Non-linear Stochastic Fractional Neutral-type Differential Systems with State Delay : A Controllability Interpretation

B.Sundaravadivoo *
Department of Mathematics, Alagappa University, Karaikudi-630 004, India.

Abstract
Aforementioned manuscript, stochastic neutral fractional systems with bounded operator having state delay is explored. We use a Grammian matrix defined by Laplace transform to get the solution representation for the considered model. Controllability results are obtained through the application of the Banach contraction principle. Finally, a numerical example shows the accuracy of the controllability parameters obtained.

Keywords. Delays; controllability; implicit derivative; integro differential equations.

1 Introduction
The design of stochastic models to explain the functions of intrinsic noise, uncertainty of natural phenomenon, extrinsic noise and variability in the system has been of rapid interest in recent years. Accordingly, the theory play a consistent elucidation of various perceived circumstances in the real world phenomena, one can refer for more details [1, 3, 4, 11, 12]. Concurrently, non-integer order differential equations has grow universally in the furthest three decades adequate to its capability to represent the complex occurrence systematically by reproducing the non-local relations in space and time. Especially, in many engineering and scientific disciplines fractional differential equation play a significant role to develop the mathematical modeling such as biological sciences, chemical sciences, physical sciences, medical sciences, industries etc.

Furthermore, in numerous medical and physical phenomena the involvement of delay features arises in the fractional differential equation. The aroused delay may be with the state-dependent one or in the non-constant delay. Basically, delay differential equations are solved by using numerical methods, asymptotic solutions and graphical tools.

On the other hand, controllability play a key role in the qualitative reaction of a dynamical system. The theory was started in the sixties of nineteenth century based on the mathematical description of the dynamical system. Generally, it can have the ability to control a dynamical system from the initial position to the desired final position with the assistance of some set of admissible controls. Hence the combination of fractional-order derivatives and as well as integrals in the theory of controllability leads to better results than the integer-order derivatives. In most of the dynamical system, the whole state of the dynamical system is not affected by the control, but it acts as a part of it. Furthermore, very often in real industrial processes it is possible to observe only a certain part of the complete state of the dynamical system. Therefore, it is very important to determine whether or not control of the complete state of the dynamical system is possible,see for references [2, 5, 6, 7, 8, 9].

The main contributions of this paper is exposed as follows:

• In the existing literature, there is no work that describes the controllability criteria of neutral stochastic fractional differential equations with delay.

*Corresponding author is B.Sundaravadivoo with e-mail: sundaravadivoon@gmail.com.
• Here controllability results were proved using algebraic approach which is very effective and easier to calculate the results.

• An example is provided to illustrate the existing theory in the available source of literature.

The contour of the manuscript is classified in this way. Section 2, provides preliminaries and solution representations. Section 3, describes the controllability theorems. Section 4, contributes an example to embellish the efficacy and utility of controllability. Eventually, epilogues are worn in Section 5.

2 Preliminaries

Notations: Let \((Y, U)\) denotes real separable Hilbert spaces and \(L\) represents bounded linear operator such that \(L : Y \rightarrow U\). Suppose that a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_s, s \geq 0, P)\) contains probability measure \(P\) on \(\Omega\), \(G\) represents the expectation corresponding to the measure \(P\). Consider the following:

(i) \(X = L_2(\Omega, \mathcal{F}_s, X)\), denotes the Hilbert space of all \(\mathcal{F}_t\)-measurable quadratically integrable random variables in terms of efficacy in \(X\).

(ii) \(H_2 \subseteq PC(J, L_2(F, X))\) comprised of all \(\mathcal{F}_t\)-measurable procedures involving values in \(X\) and equipped by \(\| \phi \|^2_{H_2} = \sup_{s \in J} E \| \phi \|^2\).

(iii) \(U_{ad} = L^2_{\mathcal{F}}(J, U)\), represents the Hilbert space of all quadratically integrable and \(\mathcal{F}_t\)-measurable processes involving values in \(U\).

Definition 2.1 [1] Let \(\alpha > 0\), \(f(t) : (0, \infty) \rightarrow \mathbb{R}\) the Caputo fractional derivative is given by

\[
(CD_0^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds
\]

Definition 2.2 [1] The Mittag-Leffler function in two parameters is given as

\[
E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \text{ for } \alpha, \beta > 0
\]

Lemma 2.3 [16] Let \(C\) be complex plane, for any \(\alpha > 0, \beta > 0\) and \(A \in \mathbb{C}^n \times n\) then

\[
L[\beta^{-1}E_{\alpha, \beta}(At^\alpha)] = s^{\alpha-\beta}(s^\alpha I - A)^{-1}, \Re(s) > \| A \|^2
\]

In general,

\[
E_{\alpha, 1}(\lambda z^\alpha) = E_{\alpha}(\lambda z^\alpha)
\]

\[
CD_0^\alpha E_{\alpha}(\lambda t^\alpha) = \lambda E_{\alpha}(\lambda t^\alpha)
\]

and \(L[E_{\alpha}(\pm at^\alpha)](s) = \frac{s^{\alpha-1}}{s^{\alpha} \pm a}\)

Assumption (A1): The operator \(A \in L(Y)\) such that \(\| A \|^2 < \frac{(2\alpha-1)\Gamma(\alpha)^2}{4\alpha}\)

Evaluate the linear neutral fractional delay differential structure as follows.

\[
CD_0^\alpha(y(s) - q(s, y(s))) = Ny(s) + Ky(s - \tau) + Bu(s) + f(s) + \sigma(s, y(s - \delta)) \frac{dw(s)}{ds}, \ s \in J = [0, T]
\]

\[
y(s) = \mu(s), s \in [-\delta, 0]
\]
The solution of system (1) is by employing (A1), we get

\[
y(s) = \mu - q(0,\mu(0)) + q(s,y(s)) + \int_0^s (s-t)^{\alpha-1}E_{\alpha,\alpha}(N(s-t)^\alpha) \\
[N\mu(0) - Nq(0,\mu(0)) + Nq(s,y(s)) + Ky(t) + Bu(t)]dt \\
+ \int_0^s (s-t)^{\alpha-1}E_{\alpha,\alpha}(N(s-t)^\alpha)\sigma(t,y(t-\delta))dw(t)
\]  

(2)

**Definition 2.4** The set \( x(s) = \{y(s), u_s\} \) is the complete state of equation (1).

**Definition 2.5** Equation (1) is completely controllable on \( J \) if for every \( y_1 \in X \) there exists \( u \in U_{ad} \) such that \( y(T) = y_1 \). The operator \( L_T : U_{ad} \to X \) is defined by

\[
L_T u = \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(N(T-s)^\alpha)Bu(s)ds \\
(L_T y) = (T-s)^{\alpha-1}B^*E_{\alpha,\alpha}(N^*(T-s)^\alpha)G\{y,F_s\}
\]

**Definition 2.6** \( W_T : Y \to Y \) is defined by

\[
W_T y = \int_0^T (T-t)^{2\alpha-2}E_{\alpha,\alpha}(N(T-t)^\alpha)BB^*E_{\alpha,\alpha}(N^*(T-t)^\alpha)G\{y,F_s\}ds
\]

where \( \ast \) represents the adjoint operator and \( W(T) \) represents the controllability Grammian operator.

**Lemma 2.7** The operator \( W_T \in L_T L_T^* \in L(Y) \) is well defined and bounded for all \( \frac{1}{2} < \alpha \leq 1 \).

**Proof.** For \( u \in U_{ad} \) and \( s \in J \) and employing Holders inequality and (A1) we get

\[
G \| L_s u \|_Y^2 = G \| \int_0^s (s-t)^{\alpha-1}E_{\alpha,\alpha}(N(s-t)^\alpha)Bu(t)dt \|^2 \\
= \sum_{k=0}^{\infty} G \| \int_0^s N_k(s-t)^{\alpha k + \alpha - 1} \frac{B u(t)dt}{\Gamma(\alpha k + \alpha)} \|^2 \\
\leq \sum_{k=0}^{\infty} G \| \int_0^s \frac{(s-t)^{2\alpha k + 2\alpha - 2}}{\Gamma(\alpha k + \alpha)} dt \|^2 \| N \|^2 \| u(t) \| \| B \|^2 \frac{1}{\Gamma(\alpha k + \alpha)} \\
\leq \| u \|_{U_{ad}}^2 T^{2\alpha - 1} \sum_{k=0}^{\infty} \frac{(\Gamma(\alpha))^2 k}{(2\alpha k + 2\alpha - 1)(2\alpha - 1)^k}
\]

where \( k = \sup_{0 \leq T \leq s} \| B \| \). If \( \alpha \in \left(\frac{1}{2}, 1\right] \) the last series converges. Hence \( L_s \) is bounded for any \( s \in J \).

**3 Controllability of Neutral Stochastic Fractional Differential Systems with Delay**

Evaluate the nonlinear neutral stochastic fractional structure with delay,

\[
C^{D^\alpha}(y(s) - q(s,y(s))) = Ny(s) + Ky(s - \delta) + Bu(s) + f(s,y(s),y(s - \delta), \int_0^s p(s,t,y(t),y(t-\delta))dt) \\
+ \sigma(s,y(s),y(s - \delta), \int_0^s h(s,t,y(t),y(t-\delta))dt) \frac{dw(s)}{dt}, \quad s \in J = [0,T] \\
y(s) = \mu(s), \quad s \in [-\delta,0]
\]  

(3)
where $N : Y \to Y$ and $K : Y \to Y$ are bounded linear operators, $B : U \to Y$ is a bounded linear operator, $q : J \times Y \times Y \to Y$ is a continuous function and $q$ is continuously differentiable, $\delta$ is a positive constant, $y \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control input, $\frac{1}{2} < \alpha \leq 1$ and the non linear function $f : J \times Y \times Y \times Y \to L^2_J, p : J \times J \times Y \times Y \to Y$ and $h : J \times J \times Y \times Y \to Y$ are continuous. $W(s)$ is a K-valued Wiener process with positive symmetric trace class covariance operator.

**Assumption (A2):** Assume that there exists constants $M_i > 0, i = 1, 2, 3, 4$ such that

$$
\| f(t, y_1, y'_1, x_1) - f(t, y_2, y'_2, x_2) \|^2 \leq M_1(\| y_1 - y_2 \|^2 + \| y'_1 - y'_2 \|^2 + \| x_1 - x_2 \|^2)
$$

$$
\| \sigma(t, y_1, y'_1, x_1) - \sigma(t, y_2, y'_2, x_2) \|^2 \leq M_2(\| y_1 - y_2 \|^2 + \| y'_1 - y'_2 \|^2 + \| x_1 - x_2 \|^2)
$$

$$
\| \int_0^t p(t, s, y_1, y'_1) - \int_0^t p(t, s, y_2, y'_2) ds \|^2 \leq M_3(\| y_1 - y_2 \|^2 + \| y'_1 - y'_2 \|^2)
$$

$$
\| \int_0^t h(t, s, y_1, y'_1) - \int_0^t h(t, s, y_2, y'_2) ds \|^2 \leq M_4(\| y_1 - y_2 \|^2 + \| y'_1 - y'_2 \|^2)
$$

$$
\| g(s, y_1) - g(s, y_2) \| \leq M_5 \| y_1 - y_2 \|^2
$$

for all $y_1, y_2, x_1, x_2 \in Y$.

Let $M_6 = \sup_{t \in J} \| \sigma(t, 0, 0, 0) \|, M_7 = \sup_{t \in J} \| \int_0^t r(t, s, 0, 0) \|$, $M_8 = \sup_{t \in J} \| h(t, s, 0, 0) \|, M_9 = \sup_{t \in J} \| f(t, 0, 0, 0) \|, M_{10} = \sup_{t \in J} \| g(t, 0) \|

For each fixed $z \in \mathcal{H}_2$, equation (3) becomes

$$
C^\alpha(y(s) - q(s, y(s))) = N_y(s) + Kg(s, z) + Bu(s) + f(s, z(s), z(s, \delta), \int_0^s p(s, t, z(t), z(t, \delta)) dt)
$$

$$
+ \sigma(s, z(s), z(s, \delta), \int_0^s h(s, t, z(t), z(t, \delta)) dt) \frac{dw(s)}{ds}, \quad s \in J = [0, T]
$$

$$
y(s) = \mu(s), \quad s \in [-\delta, 0]
$$

The solution of (4) is written by

$$
y(s) = \mu(0) - q(0, \mu(0)) + q(s, y(s)) + \int_0^s (s - t)^{\alpha - 1}E_{\alpha, \alpha}(N(s - t)^\alpha)[N\mu(0) - Nq(0, \mu(0))]
$$

$$
+ Nq(t, y(t)) + Kg(t) + Bu(t) dt + \int_0^s (s - t)^{\alpha - 1}E_{\alpha, \alpha}(N(T - t)^\alpha)
$$

$$
\times f(t, z(t), z(t, \delta), \int_0^t p(t, r, z(r), z(r, \delta)) dr + \int_0^s (s - t)^{\alpha - 1}E_{\alpha, \alpha}(N(T - t)^\alpha)
$$

$$
\times \sigma(t, z(t), z(t, \delta), \int_0^t h(t, r, z(r), z(r, \delta)) dr) dt
$$

(5)

Let $N_1 = \sup_{0 \leq s \leq T} \| E_{\alpha, \alpha}(N s^\alpha) \|^2$ and $N_2 = \sup_{0 \leq s \leq T} \| E_{\alpha, \alpha}(N^{-\alpha}) \|^2$.

**Assumption (A3):** Let $\rho_1 = \frac{8T^2\alpha}{3\alpha - 1}N_2(M_2T^{-1} + M_2M_4 + M_1M_3T + M_0)$, where $0 \leq \rho_1 < 1$. 

Theorem 3.1  Assumptions (A1) – (A3) are fulfilled and if equation (4) is controllable, then equation (3) is controllable.

Proof. Let \( y_1 \in X \) be arbitrary. Define the operator \( \psi \) on \( \mathcal{H}_2 \) by

\[
\psi(y) = \mu(0) - q(0,\mu(0)) + q(s,y(s)) + \int_0^s (s-t)^{-\alpha}E_{\alpha,\alpha}(N(s-t)^\alpha)[N\mu(0) - Nq(0,\mu(0))
+ Nq(t,y(t))]dt
+ Ky(t-\delta) + Bu(t)[dt + \int_0^s (s-t)^{-\alpha}E_{\alpha,\alpha}(N(s-t)^\alpha)
\times f(t,z(t),z(t-\delta),\int_0^t p(t,r,z(r),z(r-\delta))dr)dt + \int_0^s (s-t)^{-\alpha}E_{\alpha,\alpha}(N(s-t)^\alpha)
\times \sigma(t,z(t),z(t-\delta),\int_0^t h(t,r,z(r),z(r-\delta))dr)dw(t)\]

(6)

\( W_T \) is invertible (see [10]).

Let

\[
u(s) = (T-s)^{-\alpha}B^*E_{\alpha,\alpha}(N^*(T-s)^\alpha)G\{W_T^{-1}(y_1 - \mu(0) - q(0,\mu(0)) + q(s,y(s))
+ \int_0^T (T-t)^{-\alpha}E_{\alpha,\alpha}(N(T-t)^\alpha)[N\mu(0) - Nq(0,\mu(0)) + Nq(t,y(t))]dt
+ Ky(t-\delta) + Bu(t)[dt + \int_0^T (T-t)^{-\alpha}E_{\alpha,\alpha}(N(T-t)^\alpha)f(t,z(t),z(t-\delta),\int_0^t p(t,r,z(r),z(r-\delta))dr)dt
+ \int_0^T (T-t)^{-\alpha}E_{\alpha,\alpha}(N(T-t)^\alpha)\sigma(t,z(t),z(t-\delta),\int_0^t h(t,r,z(r),z(r-\delta))dr)dw(t))|F_s\}
\]

clearly, \( \psi(y(T)) = y_1 \). To prove that \( \psi \) maps \( \mathcal{H}_2 \) into itself.

\[
\sup_{s \in J} G \| u(s) \|^2 = \sup_{s \in J} \| (T-s)^{-\alpha}B^*E_{\alpha,\alpha}(N^*(T-s)^\alpha)G\{W_T^{-1}(y_1 - \mu(0) - q(0,\mu(0)) + q(s,y(s))
+ \int_0^T (T-t)^{-\alpha}E_{\alpha,\alpha}(N(T-t)^\alpha)[N\mu(0) - Nq(0,\mu(0)) + Nq(t,y(t))]dt
+ Ky(t-\delta) + Bu(t)[dt + \int_0^T (T-t)^{-\alpha}E_{\alpha,\alpha}(N(T-t)^\alpha)f(t,z(t),z(t-\delta),\int_0^t p(t,r,z(r),z(r-\delta))dr)dt
+ \int_0^T (T-t)^{-\alpha}E_{\alpha,\alpha}(N(T-t)^\alpha)\sigma(t,z(t),z(t-\delta),\int_0^t h(t,r,z(r),z(r-\delta))dr)dw(t))\} \|^2
\]

\[
\leq \| L_T^\alpha \|^2 \| W_T^{-1} \|^2 \| G \| y_1 \|^2 + G \| f(0) \|^2 + G \| q(0,\mu(0)) \|^2 + G \| q(s,y(s)) \|^2
+ N_2[G \| N\mu(0) \|^2 + G \| Nq(0,\mu(0)) \|^2 + G \| Nq(t,y(t)) \|^2 + G \| Ky(t) \|^2]
+ N_2 \int_0^T (T-t)^{-\alpha}dt \int_0^T (T-t)^{-\alpha}E \| f(t,z(t),z(t-\delta),\int_0^t p(t,r,z(r),z(r-\delta))dr)dt + N_2 \int_0^T (T-t)^{-\alpha}dt \int_0^T (T-t)^{-\alpha} \]
\[
\times G \| \sigma(t,z(t),z(t-\delta),\int_0^t h(t,r,z(r),z(r-\delta))dr) \|^2 ds
\]

\[
\leq \| L_T^\alpha \|^2 \| W_T^{-1} \|^2 \| G \| y_1 \|^2 + G \| \mu(0) \|^2 + G \| q(0,\mu(0)) \|^2
+ M_0 \| y(t) \|^2 + M_{10} + N_1 M \frac{T^{2\alpha}}{2\alpha - 1} + N_1 M \frac{T^{2\alpha-1}}{2\alpha - 1} = K_1 < \infty.
\]
where

\[ M = (M_1M_3 + M_1) \sup_{s \in J} G \| y(t) \|^2 + (M_1M_3 + M_1) \sup_{s \in J} G \| y(s - \delta) \|^2 + M_1M_9 + M_1M_T < \infty. \]

and

\[ M' = (M_2M_4 + M_2) \sup_{s \in J} G \| y(s) \|^2 + (M_2M_4 + M_2) \sup_{s \in J} G \| y(s - \delta) \|^2 + M_2M_8 + M_1M_9 < \infty. \]

using the assumptions, we get

\[
\sup_{s \in J} G \| \mu_y(s) \|^2 \leq 4G \| \mu(0) \|^2 + 4G \| \mu_1 \|^2 + 4M_10 + M_8 \sup_{s \in J} G \| y(s) \|^2 + 4N_2G \| K_y(t - \delta) \|^2
\]

+ \[4N_2K_1 \| B \| \frac{T^{2\alpha}}{2\alpha - 1} + 4N_2M \frac{T^{2\alpha}}{2\alpha - 1} + 4N_2M' \frac{T^{2\alpha}}{2\alpha - 1}.\]

Hence \( \psi \) maps \( \mathcal{H}_2 \) into itself. Let \( y_1, y_2 \in \mathcal{H}_2 \), we have

\[
\sup_{s \in J} G \| \psi y_1(s) - \psi y_2(s) \|^2 = \sup_{s \in J} G \left\| \int_0^s (s - t)^{\alpha - 1} E_{\alpha,\alpha}(n(s - t)^{\alpha})BL_\gamma W_T^{-1} \left\{ \int_0^T (T - \theta)^{\alpha - 1} \right. \right.
\]

\[
\times E_{\alpha,\alpha}(N(T - \theta)^{\alpha})[f(\theta, y_1(\theta), y_1(\theta - \delta), \int_0^\theta p(\theta, r, y_1(r), y_1(r - \delta))dr) - f(\theta, y_2(\theta), y_2(\theta - \delta), \int_0^\theta p(\theta, r, y_2(r), y_2(r - \delta))dr)]d\theta
\]

\[
+ \int_0^T (T - \theta)^{\alpha - 1} E_{\alpha,\alpha}(N(T - \theta)^{\alpha})[\sigma(\theta, y_1(\theta), y_1(\theta - \delta), \int_0^\theta h(\theta, r, y_1(r), y_1(r - \delta))dr)]d\theta + \int_0^s (s - t)^{\alpha - 1} \]

\[
\times E_{\alpha,\alpha}(N(s - t)^{\alpha})[k y_1(t - \delta) - k y_2(t - \delta) + g(t, y_1(t)) - g(t, y_2(t))]dt + \int_0^s (s - t)^{\alpha - 1} E_{\alpha,\alpha}(N(s - t)^{\alpha})[f(t, y_1(t), y_1(t - \delta), \int_0^t p(t, r, y_1(r), y_1(r - \delta))dr)]dt
\]

\[
+ \int_0^s (s - t)^{\alpha - 1} E_{\alpha,\alpha}(N(s - t)^{\alpha})[\sigma(t, y_1(t), y_1(t - \delta), \int_0^t h(t, r, y_1(r), y_1(r - \delta))dr)]d(t - \delta),
\]

\[
\times \int_0^t h(t, r, y_2(r), y_2(r - \delta))dr - \sigma(t, y_2(t), y_2(t - \delta), \int_0^t h(t, r, y_2(r), y_2(r - \delta))dr)]dW(t) \right\|^2
\]

\[
\leq \frac{8T^{2\alpha}}{2\alpha - 1} N_2(M_2T^{-1} + M_2M_4 + M_1M_3T + M_9) \sup_{s \in J} G \| y_1(s) - y_2(s) \|^2
\]

\[
+ \sup_{s \in J} G \| y_1(s - \delta) - y_2(s - \delta) \|^2
\]

\[
\leq \rho_1 \| y_1 - y_2 \|^2_{\mathcal{H}_2}
\]

By (A3) \( \psi \) is a contraction mapping. So, there exists a unique fixed point \( y \in \mathcal{H}_2 \) for \( \psi \). Any fixed point of \( \psi \) satisfies \( y(T) = y_1 \) for arbitrary \( y_1 \in Y \). Hence equation (3) is controllable on \( J \).
4 Controllability of Neutral Stochastic Fractional Systems with delay Implicit Derivative

Consider the following non-linear stochastic fractional system with delay,

\[ C D^\alpha (y(s) - q(s,y(s))) = N y(s) + K y(s - \delta) + Bu(s) + f(s, y(s), y(s-\delta), C D^\alpha y(s), C D^\alpha y(s-\delta)) + \sigma(s, y(s), y(s-\delta)) \frac{dw(s)}{ds}, \quad s \in J = [0, T] \]

\[ y(s) = \mu(s), \quad s \in [-\delta, 0] \]

where \( N : Y \to Y \) and \( K : Y \to Y \) are bounded linear operators, \( B : U \to Y \) is a bounded linear operator, \( q : J \times Y \times Y \to Y \) is continuous function and \( q \) is continuously differentiable, \( u \in \mathbb{R}^n \) is the control input, \( \frac{1}{2} < \alpha \leq 1 \) and the non linear function \( f : J \times Y \times Y \times Y \to Y, \sigma : J \times Y \times Y \to L^0_{\mathbb{F}_T} \) are continuous.

**Assumption (A4):** Assume that there exists constants \( R_i > 0, i = 1, 2 \) such that

\[ \| f(s, y_1, y_2, y_3, y_4) - f(s, z_1, z_2, z_3, z_4) \|^2 \leq R_1 \sum_{i=1}^4 \| y_i - z_i \|^2 \]

\[ \| \sigma(s, y_1, y_2) - \sigma(s, z_1, z_2) \|^2 \leq R_2 \sum_{i=1}^2 \| y_i - z_i \|^2 \]

for any \( s \in J \) and \( y_i \in Y, z_i \in Y (i = 1, 2, 3, 4) \).

Let \( R_3 = \sup_{s \in J} \| f(s, 0, 0, 0) \|, \quad R_4 = \sup_{s \in J} \| \sigma(s, 0) \| \).

For each fixed \( z \in \mathcal{H}^\alpha_T \), consider the corresponding linear system of (7) as

\[ C D^\alpha (y(s) - q(s,y(s))) = N y(s) + K y(s - \delta) + Bu(s) + f(s, z(s), z(s-\delta), C D^\alpha z(s), C D^\alpha z(s-\delta)) + \sigma(s, z(s), z(s-\delta)) \frac{dw(s)}{ds}, \quad s \in J = [0, T] \]

\[ y(s) = \mu(s), \quad s \in [-\delta, 0] \]

The solution of (8) is written by

\[ y(s) = \mu(0) - q(0, \mu(0)) + q(s, y(s)) + \int_0^s (s-t)^{\alpha-1} E_{\alpha,\alpha} (N(s-t)^\alpha)[N\mu(0) - Nq(0, \mu(0))] \]

\[ + Nq(t, y(t)) + K y(t - \delta) + Bu(t) + \int_0^s (s-t)^{\alpha-1} E_{\alpha,\alpha} (N(s-t)^\alpha) \]

\[ \times f(t, z(t), z(t-\delta), C D^\alpha z(s), C D^\alpha z(s-\delta)) dt \]

\[ + \int_0^s (s-t)^{\alpha-1} E_{\alpha,\alpha} (N(s-t)^\alpha) \sigma(t, z(t), z(t-\delta))dw(t) \]

**Assumption (A5):** Let \( \rho_2 = \max \{ \frac{\beta^2}{2\alpha-1} N_2 (R_2 T^{-1} + R_1), \frac{\beta^2}{2\alpha-1} N_2 R_1 \} \) where \( 0 \leq \rho < 1 \).

**Theorem 4.1** Assumptions (A1) – (A3) are fulfilled and if equation (7) is controllable. Then equation (8) is controllable.
Proof. Let $y_1 \in X$ be arbitrary. Define the operator $\psi$ on $\mathcal{H}_2$ by

$$
\psi(y) = \mu(0) - q(0, \mu(0)) + q(s, y(s)) + \int_0^s (s - t)^{\alpha - 1} E_{\alpha, \alpha}(N(s - t)^\alpha)[N\mu(0) - Nq(0, \mu(0))]
$$

$$
+ Nq(t, y(t)) + Ky(t - \delta) + Bu(t) dt + \int_0^s (s - t)^{\alpha - 1} E_{\alpha, \alpha}(N(s - t)^\alpha)
\times f(t, z(t), z(t - \delta), C D^\alpha z(s), C D^\alpha z(s - \delta)) dt
$$

$$
+ \int_0^s (s - t)^{\alpha - 1} E_{\alpha, \alpha}(N(s - t)^\alpha) \sigma(t, z(t), z(t - \delta)) dw(t)
$$

So $W_T$ is invertible (see [10]).

Define

$$
u(s) = (T - s)^{\alpha - 1} B^* E_{\alpha, \alpha}(N^*(T - s)^\alpha) G \{ (W_T^{-1}(y_1 - \mu(0)) - q(0, \mu(0))) + (s, y(s))
$$

$$+ \int_0^T (T - t)^{\alpha - 1} E_{\alpha, \alpha}(N(T - t)^\alpha)[N\mu(0) - Nq(0, \mu(0))] + Nq(t, y(t)) + Ky(t - \delta)] dt
$$

$$+ \int_0^T (T - t)^{\alpha - 1} E_{\alpha, \alpha}(N(T - t)^\alpha) f(t, z(t), z(t - \delta), C D^\alpha z(s), C D^\alpha z(s - \delta)) dt
$$

$$+ \int_0^T (T - t)^{\alpha - 1} E_{\alpha, \alpha}(N(T - t)^\alpha) \sigma(t, z(t), z(t - \delta)) dw(t) \} \mathcal{F}_s
$$

clearly, $\psi(\nu(T)) = y_1$. To prove that $\psi$ maps $\mathcal{H}_2$ into itself.

$$
\sup_s G \| u(s) \|^2 = \sup_s \| (T - s)^{\alpha - 1} B^* E_{\alpha, \alpha}(n^*(T - s)^\alpha) G \{ (W_T^{-1}(y_1 - \mu(0)) - q(0, \mu(0))) + (s, y(s))
$$

$$+ \int_0^T (T - t)^{\alpha - 1} E_{\alpha, \alpha}(N(T - t)^\alpha)[N\mu(0) - Nq(0, \mu(0))] + Nq(t, y(t))
$$

$$+ Ky(t - \delta)] dt + \int_0^T (T - t)^{\alpha - 1} E_{\alpha, \alpha}(N(T - t)^\alpha)
\times f(t, z(t), z(t - \delta), C D^\alpha z(s), C D^\alpha z(s - \delta)) dt
$$

$$+ \int_0^T (T - t)^{\alpha - 1} E_{\alpha, \alpha}(N(T - t)^\alpha) \sigma(t, z(t), z(t - \delta)) dw(t) \} \|^2
$$

$$\leq \| L_T^C \|^2 G \| W_T^{-1} \|^2 \| G \| y_1 \|^2 + G \| \mu(0) \|^2 + G \| q(0, \mu(0)) \|^2 + G \| q(s, y(s)) \|^2
$$

$$+ N_2 G \| N\mu(0) \|^2 + G \| Nq(0, \mu(0)) \|^2 + G \| Nq(t, y(t)) \|^2
$$

$$+ G \| Ky(t - \delta) \|^2] + N_2 \int_0^T (T - t)^{\alpha - 1} dt \int_0^T (T - t)^{\alpha - 1}
\times f(t, z(t), z(t - \delta), C D^\alpha z(s), C D^\alpha z(s - \delta)) dt
$$

$$+ N_2 \int_0^T (T - t)^{\alpha - 1} dt \int_0^T (T - t)^{\alpha - 1} E \| \sigma(t, z(t), z(t - \delta)) \|^2 dt
$$

$$\sup_s G \| u(s) \|^2 \leq \| L_T^C \|^2 G \| W_T^{-1} \|^2 \| G \| y_1 \|^2 + G \| \mu(0) \|^2 + G \| q(0, \mu(0)) \|^2
$$

$$+ M_9 \| y(t) \|^2 + M_{10}] + N_1 M \frac{T^{2\alpha}}{2\alpha - 1} + N_1 M' \frac{T^{2\alpha - 1}}{2\alpha - 1} = K_2 < \infty.
$$

where

$$M = \sup_{s \in J} G \| y(s) \|^2 + G \| y(s - \delta) \|^2 + G \| C D^\alpha z(s) \|^2 + G \| C D^\alpha z(s - \delta) \|^2 + R_3 < \infty.
$$

and 

$$M' = \sup_{s \in J} E \| y(s) \|^2 + R_4 < \infty.$$
using the assumptions, we get
\[
\sup_{s \in J} G \| \psi_y(s) \|^2 \leq 4G \| \mu(0) \|^2 + 4G \| y_1 \|^2 + 4M_{10} + M_0 \sup_{s \in J} G \| x(s) \|^2 + 4N_2 \| Ky(t - \delta) \|^2 \\
+ 4N_2 K_2 \| B \|^2 \frac{T^{2\alpha}}{2\alpha - 1} + 4N_2 M \frac{T^{2\alpha}}{2\alpha - 1} + 4N_2 M' \frac{T^{2\alpha}}{2\alpha - 1}
\]
Hence \( \psi \) maps \( \mathcal{H}_2 \) into itself. Let \( y_1, y_2 \in \mathcal{H}_2 \), we have
\[
\sup_{s \in J} G \| \psi_y(s) - \psi_{y_2}(s) \|^2 \leq \frac{8T^{2\alpha}}{2\alpha - 1} \sup_{s \in J} G \| y_1(s) - y_2(s) \|^2 \\
+ \sup_{s \in J} G \| y_1(s - \delta) - y_2(s - \delta) \|^2 \\
+ \frac{8T^{2\alpha}}{2\alpha - 1} \sup_{s \in J} G \| C^D y_1(s) - C^D y_2(s) \|^2 \\
+ \sup_{s \in J} G \| C^D y_1(s - \delta) - C^D y_2(s - \delta) \|^2 \\
\leq \rho_2 \| y_1 - y_2 \|_{\mathcal{H}_2}^2
\]
By (A4)\( \psi \) is a contraction mapping. So, there exists a unique fixed point \( y \in \mathcal{H}_2 \) for \( \psi \). Any fixed point of \( \psi \) satisfies \( y(T) = y_1 \) for arbitrary \( y_1 \in Y \). Hence the system (7) is controllable on \( J \).

5 Example

Example 5.1 Consider the nonlinear stochastic neutral fractional system for \( s \in [0, 1] \)
\[
C^D y_1(t) = \left( \begin{array}{c} 0.3 \\ -1.8 \\ 0.9 \end{array} \right) + \left( \begin{array}{c} 1 \ln(\cosh(y_1)) \\ e^{-s} \| x(s) \| \end{array} \right) \frac{dw(s)}{ds}
\]
\[
y(s) = \left( \begin{array}{c} 0.3 \\ 1.5 \\ -0.9 \end{array} \right),
\]
for \( s \in J \) and \( 0 < \alpha < 1 \). In matrix form
\[
A = \left( \begin{array}{c} 0.3 \\ 1.5 \\ -0.9 \end{array} \right),
\]
Here

\[ y(s) = \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix}, \sigma(s, y(s)) = \begin{pmatrix} Io(cosb(y_1)) \\ \tan^{-1}(y_2) \end{pmatrix}, \text{and } \alpha = 0.6. \]

Our aim is to steer the solution from the initial point \( x(0) \) to the final point \( x(1) \). To show the nonlinear system (11) is controllable it is enough to check the hypotheses of Theorem (4.1) are satisfied.

The operator

\[ W_T = \begin{pmatrix} 29.886 & 34.8387 \\ 34.8387 & 40.7205 \end{pmatrix}. \] (12)

Now consider

\[ \langle W_T y, y \rangle = 29.886y_1^2 + 69.6774y_1y_2 + 40.7205y_2^2 \geq \gamma (y_1^2 + y_2^2) \]

where \( 0 < \gamma \leq 0.0009 \). \( \sigma(s, y(s)) \) is Lipschitz continuous with \( M_1 = \frac{1}{300} \). The value of \( \rho = 0.7859 < 1 \). Hence the system (11) is controllable.

6 Conclusion

This paper executes the idea of controllability criteria of neutral type stochastic fractional differential systems with delays. By using the controllability Grammian matrix and the Mittag-Leffler matrix function the solution representation were obtained and the controllability results for the addressed system is proved using Banach Contraction principle. Additionally a numerical example is provided to justify the obtained theoretical results.

References

[1] A.A Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam 2006.

[2] S.A. Ammour, S. Djennoune, M. Bettayeb, A sliding mode control for linear fractional systems with input and state delays, Communications in Nonlinear Science and Numerical Simulation 14 (5) (2009) 2310-2318.

[3] B. Bonilla, M. Rivero, L. Rodriguez-Germa, J.J. Trujillo, Fractional differential equations as alternative models to nonlinear differential equations, Applied Mathematics and Computation 187 (2007) 79-88.

[4] K. Balachandran, S. Divya, L. Rodriguez-Germa, J.J. Trujillo, Relative controllability of nonlinear neutral fractional integro-differential systems with distributed delays in control, Mathematical methods in the Applied Sciences November 2014.

[5] I. Gyri, J. Wu, A neutral equation arising from compartmental systems with pipes, Journal of Dynamics and Differential Equations 3 (2) (1991) 289-311.

[6] M. Kamrani, Numerical Solution of Stochastic Fractional Differential Equations, Numerical Algorithms 68 (2015) 81-93.

[7] J. Klamka, Constrained Controllability of Semilinear Systems, Nonlinear Analysis 47 (2001) 2939-2949.
[8] J.Klamka, Controllability of Dynamical Systems: A Survey of Bulletin of the Polish Academy of Sciences Technical Sciences 61 (2013) 221-229.

[9] E.Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons Inc, New York (1978).

[10] J.Klamka, Stochastic Controllability and minimum energy control of systems with multiple delays in control, Applied Mathematics and Computation 206 (2008) 704-715.

[11] V.Lakshmikantham, A.S.Vatsala, Basic theory of fractional differential equations, Nonlinear Analysis 69 (2008) 2677-2682.

[12] R.Mabel Lizzy, K.Balachandran, M.Suvinthra, Controllability of nonlinear stochastic fractional systems with distributed delays in control, Journal of Control and Decision, 4 (3) (2017) 153-167.

[13] N.I.Mahmudov, Controllability of Linear Stochastic Systems in Hilbert Spaces, Journal of Mathematical Analysis and Applications 259 (2001) 64-82.

[14] N.I.Mahmudov, Controllability of Semilinear Stochastic Systems in Hilbert Spaces, Journal of Mathematical Analysis and Applications 288 (2003) 197-211.

[15] K.S.Miller, B.Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[16] I.Podlubny, Fractional Differential Equations of Mathematics in Science and Engineering, Technical University of Kosice, Kosice, Slovak Republic (198) 1999.

[17] R.Sakthivel, Y.Ren, Approximate controllability of fractional differential equations with state-dependent delay, Results in Mathematics 63 (2013) 949-963.

[18] R.Sakthivel, S.Suganya, S.M.Anthoni, Approximate controllability of fractional stochastic evolution equations, Computers and Mathematics with Applications 63 (2012) 660668.

[19] X.Zhang, C.Zhu, C.Yuan, Approximate controllability of impulsive fractional stochastic differential equations with state-dependent delay, Advances in Difference Equations 2015 (91) (2015) DOI 10.1186/s13662-015-0412-z.