Every graph contains a linearly sized induced subgraph with all degrees odd

Asaf Ferber*  Michael Krivelevich†

May 26, 2022

Abstract

We prove that every graph \( G \) on \( n \) vertices with no isolated vertices contains an induced subgraph of size at least \( n/10000 \) with all degrees odd. This solves an old and well-known conjecture in graph theory.

1 Introduction

We start with recalling a classical theorem of Gallai (see [2], Problem 5.17 for a proof):

**Theorem 1** (Gallai’s Theorem). Let \( G \) be any graph.

1. There exists a partition \( V(G) = V_1 \cup V_2 \) such that both graphs \( G[V_1] \) and \( G[V_2] \) have all their degrees even.

2. There exists a partition \( V(G) = V_o \cup V_e \) such that the graph \( G[V_e] \) has all its degrees even, and the graph \( G[V_o] \) has all its degrees odd.

It follows immediately from 1. that every graph \( G \) has an induced subgraph of size at least \( |V(G)|/2 \) with all its degrees even. This is easily seen to be tight by taking \( G \) to be a path.

It is natural to ask whether we can derive analogous results for induced subgraphs with all degrees odd. Some caution is required here — an isolated vertex can never be a part of a subgraph with all degrees odd. Thus we restrict our attention to graphs of positive minimum degree.

Let us introduce a relevant notation: given a graph \( G = (V, E) \), we define

\[
f_o(G) = \max\{|V_0| : G[V_0] \text{ has all degrees odd.}\},
\]

and set

\[
f_o(n) = \min\{f_o(G) \mid G \text{ is a graph on } n \text{ vertices with } \delta(G) \geq 1\}.
\]

The following is a very well known conjecture, aptly described by Caro already more than a quarter century ago [1] as “part of the graph theory folklore”:

---

*Department of Mathematics, University of California, Irvine. Email: asaff@uci.edu. Research supported in part by NSF grants DMS-1954395 and DMS-1953799.

†School of Mathematical Sciences, Tel Aviv University, Tel Aviv 6997801, Israel. Email: krivelev@tauex.tau.ac.il. Research supported in part by USA–Israel BSF grant 2018267 and by ISF grant 1261/17.
Conjecture 2. There exists a constant $c > 0$ such that for every $n \in \mathbb{N}$ we have $f_o(n) \geq cn$.

Caro himself proved [1] that $f_o(n) = \Omega(\sqrt{n})$, resolving a question of Alon who asked whether $f_o(n)$ is polynomial in $n$. The current best bound, due to Scott [3], is $f_o(n) = \Omega(n/ \log n)$. There have been numerous variants and partial results about the conjecture, we will not cover them here.

Our main result establishes Conjecture 2 with $c = 0.0001$.

Theorem 3. Every graph $G$ on $n$ vertices with $\delta(G) \geq 1$ satisfies: $f_o(G) \geq cn$ for $c = \frac{1}{10000}$.

With some effort/more accurate calculations the constant can be improved but probably to a value which is still quite far from the optimal one; we decided not to invest a substantial effort in its optimization and just chose some constants that work.

A relevant parameter was studied by Scott [4]: given a graph $G$ with no isolated vertices, let $t(G)$ be the minimal $k$ for which there exists a vertex cover of $G$ with $k$ sets, each spanning an induced graph with all degrees odd. Letting $t(n) = \min \{ t(G) \mid G \text{ is a graph on } n \text{ vertices with } \delta(G) \geq 1 \}$, Scott proved (Theorem 4 in [4]) that $\Omega(\log n) = t(n) = O(\log^2 n)$.

As indicated by Scott already, showing that $f_o(n)$ is linear in $n$ proves the following:

Corollary 4. $t(n) = \Theta(\log n)$.

For completeness, we outline its proof here.

Proof. Let $G$ be a graph on $n$ vertices with $\delta(G) \geq 1$. By a repeated use of Theorem 3, we can find disjoint sets $V_1, \ldots, V_t$ such that:

1. $V_i \subseteq V(G) \setminus \left( \bigcup_{j=1}^{i-1} V_j \right)$, and
2. all the degrees in $G[V_i]$ are odd, and
3. letting $n_i$ be the number of non-isolated vertices in $G \left[ V(G) \setminus \left( \bigcup_{j=1}^{i-1} V_j \right) \right]$, we have that $|V_i| \geq n_i/10000$.

We continue the above process as long as $n_i > 0$. Clearly, the process terminates after $t = O(\log n)$ steps. Moreover, letting $U = V(G) \setminus \left( \bigcup_{i=1}^t V_i \right)$, we have that $U$ is an independent set in $G$. Finally, as shown in the proof of Theorem 4 in [4], every independent set in such $G$ can be covered by $O(\log n)$ odd graphs. This proves that $t(n) = O(\log n)$.

To show a lower bound, we can use the following example due to Scott [4]: assume $n$ is of the form $n = s + \binom{s}{2}$. Let the vertex set of $G$ be composed of two disjoint sets: $A$ of size $s$ associated with $[s]$, and $B$ of size $\binom{s}{2}$ associated with $\binom{[s]}{2}$. The graph $G$ is bipartite with the edges defined as follows: a pair $(i, j) \in B$ is connected to both $i, j \in A$. Observe that if $U \subseteq V(G)$ spans a subgraph of $G$ with all degrees odd and containing $(i, j) \in B$, then $U$ contains exactly one of $i, j \in A$. Hence if $\mathcal{U} = \{U_1, \ldots, U_t\}$ forms a cover of $V(G)$ with subsets spanning odd subgraphs, then $\mathcal{U}$ separates the set $A$, and the minimum size of such a separating family is easily shown to be asymptotic to $\log_2 s = \Omega(\log_2 n)$.
2 Auxiliary results

The following lemma appears as Theorem 2.1 in [1]. For the convenience of the reader we provide its simple proof.

**Lemma 2.1.** For every graph $G$ we have that $f_o(G) \geq \frac{\Delta(G)}{2}$.

*Proof.* Let $v \in V(G)$ be a vertex with $d_G(v) = \Delta(G)$, and let $U \subseteq N_G(v)$ be an odd subset of size $|U| \geq \Delta(G) - 1$. Apply Gallai’s Theorem to $G[U]$ to obtain a partition $U = V_e \cup V_o$, and observe that $V_o$ must be of an even size (so in particular, $|V_e|$ is odd). If $|V_o| \geq \Delta(G)/2$, then we are done. Otherwise, define $V^* = \{v\} \cup V_e$, and observe that $G[V^*]$ has all its degrees odd and is of size at least $\Delta(G)/2$ as required. \hfill $\Box$

The next lemma appears as Theorem 1 in [3], and again, for the sake of completeness, we give its proof here.

**Lemma 2.2.** For every graph $G$ with $\delta(G) \geq 1$ we have that $f_o(G) \geq \frac{\alpha(G)}{2}$.

*Proof.* Let $I \subseteq V(G)$ be a largest independent set in $G$. Since $\delta(G) \geq 1$, every $u \in I$ has at least one neighbor in $V(G) \setminus I$.

Let $D \subseteq V(G) \setminus I$ be a smallest subset dominating all vertices in $I$. Observe that by the minimality of $D$ for every $w \in D$ there exists some $u_w \in I$ such that $N_G(u_w) \cap D = \{w\}$; let $I_D := \{u_w \mid w \in D\}$.

Let $D' \subseteq D$ be a subset of $D$ chosen uniformly at random, and let $I_0 \subseteq I \setminus I_D$ be a subset consisting of all elements $u \in I \setminus I_D$ that have an odd degree into $D'$.

Let $I_1 = \{u_w \in I_D \mid w \in D' \text{ and } w \text{ has even degree in } D' \cup I_0\}$, and observe that $G[I_0 \cup I_1 \cup D']$ is an induced subgraph of $G$ with all its degrees odd.

Finally, since $\Pr[u \in I_0] = \frac{1}{2}$, by linearity of expectation we have that

$$
\mathbb{E}[|I_0 \cup I_1 \cup D'|] = \mathbb{E}[|I_0|] + \mathbb{E}[|I_1|] + \mathbb{E}[|D'|] \geq \frac{|I| - |D|}{2} + \frac{|D|}{2} = \frac{\alpha(G)}{2}.
$$

Hence there exists a set $D'$ for which

$$
|I_0| + |I_1| + |D'| \geq \frac{\alpha(G)}{2},
$$

as desired. \hfill $\Box$

Next we argue that if $G$ contains a semi-induced matching with “nice” expansion properties, then it also has a large induced subgraph with all degrees odd.

**Lemma 2.3.** Let $G$ be a graph and let $M$ be a matching in $G$ with parts $U$ and $W$, where every vertex $w \in W$ has only one neighbor in $G$ between the vertices covered by $M$. Assume that $|N_G(U) \setminus (W \cup N_G(W))| \geq k$. Then $f_o(G) \geq \frac{k}{4}$.

*Proof.* Let $X = N_G(U) \setminus (W \cup N_G(W))$ and recall that $|X| \geq k$. Let $U_0$ be a random subset of $U$ chosen according to the uniform distribution, and let

$$
X_0 = \{x \in X : d_G(x, U_0) \text{ is odd}\}.
$$
Since $\mathbb{E}[|X_0|] = |X|/2$, it follows that there exists an outcome $U_0 \subseteq U$ for which $|X_0| \geq |X|/2 \geq k/2$. Fix such $U_0$.

Next, apply Gallai’s theorem to $G[X_0]$ to find a subset $X_1 \subseteq X_0$ with $|X_1| \geq |X_0|/2 \geq k/4$ and all degrees in $G[X_1]$ even. Finally, for every $u \in U_0$ with $d_G(u, X_1 \cup U_0)$ even, add an edge of $M$ containing $u$. Clearly, the obtained graph $G_1$ has size at least $|X_1| \geq |X|/4 \geq k/4$, and all its degrees are odd. This completes the proof. \hfill $\square$

The following simple lemma will be used several times below.

**Lemma 2.4.** Let $G$ be a bipartite graph with parts $A, B$ such that $d(b) > 0$ for every $b \in B$. Assume that $|A| \leq \alpha |B|$ for some $0 < \alpha \leq 1$. Then there is an edge $ab \in E(G)$ with $d(a) \geq \frac{d(b)}{\alpha}$.

*Proof.* We have:

$$\sum_{ab \in E(G)} \left(\frac{1}{d(b)} - \frac{1}{d(a)}\right) = \sum_{b \in B} d(b) \cdot \frac{1}{d(b)} - \sum_{a \in A} \sum_{d(a) > 0} d(a) \cdot \frac{1}{d(a)} \geq |B| - |A| \geq (1 - \alpha)|B| .$$

Hence there is $b \in B$ with

$$\sum_{a \in N_G(b)} \left(\frac{1}{d(b)} - \frac{1}{d(a)}\right) \geq 1 - \alpha .$$

It follows that there is a neighbor $a$ of $b$ for which $\frac{1}{d(b)} - \frac{1}{d(a)} \geq (1 - \alpha)\frac{1}{d(b)}$, implying $d(a) \geq \frac{d(b)}{\alpha}$ as desired. \hfill $\square$

For a graph $G = (V, E)$ and $\beta > 0$, define $L = L(G; \beta) = \{v \in V : \exists u \in V, uv \in E(G), |N(u) \setminus N(v)| \geq \beta|N(u) \cup N(v)|\}$.

We say that for $v \in L$, an edge $uv$ as above witnesses $v \in L$.

Set

$$\beta = \frac{1}{20}, \quad \gamma = \frac{1}{14}, \quad \epsilon = \frac{1}{10}. $$

The next lemma is a key part in the proof of our main theorem. Roughly speaking, the lemma asserts that if $L(G; \beta)$ is small, then $G$ contains a large induced subgraph $G[U]$ with a vertex of large degree in every connected component, allowing to find inside $U$ a large odd subgraph, using previously presented tools. We did not really pursue the goal of optimizing the constants in its statement.

**Lemma 2.5.** Let $G = (V, E)$ be a graph on $|V| = n$ vertices with $\delta(G) > 0$ and $|L(G; \beta)| \leq \gamma n$. Then $f_0(G) \geq n/61$.

*Proof.* Define

$$V_1 = \{v \in V \setminus L : d(v, L) \geq \epsilon d(v)\}, \quad V_2 = V \setminus (V_1 \cup L).$$
Suppose first that $|V_1| \geq 12|L|$. Observe that for all $v \in V_1$ we have that $d(v, L) \geq \epsilon d(v)$ and this quantity is positive by the assumption $\delta(G) > 0$. By Lemma 2.4 there exists $uv \in E(G)$ with $v \in V_1$ and $u \in L$ such that $d(u, V_1) \geq 12d(u, L) \geq 12\epsilon d(v)$ (in particular, we have $d(v) \leq \frac{5}{6}d(u)$). Therefore we have that

$$|N(u) \setminus N(v)| \geq d(u) - d(v) = (d(u) - \frac{12}{11}d(v)) + (d(v) - \frac{10}{11}d(v)) \geq \frac{1}{11}(d(u) + d(v)) > \beta|N(u) \cup N(v)|,$$

so in particular $v$ should also be in $L$ with $uv$ witnessing it — a contradiction. We conclude that

$$|V_1| < 12|L| \leq 12\gamma n,$$

and therefore $|V_2| \geq (1 - \gamma - 12\gamma)n = \frac{n}{14}$.

Let $v \notin L$. Take an edge $uv \in E(G)$. Then

$$\max\{1, d(u) - d(v)\} \leq |N(u) \setminus N(v)| \leq \beta|N(u) \cup N(v)| \leq \beta(d(u) + d(v)),$$

yielding:

$$d(u) \leq \frac{1 + \beta}{1 - \beta}d(v),$$

and

$$\beta \left(\frac{1 + \beta}{1 - \beta} + 1\right)d(v) \geq 1.$$

This shows that every vertex $v \in V \setminus L$ has degree $d(v) \geq \lceil \frac{1 - \beta}{2\beta} \rceil = 10$.

Let now $uv \in E(G)$ with $u, v \notin L$. Then

$$|N(u) \setminus N(v)|, |N(v) \setminus N(u)| \leq \beta|N(u) \cup N(v)|,$$

and hence

$$|N(u) \cap N(v)| \geq (1 - 2\beta)|N(u) \cup N(v)|. \quad (1)$$

Since

$$|N(u) \cap N(v)| \leq \min\{d(u), d(v)\} \text{ and } |N(u) \cup N(v)| \geq \max\{d(u), d(v)\},$$

it follows that

$$(1 - 2\beta)d(u) \leq d(v) \leq \frac{d(u)}{1 - 2\beta} < (1 + 3\beta)d(u). \quad (2)$$

Now, for all $v \notin L$ define $R(v) = (\{v\} \cup N(v)) \setminus L$. Notice that as $d(v) \geq 10$ we have $|R(v)| \geq (1 - \epsilon)d(v) + 1 \geq 10$ for $v \in V_2$. Suppose that $R(u) \cap R(v) \neq \emptyset$ for some $u \neq v$ where $v \in V_2$ (note that it might be that $u \in V_1$). Then for $w \in R(v) \cap R(u)$, by (1) we have

$$|N(u) \cap N(w)| \geq (1 - 2\beta)|N(u) \cup N(w)| \text{ and } |N(v) \cap N(w)| \geq (1 - 2\beta)|N(v) \cup N(w)|,$$

which implies, by the identity $|A \Delta B| = |A \cup B| - |A \cap B|$, that

$$|N(u) \Delta N(w)| \leq \frac{2\beta}{1 - 2\beta}|N(u) \cap N(w)| < 3\beta|N(u) \cap N(w)|$$

and

$$|N(v) \Delta N(w)| \leq \frac{2\beta}{1 - 2\beta}|N(v) \cap N(w)| < 3\beta|N(v) \cap N(w)|.$$
Therefore, we have
\[
|N(u) \cap N(v)| \geq |N(u) \cap N(v) \cap N(w)| \\
\geq |N(u) \cup N(v)| - |N(u) \Delta N(w)| - |N(v) \Delta N(w)| \\
> |N(u) \cup N(v)| - \frac{4\beta}{1 - 2\beta} \max\{d(u), d(v)\} \\
\geq \left(1 - \frac{4\beta}{1 - 2\beta}\right) |N(u) \cup N(v)| \geq (1 - 6\beta)|N(u) \cup N(v)|. \tag{3}
\]

Since \(v \in V_2\) we conclude that
\[
|R(u) \cap R(v)| \geq |N(u) \cap N(v)| - \epsilon d(v) \\
> \left(1 - \frac{4\beta}{1 - 2\beta} - \epsilon\right) |N(u) \cup N(v)| \\
\geq \left(1 - \frac{4\beta}{1 - 2\beta} - \epsilon\right) (|N(u) \cup R(v)| - 1) \\
\geq \frac{9}{10} \left(1 - \frac{4\beta}{1 - 2\beta} - \epsilon\right) |N(u) \cup R(v)| \\
\geq (1 - 8\beta) |N(u) \cup R(v)|,
\]
where the second to last inequality follows since \(|N(u)| \geq 10\).

Next, let \(R_1, \ldots, R_k\) be a maximal by inclusion collection of non-intersecting sets \(R(v_i), v_i \in V_2\). Due to maximality, every \(v \in V_2\) has its set \(R(v)\) intersecting with at least one of the \(R_i\)’s; moreover, the above argument shows that it can intersect only one such set. Define now
\[
U_i = \{v \notin L : R(v) \cap R_i \neq \emptyset\}.
\]
Trivially we have \(R_i \subseteq U_i\). Also, \(V_2 \subseteq \bigcup_{i=1}^{k} U_i\) due to the maximality of the family \(R_1, \ldots, R_k\).

We wish to show that all \(U_i\) are disjoint and that there are no edges in between different \(U_i\)’s. (This will add to the above stated fact that the family of \(U_i\)’s forms a cover of \(V_2\).)

To prove the latter claim, suppose that there exists an edge \(w_1w_2 \in E(G)\) for some \(w_1 \in U_i, w_2 \in U_j, 1 \leq i \neq j \leq k\). We will obtain a contradiction by showing that \(R_i \cap R_j \neq \emptyset\). Since both \(w_1, w_2 \notin L\), by (1) and (2) we conclude that
\[
|N(w_1) \cap N(w_2)| \geq (1 - 2\beta)|N(w_1) \cup N(w_2)| \quad \text{and} \quad |N(w_1)| \in (1 \pm 3\beta)|N(w_2)|.
\]
Moreover, by (3) we have
\[
|N(w_1) \cap N(v_i)| > (1 - 6\beta)|N(w_1) \cup N(v_i)|, \quad \text{and} \quad |N(w_2) \cap N(v_j)| \geq (1 - 6\beta)|N(w_2) \cup N(v_j)|.
\]
Since \(v_i, v_j \in V_2\), the above inequalities imply that
\[
|N(w_1) \cap R_i| > (1 - 6\beta - \epsilon)|N(w_1) \cup R_i|, \quad \text{and} \quad |N(w_2) \cap R_j| > (1 - 6\beta - \epsilon)|N(w_2) \cup R_j|.
\]
It follows that
\[
|N(w_1) \cap R_i| > (1 - 6\beta - \epsilon)|N(w_1)|
\]
and
\[
|N(w_2) \cap R_j| > (1 - 6\beta - \epsilon)|N(w_2)|.
\]
and recalling that
\[ |N(w_1) \cap N(w_2)| \geq (1 - 2\beta)|N(w_1) \cup N(w_2)|, \]
we conclude that \( R_i \cap R_j \neq \emptyset \) — a contradiction. In a similar way we can show that \( U_i \cap U_j = \emptyset \).

Next, suppose that \( |U_i| \geq (1 + 19\beta)|R_i| \). Then by looking at the auxiliary bipartite graph between \( R_i \) and \( U_i \) (\( v \in R_i, u \in U_i \) are connected by an edge if \( uv \in E(G) \)) and by applying Lemma 2.4 to this graph we derive that there are \( v \in R_i, u \in U_i \) with \( d(v) \geq (1 + 19\beta)d(u, R_i) \). Since \( uv \in E(G) \) and both \( u, v \not\in L \), it follows that
\[ d(v) \leq (1 + 3\beta)d(u). \]
Moreover, since \( u \in U_i \) we have:
\[ d(u, R_i) \geq (1 - 8\beta)d(u). \]

All in all, since \( d(v) \geq (1 + 19\beta)d(u, R_i) \) we conclude that
\[ (1 + 3\beta)d(u) > d(v) \geq (1 + 19\beta)d(v, R_i) > (1 + 19\beta)(1 - 8\beta)d(u) > (1 + 3\beta)d(u), \]
a contradiction.

Therefore, we can assume that \( |U_i| \leq (1 + 19\beta)|R_i| \) for all \( 1 \leq i \leq k \). Looking at the induced subgraph \( G[U_i] \), we note that it has vertex \( v_i \) of degree \( |R_i| - 1 \geq \frac{9|R_i|}{10} \geq \frac{9}{10(1 + 19\beta)}|U_i| \). By applying Lemma 2.1 to \( G[U_i] \) we find an induced odd subgraph \( O_i \) of \( G[U_i] \) of size at least \( \frac{9}{20(1 + 19\beta)}|U_i| = \frac{9|V_i|}{39} \).

Finally, since all \( U_i \)'s are disjoint, there are no edges between any two such \( U_i \)'s and since \( V_2 \subseteq \bigcup U_i \), we conclude that \( O = \bigcup_{i=1}^{k} O_i \) is an induced odd subgraph of size at least \( \frac{9|V_i|}{39} > n/61 \). This completes the proof. \( \square \)

## 3 Proof of Theorem 3

The main plan is as follows. We will grow edge by edge a matching \( M \) with sides \( U, W \) so that every \( v \in W \) has exactly one neighbor between the vertices covered by \( M \) (which is of course its mate \( u \) in the matching). Moreover, the set \( U \) has “many” neighbors outside of \( M \) not connected to \( W \). If the set of such neighbors is substantially large, then we will be able to apply Lemma 2.3 to get a large induced subgraph with all degrees odd. Otherwise we will show that either there exists a large subset of vertices \( V' \) such that \( \delta(G[V']) \geq 1 \) with small \( L(G[V']; 1/20) \) (and then we are done by Lemma 2.5), or that we can extend the matching while enlarging substantially the set of neighbors outside \( M \) not connected to \( W \). The details are given below.

We start with \( M_0 = \emptyset \), and given \( M_i, i \geq 0 \), we define
\[
X_i = N(U_i) \setminus (W_i \cup N(W_i)), \\
V_i = V \setminus N(U_i \cup W_i).
\]

In particular, we initially have \( X_0 = \emptyset \) and \( V_0 = V \). We will run our process until the first time we have \( |V_i| < n/2 \) (in particular, we may assume throughout the process that \( |V_i| \geq n/2 \)). Now, fix \( \beta = 1/20 \) and \( \gamma = 1/14 \) (same parameters as set before Lemma 2.5). Our goal is to show that \( f_0(G) \geq \frac{n}{T} \), where \( T = 10000 \). We will maintain \( |X_i| \geq \frac{|V_i|}{40} \). If at some point we reach \( |X_i| \geq \frac{4n}{T} \) then we are done by Lemma 2.3. Hence we assume \( |X_i| \leq \frac{3n}{2T} = \frac{n}{2000} \). Moreover, if \( G[V_i] \) has at least \( 2n/T \) isolated vertices, then since this set induces an independent set in \( G \), by Lemma 2.2 we are done as well.

Therefore, letting \( V_i' \subseteq V_i \) be the set of all non-isolated vertices in \( G[V_i] \), since \( |V_i| \geq n/2 \) we obtain
that $|V'_i| \geq (1 - 4/T)|V_i| \geq |V_i|/2$. We can further assume $|L(G[V'_i]; \beta)| \geq \gamma |V'_i| \geq \gamma n/4$, as otherwise by Lemma 2.5 we obtain an odd subgraph of size at least $|V'_i|/61 \geq n/244$. Our goal now is to show that under these assumptions we can add an edge to $M_i$ while maintaining $|X_{i+1}| \geq \frac{|V \setminus V_{i+1}|}{40}$.

Consider first the case where every $v \in L := L(G[V'_i]; \beta)$ satisfies $d(v, X_i) \geq d(v, V_i)/40$. By Lemma 2.4 applied to the bipartite graph between $X_i$ and $L$, using the fact that

$$|X_i| \leq \frac{4n}{T} = \frac{n}{2500} \leq \frac{|L|}{44},$$

we derive that there is an edge $xv$ with $x \in X_i$ and $v \in L$ and $d(x, L) \geq 44d(v, X_i) \geq 1.1d(v, V_i) > 0$. Then we can define $M_{i+1}$ by adding $xv$ to $M_i$ and setting $U_{i+1} := U_i \cup \{x\}$ and $W_{i+1} := W_i \cup \{v\}$. By doing so we obtain that

$$|X_{i+1}| = |N(U_{i+1}) \setminus (W_{i+1} \cup N(W_{i+1})|$$

$$\geq |N(U_i) \setminus (W_i \cup N(W_i))| + |N(x, V_i)| - |N(v, X_i)| - |N(v, V_i)|$$

$$= |X_i| + d(x, V_i) - d(v, X_i) - d(v, V_i)$$

$$\geq |X_i| + d(x, V_i) \left(1 - \frac{1}{44} - \frac{10}{11}\right)$$

$$> |X_i| + \frac{3d(x, V_i)}{44}.$$

Moreover, since we clearly have that

$$|V_{i+1}| \geq |V_i| - d(x, V_i) - d(v, V_i) \geq |V_i| - \frac{21d(x, V_i)}{11},$$

it follows that at least $\frac{3/44}{21/11} > \frac{1}{30}$ proportion of the vertices deleted from $V_i$ go to $X_{i+1}$.

In the complementary case there exists a vertex $v \in L$ with $d(v, X_i) \leq d(v, V_i)/40$. Let $uv$ be an edge in $G[V'_i]$ witnessing $v \in L$ (that is, $|N(u, V_i) \setminus N(v, V_i)| \geq \beta |N(u, V_i) \setminus N(v, V_i)|$). Then we can define $M_{i+1}$ by adding $uv$ to $M_i$, and set $U_{i+1} := U_i \cup \{u\}$ and $W_{i+1} := W_i \cup \{v\}$. In this case we have:

$$|X_{i+1}| = |N(U_{i+1}) \setminus (W_{i+1} \cup N(W_{i+1})|$$

$$\geq |N(U_i) \setminus (W_i \cup N(W_i))| + |N(u, V_i) \setminus N(v, V_i)| - |N(v, X_i)|$$

$$= |X_i| + |N(u, V_i) \setminus N(v, V_i)| - |N(v, X_i)|$$

$$\geq |X_i| + \beta |N(u, V_i) \setminus N(v, V_i)| - |N(v, X_i)|$$

$$\geq |X_i| + (\beta - \frac{1}{40})|N(u, V_i) \setminus N(v, V_i)|.$$

Moreover, since we have $|V_{i+1}| \geq |V_i| - |N(u, V_i) \setminus N(v, V_i)|$, at least $\beta - \frac{1}{40} = \frac{1}{30}$ proportion of the vertices deleted from $V_i$ go to $X_{i+1}$.

All in all, in each step, either we find an odd subgraph of size at least $\frac{n}{2}$ (in case that we have “many” isolated vertices, or that $|X_i| \geq \frac{4n}{T}$, or that $L(G[V'_i]; \beta)$ is “large”), or we can keep $X_i$ of size at least $\frac{|V \setminus V_i|}{40}$. In particular, if the latter case holds until $|V_i| < n/2$, we obtain that $|X_i| \geq \frac{n}{80}$ and we are done by Lemma 2.3. This completes the proof.

**Acknowledgement.** We would like to thank Alex Scott for his remarks, and for pointing out a serious flaw in the previous version.
References

[1] Y. Caro, On induced subgraphs with odd degrees, Discrete Mathematics 132 (1994), 23–28.

[2] L. Lovász, Combinatorial Problems and Exercises, 2nd edition, AMS Chelsea Publishing, 1993.

[3] A. D. Scott, Large induced subgraphs with all degrees odd, Combinatorics, Probability and Computing 1 (1992), 335–349.

[4] A. D. Scott, On induced subgraphs will all degrees odd, Graphs and Combinatorics 17 (2001), 539–553.