Minimal unitary representation of $SU(2, 2)$ and its deformations as massless conformal fields and their supersymmetric extensions

Sudarshan Fernando$^\ast$ and Murat Günaydin$^\dagger$

$^1$Physical Sciences Department
Kutztown University
Kutztown, PA 19530, USA

$^2$Institute for Gravitation and the Cosmos
Physics Department
Pennsylvania State University
University Park, PA 16802, USA

Abstract: We study the minimal unitary representation (minrep) of $SO(4, 2)$ over an Hilbert space of functions of three variables, obtained by quantizing its quasiconformal action on a five dimensional space. The minrep of $SO(4, 2)$, which coincides with the minrep of $SU(2, 2)$ similarly constructed, corresponds to a massless conformal scalar in four spacetime dimensions. There exists a one-parameter family of deformations of the minrep of $SU(2, 2)$. For positive (negative) integer values of the deformation parameter $\zeta$, one obtains positive energy unitary irreducible representations corresponding to massless conformal fields transforming in $(0, \zeta/2)((-\zeta/2, 0))$ representation of the $SL(2, \mathbb{C})$ subgroup. We construct the supersymmetric extensions of the minrep of $SU(2, 2)$ and its deformations to those of $SU(2, 2 | N)$. The minimal unitary supermultiplet of $SU(2, 2 | 4)$, in the undeformed case, simply corresponds to the massless $N = 4$ Yang-Mills supermultiplet in four dimensions. For each given non-zero integer value of $\zeta$, one obtains a unique supermultiplet of massless conformal fields of higher spin. For $SU(2, 2 | 4)$ these supermultiplets are simply the doubleton supermultiplets studied in hep-th/9806042.

Keywords: AdS/CFT, Minimal Unitary Representations

$^\ast$fernando@kutztown.edu
$^\dagger$murat@phys.psu.edu
Contents

1. Introduction 2

2. Quasiconformal Realizations of \( SO(d+2,4) \) and Their Minimal Unitary Representations
   2.1 Geometric realizations of \( SO(d+2,4) \) as quasiconformal groups 6
   2.2 Minimal unitary representations of \( SO(d+2,4) \) from the quantization of their quasiconformal realizations 8

3. Minimal Unitary Realization of the 4D Conformal Group \( SO(4,2) \) over the Hilbert Space of \( L^2 \) Functions of Three Variables 12

4. Change of Basis and the Minimal Unitary Realization of \( SU(2,2) \)
   4.1 Minimal unitary realization of \( su(2,2) \) 16
   4.2 From \( SO(4,2) \) to \( SU(2,2) \) 17
   4.3 \( SU(1,1)_L \) subgroup of \( SU(2,2) \) generated by the isotonic (singular) oscillators 21

5. \( SU(2) \times SU(2) \times U(1) \) Decomposition of the Minrep of \( SU(2,2) \) and the Scalar Doubleton 22

6. One Parameter Family of Deformations of the Minrep of \( SU(2,2) \) and Massless Conformal Fields in Four Dimensions 26

7. Minimal Unitary Representations of Supergroups \( SU(2,2|p+q) \)
   7.1 5-grading of \( su(2,2|p+q) \) with respect to the subalgebra \( su(1,1|p+q) \oplus u(1) \oplus so(1,1) \) 29
   7.2 3-grading of \( su(2,2|p+q) \) with respect to the compact subalgebra \( su(2|p) \oplus su(2|q) \oplus u(1) \) 33

8. Minimal Unitary Supermultiplet of \( su(2,2|4) \) and its Deformations
   8.1 Minimal Unitary Supermultiplet of \( su(2,2|4) \) 38
   8.2 Deformed minimal unitary supermultiplets of \( su(2,2|4) \) for \( \zeta \neq 0 \) 40

9. Minimal Unitary Supermultiplet of \( su(2,2|p+q) \) and its Deformations 45

10. Conclusions 46
1. Introduction

The concept of minimal unitary realizations of Lie algebras was introduced by Joseph in [1] and was inspired by the work of physicists on spectrum generating symmetry groups. Minimal unitary representation of a Lie algebra exponentiates to a unitary representation of the corresponding noncompact group over a Hilbert space of functions depending on the smallest (minimal) number of variables possible. Joseph presented the minimal realizations of the complex forms of classical Lie algebras and of $G_2$ in a Cartan-Weil basis. The existence of the minimal unitary representation of $E_{8(8)}$ using Langland’s classification was first shown by Vogan [2]. The minimal unitary representations of simply laced groups were studied by Kazhdan and Savin [3], and Brylinski and Kostant [4, 5]. The minimal representations of quaternionic real forms of exceptional Lie groups were later studied by Gross and Wallach [6]. For a review and more complete list of references on the subject in the mathematics literature prior to 2000, we refer to the lectures of Jian-Shu Li [7].

Pioline, Kazhdan and Waldron [8] reformulated the minimal unitary representations of simply laced groups given in [3] and gave explicit realizations of the simple root (Chevalley) generators in terms of pseudo-differential operators for the simply laced exceptional groups as well as the spherical vectors necessary for the construction of modular forms.

The first known geometric realization of $E_{8(8)}$ as a quasiconformal group that leaves invariant a generalized light-cone with respect to a quartic distance function in 57 dimensions was given in [9]. Quasiconformal realizations exist for various real forms of all noncompact groups as well as for their complex forms [9, 10].

Remarkably, the quantization of geometric quasiconformal action of a noncompact group leads directly to its minimal unitary representation. This was first shown explicitly for the maximally split exceptional group $E_{8(8)}$ with the maximal compact subgroup $SO(16)$, which is the U-duality group of maximal supergravity in three dimensions [11]. The minimal unitary representation of three dimensional U-duality group $E_{8(-24)}$ of the exceptional supergravity [12] by quantization of its quasiconformal realization was given in [13]. $E_{8(-24)}$ is a quaternionic real form of $E_8$ with the maximal compact subgroup $E_7 \times SU(2)$.

The quasiconformal realizations of noncompact groups correspond to natural extensions of generalized conformal realizations of some of their subgroups and were studied from a generalized spacetime point of view in [10]. The class of generalized spacetimes studied in [10] are defined by Jordan algebras of degree three that contain Minkowskian spacetimes as subspaces. For example, spacetimes defined by the generic non-simple Jordan family of Euclidean Jordan algebras of degree three describe extensions of the Minkowskian spacetimes by an extra “dilatonic” coordinate. Their quasiconformal groups are $SO(d + 2, 4)$, which contain the conformal groups $SO(d, 2)$ as subgroups. The generalized spacetimes described by simple Euclidean Jordan algebras of degree three extend the Minkowskian spacetimes in the critical dimensions ($d = 3, 4, 6, 10$) by a dilatonic and extra (2, 4, 8, 16) commuting spinorial coordinates, respectively. Their quasiconformal groups are $F_{4(4)}, E_{6(2)}, E_{7(-5)}$ and $E_{8(-24)}$, which have the generalized conformal subgroups $Sp(6, \mathbb{R}), SU^*(6), SO^*(12)$ and $E_7(-25)$, re-
respectively. The minimal unitary representations of these quasiconformal groups obtained by quantization were given in [10, 13].

In [14] a unified formulation of the minimal unitary representations of certain non-compact real forms of groups of type $A_2$, $G_2$, $D_4$, $F_4$, $E_6$, $E_7$, $E_8$ and $Sp(2n, \mathbb{R})$ was given. The minimal unitary representations of $Sp(2n, \mathbb{R})$ are simply the singleton representations. The formulation of minimal unitary representations of noncompact groups $SU(m, n)$, $SO(m, n)$, $SO^*(2n)$ and $SL(m, \mathbb{R})$ requires slight modification of the unified construction and was also given explicitly in [14]. Furthermore, this unified approach was used to define and construct the corresponding minimal representations of non-compact supergroups $G$ whose even subgroups are of the form $H \times SL(2, \mathbb{R})$ with $H$ compact.\(^1\) The unified construction with $H$ simple or Abelian leads to the minimal unitary representations of $G(3), F(4)$ and $OSp(n|2, \mathbb{R})$. The minimal unitary representations of $OSp(n|2, \mathbb{R})$ with even subgroups $SO(n) \times Sp(2, \mathbb{R})$ are the singleton representations. The minimal realization of the one parameter family of Lie superalgebras $D(2, 1; \sigma)$ with even subgroup $SU(2) \times SU(2) \times SU(1, 1)$ was also presented in [14].

Unitary representations of rank two quaternionic groups $SU(2, 1)$ and $G_{2(2)}$ induced by their geometric quasiconformal actions were studied in great detail in [15]. The set of unitary representations thus obtained include the quaternionic discrete series representations that were studied in mathematics literature using other methods [16]. In the construction of unitary representations via the quasiconformal approach [15], spherical vectors of maximal compact subgroups of $SU(2, 1)$ and $G_{2(2)}$ play a fundamental role. Authors of [15] studied the minimal unitary representations of $SU(2, 1)$ and $G_{2(2)}$ obtained by quantization as well.\(^2\) Later in [17] a unified quasiconformal realization of three dimensional U-duality groups $QConf(J)$ of all $N = 2$ MESGTs with symmetric scalar manifolds defined by Euclidean Jordan algebras $J$ of degree three was given. These three-dimensional U-duality groups are $F_{4(4)}$, $E_{6(2)}$, $E_{7(-5)}$, $E_{8(-24)}$ and $SO(n + 2, 4)$. Spherical vectors of the quasiconformal actions of all these groups with respect to their maximal compact subgroups as well as the eigenvalues of their quadratic Casimir operators were also presented in [17]. These results were then extended to the split exceptional groups $E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ and $SO(n + 3, m + 3)$ in [18].

In this paper we give a detailed study of the minrep of $SO(4, 2)$ obtained by quantizing its realization as a quasiconformal group that leaves invariant a quartic light-cone in five dimensions, its deformations and their supersymmetric extensions. The motivations for our work are manifold. First we would like to extend the results of [11,14] to construct the minimal unitary representations of more general noncompact supergroups such as $SU(n, m | p + q)$ in general. Since the group $SU(2, 2)$ is the covering group of $SO(4, 2)$, the family $SU(2, 2 | N)$ corresponds to four dimensional conformal or five dimensional anti-de Sitter superalgebras and have important applications to $AdS_5/CFT_4$ dualities [19]. The noncompact groups that are not of Hermitian symmetric type, in general, admit a unique or at most finitely many minimal unitary representations [7]. The unified approach to minimal unitary representations given

\(^1\)If $H$ is also noncompact then the supergroup $G$ does not admit any unitary representations, in general.

\(^2\)The minrep of $SU(2, 1)$ was constructed earlier in [11].
in [14] is applicable to all noncompact groups including those that are of Hermitian symmetric type. We shall extend the quasi-conformal formalism of [14] by introducing a deformation parameter ζ so as to be able to construct all the “minimal” unitary representations of SU(2, 2), which is of hermitian symmetric type. We shall refer to the representation with ζ = 0 as the minimal unitary representation and the representations with nonzero ζ as deformations of the minimal representation. For each integer value of ζ one obtains a unique unitary irreducible representation of SU(2, 2). We then extend these results to the minimal unitary representations of SU(2, 2 | N) and their deformations. Again in the supersymmetric case, each integer value of the deformation parameter ζ leads to a unique unitary supermultiplet of SU(2, 2 | N)). The minimal unitary supermultiplet of SU(2, 2 | N) and its deformations turn out to be the doubleton supermultiplets that were constructed and studied using the oscillator method [20–22] earlier. Our results extend to the minreps of SU(m, n) and of SU(m, n | p + q) and their deformations in a straightforward manner.

Now the unitary representations of SO(4, 2) or its covering group SU(2, 2) have been studied very extensively over the last half century. The so-called ladder representations of SU(2, 2) constructed using bosonic annihilation and creation operators appeared in the physics literature as early as 1960s in at least three different contexts. First, in the formulation of SO(4, 2) as a spectrum generating symmetry group of the Hydrogen atom [23–27]. Second, in hadron physics as symmetry of infinite component fields [28]. Thirdly in studies of massless wave equations in four dimensional spacetime, for which we refer to [29] and the references cited therein. A full classification of positive energy unitary representations of SU(2, 2) was given in [30], to which we refer for the earlier literature on the subject. A complete classification of all unitary representations (unitary dual) of SU(2, 2), which include the positive energy representations, was given in the mathematics literature [31]. A classification of the positive energy unitary representations of SU(2, 2 | N)) using the formalism of Kac [32, 33] was given in [34, 35].

The minimal unitary representations of symplectic groups Sp(2n, ℝ) are the singleton representations which are known as the metaplectic representations in the mathematics literature. Since the singleton representations of Sp(2n, ℝ) can be realized over the Fock space of bosonic oscillators transforming in the fundamental representation of its maximal compact subgroup U(n), they are also sometimes referred to as the “oscillator representation.” The entire Fock space created by the action of n bosonic creation operators transforming in the fundamental representation of U(n) decomposes into a direct sum of the two singleton representations of Sp(2n, ℝ). Dirac discovered the two singleton representations of the covering group Sp(4, ℝ) of four dimensional anti-de Sitter group SO(3, 2) without using oscillators and referred to them as remarkable representations of anti-de Sitter group [36]. The wave functions corresponding to the remarkable representations do not depend on the radial coordinate of the four dimensional anti-de Sitter space (AdS4), suggesting that they should be interpreted as living on the boundary of AdS4. The term singletons for these remark-

---

3We thank Professor Ivan Todorov for bringing reference [29] to our attention.
able representations of $SO(3, 2)$ was coined by Fronsdal and collaborators later [37–39], who showed that the singleton representations do not have a Poincaré limit. They also showed these representations have the additional remarkable property that by tensoring two copies of the singleton representations one obtains all the massless representations of $SO(3, 2)$ which do have a smooth Poincaré limit.

Using oscillators to construct representations of symmetry groups is a time honored tradition in physics. Here we should stress that using the oscillator method one can construct more general representations than what is commonly referred to as the “oscillator representation(s)” in the mathematics literature. For symplectic groups the term “oscillator representations” typically refers to the singleton (metaplectic) representations of $Sp(2n, \mathbb{R})$. A general method for constructing more general classes of unitary representations of noncompact groups was formulated in [40], which unified and generalized the known constructions in special cases in the physics literature. The formulation of [40] was later extended to give a general method for constructing unitary representations of noncompact supergroups in [41] using bosonic as well as fermionic oscillators. In these generalized formulations of the oscillator method the generators of noncompact groups or supergroups are realized as bilinears of an arbitrary number $P$ (colors) of sets of oscillators transforming in an irreducible representation of their maximal compact subgroups or supergroups. For symplectic groups $Sp(2n, \mathbb{R})$ the minimum possible value of $P$ is one and the resulting unitary representations are simply the singletons. If the minimum allowed value of $P_{\text{min}}$ is two, the resulting unitary representations were later referred to as doubleton representations. For example, the groups $SU(n, m)$ and $SO^*(2n)$, with maximal compact subgroups $SU(m) \times SU(n) \times U(1)$ and $U(n)$, respectively, admit doubleton representations. There exists only two singleton representations of $Sp(2n, \mathbb{R})$, for which the minimum value of $P_{\text{min}}$ is one. When the minimum allowed number $P_{\text{min}}$ of colors is two, one finds an infinite number of doubleton irreducible representations of the respective noncompact groups or supergroups. Since the general oscillator method realizes the generators as bilinear of free bosonic and fermionic oscillators, the tensoring of the resulting representations is very straightforward. Even though the singletons or doubletons themselves do not belong to the discrete series, by tensoring them one obtains unitary representations that belong, in general, to the holomorphic discrete series.

The Kaluza-Klein spectrum of IIB supergravity over the $AdS_5 \times S^5$ space was first obtained via the oscillator method by simple tensoring of the CPT self-conjugate doubleton supermultiplet of $N = 8$, $AdS_5$ superalgebra $SU(2, 2|4)$ with itself repeatedly and restricting to the CPT self-conjugate short supermultiplets of $SU(2, 2|4)$ [20]. The CPT self-conjugate doubleton supermultiplet itself decouples from the Kaluza-Klein spectrum as gauge modes. Again in [20] it was pointed out that the CPT self-conjugate doubleton supermultiplet $SU(2, 2|4)$ does not have a Poincaré limit in five dimensions and its field theory lives on the boundary of $AdS_5$ on which $SO(4, 2)$ acts as a conformal group and that the unique candidate for this theory is the four dimensional $N = 4$ super Yang-Mills theory that is conformally invariant. Analogous results were obtained for the compactifications of 11 dimensional supergravity over $AdS_4 \times S^7$ and $AdS_7 \times S^4$ with the symmetry superalgebras $OSp(8|4, \mathbb{R})$ and $OSp(8^*|4)$
in [42] and [43], respectively. These results have become an integral part of the work on AdS/CFT dualities in M/superstring theory which has seen an exponential growth since the famous paper of Maldacena [19]. The connection between the minimal representations and the more general representations of symmetry groups or supergroups obtained by tensoring them lie at the heart of AdS/CFT dualities in a true Wignerian sense. AdS/CFT dualities have also found applications in different areas of physics over the last decade. These developments show the fundamental importance of the minimal unitary representations of symmetry groups and supergroups in physics.

The plan of our paper is as follows. In section 2 we review the geometric quasiconformal realizations of groups $SO(d + 2, 4)$ as invariance groups of a light-cone with respect to a quartic distance function in $2d + 5$ dimensional space. The minimal unitary realization of the Lie algebra of $SO(d+2, 4)$, obtained by quantizing this geometric action over an Hilbert space of functions in $d + 3$ variables, is reviewed in section 3. We then specialize and study the case of $SO(4, 2)$ in detail. In section 4, we review the minimal unitary realization of $SU(2, 2)$ as a special case of $SU(n, m)$ given in [14] and show that it coincides with the minrep of $SO(4, 2)$. We give the K-type decomposition of the minrep of $SU(2, 2)$ in section 5 and show that it coincides with the K-type decomposition of scalar doubleton representation corresponding to a massless conformal scalar field in 4 dimensions [20–22].

We then show, in section 6, that there exists a one-parameter ($\zeta$) family of deformations of the minrep of $SU(2, 2)$. For every positive (negative) integer value of the deformation parameter $\zeta$ one obtains a positive energy unitary irreducible representation of $SU(2, 2)$ corresponding to a massless conformal field in four dimensions transforming in $\left(0, \frac{\zeta}{2}\right)$ ($\left(-\frac{\zeta}{2}, 0\right)$) representation of $SL(2, \mathbb{C})$ subgroup of $SU(2, 2)$. These are simply the doubleton representations of $SU(2, 2)$. They were referred to as ladder (or most degenerate discrete series) unitary representations by Mack and Todorov who showed that they remain irreducible under restriction to the Poincaré subgroup [29].

In sections 7 and 8 we give the supersymmetric extension of the minrep of $SU(2, 2)$ to the minrep $SU(2, 2|p+q)$ which has a unique irreducible unitary supermultiplet. For $SU(2, 2|4)$ the minimal unitary supermultiplet is simply the unique CPT self-conjugate massless $N = 4$ Yang-Mills supermultiplet in four dimensions [20]. For every deformed minimal unitary irreducible representation of $SU(2, 2)$ there exists a unique extension to a unitary irreducible deformed unitary supermultiplet of $SU(2, 2|p+q)$. For $SU(2, 2|4)$ these supermultiplets turn out to be precisely the doubleton supermultiplets constructed and studied in [21, 22] and correspond to massless conformal supermultiplets involving higher spin fields.

2. Quasiconformal Realizations of $SO(d + 2, 4)$ and Their Minimal Unitary Representations

2.1 Geometric realizations of $SO(d + 2, 4)$ as quasiconformal groups

Lie algebra of $SO(d + 2, 4)$ can be given a 5-graded decomposition with respect to its subal-
genera $\mathfrak{so}(d,2) \oplus \mathfrak{so}(1,1)$ \[10\]

$\mathfrak{so}(d+2, 4) = 1(-2) \oplus (d+2, 2)^{(-1)} \oplus (\Delta \oplus \mathfrak{sp}(2, \mathbb{R}) \oplus \mathfrak{so}(d,2)) \oplus (d+2,2)^{(1)} \oplus 1^{(2)}$ \[2.1\]

where $\Delta$ is the $SO(1,1)$ generator that determines the five grading and non-zero exponents $m$ label the grade of a generator

$$[\Delta, g^{(m)}] = m g^{(m)}.$$ 

Generators are realized as differential operators acting on a $(2d+5)$ dimensional space $T$ corresponding to the Heisenberg subalgebra generated by elements of $(g^{(-2)} \oplus g^{(-1)})$ subspace, whose coordinates we shall denote as $X = (X^{a}, x)$, where $X^{a}$ transform in the $(d+2,2)$ representation of $SO(d,2) \times Sp(2,\mathbb{R})$, with $a = 1, 2$ and $\mu = 1, 2, \ldots, d+2$, and $x$ is a singlet coordinate.

Let $\epsilon_{ab}$ be the symplectic metric of $Sp(2,\mathbb{R})$, and $\eta_{\mu\nu}$ the $SO(d,2)$ invariant metric ($\eta_{\mu\nu} = (-, -, \ldots, -, +, +)$). Then the quartic polynomial in $X^{a}$

$$\mathcal{I}_{4}(X) = \eta_{\mu\nu}\eta_{\rho\tau}\epsilon_{ac}\epsilon_{bd}X^{\mu,a}X^{\nu,b}X^{\rho,c}X^{\tau,d}$$ \[2.2\]

is invariant under $SO(d,2) \times Sp(2,\mathbb{R})$ subgroup.

We shall label the generators belonging to various grade subspaces as follows

$$\mathfrak{so}(d+2, 4) = K_{-} \oplus U_{\mu,a} \oplus (\Delta + M_{\mu\nu} + J_{ab}) \oplus \tilde{U}_{\mu,a} \oplus K_{+}$$ \[2.3\]

where $M_{\mu\nu}$ and $J_{ab}$ are the generators of $SO(d,2)$ and $Sp(2,\mathbb{R})$ subgroups, respectively. The infinitesimal generators of the quasiconformal action of $SO(d+2,4)$ is then given by

$$K_{+} = \frac{1}{2}(2x^{2} - \mathcal{I}_{4}) \frac{\partial}{\partial x} - \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\tau}\epsilon_{ac}\epsilon_{bd}\frac{\partial}{\partial X^{\mu,a}}\frac{\partial}{\partial X^{\nu,b}} + xX^{a}\frac{\partial}{\partial X^{a}}$$

$$U_{\mu,a} = \frac{\partial}{\partial X^{\mu,a}} - \eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\frac{\partial}{\partial x}$$

$$M_{\mu\nu} = \eta_{\mu\rho}X^{\nu,a}\frac{\partial}{\partial X^{\rho,a}} - \eta_{\nu\rho}X^{\mu,a}\frac{\partial}{\partial X^{\rho,a}}$$

$$J_{ab} = \epsilon_{ac}X^{\mu,c}\frac{\partial}{\partial X^{\mu,b}} + \epsilon_{bc}X^{\mu,c}\frac{\partial}{\partial X^{\mu,a}}$$

$$K_{-} = \frac{\partial}{\partial x} \Delta = 2x\frac{\partial}{\partial x} + X^{a}\frac{\partial}{\partial X^{a}} \tilde{U}_{\mu,a} = [U_{\mu,a}, K_{+}]$$

where $\epsilon_{ab}$ denotes the inverse symplectic metric, such that $\epsilon_{ab}\epsilon_{bc} = \delta_{a}^{c}$. Using

$$\frac{\partial \mathcal{I}_{4}}{\partial X^{\mu,a}} = -4\eta_{\mu\nu}\eta_{\lambda\rho}X^{\nu,b}X^{\lambda,c}X^{\rho,d}\epsilon_{bc}\epsilon_{ad}$$

one obtains the explicit form of $\tilde{U}_{\mu,a}$

$$\tilde{U}_{\mu,a} = \eta_{\mu\nu}\epsilon_{ad}\left(\eta_{\lambda\rho}\epsilon_{bc}X^{\nu,b}X^{\lambda,c}X^{\rho,d} - xX^{\nu,d}\right)\frac{\partial}{\partial x} + x\frac{\partial}{\partial X^{\mu,a}}$$

$$- \eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}X^{\rho,c}\frac{\partial}{\partial X^{\rho,c}} - \epsilon_{ad}\eta_{\lambda\rho}X^{\nu,d}X^{\lambda,c}\frac{\partial}{\partial X^{\mu,c}}$$

$$+ \epsilon_{ad}\eta_{\lambda\rho}X^{\rho,d}X^{\nu,b}\frac{\partial}{\partial X^{\rho,b}} + \eta_{\mu\nu}\epsilon_{bc}X^{\nu,b}X^{\rho,c}\frac{\partial}{\partial X^{\rho,a}}. $$

\[2.5\]
These generators satisfy the following commutation relations:

\[
\begin{align*}
[M_{\mu\nu}, M_{\rho\tau}] &= \eta_{\nu\rho} M_{\mu\tau} - \eta_{\mu\rho} M_{\nu\tau} + \eta_{\mu\tau} M_{\nu\rho} - \eta_{\nu\tau} M_{\mu\rho} \\
[J_{ab}, J_{cd}] &= \epsilon_{cb} J_{ad} + \epsilon_{ca} J_{bd} + \epsilon_{db} J_{ac} + \epsilon_{da} J_{bc} \\
[\Delta, K_{\pm}] &= \pm 2 K_{\pm} \quad [K_{\pm}, K_{\mp}] = \Delta \\
[\Delta, U_{\mu,a}] &= -U_{\mu,a} \quad [\Delta, \tilde{U}_{\mu,a}] = \tilde{U}_{\mu,a} \\
[U_{\mu,a}, K_{\pm}] &= \tilde{U}_{\mu,a} \quad [\tilde{U}_{\mu,a}, K_{\mp}] = -U_{\mu,a} \\
[U_{\mu,a}, U_{\nu,b}] &= 2 \eta_{\mu\nu} \epsilon_{ab} K_{-} \quad [\tilde{U}_{\mu,a}, \tilde{U}_{\nu,b}] = 2 \eta_{\mu\nu} \epsilon_{ab} K_{+} \\
[M_{\mu\nu}, U_{\rho,a}] &= \eta_{\nu\rho} U_{\mu,a} - \eta_{\mu\rho} U_{\nu,a} \quad [M_{\mu\nu}, \tilde{U}_{\rho,a}] = \eta_{\nu\rho} \tilde{U}_{\mu,a} - \eta_{\mu\rho} \tilde{U}_{\nu,a} \\
[J_{ab}, U_{\mu,c}] &= \epsilon_{cb} U_{\mu,a} + \epsilon_{ca} U_{\mu,b} \quad [J_{ab}, \tilde{U}_{\mu,c}] = \epsilon_{cb} \tilde{U}_{\mu,a} + \epsilon_{ca} \tilde{U}_{\mu,b} \\
[U_{\mu,a}, \tilde{U}_{\nu,b}] &= \eta_{\mu\nu} \epsilon_{ab} \Delta - 2 \epsilon_{ab} M_{\mu\nu} - \eta_{\mu\nu} J_{ab} \quad (2.6a) \\
\end{align*}
\]

One defines the “length” (norm) of a vector \( X = (X^{\mu,a}, x) \) as

\[
\ell (X) = \mathcal{I}_4 (X) + 2 x^2 \quad (2.7)
\]

and the “symplectic” difference of two vectors \( X \) and \( Y \) in the \((2d + 5)\) dimensional space \( T \) as

\[
\delta (X, Y) = \left( X^{\mu,a} - Y^{\mu,a}, x - y - \eta_{\mu\nu} \epsilon_{ab} X^{\mu,a} Y^{\nu,b} \right). \quad (2.8)
\]

The “quartic distance” between any two points labelled by vectors \( X \) and \( Y \) is defined as

\[
d(X, Y) := \ell (\delta (X, Y)). \quad (2.9)
\]

Under the quasiconformal action of the generators of \( SO(d + 2, 4) \) the distance function transforms as

\[
\Delta d (X, Y) = 4 d (X, Y) \\
\tilde{U}_{\mu,a} d (X, Y) = -2 \eta_{\mu\nu} \epsilon_{ab} \left( X^{\nu,b} + Y^{\nu,b} \right) d (X, Y) \\
K_{\pm} d (X, Y) = 2 (x + y) d (X, Y) \\
M_{\mu\nu} d (X, Y) = 0 \\
J_{ab} d (X, Y) = 0 \\
U_{\mu,a} d (X, Y) = 0 \\
K_{-} d (X, Y) = 0. \quad (2.10)
\]

They imply that light-like separations

\[
d (X, Y) = 0
\]
are left invariant under the quasiconformal action. In other words, quasiconformal action of \( SO(d+2,4) \) leaves the light-cone with respect to the quartic distance function invariant.

By replacing the \( SO(d,2) \) invariant metric \( \eta_{\mu\nu} \) by an \( SO(p,q) \) invariant metric in the above construction, one can obtain the quasiconformal realization of \( SO(p+2,q+2) \) in a straightforward manner. Of these noncompact real forms, only the groups of the form \( SO(n,4) \) admit quaternionic discrete series representations and the groups of the form \( SO(m,2) \) admit holomorphic discrete series representations. Of course, the group \( SO(4,2) \) admits quaternionic as well as holomorphic discrete series representations.

### 2.2 Minimal unitary representations of \( SO(d+2,4) \) from the quantization of their quasiconformal realizations

Minimal unitary representations of noncompact groups can be obtained by the quantization of their geometric realizations as quasiconformal groups \([10,11,13–15]\). In this section we shall review the minimal unitary representations of quaternionic orthogonal groups \( SO(d+2,4) \) obtained by the quantization of their geometric realizations as quasiconformal groups given in the previous section following \([10,14]\) closely. Let \( X^\mu \) and \( P_\mu \) be the quantum mechanical coordinate and momentum operators on \( \mathbb{R}^{(2,d)} \) satisfying the canonical commutation relations

\[
[X^\mu, P_\nu] = i \delta^\mu_\nu . \tag{2.11}
\]

The grade \(-2\) and \(-1\) generators of \( SO(d+2,4) \) form an Heisenberg algebra

\[
[U_{\mu,a}, U_{\nu,b}] = 2 \eta_{\mu\nu} \epsilon_{ab} K_- \tag{2.12}
\]

with \( K_- \) playing the role of the central charge. We shall relabel the generators and define

\[
U_{\mu,1} \equiv U_\mu \quad U_{\mu,2} \equiv V_\mu \tag{2.13}
\]

and realize the Heisenberg algebra (2.12) in terms of coordinate and momentum operators \( X^\mu, P_\mu \) and an extra “central charge coordinate” \( x \):

\[
U_\mu = x P_\mu \quad V^\mu = x X^\mu \quad K_- = \frac{1}{2} x^2 \tag{2.14}
\]

\[
[V^\mu, U_\nu] = 2i \delta^\mu_\nu K_- \tag{2.15}
\]

By introducing the quantum mechanical momentum operator \( p \) conjugate to the central charge coordinate \( x \)

\[
[x, p] = i \tag{2.16}
\]
one can realize the grade zero generators of $SO(d+2,4)$ as bilinears of canonically conjugate pairs of coordinates and momenta [10, 14]:

\[
M_{\mu\nu} = i \eta_{\mu\rho} X^\rho P_\nu - i \eta_{\nu\rho} X^\rho P_\mu \\
J_0 = \frac{1}{2} (X^\mu P_\mu + P_\mu X^\mu) \\
J_- = X^\mu X^\nu \eta_{\mu\nu} \\
J_+ = P_\mu P_\nu \eta^{\mu\nu} \\
\Delta = \frac{1}{2} (xp + px)
\] (2.17)

They satisfy the commutation relations

\[
[M_{\mu\nu}, M_{\rho\tau}] = \eta_{\nu\rho} M_{\mu\tau} - \eta_{\mu\rho} M_{\nu\tau} + \eta_{\mu\tau} M_{\nu\rho} - \eta_{\nu\tau} M_{\mu\rho} \\
[J_0, J_\pm] = \pm 2i J_\pm \quad [J_-, J_+] = 4i J_0 .
\] (2.18)

The coordinate $X^\mu$ and momentum $P_\mu$ operators transform contravariantly and covariantly under $SO(d,2)$ subgroup generated by $M_{\mu\nu}$, respectively, and form doublets of the symplectic group $Sp(2, \mathbb{R})$:

\[
[J_0, V^\mu] = -i V^\mu \quad [J_-, V^\mu] = 0 \quad [J_+, V^\mu] = -2i \eta^{\mu\nu} U_\nu \\
[J_0, U_\mu] = +i U_\mu \quad [J_-, U_\mu] = 2i \eta_{\mu\nu} V^\nu \quad [J_+, U_\mu] = 0
\] (2.19)

There is a normal ordering ambiguity in defining the quantum operator corresponding to the quartic invariant. We shall choose the quantum quartic invariant [10]

\[
\mathcal{I}_4 = (X^\mu X^\nu \eta_{\mu\nu})(P_\mu P_\rho \eta^{\mu\rho}) + (P_\mu P_\nu \eta^{\mu\nu})(X^\mu X^\nu \eta_{\mu\nu}) - (X^\mu P_\mu)(P_\nu X^\nu) - (P_\mu X^\mu)(X^\nu P_\nu).
\] (2.20)

Using the quartic invariant, one defines the grade +2 generator as

\[
K_+ = \frac{1}{2} p^2 + \frac{1}{4x^2} \left( \mathcal{I}_4 + \frac{(d+2)^2 + 3}{2} \right) .
\] (2.21)

Then the grade +1 generators are obtained by commutations

\[
\tilde{V}^\mu = -i [V^\mu, K_+] \quad \tilde{U}_\mu = -i [U_\mu, K_+]
\] (2.22)

which explicitly read as follows

\[
\tilde{V}^\mu = p X^\mu + \frac{1}{2x} \left( P_\nu X^\lambda X^\rho + X^\lambda X^\rho P_\nu \right) \eta^{\mu\nu} \eta_{\lambda\rho} - \frac{1}{4x} [X^\mu (X^\nu P_\nu + P_\nu X^\nu) + (X^\nu P_\nu + P_\nu X^\nu) X^\mu] \\
\tilde{U}_\mu = p P_\mu - \frac{1}{2x} (X^\nu P_\lambda P_\rho + P_\lambda P_\rho X^\nu) \eta_{\mu\nu} \eta^{\lambda\rho} + \frac{1}{4x} [P_\mu (X^\nu P_\nu + P_\nu X^\nu) + (X^\nu P_\nu + P_\nu X^\nu) P_\mu] .
\] (2.23)
The generators in \( g^{+1} \oplus g^{+2} \) subspace form an Heisenberg algebra isomorphic to \((2.12)\)
\[
\begin{align*}
\tilde{V}^{\mu}, \tilde{U}_{\nu} &= 2i \delta^{\mu\nu} K_+ \quad V^{\mu} = i \left[ \tilde{V}^{\mu}, K_- \right] \quad U_{\mu} = i \left[ \tilde{U}_{\mu}, K_- \right].
\end{align*}
\]
Commutators \([g^{-1}, g^{+1}]\) close into grade zero subspace \( g^0 \):
\[
\begin{align*}
\left[ U_{\mu}, \tilde{V}_{\nu} \right] &= i \eta_{\mu\nu} J_+ \\
\left[ V^{\mu}, \tilde{V}_{\nu} \right] &= 2 \eta^{\mu\rho} M_{\rho\nu} + i \delta^{\mu\nu} (J_0 + \Delta) \\
\left[ U_{\mu}, \tilde{U}_{\nu} \right] &= -2 \eta^{\mu\rho} M_{\rho\nu} + i \delta^{\nu\mu} (J_0 - \Delta)
\end{align*}
\]
\(\Delta\) is the generator that determines the 5-grading:
\[
\begin{align*}
\left[ K_-, K_+ \right] &= i \Delta \\
\left[ \Delta, K_\pm \right] &= \pm 2i K_\pm \\
\left[ \Delta, U_\mu \right] &= -i U_\mu \\
\left[ \Delta, V^{\mu} \right] &= -i V^{\mu}
\end{align*}
\]
The quadratic Casimir operators of subalgebras \(so(d, 2), sp(2, \mathbb{R})_J\) generated by \(J_{ab}\) of grade-zero subspace, and \(sp(2, \mathbb{R})_K\) generated by \(K_{\pm}\) and \(\Delta\) are given by
\[
\begin{align*}
M_{\mu\nu} M^{\mu\nu} &= -I_4 - 2 (d + 2) \\
J_- J_+ + J_+ J_- - 2 (J_0)^2 &= I_4 + \frac{1}{2} (d + 2)^2 \\
K_- K_+ + K_+ K_- - \frac{1}{2} \Delta^2 &= \frac{1}{4} I_4 + \frac{1}{8} (d + 2)^2.
\end{align*}
\]
They all reduce to the quartic invariant operator \(I_4\) modulo some additive constants. Furthermore, the following identity satisfied by the bilinears of grade \(\pm 1\) generators
\[
\left( U_\mu \tilde{V}^{\mu} + \tilde{V}^{\mu} U_\mu - V^{\mu} \tilde{U}_{\mu} - \tilde{U}_{\mu} V^{\mu} \right) = 2I_4 + (d + 2) (d + 6)
\]
prove the existence of a family of degree 2 polynomials in the enveloping algebra of \(so(d + 2, 4)\) that degenerate to a \(c\)-number for the minimal unitary realization, in accordance with Joseph’s theorem [1]:
\[
\begin{align*}
M_{\mu\nu} M^{\mu\nu} + \kappa_1 \left( J_- J_+ + J_+ J_- - 2 (J_0)^2 \right) + 4 \kappa_2 \left( K_- K_+ + K_+ K_- - \frac{1}{2} \Delta^2 \right) \\
- \frac{1}{2} (\kappa_1 + \kappa_2 - 1) \left( U_\mu \tilde{V}^{\mu} + \tilde{V}^{\mu} U_\mu - V^{\mu} \tilde{U}_{\mu} - \tilde{U}_{\mu} V^{\mu} \right)
\end{align*}
\]
\[
= \frac{1}{2} \left( d + 2 \right) \left( d + 2 - 4 (\kappa_1 + \kappa_2) \right)
\]
The quadratic Casimir of \(so(d + 2, 4)\) corresponds to the choice \(2\kappa_1 = 2\kappa_2 = -1\) in \((2.24)\). Hence the eigenvalue of the quadratic Casimir for the minimal unitary representation is equal to \(\frac{1}{2} (d + 2) (d + 6)\). This minimal unitary representation is realized on the Hilbert space of square integrable functions in \((d + 3)\) variables.
3. Minimal Unitary Realization of the 4D Conformal Group $SO(4, 2)$ over the Hilbert Space of $L^2$ Functions of Three Variables

With applications to AdS/CFT dualities in mind, we shall study the minrep of $SO(4, 2)$ in detail. By setting $d = 0$ in the construction of previous section we get the following 5-graded decomposition of $SO(4, 2)$ generators in the minrep:

$$so(2, 4) = K_- \oplus [U_\mu \oplus V^\mu] \oplus \{J_0, \pm \oplus M_{12}\} \oplus \{\bar{U}_\mu \oplus \bar{V}^\mu\} \oplus K_+ \quad (3.1)$$

where $\mu, \nu, \cdots = 1, 2$ and $\eta_{\mu\nu} = \delta_{\mu\nu}$. The 5-grading is determined by the $SO(1, 1)$ generator

$$\Delta = \frac{1}{2} (xp + px) .$$

On the other hand, $so(4, 2)$ has a 3-grading

$$so(2, 4) = \mathfrak{N}^- \oplus \mathfrak{N}^0 \oplus \mathfrak{N}^+ \quad (3.2)$$

with respect to the noncompact generator

$$D = \Delta + J_0 = \frac{1}{2} (xp + px + X^\mu P_\mu + P_\mu X^\mu) . \quad (3.3)$$

Explicitly we have

$$so(2, 4) = [K_- \oplus J_- \oplus V^\mu] \oplus [D \oplus B \oplus M_{\mu\nu} \oplus U_\mu \oplus \bar{V}^\mu] \oplus \{\bar{U}_\mu \oplus J_+ \oplus K_+\} \quad (3.4)$$

where $B = \Delta - J_0$ and

$$\mathfrak{N}^0 = so(3, 1) \oplus so(1, 1)_D .$$

The generators of $so(3, 1)$ subalgebra are $B, M_{\mu\nu}, U_\mu$ and $\bar{V}^\mu$. We shall refer to this as the noncompact 3-grading.

Furthermore, the Lie algebra of $so(2, 4)$ has a 3-grading with respect to the compact generator

$$H = \frac{1}{2} \left( (K_+ + K_-) + \frac{1}{2} (J_+ + J_-) \right) \quad (3.5)$$

such that

$$so(2, 4) = \mathfrak{C}^- \oplus [so(4) \oplus so(2)] \oplus \mathfrak{C}^+ . \quad (3.6)$$

In this decomposition,

$$\mathfrak{C}^0 = so(4) \oplus so(2) = [M_{\mu\nu} \oplus \left( (K_+ + K_-) - \frac{1}{2} (J_+ + J_-) \right) \oplus \left( U_\mu - \eta_{\mu\nu} \bar{V}^\nu \right) \oplus \left( \bar{U}_\mu + \eta_{\mu\nu} V^\nu \right) \oplus \frac{1}{2} \left( (K_+ + K_-) + \frac{1}{2} (J_+ + J_-) \right)$$

$$\mathfrak{C}^+ = [\Delta - i (K_+ - K_-)] \oplus [J_0 - \frac{i}{2} (J_+ - J_-)] \oplus \left[ \frac{1}{2} \left( U_\mu + \eta_{\mu\nu} \bar{V}^\nu \right) - \frac{i}{2} \left( \bar{U}_\mu - \eta_{\mu\nu} V^\nu \right) \right]$$

$$\mathfrak{C}^- = [\Delta + i (K_+ - K_-)] \oplus [J_0 + \frac{i}{2} (J_+ - J_-)] \oplus \left[ \frac{1}{2} \left( U_\mu + \eta_{\mu\nu} \bar{V}^\nu \right) + \frac{i}{2} \left( \bar{U}_\mu - \eta_{\mu\nu} V^\nu \right) \right] . \quad (3.7)$$
We shall refer to this grading as the compact 3-grading.

The \( \mathfrak{so}(4) \) generators in the subspace \( \mathfrak{c}^0 \) are given by

\[
\begin{align*}
\tilde{M}_{4\mu} &:= \frac{1}{2} \left( U_{\mu} - \eta_{\mu\nu} \tilde{V}^\nu \right) & \tilde{M}_{12} &:= -i M_{12} \\
\tilde{M}_{\mu3} &:= \frac{1}{2} \left( \tilde{U}_{\mu} + \eta_{\mu\nu} V^\nu \right) & \tilde{M}_{43} &:= \frac{1}{2} \left[ (K_+ + K_-) - \frac{1}{2} (J_+ + J_-) \right]
\end{align*}
\]

and satisfy the \( \mathfrak{so}(4) \) algebra

\[
\left[ \tilde{M}_{AB}, \tilde{M}_{CD} \right] = i \left( \delta_{AC} \tilde{M}_{BD} - \delta_{AD} \tilde{M}_{BC} - \delta_{BC} \tilde{M}_{AD} + \delta_{BD} \tilde{M}_{AC} \right)
\]

where \( A, B, \cdots = 1, 2, 3, 4 \).

To analyse the decomposition of the minimal unitary representation of \( SO(2, 4) \) into K-finite vectors of its maximal compact subgroup, let us introduce the oscillators

\[
\begin{align*}
a &:= \frac{1}{\sqrt{2}} (x + ip) & b &:= \frac{1}{\sqrt{2}} (X^1 + i P_1) & c &:= \frac{1}{\sqrt{2}} (X^2 + i P_2) \\
a^\dagger &:= \frac{1}{\sqrt{2}} (x - ip) & b^\dagger &:= \frac{1}{\sqrt{2}} (X^1 - i P_1) & c^\dagger &:= \frac{1}{\sqrt{2}} (X^2 - i P_2)
\end{align*}
\]

so that

\[
\begin{align*}
x &= \frac{1}{\sqrt{2}} (a^\dagger + a) & X^1 &= \frac{1}{\sqrt{2}} (b^\dagger + b) & X^2 &= \frac{1}{\sqrt{2}} (c^\dagger + c) \\
p &= \frac{i}{\sqrt{2}} (a^\dagger - a) & P_1 &= \frac{i}{\sqrt{2}} (b^\dagger - b) & P_2 &= \frac{i}{\sqrt{2}} (c^\dagger - c).
\end{align*}
\]

These oscillators satisfy the commutation relations:

\[
\begin{align*}
[a, a^\dagger] &= 1 & [b, b^\dagger] &= 1 & [c, c^\dagger] &= 1 \\
[x, p] &= i & [X^1, P_1] &= i & [X^2, P_2] &= i
\end{align*}
\]

The quartic invariant operator \( \mathcal{I}_4 \) takes on a simple form in terms of the oscillators \( b, c \):

\[
\mathcal{I}_4 = -2 \left( b^\dagger c - bc^\dagger \right)^2 - 4
\]

The \( \mathfrak{so}(2) \) generator in \( \mathfrak{c}^0 \), which plays the role of the AdS energy \([20–22]\), is given by:

\[
H = \frac{1}{2} \left[ (K_+ + K_-) + \frac{1}{2} (J_+ + J_-) \right]
\]

\[
= \frac{1}{2} \left[ a^\dagger a + b^\dagger b + c^\dagger c - \frac{1}{2} x^2 \left( b^\dagger c - bc^\dagger \right)^2 - \frac{1}{8} x^2 + \frac{3}{2} \right]
\]

\[ -13 - \]
while the $\mathfrak{so}(4)$ generators in terms of these oscillators become

$$\hat{M}_{12} = -iM_{12}$$
$$= -i \left( b^\dagger c - bc^\dagger \right)$$

$$\hat{M}_{43} = \frac{1}{2} \left( (K_+ + K_-) - \frac{1}{2} (J_+ + J_-) \right)$$
$$= \frac{1}{2} \left( a^\dagger a - b^\dagger b - c^\dagger c \right) - \frac{i}{4x^2} \left( b^\dagger c - bc^\dagger \right)^2 - \frac{1}{16x^2} - \frac{1}{4}$$

$$\hat{M}_{41} = \frac{1}{2} \left( U_1 - \tilde{V}^1 \right)$$
$$= \frac{i}{2} \left( ab^\dagger - a^\dagger b \right) - \frac{i}{2\sqrt{2}x} \left( b^\dagger c - bc^\dagger \right) \left( c^\dagger + c \right) + \frac{i}{4\sqrt{2}x} \left( b^\dagger + b \right)$$

$$\hat{M}_{42} = \frac{1}{2} \left( U_2 - \tilde{V}^2 \right)$$
$$= \frac{i}{2} \left( ac^\dagger - a^\dagger c \right) + \frac{i}{2\sqrt{2}x} \left( b^\dagger c - bc^\dagger \right) \left( b^\dagger + b \right) + \frac{i}{4\sqrt{2}x} \left( c^\dagger + c \right)$$

$$\hat{M}_{13} = \frac{1}{2} \left( \tilde{U}_1 + V^1 \right)$$
$$= \frac{1}{2} \left( ab^\dagger + a^\dagger b \right) - \frac{1}{2\sqrt{2}x} \left( b^\dagger c - bc^\dagger \right) \left( c^\dagger - c \right) + \frac{1}{4\sqrt{2}x} \left( b^\dagger - b \right)$$

$$\hat{M}_{23} = \frac{1}{2} \left( \tilde{U}_2 + V^2 \right)$$
$$= \frac{1}{2} \left( ac^\dagger + a^\dagger c \right) + \frac{1}{2\sqrt{2}x} \left( b^\dagger c - bc^\dagger \right) \left( b^\dagger - b \right) + \frac{1}{4\sqrt{2}x} \left( c^\dagger - c \right).$$

It is useful to list the following commutators between $a, a^\dagger$ and $1/x, 1/x^2$:

$$\begin{align*}
[a, \frac{1}{x}] &= -\frac{1}{\sqrt{2}x} \\
[a^\dagger, \frac{1}{x}] &= \frac{1}{\sqrt{2}x} \\
[a, \frac{1}{x^2}] &= -\frac{\sqrt{2}}{x^3} \\
[a^\dagger, \frac{1}{x^2}] &= \frac{\sqrt{2}}{x^3}
\end{align*}$$

(3.16)

The generators that belong to the grade $+1$ subspace $\mathfrak{c}^+$ have the following form:

$$\Delta - i(K_+ - K_-) = i a^\dagger a^\dagger + \frac{i}{2x^2} \left( b^\dagger c - bc^\dagger \right)^2 + \frac{i}{8x^2}$$

$$\frac{1}{2} \left[ \left( U_1 + \tilde{V}^1 \right) - i \left( \tilde{U}_1 - V^1 \right) \right] = i a^\dagger b^\dagger + \frac{i}{2\sqrt{2}x} \left[ c^\dagger \left( b^\dagger c - bc^\dagger \right) + \left( b^\dagger c - bc^\dagger \right) c^\dagger \right]$$

$$\frac{1}{2} \left[ \left( U_2 + \tilde{V}^2 \right) - i \left( \tilde{U}_2 - V^2 \right) \right] = i a^\dagger c^\dagger - \frac{i}{2\sqrt{2}x} \left[ b^\dagger \left( b^\dagger c - bc^\dagger \right) + \left( b^\dagger c - bc^\dagger \right) b^\dagger \right]$$

$$J_0 - \frac{i}{2} (J_+ - J_-) = i \left( b^\dagger b^\dagger + c^\dagger c^\dagger \right)$$

(3.17)
and those that belong to grade $-1$ subspaces $\mathfrak{c}^-$ are:

$$
\Delta + i (K_+ - K_-) = -i a a - \frac{i}{2 x^2} \left( b^\dagger c - bc^\dagger \right)^2 - \frac{i}{8 x^2}
$$

$$
\frac{1}{2} \left[ \left( U_1 + \tilde{V}^1 \right) + i \left( \tilde{U}_1 - V^1 \right) \right] = -i a b + \frac{i}{2 \sqrt{2} x} \left( c (b^\dagger c - bc^\dagger) + (b^\dagger c - bc^\dagger) c \right)
$$

$$
\frac{1}{2} \left[ \left( U_2 + \tilde{V}^2 \right) + i \left( \tilde{U}_2 - V^2 \right) \right] = -i a c - \frac{i}{2 \sqrt{2} x} \left[ b (b^\dagger c - bc^\dagger) + (b^\dagger c - bc^\dagger) b \right]
$$

$$
J_0 + \frac{i}{2} (J_+ - J_-) = -i (b b + c c)
$$

They satisfy

$$
[H, \mathfrak{c}^+] = + \mathfrak{c}^+ \qquad [H, \mathfrak{c}^-] = - \mathfrak{c}^-.
$$

Now the Lie algebra of $SO(4)$ is not simple and can be written as a direct sum

$$
\mathfrak{so}(4) = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R
$$

where the generators of the two $\mathfrak{su}(2)$ subalgebras are as follows:

$$
L_1 = \frac{1}{2} (\tilde{M}_{23} - \tilde{M}_{41}) \quad L_2 = \frac{1}{2} (\tilde{M}_{31} - \tilde{M}_{42}) \quad L_3 = \frac{1}{2} (\tilde{M}_{12} - \tilde{M}_{43})
$$

$$
R_1 = \frac{1}{2} (\tilde{M}_{23} + \tilde{M}_{41}) \quad R_2 = -\frac{1}{2} (\tilde{M}_{31} + \tilde{M}_{42}) \quad R_3 = -\frac{1}{2} (\tilde{M}_{12} + \tilde{M}_{43})
$$

Therefore the raising and lowering generators of the two $\mathfrak{su}(2)$'s are given by:

$$
L_+ = \frac{1}{\sqrt{2}} (L_1 + i L_2) = -\frac{i}{2 \sqrt{2}} \left[ a + \frac{i}{\sqrt{2} x} \left( b^\dagger c - bc^\dagger \right) + \frac{1}{2 \sqrt{2} x} \right] \left( b^\dagger + i c^\dagger \right)
$$

$$
L_- = \frac{1}{\sqrt{2}} (L_1 - i L_2) = \frac{i}{2 \sqrt{2}} \left[ a^\dagger + \frac{i}{\sqrt{2} x} \left( b^\dagger c - bc^\dagger \right) - \frac{1}{2 \sqrt{2} x} \right] \left( b - i c \right)
$$

$$
R_+ = \frac{1}{\sqrt{2}} (R_1 + i R_2) = \frac{i}{2 \sqrt{2}} \left[ a - \frac{i}{\sqrt{2} x} \left( b^\dagger c - bc^\dagger \right) + \frac{1}{2 \sqrt{2} x} \right] \left( b^\dagger - i c^\dagger \right)
$$

$$
R_- = \frac{1}{\sqrt{2}} (R_1 - i R_2) = -\frac{i}{2 \sqrt{2}} \left[ a^\dagger - \frac{i}{\sqrt{2} x} \left( b^\dagger c - bc^\dagger \right) - \frac{1}{2 \sqrt{2} x} \right] \left( b + i c \right)
$$

while the remaining generators are given by:

$$
L_3 = -\frac{1}{4} \left( a^\dagger a - b^\dagger b - c^\dagger c \right) + \frac{1}{8 x^2} \left( b^\dagger c - bc^\dagger \right)^2 - \frac{i}{2} \left( b^\dagger c - bc^\dagger \right) + \frac{1}{32 x^2} + \frac{1}{8}
$$

$$
R_3 = -\frac{1}{4} \left( a^\dagger a - b^\dagger b - c^\dagger c \right) + \frac{1}{8 x^2} \left( b^\dagger c - bc^\dagger \right)^2 + \frac{i}{2} \left( b^\dagger c - bc^\dagger \right) + \frac{1}{32 x^2} + \frac{1}{8}
$$

They satisfy the commutation relations:

$$
[L_+, L_-] = L_3 \quad [L_3, L_{\pm}] = \pm L_{\pm}
$$

$$
[R_+, R_-] = R_3 \quad [R_3, R_{\pm}] = \pm R_{\pm}
$$
Interestingly the two quadratic Casimir operators
\[ L^2 = L_+ L_+ + L_- L_- + L_3^2 \quad \quad R^2 = R_+ R_+ + R_- R_- + R_3^2 \] (3.25)

turn out to be equal and are given by
\[
L^2 = R^2 = \frac{1}{16} \left( a^\dagger a + b^\dagger b + c^\dagger c \right)^2 + \frac{3}{16} \left( a^\dagger a + b^\dagger b + c^\dagger c \right) - \frac{1}{64 x^2} \left( a^\dagger a + b^\dagger b + c^\dagger c \right)
\]
\[
- \frac{1}{16 x^2} \left( a^\dagger a + b^\dagger b + c^\dagger c \right) \left( b^\dagger c - bc^\dagger \right)^2 + \frac{\sqrt{2}}{32 x^3} \left( a^\dagger - a \right) \left( b^\dagger c - bc^\dagger \right)^2
\]
\[
+ \frac{\sqrt{2}}{128 x^3} \left( a^\dagger - a \right) - \frac{1}{64 x^4} \left( b^\dagger c - bc^\dagger \right)^4 + \frac{13}{128 x^4} \left( b^\dagger c - bc^\dagger \right)^2
\]
\[
- \frac{3}{32 x^2} \left( b^\dagger c - bc^\dagger \right)^2 + \frac{25}{1024 x^4} - \frac{3}{128 x^2} - \frac{7}{64}
\]
\[
= \frac{1}{16} \left[ a^\dagger a + b^\dagger b + c^\dagger c - \frac{1}{2 x^2} \left( b^\dagger c - bc^\dagger \right)^2 - \frac{1}{8 x^2} + \frac{3}{2} \right]^2 - \frac{1}{4}
\]
\[
= \frac{1}{4} (H^2 - 1) = \mathcal{J} (\mathcal{J} + 1)
\]

with
\[
\mathcal{J} = \frac{1}{2} (H - 1)
\] (3.27)

where \( H \) is the \( \mathfrak{so}(2) \) generator, as given in equation (3.14).

4. Change of Basis and the Minimal Unitary Realization of \( SU(2, 2) \)

4.1 Minimal unitary realization of \( \mathfrak{su}(2, 2) \)

Minimal unitary realizations of \( SU(n, m) \) obtained from quantization of their quasiconformal realizations were given in [14], which we review here for \( SU(2, 2) \). The Lie algebra \( \mathfrak{su}(2, 2) \) admits a 5-grading with respect to its subalgebra \( \mathfrak{su}(1, 1) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(1, 1) \):
\[
\mathfrak{su}(2, 2) = 1^{(-2)} \oplus 4^{(-1)} \oplus [\mathfrak{su}(1, 1) \oplus \mathfrak{u}(1) \oplus \Delta] \oplus 4^{(+1)} \oplus 1^{(+2)} \] (4.1)

where \( J^m, U \) and \( \Delta \) are the \( SU(1, 1) \), \( U(1) \) and \( SO(1, 1) \) generators, respectively. In [14] the corresponding generators are labelled as
\[
\mathfrak{su}(2, 2) = E \oplus (E^1, E^2, E_1, E_2) \oplus [J^m, U, \Delta] \oplus (F^1, F^2, F_1, F_2) \oplus F \] (4.2)

The covariant \( SU(1, 1) \) generators \( J^m \) are realized as bilinears of 2 pairs of oscillators \( d \) and \( g \) satisfying\(^4\)
\[
\left[ d, d^\dagger \right] = \left[ g, g^\dagger \right] = 1
\] (4.3)

as follows
\[
J^2_1 = d g \quad J^2_2 = -d^\dagger g^\dagger \quad J^1_1 = -J^2_2 = \frac{1}{2} (N_d + N_g + 1)
\] (4.4)

\(^4\)They are related to the covariant oscillators of [14] as \( a_1 = d \) and \( a^2 = g \).
where \( N_d = d^\dagger d \) and \( N_g = g^\dagger g \). Furthermore, these \( J_n^m \) can be related to \( J_{\pm,0} \) (defined in equation (2.17)) as follows:

\[
J_+ = J_1^1 + J_2^1 - J_1^2 - J_2^2 \quad J_- = J_1^1 - J_2^1 + J_1^2 - J_2^2 \quad J_0 = -i \left( J_1^1 + J_2^1 \right)
\]

The generators of \( su(2,2) \) in the minimal unitary realization take the form:

\[
J_1^1 = \frac{1}{2} (N_d + N_g + 1) \quad J_2^2 = -\frac{1}{2} (N_d + N_g + 1) \quad J_2^1 = -d^\dagger g^\dagger \quad J_1^2 = d g
\]

\[
U = N_d - N_g \quad \Delta = \frac{1}{2} (xp + px)
\]

\[
E = \frac{1}{2} x^2 \quad E^1 = x d^\dagger \quad E^2 = x g \quad E_1 = x d \quad E_2 = -x g^\dagger
\]

\[
F = \frac{1}{2} p^2 + \frac{1}{2} x^2 \left[ (N_d - N_g)^2 - \frac{1}{4} \right]
\]

\[
F_1 = d^\dagger \left[ p + \frac{i}{x} \left( N_d - N_g + \frac{1}{2} \right) \right] \quad F_2 = g \left[ p + \frac{i}{x} \left( N_d - N_g + \frac{1}{2} \right) \right]
\]

\[
F_1 = d \left[ p - \frac{i}{x} \left( N_d - N_g - \frac{1}{2} \right) \right] \quad F_2 = -g^\dagger \left[ p - \frac{i}{x} \left( N_d - N_g - \frac{1}{2} \right) \right]
\]

where \( x \) is again the singlet coordinate of the quasiconformal realization and \( p \) is its conjugate momentum.

### 4.2 From \( SO(4,2) \) to \( SU(2,2) \)

The minrep of \( SO(4,2) \) to the minrep of \( SU(2,2) \) reviewed above are related very simply by rewriting the oscillators \( d \) (\( d^\dagger \)), \( g \) (\( g^\dagger \)) in terms of \( b \) (\( b^\dagger \)), \( c \) (\( c^\dagger \)) as

\[
d = \frac{1}{\sqrt{2}} (b - ic) \quad g = \frac{1}{\sqrt{2}} (b + ic)
\]

It is trivial to verify

\[
[d, d^\dagger] = [g, g^\dagger] = 1 \quad [d, g] = [d, g^\dagger] = 0.
\]

Then

\[
X^1 P_2 - X^2 P_1 = -i \left( b^\dagger c - bc^\dagger \right) = N_d - N_g
\]

and

\[
b^\dagger b + c^\dagger c = N_d + N_g
\]

where \( N_d = d^\dagger d \) and \( N_g = g^\dagger g \). In terms of these new oscillators, the quartic invariant becomes

\[
I_4 = 2 (N_d - N_g)^2 - 4.
\]
The \( \mathfrak{so}(2) \) generator in \( \mathfrak{e}^0 \), which plays the role of the “energy” operator (Hamiltonian), becomes

\[
H = \frac{1}{2} \left[ (K_+ + K_-) + \frac{1}{2} (J_+ + J_-) \right]
\]

\[
= \frac{1}{2} \left[ N_a + N_{d} + N_{g} + \frac{1}{2} x^2 (N_{d} - N_{g})^2 - \frac{1}{8} x^2 + \frac{3}{2} \right]
\]

\[
= \frac{1}{2} \left[ N_{d} + \frac{1}{2} + \frac{1}{2} N_{g} + \frac{1}{2} N_a + \frac{1}{2} + \frac{G}{x} \right]
\]

\[
= H_d + H_g + H_\odot
\]

where

\[
G = \frac{1}{2} (N_{d} - N_{g})^2 - \frac{1}{8}.
\]

Note that \( H_d \) and \( H_g \) correspond to Hamiltonians of non-singular harmonic oscillators, while \( H_\odot \) corresponds to a singular harmonic oscillator with a potential function \( V(x) = G/x^2 \). \( H_\odot \) also arises as the Hamiltonian of conformal quantum mechanics [44] with \( G \) playing the role of coupling constant [11]. In some literature it is also referred to as the isotonic oscillator [45,46].

The lowest energy state (vacuum) of the full Hamiltonian is simply the tensor product state of the vacua of \( d \) and \( g \) type oscillators with the lowest energy state of \( H_\odot \).

Following the literature on singular or isotonic oscillators, we then introduce the operators

\[
A_L = a - \frac{L}{\sqrt{2} x}
\]

\[
A_L^\dagger = a^\dagger - \frac{L}{\sqrt{2} x}
\]

where

\[
L = N_{d} - N_{g} - \frac{1}{2}
\]

which we will refer to as singular (isotonic) oscillators.

These isotonic oscillators satisfy the following commutation relations:

\[
[A_L, A_{L'}] = -\frac{(L - L')}{2 x^2}
\]

\[
[A_L^\dagger, A_{L'}^\dagger] = +\frac{(L - L')}{2 x^2}
\]

\[
[A_L, A_{L'}^\dagger] = 1 + \frac{(L + L')}{2 x^2}
\]

In terms of these isotonic oscillators, we can write the singular harmonic oscillator of the Hamiltonian as

\[
H_\odot = \frac{1}{2} \left[ A_{L+1}^\dagger A_L + L + \frac{3}{2} \right] = \frac{1}{2} \left[ A_L A_L^\dagger + L - \frac{1}{2} \right]
\]

and the coupling constant as

\[
G = \frac{1}{2} L (L + 1).
\]
The generators of $\mathfrak{so}(4)$ in $\mathfrak{c}^{(0)}$ can then be expressed in terms of the oscillators $A_L$, $d$ and $g$ as

$$
\begin{align*}
\tilde{M}_{12} &= N_d - N_g \\
\tilde{M}_{43} &= \frac{1}{2} (N_a - N_d - N_g) + \frac{G}{2 x^2} - \frac{1}{4} \\
\tilde{M}_{41} &= \frac{i}{2\sqrt{2}} \left( a d^\dagger - a^\dagger d \right) - \frac{i}{4 x} (N_d - N_g) \left( d^\dagger - d \right) + \frac{i}{8 x} \left( d^\dagger + d \right) \\
&\quad + \frac{i}{2\sqrt{2}} \left( a g^\dagger - a^\dagger g \right) + \frac{i}{4 x} (N_d - N_g) \left( g^\dagger - g \right) + \frac{i}{8 x} \left( g^\dagger + g \right) \\
&= \frac{i}{2\sqrt{2}} \left[ A_C d^\dagger - A_{C+1}^\dagger d + A_{-(L+1)} g^\dagger - A_{-L}^\dagger g \right] \\
\tilde{M}_{42} &= \frac{1}{2\sqrt{2}} \left( a d^\dagger + a^\dagger d \right) - \frac{1}{4 x} (N_d - N_g) \left( d^\dagger + d \right) + \frac{1}{8 x} \left( d^\dagger - d \right) \\
&\quad - \frac{1}{2\sqrt{2}} \left( a g^\dagger + a^\dagger g \right) - \frac{1}{4 x} (N_d - N_g) \left( g^\dagger + g \right) - \frac{1}{8 x} \left( g^\dagger - g \right) \\
&= \frac{1}{2\sqrt{2}} \left[ A_C d^\dagger + A_{C+1}^\dagger d - A_{-(L+1)} g^\dagger - A_{-L}^\dagger g \right] \\
\tilde{M}_{13} &= \frac{1}{2\sqrt{2}} \left( a d^\dagger + a^\dagger d \right) - \frac{1}{4 x} (N_d - N_g) \left( d^\dagger + d \right) + \frac{1}{8 x} \left( d^\dagger - d \right) \\
&\quad + \frac{i}{2\sqrt{2}} \left( a g^\dagger + a^\dagger g \right) + \frac{i}{4 x} (N_d - N_g) \left( g^\dagger + g \right) + \frac{i}{8 x} \left( g^\dagger - g \right) \\
&= \frac{1}{2\sqrt{2}} \left[ A_C d^\dagger + A_{C+1}^\dagger d + A_{-(L+1)} g^\dagger + A_{-L}^\dagger g \right] \\
\tilde{M}_{23} &= -\frac{i}{2\sqrt{2}} \left( a d^\dagger - a^\dagger d \right) + \frac{i}{4 x} (N_d - N_g) \left( d^\dagger - d \right) - \frac{i}{8 x} \left( d^\dagger + d \right) \\
&\quad + \frac{i}{2\sqrt{2}} \left( a g^\dagger - a^\dagger g \right) + \frac{i}{4 x} (N_d - N_g) \left( g^\dagger - g \right) + \frac{i}{8 x} \left( g^\dagger + g \right) \\
&= -\frac{i}{2\sqrt{2}} \left[ A_C d^\dagger - A_{C+1}^\dagger d - A_{-(L+1)} g^\dagger + A_{-L}^\dagger g \right].
\end{align*}
$$

(4.18)

For reasons that will become evident later we shall work with the following linear combination of generators belonging to the grade +1 subspace $\mathfrak{c}^+$ of $\mathfrak{su}(2, 2)$:

$$
\begin{align*}
B^1 &= \Delta - i (K_+ - K_-) = i \left( a^\dagger a^\dagger - \frac{G}{x^2} \right) = i A_{-L}^\dagger A^\dagger_C = i A_{C+1}^{\dagger\dagger} A_{-(L+1)}^\dagger \\
B^2 &= \frac{1}{2} \left[ \left( U_1 + \bar{V}^1 \right) - i \left( \bar{U}_1 - V^1 \right) \right] + \frac{i}{2} \left[ \left( U_2 + \bar{V}^2 \right) - i \left( \bar{U}_2 - V^2 \right) \right] \\
&= \sqrt{2} i \left[ a^\dagger + \frac{1}{\sqrt{2} x} (N_d - N_g) - \frac{1}{2\sqrt{2} x} \right] d^\dagger = \sqrt{2} \left[ A_{-L}^\dagger \right] d^\dagger = \sqrt{2} i d^\dagger A_{-(L+1)}^\dagger
\end{align*}
$$

(4.19)
\[ B^i = \frac{1}{2} \left[ (U_1 + \tilde{V}^1) - i \left( \tilde{U}_1 - V^1 \right) \right] - \frac{i}{2} \left[ (U_2 + \tilde{V}^2) - i \left( \tilde{U}_2 - V^2 \right) \right] \]
\[ = \sqrt{2} i \left[ a^i - \frac{1}{\sqrt{2} x} (N_d - N_g) - \frac{1}{2 \sqrt{2} x} \right] g^\dagger = \sqrt{2} i A^i_{\mathcal{L}+1} g^\dagger = \sqrt{2} i g^\dagger A^i_{\mathcal{L}} \]

\[ B^4 = J_0 - \frac{i}{2} (J_+ - J_-) = 2 i d^\dagger g^\dagger \]

They satisfy the following commutation relations with the energy operator \( H \) given in equation (4.11)

\[ [H, B^i] = B^i \quad i = 1, 2, 3, 4. \quad (4.20) \]

Furthermore we have the important relation

\[ B^3 B^2 = B^4 B^1 \quad (4.21) \]

which is valid for the minrep, but is not valid in general. This constraint satisfied by the operators in the minrep will be important for its decomposition into K-finite vectors!

The \( \mathfrak{C}^- \) generators are given by

\[ B_1 = \Delta + i (K_+ - K_-) = -i \left( a a - \frac{G}{x^2} \right) = -i A_{\mathcal{L}} A_{-\mathcal{L}} = -i A_{-(\mathcal{L}+1)} A_{\mathcal{L}+1} \]
\[ B_2 = \frac{1}{2} \left[ (U_1 + \tilde{V}^1) + i \left( \tilde{U}_1 - V^1 \right) \right] - \frac{i}{2} \left[ (U_2 + \tilde{V}^2) + i \left( \tilde{U}_2 - V^2 \right) \right] \]
\[ = -\sqrt{2} i d \left[ a + \frac{1}{\sqrt{2} x} (N_d - N_g) - \frac{1}{2 \sqrt{2} x} \right] = -\sqrt{2} i d A_{-\mathcal{L}} = -\sqrt{2} i A_{-(\mathcal{L}+1)} d \quad (4.22) \]
\[ B_3 = \frac{1}{2} \left[ (U_1 + \tilde{V}^1) + i \left( \tilde{U}_1 - V^1 \right) \right] + \frac{i}{2} \left[ (U_2 + \tilde{V}^2) + i \left( \tilde{U}_2 - V^2 \right) \right] \]
\[ = -\sqrt{2} i g \left[ a - \frac{1}{\sqrt{2} x} (N_d - N_g) - \frac{1}{2 \sqrt{2} x} \right] = -\sqrt{2} i g A_{\mathcal{L}+1} = -\sqrt{2} i A_{\mathcal{L}} g \]
\[ B_4 = J_0 + \frac{i}{2} (J_+ - J_-) = -2 i g d . \]

The generators of the two \( \mathfrak{su}(2) \) subalgebras of \( \mathfrak{so}(4) \), in terms of the oscillators \( A_{\mathcal{L}}, d \) and \( g \) have the following form:

\[ L_+ = -\frac{i}{2} \left[ a - \frac{1}{\sqrt{2} x} (N_d - N_g) + \frac{1}{2 \sqrt{2} x} \right] d^\dagger = -\frac{i}{2} A_{\mathcal{L}} d^\dagger \]
\[ L_- = \frac{i}{2} \left[ a^\dagger - \frac{1}{\sqrt{2} x} (N_d - N_g) - \frac{1}{2 \sqrt{2} x} \right] d = \frac{i}{2} A^\dagger_{\mathcal{L}+1} d \quad (4.23) \]
\[ L_3 = -\frac{1}{2} (H - 1) + N_d \]
\[ R_+ = \frac{i}{2} \left[ a + \frac{1}{\sqrt{2} x} (N_d - N_g) + \frac{1}{2 \sqrt{2} x} \right] g^\dagger = \frac{i}{2} A_{-(\mathcal{L}+1)} g^\dagger \]
\[ R_- = -\frac{i}{2} \left[ a^\dagger + \frac{1}{\sqrt{2} x} (N_d - N_g) - \frac{1}{2 \sqrt{2} x} \right] g = -\frac{i}{2} A^\dagger_{-\mathcal{L}} g \]
\[ R_3 = -\frac{1}{2} (H - 1) + N_g \]
Their quadratic Casimir operators take the form

\[ L^2 = R^2 = \frac{1}{16} \left[ (N_a + N_d + N_g) + \frac{G}{x^2} + \frac{3}{2} \right]^2 - \frac{1}{4} \]

\[ = \frac{1}{4} (H^2 - 1) . \]  

(4.25)

### 4.3 SU(1,1)\(_L\) subgroup of SU(2,2) generated by the isotonic (singular) oscillators

Consider the singular harmonic oscillator part of the Hamiltonian (4.11), which in coordinate representations has the form:

\[ H_\odot = \frac{1}{2} \left[ a_1^+ a_1 + \frac{1}{2} + \frac{G}{x^2} \right] = \frac{1}{4} (x^2 + p^2) + \frac{G}{2x^2} = \frac{1}{4} \left( x^2 - \frac{\partial^2}{\partial x^2} \right) + \frac{G}{2x^2} . \]  

(4.26)

Together with the operators \( B^1 \) and \( B_1 \) belonging to \( \mathfrak{c}^+ \) and \( \mathfrak{c}^- \), respectively,

\[ B^1 = i \left( a_1^+ a_1 - \frac{G}{x^2} \right) = \frac{i}{2} (x - ip)^2 - \frac{i G}{x^2} = \frac{i}{2} \left( x^2 - 2x \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2} - 1 \right) - \frac{i G}{x^2} \]

\[ B_1 = -i \left( a a - \frac{G}{x^2} \right) = -\frac{i}{2} (x + ip)^2 + \frac{i G}{x^2} = -\frac{i}{2} \left( x^2 + 2x \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2} + 1 \right) + \frac{i G}{x^2} \]  

(4.27)

it generates a distinguished \( \mathfrak{su}(1,1)_L \) subalgebra\(^5\)

\[ [B_1, B^1] = 8 H_\odot \quad [H_\odot, B^1] = + B^1 \quad [H_\odot, B_1] = - B_1 . \]  

(4.28)

The lowest energy state \( \psi_0^{(\alpha)} (x) \) of this singular harmonic oscillator Hamiltonian must satisfy

\[ B_1 \psi_0^{(\alpha)} (x) = 0 \]  

(4.29)

whose solution is [47]

\[ \psi_0^{(\alpha)} (x) = C_0 x^\alpha e^{-x^2/2} \]  

(4.30)

where

\[ \alpha = \frac{1}{2} + \left( 2 g + \frac{1}{4} \right)^{\frac{1}{2}} \]  

(4.31)

and \( C_0 \) is a normalization constant. Note that \( g \) is defined as

\[ g = \frac{1}{2} \frac{(n_d - n_g)^2}{n_d n_g} - \frac{1}{8} \]  

(4.32)

where \( n_d \) and \( n_g \) are the eigenvalues of the number operators \( N_d \) and \( N_g \). Thus we have

\[ \alpha = \frac{1}{2} + |n_d - n_g| . \]  

(4.33)

\(^5\)This is the \( SU(1,1) \) subgroup generated by the longest root vector. Hence the subscript \( L \).
The normalizability of the state imposes the constraint
\[ \alpha \geq \frac{1}{2}. \]  
(4.34)

Clearly, \( \psi_0^{(\alpha)}(x) \) is an eigenstate of \( H_\odot \) with eigenvalue \( E_{\odot,0}^{(\alpha)} \) given by
\[ H_\odot \psi_0^{(\alpha)}(x) = E_{\odot,0}^{(\alpha)} \psi_0^{(\alpha)}(x) \quad \text{where} \quad E_{\odot,0}^{(\alpha)} = \frac{1}{4} (2\alpha + 1). \]  
(4.35)

Acting on a tensor product state of \( \psi_0^{(\alpha)} \) with the eigenstates of the number operators \( N_d \) and \( N_g \), one may obtain more eigenstates of \( H_\odot \).

The lowest “energy” normalizable eigenstate of \( H_\odot \) corresponds to the case \( n_d = n_g \) (therefore \( \alpha = \frac{1}{2} \)). All the higher “energy” eigenstates of \( H_\odot \) can be obtained from \( \psi_0^{(1/2)}(x) \) by acting on it repeatedly with the raising generator \( B^1 \),
\[ \psi_n^{(1/2)}(x) = C_n \left( B^1 \right)^n \psi_0^{(1/2)}(x) \]  
(4.36)

where \( C_n \) are normalization constants, and they have energies \( E_{\odot,n} \):
\[ H_\odot \psi_n^{(1/2)}(x) = E_{\odot,n}^{(1/2)} \psi_n^{(1/2)}(x) \]  
(4.37)

where
\[ E_{\odot,n}^{(1/2)} = E_{\odot,0}^{(1/2)} + n = \frac{1}{2} |n_d - n_g| + n + \frac{1}{2}. \]  
(4.38)

5. \( SU(2) \times SU(2) \times U(1) \) Decomposition of the Minrep of \( SU(2,2) \) and the Scalar Doubleton

Let us label the states that belong to the Fock spaces of the oscillator \( d \) as
\[ |n_d\rangle = \frac{1}{\sqrt{n_d!}} (d^\dagger)^{n_d} |0\rangle \]  
(5.1)

and similarly the states \( |n_g\rangle \) for oscillators \( g \). As a basis of the Hilbert space of the minrep we shall consider tensor product states
\[ \left| \psi_n^{(1/2)};n_d,n_g \right\rangle = \left| \psi_n^{(1/2)} \right\rangle \otimes |n_d\rangle \otimes |n_g\rangle \]  
(5.2)

where \( \left| \psi_n^{(1/2)} \right\rangle \) is the state vector corresponding to \( \psi_n^{(1/2)} \) defined above. It is an eigenstate of the energy operator \( H \) that determines the 3-grading of \( SU(2,2) \)
\[ H \left| \psi_n^{(1/2)};n_d,n_g \right\rangle = E \left| \psi_n^{(1/2)};n_d,n_g \right\rangle \]  
(5.3)

where
\[ E = \frac{1}{2} \left( n_d + \frac{1}{2} \right) + \frac{1}{2} \left( n_g + \frac{1}{2} \right) + \frac{1}{2} |n_d - n_g| + n + \frac{1}{2}. \]  
(5.4)
There exists a unique lowest energy state, namely

\[ |\psi_0^{(1/2)}(x); 0, 0\rangle \] (5.5)

that is annihilated by all four operators \( B_i \) in \( \mathfrak{c}^- \) subspace of \( \mathfrak{su}(2, 2) \) and transforms as a singlet of \( SU(2)_L \times SU(2)_R \) with energy \( E = 1 \). All the other states with higher energies can be obtained from \( |\psi_0^{(1/2)}(x); 0, 0\rangle \) by repeatedly acting on it with \( \mathfrak{c}^+ \) generators \( B^1, B^2, B^3 \) and \( B^4 \).

The commutation relations between the \( \mathfrak{su}(2)_L \) and \( \mathfrak{su}(2)_R \) generators and the operators belonging to \( \mathfrak{c}^\pm \) are as follows:

\[
\begin{align*}
[L_+, B^1] &= -\frac{i}{\sqrt{2}} B^2 & [L_-, B^1] &= 0 & [L_3, B^1] &= -\frac{1}{2} B^1 \\
[L_+, B^2] &= 0 & [L_-, B^2] &= \frac{i}{\sqrt{2}} B^1 & [L_3, B^2] &= \frac{1}{2} B^2 \\
[L_+, B^3] &= -\frac{i}{\sqrt{2}} B^4 & [L_-, B^3] &= 0 & [L_3, B^3] &= -\frac{1}{2} B^3 \\
[L_+, B^4] &= 0 & [L_-, B^4] &= \frac{i}{\sqrt{2}} B^3 & [L_3, B^4] &= \frac{1}{2} B^4
\end{align*}
\] (5.6)

\[
\begin{align*}
[R_+, B^1] &= \frac{i}{\sqrt{2}} B^3 & [R_-, B^1] &= 0 & [R_3, B^1] &= -\frac{1}{2} B^1 \\
[R_+, B^2] &= \frac{i}{\sqrt{2}} B^4 & [R_-, B^2] &= 0 & [R_3, B^2] &= \frac{1}{2} B^2 \\
[R_+, B^3] &= 0 & [R_-, B^3] &= -\frac{i}{\sqrt{2}} B^1 & [R_3, B^3] &= \frac{1}{2} B^3 \\
[R_+, B^4] &= 0 & [R_-, B^4] &= -\frac{i}{\sqrt{2}} B^3 & [R_3, B^4] &= \frac{1}{2} B^4
\end{align*}
\] (5.7)

showing that \( (B^1, B^2, B^3, B^4) \) transform in the \( \left(\frac{1}{2}, \frac{1}{2}\right) \) representation of \( \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \). Therefore, we can label them by their eigenvalues with respect to \( (L_3, R_3) \):

\[
(B^1, B^2, B^3, B^4) = T^{mn} = \left( T(-\frac{1}{2}, -\frac{1}{2}), T(+\frac{1}{2}, -\frac{1}{2}), T(-\frac{1}{2}, +\frac{1}{2}), T(+\frac{1}{2}, +\frac{1}{2}) \right)
\] (5.8)

Similarly, the generators \( B_i \) in \( \mathfrak{c}^- \) also transform in the \( \left(\frac{1}{2}, \frac{1}{2}\right) \) representation of \( \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \).

Operators \( B_i \) all commute with each other, and the \( \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \) content of the minimal representation of \( \mathfrak{so}(4, 2) \) is obtained by taking the symmetric powers of \( \left(\frac{1}{2}, \frac{1}{2}\right) \), subject to the constraint \( B^1 B^4 = B^2 B^3 \). For example, this constraint eliminates the \((0, 0)\) component.
in the tensor product \((\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})\).

\[
|\Omega\rangle = |(0, 0)\rangle \quad (E = 2)
\]

\[
T^{mn} |\Omega\rangle = \left(\frac{1}{2}, \frac{1}{2}\right) \quad (E = 4)
\]

\[
(T^{mn})^2 |\Omega\rangle = |(1, 1)\rangle \quad (E = 6)
\]

\[
(T^{mn})^3 |\Omega\rangle = \left(\frac{3}{2}, \frac{3}{2}\right) \quad (E = 8)
\]

\[
\vdots
\]

\[
(T^{mn})^P |\Omega\rangle = \left(\frac{P}{2}, \frac{P}{2}\right) \quad (E = 2P + 2)
\]

\[
\vdots
\]

where \(m, n = \pm \frac{1}{2}\).

We list all those states that form the relevant irreducible representations of \(\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R\) for the first few energy levels and their \(l_3, r_3\) quantum numbers in Table 1.

Table 1: The \(SU(2)_L \times SU(2)_R \times U(1)\) content of the minimal unitary representation of \(SO(4, 2)\).

| Irrep | State | \(|\psi_0^{(1/2)}, 0, 0\rangle| | E | N_d | N_g | \(l = r\) | \(l_3\) | \(r_3\)
|-------|-------|-----------------|---|----|----|--------|--------|--------|
| \(|(0, 0)\rangle| | | 1 | 0 | 0 | 0 | 0 | 0 |
| \(|(1/2, 1/2)\rangle| | | 2 | 0 | 0 | 1 | -1/2 | -1/2 |
| \(|(1/2, -1/2)\rangle| | | 2 | 0 | 1 | 1/2 | -1/2 |
| \(|(1/2, -1/2)\rangle| | | 2 | 0 | 1 | 1/2 | +1/2 |
| \(|(1, 1)\rangle| | | 3 | 0 | 0 | 1 | -1 | -1 |
| \(|(1, 1)\rangle| | | 3 | 1 | 0 | 1 | 0 | -1 |
| \(|(1/2, 1/2)\rangle| | | 3 | 2 | 0 | 1 | +1 | -1 |
| \(|(1/2, 1/2)\rangle| | | 3 | 0 | 1 | 1 | -1 | 0 |
| \(|(1/2, 1/2)\rangle| | | 3 | 1 | 1 | 1 | 0 | 0 |
| \(|(1/2, 1/2)\rangle| | | 3 | 2 | 1 | 1 | +1 | 0 |
| \(|(1/2, 1/2)\rangle| | | 3 | 0 | 2 | 1 | -1 | +1 |
| \(|(1/2, 1/2)\rangle| | | 3 | 1 | 2 | 1 | 0 | +1 |
| \(|(1/2, 1/2)\rangle| | | 3 | 2 | 2 | 1 | 0 | +1 |
| \(|(1/2, 1/2)\rangle| | | 4 | 0 | 0 | 3/2 | -1/2 | -3/2 |
Table 1: (continued)

| Irrep            | State             |  $E$ |  $N_d$ |  $N_p$ | $\ell = r$ |  $l_3$ |  $r_3$ |
|------------------|-------------------|-----|-------|-------|-----------|-------|-------|
| $B^2B^1B^1$      | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  1    |     | $\frac{3}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ |
| $B^2B^2B^1$      | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  2    |     | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $B^2B^2B^2$      | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  3    |     | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $B^3B^1B^1$      | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  0    |     | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $B^3B^2B^1$      | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  1    |     | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $B^3B^2B^2$      | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  2    |     | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $B^4B^1B^1$      | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  0    |     | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $B^4B^2B^1$      | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  1    |     | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $B^4B^2B^2$      | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  2    |     | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $B^4B^3B^1$      | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  0    |     | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $B^4B^3B^3$      | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  1    |     | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $B^4B^3B^3B^1$   | $|\psi_{0}^{(1/2)},0,0\rangle$ | 4   |  2    |     | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $|\langle \frac{p_f}{2}, \frac{p_{g}}{2} \rangle\rangle$ | $B^{i_1}B^{i_2} \ldots B^{i_p} |\psi_{0}^{(1/2)},0,0\rangle$ | $p + 1$ | $\ldots$ | $\frac{p}{2}$ | $\ldots$ | $\ldots$ |

Comparing the $SU(2) \times SU(2) \times U(1)$ decomposition of the minrep of $SU(2,2)$ with that of the scalar doubleton representation of $5D$ AdS group $SU(2,2)$ obtained by the oscillator method [20–22], we see that they coincide exactly. The quadratic Casimir operator of the $SU(2,2)$ is given by [10]

$$C_2 = -\frac{1}{6} J^p \cdot J^p + \frac{1}{12} \Delta^2 - \frac{1}{6} (EF + FE) - \frac{1}{6} U^2 - \frac{i}{12} (E_p F_p + F_p E_p - F_p E_p - E_p F_p)$$

(5.10)

which reduces to a c-number

$$C_2 = \frac{1}{2}$$

(5.11)

with the higher Casimirs vanishing in the minrep. They agree with the values of the Casimir operators for the scalar doubleton given in [21,22]6. Hence the minimal unitary representation

6Note that the quadratic Casimir of [21,22] differs from that of [14] by an overall factor of $-6$, i.e. $C_2^{(GMZ)} = -6C_2^{(GP)}$. 

---

---

---
of the $4D$ conformal group is nothing but the scalar doubleton representation. This representation remains irreducible under restriction to the four dimensional Poincare group and describes a massless and spinless particle [20–22, 29, 48]. We should also note that the same scalar doubleton representation $SO(4, 2)$ was used long time ago to describe the spectrum of the Hydrogen atom [23–27].

6. One Parameter Family of Deformations of the Minrep of $SU(2, 2)$ and Massless Conformal Fields in Four Dimensions

In the previous section we showed that the minrep of $SU(2, 2)$ is simply the scalar doubleton representation that describes a conformal scalar field in four dimensions. The group $SU(2, 2)$ admits infinitely many doubleton representations corresponding to $4D$ massless conformal fields of arbitrary spin [20–22, 48]. The irreducible doubletons of $SU(2, 2)$ remain irreducible under the restriction to the Poincare subgroup and describe massless particles of integer and half-integer helicity [29]. They all can be constructed by the oscillator method over the Fock space of two pairs of twistorial oscillators transforming in the spinor representation of $SU(2, 2)$. One important question is whether the doubleton representations corresponding to massless conformal fields of arbitrary spin can all be obtained from the quantization of quasiconformal action of $SU(2, 2)$. Remarkably there exists a one-parameter ($\zeta$) deformation of the construction given in the previous section such that all doubleton unitary irreducible representation of $SU(2, 2)$ can be obtained by choosing the deformation parameter to be an integer.

In the general formulation of minimal unitary representations of noncompact groups obtained by quantizing their quasiconformal realizations, the quartic invariant operator $I_4$ of the grade zero subalgebra, modulo the $SO(1, 1)$ generator that determines the 5-grading, enters in the numerator of the singular term in the grade +2 generator $F$. For the group $SU(n + 1, m + 1)$ it has the form [14]

$$F = \frac{1}{2} p^2 + \frac{1}{2 x^2} \left[ I_4 + \frac{(m + n)^2 - 1}{4} \right].$$  \hspace{1cm} (6.1)

For $SU(2, 2)$ the quartic invariant $I_4$ is related to the Casimir operator of grade zero subalgebra $su(1, 1)$ as

$$I_4 = 2 J_m^n J_n^m = (N_d - N_g)^2 - 1 = U^2 - 1.$$  \hspace{1cm} (6.2)

One parameter deformations of the minimal unitary representation are obtained by replacing the quartic invariant $I_4$ by

$$I_4 (\zeta) = (N_d - N_g + \zeta)^2 - 1.$$  \hspace{1cm} (6.3)

Then the grade +2 generator becomes

$$F (\zeta) = \frac{1}{2} p^2 + \frac{1}{2 x^2} \left[ (N_d - N_g + \zeta)^2 - \frac{1}{4} \right].$$  \hspace{1cm} (6.4)
while the negative grade generators $E$, $E^m$ and $E_m$ remaining as in the undeformed case. The grade +1 generators are modified by $\zeta$ dependent terms and are given by

$$F^1(\zeta) = d^\dagger \left[ p + \frac{i}{x} \left( N_d - N_g + \zeta + \frac{1}{2} \right) \right]$$

$$F^2(\zeta) = g \left[ p + \frac{i}{x} \left( N_d - N_g + \zeta + \frac{1}{2} \right) \right]$$

$$F_1(\zeta) = d \left[ p - \frac{i}{x} \left( N_d - N_g + \zeta - \frac{1}{2} \right) \right]$$

$$F_2(\zeta) = -g^\dagger \left[ p - \frac{i}{x} \left( N_d - N_g + \zeta - \frac{1}{2} \right) \right].$$

The only bosonic generator in $g^{(0)}$ subspace that changes under this deformation is $U$ which becomes

$$U(\zeta) = N_d - N_g + \frac{\zeta}{2}.$$

One can easily verify that all the Jacobi identities are satisfied under this deformation and the quadratic Casimir of $SU(2,2)$ takes on the value

$$C_2(\zeta) = -\frac{1}{2} \left( \frac{\zeta^2}{2} - 1 \right) \left( \frac{\zeta}{2} + 1 \right).$$

The Casimirs of the $SU(2)_L$ and $SU(2)_R$ are no longer equal under this deformation. One finds

$$L^2(\zeta) = \left[ \frac{1}{2} \left( H(\zeta) - \frac{\zeta}{2} \right) - \frac{1}{2} \right] \left[ \frac{1}{2} \left( H(\zeta) - \frac{\zeta}{2} \right) + \frac{1}{2} \right]$$

$$R^2(\zeta) = \left[ \frac{1}{2} \left( H(\zeta) + \frac{\zeta}{2} \right) - \frac{1}{2} \right] \left[ \frac{1}{2} \left( H(\zeta) + \frac{\zeta}{2} \right) + \frac{1}{2} \right]$$

where $H(\zeta)$ is the $\mathfrak{so}(2)$ generator in $\mathfrak{e}^0$ given in equation 4.11, that plays the role of the “AdS energy” operator (Hamiltonian) and determines the three grading in the compact basis, which has now become

$$H(\zeta) = \frac{1}{2} \left[ N_a + N_d + N_g + \frac{1}{2} x^2 (N_d - N_g + \zeta)^2 - \frac{1}{8} x^2 + \frac{3}{2} \right]$$

$$= \frac{1}{2} \left[ N_d + \frac{1}{2} \right] + \frac{1}{2} \left[ N_g + \frac{1}{2} \right] + \frac{1}{2} \left[ N_a + \frac{1}{2} + \frac{G(\zeta)}{x^2} \right]$$

$$= H_d + H_g + H_\circ(\zeta)$$

where

$$H_\circ(\zeta) = \frac{1}{2} \left[ a^\dagger a + \frac{1}{2} \frac{G(\zeta)}{x^2} \right].$$

with

$$G(\zeta) = \frac{1}{2} (N_d - N_g + \zeta)^2 - \frac{1}{8}.$$
Together with the operators $B^{1}(\zeta)$ and $B_{1}(\zeta)$ belonging to $\mathfrak{c}^{+}$ and $\mathfrak{c}^{-}$, respectively,

$$B^{1}(\zeta) = i \left( a^\dagger a - \frac{G(\zeta)}{x^2} \right) \quad (6.15)$$

$$B_{1}(\zeta) = -i \left( a a - \frac{G(\zeta)}{x^2} \right) \quad (6.16)$$

$H_{\odot}(\zeta)$ generates the distinguished $\mathfrak{su}(1,1)_{L}$ subalgebra:

$$[B_{1}(\zeta), B^{1}(\zeta)] = 8 H_{\odot}(\zeta)$$

$$[H_{\odot}(\zeta), B^{1}(\zeta)] = + B^{1}(\zeta)$$

$$[H_{\odot}(\zeta), B_{1}(\zeta)] = - B_{1}(\zeta) \quad (6.17)$$

We shall denote the eigenfunctions of this deformed singular harmonic oscillator Hamiltonian $H_{\odot}(\zeta)$ as $\psi^{\alpha}(x; \zeta)$

$$H_{\odot}(\zeta) \psi^{\alpha}(x; \zeta) = E_{\odot}(\zeta) \psi^{\alpha}(x; \zeta) \quad (6.18)$$

$$E_{\odot}(\zeta) = \frac{1}{4} \left[ 2 \alpha(\zeta) + 1 \right] . \quad (6.19)$$

For eigenstates that are lowest weight vectors of a unitary representation of $SU(1,1)$ we have

$$B_{1}(\zeta) \psi^{\alpha}(x; \zeta) = 0 \quad (6.20)$$

and such states take the form

$$\psi^{\alpha}(x; \zeta) = C x^{\alpha(\zeta)} e^{-x^2/2} \quad (6.21)$$

where

$$\alpha(\zeta) = \frac{1}{2} + \left( 2 g(\zeta) + \frac{1}{4} \right)^{\frac{1}{2}} = \frac{1}{2} + |n_{d} - n_{g} + \zeta| \quad (6.22)$$

and $C$ is a normalization constant. The normalizability condition requires

$$\alpha(\zeta) \geq \frac{1}{2} . \quad (6.23)$$

Therefore the fact that the normalizable states corresponding to the lowest energy eigenvalue $E_{\odot}$ of the isotonic oscillator have $\alpha = 1/2$ implies

$$n_{d} - n_{g} + \zeta = 0 . \quad (6.24)$$

This means that the deformation parameter $\zeta$ is an integer for such states. We shall denote the corresponding eigenfunction as $\psi^{(1/2)}_{0}(x; \zeta)$. There are infinitely many such states in the tensor product space $\mathcal{H}_{\odot} \otimes \mathcal{H}_{d} \otimes \mathcal{H}_{g}$.

The total energy eigenvalue $E(\zeta)$ of a tensor product state

$$\left\{ \psi^{(\alpha)}(\zeta), n_{d}, n_{d} \right\}$$
is on the other hand given by

\[ E(\zeta) = E_0(\zeta) + E_d + E_g = \frac{1}{2} |n_d - n_g + \zeta| + \frac{1}{2} (n_d + n_g) + 1. \] (6.25)

Therefore, for a given \( \zeta \) we have a unique lowest energy eigenvalue

\[ E_0(\zeta) = \frac{|\zeta|}{2} + 1. \] (6.26)

The degeneracy of this energy eigenvalue is \( |\zeta| + 1 \). These degenerate energy eigenstates transform in an irreducible representation of \( SU(2)_L \times SU(2)_R \times U(1) \). For \( \zeta = n_r \), where \( n_r \) is a positive integer, they transform in the representation \((0, \frac{n_r}{2})\) of \( SU(2)_L \times SU(2)_R \), and for \( \zeta = -n_l \), where \( n_l \) is a positive integer, they transform in the representation \( (\frac{n_l}{2}, 0) \) of \( SU(2)_L \times SU(2)_R \). The operators \( B_i \ (i = 1, 2, 3, 4) \) in grade \(-1\) subspace \( \mathcal{E}^- \) annihilate these lowest energy states for a given \( \zeta \). Let us label this finite set of states collectively as \( |\Omega\rangle \). Then

\[ B_i |\Omega\rangle = 0 \quad i = 1, 2, 3, 4. \] (6.27)

Since the states \( |\Omega\rangle \) transform irreducibly under the maximal compact subgroup \( SU(2) \times SU(2) \times U(1) \), the infinite set of states generated by the repeated action of the operators \( B^i \in \mathcal{E}^+ \) on \( |\Omega\rangle \)

\[ |\Omega\rangle , \ B^i |\Omega\rangle , \ B^i B^j |\Omega\rangle , \ldots \] (6.28)

form the (particle) basis of a positive energy unitary irreducible representation of \( SU(2,2) \) [20, 21, 40, 41]. By going to a noncompact coherent state basis labelled by 4D spacetime coordinates one can show that these unitary representations can be identified with massless conformal fields whose \( SL(2,C) \) transformation labels coincide with the \( SU(2) \times SU(2) \) labels of their lowest energy states, and the lowest energy eigenvalues \( E \) can be identified with their conformal dimensions [21, 22]. Irreducible doubleton (ladder or most degenerate discrete series) representations remain irreducible under restriction to Poincaré group and describe massless particles of arbitrary helicity \( \lambda \) [29], which is related to our deformation parameter simply as

\[ \lambda = \frac{\zeta}{2}. \] (6.29)

7. Minimal Unitary Representations of Supergroups \( SU(2,2|p+q) \)

In this section we shall construct the minimal unitary representations of the supergroups \( SU(2,2|p+q) \) and their deformations using the quasicomformal approach. Consider a set of \( p \) pairs of fermionic annihilation and creation operators labelled as \( \alpha_\mu \) and \( \alpha^\mu = (\alpha_\mu)^\dagger \) and another set of \( q \) pairs of fermionic annihilation and creation operators labelled as \( \beta_\nu \) and \( \beta^\nu = (\beta_\nu)^\dagger \), such that they satisfy the anti-commutation relations:

\[
\{ \alpha_\mu, \alpha^\nu \} = \delta^\nu_\mu \quad \{ \alpha_\mu, \alpha_\nu \} = 0 \quad \{ \alpha^\mu, \alpha^\nu \} = 0 \\
\{ \beta_\nu, \beta^\mu \} = \delta^\mu_\nu \quad \{ \beta_\nu, \beta_\mu \} = 0 \quad \{ \beta^\nu, \beta^\mu \} = 0
\] (7.1)
where $\mu, \nu = 1, \ldots, p$ and $y, z = 1, \ldots, q$. Let $N_\alpha = \alpha^\mu \alpha_\mu$ and $N_\beta = \beta^\nu \beta_\nu$ be the $\alpha$- and $\beta$-type fermionic number operators. Also denote the $\mathfrak{su}(\mathfrak{p})$ and $\mathfrak{su}(\mathfrak{q})$ generators inside $\mathfrak{su}(\mathfrak{p} + \mathfrak{q})$ by

$$A_\mu^\nu = \alpha^\nu \alpha_\mu - \frac{1}{p} \delta_\mu^\nu N_\alpha \quad B_\nu^\alpha = \beta^\nu \beta_\alpha - \frac{1}{q} \delta_\nu^\alpha N_\beta$$

so that

$$[A_\mu^\nu, A_\nu^\rho] = \delta_\mu^\rho A_\nu^\rho - \delta_\nu^\rho A_\mu^\rho \quad [B_\nu^\alpha, B_\mu^\beta] = \delta_\nu^\beta B_\mu^\beta - \delta_\mu^\beta B_\nu^\beta.$$

The remaining generators of $\mathfrak{su}(\mathfrak{p} + \mathfrak{q})$ are given by

$$C_\mu^\nu y = \alpha_\mu \beta_y \quad C_\nu^\mu y = (C_\mu^\nu y)^\dagger = \beta_\nu \alpha^\mu$$

so that

$$[C_\mu^\nu y, C_z^{\nu^\rho}] = -\delta_\mu^\rho A_\nu^\rho - \delta_\nu^\rho A_\mu^\rho \left(\frac{1}{p} N_\alpha + \frac{1}{q} N_\beta - 1\right)$$

where $\frac{1}{p} N_\alpha + \frac{1}{q} N_\beta - 1$ is the $\mathfrak{u}(1)$ generator that appears in the decomposition $\mathfrak{su}(\mathfrak{p} + \mathfrak{q}) \supset \mathfrak{su}(\mathfrak{p}) \oplus \mathfrak{su}(\mathfrak{q}) \oplus \mathfrak{u}(1)$ and determines a 3-graded decomposition of $\mathfrak{su}(\mathfrak{p} + \mathfrak{q})$.

### 7.1 5-grading of $\mathfrak{su}(2, 2 | \mathfrak{p} + \mathfrak{q})$ with respect to the subalgebra $\mathfrak{su}(1, 1 | \mathfrak{p} + \mathfrak{q}) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(1, 1)$

The Lie superalgebra $\mathfrak{su}(2, 2 | \mathfrak{p} + \mathfrak{q})$ has the following 5-graded decomposition with respect to its subalgebra $\mathfrak{su}(1, 1 | \mathfrak{p} + \mathfrak{q}) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(1, 1)$:

$$\mathfrak{su}(2, 2 | \mathfrak{p} + \mathfrak{q}) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)}$$

$$= 1^{(-2)} \oplus 2(2, \mathfrak{p} + \mathfrak{q})^{(-1)} \oplus [\mathfrak{su}(1, 1 | \mathfrak{p} + \mathfrak{q}) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(1, 1)]$$

$$\oplus 2(2, \mathfrak{p} + \mathfrak{q})^{(+1)} \oplus 1^{(+2)}$$

Using these fermionic oscillators, we define the $2(2 | \mathfrak{p} + \mathfrak{q})$ supersymmetry generators

$$S_\mu = \alpha_\mu x \quad S^\mu = (S_\mu)^\dagger = \alpha^\mu x \quad S_y = \beta_y x \quad S^y = (S_y)^\dagger = \beta^y x.$$

These supersymmetry generators, together with the bosonic generators

$$E^1 = x d^1 \quad E^2 = x g \quad E_1 = x d \quad E_2 = -x g^\dagger$$

span the grade $-1$ subspace $\mathfrak{g}^{(-1)}$. Clearly, under anticommutation, the $\mathfrak{g}^{(-1)}$ supersymmetry generators close into the $\mathfrak{g}^{(-2)}$ generator $E$:

$$\{S_\mu, S^\nu\} = 2\delta_\mu^\nu E \quad \{S_\mu, S_\nu\} = 0 \quad \{S^\mu, S^\nu\} = 0$$

$$\{S_y, S^z\} = 2\delta_y^z E \quad \{S_y, S_z\} = 0 \quad \{S^y, S^z\} = 0$$

Now based on the results of previous sections and those of [14, 21] we define the $\mathfrak{g}^{(+2)}$ generator $F$ with a deformation parameter $\zeta$ as follows

$$F = \frac{1}{2} p^2 + \frac{1}{2 x^2} \left[(N_d - N_g + N_\alpha - N_\beta + \zeta)^2 + \lambda\right]$$
where \( \zeta \) is the deformation parameter and \( \lambda \) is a constant to be determined. The \( 2(p+q) \) supersymmetry generators \( Q_\mu, Q^\mu = (Q_\mu)^\dagger, Q_y \) and \( Q^y = (Q_y)^\dagger \) in \( g^{(+1)} \) space are defined by commutation of grade \(-1\) supersymmetry generators with \( F \):

\[
Q_\mu = -i [S_\mu, F] = \alpha_\mu \left[ p - \frac{i}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2} \right) \right]
\]

\[
Q^\mu = -i [S^\mu, F] = \alpha^\mu \left[ p + \frac{i}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2} \right) \right]
\]

\[
Q_y = -i [S_y, F] = \beta_y \left[ p + \frac{i}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2} \right) \right]
\]

\[
Q^y = -i [S^y, F] = \beta^y \left[ p - \frac{i}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2} \right) \right]
\]

Requiring that

\[
\{Q_\mu, Q^{\nu}\} = 2 \delta^{\nu}_{\mu} F \quad \{Q_y, Q^2\} = 2 \delta^y_2 F
\]

fixes the constant \( \lambda = -\frac{1}{4} \). Therefore

\[
F = \frac{1}{2} p^2 + \frac{1}{2 x^2} \left( (N_d - N_g + N_\alpha - N_\beta + \zeta)^2 - \frac{1}{4} \right)
\]

The bosonic generators of \( SU(2, 2) \) in \( g^{(0)} \) subspace that are modified by the supersymmetric extension as

\[
F^1 = d^\dagger \left[ p + \frac{i}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2} \right) \right]
\]

\[
F^2 = g \left[ p + \frac{i}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2} \right) \right]
\]

\[
F_1 = d \left[ p - \frac{i}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2} \right) \right]
\]

\[
F_2 = -g^\dagger \left[ p - \frac{i}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2} \right) \right].
\]

These modifications correspond simply to a shift in the deformation parameter \( \zeta \) by \( (N_\alpha - N_\beta) \). The only generator of \( SU(2, 2) \) in \( g^{(0)} \) subspace that is modified by the supersymmetric extension is:

\[
U = N_d - N_g + \frac{1}{2} (N_\alpha - N_\beta + \zeta)
\]

which again represents a shift in the deformation parameter \( \zeta \). Under anti-commutation, the
supersymmetry generators in \( g^{(-1)} \) and \( g^{(+1)} \) close into the bosonic generators in \( g^{(0)} \):

\[
\begin{align*}
\{S_\mu, Q^\nu\} &= \delta^\nu_\mu \Delta + 2i \delta^\nu_\mu J^1_1 + 2i \delta^\nu_\mu U - 2i A^\nu_\mu - 2i \delta^\nu_\mu \left( N_d + \frac{1}{p} N_\alpha \right) \\
\{S^\nu, Q_\mu\} &= \delta^\nu_\mu \Delta - 2i \delta^\nu_\mu J^1_1 + 2i \delta^\nu_\mu U + 2i A^\nu_\mu + 2i \delta^\nu_\mu \left( N_d + \frac{1}{p} N_\alpha \right) \\
\{S_y, Q^z\} &= \delta^z_y \Delta - 2i \delta^z_y J^2_2 - 2i \delta^z_y U - 2i B^z_y - 2i \delta^z_y \left( N_g + \frac{1}{q} N_\beta \right) \\
\{S^z, Q_y\} &= \delta^z_y \Delta + 2i \delta^z_y J^2_2 + 2i \delta^z_y U + 2i B^z_y + 2i \delta^z_y \left( N_g + \frac{1}{q} N_\beta \right)
\end{align*}
\] (7.16)

where \( N_d + \frac{1}{p} N_\alpha \) and \( N_g + \frac{1}{q} N_\beta \) are the \( U(1) \) generators in \( SU(1 \mid p) \) and \( SU(1 \mid q) \), respectively.

The 4 \((p + q)\) supersymmetry generators in the \( g^{(0)} \) subspace are determined by the commutators between even (odd) generators in \( g^{(-1)} \) and odd (even) generators in \( g^{(+1)} \):

\[
\begin{align*}
\bar{Q}^1_\mu &= \alpha_\mu d^\dagger = -i \frac{1}{2} [E^1, Q^\mu] = -i \frac{1}{2} [S_\mu, F^1] \\
\bar{S}^2_\mu &= \alpha_\mu g = -i \frac{1}{2} [E^2, Q^\mu] = i \frac{1}{2} [S_\mu, F^2] \\
\bar{Q}^1 y &= \beta^y d = -i \frac{1}{2} [E_1, Q_y] = -i \frac{1}{2} [S_y, F_1] \\
\bar{S}^2 y &= \beta^y g = i \frac{1}{2} [E_2, Q_y] = i \frac{1}{2} [S_y, F_2]
\end{align*}
\] (7.17)

They satisfy the anti-commutation relations:

\[
\begin{align*}
\{ \bar{Q}^1_\mu, Q^\nu\} &= A^\nu_\mu + \delta^\nu_\mu \left( N_d + \frac{1}{p} N_\alpha \right) \\
\{ \bar{Q}^1_\mu, \bar{S}^2_\nu\} &= -\delta^\nu_\mu J^1_2 \\
\{ \bar{S}^2_\mu, \bar{S}^2_\nu\} &= -A^\nu_\mu + \delta^\nu_\mu \left( N_g + 1 - \frac{1}{p} N_\alpha \right) \\
\{ \bar{Q}^1 y, Q^1 z\} &= -B^z_y + \delta^z_y \left( N_d + 1 - \frac{1}{q} N_\beta \right) \\
\{ \bar{S}^2 y, Q^2 z\} &= B^z_y + \delta^z_y \left( N_g + \frac{1}{q} N_\beta \right) \\
\end{align*}
\] (7.18)

\[
\begin{align*}
\{ \bar{Q}^1_\mu, Q^\nu\} &= \delta^\nu_\mu F^1 \\
\{ \bar{S}^2_\mu, Q^\nu\} &= \delta^\nu_\mu F^2 \\
\{ \bar{Q}^1 y, Q^2 z\} &= \delta^z_y F_1 \\
\{ \bar{S}^2 y, Q^2 z\} &= -\delta^z_y F_2
\end{align*}
\] (7.19)
Thus the decomposition of the generators in the 5-grading of the superalgebra takes the form:

\[ E^{(-2)} \oplus [E^p, E_p, S_\mu, S^\mu, S_y, S^y]^{(-1)} \]

\[ \oplus \left[ J^p_q, U, \Delta, \mathcal{A}_\mu, B^2, C^\gamma, C_{\mu y}, \tilde{Q}^1, \tilde{Q}^2, \tilde{S}^\mu_2, \tilde{S}_2^y, \tilde{S}^2 y \right]^{(0)} \]

\[ \oplus [F^p, F_p, Q_\mu, Q^\mu, Q_y, Q^y]^{(+1)} \oplus P^{(+2)} \]

Then the quadratic Casimir of \( SU(2, 2) \) subgroup of the superalgebra in the normalization of [14] becomes

\[
\mathcal{C}_2 = -\frac{1}{6} J^p_q J^q_p + \frac{1}{12} \Delta^2 - \frac{1}{6} (EF + FE) - \frac{1}{6} U^2 - \frac{i}{12} (E_p F^p + F^p E_p - F_p E^p - E^p F_p) \]

\[
= \frac{1}{2} - \frac{1}{8} (N_\alpha - N_\beta + \zeta)^2
\]

where we have used

\[
J^p_q J^q_p = \frac{1}{2} (N_d - N_g)^2 - \frac{1}{2}
\]

\[
\Delta^2 = x^2 p^2 - 2i x p - \frac{1}{4}
\]

\[
EF + FE = \frac{1}{2} x^2 p^2 - i x p + \frac{1}{2} (N_d - N_g)^2 + \frac{1}{2} (N_\alpha - N_\beta + \zeta)^2
\]

\[
+ (N_d - N_g) (N_\alpha - N_\beta + \zeta) - \frac{5}{8}
\]

\[
U^2 = (N_d - N_g)^2 + (N_d - N_g) (N_\alpha - N_\beta + \zeta)
\]

\[
+ \frac{1}{4} (N_\alpha - N_\beta + c)^2
\]

\[
E_p F^p + F^p E_p - F_p E^p - E^p F_p = 4i (N_d - N_g)^2 + 4i (N_d - N_g) (N_\alpha - N_\beta + \zeta) + 4i.
\]

### 7.2 3-grading of \( su(2, 2 \mid p + q) \) with respect to the compact subalgebra \( su(2 \mid p) \oplus su(2 \mid q) \oplus u(1) \)

The Lie superalgebra \( su(2, 2 \mid p + q) \) can be given a 3-graded decomposition with respect to its compact subalgebra \( su(2 \mid p) \oplus su(2 \mid q) \oplus u(1) \)

\[
su(2, 2 \mid p + q) = \mathcal{C}^- \oplus \mathcal{C}^0 \oplus \mathcal{C}^+
\]

where \( \mathcal{C}^0 = su(2 \mid p) \oplus su(2 \mid q) \oplus u(1) \). The \( u(1) \) generator in \( \mathcal{C}^0 \) that defines the 3-grading is given by

\[
\mathcal{H} = \frac{1}{2} [F + E] + (J_1^1 - J_2^2) + \frac{1}{2} (\alpha^\mu \alpha_\mu - \alpha_\mu \alpha^\mu) + \frac{1}{2} (\beta^\gamma \beta_\gamma - \beta_\gamma \beta^\gamma)
\]

\[
= \frac{1}{4} (x^2 + p^2) + \frac{1}{4} x^2 \left[ (N_d - N_g + N_\alpha - N_\beta + \zeta)^2 - \frac{1}{4} \right]
\]

\[
+ \frac{1}{2} (N_d + N_g + N_\alpha + N_\beta) + \frac{2 - p - q}{4}
\]
and once again it plays the role of the “total energy” operator.

The $u(1)$ generator $H$ in $su(2, 2)$, which is the AdS energy, resides within this $\mathcal{H}$.

$$H = \frac{1}{2} \left[ (F + E) + (J_1^1 - J_2^2) \right]$$

$$= \frac{1}{4} \left( x^2 + p^2 \right) + \frac{1}{4} x^2 \left[ (N_d - N_g + N_\alpha - N_\beta + \zeta)^2 - \frac{1}{4} \right] + \frac{1}{2} (N_d + N_g + 1) \quad (7.24)$$

Recall that

$$H_\odot = \frac{1}{4} \left( x^2 + p^2 \right) + \frac{1}{4} x^2 \left[ (N_d - N_g + N_\alpha - N_\beta)^2 - \frac{1}{4} \right] \quad (7.25)$$

is the singular harmonic oscillator part of the AdS Hamiltonian $H$ and

$$H_d = \frac{1}{2} \left( N_d + \frac{1}{2} \right) \quad H_g = \frac{1}{2} \left( N_g + \frac{1}{2} \right) \quad (7.26)$$

are its non-singular parts.

The generators of $su(2)_L \oplus su(2)_R$ subalgebra in $\mathfrak{c}^0$ are given by:

$$L_+ = -\frac{i}{2\sqrt{2}} (E_1 + i F_1)$$

$$= -\frac{i}{2\sqrt{2}} d^\dagger \left[ (x + ip) - \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2} \right) \right]$$

$$L_+ = \frac{i}{2\sqrt{2}} (E_1 - i F_1)$$

$$= \frac{i}{2\sqrt{2}} d \left[ (x - ip) - \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2} \right) \right]$$

$$L_3 = \frac{1}{2} \left[ U - \frac{1}{2} (F + E) + J_1^1 \right]$$

$$= N_d - \frac{1}{2} \left[ H - \frac{1}{2} (N_\alpha - N_\beta + \zeta) - 1 \right] \quad (7.27)$$

$$R_+ = -\frac{i}{2\sqrt{2}} (E_2 + i F_2)$$

$$= \frac{i}{2\sqrt{2}} g^\dagger \left[ (x + ip) + \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2} \right) \right]$$

$$R_- = -\frac{i}{2\sqrt{2}} (E_2 - i F_2)$$

$$= -\frac{i}{2\sqrt{2}} g \left[ (x - ip) + \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2} \right) \right]$$

$$R_3 = -\frac{1}{2} \left[ U + \frac{1}{2} (F + E) + J_2^2 \right]$$

$$= N_g - \frac{1}{2} \left[ H + \frac{1}{2} (N_\alpha - N_\beta + \zeta) + 1 \right] \quad (7.28)$$
The \text{su}(2) generators satisfy the commutation relations
\begin{align}
[L_+, L_-] &= L_3 \quad [L_3, L_+] = \pm L_+ \\
[R_+, R_-] &= R_3 \quad [R_3, R_+] = \pm R_+ .
\end{align}
(7.29)

The quadratic Casimir operators of the two \text{su}(2)'s are different once again, as opposed to the non-supersymmetric non-deformed case (see equations (3.26) and (4.25)):
\begin{align}
L^2 &= \frac{1}{4} \left[ \left( H - \frac{1}{2} (N_\alpha - N_\beta + \zeta) \right)^2 - 1 \right] \\
R^2 &= \frac{1}{4} \left[ \left( H + \frac{1}{2} (N_\alpha - N_\beta + \zeta) \right)^2 - 1 \right].
\end{align}
(7.30)

Bosonic generators in \mathcal{C}^− are:
\begin{align}
B_1 &= \Delta + i (F - E) = -\frac{i}{2} \left\{ (x + ip)^2 - \frac{1}{x^2} \left( (N_d - N_g + N_\alpha - N_\beta + \zeta)^2 - \frac{1}{4} \right) \right\} \\
B_2 &= -i (E_1 + i F_1) = -i d \left[ (x + ip) + \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2} \right) \right] \quad \tag{7.31} \\
B_3 &= -i (E^2 + i F^2) = -i g \left[ (x + ip) - \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2} \right) \right] \\
B_4 &= -2i J^2_1 = -2i d g
\end{align}
while bosonic generators in \mathcal{C}^+ are:
\begin{align}
B_1 &= \Delta - i (F - E) = \frac{i}{2} \left\{ (x - ip)^2 - \frac{1}{x^2} \left( (N_d - N_g + N_\alpha - N_\beta + \zeta)^2 - \frac{1}{4} \right) \right\} \\
B_2 &= i (E^1 - i F^1) = i d^\dagger \left[ (x - ip) + \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2} \right) \right] \quad \tag{7.32} \\
B_3 &= -i (E_2 - i F_2) = i g^\dagger \left[ (x - ip) - \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2} \right) \right] \\
B_4 &= -2i J^1_1 = 2i d^\dagger g^\dagger
\end{align}
The satisfy the following commutation relations:
\begin{align}
[H, B_i] &= -B_i \\
[H, B^i] &= +B^i \quad \text{where} \quad i = 1, 2, 3, 4 \\
[B_1, B^1] &= 8 H_\odot \\
[B_2, B^2] &= 4 (H + L_3 - R_3) \quad \tag{7.33} \\
[B_3, B^3] &= 4 (H - L_3 + R_3) \\
[B_4, B^4] &= 8 (H_d + H_g)
\end{align}

Once again, we have the important relation
\begin{align}
B^3 B^2 = B^4 B^1
\end{align}
in the deformed minrep.
In this basis, the supersymmetry generators in $C^-$ are given by:

\[
S_\mu = \frac{1}{2} (S_\mu + i Q_\mu) = \frac{1}{2} \alpha_\mu \left[ (x + ip) + \frac{1}{x} \left(N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2}\right) \right]
\]

\[
\Omega_\mu = \tilde{S}_\mu^2 = \alpha_\mu g
\]

\[
S_y = \frac{1}{2} (S_y + i Q_y) = \frac{1}{2} \beta_y \left[ (x + ip) - \frac{1}{x} \left(N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2}\right) \right]
\]

\[
\Omega_y = \tilde{Q}_{1y} = \beta_y d
\]

and the supersymmetry generators in $C^+$ are given by:

\[
S^\mu = \frac{1}{2} (S^\mu - i Q^\mu) = \frac{1}{2} \alpha^\mu \left[ (x - ip) + \frac{1}{x} \left(N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2}\right) \right]
\]

\[
\Omega^\mu = \tilde{S}_2^\mu = \alpha^\mu g^\dagger
\]

\[
S^y = \frac{1}{2} (S^y - i Q^y) = \frac{1}{2} \beta^y \left[ (x - ip) - \frac{1}{x} \left(N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2}\right) \right]
\]

\[
\Omega^y = \tilde{Q}_{1y}^\dagger = \beta^y d^\dagger
\]

Note that

\[
S_\mu = (S_\mu)^\dagger \quad \Omega_\mu = (\Omega_\mu)^\dagger \quad S_y = (S_y)^\dagger \quad \Omega_y = (\Omega_y)^\dagger .
\]

These supersymmetry generators in $C^{(\pm)}$ satisfy the following (anti-)commutation relations:

\[
[\mathcal{H}, S_\mu] = - S_\mu \quad [\mathcal{H}, S^\mu] = - S^\mu \quad [\mathcal{H}, S_y] = - S_y \quad [\mathcal{H}, S^y] = - S^y \quad [\mathcal{H}, \Omega_\mu] = - \Omega_\mu \quad [\mathcal{H}, \Omega^\mu] = - \Omega^\mu \quad [\mathcal{H}, \Omega_y] = - \Omega_y \quad [\mathcal{H}, \Omega^y] = - \Omega^y
\]

\[
\{S_\mu, S^\nu\} = \delta^\nu_\mu \left( H + L_3 - R_3 \right) - A^\nu_\mu - \delta^\nu_\mu \left(N_d + \frac{1}{p} N_\alpha \right)
\]

\[
\{\Omega_\mu, \Omega^\nu\} = \delta^\nu_\mu \left( H_d + H_g \right) - A^\nu_\mu - \delta^\nu_\mu \left(N_d + \frac{1}{p} N_\alpha \right)
\]

\[
\{S_y, S^z\} = \delta^z_y \left( H - L_3 + R_3 \right) - B^z_y - \delta^z_y \left(N_g + \frac{1}{q} N_\beta \right)
\]

\[
\{\Omega_y, \Omega^z\} = \delta^z_y \left( H_d + H_g \right) - B^z_y - \delta^z_y \left(N_g + \frac{1}{q} N_\beta \right)
\]

The supersymmetry generators in grade 0 space $C^0$ are obtained by taking the commutators between fermionic (bosonic) generators in $C^-$ space and bosonic (fermionic) generators
in \( \mathcal{C}^+ \) space. They are as follows:

\[
\tilde{\mathcal{S}}_\mu = \frac{1}{2} (S_\mu - i Q_\mu) = \frac{1}{2} \alpha_\mu \left[ (x - i p) - \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2} \right) \right]
\]

\[
\tilde{\Omega}_\mu = \tilde{Q}_\mu^\dagger = \alpha_\mu d^\dagger
\]

\[
\tilde{\mathcal{S}}^\mu = \frac{1}{2} (S^\mu + i Q^\mu) = \frac{1}{2} \alpha^\mu \left[ (x + i p) - \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2} \right) \right]
\]

\[
\tilde{\Omega}^\mu = \tilde{Q}^\mu_\mu = \alpha^\mu d
\]

\[
\tilde{\mathcal{S}}_y = \frac{1}{2} (S_y - i Q_y) = \frac{1}{2} \beta_y \left[ (x - i p) + \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta + \frac{1}{2} \right) \right]
\]

\[
\tilde{\Omega}_y = \tilde{S}_2 y = \beta_y g^\dagger
\]

\[
\tilde{\mathcal{S}}^y = \frac{1}{2} (S^y + i Q^y) = \frac{1}{2} \beta^y \left[ (x + i p) + \frac{1}{x} \left( N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2} \right) \right]
\]

\[
\tilde{\Omega}^y = \tilde{S}^2 y = \beta^y g
\]

The anti-commutators between the supersymmetry generators in \( \mathcal{C}^- \) and those in \( \mathcal{C}^+ \) take the following form:

\[
\{ \tilde{\mathcal{S}}_\mu, \tilde{\mathcal{S}}^\nu \} = -2 \delta_\mu^\nu L_3 + A_\mu^\nu + \delta_\mu^\nu \left( N_d + \frac{1}{p} N_\alpha \right)
\]

\[
\{ \tilde{\Omega}_\mu, \tilde{\Omega}^\nu \} = A_\mu^\nu + \delta_\mu^\nu \left( N_d + \frac{1}{p} N_\alpha \right)
\]

\[
\{ \tilde{\mathcal{S}}_y, \tilde{\mathcal{S}}^z \} = -2 \delta_y^z R_3 + B_y^z + \delta_y^z \left( N_g + \frac{1}{q} N_\beta \right)
\]

\[
\{ \tilde{\Omega}_y, \tilde{\Omega}^z \} = B_y^z + \delta_y^z \left( N_g + \frac{1}{q} N_\beta \right)
\]

The commutators between bosonic generators in \( \mathcal{C}^- \) and supersymmetry generators in \( \mathcal{C}^+ \) are as follows:

\[
[B_1, \mathcal{S}^\mu] = -2i \tilde{\mathcal{S}}^\mu \quad [B_2, \mathcal{S}^\mu] = -2i \tilde{\mathcal{S}}^\mu \quad [B_3, \mathcal{S}^\mu] = 0 \quad [B_4, \mathcal{S}^\mu] = 0
\]

\[
[B_1, \Omega^\mu] = 0 \quad [B_2, \Omega^\mu] = 0 \quad [B_3, \Omega^\mu] = -2i \tilde{\Omega}^\mu \quad [B_4, \Omega^\mu] = -2i \tilde{\Omega}^\mu
\]

\[
[B_1, \mathcal{S}^y] = -2i \tilde{\mathcal{S}}^y \quad [B_2, \mathcal{S}^y] = 0 \quad [B_3, \mathcal{S}^y] = -2i \tilde{\mathcal{S}}^y \quad [B_4, \mathcal{S}^y] = 0
\]

\[
[B_1, \Omega^y] = 0 \quad [B_2, \Omega^y] = -2i \tilde{\Omega}^y \quad [B_3, \Omega^y] = 0 \quad [B_4, \Omega^y] = -2i \tilde{\Omega}^y
\]

The anticommutators of supersymmetry generators in \( \mathcal{C}^0 \) and those in \( \mathcal{C}^+ \) can be written...
as

\[
\begin{align*}
\{ \tilde{S}_\mu , S_\nu \} &= 0 \quad \{ \tilde{S}_\mu , \Omega_\nu \} = 0 \quad \{ \tilde{S}_\mu , S_z \} = -C^z_\mu \quad \{ \tilde{S}_\mu , \Omega_z \} = 0 \\
\{ \tilde{S}_\mu , \Omega_\nu \} &= 0 \quad \{ \tilde{S}_\mu , \Omega_z \} = 0 \quad \{ \tilde{S}_\mu , S_z \} = -C^z_\mu \\
\{ \tilde{S}_\nu , S_\nu \} &= 0 \quad \{ \tilde{S}_\nu , \Omega_\nu \} = 0 \quad \{ \tilde{S}_\nu , S_z \} = 0 \quad \{ \tilde{S}_\nu , \Omega_z \} = 0 \\
\{ \tilde{S}_\nu , \Omega_\nu \} &= 0 \quad \{ \tilde{S}_\nu , \Omega_z \} = 0 \quad \{ \tilde{S}_\nu , S_z \} = 0 \quad \{ \tilde{S}_\nu , \Omega_z \} = 0 \\
\{ \tilde{S}_\nu , \Omega_\nu \} &= 0 \quad \{ \tilde{S}_\nu , \Omega_z \} = 0 \quad \{ \tilde{S}_\nu , S_z \} = 0 \quad \{ \tilde{S}_\nu , \Omega_z \} = 0
\end{align*}
\]

(7.43)

where \( C^z_\mu = \beta^z_\alpha \alpha^\mu \) are the \( \mathfrak{su}(p + q) \) generators that belong to the \( \mathfrak{c}^+ \) subspace.

8. Minimal Unitary Supermultiplet of \( \mathfrak{su}(2, 2|4) \) and its Deformations

From equations (7.30), it follows that the quadratic Casimir operators of \( SU(2)_L \) and \( SU(2)_R \) can be written as

\[
L^2 = \mathcal{L} (\mathcal{L} + 1) \quad \quad \quad R^2 = \mathcal{R} (\mathcal{R} + 1)
\]

(8.1)

where

\[
\mathcal{L} = \frac{1}{2} \left[ H - \frac{1}{2} (N_\alpha - N_\beta + \zeta) - 1 \right] \quad \quad \quad \mathcal{R} = \frac{1}{2} \left[ H + \frac{1}{2} (N_\alpha - N_\beta + \zeta) - 1 \right].
\]

(8.2)

As we have shown earlier, \( H \) (AdS energy) is given by equation (7.24). As before we shall denote the lowest energy state of the singular (isotonic) oscillator with coordinate wave function

\[
C_0 x^\alpha e^{-x^2/2}
\]

(8.3)

as \( |\psi^{(\alpha)}\rangle \) and its tensor product with the vacua of the bosonic and fermionic oscillators as \( |\alpha; 0, 0; 0, 0\rangle \). Note that we use the notation \( |\alpha; n_d, n_g, n_\alpha, n_\beta\rangle \), where \( n_d, n_g, n_\alpha, n_\beta \) are the eigenvalues of the respective bosonic and fermionic number operators.

Clearly,

\[
d |\alpha; 0, 0; 0, 0\rangle = g |\alpha; 0, 0; 0, 0\rangle = \alpha_\mu |\alpha; 0, 0; 0, 0\rangle = \beta_y |\alpha; 0, 0; 0, 0\rangle = 0.
\]

(8.4)

We shall study the case \( p = q = 2 \). Twistorial oscillator construction of the unitary supermultiplets of \( SU(2, 2|4) \) has been studied in [20–22, 49].

8.1 Minimal Unitary Supermultiplet of \( \mathfrak{su}(2, 2|4) \)

Let us first analyze the minimal unitary supermultiplet of \( \mathfrak{su}(2, 2|4) \) for which the deformation parameter is zero

\[
\zeta = 0.
\]

(8.5)

The state \( |\frac{1}{2}; 0, 0; 0, 0\rangle \) is the unique normalizable lowest energy state annihilated by all bosonic generators \( B_i \) \( (i = 1, 2, 3, 4) \) as well as supersymmetry generators in \( \mathfrak{c}^- \) subspace. It is a singlet of \( SU(2|2) \times SU(2|2) \) subalgebra. By acting on it with the grade +1 generators
in the subspace $\mathcal{C}^+$ one obtains an infinite set of states which form a basis for the minimal unitary irreducible representation of $\mathfrak{su}(2,2|4)$. This infinite set of states can be decomposed into a finite number of irreducible representations of the even subgroup $SU(2,2) \times SU(4)$, with each irrep of $SU(2,2)$ corresponding to a massless conformal field in four dimensions.

In Table 2, we present the supermultiplet that is obtained by starting from this unique lowest weight vector

$$|\Omega\rangle = \left| \frac{1}{2}, 0, 0, 0, 0 \right\rangle$$

and acting on it with the supersymmetry generators of grade +1 space $\mathcal{C}^+$. The resultant supermultiplet is the $N = 4$ Yang-Mills supermultiplet constructed long ago in [20] which was called the CPT self-conjugate doubleton supermultiplet. In the twistorial oscillator approach the lowest weight vector $|\Omega\rangle$ is the vacuum vector $|0\rangle$ of all the oscillators in the $SU(2|2) \times SU(2|2) \times U(1)$ basis [20–22].

We should note that the positive energy unitary representations of $SU(2,2)$ are uniquely determined by the $SU(2)_L \times SU(2)_R \times U(1)$ labels $\mathcal{L}, \mathcal{R}, H$ of their lowest energy states. Thus the $SU(2,2) \times SU(4)$ decomposition of the unitary supermultiplets of $SU(2,2|4)$ given in the tables below can be read off from these labels together with the dimensions of irreps of $SU(4)$ that are listed. The eigenvalue of $H$ is simply the conformal dimension of the corresponding massless field in four dimensions.

Table 2: The minimal unitary supermultiplet of $\mathfrak{su}(2,2|4)$ with the lowest weight vector indicated with an asterisk. $SU(2)_L \times SU(2)_R \times U(1)$ labels of the unitary representations of $SU(2,2)$ are denoted as $\mathcal{L}, \mathcal{R}$ and $H$.

| State | $|\alpha; n_\alpha, n_\beta\rangle$ | $|\beta; n_\beta, n_\beta\rangle$ | $H$ | $\mathcal{H}$ | $\mathcal{L}$ | $\mathcal{R}$ | $\mathcal{R}_3$ | $SU(4)$ |
|-------|----------------------------------|----------------------------------|-----|------------|------------|------------|------------|----------|
| $\psi^\dagger | \frac{1}{2}; 0, 0, 0)$ | $\frac{1}{2}; 0, 0, 0)$ | 1 | 0 | 0 | 0 | 0 | 6 |
| $\delta^\dagger | \frac{1}{2}; 0, 0, 0)$ | $\frac{1}{2}; 0, 0, 0, 1)$ | $\frac{3}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\mathcal{T}$ |
| $\delta^\dagger | \frac{1}{2}; 0, 0, 0)$ | $\frac{1}{2}; 1, 0, 0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 |
| $\delta^\dagger | \frac{1}{2}; 0, 0, 0)$ | $\frac{1}{2}; 0, 1, 0)$ | $\frac{3}{2}$ | 1 | 0 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 4 |
| $\delta^\dagger | \frac{1}{2}; 0, 0, 0)$ | $\frac{1}{2}; 0, 1, 0)$ | 0 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 |
| $\delta^\dagger | \frac{1}{2}; 0, 0, 0)$ | $\frac{1}{2}; 1, 0, 0)$ | 2 | 2 | 1 | $-1$ | 0 | 0 | $\mathcal{T}$ |
| $\delta^\dagger | \frac{1}{2}; 0, 0, 0)$ | $\frac{1}{2}; 2, 0, 0, 0)$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $\delta^\dagger | \frac{1}{2}; 0, 0, 0)$ | $\frac{1}{2}; 1, 2, 0)$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\delta^\dagger | \frac{1}{2}; 0, 0, 0)$ | $\frac{1}{2}; 2, 0, 2)$ | 0 | 0 | 0 | 1 | 0 | 0 |
| $\delta^\dagger | \frac{1}{2}; 0, 0, 0)$ | $\frac{1}{2}; 0, 2, 0)$ | 2 | 2 | 0 | 0 | 1 | $-1$ | 1 |

Note that in [20–22] $H$ is the $\mathfrak{u}(1)$ generator corresponding to AdS$_5$ energy, which is denoted as $E$.
8.2 Deformed minimal unitary supermultiplets of $su(2,2|4)$ for $\zeta \neq 0$

When $\zeta \neq 0$, there is a multiplet of states that are annihilated by the generators in $\mathcal{C}^-$ and transform irreducibly under the subalgebra $\mathcal{C}^0$. By an abuse of terminology we shall refer to them as “lowest weight vectors” $|\Omega\rangle$ for any given non-zero integer value of $\zeta$.

In Table 3, we list all those states $|\Omega\rangle$ that are annihilated by all the generators (bosonic and fermionic) in grade $-1$ space $\mathcal{C}^-$ of $SU(2,2|4)$ for various values of $\zeta \neq 0$. For a given $\zeta$, they form an irreducible representation of $SU(2|2)_L \times SU(2|2)_R$ whose supertableau is

$$|\underline{\zeta \cdots \zeta}, 1\rangle$$

for $\zeta < 0$  

$$|1, \underline{\zeta \cdots \zeta}\rangle$$

for $\zeta > 0$.  \hspace{1cm} (8.7)

| LWV | Range | $\zeta$ | $\zeta$ | $\zeta$ | $\zeta$ | $\zeta$ | $\zeta$ |
|------|-------|--------|--------|--------|--------|--------|--------|
| $|\frac{1}{2} - \zeta; 0, 0, 0\rangle$ | $\zeta = -1, -2, -3, \ldots$ | $1 - \frac{\zeta}{2}$ | $- \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 |
| $|\frac{1}{2} + \zeta; 0, 0, 0\rangle$ | $\zeta = 1, 2, 3, \ldots$ | $1 + \frac{\zeta}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $|\frac{1}{2} - n - \zeta; n, 0, 0\rangle$ | $\zeta = -1, -2, -3, \ldots$ | $1 - \frac{\zeta}{2}$ | $- \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $n + \frac{1}{2}$ | 0 |
| $|\frac{1}{2} - m + \zeta; 0, m, 0\rangle$ | $\zeta = 1, 2, 3, \ldots$ | $1 + \frac{\zeta}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $m - \frac{1}{2}$ |
| $|\frac{1}{2} - p - \zeta; 0, 0, p\rangle$ | $\zeta = -1, -2, -3, \ldots$ | $1 - \frac{p + \zeta}{2}$ | $- \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $p + \frac{1}{2}$ | 0 |
| $|\frac{1}{2} - q + \zeta; 0, 0, q\rangle$ | $\zeta = 1, 2, 3, \ldots$ | $1 - \frac{p - \zeta}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $- \frac{1}{2}$ |
| $|\frac{1}{2} - n - q - \zeta; n, 0, p\rangle$ | $\zeta = -1, -2, -3, \ldots$ | $1 - \frac{p + \zeta}{2}$ | $- \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $n + \frac{1}{2}$ | 0 |
| $|\frac{1}{2} - m - q + \zeta; 0, m, 0\rangle$ | $\zeta = 1, 2, 3, \ldots$ | $1 - \frac{p - \zeta}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $m - \frac{1}{2}$ |

Now, for each $\zeta (\neq 0)$, we can identify separately the lowest weight vectors $|\Omega\rangle$ that are annihilated by all the generators in $\mathcal{C}^-$. For convenience, below in Tables 4, 5, 6, and 7, we list those states $|\Omega\rangle$ for $\zeta = -1, +1, -2, +2$. Clearly, in each case the possible lowest weight vectors form an irreducible representation of $SU(2|2)_L \times SU(2|2)_R$ whose supertableau is given by equation (8.7).
Table 4: States $|\Omega\rangle$ that are annihilated by all grade $-1$ generators within the minimal unitary representation space of $SU(2,2|4)$ when $\zeta = -1$. They transform in the irreducible representation $|0,0,0,0\rangle$ of $SU(2|2)_L \times SU(2|2)_R$.

| LWV  | $H$ | $\mathcal{M}$ | $L$ | $Z_3$ | $\mathcal{R}$ | $\mathfrak{R}$ |
|------|-----|-------------|-----|-------|-------------|-------------|
| $|\frac{3}{2},0,0,0\rangle$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $0$ | $0$ |
| $|\frac{1}{2},0,1,0\rangle$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $0$ |
| $|\frac{1}{2},0,0,1\rangle$ | $1$ | $\frac{1}{2}$ | $0$ | $0$ | $0$ | $0$ |

Table 5: States $|\Omega\rangle$ that are annihilated by all grade $-1$ generators within the minimal unitary representation space of $SU(2,2|4)$ when $\zeta = +1$. They transform in the irreducible representation $|0,0,1\rangle$ of $SU(2|2)_L \times SU(2|2)_R$.

| LWV  | $H$ | $\mathcal{M}$ | $L$ | $Z_3$ | $\mathcal{R}$ | $\mathfrak{R}$ |
|------|-----|-------------|-----|-------|-------------|-------------|
| $|\frac{3}{2},0,0,0\rangle$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $0$ | $0$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| $|\frac{1}{2},0,1,0\rangle$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $|\frac{1}{2},0,0,1\rangle$ | $1$ | $\frac{1}{2}$ | $0$ | $0$ | $0$ | $0$ |

Table 6: States $|\Omega\rangle$ that are annihilated by all grade $-1$ generators within the minimal unitary representation space of $SU(2,2|4)$ when $\zeta = -2$. They transform in the irreducible representation $|0,1,0,0\rangle$ of $SU(2|2)_L \times SU(2|2)_R$.

| LWV  | $H$ | $\mathcal{M}$ | $L$ | $Z_3$ | $\mathcal{R}$ | $\mathfrak{R}$ |
|------|-----|-------------|-----|-------|-------------|-------------|
| $|\frac{3}{2},0,0,0\rangle$ | $2$ | $1$ | $1$ | $-1$ | $0$ | $0$ |
| $|\frac{1}{2},1,0,0\rangle$ | $2$ | $1$ | $1$ | $0$ | $0$ | $0$ |
| $|\frac{1}{2},2,0,0\rangle$ | $2$ | $1$ | $1$ | $1$ | $0$ | $0$ |
| $|\frac{1}{2},0,1,0\rangle$ | $\frac{3}{2}$ | $1$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $0$ | $0$ |
| $|\frac{1}{2},0,0,2\rangle$ | $1$ | $1$ | $0$ | $0$ | $0$ | $0$ |
| $|\frac{1}{2},1,0,1\rangle$ | $\frac{1}{2}$ | $1$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $0$ |

Table 7: States $|\Omega\rangle$ that are annihilated by all grade $-1$ generators within the minimal unitary representation space of $SU(2,2|4)$ when $\zeta = +2$. They transform in the irreducible representation $|1,0,0\rangle$ of $SU(2|2)_L \times SU(2|2)_R$.

| LWV  | $H$ | $\mathcal{M}$ | $L$ | $Z_3$ | $\mathcal{R}$ | $\mathfrak{R}$ |
|------|-----|-------------|-----|-------|-------------|-------------|
| $|\frac{3}{2},0,0,0\rangle$ | $2$ | $1$ | $0$ | $0$ | $1$ | $-1$ |
| $|\frac{1}{2},0,1,0\rangle$ | $2$ | $1$ | $0$ | $0$ | $1$ | $0$ |
| $|\frac{1}{2},0,2,0\rangle$ | $2$ | $1$ | $0$ | $0$ | $1$ | $1$ |
Next we construct the supermultiplets that can be obtained for each \( \zeta (\neq 0) \) by starting from the above lowest weight vectors \(|\Omega\rangle\) and acting on the them with the supersymmetry generators in grade +1 space \( \mathfrak{c}^+ \).

The supermultiplet that corresponds to \( \zeta = -1 \) (given in Table 8) is exactly the doubleton supermultiplet in [21, 22] obtained by starting from the lowest weight vector \(|\Omega\rangle = |\Omega, -1\rangle\).

Next we give the doubleton supermultiplet that corresponds to \( \zeta = +1 \) in Table 9. This supermultiplet was obtained in [21, 22] by starting from the lowest weight vector \(|\Omega\rangle = |1, \Omega\rangle\).

| LWV | \( H \) | \( M \) | \( L \) | \( \Sigma_3 \) | \( \Pi \) | \( \Pi_3 \) |
|-----|-----|-----|-----|-----|-----|-----|
| \(|\frac{1}{2}, 0, 0, 0, 1\rangle\) | \(\frac{1}{2}\) | 0 | 0 | 0 | \(-\frac{1}{2}\) | 0 |
| \(|\frac{1}{2}, 0, 0, 0, 2\rangle\) | 1 | 0 | 0 | 0 | 0 | 0 |
| \(|\frac{1}{2}, 0, 1, 0, 1\rangle\) | \(-\frac{1}{2}\) | 0 | 0 | 0 | 0 | 0 |

State \( |\alpha; n_4, n_5, n_\alpha, n_\beta\rangle\) | \( H \) | \( M \) | \( L \) | \( \Sigma_3 \) | \( \Pi \) | \( \Pi_3 \) | \( SU(4) \)

\(|\frac{1}{2}, 0, 0, 0, 1\rangle^*\) | \(|\frac{1}{2}, 0, 0, 0\rangle^*\) | \(\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | 0 | 0 | 6 |

\(|\frac{1}{2}, 1, 0, 0, 0\rangle^*\) | \(|\frac{1}{2}, 1, 0, 0, 0\rangle^*\) | \(\frac{1}{2}\) | \(\frac{1}{2}\) | 0 | 0 | 0 |

\(|\frac{1}{2}, 0, 0, 0, 1\rangle\) | \(|\frac{1}{2}, 0, 0, 0\rangle\) | 2 | \(\frac{3}{2}\) | 1 | -1 | 0 | 0 | \(\Upsilon\) |

\(|\frac{1}{2}, 0, 0, 1, 0\rangle\) | \(|\frac{1}{2}, 1, 0, 1\rangle\) | 1 | 0 | 0 | 0 |

\(|\frac{1}{2}, 2, 0, 0\rangle\) | \(|\frac{1}{2}, 2, 0, 0\rangle\) | 1 | +1 | 0 | 0 |

\(|\frac{1}{2}, 0, 0, 2\rangle\) | \(|\frac{1}{2}, 1, 0, 2\rangle\) | \(\frac{3}{2}\) | \(-\frac{1}{2}\) | 0 | 0 |

\(|\frac{1}{2}, 0, 2, 0\rangle\) | \(|\frac{1}{2}, 1, 2, 0\rangle\) | \(\frac{3}{2}\) | \(+\frac{1}{2}\) | 0 | 0 |

\(|\frac{1}{2}, 0, 1, 2\rangle\) | \(|\frac{1}{2}, 1, 0, 2\rangle\) | \(\frac{3}{2}\) | \(+\frac{1}{2}\) | 0 | 0 |

\(|\frac{1}{2}, 0, 1, 0\rangle\) | \(|\frac{1}{2}, 0, 1, 0\rangle\) | 1 | \(\frac{1}{2}\) | 0 | 0 | 0 | 4 |

\(|\frac{1}{2}, 0, 2, 0\rangle\) | \(|\frac{1}{2}, 0, 2, 0\rangle\) | \(\frac{3}{2}\) | \(\frac{3}{2}\) | 0 | 0 | \(-\frac{1}{2}\) | \(+\frac{1}{2}\) | 1 |
Table 9: The doubleton supermultiplet corresponding to $\zeta = +1$. The states that are marked with an asterisk belong to $|\Omega\rangle = |1\rangle \otimes |\bar{\Omega}\rangle$.

| State $|\alpha; n_\Omega, n_{\bar{\Omega}}, n_{\alpha}, n_{\bar{\alpha}}\rangle$ | $H$ | $\mathcal{H}$ | $\mathcal{L}$ | $\mathcal{L}_3$ | $\mathfrak{R}$ | $\mathfrak{R}_3$ | SU(4) |
|---|---|---|---|---|---|---|---|
| $|\frac{3}{2}, 0, 0, 0\rangle^*$ | $|\frac{3}{2}, 0, 0, 0\rangle^*$ | $\frac{3}{2}$ | $\frac{3}{2}$ | 0 | 0 | $\frac{1}{2}$ | 6 |
| $|\frac{1}{2}, 0, 1, 0\rangle^*$ | $|\frac{1}{2}, 0, 1, 0\rangle^*$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 12 |
| $\psi^\mu \left|\frac{3}{2}, 0, 0, 0\right\rangle$ | $\left|\frac{3}{2}, 0, 0, 1\right\rangle$ | 2 | $\frac{3}{2}$ | 0 | 0 | $\frac{1}{2}$ | 1 |
| $\Omega^\mu \left|\frac{3}{2}, 0, 0, 0\right\rangle$ | $\left|\frac{1}{2}, 1, 0, 0\right\rangle$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 1 |
| $\psi^\nu \left|\frac{3}{2}, 0, 0, 0\right\rangle$ | $\left|\frac{3}{2}, 0, 0, 2\right\rangle$ | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 |
| $\Omega^\nu \left|\frac{3}{2}, 0, 0, 0\right\rangle$ | $\left|\frac{1}{2}, 0, 0, 2\right\rangle$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 7 |

The supermultiplet we obtain by taking $\zeta = -2$ (given in Table 10) corresponds to the doubleton supermultiplet in [21, 22] obtained by starting from the lowest weight vector $|\Omega\rangle = |\bar{\Omega}\rangle$, 1.

Table 10: The doubleton supermultiplet corresponding to $\zeta = -2$. The states that are marked with an asterisk belong to $|\Omega\rangle = |\bar{\Omega}\rangle$, 1.

| State $|\alpha; n_\Omega, n_{\bar{\Omega}}, n_{\alpha}, n_{\bar{\alpha}}\rangle$ | $H$ | $\mathcal{H}$ | $\mathcal{L}$ | $\mathcal{L}_3$ | $\mathfrak{R}$ | $\mathfrak{R}_3$ | SU(4) |
|---|---|---|---|---|---|---|---|
| $|\frac{3}{2}, 0, 0, 0\rangle^*$ | $|\frac{3}{2}, 0, 0, 0\rangle^*$ | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 |
| $|\frac{3}{2}, 0, 0, 0\rangle^*$ | $|\frac{3}{2}, 0, 0, 0\rangle^*$ | 1 | 0 | 0 | 0 | 6 |
| $|\frac{1}{2}, 0, 1, 0\rangle^*$ | $|\frac{1}{2}, 0, 1, 0\rangle^*$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 |
| $\psi^\nu \left|\frac{3}{2}, 0, 0, 0\right\rangle$ | $\left|\frac{3}{2}, 0, 0, 1\right\rangle$ | 2 | $\frac{3}{2}$ | 0 | 0 | 12 |
| $\Omega^\nu \left|\frac{3}{2}, 0, 0, 0\right\rangle$ | $\left|\frac{1}{2}, 0, 0, 1\right\rangle$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 7 |
| $\psi^\nu \left|\frac{3}{2}, 0, 0, 0\right\rangle$ | $\left|\frac{3}{2}, 0, 0, 1\right\rangle$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 4 |
| $\Omega^\nu \left|\frac{3}{2}, 0, 0, 0\right\rangle$ | $\left|\frac{3}{2}, 0, 0, 1\right\rangle$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 |
Finally, we give the supermultiplet obtain by taking $\zeta = +2$ in Table 11. This supermultiplet corresponds to the doubleton supermultiplet in [21,22] obtained by starting from the lowest weight vector $|\Omega\rangle = |1, \bar{\Gamma}\rangle$.

Table 11: The doubleton supermultiplet corresponding to $\zeta = +2$. The states that are marked with an asterisk belong to $|\Omega\rangle = |1, \bar{\Gamma}\rangle$.

| State | $|a; n_d, n_y; n_o, n_\beta\rangle$ | $H$ | $\mathcal{N}$ | $E_3$ | $\mathfrak{M}$ | $\mathfrak{M}_3$ | $SU(4)$ |
|-------|----------------------------------|-----|-----------|----------|-----------|-----------|----------|
| $\phi^\ast \phi^\ast | \frac{1}{2}, 0, 0, 0, 0\rangle$ | $\frac{1}{2}, 0, 0, 0, 0\rangle^\ast$ | 2 | 1 | 0 | 0 | $-\frac{1}{2}$ | 4 |
| $\phi^\ast \phi^\ast | \frac{1}{2}, 0, 1, 0, 0\rangle$ | $\frac{1}{2}, 0, 1, 0, 0\rangle^\ast$ | 0 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\mathfrak{m}$ |
| $\phi^\ast \phi^\ast | \frac{1}{2}, 0, 2, 0, 0\rangle$ | $\frac{1}{2}, 0, 2, 0, 0\rangle^\ast$ | 0 | 0 | $\frac{1}{2}$ | $+\frac{1}{2}$ | $+\frac{1}{2}$ | $\mathfrak{m}$ |
| $\phi^\ast \phi^\ast | \frac{1}{2}, 0, 0, 1, 0\rangle$ | $\frac{1}{2}, 0, 0, 1, 0\rangle^\ast$ | 2 | 1 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\mathfrak{m}$ |
| $\phi^\ast \phi^\ast | \frac{1}{2}, 0, 1, 0, 1\rangle$ | $\frac{1}{2}, 0, 1, 0, 1\rangle^\ast$ | 0 | 0 | $\frac{1}{2}$ | $+\frac{1}{2}$ | $+\frac{1}{2}$ | $\mathfrak{m}$ |
| $\phi^\ast \phi^\ast | \frac{1}{2}, 0, 0, 0, 0\rangle$ | $\frac{1}{2}, 0, 0, 0, 0\rangle^\ast$ | 3 | 3 | 0 | 0 | 2 | $-2$ | 1 |
Following this method, one can obtain all the other higher spin doubleton supermultiplets by choosing a deformation parameter $|\zeta| > 2$.

9. Minimal Unitary Supermultiplet of $\mathfrak{su}(2, 2|p + q)$ and its Deformations

It is clear that one can easily generalize the above construction of minimal unitary supermultiplet of $\mathfrak{su}(2, 2|4)$ and its deformations to those of $SU(2, 2|p$ and $q$).

Once again, when $\zeta = 0$, the state $|\frac{1}{2},0,0,0\rangle$ is the unique normalizable lowest energy state annihilated by all bosonic and fermionic generators in grade $-1$ space $\mathcal{C}^-$. Thus it forms a singlet of $SU(2|p) \times SU(2|q)$ subalgebra. By acting on it with grade $+1$ generators in $\mathcal{C}^+$, one can obtain an infinite set of states that form a basis for the minimal unitary irreducible representation of $\mathfrak{su}(2, 2|p + q)$. These infinitely many states decompose into a finite number of irreps of the even subgroup $SU(2, 2) \times SU(p + q)$, with each irrep of $SU(2, 2)$ corresponding to a massless conformal field in four dimensions.

When $\zeta \neq 0$, there are multiple states, for any given $\zeta$, that are annihilated by all bosonic and fermionic generators in grade $-1$ space $\mathcal{C}^-$ of $SU(2, 2|p + q)$. They form an irreducible representation of $SU(2|p)_L \times SU(2|q)_R$ whose supertableau is

\[
\begin{align*}
|\cdots 1\rangle & \quad \text{for } \zeta < 0 \\
|\zeta 1\cdots \rangle & \quad \text{for } \zeta > 0.
\end{align*}
\]

In Table 12, we list all such states $|\Omega\rangle$ for different values of the deformation parameter $\zeta \neq 0$.

| LWV          | Range                          | $H$  | $H_s$ | $l$  | $l_3$ | $r$  | $r_3$ |
|--------------|--------------------------------|------|-------|------|-------|------|-------|
| $|\frac{1}{2} - \zeta; 0, 0, 0\rangle$ | $\zeta = -1, -2, -3, \ldots$ | $2 - \zeta$ | $-\zeta$ | $-\frac{2}{\zeta}$ | $\frac{1}{|\zeta|}$ | $0$  | $0$  |
| $|\frac{1}{2} + \zeta; 0, 0, 0\rangle$ | $\zeta = 1, 2, 3, \ldots$     | $2 + \zeta$ | $\zeta$ | $0$  | $\frac{2}{\zeta}$ | $0$  | $0$  |
| $|\frac{1}{2} - n - \zeta; n, 0, 0\rangle$ | $n = 1, 2, \ldots, -\zeta$   | $2 - \zeta$ | $-\zeta$ | $-\frac{2}{\zeta}$ | $n + \frac{2}{\zeta}$ | $0$  | $0$  |
| $|\frac{1}{2} - m + \zeta; 0, m, 0\rangle$ | $m = 1, 2, \ldots, \zeta$     | $2 + \zeta$ | $\zeta$ | $0$  | $\frac{2}{\zeta}$ | $0$  | $0$  |
| $|\frac{1}{2} - p - \zeta; 0, 0, p\rangle$ | $p = 1, 2, \ldots, \zeta$, if $-\zeta < p$ | $2 - p - \zeta$ | $-\zeta$ | $-\frac{2 + p}{\zeta}$ | $\frac{2 + p}{|\zeta|}$ | $0$  | $0$  |
| $|\frac{1}{2} - q + \zeta; 0, 0, q\rangle$ | $q = 1, 2, \ldots, \zeta$, if $\zeta < q$ | $2 - q + \zeta$ | $\zeta$ | $0$  | $\frac{2 - q}{\zeta}$ | $0$  | $0$  |
| $|\frac{1}{2} - n - p - \zeta; n, 0, p\rangle$ | $n = 1, 2, \ldots, \zeta$, if $\zeta < p$ | $2 - p - \zeta$ | $-\zeta$ | $-\frac{2 + p}{\zeta}$ | $n + \frac{2 + p}{\zeta}$ | $0$  | $0$  |
Table 12: (continued)

| LWV | Range | $H$ | $H_s$ | $l$ | $l_3$ | $r$ | $r_3$ |
|------|-------|-----|-------|-----|-------|-----|-------|
| $\frac{1}{2} - m - q + \zeta; 0, 0, 0, q$ | $p, n = 1, 2, 3 \ldots$ $p + n \leq -\zeta$ $\zeta = 1, 2, 3, \ldots$ $q, m = 1, 2, 3, \ldots$ $q + m \leq \zeta$ | $2 - q + \zeta$ | $\zeta$ | 0 | 0 | $\frac{\zeta}{2}$ | $m - \frac{\zeta}{2}$ |

Then it is a straightforward exercise to construct the corresponding deformed minimal unitary supermultiplets of $SU(2, 2|p + q)$ for each value of $\zeta$ by acting on the lowest weight vectors $|\Omega\rangle$ with the supersymmetry generators in grade +1 space $C^+$ repeatedly.

10. Conclusions

In this paper we first studied the minrep of the four dimensional conformal group $SU(2, 2)$ and its deformations obtained by quantizing the quasiconformal realization of $SU(2, 2)$. The resulting representations correspond to massless conformal fields in four spacetime dimensions. We then extended these results to construct the minimal unitary supermultiplet of $SU(2, 2|4)$ and its deformations. The minimal unitary supermultiplet of $SU(2, 2|4)$ is simply the $N = 4$ Yang-Mills supermultiplet. For each integer value of the deformation parameter we obtained a unique supermultiplet of $SU(2, 2|4)$. These supermultiplets are simply the doubleton supermultiplets studied earlier in [20–22].

Decomposition of tensoring of minreps into irreducible unitary representations is, in general, a difficult problem. Since the decomposition of tensor products of doubleton representations into its irreducible components is relatively easier in the twistorial oscillator approach we hope to be able to use our results to solve the tensoring problem for the minreps of $SU(2, 2)$ and $SU(2, 2|p + q)$ as well as their deformations within the quasiconformal approach [50]. We hope that these results will then enable one to tackle the much harder problem of decomposition of tensor products of minreps of noncompact groups that are not of hermitian symmetric type, such as $E_{8(8)}$ or $E_{8(-24)}$.

The extension of the above results to other noncompact groups which admit positive energy unitary representations and their supersymmetric extensions, as well as their applications to AdS/CFT dualities will be the subjects of separate studies.

Acknowledgements: M.G. would like to thank Sergei Gukov and David Vogan for stimulating and informative discussions on the minimal unitary representations of noncompact groups. He would also like to acknowledge stimulating discussions with Juan Maldacena, Shiraz Minwalla, Mukund Rangamani, Augusto Sagnotti and Misha Vasiliev on the minrep of the $4D$ conformal group and thank the organizers of the ‘Fundamental Aspects of Superstring Theory 2009’ Workshop at KITP, UC Santa Barbara and of the ‘New Perspectives in String Theory 2009’ Workshop at GGI, Florence where part of this work was carried out.

S.F. would like to thank the Center for Fundamental Theory at the Institute for Gravitation...
and the Cosmos at Pennsylvania State University, where part of this work was completed, for their warm hospitality.

This work was supported in part by the National Science Foundation under grants numbered PHY-0555605 and PHY-0855356. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

[1] A. Joseph, “Minimal realizations and spectrum generating algebras,” Comm. Math. Phys. 36 (1974) 325–338.

[2] D. A. Vogan, Jr., “Singular unitary representations,” in Noncommutative harmonic analysis and Lie groups (Marseille, 1980), vol. 880 of Lecture Notes in Math., pp. 506–535. Springer, Berlin, 1981.

[3] D. Kazhdan and G. Savin, “The smallest representation of simply laced groups,” in Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), vol. 2 of Israel Math. Conf. Proc., pp. 209–223. Weizmann, Jerusalem, 1990.

[4] R. Brylinski and B. Kostant, “Lagrangian models of minimal representations of $E_6$, $E_7$ and $E_8$,” in Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993), vol. 131 of Progr. Math., pp. 13–63. Birkhäuser Boston, Boston, MA, 1995.

[5] R. Brylinski and B. Kostant, “Minimal representations, geometric quantization, and unitarity,” Proc. Nat. Acad. Sci. U.S.A. 91 (1994), no. 13, 6026–6029.

[6] B. H. Gross and N. R. Wallach, “A distinguished family of unitary representations for the exceptional groups of real rank = 4,” in Lie theory and geometry, vol. 123 of Progr. Math., pp. 289–304. Birkhäuser Boston, Boston, MA, 1994.

[7] J.-S. Li, “Minimal representations & reductive dual pairs,” in Representation theory of Lie groups (Park City, UT, 1998), vol. 8 of IAS/Park City Math. Ser., pp. 293–340. Amer. Math. Soc., Providence, RI, 2000.

[8] D. Kazhdan, B. Pioline, and A. Waldron, “Minimal representations, spherical vectors, and exceptional theta series. I,” Commun. Math. Phys. 226 (2002) 1–40, hep-th/0107222.

[9] M. Günyaydin, K. Koepsell, and H. Nicolai, “Conformal and quasiconformal realizations of exceptional Lie groups,” Commun. Math. Phys. 221 (2001) 57–76, hep-th/0008063.

[10] M. Günyaydin and O. Pavlyk, “Generalized spacetimes defined by cubic forms and the minimal unitary realizations of their quasiconformal groups,” JHEP 08 (2005) 101, hep-th/0506010.

[11] M. Günyaydin, K. Koepsell, and H. Nicolai, “The minimal unitary representation of $E_{6(8)}$,” Adv. Theor. Math. Phys. 5 (2002) 923–946, hep-th/0109005.

[12] M. Günyaydin, G. Sierra, and P. K. Townsend, “Exceptional supergravity theories and the magic square,” Phys. Lett. B133 (1983) 72.

[13] M. Günyaydin and O. Pavlyk, “Minimal unitary realizations of exceptional U-duality groups and their subgroups as quasiconformal groups,” JHEP 01 (2005) 019, hep-th/0409272.
[14] M. Günaydin and O. Pavlyk, “A unified approach to the minimal unitary realizations of noncompact groups and supergroups,” *JHEP* **09** (2006) 050, [hep-th/0604077](https://arxiv.org/abs/hep-th/0604077).

[15] M. Günaydin, A. Neitzke, O. Pavlyk, and B. Pioline, “Quasi-conformal actions, quaternionic discrete series and twistors: $SU(2,1)$ and $G_{2(2)}$,” *Commun. Math. Phys.* **283** (2008) 169–226, [0707.1669](https://arxiv.org/abs/0707.1669).

[16] B. H. Gross and N. R. Wallach, “On quaternionic discrete series representations, and their continuations,” *J. Reine Angew. Math.* **481** (1996) 73–123.

[17] M. Gunaydin and O. Pavlyk, “Spectrum Generating Conformal and Quasiconformal U-Duality Groups, Supergravity and Spherical Vectors,” [0901.1646](https://arxiv.org/abs/0901.1646).

[18] M. Gunaydin and O. Pavlyk, “Quasiconformal Realizations of $E_{6(6)}, E_{7(7)}, E_{8(8)}$ and $SO(n+3, m+3)$, $N = 4$ and $N > 4$ Supergravity and Spherical Vectors,” [0904.0784](https://arxiv.org/abs/0904.0784).

[19] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200](https://arxiv.org/abs/hep-th/9711200).

[20] M. Gunaydin and N. Marcus, “The Spectrum of the $S^5$ Compactification of the Chiral $N = 2, D = 10$ Supergravity and the Unitary Supermultiplets of $U(2,2/4)$,” *Class. Quant. Grav.* **2** (1985) L11.

[21] M. Gunaydin, D. Minic, and M. Zagermann, “4D doubleton conformal theories, CPT and II B string on $AdS(5) \times S(5)$,” *Nucl. Phys.* **B534** (1998) 96–120, [hep-th/9806042](https://arxiv.org/abs/hep-th/9806042).

[22] M. Gunaydin, D. Minic, and M. Zagermann, “Novel supermultiplets of $SU(2, 2|4)$ and the $AdS_5/CFT_4$ duality,” *Nucl. Phys.* **B544** (1999) 737–758, [hep-th/9810226](https://arxiv.org/abs/hep-th/9810226).

[23] I. A. Malkin and V. I. Man'ko *J. Nucl. Phys. (USSR)* **3** (1966) 372.

[24] Y. Nambu *Progr. Theoret. Phys. (Kyoto) Suppl.* **37, 38** (1966) 368.

[25] Y. Nambu, “Infinite-Component Wave Equations with Hydrogenlike Mass Spectra,” *Phys. Rev.* **160** (1967) 1171.

[26] A. O. Barut and H. Kleinert, “Transition Probabilities of the H-Atom from Noncompact Dynamical Groups,” *Phys. Rev.* **156** (1967) 1541.

[27] A. O. Barut and H. Kleinert, “Current Operators and Majorana Equation for the Hydrogen Atom from Dynamical Groups,” *Phys. Rev.* **157** (1967) 1180–1183.

[28] A. O. Barut and H. Kleinert, “Dynamical Group $O(4,2)$ for Baryons and the Behaviour of Form factors,” *Phys. Rev.* **161** (1967) 1464.

[29] G. Mack and I. Todorov, “Irreducibility of the ladder representations of $U(2,2)$ when restricted to the Poincare subgroup,” *J. Math. Phys.* **10** (1969) 2078–2085.

[30] G. Mack, “All Unitary Ray Representations of the Conformal Group $SU(2,2)$ with Positive Energy,” *Commun. Math. Phys.* **55** (1977) 1.

[31] A. W. Knapp and B. Speh, “Irreducible unitary representations of $SU(2,2)$,” *J. Funct. Anal.* **45** (1982), no. 1, 41–73.

[32] V. G. Kac, “Lie Superalgebras,” *Adv. Math.* **26** (1977) 8–96.

[33] V. G. Kac, “A Sketch of Lie Superalgebra Theory,” *Commun. Math. Phys.* **53** (1977) 31–64.
[34] V. K. Dobrev and V. B. Petkova, “On the Group Theoretical Approach to Extended Conformal Supersymmetry: Classification of Multiplets,” Lett. Math. Phys. 9 (1985) 287–298.

[35] V. K. Dobrev and V. B. Petkova, “All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry,” Phys. Lett. B162 (1985) 127–132.

[36] P. A. M. Dirac, “A Remarkable representation of the 3+2 de Sitter group,” J. Math. Phys. 4 (1963) 901–909.

[37] M. Flato and C. Fronsdal, “Quantum Field Theory of Singletons: The Rac,” J. Math. Phys. 22 (1981) 1100.

[38] C. Fronsdal, “The Dirac Supermultiplet,” Phys. Rev. D26 (1982) 1988.

[39] E. Angelopoulos, M. Flato, C. Fronsdal, and D. Sternheimer, “Massless Particles, Conformal Group and De Sitter Universe,” Phys. Rev. D23 (1981) 1278.

[40] M. Günaydin and C. Saclioglu, “Oscillator-like unitary representations of noncompact groups with a Jordan structure and the noncompact groups of supergravity,” Commun. Math. Phys. 87 (1982) 159.

[41] I. Bars and M. Günaydin, “Unitary Representations of Noncompact Supergroups,” Commun. Math. Phys. 91 (1983) 31.

[42] M. Günaydin and N. P. Warner, “Unitary Supermultiplets of OSp(8/4, R) and the Spectrum of the S7 Compactification of Eleven-Dimensional Supergravity,” Nucl. Phys. B272 (1986) 99.

[43] M. Günaydin, P. van Nieuwenhuizen, and N. P. Warner, “General Construction of the Unitary Representations of Anti-De Sitter Superalgebras and the Spectrum of the S4 Compactification of Eleven-Dimensional Supergravity,” Nucl. Phys. B255 (1985) 63.

[44] V. de Alfaro, S. Fubini, and G. Furlan, “Conformal invariance in quantum mechanics,” Nuovo Cim. A34 (1976) 569.

[45] J. Casahorran, “On a novel supersymmetric connection between harmonic and isotonic oscillators,” Physica A 217 (1995) 429–39. DFTUZ-94-28.

[46] J. F. Carinena, A. M. Perelomov, M. F. Ranada, and M. Santander, “A quantum exactly solvable non-linear oscillator related with the isotonic oscillator,” 2008.

[47] A. Perelomov, Generalized coherent states and their applications. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1986.

[48] B. Binegar, “Conformal superalgebras, massless representations, and hidden symmetries,” Phys. Rev. D (3) 34 (1986), no. 2, 525–532.

[49] P. Claus, M. Günaydin, R. Kallosh, J. Rahmfeld, and Y. Zunger, “Supertwistors as quarks of SU(2,2|4),” JHEP 05 (1999) 019, hep-th/9905112.

[50] S. Fernando and M. Günaydin. work in progress.