EISENSTEIN CLASSES, ELLIPTIC SOULÉ ELEMENTS
AND THE \(\ell\)-ADIC ELLIPTIC POLYLOGARITHM

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Abstract. In this paper we study systematically the \(\ell\)-adic realization of the elliptic polylogarithm in the context of sheaves of Iwasawa modules. This leads to a description of the elliptic polylogarithm in terms of elliptic units. As an application we prove a precise relation between \(\ell\)-adic Eisenstein classes and elliptic Soulé elements. This allows to give a new proof of the formula for the residue of the \(\ell\)-adic Eisenstein classes at the cusps and the formula for the cup-product construction in [HK99], which relies only on the explicit description of elliptic units. This computation is the main input in the proof of Bloch-Kato’s compatibility conjecture 6.2, needed in the proof of Tamagawa number conjecture for the Riemann zeta function.

Introduction

The purpose of this paper is twofold: on the one hand we prove a new and precise relation between \(\ell\)-adic Eisenstein classes and elliptic Soulé elements using a description of the integral \(\ell\)-adic elliptic polylogarithm in terms of elliptic units. On the other hand this relation will be used to give a new proof for the cup-product construction formula, which is the main result of [HK99] and is the main input in [Hub] to obtain a proof of Bloch-Kato’s compatibility conjecture 6.2. This new proof uses only elementary properties of elliptic units.

The explicit description of the integral \(\ell\)-adic elliptic polylogarithm in terms of elliptic units was already one of the main results in the paper [Kin01]. There we used an approach via one-motives to treat the logarithm sheaf. But the main application of the \(\ell\)-adic elliptic polylogarithm is in the context of Iwasawa theory, which makes it desirable to approach the elliptic polylogarithm systematically in this context. That such an approach is possible, is already suggested in the ground-braking paper [BL94].

In Iwasawa theory Kato, Perrin-Riou and Colmez pointed out the usefulness to work with “Iwasawa cohomology”, which is continuous Galois cohomology with values in an Iwasawa algebra. We generalize this idea to treat families of Iwasawa modules under a family of Iwasawa algebras. The main example for this is the family of Iwasawa algebras on the moduli scheme of elliptic curves, where one has in each fibre the Iwasawa algebra of the Tate module of the corresponding elliptic curve.
It is the fundamental idea of Soulé [Sou81] that twisting of units can be used to produce interesting cohomology classes. Already in Kato’s paper [Kat93] it is implicit that this twisting is related to the Iwasawa cohomology. Later Colmez used this explicitly in [Col98], where he used moment maps of $\mathbb{Q}_\ell$-measure algebras. For our approach it is crucial to develop this further by constructing the moment map at finite level. We show that in the cyclotomic case one obtains the elements defined and studied by Soulé and Deligne. Work by Soulé in the CM elliptic case and Kato’s work in [Kat04] suggest that one should carry out Soulé’s twisting construction also in the modular curve case to obtain elliptic Soulé elements. One of the main results in this paper is that these elliptic Soulé elements are essentially the $\ell$-adic Eisenstein classes in [HK99].

With the general theory of sheaves of Iwasawa modules, we obtain a concrete description of the elliptic polylogarithm in terms of the norm compatible elliptic units defined and studied by Kato [Kat04]. This gives strong ties of the elliptic polylogarithm to recent developments in Iwasawa theory and also allows many explicit computations with the $\ell$-adic elliptic polylogarithm.

As an application of the concrete description of the elliptic polylogarithm, we give a new proof of the residue computation for $\ell$-adic Eisenstein classes on the moduli scheme for elliptic curves (Corollary 5.2.3).

A second application is the evaluation of the cup-product construction used in [HK99] (and explained in this volume in [Hub]) to obtain elements in the motivic cohomology of cyclotomic fields and to prove Conjecture 6.2 in [BK90]. The approach taken here, does not need any computations of the cyclotomic polylogarithm as in [HK99]. It relies only on the concrete evaluation of the elliptic units at the cusps.

An overview of the main results in this paper is given in Section 1.

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Notations

We fix an integer \( N \geq 3 \) and let \( Y(N) \) the moduli space of elliptic curves \( \mathcal{E} \) with a full level \( N \)-structure \( \alpha : (\mathbb{Z}/N\mathbb{Z})^2 \cong \mathcal{E}[N] \). We denote by

\[ \pi : \mathcal{E} \rightarrow Y(N) \]

the universal elliptic curve. We let \( X(N) \) be the smooth compactification of \( Y(N) \) and denote by \( j : Y(N) \hookrightarrow X(N) \) the open immersion. If we fix an \( N \)-th root of unity \( \zeta_N := e^{2\pi i/N} \in \mathbb{C} \) and consider the Tate curve \( \mathcal{E}_q \) with the level structure \( \alpha : (\mathbb{Z}/N\mathbb{Z})^2 \rightarrow \mathcal{E}_q[N] \) given by \( (a, b) \mapsto q^a \zeta_N^b \). This induces a map of schemes

\[ \text{Spec} \mathbb{Q}(\zeta_N)((q^{1/N})) \rightarrow Y(N), \]
which extends to Spec\(Q(\zeta_N)[[q^{1/N}]] \to X(N)\) and a hence a map \(\infty : \text{Spec}Q(\zeta_N) \to X(N)\), whose image we call the cusp \(\infty\).

Define étale sheaves on \(Y(N)\) by

\[
\mathcal{H}_r := (R^1\pi_*\mathbb{Z}/\ell^r\mathbb{Z})^\vee \cong R^1\pi_*\mathbb{Z}/\ell^r\mathbb{Z}(1)
\]

\[
\mathcal{H} := (R^1\pi_*\mathbb{Z}_\ell)^\vee \cong R^1\pi_*\mathbb{Z}_\ell(1)
\]

\[
\mathcal{H}_{\mathbb{Q}_\ell} := (R^1\pi_*\mathbb{Q}_\ell)^\vee \cong R^1\pi_*\mathbb{Q}_\ell(1)
\]

where \((\cdot)^\vee\) denotes the \(\mathbb{Z}/\ell^r\mathbb{Z}\), \(\mathbb{Z}_\ell\) and \(\mathbb{Q}_\ell\) dual respectively. We denote by

\[
\text{Sym}_k \mathcal{H}_r, \text{Sym}_k \mathcal{H} \text{ and Sym}_k \mathcal{H}_{\mathbb{Q}_\ell}\]

the \(k\)-th symmetric power as \(\mathbb{Z}/\ell^r\mathbb{Z}\)-, \(\mathbb{Z}_\ell\)- and \(\mathbb{Q}_\ell\)-modules respectively. In the same way we denote by T\text{Sym}_k \mathcal{H}_r, T\text{Sym}_k \mathcal{H} \text{ and TSym}_k \mathcal{H}_{\mathbb{Q}_\ell}\) the functor of symmetric \(k\)-tensors as \(\mathbb{Z}/\ell^r\mathbb{Z}\)-, \(\mathbb{Z}_\ell\)- and \(\mathbb{Q}_\ell\)-modules respectively. Note that there is a canonical map

\[
\text{Sym}_k \mathcal{H}_r \to \text{TSym}_k \mathcal{H}_r,
\]

which extends to a homomorphism of graded algebras \(\text{Sym}_\cdot \mathcal{H}_r \to \text{TSym}_\cdot \mathcal{H}_r\) and similarly for \(\mathcal{H} \) and \(\mathcal{H}_{\mathbb{Q}_\ell}\).

As we will not only deal with \(\ell\)-adic sheaves, we work in the bigger abelian category of inverse systems \(\mathcal{F} = (\mathcal{F}_r)_{r \geq 1}\) of étale sheaves modulo Mittag-Leffler-zero systems (which means to work in the pro-category) and define the continuous étale cohomology in the sense of [Jan88]. This means that \(H^i(S, \mathcal{F})\) is the \(i\)-th derived functor of \(F \mapsto \lim_{\leftarrow r} H^0(S, \mathcal{F}_r)\).

More generally, one defines

\[
R^i\pi_* \mathcal{F}
\]

for a morphism \(\pi : S \to T\) to be the \(i\)-th derived functor of \(\mathcal{F} \mapsto \lim_{\leftarrow r} \pi_* \mathcal{F}_r\).

For \(\ell\)-adic sheaves, we also consider Ext-groups

\[
\text{Ext}^i_S(\mathcal{F}, \mathcal{G}),
\]

which are the right derived functors of \(\text{Hom}_S(\mathcal{F}, -)\).

Of crucial importance is the following lemma:

**Lemma 0.0.1.** Let \(\mathcal{F} = (\mathcal{F}_r)_{r \geq 1}\) be a projective system with \(H^0(S, \mathcal{F}_r)\) finite, then

\[
H^1(S, \mathcal{F}) = \lim_{\leftarrow r} H^1(S, \mathcal{F}_r).
\]

**Proof.** This follows from [Jan88 Lemma 1.15, Equation (3.1)] as the \(H^0(S, \mathcal{F}_r)\) satisfy the Mittag-Leffler condition. \(\square\)

The quotient category of the \(\ell\)-adic sheaves (or \(\mathbb{Z}_\ell\)-sheaves) by the torsion sheaves is the category of \(\mathbb{Q}_\ell\)-sheaves. In the case of projective systems of
\( \mathbb{Q}_\ell \)-sheaves \( \mathcal{F} = (\mathcal{F}_r)_{r \geq 0} \) we use the ad hoc definitions

\[
H^i(S, \mathcal{F}) := \lim_{\leftarrow \! r} H^i(S, \mathcal{F}_r)
\]

\[
\text{Ext}^i_S(\mathcal{G}, \mathcal{F}) := \lim_{\leftarrow \! r} \text{Ext}^i_S(\mathcal{G}, \mathcal{F}_r),
\]

where \( \mathcal{G} \) is just a \( \mathbb{Q}_\ell \)-sheaf.

1. **Statement of the main results**

For better orientation of the reader we give an overview of the main results in this paper and the strategy and main ingredients of the proof.

1.1. **The residue at \( \infty \) of the Eisenstein class.** We identify sections \( t : Y(N) \to \mathcal{E} \) with elements in \( (\mathbb{Z}/N\mathbb{Z})^2 \) via the universal level-\( N \)-structure on \( \mathcal{E} \). For any \( (0,0) \neq t \in (\mathbb{Z}/N\mathbb{Z})^2 \) one can define a so called Eisenstein class \( \text{Eis}^k_{\mathbb{Q}_\ell}(t) \in H^1(Y(N), \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell}(1)) \) (cf. Definition [4.2.2]). It is convenient to introduce the following notation: For any map \( \psi : (\mathbb{Z}/N\mathbb{Z})^2 \setminus \{(0,0)\} \to \mathbb{Q}_\ell \) we put

\[
\text{Eis}^k_{\mathbb{Q}_\ell}(\psi) := \sum_{t \neq e} \psi(t) \text{Eis}^k_{\mathbb{Q}_\ell}(t).
\]

It is shown in [Bla] that \( \text{Eis}^k_{\mathbb{Q}_\ell}(\psi) \) is in fact the image of a class in motivic cohomology under the regulator map. We are interested in the image of \( \text{Eis}^k_{\mathbb{Q}_\ell}(\psi) \) under the residue map

\[
\text{res}_\infty : H^1(Y(N), \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell}(1)) \to H^0(\infty, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell
\]

as defined in Definition [5.1.5]. The following result was first proved in [BL94] by a completely different method:

**Theorem 1.1.1** (See Corollary [5.2.3]). One has

\[
\text{res}_\infty(\text{Eis}^k_{\mathbb{Q}_\ell}(\psi)) = \frac{-N^k}{(k+2)k!} \sum_{(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 \setminus \{(0,0)\}} \psi(a,b) B_{k+2}(\{ a \over N \}),
\]

where \( B_{k+2} \) denotes the \( k + 2 \) Bernoulli polynomial and \( \{ a \over N \} \) is the representative in \( [0,1[ \) of \( a \over N \).

1.2. **Evaluation of the cup-product construction.** For two maps \( \phi, \psi : (\mathbb{Z}/N\mathbb{Z})^2 \setminus \{(0,0)\} \to \mathbb{Q}_\ell \) we can consider the cup-product

\[
\text{Eis}^k_{\mathbb{Q}_\ell}(\phi) \cup \text{Eis}^k_{\mathbb{Q}_\ell}(\psi) \in H^2(Y(N), \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell} \otimes \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell}(2)).
\]

The cup-product pairing \( \mathcal{H}_{\mathbb{Q}_\ell} \otimes \mathcal{H}_{\mathbb{Q}_\ell} \to \mathbb{Q}_\ell(1) \) induces a pairing

\[
\text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell} \otimes \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell} \to \mathbb{Q}_\ell(k)
\]
and we can consider the image of $\text{Eis}_k^{Q_\ell}(\psi) \cup \text{Eis}_k^{Q_\ell}(\phi)$ in $H^2(Y(N), \mathbb{Q}_\ell(k+2))$.

Let

$$\text{res}_\infty : H^2(Y(N), \mathbb{Q}_\ell(k+2)) \to H^1(\infty, \mathbb{Q}_\ell(k+1))$$

be the edge morphism in the Leray spectral sequence for $Rj_*$ using the isomorphism $\infty^* R^1 j_* \mathbb{Q}_\ell(k+2) \cong \mathbb{Q}_\ell(k+1)$.

**Definition 1.2.1.** Let $\phi_\infty, \psi : (\mathbb{Z}/N\mathbb{Z})^2 \setminus \{(0,0)\} \to \mathbb{Q}_\ell$ be two maps and suppose that $\text{res}_\infty(\text{Eis}_k^{Q_\ell}(\phi_\infty)) = 1$ and $\text{res}_\infty(\text{Eis}_k^{Q_\ell}(\psi)) = 0$. Then

$$\text{Dir}_\ell(\psi) := \text{res}_\infty(\text{Eis}_k^{Q_\ell}(\psi) \cup \text{Eis}_k^{Q_\ell}(\phi_\infty)) \in H^1(\infty, \mathbb{Q}_\ell(k+1))$$

is called the cup-product construction (compare [Hub] Definition 4.1.3.)

Note that $\text{Dir}_\ell(\psi)$ does not depend on the choice of $\phi_\infty$ (as follows from the formula in Theorem 6.1.1). The main result of this paper is:

**Theorem 1.2.2** (see Corollary 6.3.5). Let $\psi : (\mathbb{Z}/N\mathbb{Z})^2 \setminus \{(0,0)\} \to \mathbb{Q}_\ell$ be a map such that $\text{res}_\infty(\text{Eis}_k^{Q_\ell}(\psi)) = 0$. Then one has

$$\text{Dir}_\ell(\psi) = -\frac{1}{Nk!} \sum_{0 \neq b \in \mathbb{Z}/N\mathbb{Z}} \psi(0,b) \tilde{c}_{k+1}(\zeta_N^b) \in H^1(\infty, \mathbb{Q}_\ell(k+1)),$$

where $\tilde{c}_{k+1}(\zeta_N^b)$ is the modified cyclotomic Soulé-Deligne element from Definition 3.2.3.

It is explained in [Hub] how this theorem settles the compatibility conjecture 6.2. in [BK90].

The main idea in this paper (building upon our former work [Kin01]) is to describe a $\mathbb{Z}_\ell$-version of $\text{Eis}_k^{Q_\ell}(t)$ as Soulé’s twisting construction applied to elliptic units. Then all explicit computations with the Eisenstein classes are reduced to computations with the elliptic units.

1.3. Eigenshin classes and elliptic units. We explain how the Eisenstein classes are related to elliptic units and in particular how one can define these classes integrally. For this we need to introduce some notation.

Recall that the Eisenstein class is associated to a non-zero $N$-torsion section $t : Y(N) \to \mathcal{E}[N]$. Let $\ell$ be a prime number. We define the $\mathcal{E}[\ell']$-torsor $\mathcal{E}[\ell'](t)$ on the modular curve $Y(N)$ by the cartesian diagram

$$\begin{array}{ccc}
\mathcal{E}[\ell'](t) & \longrightarrow & \mathcal{E}[\ell'N] \\
pr_{\ell'} \downarrow & & \downarrow [\ell'] \\
Y(N) & \longrightarrow & \mathcal{E}[N].
\end{array}$$
Definition 1.3.1. Define the étale sheaf $\Lambda_r(\mathcal{H}_r\langle t \rangle)$ on $Y(N)$ by
\[
\Lambda_r(\mathcal{H}_r\langle t \rangle) := p_{r,t}^* \mathbb{Z}/\ell^r \mathbb{Z}.
\]
If $t = e$ is the identity section we write $\Lambda_r(\mathcal{H}_r)$.

The sheaves $\Lambda_r(\mathcal{H}_r\langle t \rangle)$ form an inverse system with respect to the trace map
\[
\Lambda_{r+1}(\mathcal{H}_{r+1}\langle t \rangle) \to \Lambda_r(\mathcal{H}_r\langle t \rangle)
\]
and we denote the resulting pro-system by
\[
\Lambda(\mathcal{H}\langle t \rangle) := (\Lambda_r(\mathcal{H}_r\langle t \rangle))_{r \geq 1}.
\]
For $t = e$ we write $\Lambda(\mathcal{H}) := (\Lambda_r(\mathcal{H}_r))_{r \geq 1}$.

Remark 1.3.2. The sheaves $\Lambda(\mathcal{H}\langle t \rangle)$ form the main example of sheaves of Iwasawa modules mentioned in the title of this paper. The connection is explained in Lemma 2.3.3.

Fix an auxiliary integer $c > 1$, which is prime to $6\ell N$. Then Kato has defined a norm-compatible unit $c\vartheta_E$ on $E \setminus E[c]$ (cf. Theorem 3.3.1). Note that for an $N$-torsion point $t \neq e$ one has $E[\ell^r]\langle t \rangle \subset E \setminus E[c]$ by our condition on $c$. Thus, we can restrict $c\vartheta_E$ to an invertible function on $E[\ell^r]\langle t \rangle$. The Kummer map (see 2.6.2) gives a class
\[
\mathcal{E}S_{c,r}^{(t)} := \partial_r(c\vartheta_E) \in H^1(E[\ell^r]\langle t \rangle, \mathbb{Z}/\ell^r \mathbb{Z}(1))
\]
and by the norm-compatibility we can define:

Definition 1.3.3. Let
\[
\mathcal{E}S_{c}^{(t)} := \lim_{r \to} \partial_r(c\vartheta_E) \in H^1(S, \Lambda(\mathcal{H}\langle t \rangle)(1)).
\]

In 2.5.2 we define a moment map
\[
\text{mom}^k : \Lambda(\mathcal{H}) \to \text{TSym}^k \mathcal{H}
\]
which gives rise to a map
\[
H^1(S, \Lambda(\mathcal{H}\langle t \rangle)(1)) \xrightarrow{\text{mom}^k} H^1(S, \text{TSym}^k \mathcal{H}(1)).
\]
We show in Proposition 2.6.8, inspired by a result of Colmez [Col98]:

Proposition 1.3.4 (see 2.6.8). The element
\[
ce_k(t) := \text{mom}^k(\mathcal{E}S_{c}^{(t)}) \in H^1(S, \text{TSym}^k \mathcal{H}(1))
\]
coincides with Soulé’s twisting construction (see [2.6.4]) applied to the norm compatible elliptic units $c\vartheta_E$ and is called the elliptic Soulé element.
Consider the image of $\text{mom}_t^k(\mathcal{E}_c(t))$ in $H^1(S, \text{TSym}^k\mathcal{H}_\ell(1))$. The isomorphism $\text{Sym}^k\mathcal{H}_\ell \to \text{TSym}^k\mathcal{H}_\ell$ induces
\begin{equation}
H^1(S, \text{Sym}^k\mathcal{H}_\ell(1)) \cong H^1(S, \text{TSym}^k\mathcal{H}_\ell(1)),
\end{equation}
which allows us to consider
\[ \widetilde{\text{mom}}_t^k(\mathcal{E}_c(t)) = \frac{1}{N^k} \text{mom}_t^k(\mathcal{E}_c(t)) \in H^1(S, \text{Sym}^k\mathcal{H}_\ell(1)). \]

**Theorem 1.3.5** (see Theorem 4.7.1). With the above notation the equality
\[ \frac{1}{N^k} c e_k(t) = \widetilde{\text{mom}}_t^k(\mathcal{E}_c(t)) = \frac{-1}{N^{k-1}} (c^2 \text{Eis}_k^k(t) - c^{-k} \text{Eis}_k^k([c]t)) \]
holds in $H^1(S, \text{Sym}^k\mathcal{H}_\ell(1))$. In particular, if $c \equiv 1 \mod N$ one has
\[ c e_k(t) = -N(c^2 - c^{-k}) \text{Eis}_k^k(t). \]
This is the desired relation between $\text{Eis}_k^k(t)$ and the elliptic Soulé element.

### 2. Sheaves of Iwasawa modules and the moment map

In this section we consider sheaves of Iwasawa modules and define the moment map. As a motivation we start by looking at the case of modules under the Iwasawa algebra.

#### 2.1. Iwasawa algebras

Fix a prime number $\ell$. Let $X$ be a totally disconnected compact topological space of the form
\[ X = \lim_{\leftarrow} X_r \]
with $X_r$ finite discrete. We denote by
\[ \mathcal{C}(X, \mathbb{Z}_\ell) := \{ f : X \to \mathbb{Z}_\ell \mid f \text{ continuous} \} \]
the continuous $\mathbb{Z}_\ell$-valued functions on $X$ together with the sup-norm $||-||_\infty$.

**Definition 2.1.1.** The space of $\mathbb{Z}_\ell$-valued measures on $X$ is
\[ \Lambda(X) := \text{Hom}_{\mathbb{Z}_\ell}(\mathcal{C}(X, \mathbb{Z}_\ell), \mathbb{Z}_\ell). \]
We also write $\Lambda_r(X) := \text{Hom}_{\mathbb{Z}_\ell}(\mathcal{C}(X_r, \mathbb{Z}_\ell), \mathbb{Z}/\ell^r\mathbb{Z})$ for the $\mathbb{Z}/\ell^r\mathbb{Z}$-valued measures on $X$. For each $\mu \in \Lambda(X)$ we write
\[ \int_X f \mu := \mu(f) \]
and for $x \in X$ we let $\delta_x \in \Lambda(X)$ be the Dirac distribution characterized by $\delta_x(f) = f(x)$.

As every continuous function in $\mathcal{C}(X, \mathbb{Z}_\ell)$ is the uniform limit of locally constant functions, we have
\[ \Lambda(X) = \lim_{\leftarrow} \text{Hom}_{\mathbb{Z}_\ell}(\mathcal{C}(X_r, \mathbb{Z}_\ell), \mathbb{Z}_\ell) = \lim_{\leftarrow} \Lambda(X_r) = \lim_{\leftarrow} \Lambda_r(X_r). \]
For a continuous map $\phi : X \to Y$ one has a homomorphism
\[(2.1.1) \phi_! : \Lambda(X) \to \Lambda(Y)\]
defined by $(\phi_! \mu)(f) := \mu(f \circ \phi)$. If $U \subset X$ is open compact one has
\[\Lambda(X) \cong \Lambda(U) \oplus \Lambda(X \setminus U).\]

Define
\[\Lambda(X) \hat{\otimes} \Lambda(Y) := \lim_{\leftarrow} \Lambda(X_r) \otimes_{\mathbb{Z}_\ell} \Lambda(Y_r)\]
then one has a canonical isomorphism
\[\Lambda(X \times Y) \cong \Lambda(X) \hat{\otimes} \Lambda(Y).\]

Let now $X = H = \varprojlim H_r$ be a profinite group. Then $\Lambda(H_r)$ is the group algebra of $H_r$ and $\Lambda(H)$ inherits a $\mathbb{Z}_\ell$-algebra structure. This algebra structure can also be defined directly by using the convolution of measures
\[\mu \ast \nu := \text{mult} \left(\mu \otimes \nu\right),\]
where $\text{mult} : H \times H \to H$ is the group multiplication. As $\delta_g \ast \delta_h = \delta_{gh}$ the map $\delta : H \to \Lambda(H)^\times$, $h \mapsto \delta_h$ is a group homomorphism.

**Definition 2.1.2.** $\Lambda(H)$ with the above $\mathbb{Z}_\ell$-algebra structure is called the *Iwasawa algebra* of $H$.

The following situation will frequently occur in the applications in this paper. Suppose that
\[0 \to H \to G \xrightarrow{q} T \to 0\]
is an exact sequence of profinite groups with $T$ finite discrete. Define for $t \in T$
\[H(t) := q^{-1}(t)\]
so that $G = \bigcup_{t \in T} H(t)$ and each $H(t)$ is an $H$-torsor, i.e., has a simply transitive $H$-action. Then
\[\Lambda(G) \cong \bigoplus_{t \in T} \Lambda(H(t))\]
is a $\Lambda(H)$-module and $\Lambda(H(t))$ is a free $\Lambda(H)$-module of rank one.

**2.2. The moment map.** In this section we consider the profinite group $H \cong \mathbb{Z}_\ell^d$ and we write
\[(2.2.1) H_r := H \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell/\ell^r \mathbb{Z}_\ell \cong (\mathbb{Z}_\ell/\ell^r \mathbb{Z}_\ell)^d\]
\[H_{Q_\ell} := H \otimes_{\mathbb{Z}_\ell} Q_\ell \cong Q_\ell^d.\]
The moment map will be a $\mathbb{Z}_\ell$-algebra homomorphism
\[\Lambda(H) \to \widetilde{\text{Sym}} H,\]
where $\widetilde{\text{Sym}} H$ is the completion of the $\mathbb{Z}_\ell$-algebra of symmetric tensors with respect to the augmentation ideal.
We start by recalling some facts about the algebra of symmetric tensors \( \text{TSym} H \). We remark right away that the right framework for the moment map is the divided power algebra \( \Gamma H \), which in our case is isomorphic to \( \text{TSym} H \). As we are interested in the relation with the symmetric algebra in the end, we found it more intuitive to work with \( \text{TSym} H \).

The algebra \( \text{TSym} H \) is graded
\[
\text{TSym} H = \bigoplus_{k \geq 0} \text{TSym}^k H
\]
and for each \( h \in H \) one has the symmetric tensor \( h^{[k]} := h^k \in \text{TSym}^k H \). This gives a divided power structure on \( \text{TSym} H \) and one has the formulae
\[
(g + h)^{[k]} = \sum_{m+n=k} g^{[m]} h^{[n]}
\]
\[
h^{[m]} h^{[n]} = \frac{(m+n)!}{m!n!} h^{[m+n]}.
\]
(2.2.2)

The map \( H \to \text{TSym}^1 H, \ h \mapsto h^{[1]} \) is an isomorphism. By the universal property of the symmetric algebra this induces an algebra homomorphism
\[
\text{Sym} H \to \text{TSym} H,
\]
which is an isomorphism after tensoring with \( \mathbb{Q}_\ell \).

From the isomorphism \( \Gamma H \cong \text{TSym} H \) it follows directly that \( \text{TSym} H \) is compatible with base change
\[
(\text{TSym} H) \otimes_{\mathbb{Z}_\ell} \mathbb{Z}/\ell^r \mathbb{Z} \cong \text{TSym} H_r
\]
and with direct sums \( \text{TSym}(H \oplus H) \cong \text{TSym} H \otimes \text{TSym} H \).

If \( (e_1, \ldots, e_d) \) is a basis of \( H \), then
\[
(e_1^{[n_1]} \cdots e_d^{[n_d]} \mid n_1 + \ldots + n_d = k)
\]
is a basis of \( \text{TSym}^k H \). Note that under the homomorphism \( \text{Sym}^k H \to \text{TSym}^k H \) one has
\[
e_1^{n_1} \cdots e_d^{n_d} \mapsto k! e_1^{[n_1]} \cdots e_d^{[n_d]}.
\]
Let \( H^\vee := \text{Hom}_{\mathbb{Z}_\ell}(H, \mathbb{Z}_\ell) \) be the dual \( \mathbb{Z}_\ell \)-module then one has a canonical isomorphism
\[
\text{Sym}^k H^\vee \cong (\text{TSym}^k H)^\vee.
\]

Let \( \text{TSym}^+ H := \bigoplus_{k>0} \text{TSym}^k H \) be the augmentation ideal. We denote by
\[
\widehat{\text{TSym}} H := \lim_{\leftarrow n} \text{TSym} H/(\text{TSym}^+ H)^n
\]
the completion of \( \text{TSym} H \) with respect to the augmentation ideal. Similarly, we also denote by \( \widehat{\text{TSym}} H_\ell \) and \( \widehat{\text{TSym}} H_{\mathbb{Q}_\ell} \) the completions with respect to the augmentation ideal.
Lemma 2.2.1. One has

\[ \hat{\text{TSym}}^\cdot H \cong \lim_{\leftarrow r} \hat{\text{TSym}}^\cdot H_r. \]

Proof. As \( \text{TSym}^k H \) is a free \( \mathbb{Z}_\ell \)-module, one has

\[ \text{TSym}^k H / (\text{TSym}^k H)^n \cong \lim_{\leftarrow r} \text{TSym}^k H_r / (\text{TSym}^k H_r)^n \]

for all \( n \geq 1 \). Taking the inverse limit over \( n \), the result follows. \( \square \)

Proposition 2.2.2. There is a unique homomorphism of \( \mathbb{Z}_\ell \)-algebras

\[ \text{mom} : \Lambda(H) \to \hat{\text{TSym}}^\cdot H, \]

which maps \( \delta_h \mapsto \sum_{k \geq 0} h^{[k]} \) and is called the moment map. It is the limit

\[ \text{mom} = \lim_{\leftarrow r} \text{mom}_r \]

of moment maps at finite level

\[ \text{mom}_r : \Lambda_r(H_r) \to \hat{\text{TSym}}^\cdot H_r \]

\[ \mu_r \mapsto \sum_{k \geq 0} \left( \sum_{h \in H_r} \mu_r(h) h^{[k]} \right). \]

Let \( (e_1, \ldots, e_d) \) be a basis of \( H \) and \( (x_1, \ldots, x_d) \) the dual basis considered as \( \mathbb{Z}_\ell \)-valued functions \( x_i : H \to \mathbb{Z}_\ell \). In terms of measures the moment map is given by

\[ \text{mom}(\mu) = \sum_{k \geq 0} \left( \sum_{n_1 + \ldots + n_d = k} \left( \int_H x_1^{n_1} \ldots x_d^{n_d} \mu e_1^{[n_1]} \ldots e_d^{[n_d]} \right) \right). \]

The projection onto the \( k \)-th component is denoted by

\[ \text{mom}^k : \Lambda(H) \to \text{TSym}^k H \]

and by \( \text{mom}_r^k : \Lambda_r(H_r) \to \text{TSym}^k H_r \) respectively.

Remark 2.2.3. The formula for the moment map in terms of measures justifies the name. For the application to sheaves of Iwasawa algebras it is the formula on finite level which is important.

Proof. The map \( \text{mom}_r : \Lambda_r(H_r) \to \hat{\text{TSym}}^\cdot H_r \) in the proposition is the algebra homomorphism induced by the group homomorphism \( h \mapsto \sum_{k \geq 0} h^{[k]} \) and the universal property of the group algebra \( \Lambda_r(H_r) \). Taking the inverse limit gives \( \text{mom} : \Lambda(H) \to \hat{\text{TSym}}^\cdot H \).

Write \( \mu \in \Lambda(H) \) as \( \mu = \lim_{\leftarrow r} \mu_r \) with \( \mu_r \in \Lambda_r(H_r) \). The dual basis \( x_1, \ldots, x_d \) considered as \( \mathbb{Z}/\ell^r \mathbb{Z} \)-linear maps \( x_i : H_r \to \mathbb{Z}/\ell^r \mathbb{Z} \) induce polynomial functions \( x_i^{n_i} : H_r \to \mathbb{Z}/\ell^r \mathbb{Z} \) and by definition

\[ \int_{H_r} x_1^{n_1} \ldots x_d^{n_d} \mu_r = \mu_r(x_1^{n_1} \ldots x_d^{n_d}) = \sum_{h \in H_r} \mu_r(h) x_1(h)^{n_1} \ldots x_d(h)^{n_d}. \]
If we observe that
\[ \sum_{n_1 + \ldots + n_d = k} x_1(h)^{n_1} \cdots x_d(h)^{n_d} e_1^{[n_1]} \cdots e_d^{[n_d]} = (x_1(h) e_1 + \cdots + x_d(h) e_d)^[k] = h^{[k]} \]
we get
\[ \sum_{n_1 + \ldots + n_d = k} \mu_r(x_1^{n_1} \cdots x_d^{n_d}) e_1^{[n_1]} \cdots e_d^{[n_d]} = \sum_{h \in H_r} \mu_r(h) h^{[k]} = \text{mom}^k(\mu_r). \]
This implies that for the measure \( \mu = \lim_{\leftarrow r} \mu_r \in \Lambda(H) \) we get
\[ \text{mom}^k(\mu) = \lim_{\leftarrow r} \text{mom}^k(\mu_r) = \lim_{\leftarrow r} \sum_{n_1 + \ldots + n_d = k} \left( \int_{H_r} x_1^{n_1} \cdots x_d^{n_d} \mu_r \right) e_1^{[n_1]} \cdots e_d^{[n_d]}, \]
which implies the desired formula for \( \text{mom}(\mu) \). \( \square \)

Note that the moment map is functorial. If \( \varphi : H \to G \) is a group homomorphism one has a commutative diagram
\[
\begin{array}{ccc}
\Lambda(H) & \xrightarrow{\text{mom}} & \hat{\text{TSym}} H \\
\phi_! \downarrow & & \downarrow \hat{\text{TSym}} (\phi) \\
\Lambda(G) & \xrightarrow{\text{mom}} & \hat{\text{TSym}} G \\
\end{array}
\]
It is a fact from classical Iwasawa theory that \( \Lambda(H) \) is isomorphic to a power series ring over \( \mathbb{Z}_\ell \) in \( d \) variables. In particular, it is a regular local ring. Let
\[ I(H) := \ker(\Lambda(H) \xrightarrow{\int_H} \mathbb{Z}_\ell) \]
be the augmentation ideal. Then from the regularity of \( \Lambda(H) \) it follows that \( I(H)^k/I(H)^{k+1} \cong \text{Sym}^k H \) and the \( k \)-th moment map factors
\[ \text{mom}^k : \Lambda(H) \to \Lambda(H)/I(H)^{k+1} \to \text{TSym}^k H. \]

**Lemma 2.2.4.** The \( k \)-th moment map induces
\[ \text{Sym}^k H \cong I(H)^k/I(H)^{k+1} \hookrightarrow \Lambda(H)/I(H)^{k+1} \xrightarrow{\text{mom}^k} \text{TSym}^k H, \]
which is just the canonical map.

**Proof.** The morphism \( \text{mom}^k \) maps an element \((\delta_{h_1} - 1) \cdots (\delta_{h_k} - 1) \in I(H)^k\) to the corresponding product taken in \( \text{TSym} H \). This implies the result. \( \square \)

Consider again the exact sequence
\[ 0 \to H \to G \xrightarrow{q} T \to 0 \]
of profinite groups with \( T \) a finite \( N \)-torsion group.

**Definition 2.2.5.** For the \( H \)-torsors \( H(t) = q^{-1}(t) \) define
\[ \text{mom}^k_t : \Lambda(H(t)) \to \text{TSym}^k H \]
to be the composition
\[ \text{mom}^k_t : \Lambda(H(t)) \xrightarrow{\text{[N]}} \Lambda(H) \xrightarrow{\text{mom}^k} \text{TSym}^k H, \]
where $|N| : G \to G$ is the $N$-multiplication, which factors through $H$.

**Remark 2.2.6.** This moment map is not independent of the choice of $N$ such that $t$ is an $N$-torsion point.

To remedy this defect consider the composition

$$\Lambda(H(t)) \xrightarrow{\text{mom}_k} \text{TSym}^k H \to \text{TSym}^k H_{Q_\ell}.$$  

**Definition 2.2.7.** The modified moment map

$$\tilde{\text{mom}}_k^t : \Lambda(H(t)) \to \text{Sym}^k H_{Q_\ell}$$

is the map composed with the inverse of the isomorphism $\text{Sym}^k H_{Q_\ell} \cong \text{TSym}^k H_{Q_\ell}$ divided by $N^k$, i.e.,

$$\tilde{\text{mom}}_k^t := \frac{1}{N^k} \text{mom}_k^t.$$

The following lemma is obvious from the definition.

**Lemma 2.2.8.** The modified moment map $\tilde{\text{mom}}_k^t$ depends only on $t$ and not on $N$.

2.3. Étale sheaves of Iwasawa modules. Consider a projective system of finite étale schemes $p_r : X_r \to S$ and let $X := \varprojlim_r X_r$. We denote by $\lambda_r : X_{r+1} \to X_r$ the finite étale maps in the projective system. We denote by $\mathcal{X}_r$ the étale sheaf associated to $X_r$ and define an étale sheaf on $S$ by

$$(2.3.1) \quad \Lambda_r(\mathcal{X}_r) := p_r^* \mathbb{Z}/\ell^r \mathbb{Z}.$$

The trace map with respect to $\lambda_r$ induces a morphism of sheaves $\lambda_{r+1}^* : \mathbb{Z}/\ell^{r+1} \mathbb{Z} \to \mathbb{Z}/\ell^r \mathbb{Z}$ which gives rise to

$$\Lambda_{r+1}(\mathcal{X}_{r+1}) = p_{r+1}^* \lambda_r^* \mathbb{Z}/\ell^{r+1} \mathbb{Z} \to p_{r+1}^* \mathbb{Z}/\ell^r \mathbb{Z} \to p_r^* \mathbb{Z}/\ell^r \mathbb{Z} = \Lambda_r(\mathcal{X}_r),$$

where the last map is reduction modulo $\ell^r$.

**Definition 2.3.1.** Define an inverse system of étale sheaves on $S$ by

$$\Lambda(\mathcal{X}) := (\Lambda_r(\mathcal{X}_r))_{r \geq 1},$$

with the above transition maps.

**Remark 2.3.2.** Note that $\Lambda(\mathcal{X})$ is not an $\ell$-adic sheaf in general.

This construction is functorial in the sense that for a morphism of inverse systems $(f_r : X_r \to Y_r)_{r \geq 1}$ the trace map induces

$$(2.3.2) \quad f_r! : \Lambda_r(\mathcal{X}_r) \to \Lambda_r(\mathcal{Y}_r)$$

and hence a map $f_! : \Lambda(\mathcal{X}) \to \Lambda(\mathcal{Y})$.

We want to explain in which sense $\Lambda(\mathcal{X})$ is a sheafification of the space of measures $\Lambda(X)$.

Let us choose a geometric point $\overline{s} : \text{Spec} K \to S$ and let $\mathcal{X}_r, \overline{s}$ be the stalk of $\mathcal{X}_r$ at $\overline{s}$. We consider $\mathcal{X}_r, \overline{s}$ as a finite set with a continuous Galois action. Immediately from the definitions we have:
Lemma 2.3.3. The stalk of $\Lambda_r(\mathcal{X}_r)$ at $\overline{x}$ is

$$\Lambda_r(\mathcal{X}_r)_{\overline{x}} \cong \Lambda_r(\mathcal{X}_r[\overline{x}]).$$

In particular, if we define $\Lambda(\mathcal{X}) := \varprojlim_r \Lambda_r(\mathcal{X}_r)$ and $\mathcal{X}[\overline{x}] := \varprojlim_r \mathcal{X}_r[\overline{x}]$ we get

$$\Lambda(\mathcal{X})_{\overline{x}} \cong \Lambda(\mathcal{X}[\overline{x}]),$$

which is the space of measures on $\mathcal{X}[\overline{x}]$ with a Galois action.

In the case where each $X_r = H_r \overset{pr}{\to} S$ a finite étale group scheme over $S$, so that $H := \varprojlim_r H_r$ is a pro-étale group scheme, the sheaves $\Lambda_r(\mathcal{H}_r)$ become sheaves of $\mathbb{Z}/\ell^r\mathbb{Z}$-modules. In fact one has

$$(p_r \times p_r)_* \mathbb{Z}/\ell^r\mathbb{Z} \cong \Lambda_r(\mathcal{H}_r) \otimes \Lambda_r(\mathcal{H}_r)$$

and the group multiplication induces a ring structure on $\Lambda_r(\mathcal{H}_r)$.

2.4. The case of torsors. The following situation will occur very frequently in this paper. Suppose we have an inverse system of finite étale group schemes on $S$

$$(2.4.1)\quad 0 \to H_r \to G_r \overset{q_r}{\to} T \to 0$$

where $T = T_r$ for all $r$ is an $N$-torsion group. For each section $t : S \to T$ we define an $H_r$-torsor $H_r\langle t \rangle$ by the cartesian diagram

$$\begin{array}{ccc}
H_r\langle t \rangle & \longrightarrow & G_r \\
p_r \times \ & | & \\
S & \overset{t}{\longrightarrow} & T.
\end{array}$$

Denote by $H := \varprojlim_r H_r$, $G := \varprojlim_r G_r$ and $H\langle t \rangle := \varprojlim_r H_r\langle t \rangle$ the associate pro-étale group schemes and by $\mathcal{H}_r$, $\mathcal{G}_r$, $\mathcal{H}_r\langle t \rangle$ and $\mathcal{H}$, $\mathcal{G}$, $\mathcal{H}\langle t \rangle$ the associated sheaves. In particular one has an exact sequence

$$\begin{array}{c}
0 \to H \to G \overset{q}{\to} T \to 0
\end{array}$$

and a cartesian diagram

$$\begin{array}{ccc}
H\langle t \rangle & \longrightarrow & G \\
p_t \ | & \\
S & \overset{t}{\longrightarrow} & T.
\end{array}$$

Each $\Lambda_r(\mathcal{H}_r\langle t \rangle)$ is a $\Lambda_r(\mathcal{H}_r)$-module of rank one and consequently the same is true for the $\Lambda(\mathcal{H})$-module $\Lambda(\mathcal{H}\langle t \rangle)$.

The sheaves $\Lambda(\mathcal{H}\langle t \rangle)$ are sheaves of Iwasawa modules under the sheaves of Iwasawa algebras $\Lambda(\mathcal{H})$. 


2.5. The sheafified moment map. In this section we describe a sheaf version of the moment maps from Proposition 2.2.2.

Let $p_r : H_r \to S$ be a finite étale group scheme which is étale locally of the form

$$H_r \cong (\mathbb{Z}/\ell^r \mathbb{Z})^d$$

for $d \geq 1$. As in (2.4.2) we consider $H_r$-torsors $p_{r,t} : H_r(t) \to S$ associated to an exact sequence

$$0 \to H_r \to G_r \to T \to 0$$

and to an $N$-torsion section $t$ of $T$. As $T$ is an $N$-torsion group by assumption, the $N$-multiplication map $[N] : G_r \to G_r$ factors through $H_r$ and we get a map of schemes

$$\tau_{r,t} : H_r \langle t \rangle \hookrightarrow G_r \quad \text{with} \quad \tau_{r,t} \circ \tau_{r,t} = \tau_{r,t}.$$ 

We interpret this as a section $\tau_{r,t} \in H^0(H_r(t), p_{r,t}^* \mathcal{H}_r)$.

Definition 2.5.1. We let $\tau_{r,t}^{[k]} \in H^0(H_r(t), p_{r,t}^* \text{TSym}^k \mathcal{H}_r)$ be the $k$-th tensor power $\tau_{r,t}$. This will also be viewed as a map of sheaves

$$\tau_{r,t}^{[k]} : \mathbb{Z}/\ell^r \mathbb{Z} \to p_{r,t}^* \text{TSym}^k \mathcal{H}_r.$$ 

Recall that for sheaves $\mathcal{F}, \mathcal{G}$ on $H_r(t)$ one has the morphism (given by the projection formula and adjunction)

$$(2.5.2) \quad p_{r,t}!(\mathcal{F} \otimes p_{r,t}^* \mathcal{G}) \cong p_{r,t}!(\mathcal{F} \otimes p_{r,t}^* p_{r,t} \mathcal{G}) \to p_{r,t}!(\mathcal{F} \otimes \mathcal{G})$$

and that $p_{r,t}! = p_{r,t}^*$ as $p_{r,t}$ is finite.

Definition 2.5.2. The sheafified moment map

$$\text{mom}_{r,t}^k : \Lambda_r(H_r(t)) \to \text{TSym}^k \mathcal{H}_r$$

is the composition ($p := p_{r,t}$)

$$p_*\mathbb{Z}/\ell^r \mathbb{Z} \xrightarrow{\text{id} \otimes \tau_{r,t}^{[k]}} p_*\mathbb{Z}/\ell^r \mathbb{Z} \otimes p_* \text{Sym}^k \mathcal{H}_r \xrightarrow{\text{tr}} p_*(\mathbb{Z}/\ell^r \mathbb{Z} \otimes \text{Sym}^k \mathcal{H}_r) \cong p_* \text{Sym}^k \mathcal{H}_r \xrightarrow{\text{tr}} \text{TSym}^k \mathcal{H}_r,$$

where tr is the trace map with respect to $p$.

From the definition it follows that the moment maps $\text{mom}_{r,t}^k$ are compatible with respect to the trace map for varying $r$.

Definition 2.5.3. Let

$$\text{mom}_{t}^k : \Lambda(\mathcal{H}_r(t)) \to \text{TSym}^k \mathcal{H}$$

be the inverse limit of $\text{mom}_{r,t}^k$. We also let

$$\text{mom}_{t}^k := \frac{1}{N^k} \text{mom}_{t}^k : H^1(S, \Lambda(\mathcal{H}_r(t))(1)) \to H^1(S, \text{Sym}^k \mathcal{H}_{Q_1}(1))$$
be the composition of the map induced by $\frac{1}{N^k}\text{mom}^k_r$ in cohomology with the inverse of the canonical isomorphism

$$H^1(S, \text{Sym}^k \mathcal{H}_Q(1)) \cong H^1(S, \text{TSym}^k \mathcal{H}_Q(1)).$$

On stalks the sheafified moment map coincides with the one defined in Definition 2.2.5.

**Lemma 2.5.4.** Let $\mathfrak{m}$ be a geometric point of $S$, then the stalk of the moment map

$$(\text{mom}^k_{r,t})_{\mathfrak{m}} : \Lambda_r(\mathcal{H}_r(t))_{\mathfrak{m}} \to \text{TSym}^k \mathcal{H}_r_{\mathfrak{m}}$$

coincides with the moment map $\text{mom}^k_{r,t}$ defined in Definition 2.2.5.

**Proof.** We have $\Lambda_r(\mathcal{H}_r(t))_{\mathfrak{m}} = \Lambda_r(\mathcal{H}_r(t)_{\mathfrak{m}})$ and we let

$$\mu_r = \sum_{\mathfrak{m} \in \mathcal{H}_r(t)_{\mathfrak{m}}} m_{\mathfrak{m}} \delta_{\mathfrak{m}}$$

be an element in $\Lambda_r(\mathcal{H}_r(t)_{\mathfrak{m}})$. We identify $(p := p_{r,t})$

$$p_*p^*\text{TSym}^k \mathcal{H}_{r,\mathfrak{m}} \cong \Lambda_r(\mathcal{H}_r_{\mathfrak{m}}) \otimes \text{TSym}^k \mathcal{H}_{r,\mathfrak{m}}$$

so that

$$(p_*\mathbb{Z}/\ell^r\mathbb{Z} \otimes p_*p^*\text{TSym}^k \mathcal{H}_{r,\mathfrak{m}})_{\mathfrak{m}} \cong \Lambda_r(\mathcal{H}_r_{\mathfrak{m}}) \otimes \Lambda_r(\mathcal{H}_r_{\mathfrak{m}}) \otimes \text{TSym}^k \mathcal{H}_{r,\mathfrak{m}}.$$  

With this identification the image of $\mu_r$ under $id \otimes \tau_{r,t}^{[k]}$ is given by

$$(2.5.3) \quad (\sum_{\mathfrak{m} \in \mathcal{H}_r(t)_{\mathfrak{m}}} m_{\mathfrak{m}} \delta_{\mathfrak{m}}) \otimes (\sum_{\mathfrak{n} \in \mathcal{H}_r(t)_{\mathfrak{m}}} \delta_{\mathfrak{n}} \otimes \tau_{r,t}^{[k]}(\mathfrak{n})).$$

The homomorphism

$$\Lambda_r(\mathcal{H}_r_{\mathfrak{m}}) \otimes \Lambda_r(\mathcal{H}_r_{\mathfrak{m}}) \otimes \text{TSym}^k \mathcal{H}_{r,\mathfrak{m}} \xrightarrow{(2.5.3)} \Lambda_r(\mathcal{H}_r_{\mathfrak{m}}) \otimes \text{TSym}^k \mathcal{H}_{r,\mathfrak{m}}$$

maps the element in (2.5.3) to

$$\sum_{\mathfrak{m} \in \mathcal{H}_r(t)_{\mathfrak{m}}} m_{\mathfrak{m}} \delta_{\mathfrak{m}} \otimes \tau_{r,t}^{[k]}(\mathfrak{m}) = \sum_{\mathfrak{m} \in \mathcal{H}_r(t)_{\mathfrak{m}}} \mu_r(\mathfrak{m}) \delta_{\mathfrak{m}} \otimes \tau_{r,t}^{[k]}(\mathfrak{m}).$$

and the trace of this is

$$\sum_{\mathfrak{m} \in \mathcal{H}_r(t)_{\mathfrak{m}}} \mu_r(\mathfrak{m}) \tau_{r,t}^{[k]}(\mathfrak{m}) = \sum_{\mathfrak{m} \in \mathcal{H}_r(t)_{\mathfrak{m}}} \mu_r(\mathfrak{m}) ([N\mathfrak{m}]^{[k]}$$

which is the desired formula.  

$\square$
2.6. **Soulé’s twisting construction and the moment map.** Let us consider the situation in (2.4.3)

\[ 0 \to H \to G \xrightarrow{\phi} T \to 0 \]

and recall the inverse system of \( H_r \)-torsors \( H_r(t) \). Denote by \( \lambda_r : H_{r+1}(t) \to H_r(t) \) the transition maps and by \( p_{r,t} : H_r(t) \to S \) the structure map.

**Definition 2.6.1.** A norm-compatible function \( \theta = (\theta_r)_{r \geq 1} \) on \( H_r(t) = (H_r(t))_{r \geq 1} \) is an inverse system of global invertible sections \( \theta_r \in \mathbb{G}_m(H_r(t)) \) such that \( \lambda_r^*(\theta_{r+1}) = \theta_r \), where \( \lambda_{rs} \) is the norm map with respect to \( \lambda_r \).

**Definition 2.6.2.** The Kummer map

\[ \partial_r : \mathbb{G}_m(H_r(t)) \to H^1(H_r(t), \mu_{\ell_r}), \]

is the boundary map for the exact sequence

\[ 0 \to \mu_{\ell_r} \to \mathbb{G}_m \xrightarrow{[\ell_r]} \mathbb{G}_m \to 0. \]

Recall the section \( \tau_{r,t}^{[k]} \in H^0(H_r(t), p_{r,t}^* \text{TSym}^k \mathcal{H}_r) \) from Definition 2.5.1.

**Definition 2.6.3.** Let

\[ s(r, k, t) := p_{r,t*}(\partial_r(\theta_r) \cup \tau_{r,t}^{[k]}) \in H^1(S, \text{TSym}^k \mathcal{H}_r(1)), \]

where we have written \( \text{TSym}^k \mathcal{H}_r(1) := \text{TSym}^k \mathcal{H}_r \otimes \mu_{\ell_r} \) as usual.

Recall from Lemma 2.2.1 that \( \lim_{\leftarrow r} \text{TSym}^k \mathcal{H}_r = \text{TSym}^k \mathcal{H} \) and denote by \( \text{red}_r : \text{TSym}^k \mathcal{H}_{r+1} \to \text{TSym}^k \mathcal{H}_r \) the reduction modulo \( \ell_r \).

**Proposition 2.6.4** (Soulé). Under the transition maps

\[ \text{red}_r : H^1(S, \text{TSym}^k \mathcal{H}_{r+1}(1)) \to H^1(S, \text{TSym}^k \mathcal{H}_r(1)) \]

one has \( \text{red}_r(s(r + 1, k, t)) = s(r, k, t) \). In particular, one gets an element

\[ s(k, t) := \lim_{\leftarrow r} s(r, k, t) \in H^1(S, \text{TSym}^k \mathcal{H}(1)). \]

We refer to this construction as Soulé’s twisting construction.

**Remark 2.6.5.** In general Soulé’s twisting construction allows also to construct elements in other Galois representations and it depends on the choice of elements in the Galois representation. Here we have fixed the tautological sections \( \tau_{r,t}^{[k]} \) of \( \text{TSym}^k \mathcal{H}_r \) to define this twist. In [HK06] one can find more general twisting constructions.
Proof. By abuse of notation we also denote by red, any map on cohomology which reduces the coefficient module modulo $\mathcal{L}$. Then one has $\text{red}_r \circ \lambda_{r*} = \lambda_{r*} \circ \text{red}_r$. We have red$_r(\tau_{r+1}^{[k]}) = \lambda_r^*(\tau_{r+1}^{[k]})$ and by assumption $\text{red}_r \circ \lambda_{r*}(\theta_{r+1}) = \lambda_{r*} \circ \text{red}_{r*}(\theta_{r+1}) = \theta_r$. Then

$$\text{red}_r(s(r + 1, k, t)) = \text{red}_r \circ p_{r+1, t*}(\partial_{r+1}(\theta_{r+1}) \cup \tau_{r+1,t}^{[k]}),$$

$$= p_{r+1, t*}(\text{red}_r(\partial_{r+1}(\theta_{r+1})) \cup \text{red}_r(\tau_{r+1,t}^{[k]}))$$

$$= p_{r,t*} \circ \lambda_{r*}(\text{red}_r(\partial_{r+1}(\theta_{r+1})) \cup \lambda^*_r(\tau_{r,t}^{[k]}))$$

$$= p_{r,t*}(\lambda_{r*} \circ \text{red}_r(\partial_{r+1}(\theta_{r+1})) \cup \tau_{r,t}^{[k]}),$$

$$= p_{r,t*}(\partial_r(\theta_r) \cup \tau_{r,t}^{[k]}),$$

$$= s(r, k, t).$$

The following identification is fundamental for the whole paper.

Lemma 2.6.6. Let $p_{r,t} : H_r(t) \to S$ be the $H_r$-torsor as above, then one has a canonical isomorphism

$$H^1(H_r(t), \mu_{r'}) \cong H^1(S, \Lambda_r(\mathcal{K}(t))(1)).$$

Proof. As $p_{r,t}$ is finite this follows from the Leray spectral sequence. \qed

With this identification we can rewrite the Kummer map and one gets a commutative diagram

(2.6.1) \[
\begin{array}{ccc}
\mathbb{G}_m(H_{r+1}(t)) & \overset{\partial_{r+1}}{\longrightarrow} & H^1(S, \Lambda_{r+1}(\mathcal{K}_{r+1}(t))(1)) \\
\downarrow \lambda_{r*} & & \downarrow \lambda_{r*} \\
\mathbb{G}_m(H_r(t)) & \overset{\partial_r}{\longrightarrow} & H^1(S, \Lambda_r(\mathcal{K}_r(t))(1)),
\end{array}
\]

where the $\lambda_{r*}$ on the right hand side is induced by the trace map $\lambda_r : \Lambda_{r+1}(\mathcal{K}_{r+1}(t)) \to \Lambda_r(\mathcal{K}_r(t))$. This diagram allows to consider the inverse limit of the $\partial_r(\theta_r)$:

Definition 2.6.7. The norm-compatible functions $\theta = (\theta_r)_{r \geq 1}$ define an element

$$S^{(t)} := \lim_{\xi} \partial_r(\theta_r) \in H^1(S, \Lambda(\mathcal{K}(t))(1)) = \lim_{\xi} H^1(S, \Lambda_r(\mathcal{K}_r(t))(1)).$$

We also let $S^{(t)} := \partial_r(\theta_r)$.

With this preliminaries we can finally explain the crucial relation between the moment map and Soulé’s twisting construction.

Proposition 2.6.8. The homomorphism

$$\text{mom}^{k}_{r,t} : H^1(S, \Lambda_r(\mathcal{K}_r(t))(1)) \to H^1(S, \text{TSym}^k \mathcal{K}_r(1))$$
induced by the moment map \( \text{mom}^k_{r,t} \) coincides with the composition

\[
H^1(S, \Lambda_r(\mathcal{H}_r(t))(1)) \cong H^1(H_r(t), \mathbb{Z}/\ell^r \mathbb{Z}(1)) \xrightarrow{\cup r'_{[k]}} H^1(H_r(t), p_r^* \text{Sym}^k \mathcal{H}_r(1)) \rightarrow H^1(S, \text{Sym}^k \mathcal{H}_r(1)).
\]

In particular, one has \( \text{mom}^k_{r,t}(S^{(t)}) = s(r, k, t) \) and in the limit

\[
\text{mom}^k_t(S^{(t)}) = s(k, t).
\]

**Proof.** Let \( p := p_r,t \) then the result follows from the commutative diagram

\[
H^1(H_r(t), \mu_{r'}) \times H^0(H_r(t), p^* \text{Sym}^k \mathcal{H}_r) \xrightarrow{\cup} H^1(H_r(t), p^* \text{Sym}^k \mathcal{H}_r(1)) \cong H^1(S, p_*(\mathbb{Z}/\ell^r \mathbb{Z} \otimes p^* \text{Sym}^k \mathcal{H}_r(1)))
\]

\[
\rightarrow H^1(S, p_* \mathbb{Z}/\ell^r \mathbb{Z} \otimes p_* p^* \text{Sym}^k \mathcal{H}_r(1))
\]

3. **Three examples**

3.1. **The Bernoulli measure and its moments.** Let \( N > 1 \) and \( t \in \mathbb{Z}/NZ \) and consider for each \( r \geq 0 \) the exact sequence

\[
0 \rightarrow \mathbb{Z}/\ell^r \mathbb{Z} \rightarrow \mathbb{Z}/\ell^r NZ \xrightarrow{q_r} \mathbb{Z}/NZ \rightarrow 0,
\]

where \( q_r \) is reduction modulo \( N \). We let \( Z_r := \mathbb{Z}/\ell^r \mathbb{Z} \) and \( Z := \mathbb{Z}_\ell \). In the notation of (2.4.3) we have \( H_r = \mathbb{Z}/\ell^r \mathbb{Z}, G_r = \mathbb{Z}/\ell^r NZ \) and \( T = \mathbb{Z}/NZ \). We define

\[
Z_r(t) := q_r^{-1}(t) = \{ x \in \mathbb{Z}/\ell^r NZ \mid x \equiv t \mod N \}
\]

so that \( Z_r(t) = H_r(t) \) in the notation of (2.4.3). We denote by

\[
Z(t) := \lim_r Z_r(t)
\]

the inverse limit. As before, each \( Z_r(t) \) is an \( Z_r \)-torsor. Recall that \( Z(0) = Z = Z_\ell \) and that \( \Lambda(Z(t)) \) is a free rank one \( \Lambda(Z) \)-module.

Let us define the Bernoulli measure in \( \Lambda(Z(t)) \). We choose, as usual, an auxiliary \( c \in \mathbb{Z} \) with \( (c, \ell^r N) = 1 \) to make the Bernoulli distribution integral (for the properties of the Bernoulli numbers we refer to [Lan90] Ch. 2, §2).

**Definition 3.1.1.** Denote by \( B_k(x) \) the \( k \)-th Bernoulli polynomial. The map

\[
B_{2,c,r}^{(t)} : Z_r(t) \rightarrow \mathbb{Z}/\ell^r \mathbb{Z}
\]

\[
x \mapsto \frac{\ell^r N}{2}(c^2 B_2(\{ \frac{x}{\ell^r N} \}) - B_2(\{ \frac{cx}{\ell^r N} \}))
\]

(3.1.1)
where for an element $x \in \mathbb{R}/\mathbb{Z}$ we write $\{x\}$ for its representative in $[0, 1]$, defines an element

$$B_{2,c,r}^{(t)} \in \Lambda_r(Z_r(t)).$$

By the distribution property of the Bernoulli polynomials the $B_{2,c,r}$ are compatible under the trace map $\Lambda_{r+1}(Z_{r+1}(t)) \to \Lambda_r(Z_r(t))$ and give rise to a measure

$$(3.1.2) \quad B_{2,c}^{(t)} := \lim_{r \to \infty} B_{2,c,r}^{(t)} \in \Lambda(Z(t)).$$

We want to compute the moments of the Bernoulli measure. Choose $e = 1 \in \mathbb{Z}_\ell$ as a basis and let $x = \text{id} : \mathbb{Z}_\ell \to \mathbb{Z}_\ell$ be the dual basis. By standard congruences for Bernoulli polynomials (see e.g. [Lan90, Theorem 2.1]) we have

$$\text{mom}_k(B_{2,c}^{(t)}) = \int_{Z(t)} x^k dB_{2,c}^{(t)}$$

$$(3.1.3) \quad = \frac{N^{k+1}}{c^k(k+2)}(c^{k+2}B_{k+2}^{(t)}(\{\frac{t}{N}\}) - B_{k+2}^{(t)}(\{\frac{ct}{N}\})).$$

Note that if $c \equiv 1 \mod N$ we get

$$(3.1.4) \quad \text{mom}_k(B_{2,c}^{(t)}) = \frac{N^{k+1}(c^{k+2} - 1)}{c^k(k+2)} B_{k+2}^{(t)}(\{\frac{t}{N}\}).$$

### 3.2. Modified cyclotomic Soulé-Deligne elements.

We review the cyclotomic elements defined by Soulé [Sou81] and Deligne [Del89] from our perspective. In the literature two kinds of Soulé-Deligne elements are in use. There are the ones used in Iwasawa theory obtained by using the norm of the field extension $\mathbb{Q}(\mu_{\ell N})/\mathbb{Q}(\mu_N)$ and the ones which come from motivic cohomology via the regulator map. These are obtained by the trace map from $[\ell^r] : \mu_{\ell N} \to \mu_N$. The relation between these two elements is essentially an Euler factor (see the discussion in [Hub]). We treat here only the later elements originating from motivic cohomology.

Let $N > 1$ and consider the exact sequence of finite étale group schemes over a base $S$

$$0 \to \mu_{\ell^r} \to \mu_{\ell^r N} \to \mu_N \to 0.$$

With the notations in (2.4.1) we have $H_r = \mu_{\ell^r}$, $G_r = \mu_{\ell^r N}$ and $T = \mu_N$. For each $1 \neq \alpha \in \mu_N(S)$ we define as in (2.4.2) the $\mu_{\ell^r}$-torsor $\mu_{\ell^r}(\alpha)$ by the cartesian diagram

$$\begin{array}{ccc}
\mu_{\ell^r}(\alpha) & \longrightarrow & \mu_{\ell^r N} \\
\downarrow_{\mu_{\ell^r}} & & \downarrow \\
S & \longrightarrow & \mu_N.
\end{array}$$

The inverse limit of these $\mu_{\ell^r}$-torsors is denoted by

$$\mathcal{T}(\alpha) := \lim_{r \to \infty} \mu_{\ell^r}(\alpha).$$
and we use the same notation for the associated sheaves. On \( G_m \setminus \{1\} \) we have the invertible function
\[
\Xi : G_m \setminus \{1\} \to G_m
\]
\[z \mapsto 1 - z\]
and it is well-known that \( \Xi \) is norm-compatible: if \([\ell^r]\) denotes the \( \ell^r \)-multiplication on \( G_m \) one has
\[
(3.2.2) \quad [\ell^r]_*(\Xi) = \Xi.
\]
Thus we can restrict \( \Xi \) to \( \mu_{\ell^r}\langle \alpha \rangle \) to get norm-compatible functions \( \theta_{r,\alpha} \) in the notation of Definition 2.6.1.

Definition 3.2.1. We let
\[
\mathcal{CS}_{r,\alpha} := \partial_r(\Xi) \in H^1(S, \Lambda_r(\mu_{\ell^r}\langle \alpha \rangle)(1))
\]
and define
\[
\mathcal{CS}^{(\alpha)} := \varprojlim_r \mathcal{CS}_{r,\alpha} \in H^1(S, \Lambda(\mathcal{F}(\alpha))(1))
\]
The section \( \tau_{r,\alpha} \) from Definition 2.5.1 is the map
\[
\tau_{r,\alpha} : \mu_{\ell^r}(\alpha) \hookrightarrow \mu_{\ell^r} \stackrel{[N]}{\longrightarrow} \mu_{\ell^r}
\]
and its \( k \)-th tensor power gives
\[
(3.2.3) \quad \tau_{r,\alpha}^{[k]} \in H^0(\mu_{\ell^r}(\alpha), Z/\ell^rZ(k)).
\]

Definition 3.2.2. Let \( 1 \neq \alpha \in \mu_N(S) \). We denote by
\[
\tilde{c}_{k+1,\alpha}(\alpha) := p_{r,\alpha*}(\partial_r(\Xi) \cup \tau_{r,\alpha}^{[k]}) \in H^1(S, Z/\ell^rZ(k + 1))
\]
the element \( s(r, k, \alpha) \) obtained by Soulé’s twisting construction.

Note that for \( S := \text{Spec} \mathbb{Q}(\mu_{\ell^r}N) \) the section \( \tau_{r,\alpha}^{[k]} \) is given by \( \beta \mapsto (\beta^N)^\otimes k \) for \( \beta \in \mu_{\ell^r}(\alpha)(S) \). Moreover, one has
\[
\mu_{\ell^r}(\alpha)(S) = \{ \beta \in \mu_{\ell^r}N(S) \mid \beta^\ell = \alpha \}.
\]
It follows that over \( S := \text{Spec} \mathbb{Q}(\mu_{\ell^r}N) \) the element \( \tilde{c}_{k+1,\alpha}(\alpha) \) is given explicitly by
\[
(3.2.4) \quad \tilde{c}_{k+1,\alpha}(\alpha) = \sum_{\beta \in \mu_{\ell^r}(\alpha)(S)} \partial_r(1 - \beta) \cup (\beta^N)^\otimes k.
\]

Definition 3.2.3. For \( 1 \neq \alpha \in \mu_N(S) \) and \( k \geq 1 \) the modified cyclotomic Soulé-Deligne element is
\[
\tilde{c}_{k+1}(\alpha) := \varprojlim_r \tilde{c}_{k+1,\alpha}(\alpha) \in H^1(S, \mathbb{Z}_\ell(k + 1)).
\]
Moreover, for a function \( \psi : \mu_N(S) \to \mathbb{Z}_\ell \) we let
\[
\tilde{c}_{k+1}(\psi) := \sum_{\alpha \in \mu_N(S)} \psi(\alpha) \tilde{c}_{k+1}(\alpha).
\]
From the general result in Proposition 2.6.8 we get the following relation between $\mathcal{CS}^{(\alpha)}$ and $\tilde{c}_{k+1}(\alpha)$ under the moment map

$$\text{mom}_\alpha^k : H^1(S, \Lambda(\mathcal{F}(\alpha))(1)) \to H^1(S, \mathbb{Z}_\ell(k+1)).$$

**Proposition 3.2.4.** In $H^1(S, \mathbb{Z}/\ell^r\mathbb{Z}(k+1))$ one has

$$\text{mom}_{r, \alpha}^k(\mathcal{CS}_r^{(\alpha)}) = \tilde{c}_{k+1,r}(\alpha)$$

and in the limit

$$\text{mom}_\alpha^k(\mathcal{CS}^{(\alpha)}) = \tilde{c}_{k+1}(\alpha)$$

For later use we need a variant of $\mathcal{CS}^{(\alpha)}$. Fix an integer $c > 1$ which is prime to $\ell N$. The function

$$(3.2.5) \quad c \Xi : \mathbb{G}_m \setminus \mu_c \to \mathbb{G}_m$$

defined by $z \mapsto (1-zc)^2(1-zc)^{-1}$. Note that the $[c]$-multiplication maps

$$[c] : \mu_\ell \langle \alpha \rangle \cong \mu_\ell \langle \alpha^c \rangle.$$

**Definition 3.2.5.** Let

$$\mathcal{CS}_c^{(\alpha)} := \partial_r(c \Xi) = c^2 \mathcal{CS}_r^{(\alpha)} - [c]^* \mathcal{CS}_r^{(\alpha^c)}$$

in $H^1(S, \Lambda_r(\mu_\ell \langle \alpha \rangle)(1))$ and define

$$\mathcal{CS}_c^{(\alpha)} := \lim_{\longleftarrow r} \mathcal{CS}_{c, r}^{(\alpha)} \in H^1(S, \Lambda(\mathcal{F}(\alpha))(1)).$$

We compute the moments of $\mathcal{CS}_c^{(\alpha)}$.

**Proposition 3.2.6.** Let $k \geq 1$ then

$$\text{mom}_\alpha^k(\mathcal{CS}_c^{(\alpha)}) = c^2 \tilde{c}_{k+1}(\alpha) - c^{-k} \tilde{c}_{k+1}(\alpha^c).$$

In particular, for $c \equiv 1 \mod N$ one has

$$\text{mom}_\alpha^k(\mathcal{CS}_c^{(\alpha)}) = \frac{c^{k+2} - 1}{c^k} \tilde{c}_{k+1}(\alpha).$$

**Proof.** There is a commutative diagram

$$\Lambda_r(\mu_\ell \langle \alpha \rangle) \xrightarrow{[c]} \Lambda_r(\mu_\ell \langle \alpha^c \rangle)$$

$$\text{mom}_{r, \alpha}^k \quad \text{mom}_{r, \alpha^c}^k$$

and as $[c] : \mu_\ell \langle \alpha \rangle \cong \mu_\ell \langle \alpha^c \rangle$ is an isomorphism, one has $[c]^*[c]^* = \text{id}$ and one computes

$$c^k \text{mom}_{r, \alpha}^k([c]^* \mathcal{CS}_r^{(\alpha^c)}) = \text{mom}_{r, \alpha^c}^k([c]^* \mathcal{CS}_r^{(\alpha^c)}) = \text{mom}_{r, \alpha^c}^k(\mathcal{CS}_r^{(\alpha^c)}) = \tilde{c}_{k+1,r}(\alpha^c)$$

with Proposition 3.2.4 and the result follows. □
3.3. **Elliptic Soulé elements.** We use the theory of norm-compatible elliptic units as developed by Kato.

First we fix an analytic uniformization of $Y(N)(\mathbb{C})$ which is the same as in [Kat04]. Note that $\sigma \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts from the left on $Y(N)$ by $\sigma v := \alpha(v\sigma)$ for all $v \in (\mathbb{Z}/N\mathbb{Z})^2$. Let $\mathbb{H} := \{\tau \in \mathbb{C} \mid \text{Im} \tau > 0\}$ be the upper half plane, then one has an analytic uniformization

$$
\nu : (\mathbb{Z}/N\mathbb{Z})^2 \times (\Gamma(N) \backslash \mathbb{H}) \xrightarrow{\sim} Y(N)(\mathbb{C})
$$

$$(a, \tau) \mapsto (\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), \alpha),$$

where $\Gamma(N) := \ker(\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/N\mathbb{Z}))$ and $\alpha$ is the level structure given by $(v_1, v_2) \mapsto \frac{a_1 v_1 + a_2}{N}$.

Recall the main theorem from [Kat04].

**Theorem 3.3.1** (Kato [Kat04] 1.10.). Let $\mathcal{E}$ be an elliptic curve over a scheme $S$ and $c$ be an integer prime to 6, then there exists a unit $c\partial_{\mathcal{E}} \in \mathcal{O}(\mathcal{E} \setminus \mathcal{E}[c])^\times$ such that

1. $\text{div}_{c\partial_{\mathcal{E}}} = c^2(0) - \mathcal{E}[c]$
2. $[d]_{c\partial_{\mathcal{E}}} = c\partial_{\mathcal{E}}$ for all $d$ prime to $c$
3. If $\varphi : \mathcal{E} \to \mathcal{E}'$ is an isogeny of elliptic curves over $S$ with $\text{deg} \varphi$ prime to $c$, then
   $$\varphi_*(c\partial_{\mathcal{E}}) = c\partial_{\mathcal{E}'}.$$
4. For $\tau \in \mathbb{H}$ and $z \in \mathbb{C} \setminus c^{-1}(\mathbb{Z}\tau + \mathbb{Z})$ let $c\partial(\tau, z)$ be the value at $z$ of $c\partial_{\mathcal{E}}$ for the elliptic curve $\mathcal{E} = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ over $\mathbb{C}$. Then
   $$c\partial(\tau, z) = q_\tau^{(c-1)/12}(-q_z)^{(c^2)/2}(1-q_z)^{c^2} \gamma_q (q_z) \gamma_q(q_z)^{-1},$$
   where $q_\tau := e^{2\pi i \tau}$, $q_z := e^{2\pi i z}$ and
   $$\gamma_q(t) := \prod_{n \geq 1} (1 - q_n^b t) \prod_{n \geq 1} (1 - q_n^b t^{-1}).$$

Note that $\gamma$ differs from Kato’s $\gamma$.

**Corollary 3.3.2.** Let $t = \frac{a}{N} + \frac{b}{N} \in \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ be an $N$-torsion point, $a, b \in \mathbb{Z}$ and let $\zeta_N := e^{2\pi i/N}$, then

$$c\partial(\tau, t) = q_\tau^{\frac{1}{12}(c^2 B_2(\{\frac{a}{N}\}) - B_2(\{\frac{ca}{N}\}))} (-\zeta_N)^{c^2} \frac{(1 - q_\tau^a \zeta_N^b)^{c^2} \gamma_q(q_\tau^{\frac{a}{N}} \zeta_N^b)}{(1 - q_\tau^{\frac{a}{N}} \zeta_N^b) \gamma_q(q_\tau^{\frac{a}{N}} \zeta_N^b)}.$$

**Proof.** This follows from Theorem 3.3.1 by writing $q_z = q_\tau^{\frac{a}{N}} \zeta_N^b$ and a straightforward computation using $B_2(x) = x^2 - x + \frac{1}{6}$, so that

$$c^2 B_2(\{\frac{a}{N}\}) - B_2(\{\frac{ca}{N}\}) = \frac{(c - c^2)a}{N} + \frac{c^2 - 1}{6}.$$

□
For the elliptic curve $\pi : E \to S$ and an integer $N > 1$ consider the exact sequence of finite étale group schemes

$$0 \to E[\ell^r] \to E[\ell^r N] \to E[N] \to 0.$$ 

In the notation of (2.4.1) we have $H_r = E[\ell^r]$, $G_r = E[\ell^r N]$ and $T = E[N]$. For each section $t \in E[N](S)$ one has the $E[\ell^r]$-torsor $H_r(t) = E[\ell^r](t)$ defined by the cartesian diagram

$$
\begin{array}{ccc}
E[\ell^r](t) & \to & E[\ell^r N] \\
p_{r,t} & & \downarrow \\
S & \to & E[N] \\
& & E.
\end{array}
$$

We denote by $H_r$ and $H_r(t)$ the sheaves associated to $E[\ell^r]$ and $E[\ell^r](t)$ respectively. We also define $H := \bigoplus_{r \geq 1} H_r$ and $H(t) := \bigoplus_{r \geq 1} H_r(t)$.

Let $c > 1$ be an integer with $(c, 6 \ell N) = 1$ and consider the function

$$c \vartheta : E \setminus E[c] \to \mathbb{G}_m.$$ 

By Theorem 3.3.1 this function is norm-compatible

$$[\ell^r]*_c \vartheta = c \vartheta.$$ 

Note that for $t \neq e$ one has $E[\ell^r](t) \subset E \setminus E[c]$, by our condition on $c$. Thus, we can restrict $c \vartheta$ to an invertible function, called $\theta_r$ in Definition 2.6.1, on $E[\ell^r](t)$.

**Definition 3.3.3.** Let

$$E^S_{c,r}(t) := \partial_r(c \vartheta) \in H^1(S, \Lambda_\vartheta(H_r(t))(1))$$

and in the limit

$$E^S_c(t) := \lim_{r \to \infty} E^S_{c,r}(t) \in H^1(S, \Lambda(H(t))(1)).$$

The section $\tau_{r,t}$ from Definition 2.5.1 is given by

$$\tau_{r,t} : E[\ell^r](t) \to E[\ell^r N] \xrightarrow{[N]} E[\ell^r].$$

Its $k$-tensor power gives

$$\tau[^k]_{r,t} \in H^0(E[\ell^r](t), T_{Sym}^k \mathcal{H}_r).$$

Soule’s twisting construction allows now to define:

**Definition 3.3.4.** With the above notations let

$$c \epsilon_{k,r}(t) := p_{r,t*}(\partial_r(c \vartheta) \cup \tau[^k]_{r,t}) \in H^1(S, T_{Sym}^k \mathcal{H}_r(1))$$

and

$$c \epsilon_k(t) := \lim_{r \to \infty} c \epsilon_{k,r}(t) \in H^1(S, T_{Sym}^k \mathcal{H}(1)).$$
We call \( c_{e_\ell}(t) \) the \textit{elliptic Soulé element}. For a function \( \psi : (E[N](S)\setminus\{e\}) \to \mathbb{Z}_\ell \) we let

\[
c_{e_\ell}(\psi) := \sum_{t \in E[N](S)\setminus\{e\}} \psi(t)c_{e_\ell}(t).
\]

Suppose that \( S \) is a scheme such that the group scheme \( E[\ell^rN] \) is isomorphic to \( (\mathbb{Z}/\ell^r\mathbb{Z})^2 \) (for example \( S = Y(\ell^rN) \)). Then one has \( \mathcal{H}_r \cong (\mathbb{Z}/\ell^r\mathbb{Z})^2 \) and the pull-back of \( c_{e_{k,r}}(t) \) to \( S \) is given explicitly by

\[
c_{e_{k,r}}(t) = \sum_{[\ell^r]Q=t} \partial_r(c_\vartheta E(Q)) \cup ([N]Q)^\otimes k \in H^1(S, \text{TSym}^k(\mathbb{Z}/\ell^r\mathbb{Z})^2(1)).
\]

The moment map is in our context

\[
\text{mom}_{k,t} : H^1(S, \Lambda_r(\mathcal{H}_r(t))(1)) \to H^1(S, \text{TSym}^k \mathcal{H}_r(1)) - \text{mom}_{k,t}
\]

or in the limit

\[
\text{mom}_{k} : H^1(S, \Lambda(\mathcal{H}(t))(1)) \to H^1(S, \text{TSym}^k \mathcal{H}(1)).
\]

From the general result Proposition 2.6.8 we get:

**Proposition 3.3.5.** One has

\[
\text{mom}_{k,t}^r(ES_{c,r}^{(t)}) = c_{e_{k,r}}(t)
\]

and

\[
\text{mom}_{k}^t(ES_{c}^{(t)}) = c_{e}(t).
\]

For later use we note:

**Lemma 3.3.6.** With the above notations, one has the relation

\[
c_{e_k}(-t) = (-1)^k c_{e_k}(t).
\]

**Proof.** The norm-compatibility of \( c_\vartheta E \) implies \([-1]_c c_\vartheta E = c_\vartheta E \) and hence

\[
[-1]_c ES_{c,r}^{(-t)} = ES_{c,r}^{(t)}.
\]

The claim follows from the commutative diagram

\[
\begin{array}{ccc}
\Lambda_r(\mathcal{H}_r(-t)) & \xrightarrow{[-1]} & \Lambda_r(\mathcal{H}_r(t)) \\
\text{mom}_{k,t}^r & & \text{mom}_{k}^t \\
\text{TSym}^k \mathcal{H}_r & \xrightarrow{(-1)^k} & \text{TSym}^k \mathcal{H}_r.
\end{array}
\]

\[\square\]

4. **Eisenstein classes, elliptic Soulé elements and the integral \( \ell \)-adic elliptic polylogarithm**

In this section we compare the elliptic Soulé elements with the Eisenstein classes. The idea consists in writing the Eisenstein classes as specializations of the elliptic polylogarithm and then to define an integral version of the elliptic polylogarithm which is directly related to the elliptic Soulé elements.
4.1. **A brief review of the elliptic logarithm sheaf.** We give a brief review of the elliptic polylogarithm and refer for more details to [Hub], the original source [BL94] or to the appendix A in [HK99].

Let \( \pi : E \to S \) be a family of elliptic curves with unit section \( e : S \to E \). We let

\[
\Lambda := \mathbb{Z}/\ell^r \mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell
\]

and consider lisse sheaves of \( \Lambda \)-modules.

A \( \Lambda \)-sheaf \( \mathcal{G} \) is unipotent of length \( n \) with respect to \( \pi \), if it has a filtration

\[
\mathcal{G} = A_0 \mathcal{G} \supset A_1 \mathcal{G} \supset \ldots \supset A_{n+1} \mathcal{G} = 0
\]

such that \( A_k \mathcal{G}/A_{k+1} \mathcal{G} \cong \pi^* \mathcal{F}_k \) for a lisse \( \Lambda \)-sheaf \( \mathcal{F}_k \) on \( S \). Beilinson and Levin show:

**Proposition 4.1.1** ([BL94] Proposition 1.2.6). There is a \( n \)-unipotent sheaf \( \log_{\Lambda}(n) \) together with a section \( 1(n) \in \Gamma(S, e^* \log_{\Lambda}(n)) \) such that for any \( n \)-unipotent \( \Lambda \)-sheaf \( \mathcal{G} \) the homomorphism

\[
\pi_* \text{Hom}_S(\log_{\Lambda}(n), \mathcal{G}) \to e^* \mathcal{G}
\]

\[\phi \mapsto \phi \circ 1(n)\]

is an isomorphism. The pair \((\log_{\Lambda}(n), 1(n))\) is unique up to unique isomorphism.

Obviously, as any \( n-1 \)-unipotent sheaf is also \( n \)-unipotent, one has transition maps \( \log_{\Lambda}(n) \to \log_{\Lambda}(n-1) \) which map \( 1(n) \to 1(n-1) \).

**Definition 4.1.2.** The pro-sheaf \((\log_{\Lambda}, 1) := (\log_{\Lambda}(n), 1(n))\) with the above transition maps is called the **elliptic logarithm sheaf**.

We review some facts about \( \log_{\Lambda} \). Let \( \mathcal{H}_\Lambda := \text{Hom}_S(R^1 \pi_* \Lambda, \Lambda) \), then one has exact sequences

\[
0 \to \pi^* \text{Sym}^n \mathcal{H}_\Lambda \to \log_{\Lambda}(n) \to \log_{\Lambda}(n-1) \to 0,
\]

which in the case that \( \Lambda = \mathbb{Q}_\ell \) induce an isomorphism

\[
e^* \log_{\mathbb{Q}_\ell}(n) \cong \prod_{k=0}^n \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell},
\]

which maps \( 1(n) \) to \( 1 \in \mathbb{Q}_\ell \). Also in the case \( \Lambda = \mathbb{Q}_\ell \) the sheaf \( \log_{\mathbb{Q}_\ell} \) admits an action of \( \mathcal{H}_{\mathbb{Q}_\ell} \)

\[
\text{mult} : \pi^* \mathcal{H}_{\mathbb{Q}_\ell} \otimes \log_{\mathbb{Q}_\ell} \to \log_{\mathbb{Q}_\ell},
\]

which on the associated graded pieces \( \pi^* \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell} \) is just the usual multiplication with \( \mathcal{H}_{\mathbb{Q}_\ell} \)

\[
\pi^* \mathcal{H}_{\mathbb{Q}_\ell} \otimes \pi^* \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell} \to \pi^* \text{Sym}^{k+1} \mathcal{H}_{\mathbb{Q}_\ell}.
\]

The most important fact about the logarithm sheaf is the vanishing of its higher direct images except the second one.
Proposition 4.1.3 ([BL94], Lemma 1.2.7). One has
\[ R^i \pi_* \log_{\Lambda} = \begin{cases} 0 & \text{if } i \neq 2 \\ R^2 \pi_* \Lambda \cong \Lambda(-1) & \text{if } i = 2. \end{cases} \]

Another important fact about the logarithm sheaf is the splitting principle, which we formulate as follows:

Proposition 4.1.4 ([BL94] 1.2.10 (vi), [HK99] Corollary A.2.6.). Let \( \varphi : E \to E' \) be an isogeny and denote by \( \log'_{Q_{\ell}} \) the logarithm sheaf of \( E' \). Then one has an isomorphism
\[ \log_{Q_{\ell}} \cong \varphi^* \log'_{Q_{\ell}}. \]
In particular, for each section \( t \in \ker \varphi(S) \) one has a canonical isomorphism
\[ t^* \log_{Q_{\ell}} \cong e'^* \log'_{Q_{\ell}} \cong e^* \log_{Q_{\ell}} = \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{Q_{\ell}}. \]

Note that in the case where \( \varphi = [N] \) the isomorphism in the proposition induces the multiplication by \( [N]^k \) on the graded pieces \( \pi^* \text{Sym}^k \mathcal{H}_{Q_{\ell}} \) of \( \log_{Q_{\ell}} \).

4.2. The elliptic polylogarithm and Eisenstein classes. The Leray spectral sequence together with Proposition 4.1.3 and the localization sequence give an isomorphism
\[ \text{Ext}^1_{E \setminus \{e\}}(\pi^* \mathcal{H}_{Q_{\ell}}, \log_{Q_{\ell}}(1)) \cong \text{Hom}_S(\mathcal{H}_{Q_{\ell}}, \prod_{n \geq 1} \text{Sym}^n \mathcal{H}_{Q_{\ell}}) \]
(see [HK99] A.3) or [Hub]).

Definition 4.2.1. The (small) elliptic polylogarithm is the class
\[ \text{pol} \in \text{Ext}^1_{E \setminus \{e\}}(\pi^* \mathcal{H}_{Q_{\ell}}, \log_{Q_{\ell}}(1)), \]
which maps to the canonical inclusion \( \mathcal{H}_{Q_{\ell}} \hookrightarrow \prod_{n \geq 1} \text{Sym}^n \mathcal{H}_{Q_{\ell}} \) under the above isomorphism (4.2.1).

Consider a non-zero \( N \)-torsion section \( t \in \mathcal{E}(S) \). If we use the isomorphism \( t^* \log_{Q_{\ell}} \cong \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{Q_{\ell}} \) from Proposition 4.1.4 we get
\[ t^* \text{pol} = (t^* \text{pol}^n)_{n \geq 0} \in \text{Ext}^1_S(\mathcal{H}_{Q_{\ell}}, \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{Q_{\ell}}(1)). \]

To get classes in \( H^1(S, \text{Sym}^n \mathcal{H}_{Q_{\ell}}(1)) \) we use the map
\[ \text{contr}_{\mathcal{H}_{Q_{\ell}}}: \text{Ext}^1_S(\mathcal{H}_{Q_{\ell}}, \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{Q_{\ell}}(1)) \to \text{Ext}^1_S(\mathcal{H}_{Q_{\ell}}, \prod_{n \geq 1} \text{Sym}^{n-1} \mathcal{H}_{Q_{\ell}}(1)) \]
defined by first tensoring an extension with \( \mathcal{H}_{Q_{\ell}}^\vee \), where \( \mathcal{H}_{Q_{\ell}}^\vee \) is the dual of \( \mathcal{H}_{Q_{\ell}} \), and then compose with the contraction map
\[ \mathcal{H}_{Q_{\ell}}^\vee \otimes \text{Sym}^n \mathcal{H}_{Q_{\ell}} \to \text{Sym}^{n-1} \mathcal{H}_{Q_{\ell}}. \]
mapping \( h^y \otimes h_1 \otimes \cdots \otimes h_n \) to \( \frac{1}{n+1} \sum_{j=1}^{n} h^y(h_j)h_1 \otimes \cdots \hat{h}_j \cdots \otimes h_n \).

**Definition 4.2.2.** Let \( N > 1 \) and \( t \in \mathcal{E}[N](S) \) be a non-zero \( N \)-torsion point, then

\[
\text{Eis}^k_{\mathbb{Q}_\ell}(t) := -N^{k-1} \text{cond}_{\mathbb{Q}_\ell}(t^* \text{pol}^{k+1}) \in H^1(S, \text{Sym}^k \mathcal{M}_{\mathbb{Q}_\ell}(1))
\]

is called the \( k \)-th Eisenstein class. If \( \psi : (\mathcal{E}[N](S) \setminus \{e\}) \to \mathbb{Q} \) is a map, we define

\[
\text{Eis}^k_{\mathbb{Q}_\ell}(\psi) := \sum_{t \in \mathcal{E}[N](S) \setminus \{e\}} \psi(t) \text{Eis}^k_{\mathbb{Q}_\ell}(t).
\]

**Remark 4.2.3.** The factor \(-N^{k-1}\) is for historical reasons as the Eisenstein classes were originally defined in a different way by Beilinson (see [Beil86, Theorem 7.3]).

### 4.3. A variant of the elliptic polylogarithm.

For the comparison of the Eisenstein classes with the elliptic Soulé elements a slight variant of the elliptic polylogarithm is useful.

The localization sequence for \( \text{Log}_{\mathbb{Q}_\ell} \) on \( \mathcal{E} \) and the closed subscheme \( \mathcal{E}[c] \) for \( c > 1 \) gives

\[
0 \to H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\text{Log}_{\mathbb{Q}_\ell}}(1)) \xrightarrow{\text{res}} H^0(\mathcal{E}[c], \mathcal{L}_{\text{Log}_{\mathbb{Q}_\ell}} |_{\mathcal{E}[c]}) \to \mathbb{Q}_\ell \to 0
\]

because \( H^1(\mathcal{E}, \mathcal{L}_{\text{Log}_{\mathbb{Q}_\ell}}(1)) = 0 \) and \( H^2(\mathcal{E}, \mathcal{L}_{\text{Log}_{\mathbb{Q}_\ell}}(1)) \cong H^2(\mathcal{E}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell \) by Proposition 4.1.3. In \( H^0(\mathcal{E}[c], \mathcal{L}_{\text{Log}_{\mathbb{Q}_\ell}} |_{\mathcal{E}[c]}) \) we have an element which maps to 0 in \( \mathbb{Q}_\ell \) as follows: We have

\[
H^0(\mathcal{E}[c], \mathbb{Q}_\ell) \subset H^0(\mathcal{E}[c], \mathcal{L}_{\text{Log}_{\mathbb{Q}_\ell}} |_{\mathcal{E}[c]})
\]

and consider \( q : \mathcal{E}[c] \to S \) and the section \( e : S \to \mathcal{E}[c] \). These morphisms induce

\[
e_* : H^0(S, \mathbb{Q}_\ell) \to H^0(\mathcal{E}[c], \mathbb{Q}_\ell)
\]

and

\[
q^* : H^0(S, \mathbb{Q}_\ell) \to H^0(\mathcal{E}[c], \mathbb{Q}_\ell).
\]

**Definition 4.3.1.** Let \( 1 \in H^0(S, \mathbb{Q}_\ell) \) be the constant section which is identically 1 on \( S \). Then we let

\[
c^2 e_*(1) - q^*(1) \in H^0(\mathcal{E}[c], \mathbb{Q}_\ell) \subset H^0(\mathcal{E}[c], \mathcal{L}_{\text{Log}_{\mathbb{Q}_\ell}} |_{\mathcal{E}[c]}).
\]

Note that \( c^2 e_*(1) - q^*(1) \) maps to 0 in \( \mathbb{Q}_\ell \) under the map in (4.3.1). We can now define the variant of the elliptic polylogarithm.

**Definition 4.3.2.** The elliptic polylogarithm \( \text{pol}_c \) associated to \( c^2 e_*(1) - q^*(1) \) is the cohomology class

\[
\text{pol}_c \in H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\text{Log}_{\mathbb{Q}_\ell}}(1))
\]

with \( \text{res}(\text{pol}_c) = c^2 e_*(1) - q^*(1) \in H^0(\mathcal{E}[c], \mathcal{L}_{\text{Log}_{\mathbb{Q}_\ell}} |_{\mathcal{E}[c]}) \).
This cohomology class is related to \( \text{pol} \) as follows. Write

\[
H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}\log_{\mathbb{Q}_\ell}(1)) \cong \text{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\mathbb{Q}_\ell, \mathcal{L}\log_{\mathbb{Q}_\ell}(1))
\]

and define a map

\[
(4.3.2) \quad \text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}} : \text{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\mathbb{Q}_\ell, \mathcal{L}\log_{\mathbb{Q}_\ell}(1)) \to \text{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathcal{H}_{\mathbb{Q}_\ell}, \mathcal{L}\log_{\mathbb{Q}_\ell}(1))
\]

by first tensoring an extension with \( \pi^* \mathcal{H}_{\mathbb{Q}_\ell} \) and then push-out with \( \text{mult} : \pi^* \mathcal{H}_{\mathbb{Q}_\ell} \otimes \mathcal{L}\log_{\mathbb{Q}_\ell} \to \mathcal{L}\log_{\mathbb{Q}_\ell} \) from Equation \((4.1.3)\). This gives

\[
\text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}}(\text{pol}_c) \in \text{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathcal{H}_{\mathbb{Q}_\ell}, \mathcal{L}\log_{\mathbb{Q}_\ell}(1)).
\]

On the other hand consider

\[
[c]^* \text{pol} \in \text{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathcal{H}_{\mathbb{Q}_\ell}, [c]^* \mathcal{L}\log_{\mathbb{Q}_\ell}(1))
\]

and use the isomorphism \( \mathcal{L}\log_{\mathbb{Q}_\ell} \cong [c]^* \mathcal{L}\log_{\mathbb{Q}_\ell} \) from Proposition \((4.1.4)\) to obtain a class in \( \text{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathcal{H}_{\mathbb{Q}_\ell}, \mathcal{L}\log_{\mathbb{Q}_\ell}(1)) \). Restriction of \( \text{pol} \) to \( \mathcal{E} \setminus \mathcal{E}[c] \) gives another class in \( \text{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathcal{H}_{\mathbb{Q}_\ell}, \mathcal{L}\log_{\mathbb{Q}_\ell}(1)) \).

**Proposition 4.3.3.** There is an equality

\[
\text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}}(\text{pol}_c) = c^2 \text{pol} |_{\mathcal{E} \setminus \mathcal{E}[c]} - c[c]^* \text{pol}
\]

in \( \text{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathcal{H}_{\mathbb{Q}_\ell}, \mathcal{L}\log_{\mathbb{Q}_\ell}(1)) \).

**Proof.** As in Equation \((4.2.1)\) we have

\[
\text{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathcal{H}_{\mathbb{Q}_\ell}, \mathcal{L}\log_{\mathbb{Q}_\ell}(1)) \subset \text{Hom}_{\mathcal{E}[c]}(\mathcal{H}_{\mathbb{Q}_\ell}, \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{\mathbb{Q}_\ell})
\]

and we have to show that the images of the elements \( \text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}}(\text{pol}_c) \) and \( c^2 \text{pol} |_{\mathcal{E} \setminus \mathcal{E}[c]} - c[c]^* \text{pol} \) in the right hand side are the same. One has two maps

\[
e_* : \text{Hom}_S(\mathcal{H}_{\mathbb{Q}_\ell}, \mathcal{H}_{\mathbb{Q}_\ell}) \to \text{Hom}_{\mathcal{E}[c]}(\mathcal{H}_{\mathbb{Q}_\ell}, \mathcal{H}_{\mathbb{Q}_\ell})
\]

and

\[
q^* : \text{Hom}_S(\mathcal{H}_{\mathbb{Q}_\ell}, \mathcal{H}_{\mathbb{Q}_\ell}) \to \text{Hom}_{\mathcal{E}[c]}(\mathcal{H}_{\mathbb{Q}_\ell}, \mathcal{H}_{\mathbb{Q}_\ell}).
\]

It follows from the definition of \( \text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}}(\text{pol}_c) \) and \( c^2 \text{pol} |_{\mathcal{E} \setminus \mathcal{E}[c]} - c[c]^* \text{pol} \) that both elements map to

\[
c^2 e_* (\text{id}) - q^*(\text{id}) \in \text{Hom}_{\mathcal{E}[c]}(\mathcal{H}_{\mathbb{Q}_\ell}, \mathcal{H}_{\mathbb{Q}_\ell})
\]

(note that the identification \( \mathcal{L}\log_{\mathbb{Q}_\ell} \cong [c]^* \mathcal{L}\log_{\mathbb{Q}_\ell} \) is multiplication by \( c \) on \( \mathcal{H}_{\mathbb{Q}_\ell} \) so that the residue of \( [c]^* \text{pol} \) is \( \frac{1}{c} \text{id}_{\mathcal{E}[c]} \)). \( \square \)
4.4. The variant of the elliptic polylogarithm and Eisenstein classes.

We are going to explain how specializations of pol are related to the Eisenstein classes.

Let \((c, N) = 1\) and recall from Definition 4.3.2 the class

\[
\text{pol}_c \in \text{Ext}^1_{\mathcal{E} \setminus \{c\}}(\mathbb{Q}_\ell, \log_{\mathbb{Q}_\ell}(1)).
\]

If we pull this back along a non-zero \(N\)-torsion section \(t \in \mathcal{E}[N](S)\), we get, using again \(t^* \log_{\mathbb{Q}_\ell} \cong \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{\mathbb{Q}_\ell}\),

\[
t^* \text{pol}_c \in \text{Ext}^1_S(\mathbb{Q}_\ell, \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{\mathbb{Q}_\ell}(1))
\]

and the \(k\)-th component gives a class

\[
t^* \text{pol}_c^k \in H^1(S, \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell}(1)).
\]

**Proposition 4.4.1.** In \(H^1(S, \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell}(1))\) we have the equality

\[
t^* \text{pol}_c^k = \frac{-1}{N^{k-1}}(c^2 \text{Eis}^k_{\mathbb{Q}_\ell}(t) - c^{-k} \text{Eis}^k_{\mathbb{Q}_\ell}([c]t)).
\]

In particular, for \(c \equiv 1 \mod N\) one has

\[
t^* \text{pol}_c^k = \frac{-1}{N^{k-1}} \frac{c^{k+2} - 1}{c^k} \text{Eis}^k_{\mathbb{Q}_\ell}(t).
\]

**Proof.** According to Proposition 4.3.3 we have

\[
\text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}}(\text{pol}_c^k) = c^2 \text{pol}_c^{k+1} |_{\mathcal{E} \setminus \{c\}} - c[c]^* \text{pol}_c^{k+1}.
\]

Taking the pull-back along \(t\) of the right hand side and applying the map \(\text{contr}_{\mathcal{H}_{\mathbb{Q}_\ell}}\) gives

\[
\frac{-1}{N^{k-1}}(c^2 \text{Eis}^k_{\mathbb{Q}_\ell}(t) - c^{-k} \text{Eis}^k_{\mathbb{Q}_\ell}([c]t))
\]

(note that the isomorphism \(\log_{\mathbb{Q}_\ell} \cong [c]^* \log_{\mathbb{Q}_\ell}\) is multiplication by \(c^{k+1}\) on \(\text{Sym}^{k+1} \mathcal{H}_{\mathbb{Q}_\ell}\) by the remark after Proposition 4.3.4 so that we have to divide by \(c^{k+1}\)). Thus it remains to show that

\[
\text{contr}_{\mathcal{H}_{\mathbb{Q}_\ell}}(t^* \text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}}(\text{pol}_c^k)) = t^* \text{pol}_c^k.
\]

But obviously we have \(\text{contr}_{\mathcal{H}_{\mathbb{Q}_\ell}} \circ t^* \text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}} = \text{contr}_{\mathcal{H}_{\mathbb{Q}_\ell}} \circ \text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}} t^*\), where the last \(\text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}}\) is now on \(\prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{\mathbb{Q}_\ell}\), which gives

\[
\text{contr}_{\mathcal{H}_{\mathbb{Q}_\ell}}(t^* \text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}}(\text{pol}_c^k)) = \text{contr}_{\mathcal{H}_{\mathbb{Q}_\ell}} \circ \text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}}(t^* \text{pol}_c^k).
\]

A direct computation shows that \(\text{contr}_{\mathcal{H}_{\mathbb{Q}_\ell}} \circ \text{mult}_{\mathcal{H}_{\mathbb{Q}_\ell}}\) is the identity map. This gives the desired result. \(\square\)
4.5. **Sheaves of Iwasawa modules and the elliptic logarithm sheaf.**

In this section we relate the elliptic logarithm sheaf to a certain sheaf of Iwasawa modules.

Write $E_r := E$ with structure map $\pi_r : E_r \to S$ and identity section $e_r : S \to E_r$. Let $p_r := [\ell^r] : E_r \to E$ be the $\ell^r$-multiplication map.

**Definition 4.5.1.** Let $\Lambda_n = \mathbb{Z}/\ell^n\mathbb{Z}$, then the geometric elliptic logarithm sheaf with coefficients in $R$ is the inverse system

$$L_{\Lambda_n} := (p_r^* \Lambda_n)$$

where the transition maps are the trace maps $p_{r+1}^* \Lambda_n \to p_r^* \Lambda_n$. Define a ring sheaf by $R_{\Lambda_n} := e^* L_{\Lambda_n}$ and let $1_n \in R_{\Lambda_n}$ be the identity section.

As $E_r$ is an $E[\ell^r]$-torsor over $E$, the sheaf $L_{\Lambda_n}$ is an $\pi^* R_{\Lambda_n}$-module, which is locally free of rank one. Denote by $I_{\Lambda_n}$ the augmentation ideal and by $I_{k\Lambda_n}$ its $k$-th power. We define

$$L^{(k)}_{\Lambda_n} := L_{\Lambda_n} \otimes_{\pi^* R_{\Lambda_n}} \pi^* (R_{\Lambda_n}/I_{k\Lambda_n}).$$

The first main result in this section is the following theorem:

**Theorem 4.5.2.** There is a canonical isomorphism

$$L^{(k)}_{\Lambda_n} \cong \log_{\Lambda_n}^{(k)}$$

which maps $1_n$ to $1_{(k)}_{\Lambda_n}$. Here $\log_{\Lambda_n}^{(k)}$ denotes the constant inverse system.

For the proof we need a characterization of lisse $\Lambda_n$-sheaves on $E$.

**Proposition 4.5.3.** Let $\mathcal{G}$ be a lisse $\Lambda_n$-sheaf on $E$. Then there is an integer $s$ such that

$$\pi_s \text{Hom}_{E}(p_r^* \Lambda_n, \mathcal{G}) \cong e^* \mathcal{G}.$$

In particular, the functor $\mathcal{G} \mapsto e^* \mathcal{G}$ induces an equivalence between the category of lisse $\Lambda_n$-sheaves on $E$ and lisse $\Lambda_n$-sheaves on $S$ with a continuous action of $R_{\Lambda_n}$.

**Proof.** As $p_r$ is finite étale, one has

$$\pi_s \text{Hom}_{E}(p_r^* \Lambda_n, \mathcal{G}) = \pi_r \text{Hom}_{E_r}(\Lambda_n, p_r^* \mathcal{G}) = \pi_r p_r^* \mathcal{G}.$$

As $\mathcal{G}$ is a lisse $\Lambda_n$-sheaf, there is an $s$ such that $p_s^* \mathcal{G}$ comes from $S$, which means $p_s^* \mathcal{G} \cong \pi_s e_s^* p_s^* \mathcal{G}$. This implies that

$$\pi_s \text{Hom}_{E}(p_s^* \Lambda_n, \mathcal{G}) = \pi_s \pi_s e_s^* p_s^* \mathcal{G} \cong e^* \mathcal{G}.$$

This isomorphism allows to define a continuous action of $R_{\Lambda_n}$ on $e^* \mathcal{G}$ (where continuous means that the action factors through some $e^* p_{ss} \Lambda_n$). The inverse functor is given by $\mathcal{F} \mapsto \pi^* \mathcal{F} \otimes_{\pi^* R_{\Lambda_n}} L_{\Lambda_n}$. \qed
Proof of Theorem 4.5.2. From Proposition 4.5.3 we get a morphism of presheaves $\mathcal{L}_\Lambda \to \mathcal{L}(k)_{\Lambda}$ corresponding to $1_{\Lambda}$. It also follows that $\mathcal{L}(k)_{\Lambda}$ is $k$-unipotent because the $R_\Lambda$-module structure on $e^*\mathcal{L}(k)_{\Lambda}$ factors through $R_\Lambda/\mathfrak{I}^{k+1}_{\Lambda}$. In particular, the above morphism factors through $\mathcal{L}(k)_{\Lambda}$. Moreover, by the definition of $\mathcal{L}(k)_{\Lambda}$ we get also a morphism $\mathcal{L}(k)_{\Lambda} \to \mathcal{L}(k)_{\Lambda}$ corresponding to $1_{\Lambda}$. It is straightforward to check that these two morphisms are inverse to each other.

Definition 4.5.4. Define the presheaf $\mathcal{L}$ by

$$\mathcal{L} := (p_r^*\Lambda_{r})_{r \geq 1}$$

where the transition maps are induced by the trace maps and the reduction modulo $\ell^r$. We also let $\mathcal{R} := e^*\mathcal{L}$ with unit section 1 and denote by $\mathcal{I} \subset \mathcal{R}$ the augmentation ideal. Finally, let

$$\mathcal{L}(k) := \mathcal{L} \otimes \pi^*\mathcal{R}_{\mathcal{I}^k+1}.$$  

Theorem 4.5.5. There is a canonical isomorphism

$$\mathcal{L}(k) \cong \mathcal{L}(k)_{\mathcal{I}^k+1}$$

which maps 1 to $1_{\Lambda}$, where $\mathcal{L}(k)_{\mathcal{I}^k+1}$ is $\mathcal{L}(k)_{\mathcal{I}^k+1}$.

Proof. There is a surjective morphism $\mathcal{L} \to \mathcal{L}_\Lambda$ and we get with Theorem 4.5.2 a map

$$\mathcal{L} \to \mathcal{L}_\Lambda \to \mathcal{L}(k)_{\Lambda} \cong \mathcal{L}(k)_{\mathcal{I}^k+1},$$

which induces a morphism $\mathcal{L} \to \mathcal{L}(k)_{\mathcal{I}^k+1}$. As already the map $\mathcal{L} \to \mathcal{L}(k)_{\mathcal{I}^k+1}$ factors through $\mathcal{L}(k)$, we get the desired morphism $\mathcal{L}(k) \to \mathcal{L}(k)_{\mathcal{I}^k+1}$, which is surjective by construction. From the isomorphism $\mathcal{L}(k) \otimes \mathcal{I}^k_{\Lambda} \cong \mathcal{L}(k)$ one deduces that $\mathcal{L}(k)$ is $k$-unipotent. By the definition of $\mathcal{L}(k)_{\mathcal{I}^k+1}$ we get a morphism in the other direction $\mathcal{L}(k)_{\mathcal{I}^k+1} \to \mathcal{L}(k)$ and one checks directly that this is inverse to $\mathcal{L}(k) \to \mathcal{L}(k)_{\mathcal{I}^k+1}$.

We now discuss the relation between the sheaves $\mathcal{L}$ and the sheaves of Iwasawa modules.

Proposition 4.5.6. Let $t : S \to \mathcal{E}$ be an $N$-torsion section, then

$$t^*\mathcal{L} \cong \Lambda(H(t)).$$

Proof. From the commutative diagram

$$\begin{array}{ccc}
\mathcal{E}[\ell^r](t) & \longrightarrow & \mathcal{E}_r \\
pr_{r,t} \downarrow & & \downarrow [\ell^r]=pr_r \\
S & \xrightarrow{t} & \mathcal{E}
\end{array}$$

we get $t^*pr_*\Lambda_r \cong pr_{r,t*}\Lambda_r$ and the result follows from the definitions. □
Finally, we relate the moment map for $\Lambda(\mathcal{H}(t))$ to the splitting principle for $\text{Log}_{\mathbb{Q}_l}$. It follows from Proposition 4.5.6 and Theorem 4.5.5 that we have a morphism

\[(4.5.2) \quad H^1(S, \Lambda(\mathcal{H}(t))(1)) \rightarrow H^1(S, t^*\text{Log}_{\mathbb{Q}_l}^{(k)}(1)) \rightarrow H^1(S, t^*\text{Log}_{\mathbb{Q}_l}^{(k)}(1)),\]

where the last morphism is the canonical map.

**Definition 4.5.7.** The **comparison map** is the inverse limit over $k$ of the maps in (4.5.2):

\[\text{comp} : H^1(S, \Lambda(\mathcal{H}(t))(1)) \rightarrow H^1(S, t^*\text{Log}_{\mathbb{Q}_l}^{(k)}(1)).\]

Recall from Proposition 4.1.4 the isomorphism

\[t^*\text{Log}_{\mathbb{Q}_l} \sim \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{\mathbb{Q}_l}.\]

**Proposition 4.5.8.** Let $t$ be an $N$-torsion section $t : S \rightarrow \mathcal{E}$. There is a commutative diagram

\[\begin{array}{ccc}
H^1(S, \Lambda(\mathcal{H}(t))(1)) & \xrightarrow{\text{comp}} & H^1(S, t^*\text{Log}_{\mathbb{Q}_l}(1)) \\
\downarrow \cong & & \downarrow \cong \\
\tilde{\text{mom}}_k H^1(S, \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{\mathbb{Q}_l}(1)) & \rightarrow & H^1(S, \text{Sym}^k \mathcal{H}_{\mathbb{Q}_l}(1)),
\end{array}\]

with $\tilde{\text{mom}}_k$ as in Definition 2.5.3.

**Proof.** As the diagram

\[\begin{array}{ccc}
\Lambda(\mathcal{H}(t)) & \rightarrow & t^*\text{Log}_{\mathbb{Z}_l}^{(k)} \\
[N]* & \downarrow & [N]* \\
\Lambda(\mathcal{H}) & \rightarrow & e^*\text{Log}_{\mathbb{Z}_l}^{(k)}
\end{array}\]

commutes, it suffices to treat the case $t = 0$. But recall that the identification $t^*\text{Log}_{\mathbb{Q}_l} \cong \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{\mathbb{Q}_l}$ is the composition

\[t^*\text{Log}_{\mathbb{Q}_l} \xrightarrow{[N]*} e^*\text{Log}_{\mathbb{Q}_l} \xrightarrow{[N]*} \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{\mathbb{Q}_l}.\]

This introduces a factor $\frac{1}{[N]^k}$ in front of $\text{Sym}^k \mathcal{H}_{\mathbb{Q}_l}$. Let $I(\mathcal{H}) \subset \Lambda(\mathcal{H})$ be the augmentation ideal. Then the isomorphism

\[\Lambda(\mathcal{H})/I(\mathcal{H})^{k+1} \cong e^*\text{Log}_{\mathbb{Z}_l}^{(k)}\]
induces isomorphisms of the associated graded pieces, which are the $\text{Sym}^n \mathcal{H}$ for $n = 0, \ldots, k$. Therefore

$$H^1(S, \text{Sym}^k \mathcal{H}(1)) \rightarrow H^1(S, \Lambda(\mathcal{H})/I(\mathcal{H})^{k+1}(1)) \cong H^1(S, e^* \log_{\mathbb{E} \ell}^{(k)})$$

$$\rightarrow H^1(S, e^* \log_{\mathbb{Q} \ell}^{(k)}) \xrightarrow{\text{pr}_k} H^1(S, \text{Sym}^k \mathcal{H}_{\mathbb{Q} \ell}(1))$$

is just the comparison map for $\text{Sym}^k \mathcal{H}$. It therefore follows from Lemma 2.2.4 that the diagram

$$\xymatrix{ H^1(S, \Lambda(\mathcal{H})(1)) \ar[d]_{\text{mom}_t} \ar[r] & H^1(S, e^* \log_{\mathbb{Q} \ell}(1)) \ar[d]^{\text{pr}_k} \\
H^1(S, \text{Sym}^k \mathcal{H}_{\mathbb{Q} \ell}(1)) & H^1(S, \text{Sym}^k \mathcal{H}_{\mathbb{Q} \ell}(1)) \ar[l] \}$$

commutes. \hfill \square

4.6. The elliptic polylogarithm and elliptic units. In this section we describe the elliptic polylogarithm in terms of Kato’s norm compatible elliptic units. This will result in a comparison of the Eisenstein classes with the elliptic Soulé elements.

Let $c$ be a positive integer with $(c, 6\ell N) = 1$. We continue to write $\Lambda_r := \mathbb{Z}/\ell^r \mathbb{Z}$ and $E_r := E$ which we consider as an étale cover over $E$ via $p_r := [\ell^r] : E_r \rightarrow E$. This induces a morphism

$$p_r : E_r \setminus E_r[\ell^r c] \rightarrow E \setminus E[c].$$

On $E \setminus E[\ell^r c]$ we have the elliptic unit $c \partial E$ from Theorem 3.3.1 We denote by

$$\Theta_{c,r} := \partial_r(c \partial_E) \in H^1(E \setminus E[\ell^r c], \Lambda_r(1)) \cong H^1(E \setminus E[c], \mathcal{L}_{\Lambda}((1))$$

the image of $c \partial_E$ under the Kummer map $\partial_r$. As the functions $c \partial_E$ are norm-compatible, we can pass to the inverse limit.

**Definition 4.6.1.** We denote by

$$\Theta_c := \varprojlim_{r} \Theta_{c,r} \in \varprojlim_{r} H^1(E \setminus E[c], \mathcal{L}_{\Lambda_r}(1))$$

the inverse limit of the classes $\Theta_{c,r}$.

Recall from Definition 3.3.3 the class

$$E \mathcal{S}_c^{(t)} \in H^1(S, \Lambda(\mathcal{H}(t))(1))$$

and from Proposition 4.5.6 the isomorphism $t^* \mathcal{L} \cong \Lambda(\mathcal{H}(t))$.

**Lemma 4.6.2.** Let $t : S \rightarrow E$ be an $N$-torsion section. Then

$$t^* \Theta_c = E \mathcal{S}_c^{(t)} \in H^1(S, \Lambda(\mathcal{H}(t))(1)).$$

**Proof.** We have $t^* \Theta_c = \partial_t(c \partial_E \mid_{E[\ell^r(t)]})$ so that the formula is clear from the definitions. \hfill \square
As in Definition 4.5.7 one can define a comparison homomorphism
\[ \text{comp} : H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}(1)) \to H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\log Q_r}(1)). \]
Recall from Definition 4.3.2 the class \( \text{pol}_c \in H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\log Q_r}(1)). \)

**Theorem 4.6.3.** Let \((c, 6\ell N) = 1\), then \( \text{comp}(\Theta_c) = \text{pol}_c \in H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\log Q_r}(1)). \)

**Proof.** Consider the commutative diagram
\[
\begin{array}{ccc}
H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}(1)) & \xrightarrow{\text{comp}} & H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\log Q_r}(1)) \\
\text{res} & & \text{res} \\
H^0(\mathcal{E}[c], \mathcal{L}|_{\mathcal{E}[c]}) & \xrightarrow{\text{comp}} & H^0(\mathcal{E}[c], \mathcal{L}_{\log Q_r}|_{\mathcal{E}[c]}).
\end{array}
\]
By definition of \( \text{pol}_c \) its image in \( H^0(\mathcal{E}[c], \mathcal{L}_{\log Q_r}|_{\mathcal{E}[c]}) \) is the element
\[ c^2 e^*_r(1) - q^*(1) \in H^0(\mathcal{E}[c], Q_r) \subset H^0(\mathcal{E}[c], \mathcal{L}_{\log Q_r}|_{\mathcal{E}[c]}). \]
To conclude the proof of Theorem 4.6.3 it suffices to compute the image of \( \text{comp}(\Theta_c) \) in \( H^0(\mathcal{E}[c], \mathcal{L}_{\log Q_r}|_{\mathcal{E}[c]}). \) For this we work at finite level and use the commutative diagram
\[
\begin{array}{ccc}
H^1(\mathcal{E} \setminus \mathcal{E}[c^r], \Lambda_r(1)) & \xrightarrow{\text{res}} & H^0(\mathcal{E}[c^r], \Lambda_r) \\
\cong & & \cong \\
H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\Lambda_r}(1)) & \xrightarrow{\text{res}} & H^0(\mathcal{E}[c], \mathcal{L}_{\Lambda_r}).
\end{array}
\]
The residue of \( \mathcal{E}_{\mathcal{L}} \) is
\[ c^2 e^*_r(1) - q^*(1) \in H^0(\mathcal{E}[c], \Lambda_r) \subset H^0(\mathcal{E}[c], \mathcal{L}_{\Lambda_r}) \]
and taking the inverse limit over \( r \) shows that \( \text{comp}(\Theta_c) \) agrees with \( \text{pol}_c \) in \( H^0(\mathcal{E}[c], \mathcal{L}_{\log Q_r}|_{\mathcal{E}[c]}). \)

4.7. **Eisenstein classes and elliptic Soulé elements.** In this section we finally prove the comparison result between Eisenstein classes and elliptic Soulé elements which is fundamental for the whole paper.

It follows from Theorem 4.6.3 that for \( t \in \mathcal{E}[N](S) \setminus \{e\} \) one has
\[ t^* \text{comp}(\Theta_c) = t^* \text{pol}_c \in \prod_{n \geq 0} H^1(S, \text{Sym}^n \mathcal{H}_{Q_r}(1)). \]

Denote by
\[ \text{pr}_k : \prod_{n \geq 0} H^1(S, \text{Sym}^n \mathcal{H}_{Q_r}(1)) \to H^1(S, \text{Sym}^n \mathcal{H}_{Q_r}(1)) \]
the projection onto the \( k \)-th component.
Theorem 4.7.1. Let \( t \in \mathcal{E}[N](S) \) be a non-zero \( N \)-torsion section. Then one has
\[
\frac{1}{N} c_k(t) = \overline{\text{mom}_i^k} (E^1(t)) = \frac{-1}{N^{k-1}}(c^2 E_{k}(t) - c^{-k} E_{k}(\langle c \rangle)).
\]

Proof. As \( \text{pr}_k(\text{comp}(\Theta_c)) = \text{pr}_k(t^* \text{pol}_c) \) by Theorem 4.6.3, this follows from Lemma 4.6.2 together with Proposition 4.4.1.

Lemma 4.7.2. The Eisenstein class \( E_{k}(t) \) is of parity \((-1)^k\), i.e., one has
\[
E_{k}(t) \equiv (-1)^k E_{k}(\langle c \rangle).
\]

In particular, \( E_{k}(\langle \psi \rangle) = 0 \) if \( \psi \) is not of parity \((-1)^k\), where we say that \( \psi \) has parity \((-1)^k\) if \( \psi(t) = (-1)^k \psi(t) \).

Proof. This follows from Lemma 3.3.6 and Theorem 4.7.1 for \( c \equiv 1 \mod N \).

5. The residue at \( \infty \) of the elliptic Soulé elements

In this section we will compute the residue at \( \infty \) of the elliptic Soulé elements and hence of the Eisenstein classes.

5.1. Definition of the residue at \( \infty \). We are going to describe several variants of the map \( \text{res}_\infty \).

Let \( \zeta_N = e^{2\pi i/N} \in \mathbb{C} \) and consider over \( \text{Spec}(\mathbb{Q}(\zeta_N)((q^{1/N})) \) the Tate curve \( \mathcal{E}_q \) with the level structure \( \alpha : (\mathbb{Z}/N\mathbb{Z})^2 \to \mathcal{E}_q[N] \) given by \( (a,b) \mapsto q^a \zeta_N^b \).

The corresponding map of schemes \( \text{Spec}(\mathbb{Q}(\zeta_N)((q^{1/N})) \to Y(N) \) induces \( \text{Spec}(\mathbb{Q}(\zeta_N)[[q^{1/N}]] \to X(N) \) and hence a map
\[
\infty : \text{Spec}(\mathbb{Q}(\zeta_N) \to X(N),
\]

whose image we call the cusp \( \infty \).

Let \( \hat{X}(N)_\infty \) be the completion of \( X(N) \) at \( \infty \), which can be identified via the above map with \( \text{Spec}(\mathbb{Q}(\zeta_N)[[q^{1/N}]] \). We denote by \( \hat{Y}(N)_\infty \) the generic fibre of \( \hat{X}(N)_\infty \) so that \( \hat{Y}(N)_\infty \cong \text{Spec}(\mathbb{Q}(\zeta_N)((q^{1/N})) \). One has a commutative diagram
\[
\begin{array}{ccc}
\hat{Y}(N)_\infty & \overset{j}{\longrightarrow} & \hat{X}(N)_\infty \\
\| & & \| \\
Y(N) & \overset{j}{\longrightarrow} & X(N)
\end{array}
\]

Note that by purity one has a canonical isomorphism \( \infty^* R^1 j_* \mathbb{Z}/\ell^r \mathbb{Z}(1) \cong \mathbb{Z}/\ell^r \mathbb{Z} \) and that \( R^2 j_* \mathbb{Z}/\ell^r \mathbb{Z}(1) = 0 \).

Definition 5.1.1. We define the residue map
\[
\text{res}_\infty : H^i(\hat{Y}(N), \mathbb{Z}/\ell^r \mathbb{Z}(1)) \to H^{i-1}(\infty, \mathbb{Z}/\ell^r \mathbb{Z})
\]
to be the morphism induced by the edge morphism of the Leray spectral sequence for \( Rj_* \).
Consider the Tate curve $E_q$ over $\hat{Y}(N)_\infty$. For each $r \geq 1$ one has an exact sequence and a commutative diagram

$$0 \longrightarrow \mu_{\ell^rN} \longrightarrow E[\ell^rN] \xrightarrow{p_r} Z/\ell^rNZ \longrightarrow 0$$

and $p_r$ induces a finite morphism

$$p_r : E[\ell^r](t) \rightarrow Z_r(p(t))$$

and hence a morphism of sheaves

$$p_r : \Lambda_r(\mathcal{H}_r(t)) \rightarrow \Lambda_r(Z_r(p(t))).$$

Note that $\Lambda_r(Z_r(p(t)))$ is a constant sheaf over $\hat{X}(N)_\infty$ so that we can consider the composition

$$H^1(\hat{Y}(N)_\infty, \Lambda_r(\mathcal{H}_r(t))(1)) \xrightarrow{p_r} H^1(\hat{Y}(N)_\infty, \Lambda_r(Z_r(t))(1)) \xrightarrow{\text{res}^\infty} H^0(\infty, \Lambda_r(Z_r(p(t)))) = \Lambda_r(Z_r(p(t))).$$

**Definition 5.1.2.** We define

$$\text{res}^\infty : H^1(\hat{Y}(N)_\infty, \Lambda_r(\mathcal{H}_r(t))(1)) \rightarrow H^0(\infty, \Lambda_r(Z_r(p(t)))) = \Lambda_r(Z_r(p(t)))$$

to be the composition of the maps in (5.1.4). We also denote by

$$\text{res}^\infty : H^1(\hat{Y}(N)_\infty, \Lambda(\mathcal{H}(t))(1)) \rightarrow \Lambda(Z(p(t)))$$

the inverse limit.

Over $\hat{Y}(N)_\infty$ one has also the exact sequence

$$0 \rightarrow Z_\ell(1) \xrightarrow{p} \mathcal{H} \xrightarrow{\lambda} Z_\ell \rightarrow 0.$$ 

**Proposition 5.1.3.** The subsheaf $\iota(Z_\ell(1)) \subset \mathcal{H}$ are the invariants of monodromy. In particular, $\iota : Z_\ell(1) \rightarrow \mathcal{H}$ induces an isomorphism $\mathcal{H}(1) \cong \infty^* j_* \mathcal{H}$ and $p : \mathcal{H} \rightarrow Z_\ell$ induces

$$\infty^* R^1 j_* \mathcal{H}(1) \cong \infty^* R^1 j_* Z_\ell(1) \cong Z_\ell.$$

**Proof.** That $\iota(Z_\ell(1)) \subset \mathcal{H}$ are the invariants of monodromy is [SGAT2], Exposé IX, Proposition 2.2.5 and (2.2.5.1)]. From the long exact sequence for $\infty^* R j_*$ we get

$$0 \rightarrow \infty^* j_* Z_\ell \rightarrow \infty^* R^1 j_* Z_\ell(1) \rightarrow \infty^* R^1 j_* \mathcal{H} \rightarrow \infty^* R^1 j_* Z_\ell \rightarrow 0.$$ 

As $\infty^* j_* Z_\ell \cong Z_\ell$ and $Z_\ell \cong \infty^* R^1 j_* Z_\ell(1)$, the first map is an isomorphism and one gets $\infty^* R^1 j_* \mathcal{H} \cong \infty^* R^1 j_* Z_\ell \cong Z_\ell(-1).$ □

**Corollary 5.1.4.** Over $\hat{Y}(N)_\infty$ the maps $\text{Sym}^k Z_\ell(1) \rightarrow \text{Sym}^k \mathcal{H}$ induced by $\iota$ and $\text{Sym}^k \mathcal{H} \rightarrow \text{Sym}^k Z_\ell$ induced by $p$ give rise to isomorphisms

$$Z_\ell(k) \cong \infty^* j_* \text{Sym}^k \mathcal{H} \quad \text{and} \quad \infty^* R^1 j_* \text{Sym}^k \mathcal{H}(1) \cong Z_\ell.$$
\textbf{Proof.} This follows by induction on $k$ from Proposition \textbf{5.1.3} and the exact sequence
\[0 \to \mathbb{Z}_\ell(k) \xrightarrow{\cdot} \text{Sym}^k \mathcal{H} \to \text{Sym}^{k-1} \mathcal{H} \to 0.\]
\[\square\]

\textbf{Definition 5.1.5.} The \textit{residue} at $\infty$ is the morphism
\[H^i(\hat{Y}(N)_\infty, \text{Sym}^k \mathcal{H}_{\ell}(1)) \to H^{i-1}(\infty, \mathbb{Q}_\ell)\]
induced from the edge morphism of the Leray spectral sequence for $Rf_*$ and the isomorphism $\infty^* R^1f_* \text{Sym}^k \mathcal{H}_{\ell}(1) \cong \mathbb{Q}_\ell$.

In the same way one defines a residue at $\infty$
\[\text{res}_\infty : H^i(Y(N), \text{Sym}^k \mathcal{H}_{\ell}(1)) \to H^{i-1}(\infty, \mathbb{Q}_\ell),\]
which obviously factors through the residue map on $H^i(\hat{Y}(N)_\infty, \text{Sym}^k \mathcal{H}_{\ell}(1))$.

The residue maps defined in Definition \textbf{5.1.2} on finite level and in Definition \textbf{5.1.5} with $\mathbb{Q}_\ell$-coefficients are compatible in the following sense.

\textbf{Lemma 5.1.6.} There is a commutative diagram
\[\begin{array}{ccc}
H^1(\hat{Y}(N)_\infty, \Lambda(\mathcal{H}(t))(1)) & \xrightarrow{\text{res}_\infty} & H^0(\infty, \Lambda(Z(p(t)))) \\
\downarrow \text{mom}_k^t & & \downarrow \text{mom}_k^{p(t)} \\
H^1(Y(N)_\infty, \text{Sym}^k \mathcal{H}_{\ell}(1)) & \xrightarrow{\text{res}_\infty} & H^0(\infty, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell.
\end{array}\]

\textit{Moreover, if one uses the isomorphism} $H^0(\infty, \Lambda(Z(p(t)))) \cong \Lambda(Z(p(t)))$ \textit{the map} $\text{mom}_k^{p(t)}$ \textit{is the composition}
\[\text{mom}_k^{p(t)} : \Lambda(Z(p(t))) \twoheadrightarrow \mathbb{Z}_\ell \xrightarrow{1/\text{NK}_k^t} \mathbb{Q}_\ell.\]

\textbf{Proof.} The functoriality of the moment map gives
\[\begin{array}{ccc}
\Lambda(\mathcal{H}(t)) & \xrightarrow{p_t} & \Lambda(Z(p(t))) \\
\downarrow \text{mom}_k^t & & \downarrow \text{mom}_k^{p(t)} \\
\text{TSym}^k \mathcal{H} & \xrightarrow{\text{TSym}_k^p} & \text{TSym}^k \mathbb{Z}_\ell \cong \mathbb{Z}_\ell
\end{array}\]

\textit{and the lemma follows from the definitions if one observes that the canonical map} $\text{Sym}^k \mathbb{Z}_\ell \to \text{TSym}^k \mathbb{Z}_\ell$ \textit{maps the generator of} $\text{Sym}^k \mathbb{Z}_\ell$ \textit{to} $k!$ \textit{times the generator of} $\text{TSym}^k \mathbb{Z}_\ell$. \[\square\]

Finally, we treat the compatibility of the Kummer map and the residue map. The scheme $Z_r(p(t))$ over $\hat{Y}(N)_\infty$ is the disjoint union of copies of $\text{Spec} \mathbb{Q}(\zeta_N)((q^{1/N}))$. An invertible function on $Z_r(p(t))$ is therefore just a collection of units in $\mathbb{Q}(\zeta_N)((q^{1/N}))$ and one can speak of the order of the unit in the uniformizing parameter $q^{1/N}$. If we denote by $\mathbb{Z}[Z_r(p(t))]$ the abelian group of maps $\varphi : Z_r(p(t)) \to \mathbb{Z}$ one gets a homomorphism
\[\text{ord}_\infty : \mathbb{G}_m(Z_r(p(t))) \to \mathbb{Z}[Z_r(p(t))].\]
The norm with respect to the finite morphism $p_r : \mathcal{E}^{[\ell^r]}(t) \to Z_r(p(t))$ induces a homomorphism

$$p_{r*} : \mathbb{G}_m(\mathcal{E}^{[\ell^r]}(t)) \to \mathbb{G}_m(Z_r(t)).$$

With these notations we have:

**Lemma 5.1.7.** The following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{G}_m(\mathcal{E}^{[\ell^r]}(t)) & \xrightarrow{\text{ord}_\infty \circ p_*} & \mathbb{Z}[Z_r(p(t))] \\
\downarrow \partial_r & & \downarrow \\
H^1(\hat{Y}(N)_\infty, \Lambda_r(\mathcal{H}_r(t))(1)) & \xrightarrow{\text{res}} & \Lambda_r[Z_r(p(t))].
\end{array}
$$

Here the right vertical arrow reduces the coefficients modulo $\ell^r$.

**Proof.** Compatibility of the Kummer map with traces and residues. □

5.2. **Computation of the residue at $\infty$ of the elliptic Soulé element.**

Recall the residue map

$$\text{res}_\infty : H^1(\hat{Y}(N)_\infty, \Lambda_r(\mathcal{H}_r(t))(1)) \to H^0(\infty, \Lambda_r(Z_r(p(t)))) \cong \Lambda_r(Z_r(p(t)))$$

from Definition 5.1.2 and the elements

$$\mathcal{E}_c^{(t)} \in H^1(\hat{Y}(N)_\infty, \Lambda_r(\mathcal{H}_r(t))(1))$$

defined in 3.3.3 and

$$B_{2,c,r}^{(p(t))} \in \Lambda_r(Z_r(p(t)))$$

defined in 3.1.1.

The residue of the elliptic Soulé elements will be deduced from the following fundamental result.

**Theorem 5.2.1.** With the above notation one has

$$\text{res}_\infty(\mathcal{E}_c^{(t)}) = B_{2,c,r}^{(p(t))}.$$ 

In particular, taking the inverse limit one has

$$\text{res}_\infty(\mathcal{E}_c^{(t)}) = B_{2,c}^{(p(t))}.$$ 

**Proof.** Recall that $\mathcal{E}_c^{(t)} = \partial_r(,v_c \mathcal{E})$ so that Lemma 5.1.7 implies that we have to compute $\text{ord}_\infty \circ p_*(,v_c \mathcal{E})$. In order to do this we perform a base change from $\hat{Y}(N)_\infty$ to $\hat{Y}(,\ell^r N)_\infty$. We introduce the shorter notation

$$T_r := \hat{Y}(,\ell^r N)_\infty = \text{Spec} \mathbb{Q}(\zeta_{\ell^r N})((q^{1/\ell^r N}))$$

for $r \geq 0$. Over $T_r$ the scheme $\mathcal{E}^{[\ell^r]}(t)$ is isomorphic to the constant scheme

$$Z_r^{2}(t) := \{(x, y) \in (\mathbb{Z}/,\ell^r N)^2 \mid [\ell^r](x, y) = t\}$$
and the map $p : Z_{2}^{2}(t) \to Z_{r}(\langle p\rangle(t))$ is simply given by the projection $pr_{1}$ onto the first coordinate: $(x, y) \mapsto x$. The base change from $T_{0}$ to $T_{r}$ induces a commutative diagram

\[
\begin{array}{ccc}
G_{m}(Z_{2}^{2}(t)_{T_{r}}) & \xrightarrow{pr_{1}} & G_{m}(Z_{r}(\langle pr_{1}\rangle(t))_{T_{r}}) \\
\downarrow & & \downarrow \\
G_{m}(E[\ell^{r}](t)_{T_{0}}) & \xrightarrow{pr} & G_{m}(Z_{r}(\langle p\rangle(t))_{T_{0}})
\end{array}
\]

where the right vertical map is the multiplication of the coefficients with $\ell^{r}$. The commutativity follows from the fact that the morphism $T_{r} \to T_{0}$ is ramified of degree $\ell^{r}$ in $q^{1/N}$. Moreover, as $pr_{1} : Z_{2}^{2}(t) \to Z_{r}(\langle pr_{1}\rangle(t))$ is unramified one has a commutative diagram

\[
\begin{array}{ccc}
G_{m}(Z_{2}^{2}(t)_{T_{r}}) & \xrightarrow{ord_{\infty}} & \mathbb{Z}[Z_{2}^{2}(t)] \\
\downarrow^{pr_{1}} & & \downarrow^{pr_{1}} \\
G_{m}(Z_{r}(\langle pr_{1}\rangle(t))_{T_{r}}) & \xrightarrow{ord_{\infty}} & \mathbb{Z}[Z_{r}(\langle pr\rangle(t))].
\end{array}
\]

For each $(x, y) \in Z_{2}^{2}(t)$ we now have to calculate the order of $(x, y)^{*}_{c}\vartheta_{E}$ at $\infty$. For this we can work on $Y(\ell^{r}N)(\mathbb{C})$. By Corollary 3.3.2 the function $(x, y)^{*}_{c}\vartheta_{E}$ is explicitly given by

\[
\frac{1}{q_{1}}(c^{2}B_{2}(\{\frac{x}{\ell^{r}N}\}) - B_{2}(\{\frac{c_{r}y}{\ell^{r}N}\})) (-\zeta_{c_{r}N}^{y-\frac{c_{r}}{y}} - \frac{1}{c_{r}} - \frac{1}{c_{r}^{N}N} - \frac{c_{y}}{c_{y}^{N}N}^{y-\frac{c_{r}}{y}}) + \frac{\gamma_{q_{1}}(\{\frac{x}{\ell^{r}N}\})^{2}}{(1 - q_{1})}.
\]

As the uniformizing parameter for $Y(\ell^{r}N)(\mathbb{C})$ at $\infty$ is $q_{r}^{1/\ell^{r}N}$, we get

\[
ord_{\infty}(x, y)^{*}_{c}\vartheta_{E} = \frac{\ell^{r}N}{2} \left( c^{2}B_{2}(\{\frac{x}{\ell^{r}N}\}) - B_{2}(\{\frac{c_{r}y}{\ell^{r}N}\}) \right) = B_{2, c_{r}}^{(p(t))}(x).
\]

To compute $pr_{1} \circ ord_{\infty}((x, y)^{*}_{c}\vartheta_{E})$ observe that for a fixed $x \in Z_{r}(\langle pr_{1}\rangle(t))$ there are $\ell^{r}$ elements $y$ with $(x, y) \in Z_{2}^{2}(t)$. As $ord_{\infty}((x, y)^{*}_{c}\vartheta_{E})$ is independent of $y$ this gives

\[
pr_{1} \circ ord_{\infty}((x, y)^{*}_{c}\vartheta_{E}) = \ell^{r}B_{2, c_{r}}^{(p(t))}(x).
\]

With diagram (5.2.1) we finally get

\[
ord_{\infty} \circ p_{*}(c_{r}\vartheta_{E}) = B_{2, c_{r}}^{(p(t))} \in \mathbb{Z}[Z_{r}(\langle p\rangle(t))].
\]

\]

From Theorem 5.2.1 we will deduce a formula for the residue of the elliptic Soulé elements.

Recall the elliptic Soulé element

\[
ce_{k}(t) = mom_{k}(E_{c}(t)) \in H^{1}(Y(N), TSym^{k}\mathcal{H}(1))
\]

from Definition 3.3.4 and consider

\[
\frac{1}{N^{k}}ce_{k}(t) = mom_{k}(E_{c}(t)) \in H^{1}(Y(N), Sym^{k}\mathcal{H}_{\mathbb{Q}_{L}}(1)).
\]
In Definition 5.1.5 and (5.1.6) we have defined the residue map
\[ \text{res}_\infty : H^1(Y(N), \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell}(1)) \to H^0(\infty, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell. \]
In the next theorem we identify \( \mathcal{E}[N] \cong (\mathbb{Z}/N\mathbb{Z})^2 \).

**Theorem 5.2.2.** Let \( t = (a, b) \in \mathcal{E}[N](Y(N)) \setminus \{e\} \), then
\[ \text{res}_\infty(c e_k(t)) = \frac{N^{k+1}}{k!(k+2)} \left( c^2 B_{k+2} \left( \frac{a}{N} \right) - c^{-k} B_{k+2} \left( \frac{ca}{N} \right) \right). \]
In particular, if \( c \equiv 1 \mod N \) one gets
\[ \text{res}_\infty(c e_k(t)) = \frac{N^{k+1}}{k!(k+2)} \left( c^2 - 1 \right) B_{k+2} \left( \frac{a}{N} \right). \]

**Proof of Theorem 5.2.2.** By Proposition 3.3.5, Lemma 5.1.6 and Theorem 5.2.1 one has
\[ \text{res}_\infty(c e_k(t)) = \frac{N^{k+1}}{k!(k+2)} \left( c^2 - 1 \right) B_{k+2} \left( \frac{a}{N} \right), \]
where the last equality is formula (3.1.3).

**Corollary 5.2.3.** Let \( S = Y(N) \) and consider the residue map from
\[ \text{res}_\infty : H^1(Y(N), \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell}(1)) \to H^0(\infty, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell, \]
then if \( t = (a, b) \in (\mathbb{Z}/N\mathbb{Z})^2 \setminus \{(0,0)\} \) one has
\[ \text{res}_\infty(\text{Eis}_{\mathbb{Q}_\ell}(t)) = \frac{N^k}{k!(k+2)} B_{k+2} \left( \frac{a}{N} \right). \]

**Proof.** This is Theorem 4.7.1 together with Corollary 5.2.2 in the case \( c \equiv 1 \mod N \).

**Remark 5.2.4.** The formula differs by a minus sign from the one in [HK99] as we have a different uniformization of the elliptic curve.

6. **The evaluation of the cup-product construction for elliptic Soulé elements**

6.1. **A different description of the cup-product construction.** Consider over \( \tilde{Y}(N)_\infty \) the sheaf \( \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell}(1) \) and the diagram
\[ \tilde{Y}(N)_\infty \xrightarrow{i} \tilde{X}(N)_\infty \xleftarrow{\iota} \infty. \]
Recall from Corollary 5.1.4 the isomorphisms
\[ \mathbb{Q}_\ell(k + 1) \cong \infty^* j_* \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell}(1) \quad \text{and} \quad \infty^* R^1 j_* \text{Sym}^k \mathcal{H}_{\mathbb{Q}_\ell}(1) \cong \mathbb{Q}_\ell. \]
The Leray spectral sequence for $Rj_*$ induces an exact sequence
\begin{equation}
0 \to H^1(\bar{X}(N)_\infty, j_*\text{Sym}^k \mathcal{H}_{Q_\ell}(1)) \to H^1(\bar{Y}(N)_\infty, \text{Sym}^k \mathcal{H}_{Q_\ell}(1)) \xrightarrow{\text{res}_{\infty}} H^0(\infty, Q_\ell) \to 0
\end{equation}
and we consider the Eisenstein class
\[ \text{Eis}_{Q_\ell}^k(\psi) \in H^1(\bar{X}(N)_\infty, j_*\text{Sym}^k \mathcal{H}_{Q_\ell}(1)). \]

The next result gives a different description of the cup-product construction.

**Theorem 6.1.1** ([HK99] Theorem 2.4.1, [Hub] Theorem 4.2.1). Assume that $\text{res}_{\infty}(\text{Eis}_{Q_\ell}^k(\psi)) = 0$ so that one can consider
\[ \text{Eis}_{Q_\ell}^k(\psi) \in H^1(\bar{X}(N)_\infty, j_*\text{Sym}^k \mathcal{H}_{Q_\ell}(1)). \]

Then
\[ \text{Dir}_t(\psi) = \infty^*\text{Eis}_{Q_\ell}^k(\psi) \]
in $H^1(\infty, \infty^*j_*\text{Sym}^k \mathcal{H}_{Q_\ell}(1)) \cong H^1(\infty, Q_{\ell}(k + 1))$.

Recall that
\[ \text{Eis}_{Q_\ell}^k(\psi) = \sum_{t \in \mathcal{E}[N]\{e\}} \psi(t)\text{Eis}_{Q_\ell}^k(t) \]
and note that it is not possible to evaluate the individual classes $\text{Eis}_{Q_\ell}^k(t)$ at $\infty$ as $\text{res}_{\infty}(\text{Eis}_{Q_\ell}^k(t)) \neq 0$.

The idea for the evaluation is as follows: Using Theorem 4.7.1, we have
\[ c_{E_k}(t) = \overline{\text{mom}}^k_x(\mathcal{E}_c(t)) = -N(c^2\text{Eis}_{Q_\ell}^k(t) - c^{-k}\text{Eis}_{Q_\ell}^k([c]t)). \]

Although $\mathcal{E}_c(t)$ still cannot be evaluated at $\infty$, we will define an auxiliary class $\mathcal{B}_c(t)$ in Definition 6.2.2 which has the same residue as $\mathcal{E}_c(t)$. The difference
\[ \mathcal{M}_c(t) := \mathcal{E}_c(t) - \mathcal{B}_c(t) \]
has then residue zero and can be evaluated at $\infty$. For this we use the description of $\mathcal{M}_c(t)$ via an explicit function via the Kummer map. The evaluation at $\infty$ is then just the evaluation of the function at $q = 0$, where $q$ is the local parameter at $\infty$. We conclude by comparing the resulting function with the one defining the Soulé-Deligne classes.

**6.2. The auxiliary class $\mathcal{B}_c(t)$**. Recall from (5.1.3) the finite morphism
\[ p_r : \mathcal{E}[\ell^r](t) \to \mathbb{Z}_r(p(t)) \]
and recall that $\hat{Y}(N)_\infty = \text{Spec} \mathbb{Q}(\zeta_N)((q^{1/N}))$.

On $\mathcal{E}[\ell^r](p(t))$ consider the function
\[ B_{p_{2,c,r}}(t) := \frac{N}{2}(c^2B_2(\{\frac{p_r(t)}{\ell^r N}\}) - B_2(\{\frac{c p_r(t)}{\ell^r N}\})) = \frac{1}{\ell^r}B_{2,c,r}^{(p(t))}(p_r(x)), \]
which defines an element in $\Lambda_r(\mathcal{H}_r(t))$, hence a global section
\[ B_{p_{2,c,r}}(t) \in H^0(\hat{Y}(N)_\infty, \Lambda_r(\mathcal{H}_r(t))). \]
Lemma 6.2.1. The elements
\[ B_{2,c,r}^{(t)} \in H^0(\hat{Y}(N)_\infty, \Lambda_r(\mathcal{H}_r(t))) \]
dis are norm-compatible, i.e., one can define
\[ B_{2,c,r}^{(t)} := \lim_{r} B_{2,c,r}^{(t)} \in H^0(\hat{Y}(N)_\infty, \Lambda(\mathcal{H}(t))). \]

Proof. Consider the push-out by \( p_1 \)
\[
\begin{array}{ccccc}
0 & \longrightarrow & \mathcal{E}[\ell] & \longrightarrow & \mathcal{E}[\ell^r+1N] \\
\downarrow & & \downarrow \phi & & \downarrow = \\
0 & \longrightarrow & \mathbb{Z}/\ell\mathbb{Z} & \longrightarrow & \tilde{\mathcal{E}}_{r+1} \\
\downarrow & & \downarrow \sigma & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}/\ell\mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell^rN\mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell^rN\mathbb{Z} \longrightarrow 0.
\end{array}
\]
This induces on the fibres over \( t \in \mathcal{E}[\mathbb{N}] \)
\[ \Lambda_{r+1}(\mathcal{H}_{r+1}(t)) \cong \Lambda_{r+1}(\tilde{\mathcal{H}}_{r+1}(t)) \cong \Lambda_{r+1}(\mathcal{H}_{r+1}(t)) \]
where \( \tilde{\mathcal{H}}_{r+1}(t) \) is the sheaf associated to the fibre of \( \tilde{\mathcal{E}}_{r+1} \) over \( t \). As \( B_{p_{2,c,r+1}}^{(t)} \)
is a pull-back from \( \mathbb{Z}/\ell^r+1N\mathbb{Z} \) the map \( g \) multiplies the element with the cardinality of the fibres of \( p_1 \), which is \( \ell \). Application of \( \sigma_1 \) to \( \ell B_{p_{2,c,r+1}}^{(t)} \) gives by the norm-compatibility of \( B_{2,c,r+1}^{(t)} \) exactly \( B_{2,c,r}^{(t)} \), which is the desired result.

\[ \square \]

Lemma 6.2.2. Let \( \eta_r \) be the invertible function on \( \mathcal{E}[\ell^r](t) \)
\[ \eta_r(x) := (q^{1/N}) B_{p_{2,c,r}}^{(t)}(x) = q^{1/N} \eta_{p_{2,c,r}}^{(t)}(p_r(x)). \]
Then the class
\[ \mathcal{B} S_{c,r}^{(t)} := \partial_r(\eta_r) \in H^1(\hat{Y}(N)_\infty, \Lambda_r(\mathcal{H}_r(t))(1)) \]
is the image of
\[ \partial_r(q^{1/N}) \otimes B_{p_{2,c,r}}^{(t)} \in H^1(\hat{Y}(N)_\infty, \Lambda_r(1)) \otimes H^0(\hat{Y}(N)_\infty, \Lambda_r(\mathcal{H}_r(t))) \]
under the cup-product. In particular, one can define
(6.2.1) \[ \mathcal{B} S_{c,r}^{(t)} := \lim_{r} \mathcal{B} S_{c,r}^{(t)} \in H^1(\hat{Y}(N)_\infty, \Lambda(\mathcal{H}(t))(1)). \]

Lemma 6.2.3. One has
\[ \text{res}_\infty(\mathcal{B} S_{c,r}^{(t)}) = B_{2,c}^{(p(t))}. \]

Proof. The residue \( \text{res}_\infty \) factors through \( p_{11} : \Lambda_r(\mathcal{H}_r(t)) \to \Lambda_r(Z_r(p(t))) \) and by definition
\[ p_{11}(B_{p_{2,c,r}}^{(t)}) = \ell^r B_{p_{2,c,r}}^{(t)} = B_{2,c,r}^{(p(t))}. \]
Using the cup-product representation of \( \mathcal{B} S_{c,r}^{(t)} \) gives the desired result. \( \square \)
Consider the moment maps
\[ \text{mom}_k^t : H^1(\hat{Y}(N)_{\infty}, \Lambda(\mathcal{H}(t))(1)) \rightarrow H^1(\hat{Y}(N)_{\infty}, \text{TSym}^k \mathcal{H}(1)). \]

**Definition 6.2.4.** We define
\[ c_b_k(t) := \text{mom}_k^t(\mathcal{BS}^{(t)}_c) \in H^1(\hat{Y}(N)_{\infty}, \text{TSym}^k \mathcal{H}(1)). \]

For a function \( \psi : \mathcal{E}[N] \setminus \{e\} \rightarrow \mathbb{Q} \) we let
\[ c_b_k(\psi) := \sum_{t \in \mathcal{E}[N] \setminus \{e\}} \psi(t) c_b_k(t). \]

**Proposition 6.2.5.** Let \( \psi \) be a function such that \( \text{res}_{\infty}(c_b_k(\psi)) = 0 \), then
\[ c_b_k(\psi) = 0 \]
in \( H^1(\hat{Y}(N)_{\infty}, \text{TSym}^k \mathcal{H}(1)). \)

**Proof.** From Lemma 6.2.2 we see that \( c_b_k(t) \) is a cup-product
\[ c_b_k(t) = (\lim_{r} \partial_r(q^{1/N})) \cup \text{mom}_k^t(Bp_{2,c}^{(t)}), \]
where \( \text{mom}_k^t(Bp_{2,c}^{(t)}) \in H^0(\hat{Y}(N)_{\infty}, \text{TSym}^k \mathcal{H}). \) The map \( p : \mathcal{H} \rightarrow \mathbb{Z}_\ell \)
induces an isomorphism
\[ H^0(\hat{Y}(N)_{\infty}, \text{TSym}^k \mathcal{H}) \cong H^0(\hat{Y}(N)_{\infty}, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell \]
because of weight reasons. The image of \( \text{mom}_k^t(\sum_{t \in \mathcal{E}[N] \setminus \{e\}} \psi(t) Bp_{2,c}^{(t)}) \) under
this isomorphism is just \( \text{res}_{\infty}(c_b_k(\psi)) \), which is zero by assumption. \( \square \)

6.3. **Evaluation at \( \infty \) of the modified elliptic Soulé element.** We modify the elliptic Soulé element \( c e_k(t) \) by subtracting the element \( c_b_k(t) \). The resulting element has no residue at \( \infty \) and hence can be evaluated.

**Definition 6.3.1.** We let
\[ \mathcal{M}S_{c,r}^{(t)} := \mathcal{E}S_{c,r}^{(t)} - \mathcal{BS}_{c,r}^{(t)} \in H^1(\hat{Y}(N)_{\infty}, \Lambda_r(\mathcal{H}(t))(1)) \]
and \( \mathcal{M}S_{c}^{(t)} := \lim_{r} \mathcal{M}S_{c,r}^{(t)}. \) Define \( \varepsilon_r := c \partial_r \eta_r^{-1} \in \mathbb{G}_m(\mathcal{E}[t](t)) \) so that
\[ \partial_r(\varepsilon_r) = \mathcal{M}S_{c,r}^{(t)}. \]

Let
\[ c\widehat{me}_k(t) := \widehat{\text{mom}}_k^t(\mathcal{M}S_{c}^{(t)}) = \frac{1}{N} (c e_k(t) - c_b_k(t)) \in H^1(\hat{Y}(N)_{\infty}, \text{Sym}^k \mathcal{H}_{Q_\ell}(1)). \]

By construction, the residue at \( \infty \) of \( c\widehat{m}e_k(t) \) is zero:

**Lemma 6.3.2.** One has \( \text{res}_{\infty}(\mathcal{M}S_{c}^{(t)}) = 0 \) hence
\[ \text{res}_{\infty}(c\widehat{m}e_k(t)) = 0, \]
so that one can consider \( c\widehat{m}e_k(t) \) as a class in
\[ H^1(\hat{X}(N)_{\infty}, j_* \text{Sym}^k \mathcal{H}_{Q_\ell}(1)). \]
Proof. This follows from Lemma \[5.1.6\], Theorem \[5.2.1\] and Lemma \[6.2.3\].

We now want to evaluate
\[
\infty^* (c \tilde{\text{me}}_k(t)) \in H^1(\infty, \mathbb{Q}_\ell(k + 1))
\]
in terms of Soulé-Deligne elements. Recall that over \( \hat{Y} (N)_\infty \) one has an exact sequence
\[
0 \to \mu_N \xrightarrow{\cdot \zeta} \mathcal{E}[N] \xrightarrow{\cdot \zeta} \mathbb{Z}/N\mathbb{Z} \to 0.
\]

**Theorem 6.3.3.** Let \( t \) be a non-zero \( N \)-torsion section of \( \mathcal{E} \). Let \( c \tilde{\text{me}}_k(t) \) be the element defined in \[6.3.1\]. If \( p(t) \neq 0 \) one has \( \infty^* (c \tilde{\text{me}}_k(t)) = 0 \). If \( p(t) = 0 \), \( t \) is in the image of \( \iota \) and will be considered as an \( N \)-th root of unity. Then the formula
\[
\infty^* (c \tilde{\text{me}}_k(t)) = \\
\frac{1}{2k!N^k} \left( \delta^2 (\tilde{c}_{k+1}(t) + (-1)^k \tilde{c}_{k+1}(t^{-1})) + e^{-k} (\tilde{c}_{k+1}(ct) + (-1)^k \tilde{c}_{k+1}(ct^{-1})) \right)
\]
holds in \( H^1(\infty, \mathbb{Q}_\ell(k + 1)) \).

The proof of this theorem is given in the next section.

**Remark 6.3.4.** In fact one can show that in \( H^1(\infty, \mathbb{Q}(k + 1)) \) the identity \( \tilde{c}_{k+1}(t^{-1}) = (-1)^k \tilde{c}_{k+1}(t) \) holds (see for example \[Del89\] 3.14.) but we do not need this fact.

The consequences for the evaluation of the cup-product construction are as follows. Identify \( \mathcal{E}[N] \cong (\mathbb{Z}/N\mathbb{Z})^2 \) and recall that
\[
\text{Eis}_{\mathcal{E}}^k(\psi) = \sum_{(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 \setminus \{(0,0)\}} \psi(a,b) \text{Eis}_{\mathcal{E}}^k(a,b).
\]

**Corollary 6.3.5.** With the above notations suppose that \( \psi \) is a function with \( \text{res}_\infty (\text{Eis}_{\mathcal{E}}^k(\psi)) = 0 \). Then
\[
\text{Dir}_\ell(\psi) = \infty^* (\text{Eis}_{\mathcal{E}}^k(\psi)) = \\
\frac{-1}{k!N} \sum_{b \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}} \psi(0,b) \tilde{c}_{k+1}(\zeta_N^b)
\]
where \( \zeta_N = e^{2\pi i/N} \).

**Proof.** By assumption we have \( \text{res}_\infty (\text{Eis}_{\mathcal{E}}^k(\psi)) = 0 \) which implies \( \text{res}_\infty (c_{b}(\psi)) = 0 \). It follows from Proposition \[6.2.5\] that
\[
\frac{1}{N^k} (c_{b}(\psi) = \sum_{(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 \setminus \{(0,0)\}} \psi(a,b) \infty^* (c \tilde{\text{me}}_k(a,b)).
\]
We now apply Theorem \[6.3.3\] and observe that \( \infty^* (c \tilde{\text{me}}_k(a,b)) = 0 \), if \( a \neq 0 \). By Lemma \[4.7.2\] we can also assume right away that \( \psi(-t) = (-1)^k \psi(t) \). Then one has
\[
\sum_{b \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}} \psi(0,b) \tilde{c}_{k+1}(\zeta_N^b) = \\
\sum_{b \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}} \psi(0,b) (-1)^k \tilde{c}_{k+1}(\zeta_N^{-b})
\]
and
\[
\sum_{b \in \mathbb{Z}/N\mathbb{Z}\setminus\{0\}} \psi(0, b) \tilde{c}_{k+1}(\zeta_N^b) = \sum_{b \in \mathbb{Z}/N\mathbb{Z}\setminus\{0\}} \psi(0, b)(-1)^k \tilde{c}_{k+1}(\zeta_N^{-cb})
\]
by substituting \( b \mapsto -b \). If we use this in the formula of Theorem 6.3.3 in the case of \( c \equiv 1 \mod N \) we get
\[
\frac{1}{N^k} \psi(c, e_k(\psi)) = \sum_{b \in \mathbb{Z}/N\mathbb{Z}\setminus\{0\}} \psi(0, b) \tilde{c}_{k+1}(\zeta_N^b).
\]
On the other hand by Theorem 4.7.1 for \( c \equiv 1 \mod N \)
\[
\frac{1}{N^k} \psi(c, e_k(\psi)) = \left(\frac{c^2 - c^{-k}}{N^{k-1}}\right) \text{Eis}_{k,T}(\zeta_N, \psi),
\]
which gives the desired result. \( \square \)

6.4. Proof of Theorem 6.3.3. We need to introduce some more notation. Over \( \hat{Y}(\mathbb{N}) \) one has
\[
\begin{array}{c}
\mu_{\ell^r N} \xrightarrow{\iota_r} \mathcal{E}[\ell^r N] \xrightarrow{\rho_r} \mathbb{Z}/\ell^r N\mathbb{Z} \\
\mu_N \xrightarrow{\iota} \mathcal{E}[N] \xrightarrow{\rho} \mathbb{Z}/N\mathbb{Z}.
\end{array}
\]

Definition 6.4.1. We denote by \( \mu_{\ell^r N} \langle t \rangle \) the fibre over the \( N \)-torsion section \( t \). We let \( \mathcal{T}_r \langle t \rangle \) be the sheaf associated to \( \mu_{\ell^r N} \langle t \rangle \) and define
\[
\mathcal{T} \langle t \rangle := \lim_{\tau} \mathcal{T}_r \langle t \rangle.
\]
In the case \( t = 0 \) we write \( \mathcal{T} := \mathcal{T} \langle 0 \rangle \).

Note that \( \mu_{\ell^r N} \langle t \rangle \) is empty if \( p(t) \neq 0 \). The maps \( \iota_r \) induce a map of sheaves
\[
(6.4.1) \quad \iota : \Lambda(\mathcal{T} \langle t \rangle) \to \Lambda(\mathcal{H} \langle t \rangle).
\]
On the other hand, pull-back by \( \iota_r \) gives a map of sheaves
\[
(6.4.2) \quad \iota^* : \Lambda(\mathcal{H} \langle t \rangle) \to \Lambda(\mathcal{T} \langle t \rangle)
\]
which is a splitting of \( \iota \). On the other hand the maps \( p_{r,t} : \mathcal{E}[\ell^r] \langle t \rangle \to \mathbb{Z}_{r}(p_r(t)) \) give rise to
\[
(6.4.3) \quad p_t : \Lambda(\mathcal{H} \langle t \rangle) \to \Lambda(\mathbb{Z}(p(t))).
\]

Proposition 6.4.2. The morphisms \( \iota \) and \( p_t \) induce isomorphisms
\[
\Lambda(\mathcal{T}) \cong \Lambda(\mathcal{H})
\]
and \( \infty^* R^1 j_* \Lambda(\mathcal{H}) \cong \Lambda(\mathbb{Z}) \).
Proof. Let $I(\mathcal{F})$, $I(\mathcal{H})$ and $I(Z)$ be the augmentation ideals of $\Lambda(\mathcal{F})$, $\Lambda(\mathcal{H})$ and $\Lambda(Z)$ respectively and $\Lambda(\mathcal{F})^{(k)}$ etc. the quotient by the $k+1$-power of the augmentation ideal. Then by induction on $k$ and Corollary 5.1.4 one has a commutative diagram

$$
0 \longrightarrow \infty^* j_* \text{Sym}^k \mathcal{H} \longrightarrow \infty^* j_* \Lambda(\mathcal{H})^{(k)} \longrightarrow \infty^* j_* \Lambda(\mathcal{H})^{(k-1)}
$$

which implies that the injective morphism $\iota_1$ is also surjective. In the same way one shows $p_1 : \infty^* R^1 j_* \Lambda(\mathcal{H}) \cong \Lambda(Z)$. \qed

With this result the Leray spectral sequence for $Rj_*$ gives

$$
H^1(\tilde{Y}(N)_{\infty}, \Lambda(\mathcal{H}(t))(1)) \longrightarrow H^0(\infty, \Lambda(Z(t)))
$$

$$
0 \longrightarrow H^1(\tilde{X}(N)_{\infty}, j_* \Lambda(\mathcal{H})(1)) \longrightarrow H^1(\tilde{Y}(N)_{\infty}, \Lambda(\mathcal{H}(t))(1)) \longrightarrow H^0(\infty, \Lambda(Z)).
$$

Recall that we want to compute

$$
\infty^*(c \tilde{m} e_k(t)) = \infty^* \widetilde{\text{mom}}^k_c(M \mathcal{E} S_c^{(t)}) = \infty^* \widetilde{\text{mom}}^k_c \circ [N]_!(M \mathcal{E} S_c^{(t)}).
$$

From Lemma 6.3.2 we get that

$$
[N]_!(M \mathcal{E} S_c^{(t)}) \in H^1(\tilde{X}(N)_{\infty}, j_* \Lambda(\mathcal{H})(1)).
$$

Consider the commutative diagram

$$
H^1(\tilde{X}(N)_{\infty}, \Lambda(\mathcal{F})(1)) \longrightarrow H^1(\tilde{X}(N)_{\infty}, j_* \Lambda(\mathcal{H})(1))
$$

$$
\text{mom}^k \downarrow \quad \text{mom}^k \downarrow

H^1(\tilde{X}(N)_{\infty}, Q_t(k+1)) \longrightarrow H^1(\tilde{X}(N)_{\infty}, j_* \text{Sym}^k \mathcal{H}_{Q_t}(1))
$$

$$
\text{mom}^k \downarrow \quad \text{mom}^k \downarrow

H^1(\infty, Q_t(k+1)) \longrightarrow H^1(\infty, \mathcal{F}_{Q_t}(k+1)).
$$

As $\iota_1$ has the splitting $\iota^*$ it follows that we have

$$
\infty^*(c \tilde{m} e_k(t)) = \infty^* \widetilde{\text{mom}}^k_c \circ \iota^* \circ [N]_!(M \mathcal{E} S_c^{(t)}) = \widetilde{\text{mom}}^k_c \circ \infty^* \iota^*(M \mathcal{E} S_c^{(t)}).
$$

At finite level we have

$$
\mathbb{G}_m(E[t^r]) \longrightarrow \mathbb{G}_m(\mu_{t^r}(t))
$$

$$
\partial_r \downarrow \quad \partial_r \downarrow

H^1(\tilde{Y}(N)_{\infty}, \Lambda_r(\mathcal{F}_r(t))(1)) \longrightarrow H^1(\tilde{Y}(N)_{\infty}, \Lambda_r(\mu_{t^r}(t))(1))
$$

which implies that we have with the notation in Definition 6.3.1

$$
\iota^*(M \mathcal{E} S^{(t)}_{c,r}) = \partial_r(\iota^*(\varepsilon_r)).
$$
Lemma 6.4.3. The function \( \iota^*(\varepsilon_r) \in \mathbb{G}_m(\mu_{\ell^r}(t)) \) extends to a function on \( \mu_{\ell^r}(t) \) over all of \( \hat{X}(N)_\infty \) (also denoted by \( \iota^*(\varepsilon_r) \)). The special fibre \( \infty^* \iota^*(\varepsilon_r) \) is the function

\[
\infty^* \iota^*(\varepsilon_r) : \mu_{\ell^r} \langle t \rangle \to \mathbb{G}_m
\]

\[
\beta \mapsto (-\beta)^{\frac{c-2}{2}} \frac{(1 - \beta)^2}{(1 - \beta^c)}. \]

Proof. Note first that there is nothing to show if \( p(t) \neq 0 \). So we assume that \( p(t) = 0 \). That \( \iota^*(\varepsilon_r) \) extends can be checked after a base extension which adjoins all \( \ell^r N \)-th roots of unity. Then by definition and Corollary 3.3.2 \( \iota^*(\varepsilon_r) \) has the form

\[
\iota^*(\varepsilon_r)(\beta) = (-\beta)^{\frac{c-2}{2}} \frac{(1 - \beta)^2}{(1 - \beta^c)} \tilde{\gamma}_q(\beta)^2 \gamma_q(\beta),
\]

where \( \beta \in \mu_{\ell^r}(t) \) and

\[
\tilde{\gamma}_q(\beta) = \prod_{n>0} (1 - q^n \beta)(1 - q^n \beta^{-1}).
\]

From this formula it is clear that \( \iota^*(\varepsilon_r) \) makes sense for \( q = 0 \), i.e., extends to \( \mu_{\ell^r} \langle t \rangle \) over all of \( \hat{X}(N)_\infty \). Putting \( q = 0 \) gives the explicit form of \( \infty^* \iota^*(\varepsilon_r) \) as claimed.

\[ \square \]

Lemma 6.4.4. Assume \( p(t) = 0 \) so that \( t \) is in the image of \( \iota \) and will be considered as an \( N \)-th root of unity. In \( H^1(\infty, \Lambda_r(\mathcal{T}(t))(1)) \) one has the identity

\[
2 \infty^* \iota^*(\mathcal{M}\mathcal{E}\mathcal{S}^{(t)}_c) = \mathcal{E}\mathcal{S}^{(t)}_c + [-1]^* \mathcal{E}\mathcal{S}^{(t-1)}_c,
\]

which gives in the limit

\[
2 \infty^* \iota^*(\mathcal{M}\mathcal{E}\mathcal{S}^{(t)}_c) = \mathcal{E}\mathcal{S}^{(t)}_c + [-1]^* \mathcal{E}\mathcal{S}^{(t-1)}_c
\]

in \( H^1(\infty, \Lambda(\mathcal{T}(t))(1)) \).

Proof. A direct computation gives

\[
\left( (-\beta)^{\frac{c-2}{2}} \frac{(1 - \beta)^2}{(1 - \beta^c)} \right)^2 = \frac{(1 - \beta)^2(1 - \beta^{-1})(1 - \beta^c)(1 - \beta^{-c})}{(1 - \beta^c)(1 - \beta^{-c})}
\]

which implies

\[
\infty^* \iota^*(\varepsilon^2_c) = e_{\Xi} \cdot (e_{\Xi} \circ [-1]),
\]

where \( e_{\Xi} \) is the function defined in (32.5). Thus, applying the Kummer map \( \partial_r \) one has

\[
2 \infty^* \iota^*(\mathcal{M}\mathcal{E}\mathcal{S}^{(t)}_c) = \mathcal{E}\mathcal{S}^{(t)}_c + [-1]^* \mathcal{E}\mathcal{S}^{(t-1)}_c
\]

because \( [-1] : \mu_{\ell^r} \langle t \rangle \cong \mu_{\ell^r} \langle t^{-1} \rangle \). \[ \square \]
To conclude the proof of Theorem 6.3.3 we have to compute
\[ \tilde{\text{mom}}^k_t(CS_c(t) + [-1]^*CS_c^{t-1}). \]
In the definition of \( \tilde{\text{mom}}^k_t \) we use the inverse of the identification \( \mathbb{Q}_\ell(k) \cong \text{Sym}^k \mathbb{Q}_\ell(1) \cong T\text{Sym}^k \mathbb{Q}_\ell(1) \cong \mathbb{Q}(k) \) which maps \( 1 \mapsto k! \). Note also that, as in the proof of Proposition 3.2.6 one has
\[ \text{mom}^k_t([-1]^*CS_c^{t-1}) = (-1)^k \tilde{c}_{k+1}(t^{-1}). \]
With formula (6.4.5) and Proposition 3.2.6 one now gets:

**Corollary 6.4.5.** One has in \( H^1(\mathbb{Q}_\ell(k + 1)) \) the identity
\[ \infty^* (\tilde{c} \text{\it{me}}_k(t)) = \frac{1}{2k!N^k} \left( c^2(\tilde{c}_{k+1}(t) + (-1)^k \tilde{c}_{k+1}(t^{-1})) + c^{-k}(\tilde{c}_{k+1}(ct) + (-1)^k \tilde{c}_{k+1}(ct^{-1})) \right). \]
This proves Theorem 6.3.3.

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