Analysis of the Multi-component Relativistic Boltzmann Equation for Electron Scattering in Big Bang Nucleosynthesis

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Big-bang nucleosynthesis (BBN) is a valuable tool to constrain the physics of the early universe and is the only probe of the radiation-dominated epoch. A fundamental assumption in BBN is that the nuclear velocity distributions obey Maxwell-Boltzmann (MB) statistics as they do in stars. In this paper, however, we suggest that there could be a difference between stellar reaction rates and BBN reaction rates, arising due to their fundamentally different environments. Specifically, the BBN epoch is characterized by a dilute baryon plasma for which the velocity distribution of nuclei is mainly determined by the dominant Coulomb elastic scattering with mildly relativistic electrons. One must therefore deduce the momentum distribution for reacting nuclei from the multi-component relativistic Boltzmann equation. However, the full multi-component relativistic Boltzmann equation has only recently been analyzed and its solution has only been worked out in special cases. Here, we construct the relativistic Boltzmann equation in the context of BBN. We also derive a Langevin model and perform relativistic Monte-Carlo simulations which clarify the baryon distribution during BBN. We show by these analyses that, contrary to our previous claim, the imposition of pressure equilibrium leads to a nuclear distribution that remains very close to MB statistics even during the most relativistic environment relevant to BBN. Hence, the predictions of standard BBN remain unchanged.

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I. INTRODUCTION

Big-bang nucleosynthesis (BBN) remains as a pillar of modern cosmology[1,2]. It provides an almost parameter free prediction of the abundances of the light isotopes $^2$H, $^3$He, $^4$He and $^7$Li formed during the first few moments of cosmic expansion. At the onset of BBN ($T \sim 10^{10}$ K) the universe is comprised of electrons, positrons, photons, neutrinos and trace amounts of protons and neutrons. Once the temperature becomes low enough ($T \sim 10^9$ K) for the formation of deuterium, most neutrons are quickly absorbed by nuclear reactions to form $^4$He nuclei along with trace amounts of $^2$H, $^3$H, $^4$He, $^7$Li and $^7$Be. These trace amounts, however, are sensitive to the detailed freeze-out of the thermonuclear reaction rates as the universe cools. In this paper we re-examine the fundamental assumptions about the BBN epoch. In particular, we analyze the multi-component relativistic thermodynamics of the BBN environment.

Although the thermodynamics of both relativistic and nonrelativistic single-component gas have been known for many decades[3], the solution of the relativistic multi-component Boltzmann equation has only recently been attempted[4,5] and transport coefficients have only been deduced for the case of equal or nearly identical-mass particles. Moreover, there has been recent interest in the possibility of a modification of the baryon distribution function from Maxwell Boltzmann (MB) statistics, in the form of Tsallis statistics[11,12], the influence of inhomogeneous primordial magnetic fields on baryons[13], non-ideal plasma effects at low temperature[14], a result of scattering from background relativistic electrons in kinetic-energy equipartition with baryons[15], and small relativistic corrections to MB distribution that arise due to nuclear kinetic drag[16].

In the work of Ref. [16], for example, the starting point was the FD distribution for baryons from which corrections were deduced. However, in Ref. [15] it was noted based upon a Langevin approximation with kinetic-energy equipartition, and also in a Monte Carlo simulation that the momentum distribution of nuclei more closely resembled the electron momentum distribution. Therefore, nuclei obeyed modified statistics when the electrons were relativistic.

The point of the present work, therefore, is to analyze the solution to the relativistic Boltzmann equation without an a prior assumption of what the baryon distribution should be. We show that the problem can be approximated as an ideal two component system of baryons immersed in a bath of relativistic electrons, for which the collision term is completely dominated by elastic scattering from relativistic electrons. We show that, contrary to our previous claim[15], the resultant baryon distribution does indeed follow MB statistics independently of the electron distribution function. This is verified by nu-
merical Monte-Carlo simulations [17] that correct for the instantaneous viscosity experienced by recoiling nuclei which was neglected in the previous simulation. We also show that the previous assumption of kinetic equipartition (though often invoked in the classical Langevin approximation) is erroneous for the relativistic primordial plasma which is in pressure equilibrium.

II. BIG BANG ENVIRONMENT

A. Nuclear Reaction Rates

The reaction rate between two species 1 and 2 can be written as [19]

\[ R = n_1 n_2 (\sigma(v)v) = n_1 n_2 \int v \sigma(v) f(v) dv \]  

where \( n_1 \) and \( n_2 \) are the number densities of the two species, \( \sigma(v) \) is the reaction cross section, \( v \) is the relative center-of-mass velocity, and \( f(v) \) is the relative velocity distribution function. In this paper we analyze the possible modification of \( f(v) \) due to the unique environment encountered during BBN. Indeed, there has been considerable recent interest in deviations of the nuclear velocity distribution as a possible solution to the over-production of lithium [10] [12].

B. Scattering in the background plasma

At the start of BBN baryons are extremely dilute in number density compared to the background of \( e^+ - e^- \) pairs and photons. The baryon-to-photon ratio (\( \eta \)) is \( \sim 10^{-9} \). Similarly, the ratio of baryons to \( e^+ - e^- \) pairs is \( \sim 10^{-9} \) during much of BBN. Hence, each nucleus undergoes scattering with a background plasma comprised of electrons, positrons and photons much more often than with other nuclei. This could be important when considering the relative velocity distribution functions \( f(v) \) for nuclear reactions. That is, the velocity distributions of nuclei will result from scattering events with the mildly relativistic background plasma [20] [21] rather than with each other.

To justify the above statement regarding the relative scattering rates we first presume the usual thermodynamic relations for photons, electrons and nuclei during BBN. (This assumption will be revisited in Section [II] and Appendix A below.) The number density of background photons is thus taken to be the usual Planck distribution:

\[ n_\gamma = \frac{g_\gamma}{2\pi^2 \hbar^3 c^3} \int_0^\infty \frac{E^2}{e^{E/kT} - 1} dE = \frac{2\xi(3)(kT)^3}{\pi^2 \hbar^3 c^3} \]  

where \( c \) is the speed of light, \( \hbar \) is the reduced Planck's constant, \( k \) is the Boltzmann constant, \( T \) is the temperature, \( g_\gamma = 2 \) is the number of photon polarization states, \( E \) is the photon energy.

Similarly, the number densities of positrons and electrons are described by a Fermi-Dirac (FD) distribution,

\[ n_\pm = \frac{g_\pm}{\pi^2 \hbar^3 c^3} \int_0^\infty \frac{p^2}{\exp\{(E \pm \mu)/kT\} + 1} dp \]  

where \( + (\mp) \) denotes positrons (electrons), \( g_\pm = 2 \) is the number of spin states, \( E = \sqrt{p^2 + m_e^2} \) is the total energy with \( m_e \) the electron rest mass, \( p \) is the three momentum, and \( \mu \) is the chemical potential for electrons. During most of BBN the chemical potential can be ignored [13].

The elastic scattering cross section for photons with nuclei (Compton scattering using the Klein-Nishina formula) is given by

\[ \frac{d\sigma}{d\cos \theta} = \frac{\pi Z^2 \alpha^2}{2v^2 p'^2 \sin^2 \frac{\theta}{2}} \left( 1 - \frac{v^2}{c^2} \sin^2 \frac{\theta}{2} \right) \]  

where \( \theta \) is the scattering angle, \( \alpha \) is the fine structure constant, \( Z \) is the nuclear charge, \( m \) is the nuclear mass, \( \omega \) and \( \omega' \) are the frequencies of the incoming and outgoing photons, respectively. From the angular integration of Eq. (4), the total reaction cross-section for a photon is

\[ \sigma \leq 60.5 \text{fm}^2 Z^2 (m_e/m)^2 \].

The elastic scattering cross-section for electrons and positrons with nuclei is given by the Mott formula

\[ \frac{d\sigma}{d\cos \theta} = \frac{\pi Z^2 \alpha^2}{2v^2 p'^2 \sin^2 \frac{\theta}{2}} \left( 1 - \frac{v^2}{c^2} \sin^2 \frac{\theta}{2} \right) \]  

where \( v \) is the velocity of the \( e^- \) or \( e^+ \) particle.

The Coulomb scattering cross-sections can be evaluated using the Mott-formula or Rutherford-formula and is known to be infinite. However, a reasonable cut-off in the impact parameter for the incoming plasma particle is given by the Debye screening length \( r_D = \sqrt{kT/4\pi n_0 e^2} \) [24]. We adopt this as the maximum impact parameter to calculate the minimum scattering angle. Using these, we obtain two realistic approximations to the Coulomb cross sections: One is simply given by the area of a circle with radius \( r_D \); while the second is based upon the Mott-formula with the upper limit defined by the minimum scattering angle.

Columns in Table I show the temperature dependence, respectively, for the ratio of number densities of electrons to photons \( n_e/n_\gamma \), the electron-to-photon elastic-scattering cross-section ratio \( \sigma_\pm/\sigma_\gamma \), for protons with cut-off radii at the Debye radius or the Mott formula minimum scattering angle, the ratio of nuclear scattering rates for electrons to photons \( \Gamma_\pm/\Gamma_\gamma \equiv n_\pm v_\pm/n_\gamma v_\gamma \), and the ratio of rates for proton elastic scattering from electrons to elastic scattering from other protons, \( \Gamma_\pm/\Gamma_p \equiv n_\pm v_\pm/n_p v_p \). It is evident from these ratios that nuclei scatter with the background \( e^- - e^+ \) pair plasma significantly more than with photons or other nuclei during BBN. Hence, nuclei are overwhelmingly thermalized by the background \( e^- - e^+ \) pair plasma, while photons and other nuclei have a negligible effect on the thermodynamics.
TABLE I: Temperature dependence of various ratios relevant to proton elastic-scattering reaction rates with $e^- - e^+$ plasma, photons and other protons. We use the minimum among the two cross section ratios (4th or 5th column) to obtain the reaction rates for $e^- - e^+$ plasma.

| $T_0$ (MeV) | $n_\pm/n_e$ | $\sigma_\pm/\sigma_e$ | $\Gamma_\pm/\Gamma_e$ | $\Gamma_\pm/\Gamma_p$ |
|-------------|--------------|------------------------|---------------------|---------------------|
| 11.6        | 1            | $5 \times 10^4$       | $10^3$              | $10^4$              |
| 11.16       | 0.1          | $10^2$                | $10^3$              | $10^{10}$           |
| 0.116       | 0.01         | $2 \times 10^{28}$    | $10^{23}$           | $10^{24}$           |

In what follows we model the response of nuclei to the dominant scattering from relativistic electrons via the relativistic Boltzmann equation. In the appendix, we give a similar Langevin derivation. Indeed, the scattering rates in Table I suggest that the physical environment for BBN is similar to that of Brownian motion.

III. RELATIVISTIC BOLTZMANN EQUATION

For our purposes we can ignore the small corrections due to the cosmic expansion, and treat the space as flat. Following [5] let us begin with a completely general mixture of $r$ constituents in a locally Minkowski space with metric tensor $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$. The fluid consists of multiple particles of mass $m_a$ with $a = 1, \ldots, r$. Each particle is characterized by space-time coordinates $x^\alpha$, $\alpha = 0, 1, 2, 3$ and momenta $p^\alpha_a = (E_a, p^\alpha_a)$, so that $E_a = \gamma m_a = \sqrt{(p^\gamma)^2 + m_a^2}$ (we adopt natural units with $c = 1$). If we restrict to the consideration of only elastic collisions, then the conservation of four momenta can be imposed

$$p^\alpha_a + p^\alpha_b = p^\alpha_a + p^\alpha_b.$$  \hfill (6)

The state of the mixture of relativistic species $r$ can be characterized by a set of one-particle distribution functions:

$$f(x, p_a, t) \equiv f_a, \quad a = 1, 2, \ldots, r.$$  \hfill (7)

The one-particle distribution function characterizing collisions of constituent $a$ with constituent $b$ satisfies a Boltzmann equation,

$$p^\alpha_a \partial_a f_a = \sum_{a=1}^{r} \int (f_a f_b - f_a f_b) F_{ab} \sigma_{ab} d\Omega \frac{d^4 p_b}{p_b^0},$$  \hfill (8)

where the right-hand side is the one-particle collision term. The quantity $F_{ab} = \sqrt{(p^\alpha_a p^\alpha_b)^2 - m_a m_b}$ is the invariant flux, while $\sigma_{ab}$ is the invariant differential elastic scattering cross section into an element of solid angle $d\Omega$ that characterizes the collision of constituent $a$ with constituent $b$.

In a multi-component plasma, one must also count the flow of momentum and energy among components in the fluid. This leads to additional constraint equations of the moments of the distribution function [5, 7]. However, as shown in Table II the collision term for nuclei is completely dominated by electron elastic scattering. Hence, one can reduce the multi-component relativistic Boltzmann equation to a two component system describing the scattering of relativistic electrons from nuclei. The identification of the thermodynamic variables can then be determined from the relativistic entropy flow as described below.

A. General Distribution Function

Denoting electrons $e$ and nuclei $n$, the relativistic baryon Boltzmann equation [8] becomes:

$$p^\alpha_a \partial_a f_n \approx \int (f_n' f_e' - f_n f_e) F_{en} \sigma_{ne} d\Omega \frac{d^4 p_e}{p_e^0}.$$  \hfill (9)

Eq. (9) differs from the usual one-particle Boltzmann equation in that the distribution is fixed by the dominant collisions with relativistic electrons. Nevertheless, from this one can immediately derive the form of the stationary solution of the Boltzmann equation for the baryons.

For the distribution to be stationary one requires that the term in brackets on the r.h.s of Eq. (9) vanish. Hence, $f_n' f_e = f_n f_e$, which implies $\ln f_n' + \ln f_e = \ln f_n + \ln f_e$. So, for both the electrons and baryons the form $\ln f = A + B \alpha p^\alpha$ is a summation invariant [5] for which the terms $A$ and $B$ are determined from the stationary values of the particle four-flow and the energy-momentum tensor. The stationary distribution for both baryons and electrons is then of the form [8]:

$$f(p) = \exp[-A - B \alpha p^\alpha].$$  \hfill (10)

or with the imposition of Fermi-Dirac statistics, one has

$$f(p) = \frac{1}{\exp[A + B \alpha p^\alpha] + 1}.$$  \hfill (11)

For simplicity in the following derivation we note that in BBN one can ignore the small (+1) correction for quantum FD statistics that only matters at low momenta which do not lead to nuclear reactions.

Next, consider the particle number-density four-current $J^\mu = n U^\mu$, with $n$ the local proper rest particle density and $U^\mu$ the particle four velocity, with $U_\mu U^\mu = -1$. Since $J^\mu$ is the only relevant four vector, one can identify $B^\mu \propto J^\mu = \zeta U^\mu$. Then ignoring the quantum statistics correction, the equilibrium distribution takes the Maxwell-Jüttner (MJ) form:

$$f_{eq}(p) = \exp[-A - \zeta (U^\alpha p_\alpha)].$$  \hfill (12)

The parameter $A$ can be identified with the chemical potential [8] which is small during BBN and so for simplicity in what follows the constant $A$ can also be ignored.
B. Entropy Flow and the Gibbs Equation

For the next step one must identify the relation between the parameter $\zeta$ and the temperature $T$. To do this one must define the thermodynamic variables via the Gibbs relation:

$$s_E = \frac{1}{T} (e - \frac{P}{n})$$

where $s_E$ is the equilibrium entropy per particle. The internal energy per particle is $e = \langle E \rangle = \langle \gamma m \rangle$, $P$ is the pressure, and $n$ is the number density. The total equilibrium relativistic entropy is deduced from the entropy flow as

$$S_E^\alpha = ns_E U^\alpha.$$  \hspace{1cm} (14)

The total entropy flow, however, must be determined from the general distribution function (Eq. 12) $f$ via [8]. Ignoring the $\pm 1$ from quantum statistics, this is just,

$$S_E^\alpha = -k \int p^\alpha f \ln \left( \frac{f h^3}{e g_s} \frac{d^3 p}{p_0} \right),$$

where $e$ is as defined above, $h$ is the Planck constant, and $g_s$ is the usual spin degeneracy factor. Insertion of Eq. (12) into Eq. (15) leads to the following expression for the entropy per particle [8].

$$s_E = \frac{k \zeta}{m} \left( e + \frac{4 \pi m^4 g_s}{3 n h^3} J_{40} \right),$$

where

$$J_{m,n}(\zeta) = \int_0^\infty \sinh^n \theta \cosh^m \theta \exp (\zeta \cosh \theta) d\theta.$$  \hspace{1cm} (17)

The relevant state variables are then:

$$n = 4\pi m^3 \frac{g_s}{h^3} J_{21},$$

$$e = m \left[ \frac{J_{22}}{J_{21}} \right],$$

$$P = 4 \pi m^4 \frac{g_s}{h^3} J_{40},$$

so that the entropy per particle is

$$s_E = \frac{k \zeta}{m} (e - \frac{P}{n}).$$

1. Relativistic non-degenerate electrons

First we consider the electrons. Early during BBN the electrons interact much more frequently with each other than with nuclei. Thus, they can essentially be treated as a single component relativistic gas. In this limit one can simply equate Eqs. (13) and (21) so that $\zeta_e = m_e/kT$. For a non-degenerate gas, the $J_{mn}$ can be related [8] to modified Bessel functions of the second kind $K_n$. In the cosmological rest frame $U^\alpha p_\alpha = E_e/m$ is the total relativistic electron energy. Hence, the Maxwell Jüttner distribution for the electrons is obtained.

$$f_e(E) = \frac{n}{4 \pi m^2 kT K_2(\zeta)} \exp (-E_e/kT).$$  \hspace{1cm} (22)

For the other thermodynamic variables one can write:

$$n_e = 4 \pi m^3 kT \frac{g_s}{h^3} K_2(m_e/kT),$$

$$e_e = m_e \left[ \frac{K_3(m_e/kT)}{K_2(m_e/kT)} \right] - kT,$$

$$P_e = 4 \pi m_e^2 (kT)^2 \frac{g_s}{h^3} K_2(m_e/kT) = n_e kT.$$  \hspace{1cm} (25)

2. Nuclei experiencing elastic collisions with electrons

The physics of the nuclei, however, is different in this scenario. The isotropic velocities of the nuclear component of the cosmic fluid during BBN are dominated (at least initially) by collisions with relativistic electrons rather than other baryons. To deduce the baryonic pressure one must consider that elastic scattering with electrons conserves momentum and energy.

The derivation of the pressure for the nuclei is straightforward. For a system of discrete point particles, the energy-momentum tensor takes the form

$$T^{\mu\nu} = \sum_a \frac{p^\mu(a)p^\nu(a)}{p^0(a)} \delta^{(3)}(\vec{x} - \vec{x}(a)),$$

where now $a$ labels each particle and $p^\mu(a) = m_a U^\mu(a)$ is the covariant four momentum, and in flat space $U^\mu = (\gamma, \gamma v^1, \gamma v^2, \gamma v^3)$.

One is only interested in the spatial components $T^{ij}$ for the derivation of pressure in the cosmological rest frame. Moreover, since the spatial components of momentum are isotropic, only diagonal components are relevant. Hence we can write

$$P_n = T_n^{ii} = \sum_a \frac{p_i(a)p_i(a)}{p^0(a)} \delta^{(3)}(\vec{x} - \vec{x}(a))$$

$$= \sum_a \gamma_a m_n (\gamma v^2)^2 \delta^{(3)}(\vec{x} - \vec{x}(a))$$

$$= \frac{1}{3} \gamma_n m_n (\gamma v^2),$$  \hspace{1cm} (27)
where the factor of 1/3 follows from the isotropy of the frame at rest w.r.t. the cosmic fluid,
\[ \langle \gamma v_s^2 \rangle = \langle \gamma v_p^2 \rangle = \langle \gamma v_e^2 \rangle = \frac{1}{3} \langle \gamma v^2 \rangle. \] (28)

A similar derivation applies to electrons, i.e.
\[ P_e = T_e = \frac{1}{3} n_e m_e (\gamma v^2). \] (29)

However, since the baryon momenta in this case arise from recoil with relativistic electrons, the baryons must be in pressure equilibrium with electrons so that the pressure per particle for each species is the same. So from Eq. (25), the pressure per baryon is:
\[ \frac{P_n}{n_n} = \frac{P_e}{n_e} = kT. \] (30)

Indeed using the Maxwell-Jüttner distribution to evaluate the average in Eq. (28) identically gives
\[ P_n = n_n kT \] (31)

### 3. Internal energy of the nuclear fluid

Having derived the pressure, the energy per baryon is almost trivial.
\[ e_n = \gamma m_n = m_n + (\gamma - 1)m_n \approx m_n + (1/2)m_n (v_n^2), \] (32)

where the latter approximation follows from the fact that in the BBN epoch, \( v_n << 1 \).

From Eqs. (27), (30), and (32) it follows that
\[ e_n = m_n + \frac{3}{2} kT. \] (33)

Hence, even in this idealized case of nuclei only experiencing elastic scattering from a distribution of relativistic electrons, the baryons have the same average kinetic energy as that of a classical Maxwell-Boltzmann gas for which \( \langle m_n v_n^2 \rangle / 2 = (3/2) kT \). Indeed, this result is independent of the electron distribution function as long as the baryons are in pressure equilibrium with electrons.

The Gibbs relation for nuclei in the relativistic electron bath is then satisfied with \( \zeta = m_n / kT \). The entropy per particle is then that of a classical gas undergoing only elastic collisions.
\[ \frac{s}{k} = \frac{1}{kT} \left( m_n + \frac{3}{2} kT + kT \right) = \frac{m_n}{kT} + 5/2. \] (34)

Then for \( v << 1 \), \( U^\alpha \approx (1, 0, 0, 0) \), the Maxwell-Jüttner distribution for nuclei reduces to the usual MB kinetic energy distribution,
\[ f_n(E) = \frac{n}{(2\pi kT)^{3/2}} \exp \left( -\frac{m_n v_n^2}{2kT} \right). \] (35)

Hence, in the limit of nuclei dominated by electron elastic scattering we conclude that the standard MB statistics emerges. Note that this result is independent of the electron distribution function. The only requirement is pressure equilibrium with the electron gas. We note here, that our previous simulation [15] came to a different conclusion. This is because in the Langevin derivation kinetic energy equipartition was assumed as is usually done in classical simulations [20]. However, pressure equilibrium as employed here is more relevant for a relativistic plasma. Also, in the Monte-Carlo simulations of that work, no accounting was made of the effects of nuclear recoil w.r.t. the background plasma as described below in Sec. III and in Ref. [17]. These two unrelated assumptions, though seemingly natural lead to a nuclear distribution function that more closely resembles the electron FD distribution than a MB distribution.

### IV. MONTE-CARLO SCATTERING SIMULATION

As a test of this derivation, Figure 1 shows a Monte-Carlo simulation [17] of the kinetic energy distribution of protons in a bath of 1 MeV FD relativistic electrons after a large number of simulated elastic scattering events. Also shown for illustration is the distribution of a classical MB gas and the FD distribution of electrons. From this it is clear that even at the highest temperatures of the BBN epoch, in the idealized case of dilute charged baryons only scattering from relativistic electrons, the baryon distribution function is very close to that of a classical Maxwell Boltzmann gas.

![Monte-Carlo histogram](image)

**FIG. 1:** Monte-Carlo histogram (blue bars) of the kinetic energy distribution of baryons scattering in a bath of relativistic $e^+ - e^-$ plasma (black line) (at kT = 1 MeV) compared to the kinetic energy distribution of a classical Maxwell-Boltzmann distribution (red line).

We simulated nuclear thermalization in a bath with temperatures and an environment relevant to BBN. This
was to numerically obtain the true kinetic energy and velocity distributions for the nuclei. Table I shows that photons play a negligible role in this process. Hence, we only needed to simulate scattering of an FD distribution of $e^- - e^+$ pairs with nuclei. During this scattering process energy is transferred to or from nuclei. The direction of transfer of momentum is governed by the angle of incoming particles, the velocity of incoming particles and the scattering angle of the outgoing electron or positron. For our simulation the angle of the incoming particles was chosen isotropically in the cosmic frame. However, this would not in general be isotropic in the nuclear rest frame due to the accumulated nuclear recoil velocity.

In Ref. [15] the sampling of electrons for collision with nuclei was performed using the distribution function $f(v)$ where $v$ is the relative velocity in the cosmological frame. The first results of this simulation seemed to indicate non-MB statistics for nuclei. However, those first simulations did not take into account the effect of the instantaneous viscosity (i.e electrons moving opposite to the nuclear direction of motion collide more frequently with the nucleus). This was corrected by sampling the electrons from the electron flux at a rate proportional to $v f(v)$, where $v$ is the relative velocity in the frame of the nucleus. With this correction the nuclear distribution function is skewed to lower energies due to the increase in the collision rate along the direction of motion. This reduced the high-energy tail of the distribution. Surprisingly, after adding this seemingly innocuous correction, the resultant distribution overlaps well with MB statistics rather than the electron FD distribution.

We randomly selected the incoming electron momentum from the FD distribution. The angle of scattering for electrons is weighted by the differential cross-section in Eq. [5]. The reactions are simulated in three dimensions. The incoming momentum of nuclei before each scattering event is given by its momentum after the previous scattering event. The scattering process is then repeated for a sufficiently large number of times ($\sim 10^7$). Note that according to Table I at $kT = 1.0$ MeV there would only be $10^{-5}$ photon scatterings for each electron scattering. Moreover, for a baryon-to-photon ratio of $\eta \sim 10^{-9}$, there would be no nucleus-nucleus scatterings during $10^7$ electron collisions. Hence, the influence of nuclear and photon scattering is negligible. This, however, is not the case in stars where the baryon density is much higher.

Nevertheless, as shown in Figure. 1 the nuclear distribution obeys MB statistics. Indeed, we have checked [17] that the nuclear distribution function is independent of the electron distribution function. Even a delta-function electron distribution function will lead to an MB distribution for nuclei.

V. DISCUSSION/CONCLUSIONS

In summary, we have shown that the thermalization of nuclei during BBN is dominated by Coulomb elastic scattering with the background mildly relativistic pair plasma. However, we have shown based upon a relativistic multi-component Boltzmann equation derivation and a Monte-Carlo simulation that the resultant nuclear distribution functions continue to obey MB statistics contrary to the claim in Ref. [15]. The reason for our previous erroneous claim can be traced to the assumption of kinetic-energy equipartition instead of pressure equilibrium, and also the neglect of the instantaneous viscosity in the motion of the baryons w.r.t. the background plasma [17].

For completeness, we reconstruct the Langevin Brownian-motion derivation from Ref. [15] with the imposition of pressure equilibrium rather than kinetic-energy equipartition. We show that this also leads to a MB distribution for the nuclear reactions.

VI. ACKNOWLEDGMENT

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Here, $m$ is the mass of the particle, $v$ is the velocity, $\lambda$ is a drag coefficient, and $R(t)$ is a noise term representing the effect of collisions with the background fluid at time $t$. The force $R(t)$ has a Gaussian probability distribution centered around $R = 0$ and the value at time $t + \tau$ does not depend on the value at time $t$, i.e.,

$$P(R) = \frac{1}{\sqrt{2\pi(R(t))^{\frac{2}{2}}}} \exp \left[ \frac{-R^{2}}{2(R(t))^{2}} \right] ,$$

$$\langle R(t) \rangle = 0 ,$$

and

$$\langle R(t)R(t + \tau) \rangle = \langle R(t) \rangle \delta(\tau) .$$

These conditions are easily satisfied in the BBN scattering environment. Note also, that it does not matter whether $R(t)$ is due to scattering from relativistic or non-relativistic particles as long as the force has a Gaussian probability distribution, the Langevin formalism can be applied to derive the distribution of the massive particle. Indeed, massive particles in a relativistic fluid do experience a random Gaussian force as has been shown in Ref. [22].

The general solution to Eq. (36) is given by

$$v(t) = v_{0}e^{-\lambda t} + \frac{1}{m} \int_{0}^{t} R(t') e^{-\frac{\lambda(t-t')}{}dt'} .$$

Even without specifying the explicit form of $R(t)$, one can deduce average properties of $v(t)$. In particular, from Eq. (40) one can take the limit as $t \to \infty$, to conclude that

$$\langle v^{2}(t) \rangle = \frac{q}{2\lambda m} ,$$

where $q = \langle R(t) \rangle ^{2} \delta(\tau)$ and $\langle R(t) \rangle ^{2}$ is the variance of $R(t)$.

Now, as shown in Eq. (30) the pressure equilibrium between the non-relativistic baryons and relativistic background requires

$$\frac{1}{2} m_{n} \langle v^{2} \rangle = \frac{3}{2} kT .$$

Then using Eq. (41) one has,

$$\frac{1}{2} m_{n} \langle v^{2} \rangle = \frac{q}{4\lambda} = \frac{3}{2} kT ,$$

so that

$$q = 6\lambda kT .$$

Hence, for a nuclide of mass $m$ in equilibrium with a relativistic background $e^{+} - e^{-}$ plasma, the distribution function can be described as the usual MB distribution

$$f(v) = \left( \frac{m}{2nK} \right)^{\frac{1}{2}} 4\pi e^{2} \exp \left( -\frac{mv^{2}}{2K} \right) ,$$

$$f(E) = 2 \left( \frac{1}{K} \right)^{\frac{1}{2}} \sqrt{\frac{E}{\pi}} \exp \left( -\frac{E}{K} \right) .$$