UNIQUENESS OF CONFORMAL MEASURES AND LOCAL MIXING FOR ANOSOV GROUPS

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ABSTRACT. In the late seventies, Sullivan showed that for a convex cocompact subgroup $\Gamma$ of $SO^+(n,1)$ with critical exponent $\delta > 0$, any $\Gamma$-conformal measure on $\partial \mathbb{H}^n$ of dimension $\delta$ is necessarily supported on the limit set $\Lambda$ and that the conformal measure of dimension $\delta$ exists uniquely. We prove an analogue of this theorem for any Zariski dense Anosov subgroup $\Gamma$ of a connected semisimple real algebraic group $G$ of rank at most 3. We also obtain the local mixing for generalized BMS measures on $\Gamma \backslash G$ including Haar measures.

Dedicated to Gopal Prasad on the occasion of his 75th birthday with respect

1. Introduction

Let $(X, d)$ be a Riemannian symmetric space of rank one and $\partial X$ the geometric boundary of $X$. Let $G = \text{Isom}^+ X$ denote the group of orientation preserving isometries and $\Gamma < G$ a non-elementary discrete subgroup. Fixing $o \in X$, a Borel probability measure $\nu$ on $\partial X$ is called a $\Gamma$-conformal measure of dimension $s > 0$ if for all $\gamma \in \Gamma$ and $\xi \in \partial X$,

$$\frac{d\gamma_* \nu}{d\nu} = e^{s(\beta_{\xi}(o,\gamma o))}$$

where $\beta_{\xi}(x,y) = \lim_{z \to \xi} d(x,z) - d(y,z)$ denotes the Busemann function.

Let $\delta > 0$ denote the critical exponent of $\Gamma$, i.e., the abscissa of the convergence of the Poincare series $\sum_{\gamma \in \Gamma} e^{-\delta d(\gamma o,o)}$. The well-known construction of Patterson and Sullivan ([9], [13]) provides a $\Gamma$-conformal measure of dimension $\delta$ supported on the limit set $\Lambda$, called the Patterson-Sullivan (PS) measure. A discrete subgroup $\Gamma < G$ is called convex cocompact if $\Gamma$ acts cocompactly on some nonempty convex subset of $X$.

**Theorem 1.1 (Sullivan).** [13] If $\Gamma$ is convex cocompact, then any $\Gamma$-conformal measure on $\partial X$ of dimension $\delta$ is necessarily supported on $\Lambda$. Moreover, the PS-measure is the unique $\Gamma$-conformal measure of dimension $\delta$.

In this paper, we extend this result to Anosov subgroups, which may be regarded as higher rank analogues of convex cocompact subgroups of rank

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Let $G$ be a connected semisimple real algebraic group and $P$ a minimal parabolic subgroup of $G$. Let $F := G/P$ be the Furstenberg boundary, and $F^{(2)}$ the unique open $G$-orbit in $F \times F$ under the diagonal action of $G$. In the whole paper, we let $\Gamma$ be a Zariski dense Anosov subgroup of $G$ with respect to $P$. This means that there exists a representation $\Phi : \Sigma \to G$ of a Gromov hyperbolic group $\Sigma$ with $\Gamma = \Phi(\Sigma)$, which induces a continuous equivariant map $\zeta$ from the Gromov boundary $\partial \Sigma$ to $F$ such that $(\zeta(x), \zeta(y)) \in F^{(2)}$ for all $x \neq y \in \partial \Sigma$. This definition is due to Guichard-Wienhard [5], generalizing that of Labourie [6].

Let $A < P$ be a maximal real split torus of $G$ and $a := \text{Lie}(A)$. Given a linear form $\psi \in a^*$, a Borel probability measure $\nu$ on $F$ is called a $(\Gamma, \psi)$-conformal measure if, for any $\gamma \in \Gamma$ and $\xi \in F$,

$$\frac{d\gamma_\ast \nu}{d\nu}(\xi) = e^{\psi(\beta_\gamma(e, \gamma))}$$

where $\beta$ denotes the $a$-valued Busemann function (see (2.1) for the definition). Let $\Lambda \subset F$ denote the limit set of $\Gamma$, which is the unique $\Gamma$-minimal subset (see [1], [7]). A $(\Gamma, \psi)$-conformal measure supported on $\Lambda$ will be called a $(\Gamma, \psi)$-PS measure. Finally, a $\Gamma$-PS measure means a $(\Gamma, \psi)$-PS measure for some $\psi \in a^*$.

Fix a positive Weyl chamber $a^+ \subset a$ and let $L_\Gamma \subset a^+$ denote the limit cone of $\Gamma$. Benoist [1] showed that $L_\Gamma$ is a convex cone with non-empty interior, using the well-known theorem of Prasad [10] on the existence of an $\mathbb{R}$-regular element in any Zariski dense subgroup of $G$. Let $\psi_T : a \to \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function of $\Gamma$ as defined in (2.2). Set

$$D_\Gamma^\ast := \{\psi \in a^* : \psi \geq \psi_T, \psi(u) = \psi_T(u) \text{ for some } u \in L_\Gamma \cap \text{int } a^+\}.$$  

As $\Gamma$ is Anosov, for any $\psi \in D_\Gamma^\ast$, there exist a unique unit vector $u \in \text{int } L_\Gamma$, such that $\psi(u) = \psi_T(u)$, and a unique $(\Gamma, \psi)$-PS measure $\nu_\psi$. Moreover, this gives bijections among

$$D_\Gamma^\ast \simeq \{u \in \text{int } L_\Gamma : \|u\| = 1\} \simeq \{\Gamma\text{-PS measures on } \Lambda\} \simeq \{\Gamma\text{-PS measures on } \Lambda\}$$

(see [4], [7]). When $G$ has rank one, $D_\Gamma^\ast = \{\delta\}$. Therefore the following generalizes Sullivan’s theorem [11]. We denote the real rank of $G$ by rank $G$, i.e., rank $G = \text{dim } a$.

**Theorem 1.4.** Let rank $G \leq 3$. For any $\psi \in D_\Gamma^\ast$, any $(\Gamma, \psi)$-conformal measure on $F$ is necessarily supported on $\Lambda$. Moreover, the PS measure $\nu_\psi$ is the unique $(\Gamma, \psi)$-conformal measure on $F$.

Our proof of Theorem 1.4 is obtained by combining the rank dichotomy theorem established by Burger, Landesberg, Lee, and Oh [2] and the local mixing property of a generalized Bowen-Margulis-Sullivan measure (Theorem 3.1), which generalizes our earlier work [4]. Indeed, our proof yields that under the hypothesis of Theorem 1.4, any $(\Gamma, \psi)$-conformal measure on $F$ is supported on the $u$-directional radial limit set $\Lambda_u$ (see (4.3)) where $\psi(u) = \psi_T(u)$. 


We end the introduction by the following:

**Open problem:** Is Theorem 1.4 true without the hypothesis $\text{rank } G \leq 3$?

## 2. Local mixing of Generalized Bowen-Margulis-Sullivan Measures

Let $G$ be a connected semisimple real algebraic group and $\Gamma \subset G$ a Zariski dense discrete subgroup. Let $P = MAN$ be a minimal parabolic subgroup of $G$ with fixed Langlands decomposition so that $A$ is a maximal real split torus, $M$ is the centralizer of $A$ and $N$ is the unipotent radical of $P$.

In [4, Prop. 6.8], we proved that local mixing of a BMS-measure on $\Gamma \backslash G/M$ implies local mixing of the Haar measure on $\Gamma \backslash G/M$. In this section, we provide a generalized version of this statement, where we replace the Haar measure by any generalized BMS-measure and also work on the space $\Gamma \backslash G$, rather than on $\Gamma \backslash G/M$. We refer to [4] for a more detailed description of a generalized BMS-measure, while only briefly recalling its definition here.

Let $a = \text{Lie}(A)$ and fix a positive Weyl chamber $a^+ < a$ so that $\log N$ consists of positive root subspaces. We also fix a maximal compact subgroup $K < G$ so that the Cartan decomposition $G = K \exp a^+ K$ holds. Denote by $\mu : G \to a$ the Cartan projection, i.e., for $g \in G$, $\mu(g) \in a$ is the unique element such that $g \in K \exp \mu(g) K$.

Denote by $L_{\Gamma} \subset \mathfrak{a}^+$ the limit cone of $\Gamma$, which is the asymptotic cone of $\mu(\Gamma)$, i.e., $L_{\Gamma} = \{ \lim t_i \mu(\gamma_i) \in \mathfrak{a}^+ : t_i \to 0, \gamma_i \in \Gamma \}$. The Furstenberg boundary $F = G/P$ is isomorphic to $K/M$ as $K$ acts on $F$ transitively with $K \cap P = M$.

The $\mathfrak{a}$-valued Busemann function $\beta : F \times G \times G \to \mathfrak{a}$ is defined as follows: for $\xi \in F$ and $g, h \in G$,

$$ \beta_\xi(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi) $$

where the Iwasawa cocycle $\sigma(g^{-1}, \xi) \in \mathfrak{a}$ is defined by the relation $g^{-1}k \in K \exp(\sigma(g^{-1}, \xi))N$ for $\xi = kP$, $k \in K$.

The growth indicator function $\psi_{\Gamma} : a^+ \to \mathbb{R} \cup \{-\infty\}$ is defined as a homogeneous function, i.e., $\psi_{\Gamma}(tu) = t\psi_{\Gamma}(u)$ for all $t > 0$, such that for any unit vector $u \in \mathfrak{a}^+$,

$$ \psi_{\Gamma}(u) := \inf_{u \in C, \text{open cones } C \subset a^+} \tau_C $$

where $\tau_C$ is the abscissa of convergence of $\sum_{\gamma \in \Gamma, \mu(\gamma) \in C} e^{-t\|\mu(\gamma)\|}$ and the norm $\| \cdot \|$ on $\mathfrak{a}$ is the one induced from the Killing form on $\mathfrak{g}$.

Denote by $u_0 \in K$ a representative of the unique element of the Weyl group $N_K(A)/M$ such that $\text{Ad}_{u_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The opposition involution $i : \mathfrak{a} \to \mathfrak{a}$ is defined by

$$ i(u) = -\text{Ad}_{u_0}(u). $$

Note that $i$ preserves $\text{int } \mathcal{L}_{\Gamma}$. 

The generalized BMS-measures $m_{\nu_1,\nu_2}$. For $g \in G$, we consider the following visual images:

$$g^+ = gP \in \mathcal{F} \quad \text{and} \quad g^- = gw_0P \in \mathcal{F}.$$ Then the map

$$gM \mapsto (g^+, g^-, b = \beta_g^{-}(e, g))$$
gives a homeomorphism $G/M \simeq \mathcal{F}^{(2)} \times a$, called the Hopf parametrization of $G/M$.

For a pair of linear forms $\psi_1, \psi_2 \in a^*$ and a pair of $(\Gamma, \nu_1)$ and $(\Gamma, \nu_2)$ conformal measures $\nu_1$ and $\nu_2$ respectively, define a locally finite Borel measure $\tilde{m}_{\nu_1,\nu_2}$ on $G/M$ as follows: for $g = (g^+, g^-, b) \in \mathcal{F}^{(2)} \times a$,

$$d\tilde{m}_{\nu_1,\nu_2}(g) = e^{\psi_1(\beta_{g^+}(e, g)) + \psi_2(\beta_{g^-}(e, g))} \ d\nu_1(g^+) d\nu_2(g^-) db,$$ (2.3)

where $db = d\ell(b)$ is the Lebesgue measure on $a$. By abuse of notation, we also denote by $\tilde{m}_{\nu_1,\nu_2}$ the $M$-invariant measure on $G$ induced by $\tilde{m}_{\nu_1,\nu_2}$. This is always left $\Gamma$-invariant and we denote by $m_{\nu_1,\nu_2}$ the $M$-invariant measure on $\Gamma \backslash G$ induced by $\tilde{m}_{\nu_1,\nu_2}$.

The generalized BMS*-measures $m_{\nu_1,\nu_2}^*$. Similarly, with a different Hopf parametrization

$$gM \mapsto (g^+, g^-, b = \beta_g^+(e, g))$$
(that is, $g^-$ replaced by $g^+$ in the subscript for $\beta$), we define the following measure

$$d\tilde{m}_{\nu_1,\nu_2}^*(g) = e^{\psi_1(\beta_{g^+}(e, g)) + \psi_2(\beta_{g^-}(e, g))} \ d\nu_1(g^+) d\nu_2(g^-) db$$ (2.4)

first on $G/M$ and then the $M$-invariant measure $d\tilde{m}_{\nu_1,\nu_2}^*$ on $\Gamma \backslash G$. One can check

$$m_{\nu_1,\nu_2}^* = m_{\nu_{2,\nu_1}^*,w_0}.$$ (2.5)

Lemma 2.6. If $\psi_2 = \psi_1 \circ i$, then $m_{\nu_{1,\nu_2}} = m_{\nu_1,\nu_2}^*$. Proof: When $\psi_2 = \psi_1 \circ i$, we can check that $m_{\nu_{2,\nu_1}^*,w_0} = m_{\nu_1,\nu_2}^*$, which implies the claim by (2.5).

PS-measures on $gN^\pm$. Let $N^- = N$ and $N^+ = w_0Nw_0^{-1}$. To a given $(\Gamma, \psi)$-conformal measure $\nu$ and $g \in G$, we define the following associated measures on $gN^\pm$: for $n \in N^+$ and $h \in N^-$,

$$d\mu_{gN^+,\nu}(n) := e^{\psi(\beta_{(gn)^+}(e, gn))} d\nu((gn)^+),$$
$$d\mu_{gN^-,\nu}(h) := e^{\psi(\beta_{(gh)^-}(e, gh))} d\nu((gh)^-).$$

Note that these are left $\Gamma$-invariant; for any $\gamma \in \Gamma$ and $g \in G$, $\mu_{\gamma gN^\pm,\nu} = \mu_{gN^\pm,\nu}$. For a given Borel subset $X \subset \Gamma \backslash G$, define the measure $\mu_{gN^+,\nu} \chi X$ on $N^+$ by

$$d\mu_{gN^+,\nu} \chi X(n) = \mathbf{1}_X([g]n) d\mu_{gN^+,\nu}(n);$$
note that here the notation $|X|$ is purely symbolic, as $\mu_{g^{N^{+}},\nu}|X|$ is not a measure on $X$. Set $P^{\pm} := MAN^{\pm}$. For $\varepsilon > 0$ and $\ast = N, N^{+}, A, M$, let $\ast_{\varepsilon}$ denote the $\varepsilon$-neighborhood of $\ast$ in $\ast$. We then set $P_{\varepsilon}^{\pm} = N_{\varepsilon}^{\pm} A_{\varepsilon} M_{\varepsilon}$.

We recall the following lemmas from \cite{4}:

**Lemma 2.7.** \cite{4} Lem. 5.6, Cor. 5.7] We have:

1. For any fixed $\rho \in C_{c}(N^{\pm})$ and $g \in G$, the map $N^{\pm} \rightarrow \mathbb{R}$ given by $n \mapsto \mu_{g^{N^{\pm}},\nu}(\rho)$ is continuous.
2. Given $\varepsilon > 0$ and $g \in G$, there exist $R > 1$ and a non-negative $\rho_{g,\varepsilon} \in C_{c}(NR)$ such that $\mu_{g^{N^{\pm}},\nu}(\rho_{g,\varepsilon}) > 0$ for all $n \in N_{\varepsilon}^{\pm}$.

**Lemma 2.8.** \cite{4} Lem. 4.2] For any $g \in G$, $a \in A$, $n_{0}, n \in N^{+}$, we have

$$d(\theta_{a}^{-1}\mu_{g^{N^{+}},\nu})(n) = e^{-\psi_{1}(\log a)}d\mu_{gan^{N^{+}},\nu}(n),$$

where $\theta : N^{+} \rightarrow N^{+}$ is given by $\theta(n) = an_{0}a^{-1}$.

**Lemma 2.9.** \cite{4} Lem. 4.4 and 4.5] For $i = 1, 2$, let $\psi_{i} \in a^{*}$ and $\nu_{i}$ a $(\Gamma, \psi_{i})$-conformal measure. Then

1. For $g \in G$, $f \in C_{c}(gN^{+}P)$, and $nham \in N^{+}NAM$,

$$m_{\nu_{1},\nu_{2}}(f) =$$

$$\int_{N^{+}} \left( \int_{NAM} f(gnham)e^{(\psi_{1}-\psi_{2})(\log a)} \, dm \right) \, da \, d\mu_{gan^{N^{+}},\nu_{2}}(h) \, d\mu_{g^{N^{+}},\nu_{1}}(n).$$

2. For $g \in G$, $f \in C_{c}(gPN^{+})$, and $hamn \in NAMN^{+}$,

$$m^{*}_{\nu_{1},\nu_{2}}(f) =$$

$$\int_{NAM} \left( \int_{N^{+}} f(ghamn) \, d\mu_{ghamn^{+},\nu_{2}}(n) \right) e^{-\psi_{2}\psi_{1}(\log a)} \, dm \, da \, d\mu_{g^{N^{+}},\nu_{2}}(h).$$

**Local mixing.** Let $P^{o}$ denote the identity component of $P$ and $\mathfrak{Y}_{\Gamma}$ denote the set of all $P^{o}$-minimal subsets of $\Gamma \backslash G$. While there exists a unique $P$-minimal subset of $\Gamma \backslash G$ given by $\{ [g] \in \Gamma \backslash G : g^{+} \in \Lambda \}$, there may be more than one $P^{o}$-minimal subset. Note that $\#\mathfrak{Y}_{\Gamma} = [P : P^{o}] = |M : M^{o}|$. Set $\Omega = \{ [g] \in \Gamma \backslash G : g^{+} \in \Lambda \}$ and write

$$3_{\Gamma} = \{ Y \cap \Omega \subset \Gamma \backslash G : Y \in \mathfrak{Y}_{\Gamma} \}.$$

Note that for each $Y \in \mathfrak{Y}_{\Gamma}$, we have $Y = (Y \cap \Omega)N$ and the collection $\{ (Y \cap \Omega)N^{+} : Y \in \mathfrak{Y}_{\Gamma} \}$ is in one-to-one correspondence with the set of $(M^{o}AN^{+})$-minimal subsets of $\Gamma \backslash G$.

In the rest of the section, we fix a unit vector $u \in \mathcal{L}_{\Gamma} \cap \text{int} a^{+}$, and set

$$a_{t} = \exp(tu) \quad \text{for } t \in \mathbb{R}.$$

We also fix

$$\psi_{1} \in a^{*} \quad \text{and} \quad \psi_{2} := \psi_{1} \circ i \in a^{*}.$$
For each $i = 1, 2$, we fix a $(\Gamma, \psi_i)$-PS measure $\nu_i$ on $\mathcal{F}$. We will assume that the associated BMS-measure $m = m_{\nu_1, \nu_2}$ satisfies the local mixing property for the $\{a_t : t \in \mathbb{R}\}$-action in the following sense:

**Hypothesis on $m = m_{\nu_1, \nu_2}$**: there exists a proper continuous function $\Psi : (0, \infty) \to (0, \infty)$ such that for all $f_1, f_2 \in C_c(\Gamma \backslash G)$,

$$\lim_{t \to +\infty} \Psi(t) \int_{\Gamma \backslash G} f_1(xa_t) f_2(x) \, dm(x) = \sum_{Z \in 3\Gamma} m|_Z(f_1) m|_Z(f_2). \quad (2.10)$$

The main goal in this section is to obtain the following local mixing property for a generalized BMS-measure $m_{\lambda_1, \lambda_2}$ from that of $m$ (note that $\lambda_1$ and $\lambda_2$ are not assumed to be supported on $\Lambda$):

**Theorem 2.11.** For $i = 1, 2$, let $\varphi_i \in \mathfrak{a}^*$ and $\lambda_i$ be a $(\Gamma, \varphi_i)$-conformal measure on $\mathcal{F}$. Then for all $f_1, f_2 \in C_c(\Gamma \backslash G)$, we have

$$\lim_{t \to +\infty} \Psi(t) e^{(\varphi_2 - \varphi_1)(t \mu)} \int_{\Gamma \backslash G} f_1(xa_t) f_2(x) \, dm^*_{\lambda_1, \lambda_2}(x) = \sum_{Z \in 3\Gamma} m^*_{\lambda_1, \lambda_2} \big|_{Z^+(f_1)} m^*_{\nu_2, \nu_1} \big|_{Z^+(f_2)}.$$

**Remark 2.12.** If $\varphi_2 = \varphi_1 \circ i$, we may replace $m^*_{\lambda_1, \lambda_2}$ by $m_{\lambda_1, \lambda_2}$ in Theorem 2.11 by Lemma 2.6. For general $\varphi_1, \varphi_2$, we get, using the identity (2.5): for all $f_1, f_2 \in C_c(\Gamma \backslash G)$, we have

$$\lim_{t \to +\infty} \Psi(t) e^{(\varphi_2 - \varphi_1)(t \mu)} \int_{\Gamma \backslash G} f_1(xa_t) f_2(x) \, dm_{\lambda_2, \lambda_1}(x) = \sum_{Z \in 3\Gamma} m_{\nu_2, \nu_1} \big|_{Z^+(f_1)} m_{\lambda_2, \lambda_1} \big|_{Z^+(f_2)}.$$

In order to prove Theorem 2.11, we first deduce equidistribution of translates of $\mu_{gN^+, \nu_1}$ from the local mixing property of $m$ (Proposition 2.13), and then convert this into equidistribution of translates of $\mu_{gN^+, \lambda_1}$ (Proposition 2.17).

**Proposition 2.13.** For any $x = [g] \in \Gamma \backslash G$, $f \in C_c(\Gamma \backslash G)$, and $\phi \in C_c(N^+)$,

$$\lim_{t \to +\infty} \Psi(t) \int_{N^+} f(xna_t) \phi(n) \, d\mu_{gN^+, \nu_1}(n) = \sum_{Z \in 3\Gamma} m|_{Z}(f) \mu_{gN^+, \nu_1}|_{Z}(\phi). \quad (2.14)$$

**Proof.** Let $x = [g]$, and $\varepsilon_0 > 0$ be such that $\phi \in C_c(N^+_{\varepsilon_0})$. For simplicity of notation, we write $d\mu_{\nu_1} = d\mu_{gN^+, \nu_1}$ throughout the proof. By Lemma 2.7 we can choose $R > 0$ and a nonnegative $\rho_{g, \varepsilon_0} \in C_c(N_R)$ such that

$$\mu_{g_{N^+}, \nu_2}(\rho_{g, \varepsilon_0}) > 0 \quad \text{for all } n \in N^+_{\varepsilon_0}.$$
Lemma 2.7. We now assume without loss of generality that
\[ f \in C_c(A_\varepsilon M_\varepsilon) \] satisfying
\[ \int_A f \, dm = 1. \]
Then, as \( \text{supp}(\tilde{\Phi}_\varepsilon) \subset C_c(G) \) and \( \Phi_\varepsilon \in C_c(\Gamma \setminus G) \) by
\[ \tilde{\Phi}_\varepsilon(g_0) := \begin{cases} \frac{\phi(n)\rho_{\varepsilon,\varepsilon_0}(h)q_\varepsilon(\varepsilon)}{\mu_{\varepsilon_\varepsilon_0}(\rho_{\varepsilon,\varepsilon_0})} & \text{if } g_0 = gnham, \\ 0 & \text{otherwise}, \end{cases} \]
and \( \Phi_\varepsilon([g]) := \sum_{\gamma \in \Gamma} \tilde{\Phi}_\varepsilon(\gamma g_0). \) Note that the continuity of \( \tilde{\Phi}_\varepsilon \) follows from Lemma 2.7. We now assume without loss of generality that \( f \geq 0 \) and define, for all \( \varepsilon > 0 \), functions \( f_\varepsilon^+ \) as follows: for all \( z \in \Gamma \setminus G \),
\[ f_\varepsilon^+(z) := \sup_{b \in N_\varepsilon^+ P_\varepsilon} f(bz) \quad \text{and} \quad f_\varepsilon^-(z) := \inf_{b \in N_\varepsilon^+ P_\varepsilon} f(bz). \]
Since \( u \in \text{int } a^+ \), for every \( \varepsilon > 0 \), there exists \( t_0(R, \varepsilon) > 0 \) such that
\[ a_t^{-1} N_R A_t \subset N_\varepsilon \quad \text{for all } t \geq t_0(R, \varepsilon). \]
Then, as \( \text{supp}(\tilde{\Phi}_\varepsilon) \subset gN_\varepsilon N_R A_\varepsilon M_\varepsilon \), we have
\[ f \left( xna_t \right) \tilde{\Phi}_\varepsilon(\varepsilon) \leq f_\varepsilon^+(xnhama_t) \tilde{\Phi}_\varepsilon(\varepsilon) \quad \text{(2.16)} \]
for all \( nham \in N_\varepsilon N_R A_\varepsilon M_\varepsilon \) and \( t \geq t_0(R, \varepsilon) \). We now use \( f_{3\varepsilon}^+ \) to give an upper bound on the limit we are interested in; \( f_{3\varepsilon}^- \) is used in an analogous way to provide a lower bound. Entering the definition of \( \Phi_\varepsilon \) and the above inequality \([2.16]\) into \([2.15]\) gives
\[ \limsup_{t \to +\infty} \Psi(t) \int_{N^+} f \left( xna_t \right) \phi(n) \, d\mu_{\varepsilon_0}(n) \leq \limsup_{t \to +\infty} \Psi(t) \int_{\Gamma \setminus G} \tilde{\Phi}_\varepsilon(\varepsilon) \, d\tilde{m}(g_0). \]

\begin{align*}
\int_{N^+} \int_{NAM} f_{3\varepsilon}^+(xnhama_t) \Phi_\varepsilon(\varepsilon) \, d\tilde{m}(g_0) \, d\mu_{\varepsilon_0}(n) \\
\leq \limsup_{t \to +\infty} \Psi(t) e^{\varepsilon ||\psi_1 - \psi_2||} \int_{N^+} \int_{NAM} f_{3\varepsilon}^+(xnhama_t) \Phi_\varepsilon(\varepsilon) \, d\tilde{m}(g_0) \\
= \limsup_{t \to +\infty} \Psi(t) e^{\varepsilon ||\psi_1 - \psi_2||} \int_{\Gamma \setminus G} f_{3\varepsilon}^+(g_0) \Phi_\varepsilon(\varepsilon) \, d\tilde{m}(g_0) \\
= \limsup_{t \to +\infty} \Psi(t) e^{\varepsilon ||\psi_1 - \psi_2||} \int_{\Gamma \setminus G} f_{3\varepsilon}^+(g_0) \Phi_\varepsilon(\varepsilon) \, d\tilde{m}(g_0),
\end{align*}
where \( \| \cdot \| \) is the operator norm on \( \mathfrak{a}^* \) and Lemma \([2.9]\) was used in the second to last line of the above calculation. By the standing assumption \([2.10]\), we have

\[
\limsup_{t \to +\infty} \Psi(t) \int_{N} f(xna_t) \phi(n) \, d\mu_{gN,\nu_2}(n) \\
\leq e^{\epsilon \| \psi_1 - \psi_2 \|} \sum_{Z \in \mathcal{F}} m|_{Z}(f^+_\epsilon) m|_{Z}(\Phi_\epsilon) \\
= e^{\epsilon \| \psi_1 - \psi_2 \|} \sum_{Z \in \mathcal{F}} m|_{Z}(f^+_\epsilon) \tilde{m} |_{Z}(\Phi_\epsilon),
\]

where \( \tilde{Z} \subset G \) is a \( \Gamma \)-invariant lift of \( Z \). Using Lemma \([2.9]\) for all \( 0 < \epsilon \ll 1 \), \( \tilde{m}|_{\tilde{Z}}(\Phi_\epsilon) \)

\[
\int_{N^+} \left( \int_{NAM} \Phi_\epsilon \mathbf{1}_{\tilde{Z}}(gnha) e^{(\psi_1 - \psi_2 \|)(\log a)} \, da \, dm \, d\mu_{gN,\nu_2}(h) \right) \, d\mu_{\nu_1}(n) \leq e^{\epsilon \| \psi_1 - \psi_2 \|} \int_{N^+} \phi(n) \mathbf{1}_{ZN}(g) \left( \int_{NAM} \rho_{\nu_1}(h) q_\epsilon(\lambda a) \, da \, dm \, d\mu_{gN,\nu_2}(h) \right) \, d\mu_{\nu_1}(n) \\
\leq e^{\epsilon \| \psi_1 - \psi_2 \|} \mathbf{1}_{\nu_1}|_{ZN}(\phi),
\]

where we have used the facts that \( \tilde{Z} \) is invariant under the right translation of identity component \( M^0 \) of \( M \), and \( \text{supp} \nu_2 = \Lambda \) as well as the identity \( \mathbf{1}_{\tilde{Z}}(gnha) = \mathbf{1}_{ZN}(g) \mathbf{1}_{\Lambda}(gN^+) \) (we remark that \( \text{supp} \nu_2 = \Lambda \) is not necessary for the upper bound as \( \mathbf{1}_{\tilde{Z}}(gnha) \leq \mathbf{1}_{ZN}(g) \), but needed for the lower bound). Since \( \epsilon > 0 \) was arbitrary, taking \( \epsilon \to 0 \) gives

\[
\limsup_{t \to +\infty} \Psi(t) \int_{N^+} f(xna_t) \phi(n) \, d\mu_{\nu_1}(n) \leq \sum_{Z \in \mathcal{F}} m|_{Z}(f) \, \mathbf{1}_{\nu_1}|_{ZN}(\phi).
\]

The lower bound given by replacing \( f^+_\epsilon \) with \( f^-_{3\epsilon} \) in the above calculations completes the proof.

\[\square\]

**Proposition 2.17.** For any \( x = [g] \in \Gamma \backslash G \), \( f \in C_c(\Gamma \backslash G) \) and \( \phi \in C_c(N^+) \),

\[
\lim_{t \to +\infty} \Psi(t) e^{(\psi_1 - \psi_1)(tu)} \int_{N^+} f(xna_t) \phi(n) \, d\mu_{gN^+,\lambda_1}(n) \\
= \sum_{Z \in \mathcal{F}} m|_{Z}(f) \, \mathbf{1}_{\nu_1}|_{ZN}(\phi).
\]

**Proof.** For \( \epsilon_0 > 0 \), set \( B_{\epsilon_0} = P_{\epsilon_0}N^+_{\epsilon_0} \). Given \( x_0 \in \Gamma \backslash G \), let \( \epsilon_0(x_0) \) denote the maximum number \( r \) such that the map \( G \to \Gamma \backslash G \) given by \( h \mapsto x_0 h \) for \( h \in G \) is injective on \( B_r \). By using a partition of unity if necessary, it suffices to prove that for any \( x_0 \in \Gamma \backslash G \) and \( \epsilon_0 = \epsilon_0(x_0) \), the claims of the proposition hold for any non-negative \( f \in C(x_0B_{\epsilon_0}) \), non-negative \( \phi \in C(N^+) \), and \( x = [g] \in x_0B_{\epsilon_0} \). Moreover, we may assume that \( f \) is given
as
\[ f([g]) = \sum_{\gamma \in \Gamma} \tilde{f}(\gamma g) \quad \text{for all } g \in G, \]
for some non-negative \( \tilde{f} \in C_c(g_0 \mathcal{B}_{\mathcal{E}_0}) \). For simplicity of notation, we write \( \mu_{\lambda_1} = \mu_{gN^+,\lambda_1} \). Note that for \( x = [g] \in [g_0] \mathcal{B}_{\mathcal{E}_0} \),
\[ \int_{N^+} f([g]na_t)\phi(n) \, d\mu_{\lambda_1}(n) = \sum_{\gamma \in \Gamma} \int_{N^+} \tilde{f}(\gamma gna_t)\phi(n) \, d\mu_{\lambda_1}(n). \quad (2.18) \]
Note that \( \tilde{f}(\gamma gna_t) = 0 \) unless \( \gamma gna_t \in g_0 \mathcal{B}_{\mathcal{E}_0} \). Together with the fact that \( \text{supp}(\phi) \subset N_{\mathcal{E}_0}^+ \), it follows that the summands in (2.18) are non-zero only for finitely many elements \( \gamma \in \Gamma \cap g_0 \mathcal{B}_{\mathcal{E}_0}a_{-t}N_{\mathcal{E}_0}^+g^{-1} \).

Suppose \( \gamma gN_{\mathcal{E}_0}^+a_t \cap g_0 \mathcal{B}_{\mathcal{E}_0} \neq \emptyset \). Then \( \gamma g a_t \in g_0 P_{\mathcal{E}_0} N^+ \), and there are unique elements \( p_t, \gamma \in P_{\mathcal{E}_0} \) and \( n_{t,\gamma} \in N^+ \) such that
\[ \gamma g a_t = g_0 p_{t,\gamma} n_{t,\gamma} \in g_0 P_{\mathcal{E}_0} N^+. \]

Let \( \Gamma_t \) denote the subset \( \Gamma \cap g_0 (P_{\mathcal{E}_0} N^+)a_{-t}^{-1}g^{-1} \). Note that although \( \Gamma_t \) may possibly be infinite, only finitely many of the terms in the sums we consider will be non-zero. This together with Lemma 2.8 gives
\[ \int_{N^+} f([g]na_t)\phi(n) \, d\mu_{\lambda_1}(n) = \sum_{\gamma \in \Gamma} \int_{N^+} \tilde{f}(\gamma gna_t)\phi(n) \, d\mu_{\lambda_1}(n) \]
\[ = \sum_{\gamma \in \Gamma_t} \int_{N^+} \tilde{f}(\gamma gna_t(a_t^{-1}na_t))\phi(n) \, d\mu_{\lambda_1}(n) \]
\[ = e^{-\varphi_1(\log a_t)} \sum_{\gamma \in \Gamma_t} \int_{N^+} \tilde{f}(\gamma gna_t n_{t,\gamma}) \phi(a_t^{-1}n) \, d\mu_{g_0 P_{\mathcal{E}_0}^+,\lambda_1}(n) \]
\[ = e^{-\varphi_1(\log a_t)} \sum_{\gamma \in \Gamma_t} \int_{N^+} \tilde{f}(g_0 p_{t,\gamma} n_{t,\gamma}) \phi(a_t^{-1}n) \, d\mu_{g_0 P_{\mathcal{E}_0}^+,\lambda_1}(n) \]
\[ = e^{-\varphi_1(\log a_t)} \sum_{\gamma \in \Gamma_t} \int_{N^+} \tilde{f}(g_0 p_{t,\gamma} n_{t,\gamma}) \phi(a_t^{-1}n_{t,\gamma}^{-1}a_t^{-1}) \, d\mu_{g_0 P_{\mathcal{E}_0}^+,\lambda_1}(n). \]

Since \( \text{supp}(\tilde{f}) \subset g_0 \mathcal{B}_{\mathcal{E}_0} \), we have
\[ \sum_{\gamma \in \Gamma_t} \int_{N^+} \tilde{f}(g_0 p_{t,\gamma} n_{t,\gamma}) \phi(a_t^{-1}n_{t,\gamma}^{-1}a_t^{-1}) \, d\mu_{g_0 P_{\mathcal{E}_0}^+,\lambda_1}(n) \]
\[ \leq \sum_{\gamma \in \Gamma_t} \left( \sup_{n \in N^+} \phi(a_t^{-1}n_{t,\gamma}^{-1}a_t^{-1}) \right) \cdot \int_{N^+} \tilde{f}(g_0 p_{t,\gamma} n_{t,\gamma}) \, d\mu_{g_0 P_{\mathcal{E}_0}^+,\lambda_1}(n). \]

Since \( u \) belongs to \( \text{int} \mathcal{L}_\Gamma \), there exist \( t_0 > 0 \) and \( \alpha > 0 \) such that
\[ a_t N_{r_0}^+a_t^{-1} \subset N_{r^0}^+ \quad \text{for all } r > 0 \text{ and } t > t_0. \]
Therefore, for all \( n \in N_+^0 \) and \( t > t_0 \), we have
\[
\phi(a_t n_t, a_t^{-1}(a_t n_t^{-1})) \leq \phi^+(a_t n_t, a_t^{-1}),
\]
(2.19)
where
\[
\phi^+(n) := \sup_{b \in N_+^0} \phi(nb) \quad \text{for all } n \in N^+, \varepsilon > 0.
\]

We now have the following inequality for \( t > t_0 \):
\[
e^{\varphi_1(\log a_t)} \int_{N^+} f([g] n a_t) \phi(n) \, d\mu_1(n)
\leq \sum_{\gamma \in \Gamma_t} \phi^+_{\varepsilon \rho}(a_t n_t, a_t^{-1}) \int_{N_0^+} \tilde{f}(g_0 p_{t, \gamma}) \, d\mu_{g_0 p_{t, \gamma} N^+, \lambda_1}(n).
\]
(2.20)

By Lemma 2.7, we can now choose \( R > 0 \) and \( \rho \in C_c(N_R^+) \) such that \( \rho(n) \geq 0 \) for all \( n \in N^+ \), and \( \mu_{g_0 p N^+, \nu_1}(\rho) > 0 \) for all \( p \in P_{\varepsilon_0} \). Define \( \tilde{F} \in C_c(g_0 P_{\varepsilon_0} N_R^+) \) by
\[
\tilde{F}(g) = \begin{cases} \frac{\rho(n)}{\mu_{g_0 p N^+, \nu_1}(\rho)} \int_{N_0^+} \tilde{f}(g_0 p_{0}) \, d\mu_{g_0 p N^+, \lambda_1}(n) & \text{if } g = g_0 p_{0} \in g_0 P_{\varepsilon_0} N_R^+ \\ 0 & \text{if } g \notin g_0 P_{\varepsilon_0} N_R^+. \end{cases}
\]

We claim that for all \( p \in P_{\varepsilon_0} \) and \( Z \in \mathcal{F} \) such that \( g_0 p^- \in \Lambda \),
\[
\int_{N^+} \tilde{F}(g_0 p_{0}) \, d\mu_{g_0 p N^+, \nu_1} |Z(n) = \int_{N_0^+} \tilde{F}(g_0 p_{0}) \, d\mu_{g_0 p N^+, \nu_1} |Z(n)
= \int_{N_0^+} (\tilde{f} 1_{Z N^+})(g_0 p_{0}) \, d\mu_{g_0 p N^+, \lambda_1}(n).
\]
(2.21)

Indeed, by the assumption \( \text{supp} \nu_1 = \Lambda \) and the fact \( \Omega \cap Z N^+ = Z \), we have the identity \( 1_Z(g_0 p_{0}) \, d\mu_{g_0 p N^+, \nu_1}(n) = 1_{Z N^+}(g_0 p) \, d\mu_{g_0 p N^+, \nu_1}(n) \) and hence
\[
\int_{N^+} \tilde{F}(g_0 p_{0}) \, d\mu_{g_0 p N^+, \nu_1} |Z(n)
= \int_{N^+} \tilde{F}(g_0 p_{0}) \, d\mu_{g_0 p N^+, \nu_1}(n)
= \int_{N^+} \frac{\rho(n)}{\mu_{g_0 p N^+, \nu_1}(\rho)} \left( \int_{N_0^+} \tilde{f}(g_0 p_{0}) \, d\mu_{g_0 p N^+, \lambda_1}(n) \right) \, d\mu_{g_0 p N^+, \nu_1}(n)
= \int_{N_0^+} (\tilde{f} 1_{Z N^+})(g_0 p_{0}) \, d\mu_{g_0 p N^+, \lambda_1}(n),
\]
(2.21)
Hence we can write
\[
\int_{N^+} \tilde{F}(g_0 p n) d\mu_{g_0 p N^+, \nu_1}(n)
\]
\[
= \sum_{Z \in \mathcal{Z}_t} \int_{N^+} \tilde{F}(g_0 p n) d\mu_{g_0 p N^+, \nu_1}|z(n)
\]
\[
= \sum_{Z \in \mathcal{Z}_t} \int_{N^+_t} \left( \tilde{f}1_{Z N^+}(g_0 p n) \right) d\mu_{g_0 p N^+, \nu_1}(n).
\]

Hence we can write
\[
\int_{N^+_t} \tilde{f}(g_0 p n) d\mu_{g_0 p N^+, \nu_1}(n)
\]
\[
= \int_{N^+} \tilde{F}(g_0 p n) d\mu_{g_0 p N^+, \nu_1}(n) + \int_{N^+_t} \tilde{h}(g_0 p n) d\mu_{g_0 p N^+, \nu_1}(n)
\]
for some \( \tilde{h} \) that vanishes on \( \bigcup_{Z \in \mathcal{Z}_t} Z N^+ \). Returning to (2.20), we now give an upper bound. We observe:
\[
e^{\varphi_1(\log a_t)} \int_{N^+} f([g] n a_t) \phi(n) d\mu_{\lambda_t}(n)
\]
\[
\leq \sum_{\gamma \in \Gamma_t} \phi^{+}_{a_0 e^{-\alpha t}}(a_t n_{t, \gamma}^{-1} a_t^{-1}) \int_{N^+_0} \tilde{f}(g_0 p t, \gamma n) d\mu_{\lambda_t}(n)
\]
\[
= \sum_{\gamma \in \Gamma_t} \phi^{+}_{a_0 e^{-\alpha t}}(a_t n_{t, \gamma}^{-1} a_t^{-1}) \int_{N^+_R} (\tilde{F} + \tilde{h})(g_0 p t, \gamma n) d\mu_{g_0 p t, \gamma N^+, \nu_1}(n)
\]
\[
= \sum_{\gamma \in \Gamma_t} \int_{N^+_R} (\tilde{F} + \tilde{h})(g_0 p t, \gamma n) \phi^{+}_{a_0 e^{-\alpha t}}(a_t n_{t, \gamma}^{-1} a_t^{-1}) d\mu_{g_0 p t, \gamma N^+, \nu_1}(n).
\]

Similarly as before, we have, for all \( t > t_0 \) and \( n \in N^+_R \),
\[
\phi^{+}_{a_0 e^{-\alpha t}}(a_t n_{t, \gamma}^{-1} a_t^{-1}) = \phi^{+}_{a_0 e^{-\alpha t}}(a_t n_{t, \gamma}^{-1} n(n^{-1} a_t^{-1})
\]
\[
\leq \phi^{+}_{(R+\varepsilon_0) e^{-\alpha t}}(a_t n_{t, \gamma}^{-1} a_t^{-1}). \tag{2.22}
\]

Hence (2.20) is bounded above by
\[
\leq \sum_{\gamma \in \Gamma_t} \int_{N^+_R} (\tilde{F} + \tilde{h})(g_0 p t, \gamma n) \phi^{+}_{(R+\varepsilon_0) e^{-\alpha t}}(a_t n_{t, \gamma}^{-1} a_t^{-1}) d\mu_{g_0 p t, \gamma N^+, \nu_1}(n)
\]
\[
= \sum_{\gamma \in \Gamma_t} \int_{N^+_R} (\tilde{F} + \tilde{h})(g_0 p t, \gamma n) \phi^{+}_{(R+\varepsilon_0) e^{-\alpha t}}(n) d((\theta_t, \gamma)^{-1})^1 \mu_{g_0 p t, \gamma N^+, \nu_1})(n)
\]
where \( \theta_t, \gamma(n) = n_{t, \gamma} a_t^{-1} n a_t \). By Lemma 2.8,
\[
d((\theta_t, \gamma)^{-1})^1 \mu_{g_0 p t, \gamma N^+, \nu_1})(n) = e^{\varphi_1(\log a_t)} d\mu_{g_0 p t, \gamma N^+, \nu_1 a_t^{-1} N^+, \nu_1}(n).
Since \( g_0 p_t, g a_t^{-1} = \gamma g \), it follows that for all \( t > t_0 \),
\[
e^{(\varphi_1 - \psi_1)(\log a_t)} \int_{N^+} f([g] n a_t) \phi(n) \, d\mu_\lambda(n)
\leq \sum_{\gamma \in \Gamma} \int_{N^+} (\tilde{F} + \tilde{h})(\gamma g a_t) \phi_+^{(R+\varepsilon_0)}(n) \, d\mu_{\gamma N^+}^\mu(n)
\leq \int_{N^+} \left( \sum_{\gamma \in \Gamma} (\tilde{F} + \tilde{h})(\gamma g a_t) \right) \phi_+^{(R+\varepsilon_0)}(n) \, d\mu_\nu(n).
\]
Define functions \( F \) and \( h \) on \( \Gamma \setminus G \) by
\[
F([g]) := \sum_{\gamma \in \Gamma} \tilde{F}(\gamma g) \quad \text{and} \quad h([g]) := \sum_{\gamma \in \Gamma} \tilde{h}(\gamma g).
\]
Then for any \( \varepsilon > 0 \) and for all \( t > t_0 \) such that \( (R + \varepsilon_0) e^{-\alpha t} \leq \varepsilon \),
\[
\Psi(t) e^{(\varphi_1 - \psi_1)(\log a_t)} \int_{N^+} f([g] n a_t) \phi(n) \, d\mu_\lambda(n)
\leq \Psi(t) \int_{N^+} (F + h)([g] n a_t) \phi_+^{(R+\varepsilon_0)}(n) \, d\mu_\nu(n).
\]
By Proposition \([2.13] \) letting \( \varepsilon \to 0 \) gives
\[
\limsup_{t \to +\infty} \Psi(t) e^{(\varphi_1 - \psi_1)(\log a_t)} \int_{N^+} f([g] n a_t) \phi(n) \, d\mu_\lambda(n)
\leq \sum_{\gamma \in \Gamma} m\gamma [F + h] \mu_\nu |_{\gamma N}(\phi).
\]
Note that \( m^* = m \) by Lemma \([2.6] \). Now, by Lemma \([2.9] \) and the fact \( \tilde{m}(\tilde{h}) = 0 \), we have
\[
m\gamma [F + h] = \tilde{m}\gamma [\tilde{F} + \tilde{h}] = \tilde{m}\gamma [\tilde{F}] = m^* [\tilde{F}]
\]
\[
= \int_P \left( \int_{N^+} \tilde{F} \tilde{Z}(g_0 h a n m) d\mu_{g_0 h a n m}^\mu_\nu(n) \right) e^{-\psi_2(\log a)} \, dm \, d\mu_{g_0 N^+}^\mu_\nu(h)
\]
\[
= \int_P \left( \int_{N^+} \tilde{F} \tilde{Z}(g_0 h a n m) d\mu_{g_0 h a n m}^\mu_\nu_\nu(n) \right) e^{-\psi_2(\log a)} \, dm \, d\mu_{g_0 N^+}^\mu_\nu_\nu(h)
\]
\[
= m_{\lambda_1, \nu_2} [\tilde{Z} N^+ (\tilde{F})] = m_{\lambda_1, \nu_2} [Z N^+ (F)]
\]
This gives the desired upper bound. Note that we have used the assumption \( \text{supp} \nu_2 = \Lambda \) in the fourth equality above to apply \([2.21] \). The lower bound can be obtained similarly, finishing the proof. \( \square \)

With the help of Proposition \([2.13] \) we are now ready to give:

**Proof of Theorem \([2.11] \)** By the compactness hypothesis on the supports of \( f_i \), we can find \( \varepsilon_0 > 0 \) and \( x_i \in \Gamma \setminus G, i = 1, \cdots, \ell \) such that the map \( G \to \Gamma \setminus G \) given by \( g \to x_i g \) is injective on \( R_{\varepsilon_0} = P_{\varepsilon_0} N_{\varepsilon_0}^+ \), and \( \bigcup_{i=1}^\ell x_i R_{\varepsilon_0/2} \) contains both \( \text{supp} f_1 \) and \( \text{supp} f_2 \). We use continuous partitions of unity
to write \( f_1 \) and \( f_2 \) as finite sums \( f_1 = \sum_{i=1}^{\ell} f_{1,i} \) and \( f_2 = \sum_{j=1}^{\ell} f_{2,j} \) with \( \text{supp} f_{1,i} \subset x_i R_{e_0/2} \) and \( \text{supp} f_{2,j} \subset x_j R_{e_0/2} \). Writing \( p = \text{ham} \in \text{NAM} \) and using Lemma 2.9,
\[
\text{dm}^*_{\lambda_1,\lambda_2} (\text{hamn}) = d\mu_{\text{hamN}+,\lambda_1}(n) e^{-\psi_2 o (\log a)} \text{dm} \text{ da} d\mu_{N,\lambda_2} (h).
\]
We have
\[
\int_{\Gamma \backslash \mathcal{G}} f_1(xa_t)f_2(x) \text{ dm}^*_{\lambda_1,\lambda_2}(x) = (2.23)
\]
\[
\sum_{i,j} \int_{R_{e_0}} f_{1,i}(x) f_{2,j}(x) \text{ d}\mu_{\text{hamN}+,\lambda_1}(n) e^{-\psi_2 o (\log a)} \text{ dm} \text{ da} d\mu_{N,\lambda_2} (h)
\]
\[
= \sum_{i,j} \int_{N_{e_0} A_{e_0} M_{e_0}} \left( \int_{N_{e_0}^+} f_{1,i}(x) f_{2,j}(x) \text{ d}\mu_{\text{hamN}+,\lambda_1}(n) \right) \times e^{-\psi_2 o (\log a)} \text{ dm} \text{ da} d\mu_{N,\lambda_2} (h).
\]
Applying Proposition 2.17 it follows:
\[
\lim_{t \to \infty} \Psi(t) e^{(\varphi_1 - \psi_1 o (\log a)} \int_{\Gamma \backslash \mathcal{G}} f_1(xa_t)f_2(x) \text{ dm}^*_{\lambda_1,\lambda_2}(x)
\]
\[
= \sum_{Z \in \mathcal{Z}} \sum_{i} m_{\lambda_1,\nu_2} |ZN^+ (f_{1,i}) \sum_{i} \int_{N_{e_0} A_{e_0} M_{e_0}} \mu_{x_1 p N^+,\nu_1} |ZN (f_{2,i}(x p \cdot )) \times e^{-\psi_2 o (\log a)} \text{ dm} \text{ da} d\mu_{N,\lambda_2} (h)
\]
\[
= \sum_{Z \in \mathcal{Z}} m_{\lambda_1,\nu_2} |ZN^+ (f_1) \sum_{i} \int_{N_{e_0} A_{e_0} M_{e_0}} \mu_{x_1 p N^+,\nu_1} (f_{2,i} |ZN (x p \cdot )) \times e^{-\psi_2 o (\log a)} \text{ dm} \text{ da} d\mu_{N,\lambda_2} (h)
\]
\[
= \sum_{Z \in \mathcal{Z}} m_{\lambda_1,\nu_2} |ZN^+ (f_1) \sum_{i} m^*_{\nu_1,\lambda_2} (f_{2,i} |ZN) = \sum_{Z \in \mathcal{Z}} m_{\lambda_1,\nu_2} |ZN^+ (f_1) m^*_{\nu_1,\lambda_2} |ZN(f_2)
\]
where the second last equality is valid by Lemma 2.9. This completes the
deproof.

3. LOCAL MIXING FOR ANOSOV GROUPS

Let \( \Gamma < G \) be a Zariski dense Anosov subgroup with respect to \( P \). For any \( u \in \text{int} \, \mathcal{L}_\Gamma \), there exists a unique
\[
\psi = \psi u \in D^*_{\Gamma},
\]
such that \( \psi(u) = \psi_T(u) \) [7] Prop. 4.4. Let \( \nu_\psi \) denote the unique \((\Gamma, \psi)\)-PS measure [7] Thm. 1.3. Similarly, \( \nu_{\psi_0} \) denotes the unique \((\Gamma, \psi \circ i)\)-PS-measure.

In this section, we deduce \((r := \dim a)\):
Theorem 3.1 (Local mixing). For $i = 1, 2$, let $\varphi_i \in a^*$ and $\lambda_{\varphi_i}$ be any $(\Gamma, \varphi_i)$-conformal measure on $F$. For any $u \in \text{int} \mathcal{L}_\Gamma$, there exists $\kappa_u > 0$ such that for any $f_1, f_2 \in C_c(\Gamma \setminus G)$, we have

$$
\lim_{t \to +\infty} t^{(r-1)/2} e^{(\varphi_1 - \psi_u)(tu)} \int_{\Gamma \setminus G} f_1(x \exp(tu)) f_2(x) \, dm_{\lambda_{\varphi_1}, \lambda_{\varphi_2}}(x) = \kappa_u \sum_{Z \in \Delta} m_{\lambda_{\varphi_1}, \nu_{\psi_u \circ i}} |ZN^+(f_1) m_{\nu_{\psi_u}, \lambda_{\varphi_2}} |ZN(f_2).
$$

Theorem 3.1 is a consequence of Theorem 2.11 since the measure $m = m_{\nu_{\psi_u}, \nu_{\psi_u \circ i}}$ satisfies the Hypothesis 2.10 by the following theorem of Chow and Sarkar.

Theorem 3.2. [3] Let $u \in \text{int} \mathcal{L}_\Gamma$. There exists $\kappa_u > 0$ such that for any $f_1, f_2 \in C_c(\Gamma \setminus G)$, we have

$$
\lim_{t \to +\infty} t^{(r-1)/2} \int_{\Gamma \setminus G} f_1(x \exp(tu)) f_2(x) \, dm_{\nu_{\psi_u \circ i}}(x) = \kappa_u \sum_{Z \in \Delta} m_{\nu_{\psi_u \circ i}} |Z(f_1) m_{\nu_{\psi_u}} |Z(f_2).
$$

Let $m_o$ denote the $K$-invariant probability measure on $F = G/P$. Then $m_o$ coincides with the $(G, 2\rho)$-conformal measure on $F$ where $2\rho$ denotes the sum of positive roots for $(\mathfrak{g}, a^+)$. The corresponding BMS measure $dx = dm_{m_o, m_o}$ is a $G$-invariant measure on $\Gamma \setminus G$. The measure $dm_{\nu_{\psi_u \circ i}}^{BR} = dm_{m_o, \nu_{\psi_u \circ i}}$ was defined and called the $N^+M$-invariant Burger-Roblin measure in [3]. Similarly, the $NM$-invariant Burger-Roblin measure was defined as $dm_{\nu_{\psi_u \circ i}}^{BR}$. In these terminologies, the following is a special case of Theorem 3.1.

Corollary 3.3 (Local mixing for the Haar measure). For any $u \in \text{int} \mathcal{L}_\Gamma$, and for any $f_1, f_2 \in C_c(\Gamma \setminus G)$, we have

$$
\lim_{t \to +\infty} t^{(r-1)/2} e^{(2\rho - \psi_u)(tu)} \int_{\Gamma \setminus G} f_1(x \exp(tu)) f_2(x) \, dx = \kappa_u \sum_{Z \in \Delta} m_{\nu_{\psi_u \circ i}}^{BR} |ZN^+(f_1) m_{\nu_{\psi_u}}^{BR} |ZN(f_2)
$$

where $\kappa_u$ is as in Theorem 3.2.

In fact, we get the following more elaborate version of the above corollary by combining the proof of [4] Theorem 7.12 and the proof of Corollary 3.3.

Theorem 3.4. Let $u \in \text{int} \mathcal{L}_\Gamma$. For any $f_1, f_2 \in C_c(\Gamma \setminus G)$ and $v \in \ker \psi_u,

$$
\lim_{t \to +\infty} t^{(r-1)/2} e^{(2\rho - \psi_u)(tu + \sqrt{t}v)} \int_{\Gamma \setminus G} f_1(x \exp(tu + \sqrt{t}v)) f_2(x) \, dx = \kappa_u e^{-I(v)/2} \sum_{Z \in \Delta} m_{\nu_{\psi_u \circ i}}^{BR} |ZN^+(f_1) m_{\nu_{\psi_u}}^{BR} |ZN(f_2)
$$

where $\kappa_u$ is as in Theorem 3.2.
where $I : \ker \psi_u \to \mathbb{R}$ is given by

$$I(v) := c \cdot \frac{\|v\|^2_2 \|u\|^2_2 - \langle v, u \rangle^2}{\|u\|^2_2}$$

(3.5)

for some inner product $\langle \cdot, \cdot \rangle_*$ and some $c > 0$. Moreover the left-hand sides of the above equalities are uniformly bounded for all $(t, v) \in (0, \infty) \times \ker \psi_u$ with $tv + \sqrt{tv} \in \mathfrak{a}^\perp$.

### 4. Proof of Theorem 1.4

Let $\Gamma < G$ be a Zariski dense Anosov subgroup with respect to $P$.

**The $u$-balanced measures.** Let $\Omega = \{ [g] \in \Gamma \backslash G : g^\perp \in \Lambda \}$. Following [2], given $u \in \interior_\Gamma$, we say that a locally finite Borel measure $m_0$ on $\Gamma \backslash G$ is $u$-balanced if

$$\limsup_{T \to +\infty} \frac{\int_0^T m_0(O_1 \cap O_1 \exp(tu)) \, dt}{\int_0^T m_0(O_2 \cap O_2 \exp(tu)) \, dt} < \infty,$$

for all bounded $M$-invariant Borel subsets $O_i \subset \Gamma \backslash G$ with $\Omega \cap \interior O_i \neq \emptyset$, $i = 1, 2$.

As an immediate corollary of Theorem 3.1 we get

**Corollary 4.1.** Let $\varphi \in \mathfrak{a}^*$. For any pair $(\lambda_\varphi, \lambda_{\varphi, i})$ of $(\Gamma, \varphi)$ and $(\Gamma, \varphi \circ i)$-conformal measures on $\mathcal{F}$ respectively, the corresponding BMS-measure $m_{\lambda_\varphi, \lambda_{\varphi, i}}$ is $u$-balanced for any $u \in \interior_\Gamma$.

**Proof.** Let $O_1, O_2$ be $M$-invariant Borel subsets such that $\Omega \cap \interior O_i \neq \emptyset$ for each $i = 1, 2$. Let $f_1, f_2 \in C_c(\Gamma \backslash G)$ be non-negative functions such that $f_1 \geq 1$ on $O_1$ and $f_2 \leq 1$ on $O_2$ and $0$ outside $O_2$. Since $\interior O_2 \cap \Omega \neq \emptyset$, we may choose $f_2$ so that $m_{\nu_{\psi_u}^*, \lambda_{\varphi, i}}(f_2) > 0$. For simplicity, we set $m_0 = m_{\lambda_\varphi, \lambda_{\varphi, i}}$. By Theorem 3.1 and using the fact that $m_0$ is $A$-quasi-invariant, we obtain that for any $u \in \interior_\Gamma$,

$$\limsup_{t \to +\infty} \frac{m_0(O_1 \cap O_1 \exp(tu))}{m_0(O_2 \cap O_2 \exp(tu))} \leq \limsup_{t \to +\infty} \frac{\int f_1(x) f_1(x \exp(-tu)) \, dm_0(x)}{\int f_2(x) f_2(x \exp(-tu)) \, dm_0(x)}$$

$$= \limsup_{t \to +\infty} \frac{\int f_1(x) f_1(x \exp(tu)) \, dm_0(x)}{\int f_2(x) f_2(x \exp(tu)) \, dm_0(x)}$$

$$= \limsup_{t \to +\infty} \frac{t^{(r-1)/2} e^{\varphi - \psi_u(tu)} \int f_1(x) f_1(x \exp(tu)) \, dm_0(x)}{t^{(r-1)/2} e^{\varphi - \psi_u(tu)} \int f_2(x) f_2(x \exp(tu)) \, dm_0(x)}$$

$$= \frac{m_{\lambda_\varphi, \lambda_{\varphi, i}}^*(f_1)}{m_{\nu_{\psi_u}^*, \lambda_{\varphi, i}}^*(f_2)} < \infty.$$  

This shows that $m_0$ is $u$-balanced.

Recall Theorem 1.4 from the introduction:
Theorem 4.2. Let \( \text{rank} G \leq 3 \). For any \( \psi \in D^*_\Gamma \), any \((\Gamma, \psi)\)-conformal measure on \( F \) is necessarily supported on \( \Lambda \). Moreover, the PS measure \( \nu_\psi \) is the unique \((\Gamma, \psi)\)-conformal measure on \( F \).

**Proof.** Let \( u \in \text{int} \mathcal{L}_\Gamma \) denote the unique unit vector such that \( \psi(u) = \psi_\Gamma(u) \), that is, \( \psi = \psi_u \). Let \( \lambda_\psi \) be any \((\Gamma, \psi)\)-conformal measure on \( F \). We claim that \( \lambda_\psi \) is supported on \( \Lambda \). The main ingredient is the higher rank Hopf-Tsuji-Sullivan dichotomy established in [2]. The main point is that all seven conditions of Theorem 1.4 of [2] are equivalent to each other for Anosov groups and \( u \in \text{int} \mathcal{L}_\Gamma \), since all the measures considered there are \( u \)-balanced by Corollary 4.1. In this proof, we only need the equivalence of (6) and (7), which we now recall.

Consider the following \( u \)-directional conical limit set of \( \Gamma \):

\[
\Lambda_u := \{ g^+ \in \Lambda : \gamma_i \exp(t_i u) \text{ is bounded for some } t_i \to +\infty \text{ and } \gamma_i \in \Gamma \}. \tag{4.3}
\]

Note that \( \Lambda_u \subset \Lambda \). For \( R > 0 \), we set \( \Gamma_u,R := \{ \gamma \in \Gamma : ||\mu(\gamma) - R u|| < R \} \). Applying the dichotomy [2, Thm. 1.4] to a \( u \)-balanced measure \( m_{\lambda_\psi, \nu_\psi \circ i} \), we deduce

**Proposition 4.4.** The following conditions are equivalent for \( \lambda_\psi \):

1. \( \lambda_\psi(\Lambda_u) = 1 \);
2. \( \sum_{\gamma \in \Gamma_u,R} e^{-\psi(\mu(\gamma))} = \infty \) for some \( R > 0 \).

On the other hand, if \( \text{rank} G \leq 3 \), we have

\[
\sum_{\gamma \in \Gamma_u,R} e^{-\psi(\mu(\gamma))} = \infty
\]

for some \( R > 0 \) [2, Thm. 6.3]. Therefore, by Proposition 4.4 we have \( \lambda_\psi(\Lambda_u) = 1 \) and hence \( \lambda_\psi \) is supported on \( \Lambda \) in this case. This finishes the proof of the first part of Theorem 1.4. The second claim follows from the first one by [7, Thm. 1.3]. \( \square \)

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