QUANTUM QUASI-SHUFFLE ALGEBRAS

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Abstract. We establish some properties of quantum quasi-shuffle algebras. They include the necessary and sufficient condition for the construction of the quantum quasi-shuffle product, the universal property, and the commutativity condition. As an application, we use the quantum quasi-shuffle product to construct a linear basis of $T(V)$, for a special kind of Yang-Baxter algebras $(V, m, \sigma)$.

1. Introduction

Quasi-shuffle algebras are a generalization of shuffle algebras. They first arose in [10] for the study of the cofree irreducible Hopf algebra built on an associative algebra. There, K. Newman and D. E. Radford constructed an associative algebra structure on $T(U)$, for an algebra $U$, by combining the multiplication of $U$ and the shuffle product of $T(U)$. These algebras have their particular interest in many branches of algebra and a number of applications have been found in the past decade. For example, they are used in the study of commutative TriDendriform algebras [7], Rota-Baxter algebras [2], and multiple zeta values [4].

After the birth of quantum groups, many algebraic objects were better understood in the more general framework of braided categories. For example, shuffle algebras, special examples of quasi-shuffle algebras, had been quantized in [13] ten years ago, and led to a more intrinsic understanding of quantum enveloping algebras. The next task was to find a suitable way to quantize the quasi-shuffle algebra. There were some attempts, for example, [1] and [4]. For a braided vector space $(V, \sigma)$, in order to study all associative algebra structures on $T(V)$ which are compatible with the "twisted" deconcatenation coproduct, [5] introduced the notion of quantum $B_\infty$-algebras. The quantum $B_\infty$-algebra provides a suitable framework for the quantization of quasi-shuffle algebras in the spirit of quantum shuffle algebras ([13]), by replacing the usual flip with a braiding. The resulting algebras, called quantum quasi-shuffle algebras, are the generalization of quantum shuffle algebras and provide Yang-Baxter algebras. Because of the importance of quasi-shuffle algebras, it seems quite reasonable to study quantum quasi-shuffle algebras for themselves as new algebraic objects, not just as special quantum $B_\infty$-algebras. This paper is the first step in this direction. As a starting point, we expect that the quantum quasi-shuffle algebra can inherit some good properties of the classical one, or have some "q-analogues" of those in the classical case. We first investigate when and how we can construct the quantum quasi-shuffle product.
on the tensor space $T(V)$ over a braided vector space $(V, \sigma)$. Universal properties always play an important role in the study of algebras. So we provide the universal property of quantum quasi-shuffle algebras in a suitable category. We also study the commutativity of the algebra and present a linear basis of $T(V)$ for a special kind of Yang-Baxter algebras $(V, m, \sigma)$ by using Lyndon words.

This paper is organized as follows. In Section 2, we recall the construction of quantum quasi-shuffle algebras and study the necessary and sufficient condition for the construction. In Section 3, we provide a universal property of quantum quasi-shuffle algebras in the category of connected twisted Yang-Baxter bialgebras and discuss the commutativity of quantum quasi-shuffle algebras. In Section 4, for a special kind of Yang-Baxter algebras $(V, m, \sigma)$, we provide a linear basis of $T(V)$ by using the quantum quasi-shuffle product and Lyndon words.

Notations

In this paper, we denote by $K$ a ground field of characteristic 0. All the objects we discuss are defined over $K$.

The symmetric group of $n$ letters $\{1, 2, \ldots, n\}$ is written by $S_n$. An $(i, j)$-shuffle is an element $w \in S_{i+j}$ such that $w(1) < \cdots < w(i)$ and $w(i+1) < \cdots < w(i+j)$. We denote by $S_{i,j}$ the set of all $(i, j)$-shuffles.

A braiding $\sigma$ on a vector space $V$ is an invertible linear map in $\text{End}(V \otimes V)$ satisfying the quantum Yang-Baxter equation on $V^{\otimes 3}$:

$$(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V) = (\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma).$$

A braided vector space $(V, \sigma)$ is a vector space $V$ equipped with a braiding $\sigma$. For any $n \in \mathbb{N}$ and $1 \leq i \leq n-1$, we denote by $\sigma_i$ the operator $\text{id}_V^{\otimes (i-1)} \otimes \sigma \otimes \text{id}_V^{\otimes (n-i-1)} \in \text{End}(V^{\otimes n})$. For any $w \in S_n$, we denote by $T_w$ the corresponding lift of $w$ in the braid group $B_n$, defined as follows: if $w = s_{i_1} \cdots s_{i_l}$ is any reduced expression of $w$, where $s_i = (i, i+1)$, then $T_w = \sigma_{i_l} \cdots \sigma_{i_1}$. This definition is well-defined (see, e.g., Theorem 4.12 in [4]). Sometimes we also use the notation $T^\sigma_w$ to indicate the action of $\sigma$.

The usual flip switching two factors is denoted by $\tau$. For a vector space $V$, we denote by $\otimes$ the tensor product within $T(V)$, and by $\boxtimes$ the one between $T(V)$ and $T(V)$ respectively.

2. Quantum quasi-shuffle algebras

We start by recalling some definitions. In the following, all algebras are assumed to be associative and unital.

Definition 1 ([3]). 1. Let $A = (A, m)$ be an algebra with product $m$ and unit $1_A$, and $\sigma$ be a braiding on $A$. We call $(A, m, \sigma)$ a Yang-Baxter algebra (YB algebra for short) if it satisfies the following conditions:

$$
\begin{cases}
(id_A \otimes m)\sigma_1\sigma_2 &= \sigma(m \otimes id_A), \\
(m \otimes id_A)\sigma_2\sigma_1 &= \sigma(id_A \otimes m),
\end{cases}
$$

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and for any \( a \in A \),

\[
\begin{align*}
\sigma(1_A \otimes a) &= a \otimes 1_A, \\
\sigma(a \otimes 1_A) &= 1_A \otimes a.
\end{align*}
\]

2. Let \( C = (C, \triangle, \varepsilon) \) be a coalgebra with coproduct \( \triangle \) and counit \( \varepsilon \), and \( \sigma \) be a braiding on \( C \). We call \( (C, \triangle, \sigma) \) a Yang-Baxter coalgebra (YP coalgebra for short) if it satisfies the following conditions:

\[
\begin{align*}
\sigma_1 \sigma_2(\triangle \otimes \text{id}_C) &= (\text{id}_C \otimes \triangle) \sigma, \\
\sigma_2 \sigma_1(\text{id}_C \otimes \triangle) &= (\triangle \otimes \text{id}_C) \sigma,
\end{align*}
\]

and

\[
\begin{align*}
(\text{id}_C \otimes \varepsilon) \sigma &= \varepsilon \otimes \text{id}_C, \\
(\varepsilon \otimes \text{id}_C) \sigma &= \text{id}_C \otimes \varepsilon.
\end{align*}
\]

These definitions give a right way to generalize the usual algebra (resp. coalgebra) structure on the tensor products of algebras (resp. coalgebras) in the following sense.

**Proposition 2** ([3], Proposition 4.2). 1. For a YB algebra \( (A, m, \sigma) \) and any \( i \in \mathbb{N} \), \( (A^{\otimes i}, m_{\sigma,i}, T_{\chi_{\sigma,i}}^\sigma) \) becomes a YB algebra with product \( m_{\sigma,i} = m_{\otimes i} \circ T_{w_i}^\sigma \) and unit \( 1_{A^i}^\otimes \), where \( \chi_{\sigma,i}, w_i \in \Sigma_{2i} \), are given by

\[
\chi_{\sigma,i} = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & 2i \\
 i+1 & i+2 & \cdots & 2i & 1 & 2 & \cdots & i \end{pmatrix},
\]

and

\[
w_i = \begin{pmatrix} 1 & 2 & 3 & \cdots & i & i+1 & i+2 & \cdots & 2i \\
 1 & 3 & 5 & \cdots & 2i-1 & 2 & 4 & \cdots & 2i \end{pmatrix}.
\]

2. For a YB coalgebra \( (C, \Delta, \sigma) \), \( (C^{\otimes i}, \Delta_{\sigma,i}, T_{\chi_{\sigma,i}}^\sigma) \) becomes a YB coalgebra with coproduct \( \Delta_{\sigma,i} = T_{w_i^{-1}}^\sigma \circ \Delta_{\otimes i} \) and counit \( \varepsilon_{\otimes i} : C^{\otimes i} \rightarrow K^{\otimes i} \simeq K \).

We call \( m_{\sigma} = m_{\sigma,2} \) the twisted algebra structure on \( A \otimes A \) and \( \Delta_{\sigma} = \Delta_{\sigma,2} \) the twisted coalgebra structure on \( C \otimes C \).

Let \( (V, \sigma) \) be a braided vector space. For any \( i, j \geq 1 \), we denote

\[
\chi_{ij} = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & i+j \\
 j+1 & j+2 & \cdots & j+1 & 1 & 2 & \cdots & j \end{pmatrix},
\]

and define \( \beta : T(V) \otimes T(V) \rightarrow T(V) \otimes T(V) \) by requiring that \( \beta_{ij} = T_{\chi_{ij}}^\sigma \) on \( V^{\otimes i} \otimes V^{\otimes j} \). For convenience, we denote by \( \beta_{0i} \) and \( \beta_{i0} \) the usual flip map \( \tau \).

Then \( (T(V), m, \beta) \) is a YB algebra, where \( m \) is the concatenation product.

Another example of YB algebras is the quantum shuffle algebra (see [3]). For a braided vector space \( (V, \sigma) \), one can construct an associative algebra structure on \( T(V) \) by: for any \( x_1, \ldots, x_{i+j} \in V \),

\[
(x_1 \otimes \cdots \otimes x_i) m_\sigma(x_{i+1} \otimes \cdots \otimes x_{i+j}) = \sum_{w \in \Sigma_{i,j}} T_w(x_1 \otimes \cdots \otimes x_{i+j}).
\]

The space \( T(V) \) equipped with \( m_\sigma \) is called the quantum shuffle algebra and denoted by \( T_\sigma(V) \). We have that \( (T_\sigma(V), \beta) \) is a YB algebra.
We define $\delta$ to be the deconcatenation on $T(V)$, i.e.,

$$\delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=0}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n).$$

We denote by $T^c(V)$ the coalgebra $(T(V), \delta, \varepsilon)$ where $\varepsilon$ is the projection from $T(V)$ to $K$. The coalgebra $T^c(V)$ is the cotensor coalgebra (see [11]) over the trivial Hopf algebra $K$. Here $V$ is a Hopf bimodule with scalar multiplication and coactions defined by $\delta_L(v) = 1 \otimes v$ and $\delta_R(v) = v \otimes 1$ for any $v \in V$. $(T^c(V), \beta)$ is a YB coalgebra.

Now we review the construction of the quantum quasi-shuffle algebra which was given as a special example of quantum $B_\infty$-algebras in [5].

Let $(V, \sigma)$ be a braided vector space and for any $p, q \geq 0$, $M_{pq} : V^{\otimes p} \otimes V^{\otimes q} \rightarrow V$ be a linear map such that

$$
\begin{cases}
    M_{00} = 0, \\
    M_{10} = \text{id}_V = M_{01}, \\
    M_{11} = M, \\
    M_{pq} = 0, \text{ otherwise.}
\end{cases}
$$

We denote

$$\hat{\otimes}_\sigma = \varepsilon \otimes \varepsilon + \sum_{n \geq 1} M^{\otimes n} \circ \Delta^{(n-1)}_\beta : T^c(V) \otimes T^c(V) \rightarrow T^c(V),$$

where $M = (M_{pq})_{p, q \geq 0}$. $\Delta_\beta = (\text{id}_{T^c(V)} \otimes \beta \otimes \text{id}_{T^c(V)}) \circ (\delta \otimes \delta)$ and $\Delta_\beta^{(n)} = (\Delta_\beta \otimes \text{id}_{T^c(V)^{\otimes (n-1)}}) \circ \Delta_\beta^{(n-1)}$ inductively. It is easy to show by induction that the summation with respect to $n$ in the above formula is finite. Indeed, since $M^{\otimes (i+j+1)} \circ \Delta_\beta^{(i+j)} = (M^{\otimes (i+j)} \circ \Delta_\beta^{(i+j-1)}) \otimes M \circ \Delta_\beta$ and the conditions for $M$, we have that $M^{\otimes (i+j+1)} \circ \Delta_\beta^{(i+j)}(x \hat{\otimes} y) = 0$, for any $x \in V^{\otimes i}$ and $y \in V^{\otimes j}$.

To illustrate the new map $\hat{\otimes}_\sigma$, we calculate a few examples. For any $u, v, w \in V$, we have

$$u \hat{\otimes}_\sigma v = (\varepsilon \otimes \varepsilon + M \circ \Delta^{(0)}_\beta + M^{\otimes 2} \circ \Delta^{(1)}_\beta)(u \hat{\otimes} v)$$

$$= M_{11}(u \hat{\otimes} v)$$

$$+ M^{\otimes 2}(1 \otimes \beta_{10}(u \hat{\otimes} 1) \hat{\otimes} v + 1 \otimes \beta_{11}(u \hat{\otimes} v) \hat{\otimes} 1$$

$$+ u \otimes \beta_{00}(1 \hat{\otimes} 1) \hat{\otimes} v + u \otimes \beta_{01}(1 \hat{\otimes} v) \hat{\otimes} 1)$$

$$= M_{11}(u \hat{\otimes} v) + (M_{01} \otimes M_{10})(1 \otimes \sigma(u \hat{\otimes} v) \hat{\otimes} 1)$$

$$+ (M_{10} \otimes M_{01})(1 \otimes \sigma(u \hat{\otimes} v) \hat{\otimes} 1)$$

$$= M_{11}(u \hat{\otimes} v) + u \hat{\otimes} v + \sigma(u \hat{\otimes} v)$$

$$= M_{11}(u \hat{\otimes} v) + u \hat{\otimes} v.$$
We denote by $u \otimes v$ the product in $V \otimes V$. Then we have the following inductive formula.

**Proposition 3.** For $i, j > 1$ and any $u_1, \ldots, u_i, v_1, \ldots, v_j \in V$, we have

\[
(u_1 \otimes \cdots \otimes u_i) \star_{\sigma} (v_1 \otimes \cdots \otimes v_j) = \left( (u_1 \otimes \cdots \otimes u_i) \star_{\sigma} (v_1 \otimes \cdots \otimes v_{j-1}) \right) \otimes v_j
\]

\[
+ \left( (u_1 \otimes \cdots \otimes u_i) \star_{\sigma(i-1,j-1)} \otimes m \right) \sigma_{i+j-2} \cdots \sigma_i (u_1 \otimes \cdots \otimes u_i \otimes v_1 \otimes \cdots \otimes v_j).
\]

(1)

**Proof.** By the definition of $\star_{\sigma}$, we have

\[
(u_1 \otimes \cdots \otimes u_i) \star_{\sigma} (v_1 \otimes \cdots \otimes v_j)
\]

\[
= (\varepsilon \otimes \varepsilon + \sum_{n \geq 1} M^{\otimes n} \circ \Delta_{\beta}^{(n-1)} \otimes M) \left( (u_1 \otimes \cdots \otimes u_i) \otimes (v_1 \otimes \cdots \otimes v_j) \right)
\]

\[
= \sum_{n=2}^{i+j} \left( (M^{\otimes(n-1)} \circ \Delta_{\beta}^{(n-2)} \otimes M) \circ \Delta_{\beta} \left( (u_1 \otimes \cdots \otimes u_i) \otimes (v_1 \otimes \cdots \otimes v_j) \right) \right)
\]

\[
= \sum_{n=2}^{i+j} \left( M^{\otimes(n-1)} \circ \Delta_{\beta}^{(n-2)} \left( (u_1 \otimes \cdots \otimes u_i) \otimes (v_1 \otimes \cdots \otimes v_{j-1}) \right) \otimes M_{01}(1 \otimes v_j) \right)
\]

\[
+ \sum_{n=2}^{i+j} \left( (M^{\otimes(n-1)} \circ \Delta_{\beta}^{(n-2)} \otimes M_{10}) \circ (\id_{V}^{\otimes i-1} \otimes \beta \otimes \id_K) \left( (u_1 \otimes \cdots \otimes u_{i-1}) \otimes u_i \otimes (v_1 \otimes \cdots \otimes v_j) \otimes 1 \right) \right)
\]

We denote by $\star_{\sigma(i,j)}$ the restriction of $\star_{\sigma}$ on $V^{\otimes i} \otimes V^{\otimes j}$. Then we have the following inductive formula.
\[ + \sum_{n=2}^{i+j} \left( (M^{\otimes (n-1)} \circ \Delta^{(n-2)}) \otimes M_{11} \right) \circ (\text{id}^{\otimes i-1} \otimes \beta_{1,j-1} \otimes \text{id}_V) \left( (u_1 \otimes \cdots \otimes u_{i-1}) \otimes v_1 \otimes \cdots \otimes v_{j-1} \otimes v_j \right) \]
\[ = \left( (u_1 \otimes \cdots \otimes u_i) \otimes (v_1 \otimes \cdots \otimes v_{j-1}) \right) \otimes v_j \]
\[ + (\otimes c) \sigma_{i+j-1} \cdots \sigma_1 (u_1 \otimes \cdots \otimes u_i \otimes v_1 \otimes \cdots \otimes v_j) \]
\[ + (\otimes c) (u_1 \otimes \cdots \otimes u_i \otimes v_1 \otimes \cdots \otimes v_j) \]

It is easy to see that \( 1 \otimes \sigma = \sigma = \sigma \otimes 1 \) for any \( x \in T^v(V) \), where \( 1 \) is the unit of \( K \). But there is no evidence that \( (T^v(V), \otimes \sigma) \) is an associative algebra. Actually, \( \otimes \sigma \) is not associative in general. If the map \( M_{11} = m \) is an associative product and is compatible with the braiding \( \sigma \) in some sense, it comes true.

**Theorem 4.** Under the above notations, \( (T^v(V), \otimes \sigma, \beta) \) is a \( YB \)-algebra if and only if \( (V, M_{11}, \sigma) \) is a \( YB \)-algebra.

**Proof.** Let \( (V, M_{11}, \sigma) \) be a \( YB \)-algebra. One can find a detailed proof in [5] for that \( (T^v(V), \otimes \sigma, \beta) \) is a \( YB \)-algebra in a more general setting. Because of the simplicity of the \( M \) here, we can provide another proof by using the relation (1). Since \( (T^v(V), \beta) \) is a \( YB \)-coalgebra, we know that \( (T^v(V)^{\otimes 2}, \Delta_{\beta}, \beta_{\otimes 2}) \) is a \( YB \)-coalgebra by Proposition 2. By using the compatibility conditions for \( M_{11} \) and \( \sigma \) and those for \( \Delta_{\beta} \) and \( \beta \), it is easy to prove that

\[
\left\{ \begin{array}{l}
\beta(\otimes \sigma \otimes \text{id}_{T^v(V)}) = (\text{id}_{T^v(V)} \otimes \sigma) \beta_1 \beta_2, \\
\beta(\text{id}_{T^v(V)} \otimes \sigma) = (\otimes \sigma \otimes \text{id}_{T^v(V)}) \beta_2 \beta_1.
\end{array} \right.
\]

Now we show that \( \otimes \sigma \) is associative, i.e., for any \( x \in V^{\otimes i}, y \in V^{\otimes k}, z \in V^{\otimes k} \), we have \( x \otimes \sigma \) \( (x \otimes \sigma \ y) \otimes \sigma \ z \). If \( j = 0 \) or \( k = 0 \), there is nothing to prove. So we assume that \( j, k \geq 1 \), \( y = y' \otimes u \) and \( z = z' \otimes v \), where \( u, v \in V \) and \( y', z' \in T^v(V) \). By the inductive formula (1) and the above compatibility conditions for \( \otimes \sigma \) and \( \beta \), it is easy to prove the statement by using induction on \( i + j + k \).

Conversely, if \( \otimes \sigma \) is associative, then for any \( u, v, w \in V \), we have

\[
(u \otimes \sigma \ v) \otimes \sigma \ w = (M_{11}(u \otimes v) + \text{id}_{\sigma}(w) \otimes \sigma) \otimes \sigma \ w = M_{11}(M_{11}(u \otimes v) \otimes w) + M_{11}(u \otimes v) \text{id}_{\sigma} \ w
\]
\[ + (\text{id}_{\sigma}(v) \otimes M_{11}) + (\text{id}_{\sigma}(u) \otimes \sigma) M_{11} \otimes \sigma + (\text{id}_{\sigma}(u) \otimes \sigma) \text{id}_{\sigma} \ w
\]
\[ + (\text{id}_{\sigma}(v) \otimes M_{11}) + (\text{id}_{\sigma}(u) \otimes \sigma) M_{11} \otimes \sigma + (\text{id}_{\sigma}(u) \otimes \sigma) \text{id}_{\sigma} \ w
\]
\[ + (\text{id}_{\sigma}(v) \otimes M_{11}) + (\text{id}_{\sigma}(u) \otimes \sigma) M_{11} \otimes \sigma + (\text{id}_{\sigma}(u) \otimes \sigma) \text{id}_{\sigma} \ w,
\]
and

\[
u \otimes \sigma \ v \otimes \sigma \ w = u \otimes \sigma \ (M_{11}(v \otimes w) + \text{id}_{\sigma} \ w)
\]
\[ = M_{11}(u \otimes M_{11}(v \otimes w)) + \text{id}_{\sigma} \ M_{11}(v \otimes w)\]
Therefore we have that 
\((u \otimes \varphi_v) \varphi w = u \varphi (v \varphi w)\) if and only if 
\[M_{11}(M_{11} \otimes \text{id}_V) + M_{11} \otimes \text{id}_V + \sigma(M_{11} \otimes \text{id}_V) + \text{id}_V \otimes M_{11} + (\text{id}_V \otimes M_{11})\sigma_1 + (M_{11} \otimes \text{id}_V)\sigma_2 + (M_{11} \otimes \text{id}_V)\sigma_2\] 
\[= M_{11}(\text{id}_V \otimes M_{11}) + \text{id}_V \otimes M_{11} + \sigma(\text{id}_V \otimes M_{11}) + M_{11} \otimes \text{id}_V + (\text{id}_V \otimes M_{11})\sigma_1 + (\text{id}_V \otimes M_{11})\sigma_2,\]
i.e., 
\[M_{11}(M_{11} \otimes \text{id}_V) + \sigma(M_{11} \otimes \text{id}_V) + (M_{11} \otimes \text{id}_V)\sigma_2\sigma_1 = M_{11}(\text{id}_V \otimes M_{11}) + \sigma(\text{id}_V \otimes M_{11}) + (\text{id}_V \otimes M_{11})\sigma_1\sigma_2.\]

By comparing the degrees of the resulting tensor vectors, we must have 
\[M_{11}(M_{11} \otimes \text{id}_V) = M_{11}(\text{id}_V \otimes M_{11}).\]

On \(V \otimes V \otimes V\), the condition \((\text{id}_V \otimes \varphi_v) \varphi w = \varphi (v \varphi w)\) implies that \((\text{id}_V \otimes M_{11})\sigma_1\sigma_2 + (\text{id}_V \otimes \text{id}_V)\sigma_1\sigma_2 = \sigma(M_{11} \otimes \text{id}_V) + \sigma(\text{id}_V \otimes \text{id}_V)\). By comparing the degrees, we get 
\[(\text{id}_V \otimes M_{11})\sigma_1\sigma_2 = \sigma(M_{11} \otimes \text{id}_V)\]. Similarly, we have 
\[(M_{11} \otimes \text{id}_V)\sigma_2\sigma_1 = \sigma(\text{id}_V \otimes M_{11}).\]

The above resulting algebra \((T^c(V), \varphi_v)\) is called the quantum quasi-shuffle algebra over \((V, M_{11}, \sigma)\).

**Remark 5.** For an algebra \((A, m)\), the quantum quasi-shuffle algebra over the trivial YB algebra \((A, m, \tau)\) is just the classical quasi-shuffle algebra over the algebra \(A\).

### 3. Universal Property and Commutativity

Let \((C, \triangle, \varepsilon)\) be a coalgebra with a preferred group-like element \(1_C \in C\). We denote \(\overline{\triangle}(x) = \triangle(x) - x \otimes 1_C - 1_C \otimes x\) for any \(x \in C\). The map \(\overline{\triangle}\) is also coassociative and called the reduced coproduct. We also denote \(\overline{C} = \text{Ker} \varepsilon\). Then 
\[C = K1_C \oplus \overline{C}\] since \(x - \varepsilon(x)1_C \in \overline{C}\) for any \(x \in C\).

**Definition 6.** \((C, \triangle, \varepsilon)\) is said to be connected if \(C = \cup_{r \geq 0} F_r C\), where
\[
F_0 C = K1_C, \\
F_r C = \{ x \in C | \overline{\triangle}(x) \in F_{r-1} C \otimes F_{r-1} C \}, \quad \text{for } r \geq 1.
\]

There is a well-known universal property for \(T^c(V)\) (see, e.g., [3]):

**Proposition 7.** Given a connected coalgebra \((C, \triangle, \varepsilon)\) and a linear map \(\phi : C \to V\) such that \(\phi(1_C) = 0\). Then there is a unique coalgebra morphism \(\overline{\phi} : C \to T^c(V)\) which extends \(\phi\), i.e., \(P_{T^c(V)} \circ \overline{\phi} = \phi\), where \(P_{T^c(V)} : T^c(V) \to V\) is the projection.
onto $V$. Explicitly, $\overline{\phi} = \varepsilon + \sum_{n \geq 1} \phi^{\otimes n} \circ \overline{\Delta}^{(n-1)}$, where $\overline{\Delta}^{(n)} = (\overline{\Delta}^{(n-1)} \circ \text{id}_C) \circ \overline{\Delta}$ inductively.

**Corollary 8.** Let $C$ be a connected coalgebra. If $\Phi, \Psi : C \to T^e(V)$ are coalgebra maps such that $\text{Pr}_V \circ \Phi = \text{Pr}_V \circ \Psi$ and $\text{Pr}_V \circ \Phi(1_C) = 0 = \text{Pr}_V \circ \Psi(1_C)$, then $\Phi = \Psi$.

We will use the above properties to provide a universal property of the quantum quasi-shuffle algebra $(T^e(V), \triangleright)$ in some category. First we describe the category in which we will work.

**Definition 9.** A quadruple $(H, \cdot, \Delta, \sigma)$ is called a twisted Yang-Baxter (YB for short) bialgebra if

1. $(H, \cdot, \sigma)$ is a YB algebra,
2. $(H, \Delta, \sigma)$ is a YB coalgebra,
3. $\cdot : H \otimes H \to H$ is a coalgebra map, where $H \otimes H$ is equipped with the twisted coalgebra structure. Or equivalently, $\Delta : H \to H \otimes H$ is an algebra map, where $H \otimes H$ is equipped with the twisted algebra structure.

From the condition 3 above, we have that $\Delta(1_H) = 1_H \otimes 1_H$.

**Examples.** 1. Let $(V, \sigma)$ be a braided vector space. Then the quantum shuffle algebra $(T_v(V), \beta)$ equipped with the deconcatenation coproduct $\delta$ is a twisted YB bialgebra (see [13]).

2. Let $(V, m, \sigma)$ be a YB algebra. Then the quantum quasi-shuffle algebra $(T^e(V), \triangleright, \sigma)$ is a twisted YB bialgebra with the deconcatenation coproduct $\delta$ (see [13]).

We denote by $CB_{YB}$ the category of connected twisted YB bialgebras. It consists of the following data:

1. the objects of $CB_{YB}$ are the twisted YB bialgebras $(H, \cdot, \Delta, \sigma)$ such that both $H$ and $H \otimes H$ are connected, where the preferred group-like elements are $1_H$ and $1_H \otimes 1_H$ respectively, and $H \otimes H$ is equipped with the twisted coalgebra structure;
2. a morphism $f$ from object $(H_1, \sigma_1)$ to object $(H_2, \sigma_2)$ is both an algebra map and a coalgebra map and satisfies that $(f \otimes f)\sigma_1 = \sigma_2(f \otimes f)$.

It is easy to see that both $(T(V), m, \delta, \beta)$ and $(T^e(V), \triangleright, \delta, \beta)$ are in $CB_{YB}$.

**Lemma 10.** Let $(V_1, \sigma_1)$ and $(V_2, \sigma_2)$ be two braided vector spaces and $f : V_1 \to V_2$ be a morphism of braided vector spaces, i.e. a linear map such that $\sigma_2(f \otimes f) = (f \otimes f)\sigma_1$. Then for any $i, j \geq 1$, $T^{\sigma_2}_{\chi_{ij}}(f^{\otimes i} \otimes f^{\otimes j}) = (f^{\otimes j} \otimes f^{\otimes i})T^{\sigma_1}_{\chi_{ij}}$.

**Proof.** We use induction on $i + j$.

When $i = j = 1$, it is trivial.

For $i + j \geq 3$, we have

$$T^{\sigma_2}_{\chi_{ij}}(f^{\otimes i} \otimes f^{\otimes j}) = (T^{\sigma_2}_{\chi_{i-1,j}} \otimes \text{id}_{V_2})(\text{id}_{V_1}^{\otimes i-1} \otimes T^{\sigma_2}_{\chi_{1,j}})(f^{\otimes i} \otimes f^{\otimes j})$$

$$= (T^{\sigma_2}_{\chi_{i-1,j}} \otimes \text{id}_{V_2})(f^{\otimes i-1} \otimes T^{\sigma_2}_{\chi_{1,j}}(f \otimes f^{\otimes j}))$$

$$= (T^{\sigma_2}_{\chi_{i-1,j}} \otimes \text{id}_{V_2})(f^{\otimes i-1} \otimes f^{\otimes j} \otimes f)(\text{id}_{V_1}^{\otimes i-1} \otimes T^{\sigma_1}_{\chi_{1,j}})$$
\[ = (f^{\otimes j} \otimes f^{\otimes i})(T_{\chi_{-1,j}}^{\sigma_1} \otimes \text{id}_{V_1})(\text{id}_{V_1}^{\otimes i-1} \otimes T_{\chi_{1,j}}^{\sigma_1}) \]
\[ = (f^{\otimes j} \otimes f^{\otimes i})T_{\chi_{1,j}}^{\sigma_1}. \]

**Lemma 11.** Let \((C, \Delta, \sigma)\) be a YB coalgebra and \(1_C\) be a group-like element of \(C\). If \(\sigma(1_C \otimes x) = x \otimes 1_C\) and \(\sigma(x \otimes 1_C) = 1_C \otimes x\) for any \(x \in C\), then we have
\[
\left\{ \begin{array}{l}
(\text{id}_C \otimes \Delta) \sigma = \sigma_1 \sigma_2 (\Delta \otimes \text{id}_C), \\
(\Delta \otimes \text{id}_C) \sigma = \sigma_2 \sigma_1 (\text{id}_C \otimes \Delta).
\end{array} \right.
\]

**Proof.** It follows direct computations. \qed

Let \((V, m, \sigma)\) be a YB algebra. We have the following universal property in \(\text{CB}_{YB}\).

**Proposition 12.** For any \((H, \cdot, \Delta_H, \alpha) \in \text{CB}_{YB}\) and a linear map \(f : H \to V\) such that \(m \circ (f \otimes f) = f \circ \Delta_H \otimes H\), \(f(1_H) = 0\) and \(f \otimes f = \sigma(f \otimes f)\), there exists a unique morphism \(\overline{f} : H \to T^c(V), \kappa, \sigma, \delta, \beta\) which extends \(f\).

**Proof.** Observe that the condition on \(f\) means that: \(\forall x, y \in H, f(xy) = f(x)f(y) + \varepsilon(x)f(y) + \varepsilon(y)f(x)\). Since \(f(1_H) = 0\) and \(H\) is connected, there is a unique coalgebra map \(\overline{f} : H \to T^c(V)\) which extends \(f\). More precisely, \(\overline{f} = \varepsilon_H + \sum_{n \geq 1} f^{\otimes n} \circ \Delta_H^{(n-1)}\).

We first prove that \(\beta(\overline{f} \otimes \overline{f}) = (\overline{f} \otimes \overline{f})\alpha\). We only need to verify it on \(\overline{H} \otimes \overline{H}\).

We have
\[
\beta(\overline{f} \otimes \overline{f}) = \beta\left( \sum_{i,j \geq 1} (f^{\otimes i} \otimes f^{\otimes j})(\Delta_H^{(i-1)} \otimes \Delta_H^{(j-1)}) \right)
\]
\[ = \sum_{i,j \geq 1} T_{\chi_{i,j}}^{\sigma_1} (f^{\otimes i} \otimes f^{\otimes j})(\Delta_H^{(i-1)} \otimes \Delta_H^{(j-1)}) \]
\[ = \sum_{i,j \geq 1} (f^{\otimes i} \otimes f^{\otimes j})T_{\chi_{i,j}}^{\sigma_1} (\Delta_H^{(i-1)} \otimes \Delta_H^{(j-1)}) \]
\[ = \sum_{i,j \geq 1} (f^{\otimes i} \otimes f^{\otimes j})(\Delta_H^{(j-1)} \otimes \Delta_H^{(i-1)}) \alpha \]
\[ = (\overline{f} \otimes \overline{f})\alpha, \]
where the third and the forth equalities follow from Lemma 10 and Lemma 11 respectively.

The next step is to prove that \(\overline{f}\) is an algebra map. We define two maps:
\[ F_1 : H \otimes H \to T^c(V), \]
\[ h \otimes g \mapsto \overline{f}(h) \kappa \sigma \overline{f}(g), \]
and
\[ F_2 : H \otimes H \to T^c(V), \]
\[ h \otimes g \mapsto \overline{f}(hg). \]

We claim that both \(F_1\) and \(F_2\) are coalgebra maps, where \(H \otimes H\) is equipped with the twisted coalgebra structure.
Indeed,
\[ \delta \circ F_1 = \delta \circ \kappa_\sigma (\overline{f} \otimes \overline{f}) = (\kappa_\sigma \otimes \kappa_\sigma) \Delta_{\beta} (\overline{f} \otimes \overline{f}) = (\kappa_\sigma \otimes \kappa_\sigma)(id_{T^-(V)} \otimes \beta \otimes id_{T^+(V)})(\delta \otimes \delta)(\overline{f} \otimes \overline{f}) \]
\[ = (\kappa_\sigma \otimes \kappa_\sigma)(id_{T^-(V)} \otimes \beta \otimes id_{T^+(V)})(\delta \otimes \delta)(\overline{f} \otimes \overline{f})(\Delta_H \otimes \Delta_H) \]
\[ = (\kappa_\sigma \otimes \kappa_\sigma)(\overline{f} \otimes \beta(\overline{f} \otimes \overline{f}) \otimes \overline{f})(\Delta_H \otimes \Delta_H) \]
\[ = (F_1 \otimes F_1)(id_H \otimes \alpha \otimes id_H)(\Delta_H \otimes \Delta_H) \]
\[ = (F_1 \otimes F_1) \Delta_\alpha. \]

And
\[ \delta \circ F_2 = \delta \circ f \circ \cdot \]
\[ = (\overline{f} \otimes \overline{f}) \circ \Delta_H \circ \cdot \]
\[ = (\overline{f} \otimes \overline{f})(\cdot \circ \cdot)(id_H \circ \alpha \otimes id_H)(\Delta_H \otimes \Delta_H) \]
\[ = (F_2 \otimes F_2) \Delta_\alpha. \]

For any \( h, g \in \mathbb{H} \), we have
\[ Pr_V \circ F_1(h \otimes g) = Pr_V(\overline{f}(h) \kappa_\sigma \overline{f}(g)) \]
\[ = Pr_V(\sum_{n \geq 1} M_{\boxtimes n} \Delta_{\beta}^{(n-1)}(\overline{f}(h) \otimes \overline{f}(g))) \]
\[ = M(\overline{f}(h) \otimes \overline{f}(g)) \]
\[ = \sum_{i,j \geq 1} M_{ij}((f^\otimes i \otimes f^\otimes j)(\overline{\Delta}_H^{(i-1)}(h) \otimes \overline{\Delta}_H^{(j-1)}(g))) \]
\[ = M_{11}(f \circ f)(h \otimes g) \]
\[ = f \circ \cdot(h \otimes g) \]
\[ = Pr_V \circ F_2(h \otimes g). \]

Now, for any \( h, g \in H \), write \( h = \overline{h} + \varepsilon(h)1 \), \( g = \overline{g} + \varepsilon(g)1 \). We have: \( h \otimes g = \overline{h} \otimes \overline{g} + \varepsilon(h)\overline{1} \otimes \overline{g} + \varepsilon(g)\overline{h} \otimes 1 + \varepsilon(h)\varepsilon(g)1 \otimes 1 \), so \( Pr_V \circ F_1(h \otimes g) = Pr_V \circ F_1(\overline{h} \otimes \overline{g}) + \varepsilon(h)f(\overline{g}) + \varepsilon(g)f(\overline{h}) \). Also, \( hg = \overline{h} \overline{g} + \varepsilon(h)\varepsilon(g)\overline{h} + \varepsilon(h)\varepsilon(g)1 \), so \( Pr_V \circ F_2(h \otimes g) = Pr_V \circ F_2(\overline{h} \otimes \overline{g}) + \varepsilon(h)f(\overline{g}) + \varepsilon(g)f(\overline{h}) \), and we have again equality. Since \( H \otimes H \) is connected with the twisted coalgebra structure, \( F_1 = F_2 \) follows from the Corollary 8. \( \square \)

Now we begin to discuss the commutativity of quantum quasi-shuffle algebras. In the classical case, if \( A \) is a commutative algebra, then the quasi-shuffle algebra built on \( A \) is also commutative (see, e.g., [11, 10]). But for quantum quasi-shuffle algebras, because of the complexity of the braiding, it is not reasonable and in fact not possible to require this even though the YB algebra is commutative in the usual sense. It is much more suitable to discuss the commutativity in the sense of braided category. This demands extra crucial conditions for the braiding and the multiplication.

**Definition 13.** A YB algebra \((A, m, \sigma)\) is called twisted commutative if \( m \circ \sigma = m \).
Examples. 1. Let \((A, m)\) be an algebra. Then the trivial YB algebra structure \((A, m, \tau)\) is twisted commutative if and only if \(A\) is commutative.

2. Let \(V\) be a vector space over \(\mathbb{C}\) with basis \(\{e_1, \ldots, e_N\}\). Take a nonzero scalar \(q \in \mathbb{C}\). We define a braiding \(\sigma\) on \(V\) by

\[\sigma(e_i \otimes e_j) = \begin{cases} e_i \otimes e_j, & i = j, \\ q^{-1}e_j \otimes e_i, & i < j, \\ q^{-1}e_i \otimes e_j + (1 - q^{-2})e_i \otimes e_j, & i > j. \end{cases}\]

Then \(\sigma\) satisfies the Iwahori's quadratic equation \((\sigma - \text{id}_V \otimes \sigma)(\sigma + q^{-2}\text{id}_V \otimes \sigma) = 0\). In fact, this \(\sigma\) is given by the \(R\)-matrix in the fundamental representation of \(U_q\mathfrak{sl}_N\). We denote \(\Lambda_\sigma(V) = T(V)/I\), where \(I\) is the ideal of \(T(V)\) generated by \(\text{Ker}(\text{id}_V \otimes \sigma)\).

By easy computation, we get that \(\text{Ker}(\text{id}_V \otimes \sigma)\) is spanned by \(\{e_i \otimes e_i, q^{-1}e_i \otimes e_j + e_j \otimes e_i | i < j\}\). We denote \(e_i \wedge \cdots \wedge e_s\) the image of \(e_i \otimes \cdots \otimes e_s\) in \(\Lambda_\sigma(V)\).

So \(\Lambda_\sigma(V)\) is an algebra generated by \(e_i\) and the relations \(e_i^2 = 0\) and \(e_j \wedge e_i = -q^{-1}e_i \wedge e_j\) if \(i < j\). And the set \(\{e_i \wedge \cdots \wedge e_s | 1 \leq i_1 < \cdots < i_p \leq N, 1 \leq p \leq N\}\) forms a linear basis of \(\Lambda_\sigma(V)\). The algebra \(\Lambda_\sigma(V)\) is called the quantum exterior algebra over \(V\).

We denote the increasing set \((i_1, \ldots, i_s)\) by \(\underline{i}\) and so on. For \(1 \leq i_1 < \cdots < i_s \leq N\) and \(1 \leq j_1 < \cdots < j_t \leq N\), we denote

\[(i_1, \ldots, i_s|j_1, \ldots, j_t) = \begin{cases} 0, & \text{if } \underline{i} \cap \underline{j} = \emptyset, \\ \{\{i_k, j_l| i_k > j_l\} - s, & \text{otherwise}. \end{cases}\]

The \(q\)-flip \(\mathcal{F} = \bigoplus_{s,t} \mathcal{F}_{s,t}: \Lambda_\sigma(V) \otimes \Lambda_\sigma(V) \to \Lambda_\sigma(V) \otimes \Lambda_\sigma(V)\) is defined by: for \(1 \leq i_1 < \cdots < i_s \leq N\) and \(1 \leq j_1 < \cdots < j_t \leq N\),

\[\mathcal{F}_{s,t}(e_{i_1} \wedge \cdots \wedge e_{i_s} \otimes e_{j_1} \wedge \cdots \wedge e_{j_t}) = (-q)^{(|i_1| - |i_1|)(|j_1| - |j_1|)} e_{j_1} \wedge \cdots \wedge e_{j_t} \otimes e_{i_1} \wedge \cdots \wedge e_{i_s}.\]

Then \((\Lambda_\sigma(V), \wedge, \mathcal{F})\) is a YB algebra. Moreover it is twisted commutative.

Lemma 14. Let \(\sigma\) be a braiding on \(V\) such that \(\sigma^2 = \text{id}_V^\otimes 2\). Then the braiding \(\beta\) on \(T(V)\) also satisfies that \(\beta^2 = \text{id}_T(V)^\otimes 2\).

Proof. We prove the statement for \(\beta_{ij}\) by using induction on \(i + j\).

When \(i = j = 1\), it is trivial since \(\beta_{11} = \sigma\).

For \(i + j \geq 3\), we have

\[\beta_{ij} \circ \beta_{ij} = (\beta_{j-1,i} \otimes \text{id}_V)(\text{id}_V^\otimes j-1 \otimes \beta_{i1})(\text{id}_V^\otimes j-1 \otimes \beta_{i1})(\beta_{i,j-1} \otimes \text{id}_V) = \text{id}_T(V)^\otimes 2.\]

If \(\sigma = \pm \tau\), then \(\sigma^2 = \text{id}_V^\otimes 2\). For a general braiding \(\sigma\), \(\sigma^2\) is not necessarily the identity map. The first nontrivial example where we nevertheless have involution is the \(q\)-flip \(\mathcal{F}\), i.e., \(\mathcal{F}^2 = \text{Id}\).

Theorem 15. Let \((V, m, \sigma)\) be a YB algebra. Then the quantum quasi-shuffle algebra \((T^c(V), \star, \beta)\) is twisted commutative if and only if \((V, m, \sigma)\) is twisted commutative and \(\sigma^2 = \text{id}_V^\otimes 2\).
Proof. If \((T^c(V), \otimes, \odot)\) is twisted commutative, then on \(V \otimes V\) we have
\[
m + \mathrm{id}_V^\otimes 2 + \sigma = m + \mathbb{M}_\sigma
\]
\[
= \otimes_{\sigma(1,1)}
\]
\[
= \otimes_{\sigma(1,1)} \circ \sigma
\]
\[
= m \circ \sigma + \sigma^2.
\]
By comparing the degrees, we have that \(m = m \circ \sigma\) and \(\sigma^2 = \mathrm{id}_V^\otimes 2\).

Conversely, we use induction on \(i + j\) where \(i\) and \(j\) are the powers of \(V^\otimes i \otimes V^\otimes j\).

When \(i = j = 1\), it is trivial.

For \(i + j \geq 3\), by the inductive relation (1), we have
\[
\otimes_{\sigma(i,j)} \circ \beta_{ij}
\]
\[
= (\otimes_{\sigma(i-1,j)} \otimes \mathrm{id}_V)(\beta_{i-1,1,j} \otimes \mathrm{id}_V)(\mathrm{id}_V^\otimes i-1 \otimes \beta_{1,j})(\mathrm{id}_V^\otimes i \otimes \beta_{1,1,j})
\]
\[
+ (\otimes_{\sigma(i-1,j-1)} \otimes \mathrm{id}_V)(\beta_{i-1,j-1} \otimes \beta_{1,i-1,1})(\mathrm{id}_V^\otimes i-1 \otimes \beta_{1,i-1,1})(\beta_{1,i-1,1,j} \otimes \mathrm{id}_V)
\]
\[
+ (\otimes_{\sigma(i-1,j-1)} \otimes m)(\beta_{i-1,j-1} \otimes \beta_{1,i-1,1} \otimes \mathrm{id}_V)
\]
\[
= (\otimes_{\sigma(i-1,j-1)} \otimes \mathrm{id}_V)(\beta_{i-1,j-1} \otimes \beta_{1,j-1} \otimes \mathrm{id}_V)
\]
\[
= \otimes_{\sigma(i,j)}.
\]

\[\square\]

4. Basis coming from Lyndon words

In this section, we use quantum quasi-shuffle products and Lyndon words to present a new linear basis of the tensor space \(T(V)\) for a special kind of YB algebras \((V, m, \sigma)\). Let \((V, m, \sigma)\) be a finite dimensional YB algebra with linear basis \((e_1, \ldots, e_N)\) and braiding of the following form: \(\sigma(e_i \otimes e_j) = q_{ij} e_j \otimes e_i\), where \(q_{ij}\)’s are powers of a fixed nonzero scalar \(q \in K\) and \(q\) is not a root of unity. For example, \((\Lambda_\alpha(V), \wedge, \mathcal{F})\) is certainly such a YB algebra.

\(T^+(V) = T(V)/K\) always has a \(K\)-linear basis
\[
(I) = \{e_{i_1} \otimes \cdots \otimes e_{i_m}| m > 0, 1 \leq i_1, \ldots, i_m \leq N\}.
\]

The length of \(e_{i_1} \otimes \cdots \otimes e_{i_m}\) is \(m\) and is denoted by \(|e_{i_1} \otimes \cdots \otimes e_{i_m}| = m\).

Given a total ordering on \(e_i\)’s, for example, say \(e_1 < e_2 < \cdots < e_N\), then there is a total ordering on \((I)\) provided by the lexicographic ordering, with the convention that \(a \leq a \otimes b\) for \(a, b \in T^+(V)\). Lyndon words of \(T^+(V)\) are defined as follows.

Definition 16. An element \(p\) in \((I)\) is called a Lyndon word if, for any splitting \(p = a \otimes b\), with \(a, b \in (I)\), one has \(p < b\).

Every \(p\) in \((I)\) has a unique factorization with respect to Lyndon words. More precisely, \(p\) can be written in a unique way as a tensor product of minimal number of Lyndon words (see [9]). We call this the standard factorization of \(p\). In fact,
Proposition 17. The set

\[(\text{II}) = \{ l_1 \otimes \cdots \otimes l_r \mid l_i \in L, l_1 \geq \cdots \geq l_r \}\]

forms a \(K\)-linear basis of \(T^+(V)\).

Proof. First we note that \((T^c(V), \ltimes)\) is a filtered algebra with

\[T^c(V)^{[n]} := \bigoplus_{i=0}^{n} V^{\otimes i}, \quad T^c(V)^{[n]} \subseteq T^c(V)^{[n+1]},\]

and

\[T^c(V)^{[m]} \ltimes T^c(V)^{[n]} \subseteq T^c(V)^{[m+n]}\]

And the quantum shuffle algebra \(T_\sigma(V)\) is a graded algebra with

\[T_\sigma^n(V) = V^{\otimes n}, \quad T_\sigma(V) = \bigoplus_{n=0}^{\infty} T_\sigma^n(V),\]

and

\[T_\sigma^m(V) \ltimes T_\sigma^n(V) \subseteq T_\sigma^{m+n}(V)\]

Moreover \(T_\sigma(V)\) is the associated graded algebra of \((T^c(V), \ltimes)\) with respect to the above filtration, since for any \(l_1 \otimes \cdots \otimes l_r \in V^{\otimes n}\),

\[l_1 \ltimes \cdots \ltimes l_r = l_1 \ltimes \cdots \ltimes l_r \mod T^c(V)^{[n-1]}\]

Hence (II) is a linear basis of \(T^+(V)\) if and only if

\[(\text{III}) = \{ l_1 \ltimes \cdots \ltimes l_r \mid l_i \in L, l_1 \geq \cdots \geq l_r \}\]

is a basis of \(T^+(V)\). For \(l_i \in L\) with \(l_1 \geq \cdots \geq l_r\), we have

\[l_1 \ltimes \cdots \ltimes l_r = al_1 \otimes \cdots \otimes l_r + \sum_{a_w \in K, w \in (I), w < l_1 \otimes \cdots \otimes l_r} a_w w,\]

where \(a\) is a scalar. After collecting the same \(l_i\)'s, we rewrite \(l_1 \otimes \cdots \otimes l_r = p_1^{\otimes n_1} \cdots p_s^{\otimes n_s}\), where \(p_i\)'s \(i \in L \subset (I)\) and \(p_1 > \cdots > p_s\). Set \(p_i = e_{j_1} \otimes \cdots \otimes e_{j_{m_i}}\) and \(Q_i = \prod_{k, l \in \{j_1, \cdots, j_{m_i}\}} q_{k,l}\) for \(1 \leq i \leq s\). Then \(a = (n_1)_{Q_1}! \cdots (n_s)_{Q_s}!\), where \((n)_\nu = \frac{n!}{(n-\nu)!}\) and \((n)_\nu! = (n)_\nu(n-1)_\nu \cdots (1)_\nu\). By the requirements for \(q_{ij}\), we have that \(a\) never vanishes. Hence the transformation matrix from (I)' to (III) is triangular with nonzero entries on its main diagonal, which implies that (III) is a basis of \(T^+(V)\).
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