ON CERTAIN TOEPLITZ OPERATORS AND ASSOCIATED COMPLETELY POSITIVE MAPS

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Abstract. We study Toeplitz operators with respect to a commuting n-tuple of bounded operators which satisfies some additional conditions coming from complex geometry. Then we consider a particular such tuple on a function space. The algebra of Toeplitz operators with respect to that particular tuple becomes naturally homeomorphic to $L^\infty$ of a certain compact subset of $\mathbb{C}^n$. Dual Toeplitz operators are characterized. En route, we prove an extension type theorem which is not only important for studying Toeplitz operators, but also has an independent interest because dilation theorems do not hold in general for $n > 2$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk while $\mathbb{D}^n$, $\overline{\mathbb{D}}^n$ and $\mathbb{T}^n$ denote the open polydisk, the closed polydisk, and the n-torus, respectively in n-dimensional complex plane for $n \geq 2$.

When does a bounded operator $X$ on a Hilbert space $\mathcal{H}$ satisfy $P^*XP = X$ for a contraction $P$ or more generally when does $X$ satisfy $\sum T_n^*XT_n = X$ where $\{T_n\}_{n \geq 1}$ is a sequence of commuting bounded operators satisfying $\sum T_n^*T_n \leq I$? This question has intrigued many investigations - from the point of completely positive (c.p.) maps and from

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the perspective of Toeplitz operators. When $P$ is the unilateral shift on the Hardy space, Brown and Halmos essentially started a new field which now flourishes as the study of Toeplitz operators.

This has been extended to higher dimensions, i.e., to the setting of the Euclidean unit ball for example, by Prunaru, in [21] and to the polydisk by many authors, see the book by Bottcher and Silverman [7] for various developments. Toeplitz operators have been studied on the Bergman space, see the survey article [3]. This note blends the two aspects. We study operators $X$ which satisfy a Toeplitz condition originating from a domain in $\mathbb{C}^n$.

The point of view of c.p. maps has greatly influenced dilation theory of one and several operators. Indeed, Sz.-Nagy's famous dilation theorem of a contraction $T$ acting on $\mathcal{H}$ can be obtained by first considering $T$ to be pure, i.e., $(T^*)^n \to 0$ strongly and then considering the canonical c.p. map $\varphi$ from the $C^*$-algebra $C^*(M_z)$ generated by the unilateral shift $M_z$ into $B(\mathcal{H})$ which sends $M_z$ to $T$. Using the Stinespring dilation, one can dilate $\varphi$ to a representation $\pi$. The operator $\pi(M_z)$ gives the required dilation. The argument extends from pure to non-pure and then to more than one operator. This scheme has been successfully generalized to a contractive tuple - by Popescu [18] in the non-commuting case and by Muller and Vasilcescu [14], Pott [19] and Arveson [2] in the commuting case.

The commuting case connects with the theory of holomorphic functions of several complex variables.

The first major theorem of this paper is a result of the above kind for a certain commuting tuple related to complex geometry. For a precise and succinct presentation of the background needed to describe it, we need a few definitions.

For $0 \leq i \leq n$, the elementary symmetric functions $s_i : \mathbb{C}^n \to \mathbb{C}$ of degree $i$ in $n$ variables are

$$s_0 := 1$$

and

$$s_i(z) := \sum_{1 \leq k_1 < k_2 < \cdots < k_i \leq n} z_{k_1} z_{k_2} \cdots z_{k_i} \text{ for all } z \in \mathbb{C}^n \text{ and for } i \geq 1.$$  

(1.1)

The symmetrization map $s : \mathbb{C}^n \to \mathbb{C}^n$ is defined by

$$s(z) := (s_1(z), s_2(z), \ldots, s_n(z)) \text{ for all } z \in \mathbb{C}^n.$$  

(1.2)

The open symmetrized polydisk is the set $G_n := s(D^n)$. Its closure is the closed symmetrized polydisk $\Gamma_n := s(D^n)$. Its distinguished boundary $b\Gamma_n$ is the symmetrized $n$–torus. Operator theory on $\Gamma_n$ was introduced in [6].

An $n$-tuple $(S_1, \ldots, S_{n-1}, P)$ of commuting bounded operators on a Hilbert space $\mathcal{H}$ is called a $\Gamma_n$–contraction, if $\Gamma_n$ is a spectral set for $(S_1, \ldots, S_{n-1}, P)$. Since $\Gamma_n$ is a polynomially convex set, this is the same as saying that for every polynomial $f$ in $n$ variables

$$\|f(S_1, \ldots, S_{n-1}, P)\| \leq \sup_{z \in \Gamma_n} |f(z)| =: \|f\|_{\infty, \Gamma_n}.$$  

It is clear from the definition that $(S_1, \ldots, S_{n-1}, P)$ is a $\Gamma_n$–contraction if and only if $(S_1^*, \ldots, S_{n-1}^*, P^*)$ is so. It is also clear that the last component $P$ of a $\Gamma_n$–contraction $(S_1, \ldots, S_{n-1}, P)$ is a contraction which follows by applying the definition of a $\Gamma_n$–contraction to the particular function that projects $\mathbb{C}^n$ to the last component. A $\Gamma_2$–contraction is what
prompted the study of such a tuple. This was initiated by Agler and Young. There are natural \( \Gamma_n \) analogues of unitaries and isometries.

Let \( S = (S_1, \ldots, S_{n-1}, P) \) be a commuting \( n \)-tuple of bounded operators on a Hilbert space \( \mathcal{H} \). We say that \( S \) is

(i) a \( \Gamma_n \)-unitary if each \( S_i \), \( i = 1, \ldots, n - 1 \) and \( P \) are normal operators and the joint spectrum \( \sigma(S) \) of \( S \) is contained in the distinguished boundary of \( \Gamma_n \);

(ii) a \( \Gamma_n \)-isometry if there exist a Hilbert space \( \mathcal{K} \) containing \( \mathcal{H} \) and a \( \Gamma_n \)-unitary \( \tilde{R} = (R_1, \ldots, R_{n-1}, U) \) on \( \mathcal{K} \) such that \( \mathcal{H} \) is a joint invariant subspace for \( \tilde{R} \) and \( \tilde{R}\mathcal{H} = S \);

(iii) a \( \Gamma_n \)-co-isometry if \( S^* \) is a \( \Gamma_n \)-isometry;

(iv) a pure \( \Gamma_n \)-isometry if \( S \) is a \( \Gamma_n \)-isometry and \( P \) is a pure isometry.

Now we are ready to state the theorem.

**Theorem 1.** If \( S = (S_1, \ldots, S_{n-1}, P) \) is a \( \Gamma_n \)-contraction on a Hilbert space \( \mathcal{H} \) such that \( P^* \) is not pure, then there exist a Hilbert space \( \mathcal{K} \), a \( \Gamma_n \)-unitary \( (R_1, \ldots, R_{n-1}, U) \) on \( \mathcal{K} \) and a bounded operator \( V : \mathcal{H} \to \mathcal{K} \) such that \( V^*V \) is the strong limit of \( (P^*)^jP^j \) as \( j \to \infty \) and

\[
VS_i = R_iV \quad \text{and} \quad VP = UV \quad \text{for each} \quad 1 \leq i \leq n - 1.
\]

Moreover, \( \mathcal{K} \) is the smallest reducing subspace for the \( \Gamma_n \)-unitary \( (R_1, \ldots, R_{n-1}, U) \) containing \( \mathcal{H} \).

If there is another \( \Gamma_n \)-unitary \( (\tilde{R}_1, \ldots, \tilde{R}_{n-1}, \tilde{U}) \) on a space \( \tilde{\mathcal{K}} \) and a bounded operator \( \tilde{V} : \mathcal{H} \to \tilde{\mathcal{K}} \) satisfying all the properties of \( V \) above, then there exists a unitary \( W : \mathcal{K} \to \tilde{\mathcal{K}} \) such that

\[
WR_i = \tilde{R}_iW, \quad WU = \tilde{U}W \quad \text{for each} \quad 1 \leq i \leq n - 1 \quad \text{and} \quad W^*\tilde{V} = V.
\]

This theorem is proved in Section 2.

It is by now folklore that a \( \Gamma_2 \)-isometry \( (T, V) \) acting on a space \( \mathcal{L} \) has a Wold decomposition, i.e., \( \mathcal{L} \) gets decomposed as \( \mathcal{L}_1 \oplus \mathcal{L}_2 \) into reducing subspaces of \( T \) and \( V \) such that on one part, say \( \mathcal{L}_1 \), the restriction \( (T|_{\mathcal{L}_1}, V|_{\mathcal{L}_1}) \) is a \( \Gamma_2 \)-unitary and on the other part, the restriction \( (T|_{\mathcal{L}_2}, V|_{\mathcal{L}_2}) \) is a pure \( \Gamma_2 \)-isometry. The structure of a pure \( \Gamma_2 \)-isometry also is clearly known, we do not get into that right now because we do not need it. The interested reader should see the paper \([1]\) by Agler and Young. It was shown in \([5]\) that given a \( \Gamma_2 \)-contraction \( (S, P) \) on a space \( \mathcal{H} \), there is an explicit construction of a \( \Gamma_2 \)-isometry \( (T, V) \) on a bigger space \( \mathcal{L} \) such that \( \mathcal{H} \) is a joint invariant subspace of \((T^*, V^*)\) and \((T^*, V^*)|_{\mathcal{H}} = (S^*, P^*)\). Such a pair \((T, V)\) is called a dilation of \((S, P)\) because

\[
S^iP^j = P_hT^iV^j|_{\mathcal{H}} \quad \text{for} \quad i = 1, 2, \ldots \quad \text{and} \quad j = 1, 2, \ldots.
\]

Further, the space \( \mathcal{K} \) is minimal, i.e., \( \mathcal{K} \) is the closure of span of \( \{V^jh : h \in \mathcal{H}, j \geq 1\} \).

Theorem \([1]\) assumes greater significance if we remember that for \( n > 2 \), a \( \Gamma_n \)-contraction does not have a \( \Gamma_n \)-unitary dilation in general, see Section 7 of \([16]\). The dilation actually fails at the level of dilating from a \( \Gamma_n \)-contraction to a \( \Gamma_n \)-isometry. However, there is always a \( \Gamma_2 \)-isometry dilation of a \( \Gamma_2 \)-contraction, as mentioned above. So, what is the
relation of the $\Gamma_n$–unitary obtained above, for $n = 2$, with the $\Gamma_2$–isometry that can always
be obtained as a dilation? The following theorem answers that.

**Theorem 2.** Let $(S, P)$ be a $\Gamma_2$–contraction on a Hilbert space $\mathcal{H}$. Assume that $P^*$ is
not pure. Let $(R, U)$ on $\mathcal{K}$ be the $\Gamma_2$–unitary associated to $(S, P)$ by Theorem 7. Then
up to unitary equivalence $(R^*, U^*)$ is the $\Gamma_2$–unitary part in the Wold decomposition of the
minimal $\Gamma_2$–isometric dilation of $(S^*, P^*)$.

This theorem is also proved in Section 2. The next definition is motivated by the paper [9] of Brown and Halmos which introduces the study of those operators $X$ which satisfy $M_z^*XM_z = X$ where $M_z$ is the unilateral shift on the Hardy space. Soon, many authors started investigating operators $X$ that satisfy $P^*XP = X$ for a contraction $P$. By von Neumann’s inequality, a contraction $P$ is an operator that has the closed unit disk as a spectral set. Since in this note, we are concerned with the symmetrized polydisk, it is natural to consider $X$ that satisfies a similar algebraic condition with respect to a $\Gamma_n$–contraction. But the defining criterion of a $\Gamma_n$–contraction is not amenable to an algebraic relation. Unlike in the case of a contraction which is characterized by $P^*P \leq I$, there is no algebraic relation that characterizes a $\Gamma_n$–contraction for $n > 2$. Hence, the next best thing is to use a property of a $\Gamma_n$–isometry. Indeed, a $\Gamma_n$–isometry $S = (S_1, \ldots, S_{n-1}, P)$,\n
apart from being commuting, satisfies $S_i = S_{n-i}^*P$ and $P^*P = I$ for all $i = 1, 2, \ldots, n - 1$.

This is what is used below to define the Brown-Halmos relations in this context. We shall give an example of a specific $\Gamma_n$–isometry on a functional Hilbert space in Section 3.

**Definition 3.** Let $\underline{S} = (S_1, \ldots, S_{n-1}, P)$ be a $\Gamma_n$–contraction on a Hilbert space $\mathcal{H}$. A
bounded operator $X$ on $\mathcal{H}$ is said to satisfy Brown-Halmos relations with respect to $\underline{S}$, if
(1.3)

$$S_i^*XP = XS_{n-i} \quad \text{and} \quad P^*XP = X \quad \text{for each} \ 1 \leq i \leq n - 1.$$ 

For a $\Gamma_n$–contraction $\underline{S} := (S_1, \ldots, S_{n-1}, P)$ on a Hilbert space $\mathcal{H}$, let $\mathcal{T}(\underline{S})$ denote the set of all operators on $\mathcal{H}$ which satisfy Brown-Halmos relations with respect to $\underline{S}$, i.e.,

$$\mathcal{T}(\underline{S}) = \{X \in B(\mathcal{H}) : S_i^*XP = XS_{n-i} \quad \text{and} \quad P^*XP = X \}.$$ 

It turns out that $\mathcal{T}(\underline{S})$ does not contain a non-zero operator, in general. However, we shall find a necessary and sufficient condition for $\mathcal{T}(\underline{S})$ to be non-zero. It is also observed that $V^*VX$ is in $\mathcal{T}(\underline{S})$, whenever $X \in B(\mathcal{H})$ commutes with all $S_i$ and $P$, where $V^*V$ is the strong limit of $P^*jP^j$, as in Theorem 1.

The second major result of this paper is the following theorem. For a subset $\mathcal{A}$ of $B(\mathcal{H})$, the notation $\mathcal{A}'$ will denote the algebra of all elements of $B(\mathcal{H})$ that commute with all elements of $\mathcal{A}$.

**Theorem 4.** For a $\Gamma_n$–contraction $\underline{S} = (S_1, \ldots, S_{n-1}, P)$ on a Hilbert space $\mathcal{H}$, let $R = (R_1, \ldots, R_{n-1}, U)$ and $V$ be as in Theorem 7. Then

(1) The map $\rho$ defined on $\{R_1, \ldots, R_{n-1}, U\}'$ by $\rho(Y) = V^*YV$, is a complete isometry
onto $\mathcal{T}(\underline{S})$;

(2) There exists a surjective unital $*$-representation $\pi : \mathcal{C}(I_\mathcal{H}, \mathcal{T}(\underline{S})) \to \{R_1, \ldots, R_{n-1}, U\}'$
such that

$$\pi \circ \rho = I;$$
(3) There exists a completely contractive, unital and multiplicative mapping
\[ \Theta : \{ S_1, \ldots, S_{n-1}, P \} \to \{ R_1, \ldots, R_{n-1}, U \} \]
defined by \( \Theta(X) = \pi(V^* VX) \) which satisfies
\[ \Theta(X) V = VX. \]

The theorem above is proved in Section 2 too. If we look at the special case when the \( \Gamma_n \)-contraction in Theorem 4 is a \( \Gamma_n \)-isometry, we obtain the following stronger version.

**Theorem 5.** Let \( S = (S_1, \ldots, S_{n-1}, P) \) be a \( \Gamma_n \)-isometry on a Hilbert space \( \mathcal{H} \). Then

1. There exists a \( \Gamma_n \)-unitary \( R = (R_1, \ldots, R_{n-1}, U) \) acting on a Hilbert space \( \mathcal{K} \) containing \( \mathcal{H} \) such that \( R \) is the minimal extension of \( S \). In fact, \( \mathcal{K} \) is the span closure of the following elements:
\[ \{ U^m h : h \in \mathcal{H}, \text{ and } m \in \mathbb{Z} \}. \]

Moreover, any operator \( X \) acting on \( \mathcal{H} \) commutes with \( S \) if and only if \( X \) has a unique norm preserving extension \( Y \) acting on \( \mathcal{K} \) commuting with \( R \).

2. An operator \( X \) is in \( \mathcal{T}(S) \) if and only if there exists a unique operator \( Y \) in the commutant of the von-Neumann algebra generated by \( \{ R_1, \ldots, R_{n-1}, U \} \) such that \( \| X \| = \| Y \| \) and \( X = P_N Y \).

3. Let \( C^*(S) \) and \( C^*(R) \) denote the unital *-algebras generated by \( \{ S_1, \ldots, S_{n-1}, P \} \) and \( \{ R_1, \ldots, R_{n-1}, U \} \), respectively and \( \mathcal{I}(S) \) denote the closed ideal of \( C^*(S) \) generated by all the commutators \( XY - YX \) for \( X, Y \in C^*(S) \cap \mathcal{T}(S) \). Then there exists a short exact sequence
\[ 0 \to \mathcal{I}(S) \to C^*(S) \to C^*(R) \to 0 \]
with a completely isometric cross section, where \( \pi_0 : C^*(S) \to C^*(R) \) is the canonical unital *-homomorphism which sends the generating set \( S \) to the corresponding generating set \( R \), i.e., \( \pi_0(P) = U \) and \( \pi_0(S_i) = R_i \) for all \( 1 \leq i \leq n - 1 \).

There are two parts of this paper. The first part deals with abstract \( \Gamma_n \)-contractions and their Toeplitz operators as described above. The second part deals with a specific \( \Gamma_n \)-isometry and its related Toeplitz operators. We now embark upon a brief description of the second part.

For \( \theta = (\theta_1, \theta_2, \ldots, \theta_n) \) with \( \theta_i \in [0, 2\pi) \) and \( 0 < r \leq 1 \), let \( re^{i\theta} = (re^{i\theta_1}, re^{i\theta_2}, \ldots, re^{i\theta_n}) \).

**Definition 6.** Let \( J_s \) denote the complex Jacobian of the symmetrization map \( s \). The Hardy space \( H^2(\mathbb{G}_n) \) of the symmetrized polydisk is the vector space of those holomorphic functions \( f \) on \( \mathbb{G}_n \) which satisfy
\[ \sup_{0 < r < 1} \int_{\mathbb{T}^n} |f \circ s(re^{i\theta})|^2 |J_s(re^{i\theta})|^2 d\theta < \infty \]
where \( d\theta = d\theta_1 d\theta_2 \cdots d\theta_n \) and \( d\theta \) is the normalized Lebesgue measure on the unit circle \( \mathbb{T} = \{ \alpha : |\alpha| = 1 \} \) for all \( i = 1, 2, \ldots, n \). The norm of \( f \in H^2(\mathbb{G}_n) \) is defined to be
\[ \| f \| = \| J_s \|^{-1} \left\{ \sup_{0 < r < 1} \int_{\mathbb{T}^n} |f \circ s(re^{i\theta})|^2 |J_s(re^{i\theta})|^2 d\theta \right\}^{1/2}, \]
where \( \| J_s \|_2 = \int_{\mathbb{T}^n} |J_s(e^{i\theta})|^2 d\theta \).

Let \( b\Gamma_n \) denote the distinguished boundary of the symmetrized polydisk. See Theorem 2.4 of [6] for a couple of characterizations of \( b\Gamma_n \). In particular, it was shown that

\[
b\Gamma_n = s(T_n).
\]

Note that \( L^2(b\Gamma_n) \) is the Hilbert space consisting of following functions

\[
\{ f : b\Gamma_n \to \mathbb{C} : \int_{\mathbb{T}^n} |f \circ s(e^{i\theta})|^2 |J_s(e^{i\theta})|^2 d\theta < \infty \}.
\]

Let \( L^\infty(b\Gamma_n) \) denote the algebra of all functions on \( b\Gamma_n \) which are bounded almost everywhere on \( b\Gamma_n \) with respect to the Lebesgue measure. For a function \( \varphi \) in \( L^\infty(b\Gamma_n) \), let \( M_\varphi \) be the operator on \( L^2(b\Gamma_n) \) defined by

\[
M_\varphi f(s_1, \ldots, s_{n-1}, p) = \varphi(s_1, \ldots, s_{n-1}, p)f(s_1, \ldots, s_{n-1}, p),
\]

for all \( f \) in \( L^2(b\Gamma_n) \). We note that the tuple \( (M_{s_1}, \ldots, M_{s_{n-1}}, M_p) \) of multiplication by coordinate functions on \( L^2(b\Gamma_n) \) is a commuting tuple of normal operators. We show that the Hardy space \( H^2(\mathbb{G}_n) \) can be embedded inside \( L^2(b\Gamma_n) \) in a canonical way. Hence one readily defines the Toeplitz operators on the symmetrized polydisk in the following way:

**Definition 7.** For a function \( \varphi \) in \( L^\infty(b\Gamma_n) \), the multiplication operator \( M_\varphi \) is called the Laurent operator with symbol \( \varphi \). The compression of \( M_\varphi \) to \( H^2(\mathbb{G}_n) \) is called the Toeplitz operator and denoted by \( T_\varphi \). Therefore

\[
T_\varphi f = PrM_\varphi f \text{ for all } f \text{ in } H^2(\mathbb{G}_n),
\]

where \( Pr \) denotes the orthogonal projection of \( L^2(b\Gamma_n) \) onto \( H^2(\mathbb{G}_n) \).

Since there are many results in this part, we pick a few to highlight here.

(1) We show that the tuple \( (T_{s_1}, \ldots, T_{s_{n-1}}, T_p) \) is a \( \Gamma_n \)-isometry on \( H^2(\mathbb{G}_n) \) and the Toeplitz operators on the symmetrized polydisk are precisely those which satisfy the Brown-Halmos relations with respect to \( (T_{s_1}, \ldots, T_{s_{n-1}}, T_p) \) on \( H^2(\mathbb{G}_n) \). In other words, the set

\[
\mathcal{T} := \{ T \in \mathcal{B}(H^2(\mathbb{G}_n)) : T_{s_i}^* TT_p = TT_{s_{n-i}} \text{ for each } 1 \leq i \leq n - 1 \text{ and } T_p^* TT_p = T \}
\]

is homeomorphic to \( L^\infty(b\Gamma_n) \).

(2) Analytic and (hence) co-analytic Toeplitz operators are characterized.

(3) Compact operators on the Hardy space are characterized in terms of certain natural shifts.

(4) Dual Toeplitz operators are characterized.

These are the contents of Section 3.

2. Part I - \( \Gamma_n \)-contractions and their Toeplitz operators

2.1. The Extension Type Theorem. In this subsection, we prove Theorem [1] by using a combination of arguments from the theory of c.p. maps, due to Prunaru [21] and Choi and Effros [9], which we list as the two lemmas below and from the theory of operators on the symmetrized polydisk \( \Gamma_n \), due to Biswas and Shyam Roy [6], which we shall quote in text.
Lemma 8. Let $\Psi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a completely positive, completely contractive and ultra-weakly continuous linear map and let $\Psi^{(j)}$ denote the $j$-power of $\Psi$ as an operator on $\mathcal{B}(\mathcal{H})$. Then there exists a completely positive and completely contractive linear map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ such that $\Phi$ is idempotent and

$$\text{Ran} \Phi = \{X \in \mathcal{B}(\mathcal{H}) : \Psi(X) = X\}.$$ 

Moreover, if $A, B \in \mathcal{B}(\mathcal{H})$ satisfy $\Psi(AXB) = A\Psi(X)B$ for all $X \in \mathcal{B}(\mathcal{H})$ then the same holds true for $\Phi$. In addition,

$$\Phi(I_\mathcal{H}) = \lim_{j \rightarrow \infty} \Psi^{(j)}(I_\mathcal{H})$$

where the limit is in the strong operator topology.

Lemma 9. Let $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a completely positive and completely contractive map such that $\Phi \circ \Phi = \Phi$. Then for all $X, Y \in \mathcal{B}(\mathcal{H})$ we have

$$\Phi(\Phi(X)Y) = \Phi(X\Phi(Y)) = \Phi(\Phi(X)\Phi(Y)).$$

Now, we start the proof of Theorem 1. First, we apply Lemma 8 to the completely positive, completely contractive and ultra-weakly continuous map $\Psi$ defined on $\mathcal{B}(\mathcal{H})$ by $\Psi(X) = P^*XP$. Consequently, we get a completely positive and completely contractive map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ such that $\Phi \circ \Phi = \Phi$ and

$$\text{Ran} \Phi = \{X \in \mathcal{B}(\mathcal{H}) : P^*XP = X\} := T(P).$$

A straightforward conclusion from the above is that $T(P)$ is fixed by $\Phi$. Indeed, if $X \in T(P)$, then, because $T(P) = \text{Ran} \Phi$, there is a $Z \in \mathcal{B}(\mathcal{H})$ such that $\Phi(Z) = X$. Hence

$$\Phi(X) = \Phi(\Phi(Z))$$

$$= \Phi(Z) \text{(because } \Phi \text{ is idempotent)}$$

$$= X.$$ 

Moreover, $\Phi(I_\mathcal{H}) = \lim_{j \rightarrow \infty} P^jP^j$ in the strong operator topology. By Lemma 9 we know that

$$\Phi(\Phi(X)Y) = \Phi(X\Phi(Y)) = \Phi(\Phi(X)\Phi(Y))$$

for all $X, Y \in \mathcal{B}(\mathcal{H})$. Let $\Phi_0$ be the restriction of $\Phi$ to the unital $C^*$-algebra $C^*(I_\mathcal{H}, T(P))$ and let $(\mathcal{K}, \pi, V)$ be the minimal Stinespring dilation of $\Phi_0$, i.e., $\mathcal{K} \supset \mathcal{H}$ is a Hilbert space, $V : \mathcal{H} \rightarrow \mathcal{K}$ is a bounded operator and $\pi$ is a representation of $C^*(I_\mathcal{H}, T(P))$ taking values in $\mathcal{B}(\mathcal{K})$ such that

$$(2.1) \quad \Phi_0(X) = V^*\pi(X)V, \text{ for every } X \in C^*(I_\mathcal{H}, T(P)).$$

Thus, if $Q$ is the limit of $P^jP^j$ as $j \rightarrow \infty$ in the strong operator topology, then $Q = \Phi_0(I_\mathcal{H}) = V^*V$. This $Q$ is obviously in $T(P)$, but it is also in $T(S)$. The proof of that will follow from the lemma below. However, before we can state and prove the lemma, we need to introduce a new concept - that of a fundamental operator of a $\Gamma_n$-contraction. A popular property of a $\Gamma_n$-contraction $(S_1, \ldots, S_{n-1}, P)$ is that it possesses fundamental operators. This means the following. Since $P$ is a contraction, consider the positive
operator $D_p = (I - P^*P)^{1/2}$. Let $D_p$ be the closure of the range of $D_p$. Now, there are $n - 1$ bounded operators $F_1, F_2, \ldots, F_{n-1}$ on $D_p$ such that $S_i - S_{n-i}^* = D_p F_i D_p$ for $i = 1, 2, \ldots, n - 1$. These are called the fundamental operators. This result can be found in both [15] and [17]. The fundamental operators play a fundamental role in constructing dilations which is not our aim here. We shall use them for proving that non-zero Toeplitz operators exist under a mild condition.

**Lemma 10.** If $(S_1, \ldots, S_{n-1}, P)$ is a $\Gamma_n$-contraction on a Hilbert space $H$, then for each $1 \leq i \leq n - 1$,

$$P^{\ast j}(S_{n-i} - S_{n-i}^*)P^j \to 0 \text{ strongly, as } j \to \infty.$$  

**Proof.** Let $h \in H$. Then using the existence of fundamental operators as mentioned above,

$$\|P^{\ast j}(S_{n-i} - S_{n-i}^*)P^j h\|^2 = \|P^{\ast j}(D_p F_{n-i} D_p)P^j h\|^2 \leq \|F_{n-i}\|^2 \|D_p P^j h\|^2 = \|F_{n-i}\|^2 (\|P^j h\|^2 - \|P^{j+1} h\|^2) \to 0.$$  

\[\Box\]

**Corollary 11.** The operator $Q$ is always in $\mathcal{T}(S)$.  

**Proof.** We show that $Q$ satisfies Brown-Halmos relation with respect to $(S_1, \ldots, S_{n-1}, P)$. It is obvious that $Q$ satisfies $P^* Q P = Q$. For the other relations in (1.3) we use Lemma 10 in the following way:

$$S_i^* Q P - Q S_{n-i} = 0$$

if and only if $\lim_j (S_i^* P^{\ast j} P - P^{\ast j} P S_{n-i}) = 0$  

if and only if $\lim_j P^{\ast j} (S_i^* P - S_{n-i}) P^j \to 0$.

\[\Box\]

Using the corollary above and commutativity of all $S_i$ and $P$, it is easy to see that all $Q S_j, j = 1, 2, \ldots, n - 1$ and $Q P$ are in $\mathcal{T}(S) \subset \mathcal{T}(P)$. Now, we are ready to construct the $\Gamma_n$-unitary. Define $(R_1, \ldots, R_{n-1}, U)$ by

$$R_i = \pi(Q S_i) \text{ for } 1 \leq i \leq n - 1 \text{ and } U = \pi(Q P).$$

The following properties of the representation $\pi$ follow by appealing to Theorem 3.1 of [20].

1. $\pi$ is a unitary such that $V P = U V$ and $K$ is the smallest reducing subspace for $U$ containing $V H$.

2. The map $\rho: \{U\}' \to \mathcal{T}(P)$ defined by $\rho(Y) = V^* Y V$, for all $Y \in \{U\}'$ is surjective and a complete isometry.

3. The representation $\pi$ obtained above satisfies $\pi \circ \rho = I$. In particular, $\pi(C^*(I_H, \mathcal{T}(P))) = \{U\}'$.

4. The linear map $\Theta: \{P\}' \to \{U\}'$ defined by $\Theta(X) = \pi(Q X)$ is completely contractive, unital and multiplicative.
We can harvest a quick corollary here, viz.,

\[(2.2) \quad \Theta(X) V = VX\]

if \(X \in \mathcal{B(H)}\) commutes with \(P\). Indeed, this follows from two inner product computations. For every \(h,k \in H\), we have

\[
\langle \pi(QX)Vh, Vk \rangle = \langle V^* \pi(QX)Vh, k \rangle \\
= \langle \Phi_0(QX)h, k \rangle \quad \text{because } \pi \text{ is the Stinespring dilation of } \Phi_0 \\
= \langle QXh, k \rangle \quad \text{because } T(P) \text{ is fixed by } \Phi_0 \\
= \langle VXh, Vk \rangle
\]

showing that \(P_{\text{Ran}} \pi(QX)V = VX\). But,

\[
\| \pi(QX)Vh \|^2 = \langle V^* \pi(X^*Q^2X)Vh, h \rangle \\
= \langle \Phi_0(X^*Q^2X)h, h \rangle \\
= \langle X^* \Phi_0(Q^2)Xh, h \rangle \quad \text{by Lemma 8} \\
= \langle X^*QXh, h \rangle \quad \text{by Lemma 9} \\
= \| VXh \|^2.
\]

Consequently, \(\pi(QX)V = VX\) for every \(X \in \{P\}'\).

Now, it has to be shown that the tuple \((R_1, \ldots, R_{n-1}, U)\) is a \(\Gamma_n\)-unitary. For this, we shall use results of Biswas and Shyam Roy who introduced operator theory on the symmetrized polydisk. A characterization of a \(\Gamma_n\)-unitary that was obtained by them in Theorem 4.2 of [6] says that \((R_1, \ldots, R_{n-1}, U)\) would be a \(\Gamma_n\)-unitary if and only if \(U\) is a unitary, \(R_i^*U = R_{n-i}\) and

\[
\left(\frac{n-1}{n}R_1, \frac{n-2}{n}R_2, \ldots, \frac{1}{n}R_{n-1}\right)
\]

is a \(\Gamma_{n-1}\)-contraction. All three criteria are fulfilled by the fact that \(\Theta\) is multiplicative (see (P4)). We shall only verify the last one. Note that \(R_i = \Theta(S_i), U = \Theta(P)\). Thus if \(f\) is any polynomial in \(n-1\) variables, then

\[
\| f\left(\frac{n-1}{n}R_1, \frac{n-2}{n}R_2, \ldots, \frac{1}{n}R_{n-1}\right) \| \\
\leq \| f\left(\frac{n-1}{n}\Theta(S_1), \frac{n-2}{n}\Theta(S_2), \ldots, \frac{1}{n}\Theta(S_{n-1})\right) \| \quad \text{because } \Theta \text{ is multiplicative} \\
\leq \| \Theta\left(f\left(\frac{n-1}{n}S_1, \frac{n-2}{n}S_2, \ldots, \frac{1}{n}S_{n-1}\right)\right) \| \quad \text{because } \Theta \text{ is a representation}.
\]

Now we use Lemma 2.7 of [6] which says that \(\left(\frac{n-1}{n}S_1, \frac{n-2}{n}S_2, \ldots, \frac{1}{n}S_{n-1}\right)\) is a \(\Gamma_{n-1}\) contraction because \((S_1, \ldots, S_{n-1}, P)\) is a \(\Gamma_n\) contraction. Thus the norm in the last line of the displayed equations is dominated by the supremum that \(|f|\) attains over \(\Gamma_{n-1}\). Hence, we have shown that \(\left(\frac{n-1}{n}R_1, \frac{n-2}{n}R_2, \ldots, \frac{1}{n}R_{n-1}\right)\) is a \(\Gamma_{n-1}\)-contraction.
Minimality of \((R_1, \ldots, R_{n-1}, U)\) follows from (\(P_1\)). To show that \(V\) intertwines \(S_i\) and \(R_i\), for each \(1 \leq i \leq n - 1\), we need only use (2.2).

Denote \(\tilde{R} = (\tilde{R}_1, \ldots, \tilde{R}_{n-1}, \tilde{U})\) and \(\hat{R} = (R_1, \ldots, R_{n-1}, U)\). It is easy to check that the operator \(\tilde{W} : K \to \hat{K}\) defined by

\[
\langle W_k, f(\tilde{R}, \tilde{R}^*)\tilde{V}h \rangle \hat{K} = \langle k, f(R, R^*)Vh \rangle \hat{K},
\]

for every \(h \in \mathcal{H}\), \(k \in K\) and polynomial \(f\) in \(z\) and \(\overline{z}\), is a unitary and has all the required properties. This completes the proof.

2.2. The unitary part of the minimal \(\Gamma_2\)-isometric dilation for \(n = 2\). The aim of this subsection is to prove Theorem 2 i.e., to show that the adjoint of the \(\Gamma_2\)-unitary obtained from Theorem 1 for a specified \(\Gamma_2\)-contraction is actually the unitary part of the minimal \(\Gamma_2\)-isometric dilation of the adjoint of the \(\Gamma_2\)-contraction up to a unitary conjugation.

Consider \((S, P)\), a \(\Gamma_2\)-contraction on a Hilbert space \(\mathcal{H}\). Let \((T_0, V_0)\) on \(\mathcal{H} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \cdots\) be the minimal \(\Gamma_2\)-isometric dilation of \((S^*, P^*)\) constructed in [5]. The actual structure of \((T_0, V_0)\) is not important, what is important is that the \(\Gamma_2\)-isometric dilation takes place on the minimal isometric dilation space of \(P^*\). The space \(\mathcal{H} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \cdots\) is indeed the space of minimal isometric dilation of \(P^*\), see Schäffer, [22].

Now we change tack a bit and return to the positive contraction \(P\), the limit of \(P^jP^j\) in the strong operator topology. Let \(h \in \mathcal{H}\). Then \(\|Q^{1/2}h\| = \|Q^{1/2}Ph\|\). This shows that there exists an isometry from \(\text{Ran}Q\) to itself with the following action:

\[
Q^{1/2}h \mapsto Q^{1/2}Ph \text{ for all } h \in \mathcal{H}.
\]

Let \(\hat{U}\) be the minimal unitary extension of that isometry. Define the operator \(\Pi : \mathcal{H} \to H^2(\mathcal{D}_P) \oplus \hat{K}\) by

\[
\Pi h = D_P(I - zP)^{-1}h \oplus Q^{1/2}h \text{ for every } h \in \mathcal{H} \text{ and } z \in \mathbb{D}.
\]

For \(z \in \mathbb{D}\), the function \((I - zP)^{-1}\) has the expansion \(\sum_{n=0}^{\infty} z^n P^n\). It takes a straightforward computation using the expansion to check that \(\Pi\) is an isometry and that

\[
\Pi P = \left( \begin{array}{cc} M_z \otimes I_{\mathcal{D}_P} & 0 \\ 0 & \hat{U} \end{array} \right) \ast \Pi.
\]

This shows that \(V_1 = \left( \begin{array}{cc} M_z \otimes I_{\mathcal{D}_P} & 0 \\ 0 & \hat{U} \end{array} \right)\) is an isometric dilation of \(P^*\). This is in fact the Wold decomposition of the isometry obtained. It can be checked that it is minimal. Recall that the minimal isometric dilation of a contraction is unique up to unitary isomorphism. Therefore there exists a unitary \(W : \mathcal{H} \oplus l^2(\mathcal{D}_{P^*}) \to H^2(\mathcal{D}_P) \oplus \hat{K}\) such that \(Wh = \Pi h\) for every \(h \in \mathcal{H}\) and \(WV_0 = V_1 W\). Let \(T_1 = W^*T_0W\). Therefore \((T_1, V_1)\) is a minimal \(\Gamma_2\)-isometric dilation of \((S^*, P^*)\). It is well-known and available in many places, see [1] for example, that when we have a \(\Gamma_2\)-isometry \((T_1, V_1)\) and when the Wold decomposition of the isometry \(V_1\) is known, \(T_1\) also is reduced accordingly. Hence with respect to the same
decomposition of the space which reduces $V_1$, we have a block form of $T_1$. Thus $T_1$ is of
the following form with respect to the decomposition $H^2(\mathcal{D}_P) \oplus \tilde{\mathcal{K}}$:

$$T_1 = \begin{pmatrix} * & 0 \\ 0 & \tilde{R} \end{pmatrix}.$$ 

This shows that the pair $(\tilde{R}, \tilde{U})$ is the $\Gamma_2$–unitary part in the Wold decomposition of the
minimal $\Gamma_2$–isometric dilation of $(S^*, P^*)$.

Now define $V : \mathcal{H} \to \tilde{\mathcal{K}}$ by $Vh = Q_{1/2}h$. Then for all $h, h'$ in $\mathcal{H}$,

$$\langle V^*Vh, h' \rangle = \langle Q_{1/2}h, Q_{1/2}h' \rangle = \langle Qh, h' \rangle.$$ 

(2.3)

Also

$$VPh = Q_{1/2}Ph = \tilde{U}^*Q_{1/2}h = \tilde{U}^*Vh.$$ 

(2.4)

And finally since $T_1 = WT_0W^*$ and $Wh = \Pi h$ for every $h \in \mathcal{H}$, we have

$$\tilde{R}^*V = \tilde{R}^*Q_{1/2} = P_{\tilde{\mathcal{K}}}T_1^*\Pi h = P_{\tilde{\mathcal{K}}}WT_0^*h$$

(2.5)

Therefore the equations (2.3), (2.4) and (2.5) show that the triple $(\tilde{\mathcal{K}}, (\tilde{R}^*, \tilde{U}^*), V)$
qualifies equally well as the $\Gamma_2$–unitary that was obtained in Theorem 1 for the $\Gamma_2$–contraction
$(S, P)$. Since that $\Gamma_2$–unitary is unique up to unitary isomorphism (the concluding part of
Theorem 1), we are done. □

2.3. The structure of $\mathcal{T}(\mathcal{S})$. The aim of this subsection is to prove Theorem 4. The
notations established so far continue to hold. Recall from the proof of Theorem 1 that $Q$
is the limit of $P_{j}^*P_{j}$ in the strong operator topology.

It turns out that the set $\mathcal{T}(\mathcal{S})$ is not always non-zero, in general. The following lemma
is a characterization of when it is non-zero.

**Lemma 12.** Let $\mathcal{S} = (S_1, \ldots, S_{n-1}, P)$ be a $\Gamma_n$–contraction on a Hilbert space $\mathcal{H}$. Then $\mathcal{T}(\mathcal{S})$ is non-zero if and only if $P_{j}^*$ is not pure, i.e., if $P_{j} \not\to 0$ strongly.

**Proof.** If $P_{j} \not\to 0$ strongly, then the non-zero operator $Q$ is in $\mathcal{T}(\mathcal{S})$ by Corollary 11.

For the converse part, suppose there is a non-zero operator in $\mathcal{T}(\mathcal{S})$. Then, in particular,
that non-zero operator is a fixed point of the c.p. map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by

$$\Psi(X) = P_{j}^*XP_{j}.$$ 

It is well-known that if $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a completely positive,
completely contractive and ultra-weakly continuous linear map, then there exists a non-zero operator $X \in \mathcal{B}(\mathcal{H})$ such that $\Psi(X) = X$ if and only if $\lim_{j \to \infty} \Psi^{(j)}(I_{\mathcal{H}}) \neq 0$ in the
strong operator topology, where $\Psi^{(j)}$ denote the $j$-power of $\Psi$ as an operator on $\mathcal{B}(\mathcal{H})$. Applying this to our $\Psi$, we are done. □

We are now ready to prove Theorem 4. We shall use all the terminology developed so far. To prove part (1) of Theorem 4 we need to show that the map $\rho$ defined on
$\{R_1, \ldots, R_{n-1}, U\}$ by $\rho(Y) = V^*YYV$ is surjective and completely isometric onto $\mathcal{T}(\mathcal{S})$. To
that end, consider the map $\rho$ that was obtained in (P2) of subsection 2.1 except that it
was defined on whole of \( \{U\}' \) and now we restrict it to \( \{R_1, \ldots, R_{n-1}, U\}' \). So complete isometry property is automatic.

Let \( Y \in \{R_1, \ldots, R_{n-1}, U\}' \). We already know from (P2) that \( P^* \rho(Y)P = \rho(Y) \) for \( Y \), i.e., \( \rho(Y) \) is in \( \mathcal{T}(P) \). We want to show that \( \rho(Y) \) is in the subset \( \mathcal{T}(\mathcal{S}) \). So, we need to verify the other Brown-Halmos relations which we do below by using the intertwining properties.

\[
S_i^* \rho(Y)P = S_i^* V^* Y V P = V^* R_i^* Y U V = V^* R_{n-i} Y V = V^* Y R_{n-i} V = \rho(Y) S_{n-i}.
\]

for every \( i = 1, 2, \ldots, n - 1 \). Thus, when \( \rho \), obtained in (P2) is restricted to the smaller algebra \( \{R_1, \ldots, R_{n-1}, U\}' \), it maps into \( \mathcal{T}(\mathcal{S}) \).

We want to show that the map \( \rho \) restricted to \( \{R_1, \ldots, R_{n-1}, U\}' \) is onto \( \mathcal{T}(\mathcal{S}) \). To that end, let \( X \in \mathcal{T}(\mathcal{S}) \). Since \( X \) is in \( \mathcal{T}(P) \), by (P2) above, we have an \( Y \) in \( \{U\}' \) such that \( \rho(Y) = V^* Y V = X \). This \( Y \) commutes with each \( R_i \). Indeed, we use the fact that \( X \) is in \( \mathcal{T}(\mathcal{S}) \). So, for each \( i = 1, 2, \ldots, n - 1 \),

\[
S_i^* V^* Y V P = V^* Y V S_{n-i} = V^* R_i^* Y U V = V^* R_{n-i} Y V = V^* Y R_{n-i} V = \rho(Y) S_{n-i}.
\]

This establishes the commutativity of \( Y \) with each \( R_i \), since \( \rho \) is an isometry, and completes the proof of part (1).

Part (2) of Theorem 4 is precisely the content of (P3) above.

For the last part of Theorem 4 we recall the \( \Theta \) that was obtained in (P4). It was defined as \( \Theta(X) = \pi(Q X) \) on a larger algebra, viz., \( \{P\}' \). The aim here is to show that if we consider its restriction to \( \{S_1, \ldots, S_{n-1}, P\}' \), then this restriction lands in \( \{R_1, \ldots, R_{n-1}, U\}' \), i.e., for each \( X \in \{S_1, \ldots, S_{n-1}, P\}' \), the operator \( \Theta(X) \) commutes with each of \( R_i \) and \( U \). That \( \Theta(X) \) commutes \( U \) is asserted in (P4). So we need to show that it commutes with each \( R_i \). This follows from part (2) of the theorem. Indeed, as observed in the Introduction, \( Q X \) is in \( \mathcal{T}(\mathcal{S}) \) and hence \( \pi \) maps it into \( \{R_1, \ldots, R_{n-1}, U\}' \). The proof of the theorem is completed by (2.2). \qed

2.4. The \( \Gamma_n \)-isometry case - the proof of Theorem 5. Let \( Q, V, \pi, \rho \) and \( R = (R_1, \ldots, R_{n-1}, U) \) be as in Theorem 4 corresponding to the \( \Gamma_n \)-isometry \( \mathcal{S} \). Then note that \( Q = I \) because \( P \) is an isometry. Since \( Q = V^* V \), we see that \( V \) is an isometry. Hence \( \mathcal{H} \) can be identified with its isometric image \( V \mathcal{H} \). Doing so, we get from the part (1) of Theorem 4 that \( R \) is a minimal unitary extension of \( \mathcal{S} \). Now let \( X \) be an operator on \( \mathcal{H} \) which commutes with \( \mathcal{S} \). Set \( Y := \pi(X) \). Then by part (3) and (4) of Theorem 4 it follows that \( Y \) commutes with \( R \) and \( Y V = V X \), that is \( Y \) is an extension of \( X \). Also since the
map $\rho$, as in the part (2) of Theorem 11, is an isometry, we get that $Y$ is a unique and norm preserving extension of $X$. That completes the proof of part (1) of this theorem.

For part (2), let $X \in \mathcal{T}(\mathcal{S})$. Set $Y := \pi(X)$. Recall that $\pi$ is the minimal Stinespring dilation of the c.p. map $\Phi_{0} = \Phi|_{C^{*}(I_{\mathcal{H}}, \mathcal{T}(\mathcal{P}))} : C^{*}(I_{\mathcal{H}}, \mathcal{T}(\mathcal{P})) \to B(\mathcal{H})$ and $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ is an idempotent c.p. map with range $\mathcal{T}(\mathcal{P})$. Then as $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}(\mathcal{P})$, it follows that $X = \Phi_{0}(X) = V^{*}YV$ and $\|X\| = \|V^{*}YV\| = \|\rho(Y)\| = \|Y\|$ as $\rho$ is an isometry. This shows that $Y$ is the unique operator in the commutant of $\{R_{1}, \ldots, R_{n-1}, U\}$ with $X = P_{\mathcal{H}}Y|_{\mathcal{H}}$ and $\|X\| = \|Y\|$.

To prove part (3), we first note that the representation $\pi_{0}$ in the statement of the theorem is actually the restriction of $\pi$ to $C^{*}(\mathcal{S})$ as the representation $\pi$ also maps the generating set $S$ of $C^{*}(\mathcal{S})$ to the generating set $R$ of $C^{*}(\mathcal{R})$. Since $\pi_{0}(S) = \mathcal{R}$, range of $\pi_{0}$ is $C^{*}(\mathcal{R})$. Therefore to prove that the following sequence

$$0 \to \mathcal{I}(\mathcal{S}) \to C^{*}(\mathcal{S}) \xrightarrow{\pi_{0}} C^{*}(\mathcal{R}) \to 0$$

is a short exact sequence, all we need to show is that $\ker \pi_{0} = \mathcal{I}(\mathcal{S})$.

We do that now. Since $\pi_{0}(C^{*}(\mathcal{S}))$ is commutative, we have $XY - YX$ in the kernel of $\pi_{0}$, for any $X, Y \in C^{*}(\mathcal{S}) \cap \mathcal{T}(\mathcal{S})$. Hence $\mathcal{I}(\mathcal{S}) \subseteq \ker \pi_{0}$. To prove the other inclusion, consider a finite product $Z_{1}$ of members of $\mathcal{S}^{*} = (S_{1}^{*}, \ldots, S_{n-1}^{*}, P^{*})$ and a finite product $Z_{2}$ of members of $\mathcal{S}$. Let $Z = Z_{1}Z_{2}$. Since $Z \in \mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}(\mathcal{P})$, we have $\Phi_{0}(Z) = Z$. Note that $\Phi_{0}(Z) = P_{\mathcal{H}}\pi_{0}(Z)|_{\mathcal{H}}$, for every $Z \in C^{*}(\mathcal{S})$. Now let $Z$ be an arbitrary finite product of members from $\mathcal{S}$ and $\mathcal{S}^{*}$. Since $\pi_{0}(\mathcal{S}) = \mathcal{R}$, which is a family of normal operators, we obtain, by Fuglede-Putnam theorem that, action of $\Phi_{0}$ on $Z$ has all the members from $\mathcal{S}^{*}$ at the left and all the members from $\mathcal{S}$ at the right. It follows from $\ker \pi = \ker \Phi_{0}$ and $\Phi_{0}$ is idempotent that $\ker \pi_{0} = \{Z - \Phi_{0}(Z) : Z \in C^{*}(\mathcal{S})\}$. Also, because of the above action of $\Phi_{0}$, if $Z$ is a finite product of elements from $\mathcal{S}$ and $\mathcal{S}^{*}$ then a simple commutator manipulation shows that $Z - \Phi_{0}(Z)$ belongs to the ideal generated by all the commutators $XY - YX$, where $X, Y \in C^{*}(\mathcal{S}) \cap \mathcal{T}(\mathcal{S})$. This shows that $\ker \pi_{0} = \mathcal{I}(\mathcal{S})$.

In order to find a completely isometric cross section, recall the completely isometric map $\rho : \pi(C^{*}(\mathcal{T}(\mathcal{P}))) \to B(\mathcal{H})$ defined by $Y \mapsto V^{*}YV$ such that $\pi \circ \rho = \text{id}_{\pi(C^{*}(\mathcal{T}(\mathcal{P}))))}$. Set $\rho_{0} := \rho|_{C^{*}(\mathcal{S})}$. Then by the definition of $\rho$ and the action of $\Phi_{0}$, it follows that $\rho_{0}(\pi(X)) = V^{*}\pi(X)V = \Phi_{0}(X) \in C^{*}(\mathcal{S})$ for all $X \in C^{*}(\mathcal{S})$. Thus $\text{Ran} \rho_{0} \subseteq C^{*}(\mathcal{S})$ and therefore is a completely isometric cross section. This completes the proof of the theorem. □

A remark on a possible strengthening of Theorem 5 is in order.

**Remark 13.** One can work with a commuting family of $\Gamma_{n}$–isometries instead of just one and obtain similar results as in Thereom 5. This was done when $n = 2$ in [4].

The following theorem has the flavour of a commutant lifting theorem. This is a special case of part (2) of Theorem 5. The reason for writing this rather simple special case separately is that it will play a significant role in the sequel.

**Theorem 14.** Let $\mathcal{S} = (S_{1}, \ldots, S_{n-1}, P)$ on $\mathcal{H}$ be a $\Gamma_{n}$–isometry and $\mathcal{R} = (R_{1}, \ldots, R_{n-1}, U)$ on $\mathcal{K}$ be its minimal $\Gamma_{n}$–unitary extension. An operator $X$ satisfies the Brown-Halmos relations with respect to $\mathcal{S}$ if and only if there exists a unique operator $Y$ in the commutant of the von-Neumann algebra generated by $\mathcal{R}$ such that $X = P_{\mathcal{H}}Y|_{\mathcal{H}}$. 


A remark on the matrix representation of the operator $Y$ in Theorem 14 is in order.

**Remark 15.** It turns out that the operator $Y$ in Theorem 14 need neither be an extension nor a co-extension of the operator $X$, in general. We shall give an example in the next section.

### 3. Part II - The Hardy space, a canonical $\Gamma_n$–contraction and its Toeplitz operators

#### 3.1. Basics of The Hardy Space.**

The Hardy space of the symmetrized polydisk was introduced in the Introduction. Here we describe two isomorphic copies of it. For that we shall make use of the following notations:

- A partition $p$ of $\mathbb{Z}$ of size $n$ is an $n$-tuple $(p_1, p_2, \ldots, p_n)$ of integers such that $p_1 > p_2 > \cdots > p_n$. Let
  \[
  [z] := \{(p_1, p_2, \ldots, p_n) : p_1 > p_2 > \cdots > p_n \text{ and } p_j \in \mathbb{Z}\},
  \]
  and
  \[
  [n] := \{(p_1, p_2, \ldots, p_n) : p_1 > p_2 > \cdots > p_n \geq 0\};
  \]
- $\Sigma_n$ denotes the permutation group of the set $\{1, 2, \ldots, n\}$.
- For $\sigma \in \Sigma_n$, $m = (m_1, m_2, \ldots, m_n) \in [z]$ and $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, let $z_\sigma := (z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(n)})$, $z^m := z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$ and $z^{m_\sigma} := z_1^{m_{\sigma(1)}} z_2^{m_{\sigma(2)}} \cdots z_n^{m_{\sigma(n)}}$.

Following notations of the Introduction, for $m \in [n]$, consider the function $a_m(z)$ obtained by anti-symmetrizing $z^m$:

\[
a_m(z) := \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) z^{m_\sigma} = \det((z_i^{m_j}))_{i,j=1}^n.
\]

The above representation of $a_m$ shows that it is zero if and only if there exist $1 \leq i < j \leq n$ such that $m_i = m_j$, - this is the reason why $[n]$ is chosen to be consisting of $n$-tuples of different integers. Note that the functions $a_m$ continue to be defined on $\mathbb{T}^n$ for $m \in [z]$. Let $L^2_\text{anti}(\mathbb{T}^n)$ denote the subspace of $L^2(\mathbb{T}^n)$ consisting of anti-symmetric functions, i.e.,

\[
f(e^{i\theta}) = \text{sgn}(\sigma) f(e^{i\theta}) \text{ a.e. with respect to the Lebesgue measure.}
\]

From the facts that the operator $\mathbb{P} : L^2(\mathbb{T}^n) \to L^2_\text{anti}(\mathbb{T}^n)$ defined by

\[
\mathbb{P}(f)(e^{i\theta}) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) f(e^{i\theta})
\]

is an orthogonal projection onto $L^2_\text{anti}(\mathbb{T}^n)$ and that the set $\{e^{im\theta} : m \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\mathbb{T}^n)$, it follows that the set

\[
\{a_p : p \in [z]\}
\]

is an orthogonal basis for $L^2_\text{anti}(\mathbb{T}^n)$. Consider the subspace

\[
H^2_\text{anti}(\mathbb{D}^n) \overset{\text{def}}{=} \{f \in H^2(\mathbb{D}^n) : f(z_\sigma) = \text{sgn}(\sigma) f(z), \text{ for all } z \in \mathbb{D}^n \text{ and } \sigma \in \Sigma_n\}.
\]
It was observed in [13] that the set
\[ \mathcal{B} = \{ a_p : p \in [n] \} \]
is an orthogonal basis for \( H^2_{\text{anti}}(\mathbb{D}^n) \). The following theorem immediately allows us to consider boundary values of the Hardy space functions.

**Theorem 16.** There is an isometric embedding of the space \( H^2(G_n) \) inside \( L^2(b\Gamma_n) \).

**Proof.** Define \( \tilde{U} : H^2(G_n) \rightarrow H^2_{\text{anti}}(\mathbb{D}^n) \) by
\[
\tilde{U}(f) = \frac{1}{||J_s||} J_s(f \circ s), \text{ for all } f \in H^2(G_n)
\]
and \( U : L^2(b\Gamma_n) \rightarrow L^2_{\text{anti}}(\mathbb{D}^n) \) by
\[
Uf = \frac{1}{||J_s||} J_s(f \circ s), \text{ for all } f \in L^2(b\Gamma_n).
\]

It was observed in [13] that \( U \) and \( \tilde{U} \) are indeed unitary operators. Also note that there is an isometry \( W : H^2_{\text{anti}}(\mathbb{D}^n) \rightarrow L^2_{\text{anti}}(\mathbb{T}^n) \) which sends a function to its radial limit. Therefore we have the following commutative diagram:

\[
\begin{array}{ccc}
H^2(G_n) & \xrightarrow{U^{-1} \circ W \circ \tilde{U}} & L^2(b\Gamma_n) \\
\downarrow \tilde{U} & & \downarrow U \\
H^2_{\text{anti}}(\mathbb{D}^n) & \xrightarrow{W} & L^2_{\text{anti}}(\mathbb{T}^n)
\end{array}
\]

\[ \square \]

The above commutative diagram enables us to study Laurent operators and Toeplitz operators on the symmetrized polydisk in an equivalent way. Let \( L^\infty_{\text{sym}}(\mathbb{T}^n) \) denote the sub-algebra of \( L^\infty(\mathbb{T}^n) \) consisting of symmetric functions, i.e., \( f(e^{i\theta} \sigma) = \text{sgn}(\sigma)f(e^{i\theta}) \) a.e., and \( \Pi_1 : L^\infty(b\Gamma_n) \rightarrow L^\infty_{\text{sym}}(\mathbb{T}^n) \) be the \(*\)-isomorphism defined by
\[
\varphi \mapsto \varphi \circ s
\]
where \( s \) is the symmetrization map as defined in (1.2). Let \( \Pi_2 : B(L^2(b\Gamma_n)) \rightarrow B(L^2_{\text{anti}}(\mathbb{T}^n)) \) denote the conjugation map by the unitary \( U \) as defined in (3.2), i.e.,
\[
T \mapsto UTU^*.
\]

**Theorem 17.** Let \( \Pi_1 \) and \( \Pi_2 \) be the above \(*\)-isomorphisms. Then the following diagram is commutative:

\[
\begin{array}{ccc}
L^\infty(b\Gamma_n) & \xrightarrow{\Pi_1} & L^\infty_{\text{sym}}(\mathbb{T}^n) \\
\downarrow i_1 & & \downarrow i_2 \\
\mathcal{B}(L^2(b\Gamma_n)) & \xrightarrow{\Pi_2} & B(L^2_{\text{anti}}(\mathbb{T}^n))
\end{array}
\]
where \( i_1 \) and \( i_2 \) are the canonical inclusion maps. Equivalently, for \( \varphi \in L^\infty(b\Gamma_n) \), the operators \( M_\varphi \) on \( L^2(b\Gamma_n) \) and \( M_{\varphi s} \) on \( L^2_{\text{anti}}(\mathbb{T}^n) \) are unitarily equivalent via the unitary \( U \).

Proof. To show that the above diagram commutes all we need to show is that \( UM_\varphi U^* = M_{\varphi s} \), for every \( \varphi \) in \( L^\infty(b\Gamma_n) \). This follows from the following computation: for every \( \varphi \) in \( L^\infty(b\Gamma_n) \) and \( f \in L^2_{\text{anti}}(\mathbb{T}^n) \),

\[
UM_\varphi U^*(f) = U(\varphi U^*f) = (\varphi \circ s) \frac{1}{\|J_s\|} J_s(U^*f \circ s) = M_{\varphi s}(f).
\]

\[\square\]

As a consequence of the above theorem, we obtain that the Toeplitz operators on the Hardy space of the symmetrized polydisk are unitarily equivalent to that on \( H^2_{\text{anti}}(\mathbb{D}^n) \).

Corollary 18. For a \( \varphi \in L^\infty(b\Gamma_n) \), \( T_\varphi \) is unitarily equivalent to \( T_{\varphi s} := P_\varphi M_{\varphi s}|_{H^2_{\text{anti}}(\mathbb{D}^n)} \), where \( P_\varphi \) stands for the projection of \( L^2_{\text{anti}}(\mathbb{T}^n) \) onto \( H^2_{\text{anti}}(\mathbb{D}^n) \).

Proof. This follows from the fact that the operators \( M_\varphi \) and \( M_{\varphi s} \) are unitarily equivalent via the unitary \( U \), which takes \( H^2(\mathbb{G}_n) \) onto \( H^2_{\text{anti}}(\mathbb{D}^n) \).

Now we turn our attention to a particular \( \Gamma_n \)-isometry and Toeplitz operators associated to it.

Proposition 19. The \( \Gamma_n \)-isometry \( (T_{s_1}, \ldots, T_{s_{n-1}}, T_p) \) on \( H^2(\mathbb{G}_n) \) has \( (M_{s_1}, \ldots, M_{s_{n-1}}, M_p) \) on \( L^2(b\Gamma_n) \) as its minimal \( \Gamma_n \)-unitary extension.

The following theorem, the main result of this section, is now an obvious consequence of Theorem 19 and Proposition 19.

Theorem 20. A Toeplitz operator on the symmetrized polydisk satisfies the Brown-Halmos relations with respect to \( (T_{s_1}, \ldots, T_{s_{n-1}}, T_p) \) and vice versa.

The following is a straightforward consequence of the characterization of Toeplitz operators obtained above.

Corollary 21. If \( T \) is a bounded operator on \( H^2(\mathbb{G}_n) \) that commutes with each entry of \( (T_{s_1}, \ldots, T_{s_{n-1}}, T_p) \), then \( T \) satisfies the Brown-Halmos relations with respect to \( (T_{s_1}, \ldots, T_{s_{n-1}}, T_p) \) and hence is a Toeplitz operator.

We end the subsection with one more isomorphic copy of the Hardy space. If \( \mathcal{E} \) is a Hilbert space, let \( \mathcal{O}(\mathbb{D}, \mathcal{E}) \) be the class of all \( \mathcal{E} \) valued holomorphic functions on \( \mathbb{D} \). Let

\[
H^2_\mathcal{E}(\mathbb{D}) = \{ f(z) = \sum a_k z^k \in \mathcal{O}(\mathbb{D}, \mathcal{E}) : a_k \in \mathcal{E} \text{ with } \|f\|^2 = \sum \|a_k\|^2 < \infty \}.
\]

Lemma 22. There is a Hilbert space isomorphism \( U_1 \) from \( H^2_{\text{anti}}(\mathbb{D}^n) \) onto the vector valued Hardy space \( H^2_\mathcal{E}(\mathbb{D}) \) where

\[
\mathcal{E} = \overline{\text{span}}\{a_\mathbf{p}(z) : \mathbf{p} \in \mathbb{N}^n \text{ such that } \mathbf{p} = (p_1, p_2, \ldots, p_{n-1}, 0) \} \subset H^2_{\text{anti}}(\mathbb{D}^n).
\]

Moreover, this unitary \( U_1 \) intertwines \( T_{s_n}(z) \) on \( H^2_{\text{anti}}(\mathbb{D}^n) \) with the unilateral shift of infinite multiplicity \( T_z \) on \( H^2_\mathcal{E}(\mathbb{D}) \).
Proof. As observed above, the set \( \{ a_p(z) : p \in [n] \} \) forms an orthogonal basis in \( \mathcal{E} \). Thus
\[
\{ z^q a_p : q \geq 0 \text{ and } p \in [n] \text{ such that } p = (p_1, p_2, \ldots, p_{n-1}, 0) \}
\]
is an orthogonal basis for \( H^2_\mathbb{D} \). On the other hand, the space \( H^2_{\text{anti}}(\mathbb{D}^n) \) is spanned by the orthogonal set \( \{ a_p(z) : p \in [n] \} \) and if \( p = (p_1, p_2, \ldots, p_{n-1}, p_n) \), then
\[
a_p(z) = (z_1 z_2 \cdots z_n)^{p_n} a_\tilde{p}(z) = T_{s_n(z)}^p a_\tilde{p}(z),
\]
where \( \tilde{p} = (p_1 - p_n, p_2 - p_n, \ldots, p_{n-1} - p_n, 0) \). Define the unitary operator from \( H^2_{\text{anti}}(\mathbb{D}^n) \) onto \( H^2_\mathbb{D} \) by
\[
a_p(z) \mapsto z^{p_n} a_\tilde{p}(z),
\]
and then extending linearly. This preserves norms because \( T_{s_n(z)} \) is an isometry on \( H^2_{\text{anti}}(\mathbb{D}^n) \) and \( T_z \) is an isometry on \( H^2_\mathbb{D} \). It is surjective and obviously intertwines \( T_{s_n(z)} \) and \( T_z \). \( \square \)

By virtue of the isomorphisms \( U \) and \( U_1 \) described above, we have the following commutative diagram:
\[
(H^2(\mathbb{G}_n), T_p) \xrightarrow{U} (H^2_{\text{anti}}(\mathbb{D}^n), T_{s_n(z)}) \xrightarrow{U_1} (H^2_\mathbb{D} \cap \mathbb{D}, T_z)
\]
i.e., the operator \( T_p \) on \( H^2(\mathbb{G}_n) \) is unitarily equivalent to the unilateral shift \( T_z \) on the vector valued Hardy space \( H^2_\mathbb{D} \) via the unitary \( U_2 \). We call \( \mathcal{E} \) the co-efficient space of the symmetrized polydisk. It is a subspace of \( H^2_\mathbb{D} \) which is naturally identifiable with the subspace of constant functions in \( H^2_\mathbb{D} \).

3.2. Properties of a Toeplitz operator. We progress with basic properties of Toeplitz operators. It is a natural question whether any of the Brown-Halmos relations implies the other. We give here an example of an operator \( Y \) which satisfies none other than the last one.

Example 23. We actually construct the example on the space \( H^2_{\text{anti}}(\mathbb{D}^n) \) and invoke the unitary equivalences we established in Section 3 to draw the conclusion. For each \( 1 \leq j \leq n - 1 \), we define operators \( Y_j \) on \( H^2_{\text{anti}}(\mathbb{D}^n) \). It is enough to define it on the basis elements, and on the basis elements its action is given by
\[
Y_j a_p = a_{p + f_j}, \text{ where } f_j = (1, \ldots, 1, 0, \ldots, 0). \tag{3.3}
\]

Note that \( M_{s_n(z)} a_p = a_{p + f_n} \) for every \( p \in [n] \). Therefore each \( Y_j \) commutes with \( M_{s_n(z)} \) on \( H^2_{\text{anti}}(\mathbb{D}^n) \). Now it follows that for \( Y_j \) to satisfy other Brown-Halmos relations, it is equivalent for it to commute with each \( T_{s_i} \). But it can be checked that \( Y_j \) commutes with none of the \( T_{s_i} \).
Lemma 24. For $\varphi \in L^\infty(b\mathbb{G}_n)$ if $T_\varphi$ is the zero operator, then $\varphi = 0$, a.e. In other words, the map $\varphi \mapsto T_\varphi$ from $L^\infty(b\mathbb{G}_n)$ into the set of all Toeplitz operators on the symmetrized polydisk, is injective.

Proof. This follows from the uniqueness part of Theorem 14. □

It is easy to see that the space $H^\infty(\mathbb{G}_n)$ consisting of all bounded analytic functions on $\mathbb{G}_n$ is contained in $H^2(\mathbb{G}_n)$. We identify $H^\infty(\mathbb{G}_n)$ with its boundary functions. In other words,

$$H^\infty(\mathbb{G}_n) = \{ \varphi \in L^\infty(b\mathbb{G}_n) : \varphi \circ s \text{ has no negative Fourier coefficients} \}$$

Definition 25. A Toeplitz operator with symbol $\varphi$ is called an analytic Toeplitz operator if $\varphi$ is in $H^\infty(\mathbb{G}_n)$. A Toeplitz operator with symbol $\varphi$ is called a co-analytic Toeplitz operator if $T_\varphi^*$ is an analytic Toeplitz operator.

Our next goal is to characterize analytic Toeplitz operators. But to be able to do that we need to define the following notion and the proposition following it.

Definition 26. Let $\varphi$ be in $L^\infty(b\mathbb{G}_n)$. The operator $H_\varphi : H^2(\mathbb{G}_n) \to L^2(b\mathbb{G}_n) \ominus H^2(\mathbb{G}_n)$ defined by

$$H_\varphi f = (I - Pr)M_\varphi f$$

for all $f \in H^2(\mathbb{G}_n)$, is called a Hankel operator.

We write down few observations about Toeplitz operators some of which will be used in the theorem following it. The proofs are not written because they go along the same line as the case of the unit disk.

Proposition 27. Let $\varphi \in L^\infty(b\mathbb{G}_n)$. Then

1. $T_\varphi^* = T_\overline{\varphi}$.
2. If $\psi \in L^\infty(b\mathbb{G}_n)$ is another function, then the product $T_\varphi T_\psi$ is another Toeplitz operator if $\overline{\varphi}$ or $\psi$ is analytic. In each case, $T_\varphi T_\psi = T_{\varphi \psi}$.
3. If $\varphi \in L^\infty(b\mathbb{G}_n)$, then $T_\varphi T_\psi - T_{\varphi \psi} = -H_\varphi^*H_\psi$.
4. If $T_\varphi$ is compact, then $\varphi = 0$.

Now we are ready to characterize Toeplitz operators with analytic symbols. In the proof of this result, we make use of the following notation:

$$[z] = \{ \mathbf{m} = (m_1, m_2, \ldots, m_n) : m_1 \geq m_2 \geq \cdots \geq m_n \}.$$

Note the difference between $[z]$ and $[[z]]$, defined in Section 3.1. For an $\mathbf{m} \in [z]$, we denote by $s_\mathbf{m}$, the symmetrization of the monomial $z^\mathbf{m}$, i.e.,

$$s_\mathbf{m}(z) = \sum_{\sigma \in \Sigma_n} z^{\mathbf{m}_\sigma}.$$

It is easy to check that the set $\{ s_\mathbf{m} : \mathbf{m} \in [z] \}$ is an orthogonal basis of $L^2_{\text{sym}}(\mathbb{T}^n)$, the space consisting of the symmetric functions in $L^2(\mathbb{T}^n)$.

Theorem 28. Let $T_\varphi$ be a Toeplitz operator. Then the following are equivalent:
(i) $T_\varphi$ is an analytic Toeplitz operator;
(ii) $T_\varphi$ commutes with $T_p$;
(iii) $T_\varphi(R\varphi T_p) \subseteq R\varphi T_p$;
(iv) $T_p T_\varphi$ is a Toeplitz operator;
(v) For each $1 \leq i \leq n - 1$, $T_\varphi$ commutes with $T_{s_i}$;
(vi) For each $1 \leq i \leq n - 1$, $T_{s_i} T_\varphi$ is a Toeplitz operator.

Proof. (i) $\Leftrightarrow$ (ii): That (i) $\Rightarrow$ (ii) is easy. To prove the other direction, we start by observing that if $m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}$ and $p = (p_1, p_2, \ldots, p_n) \in \llbracket n \rrbracket$ then

$$s_m a_p(z) = \sum_{\sigma \in \Sigma_n} a_{p+m_\sigma}(z).$$

This follows from the following computation:

$$\sum_{\sigma \in \Sigma_n} a_{p+m_\sigma}(z) = \sum_{\sigma \in \Sigma_n} \left( \sum_{\beta \in \Sigma_n} \text{sgn}(\beta) z^{(p+m_\sigma)\beta} \right)$$

$$= \sum_{\beta \in \Sigma_n} \text{sgn}(\beta) z^{p\beta} \left( \sum_{\sigma \in \Sigma_n} z^{m_\beta_{\sigma}} \right) = \sum_{\beta \in \Sigma_n} \text{sgn}(\beta) z^{p\beta} \sum_{\delta \in \Sigma_n} z^{m_\delta} = a_p s_m(z).$$

If $T_\varphi$ commutes with $T_p$, then by part (3) of Proposition \[27\] to get that $H_p^2 H_\varphi = 0$. This shows that the corresponding product of Hankel operators on $H^2_{\text{anti}}(\mathbb{D}^n)$ is also zero, that is $H_{s_n(z)}^2 H_{\varphi \circ s} = 0$. Let the power series expansion of $\varphi \circ s \in L^\infty(\mathbb{T}^n)$ be

$$\varphi \circ s(z) = \sum_{m \in \mathbb{Z}} \alpha_m s_m(z).$$

Let $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, \ldots, q_{n-1}, 0)$ be in $\llbracket n \rrbracket$. Then we have

$$0 = \langle H_{\varphi \circ s} a_p, H_{s_n(z)} a_q \rangle_{L^2(\mathbb{T}^n)}$$

$$\quad = \langle \sum_{m \in \mathbb{Z}} \alpha_m s_m(z) a_p, a_q^- \rangle_{L^2(\mathbb{T}^n)} \quad \text{[where } q^- = q - (-1, -1, \ldots, -1) \text{]}$$

$$\quad = \langle \sum_{m \in \mathbb{Z}} \alpha_m \left( \sum_{\sigma \in \Sigma_n} a_{p+m_\sigma} \right), a_q^- \rangle_{L^2(\mathbb{T}^n)}.$$ 

The above inner product survives only when, for some $\sigma \in \Sigma_n$,

$$q^- = (p_1 + m_{\sigma(1)}, \ldots, p_{n-1} + m_{\sigma(n-1)}, m_{\sigma(n)}).$$

Therefore $m$ is a possible rearrangement of the partition

$$(3.4) \quad (q_1 - p_1 - 1, \ldots, q_{n-1} - p_{n-1} - 1, -p_n - 1).$$

Note that we can catch hold of any partition $m \in \mathbb{Z}$ with the last entry negative. Hence $\alpha_m = 0$, if any coordinates of $m$ is negative, in other words, $\varphi$ is analytic in $\mathbb{G}_n$. 

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(ii) \(\Leftrightarrow\) (iii): The part (ii) \(\Rightarrow\) (iii) is easy. Conversely, suppose that \(RanT_p\) is invariant under \(T_\varphi\). Since \(RanT_p\) is closed, we have for every \(f \in H^2(\mathbb{G}_n)\),
\[
T_\varphi T_p f = T_p g_f \text{ for some } g_f \text{ in } H^2(\mathbb{G}_n).
\]

\(\Rightarrow\) \(T^*_p T_\varphi T_p f = g_f \Rightarrow T_\varphi f = g_f \) (by Theorem 20).

Hence \(T_\varphi T_p = T_p^* T_\varphi\).

(ii) \(\Leftrightarrow\) (iv): If \(T_\varphi\) commutes with \(T_p\), then \(T_p T_\varphi = T_\varphi T_p\), which is a Toeplitz operator by Proposition 27. Conversely, if \(T_p T_\varphi\) is a Toeplitz operator, then it satisfies Brown-Halmos relations, the second one of which implies that \(T_\varphi\) commutes with \(T_p\).

(i) \(\Leftrightarrow\) (v): For an analytic symbol \(\varphi\), \(T_\varphi\) obviously commutes with each \(T_s\). The proof of the converse direction is done by the same technique as in the proof of (ii) \(\Rightarrow\) (i). If \(T_\varphi\) commutes with \(T_s\), then by part (3) of Proposition 27 we have \(H^\varphi L_\varphi = 0\). Suppose \(\varphi \circ s \in L_\varphi^\infty(T^n)\) has the following power series expansion
\[
\varphi \circ s(z) = \sum_{m \in [\mathbb{G}_n]} a_m s_m(z).
\]

Let \(q = (q_1, \ldots, q_{n-1}, 0)\) and \(p = (p_1, p_2, \ldots, p_n)\) be in \([n]\) such that \(q_j - q_{j-1} = 1\) for every \(2 \leq j \leq n - 1\). Note that for this choice of \(q\), the only non-analytic term in the expression of \(s_i(z) a_q(z)\) is \(a_{q^-}\), where \(q^- = (q_1, \ldots, q_{n-i}, q_{n-i+1} - 1, \ldots, q_{n-1} - 1, -1)\).

\[
0 = \langle H_\varphi \circ s, H_\varphi^{-1} a_q \rangle_{L^2(T^n)}
\]
\[
= \langle \sum_{m \in [\mathbb{G}_n]} a_m \left( \sum_{a \in \Sigma_m} a_{p+m-a} \right), a_{q^-} \rangle_{L^2(T^n)} \quad \text{[where } q^- = q - (0, \ldots, 0, -1, \overbrace{1, \ldots, 1}^{i\text{-times}})].
\]

From this, we see that \(a_m = 0\), when \(m\) is a possible rearrangement of
\[
q_1 - p_1, \ldots, q_{n-i} - p_{n-i}, q_{n-i+1} - p_{n-i+1}, \ldots, q_{n-1} - p_{n-1} - 1, -p_n - 1).
\]

As we vary \(p\) and \(q\) with the above properties, we can catch hold of any \(m\) with the last entry negative. This proves that \(\varphi\) is analytic.

(v) \(\Leftrightarrow\) (vi): The implication (v) \(\Rightarrow\) (vi) follows from Proposition 27. Conversely suppose that \(T_{s_n}^* T_\varphi\) is a Toeplitz operator. Therefore applying Theorem 20 and the relation \(T_{s_n}^* T_{s_n} = T_{s_n}^* T_\varphi T_p = T_p^* T_{s_n} T_\varphi T_p = T_{s_n}^* T_\varphi\).

Remark 29. Since adjoint of an analytic Toeplitz operator is a co-analytic Toeplitz operator and vice versa, Theorem 28 in turn characterizes co-analytic Toeplitz operators also.

3.3. Compact Perturbation of Toeplitz Operators. In this section we first find a characterization of compact operators on \(H^2(\mathbb{G}_n)\) and then use it to characterize compact perturbation of Toeplitz operators – the so-called asymptotic Toeplitz operators.

Note that for a bounded operator \(B\) on \(H^2(\mathbb{D})\), \(B\) is a Toeplitz operator if and only if there exists \(T\) on \(H^2(\mathbb{D})\) such that \(T^* T^* \rightarrow B\) weakly on \(H^2(\mathbb{D})\). Although \(T_p\) of \(H^2(\mathbb{G}_n)\) is unitarily equivalent to \(T_\varphi\) on \(H^2_\varphi(\mathbb{D})\), a similar convergence like above does not guarantee
Lemma 30. Let $T$ and $B$ be bounded operators on $H^2(\mathbb{G}_n)$ such that $T_p^{\ast n}TT_p^n \to B$ weakly. Then $B$ is Toeplitz operator if and only if for each $1 \leq i \leq n - 1$

$$T_p^{\ast n}[T, T_{s_i}]T_p^n \to 0 \text{ weakly,}$$

where $[T, T_{s_i}]$ denotes the commutator of $T$ and $T_{s_i}$.

Proof. Note that if $T$ and $B$ are bounded operators on $H^2(\mathbb{G}_n)$ such that $T_p^{\ast n}TT_p^n \to B$ weakly, then $T_p^*BT_p = B$. Suppose $T_p^{\ast n}[T, T_{s_i}]T_p^n \to 0$ weakly. To prove that $B$ is Toeplitz, it remains to show that $B$ satisfies the other Brown-Halmos relations with respect to the $\Gamma_n$-isometry $(T_{s_1}, \ldots, T_{s_{n-1}}, T_p)$.

$$T_{s_i}BT_p = \lim T_{s_i}(T_p^{\ast n}TT_p^n)T_p = \lim T_p^{\ast n}(T_{s_i}TT_p^n)T_p = \lim T_p^{\ast n+1}T_{s_{n-i}}TT_p^{n+1} = \lim T_p^{\ast n+1}T_{s_{n-i}} = BT_{s_{n-i}}.$$ 

Conversely, suppose that the weak limit $B$ of $T_p^{\ast n}TT_p^n$ is a Toeplitz operator and hence satisfies the Brown-Halmos relations. Thus,

$$\lim T_p^{\ast n}(TT_{s_i} - T_{s_i}T)T_p^n = \lim (T_p^{\ast n}TT_p^nT_{s_i} - T_p^{\ast n}T_{s_{n-i}}TT_p^n) = \lim (T_p^{\ast n}TT_p^nT_{s_i} - T_{s_{n-i}}T_p^{\ast n-1}TT_p^{n-1}T_p) = BT_{s_i} - T_{s_{n-i}}BT_p = 0.$$ 

\[ \square \]

Before we characterize the compact operators on $H^2(\mathbb{G}_n)$, we recall the analogous result for the polydisk, discovered in [12]. The one dimensional case was proved by Feintuch [11].

For $m \geq 1$, define a completely positive map $\eta_m : \mathcal{B}(H^2(\mathbb{D}^n)) \to \mathcal{B}(\oplus_{i=1}^n H^2(\mathbb{D}^n))$ by

$$\eta_m(T) = \left( \begin{array}{c}
T_{z_1}^{m}
T_{z_2}^{m}
\vdots
T_{z_n}^{m}
\end{array} \right) T \left( \begin{array}{c}
T_{s_1}^m
T_{s_2}^m
\vdots
T_{s_n}^m
\end{array} \right).$$

Theorem 31. A bounded operator $T$ on $H^2(\mathbb{D}^n)$ is compact if and only if $\eta_m(T) \to 0$ in norm as $m \to \infty$.

This shows the importance of the forward shifts in characterizing the compact operators.

For $1 \leq j \leq n - 1$, let $Y_j$ be the operator on $H^2_{\text{ant}}(\mathbb{D}^n)$ as defined in (3.3). Let $X_j$ be the restriction of $Y_j$ to $\mathcal{E}$, the coefficient space of the symmetrized polydisk, i.e.,

$$X_j a_p = a_{p+j},$$

for every $p \in \llbracket n \rrbracket$ such that $p = (p_1, \ldots, p_{n-1}, 0)$.

Then each $Y_j$ on $H^2_{\text{ant}}(\mathbb{D}^n)$ has the following expression:

$$Y_j = \sum_{r=0}^{\infty} T_{s_n(z)}^r X_j T_{s_n(z)}^r.$$
It should be noted that each $Y_j$ is a pure isometry and the set of operators $\{Y_1, Y_2, \ldots, Y_{n-1}\}$ is doubly commuting. Consider the following operator on $E$

$$E_l := (P_E - X_1^l X_1^l) (P_E - X_2^l X_2^l) \cdots (P_E - X_{n-1}^l X_{n-1}^l).$$

Since the operators $X_1, X_2, \ldots, X_{n-1}$ are doubly commuting, the operator $E_l$ is an orthogonal projection of $E$ onto the $l$ dimensional space

$$\bigcap_{j=1}^{n-1} \ker X_j^l = \text{span}\{a_p : p = (k + (n - 2)l, k + (n - 3)l, \ldots, k, 0), 1 \leq k \leq l\}.$$

Also note that

$$E_l = P_E - \sum_{j=1}^{n-1} X_j^l X_j^l + \sum_{1 \leq j_1 < j_2 \leq n-1} X_{j_1}^l X_{j_2}^l X_{j_1}^l X_{j_2}^l + \cdots + (-1)^k \sum_{1 \leq j_1 < \cdots < j_k \leq n-1} X_{j_1}^l \cdots X_{j_k}^l X_{j_1}^l \cdots X_{j_k}^l + \cdots + (-1)^{n-1} X_1^l \cdots X_n^l X_1^l \cdots X_n^l. $$

We have the following characterization of compact operators on $H^2(\mathbb{G}_n)$ using the set of pure isometries $\{Y_1, \ldots, Y_{n-1}, T_p\}$.

**Theorem 32.** For $j \geq 1$, define completely positive maps $\eta_j : \mathcal{B}(H^2(\mathbb{G}_n)) \to \mathcal{B}(\oplus_{l=1}^n H^2(\mathbb{G}_n))$ by

$$\eta_j(T) = \begin{pmatrix} Z_1^{x_j} \\ \vdots \\ Z_{n-1}^{x_j} \\ T_p^{x_j} \end{pmatrix} T \begin{pmatrix} Z_1^{x_j} \\ \vdots \\ Z_{n-1}^{x_j} \\ T_p^{x_j} \end{pmatrix}.$$  

where $Z_l = U^* Y_l U$. A bounded operator $T$ on $H^2(\mathbb{G}_n)$ is compact if and only if $\eta(T) \to 0$ in norm.

**Proof.** The necessity follows from a straightforward application of Lemma 3.1 of [12], which states that if $P$ is a pure contraction, $T$ is a contraction and $K$ is a compact operator on a Hilbert space, then $P^* KT \to 0$ in norm as $l \to \infty$.

Conversely, suppose that $T$ is a bounded operator on $H^2(\mathbb{G}_n)$ satisfying the convergence condition. We shall conclude by finding a finite rank operators approximation of $T$. To that end note that $U^* P_E U = U^* (I - T_{s_n(z)} T_{s_n(z)}^*) U = I - T_p T_p^*$ and

$$U^* E_l U = (I - T_p T_p^*) - \sum_{k=1}^{n-1} (-1)^k \sum_{1 \leq j_1 < \cdots < j_k \leq n-1} W_{j_1}^l \cdots W_{j_k}^l W_{j_1}^l \cdots W_{j_k}^l,$$

where $W_j = U^* X_j U$. For each positive integer $l$, consider the following finite rank operators on $H^2(\mathbb{G}_n)$:

$$F_l = U^* \left( E_l + T_{s_n(z)} E_l T_{s_n(z)}^* + \cdots + T_{s_n(z)}^{l-1} E_l T_{s_n(z)}^{l-1} \right) U.$$
Using (3.6) we get $F_l$ to be the same as

$$
(I - T_p^l T_p^{*l}) - \sum_{r=0}^{l-1} T_p^r \left( \sum_{j=1}^{n-1} W_j W_j^* \right) T_p^{sr} + \cdots
$$

$$
+ (-1)^k \sum_{r=0}^{l-1} T_p^r \left( \sum_{1 \leq j_1 < \cdots < j_k \leq n-1} W_{j_1} \cdots W_{j_k} W_{j_1}^* \cdots W_{j_k}^* \right) T_p^{sr} + \cdots
$$

$$
+ (-1)^{n-1} \sum_{r=0}^{l-1} T_p^r (W_1^* \cdots W_n^* W_1^* \cdots W_n^*) T_p^{sr}.
$$

Let $P_l$ be the projection of $H^2(G_n)$ onto the space ker $T_p^{sl-1}$. Since $Z_j = \sum_{r=0}^{\infty} T_p^r W_j T_p^{sr}$,

$$
I - F_l = T_p^{l} T_p^{*l} + \sum_{j=1}^{n-1} P_l Z_j^l Z_j^{*l} P_l + \cdots + (-1)^k \sum_{1 \leq j_1 < \cdots < j_k \leq n-1} P_l Z_{j_1}^l \cdots Z_{j_k}^l Z_{j_1}^{*l} \cdots Z_{j_k}^{*l} P_l
$$

$$
+ (-1)^{n-1} P_l Z_1^l \cdots Z_{n-1}^l Z_1^{*l} \cdots Z_{n-1}^{*l} P_l.
$$

Then the operator $\tilde{F}_l = TF_l + F_l T - F_l T F_l$ is also a finite rank operator and note that $T - \tilde{F}_l = (I - F_l) T (I - F_l)$. By the above form of $(I - F_l)$ and from the hypotheses it follows that $\|T - \tilde{F}_l\| \to 0$. Hence, $T$ is compact.

**Definition 33.** A bounded operator $T$ on $H^2(G_n)$ is called an asymptotic Toeplitz operator if $T = T_\varphi + K$, for some $\varphi \in L^\infty(b\Gamma)$ and compact operator $K$ on $H^2(G_n)$.

We end this section with the following characterization of asymptotic Toeplitz operators.

**Theorem 34.** A bounded operator $T$ on $H^2(G_n)$ is an asymptotic Toeplitz operator if and only if $T_p^{sn}[T, T_r] T_p^{rn} \to 0$, $T_p^{*n} T T_p^{rn} \to B$ and $\eta_h(T - B) \to 0$.

**Proof.** If $T$ satisfies the convergence conditions as in the statement, then it follows from Lemma 30 that $B$ is a Toeplitz operator because $T_p^{sn}[T, T_r] T_p^{rn} \to 0$. Also, since $\eta_h(T - B) \to 0$, by Theorem 32 $T - B$ is a compact operator. Hence $T$ is the sum of a compact operator and a Toeplitz operator.

Conversely, suppose $T$ is an asymptotic Toeplitz operator, i.e., $T = K + T_\varphi$, where $K$ is some compact operator. Then by Theorem 32 $T_p^{*n} T T_p^{rn} \to T_\varphi$. Since $T_\varphi$ is Toeplitz, by Lemma 30 $T_p^{*n}[T, T_r] T_p^{rn} \to 0$. And finally, since $K$ is compact, by Theorem 32 $\eta_h(T - T_\varphi) \to 0$.

3.4. **Dual Toeplitz operators.** Dual Toeplitz operators have been studied for a while on the Bergman space of the unit disc $\mathbb{D}$ in [23] and on the Hardy space of the Euclidean ball $\mathbb{B}_d$ in [10]. In our setting, consider the space

$$
H^2(G_n)^\perp = L^2(b\Gamma_n) \ominus H^2(G_n).
$$

Let $(I - Pr)$ be the projection of $L^2(b\Gamma_n)$ onto $H^2(G_n)^\perp$. If $\varphi \in L^\infty(b\Gamma_n)$, define the dual Toeplitz operator on $H^2(G_n)^\perp$ by $DT_\varphi = (I - Pr) M_\varphi |_{H^2(G_n)^\perp}$. With respect to the
decomposition above,

\[
M_\varphi = \begin{pmatrix}
T_\varphi & H_\varphi^* \\
H_\varphi & DT_\varphi
\end{pmatrix}.
\]

Lemma 35. The $n$-tuple $D = (DT_{\bar{s}_1}, \ldots, DT_{\bar{s}_{n-1}}, DT_\varphi)$ is a $\Gamma_n$–isometry with the $n$-tuple $(M_{\bar{s}_1}, \ldots, M_{\bar{s}_{n-1}}, M_\varphi)$ as its minimal $\Gamma_n$–unitary extension.

Proof. It is a $\Gamma_n$–isometry because it is the restriction of the $\Gamma_n$–unitary $(M_{\bar{s}_1}, \ldots, M_{\bar{s}_{n-1}}, M_\varphi)$ to the space $H^2(G_n)$. And this extension is minimal because $M_\varphi$ is the minimal unitary extension of $DT_\varphi$. □

Theorem 36. A bounded operator $T$ on $H^2(G_n) \perp$ is a dual Toeplitz operator if and only if it satisfies the Brown-Halmos relations with respect to $D$.

Proof. The easier part is showing that every dual Toeplitz operator on $H^2(G_n) \perp$ satisfies the Brown-Halmos relations with respect to $D$. It follows from the following identities

\[ M_{\bar{s}_i}^* M_\varphi M_\varphi = M_\varphi M_{\bar{s}_i-1} \quad \text{and} \quad M_\varphi^* M_\varphi M_\varphi = M_\varphi \quad \text{for every} \ \varphi \in L^\infty(b\Gamma_n) \]

and from the $2 \times 2$ matrix representations of the operators in concern. For the converse, let $T$ on $H^2(G_n) \perp$ satisfy the Brown-Halmos relations with respect to the $\Gamma_n$–isometry $D$. By Theorem 14 and the fact that any operator that commutes with each of $M_{\bar{s}_1}, \ldots, M_{\bar{s}_{n-1}}$ and $M_\varphi$ is of the form $M_\varphi$, for some $\varphi \in L^\infty(b\Gamma_n)$, there is a $\varphi \in L^\infty(b\Gamma_n)$ such that $T$ is the compression of $M_\varphi$ to $H^2(G_n) \perp$. □

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