Spinor Representations of Surfaces in 4–Dimensional Pseudo–Riemannian Manifolds

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Abstract

Spinor representations of surfaces immersed into 4–dimensional pseudo–riemannian manifolds are defined in terms of minimal left ideals and tensor decompositions of Clifford algebras. The classification of spinor fields and Dirac operators on the immersed surfaces is given. The Dirac–Hestenes spinor field on surfaces immersed into Lorentzian manifolds and on surfaces conformally immersed into Minkowski spacetime is defined.

Mathematics Subject Classification (1991): 15A66, 53A50, 53A10, 35Q55

Keywords: Clifford algebras, spinors, minimal left ideals, spinor bundles, immersed surfaces, Dirac operators, Weierstrass representation.

1 Introduction

One of the most interesting aspects of the relationship between differential geometry of surfaces and Lie groups is a theory of spinor representations of surfaces. This interest can be explained by two reasons: firstly, the spinor representations of surfaces (SRS) are natural consequence of a fundamental relation between differential geometry and Lie groups, secondly, spinor representations as a whole are of great importance in theoretical physics. On the other hand, nowadays theory of surfaces in itself is finding increasing use in many areas of modern physics, above all in soliton theory [Sym85, Bob94, Bob99] and string theory [GM93].

However, at the present time SRS are exhaustively studied only for 3–dimensional case (in particular, for the case of conformal immersions of surfaces into a 3–dimensional euclidean space). Whereas a high–grade by content (both mathematical and physical point of view) case of immersions of surfaces into 4–dimensional manifolds still remains poorly studied, i.e. a problem of the finding
and classification of spinor representations of surfaces immersed into 4d pseudo–riemannian manifolds is opened. The present work is devoted to solving this problem.

Historically, spinor structures in the differentiable manifolds introduced by Haefliger in 1956 [Hae56] (see also [BH, Mil63]). Further, spinor structures over the Riemann surfaces are defined by Atiyah [At71] (see also [John80]). These works were served as a basis for the following construction of spinor representation of minimal surfaces immersed into a 3–dimensional space [Sul89]. The Sullivan’s results was generalized by Abresh onto a case of surfaces with constant mean curvature [Abr89]. It is well–known that conformal immersions of the minimal surfaces into a 3–dimensional euclidean space are described by a classical Weierstrass representation [Weier]. Thus, there is a close relationship between the spinor structures over the Riemann surfaces (SRS) and conformal immersions (see [KS95, KS96]). Moreover, the Weierstrass representation has a natural formulation in terms of SRS [KS95]. The following important step in this direction was made by Taimanov [Tai97a] (see also [Tai97b, Tai97c, Tai98]). In the work [Tai97a] SRS was constructed on the basis of a so–called generalized Weierstrass representation (GWR) which describes conformal immersions of generic (non–minimal) surfaces into $\mathbb{R}^3$, and also a globalization of this spinor representation was proposed by means of introduction of a spinor bundle. GWR playing a key role in these works, firstly, appears in [Eisen] and further was rediscovered by Kenmotsu [Ken79] and Konopelchenko [Kon96]. In the work [Kon96] a system of two differential equations (a so–called 2–dimensional Dirac equation) which coincides with a linear problem of a modified Veselov–Novikov hierarchy (mVN–hierarchy) has been considered along with GWR. Thus, there is a relationship between the theory of conformal immersions of surfaces into $\mathbb{R}^3$ and soliton theory, since integrable deformations of surfaces are defined by the mVN–hierarchy. Moreover, it allows to express deformations of spinor fields (smooth sections of the spinor bundles) via mVN–deformations [Tai97a]. The other important point in [Tai97a] is establishing a relation with the works of Hoffman and Osserman [HO80, HO83, HO83] on the generalized Gauss map. The following important work in the theory of SRS is a Friedrich’s paper [Fr98]. The main advantage of [Fr98] is a consideration of SRS in the framework of a theory of the Dirac operator on the spin manifolds (see [Bau81, Bär91, Fr97, Amm98]).

As noted above, spinor representations of surfaces in 4d manifolds are not studied in practice, however, at the present time in the papers of Konopelchenko and Landolfi [KL98a, Kon98, KL98b] a generalized Weierstrass representation has been considered for surfaces conformally immersed into 4d pseudo–euclidean spaces. The basic subject of these works is an extension of the results obtained in [Kon96, KT95, KT96] onto 4–dimensional spaces. At this point GWR constructed on the basis of the generalized Gauss map [HO83]. In connection with this it should be noted that a Dirac operator and SRS for surfaces conformally immersed into 4d complex space are considered recently in the framework of the
generalized Weierstrass representation \([\text{Var99a, Var99b}]\).

One of the main goals of the present research is a formulation of GWR for surfaces immersed into 4d manifolds in terms of spinor bundles. An initial point is an extension of Friedrich’s results \([\text{Fr98}]\) onto 4d pseudo–riemannian manifolds. According to widely accepted interpretation a spinor field on the manifold is understood as a smooth section of the spinor bundle. On the other hand, there exists a more profound definition introduced by Chevalley \([\text{Che54}]\) and further developed in the works \([\text{Lou81, Cru87, Cru91}]\). It is a so–called algebraic definition of the spinor field in which spinor is understood as an element of a minimal left ideal of the Clifford algebra. The advantage of this definition is obvious, since it allows to directly use basic facts and theorems of the Clifford algebras theory at the study of the spin manifolds.

The present paper is organized as follows. Since the Clifford algebras in toto are the base of the spinor representations, then the basic facts about these algebras are considered in the section 2. In the section 3 the algebraic definition of a spinor field on the surface immersed into a 3–dimensional manifold is given in terms of a minimal left ideal of the Pauli algebra. Further, a form of the spinor fields defined in the section 4 is a natural extension of the construction presented above (the section 3) onto 4d pseudo–riemannian manifolds. At this point the Clifford algebras of tangent bundles of 4d manifolds are quaternionic algebras, i.e. for any 4–dimensional Clifford algebra there exists a decomposition into the tensor product of two quaternion algebras which further associated respectively with tangent and normal bundles of the immersed surface. The main goal of the section 5 is finding of a Dirac operator on the immersed surface. In accordance with \([\text{Bau89, BFGK, Fr97}]\) we suppose that a spinor field on the ambient manifold is a real Killing spinor field. The classification of the Dirac operators on the surfaces depends on a metric of the ambient manifold and also on a metric of the immersed surface (space–like and time–like surfaces). In virtue of a close relation with theoretical physics an immersion of the surface into the Lorentzian manifold is considered in more details. The relationship between spinor fields on surfaces immersed into the Lorentzian manifold and a Dirac–Hestenes spinor field \([\text{Hes66, Hes67, Hes76}]\) (which has an important meaning in the electron theory \([\text{Kel93, DLGSC}]\)) is established. Further, in local consideration we have conformal immersions of surfaces into 4d pseudo–euclidean spaces, which are considered in the section 6.

2 Algebraic Preliminaries

In this section we will list some basic facts about Clifford algebras which relevant for our further consideration. Let \(K\) be a field of characteristic 0 (\(K = \mathbb{R}, K = \Omega, K = \mathbb{C}\)), where \(\Omega\) is a field of double numbers (\(\Omega = \mathbb{R} \oplus \mathbb{R}\)), and \(\mathbb{R}, \mathbb{C}\) are the fields of real and complex numbers, respectively. A Clifford algebra over a field
\( K \) is an algebra with \( 2^n \) basis elements: \( e_0 \) (unit of the algebra), \( e_1, e_2, \ldots, e_n \) and the products of the one–index elements \( e_{i_1 i_2 \ldots i_k} = e_{i_1} e_{i_2} \ldots e_{i_k} \). Over the field \( K = \mathbb{R} \) the Clifford algebra denoted as \( \mathcal{C}_p,q \), where the indices \( p, q \) correspond to the indices of the quadratic form

\[ Q = x_1^2 + \ldots + x_p^2 - \ldots - x_{p+q}^2 \]

of a vector space \( V \) associated with \( \mathcal{C}_p,q \). A multiplication law of \( \mathcal{C}_p,q \) defined by the following rule:

\[ e_i^2 = \sigma(q - i)e_0, \quad e_ie_j = -e_je_i, \]

where

\[ \sigma(n) = \begin{cases} -1 & \text{if } n \leq 0, \\ +1 & \text{if } n > 0. \end{cases} \]

**Theorem 1** (Chevalley [Che55]). Let \( V \) and \( V' \) be vector spaces endowed with quadratic forms \( Q \) and \( Q' \) over the field \( K \). Then a Clifford algebra \( \mathcal{C}(V \oplus V', Q \oplus Q') \) is naturally isomorphic to \( \mathcal{C}(V, Q) \otimes \mathcal{C}(V', Q') \).

An important role in the theory of Clifford algebras played the square of the volume element \( \omega = e_{12\ldots n}, n = p + q \):

\[ \omega^2 = \begin{cases} -1 & \text{if } p - q \equiv 1, 2, 5, 6 \pmod{8}, \\ +1 & \text{if } p - q \equiv 0, 3, 4, 7 \pmod{8}. \end{cases} \]

If \( p + q \) is even and \( \omega^2 = 1 \), then \( \mathcal{C}_p,q \) is called *positive* and respectively *negative* if \( \omega^2 = -1 \). Or, in accordance with (3):

\[ \mathcal{C}_p,q > 0 \quad \text{if} \quad p - q \equiv 0, 4 \pmod{8}, \]
\[ \mathcal{C}_p,q < 0 \quad \text{if} \quad p - q \equiv 2, 6 \pmod{8}. \]

**Theorem 2** (Karoubi [Kar78, prop. 3.16]). 1) If \( \mathcal{C}(V, Q) > 0 \), and \( \dim V \) is even, then

\[ \mathcal{C}(V \oplus V', Q \oplus Q') \simeq \mathcal{C}(V, Q) \otimes \mathcal{C}(V', Q'). \]

2) If \( \mathcal{C}(V, Q) < 0 \), and \( \dim V \) is even, then

\[ \mathcal{C}(V \oplus V', Q \oplus Q') \simeq \mathcal{C}(V, Q) \otimes \mathcal{C}(V', -Q'). \]

Further, let \( \mathbb{C}_n = \mathbb{C} \otimes \mathcal{C}_p,q \) and \( \Omega_{p,q} = \Omega \otimes \mathcal{C}_p,q \) be the Clifford algebras over the fields \( K = \mathbb{C} \) and \( K = \Omega \), respectively.
Theorem 3 (Rozenfel’d [Roz55]). If \( n = p + q \) is odd, then

\[
\begin{align*}
\mathcal{C}_{p,q} &\simeq \mathbb{C}_{p+q-1} \quad \text{if } p - q \equiv 1, 5 \pmod{8}, \\
\mathcal{C}_{p,q} &\simeq \Omega_{p-1,q} \\
&\simeq \Omega_{p,q-1} \quad \text{if } p - q \equiv 3, 7 \pmod{8}.
\end{align*}
\]

Example. Let us consider the algebra \( \mathcal{C}_{0,3} \). According to the theorem 3 we have \( \mathcal{C}_{0,3} \simeq \Omega_{0,2} \), where \( \Omega_{0,2} \) is an algebra of elliptic biquaternions (it is a first so-called Grassmann’s extensive algebra introduced by Clifford in 1878 [Clif78]). Since \( \Omega = \mathbb{R} \oplus \mathbb{R} \) and \( \Omega_{p,q} = \Omega \otimes \mathcal{C}_{p,q} \), we have \( \mathcal{C}_{0,3} \simeq \Omega_{0,2} \oplus \mathcal{A}_{0,2} \simeq \mathbb{H} \oplus \mathbb{H} \), where \( \mathbb{H} \) is a quaternion algebra.

Generalizing this example we obtain

\[
\begin{align*}
\mathcal{C}_{p,q} &\simeq \mathcal{C}_{p-1,q} \oplus \mathcal{C}_{p-1,q} \\
&\simeq \mathcal{C}_{p,q-1} \oplus \mathcal{C}_{p,q-1} \quad \text{if } p - q \equiv 3, 7 \pmod{8}.
\end{align*}
\]

(4)

Over the field \( \mathbb{K} = \mathbb{C} \) there is the analogous result [Ras55]

Theorem 4. When \( p + q \equiv 1, 3, 5, 7 \pmod{8} \) the Clifford algebra over the field \( \mathbb{K} = \mathbb{C} \) decomposes into a direct sum of two subalgebras:

\[
\mathbb{C}_{p+q} \simeq \mathbb{C}_{p+q-1} \oplus \mathbb{C}_{p+q-1}.
\]

A minimal left (respectively right) ideal of \( \mathcal{C}_{p,q} \) is a set of type \( I_{p,q} = \mathcal{C}_{p+q}e_{pq} \) (resp. \( e_{pq}\mathcal{C}_{p,q} \)), where \( e_{pq} \) is a primitive idempotent, i.e., \( e_{pq}^2 = e_{pq} \) and \( e_{pq} \) cannot be represented as a sum of two orthogonal idempotents, i.e., \( e_{pq} \neq f_{pq} + g_{pq} \), where \( f_{pq}g_{pq} = g_{pq}f_{pq} = 0, f_{pq}^2 = f_{pq}, g_{pq}^2 = g_{pq} \).

Theorem 5 (Lounesto [Lou81]). A minimal left ideal of \( \mathcal{C}_{p,q} \) is of the type \( I_{p,q} = \mathcal{C}_{p+q}e_{pq} \), where \( e_{pq} = \frac{1}{2}(1 + e_{1}) \ldots \frac{1}{2}(1 + e_{k}) \) is a primitive idempotent of \( \mathcal{C}_{p,q} \) and \( e_{1}, \ldots, e_{k} \) are commuting elements of the canonical basis of \( \mathcal{C}_{p,q} \) such that \( (e_{i})^2 = 1, (i = 1, 2, \ldots, k) \) that generate a group of order \( 2^k \), \( k = q - r - r \) and \( r_i \) are the Radon-Hurwitz numbers, defined by the recurrence formula \( r_{i+8} = r_i + 4 \) and

\[
\begin{array}{cccccccc}
  i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
r_i & 0 & 1 & 2 & 2 & 3 & 3 & 3 & 3
\end{array}
\]

3  The Algebraic Definition of a Spinor Field on a 3D Manifold

Let us consider a 3-dimensional oriented Riemannian manifold \( M^{3,0} \) with a fixed spin structure and also an oriented surface \( S^{2,0} \) isometrically immersed into \( M^{3,0} \),
$S^{2,0} \hookrightarrow M^{3,0}$. At this point the surface $S^{2,0}$ is understood as a Riemannian submanifold with some spin structure. Moreover, since the normal bundle of the surface (precisely speaking, hypersurface) $S^{2,0}$ is trivial, then the spin structure of $M^{3,0}$ induces a spin structure on the surface $S^{2,0} \hookrightarrow M^{3,0}$. It is well-known that a spinor field is a smooth section of the spinor bundle. Let $\Phi$ be a spinor field on the manifold $M^{3,0}$. It is obvious that the field $\Phi$ is a section of 2-dimensional spinor bundle $S = Q \times_{\text{Spin}(3,0)} \Delta_{3,0}$, since $\text{Spin}(3,0) \simeq \text{Spin}(0,3)$ and $\mathbb{C}^{+} \simeq \mathbb{H}$, then $\dim \Delta_{3,0} = 2$ (here $\mathbb{H}$ is a quaternion algebra, $\mathbb{C}^{+}$ is a Clifford algebra of all even elements). On the other hand, $\text{Spin}(3,0) \simeq Sp(1) \simeq SU(2) \simeq S^{3}$ [Port69] and $\mathbb{C}^{+} \simeq \mathbb{C} \simeq M_{2}(\mathbb{C})$. Further, it takes to find a restriction of the spinor bundle $S = Q \times_{\text{Spin}(3,0)} \Delta_{3,0}$ of the manifold $M^{3,0}$ onto a spinor bundle $S_{M^{2,0}} = Q \times_{\text{Spin}(2,0)} \Delta_{2,0}$ of the surface $M^{2,0}$ conformally immersed into $M^{3,0}$. Let $\phi$ is a spinor field on the surface $M^{2,0}$. Obviously, this field is a section of 2-dimensional spinor bundle $S = Q \times_{\text{Spin}(2,0)} \Delta_{2,0}$, since $\text{Spin}(2,0) \simeq \text{Spin}(0,2)$ and $\mathbb{C}^{+} \simeq \mathbb{C}$, then $\text{Spin}(2,0) \simeq U(1) \simeq S^{1}$. Moreover, the spinor bundle of the surface $M^{2,0}$ splits into two subbundles,

$$S = S^{+} \oplus S^{-},$$

where $S^{\pm} = Q \times_{\text{Spin}(2,0)} \Delta_{2,0}^{\pm}$. Respectively, a smooth section $\phi \in \Gamma(S)$ of the bundle $S$ has a form $\phi = \phi^{+} + \phi^{-}$, where [Fr98]

$$\phi^{+} = \frac{1}{2}(\varphi + i\xi \cdot \varphi), \quad \phi^{-} = \frac{1}{2}(\varphi - i\xi \cdot \varphi),$$

(5)

here $\xi = e_{3} = e_{1}e_{2}$, $\phi^{+}, \phi^{-}$ are so-called half–spinors (Weyl spinors) of the surface $S^{2,0} \hookrightarrow M^{3,0}$.

We will call the definition of the spinors (5) given above as a geometrical definition, where the spinor field is understood as a smooth section of the spinor bundle (this definition is widely used in [LM89, BFGK]). On the other hand, there exists an algebraic definition of the spinor field as a minimal left ideal of the Clifford algebra $\mathcal{C}_{p,q}$ (see [Che54, Lou81]). The algebraic definition in comparison with geometric definition possess a more rich structure since allows to directly use all the existing apparatus of the Clifford algebra theory.

Let us consider in details an algebraic definition of the spinors (5) as elements of a minimal left ideal $I_{3,0} = \mathcal{C}_{3,0}e_{30}$ of the Pauli algebra $\mathcal{C}_{3,0}$ ($\mathcal{C}_{3,0}$ is a Clifford algebra of a tangent bundle of the manifold $M^{3,0}$). In accordance with the theorem [B] a primitive idempotent of $\mathcal{C}_{3,0} \simeq \mathbb{C}_{2}$ has a form $e_{30} = \frac{1}{2}(1 + e_{0}) \sim \frac{1}{2}(1 + ie_{12})$, since in this case a number of commuting elements equals to $k = q - r_{q-p} = 0 - r_{-3} = 0 - (r_{5} - 4) = 1$. Further, it is obvious that $I_{3,0} = \mathcal{C}_{3,0}e_{30} \simeq \mathbb{C}_{2}e_{30} \simeq M_{2}(\mathbb{C})e_{30}$. By virtue of the isomorphism $\mathcal{C}_{3,0} \simeq \mathbb{C} \otimes \mathcal{C}_{2}$ a general element of $\mathcal{C}_{3,0}$ may be represented in the form of a following complex antiquaternion

$$\mathcal{A} = \mathcal{C}_{0,1}^{0}e_{0} + \mathcal{C}_{0,1}^{1}e_{1} + \mathcal{C}_{0,2}^{0}e_{2} + \mathcal{C}_{0,1}^{3}e_{1}e_{2},$$

(6)
where \( e_1^2 = e_2^2 = 1 \). Since \( \varphi \in I_{3,0} \), then

\[
\begin{align*}
\varphi^+ &= e_{20}^+ I_{3,0}, \\
\varphi^- &= e_{20}^- I_{3,0},
\end{align*}
\]

(7)

where

\[
\begin{align*}
e_{20}^+ &= \frac{1}{2} (1 + i e_{12}), \\
e_{20}^- &= \frac{1}{2} (1 - i e_{12})
\end{align*}
\]

are mutually orthogonal idempotents of the anti-quaternion \( \mathbb{H} \). Or, coming to matrix representations

\[
\begin{align*}
e_1 &\mapsto \begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix}, \\
e_2 &\mapsto \begin{pmatrix} 0 & i \\
-i & 0 \end{pmatrix}
\end{align*}
\]

we obtain that \( \varphi = \begin{pmatrix} \varphi_1 & 0 \\
\varphi_2 & 0 \end{pmatrix} \in I_{3,0} \cong M_2(\mathbb{C})e_{30} \) and

\[
\begin{align*}
\varphi^+ &= \frac{1}{2} (1 + i e_{12}) \varphi = \begin{pmatrix} \varphi_1 & 0 \\
0 & 0 \end{pmatrix}, \\
\varphi^- &= \frac{1}{2} (1 - i e_{12}) \varphi = \begin{pmatrix} 0 & 0 \\
\varphi_2 & 0 \end{pmatrix},
\end{align*}
\]

(8)

where

\[
\begin{align*}
\varphi_1 &= a^0 - i a^{12}, \\
\varphi_2 &= a^1 - i a^2,
\end{align*}
\]

\( a^0, a^1, a^2, a^{12} \in \mathbb{C} \). Thus, the Weyl spinors \( \varphi^+, \varphi^- \) of the surface \( S^{2,0} \hookrightarrow M^{3,0} \) are expressed via the elements of the minimal left ideal of the Pauli algebra \( \mathfrak{Cl}_{3,0} \) by the formulae (7) or (8).

\section*{4 Spinor Structures on the Immersed Surfaces}

Let \( M^{p,q} (p + q = 4) \) be a four–dimensional, real, connected, paracompact manifold and let \( T M^{p,q} \) (respectively \( T^* M^{p,q} \)) be a tangent (resp. cotangent) bundle of the manifold \( M^{p,q} \).

\textbf{Definition 1.} A Lorentzian manifold is a pair \((M^{1,3}, g)\), where \( g \) is a Lorentz metric with a signature \((+, -, -, -)\), i.e. for any \( x \in M^{1,3} \) there exists an isomorphism \( T_x M^{1,3} \cong T^*_x M^{1,3} \cong \mathbb{R}^{1,3} \), where \( \mathbb{R}^{1,3} \) is a Minkowski spacetime. 

A Majorana manifold is a pair \((M^{3,1}, g)\), where \( g \) is a metric with a signature \((+, +, +, -)\), i.e. for any \( x \in M^{3,1} \), \( T_x M^{3,1} \cong T^*_x M^{3,1} \cong \mathbb{R}^{3,1} \).

A Kleinian manifold is a pair \((M^{2,2}, g)\), where a signature of the metric \( g \) has a form \((+, +, -, -)\) and for \( x \in M^{2,2} \) follows \( T_x M^{2,2} \cong T^*_x M^{2,2} \cong \mathbb{R}^{2,2} \).
Further, it is obvious that the spin structure of the 4–dimensional manifold induces a spin structure of the immersed surface \( S^{r,s} \hookrightarrow M^{p,q} \), where \( r + s = 2, p + q = 4 \). In order to clarify this question let us consider previously a more general case. Let \( M \) be an \((n + m)\)-dimensional riemannian manifold and let \( F \hookrightarrow M \) be an \( n \)-dimensional immersed submanifold. We suppose that the both manifolds are endowed with a some spin structure. Let \( N \) be a normal bundle of the manifold \( F \hookrightarrow M \). In accordance with [Mil65], the sum of the spin structures on the tangent bundle and on the normal bundle of \( F \) coincides with the spin structure on the tangent bundle of \( M \) restricted to \( F \). Thus, for any point \( x \in F \)

we have \( T_x M = T_x F \oplus N_x \). Further, let \( \mathcal{C}(T_x M) = \mathcal{C}(T_x M, Q) \) be a Clifford algebra of the tangent space of the manifold \( M \) at the point \( x \), where \( Q \) is a quadratic form of a vector space \( \mathbb{R}^{p,q} \simeq T_x P \), with \( p + q = n + m \). In accordance with the theorem 1, it follows that \( \mathcal{C}(V \oplus V’, Q \oplus Q’) \simeq \mathcal{C}(V, Q) \otimes \mathcal{C}(V’, Q’) \), where in our case \( V \simeq T_x F, V’ \simeq N_x \), \( Q \) and \( Q’ \) are quadratic forms of the spaces \( T_x F \) and \( N_x \). Moreover, if \( \dim V \) is even we have the theorem 2: \( \mathcal{C}(T_x F \oplus N_x, Q \oplus Q’) \simeq \mathcal{C}(T_x F, Q) \otimes \mathcal{C}(N_x, Q’) \) if \( \mathcal{C}(T_x F, Q) > 0 \) and \( \mathcal{C}(T_x F \oplus N_x, Q \oplus Q’) \simeq \mathcal{C}(T_x F, Q) \otimes \mathcal{C}(N_x, -Q’) \) if \( \mathcal{C}(T_x F, Q) < 0 \). In any case the sum \( T_x F \oplus N_x \)

induces a tensor product of the corresponding Clifford algebras.

Let us return to the 4–dimensional manifolds. It is easy to see that in this case the sum \( T_x F \oplus N_x \) induces a tensor product of the quaternion algebras, \( \mathcal{C}_{r,s} \otimes \mathcal{C}_{k,t} \), where \( r + s = k + t = 2 \). At this point there exist three types of tensor factors: a quaternion algebra \( \mathcal{C}_{0,2} \), an anti–quaternion algebra \( \mathcal{C}_{2,0} \) and a pseudo–quaternion algebra \( \mathcal{C}_{1,1} \). Since a square of the volume element \( \omega = e_{12} \) of \( \mathcal{C}_{0,2} \) equals to \(-1\), then the algebra \( \mathcal{C}_{0,2} \) is negative, \( \mathcal{C}_{0,2} < 0 \). Analogously, \( \mathcal{C}_{2,0} < 0 \) and \( \mathcal{C}_{1,1} > 0 \). Thus, according to the theorem 2, we have the following decompositions for the 4–dimensional Clifford algebras:

\[
\begin{align*}
\mathcal{C}_{4,0} & \simeq \mathcal{C}_{2,0} \otimes \mathcal{C}_{0,2}, \\
\mathcal{C}_{0,4} & \simeq \mathcal{C}_{0,2} \otimes \mathcal{C}_{2,0}, \\
\mathcal{C}_{1,3} & \simeq \mathcal{C}_{1,1} \otimes \mathcal{C}_{0,2}, \\
& \simeq \mathcal{C}_{0,2} \otimes \mathcal{C}_{1,1}, \\
\mathcal{C}_{3,1} & \simeq \mathcal{C}_{1,1} \otimes \mathcal{C}_{2,0}, \\
& \simeq \mathcal{C}_{2,0} \otimes \mathcal{C}_{1,1}, \\
\mathcal{C}_{2,2} & \simeq \mathcal{C}_{2,0} \otimes \mathcal{C}_{2,0}, \\
& \simeq \mathcal{C}_{0,2} \otimes \mathcal{C}_{0,2}, \\
& \simeq \mathcal{C}_{1,1} \otimes \mathcal{C}_{1,1}.
\end{align*}
\]

Further, let \( Q \times_{\text{Spin}(r,s)} \Delta_{r,s} \) be a spinor bundle of the surface \( S^{r,s} \), where \( \Delta_{r,s} \) is a representation of the group \( \text{Spin}(r,s) \). Since \( r + s = 2 \), then the bundle \( S \) splits into two subbundles \( S = S^+ \oplus S^- \), where \( S^\pm = Q \times_{\text{Spin}(r,s)} \Delta^\pm_{r,s} \), \( \Delta^\pm_{r,s} \) are the spaces of half–spinors. The smooth section \( \psi = \psi^+ \oplus \psi^- \in \Gamma(S^+) \oplus \Gamma(S^-) \) is called a spinor field on the surface \( S^{r,s} \). On the other hand, the components of the spinor
field \( \psi \) may be represented by elements of the minimal left ideal of \( \mathfrak{A}_{r,s} \). Let us consider in details the minimal left ideals of the algebras \( \mathfrak{A}_{2,0} \cong \mathbb{R}(2) \), \( \mathfrak{A}_{0,2} \cong \mathbb{H} \), and \( \mathfrak{A}_{1,1} \cong \mathbb{R}(2) \), which are the Clifford algebras of the tangent spaces for three different types of the surfaces \( S^{2,0} \), \( S^{0,2} \), and \( S^{1,1} \). In accordance with the theorem 5, we have \( k_{\mathfrak{A}_{2,0}} = 0 - r_{-2} = 0 - (r_{0} - 4) = 1 \), \( k_{\mathfrak{A}_{0,2}} = 2 - r_{2} = 0 \), \( k_{\mathfrak{A}_{1,1}} = 1 - r_{0} = 1 \). Therefore, for the primitive idempotents of \( \mathfrak{A}_{2,0} \), \( \mathfrak{A}_{0,2} \), and \( \mathfrak{A}_{1,1} \) we obtain respectively \( e_{20} = \frac{1}{2}(1 + e_{1}) \), \( e_{02} = 1 \) and \( e_{11} = \frac{1}{2}(1 + e_{12}) \). The minimal left ideals have a form \( I_{2,0} = \mathfrak{A}_{2,0} \frac{1}{2}(1 + e_{1}) \), \( I_{0,2} = \mathfrak{A}_{0,2} \), \( I_{1,1} = \mathfrak{A}_{1,1} \frac{1}{2}(1 + e_{12}) \) and respectively for division rings we obtain \( \mathbb{K} = e_{20} \mathfrak{A}_{2,0} e_{20} = \{1\} \), \( \mathbb{K} = e_{02} \mathfrak{A}_{0,2} e_{02} = \{1, e_{1}, e_{2}, e_{12}\} \), \( \mathbb{K} = e_{11} \mathfrak{A}_{1,1} e_{11} = \{1\} \). Thus, on the surface \( S^{2,0} \) there exists a spinor field \( \psi = \psi^{+} + \psi^{-} \), where
\[
\psi^{+} = e_{20} I_{2,0}, \quad \psi^{-} = e_{20} I_{2,0},
\]
here
\[
e_{20}^{+} = \frac{1}{2}(1 + e_{0}), \quad e_{20}^{-} = \frac{1}{2}(1 - e_{0})
\]
are mutually orthogonal idempotents of \( \mathfrak{A}_{2,0} \). Analogously, on the surfaces \( S^{0,2} \) and \( S^{1,1} \) there exist spinor fields
\[
\psi^{\pm} = e_{02}^{\pm} I_{0,2}, \quad \psi^{\pm} = e_{11}^{\pm} I_{1,1},
\]
where respectively
\[
e_{02}^{\pm} = \frac{1}{2}(1 \pm i e_{12}), \quad e_{11}^{\pm} = \frac{1}{2}(1 \pm e_{1}).
\]

Further, let \( \Psi \) be a spinor field on the 4-dimensional pseudo–Riemannian manifold. The our main goal is finding of restrictions \( \Psi = \Psi|_{S^{r,s}} \) on the immersed surfaces \( S^{r,s} \hookrightarrow M^{p,q} \). First of all, the spinor field of the 4d manifold is a smooth section of the spinor bundle, \( \Psi = \Psi^{+} + \Psi^{-} \in \Gamma(S^{+}) \oplus \Gamma(S^{-}) \), where \( S^{\pm} = Q \times \text{Spin}(p,q) \Delta^{\pm}_{p,q} \), \( \Psi^{\pm} = \left( \begin{array}{c} \Psi_{1}^{\pm} \\ \Psi_{2}^{\pm} \end{array} \right) \). By analogy with the 2-dimensional case the field \( \Psi \) on \( M^{p,q} \) may be considered as an element of the minimal left ideal of the corresponding algebra \( \mathfrak{A}_{p,q} \). According to the theorem 4, primitive idempotents of the algebras \( \mathfrak{A}_{1,0} \), \( \mathfrak{A}_{2,0} \), \( \mathfrak{A}_{1,2} \), \( \mathfrak{A}_{3,1} \) have respectively a form: \( e_{40} = \frac{1}{2}(1 + e_{1}) \), \( e_{04} = \frac{1}{2}(1 + e_{123}) \), \( e_{22} = \frac{1}{2}(1 + e_{13}) \frac{1}{2}(1 + e_{24}) \), \( e_{31} = \frac{1}{2}(1 + e_{12}) \frac{1}{2}(1 + e_{1}) \), \( e_{13} = \frac{1}{2}(1 + e_{11}) \). It should be noted that the different structure of the presented idempotents may be explained by using of the Karouibi theorem 4. Indeed, for the Majorana algebra \( \mathfrak{A}_{3,1} \) by theorem 4 there exists a decomposition \( \mathfrak{A}_{3,1} \cong \mathfrak{A}_{1,1} \otimes \mathfrak{A}_{2,0} \) (or \( \mathfrak{A}_{2,0} \otimes \mathfrak{A}_{1,1} \)), which induces a product of the primitive idempotents \( e_{11} e_{20} \) (or \( e_{20} e_{11} \)). Here \( e_{11} = \frac{1}{2}(1 + e_{12}) \) and \( e_{20} = \frac{1}{2}(1 + e_{1}) \).
therefore $e_{31} = e_{11}e_{20} \sim \frac{1}{2}(1 + e_{24})\frac{1}{2}(1 + e_{1})$. In contrast with this, the spacetime algebra $\mathcal{A}_{1,3}$ has a decomposition $\mathcal{A}_{1,3} \simeq \mathcal{A}_{1,1} \otimes \mathcal{A}_{0,2}$ ($\mathcal{A}_{0,2} \otimes \mathcal{A}_{1,1}$), whence $e_{13} = e_{11}e_{20} \sim \frac{1}{2}(1 + e_{14})$, since $e_{02} = 1$. The analogous situation takes place for the algebras $\mathcal{A}_{4,0} \simeq \mathcal{A}_{2,0} \otimes \mathcal{A}_{0,2}$ and $\mathcal{A}_{0,4} \simeq \mathcal{A}_{0,2} \otimes \mathcal{A}_{2,0}$. Thus, for the minimal left ideals of the 4–dimensional Clifford algebras and their division rings we have

\[
I_{0,4} = \mathcal{A}_{0,4} \frac{1}{2}(1 + e_{123}), \quad \mathbb{K} = \{1, e_1, e_{13}, e_3\} \simeq \mathbb{H};
\]

\[
I_{1,3} = \mathcal{A}_{1,3} \frac{1}{2}(1 + e_{14}), \quad \mathbb{K} = \{1, e_2, e_3, e_{24}\} \simeq \mathbb{H};
\]

\[
I_{2,2} = \mathcal{A}_{2,2} \frac{1}{2}(1 + e_{13})\frac{1}{2}(1 + e_{24}), \quad \mathbb{K} = \{1\} \simeq \mathbb{R};
\]

\[
I_{3,1} = \mathcal{A}_{3,1} \frac{1}{2}(1 + e_{1})\frac{1}{2}(1 + e_{24}), \quad \mathbb{K} = \{1\} \simeq \mathbb{R};
\]

\[
I_{4,0} = \mathcal{A}_{4,0} \frac{1}{2}(1 + e_1), \quad \mathbb{K} = \{1, e_{23}, e_{24}, e_{34}\} \simeq \mathbb{H}.
\]

Let us consider now a spinor field of the time–like surface $S^{1,1}$ immersed into the Lorentzian manifold $M^{1,3}$. In this case the Clifford algebra of a tangent space at the point $x \in S^{1,1}$ of the manifold $M^{1,3}$ restricted to $S^{1,1}$ has a form $\mathcal{A}_{1,3} \simeq \mathcal{A}_{1,1} \otimes \mathcal{A}_{0,2}$. Let consider in more details a structure of the decomposition $\mathcal{A}_{1,1} \otimes \mathcal{A}_{0,2}$. First of all, a similar form of decomposition tells that for the algebra $\mathcal{A}_{1,3}$ there exists a transition from the real coordinates to quaternion coordinates of the form $a + b\zeta_1 + c\zeta_2 + d\zeta_1\zeta_2$, where $\zeta_1 = e_{123}$, $\zeta_2 = e_{124}$ and $\zeta_1^2 = \zeta_2^2 = (\zeta_1\zeta_2)^2 = -1$, $e_1^2 = 1, e_2^2 = e_3^2 = e_4^2 = -1$. The units $\zeta_1, \zeta_2$ are a basis of the quaternion algebra, since $\zeta_1 \sim i$, $\zeta_2 \sim j$, $\zeta_1\zeta_2 \sim k$. Therefore, a general element of the spacetime algebra $\mathcal{A}_{1,3}$ may be written as follows

\[
\mathcal{A} = \mathcal{A}_{0,1}^0 + \mathcal{A}_{1,1}^0 \zeta_1 + \mathcal{A}_{1,1}^2 \zeta_2 + \mathcal{A}_{1,1}^3 \zeta_1\zeta_2,
\]

where the every coefficient $\mathcal{A}_{1,1}^i$ is isomorphic to the pseudo–quaternion algebra $\mathcal{A}_{1,1}$:

\[
\mathcal{A}_{1,1}^0 = a + a^1e_1 + a^2e_2 + a^{12}e_{12},
\]

\[
\mathcal{A}_{1,1}^1 = -a^{123} - a^{23}e_1 - a^{13}e_2 - a^3e_{12},
\]

\[
\mathcal{A}_{1,1}^2 = -a^{124} - a^{24}e_1 + a^{14}e_2 + a^4e_{12},
\]

\[
\mathcal{A}_{1,1}^3 = -a^{34} - a^{134}e_1 - a^{34}e_2 + a^{1234}e_{12}.
\]

It is easy to verify that the units $\zeta_i$ commute with every basis element of the algebra $\mathcal{A}_{1,1}$.

Further, let $\gamma : \mathcal{A}_{1,3} \to \text{End}(I_{1,3})$ be a spinor representation of the spacetime algebra defined by the standard matrix representation

\[
\gamma_0 = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_k = -\gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix},
\]
where \( \sigma_k \) are the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

By virtue of an identity \( \mathfrak{C}_{1,3} \frac{1}{2}(1 + \gamma_0) = \mathfrak{C}_{1,3} \frac{1}{2}(1 + \gamma_0) \) \[\text{FRO90}\] the minimal left ideal of \( \mathfrak{C}_{1,3} \) takes a form \( I_{1,3} = \mathfrak{C}_{1,3} \frac{1}{2}(1 + \gamma_0) \simeq \mathfrak{C}_{3,0} \frac{1}{2}(1 + \gamma_0) \), since \( \mathfrak{C}_{3,0} \simeq \mathfrak{C}_{1,3} \).

Let \( \phi \in \mathfrak{C}_{3,0} \) be a Dirac–Hestenes spinor field and let \( \Psi \in I_{1,3} = \mathfrak{C}_{1,3} \frac{1}{2}(1 + \gamma_0) \) be a so-called mother spinor \[\text{Lou93}\], then

\[
\Psi = \phi \frac{1}{2}(1 + \gamma_0) = \begin{pmatrix} \phi_1 & -\phi_2^* & 0 & 0 \\ \phi_2 & \phi_1^* & 0 & 0 \\ \phi_3 & \phi_4^* & 0 & 0 \\ \phi_4 & -\phi_3^* & 0 & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 & -\phi_2^* & \phi_3 & \phi_4^* \\ \phi_2 & \phi_1^* & \phi_4 & -\phi_3^* \\ \phi_3 & \phi_4^* & \phi_1 & -\phi_2^* \\ \phi_4 & -\phi_3^* & \phi_2 & \phi_1^* \end{pmatrix}.
\]

Let

\[
\epsilon_{+13}^+ = \frac{1}{2}(1 + i\zeta_1 \zeta_2), \quad \epsilon_{-13}^- = \frac{1}{2}(1 - i\zeta_1 \zeta_2)
\]

be mutually orthogonal idempotents of the quaternion \( \mathbb{H} \), then by analogy with (7) we have for a spinor field \( \Psi \) of the Lorentzian manifold \( M^{1,3} \) restricted to \( S^{1,1} \) the following components

\[
\Psi_{_{\mathbb{H}_{1,1}}}^+ = \epsilon_{+13}^+ I_{1,3}, \quad \Psi_{_{\mathbb{H}_{1,1}}}^- = \epsilon_{-13}^- I_{1,3},
\]

(10)

where \( I_{1,3} \simeq \mathfrak{C}_{3,0} \frac{1}{2}(1 + \gamma_0) \simeq M_2(\mathbb{C}) \frac{1}{2}(1 + i\sigma_{12}) \), since \( \mathfrak{C}_{3,0} \simeq \mathbb{C}_2 \simeq M_2(\mathbb{C}) \).

Therefore, for the mother spinor \( \Psi \in I_{1,3} \) we have

\[
\Psi = \phi \frac{1}{2}(1 + i\sigma_{12}) = \begin{pmatrix} \phi_1^* + \phi_3^* & 0 \\ \phi_4 - \phi_2 & 0 \end{pmatrix},
\]

where \( \phi \in \mathbb{C}_2 \simeq M_2(\mathbb{C}) \) is a Dirac–Hestenes spinor field with a following matrix representation

\[
\phi = \begin{pmatrix} \phi_1^* + \phi_3^* & \phi_4 + \phi_2 \\ \phi_4 - \phi_2 & \phi_1 - \phi_3 \end{pmatrix},
\]

here

\[
\phi_1 = a^0 - ia^{12}, \quad \phi_2 = -a^{13} - ia^{23}, \quad \phi_3 = a^3 - ia^{123}, \quad \phi_4 = a^1 + ia^2.
\]

Let us define matrix representations of the quaternion units \( \zeta_1 \) and \( \zeta_2 \) as follows

\[
\zeta_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \zeta_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\]

11
then a restricted mother spinor \( \Psi_{|S^{1,1}} \) on the time–like surface \( S^{1,1} \hookrightarrow M^{1,3} \) takes a form

\[
\begin{align*}
\Psi^+_{|S^{1,1}} &= \frac{1}{2}(1 + i\zeta_1 \zeta_2)\Psi = \begin{pmatrix} \phi_1^* + \phi_4^* & 0 \\ 0 & 0 \end{pmatrix}, \\
\Psi^-_{|S^{1,1}} &= \frac{1}{2}(1 - i\zeta_1 \zeta_2)\Psi = \begin{pmatrix} 0 & 0 \\ \phi_4 - \phi_2 & 0 \end{pmatrix}.
\end{align*}
\]

(11)

Analogously, for the space–like surface \( S^{0,2} \) immersed into \( M^{1,3} \) in virtue of the decomposition \( \mathcal{C}_{1,3} \cong \mathcal{C}_0 \oplus \mathcal{C}_{1,1} \) the general element of \( \mathcal{C}_{1,3} \) can be represented by a following pseudo–quaternion

\[
\mathcal{A} = \mathcal{C}_{0,2}^0 \zeta_0 + \mathcal{C}_{1,2}^1 \zeta_1 + \mathcal{C}_{0,2}^2 \zeta_2 + \mathcal{C}_{0,2}^3 \zeta_1 \zeta_2,
\]

(12)

where \( \zeta_1 = e_{134}, \zeta_2 = e_{234} \) are pseudo–quaternion units, \( \zeta_1^2 = -1, \zeta_2^2 = 1, (\zeta_1 \zeta_2)^2 = 1 \), and an every coefficient \( \mathcal{C}_{0,2}^i \) in \((12)\) is the quaternion algebra:

\[
\begin{align*}
\mathcal{C}_{0,2}^0 &= a^0 + a^3 e_3 + a^4 e_4 + a^{34} e_{34}, \\
\mathcal{C}_{0,2}^1 &= -a^{134} - a^{14} e_3 + a^{13} e_4 - a^2 e_{34}, \\
\mathcal{C}_{0,2}^2 &= a^{234} + a^{24} e_3 + a^{23} e_4 - a^3 e_{34}, \\
\mathcal{C}_{0,2}^3 &= a^{12} + a^{123} e_3 + a^{124} e_4 + a^{1234} e_{34}.
\end{align*}
\]

Let

\[
\zeta_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \zeta_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

then for a restricted mother spinor \( \Psi_{|S^{0,2}} \) on the space–like surface \( S^{0,2} \hookrightarrow M^{1,3} \) we obtain

\[
\begin{align*}
\Psi^+_{|S^{0,2}} &= \frac{1}{2}(1 + \zeta_1 \zeta_2) I_{1,3}, \\
\Psi^-_{|S^{0,2}} &= \frac{1}{2}(1 - \zeta_1 \zeta_2) I_{1,3},
\end{align*}
\]

where \( I_{1,3} \simeq M_2(\mathbb{C}) \frac{1}{2}(1 + i\sigma_{12}) \).

Further, for the immersion \( S^{1,1} \hookrightarrow M^{3,1} \) in virtue of the decomposition \( \mathcal{C}_{3,1} \cong \mathcal{C}_{1,1} \otimes \mathcal{C}_{2,0} \) a general element of the Majorana algebra \( \mathcal{C}_{3,1} \) may be represented by a following anti–quaternion

\[
\mathcal{A} = \mathcal{C}_{1,1}^0 + \mathcal{C}_{1,1}^1 \zeta_1 + \mathcal{C}_{1,1}^2 \zeta_2 + \mathcal{C}_{1,1}^3 \zeta_1 \zeta_2,
\]

(13)

where \( \zeta_1 = e_{134}, \zeta_2 = e_{234} \) are anti–quaternion units, \( \zeta_1^2 = \zeta_2^2 = 1, (\zeta_1 \zeta_2)^2 = -1, \)

\( e_1^2 = e_2^2 = e_3^2 = 1, e_4^2 = -1 \). At this point an every coefficient \( \mathcal{C}_{1,1}^i \) in \((13)\) is
Let us show the validity of a relation
\[ E \] isomorphic to the pseudo–quaternion algebra:
\[ E \]
Since the division ring of \( \mathcal{O}_{3,1} \) is \( \mathbb{K} \cong \mathbb{R} \), then \( \mathcal{O}_{3,1} \cong \mathbb{R}(4) \) and for a spinor representation of \( \mathcal{O}_{3,1} \) we have \( \gamma : \mathcal{O}_{3,1} \to \text{End}_2(I_{3,1}) \), where \( I_{3,1} = \mathcal{O}_{3,1} e_{31} = \mathcal{O}_{3,1} \frac{1}{2}(1 + e_1) \frac{1}{2}(1 + e_{24}) \). Thus the basis of a spacetime \( S \sim I_{3,1} \) defined as follows
\[
\begin{align*}
 f_1 &= e_{31} = \frac{1}{4}(1 + e_1 + e_{24} + e_{124}), \\
f_2 &= e_{2} e_{31} = \frac{1}{4}(e_2 - e_{12} + e_4 - e_{14}), \\
f_3 &= e_{3} e_{31} = \frac{1}{4}(e_3 - e_{13} - e_{234} + e_{1234}), \\
f_4 &= e_{23} e_{31} = \frac{1}{4}(e_{23} + e_{123} - e_{34} - e_{134}).
\end{align*}
\]
In this basis the matrices \( \xi_i = \gamma(e_i) \) are
\[
\begin{align*}
 \xi_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \xi_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
 \xi_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, & \xi_4 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}
\end{align*}
\]
Let us show the validity of a relation \( \mathcal{O}_{3,1} \frac{1}{2}(1 + e_1) \frac{1}{2}(1 + e_{24}) \cong \mathcal{O}_{3,1} \frac{1}{2}(1 + e_1) \frac{1}{2}(1 + e_{24}) \) which plays a key role at the restriction of the bundle \( Q \times_{\text{Spin}(3,1)} \Delta_{3,1} \) to the bundle \( Q \times_{\text{Spin}(1,1)} \Delta_{1,1} \) of the surface \( S_{1,1} \hookrightarrow M_{3,1} \). Since \( \mathcal{O}_{3,1} = \mathcal{O}_{3,1}^+ \oplus \mathcal{O}_{3,1}^- \) and \( \mathcal{O}_{3,1} \cong M_4(\mathbb{R}) \), then in the basis \( \xi \) we obtain
\[
\mathcal{O}_{3,1} e_{31} = \mathcal{O}_{3,1}^+ e_{31} \oplus \mathcal{O}_{3,1}^- e_{31} \cong \begin{pmatrix} 0 & 0 & 0 & \phi_1 \\ 0 & 0 & 0 & \phi_2 \\ 0 & 0 & 0 & \phi_3 \\ 0 & 0 & 0 & \phi_4 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & \eta_1 \\ 0 & 0 & 0 & \eta_2 \\ 0 & 0 & 0 & \eta_3 \\ 0 & 0 & 0 & \eta_4 \end{pmatrix},
\]
where
\[
\begin{align*}
 \phi_1 &= -a_{23}^2 + a_{34}^2, & \phi_2 &= a_{13}^2 - a_{1234}^2, & \phi_3 &= -a_{12}^2 - a_{14}^2, & \phi_4 &= a_0^2 + a_{24}^2, \\
 \eta_1 &= -a_{123}^2 + a_{134}^2, & \eta_2 &= -a_3^2 + a_{234}^2, & \eta_3 &= a_2^2 + a_4^2, & \eta_4 &= a_1^2 + a_{124}^2.
\end{align*}
\]
If suppose \( \xi_i = \phi_i + \eta_i \), then from (13) follows \( C_{3,1}\epsilon_{31} \simeq C_{3,1}^+\epsilon_{31} \). Further, let us define matrix representations of the anti–quaternion units \( \zeta_1 \) and \( \zeta_2 \) as follows

\[
\zeta_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \zeta_2 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]

then for components of the restricted spinor field of the time–like surface \( S^{1,1} \hookrightarrow M^{3,1} \) we obtain

\[
\Psi^+_i |_{S^{1,1}} = \frac{1}{2}(1 + i\zeta_1\zeta_2)I_{3,1},
\]

\[
\Psi^-_i |_{S^{1,1}} = \frac{1}{2}(1 - i\zeta_1\zeta_2)I_{3,1},
\]

where \( I_{3,1} \simeq C_{3,1}^+(1 + \xi_1)\frac{1}{2}(1 + \xi_{24}) \simeq M_2(\mathbb{C})\frac{1}{2}(1 + i\sigma_{12}) \) in virtue of an isomorphism \( C_{3,1}^+ \simeq C_{1,2} \simeq \mathbb{C}_2 \). Thus, as in the case of the immersions \( S^{1,1} \hookrightarrow M^{1,3} \), \( S^{0,2} \hookrightarrow M^{1,3} \) the components (16) expressed via the Dirac–Hestenes spinors \( \phi_i \in C_2 \).

The analogous restriction take place for an immersion \( S^{2,0} \hookrightarrow M^{3,1} \). In this case in virtue of the decomposition \( C_{3,1} \simeq C_{2,0} \otimes C_{1,1} \) the general element of \( C_{3,1} \) represented by a pseudo–quaternion \( A = \sum_{i=0}^{3} C_i \zeta_i \), where \( \zeta_0 = 1, \zeta_1 = e_{124}, \zeta_2 = e_{123}, \zeta_3 = \zeta_1\zeta_2 \) and \( \zeta_1^2 = 1, \zeta_2^2 = -1, \zeta_3^2 = 1 \). At this point coefficients \( C_i \) (anti–quaternions) are

\[
C_0 = a^0 + a^1e_1 + a^2e_2 + a^{12}e_{12},
\]

\[
C_1 = a^{12} + a^{24}e_1 - a^{14}e_2 - a^4e_{12},
\]

\[
C_2 = a^{123} + a^{23}e_1 - a^{13}e_2 - a^3e_{12},
\]

\[
C_3 = a^{34} + a^{134}e_1 + a^{234}e_2 + a^{1234}e_{12}.
\]

The restricted spinor field on \( S^{2,0} \hookrightarrow M^{3,1} \) has a form \( \Psi_i |_{S^{2,1}} = (e^+ I_{3,1}, e^- I_{3,1}) \), where \( I_{3,1} \simeq C_{2,1}^+(1 + i\sigma_{12}), e^\pm = \frac{1}{2}(1 \pm i\zeta_3) \).

Let us consider now an immersion \( S^{2,0} \hookrightarrow M^{4,0} \). First of all, since the division ring of \( C_{4,0} \) is \( \mathbb{K} \simeq \mathbb{H} \) we have \( \gamma : C_{4,0} \to \text{End}_{\mathbb{H}}(I_{4,0}) \), where \( I_{4,0} = C_{4,0}^+(1 + e_1) \), and for matrices \( \mathcal{E}_i = \gamma(e_i) \) we obtain respectively

\[
\mathcal{E}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathcal{E}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]

\[
\mathcal{E}_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \mathcal{E}_4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.
\]
It is easy to verify that in the basis (17) a relation \( \mathcal{A}_{4,0} \frac{1}{2} (1 + \varepsilon_1) \simeq \mathcal{A}_{4,0} \frac{1}{2} (1 + \varepsilon_1) \) holds. Further, in virtue of \( \mathcal{A}_{4,0} \simeq \mathcal{A}_{2,0} \otimes \mathcal{A}_{0,2} \) a general element of the algebra \( \mathcal{A}_{4,0} \) can be represented by a quaternion \( \mathcal{A} = \sum_{i=0}^{3} \varepsilon_i \mathcal{A}^i \mathcal{A}_i \), where \( \varepsilon_1 = e_{123}, \varepsilon_2 = e_{124}, \varepsilon_3 = \varepsilon_1 \varepsilon_2, \varepsilon_4 = \varepsilon_2 \varepsilon_3 = \varepsilon_1^2 = \varepsilon_2^2 = -1 \). Thus, for a restricted spinor field we have

\[
\Psi_{|S^{2,0}}^+ = \frac{1}{2} (1 + i \varepsilon_3) I_{4,0},
\]

\[
\Psi_{|S^{2,0}}^- = \frac{1}{2} (1 - i \varepsilon_3) I_{4,0},
\]

where \( I_{4,0} \simeq \mathcal{A}_{4,0} \frac{1}{2} (1 + \varepsilon_1) \simeq \mathcal{A}_{0,2} \frac{1}{2} (1 + i \varepsilon_1) \), since \( \mathcal{A}_{4,0} \simeq \mathcal{A}_{0,3} \simeq \mathcal{A}_{0,2} \) (theorem 3), \( \mathcal{A}_{0,2} = \mathbb{H} \oplus \mathbb{H} \) is a semi–simple algebra (an algebra of elliptic biquaternions), \( \gamma_i \) are the matrix representations of the units of \( \mathcal{A}_{0,2} \):

\[
\gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]

Let \( \phi \in \mathcal{A}_{0,2} \) be an 'elliptic' Dirac–Hestenes spinor field with a following matrix representation

\[
\phi = \begin{pmatrix} \phi_1 + \phi_3 & -\phi_4^* - \phi_2^* \\ \phi_4 + \phi_2 & \phi_1^* + \phi_3^* \end{pmatrix},
\]

where

\[
\phi_1 = a^0 - i e a^{12}, \quad \phi_2 = e a^{23} + i a^2, \quad \phi_3 = e a^{123} - i a^3, \quad \phi_4 = a^4 + i e a^{13},
\]

\( e \) is a double unit. Then for components of the spinor field \( \Psi \in \mathcal{A}_{0,2} \frac{1}{2} (1 + i \varepsilon_1) \) on the surface \( S^{2,0} \hookrightarrow M^{4,0} \) we obtain

\[
\Psi_{|S^{2,0}}^+ = \frac{1}{2} (1 + i \varepsilon_3) \Psi = \begin{pmatrix} \phi_1 + \phi_3 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
\Psi_{|S^{2,0}}^- = \frac{1}{2} (1 - i \varepsilon_3) \Psi = \begin{pmatrix} 0 & 0 \\ \phi_4 + \phi_2 & 0 \end{pmatrix}.
\] (18)

The immersion \( S^{0,2} \hookrightarrow M^{0,4} \) is analogously defined. In this case a general element of \( \mathcal{A}_{0,4} \) is represented by an anti–quaternion \( \sum_{i=0}^{3} \varepsilon_i \mathcal{A}^i \mathcal{A}_i \), where \( \varepsilon_1 = e_{123}, \varepsilon_2 = e_{124} \). Further, since \( \mathcal{A}_{0,4} \simeq \mathcal{A}_{0,3} \simeq \mathcal{A}_{0,2} \), then \( I_{0,4} \simeq \mathcal{A}_{0,4} \frac{1}{2} (1 + \varepsilon_{123}) \simeq \mathcal{A}_{0,2} \frac{1}{2} (1 + i \varepsilon_{12}) \).

Finally, let us consider spinor fields on the surfaces immersed into the Kleinian manifold \( M^{2,2} \). First of all, for a spinor representation of the algebra \( \mathcal{A}_{2,2} \) we have \( \gamma : \mathcal{A}_{2,2} \to \text{End}_\mathbb{R} (I_{2,2}) \), where \( I_{2,2} = \mathcal{A}_{2,2} \frac{1}{2} (1 + e_{13}) \frac{1}{2} (1 + e_{24}) \), and for a
basis of the spinor space $S \simeq I_{2,2}$ we have also
\[
\begin{align*}
f_1 &= \frac{1}{4}(1 + e_{24} + e_{13} - e_{1234}), \\
f_2 &= \frac{1}{4}(e_1 + e_{124} + e_3 - e_{234}), \\
f_3 &= \frac{1}{4}(e_2 + e_4 - e_{123} + e_{134}), \\
f_4 &= \frac{1}{4}(e_{12} + e_{14} - e_{23} + e_{34}).
\end{align*}
\]

In this basis the matrices $\mathcal{E}_i = \gamma(e_i)$ are
\[
\begin{align*}
\mathcal{E}_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\mathcal{E}_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\
\mathcal{E}_3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\mathcal{E}_4 &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
\end{align*}
\] (19)

It is easy to see that in the basis (19) a relation $\mathcal{O}_{2,2}^\perp \frac{1}{4}(1 + e_{13})\frac{1}{2}(1 + e_{24}) \simeq \mathcal{O}_{2,2}^\perp \frac{1}{4}(1 + e_{23})\frac{1}{2}(1 + e_{24})$ holds. Further, for the immersion $S^{2,0} \hookrightarrow M^{2,2}$ a general element of $\mathcal{O}_{2,2}$ is represented by an anti–quaternion $\sum_{i=0}^3 \alpha_i \zeta_i$, where $\zeta_1 = e_{123}, \zeta_2 = e_{124}$, and the coefficients $\alpha_i$ (anti–quaternions) are generated by a set $\{1, e_1, e_2, e_12\}$. By virtue of an isomorphism $\mathcal{O}_{2,2}^+ \simeq \mathcal{O}_{2,1} \simeq \Omega_{2,0} \simeq \mathcal{O}_{2,0} \oplus \mathcal{O}_{2,0}$ we have for the ideal a following reduction: $I_{2,2} \simeq \mathcal{O}_{2,2}^+ \frac{1}{2}(1 + e_{13})\frac{1}{2}(1 + e_{24}) \simeq \Omega_{2,0} \frac{1}{2}(1 - i\Upsilon_{12})$, where $\Upsilon_i$ are matrix representations of the units of $\Omega_{2,0}$:
\[
\Upsilon_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Upsilon_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]
Thus, in the case of the immersion $S^{2,0} \hookrightarrow M^{2,2}$ we have a restriction of the spinor field $\Psi \in \mathcal{O}_{2,2} \frac{1}{2}(1 + e_{13})\frac{1}{2}(1 + e_{24})$ of the manifold $M^{2,2}$ onto a spinor field $\Psi|_{S^{2,0}} \in \Omega_{2,0} \frac{1}{2}(1 - i\Upsilon_{12})$ of the surface $S^{2,0}$, where $\Psi|_{S^{2,0}} = (\epsilon_{\gamma_{20}}^{+}\Psi, \epsilon_{\gamma_{20}}^{+}\Psi)$, and $\epsilon_{\gamma_{20}}^{+} = \frac{1}{2}(1 \pm i\zeta_1\zeta_2)$ are mutually orthogonal idempotents of the anti–quaternion $\sum_{i=0}^3 \alpha_i \zeta_i$.

Analogously, in the case of the immersion $S^{0,2} \hookrightarrow M^{2,2}$ the general element of $\mathcal{O}_{2,2}$ is represented by a quaternion $\sum_{i=0}^3 \alpha_i \zeta_i$, where $\zeta_1 = e_{134}, \zeta_2 = e_{234}, \zeta_3 = -\zeta_2 = -1$. At this point the quaternions $\alpha_i \zeta_i$ are generated by a set $\{1, e_3, e_4, e_{34}\}$. Further, in the case of the immersion $S^{1,1} \hookrightarrow M^{2,2}$ we have for the general element of $\mathcal{O}_{2,2}$ a following pseudo–quaternion $\sum_{i=3}^3 \alpha_i \zeta_i$, where $\zeta_1 = e_{123}, \zeta_2 = e_{134}, \zeta_3 = -\zeta_2 = 1$, $\mathcal{O}_{1,1} \simeq \{1, e_2, e_4, e_{24}\}$. Therefore, on the surfaces $S^{0,2} \hookrightarrow$
$M^{2,2}$ and $S^{1,1} \hookrightarrow M^{2,2}$ there exist restricted spinor fields $\Psi|_{S^{0,2}} = (e_{02}^+ \Psi, e_{02} \Psi)$ and $\Psi|_{S^{1,1}} = (e_{11}^+ \Psi, e_{11} \Psi)$, where $\Psi \in I_{2,2}$, $e_{02}^+ = \frac{1}{2}(1 + i \zeta_1 \zeta_2)$, $e_{11}^+ = \frac{1}{2}(1 + \zeta_1 \zeta_2)$.

Summarizing obtained above results we come to the following

**Theorem 6.** Let $P = Q \times \text{Spin}(p,q) \Delta_{p,q}$ be a spinor bundle of the 4-dimensional pseudo–riemannian manifold $M^{p,q}$ an let $\Psi \in \Gamma(P)$ be a smooth section (spinor field) of the bundle $P$. At this point the components of $\Psi$ are elements of a minimal left ideal $I_{p,q}$ of the Clifford algebra $\mathcal{C}_{p,q}$ of a tangent bundle of the manifold $M^{p,q}$. Then, a restriction of the spinor bundle $P$ of $M^{p,q}$ onto a spinor bundle $P|_{S^{p,q}}$ of a surface $S^{p,q} (p+q=2)$ immersed into $M^{p,q}$ defined as follows:

1) At the immersions $S^{1,1} \hookrightarrow M^{1,3}$, $S^{0,2} \hookrightarrow M^{1,3}$ a spinor (mother) field $\Psi \in I_{1,3} = \mathcal{C}_{1,3} \mathcal{Y}_1(1 + \gamma_0)$ on the Lorentzian manifold $M^{1,3}$ in virtue of the ideal re-duction $I_{1,3} = \mathcal{C}_{1,3} \mathcal{Y}_1(1 + \gamma_0) \simeq \mathcal{C}_{2,2}(1 + i\sigma_{12})$ induces restricted spinor fields $\Psi|_{S^{1,1}} = (e_{02}^+ \Psi, e_{02} \Psi)$ and $\Psi|_{S^{0,2}} = (e_{11}^+ \Psi, e_{11} \Psi)$ on the surfaces $S^{1,1}$ and $S^{0,2}$, where $e_{02}^+$ and $e_{11}^+$ are mutually orthogonal idempotents respectively of the quaternion $\sum_{i=0}^{3} C_{1,i} \zeta_i$ and pseudo–quaternion $\sum_{i=0}^{3} C_{2,0} \zeta_i$, by means of which (in virtue of the decompositions $C_{1,3} \simeq C_{1,1} \otimes C_{1,2}$ and $C_{1,3} \simeq C_{0,2} \otimes C_{1,1}$) is represented a general element of the spacetime algebra $C_{1,3}$, $\sigma_i$ and $\gamma_i$ are respectively the Pauli and Dirac matrices, $C_2$ is an algebra of hyperbolic biquaternions.

2) At the immersions $S^{1,1} \hookrightarrow M^{3,1}$, $S^{2,0} \hookrightarrow M^{3,1}$ a spinor field $\Psi \in I_{3,1} = \mathcal{C}_{3,1} \mathcal{Y}_1(1 + \epsilon_1)(1 + \epsilon_{24})$ on the Majorana manifold $M^{3,1}$ by virtue of the reduction $I_{3,1} = \mathcal{C}_{3,1} \mathcal{Y}_1(1 + \epsilon_1)(1 + \epsilon_{24}) \simeq \mathcal{C}_{2,2}(1 + i\sigma_{12})$ induces restricted spinor fields $\Psi|_{S^{1,1}} = (e_{02}^+ \Psi, e_{02} \Psi)$ and $\Psi|_{S^{2,0}} = (e_{11}^+ \Psi, e_{11} \Psi)$ on the surfaces $S^{1,1}$ and $S^{2,0}$, where $e_{02}^+$ and $e_{11}^+$ are mutually orthogonal idempotents respectively of the quaternion $\sum_{i=0}^{3} C_{1,i} \zeta_i$ and pseudo–quaternion $\sum_{i=0}^{3} C_{2,0} \zeta_i$, by means of which (in virtue of the decompositions $C_{3,1} \simeq C_{1,1} \otimes C_{2,0}$ and $C_{3,1} \simeq C_{0,2} \otimes C_{1,1}$) is represented a general element of the Majorana algebra $C_{3,1}$.

3) At the immersions $S^{2,0} \hookrightarrow M^{4,0}$, $S^{0,2} \hookrightarrow M^{0,4}$ spinor fields $\Psi \in I_{4,0} = \mathcal{C}_{4,0} \mathcal{Y}_1(1 + \epsilon_1)$, $\Psi \in I_{0,4} = \mathcal{C}_{0,4} \mathcal{Y}_1(1 + \epsilon_{123})$ of the manifolds $M^{4,0}, M^{0,4}$ by virtue of the reductions $I_{4,0} \simeq \mathcal{C}_{4,0} \mathcal{Y}_1(1 + \epsilon_1) \simeq \mathcal{C}_{2,2}(1 + i\epsilon_{123}) \simeq \mathcal{C}_{2,2}(1 + i\epsilon_{123})$ induce restricted spinor fields $\Psi|_{S^{2,0}} = (e_{02}^+ \Psi, e_{02} \Psi)$, $\Psi|_{S^{0,2}} = (e_{11}^+ \Psi, e_{11} \Psi)$ on the surfaces $S^{2,0}$, $S^{0,2}$ respectively, where $e_{02}^+$ and $e_{11}^+$ are mutually orthogonal idempotents of the quaternion $\sum_{i=0}^{3} C_{2,0} \zeta_i$ and anti–quaternion $\sum_{i=0}^{3} C_{2,0} \zeta_i$, by means of which (in virtue of the decompositions $C_{4,0} \simeq C_{2,0} \otimes C_{0,2}$ and $C_{0,4} \simeq C_{0,2} \otimes C_{2,0}$) represented general elements of the algebras $C_{4,0}$ and $C_{0,4}$. $\gamma_i$ are matrix representations of the units of an algebra of elliptic biquaternions $\Omega_{0,2}$.

4) At the immersions $S^{2,0} \hookrightarrow M^{2,2}$, $S^{0,2} \hookrightarrow M^{2,2}$, $S^{1,1} \hookrightarrow M^{2,2}$ a spinor field $\Psi \in I_{2,2} = \mathcal{C}_{2,2} \mathcal{Y}_1(1 + \epsilon_{13})(1 + \epsilon_{24})$ on the Kleinian manifold $M^{2,2}$ by virtue of the reduction $I_{2,2} \simeq \mathcal{C}_{2,2} \mathcal{Y}_1(1 + \epsilon_{13})(1 + \epsilon_{24}) \simeq \mathcal{C}_{2,2}(1 + i\epsilon_{123})$ induces restricted spinor fields $\Psi|_{S^{2,0}} = (e_{20}^+ \Psi, e_{20} \Psi)$, $\Psi|_{S^{0,2}} = (e_{02}^+ \Psi, e_{02} \Psi)$ and $\Psi|_{S^{1,1}} = (e_{11}^+ \Psi, e_{11} \Psi)$ on the surfaces $S^{0,2}$, $S^{2,0}$ and $S^{1,1}$, where $e_{20}^+$ and $e_{11}^+$ are mutually orthogonal idempotents respectively of the anti–quaternion $\sum_{i=0}^{3} C_{2,0} \zeta_i$, quaternion $\sum_{i=0}^{3} C_{2,0} \zeta_i$. 

17
and pseudo-quaternion \( \sum_{i=0}^{3} C_{i1}^{1} \zeta_i \), by means of which (in virtue of the decompositions \( C_{2,2} \simeq C_{2,0} \otimes C_{2,0} \), \( C_{2,2} \simeq C_{0,2} \otimes C_{0,2} \) and \( C_{2,2} \simeq C_{1,1} \otimes C_{1,1} \)) is represented a general element of the algebra \( C_{2,2} \).

**Remark 1.** In the case of the Lorentzian manifold the spinor fields \( \psi = \Psi_{SR,i} \) on the surfaces \( S_{1,1} \hookrightarrow M_{1,3} \) and \( s_{0,2} \hookrightarrow M_{1,3} \) in accordance with \([\text{Cra}85, \text{Lou}93]\) may be expressed via bilinear covariants \( \sigma, J, S, K, \omega \), i.e. \( \psi \simeq Z \eta \), where \( Z = \sigma + J + iS - i\gamma_{0123}K + \gamma_{0123} \omega, \eta \) is an arbitrary complex number, and

\[
\sigma = \psi^\dagger \gamma_0 \psi = 4 < \widetilde{\psi} \psi >_0,
J_\mu = \psi^\dagger \gamma_\mu \psi = 4 < \widetilde{\psi} \gamma_\mu \psi >_0,
S_{\mu\nu} = \psi^\dagger \gamma_0 i\gamma_{\mu\nu} \psi = 4 < \widetilde{\psi} i\gamma_{\mu\nu} \psi >_0,
K_\mu = \psi^\dagger \gamma_0 \gamma_{0123} \gamma_\mu \psi = 4 < \widetilde{\psi} i\gamma_{0123} \gamma_\mu \psi >_0,
\omega = -\psi^\dagger \gamma_0 \gamma_{0123} \psi = -4 < \widetilde{\psi} \gamma_{0123} \psi >_0.
\]

At this point the bilinear covariants satisfy to Fierz identities

\[
J^2 = \sigma^2 + \omega^2, \quad K^2 = -J^2,
J \cdot K = 0, \quad J_\lambda K = -(\omega + \gamma_{0123} \sigma) S.
\]

The spinor field \( \psi \), whose \( \sigma, J, S, K, \omega \) satisfy to Fierz identities, recovered by its bilinear covariants with an accuracy of the complex factor \( \eta \). Moreover, both in the non-null (\( \sigma, \omega \neq 0 \)) and null case (\( \sigma, \omega = 0 \)) the spinor \( \psi \) is defined by bilinear covariant \( Z \) (\( \psi = (1/4N)e^{-i\alpha} Z \eta \)), where \( N = \sqrt{< Z \eta >_0} \) which in its turn is defined by the spinor \( \psi \) as follows: \( Z = 4\psi \widetilde{\psi}^* = 4\psi \psi^* \gamma_0 \). Thus, we have a so-called boomerang \([\text{Lou}93]\). All bilinear covariants are real and have important meaning in the Dirac theory of electron. In perspective, it is of interest to consider the analogous bilinear covariants and boomerangs for the spinor fields on the surfaces immersed into 4d manifolds with signatures different from the signature of the Lorentzian manifold.

**Remark 2.** In more general case of non-orientable manifolds we come to a group \( \text{Pin}(p, q) \) which is a double covering of the structure group \( O(p, q) \) of the manifold \( M^{p,q} \). In accordance with \([\text{Dab}88, \text{BD}89]\) there exist eight double coverings of the orthogonal group \( O(p, q) \): \n
\[
\rho^{a,b,c} : \text{Pin}^{a,b,c}(p, q) \simeq \frac{(\text{Spin}_0(p, q) \otimes C^{a,b,c})}{\mathbb{Z}_2} \rightarrow O(p, q),
\]

where \( C^{a,b,c} \in \{ \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2, \mathbb{Z}_2 \otimes \mathbb{Z}_4, \mathbb{Q}_4, D_4 \} \) is a double covering of a discrete group of the space \( \mathbb{R}^{p,q} \), \( a, b, c \in \{ +, - \} \). In connection with this it is of interest to define \( \text{Pin}^{a,b,c} \)-fields (generalization of ordinary spinor fields) on surfaces immersed into the non-orientable manifolds. The classification of these fields may be easily defined with the usage of recently established relation between signatures of the spaces \( \mathbb{R}^{p,q} \) and finite groups of fundamental automorphisms of the Clifford algebras (see \([\text{Var}99c]\) theorem 10)).
5 The Dirac Operator on the Surfaces Immersed into 4D Manifolds

Let us consider now the Dirac operator on the surfaces $S^{r,s} \hookrightarrow M^{p,q}$ ($r + s = 2$, $p + q = 4$). First of all, let recall some basic facts about a theory of the Dirac operator on a spin manifold [BFGK]. Let $(M^{p,q}, g)$ be a pseudo–riemannian spin manifold and let $S = Q \times_{\text{Spin}(p,q)} \Delta_{p,q}$ be a spinor bundle of the manifold $(M^{p,q}, g)$. The Dirac operator on the manifold $(M^{p,q}, g)$ is a first order self–adjoint elliptic differential operator defined by an expression

$$D : \Gamma(S) \xrightarrow{\nabla_S} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S).$$

where $\mu$ is a so–called Clifford multiplication:

$$\mu : \mathbb{R}^n \otimes \Delta_n \rightarrow \Delta_n$$

$$x \otimes u \mapsto \mu(x \otimes u) = x \cdot u =$$

$$\begin{cases} 
\gamma_{p,q}(x)u, & \text{if } p - q = 0, 2, 4, 6 \pmod{8}; \\
\text{proj}_j \cdot \gamma_{p,q}(x)u, & \text{if } p - q = 1, 3, 5, 7 \pmod{8}.
\end{cases}$$

where $\gamma : \mathcal{O}_{p,q} \rightarrow \text{End}_\mathbb{C}(I_{p,q})$ is a spinor representation of the Clifford algebra $\mathcal{O}_{p,q}$. For the spinor bundles the Clifford multiplication is defined as follows

$$\mu : TM \otimes S \rightarrow S$$

$$x \otimes \varphi \mapsto x \cdot \varphi$$

In the case of even dimensions ($p - q \equiv (\text{mod } 0, 2, 4, 6)$) the Clifford multiplication $\mu$ exchanges the positive and negative parts of the bundle $S$. The Clifford multiplication may be also defined for the $k$–forms. The action of the $k$–form $\omega \in \Omega^k(M)$ on the spinor bundle is defined by a following local formula

$$\omega \cdot \varphi = \sum_{1 \leq i_1 < ... < i_k \leq n} \omega(s_{i_1}, \ldots, s_{i_k})s_{i_1} \cdot \ldots \cdot s_{i_k} \cdot \varphi,$$

where $(s_1, \ldots, s_n)$ is a local orthonormal basis of the manifold $(M^n, g)$, $\varphi \in \Gamma(S)$, $n = p + q$.

Further, $\nabla^S : \Gamma(S) \rightarrow \Gamma(TM \otimes S)$ in (20) is a spinor derivative, which locally is given by an expression

$$\nabla^S_X \varphi = X(\varphi) + \frac{1}{2} \sum_{1 \leq k < l \leq n} \omega_{kl}(x)s_k \cdot s_l \cdot \varphi,$$

where $\omega_{kl} = g(\nabla^M s_k, s_l)$ are the connection forms of the Levi–Civita connection $\nabla^M$ on $(M^{p,q}, g)$ with respect to a local basis $(s_1, \ldots, s_n)$, $X$ is a vector field.
It immediately follows that locally the Dirac operator may be written in the form

\[ D = \sum_{k=1}^{n} s_k \cdot \nabla^S_s. \]  

(23)

Since the all 4-dimensional manifolds are quaternionic manifolds, then in the each point of such a manifold the Clifford algebra of the tangent space is isomorphic to a quaternionic algebra, i.e. a Clifford bundle of the manifold in this case may be represented in terms of the quaternionic algebras. Indeed, in the case of even dimension, the volume element \( \omega = e_{12...n} \) is not belong to a center of the algebra \( \mathcal{A}_n \). However, when \( i \leq 2m \) we have

\[ e_{12...2m2m+k}e_i = (-1)^{2m+1-i}\sigma(i-l)e_{12...i-1i+1...2m2m+k}, \]

\[ e_i e_{12...2m2m+k} = (-1)^{i-1}\sigma(i-l)e_{12...i-1i+1...2m2m+k}, \]

where \( \sigma(n) \) are the functions of the form \( \mathbb{Z} \). Therefore, the commutativity condition of the elements \( e_{12...2m2m+k} \) and \( e_i \) is \( 2m + 1 - i \equiv i - 1 \) (mod 2). Thus, the elements \( e_{12...2m2m+1} \) and \( e_{12...2m2m+2} \) commute with all basis elements \( e_i \) whose indexes are not exceed 2\( m \). Therefore, a transition from \( \mathcal{A}_{2m} \) to \( \mathcal{A}_{2m+2} \) may be represented as transition from the real coordinates in the algebra \( \mathcal{A}_{2m} \) to quaternionic coordinates of the form \( a + b\zeta_1 + c\zeta_2 + d\zeta_1\zeta_2 \), where \( \zeta_1 \) and \( \zeta_2 \) are additional basis elements \( e_{12...2m2m+1} \) and \( e_{12...2m2m+2} \). The elements \( e_{i_1i_2...i_k}\zeta_1 \) are contain index 2\( m+1 \) and not contain index 2\( m+2 \), and the elements \( e_{i_1i_2...i_k}\zeta_2 \) are contain index 2\( m+2 \) and not contain index 2\( m+1 \). Respectively, the elements \( e_{i_1i_2...i_k}\zeta_1\zeta_2 \) are contain both indexes 2\( m+1 \) and 2\( m+2 \). Therefore, the algebras \( \mathcal{A}_{p,q+2}, \mathcal{A}_{p+2,q} \) and \( \mathcal{A}_{p+1,q+1} \) \( (p-q \equiv 0, 2, 4, 6 \mod 8) \) are isomorphic respectively to quaternionic, anti-quaternionic and pseudo-quaternionic algebras, i.e. a general element of these algebras can be represented in the form

\[ \mathcal{A}_{p,q}^0 + \mathcal{A}_{p,q}^1\zeta_1 + \mathcal{A}_{p,q}^2\zeta_2 + \mathcal{A}_{p,q}^3\zeta_1\zeta_2, \]  

(24)

where \( \zeta_1 = e_{12...2m2m+1}, \zeta_2 = e_{12...2m2m+2} \). Respectively, in dependence on the squares of the units \( \zeta_1, \zeta_2 \) the expression \( \mathcal{A}_{p,q} \) is the quaternion \( (\zeta_1^2 = \zeta_2^2 = -1) \), anti-quaternion \( (\zeta_1^2 = \zeta_2^2 = 1) \) and pseudo-quaternion \( (\zeta_1^2 = -\zeta_2^2 = 1) \). In other words, according to the theorem \( \mathbb{Z} \) we have for the quaternionic algebras the following decompositions:

\[ \mathcal{A}_{p,q+2} \cong \mathcal{A}_{0,2} \otimes \mathcal{A}_{q,p}, \]

\[ \mathcal{A}_{p+2,q} \cong \mathcal{A}_{2,0} \otimes \mathcal{A}_{q,p}, \]  

(25)

\[ \mathcal{A}_{p+1,q+1} \cong \mathcal{A}_{1,1} \otimes \mathcal{A}_{p,q}. \]

These decompositions are natural generalizations of the decompositions considered above in the section 3.
Let $\text{Spin}(2m) \subset \mathcal{O}_{2m}^*$ be a spinor group and let $\Delta_{2m} = \Delta_{2m}^+ + \Delta_{2m}^-$ be a representation of the group $\text{Spin}(2m)$, $\mathcal{O}_{2m}^*$ is a set of all invertible elements of the algebra $\mathcal{O}_{2m}$.

**Lemma 1.** The restriction of $\Delta_{2m+2}$ to $\text{Spin}(2m)$ is isomorphic to the $\text{Spin}(2m)$-representation $\Delta_{2m}$, where an action of $\zeta_1\zeta_2$ on $\Delta_{2m} = \Delta_{2m}^+ + \Delta_{2m}^-$ is defined by an expression

$$\zeta_1\zeta_2 \cdot (u^+ \oplus u^-) = (-1)^m \varepsilon u^+ - (-1)^m \varepsilon u^-.$$

Here $\varepsilon = i$ if $\zeta_1^2 = \zeta_2^2 = \pm 1$ and $\varepsilon = 1$ if $\zeta_1^2 = -\zeta_2^2 = 1$.

**Proof.** The spinor group $\text{Spin}(2m+2)$ is completely defined in terms of the algebra $\mathcal{O}_{2m+2}^*$:

$$\text{Spin}(2m+2) = \{ s \in \mathcal{O}_{2m+2}^* \mid N(s) = \pm 1 \},$$

where $s \in \mathcal{O}_{2m+2}^*$, $N : \mathcal{O}_{2m+2} \to \mathcal{O}_{2m+2}$, $N(x) = xx\bar{x}$; $\Gamma_{2m+2}^+ = \Gamma_{2m+2} \cap \mathcal{O}_{2m+2}^+$ is a special Clifford–Lipschitz group, and

$$\Gamma_{2m+2} = \{ s \in \mathcal{O}_{2m+2}^* \forall x \in \mathbb{R}^{2m+2}, sx\bar{s}^{-1} \in \mathbb{R}^{2m+2} \}.$$

Let $\rho^{2m+2} : \mathcal{O}_{2m+2} \to \text{End} E$ be a representation of the algebra $\mathcal{O}_{2m+2}$ in a vector space $E$. The representation $\rho^{2m+2}$ induces via (27) a representation of the group $\text{Pin}(2m+2) = \{ s \in \mathcal{O}_{2m+2}^* \mid N(s) = \pm 1 \}$, and also via (29) a representation $\Delta_{2m+2}$ of the group $\text{Spin}(2m+2)$. Further, in virtue of the decomposition (24) an inverse transition $\mathcal{O}_{2m+2} \to \mathcal{O}_{2m}$ induces a transition $\text{Spin}(2m+2) \to \text{Spin}(2m)$, and $\rho^{2m+2} \to \rho^{2m}$ induces a restriction $\Delta_{2m+2} \to \Delta_{2m}$ by means of mutually orthogonal idempotents (projection operators) $\varepsilon^{\pm} = \frac{1}{2}(1 \pm \varepsilon \zeta_1 \zeta_2)$. At this point $\zeta_1 \zeta_2 = e_{2m+12m+2} \mapsto e_{2m+12m+2}$ commutes with $u^+ \in \Delta_{2m}^+$ and anticommutes with $u^- \in \Delta_{2m}^-$. \qed

Further, let $M$ be an $(2m+2)$-dimensional pseudo–riemannian manifold and let $F$ be an $2m$-dimensional submanifold immersed into $M$, $F \hookrightarrow M$. We suppose that the both manifolds endowed with some spinor structure. Let $N$ be a normal bundle of the manifold $F \hookrightarrow M$, then in accordance with [Mil65] a sum of the spinor structures on the tangent bundle and on the normal bundle of the manifold $F$ coincides with the spinor structure on the tangent bundle of the manifold $F$ restricted to $F$. Let $\nabla^F$ and $\nabla^M$ be Levi–Civita connections on the manifolds $F$ and $M$, respectively. Let $\nabla^N$ be a normal connection on the bundle $N$. Denote the second fundamental form of the submanifold $F^{2m} \hookrightarrow M^{2m+2}$ as $II$. Further, let $\zeta_1$ and $\zeta_2$ be unit normal vector fields on $F^{2m} \hookrightarrow M^{2m+2}$ and let $S_F = Q_F \times_{\text{Spin}(2m)} \Delta_{2m}$ be a spinor bundle of the submanifold $F^{2m}$. 21
Lemma 2. If \( n = 2m + 2 \), then a restriction of the spinor bundle \( S \) of the manifold \((M^n, g)\) onto submanifold \( F^{n-2} \) is isomorphic to the bundle \( S_F \), where \( \zeta_1 \zeta_2 \) acts on \( S_F \) as follows

\[
\zeta_1 \zeta_2 \cdot (\psi^+ \oplus \psi^-) = (-1)^m \varepsilon \psi^+ - (-1)^m \varepsilon \psi^-,
\]

(28)

and the spinor derivative of \( \psi \in \Gamma(S) \) equals

\[
\nabla^S_X \psi = (\nabla^S_X \otimes \text{Id} + \text{Id} \otimes \nabla^S_X) \psi + \frac{1}{2} \sum_{1 \leq i_1 < \ldots < i_k \leq 2m} \langle II(X, X_i), X_i \rangle \cdot \zeta_1 \zeta_2 \cdot \psi
\]

(29)

for all \( X \in T_x F \).

Proof. The expression (28) immediately follows from the lemma 1. Further, following to [Bar98] we see that for some point \( x \in F \) and a vector field \( X \in T_x F \) the Gauss formula with respect to a decomposition \( T_x M = T_x F \oplus N_x \) gives

\[
\nabla^M_X = \begin{pmatrix}
\nabla^F_X & -II(X, \cdot) \\
II(X, \cdot) & \nabla^N_X
\end{pmatrix}.
\]

(30)

Let \( X_1, \ldots, X_{2m} \) be a local orthonormal tangent frame of the submanifold \( F^{2m} \) at the point \( x \) and let \( Y_1, Y_2 \) be a local orthonormal frame on the normal bundle \( N \) at \( x \). Then \( h := (X_1, \ldots, X_{2m}, Y_1, Y_2) \) is a local section of the tangent bundle \( P \times SO(2m+2) \) of the manifold \( M^{2m+2} \) restricted to \( F^{2m} \). Now we can to write (30) in the matrix form:

\[
\nabla^M_X - (\nabla^F_X \oplus \nabla^N_X) = \begin{pmatrix}
0 & -<II(X, \cdot), Y_j>_{i,j} \\
<II(X, \cdot), Y_j>_{i,j} & 0
\end{pmatrix}.
\]

(31)

Further, let \( \omega^F, \omega^N \) and \( \omega^M \) be respectively connection 1–forms for \( \nabla^F, \nabla^N \) and \( \nabla^M \) lifted to \( \text{Spin}(2m), \text{Spin}(2) \) and \( \text{Spin}(2m+2) \). If \( \Theta : \text{Spin}(2m+2) \to SO(2m+2) \) is an usual double covering, then (31) can be written as follows

\[
\Theta_* (\omega^M (dh \cdot X) - (\omega^F \oplus \omega^N) (dh \cdot X)) = \begin{pmatrix}
0 & -<II(X, X_i), Y_j>_{i,j} \\
<II(X, X_i), Y_j>_{i,j} & 0
\end{pmatrix}.
\]

(32)

Using the standard formula [LM89, c.42] for \( \Theta_* \) we obtain from (32)

\[
\omega^M (dh \cdot X) - (\omega^F \oplus \omega^N) (dh \cdot X) = \frac{1}{2} \sum_{i=1}^{2m} \sum_{j=1}^{2} <II(X, X_i), Y_j> e_i \cdot f_j.
\]

(33)
where \( e_1, \ldots, e_{2m} \) is a standard basis of the space \( \mathbb{R}^{2m} \), \( f_1, f_2 \) is a standard basis of the space \( \mathbb{R}^2 \).

Let \( S_M = Q \times \text{Spin}(2m+2) \Delta_{2m+2} \) be a spinor bundle of the manifold \( M^{2m+2} \).

In virtue of the decomposition \( \mathcal{O}_{2m+2} \simeq \mathcal{O}_2 \otimes \mathcal{O}_{2m} \) we have \( S_M|_F = S_F \otimes S_N \), where \( S_F = Q_F \times \text{Spin}(2m) \Delta_{2m} \), \( S_N = Q_N \times \text{Spin}(2) \Delta_2 \). Let \( \nabla^{S_M}, \nabla^{S_F} \) and \( \nabla^{S_N} \) be Levi–Civita connections on the bundles \( S_M, S_F \) and \( S_N \), respectively. Then

\[
\nabla^{S_F \otimes S_N} := \nabla^{S_F} \otimes \text{Id} + \text{Id} \otimes \nabla^{S_N}
\]

is a Levi–Civita product connection on \( S_F \otimes S_N \). At this point the equation (33) takes a form

\[
\nabla_X - \left( \nabla_X^{S_F} \otimes \text{Id} + \text{Id} \otimes \nabla_X^{S_N} \right) = \frac{1}{2} \sum_{i,j=1}^{2m} <II(X_j, X_i), Y_j > \mu(X_i \cdot Y_j),
\]

where \( \mu(X_i \cdot Y_j) \) is the Clifford multiplication defined by (21). Whence in accordance with the definition of the spinor derivative (22) and identifications \( \zeta_1 \leftrightarrow Y_1, \zeta_2 \leftrightarrow Y_2 \) follows the formula (23).

Before defining the Dirac operator (by the formula (23)), corresponding to the spinor derivative (34) it is necessary to consider the following two operators

\[
\tilde{D} = \sum_{j=1}^{2m} X_j \cdot \nabla^{S_F \otimes S_N}_{X_j}
\]

and

\[
\hat{D} = \sum_{j=1}^{2m} X_j \cdot \nabla^{S}_{X_j}.
\]

It is easy to see that both operators act on the sections of the bundle \( S_M \). Let \( H = \frac{1}{2m} \sum_{j=1}^{2m} II(X_j, X_j) \) be the mean curvature vector field of the submanifold \( F^{2m} \hookrightarrow M^{2m+2} \). Further, using (34) we obtain

\[
\hat{D} - \tilde{D} = \frac{1}{2} \sum_{i,j=1}^{2m} X_j \cdot <II(X_j, X_i), X_i > \cdot \zeta_1 \zeta_2
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{2m} \mu(X_j \cdot X_i) <II(X_j, X_i) > \cdot \zeta_1 \zeta_2.
\]

By virtue of antisymmetry, the products \( X_j \cdot X_i \) with \( i \neq j \) vanish, at this point the form \( II(X_j, X_i) \) is symmetric. Since \( X \cdot Y + Y \cdot X = 2g(X,Y) \text{Id}_{S_M} \), then

\[
\hat{D} - \tilde{D} = \frac{1}{2} \sum_{i=1}^{2m} g_{ii} H \cdot \zeta_1 \zeta_2.
\]
Let $\Psi \in I_{p,q}$ be a real Killing spinor field on the 4-dimensional pseudo–riemannian manifold $M^{p,q}$ and let $\Psi|_{S^{r,s}} = \psi = \psi^+ \oplus \psi^-$ be a restricted spinor field on the surface $S^{r,s}$ immersed into the manifold $M^{p,q}$. Then a Dirac operator of the surface $S^{r,s} \hookrightarrow M^{p,q}$ defined as follows

$$D(\psi^+) = (\alpha - \frac{1}{2} \varepsilon \beta H)\psi^-, \
D(\psi^-) = (\alpha + \frac{1}{2} \varepsilon \beta H)\psi^+,$$

where $\varepsilon = 1$ for the immersions $S^{0,2} \hookrightarrow M^{1,3}$, $S^{2,0} \hookrightarrow M^{3,1}$, $S^{1,1} \hookrightarrow M^{2,2}$, and $\varepsilon = i$ for the immersions $S^{2,0} \hookrightarrow M^{4,0}$, $S^{0,2} \hookrightarrow M^{0,4}$, $S^{1,1} \hookrightarrow M^{1,3}$, $S^{1,1} \hookrightarrow M^{3,1}$, $S^{2,0} \hookrightarrow M^{2,2}$, $S^{0,2} \hookrightarrow M^{2,2}$. $H$ is a mean curvature of the surface, $\alpha = \lambda_1 g_{11} + \lambda_2 g_{22}$, $\beta = g_{11} + g_{22}$.

Proof. In the case of the Lorentzian manifold $M^{1,3}$ we have the following immersions $S^{0,2} \hookrightarrow M^{1,3}$ and $S^{1,1} \hookrightarrow M^{1,3}$. At this point in accordance with the theorem 3 on the surfaces $S^{0,2}$ and $S^{1,1}$ there exist the spinor fields $\Psi|_{S^{0,2}} = (\psi^+, \psi^-) = (\epsilon_{11}^+ \psi^1, \epsilon_{11}^- \psi^-)$ and $\Psi|_{S^{1,1}} = (\psi^+, \psi^-) = (\epsilon_{02}^+ \psi^1, \epsilon_{02}^- \psi^2)$, respectively. Let us find a Dirac operator on the surface $S^{0,2} \hookrightarrow M^{1,3}$. First of all, in accordance with the definition (23) and the formulae (35), (36) let consider the following two operators:

$$\tilde{D} = X_1 \cdot \nabla^S_{X_1} \otimes S_N + X_2 \cdot \nabla^S_{X_2} \otimes S_N$$

and

$$D = X_1 \cdot \nabla^{S_{1,1}}_{X_1} + X_2 \cdot \nabla^{S_{1,1}}_{X_2}.$$ 

The both operators act on the section of the spinor bundle $S = Q \times_{\text{Spin}(1,3)} \Delta_{1,3}$. Using (29) and (37) we obtain

$$\tilde{D} - D = \frac{1}{2} \sum_{i,j=1}^2 X_j \cdot X_i \cdot II(X_j, X_i) \cdot \zeta_1 \zeta_2 = \frac{1}{2} \beta H \cdot \zeta_1 \zeta_2,$$

where $\beta = g_{11} + g_{22}$. Further, let $\Psi \in I_{1,3}$ be the spinor field on the Lorentzian manifold $M^{1,3}$, then from (18) follows

$$X_1 \cdot \nabla^S_{X_1} \otimes S_N (\Psi) + X_2 \cdot \nabla^S_{X_2} \otimes S_N (\Psi) = D(\psi) - \frac{1}{2} \beta H \cdot \zeta_1 \zeta_2 \cdot \psi,$$

where $D(\psi)$ is a Dirac operator of the surface $S^{0,2} \hookrightarrow M^{1,3}$ defined on the restriction $\psi = \Psi|_{S^{0,2}}$. We suppose now that the spinor field $\Psi$ on the manifold $M^{1,3}$ is a real Killing spinor, i.e. there exists such a number $\lambda \in \mathbb{R}$ that for any vector field $X \in T_x M^{1,3}$ the derivative of $\Psi$ in the direction $X$ equals

$$\nabla^S_{X} \otimes S_N (\Psi) = \lambda \cdot X \cdot \Psi.$$
Therefore, from (39) for the restriction \( \psi = \Psi|_{S^0.2} \) we have

\[
D(\psi) = \alpha \psi + \frac{1}{2} \beta H \cdot \zeta_1 \zeta_2 \cdot \psi.
\]

where \( \alpha = \lambda_1 g_{11} + \lambda_2 g_{22} \). Since \( \psi = \psi^+ \oplus \psi^- \), then in virtue of the relation (28) of the lemma \( \mathcal{B} \) from the last equation we obtain (recalling that \( \zeta_1 \) and \( \zeta_2 \) are the units of the pseudo–quaternion)

\[
D(\psi^+) = (\alpha - \frac{1}{2} \beta H) \psi^-,
\]

\[
D(\psi^-) = (\alpha + \frac{1}{2} \beta H) \psi^+.
\]  

(40)

Further, for the immersion of the time–like surface \( S^{1.1} \hookrightarrow M^{1.3} \) the analogous calculations give (at this point \( \zeta_1 \) and \( \zeta_2 \) are the quaternion units)

\[
D(\psi^+) = (\alpha - i \frac{1}{2} \beta H) \psi^-,
\]

\[
D(\psi^-) = (\alpha + i \frac{1}{2} \beta H) \psi^+.
\]

The theorem \( \mathcal{B} \) has three important particular cases

**Corollary 1.** Let \( S^{r,s} \hookrightarrow M^{p,q} \) be a minimal surface, then a Dirac operator of \( S^{r,s} \) has a form

\[
D(\psi) = \alpha \psi,
\]

where \( \psi = \psi^+ \oplus \psi^- = \Psi|_{S^{r,s}} \) is an eigenspinor on the surface \( S^{r,s} \).

**Corollary 2.** Let \( \Psi \) be a parallel spinor field \((\lambda_1 = \lambda_2 = 0)\) on the manifold \( M^{p,q} \) and let \( \psi = \Psi|_{S^{r,s}} \) be its restriction on the surface \( S^{r,s} \hookrightarrow M^{p,q} \), then a Dirac operator of \( S^{r,s} \) takes a form

\[
D(\psi^+) = -\frac{1}{2} \varepsilon \beta H \psi^-,
\]

\[
D(\psi^-) = \frac{1}{2} \varepsilon \beta H \psi^+.
\]

On the other hand, when \( \varepsilon = i \) and \( \beta = 2, \lambda_1 = \lambda_2 = 0 \) (parallel spinor field) a Dirac operator on the surface may be written in more compact form. Let consider a following spinor field

\[
\psi^p = \psi^+ - i \psi^- = \frac{1}{2}(\psi + i \cdot \zeta_1 \zeta_2 \cdot \psi) - \frac{i}{2}(\psi - i \cdot \zeta_1 \zeta_2 \cdot \psi) = \frac{1}{2}(1 - i) \psi + \frac{1}{2}(-1 + i) \cdot \zeta_1 \zeta_2 \cdot \psi,
\]

25
where $\psi \in \Omega_{2,0,\frac{1}{2}}(1-i\Upsilon_{12})$ for $S^{2,0} \hookrightarrow M^{2,2}$ and $\psi \in \Omega_{0,2,\frac{1}{2}}(1+i\Upsilon_{12})$ for $S^{2,0} \hookrightarrow M^{4,0}$. Then

$$D(\psi^o) = H\psi^o.$$  

Analogously, when $\varepsilon = i$ and $\beta = -2$, $\lambda_1 = \lambda_2 = 0$ we have for the immersions $S^{0,2} \hookrightarrow M^{2,2}$, $S^{0,2} \hookrightarrow M^{0,4}$ a following spinor field

$$\psi^* = \psi^+ + i\psi^- = \frac{1}{2}(\psi + i \cdot \zeta_1 \zeta_2 \cdot \psi) + \frac{i}{2}(\psi - i \cdot \zeta_1 \zeta_2 \cdot \psi) = \frac{1}{2}(1 + i)(\psi + \zeta_1 \zeta_2 \cdot \psi)$$

and

$$D(\psi^*) = H\psi^*.$$  

**Corollary 3.** If $\lambda_1 = \lambda_2$, then in the case of the immersions of time–like surfaces $S^{1,1} \hookrightarrow M^{p,q}$ a Dirac operator of $S^{1,1}$ is homogeneous,

$$D(\psi) = 0.$$  

**Example.** Let $S^{1,1} \hookrightarrow M^{1,3}$ be an immersion of the time–like surface into the Lorentzian manifold and let $\lambda_1 = \lambda_2$, then $\alpha = \beta = 0$ (corollary [3]) and

$$D(\psi) = 0.$$  

At this point it is easy to trace a relation with a so–called **optical geometry** [Rob61, Pen83, RT80, Num96]. Indeed, let $\phi \in \mathcal{O}^{+}_{1,3} \simeq \mathcal{O}^{3,0} \simeq \mathbb{C}_2$ be a Dirac–Hestenes spinor field an let $\Phi = E + iB = \partial \Lambda A \in \mathcal{O}^{3,0} \simeq \mathbb{C}_2$ be an electromagnetic (in general case, non–null) field, where $\partial = \partial^0 + \partial^1 e_1 + \partial^2 e_2 + \partial^3 e_3$ and $A = A^0 + A^1 e_1 + A^2 e_2 + A^3 e_3$ are partial derivative and vector–potential, respectively. By virtue of $\sigma : \mathcal{O}^{3,0} \rightarrow \text{End}_\mathbb{C}(I_{3,0})$ the element $\Phi = F_1 e_1 + F_2 e_2 + F_3 e_{12} \in \mathbb{C}_2$ in the spinor representation is defined by a following symmetric matrix

$$\Phi = \begin{pmatrix} F_1 + iF_2 & iF_3 \\ iF_3 & F_1 - iF_2 \end{pmatrix}.$$  

The determinant $\det \Phi = F_1^2 + F_2^2 + F_3^2$ vanishes if, and only if, the electromagnetic field is null, i.e. when $E \cdot B = 0$ and $E^2 = B^2$. Null electromagnetic fields play a key role in the theory of shear free congruences of null geodesics in the Lorentzian manifold and give rise to the optical geometry and a Cauchy–Riemann structure on the space of null geodesics. Expressing the Dirac–Hestenes spinor field $\phi \in \mathcal{O}^+_1$ via the null electromagnetic field, $\phi_1 = \alpha + iB_3$, $\phi_2 = -B_2 + iB_1$, $\phi_3 = E_3 + i\lambda$, $\phi_4 = E_1 + iE_2$ (see [Par92]), we find that in the case of the immersion of the time–like surface (or, light cone) $S^{1,1}$ into the Lorentzian manifold $M^{1,3}$ a restriction of the spinor field $\Psi \in M^{1,3}$ onto a spinor field $\psi = \Psi|_{S^{1,1}}$ of the surface $S^{1,1} \hookrightarrow M^{1,3}$ in accordance with (11) is expressed via the null electromagnetic field, and a Dirac operator of $S^{1,1}$ in accordance with corollary [3] is homogeneous.

Such a form of the Dirac operator corresponds to massless physical fields, which describe, as known, such particles as photon and neutrino.
Let us consider now local (spinor) representations of surfaces conformally immersed into 4–dimensional pseudo–euclidean spaces. As known, these representations are defined by so–called Gauss map (GGM) [HO80, HO83, HO85] and generalized Weierstrass representation (GWR) [KL98a]. The our main goal in this section is a realization of GGM and GWR in terms of the spinor fields introduced above in the section 4.

Let $S^{r,s} \hookrightarrow M^{p,q}$ \((r + s = 2, p + q = 4)\) be a surface endowed with some spinor structure and let $S_0$ be a connected Riemann surface with a local complex coordinate $z$. Let $P \times G$ be a principal bundle of $S_0$ with the structure group $G$ (\(G\) is a group of fractional linear transformations). Then the spinor representation of a surface $S^{r,s}$ in $M^{p,q}$ (or, locally, in $\mathbb{R}^{p,q}$) is given by the following diagramm

\[
\begin{array}{ccc}
Q \times G & \xrightarrow{\chi} & Q \times \text{Spin}(r,s) \\
\mu \downarrow & & \downarrow f \\
P \times G & \xrightarrow{\omega} & P \times \text{SO}(r,s) \\
\downarrow & & \downarrow \\
S_0 & \xrightarrow{g} & S^{r,s}
\end{array}
\]

Here $g : S_0 \rightarrow G_{2,4} \simeq Q_2 \simeq \mathbb{C}P^1 \times \mathbb{C}P^1 \simeq S^2 \times S^2$ is a generalized Gauss map.

We start a consideration with the immersion of the space–like surface $S^{2,0}$ into the manifold $M^{4,0}$. Locally, in virtue of an isomorphism $T_z M^{4,0} \simeq \mathbb{R}^{4,0}$ we have an immersion $S^{2,0} \hookrightarrow \mathbb{R}^{4,0}$. The Grassmannian of oriented two–planes in $\mathbb{R}^{4,0}$ may be identified with a quadric $Q_2 \subset \mathbb{C}P^3$, where $Q_2 \simeq S^2 \times S^2$, $S^2$ is a standard sphere of radius $1/\sqrt{2}$. In this case, according to [HO85], generalized Gauss map $g : S_0 \rightarrow G_{2,4} \simeq Q_2$ can be parametrized in terms of two complex functions $f_1$ and $f_2$ as follows

\[
\Phi(z) = (1 + f_1f_2, i(1 - f_1f_2), f_1 - f_2, -i(f_1 + f_2)), \quad (41)
\]

It is easy to see that $\sum_{k=1}^{4} \varphi_k^2 = 0$. The functions $f_1$ and $f_2$ are related by the formulae [HO85]:

\[
\left| \frac{f_{1z}}{1 + |f_1|^2} \right| = \left| \frac{f_{2z}}{1 + |f_2|^2} \right|, \\
\text{Im} \left\{ \left( \frac{f_{1z}}{f_1} - \frac{2f_1f_{1z}}{1 + |f_1|^2} \right) \bar{z} + \left( \frac{f_{2z}}{f_2} - \frac{2f_2f_{2z}}{1 + |f_2|^2} \right) \bar{z} \right\} = 0.
\]
Further, there is a natural relationship between the Gauss map (11) and generalized Weierstrass representation for surfaces, which defined as follows [KL98a]:

\[
X^1 + iX^2 = \int_{\Gamma} (-\varphi_1\varphi_2dz' + \psi_1\psi_2d\bar{z}'),
\]

\[
X^1 - iX^2 = \int_{\Gamma} (\bar{\psi}_1\psi_2dz' - \bar{\varphi}_1\varphi_2d\bar{z}'),
\]

\[
X^3 + iX^4 = \int_{\Gamma} (\varphi_1\bar{\psi}_2dz' + \psi_1\bar{\varphi}_2d\bar{z}'),
\]

\[
X^3 - iX^4 = \int_{\Gamma} (\bar{\psi}_1\varphi_2dz' + \bar{\varphi}_1\psi_2d\bar{z}'),
\]

(42)

where

\[
\psi_{\alpha z} = p\varphi_\alpha, \quad \varphi_{\alpha \bar{z}} = -p\psi_\alpha, \quad \alpha = 1, 2.
\]

(43)

\(\Gamma\) is a contour in complex plane \(\mathbb{C}\), \(\psi_\alpha, \varphi_\alpha\) are complex–valued functions. The formulae (12), (13) define a conformal immersion of the surface \(S^{2,0}\) into the space \(\mathbb{R}^{4,0}\). At this point an induced metric of \(S^{2,0}\) has a form [KL98a]:

\[
ds^2 = u_1u_2dzd\bar{z},
\]

where \(u_\alpha = |\psi_\alpha|^2 + |\varphi_\alpha|^2 (\alpha = 1, 2)\). Respectively, gaussian and mean curvature are

\[
K = -\frac{2}{u_1u_2} \log(u_1u_2)_{\bar{z}z}, \quad H^2 = 4\frac{|p|^2}{u_1u_2}.
\]

The generalized Weierstrass representation (12), (13) is related with the Gauss map (11) by means of the following substitutions:

\[
f_1 = i\frac{\bar{\psi}_1}{\varphi_1}, \quad f_2 = -i\frac{\bar{\psi}_2}{\varphi_2}.
\]

Further, in accordance with theorem 7 the Dirac operator on the surface \(S^{2,0} \hookrightarrow M^{4,0}\) has a form

\[
D(\psi^+) = (\alpha - iH)\psi^-,
\]

\[
D(\psi^-) = (\alpha + iH)\psi^+,
\]

where \(\alpha = \lambda_1 + \lambda_2, \beta = 2\), and the restricted spinor field \(\Psi|_{S^{2,0}}\) on the surface \(S^{2,0}\) according to theorem 8 and relations (15) has the form \(\Psi|_{S^{2,0}} = (\psi^+, \psi^-) = (\phi_1 + \phi_3, \phi_4 + \phi_2)\), where \(\phi_i \in \Omega_{0,2}\). Therefore, in virtue of an inverse Gauss
map $g^{-1}$ and the formulae \((18)\) a Dirac operator on the Riemann surface $S_0$ is equivalent to the following two systems:

\[
\begin{align*}
\phi_{1z} &= (\alpha - iH)\phi_4, \\
\phi_{4z}^* &= (\alpha + iH)\phi_1, \\
\phi_{3z} &= (\alpha - iH)\phi_2, \\
\phi_{2z}^* &= (\alpha + iH)\phi_3,
\end{align*}
\]

(44)

where the spinors $\phi_i \in \Omega_{0,2}$ on $S_0$ are complex-valued functions on variables $z, z^*$. Let

\[
\begin{align*}
X^1 + iX^2 &= \int_{\Gamma} (-\phi_4 \phi_2 dz + \phi_1 \phi_3 dz^*), \\
X^1 - iX^2 &= \int_{\Gamma} (\phi_1^* \phi_3 dz - \phi_4^* \phi_2^* st_2 dz^*), \\
X^3 + iX^4 &= \int_{\Gamma} (\phi_4 \phi_3^* dz + \phi_1 \phi_2^* dz^*), \\
X^3 - iX^4 &= \int_{\Gamma} (\phi_1^* \phi_2 dz + \phi_4^* \phi_3 dz^*).
\end{align*}
\]

(45)

Then formulae \((14)\), \((15)\) define a conformal immersion of the surface $S^{2,0}$ into the space $\mathbb{R}^{4,0}$. At this point an induced metric has a form

\[
ds^2 = (|\phi_1|^2 + |\phi_4|^2)(|\phi_3|^2 + |\phi_2|^2)dzdz^*.
\]

In the case of the parallel spinor field ($\lambda_1 = \lambda_2 = 0$) the formulae \((14)\), \((15)\) reduce to the generalized Weierstrass representation \((12)\), \((13)\) if suppose $p = iH$

\[
\psi_1 = \phi_1, \varphi_1 = \phi_4, \psi_2 = \phi_3, \varphi_2 = \phi_2.
\]

Analogously, when the surface $S^{0,2}$ immersed into the Kleinian manifold $M^{2,2}$ a Dirac operator on $S^{0,2} \hookrightarrow M^{2,2}$ is defined as follows (theorem 9):

\[
\begin{align*}
D(\psi^+) &= (-\alpha + iH)\psi^-, \\
D(\psi^-) &= (-\alpha - iH)\psi^+,
\end{align*}
\]

where $\alpha = \lambda_1 + \lambda_2$, $\beta = -2$, and the restricted spinor field $\Psi|_{S^{0,2}} = \psi = (\psi^+, \psi^-)$ ($\psi$ is an element of the minimal left ideal $I_{2,2} \simeq \Omega_{2,0,1/2}(1 - iT_{12})$) is expressed via the spinor $\phi \in \Omega_{2,0}$, which in the matrix representation has a form

\[
\phi = \begin{pmatrix} \phi_1 + \phi_2 & \phi_4^* - \phi_3^* \\ \phi_3 + \phi_4 & \phi_1^* - \phi_2^* \end{pmatrix},
\]

where

\[
\begin{align*}
\phi_1 &= a^0 + ia^{12}, \\
\phi_2 &= a^{13} - ia^{23}, \\
\phi_3 &= a^3 - ia^{123}, \\
\phi_4 &= a^1 + ia^2.
\end{align*}
\]

At this point $\Psi|_{S^{0,2}} = (\epsilon_0^+, \epsilon_0^- \psi) = (\phi_1 + \phi_2, \phi_3 + \phi_4)$ (theorem 8). Locally, for the every fiber $\pi^{-1}(x) = T_x M^{2,2} \simeq \mathbb{R}^{2,2}$ there exists a conformal immersion of the
surface $S^{0,2}$ into the space $\mathbb{R}^{2,2}$ defined by the following formulae:

\[
X^1 + iX^2 = \int_\Gamma (\phi_3 \phi_4 dz + \phi_1 \phi_2 dz^*),
\]

\[
X^1 - iX^2 = \int_\Gamma (\phi_1^* \phi_2^* dz + \phi_3^* \phi_2^* dz^*),
\]

\[
X^3 + iX^4 = i \int_\Gamma (\phi_3^* \phi_4 dz + \phi_3^* \phi_2 dz^*),
\]

\[
X^3 - iX^4 = -i \int_\Gamma (\phi_3 \phi_2 dz + \phi_1 \phi_2 dz^*),
\]

where

\[
\phi_{1z} = (-\alpha + iH) \phi_3, \quad \phi_{2z} = (-\alpha + iH) \phi_4,
\]

\[
\phi_{3z} = (-\alpha - iH) \phi_4, \quad \phi_{4z} = (-\alpha - iH) \phi_2.
\]

Let us consider now a most interesting case (from the viewpoint of physics) of the immersion of the space–like surface $S^{0,2}$ into the Lorentzian manifold $M^{1,3}$. According to theorem [7] for the Dirac operator on the surface $S^{0,2} \hookrightarrow M^{1,3}$ we have

\[
D(\psi^+) = (-\alpha + H) \psi^-,
\]

\[
D(\psi^-) = (-\alpha - H) \psi^+,
\]

where the restricted spinor field $\Psi|_{S^{0,2}} = \psi = (\psi^+, \psi^-)$ is expressed via the Dirac–Hestenes spinor field $\phi \in \mathbb{C}_2$ by the formulae [11], $\phi$ is the element of the minimal left ideal $I_{1,3} \simeq \mathbb{C}_2 \hat{\times} (1 + i \sigma_{12})$ (theorem [3]). Further, since the every fiber of the tangent bundle of $M^{1,3}$ is isomorphic to the Minkowski spacetime, $\pi(x) = T_x M^{1,3} \simeq \mathbb{R}^{1,3}$, then there exists (for the every fiber) a conformal immersion of the surface $S^{0,2}$ into the spacetime $\mathbb{R}^{1,3}$. This immersion may be defined as follows

\[
X^1 = \frac{1}{2} \int_\Gamma \left[ (\phi_1 \phi_4 + \phi_3 \phi_2) dz + (\phi_1^* \phi_4^* + \phi_3^* \phi_2^*) dz^* \right],
\]

\[
X^2 = \frac{1}{2} \int_\Gamma \left[ (\phi_1 \phi_4 - \phi_3 \phi_2) dz + (\phi_1^* \phi_4^* - \phi_3^* \phi_2^*) dz^* \right],
\]

\[
X^3 = \frac{i}{2} \int_\Gamma \left[ (\phi_4 \phi_3 - \phi_1 \phi_2) dz + (\phi_4^* \phi_3^* - \phi_1^* \phi_2^*) dz^* \right],
\]

\[
X^4 = \frac{1}{2} \int_\Gamma \left[ (\phi_4 \phi_3 + \phi_1 \phi_2) dz + (\phi_4^* \phi_3^* + \phi_1^* \phi_2^*) dz^* \right],
\]

where

\[
\phi_{1z} = (-\alpha + H) \phi_4, \quad \phi^*_3 = -(\alpha + H) \phi_2,
\]

\[
\phi_{3z} = (-\alpha - H) \phi_1^*, \quad -\phi_{2z} = (-\alpha - H) \phi_3^*.
\]
At this point an induced metric on the surface $S^{0.2} \hookrightarrow \mathbb{R}^{1,3}$ has a form
\[ ds^2 = |\phi_1^* \phi_2 + \phi_4 \phi_3^*|^2 dz dz^*. \]

Further, let $\Psi$ be the parallel spinor field on the manifold $M^{1,3}$, then for the system (47) we have
\[ \begin{align*}
\phi_1^* &= H \phi_4, \\
\phi_4^* &= -H \phi_1^*,
\end{align*} \quad (48) \]

It is easy to see that the every system (48) coincides with a linear problem of a modified Veselov–Novikov hierarchy. The first equation of the mVN–hierarchy has a form [Bog87]
\[ \begin{align*}
p_t + p_{zzz} + p_{zz} = 3 \rho \omega_z + 3 \bar{\rho} \bar{\omega}_z = 0, 
\end{align*} \quad (49) \]

where $\omega_z = (p^2)_z$.

**Example.** Suppose now that the surface $S^{0.2} \hookrightarrow \mathbb{R}^{1,3}$ is a surface of revolution. Then the components of the Dirac–Hestenes spinor field are defined as follows
\[ \begin{align*}
\phi_1^* &= r_1(x) \exp(\lambda y), \\
\phi_4 &= s_1(x) \exp(\lambda y),
\end{align*} \quad (50) \]

where $r_i(x), s_i(x)$ are real–valued functions, $\lambda \in \mathbb{C}$. The substitution of (50) into the systems (48), where
\[ \begin{align*}
\frac{\partial}{\partial z} &= \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \\
\frac{\partial}{\partial z^*} &= \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right),
\end{align*} \]
gives
\[ \begin{align*}
r_{1x} + i \lambda r_1 &= 2H s_1, \\
s_{1x} - i \lambda s_1 &= -2H r_1.
\end{align*} \quad (51) \]

The every system (51) is nothing but a well–known Zakharov–Shabat system [ZS71]. One–soliton solutions of ZS–system obtained via the linear Bargmann potentials [Bar49] are well studied (see [Lam80]).

Let us show that equations (50) are particular form of a canonical decomposition of the Dirac–Hestenes spinor field [Hes67]:
\[ \phi = r(x) e^{i\beta/2}, \quad (52) \]

where $r(x) = \sqrt{\rho(x)} R(x)$, $\rho(x)$ is a probability density, $R(x) \in \text{Spin}_+(1,3)$ is a Lorentz rotation, $\beta$ is a so–called Yvon–Takabayasi angle which defines a duality transformation. Since in our case the Dirac–Hestenes field is defined on
the surface and therefore depends on two variables, then \( r(x) = (x_1, 0, 0, 0) \) and \( \beta = (0, x_2, 0, 0) \), whilst the spinor \((52)\) depends on four variables \(x_1, x_2, x_3, x_4\). It is obvious that for the spinor field defined on the surface, the variables \(x_3\) and \(x_4\) play a role of the deformation parameters. For example, a dependence on an evolution parameter \(x_4 = t\) is defined by the standard procedure of the inverse scattering transform \([AS81]\). At this point in the case of the surface of revolution \((p = p(x_1), p = u/2)\) the equation \((49)\) reduces to a modified Korteweg–de Vries equation \(u_t = u_{xxx} + 3/2u^2u_x\), and a dependence of the potential \(u\) on the parameter \(t\) has a form \(u = \pm \text{sech}(\mu x - \mu^3 t)\), where \(\mu\) is a constant of integration. It allows to express a dependence of fundamental solutions (Jost functions) of ZS–systems \((51)\) and respectively the Dirac–Hestenes spinor field (in virtue of \((50)\)) on the parameter \(t\). Thus, we have a Dirac–Hestenes spinor field \(\phi \in \mathcal{O}^{+}_{1,3}\) defined on the surface of revolution (precisely speaking, solitonic surface of revolution with reflectionless potential), integrable deformations of which are defined by the mKdV–hierarchy. In connection with this it should be noted that an idea of revolution about some fixed axis has deep roots in the electron theory. For example, Uhlenbeck and Goudsmit in their fundamental paper \([UG25]\) imagine the electron as a revolving top.

**Acknowledgements**

I am deeply grateful to Prof. H. Baum, to Prof. C. Bär and Prof. P. Lounesto for sending me their interesting papers which are essentially stimulate me to write down this work.

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