A Proof Procedure for Separation Logic with Inductive Definitions and Data

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Abstract
A proof procedure, in the spirit of the sequent calculus, is proposed to check the validity of entailments between Separation Logic formulas combining inductively defined predicates denoting structures of bounded tree width and theory reasoning. The calculus is sound and complete, in the sense that a sequent is valid iff it admits a (possibly infinite) proof tree. We also show that the procedure terminates in the two following cases: (i) When the inductive rules that define the predicates occurring on the left-hand side of the entailment terminate, in which case the proof tree is always finite. (ii) When the theory is empty, in which case every valid sequent admits a rational proof tree, where the total number of pairwise distinct sequents occurring in the proof tree is doubly exponential w.r.t. the size of the end-sequent.

Keywords Separation logic · Entailment problem · Inductive theorem proving · Theory reasoning

1 Introduction

Separation Logic (SL) [19, 29], is a well-established framework for reasoning on programs manipulating pointer-based data structures. It forms the basis of several industrial-scale static program analyzers [2, 6, 10]. The logic uses specific connectives to assert that formulas are satisfied on disjoint parts of the memory, which allows for more concise and more natural specifications. Recursive data structures are specified using inductively defined predicates, which provide a specification mechanism similar to the definition of a recursive data type in an imperative programming language. Many verification tasks boil down to checking entailments between formulas built on such atoms. More precisely, the logic may be used to express pre- or post-conditions describing the shape of the data structures (linked lists,
trees, doubly linked lists, etc.) manipulated by the program, and to express structural integrity properties, such as acyclicity of linked lists, absence of dangling pointers, etc. Investigating the entailment problem for SL formulas is thus of theoretical and practical interest. In practice, it is essential to offer as much flexibility as possible, and to handle a wide class of user-defined data structures (e.g., doubly linked lists, trees with child-parent links, trees with chained leaves etc.), possibly involving external theories, such as arithmetic. In general, the entailment problem is undecidable for formulas containing inductively defined predicates [18], and a lot of effort has been devoted to identifying decidable fragments and devising proof procedures, see e.g., [1, 4, 7, 8, 14, 15]. In particular, a general class of decidable entailment problems is described in [17]. It is based on the decidability of the satisfiability problem for monadic second order logic over graphs with a bounded treewidth, for formulas involving no theory other than equality. This class is defined by restricting the form of the inductive rules, which must fulfill 3 conditions, formally defined below: the progress condition (every rule allocates a single memory location), the connectivity condition (the set of allocated locations has a tree-shaped structure) and the establishment condition (every existentially quantified variable is eventually allocated). More recently, a 2-EXPTIME algorithm was proposed for such entailments [24]. In [11] we showed that this bound is tight and in [12] we devised a new algorithm, handling more general classes of inductive definitions. The algorithms in [11, 24] work by computing some abstract representation of the set of models of SL formulas. The abstraction is precise enough to allow checking that all the models of the left-hand side are also models of the right-hand side, and also general enough to ensure termination of the entailment checking algorithm. Other approaches have been proposed to check entailments in various fragments, see e.g., [7, 15, 18]. In particular, a sound and complete proof procedure is given in [31] for inductive rules satisfying conditions that are strictly more restrictive than those in [17]. In [16] a labeled proof systems is presented for separation logic formulas handling arbitrary inductive definitions and all connectives (including negation and separated implication). Due to the expressive power of the considered language, this proof system is of course not terminating or complete in general.

In the present paper, we extend these results by defining a proof procedure in the style of sequent calculi to check the validity of entailments, using top-down decomposition rules. Induction is performed by considering infinite (rational) proof trees modeling proofs by infinite descent, as in [3]. We also tackle the combination of SL reasoning with theory reasoning, relying on external decision procedures for checking the validity of formulas in the considered theory. This issue is of uttermost importance for applications, as reasoning on data structures without taking into account the properties of the data stored in these structures has a limited scope, and the combination of SL with data constraints has been considered by several authors (see, e.g., [21, 26–28, 32]). Beside the fact that it is capable of handling theory reasoning, our procedure has several advantages over the model-based bottom-up algorithms that were previously devised [11, 24]. It is goal-oriented: the rules apply backward and reduce the considered entailments to simpler ones, until axioms are reached. The advantage is that the proof procedure is driven by the form of the current goal. The procedure is also better-suited for interactive theorem proving (e.g., the user can guide the application of the inference rules, while the procedures in [11, 24] work as “black boxes”). If the entailment is valid, then the produced proof tree serves as a certification of the result, which can be checked if needed by the user or another system, while the previous procedures [11, 24] produce no certification. Finally, the correctness proof of the algorithm is also simpler and more modular. More specifically, we establish several new results in this paper.
1. First, we show that the proof procedure is sound (Theorem 61) and complete (Theorem 65), in the sense that an entailment is valid iff it admits a proof tree. The proof tree may be infinite, hence the result does not entail that checking entailments is semi-decidable. However, this shows that the procedure can be used as a semi-decision procedure for checking non-validity (i.e., an entailment is not valid iff the procedure is stuck eventually in some branch), provided the base theory is decidable.

2. If the theory only contains the equality predicate, then we show that the entailments can be reduced to entailments in the empty theory (Theorem 42). The intuition is that all the equality and disequality constraints can be encoded in the formulas describing the shape of the data structures. This result is also described in a paper [13] accepted for presentation at ASL 2022 (a workshop with no formal proceedings).

3. By focusing on the case where the theory is empty, we show (Theorem 84) that every valid entailment admits a proof tree that is rational (i.e., has a finite number of pairwise distinct subtrees, up to a renaming of variables). Furthermore, the number of sequents occurring in the tree is at most $O(2^{2^n})$, where $n$ is the size of the initial sequent. In combination with the previous result, this theorem allows us to reestablish the 2-EXPTIME membership of the entailment problem for inductive systems satisfying the conditions above in the theory of equality [25].

4. We also show that the proof tree is finite if the inductive rules that define the predicates occurring on the left-hand side of the entailment terminate (Corollary 68).

2 Preliminaries

In this section, we define the syntax and semantics of the fragment of separation logic that is considered in the paper. Our definitions are mostly standard, see, e.g., [17, 23, 29] for more details and explanations on separation logic as well as on the conditions on the inductively defined predicates that ensure decidability of the entailment problem.

2.1 Syntax

Let $\mathcal{V}$ be a countably infinite set of variables. Let $\mathcal{P}_T$ be a set of $T$-predicates (or theory predicates, denoting relations in an underlying theory of locations) and let $\mathcal{P}_S$ be a set of spatial predicates, disjoint from $\mathcal{P}_T$. Each symbol $p \in \mathcal{P}_T \cup \mathcal{P}_S$ is associated with a unique arity $\#(p)$. We assume that $\mathcal{P}_T$ contains in particular two binary symbols $\approx$ and $\not\approx$ and a nullary symbol $false$.

Definition 1 Let $\kappa$ be some fixed\(^1\) natural number. The set of $SL$-formulas (or simply formulas) $\phi$ is inductively defined as follows:

$$\phi ::= emp \parallel x \mapsto (y_1, \ldots, y_\kappa) \parallel \phi_1 \lor \phi_2 \parallel \phi_1 \star \phi_2 \parallel p(x_1, \ldots, x_{\#(p)}) \parallel \exists x. \phi_1$$

where $\phi_1, \phi_2$ are $SL$-formulas, $p \in \mathcal{P}_T \cup \mathcal{P}_S$ and $x_1, x_2, \ldots, x_{\#(p)}, y_1, \ldots, y_\kappa$ are variables.

Note that the considered fragment (as in many works in Separation Logic) does not include standard conjunction, negation or universal quantifications. Indeed, the addition of these constructions, without any further restriction, makes entailment checking undecidable (see for instance [24]). The separating implication $\rightarrow$ is not supported either, although a

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\(^1\) Note that $\kappa$ is not considered as constant for the complexity analysis in Sect. 10: it is part of the input.
similar but more restricted connective → will be introduced below. Formulas are taken modulo associativity and commutativity of ∨ and *, modulo commutativity of existential quantifications and modulo the neutrality of emp for *.

A spatial atom is a formula that is either of the form x \mapsto (y_1, \ldots, y_r) (called a points-to atom) or p(x_1, \ldots, x_{\#(p)}) with p \in P_S (called a predicate atom). A T-atom is a formula of the form p(x_1, \ldots, x_{\#(p)}) with p \in P_T. An atom is either a spatial atom or a T-atom. A T-formula is either emp or a separating conjunction of T-atoms. A formula of the form \exists x_1, \ldots, x_n \cdot \phi \text{ (with } n \geq 0\text{)} is denoted by \exists \mathbf{x}. \phi \text{ with } \mathbf{x} = (x_1, \ldots, x_n). A formula is predicate-free (resp. disjunction-free, resp. quantifier-free) if it contains no predicate symbol in P_S (resp. no occurrence of ∨, resp. of \exists). It is in prenex form if it is of the form \exists \mathbf{x}. \phi, where \phi \text{ is quantifier-free and } \mathbf{x} \text{ is a possibly empty vector of variables. A symbolic heap is a prenex disjunction-free formula, i.e., a formula of the form } \exists \mathbf{x}. \phi, \text{ where } \phi \text{ is a separating conjunction of atoms.}

**Definition 2** A \mapsto-formula is a formula of the form \exists \mathbf{x}. (u \mapsto v * \chi), \text{ where } \chi \text{ is a } T-\text{formula.}

**Example 3** Let \emph{ils} and \emph{als} be two spatial predicates denoting increasing and acyclic nonempty list segments, respectively. The symbolic heap:\text{\emph{x}_1 \mapsto (\text{\emph{x}_2) \ast \text{\emph{x}_1}} \geq 0 \ast \emph{ils}(\text{\emph{x}_2, \text{\emph{x}_3}) \ast \emph{ils}(\text{\emph{x}_3, \text{\emph{x}_4}) \text{ denotes an increasing list of positive numbers composed by a first element } \text{\emph{x}_1}, \text{ linked to a list composed by the concatenation of two list segments, from } \text{\emph{x}_2} \text{ to } \text{\emph{x}_3} \text{ and from } \text{\emph{x}_3} \text{ to } \text{\emph{x}_4}, \text{ respectively. The atom } \text{\emph{x}_1} \mapsto (\text{\emph{x}_2}) \text{ is a points-to atom, } \emph{ils}(\text{\emph{x}_2, \text{\emph{x}_3}) \text{ and } \emph{ils}(\text{\emph{x}_3, \text{\emph{x}_4}) \text{ are predicate atoms and } \text{\emph{x}_1} \geq 0 \text{ is a } T-\text{formula (constructed using a monadic predicate stating that } \text{\emph{x}_1} \text{ is positive). The symbolic heap } \exists \text{\emph{x}_1, \text{\emph{x}_2}. (\emph{als}(\text{\emph{x}_1, \text{\emph{x}_2}) \ast \text{\emph{x}_1} \geq 0 \ast \text{\emph{x}_2} \geq 0) \text{ denotes an acyclic list segment between two positive locations.}

We denote by \text{fv}(\phi) \text{ the set of variables freely occurring in } \phi \text{ (i.e., occurring in } \phi \text{ but not within the scope of any existential quantifier). A substitution } \sigma \text{ is a function mapping variables to variables. The domain } \text{dom}(\sigma) \text{ of a substitution } \sigma \text{ is the set of variables } x \text{ such that } \sigma(x) \neq x, \text{ and we let } \text{img}(\sigma) = \sigma(\text{dom}(\sigma)). \text{ For all substitutions } \sigma, \text{ we assume that } \text{dom}(\sigma) \text{ is finite and that } \sigma \text{ is idempotent. For any expression (variable, tuple of variables or formula) } e, \text{ we denote by } e\sigma \text{ the expression obtained from } e \text{ by replacing every free occurrence of a variable } x \text{ by } \sigma(x) \text{ and by } \{x_i \leftarrow y_i \mid 1 \leq i \leq n\} \text{ (where the } x_1, \ldots, x_n \text{ are pairwise distinct) the substitution such that } \sigma(x_i) = y_i \text{ and } \text{dom}(\sigma) \subseteq \{x_1, \ldots, x_n\}. \text{ For all sets } E, \text{ card}(E) \text{ is the cardinality of } E. \text{ For all sequences or words } w, \|w\| \text{ denotes the length of } w. \text{ We sometimes identify vectors with sets, if the order is unimportant, e.g., we write } \mathbf{x} \setminus \mathbf{y} \text{ to denote the vector formed by the components of } \mathbf{x} \text{ that do not occur in } \mathbf{y}.\n
2.2 Size and Width

We assume that the symbols in P_S \cup P_T \cup V \text{ are words over a finite alphabet of a constant size, strictly greater than } 1. \text{ For any expression } e, \text{ we denote by size}(e) \text{ the size of } e, \text{ i.e., the number of occurrences of symbols in } e. \text{ We define the width of a formula as follows:}

\[
\text{width}(\phi_1 \lor \phi_2) = \max(\text{width}(\phi_1), \text{width}(\phi_2))
\]

\[
\text{width}(\exists \mathbf{x}. \phi) = \text{width}(\phi) + \text{size}(\exists \mathbf{x})
\]

2 Because we will consider transformations introducing an unbounded number of new predicate symbols, we cannot assume that the predicate atoms have a constant size.

3 Each symbol s in P_S \cup P_T \cup V \text{ is counted with a weight equal to its length } \|s\|, \text{ and all the logical symbols have weight } 1.
width(φ₁ * φ₂) = width(φ₁) + width(φ₂) + 1
width(φ) = size(φ) if φ is an atom

Note that width(φ) coincides with size(φ) if φ is disjunction-free.

2.3 Inductive Rules

The semantics of the predicates in \( \mathcal{P}_S \) is provided by user-defined inductive rules satisfying some conditions (as defined in [17]):

**Definition 4** A (progressing and connected) set of inductive rules (pc-SID) \( \mathcal{R} \) is a finite set of rules of the form

\[
p(x₁, \ldots, xₙ) \iff \exists u. x₁ \mapsto (y₁, \ldots, yₙ) * \phi,
\]

where \( \text{fv}(x₁ \mapsto (y₁, \ldots, yₙ) * \phi) \subseteq \{x₁, \ldots, xₙ\} \cup u \), \( \phi \) is a possibly empty separating conjunction of predicate atoms and \( T \)-formulas, and for every predicate atom \( q(z₁, \ldots, z_{\#q}) \) occurring in \( \phi \), we have \( z₁ \in \{y₁, \ldots, yₙ\} \). We let \( size(p(x) \iff \phi) = size(p(x)) + size(\phi) \), \( size(\mathcal{R}) = \Sigma_{\rho \in \mathcal{R}} size(\rho) \) and \( width(\mathcal{R}) = \max_{\rho \in \mathcal{R}} size(\rho) \).

In the following, \( \mathcal{R} \) always denotes a pc-SID. We emphasize that the right-hand side of every inductive rule contains exactly one points-to atom, the left-hand side of which is the first argument of the predicate symbol (this condition is referred to as the *progress* condition), and that this points-to atom contains the first argument of every predicate atom on the right-hand side of the rule (the *connectivity* condition).

**Example 5** The predicates \( \text{iils} \) and \( \text{als} \) of Example 3 are defined as follows:

\[
\begin{align*}
\text{iils}(x, y) & \iff x \mapsto (y) * x \leq y \\
\text{iils}(x, y) & \iff \exists x'. x \mapsto (x') * \text{iils}(x', y) * x \leq x' \\
\text{als}(x, y) & \iff x \mapsto (y) * x \not\approx y \\
\text{als}(x, y) & \iff \exists x'. x \mapsto (x') * \text{als}(x', y) * x \not\approx y
\end{align*}
\]

This set is progressing and connected. In contrast, the rule \( \text{iils}(x, y) \iff x \approx y \) is not progressing, because it contains no points-to atom. A possibly empty list must thus be denoted by a disjunction in our framework: \( \text{iils}(x, y) \lor (x \approx y) \).

The next definition formalizes the notion of predicate unfolding. Intuitively, \( \phi \iff_\mathcal{R} \phi' \) holds if \( \phi' \) is obtained from \( \phi \) by replacing a predicate atom by the right-hand side of one of its inductive rules.

**Definition 6** We write \( p(x₁, \ldots, x_{\#p}) \iff_\mathcal{R} \phi \) if \( \mathcal{R} \) contains a rule (up to \( \alpha \)-renaming) \( p(y₁, \ldots, y_{\#p}) \iff \psi \), where \( x₁, \ldots, x_{\#p} \) are not bound in \( \psi \), and \( \phi = \psi[yᵢ \leftarrow xᵢ \mid i \in \{1, \ldots, \#(p)\}] \).

The relation \( \iff_\mathcal{R} \) is extended to all formulas as follows: \( \phi \iff_\mathcal{R} \phi' \) if one of the following conditions holds:

(i) \( \phi = φ₁ • φ₂ \) (modulo AC, with \( • \in \{*, \lor\} \)), \( φ₁ \iff_\mathcal{R} φ'_₁ \), no free or existential variable in \( φ₂ \) is bound in \( φ'_₁ \) and \( φ' = φ'_₁ • φ₂ \);
(ii) \( \phi = \exists x. ψ, ψ \iff_\mathcal{R} ψ', x \) is not bound in \( ψ' \) and \( φ' = \exists x. ψ' \).
We denote by $\prec^+_{\mathcal{R}}$ the transitive closure of $\preceq_{\mathcal{R}}$, and by $\preceq^*_{\mathcal{R}}$ its reflexive and transitive closure. A formula $\psi$ such that $\phi \preceq^*_{\mathcal{R}} \psi$ is called an $\mathcal{R}$-unfolding of $\phi$. We denote by $\succeq_{\mathcal{R}}$ the least transitive and reflexive binary relation on $\mathcal{P}_S$ such that $p \succeq_{\mathcal{R}} q$ holds if $\mathcal{R}$ contains a rule of the form $p(y_1, \ldots , y_k(p)) \iff \psi$, where $q$ occurs in $\psi$. If $\phi$ is a formula, we write $\phi \succeq_{\mathcal{R}} q$ if $p \succeq_{\mathcal{R}} q$ for some $p \in \mathcal{P}_S$ occurring in $\phi$.

**Example 7** With the rules of Example 5, we have:

$$\text{als}(x_1, x_2) \cdot \text{als}(x_2, x_3) \preceq_{\mathcal{R}} x_1 \mapsto (x_2) \cdot x_1 \not\equiv x_2 \cdot \text{als}(x_2, x_3)$$

$$\preceq_{\mathcal{R}} \exists x'. (x_1 \mapsto (x_2) \cdot x_1 \not\equiv x_2 \cdot x_2 \mapsto (x') \cdot \text{als}(x', x_3) \cdot x_2 \not\equiv x_3)$$

$$\preceq_{\mathcal{R}} \exists x', x'' . (x_1 \mapsto (x_2) \cdot x_1 \not\equiv x_2 \cdot x_2 \mapsto (x') \cdot (x'') \cdot \text{als}(x', x_3) \cdot x' \not\equiv x_3 \cdot x_2 \not\equiv x_3)$$

$$\preceq_{\mathcal{R}} \exists x', x'' . (x_1 \mapsto (x_2) \cdot x_1 \not\equiv x_2 \cdot x_2 \mapsto (x') \cdot (x'') \cdot (x_3) \cdot x'' \not\equiv x_3 \cdot x_3 \not\equiv x_3)$$

$$\preceq_{\mathcal{R}} x' \not\equiv x_3 \cdot x_2 \not\equiv x_3)$$

Note that the number of formulas $\phi'$ such that $\phi \preceq_{\mathcal{R}} \phi'$ is finite, up to $\alpha$-renaming. Also, if $\phi \preceq_{\mathcal{R}} \phi'$ then $\text{fv}(\phi') \subseteq \text{fv}(\phi)$.

### 2.4 Semantics

**Definition 8** Let $\mathcal{L}$ be a countably infinite set of so-called locations. An SL-structure is a pair $(s, h)$ where $s$ is a store, i.e. a total function from $\mathcal{V}$ to $\mathcal{L}$, and $h$ is a heap, i.e. a partial finite function from $\mathcal{L} \to \mathcal{L}^*$ which is written as a relation: $h(\ell) = (\ell_1, \ldots , \ell_k) \iff (\ell, \ell_1, \ldots , \ell_k) \in h$. The size of a structure $(s, h)$ is the cardinality of $\text{dom}(h)$.

**Definition 9** For every heap $h$, let $\text{loc}(h) = \{ \ell_i \mid (\ell_0, \ldots , \ell_k) \in h, i = 0, \ldots , k \}$. A location $\ell$ (resp. a variable $x$) is allocated in a heap $h$ (resp. in a structure $(s, h)$) if $\ell \in \text{dom}(h)$ (resp. $s(x) \in \text{dom}(h)$). Two heaps $h_1, h_2$ are disjoint if $\text{dom}(h_1) \cap \text{dom}(h_2) = \emptyset$, in this case $h_1 \cup h_2$ denotes the union of $h_1$ and $h_2$.

Let $\models_T$ be a satisfiability relation between stores and $T$-formulas, satisfying the following properties: $s \models_T x \equiv y$ (resp. $s \models_T x \not\equiv y$) iff $s(x) = s(y)$ (resp. $s(x) \neq s(y)$), $s \models_T \text{false}$, $s \models_T \text{emp}$ for all stores $s$ and $s \models_T \chi \land \xi$ iff $s \models_T \chi$ and $s \models_T \xi$. For all $\tau$-formulas $\chi, \xi$, we write $\models_{\tau} \chi \equiv_{\tau} \xi$ if, for every store $s$ such that $s \models_{\tau} \chi$, we have $s \models_{\tau} \xi$. We write $\models_{\tau} \chi \iff_{\tau} \xi$ if the implication holds for every injective store $s$, i.e., if $s \models_{\tau} \chi \iff_{\tau} \psi$, where $\chi \iff_{\tau} \psi$ denotes the separating conjunction of all disequations $x \not\equiv x'$, with $x, x' \in \text{fv}(\chi) \cup \text{fv}(\xi)$, and $x \not\equiv x'$. We say that $\mathcal{T}$ is closed under negation if for every $\tau$-formula $\chi$, one can compute a $\tau$-formula $\chi'$ (also written $\neg_{\tau} \chi$) such that for every store $s$, $s \models_{\tau} \chi' \iff s \not\models_{\tau} \chi$. To make the rules in Sect. 8 applicable in practice it is necessary to have a procedure to check whether $\models_{\tau} \xi$. In all examples, $\mathcal{L}$ is the set of integers, and the $\tau$-formulas are arithmetic formulas, interpreted as usual (with predicates $\leq$ and $\approx$).

**Definition 10** Given formula $\phi$, a pc-SID $\mathcal{R}$ and a structure $(s, h)$, we write $(s, h) \models_{\mathcal{R}} \phi$ and say that $(s, h)$ is an $\mathcal{R}$-model (or simply a model if $\mathcal{R}$ is clear from the context) of $\phi$ if one of the following conditions holds.

- $\phi = x \mapsto (y_1, \ldots , y_k)$ and $h = \{(s(x), s(y_1), \ldots , s(y_k))\}$.
- $\phi$ is a $T$-formula, $h = \emptyset$ and $s \models_T \phi$.
- $\phi = \phi_1 \lor \phi_2$ and $(s, h) \models_{\mathcal{R}} \phi_i$, for some $i = 1, 2$. 
• \( \phi = \phi_1 \star \phi_2 \) and there exist disjoint heaps \( h_1, h_2 \) such that \( h = h_1 \uplus h_2 \) and \((s, h_i) \models R \phi_i\), for all \( i = 1, 2 \).

• \( \phi = \exists x. \phi \text{ and } (s', h) \models R \phi \), for some store \( s' \) coinciding with \( s \) on all variables distinct from \( x \).

• \( \phi = \rho(x_1, \ldots, x_{\#(p)}), p \in \mathcal{P}_S \) and \((s, h) \models R \psi \) for some \( \psi \) such that \( \phi \models R \psi \).

If \( \Gamma \) is a sequence of formulas, then we write \((s, h) \models R \Gamma \) if \((s, h) \) satisfies at least one formula in \( \Gamma \).

We emphasize that a \( T \)-formula is satisfied only in structures with empty heaps. This convention is used to simplify notations, because it avoids having to consider both standard and separating conjunctions. Note that Definition 10 is well-founded because of the progress condition: the size of \( h \) decreases at each recursive call of a predicate atom. We write \( \phi \models R \psi \) if every \( \mathcal{R} \)-model of \( \phi \) is an \( \mathcal{R} \)-model of \( \psi \) and \( \phi \equiv R \psi \) if \( \phi \models R \psi \) and \( \psi \models R \phi \).

Every formula can be transformed into prenex form using the well-known equivalences:

\[
\exists x. \phi \iff \exists x. (\phi \land x = x) \quad \land \quad \exists x. (\phi \land x \neq x) \quad \iff \exists x. (\phi \lor x = x) \quad \lor \quad \exists x. (\phi \lor x \neq x) \quad \iff \exists x. (\phi \lor x = x) \quad \land \quad \exists x. (\phi \lor x \neq x)
\]

**Example 11** Let \( \mathcal{R} \) be the set of rules in Example 5. Assume that \( \mathcal{L} \) is the set of integers \( \mathbb{Z} \).

The formula \( \phi = \ll l s(x_1, x_2) \star \ll l s(x_2, x_3) \) admits the following \( \mathcal{R} \)-model \((s, h)\), where \( s(x_1) = 1, s(x_2) = 2, s(x_3) = 4, \) and \( h = \{(1, 2), (2, 3), (3, 4)\} \).

Indeed, we have:

\[
\begin{align*}
\phi \models R & \ x_1 \mapsto (x_2) \star x_1 \leq x_2 \star \ll l s(x_2, x_3) \\
& \models R \exists x'. (x_1 \mapsto (x_2) \star x_1 \leq x_2 \star x_2 \mapsto (x') \star x_2 \leq x' \star \ll l s(x_2, x_3)) \\
& \models R \exists x'. (x_1 \mapsto (x_2) \star x_1 \leq x_2 \star x_2 \mapsto (x') \star x_2 \leq x' \star x' \mapsto x_3 \star x' \leq x_3)
\end{align*}
\]

and \((s', h) \models (x_1 \mapsto (x_2) \star x_1 \leq x_2 \star x_2 \mapsto (x') \star x_2 \leq x' \star x' \mapsto x_3 \star x' \leq x_3), \) with \( s'(x') = 3 \) and \( s'(x) = s(x) \) if \( x \neq x' \). The formula \( \ll l s(x_1, x_2) \star \ll l s(x_2, x_3) \) admits no \( \mathcal{R} \)-model. Indeed, it is clear that all the structures that satisfy \( \ll l s(x_1, x_2) \) or \( \ll l s(x_1, x_3) \) must allocate \( x_1 \), and the same location cannot be allocated in disjoint parts of the heap.

Note that the progress condition entails the following property:

**Proposition 12** For every pc-SID \( \mathcal{R} \), for every predicate \( p \) and for every structure \((s, h)\), if \((s, h) \models R p(x_1, \ldots, x_{\#(p)})\), then \( s(x_1) \in \text{dom}(h) \).

**Proof** By Definition 4, if \( p(x_1, \ldots, x_{\#(p)}) \models R \phi \), then \( \phi \) contains a points-to atom with left-hand side \( x_1 \).

**Proposition 13** Let \( \phi \) be a disjunction-free formula containing at least one spatial atom. If \((s, h) \models R \phi \) then \( h \) is nonempty.

**Proof** The proof is by induction on the set of formulas.

• If \( \phi \) is a points-to atom then it is clear that \( \text{card(dom}(h)) = 1 \).

• If \( \phi \) is a predicate atom, then there exists \( \psi \) such that \( \phi \models R \psi \) and \((s, h) \models R \psi \). By the progress condition, \( \psi \) is of the form \( \exists w. (u \mapsto (v_1, \ldots, v_k) \star \psi') \), hence there exists a subheap \( h' \) of \( h \) and a store \( s' \) such that \((s', h') \models R u \mapsto (v_1, \ldots, v_k) \). This entails that \( \text{card(dom}(h')) = 1 \) thus \( \text{card(dom}(h)) \geq 1 \).

• If \( \phi = \phi_1 \star \phi_2 \), then necessarily there exists \( i = 1, 2 \) such that \( \phi_i \) contains at least one spatial atom. Furthermore, there exist disjoint heaps \( h_1, h_2 \) such that \((s, h_i) \models R \phi_i \) and \( h = h_1 \uplus h_2 \).

• By the induction hypothesis, \( h_i \) is non empty, hence \( h \) is also non empty.

• If \( \phi = \exists x. \psi \) then \( \psi \) contains at least one spatial atom, and there exists a store \( s' \) such that \((s', h) \models R \psi \). By the induction hypothesis, we deduce that \( h \) is non empty.
2.5 Establishment

The notion of establishment [17] is defined as follows:

Definition 14 A pc-SID is established if for every atom $\alpha$, every predicate-free formula $\exists x. \phi$ such that $\alpha \equiv^*_R \exists x. \phi$, and every $x \in x$, $\phi$ is of the form $x' \mapsto (y_1, \ldots, y_k) \not\in \chi \ast \psi$, where $\chi$ is a separating conjunction of equations (possibly $\emptyset$) such that $\chi \models_T x \approx x'$.

All the rules in Example 5 are trivially established. For instance, $\mathsf{als}(x, y) \equiv \exists x'. x \mapsto (x') \ast \mathsf{als}(x', y) \ast x \not\in y$ fulfills the condition, since every predicate-free unfolding of $\mathsf{als}(x', y)$ contains a formula of the form $x' \mapsto \ldots$ and $\emptyset \models_T x' \approx x'$.

The following lemma states a key property of an established pc-SID: every location referred to in the heap is allocated, except possibly those associated with a free variable.

Lemma 15 Let $R$ be an established pc-SID and let $\phi$ be a quantifier-free symbolic heap. If $(s, h) \models_R \phi$ then $\mathsf{loc}(h) \setminus \mathsf{dom}(h) \subseteq s(\mathsf{fv}(\phi))$.

Proof By definition, $\phi \equiv^*_R \exists y. \psi$, where $\psi$ is a quantifier-free and predicate-free formula, and $(s', h) \models_R \psi$ for some store $s'$ coinciding with $s$ on all the variables not occurring in $y$. Let $\ell \in \mathsf{loc}(h) \setminus \mathsf{dom}(h)$. Since $(s', h) \models_R \psi$, necessarily $\psi$ contains an atom of the form $y_i \mapsto (y_1, \ldots, y_k)$ with $s'(y_i) = \ell$, for some $i = 1, \ldots, n$. If $y_i \in \mathsf{fv}(\psi) \subseteq \mathsf{fv}(\phi)$, then $s(y_i) = s'(y_i)$ and the proof is completed. Otherwise, $y_i \notin y$, and by the establishment condition $\psi$ contains an atom of the form $y_i' \mapsto (y'_1, \ldots, y'_k)$ and a $T$-formula $\chi$ with $\chi \models_T y_i \approx y_i'$. Since $(s', h) \models_R \psi$, we have $s' \models_T \chi$, thus $s'(y_i) = s'(y'_i)$ and $\ell \in \mathsf{dom}(h)$, which contradicts our hypothesis. \hfill \Box

In the remainder of the paper, we assume that every considered pc-SID is established.

3 Extending the Syntax

We extend the syntax of formulas by considering constructs of the form $\Phi^u_{\beta \rightarrow p(x)}[v]$, called $\mathsf{pu}$-atoms (standing for partially unfolded atoms), where $\beta$ is a possibly empty separating conjunction of predicate atoms and $p \in \mathcal{P}_S$. The intuition is that a $\mathsf{pu}$-atom is valid in a structure if there exists a partial unfolding of $p(x)$ that is true in the considered structure, and the formula $\beta$ denotes the part that is not unfolded.

Definition 16 A $\mathsf{pu}$-predicate is an expression of the form $\Phi^u_{\beta \rightarrow p(x)}$, where $\beta$ is a possibly empty separating conjunction of predicate atoms, $p \in \mathcal{P}_S$ and $u$ is a vector of pairwise distinct variables containing all the variables in $\mathsf{fv}(\beta) \cup x$. A $\mathsf{pu}$-atom is an expression of the form $\alpha[v]$, where $\alpha$ is a $\mathsf{pu}$-predicate $\Phi^u_{\beta \rightarrow p(x)}$ and $\|u\| = \|v\|$.

For every $\mathsf{pu}$-atom $\alpha[v]$, we define $\alpha[v] \sigma = \alpha[v \sigma]$, and we let $\mathsf{size}(\Phi^u_{\beta \rightarrow p(x)}[v]) = \mathsf{size}(\beta) + \mathsf{size}(p(x)) + \mathsf{size}(u) + \mathsf{size}(v) + 1$.

Note that for a $\mathsf{pu}$-atom $\alpha[v]$, $v$ is necessarily of the form $u \theta$, for some substitution $\theta$ with domain $u$, since the variables in $u$ are pairwise distinct. Note also that when applying a substitution to $\alpha[v]$ the variables occurring in the $\mathsf{pu}$-predicate $\alpha$ are not instantiated: they may be viewed as bound variables. Formally, the semantics of these constructs is defined as follows.
Definition 17 For every pc-SID \( \mathcal{R} \) and for every SL-structure \((s, h), (s, h) \models_R \Phi^u_{\beta \mapsto \bullet p(x)}[u\theta] \)
if there exists a formula of the form \( \exists y. (\beta' * \phi) \), a substitution \( \sigma \) with \( \text{dom}(\sigma) \subseteq y \cap \text{fv}(\beta') \),
and a store \( s' \) coinciding with \( s \) on all variables not occurring in \( y \) such that:

- \( p(x) \models_R \exists y. (\beta' * \phi) \), (up to AC and transformation into prefix form),
- \( \beta = \beta'\sigma \),
- \( (s', h) \models_R \Phi\sigma\theta \).

Example 18 Consider the pc-SID
\[ \mathcal{R} = \{ p(x) \iff_{p(x)} \exists z_1 z_2. x \mapsto (z_1, z_2) * q(z_1) * q(z_2),
q(x) \iff_{y \mapsto (x, x)} \}, \]
the heap \( h = \{(\ell_1, \ell_2, \ell_3), (\ell_3, \ell_3, \ell_3)\} \) and the store \( s \) such that \( s(x) = \ell_1 \) and \( s(y) = \ell_2 \).
We have \((s, h) \models_R \Phi^u_{q(y') \mapsto \bullet p(x)}[x, y] \). Indeed, it is straightforward to verify that
\[ p(x') \iff_{p(x')} \exists z_1 z_2. (q(z_1)*x' \mapsto (z_1, z_2)*z_2 \iff (z_2, z_2)), \]
using the rule above and taking into account the fact that formulas are taken modulo AC of \( * \). Hence, by letting \( \sigma = \{z_1 \mapsto y'\} \) and considering the store \( s' \) such that \( s'(z_2) = \ell_3 \) and which coincides with \( s \) otherwise, we have \((s', h) \models x \mapsto (y, z_2)*z_2 \iff (z_2, z_2) = (x' \mapsto (y', z_2)*z_2 \iff (z_2, z_2))\theta, \)
with \( \theta = \{x' \mapsto x, y' \mapsto y\} \).

Remark 19 It is clear that the semantics of an atom \( \Phi^u_{q(y') \mapsto \bullet p(x)}[u\theta] \) depends only on the variables \( x\theta \) such that \( x \) occurs in \( \beta \) or \( x \) (and the order in which those variables occur in \( u \) does not matter). Thus in the following we implicitly assume that the irrelevant variables are dismissed. In particular, in the termination proof of Sect. 10, we assume that \( \text{size}(\Phi^u_{q(y') \mapsto \bullet p(x)}[W]) = O(\text{size}(\beta) + \text{size}(p(x))) \). Also, two PU-atoms \( \alpha[u] \) and \( \alpha'[u] \) are equivalent if \( \alpha \) and \( \alpha' \) are identical up to a renaming.

A predicate atom \( p(x\theta) \) where \( x \) is a vector of pairwise disjoint variables is equivalent to
the PU-atom \( \Phi^{\text{emp}}_{\bullet p(x)}[x\theta] \), thus, in the following, we sometimes assume that predicate atoms are written as PU-atoms.

Definition 20 The relation \( \models_R \) is extended to formulas containing PU-atoms as follows:
\( \Phi^u_{q(y') \mapsto \bullet p(x)}[u\theta] \iff_{p(x)} \exists x. \psi \iff_{p(x)} p(x) \iff_{p(x)} \exists x'. (\beta' * \psi) \) and there exists a substitution \( \sigma \) such that
\( \text{dom}(\sigma) \subseteq x' \), \( \psi = \psi' \sigma \theta \), \( \beta = \beta'\sigma \) and \( x = x' \setminus \text{dom}(\sigma) \).

Example 21 Consider the rules of Example 5. Then we have:
\[ \Phi^{x',y',z'}_{\alpha \in \mathcal{S}(x', y')} \iff_{\alpha \in \mathcal{S}(x', y')} \exists x. (x' \mapsto (u) * \alpha \in \mathcal{S}(u, y') * x' \not\equiv y'), \]
hence it suffices to apply the above definition with the substitutions \( \sigma = \{u \mapsto z'\} \) and
\( \theta = \{x' \mapsto x, y' \mapsto y, z' \mapsto z\} \).

Remark 22 The semantics of \( \iff \) is similar but slightly different from that of the context predicates introduced in [12]. The difference is that the semantics uses the syntactic identity \( \beta = \beta'\sigma \) instead of a semantic equality. For instance, with the rules of Example 5,
the formula \( \Phi^{x',y'}_{\alpha \in \mathcal{S}(x', z')} \iff_{\alpha \in \mathcal{S}(x', y')}[x, y, z] \) is unsatisfiable if \( y' \not\equiv z' \), because no atom \( \alpha \in \mathcal{S}(x', z') \) can occur in an unfolding of \( \alpha \in \mathcal{S}(x', y') \). In contrast, a context predicate \( \alpha \in \mathcal{S}(x, y) \) (as defined in [12]) possibly holds in some structures \((s, h) \) with \( s(y) = s(z) \). The use of PU-predicates allows one to get rid of all equality constraints, by instantiating the variables occurring in the former. This is essential for forthcoming lemmas (see, e.g., Lemma 57). PU-atoms are also related to the notion of \( \Phi \)-trees in [24] and to the strong magic wand introduced in [22].

\footnote{We assume that \( x' \) contains no variable in \( u\theta \).}
In the following, unless specified otherwise, all the considered formulas are defined on the extended syntax. A formula containing no occurrence of the symbol → will be called a PU-free formula.

**Proposition 23** If \((s, h) \models_R \Phi^u_{\beta \rightarrow \cdot p(x_1, \ldots, x_n)[\theta]}\) then \(s(x_1 \theta) \in \text{dom}(h)\).

**Proof** By definition, there exists a formula \(\exists y.(\beta' \ast \phi)\) and a substitution \(\sigma\) such that \(p(x_1, \ldots, x_n(\rho)) \leftarrow^+ R \exists y.(\beta' \ast \phi), \beta' = \beta' \sigma, \text{dom}(\sigma) \subseteq y\) and \((\sigma', h) \models_R \phi \sigma \theta\), where \(\sigma'\) coincides with \(s\) on every variable not occurring in \(y\).

Note that \(y \cap \{x_1, \ldots, x_n(\rho)\} = \emptyset\) by Definition 6; up to α-renaming, we may assume that \(y \cap u \theta = \emptyset\). By the progress condition, \(\beta' \ast \phi\) contains an atom of the form \(x_1 \mapsto (z_1, \ldots, z_k)\). This atom cannot occur in \(\beta'\), because \(\beta = \beta' \sigma\) and \(\beta\) only contains predicate atoms by definition of PU-Atoms. Thus \(x_1 \mapsto (z_1, \ldots, z_k)\) occurs in \(\phi\), and \((\sigma'(x_1 \sigma \theta), \sigma'(z_1 \sigma \theta), \ldots, \sigma'(z_k \sigma \theta)) \in h\). Since \(x_1 \notin y\), we have \(x_1 \sigma \theta = x_1 \theta\), and since \(y \cap u \theta = \emptyset\), necessarily \(x_1 \theta \notin y\) and \(s'(x_1 \theta) = s(x_1 \theta)\). Hence \(s(x_1 \theta) \in \text{dom}(h)\). \(\square\)

**Proposition 24** If \(\psi \leftarrow_R \phi\sigma\) then \(\phi \leftarrow_R \psi \sigma\), for every substitution \(\sigma\). If \(\phi \leftarrow_R \psi\) then \(\phi \leftarrow_R \psi\) for a formula \(\psi\) such that \(\psi \sigma = \psi'\).

**Proof** The proof is by induction on the derivation. We only handle the case where \(\phi\) is a predicate atom \(p(x_1, \ldots, x_n)\) and \(\psi \leftarrow_R \psi\) (resp. \(\phi \leftarrow_R \psi\)) is alpha renamable by an immediate induction. Assume that \(\phi \leftarrow_R \psi\). Then by definition, \(\psi = \gamma\{y_i \leftarrow x_i | i = 1, \ldots, n\}\), for some rule \(p(y_1, \ldots, y_n) = \gamma\) in \(\mathcal{R}\). We deduce that \(p(x_1 \sigma, \ldots, x_n \sigma) \leftarrow_R \gamma\{y_i \leftarrow x_i \sigma | i = 1, \ldots, n\}\), thus \(\phi \sigma \leftarrow_R \psi \sigma\). Now assume that \(\phi \sigma \leftarrow_R \psi\). Then \(\psi' = \gamma\{y_i \leftarrow x_i \sigma | i = 1, \ldots, n\}\), for some rule \(p(y_1, \ldots, y_n) = \gamma\) in \(\mathcal{R}\), and w.l.o.g., we may assume up to \(\alpha\)-renaming that no variable in \(\text{dom}(\sigma)\) occurs in the rule, so that in particular, \(\gamma \sigma = \gamma\). Let \(\psi = \gamma\{y_i \leftarrow x_i | i = 1, \ldots, n\}\), then it is clear that \(\phi \leftarrow_R \psi\) and we have

\[
\psi \sigma = (\gamma\{y_i \leftarrow x_i | i = 1, \ldots, n\}) \sigma = \gamma\{y_i \leftarrow x_i \sigma | i = 1, \ldots, n\} = \psi'.
\]

**Proposition 25** If \((s, h) \models \phi\) and \(s(x) = s(x \theta)\) for every variable \(x \in \text{fv}(\phi)\), then \((s, h) \models \phi \theta\).

**Proof** The proof is by induction on the satisfiability relation. We only detail the proof when \(\phi\) is of the form \(\Phi^u_{\beta \rightarrow \cdot p(x_1, \ldots, x_n)}[\theta]\), the other cases are straightforward. Since \((s, h) \models \phi\), there exist a formula \(\exists u.(\beta' \ast \psi)\), a store \(s'\) coinciding with \(s\) on all variables not in \(u\) and a substitution \(\sigma\) such that \(p(x_1, \ldots, x_n) \leftarrow R \exists u.(\beta' \ast \psi), \text{dom}(\sigma) \subseteq u \cap \text{fv}(\beta'), (s', h) \models \psi \sigma \theta'\) and \(\beta' \sigma = \beta\). We assume (by \(\alpha\)-renaming) that \(u\) contains no variable in \(\text{dom}(\theta) \cup \text{fv}(\phi)\).

This entails that \(s'(x) = s'(x \theta)\) holds for all variables \(x \in \text{fv}(\psi \sigma \theta')\). Indeed, if \(x \in u\) then \(x \theta = x\), and otherwise, \(x \in \text{fv}(\phi)\), so that \(s'(x) = s(x) = s(x \theta) = s'(x \theta)\).

Then by the induction hypothesis we get \((s', h) \models \psi \sigma \theta'\), so that \((s, h) \models \Phi^u_{\beta \rightarrow \cdot p(x_1, \ldots, x_n)}[\theta'] = \phi \theta\). \(\square\)

### 4 Sequents

Our proof procedure handles sequents which are defined as follows.

**Definition 26** A sequent is an expression of the form \(\Phi \vdash_R \Phi_1, \ldots, \Phi_n\), where \(\mathcal{R}\) is a pc-SID, \(\Phi_0\) is a PU-free formula and \(\Phi_1, \ldots, \Phi_n\) are formulas. When \(n = 0\), the right-hand side
of a sequent is represented by □. A sequent is disjunct-free (resp. PU-free) if \( \phi_0, \ldots, \phi_n \) are disjunct-free (resp. PU-free), and established if \( \mathcal{R} \) is established. We define:

\[
\text{size}(\phi_0 \vdash_{\mathcal{R}} \phi_1, \ldots, \phi_n) = \sum_{i=0}^{n} \text{size}(\phi_i) + \text{size}(\mathcal{R}), \quad \text{fv}(\phi_1, \ldots, \phi_n) = \bigcup_{i=0}^{n} \text{fv}(\phi_i).
\]

\[
\text{width}(\phi_0 \vdash_{\mathcal{R}} \phi_1, \ldots, \phi_n) = \max\{\text{width}(\phi_1, \ldots, \phi_n), \text{width}(\mathcal{R}), \text{card}(\bigcup_{i=0}^{n} \text{fv}(\phi_i))\},
\]

\[
\text{width}'(\phi_1, \ldots, \phi_n) = \max\{\text{width}(\phi_i) \mid 0 \leq i \leq n\}
\]

Note that PU-atoms occur only on the right-hand side of a sequent. Initially, all the considered sequents will be PU-free, but PU-atoms will be introduced on the right-hand side by the inference rules defined in Sect. 8.

**Definition 27** A structure \((s, h)\) is a countermodel of a sequent \( \phi \vdash_{\mathcal{R}} \Gamma \) iff \( s \) is injective, \((s, h) \models_{\mathcal{R}} \phi \) and \((s, h) \not\models_{\mathcal{R}} \Gamma \). A sequent is valid if it has no countermodel. Two sequents are equivalent if they are both valid or both non-valid.\(^5\)

In particular, the definition of a countermodel requires that all free variables are mapped to pairwise distinct locations but imposes no constraint on the way existential variables are allocated.

**Example 28** For instance, \( \text{ils}(x_1, x_2) * x_2 \rightarrow (x_3) * x_2 < x_3 \vdash_{\mathcal{R}} \text{als}(x_1, x_3) \) is a sequent, where \( \mathcal{R} \) is the set of rules from Example 5 and \( \rightarrow \) is a \( T \)-predicate interpreted as the usual strict order on integers. It is easy to check that it is valid, since by definition of \( \text{ils} \), all the locations allocated by \( \text{ils}(x_1, x_2) \) must be less or equal to \( x_2 \), hence cannot be equal to \( x_3 \). On the other hand, the sequent \( \text{ils}(x_1, x_2) \vdash_{\mathcal{R}} \text{als}(x_1, x_2) \) is not valid: it admits the countermodel \((s, h)\), with \( s(x_1) = 0, s(x_2) = 1 \) and \( h = \{ (0, 1), (1, 1) \} \).

**Remark 29** The restriction to injective countermodels is for technical convenience only and does not entail any loss of generality, since it is possible to enumerate all the equivalence relations on the free variables occurring in the sequent and test entailments separately for each of these relations, by replacing all the variables in the same class by the same representative. The number of such relations is simply exponential w.r.t. the number of free variables, thus the reduction does not affect the overall 2-EXPTIME membership result derived from Theorem 84.

Note that the condition “\( s \) is injective” in Definition 27 could be safely replaced by the slightly weaker condition “\( s \) is injective on \( \text{fv}(\phi) \cup \text{fv}(\Gamma) \)”. Indeed, since \( \mathcal{L} \) is infinite, we can always find an injective store \( s' \) coinciding with \( s \) on all variables in \( \text{fv}(\phi) \cup \text{fv}(\Gamma) \).

Every formula \( \phi \) can be reduced to an equivalent disjunction of symbolic heaps, using the well-known equivalences (if \( x \not\in \text{fv}(\psi) \)):

\[
\phi * (\psi_1 \lor \psi_2) \equiv (\phi * \psi_1) \lor (\phi * \psi_2), \quad (\exists x. \phi) * \psi \equiv \exists x. (\phi * \psi)
\]

Consequently any sequent \( \phi \vdash_{\mathcal{R}} \Gamma \) can be reduced to an equivalent sequent of the form \((\bigvee_{i=1}^{n} \phi_i) \vdash_{\mathcal{R}} \psi_1, \ldots, \psi_m \), where \( \phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_m \) are disjunct-free. Moreover, it is clear that the latter sequent is valid iff all the sequents in the set \( \{ \phi_i \vdash_{\mathcal{R}} \psi_1, \ldots, \psi_m \mid 1 \leq i \leq n \} \) are valid. Thus we will assume in Sect. 8 that the considered sequents are disjunct-free. Note that for all \( i \), we have \( \text{width}(\phi_i \vdash_{\mathcal{R}} \psi_1, \ldots, \psi_m) \leq \text{width}(\phi \vdash_{\mathcal{R}} \Gamma) \), hence the reduction (although exponential) preserves the complexity result in Theorem 84.

\(^5\) Hence two non-valid sequents with different countermodels are equivalent.
Proposition 30  For every set of expressions (formulas, sequents or rules) $E$, if $\text{size}(e) \leq n$ for all $e \in E$, then $\text{card}(E) = O(2^{c \cdot n})$, for some constant $c$.

Proof  By definition every element $e \in E$ is a word with $\text{size}(e) \leq n$, on a vocabulary of some fixed cardinality $m$, thus $\text{card}(E) \leq m^n = 2^{n \cdot \log(m)}$. □

Proposition 31  Let $E$ be a set of expressions (formulas, sequents or rules) such that $\text{size}(e) \leq n$, for all $e \in E$. Then $\text{size}(E) \leq O(2^{d \cdot n})$, for some constant $d$.

Proof  By Proposition 30, $\text{card}(E) = O(2^{c \cdot n})$. Thus $\text{size}(E) \leq n \cdot \text{card}(E) = O(2^{d \cdot n})$ with $d = c + 1$. □

5 Allocated Variables and Roots

In this section we introduce an additional restriction on pc-SIDs, called alloc-compatibility, and we show that every pc-SID can be reduced to an equivalent alloc-compatible set. This restriction ensures that the set of free variables allocated by a predicate atom is the same in every unfolding. This property will be useful for defining some of the upcoming inference rules. Let alloc be a function mapping each predicate symbol $p$ to a subset of $\{1, \ldots, \#(p)\}$. For any disjunction-free and Pu-free formula $\phi$, we denote by alloc($\phi$) the set of variables $x \in \text{fv}(\phi)$ inductively defined as follows:

- alloc($\chi$) = $\emptyset$ if $\chi$ is a $T$-formula (or emp)
- alloc($x \mapsto (y_1, \ldots, y_k)$) = $\{x\}$
- alloc($p(x_1, \ldots, x_n)$) = $\{x_i \mid i \in \text{alloc}(p)\}$ if $p \in P_S$
- alloc($\phi_1 \text{ and } \phi_2$) = alloc($\phi_1 \cup$ alloc($\phi_2$))
- alloc($\exists x. \phi$) = alloc($\phi$) \setminus $\{x\}$

Definition 32  An established pc-SID $R$ is alloc-compatible if for all rules $\alpha \leftarrow \phi$ in $R$, we have alloc($\alpha$) = alloc($\phi$). A sequent $\Gamma \vdash_R \phi$ is alloc-compatible if $R$ is alloc-compatible.

Intuitively, alloc($\phi$) is meant to contain the free variables of $\phi$ that are allocated in the models of $\phi$. The fact that $R$ is alloc-compatible ensures that this set does not depend on the considered model of $\phi$.

Example 33  The set $R = \{p(x, y) \leftarrow x \mapsto (y), p(x, y) \leftarrow x \mapsto (y) \ast p(y, x)\}$ is not alloc-compatible. Indeed, on one hand we have $p(x, y) \leftarrow^*_R x \mapsto (y)$, and on the other hand, $p(x, y) \leftarrow^*_R x \mapsto (y) \ast y \mapsto (x)$. But alloc($x \mapsto (y)$) = $\{x\} \neq \{x, y\} = \text{alloc}(x \mapsto (y) \ast y \mapsto (x))$.

The set $R' = \{p(x, y) \leftarrow x \mapsto (y), p(x, y) \leftarrow \exists z. x \mapsto (z) \ast p(z, x)\}$ is alloc-compatible, with alloc($p$) = $\{1\}$.

Lemma 34  Let $\phi$ be a disjunction-free and Pu-free formula, and let $x \in \text{alloc}(\phi)$. If $(s, h) \models_R \phi$, and $R$ is alloc-compatible then $s(x) \in \text{dom}(h)$.

Proof  The proof is by induction on the pair $(\text{card(dom}(h)), \text{size}(\phi))$.

- If $\phi = \text{emp}$ or $\phi$ is a $T$-formula then alloc($\phi$) = $\emptyset$, which contradicts our hypothesis. Thus this case cannot occur.
- If $\phi = x' \mapsto (y_1, \ldots, y_k)$ then alloc($\phi$) = $\{x'\}$, hence $x = x'$. By definition we have $h = \{s(x'), s(y_1), \ldots, s(y_k)\}$, hence $s(x) \in \text{dom}(h)$. 

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If $\phi = p(x_1, \ldots, x_{#(p)})$ then $\text{alloc}(\phi) = \{x_i \mid i \in \text{alloc}(p)\}$, hence $x = x_i$ for some $i \in \text{alloc}(p)$. By definition we have $\phi \equiv_R \psi$ and $(s, h) \models_R \psi$; by the progress condition, $\psi$ is of the form $\exists z. (x_1 \mapsto y \cdot p_1(u_1^1, \ldots, u_{#(p_1)}^1) \cdots p_n(u_1^n, \ldots, u_{#(p_n)}^n) \cdot \chi)$, where $\chi$ is a $\mathcal{T}$-formula and $z$ is a vector of variables not occurring in $\phi$. Thus there exists a store $s'$, coinciding with $s$ on all variables not occurring in $z$, such that $(s', h) \models_R x_1 \mapsto y \cdot p_1(u_1^1, \ldots, u_{#(p_1)}^1) \cdots p_n(u_1^n, \ldots, u_{#(p_n)}^n) \cdot \chi$. Since $\mathcal{R}$ is $\text{alloc}$-compatible, $\text{alloc}(\psi) = \text{alloc}(\phi)$, hence either $x = x_1$ or there exists $j \in \{1, \ldots, n\}$ and $l \in \text{alloc}(p_j)$ such that $x_i = u_i^j$. In the first case, it is clear that $s(x) = s'(x) \in \text{dom}(h)$. In the second case, there is a proper subheap $h_j$ of $h$ such that $(s', h_j) \models_R p_j(u_1^j, \ldots, u_{#(p_j)}^j)$. We have $u_i^j \in \text{alloc}(p_j(u_1^j, \ldots, u_{#(p_j)}^j))$, hence by the induction hypothesis, $s'(u_i^j) \in \text{dom}(h_j)$, thus $s(x) = s'(u_i^j) \in \text{dom}(h)$.

If $\phi = \phi_1 \ast \phi_2$ then we have $x \in \text{alloc}(\phi_i)$, for some $i = 1, 2$. Furthermore, there exist heaps $h_i$ ($i = 1, 2$) such that $(s, h_i) \models_R \phi_i$ (for all $i = 1, 2$) and $h = h_1 \lor h_2$. By the induction hypothesis, we deduce that $s(x) \in \text{dom}(h_i)$, for some $i = 1, 2$, hence $s(x) \in \text{dom}(h)$.

If $\phi = \exists y. \psi$ then we have $x \in \text{alloc}(\psi)$, and $x \neq y$. Since there exists a store $s'$, coinciding with $s$ on all variables distinct from $y$, such that $(s', h) \models_R \psi$, by the induction hypothesis, we deduce that $s'(x) \in \text{dom}(h)$, hence $s(x) \in \text{dom}(h)$.

In the remainder of the paper, we will assume that all the considered sequents are $\text{alloc}$-compatible. This is justified by the following:

**Lemma 35** There exists an algorithm which, for every $\forall u$-free sequent $\phi \models_R \Gamma$, computes an equivalent $\text{alloc}$-compatible $\forall u$-free sequent $\phi' \models_R \Gamma'$. Moreover, this algorithm runs in exponential time and width$(\phi' \models_R \Gamma') = O(\text{width}(\phi \models_R \Gamma)^2)$.

**Proof** We associate all pairs $(p, A)$ where $p \in \mathcal{P}_S$ and $A \subseteq \{1, \ldots, \#(p)\}$ with fresh, pairwise distinct predicate symbols $p_A \in \mathcal{P}_S$, with the same arity as $p$, and we set $\text{alloc}(p_A) = A$. For each disjunction-free formula $\phi$, we denote by $\phi^*$ the set of formulas obtained from $\phi$ by replacing every predicate atom $p(x)$ by an atom $p_A(x)$ where $A \subseteq \{1, \ldots, \#(p)\}$. Let $\mathcal{R}'$ be the set of $\text{alloc}$-compatible rules of the form $p_A(x) \equiv_R \psi$, where $p(x) \equiv \phi$ is a rule in $\mathcal{R}$ and $\psi \in \phi^*$. Note that the symbols $p_A$ may be encoded by words of length $O(\|p\| + \#(p))$, thus for every $\psi \in \phi^*$ we have $\text{width}((\psi)) = O(\text{width}(\phi)^2)$, so that $\text{width}(\mathcal{R}') = O(\text{width}(\mathcal{R})^2)$.

We show by induction on the satisfiability relation that the following equivalence holds for every structure $(s, h)$: $(s, h) \models_R \phi$ iff there exists $\psi \in \phi^*$ such that $(s, h) \models_{\mathcal{R}'} \psi$. For the direct implication, we also prove that $\text{alloc}(\psi) = \{x \in \text{fv}(\phi) \mid s(x) \in \text{dom}(h)\}$.

The proof is immediate if $\phi$ is a $\mathcal{T}$-formula, since $\phi^* = \{\phi\}$, and the truth value of $\phi$ does not depend on the considered pc-SID. Also, by definition $\text{alloc}(\phi) = \emptyset$ and all the models of $\phi$ have empty heaps.

If $\phi$ is of the form $x \mapsto (y_1, \ldots, y_n)$, then $\phi^* = \{\phi\}$ and the truth value of $\phi$ does not depend on the considered pc-SID. Also, $\text{alloc}(\phi) = \{x\}$ and we have $\text{dom}(h) = \{s(x)\}$ for every model $(s, h)$ of $\phi$.

Assume that $\phi = p(x_1, \ldots, x_{#(p)})$. If $(s, h) \models_R \phi$ then there exists a formula $\gamma$ such that $\phi \equiv_R \gamma$ and $(s, h) \models_R \gamma$. By the induction hypothesis, there exists $\psi \in \gamma^*$ such that $(s, h) \models_{\mathcal{R}'} \psi$ and $\text{alloc}(\psi) = \{x \in \text{fv}(\gamma) \mid s(x) \in \text{dom}(h)\}$. Let $A = \{i \in \{1, \ldots, \#(p)\} \mid s(x_i) \in \text{dom}(h)\}$, so that $\text{alloc}(\psi) = \{x_i \mid i \in A\}$. By construction $p_A(x_1, \ldots, x_n) \equiv_R \psi$ is $\text{alloc}$-compatible, and therefore $p_A(x_1, \ldots, x_n) \equiv_{\mathcal{R}'} \psi$, which entails that $(s, h) \models_{\mathcal{R}'} p_A(x_1, \ldots, x_n)$. By definition of $A$, $\text{alloc}(p_A(x_1, \ldots, x_n)) = \{x_i \mid i \in A\}$.
\[ \{ x \in \text{fv}(\phi) \mid s(x) \in \text{dom}(h) \}. \]

Conversely, assume that \((s, h) \models \mathcal{R} \psi\) for some \(\psi \in \Phi^*\). Necessarily \(\psi\) is of the form \(p_A(x_1, \ldots, x_n)\) with \(A \subseteq \{1, \ldots, \#(p)\}\). We have \(p_A(x_1, \ldots, x_n) \iff \mathcal{R} \psi'\) and \((s, h) \models \mathcal{R} \psi'\) for some formula \(\psi'\). By definition of \(\mathcal{R}'\), we deduce that \(p(x_1, \ldots, x_n) \iff \mathcal{R} \gamma\), for some \(\gamma\) such that \(\psi \in \gamma^*\). By the induction hypothesis, \((s, h) \models \mathcal{R} p(x_1, \ldots, x_{\#(p)}))\). Since \(p(x_1, \ldots, x_{\#(p)}) = \phi\), we have the result.

- Assume that \(\phi = \phi_1 \land \phi_2\). If \((s, h) \models \mathcal{R} \phi\) then there exist disjoint heaps \(h_1, h_2\) such that \((s, h_i) \models \mathcal{R} \phi_i\), for all \(i = 1, 2\) and \(h = h_1 \uplus h_2\). By the induction hypothesis, this entails that there exist formulas \(\psi_i \in \Phi_{\mathcal{R}}^*\) for \(i = 1, 2\) such that \((s, h_i) \models \mathcal{R} \psi_i\) and \(\text{alloc}(\psi_i) = \{ x \in \text{fv}(\phi_i) \mid s(x) \in \text{dom}(h_i) \}\). Let \(\psi = \psi_1 \land \psi_2\). It is clear that \((s, h) \models \mathcal{R} \psi_1 \land \psi_2\) and \(\text{alloc}(\psi) = \text{alloc}(\psi_1) \cup \text{alloc}(\psi_2) = \{ x \in \text{fv}(\phi_1) \cup \text{fv}(\phi_2) \mid s(x) \in \text{dom}(h) \}\) = \(\{ x \in \text{fv}(\phi) \mid s(x) \in \text{dom}(h) \}\). Since \(\psi_1 \land \psi_2 \in \Phi^*\), we obtain the result.

Conversely, assume that there exists \(\psi \in \Phi^*\) such that \((s, h) \models \mathcal{R} \psi\). Then \(\psi = \psi_1 \land \psi_2\) with \(\psi_i \in \Phi_{\mathcal{R}}^*\), and we have \((s, h_i) \models \mathcal{R} \psi_i\), for \(i = 1, 2\) with \(h = h_1 \uplus h_2\). Using the induction hypothesis, we get that \((s, h_i) \models \mathcal{R} \phi_i\), hence \((s, h) \models \mathcal{R} \phi\).

- Assume that \(\phi = \exists y. \gamma\). If \((s, h) \models \mathcal{R} \phi\) then \((s', h) \models \mathcal{R} \gamma\), for some store \(s'\) coinciding with \(s\) on every variable distinct from \(y\). By the induction hypothesis, this entails that there exists \(\psi \in \gamma^*\) such that \((s', h) \models \mathcal{R} \psi\) and \(\text{alloc}(\psi) = \{ x \in \text{fv}(\gamma) \mid s'(x) \in \text{dom}(h) \}\). Then \((s, h) \models \mathcal{R} \exists y. \psi\), and we have \(\exists y. \psi \in \Phi^*\). Furthermore, \(\text{alloc}(\exists y. \psi) = \text{alloc}(\psi) \setminus \{ y \} = \{ x \in \text{fv}(\gamma) \setminus \{ y \} \mid s'(x) \in \text{dom}(h) \}\) = \(\{ x \in \text{fv}(\phi) \mid s(x) \in \text{dom}(h) \}\) holds. Conversely, assume that \((s, h) \models \mathcal{R} \psi\), with \(\psi \in \Phi^*\). Then \(\psi\) is of the form \(\exists y. \psi'\), with \(\psi' \in \gamma^*\), thus there exists a store \(s'\), coinciding with \(s\) on all variables other than \(y\) such that \((s', h) \models \mathcal{R} \psi'\). By the induction hypothesis, this entails that \((s', h) \models \mathcal{R} \psi\), thus \((s, h) \models \mathcal{R} \exists y. \psi\). Since \(\exists y. \psi = \phi\), we have the result.

Let \(\phi', \Gamma'\) be the sequence of formulas obtained from \(\phi, \Gamma\) by replacing every atom \(\alpha\) by the disjunction of all the formulas in \(\alpha^*\). It is clear that \(\text{width}(\phi' \vdash \mathcal{R} \Gamma') \leq \text{width}(\phi \vdash \mathcal{R} \Gamma)^2\). By the previous result, \(\phi' \vdash \mathcal{R} \Gamma'\) is equivalent to \(\phi \vdash \mathcal{R} \Gamma\), hence \(\phi' \vdash \mathcal{R} \Gamma'\) fulfills all the required properties. Also, since each predicate \(p\) is associated with \(2^\#(p)\) predicates \(p_A\), we deduce that \(\phi' \vdash \mathcal{R} \Gamma'\) can be computed in time \(O(2^{\text{size}(\phi^* \vdash \mathcal{R} \Gamma)})\).

We now introduce a few notations to denote variables occurring as the first argument of a predicate, including the \(\iff\) predicate:

**Definition 36** For any disjunction-free formula \(\phi\), we denote by \(\text{roots}_s(\phi)\) the multiset defined as follows: a variable occurs with multiplicity \(n\) if \(\phi\) contains \(n\) subformula occurrences of one of the forms \(x \iff (y_1, \ldots, y_n)\), \(p(x, y_1, \ldots, y_{\#(p)}-1)\) or \(\Phi^u_{\beta \rightarrow p(y_1, \ldots, y_{\#(p)})}[u\theta]\), with \(y_1\theta = x\). Similarly, we denote by \(\text{roots}_i(\phi)\) the multiset defined as follows: a variable \(x\) occurs with multiplicity \(n\) if \(\phi\) contains \(n\) atom occurrences of the form \(\Phi(q_{z_1, \ldots, z_{\#(q)}}) \rightarrow [p(y_1, \ldots, y_{\#(p)})][u\theta]\) with \(z_1\theta = x\). The variables in \(\text{roots}_s(\phi)\) are called the auxiliary roots of \(\phi\), those in \(\text{roots}_i(\phi)\) are called the main roots of \(\phi\). We let \(\text{roots}(\phi) = \text{roots}_s(\phi) \cup \text{roots}_i(\phi)\).

For instance, if \(\phi = x \iff (z) \iff [x, z] \iff [z, x, y] \iff [z, z, y]\) then \(\text{roots}_s(\phi) = \{x, y\}\) and \(\text{roots}_i(\phi) = \{z, z, z\}\).

**Proposition 37** Let \(\phi\) be a disjunction-free formula. If \((s, h) \models \mathcal{R} \phi\) and \(x \in \text{roots}_s(\phi)\) then \(s(x) \in \text{dom}(h)\). Furthermore, if \(x\) occurs twice in \(\text{roots}_s(\phi)\) then \(\phi\) is unsatisfiable.
Proof The result is an immediate consequence of Proposition 23 and of the definition of the semantics of points-to atoms. □

Definition 38 A formula φ for which roots_r(φ) contains multiple occurrences of the same variable is said to be root-unsatisfiable.

We also introduce a notation to denote the variables that may occur within a T-formula (possibly after an unfolding):

Definition 39 For every pu-free formula φ, we denote by \(fv_T(φ)\) the set of variables \(x ∈ fv(φ)\) such that there exists a formula ψ and a T-formula \(χ\) occurring in ψ such that \(φ ⇐ \psi\) and \(x ∈ fv(χ)\).

For instance, considering the rules of Example 5, we have \(fv_T(a ≤_S(x_1, x_2) ∗ x_3 ≥ 0) = \{x_1, x_2, x_3\}\), since \(a ≤_S(x_1, x_2) ∗ x_3 ≥ 0 \iff x_1 \not= x_2 ∗ x_3 ≥ 0\).

Proposition 40 For every pu-free formula φ, the set \(fv_T(φ)\) can be computed in polynomial time w.r.t. size(φ) · size(R), i.e., w.r.t. size(φ) · \(O(2^{d \cdot width(R)})\) for some constant d.

Proof It suffices to associate all predicates \(p ∈ P_S\) with subsets \(fv_T(φ)\) of \{1, . . . , #(p)\}, inductively defined as the least sets satisfying the following condition: \(i ∈ fv_T(φ)\) if there exists a rule \(p(x_1, . . . , x_{#(p)}) ⇐ φ\) in R such that φ contains either a T-formula \(χ\) with \(x_i ∈ fv(χ)\), or an atom \(q(y_1, . . . , y_{#(q)})\) with \(q ∈ P_S\) and \(x_i = y_j\) for some \(j ∈ fv_T(q)\). It is clear that these sets can be computed in polynomial time in size(R) using a standard fixpoint algorithm. Furthermore, it is easy to check, by induction on the unfolding relation, that \(x ∈ fv_T(φ)\) if either \(x ∈ fv(χ)\), for some T-formula \(χ\) occurring in \(φ\) or \(x = x_i\) for some atom \(p(x_1, . . . , x_{#(p)})\) occurring in \(φ\) and some \(i ∈ fv_T(p)\).

By definition, for every rule \(ρ ∈ R\) we have size(ρ) ≤ width(R), and by Proposition 31 there exists a constant d such that size(R) = \(O(2^{d \cdot width(R)})\).

The notation \(fv_T(φ)\) is extended to formulas \(\Phi^x_β →_α [xθ]\) as follows⁶: \(fv_T(\Phi^x_β →_α [xθ]) \defeq fv_T(αθ)\).

6 Eliminating Equations and Disequations

We show that the equations and disequations can always be eliminated from established sequents, while preserving equivalence. The intuition is that equations can be discarded by instantiating the inductive rules, while disequations can be replaced by assertions that the considered variables are allocated in disjoint parts of the heap. We wish to emphasize that the result does not follow from existing translations of SL formulas to graph grammars (see, e.g., [9, 20]), as the rules allow for disequations as well as equations. In particular the result crucially relies on the establishment property: it does not hold for non-established rules.

Definition 41 Let \(P ⊆ P_T\). A formula \(φ\) is \(P\)-constrained if for every formula \(ψ\) such that \(φ ⇐ \psi\), and for every symbol \(p ∈ P_T\) occurring in \(ψ\), we have \(p ∈ P\). A sequent \(φ ⊢_R Γ\) is \(P\)-constrained if all the formulas in \(φ, Γ\) are \(P\)-constrained.

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⁶ This set is an over-approximation of the set of variables \(x\) such that \(x ∈ fv_T(φ')\) for some predicate-free formula \(φ'\) with \(φ ⇐ φ'\), but it is sufficient for our purpose.
In particular, if \( \phi \) is \( \theta \)-constrained, then the unfoldings of \( \phi \) contain no symbol in \( \mathcal{P}_T \).

**Theorem 42** Let \( P \subseteq \mathcal{P}_T \). There exists an algorithm that transforms every \( P \)-constrained established sequent \( \phi \vdash R \Gamma \) into an equivalent (\( P \setminus \{ \approx, \not\approx \} \))-constrained established sequent \( \phi' \vdash R' \Gamma' \). This algorithm runs in exponential time and \( \text{width}(\phi') \) is polynomial w.r.t. \( \text{width}(\phi) \).

**Proof** We consider a \( P \)-constrained established sequent \( \phi \vdash R \Gamma \). This sequent is transformed in several steps, each of which is illustrated in Example 43.

**Step 1** The first step consists in transforming all the formulas in \( \phi \), \( \Gamma \) into disjunctions of symbolic heaps. Then for every symbolic heap \( \gamma \) occurring in the obtained sequent, we add all the variables freely occurring in \( \phi \) or \( \Gamma \) as parameters of every predicate symbol occurring in unfoldings of \( \gamma \) (their arities are updated accordingly, and these variables are passed as parameters to each recursive call of a predicate symbol). We obtain an equivalent sequent \( \phi_1 \vdash R_1 \Gamma_1 \), and if \( v = \text{card}(fv(\phi) \cup fv(\Gamma)) \) denotes the total number of free variables occurring in \( \phi \), \( \Gamma \), then using the fact that the size of each of these variables is bounded by \( \text{width}(\phi \vdash R \Gamma) \), we have \( \text{width}(\phi_1 \vdash R_1 \Gamma_1) \leq v \cdot \text{width}(\phi \vdash R \Gamma)^2 \). By Definition 26 we have \( v \leq \text{width}(\phi \vdash R \Gamma) \), thus \( \text{width}(\phi_1 \vdash R_1 \Gamma_1) = \mathcal{O}(\text{width}(\phi \vdash R \Gamma)^3) \).

**Step 2** All the equations involving an existential variable can be eliminated in a straightforward way by replacing each formula of the form \( \exists x.(x \approx y \ast \phi) \) with \( \phi(x \leftarrow y) \). We then replace every formula \( \exists y.\phi \) with free variables \( x_1, \ldots, x_n \) by the disjunction of all the formulas of the form

\[
\exists z.\phi \ast \exists z^* \sqrt{z \neq z'}^*,
\]

where \( \sigma \) is a substitution such that \( \text{dom}(\sigma) \subseteq y \), \( z = y \setminus \text{dom}(\sigma) \) and \( \text{img}(\sigma) \subseteq y \cup \{ x_1, \ldots, x_n \} \). Similarly we replace every rule \( p(x_1, \ldots, x_n) \leftarrow \exists y.\phi \) by the set of rules \( p(x_1, \ldots, x_n) \leftarrow \exists z.\phi \ast \exists z^* \sqrt{z \neq z'}^* \), where \( \sigma \) is any substitution satisfying the conditions above.

Intuitively, this transformation ensures that all existential variables are associated to pairwise distinct locations, also distinct from any location associated to a free variable. The application of the substitution \( \sigma \) captures all the rule instances for which this condition does not hold, by mapping all variables that are associated with the same location to a unique representative. We denote by \( \phi_2 \vdash R_2 \Gamma_2 \) the sequent thus obtained. Let \( v' \) be the maximal number of existential variables occurring in a rule in \( \mathcal{R} \). We have \( v' \leq \text{width}(\phi \vdash R \Gamma) \), since the transformation in Step 1 adds no existential variable. Since at most one disequation is added for every pair of variables, and the size of every variable is bounded by \( \text{width}(\phi \vdash R \Gamma) \), it is clear that \( \text{width}(\phi_2 \vdash R_2 \Gamma_2) = \text{width}(\phi_1 \vdash R_1 \Gamma_1) + v' \cdot (v + v') \cdot (1 + 2 \ast \text{width}(\phi \vdash R \Gamma)) = \mathcal{O}(\text{width}(\phi \vdash R \Gamma)^3) \).

**Step 3** We replace every atom \( \alpha = p(x_1, \ldots, x_n) \) occurring in \( \phi_2 \), \( \Gamma_2 \) or \( \mathcal{R}_2 \) with pairwise distinct variables \( x_{1,i} \), \( x_{m,i} \) (with \( m \leq n \) and \( i = 1 \)), by an atom \( p_{\alpha}(x_{1,i}, \ldots, x_{m,i}) \), where \( p_{\alpha} \) is a fresh predicate symbol, associated with rules of the form \( p_{\alpha}(y_{1,i}, \ldots, y_{m,i}) \leftarrow \psi\{y_i \leftarrow x_{i,i} \mid i \in \{1, \ldots, n\}\} \theta \), where \( p(y_{1,i}, \ldots, y_{m,i}) \leftarrow \psi \) is a rule in \( \mathcal{R} \) and \( \theta \) denotes the substitution \( \{x_{i,k} \leftarrow y_{i,k} \mid i \in \{1, \ldots, m\}\} \). By construction, \( p_{\alpha}(x_{1,i}, \ldots, x_{m,i}) \) is equivalent to \( \alpha \). We denote by \( \phi_3 \vdash R_3 \Gamma_3 \) the resulting sequent. It is clear that \( \phi_3 \vdash R_3 \Gamma_3 \) is equivalent to \( \phi \vdash R \Gamma \).

By induction on the derivation, we can show that all atoms occurring in an unfolding of the formulas in the sequent \( \phi_3 \vdash R_3 \Gamma_3 \) are of the form \( q(y_1, \ldots, y_{\#(q)}) \), where \( y_1, \ldots, y_{\#(q)} \) are pairwise distinct, and that the considered unfolding also contains the disequation \( y_i \not\approx y_j \),...
for all \( i \neq j \) such that either \( y_i \) or \( y_j \) is an existential variable (note that if \( y_i \) and \( y_j \) are both free then \( y_i \neq y_j \) is valid, since the considered stores are injective). This entails that the rules that introduce a trivial equality \( u \approx v \) with \( u \neq v \) are actually redundant, since unfolding any atom \( q(y_1, \ldots, y_{\theta(q)}) \) using such a rule yields a formula that is unsatisfiable. Consequently such rules can be eliminated without affecting the status of the sequent. All the remaining equations are of the form \( u \approx u \) hence can be replaced by \( \text{emp} \). We may thus assume that the sequent \( \phi_3 \vdash_{\Gamma_3} \Gamma_3 \) contains no equality. Note that by the above transformation, all existential variables must be interpreted as pairwise distinct locations in any interpretation, and also be distinct from all free variables. It is easy to see that the fresh predicates \( p_a \) may be encoded by words of size at most \( \text{width}(\phi \vdash_{\Gamma} \Gamma) \), thus \( \text{width}(\phi_3 \vdash_{\Gamma_3} \Gamma_3) \leq \text{width}(\phi \vdash_{\Gamma} \Gamma) \cdot \text{width}(\phi_2 \vdash_{\Gamma_2} \Gamma_2) = O(\text{width}(\phi \vdash_{\Gamma} \Gamma)^4) \). By Lemma 35, we may assume that \( \phi_3 \vdash_{\Gamma_3} \Gamma_3 \) is alloc-compatible (note that the transformation given in the proof of Lemma 35 does not affect the disequations occurring in the rules).

**Step 4** We now ensure that all the locations that are referred to are allocated. Consider a symbolic heap \( \gamma \) occurring in \( \phi_3, \Gamma_3 \) and any \( \mathcal{R}_3 \)-model \((s, h)\) of \( \gamma \), where \( s \) is injective. For the establishment condition to hold, the only unallocated locations in \( h \) of \( \gamma \) must correspond to locations \( s(x) \) where \( x \) is a free variable. We assume the sequent contains a free variable \( u \) such that, for every tuple \((\ell_0, \ldots, \ell_k) \in h\), we have \( s(u) = \ell_s \). This does not entail any loss of generality, since we can always add a fresh variable \( u \) to the considered problem: after Step 1, \( u \) is passed as a parameter to all predicate symbols, and we may replace every points-to atom \( z_0 \mapsto (z_1, \ldots, z_k) \) occurring in \( \phi_3, \Gamma_3 \) or \( \mathcal{R}_3 \), by \( z_0 \mapsto (z_1, \ldots, z_k, u) \) (note that this increases the value of \( \kappa \) by 1). It is clear that this ensures that \( h \) and \( u \) satisfy the above property. We also assume, w.l.o.g., that the sequent contains at least one variable \( u' \) distinct from \( u \). Note that, since \( s \) is injective, the tuple \((s(u'), \ldots, s(u'))\) cannot occur in \( h \), because its last component is distinct from \( s(u) \).

We then denote by \( \phi_4 \vdash_{\mathcal{R}_4} \Gamma_4 \) the sequent obtained from \( \phi_3 \vdash_{\Gamma_3} \Gamma_3 \) by replacing every symbolic heap \( \gamma \) in \( \phi_3, \Gamma_3 \) by

\[
\left( x \in (fv(\phi_3) \cup fv(\Gamma_3)) \setminus \text{alloc}(\gamma) \quad x \mapsto (u', \ldots, u') \right) \star \gamma.
\]

It is straightforward to check that \((s, h) \models \gamma\) iff there exists an extension \( h' \) of \( h \) such that \((s, h') \models \left( \left( x \in (fv(\phi_3) \cup fv(\Gamma_3)) \setminus \text{alloc}(\gamma) \quad x \mapsto (u', \ldots, u') \right) \star \gamma \right) \), with \( \text{loc}(h') = \text{loc}(h') = \text{dom}(h')\) and \( h'(\ell) = (s(u'), \ldots, s(u')) \) for all \( \ell \in \text{dom}(h') \setminus \text{dom}(h) \). This entails that \( \phi_4 \vdash_{\mathcal{R}_4} \Gamma_4 \) is valid if and only if \( \phi_3 \vdash_{\Gamma_3} \Gamma_3 \) is valid.

Consider a formula \( \gamma \) in \( \phi_4, \Gamma_4 \) and some satisfiable unfolding \( \gamma' \) of \( \gamma \). Thanks to the transformation in this step and the establishment condition, if \( \gamma' \) contains a (free or existential) variable \( x \) then it also contains an atom \( x' \mapsto y \) and a \( T \)-formula \( \chi \) such that \( \gamma' \models_T x \equiv x' \). But by the above transformation, if \( x, x' \) occur in the same \( T \)-formula \( \chi \) then one of the following holds; \( x \) and \( x' \) are identical; \( x \) and \( x' \) are distinct free variables (so that \( s \models_T x \neq x' \) for all injective store), or \( x \neq x' \) occurs in \( \gamma' \). The two last cases contradict the fact that \( \gamma' \) is satisfiable, since \( \gamma' \models x \approx x' \), thus \( x = x' \). Consequently, if \( \gamma' \) contains a disequation \( x_1 \neq x_2 \) with \( x_1 \neq x_2 \), then it also contains atoms \( x_1 \mapsto y_1 \) and \( x_2 \mapsto y_2 \). This entails that the disequation \( x_1 \neq x_2 \) is redundant, since it is a logical consequence of \( x_1 \mapsto y_1 \land x_2 \mapsto y_2 \). We deduce that the satisfiability status of \( \phi_4 \vdash_{\mathcal{R}_4} \Gamma_4 \) is preserved if all disequations are replaced by \( \text{emp} \).

\( \Box \)

**Example 43** We illustrate all of the steps in the proof above.
Step 1 Consider the sequent \( p(x_1, x_2) \vdash_R r(x_1) \cdot r(x_2) \), where \( R \) is defined as follows: \( R = \{ r(x) \iff x \rightarrow (x) \} \). After Step 1 we obtain the sequent \( p(x_1, x_2) \vdash_{R_1} r'(x_1, x_2) \cdot r'(x_1, x_2) \), where \( R_1 = \{ r'(x, y) \iff x \rightarrow (x) \} \).

Step 2 This step transforms the formula \( \exists y_1 \exists y_2. p(x, y_1) \cdot p(x, y_2) \) into the disjunction:

\[
\exists y_1, y_2, p(x, y_1) \cdot p(x, y_2) \cdot y_1 \neq y_2 \cdot y_1 \neq y_2 \cdot x \neq x
\]

On the other hand, the rule \( p(x) \leftarrow \exists z \cdot x \rightarrow (z) \cdot q(z, u) \) is transformed into the set:

\[
p(x) \leftarrow x \rightarrow (x) \cdot q(x, x)
\]

Step 3 Assume that \( R \) contains the rules \( p(y_1, y_2, y_3) \leftarrow y_1 \rightarrow (y_2) \cdot q(y_2, y_3) \cdot y_1 \approx y_3 \) and \( p(y_1, y_2, y_3) \leftarrow y_1 \rightarrow (y_2) \cdot r(y_2, y_3) \cdot y_1 \approx y_2 \) and consider the sequent \( p(x, y) \vdash_{R} \text{emp} \). Step 3 generates the sequent \( p_\alpha(x, y) \vdash_{R'} \text{emp} \) (with \( \alpha = p(x, y, x) \)), where \( R' \) contains the rules \( p_\alpha(y_1, y_2) \leftarrow y_1 \rightarrow (y_2) \cdot q(y_2, y_1) \cdot y_1 \approx y_1 \) and \( p_\alpha(y_1, y_2) \leftarrow y_1 \rightarrow (y_2) \cdot r(y_2, y_1) \cdot y_1 \approx y_2 \). The second rule is redundant, because \( p_\alpha(y_1, y_2) \) is used only in a context where \( y_1 \approx y_2 \) holds, and equation \( y_1 \approx y_1 \) can be discarded from the first rule.

Step 4 Let \( y = p(x, y, z, z') \cdot q(x, y, z, z') \cdot z' \rightarrow (z') \), assume \( \text{alloc}(y) = \{ x, z \} \), and consider the sequent \( y \vdash_{R} \text{emp} \). Then \( y \) is replaced by \( p(x, y, z, z', u) \cdot q(x, y, z, z', z') \cdot z' \rightarrow (z', u) \cdot u \rightarrow (x, x) \) * y \rightarrow (x, x) \) (all non-allocated variables are associated with \( x, x \), where \( x \) plays the role of the variable \( u' \) in Step 4 above). Also, every points-to atom \( z_0 \rightarrow (z_1) \) in \( R \) is replaced by \( z_0 \rightarrow (z_1, u) \).

Example 44 Consider the predicate \( \text{ls} \) defined by the rules \( \text{ls}(x, y) \leftarrow x \rightarrow (y) \), \( \text{ls}(x, y) \leftarrow \exists z \cdot x \rightarrow (z) \cdot \text{ls}(z, y) \). The (non-valid) entailment \( \text{ls}(x, y) \vdash_R \text{als}(x, y) \), where \( \text{als} \) is defined in Example 5, is transformed into

\[
(\text{ls}''(x, y, u) \cdot u \rightarrow (x, x)) \lor (y \rightarrow (x, x) \cdot \text{ls}'(x, y, u) \cdot u \rightarrow (x, x))
\]

along with the following rules, where for readability useless parameters have been removed, as well as rules with a root-unsatisfiable right-hand side:

\[
\begin{align*}
\text{ls}'(x, y, u) & \leftarrow x \rightarrow (y, u) \\
\text{ls}'(x, y, u) & \leftarrow \exists z \cdot x \rightarrow (z, u) \cdot \text{ls}'(z, y, u) \\
\text{ls}''(x, y, u) & \leftarrow \exists z \cdot x \rightarrow (z, u) \cdot \text{ls}''(z, y, u) \\
\text{ls}''(x, y, u) & \leftarrow x \rightarrow (y, u) \cdot \text{ls}'(y, u, y)
\end{align*}
\]

The atom \( \text{ls}'(x, y, u) \) (resp. \( \text{ls}''(x, y, u) \)) denotes a list segment from \( x \) to \( y \) that does not allocate \( y \) (resp. that allocates \( y \)). A variable \( u \) is added and the value of \( x \) is increased by \( 1 \) as described at Step 4. The atom \( \text{ls}(x, y) \) is replaced by the disjunction \( \text{ls}'(x, y, u) \lor \text{ls}''(x, y, u) \), by applying the transformation described in the proof of Lemma 35. The predicate \( \text{als} \) is transformed into \( \text{ls}'(x, y, u) \) at Step 4 (as disequations are removed from the rules). The atom \( y \rightarrow (x, x) \) is added since \( y \) is not allocated in \( \text{ls}'(x, y, u) \).
Example 45 Consider the (valid) entailment $\text{mls}(x, y, x) \vdash_R \text{mls}(y, x, y)$, where $\text{mls}(x, y, z)$ denotes a list from $x$ to $y$ containing $z$ and is defined by the rules:

\[ \text{mls}(x, y, z) \iff x \mapsto (y) * x \approx z \]
\[ \text{mls}(x, y, z) \iff \exists z'. x \mapsto (z') * \text{mls}(z', y) * x \approx z \]
\[ \text{mls}(x, y, z) \iff \exists z'. x \mapsto (z') * \text{mls}(z', y, z) \]

This entailment is transformed into $p(x, y, u) \vdash_R p(y, x, u)$, where (assuming $x, y, z$ are pairwise distinct) $p(x, y, u)$ denotes a list from $x$ to $y$ containing $z$, $q(x, y, u)$ denotes a list from $x$ to $y$, and $r(x, y, z, u)$ denotes a list from $x$ to $y$ containing $z$, defined by the rules (again for readability redundant parameters and rules have been removed):

\[ p(x, y, u) \iff \exists z. x \mapsto (z) * r(z, x, y, u) \]
\[ p(x, y, u) \iff x \mapsto (y, u) * q(y, x, u) \]
\[ q(x, y, u) \iff x \mapsto (y, u) \]
\[ q(x, y, u) \iff \exists z. x \mapsto (z, u) * q(z, y, u) \]
\[ r(x, y, z, u) \iff \exists z'. x \mapsto (z', u) * r(z', y, z, u) \]
\[ r(x, y, z, u) \iff x \mapsto (z, u) * q(z, y, u) \]

The predicate $p(x, y, u)$ is introduced at Step 3 to replace the atoms $\text{mls}(x, y, x)$ and $\text{mls}(y, y, x)$. The fresh variable $u$ plays no role here because all variables are allocated.

7 Heap Splitting

In this section we introduce a so-called heap splitting operation that will ensure that a given variable $x$ occurs in the main roots of a given formula. This operation preserves equivalence on the injective structures in which the considered variable $x$ is allocated (Lemma 49). It is implicitly dependent on some pc-SID $R$, which will always be clear from the context. The intuition is that if an atom $p(x_1, \ldots, x_n)$ (with $x_1 \neq x$) allocates $x$, then it must eventually call a predicate atom $q(x, y)$, where $y$ may contain variables in $\{x_1, \ldots, x_n\}$ as well as fresh variables, introduced during the unfolding. Thus $p(x_1, \ldots, x_n)$ can be written as a disjunction of formulas of the form: $\exists z. ((\Phi^u_1 q(x, y) \rightarrow p(x_1, \ldots, x_n)[u]) * q(x, y))$, where $z$ denotes the vector of fresh variables mentioned above. The transformation may then be inductively extended to any formula, using the fact that a $T$-formula allocates no location and that $x$ is allocated by a separating conjunction $\phi_1 * \phi_2$ if it is allocated by $\phi_1$ or $\phi_2$.

Definition 46 Let $\alpha = \Phi^u_1 \beta \rightarrow p(x_1, \ldots, x_n)[u \theta]$ and $x$ be a variable such that $x \neq x_1 \theta$. We assume that $u$ contains a variable $y$ such that $y \theta = x$. We denote by $\text{split}_x(\alpha)$ the set of all formulas of the form:

\[ \exists z. \left( \Phi^u_{(\beta \rightarrow q(y, y) \rightarrow p(x_1, \ldots, x_n)[u \theta])} * \Phi^u_{(\beta \rightarrow \text{mls}(y, y)[u \theta])} \right), \]

where $\beta = \beta_1 * \beta_2$; $q \in P_S$; $p \geq_R q$; $y, y$ is a vector of variables of length $\#(q)$ such that $z$ denotes the vector of variables occurring in $y$ but not in $\text{fv}(\beta) \cup \{x_1, \ldots, x_n\}$. The function

\[ y \theta = x. \]

---

7 This condition is not restrictive since a fresh variable $y$ can always be added both to $u$ and $\text{dom}(\theta)$, and by letting $y \theta = x$. Springer
is extended to disjunction-free formulas as follows (modulo prenex form and deletion of root-unsatisfiable formulas):

\[
\text{split}_x(\phi) = \emptyset \quad \text{if} \ \phi \text{ is a } T\text{-formula (possibly emp)}
\]
\[
\text{split}_x(x \mapsto y) = \{x \mapsto y\}
\]
\[
\text{split}_x(x' \mapsto y) = \emptyset \quad \text{if} \ x \not= x'
\]
\[
\text{split}_x(\alpha) = \{\alpha\} \text{ if } \alpha \text{ is a PU-atom and } \{x\} = \text{roots}_x(\alpha)
\]
\[
\text{split}_x(\phi_1 \land \phi_2) = \{\psi_1 \land \phi_2 \mid \psi_1 \in \text{split}_x(\phi_1) \cup \{\phi_1 \land \psi_2 \mid \psi_2 \in \text{split}_x(\phi_2)\}\)
\]
\[
\text{split}_x(\exists y. \phi) = \{\exists y. \psi \mid \psi \in \text{split}_x(\phi) \cup \text{split}_x(\phi[y \mapsto x])\)
\]

We emphasize that the vector \(y, y\) may contain variables from \(fv(\beta) \cup \{x_1, \ldots, x_n\}\) or fresh variables. Also, by construction, no variable in \(z\) can occur in \(\text{dom}(\theta)\), we can thus write, e.g.,
\[
\Phi^w,z\rightarrow q(y,y)[\theta]
\]
\[
\text{split}_x(\phi)[y, x] \sigma \rightarrow \text{split}_x(\Phi^u[\theta, z]) \sigma
\]
\[
\text{split}_x(\phi)[y, x] \sigma \rightarrow \text{split}_x(\Phi^u[\theta, z]) \sigma.
\]

Example 47 Let \(\phi = \exists y. \text{ls}(y, \text{nil})\) and \(\mathcal{R} = \{1 \text{ls}(u, v) \leftrightarrow u \mapsto (v), 1 \text{ls}(u, v) \mapsto \exists w. u \mapsto (w) * 1 \text{ls}(w, v)\}\). Then we have:
\[
\text{split}_x(1 \text{ls}(y, \text{nil})) = \text{split}_x(\Phi^u[\text{emp} \rightarrow \text{ls}(u, \text{nil})][y])
\]
\[
= \{\Phi^u[\text{emp} \rightarrow \text{ls}(u, \text{nil})][y, x]\}
\]
\[
= \{\Phi^u[\text{emp} \rightarrow \text{ls}(u, \text{nil})][y, x]\} * 1 \text{ls}(x, \text{nil})\}
\]

Note that, in this example, it is not useful to consider atoms other than \(\text{ls}(z, \text{nil})\) for defining \(\text{split}_x(1 \text{ls}(y, \text{nil}))\), since the only predicate occurring in the unfoldings of \(1 \text{ls}(y, \text{nil})\) are renamings of \(1 \text{ls}(z, \text{nil})\). Also, the variable \(y\) is discarded in \(\Phi^u[\text{emp} \rightarrow \text{ls}(u, \text{nil})][y]\) since it does not occur in \(\text{emp}\) or \(1 \text{ls}(z, \text{nil})\) (see Remark 19). Thus
\[
\text{split}_x(\phi) = \{1 \text{ls}(x, \text{nil}), \exists y.((\Phi^u[\text{ls}(z, \text{nil})] \rightarrow \text{ls}(u, \text{nil})[y, x]) \leftrightarrow \text{ls}(x, \text{nil}))\}.
\]

Proposition 48 Let \(\phi\) be a formula, \(x\) a variable and assume \(\text{split}_x(\phi) = \{\gamma_1, \ldots, \gamma_n\}\). Then we have \(\text{roots}_x(\gamma_i) = \{x\} \cup \text{roots}_x(\phi)\), for every \(i = 1, \ldots, n\). Thus, if \(\mathcal{R} \vdash \gamma_i\) for some \(i \in \{1, \ldots, n\}\), then \(s(x) \in \text{dom}(\mathcal{R})\).

Proof By an immediate induction on \(\phi\), inspecting the different cases in the definition of \(\text{split}_x(\phi)\) and by Proposition 37. \(\square\)

Note that if \(x \in \text{roots}_x(\phi)\) then it is straightforward to check using Proposition 48 that \(\text{split}_x(\phi) = \{\phi\}\), up to the deletion of root-unsatisfiable formulas.

Lemma 49 Let \(\phi\) be a formula, \(x\) a variable and assume that \(\text{split}_x(\phi) = \{\gamma_1, \ldots, \gamma_n\}\). For all structures \(\mathcal{R}\), if \(\mathcal{R} \vdash \gamma_i \lor \cdots \lor \gamma_n\), then \(\mathcal{R} \vdash \phi\). Moreover, if \(s(x) \in \text{dom}(\mathcal{R})\), \(s(x') \not= s(x)\) for every \(x' \not= x\) and \(\mathcal{R} \vdash \phi\), then \(\mathcal{R} \vdash \phi\), \(\mathcal{R} \vdash \gamma_i \lor \cdots \lor \gamma_n\).

Proof We proceed by induction on \(\phi\):

- Assume that \(\phi = x' \mapsto (y_1, \ldots, y_k)\) with \(x' \not= x\). Then \(n = 0\) and the first implication trivially holds. If \(\mathcal{R} \vdash \phi\), \(s(x) \in \text{dom}(\mathcal{R})\) and \(s(x) \not= s(x')\), then since \(\text{dom}(\mathcal{R}) = \{s(x')\}\), we have \(s(x) \not= \text{dom}(\mathcal{R})\), yielding a contradiction.
If \( \phi = x \mapsto (y_1, \ldots, y_k) \) then we have \( n = 1 \) and \( \gamma_1 = \phi \), thus the result is immediate.

Assume that \( \phi = \Phi^u_{\beta \mapsto p(x_1, \ldots, x_n)}[u\theta] \), where \( x_1 \theta \neq x \). We prove the two results separately:

- If \( (s, h) \models_R \gamma_1 \lor \cdots \lor \gamma_n \), then there exist \( \beta_1, \beta_2 \) and \( q(y, y) \) such that \( (s, h) \models_R \exists z. \Phi^u_{\beta_1 \mapsto q(y, y)} \circ p(x_1, \ldots, x_n)[u\theta] \ast \Phi^u_{\beta_2 \mapsto q(y, y)}[u, z\theta] \), where \( \beta = \beta_1 \ast \beta_2 \) and \( \theta, y, y \) and \( z \) fulfill the conditions of Definition 46. This entails that there exists a store \( s' \), coinciding with \( s \) on all variables not occurring in \( z \), and disjoint heaps \( h_1 \), \( h_2 \) such that \( h = h_1 \uplus h_2 \) and

\[
(s', h_1) \models_R \Phi^u_{\beta_1 \mapsto q(y, y)} \circ p(x_1, \ldots, x_n)[u\theta], \quad (s', h_2) \models_R \Phi^u_{\beta_2 \mapsto q(y, y)}[u, z\theta].
\]

By definition, there exists a formula \( \psi \) of the form \( \exists v_1. (\psi_1 \ast \beta_1 \ast q(y', y')) \) such that \( p(x_1, \ldots, x_n) \equiv_R \psi \), a substitution \( \sigma_1 \) with \( \text{dom}(\sigma_1) \subseteq v_1 \cap (fv(\beta_1') \cup \{y\} \cup y') \) and \( (\beta_1' \ast q(y', y') \ast (\psi_1 \ast \beta_1 \ast q(y, y)) \sigma_1 = \beta_1 \ast q(y, y) \), and a store \( s_1' \) coinciding with \( s' \) on all the variables not occurring in \( v_1 \) such that \( (s_1', h_1) \models_R \psi_1 \sigma_1 \theta \). Similarly, there exists a formula \( \psi' \) of the form \( \exists v_2. (\psi_2 \ast \beta_2) \) such that \( q(y, y) \equiv_R \psi' \), a substitution \( \sigma_2 \) and a store \( s_2' \) coinciding with \( s' \) on all variables not occurring in \( v_2 \) such that \( \text{dom}(\sigma_2) \subseteq v_2 \cap (fv(\beta_2')) \), \( \beta_2' \sigma_2 = \beta_2 \) and \( (s_2', h_2) \models_R \psi_2 \sigma_2 \theta \). By \( \alpha \)-renaming, we assume that the following condition (†) holds:

For \( i = 1, 2, v_i \) contains no variable in \( v_{3-i} \cup \{z\} \cup \text{img}(\sigma_{3-i}) \cup \{\psi'(\theta) \cup \text{fv}(\psi'(\theta) \cup \{x\} \cup \text{dom}(\theta)) \} \).

Since \( q(y', y') \sigma_1 = q(y, y) \) and \( q(y, y) \equiv_R \psi_2 \ast \beta_2 \), by Proposition 24, there exists a formula of the form \( \psi_2' \ast \beta_2 \) such that \( q(y', y') \equiv_R \exists v_2. (\psi_2' \ast \beta_2) \) and \( (\psi_2' \ast \beta_2) \sigma_1 = \psi_2 \ast \beta_2 \). We deduce that

\[
p(x_1, \ldots, x_n) \equiv_R \exists v_1. (\psi_1 \ast \beta_1' \ast q(y', y')) \equiv_R \exists v_1, v_2. (\psi_1 \ast \beta_1' \ast \psi_2' \ast \beta_2').
\]

Let \( \sigma = \sigma_2 \circ \sigma_1 \), note that by construction \( \text{dom}(\sigma) \subseteq (v_1 \cup v_2) \). By (†) we have

\[
(\psi_1 \ast \beta_1') \sigma = (\psi_1 \sigma_1 \ast \beta_1) \sigma_2 = \psi_1 \sigma_1 \ast \beta_1 \text{ and } (\psi_2' \ast \beta_2') \sigma = (\psi_2 \ast \beta_2') \sigma_2 = \psi_2 \sigma_2 \ast \beta_2.
\]

Let \( s'' \) be a store such that \( s''(y') = s''(x) \), \( s''(y') = s''(y) \), and that otherwise coincides with \( s_1' \) on \( v_1 \) and with \( s' \) elsewhere. By construction \( s'' \) coincides with \( s_1' \) on all the variables occurring in \( \psi_1 \sigma_1 \theta \), and since \( (s_1', h_1) \models_R \psi_1 \sigma_1 \theta \), we deduce that \( (s'', h_1) \models_R \psi_1 \sigma_1 \theta \). A similar reasoning shows that \( (s'', h_2) \models_R \psi_2 \sigma_2 \theta \) and, therefore, \( (s'', h) \models_R \psi_1 \sigma_1 \theta \ast \psi_2 \sigma_2 \theta = (\psi_1 \psi_2) \sigma \). By definition of \( \psi' \), this entails that \( (s'', h) \models_R (\psi_1 \ast \psi_2) \sigma' \theta \), where \( \sigma' \) is the restriction of \( \sigma \) to \( \text{fv}(\beta_1' \ast \beta_2') \).

Indeed, all variables \( y'' \) such that \( y'' \sigma' \neq y'' \sigma \) must occur in \( y', y' \) because \( \text{dom}(\sigma_1) \subseteq \text{fv}(\beta_1') \cup \{y\} \cup y' \) and \( \text{dom}(\sigma_2) \subseteq \text{fv}(\beta_2') \); therefore \( s''(y'' \sigma') = s''(y'' \sigma) \).

Since \( (\beta_1' \ast \beta_2') \sigma' = \beta_1' \sigma \ast \beta_2' \sigma = \beta_1 \ast \beta_2 = \beta \) and \( s'' \) coincides on variables that do not occur in \( v_1, v_2 \), we deduce that \( (s', h) \models_R \Phi^u_{\beta \mapsto p(x_1, \ldots, x_n)}[u\theta] \). But no variable from \( z \) can occur in \( \Phi^u_{\beta \mapsto p(x_1, \ldots, x_n)}[u\theta] \), hence \( s \) and \( s'' \) coincide on all variables occurring in this formula, so that \( (s, h) \models_R \Phi^u_{\beta \mapsto p(x_1, \ldots, x_n)}[u\theta] \).

- Now assume that \( (s, h) \models_R \Phi^u_{\beta \mapsto p(x_1, \ldots, x_n)}[u\theta] \), where \( x_1 \theta \neq x \), \( s(x) \in \text{dom}(h) \) and \( s(x') \neq s(x) \) for every \( x' \neq x \). Then there exists a derivation \( \psi' \) such that \( p(x_1, \ldots, x_n) \equiv_R \exists v_0. (\psi' \ast \beta') \), a store \( s' \) coinciding with \( s \) on all variables not occurring in \( w \) and a substitution \( \sigma' \) with \( \text{dom}(\sigma') \subseteq w \cap \text{fv}(\beta') \), such that...
\((s', h) \vdash_R \psi \sigma \theta \) and \(\beta = \beta' \sigma\). W.l.o.g., we assume that \(\psi\) contains no predicate atom and that \(w\) contains no variables in \(fv(\beta) \cup \{x, x_1, \ldots, x_n\} \cup dom(\theta) \cup img(\theta)\).

Since \(s(x) \in dom(h)\), necessarily \(\psi \sigma\) contains a points-to atom of the form \(x' \mapsto \) \((u_1, \ldots, u_k)\), for some variable \(x'\) such that \(s(x) = s'(x')\). Note that, by hypothesis (‡), we must have either \(x = x' \theta\) and \(x' \in \{x_2, \ldots, x_n\}\) or \(x' \in w\). In the latter case we define \(\sigma' = \sigma(x' \leftarrow y)\), where \(y\) is a variable in \(u\) such that \(y \theta = x\) (recall that \(y\) necessarily exists by the assumption of Definition 46); otherwise we let \(y = x'\) and \(\sigma' = \sigma\). Since \((s', h) \vdash_R \psi \sigma \theta\) and \(s'(y \theta) = s'(x' \sigma' \theta)\), we have \((s', h) \vdash_R \psi \sigma' \theta\). Due to the progress condition, the derivation \(D\) is necessarily of the form

\[
p(x_1, \ldots, x_n) \equiv_R^+ \exists w. (\psi \ast \beta'),
\]

where \(x' = x'' \sigma\), \(w\) is a subvector of \(w\) and \(q \in P_S\). Thus, by removing from \(D\) all the unfolding steps applied to \(q(x'', y')\) or its descendants, we obtain a derivation of the form

\[
p(x_1, \ldots, x_n) \equiv w_1. (\psi \ast \beta'_1),
\]

for some variables \(w_1\) occurring in \(w\), and formulas \(\beta'_1\) and \(\psi\) that are subformulas of \(\beta'\) and \(\psi\) respectively. Note that since \(x_1 \theta \neq x\) by hypothesis, \(q(x'', y')\) cannot coincide with \(p(x_1, \ldots, x_n)\), hence the length of this derivation is at least 1. Similarly, by keeping in \(D\) only the unfolding steps applied to \(q(x'', y')\) or its descendants, we obtain a derivation of the form

\[
q(x'', y') \equiv w_2. (\psi_2 \ast \beta'_2),
\]

where \(\beta'_2 = \beta'_1 \ast \beta'_2\), \(\psi = \psi_1 \ast \psi_2\), \(w = w_1 \cup w_2\) and \(w_1 \cap w_2 = \emptyset\). Note that \(q(x'', y')\) must be unfolded at least once, for the atom \(x'\) because no unfolding derivations can introduce new free variables to a formula. In particular, \(\beta'_1 \sigma_1 = \beta'_1 \sigma' = \beta_1\).

Let \(y = y' \sigma_1\) so that \(q(x'', y') \sigma' = q(y, y)\). Since \((s', h_1) \vdash_R \psi \sigma' \theta\) and \(s'(x' \theta) = s'(x)\), we deduce that \((s', h_1) \vdash_R \psi \sigma_1 \theta\) and \((s', h_2) \vdash_R (\psi_2 \sigma_1) \sigma_2 \theta\. Let \(z\) be the vector of all the variables in \(y \setminus (fv(\beta) \cup \{x, x_1, \ldots, x_n\})\), with no repetition. Then since \(p(x_1, \ldots, x_n) \equiv w_1. (\psi \ast q(x'', y') \ast \beta'_1)\) and \((s', h_1) \vdash_R \psi \sigma_1 \theta\), we have \((s', h_1) \vdash_R \Phi_{(u, z)}[u, z, \theta]\). Similarly, since \(q(y, y) \equiv w_2. (\psi_2 \sigma_1 \ast \beta'_2 \sigma_1)\), and \((s', h_2) \vdash_R (\psi_2 \sigma_1) \sigma_2 \theta\), we deduce that \((s', h_2) \vdash_R \Phi_{(u, z)}[u, z, \theta]\). This entails that \((s', h) \vdash_R \Phi_{(u, z)}[u, z, \theta]\). The stores \(s'\) and \(s\) coincide on all the variables not occurring in \(z\) and occurring in \(\beta_1, \beta_2, p(x_1, \ldots, x_n)\) or \(q(x, y)\). Indeed, if \(s'(u) \neq s(u)\), for some variable \(u\), then necessarily \(u \in w\) (by definition of \(s'\)), hence \(u \neq fv(\beta) \cup \{x_1, \ldots, x_n, x\}\). Thus, if \(u \in y\), then necessarily \(u \in z\). We deduce that
(s, h) ⊢ R ∃z. ((F(u, z))_{β|eq}(x, y) • p(x₁, ..., xₙ) [(u, z)θ]) • (F(u, z))_{β|eq}(x, y) [(u, z)θ]); therefore, (s, h) ⊢ R γ₁ ∨ ⋯ ∨ γₙ.

- If φ = em ∨ φ is a T-formula, then n = 0 hence (s, h) ⊭ R γ₁ ∨ ⋯ ∨ γₙ. Furthermore, if s(x) ∈ dom(h) then h is not empty hence (s, h) ⊭ R φ.
- If φ is a PU-atom and roots_r(φ) = {x} then by definition n = 1 and γ₁ = φ, thus the result is immediate.
- Assume that φ = φ₁ ∨ φ₂ and let {γ₁₁, ..., γ₁ₙ₁} = split₁(φ₁). In this case, {γ₁, ..., γₙ} = {ψ₃−i ∨ γₖ_{i} | i ∈ {1, 2}, j ∈ {1, ..., n₁}}.

If (s, h) ⊢ R γ₁ ∨ ⋯ ∨ γₙ, then there exists i ∈ {1, 2} and j ∈ {1, ..., n₁} such that (s, h) ⊢ R φ₃−i ∨ γₖ_{i}. Thus there exist disjoint heaps h₃−i, hᵢ such that h = h₃−i ∪ hᵢ, (s, h₃−i) ⊢ R φ₃−i and (s, hᵢ) ⊢ R γₖ_{i}, so that (s, hᵢ) ⊢ R γ₁ι ∨ ⋯ ∨ γₙι. By the induction hypothesis, we deduce that (s, hᵢ) ⊢ R φᵢ. Thus (s, h₃−i ∪ hᵢ) ⊢ R φ₃−i ∨ φᵢ, i.e., (s, h) ⊢ R φ.

If (s, h) ⊢ R φ, s(x) ∈ dom(h) and s(x') ≠ s(x) for every x' ≠ x, then there exist disjoint heaps h₁, h₂ such that (s, h₁) ⊢ R φᵢ for i = 1, 2, and h = h₁ ∪ h₂. Since s(x) ∈ dom(h), we must have s(x) ∈ dom(h₁), for some i = 1, 2. Then, by the induction hypothesis, we deduce that (s, h₁) ⊢ R γ₁i ∨ ⋯ ∨ γₙi. Consequently, (s, h) ⊢ R φ₃−i ∨ (γ₁i ∨ ⋯ ∨ γₙi) = ∨i=1n(φ₃−i ∨ γₖ_{i}), thus (s, h) ⊢ R γ₁ ∨ ⋯ ∨ γₙ.

- Assume that φ = ∃y. φ', where y ≠ x. Let {γ₁', ..., γₘ'} = split₁(φ') and {γ₁'', ..., γₘ''} = split₂(φ' {y ← x}), so that we have {γ₁, ..., γₙ} = {∃y. γ₁', ..., ∃y. γₘ', γ₁'', ..., γₘ''}.

If (s, h) ⊢ R γᵢ for some i = 1, ..., n, then we have either (s, h) ⊢ R ∃y. γᵢ for some j = 1, ..., m or (s, h) ⊢ R γᵢ'', for some j = 1, ..., l. In the former case, we get (s', h) ⊢ R γᵢ', for some store s' coinciding with s on all variables distinct from y, thus (s', h) ⊢ R φ' by the induction hypothesis, and therefore (s, h) ⊢ R ∃y. φ'. In the latter case, we get (s, h) ⊢ R φ' {y ← x} by the induction hypothesis, thus (s, h) ⊢ R ∃y. φ'. Conversely, if (s, h) ⊢ R φ, s(x) ∈ dom(h) and s(x') ≠ s(x), for every x' ≠ x, then either (s, h) ⊢ R φ' {y ← x} or (s', h) ⊢ R φ', for some store s' coinciding with s on all variables distinct from y, with s'(y) ≠ s(x). By the induction hypothesis, this entails that either (s, h) ⊢ R γ₁' ∨ ⋯ ∨ γₘ' or (s', h) ⊢ R γ₁' ∨ ⋯ ∨ γₘ'. We deduce that (s, h) ⊢ R γ₁ ∨ ⋯ ∨ γₙ. □

8 The Proof Procedure

In this section, we devise a proof procedure for testing the validity of sequents. This procedure is defined as a set of inference rules, operating of sequents, and deducing new valid sequents from valid premises. The inference rules, except for the axioms, are depicted in Fig. 1. The rules are intended to be applied bottom-up: a rule is applicable on a sequent φ ⊢ R Γ if there exists an instance of the rule the conclusion of which is φ ⊢ R Γ. We assume from now on that all the considered sequents are disjunction-free (as explained above, this is without loss of generality since every formula is equivalent to a disjunction of symbolic heaps) and that the formula on the left-hand side is in prenex form. We do not assume that the formulas occurring on the right-hand side of the sequents are in prenex form, because one of the inference rules (namely the Existential Decomposition rule) will actually shift existential quantifiers inside separating conjunctions. Note that none of the rules rename the existential variables occurring in the conclusion (renaming is only used on the existential variables occurring in the rules).
| Rule | Derivation |
|------|------------|
| **Sk:** | $\phi(x \leftarrow x_1) \vdash_{\mathcal{R}} \Gamma \ldots \phi(x \leftarrow x_n) \vdash_{\mathcal{R}} \Gamma \phi(x \leftarrow x') \vdash_{\mathcal{R}} \Gamma$
| | $\exists x. \phi \vdash_{\mathcal{R}} \Gamma$
| | if $\{x_1, \ldots, x_n\} = fu(\phi) \cup fu(\Gamma)$ and $x'$ is a fresh variable not occurring in $\phi$ or $\Gamma$. |
| **HF:** | $x \mapsto (y_1, \ldots, y_k) \ast \phi \vdash_{\mathcal{R}} \exists u'.(x \mapsto (z_1, \ldots, z_k) \ast \psi), \Gamma$
| | $x \mapsto (y_1, \ldots, y_k) \ast \phi \vdash_{\mathcal{R}} \exists u. (x \mapsto (z_1, \ldots, z_k) \ast \psi), \Gamma$
| | if $u \cap \{x_1, \ldots, y_k\} = \emptyset$, $dom(\sigma) \subseteq u \cap \{z_1, \ldots, z_k\}$, $dom(\sigma) \neq \emptyset$, $\forall i \in \{1, \ldots, k\} z_i \sigma = y_i$ and $u'$ is the vector of variables occurring in $u$ but not in $dom(\sigma)$. |
| **UL:** | $\phi_1 \ast \phi \vdash_{\mathcal{R}} \Gamma \ldots \phi_n \ast \phi \vdash_{\mathcal{R}} \Gamma$
| | $p(x) \ast \phi \vdash_{\mathcal{R}} \Gamma$
| | if $p \in \mathcal{P}_S$, $\{\phi_1, \ldots, \phi_n\}$ is the set of formulas such that $p(x) \leftarrow_{\mathcal{R}} \phi_i$. |
| **UR:** | $\phi \vdash_{\mathcal{R}} \exists x. (\psi_1 \ast \psi), \ldots, \exists x. (\psi_n \ast \psi), \Gamma$
| | $\phi \vdash_{\mathcal{R}} \exists x. (\alpha \ast \psi), \Gamma$
| | if $\alpha$ is a PU-atom, $\text{roots}_r(\alpha) \subseteq fu(\phi)$, $\{\psi_1, \ldots, \psi_n\}$ is the set of $\mapsto$-formulas (see Definition 2) such that $\alpha \leftarrow_{\mathcal{R}} \psi_i$ (with possibly $n = 0$). |
| **W:** | $\phi \vdash_{\mathcal{R}} \Delta$
| | $\phi \vdash_{\mathcal{R}} \psi, \Delta$
| | $\phi \vdash_{\mathcal{R}} \psi_1, \ldots, \psi_n, \Delta$ if $x \in \text{alloc}(\phi) \setminus \text{roots}_r(\psi)$, $\{\psi_1, \ldots, \psi_n\} = \text{split}_x(\psi)$. |
| **SC:** | $\phi \vdash_{\mathcal{R}} \Gamma_1 \ldots \phi \vdash_{\mathcal{R}} \Gamma_m$
| | $\phi' \vdash_{\mathcal{R}} \Gamma'_1 \ldots \phi' \vdash_{\mathcal{R}} \Gamma'_l$
| | if: (i) $\text{alloc}(\phi) \neq \emptyset$, $\text{alloc}(\phi') \neq \emptyset$; (ii) $I_1, \ldots, I_m, J_1, \ldots, J_l \subseteq \{1, \ldots, n\}$, for every $X \subseteq \{1, \ldots, n\}$, either $X \supseteq I_i$ for some $i \in \{1, \ldots, m\}$, or $\{1, \ldots, n\} \setminus X \supseteq J_j$ for some $j \in \{1, \ldots, l\}$, and for every $i, j \in \{1, \ldots, m\}$ (resp. $i, j \in \{1, \ldots, l\}$), with $i \neq j$ we have $I_i \not\subseteq I_j$ (resp. $J_i \not\subseteq J_j$); (iii) $\Gamma_i$ ($1 \leq i \leq m$) is the sequence of formulas $\psi_j$ for $j \in I_i$ and $\Gamma'_j$ (for $1 \leq i \leq l$) is the sequence of formulas $\psi'_j$ for $j \in J_i$. |
| **ED:** | $\phi \ast \phi' \vdash_{\mathcal{R}} \gamma_1, \ldots, \gamma_m, \exists y. \gamma(x \leftarrow x_1), \ldots, \exists y. \gamma(x \leftarrow x_n), \Gamma$
| | $\phi \ast \phi' \vdash_{\mathcal{R}} \exists y. \exists z. \gamma, \Gamma$
| | if $\{x_1, \ldots, x_n\} = fu(\phi) \cap fu(\phi')$, $x'$ is a fresh variable (not occurring in the conclusion) and $\{\gamma_1, \ldots, \gamma_m\}$ is a set of formulas of the form $\exists y.((\exists x. \psi) \ast \psi'(x \leftarrow x') \ast \xi)$, where $\gamma = \psi \ast \psi' \ast \chi$, $\psi' \neq \text{emp}$, both $\chi$ and $\xi$ are $\mathcal{T}$-formulas, $\xi \models_{\mathcal{T}} \chi$ and $x \not\in fu_{\mathcal{T}}(\psi') \cup fu(\xi)$. |
| **TS:** | $\phi \ast \chi \vdash_{\mathcal{R}} \Gamma$
| | $\phi \ast \chi' \vdash_{\mathcal{R}} \Gamma$
| | $\phi \vdash_{\mathcal{R}} \Gamma$
| | $\phi' \vdash_{\mathcal{T}} \chi' \vdash_{\mathcal{T}} \chi$ if $\chi$ is a $\mathcal{T}$-formula, $\chi \land \chi' \vdash_{\mathcal{T}} \chi$. |
| **TD:** | $\phi \ast \chi \vdash_{\mathcal{R}} \phi' \vdash_{\mathcal{R}} \phi' \vdash_{\mathcal{R}} \Gamma$
| | $\phi \ast \chi \vdash_{\mathcal{R}} \phi' \vdash_{\mathcal{R}} \phi' \vdash_{\mathcal{R}} \Gamma$
| | $\phi \ast \chi \vdash_{\mathcal{R}} \chi \ast \phi' \vdash_{\mathcal{R}} \Gamma$ if $\chi$ is a $\mathcal{T}$-formula and $\chi \lor \chi'$ is valid. |

Fig. 1 Inference rules
This feature will be used in the termination analysis. We now provide some explanations on these rules. We refer to Fig. 1 for the notations.

The Skolemisation rule (Sk) gets rid of existential quantifiers on the left-hand side of the sequent. The rule replaces an existential variable \( x \) by a new free variable \( x' \). Note that the case where \( x = x_i \) for some \( i = 1, \ldots, n \) must be considered apart because countermodels must be injective.

The Heap Functionality rule (HF) exploits the fact that every location refers to at most one tuple to instantiate some existential variables occurring on the right-hand side of a sequent. Note that the vector \( u' \) may be empty, in which case there is no existential quantification.

The Left Unfolding rule (UL) unfolds a predicate atom on the left-hand side. Note that the considered set of formulas is finite, up to \( \alpha \)-renaming, since \( R \) is finite. All the formulas \( \phi_i \ast \phi \) are implicitly transformed into prenex form.

The Right Unfolding rule (UR) unfolds a predicate atom on the right-hand side, but only when the unfolding yields a single points-to spatial atom. Note that this rule always applies on formulas \( \exists x. (\alpha \ast \psi) \): in the worst case, the set \( \{\psi_1, \ldots, \psi_n\} \) is empty (\( n = 0 \)), in which case the rule simply removes the considered formula from the right-hand side.

**Example 50** With the rules of Example 47, UR applies on the sequent \( x \mapsto (y) \vdash_R \exists z, x \mapsto (z) \). Indeed, \( \text{roots}_R (\Phi \mapsto_R x \mapsto (y)) = \{x\} \), \( 1_s(x, y) \iff \exists u. (x \mapsto (u) \ast 1_s(u, y)) \) and \( 1_s(u, y)(u \leftarrow z) = 1_s(z, y) \); thus \( \Phi \mapsto_R x \mapsto (z) \). Note that we also have \( 1_s(x, y) \iff x \mapsto (y) * \text{emp} \), but there is no substitution \( \sigma \) such that \( \text{emp} \sigma = 1_s(z, y) \). The rule also applies on \( x \mapsto (y) \vdash_R \Phi \mapsto_R x \mapsto (y) \), yielding \( x \mapsto (y) \vdash_R \square \), since there is no substitution \( \sigma \) with domain \( \{u\} \) such that \( 1_s(u, y) \sigma = 1_s(y, x) \).

The Weakening rule (W) allows one to remove formulas from the right-hand side.

Heap Decomposition (HD) makes use of the heap splitting operation introduced in Sect. 7; the soundness of this rule is a consequence of Lemma 49.

The Separating Conjunction Decomposition rule (SC) permits the decomposition of separating conjunctions on the left-hand side, by relating an entailment of the form \( \phi \ast \phi' \vdash_R \Gamma \) to entailments of the form \( \phi \vdash_R \Delta \) and \( \phi' \vdash_R \Delta' \). This is possible only if all the formulas in \( \Gamma \) are separating conjunctions. As we shall see, SC is always sound, but it is invertible\(^8\) only if the heap decomposition corresponding to the left-hand side coincides with that of the formulas on the right-hand side (see Definition 69). It plays a similar role to the rule (\#) defined in [31]\(^9\). For instance, assume that the conclusion is \( p(x) \ast q(y) \vdash_R p_1(x) \ast q_1(y), p_2(x) \ast q_2(y) \). Then the rule can be applied with one of the following premises:

\[
\begin{align*}
p(x) & \vdash_R p_1(x), \quad p(x) \vdash_R p_2(x),
p(y) & \vdash_R q_1(y), \quad q_2(y),
p(x) & \vdash_R p_1(x), \quad p_2(x) \vdash_R q_1(y), \quad q_2(y),
p(x) & \vdash_R \square,
q(y) & \vdash_R \square.
\end{align*}
\]

Other applications, such as the one with premises

\[
\begin{align*}
p(x) & \vdash_R p_1(x), \quad p(x) \vdash_R p_2(x),
p(y) & \vdash_R q_1(y), \quad q_2(y),
p(x) & \vdash_R \square,
q(y) & \vdash_R \square.
\end{align*}
\]

\(^8\) We recall that a rule is invertible if the validity of its conclusion implies the validity of each of its premises.

\(^9\) The key difference is that in our rule the premises are directly written into disjunctive normal form, rather than using universal quantifications over sets of indices and disjunction.
are redundant: if it is provable, then the first sequence above is also provable. The intuition of the rule is that its conclusion holds if for every model \((s, h \sqcup h')\) of \(\phi \land \phi'\) with \((s, h) \models_R \phi\) and \((s, h') \models_R \phi'\), there exists \(i = 1, \ldots, n\) such that \((s, h) \models_R \psi_i\) and \((s, h') \models_R \psi'_i\) (note that the converse does not hold in general). Intuitively, the premises are obtained by putting the disjunction \(\bigvee_{i=1}^n (s, h) \models_R \psi_i \land (s, h') \models_R \psi'_i\) into conjunctive normal form, by distributing the implication over conjunctions and by replacing the entailments of the form \((s, h) \models_R \phi \land (s, h') \models_R \phi'\) \(\implies (s, h) \models_R \Gamma \land (s, h') \models_R \Gamma'\) by the logically stronger conjunction of the two entailments \((s, h) \models_R \phi \implies (s, h') \models_R \phi'\) \(\implies (s, h') \models_R \phi'\). Also, the application conditions of the rules ensure that redundant sequents are discarded, such as \(p(x) \models_R p_1(x), p_2(x)\) w.r.t. \(p(x) \models_R p_1(x)\).

The Existential Decomposition rule \((ED)\) allows one to shift existential quantifiers on the right-hand side inside separating conjunctions. This rule is useful to allow for further applications of Rule \((SC)\).

The condition \(x \notin \text{fv}_R'(\psi')\) can be replaced by the stronger condition “\(\psi'\) is \(\emptyset\)-constrained”, which is easier to check (one does not have to compute the set \(\text{fv}_R'(\psi')\)). All the completeness results in Sect. 10 also hold with this stronger condition (and with \(\xi = \chi = \text{emp}\)). The intuition behind the rule is that \(\psi\) denotes the part of \(\gamma\) the interpretation of which depends on the value of \(x\). In most cases, this formula is unique and \(m = 1\), but there are cases where several decompositions of \(\gamma\) must be considered, depending on the unfolding.

**Example 51** Consider a sequent \(p(x, z) \ast p(y, y) \models_R \exists z.(q(x, z) \ast r(x, z))\), with the rules \(\{(p(u, v) \Leftrightarrow u \mapsto (v), q(u, v) \Leftrightarrow u \mapsto (v), r(u, v) \Leftrightarrow u \mapsto (u)\}\). Note that the interpretation of \(r(u, v)\) does not depend on \(v\). One of the premises the rule \((ED)\) yields \(p(x, z) \ast p(y, y) \models_R \exists z.(q(x, z) \ast r(x, x'))\). Afterwards, the rule \((SC)\) can be applied, yielding for instance the premises \(p(x, z) \models_R \exists z.q(x, z)\) and \(p(y, y) \models_R r(x, x')\).

The \(T\)-Simplification rule \((TS)\) allows one to simplify \(T\)-formulas, depending on some external procedure, and \(\prec\) denotes a fixed well-founded order on \(T\)-formulas. We assume that \(\chi < \chi \ast \xi\) for every formula \(\xi \neq \text{emp}\), and that \(\chi < \chi' \implies \xi \ast \chi < \xi \ast \chi'\), for every formula \(\xi\). Note that \((TS)\) is not necessary for the completeness proofs in Sect. 10.

The \(T\)-Decomposition rule \((TD)\) shifts \(T\)-formulas from the right-hand side to the left-hand side of a sequent. In particular, the rule applies with \(\chi' = \neg \chi\) if the theory is closed under negation. The completeness results in Sect. 10 hold under this requirement.

**Axioms.**

Axioms are represented in Fig. 2. The Reflexivity axiom \((R)\) gets rid of trivial entailments, which can be proven simply by instantiating existential variables on the right-hand side. For the completeness proofs in Sect. 10, the case where \(\sigma = \text{id}\) is actually sufficient. The Disjointness axiom \((D)\) handles the case where the same location is allocated in two disjoint parts of the heap. Note that (by Definition 26) the left-hand side of the sequent is \(\text{PU}\)-free, hence alloc(\(\phi\)) and alloc(\(\phi'\)) are well-defined. The \(T\)-Clash axiom \((TC)\) handles the case where the left-hand side is unsatisfiable modulo \(T\), while the Empty Heap axiom \((EH)\) applies when the left-hand side is a \(T\)-formula.

**Proof Trees**

**Definition 52** A proof tree is a possibly infinite tree, in which each node is labeled by a sequent, and if a node is labeled by some sequent \(\phi \models_R \Gamma\), then its successors are labeled by \(\phi_i \models_R \Gamma_i\) with \(i = 1, \ldots, n\), for some rule application \(\frac{\phi_1 \models_R \Gamma_1; \ldots; \phi_n \models_R \Gamma_n}{\phi \models_R \Gamma}\). A proof tree
is rational if it contains a finite number of subtrees, up to a renaming of variables. The end-sequent of a proof tree is the sequent labeling the root of the tree.

In practice one is of course interested in constructing rational proof trees. Such rational proof trees can be infinite, but they can be represented finitely. The cycles in a rational proof tree may be seen as applications of the induction principle. We provide a simple example showing applications of the rules.

**Example 53** Consider the pc-SID consisting of the following rules:

\[
\begin{align*}
   p(x) & \iff \exists y, z. x \mapsto (y, z) * p(y) * p(z) \\
   p(x) & \iff x \mapsto (x, x) \\
   q(x, u) & \iff \exists y, z. x \mapsto (y, z) * p(y) * q(z, u) \\
   q(x, u) & \iff x \mapsto (u, u)
\end{align*}
\]

The proof tree \( \tau(x) \) below admits the end-sequent \( p(x) \vdash_R \exists u. q(x, u) \):

\[
\begin{array}{c}
   x \mapsto (x, x) \vdash_R \exists u. x \mapsto (u, u) & \text{R} \\
   \vdash_R \exists u. q(x, u) & \text{UL} \\
   \frac{}{\vdash_R \exists u. q(x, u)} & \text{UL}
\end{array}
\]

where the proof tree \( \pi(x) \) with end-sequent \( \exists y, z. x \mapsto (y, z) * p(y) * p(z) \vdash_R \exists u. q(x, u) \) is defined as follows (using \( \phi \) to denote the PU-atom \( \Phi^{p(y) * q(z, u), q(x, u)}_{x, y, z, u} [x, y, z, u] \)):

\[
\begin{array}{c}
   x \mapsto (y, z) \vdash_R \exists u. (\Phi^{p(y) * q(z, u), q(x, u)}_{x, y, z, u}) & \text{R} \\
   \vdash_R \exists u. q(x, u) & \text{SC} \\
   \frac{}{\vdash_R \exists u. q(x, u)} & \text{SC}
\end{array}
\]

For the sake of readability, root-unsatisfiable formulas are removed. For example, the rule \( \text{Sk} \) applied above also adds sequents with formulas on the left-hand side such as \( x \mapsto (y, z) * p(x) * p(z) \).
Some weakening steps are also silently applied to dismiss irrelevant formulas from the
right-hand sides of the sequents.

Note that the formula \( q(x, u) \) on the right-hand side of a sequent is not unfolded unless
this unfolding yields a single points-to atom. Unfolding \( q(x, u) \) would be possible for this set
of rules because \( p(x) \) and \( q(x, u) \) share the same root, but in general such a strategy would
not terminate. Instead the rule HD is used to perform a partial unfolding of \( q(x, u) \) with a
split on variable \( y \) for the first application.

The sequent \( p(z) \vdash \exists u. q(z, u) \) is identical to the root sequent, up to a renaming
of variables. The generated proof tree is thus infinite but rational, up to a renaming of variables.

9 Soundness

We prove that the calculus is sound, in the sense that the end-sequent of every (possibly
infinite, even irrational) proof tree is valid. In the entire section, we assume that \( \mathcal{R} \) is alloc-
compatible.

Lemma 54 The conclusions of the rules \( R, D, TC \) and \( EH \) are all valid.

Proof We consider each axiom separately.

\( R \) Assume that \( (s, h) \models_R \phi * \chi \). Since \( \chi \) is a \( T \)-formula, we also have \( (s, h) \models_R \phi \). Let \( s' \)
be a store mapping each variable \( z \) to \( s(z \sigma) \). Since \( dom(\sigma) = z \), \( s \) and \( s' \) coincide on all
variables not occurring in \( z \). Since \( \phi = \psi \sigma \), we have \( (s, h) \models_R \psi \sigma \), and by definition
of \( s' \), Thus \( (s, h) \models_R \exists z. \psi \).

\( D \) Assume that \( (s, h) \models_R \phi * \phi' \), where \( x \in alloc(\phi) \cap alloc(\phi') \). Then there exist disjoint
heaps \( h_1 \) and \( h_2 \) with \( h = h_1 \cup h_2 \), \( (s, h_1) \models_R \phi \) and \( (s, h_2) \models_R \phi' \). By Lemma 34, since
\( \mathcal{R} \) is alloc-compatible and \( x \in alloc(\phi) \cap alloc(\phi') \), we have \( s(x) \in dom(h_1) \cap dom(h_2) \),
contradicting the fact that \( h_1 \) and \( h_2 \) are disjoint.

\( TC \) Assume that \( (s, h) \models_R \phi * \chi \), where \( \chi \) is a \( T \)-formula. Then by definition of the semantics
we must have \( (s, \emptyset) \models_R \chi \), hence \( \chi \) cannot be unsatisfiable.

\( EH \) If \( (s, h) \models_R \chi \) then since \( \chi \) is a \( T \)-formula necessarily \( h = \emptyset \), hence \( (s, h) \models_R \exists x_i. \xi_i \),
for some \( i = 1, \ldots, n \), by the application condition of the rule. \( \square \)

Lemma 55 The rules HF, UL, UR, W, HD, TS, TD are sound. More precisely, if \( (s, h) \) is a
countermodel of the conclusion of the rule, then it is also a countermodel of at least one of
the premises.

Proof We consider each rule separately:

HF The proof is immediate since it is clear that \( \exists u'. (x \mapsto (z_1, \ldots, z_k) * \psi) \sigma \models_R \exists u.(x \mapsto (z_1, \ldots, z_k) * \psi) \).

UL Let \( \alpha = p(x) \), and assume \( (s, h) \models_R \alpha * \phi \) and \( (s, h) \not\models_R \Gamma \). Then \( h = h_1 \cup h_2 \),
with \( (s, h_1) \models_R \alpha \) and \( (s, h_2) \models_R \phi \). By definition of the semantics of the predicate
atoms, necessarily \( (s, h_1) \models_R \psi \), for some \( \psi \) such that \( \alpha \models_R \psi \). By definition of \( \{\phi_1, \ldots, \phi_n\} \), there exists \( i \in \{1, \ldots, n\} \) such that \( \psi = \phi_i \) (modulo \( \alpha \)-renaming). We
deduce that \( (s, h_1) \models_R \phi_i \), and that \( (s, h) \models_R \phi_i * \phi \). Therefore, \( (s, h) \) is a countermodel
of \( \phi_i * \phi \models_R \Gamma \).

UR Assume that \( (s, h) \models_R \phi \) and \( (s, h) \models_R \exists x_i. (\psi_1 * \psi), \ldots, \exists x_i. (\psi_n * \psi) \), \( \Gamma \) and \( (s, h) \not\models_R \exists x_i (\alpha * \psi) \), \( \Gamma \). Then necessarily, \( (s, h) \models_R \exists x_i (\psi_i * \psi) \) holds for some \( i = 1, \ldots, n \).
Thus there exist a store \( s' \) coinciding with \( s \) on all variables not occurring in \( x \) and disjoint

\( \square \)
heaps $h_i$ (for $i = 1, 2$) such that $h = h_1 \sqcup h_2$, $(s', h_1) \models \psi_i$ and $(s', h_2) \models \psi$. Since by hypothesis $\alpha \not\models_R \psi$, we deduce that $(s', h_1) \models \alpha$, so that $(s', h_1 \sqcup h_2) \models \alpha \land \psi$, hence $(s, h) \models \exists x. (\alpha \land \psi)$, which contradicts our assumption.

The proof is immediate, since by definition every countermodel of $\phi \vdash_R \psi, \Gamma$ is also a countermodel of $\phi \vdash \Gamma$.

HD Assume that $(s, h) \models_R \phi$ and that $(s, h) \not\models_R \psi$.

By Lemma 49, since split$(\psi) = \{\psi_1, \ldots, \psi_n\}$, we deduce that $(s, h) \not\models_R \psi_1, \ldots, \psi_n$.

TS Assume that $(s, h) \models_R \phi \land \psi$, $s$ is injective and $(s, h) \not\models_R \Gamma$. Since $\chi$ is a $T$-formula, this entails that $(s, h) \models_R \phi$ and $s \models_T \chi'$, thus $s \models_T \chi'$. By the application condition of the rule we have $\chi' \models_T \chi$, hence $s \models_T \chi$ because $s$ is injective. We deduce that $(s, h) \models_R \phi \land \chi$, and that $(s, h)$ is a countermodel of $\phi \land \chi \vdash_R \Gamma$.

TD Assume that $(s, h) \models_R \phi$ and $(s, h) \not\models_R \chi \land \phi'$, $\Gamma$. We distinguish two cases. If $(s, \emptyset) \models_R \chi$ then $(s, h) \models_R \phi \land \chi$, since $h = \emptyset \sqcup \emptyset$. Furthermore, since $(s, h) \not\models_R \chi \land \phi'$, necessarily, $(s, h) \not\models_R \phi'$ and $(s, h)$ is a countermodel of $\phi \land \chi \vdash_R \phi'$, $\Gamma$. Otherwise, $(s, \emptyset) \not\models_R \chi$, hence $(s, \emptyset) \models_R \phi \land \chi'$ because $\chi \land \chi'$ is valid, and $(s, h) \models_R \phi \land \chi'$. Therefore, $(s, h)$ is a countermodel of $\phi \land \chi \vdash_R \Gamma$.

Lemma 56 The rule $SC$ is sound. More precisely, if $(s, h)$ is a countermodel of the rule conclusion, then at least one of the premises admits a countermodel $(s', h')$, where $h'$ is a proper subheap of $h$.

Proof Let $(s, h)$ be a countermodel of $\phi \land \phi' \vdash_R \psi_1 \land \psi_1', \ldots, \psi_n \land \psi_n'$, and assume that for every proper subheap $h' \subseteq h$, $(s, h')$ satisfies all the premises. Necessarily, $(s, h) \models_R \phi \land \phi'$, hence there exist heaps $h_1$ and $h_2$ such that $h = h_1 \sqcup h_2$, $(s, h_1) \models_R \phi$ and $(s, h_2) \models_R \phi'$. Furthermore, since $\phi$ and $\phi'$ both contain at least one spatial atom, both $h_1$ and $h_2$ must be nonempty by Proposition 13, and are therefore both proper subheaps of $h$. Since $(s, h) \not\models_R \psi_1 \land \psi_1', \ldots, \psi_n \land \psi_n'$, for every $x \in \{1, \ldots, n\}$, we have either $(s, h_1) \not\models_R \psi_x$ or $(s, h_2) \not\models_R \psi_x'$. By gathering all the indices $x$ satisfying the first assertion, we obtain a set $X \subseteq \{1, \ldots, n\}$ such that $x \in X \Rightarrow (s, h_1) \not\models_R \psi_x$ (†) and $x \in \{1, \ldots, n\} \setminus X \Rightarrow (s, h_2) \not\models_R \psi_x'$ (‡).

By the application condition of the rule, we have either $I_i \subseteq X$ for some $i \in \{1, \ldots, m\}$, or $J_j \subseteq \{1, \ldots, n\} \setminus X$ for some $j \in \{1, \ldots, l\}$. First assume that $I_i \subseteq X$. Then since $(s, h_1) \models_R \phi$ and $h_1$ is a proper subheap of $h$, we deduce that $(s, h_1) \models_R \bigvee_{x \in I_i} \psi_x$, which contradicts (†). Similarly if $J_j \subseteq \{1, \ldots, n\} \setminus X$, then, since $(s, h_2) \models_R \phi'$ and $h_2$ is a proper subheap of $h$, we have $(s, h_2) \models_R \bigvee_{x \in J_j} \psi_x'$, which contradicts (‡).

Lemma 57 Let $\phi$ be a formula, $x$ be a variable not occurring in $fv^T_R(\phi)$ and let $(s, h)$ be a model of $\phi$ such that $s(x) \notin loc(h)$. Then, for every store $s'$ coinciding with $s$ on all variables distinct from $x$, we have $(s', h) \models_R \phi$.

Proof The proof is by induction on the satisfiability relation. We distinguish several cases.

- If $\phi$ is a $T$-formula then $fv(\phi) = fv^T_R(\phi)$, hence by hypothesis $x \notin fv(\phi)$ and $(s', h) \models_R \phi$.

- If $\phi = x_0 \mapsto (x_1, \ldots, x_n)$ then since $(s, h) \models_R \phi$, $h = \{(s(x_0), \ldots, s(x_n))\}$, and $loc(h) = \{s(x_i) | 0 \leq i \leq \kappa\}$. Since $s(x) \notin loc(h)$ we deduce that $s(x) \neq s(x_i)$ for all $i = 0, \ldots, \kappa$, thus $x \neq x_i$ and $s'$ coincides with $s$ on $x_0, \ldots, x_n$. We conclude that $(s', h) \models_R \phi$.

- Assume that $\phi$ is of the form $\Phi^u_{\beta \rightarrow \sigma}[^u]$. Then there exist a formula $\exists y. (\beta' \land \psi)$, a substitution $\sigma$ with $dom(\sigma) \subseteq y$ and a store $s$ coinciding with $s$ on all variables not occurring in $y$ such that $\alpha \not\models_R \exists y. (\beta' \land \psi)$, $(s, h) \models_R \psi \sigma \theta$ and $\beta = \beta' \sigma$. We assume
by $\alpha$-renaming that $x \not\in y$. Let $\hat{s}'$ be a store coinciding with $\hat{s}$ on all variables distinct from $x$, and such that $\hat{s}'(x) = s'(x)$. By construction, $\hat{s}'$ coincides with $s'$ on all variables not occurring in $y$. By definition of the extension of $\lll_R \lll_R$ to formulas containing $\text{PU}$-atoms, we have $\phi \lll_R \exists y'. \psi \sigma \theta$, where $y' = y \setminus \text{dom}(\sigma)$. Assume that $x \in \text{fv}_R^T(\psi \sigma \theta)$. As $x \not\in y$, this entails that $x \in \text{fv}_R^T(\exists y'. \psi \sigma \theta)$, and by Definition 39 $x \in \text{fv}_R^T(\phi)$, which contradicts the hypothesis of the lemma. Thus $x \not\in \text{fv}_R^T(\psi \sigma \theta)$ and by the induction hypothesis we have $(\hat{s}', h) \models_R \psi \sigma \theta$. We deduce that $(s', h) \models_R \phi$.

- If $\phi = \phi_1 * \phi_2$ or $\phi = \phi_1 \lor \phi_2$ then the result is an immediate consequence of the induction hypothesis.

- If $\phi = \exists y. \psi$ then there exists a store $\hat{s}$ coinciding with $s$ on all variables distinct from $y$ such that $(\hat{s}, h) \models_R \psi$. Let $\hat{s}'$ be the store coinciding with $s'$ on $x$ and with $\hat{s}$ on all other variables. By the induction hypothesis we get $(\hat{s}', h) \models_R \psi$ thus $(s', h) \models_R \psi$. $\square$

**Lemma 58** The rules $\text{Sk}$ and $\text{ED}$ are sound. More precisely, if $(s, h)$ is a countermodel of the conclusion of the rule, then there exists a store $s'$ such that $(s', h)$ is a countermodel of the premise of the rule.

**Proof** We consider each rule separately.

**Sk** Let $(s, h)$ be a countermodel of $\exists x. \phi \vdash \Gamma$. Then $s$ is injective, $(s, h) \models_R \exists x. \phi$ and $(s, h) \not\models_R \Gamma$. This entails that there exists a store $s''$, coinciding with $s$ on all variables distinct from $x$, such that $(s'', h) \models_R \phi$. Let $\{x_1, \ldots, x_n\} = \text{fv}(\exists x. \phi) \cup \text{fv}(\Gamma)$; note that by the application condition of the rule, $x \notin \{x_1, \ldots, x_n\}$. Assume that there exists $i \in \{1, \ldots, n\}$ such that $s''(x_i) = s''(x_i) = s(x_i)$. Then $(s', h) \models_R \phi[x \leftarrow x_i]$, and since $s''$ and $s$ coincide on all variables freely occurring in $\phi[x \leftarrow x_i]$, we have $(s, h) \models_R \phi[x \leftarrow x_i]$. This entails that $(s, h)$ is a countermodel of $\phi[x \leftarrow x_i] \vdash \Gamma$, and the proof is thus completed, with $s' = s$. Otherwise, consider any injective store $\hat{s}$ coinciding with $s''$ on all variables in $\text{fv}(\exists x. \phi) \cup \text{fv}(\Gamma)$ and such that $\hat{s}(x_i) = s''(x_i)$. Since $(s'', h) \models_R \phi$ and $(s, h) \not\models_R \Gamma$ we have $(\hat{s}, h) \models_R \phi[x \leftarrow x_i]$ and $(\hat{s}, h) \not\models_R \phi$. The proof is thus completed with $s' = \hat{s}$.

**ED** Let $(s, h)$ be a countermodel of $\phi * \phi' \vdash \exists y. \exists x. \chi, \Gamma$. Then $(s, h) \models_R \phi * \phi'$ and $(s, h) \not\models_R \exists y. \exists x. \chi, \Gamma$. Let $s'$ be an injective store coinciding with $s$ on all variables distinct from $x'$ such that $s'(x')$ is a location not occurring in $\text{loc}(h)$. Since $x'$ does not occur free in the considered sequent, we have $(s', h) \models_R \phi * \phi'$, and $(s', h) \not\models_R \exists y. \exists x. \chi, \Gamma$, so that $(s', h) \not\models_R \exists y. \chi, \Gamma$, for every $i = 1, \ldots, n$. Assume that $(s', h) \models_R 3y.(3x . \psi) * \psi'[x \leftarrow x'] * \xi$, for some formulas $\psi, \psi'$ and $\chi$ such that $\chi = (\psi * \psi' * \chi)$, and $\xi$ are $\mathcal{T}$-formulas with $\xi \models \chi$, and $x \notin \text{fv}_R^T(\psi') \cup \text{fv}(\chi)$. Then there exists a store $s''$ coinciding with $s'$ on all variables not occurring in $y$, and disjoint heaps $h_1, h_2$ such that $h = h_1 \uplus h_2$, $(s'', h_1) \models_R 3y. \psi, (s'', h_2) \models_R \psi'[x \leftarrow x']$, and $(s'', h) \models \xi$. This entails that there exists a store $\hat{s}$ coinciding with $s''$ on all variables distinct from $x$ such that $(\hat{s}, h_1) \models_R \psi$. Since $x \notin \text{fv}_R^T(\psi')$, necessarily, $x' \notin \text{fv}_R^T(\psi'[x \leftarrow x'])$. By Lemma 57, since $s'(x') \not\in \text{loc}(h_2)$, this entails that $(\hat{s}, h_2) \models_R \psi'[x \leftarrow x']$. Now $(s'', h) \models \xi$ and by the application condition of the rule we have $x \notin \text{fv}(\xi)$, so that $(\hat{s}, h_2) \models_R \psi'[x \leftarrow x']$. This entails that $(\hat{s}, h) \models_R \chi$ because $\xi \models \chi$, again by the application condition of the rule. Hence we get $(\hat{s}, h) \models_R \psi * \psi' * \chi$, and $(s', h) \models_R \exists y. \exists x. (\psi * \psi'[x \leftarrow x']) = \exists y. \exists x. \chi$, which contradicts our assumption. We deduce that $(s', h)$ is a countermodel of the premise. $\square$

We introduce a measure on sequents. For every formula $\phi$ we denote by $\phi^h$ (resp. $\phi^T$) the formula obtained from $\phi$ by replacing every $T$-formula (resp. every spatial...
atom) by \( \mathsf{emp} \). We denote by \( N_3(\phi) \) the number of existential quantifications in the
prefix of \( \phi \). For all sequents \( S = \phi \vdash_R \psi_1, \ldots, \psi_n \), we denote by \( \mu(S) \) the tuple
\( \langle \text{size}(\phi^h), \{ \mu(\psi_i) \mid i \in \{1, \ldots, n\} \}, \phi^T \rangle \), where
\[
\mu'(\psi_i) = \left( \text{card}(\text{alloc}(\phi) \setminus \text{roots}_r(\psi_i)), \text{size}(\psi_i^h), N_3(\psi_i), \psi_i^T \right).
\]
The measure \( \mu \) is ordered by the lexicographic and multiset extensions of the natural ordering
on natural numbers and of the order \( \prec \) on \( T \)-formulas. We assume that all variables have the
same size and that the weights of the predicates in \( P_S \) are chosen in such a way that the size of
every predicate atom is strictly greater than that of all points-to atoms.

We say that a rule with conclusion \( \phi \vdash_R \Gamma \) decreases \( \mu \) if the inequality
\( \mu(\phi \vdash_R \Gamma) > \mu(\psi \vdash_R \Delta) \) holds for all the premises \( \psi \vdash_R \Delta \) of the rule.

**Lemma 59** All the rules, except \( \mathsf{UL} \), decrease \( \mu \).

**Proof** By an inspection of the rules (we refer to the definitions of the rules and of \( \mu \) for
notations). The result is straightforward for the rules \( \mathsf{W}, \mathsf{R}, \mathsf{D} \) and \( \mathsf{EH} \). The rule \( \mathsf{Sk} \) decreases
\( \text{size}(\phi^h) \), since it removes an existential quantifier. The rule \( \mathsf{HF} \) has no influence on the left-
hand side of the conclusion. This rule does not remove variables from \( \text{roots}_r(\psi_i) \), because it
only instantiates existential variables and by definition, no existential variable may occur
in \( \text{roots}_r(\psi_i) \). It eliminates at least one existential quantifier from the right-hand side of the
sequence since \( \text{dom}(\sigma) \neq \emptyset \), hence there is a \( \text{size}(\psi_i^h) \) that decreases strictly. Rule \( \mathsf{UR} \)
has no influence on the left-hand side of the conclusion and replaces a \( \mathsf{PU} \)-atom on the right-hand
side by the conjunction of a points-to atom and a \( T \)-formula. By the above assumption on
the weight of the predicate symbols, one of \( \text{size}(\psi_i^h) \) decreases (note that by the progress
condition the roots of \( \psi_i \) are unchanged, hence \( \text{card}(\text{alloc}(\phi) \setminus \text{roots}_r(\psi_i)) \) cannot increase).

Rule \( \mathsf{HD} \) decreases one of \( \text{card}(\text{alloc}(\phi) \setminus \text{roots}_r(\psi_i)) \), since by Proposition 48, we have
\( \text{roots}_r(\psi_i) = \{x\} \cup \text{roots}_r(\phi) \) for every \( \psi_i \in \text{split}_x(\phi) \) (furthermore, by the application
condition of the rule, there exists \( i = 1, \ldots, n \) such that \( x \in \text{alloc}(\phi) \setminus \text{roots}_r(\psi_i) \)).

For rule \( \mathsf{SC} \), we have \( \text{size}(\phi^h) < \text{size}((\phi \ast \psi_i)^h) \) and \( \text{size}(\psi_i^h) < \text{size}((\phi \ast \phi')^h) \) since both \( \phi \) and \( \phi' \)
contain a spatial atom. Rule \( \mathsf{ED} \) does not affect \( \text{alloc}(\phi) \setminus \text{roots}_r(\psi_i) \) or \( \text{size}(\psi_i^h) \), and
the rule reduces one of the \( N_3(\psi_i) \), since an existential quantifier is shifted into the scope of
a separating conjunction, by the application condition of the rule \( \psi_i' \neq \mathsf{emp} \). The rule \( \mathsf{TS} \) does
not affect \( \phi^h \) or the right-hand side and strictly decreases \( \phi^T \) by definition of the rule. The
rule \( \mathsf{TD} \) does not affect \( \phi^h \) and decreases one of the \( \psi_i^T \) (note that, since the formula \( \chi \) in
the rule is a \( T \)-formula, no new variables may be added in \( \text{alloc}(\psi_i) \)).

Let \( \mathsf{w}p(\phi) = 2 \cdot m + l \), where \( m \) (resp. \( l \)) denotes the number of occurrences of points-to
atoms (resp. of predicate atoms) in \( \phi \). Let \( \tau(\phi \vdash_R \Gamma) \) be the measure defined as follows:
\[
\tau(\phi \vdash_R \Gamma) = (N_h(\phi \vdash_R \Gamma), -\mathsf{w}p(\phi), \mu(\phi \vdash_R \Gamma)),
\]
where \( N_h(\phi \vdash_R \Gamma) \) denotes the least size of a countermodel of \( \phi \vdash_R \Gamma \), or \( \infty \) if \( \phi \vdash_R \Gamma \) is
valid (recall that, by Definition 8, the size of a structure is the cardinality of the domain of its
heap). \( \tau(\phi) \) is ordered using the lexicographic extension of the usual ordering on integers and
of the ordering on \( \mu(\phi \vdash_R \Gamma) \). Note that this order is well-founded on non-valid sequents,
because the number of points-to atoms in \( \phi \) cannot be greater than \( N_h(\phi \vdash_R \Gamma) \), since any
countermodel must satisfy the left-hand side of the sequent (hence \( \mathsf{w}p(\phi) \leq 2 \cdot N_h(\phi \vdash_R \Gamma) \)).

**Lemma 60** Let \( \Phi_i \vdash_R \Gamma_i \) be a rule application. If \( \phi \vdash_R \Gamma \) admits a countermodel,
then there exists \( i \in \{1, \ldots, n\} \) such that \( \tau(\phi_i \vdash_R \Gamma_i) < \tau(\phi \vdash_R \Gamma) \).
Proof By Lemma 54 the conclusions of all axioms are valid and do not admit any countermodel, thus the considered rule must admit at least one premise. If the rule is SC, then the result follows immediately from Lemma 56. Rule UL does not affect the least-size of countermodels by Lemma 55 and, since the rules are progressing, we have \( wp(p(x) \ast \phi) < wp(\phi_i \ast \phi) \). By Lemma 59, all the other rules decrease \( \mu \), and it is straightforward to check, by an inspection of the rules, that \( wp(\phi) \) cannot decrease (except for SC).

By Lemmas 55 and 58, there exists \( i \in \{1, \ldots, n\} \) such that \( Nh(\phi_i \vdash R_{\Gamma_i}) = Nh(\phi \vdash R_{\Gamma}) \), which entails that \( \tau(\phi_i \vdash R_{\Gamma_i}) < \tau(\phi \vdash R_{\Gamma}) \).

Theorem 61 (Soundness) Let \( R \) be an alloc-compatible pc-SID. If \( \phi \vdash R_{\Gamma} \) is the end-sequent of a (possibly infinite) proof tree, then \( \phi \vdash R_{\Gamma} \) is valid.

Proof Let \( \frac{\phi_1 \vdash R_{\Gamma_1} \ldots \phi_n \vdash R_{\Gamma_n}}{\phi \vdash R_{\Gamma}} \) be a rule application in the proof tree such that \( \phi \vdash R_{\Gamma} \) is not valid. We assume, w.l.o.g., that \( \tau(\phi \vdash R_{\Gamma}) \) is minimal. By Lemma 60, there exists \( i \) such that \( \tau(\phi_i \vdash R_{\Gamma_i}) < \tau(\phi \vdash R_{\Gamma}) \). Note that this implies that \( \phi_i \vdash R_{\Gamma_i} \) is not valid, as otherwise we would have \( Nh(\phi_i \vdash R_{\Gamma_i}) = \infty > Nh(\phi \vdash R_{\Gamma}) \), contradicting the fact that \( \tau(\phi_i \vdash R_{\Gamma_i}) < \tau(\phi \vdash R_{\Gamma}) \). Hence we get a contradiction about the minimality of \( \phi \vdash R_{\Gamma} \).

\[ \square \]

Remark 62 In general, with infinite proof trees, soundness requires some well-foundedness condition, to rule out infinite paths of invalid statements (see, e.g., [3]). In our case, no further condition is needed because every rule application either reduces the considered sequent into simpler ones or unfolds a predicate atom.

10 Completeness and Termination Results

10.1 The General Case

We first show that every valid sequent admits a (possibly infinite) proof tree. Together with Theorem 61, this result shows that the calculus can be used as a semi-decision procedure for detecting non-validity: the procedure will be “stuck” eventually in some branch, in the sense that one obtains a sequent on which no rule is applicable, iff the initial sequent is non-valid.

We shall assume that all the formulas in the considered root sequent are in prenex form. However, the sequents occurring within the proof tree will fulfill a slightly less restrictive condition, stated below:

Definition 63 A sequent \( \phi \vdash R_{\phi_1, \ldots, \phi_n} \) is quasi-prenex if \( \phi \) is in prenex form and all the formulas \( \phi_i \) that are not in prenex form are separating conjunctions of two prenex formulas.

It is easy to check that the premises of a rule with a quasi-prenex conclusion are always quasi-prenex.

Lemma 64 The rules Sk, HF, UL, HD are invertible. More precisely, if \((s, h)\) is a countermodel of one of the premises then it is also a countermodel of the conclusion.

Proof Each rule is considered separately:

**Sk** If \((s, h) \models R \bigwedge_{i=1}^n \phi[x \leftarrow x_i] \lor \phi[x \leftarrow x']\) and \((s, h) \not\models R_{\Gamma}\), then it is clear that \((s, h) \models R \exists x. \phi\), thus \((s, h)\) is a countermodel of \( \exists x. \phi \vdash R_{\Gamma} \).
Assume that \((s, h) \models \Gamma x \mapsto (y_1, \ldots, y_k) \ast \phi \) and \((s, h) \not\models \Gamma \exists u'. (x \mapsto (z_1, \ldots, z_k) \ast \psi) \sigma, \Gamma\). If \((s, h) \models \Gamma u. (x \mapsto (z_1, \ldots, z_k) \ast \psi)\), then there exist two disjoint heaps \(h_1, h_2\) and a store \(s'\) coinciding with \(s\) on all variables not occurring in \(u\) such that \(h = h_1 \cup h_2, (s', h_1) \models x \mapsto (z_1, \ldots, z_k), (s', h_2) \models \psi\). By the application condition of the rule, we have \(u \cap \{x, y_1, \ldots, y_k\} = \emptyset\), hence \(s(x) = s'(x)\). Since \((s, h) \models \Gamma x \mapsto (y_1, \ldots, y_k) \ast \phi\), we have \(h(s(x)) = (s(y_1), \ldots, s(y_k))\), thus \(s'(z_i) = s(y_i)\), for all \(i = 1, \ldots, k\). If \(z_i \in \text{dom}(\sigma)\) then by definition we have \(z_i \in \sigma = y_i\), so that \(s'(z_i) = s'(y_i) = s(y_i)\). We deduce that for all variables \(y, s'(y) = s'(y')\) and by Proposition 25, \((s', h_1) \models x \mapsto (z_1, \ldots, z_k) \ast \sigma\) and \((s', h_2) \models \psi \sigma\). We now show that no variable occurring in \((x \mapsto (z_1, \ldots, z_k) \ast \psi) \sigma\) but not in \(u'\) can occur in \(u\). Consider a variable \(z'\) that occurs in \((x \mapsto (z_1, \ldots, z_k) \ast \psi) \sigma\) but not in \(u'\). Then \(z'\) is of the form \(z\sigma\) and we distinguish two cases. If \(z \in \text{dom}(\sigma)\) then since \(u \cap \{x, y_1, \ldots, y_k\} = \emptyset\), we have the result. Otherwise, since \(u'\) is the vector of variables occurring in \(u\) but not in \(\text{dom}(\sigma)\), we deduce that \(z'\) cannot occur in \(u\) either. We deduce that \(s\) and \(s'\) coincide on all variables occurring in \((x \mapsto (z_1, \ldots, z_k) \ast \psi) \sigma\) but not in \(u'\). This entails that \((s, h) \models \Gamma \exists u'. (x \mapsto (z_1, \ldots, z_k) \ast \psi) \sigma\), which contradicts our assumption.

Assume that \((s, h)\) is a countermodel of \(\phi \ast \phi \models \Gamma\), for some \(i \in \{1, \ldots, n\}\), say, \(i = 1\). Then \((s, h) \models \Gamma \phi_1 \ast \phi\) and \((s, h) \not\models \Gamma\). We deduce that there exist disjoint heaps \(h_1, h_2\) such that \(h = h_1 \cup h_2, (s, h_1) \models \Gamma \phi_1\) and \((s, h_2) \models \phi\). By definition of the semantics of the predicate atom, since \(\alpha \models \phi_1\), we have \((s, h_1) \models \alpha \ast \phi\). Therefore, \((s, h)\) is a countermodel of \(\alpha \ast \phi \models \Gamma\).

Assume that \((s, h) \models \Gamma \phi\) and that \((s, h) \not\models \Gamma \psi_1, \ldots, \psi_n\). By the application condition of the rule, we have \(x \in \text{alloc}(\phi)\), thus, by Lemma 34, necessarily \(s(x) \in \text{dom}(h)\). Since \(s\) is injective by hypothesis, we deduce by Lemma 49 that \((s, h) \not\models \psi\).

We recall that a proof tree is irrational if it is infinite and it contains an infinite number of pairs of distinct subtrees, up to a renaming of variables.

**Theorem 65** If \(\mathcal{T}\) is closed under negation, then for all valid disjunction-free, prenex and alloc-compatible sequents \(\phi \models \Gamma\) there exists a (possibly irrational) proof tree with end-sequent \(\phi \models \Gamma\).

**Proof** It suffices to show that for every valid, disjunction-free, quasi-prenex sequent \(\phi \models \Gamma\) there exists a rule application with conclusion \(\phi \models \Gamma\) such that all the premises are valid; this ensures that an infinite proof tree can be constructed in an iterative way. Note that in particular this property holds if an invertible rule is applicable on \(\phi \models \Gamma\), since all premises are valid in this case. The sequence \(\Gamma\) can be written as \(\exists x_1. \psi_1, \ldots, \exists x_n. \psi_n\) where no \(\psi_i\) is an existential formula. If \(\phi\) contains an existential variable or a predicate atom then one of the rules \(SK\) or \(UL\) applies and both rules are invertible, by Lemma 64. Thus we may assume that \(\phi\) is of the form \(x_1 \mapsto y_1 \ast \cdots \ast x_m \mapsto y_m \ast \chi\), where \(\chi\) is a \(\mathcal{T}\)-formula. Since by Lemma 64 HD and HD are invertible, we may assume that \(\phi \models \Gamma\) is irreducible w.r.t. these rules (and also w.r.t. the axioms D, TC, R and EH). Note that for every model \((s, h)\) of \(\phi\), \(\text{card(\text{dom}(h)}) = m\). By irreducibility w.r.t. D and TC, we assume that the variables \(x_1, \ldots, x_m\) are pairwise distinct, and that \(\chi\) is satisfiable (on injective structures), so that \(\phi\) admits an injective model.

If \(m = 0\), then \(\phi\) is a \(\mathcal{T}\)-formula and every model of \(\phi\) must be of the form \((s, \emptyset)\). Let \(I\) be the set of indices \(i \in \{1, \ldots, n\}\) such that \(\psi_i\) is a \(\mathcal{T}\)-formula. Since \(\phi \models \Gamma\) is valid, for every model \((s, \emptyset)\) of \(\phi\), there exists an \(i \in \{1, \ldots, n\}\) such that \((s, \emptyset) \models \exists x_i. \psi_i\). By Proposition 13,
\( \psi_i \) cannot contain any spatial atom, and therefore \( i \in I \). Therefore, \( \phi \models \bigvee_{i \in I} \exists x_i. \psi_i \) and rule EH applies, which contradicts our assumption.

We now assume that \( m > 0 \). Since HD is not applicable on \( \phi \vdash_R \Gamma \), for every \( i = 1, \ldots, n \), and \( j = 1, \ldots, m \), \( \psi_i \) is such that \( x_j \in \text{roots}_\phi(\psi_i) \). Assume that there exists \( i \in \{1, \ldots, n\} \) such that \( \psi_i \) is of the form \( \alpha \star \phi'_i \), where \( \text{roots}_\phi(\alpha) = \{x_1\} \) and \( \alpha \) is a PU-atom. Then the rule UR applies on \( \alpha \star \phi'_i \), yielding a premise of the form \( \phi \vdash_R \gamma_1, \ldots, \gamma_p, \Gamma' \), where \( \Gamma' \) is the sequence of formulas \( \exists x_j. \psi_j \) with \( j \neq i \). We show that this premise is valid. Assume for the sake of contradiction that it admits a countermodel \((s, h)\). Then \((s, h) \models \phi \), and since \( \phi \vdash_R \Gamma \) is valid, necessarily, \((s, h) \models \Gamma \). But \((s, h) \not\models \Gamma' \), hence \((s, h) \models \exists x_i. \psi_i \) and there exists a store \( s' \) coinciding with \( s \) on all variables not occurring in \( x_i \) and has \( h_1, h_2 \) such that \( h = h_1 \cup h_2 \). This entails that \( \alpha \not\models \exists R \exists u. \psi \), and there exists a store \( s'' \) coinciding with \( s' \) on all variables not occurring in \( u \), and a substitution \( \sigma \) such that \( \text{dom}(\sigma) \subseteq u \) and \((s'', h_1) \models \psi \sigma \). W.l.o.g., we assume that \( \psi \) contains no predicate symbol. The formula \( \phi'_i \) contains at least \( m \) spatial atoms (one atom for each variable \( x_2, \ldots, x_m \)), hence \( \text{card}(\text{dom}(h_2)) \geq m - 1 \), since by Proposition 23, every such atom allocates at least one variable. We deduce that \( \text{card}(\text{dom}(h_1)) \leq \text{card}(\text{dom}(h)) - (m - 1) = 1 \). This entails by Proposition 23 that \( \psi \) contains at most one spatial atom, and since \( \psi \) does not contain any predicate symbol, this atom must be a points-to atom. Because of the progress condition, each unfolding of a predicate atom introduces exactly one points-to atom, thus the derivation from \( \alpha \) to \( 3u. \psi \) is of length 1, i.e., we have \( \alpha \not\models \exists R \exists u. \psi \). By definition of the rule UR, the formula \( \exists x_i. \exists u'. \psi \sigma \), occurs in \( \gamma_1, \ldots, \gamma_p \) (where \( u' \) is defined in the rule UR), thus \((s, h) \models \gamma_1, \ldots, \gamma_p \) and the premise is therefore valid.

We now assume that every formula \( \psi_i \) is of the form \( x_1 \mapsto z_i \star \psi'_i \). First suppose that \( z_i \cap x_i \neq \emptyset \), for some \( i = 1, \ldots, n \). If for all models \((\hat{s}, \hat{h})\) of \( \phi \) such that \( \hat{s} \) is injective, we have \((\hat{s}, \hat{h}) \not\models \exists x_i. \psi_i \), then we may apply the rule W, to remove the formula \( \exists x_i. \psi_i \) and obtain a premise that is valid. Thus we assume that \((\hat{s}, \hat{h}) \models \exists x_i. \psi_i \) for at least one model \((\hat{s}, \hat{h})\) of \( \phi \) such that \( \hat{s} \) is injective. Then we have \( \hat{h}(\hat{s}(x_1)) = \hat{s}(y_1) \), and there exists a store \( s' \) coinciding with \( \hat{s} \) on all variables not occurring in \( x_i \) such that \( \hat{h}(\hat{s}(x_1)) = \hat{s'}(z_i) \), hence \( \hat{s'}(z_i) = \hat{s}(y_1) \). Since \( \hat{s} \) is injective, this entails that there exists a substitution \( \sigma_{z_i} \) with domain \( z_i \cap x_i \) such that \( \sigma_{z_i}(z_i) = y_1 \). Since \( z_i \cap x_i \neq \emptyset \) we have \( \text{dom}(\sigma_{z_i}) \neq \emptyset \), thus the rule HF applies, and the proof is completed.

We now assume that \( z_i \cap x_i = \emptyset \), for all \( i = 1, \ldots, n \). If \( x_i \) is not empty then ED applies. Indeed, by letting \( \gamma = \psi_i \), \( m = 1 \), \( \chi = \xi = \text{emp} \), \( \psi = \psi'_i \) and \( \psi' = (x_1 \mapsto z_i) \), the application conditions of the rule are fulfilled because \( \text{emp} \models \top \text{emp} \) and \( f^R_v(x_1 \mapsto z_i) \cup fv(\xi) = \emptyset \). The application of ED shifts all variables \( x_i \) behind the formula \( \psi'_i \). Note that this application of the rule preserves the validity of the sequent, because \( x_1 \mapsto z_i \) contains no variable in \( x_i \).

We finally assume that every \( x_i \) is empty, and therefore that \( \Gamma \) is of the form \( x_1 \mapsto z_1 \star \psi'_1, \ldots, x_1 \mapsto z_n \star \psi'_n \). We distinguish two cases.

- \( m = 1 \). If there exists \( i \in \{1, \ldots, n\} \) such that \( \psi'_i \) is a \( T \)-formula (other than \( \text{emp} \)) then the rule TD applies, since \( T \) is closed under negation (by letting \( \chi = \psi'_1 \) and \( \chi' = \neg \psi'_i \)). Moreover, it is clear that the obtained sequent is valid. Now let \( i \in \{1, \ldots, n\} \) and suppose \( \psi'_i \) is not a \( T \)-formula. Then all the models of \( \psi_i \) are of size at least 2, and thus no model of \( \phi \) can satisfy \( \psi_i \), since \( \phi \) admits only models of cardinality \( m = 1 \). Similarly, if \( z_i \neq y_1 \), then for all injective models \((s, h)\) of \( \psi_i \) we have \( h(s(x_1)) = s(z_i) \neq s(y_1) \), thus \((s, h) \not\models \phi \). Since \( \phi \vdash_R \Gamma \) is valid and \( \phi \) is satisfiable, this entails that there exists \( i = 1, \ldots, n \) such that \( \psi'_i = \text{emp} \) and \( z_i = y_1 \). Then rule R applies.
\* \( m \geq 2 \). Let \( I \) be the set of indices \( i \) such that \( z_i = y_1 \) and \( \psi'_i \) contains at least one spatial atom. We apply the rule \( SC \), with the decompositions \( \phi = (x_1 \mapsto y_1) \ast (x_2 \mapsto y_2 \ast \cdots \ast x_m \mapsto y_m \ast \chi) \) and \( \psi_i = x_1 \mapsto z_1 \ast \psi'_1 \) and with the sets \( \{[i] \mid i \in I\} \) and \( I \).

It is clear that these sets satisfy the application condition of the rule: indeed, for every \( X \subseteq \{1, \ldots, n\} \), either \( X \cap I \neq \emptyset \) and then \( [i] \subseteq X \) for some \( i \in I \), or \( \{1, \ldots, n\} \setminus X \supseteq I \).

This yields the two premises: \( x_1 \mapsto y_1 \vdash x_1 \mapsto y_1 \) (more precisely this premise is obtained \( card(I) \) times) and \( \phi' \vdash \Gamma'' \), where \( \phi' = x_2 \mapsto y_2 \ast \cdots \ast x_m \mapsto y_m \ast \chi \) and \( \Gamma'' \) is the sequence of formulas \( \psi'_i \) for \( i \in I \). We prove that these premises are all valid. This is straightforward for the former one. Let \((s, h)\) be an injective model of \( \phi' \). Since the \( x_1, \ldots, x_m \) are distinct, the heaps \( h \) and \( h' = \{(s(x_1), s(y_1))\} \) are disjoint, and we have \((s, h \uplus h') \models \phi \), thus \((s, h \uplus h') \models x_1 \mapsto z_1 \ast \psi'_j \), for some \( j \in \{1, \ldots, m\} \).

This entails that \( s(z_j) = s(y_1) \) (hence \( y_1 = z_j \) since \( s \) is injective) and that \((s, h) \models \psi'_j \).

Since \( m \geq 2 \) necessarily \( h \neq \emptyset \), thus \( \psi'_j \) contains a spatial atom. Since \( y_1 = z_j \), this entails that \( j \in I \), hence the proof is completed.

The calculus is a decision procedure for sequents in which the rules defining the left-hand side terminate, if \( T \) is closed under negation and some decision procedure exists for checking entailments between \( T \)-formulas:

**Definition 66** A sequent \( \phi \vdash \Gamma \) is **left-terminating** iff for every predicate \( p \in \mathcal{P}_S \) such that \( \phi \supseteq \llbracket p \rrbracket \), we have \( p \not\supseteq \mathcal{R}p \).

**Lemma 67** Every proof tree with a left-terminating end-sequent is finite.

**Proof** Let \( \tau \) be a proof tree with a left-terminating end-sequent \( \phi \vdash \Gamma \). By hypothesis, \( \geq \llbracket \) is an order on the set of predicates \( p \) such that \( \phi \supseteq \llbracket p \rrbracket \). Thus we may assume in the definition of the measure \( \mu \) that the weight of any such predicate \( p \) is strictly greater than the size of every formula \( \psi \) such that \( p(x) \leftarrow \psi \) is a rule in \( R \). Then every application of a rule \( UL \) on an atom of the form \( p(y) \) with \( \phi \geq \llbracket p \rrbracket \) strictly decreases \( \mu \). Furthermore, since the rule \( UL \) is the only rule that can add new predicate atoms on the left-hand side of the sequent, it is easy to check that \( \phi \geq \llbracket q \rrbracket \) holds for every atom \( q(y) \) occurring on the left-hand side of a sequent in \( \tau \). Consequently, we deduce by Lemma 59 that all the rules decrease \( \mu \), which entails that \( \tau \) is finite, since \( \mu \) is well-founded.

**Theorem 68** If \( T \) is closed under negation, then for every valid disjunction-free, \( PU \)-free, alloc-compatible and left-terminating sequent \( \phi \vdash \Gamma \) there exists a finite proof tree of end-sequent \( \phi \vdash \Gamma \).

**Proof** This is an immediate consequence of Theorem 65 and Lemma 67.

10.2 Entailments Without Theories

In this section, we show that every valid, \( \emptyset \)-constrained sequent admits a rational proof tree. This shows that the calculus is a decision procedure for \( \emptyset \)-constrained sequents, and also, using the reduction in Theorem 42, for \( \{=, \neq\} \)-constrained sequents.

Before entering into technical details we provide a roadmap of the proof. Our goal is to devise a strategy that ensures termination on all valid \( \emptyset \)-constrained sequents. To this aim, we will use a principle similar to the one used in Theorem 65, which ensures that every valid sequent admits a proof, but we will refine the used strategy to ensure that the set of sequents occurring along every branch is finite (up to a renaming). The essential difference with the
proof of Theorem 65 is that we will apply the rule $SC$ eagerly, to obtain sequents that are as simple as possible. Using this rule, we will actually ensure that every sequent is eventually reduced to a sequent in which the left-hand side is an atom, before applying the unfolding rule $UL$ on this atom. This will prevent the size of the left-hand side from growing above that of the rules in $R$. This property turns out to be sufficient to ensure that the size of the sequents is bounded. Indeed, as we shall see, the formulas on the right-hand side of sequents that contain roots not occurring on the left-hand side are actually redundant and can thus be removed by weakening (see Definition 79 and Lemma 83). This entails that the number of roots is bounded, which in turns entails that the size of the formula is also bounded.

To show that such a decomposition is always possible, we first need to push some existential quantifications below separating conjunctions in the right-hand side of the considered sequents, using the rule $ED$, then apply $SC$ to eliminate separating conjunctions. Furthermore, to ensure that $SC$ indeed yields a valid premise, we also need to ensure that the decomposition of the right-hand side does not depend on the considered model of the left-hand side. This motivates Definition 69, which introduces a stronger notion of validity, formalizing this decomposition property. Then Lemma 77 shows that rule $ED$ can always be applied to obtain a sequent that is indeed strongly valid. Next, Lemma 78 shows that $SC$ may eventually be applied.

**Definition 69** A sequent $φ \vdash_R Γ$ is **strongly valid** relatively to a decomposition $φ = φ_1 ∗ φ_2$ (modulo AC) if, for every structure $(s, h_1 ∪ h_2)$ such that:

- $s$ is injective,
- $(s, h_i) =⇒_R φ_i$ and
- $x \in \text{alloc}(φ_i)$ whenever $s(x) \in \text{dom}(h_i)$ for $i = 1, 2$,

there exists a formula of the form $∃x.(ψ_1 ∗ ψ_2)$ in $Γ$ and a store $s'$ coinciding with $s$ on every variable not occurring in $x$ such that $(s', h_i) =⇒_R ψ_i$ for $i = 1, 2$.

Using Lemma 72 below, it is possible to show that if a $Θ$-constrained sequent is strongly valid then it is also valid, but the converse does not hold. The intuition is that in a strongly valid sequent, the decomposition of the heap associated with the separating conjunction on the left-hand side corresponds to a syntactic decomposition of a formula on the right-hand side. Note that the notion of strong validity is relative to a decomposition $φ = φ_1 ∗ φ_2$ which will always be clear from the context. For instance, the sequent $\exists s(x, y) * \exists s(y, z) =⇒_R \exists s(x, z)$ is valid (where $\exists s$ is defined as in Example 47), but not strongly valid, because the decomposition of the left-hand side does not correspond to any decomposition of the right-hand side. On the other hand, $\exists s(x, y) * \exists s(y, z) =⇒_R \exists u. (\exists s(x, u) * \exists s(u, z))$ is strongly valid.

**Definition 70** Let $h$ be a heap. For any mapping $η : L → L$ that is injective on $\text{dom}(h)$, we denote by $η(h)$ the heap $\{(η(ℓ_0), \ldots, η(ℓ_n)) | (ℓ_0, \ldots, ℓ_n) ∈ h\}$.

**Proposition 71** Let $(s, h)$ be a structure and let $η : L → L$ be a mapping that is injective on $\text{dom}(h)$. If $(s, h) =⇒_R ψ$ and $ψ$ is $Θ$-constrained then $(η ∘ s, η(h)) =⇒_R ψ$.

**Proof** We prove the result by induction on the satisfiability relation $=⇒_R$.

- **Assume** that $ψ = Φ^u_{β → α}[u^θ]$, $α = Φ^v_{γ}(γ * β')$, and there exists a store $s'$ coinciding with $s$ on all variables not occurring in $x$ and a substitution $σ$ such that $\text{dom}(σ) ⊆ x \cap \text{fv}(β')$, $β = β'σ$ and $(s', h) =⇒_R γσθ$. By the induction hypothesis we get $(η ∘ s', η(h)) =⇒_R γσθ$. But $η ∘ s'$ coincides with $η ∘ s$ on all variables not occurring in $x$, thus $(η ∘ s, η(h)) =⇒_R Φ^u_{β → α}[u^θ] = ψ$. 

[Springer]
• If $\psi = \emptyset$ then $h = \emptyset$, hence $\eta(h) = \emptyset$ and $(\eta \circ s, \eta(h)) \models_R \psi$.
• If $\psi = (x \mapsto (y_1, \ldots, y_k)$ and $h = \{(s(x), s(y_1), \ldots, s(y_k))\}$, then $\eta(h) = \{(\eta(s(x)), \eta(s(y_1)), \ldots, \eta(s(y_k)))\}$, so that $(\eta \circ s, \eta(h)) \models_R \phi$.

Assume that $\psi = \phi_1 \land \phi_2$ and that there exist disjoint heaps $h_1, h_2$ such that $h = h_1 \cup h_2$ and $(s, h) \models_R \phi_1$. By the induction hypothesis, we deduce that $(\eta \circ s, \eta(h_1)) \models_R \phi_1$. It is clear that $\eta(h) = \eta(h_1) \cup \eta(h_2)$, hence $(\eta \circ s, \eta(h)) \models_R \phi_1 \land \phi_2$.

• The proof is similar if $\psi = \phi_1 \lor \phi_2$ and $(s, h) \models_R \phi_1$, for some $i = 1, 2$.

Assume that $\psi = \exists x. \gamma$ and that there exists a store $s'$, coinciding with $s$ on all variables distinct from $x$, such that $(s', h) \models_R \gamma$. By the induction hypothesis, we have $(\eta \circ s', \eta(h)) \models_R \gamma$. Now, $\eta(s')$ coincides with $\eta(s)$ on all variables distinct from $x$, therefore $(\eta \circ s, \eta(h)) \models_R \psi$.$\Box$

Lemma 72 Let $\phi \vdash_R \Gamma$ be a $\emptyset$-constrained sequent, where $\phi$ is disjunction-free. If $\phi \vdash_R \Gamma$ admits a countermodel of $(s', h')$, then there exists a structure $(s, h)$ satisfying the following properties:

• $(s, h)$ is a countermodel of $\phi \vdash_R \Gamma$;
• for all variables $x$: $x \notin \text{alloc}(\phi) \implies s(x) \notin \text{dom}(h)$;
• there exists an injective mapping $\eta$ such that $s' = \eta \circ s$ and $h' = \eta(h)$.

Proof Since $\phi$ is $\emptyset$-constrained, we have $\phi \equiv_{\text{R}} \exists x. x_1 \mapsto y_1 \ast \cdots \ast x_n \mapsto y_n$ and $h' = \{(s''(x_1), s''(y_1)) \mid 1 \leq i \leq n\}$, for some store $s''$ coinciding with $s'$ on all variables not occurring in $x$. We assume, w.l.o.g., that $\mathcal{L} \setminus \text{img}(s')$ is infinite. Modulo $\alpha$-renaming, we may also assume that $\text{fv}(\phi) \cap x = \emptyset$. Since $R$ is alloc-compatible, we have

$$\text{alloc}(\phi) = \text{alloc}(\exists x. x_1 \mapsto y_1 \ast \cdots \ast x_n \mapsto y_n)$$

$$= \{x_1, \ldots, x_n\} \cap \text{fv}(\phi)$$

$$= \{x_1, \ldots, x_n\} \setminus x.$$

Let $s$ be a store mapping all variables to pairwise distinct locations not occurring in $\text{loc}(h) \cup \text{img}(s')$. Note that $s$ is injective by construction. Let $\hat{s}$ be the store mapping each variable $x \in x$ to $s''(x)$ and coinciding with $s$ on all other variables. It is clear that $\hat{s}$ is injective, thus the heap $h = \{(\hat{s}(x_1), \hat{s}(y_1)) \mid 1 \leq i \leq n\}$ is well-defined, and by construction, $(\hat{s}, h) \models_R x_1 \mapsto y_1 \ast \cdots \ast x_n \mapsto y_n$, so that $(s, h) \models_R \phi$. Consider the function $\eta$ that maps each location $s(x)$ to $\hat{s}'(x)$ and leaves all other locations unchanged. Note that $\eta$ is injective because both $s$ and $s'$ are injective, and

by definition, $s' = \eta \circ s$. Let $x$ be a variable. If $x \in x$ then $\hat{s}(x) = s''(x)$, hence $(\eta \circ \hat{s})(x) = \eta(s''(x))$. By definition of $s$, we have $s''(x) \notin \text{img}(s)$, hence $\eta(s''(x)) = s''(x)$ and we deduce that $(\eta \circ \hat{s})(x) = \eta(s)'(x)$ and again $(\eta \circ \hat{s})(x) = \eta(s)'(x)$. Thus, we deduce that $h' = \eta(h)$. If $(s, h) \models_R \Gamma$ then by Proposition 71 we have $(s', h') \models_R \Gamma$, which contradicts the definition of $(s', h')$. Thus, $(s, h)$ is a countermodel of $\phi \vdash_R \Gamma$, and by construction, for all variables $x, x \notin \text{alloc}(\phi) \implies s(x) \notin \text{dom}(h)$.$\Box$

Lemma 73 Let $\phi \vdash_R \Gamma$ be a $\emptyset$-constrained and non-valid sequent, where $\phi$ is disjunction-free. Let $s$ be an injective store and $L$ be an infinite set of locations such that $L \cap \text{img}(s) = \emptyset$. Then $\phi \vdash_R \Gamma$ admits a countermodel $(s, h)$ with $\text{dom}(h) \subseteq L \cup \text{alloc}(\phi)$.

Proof By hypothesis, there exists an injective store $s'$ and a heap $h'$ such that $(s', h') \models_R \phi$ and $(s', h') \not\models_R \Gamma$. By Lemma 72, we may assume that for all variables $x \in \text{fv}(\phi) \cup \text{fv}(\Gamma)$, if $x \notin \text{alloc}(\phi)$ then $s'(x) \notin \text{dom}(h')$. Let $\eta: \mathcal{L} \rightarrow L \cup \text{sv}((\phi) \cup \text{fv}(\Gamma))$ be a bijective mapping such that $\eta(s'(x)) = s(x)$, for all variables $x \in \text{fv}(\phi) \cup \text{fv}(\Gamma)$. Such a mapping necessarily
exists. Indeed, since \( L \) is infinite, there exists a bijection between \( L \setminus s'(fv(\phi) \cup fv(\Gamma)) \) and \( L \); and since both \( s \) and \( s' \) are injective, there exists a bijection between \( s'(fv(\phi) \cup fv(\Gamma)) \) and \( s(fv(\phi) \cup fv(\Gamma)) \). Let \( h = \eta(h') \). By construction, for every location \( \ell \in \text{dom}(h) \), we have \( \eta^{-1}(\ell) \in \text{dom}(h') \), and if \( \ell \not\in L \), then necessarily \( \ell = s(x) \) and \( \eta^{-1}(\ell) = s'(x) \), for some \( x \in fv(\phi) \cup fv(\Gamma) \). Since \( s'(x) \in \text{dom}(h') \), by \((\dagger)\) we have \( x \in \text{alloc}(\phi) \), and therefore \( \text{dom}(h) \subseteq L \cup s(\text{alloc}(\phi)) \).

We now show that \( (s, h) \) is a countermodel of \( \phi \vdash_R \Gamma \). Since \( (s', h') \vdash_R \phi \), we deduce by Proposition 71 that \( (\eta \circ s', h) \vdash_R \phi \). Moreover, since \( \eta \circ s' \) and \( s \) agree on all variables in \( fv(\phi) \) by construction, this entails that \( (s, h) \vdash_R \phi \). Assume that \( (s, h) \vdash_R \Gamma \). Applying Proposition 71 with \( \eta^{-1} \), we get \( (\eta^{-1} \circ s, \eta^{-1}(h)) \vdash_R \Gamma \). But we have \( \eta^{-1}(h) = \eta^{-1}(\eta(h')) = h' \), thus \( (\eta^{-1} \circ s, h') \vdash_R \Gamma \). Since \( \eta^{-1} \circ s \) and \( s' \) agree on all variables in \( fv(\Gamma) \) by construction, we deduce that \( (s', h') \vdash_R \Gamma \), contradicting our initial assumption. Thus \( (s, h) \not\vdash_R \Gamma \) and \( (s, h) \) is a countermodel of \( \phi \vdash_R \Gamma \).

\[ \square \]

**Definition 74** A path from \( \ell \) to \( \ell' \) in a heap \( h \) is a nonempty sequence of locations \( (\ell_1, \ldots, \ell_n) \) such that \( \ell_1 = \ell \), \( \ell_n = \ell' \) and for every \( i \in \{1, \ldots, n - 1\} \), \( \ell_{i+1} \in h(\ell_i) \).

The following proposition states an important property of pc-SIDs.

**Proposition 75** Let \( p(x_1, \ldots, x_n) \) be a predicate atom and let \( (s, h) \) be a model of \( p(x_1, \ldots, x_n) \). If \( \ell = s(x_1) \) and \( \ell' \in \text{loc}(h) \) then there exists a path from \( \ell \) to \( \ell' \) in \( h \).

**Proof** The proof is by induction on \( \text{card}(\text{dom}(h)) \). We have \( p(x_1, \ldots, x_n) \vdash_R \exists x. \phi \) for some quantifier-free formula \( \phi \), and there exists a store \( s' \) coinciding with \( s \) on all variables not occurring in \( x \), such that \( (s', h) \models \phi \). If \( \ell = \ell' \) then the result is immediate (a path of length 1 always exists from any location \( \ell \) to \( \ell \)). Assume that \( \ell \neq \ell' \). By the progress condition, \( \phi \) is of the form \(:\phi \colon (y_1, \ldots, y_k) \cdotp p_1(z_{11}, \ldots, z_{1m_1}) \cdotp \cdots \cdotp p_m(z_{m1}, \ldots, z_{m,m_m}) \cdotp \chi \cdotp \). Thus there exist disjoint heaps \( h_1, \ldots, h_m \) such that \( h = \{(s(x_1), s'(y_1), \ldots, s'(y_k))\} \cup h_1 \cup \cdots \cup h_m \) and \( (s, h_1) \models p_i(z_{1i}, \ldots, z_{mi}) \) for \( i = 1, \ldots, m \). If \( s(y_i) = \ell' \) for some \( i = 1, \ldots, k \) we have \( \ell' \in h(s(x_1)) = h(\ell) \) and the proof is completed. Otherwise, \( \ell' \) cannot occur in \( s(x_1), s'(y_1), \ldots, s'(y_k) \), hence \( \ell' \) necessary occurs in \( \text{loc}(h_i) \) for some \( i = 1, \ldots, m \), since \( \ell' \in \text{loc}(h) \) by hypothesis. By the induction hypothesis, there exists a path in \( h_i \) from \( s'(z_{1i}) \) to \( \ell' \), and by the connectivity condition \( s'(z_{1i}) \in \{s'(y_1), \ldots, s'(y_k)\} \), hence \( \ell' \) necessary occurs in \( \text{loc}(h_i) \). Thus there exists a path in \( h \) from \( \ell \) to \( \ell' \).

\[ \square \]

**Lemma 76** Let \( \phi_1 \star \phi_2 \vdash_R \Gamma \) be a valid prenex \( \emptyset \)-constrained sequent, where \( \phi_i \neq \emptyset \) for \( i = 1, 2 \), \( \phi_1 \) and \( \phi_2 \) are quantifier-free and disjunction-free. If the rule \( \text{HD} \) is not applicable on \( \phi_1 \star \phi_2 \vdash_R \Gamma \) and \( \Gamma \) is in prenex form then \( \phi_1 \star \phi_2 \vdash_R \Gamma \) is strongly valid.

**Proof** Assume that there exists a structure \( (s, h_1 \cup \cup h_2) \) such that \( (s, h_1) \models \phi_1 \) for \( i = 1, 2 \) and \( \forall x \in \text{fv}(\phi_1). s(x) \in \text{dom}(h_1) \implies x \in \text{alloc}(\ell) \). Since \( \phi_1 \star \phi_2 \vdash_R \Gamma \) is valid, and \( \Gamma \) is in prenex form, \( \Gamma \) contains a formula \( \psi = \exists x. (\psi_1 \cdotp \cdots \cdotp \psi_n) \) such that \( (s, h_1 \cup \cup h_2) \models \psi \) and each formula \( \psi_i \) (\( i \in \{1, \ldots, n\} \)) is either a points-to atom or a PNU-atom. Thus there exist a store \( s' \), coinciding with \( s \) on all variables not occurring in \( x \) and disjoint heaps \( h_1', \ldots, h_n' \) such that \( (s', h_i') \models_R \psi_i \) for \( i = 1, \ldots, n \) and \( h_1 \star h_2 = h_1' \star \cdots \star h_n' \). By \( \alpha \)-renaming, we assume that \( \text{fv}(\phi_1 \star \phi_2) \cap x = \emptyset \), so that \( s \) and \( s' \) coincide on \( \text{fv}(\phi_1 \star \phi_2) \). Note that by hypothesis \( \phi_1 \) and \( \phi_2 \) are separating conjunctions of atoms.

Assume that one of the heaps \( h_i' \) for \( i = 1, \ldots, n \) is such that \( h_i' \not\subseteq h_1 \) and \( h_i' \not\subseteq h_2 \). This entails that \( \psi_i \) cannot be a points-to atom because \( \text{card}(\text{dom}(h_i')) = 1 \) in this case, hence
that \( \psi_i \) is a PU-atom. By Proposition 37, \( s'(\text{roots}(\psi_i)) \subseteq \text{dom}(h'_i) \subseteq \text{dom}(h_1) \cup \text{dom}(h_2) \).
We assume by symmetry that \( s'(\text{roots}(\psi_i)) \subseteq \text{dom}(h_1) \), the other case is similar. Let \( \ell \in \text{dom}(h'_1) \setminus \text{dom}(h_1) \).
By Proposition 75, there exists a sequence of locations \( \ell_1, \ldots, \ell_m \) with \( \{\ell_1\} = s'(\text{roots}(\psi_i)) \), \( \ell_m = \ell \) and \( \ell_{i+1} \in h'_i(\ell_i) \) for \( i = 1, \ldots, m \).
We may assume, by considering the location \( \ell \) associated with the minimal sequence ending outside of \( \text{dom}(h_1) \),
that \( \ell_i \in \text{dom}(h_1) \) for all \( i < m \). This entails that \( \ell \in h_1(\ell_{m-1}) \) so that \( \ell \in \text{loc}(h_1) \setminus \text{dom}(h_1) \),
and by Lemma 15 we deduce that \( \ell = s(x) \) for some \( x \in \text{fv}(\phi_1) \).
Furthermore, since \( s(x) \in \text{dom}(h_2) \) we must have \( x \in \text{alloc}(\phi_2) \) by \( (\dagger) \).
Since the rule HD is not applicable, the variable \( x \) must occur in \( \text{roots}(\psi_1 \ast \cdots \ast \psi_n) \).
This entails that there exists \( j \) such that \( s(x) = h_1(\phi_j) \) by Proposition 37.
But \( x \) cannot be the main root of \( \psi_j \), because \( h_1 \) and \( h_2 \) are disjoint, \( s(x) \in \text{dom}(h_2) \) and \( s'(\text{roots}(\psi_i)) \subseteq \text{dom}(h_1) \), hence \( i \neq j \).
This contradicts the fact that \( h'_1, \ldots, h'_n \) are disjoint.

Therefore, every heap \( h'_i \) is a subheap of either \( h_1 \) or \( h_2 \). By regrouping the formulas \( \psi_i \) such that \( h'_i \subseteq h_j \) in a formula \( \psi'_j \) (for \( j = 1, 2 \)), we get a decomposition of \( \exists x.(\psi_1 \ast \cdots \ast \psi_n) \) of the form: \( \exists x.(\psi'_1 \ast \psi'_2) \), where \( s'(\psi_j) \vdash_R \psi'_j \) for \( j = 1, 2 \).
Thus \( \phi_1 \ast \phi_2 \vdash_R \Gamma \) is strongly valid.

\[ \square \]

Lemma 77 Let \( \phi_1 \ast \phi_2 \vdash_R \exists y.\exists x.\gamma, \Gamma \) be a strongly valid, \( \emptyset \)-constrained and quasi-prenex sequent, with \( \phi_i \neq \text{emp} \) for \( i = 1, 2 \).
There exists an application of the rule ED with conclusion \( \phi_1 \ast \phi_2 \vdash_R \exists y.\exists x.\gamma, \Gamma \), such that the premise is strongly valid and quasi-prenex.

\[ \text{Proof} \]
We apply the rule ED where \( \{\gamma_1, \ldots, \gamma_n\} \) is the set of all formulas satisfying the conditions of the rule, the formulas \( \phi \) and \( \phi' \) in the application condition of ED are prenex formulas and \( \chi = \text{emp} \).
Recall from the rule definition that \( \{x_1, \ldots, x_n\} = \text{fv}(\phi) \cap \text{fv}(\phi') \).
It is straightforward to verify that the premise is quasi-prenex.
Note that both \( \psi \) and \( \psi' \) are \( \emptyset \)-constrained by hypothesis.
Let \( (s, h_1 \cup h_2) \) be a structure such that for all \( i = 1, 2 \),
\( (s, h_i) \vdash_R \phi_i \) and \( \forall x \in \text{fv}(\phi_i), s(x) \in \text{dom}(\phi_i) \implies x \in \text{alloc}(\phi_i) \).

We have \( h_1 \neq \emptyset \), because \( \phi_i \neq \text{emp} \).
We show that the right-hand side of the premise contains a formula satisfying the conditions of Definition 69.
Since \( \phi_1 \ast \phi_2 \vdash_R \exists x.\gamma, \Gamma \) is strongly valid there exist a formula of the form \( \exists x.(\psi_1 \ast \psi_2) \) in \( \exists x.\gamma, \Gamma \), and a store \( s' \) coinciding with \( s \) on every variable not occurring in \( x \) such that \( (s', h_i) \vdash_R \psi_i \) for \( i = 1, 2 \).
Since the considered sequent is quasi-prenex, \( \psi_1 \) and \( \psi_2 \) are in prenex form.
If \( \exists x.(\psi_1 \ast \psi_2) \) occurs in \( \Gamma \) then the proof is completed.
Otherwise we have \( y, x = x = x \) and \( \gamma = \psi_1 \ast \psi_2 \).
If \( s'(x) = s(x_j) \), for some \( j = 1, \ldots, n \), then \( (s', h_j) \vdash_R \psi_j[x \leftarrow x_j] \),
and \( \psi_1[x \leftarrow x_j] \ast \psi_2[x \leftarrow x_j] = \gamma[x \leftarrow x_j] \) occurs on the right-hand side of the premise, by definition of the rule ED.
Thus, the result also holds in this case.
Now assume that \( s'(x) \neq s(x_j) \), for all \( j = 1, \ldots, n \).
By Lemma 15, since \( (s, h_j) \vdash_R \phi_i \), we have \( \text{loc}(h_1) \setminus \text{dom}(h_2) \subseteq s(\text{fv}(\phi_i)) \), for \( i = 1, 2 \).
Since \( s'(x) \neq s(x_j) \), for all \( j = 1, \ldots, n \),
\( s'(x) \notin s(\text{fv}(\phi_i)) \) hence \( s'(x) \notin \text{loc}(h_1) \setminus \text{dom}(h_1) \).
If \( s'(x) \in \text{loc}(h_1) \cap \text{loc}(h_2) \), then we would have \( s'(x) \in \text{dom}(h_1) \cap \text{dom}(h_2) \),
contradicting the fact that \( h_1 \) and \( h_2 \) are disjoint.
This entails that \( s'(x) \notin \text{loc}(h_1) \cap \text{loc}(h_2) \).

We assume, by symmetry, that \( s'(x) \notin \text{loc}(h_2) \). Then by Lemma 57, we deduce that \( (s'', h_2) \vdash_R \psi_2 \), for every store \( s'' \) coinciding with \( s' \) on all variables distinct from \( x \).
We also have \( (s'', h_1) \vdash_R \exists x.\psi_1 \), since \( (s', h_1) \vdash_R \psi_1 \).
By definition of the rule, the right-hand side of the premise contains a formula \( \exists y.(\exists x.\psi_1) \ast \psi_2[x \leftarrow x'] \),
hence the conditions of Definition 69 are fulfilled (note that \( \psi_2 \) cannot be emp, since \( h_2 \neq \emptyset \).

\[ \square \]

Lemma 78 Let \( \phi_1 \ast \phi_2 \vdash_R \Gamma \) be a strongly valid and \( \emptyset \)-constrained sequent.
If rules ED and D do not apply on \( \phi_1 \ast \phi_2 \vdash_R \Gamma \) then rule SC applies on \( \phi_1 \ast \phi_2 \vdash_R \Gamma \), with a valid premise.
Let $\Delta$ be the subsequence of formulas in $\Gamma$ that are not separated conjunctions, so that $\Gamma$ is of the form $\psi_1^1 \ast \psi_1^2, \ldots, \psi_n^1 \ast \psi_n^2, \Delta$. By definition of the notion of strong validity, if $(s, h_1 \uplus h_2) \models_{\mathcal{R}} \phi_1 \ast \phi_2$ and $\forall x \ (s(x) \in \text{dom}(h_1) \implies x \in \text{alloc}(\phi_1))$, then $\Gamma$ contains a formula of the form $\exists x. (\psi_1 \ast \psi_2)$ such that $(s', h_1) \models \psi_1$ and $(s', h_2) \models \psi_2$, where $s'$ coincides with $s$ on all variables not occurring in $x$. By Lemma 77, none of the formulas in $\Gamma$ may be existentially quantified, since otherwise rule ED applies. Thus $x$ is empty, $\exists x. (\psi_1 \ast \psi_2) = \psi_1 \ast \psi_2$ does not occur in $\Delta$ and there exists $i = 1, \ldots, n$ such that $\psi_1 = \psi_i^1$ and $\psi_2 = \psi_i^2$.

Let $I_1, \ldots, I_m$ denote the inclusion-minimal subsets of $\{1, \ldots, n\}$ such that for all $j = 1, \ldots, m$, $\phi_i \models_{\mathcal{R}} \bigvee_{c \in I_j} \psi_c^1$ is valid. Similarly, let $J_1, \ldots, J_l$ denote the inclusion-minimal subsets of $\{1, \ldots, n\}$ such that for all $j = 1, \ldots, l$, $\phi_2 \models_{\mathcal{R}} \bigvee_{c \in J_j} \psi_c^2$ is valid. If the rule SC applies with such sets, then, by construction all the premises are valid, hence the proof is completed. Otherwise, by the application condition of the rule, there exists a set $X$ such that $X \not\supset I_i$ for all $i = 1, \ldots, n$ and $\{1, \ldots, n\} \not\supset J_j$ for all $j = 1, \ldots, l$. This entails that $\phi_1 \models_{\mathcal{R}} \bigvee_{c \in X} \psi_c^1$ is not valid, since otherwise $X$ would contain one of the $I_i$ which are inclusion-minimal by construction. Similarly, $\phi_2 \models_{\mathcal{R}} \bigvee_{c \in \{1, \ldots, n\} \setminus X} \psi_c^2$ is not valid because otherwise $\{1, \ldots, n\} \setminus X$ would contain one of the $J_j$. Therefore, there exist injective stores $s_i$ and heaps $h_i$ for $i = 1, 2$, such that $(s_i, h_i) \models_{\mathcal{R}} \phi_i$, $(s_1, h_1) \not\models_{\mathcal{R}} \bigvee_{c \in X} \psi_c^1$ and $(s_2, h_2) \not\models_{\mathcal{R}} \bigvee_{c \in \{1, \ldots, n\} \setminus X} \psi_c^2$. Let $s$ be an injective store such that $\mathcal{L} \setminus \text{img}(s)$ is infinite, and let $L_1, L_2$ be disjoint infinite subsets of $\mathcal{L} \setminus \text{img}(s)$. By applying Lemma 73 twice on $h_1, h_2$ with $L_1$ and $L_2$ respectively, we obtain two heaps $h'_1$ and $h'_2$ such that:

- $\text{dom}(h'_i) \subseteq L_i \cup \text{sat}(\text{alloc}(\phi_i))$ for $i = 1, 2$,
- $(s, h'_i) \models_{\mathcal{R}} \phi_i$ for $i = 1, 2$,
- $(s, h'_1) \not\models_{\mathcal{R}} \bigvee_{c \in X} \psi_c^1$,
- $(s, h'_2) \not\models_{\mathcal{R}} \bigvee_{c \in \{1, \ldots, n\} \setminus X} \psi_c^2$.

In particular, since $L_1$ and $L_2$ are disjoint, we have $\text{dom}(h'_1) \cap \text{dom}(h'_2) \subseteq \text{sat}(\text{alloc}(\phi_1)) \cap \text{sat}(\text{alloc}(\phi_2))$. Assume that $\text{dom}(h'_1) \cap \text{dom}(h'_2)$ is nonempty, and contains a location $\ell$. Then for $i = 1, 2$, there exists a variable $x_i \in \text{fv}(\phi_i)$ such that $\ell = s(x_i)$. Since $s$ is injective, necessarily $x_1 = x_2$, and $\text{alloc}(\phi_1) \cap \text{alloc}(\phi_2) \neq \emptyset$; this entails that the rule $D$ is applicable, which contradicts the hypothesis of the lemma.

We deduce that $h'_1$ and $h'_2$ are disjoint. Then we have $(s, h'_1 \uplus h'_2) \models_{\mathcal{R}} \phi_1 \ast \phi_2$ and for all $c \in \{1, \ldots, n\}$, either $(s, h'_1) \not\models_{\mathcal{R}} \psi_c^1$ or $(s, h'_2) \not\models_{\mathcal{R}} \psi_c^2$. Furthermore, if $s(x) \in \text{dom}(h'_1)$, then $s(x) \in \text{sat}(\text{alloc}(\phi_1))$, thus $x \in \text{alloc}(\phi_1)$ because $s$ is injective. This contradicts the fact that $\phi_1 \ast \phi_2 \models_{\mathcal{R}} \Gamma$ is strongly valid.

\[ \square \]

**Definition 79** Let $\phi \models_{\mathcal{R}} \psi$, $\Gamma$ be a sequent. The formula $\psi$ is root-redundant if there exists a variable $x$ such that either $x \in \text{roots}_s(\psi)$ and $x \notin \text{alloc}(\phi)$, or $x \in \text{roots}_t(\psi)$ and $x \notin \text{fv}(\phi)$. This formula is variable-redundant if there exist two injective substitutions $\sigma, \theta$ such that $\psi \sigma \in \Gamma \theta$ and $(\text{dom}(\sigma) \cup \text{dom}(\theta)) \cap \text{fv}(\phi) = \emptyset$.

**Example 80** Consider the sequent $p(x, y) \vdash q(y) \ast \Phi_{p(x, y)}^{z \leftarrow x}[x, z]$, with $\text{alloc}(p) = \{1\}$. Formulas $q(y)$ and $\Phi_{p(x, y)}^{z \leftarrow x}[x, z]$ are both root-redundant. This is the case because we have $y \in \text{roots}_s(q(y))$, $y \notin \text{alloc}(p(x, y))$, $z \in \text{roots}_t(\Phi_{p(x, y)}^{z \leftarrow x}[x, z])$ and $z \notin \text{alloc}(p(x, y))$.

Now, consider the sequent $p(x, y) \vdash q(x, z) \ast q(x, u)$. The formula $q(x, z)$ is variable-redundant, since, by letting $\sigma = \{z \leftarrow u\}$ and $\theta = \text{id}$, we have $q(x, z) \sigma = q(x, u) \theta$. 
Proposition 81 If $(s, h) \models \phi$ then $s(\text{roots}_l(\phi)) \subseteq \text{loc}(h)$.

Proof The proof is by induction on the satisfiability relation. We only handle the case where $\phi = \Phi^u_{\beta \cdot q} [u\theta]$, the other cases are straightforward. By definition, there exists a formula $\exists x. (\beta^* \psi)$, a substitution $\sigma$ with $\text{dom}(\theta) \subseteq x$ and a store $s'$ coinciding with $s \circ \sigma$ on all variables not occurring in $x$ such that $\alpha \models^+ R \exists x. (\beta^* \psi)$, $\beta^* \sigma = \beta$ and $(s', h) \models \psi \theta$. Let $x \in \text{roots}_l(\phi)$. By definition, $\beta$ contains a predicate atom of the form $p(x, y)$, thus $\beta^*$ contains a predicate atom of the form $p(x', y')$, with $x' = x$. This predicate atom must be introduced in the derivation $\alpha \models^+ R \exists x. (\beta^* \psi)$ by unfolding some predicate atom $q(u, v)$.

By the progress and connectivity condition this entails that $\psi$ contains a points-to atom of the form $u \mapsto (\ldots, x', \ldots)$. Since $(s', h) \models \psi \sigma$ this entails that $s'(x') \in \text{loc}(h)$, thus $s(x) \in \text{loc}(h)$.

Proposition 82 If $(s \circ \sigma, h) \models_R \phi$ then $(s, h) \models_R \phi \sigma$.

Proof The proof is by induction on the satisfiability relation. We only handle the case where $\phi = \Phi^u_{\beta \cdot q} [u\theta]$. By definition, there exists a formula $\exists x. (\beta^* \psi)$, a substitution $\theta$ with $\text{dom}(\theta) \subseteq x$ and a store $s'$ coinciding with $s \circ \sigma$ on all variables not occurring in $x$ such that $\alpha \models^+ R \exists x. (\beta^* \psi)$, $\beta^* \theta = \beta$ and $(s', h) \models \psi \theta$. By $\alpha$-renaming, we assume that $x \cap (\text{dom}(\sigma) \cup \text{img}(\sigma)) = \emptyset$. Let $\delta$ be the store such that $\delta(x) = s(x)$ if $x \notin x$ and $\delta(x) = s'(x)$ if $x \in x$. By construction we have $s' = s \circ \sigma$.

By the induction hypothesis, we deduce that and $(s, h) \models \psi \theta$. Furthermore, $\beta^* \theta \sigma = \beta \sigma$, and $\alpha \sigma \models^+ R \exists x. (\beta^* \psi) \sigma = \exists x. (\beta \sigma \psi \sigma)$. Thus $(s, h) \models_R \phi \sigma$.

Lemma 83 Let $\phi \models_R \psi$, $\Gamma$ be a valid $\theta$-constrained disjunction-free sequent. If $\psi$ is root-unsatisfiable, root-redundant or variable-redundant, then $\phi \models_R \Gamma$ is valid.

Proof We consider the three cases separately.

- If $\psi$ is root-unsatisfiable, then $\psi$ is unsatisfiable, thus every model of $\psi$, $\Gamma$ is also a model of $\Gamma$. Therefore $\phi \models_R \Gamma$ if and only if $\phi \models_R \psi$, $\Gamma$.

- Assume that $\psi$ is root-redundant and let $(s, h)$ be an injective model of $\phi$. We show that $(s, h) \models_R \phi$. By Lemma 72, applied on any sequent $\phi \models_R \phi'$ such that $\phi'$ is unsatisfiable, there exists an injective mapping $\eta$ and a model $(s', h')$ of $\phi$ such that $s = \eta \circ s'$, $h = \eta(h')$ and $s'(x) \in \text{dom}(\eta)$.

Let $s''$ be a store coinciding with $s'$ on all the variables in $\text{fv}(\phi)$ and such that $s''(y) \notin \text{loc}(h')$, for all $y \notin \text{fv}(\phi)$. Since $s''$ and $s'$ coincide on $\text{fv}(\phi)$, necessarily $(s'', h') \models \phi$.

Also, if $s''(x) \in \text{dom}(h') \subseteq \text{loc}(h')$, then necessarily $x \in \text{fv}(\phi)$, so that $s''(x) = s'(x)$ and $x \in \text{alloc}(\phi)$. Therefore, $s''(x) \in \text{dom}(h') \implies x \in \text{alloc}(\phi)$. We distinguish two cases.

- Assume that $(s'', h') \not\models \Gamma$. Since $\phi \models_R \psi$, $\Gamma$ is valid we have $(s'', h') \models \psi$, $\Gamma$ and necessarily, $(s'', h') \models \psi$. If there exists $x \in \text{roots}_l(\psi)$ such that $x \notin \text{alloc}(\phi)$, then by Proposition 37 we have $s''(x) \in \text{dom}(h')$, which contradicts the above implication. Otherwise, since $\psi$ is root-redundant, there exists a variable $x \in \text{roots}_l(\psi) \setminus \text{fv}(\phi)$.

Since $x \in \text{roots}_l(\psi)$, we have $s''(x) \in \text{loc}(h')$ by Proposition 81, which contradicts the definition of $\phi$ because $x \notin \text{fv}(\phi)$.

- Assume that $(s'', h') \models \Gamma$.

Let $\eta'$ be the function mapping all locations of the form $s''(y)$ to $s'(y)$ and leaving all other locations unchanged. By definition, we have $s = \eta \circ \eta' \circ s'$ and for all $y \in V$, if $s''(y) \in \text{loc}(h')$ then we must have $y \in \text{fv}(\phi)$, so that $s''(y) = s'(y)$. We
deduce that $\eta'$ is the identity on every location in $loc(h')$ and that $\eta(\eta'(h')) = h$. By Proposition 71, $(\eta \circ \eta' \circ s', \eta(\eta'(h))) \models \Gamma$, i.e., $(s', h) \models \Gamma$.

- Assume that $\psi$ is variable-redundant and let $(s, h)$ be a model of $\phi$, where $s$ is injective. We show that $(s, h) \models \Gamma$. Let $s'$ be an injective store coinciding with $s$ on every variable in $fv(\phi)$ and such that $s'(x) \notin loc(h)$ for every $x \notin fv(\phi)$ (‡). It is clear that $(s', h) \models \Gamma \phi$, thus $(s', h) \models \Gamma \psi$, $\Gamma$ is valid by hypothesis. If $(s', h) \models \Gamma \psi$, then by Lemma 57 we deduce that $(s, h) \models \Gamma \psi$, and the proof is completed. Otherwise, $(s', h) \models \Gamma \psi$. By hypothesis, there exist injective substitutions $\sigma$ and $\theta$ such that $\psi(\sigma) \in \Gamma \theta$ and $(\text{dom}(\sigma) \cup \text{dom}(\theta)) \cap fv(\phi) = \emptyset$. By (‡), we have $(s' \circ \sigma)(x) = s'(x)$ for all $x$ such that $s'(x) \in loc(h)$, since in this case $x$ cannot be in $\text{dom}(\sigma)$. By Lemma 57, we deduce that $(s' \circ \sigma, h) \models \Gamma \psi$, thus $(s', h) \models \Gamma \psi \sigma$ by Proposition 82.

Since $\psi \sigma \in \Gamma \theta$, we deduce that $(s', h) \models \Gamma \theta$, hence (as $\theta$ is injective) $(s' \circ \theta, h) \models \Gamma$. If $x$ is a variable then $(s' \circ \theta)(x) \in loc(h)$ then by definition of $s'$, we have $\theta(x) \in fv(\phi)$. Since $\text{dom}(\theta) \cap fv(\phi) = \emptyset$, we deduce that $\theta(x) = x$, and $(s' \circ \theta)(x) = s'(x) = s(x)$. By Lemma 57, we conclude that $(s, h) \models \Gamma \theta$.

\textbf{Theorem 84 (Termination)} For all valid, $\emptyset$-constrained, prenex, disjoint-free, alloc-compatible and $\psi$-free sequents $\phi \vdash \Gamma$ there exists a rational proof tree with end-sequent $\phi \vdash \Gamma$.

\textit{Furthermore, the size of the proof tree is at most $O(2^{c \cdot n^3})$, where $c$ is a constant and $N = \text{width}(\phi \vdash \Gamma)$}.

\textbf{Proof} Let $A = \max\{\kappa + 1, \#(p) \mid p \in P_S\}$ and $P = \max\{|p| \mid p \in P_S\}$. We assume, w.l.o.g., that all the predicate in $P_S$ occur in $R$, so that $A = O(N)$ and $P = O(N)$. The proof tree is constructed in a similar way as in the proof of Theorem 65, except that rule $UL$ will be applied only when the left-hand side of the sequent contains a unique spatial atom. We show that there is some rule that is applicable to every valid sequent, in such a way that all the premises are valid. All the inference rules that are axioms are applied with the highest priority, whenever possible. Afterwards, the rule $W$ is applied to remove from the right-hand side of the sequents all the formulas that are root-unsatisfiable, root-redundant or variable-redundant; Lemma 83 guarantees that the validity of the sequent is preserved. Rules $SK$ and $HD$ are then applied as much as possible. If the process terminates, then we eventually obtain a sequent of the form $\psi \vdash \Delta$, such that $\psi$ is quantifier-free (by irreducibility w.r.t. $SK$) and $\text{alloc}(\psi) \subseteq \text{roots}(\psi')$, for every $\psi' \in \Delta$ (by irreducibility w.r.t. $HD$).

We now distinguish several cases, depending on the form of $\psi$. First if $\psi$ is a points-to atom, then by Theorem 65, there exists a rule application yielding valid premises, and the applied rule cannot be $UL$, since by hypothesis $\psi$ contains no predicate atom. Otherwise, if $\psi$ is a single predicate atom then we apply the rule $UL^{10}$ and validity is preserved thanks to Lemma 64. Otherwise, $\psi$ must be a separating conjunction. By Lemma 76, $\psi \vdash \Delta$ is strongly valid. Rule $ED$ is then applied as much as possible. Note that, by Lemma 77, if the prefix of a formula in $\Delta$ contains an existential variable then there exists an application of $ED$ such that the obtained sequents are strongly valid and quasi-prenex. If $ED$ is not applicable then by Lemma 78, rule $SC$ necessarily applies.

We now prove that the constructed proof tree is rational. To this purpose, we analyze the form of the sequents occurring in it. Consider any sequent $\gamma \vdash \gamma_1, \ldots, \gamma_n$ occurring in the proof tree and assume that none of the formulas $\gamma_1, \ldots, \gamma_n$ is root-unsatisfiable, root-redundant or variable-redundant. We prove the following invariant.

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\(^{10}\) Note that in the proof of Theorem 65 rule $UL$ is applied when $\psi$ contains a predicate atom. However, this strategy is not applicable here because it may produce an infinite proof tree.
Invariant 85. For every $i = 1, \ldots, n$, the following properties hold:

1. If an existential variable is the main root of a $\text{PU}$-atom in $\gamma_i$, then it is the main root of an atom occurring in the initial sequent $\phi \vdash \Gamma$. Moreover, existential variables cannot be auxiliary roots.

2. Both $\text{roots}_r(\gamma_i)$ and $\text{roots}_l(\gamma_i)$ are sets (i.e., contain at most one occurrence of each variable), and $\text{roots}(\gamma_i) \subseteq \text{fv}(\gamma)$.

3. If $x \in \text{roots}_l(\gamma_i)$ and $x \notin \text{roots}_r(\gamma_i)$ then $x \notin \text{alloc}(\gamma)$.

4. The number of variables occurring in $\gamma$ is at most $\max(N_{\text{init}}, N_R)$, where $N_{\text{init}} = \text{card}(\text{fv}(\phi) \cup \text{fv}(\Gamma))$ and $N_R$ is the maximal number of free or bound variables occurring in a rule in $\mathcal{R}$.

Proof. We assume, w.l.o.g., that $\gamma \vdash \gamma_1, \ldots, \gamma_n$ is the first sequent not satisfying these properties, along some (possibly infinite) path from the root.

1. It is straightforward to check, by inspection of the rules, that no rule can add atoms with existentially quantified roots in the premise: the only rule that can add new atoms to the right-hand side of a sequent is $\text{HD}$ (by applying the function $\text{split}()$ defined in Sect. 7), and the roots of these atoms must be free variables. Then the proof follows from the fact that the initial sequent is $\text{PU}$-free. Note that no rule can rename the existentially quantified variables occurring in the sequents; the only variables that are renamed are those occurring in the inductive rules.

2. The inclusion $\text{roots}(\gamma_i) \subseteq \text{fv}(\gamma)$ follows from the fact that the $\gamma_i$ are not root-redundant. If $\text{roots}_r(\gamma_i)$ contains two occurrences of the same variable for some $i \in \{1, \ldots, n\}$, then $\gamma_i$ is root-unsatisfiable, contradicting our assumption. Assume that a multiset $\text{roots}_l(\gamma_i)$ contains two occurrences of the same variable $x$. The initial sequent is $\text{PU}$-free, hence contains no auxiliary root, and the only rule that can add new auxiliary roots to a sequent is $\text{HD}$. Thus assume that $\gamma \vdash \gamma_1, \ldots, \gamma_n$ is obtained from a sequent of the form $\gamma \vdash \gamma'_1, \Gamma'$, by applying rule $\text{HD}$ on $\gamma'_1$ with variable $x$. By the application condition of the rule, necessarily $x \in \text{alloc}(\gamma)$. Furthermore, since by definition of the splitting operation, $\text{HD}$ introduces exactly one auxiliary root in $\gamma_i$, and $x$ occurs twice in $\text{roots}_l(\gamma'_1)$, necessarily $x \in \text{roots}_r(\gamma'_1)$. By Property 3 of the invariant, applied to the sequent $\gamma \vdash \gamma'_1, \Gamma'$ which satisfies the invariant, since $x \in \text{alloc}(\gamma)$, necessarily $x \in \text{roots}_r(\gamma'_1)$. But $\text{HD}$ also introduces an atom with main root $x$, which entails that $x$ occurs twice in $\text{roots}_r(\gamma'_1)$, hence that $\gamma_i$ is root-unsatisfiable, yielding a contradiction.

3. The only rules that can affect the roots of the right-hand side of the sequent are $\text{ED}$, $\text{HD}$, and $\text{SC}$.

- $\text{ED}$ may generate new roots by instantiating an existential variable with a free variable, however by Property 1, existential variables cannot be auxiliary roots, hence Property 3 is preserved.
- $\text{HD}$ adds a new auxiliary root $x$ to the right-hand side of the sequent. However, for each such atom, $\text{HD}$ also adds an atom with main root $x$, hence the property is preserved.
- $\text{SC}$ may remove main roots from the right-hand side of the sequent. If some premise of $\text{SC}$ does not fulfill Property 3 of the invariant, then, using the notations of the rule, there exists an index $i \in \{1, \ldots, n\}$ such that either $x \in \text{roots}_l(\psi_i)$, $x \notin \text{roots}_r(\psi_i)$ and $x \in \text{alloc}(\phi)$, or $x \in \text{roots}_l(\psi'_i)$, $x \notin \text{roots}_r(\psi'_i)$ and $x \in \text{alloc}(\phi')$. We assume by symmetry that the former assertion holds. Since $\text{alloc}(\phi) \subseteq \text{alloc}(\phi \ast \phi')$, we have $x \in \text{alloc}(\phi \ast \phi')$, which entails that $x \in \text{roots}_r(\psi'_i)$, since the conclusion satisfies Property 3. By irreducibility w.r.t. $\text{D}$, $\text{alloc}(\phi) \cap \text{alloc}(\phi') = \emptyset$, thus $x \notin \text{alloc}(\phi')$. 

\[ \square \]
This entails that the formula \( \psi'_i \) is root-redundant in all sequents with left-hand side \( \phi' \), contradicting our assumption.

4. Property 4 stems from the fact that no rule may add variables to the left-hand side of a sequent, except for \( \text{UL} \). However, \( \text{UL} \) is always applied on a sequent with a left-hand side that is an atom, which entails that the number of variables occurring on the left-hand side after any application of the rule is bounded by \( N_R \).

Properties 2, 1 and 4 in Invariant 85 entail that the number of roots in every formula \( \gamma_i \) is at most \( 2 \cdot \max(N_{\text{init}}, N_R) + N_\exists \), where \( N_\exists \) denotes the number of existential variables in the end-sequent. Indeed, a free variable may occur at most twice as a root, once as an auxiliary root and once as a main root, and an existential variable may occur at most once as a main root. Note that \( N_{\text{init}} = \mathcal{O}(N) \), \( N_R = \mathcal{O}(N) \) and \( N_\exists = \mathcal{O}(N) \). Thus the number of (free or existential) variables in \( \gamma_i \) is bounded by \( A \cdot \max(N_{\text{init}}, N_R) + N_\exists \). Hence we may assume (up to a renaming of variables) that the total number of variables occurring in the considered sequent is at most \( A \cdot \max(N_{\text{init}}, N_R) + N_\exists \). Since every such variable may be represented by a word of length \( \log(A \cdot \max(N_{\text{init}}, N_R) + N_\exists) \). Let \( \gamma'_1, \ldots, \gamma'_n \) be formulas obtained from \( \gamma_1, \ldots, \gamma_n \) by replacing all the free variables not occurring in \( fv(\gamma) \) by some unique fixed variable \( u \). The size of every expression of the form \( p(\mathbf{x}) \) occurring in \( \gamma'_1, \ldots, \gamma'_n \) is bounded by \( P = A \cdot \log(A \cdot \max(N_{\text{init}}, N_R) + N_\exists) = \mathcal{O}(N^2) \), thus the size of the formulas \( \gamma'_i \) is bounded by \( \mathcal{O}(N^3) \). By Proposition 31, the size of the set \( \{\gamma'_1, \ldots, \gamma'_n\} \) is therefore at most \( \mathcal{O}(2^{d \cdot N^3}) \) for some constant \( d \). If the size of the sequence \( \gamma_1, \ldots, \gamma_n \) is greater than \( \mathcal{O}(2^{d \cdot N^3}) \), then there must exist distinct formulas \( \gamma_i \) and \( \gamma_j \) such that \( \gamma'_i = \gamma'_j \), i.e., \( \gamma_i \) and \( \gamma_j \) are identical up to the replacement of variables not occurring in \( fv(\gamma) \). But then \( \gamma_i \) and \( \gamma_j \) would be variable-redundant, which contradicts our assumption.

Therefore, \( \Sigma_{i=1}^n \text{size}(\gamma_i) = \mathcal{O}(2^{d \cdot N^3}) \).

Necessarily \( \text{size}(\gamma) = \mathcal{O}(N) \), because no rule can increase the size of the left-hand side of the sequents above \( N \): indeed, the only rule that can add new atoms to the left-hand side is \( \text{UL} \), and this rule is applied only on single atoms, which entails that the left-hand side of the obtained sequent is of the same size as the right-hand side of one of the rules in \( \mathcal{R} \).

We deduce that the size of the sequents occurring in the proof tree is at most \( \mathcal{O}(2^{d \cdot N^3}) \) for some constant \( d \), and by Proposition 30, there are at most \( \mathcal{O}(2^{c \cdot N^3}) \) distinct sequents (for some constant \( c \), up to a renaming of variables). This entails that the constructed proof tree is rational.

\[ \square \]

### 11 Discussion

Due to the high expressive power of pc-SIDs, the conditions ensuring termination (Theorems 68 and 84) are necessarily restrictive. In the light of the undecidability result in [13], one cannot hope for more. Indeed, the entailment problem is proven to be undecidable for most theories, even for theories with a very low expressive power (such as Presburger arithmetic with only the successor function). Theorem 65 shows that, even if the conditions are not satisfied, the calculus is still useful as a semi-decision procedure to detect non-validity. The complexity of the procedure for \( \emptyset \)-constrained formulas cannot be improved significantly since entailment checking is 2-EXPTIME-hard [11]. The high complexity of entailment testing is unsatisfactory in practice. We thus plan to investigate fragments of inductive rules for which the devised calculus yields an efficient decision procedure. As emphasized by the lower-bound result in [11], which relies on very simple data structures and by the fact that
language inclusion is already EXPTIME-complete for tree automata [30], there is no hope that this can be achieved by restricting only the shape of the structures: it is also necessary to strongly restrict the class of inductive rules (forbidding for instance overlapping rules). We will also try to identify classes of entailments for which the procedure terminates for nonempty theories, under reasonable conditions on the theory.

Another problem that can be of practical interest is to extract countermodels of non-valid (or irreducible) entailments. Finally, we plan to extend the proof procedure in order to solve bi-abduction problems, a generalized form of abduction which plays a central role for the efficiency and scalability of program analysis algorithms [5].

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Declarations

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