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Global Output Feedback Stabilization of Semilinear Reaction-Diffusion PDEs

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Abstract: This paper addresses the topic of global output feedback stabilization of semilinear reaction-diffusion PDEs. The semilinearity is assumed to be confined into a sector condition. We consider two different types of actuation configurations, namely: bounded control operator and right Robin boundary control. The measurement is selected as a left Dirichlet trace. The control strategy is finite dimensional and is designed based on a linear version of the plant. We derive a set of sufficient conditions ensuring the global exponential stabilization of the semilinear reaction-diffusion PDE. These conditions are shown to be feasible provided the order of the controller is large enough and the size of the sector condition in which the semilinearity is confined into is small enough.

Keywords: Semilinear reaction-diffusion PDEs, output feedback stabilization, finite-dimensional control.

1. INTRODUCTION

The topic of feedback stabilization of linear reaction-diffusion PDEs has been intensively studied in the literature (Boskovic et al., 2001; Liu, 2003) using different approaches such as backstepping (Krstic and Smyshlyaev, 2008) and spectral reduction methods (Russeil, 1978; Coron and Trélat, 2004). The extension of these approaches to the stabilization of semilinear reaction-diffusion PDEs remains challenging. Among the reported contributions, one can find the study of stability by means of strict Lyapunov functionals (Mazenc and Prieur, 2011), control using quasi-static deformations (Coron and Trélat, 2004), state-feedback (Karafyllis and Krstic, 2019a,b; Karafyllis, 2021) or network control (Wu et al., 2019).

This paper addresses the topic of output feedback stabilization of 1-D semilinear reaction-diffusion PDEs. The case of a state-feedback was studied in (Karafyllis and Krstic, 2019a,b; Karafyllis, 2021). Using spectral reduction methods and small gain arguments, the authors derived in these works sufficient conditions on the size of the sector condition (in which the nonlinearity is confined into) so that the proposed control strategy achieves the global exponential stabilization of the plant. However, the case of the output feedback, as considered in this work, remains challenging. In this context, we consider in this paper the global output feedback stabilization of 1-D semilinear reaction-diffusion PDEs with Dirichlet/Neumann/Robin boundary conditions. Two different configurations for the actuation scheme are investigated: bounded control operator and right Robin boundary control. The measurement is selected as the left Dirichlet trace. The reported output feedback control strategy takes advantage of recent developments regarding the finite-dimensional control of parabolic PDEs (Curtain, 1982; Balas, 1988; Katz and Fridman, 2020) by leveraging control architectures introduced in (Sakawa, 1983). In particular, we adopt the enhanced procedures reported in (Lhachemi and Prieur, 2022b,e) that allow the design in a generic and systematic manner of finite-dimensional observer-based control strategies for general 1-D reaction-diffusion PDEs with Dirichlet/Neumann/Robin boundary control and Dirichlet/Neumann boundary measurements; these systematic procedures have been successfully extended to delayed boundary control (Lhachemi and Prieur, 2022a,f) and local stabilization in the presence of a saturation (Lhachemi and Prieur, 2022c,d). Assuming that the nonlinearity satisfies a sector condition, we derive a set of sufficient LMI conditions ensuring the global output feedback stabilization of the plant. We show that the derived stability conditions are always feasible when selecting the order of the observer large enough and for a size of the sector condition small enough.

The rest of the paper is organized as follows. Notations and preliminary properties are summarized in Section 2. The control design for semilinear reaction-diffusion PDEs in the case of a bounded input operator and left Dirichlet measurement is addressed in Section 3. Then, Section 4 reports the case of right Robin boundary control and left Dirichlet measurement. Finally, concluding remarks are formulated in Section 6.

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2. NOTATION AND PROPERTIES

2.1 Notation

Real spaces $\mathbb{R}^n$ are equipped with the usual Euclidean norm denoted by $\|\cdot\|$. The associated induced norms of matrices are also denoted by $\|\cdot\|$. For any two vectors $X$ and $Y$, $\text{col}(X,Y)$ represents the vector $[X^\top, Y^\top]^\top$. The space of square integrable functions on $(0,1)$ is denoted by $L^2(0,1)$, and is endowed with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$. The associated norm is denoted by $\|\cdot\|_{L^2}$. For an integer $m \geq 1$, $H^m(0,1)$ stands for the $m$-order Sobolev space and is endowed with its usual norm $\|\cdot\|_{H^m}$. For any symmetric matrix $P \in \mathbb{R}^{n \times n}$, $P \succeq 0$ (resp. $P \succ 0$) means that $P$ is positive semi-definite (resp. positive definite).

2.2 Properties of Sturm-Liouville operators

Let $\theta_1, \theta_2 \in [0, \pi/2]$, $p \in C^1([0,1])$, and $q \in C^0([0,1])$ with $p > 0$ and $q \geq 0$. The Sturm-Liouville operator (Renardy and Rogers, 2006) is defined by $A = -(p(x)f'(x) + q(x)f(x))$ on the domain $D(A) = \{ f \in H^2(0,1) : \cos(\theta_1)f(0) - \sin(\theta_1)f'(0) = 0, \cos(\theta_2)f(1) + \sin(\theta_2)f'(1) = 0 \}$. The eigenvalues $(\lambda_n)_{n \geq 1}$ of $A$ are simple, non-negative, and form an increasing sequence as $n \to +\infty$. The associated unit eigenvectors $\phi_n \in L^2(0,1)$ form a Hilbert basis. Owing to these eigenstructures, $A$ is a Riesz spectral operator (Curtain and Zwart, 2012; Delattre et al., 2003): the domain of the operator $A$ is characterized by $D(A) = \{ f \in L^2(0,1) : \sum_{n \geq 1} \lambda_n^2 \| f, \phi_n \|^2 < +\infty \}$ while $Af = \sum_{n \geq 1} \lambda_n \langle f, \phi_n \rangle \phi_n$ for all $f \in D(A)$.

Introducing constants $p_*, p^*, q^* \in \mathbb{R}$ such that $0 < p_* \leq p(x) \leq p^*$ and $0 \leq q(x) \leq q^*$ for all $x \in [0,1]$, the eigenvalues $\lambda_n$ satisfy the estimates $0 \leq \pi^2((n-1)^2)p_* \leq \lambda_n \leq \pi^2(n^2)p^* + q^*$ for all $n \geq 1$ (Orlov, 2017). If we further assume that $q > 0$, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_H^2 \leq \sum_{n \geq 1} \lambda_n \langle f, \phi_n \rangle^2 \leq (Af, f) \leq C_2 \|f\|_H^2$$

for all $f \in D(A)$. We infer, in particular, from the latter inequalities and the Riesz spectral nature of $A$ that $f(0) = \sum_{n \geq 1} \langle f, \phi_n \rangle \phi_n(0)$ for all $f \in D(A)$. Finally, if we further assume that $p \in C^2([0,1])$, we have for any $x \in [0,1]$ that $\phi_n(x) = O(1)$ and $\phi'_n(x) = O(\sqrt{\lambda_n})$ as $n \to +\infty$ (Orlov, 2017).

For an arbitrarily given integer $N \geq 1$, we define $\mathcal{R}_Nf = \sum_{n \geq N+1} \langle f, \phi_n \rangle \phi_n$.

3. DISTRIBUTED COMMAND AND DIRICHLET BOUNDARY MEASUREMENT

3.1 Problem setting and spectral reduction

Consider the reaction-diffusion PDE described by

$$z_t(t,x) = \left( p(x)z_{xx}(t,x) \right)_x = q(x)z(t,x) + f(t,x,z(t,x)) + b(x)u(t)$$

$$\cos(\theta_1)z(0) - \sin(\theta_1)z_x(0) = 0$$

$$\cos(\theta_2)z(1) + \sin(\theta_2)z_x(1) = 0$$

$$z(0,x) = z_0(x).$$

for $t > 0$ and $x \in (0,1)$. Here we have $\theta_1 \in (0,\pi/2]$, $\theta_2 \in [0,\pi/2]$, $p \in C^2([0,1])$ with $p > 0$, and $q \in C^0([0,1])$. The distributed control input is $u(t) \in \mathbb{R}$ and acts on the system via the shape function $b \in L^2(0,1)$. The state of the reaction-diffusion PDE is $z(t,\cdot) \in L^2(0,1)$ while $z_0 \in L^2(0,1)$ is the initial condition. The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is assumed to be globally Lipschitz continuous in $z$, uniformly in $(t,x)$, so that $f(\cdot,\cdot,0) = 0$.

Let $\tilde{q}_n(q) \in C^0([0,1])$ and $k \geq 0$ be such that $|f(t,x,z) - \tilde{q}(x)z(t,x)| \leq k |z(t,x)|$, $\forall t \geq 0$, $\forall x \in [0,1]$, $\forall z \in \mathbb{R}$. Inequality (3) is referred to as a sector type condition. Hence (2a) can be written as

$$z_t(t,x) = (p(x)z_{xx}(t,x))_x - \tilde{q}(x)z(t,x) + g(t,x,z(t,x)) + b(x)u(t)$$

where $\tilde{q} = q_0 - \tilde{q}$ and $g(t,x,z) = f(t,x,z) - \tilde{q}(x)z$ with $|g(t,x,z)| \leq k |z|$, $\forall t \geq 0$, $\forall x \in [0,1]$, $\forall z \in \mathbb{R}$.

(4)

The system output is selected as:

$$y(t) = z(t,0).$$

(5)

In perspective of control design, recalling that $\tilde{q} = q_0 - \tilde{q}$, and without loss of generality, we pick a function $q \in C^0([0,1])$ and a constant $q_0 \in \mathbb{R}$ such that

$$\tilde{q} = q - q_0, \quad q > 0.$$

(6)

The projection of (2) into the Hilbert basis $(\phi_n)_{n \geq 1}$ gives

$$\dot{z}_n(t) = (-\lambda_n + q_0)z_n(t) + b_n u(t) + g_n(t)$$

where $z_n(t) = (z(t,\cdot), \phi_n)$, $b_n = \langle b, \phi_n \rangle$, and $g_n(t) = \langle g(t,\cdot), z(t,\cdot), \phi_n \rangle$. Moreover, considering classical solutions, the system output (5) is expressed as:

$$y(t) = \sum_{n \geq 1} \phi_n(t)z_n(t).$$

(8)

3.2 Control design and truncated model

Let $\delta > 0$ be the desired exponential decay rate for the closed-loop system trajectories. Let an integer $N_0 \geq 1$ be such that $-\lambda_n + q_0 < -\delta < 0$ for all $n \geq N_0 + 1$. We introduce an arbitrary integer $N \geq N_0 + 1$ that will be specified later. The control strategy is described by:

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_0)\hat{z}_n(t) + b_n u(t)$$

(9a)

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_0)\hat{z}_n(t) + b_n u(t), \quad N_0 + 1 \geq n \leq N$$

(9b)

$$u(t) = \sum_{n=1}^{N_0} k_n \hat{z}_n(t)$$

(9c)

where $l_n, k_n \in \mathbb{R}$ are the observer and feedback gains.

In preparation for stability analysis, we need to build a truncated model capturing the $N$ first modes of the PDE (2) as well as the dynamics of the controller (9). To do so, let us define the error of observation $e_n = z_n - \hat{z}_n$, the scaled error of observation $\mathcal{E}_n = \sqrt{\lambda_n}e_n$, and the vectors $\mathcal{Z}_{N_0} = [\hat{e}_1 \ldots \hat{e}_{N_0}]^\top$, $\mathcal{E}_{N_0} = [e_1 \ldots e_{N_0}]^\top$, $\mathcal{Z}^\top = [\hat{z}_{N_0+1} \ldots \hat{z}_N]^\top$, $\mathcal{E}^\top = [\hat{e}_{N_0+1} \ldots \hat{e}_N]^\top$, $R_1 = [g_1 \ldots g_{N_0}]^\top$, $\tilde{R}_2 = [\sqrt{\lambda_{N_0+1}}g_{N_0+1} \ldots \sqrt{\lambda_N}g_N]^\top$, and $R = \text{col}(R_1, \tilde{R}_2)$. We also introduce the matrices defined by $A_0 = \text{diag}(-\lambda_1 + q_0, \ldots, -\lambda_{N_0} + q_0)$, $A_1 = \ldots$
diag\left(-\lambda_{N+1}, \ldots, -\lambda_N, \ldots, -\lambda_1, q_c\right), \quad B_0 = \begin{bmatrix} b_1 & \cdots & b_{N_0} \end{bmatrix}^T, \\
B_1 = \begin{bmatrix} b_{N_0+1} & \cdots & b_N \end{bmatrix}^T, \quad C_0 = [\phi_1(0) \cdots \phi_N(0)], \quad \hat{C}_1 = \frac{\phi_{N+1}(0)}{\sqrt{\lambda_{N+1}}} \cdots \frac{\phi_N(0)}{\sqrt{\lambda_N}}, \quad K = [k_1 \cdots k_N], \quad L = [l_1 \cdots l_N]^T.

With $X = \text{col}\left(\hat{Z}^{N_0}, \hat{E}^{N_0}, \hat{Z}^{N-N_0}, \hat{E}^{N-N_0}\right)$, we obtain from (7) and (9) that $u = K \hat{Z}^{N_0}$ and

$$
\dot{X} = FX + L\zeta + GR
$$

where $\zeta = \sum_{n\geq N+1} \phi_n(0)z_n$,

$$
F = \begin{bmatrix} \left[\begin{array}{ccc} A_0 + B_0 K & L C_0 & 0 \\
0 & A_0 - L C_0 & -\hat{L}_C \\
B_1 K & 0 & A_1 \\
0 & 0 & 0 \\
& & 0 
\end{array}\right] & G = \begin{bmatrix} 0 \\
I \\
0 \\
0 \\
0 
\end{bmatrix}
$$

and $L = \text{col}(L, -L, 0, 0)$. Defining $\hat{K} = [K \ 0 \ 0 \ 0]$, we also have that

$$
u = \hat{K}X.
$$

We finally define the matrices

$$
\Lambda = \text{diag}(\lambda_{N+1}, \ldots, \lambda_N), \quad \Omega = \begin{bmatrix} I & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
0 & 0 & 0 & 0 
\end{bmatrix}
$$

and $\hat{\Lambda} = \text{diag}(I, \Lambda)$. In particular, we have that $\Omega \geq max(1, 1/\lambda_N)I$ and $\hat{\Lambda}^{-1} \geq \min(1, 1/\lambda_N)I$.

### 3.3 Stability result

**Theorem 1.** Let $\theta_1 \in [0, \pi/2], \theta_2 \in [0, \pi/2], p \in C^2([0, 1])$ with $p > 0$, $\tilde{q}_0 \in C^0([0, 1])$, and $b \in L^2(0, 1)$. Let $f : \mathbb{R}^+ \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ be globally Lipschitz continuous in $z$, uniformly in $(t, x)$, so that $f(t, \cdot, 0) = 0$. Let $\tilde{q}_0 \in C^0([0, 1])$ and $k_f > 0$ be so that (3) holds. Let $g \in C^0([0, 1])$ and $q_c \in \mathbb{R}$ be such that (6) holds. Let $\delta > 0$ and $N_0 > 1$ be such that $-\tilde{\lambda}_n + q_c < -\delta$ for all $n \geq N_0 + 1$. Assume that $b_1 \neq 0$ for all $1 \leq n \leq N_0$. Let $K \in \mathbb{R}^{1 \times N_0}$ and $L \in \mathbb{R}^{N_0}$ be such that $A_0 + B_0 K$ and $A_1 - L C_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. For a given $N \geq N_0 + 1$, assume that there exist a symmetric positive definite $P \in \mathbb{R}^{2N \times 2N}$, positive real numbers $\alpha_1, \alpha_2 > 1$ and $\beta, \gamma > 0$ such that

$$
\Theta_1 \preceq 0, \quad \Theta_2 \leq 0
$$

where

$$
\Theta_1 = \begin{bmatrix} \Theta_{1,1} & P \mathcal{L} & PG \\
\mathcal{L}^T P & -\beta & 0 \\
P^T G \mathcal{P}^T & 0 & -\alpha_2 \gamma N^{-1} \end{bmatrix},
$$

$\Theta_{1,1} = F^TP + PP + 2\delta P + b_1 \mathcal{L} \mathcal{R} \mathcal{L}^T \mathcal{R} + \alpha_1 \gamma \mathcal{L} \mathcal{R} \mathcal{L}^T \mathcal{R} + \alpha_2 \gamma^2 \Omega$,

$$
\Theta_2 = 2\gamma \left\{ \left[1 - \frac{1}{2} \left( \frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) \right] \lambda_{N+1} + q_c + \delta \right\} + \beta \mathcal{M}_\phi + \frac{\alpha_2 \gamma^2 \hat{\Omega}}{\lambda_{N+1}}
$$

with $\mathcal{M}_\phi = \sum_{n\geq N+1} \phi_n(0)^2 \hat{\Omega}$. Then, considering the closed-loop system composed of the plant (2) with the system output (5) and the controller (9), there exists $M > 0$ such that for any initial conditions $z_0 \in H^2(0, 1)$ and $\hat{z}_n(0) \in \mathbb{R}^N$

1. This implies that $(A_0, B_0)$ satisfies the Kalman condition. Note that $(A_0, C_0)$ satisfies the Kalman condition by arguments from Lhachemi and Prieur (2022).
\( \alpha_2 = N^3 \), and \( k_f = 1/N^2 \). Hence it can be seen that
\[ \Theta_2 \to -\infty \text{ as } N \to +\infty. \]
Moreover, since \( \|\tilde{K}\| = \|K\| \), \( \|\bar{L}\| = \sqrt{2}\|L\| \), and \( \|G\| = 1 \) are constants independent of \( N \), \( \|P\| = O(1) \) as \( N \to +\infty \), and \( \Omega \leq 2 \max(1,1/\lambda_{N+1})I \) and \( -\tilde{K}^{-1} \preceq -\min(1,1/\lambda_N)I \), the Schur complement shows that \( \Theta_1 \leq 0 \) for \( N \) selected to be large enough. This completes the proof.

**Remark 2.** For a fixed order \( N \geq N_0 + 1 \), let us arbitrarily fix the value of the decision variable \( \gamma > 0 \) (following the last part of the proof of Theorem 1), the obtained constraints remain feasible for \( N \) large enough and \( k_f > 0 \) small enough). Now \( \Theta_1 \leq 0 \) takes the form of a LMI w.r.t. the decision variables \( P, \alpha_1, \alpha_2, \beta \) while, using Schur complement, \( \Theta_2 \leq 0 \) is equivalent to the LMI formulation:
\[
\begin{bmatrix}
\mu & \sqrt{\gamma \lambda_{N+1}} \\
\sqrt{\gamma \lambda_{N+1}} & -\alpha_1 & \sqrt{\gamma \lambda_{N+1}} & -\alpha_2
\end{bmatrix} \preceq 0 \text{ with }
\mu = 2\gamma(-\lambda_{N+1} + q_c + \delta) + \beta M_0 + \alpha_2 \gamma k_f^2/\lambda_{N+1}.
\]
A similar remark applies to the constraints of Theorem 3.

4. ROBIN BOUNDARY CONTROL AND DIRICHLET BOUNDARY MEASUREMENT

4.1 Problem setting and spectral reduction

We now consider the boundary stabilization of the reaction-diffusion PDE described by
\[
z(t,x) = (p(x)z_x(t,x))_x - q_0(x)z(t,x) + f(t,x,z(t,x))
\]
(14a)
\[
\cos(\theta_1)z(t,0) - \sin(\theta_1)z_x(t,0) = 0 \quad \text{for } t > 0 \text{ and } x \in (0,1)
\]
(14b)
\[
\cos(\theta_2)z(t,1) + \sin(\theta_2)z_x(t,1) = u(t)
\]
(14c)
\[
z(0,x) = z_0(x)
\]
(14d)

where \( t > 0 \) and \( x \in (0,1) \) where \( p \in C^2([0,1]) \) and \( \theta_i \in (0,\pi/2) \). The boundary measurement is selected as the left Dirichlet trace (5). We still assume that there exist \( \tilde{q} \in C^0([0,1]) \) and \( k_f > 0 \) so that (3) holds. Hence we define \( \tilde{q} = \tilde{q}_0 - \tilde{q} \) and \( g(t,x,z) = f(t,x,z) - \tilde{q}(x)z \).

Consider the change of variable
\[
w(t,x) = z(t,x) - \frac{x^2}{\cos \theta_2 + 2 \sin \theta_2} u(t)
\]
(15)

so that, introducing \( v = \tilde{u} \), the PDE (14) is equivalently rewritten under the following homogeneous representation:
\[
u(t) = v(t)
\]
(16a)
\[
w(t,x) = (p(x)w_x(t,x))_x - q_0(x)w(t,x) + g(t,x,z(t,x))
\]
(16b)
\[
+ a(x)u(t) + b(x)v(t)
\]
\[
\cos(\theta_1)w(t,0) - \sin(\theta_1)w_x(t,0) = 0
\]
(16c)
\[
\cos(\theta_2)w(t,1) + \sin(\theta_2)w_x(t,1) = 0
\]
(16d)
\[
w(0,x) = w_0(x).
\]
(16e)

where \( a(x) = \frac{1}{\cos \theta_2 + 2 \sin \theta_2} (2p(x) + 2rp'(x) - x^2\tilde{q}(x)) \),
\( b(x) = -\frac{x^2}{\cos \theta_2 + 2 \sin \theta_2} \) and \( w_0(x) = z_0(x) - \frac{x^2}{\cos \theta_2 + 2 \sin \theta_2} u_0(x) \). Note that we introduced \( q \in C^0([0,1]) \) and \( q_0 \in \mathbb{R} \) so that (6) holds. Finally, we define the coefficients of projection \( z_n(t) = \langle z(t,\cdot) , \phi_n \rangle \),
\( w_n(t) = \langle w(t,\cdot) , \phi_n \rangle \), \( a_n = \langle a, \phi_n \rangle \), \( b_n = \langle b, \phi_n \rangle \), and \( g_n(t) = \langle g(t,\cdot,z(t,\cdot)), \phi_n \rangle \) from (15) we deduce that
\[
w_n(t) = \zeta_n(t) + b_n v(t), \quad n \geq 1.
\]
(17)

Moreover, the projection of (16) into the Hilbert basis \( \langle \phi_n \rangle_{n \geq 1} \) gives \( \hat{u} = 0 \) and
\[
w_n(t) = -\lambda_n + q_c w_n(t) + a_n u(t) + b_n v(t) + g_n(t)
\]
(18)

with \( w(t,\cdot) = \sum_{n \geq 1} w_n(t) \phi_n \).

Inserting (17) into (18), the projection of (14) gives
\[
\hat{z}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t) + g_n(t)
\]
(19)

where \( \beta_n = a_n + (\gamma_n - q_c) b_n = p(1 - (1 - \lambda_n)\phi_n(x) + \sin(\theta_2)\phi_n(1)) \).

Finally, considering classical solutions, the system output (5) is expressed as:
\[
y(t) = z(t,0) = w(t,0) = \sum_{n \geq 1} \hat{\phi}_n(0) w_n(t) \phi_n(t)
\]
(20)

4.2 Control design and truncated model

Let \( \delta > 0 \) be the desired exponential decay rate for the closed-loop system trajectories. Let an integer \( N_0 \geq 1 \) be such that \(-\lambda_n + q_c < -\delta < 0 \) for all \( n \geq N_0 + 1 \).

For an arbitrary integer \( N \geq N_0 + 1 \), the control strategy reads:
\[
\hat{z}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t)
\]
(21a)

\[
\hat{z}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t)
\]
(21b)

\[
\hat{z}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t), \quad N_0 + 1 \leq n \leq N
\]
(21c)

\[
u(t) = \sum_{n = 1}^{N_0} k_n \hat{z}_n(t)
\]
(21d)

where \( l_n, k_n \in \mathbb{R} \) are the observer and feedback gains, respectively.

To build the truncated model, we adopt the notations of Subsection 3.2 except that we introduce the notations \( \hat{z}_n = z_n/\lambda_n \) along with the vector \( \hat{Z}^{N-N_0} = [\hat{z}_{N+1} \ldots \hat{z}_N]^T \) and the matrices \( \hat{\mathcal{B}}_0 = [\beta_1 \ldots \beta_N]^T \) and \( \hat{\mathcal{B}}_1 = \left[ \begin{array}{c} \beta_{N+1} \\ \lambda_{N+1} \end{array} \right] \left[ \begin{array}{c} \beta_N \\ \lambda_N \end{array} \right]^T \). Hence, introducing the state vector \( \hat{X} = \text{col} (\hat{Z}^{N-N_0}, \hat{Z}^{N-N_0}, \hat{Z}^{N-N_0}, \hat{E}^{N-N_0}) \) we deduce similarly to the developments of Subsection 3.2 that the following truncated dynamics holds:
\[
\hat{X} = FX + L\zeta + GR
\]
(22)

where
\[
F = \begin{bmatrix} A_0 + \mathcal{B}_1 K & LC_0 & 0 & L\hat{C}_1 \\ 0 & A_0 - LC_0 & 0 & -L\hat{C}_1 \\ 0 & 0 & A_1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}
\]

and \( L = \text{col}(L, -L, 0, 0) \). We define the matrices
\[
\Lambda = \text{diag}(\lambda_{N+1}, \ldots, \lambda_N), \quad \Omega = \begin{bmatrix} I & I & 0 & 0 \\ I & I & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & A^2 & A^{1/2} \\ 0 & A^{1/2} & A^{-1} \end{bmatrix}
\]

and \( \Lambda = \text{diag}(I, \lambda) \). These matrices which are such that
\[
\Omega \leq 2 \max(1,1/\lambda_{N+1}, \lambda_N^2)I \text{ and } \Lambda^{-1} \leq \min(1,1/\lambda_N)I
\]

Finally, introducing \( \tilde{X} = \text{col}(X, \zeta, R) \), we have
\[
v = u = K\tilde{X}^{N_0} = E\tilde{X}
\]
(23)
where $E = K \begin{bmatrix} A_0 + 2\mathbb{B}_0 K & L C_0 \end{bmatrix} 0 \ L \tilde{C}_1 \ L_0 \end{bmatrix}$.

### 4.3 Stability result

**Theorem 3.** Let $\theta_1 \in (0, \pi/2]$, $\theta_2 \in [0, \pi/2]$, $p \in C^{2}(0, 1]$ with $p > 0$, and $q_0 \in C^2([0, 1])$ be such that (6) holds. Let $\delta > 0$ be such that (3) holds. Let $q \in C^2([0, 1])$ and $k_f > 0$ be such that (6) holds. Let $\delta > 0$ and $N_0 \geq 1$ be such that $-\lambda_n < \delta < \delta$ for all $n \geq N_0 + 1$. Let $K \in \mathbb{R}^{1 \times N_0}$ and $L \in \mathbb{R}^{N_0 \times 2}$ be such that $A_0 + 2\mathbb{B}_0 K$ and $A_0 - L C_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. For a given $N \geq N_0 + 1$, assume that there exist a symmetric positive definite $P \in \mathbb{R}^{2N \times 2N}$, positive real numbers $\alpha_1, \alpha_2, \alpha_3 > 3/2$ and $\beta, \gamma > 0$ such that

$$\Theta_1 \leq 0, \quad \Theta_2 \leq 0 \quad (24)$$

where

$$\Theta_1 = \begin{bmatrix} \Theta_{1,1,1} & \mathcal{P} L & \mathcal{P} G \\ \mathcal{P} G^T & 0 & -\alpha_3 \mathcal{K}^\top \mathcal{K} \end{bmatrix} + \alpha_2 \gamma \| R_n b_n \|_{L_2}^2 \mathcal{E}^\top \mathcal{E}$$

$$\Theta_{1,1,1} = F^T P + P F + 2\delta P \mathcal{F} + \alpha_1 \gamma \| R_n b_n \|_{L_2}^2 \mathcal{K}^\top \mathcal{K}$$

$$+ 2 \alpha_3 \mathcal{K}^\top \mathcal{K} \mathcal{K}^\top \mathcal{K}$$

$$+ 2 \gamma \left\{ \frac{1}{2} - \frac{3}{2} \sum_{i=1}^{N} \alpha_i \right\} \lambda_n + \mathcal{Q} + \delta \right) + \beta \mathcal{M}_0 + \frac{2 \alpha_3 \gamma^2 \mathcal{K}^\top \mathcal{K}^\top \mathcal{K}}{\lambda_{N+1}} \Theta_2 \quad (24)$$

with $\mathcal{M}_0 = \sum_{n=N+1}^\infty \frac{2}{\lambda_n}$. Then, considering the closed-loop system composed of the plant (14) with the system output (5) and the controller (21), there exists $M > 0$ such that for any initial conditions $z_0 \in H^2(0, 1)$ and $\tilde{z}_n(0) \in \mathbb{R}$ so that $\cos(\theta_1) z_0(0) - \sin(\theta_1) z_0(0) = 0$ and $\cos(\theta_2) z_0(0) + \sin(\theta_2) z_0(0) = K \tilde{z}_n(0)$, the system trajectory satisfies

$$\left\| z(t) \right\|_{H^1}^2 + \sum_{n=1}^{N} \tilde{z}_n(t)^2 \leq M e^{-2\delta t} \left\| z_0 \right\|_{H^1}^2 + \sum_{n=1}^{N} \tilde{z}_n(0)^2$$

for all $t > 0$. Moreover, when selecting $N$ to be sufficiently large, there exists $k_f > 0$ (small enough) so that the constraints (24) are feasible.

**Proof.** Consider the Lyapunov functional defined for $X \in \mathbb{R}^{2N}$ and $w \in D(A)$ by $V(X, w) = X^\top P X + \gamma \sum_{n=N+1} \lambda_n w_n^2$. Note that, following (Lhachemi and Prieur, 2022c), the first term of the Lyapunov functional captures the $N$ first modes of the PDE device in original $z$ coordinates (14) while the second term captures the residual modes (i.e., for $n \geq N+1$) in homogeneous $w$ coordinates (16). The computation of the time derivative of $V$ along the system trajectories (18) and (22) reads

$$\dot{V} + 2\delta V = X^\top \left\{ F^T P + P F + 2\delta P \mathcal{F} \right\} X + 2 X^\top P \mathcal{L}$$

$$+ 2 X^\top P G R + 2 \gamma \sum_{n=N+1} \lambda_n (\lambda_n - \lambda_n + q_n + \delta) w_n^2$$

$$+ 2 \gamma \sum_{n=N+1} \lambda_n (a_n u + b_n v + g_n) w_n.$$
The maximal value of $k_f > 0$ (corresponding to the size of the sector condition (3) in which the nonlinearity $f(t, x, z)$ is confined into) for which the stability of the closed-loop system is ensured by applying the theorems of this paper is detailed in Tab. 1 for the two studied configurations and for different values of the order of the observer.

| Dimension of the observer | $N = 2$ | $N = 3$ | $N = 4$ | $N = 5$ | $N = 6$ |
|---------------------------|---------|---------|---------|---------|---------|
| Theorem 1                 | $k_f = 1.99$ | $k_f = 2.32$ | $k_f = 2.45$ | $k_f = 2.54$ | $k_f = 2.59$ |
| Theorem 3                 | $k_f = 1.93$ | $k_f = 2.14$ | $k_f = 2.21$ | $k_f = 2.25$ | $k_f = 2.27$ |

Table 1. Maximum value of $k_f$ for the sector condition (3) obtained for different dimensions $N$ of the observer.

6. CONCLUSION

This paper solved the problem of output feedback stabilization of semilinear reaction-diffusion PDEs for which the semilinearity is assumed to be confined into a sector condition. It is worth noting that even if the developments of this paper have been focused on a Dirichlet boundary measurement, the case of a Neumann boundary measurement can also be handled using the framework of this paper augmented with the procedure reported in Lhachemi and Prieur (2022b,e).

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