Synthesizing Linked Data Under cardinality and Integrity Constraints

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ABSTRACT
The generation of synthetic data is useful in multiple aspects, from testing applications to benchmarking to privacy preservation. Generating the links between relations, subject to cardinality constraints (CCs) and integrity constraints (ICs) is an important aspect of this problem. Given instances of two relations, where one has a foreign key dependence on the other and is missing its foreign key (FK) values, and two types of constraints: (1) CCs that apply to the join view and (2) ICs that apply to the table with missing FK values, our goal is to impute the missing FK values such that the constraints are satisfied. We provide a novel framework for the problem based on declarative CCs and ICs. We further show that the problem is NP-hard and propose a novel two-phase solution that guarantees the satisfaction of the ICs. Phase I yields an intermediate solution accounting for the CCs alone, and relies on a hybrid approach based on CC types. For one type, the problem is modeled as an Integer Linear Program. For the others, we describe an efficient and accurate solution. We then combine the two solutions. Phase II augments this solution by incorporating the ICs and uses a coloring of the conflict hypergraph to infer the values of the FK column. Our extensive experimental study shows that our solution scales well when the data and number of constraints increases. We further show that our solution maintains low error rates for the CCs.

1 INTRODUCTION
In recent years, we have witnessed an increase in data-centric applications that call for efficient testing over reliable databases with certain desired qualities [11, 31]. Existing benchmarks such as TPC-H [40, 51] may not possess the desired characteristics for testing a specific application as they may not have the needed statistical qualities or the correct Integrity Constraints (ICs). The field of data generation [5, 9, 19, 21, 24, 34, 41, 43] has proven effective in this respect. Two prominent challenges in this field are: (1) the generation of links between different tables, i.e., aligning foreign keys with primary keys based on Cardinality Constraints (CCs) [5], and (2) ensuring that the data will satisfy a set of expected ICs [47].

In particular, when the real data is sensitive and access to it is heavily regulated, users often need to wait months or years to get access to the real data before they can even start writing data analysis programs. One solution is to generate realistic synthetic data that satisfies some CCs and ICs so that users can: (a) start writing code to analyse the data, (b) test it locally, and (c) evaluate whether access to the data would be useful for their purposes even before they get access to the real data. However, current methods for generating synthetic data under privacy constraints (especially state-of-the-art standards like differential privacy [17]) do not handle data with a combination of CCs or statistical constraints and ICs. Most, (e.g., [23, 46, 55]), only handle statistical constraints.

Furthermore, there has been a lot of recent work on answering count queries under differential privacy (e.g., Matrix mechanism [29], HDMM [35]) and in particular over relational databases [27]. A key challenge when answering queries especially over relational databases is that of consistency – are the answers outputted by a differentially private algorithm consistent with some underlying database? While there is work on using inference to enforce consistency when all the count queries are over a single view of the underlying database [22], these techniques do not extend to the case when: (a) the underlying database is relational and query answers are over several joined views of the relations, and (b) when the underlying database needs to satisfy some ICs. One solution to this problem is to find a database that is consistent with the query answers and the ICs, and answer queries from it. While techniques for finding such a consistent database are known for single tables without ICs [7, 22, 28], no such techniques are known when there are multiple tables in a relational database with ICs.

Moreover, DBMS testing and other applications may require databases that conform to both CCs and ICs to make them more realistic [5, 47]. For instance, consider a table with the attributes A and B. A query grouping over attributes A and B could return as many tuples as the cross product of the active domains of A and B. However, if there is a Functional Dependency A → B, then the output size of the group-by query is only the maximum of the active domains of the two attributes. Thus, the presence of ICs can significantly impact the performance characteristics of queries.

Figure 1: Database D with FK h_{id} missing from R_1

In this paper, we investigate the problem of generating the links between database tables based on a set of linear CCs and a set of ICs. Formally, we consider two relations, R_1 and R_2, where R_1 has a foreign key dependence on R_2 and is missing all values in its foreign key column FK. The goal is to impute FK in R_1 based on the given CCs and ICs. Importantly, this problem and our solutions can be extended to relational databases with a snowflake schema [13], by focusing on pairs of relations linked by foreign key joins.

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We model the problem, give a theoretical analysis, and provide a solution for the generation of foreign keys for existing database relations while ensuring the satisfaction of a set of ICs and reducing the error of a set of CCs. Next, we give our main contributions.

**Model and Theoretical Results:** We define the problem of C-Extension whose input is a relation \( R_1 \) with an unknown foreign key dependence on a relation \( R_2 \), i.e., the FK column in \( R_1 \) is missing, and a set of CCs and ICs. For the CCs, we define and use linear CCs that apply to \( R_1 \approx R_2 \), based on [5]. For the ICs, we define a type of Denial Constraints (DCs) \([14, 16]\), called Foreign Key DCs, that applies to \( R_1 \) and forbids tuples from having the same FK value under specified conditions. We then show that C-Extension is NP-hard in data complexity. This result leads us to a two-phase heuristic solution that still ensures the satisfaction of all DCs, while tolerating possible errors in the CC counts.

**Solution:** Our solution can be split into two phases: (1) first phase (Section 4) is designed for the completion of a view \( V_{join} \) based on CCs, where \( V_{join} \) represents \( R_1 \approx R_2 \) and is initialized with a copy of \( R_1 \) (without the FK column) along with an empty column per non-key column in \( R_2 \) (due to foreign key dependence, \( |R_1| = |V_{join}| \)), and (2) second phase (Section 5) uses the generated view \( V_{join} \) to complete the FK column in \( R_1 \) so that the DCs are satisfied.

**Phase I:** We provide a novel description of CC relationships that allows for \( V_{join} \) to be completed efficiently and precisely under specific conditions (presented in Section 3.1). We further devise algorithms for this case and the general case:

- For the general case, we devise an algorithm that models the CCs and the tuples in \( V_{join} \) as an Integer Linear Program (inspired by [5]). From its solution, we greedily infer the values in \( V_{join} \) for the attributes that come from \( R_2 \).
- For the special case, we devise a novel algorithm based on relationships between the CCs. We show that if the CCs have containment or disjointness relationships between them (defined in Section 4.2), then we can find an exact completion of \( V_{join} \) without any errors, provided one exists.

Our approach is a hybrid of these two solutions that employs the first solution for the subset of CCs that does not fit the special case, and employs the second solution for the subset of CCs that does.

Another novelty in our solution exploits the fact that the all-way marginals for \( R_1 \), i.e., counts of tuples with different combinations of values in \( R_1 \)'s non-key columns, have the same counts in \( V_{join} \). Thus, we augment the input set of CCs to improve accuracy.

**Phase II:** For the second phase, we employ the concept of a conflict hypergraph \([16]\) and use a novel algorithm based on hypergraph coloring. We model the tuples in \( R_1 \) as vertices and connect by an edge every set of tuples that will violate a DC if assigned the same foreign key. Thus, colors represent the values that the foreign keys can take in \( R_1 \), and a proper coloring represents a mapping of tuples to foreign keys that does not violate any DC. Due to the previous stage that considered \( R_1 \approx R_2 \) tuples in \( R_1 \) have a certain list of permitted colors. This version of the graph coloring problem is called List Coloring \([2]\) and is known to be NP-hard. To color the graph, we use a greedy coloring algorithm that considers vertices in descending order by degrees. The algorithm skips vertices whose list of permitted colors is subsumed by the colors assigned to their neighbors. We ensure a proper coloring by adding the least number of new colors for the skipped vertices. Adding colors beyond the permitted lists corresponds to artificially adding tuples in \( R_2 \).

**Experimental Evaluation** We have implemented our solution and performed a comprehensive set of experiments on a dataset derived from the 2010 U.S. Decennial Census \([44]\). We have evaluated our solution in terms of accuracy and scalability in various scenarios, several of which were used for comparison with a baseline based on [5]. We further examined the runtime breakdown of our approach, presenting the runtimes of phases I and II in our solution. Our results indicate that our solution incurs relatively small error for CCs and no error for DCs (as guaranteed by our theoretical analysis). Moreover, our algorithms scale well for large data sizes, and large and complex sets of CCs and DCs. For increasing data scales, our approach was 17 times faster on average across different cases than the baseline we compare to.
We now define the basic concepts used throughout the paper, and the C-Extension problem.

**Relations in a Database:** Let \( R_1 \) and \( R_2 \) be relations over the schema attributes \((K_1, A_1, \ldots, A_p, FK)\) and \((K_2, B_1, \ldots, B_q)\), respectively. An attribute \( A_i \) of \( R_1 \) may also be called a column and is denoted by \( R_i.A_j \) where \( t \in R_i \) denotes a tuple in \( R_i \) and \( t.A_j \) denotes the cell of column \( A_j \) in tuple \( t \). The last column in \( R_1 (FK) \) is a foreign key column that gets its values from the key column \( K_2 \) in \( R_2 \). The view \( V_{\text{Join}} = R_1 \bowtie_{FK=K_2} R_2 \) denotes the join of the two relations. If all values of a column \( A_j \) are missing, it is called a missing column.

**Example 2.1.** Consider a database \( D \) with two relations \( R_1 \) and \( R_2 \) as shown in Figure 2. \( R_1.h_{id} \) is a missing column. The first row in \( R_1 \) says that \( t_1.Age \) is 75, \( t_1.Rel \) is Owner and \( t_1.Multi-lin\)g is 0.

**Foreign Key Denial Constraints:** DCs [14] are a general form of constraints that can be written as a negated First Order Logic statement. DCs can express several types of integrity constraints like functional dependencies and conditional functional dependencies [10]. In this paper, we restrict our attention to DCs that contain a primary key column \( K \) and \( FK \).

**Definition 2.2 (Foreign Key DC).** A Foreign Key DC on a relation \( R(K_1, A_1, \ldots, A_p, FK) \) is defined as the following FOL statement:

\[
\forall t_1, t_2, \ldots, t_n. \neg (p_1 \land \cdots \land p_n)
\]

where \( p_q = t_1.A_j \land t_2.A_j \) or \( p_q = t_1.A_j \land c \), for \( t_1, t_2 \in R \), \( p \geq 2 \), \( c \in \{=, <, >, \leq, \geq \} \), \( c \) and \( k \) are constants, and \( p_n = (t_1.FK = \cdots = t_k.FK) \).

We use the terms Foreign Key DC and DC interchangeably.

**Example 2.3.** \( DC_{\text{O,O}} \) (Figure 2a), which states that two homeowners cannot be in the same home, can be formulated as follows:

\[
\forall t_1, t_2 \in R_1. \neg (t_1.Rel = t_2.Rel = \text{Owner} \land t_1.h_{id} = t_2.h_{id})
\]

Note that the restriction to Foreign Key DC means that all constraints are on people that are in the same household.

**Linear Cardinality Constraints:** CCs form the second class of constraints that allows for the specification of the number of tuples that should possess a certain set of attribute values, which can be expressed as a selection condition. As standard in previous work [5, 36], we restrict our attention to linear CCs.

**Definition 2.4 (Linear CC, adapted from [5]).** A Linear CC over a database \( D \) consisting of relations \( R_1(K_1, A_1, \ldots, A_p, FK) \) and \( R_2(K_2, B_1, \ldots, B_q) \) is defined as follows:

\[
|\sigma_\varphi(R_1 \bowtie_{FK=K_2} R_2)| = k
\]

where \( \varphi \) is a Boolean selection predicate over a subset of (non-key) attributes in \( D \), and \( k \in \mathbb{N} \).

In the rest of the paper, we only refer to conjunctive selection predicates with conjuncts of the form \( A_i \land c \), where \( c \in \{=, <, >, \leq, \geq \} \) and \( c \) is in the domain of column \( A_i \), though our algorithms can be extended to conditions that contain disjunction as well.

**Example 2.5.** \( CC_{\text{O}} \) (Figure 2b), which states that the number of homeowners \((Rel = \text{Owner})\) living in \( Area = \text{Chicago} \) must equal 4, can be written as: \( |\sigma_{\text{Rel=Owner,Area=Chicago}}(R_1) \bowtie_{FK=K_2} R_2| = 4 \).

We denote by \( R \models \varphi \) the fact that relation \( R \) meets constraint \( \varphi \).

**Problem Definition:** We now formally define the C-Extension problem and discuss its intractability.

**Definition 2.6 (C-Extension).** Let \( R_1(K_1, A_1, \ldots, A_p, FK) \) and \( R_2(K_2, B_1, \ldots, B_q) \) be two relations, where \( R_1.FK \) is a foreign key mapped from \( R_2.FK \) and is empty. Let \( S_{\text{DC}} \) denote the set of DCs over \( R_1 \) and let \( S_{\text{CC}} \) denote the set of linear CCs over the foreign key join between \( R_1 \) and \( R_2 \). C-Extension is the problem of completing all values in \( R_1.FK \) to create \( R \) so that (1) \( \forall \sigma \in S_{\text{DC}}, R_1 \models \sigma \), (2) \( \forall \sigma \in S_{\text{CC}}, R_1 \bowtie_{FK=K_2} R_2 \models \sigma \).

**Example 2.7.** Reconsider relations \( R_1 \) and \( R_2 \) in Figure 1, and DCs and CCs in Figure 2. A solution \( R \) for the C-Extension problem as defined by these relations and constraints is shown in Figure 3.

![Table 1: Persons (rel. \( R_1 \))](data:image/png;base64,iVBORw0KGgoAAAANSUhEUgAAAgAAAAAgCAYAAABAcJqlAAAAAXNSR0IArs4c6QAAAARnQU1BAACxjwv8YQUAAAAJcEhZcwAADsQAA7EAk尔.jpg)

Figure 3: Relation \( R_1 \) from Figure 1 with FK \( h_{id} \) filled-in to satisfy DCs and CCs given in Figure 2

The decision version of C-Extension is given by the same setting as in Definition 2.6. The output is 1 if there exists a completion of \( R_1.FK \) such that all DCs and CCs are satisfied, and 0 otherwise.

**Proposition 2.8.** The decision problem version of C-Extension is \( \text{NP} \)-hard in data complexity.

**Proof Sketch.** We describe a reduction from NAE-3SAT to C-Extension. In the NAE-3SAT problem, we are given a 3-CNF formula \( \varphi \) and asked whether there is a satisfying assignment to \( \varphi \) with every clause having at least one literal with the value False. Given a 3-CNF formula \( \varphi = C_1 \land \cdots \land C_m \), where \( x_1, \ldots, x_m \) are the propositional variables in \( \varphi \), construct a relation \( R_1(Var, \alpha, Cls, Chosen) \), where \( Chosen \) is missing all values, and \( Var, \alpha, Cls \) columns take values:

(1) \((x_i, 1, C_j, ?)\) if making \( x_i \) True makes \( C_j \) True

(2) \((x_i, 0, C_j, ?)\) if making \( x_i \) False makes \( C_j \) True

We define \( S_{\text{DC}} \) to be the set with the following two DCs:

(1) \( \forall t_1, t_2, \neg (t_1.Var=t_2.Var \land t_1.\alpha \neq t_2.\alpha \land t_1.Chosen=t_2.Chosen) \)

(2) \( \forall t_1, t_2, \neg (t_1.Cls=t_2.Cls \land t_1.Chosen=t_2.Chosen \land t_1.Chosen=t_2.Chosen) \)

CCs are not needed in the reduction. The goal is to complete the missing column \( Chosen \) in \( R_1 \). We define \( R_2 \) as containing two columns: a primary key column \( Chosen \), and another column \( E \). \( R_2 \) contains the tuples \((0, a)\) and \((1, b)\), i.e., the domain for \( Chosen \) is \([0, 1]\). Intuitively, \( Chosen \) encodes the satisfying assignment for \( \varphi \) by assigning values to each tuple, where \( t.Chosen=1 \) if the assignment should be \( t.Var=1.a \).

The full proofs are detailed in Section A.1 of the appendix.
3 SOLUTION OVERVIEW

Our solution proceeds in two phases as seen in Figure 4. In phase I, we consider the view $V_{\text{join}}$ representing the join of the two relations $R_1$ and $R_2$, where $R_1$ has a foreign key dependence on $R_2$, and initialize it with (non-FK) columns from $R_1$ and an empty column per non-key column from $R_2$. We infer these values based on the CCs by a hybrid approach that uses both ILP [5] and a more efficient and accurate procedure for special cases. In phase II, we impute $R_1.FK$ by modeling the problem as a conflict hypergraph using the DCs, and coloring it based on the inferred values in $V_{\text{join}}$.

### 3.1 Overview of the First Phase

Due to the foreign key dependence (Definition 2.6), we define $V_{\text{join}}$ over the columns $K_1, A_1, \ldots, A_p, B_1, \ldots, B_q$ such that $t \in R_1$ implies that there is a single $t' \in V_{\text{join}}$ with $t \cdot K_i = t' \cdot K_i$ and $V_1 \leq i \leq p, t \cdot A_i = t' \cdot A_i$ with additional $B_1, \ldots, B_q$ entries that are initially all empty because FK is missing in $R_1$. Therefore, $|V_{\text{join}}| = |R_1|$. Our goal is to complete these columns based on the CCs.

**Example 3.1.** Reconsider $R_1$ and $R_2$ shown in Figure 1 and the CCs in Figure 2b. The join view $V_{\text{join}}$ is $R_1$ as it appears in Figure 1 (without $h_{id}$) with an empty Area column (as this is the schema of $R_1 \bowtie R_2$). Due to the foreign key dependency, we have $|V_{\text{join}}| = |R_1|$, and $V_{\text{join}}$ contains a tuple for each $R_1$ tuple with the same values as in $R_1$ and an empty Area value. The reason is that the FK values are missing in $R_1$. Our goal is to fill-in $V_{\text{join}}$ so that the CCs are satisfied.

We give a short description of our solution for completing $V_{\text{join}}$.

**Solution as an ILP (Section 4.1, green box in Figure 4):** Given a set of CCs on $V_{\text{join}}$, we model the problem of completing the missing columns as a system of linear equations with variables accounting for counts of different tuples needed in $V_{\text{join}}$ to satisfy the CCs. Thus, the variables must take non-negative integer values. We artificially add to $S_{CC}$ all-way marginals (using the idea of intervalization from [5] that is explained in Section 4) from $R_1$ to enhance the accuracy of the solution. For example, based on CCs given in Example 1.1, $|\sigma_{Age \leq 24 \land Rel = \text{Spouse} \land Multi-ling = \text{a}}| = 1$ gets added to $S_{CC}$. We then assign $B_1, \ldots, B_q$ values to the tuples in $V_{\text{join}}$ based on the solution returned by an ILP solver.

**Using CC Relationships for Special Cases (Section 4.2, blue box in Figure 4):** We give a novel description of the relationships between CCs based on their selection conditions, defining CC containment, disjointness and intersection. In the case where there are no intersecting CCs and no disjunctions, we give an algorithm to complete $V_{\text{join}}$ that models the containment and disjointness of CCs as a Hasse diagram [52] that it recurses on bottom-up to fill-in $V_{\text{join}}$. Any leftover $V_{\text{join}}$ tuples without $B_1, \ldots, B_q$ values are randomly assigned a combination that cannot cause a new contribution towards the target count of any CC. However, if no such combinations are available, then the leftover $V_{\text{join}}$ tuples cannot be completed. We refer to these as invalid tuples.

**Hybrid Approach (Section 4.3):** In the absence of intersecting CCs, the solution decomposes cleanly as seen above. This motivates the hybrid approach that combines ideas from both cases to achieve better runtime and accuracy when some CCs intersect. We start by labeling each pair of CCs as disjoint, contained or intersecting. For all CCs that do not intersect or contain any intersecting CCs, we use the approach from Section 4.2, and for the rest, we use the ILP approach from Section 4.1. Lastly, as seen above in the special case, we may end up with some invalid tuples.

### 3.2 Overview of the Second Phase

After filling-in the columns of $V_{\text{join}}$ that originate in $R_2 (B_1, \ldots, B_q)$, we turn to reverse-engineering $R_1$ from $V_{\text{join}}$. This phase uses conflict hypergraphs [16] to represent possible DC violations.

**Conflict Hypergraph (Section 5.1, red box in Figure 4):** We use the notion of conflict hypergraph for the tuples of $R_1$ based on the DCs. Given a DC, we construct an edge for all the sets of tuples that cannot get the same foreign key value due to that DC.

**Example 3.2.** Consider the relation $R_1$ depicted in Figure 1 and the first DC in Figure 2a. Suppose the first two tuples are assigned the same Area value in $V_{\text{join}}$. Thus, the conflict hypergraph will have an edge containing the tuples with $p_{id} = 1$ and $p_{id} = 2$ since they are both owners and cannot be in the same household (the $h_{id}$ value). The conflict hypergraph of our running example is depicted in Figure 7.

**List Coloring (Section 5.1, orange box in Figure 4):** Proper coloring of the hypergraph ensures that there must be at least two vertices in each edge with distinct colors. Thus, modeling each FK value as a color and each tuple as a vertex allows us to prove that a proper coloring results in an assignment of FK values that satisfies the DCs. The values in $V_{\text{join}}$ filled-in by the previous phase induce a list of possible FK values, and thus colors, for $R_1$ tuples. Finding a proper coloring such that each vertex assumes a color from its predefined list is called List Coloring [2] and is NP-hard. We thus propose a greedy coloring algorithm based on vertex degree.

**Algorithm for Satisfying the DCs (Section 5.2):** The size of the conflict hypergraph can be very large and thus may cause a significant slowdown in practice. Therefore, we partition $R_1$ into smaller sets with the same $B_1, \ldots, B_q$ values and construct a conflict hypergraph for each set separately. For each non-invalid tuple, $V_{\text{join}}$ contains $B_1, \ldots, B_q$ values, so we can use our greedy coloring algorithm to find a coloring for them. We color invalid tuples at the end using all FK values as candidates. This phase may result in the addition of extra tuples to $R_2$ (the second output in Figure 4).

### 4 FIRST PHASE: SOLVING CCS

In this section, we focus on the first phase. Given two relations $R_1(K_1, A_1, \ldots, A_p, FK)$ and $R_2(K_2, B_1, \ldots, B_q)$, we wish to satisfy a set $S_{CC}$ of CCs over the join view $V_{\text{join}} = R_1 \bowtie_{FK=K_2} R_2$. <ref>Figure 4: Solution Overview</ref>
4.1 Solution as an ILP

We give a two-part solution in Algorithm 1 where we: (1) model the CCs as a system of linear equations and solve it using an ILP solver, and (2) greedily fill-in $B_1, \ldots, B_q$ values for each tuple in $V_{\text{join}}$. The first part (lines 2–14) is inspired by [5]. Each variable represents the number of tuples with a specific combination of $A_1, \ldots, A_p, B_1, \ldots, B_q$ values in $V_{\text{join}}$. Each CC is written as a sum of the variables whose associated tuples satisfy its selection condition. We now introduce the notion of intervalization [5].

**Intervalization:** Creating a variable for every combination of values in the cross product of the full domains of all the $p+q$ (non-key) columns in $V_{\text{join}}$ would give a very large ILP. We augment the notion of intervalization [5] so that it will not only assist in reducing the number of variables based on the intervals of values in $S_{CC}$, but also use only the combinations of $A_1, \ldots, A_p$ values already in $R_1$. We call this *binning* the distinct $(A_1,\ldots,A_p)$ values in $R_1$.

In the system of equations $Ax = b$, row $r_1$ (in $A$) corresponds to $CC_1$ and row $b_1$ (in $b$) stores $CC_1$’s target count. We create the vector $x$ of variables by putting bins with the same $B_1, \ldots, B_q$ values as contiguous elements (see Example 4.1). Since input CCs are linear, each element in $A$ is 0 or 1. The goal is to solve for an $x$ with non-negative integer entries (line 14). Such a solution can be obtained if there exists a solution to C-Extension where $R_1 \supseteq F_{K=K_2} R_2$ satisfies $S_{CC}$. In the second part (lines 15–17), we fill-in the $B_1, \ldots, B_q$ values greedily. For each assignment $x_1 = v_1$, we find at most $v_1$ tuples (with empty $B_1, \ldots, B_q$ cells) in $V_{\text{join}}$ that satisfy $x_1$’s selection condition on $R_1$, and fill-in their $B_1, \ldots, B_q$ values as encoded by $x_1$.

**Example 4.1.** Consider relations $R_1$ and $R_2$ in Figure 1, CCs in Figure 2b and $V_{\text{join}}$, described in Example 3.1. Intervalization splits Age into [0, 24] and [25, 114] due to $CC_3$ (all other columns are categorical). Even though $R_1$ contains multiple tuples for multi-lingual homeowners with age greater than 24, it suffices to look at those with Age in [0, 24] and [25, 114] Importantly, for the given instance, we only need to keep track of the following tuple types: (1) Age $\in$ [25, 114], Rel $=$ Owner, Multi-ling $=$ 0, (2) Age $\in$ [0, 24], Rel $=$ Spouse, Multi-ling $=$ 0, (3) Age $\in$ [0, 24], Rel $=$ Child, Multi-ling $=$ 1, and (4) Age $\in$ [25, 114], Rel $=$ Owner, Multi-ling $=$ 1. Here, vector $x$ uses a copy of these four bins with $Area = Chicago$ in $x_1$ and $Area = NYC$ in $x_2$. Without the idea of binning, we would need 16 variables because Area can take 2 distinct values and $R_1$ contains 8 unique tuples. Finally, we iterate through each $CC_1 \in S_{CC}$ and add rows $r_1$ and $b_1$ in $A$ and $b$, resp. For $CC_1$, $r_1 = \{1, 0, 0, 0\}$ and $b_1 = 4$ because only $x_1$ and $x_2$ match the selection conditions in $CC_1$; similarly for other CCs. Hence, $Ax = b$ has a solution given by $x_1 = 2, x_2 = 1, x_3 = 2, x_4 = 2, x_5 = 1, x_6 = 0, x_7 = 0$ and $x_8 = 1$. Finally, we iterate through $x_1$’s to find $V_{\text{join}}$ tuples which satisfy its selection condition and assign the matching $Area$ value that gives the view in Figure 5. E.g., we find two tuples in $V_{\text{join}}$ with Age $\in$ [25, 114], Rel $=$ Owner and Multi-ling $=$ 0 for $x_1$ and assign $Area = Chicago$.

**Augmenting with All-Way Marginals:** When $A$ is sparse, some $x_1$ values in the solution may not match the true counts. Despite such discrepancies, we can complete several tuples in $V_{\text{join}}$ because we update at most as many tuples as the value of $x_1$ in the solution. The order of updates may also impact which subset of $V_{\text{join}}$ tuples gets specific $B_1, \ldots, B_q$ values. For example, another solution to the ILP in Example 4.1 is given by $x_1 = 0, x_2 = 3, x_3 = 0, x_4 = 4, x_5 = \ldots$ $x_6 = x_7 = 0, x_8 = 2$. This assigns $Area = Chicago$ to tuples with $pid = 2, 4, 5, 9$ in $V_{\text{join}}$. However, the remaining tuples do not get any $Area$ value and no CC in $CC_3$ gets satisfied in $V_{\text{join}}$. We overcome this issue by using both $S_{CC}$ and all-way marginals over $A_1, \ldots, A_p$ from $R_1$ when solving the ILP (see the discussion about the baseline’s CC accuracy in Section 6). The solution reported in Example 4.1 was computed with all-way marginals.

**Complexity:** The complexity of Algorithm 1 is $O(\|S_{CC}\| \cdot m + S)$, where $S_{CC}$ contains CCs in $CC_3$ along with the marginals, and $m$ is the number of variables that is upper-bounded by the number of tuples in $R_1$ times the product of the sizes of the active domains of $B_1, \ldots, B_q$ in $R_2$. Lastly, $S$ is the time complexity of the ILP solver.
4.2 Efficient Algorithm for Special CC Types

In practice, Algorithm 1 may incur slow runtimes as generating and solving the system of equations is time consuming, even with state-of-the-art ILP solvers (as shown in Section 6). Thus, we describe a model for relationships between the CCs in \( S_{CC} \) and devise an algorithm to better tackle \( V_{join} \) completion in specific scenarios.

**Algorithm 2** that outputs an exact solution.

Now, we focus on the set \( R \) of columns, where there are no intersecting CCs present and describe conditions on \( R \) to identify the number of tuples that satisfy the selection condition on \( R \) and the conditions on \( R \) are disjoint.

We denote this by \( CC_1 \cap CC_2 = \emptyset \).

**Note** that we also consider pairs of CCs with the same \( R \), but disjoint \( R \) selection conditions as disjoint. For a pair \((CC_i, CC_j)\) of such CCs, assigning \( B_1, \ldots, B_q \) values to tuples that contribute to the count of \( CC_j \) should not limit the set of tuples available for \( CC_j \), if a solution exists. We label such pairs similarly to a pair of disjoint CCs. Next, we define the notion of CC containment.

**Definition 4.3.** Let \( CC_i, CC_j \in S_{CC} \) such that \( CC_i \cap CC_j = \emptyset \) and \( CC_j \cap CC_j = \emptyset \). Then, \( CC_i \) is contained in \( CC_j \), denoted \( CC_i \subseteq CC_j \), if \( \phi_i \) uses a (non-)strict superset of attributes in \( \phi_j \) and for each common attribute, the values in \( CC_i \) are a subset of the corresponding values in \( CC_j \).

Intuitively, if \( CC_i \) is contained in \( CC_j \), then \( CC_i \) is more restrictive than \( CC_j \), and assigning a tuple \( t \in R_1 \) values in \( B_1, \ldots, B_q \) that satisfy the selection condition in \( CC_j \) will also satisfy the selection condition in \( CC_i \). This observation defines a partial order on \( S_{CC} \) which we utilize later to find a solution for CCs.

**Definition 4.4.** CCs \( CC_i, CC_j \in S_{CC} \) are said to be intersecting if they are neither disjoint nor does one contain the other. We denote this by \( CC_i \cap CC_j \neq \emptyset \).

**Example 4.5.** Assume \( R_1 \) (or \( V_{join} \)) contains 10 tuples with \( Age \in [10,30] \), 20 with \( Age \in [30,50] \), and 50 with \( Age \in [50,70] \). Let:

\[
\begin{align*}
CC_1 & : |\{\text{Age}\} \cap R_1| = 20 \\
CC_2 & : |\{\text{Age}\} \cap R_1| = 30 \\
CC_3 & : |\{\text{Age}\} \cap R_1| = 30 \\
CC_4 & : |\{\text{Age}\} \cap R_1| = 30 \\
\end{align*}
\]

If all tuples with \( Age \in [30,50] \) get assigned \( Area = NYC \), \( CC_1 \) cannot be satisfied. Even when \( Area = Chicago \) in \( CC_2 \), it is unclear how many tuples with \( Age \in [30,50] \) can be assigned \( Area = Chicago \).

**Solution Without Intersecting CCs:** Now, we focus on the setting where there are no intersecting CCs present and describe Algorithm 2 that outputs an exact solution.

We use the notion of a Hasse diagram \([52]\), denoted by \( H = (V, E) \), to encode the containment relationships between the CCs in \( S_{CC} \). We refer to each connected component in the undirected version of \( H \) as a diagram. Within each diagram, the CC that is not contained in any other CC is referred to as the maximal element.

Algorithm 2 is given the view \( V_{join} \) with missing \( B_1, \ldots, B_q \) columns, \( S_{CC} \) and the Hasse diagram \( H \) describing the containment relations in \( S_{CC} \). We denote by \( V(H) \) and \( E(H) \) the collective set of all nodes and edges of the diagrams in \( H \). The algorithm operates recursively with a single base case – if all the CCs in \( S_{CC} \) are disjoint, i.e., \( E(H) \) is empty (line 2), then it simply chooses \( k \) tuples that can contribute to each \( CC_j \in S_{CC} \) and completes their \( B_1, \ldots, B_q \) values given by \( CC_j \). When the base case is not met, for each \( H \in H \), the algorithm makes a recursive call on each child of the maximal element \( m \) in \( H \) (lines 9–11) to get the resulting view of the sub-diagram and then finds the remaining number of tuples that will get \( CC_m \) to its target count (lines 12–13). Finally, in the loop in line 15, the algorithm completes any missing values in the tuples while ensuring that these values do not add to the count of any \( CC \in S_{CC} \) by finding combinations that are not specified in \( S_{CC} \).

**Algorithm 2:** Complete \( V_{join} \) - Non-intersecting CCs

**Input:** \( V_{join} \) - View to complete, \( S_{CC} \) - Set of CCs, \( H \) - Set of diagrams encoding CC containment

**Output:** \( V_{join} \) - \( B_1, \ldots, B_q \) values filled-in

1. \( \forall i \in V(H), \sigma_i, k_i \leftarrow \) selection condition on \( R_1 \), count;
2. if \( E(H) = \emptyset \) then
3. for each \( i \in V(H) \) do
4. Find \( k_i \) tuples in \( V_{join} \) (without \( B_1, \ldots, B_q \) values) that satisfy \( \sigma_i \);
5. Assign \( B_1, \ldots, B_q \) values;
6. return \( V_{join} \);
7. for each \( H \in H \) do
8. \( m \leftarrow \) maximal elem. in \( H \);
9. for each \( c \in \text{children}(m) \) do
10. \( H_c \leftarrow \) sub-diagram with maximal elem. \( c \);
11. \( V_{join} = \text{Algorithm 2}(V_{join}, S_{CC}, (H_c)) \);
12. Find \( k_m = \sum_{c \in \text{children}(m)} k_c \) tuples in \( V_{join} \) that satisfy \( \sigma_m \wedge \neg \sigma_c \);
13. Assign \( B_1, \ldots, B_q \) values from \( CC_m \);
14. \( \text{combo-unused} \leftarrow \) list of combinations in \( R_2 \) columns that are not relevant to \( S_{CC} \);
15. for each \( t \in V_{join} \) do
16. if \( t.B_1, \ldots, B_q \) values are missing then
17. Assign a combination of values from \( \text{combo-unused} \);
18. return \( V_{join} \);

**Example 4.6.** Reconsider CCs 1–4 in Figure 6. The set \( H = \{H_1, H_2, H_3\} \), where \( H_1 \) and \( H_2 \) contain only \( CC_1 \) and \( CC_2 \), respectively, and \( H_3 \) is a diagram composed of one edge from \( CC_3 \) to \( CC_4 \). Algorithm 2 gets \( H \) along with \( V_{join} \) and \( S_{CC} \) as input. It assigns \( CC_i \)'s selection condition on \( R_1 \) and target count to \( \sigma_i \) and \( k_i \), for all \( i \). Then, it checks the condition in line 2, which does not hold as we have the edge \((CC_3, CC_4)\). Thus, it goes to the loop in line 7 to iterate over the three diagrams. For \( H_3 \), the maximal element is \( CC_3 \), so Algorithm 2 recursively calls itself for the sub-diagram containing only \( CC_4 \) (line 11) and finds 16 tuples such that \( Age \in [18,24] \) and \( Multi-ling = 0 \), and assigns to them \( Area = Chicago \) (lines 3–5). It then returns from the recursive call to find \( 100 - 16 = 84 \) tuples with \( Age \in \)
We add CCs with the following selection predicates: (1) $\text{Age} \in [10, 14]$ (Area $\in [50, 60]$ and Multi-ling $= 0$) and assign to them $\text{Area} = \text{Chicago}$ (Area $= \text{NYC}$). Here, $\text{combo}_\text{unused}$ contains values from Area’s domain except Chicago and NYC which get used in $S_{CC}$. If there are any tuples in $V_{\text{Join}}$ without an assignment (see loop on line 15), we assign to each a value chosen from $\text{combo}_\text{unused}$. At the end of the algorithm, any tuple in $V_{\text{Join}}$ without $B_1, \ldots, B_q$ values is randomly assigned a combination of values that is not used in $S_{CC}$ (line 17). We refer to these tuples as invalid tuples if no such combination is available. Observe that the matching tuples in $R_1$ do not have an FK to $K_2$ mapping, i.e., if there is a tuple in $V_{\text{Join}}$ that is missing an assignment, also called an invalid tuple, then $V_{\text{Join}}$ does not give a set of candidate $K_2$ values that could be assigned in its FK cell. We will handle such tuples in Section 5.2.

**Proposition 4.7.** If $S_{CC}$ does not contain intersecting CCs and there exists a join view $V_{\text{Join}}$ that satisfies all CCs in $S_{CC}$, then Algorithm 2 finds such a view.

**Complexity:** The complexity of Algorithm 2 is $O(|S_{CC}|^2 \cdot d_1 + |S_{CC}| \cdot (\max_i |\text{dom}_a(B_i)|)d_2 + |S_{CC}| \cdot |V_{\text{Join}}|)$, where $d_1, d_2$ are the number of columns in $V_{\text{Join}}$ and $R_2$, $\text{dom}_a(B_i)$ is the active domain of $R_2, B_i$. The first term is for computing the relationships between CCs and recursing on the Hasse diagrams (lines 7–11), second term is for constructing $\text{combo}_\text{unused}$ (line 14) and third term is for lines 1–6, 12–13 and choosing a random value per tuple in lines 15–17. In practice, we only consider columns used in $S_{CC}$ instead of $d_2$.

### 4.3 Hybrid Approach

In many cases, $S_{CC}$ contains a combination of disjoint, contained, and intersecting CCs, so we combine Algorithms 1 and 2.

We start by constructing Hasse diagram based on containment relationship between pairs of CCs in $S_{CC}$. Next, we iterate through each diagram $H \in \mathcal{H}$, and discard $H$ if it contains intersecting CCs. Note that the absence of an edge in the Hasse diagram does not guarantee the lack of intersection at the beginning of phase I (demonstrated by Example 4.5 where the Hasse diagram starts out as two nodes without an edge, but the CCs represented by these nodes do intersect). Therefore, we keep track of which CCs intersect to then discard the affected diagrams (set $S_2$) and run Algorithm 2 on the remaining diagrams (set $S_1$). In particular, $\forall C_{i} \in S_1, CC_i \in S_2, CC_i \cap CC_j = \emptyset, CC_i \not\subset CC_j$ and $CC_j \not\subset CC_i$. We then run Algorithm 2 for CCs in $S_1$, and Algorithm 1 for those in $S_2$.

As seen above, it is possible that some tuples may not have a $B_1, \ldots, B_q$ assignment in $V_{\text{Join}}$. Let $S_1$ be the set of these tuples that are dealt with using $\text{combo}_\text{unused}$ as described in Example 4.6. If $|\text{combo}_\text{unused}| = 0$, then all tuples in $S_1$ are invalid tuples.

**Augmenting with Modified Marginals** Our approach guarantees that the partial solution returned by Algorithm 2 satisfies $S_1$ exactly. In comparison to how we augment $S_{CC}$ with marginals in Section 4.1 before solving the ILP, we now want the scope of the marginals being added to be limited to the tuples that are relevant for the CCs in $S_1$, for example, let $S_{CC} = \{CC_1, CC_3\}$ from Figure 2b. We add CCs with the following selection predicates: (1) $\text{Age} \leq 10, 14$ and Multi-ling $= 0$, and assigns to them Area = Chicago (lines 12–13). For $H_1$ ($H_2$) the maximal element is $CC_1$ ($CC_2$), the algorithm then performs a recursive call to itself with an empty diagram, and returns from the call to select 20 (25) tuples that have $\text{Age} \in [10, 14]$ ($\text{Age} \in [50, 60]$ and Multi-ling $= 0$) and assign to them Area = Chicago (Area = NYC). Here, $\text{combo}_\text{unused}$ contains values from Area’s domain except Chicago and NYC which get used in $S_{CC}$. If there are any tuples in $V_{\text{Join}}$ without an assignment (see loop on line 15), we assign to each a value chosen from $\text{combo}_\text{unused}$. For example, let $S_{CC} = \{CC_1, CC_3\}$ from Figure 2b. We add CCs with the following selection predicates: (1) $\text{Age} \leq 10, 14$ and Multi-ling $= 0$, and assigns to them Area = Chicago (lines 12–13). For $H_1$ ($H_2$) the maximal element is $CC_1$ ($CC_2$), the algorithm then performs a recursive call to itself with an empty diagram, and returns from the call to select 20 (25) tuples that have $\text{Age} \in [10, 14]$ ($\text{Age} \in [50, 60]$ and Multi-ling $= 0$) and assign to them Area = Chicago (Area = NYC). Here, $\text{combo}_\text{unused}$ contains values from Area’s domain except Chicago and NYC which get used in $S_{CC}$. If there are any tuples in $V_{\text{Join}}$ without an assignment (see loop on line 15), we assign to each a value chosen from $\text{combo}_\text{unused}$. At the end of the algorithm, any tuple in $V_{\text{Join}}$ without $B_1, \ldots, B_q$ values is randomly assigned a combination of values that is not used in $S_{CC}$ (line 17). We refer to these tuples as invalid tuples if no such combination is available. Observe that the matching tuples in $R_1$ do not have an FK to $K_2$ mapping, i.e., if there is a tuple in $V_{\text{Join}}$ that is missing an assignment, also called an invalid tuple, then $V_{\text{Join}}$ does not give a set of candidate $K_2$ values that could be assigned in its FK cell. We will handle such tuples in Section 5.2.

**Proposition 4.7.** If $S_{CC}$ does not contain intersecting CCs and there exists a join view $V_{\text{Join}}$ that satisfies all CCs in $S_{CC}$, then Algorithm 2 finds such a view.

**Complexity:** The complexity of Algorithm 2 is $O(|S_{CC}|^2 \cdot d_1 + |S_{CC}| \cdot (\max_i |\text{dom}_a(B_i)|)d_2 + |S_{CC}| \cdot |V_{\text{Join}}|)$, where $d_1, d_2$ are the number of columns in $V_{\text{Join}}$ and $R_2$, $\text{dom}_a(B_i)$ is the active domain of $R_2, B_i$. The first term is for computing the relationships between CCs and recursing on the Hasse diagrams (lines 7–11), second term is for constructing $\text{combo}_\text{unused}$ (line 14) and third term is for lines 1–6, 12–13 and choosing a random value per tuple in lines 15–17. In practice, we only consider columns used in $S_{CC}$ instead of $d_2$.

5 SECOND PHASE: ADDING DCS

We start by presenting our model for conflict hypergraph for FK DCs and then use it to describe the solution for DCs. In short, our approach is to reverse-engineer $R_1$ from $V_{\text{Join}}$ so that joining it with $R_2$ recovers $V_{\text{Join}}$, and $R_1$ satisfies all DCs in $S_{DC}$.

### 5.1 Conflict Hypergraphs and List Coloring

We slightly augment the notion of conflict hypergraphs to illustrate possible foreign Key DC violations caused by subsets of $R_1$ tuples.

**Definition 5.1 (Conflict Hypergraph for Foreign Key DCs).** A conflict hypergraph for $R_1$ and $S_{DC}$ is defined as $G = (V, E)$ where $V$ is the set of tuples in $R_1$ and $\epsilon = \{t_1, \ldots, t_k\} \in E$ if there is a foreign Key DC of the form $\lnot(\varphi(t_1, \ldots, t_k) \land t_1.\text{FK} = \ldots = t_k.\text{FK})$ such that $\varphi(t_1, \ldots, t_k)$ evaluates to True.

It suffices to consider only $\varphi(t_1, \ldots, t_k)$ in the DCs when adding edges because FK is initially missing. Abusing notation, we denote a set of tuples $T$ violating $\varphi(t_1, \ldots, t_k)$ in a given DC $\sigma_1$ by $T \varphi_{\sigma_1}$ $\sigma_1$ (we will use this notation in Algorithm 4).

Next, we give the connection between conflict hypergraph coloring and FK assignment in $R_1$, so a proper coloring satisfies DCs.

**Proposition 5.2.** Given an instance of C-Extension, a coloring of the conflict hypergraph gives an assignment to all cells of the missing FK column in $R_1$ such that all DCs are satisfied.

We now turn to the problem of inferring FK values in $R_1$ from the completed $V_{\text{Join}}$. Each tuple in $R_1$ can have multiple options for foreign key values that lead to $V_{\text{Join}}$ obtained in phase I. This establishes a list of possible colors, also referred to as candidate colors, for each vertex in the conflict graph. This problem is called List Coloring [2, 25]. It is a generalization of $k$-coloring, and is thus NP-hard. Hence, we use a heuristic approach, described by Algorithm 3, to color the vertices in a non-increasing order by degree, coloring as many vertices as possible. In Section 5.2, we describe how to color the vertices that remain uncolored by Algorithm 3.

Algorithm 3 takes as input the conflict hypergraph $G_c$, a mapping $c$ from vertices to colors (initially empty) and a list of candidate colors $L$. It can be called on a graph with a partial color assignment (used in Algorithm 4 in Section 5.2). Initially, $s$ is an empty list...
that is used to store skipped vertices, and \( l \) is the list of uncolored vertices sorted in non-increasing order by degree in \( G_C \) (lines 2–3). In lines 4–12, we find a list of permissible colors for each \( v \in l \), i.e., those vertices in \( G_C \) that have not been given a color in the input color map \( c \). If vertex \( v \) belongs to an edge \( e \) where all vertices other than \( v \) in \( e \) have the same color \( c \), then \( c \) is a forbidden color for \( v \). Next, the algorithm assigns the “smallest” available color to \( v \) in line 10. Otherwise, \( v \) gets added to \( s \) and remains uncolored (line 12). Finally, color map \( c \) and list of skipped vertices \( s \) are returned.

**Algorithm 3**: Largest-first list coloring

**Input**: \( G_C \) - conflict hypergraph with color choices per vertex, \( c \) - a map from vertices to colors so far, \( L \) - list of candidate colors

**Output**: \( c \) - updated coloring that builds on the input coloring, \( s \) - list of skipped vertices

```plaintext
1 Function ColoringLF(G_c, c, L):
2     s ← ∅;
3     l ← sortDescendingDeg([v ∈ V[G_c] | v /∈ c]);
4     for v ∈ l do
5         forbidden ← ∅;
6         for e ∈ e do
7             if ∃uv ∈ v ∈ e. c[u] = c then
8                 forbidden.add(c);
9             if L ∖ forbidden ≠ ∅ then
10                c[v] ← min(L ∖ forbidden);
11             else
12                s ← s ∪ {v};
13     return c, s
```

**Example 5.3.** Reconsider the view \( V_{join} \) and DCs shown in Figures 5 and 2a, respectively. Figure 7 (including the dashed edges) gives the resulting conflict graph \( G_C \). For example, there is an edge between vertices 1 and 2 because \( t_1, \text{Rel}=12, \text{Rel} = \text{Owner} \), so assigning them the same FK value would violate DC \( D_{O,O} \). Here, \( l = \{2, 1, 3, 4, 8, 9, 5, 6, 7\} \).

Thus, Algorithm 3 returns: \( c[1] = 2, c[2] = 1, c[3] = 3, c[4] = 4, c[5] = 3, c[6] = 2, c[7] = 2, c[8] = 5 \) and \( c[9] = 6 \).

**Complexity**: The complexity of Algorithm 3 is \( O(|V| \cdot \log |V| + |V| \cdot |E|) \) since, the algorithm sorts all vertices by degree (line 3) and then traverses all edges adjacent to each vertex.

5.2 Algorithm for DCs

We describe Algorithm 4 as the last step in solving C-Extension by completing \( R_1, FK \). In Section 4, we showed how to complete \( V_{join} \) by assigning values in the columns that came from \( R_2 \). For a tuple \( t \in V_{join} \) with values \( t.B_i = b_i, 1 ≤ i ≤ q \), the candidate FK values for the corresponding tuple in \( R_1 \) are given by \( \pi_{R_1}(\sigma_{R_2=b_i} R_2) \). We begin with an optimization that we employ in the algorithm.

**Optimization**: Working with a single conflict hypergraph when \( R_1 \) contains a large number of tuples would not scale since the hypergraph can form one clique in the worst-case. However, observe that we can partition the filled-in \( V_{join} \) and \( R_2 \) by \( B_1, \ldots, B_q \) values into sets, and only consider conflict hypergraphs within each set because the candidate FK values are disjoint across sets.

**Example 5.4.** Reconsider relations \( R_1 \) and \( R_2 \), and view \( V_{join} \) shown in Figures 1 and 5, respectively. In \( V_{join} \), the tuples have \( \text{Area} = \text{Chicago or Area} = \text{NYC} \). Note that the set of candidate keys for tuples with \( \text{Area} = \text{Chicago} \) comprises of values in \( \pi_{R_1}(\sigma_{\text{Area}=\text{Chicago}} R_2) \) that is disjoint from those in \( \pi_{R_1}(\sigma_{\text{Area}=\text{NYC}} R_2) \). This eliminates edges that would have been added to the conflict graph if we were to consider all vertices at once (shown as dashed edges in Figure 7). After partitioning \( V_{join} \) by \( B_1, \ldots, B_q \) values and using the DCs in Figure 2a, we get two conflict graphs: (1) with vertices for tuples \( t_1, \ldots, t_7 \), and (2) with vertices for tuples \( t_8 \) and \( t_9 \). There is an edge between a pair of vertices when the corresponding tuples would violate a DC if assigned the same \( h_{id} \) value, and these are shown as solid edges in Figure 7.

**Algorithm 4**: Complete \( R_1, FK \) column using \( V_{join} \)

**Input**: View \( V_{join}(K_1, A_1, \ldots, A_p, B_1, \ldots, B_q) \)

Relations \( R_1(K_1, A_1, \ldots, A_p, FK) \) and \( R_2(K_2, B_1, \ldots, B_q) \), \( S_{DC} \) - set of DCs on \( R_1 \)

**Output**: \( R_1 \) - copy of \( R_1 \) with FK column filled-in, \( R_2 \) - updated copy of \( R_2(K_2, B_1, \ldots, B_q) \)

```plaintext
1 c_all ← ∅, R_1 ← copy of R_1, R_2 ← copy of R_2;
2 for v = (b_1, ..., b_q) ∈ π_{B_1,...,B_q} V_{join} do
3     P_0 = {t ∈ V_{join} | 1 ≤ i ≤ q. t.B_i = b_i};
4     V ← 0, E ← ∅, c ← ∅;
5     L = π_{K_1,σ_{B_i=b_i} R_2} R_2; 
6     for t_j ∈ P_0 do
7         V ← V ∪ {t_j};
8     for T ⊆ P_0 s.t. ∃σ ∈ S_{DC}. T ⊆ T \ σ do
9         E ← E ∪ {T};
10        c, s ← ColoringLF(G_c = (V, E), c, L);
11        c_new ← |s| number of new colors;
12        c, s ← ColoringLF(G_c = (V, E), c, L_new);
13        for color c_new in L_new that gets used do
14            Add tuple t_new in R_2 with t_new.K_2 = c_new and t_new.B_i = b_i (1 ≤ i ≤ q);
15            c_all ← c_all ∪ c;
16        call solveInvalidTuples(V_{join}, S_{DC}, S_{DC}, R_2);
17        ∀t_j ∈ V_{join}, t_j′ ∈ R_1. t_j.K_1 = t_j′.K_1 set t_j′.FK = c_all[{t_j}];
18        return R_1, R_2;
```

Algorithm 4 gets as input the view \( V_{join} \) outputted by the algorithm described in Section 4.3, relations \( R_1 \) (with missing FK values) and \( R_2 \), and set \( S_{DC} \). It outputs \( R_1, i.e., R_1 \) with values in the FK column and \( R_2 \) i.e., \( R_2 \) with possible additional tuples (as described next). The algorithm can be divided into three parts: (1) coloring the tuples that were assigned \( B_1, \ldots, B_q \) values in \( V_{join} \), (2) coloring the invalid tuples, i.e., tuples that were not assigned \( B_1, \ldots, B_q \) values in \( V_{join} \), and (3) coloring any skipped tuples (defined in Section 5.1). Algorithm 4 maintains a map from tuples to their list of colors in \( c \) and tracks the overall coloring in \( c_all \). Eventually, \( c_all \) has a color for every vertex that is used to complete FK in \( R_1 \) (lines 17, 18).
In lines 2–15, the algorithm iterates over each set of tuples with the same \(B_1, \ldots, B_q\) value. Given a vector \(v = (b_1, \ldots, b_q)\) of \(q\) constants, it iterates over tuples in sets given by \(P_r = \{ t \in V_{\text{join}} \mid \forall 1 \leq i \leq q \text{ such that } b_i = v_i \}\). For each \(P_r\), it generates the conflict hypergraph \(G_r\) as follows: a node \(v_j\) per tuple \(t_j\), a list \(L\) of candidate colors given by the keys from tuples in \(P_r\) with values \(v_j \cdot B_1, \ldots, v_j \cdot B_q\), and an edge per set of tuples in \(P_r\) that violates \(\varphi\) in some DC. Next, \(G_r\), \(c\) and \(L\) are inputted to Algorithm 3, which outputs a partial coloring \(c\) and a list of skipped vertices \(s\). Vertices in \(s\) are colored using at most \(|s|\) new colors (lines 11-14), resulting in insertion of tuples in \(P_r\) because colors correspond to primary keys in \(P_r\).

Procedure \text{solveInvalidTuples} (line 16) handles the invalid tuples (defined in Section 4.2). Since these do not have \(B_1, \ldots, B_q\) values in \(V_{\text{join}}\), the corresponding \(P_r\) tuples are missing FK values because we have not yet considered them in any conflict hypergraphs. We construct a hypergraph for tuples in \(V_{\text{join}}\) with edges directed to only the vertices for invalid tuples. However, the set \(S\) outputted by Algorithm 3 may contain invalid tuples that had to be skipped for a lack of available colors. Our strategy for coloring this is to assign to each a combination of \(B_1, \ldots, B_q\) values that minimizes the error stemming from the CCs (defined in Section 6), and generate a tuple in \(P_r\) with a fresh key and the chosen \(B_1, \ldots, B_q\) values. Finally, \(R_1.FK\) values are assigned based on the coloring \(c_{\text{all}}\) (line 17).

**Proposition 5.5.** Given \(V_{\text{join}}\), \(R_1, R_2, S_{\text{DC}}\), Algorithm 4 outputs relations \(\hat{R}_1, \hat{R}_2\) such that \(\hat{R}_2\) is a copy of \(R_2\), possibly with more tuples, and \(\hat{R}_1\) is a copy of \(R_1\) with all the values in the FK column completed such that \(\forall \sigma \in S_{\text{DC}}, R_1 \equiv \sigma\), and \(R_1 \equiv_{\text{FK} = \sigma} \hat{R}_1 \equiv_{\text{FK} = \sigma} \hat{R}_2 \equiv V_{\text{join}}\).

**Complexity:** The complexity of Algorithm 4 is \(O(n \cdot |S_{\text{DC}}| \cdot \binom{|T|}{2})\), where \(|R_1| = n\), and \(T\) is the number of tuples involved in the largest DC (assumed to be a constant), since the number of edges of each vertex can be at most \(\binom{|T|}{2}\). The \(n\) component stands for the possible need to iterate over all tuples in \(V_{\text{join}}\) in line 2, the \(|S_{\text{DC}}|\) component stands for the possible need to iterate over all DCs when checking the condition in line 8, and the \(\binom{|T|}{2}\) component is added due to the need to iterate over all subsets of \(P_r\) that may satisfy a DC in lines 8–9. Since Algorithm 3 (lines 10, 12) has a complexity of \(O(n \cdot \binom{|T|}{2})\), and the loop (line 13) has complexity of \(O(n)\), they are not presented in the overall complexity of the algorithm. Note that \(\bigcup_{j=1}^{m} P_j = V_{\text{join}}\), where \(m\) is the number of iterations and \(P_j\) is the set \(P_j\) generated in iteration \(j\).

**Extending the solution to snowflake schemas:** Our solution can be generalized to snowflake schemas in a manner similar to [5]. The idea is to start from the fact table (the central table) as \(R_1\) and a table connected to it as \(R_2\), i.e. going from the inside out in a Breadth-First Search manner. In every step, we include the previously completed tables in \(R_1\), allowing CCs that span over the join view of multiple tables. This ensures that tuples are (possibly) added to the relation in the role of \(R_2\) only once, since in the next step it would be considered as \(R_1\) and thus maintain the foreign key dependency from the previous steps.

**Example 5.6.** Consider a central Students table with two foreign key dependencies of a Majors table and a Courses table, and another foreign key connection to a Departments table through Majors:

The steps of the algorithm are as follows:

| Step | \(R_1\) | \(R_2\) |
|------|--------|--------|
| 1    | Students | Majors |
| 2    | Students \(\bowtie\) Majors | Courses |
| 3    | Students \(\bowtie\) Majors \(\bowtie\) Courses | Departments |

At each step we can therefore consider CCs over all tables we have considered so far. For example, in step 2, we can consider CCs over ((Students \(\bowtie\) Majors) \(\bowtie\) Courses) and not just over Students \(\bowtie\) Courses. Note that in the first step, we might have added artificial tuples to the Majors table. These are added without an FK value that connects them to the Departments table so we account for them in the last step of connecting the Majors and Departments tables, making sure that the DCs that apply to the Majors table are satisfied.

**6 EXPERIMENTS**

We analyze the performance of our (hybrid) approach, and compare it with a baseline algorithm (based on [5]) in these terms:

1. Accuracy and runtime comparison between the baseline and our approach as data grows for fixed \(S_{\text{DC}}\) and two settings of \(S_{\text{CC}}\) (based on Section 4.2): (i) \(S_{\text{CC}}\) with no intersecting CCs, and (ii) \(S_{\text{CC}}\) with intersecting CCs.
2. Accuracy and runtime comparison between the baseline and our approach for fixed data and combinations of good and bad \(S_{\text{DC}}\) and \(S_{\text{CC}}\). Good \(S_{\text{DC}}\) creates zero cliques in conflict graphs and good \(S_{\text{CC}}\) contains zero intersecting CCs.
3. Runtime performance of our approach when data and \(S_{\text{DC}}\) are kept fixed but the size of good \(S_{\text{CC}}\) and bad \(S_{\text{CC}}\) varies.
4. Runtime performance of our approach for fixed data, and good \(S_{\text{DC}}\) and \(S_{\text{CC}}\) as the number of columns in \(R_2\) grows.

We implemented our solution and baseline in Python 3.6.9 and Pandas DataFrame interface [38] on Tensor TXR231-1000R D126 Intel(R) Xeon(R) CPU E5-2640 v4 2.40GHz CPU with 512 GB (40 cores) of RAM. We use the standard PuLP [37] and NumPy Libraries for the ILP, and NetworkX [6] to construct and color conflict graphs.

**A summary of our findings:**

1. Our approach satisfies all CCs in the absence of intersecting CCs with no error. Additionally, our approach satisfies all DCs (as guaranteed by our theoretical analysis), whereas the baseline does not (Figures 8–10). Overall, our approach has the shortest runtime (Figure 11a) and achieves better accuracy for CCs and DCs together. Additionally, augmenting the input set of CCs with marginals over the non-key attributes in \(R_1\) improves accuracy for CCs. We also find that the time spent by the baseline on the ILP solver alone is comparable to the total time taken by our approach for larger data scales.

2. At a fixed data scale and for good and bad settings of DCs and CCs, where good DCs do not create cliques in conflict graphs and good CCs do not intersect, our approach has the shortest runtime. Its best performance is for good DCs and good CCs. In comparison, using bad DCs is slower because conflict graphs become denser, and using bad CCs is even slower because of the ILP solver.
We now describe the experimental setup (summarized in Table 3) and define the error measures that are used to evaluate accuracy. We vary the database size (Table 1), DCs and CCs (Sec. A.2) to examine the scalability and accuracy of our solution. In Section 6.2, the errors and runtimes are averaged over 3 independent runs.

**Data.** We perform experiments on a dataset that is derived from the 2010 U.S. Decennial Census [44] comprising of two relations *Persons* *(pid, Rel, Age, Multi-ling, hid)* and *Housing* *(hid, Tenure, Area)*, with *Persons* *(R₁)* missing all values in its foreign key column *hid*. The different data scales are given in Table 1. By construction, both *Persons* and *Housing* contain the same number of tuples. We also consider up to 10 (non-key) columns in *Housing*, where we go from *(Tenure, Area)* to *(Tenure, County, Area, St)*, add *(Div, Reg)* and then add binary attributes *(Water, Bath)* followed by *(Fridge, Stove)*. Note that values in *Div* and *Reg* are determined by the *St* value.

**Denial Constraints.** *S* is the set of DCs (Sec. A.2) that not only gives the permissible age gap between a homeowner *(Rel = Owner)* and other members in the same home, but also limits the number of homeowners, spouses and unmarried partners per home. *S* contains first 8 DCs, none of which create cliques in conflict graphs.

**Cardinality Constraints.** We use two sets of CCs, *S* with 1001 CCs each (Sec. A.2). We assume that each input CC specifies a condition on an attribute from both *R₁* and *R₂*, *S* contains CCs with intersecting *Age* intervals, but *S* does not.

**Error Measures.** We measure *relative CC error* as \[ \frac{c_{\text{err}}}{c_{\text{act}}} \] where \( c_{\text{err}} \) and \( c_{\text{act}} \) are CC’s (in *S*) counts in the solution and input. We use a threshold of 10 in the denominator because some CCs have a target count of 0 for small data scales. We report the median relative CC errors in Figures 8-10, where \( y = 1 \) represents 100% error. We measure *DC error* as the fraction of tuples in *R₁* that violate a DC in *S* which, e.g., if *hid* value in the first two tuples in *Persons* relation in Figure 3 was 2, then the DC error would be 2/9.

**Baseline.** Arasu et al. [5] focuses on the generation of synthetic databases with snowflake schema, where all joins are foreign key joins. This work considers CCs alone (no DCs) and imputes FK using *Vjoin*. Motivated by this work, we establish the two baseline versions given below (Section 7 surveys more related works).

1. **Baseline:** First, we use Algorithm 1 (without the for loop on line 8) to fill-in tuples in *Vjoin*. Any *Vjoin* tuple without an assignment is completed by randomly assigning values in \( B₁, \ldots, B_q \). In phase II, we randomly assign a value from the candidate FK values given by *Vjoin* for each tuple in *R₁*.

2. **Baseline with marginals:** We also study the impact of augmenting *S* with all *Age-Rel-Multi-ling* (all-way) marginals from *Persons*, where domains of numerical attributes are broken using intervalization [5] on *S*. Note that the marginals have equal target counts in *Persons* and *Vjoin* by construction. They ensure that each variable participates in the ILP, and is thus assigned a value in the solution. We find this fills in all *Vjoin* tuples. We refer to this algorithm as baseline with marginals that uses Algorithm 1 for phase I, followed by random assignment in FK using *Vjoin* for phase II. Hence, it falls in-between the baseline and our approach (Section 4.3).

### Table 1: Data scales given by the number of tuples

| Scale | Persons table | Housing table | Vjoin |
|-------|---------------|---------------|-------|
| 1x    | 25,099        | 9,820         | 25,099|
| 2x    | 50,039        | 19,640        | 50,039|
| 5x    | 124,746       | 49,100        | 124,746|
| 10x   | 249,259       | 98,200        | 249,259|
| 40x   | 1,015,686     | 392,800       | 1,015,686|
| 80x   | 2,043,975     | 785,600       | 2,043,975|
| 120x  | 3,064,328     | 1,178,400     | 3,064,328|
| 160x  | 4,097,471     | 1,571,200     | 4,097,471|

### Table 2: Datasets used in experiments, with details about the data scales, DCs and CCs given in Table 1 and Section A.2

| Dataset no. | Data Scale | DCs | CCs |
|-------------|------------|-----|-----|
| 1-5         | 1x to 40x  | *S*  | *S*  |
| 6-10        | 1x to 40x  | *S*  | *S*  |
| 11          | 10x        | *S*  | *S*  |
| 12          | 10x        | *S*  | *S*  |
| 13 - 17     | 10x        | *S*  | *S*  |
| 18 - 22     | 10x        | *S*  | *S*  |
| 23 - 26     | 10x        | *S*  | *S*  |
| 27 - 30     | 10x        | *S*  | *S*  |
| 31 - 34     | 10x (4, 6.8.10 non-key *R₁* columns) | *S* | *S* |

### Table 3: Experimental settings for Figures 8-13. Table 2 contains details about the input datasets

| Experiment | Figure | Algorithm | Input datasets |
|------------|--------|-----------|----------------|
| Accuracy Exp. | 8a Baselines vs Hybrid | 1-5 |
| 8b Baselines vs Hybrid | 6-10 |
| 9 Baselines vs Hybrid | 10 |
| 10 Baselines vs Hybrid | 11, 12, 4, 9 |
| Scalability Exp. | 11a Baselines vs Hybrid | 9, 10 |
| 11b Hybrid | 11, 12, 24, 26, 27, 30 |
| 12 Hybrid | 11, 31, 34 |
| 13 Hybrid | 17, 22 |

### 6.2 Experimental Findings

We discuss results and address aspects raised at the start of Section 6. **Our approach vs baselines - Accuracy.** We consider the experimental setup from Table 3 for the accuracy experiments, and detail our results in Figures 8-10. Our approach always satisfies all DCs, and all CCs in *S*. For *S* (Table 8b), the median CC error is 0 but the smallest and largest average errors are 0.048 and 0.093 due to limitations in augmenting *S* (Section 4.3). In contrast, the baseline gives median CC and DC errors between 0.233-0.580 and 0.228-0.373, whereas baseline with marginals satisfies all CCs but gives DC errors between 0.402-0.510. We take a closer look at the relative CC errors for data Scale 40x and *S* in Figure 9. Note that DCs are used only after *Vjoin* is partitioned by *B₁, ..., B_q* values, so CCs affect the quality of the solution given by Algorithm 4.

Next, we look at combinations of good and bad cases of DCs and CCs for data at Scale 10x. Again, our approach satisfies all DCs and gives a median CC error of 0. Here, half the CCs were passed into Algorithm 2 that satisfies CCs exactly. The remaining CCs are augmented (Section 4.3) to improve accuracy for Algorithm 1.
Figure 8: Error rate comparison between the baseline, baseline with marginals and hybrid as data grows from Scale $1\times$ to $40\times$ and: (a) $\text{all}_{\text{DC}}$ (12 DCs) and $\text{good}_{\text{CC}}$ (1001 CCs) are used, and (b) $\text{all}_{\text{DC}}$ (12 DCs) and $\text{bad}_{\text{CC}}$ (1001 CCs) are used.

Find that most CCs have a relative error of 0. For $\text{bad}_{\text{CC}}$, the average CC error given by our approach is 0.0735. In contrast, baseline gives CC errors between 0.537-0.618 and DC errors between 0.079-0.305, whereas baseline with marginals satisfies all CCs but gives DC errors between 0.004-0.510 due to random assignment in $R_1.FK$.

Our approach vs baselines - Runtime. We consider the experimental setup as given in Table 3 for the scalability experiments. The total runtimes at data Scales $10\times$ and $40\times$ are given in Figure 11a. Observe that the time spent on phase II by the baseline is minimal because it randomly assigns $FK$ values, whereas our approach colors conflict graphs to satisfy all DC exactly. In our approach, Algorithm 1 and 4 are the bottlenecks taking 25.23% and 72.74% of the total runtime for $\text{all}_{\text{DC}}$ and $\text{bad}_{\text{CC}}$ at data Scale $40\times$.

At data Scale $40\times$, the runtimes for completing $V_{\text{join}}$ for $\text{good}_{\text{CC}}$ and $\text{bad}_{\text{CC}}$ are as follows: (1) baseline takes 5.88 hours and 6.07 hours, (2) baseline with marginals takes 9.75 hours and 10.15 hours, and (3) our approach takes 7.79 minutes and 1.48 hours. The corresponding total runtimes are: (1) 6.19 hours and 6.38 hours, (2) 10.04 hours and 10.49 hours, and (3) 1.06 hours and 5.43 hours hours. Our approach has the shortest runtime in completing $V_{\text{join}}$, because we take advantage of the relationships between the CCs to separate out intersecting CCs from $S_{\text{CC}}$ which reduces the time to solve the ILP. In contrast, the baseline creates one large ILP with all CCs (with or without all-way marginals). In addition, our approach does not need the ILP solver for $\text{good}_{\text{CC}}$, further improving the runtime.

Note that the baselines do not take into account the DCs and randomly assign $FK$ values based on filled-in $V_{\text{join}}$, so solving the ILP dominates the time to populate $b_{\text{id}}$ in Persons ($R_1$). Our approach has the shortest runtime for $S_{\text{DC}}^{\text{good}_{\text{CC}}}$ (at 5.17 minutes), followed by $S_{\text{DC}}^{\text{bad}_{\text{CC}}}$, $S_{\text{DC}}^{\text{good}_{\text{CC}}}$, and $S_{\text{DC}}^{\text{bad}_{\text{CC}}}$ (at 1.36 hours). Intuitively, for fixed data, completing $V_{\text{join}}$ is faster for $\text{good}_{\text{CC}}$ and conflict graphs are more likely to have fewer edges for $\text{good}_{\text{CC}}$. For $S_{\text{DC}}^{\text{good}_{\text{CC}}}$ and $S_{\text{DC}}^{\text{bad}_{\text{CC}}}$: (1) baseline took 4.84-5.14 hours, and (2) baseline with marginals took close to 8 hours. Baseline ran faster because it runs the ILP solver without the marginals.

Larger data scales - Runtime. We examine our solution’s runtime for larger data scales where $\text{good}_{\text{CC}}$ is used with $S_{\text{DC}}^{\text{good}_{\text{CC}}}$ vs $S_{\text{DC}}^{\text{bad}_{\text{CC}}}$ (see Figure 11b). We find that our solution scales well, taking a total of 9.3 hours for $\text{good}_{\text{CC}}$ and 10.46 hours for $\text{bad}_{\text{CC}}$ at data Scale $160\times$.
2 to 10. Our approach takes a total of 5.17 minutes for 2 columns and 38.66 minutes for 10 columns because we only consider the columns that are used in $S_{CC}$ for $\text{combo}\_\text{unused}$ (Algorithm 2).

**Increasing the number of CCs - Runtime and accuracy.** Here we study the effect of the size of $S_{CC}$ on the runtime and error in our approach (the last row in Table 3). The breakdown of runtimes for 900 CCs chosen from $S_{CC}^{good}$ and $S_{CC}^{bad}$ is given in Figure 13.

![Figure 13: Runtime breakdown of the hybrid approach for data Scale 10x with $S_{DC}^{all}$ (12 DCs) and 900 CCs from $S_{CC}^{good}$ or $S_{CC}^{bad}$ (overall we have 1001 CCs for both)](image)

As more CCs are used, the time spent on labeling pairs of CCs as disjoint, contained or intersecting increases. Since $S_{CC}^{good}$ contains no intersecting CCs, the ILP solver is not used and the runtime is faster. Algorithm 2 takes 1.42 minutes for 500 CCs and 1.78 minutes for 900 CCs. More CCs not only cause more updates to $V_{\text{join}}$, but may also add CCs where Area is used without Tenure, creating tuples with a partial assignment in $V_{\text{join}}$ that are completed in line 17 of the algorithm. For 900 CCs, the total runtime is 6.65 minutes, of which 4.87 minutes are spent in filling-in $R_1$ (Algorithm 4).

When more CCs are chosen from $S_{CC}^{bad}$, we see an increase in the number of CCs passed to Algorithm 1 that makes the ILP solver slower. Algorithm 2 takes 1.21 minutes for 500 CCs and 1.36 minutes for 900 CCs, whereas Algorithm 1 takes 25.99 minutes for 500 CCs and 1.06 hours for 900 CCs. For 900 CCs, the total runtime is 1.23 hours, of which 8.77 minutes are spent in completing $R_1$.

Our approach satisfies all DCs, and CCs in $S_{CC}^{good}$. The median and average CC error rates are 0 and 0.034-0.092, resp. We propose an optimization for the (bottleneck) coloring step in Section A.3.

7 RELATED WORK

Data generation has been the focus of multiple works, e.g., [4, 5, 9, 11, 12, 19, 21, 24, 31, 34, 41, 43, 45, 50]. The main novelty of this paper is the generation of foreign keys for existing database relations while reducing the error of a set of CCs and ensuring the satisfaction of a set of DCs that relate to the foreign key attribute.

A prominent line of work uses CCs to define the desired parameters of the generated data [5, 9, 43]. QAGen [9] was among the first system that focused on data generation in a query-aware fashion. The target application was to test the performance of a database management system (DBMS) when given a database schema, one parametric Conjunctive Query and a collection of constraints on each operator. MyBenchmark [31] extends [9] by generating a set of database instances that approximately satisfies the cardinality expectations from a set of query results. HYDRA [43] uses a declarative approach that allows for the generation of a database summary that can be used for dynamically generating data for query execution. Arasu et. al. [5] proposed a framework that supports multiple CCs and generates data using a graphical model that converts the CCs to equations, using the concept of intervalization for efficient computations. Indeed, we have drawn on this work for Algorithm 1. These approaches allow for complex CCs, whereas our approach allows for DCs as well. A recent work [47] has proposed a solution for generating multiple data samples using a seed sample of the data (generated by previous work [48]), statistical constraints and data validity constraints specified in OCL [1]. UpSizeR [50] has focused on scaling the database while maintaining foreign key constraints. Data generation from the database schema and statistical information has also been studied [41, 45].

The field of data privacy [17, 27, 29, 33, 35, 49, 53] typically gives mechanisms that generate query answers that do not expose features of the underlying private data, rather than generate the data itself. Some works [22, 55] focus on providing consistent query answers, but none, to our knowledge, consider queries over linked data that guarantee the satisfaction of a set of ICs. Yahalom et. al. [54] developed a framework for converting production data into test data by modeling it as a constrained satisfaction problem (CSP) using specific constraints that can be expressed as part of the CSP.

Finally, DCs (without CCs) have been mainly explored in relation with data cleaning [3, 8, 14, 16, 18, 20, 26, 42]. Previous work on the subject has focused on two main approaches: (1) repairing attribute values in cells [8, 16, 42] and (2) tuple deletion [14, 20, 32]. In this context, there has been previous work on automatically discovering DCs from the complete data [15, 30, 39]. We consider DCs based on the FK column, which is missing. In many scenarios, as is the premise in many data cleaning works (e.g., [14, 16, 20, 42]), such DCs can be naturally inferred from the schema or from domain knowledge. As in data cleaning, the constraints can be formulated by the users as logical statements [42] or as SQL queries [20].

8 CONCLUSIONS AND LIMITATIONS

We have defined the problem of generating links between database relations using linear CCs and foreign key DCs, and proved that it is intractable. Therefore, we have shown a novel two-phase heuristic solution. Our solution first considers the CCs, with a hybrid approach that combines an ILP-based solution and a solution based on specific relationships between the CCs. Second, our approach utilizes a version of conflict graph coloring in order to find a completion of the tuples that satisfies all DCs. Our experimental results show that our solution is both accurate and scalable.

There are many intriguing directions for future work. First, our solution focuses on linear CCs and a subset of DCs. Finding a solution when the constraints include non-linear CCs (e.g., CCs on the number of rows that share the same foreign key) and general DCs (e.g., DCs on tuples that do not share a foreign key) is an important extension of our approach. Second, in phase I, we assume foreign key dependence that induces a one-to-one mapping between the tuples of $R_1$ and the tuples of $V_{\text{join}}$. Examining other join dependencies that do not have this property is an interesting direction of exploration. Third, in phase II, tuples may be artificially added to $R_2$ due to the coloring algorithm. Some scenarios may not allow such augmentation and thus require different solutions. Finally, the extension of our solution to non-relational databases, such as graph databases and wide-column stores is another subject of future study.

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A APPENDIX

We next give more details about the paper, including the full proofs of all propositions, details about our experimental settings and an outline for an optimization of the coloring process.

A.1 Proofs

We next detail the full proofs for all of the propositions in the paper.

Proof of Proposition 2.8. We give a reduction from NAE-3SAT\(^1\) to C-Extension. In the NAE-3SAT problem, we are given a 3-CNF formula \(\varphi\) and asked whether there is a satisfying assignment to \(\varphi\) with every clause having at least one literal with the value False. Given a 3-CNF formula \(\varphi = C_1 \land \ldots \land C_n\), where \(x_1, \ldots, x_m\) are the propositional variables in \(\varphi\), construct a relation \(R(V, \alpha, \text{Cls}, \text{Chosen})\), where \(\text{Chosen}\) is missing all values, and \(\text{Var}, \alpha, \text{Cls}\) attributes take the following values:

1. \((x_i, 1, C_j, ?)\) if making \(x_i\) True makes \(C_j\) True
2. \((x_i, 0, C_j, ?)\) if making \(x_i\) False makes \(C_j\) True

We define \(S_{DC}\) to be the set with the following two DCs:

1. \(\forall t_1, t_2. \neg(t_1.\text{Var} = t_2.\text{Var} \land t_1.\alpha \neq t_2.\alpha \land t_1.\text{Chosen} = t_2.\text{Chosen})\)
2. \(\forall t_1, t_2, t_3. \neg(t_1.\text{Cls} = t_2.\text{Cls} = t_3.\text{Cls} \land t_1.\text{Chosen} = t_2.\text{Chosen} = t_3.\text{Chosen})\)

The goal is to complete the missing column \(\text{Chosen}\) in \(R_1\). CCs are not needed in the reduction. We define \(R_2\) as containing two attributes: a primary key column \(\text{Chosen}\), and another column \(E\). \(R_2\) contains the tuples \((0, a)\) and \((1, b)\), i.e., the domain for \(\text{Chosen}\) is \([0, 1]\). DC (1) makes sure that if a tuple of the form \((x_i, 1, C_j)\) is chosen for the assignment, then \((x_i, 0, C_j)\) cannot be chosen as well and versa, and DC (2) enforces that for each clause, at least one literal in that clause will have the value True and at least one literal will have value False. We now show that a satisfying assignment for \(\varphi\) exists iff there is a solution to C-Extension.

(\(\Rightarrow\)) Assume there is a satisfying assignment \(\alpha\) to \(\varphi\) such that each clause contains a literal assigned to False. For each variable \(x_i\) mapped to True, for tuples of the form \((x_i, 1, C_j, ?)\) and \((x_i, 0, C_j, ?)\) we fill-in \((x_i, 1, C_j, 1)\) and \((x_i, 0, C_j, 0)\), i.e., a tuple will be completed with a \(\text{Chosen}\) value of 1 iff \(\alpha\) value is 1. If \(x_i\) was mapped to False, we do the opposite. We need to show that DCs (1) and (2) are satisfied. For DC (1), since the procedure described above gives only one of the tuples of the form \((x_i, 0, C_j, ?)\) the value 1 and the other gets the value 0 according to \(\alpha\), this DC is satisfied. For DC (2), assume we have \((x_1, 1, C_j, ?), (x_2, 1, C_j, ?), (x_3, 0, C_j, ?)\) (w.l.o.g. it could also be a different combination). Then, \(\alpha\) cannot map the variables \(x_1, x_2, x_3\) to True and \(x_3\) to False. Hence, the \(\text{Chosen}\) attribute of at least one of the above tuples will be 0, satisfying DC (2). Thus, \(\alpha\) defines a solution to C-Extension.

(\(\Leftarrow\)) Assume there is a solution to C-Extension. Define the assignment \(\alpha\) as follows. if \((x_1, 1, C_j, 1)\) then \(\alpha(x_1) = \text{True}\) and otherwise, \(\alpha(x_1) = \text{False}\). We prove that \(\alpha\) is a proper assignment, i.e., each variable is mapped to either True or False and not both and we further prove that \(\alpha\) satisfies \(\varphi\) and each clause contains a literal mapped to False by \(\alpha\). First, since the solution satisfies DC (1), two tuples containing the same variable name \(x_i\) and different assignment values will be given different \(\text{Chosen}\) values, i.e., \((x_1, 1, C_j, 1)\) which makes \(x_1\) True and \((x_1, 0, C_j, 0)\) which makes \(x_1\) False, therefore, \(\alpha\) is a proper assignment. Second, since the solution satisfies DC (2), for every three tuples of the form \(t_1 = (x_1, 1, C_j, ?)\), \(t_2 = (x_2, 1, C_j, ?)\), \(t_3 = (x_3, 0, C_j, ?)\) (w.l.o.g.), it holds that at least one of the values \(t_1[\text{Chosen}], t_2[\text{Chosen}], t_3[\text{Chosen}]\) is different than the rest, i.e., there is at least one 1 value and one 0 value in the tuples for \(C_j\). The three tuples mentioned above correspond to the clause \(C_j = (x_1 \lor x_2 \lor \neg x_3)\) (recall that in our reduction the second attribute suggests the assignment of \(x_i\) that will satisfy the clause \(C_j\), so since \(x_1 = \text{True}\) will satisfy \(C_j\), we initially have \((x_i, 1, C_j, ?)\)). Assume w.l.o.g., that \(t_1.\text{Chosen} = 1\), and \(t_2.\text{Chosen} = t_3.\text{Chosen} = 0\), then \(C_j\) contains the literals \(x_1 = \text{True}, x_2 = \text{False}\), and \(x_3 = \text{True}\), thus having one literal mapped to True and one mapped to False. It is possible to verify that this mapping works for the other three combinations of literals as well ((\(x_1 \lor x_2 \lor x_3\), \(x_1 \lor \neg x_2 \lor \neg x_3\), \(\neg x_1 \lor \neg x_2 \lor x_3\)). This implies that \(\alpha\) both satisfies \(\varphi\) (since every clause contains at least one variable that satisfies it) and maps at least one variable in each clause to False.

\[\square\]

Proof of Proposition 4.7.

Lemma A.1. If every pair of CCs in \(S_{CC}\) is disjoint, Algorithm 2 fills-in \(V_{join}\) in polynomial time and \(V_{join} \models \sigma, \forall \sigma \in S_{CC}\).

Proof. Assume that every pair of CCs in \(S_{CC}\) is disjoint. If \(CC_l\) and \(CC_j\) are disjoint, then (by definition) either the selection conditions on \(R_1\) columns are disjoint or they are identical on \(R_1\) columns but disjoint on \(R_2\) columns. Let us examine a pair \(CC_l\cdot CC_j\) of CCs that is disjoint based on the first condition. Here, the set of \(V_{join}\) tuples that can contribute towards the target count of \(CC_l\) is disjoint from that of \(CC_j\). Thus, satisfying one cannot affect the availability of tuples for the other. Another possibility is that \(CC_l\) and \(CC_j\) are disjoint based on the second condition. Here, the set of \(V_{join}\) tuples that can contribute to both is the same but must contain at least as many tuples as the sum of the target counts for the two CCs since a \(V_{join}\) exists that satisfies \(S_{CC}\).

Therefore, given that \(S_{CC}\) contains only disjoint CCs, we go straight to line 2 and then to the loop on line 3, where we first check as many tuples in \(V_{join}\) as the target count for \(CC_l\) and then assign the corresponding values from the selection condition in \(CC_l\) to these tuples in line 5. Finally, if some tuples are missing some or all values from \(R_2\) (the \(B_1, \ldots, B_q\) values), combinations of values in \(R_2\) columns that are not relevant to \(S_{CC}\) are assigned to them in lines 15–17. These do not add to the counts of the CCs since they use fresh combinations that do not appear in the selection conditions of the CCs in \(S_{CC}\).

\[\square\]

We prove the proposition by induction on the size of the diagrams \(H \in \mathcal{H}, \; H = (V_H, E_H)\).

If \(|E_H| = 0\), the first case checked in Algorithm 2 applies and by Lemma A.1, we are done.

For the inductive hypothesis, assume the proposition holds true for all diagrams with size \(< n\). We prove it for a diagram with size \(n\). If the base case does not hold, the algorithm identifies the maximal element, \(CC_m\), of \(H\) in line 8. Then, for each one of \(CC_m\)’s children, \(c\), it solves the problem for the sub-diagram whose maximal element

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\(^1\)See https://en.wikipedia.org/wiki/Not-all-equal_3-satisfiability for details
is c. Based on the induction assumption, the completion of $V_{\text{join}}$ returned from this call satisfies all the CCs in these sub-diagrams (the children of $m$ are pairwise-disjoint). We first find the remaining number of tuples needed for $CC_m$ after the CCs of its children have been satisfied in line 12 and then in line 13 assign the $B_1, \ldots, B_q$ values given by $CC_m$. Finally, as shown in Lemma A.1, the values assigned to the tuples in lines 15–17 do not add to the count of any of the CCs. Thus, we find a solution that satisfies all of the CCs whose vertices are in $H$. \hfill \square

Proof of Proposition 5.2. Given relation $R_1$ with an empty FK column and the set $S_{DC}$ of FK DCs, consider the conflict hypergraph $G$. Suppose we have a proper coloring of $G$, and we assign each tuple $t \in R_1$, $t.FK = c(t)$.

We now show that this satisfies the DCs. Consider a DC $\sigma = \forall t_1, \ldots, t_k. \neg(\phi(t_1, \ldots, t_k) \land t_1.FK = \cdots = t_k.FK)$, and a set of tuples $t_1, \ldots, t_k$ such that $\phi(t_1, \ldots, t_k) = \text{True}$. Therefore, there is an edge $(t_1, \ldots, t_k)$ in $G$. Since $c$ is a proper coloring of $G$, at least one of the tuples $t_1, \ldots, t_k$ gets a different color than the rest. Suppose this tuple is $t_i$. Thus, $t_i.FK \neq t_j.FK$ for all $1 \leq j \neq i, j \neq i$, and $t_1, \ldots, t_k$ do not violate $\sigma$. Since this is true for all such sets of tuples, $\sigma$ is satisfied by such an assignment of FK values. \hfill \square

Proof of Proposition 5.5. First, we show that $\forall \sigma \in S_{DC}, \hat{R}_1 \models c$. The algorithm starts by constructing the conflict hypergraph, $G_c$, in lines 2–9, where each tuple in $V_{\text{join}}$ with the same combination of $B_1, \ldots, B_q$ values is added as a vertex. Note that these vertices represent the tuples in $R_1$. Each set of vertices that violates the first part of an FK DC is added as an edge. According to Proposition 5.2, in line 10, the colored vertices represent tuples that now have FK values that satisfy the DCs that they are involved in. $s$ now contains all vertices that could not be colored. In lines 11–13, the algorithm colors the vertices $t \in s$ using a fresh color $c[t]$ so all colored vertex combinations still do not violate any FK DC over $\hat{R}_1$. Finally, in line 16 we construct a secondary conflict hypergraph and color the invalid vertices according to it, making sure that none of these tuples violate a DC over $\hat{R}_1$.

We now prove that $\hat{R}_1 \models_{FK=K_1} \hat{R}_2 = V_{\text{join}}$. Prior to line 13, we do not artificially add tuples to $\hat{R}_2$, so it is clear that $\hat{R}_1 \models_{FK=K_1} \hat{R}_2 = V_{\text{join}}$. Suppose we color $t \in \hat{R}_1$ with a new color $c[t]$ (i.e., $t.FK = c[t]$) in line 12 and its corresponding tuple in $V_{\text{join}}$ is $t_f$. In the loop on line 13 we add a tuple, $t_{\text{new}}$, to $\hat{R}_2$ with $B_1, \ldots, B_q$ values as in $t_f$ and give it the primary key value $K_2 = c[t]$ (which is a fresh value not used in other tuples). Thus, joining $t$ with $t_{\text{new}}$ will give us $t_f$, which already exists in $V_{\text{join}}$. A similar procedure is performed in line 16 for the invalid vertices. Therefore, the added tuple does not add new tuples to $V_{\text{join}}$, and we still have $\hat{R}_1 \models_{FK=K_1} \hat{R}_2 = V_{\text{join}}$.

\hfill \square

### A.2 Constraints Used in Section 6

We detail the CCs and DCs used in our experimental study. Table 4 depicts the DCs, and Table 5 depicts the CCs.

### A.3 Parallelizing the Coloring Processes

In Section 5.2, we describe an optimization that allows for the conflict hypergraph to be split according to the filled-in $V_{\text{join}}$ and $R_2$ relations by the $B_1, \ldots, B_q$ values. As a result, we color each component of the hypergraph individually with Algorithm 3 (in lines 10 and 12) in different iterations of the loop in line 2. For example, in our experiments with Dataset no. 11 in Table 2, the conflict hypergraph was split into 3805 – 3813 components. Thus, it is possible to parallelize the coloring process performed in each iteration of the loop in line 2 in Algorithm 4 and color separate components obtained in different iterations on different machines.