Convergence of solutions of mixed stochastic delay differential equations with applications

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Abstract
The paper is concerned with a mixed stochastic delay differential equation involving both a Wiener process and a $\gamma$-H"older continuous process with $\gamma > 1/2$ (e.g. a fractional Brownian motion with Hurst parameter greater than $1/2$). It is shown that its solution depends continuously on the coefficients and the initial data. Two applications of this result are given: the convergence of solutions to equations with vanishing delay to the solution of equation without delay and the convergence of Euler approximations for mixed stochastic differential equations. As a side result of independent interest, the integrability of solution to mixed stochastic delay differential equations is established.

Keywords: Mixed stochastic differential equation, stochastic delay differential equation, convergence of solutions, fractional Brownian motion, vanishing delay, Euler approximation

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Introduction
In this paper we consider a multidimensional mixed stochastic delay differential equation

$$dX(t) = a(t, X_t)dt + b(t, X_t)dW(t) + c(t, X_t)dZ(t), \quad t \in [0, T].$$

(1)

In this equation the coefficients $a, b, c$ depend on the past $X_s = \{X(s + u), u \in [-r, 0]\}$ of the process $X$, and the initial condition is thus given by $X(t) = \eta(t), \quad t \in [-r, 0]$, where $\eta: [-r, 0] \to \mathbb{R}$ is some function. Equation (1) is driven by two random processes: a standard Wiener process $W$ and a process $Z$, whose trajectories are $\gamma$-H"older continuous with $\gamma > 1/2$. The process $Z$ is usually a long memory process, e.g. a fractional Brownian motion $B^H$ with the Hurst parameter $H > 1/2$.

It\text{"o} stochastic delay differential equations, i.e. those with $c = 0$, were investigated in many articles, see \cite{9, 13} and references therein. Fractional stochastic delay differential equations, in which $b = 0$ and $Z = B^H$, were considered in only few papers. For $H > 1/2$, the existence and uniqueness of solution under different sets of assumptions was established in \cite{1, 2, 3, 4, 5, 8}. In the case $H > 1/3$, the existence and uniqueness of solution was shown in \cite{14} for coefficients of the form $f(X(t), X(t - r_1), X(t - r_2), \ldots)$.

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The existence and uniqueness of solution to equation (1) was established in [15], where also the finiteness of moments was shown under the additional assumption that the coefficient $b$ is bounded.

Mixed equations without delay were considered in articles [6, 7, 11, 12, 16, 17].

In this article we prove that if the coefficients and initial conditions of mixed stochastic delay differential equations converge, then their solutions converge uniformly in probability. We give two applications of this result. First we show that when the delay vanishes, solutions of mixed stochastic delay differential equations converge uniformly in probability to a solution of equation without delay. Then we establish the uniform convergence of Euler approximations for mixed stochastic differential equation towards its solution. We also extend the results of [16] about the integrability of solution to (1), dropping the assumption that $b$ is bounded.

The paper is organized as follows. Section 1 provides necessary information about pathwise stochastic integration and describes the notation used in the article. Section 2 contains the main result of the article about convergence of solutions to mixed stochastic delay differential equations with convergent coefficients and initial conditions. It also contains a result of independent interest about the integrability of solution to mixed stochastic delay differential equation. Section 3 is devoted to applications of the convergence results. In Subsection 3.1 a sequence of mixed stochastic delay differential equations is considered with delay horizon converging to zero, and it is shown that their solutions converge to a solution of equation without delay. In Subsection 3.2 it is proved that the Euler approximations for mixed stochastic differential equation (without delay) converge to its solution, as the mesh of partition goes to zero. Appendix contains an auxiliary result about convergence of solutions to Itô stochastic delay differential equations with random coefficients, which is used in the proof of main theorem.

1. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P} = \{\mathcal{F}_t, t \geq 0\}, P)$ be a standard stochastic basis.

Throughout the article, $|\cdot|$ will denote the absolute value of a real number, the Euclidean norm of a vector, or the operator norm of a matrix. The symbol $C$ will denote a generic constant, whose value may change from one line to another. To emphasize its dependence on some parameters, we will put them into subscripts.

We will need the notion of generalized fractional Lebesgue–Stieltjes integral. Below we give only basic information on the integral, further details and proofs can be found in [18].

Let $f, g: [a, b] \to \mathbb{R}, \alpha \in (0, 1)$. Define the forward and backward fractional Riemann–Liouville derivatives

$$(D^\alpha_{a+} f)(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int^x_a \frac{f(x) - f(u)}{(x-u)^{1+\alpha}} du \right),$$

$$(D^\alpha_{b-} g)(x) = \frac{e^{i\pi \alpha}}{\Gamma(\alpha)} \left( \frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int^b_x \frac{g(x) - g(u)}{(u-x)^{2-\alpha}} du \right),$$

where $x \in (a, b)$.

The generalized Lebesgue–Stieltjes integral is defined as

$$\int^b_a f(x) dg(x) = e^{i\pi \alpha} \int^b_a (D^\alpha_{a+} f)(x) (D^\alpha_{b-} g)(x) dx$$

provided the integral in the right-hand side exists. For functions $f, g: [a, b] \to \mathbb{R}$ and a number
\( \alpha \in (0, 1) \) define
\[
\|f\|_{1, \alpha; [a, b]} = \int_a^b \left( \frac{|f(a)|}{(t - a)^\alpha} + \int_a^t \frac{|f(t) - f(s)|}{(t - s)^{1+\alpha}} \, ds \right) \, dt, \tag{2}
\]
\[
\|g\|_{0, \alpha; [a, b]} = \sup_{a \leq s \leq t \leq b} \left( \frac{|g(t) - g(s)|}{(t - s)^{1-\alpha}} + \int_s^t \frac{|g(u) - g(s)|}{(u - s)^{2-\alpha}} \, du \right). \tag{3}
\]

Note that only the first expression defines a norm, the second defines a seminorm. The following fact is evident: if \( \|f\|_{1, \alpha; [a, b]} < \infty \) and \( \|g\|_{0, \alpha; [a, b]} < \infty \), then the generalized Lebesgue–Stieltjes integral is well defined and admits an estimate
\[
\left| \int_a^b f(x) \, dg(x) \right| \leq \frac{1}{\Gamma(\alpha)\Gamma(1 - \alpha)} \|f\|_{1, \alpha; [a, b]} \|g\|_{0, \alpha; [a, b]}. \tag{4}
\]
We will also need an estimate in terms of Hölder norms. Namely, if \( f \in C^\lambda[a, b], \ g \in C^\mu[a, b] \) with \( \lambda \in (0, 1), \mu \in (0, 1) \) and \( \lambda + \mu > 1 \), then the generalized Lebesgue–Stieltjes integral is well defined and coincides with the limit of Riemann–Stieltjes integral sums. Moreover, the Young–Love inequality holds:
\[
\left| \int_a^b f(s) \, dg(s) \right| \leq C_{\lambda, \mu} \|g\|_{a, b, \mu} \left( \|f\|_{a, b, \infty} + \|f\|_{a, b, \lambda}(b - a)^\lambda \right)(b - a)\mu,
\]
where \( \|f\|_{a, b, \infty} = \sup_{x \in [a, b]} |f(x)| \) is the supremum norm on \([a, b]\), and
\[
\|f\|_{a, b, \lambda} = \sup_{a \leq x < y \leq b} \frac{|f(y) - f(x)|}{(y - x)^\lambda}
\]
is the Hölder seminorm.

2. Limit theorem for mixed equation

Fix some \( T > 0 \) and \( r > 0 \) (they will play a role of time horizon and delay horizon, respectively) and denote by \( \mathcal{C} = C([-r, 0]; \mathbb{R}^d) \) the space of continuous \( \mathbb{R}^d \)-valued functions defined on the interval \([-r, 0]\). This space is a Banach space with the supremum norm \( \|\psi\|_{\mathcal{C}} = \max_{s \in [-r, 0]} |\psi(s)|, \ \psi \in \mathcal{C} \).

In order to introduce the dependence on past, for a stochastic process \( \xi = \{\xi(t), t \in [-r, T]\} \) define a segment \( \xi_t \in \mathcal{C} \) at the point \( t \in [0, T] \) by \( \xi_t(u) = \xi(s + u), u \in [-r, 0] \).

Consider the following sequence of mixed stochastic delay differential equations (SDDEs) in \( \mathbb{R}^d \) indexed by \( n \geq 0 \):
\[
X^n(t) = \eta^n(0) + \int_0^t a^n(s, X^n_s) \, ds + \sum_{i=1}^m \int_0^t b^n_i(s, X^n_s) \, dW_i(s) + \sum_{j=1}^l \int_0^t c^n_j(s, X^n_s) \, dZ_j(s), t \in [0, T];
\]
\[
X^n(t) = \eta^n(t), t \in [-r, 0]. \tag{5}
\]
Here \( a^n, b^n_i, c^n_j : [0, T] \times \mathcal{C} \to \mathbb{R}^d, \ i = 1, \ldots, m, \ j = 1, \ldots, l \) are measurable functions; \( Z = \{Z(t), t \in [0, T]\} \) is an \( \mathbb{F} \)-adapted process in \( \mathbb{R}^d \) such that its trajectories are almost surely Hölder continuous of order \( \gamma > 1/2 \); \( W = \{W(t), t \in [0, T]\} \) is an \( \mathbb{F} \)-Wiener process in \( \mathbb{R}^m \), the “initial condition” \( \eta : [0, T] \to \mathbb{R}^d \) is non-random. The integral w.r.t. the Wiener process is understood as the usual Itô integral, which is well defined provided that the integrand belongs to \( L^2[0, T] \) a.s. The integral w.r.t. is understood in generalized Lebesgue–Stieltjes sense.

We will assume the following about the coefficients of equations \( (5) \):
H1. For all \( \psi \in C, t \in [0, T] \),
\[
|a^n(t, \psi)| + |b^n(t, \psi)| + |c^n(t, \psi)| \leq K(1 + \|\psi\|_C).
\]

H2. For all \( t \in [0, T], \psi \in C \), \( c^n \) has a Fréchet derivative \( \partial_{\psi}c^n(t, \psi) \) belonging to the space \( L(C, \mathbb{R}^d) \) of bounded linear operators from \( C \) to \( \mathbb{R}^d \), and this derivative is bounded uniformly in \( t \in [0, T], \psi \in C \):
\[
\|\partial_{\psi}c^n(t, \psi)\|_{L(C, \mathbb{R}^d)} \leq K.
\]

H3. The functions \( a^n, b^n \) and \( \partial_{\psi}c^n \) are locally Lipschitz continuous in \( \psi \): for any \( R > 0, t \in [0, T] \), and all \( \psi_1, \psi_2 \in C \) with \( \|\psi_1\|_C \leq R, \|\psi_2\|_C \leq R \),
\[
|a^n(t, \psi_1) - a^n(t, \psi_2)| + |b^n(t, \psi_1) - b^n(t, \psi_2)| + \|\partial_{\psi}c^n(t, \psi_1) - \partial_{\psi}c^n(t, \psi_2)\|_{L(C, \mathbb{R}^d)} \leq K_R \|\psi_1 - \psi_2\|_C.
\]

H4. The functions \( c^n \) and \( \partial_{\psi}c^n \) are Hölder continuous in \( t \): for some \( \beta \in (1 - \gamma, 1) \) and for all \( t_1, t_2 \in [0, T], \psi \in C \),
\[
|c^n(t_1, \psi) - c^n(t_2, \psi)| \leq K|t_1 - t_2|^{\beta}(1 + \|\psi\|_C), \quad \|\partial_{\psi}c^n(t_1, \psi) - \partial_{\psi}c^n(t_2, \psi)\|_{L(C, \mathbb{R}^d)} \leq K|t_1 - t_2|^{\beta}.
\]

H5. The initial condition \( \eta^n \) is a Hölder continuous function: for some \( \theta \in (1 - \gamma, 1/2) \) and for all \( t_1, t_2 \in [0, T] \),
\[
|\eta^n(t_1) - \eta^n(t_2)| \leq K|t_1 - t_2|^{\theta}.
\]

Fix some \( \alpha \in (1 - \gamma, 1/2) \), denote \( h(t, s) = (t - s)^{-1 - \alpha} \) and define for an either real- or vector-valued function \( f \)
\[
\|f\|_{\infty, t} = \sup_{s \in [-r, t]}|f(s)|, \\
\|f\|_{1, t} = \int_0^t \|f(\cdot + t - s) - f(\cdot)\|_{\infty, s} h(t, s) ds, \\
\|f\|_t = \|f\|_{\infty, t} + \|f\|_{1, t}.
\]

Also denote for brevity \( \|f\|_{0,t} = \|f\|_{0,0;[0,t]} \). It is proved in \cite{[15], Theorem 4.1} that under the assumptions H1–H5 equation \( (5) \) has a unique solution, which is an \( \mathbb{F} \)-adapted process \( X \) such that \( \|X\|_T < \infty \) a.s., and \( (5) \) holds almost surely for all \( t \in [0, T] \).

In the following we will abbreviate equations \( (5) \) and their ingredients as
\[
X^n(t) = \eta^n(0) + \int_0^t a^n(s, X^n_s) ds + \int_0^t b^n(s, X^n_s) dW(s) + \int_0^t c^n(s, X^n_s) dZ(s). \tag{6}
\]

We will need some auxiliary results from \cite{[15]}, which we for convenience state here in a slightly modified form.

**Lemma 2.1.** Let the coefficients of equation
\[
Y(t) = \eta(0) + \int_0^t a(s, Y_s) ds + \int_0^t b(s, Y_s) dW(s) + \int_0^t c(s, Y_s) dZ(s), \quad t \in [0, T],
\]
\[
Y(t) = \eta(t), \quad t \in [-r, 0]
\]
satisfy H1–H5, and let \( A_M = \{\|Z\|_{0:T} \leq M\} \) for \( M \geq 1 \). Then for each \( p \geq 1 \) there is a constant \( C = C_{M,p,r,T,K,\alpha,\eta(0)} \) such that
\[
\mathbb{E}[\|Y\|_T^p 1_{A_M}] \leq C.
\]
Lemma 2.2. Let the coefficients of equations

\[ Y^i(t) = \eta(0) + \int_0^t a(s, Y^i_s)ds + \int_0^t b(s, Y^i_s)dW(s) + \int_0^t c(s, Y^i_s)dZ(s), \ t \in [0, T], i = 1, 2, \]

\[ Y^i(t) = \eta(t), \ t \in [-r, 0] \]

satisfy H1–H5, and let \( A_{M, R} = \{ \|Z^1\|_{0, T} \leq M, \|Z^2\|_{0, T} \leq M, \|Y^1\|_T \leq R, \|Y^2\|_T \leq R \} \) for \( M \geq 1, \]

\( R \geq 1. \) Then for each \( p \geq 4/(2\alpha) \) there is a constant \( C = C_{M, R, p, r, K, K_\alpha} \) such that

\[ E \left[ \left\| Y^1 - Y^2 \right\|_{0, T}^p 1_{A_{M, R}} \right] \leq CE \left[ \left\| Z^1 - Z^2 \right\|_{0, T}^p 1_{A_{M, R}} \right]. \]

We will also need a result about the finiteness of moments of \( \|Y\|_{0, T, \infty} \).

Theorem 2.1. Let the coefficients of equation

\[ Y(t) = \eta(0) + \int_0^t a(s, Y_s)ds + \int_0^t b(s, Y_s)dW(s) + \int_0^t c(s, Y_s)dZ(s), \ t \in [0, T], \]

\[ Y(t) = \eta(t), \ t \in [-r, 0] \]

satisfy H1–H5, and the process \( \{Z(t), t \in [0, T]\} \) be such that

\[ E \left[ \exp \left\{ \left\| Z \right\|_{0, T, \infty}^{1/\gamma} \right\} \right] \leq L. \]

Then for any \( p > 0 \) there is a constant \( C = C_{p, r, T, K, L, \eta(0)} \) such that

\[ E \left[ \left\| Y \right\|_{0, T, \infty}^p \right] \leq C. \]

Proof. This proof closely follows that of Theorem 1 in [16]. Throughout the proof, \( C \) will denote a generic constant depending on \( p, r, T, K, L, \eta(0) \).

Fix some \( N \geq 1, \) \( R \geq 1 \) and denote \( \tau_{N, R} = \min \left\{ t \geq 0 : \left\| Z \right\|_{0, t, \infty} \geq R \text{ or } \left\| Z \right\|_{0, t, \gamma} \geq N \right\} \),

\[ Y^{N, R}(t) = Y(t \wedge \tau_{N, R}), \ 1_t = 1_t = 1_{\{ t \leq \tau_{N, R} \}}. \] Put \( I^a_t = \int_0^t a(s, Y_s^{N, R}) 1_s ds, \ I^b_t = \int_0^t b(s, Y_s^{N, R}) 1_s dW(s), \)

\[ I^c_t = \int_0^t c(s, Y_s^{N, R}) 1_s dZ(s). \]

Take a number \( \Delta \in (0, 1) \), whose value will be specified later. For \( t \in [0, T] \) and \( u, v \) such that

\( 0 \leq u < v \leq u + \Delta \leq t \) write

\[ \left| Y^{N, R}(v) - Y^{N, R}(u) \right| \leq \left| I^a_v - I^a_u \right| + \left| I^b_v - I^b_u \right| + \left| I^c_v - I^c_u \right|. \]

Estimate first

\[ \left| I^a_v - I^a_u \right| \leq \int_u^v \left| a(z, Y_z^{N, R}) \right| dz \leq C \int_u^v \left( 1 + \left\| Y_z^{N, R} \right\|_c \right) dz \]

\[ \leq C \left( 1 + \left\| Y^{N, R} \right\|_{0, t, \infty} + \left\| \eta \right\|_c \right) (v - u) \leq C \left( 1 + \left\| Y^{N, R} \right\|_{0, t, \infty} \right) (v - u), \]

where in the last step we have used the following simple observation: \( \|\eta\|_c \leq \|\eta(0)\| + r^\theta \|\eta\|_{-r, 0, \theta} \leq C + K(r + 1) \leq C. \)
From the Young–Love inequality it follows that
\[ |I_v^c - I_u^c| \leq CN \left( \| c(\cdot, Y^N_{t,\theta}) \|_{u,v,\infty} + \| c(\cdot, Y^N_{t,\theta}) \|_{u,v,\theta} (v-u)\theta \right) (v-u)\gamma. \]

Further, from the linear growth assumption
\[ \| c(\cdot, Y^N_{t,\theta}) \|_{u,v,\infty} \leq C \left( 1 + \| Y^N_{t,\theta} \|_{0,v,\infty} + \| \eta \|_c \right) \leq C \left( 1 + \| Y^N_{t,\theta} \|_{0,t,\infty} \right). \]

Define
\[ \| f \|_{a,b,\Delta,\theta} = \sup_{\substack{a \leq x < y \leq b, \\|y - x\| \leq \Delta}} \frac{|f(y) - f(x)|}{(y-x)^{\theta}}. \]

Since
\[ |c(x, Y^N_{x,\theta}) - c(y, Y^N_{y,\theta})| \leq |c(x, Y^N_{x,\theta}) - c(y, Y^N_{x,\theta})| + |c(y, Y^N_{x,\theta}) - c(y, Y^N_{y,\theta})| \leq C \left( \| x - y \|^{\beta} (1 + \| Y^N_{x,\theta} \|_c) + \| Y^N_{x,\theta} - Y^N_{y,\theta} \|_c \right), \]
then
\[ \| c(\cdot, Y^N_{t,\theta}) \|_{u,v,\theta} \leq C \left( (v-u)^{\beta-\theta} (1 + \| Y^N_{t,\theta} \|_{0,v,\infty} + \| \eta \|_c) + \| Y^N_{t,\theta} \|_{0,t,\Delta,\theta} + \| \eta \|_{-r,0,\theta} \right) \leq C \left( 1 + (v-u)^{\beta-\theta} (1 + \| Y^N_{t,\theta} \|_{0,t,\infty}) + \| Y^N_{t,\theta} \|_{0,t,\Delta,\theta} \right). \]

Therefore,
\[ |I_v^c - I_u^c| \leq CN \left( 1 + \| Y^N_{t,\theta} \|_{0,t,\infty} + \| Y^N_{t,\theta} \|_{0,t,\Delta,\theta} (v-u)\theta \right) (v-u)\gamma. \]

Collecting the above estimates, we get
\[ \| Y^N_{t,\theta} \|_{0,t,\Delta,\theta} \leq C \left( 1 + \| Y^N_{t,\theta} \|_{0,t,\infty} \right) \Delta^{1-\theta} + \| I^b \|_{0,t,\Delta,\theta} \leq C \left( 1 + \| Y^N_{t,\theta} \|_{0,t,\infty} \right) \Delta^{\gamma-\theta} + \| Y^N_{t,\theta} \|_{0,t,\Delta,\theta} \Delta^\gamma \]
\[ \leq \| I^b \|_{0,t,\Delta,\theta} + K'N \left( 1 + \| Y^N_{t,\theta} \|_{0,t,\infty} \right) \Delta^{\gamma-\theta} + \| Y^N_{t,\theta} \|_{0,t,\Delta,\theta} \Delta^\gamma \]

with certain non-random constant $K'$.

Suppose that $\Delta \leq (2K'N)^{-1/\gamma}$. Then
\[ \| Y^N_{t,\theta} \|_{0,t,\Delta,\theta} \leq 2 \| I^b \|_{0,t,\Delta,\theta} + 2K'N \left( 1 + \| Y^N_{t,\theta} \|_{0,t,\infty} \right) \Delta^{\gamma-\theta}. \] (7)

Let $s \in [0 \vee (t - \Delta), t]$. Then from the obvious inequality
\[ \| Y^N_{t,\theta} \|_{0,t,\infty} \leq \| Y^N_{t,\theta} \|_{0,s,\infty} + \| Y^N_{t,\theta} \|_{s,t,\theta} (t-s)\theta \leq \| Y^N_{t,\theta} \|_{0,s,\infty} + \| Y^N_{t,\theta} \|_{0,t,\Delta,\theta} \Delta^\theta \]
using (7), we obtain
\[ \| Y^N_{t,\theta} \|_{0,t,\infty} \leq \| Y^N_{t,\theta} \|_{0,s,\infty} + 2 \left( \| I^b \|_{0,t,\Delta,\theta} + K'N \right) (t-s)\theta + 2K'N \| Y^N_{t,\theta} \|_{0,t,\infty} (t-s)^\gamma \]
\[ \leq \| Y^N_{t,\theta} \|_{0,s,\infty} + 2 \left( \| I^b \|_{0,t,\Delta,\theta} + K'N \right) \Delta^\theta + 2K'N \| Y^N_{t,\theta} \|_{0,t,\infty} \Delta^\gamma. \]
Assuming further that $\Delta \leq (4K'N)^{-1/\gamma}$, we get
\[
\|Y^{N,R}\|_{0,t,\infty} \leq 2 \|Y^{N,R}\|_{0,s,\infty} + 4 \left( \|I^b\|_{0,t,\Delta,\theta} + K'N \right) \Delta^\theta.
\]
Hence we derive for any $p > 1$ that
\[
E \left[ \|Y^{N,R}\|_{0,t,\infty}^p \right] \leq C \left( E \left[ \|Y^{N,R}\|_{0,s,\infty}^p \right] + E \left[ \|I^b\|_{0,t,\Delta,\theta}^p \right] \Delta^\theta + N^p \right) .
\tag{8}
\]
Take some $\kappa \in (\theta, 1/2)$. Obviously, $\|I^b\|_{0,t,\Delta,\theta} \leq \Delta^{\kappa-\theta} \|I^b\|_{0,t,\kappa}$. Assuming that $p > (1/2 - \kappa)^{-1}$ and using the Garsia–Rodemich–Rumsey inequality, we get
\[
E \left[ \|I^b\|_{0,t,\kappa}^p \right] \leq C \int_0^t \int_0^t E \left[ \left| I^b(x) - I^b(y) \right|^p \right] |x-y|^{-p\kappa-2} dx dy.
\]
Plugging this estimate into (8), we arrive at the inequality
\[
E \left[ \|Y^{N,R}\|_{0,t,\infty}^p \right] \leq K'_p \left( E \left[ \|Y^{N,R}\|_{0,s,\infty}^p \right] + E \left[ \|Y^{N,R}\|_{0,t,\infty}^p \right] \Delta^{\kappa} + N^p \right)
\]
with certain constant $K'_p$. Assuming that $\Delta \leq (2K'_p)^{-1/(p\kappa)}$, we get
\[
E \left[ \|Y^{N,R}\|_{0,t,\infty}^p \right] \leq 2K'_p \left( E \left[ \|Y^{N,R}\|_{0,s,\infty}^p \right] + N^p \right). \tag{9}
\]
Finally, put $\Delta = \min \{(4K'N)^{-1/\gamma}, (2K'_p)^{-1/(p\kappa)}\}$. Splitting the segment $[0, T]$ into $[T/\Delta] + 1$ parts of length at most $\Delta$, we obtain from the estimate (9) that
\[
E \left[ \|Y^{N,R}\|_{0,T,\infty}^p \right] \leq (2K'_p + 1)^{T/\Delta + 1} (\|\eta(0)\|^p + N^p) \leq C \exp \left\{ CN^{1/\gamma} \right\}.
\]
Letting $R \to \infty$ and using the Fatou lemma, we get
\[
E \left[ \|X\|_{0,T,\infty}^p \|Z\|_{0,T,\gamma} \leq N \right] \leq K'_p \exp \left\{ K'_p N^{1/\gamma} \right\}
\]
with some constant $K'_p$. Denote $\xi = \|X\|_{0,T,\infty}^p, \eta = \|Z\|_{0,T,\gamma}$ and write
\[
(E [\xi^p])^2 \leq E \left[ \exp \left\{ 2K'_p^p \eta^{1/\gamma} \right\} \right] E \left[ \xi^{2p} \exp \left\{ -2K'_p \eta^{1/\gamma} \right\} \right]
\]
\[
\leq C \sum_{n=1}^{\infty} E \left[ \xi^{2p} \exp \left\{ -2K'_p \eta^{1/\gamma} \right\} \mathbb{1}_{\eta \in [n-1,n)} \right]
\]
\[
\leq C \sum_{n=1}^{\infty} \exp \left\{ -2K'_p (n - 1)^{1/\gamma} \right\} E \left[ \xi^{2p} \mathbb{1}_{\eta \in [n-1,n)} \right]
\]
\[
\leq C \sum_{n=1}^{\infty} \exp \left\{ -2K'_p (n - 1)^{1/\gamma} \right\} \exp \left\{ K'_p n^{1/\gamma} \right\} < \infty,
\]
where the last constant depends on the parameters specified. The proof is now complete. \qed
We will impose the following assumptions concerning the convergence.

C1. Convergence of the coefficients: For all \( \psi \in \mathcal{C} \), \( t \in [0, T] \),
    \[
a^n(t, \psi) \to a^0(t, \psi), \quad b^n(t, \psi) \to b^0(t, \psi), \quad c^n(t, \psi) \to c^0(t, \psi), \quad n \to \infty.
    \]

C2. Convergence of the initial conditions:
    \[
    \|\eta^n - \eta^0\|_\mathcal{C} \to 0, \quad n \to \infty.
    \]

**Theorem 2.2.** Under assumptions H1–H5 and C1–C2, the following convergence in probability takes place:
    \[
    \|X^n - X^0\|_{\infty, T} \overset{\text{P}}{\to} 0, \quad n \to \infty.
    \]

*Proof.* Obviously, we can assume without loss of generality that \( Z(0) = 0 \). Let for \( N \geq 1 \), \( x \in \mathbb{R}^d \)
    \[
h_N(x) = \begin{cases} 
    x, & |x| \leq N, \\
    N\frac{x}{|x|}, & |x| > N.
    \end{cases}
    \]

Define the following sequence of smooth approximations of \( Z \):
    \[
    Z^N(t) = N \int_{(t-1/N)\wedge 0}^t h_N(Z(s)) \, ds, \quad N \geq 1.
    \]

Consider auxiliary stochastic differential equations
    \[
    X^{n,N}(t) = \eta^n(0) + \int_0^t a^n(s, X^{n,N}_s) \, ds + \int_0^t b^n(s, X^{n,N}_s) \, dW(s) + \int_0^t c^n(s, X^{n,N}_s) \, dZ^N(s), \quad t \in [0, T];
    \]
    \[
    X^{n,N}(t) = \eta^n(t), \quad t \in [-r, 0].
    \]

Since \( Z^N \) is absolutely continuous,
    \[
dZ^N(t) = N \left( h_N(Z(t)) - h_N(Z((t - 1/N) \wedge 0)) \right) dt =: \dot{Z}^N(t) \, dt 
    \]
we can rewrite the equation (10) as an Itô delay differential equation with random coefficients
    \[
    X^{n,N}(t) = \eta^n(0) + \int_0^t f^{n,N}(s, X^{n,N}_s) \, ds + \int_0^t b^n(s, X^{n,N}_s) \, dW(s), \quad t \in [0, T];
    \]
    \[
    X^{n,N}(t) = \eta^n(t), \quad t \in [-r, 0],
    \]
where \( f^{n,N}(t, \psi) = a^n(t, \psi) + c^n(t, \psi) \dot{Z}^N(t), \quad t \in [0, T], \quad \psi \in \mathcal{C} \). Obviously, the coefficients of these equations satisfy assumptions of Theorem A.1 from Appendix A. Therefore,
    \[
    \|X^{n,N} - X^{0,N}\|_{\infty, T} \overset{\text{P}}{\to} 0, \quad n \to \infty.
    \]

Now write for \( \varepsilon > 0 \)
    \[
    \mathbb{P} \left( \|X^n - X^0\|_{\infty, T} > \varepsilon \right) \leq \mathbb{P} \left( \|X^n - X^{n,N}\|_{\infty, T} > \varepsilon/3 \right)
    + \mathbb{P} \left( \|X^{n,N} - X^{0,N}\|_{\infty, T} > \varepsilon/3 \right) + \mathbb{P} \left( \|X^{0,N} - X^0\|_{\infty, T} > \varepsilon/3 \right).
    \]
Hence,

\[
\limsup_{n \to \infty} P \left( \|X^n - X^0\|_{\infty,T} > \varepsilon \right) \leq 2 \sup_{n \geq 0} P \left( \|X^n - X^{n,N}\|_{\infty,T} > \varepsilon/3 \right) =: 2 \sup_{n \geq 0} P(A_{n,N,\varepsilon}).
\]

We need to show that \( \sup_{n \geq 0} P(A_{n,N,\varepsilon}) \to 0 \) as \( N \to \infty \). To this end, write for any \( M > 0 \), \( R > 0 \)

\[
P(A_{n,N,\varepsilon}) \leq P \left( A_{n,N,\varepsilon}, \|X^n,N\|_{T} \leq R, \|X^n\|_{T} \leq R, \|Z^N\|_{0:T} \leq M, \|Z\|_{0:T} \leq M \right)
\]

\[
+ P \left( \|X^n,N\|_{T} > R, \|Z^N\|_{0:T} \leq M \right) + P \left( \|X^n\|_{T} > R, \|Z\|_{0:T} \leq M \right)
\]

\[
+ P \left( \|Z^N\|_{0:T} > M \right) + P \left( \|Z\|_{0:T} > M \right).
\]

Since \( \|Z^N - Z\|_{0:T} \to 0 \), \( N \to \infty \) a.s. (it can be proved similarly to Lemma 2.1 in [12]), it follows easily from Lemma 2.2 that

\[
\sup_{n \geq 0} P \left( A_{n,N,\varepsilon}, \|X^n,N\|_{T} \leq R, \|X^n\|_{T} \leq R, \|Z^N\|_{0:T} \leq M, \|Z\|_{0:T} \leq M \right) \to 0, \; N \to \infty.
\]

Further, from Lemma 2.1 we have with the help of Chebyshev inequality that

\[
\sup_{n \geq 0} \left( \sup_{N \geq 1} P \left( \|X^n,N\|_{T} > R, \|Z^N\|_{0:T} \leq M \right) + P \left( \|X^n\|_{T} > R, \|Z\|_{0:T} \leq M \right) \right) \to 0, \; R \to \infty.
\]

Finally,

\[
\sup_{N \geq 1} \left( \sup_{n \geq 0} P \left( \|Z^N\|_{0:T} > M \right) + P \left( \|Z\|_{0:T} > M \right) \right) \to 0, \; M \to \infty.
\]

Thus, we arrive to

\[
\sup_{n \geq 0} P(A_{n,N,\varepsilon}) \to 0, \; N \to \infty,
\]

as required.

\[\square\]

**Corollary 2.1.** Assume that coefficients of \( \Box \) satisfy assumptions H1–H5 and

\[
E \left[ \exp \{ \|Z\|_{0,T,\gamma}^{1/\gamma} \} \right] < \infty.
\]

Then for all \( p \geq 1 \)

\[
E \left[ \|X^n - X^0\|_{0,T,\infty}^p \right] \to 0, \; n \to \infty.
\]

**Proof.** Theorem 2.2 implies the boundedness of sequence \( \{ \|X^n - X^0\|_{0,T,\infty}^q, \; n \geq 1 \} \) in \( L^q(\Omega) \) for all \( q > p \). Therefore, the statement of the corollary follows from Theorem 2.2 thanks to the uniform integrability. \[\square\]

3. **Applications**

3.1. **Vanishing delay**

Let, as above, the process \( \{ Z(t), t \geq 0 \} \) be an \( F \)-adapted process in \( \mathbb{R}^l \) with \( \gamma \)-Hölder continuous paths, \( \gamma > 1/2 \), \( \{ W(t), t \geq 1 \} \) be a standard \( F \)-Wiener process in \( \mathbb{R}^m \).
Consider the following sequence of equations with delay in $\mathbb{R}^d$.

$$
X^n(t) = \eta(0) + \int_0^t a^V(s, X^n(s), X^n(s - \tau_n))ds + \int_0^t b^V(s, X^n(s), X^n(s - \tau_n))dW(s) \\
+ \int_0^t c^V(s, X^n(s), X^n(s - \tau_n))dZ(s),
$$

(13)

$$
X^n(t) = \eta(t), t \in [-\tau_n, 0].
$$

Here $a^V: [0, T] \times \mathbb{R}^{2d} \to \mathbb{R}^d$, $b^V_i: [0, T] \times \mathbb{R}^{2d} \to \mathbb{R}^d$, $i = 1, \ldots, m$, $c^V_j: [0, T] \times \mathbb{R}^{2d} \to \mathbb{R}$, $j = 1, \ldots, l$, $\eta: [0, T] \to \mathbb{R}^d$ are measurable functions; \{\tau_n, n \geq 1\} is a sequence of positive numbers with $\tau_n < r$.

The assumptions about the coefficients are similar to H1–H5.

HV1. For all $t \in [0, T]$, $x, y \in \mathbb{R}^d$

$$
|a^V(t, x, y)| + |b^V(t, x, y)| + |c^V(t, x, y)| \leq K(1 + |x| + |y|).
$$

HV2. For all $t \in [0, T]$, $x, y \in \mathbb{R}^d$ there exist bounded derivatives $\partial_x c^V(t, x, y), \partial_y c^V(t, x, y)$:

$$
|\partial_x c^V(t, x, y)| + |\partial_y c^V(t, x, y)| \leq K.
$$

HV3. For all $t \in [0, T], R > 1,$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ with $|x_i| \leq R$, $|y_i| \leq R$, $i = 1, 2$,

$$
\begin{align*}
|a^V(t, x_1, y_1) - a^V(t, x_2, y_2)| + |b^V(t, x_1, y_1) - b^V(t, x_2, y_2)| \\
+ |\partial_x c^V(t, x_1, y_1) - \partial_x c^V(t, x_2, y_2)| + |\partial_y c^V(t, x_1, y_1) - \partial_y c^V(t, x_2, y_2)| \\
\leq K_R(|x_1 - x_2| + |y_1 - y_2|).
\end{align*}
$$

HV4. There exists $\beta \in (1 - \gamma, 1)$ such that for all $s, t \in [0, T], x, y \in \mathbb{R}^d$

$$
\begin{align*}
|c^V(t_1, x, y) - c^V(t_2, x, y)| &\leq K|t_1 - t_2|^{\beta} (1 + |x| + |y|), \\
|\partial_x c^V(t_1, x, y) - \partial_x c^V(t_2, x, y)| + |\partial_y c^V(t_1, x, y) - \partial_y c^V(t_2, x, y)| &\leq K|t_1 - t_2|^{\beta}.
\end{align*}
$$

HV5. The initial condition $\eta$ is a Hölder continuous function: for some $\theta \in (1 - \gamma, 1/2)$ and for all $t_1, t_2 \in [0, T]$

$$
|\eta(t_1) - \eta(t_2)| \leq K|t_1 - t_2|^\theta.
$$

From Theorem 2.2 we deduce the following result about vanishing delay convergence.

**Theorem 3.1.** Assume that the coefficients of equations (13) satisfy HV1–HV5, and $\tau_n \to 0$, $n \to \infty$. Then we have the following uniform convergence:

$$
\|X^n - X\|_{\infty, T} \xrightarrow{p} 0, n \to \infty,$$

to the solution $X$ of equation

$$
X(t) = \eta(0) + \int_0^t a^V(s, X(s), X(s))ds + \int_0^t b^V(s, X(s), X(s))dW(s) + \int_0^t c^V(s, X(s), X(s))dZ(s).
$$

**Proof.** Set $\tau_0 = 0$ and define for $n \geq 0$ the following sequence of functions $a^n(s, \psi) = a^V(s, \psi(0), \psi(-\tau_n))$, $b^n_i(s, \psi) = b^V_i(s, \psi(0), \psi(-\tau_n))$, $i = 1, \ldots, m$, $c^n_j(s, \psi) = c^V_j(s, \psi(0), \psi(-\tau_n))$, $j = 1, \ldots, l$, where $t \in [0, T], \psi \in C$. These coefficients are easily seen to satisfy assumptions H1–H4. Moreover, since the $a^V, b^V, c^V$ are continuous, we have for any $t \in [0, T], \psi \in C$ the convergence $a^n(t, \psi) \to a^0(t, \psi)$, $b^n(t, \psi) \to b^0(t, \psi)$, $c^n(t, \psi) \to c^0(t, \psi)$ as $n \to \infty$. The proof is finished by observing that for such coefficients the solutions to (5) coincide with those of (13) and applying Theorem 2.2. $\square$
3.2. Euler approximations

Consider now a standard mixed stochastic differential equation

\[ X(t) = X(0) + \int_0^t a^E(s, X(s))ds + \int_0^t b^E(s, X(s))dW(s) + \int_0^t c^E(s, X(s))dZ(s). \] (14)

Here \( a^E: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, b^E_i: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, i = 1, \ldots, k, \) and \( c^E_j: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, j = 1, \ldots, l, \) satisfy the following assumptions.

HE1. For all \( t \in [0, T], \) \( x \in \mathbb{R}^d \)

\[ |a^E(t, x)| + |b^E(t, x)| + |c^E(t, x)| \leq K(1 + |x|). \]

HE2. For all \( t \in [0, T], \) \( x \in \mathbb{R}^d \) there exists a bounded derivative \( \partial_x c^E(t, x): \)

\[ |\partial_x c^E(t, x)| \leq K. \]

HE3. For all \( t \in [0, T], \) \( R > 1, \) and \( x_1, x_2 \in \mathbb{R}^d \) with \( |x_1| \leq R, |x_2| \leq R \)

\[ |a^E(t, x_1) - a^E(t, x_2)| + |b^E(t, x_1) - b^E(t, x_2)| + |\partial_x c^E(t, x_1) - \partial_x c^E(t, x_2)| \leq K_R |x_1 - x_2|. \]

HE4. There exists \( \beta \in (1 - \gamma, 1) \) such that for all \( s, t \in [0, T], x, y \in \mathbb{R}^d \)

\[ |c^E(t_1, x) - c^E(t_2, x)| \leq K |t_1 - t_2|^{\beta} (1 + |x|), \]

\[ |\partial_x c^E(t_1, x) - \partial_x c^E(t_2, x)| \leq K |t_1 - t_2|^{\beta}. \]

Euler approximations for the solution of (14) are constructed as follows. For \( n \geq 1 \) define \( \delta = T/n \) and consider a uniform partition of \([0, T]: t^n_k = k\delta, \) \( k = 0, 1, \ldots, n. \) Define recursively

\[
X^n(0) = X(0), \\
X^n(t^n_{k+1}) = X^n(t^n_k) + a^E(t^n_k, X^n(t^n_k))\delta + b^E(t^n_k, X^n(t^n_k)) (W(t^n_{k+1}) - W(t^n_k)) \\
+ c^E(t^n_k, X^n(t^n_k)) (Z(t^n_{k+1}) - Z(t^n_k)), \quad k \geq 0.
\]

Denoting \( t^n(s) = \max \{ t^n_k : t^n_k \leq s \}, \) we can interpolate the values of approximations with

\[
X^n(t) = X(0) + \int_0^t a^E(t^n(s), X^n(t^n(s)))ds + \int_0^t b^E(t^n(s), X^n(t^n(s)))dW(s) \\
+ \int_0^t c^E(t^n(s), X^n(t^n(s)))dZ(s).
\]

This can be considered as an equation with delay, however, its coefficient \( c^n(s, \psi) := c^E(t^n(s), \psi(t^n(s))) \)

does not satisfy the assumption H4, so one needs to prove analogues of Lemmas 2.1 and 2.2. This can be done with a slight modification of corresponding arguments in [15]: we skip the proof as this is not our main interest here. Thus, we have the following result.

**Theorem 3.2.** Assume that the coefficients of equations (14) satisfy HE1–HE4. Then we have the following uniform convergence of Euler approximations:

\[
\sup_{t \in [0, T]} |X^n(t) - X(t)| \overset{P}{\to} 0, \quad n \to \infty.
\]

**Remark 3.1.** While establishing the convergence of approximations, Theorem 3.2 tells nothing about the rate of convergence. In [10], the rate of convergence was established for a one-dimensional equation driven with \( Z = B^H \) under more restrictive assumptions on the coefficients.
Appendix A. Limit theorem for Itô delay equations

Here we prove a convergence result for Itô SDDEs. Consider a sequence of stochastic delay differential equations in $\mathbb{R}^d$:

$$
Y^n(t, \omega) = \theta^n(0, \omega) + \int_0^t f^n(s, Y^n_s, \omega)ds + \sum_{i=1}^m \int_0^t g^n_i(s, Y^n_s, \omega)dW_i(s),
$$

(A.1)

or, shortly,

$$
Y^n(t) = \theta^n(0) + \int_0^t f^n(s, Y^n_s)ds + \int_0^t g^n(s, Y^n_s)dW(s),
$$

with $\mathcal{F}_0$-measurable initial conditions $Y^n(t, \omega) = \theta^n(t, \omega), t \in [-r, 0]$. These equations are similar to (5), but they do not contain a part with the process $Z$. Another difference is that the coefficients of (A.1) are random. We will impose the following assumptions on them.

HA1. For all $\psi \in C, t \in [0, T] \ f^n(t, \psi)$ and $g^n(t, \psi)$ are $\mathcal{F}_t$-measurable.

HA2. For all $\psi \in C, t \in [0, T]$ and a.a. $\omega \in \Omega$

$$
|f^n(t, \psi, \omega)| + |g^n(t, \psi, \omega)| \leq K(1 + \|\psi\|_C).
$$

HA3. The functions $f^n$ and $g^n$ are locally Lipschitz continuous in $\psi$: for all $R > 1, t \in [0, T], \text{a.a.} \ \omega \in \Omega,$ and any $\psi_1, \psi_2 \in C$ with $\|\psi_1\|_C \leq R, \|\psi_2\|_C \leq R$,

$$
|f^n(t, \psi_1, \omega) - f^n(t, \psi_2, \omega)| + |g^n(t, \psi_1, \omega) - g^n(t, \psi_2, \omega)| \leq K_R \|\psi_1 - \psi_2\|_C.
$$

The unique solvability can be shown by slight modification of arguments in [13, Theorem I.2] and [9, Chapter 5, Theorem 2.5]. Moreover, there is a uniform integrability of solutions, which we state below.

**Proposition A.1.** Under assumptions HA1–HA2, for any $p \geq 2$

$$
E \left[ \|Y^n\|_{\infty, T}^p \mid \mathcal{F}_0 \right] \leq C_p(1 + \|\theta^n\|_C^p).
$$

(A.2)

**Proof.** Let $M^n(t) = \sup_{s \in [0, t]} |Y^n(s)|^p$. For a fixed $R > 0$, denote $1_t = 1_{M^n(t) \leq R}$. Also abbreviate $E_0[\cdot] = E[\cdot \mid \mathcal{F}_0]$.

Estimate

$$
M^n(t) \leq C_p \left( |\theta(0)|^p + \sup_{s \in [0, t]} \left| \int_0^s f^n(u, Y^n_u)du \right|^p + \sup_{s \in [0, t]} \left| \int_0^s g^n(u, Y^n_u)dW(u) \right|^p \right)
$$

$$
\leq C_p \left( |\theta(0)|^p + \int_0^t |f^n(s, Y^n_s)|^p ds + \sup_{s \in [0, t]} \left| \int_0^s g^n(u, Y^n_u)dW(u) \right|^p \right)
$$

Therefore,

$$
E_0[M^n(t)1_t] \leq C_p \left( |\theta(0)|^p + I_t^0 + I_t^1 \right),
$$

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where

\[ I^a_t = \int_0^t E_0 \left[ |f^n(s, Y^n_s)|^p \right] ds, \]

\[ I^b_t = E_0 \left[ \sup_{s \in [0,t]} \left| \int_0^s g^n(u, Y^n_u) dW(u) \right|^p \right] 1_t. \]

Since the events \( \{M^n_t \leq R\} \) are decreasing in \( t \), we can estimate

\[ I^a_t \leq \int_0^t E_0 \left[ |f^n(s, Y^n_s)|^p \right] ds \leq C_p \int_0^t E_0 \left[ (1 + \|Y^n_s\|_C^p) 1_s \right] ds \]
\[ \leq C_p \int_0^t E_0 \left[ (1 + \|\theta^n\|_C^p + M^n(s)) 1_s \right] ds \leq C_p \left( 1 + \|\theta^n\|_C^p + \int_0^t E_0 [M^n(s) 1_s] ds \right). \]

Further, with the help of the Burkholder–Gundy–Davis inequality, we obtain

\[ I^b_t \leq C_p E_0 \left[ \sup_{s \in [0,t]} \left| \int_0^s g^n(u, Y^n_u) dW(u) \right|^p \right] \leq C_p \left( \left( \int_0^t g^n(s, Y^n_s)^2 1_s ds \right)^{p/2} \right) \]
\[ \leq C_p \int_0^t E_0 \left[ |g^n(s, Y^n_s)|^p 1_s \right] ds \leq C_p \left( 1 + \|\theta^n\|_C^p + \int_0^t E_0 [M^n(s) 1_s] ds \right), \]

where the last inequality is obtained the same way as for \( I^a_t \).

Consequently,

\[ E_0 [M^n(t) 1_t] \leq C_p \left( 1 + \|\theta^n\|_C^p + \int_0^t E_0 [M^n(s) 1_s] ds \right). \]

By the Gronwall lemma,

\[ E_0 [M^n(t) 1_t] \leq C_p (1 + \|\theta^n\|_C^p). \]

Now letting \( R \to \infty \) and using the Fatou lemma, we arrive at the statement. \( \square \)

Concerning the convergence, we will assume the following.

CA1. Pointwise convergence of the coefficients in probability: for all \( \psi \in \mathcal{C}, t \in [0, T] \)

\[ f^n(t, \psi) \xrightarrow{P} f^0(t, \psi), \quad g^n(t, \psi) \xrightarrow{P} g^0(t, \psi), \quad n \to \infty. \]

CA2. Convergence of initial conditions in probability:

\[ \|\theta^n - \theta^0\|_C \xrightarrow{P} 0, \quad n \to \infty. \]

**Theorem A.1.** Under assumptions HA1–HA3 and CA1–CA2, the following convergence in probability takes place:

\[ \|Y^n - Y^0\|_{\infty, T} \xrightarrow{P} 0, \quad n \to \infty. \]

Moreover, if additionally for some \( p \geq 2 \) \( E \left[ \|\theta^n - \theta^0\|_C^p \right] \to 0, \quad n \to \infty, \) then

\[ E \left[ \|Y^n - Y^0\|_{\infty, T}^p \right] \to 0, \quad n \to \infty. \]

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Proof. It is enough to prove that any subsequence of $\|Y^n - Y^0\|_{\infty,T}$ contains a subsequence converging to zero in probability. Therefore, it can be assumed without loss of generality that $\|\theta^n - \theta^0\|_C \to 0$, $n \to \infty$, a.s.

Denote for some $p \geq 2$ $\Delta^n_t = \|Y^n - Y^0\|_{\infty,t}$ and abbreviate, as in Proposition A.1, $E_0 [\cdot] = E [\cdot | F_0]$. Write

$$E_0 [\Delta^n_t] \leq C_p \left( \|\theta^n(0) - \theta^0(0)\|^p + E_0 \left[ \sup_{s \in [0,t]} \left\| \int_0^s \left( f^n(u,Y^n_u) - f^0(u,Y^0_u) \right) du \right\|^p \right] \right)$$

$$+ E_0 \left[ \sup_{s \in [0,t]} \left\| \int_0^s \left( g^n(u,Y^n_u) - g^0(u,Y^0_u) \right) dW(u) \right\|^p \right] =: C_p \left( \|\theta^n(0) - \theta^0(0)\|^p + J^n_t + J^n_t^b \right).$$

Estimate separately

$$J^n_t \leq C_p \int_0^t E_0 \left[ \left\| f^n(s,Y^n_s) - f^0(s,Y^0_s) \right\|^p \right] ds$$

$$\leq C_p \int_0^t E_0 \left[ \left\| f^n(s,Y^n_s) - f^n(s,Y^0_s) \right\|^p + \left\| f^n(s,Y^0_s) - f^0(s,Y^0_s) \right\|^p \right] ds.$$

Now

$$\int_0^t E_0 \left[ \left\| f^n(s,Y^n_s) - f^n(s,Y^0_s) \right\|^p \right] ds \leq C_p \int_0^t E_0 \left[ \left\| Y^n_s - Y^0_s \right\|^p \right] ds$$

$$\leq C_p \int_0^t E_0 \left[ \|\theta^n - \theta^0\|^p + \Delta^n_s \right] ds \leq C_p \left( \|\theta^n - \theta^0\|^p + \int_0^t E_0 \left[ \Delta^n_s \right] ds \right).$$

Thus,

$$J^n_t \leq C_p \left( \|\theta^n - \theta^0\|^p + \int_0^t E_0 \left[ \Delta^n_s \right] ds + \int_0^t E_0 \left[ \left\| f^n(s,Y^0_s) - f^0(s,Y^0_s) \right\|^p \right] ds \right).$$

A similar estimate for $J^n_t^b$ is obtained with the help of the Burkholder–Gundy–Davis inequality. Therefore, we arrive to

$$E_0 [\Delta^n_t] \leq C_p \left( \|\theta^n - \theta^0\|^p + \int_0^t E_0 \left[ \Delta^n_s \right] ds + \int_0^t E_0 \left[ \left\| f^n(s,Y^0_s) - f^0(s,Y^0_s) \right\|^p \right] ds \right). \quad (A.3)$$

Thanks to the linear growth assumption HA2 and to Proposition A.1,

$$E_0 \left[ \left\| f^n(s,Y^0_s) - f^0(s,Y^0_s) \right\|^p \right] \leq C_p (1 + \|\theta^n\|^p).$$

Hence, in view of the dominated convergence theorem and assumption CA1, the last term in (A.3) vanishes as $n \to \infty$. Also by Proposition A.1, $\lim sup_{n \to \infty} E_0 [\Delta^n_s] \leq C_p (1 + \lim sup_{n \to \infty} \|\theta^n\|^p) < \infty$.

Thus, taking $\lim sup_{n \to \infty}$ in (A.3), we obtain

$$\lim sup_{n \to \infty} E_0 [\Delta^n_t] \leq C_p \int_0^t \lim sup_{n \to \infty} E_0 [\Delta^n_s] ds.$$

By the Gronwall lemma, $\lim sup_{n \to \infty} E_0 [\Delta^n_t] = 0$, yielding the first statement of the theorem. The second statement is obtained by taking expectation in equation (A.3) and repeating the argument following that equation. \qed
References

[1] Besalú, M., Rovira, C., 2012. Stochastic delay equations with non-negativity constraints driven by fractional Brownian motion. Bernoulli 18 (1), 24–45.

[2] Boufoussi, B., Hajji, S., 2011. Functional differential equations driven by a fractional Brownian motion. Comput. Math. Appl. 62 (2), 746–754.

[3] Caraballo, T., Garrido-Atienza, M. J., Taniguchi, T., 2011. The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion. Nonlinear Anal. 74 (11), 3671–3684.

[4] Ferrante, M., Rovira, C., 2006. Stochastic delay differential equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Bernoulli 12 (1), 85–100.

[5] Ferrante, M., Rovira, C., 2010. Convergence of delay differential equations driven by fractional Brownian motion. J. Evol. Equ. 10 (4), 761–783.

[6] Guerra, J., Nualart, D., 2008. Stochastic differential equations driven by fractional Brownian motion and standard Brownian motion. Stoch. Anal. Appl. 26 (5), 1053–1075.

[7] Kubilius, K., 2002. The existence and uniqueness of the solution of an integral equation driven by a $p$-semimartingale of special type. Stochastic Process. Appl. 98 (2), 289–315.

[8] León, J. A., Tindel, S., 2012. Malliavin calculus for fractional delay equations. J. Theoret. Probab. 25 (3), 854–889.

[9] Mao, X., 2007. Stochastic differential equations and applications. 2nd ed. Chichester: Horwood Publishing.

[10] Mishura, Y. S., Shevchenko, G. M., 2011. Rate of convergence of Euler approximations of solution to mixed stochastic differential equation involving Brownian motion and fractional Brownian motion. Random Oper. Stoch. Equ. 19 (4), 387–406.

[11] Mishura, Y. S., Shevchenko, G. M., 2011. Stochastic differential equation involving Wiener process and fractional Brownian motion with Hurst index $H > 1/2$. Comm. Statist. Theory Methods 40 (19–20), 3492–3508.

[12] Mishura, Y. S., Shevchenko, G. S., 2012. Mixed stochastic differential equations with long-range dependence: Existence, uniqueness and convergence of solutions. Comput. Math. Appl. 64 (10), 3217–3227.

[13] Mohammed, S.-E. A., 1998. Stochastic differential systems with memory: theory, examples and applications. In: Stochastic analysis and related topics, VI (Geilo, 1996). Vol. 42 of Progr. Probab. Birkhäuser Boston, Boston, MA, pp. 1–77.

[14] Neuenkirch, A., Nourdin, I., Tindel, S., 2008. Delay equations driven by rough paths. Electron. J. Probab. 13, 2031–2068.

[15] Shevchenko, G. M., 2013. Mixed stochastic delay differential equations. Theory Probab. Math. Stat. 89, 169–182.
[16] Shevchenko, G. M., 2014. Integrability of solutions to mixed stochastic differential equations. J. Math. Sci. 198 (4), 457–468.

[17] Shevchenko, G. M., Shalaiko, T. O., 2013. Malliavin regularity of solutions to mixed stochastic differential equations. Stat. Probab. Letters 83 (12), 2638–2646.

[18] Zähle, M., 1998. Integration with respect to fractal functions and stochastic calculus. I. Probab. Theory Relat. Fields 111 (3), 333–374.