)}$. Note $Y_{(3i)}'$ is isomorphic with $Y_{(3i)}$. So $Y_{(3i)}'$ (\subset $Y_{(1i)}'$) is isomorphic to $Y_{(3i)}$ (\subset $Y_{(3i)}$). Note that $Y_{(ji)}'$ (\subset $Y'_{(ji)}$) is the blow-up of $Y_{(ji)}'$ at the points $C_i \cap \tau$. Since $C_1 \cap \tau = C_2 \cap \tau$, $Y_{(21)}'$ (\subset $Y_{(11)}'$) and $Y_{(12)}'$ (\subset $Y_{(22)}'$) are isomorphic. Let $c_{i3}$ be the proper transform of $c_{3}$ on $Y_{(21)}$ (\subset $Y_{(11)}$) in the sequential blow-up $Y_1' \to Y_1$. Since $C_1 \cap \tau = C_2 \cap \tau$ and $Y_{(21)}'$ (\subset $Y_{(11)}'$) is the blow-up of $Y_{(21)}$ at the points $C_1 \cap \tau$, the effective divisor $C_3' := \sum c_{i3}$ does not meet with $\tau'$, where $\tau' = Y_{(21)}' \cap Y_{(31)}'$ and accordingly does not meet with $Y_{(31)}'$. In sum, the divisor $C_3'$ is disjoint with $Y_{(31)}'$. Let $\pi_{11} : Y_1'' \to Y_1'$ be the sequential blow-up along $c_{13}, c_{23}, \cdots, c_{a3}$ on $Y_{(21)}'$, $E_{12}''$ be the proper transform of $E_{12}$ and $E_{13}''$ be the exceptional divisor over $c_{13}$. Then the sequential blow-up $Y_1'' \to Y_1'$ does not change $Y_{(31)}'$, i.e. the proper transform $Y_{(31)}''$ in $Y_1''$ of $Y_{(31)}'$ is isomorphic with $Y_{(31)}'$. Since the curves $c_{13}$'s lie on $Y_{(21)}'$, the proper transform $Y_{(21)}''$ in $Y_1''$ of $Y_{(21)}'$ is also isomorphic with $Y_{(21)}'$. Let us
try to make a normal crossing variety by pasting $Y''_1, Y'_2, Y_3$. We note that $Y''_{(21)} \simeq Y'_{(12)}, Y'_{(32)} \simeq Y_{(23)}$ and $Y_{(13)} \simeq Y''_{(31)}$. Moreover those isomorphisms induce isomorphisms

$$Y''_{(21)} \cap Y''_{(31)} \simeq Y'_{(12)} \cap Y'_{(32)} \simeq Y_{(13)} \cap Y''_{(32)}.$$

Hence we can make a normal crossing variety $\tilde{Y}$ such that there is a normalization

$$\psi : Y''_1 \sqcup Y'_2 \sqcup Y_3 \to \tilde{Y}$$

with $\psi(Y''_1) = \tilde{Y}_1$, $\psi(Y'_2) = \tilde{Y}_2$, $\psi(Y_3) = \tilde{Y}_3$. We simply identify $Y''_1$ with $\tilde{Y}_1$, $Y'_2$ with $\tilde{Y}_2$ and $Y_3$ with $\tilde{Y}_3$. Let $\tilde{D}_i = \tilde{Y}_{jk}$ for $\{i,j,k\} = \{1,2,3\}$.

We also note that $Y''_{(21)} \simeq Y'_{(12)}, Y'_{(32)} \simeq Y_{(23)}$ and $Y_{(13)} \simeq Y''_{(31)}$. Hence we can make a normal crossing variety $\tilde{Y}$ by pasting $Y'_1, Y'_2, Y_3$ ($\tilde{Y}_1 = Y'_1, \tilde{Y}_2 = Y'_2, \tilde{Y}_3 = Y_3$). Then there are natural maps

$$\tilde{Y} \xrightarrow{\pi_1} Y' \xrightarrow{\pi} Y.$$

Note that $\pi|_{\tilde{Y}_i} = \pi_i$ for $i = 1,2$ and $\pi'|_{\tilde{Y}_1} = \pi_1'$. As before, let $\tilde{D}_i = \tilde{Y}_{jk}$ for $\{i,j,k\} = \{1,2,3\}$.

Since $\tilde{Y}_i, \tilde{Y}_{ij}$ are birational to $Y_i, Y_{ij}$ respectively, $\tilde{Y}$ also satisfies (2) in Condition 3.1.

**Theorem 5.3.** $\tilde{Y}$ also satisfies (4) in Condition 3.1 (i.e. it has trivial dualizing sheaf). Furthermore If $\{C_1, C_2, C_3\}$ is a collective normal class, then $\tilde{Y}$ has trivial collective normal class.

**Proof.** $Y_{(ji)} + Y_{(ki)}$ is an anticanonical normal crossing divisor of $Y_i$ for each $\{i,j,k\} = \{1,2,3\}$ and so $\tilde{Y}_{(13)} + \tilde{Y}_{(23)}$ is an anticanonical normal crossing divisor of $\tilde{Y}_3$. By (3) in Lemma 5.1, $Y_{(ji)} + Y_{(ki)}$ is an anticanonical normal crossing divisor of $\tilde{Y}_i$ for $i = 1,2$ and $\tilde{Y}_{(21)} + \tilde{Y}_{(31)}$ is an anticanonical normal crossing divisor of $\tilde{Y}_1$. Since $\tilde{Y}_2 = \tilde{Y}_2, \tilde{Y}_{(12)} + \tilde{Y}_{(31)}$ is an anticanonical normal crossing divisor of $\tilde{Y}_2$. In sum, $\tilde{Y}_{(ji)} + \tilde{Y}_{(ki)}$ is an anticanonical normal crossing divisor of $\tilde{Y}_i$ for any $\{i,j,k\} = \{1,2,3\}$ and accordingly $\tilde{Y}$ has trivial dualizing sheaf.

Now assume that $\{C_1, C_2, C_3\}$ is a collective normal class of $Y$. Firstly we show that the following claim.

**Claim.** $\{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3\}$ is a collective normal class of $\tilde{Y}$, where $\tilde{C}_1, \tilde{C}_2$ are the zero divisors on $\tilde{Y}_{23}, \tilde{Y}_{31}$ respectively and $\tilde{C}_3 = c_{13} + c_{23} + \cdots + c_{\alpha,3}$.

**Proof of Claim.** Note $N_\tilde{Y}(Y_{i3}) = Y_{(i3)}|_{Y_{i3}} + Y_{(3i)}|_{Y_{i3}} + Y_{(ji3)} \sim C_j$ for $\{i,j\} = \{1,2\}$. By (1) in Lemma 5.1, for $\{i,j\} = \{1,2\}$

$$\tilde{Y}_{(3i)}|_{\tilde{Y}_{(3i)}} \sim (\pi_i|_{\tilde{Y}_{i3}})^*(Y_{(3i)}|_{Y_{(3i)}} - c_{1j} - c_{2j} - \cdots - c_{\alpha,j}) = (\pi_i|_{\tilde{Y}_{i3}})^*(Y_{(3i)}|_{Y_{(3i)}} - C_j).$$
And note \((\pi_i|_{\tilde{Y}_{12}})^*(Y_{(12)}|_{Y_{12}}) = \tilde{Y}_{(12)}|_{\tilde{Y}_{12}}\) and \((\pi_i|_{\tilde{Y}_{12}})^*(Y_{(j12)}) = \tilde{Y}_{(j12)}\). So we have
\[
N_{\tilde{Y}}(\tilde{Y}_{13}) = \tilde{Y}_{(i3)}|_{\tilde{Y}_{13}} + \tilde{Y}_{(3i)}|_{\tilde{Y}_{13}} + \tilde{Y}_{(j13)} \\
\sim (\pi_i|_{\tilde{Y}_{13}})^*(Y_{(i3)}|_{Y_{13}}) + (\pi_i|_{\tilde{Y}_{13}})^*(Y_{(3i)}|_{Y_{i3}} - C_j) + (\pi_i|_{\tilde{Y}_{13}})^*(Y_{(j13)}) \\
= (\pi_i|_{\tilde{Y}_{13}})^*(Y_{(i3)}|_{Y_{13}}) + Y_{(3i)}|_{Y_{i3}} - C_j + Y_{(j13)} \\
\sim (\pi_i|_{\tilde{Y}_{13}})^* (N_{\tilde{Y}}(\tilde{Y}_{13}) - C_j) \\
= (\pi_i|_{\tilde{Y}_{13}})^*(0) \\
= 0.
\]

Note
\[
(\pi_i|_{\tilde{Y}_{12}})^*(Y_{(12)}|_{Y_{12}}) = \tilde{Y}_{(12)}|_{\tilde{Y}_{12}}, (\pi_i|_{\tilde{Y}_{12}})^*(Y_{(21)}|_{Y_{12}}) = \tilde{Y}_{(21)}|_{\tilde{Y}_{12}}
\]
and
\[
(\pi_i|_{\tilde{Y}_{12}})^*(Y_{(312)}) = \tilde{Y}_{(312)} + (\sum_l E_{\bar{t}l})|_{\tilde{Y}_{12}}.
\]

We have \((\sum_l E_{\bar{t}l})|_{\tilde{Y}_{12}} = (\sum_l E_{\bar{t}l})|_{\tilde{Y}_{12}}\) because of Condition 5.2. Hence
\[
N_{\tilde{Y}}(\tilde{Y}_{12}) = \tilde{Y}_{(12)}|_{\tilde{Y}_{12}} + \tilde{Y}_{(21)}|_{\tilde{Y}_{12}} + \tilde{Y}_{(312)} \\
\sim (\pi_i|_{\tilde{Y}_{12}})^*(Y_{(12)}|_{Y_{12}}) + (\pi_i|_{\tilde{Y}_{12}})^*(Y_{(21)}|_{Y_{12}}) + (\pi_i|_{\tilde{Y}_{12}})^*(Y_{(312)}) - (\sum_l E_{\bar{t}l})|_{\tilde{Y}_{12}} \\
= (\pi_i|_{\tilde{Y}_{12}})^*(Y_{(12)}|_{Y_{12}} + Y_{(21)}|_{Y_{12}} + Y_{(312)}) - (\sum_l E_{\bar{t}l})|_{\tilde{Y}_{12}} \\
\sim (\pi_i|_{\tilde{Y}_{12}})^*(C_3) - (\sum_l E_{\bar{t}l})|_{\tilde{Y}_{12}} \\
= \tilde{C}_3,
\]

which finishes the proof of claim.

Now let us complete the proof of the theorem. \(N_{\tilde{Y}}(\tilde{Y}_{23}) = 0\) implies \(N_{\tilde{Y}}(\tilde{Y}_{23}) = 0\) because \(\tilde{Y}_2 = \tilde{Y}_2\) and \(\tilde{Y}_3 = \tilde{Y}_3\). \(N_{\tilde{Y}}(\tilde{Y}_{13}) = 0\) implies \(N_{\tilde{Y}}(\tilde{Y}_{13}) = 0\). Since the blow-up centers \(C_{13}'\)'s are disjoint from \(\tilde{Y}_{13}\). Now it remains to show \(N_{\tilde{Y}}(\tilde{Y}_{23}) = 0\). This can be showed similarly as before.

\[
N_{\tilde{Y}}(\tilde{Y}_{12}) = \tilde{Y}_{(12)}|_{\tilde{Y}_{12}} + \tilde{Y}_{(21)}|_{\tilde{Y}_{12}} + \tilde{Y}_{(312)} \\
\sim (\pi_1'|_{\tilde{Y}_{12}})^*(Y_{(12)}|_{Y_{12}}) + (\pi_1'|_{\tilde{Y}_{12}})^*(Y_{(21)}|_{Y_{12}}) - C_3' + \tilde{Y}_{(312)} \\
= (\pi_1'|_{\tilde{Y}_{12}})^*(Y_{(12)}|_{Y_{12}} + Y_{(21)}|_{Y_{12}} - C_3' + \tilde{Y}_{(312)} \\
\sim (\pi_1'|_{\tilde{Y}_{12}})^*(N_{\tilde{Y}}(\tilde{Y}_{12}) - C_3') \\
= (\pi_1'|_{\tilde{Y}_{12}})^*(0) \\
= 0.
\]

To have the projectivity, we introduce another condition for \(C_1, C_2, C_3\):
**Condition 5.4.**

(1) \( \alpha_1 = \alpha_2 = \alpha_3 \) and
\[
c_1 \cap \tau = c_2 \cap \tau = c_3 \cap \tau
\]
for each \( l \). Let \( \alpha = \alpha_1 \).

(2) For each \( l = 1, 2, \cdots, \alpha \) and distinct \( i, j = 1, 2, 3 \), there is a divisor \( G_{ij} \) of \( Y_i \) such that \( G_{ij} \mid D_j \) is linearly equivalent to \( c_{lj} \).

**Theorem 5.5.** If \( C_i \)'s satisfy Condition 5.4, then \( \tilde{Y} \) is projective.

**Proof.** Let \( H_1, H_2, H_3 \) be ample divisors on \( Y_1, Y_2, Y_3 \) respectively such that \( H_i \mid Y_{ij} \sim H_j \mid Y_{ij} \). Then by (2) in Lemma 5.1, for each \( \{i, j\} = \{1, 2\} \), there is a positive number \( m \) such that
\[
\tilde{H}_i := n\pi_i^*(H_i) - \sum_{l=1}^{\alpha} m^{\alpha-l} E_{ij}
\]
is ample on \( \tilde{Y}_i \) for sufficiently large \( n \). Also the divisor
\[
\tilde{H}_3 := nH_3 - \sum_{l=1}^{\alpha} m^{\alpha-l} G_{l3}
\]
is ample on \( \tilde{Y}_3 \) for sufficiently large \( n \). Hence one can choose some \( n \) such that the divisor \( \tilde{H}_i \) is ample on \( \tilde{Y}_i \) for each \( i = 1, 2, 3 \). Firstly Condition 5.2 implies \( \tilde{H}_1 \mid D_3 \sim \tilde{H}_2 \mid D_3 \). The condition (2) in Condition 5.4 implies \( \tilde{H}_1 \mid D_1 \sim \tilde{H}_3 \mid D_j \) for distinct \( i, j = 1, 2 \). So we showed that \( \tilde{Y} \) is projective.

Let \( \tilde{G}_{i2} = \pi_2^*(G_{i2}) - E_{i1} \) for \( l = 1, 2, \cdots, \alpha \). Then
\[
\tilde{G}_{i2} \mid D_3 \sim c_{l3}.
\]
Again, by (2) in Lemma 5.1, there is some positive integer \( m' \) such that the divisors
\[
\tilde{H}_1 := \tilde{n}\pi_1^*(\tilde{H}_1) - \sum_{l=1}^{\alpha} m'^{\alpha-l} E_{i3}
\]
and
\[
\tilde{H}_2 := \tilde{n}\tilde{H}_2 - \sum_{l=1}^{\alpha} m'^{\alpha-l} \tilde{G}_{i3}
\]
that are ample on \( \tilde{Y}_1 \) and \( \tilde{Y}_2 \) respectively for sufficiently large \( \tilde{n} \). Let \( \tilde{H}_3 = \tilde{n}\tilde{H}_3 \). Then \( \tilde{H}_i \) is ample on \( \tilde{Y}_i \) and \( \tilde{H}_i \mid D_j \sim \tilde{H}_3 \mid D_j \) for distinct \( i, j = 1, 2 \). So we showed that \( \tilde{Y} \) is also projective.

\( \square \)

**Lemma 5.6.** If \( C_i \)'s satisfy Condition 5.4,
\[
h^2(\tilde{Y}) = h^2(Y) + 2\alpha.
\]
Proof. The maps
\[ \tilde{Y} \xrightarrow{\pi_1} \tilde{Y} \xrightarrow{\pi} Y. \]
induces pullbacks
\[ H^2(Y_i, \mathbb{Q}) \to H^2(\tilde{Y}_i, \mathbb{Q}), H^2(D_j, \mathbb{Q}) \to H^2(\tilde{D}_j, \mathbb{Q}) \]
and
\[ H^2(Y_i, \mathbb{Q}) \to H^2(\tilde{Y}_i, \mathbb{Q}), H^2(D_j, \mathbb{Q}) \to H^2(\tilde{D}_j, \mathbb{Q}), \]
which altogether make the following commutative diagram.

\[ \begin{array}{ccc}
H^2(\tilde{Y}_1, \mathbb{Q}) \oplus H^2(\tilde{Y}_2, \mathbb{Q}) \oplus H^2(\tilde{Y}_3, \mathbb{Q}) & \xrightarrow{\mu} & H^2(\tilde{D}_1, \mathbb{Q}) \oplus H^2(\tilde{D}_2, \mathbb{Q}) \oplus H^2(\tilde{D}_3, \mathbb{Q}) \\
\downarrow{g} & & \downarrow{\mu} \\
H^2(Y_1, \mathbb{Q}) \oplus H^2(Y_2, \mathbb{Q}) \oplus H^2(Y_3, \mathbb{Q}) & \xrightarrow{\mu} & H^2(D_1, \mathbb{Q}) \oplus H^2(D_2, \mathbb{Q}) \oplus H^2(D_3, \mathbb{Q}) \\
\downarrow{f} & & \downarrow{\mu} \\
H^2(Y_1, \mathbb{Q}) \oplus H^2(Y_2, \mathbb{Q}) \oplus H^2(Y_3, \mathbb{Q}) & \xrightarrow{\mu} & H^2(D_1, \mathbb{Q}) \oplus H^2(D_2, \mathbb{Q}) \oplus H^2(D_3, \mathbb{Q}) 
\end{array} \]

Firstly by Proposition 3.3, \( \ker \mu \simeq H^2(Y, \mathbb{Q}) \), \( \ker \tilde{\mu} \simeq H^2(\tilde{Y}, \mathbb{Q}) \) and \( \ker \tilde{\mu} \simeq H^2(\tilde{Y}, \mathbb{Q}) \). Note also that the maps \( f, g \) are injective. We can regard divisors as elements of \( H^2(Y_i, \mathbb{Q}), H^2(D_j, \mathbb{Q}) \). It is easy to check
\[ E_l := (E_{l1}, E_{l2}, G_{l3}) \in \ker \tilde{\mu}. \]
Condition 5.2 and the connectivity condition imply
\[ \ker \tilde{\mu} = f(\ker \mu) \oplus \langle E_1, E_2, \cdots, E_\alpha \rangle \]
and so
\[ \dim(\ker \tilde{\mu}) = \dim(f(\ker \mu)) + \alpha = h^2(Y) + \alpha. \]
It is also easy to check
\[ E'_l := (E'_{l1}, G'_{l2}, 0) \in \ker \tilde{\mu}. \]
Again the connectivity condition implies
\[ \ker \tilde{\mu} = g(\ker \tilde{\mu}) \oplus \langle E'_1, E'_2, \cdots, E'_\alpha \rangle \]
and
\[ \dim \ker(\tilde{\mu}) = \dim(g(\ker \tilde{\mu})) + \alpha = (h^2(Y) + \alpha) + \alpha = h^2(Y) + 2\alpha. \]
So we are done. \( \square \)

Combining Theorem 5.3, 5.5, we have the following corollary, which is the most important theorem in this note.

**Corollary 5.7.** Suppose that \( \{C_1, C_2, C_3\} \) is a collective normal class of \( Y \), satisfying Condition 5.2, 5.4. Then \( \tilde{Y} \) also satisfies Condition 3.1 and it is \( d \)-semistable. \( \tilde{Y} \) is smoothable to a Calabi–Yau threefold \( M_{\tilde{Y}} \) and hence it is a Calabi–Yau threefold of type III.
Theorem 5.8. Assume all the conditions in Corollary 5.7 and let $M_{\tilde{Y}}$ be the smoothing of $\tilde{Y}$ with smooth total space, then

$$h^{1,1}(M_{\tilde{Y}}) = h^2(Y) + 2\alpha - 2$$

and

$$e(M_{\tilde{Y}}) = \sum_i e(Y_i) - 2 \sum_j e(D_j) + 3e(\tau) + \sum_{i,l} e(c_i) - 2\gamma,$$

where $\gamma = e(\tau \cap C_1)$.

Proof. By Corollary 3.4 and Lemma 5.6

$$h^{1,1}(M_{\tilde{Y}}) = \dim(\ker \tilde{\mu}) - 2 = h^2(Y) + 2\alpha - 2.$$

By Proposition 3.2,

$$e(M_{\tilde{Y}}) = e(\tilde{Y}_1) + e(\tilde{Y}_2) + e(\tilde{Y}_3) - 2 \left( e(\tilde{D}_1) + e(\tilde{D}_2) + e(\tilde{D}_3) \right) + 3e(\tilde{\tau})$$

$$= (e(\tilde{Y}_1) + \sum_l e(c_{i,l})) + e(\tilde{Y}_2) + e(\tilde{Y}_3) - 2 (e(D_1) + e(D_2) + (e(D_3) + \gamma)) + 3e(\tau)$$

$$= ((e(Y_1) + \sum_l e(c_2)) + \sum_l e(c_3)) + (e(Y_2) + \sum_l e(c_3)) + e(Y_3)$$

$$- 2 (e(D_1) + e(D_2) + e(D_3) + \gamma)) + 3e(\tau)$$

$$= e(Y_1) + e(Y_2) + e(Y_3) - 2 (e(D_1) + e(D_2) + e(D_3)) + 3e(\tau) + \sum_{i,l} e(c_i) - 2\gamma.$$

Since $h^{1,2}(M_{\tilde{Y}}) = h^{1,1}(M_{\tilde{Y}}) - \frac{1}{2} e(M_{\tilde{Y}})$, we have determined all the Hodge numbers of $M_{\tilde{Y}}$.

Now return to Example 4.2. One can choose curves

$$C_i = c_{1i} + c_{2i} + \cdots + c_{ai}$$

with $c_{li}$ belonging to the linear system $|O_{D_i}(a_l)|$ such that $C_1, C_2, C_3$ satisfies Condition 5.2, Condition 5.4 and the conditions in Lemma 5.6. If $a_1 + a_2 + \cdots + a_\alpha = 5$, then $\{C_1, C_2, C_3\}$ is a collective normal curve of $Y$. As described, we can build a normal crossing variety $\tilde{Y}$ with respect to $C_i$’s, where we let $G_{li} = O_Y(a_l)$. Then $\tilde{Y}$ is projective and $d$-semistable and so it is smoothable to a Calabi–Yau threefold $M_{\tilde{Y}}$. Noting $h^2(Y) = 1$ and $\gamma = 15$, we have

$$h^{1,1}(M_{\tilde{Y}}) = 2\alpha - 1,$$

$$e(M_{\tilde{Y}}) = -70 + \sum_l (3-a_l)a_l + 2 \sum l 3(1-a_l)a_l$$

$$= -25 - 7 \sum l a_l^2.$$
and so
\[ h^{1,2}(M_Y) = h^{1,1}(M_Y) - \frac{1}{2} e(M_Y) = \frac{1}{2} (4\alpha + 7 \sum a_l^2 + 23). \]

Note that Hodge numbers of \( M_{\tilde{Y}} \) depend on \( a_l \)'s but not on the ordering of \( a_l \)'s. Let us take some examples. For the both cases of \((a_1, a_2) = (1, 4)\) and \((4, 1)\), we have the same Hodge numbers are \((h^{1,1}, h^{1,2}) = (3, 75)\). But the sequential blow-up depends on the ordering of blow-ups and so one can expect that smoothed Calabi–Yau threefolds are different. Let \( \tilde{Y}^{14} \) and \( \tilde{Y}^{41} \) be normal crossing varieties for \((a_1, a_2) = (1, 4)\) and \((4, 1)\) respectively. By using the technics in §6, §7 in [11], one can show that the ternary cubic forms on \( H^2(M_{\tilde{Y}^{14}}, \mathbb{Z}) \), \( H^2(M_{\tilde{Y}^{41}}, \mathbb{Z}) \) are different. There is another variation. Let \( Y_1 \) be a cubic threefold in \( \mathbb{P}^4 \) and \( Y_2, Y_3 \) be hyperplanes in \( \mathbb{P}^4 \) respectively. Then \( Y = Y_1 \cup Y_2 \cup Y_3 \) is the same normal crossing with the previous one and only the index ordering is changed. However if we build \( \tilde{Y} \), it is different from the previous one because \( Y_1 \) is sequentially blow up twice and \( Y_2 \) is sequentially blow up once — choosing different index orders of \( Y_i \)'s changes \( \tilde{Y} \) and gives us non-homeomorphic Calabi–Yau threefolds \( M_{\tilde{Y}} \)'s with same Hodge numbers. In this way, one can build several non-homeomorphic Calabi–Yau threefolds with same Hodge numbers. For the rest of examples, we will give only Hodge numbers of \( M_{\tilde{Y}} \)'s:

| \((a_1, a_2, \ldots, a_\alpha)\) | \((1,1,1,1)\) | \((1,1,1,2)\) | \((1,1,3)\) | \((1,4)\) | \((1,2,2)\) | \((2,3)\) | \((5)\) |
|---|---|---|---|---|---|---|---|
| \((h^{1,1}, h^{1,2})\) | \((9, 39)\) | \((7, 44)\) | \((5, 56)\) | \((3, 75)\) | \((5, 49)\) | \((3, 61)^*\) | \((1, 101)\) |

In the table, ‘*’ means (also will mean later) that such Hodge pairs do not overlap with those of Calabi–Yau threefolds in toric construction ([1, 7, 8]) or examples recently constructed in [12, 13]. The Hodge pair \((1, 101)\) comes from the quintic threefold in \( \mathbb{P}^4 \) which is a toric variety. Except for this case, there are multiple non-homeomorphic Calabi–Yau threefolds having the Hodge numbers as previously explained and although other pairs of Hodge numbers appear in the toric construction, probably those Calabi–Yau threefolds are different from ones of same Hodge numbers in the toric construction — there seems no reason that they are the same ones. One possible way of distinguishing them is comparing the cubic forms on their second integral cohomology classes.

6. More examples

Applying technics in the previous sections, we construct more \( d \)-semistable Calabi–Yau threefolds of type III. In each of the following examples, we start with a three-dimensional normal crossing variety \( Y = Y_1 \cup Y_2 \cup Y_3 \) with a divisor \( C_i = c_{1i} + c_{2i} + \ldots + c_{\alpha i} \) on \( D_i \) for \( i = 1, 2, 3 \) such that

1. \( Y \) satisfies Condition 3.1,
2. \( c_{ji} \)'s are distinct irreducible smooth curves on \( D_i \) and
(3) \( \{C_1, C_2, C_3\} \) is a collective normal class of \( Y \), satisfying Condition 5.2, 5.4.

As we did in the previous section, we make a new normal crossing \( \tilde{Y} \), by pasting sequential blow-ups of \( Y_i \)'s with respect to the curves \( c_{ij} \)'s, which is now smoothable to a Calabi–Yau threefold \( M_{\tilde{Y}} \). We give the Hodge numbers of \( M_{\tilde{Y}} \)'s in each example.

**Example 6.1.** Let \( S, S' \) be smooth quadratic hypersurfaces in \( \mathbb{P}^3 \) that intersect each other transversely. We note that \( S, S' \) are projectively isomorphic. So there is an isomorphism \( \phi : \mathbb{P}^3 \to \mathbb{P}^3 \) such that \( \phi(S) = S' \). Let \( \phi' : S \to S' \) be the restriction of \( \phi \) to \( S \). Let \( Y_1, Y_2, Y_3 \) be copies of \( \mathbb{P}^3 \) and \( S_i, S_i' \) be copies in \( Y_i \) of \( S, S' \). Identifying \( S_1 \) with \( S_2' \), \( S_2 \) with \( S_3' \) and \( S_3 \) with \( S_1' \) via copies of the isomorphism \( \phi' \), we have a normal crossing variety \( Y = Y_1 \cup Y_2 \cup Y_3 \).

It is easy to check that \( Y \) is projective and that have a trivial dualizing sheaf. \((O_{D_1}(6), O_{D_2}(6), O_{D_3}(6))\) is the collective normal class of \( Y \). One can choose curves \( C_i = c_{1i} + c_{2i} + \cdots + c_{ai} \) with \( c_{li} \) belonging to the linear system \( O_{D_i}(a_i) \) such that \( C_1, C_2, C_3 \) satisfies Condition 5.2, Condition 5.4 and the conditions in Lemma 5.6. If \( a_1 + a_2 + \cdots + a_\alpha = 6 \), then \( \{C_1, C_2, C_3\} \) is a collective normal curve of \( Y \). As described, we build a normal crossing variety \( \tilde{Y} \) with respect to \( C_i \)'s. Then \( \tilde{Y} \) is projective and \( d \)-semistable and so it is smoothable to a Calabi–Yau threefold \( M_{\tilde{Y}} \). Its dual complex is two-dimensional, so it is a \( d \)-semistable Calabi–Yau threefold of type III. Noting \( h^2(Y) = 1 \) and \( \gamma = 24 \), we have

\[
h^{1,1}(M_{\tilde{Y}}) = 2\alpha - 1, \\
e(M_{\tilde{Y}}) = -60 + \sum_l 3 \cdot 2(2 - a_l)a_l = 12 - 6 \sum_l a_l^2
\]

and so

\[
h^{1,2}(M_{\tilde{Y}}) = h^{1,1}(M_{\tilde{Y}}) - \frac{1}{2}e(M_{\tilde{Y}}) = \frac{1}{2}(4\alpha + 6 \sum_l a_l^2 - 14).
\]

The Hodge number numbers of \( M_{\tilde{Y}} \) are as follows:

| \((a_1, a_2, \cdots, a_\alpha)\) | \((h^{1,1}, h^{1,2})\) |
|----------------|----------------|
| (1,1,1,1,1,1) | (11,23) |
| (1,1,1,1,2) | (9,27) |
| (1,1,1,3) | (7,37) |
| (1,1,2,2) | (7,31) |
| (1,1,4) | (5,53) |
| (1,2,3) | (5,41) |
| (2,2,2) | (5,35) |
| (1,5) | (3,75) |
| (2,4) | (3,57) |
| (3,3) | (3,51) |
| (6) | (1,103) |
Example 6.2. Now consider a normal crossing $Y$ of smooth hypersurfaces $Y_1, Y_2$ and $Y_3$ in a smooth quartic hypersurface in $\mathbb{P}^5$ with degrees 1, 1, and 2 respectively. Clearly $Y$ is projective and that have a trivial dualizing sheaf. $(\mathcal{O}_{D_1}(4), \mathcal{O}_{D_2}(4), \mathcal{O}_{D_3}(4))$ is the collective normal class of $Y$. One can choose curves $C_i = c_{i1} + c_{i2} + \cdots + c_{i\alpha}$ with $c_{i\alpha}$ belonging to the linear system $\mathcal{O}_{D_i}(a_i)$ such that $C_1, C_2, C_3$ satisfies Condition 5.2, Condition 5.4 and the conditions in Lemma 5.6. If $a_1 + a_2 + \cdots + a_\alpha = 3$, then $\{C_1, C_2, C_3\}$ is a collective normal curve of $Y$. Build a normal crossing variety $\tilde{Y}$ with respect to $C_i$’s. Then $\tilde{Y}$ is a $d$-semistable Calabi–Yau threefold of type III. Note $h^2(Y) = 1$ and $\gamma = 16$. The Hodge numbers numbers of $M_{\tilde{Y}}$ are as follows:

| $(a_1, a_2, \cdots, a_\alpha)$ | $(h^{1,1}, h^{1,2})$ |
|---------------------------------|----------------------|
| (1, 1, 1)                      | (7, 35)              |
| (1, 1, 2)                      | (5, 43)              |
| (1, 3)                         | (3, 61)*             |
| (2, 2)                         | (3, 51)              |
| (4)                            | (1, 89)*             |

Example 6.3. Let $Y$ be a normal crossing of smooth hypersurfaces $Y_1, Y_2$ and $Y_3$ in a smooth cubic hypersurface in $\mathbb{P}^5$ with degree three. $(\mathcal{O}_{D_1}(3), \mathcal{O}_{D_2}(3), \mathcal{O}_{D_3}(3))$ is the collective normal class of $Y$. Choose curves $C_i = c_{i1} + c_{i2} + \cdots + c_{i\alpha}$ with $c_{i\alpha}$ belonging to the linear system $\mathcal{O}_{D_i}(a_i)$ such that $a_1 + a_2 + \cdots + a_\alpha = 3$, then $\{C_1, C_2, C_3\}$ is a collective normal curve of $Y$. Build a normal crossing variety $\tilde{Y}$ with respect to $C_i$’s. Then $\tilde{Y}$ is a $d$-semistable Calabi–Yau threefold of type III. Note $h^2(Y) = 1$ and $\gamma = 9$. The Hodge number numbers of $M_{\tilde{Y}}$ are as follows:

| $(a_1, a_2, \cdots, a_\alpha)$ | $(h^{1,1}, h^{1,2})$ |
|---------------------------------|----------------------|
| (1, 1, 1)                      | (5, 50)              |
| (1, 2)                         | (3, 57)              |
| (3)                            | (1, 73)*             |

Example 6.4. Let $Y$ be a normal crossing of smooth hypersurfaces $Y_1, Y_2$ and $Y_3$ of degree one in a smooth complete intersections two quadrics in $\mathbb{P}^6$. $(\mathcal{O}_{D_1}(3), \mathcal{O}_{D_2}(3), \mathcal{O}_{D_3}(3))$ is the collective normal class of $Y$. Choose curves $C_i = c_{i1} + c_{i2} + \cdots + c_{i\alpha}$ with $c_{i\alpha}$ belonging to the linear system $\mathcal{O}_{D_i}(a_i)$ such that $a_1 + a_2 + \cdots + a_\alpha = 3$ and build a normal crossing variety $\tilde{Y}$ with respect to $C_i$’s. Then $\tilde{Y}$ is a $d$-semistable Calabi–Yau threefold of type III. Note $h^2(Y) = 1$ and $\gamma = 12$. The Hodge number numbers of $M_{\tilde{Y}}$ are as follows:

| $(a_1, a_2, \cdots, a_\alpha)$ | $(h^{1,1}, h^{1,2})$ |
|---------------------------------|----------------------|
| (1, 1, 1)                      | (5, 41)              |
| (1, 2)                         | (3, 51)              |
| (3)                            | (1, 73)              |
Example 6.5. Let $Y$ be a normal crossing of smooth hypersurfaces $Y_1, Y_2$ and $Y_3$ of degree one in a section of the Grassmannian $\mathrm{Gr}(2, 5)$ embedded by Plücker by a subspace of codimension two. $(\mathcal{O}_{D_1}(3), \mathcal{O}_{D_2}(3), \mathcal{O}_{D_3}(3))$ is the collective normal class of $Y$. Choose curves $C_i = c_{1i} + c_{2i} + \cdots + c_{ai}$ with $c_{li}$ belonging to the linear system $\mathcal{O}_{D_i}(a_i)$ such that $a_1 + a_2 + \cdots + a_\alpha = 3$ and build a normal crossing variety $\tilde{Y}$ with respect to $C_i$’s. Then $\tilde{Y}$ is projective and $d$-semistable and so it is smoothable to a Calabi–Yau threefold $M_{\tilde{Y}}$. Note $h^2(Y) = 1$ and $\gamma = 15$. The Hodge numbers of $M_{\tilde{Y}}$ are as follows:

| $(a_1, a_2, \cdots, a_\alpha)$ | $(h^{1,1}, h^{1,2})$ |
|--------------------------------|-------------------|
| (1, 1, 1)                      | (5, 35)           |
| (1, 2)                         | (3, 48)*          |
| (3)                            | (1, 76)*          |

Example 6.6. Let $Y$ be a normal crossing of smooth hypersurfaces $Y_1, Y_2$ and $Y_3$ in $\mathbb{P}^2 \times \mathbb{P}^2$ with bi-degree $(1, 1)$. $(\mathcal{O}_{D_1}(3, 3), \mathcal{O}_{D_2}(3, 3), \mathcal{O}_{D_3}(3, 3))$ is the collective normal class of $Y$. One can choose curves $C_i = c_{1i} + c_{2i} + \cdots + c_{ai}$ with $c_{li}$ belonging to the linear system $\mathcal{O}_{D_i}(a_i, b_i)$ such that $C_1, C_2, C_3$ satisfies Condition 5.2, Condition 5.4 and the conditions in Lemma 5.6. If $a_1 + a_2 + \cdots + a_\alpha = 3$ and $b_1 + b_2 + \cdots + b_\alpha = 3$, then $\{C_1, C_2, C_3\}$ is a collective normal curve of $Y$. As described, we build a normal crossing variety $\tilde{Y}$ with respect to $C_i$’s. Then $\tilde{Y}$ is projective and $d$-semistable and so it is smoothable to a Calabi–Yau threefold $M_{\tilde{Y}}$. Note $h^2(Y) = 2$ and $\gamma = 18$. Note

\[ e(Y_i) = 6, e(D_i) = 6. \]

So $\sum_i e(Y_i) - 2\sum_i e(D_i) - 2\gamma = 18 - 4 \cdot 18 = -54$. We have $h^{1,1}(M_{\tilde{Y}}) = 2\alpha$,

\[
e(M_{\tilde{Y}}) = -54 + \sum_l 3(3a_l - a_l^2 + 3b_l - b_l^2 - 4a_l b_l) = -3 \sum_l (a_l^2 + b_l^2 + 4a_l b_l)
\]

and so

\[
h^{1,2}(M_{\tilde{Y}}) = h^{1,1}(M_{\tilde{Y}}) - \frac{1}{2}e(M_{\tilde{Y}}) = \frac{1}{2}(4\alpha + 6 \sum_l a_l^2 - 14).
\]

The Hodge number numbers of $M_{\tilde{Y}}$ are as follows:
One can choose collective normal curves of type III that consists of more than three components. Let \( \tilde{Y} \) be a projective Gorenstein variety of dimension four. Let \( Y = Y_1 \cup Y_2 \cup Y_3 \) be normal crossing hypersurface in \( Z \) such that \( Y_1 + Y_2 + Y_3 \sim -K_Z \). Then \( Y \) is projective and have trivial dualizing sheaf. We assume that \( Y \) satisfies the homological condition (3) in Theorem 2.2. Suppose that one can choose collective normal curves \( C_i = c_{i1} + c_{i2} + \cdots + c_{ai} \) satisfying Condition 5.2, Condition 5.4 and the conditions in Lemma 5.6. Then we can build a normal crossing variety \( \tilde{Y} \) with respect to \( C_i \)'s, which is a \( d \)-semistable Calabi–Yau threefold of type III.

Now let us consider some examples of \( d \)-semistable Calabi–Yau threefolds of type III that consists of more than three components. Let \( Y = Y_1 \cup Y_2 \cup Y_3 \) 

| \((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\) | \((h^{1,1}, h^{1,2})\) |
|-----------------------------|--------------------|
| \((3,3)\)                  | (2, 83)            |
| \((1,0), (2,3)\)          | (4, 61)            |
| \((1,3), (2,0)\)          | (4, 43)*           |
| \((0,3), (3,0)\)          | (4, 31)*           |
| \((0,2), (3,1)\)          | (4, 43)*           |
| \((1,2), (2,1)\)          | (4, 43)*           |
| \((1,1), (2,2)\)          | (4, 49)            |
| \((0,1), (3,2)\)          | (4, 61)            |
| \((1,0), (1,0), (1,3)\)   | (6, 42)            |
| \((0,3), (1,0), (2,0)\)   | (6, 27)            |
| \((0,2), (1,0), (2,1)\)   | (6, 33)            |
| \((1,0), (1,1), (1,2)\)   | (6, 36)            |
| \((0,1), (1,0), (2,2)\)   | (6, 45)            |
| \((0,2), (1,1), (2,0)\)   | (6, 27)            |
| \((0,1), (1,2), (2,0)\)   | (6, 33)            |
| \((0,1), (0,2), (3,0)\)   | (6, 27)            |
| \((0,1), (0,1), (3,1)\)   | (6, 42)            |
| \((0,1), (1,1), (2,1)\)   | (6, 36)            |
| \((1,1), (1,1), (1,1)\)   | (6, 33)            |
| \((0,3), (1,0), (1,0), (1,0)\) | (8, 26) |
| \((0,2), (1,0), (1,0), (1,1)\) | (8, 26) |
| \((0,1), (1,0), (1,0), (1,2)\) | (8, 32) |
| \((0,1), (0,2), (1,0), (2,0)\) | (8, 23)* |
| \((0,1), (0,1), (1,0), (2,1)\) | (8, 32) |
| \((0,1), (1,0), (1,1), (1,1)\) | (8, 29) |
| \((0,1), (0,1), (1,1), (2,0)\) | (8, 26) |
| \((0,1), (0,1), (0,1), (3,0)\) | (8, 26) |
| \((0,1), (0,2), (1,0), (1,0), (1,0)\) | (10, 22) |
| \((0,1), (0,1), (1,0), (1,0), (1,1)\) | (10, 25) |
| \((0,1), (0,1), (0,1), (1,0), (2,0)\) | (10, 22) |
| \((0,1), (0,1), (0,1), (1,0), (1,0), (1,0)\) | (12, 21) |
be any $d$-semistable Calabi–Yau threefold of type III — we already constructed several of such examples. Then there is a semistable degeneration $\mathcal{X} \to \Delta$ of Calabi–Yau threefolds whose central fiber $\mathcal{X}_0$ is $Y$. Let $\tau$ be the triple curve in $Y$ and $\mathcal{X}' \to \mathcal{X}$ be the blow-up along $\tau$. Now our new degeneration $\mathcal{X}' \to \Delta$ is not semistable because the central fiber is not reduced. However a base extension $\Delta' \to \Delta$ by $t \mapsto t^3$ gives us a semistable degeneration $\mathcal{X}' \times_{\Delta} \Delta' \to \Delta'$ (Mumford’s semistable reduction). Now the central fiber is a normal crossing variety $Y' = Y'_1 \cup Y'_2 \cup Y'_3 \cup F$, where $F$ is the exceptional divisor of the blow-up. Note that $F$ is a $\mathbb{P}^1$-bundle over $\tau$. The dual complex of $Y'$ has three triangles as its maximal cells and so $Y'$ is $d$-semistable Calabi–Yau threefolds of type III. By doing base changes after blowing up the triple loci of the central fiber, one can add as many components to central fiber as one wants. All the newly added components are $\mathbb{P}^1$-bundles over elliptic curves. Note a $\mathbb{P}^1$-bundle over elliptic curve does not satisfy the cohomological condition (3) in Theorem 2.2 $-(h^1(F, O_F) \neq 0)$. So those $d$-semistable Calabi–Yau threefolds do not fit into the situation in Theorem 2.2. This may imply that Theorem 2.2 needs some generalization.

The author is very thankful to the referee for making several valuable suggestions for the initial draft of this note. This work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2017R1D1A2B03029525) and Hongik University Research Fund.

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