Preparation of quantum correlations assisted by a steering Maxwell demon

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A Maxwell demon can reduce the entropy of a quantum system by performing measurements on its environment. The nonsignaling theorem prevents the demon from affecting the average state of the system. We study the preparations of quantum correlations from a system qubit and an auxiliary qubit, assisted by a demon who obtains information of the system qubit from measurements on its environment. The demon can affect the postmeasured states of system by choosing different measurements, which establishes the relationships between quantum steering and other correlations in the thermodynamic framework. We present the optimal protocols for creating mutual information, entanglement and Bell-nonlocality. These maximal correlations are found to relate exactly to the steerable boundary of the system-environment state with maximally mixed marginals. We also present upper bounds of the prepared correlations by utilizing classical environment-system correlation, which can be regarded as steering-type inequalities bounding the correlations created with the aid of classical demons.

I. INTRODUCTION

The connection between thermodynamics and information provides a different angle of view to understand the physical world. In the history of this topic, the Maxwell’s demon, first introduced by Maxwell in 1871 [1], has played an important role. The Maxwell demon is a creature who can reduce the entropy of a system, by observing its microstates, without performing any work on it. Szilárd [2] presented a one-molecule heat engine assisted by a Maxwell demon measuring the (binary) position of the molecule. His model showed an explicit connection between information and physics that, one can extract work \( W = kT \log 2 \) from the one-molecule system at a temperature \( T \) by using 1 bit information acquired by the demon.

In the field of quantum thermodynamics [3, 4], many quantum versions of the Maxwell demon and Szilárd engine have been presented, to investigate the role of quantumness in thermodynamics and the interplay between quantum information and thermodynamics [5–15]. The definitions of these models rely on the division between the quantum and classical worlds. For instance, in Zurek’s division [6], a quantum demon is the one who can perform global measurements on composite systems, while a classical demon is local. On the other hand, quantum correlations in thermodynamics have gotten a lot of attention, as they are the most profound quantum features and deeply connected to quantum information. The thermodynamic cost and fundamental limitations for preparation of quantum correlations were studied under different conditions [16, 17]. The correlations in turn can be used to enhance the extraction of work [6, 10, 11, 18–24]. The Maxwell demons and Szilárd engines often played key roles in these works, such as the studies of work deficit [18], discord [6, 22–24] and steering heat engines [10, 11].

Measurements on a quantum system would in general disturb its state, and thus affect its energy. This actually provides a different paradigm in quantum thermodynamics in which measurement apparatuses are used to fuel engines [8, 25]. That is, Maxwell demons directly measuring quantum systems lack a basic feature of their classical counterparts: acquiring information but without affecting the state. The difficulty was overcome in the version of quantum Szilárd engine presented by Beyer et al. [10], where a demon obtains the information of a system from measurements on its environment. The average state of the system was protected by the nonsignaling theorem. Their approach connects the thermodynamic task of work extraction with the quantum steering, which is a kind of quantum correlation lying between Bell nonlocality and entanglement [26]. Here, the term steering, introduced by Schrödinger [27], means that the demon can project the system into different states by choosing it’s measurements on the environment. The demon’s ability of steering can be convincingly demonstrated, only when the postmeasured states of the system cannot be described by a local-hidden-state (LHS) model. In this case, the state of system and environment is said to have quantum steering from the environment to the system [26], and the demon is termed truly quantum by Beyer et al. [10].

In this work we investigate the process for creating quantum correlations from a system qubit and an auxiliary qubit assisted by a Maxwell demon measuring its environment. This is based on the consideration that the information acquired by demon to extract work in [10] can certainly be used for creating correlations. Our processes connect the quantum steering with other correlations in the thermodynamic framework. For an arbitrary set of observables on the environment, we show that the maximums of quantum mutual information, entanglement and Bell-nonlocality allowed between the system and ancilla are all monotone increasing functions.

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of the average length of Bloch vectors in the postmeasured states for the system qubit. When the dimension of the Hilbert space of the environment can be measured by the demon is 2, these maximal correlations are related exactly to the steerable boundary of the system-environment state with maximally mixed marginals. We also present upper bounds of the prepared correlations for unsteering demons. These can be regarded as steering-type inequalities bounding the correlations created with the aid of classical environment-system correlation.

In the next section, we study the optimal protocols of quantum correlations preparation, with brief introductions to the correlation measures. In Sec. III, we deal with the case that the demon performs measurements on a two-dimensional Hilbert space of the environment, to show the advantage of a quantum demon. Finally, a summary of our results and some outlooks are given in Sec. IV.

II. CORRELATIONS PREPARATION

We begin by introducing the procedure for quantum correlations preparation assisted by the demon, which is shown in Fig. 1. Suppose that $S$ is a system qubit. Ella is the Maxwell demon who can perform measurements on the environment $E$ of $S$. Here, $E$ should be understood as a part (a subsystem or a subspace of the Hilbert space) of the whole environment of $S$, which Ella is able to measure. Bob is the operator manipulating the system qubit and an auxiliary qubit $A$. The task of Bob is to create quantum correlations from his two qubits, in an uncorrelated initial state, by applying a global unitary on them. In this work, we study three types of correlations between $S$ and $A$: Total correlation measured by quantum mutual information $I$ [28], entanglement measured by concurrence $C$ [29] and negativity $N$ [30], and Bell-nonlocality measured by the maximal violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality $B$ [31].

Without loss of generality, we set the initial state of $S$ to be

$$\tau_s = \frac{1}{2}(\mathbb{I} + \eta \sigma_z),$$

with $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ being the third Pauli operator. The whole state of $E$ and $S$ is $\rho_{se}$, with $\rho_s = \text{Tr}_e\rho_{se} = \tau_s$. Suppose $\{M_n, |n = 1, 2, \ldots\}$ is the set of observables Ella can measure on $E$, and $M_{n|k}$ with $k = 0, 1, \ldots$ denote the positive operator-valued measurement elements of $M_n$.

In each round, Bob generates a value of $n$ with a probability $q_n$ and sends it to Ella. Then, Ella performs $M_n$ on $E$. The probability of the outcome $k$ is $p_{n|k} = \text{Tr}[(\mathbb{I} \otimes M_{n|k})\rho_{se}]$, and the corresponding collapsed state of the system is $\rho_{n|k} = \text{Tr}_e[(\mathbb{I} \otimes M_{n|k})\rho_{se}] / p_{n|k}$. Each $M_n$ leads to a decomposition of the initial state as $\tau_s = \sum_k p_{n|k} p_{n|k}$. Ella informs Bob of her outcome. According to the outcome, Bob performs a global unitary $U_{n|k}$ on $S$ and the auxiliary qubit $A$ in his hands, which are in the initial state $\tau_s \otimes \rho_A$. We assume that another observer, Charlie, receiving the two-qubit state prepared by Bob, knows in advance the details of the procedure but is ignorant of the values of $n$ and $k$ in a specific run. Therefore, the final state of $S$ and $A$ received by Charlie is

$$\xi_{sa} = \sum_{n,k} q_n p_{n|k} \left(U_{n|k}\rho_{n|k} \otimes \rho_A U_{n|k}^\dagger\right),$$

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A. Without the demon

Let us begin with the case without the help of Ella as a preview. We consider an arbitrary initial state of the system qubit

$$\rho_s = \frac{1}{2} (\mathbb{I} + \vec{r} \cdot \vec{\sigma}),$$

with the Bloch vector $|\vec{r}| = r \in [0, 1]$ and $\vec{\sigma}$ being the vector of Pauli matrices. Specifically, in the following, we show all the maximal correlations of

$$\zeta_{sa} = U \rho_s \otimes \rho_A U^\dagger,$$

among all the global unitaries $U$ and initial states of $A$ $\rho_A$, are monotonic increasing functions of $r$. To reach these maximums, the initial state of $A$ can be chosen as $\rho_A = |0\rangle\langle 0|$, and the global unitaries $U$ can be implemented.
in two steps: (1) a local unitary diagonalizing $\rho_s$ into $\frac{1}{\sqrt{2}}(1 + r\sigma_z)$; (2) a global unitary $U_0$ such that $U_0|00\rangle = |\psi_+\rangle$ and $U_0|10\rangle = |\psi_-\rangle$ for entanglement while $U_0|10\rangle = |\psi_-\rangle$ for mutual information and Bell-nonlocality. Here $|\psi_{\pm}\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ are two of the Bell states.

**Mutual information.** The quantum mutual information of a bipartite state $\rho_{\alpha\beta}$ is defined as

$$I(\rho_{\alpha\beta}) = H(\rho_{\alpha}) + H(\rho_{\beta}) - H(\rho_{\alpha\beta}),$$

where $\rho_{\alpha} = Tr_{\beta}\rho_{\alpha\beta}$ and $\rho_{\beta} = Tr_{\alpha}\rho_{\alpha\beta}$ are the reduced states of subsystems $\alpha$ and $\beta$, respectively, and $H(\rho) = -Tr(\rho \log \rho)$ is the von Neumann entropy of $\rho$. It measures the total correlation between the two subsystems, which does not distinguish classical correlation from quantum one [32].

The total entropy of the whole state of $S$ and $A$ is conserved under the global unitary $U$ in Eq. (4), i.e.,

$$H(\zeta_{sa}) = H(\rho_s \otimes \rho_a) = H(\rho_s) + H(\rho_a).$$

It can be minimized by choosing $\rho_a = |0\rangle|0\rangle$ when the initial state $S$ is fixed. The local entropies of $\zeta_{sa}$ are upper bounded by the dimension of the two subsystems as $H(\zeta_{sa}) \leq 2$ and $H(\zeta_{sa}) \leq 2$. The two equalities hold when the eigenstates of $\zeta_{sa}$ are maximally entangled states. These lead to the maximum of the mutual information as

$$\max_{\{U,\rho_a\}} I(\zeta_{sa}) = 2 \log 2 - H(\rho_s) = 2 \log 2 - h(r),$$

where $h(r) = -\frac{1+r}{1+r} \log \frac{1+r}{1+r} - \frac{1-r}{1-r} \log \frac{1-r}{1-r}$. It is by the whole state $\zeta_{sa} = \frac{1+r}{2}\langle \psi_+ | \psi_+ \rangle + \frac{1-r}{2}| \psi_- \rangle \langle \psi_- |$.

**Entanglement.** Concurrence and negativity are the two most widely used measures of entanglement in two-qubit states. The former leads to a computable formula for entanglement of formation in the two-qubit case [29].

The latter is closely related to partial transpose criterion of entanglement [30]. Both of them do not increase on average, which is the condition of convexity usually satisfied by known entanglement measures [33]. Based on the convexity, one can easily draw a conclusion that the maximal concurrence and negativity can be simultaneously reached by choosing $\rho_a = |0\rangle|0\rangle$. Without loss of generality, we suppose $\rho_a = p_0|0\rangle|0\rangle + p_1|1\rangle|1\rangle$ with $p_{0,1} \in [0,1]$ and $p_0 + p_1 = 1$, and only show the computation procedure for $C$ as

$$C(\zeta_{sa}) \leq p_0 C(U|\rho_s \otimes |0\rangle|0\rangle U^\dagger) + p_1 C(U|\rho_s \otimes |1\rangle|1\rangle U^\dagger) \leq p_0 \max_{\{U\}} C(U|\rho_s \otimes |0\rangle|0\rangle U^\dagger) + p_1 \max_{\{U\}} C(U|\rho_s \otimes |1\rangle|1\rangle U^\dagger) = \max_{\{U\}} C(U|\rho_s \otimes |0\rangle|0\rangle U^\dagger).$$

Then, both of the measures of entanglement for the rank 2 state $U|\rho_s \otimes |0\rangle|0\rangle U^\dagger$ can be maximized among global unitaries $U$ by $\zeta_{sa} = U|\rho_s \otimes |0\rangle|0\rangle U^\dagger = \frac{1+r}{2}| \psi_+ \rangle \langle \psi_+ | + \frac{1-r}{2}| \psi_- \rangle \langle \psi_- |$.

$$\max_{\{U,\rho_a\}} C(\zeta_{sa}) = \frac{1}{2}(1 + r),$$

$$\max_{\{U,\rho_a\}} N(\zeta_{sa}) = \frac{1}{2} \left[ \sqrt{2(1 + r^2)} - 1 + r \right].$$

It is interesting that, the final state $\zeta_{sa}$ above has the minimal negativity for a fixed concurrence [36].

**Bell-nonlocality.** Bell-nonlocality exists in the states whose outcomes of local measurements do not admit by any local-hidden-variable models, which can be witnessed by the violation of Bell-type inequalities. We adopt the maximal quantum violation of the CHSH inequality, $B$, as the degree of Bell-nonlocality for two-qubit systems.

For a two-qubit state $\rho$ with spin correlation matrix $T$, whose elements $T_{ij} = Tr(\sigma_i \otimes \sigma_j \rho) (i,j = 1,2,3)$, $B(\rho) = 2\sqrt{t_1^2 + t_2^2}$ [37]. Here, $t_1^2$ and $t_2^2 \in [0,1]$ are the two largest eigenvalues of $T^T T$. The amount $B(\rho) > 2$ demonstrates the Bell-nonlocality of $\rho$.

In the region of nonlocality, for a fixed linear entropy $S_L(\rho) = \frac{1}{2}(1 + Tr\rho^2)$, $B(\rho)$ is maximized by the rank 2 states mixed by any two of the Bell states [38]. And, the maximal $B$ decreases with $S_L$. Therefore, for a given $\rho_s$, the maximal Bell-nonlocality that can be created is

$$\max_{\{U,\rho_a\}} B(\zeta_{sa}) = 2\sqrt{(1 + r^2)},$$

which is reached by $\rho_a = |0\rangle|0\rangle$ and $\zeta_{sa} = \frac{1+r}{2}| \psi_+ \rangle \langle \psi_+ | + \frac{1-r}{2}| \psi_- \rangle \langle \psi_- |$.

**B. Assisted by the demon**

Now, we allow Ella to participate. For fixed $\{q_{n}\}$ and $\{M_{n}\}$, we define the average length of the Bloch vectors $\tilde{r}_{n|k}$ of $\rho_{n|k}$ in Eq. (2) as $\tilde{r} = \sum_{n,k} q_n \rho_{n|k} r_{n|k}$. The maximal quantum correlations of $\zeta_{sa}$ can be obtained by replacing $r$ in Eqs. (6) and (8)-(10) with $\tilde{r}$. These maxima can be reached by optimizing each of the terms $U_{n|k} \rho_{n|k} \otimes \rho_{n|k} U_{n|k}^\dagger$ in the ways given in the above subsection. Namely, one chooses $\rho_a = |0\rangle|0\rangle$ and $U_{n|k}$, and transforms $\rho_{n|k} \otimes |0\rangle|0\rangle$ into the mixtures of $|\psi_+\rangle$ and $|\psi_-\rangle$ for maximal $I$ and $B$ while into the mixtures of $|\psi_+\rangle$ and $|\psi_-\rangle$ for maximal entanglement. These are nontrivial, as the quantum correlations are nonlinear functions of the state. The details are in the following.

**Mutual information.** The minimum of the entropy for $\zeta_{sa}$ can be achieved by setting: (1) $\rho_a = |0\rangle|0\rangle$; (2) the elements are diagonal in the same set of basis as $U_{n|k} \rho_{n|k} \otimes |0\rangle|0\rangle U_{n|k}^\dagger = \sum_{\lambda} \lambda_{n|k,\lambda} |\phi_{\lambda}\rangle \langle \phi_{\lambda} |$ with $\lambda_{n|k,\lambda} \geq \lambda_{n|k,0}$. The first point can be easily proved by using the concavity property of the von Neumann entropy [28], as the calculation of concurrence in Eq. (7). Here, we omit the procedure for brevity. The second point can be derived based on the Lemma 1 in Appendix A. Namely, we set $X_{n|k} = g_n \rho_{n|k} (\mathbb{I} - U_{n|k} \rho_{n|k} \otimes |0\rangle|0\rangle U_{n|k}^\dagger)$.
and \( X = \sum_{n,k} X_{n,k} = \mathbb{1} - \xi_{sa} \). The von Neumann entropy for the two-qubit state can be written as

\[
\mathcal{H}(\xi_{sa}) = 3 - \sum_{n=2}^{+\infty} \frac{1}{n(n-1)} \text{Tr}(X^n).
\]

(11)

All the terms \( \text{Tr}(X^n) \) can be simultaneously maximized by the above condition (2).

Then, the entropies of two reduced states of \( \xi_{sa} \) can be maximized by transforming \( |\phi_0\rangle \) and \( |\phi_1\rangle \) into the two Bell states \( |\phi_+\rangle \) and \( |\phi_-\rangle \) without affecting entropy for \( \xi_{sa} \). These lead to the maximal mutual information as

\[
\max_{\{U_{n|k},\rho_{nk}\}} \mathcal{I}(\xi_{sa}) = 2 \log 2 - h(\bar{r}).
\]

(12)

Entanglement.— One can still use the procedure in Eq. (7) to restrict in the initial state of \( A \) as \( \rho_a = |0\rangle \langle 0| \). The convexity of concurrence further gives

\[
\mathcal{C}(\xi_{sa}) = \sum_{n,k} q_n p_{nk} \mathcal{C}(U_{n|k} \rho_{nk} \otimes |0\rangle \langle 0| U_{n|k}^\dagger).
\]

(13)

When the states \( U_{n|k} \rho_{nk} \otimes |0\rangle \langle 0| U_{n|k}^\dagger = \frac{1+\tau_{nk}}{2} |\psi_+\rangle \langle \psi_+| + \frac{1-\tau_{nk}}{2} |0\rangle \langle 0| \), each concurrence of the right-hand side reaches its maximum, and meanwhile, the equality holds. Then, the maximum of concurrence is given by

\[
\max_{\{U_{n|k},\rho_{nk}\}} \mathcal{C}(\xi_{sa}) = \frac{1}{2} (1 + \bar{r})
\]

(14)

These choices simultaneously maximize the negativity as

\[
\max_{\{U_{n|k},\rho_{nk}\}} \mathcal{N}(\xi_{sa}) = \frac{1}{2} \sqrt{2(1 + \bar{r}^2) - 1 + \bar{r}}.
\]

(15)

We give the details in Appendix B.

Bell-nonlocality.— The optimization of the Bell-nonlocality measured by \( B \) can be solved by minimizing the linear entropy again. The minimum of \( S_L(\xi_{sa}) \) is achieved under the same two conditions above for the von Neumann entropy, which can be proved by using a similar procedure. Then, the state \( \xi_{sa} \) is rank 2. By transforming its eigenstates into \( |\phi_+\rangle \) and \( |\phi_-\rangle \), Bob obtains the maximal Bell-nonlocality assisted by the demon as

\[
\max_{\{U,\rho_a\}} \mathcal{B}(\xi_{sa}) = 2\sqrt{1+\bar{r}^2}.
\]

(16)

C. Optimal measurements of the demon

It would be interesting to find out Ella’s optimal measurements, which maximize \( \bar{r} \). Because of the convex form of \( \bar{r} = \sum_{n,k} q_n p_{nk} r_{nk} |k|_1 \), the maximum of \( \bar{r} \) occurs at \( q_n = \{0,1\} \). That is, the maximum is equal to \( \max_{\{n\}} \sum_{k} p_{nk} r_{nk} |k|_1 \).

When \( \rho_{se} \) is pure, any von Neumann measurement on \( E \) can project \( S \) into a pure local state, and thereby \( \bar{r} = 1 \) reaches its maximum. However, for mixed states between \( E \) and \( S \), it is difficult to uniformly optimize Ella’s measurement, even if for the two-qubit case (i.e., \( E \) is two-dimensional) under local von Neumann measurements. We give some results for this simple case below.

A general form of the two-qubit state, whose reduced state of \( S \) part is \( \tau_s \) in Eq. (1), can be expressed as

\[
\rho_{se} = \frac{1}{4}(|1+\eta \sigma_z \otimes 1+\eta \otimes \sigma_\cdot \bar{b} + \sum_{ij} T_{ij} \sigma_i \otimes \sigma_j|),
\]

(17)

where \( \bar{b} \) is the Bloch vector on Ella’s side and \( T \) is the \( 3 \times 3 \) spin correlation matrix. An observable of Ella can be labeled by a unit vector as \( M_{\bar{n}} = \bar{n} \cdot \sigma \), and its elements are two projectors

\[
M_{\bar{n}|k} = \frac{1}{2}(1 + k \bar{n} \cdot \bar{\sigma})
\]

(18)

corresponding to the outcomes \( k = \pm 1 \). After her measurements, the system qubit is left in the unnormalized state

\[
\rho_{\bar{n}|k} = \frac{1}{2} \left[ \mathbb{1} + \eta \sigma_z + k(T \bar{n} \cdot \bar{\sigma}) \right].
\]

(19)

with the measurement probability \( p_{\bar{n}|k} = \frac{1}{2}(1 + k \bar{b} \cdot \bar{n}) \).

Then, one can obtain the the maximal \( \bar{r} \) as

\[
\bar{r} = \max_{\{\bar{n}\}} \left( |\bar{\eta} + T \bar{n}| + |\bar{\eta} - T \bar{n}| \right),
\]

(20)

where \( \bar{\eta} = (0,0,\eta) \).

When the Bloch vector \( \eta = 0 \), it is obvious that the maximum of \( \bar{r} \) is the maximal absolute value of the eigenvalues of \( T \). The optimal \( \bar{n} \) is the corresponding eigenvector. However, when \( \eta \neq 0 \), the optimal solution depends on both \( \eta \) and \( T \). We adopt the state in the case study of Beyer et al. [10] as an example, which is

\[
\rho(p,\eta) = p |\Psi\rangle \langle \Psi| + (1-p) \rho_{cl},
\]

(21)

with the two components

\[
|\Psi\rangle = \sqrt{\frac{1+\eta}{2}}|0\rangle + \sqrt{\frac{1-\eta}{2}}|1\rangle,
\]

(22)

and

\[
\rho_{cl} = \frac{1+\eta}{2} |0\rangle \langle 0| \otimes |0\rangle \langle 0| + \frac{1-\eta}{2} |1\rangle \langle 1| \otimes |1\rangle \langle 1|.
\]

(23)

Its reduced state of the system qubit is \( \tau_s \) in (1), and correlation matrix \( T = \text{Diag} \left( p_1 \sqrt{1 - \eta^2} - p_2 \sqrt{1 - \eta^2} \right) \). Ella’s optimal measurement is \( M_{\bar{n}} = \sigma_z \), which reaches the maximal \( \bar{r} = 1 \). Replacing the projectors \( |0\rangle \langle 0| \) and \( |1\rangle \langle 1| \) of \( E \) in \( \rho_{se} \) with \( |+\rangle \langle +| \) and \( |-\rangle \langle -| \), one can directly find that \( \bar{r} \) can reaches 1 only if \( p = 0 \) or 1, and the
optimal measurement is no longer \( \sigma_z \) when \( p \neq 0 \).

### III. CLASSICAL AND QUANTUM DEMONS

We continue to consider the simple case of a two-dimensional \( \mathcal{E} \) under von Neumann measurements, to succinctly show the different effects between a quantum demon and a classical one. The maximal prepared correlations are monotonic increasing functions of the average length of the Bloch vectors in postmeasured states of the system qubit. Therefore, the length change of the Bloch vector, \( \Delta r = \bar{r} - \eta \), measures the quantum correlations enhanced by the participation of the demon. In this part, we adopt the increase in the created entanglement

\[
\Delta C = \frac{1}{2}(\bar{r} - \eta),
\]

which is proportional to \( \Delta r \), as a figure of merit of the demon.

The operator, Bob, is assumed to know the form of state \( \rho_{se} \) to choose Ella’s measurements and perform his optimal operations. The triple \((\eta, T, \vec{b})\) is his a priori knowledge of the system qubit and its environment. Ella’s measurements and outcomes convert this information to the system \( \mathcal{S} \), as shown in the form of \( \rho_{\tilde{u}|k} \) in Eq. (19), which leads to the enhancement of created correlations.

The case with the Bloch vectors \( \eta = 0 \) and \( \vec{b} = 0 \) is notable, in demonstrating the relationship between the created correlations and the quantumness in \( \rho_{se} \). That is, the system \( \mathcal{S} \) is in the high temperature limit \( T \to \infty \) and the local state of \( \mathcal{E} \) is completely unknown. Such a \( \rho_{se} \) is equivalent to a Bell diagonal state \([39–41]\) under von Neumann measurements, to consider that a physical Bloch vector is not longer than the assistance of the demon can serve as criterions and measures for steerability of the state \( \rho_{se} \) in this case.

For a general two-qubit \( \rho_{se} \), we focus on the situation with two observables. This is the minimum number of observables to show the advantage of a quantum Maxwell demon. In addition, we assume that Bob requires the two measurement directions to satisfy \( T\vec{u}_1 \perp T\vec{u}_2 \), according to his a priori knowledge of the matrix \( T \). That is, the changes of direction of the Bloch vector of \( \mathcal{S} \) affected by Ella’s measurements are perpendicular to each other. These two points conform to the intuitive understanding from the classically correlated state \( \rho_{cl} \) in Eq. (23). It cannot be distinguished from the fully quantum state \( |\Psi\rangle \) in Eq. (22) by the measurement on \( \sigma_z \), which leads to the maximum \( \bar{r} = 1 \). A feature of \( \rho_{cl} \) is that, corresponding to any measurement on \( \mathcal{E} \), the change of the Bloch vector of \( \mathcal{S} \) is along the z axis. For the case with two observables onto the two general states (17), chosen by Bob with equal probabilities, one can directly obtain that the entanglement enhanced by Ella is

\[
\Delta C = \frac{1}{8}\left( -4\eta + \sum_{i=1,2;k=\pm 1} |\vec{n} + kT\vec{n}_i| \right).
\]

Below we present an upper bound on the enhanced concurrence under this condition for classical (unsteerable) demons.

A LHS model admitted by a two-qubit state \( \rho_{se} \) can be identified with a hidden Bloch vector \( \vec{\lambda} \) with a distribution \( \omega(\vec{\lambda}) \), and a function \( f(\vec{n}, \vec{\lambda}) \in [-1, 1] \) of \( \vec{\lambda} \) and the measurement direction \( \vec{n} \) \([41]\). They satisfy

\[
\begin{align*}
\int \omega(\vec{\lambda})(\mathbf{1} + \vec{\lambda} \cdot \vec{\sigma}) d\vec{\lambda} &= \mathbf{1} + \eta \sigma_z, \\
\int \omega(\vec{\lambda})f(\vec{n}, \vec{\lambda})(\mathbf{1} + \vec{\lambda} \cdot \vec{\sigma}) d\vec{\lambda} &= (\vec{n} \cdot \vec{b}) \mathbf{1} + (T\vec{u}) \cdot \vec{\sigma},
\end{align*}
\]

where the integral is over the Bloch sphere and \( d\vec{\lambda} \) is the surface element. We denote \( T\vec{u}_1 = \vec{\alpha}_1 \) and \( T\vec{u}_2 = \vec{\alpha}_2 \). The average length of Bloch vectors in Eq. (26) satisfies

\[
\bar{r} \leq \frac{1}{2}\left( \sqrt{\eta^2 + \alpha_1^2} + \sqrt{\eta^2 + \alpha_2^2} \right),
\]

and the two changes

\[
\alpha_i^2 = \int \omega(\vec{\lambda}) f(\vec{n}_i, \vec{\lambda}) (\vec{\lambda} \cdot \vec{\alpha}_i) d\vec{\lambda},
\]

with \( i = 1, 2 \).

The amount of \( \alpha_i^2 \) is upper bounded by \( f(\vec{n}_i, \vec{\lambda}) = \text{sgn}(\vec{\lambda} \cdot \vec{\alpha}_i) \), where \( \text{sgn} \) is the sign function. Without loss of generality, we choose \( \vec{\alpha}_1 = \vec{i} \) and \( \vec{\alpha}_2 = \vec{j} \), where \( \vec{i} \) and \( \vec{j} \) are unit vectors in the \( x \) and \( y \) directions, respectively. When \( f(\vec{n}_i, \vec{\lambda}) = \text{sgn}(\vec{\lambda} \cdot \vec{\alpha}_i) \), the integral (29) is invariant under the inversions of the distribution that

\[
\begin{align*}
\omega(\lambda_1, \lambda_2, \lambda_3) &\to \omega(\lambda_1, \lambda_2, \lambda_3) \\
\omega(\lambda_1, \lambda_2, \lambda_3) &\to \omega(-\lambda_1, \lambda_2, \lambda_3),
\end{align*}
\]

Consequently, the maximum of the right-hand side of Eq. (28) can always be found among the distributions which are invariant under the above two inversions. Then, \( \alpha_i^2 \) are determined by \( \bar{q} = \int_{\lambda_1 \geq 0, \lambda_2 \geq 0} \omega(\vec{\lambda}) d\vec{\lambda} \). An upper bound of the average length of Bloch vectors is given by \( \bar{q} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \), which is \( \bar{r} \leq \sqrt{\frac{1}{2} + \eta^2} \). Considering that a physical Bloch vector is not longer than 1, one can conclude that the entanglement enhanced by
quantum region in the parameter space of \((p, \eta)\) under the measurements on \(M_{\sigma_1} = \sigma_x\) and \(M_{\sigma_2} = \sigma_z\) with equal probabilities. The solid lines show enhanced concurrence \(\Delta C\) with the parameter \(\eta = 0.1, \eta = 0.35,\) and \(\eta = 0.6\) from top to bottom. The dashed bound is obtained by comparing \(\Delta C\) to the inequality (30).

a classical demon is bounded by

\[
\Delta C_{cl} \leq \frac{1}{2} \left( -\eta + \min \left\{ \sqrt{\eta^2 + \frac{1}{2}}, 1 \right\} \right). \tag{30}
\]

We now compare the bound to the enhanced concurrence for \(\rho(p, \eta)\) in Eq. (21). Under the condition that \(T\vec{n}_1 \perp T\vec{n}_2\), the optimal two measurements on \(\rho(p, \eta)\) to create the maximal quantum correlations can be found to be \(M_{\sigma_1} = \sigma_x\) and \(M_{\sigma_2} = \sigma_z\) by numerical calculation. Then, the enhanced concurrence is given by \(\Delta C = [1 - 2\eta + \sqrt{\eta^2 + p^2(1 - \eta^2)}]/4\). One can draw the quantum region in the parameter space of \((p, \eta)\), or equivalently the space of \(p\) and \(\Delta C\). As shown in Fig. 2, the quantumness in a pure state \(|\Psi\rangle\) with a larger \(\eta\) is more fragile under the mixture of classical state, although an arbitrary \(|\Psi\rangle\) leads to \(\bar{r} = 1\). To demonstrate the quantumness, the enhanced entanglement increases with the proportion of classical state.

IV. SUMMARY

We studied the preparation of quantum correlations from a system qubit and an auxiliary qubit, assisted by a Maxwell demon who obtains information of the thermal qubit from measurements on its environment. These processes avoid the disturbance to average state of the system by direct measurements, and establish the relationships between quantum steering and other correlations in the thermodynamic framework. We derived the optimal operations between the system and the auxiliary to create the maximal mutual information, entanglement and Bell-nonlocality. The maximums are monotonic increasing functions of the average length of the Bloch vectors in postmeasured states of the system qubit. A critical value of the average length naturally corresponds to the necessary and sufficient condition for steerable ability in the case with maximally mixed marginals. We also presented an upper bound of the average length for unsteerable environment-system correlation, which can be regarded as a steering-type inequality demonstrating the quantumness of the Maxwell demon.

It would be interesting to consider extensions of the current results in several directions. On the theoretical side, one can try to derive more general relationships between the preparation of quantum correlations and the quantum steering, especially in multipartite systems and in the processes with thermodynamic cycles. And, the result in the case with maximally mixed marginals suggests that, it is possible to find an operational interpretation in thermodynamic tasks of the necessary and sufficient conditions for steerable ability of general two-qubit states [44–46]. Experimentally, we hope that the processes studied in this paper can be implemented in laboratories with the recent techniques developed in spin systems [11].

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Appendix A: Lemma 1

Lemma 1: Let \(\{X_j\}\) be a set of Hermitian semipositive definite operators in the \(d\)-dimension Hilbert space, with the spectral decomposition \(X_j = \sum_{i=1}^d \lambda_i^{(j)} |\phi_i^{(j)}\rangle \langle \phi_i^{(j)}|\) and \(\lambda_1^{(j)} \geq \lambda_2^{(j)} \geq \ldots \geq \lambda_d^{(j)}\). For given spectrums \(\{\lambda_i^{(j)}\}\), the maximum of \(\text{Tr}(\sum_j X_j)^n\) \((n = 2, 3, \ldots)\) occurs when the eigenstates satisfy \(|\langle \phi_i^{(j)}| \phi_j^{(j')}\rangle| = \delta_{i,j'}\).

Proof. For given spectrums \(\{\lambda_i^{(j)}\}\), the amount of \(\text{Tr}(\sum_j X_j)^n\) depends on the cross terms in the form \(\text{Tr}(X_j^k Y),\) where \(1 \leq k \leq d - 1\) and \(Y\) is a product of \(X_j^{k'}\) with \(j' \neq j\) and \(\sum_j k_j' + k = n\). The operator \(Y\) is semipositive definite. Suppose \(Y = \sum_{i=1}^d y_i |y_i\rangle \langle y_i|\) and \(y_1 \geq y_2 \ldots \geq y_d\). One can directly calculate and obtain

\[
\text{Tr}(X_j^k Y) = \sum_{i,i'} \lambda_i^{(j)} y_{i'} P_{ii'}, \tag{A1}
\]

where \(P_{ii'} = |\langle \phi_i^{(j)}| y_{i'}\rangle|^2\) is a doubly stochastic matrix. For given \(\lambda_i^{(j)}\) and \(y_{i'}\), \(\text{Tr}(X_j^k Y)\) is maximized by \(P_{ii'} = \delta_{i,i'}\). For all the cross terms and all the choices of the subscript \(j\), the above maximization can be simultaneously achieved by \(|\langle \phi_i^{(j)}| \phi_j^{(j')}\rangle| = \delta_{i,j'}\).

Appendix B: Optimization of negativity

For given eigenvalues \( \{ \lambda_i \} \) of two-qubit states in nonascending order, the maximal negativity is given by [35]

\[
N_{\text{max}} = \max \{ 0, \sqrt{(\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_4)^2 - \lambda_2 - \lambda_4} \}.
\] (B1)

It is reached by

\[
\rho = |\psi_+\rangle \langle \psi_+| + |\psi_-\rangle \langle \psi_-| + |\lambda_4| |\lambda_4| (10).\] (B2)

By using this result, one can derive the maximal negativity for a given the maximal eigenvalue, \( \lambda_1 \in [1/4, 1] \).

We define four lines on the plane of \((\lambda_3, \lambda_4)\) as

(a) \( \lambda_4 = 0 \),
(b) \( \lambda_4 = \lambda_3 \),
(c) \( \lambda_4 = 1 - \lambda_1 - 2\lambda_3 \),
(d) \( \lambda_4 = 1 - 2\lambda_1 - \lambda_3 \).

They are equivalent to the four equals signs in 0 \(\leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \) in order. For a fixed value of \( \lambda_1 \), there are three different situations for the physical region on the plane of \((\lambda_3, \lambda_4)\) as: (i) a triangle defined by lines (a), (b), and (c), when \( \lambda_1 \in [1/2, 1] \); (ii) a quadrilateral defined by lines (a), (b), (c), and (d), when \( \lambda_1 \in [1/3, 1/2] \); (iii) a triangle defined by lines (b), (c), and (d), when \( \lambda_1 \in [1/4, 1/3] \).

Because of the convexity of negativity, the maximum occurs on the vertices of the triangles or quadrilateral. Calculating the value of \( N_{\text{max}} \) on these vertices, one obtains

\[
\tilde{N}_{\text{max}} = \max \{ 0, \sqrt{10\lambda_1^2 - 6\lambda_1 + 1 - \lambda_4}, \sqrt{2\lambda_1^2 - 2\lambda_1 + 1 + \lambda_1 - 1} \}.
\] (B3)

It is a monotonic increasing function of \( \lambda_1 \).

The choice with \( \rho_a = |0\rangle \langle 0| \) and \( U_{n|k}\rho_{n|k} \otimes |0\rangle \langle 0| U_{n|k}^\dagger = (1 + e^{i\theta/k}) |\psi_+\rangle \langle \psi_+| + (1 - e^{i\theta/k}) |0\rangle \langle 0| \) maximizes the maximal eigenvalue of \( \xi_{\text{max}} \), and meanwhile leads to \( N(\xi_{\text{max}}) = N_{\text{max}} = \sqrt{2\lambda_1^2 - 2\lambda_1 + 1 + \lambda_1 - 1} \). Here \( \lambda_1 = \frac{\sqrt{n}}{2} \in [1/2, 1] \). Hence, this optimizes the amount of negativity.

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