Recursion Operator and Bäcklund Transformation for Super mKdV Hierarchy

A.R. Aguirre, J.F. Gomes, A.L. Retore, N.I. Spano, and A.H. Zimerman

Abstract In this paper we consider the \( \mathcal{N} = 1 \) supersymmetric mKdV hierarchy composed of positive odd flows embedded within an affine \( \hat{sl}(2,1) \) algebra. Its Bäcklund transformations are constructed in terms of a gauge transformation preserving the zero curvature representation. The recursion operator relating consecutive flows is derived and shown to relate their Backlund transformations.

1 Introduction

The algebraic formulation for integrable hierarchies presents itself as a powerful framework in order to discuss its integrable properties, symmetries and soliton solutions. In particular the supersymmetric mKdV hierarchy consists of a set of time evolution (flows) equations obtained from a zero curvature representation involving a two dimensional gauge potential lying within an affine \( \hat{sl}(2,1) \) Kac-Moody algebra and a common infinite set of conservation laws [1, 2, 3].

Moreover, Bäcklund transformation can be employed to construct an infinite sequence of solitons solutions by purely superposition principle and also

A.R. Aguirre
Instituto de Física e Química, Universidade Federal de Itajubá - IFQ/UNIFEI, Av. BPS 1303, 37500-903, Itajubá, MG, Brasil. e-mail: alexis.roaaguirre@unifei.edu.br

A.L. Retore
Physics Department of the University of Miami, Coral Gables, FL 33124 USA. e-mail: retore@ift.unesp.br

N.I. Spano, J.F. Gomes, and A.H. Zimerman
Instituto de Física Teórica - IFT/UNESP, Rua Dr. Bento Teobaldo Ferraz 271, Bloco II, 01140-070, São Paulo, Brasil. e-mail: jfg@ift.unesp.br, natyspano@ift.unesp.br, zimerman@ift.unesp.br
to link nonlinear equations to canonical forms as discussed for many examples in [4]. For the supersymmetric mKdV hierarchy, the Bäcklund transformation was derived for the entire hierarchy by an universal gauge transformation that preserves the zero curvature representation and henceforth the equations of motion [1]. The results obtained in [2, 5, 6] for the super sinh-Gordon were extended to the entire smKdV hierarchy by the construction of a Bäcklund-gauge transformation which connects two field configurations of the same equations of motion [1]. Such structure was first introduced in [7, 8] for the bosonic sine-Gordon theory in order to describe integrable defects in the sense that two solitons solutions are interpolated by a defect, as a set of internal boundary conditions derived from a Lagrangian density located at certain spatial position connecting two types of solutions. The integrability of the model is guaranteed by the gauge invariance of the zero curvature representation.

The \( \mathcal{N} = 1 \) supersymmetric modified Korteweg de-Vries (smKdV) hierarchy in the presence of defects was investigated in [1] through the construction of gauge transformation in the form of a Bäcklund-defect matrix approach. Firstly, we employ the defect matrix associated to the hierarchy which turns out to be the same as for the super sinh-Gordon (sshG) model. The method is general for all flows and as an example we have derived explicitly the Bäcklund equations in components for the first few flows of the hierarchy, namely \( t_1, t_3 \) and \( t_5 \). Finally, this super Bäcklund transformation is employed to introduce type I defects for the supersymmetric mKdV hierarchy.

In this note we propose an alternative derivation for the Bäcklund transformation obtained in [1] by employing a recursion operator. For the bosonic case of the mKdV hierarchy the recursion operator was constructed in [12] and it relates equations of motion for two consecutive time evolutions. We show that the same philosophy can be applied to the supersymmetric mKdV hierarchy to relate Bäcklund transformations for two consecutive flows.

In what follows, we first derive the recursion operator for the supersymmetric mKdV hierarchy directly from the zero curvature representation. For technical reasons we change variables \( u(x, t\mathcal{N}) \) of mKdV equation as \( u(x, t\mathcal{N}) = \partial_x \phi(x, t\mathcal{N}) \) which seems more suitable to deal with Bäcklund transformations. We next conjecture that the Bäcklund transformations for consecutive flows are also related by the same recursion operator. In fact we verify our conjecture for the first few flows generated by \( t_1, t_5 \) and \( t_3 \).

2 The smKdV hierarchy

An integrable hierarchy can be obtained from the zero curvature condition

\[
[\partial_x + A_x, \partial_{t\mathcal{N}} + A_{t\mathcal{N}}] = 0
\] (1)
where, $A_x$ and $A_{t_N}$ are the Lax pair lying into an affine Kac-Moody super-algebra ($\hat{G}$) and $t_N$ represents the time flow of an integrable equation.

Another important key ingredient to construct an integrable hierarchy is a grading operator $Q$ and a constant grade one element $E^{(1)}$ that decomposes the affine superalgebra into the following subspaces

$$\hat{G} = \oplus \hat{G}_m = K(E) \oplus M(E)$$

where $m$ is the degree of the subspace $\hat{G}_m$ according to $Q$, i.e., $[Q, \hat{G}_m] = m\hat{G}_m$, $K(E) = \{x \in \hat{G} \mid [x, E^{(1)}] = 0\}$ is the kernel of $E^{(1)}$ and $M(E)$ is its complement (image).

Now we can define the Lax pair as

$$A_x = E^{(1)} + A_0 + A_{1/2},$$

$$A_{t_N} = D_N^{(N)} + D_N^{(N-1/2)} + \ldots + D_N^{(1/2)} + D_N^{(0)},$$

where $A_0 \in \hat{G}_0 \cap M(E)$, $A_{1/2} \in \hat{G}_{1/2} \cap M(E)$ with their respective components defining the bosonic and fermionic fields of the theory and $D_N^{(m)} \in \hat{G}_m$.

The set of equations (5) can be recursively solved yielding the time evolution equations for the fields in $A_0$ and $A_{1/2}$ as the zero and 1/2 grade components, respectively.

In particular, the construction of the supersymmetric mKdV hierarchy is based on the judicious construction of an affine subalgebra of $\hat{G} = \hat{gl}(2,1)$, with the principal gradation operator $Q = 2d + \frac{1}{2} h_1^{(0)}$ and the constant element $E^{(1)} = K^{(1)} + K^{(2)}$. Its generators may be regrouped as
\[ F_1^{(2n+\frac{1}{2})} = E_{\alpha_1+\alpha_2}^{(n+\frac{1}{2})} - E_{\alpha_2}^{(n+1)} + E_{-\alpha_1-\alpha_2}^{(n+1)} - E_{-\alpha_2}^{(n+\frac{1}{2})}, \]
\[ F_2^{(2n+\frac{1}{2})} = -E_{\alpha_1+\alpha_2}^{(n)} + E_{\alpha_2}^{(n+\frac{1}{2})} + E_{-\alpha_1-\alpha_2}^{(n+\frac{1}{2})} - E_{-\alpha_2}^{(n)}, \]
\[ G_1^{(2n+\frac{1}{2})} = E_{\alpha_1+\alpha_2}^{(n)} + E_{\alpha_2}^{(n+\frac{1}{2})} + E_{-\alpha_1-\alpha_2}^{(n)} + E_{-\alpha_2}^{(n)}. \]

and decomposed as follows (see [9] for details),

\[ M_{bos} = \{ M_2^{(2n)}, M_1^{(2n+1)} \}, \quad M_{fer} = \{ G_1^{(2n+\frac{1}{2})}, G_2^{(2n+\frac{1}{2})} \}, \]
\[ K_{bos} = \{ K_1^{(2n+1)}, K_2^{(2n+1)} \}, \quad K_{fer} = \{ F_1^{(2n+\frac{1}{2})}, F_2^{(2n+\frac{1}{2})} \} \]

Notice that the fermionic generators \( F_i \) and \( G_i, i = 1, 2 \) lying in the Kernel and Image respectively display an explicit \( Z_2 \) structure in their affine indices, in the sense that the semi integers indices \( N + 1/2 \) are decomposed according to \( 2n + 1/2 \) and \( 2n + 3/2 \) disjoints subsets. Another \( Z_2 \) structure arises, now decomposing the integers \( N \) into odd \( (2n + 1) \) and even \( (2n) \) subsets. Assign to the bosonic generators \( \{ K_1, K_2, M_1 \} \) and \( \{ M_2 \} \) the grades \( 2n + 1 \) and \( 2n \) respectively. The affine algebra displayed in the appendix is shown to close consistently with the \( Z_2 \) structures described above.

The \( x \) part of the Lax pair is then constructed from \( A_0 = u(x,t)M_2^{(0)} \) and \( A_{1/2} = \psi(x,t)G_1^{(1/2)} \).

The first equation in the system (5) implies that \( D_N^{(N)} \in K(E) \) and hence \( N = 2n + 1 \). In order to solve equations in (5) we expand \( D_N^{(m)} \) according to its bosonic or fermionic character using latin or greek coefficients, respectively following the grading given in equation (7) ¹, i.e.,

¹ Moreover we use \( \alpha_m, \beta_m \) for grades \( m = 2n + 1/2 \) and \( \gamma_m, \delta_m \) for \( m = 2n + 3/2 \) while \( \alpha_m, \beta_m, \gamma_m, \delta_m \) for \( m = 2n \).
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where the $a_m, b_m, c_m, d_m$ and $\alpha_m, \beta_m, \gamma_m, \delta_m$ are functionals of the fields $u$ and $\psi$.

Substituting this parameterization in the equation (5), one solve recursively for all $D^{(m)}$, $m = 0, \cdots, N$. Starting with the highest grade equation in (5) in which $N = 2n + 1$,

$$[K_1^{(1)} + K_2^{(1)}, a_{2n+1}K_1^{(2n+1)} + b_{2n+1}K_2^{(2n+1)} + c_{2n+1}M_1^{(2n+1)}] = 0 \quad (9)$$

We obtain after using the comutation relations given in the appendix that $c_{2n+1} = 0$. Now substituting this result in the next equation in (5), i.e, the equation for degree $N + 1/2$ we get,

$$\beta_{2n+ \frac{1}{2}} = \frac{1}{2} \psi(a_{2n+1} + b_{2n+1}) \quad (10)$$

From the equation for degree $N$ we find that $a_{2n+1}, b_{2n+1}$ are constants and $d_2 = u_a2n+1 + \psi\alpha_{2n+\frac{1}{2}}$. Proceeding in this way until we reach the equation for degree $N - 2$, we get

$$D_{N}^{(2n+\frac{3}{2})} = \gamma_{2n+\frac{1}{2}}F_1^{(2n+\frac{3}{2})} + \delta_{2n+\frac{1}{2}}G_2^{(2n+\frac{3}{2})},$$

$$D_{N}^{(2n+1)} = a_{2n+1}K_1^{(2n+1)} + b_{2n+1}K_2^{(2n+1)} + c_{2n+1}M_1^{(2n+1)},$$

$$D_{N}^{(2n+\frac{5}{2})} = \alpha_{2n+\frac{1}{2}}F_2^{(2n+\frac{5}{2})} + \beta_{2n+\frac{1}{2}}G_1^{(2n+\frac{5}{2})},$$

$$D_{N}^{(2n)} = d_{2n}M_2^{(2n)},$$

$$D_{N}^{(2n-\frac{1}{2})} = \gamma_{2n-\frac{1}{2}}F_1^{(2n-\frac{1}{2})} + \delta_{2n-\frac{1}{2}}G_2^{(2n-\frac{1}{2})},$$

$$D_{N}^{(2n-1)} = a_{2n-1}K_1^{(2n-1)} + b_{2n-1}K_2^{(2n-1)} + c_{2n-1}M_1^{(2n-1)},$$

$$D_{N}^{(2n-\frac{3}{2})} = \alpha_{2n-\frac{1}{2}}F_2^{(2n-\frac{3}{2})} + \beta_{2n-\frac{1}{2}}G_1^{(2n-\frac{3}{2})},$$

$$D_{N}^{(2n-2)} = d_{2n-2}M_2^{(2n-2)}.$$
\( (N - 1/2) : \quad \partial_x \alpha_{2n+\frac{1}{2}} - u \beta_{2n+\frac{1}{2}} + \psi d_{2n} = 0 \)
\( \partial_x \beta_{2n+\frac{1}{2}} - u \alpha_{2n+\frac{1}{2}} + 2 \delta_{2n-\frac{1}{2}} = 0 \)  
(11)

\( (N - 1) : \quad \partial_x d_{2n} - 2c_{2n-1} + 2\psi \gamma_{2n-\frac{1}{2}} = 0 \)  
(12)

\( (N - 3/2) : \quad \partial_x \gamma_{2n-\frac{3}{2}} - u \delta_{2n-\frac{3}{2}} + \psi c_{2n-1} = 0 \)
\( \partial_x \delta_{2n-\frac{3}{2}} - u \gamma_{2n-\frac{3}{2}} + 2 \beta_{2n-\frac{3}{2}} - \psi (a_{2n-1} + b_{2n-1}) = 0 \)  
(13)

\( (N - 2) : \quad \partial_x a_{2n-1} + 2u c_{2n-1} - 2 \psi \beta_{2n-\frac{3}{2}} = 0 \)
\( \partial_x b_{2n-1} + 2 \psi \beta_{2n-\frac{3}{2}} = 0 \)
\( \partial_x c_{2n-1} - 2d_{2n-2} + 2u a_{2n-1} + 2 \psi a_{2n-\frac{3}{2}} = 0 \)  
(14)

The subsequent equations are all similar to the set above, in the sense that the equations for even grade will correspond to (12), the odd ones will be similar to the set in (14).

For the semi-integer degree equations the following combinations are allowed: if \( (N - \frac{1}{2} - 2m) \) then it corresponds to the set (11) and if the grade can be written as \( (N - \frac{1}{2} - 2m - 1) \) it seems like (13) where \( m \in \mathbb{Z}_+ \).

Then for a specific \( n \in \mathbb{Z}_+ \) these results can be written in the following way,

\[ c_{2n+1} = 0, \quad \beta_{2n+\frac{1}{2}} = \frac{\sqrt{\psi}}{2} (a_{2n+1} + b_{2n+1}), \]
\[ a_{2n+1} = \text{constant} \quad b_{2n+1} = \text{constant} \]
\[ d_{2n} = u a_{2n+1} + \sqrt{\psi} \alpha_{2n+\frac{1}{2}} \]
\( \partial_x a_{2n+\frac{1}{2}} - u \beta_{2n+\frac{1}{2}} + \sqrt{\psi} d_{2n+1-j} = 0 \) \hspace{1cm} \text{(odd) j} \]
\( \partial_x \beta_{2n+\frac{1}{2}} - u \alpha_{2n+\frac{1}{2}} + 2 \delta_{2n+\frac{1}{2}-j} = 0 \) \hspace{1cm} \text{(odd) j} \]
\( \partial_x d_{2n+1-j} - 2c_{2n-j} + 2 \sqrt{\psi} \gamma_{2n+\frac{1}{2}-j} = 0 \) \hspace{1cm} \text{(odd) j} \]
\( \partial_x \gamma_{2n+1-j} - u \delta_{2n+\frac{1}{2}-j} + \sqrt{\psi} c_{2n+1-j} = 0 \) \hspace{1cm} \text{(even) j} \]
\( \partial_x \delta_{2n+\frac{1}{2}-j} - u \gamma_{2n+\frac{1}{2}-j} + 2 \beta_{2n+\frac{1}{2}-j} - \sqrt{\psi} (a_{2n+1-j} + b_{2n+1-j}) = 0 \) \hspace{1cm} \text{(even) j} \]
\( \partial_x a_{2n+1-j} + 2u c_{2n+1-j} - 2 \sqrt{\psi} \beta_{2n+\frac{1}{2}-j} = 0 \) \hspace{1cm} \text{(even) j} \]
\( \partial_x b_{2n+1-j} + 2 \sqrt{\psi} \beta_{2n+\frac{1}{2}-j} = 0 \) \hspace{1cm} \text{(even) j} \]
\( \partial_x c_{2n+1-j} - 2d_{2n-j} + 2u a_{2n+1-j} + 2 \sqrt{\psi} a_{2n+\frac{1}{2}-j} = 0 \) \hspace{1cm} \text{(even) j} \]

(15)

where \( j = 1, \ldots, 2n \).

We proceed in this way until the grade \( (1/2) \) equation in (5) to get

\[ \partial_x \alpha_{\frac{1}{2}} = u \beta_{\frac{1}{2}} - \psi d_0 \]  
(16)

\[ \partial_x \beta_{\frac{1}{2}} = \partial_x \beta_{\frac{1}{2}} - u \alpha_{\frac{1}{2}} \]  
(17)
and the zero grade equation to obtain
\[ \partial_{t_{2n+1}} u = \partial_x d_0 \] (18)

Therefore the problem is to recursively solve this set of equations, finding the respective coefficients for a given value of \( n \) and then substitute them in (18) and (17) to obtain the time evolution of the fields \( u, \bar{\psi} \). For example, if \( n = 0 \) the equations of motion are
\[ \partial_{t_1} \bar{\psi} = \partial_x \bar{\psi}, \quad \partial_{t_1} u = \partial_x u \] (19)

For \( n = 1 \) we have the supersymmetric mKdV equation
\begin{align*}
4 \partial_{t_3} u &= u_{3x} - 6u^2 u_x + 3i\bar{\psi} \partial_x (u \bar{\psi}_x), \quad (20) \\
4 \partial_{t_3} \bar{\psi} &= \bar{\psi}_{3x} - 3u \partial_x (u \bar{\psi}). \quad (21)
\end{align*}

Then, for \( n = 2 \) we have
\begin{align*}
16 \partial_{t_5} u &= u_{5x} - 10(u_x)^3 - 40u(u_x)(u_{2x}) - 10u^2(u_{3x}) + 30u^4(u_x) \\
&\quad + 5i\bar{\psi} \partial_x (u \bar{\psi}_{3x} - 4u^3 \bar{\psi}_x + u_x \bar{\psi}_{2x} + u_{2x} \bar{\psi}_x) + 5i\bar{\psi}_x \partial_x (u \bar{\psi}_{2x}), \quad (22) \\
16 \partial_{t_5} \bar{\psi} &= \bar{\psi}_{5x} - 5u \partial_x (u \bar{\psi}_{2x} + 2u_x \bar{\psi}_x + u_{2x} \bar{\psi}) + 10u^2 \partial_x (u^2 \bar{\psi}) \\
&\quad - 10(u_x) \partial_x (u_x \bar{\psi}). \quad (23)
\end{align*}

3 Recursion operator for smKdV hierarchy

We shall now consider the construction of a set of supersymmetric integrable equations by solving the system in (15). Since the solution of (15) is similar for all values of \( n \) it is expected that there exists a connection among the time flows. The recursion operator is the mathematical object responsible for such connection and will be constructed in this section.

In order to see this we consider the equations for \( N = 2n+1 \) and \( N = 2n+3 \)
solution, in such a way that we can make the following useful identifications,

\( N \) can be solved in terms of the coefficients for \( t \),

\[ \beta_{2n+\frac{1}{2}} = \bar{\psi} \]

\[ d_{2n} = u + \bar{\psi} a_{2n+\frac{1}{2}} \]

\[ \partial_x a_{2n+\frac{1}{2}} - u \beta_{2n+\frac{1}{2}} + \bar{\psi} d_{2n} = 0, \]

\[ \partial_x \beta_{2n+\frac{1}{2}} - u \alpha_{2n+\frac{1}{2}} + 2 \delta_{2n-\frac{1}{2}} = 0, \]

\[ \partial_x d_{2n} - 2c_{2n-1} + 2 \psi \gamma_{2n-\frac{1}{2}} = 0, \]

\[ \partial_x a_{1/2} - u \beta_{1/2} + \bar{\psi} d_0 = 0, \]

Notice that until the equation (32) the alligned equations have the same solution, in such a way that we can make the following useful identifications,

\[ d_{2n+3} = d_0 \bigg|_{2n+1}, \quad \beta_{5/2} \bigg|_{2n+3} = \beta_{1/2} \bigg|_{2n+1}, \quad \alpha_{5/2} \bigg|_{2n+3} = \alpha_{1/2} \bigg|_{2n+1}. \]

The case for \( N = 2n+3 \) has eight additional equations (32)-(39), which can be solved in terms of the coefficients for \( N = 2n+1 \) by the relations (42). Then we will be able to relate the time evolution equations for \( t_{2n+3} \) to the time evolution equations for \( t_{2n+1} \).

Starting with the equation (32) by using (42) we get

\[ \delta_{3/2} \bigg|_{2n+3} = -\frac{1}{2} \partial_{t_{2n+1}} \bar{\psi}. \]
we obtain that the recursion operator and Bäcklund transformation for smKdV hierarchy are given by
\[
\begin{align*}
\alpha_{1/2} \bigg|_{2n+3} &= \frac{1}{4} \int dx (u \partial_x \partial_{t_{2n+1}} \bar{\psi} - \bar{\psi} \partial_x \partial_{t_{2n+1}} u) \\
&\quad + \frac{1}{2} \int dx' u \partial_{t_{2n+1}} u - \bar{\psi} \partial_x \partial_{t_{2n+1}} \bar{\psi} \\
&\quad - \frac{1}{4} \int dx' (u^2 - \bar{\psi} \partial_x \bar{\psi}) \int dx \partial_{t_{2n+1}} (u \bar{\psi}).
\end{align*}
\]

Finally putting these coefficients in the equations of motion (40) and (41) we obtain that the \( t_{2n+3} \) equation of the smKdV hierarchy is given by
\[
\begin{align*}
\frac{\partial u}{\partial t_{2n+3}} &= R_1 \frac{\partial u}{\partial t_{2n+1}} + R_2 \frac{\partial \bar{\psi}}{\partial t_{2n+1}}, \\
\frac{\partial \bar{\psi}}{\partial t_{2n+3}} &= R_3 \frac{\partial u}{\partial t_{2n+1}} + R_4 \frac{\partial \bar{\psi}}{\partial t_{2n+1}}.
\end{align*}
\]
\[ R_2 = \frac{i}{2} u \bar{\psi} \bar{\psi} D - \frac{i}{2} u \bar{\psi}_x - \frac{i}{4} u_x \bar{\psi} + \frac{i}{4} u^2 \bar{\psi} \bar{\psi} D^{-1} u + \frac{i}{2} u_x \bar{\psi} D^{-1} \bar{\psi} D + \frac{i}{2} u_x D^{-1} u \bar{\psi} D^{-1} u \\
\quad - \frac{i}{4} \bar{\psi}_x D^{-1} u + \frac{i}{4} \bar{\psi}_x D^{-1} - u \bar{\psi} D^{-1} u + \frac{i}{2} \bar{\psi}_x D^{-1} u \bar{\psi} D^{-1} \bar{\psi} D \\
\quad - \frac{1}{4} \bar{\psi}_x D^{-1} \bar{\psi} \bar{\psi}_x D^{-1} u, \quad (53) \]

\[ R_3 = \frac{3}{4} u \bar{\psi} - \frac{1}{4} u_x D^{-1} \bar{\psi} D^{-1} u + \frac{i}{2} \bar{\psi}_x D^{-1} - u \bar{\psi} D^{-1} \bar{\psi} D + \frac{1}{4} u D^{-1} \bar{\psi} D \\
\quad - \frac{1}{2} u D^{-1} u \bar{\psi} D^{-1} u - \frac{i}{4} u D^{-1} \bar{\psi} \bar{\psi}_x D^{-1} \bar{\psi} + \frac{1}{4} u D^{-1} u^2 D^{-1} \bar{\psi}, \quad (54) \]

\[ R_4 = \frac{1}{4} D^2 - \frac{1}{4} u^2 - \frac{1}{4} u D^{-1} u - \frac{1}{4} u D^{-1} u D + \frac{1}{4} u D^{-1} u^2 D^{-1} u \\
\quad - \frac{i}{4} u D^{-1} \bar{\psi} \bar{\psi}_x D^{-1} u + \frac{i}{2} \bar{\psi}_x D^{-1} u \bar{\psi} D^{-1} u + \frac{i}{2} u D^{-1} u \bar{\psi} D^{-1} \bar{\psi} D. \quad (55) \]

where \( D = \partial_x \) and \( D^{-1} \) is its inverse.

In terms of \( u = \phi_x \), we get

\[ \frac{\partial \phi}{\partial t_{2n+3}} = R_1 \frac{\partial \phi}{\partial t_{2n+1}} + R_2 \frac{\partial \psi}{\partial t_{2n+1}}, \quad \frac{\partial \psi}{\partial t_{2n+3}} = R_3 \frac{\partial \phi}{\partial t_{2n+1}} + R_4 \frac{\partial \psi}{\partial t_{2n+1}} \quad (56) \]

where \( R_1 = D^{-1} R_1 D, \ R_2 = D^{-1} R_2, \ R_3 = R_3 D, \ R_4 = R_4, \) with

\[ R_1 = \frac{1}{4} D^2 - \phi_x^2 - \phi_{2x} D^{-1} \phi_x + \frac{i}{4} \bar{\psi} \bar{\psi}_x + \frac{i}{4} \phi_x^2 \bar{\psi} D^{-1} \bar{\psi} - \frac{i}{4} \bar{\psi}_x D^{-1} \bar{\psi} \\
\quad - \frac{i}{4} \bar{\psi}_x D^{-1} \bar{\psi} D - \frac{i}{4} \psi_x D^{-1} \phi_x^2 D^{-1} \bar{\psi} + \frac{i}{2} \bar{\psi}_x D^{-1} \phi_x \bar{\psi} D^{-1} \bar{\psi} \\
\quad - \frac{1}{4} \bar{\psi}_x D^{-1} \bar{\psi} \bar{\psi}_x D^{-1} \bar{\psi} + \frac{i}{2} \phi_x D^{-1} \phi_x D^{-1} \bar{\psi}, \quad (57) \]

\[ R_2 = \frac{i}{2} \phi_x \bar{\psi} \bar{\psi} D - \frac{i}{2} \phi_x \bar{\psi} - \frac{i}{4} \phi_{2x} \bar{\psi} + \frac{i}{4} \phi_x^2 D^{-1} \phi_x + \frac{i}{2} \phi_{2x} D^{-1} \bar{\psi} D - \frac{i}{4} \bar{\psi}_x D^{-1} \phi_x \\
\quad + \frac{i}{4} \bar{\psi}_x D^{-1} \phi_x D - \frac{i}{4} \psi_x D^{-1} \phi_x^2 D^{-1} \bar{\psi} + \frac{1}{2} \bar{\psi}_x D^{-1} \phi_x \bar{\psi} D^{-1} \bar{\psi} \\
\quad - \frac{1}{4} \bar{\psi}_x D^{-1} \bar{\psi} \bar{\psi}_x D^{-1} \phi_x + \frac{i}{2} \phi_{2x} D^{-1} \phi_x D^{-1} \bar{\psi}, \quad (58) \]

\[ R_3 = \frac{3}{4} \phi_x \bar{\psi} - \frac{1}{4} \phi_{2x} D^{-1} \bar{\psi} - \frac{1}{4} \psi_x D^{-1} \phi_x + \frac{i}{2} \bar{\psi}_x D^{-1} \phi_x \bar{\psi} D^{-1} \bar{\psi} + \frac{i}{4} \phi_x D^{-1} \bar{\psi} D \\
\quad - \frac{i}{2} \phi_{2x} D^{-1} \phi_x D^{-1} \bar{\psi} - \frac{i}{4} \phi_x D^{-1} \bar{\psi} \bar{\psi}_x D^{-1} \bar{\psi} + \frac{1}{4} \phi_x D^{-1} \phi_x^2 D^{-1} \bar{\psi}, \quad (59) \]
\[ R_4 = \frac{1}{4} D^2 - \frac{1}{4} \phi_x^2 - \frac{1}{4} \phi_{xx} D^{-1} \phi_x - \frac{1}{4} \phi_x D^{-1} \phi_x D + \frac{1}{4} \phi_x D^{-1} \phi_x^2 D^{-1} \phi_x \]

\[ - i \frac{1}{4} \phi_x D^{-1} \bar{\psi} \bar{\psi} D^{-1} \phi_x + \frac{i}{2} \bar{\psi} D^{-1} \phi_x \bar{\psi} D^{-1} \phi_x + \frac{i}{2} \phi_x D^{-1} \phi_x \bar{\psi} D^{-1} \bar{\psi} D \]

(60)

We have explicitly checked that by employing equation (51) for \( n = 0 \) we recover the smKdV equation (20), (21). Also it was verified that (51) for \( n = 1 \), yields the \( t_5 \) flow of the hierarchy (22), (23) as predicted.

4 The Bäcklund transformations for the smKdV hierarchy

In this section we will start by reviewing the systematic construction of the Bäcklund transformation for the smKdV hierarchy, based on the invariance of the zero curvature equation (1) under the gauge transformation,

\[ \partial_\mu K = KA_\mu (\phi_1, \bar{\psi}_1) - A_\mu (\phi_2, \bar{\psi}_2) K, \]

(61)

where \( A_1 = A_x \), \( A_0 = A_{t_{2n+1}} \), \( \partial_1 = \partial_x \), \( \partial_0 = \partial_{t_{2n+1}} \) and assuming the existence of a defect matrix \( K(\phi_1, \bar{\psi}_1, \phi_2, \bar{\psi}_2) \) which maps a field configuration \( \{\phi_1, \bar{\psi}_1\} \) into another \( \{\phi_2, \bar{\psi}_2\} \).

It is important to point out that the spatial Lax operator \( A_x \) is common to all members of the smKdV hierarchy and is given by

\[ A_x = \begin{pmatrix} \lambda^{1/2} - \phi_x & -1 & \sqrt{i} \bar{\psi} \\ -\lambda & \lambda^{1/2} + \phi_x & \sqrt{i} \lambda^{1/2} \bar{\psi} \\ \sqrt{i} \lambda^{1/2} \bar{\psi} & \sqrt{i} \bar{\psi} & 2 \lambda^{1/2} \end{pmatrix}, \]

(62)

Moreover, based on this fact it has been shown that the spatial component of the Bäcklund transformation, and consequently the associated defect matrix, are also common and henceforth universal within the entire bosonic hierarchy [10, 11]. This agrees More recently, in [1], this result has been extended to the supersymmetric mKdV hierarchy with the following defect matrix

\[ K = \begin{pmatrix} \lambda^{1/2} & -\frac{2}{\omega} e^{\phi_+} \lambda^{-1/2} & -\frac{2\sqrt{\lambda}}{\omega} e^{\phi_+} \lambda^{1/2} f_1 \\ -\frac{2}{\omega} e^{-\phi_+} \lambda^{1/2} & \lambda^{1/2} & -\frac{2\sqrt{\lambda}}{\omega} e^{-\phi_+} \lambda f_1 \\ \frac{2\sqrt{\lambda}}{\omega} e^{-\phi_+} f_1 \lambda^{1/2} & \frac{2\sqrt{\lambda}}{\omega} e^{\phi_+} f_1 & \frac{2}{\omega} + \lambda^{1/2} \end{pmatrix} \]

(63)
where \( \phi_\pm = \phi_1 \pm \phi_2 \), \( \omega \) represents the Bäcklund parameter, and \( f_1 \) is an auxiliary fermionic field.

We can now substitute (62) and (63) in the \( x \)-part of the gauge transformation (61), to get

\[
\partial_x \phi_- = \frac{4}{\omega^2} \sinh(\phi_+) - \frac{2i}{\omega} \sinh \left( \frac{\phi_+}{2} \right) f_1 \tilde{\psi}_+, \tag{64}
\]

\[
\tilde{\psi}_- = \frac{4}{\omega} \cosh \left( \frac{\phi_+}{2} \right) f_1, \tag{65}
\]

\[
\partial_x f_1 = \frac{1}{\omega} \cosh \left( \frac{\phi_+}{2} \right) \tilde{\psi}_+. \tag{66}
\]

which is the spatial part of the Bäcklund transformations, where we have denoted \( \psi_\pm = \psi_1 \pm \psi_2 \).

The corresponding temporal part of the Bäcklund is obtained from the gauge transformation in (61) for \( \mu = 0 \), i.e.,

\[
\partial_{t_{2n+1}} K = KA_{t_{2n+1}}(\phi_1, \tilde{\psi}_1) - A_{t_{2n+1}}(\phi_2, \tilde{\psi}_2) K. \tag{67}
\]

where we consider the corresponding temporal part of the Lax pair \( A_{t_{2n+1}} \).

Now, we will consider some examples.

For \( n = 0 \) we have that \( A_x = A_1 \) so the temporal part of the Bäcklund is,

\[
\partial_t \phi_+ = \partial_x \phi_+, \tag{68}
\]

\[
\partial_t f_1 = \partial_x f_1. \tag{69}
\]

This implies that

\[
\partial_t \phi_- = \partial_x \phi_. \tag{70}
\]

Then, for \( n = 0 \) the \( x \) and \( t \) components of the Bäcklund are the same. The next non-trivial example is the smKdV equation \( (n = 1) \),

\[
A_{t_3} = \begin{pmatrix}
  p_0 + \lambda^{1/2} p_{1/2} - \lambda \phi_x + \lambda^{3/2} & p_+ - \lambda & \mu_+ + \lambda^{1/2} \nu_+ + \lambda \sqrt{6} \mu \\
  -p_0 - \lambda^{1/2} p_{1/2} + \lambda \phi_x + \lambda^{3/2} & -p_+ - \lambda & \lambda^{1/2} \mu_- - \lambda \nu_- + \lambda^{3/2} \sqrt{6} \nu \\
  \lambda^{1/2} \mu_- - \lambda \nu_- + \lambda^{3/2} \sqrt{6} \mu & \lambda^{1/2} \mu_+ - \lambda \nu_+ + \lambda \sqrt{6} \nu & 2 \lambda^{1/2} p_{1/2} + 2 \lambda^{3/2}
\end{pmatrix}, \tag{71}
\]

where

\[
p_0 = \frac{1}{4} \left( 2(\phi_x)^3 - \phi_{3x} - 3i \phi_x \bar{\psi} \partial_x \bar{\psi} \right), \quad p_{1/2} = -\frac{i}{2} \bar{\psi} \partial_x \bar{\psi}, \quad \nu_\pm = \sqrt{i} \left( \partial_x \bar{\psi} \pm \bar{\psi} \partial_x \phi \right)
\]

\[
p_\pm = \frac{1}{2} \left( \phi_{2x} \pm (\phi_x)^2 + i \bar{\psi} \partial_x \bar{\psi} \right), \quad \mu_\pm = \sqrt{i} \left( \partial_x \bar{\psi} \pm \phi_x \partial_x \bar{\psi} \pm \bar{\psi} \phi_{2x} - 2 \bar{\psi} (\phi_x)^2 \right). \tag{72}
\]
4 \partial_t \phi_+ = \frac{i}{\omega} \left[ \phi_{2x}^{(+)} \cosh \left( \frac{\phi_+}{2} \right) - \left( \phi_x^{(+)} \right)^2 \sinh \left( \frac{\phi_+}{2} \right) \right] \psi_1 f_1 - \frac{32}{\omega^6} \sinh^3 \phi_+ \\
- \frac{i}{\omega^2} \left[ \phi_x^{(+)} \cosh \left( \frac{\phi_+}{2} \right) \bar{\psi}_1^{(+)} - 2 \sinh \left( \frac{\phi_+}{2} \right) \bar{\psi}_1^{(+)} \right] f_1 \\
+ \frac{2}{\omega^2} \left[ 2 \phi_{2x}^{(+)} \cosh \phi_+ - \left( \phi_x^{(+)} \right)^2 \sinh \phi_+ + i \bar{\psi}_1^{(+)} \bar{\psi}_1^{(+)} \sinh \phi_+ \right] \\
- \frac{96i}{\omega^5} \left[ \sinh \left( \frac{\phi_+}{2} \right) + 4 \sinh^3 \left( \frac{\phi_+}{2} \right) + 3 \sinh^5 \left( \frac{\phi_+}{2} \right) \right] \bar{\psi}_1 f_1, \\
(73)

4 \partial_{t_1} f_1 = \frac{1}{2\omega} \cosh \left( \frac{\phi_+}{2} \right) \left[ 2 \psi_2^{(+)} - \bar{\psi}_1^{(+)} \left( \phi_x^{(+)} \right)^2 \right] + \frac{12}{\omega^5} \sinh^2 \phi_+ \cosh \left( \frac{\phi_+}{2} \right) \bar{\psi}_1 \\
+ \frac{1}{2\omega} \sinh \left( \frac{\phi_+}{2} \right) \left[ \bar{\psi}_1^{+} \phi_2^{(+)} - \phi_x^{(+)} \bar{\psi}_1^{(+)} \right] - \frac{12}{\omega^4} \sinh \phi_+ \cosh^2 \left( \frac{\phi_+}{2} \right) \phi_x^{(+)} f_1, \\
(74)

the corresponding temporal part of the Bäcklund transformation for the smKdV equation, where \( \phi_i^{(\pm)} = \partial^i x \phi_{\pm} \) and \( \bar{\psi}_i^{(\pm)} = \partial^i x \bar{\psi}_{\pm} \). In [1] this procedure have been applied to obtain these transformations for the \( t_5 \) member of the smKdV hierarchy.

5 Recursion operator for the Bäcklund transformations

In this section we will extend the idea of recursion operator to generate the Bäcklund transformation for smKdV hierarchy as an alternative method.

In order to construct the recursion operator for the Bäcklund transformations we consider two different solutions of the equation (56) as

\[
\frac{\partial \phi_1}{\partial t_{2n+3}} = R_1^{(1)} \frac{\partial \phi_1}{\partial t_{2n+1}} + R_2^{(1)} \frac{\partial \bar{\psi}_1}{\partial t_{2n+1}}, \quad \frac{\partial \bar{\psi}_1}{\partial t_{2n+3}} = R_1^{(1)} \frac{\partial \phi_1}{\partial t_{2n+1}} + R_4^{(1)} \frac{\partial \bar{\psi}_1}{\partial t_{2n+1}}
\]

\[
(75)
\]

\[
\frac{\partial \phi_2}{\partial t_{2n+3}} = R_1^{(2)} \frac{\partial \phi_2}{\partial t_{2n+1}} + R_2^{(2)} \frac{\partial \bar{\psi}_2}{\partial t_{2n+1}}, \quad \frac{\partial \bar{\psi}_2}{\partial t_{2n+3}} = R_1^{(2)} \frac{\partial \phi_2}{\partial t_{2n+1}} + R_4^{(2)} \frac{\partial \bar{\psi}_2}{\partial t_{2n+1}}
\]

\[
(76)
\]

where \( R_p^{(i)} = R_i \left( \phi_i^{(p)}, \phi_x^{(p)}, \bar{\psi}_x^{(p)}, \bar{\psi}_x^{(p)}, \bar{\psi}_2^{(p)} \right), \quad i = 1, ..., 4, \quad p = 1, 2. \)

And take the following combination of these solutions

\[
2 \partial_{t_{2n+3}} \phi_- = \left( R_1^{(1)} + R_1^{(2)} \right) \partial_{t_{2n+1}} \phi_- + \left( R_2^{(1)} + R_2^{(2)} \right) \partial_{t_{2n+1}} \bar{\psi} - \left( R_1^{(1)} - R_1^{(2)} \right) \partial_{t_{2n+1}} \phi_+ + \left( R_2^{(1)} - R_2^{(2)} \right) \partial_{t_{2n+1}} \bar{\psi}_+, \quad (77)
\]
At this point, we conjecture that the equations (77) and (78) correspond to the temporal part of the super Bäcklund transformation for an super integrable equation specified by $n$. We note that as well as the consecutive equations of motion within the hierarchy are connected by the same recursion operator, here the same occurs to the Bäcklund transformations. In order to clarify this hypothesis we next consider some examples.

For $n = 0$ we have

\[ 2\partial_{t_2n+3} \tilde{\psi}_- = \left( R_3^{(1)} + R_3^{(2)} \right) \partial_{t_2n+1} \phi_- + \left( R_4^{(1)} + R_4^{(2)} \right) \partial_{t_2n+1} \tilde{\psi}_- + \left( R_3^{(1)} - R_3^{(2)} \right) \partial_{t_2n+1} \phi_+ \]

(78)

By using (68)-(70) we recover equations (73) and (74), ie the time component of the Bäcklund transformantions for $n = 2$ (smKdV).

Next we consider the case for $n = 2$ and using again (73)-(74) we obtain from (77),

\[ 16\partial_{t_2} \phi_- = \phi^{(-)}_{5x} + \frac{3}{8} \left( \phi^{(-)}_x \right)^5 - \frac{5}{2} \phi^{(-)}_x \left( \phi^{(-)}_{2x} \right)^2 + \left( \phi^{(+)}_{2x} \right)^2 - 5\phi^{(-)}_{2x} \phi^{(+)}_{2x} \phi^{(+)_{2x}} \]

\[ + \frac{5i}{4} \left( \psi^- \psi^+(x) + \psi^+ \psi^-(x) \right) \left( \phi^{(-)}_{3x} - \phi^{(-)}_x \right) \left( \phi^{(-)}_x \right)^2 + 3 \left( \phi^{(+)}_x \right)^2 \]

\[ + \frac{5i}{4} \left( \psi^- \psi^+(x) + \psi^+ \psi^-(x) \right) \left( \phi^{(+)_{3x}} - \phi^{(+)_{2x}} \right) \left( \phi^{(+)_{2x}} \right)^2 + 3 \left( \phi^{(-)}_{2x} \right)^2 \]

\[ + \frac{5i}{4} \left( \psi^- \phi^{(-)}_x + \psi^+ \phi^{(+)}_x \right) + \frac{5i}{4} \left( \psi^- \phi^{(-)}_{2x} + \psi^+ \phi^{(+)_{2x}} \right) \]

\[ + \frac{15}{4} \left( \phi^{(-)}_x \right)^3 + \frac{5i}{4} \left( \phi^{(-)}_{2x} \right)^2 + \frac{5i}{4} \left( \phi^{(-)}_{3x} \right)^2 + \frac{5i}{4} \left( \phi^{(-)}_{2x} \right)^2 + \frac{5i}{4} \left( \phi^{(+)}_{2x} \right)^2 \]

(81)

And for the equation (78) we get
Recursion Operator and Bäcklund Transformation for smKdV Hierarchy

16\partial_2 \bar{\psi}_- = \bar{\psi}_{5x} - \frac{5}{4} \bar{\psi}_- \left( \phi_x \phi_{4x} + \phi_x^{(+)} \phi_{4x}^{(+)} \right) - \frac{5}{4} \bar{\psi}_+ \left( \phi_x \phi_{4x}^{(-)} + \phi_x^{(+)} \phi_{4x}^{(-)} \right)
- \frac{5}{4} \left( \bar{\psi}_- \phi_{2x}^{(+)} + \bar{\psi}_+ \phi_{2x}^{(-)} \right) \left[ 2\phi_{3x}^{(+)} - \phi_x^{(+)} \left( \phi_x^{(+)2} + 3\phi_x^{(-)2} \right) \right]
- \frac{5}{4} \left( \bar{\psi}_- \phi_{2x}^{(-)} + \bar{\psi}_+ \phi_{2x}^{(+)} \right) \left[ 2\phi_{3x}^{(-)} - \phi_x^{(-)} \left( \phi_x^{(-)2} + 3\phi_x^{(+)2} \right) \right]
- \frac{5}{8} \bar{\psi}_x^{(-)} \phi_x^{(+)2} \left[ 6\phi_x^{(+)2} + \phi_x^{(-)} \left( \phi_x^{(-)} + 6\phi_x^{(+)2} \right) \right]
- \frac{5}{8} \bar{\psi}_x^{(-)} \phi_x^{(-)} \left[ 6\phi_x^{(-)2} + \phi_x^{(+)2} \left( 4\phi_x^{(-)} \phi_x^{(+)2} + 3\phi_x^{(+)2} \phi_x \right) \right]
- \frac{5}{4} \bar{\psi}_x^{(+)} \phi_x^{(+)2} \left[ 3\phi_x^{(-)2} - \left( (\phi_x^{(-)})^2 + (\phi_x^{(+)})^2 \right) \right]
- \frac{15}{4} \bar{\psi}_x^{(+)} \phi_x^{(+)2} \phi_x^{(-)} + \phi_x^{(-)} \phi_x^{(+)2} \phi_x^{(+)2} - \frac{5}{4} \bar{\psi}_x^{(-)} \phi_x^{(-)} \phi_x^{(+)} \phi_x^{(+)} \phi_x^{(+)2}
- \frac{15}{4} \bar{\psi}_x^{(-)} \phi_x^{(-)} \phi_x^{(+)2} + \phi_x^{(-)} \phi_x^{(+)2} \phi_x^{(+)2} \phi_x^{(+)} \phi_x^{(+)2}
(82)

Now, using the x-part of the Bäcklund transformation (64)-(66) in these two equations we end up with the corresponding Bäcklund transformation for n = 3, that was obtained in [1].

Conclusions

In this note we have considered a hierarchy of supersymmetric equations of motion underlined by an affine construction of a Kac-Moody algebra \(\hat{sl}(2,1)\). These equations of motion were shown to be related by a recursion operator that maps consecutive time flows. Moreover, it was shown that the Bäcklund transformation follows the same relation generated by the recursion operator. Such framework provides a general and systematic method of constructing Bäcklund transformations for the entire hierarchy completing and clarifying the question raised in ref. [1].

An interesting point we would like to point out and is still under investigation is about the construction of the underlying affine Kac-Moody algebra. The key ingredient in the affinization was the decomposition of the integers and semi integers numbers according to a \(Z_2\) structure assigned to bosonic and fermionic generators defined in (6). For higher rank algebras we expect to systematize such construction decomposing both integers and semi-integers in disjoint subsets compatible with the closure of the algebra, e.g. \(Z_k\) for \(\hat{sl}(k,1)\).
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Appendix

Here we resume the commutation and Anti-commutation relations for the $\mathfrak{sl}(2,1)$ affine Lie superalgebra

\[
\begin{align*}
[K_1^{(2n+1)}, K_2^{(2m+1)}] &= 0, \\
[M_1^{(2n+1)}, K_1^{(2m+1)}] &= 2M_2^{2(n+m+1)} + (n + m)\delta_{n+m+1,0}\hat{c}, \\
[M_1^{(2n+1)}, K_2^{(2m+1)}] &= 0, \\
[K_2^{(2n+1)}, K_2^{(2m+1)}] &= -(n - m)\delta_{n+m+1,0}\hat{c}, \\
[M_2^{(2n)}, K_1^{(2m+1)}] &= 2M_1^{2(n+m)+1}, \\
[M_2^{(2n)}, K_2^{(2m+1)}] &= 0, \\
[M_1^{(2n+1)}, M_2^{(2m+1)}] &= -2K_1^{2(n+m)+1}, \\
[M_1^{(2n+1)}, M_1^{(2m+1)}] &= -(n - m)\delta_{n+m+1,0}\hat{c}, \\
[M_2^{(2n)}, M_2^{(2m)}] &= (n - m)\delta_{n+m,0}\hat{c}, \\
[K_1^{(2n+1)}, K_1^{(2m+1)}] &= (n - m)\delta_{n+m+1,0}\hat{c}, \\
[F_1^{(2n+3/2)}, K_1^{(2m+1)}] &= -[F_1^{(2n+3/2)}, K_2^{(2m+1)}] = F_2^{2(n+m+1)+1/2}, \\
[F_2^{(2n+1/2)}, K_1^{(2m+1)}] &= -[F_2^{(2n+1/2)}, K_2^{(2m+1)}] = F_1^{2(n+m)+3/2}, \\
[M_1^{(2n+1)}, F_1^{(2m+3/2)}] &= G_2^{2(n+m)+1/2}, \\
[M_1^{(2n+1)}, F_2^{(2m+3/2)}] &= [M_2^{(2n)}, F_1^{(2m+3/2)}] = G_2^{2(n+m)+3/2}, \\
[M_2^{(2n)}, F_2^{(2m+1/2)}] &= -G_2^{2(n+m)+1/2}, \\
[G_1^{(2n+1/2)}, K_1^{(2m+1)}] &= -G_2^{2(n+m)+3/2}, \\
[G_1^{(2n+1/2)}, K_2^{(2m+1)}] &= -G_2^{2(n+m)+3/2}, \\
[G_2^{(2n+3/2)}, K_1^{(2m+1)}] &= -G_1^{2(n+m+1)+1/2}, \\
[G_2^{(2n+3/2)}, K_2^{(2m+1)}] &= -G_1^{2(n+m+1)+1/2}, \\
[M_1^{(2n+1)}, G_1^{(2m+1/2)}] &= -F_2^{2(n+m)+3/2}, \\
[M_2^{(2n+1)}, G_2^{(2m+3/2)}] &= -F_2^{2(n+m+1)+1/2}, \\
[M_2^{(2n)}, G_1^{(2m+1/2)}] &= -F_2^{2(n+m)+1/2}, \\
[M_2^{(2n)}, G_2^{(2m+3/2)}] &= -F_1^{2(n+m)+3/2}, \\
\{F_1^{(2n+3/2)}, F_2^{(2m+1/2)}\} &= [(2n + 1) - 2m]\delta_{n+m+1,0}\hat{c},
\end{align*}
\]
\begin{align}
\{ F_1^{(2n+3/2)}, F_1^{(2m+3/2)} \} &= 2(K_2^{2(n+m+1)+1} + K_1^{2(n+m+1)+1}), \\
\{ F_2^{(2n+1/2)}, F_2^{(2m+1/2)} \} &= -2(K_2^{2(n+m)+1} + K_1^{2(n+m)+1}), \\
\{ F_2^{(2n+1/2)}, G_1^{(2m+1/2)} \} &= 2M_1^{2(n+m)+1}, \\
\{ F_1^{(2n+3/2)}, G_2^{(2m+3/2)} \} &= -2M_1^{2(n+m+1)+1}, \\
\{ F_1^{(2n+3/2)}, G_1^{(2m+1/2)} \} &= 2M_2^{2(n+m+1)} + [(2n + 1) + 2m]\delta_{n+m+1,0\hat{c}}, \\
\{ F_2^{(2n+1/2)}, G_2^{(2m+3/2)} \} &= -2M_2^{2(n+m)+1} - [2n + 2m + 1]\delta_{n+m+1,0\hat{c}}, \\
\{ G_1^{(2n+1/2)}, G_2^{(2m+1/2)} \} &= [2n - (2m + 1)]\delta_{n+m+1,0\hat{c}}, \\
\{ G_1^{(2n+3/2)}, G_1^{(2m+1/2)} \} &= 2(K_2^{2(n+m)+1} - K_1^{2(n+m)+1}), \\
\{ G_2^{(2n+3/2)}, G_2^{(2m+3/2)} \} &= -2(K_2^{2(n+m+1)+1} - K_1^{2(n+m+1)+1}).
\end{align}

(83)

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