Bifurcation analysis of epidemic model waning immunity

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Abstract. In this paper, we analyse the bifurcation of epidemic models. Those model is waning immunity. First, the stability of those system is analyzed by analyzing the stability of critical points. The threshold number that associated with the stability was determined. Later, the existence of bifurcation was analyzed. The result show that bifurcation exist and threshold number is considered as bifurcation parameter. Numerical simulation is given to confirm the analytical results by showing the graphic solutions and phase portrait.

1. Introduction

The spread of infectious diseases has always been a threat to public health. There are millions people die caused infectious diseases in the world. Infectious diseases can be transmitted through direct or indirect contact between an infected individual and a susceptible individual. The epidemic mathematical model describes the dynamics of infectious disease. Those model reflects the process of spreading of the disease[1,2].

The first epidemic model was developed by Kermarck-McKendrick in the late 1920s. The model is described as SIR (susceptible, infected, recovered) model which is a system of ordinary differential equations. The population divided by three sub population, that are susceptible (S), infected (I), and recovered (R) individuals. The model is given as follows

\[
\frac{dS}{dt} = -\frac{\beta SI}{N}, \quad \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I, \quad \frac{dR}{dt} = \gamma I,
\]

where \(\beta > 0\) is the infection rate and \(\gamma > 0\) is the recovery rate. The total population \(N = S + I + R\) is constant. Furthermore, the basic reproduction number, denoted as \(R_0\) and defined as the epidemic threshold, was introduced. The number indicates either the population is free of disease or disease invades the population [3].

There are many models of epidemic model, one of the models is SIRS. The model shows individuals change the status as susceptible become infective, infective become recovered and recovered become susceptible again. Qualitative analysis for a family of SIRS type epidemiological models was studied [4]. Stability analysis of a kind of SIRS epidemic model was discussed [5,6].

A bifurcation of a dynamical system is an interesting phenomenon in epidemic model analysis. This phenomenon is defined as a qualitative change in its dynamics by varying parameters. At a point of bifurcation, stability may be gained or lost [7,8]. A bifurcation of a dynamical system which is SIRS epidemic model was discussed [9,10,11]. These studies analyzed existence bifurcation of epidemic models.
In this paper, first, a SIRS epidemic model which is waning immunity model are discussed. Stability of the free of disease critical point and the endemic critical point is analyzed. The stability analysis of the models is associated with the threshold number. Later, the existence of bifurcation was analyzed.

2. The Mathematical Model

The epidemic model waning immunity describes spread of disease in the population that individuals will have immunity losts for a limited period before waning such that the individual is once again susceptible. The model is known as SIRS (susceptible, infected, recovered, susceptible) \(^{(12)}\). The model was reformulated based on the following assumptions:

- The system is closed.
- The population is fixed.
- Becoming infected is the only way an individual can leave the susceptible group. Becoming recovered is the only way an individual can leave the infected group. Becoming susceptible again is the only way an individual can leave the recovered group.
- The population is homogeneously mixed.

The three basic notations of the model:

- \( S(t) \) is the number of susceptible individuals at time \( t \).
- \( I(t) \) is the number of infected individuals at time \( t \).
- \( R(t) \) is the number of recovered individuals at time \( t \).

Hence, system of equations for epidemic model waning immunity can be written as:

\[
\begin{align*}
\frac{dS}{dt} &= -\frac{\beta SI}{N} + \omega R \\
\frac{dI}{dt} &= \frac{\beta SI}{N} - \gamma I \\
\frac{dR}{dt} &= \gamma I - \omega R
\end{align*}
\]

where \( \beta > 0, \gamma > 0, \) and \( \omega > 0 \) are the infection rate, the recovery rate, and the rate at which immunity is lost and recovered individuals move into the susceptible class, respectively. The total population \( N = S + I + R \) is constant.

Let \( s = \frac{S}{N}, i = \frac{I}{N}, \) and \( r = \frac{R}{N} \) denote the proportion of the classes \( S, I, \) and \( R \) in the population, respectively. The system equations (4), (5), and (6) become

\[
\begin{align*}
\frac{ds}{dt} &= -\beta si + \omega r \\
\frac{di}{dt} &= \beta si - \gamma i \\
\frac{dr}{dt} &= \gamma i - \omega r
\end{align*}
\]

with the initial conditions

\( s(0) \geq 0, i(0) \geq 0, r(0) \geq 0. \)

By considering the total population density, we have

\( s + i + r = 1 \Rightarrow r = 1 - s - i. \)

Therefore it is enough to consider

\[
\begin{align*}
\frac{ds}{dt} &= -\beta si + \omega(1 - s - i), \\
\frac{di}{dt} &= (\beta s - \gamma)i.
\end{align*}
\]
3. The Stability Analysis
There are two critical points of the system equations (10) and (11). The one is disease free critical point, \( E_0 = (1, 0) \), and the other one is endemic critical point, \( E_1 = \left( \frac{\gamma}{\beta}, \frac{\omega(\beta - \gamma)}{\beta(\gamma + \omega)} \right) \). Let the threshold number is \( R_0 = \frac{\beta}{\gamma} \).

Jacobian matrix of the equations (10) and (11) is

\[
J = \begin{bmatrix}
-\omega - \beta i & -\omega - \beta s \\
\beta i & \beta s - \gamma
\end{bmatrix}.
\]

The Jacobian matrix evaluated at the disease free critical point, \( E_0 \), that is

\[
J(E_0) = \begin{bmatrix}
-\omega & -\omega - \beta \\
0 & \beta - \gamma
\end{bmatrix}.
\]

Eigenvalues of the Jacobian matrix \( J(E_0) \) are \( \lambda_1 = -\omega \) and \( \lambda_2 = \beta - \gamma \). It means that the disease free critical point \( E_0 \) is asymptotically stable for \( \beta - \gamma < 0 \) or \( R_0 < 1 \) and it is unstable for \( \beta - \gamma > 0 \) or \( R_0 > 1 \).

The Jacobian matrix evaluated at the endemic critical point, \( E_1 \), that is

\[
J(E_1) = \begin{bmatrix}
-\omega + \frac{\omega(\gamma - \beta)}{\gamma + \omega} & -\omega - \gamma \\
-\frac{\omega(\gamma - \beta)}{\gamma + \omega} & 0
\end{bmatrix}.
\]

Eigenvalues of the Jacobian matrix \( J(E_1) \) are

\[
\lambda_1 = \frac{1}{2} \frac{(-\omega(\omega + \beta)) + \sqrt{\omega^4 + (4\gamma - 2\beta)\omega^3 + (\beta^2 - 8\beta\gamma + 8\gamma^2)\omega^2 + (-4\beta\gamma^2 + 4\gamma^3)\omega}}{\gamma + \omega}, \quad \text{and}
\]

\[
\lambda_2 = -\frac{1}{2} \frac{(-\omega(\omega + \beta)) + \sqrt{\omega^4 + (4\gamma - 2\beta)\omega^3 + (\beta^2 - 8\beta\gamma + 8\gamma^2)\omega^2 + (-4\beta\gamma^2 + 4\gamma^3)\omega}}{\gamma + \omega}.
\]

It means that the stability of the endemic critical point \( E_1 \) is determined by the given parameter \( \beta, \gamma \), and \( \omega \).

4. Bifurcation Analysis of The Models
A bifurcation of a dynamical system occurs when there is change of the parameter value of a system. We can examine that \( R_0 = 1 \) is the bifurcation point of the model. Bifurcation diagram of system equations (10) and (11) is given by

Figure 1. Bifurcation diagram of system equations (10) and (11)
Figure 1 shows the critical points (s in this case) versus the bifurcation parameter ($R_0$) taking into account stability. The solid line represents the stable critical point and the dash line represents the unstable critical point. We know that for $s = 1$, the disease free critical point is asymptotically stable when $R_0 < 1$ and unstable when $R_0 > 1$. For $s = \frac{\gamma}{\beta}$, the endemic critical point is asymptotically stable when $R_0 > 1$ and unstable when $R_0 < 1$.

5. Numerical Simulation

The analytical results obtained in the previous section are confirmed by numerical simulation using MATLAB Software. The parameters and the initial condition are taken:

$\beta = 0.16, \gamma = 0.07, \omega = \frac{1}{10^7}, S(0) = \frac{267941010}{267941012}$ and $I(0) = \frac{2}{267941012}$.

Thus, we obtain the endemic critical point, $E_1 = (0.45,0.07)$, the basic reproduction number $R_0 = 2.29$, and the eigenvalues $\lambda_1 = -0.01 + 0.03i, \lambda_2 = -0.01 - 0.03i$. The solution and phase portrait of the system equations (10) and (11) are show in Figure 2 and Figure 3, respectively.

![Figure 2. Graphic solutions of the system equations (10) and (11)](image1)

![Figure 3. Phase portrait of system equations (10) and (11)](image2)
Figure 2 shows the proportion of the susceptible individual. It converges and stable to the endemic critical point, $E_1$. Figure 3 shows that the disease free critical point $E_0 = (1,0)$ is unstable node because the movement of the vector field away from the disease free critical point $E_0 = (1,0)$. On the other hand, the endemic critical point $E_1 = (0.45,0.07)$ is stable spiral because the movement of the vector field which is close to the endemic critical point $E_1 = (0.45,0.07)$.

6. Conclusions

In this study, a mathematical epidemic model which is a SIRS model was discussed. Our results have shown that the disease free critical point ($E_0$) is asymptotically stable if $R_0 < 1$. On the other hand, the endemic critical point ($E_1$) is asymptotically stable if $R_0 > 1$. Our result also shows that the threshold number, $R_0 = 1$, is the bifurcation parameter of the model.

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