What is integrability of discrete variational systems?

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Based on:

1. Yu.B. Suris. *Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms.* arXiv:1212.3314 [math-ph].

2. R. Boll, M. Petrera, Yu.B. Suris. *What is integrability of discrete variational systems?* arXiv:1307.0523 [math-ph].
Origin of the theory (1)

- S. Lobb, F.W. Nijhoff. *Lagrangian multiforms and multidimensional consistency*, J. Phys. A: Math. Theor. 42 (2009) 454013: closedness of Lagrangian 2-form on solutions of systems of quad-equations (for a part of ABS list).
- Proved by Bobenko–Suris for the whole ABS list, generalized by Boll–Suris for asymmetric systems of quad-equations.
- Subsequent work by Lobb, Nijhoff et al. on:
  - multi-field 2D systems;
  - dKP, the fundamental 3D discrete integrable system;
  - Calogero-Moser, an important 1D integrable system.
■ Theory of pluriharmonic functions: $f : \mathbb{C}^m \rightarrow \mathbb{R}$ minimizes the Dirichlet functional $E_\Gamma = \int_\Gamma |(f \circ \Gamma)_z|^2 dz \wedge d\bar{z}$ along any holomorphic curve in its domain $\Gamma : \mathbb{C} \rightarrow \mathbb{C}^m$.

■ Baxter’s $Z$-invariance of solvable models of statistical mechanics: invariance of the partition function under elementary local transformations of the underlying graph. In quasi-classical limit, partition function turns into the action functional (made mathematically precise for a number of models by Bazhanov-Mangazeev-Sergeev).

■ Variational symmetries (classical work by E. Noether).
Let $\mathcal{L}$ be a discrete $d$-form, depending on some field $x: \mathbb{Z}^m \to \mathcal{X}$.

To an arbitrary oriented $d$-dimensional manifold $\Sigma$ in $\mathbb{Z}^m$, there corresponds the action functional, which assigns to $x|_{V(\Sigma)}$ the number $S_\Sigma = \int_\Sigma \mathcal{L}$.

The field $x: V(\Sigma) \to \mathcal{X}$ is a critical point of $S_\Sigma$, if at any interior point $n \in V(\Sigma)$, we have $\partial S_\Sigma / \partial x(n) = 0$ (discrete Euler-Lagrange equations for the action $S_\Sigma$).

The field $x: \mathbb{Z}^m \to \mathcal{X}$ solves the pluri-Lagrangian problem for the Lagrangian 2-form $\mathcal{L}$ if, for any $d$-dimensional manifold $\Sigma$ in $\mathbb{Z}^m$, the restriction $x|_{V(\Sigma)}$ is a critical point of the corresponding action $S_\Sigma$. 
Theorem. Function $x : \mathbb{R}^m \to \mathcal{X}$ solves the pluri-Lagrangian problem for

$$\mathcal{L}(x, x_{t_1}, \ldots, x_{t_m}) = \sum_{\alpha=1}^{m} L_{\alpha}(x, x_{t_1}, \ldots, x_{t_m}) dt_{\alpha},$$

iff the following multi-time Euler-Lagrange equations are satisfied:

▸ $\frac{\partial L_{\alpha}}{\partial v_{\beta}}(x, x_{t_1}, \ldots, x_{t_m}) = 0$ for $1 \leq \alpha \neq \beta \leq m$;

▸ $\frac{\partial L_1}{\partial v_1}(x, x_{t_1}, \ldots, x_{t_m}) = \ldots = \frac{\partial L_m}{\partial v_m}(x, x_{t_1}, \ldots, x_{t_m}) =: p$.

▸ $\frac{\partial p}{\partial t_{\alpha}} = \frac{\partial L_{\alpha}}{\partial x}(x, x_{t_1}, \ldots, x_{t_m})$. 

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Results: continuous time, $d = 1$ (continued)

**Definition.** Lagrangian 1-form $\mathcal{L}$ is *Legendre transformable*, if the first two sets of the above EL equations can be solved for $x_{t\alpha} = x_{t\alpha}(x, p)$ ($\alpha = 1, \ldots, m$).

**Theorem.** For a Legendre transformable $\mathcal{L}$, set

$$H_\alpha(x, p) = \langle p, x_{t\alpha} \rangle - L_\alpha(x, x_{t1}, \ldots, x_{tm}), \quad \alpha = 1, \ldots, m.$$  

Then $x, p : \mathbb{R}^m \to X$ satisfy Hamiltonian equations of motion:

$$\frac{\partial x}{\partial t_{\alpha}} = \frac{\partial H_\alpha}{\partial p}, \quad \frac{\partial p}{\partial t_{\alpha}} = -\frac{\partial H_\alpha}{\partial x}, \quad \alpha = 1, \ldots, m.$$  

These Hamiltonian flows *commute*, so that the pairwise Poisson brackets are constant:

$$\{H_\alpha, H_\beta\} = h_{\alpha\beta} = \text{const}.$$
Theorem. Conversely, for any $m$ commuting Hamiltonian flows, one can find a pluri-Lagrangian 1-form $L$. In particular, if $\det(\frac{\partial^2 H_1}{\partial p^2}) \neq 0$, one can set:

$$L_1(x, x_{t_1}) = \langle p, x_{t_1} \rangle - H_1(x, p),$$

$$L_\alpha(x, x_{t_1}, x_{t_\alpha}) = \langle p, x_{t_\alpha} \rangle - H_\alpha(x, p), \quad \alpha = 2, \ldots, m$$

(with $p$ on r.h.s. expressed through $x, x_{t_1}$).

Theorem. On solutions of EL equations, we have:

$$\frac{\partial}{\partial t_\alpha} L_\beta(x, x_{t_1}, \ldots, x_{t_m}) - \frac{\partial}{\partial t_\beta} L_\alpha(x, x_{t_1}, \ldots, x_{t_m}) = h_{\alpha\beta}.$$

In particular, if $H_\alpha$ are in involution, then $L$ is closed on solutions of EL equations.
Theorem. Function $x : \mathbb{Z}^m \to X$ solves the pluri-Lagrangian problem for $\mathcal{L}(n, n + e_\alpha) = \Lambda_\alpha(x, x_\alpha)$, iff the following multi-time discrete EL equations are satisfied:

$$\frac{\partial \Lambda_\alpha(x_{-\alpha}, x)}{\partial x} + \frac{\partial \Lambda_\beta(x, x_\beta)}{\partial x} = 0,$$

$$\frac{\partial \Lambda_\alpha(x_{-\alpha}, x)}{\partial x} - \frac{\partial \Lambda_\beta(x_{-\beta}, x)}{\partial x} = 0,$$

$$\frac{\partial \Lambda_\alpha(x, x_\alpha)}{\partial x} - \frac{\partial \Lambda_\beta(x, x_\beta)}{\partial x} = 0.$$
In other words, if there exists \( p : \mathbb{Z}^m \rightarrow X \) satisfying

\[
p = \frac{\partial \Lambda_\alpha(x, x_\alpha)}{\partial x} = -\frac{\partial \Lambda_\beta(x_{-\beta}, x)}{\partial x}, \quad \alpha, \beta = 1, \ldots, m.
\]

**Theorem.** Discrete EL equations are consistent, iff symplectic maps \( F_\alpha : (x, p) \mapsto (x_\alpha, p_\alpha) \) defined by

\[
p = \frac{\partial \Lambda_\alpha(x, x_\alpha)}{\partial x}, \quad p_\alpha = -\frac{\partial \Lambda_\alpha(x, x_\alpha)}{\partial x_\alpha},
\]

commute, \( F_\alpha \circ F_\beta = F_\beta \circ F_\alpha \).
Theorem. On solutions of discrete EL equations, we have:

\[\Lambda_\alpha(x, x_\alpha) + \Lambda_\beta(x_\alpha, x_{\alpha\beta}) - \Lambda_\alpha(x_\beta, x_{\alpha\beta}) - \Lambda_\beta(x, x_\beta) = \ell_{\alpha\beta} = \text{const.}\]

In particular, if all \(\ell_{\alpha\beta} = 0\) then \(L\) is closed on solutions of discrete EL equations.

Theorem. For a one-parameter family of commuting symplectic maps (Bäcklund transformations)

\[F_\lambda: \quad p = \frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial x}, \quad \tilde{p} = -\frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial \tilde{x}},\]

\(L\) is closed on solutions of discrete EL equations iff

\[\frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial \lambda}\]

is a common integral of motion for all \(F_\mu\) (spectrality property of Kuznetsov–Sklyanin).
Problem 1. Do discrete Laplace-type systems exhibit some multi-dimensionally consistency?

For instance, for discrete time relativistic Toda-type systems

\[ g(\tilde{x}_k - x_k) - g(x_k - \tilde{x}_k) = f(x_{k+1} - x_k) - f(x_k - x_{k-1}) \]
\[ + h(x_{k+1} - x_k) - h(x_k - \tilde{x}_{k-1}). \]
Problem 2. What are the natural conditions for validity of the closure relation for action in multi-dimensions?

Lobb–Nijhoff proved closedness of $\mathcal{L}$ on solutions of quad-equations. However, quad-equations are non-variational (action provides a weak Lagrangian formulation only). Would be appealing to prove closedness of $\mathcal{L}$ on solutions of discrete EL equations on arbitrary surface $\Sigma$. How to handle them?
3D-corners

(a) 3D-corner at $n$

(b) 3D-corner at $n + e_i$

(c) 3D-corner at $n + e_i + e_j$

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**Lemma.** The flower of any interior vertex of an oriented quad-surface in $\mathbb{Z}^m$ can be represented as a sum of (oriented) 3D-corners in $\mathbb{Z}^{m+1}$.

- **(a) A flower in $\Sigma$**
- **(b) Extension of the flower into an additional coordinate direction**

**Corollary.** Any discrete EL equation for any surface is a sum of discrete EL equations for 3D-corners.
**Definition.** For a given discrete 2-form $\mathcal{L}$, the *system of corner equations* consists of discrete EL equations for all possible 3D-corners in $\mathbb{Z}^m$. If we denote

$$S^{ijk} = d\mathcal{L}(\sigma_{ijk}) = \Delta_k \mathcal{L}(\sigma_{ij}) + \Delta_i \mathcal{L}(\sigma_{jk}) + \Delta_j \mathcal{L}(\sigma_{ki}),$$

then system of corner equations consists of eight equations

$$\frac{\partial S^{ijk}}{\partial x} = 0, \quad \frac{\partial S^{ijk}}{\partial x_i} = 0, \quad \frac{\partial S^{ijk}}{\partial x_j} = 0, \quad \frac{\partial S^{ijk}}{\partial x_k} = 0,$$

$$\frac{\partial S^{ijk}}{\partial x_{ij}} = 0, \quad \frac{\partial S^{ijk}}{\partial x_{jk}} = 0, \quad \frac{\partial S^{ijk}}{\partial x_{ik}} = 0, \quad \frac{\partial S^{ijk}}{\partial x_{ijk}} = 0$$

for each cube $\sigma_{ijk}$. Symbolically: $\delta(d\mathcal{L}) = 0$, where $\delta$ stands for the “vertical” differential.
Thus, corner equations encompass \textit{all} discrete EL equations for \textit{any} surface $\Sigma$.

**Definition.** System of corner equations is called \textit{consistent}, if, for any cube $\sigma_{ijk}$, it has the minimal possible rank 2, i.e., if exactly two of these eight equations are independent.

**Theorem.** For any triple $i, j, k$, action $S^{ijk}$ over an elementary cube is constant on solutions of corner equations:

$$S^{ijk}(x, \ldots, x_{ijk}) = c^{ijk} = \text{const}$$

$$\text{(mod } \partial S^{ijk}/\partial x = 0, \ldots, \partial S^{ijk}/\partial x_{ijk} = 0\text{).}$$

Most interesting case: all $c^{ijk} = 0$, i.e., $\mathcal{L}$ closed on solutions of corner equations. (Why most interesting?)
Particular case: 3-point 2-form

For ABS equations:

\[ \mathcal{L}(\sigma_{ij}) = \mathcal{L}(x, x_i, x_j; \alpha_i, \alpha_j) = \mathcal{L}(x, x_i; \alpha_i) - \mathcal{L}(x, x_j; \alpha_j) - \Lambda(x_i, x_j; \alpha_i, \alpha_j). \]

For such \( \mathcal{L} \), action \( S^{ijk} \) depends neither on \( x \) nor on \( x_{ijk} \):

![Diagram showing a 3-point 2-form configuration with points labeled as \( x, x_i, x_j, x_{ij}, x_{ik}, x_{jk}, x_{ijk} \).]
Corner equations for a 3-point 2-form

**four-leg, five-point equations:**

\[
\psi(x_i, x_{ij}; \alpha_j) - \psi(x_i, x_{ik}; \alpha_k) - \phi(x_i, x_{k}; \alpha_i, \alpha_k) + \phi(x_i, x_{j}; \alpha_i, \alpha_j) = 0,
\]

\[
\psi(x_{ij}, x_i; \alpha_j) - \psi(x_{ij}, x_j; \alpha_i) - \phi(x_{ij}, x_{ik}; \alpha_j, \alpha_k) + \phi(x_{ij}, x_{jk}; \alpha_i, \alpha_k) = 0.
\]

Here, we introduced the notation

\[
\psi(x, y; \alpha) = \frac{\partial L(x, y; \alpha)}{\partial x}, \quad \phi(x, y; \alpha, \beta) = \frac{\partial \Lambda(x, y; \alpha, \beta)}{\partial x}.
\]

(a) 4-leg corner equation at \(x_i\)

(b) 4-leg corner equation at \(x_{ij}\)

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What is integrability of discrete variational systems?
From corner equations to planar Laplace type equations

Figure 7: Sum of four corner equations (to be matched at $x$) results in a planar seven-point equation of the relativistic Toda type.

To phrase this on the more concrete level of equations, we observe that each of the corner equations ($E_i$, $E_{ij}$) is a difference of the corresponding three-leg forms of two quad-equations. For instance, equation ($E_i$) is a difference of

$$
\psi(x_i, x_{ij}; \alpha_j) + \phi(x_i, x_j; \alpha_i, \alpha_j) - \psi(x_i, x; \alpha_i) = 0,
$$

(8)

$$
\psi(x_i, x_{ik}; \alpha_k) + \phi(x_i, x_k; \alpha_i, \alpha_k) - \psi(x_i, x; \alpha_i) = 0,
$$

(9)

which are three-leg forms, centered at $x_i$, of the quad-equations on the elementary squares $(x, x_i, x_{ij}, x_j)$, resp. $(x, x_i, x_{ik}, x_k)$. Now we do not require that these three-leg equations hold, but only that their four-leg difference is satisfied.

Theorem 7.

For the discrete 2-forms $L$ given in [18, 12] for multi-dimensionally consistent quad-equations of the ABS list, the corresponding systems of corner equations are consistent, as well. Moreover, the 2-form $L$ is closed on solutions of the corner equations.

Proof.

Consider initial data at four out of six vertices of an elementary 3D cube with one or two indices. For the sake of concreteness, let them be the values $x_i$, $x_j$, $x_{ij}$, $x_{ik}$.

Find the other two by using two of the corner equations. In our example these should be equations ($E_i$) delivering $x_k$ and ($E_{ij}$) delivering $x_{jk}$. Now we define an auxiliary field $x$ by requiring that the quad-equation on the face $(x, x_i, x_{ij}, x_j)$ be fulfilled. In other words, equation (8) is used to define $x$.

Note that this value $x$ is "alien" in the sense that it has nothing to do neither with the solution of the system of corner equations (which does not contain the corner equation at the vertex corresponding to the 3D cube under consideration) nor with the solution of discrete Euler-Lagrange equations for any 2D surface, even if it contains the vertex $x$. This "alien" value of $x$ satisfies also the quad-equation on the face $(x, x_i, x_{ik}, x_k)$ in its three-leg form (9), as it follows from the

...
Consistency of corner equations in case of a 3-point 2-form $\mathcal{L}$: six equations per cube, precisely two of them independent (minimal possible rank 2).

**Theorem.** For the discrete 2-forms $\mathcal{L}$ for quad-equations of the ABS list, the corresponding systems of corner equations are consistent, as well. Moreover, the 2-form $\mathcal{L}$ is closed on solutions of the corner equations.
Theorem. For the discrete 2-forms $\mathcal{L}$ for quad-equations of the types $Q1$, $Q3_{\delta=0}$, $H1$, $H2$, $H3$, solutions of the corresponding systems of corner equations on an elementary 3D cube satisfy the respective octahedron relations.

Moreover, all six corner equations within a cube are equivalent modulo the octahedron relation:

- if any two of the corner equations are satisfied, then the respective octahedron relation is satisfied, as well;
- if any one of the corner equations and the octahedron relation are satisfied, then all other five corner equations are satisfied, as well.
Examples of octahedron relations

Q1$_{\delta=0}$:

\[\frac{(x_{ij} - x_i)(x_{jk} - x_j)(x_{ki} - x_k)}{(x_{ij} - x_j)(x_{jk} - x_k)(x_{ki} - x_i)} = 1.\]

Q1$_{\delta=1}$:

\[\frac{(x_{ij} - x_i + \alpha_j)(x_{jk} - x_j + \alpha_k)(x_{ki} - x_k + \alpha_i)}{(x_{ij} - x_j + \alpha_i)(x_{jk} - x_k + \alpha_j)(x_{ki} - x_i + \alpha_k)} = 1.\]

H1:

\[x_{ij}(x_i - x_j) + x_{jk}(x_j - x_k) + x_{ki}(x_k - x_i) = 0.\]

H3:

\[\alpha_i x_j x_{ij} - \alpha_j x_i x_{ij} + \alpha_j x_k x_{jk} - \alpha_k x_j x_{jk} + \alpha_k x_i x_{ki} - \alpha_i x_k x_{ki} = 0.\]
Consider a 3-point 2-form $\mathcal{L}$ with the functions $L$ and $\Lambda$ defined by
\[
\frac{\partial}{\partial x} L(x, y; \alpha) = \log (\alpha - e^{y-x}) = -\frac{\partial}{\partial y} L(x, y; \alpha),
\]
\[
\frac{\partial}{\partial x} \Lambda(x, y; \alpha, \beta) = \log \frac{\alpha - \beta e^{y-x}}{\beta - \alpha e^{y-x}} = -\frac{\partial}{\partial y} \Lambda(x, y; \alpha, \beta).
\]

Notice the corresponding (asymmetric) discrete Toda equation:
\[
\frac{e^{x_{k+1}-x_k} - \alpha}{e^{x_k-x_{k-1}} - \alpha} = \frac{e^{x_k+1-x_k} - \beta}{e^{x_k-x_{k-1}} - \beta} \cdot \frac{\beta e^{x_k+1-x_k} - \alpha}{\alpha e^{x_k+1-x_k} - \beta} \cdot \frac{\alpha e^{x_k-x_{k-1}} - \beta}{\beta e^{x_k-x_{k-1}} - \alpha}.
\]
Theorem.

- The system of corner equations for $L$ is consistent. Any two of the corner equations within one elementary 3D cube are equivalent modulo *octahedron relation* ($X = e^X$)

$$\frac{\alpha_j X_{ij} - \alpha_k X_{ik}}{X_i} + \frac{\alpha_k X_{jk} - \alpha_i X_{ij}}{X_j} + \frac{\alpha_i X_{ik} - \alpha_j X_{jk}}{X_k} = 0.$$ 

- 2-form $L$ is closed on solutions of the system of corner equations.

- Corner equations for $L$ cannot be represented as differences of the three-leg forms of multi-affine quad-equations.
Conclusions

- What is integrability of discrete (and continuous) variational systems? Pluri-Lagrangian property!
- (Almost) closedness of the Lagrangian form ($d\mathcal{L} = \text{const}$) on solutions of the pluri-Lagrangian system built-in!
- Relation of closedness of the Lagrangian form on solutions to more traditional notions of integrability established for $d = 1$. For $d \geq 2$ highly desirable!
- Classification of pluri-Lagrangian systems looks promising!
- We discovered corner equations in the study of superposition formulas for discrete Toda-type systems. See R. Boll, M. Petrera, Yu.B. Suris. *Multi-time Lagrangian 1-forms for families of Bäcklund transformations. Toda-type systems*. J. Phys. A: Math. Theor. **46** (2013) 275204, as well as the poster by R. Boll!