Approximation Algorithms For The Euclidean Dispersion Problems

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Abstract

In this article, we consider the Euclidean dispersion problems. Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of $n$ points in $\mathbb{R}^2$. For each point $p \in P$ and $S \subseteq P$, we define $\text{cost}_\gamma(p, S)$ as the sum of Euclidean distance from $p$ to the nearest $\gamma$ point in $S \setminus \{p\}$. We define $\text{cost}_\gamma(S) = \min_{p \in S} \{\text{cost}_\gamma(p, S)\}$ for $S \subseteq P$. In the $\gamma$-dispersion problem, a set $P$ of $n$ points in $\mathbb{R}^2$ and a positive integer $k \in [\gamma + 1, n]$ are given. The objective is to find a subset $S \subseteq P$ of size $k$ such that $\text{cost}_\gamma(S)$ is maximized. We consider both 2-dispersion and 1-dispersion problem in $\mathbb{R}^2$. Along with these, we also consider 2-dispersion problem when points are placed on a line.

In this paper, we propose a simple polynomial time $(2\sqrt{3} + \epsilon)$-factor approximation algorithm for the 2-dispersion problem, for any $\epsilon > 0$, which is an improvement over the best known approximation factor $4\sqrt{3}$ [Amano, K. and Nakano, S. I., An approximation algorithm for the 2-dispersion problem, IEICE Transactions on Information and Systems, Vol. 103(3), pp. 506-508, 2020]. Next, we develop a common framework for designing an approximation algorithm for the Euclidean dispersion problem. With this common framework, we improve the approximation factor to $2\sqrt{3}$ for the 2-dispersion problem in $\mathbb{R}^2$. Using the same framework, we propose a polynomial time algorithm, which returns an optimal solution for the 2-dispersion problem when points are placed on a line. Moreover, to show the effectiveness of the framework, we also propose a 2-factor approximation algorithm for the 1-dispersion problem in $\mathbb{R}^2$.

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1 Introduction

The facility location problem is one of the extensively studied optimization problems. Here, we are given a set of locations on which facilities can be placed and a positive integer $k$, and the goal is to place $k$ facilities on those locations so that a specific objective is satisfied. For example, the objective is to place these facilities such that their closeness is undesirable. Often, this closeness measured as a function of the distances between a pair of facilities. We refer to such facility location problem as a dispersion problem. More specifically, we wish to minimize the interference between the placed facilities. The most studied dispersion problem is the max-min dispersion problem.

In the max-min dispersion problem, we are given a set $P = \{p_1, p_2, \ldots, p_n\}$ of $n$ locations, the non-negative distances between each pair of locations $p, q \in P$, and a positive integer $k$ ($k \leq n$). Here, $k$ refers to the number of facilities to be opened and distances are assumed to be symmetric. The objective is to find a $k$ size subset $S \subseteq P$ of locations such that $\text{cost}(S) = \min\{d(p, q) \mid p, q \in S\}$ is maximized, where $d(p, q)$ denotes the distance between $p$ and $q$. This problem is known as 1-dispersion problem in the literature. In this article, we consider a variant of the max-min dispersion problem. We refer to it as a 2-dispersion problem. Now, we define 2-dispersion problem as follows:

2-dispersion problem: Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of $n$ points in $\mathbb{R}^2$. For each point $p \in P$ and $S \subseteq P$, we define $\text{cost}_2(p, S)$ as the sum of Euclidean distance from $p$ to the first closest point in $S \setminus \{p\}$ and the second closest point in $S \setminus \{p\}$. We also define $\text{cost}_2(S) = \min_{p \in S}\{\text{cost}_2(p, S)\}$ for each $S \subseteq P$. In the 2-dispersion problem, a set $P$ of $n$ points in $\mathbb{R}^2$ and a positive integer $k \in [3, n]$ are given. The objective is to find a subset $S \subseteq P$ of size $k$ such that $\text{cost}_2(S)$ is maximized.

We find an immense number of applications for the dispersion problem in the real world. The situation in which we want to open chain stores in a community has generated our interest in the dispersion issue. In order to eliminate/prevent self-competition, we need to open stores far away from each other. Another situation in which the issue of dispersion occurs is installing hazardous structures, such as nuclear power plants and oil tanks. These facilities need to be dispersed to the fullest degree possible so that an accident at one of the facilities would not affect others. The dispersion problem also has its application in information retrieval where we need to find a small subset of data with some desired variety from an extensive data set such that a small subset is a reasonable sample to overview the large data set.
2 Related Work

In 1977, Shier [10] in his study on the $k$-center problem on a tree, studied the max-min dispersion problem on trees. In 1981, Chandrasekharan and Daughety [7] studied max-min dispersion problem on a tree network. The max-min dispersion problem is NP-hard even when the distance function satisfies triangular inequality [8]. Wang and Kuo [12] introduced the geometric version of the max-min dispersion problem. They consider the problem in the $d$-dimensional space where the distance between two points is Euclidean. They proposed a dynamic programming that solves the problem for $d = 1$ in $O(kn)$ time. They also proved that the problem is NP-hard for $d = 2$. In [13], White studied the max-min dispersion problem and proposed a 3-factor approximation result. Later in 1994, Ravi et al. [11] also studied max-min dispersion problem where they proposed a 2-factor approximation algorithm when the distance function satisfies triangular inequality. Moreover, they showed that when distance satisfies the triangular inequality, the problem cannot be approximated within the factor of 2 unless $P = NP$.

Recently, in [1], the exact algorithm for the problem was shown by establishing a relationship between the max-min dispersion problem and the maximum independent set problem. They proposed an $O(n^{w/3} \log n)$ time where $w < 2.373$. In [1], Akagi et al. also studied two special cases where a set of $n$ points lies on a line and a set of $n$ points lies on a circle separately. They proposed a polynomial time exact algorithm for both special cases.

The other popular variant of the dispersion problem is max-sum $k$-dispersion problem. Here, the objective is to maximize the sum of distances between $k$ facilities. Erkut [8] idea’s can be adapted to show that the problem is NP-hard. Ravi et al. [11] gave a polynomial time exact algorithm when the points are placed on a line. They also proposed a 4-factor approximation algorithm if the distance function satisfies triangular inequality. In [11], they also proposed a $(1.571 + \epsilon)$-factor approximation algorithm for 2-dimensional Euclidean space, where $\epsilon > 0$. In [5] and [9], the approximation factor of 4 was improved to 2. One can see [4] and [6] for other variations of the dispersion problems. In comparison with max-min dispersion (1-dispersion) problem, a handful amount of research has been done in 2-dispersion problem. Recently, in 2018, Amano and Nakano [2] proposed a greedy algorithm, which produces an 8-factor approximation result. In 2020, [3] they analyzed the same greedy algorithm proposed in [2] and proposed a $4\sqrt{3}(\approx 6.92)$-factor approximation result.
2.1 Our Contribution

In this article, we first consider the 2-dispersion problem in $\mathbb{R}^2$ and propose a simple polynomial time $(2\sqrt{3} + \epsilon)$-factor approximation algorithm for any $\epsilon > 0$. The best known result in the literature is $4\sqrt{3}$-factor approximation algorithm [3]. We also develop a common framework that improves the approximation factor to $2\sqrt{3}$ for the same problem. We present a polynomial time optimal algorithm for 2-dispersion problem if the input points lies on a line. Though a 2-factor approximation algorithm available in the literature for the 1-dispersion problem in $\mathbb{R}^2$ [11], but to show the effectiveness of the proposed common framework, we propose a 2-factor approximation algorithm for the 1-dispersion problem using the developed framework.

2.2 Organization of the Paper

The remainder of the paper is organized as follows. In Section 3 we propose a $(2\sqrt{3} + \epsilon)$-factor approximation algorithm for the 2-dispersion problem in $\mathbb{R}^2$, where $\epsilon > 0$. In Section 4 we propose a common framework for the dispersion problem. Using the framework, followed by $2\sqrt{3}$-factor approximation result for the 2-dispersion problem in $\mathbb{R}^2$, a polynomial time optimal algorithm for the 2-dispersion problem on a line and 2-factor approximation result for the 1-dispersion problem in $\mathbb{R}^2$. Finally, we conclude the paper in Section 5.

3 $(2\sqrt{3} + \epsilon)$-Factor Approximation Algorithm

In this section, we propose a $(2\sqrt{3} + \epsilon)$-factor approximation algorithm for the 2-dispersion problem, for any $\epsilon > 0$. Actually, we consider the same algorithm proposed in [3], but using different argument, we will show that for any $\epsilon > 0$, it is a $(2\sqrt{3} + \epsilon)$-factor approximation algorithm. For completeness of this article, we prefer to discuss the algorithm briefly as follows. Let $I = (P, k)$ be an arbitrary instance of the 2-dispersion problem, where $P = \{p_1, p_2, \ldots, p_n\}$ is the set of $n$ points in $\mathbb{R}^2$ and $k \in [3, n]$ is a positive integer. Initially, we choose a subset $S_3 \subseteq S$ of size 3 such that $\text{cost}_2(S_3)$ is maximized. Next, we add one point $p \in P$ into $S_3$ to construct $S_4$, i.e., $S_4 = S_3 \cup \{p\}$, such that $\text{cost}_2(S_4)$ is maximized and continues this process up to the construction of $S_k$. The pseudo code of the algorithm is described in Algorithm 1.

**Theorem 3.1.** For any $\epsilon > 0$, Algorithm 1 produces $(2\sqrt{3} + \epsilon)$-factor approximation result in polynomial time.
Algorithm 1 GreedyDispersionAlgorithm($P, k$)

**Input:** A set $P = \{p_1, p_2, \ldots, p_n\}$ of $n$ points, and a positive integer $k (3 \leq k \leq n)$.  

**Output:** A subset $S_k \subseteq P$ of size $k$.

1: Compute $\{p_{i1}, p_{i2}, p_{i3}\} \subseteq P$ such that $cost_2(S_3)$ is maximized. 
2: $S_3 = \{p_{i1}, p_{i2}, p_{i3}\}$ 
3: for $(j = 4, 5, \ldots, k)$ do 
4: Let $p \in P \setminus S_{j-1}$ such that $cost_2(S_{j-1} \cup \{p\})$ is maximized. 
5: $S_j \leftarrow S_{j-1} \cup \{p\}$ 
6: end for 
7: return ($S_k$)

**Proof.** Let $I = (P, k)$ be an arbitrary input instance of the 2-dispersion problem, where $P = \{p_1, p_2, \ldots, p_n\}$ is the set of $n$ points and $k$ is a positive integer. Let $S_k$ and $OPT$ be the output of Algorithm 1 and optimum solution, respectively, for the instance $I$. To prove the theorem, we have to show that $\frac{cost_2(OPT)}{cost_2(S_k)} \leq 2\sqrt{3} + \epsilon$. Here we use induction to show that $cost_2(S_i) \geq \frac{cost_2(OPT)}{2\sqrt{3} + \epsilon}$ for each $i = 3, 4, \ldots, k$. Since $S_3$ is an optimum solution for 3 points (see line number 2 of Algorithm 1), therefore $cost_2(S_3) \geq cost_2(OPT) \geq \frac{cost_2(OPT)}{2\sqrt{3} + \epsilon}$ holds. Now, assume that the condition holds for each $i$ such that $3 \leq i < k$. We will prove that the condition holds for $(i + 1)$ too.

Now, we define a disk $D_i$ centered at each $p_i \in P$ as follows: $D_i = \{p_\ell \in \mathbb{R}^2 | d(p_i, p_\ell) \leq \frac{cost_2(OPT)}{2\sqrt{3} + \epsilon}\}$. Let $D^*$ be a set of disks corresponding to each point in $OPT$. A point $p_j$ is contained in $D_i$, if $d(p_i, p_j) \leq \frac{cost_2(OPT)}{2\sqrt{3} + \epsilon}$.

**Lemma 3.2.** For any point $p_i \in P$, $|D_i \cap OPT| \leq 2$.

**Proof.** On the contrary assume that three points $p_a, p_b, p_c \in D_i \cap OPT$. Let $S = \{p_a, p_b, p_c\}$. Without loss of generality assume that $cost_2(p_a, S) \leq cost_2(p_b, S)$ and $cost_2(p_a, S) \leq cost_2(p_c, S)$, i.e., $d(p_a, p_b) + d(p_a, p_c) \leq d(p_a, p_b) + d(p_b, p_c)$ and $d(p_a, p_b) + d(p_a, p_c) \leq d(p_a, p_c) + d(p_b, p_c)$, which leads to $d(p_a, p_b) \leq d(p_a, p_c)$ and $d(p_b, p_a) \leq d(p_b, p_c)$. We notice that maximizing $d(p_a, p_b) + d(p_a, p_c)$ results in minimizing $d(p_b, p_c)$(see Figure 1). The minimum value of $d(p_b, p_c)$ is $\sqrt{3} \frac{cost_2(OPT)}{2\sqrt{3} + \epsilon}$ as both $d(p_a, p_b)$ and $d(p_a, p_c)$ is less than equal to $d(p_b, p_c)$. Therefore, from the packing argument inside a disk, $d(p_a, p_b) + d(p_a, p_c)$ is maximum if $p_a, p_b, p_c$ are on an equilateral triangle and on the boundary of the disk $D_i$. Then, $cost_2(S) \leq d(p_a, p_b) + d(p_a, p_c) \leq \sqrt{3} \frac{cost_2(OPT)}{2\sqrt{3} + \epsilon} + \sqrt{3} \frac{cost_2(OPT)}{2\sqrt{3} + \epsilon} = 2\sqrt{3} \frac{cost_2(OPT)}{2\sqrt{3} + \epsilon} < cost_2(OPT)$, which leads to a contradiction to the optimal value $cost_2(OPT)$. Therefore for any $p_i \in P$, $D_i$ contains at most two points from the optimal set $OPT$.

**Lemma 3.3.** For some $p_j \in OPT$, $|D_j \cap S_i| < 2$.  

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Proof. On the contrary assume that there does not exist any $j \in [1, k]$ such that $|D_j \cap S_i| < 2$. Let $D^* = \{ D_i \mid p_i \in OPT \}$. Construct a bipartite graph $H(S_i \cup D^*, \mathcal{E})$ as follows: (i) $S_i$ and $D^* = \{ D_1, D_2, \ldots, D_k \}$ are two partite vertex sets, and (ii) for $u'_i \in S_i$, $(u'_i, D_j) \in \mathcal{E}$ if and only if $u'_i$ is contained in $D_j$. According to assumption, each disk $D_j$ contains at least 2 points from $S_i$. Therefore, the total degree of the vertices in $D^*$ in $H$ is at least $2k$. Note that $|D^*| = k$. On the other hand, the total degree of the vertices in $S_i$ in $H$ is at most $2 \times |S_i|$ (see Lemma 3.2). Since $|S_i| < k$ (based on the assumption of the induction hypothesis), the total degree of the vertices in $S_i$ in $H$ is less than $2k$, which leads to a contradiction that the total degree of the vertices in $D^*$ in $H$ is at least $2k$. Thus, there exist at least one $p_j \in OPT$ such that $|D_j \cap S_i| < 2$.

Without loss of generality, assume that disk $D_j \in D^*$ has at most one point from the set $S_i$. Suppose $D_j$ contains only one point of the set $S_i$, then the distance of $p_j$ to the second closest point in $S_i$ is greater than $\frac{\text{cost}_2(OPT)}{2\sqrt{3+\epsilon}}$ (see Figure 2). Also, from triangular inequality $d(p_i, p_j) + d(p_i, p_i) > \frac{\text{cost}_2(OPT)}{2\sqrt{3+\epsilon}}$ for each point $p_i \in S_i$. So, we can add the point $p_j \in OPT$ to the set $S_i$ to construct set $S_{i+1}$. Here, $S_{i+1} = S_i \cup \{ p_j \}$. Therefore, the cost of $S_{i+1} \geq \frac{\text{cost}_2(OPT)}{2\sqrt{3+\epsilon}}$.

Now, assume that $D_j$ does not contain any point from the set $S_i$, then the distance of the point $p_j \in OPT$ to any point of $S_i$ is greater than $\frac{\text{cost}_2(OPT)}{2\sqrt{3+\epsilon}}$. By adding the point $p_j$ in the set $S_i$, we construct the set $S_{i+1}$, which leads to the $\text{cost}_2(S_{i+1}) \geq \frac{\text{cost}_2(OPT)}{2\sqrt{3+\epsilon}}$. Since our algorithm chooses a point (see line number 4 of Algorithm 1) that maximizes $\text{cost}_2(S_{i+1})$, therefore algorithm will always choose a point in the iteration $i + 1$ such that $\text{cost}_2(S_{i+1}) \geq \frac{\text{cost}_2(OPT)}{2\sqrt{3+\epsilon}}$.

By the help of Lemma 3.2 and Lemma 3.3 we can conclude that the $\text{cost}_2(S_{i+1}) \geq \frac{\text{cost}_2(OPT)}{2\sqrt{3+\epsilon}}$ and thus condition holds for $(i + 1)$ too. Therefore, for any $\epsilon > 0$, Algorithm 1 produces $(2\sqrt{3} + \epsilon)$-factor approximation result in polynomial time.

\[\text{cost}_2(S_{i+1}) \geq \frac{\text{cost}_2(OPT)}{2\sqrt{3+\epsilon}}\]
4 An Algorithm for the Dispersion Problem

In this section, we propose an algorithm for the dispersion problem. It is a common algorithm for 1-dispersion, 2-dispersion problem in $\mathbb{R}^2$ and 1-dispersion/2-dispersion problem in $\mathbb{R}$. Input of the algorithm are (1) a set $P = \{p_1, p_2, \ldots, p_n\}$ of $n$ points, (2) an integer $\gamma (= 1$ or $2$) for the $\gamma$-dispersion problem, and (3) an integer $k (\gamma + 1 \leq k \leq n)$. In the first line of the algorithm, we set the value of a constant $\lambda$. If $\gamma = 2$ and points are in $\mathbb{R}^2$ (resp. $\mathbb{R}$), then we set $\lambda = 2\sqrt{3}$ (resp. $\lambda = 1$), and if $\gamma = 1$ and points are in $\mathbb{R}^2$, then we set $\lambda = 2$. We prove that the algorithm is $\lambda$-factor approximation algorithm.

We use $S_i (\subseteq P)$ to denote a set of size $i$. We start algorithm with $S_{\gamma+1} \subseteq P$ containing $\gamma + 1$ points as a solution set. Next, iteratively we add one by one point from $P$ into the solution set to get a final solution set, i.e., if we have a solution set $S_i$ of size $i$, then we add one more point into $S_i$ to get solution set $S_{i+1}$ of size $i + 1$. Let $\alpha = \text{cost}_\gamma(S_i)$. Now, we add a point from $P \setminus S_i$ into $S_i$ to get solution set $S_{i+1}$ such that $\text{cost}_\gamma(S_{i+1}) \geq \frac{\alpha}{\lambda}$. We stop this iterative method if we have $S_k$ or no more point addition is possible. We repeat the above process for each distinct $S_{\gamma+1} \subseteq P$ and report the solution for which the $\gamma$-dispersion cost value is maximum.

4.1 $2\sqrt{3}$-Factor Approximation Result for the 2-Dispersion Problem

Let $S^* \subseteq P = \{p_1, p_2, \ldots, p_n\}$ be an optimal solution for a given instance $(P, k)$ of the 2-dispersion problem and $S_k \subseteq P$ be a solution returned by greedy Algorithm 2 for the given instance, provided $\gamma = 1$ as an additional input. A point $s_o^* \in S^*$ is said to be a solution point if $\text{cost}_2(S^*)$ is defined by $s_o^*$, i.e., $\text{cost}_2(S^*) = d(s_o^*, s_r^*) + d(s_o^*, s_t^*)$ such that (i) $s_r^*, s_t^* \in S^*$, and (ii) $s_r^*$ and $s_t^*$ are the first and second closest points of $s_o^*$ in $S^*$, respectively. We call $s_r^*$, $s_t^*$ as supporting points. Let $\alpha = \text{cost}_2(S^*)$. In this problem, the value of $\lambda$ is $2\sqrt{3}$ (line number 1 of Algorithm 2).

Lemma 4.1. The triangle formed by three points $s_o^*, s_r^*$ and $s_t^*$ does not contain any point in $S^* \setminus \{s_o^*, s_r^*, s_t^*\}$, where $s_o^*$ is the solution point, and $s_r^*, s_t^*$ are supporting points.
Algorithm 2 Dispersion Algorithm($P, k, \gamma$)

**Input:** A set $P$ of $n$ points, a positive integer $\gamma$ and an integer $k$ such that $\gamma + 1 \leq k \leq n$

**Output:** A subset $S_k \subseteq P$ such that $|S_k| = k$ and $\beta = cost_\gamma(S_k)$.

1. If $\gamma = 2$ (resp. $\gamma = 1$), then $\lambda \leftarrow 2\sqrt{3}$ (resp. $\lambda \leftarrow 2$), and if points are on a line then $\lambda \leftarrow 1$.
2. $\beta \leftarrow 0$ // Initially, $cost_\gamma(S_k) = 0$
3. for each subset $S_{\gamma+1} \subseteq P$ consisting of $\gamma + 1$ points do
   4. Set $\alpha \leftarrow cost_\gamma(S_{\gamma+1})$
   5. Set $\rho \leftarrow \alpha / \lambda$
   6. if $\rho > \beta$ then
      7. flag $\leftarrow 1$, $i \leftarrow \gamma + 1$
      8. while $i < k$ and flag $\neq 0$ do
         9. flag $\leftarrow 0$
         10. choose a point $p \in P \setminus S_i$ (if possible) such that $cost_\gamma(S_i \cup \{p\}) \geq \rho$ and $cost_\gamma(p, S_i) = \min_{q \in P \setminus S_i} cost_\gamma(q, S_i)$.
      11. if such point $p$ exists in step 10 then
         12. $S_{i+1} \leftarrow S_i \cup \{p\}$
         13. $i \leftarrow i + 1$, flag $\leftarrow 1$
      14. end if
   15. end while
   16. if $i = k$ then
      17. $S_k \leftarrow S_i$ and $\beta \leftarrow \rho$
   18. end if
5. end for
21. return $(S_k, \beta)$

**Proof.** Suppose there exist a point $s^*_m \in S^*$ inside the triangle formed by $s^*_o$, $s^*_r$ and $s^*_t$. Now, if $d(s^*_o, s^*_r) \geq d(s^*_o, s^*_t)$ then $d(s^*_o, s^*_r) + d(s^*_o, s^*_m) < d(s^*_o, s^*_r) + d(s^*_o, s^*_t)$ which contradict the optimality of $cost_2(S^*)$. A similar argument will also work for $d(s^*_o, s^*_r) < d(s^*_o, s^*_t)$. \qed

In this problem, $\rho = \frac{\alpha}{\lambda} = \frac{cost_2(S^*)}{2\sqrt{3}}$. We define a disk $D_i$ centered at $p_i \in P$ as follows: $D_i = \{p_j \in \mathbb{R}^2 | d(p_i, p_j) \leq \rho\}$. Let $D = \{D_i \mid p_i \in P\}$. Let $D^*$ be the subsets of $D$ corresponding to disks centered at points in $S^*$. A point $p_j$ is properly contained in $D_i$, if $d(p_i, p_j) < \rho$, whereas if $d(p_i, p_j) \leq \rho$, then we say that point $p_j$ is contained in $D_i$.

**Lemma 4.2.** For any point $p \in P$, if $D^p = \{q \in \mathbb{R}^2 \mid d(p, q) \leq \rho\}$ then $D^p$ properly contains at most two points of the optimal set $S^*$.

**Proof.** On the contrary assume that three points $p_a, p_b, p_c \in S^*$ such that $p_a, p_b, p_c$ are properly contained in $D^p$. Using the similar arguments discussed in the proof of Lemma 3.2, $cost_2(\{p_a, p_b, p_c\})$ is maximum if $p_a, p_b, p_c$ are on equilateral triangle inside $D^p$. Therefore, $d(p_a, p_b) = d(p_a, p_c) = d(p_b, p_c)$. Now, $cost_2(\{p_a, p_b, p_c\}) = d(p_a, p_b) + d(p_a, p_c) < 

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\[ \sqrt{3}\rho + \sqrt{3}\rho = 2\sqrt{3}\rho = \text{cost}_2(S^*). \] Therefore, \( p_a, p_b, p_c \in S^* \) and \( \text{cost}_2(\{p_a, p_b, p_c\}) < \text{cost}_2(S^*) \) leads to a contradiction. Thus, the lemma.

\[ \square \]

**Lemma 4.3.** For any three points \( \{p_a, p_b, p_c\} \in S^* \), there does not exist any point \( s \in \mathbb{R}^2 \) such that \( s \) is properly contained in \( D_a \cap D_b \cap D_c \).

**Proof.** On the contrary assume that \( s \) is properly contained in \( D_a \cap D_b \cap D_c \). This implies \( d(p_a, s) < \rho, d(p_b, s) < \rho \) and \( d(p_c, s) < \rho \). Therefore, the disk \( D^* = \{q \in \mathbb{R}^2 \mid d(s, q) \leq \rho\} \) properly contains three points \( p_a, p_b \) and \( p_c \), which is a contradiction to Lemma 4.2. Thus, the lemma.

\[ \square \]

**Corollary 4.4.** For any point \( p \in P \), if \( D' \subseteq D^* \) is the subset of disks that contains \( p \), then \( |D'| \leq 3 \) and \( p \) lies on the boundary of each disk in \( D' \).

**Proof.** Follows from Lemma 4.3.

**Corollary 4.5.** For any point \( p \in P \), if \( D'' \subseteq D^* \) is the subset of disks that properly contains point \( p \), then \( |D''| \leq 2 \).

**Proof.** Follows from Lemma 4.3 and Corollary 4.4.

**Lemma 4.6.** Let \( S \subseteq P \) be a set of points such that \( |S| < k \). If \( \text{cost}_2(S) \geq \rho \), then there exists at least one disk \( D_j \in D^* = \{D_1, D_2, \ldots, D_k\} \) that properly contains at most one point from the set \( S \).

**Proof.** On the contrary assume that each \( D_j \in D^* \) properly contains at least two points from the set \( S \). Construct a bipartite graph \( G(S \cup D^*, E) \) as follows: (i) \( S \) and \( D^* \) are two partite vertex sets, and (ii) for \( u \in S \), \( (u, D_j) \in E \) if and only if \( u \) is properly contained in \( D_j \). According to assumption, each disk \( D_j \) contains at least 2 points from the set \( S \). Therefore, the total degree of the vertices in \( D^* \) in \( G \) is at least \( 2k \). Note that \( |D^*| = k \).

On the other hand, the total degree of the vertices in \( S \) in \( G \) is at most \( 2 \times |S| \) (see Corollary 4.5). Since \( |S| < k \), the total degree of the vertices in \( S \) in \( G \) is less than \( 2k \), which leads to a contradiction that the total degree of the vertices in \( D^* \) in \( G \) is at least \( 2k \). Thus, there exist at least one disk \( D_j \in D^* \) such that the disk \( D_j \) properly contains at most one point from the set \( S \).

**Theorem 4.7.** Algorithm \( \square \) produces a \( 2\sqrt{3} \)-factor approximation result for the \( 2 \)-dispersion problem in \( \mathbb{R}^2 \).
Proof. Since it is a 2-dispersion problem, so $\gamma = 2$ and set $\lambda = 2\sqrt{3}$ in line number 3 of Algorithm 2. Now, assume $\alpha = \text{cost}_2(S^*)$ and $\rho = \frac{\alpha}{\gamma} = \frac{\text{cost}_2(S^*)}{2\sqrt{3}}$, where $S^*$ is an optimum solution. Here, we show that Algorithm 2 returns a solution set $S_k$ of size $k$ such that $\text{cost}_2(S_k) \geq \rho = \frac{\text{cost}_2(S^*)}{2\sqrt{3}}$. More precisely, we show that Algorithm 2 returns a solution set $S_k$ of size $k$ such that $\text{cost}_2(S_k) \geq \frac{\text{cost}_2(S^*)}{2\sqrt{3}}$ and $S_k \supseteq \{s^*_o, s^*_r, s^*_t\}$, where $s^*_o$ is the solution point and $s^*_r$ and $s^*_t$ are supporting points, i.e., $\text{cost}_2(S^*) = d(s^*_o, s^*_r) + d(s^*_o, s^*_t)$. Now, consider the case when $S_3 = \{s^*_o, s^*_r, s^*_t\}$ in line number 3 of Algorithm 2 then it computes a solution $S_k$ of size $k$ such that $\text{cost}_2(S_k) \geq \frac{\text{cost}_2(S^*)}{2\sqrt{3}}$. Note that any other solution returned by Algorithm 2 has a 2-dispersion cost better than $\frac{\text{cost}_2(S^*)}{2\sqrt{3}}$. Therefore, it is sufficient to prove that if $S_3 = \{s^*_o, s^*_r, s^*_t\}$ in line number 3 of Algorithm 2 then the size of $S_k$ (updated) in line number 12 of Algorithm 2 is $k$ as every time Algorithm 2 added a point (see line number 12) into the set with the property that 2-dispersion cost of the updated set is greater than or equal to $\frac{\text{cost}_2(S^*)}{2\sqrt{3}}$. Therefore, we consider $S_3 = \{s^*_o, s^*_r, s^*_t\}$ in line number 3 of Algorithm 2.

We use induction to establish the condition $\text{cost}_2(S_i) \geq \rho$ for each $i = 3, 4, \ldots k$. Since $S_3 = S^*_3$, therefore $\text{cost}_2(S_3) = \text{cost}_2(S^*_3) = \alpha > \rho$ holds. Now, assume that the condition $\text{cost}_2(S_i) \geq \rho$ holds for each $i$ such that $3 \leq i < k$. We will prove that the condition $\text{cost}_2(S_{i+1}) \geq \rho$ holds for $(i+1)$ too.

Let $D^*$ be the set of disks centered at the points in $S^*$ such that the radius of each disk is $\rho$. Since $i < k$ and $S_i \subseteq P$ with condition $\text{cost}_2(S_i) \geq \rho$, there exist at least one disk, say $D_j \in D^*$ that properly contains at most one point in $S_i$ (see Lemma 4.6). We will show that $\text{cost}_2(S_{i+1}) = \text{cost}_2(S_i \cup \{p_j\}) \geq \rho$, where $p_j$ is the center of the disk $D_j$. Suppose, $D_j$ contains only one point $p_x \in S_i$, then $p_x$ is the first closest point of $p_j$ in the set $S_i$. Now, by Corollary 4.4 and by Lemma 4.6 we claim the second closest point $p_t$ of $p_j$ in the set $S_i$ may lie (1) on the boundary of the disk $D_j$ (see Figure 3(a)) or (2) outside of the disk $D_j$ (see Figure 3(b)).

![Figure 3](image_url)

Figure 3: (a) $p_t$ lies on the boundary of the disk $D_j$ and (b) $p_t$ lies outside of the disk $D_j$

Since $d(p_j, p_t) \geq \rho$ for both the above mentioned cases, therefore $\text{cost}_2(p_j, S_i) \geq \rho$. 10
Also, from triangular inequality \( d(p_x, p_y) + d(p_y, p_z) \geq d(p_x, p_z) \geq \rho \) for each point \( p_i \in S_i \). So, we can add the point \( p_j \) to the set \( S_i \) to construct set \( S_{i+1} \). Here, \( S_{i+1} = S_i \cup \{p_j\} \). Therefore, \( cost_2(S_{i+1}) \geq \rho \).

Now, if \( D_j \) does not properly contain any point from the set \( S_i \), then the distance of \( p_j \) to any point of the set \( S_i \) is greater than or equal to \( \rho \). Since there exists at least one point \( p_j \in P \setminus S_i \) such that \( cost_2(S_{i+1}) = cost_2(S_i \cup \{p_j\}) \geq \rho \), therefore Algorithm 2 will always choose a point (see line number 10 of Algorithm 2) in the iteration \( i + 1 \) such that \( cost_2(S_{i+1}) \geq \rho \).

So, we can conclude that \( cost_2(S_{i+1}) \geq \rho \) and thus condition holds for \( (i + 1) \) too.

Therefore, Algorithm 2 produces a set \( S_k \) of size \( k \) such that \( cost_2(S_k) \geq \rho \). Since \( \rho \geq \frac{cost_2(S^*)}{2\sqrt{3}} \), Algorithm 2 produces \( 2\sqrt{3} \)-factor approximation result for the 2-dispersion problem.

\( \square \)

### 4.2 2-Dispersion Problem on a Line

In this section, we discuss the 2-dispersion problem on a line \( L \). Let the point set \( P = \{p_1, p_2, \ldots, p_n\} \) be on a horizontal line arranged from left to right. Let \( S_k \subseteq P \) be a solution returned by Algorithm 2 and \( S^* \subseteq P \) be an optimal solution. Note that, the value of \( \gamma \) is 2 and the value of \( \lambda \) (line number 1 of Algorithm 2) is 1 in this problem. Let \( s_o^* \) be a solution point and \( s_i^*, s_t^* \) be supporting points, i.e., \( cost_2(S^*) = d(s_o^*, s_i^*) + d(s_o^*, s_t^*) \). Let \( S_3 = \{s_o^*, s_i^*, s_t^*\} \). We show that if \( S_3 = S_3 \) in line number 3 of Algorithm 2 then \( cost_2(S_3) = cost_2(S^*) \). Let \( S^* = \{s_1^*, s_2^*, \ldots, s_k^*\} \) are arranged from left to right.

**Lemma 4.8.** Let \( S^* \) be an optimal solution. If \( s_o^* \) is the solution point and \( s_i^*, s_t^* \) are supporting points, then both points \( s_i^* \) and \( s_t^* \) cannot be on the same side on the line \( L \) with respect to \( s_o^* \) and three points \( s_r^*, s_o^*, s_t^* \) are consecutive on the line \( L \) in \( S^* \).

\[
\begin{array}{c}
  s_r^* \quad s_i^* \quad s_t^* \quad s_o^* \\
\end{array}
\]

Figure 4: \( s_i^*, s_t^* \) on left side of \( s_o^* \)

**Proof.** On the contrary assume that both \( s_i^* \) and \( s_t^* \) are on the left side of \( s_o^* \), and \( s_r^* \) lies between \( s_i^* \) and \( s_o^* \) (see Figure 4). Now, \( d(s_r^*, s_o^*) + d(s_o^*, s_i^*) < d(s_o^*, s_t^*) + d(s_o^*, s_r^*) \) which leads to a contradiction that \( s_o^* \) is a solution point, i.e., \( cost_2(S^*) = d(s_o^*, s_i^*) + d(s_o^*, s_t^*) \). Now, suppose \( s_i^*, s_o^*, s_t^* \) are not consecutive in \( S^* \). Let \( s^* \) be the point in \( S^* \) such that either \( s^* \in (s_i^*, s_o^*) \) or \( s^* \in (s_o^*, s_t^*) \). If \( s^* \in (s_i^*, s_o^*) \), then \( d(s_o^*, s_i^*) + d(s_o^*, s^*) + d(s_o^*, s_t^*) > d(s_o^*, s_i^*) + d(s_o^*, s_t^*) \), which leads to a contradiction that \( s_i^* \) is a supporting point. Similarly, we can show that
if \( s^* \in (s^*_o, s^*_i) \), then \( s^*_i \) is not a supporting point. Thus, \( s^*_r, s^*_o, s^*_i \) are consecutive points on the line \( L \) in \( S^* \).

\[ \square \]

Lemma 4.8 says that if \( s^*_o \) is a solution point, then \( s^*_o \) and \( s^*_o \) are supporting points as \( s^*_1, s^*_2, \ldots, s^*_k \) are arranged from left to right.

**Lemma 4.9.** Let \( S_3 = \{ s^*_o, s^*_r, s^*_i \} \) and \( \alpha = \text{cost}_2(S_3) \). Now, if \( S_i = S_{i-1} \cup \{ p_i \} \) constructed in line number 12 of Algorithm 2, then \( \text{cost}_2(S_i) = \alpha \).

**Proof.** We use induction to prove \( \text{cost}_2(S_i) = \alpha \) for \( i = 4, 5, \ldots, k \).

**Base Case:** Consider the set \( S_4 = S_3 \cup \{ p_4 \} \) constructed in line number 12 of Algorithm 2. If \( s^*_o \) is a solution points, and \( s^*_o, s^*_i \) are supporting points and \( \text{cost}_2(p_4, S_4) \geq \alpha \), then \( p_4 \notin [s^*_o-1, s^*_o+1] \) (otherwise one of \( s^*_o-1 \) and \( s^*_o+1 \) will not be supporting point). This implies \( p_4 \) either lies in \( [p_1, s^*_o-1] \) or \( (s^*_o, p_4) \). Assume \( p_4 \in (s^*_o-1, p_4) \). In Algorithm 2, we choose \( p_4 \) such that \( \text{cost}_2(p_4, S_4) \geq \alpha \) (see line number 10 of Algorithm 2) and \( \text{cost}_2(p_4, S_4) = \min_{q \in P \setminus S_3} \text{cost}_2(q, S_4) \). Therefore, \( p_4 \in (s^*_o-1, s^*_o+2] \). Let \( S^*_i = \{ s^*_1, s^*_2, \ldots, s^*_o \} \cup S_4 \cup \{ s^*_o+3, s^*_o+4, \ldots, s^*_k \} \). Suppose \( p_4 = s^*_o+2 \) and we know that \( S_3 = S^*_i \) then \( S^*_i = S^* \). So, \( \text{cost}_2(S^*_i) = \text{cost}_2(S^*) = \alpha \). This implies \( \text{cost}_2(S^*_i) = \alpha \).

Now assume that \( p_4 \in (s^*_o+1, s^*_o+2] \), then also we will show that \( \text{cost}_2(S^*_i) = \alpha \). We calculate \( \text{cost}_2(p_4, S^*_i) = d(p_4, s^*_o+1) + d(p_4, s^*_o+3) = d(s^*_o+2, s^*_o+1) + d(s^*_o+2, s^*_o+3) \geq \alpha \) and \( \text{cost}_2(s^*_o+3, S^*_i) = d(s^*_o+3, p_4) + d(s^*_o+3, s^*_o+1) \geq d(s^*_o+3, s^*_o+2) + d(s^*_o+3, s^*_o+4) \geq \alpha \) (see Figure 5). Thus if \( p_4 \in (s^*_o+1, s^*_o+2] \), then \( \text{cost}_2(S^*_i) = \alpha \). Therefore, if \( k \geq 4 \), then \( p_4 \) exists and \( \text{cost}_2(S_4) = \alpha \). Similarly, we can prove that if \( p_4 \in [p_1, s^*_o-1] \), then \( \text{cost}_2(S^*_i) = \alpha \), where \( S^*_i = \{ s^*_1, s^*_2, \ldots, s^*_o \} \cup S_4 \cup \{ s^*_o+3, s^*_o+4, \ldots, s^*_k \} \). In this case also \( p_4 \) exists and \( \text{cost}_2(S_4) = \alpha \).

\[ s^*_o-1 \quad s^*_i \quad s^*_o+1 \quad p_4 \quad s^*_o+2 \quad s^*_o+3 \quad s^*_o+4 \]

**Figure 5:** Snippet of \( S^*_i \)

Now, assume that \( S_i = S_{i-1} \cup \{ p_i \} \) for \( i < k \) such that \( \text{cost}_2(S^*_i) = \alpha \) and \( \text{cost}_2(S_i) = \alpha \) where \( S^*_i = \{ s^*_1, s^*_2, \ldots, s^*_o \} \cup S_i \cup \{ s^*_o, s^*_o+1, \ldots, s^*_k \} \). If \( p_i \in (s^*_o, p_n) \), then \( s^*_o \) is the left most point in the right of \( p_i \) and \( u \geq k - (i + k - v + 1) = v - i + 1 \) with each point of \( S_i \) are on the right side of \( s^*_o \) (see Figure 6(a)) and if \( p_i \in (p_1, s^*_o) \), then \( s^*_o \) is the right most point in the left of \( p_i \) where \( v \geq u + i + 1 \) (see Figure 6(b)).

We prove that \( \text{cost}_2(S_{i+1}) = \alpha \), where \( S_{i+1} = S_i \cup \{ p_i \} \). It follows from the fact that size of \( S_i \) is less than \( k \), and the set \( \{ s^*_1, s^*_2, \ldots, s^*_o \} \cup \{ s^*_o, s^*_o+1, \ldots, s^*_k \} \neq \phi \) and the similar arguments discussed in the base case.

\[ \square \]
Lemma 4.10. The running time of Algorithm 2 on line is $O(n^4)$.

Proof. Since it is a 2-dispersion problem on a line, so algorithm starts by setting $\lambda = 1$ in line number 1 of Algorithm 2 and then compute solution set for each distinct $S_3 \subseteq P$ independently. Now, for each $S_3$, algorithm selects a point iteratively based on greedy choice (see line number 10 of Algorithm 2). Now, for choosing remaining $(k - 3)$ points, the total amortize time taken by the algorithm is $O(n)$. So, the overall time complexity of Algorithm 2 on line consisting of $n$ points is $O(n^4)$.

Theorem 4.11. Algorithm 2 produces an optimal solution for the 2-dispersion problem on a line in polynomial time.

Proof. Follows from Lemma 4.9 that $\text{cost}_2(S_i) = \alpha = \text{cost}_2(S^*_i)$ for $3 \leq i \leq k$, where $S_3 = \{s^*_o, s^*_r, s^*_i\}$. Therefore, $\text{cost}_2(S_k) = \alpha$. Also, Lemma 4.10 says that Algorithm 2 computes $S_k$ in polynomial time. Thus, the theorem.

4.3 1-Dispersion Problem in $\mathbb{R}^2$

In this section, we show the effectiveness of Algorithm 2 by showing 2-factor approximation result for the 1-dispersion problem in $\mathbb{R}^2$. Here, we set $\gamma = 1$ as input along with input $P$ and $k$. We also set $\lambda = 2$ in line number 1 of the algorithm 2.

Let $S^*$ be an optimal solution for a given instance $(P, k)$ of 1-dispersion problem and $S_k \subseteq P$ be a solution returned by our greedy Algorithm 2 provided $\gamma = 1$ as an additional input. Let $s^*_o \in S^*$ a solution point, i.e., $\text{cost}_1(S^*) = d(s^*_o, s^*_r)$ such that $s^*_r$ is the closest points of $s^*_o$ in $S^*$. We call $s^*_r$ as supporting point. Let $\alpha = d(s^*_o, s^*_r)$ and $\rho = \frac{\alpha}{2}$.

We define a disk $D_i$ centered at $p_i \in P$ as follows: $D_i = \{p_j \in \mathbb{R}^2 | d(p_i, p_j) \leq \rho\}$. Let $D = \{D_i \mid p_i \in P\}$. Let $D^*$ be the subsets of $D$ corresponding to disks centered at points in $S^*$. If $d(p_i, p_j) < \rho$, then we say that $p_j$ is properly contained in $D_i$ and if $d(p_i, p_j) \leq \rho$, then we say that $p_j$ is contained in $D_i$.
Lemma 4.12. For any point $s \in P$, if $D^* = \{q \in \mathbb{R}^2 \mid d(s, q) \leq \rho\}$ then $D^*$ properly contains at most one point of the optimal set $S^*$.

Proof. On the contrary assume that $p_a, p_b \in S^*$ such that $p_a, p_b$ are properly contained in $D^*$. If two points $p_a$ and $p_b$ are properly contained in $D^*$, then $d(p_a, p_b) < d(p_a, s) + d(p_b, s) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$, which leads to a contradiction to the optimality of $S^*$. Thus, the lemma.

Lemma 4.13. For any two points $p_a, p_b \in S^*$, there does not exist any point $s \in \mathbb{R}^2$ that is properly contained in $D_a \cap D_b$.

Proof. On the contrary assume that $s$ is properly contained in $D_a \cap D_b$. This implies $d(p_a, s) < \frac{\alpha}{2}$ and $d(p_b, s) < \frac{\alpha}{2}$. Therefore, the disk $D^* = \{q \in \mathbb{R}^2 \mid d(s, q) \leq \rho\}$ properly contains two points $p_a$ and $p_b$, which is a contradiction to Lemma 4.12. Thus, the lemma.

Corollary 4.14. For any point $s \in P$, if $D' \subseteq D^*$ is the set of disks that contain $s$, then $|D'| \leq 2$ and $s$ lies on the boundary of both the disk in $D'$.

Proof. Follows from Lemma 4.13.

Corollary 4.15. For any point $s \in P$, if $D'' \subseteq D^*$ be a subset of disks that properly contains point $s$, then $|D''| \leq 1$.

Proof. Follows from Lemma 4.13 and Corollary 4.14.

Lemma 4.16. Let $M \subseteq P$ be a set of points such that $|M| < k$. If $\text{cost}_2(M) \geq \frac{\alpha}{2}$, then there exists at least one disk $D_j \in D^* = \{D_1, D_2, \ldots, D_k\}$ that does not properly contain any point from the set $M$.

Proof. On the contrary, assume that each $D_j \in D^*$ properly contains at least one point from the set $M$. Construct a bipartite graph $G(M \cup D^*, \mathcal{E})$ as follows: (i) $M$ and $D^*$ are two partite vertex sets, and (ii) for $u \in M$, $(u, D_j) \in \mathcal{E}$ if and only if $u$ is properly contained in $D_j$. According to assumption, each disk $D_j$ contains at least 1 points from the set $M$. Therefore, the total degree of the vertices in $D^*$ in $G$ is at least $k$. Note that $|D^*| = k$. On the other hand, the total degree of the vertices in $M$ in $G$ is at most $|M|$ (see Corollary 4.15). Since $|M| < k$, the total degree of the vertices in $M$ in $G$ is less than $k$, which leads to a contradiction that the total degree of the vertices in $D^*$ in $G$ is at least $k$. Thus, there exist at least one disk $D_j \in D^*$ such that $D_j$ does not properly contain any point from the set $M$.

Theorem 4.17. Algorithm produces a 2-factor approximation result for the 1-dispersion problem in $\mathbb{R}^2$. 

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Proof. Since it is a 1-dispersion problem, so γ = 1 and set λ = 2 in line number 1 of the algorithm. Now, assume α = \(\text{cost}_1(S^*)\) and \(\rho = \frac{\alpha}{\lambda} = \frac{\text{cost}_1(S^*)}{2}\), where \(S^*\) is the optimum solution. Here, we show that Algorithm 2 returns a solution set \(S_k\) of size \(k\) such that \(\text{cost}_1(S_k) \geq \rho\). More precisely, we show that Algorithm 2 returns a solution \(S_k\) of size \(k\) such that \(\text{cost}_1(S_k) \geq \rho\) and \(S_k \supseteq \{s^*_o, s^*_r\}\), where \(s^*_o\) is the solution point and \(s^*_r\) is the supporting point. Our objective is to show that if \(S_2 = \{s^*_o, s^*_r\}\) in line number 3 of Algorithm 2 then it computes a solution \(S_k\) of size \(k\) such that \(\text{cost}_1(S_k) \geq \rho\). Note that any other solution returned by Algorithm 2 has a 1-dispersion cost better than \(\frac{\text{cost}_1(S^*)}{2}\). Therefore, it is sufficient to prove that if \(S_2 = \{s^*_o, s^*_r\}\) in line number 3 of Algorithm 2 then the size of \(S_k\) (updated) in line number 17 of Algorithm 2 is \(k\) as every time Algorithm 2 added a point (see line number 12) into the set with the property that 1-dispersion cost of the updated set is greater than or equal to \(\rho = \frac{\text{cost}_1(S^*)}{2}\). Therefore, we consider \(S_2 = \{s^*_o, s^*_r\}\) in line number 3 of Algorithm 2.

We use induction to establish the condition \(\text{cost}_1(S_i) \geq \rho\) for each \(i = 3, 4, \ldots k\). Since \(S_2 = S^*_2\), therefore \(\text{cost}_1(S_2) = \text{cost}_1(S^*_2) = \alpha\) holds. Now, assume that the condition \(\text{cost}_1(S_i) \geq \rho\) holds for each \(i\) such that \(3 \leq i < k\). We will prove that the condition \(\text{cost}_1(S_{i+1}) \geq \rho\) holds for \((i+1)\) too.

Let \(D^*\) be the set of disks centered at the points in \(S^*\) such that the radius of each disk be \(\rho = \frac{\alpha}{2}\). Since \(i < k\) and \(S_i \subseteq P\) with condition \(\text{cost}_1(S_i) \geq \rho = \frac{\alpha}{2}\), there exist at least one disk, say \(D_j \in D^*\) that does not contain any point from the set \(S_i\) (see Lemma 4.16). We will show that \(\text{cost}_1(S_{i+1}) = \text{cost}_1(S_i \cup \{p_j\}) \geq \rho\), where \(p_j\) is the center of the disk \(D_j\).

Now, if \(D_j\) does not properly contain any point from the set \(S_i\), then the closest point of \(p_j \in S^*\) may lie on the boundary of the disk \(D_j\) (by Corollary 4.14) or outside the disk \(D_j\) (by Lemma 4.16). In both the cases, distance of \(p_j\) to any point of the set \(S_i\) is greater than or equal to \(\rho\) (see Figure 7). Since there exists at least one point \(p_j \in P \setminus S_i\) such that \(\text{cost}_1(S_{i+1}) = \text{cost}_1(S_i \cup \{p_j\}) \geq \rho\), therefore Algorithm 2 will always choose a point (see line number 10 of Algorithm 2) in the iteration \(i+1\) such that \(\text{cost}_1(S_{i+1}) \geq \rho\).

So, we can conclude that \(\text{cost}_1(S_{i+1}) \geq \rho\) and thus condition holds for \((i+1)\) too.

![Figure 7: Second closest point of p_j lies on boundary of D_j or outside of D_j.](image)

Therefore, Algorithm 2 produces a 2-factor approximation result for the 1-dispersion problem in \(\mathbb{R}^2\). \(\square\)
5 Conclusion

In this article, we proposed a \((2\sqrt{3}+\epsilon)\)-factor approximation algorithm for the 2-dispersion problem in \(\mathbb{R}^2\), where \(\epsilon > 0\). The best known approximation factor available in the literature is \(4\sqrt{3}\) \([3]\). Next, we proposed a common framework for the dispersion problem. Using the framework, we further improved the approximation factor to \(2\sqrt{3}\) for the 2-dispersion problem in \(\mathbb{R}^2\). We studied the 2-dispersion problem on a line and proposed a polynomial time algorithm that returns an optimal solution using the developed framework. Note that, for the 2-dispersion problem on a line, one can propose a polynomial time algorithm that returns an optimal value in relatively low time complexity, but to show the adaptability and flexibility of our proposed framework, we presented an algorithm for the same problem using the developed framework. We also proposed a 2-factor approximation algorithm for the 1-dispersion problem using the proposed common framework to show effectiveness of the framework.

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