Quantum Superposition Principle and Geometry

Alejandro Corichi\textsuperscript{1,2,*}

\textsuperscript{1}Instituto de Ciencias Nucleares
Universidad Nacional Autónoma de México,
A. Postal 70-543, México D.F. 04510, México.

\textsuperscript{2}Instituto de Matemáticas
Universidad Nacional Autónoma de México,
A. Postal 61-3, Morelia, Michoacán 58090, México

Abstract

If one takes seriously the postulate of quantum mechanics in which physical states are \textit{rays} in the standard Hilbert space of the theory, one is naturally lead to a geometric formulation of the theory. Within this formulation of quantum mechanics, the resulting description is very elegant from the geometrical viewpoint, since it allows to cast the main postulates of the theory in terms of two geometric structures, namely a symplectic structure and a Riemannian metric. However, the usual superposition principle of quantum mechanics is not naturally incorporated, since the quantum state space is non-linear. In this note we offer some steps to incorporate the superposition principle within the geometric description. In this respect, we argue that it is necessary to make the distinction between a \textit{projective superposition principle} and a \textit{decomposition principle} that extend the standard superposition principle. We illustrate our proposal with two very well known examples, namely the spin 1/2 system and the two slit experiment, where the distinction is clear from the physical perspective. We show that the two principles have also a different mathematical origin within the geometrical formulation of the theory.

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\*Electronic address: corichi@nucleares.unam.mx, corichi@matmor.unam.mx
I. INTRODUCTION

It has been known for some time that Quantum Mechanics, with all its postulates, can be put into geometric language. For details see [1-4]. To begin with, let us recall the geometrical formalism for systems with a finite dimensional Hilbert space. The generalization to the infinite dimensional case is straightforward [3]. Denote by $\mathcal{P}$ the space of rays in the Hilbert space $\mathcal{H}$. That is, given two states $|\phi\rangle$ and $|\psi\rangle$ in $\mathcal{H}$ such that they are proportional $|\psi\rangle = \alpha |\phi\rangle$ for $\alpha \in \mathbb{C}$, then both vectors belong to the same equivalence class $[|\psi\rangle] \in \mathcal{P}$. In the finite dimensional case $\mathcal{P}$ will be the complex projective space $\mathbb{C}P^{n-1}$, since $\mathcal{H}$ can be identified with $\mathbb{C}^n$.

It is convenient to view $\mathcal{H}$ as a real vector space equipped with a complex structure (recall that a complex structure $J$ is a linear mapping such that $J^2 = -1$). Let us decompose the Hermitian inner product into real and imaginary parts,

$$\langle \Psi | \Phi \rangle = G(\Psi, \Phi) - i \Omega(\Psi, \Phi),$$

where $G$ is a Riemannian inner product on $\mathcal{H}$ and $\Omega$ is a symplectic form.

Let us restrict our attention to the sphere $S$ of normalized states. The true space of states is given by the quotient of $S$ by the $U(1)$ action of states the differ by a ‘phase’, i.e. the projective space $\mathcal{P}$. The complex structure $J$ is the generator of the $U(1)$ action ($J$ plays the role of the imaginary unit $i$ when the Hilbert space is taken to be real). Since the phase rotations preserve the norm of the states, both the real and imaginary parts of the inner product can be projected down to $\mathcal{P}$.

Therefore, the structure on $\mathcal{P}$ which is induced by the Hermitian inner product is given by a Riemannian metric $g$ and a symplectic two-form $\Omega$. The pair $(g, \Omega)$ defines a Kähler structure on $\mathcal{P}$ (Recall that a Kähler structure is a triplet $(M, g, \Omega)$ where $M$ is a complex manifold (with complex structure $J$), $g$ is a Riemannian metric and $\Omega$ is a symplectic two-form, such that they are compatible).

The space $\mathcal{P}$ of quantum states has then the structure of a Kähler manifold, so, in particular, it is a symplectic manifold and can be regarded as a ‘phase space’ by itself. It turns out that the quantum dynamics can be described by a ‘classical dynamics’, that is, with the same symplectic description that is used for classical mechanics. Let us see how it works. In quantum mechanics, Hermitian operators on $\mathcal{H}$ are generators of unitary transformations (through exponentiation) whereas in classical mechanics, generators of canonical transformations are real valued functions $f : \mathcal{P} \to \mathbb{R}$. We would like then to associate with each operator $F$ on $\mathcal{H}$ a function $f$ on $\mathcal{P}$. There is a natural candidate for such function: $f := \langle F \rangle_S$ (denote it by $f = \langle F \rangle$). The Hamiltonian vector field $X_f$ of such a function is a Killing field of the Riemannian metric $g$. The converse also holds, so there is a one to one correspondence between self-adjoint operators on $\mathcal{H}$ and real valued functions (‘quantum observables’) on $\mathcal{P}$ whose Hamiltonian vector fields are symmetries of the Kähler structure.

There is also a simple relation between a natural vector field on $\mathcal{H}$ generated by $F$ and the Hamiltonian vector field associated to $f$ on $\mathcal{P}$. Consider on $S$ a ‘point’ $\psi$ and an operator $F$ on $\mathcal{H}$. Define the vector $X_F|_\psi := \frac{d}{dt} \exp[-i F t]|_{t=0} = -i F \psi$. This is the generator of a one parameter family (labelled by $t$) of unitary transformation on $\mathcal{H}$. Therefore, it preserves the Hermitian inner-product. The key result is that $X_F$ projects down to $\mathcal{P}$ and the projection is precisely the Hamiltonian vector field $X_f$ of $f$ on the symplectic manifold $(\mathcal{P}, \Omega)$.

Dynamical evolution is generated by the Hamiltonian vector field $X_h$ when we choose as our observable the Hamiltonian $h = \langle H \rangle$. Thus, Schrödinger evolution is described by Hamiltonian dynamics, exactly as in classical mechanics.
One can define the Poisson bracket between a pair of observables \((f, g)\) from the inverse of the symplectic two form \(\Omega^{ab}\),
\[
\{f, g\} := \Omega(X_g, X_f) = \Omega^{ab}(\partial_a f)(\partial_b g). 
\]  
(1.2)
The Poisson bracket is well defined for arbitrary functions on \(P\), but when restricted to observables, we have,
\[
\langle -i[F, G] \rangle = \{f, g\}. 
\]  
(1.3)
This is in fact a slight generalization of Ehrenfest theorem, since when we consider the ‘time evolution’ of the observable \(f\) we have the Poisson bracket \(\{f, h\} = \dot{f}\),
\[
\dot{f} = \langle -i[F, H] \rangle. 
\]  
(1.4)

As we have seen, the symplectic aspect of the quantum state space is completely analogous to classical mechanics. Notice that, since only those functions whose Hamiltonian vector fields preserve the metric are regarded as ‘quantum observables’ on \(P\), they represent a very small subset of the set of functions on \(P\).

There is another facet of the quantum state space \(P\) that is absent in classical mechanics: Riemannian geometry. Roughly speaking, the information contained in the metric \(g\) has to do with those features which are unique to the quantum description, namely, those related to measurement and ‘probabilities’. We can define a Riemannian product \((f, g)\) between two observables as
\[
(f, g) := g(X_f, X_g) = g^{ab}(\partial_a f)(\partial_b g). 
\]  
(1.5)
This product has a very direct physical interpretation in terms of the dispersion of the operator in the given state:
\[
(f, f) = 2(\Delta F)^2. 
\]  
(1.6)
Therefore, the length of \(X_f\) is the uncertainty of the observable \(F\).

The metric \(g\) has also an important role in those issues related to measurements. Note that eigenvectors of the Hermitian operator \(F\) associated to the quantum observable \(f\) correspond to points \(\phi_i\) in \(P\) at which \(f\) has local extrema. These points correspond to zeros of the Hamiltonian vector field \(X_f\), and the eigenvalues \(f_i\) are the values of the observable \(f_i = f(\phi_i)\) at these points.

If the system is in the state \(\Psi\), what are the probabilities of measuring the eigenvalues \(f_i\)? The answer is strikingly simple: measure the geodesic distance given by \(g\) from the point \(\Psi\) to the point \(\phi_i\) (denote it by \(d(\Psi, \phi_i)\)). The probability of measuring \(f_i\) is then,
\[
P_i(\Psi) = \cos^2 [d(\Psi, \phi_i)]. 
\]  
(1.7)
Therefore, a state \(\Psi\) is more likely to ‘collapse’ to a nearby state than to a distant one when a measurement is performed. This ends our brief review of the geometric formulation of quantum Mechanics (GFQM).

It is important to note that, in most treatments of the Geometric Formulation of Quantum Mechanics, the superposition principle is not discussed. The main obvious reason being that the space \(P\) is not linear. That is, the sum of two states \(\Psi = |\Psi\rangle\) and \(\Phi = |\Phi\rangle\) is not well defined. That is,
\[
[|\Psi\rangle + |\Phi\rangle] \neq [|\Psi\rangle + |\Phi\rangle],
\]
where \(|\Psi\rangle\) and \(|\Psi'\rangle\) belong to the same equivalence class. That is, the sum in the Hilbert space depends on the representative on each equivalence class, and therefore one cannot project it to the space \(P\) of quantum states.

At first sight it might seem that there is some incompatibility between the standard formulation of Quantum Mechanics in terms of linear spaces and the Geometric formulation, given the prominent position that the superposition principle holds in most treatments of the subject. In this contribution, we shall reexamine the superposition principle and discuss two different aspects that should, from our perspective, be distinguished. The first one is what we call the projective ‘non-linear’ superposition principle and the second one what we would like to call principle of decomposition. We shall illustrate the difference by means of the most discussed systems: spin 1/2 system for the first principle, and the double slit experiment for the second one. In the last part of the paper, we show by means of a geometrical argument that the relation between geodesic distance as given by the Riemannian metric and transition probability holds in general, and allows us to picture the general structure of the quantum space \(P\).

In what follows we shall present the two aspects of the superposition principle that we feel need to be considered separately, in view of the geometric (non-linear) nature of the space of states \(P\).

II. PROJECTIVE SUPERPOSITION PRINCIPLE

In the following two sections we shall present the two aspects of the superposition principle that we feel need to be considered separately, in view of the geometric (non-linear) nature of the space of states \(P\). First we shall focus our attention in what we have called the projective ‘non-linear’ superposition principle and in the next section we shall consider the decomposition principle.

In a linear space like the Hilbert Space \(\mathcal{H}\) the sum of two vectors is of course well defined. Thus, one has a binary operation \((\mathcal{H}, \mathcal{H}, +)\) from \(\mathcal{H} \times \mathcal{H}\) to \(\mathcal{H}\). The second operation one can define is multiplication by a scalar, which in the case of a complex vector space means multiplication by a complex number \(\alpha \in \mathbb{C}\). The combination of these two operations is manifested in the linear superposition of two vectors. That is, given \(|\psi\rangle\) and \(|\phi\rangle\) \(\in \mathcal{H}\), and a pair of complex numbers \((\alpha, \beta)\), one constructs,

\[
\alpha |\phi\rangle + \beta |\psi\rangle.
\]

Thus, for each choice of \((\alpha, \beta)\), one gets a new vector, and all the possible linear combinations form the Span of \((|\psi\rangle, |\phi\rangle)\), which in this case is isomorphic to \(\mathbb{C}^2\). Thus, for fixed ‘basis vectors’ \((|\psi\rangle, |\phi\rangle)\), one can think of the linear superposition as a mapping from \(\mathbb{C}^2\) to \(\mathcal{H}\) whose image is a 2-dimensional subspace.

The state space \(P\) is a non-linear space. Thus, one can not hope to define a sum of two states. However, as we shall argue, what one can define is a generalization of the concept of linear combination in the Hilbert. We shall refer to this generalization, as the non-linear superposition. The basic idea is the following: First consider two orthogonal vectors \(|\psi\rangle\) and \(|\phi\rangle\) in \(\mathcal{H}\). (If they are not orthogonal, one can always construct an orthogonal set by the Gram-Schmidt procedure.) Let us now define in \(\mathcal{H}\) the following operation. Given \(|\psi\rangle\) and \(|\phi\rangle\) in \(\mathcal{H}\), and a complex number \(z \in \mathbb{C}\), define

\[
|\psi\rangle \stackrel{z}{+} |\phi\rangle := |\psi\rangle + z |\phi\rangle.
\] (2.1)
We can now project the state to \( \mathcal{P} \) and get
\[
|\psi\rangle \oplus |\phi\rangle,
\]
as the non-linear combination of \( |\phi\rangle \) and \( |\psi\rangle \) with parameter \( z \). Several remarks are in order. First, we know from the geometrical description of a two state system, such as the spin \( 1/2 \) system described in Ref. [6], that the projective space \( \mathbb{C}P^1 \) one gets starting from \( \mathcal{H} = \mathbb{C}^2 \) is topologically a sphere. Then, the number \( z \) should be thought of as a (Riemann) coordinate on the sphere. Note that the ‘origin’ of the sphere corresponds to the state \( \Psi := |\psi\rangle \) and the ‘point at infinity’ corresponds to \( \Phi := |\phi\rangle \). Thus, given the basis vectors \( |\phi\rangle \) and \( |\psi\rangle \), there is ‘a sphere’s worth’ of possible non-linear superpositions of them, one for each point on the sphere \( z \).

Second, note that there seems to be an ambiguity in the mapping between quantum states in \( \mathcal{P} \) and coordinate \( z \). That can be seen by considering another state \( |\phi'\rangle = e^{i\lambda}|\phi\rangle \) to define the non-linear superposition. Then, of course, the same state \( \Psi \in \mathcal{P} \) that had coordinate \( z \) will now have coordinate \( e^{-i\lambda}z \).\(^1\) Thus, one should fix once and for all the vectors \( |\psi\rangle \) and \( |\phi\rangle \) with respect to which the construction is defined. Then, there exists a one-to-one correspondence between states and complex coordinate \( z \). The apparent ambiguity in the correspondence between states and coordinates in nothing but the freedom in choosing complex coordinates for the Riemann sphere with the zero and infinity fixed, namely, the freedom to choose a ‘real section’ of the sphere. Geometrically these freedom corresponds to the freedom in choosing different complex structures in the projective space \( \mathcal{P} \), which in the case of \( \mathbb{C}P^1 \) reduces to a \( U(1) \) freedom. This freedom should be thought of not as ‘gauge’, but rather as a symmetry of the geometric description.

Finally, note that we can think of the sphere \( \mathcal{S} := \{ \xi \in \mathcal{P} / (|\psi\rangle \oplus |\phi\rangle) \forall z \in \mathbb{C} \} \) as the non-linear span of the states \( \Psi \) and \( \Phi \) in \( \mathcal{P} \). Note that this submanifold is independent of the representatives chosen (and in fact does not require the original vectors to be orthogonal). Therefore, one can conclude that, given any two states in \( \mathcal{P} \), there exists a canonical sphere \( \mathcal{S} \subset \mathcal{P} \) containing them. This conclusion seems to contradict basic intuition, say in \( \mathbb{R}^3 \), which states that there is in fact an infinite number of spheres passing through any two points. The extra (hidden) constraint in the case of the quantum state space \( \mathcal{P} \) is that these spheres are always normalized to have \( \text{Area}(\mathcal{S}) = \pi \). Then, one is concluding that there exists a unique normalized sphere embedded in \( \mathcal{P} \) containing any two points. In the algebraic geometric language used in Ref. [4] this sphere corresponds to an ‘algebraic curve’.

This last observation is of particular importance for the following reason. In the Hilbert space description of quantum mechanics, linearity plays an important role, particularly in what is known as the superposition principle. In its simplest form one could phrase it as saying that given two vectors one can define superposition of them (via a linear combination) and define new vectors. Geometrically, one is placing importance to the span of the two vectors which is a complex plane. In the non-linear, geometric description given by \( \mathcal{P} \), the role of the 2-dimensional complex plane is taken now by the one (complex) dimensional sphere corresponding to the non-linear span. The relevance goes further than just being able to describe the superposition principle. As we have seen before, transition probabilities

\(^1\) The other possibility, namely to change the phase of \( |\psi\rangle \), will result in a different state in \( \mathcal{P} \) but with the same coordinate \( z \). This corresponds to an active diffeomorphism as opposed to the passive one discussed before.
to go from a state to the eigenstate of the observable being measured are given by (a simple function of) the geodesic distance along $\mathcal{P}$. What is in a sense unexpected and surprising is that if we consider two arbitrary points $p$ and $q$ on the state space $\mathcal{P}$, and we want to consider the geodesic distance from $p$ to $q$ in order to compute probabilities, it suffices to consider the canonical sphere $\mathcal{S}$ passing through them and compute the geodesic distance along the sphere. That is, the geodesic (with respect to the full metric) on $\mathcal{P}$ going from $p$ to $q$ lies entirely within $\mathcal{S}$! In other words, the spheres spanned by the two states are always totally geodesic. The proof of this fact is done in the Appendix A.

It is rather easy to show that, indeed, the description for superimposed states we have constructed is consistent with the usual, and very well known facts about ordinary quantum mechanics, for instance, in the context of a Stern-Gerlach experiment for a spin $1/2$ system. Physically, what is important to realize is that one is able to prepare and construct the state in any possible (non-linear) superposition state, that is, on any point on the two-sphere. This is because we can choose to prepare the state (of say a beam of neutrons) by aligning the Stern-Gerlach apparatus along any possible orientation. Furthermore, this is the only choice one can make in preparing the state, and therefore there is a one to one correspondence with the state space $\mathcal{P}$. Thus, to conclude, even when one can not add two states in $\mathcal{P}$, there is a precise sense in which there is a non-linear superposition of any two states (or more by iterating the procedure). One should finally note that a similar version of the ‘projective superposition principle’ was independently developed in [9].

In the next Section we shall consider the other physical principle that we think should be distinguished within the geometric framework and that is normally associated to the (linear) superposition principle in Hilbert space. For that purpose, we shall consider the physical situation of a double slit experiment.

III. DECOMPOSITION PRINCIPLE

One of the main difference between the classical and quantum description of physical systems has to do with the way in which probabilities are computed for different situations. In classical probability theory, the probabilities of two disconnected events are added when the outcome of the ‘experiment’ is the same. In quantum mechanics one adds probability amplitudes which are complex numbers, and at the end, one computes the square of the modulus in order to find probabilities. This last procedure brings in interference effects that are so notorious in Quantum Mechanics.

Let us at this point consider the most common example in which quantum interference is known to exist, namely, the two slit experiment. The purpose of analyzing this system is to point out some subtleties that we feel should be addressed when analyzing this physical situation within the geometrical description. In particular, we would like to differentiate this situation with the ordinary superposition of states (be it linear or its non-linear, projective generalization). For this reason we have decided to refer to it as the decomposition principle.

Let us now recall the basic setting. One assumes that there is a particle source and a screen where the particles are to be detected. In between the source and the screen one places a wall with two idealized slits. Let us call them 1 and 2. If the system initially is described by the state vector $|\psi\rangle$, then the complex number $\langle x|\psi\rangle$ is the ‘probability amplitude’ for a particle to hit the screen at the point $x$. The probability (density) for the particle to be measured at point $x$ will be the norm squared of $\langle x|\psi\rangle$. As a first step, one says that the
number $\langle x|\psi \rangle$ is of the form,

$$\langle x|\psi \rangle = \phi_1 + \phi_2,$$

where $\phi_1$ is interpreted as the ‘wave function of the particle’ passing through slit 1, and similarly for $\phi_2$. In order to arrive to such an expression, one can use ‘Feynman’s second and third general principles’ [8], which imply that the presence of the wall with two slits can be represented as a pair of projection operators $P_1$ and $P_2$ such that the action of putting an intermediate wall can be written as $P_{\text{wall}} = P_1 + P_2$. Since we assume that $P_i \cdot P_j = \delta_{ij} P_j$, then $P_2^{\text{wall}} = P_{\text{wall}}$ so we also have that $P_{\text{wall}}$ is a projection operator. Then, the transition probability is of the form,

$$\langle x|\tilde{\psi} \rangle = \langle x|P_1 + P_2|\psi \rangle = \langle x|P_1|\psi \rangle + \langle x|P_2|\psi \rangle \tag{3.1}$$

We now see that one can identify the numbers $\phi_i$ with $\phi_i = \langle x|P_i|\psi \rangle$, for $i = 1, 2$. The quantum interference is then associated with the Real part of the complex number $\phi_1 \phi_2$.

The important thing is to note that this number is independent of the phase and normalization of the state vector $|\psi\rangle$, so it can be projected to the space $P$. Furthermore, if the projection operator $P$ are, in the general case, of the form

$$P = \sum_i |\psi_i\rangle \langle \psi_i| \tag{3.2}$$

for any orthogonal set of vectors $|\psi_i\rangle$ in $H$, then the operator $P$ would also be independent of the ‘phase’ of each $|\psi_i\rangle$, and it can be projected unambiguously to $P$. In this respect the physical situation is very different to the case of superposition of two states. Here, one can not change independently two states that are to be composed (as is the case for the preparation of the state in the Stern Gerlach experiment of the previous section), but the only thing one can do is to change, for instance, the phase of the original state $|\psi\rangle$. Since this does not affect the interference pattern, and can thus be projected down to the quantum space $P$, we are in a rather different physical situation as before. The end result is that the interference term in the probability is invariant under change of phase of the original wave-function representing the incoming beam. It is important here to stress that the linearity in the ‘superposition’ is now manifest in the properties of the projection operators that have a linear structure. Thus, even when in the standard presentations of the subject, the states seem to be superposed, what one is actually doing is defining the operator $P_{\text{wall}}$ as the linear addition of projection operators representing the different possible outcomes of the experiment. Thus, instead of superposing two states, what one is doing is to decompose the state by means of projection operators.

The next question that comes to mind is how to interpret this process geometrically. That is, how can we visualize this ‘quantum interference’ in terms of the geometrical objects available in $P$. In particular, how can we understand in the geometrical picture the fact that we seem to retain the linear structure of addition of amplitudes even in the non-linear space $P$. The answer lies in the fact that we are not adding states in order to find the amplitude, but rather adding projection operators.

Let us now understand the geometric properties of these operators derived from its algebraic structure and the nature of the geometry of $P$. First we need to recall the dual nature that Hermitian operators have. The first obvious feature about a Hermitian operator $\hat{F}$ is that its action on a vector $|\phi\rangle$, will yield a new vector $|\phi'\rangle = \hat{F} \cdot |\phi\rangle$. Since the action is linear and $\hat{F} \cdot \alpha |\phi\rangle = \alpha \hat{F} \cdot |\phi\rangle$, this means that the action commutes with the projection to $P$ and
it induces a finite mapping, $\hat{F} : \mathcal{P} \to \mathcal{P}$. The problem with this basic fact is that this finite mapping on $\mathcal{P}$ does not preserve any of the geometric structures on $\mathcal{P}$, that is, the symplectic structure nor the metric. That is, it is not a symplectomorphism of $\Omega$ nor an isometry of $g$. Nevertheless, there is a precise sense in which the operator $\hat{F}$ is related to a geometric invariant quantity. For this one should recall that the operator $U_{\hat{F}}(\lambda) = \exp(-i \lambda \hat{F})$, being an unitary operator represents a one parameter family of symplectomorphisms (and isometries) of $\Omega$ (and $G$) on $\mathcal{H}$ that gets projected down to $\mathcal{P}$ (and remaining a symmetry), when acting on vectors of $\mathcal{H}$. The curve passes through the point $\Psi$, for $\lambda = 0$ and has, as its tangent vector in $\mathcal{H}$, the object,

$$V_{\hat{F}}(\Psi) = \frac{d}{d\lambda} (U_{\hat{F}}(\lambda) \cdot |\Psi\rangle)_{|\lambda=0} \quad (3.3)$$

which is just given by,

$$V_{\hat{F}}(\Psi) = -i \hat{F} \cdot |\Psi\rangle \quad (3.4)$$

The vector $V_{\hat{F}}(\Psi)$ in $T_{\Psi}\mathcal{H}$ get projected down precisely to the Hamiltonian vector field $X_{\hat{F}}$ in $T_{\Psi}\mathcal{P}$. Thus, associated to any Hermitian operator in $\mathcal{H}$ there is a vector field on $\mathcal{P}$ that is a symmetry of both the symplectic structure and the metric. Symmetries of the geometric structures (i.e. vector field preserving the geometric structures) form an algebra, where one can take linear combinations with constant coefficient (as opposed to scalar fields) and commutators. This nice geometric property is true for any Hermitian operator.

We return then to the decomposition principle and the question of why it seems that we are adding wave functions. What we have argued here is that this property is due to the fact that what one is doing is to decompose the original state into several ‘parts’ by means of projection operators. These operators are self-adjoint and therefore have a nice and clean geometrical interpretation as vectors that can be added. Indeed, projection operators have a special algebraic structure somewhat different to that inherited from is Hermitean nature. If we have to projection operators $P_{1,2}$, the only linear combination $\alpha P_{1} + \beta P_{2}$ that yields again a projection operator (satisfying $P^{2} = P$) is for $\alpha$ and $\beta$ to be 0 or 1 (so they do not form an algebra). Thus, projection operators in fact have to be added.\footnote{Furthermore, if the projection operators are of the form (3.2) then the individual operators commute amongst themselves $[P_{i}, P_{j}] = 0$, the corresponding real functions on $\mathcal{P}$, $f_{i} := \langle \psi | P_{i} | \psi \rangle / \langle \psi | \psi \rangle$, Poisson commute amongst themselves, an their associated vector fields also commute.} With this discussion we have to conclude that there is no inconsistency with the absence of linear addition of states in $\mathcal{P}$. The interference patterns arises as usual when computing the modulus squared of the final quantum amplitude $\langle x | \psi \rangle$, where the terms in the sum $\sum_{i} \langle x | P_{i} | \psi \rangle$ interfere in the usual way.

IV. DISCUSSION

We have considered in this paper the issue of the superposition principle of quantum mechanics within the context of the geometric formulation of the theory. The superposition principle is many times put at the forefront of the formulation of quantum theory (see for instance [7, 8]), but on the other hand the theory has been for a long time recognized to be about states, or rays in the Hilbert space [7]. The fact that the quantum state space
\(\mathcal{P}\) is non-linear possesses the challenge of how to accommodate for the linear superposition principle. In this contribution we have put forward two basic proposals. The first one is that one should distinguish between two different physical situations. The first one, as is the case of a spin 1/2 particle, one can have full control over the preparation of the state that in this case can be polarized along any direction is space. This amounts to the freedom of (non-linearly) superposing the \(|+\rangle\) and \(|-\rangle\) states. The second physical situation is when there is no such freedom in preparing the state such as in the case of an incoming beam (of photons) in the double slit experiment. In this situation the freedom is in the possibility of having no intermediate screen, or one with different number of slits. We have argued that in this case, the relevant object to represent the barrier with the slits is via a projection operator, which is the linear addition of projection operators representing each of the slits. The end result is that the interference term in the probability is invariant under change of phase of the original wave-function representing the incoming beam. The linearity in the ‘superposition’ is now manifest in the properties of the projection operator that have a linear structure.

From the mathematical point of view, both principles are due to different geometrical properties of the space of states \(\mathcal{P}\). The nonlinear superposition is the best one can do in the absence of linear superposition of states in the state space \(\mathcal{P}\). The fact that in the two slit experiment one seems to be able to add wave functions is not due to a linear addition of states, but to the fact that projection operators are to thought as vectors on \(\mathcal{P}\), and vectors can indeed be added. Furthermore, projection operators cannot be ‘linearly spanned’ with arbitrary coefficients but have to be added. Needles to say, still further investigation is needed to unravel the physical distinction between the two principles that we are here proposing.

Another aspect of the geometric description of the theory deals with the structure of the state space \(\mathcal{P}\). In the Appendix A we show that the spheres of fixed radius \(r = 1/2\), that correspond to the state space of a spin 1/2 particle, or \(q\)-bit, is fundamental in understanding the general state space \(\mathcal{P}\). Given any two states in \(\mathcal{P}\) there is a unique sphere that passes through them. Furthermore, this submanifold is totaly geodesic, in the sense that the geodesic between the two points in \(\mathcal{P}\) lies entirely in the sphere defined by them, and the geodesic distance defines the transition probabilities entirely. The proof here presented is complementary to those presentations already available [3]. In Appendix B we show another feature of the geometric formulation, namely the fact that there is a precise sense in which the canonical operators \(\hat{q}\) and \(\hat{p}\) commute. The origin of Heisenberg’s uncertainty principle is clarified from the geometric perspective.

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APPENDIX A: GEODESIC DISTANCE IS THE TRANSITION PROBABILITY

In this part we shall focus on the general geometric structure of the state space \(\mathcal{P}\), and in particular, in the way in which the projection of the linear spans of any two vectors are
embedded in the space $\mathcal{P}$. This geometrical property is fundamental to prove the close relation between geodesic distance and transition probability. In particular, we shall present a new proof that, in fact, the transition probability is given by the geodesic through the formula,

$$ P(\phi, \psi) = \cos^2(d(\phi, \psi)) , $$

with $d(\phi, \psi)$ the geodesic distance between the two states on $\mathcal{P}$.

The structure of the proof is as follows. First, one has to convince oneself that on the sphere, that is, for the spin 1/2 case, the ‘transition probability’ $|\langle \phi | \psi \rangle|^2$ is in fact equal to $\cos^2(d(\phi, \psi))$ (see for instance [6] for a discussion of the geometry of $\mathcal{P}$). The term transition probability is in the sense of Feynman. In particular, the states need not be eigenstates of any particular operator. For the case of the sphere it has been shown explicitly that in fact the relation (A1) holds.

The next step is to recognize that there are preferred submanifolds in the case of larger state spaces. This is in a sense a remanent of the linear structure in the Hilbert space. These submanifolds correspond to the projections to $\mathcal{P}$ of the subvector spaces in the Hilbert. In particular, the two-dimensional (complex) planes in $\mathcal{H}$ (the span of any two non-collinear vectors) are projected to $\mathbb{C}P^1$ (spheres) embedded in the state space $\mathcal{P}$. Similarly, 3 dimensional (complex) planes in $\mathcal{H}$ are projected to $\mathbb{C}P^2$s in $\mathcal{P}$. These submanifolds have the property, as we will show below, that the geodesics of the total space between any two points lying on them (on the sphere, for example) lie completely inside the submanifold. That is, these are ‘totally geodesic’ submanifolds.

The next task is to show that indeed these preferred submanifolds are totally geodesic. If we prove this, then we would be finished since that would mean that in order to build the geodesic in $\mathcal{P}$ from $\psi$ to $\phi$, we just have to consider the two-sphere they span and consider the geodesics on it. Since we know that geodesics along spheres give us the correct probability we are finished.

The question that immediately arises is: how do we prove that the spheres are totally geodesic? First, let us consider the simplest case, namely, a spin 1 system. This is represented by a vector in $\mathbb{C}^3$,

$$ (\alpha, \beta, \gamma) \in \mathbb{C}^3. $$

A point in $\mathbb{C}P^2$ represents a state. Let us now restrict our attention to state vectors that have $\gamma = 0$. That is, we consider the 2-dimensional plane spanned by $(1, 0, 0)$ and $(0, 1, 0)$ in $\mathbb{C}^3$. It gets projected on to a two dimensional submanifold in the state space (a sphere). Now, it is easy to see that we can do any calculation in this subspace, including a transition amplitude $\langle \psi | \phi \rangle$ and everything involves only the two first coordinates, just as if we were in the spin 1/2 case! Consider two vectors $(\alpha, \beta, 0)$ and $(\delta, \gamma, 0)$, then the transition probability is $|\alpha \delta + \beta \gamma|^2$, that is the probability we would have in the spin 1/2 case for states $(\alpha, \beta)$ and $(\delta, \gamma)$.

What we shall show is that $\mathbb{C}P^1$ is embedded in $\mathbb{C}P^2$ in a totally geodesic fashion. As a first step, let us recall what the condition is for a submanifold to be totally geodesic. Let $\eta^a$ be a tangent vector to the submanifold $N$ embedded in $\mathcal{M}$. This means that it is orthogonal to all normals to $N$. In our case, as real manifolds, the two-dimensional sphere is embedded in the 4-dimensional $\mathbb{C}P^2$. Thus, there are two independent normals $n_a$ and $m_a$. Let us assume that they are orthogonal $(n \cdot m) = 0$. Then we have that $\eta^a n_a = \eta^b m_a = 0$. Let us now assume that we start with the vector $\eta^a$ at point $p$ and parallel transport it along itself. That is, it satisfies the equation

$$ \eta^a \nabla_a \eta^b = 0 $$

(A2)
where $\nabla$ is the covariant derivative compatible with the metric $g_{ab}$ in the full space. If we want to ensure that the geodesic continues to be tangent to the sub-manifold, we should impose the condition that $0 = \eta^a \nabla_a (\eta^b n_b)$ for all normals to $N$, This implies that,

$$0 = \eta^a \nabla_a (\eta^b n_b) = n_b \eta^a (\nabla_a \eta^b) + \eta^a \eta^b \nabla_a n_b$$  \quad (A3)

The first term vanishes due to the geodesic equation. In the second term we can consider arbitrary vectors tangent to $N$ so we can conclude that,

$$K^{(1)}_{ab} := \nabla_{(a} n_{b)} = 0$$  \quad (A4)

and

$$K^{(2)}_{ab} := \nabla_{(a} m_{b)} = 0$$  \quad (A5)

Where the underline denotes pullback to $N$. These are the two ‘extrinsic curvatures’ of $N$ (recall that it has co-dimension two). We can also write the induced metric on $N$ by,

$$q_{ab} := g_{ab} - m_a m_b - n_a n_b$$  \quad (A6)

with $g_{ab}$ the metric on $M$.

The conditions (A4,A5) of the extrinsic curvatures can be rewritten in terms of the induced metric as follows,

$$\mathcal{L}_n q_{ab} = \mathcal{L}_m q_{ab} = 0$$  \quad (A7)

Where $n^a$ and $m^a$ are orthogonal vectors not necessarily normalized.

Let us now consider the concrete case under consideration. We have $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2$. As is well known, the metric in $\mathbb{C}P^n$ coming from the reduction from the Hilbert $\mathbb{C}^n$ is given by the Fubini-Study metric [4]. If we consider coordinates $(z_0, z_1, z_2)$ in $\mathbb{C}^3$ and homogeneous coordinates on $\mathbb{C}P^2$ given by $t_1 = \frac{z_1}{z_0}$, $t_2 = \frac{z_2}{z_0}$ (valid whenever $z_0 \neq 0$), the metric $g_{ab}$ defines the line element,

$$ds^2 = \frac{(1 + t_1 t_i) (dt^i dt_j) - (\bar{t}^i dt) (\bar{t}^j dt)}{(1 + t^i t_j)^2},$$  \quad (A8)

with $i = 1, 2$. In our example we are considering the embedding $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2$ defined by $z_2 = 0$, that is, $t_2 = 0$. The induced metric on $N = \mathbb{C}P^1$ is,

$$ds^2 = \frac{(1 + t_1 t^1) (dt^1 dt_1) - (t_1 dt_1) (t^1 dt^1)}{(1 + t^1 t_1)^2}.$$  \quad (A9)

We can also consider the metric $g_{ab}$ as a (two-parameter) ‘foliation’ of the metric induced on $z_2 = \text{cont.}$ surfaces. If we now write the complex coordinates in terms of real coordinates, $t_1 = u_1 + iv_1$ ; $t_2 = u_2 + iv_2$, and consider the induced metric on such surfaces we get,

$$ds^2 = \frac{(1 + u_2^2 + v_2^2)}{(1 + u_1^2 + u_2^2 + v_1^2)} (du_1^2 + dv_1^2).$$  \quad (A10)

We can now see whether the condition that the $g_{ab}$ metric is Lie-dragged by the orthogonal vectors is true or not. It is easy to see that whenever $t_2 = 0$, the base vectors $(\partial/\partial v_2)^a$ and $(\partial/\partial u_2)^a$ are orthogonal to the $u_2 = v_2 = 0$ ($N$) surface. Therefore, the submanifold will be
totally geodesic if and only if $\mathcal{L} \left( \frac{\partial}{\partial u_2} \right) q_{ab} = 0$. The result is then straightforward,

$$\mathcal{L} \left( \frac{\partial}{\partial v_2} \right) q_{ab} = \frac{2v_2(1 + u_1^2 + v_1^2)}{(1 + u_1^2 + v_1^2 + u_2^2 + v_2^2)} \left( \nabla_a u_1 \nabla_b u_1 + \nabla_a u_2 \nabla_b u_2 \right). \quad (A11)$$

Thus,

$$\mathcal{L} \left( \frac{\partial}{\partial v_2} \right) q_{ab} \bigg|_{u_2=v_2=0} = 0, \quad (A12)$$

and similarly for the Lie-derivative along $\frac{\partial}{\partial u_2}$.

To conclude, we have shown that the embedding $\mathbb{C}P^2 \hookrightarrow \mathbb{C}P^3$ defined by the condition $u_2 = v_2 = 0$ is totally geodesic. As the discussion above demonstrates, this implies that the total geodesics of the $g_{ab}$ metric on $\mathbb{C}P^3$ between two points on $\mathbb{C}P^2$ lie entirely on it.

We can now go from $\mathbb{C}P^3$ to $\mathbb{C}P^\infty$, which is in a sense, the Hilbert space of ordinary quantum mechanical systems (for the subtleties see [5]). The basic idea is the same and it is straightforward to see that a $\mathbb{C}P^1$ constructed by setting all but two coordinates equal to zero in the Hilbert, is totally geodesic.

Let us end this section with a summary of the results that we have found:

1. Every pair of points on $\mathcal{P}$ define a unique sphere embedded in $\mathcal{P}$ that passes through them. We can say that the sphere is spanned by the pair of points.

2. Geodesics of $\mathcal{P}$ connecting any two points of this sphere lie entirely on it. That is, all these spheres are totally geodesic.

3. The transition probability between these two points is given by $\cos^2(d)$.

This ends our proof of this geometrical result.

APPENDIX B: $[\hat{q}, \hat{p}]$ AND HEISENBERG’S UNCERTAINTY PRINCIPLE

In this part we show some interesting facts about the geometric formulation. First let us consider a system that in its classical description is given by a phase space of the form $\Gamma = (q^i, p_j) = \mathbb{R}^{2n}$, that it, it has a linear structure. The usual canonical Poisson Brackets are such that $\{q^i, p_j\} = \delta^i_j$, that get promoted to the CCR of the form $[\hat{q}^i, \hat{p}_j] = i\hbar \delta^i_j$. We can identify, as relevant vector fields in the classical phase space, the Hamiltonian vector fields associated to the canonical coordinates, responsible for linear translations on $\Gamma$, namely,

$$X^a_{q^i} = \omega^{ab} \partial_b q^i = -\frac{\partial}{\partial p_j} \quad ; \quad X^a_{p_j} = \omega^{ab} \partial_b p_j = \frac{\partial}{\partial q^i}. \quad (B1)$$

The finite action of these vector field are finite translations along the coordinates of the phase space. Since these are linear actions, one expects those vector fields to commute. It is easy to see that they indeed commute:

$$[X^a_{q^i}, X^b_{p_j}] = X^a_{[q^i, p_j]} = 0 \quad (B2)$$

These vectors for the (phase space generalization) of the Galileo algebra. It is quite natural to ask about the corresponding algebra of vectors in the quantum theory. The geometrical
formulation of the theory allows us to pose the question in a natural way. First, we know that the unitary operators $U(\lambda)$ and $V(\mu)$ given by

$$U_i(\lambda) = \exp(-i\lambda \hat{q}_i) \quad : \quad V_j(\mu) = \exp(-i\mu \hat{p}_j)$$

which are the generators of the so-called Weyl algebra. They represent the finite unitary transformations on $\mathcal{H}$ that project down to a symmetry of $(\mathcal{P}, g, \Omega)$. At any point $\Psi \in \mathcal{P}$ the vector field that generate the curves are the projections of the vectors on $\mathcal{H}$

$$V_{\hat{q}_i}(\Psi) = -i \hat{q}_i \cdot |\Psi\rangle$$

and,

$$V_{\hat{p}_i}(\Psi) = -i \hat{p}_i \cdot |\Psi\rangle$$

Now, in $\mathcal{H}$, the vector fields have as a commutator,

$$[V_{\hat{q}_i}, V_{\hat{p}_j}] = [\hat{q}_i, \hat{p}_j] \cdot |\Psi\rangle = -i\hbar |\Psi\rangle$$  \hspace{1cm} (B3)

But the projection of the vector field associated to ($i$ times) the identity operator is zero. Thus, in $\mathcal{P}$ we have,

$$[X_{\hat{q}_i}, X_{\hat{p}_j}] = X_{\{q,p\}} = 0$$  \hspace{1cm} (B4)

That is, there is precise sense in which the operators $\hat{q}_i$ and $\hat{p}_i$ commute in $\mathcal{P}$: the only meaningful geometrical interpretation for a Hermitian operator $\hat{F}$ on $\mathcal{P}$ is as a vector, which can be both interpreted as the Hamiltonian vector field $X_f$ of $f = \langle \hat{F} \rangle$ or as the projection of the tangent vector to $\exp(-i\lambda \hat{F})$. For the Hermitian operator corresponding to the commutator, the associated vector on $\mathcal{P}$ vanishes.

How can we then make contact with the Heisenberg uncertainty principle for $\hat{q}$ and $\hat{p}$ that is normally attributed to the CCR? For that, we need to recall that there is a stronger inequality for the product $(\Delta \hat{F})(\Delta \hat{M})$ for two Hermitian operators $(\hat{F}, \hat{M})$ on $\mathcal{H}$, that is given (on $\mathcal{P}$) by [3],

$$(\Delta \hat{F})(\Delta \hat{M}) \geq \{f, m\} + g(X_f, X_m)$$  \hspace{1cm} (B5)

In the case of the CCR, we have seen that there is a precise sense in which the canonical observables commute, even when the first term on the right hand side does not vanish. We have then that the uncertainty in the measurement of position and momenta does not come from the commutator only (associated to $\Omega$ on $\mathcal{P}$) but also from the other geometric structure, namely the metric $g$. This is in a sense consistent with the fact that the metric $g$ is responsible for the 'most quantum' behavior of the system, namely transition probabilities and quantum fluctuations. Nevertheless, it is somewhat surprising that, in the geometric formulation of the theory, the origin of the textbook uncertainty relations comes not only from the quantum commutator but from the other product available as well.
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