Charged dilaton black hole with multiple Liouville potentials and gauge fields

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Abstract

A solution to an Einstein–Maxwell–dilaton-type theory with $M$ Liouville potentials and $N$ gauge fields is presented, where $M$ and $N$ are arbitrary integers. This exact solution contains, as special cases, the Lifshitz black hole and the topological dilaton black hole. The thermodynamic behaviour of the solution is found to be similar to that of the Lifshitz black hole, where a phase transition may occur for sufficiently small charge in the canonical ensemble, or sufficiently small potential in the grand canonical ensemble.

1 Introduction

In the various applications of General Relativity, spacetimes which are non-asymptotically flat have gained increasing interest. Perhaps the most notable is the asymptotically Anti-de Sitter spacetimes which are central to the gauge/gravity correspondence, string theory, and quantum gravity. Extensions of these and other related ideas have subsequently led to spacetimes of other asymptotics.

More recently, attention has been turned to spacetimes which may serve as a gravitational dual to non-relativistic field theories [1,2]. (For a review and references, see the reviews by Taylor [3,4] and Park [5].) Such theories require an anisotropic scaling between space and time in the form

$$t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x},$$

(1)

where $t$ denotes the time coordinate and $\vec{x}$ denotes the $(d - 1)$-dimensional spatial coordinates. The spacetime which provides such a scaling at its boundary is the Lifshitz
spacetime \cite{2} and its subsequent inclusion of a black hole \cite{6} provides a dual to finite-temperature non-relativistic field theories.

The analytical black hole solutions in Lifshitz spacetimes require the support of various matter fields. The approach adopted by \cite{2,6–8} is to use a coupled one-form and two-form gauge field. The Einstein-Proca theory was used in \cite{9,10}. The Lifshitz black hole can also be constructed in Lovelock gravity \cite{11} and higher-curvature gravity \cite{12,13}.

Of relevance to this paper is the thread where an Einstein-Maxwell-dilaton (EMd)-type action is used. In Ref. \cite{14}, an EMd-type action with multiple $U(1)$ gauge fields was used to construct a charged spherical black hole in Lifshitz spacetime. There, the authors have found that distinct $U(1)$ fields are required to support the various curvature structures of the solution. In particular, one $U(1)$ field is necessary to support the Lifshitz spacetime, even in the absence of the black hole \cite{3}, a second $U(1)$ field is needed to support a spherical topology of the horizon, and finally a third gauge field is used to charge up the black hole itself. Subsequent inclusions of a fourth gauge field onwards will then contribute to different $U(1)$ charges on the black hole. Hence, out of $N$ distinct gauge fields, two of them are locked by the structure of the spacetime, while the remaining $N − 2$ are freely parameterised (within constraints) charges carried by the black hole itself. The holographic and thermodynamic consequences of these solutions were studied in \cite{15,16}, in addition to the original analysis performed by \cite{14}.

Another kind of non-asymptotically flat black hole solution that was well studied are solutions to EMd-theories in which the dilaton has a non-trivial potential. When the potential has a Liouville (exponential) form, exact solutions were found by Chan et al. \cite{17}, where they considered the potential which is a sum of up to two Liouville terms. A similar intuition can be carried over from the Lifshitz black hole with multiple gauge fields, namely that each Liouville potential is used to ‘support’ a particular aspect of the spacetime curvature. Further properties and thermodynamics of these solutions were studied by \cite{18–20}. More general features of spacetimes under this potential were considered in Refs. \cite{21–23}.

In the present paper, we wish to study an EMd-like framework that unifies the Lifshitz and dilaton black holes. This is straightforwardly achieved by combining the ingredients of the aforementioned Lifshitz and dilaton solutions into a single action. Curiously, the asymptotic behaviour of the dilaton spacetimes appear very similar to those of Lifshitz, though such solutions have not been considered by the non-relativistic holography literature. A special case of the solution presented in this paper contains the Lifshitz and dilaton black hole as special cases, and comes with a parameter $\nu$ that interpolates between the two.

More specifically, we solve an EMd-type action with $M$ Liouville potentials and $N$ gauge fields. Just as in the Lifshitz and dilaton black holes, each of the $M + N$ ingredients are used to support various curvature structures of the spacetime, which are reflected by
the different terms of the metric component $g_{tt}$. Two of the notable terms are related to the spherical topology of the horizon and another to its asymptotic structure. In general, a gauge field and a Liouville potential may ‘co-operatively support’ the same term. For Lifshitz holography, this may possibly alleviate a problematic issue of a gauge field diverging at the boundary by replacing it with a Liouville potential.

This paper is organised as follows. In Sec. 2, the equations of motion are presented and the exact solution is derived. Some physical and geometrical properties of the solution will be studied in Sec. 3, followed by an elementary thermodynamic analysis in Sec. 4. The paper concludes in Sec. 5.

2 Derivation of the solution

We consider an Einstein-Maxwell-dilaton-type theory consisting of $N$ $U(1)$ gauge fields and a dilaton potential that is a sum of $M$ exponential terms,

$$I = \frac{1}{16\pi G} \int_M d^Dx \sqrt{-g} \left( R - (\nabla \varphi)^2 - \sum_{i=1}^N e^{-\alpha_i \varphi} F_i^2 - V(\varphi) \right) + \frac{1}{8\pi G} \int_{\partial M} d^{D-1}x \sqrt{-\gamma} K,$$

(2)

where the boundary $\partial M$ is taken to be the hyper-surface orthogonal to a space-like normal $n^\mu$, and $\gamma_{\mu\nu}$ is the induced metric on the boundary. The extrinsic curvature is then $K_{\mu\nu} = \gamma_{\mu\lambda} \nabla^\lambda n_\nu$. Here we have denoted $F_i^2 = (F_i)_{\mu\nu}(F_i)^{\mu\nu}$, with each of the two-form fields $F_i$ arise from the exterior derivative of their respective one-form potentials, $F_i = dA_i$. Each of these fields are coupled to the dilaton $\varphi$ via their respective coupling parameters $\alpha_i$.

We shall consider potentials $V(\phi)$ where it is a sum of $M$ exponential terms,

$$V(\varphi) = \sum_{j=1}^M 2\Lambda_j e^{2\beta_j \varphi}.$$

(3)

Extremising the action gives the Einstein-Maxwell-dilaton equations

$$R_{\mu\nu} = \nabla_\mu \nabla_\nu \varphi + \frac{1}{D-2} V g_{\mu\nu} + \sum_{i=1}^\infty \left[ 2e^{-2\alpha_i \varphi} (F_i)_{\mu\lambda}(F_i)^{\lambda\nu} - \frac{1}{D-2} e^{-2\alpha_i \varphi} F_i^2 g_{\mu\nu} \right],$$

(4)

$$\nabla_\mu (e^{-2\alpha_i \varphi} F_i^{\mu\nu}) = 0,$$

(5)

$$\nabla^2 \varphi = \frac{1}{2} \frac{dV}{d\varphi} - \sum_{i=1}^N \alpha_i e^{-2\alpha_i \varphi} F_i^2.$$

(6)
To solve these equations, we shall begin with an ansatz of the form

\[
\begin{align*}
\text{ds}^2 &= -f(r) \text{d}t^2 + h(r) \text{d}r^2 + r^2 \tilde{g}_{ab} \text{d}\sigma^a \text{d}\sigma^b, \\
A_i &= \chi_i(r) \text{d}t, \quad i = 1, \ldots, N, \\
\varphi &= \varphi(r),
\end{align*}
\]

where \(f(r), h(r), \chi_i(r),\) and \(\varphi(r)\) are functions that depend on \(r\) only. The metric \(\tilde{g}_{ab}\) is a \((D - 2)\)-dimensional space of constant unit curvature \(k = 0, \pm 1\). We shall keep \(k\) unspecified for most of our derivation, though we will mainly be interested in black holes of spherical topology, for which \(k = 1\). In this form, the equations of motion become

\[
\begin{align*}
- \frac{1}{2\sqrt{f}h} \left( \frac{f'}{\sqrt{f}h} \right)' - \frac{(D - 2)f'}{2rh} &= \sum_{j=1}^{\Lambda} \frac{2\Lambda_j e^{2\beta_j \varphi}}{D - 2} - \sum_{i=1}^{N} \frac{2(D - 3)}{D - 2} \frac{1}{h} e^{-2\alpha_i \varphi} \chi_i^2, \\
- \frac{1}{2\sqrt{f}h} \left( \frac{h'}{\sqrt{f}h} \right)' + \frac{(D - 2)h'}{2rh^2} &= \sum_{j=1}^{\Lambda} \frac{2\Lambda_j e^{2\beta_j \varphi}}{D - 2} - \sum_{i=1}^{N} \frac{2(D - 3)}{D - 2} \frac{1}{h} e^{-2\alpha_i \varphi} \chi_i^2 + \frac{1}{h}\varphi'^2,
\end{align*}
\]

\[
\begin{align*}
\frac{D - 3}{r^2} & \left( k - \frac{1}{h} \right) + \frac{1}{2rh} \left( \frac{h'}{h} - \frac{f'}{f} \right) = \sum_{j=1}^{\Lambda} \frac{2\beta_j \Lambda_j e^{2\beta_j \varphi}}{D - 2} + \sum_{i=1}^{N} \frac{2}{D - 2} \frac{1}{h} e^{-2\alpha_i \varphi} \chi_i^2, \\
\left( \sqrt{\frac{r^{D-2}}{f} e^{-2\alpha_i \varphi} \chi_i} \right)' &= 0, \quad i = 1, \ldots, N \\
- \frac{1}{\sqrt{f}h} \sqrt{r^{D-2}} \varphi' &= \sum_{j=1}^{\Lambda} 2\beta_j \Lambda_j e^{2\beta_j \varphi} + \sum_{i=1}^{N} \frac{2\alpha_i}{h} e^{-2\alpha_i \varphi} \chi_i^2.
\end{align*}
\]

Taking the difference between (10) and (11) gives

\[
\frac{D - 2}{2r} \left( \frac{h'}{h} + \frac{f'}{f} \right) = \varphi'^2.
\]

If we further assume

\[
h = \frac{2^n}{f},
\]

for some constant \(n\), then Eq. (15) can be solved to give

\[
\varphi = \pm \delta \ln r + \varphi_0, \quad \delta = \sqrt{(D - 2)n},
\]

where \(\varphi_0\) is an integration constant. The Maxwell equation (13) can then be integrated to give

\[
\chi_i' = \lambda_i e^{2\alpha_i \varphi_0} r^{-(D - 2 - n + 2\alpha_i \delta)},
\]
where $\lambda_i$ are their respective integration constants. What remains is to determine $f$. To this end, let us assume the ansatz

$$f = \sum_{a=0}^{M+N-p} m_a r^{c_a},$$

(19)

where $m_a$ and $c_a$ are constants. Then the remaining Einstein and dilaton equations are

$$- \frac{1}{2} \sum_{a=0}^{M+N-p} (D - 3 - n + c_a)c_a m_a r^{c_a - 2 - 2n} = \sum_{j=1}^{M} \frac{2\Lambda_j e^{2\beta_j \varphi_0}}{D - 2} r^{-2\beta_j \delta} - \sum_{i=1}^{N} \frac{2(D - 3)\lambda_i^2 e^{-2\alpha_i \varphi_0}}{D - 2} r^{-2\alpha_i \delta - 2(D - 2)},$$

(20a)

$$\frac{(D - 3)k}{r^2} - \sum_{a=0}^{M+N-p} (D - 3 - n + c_a)m_a r^{c_a - 2 - 2n} = \sum_{j=1}^{M} \frac{2\Lambda_j e^{2\beta_j \varphi_0}}{D - 2} r^{-2\beta_j \delta} + \sum_{i=1}^{N} \frac{2\lambda_i^2 e^{-2\alpha_i \varphi_0}}{D - 2} r^{-2\alpha_i \delta - 2(D - 2)},$$

(20b)

$$- \sum_{a=0}^{M+N-p} \delta (D - 3 - n + c_a) m_a r^{c_a - n - 2} = \sum_{j=1}^{M} 2\beta_j \Lambda_j e^{2\beta_j \varphi_0} r^{-2\beta_j \delta} + \sum_{i=1}^{N} 2\alpha_i \lambda_i^2 e^{-2\alpha_i \varphi_0} r^{-2\alpha_i \delta - 2(D - 2)}.$$  

(20c)

Our strategy is simply to fix the coefficients and exponents of $r$ so that the equations are solved consistently term-by-term. In particular, let us consider the general possibility that on the right hand sides of Eq. (20), there are $p$ terms involving $\Lambda_l$ and $F_l$ which have common exponent for $r$, hence $M - p$ terms involving only $\Lambda_j$ and $F_l$ which have common exponent for $r$, hence $M - p$ terms involving only $\Lambda_j$ and $F_l$. The total number of terms are $M + N - p$. The number of terms chosen in the ansatz (19) is $M + N - p + 1$, the extra term being the Schwarzschild-like mass term, which we expect to still be present upon switching off all the matter fields.

In practice, only Eqs. (20a) and (20b) are sufficient to completely fix all the constants upon matching the exponents and coefficients of $r$. Then, the results must be checked to be consistent with Eq. (20c). Upon renaming the constants $m_a$ and $c_a$ with more familiar symbols, the result is

$$f = Br^{2n} - \frac{\mu}{r^{D-3-n}} + \sum_{l=1}^{p} \frac{r^{2+2n-2\alpha_l}}{L_l^2} + \sum_{i=p+1}^{N-1} \frac{q_i^2}{r^{2(D-3)}} + \sum_{j=p+1}^{M-1} \frac{r^2}{\ell_j^2},$$

(21)
where the parameters $B$, $L_l$, and $\ell_i$ are related to the Liouville potential strengths by

$$2\Lambda_l e^{2\beta_l \varphi_0} = -(D - 1 + n - 2\nu_l)(D - 2 + n - \nu_l) \frac{L_l^2}{\epsilon_l^2}, \quad \beta_l = \frac{\nu_l}{\sqrt{(D - 2)n}}, \quad l = 1, \ldots, p,$$

$$2\Lambda_j e^{2\beta_j \varphi_0} = -(D - 2)(D - 1 - n) \frac{\ell_j^2}{\rho_j^2}, \quad \beta_j = \sqrt{\frac{n}{D - 2}}, \quad j = p + 1, \ldots, M - 1,$$

$$B = \frac{(D - 3)^2 k}{(D - 3 + n)} + \frac{2\Lambda_M e^{2\beta_M \varphi_0}}{(D - 3 + n)^2}, \quad \beta_M = \frac{1}{\sqrt{(D - 2)n}}. \quad (22a)$$

The gauge potentials and their associated coupling exponents $\alpha_i$ are obtained upon integrating the Maxwell equations again, which gives

$$\chi_l = e^{\alpha_l \varphi_0} \sqrt{\frac{n - \nu_l}{2(D - 1 + n - 2\nu_l)}} \frac{r^{D - 1 + n - 2\nu_l}}{L_l} + \Phi_l, \quad \alpha_l = \frac{\nu_l - (D - 2)}{\sqrt{(D - 2)n}}, \quad l = 1, \ldots, p,$$

$$\chi_i = -e^{-\alpha_i \varphi_0} \sqrt{\frac{D - 2}{2(D - 3 + n)}} q_i r^{-(D - 3 + n)} + \Phi_i, \quad \alpha_i = \sqrt{\frac{n}{D - 2}}, \quad i = p + 1, \ldots, N - 1,$$

$$\chi_N = \frac{e^{\alpha_N \varphi_0}}{\sqrt{2(D - 3 + n)}} \rho N r^{D - 3 + n} + \Phi_N, \quad \alpha_N = -\frac{D - 3}{\sqrt{(D - 2)n}}, \quad (23a)$$

$$\chi_i = \frac{e^{\alpha_i \varphi_0}}{\sqrt{2(D - 3 + n)}} \rho N r^{D - 3 + n} + \Phi_i, \quad \alpha_i = \sqrt{\frac{n}{D - 2}}, \quad i = p + 1, \ldots, N - 1,$$

where we have introduced an abbreviation $\rho$ which is related to $B$ and $\Lambda_M$ via

$$B = \frac{(D - 3)^2 k - 2\Lambda_M e^{2\beta_M \varphi_0}}{(D - 3 + n)^2} = \frac{(D - 3)k - (D - 3 + n)\rho^2}{(1 - n)(D - 3 + n)}. \quad (24)$$

Also, $\Phi_l$, $\Phi_i$, and $\Phi_N$ above are the integration constants upon the final integration of the Maxwell’s equations.

We recover known solutions by the following choices of the parameters:

- **Lifshitz spherical/planar black hole**: The Lifshitz black hole is recovered by setting $\nu_l = 0$ and $\Lambda_j = 0$. The latter condition is tantamount to having $\ell_j^2 \to 0$ and all the terms proportional to $r^2$ in $f$ will vanish. Then, with $\nu_l = 0$, the $p$ gauge fields are redundant since they all give rise to the same term proportional to $r^2 + 2n$, so we might well consider $p = 1$. If we further let $z = n + 1$, the function $f$ can be rewritten as

$$f = \frac{r^{2z}}{L^2} \left[ \frac{(D - 3)^2 k}{(D - 3 + n)^2} \frac{L^2}{r^{D - 2 - z}} + 1 - \frac{\mu}{r^{D - 2 - z}} + \sum_{i=2}^{N-1} \frac{q_i^2}{r^{2(D - 2)}} \right]. \quad (25)$$

We see that the term in the square brackets is precisely the function denoted as $b_k$ in [14] and that $z$ is the familiar Lifshitz exponent. Along with rest of the solution,
we have recovered the Lifshitz spherical/planar black hole derived by [14].

- **The dilaton black hole:** If we set \( \nu_l = n \), then the gauge fields \( \chi_l \) where \( l = 1, \ldots, p \) are switched off. Then, Eq. (22a) becomes the same as Eq. (22b), \( \beta_l = \beta_j = \sqrt{\frac{n}{D-2}} \).

Then \( M - 1 \) terms in the dilaton potential become identical and hence redundant, because

\[
V = \sum_{l=1}^{p} 2\Lambda_l e^{2\beta_l \varphi} + \sum_{i=p+1}^{M-1} 2\Lambda_j e^{2\beta_j \varphi} + 2\Lambda_M e^{2\beta_M \varphi}
= 2 \left( \sum_{j=1}^{p} \Lambda_j + \sum_{i=p+1}^{M-1} \Lambda_i \right) e^{2\sqrt{\frac{n}{D-2}} \varphi} + 2\Lambda_M e^{2\beta_M \varphi}.
\]

(26)

We can then rename \( \sum_{j=1}^{p} \Lambda_j + \sum_{i=p+1}^{M-1} \Lambda_i = \Lambda_1 \), and \( M = 2 \) suffices. We then have

\[
2\Lambda_1 e^{2\beta_1 \varphi_0} = -(D-2)(D-1-n)\ell^{-2}, \quad \beta_1 = \sqrt{\frac{n}{D-2}}.
\]

(27)

Further introducing the transformation

\[
r = \rho^{\frac{1}{1+n}}, \quad t = (1+n)\hat{t},
\]

(28)

the solution becomes

\[
ds^2 = -U d\hat{t}^2 + U^{-1} d\rho^2 + \rho^{\frac{4n}{1+n}} \hat{\gamma}_{ab} d\theta^a d\sigma^b,
\]

(29)

\[
U = (1+n)^2 \left[ B \rho^{\frac{2n}{1+n}} - \mu \rho^{\frac{n-(D-3)}{1+n}} + \sum_{i=1}^{N-1} q_i^2 \rho^{\frac{2(D-3)}{1+n}} + \rho^{\frac{4}{1+n}} \right].
\]

(30)

At the moment, the solution still contains \( N - 1 \) non-zero \( U(1) \) charges. If we set all but one of them to zero, we recover the charged dilaton black hole with two exponential potentials originally obtained in Ref. [17].

The intuition we have learned in solving the equations of motion with this ansatz is that all but two terms of \( f \) in (21) require a source which is either a Liouville potential, gauge field, or both. Explicitly, the terms \( q_i^2 / r^{2(D-3)} \) are supported by a \( U(1) \) gauge field, \( r^2 / \ell_j^2 \) are supported by a Liouville potential, and \( r^{2+2n-2n} / \ell_j^2 \) are supported by both a gauge field and Liouville potential. Turning off any of the fields or potentials appropriately will set its corresponding terms to zero.

The first two terms of \( f \) are \( Br^{2n} - \mu / r^{D-3-n} \), which reduces to the usual Schwarzschild-Tangherlini form \( 1 - \mu / r^{D-3} \) when all the matter fields are turned off. The second term is related to the black hole mass and is not affected by the presence of the matter fields, though the presence of the \( M \)-th potential and \( N \)-th gauge field modifies the particular value of \( B \) in accordance to Eq. (24).
3 Physical and geometrical properties

For concreteness in the rest of the paper, we shall henceforth consider the simplest non-trivial case \( M = N = 3 \), where the solution is explicitly given by

\[
ds^2 = -f dt^2 + r^{2n} f^{-1} dr + r^2 \tilde{\gamma}_{ab} d\sigma^a d\sigma^b,
\]

\[
f = B r^{2n} - \frac{\mu}{r^{D-3-n}} + \frac{r^{2+2n-2\nu}}{L^2} + \frac{\eta^2}{r^{2(D-3)}} + \frac{r^2}{\ell^2},
\]

\[
\varphi = -\sqrt{(D-2)n} \ln r + \varphi_0.
\]

\[
A_1 = \left( e^{\alpha_1 \varphi_0} \sqrt{\frac{n - \nu}{2(D-1+n-2\nu)} \frac{r^{D-1+n-2\nu}}{L} + \Phi_1} \right) dt, \quad \alpha_1 = \frac{\nu - (D-2)}{\sqrt{(D-2)n}}.
\]

\[
A_2 = \left( e^{-\alpha_2 \varphi_0} \sqrt{\frac{D-2}{2(D-3+n)}} q r^{(D-3+n)} + \Phi_2 \right) dt, \quad \alpha_2 = \frac{n}{\sqrt{D-2}},
\]

\[
A_3 = \left( e^{\alpha_3 \varphi_0} \sqrt{\frac{D-2}{2(D-3+n)}} q r^{D-3+n} + \Phi_3 \right) dt, \quad \alpha_3 = -\frac{D-3}{\sqrt{(D-2)n}}.
\]

\[
2\Lambda_1 e^{2\beta_1 \varphi_0} = -(D-1+n-2\nu)(D-2+n-\nu)L^{-2}, \quad \beta_1 = \frac{\nu}{\sqrt{(D-2)n}},
\]

\[
2\Lambda_2 e^{2\beta_2 \varphi_0} = -(D-2)(D-1+n-\ell^2), \quad \beta_2 = \frac{n}{\sqrt{D-2}},
\]

\[
B = \frac{(D-3)^2 k - 2\Lambda_3 e^{2\beta_3 \varphi_0}}{(D-3+n)^2} = \frac{(D-3)k - (D-3+n)\rho^2}{(1-n)(D-3+n)}.
\]

We are keeping \( k \) unspecified at the moment, though we will be mainly interested in horizons of spherical topology for which \( k = 1 \).

Let first account for the independent parameters of this solution. The first two of these are straightforwardly \( L^2 \) and \( \ell^2 \), which fixes \( \Lambda_1 \) and \( \Lambda_2 \) via Eq. (31g) and (31h). If we wish to consider the cases where \( \Lambda_1 \) and/or \( \Lambda_2 \) are positive, we replace \( L^2 \to -L^2 \) and/or \( \ell^2 \to -\ell^2 \). Next we have \( n \) which controls the strength of the dilaton \( \varphi \). The parameter \( \nu \) determines the strength of the gauge potential \( A_1 \) together with \( n \). The parameter \( q \) intertwines the two equalities in (31i). Hence there should be one free parameter among them. \( n \) has already been accounted for. Solving for \( \Lambda_3 \), or \( \rho \), we find

\[
2\Lambda_3 e^{2\beta_3 \varphi_0} = \frac{(D-3+n)^2 \rho^2 - n(D-2)(D-3)k}{1-n}, \quad (32)
\]

\[
\rho^2 = \frac{(1-n)2\Lambda_3 e^{2\beta_3 \varphi_0} + n(D-2)(D-3)k}{(D-3+n)^2}. \quad (33)
\]

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Without loss of generality, let us choose $\Lambda_3$ as the independent parameter. Subsequently $\rho$ will be determined from Eq. (33). With these two quantities will then fix $B$ via Eq. (31).

Finally, $\mu$ is not related to any of the matter fields and is obviously the Schwarzschild-like term responsible for the mass of the black hole. For the purposes of analysis, it will be more convenient to express quantities in terms of the horizon radius $r_+$, where $f(r_+) = 0$ and $f'(r_+) \geq 0$. From this we have

$$\mu = \frac{r_+^{D-1+n-2\nu} L^2}{\ell^2} + \frac{r_+^{D-1-n} \ell^2}{\ell^2} + B r_+^{D-3+n} + q^2 r_+^{(D-3+n)}.$$  (34)

Therefore we shall use $r_+$ to parametrise the mass with (34).

Having the parameters accounted for, we shall regard our solution (31) as being parametrised by the following seven quantities:

$$(r_+, q, \ell^2, \Lambda_3, n, \nu),$$  (35)

where Eq. (31d) indicates that $\nu$ must take the range $\nu \leq n$. Also, Eq. (32) forbids $n = 1$, unless $\Lambda_3 = 0$. In the ranges where all the parameters are well-defined, the function $f$ typically has three roots, one of which is our parameter $r_+$ as intended. The other two are $r_-$ and $r_c$, which are the inner and cosmological horizon, respectively and

$$r_- \leq r_+ \leq r_c,$$  (36)

where $r_-$ is non-zero if $q$ is non-zero and $r_c$ is finite if $\Lambda_1$ and/or $\Lambda_2$ are positive. The spacetime will be static with Lorentzian signature $(-, +, \ldots, +)$ in the range $r_+ < r < r_c$ and $r < r_-$. As we will show below, the latter region contains a curvature singularity at $r = 0$, and hence we shall mainly be interested in physical quantities measurable by observers at $r_+ < r < r_c$, where this range is understood to include the case $r_c = \infty$ for negative $\Lambda_1$ and $\Lambda_2$.

The surface gravity is given by $\kappa = \frac{1}{2} r_+^{-n} f'(r_+)$, which can be derived by the usual trick of removing the conical singularity in the Euclideanised metric where $t \to i\tau$. The result is

$$\kappa = \frac{1}{2} \left[ (D - 1 + n - 2\nu) L^2 r_+^{1+n-2\nu} + (D - 1 - n) \ell^{-2} r_+^{1-n} + (D - 3 + n) B r_+^{1+n} - (D - 3 + n) q^2 r_+^{-(2D-5+n)} \right].$$  (37)

The extremal case occurs when $\kappa = 0$, which is obtained by decreasing the horizon radius

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1This latter condition is to preclude the possibility of another root of $f$ due to a possible cosmological horizon.
to \( r_+ = r_e \), where \( r_e \) satisfies the equation
\[
q^2 r_e^{-(D-3+n)} = \frac{D-1+n-2\nu}{D-3+n} L^{-2} r_e^{D-1+n-2\nu} + \frac{D-1+n}{D-3+n} \epsilon^{-2} r_e^{D-1-n} + Br_+^{D-3+n}. \tag{38}
\]

The corresponding mass parameter in the extremal case is
\[
\mu_e = \frac{2(D-2+n-\nu)}{D-3+n} \epsilon^{-2} r_e^{D-1+n-2\nu} + \frac{2(D-2)}{D-3+n} \epsilon^{-2} r^{D-1-n} + 2B_+^{D-3+n}. \tag{39}
\]

The horizon area, which will be crucial in the thermodynamic analysis in the next section, is
\[
A = r_+^{D-2} \Omega, \tag{40}
\]
where \( \Omega = \int d^{D-2} \sqrt{\gamma} \).

Briefly looking at some curvature invariants, the Ricci and Kretschmann scalars are
\[
R = -\frac{1}{r^n} \left( \frac{f'}{r^n} \right)' + \frac{(D-2)}{r^{1+n}} \left( \frac{2n}{r} f - 2f' \right) + \frac{(D-3)(D-2)}{r} \left( k - \frac{f}{r^n} \right), \tag{41}
\]
\[
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{1}{r^{2n}} \left[ f'' 2n - \frac{2n}{r} f'' f' + \frac{n^2}{r^2} f'^2 \right] + \frac{D-2}{r^{2n+2}} f^2 \left( \frac{2f'^2}{f^2} + \frac{4n^2}{r^2} - \frac{4n f'}{r} \right) + \frac{2(D-2)(D-3)}{r^4} \left( k - \frac{1}{r^{2n}} f' \right)^2, \tag{42}
\]
we find that a curvature singularity occurs for \( r = 0 \), and it persists in the ‘no black hole’ case \( \mu = 0 \), which is similar to the pure Lifshitz spacetime, as well as the zero mass limit of the dilaton black hole with Liouville potentials.

To calculate the mass, we adopt the boundary stress tensor procedure of [24], where we consider a fixed \( r \) boundary \( \partial M \) with a unit normal \( n^\mu = r^{-n} \sqrt{\gamma} \). The induced metric at the boundary is \( \gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \). The boundary stress tensor is the variation of the action with respect to \( \gamma \):
\[
T_{\mu\nu} = \frac{2}{\sqrt{|\gamma|}} \frac{\delta I}{\delta \gamma_{\mu\nu}} = -\frac{1}{8\pi G} (K_{\mu\nu} - K \gamma_{\mu\nu}). \tag{43}
\]
evaluated at some \( r = r_b \). In general, this term diverges as \( r_b \to \infty \), therefore we need to subtract from it an appropriately chosen background. We shall take our background to be the spacetime \( g_{\mu\nu}^0 \) with \( \mu = 0 \). Taking care to ensure that spacetime matches with the background at the boundary, a finite boundary stress tensor is obtained by
\[
\hat{T}_{\mu\nu} = T_{\mu\nu} - T_{\mu\nu}^0, \tag{44}
\]
where \( T_{\mu\nu}^0 \) is the stress tensor associated with the background spacetime. The mass is
then the conserved charge associated with the time-like Killing vector \( \xi^\mu = \delta^\mu_t \),

\[
M = \oint_{\partial M} d^{D-2}x \sqrt{\tilde{g}} \xi^\mu \xi^\nu \tilde{T}_{\mu\nu} = \frac{\Omega}{16\pi G} (D-2) \mu. \tag{45}
\]

In the topological dilaton black hole case with \( \nu = n \), this agrees with the mass obtained by the counter-term method by Cai and Ohta \cite{19}, and also agrees in the Lifshitz case (\( \nu = 0 \)) with the calculation by Tarrio and Vandoren \cite{14} wherein the Komar integral method was used.

In the canonical ensemble which we will consider in the thermodynamic analysis, the charge of the spacetime is fixed. In that case a more appropriate background would be the extremal spacetime \( g^e_{\mu\nu} \). The mass of the black hole measured against this extremal background is then

\[
\Delta M = \sum \frac{\Omega}{16\pi G} (D-2)(\mu - \mu_e). \tag{46}
\]

The respective \( U(1) \) charges are calculated by

\[
Q_i = \frac{1}{4\pi G} \oint d^{D-2}\sigma \sqrt{\gamma} e^{-2\alpha_i^e} (F_i)_{\mu\nu} n^\mu \xi^\nu, \tag{47}
\]

for \( i = 1, 2, \) and \( 3 \). Calculating the Maxwell tensors from the potentials given in Eqs. (31d), (31e) and (31f), the charges are explicitly

\[
Q_1 = \frac{\Omega}{4\pi G} e^{-\alpha_1^e} \sqrt{\frac{1}{2}(n - \nu)(D - 1 + n - 2\nu)} \frac{1}{L}, \tag{48}
\]

\[
Q_2 = \frac{\Omega}{4\pi G} e^{-\alpha_2^e} \sqrt{\frac{1}{2}(D - 2)(D - 3 + n)} q, \tag{49}
\]

\[
Q_3 = \frac{\Omega}{4\pi G} e^{-\alpha_3^e} \sqrt{\frac{1}{2}(D - 3 + n)} \rho. \tag{50}
\]

We shall also fix the respective gauge potentials so that \( A_i \) for each \( i \) are zero at the horizon. This gives

\[
\Phi_1 = -e^{\alpha_1^e} \sqrt{\frac{n - \nu}{2(D - 1 + n - 2\nu)}} L, \tag{51}
\]

\[
\Phi_2 = e^{\alpha_2^e} \sqrt{\frac{D - 2}{2(D - 3 + n)}} q, \tag{52}
\]

\[
\Phi_3 = -e^{\alpha_3^e} \sqrt{\frac{1}{2}(D - 3 + n)} L. \tag{53}
\]

It can be checked that the physical quantities discussed above satisfies the first law of
black-hole mechanics
\[ dM = \frac{\kappa dA}{8\pi G} + \Phi_2 dQ_2, \quad \text{(54)} \]
which is just the first law of thermodynamics to be used in the next section. Here, we have followed the reasoning of [14] in that the terms \( \Phi_1 dQ_1 \) and \( \Phi_3 dQ_3 \) is not present as these quantities are fixed by the asymptotic structure of the spacetime and is not expected to participate in the mechanics of the black hole itself.

4 Thermodynamics

4.1 The thermodynamic variables
In this section, we shall consider the thermodynamic behaviour of the black hole in the case \( M = N = 3 \). Similar to the Lifshitz black hole of [14], the first and third gauge fields, \( A_1 \) and \( A_3 \) are fixed by the asymptotic structure of the spacetime, and does not participate in the mechanics of the black hole, as is also evident in Eq. (54). This is compounded by the fact that they are diverging at the boundary and may not have an appropriate holographic dual, if one were considering Lifshitz holography.

In [14], the presence of the gauge fields are a necessary evil where \( A_1 \) was needed to support a Lifshitz asymptotic structure, and \( A_3 \) is needed to hold a horizon of spherical topology. The latter condition is translated to a contribution to \( B \) in our Eq. (31). However, in our case, \( B \) also receives a contribution from \( \Lambda_3 \). Therefore, in our case, \( A_3 \) is no longer crucially needed and we can then set
\[ \rho = 0. \quad \text{(55)} \]
On the other hand, \( A_1 \) is still needed by our solution, which we shall consider fixed and does not participate in the thermodynamics and mechanics of the black hole.

The temperature and entropy of the black hole are obtained by its usual identification with its surface gravity and horizon area respectively:
\[ T = \frac{\kappa}{2\pi}, \quad S = \frac{A}{4G}. \quad \text{(56)} \]
As discussed above, \( A_2 \) is the only \( U(1) \) field participating in the thermodynamics of the black hole, and we shall henceforth drop the subscripts from its associated quantities.

In the grand canonical ensemble, the gauge potential is held fixed at the boundary at value \( \Phi \), and serves as the variable conjugate to the charge \( Q \). We shall also set \( \varphi_0 \) to
zero so that $\Phi$ is explicitly

$$
\Phi = \sqrt{\frac{D-2}{2(D-3+n)}} q r_+^{-(D-3+n)}.
$$

(57)

The energy in this ensemble is taken to be the black hole mass $M$ as given by (45). Then, the first law in (54) now has an explicit thermodynamic form:

$$
dE = TdS + \Phi dQ.
$$

(58)

In the canonical ensemble, it is the charge $Q$ is held fixed. Therefore we take its conjugate variable to be the difference of the potential with the extremal case, $\Phi - \Phi_e$, where $\Phi_e = \sqrt{\frac{D-2}{2(D-3+n)}} q r_+^{-(D-3+n)}$. Also, the energy should correspondingly be the difference with the extremal case, $\Delta E = \Delta M$, where $\Delta M$ is given in Eq. (46). In this case, the first law reads

$$
d\Delta E = TdS.
$$

(59)

### 4.2 Canonical ensemble

The relevant thermodynamic potential in the canonical ensemble is the Helmholtz free energy

$$
\mathcal{F} = \Delta E - TS
$$

$$
= \frac{\Omega}{16\pi G} \left[ - (1 + n - 2\nu) \ell^{-2} r_+^{1+n-2\nu} - (1 - n) \ell^{-2} r_+^{1-n} \\
+ (1 - n) B r_+^{D-3+n} + (2D - 5 + n) q^2 r_+^{-(D-3+n)} \\
- (D - 2) \left( \frac{D - 1 + n - 2\nu}{D - 3 + n} L^{-2} r_e^{D-1+n-2\nu} + \frac{D - 1 - n}{D - 3 + n} \ell^{-2} r_+^{D-1-n} + B r_e^{D-3+n} \right) \right].
$$

(60)

Let us also write the temperature here explicitly for convenient reference:

$$
T = \frac{1}{4\pi} \left[ (D - 1 + n - 2\nu) L^{-2} r_+^{1+n-2\nu} + (D - 1 - n) \ell^{-2} r_+^{1-n} + (D - 3 + n) B r_+^{1+n} \\
- (D - 3 + n) q^2 r_+^{(2D-5+n)} \right].
$$

(61)

The heat capacity at constant charge is

$$
C_Q = T \left( \frac{\partial S}{\partial T} \right)_Q = \frac{\partial M/\partial r_+}{\partial T/\partial r_+},
$$

(62)
where
\[
\begin{align*}
\frac{\partial M}{\partial r_+} &= \frac{\Omega(D - 2)}{4G} r_+^{D-3} T, \\
\frac{\partial T}{\partial r_+} &= \frac{1}{4\pi} r_+^{-1} \left[ (1 + n - 2\nu)(D - 1 + n - 2\nu)L^{-2} r_+^{1+n-2\nu} + (1 - n)(D - 1 - n)\ell^{-2} r_+^{1-n} \\
&\quad - (1 - n)(D - 3 + n)B r_+^{-(1-n)} + (2D - 5 + n)(D - 3 + n)q^2 T_+^{-(2D-5+n)} \right].
\end{align*}
\]

Recall that for fixed \( n \), the parameter \( \nu \) ranges from \( \nu = 0 \) to \( \nu = n \). The former case being the Lifshitz black hole and the latter case corresponds to a charged dilaton black hole with two exponential potentials. Indeed, the study of the temperature and free energies show that both cases share similar thermodynamic behaviour connected by a continuously varying parameter \( \nu \).

In particular, for sufficiently small charge, there exist a possible range where three branches of solutions with different \( r_+ \) share the same temperature. This is reflected in Eq. (61) where the equation \( T_0 = T(r_+) \) has possibly multiple roots for some \( T_0 \). As \( q \) is increased until \( q_{\text{crit}} \), the roots will coalesce to a single point, after which for \( q > q_{\text{crit}} \) there will be a unique \( r_+ \) for every \( T_0 \). Given some spacetime parameters \( (n, \nu, L^2, \ell^2, B) \) at some dimensionality \( D \), the value of \( q_{\text{crit}} \) is determined by the condition
\[
\frac{\partial T}{\partial r_+} = \frac{\partial^2 T}{\partial r_+^2} = 0.
\]

Fig. 1 shows a representative example for the case \( n = 0.6, L^2 = 2, \ell^2 = 2, \) and \( D = 4. \) In this case, solving Eq. (65) gives \( q_{\text{crit}} = 0.0588. \) For a charge less than \( q_{\text{crit}} \), there are three co-existing branches of black holes with temperature \( 0.3931 \leq T \leq 0.4508 \), which can be seen in in the solid curve in the left-hand plot in Fig. 1. The plot of \( \mathcal{F} \) vs \( T \) for this charge is shown in the solid curve on the right-hand plot of Fig. 1, where we can see it’s characteristic swallow-tail structure. At \( q = q_{\text{crit}} \), the three branches collapse to a point and the swallowtail in the \( \mathcal{F}-T \) curve shrinks to a single kink. Further increasing beyond \( q > q_{\text{crit}} \), the kink disappears and \( \mathcal{F} \) becomes a smooth function of \( T \).

Let us explore another case of \( q < q_{\text{crit}} \) in further detail, this time with \( n = 0.4, \nu = 0.3, L^2 = \ell^2 = 2 \) and \( D = 4. \) The three branches are labelled explicitly in Fig. 2. Branch 1 is a low-temperature solution which starting from the extremal case \( r_+ = r_e \) up to \( r_+ = 0.2209 \) whereas Branch 2 is an unstable one at \( 0.2209 < r_+ < 0.5920 \), and Branch 3 is for \( r_+ > 0.5920 \). From Fig. 2, we see that the three branches can co-exist in the temperature range \( 0.3290 \leq T \leq 0.3606 \), and that a first-order phase transition may occur in this range \([14,25]\).

This behaviour is further corroborated by the heat capacity. The left-hand plot of Fig. 3 shows the heat capacity \( C_Q \) vs \( r_+ \). The heat capacities of Branches 1 and 3 are

\[\text{To avoid clutter, we shall display numerical figures up to five significant figures.}\]
Figure 1: Plots of $T$ vs $r_+$ (left) and $\mathcal{F}$ vs $T$ (right) for the case $n = 0.6$, $\nu = 0.3$, $L^2 = \ell^2 = 2$, and $D = 4$. In this case, the critical charge has the value $q_{\text{crit}} = 0.0588$.

Figure 2: Plots of $T$ vs $r_+$ (left) and $\mathcal{F}$ vs $T$ (right) for $q = \sqrt{0.3} q_{\text{crit}}$ in the case $n = 0.4$, $\nu = 0.3$, $L^2 = \ell^2 = 2$, and $D = 4$. In this case, the critical charge has the value $q_{\text{crit}} = 0.0868$. 
Figure 3: Plots of $C_Q$ vs $r_+$ (left) and $C_Q$ vs $T$ (right). The parameter choices are the same as for Fig. 2.

Figure 4: Plots of $q_{\text{crit}}$ against $\nu$ for various $n$. Note that the allowed ranges of $\nu$ are $\nu \leq n$, and hence each curve terminates at their respective $\nu = n$.

positive while it is negative for Branch 2, and they are separated by discontinuities. The plot of $C_Q$ vs $T$ is shown in the right-hand plot of Fig. 3, where we observe the co-existing branches in the aforementioned temperature range.

The thermodynamics typically show similar qualitative behaviour for $0 \leq \nu \leq n$, where $\nu$ determines the specific value of $q_{\text{crit}}$. In Fig. 4, some values of $q_{\text{crit}}$ as a function of $\nu$ are shown for different $n$. Furthermore, as $\nu$ ranges from $\nu = 0$ to $\nu = n$, these observations connect the thermodynamics of the Lifshitz black hole [14] to that of a charged dilaton black hole with two Liouville potentials [26, 27].
4.3 Grand canonical ensemble

In the grand canonical ensemble, the gauge potential $\Phi$ is held fixed instead of $Q$. In this case, the relevant thermodynamic potential is the Gibbs free energy

$$W = E - TS - \Phi Q$$

$$= \frac{1}{16\pi G} \left[- (1 + n - 2\nu)L^2 r_+^{D-1+n-2\nu} - (1 - n)\ell^2 r_+^{D-1-n} + (1 - n) \left(B - \frac{2(D - 3 + n)}{D - 2} \Phi^2\right) r_+^{D-3+n}\right].$$

(66)

In this ensemble, the system should be parametrised by $\Phi$ instead of $q$. Therefore the temperature is expressed as

$$T = \frac{1}{4\pi} \left[(D - 1 + n - 2\nu) L^2 r_+^{1+n-2\nu} + (D - 1 - n)\ell^2 r_+^{1-n} + (D - 3 + n) \left(B - \frac{2(D - 3 + n)}{D - 2} \Phi^2\right) r_+^{-1+n}\right].$$

(67)

The above relations imply the existence of a critical potential $\Phi_{\text{crit}}$, where

$$\Phi_{\text{crit}} = \sqrt{\frac{(D - 2)B}{2(D - 3 + n)}}.$$  

(68)

For $\Phi < \Phi_{\text{crit}}$, the third term in Eq. (67) is negative, and this allows the possibility two branches of solution with the same temperature. The free energies of these two branches meet at a discontinuous point of the $W$-$T$ curve, as shown in Fig. 5, where the numerical values are $n = 0.6$, $\nu = 0.3$, $L^2 = \ell^2 = 2$, and $D = 4$. Increasing the potential to $\Phi_{\text{crit}}$, we see that the low-temperature branch vanishes by collapsing into the point $(r_+, T) = (0, 0)$ and only one smooth branch $W$-$T$ curve remains. This behaviour persists for various values of $\nu$ in $0 \leq \nu \leq n$, and is similar to the thermodynamics observed in the Riessner–Nordström–Anti-de Sitter black hole [25] as well as the spherical Lifshitz black hole [14].

5 Conclusion

In summary, we have considered an Einstein–Maxwell–dilaton-type theory consisting of $M$ Liouville potentials and $N U(1)$ gauge fields. With an appropriate ansatz, the equations of motion are solved for arbitrary $M$ and $N$.

The ansatz was chosen based on the assumption that each term in the metric function $g_{tt}$ is supported by a potential and gauge field, allowing for the possibility that a potential and gauge field may support the same term. By switching off the appropriate potentials and gauge fields, we recover the Lifshitz and dilaton black holes. The solution with
Figure 5: Plots of $T$ vs $r_+$ (left) and $W$ vs $T$ for the case $n = 0.6$, $\nu = 0.3$, $L^2 = \ell^2 = 2$, and $D = 4$. For these values, the critical potential is $\Phi_c = 0.9882$.

$M = N = 3$ corresponds to a metric that interpolates between the two.

Some basic thermodynamic quantities have been studied. In the canonical ensemble, solutions with sufficiently small charge may have three co-existing branches with the same temperature, and that a phase transition may occur among them. In the grand canonical ensemble, solutions with sufficiently small potential, there are two branches with the same temperature. This behaviour shares similar features to the Lifshitz black hole as well as the dilaton black hole to which it interpolates, thus bridging the results of [14] for the Lifshitz black hole and [26, 27] for the dilaton black hole.

Our solution inherits similar problematic features of the Lifshitz and dilaton black holes. In particular, the dilaton $\phi$ diverges at the boundary. It was argued in [14] that this issue might be resolved by an appropriate embedding of the model in string theory. Secondly, the unusual asymptotic structure, being non-asymptotically flat and non-(A)dS complicates the issue of calculations at the boundary. For instance, the choice of background required to renormalise the divergence of the Euclidean action is ambiguous. Instead, a background-independent counter-term was introduced in the Lifshitz case [28] and similarly for the dilaton black hole [19]. However, in the dilaton case, the boundary stress tensor remains ill-defined despite having a finite action [19].

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