MOVING PLANES, JACOBI CURVES AND THE DYNAMICAL APPROACH TO FINSLER GEOMETRY

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ABSTRACT. We express invariants of Finsler manifolds in a geometrical way by means of using moving planes and their associated Jacobi curves, which are curves in a fixed homogeneous Grassmann manifold. Some applications are given.

1. INTRODUCTION

A common way of writing computations in Finsler geometry is through some extension of the Levi-Civita calculus of Riemannian geometry. However, since there cannot be a Levi-Civita connection in Finsler geometry (for reasonable notions of connection with metric compatibility and torsion freeness, if a Finsler manifold admits such a connection it is actually Riemannian), there is a plethora of connections (Berwald, Cartan, Chern and Rund, . . . ) where each one of them is defined by partial compatibilities and torsion freeness. While this connection formalism has led to important developments, there are contexts where a different point of view can shed new light.

An alternative approach, of a more dynamical flavor, to the geometry of sprays and Finsler metrics consists in regarding the local differential invariants of sprays and Finsler metrics as local invariants (under the action of the appropriate group of diffeomorphisms) of the following type of geometric data on a manifold:

Definition 1.1. A moving plane on a smooth manifold $X$ is a triplet $\mathcal{P} = (\Delta_r, \Delta_k, \Phi_t)$, where

1. $\Delta_k \subset \Delta_r$ are distributions on $X$ with dimensions $k$ and $r$, respectively.
2. $\Phi_t$ is a flow in $X$ which leaves $\Delta_r$ invariant.

For instance, the prototypical examples that motivated this paper are the cases where (we refer to §2.1 for precise definitions)

1. $X$ is the tangent bundle without the zero section $TM\setminus 0$ of a manifold $M^n$, $\Delta_{2n}$ is the full tangent distribution, $\Delta_n$ is the vertical distribution $VTM$, and $\Phi_t$ is the flow corresponding to a spray $S$ on $M$.
2. $X$ is the unit co-sphere bundle $\Sigma^*_F M$ of a Finsler manifold $(M^n, F)$, $\Delta_{2n-2}$ is the canonical contact distribution on $\Sigma^*_F M$, $\Delta_{n-1}$ is the vertical distribution $V\Sigma^*_F M$ and $\Phi_t$ is the restriction to $\Sigma^*_F M$ of the co-geodesic flow of $F$.

This approach is implicit in the pioneering works of Grifone [17] and Foulon [15] where, for instance, the classical notions of Ehresman connection and curvature
endomorphism from the theory of second order differential equations and Finsler metrics, are recovered by considering the so-called almost tangent structure (in the case \[17\]) and the vertical endomorphism (in the case \[15\]) and their successive Lie derivatives along the geodesic vector field.

Back to the moving plane setting, the infinitesimal action of the flow \(\Phi_t\) on the distribution \(\Delta_k\) gives rise, for each \(x \in X\), to a curve

\[
\ell_x(t) = (\Phi_t^* \Delta_k)(x) = d\Phi_{-t}(\Phi_t(x))\Delta_k(\Phi_t(x))
\]

of \(k\)-dimensional subspaces of the fixed vector space \(\Delta_r\): that is, \(\ell_x(t)\) is a curve on the Grassmannian manifold \(\text{Gr}_k(\Delta_r(x))\), called the Jacobi curve of \(\mathcal{P}\) based at \(x\). In the above examples, the Jacobi curves live on half-Grassmannians \(\text{Gr}_n(\mathbb{R}^{2n})\) and on Lagrangian Grassmannians \(\Lambda(\mathbb{R}^{2n})\), respectively. It is well-known that the topology of curves of Lagrangian subspaces successfully describes conjugacy of geodesics via the Maslov index theory \[25\]. As we will show here, the local geometry of Jacobi curves also describes relevant local invariants of sprays and Finsler metrics, in particular the invariants related to variational phenomena; by this we mean, for example, the Jacobi endomorphism \(Y \mapsto R(Y, T)T\) which appears in the Jacobi equation and leads to the definition of flag curvature.

To the best of our knowledge, this was first noticed by Adhout \[4\] in the case of Riemannian geodesic flows; there, by identifying an important generic property of curves of Lagrangian subspaces (the fanning property, later extended to curves on \(\text{Gr}_n(\mathbb{R}^{2n})\) in \[7\]), the author uncovers the local invariants as linear symplectic invariants of the Jacobi curve. On the other hand, the geometry of curves on \(\text{Gr}_n(\mathbb{R}^{2n})\) and \(\Lambda(\mathbb{R}^{2n})\), under the action of the general linear and symplectic groups, is a beautiful subject in itself. As has been shown in \[7\], the behaviour of the class of fanning curves can be completely described, in the spirit of Cartan-Klein, by a set of linear invariants. As we shall show here, the formalism of \[7\] applied to the Jacobi curves of the above examples gives us the desired local invariants. This gives a unified treatment of the approaches of Grifone, Foulon and Adhout, and can be viewed as a Cartan-Klein geometrization of them. This point of view leads to some applications to Finsler geometry that we now describe:

**An O’Neill formula for Finsler submersions.** A fundamental tool in the study of curvature properties of Riemannian manifolds is the O’Neill tensors and associated O’Neill formulas \[24\], which relate curvatures of the total space and the base of Riemannian submersions; see for example \[20\] for a description of its use in the study of non-negative curvature. We give an O’Neill formula for Finsler manifolds expressed in terms of invariants of the Jacobi curve. As is common in Finsler geometry, the results are interesting even for Riemannian manifolds: the standard proof and applications of O’Neill formulas are given as algebraic manipulations of the Levi-Civita connection, whereas the Jacobi curve gives an O’Neill formula as a quantification of the relationship, as a symplectic reduction, of the geodesic flows of the total space and the base \[6\]. In addition to curvature bounds applications, the fine details of the O’Neill tensor allows the consideration of rigidity results of special submersions \[14\, 19\] and the original rigidity results of O’Neill (theorem 4 of \[24\]), which would be quite interesting to generalize to the Finslerian setting.

**A characterization of the sign of flag curvature.** An important area of Riemannian geometry is the construction of examples of manifolds with sign properties
of the sectional curvature, for example manifolds of positive (resp. negative) sectional curvature and their associated relaxed conditions non-negative (resp. non-positive), see e.g. [32]. This interest has spread to Finsler manifolds [26], and the study of examples has begun with the homogeneous case (see [33] for a survey). We give a dynamical characterization of the sign of flag curvature in terms of the Jacobi curve, or, more precisely, in terms of the horizontal curve, which is another curve in the (Lagrangian) Grassmannian canonically produced from the Jacobi curve.

**The flag curvature of a class of projectively related Finsler metrics.** One area where Finsler geometry is completely different from Riemannian geometry is inverse problems, where in the Finsler case there is typically a rich moduli space (specially in the non-symmetric case), whereas there is rigidity in the Riemannian case, for example, in Hilbert’s Fourth Problem [5] and projectively flat metrics of constant curvature [8]. In this spirit, two Finsler metrics are projectively related if they share the same geodesics up to reparametrization. An important transformation that does not change the projective class of a metric is the addition of a closed 1-form. We describe how the Jacobi curve furnishes a formula relating the flag curvature of a metric with that of its deformation by a closed 1-form.

**The flag curvature of Katok perturbations.** In 1973 A. Katok constructed examples of a non-symmetric Finsler metric on the sphere $S^2$ with only two prime closed geodesics; the geometry of these metrics has been nicely described in [31] and a standard Finsler description is given in [28]. It is well-known that these metrics have constant curvature (see, e.g. Foulon [16] or §11 of Rademacher [26]). We present a proof of this property, due to J.C. Álvarez, that proceeds by showing that the Jacobi curves of the original metric and of the Katok-perturbed one are equivalent under a linear-symplectic transformation, thus having the same invariants.

**Remark 1.2.** The local geometry of the Jacobi curve has also been intensively studied with motivation coming from Control Theory and Sub-Riemannian geometry; see [2] and the references therein for a contemporary account, and the appendix of [4] for comparison of the approaches to the invariants. In particular, in [5], there is a reduction procedure similar to the one we use for giving the Finslerian version of the O’Neill tensor and associated formula.

**Remark 1.3.** Moving planes and their Jacobi curves in half-Grassmannians are specially adapted to Finsler geometry; however this concept can be generalized and applied to other situations: one can consider for example a whole linear flag of distributions $\Delta_{k_1} \subset \Delta_{k_2} \subset \cdots \subset \Delta_r$, and its associated Jacobi curve in a fixed flag manifold. This situation appears in the study of higher order variational problems, where the $\Delta_{k_i}$ are kernels of the derivative of the projections of the jet spaces of curves $\pi : J^s(\mathbb{R}, M) \to J^r(\mathbb{R}, M)$ for adequate $s > r$. See [10, 11, 12, 13].

After this introduction, the paper is organized as follows: we give some preliminaries in [2] in order to fix language and make the paper reasonably self-contained. In [3] we establish how curvature invariants are expressed in terms of moving planes and their associated Jacobi fields, by relating these invariants with those obtained by the dynamic method and Finsler connections; in particular, we recover the flag curvature in Theorem 3.12. The rest of the sections of the paper correspond to each of the aforementioned applications.
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2. Preliminaries

The two ends that this paper aims to connect are, on one side, the global invariants of Finsler manifolds, and on the other side, the invariants of curves in a fixed Grassmann manifold viewed as a homogenous space. Sections 2.1 and 2.2 correspond respectively to the necessary preliminaries of each side.

2.1. Sprays and Finsler manifolds.

2.1.1. Notations and the structure of the tangent bundle. We shall denote by $T M\setminus 0$ the tangent bundle of a manifold with the null section removed, and by $\pi$ and $\rho$ the projection maps $\pi : T M\setminus 0 \to M$, $\rho : T(T M\setminus 0) \to T M\setminus 0$. The latter contains as a vector subbundle the vertical tangent bundle
\[
\rho : V T M \to T M\setminus 0,
\]
whose fibers are the tangent spaces of the fibers of $\pi$. We shall call vertical vector fields on $T M\setminus 0$ the sections of (2.1). The vertical lift at a given $w \in T M\setminus 0$ is the tautological isomorphism
\[
i_w : T\pi(w)M \to V_w T M, \quad i_w(u) = (d/dt)|_{t=0}(w + t \cdot u),
\]
where the name of these isomorphisms stems from the fact that $i_w$ furnishes canonical lifts of a vector fields $U$ on $M$ to vertical fields $U^\rho$ on $T M\setminus 0$; the same procedure also gives vertical lifts of vector fields defined along curves in $M$. The canonical vector field $C$ on $T M\setminus 0$ is defined by $C_w = i_w(w)$.

We remark that analogous constructions apply to the punctured co-tangent bundle $\tau : T^*M\setminus 0 \to M$: a vertical distribution $VT^*M$ on $T^*M\setminus 0$, tautological isomorphisms $i_\xi : T_{\tau(\xi)}M \to V_{\xi}T^*M$, and the canonical vector field $C^*$ on $T^*M\setminus 0$ are defined as before.

With this tool in hand, we can define

**Definition 2.1.** The almost-tangent structure of $T M\setminus 0$ is the section $J$ of $\text{End}(T(T M\setminus 0)) \to T M\setminus 0$ defined by
\[
J(X) = i_w(d\pi(w)X), \quad \text{for } w = \rho(X).
\]

Observe that $J$ has both kernel and image equal to $VT M$.

**Definition 2.2.** A second order differential equation (SODE) on $M$ is a smooth vector field $S$ on $T M\setminus 0$ such that $J(S) = C$. This means that the integral curves of $S$ are of the form $t \mapsto \gamma(t)$, for some class of curves $\{\gamma\}$ in $M$. If furthermore $[C, S] = S$, then $S$ is called a spray, in which case the curves $\{\gamma\}$ are the geodesics of $S$. 

In natural local coordinates \((x, y)\) for \(T\mathcal{M}\), i.e. \((x, y)\) are induced from local coordinates \(x\) for \(M\), a SODE assumes the form

\[
S = \sum_i y_i \frac{\partial}{\partial x_i} - 2 \sum_i G_i(x, y) \frac{\partial}{\partial y_i},
\]

for certain smooth functions \(G_i\) that are positively homogeneous of degree 2 in \(y\) if, and only if, \(S\) is a spray. The following basic property (cf. \[17, \text{Prop. I.7}\]) will be essential later. For the sake of completeness we have included a proof.

**Lemma 2.3.** If \(S\) is a SODE on \(M\) and \(X\) is a vertical vector field on \(T\mathcal{M}\), then \(J([X, S]) = X\).

**Proof.** Since \(J\) vanishes on vertical vectors, \(J([X, S]) - X\) is \(C^\infty(T\mathcal{M})\)-linear in the sections \(X\) of (2.1). Relatively to natural local coordinates \((x, y)\), \(J(\partial/\partial y_i) = 0\), \(J(\partial/\partial x_i) = \partial/\partial y_i\), and, from (2.3), \([\partial/\partial y_i, S] = \partial/\partial x_i - 2 \sum_j (\partial G_j/\partial y_i) \partial/\partial y_j\).

Therefore, \(J([\partial/\partial y_i, S]) - \partial/\partial y_i = 0\). \(\square\)

2.1.2. Finsler manifolds.

**Definition 2.4.** A Finsler metric on a smooth manifold \(M\) is a function \(F: T\mathcal{M} \to [0, \infty)\) that is smooth on \(T\mathcal{M}\) and that restricts to a Minkowski norm \(F_m\) on each tangent space \(T_m\mathcal{M}\). This means that

1. \(F(v) = 0\) if, and only if \(v = 0\);
2. \(F(\lambda v) = \lambda F(v)\), if \(\lambda \geq 0\);
3. For every \(v \in T\mathcal{M}\), the second fiber-derivative of \((1/2)F^2\) at \(v\),

\[
g_F(v) := (1/2)d^2_F F^2(v),
\]

is a positive-definite inner product on \(T_{\pi(v)}\mathcal{M}\).

The inner product (2.4) is usually referred to as the fundamental tensor of \(F\) at a \(v\).

An important concept attached to a Finsler metric is the notion of *dual*.

**Definition 2.5.** The dual of a Finsler metric \(F\) on \(M\) is the function \(F^* : T^*\mathcal{M} \to \mathbb{R}\) obtained by fiberwise taking the dual of the Minkowski norms \(F_m\), that is,

\[
(F^*)(\xi) = \sup_{F_m(v) = 1} \xi(v).
\]

Alternatively, the dual \(F^*\) of \(F\) is obtained by composing \(F\) with the inverse of its Legendre transformation. The latter is the diffeomorphism \(L_F : T\mathcal{M} \to T^*\mathcal{M}\),

\[
L_F(v) = (1/2)d_F F^2(v) = g_F(v)(v, \cdot).
\]

We remark that, from the homogeneity, the fiber derivative of \(L_F\) is given by

\[
d_F L_F(v)w = g_F(v)(w, \cdot).
\]

2.1.3. The Hamiltonian point of view.

*The co-tangent bundle setting.* Let us begin by recalling

**Definition 2.6.** The canonical 1-form of \(T^*M\) is the 1-form \(\alpha\) on \(T^*M\) defined by

\[
\alpha_\xi(X) = \xi(d\tau(\xi)X).
\]

The 2-form \(\omega = -d\alpha\) defines the so-called canonical symplectic structure of \(T^*M\).
We remark that one can recover \( \alpha \) from \( \omega \) and the tautological vector field \( C^* \) via
\[
\alpha = -i_{C^*} \omega.
\]

Given a Finsler metric \( F \), let us consider the Hamiltonian function
\[
(2.8) \quad \alpha = \frac{1}{2}(F^*)^2 : T^*M \setminus 0 \to \mathbb{R}
\]
on the symplectic manifold \((T^*M \setminus 0, \omega)\).

**Definition 2.7.** We shall call \textit{co-geodesic vector field} of \( F \), and denote by \( S^*_F \) (or simply \( S^* \)), the Hamiltonian vector field of (2.8); that is, \( S^*_F \) is the vector field on \( T^*M \setminus 0 \) defined by
\[
(1/2)\text{d}(F^*)^2 = \omega(\cdot, S^*_F).\]
The corresponding flow \( \Phi^{S^*_F}_t \) is the \textit{co-geodesic flow} of \( F \). Observe that since (2.8) is positively homogeneous of degree 2, then \([C^*, S^*_F] = S^*_F\).

Being the Hamiltonian flow of (2.8), the co-geodesic flow preserves \( \omega \) and leaves invariant every level set of \( F^* \). In particular, it restricts to a flow on the \textit{unit co-sphere bundle}
\[
(2.10) \quad \Sigma^*_F M = (F^*)^{-1}(1).
\]
We shall still let \( S^*_F \) and \( \Phi^{S^*_F}_t \) denote their restrictions to (2.10). In the following, \( \alpha \) and \( \omega \) will mean their pull-backs to (2.10). The contact geometry of \( F \) is described by

**Proposition 2.8.** The 1-form \( \alpha \) is a contact form on (2.10); this means that \( \omega \) is non-degenerate (hence, induces a symplectic structure) on the so-called contact distribution \( \ker(\alpha) \). Furthermore, the vector field \( S^*_F \) is the Reeb vector field of \((\Sigma^*_F M, \alpha)\), that is, it is the unique vector field such that
\[
i_{S^*_F} \omega = 0, \quad \alpha(S^*_F) = 1.
\]

**Remark 2.9.** For future reference, we remark that \( T^*M, C^*, \alpha, \omega, S^*_F, \Sigma^*_F M \) fit in the following abstract setting. Let \((X, \omega, S)\) be a symplectic manifold endowed with a vector field \( S \) generating a symplectic flow \( \Phi^S_t \), and let \( \Sigma \subset X \) be a \( \Phi^S_t \)-invariant hypersurface such that

1. \( \Sigma \) is of contact type with respect to a Liouville vector field \( C \); this means that \((\text{cf.}[23]) \) \( C \) is a vector field defined in a neighborhood of \( \Sigma \) that is everywhere transverse to \( \Sigma \) and such that \([C, \omega] = \omega\).
2. \( S \) generates the characteristic distribution of \( \Sigma \), i.e. \( T\Sigma = \ker(i_{S}\omega|_{\Sigma}) \).
3. \( S \) satisfies the homogeneity \([C, S] = S\).

In this setting, \( \alpha := -i_C \omega \) pulls back to a contact form on \( \Sigma \), still denoted by \( \alpha \). Moreover, \( \Phi^S_t \) restricts to an exact contact flow on \((\Sigma, \alpha)\), i.e. \( \Phi^S_t \alpha = \alpha \) for all \( t \), and \(-d\alpha \) and \( \omega \) restrict to the same symplectic structure on the contact distribution \( \ker(\alpha) \).

**The tangent bundle setting.** We shall let \( \alpha_F \) and \( \omega_F \) be the pull-backs of \( \alpha \) and \( \omega \) by the Legendre transformation \( \mathcal{L}_F \). Observe that, from (2.8) and (2.7),
\[
(\alpha_F)_*(X) = g_F(v)(v, d\pi(v)X).
\]
The pull-back of $S_F^\ell$ by $\mathcal{L}_F$ is a spray on $M$, the so-called geodesic spray $S_F$ of $F$, and the corresponding flow $\Phi^S_F$ is the geodesic flow of $F$. It follows that $\Phi^S_F$ preserves $\omega_F$ and

$$\mathcal{L}_F \circ \Phi^S_F = \Phi^S_F \circ \mathcal{L}_F.$$  

As in the co-tangent case, $\alpha_F$ pulls-back to a contact form, still denoted by $\alpha_F$, on the unit sphere bundle $\Sigma_F M = F^{-1}(1)$, and $S_F$ restricts to the Reeb field of $(\Sigma_F M, \alpha_F)$, still denoted by $S_F$. The Legendre transformation $\mathcal{L}_F$ relates both contact geometries.

2.2. The geometry of fanning curves. In this section we summarize the invariants of curves in the half-Grassmannians and Lagrangian Grassmannians constructed in [7].

2.2.1. Fanning curves on $\text{Gr}_n(V)$. Let $V$ be a 2$n$-dimensional real vector space. A smooth curve $\ell(t)$ on the Grassmannian manifold $\text{Gr}_n(V)$ of $n$-dimensional subspaces of $V$ is fanning if, upon identifying the tangent spaces $T_t\text{Gr}_n(V)$ with the spaces of linear maps from $\ell$ to $V/\ell$, each velocity vector $\dot{\ell}(t)$ is an invertible linear map; this is a non-degeneracy condition satisfied by an open and dense set of smooth curves. The set of fanning curves is acted upon by the general linear group $\text{GL}(V)$ and it turns out that, with respect to the prolonged action of $\text{GL}(V)$ on the space $J^1_\ell(\mathbb{R}; \text{Gr}_n(V))$ of one-jets of fanning curves on $\text{Gr}_n(V)$ and the adjoint action of $\text{GL}(V)$ on $\mathfrak{gl}(V)$, all the equivariant maps

$$J^1_\ell(\mathbb{R}; \text{Gr}_n(V)) \to \mathfrak{gl}(V)$$

are of the form $aI + bF$, $a, b \in \mathbb{R}$, where $I$ is the identity of $V$ and the fundamental endomorphism $F$ can be described in terms of frames as follows.

2.2.2. Frames and the Fundamental endomorphism. If $A(t) = (a_1(t), \ldots, a_n(t))$ is a frame for $\ell(t)$, i.e. $a_1(t), \ldots, a_n(t)$ are smooth curves on $V$ spanning $\ell(t)$, then the condition of being fanning is equivalent to requiring that

$$(A(t), \dot{A}(t)) = (a_1(t), \ldots, a_n(t), \dot{a}_1(t), \ldots, \dot{a}_n(t))$$

be a frame for $V$. In general, we shall call a smooth curve $a(t)$ on $V$ satisfying $a(t) \in \ell(t)$ for all $t$ a section of $\ell(t)$. The following definition does not depend on the choice of frame for $\ell(t)$.

Definition 2.10. The fundamental endomorphism of the fanning curve $\ell(t)$ is the curve $F(t) \in \text{End}(V)$ defined in the basis $(a_1(t), \ldots, a_n(t), \dot{a}_1(t), \ldots, \dot{a}_n(t))$ by

$$F(t)a_i(t) = 0 \ , \ F(t)\dot{a}_i(t) = a_i(t).$$

Remark 2.11. It is customary to abbreviate the notation in situations like the one above by $F(t)A(t) = 0$ , $F(t)\dot{A}(t) = A(t)$.

The main thrust of [7] is that the geometry of fanning curves under the action of $\text{GL}(V)$ is completely described by $F(t)$ and its derivatives $\dot{F}(t), \ddot{F}(t)$. 

2.2.3. The horizontal curve and the horizontal derivative. The derivative $\dot{\Phi}(t)$ is a curve of reflections whose -1 eigenspace is $\ell(t)$. The 1-eigenspaces at each $t$ form thus a curve $h(t)$ on $Gr_n(V)$, called the horizontal curve of $\ell(t)$. The projection operators corresponding to the decomposition
\begin{equation}
V = \ell(t) \oplus h(t)
\end{equation}
are denoted by $P_h(t) = \frac{1}{2}(I + \dot{\Phi}(t))$, $P_e(t) = I - P_h(t)$.

**Definition 2.12.** The horizontal derivative at time $t = \tau$ is the isomorphism
\begin{equation}
H(\tau) : \ell(\tau) \to h(\tau), 
H(\tau)v = P_h(\tau)\dot{u}(\tau),
\end{equation}
for $a : I \to V$ any section of $\ell(t)$ with $a(\tau) = v$. The horizontal derivative of a frame $A(t)$ for $\ell(t)$ is thus a frame for $h(t)$, denoted by
$$
H(t) = H(t)A(t).
$$

We remark that the inverse of (2.12) is the restriction of $\Phi(t)$ to $h(t)$,
\begin{equation}
H(t)^{-1} = \Phi(t)|_{h(t)} : h(t) \to \ell(t).
\end{equation}

Given a frame $A(t)$ for $\ell(t)$, the fanning condition implies that there exist curves of $n \times n$ matrices $P(t)$ and $Q(t)$ such that
\begin{equation}
\dot{A}(t) + \dot{\Phi}(t)P(t) + A(t)Q(t) = 0.
\end{equation}
The frame is called normal if $P = 0$, which in turn is equivalent to $H(t) = \dot{A}(t)$.

2.2.4. The Jacobi endomorphism and the Schwarzian. Since $\dot{\Phi}(t)$ is a curve of reflections, its derivative $\ddot{\Phi}(t)$ interchanges the decomposition (2.11). The Jacobi endomorphism of $\ell(t)$ is the curve $K(t)$ on $\text{End}(V)$ defined by
\begin{equation}
K(t) = \frac{1}{4}\ddot{\Phi}(t)^2 = \dot{P}(t)^2.
\end{equation}
A nice description of $K(t)$ is given in terms of the Schwarzian $\{A(t), t\}$ of a frame $A(t)$. If $P(t)$ and $Q(t)$ are as in (2.14), then $\{A(t), t\}$ is defined by
\begin{equation}
\{A(t), t\} = 2Q(t) - (1/2)P(t)^2 - \dot{P}(t).
\end{equation}
Note that if $A(t)$ is normal, then
$$
\dot{A}(t) = -(1/2)A(t)\{A(t), t\}.
$$

**Proposition 2.13.** Given a frame $A(t)$ for $\ell(t)$, the matrices of $(1/2)\ddot{\Phi}(t) = -\dot{P}(t)$ and $K(t)$ in the basis $(A(t), H(t))$ are, respectively,
\begin{equation}
\begin{pmatrix}
O & -(1/2)\{A(t), t\}
\end{pmatrix},
\begin{pmatrix}
(1/2)\{A(t), t\} & O
\end{pmatrix}
\begin{pmatrix}
O
\end{pmatrix}.
\end{equation}

2.2.5. Fanning curves of Lagrangian subspaces. Let us now suppose that $V$ is endowed with a symplectic form $\omega$. Recall that a subspace $\ell \subseteq V$ is called Lagrangian if $\ell = \ell^\omega := \{u \in V : \omega(u, v) = 0 \text{ for all } v \in \ell\}$, and the collection of all such subspaces forms a submanifold $\Lambda(V, \omega)$, or simply $\Lambda(V)$, of $Gr_n(V)$, the so-called Lagrangian Grassmannian of $V$. For each $\ell \in \Lambda(V)$ there is a canonical identification
\begin{equation}
T_\ell \Lambda(V) \cong \text{Bil}_{\text{sym}}(\ell),
\end{equation}
through which the velocity vectors of a smooth curve $\ell : I \subseteq \mathbb{R} \to \Lambda(V)$ are regarded as symmetric bilinear forms. Concretely,
Definition 2.14. The Wronskian at time $t = \tau$ of a smooth curve $\ell : I \subseteq \mathbb{R} \to \Lambda(V)$ is the symmetric bilinear form $W(\tau) \in \text{Bil}_{\text{sym}}(\ell(\tau))$ given by $W(\tau)(u, v) = \omega(u, \dot{a}(\tau))$, for $a : I \to V$ any section of $\ell(t)$ with $a(\tau) = v$.

In this setting, the condition for a curve $\ell : I \subseteq \mathbb{R} \to \Lambda(V)$ to be fanning is equivalent to $W(t)$ being non-degenerate for all $t$. Furthermore,

Proposition 2.15. For a fanning curve $\ell(t)$ on $\Lambda(V)$, the following hold:

1. The fundamental endomorphism $F(t)$ takes values in the Lie algebra $\mathfrak{sp}(V)$.
2. The horizontal curve $h(t)$ consists of Lagrangian subspaces.
3. The restriction of $K(t)$ to $\ell(t)$ is symmetric with respect to $W(t)$.

2.2.6. Transformation properties. Fanning curves on $\text{Gr}_n(V)$, resp. $\Lambda(V)$, are naturally acted upon by $\text{GL}(V)$, resp. $\text{SP}(V)$, and by the group $\text{Diff}(\mathbb{R})$ of diffeomorphisms of $\mathbb{R}$ via reparametrization.

Proposition 2.16. Let $\ell(t)$ be a fanning curve on $\text{Gr}_n(V)$, resp. $\Lambda(V)$. Given $T \in \text{GL}(V)$, resp. $\text{SP}(V)$, and $s \in \text{Diff}(\mathbb{R})$, then

1. The fundamental endomorphism, the Wronskian, and the Jacobi endomorphism of $T\ell(t)$ are, respectively, $T\ell(t)T^{-1}$, $(T\ell(t))_\tau W(t)$ and $T\ell(t)T^{-1}$.
2. The fundamental endomorphism, the Wronskian, and the Jacobi endomorphism of $\ell(s(t))$ are, respectively, $\dot{s}(t)\dot{F}(s(t))$, $\dot{s}(t)W(s(t))$ and

$$\dot{s}(t)^2K(s(t)) + \frac{1}{2}\{s(t), t\}I,$$

where $\{s(t), t\} = (d/dt)(\dot{s}^{-1}\dot{s}) - (1/2)(\dot{s}^{-1}\dot{s})^2$ is the Schwarzian derivative of $s(t)$.

3. MOVING PLANES, JACOBI CURVES AND THEIR INVARIANTS

Let us consider a moving plane $\mathcal{P}$ on a smooth manifold $X$, of the type

$$(3.1) \quad \mathcal{P} = (\Delta_{2n}, \Delta_n, \Phi_1),$$

and let $S$ be the vector field on $X$ that generates $\Phi_t$. In particular, we will also be interested in the cases where

(I) $X = (X^{2n}, \omega)$ is a symplectic manifold, $\Delta_{2n} = TX$, $\Delta_n$ is a Lagrangian distribution on $X$ (i.e. each $\Delta_n(x)$ is a Lagrangian subspace of $T_xX$), and $\Phi_t$ is a symplectic flow (i.e. $(\Phi_t)^*\omega = \omega$).

(II) $X = (X^{2m+1}, \omega)$ is an exact contact manifold, in which case we let $\omega = dx$, $\Delta_{2n}$ is the contact distribution $\ker(\alpha)$, $\Delta_n$ is a Legendrian distribution $\mathcal{L}$ (i.e. each $\mathcal{L}_x$ is a Lagrangian subspace of $(\ker(\alpha_x), \omega_x)$, and $\Phi_t$ is an exact contact flow (i.e. $(\Phi_t)^*\alpha = \alpha$).

It then follows that the Jacobi curve $\ell_x(t)$ of $\mathcal{P}$, based at a given $x \in X$ (recall (I)), is a curve in the half-Grassmannian $\text{Gr}_n(\Delta_{2n}(x))$ and that, in cases (I) and (II), $\ell_x(t)$ takes values on the Lagrangian Grassmannian $\Lambda(V)$, where $V = (\Delta_{2n}(x), \omega_x)$.

Example 3.1. The examples to keep in mind are provided by the geodesic flows of sprays and Finsler metrics. Let $S$ be a spray on $M^n$.

1. The action of $\Phi_t^S$ on the vertical distribution $\mathcal{V}TM$ gives rise to the moving plane

$$(3.2) \quad \mathcal{P} = (T(TM\setminus 0), \mathcal{V}TM, \Phi_t^S).$$
(2) Suppose $S$ is the geodesic spray $S_F$ of a Finsler metric $F$. The canonical 1-form $\alpha$ pulls-back to the null form on each fiber of $\tau : T^* M \to M$, and so does $\omega$. In particular, $\gamma T^* M$ and, hence, $\gamma T M$ are Lagrangian distributions on $(T^* M \setminus 0, \omega)$ and $(T M \setminus 0, \omega_F)$, respectively. Furthermore, the flows $\Phi^S_t$ and $\Phi^F_t$ are symplectic. Therefore, (3.3) is of type (I) with respect to $\omega_F$, and

$$\mathcal{P} = (T(T^* M \setminus 0), \gamma T^* M, \Phi^S_t)$$

is of type (I) on $(T^* M \setminus 0, \omega)$.

(3) Still in the Finslerian setting, the tangent spaces to the fibers of $\Sigma_F M \to M$ and $\Sigma^* F M \to \Sigma M$ define, respectively, the vertical distributions $\gamma \Sigma_F M$ and $\gamma \Sigma^* F M$. As before, these are Legendrian distributions on $(\Sigma_F M, \alpha_F)$ and $(\Sigma^* F M, \alpha)$. We therefore obtain moving planes of type (II) on these contact manifolds,

$$\mathcal{P}^c = (\ker(\alpha_F), \gamma \Sigma_F M, \Phi^S_t), \quad \mathcal{P} = (\ker(\alpha), \gamma \Sigma^* F M, \Phi^F_t).$$

Given a frame $U_1, \ldots, U_n$ for $\Delta_n$ defined around a given point $x \in X$, a frame for the corresponding Jacobi curve $\ell_x(t)$ is obtained by setting

$$a_i(t) = (\Phi^* U_i(x), i = 1, \ldots, n, \text{ so that we conclude}$$

$$\ell_x(t) = (\Phi^* U_i)(x), \quad \text{for the corresponding Jacobi curve } \ell_x(t) \text{ is obtained by setting}$$

(3.3)$$a_i(t) = (\Phi^* U_i(x), i = 1, \ldots, n, \text{ so that we conclude}$$

Lemma 3.2. The Jacobi curve $\ell_x(t)$ is fanning if, and only if, along the flow line $t \mapsto \Phi_t(x)$,

(3.4)$$U_1, \ldots, U_n, [S, U_1], \ldots, [S, U_n]$$

constitute a frame for $\Delta_{2n}$. In particular, this condition on (3.4) does not depend on the choice of the local frame $U_1, \ldots, U_n$.

Definition 3.3. We shall call the moving plane $\mathcal{P}$ regular if (3.3) are a local frame for $\Delta_{2n}$ whenever $U_1, \ldots, U_n$ are a local frame for $\Delta_n$.

For a regular moving plane (3.1), we shall denote by $F_x(t), P_{\ell_x(t)}, K_x(t), h_x(t)$ and, in cases (I) and (II), $W_x(t)$ the invariants of the fanning curve $\ell_x(t)$, for $x \in X$. Evaluating at $t = 0$ and by varying $x$, one thus obtains, respectively, sections $\mathcal{F}, \mathcal{P}_{\Delta n}, \mathcal{K}$ of $\text{End}(\Delta_{2n}) \to X$, a distribution $\mathcal{H} \subset \Delta_{2n}$ and a section $\mathcal{W}$ of $\text{Bil}(\Delta_{2n}) \to X$.

Lemma 3.4. Along an orbit $t \mapsto \Phi_t(x), \mathcal{H}_{\Phi_t(x)}, \mathcal{F}_{\Phi_t(x)}, (\mathcal{P}_{\Delta_{2n}})_{\Phi_t(x)}, \mathcal{K}_{\Phi_t(x)}$ and $\mathcal{W}_{\Phi_t(x)}$ correspond to $h_x(t), F_x(t), P_{\ell_x(t)}, K_x(t)$ and $W_x(t)$ via the isomorphisms

$$d\Phi_t(x)|_{\Delta_{2n}} : \Delta_{2n}(x) \to \Delta_{2n}(\Phi_t(x)), \quad d\Phi_t(x)|_{\ell_x(t)} : \ell_x(t) \to \Delta_{n}(\Phi_t(x)).$$

Proof. Just note that $d\Phi_t(x)\ell_x(s) = \ell_{\Phi_t(x)}(s - t)$ and apply Proposition 2.16. □

Reduction by a contact type hypersurface. Let $X^{2n}, \omega, S, \Sigma^{2n-1}, C, \alpha$, be as in Remark 2.9. Let, furthermore, $\Delta_n$ be a Lagrangian distribution on $X$ such that $\Delta_n \subset \ker(\alpha)$ and $C \subset \Delta_n$, so that $L := \Delta_n \cap T \Sigma$ is a Legendrian distribution on $\Sigma$. Then,
Proposition 3.5. Given \( x \in \Sigma \), let \( \ell_x(t) \in \Lambda(T_x X) \) and \( \ell^c_x(t) \in \Lambda(\ker(\alpha_x)) \) be the Jacobi curves of the moving planes \( \mathcal{P} = (TX, \Delta_n, \Phi^S_X) \) and \( \mathcal{P}^c = (\ker(\alpha), L, \Phi^S_X|_\Sigma) \), on \( X \) and \( \Sigma \) respectively, based at \( x \), and let \( W_x(t) \) and \( W^c_x(t) \) be their Wronskians. Then,

1. We have a \( W_x(t) \)-orthogonal decomposition

\[
\ell_x(t) = \ell^c_x(t) \oplus \text{span}[C_x - tS_x],
\]

and the restriction of \( W_x(t) \) to \( \ell^c_x(t) \) is equal to \( W^c_x(t) \).

2. \( \mathcal{P} \) is regular in a neighborhood of \( \Sigma \) if, and only if, \( \mathcal{P}^c \) is regular. This being the case, the horizontal curves \( h_x(t), h^c_x(t) \), and the Jacobi endomorphisms \( K_x(t), K^c_x(t) \), of \( \ell_x(t) \) and \( \ell^c_x(t) \), satisfy

\[
h^c_x(t) = h_x(t) \cap \ker(\alpha_x), \quad K_x(t)|_{L_x} = K^c_x(t)|_{L_x}, \quad K_x(t)(C_x - tS_x) = 0.
\]

Proof. By hypothesis, we can choose a local frame for \( \Delta_n \) around \( x \), \( U_1, \ldots, U_n \), such that \( U_n = C \) and that, along \( \Sigma, U_1, \ldots, U_{n-1} \) is a frame for \( L \). Let \( \mathcal{A}^c(t) \) and \( \Lambda(t) = (\mathcal{A}^c(t), a_n(t)) \) be the corresponding frames for \( \ell^c_x(t) \) and \( \ell_x(t) \), respectively. It follows from \( [C, S] = S \) and \( \Phi^* S = S \) that

\[
\dot{a}_n(t) = (d/dt)(\Phi_t^* C)_x = (\Phi_t^* [S, C])_x = -S_x.
\]

Since \( a_n(0) = C_x \), we obtain \( a_n(t) = C_x - tS_x \) and \ref{3.5} follows. Observe that we have a direct sum decomposition \( T_x \Sigma = \ker(\alpha_x) \oplus \text{span}[S_x] \oplus \text{span}[C_x] \). Since \( \text{span}(\mathcal{A}^c(t), \mathcal{A}^c(t)) \subseteq \ker(\alpha_x) \), and \( a_n(t) = C_x - tS_x \), it thus follows that \( \ell_x(t) \) is fanning if, and only if, \( \ell^c_x(t) \) is fanning. Being the case, let \( P(t), Q(t) \), and \( P^c(t), Q^c(t) \) be given by \ref{2.13} with respect to \( \mathcal{A}(t) \) and \( \mathcal{A}^c(t) \), respectively. Since \( \dot{a} = 0 \), it follows that \( P = \text{diag}(P^c, 0) \) and \( Q = \text{diag}(Q^c, 0) \). Recalling \ref{2.15}, we conclude that \( \{\mathcal{A}(t), t\} = \text{diag}([\mathcal{A}^c(t), t], 0) \). The assertion about the Jacobi endomorphisms follows now from Proposition \ref{2.13}. The ones about the Wronskians and the horizontal curves are analogues. \( \square \)

3.1. Expressions in terms of Lie brackets. The objects \( \mathcal{F}, \mathcal{H}, \mathcal{K} \) and \( \mathcal{W} \) can be described in terms of taking Lie brackets with the vector field \( S \). Firstly, if \( \mathcal{T} \) is a section of \( \text{End}(\Delta_{2n}) \to X \), the Lie derivative \( [S, \mathcal{T}] \) is defined and it holds that

\[
\frac{d}{dt}(\Phi_t)^* \mathcal{T} = (\Phi_t)^*[S, \mathcal{T}].
\]

It follows from this, \ref{3.3}, and \ref{2.2.1} that

1. The endomorphism \( \mathcal{F} \) is characterized by

\[
\mathcal{F}(U_i) = 0, \quad \mathcal{F}([S, U_i]) = U_i,
\]

\( i = 1, \ldots, n \), for any local frame \( U_1, \ldots, U_n \) for \( \Delta_n \).

2. The Lie derivative \( [S, \mathcal{F}] \) is a section of reflections across \( \mathcal{H} \).

3. \( \mathcal{K} \) is the square of \((1/2)[S, [S, \mathcal{F}]] = -[S, \mathcal{P}_{\Delta_n}] \). Furthermore, let \( H \) be the section of \( \text{Iso}(\Delta_n, \mathcal{H}) \to X \) corresponding to \ref{2.12}, so that

\[
H(U) = \mathcal{P}_H([S, U]),
\]

for \( U \) a vector field tangent to \( \Delta_n \). Then,

\[
\mathcal{K}|_{\Delta_n} = [S, \mathcal{P}_{\Delta_n}]|_{\mathcal{H}} \circ H.
\]
Applying (3.7) to a vector field $U$ tangent to $\Delta_n$ and using that $P_{\Delta_n}$ vanishes on $\mathcal{H}$, one obtains

\begin{equation}
K(U) = -P_{\Delta_n}(\{S, H(U)\}).
\end{equation}

(3.8)

(4) In cases (I) and (II), given vector fields $U, V$ tangent to $\Delta_n$, then

\begin{equation}
W(U, V) = \omega(U, [S, V]).
\end{equation}

(3.9)

3.2. The Jacobi curves associated to sprays and Finsler metrics. Let us now come back to the moving planes from Example 3.1. Throughout this section, let $S$ be fixed a spray on $M^n$.

**Lemma 3.6.** The moving plane (3.2) is regular.

**Proof.** Let $X_1, \cdots, X_n$ be a local frame for $VTM$. Since the almost-tangent structure $\mathcal{J}$ satisfies (cf. Lemma 2.3)

\begin{equation}
\mathcal{J}(X_i) = 0, \quad \mathcal{J}([S, X_i]) = -X_i,
\end{equation}

a linear dependence relation among $X_1, \cdots, X_n, [S, X_1], \cdots, [S, X_n]$ would give a linear dependence relation among $X_1, \cdots, X_n$. □

Let, therefore, $F, H, K$ be the corresponding differential invariants of $\mathcal{P}$. From (3.10) and (3.6) we obtain

\begin{equation}
F = -J.
\end{equation}

(3.11)

In particular, since $[S, F]$ consists of reflections across $\mathcal{H}$, we recover the following result [17, Prop. I.41].

**Corollary 3.7.** The Lie derivative $\Gamma_S := -[S, \mathcal{J}]$ is a section of reflections of $\text{End}(T(TM\setminus 0)) \to TM\setminus 0$ such that $\ker(\Gamma_S + I) = VTM$. \[\]

The section $\Gamma_S$ is an example of a connection on $M$ in the sense of Grifone (cf. [17, Def. I.14]): indeed, $\Gamma_S$ is the canonical connection associated to the spray $S$. The corresponding Ehresmann connection on $TM\setminus 0$, given by the 1-eigenspaces of $\Gamma_S$, is the so-called horizontal tangent bundle (associated to $S$), $\mathcal{H}TM = \ker([S, \mathcal{J}] - I)$, so that

\begin{equation}
T(TM\setminus 0) = \mathcal{H}TM \oplus VTM.
\end{equation}

(3.12)

Therefore, we have recovered $\mathcal{H}TM$ as the horizontal distribution $\mathcal{H}$ of $\mathcal{P}$,

\begin{equation}
\mathcal{H}TM = \mathcal{H},
\end{equation}

(3.13)

and we can unambiguously denote by $\mathcal{P}_\mathcal{H}$ and $\mathcal{P}_V$ the projections relative to (3.12).

Note that the homogeneity $[C, S] = S$ of $S$ implies that $S$ is tangent to $\mathcal{H}TM$. \[\]

In terms of Jacobi curves: fixing a non-zero vector $v \in T_mM$, let $\gamma : I \subseteq \mathbb{R} \to M$ be the geodesic of $S$ with $\dot{\gamma}(0) = v$, and let

$\ell_v : I \subseteq \mathbb{R} \to \text{Gr}_n(T_vTM)$

be the Jacobi curve of $\mathcal{P}$ based at $v$. We have shown that

**Proposition 3.8.** Under the isomorphism $d\Phi^S_t : T_vTM \to T_{\gamma(t)}M$, the endomorphism $-\mathcal{J}_{\gamma(t)}$ corresponds to $\mathcal{F}_v(t)$ and, thus, $(\Gamma_S)_{\gamma(t)}$ corresponds to $\mathcal{F}_v(t)$. Therefore, $\mathcal{H}_{\gamma(t)}TM = d\Phi^S_t(v)h_v(t)$. \[\]
Next we show how the notions of covariant derivative and curvature endomorphism along \( \gamma \), associated to \( S \), can be recovered in this setting. We refer the reader to [33] for the definitions of those concepts as well as for the proofs of the following results.

Consider, for each \( t \), the isomorphism
\[
\ell_{v,t} := \ell_{\gamma(t)}^{-1} \circ d\Phi^t_\gamma(v) : \ell_v(t) \to T_{\gamma(t)}M.
\]
For \( t = 0 \) this is just the tautological isomorphism \( \ell_v : \ell_v(0) = \mathcal{V}_vTM \to T_vM \).

**Proposition 3.9.** The endomorphisms \( K_v(t)\ell_v(t) : \ell_v(t) \to \ell_v(t) \) and \( R_{\gamma(t)} : T_{\gamma(t)}M \to T_{\gamma(t)}M \) correspond under (3.14).

It therefore follows from Proposition [2.13] that, given a frame \( V_1, \ldots, V_n \in \mathfrak{X}(\gamma) \), if \( \mathcal{A}(t) \) is the corresponding frame for \( \ell_v(t) \), then the matrix of \( R_{\gamma(t)} \) with respect to that frame is \( (1/2)\{\mathcal{A}(t), t\} \).

**Proposition 3.10.** Given \( V \in \mathfrak{X}(\gamma) \), let \( a(t) \in \ell_v(t) \) correspond to \( V \) via (3.14). Then \( D^vV/\partial t \) corresponds to \( P_{\ell_v(t)}a(t) \) via (3.14).

**3.2.1. The case of a Finsler metric.** Let us now suppose that \( S \) is the geodesic spray of a Finsler metric \( F \) on \( M \).

In this case, \( \ell_v(t) \) takes values in \( \Lambda(T_vTM) \) if we regard \( \mathcal{P} \) as of type (I) with respect to \( \omega_F \).

**Proposition 3.11.** The Wronskian \( W_v(t) \) of \( \ell_v(t) \) corresponds, under (3.14), to the fundamental tensor \( g_F(\dot{\gamma}(t)) \) of \( F \) at \( \dot{\gamma}(t) \).

**Proof.** This is equivalent to show that, given vector fields \( U, V \) on \( M \), then \( W(V^v, U^v)(w) = g_F(w)(V, U) \), for \( W \) the section of \( \text{Bil}_\text{sym}(VTM) \to TM \setminus 0 \) associated to \( \mathcal{P} \). On one hand, from (3.30)
\[
W(V^v, U^v) = -\alpha_F\left( V^v, [S, U^v] \right) = [S, U^v]\left( \alpha_F(V^v) \right) - V^v\left( \alpha_F([S, U^v]) \right) - \alpha_F(\left( [S, U^v], V^v \right)).
\]
On the other hand, Lemma [2.23] implies that \( [S, U^v] \) is \( \pi \)-related to \( -U \) (i.e., \( d\pi[S, U^v] = -U \)). From this it follows that \( [S, U^v], V^v \) is vertical and that \( \alpha_F([S, U^v]) \) is the function \( u \mapsto -F_F(w)U \). Since \( \alpha_F \) vanishes on vertical vectors and \( d_FF_F(w) = g_F(w)(V, \cdot) \) we therefore obtain \( W(U^v, V^v)(w) = -V^v(\alpha_F([S, U^v]))(w) = g_F(w)(V, U) \).

As a corollary of this and Proposition 3.9 we get the flag curvature in terms of the Jacobi curve:

**Theorem 3.12.** Given a 2-plane \( \Pi = \text{span}[v, u] \) in \( T_{\gamma(t)}M \), with \( g_F(v, u) = 0 \), let \( a \in \ell(0) = \mathcal{V}_vTM \) be \( a = \ell_v(u) \). Then,
\[
K_F(v, \Pi) = \frac{1}{F(v)^2} \frac{W_v(0)(K_v(0)a, a)}{W_v(0)(a, a)}
\]
The co-tangent setting. Let \( \xi = L_F(v) \) and let \( L_{\xi}(t) \in \Lambda(T_{\xi(t)}T^*M) \) be the Jacobi curve of \( \mathcal{P}^* \) based at \( \xi \). With the help of the Legendre transformation \( L_F \), one obtains an isomorphism
\[
\ell_{\xi,t} := \left( d_FL_F(\dot{\gamma}(t)) \right)^{-1} \circ L_{\xi(t)}^{-1} \circ d\Phi^t_\xi(\xi) : \ell_\xi(t) \to T_{\gamma(t)}M.
\]
Note from (2.10) that, for \( t = 0 \), (3.13) is the inverse of 
\[
T_m M \rightarrow \ell_\xi(0) = \nu_\xi T^*M, \quad w \mapsto i_\xi(g_F(v)(w, \cdot)).
\]

Now, since \( \mathcal{L}_F \) is a symplectic diffeomorphism that maps the data in \( \mathcal{P} \) to the ones in \( \mathcal{P}^*_\xi \), then
\[
(3.16) \quad d\mathcal{L}_F(v) : T_v TM \rightarrow T_\xi T^*M
\]
is a symplectic isomorphism mapping \( \ell_v(t) \) to \( \ell_\xi(t) \). In particular, it follows from Proposition 2.10 that \( F_v(t), W_v(t), h_v(t), K_v(t) \), correspond to \( F_\xi(t), W_\xi(t), h_\xi(t), K_\xi(t) \), under (3.16). Therefore, \( W_\xi(t) \) and \( K_\xi(t) \) correspond to \( g_F(\tilde{y}(t)) \) and \( R_\gamma(t) \), respectively, under (3.16).

The contact setting. Suppose \( F(v) = 1 \), hence \( F^*(\xi) = 1 \), and let
\[
\ell_v(t) \in \Lambda(\ker(\alpha_F)_v), \quad \ell_\xi(t) \in \Lambda(\ker(\alpha_\xi))
\]
be the Jacobi curves of \( \mathcal{P}^c \) and \( \mathcal{P}^*_\xi \) based at \( v \) and \( \xi \), respectively. Observe that \( \mathcal{P} \) and \( \mathcal{P}^c \), as well as \( \mathcal{P}^*_\xi \) and \( \mathcal{P}^*_{\xi^c} \), fit within the setting in Proposition 3.3. Therefore, since (2.2) maps \( \ker g_F(w)(w, \cdot) \) onto \( V_w \Sigma_F M \) (for \( w \in \Sigma_F M \)), then (3.14) and (3.15) restrict to isomorphisms
\[
(3.17) \quad \nu_v, t|_{\ell_v(t)} : \ell_v(t) \rightarrow \ker g_F(\dot{\gamma}(t))(\dot{\gamma}(t), \cdot)
\]
\[
\nu_\xi, t|_{\ell_\xi(t)} : \ell_\xi(t) \rightarrow \ker g_F(\dot{\gamma}(t))(\dot{\gamma}(t), \cdot)
\]
under which \( W_v^c(t), W_\xi^c(t), \) and \( K_v^c(t), K_\xi^c(t) \), respectively, correspond to the restrictions of \( g_F(\dot{\gamma}(t)) \) and \( R_\gamma(t) \) to \( \ker g_F(\dot{\gamma}(t))(\dot{\gamma}(t), \cdot) \). Also, the horizontal curves \( h_v^c(t) \), for \( v \in \Sigma_F M \), give rise to the standard horizontal distribution on \( \Sigma_F M \),
\[
(3.18) \quad \mathcal{H}\Sigma_F M = \mathcal{HT}M \cap \ker(\alpha_F).
\]

3.3. Invariants from the connections point of view. The linear connections arising in the theory of sprays and Finsler metrics are naturally defined on the vertical tangent bundle (2.1). As shown in [29], the classical connections of Berwald, Cartan, Chern and Rund, and Hashiguchi are examples of linear connections \( \nabla \) on (2.1) satisfying the following two conditions (recall from Corollary 3.7 the definition of \( \Gamma_S \))

\[ \mathbf{L.} \ \nabla \ \text{is lift of the connection } \Gamma_S, \ \text{i.e. given } X \in T(TM\setminus 0), \ \text{then} \]
\[ \nabla_X C = \mathcal{P}_V(X). \]

\[ \mathbf{T.} \ \ T(S, X) = 0 \ \text{for all } X \in T(TM\setminus 0); \ \text{here, the torsion } T \ \text{of } \nabla \ \text{is the } VTM\text{-valued tensor field on } TM\setminus 0 \ \text{defined (in terms of vector fields) by} \]
\[ (3.20) \quad T(X, Y) = \nabla_X J(Y) - \nabla_Y J(X) - J([X, Y]). \]

On the other hand, the above conditions on a linear connection \( \nabla \) guarantee that the covariant derivatives and the curvature endomorphism on \( M \) induced by \( \nabla \), as defined next, are intrinsic to the spray \( S \).

3.3.1. The covariant derivative, the curvature endomorphism and the flag curvature. Throughout this section, let \( \nabla \) be fixed a connection on (2.1) satisfying \( \mathbf{L} \) and \( \mathbf{T} \). For a smooth curve \( \gamma : I \subseteq \mathbb{R} \rightarrow M \), we let \( \mathbf{X}(\gamma) \) denote the space of vector fields along \( \gamma \).
Definition 3.13. Given a smooth curve \( \gamma : I \subseteq \mathbb{R} \to M \), with \( \gamma(t_0) = m \), and non-null vector \( w \in T_mM \), the map \( D^w/\gamma : \mathfrak{X}(\gamma) \to T_mM \) is defined by
\[
\frac{D^w V}{dt} = i_w^{-1}\left( \nabla^w \right)(t_0),
\]
where \( V^w \) is the vertical lift of \( V \) along the horizontal lift \( \gamma \). The covariant derivative of \( \gamma \) through \( w \) at \( t = t_0 \) (i.e. \( \gamma \) is the lift of \( \gamma \) that is tangent to \( \mathcal{H}M \) and \( \gamma(t_0) = w \)).

By considering a nowhere null vector field \( W \in \mathfrak{X}(\gamma) \), one thus obtains a map \( D^W/\gamma : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma) \) that satisfies the properties of a covariant derivative.

Proposition-Definition 3.14. If \( \gamma \) is a regular curve, then the map
\[
D^\gamma/\gamma : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)
\]
does not depend on the choice of \( \nabla \), but only on \( S \). This is the covariant derivative along \( \gamma \) associated to \( S \).

By using vertical and horizontal lift operations one can bring the curvature tensor of \( \nabla \),
\[
\mathcal{R}(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,
\]
down to \( M \) so as to define, for given \( m \in M \) and non-null vector \( w \in T_mM \), a tri-linear map \( R_w : T_mM \times T_mM \times T_mM \to T_mM \) by
\[
R_w(u,v)z = i_w^{-1} \mathcal{R}(u^w, v^w)z^w,
\]
where the vertical and horizontal lifts are at \( w \). The following is a consequence of Proposition 3.15 which we shall prove in \( \S \).

Proposition-Definition 3.15. The endomorphism \( \mathcal{R}_w : T_mM \to T_mM \) defined by \( \mathcal{R}_w(v) = R_w(w,v)w \) does not depend on the choice of \( \nabla \), but only on \( S \). This is the curvature endomorphism of \( S \) in the direction \( w \).

Let us now suppose that \( S \) is the geodesic spray of a Finsler metric \( F \) on \( M \).

Proposition-Definition 3.16. The curvature endomorphism \( \mathcal{R}_w \) is symmetric with respect to \( g_F(w) \). As a consequence, given a 2-dimensional subspace \( \Pi \subset T_mM \) containing \( w \), say \( \Pi = \text{span}[w,u] \), then the following quantity
\[
K_F(w,\Pi) = \frac{g_F(w)(\mathcal{R}_w(u),u)}{g_F(w)(w,w)g_F(w)(u,u) - g_F(w)(w,u)^2}
\]
does not depend on \( u \) but only on the flag \( (w,\Pi) \). This is the so-called flag curvature of the flag \( (w,\Pi) \).

Proof. By Proposition 3.3 to be proved below, and Proposition 3.11 the statement about \( \mathcal{R}_w \) is nothing but a manifestation of the symmetry of the Jacobi endomorphism stated in (3) of Proposition 2.14.

3.3.2. Proofs of Propositions 3.14 and 3.16.

Let \( V \) be a vector field on \( M \). Since \( V^h \) is \( \pi \)-related to \( V \), then
\[
\mathcal{J}(V^h) = V^v.
\]

Let, as in \( \S \), \( H : \mathcal{V}TM \to \mathcal{H}TM \) be the bundle isomorphism corresponding to the horizontal derivative. From (2.13) and (3.11) we have \( H^{-1} = -\mathcal{J}_{\mathcal{H}TM} \). It follows from this and (3.22) that
\[
H(V^v) = -V^h.
\]
Substituting this in (3.8) gives us
\begin{equation}
\mathcal{K}(V^v) = \mathcal{P}_V([S, V^h]).
\end{equation}
Let us now compute \( R_{\gamma(t)}(V) \). We have \( \dot{\gamma}(t)^v = C_{\gamma(t)} \) and \( \dot{\gamma}(t)^h = S_{\gamma(t)} \), since \( d\pi(\dot{\gamma}(t))S = \dot{\gamma}(t) \) and \( S \) is horizontal. Thus
\[ R_{\gamma(t)}(V) = i_{\dot{\gamma}(t)}^{-1} \mathcal{R}(S, V^h) C. \]
On the other hand, it follows from L. that \( \nabla_S C = \mathcal{P}_V(S) = 0 \), \( \nabla_{V^h} C = \mathcal{P}_V(V^h) = 0 \) and \( \nabla_{[S, V^h]} C = \mathcal{P}_V([S, V^h]) \). Therefore,
\[ R_{\gamma(t)}(V) = i_{\dot{\gamma}(t)}^{-1} \mathcal{P}_V([S, V^h]) = i_{\dot{\gamma}(t)}^{-1} \mathcal{K}(V^v). \]
This proves Proposition 3.9.

As for Proposition 3.10, note that since \( \gamma \) is a geodesic of \( S \) and \( S \) is horizontal, then \( \dot{\gamma} : I \subseteq \mathbb{R} \to TM \setminus 0 \) is a horizontal lift of \( \gamma \) and, thus,
\[ \frac{D\dot{V}}{dt} = i_{\dot{\gamma}}^{-1} \nabla_S V^v. \]
On the other hand, by substituting \( J(S) = C \), \( J(V^h) = V^v \), and \( \nabla_{V^h} C = 0 \) in the equality \( T(S, V^h) = 0 \), we obtain \( \nabla_S V^v = J([S, V^h]) \). Therefore, since \( d\Phi_{-\ell}(\Phi_{\ell}(v))^V^h = -H_c(t)a(t) \) (this follows from (3.24)), we have
\[
\begin{align*}
d\Phi_{-\ell}(\Phi_{\ell}(v))\nabla_S V^v &= -F_c(t) \frac{d}{dt}(-H_c(t)a(t)) \\
&= \frac{d}{dt}(F_c(t)H_c(t)a(t)) - \dot{F}_c(t)H_c(t)a(t) \\
&= \dot{a}(t) - H_c(t)a(t) \\
&= P_{\ell}c(t)a(t),
\end{align*}
\]
where we have used (2.13). The result follows.

4. An O’Neill formula for the flag curvatures in an isometric submersion via symplectic reduction of fanning curves

In this section we shall see how a theory of symplectic reductions of fanning curves, as developed in [30], leads to an O’Neill type formula for flag curvatures in a Finsler submersion. As remarked in the introduction, a similar theory of symplectic reductions has been developed in [3] and applied to some problems from mechanics.

4.1. Symplectic reduction of fanning curves. We begin by summarizing the results from [30] we shall need, and refer the reader to that work for more details.

4.1.1. Linear symplectic reduction. A subspace \( \mathcal{W} \subseteq V \) is said to be coisotropic if \( \mathcal{W}^\omega \subseteq \mathcal{W} \). For such a subspace \( \mathcal{W} \), the (restriction of) the symplectic form \( \omega \) descends to a symplectic form \( \omega_R \) on \( \mathcal{W}/\mathcal{W}^\omega \) and the symplectic space \( (\mathcal{W}/\mathcal{W}^\omega, \omega_R) \) is the so-called linear symplectic reduction of \( V \) by \( \mathcal{W} \). Furthermore, if \( \ell \subseteq V \) is a Lagrangian subspace, then \( \pi(\ell \cap \mathcal{W}) \) is a Lagrangian subspace of \( \mathcal{W}/\mathcal{W}^\omega \), where \( \pi : \mathcal{W} \to \mathcal{W}/\mathcal{W}^\omega \) is the quotient map. We shall use the notation \( \ell_R = \pi(\ell \cap \mathcal{W}) \). Therefore, fixed a coisotropic subspace \( \mathcal{W} \), one has a symplectic reduction map
\begin{equation}
\lambda : \Lambda(V) \to \Lambda(\mathcal{W}/\mathcal{W}^\omega) , \lambda(\ell) = \ell_R.
\end{equation}
Consider the following open and dense subset $\mathcal{U} \subset \Lambda(V)$, $\mathcal{U} = \{\ell : \ell \cap \mathbb{W}^\omega = \{0\}\}$.
For $\ell \in \mathcal{U}$, one has an isomorphism
\begin{equation}
\pi|_{\ell \cap \mathbb{W}} : \ell \cap \mathbb{W} \to \ell_R.
\end{equation}

**Lemma 4.1.** The map (4.1) is smooth on $\mathcal{U}$. Furthermore, given $\ell \in \mathcal{U}$, upon identifying $\ell_R$ with $\ell \cap \mathbb{W}$ via (4.2), the derivative $d\lambda(\ell) : \text{Bil}_\text{sym}(\ell) \to \text{Bil}_\text{sym}(\ell \cap \mathbb{W})$ is the restriction map.

4.1.2. The symplectic reduction of a fanning curve. Let $\mathbb{W} \subset V$ be a fixed coisotropic subspace and $\ell : I \subseteq \mathbb{R} \to \Lambda(V)$ a fanning curve such that for all $t$,

i. $\ell(t) \cap \mathbb{W}^\omega = \{0\}$,

ii. the Wronskian $W(t)$ is non-degenerate on $\ell(t) \cap \mathbb{W}$.

In this setting, it follows from Lemma 4.1 that the symplectic reduction of $\ell(t)$ by $\mathbb{W}$ is a smooth fanning curve \[ \ell_R := \lambda \circ \ell : I \to \Lambda(\mathbb{W}/\mathbb{W}^\omega). \]

**Definition 4.2.** For each $t$, we let $h(t)$ be $\ell(t) \cap \mathbb{W}$, and let $\mathfrak{v}(t) \subset \ell(t)$ be its $W(t)$-orthogonal subspace. Since $W(t)$ is non-degenerate on $h(t)$, then
\begin{equation}
\ell(t) = h(t) \oplus \mathfrak{v}(t).
\end{equation}

With respect to the decomposition $V = h(t) \oplus \mathfrak{v}(t) \oplus \mathfrak{h}(t)$, the projectors onto $h(t)$ and $\mathfrak{v}(t)$ are denoted by $P_h(t)$ and $P_v(t)$, respectively.

It follows from Lemma 4.1 that for each $t$ the quotient map $\pi$ restricts to an isomorphism
\begin{equation}
\pi|_{h(t)} : h(t) \to \ell_R(t)
\end{equation}
that pulls back the Wronskian $W_R(t)$ of $\ell_R(t)$ to the restriction of $W(t)$ to $h(t)$.

4.1.3. The O'Neil endomorphism. The set of fanning curves on $\Lambda(V)$ satisfying i. and ii. above is acted upon by the group $\text{SP}_\mathbb{W}(V) = \{T \in \text{SP}(V) : T(\mathbb{W}) = \mathbb{W}\}$ and so is the space $J^1_{f,\mathbb{W}}(\mathbb{R}; \Lambda(V))$ of 1-jets of such curves. A natural equivariant map
\[ J^1_{f,\mathbb{W}}(\mathbb{R}; \Lambda(V)) \to \text{sp}_\mathbb{W}(V) \]
is obtained by considering, for a given fanning curve $\ell(t) \in \Lambda(V)$ satisfying i. and ii., the endomorphisms
\[ F_h(t) := P_h(t) \circ F(t). \]
As for the first derivative $\dot{F}_h(t)$, one has

**Lemma 4.3.** Let $A(t)$ be a frame for $\ell(t)$. With respect to the basis $(A(t), H(t))$, the matrix of $F_h(t)$ has the block form
\[ \left( \begin{array}{cc} -C_1(t) & C_2(t) \\ 0 & C_1(t) \end{array} \right), \]
where $C_1(t)$ is the matrix of $P_h(t)|_{\ell(t)}$ in the basis $A(t)$. As for the block $C_2(t)$,

1. Denoting still by $W(t)$ the matrix of the Wronskian of $\ell(t)$ in the basis $A(t)$, then $C_2(t)W(t)^{-1}$ is symmetric.
2. If $A(t) = (A_h(t), A_v(t))$, where $A_h(t)$ and $A_v(t)$ are frames for $h(t)$ and $\mathfrak{v}(t)$, respectively, then
\[ A(t)C_2(t) = (P_v(t)\dot{A}_h(t), -P_h(t)\dot{A}_v(t)). \]
Definition 4.4. The O’Neill endomorphism, at time $\tau$, of the pair $(\ell(t), \mathcal{W})$ is the $W(\tau)$-symmetric endomorphism

$$A(\tau) : \ell(\tau) \rightarrow \ell(\tau)$$

whose matrix with respect to a frame $A(\tau)$ for $\ell(\tau)$ is the matrix $C_2(\tau)$ from Lemma 4.3. Therefore, given frames $A_h(t)$ and $A_v(t)$ for $h(t)$ and $v(t)$, respectively, then

$$(4.5) \quad A(t)A_h(t) = P_v(t)A_h(t)$$
$$A(t)A_v(t) = -P_h(t)A_v(t).$$

The importance of $A(t)$ is described in the way it relates the Jacobi endomorphism $\mathbf{K}(t)$ of $\ell_R(t)$ with the “$h$-component” of the Jacobi endomorphism $\mathbf{K}(t)$ of $\ell(t)$:

Theorem 4.5. Given $a \in h(t)$, let $\mathcal{P}$ denote its image under the isomorphism $\mathbf{4.4}$. Then,

$$W_R(t)(\mathbf{K}_R(t)\mathcal{P}, \mathcal{P}) = W(t)(\mathbf{K}(t)a, a) + 3W(t)(A(t)a, A(t)a).$$

4.2. Isometric submersions of Finsler manifolds. In this section we shall briefly collect some definitions and results from [6].

Definition 4.6. Given Finsler manifolds $(M, F_1)$ and $(N, F_2)$, a submersion

$$(4.7) \quad f : M \rightarrow N$$

is said to be isometric if, for every $m \in M$, the derivative $df(m) : T_mM \rightarrow T_{f(m)}N$ maps the closed unit ball of $(F_1)_m$ onto the closed unit ball of $(F_2)_m$.

Remark 4.7. This concept can be alternatively stated as follows: for all $m \in M$, the derivative $df(m) : T_mM \rightarrow T_{f(m)}N$ induces an isometry between $T_{f(m)}N$ and the quotient $T_mM/\ker df(m)$, endowed with the quotient norm

$$|v|_{\text{quotient}} = \min_{w \in \ker df(m)} F_1(v + w).$$

For an isometric submersion one defines the horizontal cone at a given $m$ as the set

$$\mathcal{H}_m = \{ v \in T_mM \setminus 0 : F_1(v) = F_2(df(m)v) \},$$

that is, the elements of the horizontal cone are the non-zero vectors realizing the quotient norm above.

Denoting by $\mathcal{V}_m$ the kernel of $df(m)$, one has, for each $v \in \mathcal{H}_m$, a $g_{F_1}(v)$-orthogonal decomposition

$$T_mM = T_v\mathcal{H}_m \oplus \mathcal{V}_m$$

and the derivative $df(m)$ restricts to an isometry

$$(4.8) \quad df(m) : (T_v\mathcal{H}_m, g_{F_1}(v)) \rightarrow (T_{f(m)}N, g_{F_2}(u))$$

for $u = df(m)v$.

An immersed curve $\gamma : I \subseteq \mathbb{R} \rightarrow M$ is said to be horizontal if $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$ for every $t \in I$. If $\gamma$ is a geodesic, this condition holds once it holds for some $t_0 \in I$. 
4.3. The point of view of symplectic reductions. A submanifold $P$ of a symplectic manifold $(Q, \omega)$ is co-isotropic if, for every $p \in P$, $T_p P$ is a co-isotropic subspace of $T_p Q$. In this case, the distribution $p \mapsto T_p P^\omega$ on $P$ is integrable. When the space of leaves $P_R$ of the corresponding foliation has a smooth structure, the pull-back of $\omega$ to $P$ descends to a symplectic structure on $P_R$; we refer to [1] for more details. This procedure has been applied in [6] to obtain a symplectic description of an isometric submersion that, by passing from co-tangent to tangent bundles via the Legendre transformations, goes as follows:

**Definition 4.8.** The co-normal bundle of the isometric submersion (4.7) is the submanifold of $TM \setminus 0$ given by the union of all horizontal cones, and shall be denoted by $N$. The derivative of $f$ restricts to a map $\nu = f_*|_N : N \to TN \setminus 0$.

**Proposition 4.9.** The co-normal bundle $N$ is a co-isotropic submanifold of $(TM \setminus 0, \omega_{F_1})$ with smooth space of leaves $N_R$. The map $\nu$ above is constant on the leaves and the induced map $\nu : N_R \to (TN \setminus 0, \omega_{F_2})$ is a symplectic diffeomorphism. Furthermore, the geodesic flow $\Phi^t_{F_1}$ of $F_1$ leaves $N$ invariant and its restriction to $N$ descends to a flow in $N_R$ which corresponds, under $\nu$, to the geodesic flow of $F_2$, $\Phi^t_{F_2}$.

In particular, it follows from the proposition above that given $v \in N$, and letting $u = f_* v$, the map

$$\lambda_v : \Lambda(T_v TM) \to \Lambda(T_u TN), \lambda(\ell) = d\nu(v)(\ell \cap T_v N)$$

is well-defined and is the symplectic reduction map (4.1) with respect to the co-isotropic subspace $T_v N \subset T_v TM$. Observe that

$$\lambda_v(V_v TM) = V_u TN;$$

indeed, this follows from the following lemma whose straightforward proof will be omitted.

**Lemma 4.10.** Let $m \in M$, $v \in H_m$, and $u = df(m)v$. Then,

1. $V_v TM \cap T_v N = T_v H_m$.
2. The map $d\nu(v)|_{T_v H_m}$ is equal to $i_u \circ df(m)|_{T_v H_m} : T_v H_m \to V_u TN$.

4.4. The Jacobi curves. We now compare the Jacobi curves of the total space and the base space of an isometric submersion, based on [30]. This will furnish the desired O’Neill formula.

Let $\gamma : I \subseteq \mathbb{R} \to M$ be fixed a unit-speed horizontal geodesic, with $\dot{\gamma}(0) = v$, and consider, as in [3,2] the Jacobi curves associated to $F_1$ and $F_2$, based at $v$ and $u = f_* v$, respectively,

$$\ell_v(t) \in \Lambda(T_v TM), \ell_u(t) \in \Lambda(T_u TN).$$

**Proposition 4.11.** We have that $\ell_u = \lambda_v \circ \ell_v$.

**Proof.** This follows from the statement about the flows in Proposition 4.9 and the fact that $\lambda_w(V_w TM) = V_{f_* w} TN$ for all $w \in N$. □

**Lemma 4.12.** For all $t$, $\ell_v(t) \cap T_v N^\omega = \{0\}$. 
Lemma 4.13. To the decomposition (4.10) and the O'Neill endomorphism $\ell$.

Proof. Since $T_N$ is invariant by the derivative of $\Phi^F_1$, the same is true of $T_N^\omega$. Therefore, $d\Phi^F_1(v)(\ell_v(t) \cap T_vN^\omega) = \mathcal{V}_v(t)TM \cap T_v\mathcal{N}^\omega$. Let us show that $\mathcal{V}_wTM \cap T_w\mathcal{N}^\omega = \{0\}$ for all $w \in \mathcal{N}$. On one hand, from the first part of Proposition 4.10 we have $T_w\mathcal{N}^\omega = \ker dv(w)$. On the other hand, $\mathcal{V}_wTM \cap T_w\mathcal{N}^\omega \subset \mathcal{V}_wTM \cap T_w\mathcal{N} = T_w\mathcal{H}_m$ and $\ker dv(w)|_{T_w\mathcal{H}_m} = \{0\}$ since, by Lemma 4.10, $dv(w)|_{T_w\mathcal{H}_m} = i_{f_w}w \circ df(m)|_{T_w\mathcal{H}_m}$. The result follows.

The above lemma says that the pair $(\ell_v(t), T_v\mathcal{N})$ fulfills the conditions in (4.1.2) (condition ii.) automatically holds since the Wronskian $W_v(t)$ is positive-definite). Therefore, $\ell_v(t)$ decomposes as

(4.9) $\ell_v(t) = h_v(t) \oplus v_v(t)$

and the O'Neill endomorphism $A_v(t)$ is defined.

Lemma 4.13. Under the isomorphism (3.14), the decomposition (4.9) corresponds to the decomposition

(4.10) $T_{\gamma(t)}M = T_{\gamma(t)}\mathcal{H}_{\gamma(t)} \oplus \mathcal{V}_{\gamma(t)}$.

Proof. From $d\Phi^F_1(v)T_v\mathcal{N} = T_{\gamma(t)}\mathcal{N}$ and Lemma 4.10 we obtain

$$d\Phi^F_1(v)h_v(t) = d\Phi^F_1(v)(\ell_v(t) \cap T_v\mathcal{N}) = \mathcal{V}_v(t)TM \cap T_v\mathcal{N} = T_v\mathcal{H}_v(t).$$

This proves the assertion about $h_v(t)$. The assertion about $v_v(t)$ then follows since the decompositions (4.10) and (4.9) are orthogonal with respect to $g_{F_1}(\gamma(t))$ and $W_v(t)$, respectively, and these inner products correspond under (3.14). □

Given a unit vector $w \in T_v\mathcal{H}_m$, with $g_{F_1}(v, w) = 0$, let us denote $a = i_v(w) \in h_v(0)$ and $\overline{a} = dv(a) \in \ell_a(0)$. From (2) of Lemma 4.10 we have $\overline{a} = i_u(f_wv)$ and, since (4.8) is an isometry, $F_2(f_wv) = 1$ and $g_{F_2}(v, w) = 0$. On the one hand, denoting $\Pi = \text{span}[v, w]$ then Theorem 3.12 gives us

$$K_{F_1}(v, \Pi) = W_v(0)(K_v(0)a, a)$$

$$K_{F_2}(u, f_w\Pi) = W_u(0)(K_u(0)a, a).$$

On the other hand, it follows from Theorem 4.5 that

$$W_u(0)(K_u(0)a, a) = W_v(0)(K_v(0)a, a) + 3W_v(0)(A_v(0)a, A_v(0)a).$$

Therefore,

Theorem 4.14. Let $A(t)$ correspond to $A_v(t)$ under (3.14). Then,

$$K_{F_2}(u, f_w\Pi) = K_{F_1}(v, \Pi) + 3g_{F_1}(v)(A(0)w, A(0)w).$$

Observe that the expressions (4.5) and (4.6) and Proposition 3.10 imply that

$$A(t)\mathcal{V}(t) = P_{\mathcal{V}}(t)\left(\frac{D}{dt}P_{\mathcal{V}}(t)V(t)\right) + P_{\mathcal{V}}(t)\left(\frac{D}{dt}P_{\mathcal{V}}(t)V(t)\right),$$

where $V \in \mathcal{X}(\gamma)$, and $P_{\mathcal{V}}(t)$ and $P_{\mathcal{V}}(t)$ are the projections onto $T_{\gamma(t)}\mathcal{H}_{\gamma(t)}$ and $\mathcal{V}_{\gamma(t)}$, respectively, with respect to (4.11).
5. A DYNAMICAL CHARACTERIZATION OF THE SIGN OF FLAG CURVATURE

Definition 5.1. A Legendrian distribution \( \mathcal{L} \) on \( \Sigma_F M \) is said to have the positive (resp. negative) twist property if, for every \( v \in \Sigma_F M \), the curve of Lagrangian subspaces

\[
(5.1) \quad t \mapsto d\Phi^F_t(\dot{\gamma}(t)) (\mathcal{L}_{\dot{\gamma}(t)}) \in \Lambda^0(\ker(\alpha_F)_v),
\]

where \( \gamma(t) \) is the geodesic with \( \dot{\gamma}(0) = v \), has positive-definite (resp. negative-definite) Wronskian for all \( t \).

Remark 5.2. Pointing toward the Maslov index theory, the above property has the following reformulation: over \( \Sigma_F M \) there is a fiber bundle \( \Lambda(\Sigma_F M) \to \Sigma_F M \) whose fiber over a given \( v \) is \( \Lambda(\ker(\alpha_F)_v) \). Observe that the flow \( \Phi^F_t \) lifts in a canonical way to a flow \( \hat{\Phi}^F_t : \Lambda(\Sigma_F M) \to \Lambda(\Sigma_F M) \). Given a Legendrian distribution \( \mathcal{L} \) on \( \Sigma_F M \), its Maslov cycle is the subset \( \Lambda_{\geq 1}(\mathcal{L}) \subset \Lambda(\Sigma_F M) \),

\[
\Lambda_{\geq 1}(\mathcal{L}) = \{(v, \ell) : \ell \cap \mathcal{L}_v \neq \{0\}\}.
\]

This is a stratified submanifold of co-dimension 1 with a natural co-orientation given by using the identification \((2.10)\). The positive twist property for \( \mathcal{L} \) is then equivalent to requiring that, for all \( \ell \in \Lambda(\Sigma_F M) \), if the flow line of \( \hat{\Phi}^F_t \) through \( \ell \) crosses \( \Lambda_{\geq 1}(\mathcal{L}) \), it does so pointing toward the co-orientation of \( \Lambda_{\geq 1}(\mathcal{L}) \).

We shall prove

Proposition 5.3. \((M, F)\) has positive (resp. negative) flag curvature if, and only if, the horizontal bundle \( \mathcal{H}_\Sigma F M \) (see \((3.10)\)) has the positive (resp. negative) twist property.

Positiveness (resp. negativeness) of the flag curvature means positiveness (resp. negativeness) of the quadratic form

\[
g_F(\dot{\gamma}(t))(\mathbf{R}_{\dot{\gamma}(t)}, \cdot) : \ker g_F(\dot{\gamma}(t))(\dot{\gamma}(t), \cdot) \to \mathbb{R}
\]

for all \( t \) and \( v \). Recall \((3.2.1)\) the curve \((5.1)\) is the horizontal curve \( h^c(t) \) of \( \mathcal{P} \) when \( \mathcal{L} = \mathcal{H}_\Sigma F M \), and the above quadratic form corresponds to \( W^c(t)(\mathbf{K}_h^c(t), \cdot) \) under \((3.17)\). Therefore, the above proposition follows at once of the following general property of fanning curves.

Proposition 5.4. Let \( W_h(t) \) denote the Wronskian of the horizontal curve \( h(t) \) of a fanning curve \( \ell(t) \in \Lambda(V) \). Then, given \( t \) and \( u, v \in \ell(t) \),

\[
W_h(t)(\mathbf{H}(t)u, \mathbf{H}(t)v) = W(t)(\mathbf{K}(t)u, v).
\]

Proof. By choosing linear symplectic coordinates, we can suppose \((V, \omega) = (\mathbb{R}^{2n}, \omega_0)\) where \( \omega_0(u, v) = u^T J v \) and \( J \) is the standard complex structure of \( \mathbb{R}^{2n} \). Given a frame \( \mathcal{A}(t) \) for \( \ell(t) \), the matrices of \( W(t) \) and \( W_h(t) \) in the basis \( \mathcal{A}(t) \) and \( \mathcal{H}(t) \) are, respectively,

\[
\mathcal{A}(t)^T J \mathcal{A}(t) = \mathcal{H}(t)^T J \mathcal{H}(t); \quad \mathcal{A}(t)^T \mathcal{A}(t) = (1/2)\mathcal{A}(t)^T \mathcal{A}(t) \{ \mathcal{A}(t), t \} \quad \text{and, therefore,}
\]

\[
\mathcal{H}(t)^T J \mathcal{H}(t) = -(1/2) \mathcal{A}(t)^T \mathcal{A}(t) \{ \mathcal{A}(t), t \}.
\]

On the other hand, since the matrix of \( \mathbf{K}(t) \) in the basis \( \mathcal{A}(t) \) is \((1/2)\{ \mathcal{A}(t), t \} \) (cf. Proposition \((3.9)\)), the matrix of \( W(t)(\mathbf{K}(t)|_{\ell(t)}^{}, \cdot) \) in the basis \( \mathcal{A}(t) \) is given by
\(-\frac{1}{2})A(t)J\{A(t), t\}. \) This shows that the matrix of \(W(t)K(t)|_{\ell(t)}\), in a
basis \(A(t)\) is equal to the matrix of \(W_0(t)\) in the basis \(H(t) = H(t)A(t)\) provided
that the frame \(A(t)\) is normal. The result now follows from the fact that given \(\tau\)
and a basis \(B\) \(\ell(\tau)\), there is a unique normal frame \(A(t)\) with \(A(\tau) = B\). \(\square\)

6. The Flag Curvature of a Class of Projectively Related Finsler Metrics

6.1. Statement of the result. Let \((M, F_0)\) be a Finsler manifold and \(\theta\) a smooth
1-form on \(M\) such that

\(\text{(i) } F_0^*(m) < 1 \text{ for all } m \in M,\)
\(\text{(ii) } d\theta = 0.\)

The first condition ensures that the following deformation of \(F_0,\)

\[ F = F_0 + \theta : TM \to \mathbb{R}, \]

defines a Finsler metric on \(M\) (this follows, for instance, from the proof of Lemma
below), and the closedness of \(\theta\) implies that \(F\) and \(F_0\) share the same un-
parametrized geodesics since the associated arc-length functionals have the same extremals.

We shall prove the following relation between the flag curvatures of \(F_0\) and \(F\).

**Theorem 6.1.** The map

\[ \Psi(v) = \mathcal{L}_{F_0}^{-1}\left(\mathcal{L}_F(v) - \theta\right) \]

restricts to a diffeomorphism from \(\Sigma_{F_0}M\) onto \(\Sigma_F M\). Given a 2-plane \(\Pi \subset T_m M,\)
\(\Pi = \text{span}[v, w],\) where \(v \in \Sigma_{F_0} M\) and \(w \in T_v(\Sigma_{F_0} M \cap T_m M) = \ker g_{F_0}(v)(v, \cdot),\)
let \(\bar{\Pi} = \text{span}[u, \bar{w}] \subset T_m M\) be the 2-plane where \(u = \Psi(v)\) and \(\bar{w} = d\Psi(v)w.\)
Denoting by \(\phi : \Sigma_{F_0} M \to \mathbb{R}\) the function \(\phi(z) = 1/(1 + \theta(z))\), then

\[ K_F(v, \Pi) = \phi(u)^2K_{F_0}(u, \bar{\Pi}) - \frac{1}{2}\left[\frac{1}{2}S_{F_0}(\phi)^2 - \phi S_{F_0}(S_{F_0}(\phi))\right](u), \]

where \(S_{F_0}\) is the geodesic spray of \(F_0\). Alternatively, if \(h\) is a primitive for \(\theta\) around
\(m\) and if we let \(f(t) = t + h(\gamma_u(t))\), where \(\gamma_u\) is the \(F_0\)-geodesic with \(\gamma_u(0) = u,\)
then

\[ K_F(v, \Pi) = \frac{1}{f(0)^2}\left[K_{F_0}(u, \bar{\Pi}) - \frac{1}{2}\{f(t), t\}_{l=0}\right], \]

where \(\{f(t), t\} = (d/dt)(\hat{f}^{-1}\hat{f}) - (1/2)(\hat{f}^{-1}\hat{f})^2\) is the Schwarzian derivative of \(f(t)\).

**Remark 6.2.** It follows easily from the definition of the map \(\Psi\) that \(\bar{w}\) is determined
by the equality \(g_{F_0}(u)(\bar{w}, \cdot) = g_F(v)(w, \cdot).\)

6.2. Preliminaries. Throughout, \(S_{F_0}, S_F,\) and \(S^*_{F_0}, S^*_{F}\), shall denote the geodesic
sprays and co-geodesic vector fields, respectively, of \(F_0\) and \(F\), viewed as vector
fields on \(\Sigma_{F_0} M, \Sigma_F M,\) and \(\Sigma^*_{F_0} M, \Sigma^*_{F} M.\)

**Lemma 6.3.** We have that

\[ \Sigma_F^* M = \{\xi \in T^* M : F_0^*(\xi - \theta) = 1\} = \theta + \Sigma_{F_0}^* M \]
Proof. Since $F^{*}_{\theta}(\theta) < 1$, then $\theta + \Sigma F^{*}_{\theta} M$ is the unit co-sphere bundle $\Sigma F^{*}_{\theta} M$ of some Finsler metric $F$ on $M$. To see that $\tilde{F} = F$, let $v \in T_{m} M$ and compute:

$$\tilde{F}(v) = (\tilde{F}^{*})^{\ast}(v)$$

$$= \sup\{\xi(v) : \xi \in \Sigma F^{*}_{\theta} M \cap T_{m} M = \theta_{m} + \Sigma F^{*}_{\theta} M \cap T_{m} M\}$$

$$= \theta_{m}(v) + \sup\{\xi(v) : \xi \in \Sigma F^{*}_{\theta} M \cap T_{m} M\} \quad = \theta_{m}(v) + F_{0}(v).$$

If we introduce the magnetic Hamiltonian $H_{m} : T^{*} M \setminus 0 \to \mathbb{R}$,

$$H_{m}(\xi) = (1/2)(F^{*}_{\theta})^{2}(\xi - \theta),$$

then (6.3) says that the energy level $1/2$ of $H_{m}$ is

$$H_{m}^{-1}(1/2) = \Sigma F^{*}_{\theta} M.$$

Let us follow the terminology in [1, Chap. 3]. The Hamiltonian $H_{m}$ corresponds to the Lagrangian function $L_{m} : TM \setminus 0 \to \mathbb{R}$, $L_{m}(v) = (1/2)F_{0}(v)^{2} + \theta(v)$; that is, the Legendre transformation $\mathcal{L}_{m} : TM \setminus 0 \to T^{*} M \setminus 0$ of $L_{m}$, which one computes easily as

$$\mathcal{L}_{m}(v) = \mathcal{L}_{F_{0}}(v) + \theta,$$

is a diffeomorphism and $H_{m} \circ \mathcal{L}_{m} = E_{m}$, where the energy $E_{m}$ of $L_{m}$ computes as $E_{m} = (1/2)(F_{0})^{2}$. It follows that $\mathcal{L}_{m}$ restricts to a diffeomorphism

$$\mathcal{L}_{m} : E_{m}^{-1}(1/2) = \Sigma F_{0} M \to H_{m}^{-1}(1/2) = \Sigma F^{*}_{\theta} M$$

whose inverse, pre-composed with the diffeomorphism $\mathcal{L}_{F} : \Sigma F^{*} M \to \Sigma F^{*}_{\theta} M$, is the map (6.1):

$$\Psi = \mathcal{L}^{-1}_{m} \circ \mathcal{L}_{F} : \Sigma F^{*} M \to \Sigma F_{0} M.$$

Lemma 6.4. If $\phi$ is the function in Theorem [6.7] then

$$\Psi \circ S_{F} = \phi S_{F_{0}}.$$

Proof. Let $X_{H_{m}}$ be the restriction to $H_{m}^{-1}(1/2)$ of the Hamiltonian vector field of $H_{m}$. Since the Hamiltonians $H_{m}$ and $(1/2)(F^{*})^{2}$ have the same energy level $1/2$, it follows easily that, on that level, their Hamiltonian vector fields must differ by a multiplicative function $\lambda : \Sigma F^{*} M \to \mathbb{R}$,

$$S_{F} = \lambda X_{H_{m}}.$$

On the other hand, $S_{F}^{*}$ is $\mathcal{L}_{F}$-related to $S_{F}$ and, letting $X_{E_{m}}$ be the restriction to $E_{m}^{-1}(1/2) = \Sigma F_{0} M$ of the Euler-Lagrange vector field of $L_{m}$, $X_{H_{m}}$ is $\mathcal{L}_{m}$-related to $X_{E_{m}}$. Therefore, $\Psi \circ S_{F} = (\lambda \circ \mathcal{L}_{m}) X_{E_{m}}$. It remains to show that $X_{E_{m}} = S_{F_{0}}$ and $\lambda \circ \mathcal{L}_{m} = \phi$. The former is a consequence of the closedness of $\theta$ since $L_{m}$ differs from (1/2)(F_{0})^{2} by $\theta$ (cf. [1 Prop. 3.5.18]). As for the latter, applying the canonical 1-form $\alpha$ to (6.9), and recalling that $\alpha(S_{F}^{*}) = 1$, then

$$\lambda = 1/\alpha(X_{H_{m}}).$$

On the other hand, since $X_{H_{m}}$ and $X_{E_{m}}$ are $\mathcal{L}_{m}|_{\Sigma F_{0} M}$-related, and

$$\mathcal{L}_{m} \ast \alpha = \mathcal{L}_{F_{0}} \ast \alpha + \pi \ast \theta$$


as follows from (6.6), we have for $v \in \Sigma_{F_0} M$,
\[ \alpha_{\mathcal{L}_m}(X_{H_m}) = (\mathcal{L}_m^* \alpha)_v(X_{E_m}) = (\mathcal{L}_{F_0}^* \alpha)_v(X_{E_m}) + (\pi^* \theta)_v(X_{E_m}) \]
\[ = g_{F_0}(v)(v, d\pi(v)X_{E_m}) + \theta(d\pi(v)X_{E_m}). \]
Using now that $X_{E_m}$ is a SODE, the above expression is $g_{F_0}(v)(v, \theta(v)) + 1 + \theta(v)$ and the equality $\lambda \circ \mathcal{L}_m|\Sigma_{F_0} M = \phi$ follows now from (6.10).

6.3. **Proof of Theorem 6.1.** Consider, as in (3.2.1) the Jacobi curves
\[ \ell_c^\nu(t) \in \Lambda(\ker(\alpha_F)_v, \omega_F), \ell_c^\nu(u) \in \Lambda(\ker(\alpha_{F_0})_u, \omega_{F_0}), \]
based at $v$ and $u$, associated to $F$ and $F_0$, respectively. We shall break up the proof in several simple steps.

I. The map $\Psi : (\Sigma_F M, \alpha_F) \rightarrow (\Sigma_{F_0} M, \Psi_* \alpha_F)$ is a fiber-preserving exact contact diffeomorphism and, by Lemma 6.4, $\Psi_* S_F = \phi S_{F_0}$. Moreover, $d(\Psi_* \alpha_F) = \omega_{F_0}$; for, it follows successively from the definition (6.7) of $\Psi$, the definitions of (6.15) and (6.11) that $\Psi$ is a symplectic isomorphism such that $\Psi(\ker(\alpha_F)_v, \omega_F)$ generates the kernel of $\Psi_* \alpha_F$. Therefore, the derivative $d\Psi(v)$ restricts to a symplectic isomorphism
\[ d\Psi(v) : (\ker(\alpha_F)_v, \omega_F) \rightarrow (\ker(\Psi_* \alpha_F)_u, \omega_{F_0}) \]
that maps $\ell^\nu_u(t)$ to the Jacobi curve $\hat{\ell}^\nu_u(t) \in \Lambda(\ker(\Psi_* \alpha_F)_u, \omega_{F_0})$ of the moving plane $(\ker(\Psi_* \alpha_F), \gamma \Sigma_{F_0} M, \Phi^{\phi S_{F_0}}_t)$ defined on the exact contact manifold $(\Sigma_{F_0} M, \Psi_* \alpha_F)$.

II. The flow $\Phi^{\phi S_{F_0}}_t$ is a reparametrization of $\Phi^{S_{F_0}}_t$; more precisely, if $\eta_\nu(t) = \eta(t, u)$ denotes the solution, defined for $(t, u)$ on some neighborhood of $\{0\} \times \Sigma_{F_0} M$, to
\[ \frac{\partial \eta}{\partial t}(t, u) = \phi(\Phi^{S_{F_0}}_{\eta(t, u)}(u)) , \quad \eta(0, u) = 0, \]
then
\[ \Phi^{\phi S_{F_0}}_t(u) = \Phi^{S_{F_0}}_{\eta(t, u)}(u). \]
It follows from this and from a straightforward computation that the derivative of $\Phi^{\phi S_{F_0}}_t$ at $\Phi^{\phi S_{F_0}}_t(u)$ takes the form
\[ d\Phi^{\phi S_{F_0}}_t(\Phi^{\phi S_{F_0}}_t(u)) = d\Phi_{-\eta(t, u)}^{S_{F_0}}(\Phi^{S_{F_0}}_{\eta(t, u)}(u)) + \zeta \otimes (S_{F_0})_u \]
for some $\zeta \in T^* u(\Sigma_{F_0} M)$.

III. Let $\text{pr}_u : T_u \Sigma_{F_0} M \rightarrow \ker(\alpha_{F_0})_u$ be the projection map with kernel generated by $(S_{F_0})_u$. Since one also has $T_u \Sigma_{F_0} M = \ker(\Psi_* \alpha_F)_u \oplus \text{span}[(S_F)_u]$ and $S_{F_0}$ generates the kernel of $\omega_{F_0}$, then $\text{pr}_u$ restricts to a symplectic isomorphism
\[ \text{pr}_u : (\ker(\Psi_* \alpha_F)_u, \omega_{F_0}) \rightarrow (\ker(\alpha_{F_0})_u, \omega_{F_0}). \]
Recalling the definitions of $\ell^\nu_u(t)$ and $\hat{\ell}^\nu_u(t)$, it follows from (6.15) that $\text{pr}_u(\hat{\ell}^\nu_u(t)) = \ell^\nu_u(\eta_\nu(u)) (t)$. Therefore, the composition of (6.12) with (6.16),
\[ T = \text{pr}_u \circ d\Psi(v) : (\ker(\alpha_F)_v, \omega_F) \rightarrow (\ker(\alpha_{F_0})_u, \omega_{F_0}), \]
is a symplectic isomorphism such that
\[ T\ell^\nu_v(t) = \ell^\nu_u(\eta_\nu(u)(t)). \]
IV. Observe that $pr_{u}$ is the identity on $V_{u}\Sigma_{F_{0}}M$, so $Tw = \tilde{w}$. Applying Proposition 2.16 to (6.17) one obtains

$$W_{\xi}(0)(K_{\xi}(0)w, w) = \dot{\eta}_{u}(0)W_{\xi}(0)(K_{\xi}(0)\tilde{w}, \tilde{w}) + \frac{1}{2}\{\eta_{u}(t), t\}|_{t=0}$$

and therefore

$$K_{F}(v, \Pi) = \dot{\eta}_{u}(0)^{2}K_{F}(u, \bar{\Pi}) + \frac{1}{2}\{\eta_{u}(t), t\}|_{t=0}.$$  

It remains to compute $\dot{\eta}_{u}(0)$ and $\{\eta_{u}(t), t\}|_{t=0}$. From (6.13) and (6.14) one has $\dot{\eta}_{u}(t) = \phi(\Phi_{t}S_{F_{0}}(u))$. Hence,

(i) $\dot{\eta}_{u}(0) = \phi(u)$

(ii) $\dot{\eta}_{u}(0) = \phi S_{F_{0}}(\phi)_{|u}$

(iii) $\dot{\eta}_{u}(0) = \phi S_{F_{0}}(\phi)_{|u} = \phi S_{F_{0}}(\phi)_{|u} + \phi^{2}S_{F_{0}}(S_{F_{0}}(\phi))_{|u}$

Therefore $\{\eta_{u}(t), t\}|_{t=0} = \phi S_{F_{0}}(S_{F_{0}}(\phi))_{|u} - (1/2)S_{F_{0}}(\phi)^{2}_{|u}$ and (6.2) follows.

7. THE FLAG CURVATURE OF KATOK PERTURBATIONS

Let $(M, F)$ be a Finsler manifold and $V$ a vector field on $M$ such that $F(V_{m}) < 1$ for all $m \in M$. Regarding $V$ as a function

$$V : T^{*}M \rightarrow R, \ V(\xi) = \xi(V_{r}(\xi)),$$

there exists a unique Finsler metric $\tilde{F}$ on $M$ whose dual $\tilde{F}^{*}$ is given by

$$\tilde{F}^{*} = F^{*} + V.$$

**Definition 7.1.** In the case where $V$ is a Killing vector field for $F$, that is, its flow $\Phi_{t}^{V}$ satisfies $(\Phi_{t}^{V})^{*}F = F$ for all $t$, we shall call $\tilde{F}$ the *Katok perturbation* of $F$ by $V$.

Although the computations of the flag curvature in the more general cases of perturbations by *homothetic* vector fields and even for *conformal* vector fields have been done ([22] and [21], resp.), a proof via fanning curves of the theorem below is particularly simple and elegant and shall, thus, be presented here.

**Theorem 7.2** (Foulon [16]). Let $\tilde{F}$ be a Katok perturbation of $F$. If $K_{F} \equiv 1$, then $K_{\tilde{F}} \equiv 1$.

**7.1. Proof of Theorem 7.2.** We shall denote by $\alpha_{F}$ and $\alpha_{\tilde{F}}$ the contact 1-forms on $\Sigma^{*}_{F}M$ and $\Sigma^{*}_{\tilde{F}}M$, respectively, and let $\omega_{F} = -d\alpha_{F}$ and $\omega_{\tilde{F}} = -d\alpha_{\tilde{F}}$. Let $X$ be the Hamiltonian vector field of (7.1). As pointed out in [31], the Hamiltonian flow $\Phi_{t}^{X}$ is pulling-back by $\Phi_{t}^{V}$.

$$\Phi_{t}^{X} = (d\Phi_{t}^{V})^{*} : T^{*}M \rightarrow T^{*}M.$$

Since $V$ is a Killing vector field of $F$, it follows that $\Sigma^{*}_{F}M$ is invariant by $\Phi_{t}^{X}$ and, hence, $X$ is tangent to $\Sigma^{*}_{F}M$. Also, since $F^{*}$ is constant on the orbits of $X$, we have the commutation of the flows $\Phi_{t}^{S_{F}^{*}}$ and $\Phi_{t}^{X}$.

$$[S_{F}^{*}, X] = 0.$$  

We shall still denote by $X$ and $V$ the restrictions of $X$ and $\Phi_{t}^{V}$ to $\Sigma^{*}_{F}M$.

Consider the diffeomorphism

$$\Psi : \Sigma^{*}_{F}M \rightarrow \Sigma^{*}_{\tilde{F}}M, \ \Psi(\xi) = \frac{1}{F^{*}(\xi)}\xi.$$
From the definitions, one easily computes

\begin{equation}
\Psi_{\ast} \alpha_{\hat{F}} = \frac{1}{F_{\ast}} \alpha_{F}.
\end{equation}

**Lemma 7.3.** We have that \( \Psi_{\ast} S_{F}^{\ast} = S_{F}^{\ast} + X \).

**Proof.** All we have to show is that

\begin{equation}
i_{S_{F}^{\ast} + X} \cdot (\Psi_{\ast} \alpha_{\hat{F}}) = 0 , \quad (\Psi_{\ast} \alpha_{F})(S_{F}^{\ast} + X) = 1.
\end{equation}

Observe that \( \alpha_{F}(X) = V \) since (2.2) implies that \( X \) is \( \tau \)-related to the vector field \( V \). Thus, since \( \alpha_{F}(S_{F}^{\ast}) = 1 \) and, as functions on \( \Sigma_{F}^{\ast} M \), \( \tilde{F}^{\ast} = F^{\ast} + V = 1 + V \), the second equality in (7.6) follows from (7.5). By taking derivatives in (7.5), and using that \( i_{S_{F}^{\ast} \omega_{F}} = 0 \), \( i_{X} \omega_{F} = dV \), and \( \alpha_{F}(S_{F}^{\ast} + X) = F^{\ast} \), we obtain successively,

\begin{align*}
i_{S_{F}^{\ast} + X} \cdot (\Psi_{\ast} \alpha_{\hat{F}}) & = (1/\tilde{F}^{\ast})^{2} i_{S_{F}^{\ast} + X} (dV \wedge \alpha_{F}) + (1/\tilde{F}^{\ast}) i_{S_{F}^{\ast} + X} \omega_{F} \\
& = (1/\tilde{F}^{\ast})^{2} ((S_{F}^{\ast} + X)(V) \alpha_{F} - \alpha_{F}(S_{F}^{\ast} + X) dV) + (1/\tilde{F}^{\ast}) i_{X} \omega_{F} \\
& = (1/\tilde{F}^{\ast})^{2} (S_{F}^{\ast}(V) \alpha_{F} - \tilde{F}^{\ast} dV) + (1/\tilde{F}^{\ast}) dV.
\end{align*}

On the other hand, the commutativity of the flows \( \Phi_{t}^{S_{F}^{\ast}} \) and \( \Phi_{t}^{X} \) gives us \( S_{F}^{\ast}(V) = 0 \). The result follows. \( \square \)

The lemma above and (7.3) imply, respectively,

\[ \Psi \circ \Phi_{t}^{S_{F}^{\ast}} = \Phi_{t}^{S_{F}^{\ast} + X} \circ \Psi = (\Phi_{t}^{X} \circ \Phi_{t}^{S_{F}^{\ast}}) \circ \Psi. \]

On the other hand, \( \Psi \) is fiber-preserving, and the same is true of \( \Phi_{t}^{X} \) since it is \( \tau \)-related to a flow on \( M \). Therefore, if \( \ell_{c}^{\xi}(t) \in \Lambda(\ker(\alpha_{\hat{F}})_{\xi}) \) and \( \ell_{c}^{\eta}(t) \in \Lambda(\ker(\alpha_{F})_{\eta}) \) denote, as in (3.2.1), the Jacobi curves associated to \( \hat{F} \) and \( F \), respectively, based at \( \xi \in \Sigma_{F}^{\ast} M \) and \( \eta = \Psi(\xi) \), we have shown

**Proposition 7.4.** \( d\Psi(\xi) \) restricts to an isomorphism \( T : \ker(\alpha_{\hat{F}})_{\xi} \to \ker(\alpha_{F})_{\eta} \) such that

\begin{equation}
T \ell_{c}^{\xi}(t) = \ell_{c}^{\eta}(t).
\end{equation}

**Proof of Theorem 7.2.** The hypothesis \( K_{F} \equiv 1 \) means that \( K_{c}^{\eta}(t) \equiv \mathbf{Id} \) for all \( \eta \in \Sigma_{F}^{\ast} M \). Applying Proposition 2.19 to (7.7), we obtain \( K_{c}^{\xi}(t) \equiv \mathbf{Id} \) for all \( \xi \in \Sigma_{F}^{\ast} M \) and, therefore, \( K_{\hat{F}} \equiv 1 \). \( \square \)

**References**

[1] R. Abraham and J. E. Marsden, *Foundations of mechanics*, Benjamin/Cummings Publishing Co. (1978)
[2] A. Agrachev, D. Barilari and L. Rizzi, *Curvature: a variational approach*, Memoirs of the A.M.S., to appear, [arXiv:1306.5318v5 [math.DG]], (2013).
[3] A. Agrachev, N. Chtcherbakova and I. Zelenko, *On Curvatures and Focal Points of Distributions of Dynamical Lagrangian Distributions and their Reductions by First Integrals*, J. Dyn. Control. Syst., 11 (2005), pp. 297–327.
[4] S. Aïdoud, *Fanning curves of Lagrangian manifolds and geodesic flows*, Duke Math. J., 59 (1999), pp. 537–552.
[5] J. C. Álvarez-Paiva, *Symplectic Geometry and Hilbert’s Fourth Problem*, J. Differential Geom. Volume 69, Number 2 (2005), pp. 353–378.
[6] J. C. Álvarez Paiva and C. E. Durán, *Isometric submersions of Finsler manifolds*, Proc. Amer. Math. Soc. 129 (2001), no. 8, pp. 2409–2417.
[7] J. C. Álvarez Paiva and C. E. Durán, Geometric invariants of fanning curves, Adv. in Appl. Math., 42 (2009), pp. 290–312.
[8] R. L. Bryant, Projectively flat Finsler 2-spheres of constant curvature, Sel. math., New ser. (1997) 3: 161. doi:10.1007/s000290050009.
[9] J. Cheeger and D. Ebin, Comparison Theorems in Riemannian Geometry, North-Holland (1975).
[10] M. Crampin, W. Sarlet and F. Cantrijn, Higher-Order Differential Equations and Higher-Order Lagrangian Mechanics, Math. Proc. Cambridge Philos. Soc., 99, 565–587, (1986).
[11] M. de León and P. R. Rodrigues, Generalized Classical Mechanics and Field Theory, Elsevier, (1985).
[12] C. E. Durán and D. Otero, The projective symplectic geometry of higher order variational problems: Minimality conditions, J. Geom. Mech., 8(3) (2016), pp. 305–322.
[13] C. E. Durán and C. Peixoto, Geometry of Fanning Curves in Divisible Grassmannians, Differential Geom. Appl., Volume 49, (2016), pp. 447–472.
[14] C. E. Durán and L. D. Sperança, Rigidity of flat sections on non-negatively curved pullback submersions, Manuscripta Math. 147, (2015) pp. 511–525.
[15] P. Foulon, Géométrie des équations différentielles du second ordre, Ann. Inst. H. Poincaré Phys. Théor., 45 (1986), pp. 1–28.
[16] P. Foulon and R. Ruggiero, A first integral for $C^\infty$, $k$-basic Finsler surfaces and applications to rigidity, Proc. Amer. Math. Soc. 144 (2016), pp. 3847–3858.
[17] J. Grifone, Structure presque-tangente et connexions. I, Ann. Inst. Fourier (Grenoble), 22 (1972), pp. 287–334.
[18] J. Grifone, Structure presque-tangente et connexions. II, Ann. Inst. Fourier (Grenoble) 22 (1972), no. 3, 291–338.
[19] D. Gromoll and K. Grove The low-dimensional metric foliations of Euclidean spheres J. Differential Geom. Volume 28, Number 1 (1988), pp. 143–156.
[20] D. Gromoll and G. Walsch, Metric Foliations and Curvature, Birkhauser, (2009).
[21] L. Huang and X. Mo, On the flag curvature of a class of Finsler metrics produced by the navigation problem, Pacific J. Math. 277 (2015), no. 1, 149–168.
[22] M. Javaloyes and H. Vitório, Zermelo navigation in pseudo-Finsler metrics, arXiv:1412.0465.
[23] D. McDuff and D. Salamon, Introduction to symplectic topology, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, (1998).
[24] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J. 13 (1966), pp. 459–469.
[25] P. Piccione and D. V. Tausk, A students guide to symplectic spaces, Grassmannians and Maslov index. Publicaes Matemticas do IMPA, p. xiv+301. [IMPA Mathematical Publications], Instituto de Matemtica Pura e Aplicada (IMPA), Rio de Janeiro (2008).
[26] H. B. Rademacher, Non-reversible Finsler metrics of positive curvature, In: A sampler of Riemann-Finsler geometry. Eds.: D.Bao, R.Bryant, S.S.Cheern, Z.Shen, Math.Sciences Res. Inst. Series 50, Cambridge Univ. Press (2004), pp. 261–302.
[27] Z. Shen, Differential Geometry of Spray and Finsler Spaces, Kluwer Academic Publishers, Dordrecht, (2001).
[28] Z. Shen, Two-dimensional Finsler metrics with constant flag curvature, Manuscripta Math. 109, no. 3, (2001), pp. 349–366.
[29] H. Vitório, A Unified Approach to the Theory of Connections in Finsler Geometry, Bull. Braz. Math. Soc., (New Series), (2016). doi:10.1007/s00574-016-0014-8.
[30] H. Vitório, Geometria de curvas fanning e de suas reduções simpléticas, Ph.D. thesis, Universidade estadal de Campinas, (2010).
[31] W. Ziller, Geometry of the Katok examples, Ergodic Theory Dynam. Systems, 3 (1983), pp. 135–157.
[32] W. Ziller, Examples of Riemannian manifolds with nonnegative sectional curvature , in: Metric and Comparison Geometry, Surv. Diff. Geom. 11 , ed. K.Grove and J.Cheeger, (2007), pp.] 63–102.
[33] M. Xu and S. Deng, Recent progress on homogeneous Finsler spaces with positive curvature, arXiv:1612.08372 [math.DG].
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