Quantum Link Models with Many Rishon Flavors and with Many Colors

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Quantum link models are a novel formulation of gauge theories in terms of discrete degrees of freedom. These degrees of freedom are described by quantum operators acting in a finite-dimensional Hilbert space. We show that for certain representations of the operator algebra, the usual Yang-Mills action is recovered in the continuum limit. The quantum operators can be expressed as bilinears of fermionic creation and annihilation operators called rishons. Using the rishon representation the quantum link Hamiltonian can be expressed entirely in terms of color-neutral operators. This allows us to study the large $N_c$ limit of this model. In the 't Hooft limit we find an area law for the Wilson loop and a mass gap. Furthermore, the strong coupling expansion is a topological expansion in which graphs with handles and boundaries are suppressed.

1. Introduction

The quantum link formulation of lattice gauge theories is an approach which is very different from the standard framework based on a Euclidean path integral \cite{1}. Since quantum links are discrete variables, i.e. operators that act in a finite dimensional Hilbert space, it is hoped that the quantum link formulation will allow numerical simulations using cluster algorithms. Meron-cluster algorithms \cite{2,3}, for instance, proved to be superior to standard methods for a wide range of models, such as the 2-d O(3) model with $\theta = \pi$ and with a nonzero chemical potential $\mu$, the Potts model of QCD, and various fermionic systems with a sign problem, like the attractive Hubbard model.

Furthermore quantum link models have several features that go beyond the standard formulation. For example, a quantum link can be written as a bilinear of fermionic creation and annihilation operators \cite{5},

$$\hat{U}^{ij}_{x,\mu} = \hat{c}^\alpha_{x,\mu} \hat{c}^{i\alpha\dagger}_{x+\vec{\mu},-\mu}.$$  \hfill (1)

These fermions are called rishons. This splitting of the link variable has no analogue in Wilson’s formulation of lattice QCD. In the following, we make use of the rishon representation to investigate the large $N_c$ limit of the quantum link model. Using coherent state techniques we derive a new effective theory for large $N_c$ gauge theory.

2. The quantum link model

The quantum link model is described by a Hamilton operator $\hat{H}$, which evolves the system in an additional fifth dimension \cite{4,11}. The partition function and correlation functions are given by operator traces, e.g. $Z = \text{Tr} \ e^{-\beta \hat{H}}$, where $\beta$ is the extent of the additional dimension. This extra dimension should eventually disappear via dimensional reduction, thereby giving rise to the 4-d target theory. In the next section, we will discuss under what circumstances this will happen.
The Hamilton operator is obtained from Wilson’s plaquette action by replacing the elements of $SU(N_c)$ link matrices — ordinary complex numbers — by operators acting in a finite-dimensional Hilbert space,

$$
\hat{H} = J \sum_{x, \mu \neq \nu} \text{tr} \left( \hat{U}_{x,\mu} \hat{U}_{x+\mu,\nu} \hat{U}_{x+\nu,\mu} \hat{U}_{x,\nu} \right), \quad (2)
$$

Here, the trace sums only over the color indices. The commutation relations of the quantum link operators are determined by the requirement that the Hamilton operator (2) be invariant under $SU(N_c)$ gauge transformations.

Gauge invariance is expressed as $[\hat{H}, \hat{G}^a] = 0$. The generators of gauge transformations satisfy $[\hat{G}^a_x, \hat{G}^b_y] = 2i f^{abc} \delta_{xy} \delta_{xy}$, where $f^{abc}$ are the structure constants of $SU(N_c)$. Writing $\hat{G}^a_x = \sum_\mu (\hat{L}^a_x + \hat{R}^a_{x-\mu})$, we find the following commutation relations between $\hat{R}, \hat{L}, \hat{U}$ and $\hat{U}^\dagger$ ($x$ and $\mu$ indices suppressed):

$$
\begin{align*}
[\hat{R}, \hat{R}^\dagger] &= 2i f^{abc} \hat{R}^c, \\
[\hat{L}, \hat{R}^\dagger] &= \hat{U} \lambda^a, \\
[\hat{R}, \hat{U}] &= \hat{U} \lambda^a, \\
[\hat{R}^\dagger, \hat{U}] &= -\lambda^a \hat{U}, \\
[\hat{L}, \hat{U}] &= -\lambda^a \hat{U}\dagger.
\end{align*} \quad (3)
$$

Here, the $\lambda^a$ are $SU(N_c)$ generators in the defining representation. The commutation relations between the $\hat{U}$’s and $\hat{U}^\dagger$’s are as yet undetermined. The smallest algebra under which the above relations close is $SU(2N_c)$, provided we postulate the proper commutation relations between the $\hat{U}$’s and $\hat{U}^\dagger$’s. Hence, an $SU(2N_c)$ algebra is associated with each link. Actually, this construction leads to a $U(N_c)$ gauge theory, and in order to break the unwanted $U(1)$ symmetry and obtain an $SU(N_c)$ gauge theory, one can add a determinant term to $\hat{H}$,

$$
\hat{H}' = J' \sum_{x, \mu} \left[ \text{det} \hat{U}_{x,\mu} + \text{det} \hat{U}^\dagger_{x,\mu} \right], \quad (4)
$$

thus breaking the unwanted $U(1)$ symmetry.

The $SU(2N_c)$ generators may be expressed as products of fermionic creation and annihilation operators, which obey canonical anti-commutation relations:

$$
\begin{align*}
\hat{U}^\dagger_{x,\mu} &= \hat{c}^{\alpha \dagger}_{x,\mu} \hat{c}^{\beta}_{x+\mu,\mu}, \\
\hat{U}_{x,\mu} &= \hat{c}^{\alpha}_{x,\mu} \hat{c}^{\beta \dagger}_{x+\mu,\mu}, \\
\hat{R}^a &= \hat{c}^{\alpha \dagger}_{x,\mu} \lambda^a_{ij} \hat{c}^{\beta}_{x+\mu,\mu}, \\
\hat{R}^a &= \hat{c}^{\alpha}_{x,\mu} \lambda^a_{ij} \hat{c}^{\beta \dagger}_{x+\mu,\mu}.
\end{align*} \quad (5)
$$

These fermions are called rishons, and in addition to the color index $i$ they carry a rishon flavor index $\alpha$, which runs from 1 to $M$. In fact, their introduction corresponds to the choice of a particular representation for the $SU(2N_c)$ algebra, represented by a rectangular Young tableau with $N_c$ rows and $M$ columns,

![Young tableau](image)

We always choose the representation with vertical length $N_c$. This so-called half-filling constraint is equivalent to fixing the total number of rishons on a link to be $N_c M$,

$$
\sum_i (\hat{c}^{\alpha \dagger}_{x,\mu} \hat{c}^\beta + \hat{c}^{\alpha \dagger}_{x+\mu,\mu} \hat{c}^\beta) = \delta^{\alpha \beta} N_c. \quad (6)
$$

For these representations the determinant term in (4) is non-trivial and hence the unwanted $U(1)$ symmetry is broken.

3. The low-energy effective theory

The physics of the quantum link model is completely determined, once the representation for the $SU(2N_c)$ quantum link algebra has been chosen. Having picked a particular type of representation in the previous section, we would like to investigate the dynamics of the theory. In particular, we are interested in determining whether the theory undergoes dimensional reduction and what kind of effective theory is described in the long-distance limit. To this end, we start by setting up a coherent state path integral. Letting $M$ get large, this will allow us to do a semi-classical expansion around the classical ground state, whence we recover Yang-Mills theory as a low-energy effective action of the theory (5). Generalized coherent states for $SU(2N_c)$ are generated by acting with all group elements on the
highest weight state of the chosen representation $C$. In our case, the highest weight state is given by

$$|\psi_0\rangle = C \left[e^{ab\cdots}e^{a_1\cdots}e^{b_1\cdots}\right]^M |0\rangle,$$

where there are $N_c$ creation operators inside the square bracket. The indices $a, b, \ldots$ run from 1 to $N_c$, and $\alpha$ runs from 1 to $M$. Collectively denoting the $SU(2N_c)$ generators of eq. (3) by $S^{ij}$, coherent states are given by

$$|q\rangle = \exp \left(-q^{ij} \hat{S}^{ij} + q^{ij*} \hat{S}^{ij}\right) |\psi_0\rangle,$$

for $1 \leq j \leq N_c$ and $N_c + 1 \leq i \leq 2N_c$. These states have a number of nice properties. Most importantly, the identity operator can be resolved as

$$\mathbb{I} = \int dq |q\rangle \langle q|.$$

Furthermore, we have the following expression for diagonal matrix elements of $SU(2N_c)$ generators between coherent states,

$$\langle q| \hat{S}^{ij} |q\rangle = (M/2) Q^{ij},$$

where the matrix $Q$ parameterizes the coset space $SU(2N_c)/[SU(N_c) \times SU(N_c) \times U(1)]$.

$$Q = \exp \left(\begin{array}{cc} 0 & q^i \\ q & 0 \end{array}\right) \left(\begin{array}{cc} 1_N & 0 \\ 0 & -1_N \end{array}\right) \exp \left(\begin{array}{cc} 0 & -q^i \\ q & 0 \end{array}\right).$$

Note that this coset space arises, because the stability subgroup of $|\psi_0\rangle$ is $SU(N_c) \times SU(N_c) \times U(1)$. Using these properties, it is straightforward to set up a coherent state path integral in the standard way, which is an integral over $GL(N, \mathbb{C})$ matrices $q$.

In order to derive the low-energy effective theory, we do a semi-classical expansion around the classical ground state. The validity of this expansion depends on the existence of an intermediate momentum cut-off scale, which is much larger than the inverse correlation length and much smaller than the inverse lattice spacing, $\xi^{-1} \ll \Lambda \ll a^{-1}$. Then, dominant contributions to the path integral are slowly varying on this intermediate scale, and higher order terms can be ignored. Below, we will see that such a cut-off scale exists for large enough values of $M$, i.e. large enough representations of the quantum link algebra. To expand around the minimum of the action, we decompose the matrices $q = bu$ into a Hermitian part $b$ and a unitary part $u$, and write

$$u_\mu = \exp(-ia^2(\theta_\mu/N)\mathbb{I} - iaA_\mu),$$

and $b_\mu = (\pi/4)\mathbb{I} + a^2 E_\mu$, and expand in powers of the lattice spacing $a$, keeping only terms up to order $a^4$. We then integrate out the quadratic fluctuations $E^2$, to obtain an effective five-dimensional action,

$$S[A] = \int_0^\beta dx \int d^4x \frac{1}{4e^2} \left(\begin{array}{cc} \text{tr} F_{\mu\nu}^2 + \frac{1}{e^2} \text{tr} (\partial_5 A_\mu)^2 \right).$$

Here, $e = 8/(M^4J)$ is the 5-d gauge coupling, and $c = (Ma/2\sqrt{\gamma}) J$ is the “velocity of light”. The constant $\gamma$ is given by $\gamma = 3JM^4 + 4J/(M/2)^{N_c}$. Notice that there is no $A_5$ field in the action. This is due to the fact, that we did not impose Gauss’ law, as we have no need for 5-dimensional gauge invariance. The 4-dimensional gauge invariance is intact, since $A_4 \neq 0$. Taking the extent $\beta$ of the fifth dimension to be infinite, we nevertheless have a 5-dimensional gauge theory in $A_5 = 0$ gauge. Note that a five-dimensional gauge theory can exist in a non-Abelian Coulomb phase. This is generically the case for lattice gauge theories such as ours, and this phase is characterized by massless gluons $A_\mu$ and hence an infinite correlation length $\xi$. When $\beta$ is made finite, the confinement hypothesis requires that gluons pick up a non-perturbatively generated mass. However, $\xi$ is still exponentially larger than $\beta$,

$$\xi \propto \exp[24\pi^2\beta/(11N_c e^2)],$$

and the theory is dimensionally reduced to 4 dimensions,

$$S[A] = \int d^4x \frac{\beta}{4e^2} \left(\text{tr} F_{\mu\nu}^2 + \ldots \right).$$

This is the standard gauge field action with the 4-d gauge coupling $g$ given by $1/g^2 = \beta/e^2$. Thus,
the continuum limit $g \to 0$ is approached by sending the extent $\beta$ to infinity, and we see that the low-energy dynamics of the quantum link model describes ordinary 4-d Yang-Mills theory.

Notice that the dependence of $e^2$ on $M$ leads to a large correlation length $\xi$ for large $M$. At fixed $\beta$, this guarantees the existence of an intermediate momentum cut-off scale mentioned above, if we choose $\beta$ large enough, and hence, the semi-classical expansion is valid in this case.

4. The quantum $\Phi$–model and large $N_c$

Consider the Hamilton operator of the $U(N_c)$ quantum link model in (14), without the determinant term, i.e. $J' = 0$. Writing the link operators in terms of rishons, the product of four quantum links becomes a product of eight rishon operators. We can thus group together rishon operators at each lattice point to form $\Phi$-operators:

$$\hat{\Phi}^{\alpha \beta}_{x,\mu,\nu} = \sum_i e^{i\alpha c_{x,\mu}c_{x,\nu}^{\beta \dagger}}.$$  \hfill (16)

These operators transform as color singlets since color indices are summed over. In terms of these operators the Hamilton operator reads

$$\hat{H} = -J \times \sum_{x,\mu \neq \nu} \text{tr}(\hat{\Phi}_{x,\nu,\mu}^{\dagger} \hat{\Phi}_{x,\mu,\nu} + \hat{\Phi}_{x,\nu,\mu}^{\dagger} \hat{\Phi}_{x,\nu,\mu} + \hat{\Phi}_{x,\nu,\mu}^{\dagger} \hat{\Phi}_{x,\nu,\mu} - \hat{\Phi}_{x,\nu,\mu} \hat{\Phi}_{x,\nu,\mu}) \, .$$  \hfill (17)

Here the trace sums over rishon flavor indices. The sign appears because we have to commute the last – fermionic – rishon operator an odd number of times.

The Hamiltonian in terms of the $\Phi$’s has a local $U(M)$ flavor symmetry. The color symmetry, on the other hand, is completely hidden in the definition of the $\Phi$-operators.

As already mentioned, a $U(2N_c)$ algebra is associated with each link of the lattice when working with the quantum link operators $\hat{U}, \hat{U}^\dagger$. Using the anti-commutation relations between the rishon operators one can work out the commutation relations between the $\Phi$-operators. One finds that they generate a $U(2dM)$ algebra at each lattice point, where $d$ is the number of physical space-time dimensions.

Moreover, fixing the number of rishons at each lattice site to be $dMN_c$ (recall half-filling) is equivalent to choosing a representation for $U(2dM)$ with the rectangular Young-tableau

What is the advantage of working with the $\Phi$-operators? By using the $\Phi$-operators the roles of $M$ and $N_c$ are reversed: the number of colors $N_c$ determines the size of the representation of $U(2dM)$. In the last section, we derived a 4-d low-energy effective theory of the quantum link model for large representations. We recover 4-d Yang-Mills theory for large $M$. This suggests the use of similar methods to derive an effective theory for the $\Phi$-model and obtain the large $N_c$ limit of pure gauge theory!

It turns out that one can indeed take the limit $N_c \to \infty$ in the $\Phi$–model for any value of $M$. But before getting too excited about this, one should bear in mind that in the ’t Hooft limit of gauge theories $g^2N_c$ is kept fixed as $N_c \to \infty$.

To determine what this implies for $M$, recall the 5-d effective action describing the low-energy dynamics of the quantum link model (13). As already pointed out, the parameters $c$ and $c'$ of the five dimensional theory are functions of $N_c$ and $M$ and determine the parameters of the dimensionally reduced 4-d theory.

The ’t Hooft limit of the 5-d theory is given by $c^2N_c = \text{const.}$ and $c = \text{const.}$ as $N_c \to \infty$. Unfortunately, this implies that $M \propto N_c$, and so the ’t Hooft limit requires $M \to \infty$ as well.

5. The classical $\phi$–model

The classical $\phi$-model is the 4-d effective theory of the quantum $\Phi$-model in the limit $N_c \to \infty$, and after dimensional reduction. It’s derivation is
elements of the coset space are not independent. They are constrained to be direction.

− positive and negative values. The negative index to each lattice point. The Greek indices can take the resulting 4-d lattice theory.

ent state techniques. We skip the details concerning the coherent states and present directly the resulting 4-d lattice theory.

The 4-d φ-model assigns a field variable

\[ \phi_{\mu,\nu}(x) \in GL(M, \mathbb{C}), \quad \mu, \nu = \pm 1, \ldots, \pm 4 \] (18)

to each lattice point. The Greek indices can take positive and negative values. The negative index \(-\mu\) refers to the link ‘pointing’ in the opposite \(\mu\) direction.

The field variables \(\phi_{\mu,\nu}(x)\) at a lattice point are not independent. They are constrained to be elements of the coset space \(U(2dM)/[U(dM) \times U(dM)]\),

\[ \phi = UJU^\dagger, \] (19)

where \(U\) is an element of \(U(2dM)\) and \(J\) being the diagonal matrix \(\text{diag}(1, \ldots, 1, -1, \ldots, -1)\) with \(dM\) 1’s and -1’s, respectively.

The action contains two parts. The first term reads

\[ S_1[\phi] = \beta \sum_{x, \mu \neq \nu} \text{tr} P_{\mu\nu}(x) + \text{h.c.} \] (20)

with \(P_{\mu\nu}\) defined as the product of four φ’s at the corners of the basic plaquettes, explicitly:

\[ P_{\mu,\nu}(x) = \phi_{\mu,\nu}(x)\phi_{-\nu,\mu}(x + a\hat{\nu}) \times \] (21)

\[ \phi_{-\mu,\nu}(x + a\hat{\nu} + a\hat{\mu})\phi_{\mu,-\nu}(x + a\hat{\mu}). \]

This term directly stems from the plaquette term in the Hamiltonian of the quantum link model.

The half-filling constraint for the \(U(2dM)\) representations of the quantum \(\Phi\) model results in a constraint for the \(\phi\)-fields at neighboring lattice points,

\[ \sum_{d \geq \mu \geq 1} \left( \phi_{\mu,\mu}(x) + \phi_{-\mu,-\mu}(x + a\hat{\mu}) \right) = 0. \] (22)

This constraint is encouraged by adding a second part to the gauge action,

\[ S_2[\Phi] = \lambda \sum_{x, \mu} \text{tr} \left[ \phi_{\mu,\mu}(x) + \phi_{-\mu,-\mu}(x + a\hat{\mu}) \right]^2. \] (23)

This λ-term imposes a soft version of the half-filling constraint.

A crucial feature of the total action \(S_1 + S_2\) is its invariance under local \(U(M)\) rotations. It is obvious from (23) and (24) that the action is invariant under the transformation

\[ \phi_{\mu,\nu}(x) \rightarrow g\mu(x)\phi_{\mu,\nu}(x)g\nu(x)^{-1}, \] (24)

if we define \(g_{-\nu}(x) = g_{\nu}(x - a\hat{\nu})\) for the non-positive values \(-1, \ldots, -d\) of \(\nu\).

To complete our definition of the φ-model we need to specify an integration measure for the functional integral. In view of (17) it is natural to define

\[ \mathcal{D}[\phi] = \prod_x d\phi(x), \] (25)

where \(d\phi(x)\) is the Haar measure of the group \(U(2dM)\). Obviously this measure is invariant under gauge transformations.

With the measure at hand we define the partition function

\[ Z = \int \mathcal{D}[\phi] e^{-S[\phi]} \] (26)

and correlation functions

\[ \langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\phi] O e^{-S[\phi]} \] (27)

of any product \(O\) of the φ-fields.

As discussed before, the φ-model we have defined here should be equivalent to large \(N_c\) pure gauge theory in the limit \(M \rightarrow \infty\). Sending \(M\) to infinity, however, we have to keep

\[ \beta' = \frac{\beta}{M}, \quad \lambda' = \frac{\lambda}{M}, \] (28)

finite, similar to the ’t Hooft limit.

6. Strong coupling results

As a first step in the investigation of the φ-model, we looked at its strong coupling expansion. In this expansion the Boltzmann weight \(e^{-S[\phi]}\) in the functional integral is expanded in powers of \(\beta\) and \(\lambda\). This leads to a power series expansion of the observable for any finite \(M\). In
the end we have taken the infinite $M$ limit. We find the following results:

1. At strong coupling the $\phi$-model has a mass gap. In the limit $M \to \infty$ the lowest glue ball mass at leading order is given by
   \[ m = -2 \ln \frac{\beta'}{8^3}. \]

2. An area law is found for the Wilson loop and in the limit $M \to \infty$ the string tension at leading order is given by
   \[ \sigma = -\ln \frac{\beta'}{8^3}. \]

This result is essentially a consequence of the $U(M)$ rishon flavor symmetry. To obtain a non-zero expectation value, the expansion of the Boltzmann factor needs to generate at least the plaquettes inside the Wilson loop. This is analogous to the strong coupling expansion in conventional pure lattice gauge theory.

3. The strong coupling expansion is a topological expansion. Strong coupling graphs with $H$ handles and $B$ boundaries (or quark loops) are suppressed by inverse powers of $M$,
   \[ (1/M)^{2H+B-2}. \] (29)
To leading order only planar diagrams contribute.

From these results we conclude that the $\phi$-model shares some of the qualitative features of large $N_c$ gauge theory, known from the conventional approach \[11\]. Note, however, that it seems almost impossible to come up with the $\phi$-model as a reformulation of large $N_c$ gauge theory without the detour via the quantum link formulation.

7. Conclusions and Outlook

In conclusion, we showed analytically, that ordinary 4-d Yang-Mills theory is the low-energy effective theory of the quantum link model, provided the representation of the quantum link algebra is sufficiently large. Hence the quantum link approach is a valid formulation of gauge field theory.

The large $N_c$ limit of quantum link models can be investigated in terms of color singlet fields. This so-called $\phi$-model exhibits important qualitative features of large $N_c$ pure gauge theory like a mass gap and an area law for the Wilson loop.

There are several directions for future research. It would be interesting to study the $\phi$-model at weak coupling and compute the $\beta$-function to see if this model is asymptotically free, as expected. Furthermore, numerical simulations of the $\phi$-model for various values of $M$ may lead to further insight into the model. It is quite possible, that fairly small values of $M$ are already sufficient for studying the large $N_c$ limit. In order to study full QCD, fermions should be included. By pairing rishon and quark operators, the theory can be completely bosonized \[12\] which would be a significant advantage in numerical simulations.

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