Prescribing a heat flux coming from a wave equation

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Abstract
What happens when one prescribes a heat flux which is proportional to the Neumann data of a solution of the wave equation in the whole space on the surface of a heat conductive body? It is shown that there is a difference in the asymptotic behaviour of the indicator function in the most recent version of the time domain enclosure method which aims at extracting information about an unknown cavity embedded in the body.
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1 Introduction
Let Ω be a bounded domain of $\mathbb{R}^3$ with $C^2$-boundary. Let $D$ be a nonempty bounded open subset of $\Omega$ with $C^2$-boundary such that $\Omega \setminus D$ is connected. Let $0 < T < \infty$.

Given $f = f(x, t), (x, t) \in \partial \Omega \times ]0, T[ \text{ which belongs to } L^2(0, T; H^{-1/2}(\partial \Omega))$ let $u = u_f(x, t), (x, t) \in (\Omega \setminus D) \times ]0, T[$ denote the weak solution of the following initial boundary value problem for the heat equation:

\[
\begin{aligned}
(\partial_t - \Delta)u &= 0 \quad \text{in } (\Omega \setminus D) \times ]0, T[ , \\
u(x, 0) &= 0 \quad \text{in } \Omega \setminus D , \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial D \times ]0, T[ , \\
\frac{\partial u}{\partial \nu} &= f(x, t) \quad \text{on } \partial \Omega \times ]0, T[ .
\end{aligned}
\]

(1.1)

We use the same symbol $\nu$ to denote both the outer unit normal vectors of both $\partial D$ and $\partial \Omega$. The solution class is the same one as [10] which employs the weak solution in [1]. The $u$ satisfies, for a positive constant $C_T$ being independent of $f$

\[
\|u\|_{L^2(0,T; H^1(\Omega \setminus D))} + \|\partial_t u\|_{L^2(0,T; H^1(\Omega \setminus D))} + \|u(\cdot, T)\|_{L^2(\Omega \setminus D)} \leq C_T \|f\|_{L^2(0,T; H^{-1/2}(\partial \Omega))}.
\]

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See Subsection 2.1 in [10] for more information about the direct problem.

This paper is concerned with the inverse obstacle problem described below.

**Problem.** Fix $T$. Assume that the set $D$ is unknown. Extract information about the location and shape of $D$ from $u_f(x,t)$ which corresponds to a suitable known $f$ and is given for all $x \in \partial \Omega$ and $t \in \]0, T\].

Using the time domain enclosure method originary developed in [3], in the previous papers [10, 5, 11, 12] we have considered the problem above (see also [9] for a system). The prescribed heat flux $f$ takes the form

$$f(x,t) = \frac{\partial v}{\partial \nu}(x) \varphi(t), \quad (x,t) \in \partial \Omega \times \]0, T\]$$

where $v$ is a special solution of the modified Helmholtz equation $(\Delta - \tau)v = 0$ with $\tau > 0$ in a domain enclosing $\overline{\Omega}$ and say, $\varphi(t) \sim t^m$ as $t \downarrow 0$ with $m$ being a nonnegative integer. Note that $f$ depends on $\tau$ and thus, in this sense, the observation data $u_f(x,t)$, $(x,t) \in \partial \Omega \times \]0, T\]$ are infinitely many. Using the data we constructed the indicator function

$$\tau \mapsto \int_{\partial \Omega} \int_0^T e^{-\tau t} \left( u_f(x,t) \frac{\partial v}{\partial \nu}(x) - v(x)f(x,t) \right) dt \, dS.$$ 

From the asymptotic behaviour of this indicator function we extracted several information about the geometry of $D$, more precisely, the distance of an arbitrary point outside of $\Omega$ to $D$, the value of the support function of $D$ at a given direction, the minimum sphere that encloses $D$ with an arbitrary given center point. Note that in [5, 11] the governing equation was the heat equation with a variable coefficient having discontinuity, however, it is easy to see that the present case also can be covered without difficulty.

Recently, using a new version of the time domain enclosure method developed in [7], in [8] the author introduced another substitution of $f$:

$$f(x,t) = \frac{\partial \Theta}{\partial \nu}(x,t), \quad (x,t) \in \partial \Omega \times \]0, T\]$$

where $\Theta = \Theta(x,t)$, $(x,t) \in \mathbb{R}^3 \times \]0, T\]$ is the solution of the Cauchy problem for the heat equation

$$(\partial_t - \Delta)\Theta = 0$$

in the whole space with a special initial data supported on an arbitrary closed ball outside of $\Omega$. Note that $f$ does not depend on any parameter except for the ball. The result in [8] says that the data $u_f$ on $\partial \Omega \times \]0, T\]$ yields the distance of the ball $B$ to $D$ from the explicit formula

$$\lim_{\tau \to \infty} \frac{1}{\sqrt{\tau}} \log I(\tau) = -2 \text{dist} (D, B),$$

where

$$I(\tau) = \int_{\partial \Omega} \left( \int_0^T e^{-\tau t} (u_f(x,t) - \Theta(x,t)) \frac{\partial}{\partial \nu} \int_0^T e^{-\tau t} \Theta(x,t) dt \right) dS$$

and $\text{dist} (D, B) = \inf_{x \in D, y \in B} |x - y|$.

Needless to say, there should be other possibilities of choosing a suitable heat flux which yields the geometry of $D$ since the function space $L^2(0,T; H^{-1/2}(\partial \Omega))$ is large.
Here we have a naive question: if one replaces $\Theta$ on (1.2) with a solution of another type of equation what happens on the asymptotic behaviour of the indicator function above? This is the subject of this paper. In this paper, as another type of equation we choose the wave equation. This choice comes from the finite propagation speed of the signal governed by the wave equation unlike the heat equation. How does the wave interact on the surface of a heat conductive body? Can one extract information about the geometry of $D$ from the data $u_f$ on $\partial\Omega \times [0, T]$ by using the idea of the enclosure method under such a choice of $f$? It should be pointed that in [11], the distance of $\partial\Omega$ to $D$ is also given by using the data $u_f$ on $\partial\Omega \times [0, T]$ for an arbitrary $f$ having a positive lower bound. However, the $f$ is not necessary a solution of any equation. See also [12] for a result in the enclosure method using a single $f$ which is independent of solutions of any equation.

Now let us describe the results in this paper. Let $B$ be an open ball satisfying $\overline{B} \cap \overline{\Omega} = \emptyset$. We assume that the radius $\eta$ of $B$ is very small. Let $\chi_B$ denote the characteristic function of $B$. Let $v = v_{B, \lambda}$ be a solution of

$$
\begin{cases}
(\lambda^2 \partial_t^2 - \Delta)v = 0 & \text{in } \mathbb{R}^3 \times [0, T], \\
v(x, 0) = 0 & \text{in } \mathbb{R}^3, \\
\partial_t v(x, 0) = \Psi_B(x) & \text{in } \mathbb{R}^3,
\end{cases}
$$

(1.3)

where $\lambda$ is a positive constant and

$$
\Psi_B(x) = (\eta - \|x - p\|)\chi_B(x), \ x \in \mathbb{R}^3
$$

and $p$ denotes the center of $B$. Note that the function $\Psi_B$ belongs to $H^1(\mathbb{R}^3)$ since $\nabla \Psi_B(x) = -\frac{x - p}{\|x - p\|}\chi_B(x)$ in the sense of distribution. The solution $v_B$ of (1.3) is constructed by using the theory of $C_0$-semigroups [13]. The class where $v_B$ belongs to is the following:

$$C^2([0, T], L^2(\mathbb{R}^3)) \cap C^1([0, T], H^1(\mathbb{R}^3)) \cap C([0, T], H^2(\mathbb{R}^3)).$$

Needless to say, $v_B$ has an explicit analytical expression, however, we never make use of such expression in the time domain. We need just the existence of $v_B$ in function spaces indicated above.

The following function is the special $f$ in the problem mentioned above:

$$f_{B, \lambda} = f_{B, \lambda}(\cdot, t) = \frac{\partial}{\partial \nu}v_{B, \lambda}(\cdot, t), \ t \in [0, T].$$

(1.4)

Now we construct the solution $u = u_f$ of (1.1) by prescribing $f = f_{B, \lambda}$ and define

$$w_{B, \lambda}(x, \tau) = \int_0^\tau e^{-\tau t}u_f(x, t) \, dt, \ x \in \Omega \setminus \overline{D}, \ \tau > 0,$n

(1.5)

and

$$w_{B, \lambda}^0(x, \tau) = \int_0^\tau e^{-\tau t}v_{B, \lambda}(x, t) \, dt, \ x \in \mathbb{R}^3, \ \tau > 0.$$

(1.6)

We define

$$I_{\partial\Omega}(\tau; B, \lambda) = \int_{\partial\Omega} (w_{B, \lambda} - w_{B, \lambda}^0) \frac{\partial w_{B, \lambda}}{\partial \nu} \, dS, \ \tau > 0.$$
This is the indicator function in the enclosure method discussed in this paper. Since we have
\[ \frac{\partial w_{B,\lambda}}{\partial \nu} = \frac{\partial w_{B,\lambda}^0}{\partial \nu}, \]
this indicator function has the form
\[ I_{\partial \Omega}(\tau; B, \lambda) = \int_{\partial \Omega} \left( w_{B,\lambda} \frac{\partial w_{B,\lambda}^0}{\partial \nu} - w_{B,\lambda}^0 \frac{\partial w_{B,\lambda}}{\partial \nu} \right) dS. \]

This indicator function can be computed from the response \( u_f \) on \( \partial \Omega \) over the time interval \([0, T]\) which is the solution of (1.1) with \( f = f_B \).

**Theorem 1.1 (Hardening).**

(i) If \( T \) satisfies
\[ T > \lambda \text{ dist}(\Omega, B) \] (1.7)
then, there exists a positive number \( \tau_0 \) such that \( I_{\partial \Omega}(\tau; B, \lambda) > 0 \) for all \( \tau \geq \tau_0 \), and we have
\[ \lim_{\tau \to \infty} \frac{1}{\sqrt{\tau}} \log I_{\partial \Omega}(\tau; B, \lambda) = -2 \lambda \text{ dist}(\Omega, B). \] (1.8)

(ii) We have
\[ \lim_{\tau \to \infty} e^{\tau T} I_{\partial \Omega}(\tau; B, \lambda) = \begin{cases} \infty & \text{if } T > 2\lambda \text{ dist}(\Omega, B), \\ 0 & \text{if } T < 2\lambda \text{ dist}(\Omega, B). \end{cases} \] (1.9)

(iii) If \( T = 2\lambda \text{ dist}(\Omega, B) \), then \( e^{\tau T} I_{\partial \Omega}(\tau; B, \lambda) = O(\tau^3) \) as \( \tau \to \infty \).

As we have already known from [3, 5, 10, 11, 12] there should be various possibilities of the choice of the Neumann data \( f \) in (1.1) to extract information about the geometry of an unknown cavity from \( u \) on \( \partial \Omega \times [0, T] \). Theorem 1.1 shows that if one chooses the Neumann data \( f = f_{B,\lambda} \) which comes from the solution of the wave equation with a fixed parameter \( \lambda \), then, the leading profile of the indicator function does not yield any information about the cavity. The choice was bad! Note also that from (1.8) we have
\[ \lim_{\tau \to \infty} \frac{1}{\sqrt{\tau}} \log I_{\partial \Omega}(\tau; B, \lambda) = -\infty. \]

However, there is another possibility of the choice of \( f \) in (1.1). Given \( \tau > 0 \) choose \( \lambda > 0 \) in (1.4) in such a way that
\[ \lambda^2 \tau^2 - \tau = 0, \] (1.10)
that is
\[ \lambda = \frac{1}{\sqrt{\tau}}. \] (1.11)

Then, we have

**Theorem 1.2 (Penetrating).**

(i) Let \( T \) be an arbitrary positive number. Then, there exists a positive number \( \tau_0 \) such that \( I_{\partial \Omega}(\tau; B, \frac{1}{\sqrt{\tau}}) > 0 \) for all \( \tau \geq \tau_0 \), and we have
\[ \lim_{\tau \to \infty} \frac{1}{\sqrt{\tau}} \log I_{\partial \Omega}(\tau; B, \frac{1}{\sqrt{\tau}}) = -2 \text{ dist}(D, B). \] (1.12)
(ii) We have
\[
\lim_{\tau \to \infty} e^{\sqrt{\tau} T} I_{\partial \Omega}(\tau; B, \frac{1}{\sqrt{\tau}}) = \begin{cases} 
\infty & \text{if } T > 2 \text{dist} (D, B), \\
0 & \text{if } T < 2 \text{dist} (D, B). 
\end{cases} \tag{1.13}
\]

(iii) If \( T = 2 \text{dist} (D, B) \), then \( e^{\sqrt{\tau} T} I_{\partial \Omega}(\tau; B, \frac{1}{\sqrt{\tau}}) = O(1) \) as \( \tau \to \infty \).

Note that, in this thereom the Neumann data \( f \) is given by (1.4) with \( \lambda \) given by (1.11). Thus, the input data used in Theorem 1.2 vary as \( \tau \to \infty \) and in this sense, they are infinitely many unlike those of Theorem 1.1. The equation (1.2) with \( \lambda \) given by (1.11) becomes the wave equation with propagation speed \( \sqrt{\tau} \):
\[
(\partial_t^2 - \tau \Delta) v = 0. \tag{1.14}
\]

So, this shoulde be called the enclosure method for the heat equation using a solution of the wave equation with growing propagation speed. One may consider the limit in Theorem 1.2 is a kind of non-relativistic limit. Since the speed of (1.14) grows to infinity, we do not need the waiting time for collecting the observation data unlike (1.7). Note that the role of \( T \) in (1.12) and (1.13) is different. In (1.12) \( T \) is an arbitrary, however, to get \( \text{dist} (D, B) \) by using (1.13) only we need, all \( T \in ]0, T_0[ \) with \( T_0 > 2 \text{dist} (D, B) \).

It seems that the equation (1.10) means the vanishing of an obstruction which prevents the temperature field generated by the flux (1.4) from entering deep inside of the body \( \Omega \setminus \overline{D} \). As an evidence we have different formulae (1.8) and (1.12). See (2.6) and (2.14) in Section 2 for an explicit role of equation (1.10).

However, we have a question about (1.11). If \( \lambda \) does not satisfy (1.11) exactly, then what happens on the asymptotic behavior of the indicator function? Here instead of (1.11) we choose the case when \( \lambda \) is given by
\[
\lambda = \sqrt{\frac{c}{\tau}}, \tag{1.15}
\]
where \( c \) is a positive constant. Then (1.10) becomes
\[
\lambda^2 \tau^2 - \tau = (c - 1) \tau
\]
and equation (1.14)
\[
(\partial_t^2 - \frac{\tau \Delta}{c}) v = 0.
\]
This equation also has a growing propagation speed as \( \tau \to \infty \).

In the following result we show that the indicator function \( I_{\partial \Omega}(\tau; B, \sqrt{\frac{c}{\tau}}) \) has a different asymptotic behavior across \( c = 1 \).

**Theorem 1.3 (Discontinuity across \( c = 1 \)).**

(i) Let \( T \) be an arbitrary positive number. Let \( \pm (c - 1) > 0 \). Then, there exists a positive number \( \tau_0 \) such that \( \pm I_{\partial \Omega}(\tau; B, \sqrt{\frac{c}{\tau}}) > 0 \) for all \( \tau \geq \tau_0 \), and we have
\[
\lim_{\tau \to \infty} \frac{1}{\sqrt{\tau}} \log \left| I_{\partial \Omega}(\tau; B, \sqrt{\frac{c}{\tau}}) \right| = -2\sqrt{c} \text{dist} (\Omega, B). \tag{1.16}
\]

\( \text{proof} \)

The proof of Theorem 1.3 follows from (1.10) and (1.11) for \( \lambda = \sqrt{\frac{c}{\tau}} \).
(ii) Let $\pm(c-1) > 0$. We have
\[
\lim_{\tau \to \infty} e^{\sqrt{\tau} T} I_{\partial \Omega}(\tau; B, \sqrt{\tau}) = \begin{cases}
\pm \infty & \text{if } T > 2\sqrt{c} \text{ dist}(\Omega, B), \\
0 & \text{if } T < 2\sqrt{c} \text{ dist}(\Omega, B).
\end{cases}
\] (1.17)

(iii) If $T = 2\sqrt{c} \text{ dist}(D, B)$, then $e^{\sqrt{T} T} I_{\partial \Omega}(\tau; B, \sqrt{\tau}) = O(1)$ as $\tau \to \infty$.

(iv) Let $c > 0$. We have
\[
c - 1 \leq \liminf_{\tau \to \infty} \frac{I_{\partial \Omega}(\tau; B, \sqrt{\tau})}{\tau \int_{\Omega} |w_0|^2 dx} \leq \limsup_{\tau \to \infty} \frac{I_{\partial \Omega}(\tau; B, \sqrt{\tau})}{\tau \int_{\Omega} |w_0|^2 dx} \leq (c-1)c.
\] (1.18)

Note that in (1.18) the case when $c = 1$ is also covered. From Theorems 1.2 and 1.3 we see that the asymptotic behaviour of indicator function $I_{\partial \Omega}(\tau; B, \sqrt{\tau})$ has a jump discontinuity at $c = 1$. Only at $c = 1$ one can extract information about the cavity from the leading profile of the indicator function as $\tau \to \infty$.

**Remark 1.1.** Instead of $f = f_{B, \lambda}$ given by (1.4) prescribe the $f$ in (1.1) as
\[
f = kf_{B, \lambda},
\]
where $k$ is a non zero real constant. The new indicator function should be
\[
\tilde{I}_{\partial \Omega}(\tau; B, \lambda) = \int_{\partial \Omega} \left( \tilde{w}_{B, \lambda} \frac{\partial w_0}{\partial \nu} - w_0 \frac{\partial \tilde{w}_{B, \lambda}}{\partial \nu} \right) dS,
\]
where the function $\tilde{w}_{B, \lambda}$ is given by (1.5) with $f = kf_{B, \lambda}$. Since we have
\[
uf_{B, \lambda} = kuf_{B, \lambda},
\]
once obtains $\tilde{w}_{B, \lambda} = kw_{B, \lambda}$. This yields
\[
\tilde{I}_{\partial \Omega}(\tau; B, \lambda) = kI_{\partial \Omega}(\tau; B, \lambda).
\]
Thus, everything is reduced to studying the case when $k = 1$.

Before closing Introduction, we describe some estimates on $v_{B, \lambda}$ and $u_f$ with $f = f_{B, \lambda}$ given by (1.4) which are employed in Section 2.

The $v_{B, \lambda}$ is given by a scaling of the classical wave equation. More precisely, let $v_0 = v_0(x, s)$ solve
\[
\begin{cases}
(\partial_s^2 - \Delta)v = 0 & \text{in } \mathbb{R}^3 \times ]0, \infty[, \\
v(x, 0) = 0 & \text{in } \mathbb{R}^3, \\
\partial_s v(x, 0) = \Psi_B(x) & \text{in } \mathbb{R}^3.
\end{cases}
\]
Then the $v_{B, \lambda}$ is given by
\[
v_{B, \lambda}(x, t) = \lambda v_0(x, \frac{1}{\lambda} t), \quad (x, t) \in \mathbb{R}^3 \times ]0, T[.
\]
Thus $f_{B, \lambda}$ is given by

$$f_{B, \lambda}(x, t) = \lambda \frac{\partial}{\partial \nu} v_0(x, \frac{1}{\lambda} t), \quad (x, t) \in \partial \Omega \times [0, T].$$

Using the Fourier transform of $v_0$ with respect to $x \in \mathbb{R}^3$, we have, for all $s > 0$

$$\frac{1}{s} \|v_0(\cdot, s)\|_{L^2(\mathbb{R}^3)} + \|\partial_s v_0(\cdot, s)\|_{L^2(\mathbb{R}^3)} \leq 2\|\Psi_B\|_{L^2(\mathbb{R}^3)} \equiv C_B.$$ 

Thus, one has

$$\frac{1}{T} \|v_{B, \lambda}(\cdot, T)\|_{L^2(\mathbb{R}^3)} + \|\partial_t v_{B, \lambda}(\cdot, T)\|_{L^2(\mathbb{R}^3)} \leq C_B. \quad (1.19)$$

Moreover, the Fourier transform of $v_0$ with respect to $x \in \mathbb{R}^3$ yields also

$$\|v_0(\cdot, s)\|_{H^1(\mathbb{R}^3)} \leq C_B \sqrt{s^2 + 3}.$$ 

This together with the trace theorem yields

$$\|f_{B, \lambda}(\cdot, t)\|_{H^{1/2}(\partial \Omega)} \leq C_{\Omega} C_B \sqrt{T^3 + 3\lambda^2}$$ 

and hence

$$\|f_{B, \lambda}\|_{L^2(0, T; H^{-1/2}(\partial \Omega))} \leq C_{\Omega} C_B \sqrt{T^3 + 3\lambda^2 T}.$$ 

Thus one gets

$$\|u_f(\cdot, T)\|_{L^2(\Omega \setminus \overline{D})} \leq C \sqrt{T^3 + 3\lambda^2 T}, \quad (1.20)$$

where $C = C_T C_{\Omega} C_B$. Note that $C$ is independent of $\lambda$.

## 2 Proof of Theorems

In this section, for simplicity of description we always write

$$w = w_{B, \lambda}, \quad w_0 = w_{B, \lambda}^0, \quad R = w - w_0,$$

where $w_{B, \lambda}$ and $w_{B, \lambda}^0$ are given by (1.5) and (1.6), respectively.

### 2.1 A decomposition formula of the indicator function

It follows from (1.1) that $w$ satisfies

$$\begin{cases}
(\Delta - \tau)w = e^{-\tau T} u(x, T) \quad \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial w}{\partial \nu} = \frac{\partial w_0}{\partial \nu} \quad \text{on } \partial \Omega, \\
\frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial D.
\end{cases}$$
Rewrite this as
\[
\begin{aligned}
(\Delta - \lambda^2 \tau^2)w &= e^{-\tau T}F \quad \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial w}{\partial \nu} &= \frac{\partial w_0}{\partial \nu} \quad \text{on } \partial \Omega, \\
\frac{\partial w}{\partial \nu} &= 0 \quad \text{on } \partial D,
\end{aligned}
\]  
(2.1)

where
\[
F = F(x, \tau) = u(x, T) + e^{\tau T} (\tau - \lambda^2 \tau^2)w, \quad x \in \Omega \setminus \overline{D}.
\]  
(2.2)

It follows from (1.3) that the \(w_0\) satisfies
\[
(\Delta - \lambda^2 \tau^2)w_0 + \lambda^2 \Psi_B = e^{-\tau T} \lambda^2 F_0 \quad \text{in } \mathbb{R}^3,
\]  
(2.3)

where
\[
F_0 = F_0(x, \tau) = \partial_t v_{B, \lambda}(x, T) + \tau v_{B, \lambda}(x, T) \quad \text{in } \mathbb{R}^3.
\]

Note that, from (1.19) we have
\[
\|F_0\|_{L^2(\mathbb{R}^3)} \leq C(1 + \tau),
\]  
(2.4)

where \(C\) is a positive constant being independent of \(\lambda\).

Then, integration by parts together with (2.1) and (2.3) in \(\Omega\) yields
\[
\int_{\partial \Omega} \left( \frac{\partial w_0}{\partial \nu} w - \frac{\partial w}{\partial \nu} w_0 \right) dS = \int_{\partial D} w \frac{\partial w_0}{\partial \nu} dS + e^{-\tau T} \int_{\Omega \setminus \overline{D}} (\lambda^2 F_0 w - F w_0) dx,
\]
and hence
\[
I_{\partial \Omega}(\tau; B, \lambda) = \int_{\partial D} w \frac{\partial w_0}{\partial \nu} dS + e^{-\tau T} \int_{\Omega \setminus \overline{D}} (\lambda^2 F_0 w - F w_0) dx.
\]  
(2.5)

This is the first representation of the indicator function. Next we decompose the first term on the right-hand side of (2.5). The result yields the following decomposition formula.

**Proposition 2.1.** We have
\[
I_{\partial \Omega}(\tau; B, \lambda) = \int_{\Omega} (\lambda^2 \tau^2 - \tau) |w_0|^2 \, dx + J_h(\tau) + E_h(\tau) + R(\tau),
\]  
(2.6)

where
\[
J_h(\tau) = \int_D (|\nabla w_0|^2 + \tau |w_0|^2) \, dx, 
\]  
(2.7)

\[
E_h(\tau) = \int_{\Omega \setminus \overline{D}} (|\nabla R|^2 + \tau |R|^2) \, dx
\]  
(2.8)

and
\[
R(\tau) = e^{-\tau T} \left\{ \int_D \lambda^2 F_0 w_0 dx + \int_{\Omega \setminus \overline{D}} u(x, T) R dx + \int_{\Omega \setminus \overline{D}} (\lambda^2 F_0 - u(x, T)) w_0 dx \right\}.
\]  
(2.9)

**Proof.** First we show that
\[
I_{\partial \Omega}(\tau; B, \lambda) = J(\tau) + E(\tau) + \tilde{R}(\tau),
\]  
(2.10)
where

\[
J(\tau) = \int_D (|\nabla w_0|^2 + \lambda^2 \tau^2 |w_0|^2) \, dx,
\]

\[
E(\tau) = \int_{\Omega \setminus \overline{D}} (|\nabla R|^2 + \lambda^2 \tau^2 |R|^2) \, dx
\]

and

\[
\tilde{R}(\tau) = e^{-\tau T} \left\{ \int_D \lambda^2 F_0 w_0 \, dx + \int_{\Omega \setminus \overline{D}} FR \, dx + \int_{\Omega \setminus \overline{D}} (\lambda^2 F_0 - F) w_0 \, dx \right\}.
\]

The proof of (2.10) is now standard in the enclosure method, however, in the next section we make use of an equation appearing in the proof. So for the reader’s convenience we present the proof.

Since \(\overline{B} \cap \overline{\Omega} = \emptyset\), we have that the \(R\) satisfies

\[
\begin{aligned}
\Delta R &= e^{-\tau T} \{ F - \lambda^2 F_0 \} \quad \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial R}{\partial \nu} &= 0 \quad \text{on } \partial \Omega, \\
\frac{\partial R}{\partial \nu} &= -\frac{\partial w_0}{\partial \nu} \quad \text{on } \partial D.
\end{aligned}
\]

Then, one can write

\[
\int_{\partial \overline{D}} w \frac{\partial w_0}{\partial \nu} \, dS = \int_{\partial \overline{D}} \frac{w_0}{\partial \nu} \, dS - \int_{\partial \overline{D}} \frac{\partial R}{\partial \nu} \, dS.
\]

It follows from (2.3) in \(D\) that

\[
\int_{\partial \overline{D}} \frac{w_0}{\partial \nu} \, dS = \int_D (|\nabla w_0|^2 + \lambda^2 \tau^2 |w_0|^2) \, dx + e^{-\tau T} \int_D \lambda^2 F_0 w_0 \, dx.
\]

It follows from (2.12) that

\[
- \int_{\partial \overline{D}} \frac{\partial R}{\partial \nu} \, dS = \int_{\partial (\Omega \setminus \overline{D})} \frac{\partial R}{\partial \nu} \, dS = \int_{\Omega \setminus \overline{D}} (|\nabla R|^2 + \lambda^2 \tau^2 |R|^2) \, dx + e^{-\tau T} \int_{\Omega \setminus \overline{D}} (F - \lambda^2 F_0) R \, dx.
\]

Thus we obtain

\[
\int_{\partial D} w \frac{\partial w_0}{\partial \nu} \, dS
\]

\[
= \int_D (|\nabla w_0|^2 + \lambda^2 \tau^2 |w_0|^2) \, dx + \int_{\Omega \setminus \overline{D}} (|\nabla R|^2 + \lambda^2 \tau^2 |R|^2) \, dx
\]

\[
+ e^{-\tau T} \left\{ \int_D \lambda^2 F_0 w_0 \, dx + \int_{\Omega \setminus \overline{D}} (F - \lambda^2 F_0) R \, dx \right\}.
\]
Then a combination of this and (2.5) yields (2.10).

Substituting (2.2) into the right-hand side on (2.11), we can rewrite (2.11) as

\[
\tilde{R}(\tau) = e^{-\tau T} \left\{ \int_D \lambda^2 F_0 w_0 dx + \int_{\Omega \setminus D} u(x, T) Rd x + \int_{\Omega \setminus D} (\lambda^2 F_0 - u(x, T)) w_0 dx \right\} \\
+ \int_{\Omega \setminus D} (\tau - \lambda^2 \tau^2) (|R|^2 - |w_0|^2) \, dx \\
= e^{-\tau T} \left\{ \int_D \lambda^2 F_0 w_0 dx + \int_{\Omega \setminus D} u(x, T) Rd x + \int_{\Omega \setminus D} (\lambda^2 F_0 - u(x, T)) w_0 dx \right\} \\
+ \int_{\Omega \setminus D} (\tau - \lambda^2 \tau^2) |R|^2 \, dx - \int_{\Omega} (\tau - \lambda^2 \tau^2) |w_0|^2 \, dx + \int_{D} (\tau - \lambda^2 \tau^2) |w_0|^2 \, dx.
\]

Then (2.10) becomes (2.6).

\[\square\]

2.2 Estimating indicator functions

First we give a rough estimate of \( E_h(\tau) \) from above in terms of \( J_h(\tau) \) and \( \|w_0\|_{L^2(\Omega \setminus D)} \).

Lemma 2.1. Let \( \epsilon > 0 \). We have, as \( \tau \to \infty \)

\[
E_h(\tau) \leq C_1 (\lambda^4 \tau^3 + 1) J_h(\tau) + C_2(\epsilon) M(\lambda, \tau) e^{-2\tau T} \\
+ \frac{1 + \epsilon}{\tau} \int_{\Omega \setminus D} (\lambda^2 \tau^2 - \tau)^2 |w_0|^2 \, dx,
\]

where \( C_1 \) and \( C_2(\epsilon) \) are positive constants being independent of \( \lambda \) and \( \tau \);

\[
M(\lambda, \tau) = \lambda^4 (1 + \tau)^2 + \frac{1}{\tau} \{(1 + \lambda^2) + \lambda^4 (1 + \tau)^2\}.
\]

Proof. It follows from the boundary condition on \( \partial D \) in (2.12) and (2.13) that

\[
\int_{\Omega \setminus D} \left( |\nabla R|^2 + \lambda^2 \tau^2 |R|^2 + e^{-\tau T} (F - \lambda^2 F_0) R \right) \, dx \\
= \int_{\partial D} \frac{\partial w_0}{\partial n} Rd S.
\]
Using (2.2), we have

\[
\lambda^2 \tau^2 |R|^2 + e^{-\tau T}(F - \lambda^2 F_0)R
\]

\[
= \lambda^2 \tau^2 |R|^2 + (\tau - \lambda^2 \tau^2)w R + e^{-\tau T}(u(x, T) - \lambda^2 F_0)R
\]

\[
= \lambda^2 \tau^2 |R|^2 + (\tau - \lambda^2 \tau^2)(w_0 + R)R + e^{-\tau T}(u(x, T) - \lambda^2 F_0)R
\]

\[
= \tau |R|^2 + (\tau - \lambda^2 \tau^2)w_0 R + e^{-\tau T}(u(x, T) - \lambda^2 F_0)R
\]

\[
= \tau \left( R + \frac{(\tau - \lambda^2 \tau^2)w_0 + e^{-\tau T}(u(x, T) - \lambda^2 F_0)}{2\tau} \right)^2
\]

\[
\leq \frac{|(\tau - \lambda^2 \tau^2)w_0 + e^{-\tau T}(u(x, T) - \lambda^2 F_0)|^2}{4\tau}
\]

and thus

\[
\int_{\Omega \setminus D} \left( |\nabla R|^2 + \tau \left| R + \frac{(\tau - \lambda^2 \tau^2)w_0 + e^{-\tau T}(u(x, T) - \lambda^2 F_0)}{2\tau} \right|^2 \right) \, dx
\]

\[
= \int_{\partial D} \frac{\partial w_0}{\partial \nu} R dS + \frac{1}{4\tau} \int_{\Omega \setminus D} |(\tau - \lambda^2 \tau^2)w_0 + e^{-\tau T}(u(x, T) - \lambda^2 F_0)|^2 \, dx.
\]

This yields

\[
\frac{1}{2} \int_{\Omega \setminus D} (|\nabla R|^2 + \tau |R|^2) \, dx
\]

\[
\leq \int_{\partial D} \frac{\partial w_0}{\partial \nu} R dS + \frac{1}{2\tau} \int_{\Omega \setminus D} |(\tau - \lambda^2 \tau^2)w_0 + e^{-\tau T}(u(x, T) - \lambda^2 F_0)|^2 \, dx
\]

and thus

\[
\int_{\Omega \setminus D} (|\nabla R|^2 + \tau |R|^2) \, dx
\]

\[
\leq 2 \int_{\partial D} \frac{\partial w_0}{\partial \nu} R dS + \frac{1}{\tau} \int_{\Omega \setminus D} |(\tau - \lambda^2 \tau^2)w_0 + e^{-\tau T}(u(x, T) - \lambda^2 F_0)|^2 \, dx.
\]

From (1.20) and (2.4) we have

\[
\|u(x, T) - \lambda^2 F_0\|_{L^2(\Omega \setminus D)} \leq C_3(\sqrt{1 + \lambda^2} + \lambda^2(1 + \tau)),
\]

where $C_3$ is a positive constant being independent of $\lambda$ and $\tau$. This together with (2.8) and the inequality

\[
(a + b)^2 \leq (1 + \epsilon)a^2 + (1 + 4\epsilon^{-1})b^2, \quad a > 0, \quad b > 0,
\]

yields

\[
E_h(\tau) \leq 2 \int_{\partial D} \frac{\partial w_0}{\partial \nu} R dS + \frac{1 + \epsilon}{\tau} \int_{\Omega \setminus D} (\lambda^2 \tau^2 - \tau)^2 |w_0|^2 \, dx + C_4 C_3^2 K(\lambda, \tau) e^{-2\tau T},
\]

(2.15)
where $C_\epsilon = 1 + 4\epsilon^{-1}$ and

$$K(\lambda, \tau) = \frac{1}{\tau}(\sqrt{1 + \lambda^2 + \lambda^2(1 + \tau)^2}).$$

By the trace theorem [2], one can choose a positive constant $C = C(D, \Omega)$ and $\tilde{R} \in H^1(D)$ such that $\tilde{R} = R$ on $\partial D$ and $\|\tilde{R}\|_{H^1(D)} \leq C\|R\|_{H^1(\Omega \setminus D)}$. Then, we have

$$\int_{\partial D} \frac{\partial w_0}{\partial \nu} R dS = \int_{\partial D} \frac{\partial w_0}{\partial \nu} \tilde{R} dS = \int_D (\Delta w_0) \tilde{R} dx + \int_D \nabla w_0 \cdot \nabla \tilde{R} dx \quad \text{and} \quad \int_D \lambda^2 \tau^2 w_0 \tilde{R} dx + \int_D \lambda^2 \tau^2 |w_0|^2 dx + e^{-\tau T} \int_D \lambda^2 F_0 \tilde{R} dx.$$

Note that in the last step, we have made use of equation (2.3) on $D$. Then the choice of $\tilde{R}$ and (2.4) yield

$$\left| \int_{\partial D} \frac{\partial w_0}{\partial \nu} R dS \right| \leq C\|R\|_{H^1(\Omega \setminus D)} \left( \lambda^2 \tau^2 \|w_0\|_{L^2(D)} + \|\nabla w_0\|_{L^2(D)} + C_4 \lambda^2 (1 + \tau)e^{-\tau T} \right).$$

Here we note that $\|R\|_{H^1(\Omega \setminus D)} \leq E_h(\tau)^{1/2}$ for all $\tau \geq 1$, and $\|w_0\|_{L^2(D)} \leq \tau^{-1/2} J_h(\tau)^{1/2}$, $\|\nabla w_0\|_{L^2(D)} \leq J_h(\tau)^{1/2}$ for all $\tau > 0$. From these, (2.15) and (2.16) we obtain

$$E_h(\tau) \leq C'E_h(\tau)^{1/2} \left\{ (\lambda^2 \tau^3 + 1)J_h(\tau)^{1/2} + C_4 \lambda^2 (1 + \tau)e^{-\tau T} \right\}$$

$$+ \frac{1 + \epsilon}{\tau} \int_{\Omega \setminus B} (\lambda^2 \tau^2 - \tau^2 |w_0|^2 dx + C_\epsilon C_3^2 K(\lambda, \tau)e^{-2\tau T},$$

where $C'$ is a positive constant being independent of $\lambda$ and $\tau$. Now a standard argument yields (2.14).

\[\square\]

**Remark 2.1.** It is very important to have the factor $1 + \epsilon$ in the third term of the right-hand side on (2.14). This yields the upper bound (2.30) for the indicator function $I_{\partial D}(\tau; B, \sqrt{\lambda}).$

Next we describe local upper and lower estimates for $w_0$.

**Lemma 2.2 (Propagation estimates).** Let $U$ be an arbitrary bounded open subset of $\mathbb{R}^3$ such that $\overline{B} \cap \overline{U} = \emptyset$.

(i) We have

$$\lambda \tau \|w_0\|_{L^2(U)} + \|\nabla w_0\|_{L^2(U)} \leq C \left\{ \lambda^3 \tau e^{-\tau \lambda \operatorname{dist}(U, B)} + \frac{\lambda(1 + \tau)}{\tau} e^{-\tau T} \right\},$$

where $C_\epsilon$ is a positive constant being independent of $\lambda$ and $\tau$. Now a standard argument yields (2.14).
where $C$ is a positive constant being independent of $\lambda$ and $\tau$.

(ii) Let $\partial U$ be $C^2$. Fix $\lambda$ and let $T$ satisfy

$$T > \lambda \text{dist}(U, B).$$

Then, there exist positive constants $\tau_0$ and $C$ such that, for all $\tau \geq \tau_0$

$$\tau^{12} e^{2\tau \lambda \text{dist}(U, B)} \int_U |w_0|^2 \, dx \geq C. \quad (2.18)$$

(iii) Let $\partial U$ be $C^2$. Let $\lambda = \sqrt{\frac{C}{\tau}}$ with a positive constant $c$. Let $T$ be an arbitrary positive number. Then, there exist positive constants $\tau_0$ and $C$ such that, for all $\tau \geq \tau_0$

$$\tau^8 e^{2\sqrt{\tau} \text{dist}(U, B)} \int_U |w_0|^2 \, dx \geq C. \quad (2.19)$$

Proof. We set

$$\epsilon_0 = e^{\tau T} (w_0 - v_0),$$

where $v_0 \in H^1(\mathbb{R}^3)$ is the solution of

$$(\Delta - \lambda^2 \tau^2) v_0 + \lambda^2 \Psi_B = 0 \quad \text{in} \quad \mathbb{R}^3.$$ 

The $v_0$ has the explicit form

$$v_0(x) = \frac{\lambda^2}{4\pi} \int_B \frac{e^{-\lambda |x-y|}}{|x-y|} (\eta - |y-p|) \, dy. \quad (2.20)$$

We have

$$w_0 = v_0 + e^{-\tau T} \epsilon_0$$

and, from (2.3),

$$(\Delta - \lambda^2 \tau^2) \epsilon_0 = \lambda^2 \Psi_B \quad \text{in} \quad \mathbb{R}^3. \quad (2.21)$$

Then, from (2.4) and (2.22) we can easily see that

$$\lambda \tau \|\epsilon_0\|_{L^2(\mathbb{R}^3)} + \|\nabla \epsilon_0\|_{L^2(\mathbb{R}^3)} \leq C_5 \frac{\lambda (1 + \tau)}{\tau}, \quad (2.22)$$

where $C_5$ is a positive constant being independent of $\lambda$ and $\tau$.

The expression (2.21) for $v_0$ yields

$$\lambda \tau \|v_0\|_{L^2(\mathbb{R}^3)} + \|\nabla v_0\|_{L^2(\mathbb{R}^3)} \leq C_6 \lambda^3 \tau e^{-\tau \lambda \text{dist}(U, B)}, \quad (2.23)$$

where $C_6$ is a positive constant being independent of $\lambda$ and $\tau$. A combination of (2.22) and (2.24) gives (2.17).

It follows from (2.23) that

$$\int_U |w_0|^2 \, dx \geq \frac{1}{2} \int_U |v_0|^2 \, dx - C_5 \left( \frac{1 + \tau}{\tau} \right)^2 \frac{e^{-2\tau T}}{\tau^2}. \quad (2.24)$$
From Appendix in [7], we know that (2.21) has the expression

\[ v_0(x) = \frac{1}{\tau^4 \lambda^2} e^{-\tau \lambda |x-p|} (-2 \cosh(\tau \lambda \eta) + \tau \lambda \eta \sinh(\tau \lambda \eta) + 2). \]

This yields, for sufficiently large \( \tau \)

\[
\int_U |v_0|^2 \, dx \geq \begin{cases} 
C_7^2 \tau^{-6} \int_U \frac{e^{-2\tau \lambda (|x-p|-\eta)}}{|x-p|^2} \, dx & \text{if } \lambda \text{ is fixed}, \\
C_8^2 \tau^{-5} \int_U \frac{e^{-2\sqrt{\tau} \sqrt{|x-p|-\eta}}}{|x-p|^2} \, dx & \text{if } \lambda = \sqrt{\frac{c}{\tau}},
\end{cases}
\]

(2.25)

where \( C_7 \) is a positive constant being independent of \( \tau \); \( C_8 \) being independent of \( \tau \) and \( \lambda \). Assume that \( \partial U \) is \( C^2 \). In [4, 6] we have already proved that, there exist positive constants \( \tau_0 \) and \( C' \) such that, for all \( \tau \geq \tau_0 \)

\[
\tau^6 e^{2\tau \text{dist}(U,B)} \int_U \frac{e^{-2\tau (|x-p|-\eta)}}{|x-p|^2} \, dx \geq C'
\]

and thus

\[
\tau^3 e^{2\sqrt{\tau} \sqrt{\text{dist}(U,B)}} \int_U \frac{e^{-2\sqrt{\tau} \sqrt{|x-p|-\eta}}}{|x-p|^2} \, dx \geq C'.
\]

Now applying these to the right-hand side on (2.25), we see the validity of (ii) and (iii).

\[ \square \]

**Remark 2.2.** In the proof of (2.17) the estimate (2.24) is essential. For this purpose, we made use of the expression of \( v_0 \) given by (2.21) only.

Here we describe preliminary estimates for the indicator function.

**Lemma 2.3.** Let \( T \) be an arbitrary positive number.

(i) Fix \( \lambda > 0 \). We have, as \( \tau \to \infty \)

\[
I_{\partial \Omega}(\tau; B, \lambda) = O(\tau^3 e^{-2\tau \lambda \text{dist}(\Omega,B)} + \tau^3 e^{-2\tau T} + \tau e^{-\tau T} e^{-\tau \lambda \text{dist}(\Omega,B)})
\]

(2.26)

and

\[
I_{\partial \Omega}(\tau; B, \lambda) \geq \int_\Omega (\lambda^2 \tau^2 - \tau)|w_0|^2 \, dx + O(\tau e^{-\tau T} e^{-\tau \lambda \text{dist}(\Omega,B)} + \tau e^{-2\tau T}).
\]

(2.27)

(ii) We have, as \( \tau \to \infty \)

\[
I_{\partial \Omega}(\tau; B, \frac{1}{\sqrt{T}}) = O(e^{-\sqrt{\tau} \text{dist}(D,B)} + \tau e^{-\tau T} e^{-\sqrt{\tau} \text{dist}(\Omega,B)})
\]

(2.28)

and

\[
I_{\partial \Omega}(\tau; B, \frac{1}{\sqrt{T}}) \geq \tau \int_D |w_0|^2 \, dx + O(\tau^{-1} e^{-\tau T} e^{-\sqrt{\tau} \text{dist}(\Omega,B)}).
\]

(2.29)

(iii) Let \( \epsilon \) be an arbitrary positive number. We have, as \( \tau \to \infty \)

\[
I_{\partial \Omega}(\tau; B, \sqrt{\frac{c}{\tau}}) \leq \tau(1 + \epsilon)(c - 1) \left( c - \frac{\epsilon}{1 + \epsilon} \right) \int_\Omega |w_0|^2 \, dx + O(\tau e^{-\tau T} e^{-2\sqrt{\tau} \sqrt{\text{dist}(\Omega,B)}})
\]

(2.30)
\[ I_{\partial \Omega}(\tau; B, \sqrt{\frac{c}{\tau}}) \]
\[ \geq \tau(c - 1) \int_{\Omega} |w_0|^2 \, dx + O(\tau e^{-\tau T} e^{-2\sqrt{\tau} \tau \text{dist}(\Omega, B)}) \]  

\textbf{Proof.} First we give a proof of (2.26) and (2.27). From (2.17) with \(* = D, \Omega \setminus \overline{D}\) we have

\[ \|w_0\|_{L^2(\star)} = O(\tau^{-\lambda \text{dist}(\star, B)} + \tau^{-1} e^{-\tau T}) \]  

(2.32)

and

\[ \|\nabla w_0\|_{L^2(\star)} = O(\tau e^{-\tau \lambda \text{dist}(\star, B)} + e^{-\tau T}) \].

Thus (2.7) gives

\[ J_h(\tau) = O(\tau^5 e^{-2\tau \lambda \text{dist}(D, B)} + \tau^3 e^{-2\tau \lambda \text{dist}(\Omega, B)} + \tau^3 e^{-2\tau T}) \]  

(2.33)

Since \( \text{dist}(D, B) > \text{dist}(\Omega, B) \), this yields

\[ E_h(\tau) = O(\tau^3 e^{-2\tau \lambda \text{dist}(\Omega, B)} + \tau^3 e^{-2\tau T}) \]  

(2.34)

Since \( \tau \|R\|^2_{L^2(\Omega \setminus \overline{D})} \leq E_h(\tau) \), (2.34) gives

\[ \|R\|_{L^2(\Omega \setminus \overline{D})} = O(\tau e^{-\tau \lambda \text{dist}(\Omega, B)} + \tau e^{-\tau T}) \].

This together with (1.20) gives

\[ \int_{\Omega \setminus \overline{D}} u(x, T) R \, dx = O(\tau e^{-\tau \lambda \text{dist}(\Omega, B)} + \tau e^{-\tau T}) \].

From (2.32), (1.20) and (2.4), we obtain

\[ \int_D \lambda^2 F_0 w_0 \, dx = O(\tau e^{-\tau \lambda \text{dist}(D, B)} + e^{-\tau T}) \]

and

\[ \int_{\Omega \setminus \overline{D}} (\lambda^2 F_0 - u(x, T)) w_0 \, dx = O(\tau e^{-\tau \lambda \text{dist}(\Omega, B)} + e^{-\tau T}) \].

Applying these to the right-hand side on (2.9), we obtain

\[ \mathcal{R}(\tau) = O(\tau e^{-\tau T} e^{-\tau \lambda \text{dist}(\Omega, B)} + \tau e^{-2\tau T}) \]  

(2.35)

Now applying (2.33), (2.34) and (2.35) to (2.6), we obtain

\[ I_{\partial \Omega}(\tau; B, \lambda) \]

\[ = \int_{\Omega} (\lambda^2 \tau^2 - \tau) |w_0|^2 \, dx \]

\[ + O(\tau^3 e^{-2\tau \lambda \text{dist}(\Omega, B)} + \tau^3 e^{-2\tau T} + \tau e^{-\tau T} e^{-\tau \lambda \text{dist}(\Omega, B)}) \]
From (2.6) and (2.40) we have
\[ I_{\partial \Omega}(\tau; B, \lambda) \geq \int_{\Omega}(\lambda^2 \tau^2 - \tau)|w_0|^2 \, dx + \mathcal{R}(\tau). \]

Now from (2.35) we obtain (2.26).

Next we give a proof of (2.28), (2.29), (2.30) and (2.31). Let \( \lambda \) be given by (1.15) which covers (1.11) as a special case. From (2.17) with \( \star = D, \Omega \setminus \overline{B} \) we have
\[ \|w_0\|_{L^2} = O(\tau^{-1} e^{-\sqrt{\tau}} \sqrt{\tau} \text{dist}(\star, B) + \tau^{-1} e^{-\tau T}) \] (2.36)
and
\[ \|\nabla w_0\|_{L^2} = O\left(\frac{1}{\sqrt{\tau}} \tau^{-1} e^{-\sqrt{\tau}} \sqrt{\tau} \text{dist}(\star, B) + \frac{1}{\sqrt{\tau}} e^{-\tau T}\right). \]

Thus one gets
\[ J_h(\tau) = O(\tau^{-1} e^{-2\sqrt{\tau}} \sqrt{\tau} \text{dist}(D, B) + \tau^{-1} e^{-2\tau T}). \] (2.37)

Moreover, from (2.14) we have
\[ E_h(\tau) \leq C_1(\tau + 1) J_h(\tau) + (1 + \epsilon)(c - 1)^2 \tau \int_{\Omega}|w_0|^2 \, dx + O(e^{-2\tau T}) \] (2.38)
and thus
\[ E_h(\tau) = O(e^{-2\sqrt{\tau}} \text{dist}(D, B) + e^{-2\tau T} + (c - 1)^2 \tau^{-1} e^{-2\sqrt{\tau}} \sqrt{\tau} \text{dist}(\Omega, B)). \] (2.39)

This yields
\[ \|R\|_{L^2(\Omega \setminus \overline{B})} = O\left(\frac{e^{-\sqrt{\tau}} \text{dist}(D, B)}{\sqrt{\tau}} + \frac{e^{-\tau T}}{\sqrt{\tau}} + |c - 1| e^{-\sqrt{\tau}} \sqrt{\tau} \text{dist}(\Omega, B)\right). \]

Then, from (2.9) we have
\[ \mathcal{R}(\tau) = O(e^{-\tau T} \left(\frac{1}{\tau} e^{-\sqrt{\tau}} \sqrt{\tau} \text{dist}(D, B) + \frac{1}{\tau} e^{-\tau T} + \frac{e^{-\sqrt{\tau}} \sqrt{\tau} \text{dist}(D, B)}{\sqrt{\tau}} + \frac{e^{-\tau T}}{\sqrt{\tau}} \right. \]
\[ + \frac{1}{\tau} e^{-\sqrt{\tau}} \sqrt{\tau} \text{dist}(\Omega, B) + \frac{1}{\tau} e^{-\tau T}) \right) \]
\[ + O(|c - 1| e^{-\tau T} e^{-\sqrt{\tau}} \sqrt{\tau} \text{dist}(\Omega, B)) \]
\[ = O(\tau^{-1} e^{-\tau T} e^{-\sqrt{\tau}} \sqrt{\tau} \text{dist}(\Omega, B)) + O(|c - 1| e^{-\tau T} e^{-\sqrt{\tau}} \sqrt{\tau} \text{dist}(\Omega, B)) \]
\[ = O(\tau^{-1} e^{-\tau T} e^{-\sqrt{\tau}} \sqrt{\tau} \text{dist}(\Omega, B)). \] (2.40)

From (2.6) and (2.40) we have
\[ I_{\partial \Omega}(\tau; B, \sqrt{\frac{c}{\tau}}) = J_h(\tau) + (c-1) \int_{\Omega} |w_0|^2 \, dx + E_h(\tau) + O(\tau^{-1} e^{-\tau T} e^{-\sqrt{\tau}} \sqrt{\tau} \text{dist}(\Omega, B)). \] (2.41)
Let \( c = 1 \). Applying (2.36) with \( \star = \Omega \), (2.37) and (2.39) to the right-hand side on (2.41) we obtain (2.28). (2.29) is now clear.

Next consider the case when \( c \neq 1 \). Since we have
\[
(c - 1) + (1 + \epsilon)(c - 1)^2 = (1 + \epsilon)(c - 1) \left( c - \frac{\epsilon}{1 + \epsilon} \right),
\]
from (2.38) one gets
\[
\tau(c - 1) \int_\Omega |w_0|^2 \, dx + E_h(\tau) \leq C_1 (\tau + 1) J_h(\tau) + (1 + \epsilon)(c - 1)
\]
\[
\times \left( c - \frac{\epsilon}{1 + \epsilon} \right) \tau \int_\Omega |w_0|^2 \, dx + O(e^{-2\tau T}).
\]
This together with (2.41) and (2.37) yields (2.30). (2.31) is an easy consequence of (2.41).

\[\text{\Box}\]

2.3 Proof of Theorems 1.1 and 1.2

First we give a proof of Theorem 1.1. Let \( T \leq 2\lambda \text{dist} (\Omega, B) \). It follows from (2.26) that
\[
e^{\tau T} I_{\partial \Omega}(\tau; B, \lambda) = O(\tau^3 e^{-\tau(2\lambda \text{dist}(\Omega, B) - T)} + \tau^3 e^{-\tau T} + \tau e^{-\tau \lambda \text{dist}(\Omega, B)}).
\]
This yields (1.9) in the case of \( T < 2\lambda \text{dist}(\Omega, B) \) and (iii).

Next let \( T \) satisfy (1.7), i.e., (2.18) with \( U = \Omega \). Then, we have (2.19) with \( U = \Omega \). This together with (2.27) yields: there exist positive constants \( C \) and \( \tau_0 \) such that, for all \( \tau \geq \tau_0 \)
\[
\tau^{10} e^{2\tau \lambda \text{dist}(\Omega, B)} I_{\partial \Omega}(\tau; B, \lambda) \geq C + O(\tau^{11} e^{-\tau(\lambda \text{dist}(\Omega, B))}).
\]
(2.42)

A combination of (2.26) and (2.42) yields (1.8). (ii) in the case of \( T > 2\lambda \text{dist}(\Omega, B) \) is a direct consequence of (1.8) since \( T \) satisfies (1.7) in this case.

The proof of Theorem 1.2 is as follows. From (2.20) with \( U = D \) and (2.29) we have
\[
\tau^7 e^{2\sqrt{\tau \text{dist}(D, B)}} I_{\partial \Omega}(\tau; B, \frac{1}{\sqrt{\tau}}) \geq C + O(\tau^6 e^{-\tau T} e^{\sqrt{\tau(\text{dist}(D, B) - \text{dist}(\Omega, B))}}).
\]
(2.43)

Note that there is no restriction on \( T \). Now it is easy to see that a combination of (2.28) and (2.43) yields the validity of Theorem 1.2.

2.4 Proof of Theorem 1.3

Using (2.30) and (2.31), we see that (i), (ii) and (iii) can be easily derived as those of Theorem 1.2. (1.18) in the case when \( c = 1 \) is a direct consequence of Theorem 1.2 (i) and (2.20) with \( U = \Omega \). Let \( c \neq 1 \). It follows from (2.20) with \( U = \Omega \), (2.30) and (2.31) that
\[
\limsup_{\tau \to \infty} \frac{I_{\partial \Omega}(\tau; B, \sqrt{\frac{c}{\tau}})}{\tau \int_\Omega |w_0|^2 \, dx} \leq (1 + \epsilon)(c - 1) \left( c - \frac{\epsilon}{1 + \epsilon} \right).
\]
(2.44)
and
\[
\liminf_{\tau \to \infty} \frac{I_{\partial \Omega}(\tau; B, \sqrt{c/\tau})}{\tau \int_{\Omega} |w_0|^2 \, dx} \geq c - 1. \tag{2.45}
\]

Since the left-hand side on (2.44) is independent of \( \epsilon \) one gets
\[
\limsup_{\tau \to \infty} \frac{I_{\partial \Omega}(\tau; B, \sqrt{c/\tau})}{\tau \int_{\Omega} |w_0|^2 \, dx} \leq (c - 1)c.
\]

This completes the proof of (1.18).

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