HOCHSCHILD HOMOLOGY AND THE DERIVED DE RHAM COMPLEX
REVISITED

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Abstract. We characterize two objects by universal property: the derived de Rham complex and Hochschild homology together with its Hochschild–Kostant–Rosenberg filtration. This involves endowing these objects with extra structure, built on notions of “homotopy-coherent cochain complex” and “filtered circle action” that we study here. We use these universal properties to give a conceptual proof of the statements relating Hochschild homology and the derived de Rham complex, in particular giving a new construction of the filtrations on cyclic, negative cyclic, and periodic cyclic homology that relate these invariants to derived de Rham cohomology.

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§1. INTRODUCTION

One of the basic cohomological invariants in algebraic geometry is algebraic de Rham cohomology: for A a commutative ring and B a commutative A-algebra, this is given by the cohomology of the algebraic de Rham complex

$$\Omega^*_B/A = \{ \Omega^0_{B/A} \to \Omega^1_{B/A} \to \Omega^2_{B/A} \to \cdots \}.$$  

This cochain complex has the structure of a strictly commutative differential graded A-algebra\(^1\), and is characterized up to isomorphism as such by the following universal property:

(†) For any strictly commutative differential graded A-algebra \(X^*\), a map of commutative A-algebras \(B \simeq \Omega^0_{B/A} \to X^0\) extends uniquely to a map of strictly commutative differential graded A-algebras \(\Omega^*_B/A \to X^*\).

This paper proves two analogues of (†): we characterize the derived de Rham complex and the HKR-filtered Hochschild homology of B over A by similar universal properties. The meaning of this terminology and the features of these universal properties will be explained in the remainder of this introduction: in §1.1, we overview our result on the derived de Rham complex (Theorem 1.1.1); in §1.2, we overview our result on HKR-filtered Hochschild homology (Theorem 1.2.3) and explain how these universal properties together supply a conceptual proof of the known relationship between Hochschild (and negative and periodic cyclic) homology and derived de Rham cohomology (Theorem 1.2.1), which is what motivated this work.

The universal properties of the derived de Rham complex and HKR-filtered Hochschild homology, while analogous to (†) in a natural manner, involve significantly more intricate types of algebraic structure, which in particular are \(\infty\)-categorical in nature. Formulating these structures is a primary task of the paper. This necessitates two notices regarding the remainder of the introduction: we will immediately begin using \(\infty\)-categorical language, and we will content ourselves with giving somewhat informal sketches of the main definitions and results, pointing to the body of the paper for their precise articulation.

§1.1. The derived de Rham complex. In this paper, we will be interested in Hodge-completed derived de Rham cohomology, a variant of algebraic de Rham cohomology that naturally lives in the context of derived algebraic geometry: given a simplicial commutative ring A and a simplicial commutative A-algebra B, the Hodge-completed derived de Rham cohomology of B over A is an \(E_\infty\)-A-algebra \(dR^\wedge_{B/A}\), equipped with a complete \(\mathbb{Z}_{\geq 0}\)-indexed decreasing filtration \(dR^\wedge_{B/A}\) called the Hodge filtration.

If A is an ordinary commutative ring and B is a smooth A-algebra, then \(dR^\wedge_{B/A}\) is equivalent to the \(E_\infty\)-A-algebra underlying the commutative differential graded A-algebra \(\Omega^*_{B/A}\), and the Hodge filtration \(dR^\wedge_{B/A}\) is induced by the brutal filtration \(\Omega^*_{B/A}\), given by

$$\Omega^*_{B/A} = \{ \cdots \to \Omega^i_{B/A} \to \Omega^{i+1}_{B/A} \to \cdots \} \quad \text{for } i \in \mathbb{Z}_{\geq 0}.$$  

In particular, in this smooth situation, the associated graded pieces of the Hodge filtration are given by shifts of the modules of differential forms, \(\Omega^i_{B/A}[-i]\). In general, the associated graded pieces of \(dR^\wedge_{B/A}\) are given by shifts of modules of \(\text{"derived differential forms"}, L\Omega^i_{B/A}[-i]\), with \(L\Omega^i_{B/A}\) being an alternative notation for the cotangent complex \(L_{B/A}\), and more generally \(L\Omega^i_{B/A}\) denoting the derived \(i\)-th exterior power of \(L_{B/A}\).

Here we give an alternative perspective on Hodge-completed derived de Rham cohomology and its Hodge filtration: namely, we formulate a derived version of the strictly commutative differential graded A-algebra \(\Omega^*_{B/A}\), with terms \(L\Omega^i_{B/A}\) in place of \(\Omega^i_{B/A}\), and characterized by a universal property analogous to (†) above. As with \(\Omega^*_{B/A}\), this object has an “underlying \(E_\infty\)-A-algebra” that

\(^1\)The modifier “strictly” refers to the property that \(x^2 = 0\) whenever \(x \in \Omega^i_{B/A}\) for \(i\) odd.
carries the information of its “cohomology” and is equipped with a “brutal filtration”, and now these recover Hodge-completed derived de Rham cohomology and its Hodge filtration for all maps of simplicial commutative rings \( A \to B \).

To realize the vague idea suggested in the previous paragraph, we define the following over any simplicial commutative ring \( A \):

- The notion of an \( h_* \)-differential graded \((h_* \text{-} dg)\) \( A \)-module \( X^* \). This is a type of “homotopy-coherent cochain complex”, consisting of an underlying graded \( A \)-module \( X^* = \{X^i\}_{i \in \mathbb{Z}} \), together with differentials \( d : X^i[1] \to X^{i+1} \) for \( i \in \mathbb{Z} \) that square to zero up to coherent homotopy (the shift in the differentials is the source of the “\( h_* \)” in the terminology). These are precisely defined as graded modules over a certain split square-zero graded algebra \( D_* \), whose underlying \( A \)-module is equivalent to \( A \oplus A[1] \). When \( A \) is an ordinary commutative ring, any ordinary cochain complex of \( A \)-modules \( M^* \) determines an \( h_* \)-\( A \)-module \( X^* \), with underlying graded object given by \( X^i \cong M^i[i] \).

- The cohomology type \([X^*]\) of an \( h_* \)-\( A \)-module \( X^* \). This is an \( A \)-module equipped with a complete filtration \([X^*]^{|*}\) whose associated graded pieces are given by \( X^i[-2i] \). In the case that \( A \) is an ordinary commutative ring and \( X^* \) comes from an ordinary cochain complex of \( A \)-modules \( M^* \) in the manner described above, the cohomology type \([X^*]\) is simply the object in the derived category represented by \( M^* \), equipped with the brutal filtration (whose associated graded pieces are \( M^*[−i] \)).

- The notion of a simplicial commutative algebra structure on a \( h_* \)-\( A \)-module. We refer to \( h_* \)-\( A \)-modules equipped with this structure as \( h_* \)-\( A \)-algebras. If \( X^* \) is an \( h_* \)-\( A \)-algebra, then the zeroth graded piece \( X^0 \) canonically carries the structure of a simplicial commutative \( A \)-algebra and the filtered cohomology type \([X^*]^{|*}\) canonically carries the structure of a filtered \( E_{\infty} \)-\( A \)-algebra.

With these notions in place, we prove the following result:

**Theorem 1.1.1** (see §5.3). Let \( A \) be a simplicial commutative ring and let \( B \) be a simplicial commutative \( A \)-algebra. Then:

(a) There is an initial object \( L\Omega^i_{B/A} \) in the \( \infty \)-category of \( h_* \)-\( A \)-algebras equipped with a map of simplicial commutative \( A \)-algebras \( B \to X^0 \).

(b) For \( i \geq 0 \), there are equivalences \( L\Omega^i_{B/A} \cong L\Omega^i_{B/A}[i] \), and for \( i < 0 \), we have \( L\Omega^i_{B/A} \cong 0 \). Under these equivalences, the first differential \( L\Omega^0_{B/A} \to L\Omega^1_{B/A}[-1] \) is given by the universal derivation \( d : B \to L\Omega^1_{B/A} \).

(c) There is an equivalence of filtered \( E_{\infty} \)-\( A \)-algebras \( [L\Omega^*_{B/A}]^{|*} \cong dR^\wedge_{B/A} \).

The object \( L\Omega^*_{B/A} \) of Theorem 1.1.1 is what we refer to in this paper as the derived de Rham complex of \( B \) over \( A \). Its definition in statement (a) of the theorem is the promised universal property analogous to (i), and statement (c) in the theorem explains the way in which this object recovers Hodge-completed derived de Rham cohomology and its Hodge filtration.

### §1.2. Hochschild homology.

Another family of cohomological invariants, related to the de Rham invariants discussed above, arises from the theory of Hochschild homology. For \( A \) a simplicial commutative ring and \( B \) a simplicial commutative \( A \)-algebra, the Hochschild homology of \( B \) over \( A \) is a simplicial commutative \( A \)-algebra with \( S^1 \)-action, \( HH(B/A) \). To this object with \( S^1 \)-action, we may apply the homotopy fixed points construction to obtain negative cyclic homology \( HC^{-}(B/A) \) and the Tate construction to obtain periodic cyclic homology \( HP(B/A) \). The relationship between these invariants and the de Rham invariants mentioned above manifests in the form of certain filtrations, as described in the following result.

**Theorem 1.2.1.** Let \( A \) be a simplicial commutative ring and let \( B \) be a simplicial commutative
A-algebra. Then:

(a) There is a complete $\mathbb{Z}_{\geq 0}$-indexed decreasing filtration $\text{fil}^i \text{HH}(B/A)$ on $\text{HH}(B/A)$ with associated graded pieces given by

$$\text{gr}^i \text{HH}(B/A) \cong L\Omega^i_{B/A}[i] \quad (i \in \mathbb{Z}_{\geq 0}).$$

(b) There are complete decreasing $\mathbb{Z}$-indexed filtrations $\text{fil}^i \text{HC}^{-}(B/A)$ and $\text{fil}^i \text{HP}(B/A)$ on $\text{HC}^{-}(B/A)$ and $\text{HP}(B/A)$, respectively, with associated graded pieces given by

$$\text{gr}^i \text{HC}^{-}(B/A) \cong \text{dR}^{\text{uni}^\mathbb{Z}_i}[2i] \quad \text{and} \quad \text{gr}^i \text{HP}(B/A) \cong \text{dR}^B_{B/A}[2i] \quad (i \in \mathbb{Z}).$$

These filtrations are exhaustive under the assumptions that $B$ is truncated (i.e. $\pi_k(B) = 0$ for $k > 0$) and that the Tor-amplitude of $L_{B/A}$ over $B$ is contained in $[0, 1]$.

Part (a) of Theorem 1.2.1 goes back to the classical theorem of Hochschild–Kostant–Rosenberg: the latter is equivalent to the special case of the former where $A$ is an ordinary commutative ring and $B$ is a smooth $A$-algebra; one can then bootstrap to the general statement of (a) using formal/categorical arguments, as explained by Bhatt–Morrow–Scholze in [BMS18, §2.2]. For this reason, the filtration in Theorem 1.2.1 is referred to as the HKR filtration on Hochschild homology.

Part (b) was proved by Antieau [Ant18], after being conjectured by Bhatt–Morrow–Scholze, who proved a $p$-adic variant of the result in [BMS18].

In this paper, we shall give a new proof of Theorem 1.2.1, i.e. we’ll find new constructions of the filtrations appearing in Theorem 1.2.1. Firstly, we construct HKR-filtered Hochschild homology, refined by additional $S^1$-equivariant and multiplicative structures, via a universal characterization (which crucially refers to this extra structure). Secondly, we use this additional $S^1$-equivariant structure to construct the filtrations on negative and periodic cyclic homology.

Before elaborating on our universal property for HKR-filtered Hochschild homology, let us recall that Hochschild homology itself is characterized by a universal property. Namely, for $A$ a simplicial commutative ring and $B$ a simplicial commutative $A$-algebra, we have:

(†) For any simplicial commutative $A$-algebra with $S^1$-action $X$, a map of commutative $A$-algebras $B \to X$ extends uniquely to an $S^1$-equivariant map of simplicial commutative $A$-algebras $\text{HH}(B/A) \to X$.²

Our universal property for HKR-filtered Hochschild homology is an interpolation between (†) and the universal property of the derived de Rham complex described in §1.1. To this end, we define the following over any base simplicial commutative ring $A$:

- The notion of a $(\mathbb{Z}$- or $\mathbb{Z}_{\geq 0}$-indexed) filtered $A$-module with filtered $S^1$-action. Roughly speaking, a filtered $S^1$-action on a filtered $A$-module is “an $S^1$-action that increases the filtration degree” (analogous to the differential of a cochain complex increasing the grading degree). Rigorously, these are defined as filtered modules over a filtered $A$-algebra $T_{fil}$, which we refer to as the $A$-linear filtered circle. The underlying $A$-algebra of $T_{fil}$ is the group algebra $A[S^1]$, and the associated graded object of $T_{fil}$ is the graded algebra $D_+$ appearing in the definition of $h_+$-dg objects. It follows that a filtered $S^1$-action on a filtered $A$-module determines an $S^1$-action on its underlying $A$-module and an $h_+$-dg structure on its associated graded $A$-module.

- The notion of a simplicial commutative algebra structure on a filtered $A$-module with filtered $S^1$-action. We refer to filtered $A$-modules with filtered $S^1$-action equipped with this structure as filtered simplicial commutative $A$-algebras with filtered $S^1$-action. If $X$ is a filtered simplicial commutative $A$-algebra with filtered $S^1$-action, then the underlying object of $X$ canonically carries the structure of a simplicial commutative $A$-algebra with $S^1$-action and the associated graded object $\text{gr}(X)$ canonically carries the structure of a $h_+$-dg simplicial commutative $A$-algebra.

²It is perhaps worth emphasizing here that, when we refer to simplicial commutative rings, we always mean to work with the $\infty$-category of such objects. In particular, the universal property (†) is a properly $\infty$-categorical one, characterizing $\text{HH}(B/A)$ up to equivalence, in contrast with the universal property (†).
A-algebra.

Remark 1.2.2 (The filtered circle). Our ability to define the notion of a filtered simplicial commutative A-algebra with filtered \(S^1\)-action—the key notion for the universal property of HKR-filtered Hochschild homology—boils down to exhibiting a sufficient amount of structure on the filtered circle \(\mathbb{T}_{\text{fil}}\). We need this object not just as a filtered algebra but as a suitably structured filtered bialgebra. This construction of the filtered circle is a key component in this paper. (The same comments apply to the object \(\mathbb{D}_+\) in the graded setting, though that construction is weaker in the sense that it may be recovered from the filtered circle but not vice-versa.)

With these notions in place, we prove the following result:

Theorem 1.2.3 (see §6.2). Let \(A\) be a simplicial commutative ring and let \(B\) be a simplicial commutative \(A\)-algebra. Then:

(a) There is an initial object \(\text{HH}_{\text{fil}}(B/A)\) in the \(\infty\)-category of \(\mathbb{Z}_{\geq 0}\)-indexed filtered simplicial commutative rings with filtered \(S^1\)-action \(X^*\) equipped with a map of simplicial commutative \(A\)-algebras \(B \to X^0\).

(b) There is an equivalence of simplicial commutative \(A\)-algebras with \(S^1\)-action \(\text{HH}_{\text{fil}}(B/A)^0 \simeq \text{HH}(B/A)\), i.e. the underlying object of \(\text{HH}_{\text{fil}}(B/A)\) is the Hochschild homology of \(B\) over \(A\).

(c) There is an equivalence of \(h_\ast\)-dg simplicial commutative \(A\)-algebras \(\text{gr}(\text{HH}_{\text{fil}}(B/A)) \simeq L\Omega_B^* A\), i.e. the associated graded object of \(\text{HH}_{\text{fil}}(B/A)\) is the derived de Rham complex of \(B\) over \(A\).

(d) As a filtered object, \(\text{HH}_{\text{fil}}(B/A)\) is complete.

Statements (a–c) of Theorem 1.2.3 are completely formal once the definitions have been made, conceptualizing the relationship between Hochschild homology and the derived de Rham complex. Statement (d) requires the same basic computation going into the proof of the HKR theorem. From all the statements together we certainly in particular recover Theorem 1.2.1(a), and in fact one can immediately deduce that the filtration on \(\text{HH}(B/A)\) determined by the object \(\text{HH}_{\text{fil}}(B/A)\) agrees with the HKR filtration. Properly speaking, it is the object \(\text{HH}_{\text{fil}}(B/A)\) that we mean to refer to by the terminology “HKR-filtered Hochschild homology.”

To furthermore recover Theorem 1.2.1(b), we also define, for \(A\) a simplicial commutative ring and \(X\) a filtered \(A\)-module with filtered \(S^1\)-action, the filtered fixed points \(X^{T_{\text{fil}}}\) and filtered Tate construction \(X^{T_{\text{fil}}}\). These are both filtered \(A\)-modules, which can be regarded as filtrations on the usual fixed points and Tate construction of the \(A\)-module with \(S^1\)-action underlying \(X\). We then prove the following result:

Proposition 1.2.4 (see §6.3). Let \(A\) be a simplicial commutative ring and let \(X\) be a filtered \(A\)-module with filtered \(S^1\)-action. Then, for \(i \in \mathbb{Z}\), there are equivalences

\[
\text{gr}^i(X^{T_{\text{fil}}}) \simeq |\text{gr}(X)|^{|\mathbb{Z}|}[2\ell], \quad \text{gr}^i(X^{T_{\text{fil}}}) \simeq |\text{gr}(X)|[2\ell]
\]

(where we are regarding \(\text{gr}(X)\) as an \(h_\ast\)-dg \(A\)-module, so that \(|\text{gr}(X)|\) denotes the cohomology type of \(\text{gr}(X)\) and \(|\text{gr}(X)|\) the brutal filtration thereon).

It follows from Theorem 1.2.3 and Proposition 1.2.4 (together with a bit of extra analysis regarding completeness and exhaustiveness) that applying filtered fixed points and filtered Tate construction to HKR-filtered Hochschild homology \(\text{HH}_{\text{fil}}(B/A)\) produces filtrations on negative and periodic cyclic homology as specified by Theorem 1.2.1(b). It is possible to show that these filtrations agree with those constructed in [Ant18], but we do not spell this out in the paper.

§1.3. Outline. In general, we refer to the beginning of each section and subsection of the paper for description and motivation of its contents. However, this paper being somewhat lengthy and at times technical, let us give a brief guide to the material. The main definitions and results of the paper are contained in §§5–6; this was summarized in §§1.1–1.2, so we say no more about it here.
The earlier §§2–4 are devoted to preliminary categorical and algebraic material that are needed to formulate and work with the structures studied later. Two pieces of this preliminary material may be of particular interest:

- In §3.2, we discuss the relationship between filtered objects in a stable presentable symmetric monoidal ∞-category $\mathcal{C}$ and $h$.-differential graded objects in $\mathcal{C}$; the latter is another flavor of “homotopy-coherent cochain complexes” playing a role in this paper (in addition to the $h_+\text{-dg}$ objects discussed in §1.1). This relationship underlies our notion of “cohomology type” for homotopy-coherent cochain complexes and hence the proof of Proposition 1.2.4. Additionally, it supplies a clean construction of the Beilinson t-structure on filtered objects (Proposition 3.3.11).

- In §4, we discuss the theory of derived commutative rings, an extension of the theory of simplicial commutative rings to the nonconnective setting, which we use to formulate the structure of interest on the filtered circle. This theory (though not our choice of terminology) is due to Mathew, and is being investigated in work-in-progress by Bhatt–Mathew [BM]. However, as their work has yet to appear in writing, we give a complete account of the aspects of the theory needed here.

§1.4. Related work. The relationship between Hochschild homology and the de Rham complex simplifies substantially in characteristic zero. For example, the filtrations of Theorem 1.2.1 are canonically split when $A$ is a $\mathbb{Q}$-algebra. In this characteristic zero setting, results essentially equivalent to the ones of this paper appear in earlier work of Toën–Vezzosi [TV09], and there are related results due to Ben-Zvi–Nadler [BZN12]. However, a crucial ingredient in these previous treatments is a certain formality result of the circle over $\mathbb{Q}$ that breaks down in positive and mixed characteristic. In this paper, we accommodate this non-formality and prove integral results. Our perspective should be viewed as building on and refining the viewpoint of Toën–Vezzosi in particular.

Some of the main ideas in this paper were developed independently and concurrently by Moulins–Robalo–Toën. In [MRT19], they construct a “filtered circle” essentially equivalent to ours, and use it just as we do here to construct the filtrations on Hochschild homology and negative cyclic homology discussed above. Of course, there are also differences between their paper and this one; let us highlight a few:

- There is some difference in formalism: [MRT19] employs the language of derived algebraic geometry, while here we stick to (derived) algebra. For example, in [MRT19], the derived stacks $BG_m$ and $A^1/G_m$ are used to work with graded and filtered objects, whereas here we do so directly. Accordingly, the filtered circle takes the form of a derived stack in [MRT19]; that object should be the spectrum (in an appropriate derived algebro-geometric sense) of the filtered circle studied in this paper.

- Most importantly, the two papers give rather different constructions of the filtered circle. Here we use the Postnikov filtration construction and demonstrate that this preserves the necessary structure, while [MRT19] uses Toën’s description of the affinization of $S^1$ (regarded as a derived stack), which relates it at each prime $p$ to the scheme of $p$-typical Witt vectors. The latter is closer in spirit to the work of Ben-Zvi–Nadler cited above, and offers its own insights into the nature of the filtered circle.

The approach here has the advantages of involving no computations and working directly over $\mathbb{Z}$ (the results in [MRT19] are proved only over $\mathbb{Z}_{(p)}$, though they do indicate how to make the desired construction over $\mathbb{Z}$ from their perspective; see also the related further work of Toën [Toë19]). Our construction also allows for comparison, as the universal property of the Postnikov filtration construction gives uniqueness results (see Theorem 6.1.5).

- The results of [MRT19] involve an object equivalent to our derived de Rham complex $L\Omega^*_{/A}$, but there the relationship between this object and the usual notion of (Hodge-completed)
derived de Rham cohomology is not discussed. This is addressed here in our study of the
“cohomology” of homotopy-coherent cochain complexes, or in other words in our results
relating filtered objects with \( h_* \) cochain complexes and \( h_* \) cochain complexes with \( h_* \) cochain complexes.

- There are applications of these ideas discussed in [MRT19] that are not mentioned at all here,
e.g. to the theory of shifted symplectic structures in positive characteristic.

Finally, an alternative formulation of the notion of “homotopy-coherent cochain complex” was
considered by Joyal [Joy08, §35], and is studied in recent work of Walde [Wal19] and in forthcoming
work of Ariotta [Ari]. In particular, Ariotta gives another perspective on the relationship between
cochain complexes and filtered objects, discussed here in §3.2.

§1.5. Acknowledgements. It is a pleasure to thank Ben Antieau, Tony Feng, Søren Galatius,
Joj Helfer, Lars Hesselholt, Jacob Lurie, Akhil Mathew, Thomas Nikolaus, and Allen Yuan for
effecting conversations related to this work, and Ben Antieau, Søren Galatius, Akhil Mathew, and Bertrand Toën for feedback on an earlier draft of the paper. I would also like to
acknowledge Tasos Moulinos and Marco Robalo for discussing some of the material presented here
and its relation to their joint work with Toën. Finally, I am grateful for the support provided by
the NSF Graduate Research Fellowship and the hospitality of the institutions where this work was
carried out: Stanford University, the University of Copenhagen (including support by the Danish
National Research Foundation through the Centre for Symmetry and Deformation (DNRF92)),
and my parents’ home.

§1.6. Notation and terminology. Throughout the paper, we use the language of higher category
theory and higher algebra developed by Lurie [L-HTT; L-HA]. All statements and constructions
should be understood in a homotopy-invariant sense. We will often cite the adjoint functor theorem,
by which we mean [L-HTT, Corollary 5.5.2.9], and the Barr–Beck–Lurie theorem, by which we mean [L-HA, Theorem 4.7.3.5].

We let \( \text{Spc} \) denote the \( \infty \)-category of spaces and \( \text{Spt} \) the \( \infty \)-category of spectra; we let \( \text{Cat}_{\infty} \)
denote the \( \infty \)-category of small \( \infty \)-categories; we let \( \text{Pr}^{\infty} \) denote the \( \infty \)-category of presentable \( \infty \)-categories and colimit-preserving functors, which we often regard as a symmetric monoidal
\( \infty \)-category by [L-HA, §4.8.1], so that commutative algebra objects of \( \text{Pr}^{\infty} \) may be identified with
presentable symmetric monoidal \( \infty \)-categories. We implicitly regard every ordinary category as an
\( \infty \)-category. We also often implicitly regard abelian groups as spectra and commutative rings as simplicial commutative rings or \( E_{\infty} \)-rings; in particular, for \( A \) a commutative ring, \( \text{Mod}_A \) denotes the \( \infty \)-category of \( A \)-module spectra.

For \( \mathcal{C} \) a small \( \infty \)-category, we let \( \mathcal{P}(\mathcal{C}) \) denote the \( \infty \)-category \( \text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc}) \) of presheaves
(of spaces) on \( \mathcal{C} \). For \( \mathcal{C} \) a small \( \infty \)-category admitting coproducts, we let \( \mathcal{P}_{\Sigma}(\mathcal{C}) \) denote the full
subcategory of \( \mathcal{P}(\mathcal{C}) \) spanned by the product-preserving functors \( \mathcal{C}^{\text{op}} \to \text{Spc} \).

For \( \mathcal{C} \) an \( \infty \)-category, we let \( \text{End}(\mathcal{C}) \) denote the \( \infty \)-category of endofunctors on \( \mathcal{C} \). We often regard \( \text{End}(\mathcal{C}) \) as a monoidal \( \infty \)-category via the composition monoidal structure, over which \( \mathcal{C} \)
is left-tensored (via evaluation). A (co)monad on \( \mathcal{C} \) is a (co)algebra object \( T \) of \( \text{End}(\mathcal{C}) \). Given a (co)monad \( T \) on \( \mathcal{C} \), we may consider \( T \)-(co)module objects of \( \mathcal{C} \) (note that others often refer to these as \( T \)-(co)algebras).

At various points, we will invoke the following notion of endomorphism objects, introduced in
[L-HA, §4.7.1]. Let \( \mathcal{E} \) be a monoidal \( \infty \)-category and let \( \mathcal{C} \) an \( \infty \)-category left tensored over \( \mathcal{E} \).
Given an object \( X \in \mathcal{E} \), an endomorphism object for \( X \) in \( \mathcal{E} \) is an object \( E \in \mathcal{E} \) equipped with a map
\( \epsilon : E \otimes X \to X \) in \( \mathcal{E} \), such that, for every \( E' \in \mathcal{E} \), the map
\[
\text{Map}_{\mathcal{E}}(E', E) \to \text{Map}_{\mathcal{E}}(E' \otimes X, X)
\]
induced by \( \epsilon \) is an equivalence. It is shown in loc. cit. that, in this situation, \( E \) canonically promotes
to an algebra in $\mathcal{E}$ and the map $\epsilon$ canonically promotes to a left $E$-module structure on $X$, such that, for any $E' \in \text{Alg}(\mathcal{E})$, the induced map

$$\text{Map}_{\text{Alg}(\mathcal{E})}(E', E) \to \text{LMod}_{E'}(\mathcal{E}) \times E \{X\}$$

is an equivalence. There is an evident dual notion of coendomorphism objects, which we will also make use of.

Finally, throughout, we let $S^1$ denote the circle, regarded as a group object in $\text{Spc}$, and we let $BS^1$ denote its classifying space.

### §2. Bialgebras

Let $Z[S^1]$ denote the group algebra over $\mathbb{Z}$ on $S^1$. Recall that there is a canonical equivalence of $\infty$-categories

$$\text{Mod}_{Z[S^1]} \simeq \text{Fun}(BS^1, \text{Mod}_{\mathbb{Z}}).$$

The symmetric monoidal structure on $\text{Mod}_{\mathbb{Z}}$ induces a pointwise symmetric monoidal structure on the right-hand side (encoding the tensor product of $S^1$-representations), which transports under the above equivalence to a symmetric monoidal structure on $\text{Mod}_{Z[S^1]}$. This is not given by the relative tensor product over $Z[S^1]$, but there is a way of producing this symmetric monoidal structure on $\text{Mod}_{Z[S^1]}$ purely in terms of structure on the group algebra. The important point is that $Z[S^1]$ is not just an algebra, but moreover a cocommutative bialgebra: that is, it carries a cocommutative coalgebra structure compatible with its algebra structure. For example, the diagonal map $S^1 \to S^1 \times S^1$ induces a map of algebras $\Delta : Z[S^1] \to Z[S^1] \times Z[S^1] \cong Z[S^1] \otimes Z[S^1]$. This leads to the desired tensor product of $Z[S^1]$-modules as follows: given two $Z[S^1]$-modules $M$ and $N$, the $Z$-linear tensor product $M \otimes Z N$ canonically has the structure of a $Z[S^1]$-module, and we restrict this to a $Z[S^1]$-module structure using the map $\Delta$.

In §§5–6, we will be dealing with structures that are analogous to $S^1$-actions—namely, homotopy-coherent cochain complexes and filtered $S^1$-actions—but that cannot be formulated using functor categories, rather only as module categories in the linear setting. The purpose of this section is to prepare for those situations, by studying bialgebras in the general context of a (stable) presentable symmetric monoidal $\infty$-category. Our aim is to explain how certain features of representation categories like $\text{Fun}(BS^1, \text{Mod}_{\mathbb{Z}})$, e.g. the tensor product discussed above, carry through to categories of modules (or comodules) over bialgebras in this generality.

This section is organized as follows: in §2.1, we state the basic definitions of bialgebras in the $\infty$-categorical setting; in §2.2, we construct the symmetric monoidal structure on the category of modules over a cocommutative bialgebra informally sketched above; in §2.3, we discuss what happens when one dualizes a bialgebra; and in §2.4, we formulate generalizations to this setting of the orbits, fixed points, and Tate construction for group actions.

### §2.1. Definitions

Fix a symmetric monoidal $\infty$-category $\mathcal{C}$. Let us begin by recalling the definition of coalgebra and comodule objects of $\mathcal{C}$. In what follows, recall that the symmetric monoidal structure on $\mathcal{C}$ determines a symmetric monoidal structure on $\mathcal{C}^{\text{op}}$.

**Definition 2.1.1.** A coalgebra object of $\mathcal{C}$ is an algebra object of $\mathcal{C}^{\text{op}}$. We let $\text{cAlg}(\mathcal{C})$ denote the $\infty$-category $\text{Alg}(\mathcal{C}^{\text{op}})^{\text{op}}$, and refer to this as the $\infty$-category of coalgebra objects of $\mathcal{C}$. Given a coalgebra object $A$ of $\mathcal{C}$, a left $A$-comodule object of $\mathcal{C}$ is a left $A$-module object of $\mathcal{C}^{\text{op}}$. We let $\text{cLMod}_A(\mathcal{C})$ denote the $\infty$-category $\text{LMod}_A(\mathcal{C}^{\text{op}})^{\text{op}}$, and refer to this as the $\infty$-category of left $A$-comodule objects of $\mathcal{C}$.

Similarly, a cocommutative coalgebra object of $\mathcal{C}$ is a commutative algebra object of $\mathcal{C}^{\text{op}}$. We let $\text{cCAlg}(\mathcal{C})$ denote the $\infty$-category $\text{CAlg}(\mathcal{C}^{\text{op}})^{\text{op}}$, and refer to this as the $\infty$-category of cocommutative coalgebra objects of $\mathcal{C}$.
commuting with the (symmetric monoidal) forgetful functors to cocommutative bialgebra objects of \( \mathcal{C} \).

Now, there are two natural options for encoding the idea of an object of \( \mathcal{C} \) equipped with compatible algebra and coalgebra structures. Noting that there are canonical symmetric monoidal structures on the \( \infty \)-categories \( \text{Alg}(\mathcal{C}) \) and \( \text{cAlg}(\mathcal{C}) \), we may consider both coalgebra objects of \( \text{Alg}(\mathcal{C}) \) and algebra objects of \( \text{cAlg}(\mathcal{C}) \). In this paper, we will only deal with the situation where at least one of the algebra and coalgebra structures is commutative. In these cases, we can show that the two options are equivalent:

**Proposition 2.1.2.** There exist unique equivalences of symmetric monoidal \( \infty \)-categories

\[
\text{cAlg}(\text{Alg}(\mathcal{C})) \xrightarrow{\sim} \text{Alg}(\text{cAlg}(\mathcal{C})), \quad \text{Alg}(\text{cCAlg}(\mathcal{C})) \xrightarrow{\sim} \text{cCAlg}(\text{Alg}(\mathcal{C})),
\]

\[
\text{CAlg}(\text{cCAlg}(\mathcal{C})) \xrightarrow{\sim} \text{cCAlg}(\text{CAlg}(\mathcal{C}))
\]

commuting with the (symmetric monoidal) forgetful functors to \( \mathcal{C} \).

**Proof.** The third equivalence is \([L-AV, \text{Corollary 3.3.4}]\).

Let us address the first equivalence. The symmetric monoidal forgetful functor \( \text{CAlg}(\mathcal{C}) \to \mathcal{C} \) induces a symmetric monoidal functor \( G : \text{cAlg}(\text{CAlg}(\mathcal{C})) \to \text{cAlg}(\mathcal{C}) \). We also have a symmetric monoidal forgetful functor \( G' : \text{CAlg}(\text{cAlg}(\mathcal{C})) \to \text{cAlg}(\mathcal{C}) \). Since the symmetric monoidal structure on \( \text{CAlg}(\mathcal{C}) \) is cocartesian, so is the one on \( \text{cAlg}(\text{CAlg}(\mathcal{C})) \), implying that there is a unique symmetric monoidal functor \( U : \text{cAlg}(\text{CAlg}(\mathcal{C})) \to \text{CAlg}(\text{cAlg}(\mathcal{C})) \) equipped with an equivalence \( G' \circ U \simeq G \). We wish to show that \( U \) is an equivalence.

Let us first consider the case that \( \mathcal{C} \) is presentable. Then the functors \( G \) and \( G' \) admit left adjoints, both given by formation of symmetric algebras in \( \mathcal{C} \). Moreover, \( G \) and \( G' \) are both monadic, so it follows from \([L-HA, \text{Corollary 4.7.3.16}]\) that \( U \) is an equivalence.

Suppose now that \( \mathcal{C} \) is not presentable. Enlarging the universe if necessary, we may assume that \( \mathcal{C} \) is small. Then we may consider the symmetric monoidal Yoneda embedding \( \mathcal{C} \to \mathcal{P}(\mathcal{C}) \), where \( \mathcal{P}(\mathcal{C}) \) is equipped with the Day convolution symmetric monoidal structure (see \([L-HA, \text{Corollary 4.8.1.12}]\)). As \( \mathcal{P}(\mathcal{C}) \) is a presentable symmetric monoidal \( \infty \)-category, the preceding argument implies that we have an equivalence \( \text{cAlg}(\text{CAlg}(\mathcal{P}(\mathcal{C}))) \xrightarrow{\sim} \text{CAlg}(\text{cAlg}(\mathcal{P}(\mathcal{C}))) \). This restricts to the desired equivalence \( \text{cAlg}(\text{CAlg}(\mathcal{C})) \xrightarrow{\sim} \text{CAlg}(\text{cAlg}(\mathcal{C})) \) upon pulling back from \( \mathcal{P}(\mathcal{C}) \) to \( \mathcal{C} \).

Establishing the second equivalence is similar, except that, for \( \mathcal{C} \) presentable, we invoke cartesian-ness instead of cocartesianness, the right adjoints to the forgetful functors \( \text{Alg}(\text{cCAlg}(\mathcal{C})) \to \text{Alg}(\mathcal{C}) \) and \( \text{cCAlg}(\text{Alg}(\mathcal{C})) \to \text{Alg}(\mathcal{C}) \) (which exist by \([L-AV, \text{Corollary 3.1.5}]\)) instead of left adjoints, and the dual comonadic form of \([L-HA, \text{Corollary 4.7.3.16}]\).

**Definition 2.1.3.** We set \( \text{bAlg}_{\mathcal{C}}(\mathcal{C}) := \text{CAlg}(\text{cAlg}(\mathcal{C})) \), and implicitly identify this with the \( \infty \)-category \( \text{cAlg}(\text{CAlg}(\mathcal{C})) \) whenever convenient using the equivalence of Proposition 2.1.2. We refer to objects of \( \text{bAlg}_{\mathcal{C}}(\mathcal{C}) \) as *commutative bialgebra objects of \( \mathcal{C} \)*. Similarly, we set \( \text{bAlg}_{\mathcal{C}}^{\mathcal{C}}(\mathcal{C}) := \text{cCAlg}(\text{Alg}(\mathcal{C})) \), which we implicitly identify with \( \text{Alg}(\text{cCAlg}(\mathcal{C})) \), and whose objects we refer to as *cocommutative bialgebra objects of \( \mathcal{C} \)*; and we set \( \text{bAlg}_{\mathcal{C}}^{\mathcal{C}}(\mathcal{C}) := \text{CAlg}(\text{cCAlg}(\mathcal{C})) \), which we implicitly identify with \( \text{cCAlg}(\text{CAlg}(\mathcal{C})) \), and whose objects we refer to as *bicommutative bialgebra objects of \( \mathcal{C} \)*.

**Example 2.1.4.** Suppose that the symmetric monoidal structure on \( \mathcal{C} \) is cartesian. Then the forgetful functor \( \text{cCAlg}(\mathcal{C}) \to \mathcal{C} \) is a (symmetric monoidal) equivalence, so the same is true of the forgetful functor \( \text{Alg}(\text{cCAlg}(\mathcal{C})) \to \text{Alg}(\mathcal{C}) \). As \( \text{Alg}(\mathcal{C}) \) is equivalent to the \( \infty \)-category of monoid objects in \( \mathcal{C} \), we deduce that any monoid object of \( \mathcal{C} \) canonically determines a cocommutative bialgebra object of \( \mathcal{C} \).
§2.2. Tensor products. In this subsection, we construct the canonical tensor product of modules over a cocommutative bialgebra, generalizing the tensor product of group representations, making precise the informal discussion from the beginning of the section. It will actually be more convenient to discuss the construction in the dual context of comodules over a commutative bialgebra.

Out of convenience, we will make most statements in this subsection in the setting of small ∞-categories; however, everything goes through relative to an arbitrary universe, so we will feel free to apply the results of this subsection to large ∞-categories. We regard $\text{Cat}_\infty$ as a symmetric monoidal ∞-category via the cartesian symmetric monoidal structure. We identify small (symmetric) monoidal ∞-categories with (commutative) algebra objects of $\text{Cat}_\infty$. Furthermore, for $\mathcal{C}$ a small monoidal ∞-category, we identify small ∞-categories right tensored over $\mathcal{C}$ with right $\mathcal{C}$-modules in $\text{Cat}_\infty$. If $\mathcal{C}$ is a small symmetric monoidal ∞-category, then $\text{RMod}_\mathcal{C}(\text{Cat}_\infty)$ carries a canonical symmetric monoidal structure given by the relative tensor product construction, so that commutative algebra objects of $\text{RMod}_\mathcal{C}(\text{Cat}_\infty)$ may be identified with small symmetric monoidal ∞-categories $\mathcal{D}$ under $\mathcal{C}$.

The following result encapsulates the main content of this subsection.

Proposition 2.2.1. Let $\mathcal{C}$ be a small symmetric monoidal ∞-category. Then the construction $A \mapsto \text{cLMod}_A(\mathcal{C})$ canonically extends to a symmetric monoidal functor $\mu_\mathcal{C} : \text{cAlg}(\mathcal{C}) \to \text{RMod}_\mathcal{C}(\text{Cat}_\infty)$.

Before proving this, let us explain how it supplies the symmetric monoidal structures we are looking for.

Construction 2.2.2. Let $\mathcal{C}$ be a small symmetric monoidal ∞-category and let $A$ be a commutative bialgebra object of $\mathcal{C}$. Using Proposition 2.2.1, we obtain a symmetric monoidal structure on the ∞-category $\text{cLMod}_A(\mathcal{C})$, together with one on the forgetful functor $\text{cLMod}_A(\mathcal{C}) \to \mathcal{C}$, as follows.

Note that $\text{cAlg}(\mathcal{C})$ admits a final object, namely the trivial coalgebra $1$, with $\text{cLMod}_1(\mathcal{C}) \simeq \mathcal{C}$. It follows that the symmetric monoidal functor $\mu_\mathcal{C}$ of Proposition 2.2.1 factors canonically through a symmetric monoidal functor $\mu'_\mathcal{C} : \text{cAlg}(\mathcal{C}) \to \text{RMod}_\mathcal{C}(\text{Cat}_\infty)/\mathcal{C}$. This induces a functor on commutative algebra objects

$$\text{bAlg}_\mathcal{C}(\mathcal{C}) \simeq \text{CAlg}(\text{cAlg}(\mathcal{C})) \to \text{CAlg}(\text{RMod}_\mathcal{C}(\text{Cat}_\infty)/\mathcal{C}) \simeq \text{CAlg}(\text{Cat}_\infty)_{/\mathcal{C}}.$$  

Applying this functor to $A \in \text{bAlg}_\mathcal{C}(\mathcal{C})$ gives the desired symmetric monoidal functor $\text{cLMod}_A(\mathcal{C}) \to \mathcal{C}$.

Variant 2.2.3. Let $\mathcal{C}$ be a small symmetric monoidal ∞-category and let $A$ be a cocommutative bialgebra object of $\mathcal{C}$. Applying Construction 2.2.2 with $\mathcal{C}^{\text{op}}$ in place of $\mathcal{C}$, we obtain a symmetric monoidal structure on the ∞-category $\text{LMod}_A(\mathcal{C})$, together with one on the forgetful functor $\text{LMod}_A(\mathcal{C}) \to \mathcal{C}$.

We now move on to the proof of Proposition 2.2.1. The key constructions needed for this are very similar to those of [L-HA, §§4.8.4–4.8.5]; we will omit details here.

Notation 2.2.4. The construction $\mathcal{C} \mapsto \text{cAlg}(\mathcal{C})$ determines a functor $\text{cAlg} : \text{Alg}(\text{Cat}_\infty) \to \text{Cat}_\infty$, classified by a cocartesian fibration that we denote by $\text{Cat}_\infty^{c\text{Alg}} \to \text{Alg}(\text{Cat}_\infty)$. We identify objects of $\text{Cat}_\infty^{c\text{Alg}}$ with pairs $(\mathcal{C}, A)$ where $\mathcal{C}$ is a small monoidal ∞-category and $A$ is a coalgebra object of $\mathcal{C}$.

For the sake of consistent notation, let $\text{Cat}_\infty^{\text{RMod}}$ denote the ∞-category $\text{RMod}(\text{Cat}_\infty)$ of pairs $(\mathcal{C}, \mathcal{M})$ where $\mathcal{C}$ is a small monoidal ∞-category and $\mathcal{M}$ is a small ∞-category right tensored over $\mathcal{C}$. We have a cocartesian fibration $\text{Cat}_\infty^{\text{RMod}} \to \text{Alg}(\text{Cat}_\infty)$ sending $(\mathcal{C}, \mathcal{M}) \mapsto \mathcal{C}$.

Remark 2.2.5. The ∞-categories $\text{Cat}_\infty^{c\text{Alg}}$ and $\text{Cat}_\infty^{\text{RMod}}$ admit finite products, given by the formulas

$$\prod_{i \in I}(\mathcal{C}_i, A_i) \simeq (\prod_{i \in I} \mathcal{C}_i, \{A_i\}_{i \in I}) \quad \text{and} \quad \prod_{i \in I}(\mathcal{C}_i, \mathcal{M}_i) \simeq (\prod_{i \in I} \mathcal{C}_i, \prod_{i \in I} \mathcal{M}_i)$$

3The relative tensor product construction goes through because the cartesian product in $\text{Cat}_\infty$ commutes with colimits in each variable, which follows from the fact that $\text{Cat}_\infty$ is cartesian closed.
respectively. We regard $\text{Cat}^{\text{cAlg}}$ and $\text{Cat}^{\text{RMod}}$ as symmetric monoidal $\infty$-categories via the cartesian symmetric monoidal structures.

**Construction 2.2.6.** Given a monoidal $\infty$-category $\mathcal{C}$ and a coalgebra object $A$ of $\mathcal{C}$, the $\infty$-category $\text{cLMod}_A(\mathcal{C})$ is canonically right tensored over $\mathcal{C}$. Moreover, this construction canonically extends to a functor $\mu : \text{Cat}^{\text{cAlg}} \to \text{Cat}^{\text{RMod}}$ over $\text{Alg}(\text{Cat}_\infty)$ sending $(\mathcal{C}, A) \mapsto (\mathcal{C}, \text{cLMod}_A(\mathcal{C}))$. The functor $\mu$ evidently preserves finite products, and we thus regard it as a symmetric monoidal functor.

**Proof of Proposition 2.2.1.** We may view the small symmetric monoidal $\infty$-category $\mathcal{C}$ as a commutative algebra object of $\text{Alg}(\text{Cat}_\infty)$. This determines symmetric monoidal structures on the fibers

$$\text{Cat}_\infty^{\text{cAlg}} \times_{\text{Alg}(\text{Cat}_\infty)} \{\mathcal{C}\} \simeq \text{cAlg}(\mathcal{C}) \quad \text{and} \quad \text{Cat}_\infty^{\text{RMod}} \times_{\text{Alg}(\text{Cat}_\infty)} \{\mathcal{C}\} \simeq \text{RMod}_\mathcal{C}(\text{Cat}_\infty),$$

which identify canonically with the usual ones (cf. [L-HA, Remark 4.8.5.19]). Thus, the symmetric monoidal functor $\mu$ of Construction 2.2.6 restricts to the desired symmetric monoidal functor $\mu_\mathcal{C} : \text{cAlg}(\mathcal{C}) \to \text{RMod}_\mathcal{C}(\text{Cat}_\infty)$, with $\mu_\mathcal{C}(A) \simeq \text{cLMod}_A(\mathcal{C})$. \qed

**Remark 2.2.7.** Let $\mathcal{C}$ be a small symmetric monoidal $\infty$-category. The proof of Proposition 2.2.1 shows that Construction 2.2.2 exhibits the following functoriality properties:

(a) Let $A \to A'$ be a map of commutative bialgebras in $\mathcal{C}$. Then there is a canonical symmetric monoidal structure on the corestriction functor $\text{cLMod}_A(\mathcal{C}) \to \text{cLMod}_{A'}(\mathcal{C})$, compatible with the symmetric monoidal forgetful functors to $\mathcal{C}$.

(b) Let $D$ be another small symmetric monoidal $\infty$-category and $F : \mathcal{C} \to D$ a symmetric monoidal functor. Let $A$ be a commutative bialgebra in $\mathcal{C}$. Then there is an induced commutative bialgebra structure on $F(A)$, and the functor $F' : \text{cLMod}_A(\mathcal{C}) \to \text{cLMod}_{F(A)}(D)$ determined by $F$ is canonically symmetric monoidal.

**Example 2.2.9.** Let $G$ be a group object of $\text{Spc}$; by Example 2.1.4, we may regard $G$ as a cocommutative bialgebra object of $\text{Spc}$. Let $\mathcal{C}$ be a presentable symmetric monoidal $\infty$-category. Then we have a unique colimit-preserving symmetric monoidal functor $F : \text{Spc} \to \mathcal{C}$, giving us a cocommutative bialgebra $F(G)$ in $\mathcal{C}$. From Variant 2.2.3, we obtain a symmetric monoidal structure on $\text{LMod}_{F(G)}(\mathcal{C})$. We claim that there is a canonical symmetric monoidal equivalence between this and $\text{Fun}(BG, \mathcal{C})$ (with the pointwise tensor product, and where $BG$ denotes the classifying space of $G$).

To see this, first observe that we have canonical equivalences $\text{LMod}_{F(G)}(\mathcal{C}) \simeq \mathcal{C} \otimes \text{LMod}_G(\text{Spc})$ and $\text{Fun}(BG, \mathcal{C}) \simeq \mathcal{C} \otimes \text{Fun}(BG, \text{Spc})$ in $\text{Alg}(\text{Pr}_L)$ (the former as mentioned in Remark 2.2.7 and the latter following from [L-HA, Proposition 4.8.1.17]). It therefore suffices to treat the case $\mathcal{C} = \text{Spc}$. In this case, the symmetric monoidal structures on $\text{LMod}_G(\text{Spc})$ and $\text{Fun}(BG, \text{Spc})$ are cartesian, so the standard equivalence $\text{LMod}_G(\text{Spc}) \simeq \text{Fun}(BG, \text{Spc})$ promotes uniquely to a symmetric monoidal equivalence.
We close this subsection with an alternative characterization of commutative algebra objects for the symmetric monoidal structures constructed above.

**Proposition 2.2.10.** In the situation of Construction 2.2.2, there is a canonical equivalence of ∞-categories

\[
cLMod_A(CAlg(\mathcal{C})) \simeq CAlg(cLMod_A(\mathcal{C}))
\]

commuting with the forgetful functors to \(\mathcal{C}\), where on the left-hand side we regard the commutative bialgebra \(A\) as a coalgebra object of \(CAlg(\mathcal{C})\).

**Proof.** Using the fact that the forgetful functor \(CAlg(CAlg(\mathcal{C})) \to CAlg(\mathcal{C})\) is an equivalence, we may regard \(A\) as a commutative bialgebra object of \(CAlg(\mathcal{C})\) (which is carried to the original commutative bialgebra object \(A\) of \(\mathcal{C}\) under the forgetful functor \(CAlg(\mathcal{C}) \to \mathcal{C}\)). By Remark 2.2.7(b), we have a canonical symmetric monoidal functor \(\alpha : cLMod_A(CAlg(\mathcal{C})) \to cLMod_A(\mathcal{C})\). The symmetric monoidal structure on \(CAlg(\mathcal{C})\) is cocartesian, from which we deduce that the same is true of \(cLMod_A(CAlg(\mathcal{C}))\) (as the forgetful functor \(cLMod_A(CAlg(\mathcal{C})) \to CAlg(\mathcal{C})\) is symmetric monoidal and preserves coproducts). It follows that the functor \(\alpha\) factors uniquely through the forgetful functor \(CAlg(cLMod_A(\mathcal{C}))\), giving us a commutative diagram of ∞-categories

\[
\begin{array}{ccc}
\text{cLMod}_A(CAlg(\mathcal{C})) & \longrightarrow & \text{CAlg}(cLMod_A(\mathcal{C})) \\
\downarrow \quad & & \downarrow \\
\text{cLMod}_A(\mathcal{C}). & &
\end{array}
\]

We wish to show that the horizontal arrow in the above diagram is an equivalence. Using a reduction similar to the one made in the proof of Proposition 2.1.2, we may move to the situation that \(\mathcal{C}\) is a presentable symmetric monoidal ∞-category. Then, by the Barr–Beck–Lurie theorem, the two diagonal arrows in the diagram are monadic functors, with left adjoints both given by the formation of symmetric algebras in \(\mathcal{C}\) (here we are using that the symmetric monoidal structure on \(cLMod_A(\mathcal{C})\) is presentable, as pointed out in Remark 2.2.8). It thus follows from [L-HA, Corollary 4.7.3.16] that the horizontal arrow is an equivalence. \(\square\)

§2.3. Duality. Let \(\mathcal{C}\) be a symmetric monoidal ∞-category and let \(\mathbb{1} \in \mathcal{C}\) denote the unit object. Recall that an object \(X \in \mathcal{C}\) is called dualizable if there exists an object \(X^\vee \in \mathcal{C}\) and maps \(\epsilon : X \otimes X^\vee \to \mathbb{1}\) and \(\eta : X^\vee \otimes X \to \mathbb{1}\) such that the compositions

\[
\begin{align*}
& X \simeq \mathbb{1} \otimes X \xrightarrow{\eta \otimes id} X \otimes X^\vee \otimes X \xrightarrow{id \otimes \epsilon} X \otimes \mathbb{1} \simeq X, \\
& X^\vee \simeq X^\vee \otimes \mathbb{1} \xrightarrow{id \otimes \eta} X^\vee \otimes X \otimes X^\vee \xrightarrow{\epsilon \otimes id} \mathbb{1} \otimes X^\vee \simeq X^\vee
\end{align*}
\]

are homotopic to the identity. If \(X\) is dualizable, the object \(X^\vee\) is determined up to equivalence and referred to as the dual of \(X\).

Our goal in this subsection is to understand the behavior of bialgebra structures under dualization. Similar to the previous subsection, we shall work under the assumption that \(\mathcal{C}\) is small, but this restriction is only apparent.

**Notation 2.3.1.** We let \(\mathcal{C}_{\text{fd}}\) denote the full subcategory of \(\mathcal{C}\) spanned by the dualizable objects; it contains \(\mathbb{1}\) and is closed under tensor products, hence inherits a symmetric monoidal structure from \(\mathcal{C}\). Dualization then determines a functor \((\cdot)^\vee : \mathcal{C}_{\text{fd}} \to \mathcal{C}_{\text{fd}}^{\text{op}}\).

**Proposition 2.3.2.** There exists a symmetric monoidal functor

\[
\beta : cAlg(\mathcal{C}_{\text{fd}}) \to Alg(\mathcal{C}_{\text{fd}})^{\text{op}} \simeq cAlg(\mathcal{C}_{\text{fd}}^{\text{op}})
\]
uniquely determined by the commuting of the following diagram of symmetric monoidal $\infty$-categories:

$$
\begin{aligned}
\text{cAlg}(\mathcal{C}_{\text{fd}}) & \longrightarrow \text{cAlg}(\mathcal{C}) & \mu' \longrightarrow \text{RMod}_{\mathcal{C}}(\text{Cat}_{\infty})_{/\mathcal{C}} \\
\beta \downarrow & & \downarrow (-)^{op} \\
\text{cAlg}(\mathcal{C}_{\text{fd}}^{\text{op}}) & \longrightarrow \text{cAlg}(\mathcal{C}_{\text{fd}}^{\text{op}}) & \mu'^{op} \longrightarrow \text{RMod}_{\mathcal{C}^{\text{op}}}(\text{Cat}_{\infty})_{/\mathcal{C}^{\text{op}}}
\end{aligned}
$$

(where the symmetric monoidal functors $\mu'(-)$ are as in Construction 2.2.2). Moreover, the diagram of $\infty$-categories

$$
\begin{aligned}
\text{cAlg}(\mathcal{C}_{\text{fd}}) & \longrightarrow \text{cAlg}(\mathcal{C}_{\text{fd}}^{\text{op}}) \\
\beta \downarrow & & \downarrow (-)^{\gamma} \\
\mathcal{C}_{\text{fd}} & \longrightarrow \mathcal{C}_{\text{fd}}^{\text{op}}
\end{aligned}
$$

canonical equivalence (where the vertical arrows are the forgetful functors).

**Corollary 2.3.3.** Let $A$ be a dualizable coalgebra (resp. commutative bialgebra) object of $\mathcal{C}$. Then there exists a unique algebra (resp. cocommutative bialgebra) structure on $\mathcal{C}$ uniquely determined by the commuting of the following diagram of symmetric monoidal $\infty$-categories $\text{cAlg}_A(\mathcal{C}) \simeq \text{LMod}_{\mathcal{A}^{\text{op}}}(\mathcal{C})$ commuting with the forgetful functors to $\mathcal{C}$.

**Remark 2.3.4.** It is immediate from Proposition 2.3.2 that the composite

$$
\text{cAlg}(\mathcal{C}_{\text{fd}}) \xrightarrow{\beta} \text{Alg}(\mathcal{C}_{\text{fd}})^{\text{op}} \xrightarrow{\beta} \text{cAlg}(\mathcal{C}_{\text{fd}})
$$

is canonically equivalent to the identity, and hence that $\beta$ is in fact a symmetric monoidal equivalence.

The proof of Proposition 2.3.2 will require some preliminaries. In the following two results, we will denote objects of the $\infty$-category $\text{RMod}_{\mathcal{C}}(\text{Cat}_{\infty})_{/\mathcal{C}}$ by pairs $(\mathcal{M}, U)$, where $\mathcal{M}$ is an $\infty$-category right tensored over $\mathcal{C}$ and $U$ is a $\mathcal{C}$-linear functor $U : \mathcal{M} \to \mathcal{C}$, and sometimes will omit $U$ when it may be understood from context. Also, note that if $U$ admits a right adjoint $G : \mathcal{C} \to \mathcal{M}$, then $G$ is canonically lax $\mathcal{C}$-linear, and we may consider the property that $G$ is in fact strictly $\mathcal{C}$-linear ([L-HA, Example 7.3.2.8 and Remark 7.3.2.9]).

**Lemma 2.3.5.** Let $(\mathcal{M}, U)$ be an object of $\text{RMod}_{\mathcal{C}}(\text{Cat}_{\infty})_{/\mathcal{C}}$ such that $U$ admits a right adjoint $G : \mathcal{C} \to \mathcal{M}$ that is $\mathcal{C}$-linear. Then there is a coalgebra $\mathcal{A} \in \text{cAlg}(\mathcal{C})$ equipped with a map $\alpha : \mathcal{M} \to \text{cLMod}_{\mathcal{A}}(\mathcal{C})$ in $\text{RMod}_{\mathcal{C}}(\text{Cat}_{\infty})_{/\mathcal{C}}$, such that, for any $B \in \text{cAlg}(\mathcal{C})$, the map

$$
\text{Map}_{\text{cAlg}(\mathcal{C})}(A, B) \to \text{Map}_{\text{RMod}_{\mathcal{C}}(\text{Cat}_{\infty})_{/\mathcal{C}}}(\mathcal{M}, \text{cLMod}_{\mathcal{B}}(\mathcal{C}))
$$

induced by $\alpha$ is a homotopy equivalence.

**Proof.** Let $T := UG \in \text{End}(\mathcal{C})$. Recall from [L-HA, §4.7.3] that:

- the natural transformation $\eta' : U \to TU = UGU$ induced by the unit transformation $\eta : \text{id}_{\mathcal{M}} \to GU$ exhibits $T$ as a coendomorphism object for $U \in \text{Fun}(\mathcal{M}, \mathcal{C})$, where we regard $\text{Fun}(\mathcal{M}, \mathcal{C})$ as left tensored over the monoidal $\infty$-category $\text{End}(\mathcal{C})$ via postcomposition;

- this determines a comonad structure on $T$ and a factorization of $U$ through the forgetful functor $\text{cLMod}_{T}(\mathcal{C}) \to \mathcal{C}$, such that, for any other comonad $T' \in \text{cAlg}(\text{End}(\mathcal{C}))$, the induced map

$$
\text{Map}_{\text{cAlg}(\text{End}(\mathcal{C}))}(T, T') \to \text{Map}_{(\text{Cat}_{\infty})_{/\mathcal{C}}}(\mathcal{M}, \text{cLMod}_{T'}(\mathcal{C}))
$$

is a homotopy equivalence.

In our situation, the functors $U, G, T$ and natural transformations $\eta, \eta'$ are all canonically $\mathcal{C}$-linear, and the same argument may be carried out with $\text{End}(\mathcal{C})$ replaced by the monoidal $\infty$-category $\text{End}_{\mathcal{C}}(\mathcal{C})$ of $\mathcal{C}$-linear endofunctors and $(\text{Cat}_{\infty})_{/\mathcal{C}}$ replaced by $\text{RMod}_{\mathcal{C}}(\text{Cat}_{\infty})_{/\mathcal{C}}$. Since we have a canonical equivalence of monoidal $\infty$-categories $\text{End}_{\mathcal{C}}(\mathcal{C}) \simeq \mathcal{C}$, this gives the claim. □
Proposition 2.3.6. The functor $\mu^\phi_{\mathcal{C}} : \mathrm{cAlg}(\mathcal{C}) \to \mathrm{RMod}_C(\mathcal{C})$ is fully faithful. Its essential image consists of those objects $(\mathcal{M}, U) \in \mathrm{RMod}_C(\mathcal{C})$ satisfying the following properties:

(a) $U$ is comonadic, in particular admits a right adjoint $G : \mathcal{C} \to \mathcal{M}$;

(b) $G$ is $\mathcal{C}$-linear.

Proof. Let $\mathrm{RMod}_C(\mathcal{C})^{\mathcal{C}}_0$ denote the full subcategory of $\mathrm{RMod}_C(\mathcal{C})^{\mathcal{C}}$ spanned by those objects $(\mathcal{M}, U)$ such that $U$ admits a $\mathcal{C}$-linear right adjoint $G$. The functor $\mu^\phi_{\mathcal{C}}$, factors through this subcategory, so let us abuse notation and now regard $\mu^\phi_{\mathcal{C}}$ as a functor $\mathrm{cAlg}(\mathcal{C}) \to \mathrm{RMod}_C(\mathcal{C})^{\mathcal{C}}_0$. It follows from Lemma 2.3.5 that $\mu^\phi_{\mathcal{C}}$ admits a left adjoint $\nu^\phi_{\mathcal{C}} : \mathrm{RMod}_C(\mathcal{C})^{\mathcal{C}}_0 \to \mathrm{cAlg}(\mathcal{C})$. It is easy to see that the counit transformation $\nu^\phi_{\mathcal{C}} \mu^\phi_{\mathcal{C}} \cong \mathrm{id}_{\mathrm{cAlg}(\mathcal{C})}$ is an equivalence, implying that $\mu^\phi_{\mathcal{C}}$ is fully faithful, and that the unit transformation $\mu^\phi_{\mathcal{C}} \nu^\phi_{\mathcal{C}}$ is an equivalence on an object $(\mathcal{M}, U)$ if and only if $U$ is comonadic, implying that the essential image of $\Psi^\phi_{\mathcal{C}}$ is as claimed. □

We need one more lemma to prove Proposition 2.3.2.

Lemma 2.3.7. Let $A \in \mathrm{cAlg}(\mathcal{C}^{\mathcal{A}})$ be a dualizable coalgebra object of $\mathcal{C}$. Then the forgetful functor $U : \mathrm{cLMod}(\mathcal{C}) \to \mathcal{C}$ is monadic, in particular admits a left adjoint $F : \mathcal{C} \to \mathrm{cLMod}(\mathcal{C})$. Moreover, there is a canonical equivalence between the composition $UF$ and the functor $A^\vee \otimes - : \mathcal{C} \to \mathcal{C}$.

Proof. By a reduction similar to the one made in Proposition 2.1.2, we may instead work in the situation that $\mathcal{C}$ is a presentable symmetric monoidal $\infty$-category. Since $U$ is a forgetful functor from a comodule category, it preserves colimits. As $A$ is dualizable, the functor $A \otimes - : \mathcal{C} \to \mathcal{C}$ preserves limits, implying that $U$ also preserves limits. It follows from the adjoint functor theorem that $U$ admit a left adjoint $F$ and from the Barr–Beck–Lurie theorem that $U$ is monadic.

Let $G : \mathcal{C} \to \mathrm{cLMod}(\mathcal{C})$ denote the right adjoint of $U$, which is given by tensoring with $A$, so that $UG$ is naturally equivalent to the functor $A \otimes - : \mathcal{C} \to \mathcal{C}$. It is then immediate from $UF$ being left adjoint to $UG$ that $U \simeq A^\vee \otimes -$. □

Proof of Proposition 2.3.2. By Proposition 2.3.6, $\mu^\phi_{\mathcal{C}}$ and $\mu^\phi_{\mathcal{C}}^{\mathcal{C}}$ are fully faithful. Thus, uniqueness of $\beta$ is automatic, and to prove existence, it suffices to show for $A \in \mathrm{cAlg}(\mathcal{C}^{\mathcal{A}})$ that $\mathrm{cLMod}(\mathcal{C})^{\mathcal{A}}$ lies in the essential image of $\mu^\phi_{\mathcal{C}}^{\mathcal{A}}$. Using the description of the essential image in Proposition 2.3.6 (and unravelling opposites), we need to show that the forgetful functor $U : \mathrm{cLMod}(\mathcal{C}) \to \mathcal{C}$ admits a $\mathcal{C}$-linear left adjoint $F : \mathcal{C} \to \mathrm{cLMod}(\mathcal{C})$, that $U$ is monadic, and that the $\mathcal{C}$-linear composite $UF$ is given by tensoring with a dualizable object of $\mathcal{C}$. But this follows from Lemma 2.3.7. Since Lemma 2.3.7 in fact gives a natural identification $U \simeq A^\vee \otimes -$, we deduce a natural identification between the underlying object in $\mathcal{C}$ of $\beta(A)$ with $A^\vee$. □

§2.4. The Tate construction. Let $X$ be a spectrum with $S^1$-action, i.e. a diagram $BS^1 \to \text{Spt}$. The homotopy orbits $X_{hS^1}$ and homotopy fixed points $X^{hS^1}$ are defined as the colimit and limit, respectively, of the diagram. There is a canonical norm map $\text{Nm}_X : X_{hS^1}[1] \to X^{hS^1}$ relating these two constructions, and the Tate construction $X^{tS^1}$ is defined as the cofiber of $\text{Nm}_X$. In the case that $X$ arises from the Hochschild homology construction, $X_{hS^1}$ is cyclic homology, $X^{hS^1}$ is negative cyclic homology, and $X^{tS^1}$ is periodic cyclic homology.

The goal of this subsection is to formulate these notions in a more general context that will apply not just to objects with $S^1$-action but to filtered objects with filtered $S^1$-action (and in fact to homotopy-coherent cochain complexes as well). Throughout this subsection, we work in the following general context:

- We let $\mathcal{C}$ be a stable presentable symmetric monoidal $\infty$-category. We denote the unit object by $1$. Presentability ensures that the monoidal structure is closed; we denote internal mapping objects in $\mathcal{C}$ by $\text{Map}(\cdot, \cdot)$. 

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We let $A$ be a cocommutative bialgebra object of $\mathcal{C}$. We regard $\text{LMod}_A(\mathcal{C})$ as a presentable symmetric monoidal $\infty$-category via Variant 2.2.3 (and Remark 2.2.8), and denote the tensor product simply by $\otimes$. Note however that we will also be using the relative tensor product over $A$, which we denote by $\odot_A$. In $\text{LMod}_A(\mathcal{C})$ too we have $\mathcal{C}$-valued mapping objects, which we denote by $\text{Map}_A(-,-)$: these are determined by natural equivalences

$$\text{Map}_\mathcal{C}(T, \text{Map}_A(X, Y)) \simeq \text{Map}_{\text{LMod}_A(\mathcal{C})}(X \otimes T, Y).$$

- We assume that $A$ is dualizable in $\mathcal{C}$. We regard $A^\vee$ as an $A$-module by virtue of it being the image of $1$ under the functor $\mathcal{C} \to \text{LMod}_A(\mathcal{C})$ right adjoint to the forgetful functor (this follows from the categorical dual of Lemma 2.3.7).

- We assume given an invertible object $\omega_A \in \mathcal{C}$ and an equivalence of $A$-modules $\alpha: A^\vee \simeq A \otimes \omega_A^{-1}$. The last stipulation on $A^\vee$ is motivated by the example of $\mathcal{C} = \text{Spt}$ and $A = S[S^1]$. Namely, one way to understand the source of the shift in the norm map for spectra with $S^1$-action described above is that this stipulation is satisfied in this example with $\omega_A \simeq S[1]$ (by Atiyah duality).

Let us now begin by defining generalizations of the orbits, fixed points, norm map, and Tate construction for $A$-modules in $\mathcal{C}$.

**Notation 2.4.1 (Orbits and fixed points).** Let $\epsilon: A \to 1$ denote the counit of the bialgebra structure on $A$. Restriction in $\epsilon$ determines a functor $\rho: \mathcal{C} \to \text{LMod}_A(\mathcal{C})$ that preserves limits and colimits, hence admits left and right adjoints by the adjoint functor theorem. We will denote these adjoints by $(-)_A: \text{LMod}_A(\mathcal{C}) \to \mathcal{C}$ and $(-)^A: \text{LMod}_A(\mathcal{C}) \to \mathcal{C}$, respectively; they are given by the formulas

$$X_A = X \otimes_A 1 \quad \text{and} \quad X^A \simeq \text{Map}_A(1, X).$$

**Remark 2.4.2.** Let $B_* : \Delta^{op} \to \text{LMod}_A(\mathcal{C})$ denote the bar resolution computing $1 \otimes_A A \simeq 1$, so that $B_n = A \otimes_{n+1}$ for $n \geq 0$. Then, for any $X \in \text{LMod}_A(\mathcal{C})$, we obtain a cosimplicial diagram $\text{Map}_A(B_*, X): \Delta \to \text{LMod}_A(\mathcal{C})$, with

$$\text{Map}_A(B_n, X) \simeq \text{Map}_A(A \otimes_{n+1}, X) \simeq \text{Map}(A \otimes n, X) \simeq (A^\vee)^n \odot X,$$

and whose limit is $\text{Map}_A(1, X) \simeq X^A$.

**Remark 2.4.3.** It follows from Remark 2.2.7 that the restriction functor $\rho: \mathcal{C} \to \text{LMod}_A(\mathcal{C})$ is canonically symmetric monoidal, implying that the right adjoint $(-)^A: \text{LMod}_A(\mathcal{C}) \to \mathcal{C}$ is canonically lax symmetric monoidal.

**Construction 2.4.4 (Norm map and Tate construction).** Let $\eta: 1 \to A^\vee$ denote the map of $A$-modules corresponding to the identity map under the equivalence $\text{Map}_{\text{LMod}_A(\mathcal{C})}(1, A^\vee) \simeq \text{Map}_\mathcal{C}(1, 1)$. For any $X \in \text{LMod}_A(\mathcal{C})$, this induces a map of $A$-modules

$$\omega_A \otimes X_A \simeq \omega_A \otimes (X \otimes_A 1)$$

$$\xrightarrow{\eta} \omega_A \otimes (X \otimes_A A^\vee)$$

$$\xrightarrow{\alpha} \omega_A \otimes (X \otimes_A (A \otimes \omega_A^{-1}))$$

$$\simeq \omega_A \otimes X \otimes \omega_A^{-1}$$

$$\simeq X.$$

Since the left-hand side is a module over $1$ in $\text{LMod}_A(\mathcal{C})$, this factors uniquely through a map

$$\text{Nm}_X: \omega_A \otimes X_A \to X^A$$

in $\mathcal{C}$, which we refer to as the **norm map**. We then define the **Tate construction** of $X$ as

$$X^{TA} := \text{cofib}(\text{Nm}_X).$$

Note that these constructions are evidently functorial in $X$, in the sense that they determine functors $\text{Nm}: \Delta^1 \times \text{LMod}_A(\mathcal{C}) \to \mathcal{C}$ and $(-)^{TA}: \text{LMod}_A(\mathcal{C}) \to \mathcal{C}$.
Remark 2.4.5. As \((-)^A : \text{LMod}_A(\mathcal{C}) \to \mathcal{C}\) is a right adjoint functor between presentable \(-\)-categories, it is \(\kappa\)-accessible for some regular cardinal \(\kappa\). Since \(\omega_A \otimes (-)_A : \text{LMod}_A(\mathcal{C}) \to \mathcal{C}\) preserves all colimits, it follows that the Tate construction \((-)^{1A} : \text{LMod}_A(\mathcal{C}) \to \mathcal{C}\) is also \(\kappa\)-accessible.

Next, imitating [NS18, §I.3], we characterize the above Tate construction, or more precisely the natural transformation \((-)^A \to (-)^{1A}\), by a universal property involving the behavior of these functors on induced \(A\)-modules (see Definition 2.4.6 below). As in op. cit., this will show that the functor \((-)^{1A}\) inherits a lax symmetric monoidal structure from the one on the functor \((-)^A\) (Remark 2.4.3). It will also show that the definitions here agree with the usual ones in the setting of group actions.

Definition 2.4.6. We let \(\text{LMod}^{\text{ind}}_A(\mathcal{C})\) denote the smallest stable full subcategory of \(\text{LMod}_A(\mathcal{C})\) containing the objects \(A \otimes X\) for \(X \in \mathcal{C}\). We refer to the objects of \(\text{LMod}^{\text{ind}}_A(\mathcal{C})\) as induced \(A\)-modules.

Remark 2.4.7. The subcategory \(\text{LMod}^{\text{ind}}_A(\mathcal{C}) \subseteq \text{LMod}_A(\mathcal{C})\) is a \(\otimes\)-ideal: that is, given an induced \(A\)-module \(X\) and an arbitrary \(A\)-module \(Y\), the tensor product \(X \otimes Y\) is also induced. It is enough to show this in the case that \(X = A \otimes X_0\) for \(X_0 \in \mathcal{C}\). Assuming this, the canonical map \(X_0 \to X\) induces a map \(X_0 \otimes Y \to X \otimes Y\) in \(\mathcal{C}\), which extends uniquely to a map \(u : A \otimes (X_0 \otimes Y) \to X \otimes Y\) in \(\text{LMod}_A(\mathcal{C})\). It is easy to see that \(u\) is an equivalence, which proves the claim.

Lemma 2.4.8. For \(X \in \text{LMod}^{\text{ind}}_A(\mathcal{C})\), we have \(X^{1A} \simeq 0\).

Proof. As \((-)^{1A}\) is an exact functor, it suffices to prove this in the case that \(X = A \otimes X_0\) for \(X_0 \in \mathcal{C}\). We then have the chain of equivalences

\[
\omega_A \otimes X_A \simeq \omega_A \otimes X_0 \\
\simeq \text{Map}(1, \omega_A \otimes X_0) \\
\simeq \text{Map}_A(1, A^Y \otimes \omega_A \otimes X_0) \\
\simeq \text{Map}_A(1, A \otimes X_0) \\
\simeq \text{Map}_A(1, X) \\
\simeq X^{1A}.
\]

It is straightforward to check that the composite equivalence is homotopic to \(\text{Nm}_X\), implying that \(X^{1A} = \text{cofib}(\text{Nm}_X) \simeq 0\).

Lemma 2.4.9. Suppose that \(\mathcal{C}\) is compactly generated. Then, for all \(X \in \text{LMod}_A(\mathcal{C})\), the canonical maps

\[
\left[ \colim_{Y \in \text{LMod}^{\text{ind}}_A(\mathcal{C})/X} Y \right] \to X,
\]

\[
\left[ \colim_{Y \in \text{LMod}^{\text{ind}}_A(\mathcal{C})/X} \text{cofib}(Y \to X)^A \right] \to \left[ \colim_{Y \in \text{LMod}^{\text{ind}}_A(\mathcal{C})/X} \text{cofib}(Y \to X)^{1A} \right]
\]

are equivalences.

Proof. We begin with the first map. Let us choose a set \(\{W_s\}_{s \in S}\) of compact generators for \(\mathcal{C}\). We may assume that the set is closed under shifts, so that it suffices to show for each \(s \in S\) that the induced map of abelian groups

\[
\colim_{Y \in \text{LMod}^{\text{ind}}_A(\mathcal{C})/X} \pi_0 \text{Map}_C(W_s, Y) \to \pi_0 \text{Map}_C(W_s, \colim_{Y \in \text{LMod}^{\text{ind}}_A(\mathcal{C})/X} Y) \to \pi_0 \text{Map}_C(W_s, X)
\]

is an isomorphism; here the first map is an isomorphism because \(W_s\) is compact and the \(\infty\)-category \(\text{LMod}^{\text{ind}}_A(\mathcal{C})/X\) is filtered (as it admits finite colimits). We show that the map is surjective and injective. Surjectivity follows from the fact that any map \(W_s \to X\) factors as the composite \(W_s \to A \otimes W_s \to X\), the second map being an object of \(\text{LMod}^{\text{ind}}_A(\mathcal{C})/X\). For injectivity, the colimit
being filtered implies that it suffices to see that, given \( Y \in \text{LMod}^{\text{ind}}_A(\mathcal{C})/X \) and a map \( \eta : W_s \to Y \) whose composite with the map \( Y \to X \) is nullhomotopic, there is some map \( Y \to Y' \) in \( \text{LMod}^{\text{ind}}_A(\mathcal{C})/X \) whose composite with \( \eta \) is nullhomotopic. We may take \( Y' \) to be the cofiber of the induced map \( \eta' : A \otimes W_s \to Y \), which lies in \( \text{LMod}^{\text{ind}}_A(\mathcal{C}) \) since both \( A \otimes W_s \) and \( Y \) do, and through which the map \( Y \to X \) factors since the composite of \( \eta' \) with \( Y \to X \) is nullhomotopic.

We now address the second map. Its fiber is given by

\[
\text{colim}_{Y \in \text{LMod}^{\text{ind}}_A(\mathcal{C})/X} \text{cofib}(Y \to X)_A = \left( \text{colim}_{Y \in \text{LMod}^{\text{ind}}_A(\mathcal{C})/X} \text{cofib}(Y \to X) \right)_A,
\]

the equivalence resulting from the fact that the functor \((-)_A\) preserves colimits. But this vanishes, since

\[
\text{colim}_{Y \in \text{LMod}^{\text{ind}}_A(\mathcal{C})/X} \text{cofib}(Y \to X) \simeq \text{cofib}\left( \text{colim}_{Y \in \text{LMod}^{\text{ind}}_A(\mathcal{C})/X} Y \to X \right) \simeq 0,
\]

the first equivalence using that colimits commute with cofibers and that

\[
X = \text{colim}_{Y \in \text{LMod}^{\text{ind}}_A(\mathcal{C})/X} X
\]

since \( \text{LMod}^{\text{ind}}_A(\mathcal{C})/X \) is filtered, and the second equivalence following from the first part of the proposition.

\[\square\]

**Proposition 2.4.10.** Suppose that \( \mathcal{C} \) is compactly generated. Then there is a unique pair of lax symmetric monoidal structure on the Tate construction \((-)^A : \text{LMod}_A(\mathcal{C}) \to \mathcal{C} \) and lax symmetric monoidal structure on the natural transformation \((-)^A \to (-)^A \). Moreover, if \( \kappa \) is any regular cardinal such that the functors \((-)^A, (-)^A : \text{LMod}_A(\mathcal{C}) \to \mathcal{C} \) are \( \kappa \)-accessible (Remark 2.4.5), this exhibits \((-)^A \) as the initial, \( \kappa \)-accessible, exact lax symmetric monoidal functor \( F : \text{LMod}_A(\mathcal{C}) \to \mathcal{C} \) under \((-)^A \) satisfying \( F(X) \simeq 0 \) for all \( X \in \text{LMod}^{\text{ind}}_A(\mathcal{C}) \).

**Proof.** This follows from Remark 2.4.7 and Lemmas 2.4.8 and 2.4.9 using the the results of [NS18, §1.3].

\[\square\]

**Remark 2.4.11.** It follows from Proposition 2.4.10 that, in the case \( \mathcal{C} = \text{Spt} \) and \( A = \mathbb{S}[G] \) for \( G \) any finite group or compact abelian Lie group, the norm map and Tate construction defined in this section agree with the usual ones under the symmetric monoidal equivalence \( \text{LMod}_A(\mathcal{C}) \simeq \text{Fun}(BG, \text{Spt}) \) (Example 2.2.9), as it is shown in [NS18, §§1-3-1.4] that the latter enjoy the same universal property.

**Remark 2.4.12** (Naturality). Let \( \mathcal{D} \) be another stable presentable symmetric monoidal \( \infty \)-category and let \( F : \mathcal{C} \to \mathcal{D} \) be a symmetric monoidal functor. Then \( F(A) \) is a cocommutative bialgebra in \( \mathcal{D} \) satisfying the same hypotheses as \( A \), in particular with \( \omega_{F(A)} \simeq F(\omega_A) \). By Remark 2.2.7, \( F \) determines a symmetric monoidal functor \( \text{LMod}_A(\mathcal{C}) \to \text{LMod}_{F(A)}(\mathcal{D}) \), which here we will just denote by \( F \). This begets a natural transformation \( \phi_A : F(X)_F(A) \to F(X_A) \) and a lax symmetric monoidal natural transformation \( \phi^A : F(X^A) \to F(X)^{F(A)} \). It is straightforward to check that the composite

\[
F(X)_F(A) \xrightarrow{\phi_A} F(X_A) \xrightarrow{F(Nm)} F(X^A) \xrightarrow{\phi^A} F(X)^{F(A)}
\]

is canonically homotopic to \( \text{Nm}_{F(X)} \).

Suppose now that \( F \) preserves colimits. Then \( \phi_A \) is an equivalence, and \( \phi^A \) and \( \phi_A^{-1} \) induce a natural transformation \( \phi^A : F(X^{1A}) \to F(X)^{F(1A)} \).

Assume next that \( \mathcal{C} \) and \( \mathcal{D} \) are compactly generated, so that the functors \((-)^{1A} \) and \( (-)^{F(1A)} \) obtain lax symmetric monoidal structures from Proposition 2.4.10. We claim that \( \phi^{1A} \) is canonically a transformation of lax symmetric monoidal functors. To see this, let \( G : \mathcal{D} \to \mathcal{C} \) denote the right adjoint to \( F \), which exists by the adjoint functor theorem. The symmetric monoidal structure on \( F \) determines a lax symmetric monoidal structure on \( G \), and it suffices to promote the adjoint
transformation $X^A \to G(F(X)^{tF(A)})$ to one of lax symmetric monoidal functors. Since $F$ sends induced $A$-modules to induced $F(A)$-modules, this follows from Proposition 2.4.10 (where we choose our regular cardinal $\kappa$ large enough such that $G$ is $\kappa$-accessible).

Finally, if we assume moreover that $F$ preserves limits, then we deduce from Remark 2.4.2 that $\phi^A$ is an equivalence, and hence $\phi^{tA}$ is as well.

§3. Gradings and filtrations

Graded and filtered algebraic structures are central in this paper. This section is devoted to generalities concerning graded and filtered objects in a stable presentable symmetric monoidal $\infty$-category, following [L-Rot, §3] to a great extent. In §3.1, we set out our basic framework for working with graded and filtered objects; in §3.2, we discuss the relationship between graded and filtered objects as mediated by the “associated graded” construction (an instance of Koszul duality, and the first appearance of “homotopy-coherent cochain complexes” in the paper); in §3.3, we discuss various t-structures in the graded and filtered settings that will be useful later on (including a new and more general perspective on the “Beilinson t-structure” furnished by the results of §3.2).

§3.1. Definitions.

Notation 3.1.1. We regard the set of integers $\mathbb{Z}$ as a category via the standard partial order $\leq$, and we let $\mathbb{Z}_{ds}$ denote the set of integers regarded as a discrete category (i.e. with only identity morphisms). We use the following notation for any presentable symmetric monoidal $\infty$-category $\mathcal{C}$:

(a) Let $\text{Gr}(\mathcal{C})$ denote the $\infty$-category $\text{Fun}(\mathbb{Z}_{ds}, \mathcal{C})$, which we regard as a (stable presentable) symmetric monoidal $\infty$-category via the Day convolution symmetric monoidal structure coming from addition on $\mathbb{Z}$. We refer to objects of $\text{Gr}(\mathcal{C})$ as graded objects of $\mathcal{C}$, and sometimes denote them by $X^\ast$ or $\{X^n\}_{n \in \mathbb{Z}}$. We denote the Day convolution tensor product of graded objects by $\otimes$; it is given by the formula

$$(X^\ast \otimes Y^\ast)^n \simeq \bigsqcup_{i+j=n} X^i \otimes Y^j.$$  

For $n \in \mathbb{Z}$, restriction along the inclusion $\{n\} \to \mathbb{Z}_{ds}$ defines an evaluation functor $ev^n : \text{Gr}(\mathcal{C}) \to \mathcal{C}$, sending $X^\ast \mapsto X^n$. Each of these admits a fully faithful left adjoint insertion functor $\text{ins}^n : \mathcal{C} \to \text{Gr}(\mathcal{C})$, given by left Kan extension along the same inclusion, or concretely by the formula

$$\text{ins}^n(X)^m \simeq \begin{cases} X & m = n \\ 0 & \text{otherwise}. \end{cases}$$

Note that right Kan extension would produce the same result, i.e. $\text{ins}^n$ is also right adjoint to $ev^n$. We note that $\text{ins}^0$ has a canonical symmetric monoidal structure, since $\{0\} \to \mathbb{Z}$ is a map of commutative monoids, and hence $ev^0$ has a canonical lax symmetric monoidal structure.

(b) We let $\text{Fil}(\mathcal{C})$ denote the $\infty$-category $\text{Fun}(\mathbb{Z}^{op}, \mathcal{C})$, which we again regard as a (stable presentable) symmetric monoidal $\infty$-category via the Day convolution structure coming from addition on $\mathbb{Z}$. We refer to objects of $\text{Fil}(\mathcal{C})$ as filtered objects of $\mathcal{C}$, and sometimes denote them by $X^\ast$ or depict them by diagrams

$$\cdots \to X^2 \to X^1 \to X^0 \to X^{-1} \to X^{-2} \to \cdots.$$  

We denote the Day convolution tensor product of filtered objects by $\otimes$; it is given by the formula

$$(X^\ast \otimes Y^\ast)^n \simeq \operatorname{colim}_{i+j \geq n} X^i \otimes Y^j.$$  

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Just as for graded objects, for \( n \in \mathbb{Z} \), restriction along the inclusion \( \{n\} \to \mathbb{Z} \) defines an 
**evaluation functor** \( \text{ev}^n : \text{Fil}(\mathcal{C}) \to \mathcal{C} \), sending \( X^* \mapsto X^n \). And again each of these admits a fully 
faithful left adjoint **insertion functor** \( \text{ins}^n : \mathcal{C} \to \text{Fil}(\mathcal{C}) \) given by left Kan extension, here given 
by the formula

\[
\text{ins}^n(X)^m = \begin{cases} X & m \leq n \\ 0 & \text{otherwise} \end{cases}
\]

(when we need to refer to both the filtered and graded insertion functors in proximity, we 
will add extra decorations to clarify), with \( \text{ins}^0 \) again having a canonical symmetric monoidal 
structure and \( \text{ev}^0 \) a canonical lax symmetric monoidal structure.

(c) Restriction along the evident functor \( \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\text{op}} \) defines a functor \( \text{und} : \text{Fil}(\mathcal{C}) \to \text{Gr}(\mathcal{C}) \), 
extracting the **underlying graded object** of a filtered object. Given a filtered object \( X^* \), we may 
denote its underlying graded object by \( X^* \).

(d) We have a functor \( \text{gr} : \text{Fil}(\mathcal{C}) \to \text{Gr}(\mathcal{C}) \) extracting the **associated graded object** of filtered object; 
it is given by the formula \( \text{gr}(X)^n = \text{cofib}(X^{n+1} \to X^n) \).

(e) The functor \( \text{und} : \text{Fil}(\mathcal{C}) \to \text{Gr}(\mathcal{C}) \) admits a left adjoint, given by left Kan extension along 
the inclusion \( \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\text{op}} \). We denote this left adjoint by \( \text{spl} : \text{Gr}(\mathcal{C}) \to \text{Fil}(\mathcal{C}) \); it is described 
concretely by the formula \( \text{spl}(X)^i = \bigoplus_{j \geq i} X^j \). We say a filtered object \( X \in \text{Fil}(\mathcal{C}) \) is **split** if there 
exists an equivalence \( \text{spl}(\text{gr}(X)) \cong X \in \text{Fil}(\mathcal{C}) \).

(f) If \( X \) is a graded (resp. filtered) object of \( \mathcal{C} \), then, for \( n \in \mathbb{Z} \), we let \( X(n) \) denote the graded (resp. 
filtered) object defined by \( X(n)^m = X^{m-n} \). Note that, in both the graded and filtered settings, 
for each \( n \in \mathbb{Z} \), we have natural equivalences \( \text{ins}^n(X) = \text{ins}^0(X)(n) \) for \( X \in \mathcal{C} \). We will sometimes 
leave the functor \( \text{ins}^0 \) implicit, i.e. by default regard objects of \( \mathcal{C} \) as graded/filtered objects in 
grading/filtration 0, and by this token write \( X(n) \) to mean \( \text{ins}^n(X) \) for \( X \in \mathcal{C} \) and \( n \in \mathbb{Z} \).

**Variant 3.1.2.** We will also consider **nonnegative** graded and filtered objects. These are defined 
by replacing the set of integers \( \mathbb{Z} \) in Notation 3.1.1 with the set of nonnegative integers \( \mathbb{Z}_{\geq 0} \). If 
\( \mathcal{C} \) is a presentable symmetric monoidal \( \infty \)-category, we will denote the resulting \( \infty \)-categories of 
nonnegatively graded and filtered objects by \( \text{Gr}^{\geq 0}(\mathcal{C}) \) and \( \text{Fil}^{\geq 0}(\mathcal{C}) \). The rest of the constructions 
described in Notation 3.1.1 (the symmetric monoidal structures, the evaluation and insertion 
functors, the underlying and associated graded functors) also go through in the nonnegative setting, 
and we will use the same notation for them.

There are evident restriction functors \( \text{Gr}(\mathcal{C}) \to \text{Gr}^{\geq 0}(\mathcal{C}) \) and \( \text{Fil}(\mathcal{C}) \to \text{Fil}^{\geq 0}(\mathcal{C}) \), both of which 
we will denote by \( \text{ev}^{\geq 0} \). Similar to the evaluation functors discussed in Notation 3.1.1, these have 
fully faithful left adjoints, both of which we will denote by \( \text{ins}^{\geq 0} \) (and again, in the graded setting 
this is also right adjoint to \( \text{ev}^{\geq 0} \)). Similar to \( \text{ins}^0 \), the functors \( \text{ins}^{\geq 0} \) are canonically symmetric 
monoidal, and hence \( \text{ev}^{\geq 0} \) canonically lax symmetric monoidal. We often implicitly identify \( \text{Gr}^{\geq 0}(\mathcal{C}) \) 
and \( \text{Fil}^{\geq 0}(\mathcal{C}) \) with full subcategories of \( \text{Gr}(\mathcal{C}) \) and \( \text{Fil}(\mathcal{C}) \) via the embeddings \( \text{ins}^{\geq 0} \).

**Notation 3.1.3.** Let \( \mathcal{C} \) be a presentable symmetric monoidal \( \infty \)-category, and regard \( \text{Gr}(\mathcal{C}) \) 
and \( \text{Fil}(\mathcal{C}) \) as such as in Notation 3.1.1. We will refer to commutative algebras in \( \text{Gr}(\mathcal{C}) \) as 
**graded commutative algebras in** \( \mathcal{C} \), and denote \( \text{CAlg}(\text{Gr}(\mathcal{C})) \) by \( \text{GrCAlg}(\mathcal{C}) \). Similarly, we refer to 
commutative algebras in \( \text{Fil}(\mathcal{C}) \) as **filtered commutative algebras in** \( \mathcal{C} \), and denote \( \text{CAlg}(\text{Fil}(\mathcal{C})) \) by \( \text{FilCAlg}(\mathcal{C}) \). We also have **nonnegatively** graded and filtered commutative algebras in \( \mathcal{C} \), the 
\( \infty \)-categories of which we denote by \( \text{Gr}^{\geq 0}\text{CAlg}(\mathcal{C}) \) and \( \text{Fil}^{\geq 0}\text{CAlg}(\mathcal{C}) \) respectively.

**Remark 3.1.4.** In [L-Rot], the notions in Notation 3.1.1 are discussed only in the case \( \mathcal{C} = \text{Spt} \). 
However, this is the universal case in the stable setting: for any stable presentable symmetric 
monoidal \( \infty \)-category \( \mathcal{C} \), there are canonical equivalences \( \text{Gr}(\text{Spt}) \otimes \mathcal{C} \to \text{Gr}(\mathcal{C}) \) and \( \text{Fil}(\text{Spt}) \otimes \mathcal{C} \to \text{Fil}(\mathcal{C}) \) in \( \text{CAlg}(\text{Pr}^L) \). Moreover, the functors \( \text{und, gr} : \text{Fil}(\mathcal{C}) \to \text{Gr}(\mathcal{C}) \) are obtained by tensoring 
the functors \( \text{und, gr} : \text{Fil}(\text{Spt}) \to \text{Gr}(\text{Spt}) \) with \( \mathcal{C} \in \text{Pr}^L \).
§3.2. Koszul duality. Our goal in this subsection is to analyze the relationship between filtered and graded objects in a stable presentable symmetric monoidal ∞-category $\mathcal{C}$. The analysis can be framed by the following question: To what extent, and how, can one recover a filtered object $X \in \text{Fil}(\mathcal{C})$ from its associated graded object $\text{gr}(X) \in \text{Gr}(\mathcal{C})$? The answer turns out to fit into the framework of Koszul (or bar-cobar) duality. We’ll begin with a discussion of the general framework, and subsequently explain how it applies to graded and filtered objects.

Let us first recall the basic features of Koszul duality between augmented algebras and coalgebras, following [L-HA, §5.2.2].

**Recollection 3.2.1.** Let $\mathcal{C}$ be a monoidal ∞-category admitting geometric realizations. Let $A$ be an augmented algebra object of $\mathcal{C}$. We let $\text{Bar}(A) \in \mathcal{C}$ denote the bar construction on $A$: this is the geometric realization of a canonical simplicial diagram $\Delta^{op} \to \mathcal{C}$ that on objects sends $[n] \to A^{\otimes n}$, and under the assumption that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations, this computes the relative tensor product $\mathcal{C} \otimes \mathcal{C}$ (where $\mathcal{C}$ denotes the unit object).

The bar construction furthermore satisfies the following universal property. Let $\text{A}B\text{Mod}_A(\mathcal{C})$ denote the ∞-category of $A$-$A$-bimodule objects of $\mathcal{C}$ and let $\rho : \mathcal{C} \to \text{A}B\text{Mod}_A(\mathcal{C})$ denote the restriction functor induced by the augmentation of $A$. Then there is a canonical map $A \to \rho(\text{Bar}(A))$ in $\text{A}B\text{Mod}_A(\mathcal{C})$ such that the induced map

$$\text{Map}_\mathcal{C}(\text{Bar}(A), X) \to \text{Map}_{\text{A}B\text{Mod}_A(\mathcal{C})}(A, \rho(X))$$

is a homotopy equivalence for all $X \in \mathcal{C}$.

We then have the formally dual notion: assume instead that $\mathcal{C}$ admits totalizations and let $B$ be an augmented coalgebra object of $\mathcal{C}$. Then we let $\text{Cobar}(B) \in \mathcal{C}$ denote the cobar construction on $B$, which is the totalization of a canonical cosimplicial diagram $\Delta^{op} \to \mathcal{C}$ that on objects sends $[n] \to B^{\otimes n}$, and satisfies a universal property dual to the one stated for the bar construction.

Finally, we recall that, assuming $\mathcal{C}$ admits both geometric realizations and totalizations, the constructions $A \mapsto \text{Bar}(A)$ and $B \mapsto \text{Cobar}(B)$ canonically promote to a pair of adjoint functors

$$\text{Bar} : \text{Alg}_{\text{aug}}(\mathcal{C}) \rightleftarrows \text{cAlg}_{\text{aug}}(\mathcal{C}) : \text{Cobar},$$

where $\text{Alg}_{\text{aug}}(\mathcal{C})$ and $\text{cAlg}_{\text{aug}}(\mathcal{C})$ denote the ∞-categories of augmented algebra objects and augmented coalgebra objects of $\mathcal{C}$ respectively. In particular, the bar (resp. cobar) construction on an augmented algebra (resp. coalgebra) carries a canonical coalgebra (resp. algebra) structure.

**Remark 3.2.2.** Let $\mathcal{C}$ be a symmetric monoidal ∞-category admitting geometric realizations, and assume that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations. Let $A$ be an augmented commutative algebra object of $\mathcal{C}$. Since the forgetful functor $\text{Alg}(\text{CAlg}(\mathcal{C})) \to \text{CAlg}(\mathcal{C})$ is an equivalence, we may regard $A$ as an augmented algebra object in $\text{CAlg}(\mathcal{C})$, and hence perform the bar construction in $\text{CAlg}(\mathcal{C})$ to obtain $\text{Bar}(A)$ as an object in $\text{cAlg}(\text{CAlg}(\mathcal{C}))$, i.e. a commutative bialgebra object of $\mathcal{C}$. Since the forgetful functor $\text{CAlg}(\mathcal{C}) \to \mathcal{C}$ preserves geometric realizations, the underlying coalgebra of $\text{Bar}(A)$ agrees with the bar construction on underlying augmented algebra of $A$.

The following proposition supplies an alternative construction of a coalgebra structure on bar construction of an augmented algebra.\(^4\)

**Proposition 3.2.3.** Let $\mathcal{C}$ be a monoidal ∞-category admitting geometric realizations. Let $\mathcal{M} := \text{LMod}_A(\mathcal{C})$, which we regard as right tensored over $\mathcal{C}$. Then $\text{Bar}(A) \in \mathcal{C}$ is a coendomorphism object for $\mathcal{I} \in \mathcal{M}$.

\(^4\)I learned this statement from lecture notes of Lurie. Unfortunately, I was unable to prove that this alternative coalgebra structure agrees with the one constructed in [L-HA, §5.2.2] in general (though this must be the case). Fortunately, this will be a non-issue in the particular situation in which we will invoke both perspectives (the proof of Proposition 3.2.5).
Proof. Consider the canonical map $A \rightarrow \rho(\text{Bar}(A))$ discussed in Recollection 3.2.1 and the unit map $1 \rightarrow A$. These induce homotopy equivalences

$$\text{Map}_C(\text{Bar}(A), X) \cong \text{Map}_{A \text{BMod}_A(C)}(A, \rho(X)) \cong \text{Map}_{\text{LMod}_A(C)}(1, \rho'(X)),$$

where $\rho : C \rightarrow A \text{BMod}_A(C)$ and $\rho' : C \rightarrow \text{LMod}_A(C)$ denote the forgetful functors induced by the augmentation of $A$. Noting that $\rho'(X)$ is naturally equivalent to the tensoring of $1 \in M$ with $X \in C$, this exhibits $\text{Bar}(A)$ as a coendomorphism object for $1 \in M$.

What we will actually be relevant for us is the dual statement for the cobar construction:

**Variant 3.2.4.** Let $\mathcal{C}$ be a monoidal $\infty$-category admitting totalizations. Let $B$ be an augmented coalgebra in $\mathcal{C}$. Let $\mathcal{M} := \text{cLMod}_B(\mathcal{C})$, which we regard as right tensored over $\mathcal{C}$. Then $\text{Cobar}(B) \in \mathcal{C}$ is an endomorphism object for $1 \in M$.

We now come to the Koszul duality result that we are ultimately interested in here, taking place at the level of categories of modules and comodules:

**Proposition 3.2.5.** Let $\mathcal{C}$ be a presentable symmetric monoidal $\infty$-category. Let $1 \in \mathcal{C}$ denote the unit object. Let $A$ be an augmented commutative algebra object of $\mathcal{C}$. Assume that the augmentation exhibits $1$ as a dualizable $A$-module, so that $\text{Bar}(A) \cong 1 \otimes_A 1$ is dualizable in $\mathcal{C}$. We regard $\text{Bar}(A)$ as a commutative bialgebra object of $\mathcal{C}$ by Remark 3.2.2, and thereby regard $\text{cLMod}_{\text{Bar}(A)}(\mathcal{C})$ as a presentable symmetric monoidal $\infty$-category by Construction 2.2.2 and Remark 2.2.8. The following statements then hold:

(a) The symmetric monoidal functor $1 \otimes_A - : \text{Mod}_A(\mathcal{C}) \rightarrow \mathcal{C}$ canonically lifts along the (symmetric monoidal) forgetful functor $U : \text{cLMod}_{\text{Bar}(A)} \rightarrow \mathcal{C}$ to a symmetric monoidal functor $F : \text{Mod}_A(\mathcal{C}) \rightarrow \text{cLMod}_{\text{Bar}(A)}(\mathcal{C})$.

(b) The functor $F$ in (a) admits a fully faithful right adjoint $G : \text{cLMod}_{\text{Bar}(A)}(\mathcal{C}) \rightarrow \text{Mod}_A(\mathcal{C})$.

(c) If $\mathcal{C}$ is stable, then the essential image of the functor $G$ in (b) is the full subcategory of $\text{Mod}_A(\mathcal{C})$ spanned by the 1-local objects, in the sense of Bousfield localization.

**Proof.** We begin with (a). By the results of [L-HA, §4.8.5], the symmetric monoidal structure on $\text{cLMod}_{\text{Bar}(A)}(\mathcal{C})$ determines a commutative algebra structure on the endomorphism object $E := \text{End}_{\text{cLMod}_{\text{Bar}(A)}}(1) \in \mathcal{C}$ and the symmetric monoidal forgetful functor $U$ induces a map of commutative algebras $E \rightarrow 1$. Moreover, producing the desired symmetric monoidal functor $F$ factoring $1 \otimes_A - : \text{Mod}_A(\mathcal{C}) \rightarrow \mathcal{C}$ is equivalent to producing a map of augmented commutative algebras $A \rightarrow E$. To do so, it suffices to exhibit an equivalence of augmented commutative algebras $E \cong \text{Cobar}(\text{Bar}(A))$, since there is a canonical map of augmented commutative algebras $A \rightarrow \text{Cobar}(\text{Bar}(A))$ determined by the bar-cobar adjunction described in Recollection 3.2.1 (applied in $\text{CAlg}(\mathcal{C})$).

Let $E' := \text{Cobar}(\text{Bar}(A))$. Let

$$\rho : \mathcal{C} \rightarrow \text{cLMod}_{\text{Bar}(A)}(\mathcal{C}), \quad \rho' : \text{CAlg}(\mathcal{C}) \rightarrow \text{cLMod}_{\text{Bar}(A)}(\text{CAlg}(\mathcal{C}))$$

denote the corestriction functors induced by the augmentation $1 \rightarrow \text{Bar}(A)$. By Variant 3.2.4, $E'$ is an endomorphism object of $1 \in \text{cLMod}_{\text{Bar}(A)}(\text{CAlg}(\mathcal{C}))$. Thus, the image under $\rho'$ of the augmentation $E \rightarrow 1$ in $\text{CAlg}(\mathcal{C})$ determines a map of augmented commutative algebras $\alpha : E \rightarrow E'$. Conversely, using the universal property of $E$ as an endomorphism object, we obtain a canonical map $\beta : E' \rightarrow E$ in $\mathcal{C}$. One can check that the map $\rho(E) \cong 1 \otimes E \rightarrow 1$ in $\text{cLMod}_{\text{Bar}(A)}(\mathcal{C})$ exhibiting $E$ as the endomorphism object of $1$ is obtained by applying $\rho$ to the augmentation $E \rightarrow 1$ in $\mathcal{C}$, from which we deduce that the composition $\beta \circ \alpha$ is homotopic to the identity. It now suffices to show that $\beta$ is an equivalence, as then $\alpha$ is too. This follows from the chain of equivalences, natural in
\[
X \in \mathcal{C},
\]

\[
\text{Map}_\mathcal{C}(X, E') \cong \text{Map}_{\text{CAlg}(\mathcal{C})}(\text{Sym}(X), E')
\]

\[
\cong \text{Map}_{\text{clMod}_{\text{Bar}(A)}}(\text{CAlg}(\mathcal{C}))(\rho'(\text{Sym}(X)), 1)
\]

\[
\cong \text{Map}_{\text{clMod}_{\text{Bar}(A)}}(\mathcal{C})(\rho(X), 1)
\]

\[
\cong \text{Map}_\mathcal{C}(X, E),
\]

where \(\text{Sym} : \mathcal{C} \to \text{CAlg}(\mathcal{C})\) denotes the left adjoint to the forgetful functor (the composite equivalence is indeed induced by \(\beta\)).

Statement (b) follows from applying (the dual form of) [L-HA, Corollary 4.7.3.16] to the following commutative diagram of \(\infty\)-categories:

\[
\begin{array}{ccc}
\text{Mod}_A(\mathcal{C}) & \xrightarrow{F} & \text{clMod}_{\text{Bar}(A)}(\mathcal{C}) \\
\downarrow^{\mathbb{1}_{\mathbb{S}_A} -} & & \downarrow^{U} \\
\mathcal{C} & \xleftarrow{\mathbb{1}} & \text{clMod}_{\text{Bar}(A)}(\mathcal{C})
\end{array}
\]

For statement (c), observe that \(F\) vanishes on \(\mathbb{1}\)-acyclic objects, as \(U\) is conservative. It follows immediately that the essential image of \(G\) is contained in the Bousfield localization \(\text{Mod}_A(\mathcal{C})_1\) of \(\text{Mod}_A(\mathcal{C})\) at \(1\), and that \(F\) factors through the localization. The resulting adjunction \(\text{Mod}_A(\mathcal{C})_1 \xrightarrow{\cong} \text{clMod}_{\text{Bar}(A)}(\mathcal{C})\) is an equivalence, since the right adjoint remains fully faithful and now by construction the left adjoint is conservative.

**Remark 3.2.6 (Naturality).** In the situation of Proposition 3.2.5, suppose given another augmented commutative algebra \(B\), with augmentation exhibits \(\mathbb{1}\) as a dualizable \(B\)-module, so that we also obtain a symmetric monoidal functor \(F' : \text{Mod}_B(\mathcal{C}) \to \text{clMod}_{\text{Bar}(B)}(\mathcal{C})\). It follows from the proof of Proposition 3.2.5 that, for any map of augmented commutative algebras \(\phi : A \to B\), the diagram of symmetric monoidal \(\infty\)-categories

\[
\begin{array}{ccc}
\text{Mod}_A(\mathcal{C}) & \xrightarrow{F} & \text{clMod}_{\text{Bar}(A)}(\mathcal{C}) \\
\downarrow^{B\mathbb{1}_{\mathbb{S}_A} -} & & \downarrow^{\text{clMod}_{\text{Bar}(B)}(\mathcal{C})} \\
\text{Mod}_B(\mathcal{C}) & \xleftarrow{F'} & \text{clMod}_{\text{Bar}(B)}(\mathcal{C})
\end{array}
\]

commutes, where the right-hand vertical arrow is the corestriction functor of Remark 2.2.7(c) determined by the map of commutative bialgebras \(\text{Bar}(\phi) : \text{Bar}(A) \to \text{Bar}(B)\).

We now explain how Proposition 3.2.5 applies to the question of recovering a filtered object from its associated graded object. For the remainder of the subsection, we let \(\mathcal{C}\) be a stable presentable symmetric monoidal \(\infty\)-category and \(\mathbb{1}\) the unit object of \(\mathcal{C}\).

**Notation 3.2.7.** Let \(S^\text{gr} \in \text{Gr}(\text{Spt})\) and \(S^\text{fil} \in \text{Fil}(\text{Spt})\) denote the unit objects in graded and filtered spectra. Set \(S^\text{gr}[t] := \text{und}(S^\text{fil}) \in \text{CAlg}(\text{Gr}(\text{Spt}))\), the commutative algebra structure coming from the canonical lax symmetric monoidal structure on the functor \(\text{und} : \text{Fil}(\mathcal{C}) \to \text{Gr}(\text{Spc})\) (see [L-Rot, Remark 3.1.3]). There is a canonical augmentation \(\epsilon : S^\text{gr}[t] \to S^\text{gr}\) in \(\text{CAlg}(\text{Gr}(\text{Spt}))\) ([L-Rot, Remark 3.2.6]).

Now let \(1^\text{gr} \in \text{Gr}(\mathcal{C})\) and \(1^\text{fil} \in \text{Fil}(\mathcal{C})\) denote the analogous graded and filtered unit objects of \(\mathcal{C}\), and let \(1^\text{gr}[t] := \text{und}(1^\text{fil}) \in \text{CAlg}(\text{Gr}(\mathcal{C}))\). These are the images of \(S^\text{gr}, S^\text{fil}, S^\text{gr}[t]\) under the canonical symmetric monoidal functors \(\text{Gr}(\text{Spt}) \to \text{Gr}(\mathcal{C})\) and \(\text{Fil}(\text{Spt}) \to \text{Fil}(\mathcal{C})\). We let \(\epsilon : 1^\text{gr}[t] \to 1^\text{gr}\) denote the image of the map \(\epsilon : S^\text{gr}[t] \to S^\text{gr}\) of the previous paragraph under the canonical functor \(\text{Gr}(\text{Spt}) \to \text{Gr}(\mathcal{C})\).

**Remark 3.2.8.** Observe that the fiber of the augmentation \(\epsilon : 1^\text{gr}[t] \to 1\) is equivalent to \(1^\text{gr}[t](-1)\). This implies that \(1\) is dualizable as a \(1^\text{gr}[t]\)-module.

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Theorem 3.2.14. There is a canonical equivalence of symmetric monoidal ∞-categories
\[ \text{Fil}(\mathcal{C}) \xrightarrow{\cong} \text{Mod}_{1^{\text{st}}[t]}(\text{Gr}(\mathcal{C})). \]
Moreover, the composition
\[ \text{Fil}(\mathcal{C}) \xrightarrow{\cong} \text{Mod}_{1^{\text{st}}[t]}(\text{Gr}(\mathcal{C})) \xrightarrow{1^{\text{st}} \otimes 1^{\text{st}}[t]} \text{Mod}_{1^{\text{st}}}(\text{Gr}(\mathcal{C})) \cong \text{Gr}(\mathcal{C}) \]
is canonically equivalent to the associated graded functor \( \text{gr} : \text{Fil}(\mathcal{C}) \to \text{Gr}(\mathcal{C}) \).

Proof. For \( \mathcal{C} = \text{Spt} \), this is proven in [L-Rot, §3.2]. The general statement follows from this case by tensoring with \( \mathcal{C} \) in \( \text{Pr}_L \) (see Remark 3.1.4).

Remark 3.2.10. Proposition 3.2.9 is an incarnation of the classical “Rees algebra” construction.

It follows from Remark 3.2.8 and Proposition 3.2.9 that \( \text{Fil}(\mathcal{C}) \) and \( \text{Gr}(\mathcal{C}) \) fit into an example of the situation of Proposition 3.2.5. Let us unravel what Proposition 3.2.5 says in this example. We first analyze the relevant bar construction.

**Notation 3.2.11.** We let \( D^\vee \) denote the bar construction \( \text{Bar}(1^{\text{st}}[t]) \), which we regard as a commutative bialgebra in \( \text{Gr}(\mathcal{C}) \) (Remark 3.2.2). We let \( D_- \) denote the dual \( \text{Bar}(1^{\text{st}}[t])^\vee \), which we regard as a cocommutative bialgebra in \( \text{Gr}(\mathcal{C}) \) (Corollary 2.3.3).

We can compute what these objects look like. Under the equivalence \( \text{Fil}(\mathcal{C}) \cong \text{Mod}_{1^{\text{st}}[t]}(\text{Gr}(\mathcal{C})) \) of Proposition 3.2.9, the \( 1^{\text{st}}[t] \)-module 1 corresponds to the unique filtered object \( A \in \text{Fil}(\mathcal{C}) \) with \( A^0 = 1 \) and \( A^n = 0 \) for \( n \neq 0 \) (see [L-Rot, Notation 3.2.4]). It follows from Proposition 3.2.9 that in \( \text{Gr}(\mathcal{C}) \) have
\[ D^\vee_\ast \cong 1^{\text{st}} \otimes 1^{\text{st}}[t] \cong \text{gr}(A), \]
from which we calculate
\[ (D^\vee)^n \cong \begin{cases} 1 & \text{if } n = 0 \\ 1[1] & \text{if } n = -1 \\ 0 & \text{otherwise}, \end{cases} \text{ and hence } D^\vee_0 \cong \begin{cases} 1 & \text{if } n = 0 \\ 1[-1] & \text{if } n = 1 \\ 0 & \text{otherwise}. \end{cases} \]
Regarding the bialgebra structures on these, the unit and counit maps on each are given by the displayed equivalences in degree 0, and the multiplication and comultiplication maps are then uniquely determined in the homotopy category \( \text{hGr}(\mathcal{C}) \); for example, the multiplications are the standard square-zero ones. However, the further homotopy-coherent structure in \( \text{Gr}(\mathcal{C}) \) itself is less transparent in general.

We next characterize the relevant Bousfield localization.

**Definition 3.2.12.** We say a filtered object \( X \in \text{Fil}(\mathcal{C}) \) is complete if \( \lim_{n \to \infty} X^n = 0 \). We denote by \( \text{Fil}^\wedge(\mathcal{C}) \) the full subcategory of \( \text{Fil}(\mathcal{C}) \) spanned by the complete objects.

**Lemma 3.2.13 ([GP18, Lemma 2.15]).** The inclusion \( \text{Fil}^\wedge(\mathcal{C}) \to \text{Fil}(\mathcal{C}) \) admits a left adjoint \( (-)^\wedge : \text{Fil}(\mathcal{C}) \to \text{Fil}^\wedge(\mathcal{C}) \). Given a map \( X \to Y \) in \( \text{Fil}(\mathcal{C}) \), the induced map \( X^\wedge \to Y^\wedge \) is an equivalence if and only if the induced map \( \text{gr}(X) \to \text{gr}(Y) \) is an equivalence in \( \text{Gr}(\mathcal{C}) \).

We finally arrive at the main result of this subsection.

**Theorem 3.2.14.** There is a canonical equivalence of symmetric monoidal ∞-categories \( \overline{\text{gr}} : \text{Fil}^\wedge(\mathcal{C}) \to \text{LMod}_{D_-}(\text{Gr}(\mathcal{C})) \) making the diagram
\[
\begin{array}{ccc}
\text{Fil}(\mathcal{C}) & \xrightarrow{\text{gr}} & \text{Gr}(\mathcal{C}) \\
\downarrow \text{(-)^\wedge} & & \downarrow \text{U} \\
\text{Fil}^\wedge(\mathcal{C}) & \xrightarrow{\overline{\text{gr}}} & \text{LMod}_{D_-}(\text{Gr}(\mathcal{C}))
\end{array}
\]

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**commute (here $U$ is the forgetful functor).**

**Proof.** Noting that Corollary 2.3.3 gives an equivalence $\operatorname{LMod}_{D_+}(\operatorname{Gr}(\mathcal{C})) \simeq \operatorname{cLMod}_{D_+}(\operatorname{Gr}(\mathcal{C}))$ commuting with the forgetful functors to $\operatorname{Gr}(\mathcal{C})$, this follows from combining Propositions 3.2.5 and 3.2.9, Remark 3.2.8, and Lemma 3.2.13.

**Remark 3.2.15.** More informally, what Theorem 3.2.14 says is that, given a filtered object $X \in \operatorname{Fil}(\mathcal{C})$, what can be recovered from the associated graded object $\operatorname{gr}(X)$ is the completion $X^\wedge^\mathbb{A}$, and the extra data necessary to perform this recovery is a $D_-$-module structure on $\operatorname{gr}(X)$.

In view of the description of $D_-$ in Notation 3.2.11, in the homotopy category $h\operatorname{Gr}(\mathcal{C})$, the $D_-$-module structure on $\operatorname{gr}(X)$ determines a type of cochain complex object, with differentials of the form $\operatorname{gr}^i(X)[-1] \to \operatorname{gr}^{i+1}(X)$. Unravelling the constructions made above, one can check that the differentials are induced by the boundary maps in the cofiber sequence $X^{i+1} \to X^i \to \operatorname{gr}^i(X)$, recovering the cochain complex recorded explicitly in [L-HA, Remark 1.2.2.3].

Accordingly, we may think of a $D_-$-module structure in $\operatorname{Gr}(\mathcal{C})$ as a kind of “homotopy-coherent cochain complex” structure. We will return to this idea in §5.

### §3.3. t-structures

In this subsection, we define various t-structures for graded and filtered objects that will come in handy later in the paper.\(^5\)

**Definition 3.3.1.** Let $\mathcal{C}$ be a stable presentable symmetric monoidal $\infty$-category. By a **compatible t-structure** on $\mathcal{C}$, we will mean a t-structure $(\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0})$ on the underlying stable $\infty$-category of $\mathcal{C}$ satisfying the following properties:

(a) $\mathcal{C}_{\leq 0}$ is closed under filtered colimits in $\mathcal{C}$;
(b) the unit object $1 \in \mathcal{C}$ lies in $\mathcal{C}_{\geq 0}$;
(c) if $X, Y \in \mathcal{C}_{\leq 0}$, then $X \otimes Y \in \mathcal{C}_{\geq 0}$.

Below, we will state that various constructions give compatible t-structures, but omit verifying the t-structure and compatibility axioms, as they will always be straightforward to check.

For the remainder of this subsection, we fix a stable presentable symmetric monoidal $\infty$-category $\mathcal{C}$, equipped with a compatible t-structure $(\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0})$. We begin by discussing ways in which $\operatorname{Gr}(\mathcal{C})$ inherits a t-structure from $\mathcal{C}$.

**Construction 3.3.2.** We define the following three t-structures on $\operatorname{Gr}(\mathcal{C})$:

(a) Let $\operatorname{Gr}(\mathcal{C})_{\leq 0}$ be the full subcategory of $\operatorname{Gr}(\mathcal{C})$ consisting of those graded objects $X^*$ for which $X^n \in \mathcal{C}_{\leq 0}$ for all $n \in \mathbb{Z}$. Let $\operatorname{Gr}(\mathcal{C})_{\geq 0}$ be the full subcategory of $\operatorname{Gr}(\mathcal{C})$ consisting of those graded objects $X^*$ for which $X^n \in \mathcal{C}_{\geq 0}$ for all $n \in \mathbb{Z}$. Then $(\operatorname{Gr}(\mathcal{C})_{\leq 0}, \operatorname{Gr}(\mathcal{C})_{\geq 0})$ is a compatible t-structure on $\operatorname{Gr}(\mathcal{C})$, which we will refer to as the **neutral t-structure**.

(b) Let $\operatorname{Gr}(\mathcal{C})_{\geq 0}$ be the full subcategory of $\operatorname{Gr}(\mathcal{C})$ consisting of those graded objects $X^*$ for which $X^n \in \mathcal{C}_{\geq 0}$ for all $n \in \mathbb{Z}$. Let $\operatorname{Gr}(\mathcal{C})_{\leq 0}$ be the full subcategory of $\operatorname{Gr}(\mathcal{C})$ consisting of those graded objects $X^*$ for which $X^n \in \mathcal{C}_{\leq 0}$ for all $n \in \mathbb{Z}$. Then $(\operatorname{Gr}(\mathcal{C})_{\geq 0}, \operatorname{Gr}(\mathcal{C})_{\leq 0})$ is a compatible t-structure on $\operatorname{Gr}(\mathcal{C})$. We will refer to this as the **positive t-structure** on $\operatorname{Gr}(\mathcal{C})$, and may write $\operatorname{Gr}(\mathcal{C})_+$ to denote $\operatorname{Gr}(\mathcal{C})$ equipped with this t-structure.

(c) Let $\operatorname{Gr}(\mathcal{C})_{\geq 0}$ be the full subcategory of $\operatorname{Gr}(\mathcal{C})$ consisting of those graded objects $X^*$ for which $X^n \in \mathcal{C}_{\geq 0}$ for all $n \in \mathbb{Z}$. Let $\operatorname{Gr}(\mathcal{C})_{\leq 0}$ be the full subcategory of $\operatorname{Gr}(\mathcal{C})$ consisting of those graded objects $X^*$ for which $X^n \in \mathcal{C}_{\leq 0}$ for all $n \in \mathbb{Z}$. Then $(\operatorname{Gr}(\mathcal{C})_{\geq 0}, \operatorname{Gr}(\mathcal{C})_{\leq 0})$ is a compatible t-structure on $\operatorname{Gr}(\mathcal{C})$. We will refer to this as the **negative t-structure** on $\operatorname{Gr}(\mathcal{C})$, and may write $\operatorname{Gr}(\mathcal{C})_-$ to denote $\operatorname{Gr}(\mathcal{C})$ equipped with this t-structure.

\(^5\)The material in this subsection grew in part out of a conversation with Ben Antieau.
Remark 3.3.3. The hearts $\text{Gr}(\mathcal{E})^\circ$, $\text{Gr}(\mathcal{E})^1$, and $\text{Gr}(\mathcal{E})^2$ of the three t-structures defined in Construction 3.3.2 can each be identified with the category $\text{Gr}(\mathcal{E})^0 = \text{Fun}(\mathbb{Z}^{ds}, \mathcal{E})$, via the functors described the following formulas:

$$\{X^n\}_{n \in \mathbb{Z}} \in \text{Gr}(\mathcal{E}) \quad \mapsto \quad \begin{cases} \{X^n\}_{n \in \mathbb{Z}} \in \text{Gr}(\mathcal{E})^0 \\ \{X^n[2n]\}_{n \in \mathbb{Z}} \in \text{Gr}(\mathcal{E})^1 \\ \{X^n[-2n]\}_{n \in \mathbb{Z}} \in \text{Gr}(\mathcal{E})^2. \end{cases}$$

As the three t-structures are all compatible with the symmetric monoidal structure on $\text{Gr}(\mathcal{E})$, there are induced symmetric monoidal structures on each of their hearts, which under the above identifications determine symmetric monoidal structures on $\text{Gr}(\mathcal{E})^0$. For the positive and negative t-structures, this recovers the usual symmetric monoidal structure considered on graded objects in an abelian category, with tensor product again given by Day convolution but with the symmetry isomorphism incorporating the Koszul sign convention (for example, a commutative algebra object for this symmetric monoidal structure is a graded commutative algebra in $\mathcal{E}^0$ as usually defined). However, the appearance of this sign convention depended on our choices of shifts in the t-structures $\text{Gr}(\mathcal{E})_a$: for the natural t-structure, we obtain the Day convolution symmetric monoidal structure on $\text{Gr}(\mathcal{E})^0$ with no Koszul sign convention involved. Since these two symmetric monoidal structures on $\text{Gr}(\mathcal{E})^0$ will both arise later on, we will distinguish between them by denoting the former (with the Koszul sign rule) by $\text{Gr}(\mathcal{E})^0\kappa$, and the latter (without the Koszul sign rule) by $\text{Gr}(\mathcal{E})^{0\circ}$.

Mediating between the positive and negative t-structures on graded objects will turn out to be an important aspect of the story we will tell later on. The necessary translation is encapsulated in the following result.

Proposition 3.3.4. Let $\mathcal{E}$ be a $\mathbb{Z}$-linear stable presentable symmetric monoidal $\infty$-category (in the sense of [L-SAG, Definition D.1.2.1]). Then the two constructions

$$\{X^n\}_{n \in \mathbb{Z}} \in \text{Gr}(\mathcal{E}) \quad \mapsto \quad \begin{cases} \{X^n[2n]\}_{n \in \mathbb{Z}} \in \text{Gr}(\mathcal{E})^0 \\ \{X^n[-2n]\}_{n \in \mathbb{Z}} \in \text{Gr}(\mathcal{E})^1 \\ \{X^n\}_{n \in \mathbb{Z}} \in \text{Gr}(\mathcal{E})^2. \end{cases}$$

canonical extend to a pair of mutually inverse symmetric monoidal t-exact equivalences

$$[2\ast]: \text{Gr}(\mathcal{E})^{-} \xRightarrow{\cong} \text{Gr}(\mathcal{E})_{+}, [-2\ast],$$

such that the induced equivalences

$$\text{Gr}(\mathcal{E})^0 \simeq \text{Gr}(\mathcal{E})^1 \cong \text{Gr}(\mathcal{E})^2 \simeq \text{Gr}(\mathcal{E}^0)$$

are homotopic to the identity.

Proof. We first extend the assignments $n \mapsto \mathbb{Z}[2n]$ and $n \mapsto \mathbb{Z}[-2n]$ to maps of $\mathcal{E}_{\infty}$-spaces $\phi, -\phi: \mathbb{Z}_{ds}^{\infty} \rightarrow \text{Pic}(\mathbb{Z})$, where $\text{Pic}(\mathbb{Z})$ denote the Picard space of invertible $\mathbb{Z}$-module spectra.$^6$ There is a fiber sequence of $\mathcal{E}_{\infty}$-spaces $\text{Pic}(\mathbb{Z}) \rightarrow \mathbb{Z} \xrightarrow{\kappa} B^2(\mathbb{Z}/2)$, coming from the Postnikov sequence at the level of spectra,

$$\xymatrix{ \pi_1(\text{pic}(\mathbb{Z}))[1] \ar[r]^-{\tau_{\leq 1}} \ar@{^{(}->}[d] \ar@{^{(}->}[u] & \tau_{\leq 1} \text{pic}(\mathbb{Z}) \ar[r]^-{\tau_{\leq 0}} \ar@{^{(}->}[d] & \tau_{\leq 0} \text{pic}(\mathbb{Z}) \ar@{^{(}->}[d] \ar[r] & \mathbb{Z}/2[1] \ar[r] & \text{pic}(\mathbb{Z}) \ar[r] & \mathbb{Z}, }$$

where $\text{pic}(\mathbb{Z})$ denotes the connective delooping of $\text{Pic}(\mathbb{Z})$. The composite maps

$$\mathbb{Z}^{ds}_{\kappa} \cong B^2(\mathbb{Z}/2), \quad \text{or equivalently} \quad \mathbb{Z}_{\kappa}^{ds} \xrightarrow{\cong} B^2(\mathbb{Z}/2) \xrightarrow{\kappa} B^2(\mathbb{Z}/2),$$

$^6$This cannot be done over the sphere spectrum $S$: the map $\mathbb{Z}^{ds}_{\kappa} \rightarrow \text{Pic}(S)$ sending $n \mapsto S[2n]$ can be given an $E_2$-structure but not an $E_3$-structure. However, it is not necessary to pass all the way to $\mathbb{Z}$ to succeed: for example, it is possible to do so (and thus the proposition goes through) already over $\text{MU}$.  

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We let $X$ where the second map comes from the §3.2. [Ant18]. We give an alternative explanation/construction of this t-structure using the results of §3.3. Let $\tau \colon X \to X$ denote the connective cover functor with respect to the Postnikov t-structure (Construction 3.3.6). Construction 3.3.7. Let $\delta \colon X \to X$ and let $\delta^\ast \colon X \to X$ be the full subcategory of Fil($\mathcal{C}$) consisting of those filtered objects $X^+$ for which $X^n \in \mathcal{C}_{\geq n}$ for all $n \in \mathbb{Z}$, and let Fil($\mathcal{C}$)$_{\geq 0}$ be the full subcategory of Fil($\mathcal{C}$) consisting of those filtered objects $X^+$ for which $X^n \in \mathcal{C}_{\leq 0}$ for all $n \in \mathbb{Z}$. We refer to this as the $\textit{passive}$ t-structure on Fil($\mathcal{C}$).

The next t-structure on Fil($\mathcal{C}$) is related to the positive t-structure on Gr($\mathcal{C}$).

Construction 3.3.8. We define a compatible t-structure (Fil($\mathcal{C}$)$_{\leq 0}$, Fil($\mathcal{C}$)$_{\geq 0}$) on Fil($\mathcal{C}$) as follows: let Fil($\mathcal{C}$)$_{\leq 0}$ be the full subcategory of Fil($\mathcal{C}$) consisting of those filtered objects $X^+$ for which $X^n \in \mathcal{C}_{\leq n}$ for all $n \in \mathbb{Z}$, and let Fil($\mathcal{C}$)$_{\geq 0}$ be the full subcategory of Fil($\mathcal{C}$) consisting of those filtered objects $X^+$ for which $X^n \in \mathcal{C}_{\geq n}$ for all $n \in \mathbb{Z}$. We refer to this as the $\textit{neutral}$ t-structure on Fil($\mathcal{C}$).

The last t-structure on Fil($\mathcal{C}$) that will be relevant in this paper is related to the negative t-structure on Gr($\mathcal{C}$): it is the $\textit{Beilinson t-structure}$, as discussed in [BMS18, §5.1] and employed in [Ant18]. We give an alternative explanation/construction of this t-structure using the results of §3.2.
Notation 3.3.9. Let $DG(C^\circ)$ denote the ordinary category of cochain complexes in the abelian category $C^\circ$. We regard $DG(C^\circ)$ as a symmetric monoidal category in the standard manner (in particular with the Koszul sign rule imposed).

Construction 3.3.10. Recall from Construction 3.3.2 that we have the negative t-structure on $Gr(C)$, whose connective part $Gr(C)_{\geq 0}$ consists of those graded objects $X^*$ such that $X^n \in C_{2-n}$ for all $n \in \mathbb{Z}$. Let $D_-$ be as in Notation 3.2.11. Observe that $D_- \in Gr(C)_{\geq 0}$, so that we have a full subcategory $LMod_{D_-}(Gr(C)_{\geq 0}) \subseteq LMod(D_-(Gr(C)))$; it follows from $[L-HA$, Proposition 1.4.4.11] that this is the connective part of a t-structure on $LMod(D_-(Gr(C)))$. The heart is given by $LMod_{D_-}(Gr(C)_{\geq 0})$, which under the equivalence $Gr(C)_{\geq 0} \cong Gr(C^\circ)$ can be identified with the symmetric monoidal category $DG(C^\circ)$ of cochain complexes in $C^\circ$.

Proposition 3.3.11. Let $Fil(C)_B$ denote the full subcategory of $Fil(C)$ spanned by those filtered objects $X$ satisfying $gr(X) \in Gr(C)_{\geq 0}$, i.e. $gr(X) \in C_{2-n}$ for all $n \in \mathbb{Z}$. Then $Fil(C)_{\geq 20}$ is the connective part of a compatible t-structure $(Fil(C)_B, Fil(C)_G)$ on $Fil(C)$, whose heart is canonically equivalent to the symmetric monoidal category $DG(C^\circ)$ of cochain complexes in $C^\circ$.

Proof. We first observe that it suffices to prove the claim with $Fil(C)$ replaced by $Fil^*(C)$: this follows from $[L-HA$, Proposition 1.2.1.16], using the facts that $Fil^*(C)$ is closed under extensions in $Fil(C)$ and that the natural map $X \to X^\circ$ is an equivalence on associated graded objects. For $Fil^*(C)$, the claim follows from Construction 3.3.10 using the equivalence $Fil^*(C) \cong LMod_{D_-(Gr(C))}$ of Theorem 3.2.14.

Notation 3.3.12. We refer to the t-structure $(Fil(C)_B, Fil(C)_G)$ of Proposition 3.3.11 as the Beilinson t-structure on $Fil(C)$. The identification of the heart of this t-structure supplies a canonical lax symmetric monoidal functor $DG(C^\circ) \to Fil^*(C)$. We will denote the image of a cochain complex $M \in DG(C^\circ)$ under this functor by $|M|^{\geq 0}$ and we let $|M| = \text{colim}(|M|^{\geq 0}) \in C$.

Remark 3.3.13. Let $M \in DG(C^\circ)$. Let $M^{\geq 0} \in \text{Fun}(Z^\text{op}, DG(C^\circ))$ denote the (decreasing) brutal filtration of $M$, so that, for $i \in \mathbb{Z}$, the graded terms of $M^{\geq 0}$ are given by

$$\begin{align*}
(M^{\geq 0})^j &= \begin{cases} 
M^i & j \geq i \\
0 & \text{otherwise}
\end{cases}.
\end{align*}$$

There is a canonical natural equivalence $|M|^{\geq 0} \cong |M|^{\geq 0}$ of functors $DG(C^\circ) \to Fil(C)$.

Example 3.3.14. Suppose that $C = Mod_A$ for $A$ a commutative ring. Then, for $M \in DG(Mod_A\^\circ)$, the object $|M| \in Mod_A$ is equivalent to the Eilenberg-MacLane object represented by $M$, and, by Remark 3.3.13, the filtration $|M|^{\geq 0} \in Fil^*(Mod_A)$ on $|M|$ is the brutal filtration.

§4. Derived commutative rings

The constructions and results of §2 allow us to reformulate the notion of an $E_\infty$-$Z$-algebra with $S^1$-action purely in terms of the group algebra $Z[S^1]$ or its dual $Z^{S^1}$. Namely, we have equivalences of $\infty$-categories

$$\text{Fun}(BS^1, CAlg_Z) \cong CAlg(Mod_{Z[S^1]}) \cong cMod_{Z^{S^1}}(CAlg_Z),$$

where in the middle term we regard $Mod_{Z[S^1]}$ as a symmetric monoidal $\infty$-category using the cocommutative bialgebra structure on $Z[S^1]$, and in the right term we regard $Z^{S^1}$ as a commutative bialgebra over $Z$ (this will be spelled out more carefully in §6). These alternative perspectives will be crucial for us later in the paper, but we will in fact need even more: in addition to $E_\infty$-algebra structures, we will be interested in simplicial commutative algebraic structures. Thus, for example,
we would like to similarly rewrite the $\infty$-category $\text{Fun}(BS^1, \text{CAlg}_{2}^{\Delta})$ of simplicial commutative rings with $S^1$-action in terms of the group algebra $\mathbb{Z}[S^1]$.

It turns out that this is more naturally formulated with the dual $Z^{S^1}$; we would like $Z^{S^1}$ to be a “simplicial commutative bialgebra”, i.e. a coalgebra object in $\text{CAlg}_{2}^{\Delta}$, and then for there to be an equivalence of $\infty$-categories $\text{Fun}(BS^1, \text{CAlg}_{2}^{\Delta}) \simeq \text{cLMod}_{\mathbb{Z}^{S^1}}(\text{CAlg}_{2}^{\Delta})$. However, this wish is nonsensical as stated: simplicial commutative rings are by definition connective objects, while $Z^{S^1}$ is not.

The goal of this section is to introduce some theory that allows us to make sense of the above wish. Namely, we define the notion of a derived commutative ring. We will make the definition in a certain axiomatic context in §4.2, and then in §4.3 explain the example contexts of interest, including the basic setting of $Z$-modules involved above as well as the graded and filtered settings needed in §§5–6. The definition is formulated using the language of monads, some preliminary discussion of which will be necessary in §4.1. In §4.4, we give a definition of the cotangent complex in the setting of derived commutative rings, generalizing the cotangent complex from the setting of simplicial commutative rings; our main aim there is really to give a recharacterization of the latter that will be useful for studying the derived de Rham complex in §5.3. Finally, in §4.5, we discuss some key facts about coconnective derived commutative rings that will be needed to construct the graded and filtered analogues of $Z^{S^1}$ in §§5–6.

Let us also repeat what was mentioned in §1.3: while we are giving a complete account here of the definition of derived commutative rings, the theory is due to Mathew and work-in-progress of Bhatt–Mathew [BM].

§4.1. Monad miscellany. The following example of a monad will be a key player throughout much of this section:

Construction 4.1.1 (Symmetric algebra monad). Let $\mathcal{C}$ be a presentable symmetric monoidal $\infty$-category. Then the forgetful functor $G : \text{CAlg}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $F : \mathcal{C} \to \text{CAlg}(\mathcal{C})$. We let $\text{Sym}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ denote the monad associated to this adjunction, and refer to it as the symmetric algebra monad on $\mathcal{C}$. Recall that the functor $\text{Sym}_{\mathcal{C}}$ preserves sifted colimits, and can be described by the formula $\text{Sym}_{\mathcal{C}}(X) = \amalg_{j \geq 0}(X^{\otimes j})_{S_j}$. Recall also that $G$ is monadic (by the Barr–Beck–Lurie theorem), so that we have a canonical equivalence of $\infty$-categories $\text{CAlg}(\mathcal{C}) \simeq \text{LMod}_{\text{Sym}_{\mathcal{C}}}(\mathcal{C})$.

In the situation of Construction 4.1.1, observe that the functor $\text{Sym}_{\mathcal{C}}$ has a canonical filtration by functors $\text{Sym}_{\mathcal{C}}^{S_j} : \mathcal{C} \to \mathcal{C}$ for $i \geq 0$, given by $\text{Sym}_{\mathcal{C}}^{S_j}(X) = \amalg_{0 \leq j \leq i}(X^{\otimes j})_{S_j}$. In §4.2, it will be useful to be able to control $\text{Sym}_{\mathcal{C}}$ as a monad in terms of this filtration. The bulk of this subsection is concerned with setting up the definitions and constructions needed to do this. We begin with the following definition.

Definition 4.1.2. Let $\mathcal{C}$ be an $\infty$-category. A filtered monad on $\mathcal{C}$ is a lax monoidal functor $Z_{\geq 0} \to \text{End}(\mathcal{C})$, where $Z_{\geq 0}$ denotes the partially ordered set of nonnegative integers regarded as a monoidal category via multiplication. More generally, if $\mathcal{E}$ is a monoidal full subcategory of $\text{End}(\mathcal{C})$, we will refer to algebra objects of $\mathcal{E}$ as $\mathcal{E}$-monads and to lax monoidal functors $Z_{\geq 0} \to \mathcal{E}$ as filtered $\mathcal{E}$-monads.

Notation 4.1.3. Let $\mathcal{C}$ and $\mathcal{E}$ be as in Definition 4.1.2. We let $\mathcal{E}^\circ \to \text{Assoc}^\circ$, denote the fibration of $\mathcal{E}$-operads encoding the monoidal structure on $\mathcal{E}$, and we let $\text{Fun}(Z_{\geq 0}^\circ, \mathcal{E}^\circ)^* \to \text{Assoc}^\circ$ denote the fibration of $\mathcal{E}$-operads obtained by applying the Day convolution construction ([L-HA, Construction 2.2.6.7]) to the monoidal $\mathcal{E}$-categories $Z_{\geq 0}^\circ$ and $\mathcal{E}$. We note that algebra objects of $\text{Fun}(Z_{\geq 0}^\circ, \mathcal{E}^\circ)^*$ can be identified with lax monoidal functors $Z_{\geq 0} \to \mathcal{E}$, i.e. filtered $\mathcal{E}$-monads.

Proposition 4.1.4. Let $\mathcal{C}$ and $\mathcal{E}$ be as in Definition 4.1.2. Assume moreover that:

(a) $\mathcal{C}$ admits all small colimits;
(b) \( E \) is closed under sequential colimits, so that we have a colimit functor \( \text{colim} : \text{Fun}(\mathbb{Z}_{\geq 0}, E) \to E \), left adjoint to the diagonal functor \( \delta : \text{Fun}(\mathbb{Z}_{\geq 0}, E) \to E \);

(c) each \( F \in E \) commutes with sequential colimits.

Then the adjunction \( \text{colim} : \text{Fun}(\mathbb{Z}_{\geq 0}, E) \nvdash \delta : \text{Fun}(\mathbb{Z}_{\geq 0}, E) \) canonically lifts to a relative adjunction of \( \infty \)-operads \( \text{Fun}(\mathbb{Z}_{\geq 0}, E^\circ)^* \nvdash E^\circ \) over \( \text{Assoc}^\circ \). In particular, the colimit of a filtered \( E \)-monad is canonically an \( E \)-monad.

**Proof.** We begin with the right adjoint functor, i.e. the diagonal \( \delta : E \to \text{Fun}(\mathbb{Z}_{\geq 0}, E) \). Recall that \( \delta \) corresponds under the equivalence \( \text{Fun}(E, \text{Fun}(\mathbb{Z}_{\geq 0}, E)) \cong \text{Fun}(E \times \mathbb{Z}_{\geq 0}, E) \) to the projection functor. Similarly, the projection map of \( \infty \)-operads \( E^\circ \times_{\text{Assoc}^\circ} \mathbb{Z}_{\geq 0} \to E^\circ \) corresponds uniquely under the universal property of Day convolution ([L-HA, Definition 2.2.6.1] to a map of \( \infty \)-operads \( G : E^\circ \to \text{Fun}(\mathbb{Z}_{\geq 0}, E^\circ)^* \) over \( \text{Assoc}^\circ \), which lifts \( \delta \).

Now, to prove the desired claim, it suffices to show that \( G \) admits a left adjoint relative to \( \text{Assoc}^\circ \). It follows from [L-HA, Corollary 2.2.6.14] that \( \text{Fun}(\mathbb{Z}_{\geq 0}, E^\circ)^* \to \text{Assoc}^\circ \) is a locally cocartesian fibration of \( \infty \)-operads, so it will be enough to check that criterion (2) of [L-HA, Proposition 7.3.2.11] is satisfied (note that the first criterion amounts in our situation to the existence of the colimit functor \( \text{Fun}(\mathbb{Z}_{\geq 0}, E) \to E \) that we are studying). Unravelling the definitions, this follows from our hypothesis that each \( F \in E \) commutes with sequential colimits.

We can now state the main goal of this subsection a bit more precisely: in the situation of Construction 4.1.1, we would like to construct a certain filtered monad \( \text{Sym}_{\ell}^\circ \) on \( \ell \), whose colimit recovers the symmetric algebra monad \( \text{Sym}_{\ell} \). Doing so will require some preliminaries, involving the theory of symmetric sequences and the concomitant theory of operads. Let us begin by reviewing the basic notions.

**Recollection 4.1.5.** Let \( F \) denote the groupoid whose objects are finite sets and whose morphisms are bijections, regarded as a symmetric monoidal category via disjoint union. For any presentable symmetric monoidal \( \infty \)-category \( \ell \), we let \( \text{SSeq}(\ell) \) denote the \( \infty \)-category \( \text{Fun}(F, \ell) \) of symmetric sequences in \( \ell \), regarded as a (presentable) symmetric monoidal \( \infty \)-category via Day convolution. For \( A \in \text{SSeq}(\ell) \) and \( n \in \mathbb{Z}_{\geq 0} \), we let \( A_n \in \ell \) denote the value of \( A \) on the finite set \( \{1, \ldots, n\} \).

Left Kan extension along the inclusion \( \{0\} \to F \) determines a fully faithful, colimit preserving symmetric monoidal embedding \( i : \ell \to \text{SSeq}(\ell) \). Composing the Yoneda embedding

\[
F \cong F^{op} \to \text{Fun}(F, \text{Spc}) = \text{SSeq}(\text{Spc})
\]

with the unique map \( \text{Spc} \to \ell \) in \( \text{CAlg}(\mathbb{P}^{1}_{\ell}) \), we obtain a functor \( y : F \to \text{SSeq}(\ell) \).

There is an additional (non-symmetric) monoidal structure on \( \text{SSeq}(\ell) \), referred to as the composition monoidal structure. This arises as follows. For any \( D \in \text{CAlg}(\mathbb{P}^{1}_{\ell}) \), evaluation at \( y(\{1\}) \) determines an equivalence of \( \infty \)-categories

\[
\text{Fun}_{\text{CAlg}(\mathbb{P}^{1}_{\ell})}^{\ell}(\text{SSeq}(\ell), D) \sim D,
\]

where the left-hand side denote the \( \infty \)-category of colimit-preserving symmetric monoidal functors under \( \ell \). Taking \( D = \text{SSeq}(\ell) \), we obtain an equivalence of \( \infty \)-categories \( \text{End}_{\text{CAlg}(\mathbb{P}^{1}_{\ell})}^{\ell}(\text{SSeq}(\ell)) \cong \text{SSeq}(\ell) \). Transporting the composition monoidal structure on the left hand side across this equivalence defines a monoidal structure on \( \text{SSeq}(\ell) \), and we then take the reverse monoidal structure to obtain the composition monoidal structure on \( \text{SSeq}(\ell) \). The tensor product of the composition monoidal structure is referred to as the composition product, denoted by \( (A, B) \mapsto A \circ B \), and given concretely by the formula

\[
A \circ B \simeq \prod_{n \geq 0} (A_n \otimes B^{\otimes n}) \Sigma_n,
\]

where \( \otimes \) denotes the tensoring of \( \text{SSeq}(\ell) \) over \( \ell \) (determined by the symmetric monoidal embedding \( i ) and \( \otimes \) denotes the Day convolution product. We refer to algebra objects of \( \text{SSeq}(\ell) \) with respect
to the composition monoidal structure as *operads in* \( \mathcal{C} \), and let \( \text{Op}(\mathcal{C}) \) denote the \( \infty \)-category \( \text{Alg}(\text{SSeq}(\mathcal{C})) \) of operads in \( \mathcal{C} \).

Finally, we let \( \theta : \text{SSeq}(\mathcal{C}) \to \text{End}(\text{SSeq}(\mathcal{C})) \) denote the monoidal functor determined by the composition product, sending \( A \mapsto A \circ - \). We also denote the induced functor on algebra objects \( \text{Op}(\mathcal{C}) \to \text{Alg}(\text{End}(\text{SSeq}(\mathcal{C}))) \) by \( \theta \). For any \( A \in \text{SSeq}(\mathcal{C}) \), the endofunctor \( \theta(A) \) preserve the essential image of the embedding \( \iota : \mathcal{C} \to \text{SSeq}(\mathcal{C}) \), and hence restricts to an endofunctor on \( \mathcal{C} \).

**Construction 4.1.6.** Let \( \mathcal{C} \) be a presentable symmetric monoidal \( \infty \)-category. Let \( A \in \text{SSeq}(\mathcal{C}) \) denote the constant symmetric sequence with value the unit object \( 1 \in \mathcal{C} \). We will construct an operad structure on \( A \), together with an equivalence of monads \( \theta(A) \simeq \text{Sym}^{\text{SSeq}}(\mathcal{C}) \).

We use ideas similar to those used in the proof of Lemma 2.3.5. As there, let us begin by recalling how the monad structure on \( \text{Sym}^{\text{SSeq}}(\mathcal{C}) \) is defined. Let \( G : \text{CAlg}(\text{SSeq}(\mathcal{C})) \to \text{SSeq}(\mathcal{C}) \) denote the forgetful functor, and let \( F \) denote its left adjoint, so that \( \text{Sym}^{\text{SSeq}}(\mathcal{C}) \) is given as a functor by the composition \( G \circ F \). Regarding the functor category \( \text{Fun}(\text{CAlg}(\text{SSeq}(\mathcal{C})), \text{SSeq}(\mathcal{C})) \) as left tensored over the monoidal \( \infty \)-category \( \text{End}(\text{SSeq}(\mathcal{C})) \) via postcomposition, one checks that \( \text{Sym}^{\text{SSeq}}(\mathcal{C}) = G \circ F \) is an endomorphism object for \( G \), hence carries a canonical algebra (i.e. monad) structure.

We now refine this construction. Let \( \mathcal{E} := \text{End}_{\text{CAlg}(\mathcal{C})}(\text{SSeq}(\mathcal{C})) \). We regard \( \mathcal{E} \) as a monoidal \( \infty \)-category, and we regard both \( \text{SSeq}(\mathcal{C}) \) and \( \text{CAlg}(\text{SSeq}(\mathcal{C})) \) as left tensored over \( \mathcal{E} \) via evaluation, so that the forgetful functor \( G \) is canonically \( \mathcal{E} \)-linear. It follows that the left adjoint \( F \) is canonically \( \mathcal{E} \)-linear, and it is easy to check that it is in fact \( \mathcal{E} \)-linear. Carrying out the same construction above in the \( \mathcal{E} \)-linear setting, we deduce that \( \text{Sym}^{\text{SSeq}}(\mathcal{C}) \) is canonically an \( \mathcal{E} \)-linear monad, i.e. an algebra object in the \( \infty \)-category of \( \mathcal{E} \)-linear endofunctors of \( \text{SSeq}(\mathcal{C}) \).

Now recall from Recollection 4.1.5 that \( \mathcal{E} \simeq \text{SSeq}(\mathcal{C}) \). Under this equivalence the left action of \( \mathcal{E} \) on \( \text{SSeq}(\mathcal{C}) \) identifies with the right action of \( \text{SSeq}(\mathcal{C}) \) itself via the composition monoidal structure. It follows that any \( \mathcal{E} \)-linear endofunctor (resp. monad) on \( \text{SSeq}(\mathcal{C}) \) is given by \( B \mapsto A' \circ B \) for some \( A' \in \text{SSeq}(\mathcal{C}) \) (resp. \( A' \in \text{Op}(\mathcal{C}) \)). By inspection, the symmetric sequence underlying the operad giving \( \text{Sym}^{\text{SSeq}}(\mathcal{C}) \) is constant with value \( 1 \in \mathcal{C} \).

**Construction 4.1.7.** Let \( \mathcal{C} \) and \( \text{SSeq}(\mathcal{C}) \) be as in Recollection 4.1.5. For \( i \in \mathbb{Z}_{\geq 0} \), let \( \text{SSeq}^{\leq i}(\mathcal{C}) \subseteq \text{SSeq}(\mathcal{C}) \) denote the full subcategory spanned by those symmetric sequences \( A \) such that \( A_j \) is an initial object of \( \mathcal{C} \) for all \( j > i \).

Now consider the \( \infty \)-category \( \text{Fun}(\mathbb{Z}_{\geq 0}, \text{SSeq}(\mathcal{C})) \), and let \( \text{SSeq}^{\leq i}(\mathcal{C}) \) denote the full subcategory of \( \text{Fun}(\mathbb{Z}_{\geq 0}, \text{SSeq}(\mathcal{C})) \) spanned by those functors \( F \) such that \( F(i) \in \text{SSeq}^{\leq i}(\mathcal{C}) \) for all \( i \in \mathbb{Z}_{\geq 0} \). The inclusion \( \phi : \text{SSeq}^{\leq i}(\mathcal{C}) \to \text{Fun}(\mathbb{Z}_{\geq 0}, \text{SSeq}(\mathcal{C})) \) admits a right adjoint \( \psi : \text{Fun}(\mathbb{Z}_{\geq 0}, \text{SSeq}(\mathcal{C})) \to \text{SSeq}^{\leq i}(\mathcal{C}) \), given by the formula

\[
\psi(F)(i)_{j} = \begin{cases} F(i) & j \leq i \\ \emptyset & \text{otherwise,} \end{cases}
\]

for \( i, j \in \mathbb{Z}_{\geq 0} \), where \( \emptyset \) denotes the initial object of \( \mathcal{C} \).

Let \( p : \text{Fun}(\mathbb{Z}_{\geq 0}, \text{SSeq}(\mathcal{C})) \to \text{Assoc}^{\leq} \) denote the fibration of \( \infty \)-operads obtained by applying the Day convolution construction for the multiplication monoidal structure on \( \mathbb{Z}_{\geq 0} \) and the composition monoidal structure on \( \text{SSeq}(\mathcal{C}) \). The composition product on \( \text{SSeq}(\mathcal{C}) \) preserves sequential colimits in each variable, so that, by [L-HA, Corollary 2.2.6.14], \( p \) is a locally cocartesian fibration. Concretely, this means that, for every finite linearly ordered set \( S \simeq \{1, \ldots, n\} \) (with \( n = 0 \) for \( S = \emptyset \)), we have a well-defined Day convolution product

\[
\bigotimes_{k=1}^{n} : \prod_{k=1}^{n} \text{Fun}(\mathbb{Z}_{\geq 0}, \text{SSeq}(\mathcal{C})) \to \text{Fun}(\mathbb{Z}_{\geq 0}, \text{SSeq}(\mathcal{C})),
\]

given by the formula

\[
(\bigotimes_{k=1}^{n} F_k)(i) = \colim_{\sum_{k=1}^{n} i_k} F_1(i_1) \circ \cdots \circ F_k(i_k)
\]
(see [L-HA, Remark 2.2.6.15]). Observe that, for $n \geq 0$ and $F_1, \ldots, F_n \in \text{SSeq}^{s*}(\mathcal{C})$, we have $\bigotimes_{1 \leq i \leq n} F_k \in \text{SSeq}^{s*}(\mathcal{C})$ as well.

Let $\text{SSeq}^{s*}(\mathcal{C}) \subseteq \text{Fun}(Z_{\geq 0}^*, \mathcal{C})^\otimes$ denote the full subcategory determined by $\text{SSeq}(\mathcal{C}) \in \text{Fun}(Z_{\geq 0}, \mathcal{C})$ (as defined in [L-HA, §2.2.1]). Noting that the statement and proof of [L-HA, Proposition 2.2.1.1] go through with “cocartesian fibration” replaced by “locally cocartesian fibration”, we deduce from the observation above that the functor $\psi : \text{Fun}(Z_{\geq 0}^*, \mathcal{C})^\otimes \to \text{SSeq}^{s*}(\mathcal{C})^\otimes$.

Finally, we will denote the composition

$\text{SSeq}(\mathcal{C})^\otimes \xrightarrow{\delta} \text{Fun}(Z_{\geq 0}^*, \mathcal{C})^\otimes \xrightarrow{\psi'} \text{SSeq}^{s*}(\mathcal{C})^\otimes \subseteq \text{Fun}(Z_{\geq 0}^*, \mathcal{C})^\otimes$.

by $(\cdot)^{s*}$; here $\delta$ is the diagonal map, constructed as in the proof of Proposition 4.1.4. We will also denote by $(\cdot)^{s*}$ the induced functor on algebra objects

$\text{Op}(\mathcal{C}) = \text{Alg}(\text{SSeq}(\mathcal{C})) \to \text{Alg}(\text{Fun}(Z_{\geq 0}^*, \mathcal{C})^\otimes) \approx \text{Fun}^{\text{lax}}((Z_{\geq 0}^*, \mathcal{C})^\otimes)$.

We finally come to the main construction of the subsection.

**Construction 4.1.8 (Symmetric powers filtered monad).** Let $\mathcal{C}$ be a presentable symmetric monoidal $\infty$-category. Let $\mathcal{E} \subseteq \text{End}(\mathcal{C})$ denote the full subcategory spanned by those endofunctors of $\mathcal{C}$ that preserve sifted colimits. We will construct a filtered $\infty$-monad $\text{Sym}_{\mathcal{E}}^{s*}$, whose filtered pieces $\text{Sym}_{\mathcal{E}}^{s_j}$ are given by the formula $\text{Sym}_{\mathcal{E}}^{s_j}(X) \approx \bigcup_{b \in \mathcal{E}(E)} (X^\otimes)^b$, together with an equivalence of $\infty$-monads $\text{colim} (\text{Sym}_{\mathcal{E}}^{s*}) \simeq \text{Sym}_{\mathcal{E}}$. We will refer to $\text{Sym}_{\mathcal{E}}^{s*}$ as the symmetric powers filtered monad on $\mathcal{C}$.

Let $\text{SSeq}(\mathcal{C})$ be as in Recollection 4.1.5 and let $\mathcal{E}'$ denote the (monoidal) full subcategory of $\text{End}(\text{SSeq}(\mathcal{C}))$ spanned by those endofunctors of $\text{SSeq}(\mathcal{C})$ that preserve sifted colimits and that preserve the image of the embedding $\iota : \mathcal{E} \to \text{SSeq}(\mathcal{C})$. We have a monoidal restriction functor $\rho : \mathcal{E}' \to \mathcal{E}$ that carries the $\mathcal{E}'$-monad $\text{Sym}_{\text{SSeq}(\mathcal{C})}$ to the $\mathcal{E}$-monad $\text{Sym}_{\mathcal{C}}$.

The monoidal functor $\theta : \text{SSeq}(\mathcal{C}) \to \text{End}(\text{SSeq}(\mathcal{C}))$ factors through $\mathcal{E}'$. By Construction 4.1.6, we have an operad $A \in \text{Op}(\mathcal{C})$ with an equivalence of monads $\theta(A) \simeq \text{Sym}_{\text{SSeq}(\mathcal{C})}$. Applying Construction 4.1.7, we obtain a lax monoidal functor $A^{s*} : Z_{\geq 0}^* \to \text{SSeq}(\mathcal{C})^\otimes$. We let $\text{Sym}_{\mathcal{C}}^{s*}$ denote the composite lax monoidal functor

$Z_{\geq 0}^* \xrightarrow{A^{s*}} \text{SSeq}(\mathcal{C})^\otimes \xrightarrow{\theta} \mathcal{E}' \xrightarrow{\rho} \mathcal{E}$.

This filtered $\mathcal{E}$-monad $\text{Sym}_{\mathcal{C}}^{s*}$ fulfills the requirements delineated above.

We close this subsection by recording two unrelated results about monads that will be needed later in the section.

**Proposition 4.1.9.** Let $\tau : \mathcal{C} \xrightarrow{\mathcal{C}_0} : \iota$ be a localization of $\infty$-categories, i.e. an adjunction where the right adjoint $\iota$ is fully faithful. Let $T$ be a monad on $\mathcal{C}$ such that the unit transformation $\iota \circ T \to \tau \circ T \circ \iota \circ \tau$ induces an equivalence $\tau \circ T \to \tau \circ T \circ \iota \circ \tau$. Then there is an induced monad structure on the composite $T_0 := T \circ T \circ \iota \in \text{End}(\mathcal{C}_0)$ and an induced localization $\tau : \text{LMod}_T(\mathcal{C}) \to \text{LMod}_{T_0}(\mathcal{C}_0)$ of $\mathcal{C}_0$, where the embedding $\iota : \text{LMod}_{T_0}(\mathcal{C}_0) \to \text{LMod}_T(\mathcal{C})$ identifies $\text{LMod}_{T_0}(\mathcal{C}_0)$ with the fiber product $\text{LMod}_T(\mathcal{C}) \times \mathcal{C}_0$.

**Proof.** Let $\mathcal{E}$ denote the full subcategory of $\text{End}(\mathcal{C})$ spanned by those endofunctors satisfying the property that the natural transformation $\tau \circ T \to \tau \circ T \circ \iota \circ \tau$ is an equivalence. Then $\mathcal{E}$ is closed under composition in $\text{End}(\mathcal{C})$ and contains $\iota \circ \mathcal{C}_0$, so we may regard $\mathcal{E}$ as a monoidal $\infty$-category over which $\iota$ is left tensored. It follows from the definition of $\mathcal{E}$ that the localization $\tau : \mathcal{C} \xrightarrow{\mathcal{C}_0} : \iota$ is compatible with this tensoring over $\mathcal{E}$, in the sense of [L-HA, Definition 2.2.1.6] (for the case that the $\infty$-operad $\mathcal{O}^\otimes$ is the left module $\infty$-operad $\text{LM}_\mathcal{O}^\otimes$ of §4.2.1 in op. cit.), determining a canonical left tensoring of $\mathcal{C}_0$ over $\mathcal{E}$. After unravelling definitions, this gives the claim. □
Proposition 4.1.10. Let $\kappa$ be a regular cardinal and let $\mathcal{C}$ be a $\kappa$-presentable $\infty$-category. Let $T$ be a monad on $\mathcal{C}$ that commutes with sifted colimits. Then $\text{LMod}_T(\mathcal{C})$ is a $\kappa$-presentable $\infty$-category.

Proof. We first argue that $\text{LMod}_T(\mathcal{C})$ admits small colimits. The hypothesis that $T$ commutes with sifted colimits ensures that $\text{LMod}_T(\mathcal{C})$ admits sifted colimits (which are preserved by the forgetful functor $\text{LMod}_T(\mathcal{C}) \to \mathcal{C}$), by [L-HA, Corollary 4.2.3.5]. It must admit an initial object as the functor $T: \mathcal{C} \to \text{LMod}_T(\mathcal{C})$ is a left adjoint and hence preserves initial objects. It’s enough then to see that any pair of objects $A, B \in \text{LMod}_T(\mathcal{C})$ admit a coproduct in $\text{LMod}_T(\mathcal{C})$. Using the bar resolutions $A \simeq \text{colim}_{[n] \in \Delta_{cp}} T^{(n+1)}(A)$ and $B \simeq \text{colim}_{[n] \in \Delta_{cp}} T^{(n+1)}(B)$, where $T^{(k)}$ denotes the $k$-fold iterated functor, we may reduce to the case that $A = T(X)$ and $B = T(Y)$ for $X, Y \in \mathcal{C}$. But then we may again use the fact that $T: \mathcal{C} \to \text{LMod}_T(\mathcal{C})$ is a left adjoint to conclude that $A$ and $B$ admit a coproduct.

We now want to show that $\text{LMod}_T(\mathcal{C})$ admits a small set of $\kappa$-compact generators. Let $\mathcal{C}_0$ denote the full subcategory of $\kappa$-compact objects in $\mathcal{C}$. If $\kappa = \omega$, define $\mathcal{A}_0$ to be the full subcategory of $\text{LMod}_T(\mathcal{C})$ spanned by objects that can be obtained as colimits of diagrams $A_\omega: \Delta_{cp}^\omega \rightarrow \text{LMod}_T(\mathcal{C})$, for some $m \geq 0$, such that for each $0 \leq n \leq m$ we have $A_n \simeq T(X_n)$ for some $X_n \in \mathcal{C}_0$; and if $\kappa > \omega$, define $\mathcal{A}_0$ to be the full subcategory of $\text{LMod}_T(\mathcal{C})$ spanned by objects that can be obtained as colimits of diagrams $A_\omega: \Delta_{cp}^\omega \rightarrow \text{LMod}_T(\mathcal{C})$ such that for each $n \geq 0$ we have $A_n \simeq T(X_n)$ for some $X_n \in \mathcal{C}_0$. We make the following observations:

- Since $\mathcal{C}$ is $\kappa$-presentable, $\mathcal{C}_0$ is essentially small, which implies that $\mathcal{A}_0$ is essentially small.
- It is immediate from $T: \mathcal{C} \rightarrow \text{LMod}_T(\mathcal{C})$ being left adjoint to the forgetful functor that $T(X)$ is a $\kappa$-compact object in $\text{LMod}_T(\mathcal{C})$ for any $X \in \mathcal{C}_0$. It follows that any $\kappa$-small colimit in $\text{LMod}_T(\mathcal{C})$ of objects of the form $T(X)$ for $X \in \mathcal{C}_0$ is $\kappa$-compact. In particular, any object of $\mathcal{A}_0$ is $\kappa$-compact.
- Let $A \in \text{LMod}_T(\mathcal{C})$. Since $\mathcal{C}$ is $\kappa$-presentable, there is a $\kappa$-filtered diagram $\{X_i\}_{i \in \mathcal{I}}$ in $\mathcal{C}_0$ whose colimit in $\mathcal{C}$ is equivalent to (the underlying object in $\mathcal{C}$ of) $A$. The bar resolution and $T$ commuting with $\kappa$-filtered colimits then gives us equivalences in $\text{LMod}_T(\mathcal{C})$:

$$A \simeq \text{colim}_{[n] \in \Delta_{cp}} T^{(n+1)}(A)$$

$$\simeq \text{colim}_{\{i \in \mathcal{I}\}} \text{colim}_{[n] \in \Delta_{cp}} T^{(n+1)}(X_i)$$

$$\simeq \text{colim}_{i \in \mathcal{I}} \text{colim}_{[n] \in \Delta_{cp}} T^{(n+1)}(X_i)$$

$$\simeq \text{colim}_{i \in \mathcal{I}} \text{colim}_{m \rightarrow \infty} \text{colim}_{[n] \in \Delta_{cp}} T^{(n+1)}(X_i).$$

The penultimate expression demonstrates that $A$ lies in the full subcategory of $\text{LMod}_T(\mathcal{C})$ generated under $\kappa$-filtered colimits by $\mathcal{A}_0$ when $\kappa > \omega$, and the last expression does so when $\kappa = \omega$.

We thus conclude that $\mathcal{A}_0$ is an essentially small subcategory of $\kappa$-compact objects generating $\text{LMod}_T(\mathcal{C})$, as desired. 

§4.2. The general definition. Our goal in this subsection is to define derived commutative rings. Before we begin, let us briefly outline the idea. As mentioned earlier, the goal here is to enlarge the $\infty$-category of simplicial commutative rings $\text{CAlg}_\mathbb{Z}^\Delta$ to allow for nonconnective objects. To do so, we look at $\text{CAlg}_\mathbb{Z}^\Delta$ from the vantage of the fact that its forgetful functor the $\infty$-category $\text{Mod}_\mathbb{Z}^\Sigma$ of connective $\mathbb{Z}$-modules is monadic: that is, it admits a left adjoint, and $\text{CAlg}_\mathbb{Z}^\Delta$ can be identified with $\infty$-category of modules over the monad associated to this adjunction. We denote this monad by $\text{LSym}_\mathbb{Z}$, and refer to it as the derived symmetric algebra monad on $\text{Mod}_\mathbb{Z}^\Sigma$. It is uniquely characterized by the facts that it preserves sifted colimits and that on finite free $\mathbb{Z}$-modules it is given by the formation of ordinary symmetric/polynomial algebras (not of free $E_\infty$-algebras). Our
strategy will be to construct an extension of the derived symmetric algebra monad from \( \text{Mod}^{\Sigma}_{\mathbb{Z}} \) to all of \( \text{Mod}_{\mathbb{Z}} \), and then define derived commutative rings as modules for this extended monad.

In this subsection, we will work not just in the setting of \( \mathbb{Z} \)-modules, but in the general context of an \( \infty \)-category \( \mathcal{C} \) subject to certain axioms. For our purposes in this paper, the point of making the definition in this generality is so that it may also be applied in the graded and filtered settings, i.e. not just for \( \mathcal{C} = \text{Mod}_{\mathbb{Z}} \) but for \( \mathcal{C} = \text{Gr}(\text{Mod}_{\mathbb{Z}}) \) and \( \mathcal{C} = \text{Fil}(\text{Mod}_{\mathbb{Z}}) \) as well. We will discuss in \( \S 4.3 \) how these settings fit into the framework of this subsection.

Let us now begin by formulating the axiomatic context in which we will work.

**Definition 4.2.1.** A derived algebraic context consists of a stable presentable symmetric monoidal \( \infty \)-category \( \mathcal{C} \), a compatible t-structure \( (\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}) \) (Definition 3.3.1), and a small full subcategory \( \mathcal{C}^{0} \subseteq \mathcal{C}^{\Sigma} \), satisfying the following properties:

(a) The t-structure is right complete. Note also that the compatibility of the t-structure implies that there is an induced presentable symmetric monoidal structure on \( \mathcal{C}^{\Sigma} \), with tensor product given by \( X \otimes_{\mathcal{C}^{0}} Y \simeq \pi_{0}(X \otimes_{\mathcal{C}} Y) \).

(b) The subcategory \( \mathcal{C}^{0} \) is a symmetric monoidal subcategory of \( \mathcal{C} \) and is closed under the formation of \( \mathcal{C}^{\Sigma} \)-symmetric powers: that is, for \( X \in \mathcal{C}^{0} \) and \( n \geq 0 \), we have that \( \text{Sym}_{\mathcal{C}^{0}}^{n}(X) \in \mathcal{C}^{0} \) as well.

(c) The subcategory \( \mathcal{C}^{0} \) is closed under the formation of finite coproducts in \( \mathcal{C} \) and its objects form a set of compact projective generators for \( \mathcal{C}_{\geq 0} \).

We will usually abusively refer to just \( \mathcal{C} \) as a derived algebraic context, leaving the rest of the structure implicit.

There is a natural notion of morphism between derived algebraic contexts \( \mathcal{C} \) and \( \mathcal{D} \), namely a colimit-preserving, right t-exact symmetric monoidal functor \( F : \mathcal{C} \to \mathcal{D} \) such that \( F(\mathcal{C}^{0}) \subseteq \mathcal{D}^{0} \).

**Remark 4.2.2.** The entire structure of a derived algebraic context \( \mathcal{C} \) is uniquely determined by the constituent piece \( \mathcal{C}^{0} \). Firstly, by the projective generation assumption, we have a canonical symmetric monoidal equivalence \( \mathcal{C}_{\geq 0} \simeq \mathcal{P}_{\Sigma}(\mathcal{C}^{0}) \), where \( \mathcal{P}_{\Sigma}(\mathcal{C}^{0}) \) inherits a symmetric monoidal structure from \( \mathcal{C}^{0} \) by [L-HA, Proposition 4.8.1.10]. And secondly, by the right completeness assumption, we have a canonical equivalence \( \mathcal{C} \simeq \text{Spt}(\mathcal{C}_{\geq 0}) \) (with the symmetric monoidal structure again inherited). These observations imply that \( \mathcal{C}^{\Sigma} \) may be canonically identified with the category \( \mathcal{P}_{\Sigma}(\mathcal{C}^{0}; \text{Set}) \) of product-preserving functors \( (\mathcal{C}^{0})^{\text{op}} \to \text{Set} \).

For the remainder of this subsection, we fix a derived algebraic context \( \mathcal{C} \).

**Notation 4.2.3.** If \( A \) and \( B \) are \( \infty \)-categories admitting sifted colimits, we let \( \text{Fun}^{S}(A, B) \) denote the full subcategory of \( \text{Fun}(A, B) \) spanned by those functors that preserve sifted colimits.

Recall from [L-HTT, §5.5.8] that \( \mathcal{C}^{0} \) projectively generating \( \mathcal{C}_{\geq 0} \) implies that, for any \( \infty \)-category \( \mathcal{D} \) admitting sifted colimits, the restriction functor \( \text{Fun}^{S}(\mathcal{C}, \mathcal{D}) \to \text{Fun}(\mathcal{C}^{0}, \mathcal{D}) \) is an equivalence of \( \infty \)-categories. We denote the image of a functor \( F : \mathcal{C}^{0} \to \mathcal{D} \) under the inverse to this equivalence by \( \mathcal{L}F : \mathcal{C}_{\geq 0} \to \mathcal{D} \), and refer to this as the left derived functor of \( F \).

**Example 4.2.4.** Let \( F : \mathcal{C}^{0} \to \mathcal{C}_{\geq 0} \) denote the functor sending \( X \mapsto \text{Sym}_{\mathcal{C}^{0}}(X) \). We denote the left derived functor \( \mathcal{L}F \) by \( \text{LSym}_{\mathcal{C}^{0}}^{\mathcal{C}_{\geq 0}} : \mathcal{C}_{\geq 0} \to \mathcal{C}_{\geq 0} \), and refer to this as the derived symmetric algebra functor on \( \mathcal{C}_{\geq 0} \). Replacing \( \text{Sym}_{\mathcal{C}^{0}} \) with \( \text{Sym}_{\mathcal{C}^{0}}^{i} \) for \( i \geq 0 \), we similarly obtain derived symmetric powers functors \( \text{LSym}^{i}_{\mathcal{C}^{0}} \) on \( \mathcal{C}_{\geq 0} \).

As we will see below, the derived symmetric algebra functor \( \text{LSym}^{\mathcal{C}^{0}}_{\mathcal{C}_{\geq 0}} \) has a canonical monad structure. Our goal is to construct an extension of this monad to \( \mathcal{C} \). We will do so following an approach due to Mathew, via the theory of functor calculus\(^7\), so let us begin by reviewing the

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\(^7\)There is also an earlier approach to extending the derived symmetric powers functors due to Illusie ([Ill71]).
necessary aspects of this theory. It will be apparent further below, but let us note now that we are largely following [BM19, §3] here.

**Definition 4.2.5.** For \( n \geq 0 \), let \( \mathcal{P}_n \) denote the power set of \( \{0, \ldots, n\} \), and for \( 0 \leq m \leq n + 1 \), let \( \mathcal{P}_n^\leq m \) (resp. \( \mathcal{P}_n^\geq m \)) denote the subset of \( \mathcal{P}_n \) consisting of those subsets of \( \{0, \ldots, n\} \) of cardinality at most (resp. at least) \( m \). We regard \( \mathcal{P}_n, \mathcal{P}_n^\leq m, \mathcal{P}_n^\geq m \) as categories via their partial ordering by inclusion.

Given an \( \infty \)-category \( A \), an \( n \)-cube in \( A \) is a diagram \( \chi : \mathcal{P}_n \to A \). We say an \( n \)-cube \( \chi \) is:
- **cocartesian** if it is a colimit diagram, i.e. exhibits \( \chi(\{0, \ldots, n\}) \) as a colimit of the diagram \( \chi|_{\mathcal{P}_n^{\leq n}} \);
- **cartesian** if it is a limit diagram, i.e. exhibits \( \chi(\varnothing) \) a a limit of the diagram \( \chi|_{\mathcal{P}_n^{\geq 1}} \);
- **strongly cocartesian** if it is left Kan extended from its restriction to \( \mathcal{P}_n^\leq 1 \).

**Remark 4.2.6.** If \( A \) is a stable \( \infty \)-category, then an \( n \)-cube in \( A \) is cartesian if and only if it is cocartesian ([L-HA, Proposition 1.2.4.13]).

**Definition 4.2.7.** Let \( A \) be an \( \infty \)-category admitting finite colimits and let \( \mathcal{B} \) be a stable \( \infty \)-category.

- For \( n \geq 0 \), a functor \( F : A \to \mathcal{B} \) is called **\( n \)-excisive** if it carries strongly cocartesian \( n \)-cubes in \( A \) to cocartesian (equivalently cartesian by Remark 4.2.6) \( n \)-cubes in \( \mathcal{B} \). We let \( \text{Exc}_n(A, \mathcal{B}) \) denote the full subcategory of \( \text{Fun}(A, \mathcal{B}) \) spanned by the functors that are \( n \)-excisive.
- A functor \( F : A \to \mathcal{B} \) is called **excisively polynomial** if it is \( n \)-excisive for some \( n \geq 0 \). We let \( \text{Fun}_{\text{poly}}(A, \mathcal{B}) \) denote the full subcategory of \( \text{Fun}(A, \mathcal{B}) \) spanned by the functors that are excisively polynomial.

**Example 4.2.8.** Let \( \mathcal{B} \) be a stable symmetric monoidal \( \infty \)-category. Then the functor \( \text{Sym}^\leq i : \mathcal{B} \to \mathcal{B} \) is \( i \)-excisive.

**Remark 4.2.9.** Let \( \mathcal{B}, \mathcal{B}', \mathcal{B}'' \) be stable \( \infty \)-categories. If \( F : \mathcal{B} \to \mathcal{B}' \) is an \( n \)-excisive functor and \( G : \mathcal{B}' \to \mathcal{B}'' \) is an \( m \)-excisive functor, then the composition \( GF \) is \( mn \)-excisive (see for example [McC, Lemma 7.5]). It follows that the collection of excisively polynomial endofunctors on a stable \( \infty \)-category \( \mathcal{B} \) is closed under composition.

**Definition 4.2.10.** Let \( A \) and \( \mathcal{B} \) be additive \( \infty \)-categories and assume \( \mathcal{B} \) is idempotent complete. We inductively say that a functor \( F : A \to \mathcal{B} \) is:
- **of degree** \( 0 \) if \( F \) is constant;
- **of degree** \( n \) for \( n \geq 1 \) if, for each \( X \in A \), the difference functor \( D_XF : A \to \mathcal{B} \) defined by \( D_XF(Y) := \text{fib}(F(X \oplus Y) \to F(Y)) \) is of degree \( n - 1 \) (note that this fiber is guaranteed to exist by the idempotent completeness assumption).

We say that a functor \( F : A \to \mathcal{B} \) is **additively polynomial** if it is of degree \( n \) for some \( n \geq 0 \), and let \( \text{Fun}_{\text{poly}}(A, \mathcal{B}) \) denote the full subcategory of \( \text{Fun}(A, \mathcal{B}) \) spanned by the additively polynomial functors.

**Example 4.2.11.** Let \( \mathcal{B} \) be an additive symmetric monoidal \( \infty \)-category. Then the functor \( \text{Sym}^\leq i : \mathcal{B} \to \mathcal{B} \) is of degree \( i \).

**Remark 4.2.12.** Let \( \mathcal{B}, \mathcal{B}', \mathcal{B}'' \) be additive \( \infty \)-categories. If \( F : \mathcal{B} \to \mathcal{B}' \) is a functor of degree \( n \) and \( G : \mathcal{B}' \to \mathcal{B}'' \) is a functor of degree \( m \), then the composition \( GF \) is of degree \( mn \). It follows that the collection of additively polynomial endofunctors on an additive \( \infty \)-category \( \mathcal{B} \) is closed under composition.

**Notation 4.2.13.** The functor category decorations introduced in Notation 4.2.3 and Definitions 4.2.7 and 4.2.10 will also be applied to endomorphism categories, with the same meaning.
They may also be applied in conjunction, with the obvious meaning. For example, \( \text{End}^{\Sigma}_{e\text{-poly}}(\mathcal{C}) \) denotes the \( \infty \)-category of functors \( F : \mathcal{C} \rightarrow \mathcal{C} \) that are excisively polynomial and preserve sifted colimits.

**Proposition 4.2.14.** Let \( \mathcal{D} \) be a stable \( \infty \)-category admitting small colimits and let \( F : \mathcal{C}^0 \rightarrow \mathcal{D} \) be a functor of degree \( n \) for some \( n \geq 0 \). Then the left derived functor \( LF : \mathcal{C}_{20} \rightarrow \mathcal{D} \) is \( n \)-excisive.

**Proof.** See the proof of [BM19, Proposition 3.34]. \( \square \)

The following result is the key for extending functors defined on \( \mathcal{C}_{20} \) to all of \( \mathcal{C} \).

**Proposition 4.2.15.** Let \( \mathcal{D} \) be a stable \( \infty \)-category admitting small colimits. Then the restriction functor

\[
\text{Fun}^{\Sigma}_{e\text{-poly}}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}^{\Sigma}_{e\text{-poly}}(\mathcal{C}_{20}, \mathcal{D})
\]

is an equivalence of \( \infty \)-categories.

**Proof.** It will suffice to show for each fixed \( n \geq 0 \) that the restriction functor

\[
\psi : \text{Exc}^\Sigma_n(\mathcal{C}, \mathcal{D}) \rightarrow \text{Exc}^\Sigma_n(\mathcal{C}_{20}, \mathcal{D})
\]

is an equivalence. We closely follow the proof of [BM19, Theorem 3.35].

We first claim that \( \psi \) has a left adjoint \( \phi \), which sends \( F \in \text{Exc}^\Sigma_n(\mathcal{C}_{20}, \mathcal{D}) \) to \( P_n(F \circ \tau_{20}) \), where \( P_n : \text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Exc}_n(\mathcal{C}, \mathcal{D}) \) denotes the \( n \)-excisive approximation functor. This is straightforward to see once we know that \( F_n(F \circ \tau_{20}) \) preserves sifted colimits, assuming \( F \) does. It preserves filtered colimits because \( \tau_{20} \) does and \( P_n \) preserves this property (as a result of Goodwillie’s explicit description of \( P_n \)). By [BM19, Proposition 3.36], this implies it preserves sifted colimits.

Now, the unit map \( \text{id} \rightarrow \psi \circ \phi \) is an equivalence, implying that \( \phi \) is fully faithful. Thus, to prove that \( \psi \) is an equivalence, it suffices to prove that \( \psi \) is conservative. We can see this in two steps: firstly, restriction to \( \mathcal{C}_{2-\infty} := \bigcup_{k \geq 0} \mathcal{C}_{2-k} \) is conservative, because our functors in particular preserve sequential colimits and, for any \( X \in \mathcal{C} \), we have \( X \cong \operatorname{colim}_{m \rightarrow \infty} \tau_{2-m}(X) \) by our assumption that the \( t \)-structure on \( \mathcal{C} \) is right complete; secondly, restriction from \( \mathcal{C}_{2-\infty} \) to \( \mathcal{C}_{20} \) is conservative by the argument in the last paragraph of the proof of [BM19, Theorem 3.35]. \( \square \)

**Remark 4.2.16.** Let \( \mathcal{D} \) be a stable \( \infty \)-category admitting small colimits and let \( F : \mathcal{C} \rightarrow \mathcal{D} \) be a functor that preserves filtered colimits and is excisively polynomial. By [BM19, Proposition 3.36], \( F \) preserves \( n \)-skeletal totalizations for any \( n \geq 0 \). This provides a way of understanding the behavior of \( F \) on certain objects in \( \mathcal{C}_{20} \) in terms of its behavior on \( \mathcal{C}^0 \): namely, if \( X \in \mathcal{C} \), then \( F(X) \cong \lim F(X^\sharp) \). For example, in the case that \( \mathcal{C} = \text{Mod}_Z \), this may be applied when \( X \) is any perfect coconnective \( Z \)-module.

We need just a bit more of preparation to construct the derived symmetric algebra monad on \( \mathcal{C} \).

**Lemma 4.2.17.** Let \( F : \mathcal{C}_{20} \rightarrow \mathcal{C}_{20} \) be a functor preserving sifted colimits. Then the truncation map \( X \rightarrow \pi_0(X) \) induces an equivalence \( \pi_0(F(X)) \cong \pi_0(F(\pi_0(X))) \) for all \( X \in \mathcal{C}_{20} \).

**Proof.** Let \( G := \pi_0 F : \mathcal{C}_{20} \rightarrow \mathcal{C}^0 \). Then \( G \) preserves sifted colimits, hence is the left derived functor of its restriction \( G_0 := G|_{\mathcal{C}^0} \). On the other hand, as we have \( \mathcal{C}^0 \cong \mathcal{P}_{\Delta}(\mathcal{C}^0; \text{Set}) \) (Remark 4.2.2), there is a unique sifted colimit-preserving functor \( G' : \mathcal{C}^0 \rightarrow \mathcal{C}^0 \) with restriction \( G'|_{\mathcal{C}^0} \cong G_0 \), and from this we obtain that \( G' \circ \pi_0 : \mathcal{C}_{20} \rightarrow \mathcal{C}^0 \) is also the left derived functor of \( G_0 \). We thus have \( G \cong G' \), and the claim follows. \( \square \)

**Remark 4.2.18.** Consider the following two full subcategories of \( \text{End}^{\Sigma}(\mathcal{C}_{20}) \):

- \( \text{End}_{0}^{\Sigma}(\mathcal{C}_{20}) \), spanned by those endofunctors that, in addition to preserving sifted colimits, preserve the subcategory \( \mathcal{C}^0 \subseteq \mathcal{C}_{20} \);
• \( \text{End}_{\mathcal{C}}^{\leq}(\mathcal{C}_{\geq}) \), spanned by those endofunctors that, in addition to preserving sifted colimits, satisfy \( \pi_0(F(X)) \in \mathcal{C}^0 \) for all \( X \in \mathcal{C}^0 \).

The former is evidently a monoidal subcategory, and it follows from Lemma 4.2.17 that the latter is a monoidal subcategory. We also have an inclusion \( \text{End}_{\mathcal{C}}^{\geq}(\mathcal{C}_{\leq}) \subseteq \text{End}_{\mathcal{C}}^{\leq}(\mathcal{C}_{\geq}) \).

This inclusion of \( \infty \)-categories admits a left adjoint \( \tau : \text{End}_{\mathcal{C}}^{\leq}(\mathcal{C}_{\leq}) \to \text{End}_{\mathcal{C}}^{\geq}(\mathcal{C}_{\geq}) \): identifying sifted colimit–preserving functors \( \mathcal{C}_{\leq} \to \mathcal{C}_{\leq} \) with functors \( \mathcal{C}^0 \to \mathcal{C}_{\leq} \), the left adjoint \( \tau \) is given by composition with \( \pi_0 \). The monoidal structure on the inclusion endows the left adjoint \( \tau \) with an oplax monoidal structure, and it follows from Lemma 4.2.17 that it is in fact strictly monoidal.

We now come to the crucial constructions of this subsection. Note that we are forced to work with the filtrations \( \text{Sym}^{\leq} \) and \( \text{LSym}^{\leq} \), and the notion of filtered monads defined in §4.1.1, because the filtered pieces are excisively polynomial (hence fall under the purview of Proposition 4.2.15) but their colimits \( \text{Sym} \) and \( \text{LSym} \) are not.

**Construction 4.2.19** (Derived symmetric powers filtered monad). Let \( \mathcal{E} \subseteq \text{End}(\mathcal{C}) \) denote the full subcategory spanned by those endofunctors that are excisively polynomial, preserve sifted colimits, and preserve the full subcategory \( \mathcal{C}_{\leq} \subseteq \mathcal{C} \). We will construct a filtered \( \mathcal{E} \)-monad \( \text{LSym}_{\mathcal{C}}^{\leq} \), together with natural equivalences \( \text{LSym}_{\mathcal{C}}^{\leq} |_{\mathcal{C}_{\geq}} \simeq \text{Sym}_{\mathcal{C}_{\geq}}^{\leq} \) for all \( i \geq 0 \), as well as a map of filtered \( \mathcal{E} \)-monads \( \theta^{\leq} : \text{Sym}_{\mathcal{C}}^{\leq} \to \text{LSym}_{\mathcal{C}}^{\leq} \), where \( \text{LSym}_{\mathcal{C}}^{\leq} \) is the symmetric powers filtered monad on \( \mathcal{C} \) (Construction 4.1.8). We will refer to the filtered monad \( \text{LSym}_{\mathcal{C}}^{\leq} \) as the derived symmetric powers filtered monad on \( \mathcal{C} \).

Let \( \mathcal{E}' \subseteq \text{End}(\mathcal{C}_{\leq}) \) denote the full subcategory spanned by those endofunctors that preserve sifted colimits and whose composition with the inclusion \( \mathcal{C}_{\leq} \subseteq \mathcal{C} \) is excisively polynomial; \( \mathcal{E}' \) is a monoidal subcategory, i.e., contains the identity functor and is closed under composition. It follows from Proposition 4.2.15 that the (monoidal) restriction functor \( \mathcal{E} \to \mathcal{E}' \) is an equivalence. Noting that this restriction carries \( \text{Sym}_{\mathcal{C}}^{\leq} \) to \( \text{Sym}_{\mathcal{C}_{\leq}}^{\leq} \) (invoking Example 4.2.8), it will be enough to make the construction with \( \mathcal{C}_{\leq} \) in place of \( \mathcal{C} \).

Now, using the notation of Remark 4.2.18, we have \( \text{Sym}_{\mathcal{C}_{\leq}}^{\leq} \in \text{End}_{\mathcal{C}}^{\leq}(\mathcal{C}_{\leq}) \) for all \( i \geq 0 \), and we define \( \text{LSym}_{\mathcal{C}_{\leq}}^{\leq} = \tau(\text{Sym}_{\mathcal{C}_{\leq}}^{\leq}) \). The monoidality of \( \tau \) implies that \( \text{LSym}_{\mathcal{C}_{\leq}}^{\leq} \) inherits a filtered monad structure from \( \text{Sym}_{\mathcal{C}_{\leq}}^{\leq} \), and the description of \( \tau \) in Remark 4.2.18 gives natural equivalences

\[
\text{LSym}_{\mathcal{C}_{\leq}}^{\leq}(X) \simeq \pi_0(\text{Sym}_{\mathcal{C}_{\leq}}^{\leq}(X)) \simeq \text{Sym}_{\mathcal{C}}^{\leq}(X)
\]

for \( X \in \mathcal{C}^0 \) and \( i \geq 0 \), as desired. Since the functors \( \text{LSym}_{\mathcal{C}_{\leq}}^{\leq} \) preserve sifted colimits, they must be the left derived functors of the functors \( \text{Sym}_{\mathcal{C}_{\leq}}^{\leq} \), by Proposition 4.2.14 and Example 4.2.11, this implies that their compositions with the inclusion \( \mathcal{C}_{\leq} \subseteq \mathcal{C} \) are excisively polynomial. Thus, \( \text{LSym}_{\mathcal{C}_{\leq}}^{\leq} \) is indeed a filtered \( \mathcal{E}' \)-monad.

Finally, the unit map of the adjunction defining \( \tau \) gives us the desired map of filtered monads \( \theta^{\leq} : \text{Sym}_{\mathcal{C}_{\leq}}^{\leq} \to \text{LSym}_{\mathcal{C}_{\leq}}^{\leq} \), finishing the construction.

**Construction 4.2.20** (Derived symmetric algebra monad). We let \( \text{LSym}_{\mathcal{C}} \) denote the colimit of the derived symmetric powers filtered monad \( \text{LSym}_{\mathcal{C}}^{\leq} \) of Construction 4.2.19. We regard \( \text{LSym}_{\mathcal{C}} \) as a (sifted colimit–preserving) monad on \( \mathcal{C} \) by Proposition 4.1.4, and refer to it as the derived symmetric algebra monad on \( \mathcal{C} \).

**Remark 4.2.21.** By Propositions 4.2.14 and 4.2.15, for each \( i \geq 0 \), there is a unique sifted colimit–preserving, excisively polynomial functor \( \text{LSym}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \) that restricts to the derived symmetric power functor \( \text{LSym}_{\mathcal{C}}^{\leq} \) (Example 4.2.4) on \( \mathcal{C}_{\leq} \). Examining Constructions 4.2.19 and 4.2.20, we see that the equivalence \( \text{Sym}_{\mathcal{C}_{\leq}}^{\leq} \simeq \bigoplus_{0 \leq j \leq i} \text{Sym}_{\mathcal{C}^0}^{\leq} \) determine equivalences \( \text{LSym}_{\mathcal{C}}^{\leq} \simeq \bigoplus_{0 \leq j \leq i} \text{LSym}_{\mathcal{C}}^{j} \), and hence an equivalence \( \text{LSym}_{\mathcal{C}} \simeq \bigoplus_{i \geq 0} \text{LSym}_{\mathcal{C}}^{j} \).

We are now able to make the main definition of this section.
Definition 4.2.22. A derived commutative algebra object of $\mathcal{C}$ is a module over the derived symmetric algebra monad $\text{LSym}_{\mathcal{C}}$ on $\mathcal{C}$ (Construction 4.2.20). We let $\text{DAlg}(\mathcal{C})$ denote the $\infty$-category $\text{LMod}_{\text{LSym}_{\mathcal{C}}}(\mathcal{C})$ of derived commutative algebra objects of $\mathcal{C}$.

Remark 4.2.23. By Proposition 4.1.10, the $\infty$-category $\text{DAlg}(\mathcal{C})$ is presentable, in particular admits small limits and colimits.

Remark 4.2.24. Let $\text{DAlg}(\mathcal{C})^c$ denote the fiber product $\text{DAlg}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{s0}$, and let $\text{DAlg}(\mathcal{C})^\circ$ denote the fiber product $\text{DAlg}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}^\circ$. These $\infty$-categories admit the following alternative descriptions:

- Let $\mathcal{D}^0$ denote the full subcategory of $\text{DAlg}(\mathcal{C})^c$ spanned by the objects $\text{LSym}_{\mathcal{C}}(X)$ for $X \in \mathcal{C}^0$. Then $\text{DAlg}(\mathcal{C})^c$ is projectively generated by the objects of $\mathcal{D}^0$, so that there is a canonical equivalence $\text{DAlg}(\mathcal{C})^c \simeq \mathcal{P}_{\mathcal{C}}(\mathcal{D}^0)$. This follows from [L-HA, Corollary 4.7.3.18].

- There is a canonical equivalence $\text{DAlg}(\mathcal{C})^\circ \simeq \text{CAlg}(\mathcal{C}^\circ)$. To see this, consider the localization $\pi_0 : \mathcal{C}_{s0} \to \mathcal{C}^\circ : \iota$. Using reasoning similar to that in the proof of Construction 4.2.19, one can see that the monad $\text{LSym}_{\mathcal{C}_{s0}}$ on $\mathcal{C}_{s0}$ satisfies the hypothesis of Proposition 4.1.9 for this localization, and that the induced monad on $\mathcal{C}^\circ$ canonically identifies with the symmetric algebra monad $\text{Sym}_{\mathcal{C}^\circ}$ on $\mathcal{C}^\circ$. Proposition 4.1.9 then implies the desired equivalence

$$\text{DAlg}(\mathcal{C})^\circ \simeq \text{LMod}_{\text{Sym}_{\mathcal{C}^\circ}}(\mathcal{C}^\circ) \simeq \text{CAlg}(\mathcal{C}^\circ).$$

Remark 4.2.25. The above definition of derived commutative algebra objects is natural in the following sense: a morphism of derived algebraic contexts $F : \mathcal{C} \to \mathcal{D}$ determines a colimit-preserving functor $F' : \text{DAlg}(\mathcal{C}) \to \text{DAlg}(\mathcal{D})$ such that the diagrams

$$\begin{array}{ccc}
\text{DAlg}(\mathcal{C}) & \xrightarrow{F'} & \text{DAlg}(\mathcal{D}) \\
U_{\mathcal{C}} \downarrow & & \downarrow U_{\mathcal{D}} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\text{DAlg}(\mathcal{C}) & \xrightarrow{F'} & \text{DAlg}(\mathcal{D}) \\
L\text{Sym}_{\mathcal{C}} \downarrow & & \downarrow L\text{Sym}_{\mathcal{D}} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}$$

canonically commute (where the vertical arrows in the first diagram are the forgetful functors, and in the second diagram are their left adjoints). Note that the commuting of the second diagram is equivalent to the commuting of the diagram of right adjoints

$$\begin{array}{ccc}
\text{DAlg}(\mathcal{C}) & \xleftarrow{G'} & \text{DAlg}(\mathcal{D}) \\
U_{\mathcal{C}} \downarrow & & \downarrow U_{\mathcal{D}} \\
\mathcal{C} & \xleftarrow{G} & \mathcal{D}
\end{array}$$

(F and $F'$ admit right adjoints $G$ and $G'$ by the adjoint functor theorem).

More generally, similar statements hold when the morphism $F : \mathcal{C} \to \mathcal{D}$ is replaced by any diagram of derived algebraic contexts. This general form of naturality can be proved by running through the constructions and results of this section with a bit more care, replacing the single derived algebraic context $\mathcal{C}$ with any diagram of such.

We now compare derived commutative algebra objects of $\mathcal{C}$ with $E_\infty$-algebra objects of $\mathcal{C}$.

Notation 4.2.26. Let $\theta : \text{Sym}_{\mathcal{C}} \to \text{LSym}_{\mathcal{C}}$ denote the map of monads obtained by taking the colimit of the map of filtered monads $\theta^*$ of Construction 4.2.19, using the equivalence $\text{Sym}_{\mathcal{C}} \simeq \text{colim}(\text{Sym}_{\mathcal{C}}^*)$ of Construction 4.1.8. Then let $\Theta : \text{DAlg}(\mathcal{C}) \to \text{CAlg}(\mathcal{C})$ denote the restriction functor induced by $\theta$ on $\infty$-categories of modules, using the canonical equivalence $\text{CAlg}(\mathcal{C}) \simeq \text{LMod}_{\text{Sym}_{\mathcal{C}}}(\mathcal{C})$. Following [L-SAG, §25], for $A \in \text{DAlg}(\mathcal{C})$, we let $A^\circ$ denote $\Theta(A) \in \text{CAlg}(\mathcal{C})$ and refer to this as the underlying $E_\infty$-algebra of $A$. When less precision is acceptable, we will just write $A$ instead of $A^\circ$.

Proposition 4.2.27. The functor $\Theta : \text{DAlg}(\mathcal{C}) \to \text{CAlg}(\mathcal{C})$ of Notation 4.2.26 preserves small limits and colimits.
Proof. Limits and sifted colimits are preserved by $\Theta$ because they are computed in both $\text{DAlg}(\mathcal{C})$ and $\text{CAlg}(\mathcal{C})$ at the level of underlying objects in $\mathcal{C}$. The initial object is preserved by $\Theta$ because $\text{Sym}_{\mathcal{C}}$ and $\text{LSym}_{\mathcal{C}}$ preserve initial objects (being left adjoints) and we have $\text{Sym}_{\mathcal{C}}(\emptyset) = \mathbb{1}_{\mathcal{C}} = \text{LSym}_{\mathcal{C}}$, where $\mathbb{1}_{\mathcal{C}}$ denotes the unit object of $\mathcal{C}$ (the latter equivalence uses the hypothesis that $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}^0$). It now only remains to show that binary coproducts are preserved.

Recall that the coproduct in $\text{CAlg}(\mathcal{C})$ is given by the tensor product $\otimes_{\mathcal{C}}$, and let us temporarily denote the coproduct in $\text{DAlg}(\mathcal{C})$ by $\boxtimes_{\mathcal{C}}$. We wish to show that, for $A, B \in \text{DAlg}(\mathcal{C})$, the canonical map $\Theta(A) \otimes_{\mathcal{C}} \Theta(B) \to \Theta(A \boxtimes B)$ is an equivalence. Remembering that $\text{DAlg}(\mathcal{C})$ is the $\infty$-category of algebras for the monad $\text{LSym}_{\mathcal{C}}$, we have the canonical bar resolutions $A \simeq \text{colim}_{[n] \to \Delta^{op}} \text{LSym}_{\mathcal{C}}^{(n+1)}(A)$ and $B \simeq \text{colim}_{[n] \to \Delta^{op}} \text{LSym}_{\mathcal{C}}^{(n+1)}(B)$, where $\text{LSym}_{\mathcal{C}}^{(n)}$ denotes the $k$-fold iterated functor. As $\otimes_{\mathcal{C}}$ and $\boxtimes_{\mathcal{C}}$ commute with geometric realizations in each variable and $\Theta$ commutes with geometric realizations, this reduces us to the case that $A \simeq \text{LSym}_{\mathcal{C}}(X)$ and $B \simeq \text{LSym}_{\mathcal{C}}(Y)$ for $X, Y \in \mathcal{C}$.

In this special case, we have a canonical equivalence of derived commutative algebras

$$\text{LSym}_{\mathcal{C}}(X) \boxtimes_{\mathcal{C}} \text{LSym}_{\mathcal{C}}(Y) \to \text{LSym}_{\mathcal{C}}(X \oplus Y),$$

since $\text{LSym}_{\mathcal{C}} : \mathcal{C} \to \text{DAlg}(\mathcal{C})$ is a left adjoint and hence preserves coproducts. We thus wish to show that the canonical map $\text{LSym}_{\mathcal{C}}(X) \otimes_{\mathcal{C}} \text{LSym}_{\mathcal{C}}(Y) \to \text{LSym}_{\mathcal{C}}(X \oplus Y)$ is an equivalence. One can show that this map may be naturally obtained as the colimit of a filtered map $\text{LSym}_{\mathcal{C}}^\omega(X) \otimes_{\mathcal{C}} \text{LSym}_{\mathcal{C}}^\omega(Y) \to \text{LSym}_{\mathcal{C}}^\omega(X \oplus Y)$, where we regard $\text{LSym}_{\mathcal{C}}^\omega$ as a functor $\mathcal{C} \to \text{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{C})$ and $\otimes_{\mathcal{C}}$ denotes the Day convolution tensor product in $\text{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{C})$. Since each filtered piece of $\text{LSym}_{\mathcal{C}}^\omega$ is excisively polynomial and preserves sifted colimits, we may reduce first by Proposition 4.2.15 to the case that $X, Y \in \mathcal{C}_{\geq 0}$ and then to the case that $X, Y \in \mathcal{C}^0$. In this last case, the claim is immediate from the construction of $\text{LSym}_{\mathcal{C}}$.

\[\square\]

Notation 4.2.28. Let $A$ be a derived commutative algebra object of $\mathcal{C}$.

(a) By an $A$-module we mean an $A^\circ$-module object of $\mathcal{C}$, and we let $\text{Mod}_A(\mathcal{C})$ denote the $\infty$-category $\text{Mod}_{A^\circ}(\mathcal{C})$.

(b) A derived commutative $A$-algebra is a derived commutative algebra object $B$ of $\mathcal{C}$ equipped with a map $\phi : A \to B$ in $\text{DAlg}(\mathcal{C})$. We let $\text{DAlg}_A(\mathcal{C})$ denote the $\infty$-category $\text{DAlg}(\mathcal{C})_{A/}$ of derived commutative $A$-algebras, which is presentable (by [L-HTT, Proposition 5.5.3.10], since $\text{DAlg}(\mathcal{C})$ is presentable).

(c) Let $\text{CAlg}_{A^\circ}(\mathcal{C})$ denote the $\infty$-category $\text{CAlg}(\mathcal{C})_{A^\circ/}$ of $E_{\infty}$-$A$-algebras in $\mathcal{C}$. The functor $\Theta$ induces a functor $\Theta_A : \text{DAlg}_A(\mathcal{C}) \to \text{CAlg}_{A^\circ}(\mathcal{C})$, which also preserves small limits and colimits. From this we obtain a forgetful functor $\text{DAlg}_A(\mathcal{C}) \to \text{Mod}_A(\mathcal{C})$ which preserves limits and sifted colimits; this forgetful functor admits a left adjoint $\text{LSym}_A : \text{Mod}_A(\mathcal{C}) \to \text{DAlg}_A(\mathcal{C})$ by the adjoint functor theorem, and this adjunction is monadic by the Barr–Beck–Lurie theorem.

(d) We regard $\text{DAlg}_A(\mathcal{C})$ as a symmetric monoidal $\infty$-category via the cocartesian symmetric monoidal structure. We denote the coproduct by $\boxtimes_A$, which is reasonable since the functor $\Theta_A : \text{DAlg}_A(\mathcal{C}) \to \text{CAlg}_{A^\circ}(\mathcal{C})$ preserves coproducts, so that, at the level of underlying $A$-modules or $E_{\infty}$-algebras, $\boxtimes_A$ is given by the relative tensor product $\otimes_{A^\circ}$. In other words, $\Theta_A$ and the forgetful functor $\text{DAlg}_A(\mathcal{C}) \to \text{Mod}_A(\mathcal{C})$ are canonically symmetric monoidal.

When it is safe and convenient to do so, we will drop $\mathcal{C}$ from the notation established above; for example, we may denote $\text{Mod}_A(\mathcal{C})$ simply by $\text{Mod}_A$ and $\text{DAlg}_A(\mathcal{C})$ by $\text{DAlg}_A$.

Remark 4.2.29. Let $A \to B$ be a map in $\text{DAlg}(\mathcal{C})$. The resulting restriction functor $\text{DAlg}_B(\mathcal{C}) \to \text{DAlg}_A(\mathcal{C})$ has a left adjoint $B \otimes_A - : \text{DAlg}_A(\mathcal{C}) \to \text{DAlg}_B(\mathcal{C})$ given by coproduct in $\text{DAlg}_A(\mathcal{C})$ (or equivalently pushout in $\text{DAlg}(\mathcal{C})$) with $B$. Moreover, there is a canonical natural equivalence of derived commutative $B$-algebras $B \boxtimes_A \text{LSym}_A(M) \simeq \text{LSym}_B(B \otimes_A M)$ for $M \in \text{Mod}_A(\mathcal{C})$ (this is immediate from considering the right adjoint functors).

We close this subsection by discussing the following definition, which was motivated at the
We construct a monad canonically commutes (the vertical arrows being the forgetful functors), together with an equivalence $\infty$-category of derived commutative bialgebra objects of $\mathcal{C}$ and $\infty$-category of derived bicommutative bialgebra objects of $\mathcal{C}$ respectively.

**Definition 4.2.30.** A derived commutative (resp. derived bicommutative) bialgebra object of $\mathcal{C}$ is a coalgebra (resp. cocommutative coalgebra) object of $DAlg(\mathcal{C})$. We set $bAlg_{DL}(\mathcal{E}) := cAlg(DAlg(\mathcal{E}))$ and $bAlg_{DL}(\mathcal{E}) := cCAlg(DAlg(\mathcal{E}))$, and refer to these as the $\infty$-category of derived commutative bialgebra objects of $\mathcal{C}$ and $\infty$-category of derived bicommutative bialgebra objects of $\mathcal{C}$ respectively.

**Remark 4.2.31.** A derived commutative (resp. derived bicommutative) bialgebra object $A$ of $\mathcal{C}$ has an underlying commutative (resp. bicommutative) bialgebra object $A^o$ (which again we may just denote by $A$).

Let us now explain why, given a derived commutative bialgebra object $A$ of $\mathcal{C}$, we may lift the derived symmetric algebra monad $LSym$ from $\mathcal{C}$ to the comodule category $cLMod_A(\mathcal{C})$, allowing us to make sense of derived commutative algebra objects in this comodule category. This is analogous to what was discussed in §2.2, namely that the underlying commutative bialgebra structure on $A$ determines a symmetric monoidal structure on $cLMod_A(\mathcal{C})$, allowing us to make sense of $E_\infty$-algebra objects in this comodule category.

**Construction 4.2.32.** Let $A \in bAlg_{DL}(\mathcal{E})$, which we note has an underlying coalgebra object in $\mathcal{C}$. We construct a monad $LSym$ on the $\infty$-category $cLMod_A(\mathcal{C})$ such that the diagram

$$
\begin{array}{ccc}
\text{cLMod}_A(\mathcal{C}) & \xrightarrow{LSym} & \text{cLMod}_A(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{LSym} & \mathcal{C}
\end{array}
$$

canonical commutes (the vertical arrows being the forgetful functors), together with an equivalence of $\infty$-categories $LMod_{LSym}(cLMod_A(\mathcal{C})) \simeq cLMod_A(DAlg(\mathcal{C}))$.

Consider the free-forget adjunction $F : \mathcal{C} \rightleftarrows DAlg(\mathcal{C}) : G$, whose associated monad is $LSym$ by definition of $DAlg(\mathcal{C})$. Since the right adjoint $G$ is canonically symmetric monoidal, the left adjoint $F$ is canonically oplax symmetric monoidal, and there is an induced adjunction $F' : cLMod_A(\mathcal{C}) \rightleftarrows cLMod_A(DAlg(\mathcal{C})) : G'$. We define the monad $LSym$ on $cLMod_A(\mathcal{C})$ to be the derived commutative bialgebra structure on $\mathcal{C}$ begins with what was discussed in §2.2, namely that the underlying commutative bialgebra structure on $A$ determines a symmetric monoidal structure on $cLMod_A(\mathcal{C})$, allowing us to make sense of $E_\infty$-algebra objects in this comodule category.

**Variant 4.2.33.** Let $B$ be a dualizable cocommutative bialgebra object of $\mathcal{C}$. By Corollary 2.3.3, there is an induced commutative bialgebra structure on $B^v$ for which we have an equivalence of symmetric monoidal $\infty$-categories $\alpha : LMod_B(\mathcal{C}) \simeq LMod_{B^v}(\mathcal{C})$, where the symmetric monoidal structures are as defined in §2.2.

Suppose we are given a lift of the commutative bialgebra structure on $B^v$ to a derived commutative bialgebra structure. Then, using Construction 4.2.32 and the equivalence $\alpha$, we obtain a derived symmetric algebra monad $LSym$ on $LMod_B(\mathcal{C})$, compatible with that on $\mathcal{C}$. We set $DAlg(LMod_B(\mathcal{C})) := LMod_{LSym}(LMod_B(\mathcal{C}))$, and refer to objects of this $\infty$-category as derived commutative algebra objects of $LMod_B(\mathcal{C})$. Note that we still have an underlying $E_\infty$-algebra functor:

$$
DAlg(LMod_B(\mathcal{C})) \simeq cLMod_{B^v}(DAlg(\mathcal{C})) \xrightarrow{\Theta} cLMod_{B^v}(CAlg(\mathcal{C})) \simeq CAlg(LMod_B(\mathcal{C}))
$$

where the first equivalence is from Construction 4.2.32 and the last equivalence is from Proposition 2.2.10.

The above discussion reformulated the $\infty$-category $cLMod_A(DAlg(\mathcal{C}))$ as the $\infty$-category $DAlg(LMod_B(\mathcal{C}))$ (where $A$ is a dualizable derived commutative bialgebra and $B$ is its dual).
This rephrasing in terms of modules is not strictly necessary for us, but it is somewhat comforting psychologically, and makes the end of the following remark a bit clearer. (Conversely, we should emphasize that it is not clear how to define $\text{DAlg}(\mathcal{LMod}_{B}(\mathcal{C}))$ without passing to the dual comodule perspective.)

**Remark 4.2.34.** Let $F: \mathcal{C} \to \mathcal{D}$ be a morphism of derived algebraic contexts. Let $B$ be a dualizable cocommutative bialgebra object of $\mathcal{C}$ equipped with a lift of $B^\vee$ to $\text{bAlg}_{B}(\mathcal{C})$, as in Variant 4.2.33. Then $F(B)$ and $F(B)^\vee = F(B^\vee)$ inherit the same structure in $\mathcal{D}$, and we have an induced functor $F_B: \mathcal{LMod}_{B}(\mathcal{C}) \to \mathcal{LMod}_{F(B)}(\mathcal{D})$. By Remark 4.2.25, there is also an induced functor $\text{DAlg}(\mathcal{LMod}_{B}(\mathcal{C})) \to \text{DAlg}(\mathcal{LMod}_{F(B)}(\mathcal{D}))$, which here we denote by $F'_B$, making the diagrams

$$\begin{align*}
\text{DAlg}(\mathcal{LMod}_{B}(\mathcal{C})) & \xrightarrow{F'_B} \text{DAlg}(\mathcal{LMod}_{F(B)}(\mathcal{D})) \\
\mathcal{LMod}_{B}(\mathcal{C}) & \xrightarrow{F_B} \mathcal{LMod}_{F(B)}(\mathcal{D})
\end{align*}$$

and

$$\begin{align*}
\text{DAlg}(\mathcal{LMod}_{B}(\mathcal{C})) & \xrightarrow{F'_B} \text{DAlg}(\mathcal{LMod}_{F(B)}(\mathcal{D})) \\
\mathcal{LMod}_{B}(\mathcal{C}) & \xrightarrow{F_B} \mathcal{LMod}_{F(B)}(\mathcal{D})
\end{align*}$$

commute. Note that we may pass to right adjoints in the second diagram, and since the right adjoint of $F_B$ is induced by the right adjoint $G: \mathcal{D} \to \mathcal{C}$ of $F$, this shows that the right adjoint of $F'_B$ is also induced by $G$.

§4.3. Examples of contexts. In this subsection, we catalogue the derived algebraic contexts that will be relevant later in the paper, and establish some abbreviated notation in these specific contexts for the notions defined in §4.2.

**Example 4.3.1.** The fundamental example of a derived algebraic context is obtained by taking $\mathcal{C} = \text{Mod}_{Z}$, which we regard as equipped with the usual symmetric monoidal structure and t-structure, and taking $\mathcal{C}^0$ to be the full subcategory spanned by the finite free $Z$-modules. We refer to derived commutative algebra objects of $\text{Mod}_{Z}$ simply as derived commutative rings, let $\text{DAlg}_{Z}$ denote $\text{DAlg}(\text{Mod}_{Z})$, and abbreviate the derived symmetric algebra monad $\text{LSym}_{\text{Mod}_{Z}}$ to $\text{LSym}_{Z}$.

**Remark 4.3.2.** In addition to being the fundamental example, $\text{Mod}_{Z}$ is the initial example of a derived algebraic context $\text{Mod}_{Z}$: that is, for any derived algebraic context $\mathcal{C}$, there is a unique morphism of derived algebraic contexts $\text{Mod}_{Z} \to \mathcal{C}$.

**Remark 4.3.3.** It follows from Remark 4.2.24 that the $\infty$-category $\text{DAlg}_{Z}^\text{cn} := \text{DAlg}_{Z} \times_{\text{Mod}_{Z}} \text{Mod}_{Z}^\text{cn}$ of connective derived commutative rings is canonically equivalent to the $\infty$-category of simplicial commutative rings, as described for example in [L-SAG, §25].

We now explain how to obtain derived algebraic contexts in the graded and filtered settings.

**Construction 4.3.4.** Let $\mathcal{C}$ be a derived algebraic context. Then we regard the $\infty$-category $\text{Gr}(\mathcal{C})$ of graded objects of $\mathcal{C}$ as a derived algebraic context by equipping it with the Day convolution symmetric monoidal structure (Notation 3.1.1) and the neutral t-structure (Construction 3.3.2), and taking $\text{Gr}(\mathcal{C})^0 \subseteq \text{Gr}(\mathcal{C})$ to be full subcategory spanned by finite coproducts of the objects
X(n) = \text{ins}^n(X) for X \in \mathcal{C} and n \in \mathbb{Z}. We regard the \infty-category \text{Fil}(\mathcal{C}) of filtered objects of \mathcal{C} as a derived algebraic context in the analogous way, and similarly for the nonnegative variants \text{Gr}^{\geq 0}(\mathcal{C}) and \text{Fil}^{\geq 0}(\mathcal{C}) (Variant 3.1.2).

We will refer to derived commutative algebra objects of \text{Gr}(\mathcal{C}) as \textit{graded derived commutative algebra objects of} \mathcal{C}, and denote D\text{Alg}(\text{Gr}(\mathcal{C})) by GrD\text{Alg}(\mathcal{C}). Similarly, we refer to derived commutative algebra objects of \text{Fil}(\mathcal{C}) as \textit{filtered derived commutative algebra objects of} \mathcal{C}, and denote D\text{Alg(Fil}(\mathcal{C})) by FilD\text{Alg}(\mathcal{C}). We similarly have \textit{nongraded} graded and filtered derived commutative objects of \mathcal{C}, the \infty-categories of which we denote by Gr^{0}\text{DA}lg(\mathcal{C}) and Fil^{0}\text{DA}lg(\mathcal{C}) respectively. In all these cases, we adopt the same notation for the relative notions of Notation 4.2.28; moreover, as mentioned there, we will sometimes drop \mathcal{C} from the notation in the relative situation.

Remark 4.3.5. In the situation of Construction 4.3.4, note that, for X \in \mathcal{C} and m, n \in \mathbb{Z}, we have Sym^n_{\text{Gr}(\mathcal{C})}(X(n)) \simeq Sym^n_{\mathcal{C}}(X)(mn). It follows that, for all X \in \mathcal{C}, we have LSym^n_{\text{Gr}(\mathcal{C})}(X(n)) \simeq LSym^n_{\mathcal{C}}(X)(mn). The same holds with \text{Gr}(\mathcal{C}) replaced by \text{Fil}(\mathcal{C}).

Remark 4.3.6. The functor \text{ins}^0 : \mathcal{C} \to \text{Gr}(\mathcal{C}) is a morphism of derived algebraic contexts, with right adjoint ev^0 : \mathcal{C} \to \text{Gr}(\mathcal{C}). It follows from Remark 4.2.25 that there is an induced adjunction ins^0 : D\text{Alg}(\mathcal{C}) \rightleftarrows \text{GrDAlg}(\mathcal{C}) : ev^0. Informally speaking, this says that the zeroth graded piece of a graded derived commutative algebra object of \mathcal{C} is canonically a derived commutative algebra object of \mathcal{C}, and conversely any derived commutative algebra object of \mathcal{C} can be regarded as a graded derived commutative algebra in \mathcal{C} concentrated in grading-degree 0. We will often do the latter implicitly.

Similar statements hold with Gr(\mathcal{C}) replaced by Fil(\mathcal{C}) or the nonnegative variants of these, as well as for the adjunction ins^{0} : \text{Gr}^{0}(\mathcal{C}) \rightleftarrows \text{Gr}(\mathcal{C}) : ev^{0}, and so on. The same can also be said for the colimit functor colim : Fil(\mathcal{C}) \to \mathcal{C} and the associated graded functor \text{gr} : Fil(\mathcal{C}) \to \text{Gr}(\mathcal{C}), which both determine morphisms of derived algebraic contexts.

Before discussing the final construction of interest, we must spell out when a localization of a derived algebraic context inherits a derived algebraic context structure.

Construction 4.3.7. Let \mathcal{C} be a derived algebraic context and let \mathcal{D} be a full subcategory of \mathcal{C} satisfying the following conditions:

(a) \mathcal{D} is stable and presentable, and the inclusion \iota : \mathcal{D} \to \mathcal{C} admits a left adjoint \lambda : \mathcal{C} \to \mathcal{D}.

(b) For X \in \mathcal{C}, if X is contained in \mathcal{D}, then \tau_{\leq 0}(X), \tau_{\geq 0}(X) are also contained in \mathcal{D}, so that the t-structure on \mathcal{C} restricts to a t-structure on \mathcal{D}. Moreover, \lambda is right t-exact.

(c) The left adjoint \lambda is compatible with the symmetric monoidal structure on \mathcal{C} (in the sense of [L-HA, Definition 2.2.1.6]). It follows that \mathcal{D} inherits a presentable symmetric monoidal structure for which \lambda is canonically symmetric monoidal.

(d) Set \mathcal{D}^0 := \lambda(\mathcal{C}^0). Then the objects of \mathcal{D}^0 form a set of compact projective generators for D_{\geq 0}.

Then the above structure makes \mathcal{D} a derived algebraic context and \lambda a morphism of derived algebraic contexts, so that there is an induced localizing adjunction \lambda : \text{DA}lg(\mathcal{C}) \rightleftarrows \text{DA}lg(\mathcal{D}) : \iota.

Example 4.3.8. Let \mathcal{C} be a derived algebraic context, and regard \text{Gr}^{0}(\mathcal{C}) as a derived algebraic context as in Construction 4.3.4. Let Gr^{(0,1)}(\mathcal{C}) denote the \infty-category \mathcal{C} \times \mathcal{C} of pairs (X^0, X^1) of objects in \mathcal{C}. We have a fully faithful embedding ins^{(0,1)} : \text{Gr}^{(0,1)}(\mathcal{C}) \to \text{Gr}^{0}(\mathcal{C}) identifying Gr^{(0,1)}(\mathcal{C}) with the full subcategory of \text{Gr}^{0}(\mathcal{C}) spanned by those graded objects X^n for which X^n = 0 for all n \not\in \{0, 1\}. This embedding has a left adjoint given by the functor ev^{(0,1)} : \text{Gr}^{0}(\mathcal{C}) \to \text{Gr}^{(0,1)}(\mathcal{C}) that sends X^1 \to (X^0, X^1). This localization satisfies the conditions of Construction 4.3.7, hence equips Gr^{(0,1)}(\mathcal{C}) with the structure of a derived algebraic context. We note in particular that the tensor product on Gr^{(0,1)}(\mathcal{C}) is given by the construction

(((X^0, X^1), (Y^0, Y^1)) \to (X^0 \otimes Y^0, (X^0 \otimes Y^1) \oplus (X^1 \otimes Y^0))).
We let $\text{Gr}^{(0,1)}\text{DAlg}(\mathcal{C})$ denote the $\infty$-category $\text{DAlg}(\text{Gr}^{(0,1)}(\mathcal{C}))$, and similarly for the relative situation.

§4.4. The cotangent complex. In this subsection, we continue to work with a fixed derived algebraic context $\mathcal{C}$. Our goal is to give a definition of relative cotangent complexes for derived commutative algebra objects of $\mathcal{C}$, generalizing and reformulating the usual cotangent complex in the setting of simplicial commutative rings in a manner that will be convenient in §5.3. We follow the usual pattern: we first define trivial square-zero extensions, then define derivations as maps into these, and finally define the cotangent complex as a representing object for derivations.

Notation 4.4.1. Let $\text{CAlgMod}(\mathcal{C})$ denote the $\infty$-category of pairs $(A, M)$ where $A$ is an $E_{\infty}$-algebra in $\mathcal{C}$ and $M$ is an $A$-module in $\mathcal{C}$, and let $\text{DAlgMod}(\mathcal{C})$ denote the $\infty$-category of pairs $(A, M)$ where $A$ is a derived commutative ring in $\mathcal{C}$ and $M$ is an $A$-module in $\mathcal{C}$.

In the following two results, we regard $\text{Gr}^{(0,1)}(\mathcal{C}) = \mathcal{C} \times \mathcal{C}$ as a derived algebraic context as in Example 4.3.8; we let $\text{Gr}^{(0,1)}\text{CAlg}(\mathcal{C})$ denote $\text{CAlg}(\text{Gr}^{(0,1)}(\mathcal{C}))$.

Lemma 4.4.2. Then there is a canonical equivalence of $\infty$-categories

$$\alpha : \text{Gr}^{(0,1)}\text{CAlg}(\mathcal{C}) \xrightarrow{\sim} \text{CAlgMod}(\mathcal{C})$$

commuting with the forgetful functors to $\text{Gr}^{(0,1)}(\mathcal{C})$. 8

Proof. The embedding $\text{ins}^{(0,1)} : \text{Gr}^{(0,1)}(\mathcal{C}) \to \text{Gr}^{20}(\mathcal{C})$ induces an embedding $\iota : \text{Gr}^{(0,1)}\text{CAlg}(\mathcal{C}) \to \text{Gr}^{20}\text{CAlg}(\mathcal{C})$. We will define a functor $\beta : \text{Gr}^{20}\text{CAlg}(\mathcal{C}) \to \text{CAlgMod}(\mathcal{C})$ and then define the desired functor $\alpha$ to be the composition $\beta \circ \iota$.

To define $\beta$, recall that $\text{Gr}^{20}\text{CAlg}(\mathcal{C})$ can be identified with the $\infty$-category of lax symmetric monoidal functors $X : \mathbb{Z}_{20}^{\text{ds}} \to \mathcal{C}$, where $\mathbb{Z}_{20}^{\text{ds}}$ is regarded as a symmetric monoidal category via addition ([L-HA, Example 2.2.6.9]). Observe that $0 \in \mathbb{Z}_{20}^{\text{ds}}$ has a (unique) commutative algebra structure and $1 \in \mathbb{Z}_{20}^{\text{ds}}$ a (unique) module structure over $0$. It follows that a lax symmetric monoidal structure on a functor $X : \mathbb{Z}_{20}^{\text{ds}} \to \mathcal{C}$ determines a commutative algebra structure on $X^0$ and an $X^0$-module structure on $X^1$. This construction defines our functor $\beta : \text{Gr}^{20}\text{CAlg}(\mathcal{C}) \to \text{CAlgMod}(\mathcal{C})$.

Now consider the composition

$$\alpha := \beta \circ \iota : \text{Gr}^{(0,1)}\text{CAlg}(\mathcal{C}) \to \text{CAlgMod}(\mathcal{C}).$$

This evidently commutes with the forgetful functors to $\text{Gr}^{(0,1)}(\mathcal{C})$. We finish by showing that $\alpha$ is an equivalence of $\infty$-categories. Note that both forgetful functors admit left adjoints,

$$\mathcal{F} = \text{Sym}_{\text{Gr}^{(0,1)}(\mathcal{C})} : \text{Gr}^{(0,1)}(\mathcal{C}) \to \text{Gr}^{(0,1)}\text{CAlg}(\mathcal{C}), \quad \mathcal{F}' : \text{Gr}^{(0,1)}(\mathcal{C}) \to \text{CAlgMod}(\mathcal{C}),$$

where $\mathcal{F}'(X^0, X^1) = (\text{Sym}_\mathcal{C}(X^0), \text{Sym}_\mathcal{C}(X^0) \otimes_\mathcal{C} X^1)$; moreover, both adjunctions are monadic. Unravelling the definition of the symmetric powers $\text{Sym}^{n}_{\text{Gr}^{(0,1)}(\mathcal{C})}$, one can see that the canonical map $\alpha \circ \mathcal{F} \to \mathcal{F}'$ is an equivalence. It then follows from [L-HA, Corollary 4.7.3.16] that $\alpha$ is an equivalence.

Lemma 4.4.3. There is a canonical equivalence of $\infty$-categories

$$\alpha' : \text{Gr}^{(0,1)}\text{DAlg}(\mathcal{C}) \xrightarrow{\sim} \text{DAlgMod}(\mathcal{C})$$

commuting with the forgetful functors to $\text{Gr}^{(0,1)}(\mathcal{C})$.

Proof. By our definition of modules over derived commutative rings (Notation 4.2.28), we have an equivalence of $\infty$-categories $\text{DAlgMod}(\mathcal{C}) \simeq \text{DAlg}(\mathcal{C}) \times_{\text{CAlg}(\mathcal{C})} \text{CAlgMod}(\mathcal{C})$. In terms of this equivalence, we define the functor $\alpha'$ via the functor $\text{Gr}^{(0,1)}\text{DAlg}(\mathcal{C}) \to \text{DAlg}(\mathcal{C})$ induced by

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8This result goes through for any presentable symmetric monoidal $\infty$-category $\mathcal{C}$.
ev₀ : Gr^{(0,1)}(C) → C (see Remark 4.3.6, which applies also with Gr(C) replaced by Gr^{(0,1)}(C)) and the composition

Gr^{(0,1)}DAlg(C) → Gr^{(0,1)}CAlg(ε) → CAlgMod(C),

where α is the equivalence of Lemma 4.4.2. That α' is an equivalence follows from an argument similar to the one used in Lemma 4.4.2 to show that α is an equivalence.

**Notation 4.4.4.** Let D' ∈ GrCAlg(C^0) denote the ordinary trivial square-zero commutative algebra 1_ε ⊗ 1_{ε(-1)}, where 1_ε denotes the unit object of C. By Remark 4.2.24, we have a canonical embedding GrCAlg(C^0) → GrDAlg(ε), via which we regard D' as an object of GrDAlg(ε).

**Construction 4.4.5** (Trivial square-zero extensions). Let U : DAAlgMod(ε) → DAAlg(ε) and U' : DAAlgMod(ε) → Mod(ε) denote the functors given by U(A, M) := A and U'(A, M) := M. Define G : DAAlgMod(ε) → DAAlg(ε) to be the composite functor

DAAlgMod(ε) → Gr^{(0,1)}DAAlg(ε) → GrDAlg(ε) → GrDAlg(ε) → DAAlg(ε).

Then we have:
- natural transformations η : U → G and ε : G → U, induced by the unit and augmentation maps η_D' : 1_{Gr(C)} → D' and ε_D' : D' → 1_{Gr(C)} in CAlg(Gr(C)^0) ⊆ GrDAlg(ε);
- a homotopy ε ∘ η = id_{U}, induced by the equality ε_D' ∘ η_D' = id_{1_{Gr(C)}};
- a natural equivalence fib(ε) ∼ U, induced by the equivalence fib(ε_D') ∼ 1_{ε(-1)}.

These data determine a natural equivalence G(A, M) = A ⊕ M in C for (A, M) ∈ DAAlgMod(ε). We will thus denote G(A, M) ∈ DAAlg(ε) itself by A ⊕ M, and refer to it as the **trivial square-zero extension of** A **by** M.

**Notation 4.4.6.** For A a derived commutative algebra object of C, we let DAAlgMod_A denote the ∞-category DAAlgMod(ε)_{(A,0)}/ of pairs (B, M) where B is a derived commutative A-algebra and M is a B-module. Since the functors U and G in Construction 4.4.5 both send (A, 0) → A, they induce functors DAAlgMod_A → DAAlg_A, which we abusively also denote by U and G.

**Definition 4.4.7.** Let A be a derived commutative algebra object C. For B a derived commutative A-algebra and M a B-module, we define an **A-linear derivation of B into M** to be a map δ : B → B ⊕ M in (DAAlg_A)/B. We often abusively identify a derivation δ with the map of A-modules d : B → M obtained by composing with the projection B ⊕ M → M.

**Proposition 4.4.8.** Let A be a derived commutative algebra object of C. Then the trivial square-zero extension functor G : DAAlgMod_A → DAAlg_A preserves limits and sifted colimits, hence admits a left adjoint L.

**Proof.** It suffices to show that the composite of G with the forgetful functor DAAlg_A → C preserves limits and sifted colimits. But we know from Construction 4.4.5 that this composite is given by the construction (A, M) → A ⊕ M, so this is clear. It then follows from the adjoint functor theorem that G admits a left adjoint.

**Remark 4.4.9.** In the situation of Proposition 4.4.8, we have a natural equivalence U ∘ L = id_{DAAlg_A}, where U : DAAlgMod_A → DAAlg_A is as defined in Construction 4.4.5. To see this, note that U admits a right adjoint Z : DAAlg_A → DAAlgMod_A, given on objects by Z(B) = (B, 0). The natural transformation η of Construction 4.4.5 induces an natural equivalence id_{DAAlg_A} ∼ G ∘ Z, and passing to left adjoints gives the desired natural equivalence U ∘ L ∼ id_{DAAlg_A}.

**Construction 4.4.10** (Cotangent complex). Let A be a derived commutative algebra object of C and let B be a derived commutative A-algebra. It follows from Remark 4.4.9 that there is a unique

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9Our choice of notation here will make more sense in the context of §5.
B-module $L_{B/A} \in \text{Mod}_B$ equipped with an equivalence $(B, L_{B/A}) \simeq L(B)$ in $\text{DAlgMod}_A$, where $L$ is the left adjoint functor of Proposition 4.4.8. We refer to $L_{B/A}$ as the \textit{cotangent complex} of $B$ over $A$. By definition, $L_{B/A}$ comes equipped with an $A$-linear derivation $d : B \to L_{B/A}$; we refer to $d$ as the \textit{universal $A$-linear derivation} of $B$ (or just the \textit{universal derivation}, if context makes the abbreviation appropriate).

\textbf{Example 4.4.11.} Let $A$ be a derived commutative algebra object of $\mathcal{C}$, let $M \in \text{Mod}_A$, and let $B := \text{LSym}_A(M) \in \text{DAlg}_A$. Then there is a canonical equivalence of $B$-modules $L_{B/A} \simeq B \otimes_A M$.

\textbf{Remark 4.4.12.} Let $\text{DAlgMod}(\mathcal{C})^{cn}$ denote the full subcategory of $\text{DAlgMod}(\mathcal{C})$ spanned by those pairs $(A, M)$ where $A$ and $M$ are connective. This $\infty$-category is projectively generated by the full subcategory $\text{DAlgMod}(\mathcal{C})^0$ spanned by those pairs $(A, M)$ where $A \simeq \text{LSym}_c(X) \simeq \text{Sym}_{c^0}(X)$ for some $X \in \mathcal{C}^0$ and $M \simeq A^{\otimes n}$ for some $n \geq 0$. Since the trivial square-zero extension functor $G$ of Construction 4.4.5 preserves sifted colimits, the restriction $G|_{\text{DAlgMod}(\mathcal{C})^{cn}}$ is the left derived functor of the restriction $G|_{\text{DAlgMod}(\mathcal{C})^0}$. This latter restriction is valued in the full subcategory $\text{CAlg}(\mathcal{C}) = \text{DAlg}(\mathcal{C})^{cn} \subseteq \text{DAlg}(\mathcal{C})$, and it is easy to see that this canonically identifies with the ordinary trivial square-zero extension construction.

It follows that, in the case $\mathcal{C} = \text{Mod}_Z$, the notions of trivial square-zero extension, derivation, and cotangent complex agree with the usual notions for simplicial commutative rings (as described in [L-SAG, §25]), under the equivalence $\text{DAlg}_Z^{cn} \simeq \text{CAlg}_Z^\Delta$ of Remark 4.3.3.

\section*{§4.5. Connectivity in negative degrees.} In §4.2, we extended the derived symmetric algebra monad $\text{LSym}_Z$ from connective $Z$-modules to all $Z$-modules, and thereby defined the notion of possibly nonconnective derived commutative rings. In order to construct the nonconnective derived commutative rings of interest in §§5–6, we will need to know a little bit about how this extended monad behaves on nonconnective modules. The purpose of this subsection is to record the relevant facts regarding this behavior, the main results being Propositions 4.5.6 and 4.5.8.

\textbf{Lemma 4.5.1.} Let $\mathcal{C}$ be a stable $\infty$-category equipped with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Let $\chi : \mathbb{P}_n \to \mathcal{C}$ be a cartesian $n$-cube in $\mathcal{C}$ for some $n \geq 0$ (see Definition 4.2.5). Suppose that there is some integer $m$ such that $\chi(S)$ is $m$-connective for every nonempty $S \in \mathbb{P}_n$. Then $\chi(\varnothing)$ is $(m - n)$-connective.

\textbf{Proof.} We proceed by induction on $n$, the case $n = 0$ being trivial. For $n \geq 1$, we may identify $\mathbb{P}_n$ with $\mathbb{P}_{n-1} \times \Delta^1$ and thereby regard $\chi$ as a natural transformation $\alpha : \chi' \to \chi''$ of $(n - 1)$-cubes. By [L-HA, Lemma 1.2.4.15], $\chi$ being a cartesian $n$-cube implies that the cofib$(\alpha)$ is a cartesian $(n - 1)$-cube, and the connectivity assumption on $\chi$ implies that cofib$(\alpha)(T)$ is $m$-connective for all nonempty $T \in \mathbb{P}_{n-1}$. We deduce from the inductive hypothesis that cofib$(T)(\varnothing)$ is $(m - n + 1)$-connective. It follows that $\chi(\varnothing)$ is the fiber of a map from an $m$-connective object to an $(m - n + 1)$-connective object, implying that it must be $(m - n)$-connective, as desired.

\textbf{Lemma 4.5.2.} For $k \leq 0$ and $n \geq 0$, the canonical map $\theta^n : \text{Sym}_Z^n(Z[k]) \to \text{LSym}_Z^n(Z[k])$ is $(nk + 1)$-connective.

\textbf{Proof.} Fix $n \geq 0$ and let $F_k$ denote the fiber of $\theta^n : \text{Sym}_Z^n(Z[k]) \to \text{LSym}_Z^n(Z[k])$ for $k \leq 0$. We wish to show that $F_k$ is $(nk + 1)$-connective. The case $k = 0$ is clear. We may then deduce the claim for $k \leq -1$ by (descending) induction, using Lemma 4.5.1 and the fact that $\text{Sym}_Z^n$ and $\text{LSym}_Z^n$ are n-excisive.

\textbf{Lemma 4.5.3.} The graded derived symmetric algebra monad $\text{LSym}_Z : \text{Gr}(\text{Mod}_Z) \to \text{Gr}(\text{Mod}_Z)$ preserves the full subcategory $\text{Gr}^{[0]}(\text{Mod}_Z)^{\geq 0} \subseteq \text{Gr}(\text{Mod}_Z)$ consisting of those graded modules $X^n$ where $X^n \simeq 0$ for $n > 0$ and $X^n$ is $n$-connective for $n \leq 0$.

\textbf{Proof.} The subcategory $\text{Gr}^{[0]}(\text{Mod}_Z)^{\geq 0}$ is closed under colimits and tensor products and has compact projective generators given by finite coproducts of the objects $Z[k](k)$ for $k \leq 0$. Since
LSym\_Z preserves sifted colimits and sends coproducts to tensor products, it therefore suffices to check that LSym(Z[k](k)) lies in Gr^0(ModZ)\_{\geq 0} for k \leq 0.

By virtue of Remarks 4.2.21 and 4.3.5, this amounts to showing that LSym^n_\Z(\Z[k]) ∈ ModZ is nk-connective for each k ≤ 0 and n ≥ 0. This is true when LSym^n_\Z replaced by Sym^n_\Z, and it then follows from Lemma 4.5.2 that it is true for LSym^n_\Z as well.

**Notation 4.5.4.** Let

\[ \ast : \text{Gr}(\text{Mod}^n_\Z) \cong \text{Gr}(\text{ModZ})^n_\ast : [-\ast] \]

denote the equivalence discussed in Remark 3.3.3 (so the functor \( \ast \) sends \( \{X^n\}_{n \in \Z} \mapsto \{X^n[n]\}_{n \in \Z} \)). The fully faithful embedding \( \iota : \text{Gr}^0(\text{Mod}^n_\Z) \to \text{Gr}^0(\text{ModZ})\_{\geq 0} \), given by the composition

\[ \text{Gr}^0(\text{Mod}^n_\Z) \overset{[\ast]}{\longrightarrow} \text{Gr}^0(\text{ModZ})^n_\ast \overset{\tau}{\longrightarrow} \text{Gr}^0(\text{ModZ})\_{\geq 0} \]

admits a left adjoint \( \tau : \text{Gr}^0(\text{ModZ})_{\geq 0} \to \text{Gr}(\text{Mod}^n_\Z) \), given by the composition

\[ \text{Gr}^0(\text{ModZ})_{\geq 0} \overset{\tau}{\longrightarrow} \text{Gr}(\text{Mod}^n_\Z)^n_\ast \overset{[\ast]^{-1}}{\longrightarrow} \text{Gr}(\text{Mod}^n_\Z). \]

**Lemma 4.5.5.** The following diagram of ∞-categories canonically commutes:

\[ \begin{array}{ccc}
\text{Gr}^0(\text{ModZ})_{\geq 0} & \xrightarrow{\text{LSym}_\Z} & \text{Gr}^0(\text{ModZ})_{\geq 0} \\
\downarrow{\tau} & & \downarrow{\tau} \\
\text{Gr}^0(\text{Mod}^n_\Z) & \xrightarrow{\text{CSym}_\Z} & \text{Gr}^0(\text{Mod}^n_\Z);
\end{array} \]

here the vertical arrows are as defined in Notation 4.5.4, the functor LSym\_Z is the restriction of the graded derived symmetric algebra monad provided by Lemma 4.5.3, and CSym\_Z denotes the ordinary graded symmetric algebra functor (i.e. the symmetric algebra functor for the Koszul symmetric monoidal structure Gr(\text{Mod}^n_\Z)\_{\geq 0}).

**Proof.** We argue in a fashion similar to the one used to prove Lemma 4.5.3. For \( M ∈ \text{Gr}^0(\text{ModZ})_{\geq 0} \), the unit map \( M \to \text{LSym}_\Z(M) \) induces a natural map \( \tau(M) \to \tau(\text{LSym}_\Z(M)) \). Observing that \( \tau \) is symmetric monoidal (for the Koszul symmetric monoidal structure on the target), the natural E∞-algebra structure on LSym\_Z(M) induces a natural graded commutative algebra structure on \( \tau(\text{LSym}_\Z(M)) \), and hence the previous map extends to a natural map \( \text{CSym}_\Z(\tau(M)) \to \tau(\text{LSym}_\Z(M)) \). We wish to show that this is an equivalence. The source and target functors preserve sifted colimits and send direct sums to tensor products, reducing us to the case that \( M = \Z[k](k) \) for some \( k ≤ 0 \). Now, the statement holds with LSym\_Z replaced by Sym\_Z, and we conclude by applying Lemma 4.5.2.

**Proposition 4.5.6.** The functors \( \iota \) and \( \tau \) of Notation 4.5.4 induced an equivalence between the following two ∞-categories:

- the full subcategory of GrDAlg\_Z spanned by those graded derived commutative rings \( X^\ast \) such that \( X^n \cong 0 \) for \( n > 0 \) and \( X^n \) has homotopy concentrated in degree \( n \) for \( n ≤ 0 \);
- the full subcategory of CAlg(Gr(\text{Mod}^n_\Z)\_{\geq 0}) spanned by those ordinary graded commutative rings \( Y^\ast \) such that \( Y^n \cong 0 \) for \( n > 0 \).

**Proof.** By Lemma 4.5.5, we have a natural equivalence \( \tau \circ \text{LSym}_\Z \circ \iota \cong \text{CSym}_\Z \). It follows from Proposition 4.1.9 that the monad structure on LSym\_Z determines a monad structure on CSym\_Z, and it is easy to see that this agrees with the usual graded symmetric algebra monad. Thus, Proposition 4.1.9 also tells us that \( \tau \circ \iota \) induces a localizing adjunction on module categories over the monads LSym\_Z and CSym\_Z. Unravelling definitions, this gives the claim.

Lemma 4.5.3 has the following analogue in the filtered setting, which is proved in exactly the same way.
Lemma 4.5.7. The filtered derived symmetric algebra monad $\LSym_Z : \Fil(\Mod_Z) \to \Fil(\Mod_Z)$ preserves the full subcategory $\Fil^{\geq 0}(\Mod_Z)^{\leq 0} \subseteq \Fil(\Mod_Z)$ consisting of those filtered modules $X^*$ where $X^n \cong 0$ for $n > 0$ and $X^n$ is $n$-connective for $n \leq 0$.

And from this it is straightforward to deduce the following analogue of Proposition 4.5.6 in the filtered setting.

**Proposition 4.5.8.** Let $\Mod^{\geq n}_Z$ denote the full subcategory of $\Mod_Z$ spanned by the coconnective $\mathbb{Z}$-modules, and let $\DAlg^{ccn}_Z$ be defined similarly. Then the Postnikov filtration functor $\tau_* : \Mod_Z \to \Fil(\Mod_Z)$ (Construction 3.3.7) induces a fully faithful embedding $\tau_* : \DAlg^{ccn}_Z \to \Fil\DAlg_Z$.

## §5. Homotopy-coherent cochain complexes

As discussed in §1.1, one of the main goals of this paper is to formulate generalizations of the notions of cochain complex and (strictly) commutative differential graded algebra to the world of higher algebra, and to use these generalizations to characterize the derived de Rham complex by universal property. We will accomplish this goal in this section, proving Theorem 1.1.1. Before beginning, let us briefly motivate the path we will take.

The data comprising an ordinary cochain complex of abelian groups may be broken into two pieces:

(a) an underlying graded abelian group;

(b) a degree-one endomorphism of this graded abelian group that squares to zero.

It is straightforward to generalize the first piece to the higher setting: we replace the graded abelian group with, say, a graded spectrum. It is also straightforward to contemplate a degree-one endomorphism of a graded spectrum. However, in the higher setting, it no longer makes good sense to consider the property that such a map square to zero. Rather, we must consider the further data of a nullhomotopy of the squared map, followed by an infinite cascade of further coherence data.

To encode all this data systematically, we adopt an alternative perspective on (b): a differential on a graded abelian group $X^*$ is equivalent to a graded module structure over the graded ring $\mathbb{Z}[\epsilon]$ with $\epsilon$ in grading $1$ and $\epsilon^2 = 0$. As discussed in §2, from this perspective, the tensor product of cochain complexes arises from the canonical cocommutative bialgebra structure on $\mathbb{Z}[\epsilon]$. Our strategy will be to locate an appropriate version of this bialgebra in the homotopical setting, and then analogously define homotopy-coherent cochain complexes as modules over it.

In fact, we already located a version of this graded bialgebra in §3.2, namely the object $\mathbb{D}_*$ of Notation 3.2.11 (see Remark 3.2.15). This was defined in the context of any stable presentable symmetric monoidal $\infty$-category $\mathcal{C}$, in particular the universal example $\mathcal{C} = \Spt$, i.e. “over the sphere spectrum”. While this object is certainly of interest to us, it is not the end of the story. Recall that a $\mathbb{D}_*$-module structure on a graded object $X^*$ encodes differentials of the form $X^i[-1] \to X^{i+1}$, i.e. which decrease homotopy-degree by 1; we will refer to these as $h_*$-differentials. On the other hand, circle actions naturally give rise to (non-graded) differentials of the form $X[1] \to X$, i.e. that increase homotopy-degree by 1; we will refer to these as $h_*$-differentials. Thus, in order to make contact with circle actions, we would like a variant bialgebra $\mathbb{D}_+$ whose modules come equipped with $h_*$-differentials. It is actually not possible to construct $\mathbb{D}_+$ over $\mathbb{S}$ (as a cocommutative bialgebra), but it is fairly simple to do so over $\mathbb{Z}$, which suffices for the purposes of this paper. (There is also a simpler construction of $\mathbb{D}_+$ over $\mathbb{Z}$.)

Furthermore, it turns out that it is in the context of $h_*$-differentials that the universal property of the derived de Rham complex is naturally formulated. (This “coincidence” is in some sense the main subject of this paper.) This is related to the fact that $\mathbb{D}_*$ carries more structure than $\mathbb{D}_+$: we will see that the dual $\mathbb{D}_*^\vee$ is canonically a derived commutative bialgebra, allowing us to formulate the notion of an $h_*$-differential graded derived commutative algebra, our derived analogue of strictly...
commutative differential graded algebras.

This section is organized as follows. In §5.1, we will define the basic notions of $h_+$- and $h_-$-differential graded objects, and explain how to shift between the two worlds. In §5.2, we will define the “homotopy-coherent cohomology” of our homotopy-coherent cochain complexes, and use the Koszul dual perspective on $h_-$-differentials from §3.2 to understand this. Finally, in §5.3, we define and analyze the derived de Rham complex; this goes through in the generality of any derived algebraic context $\mathcal{C}$, and we explain how it recovers the existing notion of Hodge-completed derived de Rham cohomology in the case $\mathcal{C} = \text{Mod}_Z$.

§5.1. H-plus and h-minus differentials. We begin by defining the bialgebras over $\mathbb{Z}$ encoding the notions of differential graded objects that we will consider. For this, we recall from Remark 3.3.3 that we have identifications of $\text{Gr}(\text{Mod}_Z^\infty)$ with the hearts of the positive and negative $t$-structures on $\text{Gr}(\text{Mod}_Z)$ (Construction 3.3.2), determining embeddings $\iota_\pm : \text{Gr}(\text{Mod}_Z^\infty) \to \text{Gr}(\text{Mod}_Z)$, which are canonically lax symmetric monoidal with respect to the Koszul symmetric monoidal structure $\text{Gr}(\text{Mod}_Z^\infty)^{\otimes_k}$ on the source.

Construction 5.1.1. We construct a pair of bicocommutative bialgebra objects $\mathbb{D}_+$ and $\mathbb{D}_-$ in $\text{Gr}(\text{Mod}_Z)$, with underlying objects given by

$$\mathbb{D}_+ \simeq \mathbb{Z} \oplus \mathbb{Z}[\pm 1](1).$$

First consider the trivial square-zero algebra $\mathbb{Z}[\epsilon] = \mathbb{Z}[\{\epsilon\}/(\epsilon^2)]$, which we regard as a bicocommutative bialgebra in $\text{Gr}(\text{Mod}_Z^\infty)^{\otimes_k}$ with the element $\epsilon$ in grading-degree 1 and the comultiplication sending $\epsilon \mapsto \epsilon \otimes 1 - 1 \otimes \epsilon$. We then define

$$\mathbb{D}_+: = \iota_+(\mathbb{Z}[\epsilon]).$$

These have underlying graded objects as written above by definition of $\iota_+$, and inherit the bicocommutative bialgebra structure from $\mathbb{Z}[\epsilon]$ since the lax symmetric monoidal structures on the functors $\iota_+$ are strictly symmetric monoidal when restricted to the full subcategory of $\text{Gr}(\text{Mod}_Z^\infty)$ spanned by those graded abelian groups $X^*$ such that $X^i$ is free for all $i \in \mathbb{Z}$.

Remark 5.1.2. We have now constructed two cocommutative bialgebra objects in $\text{Gr}(\text{Mod}_Z)$ named $\mathbb{D}_-$: one just above in Construction 5.1.1, and one earlier in Notation 3.2.11. However, there is no ambiguity: there is a unique equivalence between the two. This follows from the fact that the earlier one also has underlying object $\mathbb{Z} \oplus \mathbb{Z}[-1](1)$, hence lies in the essential image of the embedding $\iota_-$, and can thereby be uniquely identified with $\mathbb{Z}[\epsilon]$ in $\text{Gr}(\text{Mod}_Z^\infty)$.

Notation 5.1.3. If $\mathcal{C}$ is any $\mathbb{Z}$-linear stable presentable symmetric monoidal $\infty$-category, the structure map $\text{Mod}_Z \to \mathcal{C}$ induces a symmetric monoidal functor $\text{Gr}(\text{Mod}_Z) \to \text{Gr}(\mathcal{C})$. We will abusively denote the image of $\mathbb{D}_+$ under this functor also by $\mathbb{D}_+$.

With the bialgebras $\mathbb{D}_+$ in hand, we may formulate the desired notions of homotopy-coherent cochain complex:

Definition 5.1.4. Let $\mathcal{C}$ be a $\mathbb{Z}$-linear stable presentable symmetric monoidal $\infty$-category. We define

$$\text{DG}_+(\mathcal{C}) := \text{Mod}_{\mathbb{D}_+}(\text{Gr}(\mathcal{C})), \quad \text{DG}_-(\mathcal{C}) := \text{Mod}_{\mathbb{D}_-}(\text{Gr}(\mathcal{C})).$$

which we regard as presentable symmetric monoidal $\infty$-categories using the cocommutative bialgebra structures on $\mathbb{D}_+$ and $\mathbb{D}_-$ (see Variant 2.2.3). We refer to objects of $\text{DG}_+(\mathcal{C})$ (resp. $\text{DG}_-(\mathcal{C})$) as $h_+$ (resp. $h_-$) cochain complexes in $\mathcal{C}$ or $h_+$-differential (resp. $h_-$-differential) graded objects of $\mathcal{C}$. We will sometimes use the notation $X^*$ to refer to an $h_+$ or $h_-$ cochain complex and then write $X^*$ for its underlying graded object.
Remark 5.1.5. The above definition of $h_-$ cochain complexes does not rely on the $\mathbb{Z}$-linearity assumption on $\mathcal{C}$. That is, the same definition can be made in general using the construction of $D_-$ from §3.2, although in this generality we should write $DG_- := L\text{Mod}^\omega_\bullet(\text{Gr}(\mathcal{C}))$, as $D_-$ does not have a commutative structure. With this notation, the Koszul duality result Theorem 3.2.14 says that we have $\text{Fil}^\vee(\mathcal{C}) \simeq DG_-(\mathcal{C})$.

The symmetric monoidal structures on $DG_+(\mathcal{C})$ allow one to consider homotopy-coherent analogues of commutative differential graded algebras, namely commutative algebra objects in these $\omega$-categories. We now explain how, in the $h_-$ setting, we may formulate a stronger notion, analogous to strictly commutative differential graded algebras, using the theory of derived commutative algebras from §4.

Notation 5.1.6. Observe that $D_+$ is dualizable as an object of $\text{Gr}(\text{Mod}_{\mathbb{Z}})$. We let $D_+^\vee \simeq \mathbb{Z} \oplus \mathbb{Z}[-1](-1)$ denote its dual, and regard this as a bicommutative bialgebra object in $\text{Gr}(\text{Mod}_{\mathbb{Z}})$ by Corollary 2.3.3.

Proposition 5.1.7. There is a unique derived bicommutative bialgebra structure (Definition 4.2.30) on $D_+^\vee$ in $\text{Gr}(\text{Mod}_{\mathbb{Z}})$ promoting its bicommutative bialgebra structure.

Proof. This is an immediate consequence of Proposition 4.5.6.

Remark 5.1.8. It follows from Proposition 5.1.7 that, for any derived algebraic context $\mathcal{C}$, we have a canonical derived bicommutative bialgebra structure on $D_+^\vee$ in $\text{Gr}(\mathcal{C})$. Moving forward, we will regard $D_+^\vee$ as equipped with this structure.

Remark 5.1.9. Let $D_+^\vee$ denote the unshifted trivial square-zero algebra $\mathbb{Z} \oplus \mathbb{Z}(-1)$, regarded as an object of $\text{GrDAlg}_{\mathbb{Z}}$ as in Notation 4.4.4. Ignoring the coalgebra structure on $D_+^\vee$, there is a canonical equivalence $D_+^\vee \simeq \mathbb{Z} \times_{D^\vee} \mathbb{Z}$ in $\text{GrDAlg}_{\mathbb{Z}}$: the two objects clearly have equivalent underlying graded $\mathbb{Z}$-modules, and it is easy to promote this to an equivalence of graded derived commutative rings because both objects lie in the essential image of $i_+$ and hence can be compared in the ordinary category of graded commutative rings.

The following is the key definition for formulating the universal property of the derived de Rham complex (see §5.3).

Definition 5.1.10. Let $\mathcal{C}$ be a derived algebraic context and let $A$ be a derived commutative algebra object of $\mathcal{C}$. Let $\text{GrDAlg}_A$ denote the $\omega$-category of graded derived commutative $A$-algebras (Construction 4.3.4). We let $DG_+, DAlg_A$ denote the $\omega$-category $\text{cMod}_{D^\vee}(\text{GrDAlg}_A)$, and refer to objects of this $\omega$-category as $h_+$-differential graded derived commutative $A$-algebras in $\mathcal{C}$.

Remark 5.1.11. In the context of Definition 5.1.10, the discussion of Variant 4.2.33 allows us to rewrite the $\omega$-category $DG_+, DAlg_A$ as $DAlg(DG_+(\text{Mod}_A))$: that is, we may think of $h_+$-differential graded derived commutative $A$-algebras as derived commutative algebra objects of the $\omega$-category of $h_+$ cochain complexes of $A$-modules.

Finally, let us comment on passing between the $h_+$ and $h_-$ settings.

Remark 5.1.12. Recall from Proposition 3.3.4 that we have symmetric monoidal equivalences $[\pm 2*]: \text{Gr}(\text{Mod}_{\mathbb{Z}}) \to \text{Gr}(\text{Mod}_{\mathbb{Z}})$. It follows from the statement in loc. cit. regarding the hearts that these equivalences send the bicommutative bialgebra $D_+$ to the bicommutative bialgebra $D_+$. This implies that, for $\mathcal{C}$ a $\mathbb{Z}$-linear presentable symmetric monoidal $\omega$-category, the symmetric monoidal equivalence $[\pm 2*]: \text{Gr}(\mathcal{C}) \to \text{Gr}(\mathcal{C})$ induces a symmetric monoidal equivalence $[\pm 2*]: DG_+(\mathcal{C}) \to DG_+(\mathcal{C})$.

Remark 5.1.13. Let $\mathcal{C}$ be a $\mathbb{Z}$-linear stable presentable symmetric monoidal $\omega$-category equipped with a compatible t-structure $(\mathcal{E}_{\geq 0}, \mathcal{E}_{< 0})$. Then the fact that $D_+$ lies in the heart of $\text{Gr}(\text{Mod}_{\mathbb{Z}})$,
corresponding to the ordinary graded bialgebra $\mathcal{D} = \mathbb{Z}[\epsilon]$, implies that the t-structures of $\Gr(\mathcal{C})_\pm$ induces t-structures $(\DG_+(\mathcal{C})_{2\mathbb{N}}, \DG_+(\mathcal{C})_{2\mathbb{N}})$ on $\DG_+(\mathcal{C})$ with the following properties:

(a) An $h_\pm$ cochain complex $X^* \in \DG_+(\mathcal{C})$ is connective if and only if its underlying graded object is connective in $\Gr(\mathcal{C})_\pm$, i.e. $X^n \in \mathcal{C}_{\pm n}$ for all $n \in \mathbb{Z}$.

(b) There is a canonical symmetric monoidal identification of $\DG_+(\mathcal{C})^\mathcal{C}$ with the category of
ordinary cochain complexes in $\mathcal{C}^\mathcal{C}$ (induced by the same functors as in the identification between $\Gr(\mathcal{C})^\mathcal{C}$ with $\Gr(\mathcal{C})$, described in Remark 3.3.3).

c) The equivalence $[\pm 2\ast] : \DG_+(\mathcal{C}) \to \DG_+(\mathcal{C})$ discussed in Remark 5.1.12 is t-exact.

In the $h_-$ setting, this t-structure was discussed earlier (Construction 3.3.10).

§5.2. Cohomology. Given an ordinary cochain complex of abelian groups $M$, one is often interested in its cohomology groups $H^i(M) = \ker(d^i : M^i \to M^{i+1})/\operatorname{im}(d^{i-1} : M^{i-1} \to M^i)$. Our goal in this subsection is to study the analogous construction for the homotopy-coherent cochain complexes we introduced in §5.1.

Let us first observe that, for $M$ an ordinary cochain complex as above, we can rewrite $H^i(M)$ as $\ker(\nu_M^i)$, where $\nu_M^i$ denotes the canonical map $\ker(d^{i+1}) \to \ker(d^i)$. Moreover, if we regard $M$ as a graded module over $\mathcal{D} := \mathbb{Z}[\epsilon]$, then we can reinterpret this as a graded map

$$\nu_M : (M \otimes_{\mathbb{Z}} \mathcal{D})(-1) \to \Hom_{\mathcal{D}}(\mathcal{D}, M).$$

One can check that this map agrees with the norm map $N_{M\mathbb{Z}}$ defined in Construction 2.4.4 (over the bialgebra $\mathcal{D}$), so that taking its cokernel can be regarded as an analogue in ordinary algebra of the Tate construction. It is thus natural to consider the Tate construction in the homotopy-coherent setting. This will be relevant later on, when we want to understand the associated graded of the Tate construction for filtered circle actions.

For the remainder of this section, we let $\mathcal{C}$ be a $\mathbb{Z}$-linear stable presentable symmetric monoidal $\infty$-category. Note however that it is only the statements about $h_\pm$ cochain complexes in this subsection that rely on the $\mathbb{Z}$-linearity assumption; all statements and proofs concerning only $h_-$ cochain complexes go through over $\mathbb{S}$.

Remark 5.2.1. The cocommutative bialgebras $\mathcal{D}_\pm$ in $\Gr(\mathcal{C})$ satisfy the assumptions of §2.4, so we have a norm map and Tate construction for objects of $\LMod_{\mathcal{D}_\pm}(\Gr(\mathcal{C})) \simeq \DG_+(\mathcal{C})$. Indeed, $\mathcal{D}_\pm$ is dualizable in $\Gr(\mathcal{C})$ and we have a canonical equivalence of $\mathcal{D}_\pm$-modules $\mathcal{D}_\pm^\ast \simeq \mathcal{D}_\pm([\ast])(-1)$, so that we can take $\omega_{\mathcal{D}_\pm} = [\pm 1](1)$ (in the notation of loc. cit.).

Definition 5.2.2. Let $X \in \DG_+(\mathcal{C})$. We let $H^\ast(X) \in \Gr(\mathcal{C})$ denote the Tate construction $X^{\otimes \mathcal{D}_\pm}$ of Construction 2.4.4, and refer to $H^\ast(X)$ as the cohomology of $X$.

Remark 5.2.3. Suppose that $\mathcal{C}$ is compactly generated (which implies that $\Gr(\mathcal{C})$ is compactly generated). Then, by Proposition 2.4.10, the functor $H^\ast(-) : \DG_+(\mathcal{C}) \to \Gr(\mathcal{C})$ is canonically lax symmetric monoidal. In particular, given a commutative algebra in $\DG_+(\mathcal{C})$, its cohomology $H^\ast(X)$ is canonically a commutative algebra in $\Gr(\mathcal{C})$.

We now seek to understand this notion of cohomology. Recall that the cohomology of an ordinary cochain complex of abelian groups $M$ depends only on the object it represents in the derived category, or equivalently the associated Eilenberg-MacLane object $[M] \in \Mod_{\mathcal{C}}$; namely, we have $H^i(M) \simeq \pi_{-i}([M])$. Analogous to the homotopy type of a topological space, let us refer to $[M]$ as the cohomology type of $M$. We will understand the cohomology of homotopy-coherent cochain complexes $X$ by observing an analogous phenomenon: that is, there is an underlying “cohomology type” $[X]$ that contains less information than $X$ itself but completely determines the cohomology $H^\ast(X)$.
**Definition 5.2.4.** Let $X \in \text{DG}^{-}(\mathcal{C})$. We let $|X|^{2*} \in \text{Fil}^{+}(\mathcal{C})$ denote the image of $X$ under the inverse to the equivalence $\overline{\delta} : \text{Fil}^{+}(\mathcal{C}) \to \text{DG}^{-}(\mathcal{C})$ of Theorem 3.2.14, and we define $|X| := \text{colim}(|X|^{2*}) \in \mathcal{C}$. We then translate these definitions to objects of $\text{DG}^{-}(\mathcal{C})$ using the equivalence $[-2*] : \text{DG}^{+}(\mathcal{C}) \simeq \text{DG}^{-}(\mathcal{C})$ (Remark 5.1.12): that is, for $X \in \text{DG}^{+}(\mathcal{C})$, we define $|X|^{2*} := |X[-2*]|^{2*} \in \text{Fil}^{-}(\mathcal{C})$ and $|X| := |X[-2*]| \in \mathcal{C}$. In both cases, we refer to $|X|$ as the cohomology type of $X$ and $|X|^{2*}$ as the brutal filtration on the cohomology type of $X$.

**Remark 5.2.5.** The constructions in Definition 5.2.4 are a generalization of extracting from an ordinary cochain complex the object it represents in the derived category and its brutal filtration (see Notation 3.3.12 and Example 3.3.14).

We can now state the main result of this subsection.

**Proposition 5.2.6.** Let $\delta_{gr} : \mathcal{C} \to \text{Gr}(\mathcal{C})$ denote the diagonal functor. Then:

(a) For $X \in \text{DG}^{-}(\mathcal{C})$, there is a natural equivalence $H^{*}(X) \simeq \delta_{gr}(|X|)$.

(b) For $X \in \text{DG}^{+}(\mathcal{C})$, there is a natural equivalence $H^{*}(X) \simeq \delta_{gr}(|X|)[2*]$, where $[2*] : \text{Gr}(\mathcal{C}) \to \text{Gr}(\mathcal{C})$ is as defined in Proposition 3.3.4.

It will be useful to set some notation and isolate some steps of the proof into a lemma, as we will need to refer to them again in §6.3.

**Notation 5.2.7.** Let $X \in \text{DG}_{\pm}(\mathcal{C})$. Let $\delta_{fil} : \mathcal{C} \to \text{Fil}(\mathcal{C})$ denote the diagonal functor. We have a canonical map $|X|^{2*}(-1) \to \delta_{fil}(|X|)$ in $\text{Fil}(\mathcal{C})$, and we let $|X|^{2*} \in \text{Fil}(\mathcal{C})$ denote the cofiber of this map, so that $|X|^{2*} \simeq \text{cofib}(|X|^{2*+1} \to |X|)$.

**Lemma 5.2.8.** For $X \in \text{DG}_{\pm}(\mathcal{C})$, there are natural equivalences

\[
X^{D^{-}} \simeq \text{und}(|X|^{2*}) \quad \text{and} \quad X_{D^{-}} \simeq \text{und}(|X|^{2*}),
\]

where $X_{D^{-}}, X^{D^{-}} \in \text{Gr}(\mathcal{C})$ are as defined in Notation 2.4.1 and $\text{und} : \text{Fil}(\mathcal{C}) \to \text{Gr}(\mathcal{C})$ as in Notation 3.1.1(c).

**Proof.** The functors $(-)^{D^{-}}$ and $(-)^{D^{-}}$ are defined as the left and right adjoints to the restriction functor $\rho : \text{Gr}(\mathcal{C}) \to \text{LMod}_{D^{-}}(\text{Gr}(\mathcal{C})) \simeq \text{DG}^{-}(\mathcal{C})$. It follows from Remark 3.2.6 that the diagram

\[
\begin{array}{ccc}
\text{Gr}(\mathcal{C}) & \xrightarrow{\rho} & \text{LMod}_{D^{-}}(\text{Gr}(\mathcal{C})) \\
\downarrow^{1^{\text{gr}}[t]} \quad & & \quad \downarrow \scriptstyle{F} \\
\text{Mod}_{1^{\text{gr}}[t]}(\text{Gr}(\mathcal{C})) & \xrightarrow{}\rightarrow & \text{Gr}(\mathcal{C})
\end{array}
\]

commutes, where $F$ is the symmetric monoidal functor of Proposition 3.2.5. Under the equivalence $\text{Fil}(\mathcal{C}) \simeq \text{Mod}_{1^{\text{gr}}[t]}(\text{Gr}(\mathcal{C}))$ of Proposition 3.2.9, the forgetful functor $\text{Mod}_{1^{\text{gr}}[t]}(\text{Gr}(\mathcal{C})) \to \text{Gr}(\mathcal{C})$ identifies with the functor $\text{und} : \text{Fil}(\mathcal{C}) \to \text{Gr}(\mathcal{C})$, so the base change functor $1^{\text{gr}}[t] \otimes - : \text{Gr}(\mathcal{C}) \to \text{Mod}_{1^{\text{gr}}[t]}(\text{Gr}(\mathcal{C}))$ identifies with the left adjoint $\text{spl} : \text{Gr}(\mathcal{C}) \to \text{Fil}(\mathcal{C})$. We thus obtain a commutative diagram

\[
\begin{array}{ccc}
\text{Gr}(\mathcal{C}) & \xrightarrow{\rho} & \text{LMod}_{D^{-}}(\text{Gr}(\mathcal{C})) \\
\downarrow^{\text{spl}} \quad & & \quad \downarrow \scriptstyle{\text{Fil}(\mathcal{C})} \\
\text{Fil}(\mathcal{C}) & \xrightarrow{(-)^{\text{gr}}} & \text{Fil}^{+}(\mathcal{C}) \\
\end{array}
\]

where the bottom row is as in Theorem 3.2.14. Passing to right adjoints, we deduce that $X^{D^{-}} \simeq \text{und}(|X|^{2*})$ for $X \in \text{DG}_{\pm}(\mathcal{C})$, as desired.
We now address $X_{\mathcal{D}_-}$. Fix any $i \in \mathbb{Z}$ and $Y \in \mathcal{C}$. We have

$$\text{Map}_c(X_{\mathcal{D}_-}^i, Y) \simeq \text{Map}_{Gr(\mathcal{C})}(X_{\mathcal{D}_-}, \text{ins}^i_{gr}(Y))$$

$$\simeq \text{Map}_{DG_-(\mathcal{C})}(X, \rho(\text{ins}^i_{gr}(Y)))$$

$$\simeq \text{Map}_{Fil(\mathcal{C})}([X]^\otimes_*, \text{sp}(\text{ins}^i_{gr}(Y))^\wedge)$$

$$\simeq \text{Map}_{Fil(\mathcal{C})}([X]^\otimes_*, \text{ins}^i_{fil}(Y)),$$

where the penultimate equivalence comes from the commutative diagram above, and, for the sake of clarity, we are using $\text{ins}^i_{gr}$ and $\text{ins}^i_{fil}$ rather than just $\text{ins}^i$ to denote the respective insertion functors $\mathcal{C} \to \text{Gr}(\mathcal{C})$ and $\mathcal{C} \to \text{Fil}(\mathcal{C})$, which we recall are given by left Kan extension along the inclusions $\{i\} \to \mathbb{Z}^{op}$ and $\{i\} \to \mathbb{Z}^{op}$. We now use the canonical cofiber sequence

$$\text{ins}^i_{fil}(Y) \to \delta_{fil}(Y) \to \text{coins}^{i+1}_{fil}(Y),$$

where $\text{coins}^{i+1}_{fil} : \mathcal{C} \to \text{Fil}(\mathcal{C})$ does not refer to currency but rather denotes right Kan extension along the inclusion $\{i+1\} \to \mathbb{Z}^{op}$, described concretely by the formula

$$\text{coins}^{i+1}_{fil}(Y) = \begin{cases} Y & j \geq i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Combining this cofiber sequence with the above equivalences, we find

$$\text{Map}_c(X_{\mathcal{D}_-}^i, Y) \simeq \text{fib}(\text{Map}_{Fil(\mathcal{C})}([X]^{\otimes_*}, \delta_{fil}(Y)) \to \text{Map}_{Fil(\mathcal{C})}([X]^{\otimes_*}, \text{coins}^{i+1}_{fil}(Y)))$$

$$\simeq \text{fib}(\text{Map}_c([X], Y) \to \text{Map}_c([X]^{\otimes_{i+1}}, Y)).$$

This gives us the desired equivalence $X_{\mathcal{D}_-}^i \simeq \text{fib}([X]^{\otimes_{i+1}} \to [X]) \simeq [X]^{\otimes i}$, finishing the proof. \qed

Proof of Proposition 5.2.6. First note that (b) follows from (a), since the Tate construction transports across the equivalence $[2\ast] : DG_-(-, \mathcal{C}) \to DG_+(\mathcal{C})$, by Remark 2.4.12. Now let us prove (a). Let $X \in DG_-(\mathcal{C})$. The Tate construction $H^i(X) = X^{D_+}$ is given by the cofiber of the norm map $N_{M_X} : \omega_{D_-} \otimes X_{D_-} \to X^{D_-}$. Recall from Remark 5.2.1 that $\omega_{D_-} \simeq [1](1)$. Unravelling definitions shows that, under the equivalences $X_{D_-} \simeq \text{und}([X]^{\otimes _{2i} \ast})$ and $X^{D_-} \simeq \text{und}([X]^{\otimes _{2i} \ast})$ of Lemma 5.2.8, $N_{M_X}$ is given by applying $\text{und} : \text{Fil}(\mathcal{C}) \to \text{Gr}(\mathcal{C})$ to the canonical map

$$[X]^{\otimes _{2i} \ast}[-1](1) \simeq \text{fib}([X]^{\otimes _{2i} \ast} \to \delta_{fil}([X])) \to [X]^{\otimes _{2i} \ast}.$$ 

The cofiber is thus canonically equivalent to $\text{und}(\delta_{fil}([X])) \simeq \delta_{gr}(X)$, as desired. \qed

Remark 5.2.9. The equivalence $(-)^{D_-} \simeq \text{und}([-]^{\otimes _{2i} \ast})$ constructed in the proof of Lemma 5.2.8 is evidently one of lax symmetric monoidal functors. Assuming $\mathcal{C}$ is compactly generated, it follows from the uniqueness statement for the lax symmetric monoidal structure on the Tate construction (Proposition 2.4.10) that the equivalences of Proposition 5.2.6 also respect lax symmetric monoidal structures.

§5.3. The derived de Rham complex. We now formulate the universal property of the derived de Rham complex. Throughout this subsection, we work over a fixed derived commutative algebra $A$ object of a fixed derived algebraic context $\mathcal{C}$. The reader should feel free to imagine $\mathcal{C} = \text{Mod}_{\mathbb{Z}}$ to fix ideas.

Notation 5.3.1. We let $DG^{gr}_{\mathbb{Z}} \mathcal{D}_{\mathbb{A}}$ denote the fiber product $DG_{\mathbb{Z}} \times_{Gr \mathcal{D}_{\mathbb{A}}} Gr^{gr} \mathcal{D}_{\mathbb{A}}$, i.e. the $\infty$-category of nonnegative $h_\ast$-differential graded derived commutative $A$-algebras (see Definition 5.1.10 and Construction 4.3.4 for the definitions of the constituent terms of the fiber product).

Proposition 5.3.2. The composite of the forgetful functor $U : DG^{gr}_{\mathbb{Z}} \mathcal{D}_{\mathbb{A}} \to Gr^{gr} \mathcal{D}_{\mathbb{A}}$ with the evaluation functor $ev^0 : Gr^{gr} \mathcal{D}_{\mathbb{A}} \to \mathcal{D}_{\mathbb{A}}$ admits a left adjoint.
Theorem 5.3.6. For a derived commutative $A$-algebra, there is a canonical equivalence of graded derived commutative $B$-algebras

$$LΩ^{*}_{B/A} \simeq LSym_B(L_{B/A}[1](1)),$$

where $L_{B/A}$ denotes the cotangent complex of $B$ over $A$ (Construction 4.4.10). In particular, there are canonical equivalences of $B$-modules $LΩ^{*}_{B/A} \simeq (Λ^i_B L_{B/A})[i]$ for $i \geq 0$. Furthermore, under these equivalences, the first differential

$$B \simeq LΩ^{0}_{B/A} \rightarrow LΩ^{1}_{B/A} [-1] \simeq L_{B/A}$$

of the $h_*$ cochain complex $LΩ^{*}_{B/A}$ is given by the universal $A$-linear derivation of $B$.

Remark 5.3.7. Theorem 5.3.6 in particular tells us that, for $B \in \text{DAlg}_A$, the unit map $B \rightarrow ev^0(LΩ^{*}_{B/A})$ is an equivalence of derived commutative $A$-algebras. This implies that the derived de Rham complex functor $LΩ^{*}_{/A} : \text{DAlg}_A \rightarrow DG^0_{\ast} \text{DAlg}_A$ is fully faithful.\(^{10}\)

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\(^{10}\)I thank Ben Antieau for suggesting this remark.
Before proving Theorem 5.3.6, let us see how we may apply it to recover Hodge-completed derived de Rham cohomology from our derived de Rham complex. We first recall the definition of the former (going back to work of Illusie [Ill72], and studied in particular by Bhatt [Bha12]).

**Recollection 5.3.8.** Suppose that $\mathcal{C} = \text{Mod}_\mathbb{Z}$ and that $A$ is an ordinary commutative ring. Let $\text{Fil}^\circ \text{CAlg}_A$ denote the full subcategory of $\text{FilCAlg}_A$ (the category of filtered $E_{\infty}$-$A$-algebras) spanned by the complete filtered objects. We recall the definition of the functor

$$dR^\wedge_{-/A} : \text{DAlg}^{cn}_A \to \text{Fil}^\circ \text{CAlg}_A$$

taking a connective derived commutative $A$-algebra (equivalently simplicial commutative $A$-algebra) to its Hodge-filtered Hodge-completed derived de Rham cohomology.

Recall that $\text{DAlg}^{cn}_A = \text{CAlg}_A^\Delta$ is projectively generated by the full subcategory $\text{Poly}_A$ spanned by the finitely generated polynomial $A$-algebras (Remark 4.2.24). We have a functor

$$\Omega^*_A : \text{Poly}_A \to \text{CAlg}(\text{DG}(\text{Mod}^\Delta_A))$$

assigning to a finitely generated polynomial $A$-algebra $B$ the algebraic de Rham complex $\Omega^*_B$, regarded as an ordinary commutative differential graded $A$-algebra. Composing with the functor $| - |^{2*} : \text{DG}(\text{Mod}^\Delta_A) \to \text{Fil}^\wedge(\text{Mod}_A)$ of Notation 3.3.12 (which is lax symmetric monoidal, hence induces a functor on commutative algebra objects), we obtain a functor $F = |\Omega^*_A|^{2*} : \text{Poly}_A \to \text{Fil}^\circ \text{CAlg}_A$.

The functor $dR^\wedge_{-/A} : \text{DAlg}^{cn}_A \to \text{Fil}^\circ \text{CAlg}_A$ is defined to be the left derived functor of $F$: that is, $dR^\wedge_{-/A}$ is the unique such functor preserving sifted colimits and restricting to $F$ on $\text{Poly}_A$.

**Hodge-completed derived de Rham cohomology** is then defined by taking colimits: for $B \in \text{DAlg}^{cn}_A$, we set $dR^\wedge_{B/A} := \text{colim}(dR^\wedge_{B/A}) \in \text{CAlg}_A$.

In the following statement, we apply the cohomology type $| - |$ and brutal filtration $| - |^{2*}$ constructions of Definition 5.2.4 to the derived de Rham complex.

**Corollary 5.3.9.** Suppose that $\mathcal{C} = \text{Mod}_\mathbb{Z}$ and that $A$ is an ordinary commutative ring. Then there is a canonical natural equivalence between the functors $|\Omega^*_A|^{2*}, dR^\wedge_{-/A} : \text{DAlg}^{cn}_A \to \text{Fil}^\circ \text{CAlg}_A$, and hence between their colimits $|\Omega^*_A|^{2*}, dR^\wedge_{-/A} : \text{Poly}_A \to \text{CAlg}(\text{DG}(\text{Mod}^\Delta_A))$.

**Proof.** Observe that $|\Omega^*_A|^{2*}$ preserves sifted colimits (in fact all colimits), so that it suffices to construct such an equivalence on the restriction of these functors to $\text{Poly}_A \subseteq \text{DAlg}^{cn}_A$. Passing through the (symmetric monoidal) equivalence $\text{Fil}^\circ(\text{Mod}_A) \cong \text{DG}_-(\text{Mod}_A)$ of Theorem 3.2.14, it suffices to produce a natural equivalence between the compositions

$$\text{Poly}_A \xrightarrow{\Omega^*_A} \text{CAlg}(\text{DG}_+(\text{Mod}_A)) \xrightarrow{|-2*|} \text{CAlg}(\text{DG}_-(\text{Mod}_A)),$$

$$\text{Poly}_A \xrightarrow{\Omega^*_A} \text{CAlg}(\text{DG}(\text{Mod}^\Delta_A)) \xrightarrow{|-*|} \text{CAlg}(\text{DG}_-(\text{Mod}_A)),$$

where the functor $|-*|$ is induced by the equivalence $|-*| : \text{DG}_+(\text{Mod}_A) \to \text{DG}_-(\text{Mod}_A)$ of Remark 5.1.12, and the functor $|2*|$ is induced by the fully faithful embedding $|2*| : \text{DG}(\text{Mod}^\Delta_A) \to \text{DG}_-(\text{Mod}_A)$ whose essential image is the heart of the $t$-structure on the target (Remark 5.1.13). Now, since $L_{B/A}$ is a free $A$-module for $B \in \text{Poly}_A$ (and $A$ is discrete), the equivalences $L\Omega^*_{B/A} = (A_{B/A}[i])$ of Theorem 5.3.6 imply that the first composition factors through the latter embedding via the composition

$$\text{Poly}_A \xrightarrow{\Omega^*_A} \text{CAlg}(\text{DG}_+(\text{Mod}_A)) \xrightarrow{|-*|} \text{CAlg}(\text{DG}(\text{Mod}^\Delta_A)),$$

It remains to produce a natural equivalence between this last composite and $\Omega^*_A$. But this is immediate from the identification of $L\Omega^*_{B/A}$ and the first differential of $L\Omega^*_{B/A}$ in Theorem 5.3.6.  

\begin{flushright} $\square$\end{flushright}
Lemma 5.3.11. The composite functor
\[ \text{Gr}^{20}\text{DAlg}_A \xrightarrow{\text{ev}^{(0,1)}} \text{Gr}^{(0,1)}\text{DAlg}_A \xrightarrow{\alpha'} \text{DAlgMod}_A \]

admits a left adjoint \( F : \text{DAlgMod}_A \to \text{Gr}^{20}\text{DAlg}_A \), which on objects sends \((B, M) \mapsto \text{LSym}_B(M(1))\); here \(\text{Gr}^{(0,1)}\text{DAlg}_A\) is as defined in Example 4.3.8 and \(\alpha'\) denotes the equivalence of Lemma 4.4.3.

Proof. For \((B, M) \in \text{DAlgMod}_A\), there is an equivalence
\[ (B, M) \xrightarrow{\sim} (\alpha' \circ \text{ev}^{(0,1)})(\text{LSym}_B(M(1))) \]
in \(\text{DAlgMod}_A\), inducing a commutative diagram of spaces
\[
\begin{array}{ccc}
\text{Map}_{\text{Gr}^{0}\text{DAlg}_A}(\text{LSym}_B(M(1)), C) & \xrightarrow{f} & \text{Map}_{\text{DAlgMod}_A}((B, M), (C^0, C^1)) \\
\, \downarrow{p} & \swarrow{q} & \\
\text{Map}_{\text{DAlg}_A}(B, C^0). & & \\
\end{array}
\]

for any \(C \in \text{Gr}^{0}\text{DAlg}_A\). We wish to show that \(f\) is an equivalence. This follows from the fact that \(f\) induces an equivalence on the fibers of \(p\) and \(q\) over any \(\phi \in \text{Map}_{\text{DAlg}_A}(B, C^0)\), both fibers being equivalent to the space \(\text{Map}_{\text{Mod}_A}(M, C^1)\) (where \(C^1\) is regarded as an \(B\)-module via restriction in \(\phi\)). \(\square\)

Lemma 5.3.12. (a) The forgetful functor \(U : \text{DG}_+ \text{DAlg}_A \to \text{GrDAlg}_A\) admits a right adjoint \(V : \text{GrDAlg}_A \to \text{DG}_+ \text{DAlg}_A\), and there is a canonical natural equivalence \(U(V(B)) \cong B \otimes D'_+\) for \(B \in \text{GrDAlg}_A\).

(b) The natural transformation between the composites
\[
\text{GrDAlg}_A \xrightarrow{\text{ev}^{(0,1)}} \text{Gr}^{(0,1)}\text{DAlg}_A \xrightarrow{\text{ins}^{(0,1)}} \text{GrDAlg}_A \xrightarrow{\text{ev}^0} \text{DAlg}_A,
\]

induced by the unit transformation \(\text{id} \to \text{ins}^{(0,1)} \circ \text{ev}^{(0,1)}\) is an equivalence.

(c) There is a canonical natural equivalence between the following two composite functors:
\[
\text{Gr}^{(0,1)}\text{DAlg}_A \xrightarrow{\text{ins}^{(0,1)}} \text{GrDAlg}_A \xrightarrow{\Omega} \text{DAlgMod}_A \xrightarrow{G} \text{DAlg}_A;
\]
\[
\text{Gr}^{(0,1)}\text{DAlg}_A \xrightarrow{\alpha'} \text{DAlgMod}_A \xrightarrow{\Omega} \text{DAlgMod}_A \xrightarrow{G} \text{DAlg}_A;
\]

here \(G\) denotes the trivial square-zero extension functor (Construction 4.4.5) and \(\Omega\) denotes the functor \((B, M) \mapsto (B, M[-1])\).
Proof. (a) The right adjoint $V$ forms the cofree comodule over $D_i^+$; that this adjoint exists and is
given by tensoring with $D_i^+$ follows from the formal duals of the statements about free modules in
[L-HA, §4.2.4].

(b) This follows from the natural equivalence $V(B)^0 \cong (B \oplus D_i^+)^0 \cong B^0 \oplus B^1[-1]$ in $\text{Mod}_A$.

(c) Recall from Remark 5.1.9 that we have an equivalence of graded derived commutative $A$-algebras
$D_i^+ \cong A \times_{D_i^+} A$, and that $G$ is defined as the composite

$$D\text{AlgMod}_A \xrightarrow{(\alpha')^{-1}} Gr^{(0,1)}D\text{Alg}_{\text{SA}} \xrightarrow{\text{ins}^{(0,1)}} GrD\text{Alg}_{\text{SA}} \xrightarrow{- \otimes D^\vee} GrD\text{Alg}_{\text{SA}} \xrightarrow{ev^0} D\text{Alg}_{\text{SA}}.$$ 

From this we obtain a natural equivalence between the first composite in the statement with
the functor $ev^0 \times_G ev^0$, which is easily seen to be equivalent to the second composite in the
statement.

Proof of Theorem 5.3.6. We will contemplate the following diagram of $\infty$-categories:

$$
\begin{tikzcd}
DG_{+} \text{DAlg}_{\text{SA}} \arrow{l}[swap]{\text{ins}^{(0,1)}} \arrow{r}{ev^0} \arrow{d}{L} & DG_{+} \text{DAlg}_{\text{SA}} \arrow{l}[swap]{U} \arrow{d}{V} \\
\text{DAAlg}_{\text{SA}} \arrow{r}{ev^0} & \text{GrDAlg}_{\text{SA}} \\
\text{L} \Omega^*_A \arrow{r}{\alpha'} \arrow{d}{L} & \text{DAAlgMod}_A \arrow{r}{\Sigma} \arrow{d}{\text{ins}^{(0,1)}} \arrow{u}{G} & \text{DAAlgMod}_A \arrow{r}{F} \arrow{u}{\text{F}} & \text{GrDAlg}_{\text{SA}} \arrow{r}{ev^0} \arrow{u}{V} & \text{Gr}^{(0,1)}\text{DAAlg}_{\text{SA}} \arrow{u}{ev^0}
\end{tikzcd}
$$

Let us explain what’s going on in the diagram:

- The adjacent pairs of arrows denote adjunctions, with the left-hand or upper arrow of the
pair denoting the left adjoint.
- The adjoint pair $L \dashv G$ is that of §4.4, i.e. $G$ is the trivial square-zero extension functor and
$L$ is the cotangent complex functor $B \mapsto (B, L_B/A)$.
- The functor $\alpha'$ is the equivalence of Lemma 4.4.3 (so its adjoint is the inverse equivalence),
and the functor $F$ is the left adjoint of Lemma 5.3.11.
- The adjoint pair $U \dashv V$ is that of Lemma 5.3.12(a).
- As in Lemma 5.3.12(c), the functor $\Omega$ is given by $(B, M) \mapsto (B, M[1])$, and so its left adjoint
$\Sigma$ is given by $(B, M) \mapsto (B, M[1])$.

Now, we find the functor $L \Omega^*_A : \text{DAAlg}_{\text{SA}} \to \text{GrDAlg}_{\text{SA}}$ by following the left adjoints in the diagram
from the bottom left to the top right. And, by the description of $F$ in Lemma 5.3.11, we
find the functor $L \text{Sym}_\omega(L_{/A}[1](1)) : D\text{Alg}_{\text{SA}} \to \text{GrDAlg}_{\text{SA}}$ by following the left adjoints from
the bottom left to the center of the diagram and then to the top right. We will produce a natural
equivalence between these two functors by producing one between the corresponding composites of
right adjoints. To do so, it suffices to see that the diagram commutes when we consider only the
right adjoints in the adjoint pairs: the outer rectangle commutes by Lemma 5.3.12(b); the inner
triangle commutes by definition of $F$; and the inner trapezoid commutes by Lemma 5.3.12(c).

In sum, we have produced an equivalence $L \Omega^*_A \cong L \text{Sym}_\omega(L_{/A}[1](1))$ in $\text{GrDAlg}_{\text{SA}}$. It is not
difficult to check that this canonically promotes to an equivalence in $\text{GrDAlg}_{\text{SA}}$. The equivalences
$L \Omega^*_A \cong (A_B L_{/A})[i]$ then follow from Remark 4.3.5 and the definition of the derived exterior
powers (Notation 5.3.5).

Finally, let us address the claim about the first differential of $L \Omega^*_A$. Unravelling definitions, it is
given by the unit map

$$B \to (ev^0 \circ V \circ U)(L \Omega^*_A)$$
of the adjunction obtained by going up around the left of the diagram. Since the diagram commutes, this is the same as the unit map
\[ B \to (G \circ \Omega \circ \alpha' \circ \ev(0,1) \circ \Sigma \circ L)(B), \]
of the adjunction obtained by going up through the middle of the diagram. The latter map is clearly the universal derivation.

\[ \square \]

§6. Filtered circle actions

Let \( X \) be a \( \mathbb{Z} \)-module with \( S^1 \)-action. Consider the Postnikov filtration \( \tau_{2*}(X) \),
\[ \cdots \to \tau_{24}(X) \to \tau_{23}(X) \to \tau_{20}(X) \to \tau_{2-1}(X) \to \tau_{2-2}(X) \to \cdots. \]

By functoriality of this construction, the filtered \( \mathbb{Z} \)-module \( \tau_{2*}(X) \) inherits an \( S^1 \)-action, or in other words, the filtration is \( S^1 \)-equivariant. In the case that \( X = \text{HH}(B/A) \) for \( A \) a commutative ring and \( B \) a smooth commutative \( A \)-algebra, the above filtration is the HKR filtration, and this \( S^1 \)-equivariant structure has been observed and used in previous work, e.g. in [Ant18].

In this section, we study a more refined \( S^1 \)-equivariant structure in the filtered setting, referred to here as a \textit{filtered} \( S^1 \)-action, which in particular exists on \( \tau_{2*}(X) \) in the situation above, and hence on the HKR filtration on Hochschild homology. The idea is that the circle action “increases the filtration degree”, analogous to the differential of a cochain complex increasing the grading degree. More precisely, we adopt the perspective that the \( S^1 \)-action on \( X \) is equivalent to a module structure on \( X \) over the group algebra \( \mathbb{Z}[S^1] \), and we observe that the Postnikov filtration construction carries this to a filtered module structure on \( \tau_{2*}(X) \) over the filtered algebra \( \tau_{2*}(\mathbb{Z}[S^1]) \). We refer to this last object as the \textit{(\( \mathbb{Z} \)-linear) filtered circle}, and the notion of filtered \( S^1 \)-action mentioned above is by definition taken to mean a filtered module structure over the filtered circle.

This section is organized as follows. In §6.1, we flesh out the construction of the filtered circle and definition of filtered circle actions sketched above. In §6.2, we use this definition to characterize HKR-filtered Hochschild homology by universal property, proving Theorem 1.2.3. In §6.3, we analyze the orbits, fixed points, and Tate constructions in the setting of filtered circle actions, and use these to construct filtrations on cyclic, negative cyclic, and periodic cyclic homology from HKR-filtered Hochschild homology. Finally, in §6.4, we explain how our construction of the filtrations on Hochschild, cyclic, negative cyclic, and periodic cyclic homology allows for a simple analysis of their interaction with the Adams operations that exist on these invariants.

§6.1. The \( \mathbb{Z} \)-linear filtered circle. Before filtering it, let us first discuss the \( \mathbb{Z} \)-linear circle itself.

\textbf{Notation 6.1.1.} To simplify notation, we will let \( T \) denote the group algebra \( \mathbb{Z}[S^1] \), i.e. the image of \( S^1 \) under the unique colimit-preserving symmetric monoidal functor \( \text{Spc} \to \text{Mod}_\mathbb{Z} \). As discussed in Example 2.2.9, the group structure on \( S^1 \) induces a bicommutative bialgebra structure on \( T \).

Note that \( T \) is dualizable: the dual \( T^\vee \) is given by the function spectrum \( \mathbb{Z}^{S^1} \), which can be described as the limit of the constant diagram \( S^1 \to \text{Mod}_\mathbb{Z} \) with value \( \mathbb{Z} \), or as the image of \( S^1 \) under the unique limit preserving functor \( \mathbb{Z}^{(-)}: \text{Spc}^{op} \to \text{Mod}_\mathbb{Z} \) sending \( \text{pt} \mapsto \mathbb{Z} \). By Corollary 2.3.3, the bicommutative bialgebra structure on \( T \) determines a bicommutative bialgebra structure on \( T^\vee \).

If \( \mathcal{C} \) is any \( \mathbb{Z} \)-linear stable presentable symmetric monoidal \( \infty \)-category, we will abusively denote the images of \( T \) and \( T^\vee \) under the structure map \( \text{Mod}_\mathbb{Z} \to \mathcal{C} \) also by \( T \) and \( T^\vee \). We regard these images also as bicommutative bialgebras; by Example 2.2.9, we have a canonical identification of symmetric monoidal \( \infty \)-categories \( \text{Mod}_T(\mathcal{C}) \simeq \text{Fun}(\text{BS}^1, \mathcal{C}) \).

\textbf{Construction 6.1.2.} We construct a derived bicommutative bialgebra structure (Definition 4.2.30) on \( T^\vee \) in \( \text{Mod}_\mathbb{Z} \) promoting its bicommutative bialgebra structure.
By definition of the bicommutative bialgebra structure on \( T \) and the fact that dualization determines an equivalence \( \mathbf{bAlg}^{c}_{\text{op}}((\text{Mod}_{\mathbb{Z}})_{\text{id}}) \cong \mathbf{bAlg}^{c}_{\text{op}}((\text{Mod}_{\mathbb{Z}})_{\text{id}}) \) (see Remark 2.3.4), \( T^{\vee} \) can be described, with its bicommutative bialgebra structure, as the image of the commutative monoid \( S^{1} \) under the finite limit–preserving symmetric monoidal functor \( Z^{-}(1) : \text{Spc}^{\text{op}} \rightarrow \mathbf{CAlg}_{\mathbb{Z}} \), where \( \text{Spc}^{\text{fin}} \subseteq \text{Spc} \) is the full subcategory spanned by the finite spaces. This description/construction lifts immediately along the limit-preserving symmetric monoidal forgetful functor \( \Theta : \mathbf{DAlg}_{\mathbb{Z}} \rightarrow \mathbf{CAlg}_{\mathbb{Z}} \), proving the claim.

**Remark 6.1.3.** It follows from Construction 6.1.2 that, for any derived algebraic context \( \mathcal{C} \), we have a canonical derived bicommutative bialgebra structure on \( T^{\vee} \) in \( \mathcal{C} \).

This allows us to realize the wish described at the start of §4. Namely, for any derived commutative algebra object \( A \) of \( \mathcal{C} \), we may construct an equivalence of \( \infty \)-categories \( \text{Fun}(\text{BS}^{1}, \mathbf{DAlg}_{\mathbb{A}}) \cong \mathbf{cMod}_{T}^{\vee}(\mathbf{DAlg}_{\mathbb{A}}) \), as follows.

We begin with the constant functor \( f : (\text{BS}^{1})^{\text{op}} \rightarrow \mathbf{cMod}(\mathbf{DAlg}_{\mathbb{A}}) \) with value \( A \). This extends uniquely to a colimit preserving functor \( f^{\vee} : \text{Fun}(\text{BS}^{1}, \text{Spc}) \rightarrow \mathbf{cMod}(\mathbf{DAlg}_{\mathbb{A}}) \). Next, observe that the unique group map \( \mathbb{Z}^{1} \rightarrow \mathbb{pt} \) induces a map of derived commutative bialgebras \( 1_{\mathbb{C}} \rightarrow T^{\vee} \), determining a colimit-preserving corestriction functor \( g : \mathbf{DAlg}_{\mathbb{A}} \rightarrow \mathbf{cMod}(\mathbf{DAlg}_{\mathbb{A}}) \). Together, \( f^{\vee} \) and \( g \) induce a colimit-preserving functor \( F : \text{Fun}(\text{BS}^{1}, \mathbf{DAlg}_{\mathbb{A}}) \cong \text{Fun}(\text{BS}^{1}, \text{Spc}) \otimes \mathbf{DAlg}_{\mathbb{A}} \rightarrow \mathbf{cMod}(\mathbf{DAlg}_{\mathbb{A}}) \). One can check that \( F \) commutes with the two forgetful functors from the source and target to \( \mathbf{DAlg}_{\mathbb{A}} \), and one can then conclude that \( F \) is an equivalence using (the categorical dual of) [L-HA, Corollary 4.7.3.16].

We now construct the filtered circle, and then use it to define filtered circle actions.

**Notation 6.1.4.** Let \( \tau_{Z} : \text{Mod}_{\mathbb{Z}} \rightarrow \text{Fil}(\text{Mod}_{\mathbb{Z}}) \) denote the Postnikov filtration functor (Construction 3.3.7). We set \( T_{\text{fil}} := \tau_{Z}(T) \in \text{Fil}(\text{Mod}_{\mathbb{Z}}) \) and \( T_{\text{fil}}^{\vee} := \tau_{Z}(T^{\vee}) \in \text{Fil}(\text{Mod}_{\mathbb{Z}}) \).

**Theorem 6.1.5.** (a) There exist unique bicommutative bialgebra structures on \( T_{\text{fil}} \) and \( T_{\text{fil}}^{\vee} \) promoting the bicommutative bialgebra structures on \( T \) and \( T^{\vee} \).

(b) There exists a unique derived bicommutative bialgebra structure on \( T_{\text{fil}}^{\vee} \) promoting the derived bicommutative bialgebra structure on \( T^{\vee} \).

To be clear, in the above statement, by “promoting”, we mean recovering the latter upon applying the functor colim : \( \text{Fil}(\text{Mod}_{\mathbb{Z}}) \rightarrow \text{Mod}_{\mathbb{Z}} \).

**Proof.** Recall from Construction 3.3.7 that the functor \( \tau_{Z} : \text{Mod}_{\mathbb{Z}} \rightarrow \text{Fil}(\text{Mod}_{\mathbb{Z}}) \) is canonically lax symmetric monoidal and from Remark 3.3.8 that \( \tau_{Z} \) is fully faithful. Let \( \text{Mod}^{\text{lf}}_{\mathbb{Z}} \) denote the full subcategory of \( \text{Mod}_{\mathbb{Z}} \) spanned by the quasi-free objects, i.e., those of the form \( \Theta_{n} \mathbb{Z}[n] \). Note that this is a symmetric monoidal subcategory, and observe that the restriction of \( \tau_{Z} \), to \( \text{Mod}^{\text{lf}}_{\mathbb{Z}} \) is in fact strictly symmetric monoidal. We deduce that \( \tau_{Z} \) embeds \( \text{Mod}^{\text{lf}}_{\mathbb{Z}} \) as a symmetric monoidal full subcategory of \( \text{Fil}(\text{Mod}_{\mathbb{Z}}) \). Statement (a) follows, since \( T \) and \( T^{\vee} \) are quasi-free.

Statement (b) follows from similar reasoning. Namely, if we let \( \text{DAAlg}^{\text{lf,ccn}}_{\mathbb{Z}} \) denote the full subcategory of \( \text{DAAlg}_{\mathbb{Z}} \) spanned by those derived commutative rings that are coconnective and whose underlying \( \mathbb{Z} \)-module are quasi-free, then it follows from Proposition 4.5.8 that \( \tau_{Z} \) induces a symmetric monoidal embedding \( \text{DAAlg}^{\text{lf,ccn}}_{\mathbb{Z}} \rightarrow \text{FilDAAlg}_{\mathbb{Z}} \). This implies the claim, since \( T^{\vee} \in \text{DAAlg}^{\text{lf,ccn}}_{\mathbb{Z}} \).

**Notation 6.1.6.** If \( \mathcal{C} \) is any \( \mathbb{Z} \)-linear stable presentable symmetric monoidal \( \infty \)-category, the structure map \( \text{Mod}_{\mathbb{Z}} \rightarrow \mathcal{C} \) induces a symmetric monoidal functor \( \text{Fil}(\text{Mod}_{\mathbb{Z}}) \rightarrow \text{Fil}(\mathcal{C}) \). We will denote the images of \( T_{\text{fil}} \) and \( T_{\text{fil}}^{\vee} \) under this functor also by \( T_{\text{fil}} \) and \( T_{\text{fil}}^{\vee} \), and regard these as bicommutative bialgebras by virtue of Theorem 6.1.5(a). The uniqueness statement in loc. cit. implies that the bicommutative bialgebra structures on \( T_{\text{fil}} \) and \( T_{\text{fil}}^{\vee} \) are dual in the sense of Corollary 2.3.3.
Definition 6.1.7. Let \( \mathcal{C} \) be a \( \mathbb{Z} \)-linear stable presentable symmetric monoidal \( \infty \)-category. We let \( \text{Fil}_{S^1}(\mathcal{C}) \) denote the \( \infty \)-category \( \text{Mod}_{\text{fil}}(\text{Fil}(\mathcal{C})) \), and refer to objects of this \( \infty \)-category as \textit{filtered objects of} \( \mathcal{C} \) \textit{with filtered} \( S^1 \)-\textit{action}. We regard \( \text{Fil}_{S^1}(\mathcal{C}) \) as a symmetric monoidal \( \infty \)-category by Proposition 2.2.1, using the cocommutative bialgebra structure on \( \text{Fil}_{\text{fil}} \). We refer to commutative algebra objects of \( \text{Fil}_{S^1}(\mathcal{C}) \) as \textit{filtered commutative algebras in} \( \mathcal{C} \) \textit{with filtered} \( S^1 \)-\textit{action}, and we denote the \( \infty \)-category \( \text{CAlg}(\text{Fil}_{S^1}(\mathcal{C})) \) by \( \text{Fil}_{S^1}\text{CAlg}(\mathcal{C}) \).

The following is the key definition for formulating the universal property of HKR-filtered Hochschild homology (see §6.2).

Definition 6.1.8. Let \( \mathcal{C} \) be a derived algebraic context. By Theorem 6.1.5(b), we may regard \( T_{\text{fil}}^\vee \) as a derived bicommutative bialgebra in \( \text{Fil}(\mathcal{C}) \). Let \( A \) be a derived commutative algebra object of \( \mathcal{C} \) and let \( \text{FilDAlg}_A \) denote the \( \infty \)-category of filtered derived commutative algebra objects of \( \mathcal{C} \) (Construction 4.3.4). Then we let \( \text{Fil}_{S^1}\text{DAlg}_A \) denote the \( \infty \)-category \( \text{cMod}_{T_{\text{fil}}^\vee}(\text{FilDAlg}_A) \), and we refer to objects of this \( \infty \)-category as \textit{filtered derived commutative} \( A \)-\textit{algebras with filtered} \( S^1 \)-\textit{action}.

Remark 6.1.9. In the context of Definition 6.1.8, the discussion of Variant 4.2.33 allows us to rewrite the \( \infty \)-category \( \text{Fil}_{S^1}\text{DAlg}_A \) as \( \text{DAlg}(\text{Fil}_{S^1}(\text{Mod}_A)) \); that is, we may think of filtered derived commutative \( A \)-algebras with filtered \( S^1 \)-action as derived commutative algebra objects of the \( \infty \)-category of filtered \( A \)-modules with filtered \( S^1 \)-action.

To understand the structure of a filtered \( S^1 \)-action on a filtered object \( X^* \), it is helpful to examine what structure is inherited by the underlying object \( \text{colim}(X^*) \) and the associated graded object \( \text{gr}(X^*) \); we close the subsection by discussing this.

Remark 6.1.10. Let \( \mathcal{C} \) be a stable presentable \( \mathbb{Z} \)-linear symmetric monoidal \( \infty \)-category. Consider the colimit functor \( \text{colim} : \text{Fil}(\mathcal{C}) \to \mathcal{C} \), with right adjoint given by the diagonal functor \( \delta : \mathcal{C} \to \text{Fil}(\mathcal{C}) \). By construction, we have an equivalence of bicommutative bialgebras \( \text{colim}(T_{\text{fil}}) \simeq T^\vee \) in \( \mathcal{C} \). It follows that there is an induced adjunction

\[
\text{Fil}_{S^1}(\mathcal{C}) \xrightarrow{\text{colim}} \text{Mod}_T(\mathcal{C}) \simeq \text{Fun}(\text{BS}^1, \mathcal{C})
\]

with the left adjoint symmetric monoidal so that this further determines an adjunction on commutative algebra objects

\[
\text{Fil}_{S^1}\text{CAlg}(\mathcal{C}) \xrightarrow{\text{colim}} \text{CAlg}(\text{Mod}_T(\mathcal{C})) \simeq \text{Fun}(\text{BS}^1, \text{CAlg}(\mathcal{C})).
\]

If \( \mathcal{C} \) is a derived algebraic context, then moreover we have an equivalence of derived bicommutative bialgebras \( \text{colim}(T_{\text{fil}}^\vee) \simeq T^\vee \) in \( \mathcal{C} \), and it follows from Remark 4.2.34 that, for any \( A \in \text{DAlg}(\mathcal{C}) \), we also obtain an induced adjunction

\[
\text{Fil}_{S^1}\text{DAlg}_A \xrightarrow{\text{colim}} \text{DAlg}(\text{Mod}_T(\text{Mod}_A)) \simeq \text{Fun}(\text{BS}^1, \text{DAlg}_A),
\]

where the last equivalence is as discussed in Remark 6.1.3.

We can summarize the above discussion informally as follows: a filtered object with filtered \( S^1 \)-action has an underlying object with \( S^1 \)-action and the constant filtration of an object with \( S^1 \)-action carries a filtered \( S^1 \)-action, and moreover both of these constructions are compatible with commutative or derived commutative algebra structures.

Remark 6.1.11. Slightly modifying the discussion in Remark 6.1.10 allows us to make precise the motivating example of filtered \( S^1 \)-actions sketched at the beginning of the section, as follows. Note that \( T_{\text{fil}} \) is connective for the Postnikov t-structure on \( \text{Fil}(\mathcal{C}) \) (Construction 3.3.6). The restricted colimit functor \( \text{colim} : \text{Fil}(\mathcal{C})_{\geq 0} \to \mathcal{C} \) has right adjoint given by the Postnikov filtration functor \( \tau_{2*} : \mathcal{C} \to \text{Fil}(\mathcal{C})_{\geq 0} \) (Construction 3.3.7). Setting \( \text{Fil}_{S^1}(\mathcal{C})_{\geq 0} : = \text{Fil}_{S^1}(\mathcal{C}) \times_{\text{Fil}(\mathcal{C})} \text{Fil}(\mathcal{C})_{\geq 0} \), we obtain an
adjunction

\[
\text{Fil}_{S^1}(\mathcal{C}) \Rightarrow \text{mod}_T \Rightarrow \text{Mod}_T(\mathcal{C}) \Rightarrow \text{Fun}(BS^1, \mathcal{C})
\]

with the left adjoint symmetric monoidal so that this further determines an adjunction on commutative algebra objects, as in Remark 6.1.10. This shows that, for \( X \in \text{Fun}(BS^1, \mathcal{C}) \), the Postnikov filtration \( \tau_{\ast}(X) \) carries a canonical equivalence of bi-commutative bialgebras \( \text{gr}(\tau_{\ast}(X)) \cong D_{+} \) in \( \text{Gr}(\mathcal{C}) \); this is immediate from the construction of \( D_{+} \) (Construction 5.1.1). It follows that there is an induced adjunction

\[
\text{Fil}_{S^1}(\mathcal{C}) \xrightarrow{\text{gr}} \text{DG}_{+}(\mathcal{C})
\]

with the left adjoint symmetric monoidal so that this further determines an adjunction on commutative algebra objects

\[
\text{Fil}_{S^1} \text{CAlg}(\mathcal{C}) \xrightarrow{\text{gr}} \text{DG}_{+} \text{CAlg}(\mathcal{C}).
\]

If \( \mathcal{C} \) is a derived algebraic context, then \( \text{gr} \) is a morphism of derived algebraic contexts and it follows from Proposition 5.1.7 that we moreover have an equivalence of derived bicommutative bialgebras \( \text{gr}(\tau_{\ast}(X)) \cong D_{+} \) in \( \text{Gr}(\mathcal{C}) \). It then follows from Remark 4.2.34 that, for any \( A \in \text{DAAlg}(\mathcal{C}) \), we also obtain an induced adjunction

\[
\text{Fil}_{S^1} \text{DAAlg}_{A} \xrightarrow{\text{gr}} \text{DG}_{+} \text{DAAlg}_{A}.
\]

We can summarize the above discussion informally as follows: the associated graded of a filtered object with filtered \( S^1 \)-action is canonically an \( h_{+} \) cochain complex, compatibly so with commutative or derived commutative algebra structures, and conversely for the right adjoint construction \( \zeta \).

### §6.2. HKR-filtered Hochschild homology

In this subsection, we explain how the constructions and definitions of §6.1 supply a direct construction, and universal characterization, of Hochschild homology together with its HKR filtration. Let us begin by recalling the definition of Hochschild homology; here stated in the setting of derived commutative \( A \)-algebras for a fixed derived commutative algebra \( A \) in a fixed derived algebraic context \( \mathcal{C} \).

**Definition 6.2.1.** Consider the \( \infty \)-category \( \text{Fun}(BS^1, \text{DAAlg}_{A}) \) of derived commutative \( A \)-algebras with \( S^1 \)-action. We have a functor \( U : \text{Fun}(BS^1, \text{DAAlg}_{A}) \to \text{DAAlg}_{A} \) forgetting the \( S^1 \)-action, given by restriction to the basepoint \( b : pt \to BS^1 \). This functor admits a left adjoint, given by left Kan extension along \( b \); we denote this left adjoint by \( \text{HH}(\cdot / A) : \text{DAAlg}_{A} \to \text{Fun}(BS^1, \text{DAAlg}_{A}) \). For \( B \in \text{DAAlg}_{A} \), we refer to \( \text{HH}(B/A) \) as the *Hochschild homology of \( B \) over \( A \).*

**Remark 6.2.2.** Let \( B \in \text{DAAlg}_{A} \). It follows from the pointwise formula for left Kan extension that, after forgetting the \( S^1 \)-action, the Hochschild homology \( \text{HH}(B/A) \) is given by the colimit of the
constant diagram $S^1 \to \text{DAlg}_A$ with value $B$. Equivalently, it is given by the image of $S^1$ under the unique colimit preserving functor $B^\otimes_{A^\vee} : \text{Spc} \to \text{DAlg}_A$ sending $pt \mapsto B$. From this perspective, the $S^1$-action on $\text{HH}(B/A) = B^\otimes_{A^\vee} S^1$ is induced by the $S^1$-action on $S^1$ itself. Note also that the equivalence $S^1 \simeq pt \cup_{S^0} pt$ in $\text{Spc}$ determines an equivalence $\text{HH}(B/A) \simeq B^\otimes_{B_{S^0}B} B$ in $\text{DAlg}_A$.

We now define a filtered enhancement of Hochschild homology by replacing $S^1$-actions with filtered $S^1$-actions.

**Notation 6.2.3.** We let $\text{Fil}_S^0 \text{DAlg}_A$ denote the fiber product $\text{Fil}_S^1 \text{DAlg}_A \times_{\text{FilDAlg}_A} \text{Fil}_S^0 \text{DAlg}_A$, i.e. the $\infty$-category of nonnegative filtered derived commutative $A$-algebras with filtered $S^1$-action (see Definition 6.1.8 and Construction 4.3.4 for the definitions of the constituent terms of the fiber product).

**Proposition 6.2.4.** The composition of the forgetful functor $U: \text{Fil}_S^0 \text{DAlg}_A \to \text{Fil}_S^0 \text{DAlg}_A$ with the evaluation functor $\text{ev}^0: \text{Fil}_S^0 \text{DAlg}_A \to \text{DAlg}_A$ admits a left adjoint.

**Proof.** Similar to the proof of Proposition 5.3.2. \hfill \square

**Definition 6.2.5.** We shall denote the left adjoint functor given by Proposition 6.2.4 by $\text{HH}_{fil}(-/A): \text{DAlg}_A \to \text{Fil}_S^0 \text{DAlg}_A$.

For $B \in \text{DAlg}_A$, we refer to $\text{HH}_{fil}(B/A)$ as the *HKR-filtered Hochschild homology of $B$ over $A$*.

This definition of $\text{HH}_{fil}(-/A)$ should be viewed as an interpolation between the definition of Hochschild homology (Definition 6.2.1) and the definition of the derived de Rham complex (Definition 5.3.3). The following (completely formal) result says that it is an interpolation in a precise mathematical sense too.

**Theorem 6.2.6.** For $B \in \text{DAlg}_A$, there are canonical natural equivalences

$$
\text{ev}^0(\text{HH}_{fil}(B/A)) \simeq \text{HH}(B/A) \quad \text{in Fun}(BS^1, \text{DAlg}_A),
$$

$$
\text{gr}(\text{HH}_{fil}(B/A)) \simeq L\Omega^*_B/A \quad \text{in DG, } \text{DAlg}_A.
$$

**Proof.** We contemplate the following diagram of $\infty$-categories:

\begin{equation*}
\begin{array}{ccc}
\text{Fun}(BS^1, \text{DAlg}_A) & \xrightarrow{\text{ev}^0} & \text{Fil}_S^0 \text{DAlg}_A & \xleftarrow{\text{gr}} & \text{DG}^2_\text{DAlg}_A \\
\text{HH}(-/A) & \xleftarrow{U} & \text{HH}_{fil}(-/A) & \xrightarrow{\text{ev}^0} & \text{DAlg}_A \\
& & \text{HH}(-/A) & \xrightarrow{\text{ev}^0} & \text{DAlg}_A \\
\end{array}
\end{equation*}

Here, the adjoint pair $\text{HH}(-/A) \dashv U$ is as defined in Definition 6.2.1, the adjoint pair $\text{HH}_{fil}(-/A) \dashv \text{ev}^0$ is as defined in Definition 6.2.5, the adjoint pair $L\Omega^*_B/A \dashv \text{ev}^0$ is as defined in Definition 5.3.3, and the two horizontal adjoint pairs are as described in Remarks 6.1.10 and 6.1.12 (note that $\text{ev}^0 \simeq \text{colim}$ on the nonnegative filtered category $\text{Fil}_S^0 \text{DAlg}_A$). The claim amounts to showing that the diagram of left adjoints commutes, but it is easy to see that the diagram of right adjoints commutes. \hfill \square

We now address completeness of HKR-filtered Hochschild homology (as defined above).

**Lemma 6.2.7.** Let $M \in \text{Mod}_A$ and let $B := \text{LSym}_A(M) \in \text{DAlg}_A$. Then the filtered object $\text{HH}_{fil}(B/A)$ is split (Notation 3.1.1(e)).
\textbf{Proof.} Set }$$\text{Fil}^{\geq 0}_{S}(\text{Mod}_{A}) = \text{Fil}_{S}(\text{Mod}_{A}) \times_{\text{Fil}(\text{Mod}_{A})} \text{Fil}^{\geq 0}(\text{Mod}_{A}).$$ \text{ Consider the commutative diagram of } \infty\text{-categories}

\[
\begin{array}{ccc}
\text{Fil}^{\geq 0}_{S}(\text{Mod}_{A}) & \xleftarrow{\text{ev}^{0}} & \text{Fil}^{\geq 0}_{S}(\text{DAlg}_{A}) \\
\text{ev}^{0} \downarrow & & \downarrow \text{ev}^{0} \\
\text{Mod}_{A} & \xleftarrow{=} & \text{DAlg}_{A},
\end{array}
\]

where the leftward arrows are the forgetful functors. Passing to left adjoints, we obtain the commutative diagram

\[
\begin{array}{ccc}
\text{Fil}^{\geq 0}_{S}(\text{Mod}_{A}) & \xrightarrow{\text{LSym}_{A}} & \text{Fil}^{\geq 0}_{S}(\text{DAlg}_{A}) \\
\tau_{\text{fil}}@\text{ins}^{0}(-) \uparrow & & \uparrow \text{HH}_{\text{fil}}(-/A) \\
\text{Mod}_{A} & \xrightarrow{\text{LSym}_{A}} & \text{DAlg}_{A}.
\end{array}
\]

As $$\tau_{\text{fil}} \simeq 1_{c} \oplus \mathbb{F}_{c}[1](1),$$ this gives us an equivalence

\[
\text{HH}_{\text{fil}}(B/A) \simeq \text{LSym}_{A}(M \oplus M[1](1))
\]

\[
\simeq \text{LSym}_{A}(M) \otimes_{A} \text{LSym}_{A}(M[1](1))
\]

\[
= B \otimes_{A} \text{LSym}_{A}(M[1](1))
\]

\[
\simeq \text{LSym}_{B}((B \otimes_{A} M)[1](1))
\]

in $$\text{Fil}^{\geq 0}_{S}(\text{DAlg}_{A}).$$ Combining this with the equivalences

\[
\text{gr}(\text{HH}_{\text{fil}}(B/A)) \simeq \Omega_{B/A}^{*} \simeq \text{LSym}_{B}((B_{A}[1](1))
\]

in $$\text{Gr}^{\geq 0}_{S}(\text{DAlg}_{A})$$ from Theorems 6.2.6 and 5.3.6 and the equivalence $$L_{B/A} \simeq B \otimes_{A} M$$ (Example 4.4.11), we deduce the claim. \hfill \Box

\textbf{Warning 6.2.8.} It does not follow from Lemma 6.2.7 that $$\text{HH}_{\text{fil}}(B/A)$$ is split for all $$B \in \text{DAlg}_{A}.$$ The splitting produced in the proof of Lemma 6.2.7 depends on the identification $$B \simeq \text{LSym}_{A}(M),$$ hence is functorial in $$M \in \text{Mod}_{A}$$ but not in $$B \in \text{DAlg}_{A}.$$

\textbf{Proposition 6.2.9.} Suppose that $$A$$ is connective and that the $$t$$-structure on $$\mathcal{C}$$ is left separated. Then, for every connective derived commutative $$A$$-algebra $$B,$$ the filtered object $$\text{HH}_{\text{fil}}(B/A)$$ is complete.

\textbf{Proof.} We will show, for such $$A$$ and $$B,$$ that $$\text{HH}_{\text{fil}}(B/A)^{i}$$ is $$i$$-connective for all $$i \geq 0,$$ i.e. $$\text{HH}_{\text{fil}}(B/A)$$ is connective in the Postnikov $$t$$-structure on $$\text{Fil}(\text{Mod}_{A});$$ this implies the claim by the left separatedness hypothesis. Let $$X \subseteq \text{DAlg}_{A}^{cn}$$ denote the full subcategory spanned by those $$B$$ for which $$\text{HH}_{\text{fil}}(B/A)$$ satisfies this connectivity condition. By Theorems 6.2.6 and 5.3.6, we know that $$\text{gr}^{i}\text{HH}_{\text{fil}}(B/A)$$ is $$i$$-connective for all $$B \in \text{DAlg}_{A}^{cn}.$$ Combining this with Lemma 6.2.7, we see that $$X$$ contains $$\text{LSym}_{A}(M)$$ for all $$M \in \text{Mod}_{A}^{cn}.$$ Since $$X$$ is closed under colimits, this implies that $$X = \text{DAlg}_{A}^{cn},$$ as any object of $$\text{DAlg}_{A}^{cn}$$ can be written as a geometric realization of such free objects (e.g. by the bar resolution). \hfill \Box

Combining Proposition 6.2.9 with Theorem 6.2.6 (and specializing to the context $$\mathcal{C} = \text{Mod}_{Z}),$$ we have now fully proved Theorem 1.2.3. We may also verify that our definition of the HKR filtration agrees with the usual one:

\textbf{Remark 6.2.10.} Suppose that $$\mathcal{C} = \text{Mod}_{Z}$$ and that $$A$$ is an ordinary commutative ring. Consider the composition of functors

\[
(\ast) \quad \text{DAlg}_{A}^{cn} \xrightarrow{\text{DAlg}_{A}} \text{DAlg}_{A} \xrightarrow{\text{HH}_{\text{fil}}(-/A)} \text{Fil}^{\geq 0}_{S}(\text{DAlg}_{A}) \xrightarrow{U} \text{FilCAlg}_{A},
\]

where $$U$$ denotes the forgetful functor. Each of these functors preserves colimits, so the composite is the left derived functor of its restriction to the full subcategory $$\text{Poly}_{A} \subseteq \text{DAlg}_{A}^{cn}$$ spaned by the
finitely generated polynomial $A$-algebras. By virtue of Theorem 6.2.6 and Proposition 6.2.9, the restriction to $\text{Poly}_A$ factors through the full subcategory $(\text{FilCAlg}_A)^{P}_{\leq 0} \subseteq \text{FilCAlg}_A$ spanned by those objects that are connective for the Postnikov $t$-structure. Using the adjunction
\[
\colim : (\text{FilCAlg}_A)^{P}_{\leq 0} \rightleftarrows \text{CAlg}_A : \tau_{2*},
\]
we obtain a natural comparison map $\alpha : \text{HH}_{\text{fil}}(B/A) \rightarrow \tau_{2*}(\colim(\text{HH}_{\text{fil}}(B/A)))$ for $B \in \text{Poly}_A$. It follows from Theorem 6.2.6 that the target can be identified with $\tau_{2*}(\text{HH}(B/A))$ and that $\alpha$ is an equivalence (using that $L_{B/A}$ is a free $B$-module for $B \in \text{Poly}_A$). We conclude that the functor $(\ast)$ agrees with the usual definition of the HKR filtration on Hochschild homology for simplicial commutative $A$-algebras.

§6.3. Filtered orbits, fixed points, and Tate construction. In this subsection, we specialize the constructions of §2.4 to obtain notions of orbits, fixed points, and the Tate construction in the setting of filtered $S^1$-actions, and discuss their behavior upon taking colimits and associated graded objects. Applying these constructions to HKR-filtered Hochschild homology, we will obtain filtrations on cyclic, negative cyclic, and periodic cyclic homology.

Remark 6.3.1. Recall that there is a canonical equivalence of $T$-modules $T^y \cong T[-1]$. The (lax symmetric monoidal) Postnikov filtration functor $\tau_{2*}$ carries this to an equivalence of $T_{\text{fil}}$-modules $T_{\text{fil}}^y \cong T_{\text{fil}}[-1][-1]$. It follows that, for any stable presentable $\mathbb{Z}$-linear symmetric monoidal $\infty$-category $\mathcal{C}$, the cocommutative bialgebra $T_{\text{fil}}$ in $\text{Fil}(\mathcal{C})$ satisfies the assumptions of §2.4, where we take $\omega_{\text{fil}} = 1_{\mathcal{C}}[1](1)$ for $1_{\mathcal{C}}$ the unit object of $\mathcal{C}$. Thus, we have a norm map and Tate construction for filtered objects of $\mathcal{C}$ with filtered $S^1$-action.

Notation 6.3.2. Let $\mathcal{C}$ be a stable presentable $\mathbb{Z}$-linear symmetric monoidal $\infty$-category. For $X \in \text{Fun}(BS^1, \mathcal{C}) \simeq \text{Mod}_T(\mathcal{C})$, we let
\[
X_T, \quad X^T, \quad X^{tT} \in \mathcal{C}
\]
denote the orbits, fixed points, and Tate construction, using the definitions of §2.4 for the cocommutative bialgebra $T$ in $\mathcal{C}$; these agree with the usual homotopy $S^1$-orbits, homotopy $S^1$-fixed points, and $S^1$-Tate construction. For $X \in \text{FilS}_1(\mathcal{C})$, we let
\[
X_{T_{\text{fil}}}, \quad X^{T_{\text{fil}}}, \quad X^{tT_{\text{fil}}} \in \text{Fil}(\mathcal{C})
\]
denote the orbits, fixed points, and Tate construction, using the definitions of §2.4 for the cocommutative bialgebra $T_{\text{fil}}$ in $\text{Fil}(\mathcal{C})$; we refer to these as the filtered $S^1$-orbits, filtered $S^1$-fixed points, and filtered $S^1$-Tate construction.

Proposition 6.3.3. Let $\mathcal{C}$ be a stable presentable $\mathbb{Z}$-linear symmetric monoidal $\infty$-category. Then, for $X \in \text{FilS}_1(\mathcal{C})$, there are canonical natural equivalences
\[
\text{gr}(X_{T_{\text{fil}}}) \cong \text{und}(\text{gr}(X)[z^*])[2*],
\]
\[
\text{gr}(X^{T_{\text{fil}}}) \cong \text{und}(\text{gr}(X)[z^*])[2*],
\]
\[
\text{gr}(X^{tT_{\text{fil}}}) \cong \delta(\text{gr}(X))[2*]
\]
in $\text{Gr}(\mathcal{C})$, where:

- we regard $\text{gr}(X)$ as an object of $\text{DG}_+(\mathcal{C})$ by Remark 6.1.12;
- the functors $| - |^*: - \rightarrow | - |^*: \text{DG}_+(\mathcal{C}) \rightarrow \text{Fil}(\mathcal{C})$ and $- | - : \text{DG}_+(\mathcal{C}) \rightarrow \mathcal{C}$ are as defined in Definition 5.2.4;
- the functor $\text{und}: \text{Fil}(\mathcal{C}) \rightarrow \text{Gr}(\mathcal{C})$ is as defined in Notation 3.1.1(c);
- $\delta$ denotes the diagonal functor $\mathcal{C} \rightarrow \text{Gr}(\mathcal{C})$;
- the functor $[2*]: \text{Gr}(\mathcal{C}) \rightarrow \text{Gr}(\mathcal{C})$ is as defined in Proposition 3.3.4.
Proof. The symmetric monoidal functor \(gr: \text{Fil}(\mathcal{C}) \to \text{Gr}(\mathcal{C})\) preserves all small limits and colimits, so, by Remark 2.4.12, we have canonical natural equivalences

\[
gr(X^\text{Fil}) \simeq gr(X)_\text{D}, \quad gr(X^{t\text{Fil}}) \simeq gr(X)_\text{D^r}, \quad gr(X^{t\text{Fil}}) \simeq gr(X)^\text{D^r}.
\]

The claim now follows from Proposition 5.2.6 and Lemma 5.2.8.

Remark 6.3.4. In the situation of Proposition 6.3.3, it follows from Remark 5.2.9 that the natural equivalence \(gr(X^{t\text{Fil}}) \simeq \text{und}(\text{gr}(X^{[2^*]}))[2^*]\) is canonically one of lax symmetric monoidal functors, and that the same goes for the equivalence \(gr(X^{t\text{Fil}}) \simeq \delta(\text{gr}(X))[2^*]\) assuming that \(\mathcal{C}\) is compactly generated.

Proposition 6.3.5. Let \(\mathcal{C}\) be a stable presentable \(\mathbb{Z}\)-linear symmetric monoidal \(\infty\)-category. Then, for \(X \in \text{Fil}_{\mathbb{S}_1}(\mathcal{C})\), there are a canonical natural equivalence

\[
\text{colim}(X^{t\text{Fil}}) \simeq \text{colim}(X)^T
\]

and canonical natural transformations

\[
\text{colim}(X^{t\text{Fil}}) \to \text{colim}(X)^T, \quad \text{colim}(X^{t\text{Fil}}) \to \text{colim}(X)^T,
\]

where here we regard \(\text{colim}(X)\) as an object of \(\text{Fun}(B\mathbb{S}_1, \mathcal{C}) \simeq \text{Mod}_T(\mathcal{C})\) by Remark 6.1.10. In the situation that \(\mathcal{C}\) is equipped with a right separated \(t\)-structure \((\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})\), the latter two transformations are equivalences whenever \(X\) satisfies the following two conditions:

(a) \(X\) is nonnegatively filtered, i.e. \(\text{gr}^i(X) = 0\) for all \(i < 0\);

(b) there exists an \(a \geq 0\) such that \(\text{gr}^i(X) \in \mathcal{C}_{\leq i + a}\) for all \(i \geq 0\).

Proof. Everything except the last sentence is immediate from Remark 2.4.12. Let us address the last sentence. Suppose given a right separated \(t\)-structure on \(\mathcal{C}\) and \(X \in \text{Fil}_{\mathbb{S}_1}(\mathcal{C})\) satisfying (a) and (b). Set \(Y = \text{colim}(X) \simeq X^0\). It suffices to show that the natural map \(\text{colim}(X^{t\text{Fil}}) \to Y^T\) is an equivalence; we will show that its cofiber \(Z\) vanishes. By Remark 2.4.2, we have

\[
Y^T \simeq \text{lim}( Y \Rightarrow Y \oplus Y[-1] \Rightarrow Y \oplus (Y[-1])^\oplus \oplus Y[-2] \Rightarrow \cdots )
\]

and, for each \(n \geq 0\), we have

\[
(X^{t\text{Fil}})^{-n} \simeq \text{lim}( X^{-n} \Rightarrow X^{-n} \oplus X^{-n+1}[-1] \Rightarrow X^{-n} \oplus (X^{-n+1}[-1])^\oplus \oplus X^{-n+2}[-2] \Rightarrow \cdots )
\]

Consider the cofiber \(Z^{-n}\) of the natural map \((X^{t\text{Fil}})^{-n} \to Y^T\), which in terms of the above formulas is induced by the canonical maps \(\theta^k: X^k \Rightarrow Y\). Our assumptions on \(X\) imply that \(\theta^k\) is an equivalence for \(k \leq 0\) and that cofib(\(\theta^k\)) is \((2(k - 1) + a)\)-coconnective for \(k \geq 1\). It follows that \(Z^{-n}\) is the limit of a cosimplicial diagram which is zero up through level \(n\), and at level \((n + i)\) for \(i \geq 1\) is \((i + a - n - 2)\)-coconnective (here \(i + a - n - 2\) appears as \(2(i - 1) + a - (n + i)\)). Examining the Bousfield–Kan spectral sequence, we deduce that the limit \(Z^{-n}\) is \((a - 2n - 2)\)-coconnective. Hence \(Z \simeq \text{colim}_{n \to \infty} Z^{-n} \simeq 0\), as desired.

Remark 6.3.6. In the situation of Proposition 6.3.5, recall from Remark 2.4.12 that the natural transformation \(\text{colim}(X^{t\text{Fil}}) \to \text{colim}(X)^T\) is canonically one of lax symmetric monoidal functors, and the same goes for the transformation \(\text{colim}(X^{t\text{Fil}}) \to \text{colim}(X)^{tT}\) assuming that \(\mathcal{C}\) is compactly generated.

Proposition 6.3.7. Let \(\mathcal{C}\) be a stable presentable \(\mathbb{Z}\)-linear symmetric monoidal \(\infty\)-category. Let \(X \in \text{Fil}_{\mathbb{S}_1}(\mathcal{C})\). Then:

(a) If the underlying filtered object of \(X\) is complete, the same holds for \(X^{t\text{Fil}} \in \text{Fil}(\mathcal{C})\).

Assuming that \(\mathcal{C}\) is equipped with a \(t\)-structure \((\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})\), we also have the following:
We may understand these filtered objects using the results of this subsection, together with the Adams operations on the graded pieces of the filtrations on Hochschild homology and negative cyclic homology discussed in the previous two subsections. The same ideas can be applied to cyclic and periodic cyclic homology, but we shall leave this to the interested reader.

\textbf{§6.4. Application: Adams operations.} Our goal in this subsection will be to analyze the effect of the Adams operations on the graded pieces of the filtrations on Hochschild homology and negative cyclic homology discussed in the previous two subsections. The same ideas can be applied to cyclic and periodic cyclic homology, but we shall leave this to the interested reader.

Let us begin by recalling how Adams operations are defined on Hochschild homology. Throughout this subsection, we fix an integer $\ell \geq 2$, a derived algebraic context $\mathcal{C}$ such that $\ell$ is invertible in $\pi_0 \text{End}_{\mathcal{C}}(1_{\mathcal{C}})$ (where $1_{\mathcal{C}}$ denotes the unit object of $\mathcal{C}$), and a map of derived commutative algebras $A \to B$ in $\mathcal{C}$. 

\begin{itemize}
  \item[(b)] If $X^i$ is $i$-connective for all $i \geq 0$, the same holds for $X_{t_0}^i$, where $\mathcal{C}$ is left separated, this condition implies that $X_{t_0}^i$ is complete, which in combination with (a) implies that $X^t_{t_0}$ is complete.
\end{itemize}

\textbf{Proof.} Statement (a) follows from the formula in Remark 2.4.2 for $X^t_{t_0}$ and the fact that completeness is stable under the formation of limits. Statement (b) follows similarly using the formula for the relative tensor product $X_{t_0}^i = X \otimes_{t_0} (X^t_{t_0})$ as a geometric realization and the fact that $i$-connectivity is stable under the formation of colimits for any $i \geq 0$.

We now explain how these results supply a proof of Theorem 1.2.1(b).

\textbf{Example 6.3.8.} Let $\mathcal{C}$ be a derived algebraic context. Let $A \in \text{DAAlg}(\mathcal{C})$ and let $B$ be a derived commutative $A$-algebra. Then cyclic, negative cyclic, and periodic cyclic homology of $B$ over $A$ are defined as follows:

\[
\begin{align*}
\text{HC}(B/A) &:= \text{HH}(B/A)_{T}, \\
\text{HC}^-(B/A) &:= \text{HH}(B/A)^T, \\
\text{HP}(B/A) &:= \text{HH}(B/A)^\Delta T,
\end{align*}
\]

where $\text{HH}(B/A)$ denotes the Hochschild homology of $B$ over $A$ (Definition 6.2.1). Now let $\text{HH}_{fil}(B/A)$ denote HKR-filtered Hochschild homology (Definition 6.2.5), and define

\[
\begin{align*}
\text{HC}_{fil}(B/A) &:= \text{HH}_{fil}(B/A)_{t_0}, \\
\text{HC}^-_{fil}(B/A) &:= \text{HH}_{fil}(B/A)^{t_0}, \\
\text{HP}_{fil}(B/A) &:= \text{HH}_{fil}(B/A)^{\Delta t_0}.
\end{align*}
\]

We may understand these filtered objects using the results of this subsection, together with the identifications $\colim(\text{HH}_{fil}(B/A)) \simeq \text{HH}(B/A)$ and $\text{gr}(\text{HH}_{fil}(B/A)) \simeq L\Omega^*_{B/A}$ of Theorem 6.2.6.

Firstly, Proposition 6.3.5 supplies a canonical equivalence and maps

\[
\colim(\text{HC}_{fil}(B/A)) \simeq \text{HC}(B/A),
\]

\[
\colim(\text{HC}^-_{fil}(B/A)) \to \text{HC}^-(B/A), \quad \colim(\text{HP}_{fil}(B/A)) \to \text{HP}(B/A),
\]

and says that the latter two maps are equivalences whenever there exists an $a \geq 0$ such that $\text{gr}^i(\text{HH}_{fil}(B/A)) \simeq L\Omega^*_B \simeq (\Lambda_B^i L_B/A)[i]$ is $(2i + a)$-truncated for all $i \geq 0$. In the case $\mathcal{C} = \text{Mod}_Z$, one can check that this condition holds whenever $A$ and $B$ are connective, $B$ is truncated (i.e. $\pi_k(B) \simeq 0$ for $k \gg 0$), and $L_B/A$ has Tor-amplitude contained in $[0, 1]$; for example, this includes the case where $A$ is an ordinary commutative ring and $B$ is quasi-smooth over $A$ (in the sense of [L-DAG, Definition 3.4.15]).

Secondly, Proposition 6.3.3, together with the equivalence $\text{gr}(\text{HH}_{fil}(B/A)) \simeq L\Omega^*_{B/A}$, supplies canonical identifications

\[
\text{gr}(\text{HC}_{fil}(B/A)) \simeq dR^\wedge_{B/A}[2*], \quad \text{gr}(\text{HC}^-_{fil}(B/A)) \simeq dR^\wedge_{B/A}[2*], \quad \text{gr}(\text{HP}_{fil}(B/A)) \simeq dR^\wedge_{B/A}[2*],
\]

where $dR^\wedge_{B/A}$ and $dR^\wedge_{B/A}$ denote Hodge-completed derived de Rham cohomology of $B$ over $A$ and the Hodge filtration thereon (see Corollary 5.3.9 and Remark 5.3.10); here we have left the underlying graded functor (und) and the diagonal functor (δ) implicit.

Finally, Proposition 6.3.7, together with Proposition 6.2.9, implies that the filtered objects $\text{HC}_{fil}(B/A)$, $\text{HC}^-_{fil}(B/A)$, and $\text{HP}_{fil}(B/A)$ are complete whenever $A$ and $B$ are connective.
Notation 6.4.1. Let $\mathcal{D}$ be a symmetric monoidal $\infty$-category and let $T$ be a bicommutative bialgebra in $\mathcal{D}$. We let $[\ell] : T \to T$ denote the $\ell$-th power map of bicommutative bialgebras, obtained by composing the $\ell$-fold comultiplication map $T \to T^{\otimes \ell}$ with the $\ell$-fold multiplication map $T^{\otimes \ell} \to T$.

The cases of interest to us are when $\mathcal{D} = \text{Spc}$ and $T = S^1$, when $\mathcal{D} = \mathcal{E}$ and $T = T$, when $\mathcal{D} = \text{Fil}(\mathcal{E})$ and $T = T_{\text{fil}}$, and when $\mathcal{D} = \text{Gr}(\mathcal{E})$ and $T = D$.\[\]

Notation 6.4.2. Let $\mathcal{D}$ be a symmetric monoidal $\infty$-category, let $T$ be a bicommutative bialgebra in $\mathcal{D}$, and let $\phi : T \to T$ be a map of bicommutative bialgebras in $\mathcal{D}$. Then we denote the corresponding restriction functor $\text{Mod}_A(\mathcal{D}) \to \text{Mod}_A(\mathcal{D})$ by $M \mapsto M^\phi$. Recall from Remark 2.2.7 that this restriction functor is canonically symmetric monoidal.

In the situation that $T$ is dualizable, we have from §2.3 a dual bicommutative bialgebra structure on $T^\vee$ and a symmetric monoidal equivalence $\text{Mod}_T(\mathcal{D}) \simeq \text{cMod}_{T^\vee}(\mathcal{D})$. Under this equivalence, restriction in $\phi$ corresponds to corestriction in the dual map of bicommutative bialgebras $\phi^\vee : T^\vee \to T^\vee$. If $\mathcal{D}$ is in fact a derived algebraic context, $T^\vee$ a derived bicommutative bialgebra in $\mathcal{D}$, and $\phi^\vee$ a map of such, there is in addition an induced restriction functor on derived commutative algebra objects, $(-)^\phi : \text{DAlg}(\text{Mod}_A(\mathcal{D})) \to \text{DAlg}(\text{Mod}_A(\mathcal{D}))$ (our notation here is as in Variant 4.2.33). This applies in the last three examples listed in Notation 6.4.1.

Construction 6.4.3. Restricting $\text{HH}(B/A) \in \text{Fun}(B^1, \text{DAlg}_A)$ along the $\ell$-th power map $[\ell] : S^1 \to S^1$ gives an object $\text{HH}(B/A)[\ell] \in \text{Fun}(B^1, \text{DAlg}_A)$, whose underlying derived commutative algebra is the same as that of $\text{HH}(B/A)$ (i.e. the difference is only in the $S^1$-actions). Thus, the canonical map of derived commutative algebras $A \to \text{HH}(B/A)$ can also be regarded as a map of derived commutative algebras $A \to \text{HH}(B/A)[\ell]$. By the definition/universal property of $\text{HH}(B/A)$, this extends to a unique map $\psi^\ell : \text{HH}(B/A) \to \text{HH}(B/A)[\ell]$ in $\text{Fun}(B^1, \text{DAlg}_A)$, which we refer to as the $\ell$-th Adams operation on $\text{HH}(B/A)$.

We can run the same construction in the filtered setting to see that the Adams operation is compatible with the HKR filtration on Hochschild homology:

Proposition 6.4.4. The $\ell$-th Adams operation $\psi^\ell : \text{HH}(B/A) \to \text{HH}(B/A)[\ell]$ promotes uniquely to a map $\psi^\ell_{\text{fil}} : \text{HH}_{\text{fil}}(B/A) \to \text{HH}_{\text{fil}}(B/A)[\ell]$ in $\text{Fil}_1^0 \text{DAlg}_A$.

Proof. This is immediate from the universal property of $\text{HH}_{\text{fil}}(B/A)$, or in other words from the equivalences
\[
\text{Map}_{\text{Fil}_1^0 \text{DAlg}_A}(\text{HH}_{\text{fil}}(B/A), \text{HH}_{\text{fil}}(B/A)[\ell]) \simeq \text{Map}_{\text{DAlg}_A}(B, \text{ev}^0(\text{HH}_{\text{fil}}(B/A)[\ell])) \\
\simeq \text{Map}_{\text{DAlg}_A}(B, \text{HH}(B/A)[\ell]) \\
\simeq \text{Map}_{\text{Fun}(B^1, \text{DAlg}_A)}(\text{HH}(B/A), \text{HH}(B/A)[\ell]).
\]

We now explain why the Adams operations on Hochschild homology induce Adams operations on negative cyclic homology.

Lemma 6.4.5. Let $\mathcal{D}$ be a presentable symmetric monoidal $\infty$-category, let $T$ be a cocommutative bialgebra in $\mathcal{D}$, and let $\phi : T \to T$ be an equivalence of cocommutative bialgebras in $\mathcal{D}$. Then there is a canonical natural lax symmetric monoidal equivalence $M^\phi \simeq (M^\phi)^T$ for $M \in \text{LMod}_T(\mathcal{E})$, where $(-)^T$ denotes the fixed point functor of Notation 2.4.1.

Proof. This is an immediate consequence of $\phi$ being an equivalence.\[\]

Lemma 6.4.6. The $\ell$-th power maps $[\ell] : T \to T$, $[\ell] : T_{\text{fil}} \to T_{\text{fil}}$, $[\ell] : D_+ \to D_+$ are all equivalences, in $\mathcal{E}$, $\text{Fil}(\mathcal{E})$, and $\text{Gr}(\mathcal{E})$ respectively.
Proof. We will just argue for the $\ell$-th power map on $T$; the others can be addressed by the same reasoning (or alternatively follow immediately from this first case). Let $R = \mathbb{Z}[1/\ell]$. Since we have assumed that $\ell$ is invertible in $\pi_0 \End_{\mathcal{C}}(1_{\mathcal{C}})$, the unique morphism of derived algebraic contexts $\Mod_{\mathcal{C}} \to \mathcal{C}$ factors through $\Mod_R$. It is thus enough to consider the case $\mathcal{C} = \Mod_R$. Here we have $T = R[S^1]$, and the map $[\ell] : T \to T$ is an equivalence because it induces isomorphisms on homotopy groups (namely the identity on $\pi_0$ and multiplication by $\ell$ on $\pi_1$).

Construction 6.4.7. Let $\psi^\ell : \HH(B/A) \to \HH(B/A)[\ell]$ be the Adams operation of Construction 6.4.3. Passing to $S^1$-fixed points, we obtain a map

$$\HH^{-}(B/A) = \HH(B/A)^T \to (\HH(B/A)[\ell])^T \simeq \HH(B/A)^T = \HH^{-}(B/A)$$

in $\Alg_{\mathbb{A}}$, where the equivalence comes from Lemmas 6.4.5 and 6.4.6. We denotes this map also by $\psi^\ell$, and refer to it as the $\ell$-th Adams operation on $\HH^{-}(B/A)$.

Again, we may repeat the above construction in the filtered setting to see that the Adams operation is compatible with the filtration on negative cyclic homology.

Construction 6.4.8. Let $\psi^\ell_i : \HH^{-i}(B/A) \to \HH^{-i}(B/A)[\ell]$. Applying filtered $S^1$-fixed points, we obtain a map $\psi^\ell_i : \HH^{-i}(B/A) \to \HH^{-i}(B/A)$, which is compatible with the Adams operation $\psi^\ell : \HH^{-}(B/A) \to \HH^{-}(B/A)$ of Construction 6.4.7 in the sense that the following diagram commutes:

$$\begin{array}{ccc}
\colim(\HH^{-i}(B/A)) & \xrightarrow{\psi^\ell_i} & \colim(\HH^{-i}(B/A)) \\
\downarrow & & \downarrow \\
\HH^{-}(B/A) & \xrightarrow{\psi^\ell} & \HH^{-}(B/A).
\end{array}$$

We would now like to identify the effect of the maps $\psi^\ell_i$ of Proposition 6.4.4 and Construction 6.4.8 upon passage to associated graded objects. What we will show is that, for each $i \in \mathbb{Z}$, the effect on $gr_i$ is given by multiplication by $\ell^i$. It will be useful to first formulate this graded multiplication map in a precise and structured way.

Construction 6.4.9. Let $u$ be a unit in $\mathbb{Z}[1/\ell]$ (the relevant examples being $u = \ell^{i+1}$). We will construct a natural isomorphism of symmetric monoidal functors

$$\{u^\ast\} : id_{\Gr(\mathcal{C})} \to id_{\Gr(\mathcal{C})},$$

whose evaluation on any graded object $X^i \in \Gr(\mathcal{C})$ is the endomorphism given by multiplication by $u^i$ on $X^i$ for each $i \in \mathbb{Z}$.

Consider the pointwise tensor product functor $\otimes : \Gr(\mathcal{C}) \times \Gr(\mathcal{C}) \to \Gr(\mathcal{C})$, given by the formula $(X^i \otimes Y^j)^n \simeq X^n \otimes Y^n$. We claim that this has a canonical lax symmetric monoidal structure, where we still regard (each copy of) $\Gr(\mathcal{C})$ as equipped with the Day convolution symmetric monoidal structure. By the universal property of Day convolution, it is equivalent to produce a lax symmetric monoidal structure on the corresponding functor $\mathbb{Z}^{ds} \times \Gr(\mathcal{C}) \times \Gr(\mathcal{C}) \to \mathcal{C}$, which we may obtain by factoring it as the following composition of lax symmetric monoidal functors:

$$\begin{array}{c}
\mathbb{Z}^{ds} \times \Gr(\mathcal{C}) \times \Gr(\mathcal{C}) \xrightarrow{\Delta \otimes id} \mathbb{Z}^{ds} \times \mathbb{Z}^{ds} \times \Gr(\mathcal{C}) \times \Gr(\mathcal{C}) \\
\simeq \mathbb{Z}^{ds} \times \Gr(\mathcal{C}) \times \mathbb{Z}^{ds} \times \Gr(\mathcal{C}) \\
\xrightarrow{id \otimes ev} \mathcal{C} \times \mathcal{C} \\
\xrightarrow{\otimes} \mathcal{C}
\end{array}$$

The preceding claim implies that, for any commutative algebra object $X$ of $\Gr(\mathcal{C})$, the functor $X \otimes - : \Gr(\mathcal{C}) \to \Gr(\mathcal{C})$ given by pointwise tensor product with $X$ is canonically lax symmetric.
monoidal, and that this extends to functor
\[ \tau : \text{GrCAlg}(\mathcal{C}) \to \text{End}^{\text{lax}}(\text{Gr}(\mathcal{C})), \]
where the right-hand side denotes the \( \infty \)-category of lax symmetric monoidal endofunctors of \( \text{Gr}(\mathcal{C}) \). Let \( A \in \text{GrCAlg}(\mathcal{C}) \) denote the constant graded object with value \( 1_{\mathcal{C}} \), regarded as a commutative algebra in the canonical manner. Then we have a canonical equivalence \( \tau(A) \cong \text{id}_{\text{Gr}(\mathcal{C})} \), so that \( \tau \) determines a map of spaces
\[ \text{End}_{\text{GrCAlg}(\mathcal{C})}(A) \to \text{End}_{\text{End}^{\text{lax}}(\text{Gr}(\mathcal{C}))}(\text{id}_{\text{Gr}(\mathcal{C})}). \]
Now recall (see [L-HA, Example 2.2.6.9]) that \( \text{GrCAlg}(\mathcal{C}) \) is canonically equivalent to the \( \infty \)-category \( \text{Fun}^{\text{lax}}(\mathcal{Z}^{d_{s}}, \mathcal{C}) \) of lax symmetric monoidal functors \( \mathcal{Z}^{d_{s}} \to \mathcal{C} \). Under this equivalence, \( A \) corresponds to the constant lax symmetric monoidal functor with value \( 1_{\mathcal{C}} \). It follows that there is an induced equivalence of spaces
\[ \text{End}_{\text{GrCAlg}(\mathcal{C})}(A) \cong \text{Map}_{\text{CAlg}}(\mathcal{Z}^{d_{s}}, \text{End}_{\mathcal{C}}(1_{\mathcal{C}})), \]
where, on the right-hand side, \( \text{End}_{\mathcal{C}}(1_{\mathcal{C}}) \) is regarded as an \( E_{\infty} \)-space via the multiplicative structure determined by the symmetric monoidal structure on \( \mathcal{C} \). We thus obtain a map
\[ \sigma : \text{Map}_{\text{CAlg}}(\mathcal{Z}, \text{End}_{\mathcal{C}}(1_{\mathcal{C}})) \to \text{End}_{\text{End}^{\text{lax}}(\text{Gr}(\mathcal{C}))}(\text{id}_{\text{Gr}(\mathcal{C})}). \]

The preceding map allows us to make the desired construction. Let \( R := \mathbb{Z}[1/\ell] \). As used in the proof of Lemma 6.4.6, we have an \( R \)-linear structure on \( \mathcal{C} \), determining a map of \( E_{\infty} \)-spaces
\[ \lambda : R^{\times} \cong \text{End}_{\text{Mod}}(R) \to \text{End}_{\mathcal{C}}(1_{\mathcal{C}}). \]
Precomposing this with the map of commutative monoids \( \mathbb{Z} \to R^{\times} \) sending \( 1 \mapsto u \) and then applying the map \( \sigma \) above produces the desired natural isomorphism \( \{ u^{*} \} \in \text{End}_{\text{End}^{\text{lax}}(\text{Gr}(\mathcal{C}))}(\text{id}_{\text{Gr}(\mathcal{C})}). \)

**Remark 6.4.10.** As the natural isomorphism \( \{ u^{*} \} \) of Construction 6.4.9 is one of symmetric monoidal functors, if \( X \) is a commutative algebra object or bicommutative bialgebra object of \( \text{Gr}(\mathcal{C}) \), the endomorphism \( \{ u^{*} \} : X \to X \) is canonically a map of such objects. The same statements go through for derived commutative algebra objects and derived bicommutative bialgebra objects by virtue of the naturality stated in Remark 4.2.25.

**Remark 6.4.11.** The \( \ell \)-th power map \( [\ell] : \mathbb{D}_{+} \to \mathbb{D}_{+} \) is equivalent to the endomorphism \( \{ \ell^{*} \} : \mathbb{D}_{+} \to \mathbb{D}_{+} \) of Construction 6.4.9 as maps of bicommutative bialgebras, and dually \( [\ell] : \mathbb{D}_{c}^{\ell} \to \mathbb{D}_{c}^{\ell} \) is equivalent to \( \{ \ell^{-1}\ell^{*} \} : \mathbb{D}_{c}^{\ell} \to \mathbb{D}_{c}^{\ell} \) as maps of derived bicommutative bialgebras. These identifications reduce to straightforward calculations since \( \mathbb{D}_{+} \) and \( \mathbb{D}_{c}^{\ell} \) lie in the ordinary category \( \text{Gr}(\mathcal{C})^{\mathbb{Z}} \) (and using Proposition 4.5.6).

These statements imply that there is an induced natural isomorphism of symmetric monoidal functors \( \{ \ell^{*} \} : \text{id}_{\text{DG}_{\ell}(\mathcal{C})} \to \text{id}_{\text{DG}_{\ell}(\mathcal{C})} \), which moreover respects derived commutative algebra structures.

**Proposition 6.4.12.** Let \( \psi_{\text{fil}}^{\ell} : \text{HH}_{\text{fil}}(B/A) \to \text{HH}_{\text{fil}}(B/A)[\ell] \) and \( \psi_{\text{C}_{\text{fil}}}^{\ell} : \text{HC}_{\text{fil}}^{*}(B/A) \to \text{HC}_{\text{fil}}^{*}(B/A) \) be the maps of Proposition 6.4.4 and Construction 6.4.8. Then the associated graded maps
\[ \text{gr}(\psi_{\text{fil}}^{\ell}) : \text{gr}(\text{HH}_{\text{fil}}(B/A)) \to \text{gr}(\text{HH}_{\text{fil}}(B/A)[\ell]) \cong \text{gr}(\text{HH}_{\text{fil}}(B/A))[\ell], \]
\[ \text{gr}(\psi_{\text{mil}}^{\ell}) : \text{gr}(\text{HC}_{\text{fil}}^{*}(B/A)) \to \text{gr}(\text{HC}_{\text{fil}}^{*}(B/A)) \]
are each homotopic to the map \( \{ \ell^{*} \} \).

**Proof.** We first prove the statement for Hochschild homology. By Theorem 6.2.6, we have an equivalence \( \text{gr}(\text{HH}_{\text{fil}}(B/A)) \cong L_{\Omega}^{*}_{B/A} \), through which we may identify the maps \( \text{gr}(\psi_{\text{fil}}^{\ell}) : \text{gr}(\text{HH}_{\text{fil}}(B/A)) \to \text{gr}(\text{HH}_{\text{fil}}(B/A))[\ell] \) with a map \( \phi : L_{\Omega}^{*}_{B/A} \to (L_{\Omega}^{*}_{B/A})[\ell] \). By construction of \( \psi_{\text{fil}}^{\ell} \), the map \( \phi \) is one of differential graded derived commutative \( A \)-algebras, which on zeroth graded pieces is the identity
map $id_B : B \to B$. By the definition/universal property of $L\Omega^*_{fil/A}$, this uniquely characterizes $\phi$. On the other hand, it follows from Remark 6.4.11 that $\{\ell^*\}$ is also such a map, so we must have $\phi \sim \{\ell^*\}$, as desired.

We now prove the statement for negative cyclic homology. Recall that the map $\psi^\ell_{fil} : HC^-_{fil}(B/A) \to HC^-_{fil}(B/A)$ was obtained from the map $\psi^\ell_{fil} : HH_{fil}(B/A) \to HH_{fil}(B/A)[\ell]$ by passing to filtered $S^1$-fixed points. It follows that the map $gr(\psi^\ell_{fil}) : gr(HC^-_{fil}(B/A)) \to gr(HC^-_{fil}(B/A))$ is obtained from the map $gr(\psi^\ell_{fil}) : gr(HH_{fil}(B/A)) \to gr(HH_{fil}(B/A))[\ell]$ by passing to $D_+\text{-fixed points}$, hence is the unique such map $\phi'$ fitting into a commutative diagram

$$
\begin{array}{ccc}
\rho(gr(HC^*_{fil}(B/A))) & \longrightarrow & gr(HH_{fil}(B/A)) \\
\rho(gr(HC^-_{fil}(B/A))) & \sim & \rho(gr(HC^-_{fil}(B/A))[\ell] \\
\end{array}
$$

in $DG_+\mathcal{C}Alg(\mathcal{C})$, where $\rho : Gr(\mathcal{C}) \to DG_+(\mathcal{C})$ is the restriction functor (along the counit $D_+ \to 1_\mathcal{C}$) and the displayed equivalence is that of Lemma 6.4.5. We have seen that $\phi \sim \{\ell^*\}$, and it follows from $\{\ell^*\}$ being a natural transformation of symmetric monoidal functors that we must have $\phi' \sim \{\ell^*\}$ as well, finishing the proof.

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