Towards the Super Yang-Mills Theory on the Lattice

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\textbf{Abstract:} We present an entirely new approach towards a realization of the supersymmetric Yang-Mills theory on the lattice. The action consists of the staggered fermion and the plaquette variables distributed in the Euclidean space with a particular pattern. The system is shown to have fermionic symmetries relating the fermion and the link variables.

\textbf{Keywords:} Lattice Gauge Field Theories, Nonperturbative Effects, Supersymmetry and Duality
1. Introduction

The vacuum structure of 4 dimensional super Yang-Mills (SYM) theory has been studied by many authors after the works by Seiberg and Witten\cite{1}. In their approach, dualities, the holomorphy and symmetries of low energy theories play crucial roles.

The lattice gauge theory has proved to be a powerful formulation for gauge theories and made definite success for non-supersymmetric theories. Many efforts have been paid to formulate the SYM theory on the lattice with the expectation to provide a non-perturbative method. However this hope has not been materialized yet due to various difficulties against a realization of the SUSY algebra, such as: 1) a realization of a massless Majorana fermion on the lattice; 2) the absence of the Leibniz rule; 3) we may also count the absence of the Bianchi identity in lattice formulations\cite{3}, which is crucial to show the SUSY invariance of the continuum SYM.

Among various attempts to formulate the SYM on the lattice, here let us recall a couple of them. Curci and Veneziano\cite{4} proposed a formulation based on the Wilson fermion. They claimed that the SUSY WT identity as well as chiral WT identity would be fulfilled by the fine-tuning of a fermion mass, ie, once the mass counter term is appropriately chosen. This claim has been confirmed at the one-loop level in perturbation theory\cite{5}, but there is no proof up to now that it should hold in all orders. In recent years, some numerical studies have been performed for their approach\cite{6}. However the results are not so clearly favorable for the presence of SUSY. In this approach the fine-tuning is expected to produce a massless Majorana fermion.

\footnote{See\cite{2} for earlier works.}
A latest attempt is based on the Domain Wall fermion (DWF)\cite{7,8}. The authors took the viewpoint that, except the gaugino mass term, any SUSY breaking interaction is irrelevant around a SUSY fixed point when we have the gauge symmetry and the right degrees of freedom. To realize the latter condition, the DWF was employed to produce a massless Majorana fermion. In this approach, a SUSY invariant fixed point is assumed for the discussion of the continuum limit. Its validity or plausibility is yet to be clarified. No evident relation is known between the fixed point and the DWF approach.

Both of the above approaches use a "mass" term to remove the fermion doublers. If the SUSY requires that the gauge sector is influenced by the fermion mass term, it may be hard to keep the gauge invariance. So we choose another way to avoid the doubling problem, i.e., the staggered fermion\cite{9}. It is not a new idea to use the staggered fermion to realize the SUSY. For earlier attempts, see Refs. \cite{10}.

In this paper, we propose an entirely new lattice theory with exact fermionic symmetries which relate the gaugino and the gauge degrees of freedom with Grassmann odd parameters. An overview of its construction and its properties will be described in the next section. The rest of this paper is organized as follows. In the sections 3 and 4, we present a model defined on a single hypercube, to be addressed as one cell model, and show that it has an exact fermionic symmetry (preSUSY). In sections 5 and 6, the model is extended to the entire Euclidean space, so that the preSUSY survives. The last section is devoted to the summary and discussion.

2. An overview of our models

It would be appropriate to present an overview of our approach and strategy before going into the detailed explanation of our models.

We start our construction from a lattice theory defined on a single hypercube. This model will be addressed as the one cell model. For this model, written with link variables and site variables (real fermions in the adjoint representation), we find an exact symmetry which mixes the link and site variables. This symmetry will be called the preSUSY.

Although our theory contains the staggered fermion, it is different from the earlier works listed in Ref. \cite{10}. Our model is defined on a hypercube or a cell and its preSUSY transformation is clearly different from the SUSY transformation in the latter approach.

The gauge part of the action $S_g$ is written by the ordinary plaquette variables. The fermion action $S_f$ consists of terms which corresponds to fermions sitting at two neighboring sites connected by link variables. The coefficients for the terms in $S_f$ and those in the preSUSY transformations on the link variables and fermions, denoted by $\delta_U$ and $\delta_\psi$ respectively, are to be determined to realize the preSUSY invariant action. The preSUSY transformation of the action consists of fermion cubed terms $\delta_U S_f$ and linear terms. The vanishing of the former terms relate coefficients in $\delta_U$ and $S_f$. The condition allows us to have an interesting fermion action, that is the staggered (Majorana) fermion coupled to link variables. From the vanishing of the fermion linear terms we may find a condition relating the remaining coefficient in $\delta_\psi$ to the rest. In order to keep the preSUSY invariance even at the quantum level, the path integral measure should be invariant as well. The Haar
measure for the link variables and the Grassmannian measure for the Majorana fermions is to be invariant as a whole. It will be found that the condition for $\delta U S_f = 0$ is sufficient to show the invariance of the measure.

We extend the preSUSY invariant one cell model to the entire (Euclidean) space and find an interacting cell model. A naive attempt to extend the model tends to end up with the uninteresting “SUSY” transformations: in the naive continuum limit they become $O(a)$, thus the symmetry is a lattice artifact. The extension presented here avoids the above mentioned difficulty owing to a non-trivial space structure.

The fermion part of the action $S_f$ is extended for all the links. The non-trivial structure is introduced for the gauge part. We classify the hypercubes in the entire space into three categories, the E-cells, O-cells and the rest. The two dimensional example is shown in Fig. 3. We introduce the plaquette variables only for E- and O-cells. Those for other hypercubes are missing. A link variable or a plaquette variable is associated with either an E-cell or O-cell, while the fermion as a site variable belongs to both E- and O-cells. Accordingly each term in the action belongs to either an E- or an O-cell, and the interactions between cells are present due to the two-sided nature of the fermions. The discrete translational invariance by $2a$ is present by construction. The condition of the preSUSY invariance of the action relates coefficients defined for neighboring cells. Due to this condition, the number of independent parameters in the preSUSY transformation are reduced drastically.

3. One cell model

Consider a cell or hypercube in D dimensional Euclidean space. The coordinates of the sites may be written as $r \equiv (r_1, r_2, \cdots)$ with $r_i = 0$ or 1. In this paper, we set the lattice spacing as $a = 1$.

The gauge action on this cell is given as

$$S_g = -\beta \sum_{0<\mu<\nu} \sum_{n=r(\mu\nu)} \text{tr} \left(U_{n,\mu\nu} + U_{n,\nu\mu}\right)$$

(3.1)

where $r(\mu\nu)$ implies sites with $r^\mu = r^\nu = 0$. $U_{n,\mu\nu}$ is a plaquette variable which goes first to the $\mu$ direction and then to the $\nu$ direction starting from the site $n$. In order for the plaquette to be on the cell, the starting site $n$ is restricted to that with $r^\mu = r^\nu = 0$.

Our action of the real staggered fermion is

$$S_f = \sum_{0<\rho} \sum_{n=r(\rho)} b_\rho(n) \text{tr} \left(\psi_{n,\rho} U_{n,\rho} \psi_{n+\rho}^\dagger U_{n,\rho}^\dagger\right) \equiv \sum_{0<\rho} \sum_{n=r(\rho)} b_\rho(n) \text{tr} \left(\psi_{n,\rho} \xi_{n,\rho}\right).$$

(3.2)

Here the coefficient $b_\rho(n)$ is a sign factor for the staggered fermion. We also introduced new notations $r(\rho) \equiv (r_1, \cdots, r_{\rho-1}, r_{\rho} = 0, r_{\rho+1}, \cdots)$ and $\xi_{n,\rho} \equiv U_{n,\rho} \psi_{n+\rho}^\dagger U_{n,\rho}^\dagger$. It is convenient to treat two fermions in (3.2) in a symmetric manner. So let us write the fermion action as

$$S_f = \sum_{0<\rho} \sum_{n=r(\rho)} b_{-\rho}(n+\rho) \text{tr} \left(\psi_{n+\rho} U_{n,\rho}^\dagger \psi_{n,\rho}\right) \equiv \sum_{0<\rho} \sum_{n=r(\rho)} b_{-\rho}(n+\rho) \text{tr} \left(\psi_{n+\rho} \xi_{n+\rho,\rho}^{-}\right).$$
Here $b_{-\rho}(r(\hat{\rho}) + \hat{\rho}) \equiv -b_{\rho}(r(\hat{\rho}))$ and $\xi_{n+\rho}^{-\rho} \equiv U_{n,\rho}^\dagger \psi_n U_{n,\rho}$. Later we extend the fermion action to the entire Euclidean space. The sign factor $b_\mu(n)$ should be properly defined in the whole space so that the action describes the staggered fermion.

We assume the form of preSUSY transformation and find conditions among coefficients required for the invariance of the action. Let us write down our ansatz for the preSUSY transformation. For the link variable, it is

$$\delta U_{n,\mu} = (\alpha \cdot \xi)_{n,\mu} U_{n,\mu} + U_{n,\mu}(\alpha \cdot \xi)_{n+\hat{\mu},\mu}$$

(3.3)

where

$$(\alpha \cdot \xi)_{n,\mu} \equiv \sum_\rho \alpha^{\rho[n]}_{n,\mu} \xi^{\rho[n]}_{n,\mu}$$

(3.4)

and $\rho[n] \equiv (-1)^n \rho$. The transformation parameters $\alpha^{\rho[n]}_{n,\mu}$ are Grassmann odd. The preSUSY transformation of fermion fields is assumed to be,

$$\delta \psi_n = \frac{1}{2} \sum_{0<\mu,\nu} C_{n}^{(\mu\nu)[n]} (U_{n,(\mu\nu)[n]} - U_{n,(\nu\mu)[n]})$$

(3.5)

where $(\mu\nu)[n] \equiv \mu[n]\nu[n]$. The parameters $C_{n}^{(\mu\nu)[n]}$ are also Grassmann odd and antisymmetric under the exchange of $\mu$ and $\nu$.

Note that coefficients $C_{n}^{(\mu\nu)[n]}$ and $\alpha^{\nu[n]}_{n,\mu}$ have indices referring to a site $n$ and the face defined by two directions, $\mu$ and $\nu$. So for a given site and a given face, we introduce four independent parameters by $\alpha^{\nu[n]}_{n,\mu}$ while we do only one by $C_{n}^{(\mu\nu)[n]}$. These counting will be useful for later discussion. From now on we may call these parameters $\alpha$-parameters and $C$-parameters for convenience.

Some explanations are in order on our ansatz for the preSUSY transformation. The form of fermion transformation in (3.3) is motivated by the continuum SUSY transformation, where the fermion is related to the field strength with its Lorentz indices contracted with the gamma matrices. In order to cancel terms produced by the fermion transformation acting on $S = S_g + S_f$, we chose eq. (3.3) as the transformation of the link variable. Naturally, we expect that the coefficients of transformations tend to the gamma matrices in a continuum limit: $C_{n}^{(\mu\nu)[n]} \propto \gamma^{(\mu\nu)}$ and $\alpha^{\nu[n]}_{n,\mu} \propto \gamma^{\mu}$.

In the next section, we derive relations between the coefficients in the action and the parameters in the preSUSY transformation by requiring the invariance of the action. We will find that the relations may be solved for any given coefficients in the action, $\beta$ and $b_\mu(n)$. So let us assume that these are appropriately chosen (non-vanishing) coefficients.

4. preSUSY invariance of one cell model

We derive relations on the coefficients in our action and preSUSY transformation. The transformation of the action consists of three terms,

$$\delta S = \delta U S_g + \delta U S_f + \delta \psi S_f.$$

(4.1)
Here $\delta U$ and $\delta \psi$ denote the transformations of link variables and fermions given in eqs. (3.3) and (3.5), respectively. The first term of (4.1) consists of fermion cubed terms and the rest linear terms. So they must vanish separately, $\delta U S_f = 0$ and $\delta U S_g + \delta \psi S_f = 0$. We also study the invariance of the path integral measure.

After a straightforward but tedious calculation, we find that the condition $\delta U S_f = 0$ gives us

$$(-)^{r_{\nu}} \frac{\alpha^{\nu[r]}_{r_{\mu}}}{b_{\nu}(r)} + (-)^{r_{\mu}} \frac{\alpha^{\mu[r]}_{r_{\nu}}}{b_{\mu}(r)} = 0. \quad (4.2)$$

In Fig. 1, we present a graphical representation of a term in $\delta U S_f$. Two terms of eq. (4.2) correspond to two different ways to obtain this graph.

Let us consider its implication. Eq. (4.2) simply tells us that the combination $(-)^{r_{\nu}} \frac{\alpha^{\nu[r]}_{r_{\mu}}}{b_{\nu}(r)}$ is anti-symmetric under the exchange of $\mu$ and $\nu$. In particular, the relation with $\mu = \nu$ leads to the vanishing of the diagonal element of $\alpha$-parameter. Thus we learn from eq. (4.2) that there is only one independent element in $\alpha^{\nu[r]}_{r_{\mu}}$ for the site $r$ and the face determined by $\mu$ and $\nu$.

The other condition $\delta U S_g + \delta \psi S_f = 0$ contains new transformation parameters $C_r^{(\mu\nu)[r]}$. We find two different relations,

$$b_{\rho}(r) C_r^{(\mu\nu)[r]} + b_{\nu}(r_d) C_{r_d}^{(\mu\nu)[r_d]} = \beta \left( (-)^{r_{\nu}} \alpha^{\rho[r]}_{r_{\mu}} - (-)^{r_{\nu}} \alpha^{\rho[r]}_{r_{\nu}} \right), \quad (4.3)$$

$$b_{\mu}(r) C_r^{(\mu\nu)[r]} + b_{\nu}(r_d) C_{r_d}^{(\mu\nu)[r_d]} = -\beta \left( (-)^{r_{\mu}} \alpha_{r_d,\nu}^{\rho[r]} + (-)^{r_{\nu}} \alpha_{r_d,\mu}^{\rho[r]} \right). \quad (4.4)$$

In eqs. (4.3) and (4.4), the lhs (rhs) comes from $\delta \psi S_f \left( \delta U S_g \right)$. The fermion action (3.2) consists of terms with two $\psi$ connected by link variables. Since the transformation (3.3) replaces one of $\psi$ by a plaquette variable, generically the corresponding term may be expressed by a three dimensional figure shown in Fig. 2(a). In $\delta U S_g + \delta \psi S_f$, the terms with $\rho \neq \mu, \nu$ give us the relation (4.3). When $\rho = \mu$ or $\nu$, we have to look at the cancellation condition with some care. As shown in Fig. 2(b), the figure becomes two dimensional and the same figure may be produced by the transformation of fermions located at $r$ as well as $r_d$. Here $r_d \equiv (r_1, r_2, \cdots, 1 - r_{\mu}, \cdots, 1 - r_{\nu}, \cdots)$ is the site diagonal to $r$ on the $\mu \nu$ face including the site $r$. We find eq. (4.4), in which coefficients associated with two sites $r$ and $r_d$ are related.
Using (4.2), we may rewrite (4.3) as

\[ C(\mu\nu)[r] = \beta(-)^{1+r\rho} \left[ \frac{\alpha[r]}{b\mu(r)} - \frac{\alpha[r]}{b\nu(r)} \right]. \]  \hspace{1cm} (4.5)

In eq. (4.3), the index \( \rho \) on the rhs could take any value. Choosing \( \rho = \mu \) and \( \nu \), we find the equalities,

\[ C(\mu\nu)[r] = \beta(-)^{r\mu} \frac{\nu[r]}{b\nu(r)} = -\beta(-)^{r\nu} \frac{\mu[r]}{b\mu(r)}. \]  \hspace{1cm} (4.6)

Eq. (4.4) is found to produce the same relations and does not carry any further information. From eq. (4.6), we learn that the \( C \)-parameters are written by the \( \alpha \)-parameters or vice versa. By combining this with the result in the previous section, we find that there is one independent parameter for a site and a face. So there are \( D \) independent parameters for a site. However this counting is not sufficient since a further restriction is present. From (4.5) we easily find the cyclic relation

\[ C(\mu\nu)[r] + C(\nu\lambda)[r] + C(\lambda\mu)[r] = 0 \]  \hspace{1cm} (4.7)

for any combination of \( \mu, \nu \) and \( \lambda \).

Eq. (4.7) is a “local” relation; it holds at each site. So let us consider it at the origin for convenience. By ignoring the site index, it is written simply as

\[ C^{\mu\nu} + C^{\nu\lambda} + C^{\lambda\mu} = 0. \]  \hspace{1cm} (4.8)

Observing that \( C \) are antisymmetric with respect to the upper indices, we use the following analogy to find the number of independent \( C \). Take \( D \) independent points in \( D \) dimensional (or larger) space and name them as 1, 2, 3, \cdots. Consider the vectors connecting the points, \( \vec{12} = -\vec{21} \) etc. There are \( D^2 \) of them altogether. Obviously, these vectors satisfy the relation similar to (4.8): the cyclic relation simply implies that three vectors connecting three points form a triangle. Including the antisymmetric property, we may identify \( C^{\mu\nu} \) and \( \vec{\mu}\vec{\nu} \) for solving eq. (4.8). From this identification, we clearly see that there are \( D - 1 \) independent \( C \)-parameters at a site.

Before closing this section we would like to confirm the invariance of the path integral measure at the first order of the transformation parameters. We will soon find that there
appears no new relation from the invariance. Actually the vanishing of the diagonal elements of the \( \alpha \)-parameter, concluded from eq. (4.2), is enough to show the invariance of the path integral measure.

Let us express eqs. (3.3) and (3.5) schematically as
\[
\delta U = F(\alpha, \psi, U), \\
\delta \psi = G(C, U).
\]

For the present discussion, we will see it convenient to achieve the transformation by taking the following two steps,
\[
\begin{align*}
\left( \begin{array}{c} U \\ \psi \end{array} \right) & \rightarrow \left( \begin{array}{c} U + F(\alpha, \psi - \delta \psi, U) \\ \psi \end{array} \right) \\
& \rightarrow \left( \begin{array}{c} U + F(\alpha, \psi, U) \\ \psi + \delta \psi \end{array} \right).
\end{align*}
\]

(4.9)
The path integral measure may be written schematically as \( \prod d\psi \prod dU \). The Haar measure \( dU \) for each link variable is invariant under the action of a unitary matrix from the left (or the right). In the second step, we transform only the fermions. The path integral measure is obviously invariant in this step. So we are left to study the first step. In the lowest order in the transformation parameters, it is the transformation only of the link variables, \( \delta U = F(\alpha, \psi, U) \). It is simply the infinitesimal form of unitary transformations acted both from the left and right, if \((\alpha \cdot \xi)\) is pure imaginary and it does not contain the transformed variable \( U_{n,\mu} \) itself. The latter condition may be expressed as
\[
\alpha_{n,\mu}^\mu = \alpha_{n+\hat{\mu},\mu}^{-\mu} = 0,
\]
the vanishing of the diagonal elements.

5. Interacting cell model

Although we showed that our one cell model is invariant under the Grassmannian transformation, it is highly non-trivial whether we may extend it to the entire D-dimensional space. Here we present one successful way of its extension.

Take a site with all its coordinate elements as even (or odd) integers. This site will be addressed as an even (or odd) reference point and its coordinate is indicated by \( N \). Starting from a reference point, we may form a cell, a hypercube, with unit vectors towards positive directions. A cell formed from an even (or odd) reference point is called an E-cell (O-cell). The two dimensional example is a traditional pattern\(^2\) shown in Fig. 2. We put copies of the one cell model on the E-cells as well as on the O-cells. Owing to our restriction on the reference points, there appear the spaces without the cell structure.

The link variables belong either to an E-cell or to an O-cell, while a fermion is on the site and associated with a pair of neighboring E-cell and O-cell. The action of this interacting cell model is a simple extension of the one cell model. The interaction between cells are described solely by the fermion action. The link variables in a pair of neighboring

\(^2\)This is called “Ichimatsu pattern” in the Japanese tradition.
cells interact through the fermion located at the shared site. To be described below, we modify our ansatz for the preSUSY transformation by including the contributions from E-cells and O-cells.

The sites on a cell have the coordinates denoted as \( n = N + r \). Here \( r \) is the relative coordinate from the reference point \( N \): the component of \( r \) is either 1 or 0. As for the one cell model, we denote by \( r(\hat{\mu}) \) a relative coordinate with \( r_\mu = 0 \); similarly \( r(\hat{\mu}\hat{\nu}) \) denotes a relative coordinate with \( r_\mu = 0 \) and \( r_\nu = 0 \).

A site \( n \) belongs to two cells and its coordinate may be written in two different ways,

\[
 n = N + r = N' + r',
\]

where \( N' \equiv N - e + 2r \) and \( r' \equiv e - r \) with \( e \equiv (1,1,1,1,\cdots) \). It is easy to confirm that \( N \) and \( N' \) are the reference points of two neighboring cells. In the following we use the notations \( n \equiv N + r \) and \( n' \equiv N' + r' \) to represent the same site but in reference to two different cells.

![Figure 3: Ichimatsu pattern. The thick (thin) cells are E-(O-)cells. The point P is shared by two neighbours cells.](image)

The action for the interacting model is easily obtained from (3.1) and (3.2). The generic notation for the site \( n \) is now to be understood as the sum of the reference and the relative coordinates, \( n = N + r \). The action is

\[
 S_g = -\beta \sum_N \sum_{0<\mu<\nu} \sum_{n=N+r(\hat{\mu}\hat{\nu})} \text{tr} \left( U_{n,\mu\nu} + U_{n,\nu\mu} \right), \quad (5.2)
\]

\[
 S_f = \sum_N \sum_{0<\rho} \sum_{n=N+r(\hat{\rho})} b_\rho(n) \text{tr} \left( \psi_n U_{n,\rho} \psi_{n+\rho} U_{n,\rho}^\dagger \right). \quad (5.3)
\]

The preSUSY transformation is modified as well. On the link variable, it takes the same form as the one cell model given in eq. (3.3), but now \( (\alpha \cdot \xi)_{n,\mu} \) on the rhs is to be understood as follows,

\[
 (\alpha \cdot \xi)_{n,\mu} = \sum_{\rho} \left( \alpha_{n,\mu}^{\rho[n]} \xi_{n}^{\rho[n]} + \tilde{\alpha}_{n,\mu}^{-\rho[n]} \xi_{n}^{-\rho[n]} \right). \quad (5.4)
\]

For the one cell model, \( \rho[n] = (-)^{n_r} \rho \) is chosen so that the fermion in \( \xi_{n}^{\rho[n]} \) stays inside the cell. So the fermions in the second term of (5.4) are in the neighboring cell. A new set of \( \tilde{\alpha} \)-parameters are introduced accordingly.
Since a fermion variable is associated with two neighboring cells, a natural extension of eq. (3.5) is to include plaquette variables of those cells. We modify the transformation accordingly,

\[ \delta \psi_n = \sum_{0<\mu<\nu} \left[ C_n^{(\mu\nu)[r]} (U_{n,(\mu\nu)[r]} - U_{n,(\nu\mu)[r]}) \right]_{n=N+r} 
+ \sum_{0<\mu<\nu} \left[ C_{n'}^{(\mu\nu)[r']} (U_{n',(\mu\nu)[r']} - U_{n',(\nu\mu)[r']}) \right]_{n'=N'+r'} . \tag{5.5} \]

6. preSUSY invariance of interacting cell model

We take the variation of the action given in (5.2) and (5.3) under the modified preSUSY transformation. From now on, in writing down the cancellation conditions, let us use \( n = N+r \) and \( n' = N'+r' \) to represent the coordinates in the E-cell and O-cell, respectively.

Consider the fermion cubed terms in \( \delta S \), or \( \delta_U S_f \). Even in the interacting cell models, the terms have the same graphical representation as shown in Fig. 3. Imagine to draw the diagram on the pattern in Fig. 2. We may put it within a single cell or over a pair of neighboring cells. When it is drawn within a single cell, the corresponding vanishing condition is the same as the one cell model. The condition is on the \( \alpha \)-parameters in the first term of eq. (5.4).

\[ (-)^{r_\nu} \frac{\alpha_{n,\mu}^{\nu[r]}}{b_\nu(n)} + (-)^{r_\mu} \frac{\alpha_{n,\nu}^{\mu[r]}}{b_\mu(n)} = 0, \]
\[ (-)^{r_{n'}^{\nu'}} \frac{\alpha_{n',\mu}^{\nu'[r']}}{b_\nu(n')} + (-)^{r_{n'}^{\mu'}} \frac{\alpha_{n',\nu}^{\mu'[r']}}{b_\mu(n')} = 0. \tag{6.1} \]

When we put the diagram in Fig. 1 over a pair of neighboring cells, we find relations relating newly introduced \( \tilde{\alpha} \)-parameters.

\[ (-)^{r_{n'}^{\nu}} b_\mu(n') \tilde{\alpha}_{n',\mu}^{-\nu[r']} + (-)^{r_{n'}^{\mu'}} b_\nu(n') \tilde{\alpha}_{n',\nu}^{-\mu[r']} = 0. \tag{6.2} \]

Eq. (6.2) relates the \( \tilde{\alpha} \)-parameters from two neighboring cells and reduces the independent parameters to the half.

As for the fermion linear condition, we again have the same conditions as the one cell model,

\[ C_n^{(\mu\nu)[r]} = \beta (-)^{r_\nu} \frac{\alpha_{n,\mu}^{\nu[r]}}{b_\nu(n)} = -\beta (-)^{r_\mu} \frac{\alpha_{n,\nu}^{\mu[r]}}{b_\mu(n)}, \]
\[ C_{n'}^{(\mu\nu)[r']} = \beta (-)^{r_{n'}^{\nu'}} \frac{\alpha_{n',\mu}^{\nu'[r']}}{b_\nu(n')} = -\beta (-)^{r_{n'}^{\mu'}} \frac{\alpha_{n',\nu}^{\mu'[r']}}{b_\mu(n')} . \tag{6.3} \]

The remaining conditions from \( \delta_U S_f + \delta_\psi S_f = 0 \) relate the \( C \)-parameters to the new \( \tilde{\alpha} \)-parameters

\[ b_\mu(n) C_n^{(\mu\nu)[r']} = \beta \left( (-)^{r_{n'}^{\nu'}} \tilde{\alpha}_{n',\mu}^{-\rho[r']} - (-)^{r_{n'}^{\mu'}} \tilde{\alpha}_{n',\nu}^{-\rho[r']} \right), \tag{6.4} \]
\[ b_\nu(n') C_n^{(\mu\nu)[r]} = \beta \left( (-)^{r_{n'}^{\nu}} \tilde{\alpha}_{n,\mu}^{-\rho[r]} - (-)^{r_{n'}^{\mu}} \tilde{\alpha}_{n,\nu}^{-\rho[r]} \right). \tag{6.5} \]
In the above we wrote two almost the same equations. They differ only in the indices. Recall that the indices with a prime is associated with the O-cell, those without a prime is for the neighboring E-cell.

Now let us count the number of independent parameters for the interacting cell model. We may rewrite eq. (6.4) for \( \tilde{\alpha}^{\rho}[r] \). Thus the parameter with different lower index \( \mu \) are related each other. From (6.4) we see the same holds for \( \tilde{\alpha}_{n,\mu}^{\rho} \). Taking into account of the fact that the \( \alpha \)-parameters for the neighboring cells are related by (6.2), the above stated observations imply that there is only one independent \( \tilde{\alpha} \)-parameter at the site, say \( \tilde{\alpha}_{n,1}^{\rho} \).

Including the consideration for the one cell model, we conclude that there are \( 2D - 1 \) independent parameters at each site: \( D - 1 \) \( C \)-parameters from two neighboring cells and plus one \( \tilde{\alpha} \)-parameter just stated. Let us write the rest of parameters in terms of the independent parameters, \( C_n^{(\mu_1)[r]}, C_{n'}^{(\mu_1)[r']} \) and \( \tilde{\alpha}_{n,1}^{\rho} \),

\[
\begin{align*}
C_{n}^{(\mu)[r]} &= C_{n}^{(\mu_1)[r]} - C_{n}^{(\nu_1)[r]}, \\
C_{n'}^{(\mu)[r']} &= C_{n'}^{(\mu_1)[r']} - C_{n'}^{(\nu_1)[r']}, \\
\alpha^{(\rho)[r]}_{n,\mu} &= (-)^{r_\mu} \frac{b_\nu(n)}{\beta} \left(C_{n}^{(\mu_1)[r]} - C_{n}^{(\nu_1)[r]}\right), \\
\alpha^{(\rho)[r']}_{n',\mu} &= (-)^{r'_{\mu}} \frac{b_\nu(n')}{\beta} \left(C_{n'}^{(\mu_1)[r']} - C_{n'}^{(\nu_1)[r']}\right), \\
\tilde{\alpha}^{-\rho}[r] &= \frac{b_\nu(n')}{b_1(n)} (-)^{r'_{\mu}} \tilde{\alpha}^{-\rho}_{n',1} + \frac{b_\nu(n)}{b_1(n)} (-)^{r_{\mu}} \tilde{\alpha}^{-\rho}_{n,1} + (-)^{r_{\mu}} \frac{b_\nu(n)}{\beta} \left(C_{n}^{(\mu_1)[r]} - C_{n}^{(\nu_1)[r]}\right), \\
\tilde{\alpha}^{-\rho}[r'] &= \frac{b_\nu(n)}{b_1(n)} (-)^{r_{\mu}} \tilde{\alpha}^{-\rho}_{n',1} + \frac{b_\nu(n')}{b_1(n')} (-)^{r'_{\mu}} \tilde{\alpha}^{-\rho}_{n,1} + (-)^{r'_{\mu}} \frac{b_\nu(n')}{\beta} \left(C_{n'}^{(\mu_1)[r']} - C_{n'}^{(\nu_1)[r']}\right).
\end{align*}
\]

A comment is in order on the invariance of the path integral measure. It is easy to realize that the problem reduces to that for the one cell model. Note that the new terms in the transformation in eqs. (5.4) relate the variables in different cells. So there is no new contribution to the Jacobian beyond the one cell model.

7. Summary and Discussion

In this paper we presented a lattice model with an exact fermionic symmetry. By requiring the invariance of the action, we derived relations among the coefficients in the action and the parameters in the preSUSY transformation. We found a finite number of independent transformation parameters on each site by solving the relations. The path integral measure is found to be invariant if these relations are satisfied. Note that, in solving the relations, no assumption was made for the coefficients in the action. In particular we did not need an expression of \( b_\mu(n) \) explicitly. Of course, we understand that it is non-zero and satisfies the restriction for the reality of the action.

By using the preSUSY invariance, we may derive the corresponding Ward-Takahashi identities, exactly. For example, a plaquette-plaquette correlation is equal to some fermion-fermion correlations.

Obviously there are open questions in our formalism: 1) the relation between the continuum SUSY and the local fermionic symmetry reported in this paper; 2) the recovery
of the spinor structure for the real staggered fermion $\psi_n$; 3) we also need some explanation of the peculiar lattice structure. Although we have not reached the complete solution to all of these problems, let us emphasize that it is this lattice structure which allows us to have the fermionic symmetry. This is very important. Because, a naive attempt to construct a model over the whole lattice often encounters the situation that a fermionic symmetry tends to an $O(a)$ symmetry in a naive continuum limit. We have avoided this uninteresting situation owing to the lattice structure chosen in this paper. The details on this point will be reported in our forthcoming paper [1].

As the reader may have noticed, the interaction in our model goes off when we remove the fermions. In the paper [1], we extend our consideration in the present paper and show that another model may be constructed naturally which does not have this peculiar nature.

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