Abstract
Recently, the problem of allocating one resource per agent with initial endowments (house markets) has seen a renewed interest: indeed, while in the general domain Top Trading Cycle (Shapley and Scarf, 1974) is known to be the only procedure guaranteeing Pareto-optimality, individual rationality, and strategy proofness (Ma, 1994), the situation differs in single-peaked domains. Bade (2019) presented the Crawler, an alternative procedure enjoying the same properties (with the additional advantage of being implementable in obviously dominant strategies); while Damamme et al. (2015) showed that allowing mutually beneficial swap-deals among the agents was already enough to guarantee Pareto-optimality. In this paper we significantly deepen our understanding of this decentralized procedures: we show in particular that the single-peaked domains happen to be “maximal” if one wishes to guarantee this convergence property. Interestingly, we also observe that the set of allocations reachable by swap-deals always contains the outcome of the Crawler. To further investigate how these different mechanisms compare, we pay special attention to the average and minimum rank of the resource obtained by the agents in the outcome allocation. We provide theoretical bounds on the loss potentially induced by these procedures with respect to these criteria, and complement these results with an extensive experimental study which shows how different variants of swap dynamics behave. In fact, even the simplest dynamics exhibit very good results, and it is possible to further guide the process towards our objectives, if one is ready to sacrifice a bit in terms of decentralization. On our way, we also show that a simple variant of the Crawler allows to check efficiently that an allocation is Pareto-optimal in single-peaked domains.

Keywords: Distributed Artificial Intelligence; Fair allocation; Decentralized resource allocation

1. Introduction
Fair allocation is a research agenda that has been extensively studied in the recent years. It is a particularly dynamic field in both artificial intelligence (Brandt et al., 2016, Part II) and economics (Moulin, 2018). It investigates the issue of allocating a set of objects to a set of agents while taking into account their preferences. Two distinct settings are usually considered, depending on whether the objects are divisible or not. In this work, we focus on the latter, assuming that every object is indivisible hence can only be allocated entirely and to a single agent. The setting we are interested in is even more restricted as we also assume there is exactly one resource per agent. This setting is usually referred to as house markets (Shapley and Scarf, 1974).

In this basic but very common setting, the celebrated top trading cycle (TTC) algorithm (Shapley and Scarf, 1974) is known to satisfy numerous key desirable properties: Pareto-efficiency, strategy-proofness and individual rationality (Shapley and Scarf, 1974; Roth, 1982). This procedure is in fact the only one to satisfy these three properties in house markets with strict preference orders, i.e. when agents express strict linear orders over the resources (Ma, 1994).

An interesting way to get around Ma’s result is to consider different preference domains. Aziz and De Keijzer (2012) extended the preference domain of strict linear orders by allowing for indifferences between the items. It is also possible to investigate relevant domain restrictions such as the single-peaked domain (Black, 1948; Arrow, 1951). In this domain, preferences are decreasing the further you go from the most preferred resource of each agent. This domain restriction is commonly encountered in various real-world problems where some characteristics of the resources inherently define a common axis: political trends of candidates in elections (Bruner and Lackner, 2015), storage capacities for hard-disks, sizes for clothes... In politics for instance, preferences usually decrease when considering candidates further away on the political spectrum, from the most preferred candidate. In resource allocation, single-peaked preferences are also particularly relevant. Take for instance agents who are looking for houses in a street which has a metro station at one of its end, and a bike rental platform at the other end. The agents’ preferences are likely to be single-peaked depending on their favorite means of transportation. In this paper, we will focus on this natural preference domain.

Recently, Bade (2019) has also considered house markets under single-peaked preferences and introduced a new allocation procedure called the Crawler that is efficient, strategy-proof and individually rational. It is more-
over strictly different from TTC, hence weakening Ma’s result under single-peaked preferences. In addition to satisfy the same properties as TTC, the Crawler is also easier to understand in the sense of obviously dominant strategies (Li, 2017). Bade (2019) proved that the Crawler can be implemented using obviously dominant strategies while TTC can not be, even on the single-peaked domain.

Nevertheless, both the TTC algorithm and the Crawler suffer from some drawbacks of centralized procedures: they rely on a benevolent central authority to proceed. Depending on the context, either more stringent guarantees in terms of strategy-proofness, or more decentralization, may be desirable.

Centralized procedures may be perceived as less fair, in particular their outcomes are less acceptable by the agents (Leventhal, 1980; Thibaut and Walker, 1975; Van den Bos et al., 1997). Moreover these procedures require rather advanced communication and coordination protocols. Indeed both procedures can potentially involve long cycles of resource reallocations between agents. In some real life scenarios such cycles may not be acceptable, for example in the kidney exchange problem they are impossible to implement while taking into account the time constraints (kidney exchange programs usually restrict exchange cycles to sizes two or three (Roth et al., 2005)). The probability of failure also increases with the size of the cycle. This issue is orthogonal to the question of decentralization: centralized procedures can be designed (and are, in the case of kidney exchanges) so as to restrict the size of cycles involved.

An alternative approach is based on decentralized dynamics based on local exchanges between agents. In this model, long cycles pose real challenges (Rosenschein and Zlotkin, 1994) as they involve distributed coordination among numerous agents in order to exchange the resources. In a related version of this paper, Damamme et al. (2015) presented and analysed the simplest possible local deals: the swap-deals. There, agents randomly meet each other, in a pairwise fashion, and exchange their resources if they both benefit from it. The process iterates until a stable state, an equilibrium, is reached. On the single-peaked domain, Damamme et al. (2015) showed that the swap-deal procedure is efficient: every allocation stable with respect to swap-deals is Pareto optimal. A similar link between local deals and global properties in house markets was recently made by Kondratenv and Nesterov (2019), who established that a matching is popular if and only if no (majority) improving exchanges between 3 agents exist.

1.1. Contribution

This paper focuses on house markets with single-peaked preferences. In this setting we first discuss the Crawler procedure (Bade, 2019). Inspired by the Crawler we introduce the Diver, a procedure that checks whether an allocation is Pareto-optimal or not, which is asymptotically optimal in terms of communication complexity.

Next, we investigate swap-deal procedures. We significantly extend Damamme et al. (2015) results on the swap-deal procedure by showing that the single-peaked domain is a maximal domain on which the procedure is efficient. Although not all allocations Pareto-dominating the initial allocation can be reached by swap-deals, it turns out, interestingly, that outcome of the Crawler always can. We then investigate the cost of such decentralized procedure, when the objective is either to maximize the average or minimum rank of the resource obtained by the agents. We show that the bounds found in Damamme et al. (2015) for more general settings are also valid in the single-peaked domain.

Finally, we explore experimentally the different procedures discussed in this paper. These experiments highlight that the swap-deal procedure performs particularly well for these objectives. We further show that very few number of swap-deals are needed before reaching a stable allocation emphasizing the interest for such procedures.

1.2. Outline of the paper

We first present some related works (Section 2) and we introduce our model and the different criteria used to assess both the allocations and the procedures in Section 3. Section 4 presents the state-of-art allocation procedures discussed in this paper. The Diver procedure is explored in Section 5 while the swap-deal procedure is investigated in Sections 6 and 7. Section 8 deals with the price of decentrality. The experimental analysis is presented in Section 9. Finally a discussion over our results and a conclusion are given in Section 10.

2. Related works

The theory of fair division has been introduced by Steinhaus (1948) who defined “the Problem of Fair Division”. This seminal work has led to an extensive literature in economics (see Young (1995); Moulin (2004) or Moulin (2018) for complete surveys). Most of these works focus on settings with divisible resources and/or allow for monetary transfers between the agents. Henry (1970) and Shapley and Scarf (1974) are among the first ones to consider indivisible resources. While in the former an additional divisible resource plays the role of money, in the latter all the resources are assumed to be indivisible.

Fair division was introduced later in computer science and artificial intelligence, through the study of the cake cutting problems (Brams and Taylor, 1996): a divisible and heterogeneous resource is to be divided among a finite set of agents. A wide literature has been developed since then, enriching the economists’ approach by mainly focusing on the computational issues (Chevaleyre et al., 2006). Another line of research has paid particular attention to settings involving indivisible resources as it poses more technical issues. We refer the reader to the following surveys on this literature: Nguyen et al. (2013), Bouveret et al. (2016) and Lang and Rothe (2016).

In the present paper we focus on the model defined by Shapley and Scarf (1974), called house market or assignment problem, in which there are exactly as many indivisible resources as agents and no money. Shapley and Scarf (1974) defined the top-trading cycle (TTC) procedure which has been extensively studied (Roth, 1982) and shown to be unique when preferences are strict (Ma, 1994).
Bogomolnaia and Moulin (2001) renewed the interest for the assignment problem in the economic literature by investigating randomized procedures. Subsequent work by Kasajima (2013) introduced single-peaked preferences in their setting. Hougaard et al. (2014) later considered deterministic and probabilistic solutions under single-peaked preferences.

Aziz and De Keijzer (2012) defined a set of Pareto-efficient procedures generalizing the TTC algorithm when allowing for indifferences, which however include procedures that are not strategy-proof. Plaxton (2013) and Saban and Sethuraman (2013) independently proposed general frameworks for efficient and strategy-proof generalization of the TTC procedure with indifferences.

Bade (2019) explored another direction by restricting preferences to single-peaked domains. In this case she presents the Crawler that satisfies the same properties as TTC hence overcoming Ma’s result (Ma, 1994) on the single-peaked domain.

Following this line of work, we assume single-peaked preferences in this paper. This domain of preferences has been introduced by Black (1948) and Arrow (1951). It has been more specifically studied in voting theory and now are a very common domain of preferences (Moulin, 1991; Elkind et al., 2017). Numerous works and now are a very common domain of preferences. It has been more specifically studied in voting theory and now are a very common domain of preferences (Moulin, 1991; Elkind et al., 2017).

The rank would correspond to the Borda score an agent gives to a resource. Any subset \( D \subseteq R \) is then called a preference domain. Among individually rational agents. Endriss and Maudet (2005) and Aziz et al. (2016) investigated different complexity problems in this setting. Endriss et al. (2006) and Chevaleyre et al. (2010) respectively characterized the class of deals and the class of preferences required to reach socially optimal allocations. Chevaleyre et al. (2007) focused on reaching efficient and envy-free allocations. Similar procedures were also introduced in the area of two-sided matching (Roth and Vate, 1990; Ackermann et al., 2011).

The idea of using swap deals was explored for instance by Abbassi et al. (2013), who studied barter exchange networks. Gourvès et al. (2017) and Saffidine and Wilczynski (2018) studied dynamics of swap-deals by considering an underlying social network constraining the possible interactions of the agents, and focusing on complexity issues. These results were recently extended by Bentert et al. (2019).

3. Preliminaries

We start by presenting the basic components of our model and the different fairness and efficiency concepts discussed in this paper.

3.1. The model

Let us consider a set \( N = \{a_1, \ldots, a_n\} \) of agents and a set \( R = \{r_1, \ldots, r_n\} \) of resources. An allocation \( \pi = (\pi_{a_1}, \ldots, \pi_{a_n}) \) is a vector of \( R^n \) whose components \( \pi_{a_i} \in R \) represent the resource allocated to \( a_i \).

Agents are assumed to express their preferences over the resources through complete linear orders. Using ordinal preferences is a popular trend in fair division (Brams et al., 2003; Bouveret et al., 2010; Aziz et al., 2015). We denote \( \succ_i \) agent \( a_i \)'s preferences, where \( r_1 \succ_i r_2 \) means that \( a_i \) prefers \( r_1 \) over \( r_2 \).

A preference profile \( L = \{\succ_i| a_i \in N\} \) is then a set of linear orders, one for each agent. For a given linear order \( \succ \), we use \( \text{top}(\succ) \) to denote its top-ranked item: \( \forall r \in R, \text{top}(\succ) \succeq r \). Similarly, \( \text{snd}(\succ) \) refers to the second most preferred resource in \( \succ \). With a slight abuse of notation we will write \( \text{top}(a_i) \) and \( \text{snd}(a_i) \) to refer to \( \text{top}(\succ_i) \) and \( \text{snd}(\succ_i) \). When it is not clear from the context we will subscript these notations to specify the resource set, for instance \( \text{top}_R(a_i) \) is the most preferred resource for agent \( a_i \) among resources in \( R \subseteq R \). Given resource \( r \in R \) and an agent \( a_i \in N \), we use \( \text{rank}_{a_i}(r) \) to refer to the rank of \( r \) in \( \succ_i \), that is \( n \) for \( \text{top}(a_i) \), \( n - 1 \) for \( \text{snd}(a_i) \) and so on?

An instance is a vector \( I = (N, R, L, \pi^0) \) composed of a set of agents \( N \), a set of resources \( R \), a preference profile \( L \) and an initial allocation \( \pi^0 \).

In some settings, natural properties of the agents’ preferences can be identified, thus restricting the set of possible preference orderings. The notion of preference domain formalizes these restrictions. For a set of resources \( R \), we denote by \( L_R \) the set of all linear orders over \( R \). Any subset \( D \subseteq L_R \) is then called a preference domain.

\footnote{The rank would correspond to the Borda score an agent gives to a resource.}
We say that an instance $I = (\mathcal{N}, \mathcal{R}, L, \pi^0)$ is defined over a preference domain $D$ if $L \subseteq D$ i.e. the preferences of the agents are selected inside $D$.

3.2. Single-peaked preferences

Under single-peaked preferences, the agents are assumed to share a common axis $\prec$ over the resources and individual rankings are defined with respect to this axis.

**Definition 1.** Let $\mathcal{R}$ be a set of resources and $\prec$ a linear order (i.e. the axis) over $\mathcal{R}$. We say that a linear order $\succ$ is single-peaked with respect to $\prec$ if we have:

$$\forall (r_1, r_2) \in \mathcal{R}^2\text{ s.t. } r_2 \prec r_1 \prec \text{top}(\succ), \quad \text{or, } \text{top}(\succ) \prec r_1 \prec r_2 \implies r_1 \succ r_2.$$ 

In other words, $\succ$ is single-peaked over $\prec$ if $\succ$ is decreasing on both the left and the right sides of top$(\succ)$, where left and right are defined by $\prec$.

For a given linear order $\prec$, we call $\mathcal{SP}_\prec$ the set of all the linear orders single-peaked with respect to $\prec$:

$$\mathcal{SP}_\prec = \{\succ \in \mathcal{LR} \mid \succ \text{ is single-peaked w.r.t. } \prec\}. $$

A preference domain $D$ is called single-peaked if and only if $D \subseteq \mathcal{SP}_\prec$ for a given $\prec$. An instance $I$ is said to be single-peaked if it is defined over a single-peaked preference domain.

Ballester and Haeringer (2011) provided a characterization of single-peaked domains. In particular, they gave a necessary condition for a domain to be single-peaked: it should be worst-restricted (Sen, 1966).

**Definition 2.** An instance $I = (\mathcal{N}, \mathcal{R}, L, \pi^0)$ is worst-restricted if for any triplet of resources $(r_x, r_y, r_z) \in \mathcal{R}^3$, one of them is never ranked last in the profile $L$ restricted to these three resources.

**Proposition 1** (Ballester and Haeringer (2011)). If an instance is single-peaked then it is worst-restricted.

Let us illustrate the single-peaked domain with a simple example.

**Example 1.** Let us consider the following linear orders defined over 3 resources.

\[
\begin{align*}
\succ_1: & \quad r_1 \succ r_2 \succ r_3 \\
\succ_2: & \quad r_3 \succ r_2 \succ r_1 \\
\succ_3: & \quad r_2 \succ r_3 \succ r_1 \\
\succ_4: & \quad r_2 \succ r_4 \succ r_3 \succ r_1
\end{align*}
\]

One can check that these orders define a single-peaked preference profile with respect to $\prec$ defined as: $r_1 \prec r_2 \prec r_3$. In fact these orders exactly correspond to $\mathcal{SP}_\prec$. However, let us consider the following preferences.

\[
\begin{align*}
\succ_1: & \quad r_1 \succ r_2 \succ r_3 \\
\succ_2: & \quad r_3 \succ r_2 \succ r_1 \\
\succ_3: & \quad r_2 \succ r_3 \succ r_1 \succ r_4
\end{align*}
\]

3.3. Efficiency and fairness criteria for an allocation

There exists an extensive literature investigating how to define the efficiency of an allocation (see Chevaleyre et al. (2006) and Thomson (2016) for some surveys). The gold standard in terms of efficiency of an allocation is Pareto-optimality. We say that an allocation $\pi'$ Pareto-dominates another allocation $\pi$ if all the agents are weakly better off in $\pi'$ than in $\pi$, with at least one agent being strictly better off. An allocation $\pi$ is then called Pareto-optimal if and only if there does not exist an allocation $\pi'$ that Pareto-dominates it.

The efficiency of an allocation can also be evaluated by measuring the average rank (ark) of the resources held by the agents defined as:

$$\text{ark}(\pi) = \sum_{a_i \in \mathcal{N}} \text{rank}_{a_i}(\pi_{a_i}).$$

Maximizing the average rank is equivalent in our case to maximizing the utilitarian social welfare which is a widely used measure of efficiency that dates back to the idea of utilitarianism.

However, maximizing the average rank or searching for Pareto-efficient solutions may not be satisfactory as it can lead to particularly unfair allocations. The allocation in which one agent receives all the resources and the others nothing is Pareto-optimal but is unarguably unfair. For this reason, many fairness criteria have been introduced.

In this paper, we will focus on maximizing the minimum rank (mrk) of the resources held by the agents defined as:

$$\text{mrk}(\pi) = \min_{a_i \in \mathcal{N}} \text{rank}_{a_i}(\pi_{a_i}).$$

When using the rank as cardinal utility function, the minimum rank is equivalent to the egalitarian welfare. Maximizing the minimum rank follows Rawls' principle of maximizing the welfare of the worst-off (Rawls, 1971). It has been introduced by Pazner and Schmeidler (1978) and is now a very common rule in fair division (Thomson, 1983; Sprumont, 1996; Nguyen et al., 2014).

3.4. Properties of the procedures

We say that a procedure returning Pareto-efficient allocations is Pareto-efficient. Moreover, when dealing with procedures that take an initial allocation as an input, a very common efficiency criteria is that of individual rationality. A procedure is said to be individually rational if, in the final allocation, no agent is assigned an object less preferred than the one she held in the initial allocation. Let us illustrate it on an example.

**Example 2.** Let us consider the following instance with 5 agents and 5 resources. The preferences are presented below, they are single-peaked with respect to $r_1 \prec r_2 \prec r_3 \prec r_4 \prec r_5$. The initial allocation $\pi^0 = (r_5, r_1, r_3, r_4, r_2)$ is defined by the
underlined resources.

\[
\begin{align*}
\alpha_1 : & \ r_1 \succ r_2 \succ r_3 \succ r_4 \succ r_5 \\
\alpha_2 : & \ r_5 \succ r_4 \succ r_2 \succ r_3 \succ r_1 \\
\alpha_3 : & \ r_3 \succ r_2 \succ r_1 \succ r_4 \succ r_5 \\
\alpha_4 : & \ r_4 \succ r_3 \succ r_2 \succ r_5 \succ r_1 \\
\alpha_5 : & \ r_4 \succ r_5 \succ r_5 \succ r_4 \succ r_2 \succ r_5 \succ r_1 
\end{align*}
\]

The allocation \( \pi_0 \) is not Pareto-optimal as it is Pareto-dominated by the squared allocation \( \pi = \langle r_1, r_5, r_3, r_4, r_2 \rangle. \) We have \( \text{ark}(\pi) = 22 \) and \( \text{mark}(\pi) = 2. \) Note that the allocation \( \pi' = \langle r_1, r_5, r_3, r_2, r_4 \rangle \) would yield \( \text{ark}(\pi') = 23 \) and \( \text{mark}(\pi') = 3 \) but violates individual rationality for agent \( \alpha_4. \)

Another desirable property is strategy-proofness. An allocation procedure is strategy-proof if no agent has an incentive to misreport her preferences, that is, no one can obtain a better outcome by reporting false preferences.

4. Centralized allocation procedures for house market

This section introduces the centralized allocation procedures that will be studied in the rest of the paper. First, we describe local deals that will be the basic component of our procedures. Then, we show how the Top Trading Cycle algorithm can be described accordingly to these deals and finally we present the Crawler algorithm.

4.1. Improving deals

Following the work of (Sandholm, 1998), we consider procedures where, departing from an initial allocation, the agents negotiate deals so as to obtain more preferred resources. Hence, agents make local improving deals (or exchanges) until they reach a stable allocation, that is, an allocation where no improving deal is possible. In our context, the transfer of a resource from one agent to another will be balanced by another resource to compensate the loss of welfare. Another line of research consists in assuming that some monetary transfers can compensate disadvantageous deals (Sandholm, 1998; Endriss et al., 2006; Chevaleyre et al., 2017). In this paper, we assume that such monetary compensation is either not possible or not desirable, for instance because of ethical reasons (Abraham et al., 2007).

A deal consists in an exchange cycle \( \mu = \langle a_1, \ldots, a_k \rangle \), where \( a_i \in N, \forall i \in [1, k] \). Such deals model exchanges where agent \( a_i \) gives her resource to agent \( a_{i+1} \) for each \( i \in [1, k-1] \) and agent \( a_k \) gives her resource to agent \( a_1 \). For the particular case of deals involving only two agents, \( k = 2 \), we will talk about swap-deals. In our house market model, it has to be noticed that any reallocation (permutation of resources) can be implemented as a collection of disjoint cycle-deals (Shapley and Scarf, 1974).

Definition 3. Let \( \pi \) be an allocation, and \( \mu = \langle a_1, \ldots, a_k \rangle \) a deal involving \( k \) agents. The allocation \( \pi[\mu] \) obtained by applying the deal \( \mu \) to \( \pi \) is defined by:

\[
\begin{align*}
\pi[\mu]_{a_i} & = \pi_{a_{i+1}} & \text{if } i \in [2, k], \\
\pi[\mu]_{a_1} & = \pi_{a_k} & \text{otherwise}.
\end{align*}
\]

A deal is said to be improving if \( \pi[\mu]_{a_i} \succ \pi_{a_i} \) for every agent \( a_i \) involved in \( \mu \).

Note that a procedure applying only improving deals trivially satisfies individual rationality.

Given an allocation \( \pi \), we denote by \( C_k(\pi) \), \( k \geq 2 \), the set of all the improving deals of size at most \( k \) that can be applied from \( \pi \):

\[ C_k(\pi) = \{ \mu \mid \mu \text{ is an improving deal and } |\mu| \leq k \}. \]

Definition 4. An allocation \( \pi \) is stable with respect to \( C_k \), \( k \geq 2 \) if \( C_k(\pi) = \emptyset \).

It can be observed that any Pareto-efficient allocation is stable with respect to \( C_\mu \). Indeed, in a Pareto-efficient allocation no deal can be improving.

4.2. Gale’s Top Trading Cycle algorithm

When investigating resource allocation in house market, the Top Trading Cycle algorithm (TTC) (Shapley and Scarf, 1974), attributed to David Gale, is well known to satisfy the three main desirable properties of an allocation procedure: Pareto-optimality, individual rationality and strategy-proofness.

The TTC algorithm takes as input an instance \( I = \langle N, \mathcal{R}, L, \pi^0 \rangle \) and proceeds as follow. The algorithm maintains a set of available agents \( N \) and a set of available resources \( R \) where initially \( R = \mathcal{R} \) and \( N = N \). At each step of the algorithm, a directed bipartite graph \( G = (V, E) \), with \( V = N \cup R \), is defined. The nodes of \( G \) represent agents in \( N \) and resources in \( R \), and the set of edges \( E \) is such that:

- there is a directed edge \( (a_i, r_i) \) between \( a_i \) and \( r_i \) if and only if \( r_i \) is the top resource in \( a_i \); and
- there is a directed edge \( (r_i, a_i) \) between \( r_i \) and \( a_i \) if and only if \( r_i \) is the owner in \( a_i \).

Figure 1: Bipartite graph created by the TTC algorithm for \( \pi^0 \) as defined in Example 2

Note that there always exists at least one cycle in \( G \) and cycles correspond to improving cycle-deals. The cycle-deals constructed can be of size 1 if an agent already owns her top resource in \( R \). The TTC algorithm selects one of the cycles \( \mu \) in \( G \), the agents and resources involved in \( \mu \) are then removed from \( N \) and \( R \) to obtain a new graph \( G' \) with the new available agents and resources. The process is iterated on the new graph \( G' \) and \( \pi'[\mu] \) until reaching an empty graph.
Example 3. Let us consider the instance defined in Example 2, the first graph constructed during the TTC procedure is depicted in Figure 1. The resource and the agents are represented so that the cycles in G appear clearly. The final allocation computed by TTC is \( \pi = \langle r_1, r_5, r_3, r_4, r_2 \rangle \), the underlined allocation in Example 2. The cycle-deals that can be applied are \( \mu_1 = \langle a_1, a_2 \rangle \) and \( \mu_2 = \langle a_3 \rangle \). The last one is a specific deal involving only one agent which means that the agent keeps her resource. Once these deals performed, there is no other improving cycle-deal between agents \( a_4 \) and \( a_5 \).

At this stage, it is important to recall that TTC makes very few assumptions on the preferences domain. It only assumes that each agent’s preferences are defined as complete strict linear orders.

4.3. The Crawler

Bade (2019) proposed a new mechanism for resource allocation problems in house markets under single-peaked preferences. This mechanism has the same guarantees as TTC: Pareto-optimality, individual rationality and strategy-proofness.

The agents are initially ordered along the single-peaked axis according to the resource they initially hold. The first agent is the one holding the resource on the left side of the axis and the last agent is the one holding the resource on the right side of the axis. \( R \) is the list of available resources ordered according to the single-peaked axis. \( N \) is the list of available agents such as the \( i \)th agent of the list is the one who holds the \( i \)th resource in \( R \). The algorithm then screens the agent from left to right and check, for each agent \( a_i \), whether the peak of \( a_i \) (her top preferred resource among the ones available in \( R \)) is on her right:

- If the peak of \( a_i \) is on her right, the algorithm moves to the next agent on the right.
- If \( a_i \) holds her peak \( r_i \), it is assigned to \( a_i \), \( a_j \) and \( r_i \) are then removed from \( N \) and \( R \). The algorithm restarts screening the agents from the left side of the axis.
- If the peak \( r_j \) of \( a_i \) is on her left, she picks \( r_j \). Let \( t^* \) be the index of \( r_j \) and \( t \) the index of \( a_i \) (we have \( t^* < t \)). Then, all the agents between \( t^* \) and \( t - 1 \) receive the resource held by the agent on their right (the resources “crawl” towards the left). \( a_i \) and \( r_j \) are then removed from \( N \) and \( R \). The algorithm restarts screening the agents from the left side of the axis.

The algorithm terminates when \( N \) and \( R \) are empty.

A formal description of the procedure is given in Algorithm 1. Note that we make use of the sub-procedure \( \text{pick}(a_i, r, N, R, \pi) \) which simply assigns the specified resource \( r \) to the specified agent \( a_i \) in the allocation \( \pi \), and then removes the agent and the resource from the lists of available agents and resources, \( N \) and \( R \) respectively. Since the list of resources is ordered following the single-peaked axis and the \( i \)th agent in \( N \) corresponds to the owner of the \( i \)th resource in \( R \), the removal of \( r \) and \( a_i \) is in fact equivalent to assigning \( r \) to agent \( a_i \) and crawling the resources from right to left.

Algorithm 1: The Crawler procedure

\begin{algorithm}
\caption{The Crawler procedure}
\begin{algorithmic}[1]
\State \textbf{Input:} An instance \( I = (N, R, L, \pi^0) \) single-peaked with respect to \(<\)
\State \textbf{Output:} \( \pi \) an allocation
\State \( \pi \leftarrow \text{empty allocation} \)
\State \( R \leftarrow \text{list of resources sorted according to} < \)
\State \( N \leftarrow \text{list of agents such as the} i \text{th agent of the list is} \)
\State \( \text{the one who initially holds the} i \text{th resource in} R \)
\While \( N \neq \emptyset \)
\State \( t^* \leftarrow |N| \)
\For \( t = 0 \rightarrow |N| - 1 \)
\If \( r_t \succ r_{t+1} \) then /* no crawl */
\State \( t^* \leftarrow t \)
\EndIf
\Break
\EndFor
\EndWhile
\State \( r \leftarrow \text{top}_R(a_i) \)
\State \( \text{pick}(a_i, r, N, R, \pi) \)
\State \Return \( \pi \)
\end{algorithmic}
\end{algorithm}

As observed by (Bade, 2019), the Crawler always terminates. We also show that the Crawler runs in quadratic time.

Proposition 2. The Crawler procedure terminates and its complexity is in \( O(n^2) \) where \( n \) is the number of agents.

Proof. Termination is proved by observing that \( |N| \) is strictly decreasing at each step of the main while loop. This loop is applied at most \( n \) times and each step of the loop requires at most \( O(n) \) elementary operations. The time complexity is then in \( O(n^2) \).

Let us illustrate the execution of the Crawler on the instance of Example 2.

Example 4. Let us return once again to Example 2. The execution of the Crawler is presented in Figure 2. First, agent \( a_3 \) is the first agent whose top is not on her right, she thus receives her top \( r_3 \). The second step matches agent \( a_4 \) to \( r_2 \). On the third step, agents \( a_2 \) and \( a_5 \) both have their top on the right but the last agent \( a_1 \) has her top on her left. \( a_1 \) is then matched to her top \( r_1 \). Agent \( a_5 \) is matched to \( r_5 \) on the fourth step. Finally \( a_2 \) is assigned resource \( r_2 \).

At each step \( i \) of the procedure, the improving deal \( \mu_i \) is applied: \( \mu_1 = \langle a_3 \rangle , \mu_2 = \langle a_4 \rangle , \mu_3 = \langle a_1, a_5, a_2 \rangle , \mu_4 = \langle a_5 \rangle \) and \( \mu_5 = \langle a_2 \rangle \).

One can observe that on this example the allocation returned by the Crawler is not the same as the one returned by TTC. Both procedures lead to the same minimum rank \( 

5. Checking Pareto-optimality: the Diver

If our objective is simply to check Pareto-optimality of a given allocation, one question to investigate is whether we can gain advantage from the single-peaked domain or not. Observe first that the Crawler could be used for that purpose as it would return the initial allocation.
Observation 1. Let \( \pi \) be a Pareto-optimal allocation, then the Crawler returns \( \pi \) when applied with \( \pi \) as the initial allocation.

Proof. The proof is straightforward since the Crawler is individually rational and Pareto-optimal.

However, the procedure would not enjoy better complexity guarantees in that case. This is in contrast with TTC which stops if it fails to find an improving cycle in the first step. Hence, the complexity of the Pareto-optimality test based on TTC would be only \( O(n^2) \). The Crawler has thus the same worst-case running time as TTC when used for that purpose. A worst-case instance can be described as follows: suppose that all the agents (ordered from left to right), have the next resource on their right as their top, except for the last one who likes her own resource. In that case, at each step, the Crawler would go through all agents before realizing that the last one wants to keep her resource.

We now present a slight variant of the Crawler, called the Diver, which allows to check Pareto-optimality of the initial allocation more efficiently. We first informally present the modified procedure, which proceeds in a single screening of the agents. The key difference with the Crawler is that the procedure does not start a new screening once an agent picks a resource: it only checks whether the last agent who was happy to crawl for this resource now agrees to dive to the next one.

At each step, the central entity asks the agent whether she wishes to:

1. pick her current resource;
2. pass (expressing that she is happy to dive to the next resource); or
3. pick a smaller resource.

Note that, each time an agent picks a resource, the central entity communicates this information to the other remaining agents so that they can update their list of available resources.

In case (1), the agent (and her resource) are removed and we enter a sub-protocol \( \varphi' \) where the center asks the previous agents, one by one, whether they still agree to dive to the next resource. This sub-protocol stops as soon as one agent says yes, or there are no more agents to consider. All the agents who said ‘no’ pick their resources and are themselves removed (with their resource). Note that in case (3), we have the guarantee that there is indeed a better resource available, otherwise the agent would have picked her own resource (case 1). As soon as an agent says she wants a smaller resource, the protocol stops and returns ‘not PO’. Alternatively, if the screening goes through all the agents, then (as we shall prove) all the agents left with their own resource, and the protocol returns ‘PO’.

The protocol is formally described in Algorithm 2. Note that, unlike with the Crawler, the lists \( N \) and \( R \) don’t need to be updated, but instead we record the list of agents who crawl (or dive). The sub-procedure \( \text{pick}(a_i, r) \) simply assigns resource \( r \) to agent \( i \), while \( \text{pick}(a_i, r, D) \) does the same, and removes agent \( i \) from the list \( D \) of agents who crawl.

Algorithm 2: The Diver procedure

Input: An instance \( I = (N, R, L, \pi^0) \) single-peaked with respect to \( \prec \)

Output: PO if \( \pi^0 \) is Pareto-optimal and not PO otherwise

1. \( \pi \leftarrow \) list of pairs \((a_i, r)\) such that agent \( a_i \) holds resource \( r \) in \( \pi^0 \), sorted according to \( \prec \) for the resources
2. \( D \leftarrow \emptyset \): list of agents who crawl or dive
3. for \((a_i, r_i)\) in \( \pi \) do
4. if top\( _D(a_i) = r_i \) then /\* pick your top */
5. \( \text{pick}(a_i, r_i) \)
6. for \( a_j \) in reverse\( (D) \) do
7. if \( r_j \succ_a, r_{j+1} \) then /\* if you don’t dive, pick your resource */
8. \( \text{pick}(a_j, r_j, D) \)
9. else
10. break
11. end
12. end
13. else if \( r_i \succ_a, r_{i+1} \) then /\* your top is on your left: not PO */
14. return not PO
15. else /* crawl */
16. \( D \leftarrow D.\text{append}(a_i) \)
17. end
18. end
19. return PO

Example 5. Coming back to Example 2, by applying the Diver to the initial allocation \( \pi^0 \), the agents are first sorted as follows:

\[
\begin{array}{cccccc}
| r_1 & r_2 & r_3 & r_4 & r_5 |
|---|---|---|---|---|
| a_2 & a_5 & a_3 & a_4 & a_1 |
\end{array}
\]

The Diver screens the agent from left to right and asks each agent her wish:

- \( a_2 \) passes;
- \( a_5 \) passes;
• \(a_3\) picks her current resource, \(a_5\) still agrees to pass;
• \(a_4\) picks her current resource, \(a_5\) still agrees to pass;
• \(a_1\) wants to pick a smaller resource \((r_1) \to \) the Diver returns ‘not PO’.

Now let us consider the allocation \(\pi = (r_1, r_5, r_2, r_4, r_3)\) leading to the following order:

\[
\begin{array}{cccccc}
  r_1 & r_2 & r_3 & r_4 & r_5 \\
 a_2 & a_3 & a_4 & a_5 & a_1
\end{array}
\]

Again, the Diver screens the agent from left to right and asks each agent her wish:
• \(a_1\) picks her current resource;
• \(a_3\) passes;
• \(a_5\) passes;
• \(a_4\) picks her current resource, \(a_5\) still agrees to pass;
• \(a_2\) picks her current resource, \(a_5\) picks her current resource, \(a_3\) picks her current resource \(\to \) all the agents left with their resource and the Diver returns ‘PO’.

**Theorem 1.** The Diver terminates in \(O(n \log n)\) and returns whether the initial assignment is Pareto-optimal or not.

**Proof.** Termination is obvious since the procedure proceeds in a single main screening. We first show that the procedure is sound. First observe that when the Diver returns ‘PO’, all the agents must have picked their own resource. Indeed, consider the last agent in the order: this agent picked her resource (otherwise the procedure would have returned ‘not PO’). But now the previous available agent in backward must also pick her resource next (as there are no more possibility to dive), and so on until there are no agents remaining. Now, following the argument used in Bade (2019), consider all the agents who picked their resource during this process, in the order they picked it: they clearly all picked their best available resource. The obtained matching is thus indeed Pareto-optimal. On the other hand, when the Diver returns ‘not PO’, there is indeed an improving cycle, consisting of the agent (say, \(a_j\)) who chose a resource on her left, and all the agents, from the owner of this resource to \(a_j\), who are not matched yet.

In terms of complexity, sorting the agents according to the single-peaked order can be done in \(O(n \log n)\). Now for the main loop of the procedure: in the reverse loop, note that if \(k + 1\) agents are screened backwards, then \(k\) agents are removed for good. Thus through the entire procedure the reverse loop involves \(O(n)\) steps, and thus the main loop takes \(O(n)\) as well, which is dominated by the cost of sorting.

The same line of analysis allows us to derive a result regarding the amount of communication induced by the procedure. As we do not consider communication from the center to the agents, the cost of communicating the single-peaked order to agents is not counted here, and we see that the Diver only requires a linear number of bits (in the number of agents) to be communicated.

**Theorem 2.** The Diver requires \(4n\) bits of communication.

**Proof.** The key is to observe that sub-protocol \(p'\) requires overall \(n + n\) bits, as there may only be \(n\) agents saying ‘no’ and \(n\) agents saying ‘yes’ throughout the whole run of the Diver. In the main loop of the protocol, the question requires 2 bits to be answered. This makes overall \(2n + 2n = 4n\) bits.

Thus, only \(\Omega(n)\) bits of communication are needed. Intuitively, it seems unlikely that we can improve on this protocol. We now show that this is asymptotically optimal.

More formally, given an instance \(I = (\mathcal{N}, R, L, \pi^0)\) single-peaked with respect to \(\prec\), we consider the problem CheckPO whose answer is yes if and only if \(\pi^0\) is Pareto-optimal. We assume that \(\prec\) is known to the agents. Without loss of generality, we consider that \(\forall i \in \mathcal{N}, \pi^0_i = r_i\).

To prove the optimality of our bound, we will exhibit a straightforward fooling set (Kushilevitz and Nisan, 1996) to formally establish that the Crawler matches the lower bound for CheckPO. We consider strict preferences for the agents written \(\succ_i^L\) in a profile \(L\), and by a slight abuse of notation we write \(\succ_i^L\) to say that \(\succ_i\) is the preference of agent \(a_i\) in either \(L\) or \(L'\). In our context, the fooling set will be a collection of profiles \(F = \{L_1, \ldots, L_K\}\) such that:

1. for any \(i \in \{1, \ldots, K\}\) CheckPO’s answer on \((\mathcal{N}, R, L, \pi^0)\) is yes.
2. for any \(i \neq j\), there exists \(L' = \langle \succ_i^{L_1}, \ldots, \succ_i^{L_n} \rangle\), such that CheckPO’s answer on \((\mathcal{N}, R, L', \pi^0)\) is no.

By a standard result in communication complexity, it is known that \(\log |F|\) is a lower bound on the communication complexity of the problem (Kushilevitz and Nisan, 1996).

**Proposition 3.** In the single-peaked domain, the communication complexity of CheckPO is \(\Omega(n)\).

**Proof.** Let us call a consensual profile the profile where \(\succ_i \succ_j\) for any \((i, j)\), i.e. all the agents have the same linear orders over preferences. The consensual linear order will be denoted by \(\succ\). We claim that the set \(F\) of the \(2^{n-1}\) (single-peaked) consensual profiles constitutes a fooling set.

To show this, first observe that in any such profile, the original assignment \(\pi^0\) is indeed Pareto-optimal. Indeed, in a consensual profile, no trading cycle is possible. Hence the aforementioned condition 1. of a fooling set is satisfied.

Now to show that we can fool the function, consider any pair of profiles \((L_i, L_j)\). Because these profiles are different, there must exist at least one pair of resources \((r_p, r_q)\) such that \(r_p \succ_j r_q\) in \(L_i\), while \(r_q \succ_p r_p\) in \(L_j\) (it is true for all agents since the profiles are consensual). Now consider the agent \(a_p\) (resp. \(a_q\)) holding \(r_p\) (resp. \(r_q\)) in \(\pi^0\). Let us now consider a mixed profile \(L'\) such that:

\[
\forall k \in \mathcal{N}, \succ_k^{L'} = \begin{cases} \succ_k^{L_i} & \text{if } k \neq p, \\ \succ_k^{L_j} & \text{if } k = p. \end{cases}
\]
Observe that now \( a_p \) and \( a_q \) have opposite preferences for \( r_p \) and \( r_q \) and would prefer to swap, i.e. \( \text{Crawler} \)’s answer on \( \langle N, R, L', \pi^0 \rangle \) is no. This concludes the proof. \( \square \)

6. Swap-deal procedures

Both TTC and the Crawler require a central entity to run the procedure with the drawbacks that were stated in the introduction. Moreover, they often require long cycles to be implemented. As mentioned previously, such deals may be complicated to implement or may not be desirable. In decentralized settings, a natural approach consists in departing from an initial allocation and letting the agents negotiate improving cycle-deals involving at most \( k \) agents until reaching an allocation stable with respect to \( C_k \). We call such a procedure the \( C_k \)-procedure.

In general, these procedures are not deterministic: from a same initial allocation, many different stable allocations can be reached. Let us see it with a simple example.

Example 6. Coming back to Example 2, by applying the deal \( \langle a_1, a_2 \rangle \) to \( \pi^0 \) (the squared allocation) we reach the underlined allocation \( \pi \) that is stable with respect to swap-deals and Pareto-optimal.

However, applying the improving deals \( \langle a_1, a_5 \rangle \) and then \( \langle a_1, a_2 \rangle \) yields to the allocation \( \langle r_1, r_2, r_3, r_1, r_5 \rangle \), that is also Pareto-optimal, it moreover corresponds to the allocation returned by the Crawler.

This example emphasizes the importance of specifying how improving deals are selected when several ones are available. Most of our results do not rely on any specific selection dynamic, however for the experiments presented in Section 9, such dynamics will be defined.

In this paper, we will more specifically focus on the simplest version of cycle-deals: bilateral deals i.e. deals involving exactly two agents (also denoted as \( C_2 \)). This type of deals has the advantage of being easy to implement since it does not require many agents to meet each other and to coordinate.

Since TTC and the Crawler both provide desirable guarantees, under single-peaked preferences, on Pareto-optimality, individual rationality and strategy-proofness, it is natural to investigate whether the \( C_2 \) procedure also provides such good properties.

First, observe that the swap-deal procedure is individually rational: throughout the procedure only improving swap-deals are performed, hence an agent can only improve her satisfaction at each step of the procedure. In the “worst-case” an agent will not perform any deal during the procedure and will own the same resource as in the initial allocation. This is true for every \( C_k \)-procedures.

However, we show here that the swap-deal procedure is not strategy-proof.

Proposition 4. The swap-deal procedure is not strategy-proof.

Proof. Let us consider the following instance single-peaked with respect to \( r_1 \preceq r_2 \preceq r_3 \) and where the initial allocation \( \pi^0 \) is represented by the underlined resources.

\[
\begin{align*}
\langle a_1, a_2, a_3 \rangle & : r_2 \succ_1 r_3 \succ_1 r_1 \quad a_2 : r_2 \succ_2 r_3 \succ_2 r_1 \\
\langle a_1, a_2, a_3 \rangle & : r_1 \succ_3 r_2 \succ_3 r_3 \quad a_4 : r_1 \succ_3 r_3 \succ_3 r_2 \\
\langle a_1, a_2, a_3 \rangle & : r_2 \succ_1 r_3 \succ_1 r_2 \quad a_5 : r_2 \succ_2 r_3 \succ_2 r_2
\end{align*}
\]

In \( \pi^0 \), \( \langle a_2, a_3 \rangle \) is an improving swap-deal, it leads to the allocation \( \pi = \langle r_3, r_2, r_1 \rangle \).

Let us now consider the following preference profile \( L' \) in which \( a_1 \) misreports her preferences, inverting the order between \( r_2 \) and \( r_3 \),

\[
\begin{align*}
\langle a_1, a_2, a_3 \rangle & : r_2 \succ_1 r_1 \succ_1 r_3 \\
\langle a_1, a_2, a_3 \rangle & : r_2 \succ_2 r_3 \succ_2 r_1 \\
\langle a_1, a_2, a_3 \rangle & : r_1 \succ_3 r_3 \succ_3 r_2
\end{align*}
\]

Note that the preference profile is still single-peaked with respect to \( \prec \).

In this scenario, two improving deals are possible from \( \pi^0 \): \( \mu_1 = \langle a_2, a_3 \rangle \) as before and a new deal \( \mu_2 = \langle a_1, a_2 \rangle \). If \( \mu_2 \) is performed, \( a_1 \) can later exchange \( r_1 \) against \( r_2 \) with \( a_3 \) thus owning her real most-preferred resource.

Let us suppose that \( \mu_2 \) will be performed with probability \( p \) and \( \mu_1 \) with probability \( 1 - p \). Agent \( a_1 \) will then receive \( \text{top}(a_1) \) with probability \( p \) or do not perform any deal with probability \( 1 - p \). Overall, agent \( a_1 \) has a strict incentive to lie as long as \( p > 0 \), for \( p = 0 \) her lie will not affect her outcome.

This proves that the swap-deal procedure is not strategy-proof. \( \square \)

The swap-deal procedure does not satisfy the desirable property of strategy-proofness while the Crawler and the TTC algorithm do. One can view it as a cost emerging from the decentrality of the swap-deal procedure. This idea of a trade-off between decentrality and satisfaction of desirable properties has already been observed in the literature (Herreiner and Puppe, 2002; Brams et al., 2012) and is explored in more depth in Section 8.

7. Efficiency of the swap-deal procedures

To complete the picture, we investigate the efficiency of the swap-deal procedure by showing that any allocation stable with respect to swap-deals is Pareto-optimal. Then, we show that the single-peaked domain is maximal for this assertion.

7.1. Pareto-optimality of the swap-deal procedure

We show here that any allocation reached by the swap-deal procedure is Pareto-optimal.

Theorem 3. In a single-peaked domain, every allocation \( \pi \) stable with respect to \( C_2 \) is stable with respect to \( C_n \).

Proof. Let us consider, toward contradiction, an allocation \( \pi \) stable with respect to \( C_2 \) but not with respect to \( C_n \). As \( \pi \) is not stable w.r.t. \( C_n \) there exists an improving deal \( \mu = \langle a_1, \ldots, a_k \rangle \), with \( 2 < k \leq n \), in \( \pi \). Let us show by induction of the size of \( \mu \), denoted by \( k \), that such an improving deal cannot exist.
First, note that as $\mu$ is an improving deal, we have:

$$\begin{align*}
\pi_{a_{i-1}} &\succ \pi_{a_{i}}, \quad \forall a_i \in \mu, a_i \neq a_1 \\
\pi_{a_k} &\succ \pi_{a_1}.
\end{align*} \tag{1}$$

Then, observe that as $C_2(\pi) = \emptyset$, we have:

$$\begin{align*}
\pi_{a_i} &\succ \pi_{a_{i+1}}, \quad \forall a_i \in \mu, a_i \neq a_k \\
\pi_{a_k} &\succ \pi_{a_1},
\end{align*} \tag{2}$$

otherwise, based on (1), an improving swap-deal would exist in $\pi$. Base case $(k = 3)$ Let us consider $\mu = \langle a_1, a_2, a_3 \rangle$. From (1) and (2), we obtain:

$$\begin{align*}
a_1 : & \pi_{a_3} \succ \pi_{a_1} \succ \pi_{a_2}, \\
a_2 : & \pi_{a_3} \succ \pi_{a_2} \succ \pi_{a_1}, \\
a_3 : & \pi_{a_2} \succ \pi_{a_3} \succ \pi_{a_1}.
\end{align*}$$

The triplet of resources $\langle \pi_{a_1}, \pi_{a_2}, \pi_{a_3} \rangle$ is thus a witness of the violation of the worst-restrictedness (WR) condition of a profile to be single-peaked (Proposition 1): all the three resources are ranked last by an agent when we restrict our attention to these resources. The contradiction if thus set.

**Induction step** Suppose now that $\pi$ is stable with respect to $C_{k-1}$, we show that no improving deal of size $k$ exists in $\pi$. From this induction hypothesis, we get that:

$$\begin{align*}
\pi_{a_i} &\succ \pi_{a_j}, \quad \forall a_i, a_j \in \mu, a_i \neq a_j, \forall a_i, a_j \notin \{a_{i-1}, a_i\} \\
\pi_{a_i} &\succ \pi_{a_j}, \quad \forall a_i, a_j \in \mu, a_i \neq a_j \notin \{a_k, a_1\}.
\end{align*} \tag{3}$$

Indeed if this condition was not satisfied, then there would exist two agents $a_i$ and $a_j$ that are not next to one another in $\mu$ such that $\pi_{a_i} \succ \pi_{a_j}$. It would then have been possible to "cut" $\mu$ between those two agents so that $a_i$ receives $\pi_{a_i}$. The new cycle deal obtained would also have been improving and then an improving deal of size strictly smaller than $k$ would exist.

Let us now consider a triplet of resources $O = \langle \pi_{a_{u-1}}, \pi_{a_u}, \pi_{a_{u+1}} \rangle$ such that $\pi_{a_u}$ is ranked last by a given agent when restricting preferences to $O$. Then from (1), (2) and (3) we get:

$$\begin{align*}
a_w : & \pi_{a_{u-1}} \succ \pi_{a_w} \succ \pi_{a_{u+1}}, \\
a_{w+1} : & \pi_{a_w} \succ \pi_{a_{w+1}} \succ \pi_{a_{u-1}}.
\end{align*}$$

Hence when restricting preferences to $O$, for every resource in $O$, there exists an agent ranking it last among $O$’s resources. This violates the worst-restrictedness condition of the single-peaked profile and sets the contradiction.

This Theorem states that the $C_k$ stable hierarchy collapses at the $C_2$ level when preferences are single-peaked and in a house-market setting. It is particularly interesting for us as it provides a characterization of Pareto-optimality, as observed in Subsection 4.1.

**Corollary 1.** In a single-peaked domain and house-market, an allocation $\pi$ is Pareto-optimal if and only if it is stable with respect to $C_2$.

Stating this result in terms of stability with respect to $C_0$ and not just Pareto-optimality is particularly interesting as it can be shown that the same result holds for more general settings than house markets (Beynier et al., 2019).

As we have proved that the allocation reached by swap-deals is Pareto-optimal, a natural question is then whether every allocation that Pareto-dominates the initial allocation can be reached by swap-deals. This is answered by the negative by Proposition 5.

**Proposition 5.** There exists instances $I = \langle \mathcal{N}, \mathcal{R}, L, \pi^0 \rangle$ for which there is an allocation $\pi$ that Pareto-dominates $\pi^0$ and that cannot be reached by a sequence of improving swap-deals.

**Proof.** Let us consider the following instance where the initial allocation $\pi^0 = \langle r_3, r_2, r_1 \rangle$ is the one in the underlined boxes.

$$\begin{align*}
a_1 : & r_1 \succ r_2 \succ r_3, \\
a_2 : & r_1 \succ r_2 \succ r_3, \\
a_3 : & r_2 \succ r_3 \succ r_1.
\end{align*}$$

The allocation $\pi = \langle r_2, r_1, r_3 \rangle$, the boxed one, is Pareto-optimal. However from $\pi^0$ only two deals are possible: $\mu_1 = \langle a_1, a_3 \rangle$ that reaches allocation $\langle r_1, r_2, r_3 \rangle$, or $\mu_2 = \langle a_2, a_3 \rangle$ that leads to $\langle r_3, r_1, r_2 \rangle$. No sequence of improving swap-deals can thus reach $\pi$.  

It is however interesting to note that the allocation returned by the Crawler can always be reached through improving swap-deals.

**Proposition 6.** Let $I = \langle \mathcal{N}, \mathcal{R}, L, \pi^0 \rangle$ be an instance and let $\pi^C$ be the allocation returned by the Crawler. Then $\pi^C$ is reachable by swap-deals from $\pi^0$.

**Proof.** We show that every cycle-deal of the Crawler can be implemented as a sequence of improving swap-deals. For clarity reasons and without loss of generality, we assume that each agent $a_i$ currently holds resource $r_j$. Let us consider a step $i$ of the procedure where the agent $a_i$ picks resource $r_k$ currently held by $a_i$. From the definition of the procedure, $a_k$ is on the left of $a_i$ (with respect to the single-peaked axis) and $a_k$ has already been considered at this step before considering $a_i$. In fact, all the agents between $a_k$ (included) and $a_i$ (excluded) on the single-peaked axis have already been considered at step $i$ before reaching $a_i$. Moreover, all these agents have passed their turn because their peak is on their right. In other words, each agent $a_j$ between $a_k$ (included) and $a_{j+1}$ (excluded) prefers the resource held by the agent $a_{j+1}$ on her right. So, $\forall j \in \{k, i-1\} : \forall j, r_j \succ r_{j+1}$.

Let $\mu_i$ be the deal implemented by the crawler. $\mu_i = \langle a_i, a_{i-1}, \ldots, a_{k+1}, a_k \rangle$. In this deal, $a_k$ gives her resource $r_k$ to $a_i$ and all the other agents of the deal give their resource to the next agent in the sequence which is the agent on their left with respect to the single-peaked axis. The decomposition of $\mu_i$ into a sequence of swap-deals consists in using agent $a_i$ as a hub for the exchanges of resources. Agent $a_i$ first swaps with $a_{i-1}$ then, $a_i$ swaps with $a_{i-2}$ and so on until $a_i$ swaps with $a_k$. At then end,
\(a_i\) holds \(r_k\) and each other agent \(a_j\) in \(\mu_i\) holds the resource initially held by \(a_{j+1}\). The sequence of swap-deals is thus equivalent to \(\mu_i\).

We now show that all these swap-deals are improving exchanges. In the first deal \((a_i, a_{i-1})\), \(a_{i-1}\) receives the resource \(r_j\) held by \(a_i\) that she prefers to her current resource \(r_{i-1}\) (as shown previously \(r_j \succ r_{i-1}\)). \(a_i\) receives the resource \(r_{i-1}\) held by \(a_{i-1}\) that she prefers to her current resource since her peak is on the left of \(r_{i-1}\) (it is held by \(a_k\) i.e. \(\forall j \in \{k, i-1\}: r_j \succ r_{j+1}\)). Concerning the next swap-deals \((a_i, a_j)\) with \(j \in \{k, i-2\}\), \(a_i\) exchanges \(r_{j+1}\) that she obtained from her previous swap-deal, against \(r_j\) held by \(a_j\). \(a_j\) receives the resource \(r_{j+1}\) that she prefers to \(r_{j+1}\) (since \(\forall j \in \{k, i-1\}: r_j \succ r_{j+1}\)). All the swap-deals are thus improving.

\[\Box\]

7.2. Maximalty of the single-peaked domain

We show here that the single-peaked domain is maximal for the swap-procedure of the swap-deal-efficiency of the swap-procedure: for every preference domain strictly larger than \(\mathcal{SP}_\prec\), for a given linear order \(\prec\), there exists an instance such that the swap-procedure does not reach a Pareto-optimal allocation.

**Theorem 4.** Let \(\mathcal{R}\) be a set of resources and \(\prec\) a linear order over \(\mathcal{R}\). For every preference domain \(D\) such that \(\mathcal{SP}_\prec \subset D\), there exists an instance \(I = (\mathcal{N}, \mathcal{R}, L, \pi^0)\) defined over \(D\) such that the swap-procedure on \(I\) does not reach a Pareto-optimal allocation.

**Proof.** Let us construct an instance \(I = (\mathcal{N}, \mathcal{R}, L, \pi^0)\) defined over \(D\) such that swap-procedure on \(I\) does not reach a Pareto-optimal allocation.

Without loss of generality and for the ease of the reader, let us assume that \(\prec\) is such that \(r_1 \prec r_2 \ldots \prec r_n\).

Observe that, for every resource \(r_i \in \mathcal{R}\) there exists two linear orders \(\succ_1\) and \(\succ_2\) that are single-peaked with respect to \(\prec\) and such that \(top(\succ_1) = top(\succ_2) = r_i\) and \(sdn(\succ_2) = r_{i-1}\).

Moreover, as \(\mathcal{SP}_\prec \subset D\), there exists a linear order \(\succ^*\) in \(D\) that is not single-peaked with respect to \(\prec\), hence there exists two resources \((r_{s-1}, r_s) \in \mathcal{R}^2\) such that:

\[
\begin{align*}
    r_s \prec r_{s-1} & \prec top(\succ^*), \\
    or \quad top(\succ^*) & \prec r_{s-1} \prec r_s \\
\end{align*}
\]

Without loss of generality, let us assume that \(top(\succ^*) \prec r_{s-1} \prec r_s\). We will use index \(i\) to refer to the top ranked resource in \(\succ^*\): \(r_i = top(\succ^*)\).

The preference profile \(L = \{\succ_i, a_i \in \mathcal{N}\}\) is then defined as follows:

\[
\begin{align*}
    \succ_1 & = \succ^*, \\
    \succ_i & = \succ_{i-1}, & \forall a_i \in \mathcal{N}, i \in [2, t], \\
    \succ_1 & = \succ^*, & \forall a_i \in \mathcal{N}, i \in [t + 1, s], \\
    \succ_i & = \succ^*, & \forall a_i \in \mathcal{N}, i \in [s + 1, n].
\end{align*}
\]

Let us now describe the initial allocation \(\pi^0\):

\[
\begin{align*}
    \pi^0_{a_1} & = r_s, \\
    \pi^0_{a_i} & = top(a_i), & \forall a_i \in \mathcal{N}, i \in [2, t], \\
    \pi^0_{a_i} & = sdn(a_i), & \forall a_i \in \mathcal{N}, i \in [t + 1, s], \\
    \pi^0_{a_i} & = top(a_i), & \forall a_i \in \mathcal{N}, i \in [s + 1, n].
\end{align*}
\]

To get a better understanding, of the instance \(I = (\mathcal{N}, \mathcal{R}, L, \pi^0)\) constructed in this proof, Figure 3 presents the preference profile \(L\) and the initial allocation \(\pi^0\).

We claim that \(\pi^0\) is stable with respect to \(C_2\) but not Pareto-optimal. Let us first show that \(C_2(\pi^0) = \emptyset\).

First observe that every agent \(a_i\), \(i \in [2, t] \cup [s + 1, n]\) owns her top resource, hence can not be involved in an improving swap-deal.

Consider then agent \(a_s\), \(i \in [t + 1, s - 1]\), she owns her second most preferred resource which she would only trade in exchange of a most preferred resource, owned by agent \(a_{i+1}\). However, agent \(a_{i+1}\) is not interested in \(\pi_{a_s}\), hence no improving swap-deal is possible.

Finally, let us consider agent \(a_s\), who owns her second most preferred resource \(r_{s-1}\) and whose top resource \(r_s\) is owned by agent \(a_1\). By the hypothesis that \(a_1\)'s preferences are not single-peaked, we have \(r_s \succ \_ r_{s-1}\). Once again there is no improving swap-deal involving agent \(a_s\).

Overall, we have proved that \(C_2(\pi^0) = \emptyset\), hence the swap-deal procedure on \(I\) returns \(\pi^0\). Let us show now that it is not Pareto-optimal. Observe the allocation in which every agent receives her top resource is feasible as no couple of agents have the same top resource, this allocation clearly Pareto-dominates \(\pi^0\).

To conclude the proof, we have constructed an instance \(I = (\mathcal{N}, \mathcal{R}, L, \pi^0)\) defined over \(D\) such that the swap-deal procedure returns a Pareto-dominated allocation when applied on \(I\).

\[\Box\]

This result is particularly interesting since it shows that the single-peaked domain captures in a "tight" way...
the domain under which the swap-procedure is Pareto-efficient (in the vein of similar results obtained in Chevalley et al. (2010) in different settings). The single-peaked domain is however not a characterization of this fact, there exists some domains that are not single-peaked but on which the swap-deal procedure returns a Pareto-optimal allocation.

Example 7. Let us consider the following preference profile:

\[ a_1 : r_1 \succ r_2 \succ r_3 \]
\[ a_2 : r_1 \succ r_3 \succ r_2 \]
\[ a_3 : r_3 \succ r_2 \succ r_1 \]

This profile is not single-peaked over any linear order \( \prec \): the triplet \( (r_1, r_2, r_3) \) is a witness of the violation of the worst-restrictiveness condition (Proposition 1), however we can show there exists some domains that are not single-peaked efficient (in the vein of similar results obtained in Chevalley et al. (2010) in indifferent settings). The single-peaked assumption does not contradict Theorem 4. The initial allocation \( \pi = \{ r_2, r_3, r_4, \cdots, r_{n-1}, r_n, r_1 \} \) is the underlying allocation. From this allocation \( \pi^0 \), it is possible to reach \( \pi = \langle r_{n-1}, r_2, r_3, \cdots, r_{n-2}, r_n, r_1 \rangle \) (the shaded allocation) by performing \( n-3 \) swap-deals between \( a_1 \) and \( a_n, \forall i \in [2, n-2] \) (starting by the swap-deal between \( a_1 \) and \( a_2 \) and finishing with the swap-deal \( \langle a_1, a_{n-2} \rangle \)). From \( \pi^0 \), it is also possible to reach \( \pi^* = \langle r_n, r_1, r_2, \cdots, r_{n-3}, r_{n-2}, r_{n-1} \rangle \) (the white squared allocation) which is both Pareto-optimal and optimal in terms of average rank. The sequence of swap-deals leading to such an allocation depends on the number of agents. If the number of agents is even, we just have to perform sequence of coupled swap-deals \( \langle a_2, a_n \rangle \) and \( \langle a_{2n+1}, a_1 \rangle \), \( \forall i \in [1, \frac{n}{2}-1] \). If the number of agents is odd we just have to change the final swaps. We still perform the sequence of couples of swap-deals described earlier but at the end of the swap-deal concerning \( a_{n-3} \) we perform the three final swap-deals \( \langle a_{n-2}, a_n \rangle, \langle a_{n-1}, a_1 \rangle \) and \( \langle a_{n-1}, a_{n-2} \rangle \) instead of \( \langle a_{n-2}, a_1 \rangle \).

The gap between \( \pi^* \) and \( \pi \), in terms of our criteria, shows the price induced by individual rationality alone. Indeed, it could be that \( \pi \) is the initial allocation, in which case any mechanism respecting individual rationality would return \( \pi \) as the outcome and thus exhibit this gap in terms of our criteria. This is specific to our decentralized approach.

8. The cost of decentralized procedures

It has become common to measure the cost induced by a decentralized approach when compared to a centralized one, by invoking the price of anarchy (PoA) (Koutsoupias and Papadimitriou, 1999; Anshelevich et al., 2013). Technically, it amounts to compute the (worst-case, over all instances) ratio of the worst stable outcome over the social welfare optimum.

We take inspiration from this notion to understand how bad can be the decentralized myopic dynamics presented, in terms of the criteria studied. In our analysis, three allocations are presented: two allocations \( \pi^* \) and \( \pi \) which happen to stand on the Pareto front, but exhibit a large gap in terms of our criteria \( \pi^* \) being the optimal in terms of the criteria considered, and \( \pi \) being as far as possible from this value); together with a third allocation \( \pi^0 \) which happens to be Pareto-dominated by both \( \pi^* \) and \( \pi \).

We see that \( \text{ar}k(\pi^*) = \min(n-1) + (n-1)/2 \), whereas \( \text{ar}k(\pi^0) = n(n+1)/2 \). This instance shows that (asymptotically) the PoA is \( \geq 2 \). In Damamme et al. (2015) it is shown that 2 is a lower bound for the PoA of swap procedures. We thus conclude that PoA is 2, as in the general domain.
**Proposition 8.** The price of anarchy in terms of minimum rank of the swap-deal procedure is in $\Theta(n)$ in the single-peaked domain.

**Proof.** Let us consider the following single-peaked instances involving $n$ agents. The initial allocation $\pi^0 = (r_{n-1}, r_1, r_2, \ldots, r_{n-2}, r_n)$ is the undominated allocation. From this allocation $\pi^0$, it is possible to reach $\pi = (r_1, r_2, r_3, \ldots, r_{n-1}, r_n)$ (the shaded allocation) by a sequence of couples of swap-deals $\langle a_{n-1}, a_n \rangle$ and $\langle a_{n-2}, a_1 \rangle$ if $n$ is even. If $n$ is odd, we perform the same sequence of couples of swap-deals but $\forall i \in [n, n-2, \ldots, 7, 5]$ and by adding two final swap-deals $\langle a_2, a_n \rangle$ and $\langle a_1, a_n \rangle$. From $\pi^0$, it is also possible to reach $\pi^* = (r_1, r_3, r_4, \ldots, r_n, r_2)$ (the white squared allocation) which is both Pareto-optimal and optimal in terms of average rank. This allocation is reached by performing $n-2$ swap-deals between $a_1$ and $a_i \forall i \in [n-1, 2]$.

$$
\begin{align*}
\pi_1 & : r_1 \succ r_2 \succ r_3 \succ \ldots \succ r_{n-1} \succ r_n \\
\pi_2 & : r_2 \succ r_3 \succ r_4 \succ \ldots \succ r_{n-1} \succ r_n \\
\pi_3 & : r_3 \succ r_4 \succ r_1 \succ \ldots \succ r_{n-1} \succ r_n \\
\pi_4 & : r_4 \succ r_5 \succ r_3 \succ \ldots \succ r_{n-1} \succ r_n \\
\pi_5 & : \ldots \succ r_1 \succ \ldots \succ r_4 \succ \ldots \succ r_{n-1} \succ r_n \\
\pi_6 & : r_2 \succ r_1 \succ r_3 \succ \ldots \succ r_{n-1} \succ r_n
\end{align*}
$$

It is easy to see that $\text{mrk}(\pi^*) = n-1$, while $\text{mrk}(\pi) = 1$, thus $\text{PoA} = \Omega(n)$. Clearly, the PoA cannot be worse, thus it is $\Theta(n)$.

9. Experimental analysis

In this section, we investigate experimentally the swap-deal procedure ($C_2$) and compare it with the Top Trading Cycle algorithm and the Crawler procedure. As proved in Theorem 3, $C_2$ procedures always return a Pareto-Optimal allocation under single-peaked preferences. In section 8, we described theoretical results dealing with the price of anarchy. However, PoA considers the worst-case ranks that can be obtained by $C_2$ procedures. Another relevant question is to study the ranks obtained in practice by these procedures. In the following experiments, we study the average rank and the minimum rank of the allocations reached by the different procedures in comparison with the optimal values that can be computed in a centralized way.

9.1. Experimental protocol

We consider and compare six distinct procedures: the three procedures investigated in the paper (swap-deal procedure, Crawler and TTC), the procedure $C_3$ that realizes improving deals of size 2 or 3 until stability, and the two allocation rules $\text{max mrk IR}$ and $\text{max mrk IR}$ which respectively returns the allocation maximizing the average and the minimum ranks while being individually rational. For the last two allocation rules, the allocation returned can be computed using the technique defined by Garfinkel (1971).

Different methods can be envisioned to generate single-peaked preferences. We consider impartial culture for single-peaked domain (IC-SP) and uniform peak for the single-peaked domain (UP-SP). Single-peaked preferences under impartial culture (IC-SP) are drawn using the method proposed by Walsh (2015). Given an axis, single-peaked preferences are built by a recursive method from the end (i.e. the worst resource of the agent) to the top resource. At each iteration, the next resource in the preference order is randomly selected between the two extremes of the axis. The selected resource is then removed from the axis and so on until the axis is empty. In the uniform peak culture (UP-SP), presented by Conitzer (2009), preferences are constructed by first picking uniformly at random a resource to be the peak. The second-highest ranked resource is chosen with equal probability from the two adjacent alternatives, and so on until a full order is obtained.

For each experiment, we varied the size of the instances from $n = 2$ to $n = 60$ and we randomly generated 1000 instances for each size. For each instance, we ran the six procedures mentioned above and we computed the efficiency ratio of the outcomes averaged over the 1000 instances. The efficiency ratio is defined as the ratio between the rank realized by the procedure and the maximal rank achievable for the instance without imposing the individual rationality constraint (unlike the two allocation rules "max mrk IR" and "max mrk IR"). We are thus comparing relative efficiency of each procedure. Figures 4, 8 and 9 describe efficiency ratios for the minimum rank and the average rank. We also compare the execution efficiency of the procedures by recording the number of deals performed until convergence (see Figures 5 and 6) and the lengths of the cycle-deals (see Figure 7).

9.2. Selection dynamics for the swap-deal procedures

Concerning the $C_2$ procedure, we investigated several dynamics to select the improving deal to be performed among all possible improving deals. We more specifically focused on the following dynamics:

- **Round Robin over the agents ($C_2$ RRA):** agents are considered following the sequence $(a_1, \ldots, a_n)$, the first improving swap-deal found is implemented. We thus consider swaps between pairs of agents in the following order: $(a_1, a_2), (a_1, a_3), \ldots, (a_1, a_n), (a_2, a_3), \ldots, (a_{n-1}, a_n)$.

- **Uniform ($C_2$ U):** a swap-deal is selected uniformly at random among all possible improving swap-deals.

- **Priority to new ($C_2$ PN):** a swap-deal is selected uniformly at random among all the improving swap-deals involving pairs of agents who have never performed a deal yet. If no such pair of agents exists all the improving swap-deals are considered and one deal is selected uniformly at random. This dynamic favours agents who have not exchanged yet.

- **Priority to the worst-off agent ($C_2$ PW):** agents are ordered considering the rank of their resource from the one with the lowest rank to the one with the highest rank. The first improving swap-deal found is implemented.
• **Round Robin over pairs of agents** (C2 RRP): pairs of agents are considered in the following sequence \((a_1, a_2), (a_3, a_4), \ldots, (a_{n-1}, a_n), (a_1, a_3), (a_2, a_4) \ldots (a_1, a_n)\), the first improving swap-deal found is performed. This dynamics reduces the impact of the agent ordering used in Round Robin.

It has to be noticed that implementing these dynamics in a decentralized way may require some additional communication between the agents. Hence, round robin dynamics require the agents to know which pairs of agents have already been considered. The dynamics giving priority to the worst-off requires each agent to know the ranks of the other agents.

We first investigate whether the selection dynamics for the swap-deal procedure C2 influences the efficiency of the outcomes and the efficiency of the procedure. We thus compared the dynamics in terms of efficiency and in terms of number of swaps performed before reaching a stable allocation.

Figure 4 presents the efficiency ratio for each selection dynamic for IC-SP preferences (left side) and UP-SP preferences (right side). Regarding the average rank of the outcomes (lower part of Figure 4), all the dynamics obtain very good results (above 90% under IC-SP and above 96% under UP-SP). It can also be observed that all dynamics give similar values except C2 PW which gives better results. Regarding the minimum rank (upper part of Figure 4), C2 PW also allows for reaching significantly better allocations. In fact, this dynamic favours deals between low ranked agents and thus tends to improve the rank of the poorest agents. On the contrary, Round Robin dynamics tends to always favour the same agents and thus often leads to lower minimum rank.

We also investigate how the various dynamics of C2 influence the number of swaps performed by the agents. On Figure 5, solid lines represent the mean number of swap-deals performed when varying the size of the instances. The dynamics that are not represented all performed a number of swaps similar to either C2 RRA or C2 U so, they are not plotted for readability reasons. Dotted lines represent the highest and the lowest numbers of swaps registered for an instance of a given size (averaged over 1000 randomly generated instances).

It can be observed that all the dynamics lead to quite the same average number of swaps. However, the number of swaps performed under UP-SP is significantly higher than under IC-SP. This phenomenon is related to the method used to generate single-peaked preferences. As already mentioned by (Walsh, 2015), the probabilities of the preference orders significantly differ from one method to another. Under IC-SP, each single-peaked preference order has a uniform probability \(\frac{1}{n!}\) to be selected. On the contrary, under UP-SP, probabilities over preference orders are not uniform. In fact, the peak is uniformly drawn (with probability \(\frac{1}{n}\)) and single-peaked preferences are then built from this peak. Since, there is only one preference order for each peak at the ends of the axis, these preference orders are more likely to be returned than preference orders with a peak in the middle of the axis.

### Table 1: Linear regression of the maximum size of the deals over the number of agents.

|                | TTC IC-SP | TTC UC-SP | Crawler IC-SP | Crawler UC-SP |
|----------------|-----------|-----------|---------------|---------------|
| \(\beta_0\)    | 4.64      | 4.73      | 4.27          | 2.86          |
| \(\beta_1\)    | 0.15      | 0.36      | 0.20          | 0.56          |
| \(R^2\)        | 0.7872    | 0.9032    | 0.883         | 0.9862        |
| p-value         | < 0.001   | < 0.001   | < 0.001       | < 0.001       |

The axis is in a linear regression of the maximum size of the deals over the number of agents. \(\beta_0\) and \(\beta_1\) are the coefficients of the regression: \(\text{maxSize} = \beta_0 + \beta_1 \times \text{NbAgent}\), \(R^2\) the coefficient of determination and the p-value is the one of the model.

9.3. Centralized and decentralized procedures

The number of deals performed by C2 can be compared with the number of deals induced by the TTC algorithm or the Crawler algorithm. As shown in Figure 6, the last two procedures perform less exchanges than the C2 procedures since they allow for larger sizes of cycle-deals. Nevertheless, the sizes of the deals can be extremely large as depicted in Figure 7. Although the average size of the deals is quite low as shown by the solid lines in Figure 7, the size of the largest cycles can be large compared to the number of agents (dotted lines in Figure 7). Hence, a cycle-deal may involve more than half (resp. 35%) of the agents under UC-SP and a fifth (resp. 15%) of the agents under UP-SP for the Crawler (resp. TTC). The linear regressions explaining these values are presented in Table 1. Such cycle-deals can be difficult to implement in practice since they require coordination between a large number of agents that may be hard to achieve as debated in Section 1.
Figure 4: Mean efficiency ratio for each dynamic of the swap-deal procedure and for each preference culture.

Figure 5: Mean number of swaps performed, the filled area represents the range, from the minimum to the maximum.

Figure 6: Mean number of cycles performed by TTC and the Crawler, the filled area represents the range, from the minimum to the maximum.
We now turn our attention to the efficiency of the $C_2$ procedures compared to the other centralized procedures. The previous section showed a clear gap between the dynamics giving priority to the agent with the lowest utility compared to the others ($C_2$ PW). We thus focus on this dynamics when studying $C_2$. We also keep track of the $C_2$ U dynamics as it can be implemented in a fully decentralized way without additional communication between the agents.

We first report on the experiments about the average minimum rank of the outcomes (upper part of Figure 8). The first observation is that the results under UP-SP are significantly lower than the ones obtained under IC-SP. Under UP-SP, $C_2$ PW reaches ratios over 80% whereas it goes below 40% under IC-SP for 60 agents. Again, the way the single-peaked are built and the correlation between the preferences of the agents have a major impact on the performances of all procedures.

Another important observation is that $C_2$ performs significantly better than the TTC algorithm and the Crawler algorithm. $C_2$ also outperforms $C_3$ when considering the minimum rank. The performances of TTC are not surprising since this algorithm implements the best cycle and then discards the resources and the agents involved in the deal. This significantly limits the range of possible deals for the remaining agents and tends to disadvantage low-ranked agents. On the other hand $C_2$ PW favours low ranked agents and gives more opportunities to these agents to exchange their initial resources. Higher minimum ranks are then obtained. However, it can be observed that even when the $C_2$ U dynamics is considered, swap deals outperform the TTC algorithm. Indeed, even $C_2$ U gives more chances for the agents with low-ranked resources to perform some improving deals.

It can also be observed that $C_3$ improves little over $C_2$ when considering the same dynamics (Uniform selection of the exchanges in this case). Slightly increasing the size of the deals leads to few improvements while it raises more complex coordination issues.

We then study the average rank of the outcomes obtained by the different procedures (lower part of Figure 8). It can first be observed that $C_2$ also gives very good results when considering the average rank and $C_2$ PW outperforms all the other methods. All methods provide outcomes with high average rank (above 89% under IC-SP and above 96% under UP-SP). In particular $C_2$ outperforms TTC and the Crawler under both cultures. Again, it can be noticed that there is no significant difference between $C_2$ and $C_3$ when considering the same dynamics and there is no interest to consider slightly larger deals.

These experiments promote the relevance of the $C_2$ procedure: besides being simple to implement, $C_2$ also provides very good results both in terms of average rank and minimum rank.

We conclude this section with a comparison between the Top Trading Cycle algorithm and the Crawler algorithm. As it can be seen on Figure 9, both algorithms give quite similar performances regarding the minimum rank and the average rank. Indeed, on Figure 9 both curves almost always coincide. However, as depicted in Figure 6, the Crawler algorithms performs much more deals than TTC algorithm. The size of the deals is also larger when implementing the Crawler algorithm. The larger number of deals and the larger sizes of deals performed by the Crawler are related to the fact that agents are ordered with respect to the resource they initially hold and with respect to the order of these resources on the single-peaked axis. Based on this order, the Crawler only considers deals $\mu = \langle a_1, \ldots, a_k \rangle$ such that $a_i$ and $a_{i+1}$ (with $i \in \{1, 2, \ldots, k-1\}$) are owners of adjacent resources on the axis. Hence, TTC allows a larger range of cycle-deals than the Crawler. Of course, neither TTC or the Crawler were designed to optimize these objectives: their interest rely in the strong strategy-proofness that they offer (and, for the Crawler, on the fact that it can even be implemented in obviously dominant strategies (Bade, 2019)). Still, we believe these findings provide interesting insights regarding the nature of outcomes they provide.

10. Discussion and conclusion

This paper considered the fair division problem of indivisible resources in the restricted setting of a house market and under single-peaked preferences. We first
Figure 8: Mean efficiency ratio for each procedure and for each preference domain.

Figure 9: Mean efficiency ratio for the Crawler and the TTC algorithm, the filled are represents the standard deviation.
focused on centralized procedures with a particular interest for the Crawler procedure, and showed that it can be adapted to check Pareto-optimality with optimal communication complexity using the Diver procedure. We then turned our attention to decentralized procedures and focused on swap-deals. The efficiency (in terms of Pareto-optimality) of swaps-deals has been proved and we showed that the single-peaked domain is maximal for the swap-deal procedure. To refine our analysis, we also concentrated on two notions: the average rank and the minimum rank of the resources obtained by the agents. None of the procedures discussed in this paper are specifically designed for optimizing these ranks, even though these notions capture very natural notions of efficiency (for the average rank) and fairness (for the min rank). It thus seems important to study how these allocation procedures behave on that respect. Of course, moving from centralized to decentralized procedures incurs a cost that we analyzed through the price of anarchy, but in fact individual rationality alone incurs similar costs. To complement these theoretical bounds, we ran experiments which demonstrated that such swap dynamics provide in practice very good results. In particular, using the “priority for the worst off” dynamics provides a particularly fair and efficient swap-deal procedure, but it involves giving away in terms of distribution, as agents must coordinate to identify the agent which should deal next (besides triggering obvious issues of manipulation). Still, simpler (less informed) dynamics already offer very satisfying results.

To go further with the experiments, it would be interesting to use real data. Our attempt to use data from Preflib (Mattei and Walsh, 2013) was not successful as there is no dataset that is single-peaked when there are more than 5 agents. Getting such preferences would be an interesting way to confirm our results.

Regarding the model itself, we observe that Bade (2019) extended the Crawler procedure to single-peaked domains with indifferences. Whether our results with swap-deal procedures could be similarly generalized is an avenue for future research.

Overall, this paper raises the exciting issue of giving a characterization of rules that are efficient, individually rational and strategy-proof for the single-peaked domain. Such characterization would follow the spirit of the idea developed by Sprumont (1991). It would also be interesting to tackle the characterization of the swap-deal procedure efficiency, that is, giving a precise characterization of the domain on which the swap-deal procedure is efficient. Our maximality result is a significant step in this sense, it would be nice to complete the picture.

Acknowledgements

We thank Sophie Bade, Yann Chevaleyre, Anastasia Damamme, and Julien Lesca, for discussions related to this topic.

References

Abbassi, Z., Lakshmanan, L.V., Xie, M., 2013. Fair recommendations for online barter exchange networks., in: Proceedings of the 16th International Workshop on the Web and Databases (WebDB), pp. 43–48.

Abraham, D.J., Blum, A., Sandholm, T., 2007. Clearing algorithms for barter exchange markets: Enabling nationwide kidney exchanges, in: Proceedings of the 8th ACM Conference on Electronic Commerce (ACM-EC), pp. 295–304.

Ackermann, H., Goldberg, P.W., Mirrokni, V.S., Röglin, H., Vöcking, B., 2011. Uncoordinated two-sided matching markets. SIAM Journal on Computing 40, 92–106.

Anshelevich, E., Das, S., Naamad, Y., 2013. Anarchy, stability, and utopia: creating better matchings. Autonomous Agents and Multi-Agent Systems 26, 120–140.

Arrow, K.J., 1951. Social choice and individual values .

Aziz, H., Biró, P., Lang, J., Lesca, J., Monnot, J., 2016. Optimal reallocation under additive and ordinal preferences, in: Proceedings of the 15th International Joint Conference on Autonomous Agents and Multi-Agent Systems (AAMAS), pp. 402–410.

Aziz, H., De Keijzer, B., 2012. Housing markets with indifferences: A tale of two mechanisms, in: Proceedings of the 26th AAAI Conference on Artificial Intelligence (AAAI).

Aziz, H., Gaspers, S., Mackenzie, S., Walsh, T., 2015. Fair assignment of indivisible objects under ordinal preferences. Artificial Intelligence 227, 71–92.

Aziz, H., Hougaard, J.L., Moreno-Ternero, J.D., Østerdal, L.P., 2017. Computational aspects of assigning agents to a line. Mathematical Social Sciences 90, 93–99.

Ballester, M.A., Haeringer, G., 2011. A characterization of the single-peaked domain. Social Choice and Welfare 36, 305–322.

Bentert, M., Chen, J., Froese, V., Woeginger, G.J., 2019. Good things come to those who swap objects on paths. CoRR abs/1905.04219.

Beynier, A., Bouveret, S., Lemaître, M., Maudet, N., Rey, S., Shams, P., 2019. Efficiency, sequenceability and deal-optimality in fair division of indivisible goods, in: Proceedings of the 18th International Joint Conference on Autonomous Agents and Multi-Agent Systems (AAMAS).

Black, D., 1948. On the rationale of group decision-making. Journal of political economy 56, 23–34.

Bogomolnaia, A., Moulin, H., 2001. A new solution to the random assignment problem. Journal of Economic theory 100, 295–328.

Van den Bos, K., Lind, E.A., Vermunt, R., Wilke, H.A., 1997. How do I judge my outcome when I do not know the outcome of others? the psychology of the fair process effect. Journal of personality and social psychology 72, 1034.

Bouveret, S., Chevaleyre, Y., Maudet, N., 2016. Fair allocation of indivisible goods., in: Brandt, F., Conitzer, V., Endriss, U., Lang, J., Procaccia, A.D.E. (Eds.), Handbook of Computational Social Choice. Cambridge University Press.

Bouveret, S., Endriss, U., Lang, J., 2010. Fair division under ordinal preferences: Computing envy-free allocations of indivisible goods., in: Proceedings of the 19th European Conference on Artificial Intelligence (ECAI), pp. 387–392.

Bouveret, S., Lang, J., 2011. A general elicitation-free protocol for allocating indivisible goods, in: Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI).

Brams, S.J., Edelman, P.H., Fishburn, P.C., 2003. Fair division of indivisible items. Theory and Decision 55, 147–180.

Brams, S.J., Kilgour, D.M., Klamler, C., 2012. The undercut procedure: an algorithm for the envy-free division of indivisible items. Theory and Decision 55, 147–180.

Brams, S.J., Kilgour, D.M., Klamler, C., 2012. The undercut procedure: an algorithm for the envy-free division of indivisible items. Theory and Decision 55, 147–180.

Brams, S.J., Taylor, A.D., 1996. Fair Division: From cake-cutting to dispute resolution. Cambridge University Press.

Brams, S.J., Taylor, A.D., 2000. The win-win solution: guaranteeing fair shares to everybody. WW Norton & Company.

Brandt, F., Conitzer, V., Endriss, U., Lang, J., Procaccia, A.D., 2016. Handbook of computational social choice. Cambridge University Press.

Brunner, M.L., Lackner, M., 2015. On the likelihood of single-peaked preferences. CoRR abs/1505.05852.

Chevaleyre, Y., Dunne, P.E., Endriss, U., Lang, J., Lemaître, M., Maudet, N., Padget, J., Phelps, S., Rodriguez-Aguilar, J.A., Sousa, P., 2006. Issues in multiagent resource allocation. Informatica 30.

Chevaleyre, Y., Endriss, U., Estivie, S., Maudet, N., 2007. Reaching envy-
free states in distributed negotiation settings, in: Proceedings of the 20th International Joint Conference on Artificial Intelligence (IJCAI), pp. 1239–1244.

Chevaleyre, Y., Endriss, U., Maudet, N., 2010. Simple negotiation schemes for agents with simple preferences: Sufficiency, necessity and maximality. Autonomous Agents and Multi-Agent Systems 20, 234–259.

Chevaleyre, Y., Endriss, U., Maudet, N., 2017. Distributed Fair Allocation of Indivisible Goods. Artificial Intelligence 242, 1–22.

Ching, S., 1994. An alternative characterization of the uniform rule. Social Choice and Welfare 11, 131–136.

Conitzer, V., 2009. Elliciting single-peaked preferences using comparison queries. Journal of Artificial Intelligence Research 35, 161–191.

Damamme, A., Beynert, A., Chevaleyre, Y., Maudet, N., 2015. The power of swap deals in distributed resource allocation, in: Proceedings of the 14th International Joint Conference on Autonomous Agents and Multi-Agent Systems (AAMAS), pp. 625–633.

Elkind, E., Läckner, M., Peters, D., 2017. Structured preferences, in: E., U. (Ed.), Trends in Computational Social Choice, pp. 187–207.

Endriss, U., Maudet, N., 2005. On the communication complexity of multilateral trading. Autonomous Agents and Multi-Agent Systems 11, 91–107.

Endriss, U., Maudet, N., Sadri, F., Toni, F., 2006. Negotiating socially optimal allocations of resources. Journal of artificial intelligence research.

Garfinkel, R.S., 1971. An improved algorithm for the bottleneck assignment problem. Operations Research 19, 1747–1751.

Gourvès, L., Lesca, J., Wilczynski, A., 2017. Object allocation via swaps along a social network, in: Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI), pp. 213–219.

Henry, C., 1970. Indivisibilités dans une économie d’échanges. Econométrie, 542–558.

Herein, D., Puppe, C., 2002. A simple procedure for finding equitable allocations of indivisible goods. Social Choice and Welfare 19, 415–430.

Hougaard, J.L., Moreno-Ternero, J.D., Østelder, L.P., 2014. Assigning agents to a line. Games and Economic Behavior 87, 539–553.

Kasajima, Y., 2019. Minimal envy and popular matching. arXiv preprint arXiv:1902.08003.

Koutsoupias, E., Papadimitriou, C., 1999. Worst-case equilibria, in: Meinel, C., Tison, S. (Eds.), STACS 99, Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 404–413.

Kushilevitz, E., Nisan, N., 1996. Communication Complexity. Cambridge University Press.

Lang, J., Roth, J., 2016. Fair division of indivisible goods, in: Roth, J. (Ed.), Economics and Computation, pp. 493–550.

Leventhal, G.S., 1980. What should be done with equity theory?, in: Social exchange. Springer, pp. 27–55.

Li, S., 2017. Obviously strategy-proof mechanisms. American Economic Review 107, 2527–2537.

Ma, J., 1994. Strategy-proofness and the strict core in a market with indivisibilities. International Journal of Game Theory 23, 75–83.

Mattei, N., Walsh, T., 2013. Preflib: A library for preferences http://www.preflib.org, in: Proceedings of the 3rd International Conference on Algorithmic Decision Theory (ADT), Springer, pp. 259–270.

Moulin, H., 1991. Axioms of cooperative decision making. 15, Cambridge university press.

Moulin, H., 2004. Fair division and collective welfare. MIT press.

Moulin, H., 2018. Fair division in the age of internet.

Nguyen, N.T., Nguyen, T.T., Roos, M., Roth, J., 2014. Computational complexity and approximability of social welfare optimization in multiagent resource allocation. Autonomous agents and multi-agent systems 28, 256–289.

Nguyen, T.T., Roos, M., Roth, J., 2013. A survey of approximability and inapproximability results for social welfare optimization in multiagent resource allocation. Annals of Mathematics and Artificial Intelligence 68, 65–90.

Pazner, E.A., Schmeidler, D., 1978. Egalitarian equivalent allocations: A new concept of economic equity. The Quarterly Journal of Economics 92, 671–687.

Plaxton, C.G., 2013. A simple family of top trading cycles mechanisms for housing markets with indifferences.

Rawls, J., 1971. A theory of justice. Harvard university press.

Rosenschein, J.S., Zlotkin, G., 1994. Rules of encounter: designing conventions for automated negotiation among computers. MIT press.

Roth, A.E., 1982. Incentive compatibility in a market with indivisible goods. Economics letters 9, 127–132.

Roth, A.E., Sonmez, T., Ünver, M.U., 2005. Pairwise kidney exchange. Journal of Economic theory 125, 151–188.

Roth, A.E., Vate, J.H.V., 1990. Random paths to stability in two-sided matching. Econometrica, 1475–1480.

Saban, D., Sethuraman, J., 2013. House allocation with indifferences: a generalization and a unified view, in: Proceedings of the 14th ACM Conference on Electronic Commerce (ACM-EC), ACM, pp. 803–820.

Saffidine, A., Wilczynski, A., 2018. Constrained swap dynamics over a social network in distributed resource reallocation, in: Proceedings of the 11th International Symposium on Algorithmic Game Theory (SAGT), pp. 213–225.

Sandholm, T., 1998. Contract types for satisficing task allocation, in: Proceedings of the Spring Symposium of AAAI Conference on Artificial Intelligence (AAAI), pp. 23–25.

Sen, A.K., 1966. A possibility theorem on majority decisions. Econometrica, 491–499.

Shapley, L., Scarf, H., 1974. On cores and indivisibility. Journal of mathematical economics 1, 23–37.

Sprumont, Y., 1991. The division problem with single-peaked preferences: a characterization of the uniform allocation rule. Econometrica, 509–519.

Sprumont, Y., 1996. Axiomatizing ordinal welfare egalitarianism when preferences may vary. Journal of Economic Theory 68, 77–110.

Steinhaus, H., 1948. The problem of fair division. Econometrica 16, 234–259.

Thibault, J.W., Walker, L., 1975. Procedural justice: A psychological analysis. L. Erlbaum Associates.

Thomson, W., 1983. Problems of fair division and the egalitarian solution. Journal of Economic Theory 31, 211–226.

Thomson, W., 1994a. Consistent solutions to the problem of fair division when preferences are single-peaked. Journal of Economic Theory 63, 219–245.

Thomson, W., 1994b. Resource-monotonic solutions to the problem of fair division when preferences are single-peaked. Social Choice and Welfare 11, 205–223.

Thomson, W., 2016. Introduction to the theory of fair allocation, in: Brandt, F., Conitzer, V., Endriss, U., Lang, J., Procaccia, A.D. (Eds.), Handbook of Computational Social Choice. Cambridge University Press, p. 261–283.

Walsh, T., 2015. Generating single peaked votes. arXiv preprint arXiv:1503.02766.

Young, H.P., 1995. Equity: in theory and practice. Princeton university press.