A method to study complex systems of mesons in Lattice QCD

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Abstract

Finite density systems can be explored with Lattice QCD through the calculation of multi-hadron correlation functions. Recently, systems with up to 12 π$^+$'s or K$^+$'s have been studied to determine the 3-π$^+$ and 3-K$^+$ interactions, and the corresponding chemical potentials have been determined as a function of density. We derive recursion relations between correlation functions that allow this work to be extended to systems of arbitrary numbers of mesons and to systems containing many different types of mesons, such as π$^+$'s, K$^+$'s, D$^0$'s and B$^+$'s. These relations allow for the study of finite-density systems in arbitrary volumes, and for the study of high-density systems.
I. INTRODUCTION

An important goal of Lattice QCD (LQCD) is to calculate, with quantifiable uncertainties, the properties and interactions of systems comprised of multiple hadrons directly from QCD. The last few years have seen the first calculations of three-baryon systems in QCD [1] (ΞΞn and the triton or 3He), a four-baryon system in quenched QCD [2] (the α-particle), and the three-π± [3, 4] and three-K± [5] interactions from the calculation of systems involving up to twelve π±’s [3, 4] and K±’s [5] respectively. While all of these calculations were at unphysical values of the light quark masses due to the limited computational resources, they represent a significant step forward in a QCD-based understanding of the complex hadronic systems that dominate nature.

The study of multi-meson systems comprised of one or more species will provide important insights into the structure of dense forms of matter that may arise in astrophysical settings. Further, they will provide insight into the phase structure of QCD, and strongly interacting many-body systems in general. Finite density systems of mesons have been studied in LQCD using an appropriate chemical potential [7–9]. However, as shown in Refs. [3–5], one can also study these systems as a function of density and chemical potential by explicitly considering LQCD correlation functions with increasing numbers of mesons. For instance, the isospin chemical potential has been determined as a function of isospin density from systems of π±’s [4] by measuring the ground-state energies of different numbers of mesons in a fixed volume, and forming discrete differences, e.g. \( \mu_I \sim \frac{dE}{dn} \sim \frac{E_n+j-E_n}{j} \).

Lattice QCD calculations of systems involving multiple hadrons, such as nuclei or systems of multiple mesons, necessarily involve large numbers of contractions between quark field operators which naively grow as the product of the factorial of the number of each flavor of quark present in the system. For instance, a simple interpolating field for the proton is comprised of two up-quarks and one down-quark, and therefore the number of independent contractions required in the computation of the proton correlation function is \( N_p = 2! \cdot 1! = 2 \). The proton-proton correlation function requires \( N_{pp} = 4! \cdot 2! = 48 \), the triton (pnn) correlation function (or, equivalently in the isospin limit, 3He) requires \( N_{pnn} = 4! \cdot 5! = 2880 \), and the α-particle (ppnn) requires \( N_{ppnn} = 6! \cdot 6! = 518400 \). In the first calculation of three-baryon systems [1], the ΞΣΣ and the triton, the number of measurements of the correlation function that could be made was limited, not by the number of gauge-field configurations or quark propagators that could be computed, but by the number of contractions that could be performed with the available computational resources (even after identifying identical and vanishing contributions). The same limitation exists for the calculation of systems involving large numbers of mesons. The actual number of contractions required for such systems can be substantially reduced by exploiting the symmetry of the contractions [1–5] (identifying redundant contributions), or by using different sources (e.g. using only the upper two components of the quark field operators [2]). However, even with these simplifications, the number of contractions does not scale polynomially with the number of hadrons to large systems, and the calculation of contractions remains a significant roadblock to the exploration of multi-hadron systems with LQCD.

In this work, we develop recursion relations among contractions that allow for the calculation of correlation functions corresponding to systems with arbitrary numbers of mesons.¹

¹ In this work, we limit our discussion to mesonic systems that do not involve creation and annihilation of the same flavor of quark field at the same Euclidean time. We also only consider pseudoscalar mesons for
The correlation function of the \((\mathcal{N}+1)\)-meson system is related to that of the \(\mathcal{N}\)-meson system by a small number of matrix and scalar multiplications using the recursion relations. The recursion makes use of the fact that many of the contractions required for the \((\mathcal{N}+1)\)-meson system have already been calculated in the construction of the \(\mathcal{N}\)-meson system. The simplest recursion relations for a single species of meson are developed in Section II, and the generalizations to two species and to many species are presented in Sections III and IV. As the repeated use of a quark propagator from a single source limits the number of mesons in the system to be \(\mathcal{N} \leq N_c N_s = 12\) (where \(N_c\) and \(N_s\) are the number of colors and spinor components, respectively), we present recursion relations for systems arising from two sources in Section V A and from multiple sources in Section V B. These two extensions of the original recursions are finally combined into a recursion relation that allows for systems with arbitrary numbers of mesons of arbitrary species (see footnote 1) to be computed from propagators from many different sources. This is presented in Section VI. The recursive approach offers a significant speedup for intermediate-sized systems and allows for the investigation of larger systems that are otherwise impractical. Section VII is a concluding discussion of such computational aspects and summarizes the broader perspective of this approach.

II. SINGLE SPECIES MULTI-MESON SYSTEMS FROM ONE SOURCE

Let us begin by considering multi-pion systems that are composed of \(n\)-\(\pi^+\)'s for which the correlation functions are produced from a single light-quark propagator. As there are \(N_s = 4\) Dirac indices and \(N_c = 3\) color indices associated with each quark field (on a given lattice site), there are \(N_s \times N_c = 12\) independent components in each quark-field and hence a single light-quark propagator can be used to generate systems containing up to 12 \(\pi^+\)'s. To calculate systems with \(n > 12\), additional distinct light-quark propagators must be calculated, as discussed below. A correlation function for a system of \(n < 12\) \(\pi^+\)'s has the form

\[
C_{n\pi^+}(t) = \left\langle \left( \sum_x \pi^+(x,t) \right)^n \left( \pi^-(0,0) \right)^n \right\rangle ,
\]

where the operator \(\pi^+(x,t)\) denotes a quark-level operator \(\pi^+(x,t) = \bar{d}(x,t) \gamma_5 u(x,t)\). Naively, there are \(N_c^2 N_u! = (n!)^2\) independent contractions that contribute to this correlation function, which for the \(n = 12\) system corresponds to a total number of \(\sim 2.3 \times 10^{17}\). By the symmetry of the correlation function, with all propagators originating from a single source, all of the contractions of either the up- or down-quark fields are the same, leaving only \(n!\) contractions to be evaluated, which for \(n = 12\) is \(\sim 4.8 \times 10^8\). However, considering how the contractions can be grouped by permutations, there are far fewer independent contractions that must be performed.

As the sources and the sinks of each meson are identical, a twelve component Grassmann variable, \(\eta\), can be introduced in order to write the correlation function in eq. (1) as

\[
C_{n\pi^+}(t) = n! \left\langle \left( \eta_i A_{ij}(t) \eta_j \right)^n \right\rangle , \quad A_{ij}(t) = \sum_x \left[ S(x,t;0,0) \right]_{ik} \left[ S^\dagger(x,t;0,0) \right]_{kj} \tag{2}
\]

simplicity, however, there are no conceptual difficulties in including other types of mesons. To be specific, we focus on mesons with the the quantum numbers of \(\bar{7}_{\gamma 5}u\) with \(q \neq u\).
where \( S(x, t; 0, 0) \) is the light-quark propagator from the source located at \((0, 0)\) to the sink at \((x, t)\) and we have used the relation \( S(x, t; y, t') = \gamma_5 S^\dagger(y, t'; x, t) \gamma_5 \). \( A(t) \) is a time-dependent \( 12 \times 12 \) matrix and the indices in eq. (2) are combined spinor-color indices running over \( i = 1, \ldots, 12 \). Repeated spinor-color indices imply summation. From the anti-commuting nature of of the \( \eta \), it follows that

\[
C_{n\pi^+}(t) = (-)^n \frac{n!}{(N-n)!} \epsilon^{a_1 \cdots a_{N-n} a_1 \cdots a_n} \epsilon_{a_1 \cdots a_{N-n} \beta_1 \cdots \beta_n} \left[ A(t) \right]_{a_1 \beta_1} \cdots \left[ A(t) \right]_{a_n \beta_n}
\]

(3)

where \( N = N_s \times N_c \), and the indices \( a_i, \alpha_i, \) and \( \beta_i \) are summed. In the case of interest, \( N = 12 \), but the relations that we derive are true for arbitrary values of \( N \). An important building block for the contractions \( C_{n\pi^+} \) is a partly contracted object \( R_n \) whose spinor-color trace is (up to an irrelevant combinatorial factor) equivalent to the contraction. Formally this is defined via the functional relation

\[
[R_n]_{ij} = \bar{u}_i(0)d_k(0)\frac{\delta}{\delta d_k(0)}\frac{\delta}{\delta \pi_j(0)}C_{n\pi^+}
\]

(4)

The correlation functions in eq. (3) can be related to sums of traces over \( A(t) \) as

\[
\det [1 + \lambda A] = \frac{1}{N!} \sum_j^N N C_j \lambda^j \epsilon^{a_1 \cdots a_{N-j} a_1 \cdots a_j} \epsilon_{a_1 \cdots a_{N-j} \beta_1 \cdots \beta_j} \left[ A(t) \right]_{a_1 \beta_1} \cdots \left[ A(t) \right]_{a_j \beta_j}
\]

\[
= \exp (\text{Tr} \left[ \log [1 + \lambda A] \right]) = \exp \left( \text{Tr} \left[ \sum_{p=1}^\infty \frac{(-)^{p-1}}{p} \lambda^p A^p \right] \right)
\]

\[
= 1 + \lambda \langle A \rangle + \frac{\lambda^2}{2!} (\langle A \rangle^2 - \langle A^2 \rangle) + \frac{\lambda^3}{3!} (\langle A \rangle^3 - 3\langle A^2 \rangle\langle A \rangle + 2\langle A^3 \rangle)
\]

\[
+ \ldots
\]

\[
= \sum_j^N \frac{1}{j!} \lambda^j \langle R_j \rangle = \sum_j^N (-)^j \left( \frac{1}{j!} \right)^2 \lambda^j C_{j\pi^+}
\]

(5)

with the explicit expressions for systems with \( n \leq 13 \) given in the Appendix of Ref. [4]. In the last line \( \langle R_j \rangle \equiv \text{Tr}[R_n] \) is the Dirac and color trace. As the l.h.s. of eq. (5) is an order-\( N \) polynomial in \( \lambda \), the \( \langle R_j \rangle \) is 0 (and hence \( C_{j\pi^+} = 0 \) \( \forall j > N \)). For \( n = 12 \), there are approximately 80 independent terms that must be summed (resulting from the partition of 12 objects) [3], requiring approximately \( 10^3 \) calculations, which is significantly smaller than the naive number of \( \sim 2.3 \times 10^{17} \) and the improved number of \( \sim 4.8 \times 10^8 \). One sees that large coefficients appear in the expansion of \( \langle R_n \rangle \) for large values of \( n \), leading to significant cancellations among terms, and the need to use high precision arithmetic libraries in the numerical calculation of such correlation functions. For large \( n \), the number of terms that must be evaluated behaves as \( \frac{1}{2\sqrt{2n\pi}} e^{\pi\sqrt{2n/3}} \) [6], which scales poorly to systems involving a large number of \( \pi^+ \)’s.

### A. Ascending Recursion Relations

The objects \( R_n \) in eq. (5) are \( N \times N \) matrices, and their trace is proportional to the contraction associated with the \( n \)-pion system. The matrices themselves correspond to the
contractions in the \( n \)-pion system with one up-quark and one anti-up-quark remaining uncontracted. Therefore, the contraction associated with the \((n+1)\)-pion system can be found by contracting \( A \) with \( R_n \) in all possible ways. There are two independent contractions of \( A \) and \( R_n \), and their coefficients can be determined by requiring that the \( \langle R_n \rangle \) reproduce the multi-\( \pi^+ \) contractions given in Ref. [4]. It is straightforward to show that the object \( R_{n+1} \) associated with the \((n+1)\)-\( \pi^+ \) system is related to that of the \( n\)-\( \pi^+ \) system through

\[
R_{n+1} = \langle R_n \rangle A - n R_n A. \tag{6}
\]

In order for the recursion relation in eq. (6) to be useful for LQCD calculations, a starting point (starting contraction) must be identified. An obvious starting point is the contraction associated with the single-\( \pi^+ \) system, \( n = 1 \), for which \( R_1 = A \), and \( \langle R_1 \rangle = \langle A \rangle \). This can be used as the starting point of ascending recursion relations that determine \( \langle R_{n+1} \rangle \) from \( R_n \). On the other hand, a less obvious starting point is that \( R_{N+1} = 0 \), induced by the Pauli-principle, which will yield descending recursion relations from which \( R_{n-1} \) can be determined from \( R_n \).

The initial condition for the ascending recursion relation is (beyond \( \langle R_0 \rangle = 1 \))

\[
R_1 = A \quad , \quad \langle R_1 \rangle = \langle A \rangle , \tag{7}
\]

The correlation function for the 2-\( \pi^+ \) system is

\[
R_2 = \langle R_1 \rangle A - R_1 A = \langle A \rangle A - A^2
\]

\[
\langle R_2 \rangle = \langle A \rangle^2 - \langle A^2 \rangle , \tag{8}
\]

which agrees with the result in Ref. [4]. The correlation function for the 3-\( \pi^+ \) system is

\[
R_3 = \langle R_2 \rangle A - 2 R_2 A = \langle A \rangle^2 A - \langle A^2 \rangle A - 2 \langle A \rangle A^2 + 2 A^3
\]

\[
\langle R_3 \rangle = \langle A \rangle^3 - 3 \langle A^2 \rangle \langle A \rangle - 2 \langle A^3 \rangle , \tag{9}
\]

also in agreement with Ref. [4]. Repeated application of the recursion relation recovers all of the contractions given explicitly in Ref. [4].

**B. Descending Recursion Relations**

The ascending recursion relation enables a sequential calculation of the correlation functions for systems containing \( n \)-\( \pi^+ \)’s for \( n \leq N \) from a single light-quark propagator. For \( n > N \) the correlations functions all vanish due to the Pauli-principle, which is implemented by eq. (5). As \( R_{N+k} = 0 \) for \( k > 0 \) for an arbitrary matrix \( A \), it is obvious from the recursion relation, eq. (6), that \( R_N \propto I_N \), where \( I_N \) is the \( N \times N \) identity matrix. It then follows from eq. (5) that

\[
R_N = (N-1)! \det(A) I_N ,
\]

\[
\langle R_N \rangle = N! \det(A) . \tag{10}
\]

The fact that \( R_N \propto I_N \) and \( R_{N+1} = 0 \) allows one to construct descending recursion relations by working with “holes” in the “closed-shell” of \( R_N \). Multiplying the recursion relation in eq. (6) by \( A^{-1} \) on the right yields

\[
R_{n+1} A^{-1} = \langle R_n \rangle I_N - n R_n
\]

\[
\langle R_{n+1} A^{-1} \rangle = (N-n) \langle R_n \rangle , \tag{11}
\]
from which it follows that
\[ R_{n-1} = \frac{1}{n-1} \left[ \frac{1}{N+1-n} \langle R_n A^{-1} \rangle I_N - R_n A^{-1} \right] , \] (12)
and therefore provides a descending recursion relation where \( A^{-1} \) is interpreted as a \( \pi^+ \)-hole (while \( A \) is interpreted as a \( \pi^+ \)). Applying this recursion relation to the result in eq. (10) produces
\[ R_{N-1} = (N-2)! \det(A) \left[ \langle A^{-1} \rangle I_N - A^{-1} \right] \]
\[ \langle R_{N-1} \rangle = (N-1)! \det(A) \langle A^{-1} \rangle , \] (13)
and further application of the recursion relation to the result in eq. (13) produces
\[ R_{N-2} = \frac{(N-3)!}{2} \det(A) \left[ \langle A^{-1} \rangle^2 I_N - \langle (A^{-1})^2 \rangle I_N - 2 \langle A^{-1} \rangle A^{-1} + 2 (A^{-1})^2 \right] \]
\[ \langle R_{N-2} \rangle = \frac{(N-2)!}{2} \det(A) \left[ \langle A^{-1} \rangle^2 - \langle (A^{-1})^2 \rangle \right] . \] (14)
It is interesting to note that the \( \langle R_{N-k} \rangle \) have the same form in terms of the \( A^{-1} \) as the \( \langle R_k \rangle \) do in terms of \( A \) (modulo factors of \( \det(A) \) and numerical factors depending upon \( k, N \)). This observation makes it obvious that
\[ \langle R_{N-k} \rangle = \frac{(N-k)!}{k!} \det(A) \langle R_k(A^{-1}) \rangle , \] (15)
where the recursion \( R_{n+1}(w) \) is defined by
\[ R_{n+1}(w) = \langle R_n(w) \rangle w - n R_n w . \] (16)

III. TWO SPECIES MULTI-MESON SYSTEMS FROM ONE SOURCE

The recursion relations that allow for the computation of correlation functions for systems composed of \((n+1)\)-\( \pi^+ \)'s from systems composed of \(n-\pi^+\) can be extended to construct the correlation functions composed of both \( \pi^+ \)'s and \( K^+ \)'s. A correlation function for a system composed of \( n_{\pi} \) \( \pi^+ \)'s and \( n_K \) \( K^+ \)'s is
\[ C_{\{n_{\pi}\pi^+, n_KK^+\}}(t) = \left\langle \left( \sum_x \pi^+(x,t) \right)^{n_{\pi}} \left( \sum_x K^+(x,t) \right)^{n_K} \right\rangle , \] (17)
where the operator \( K^+(x,t) \) denotes a quark-level operator \( K^+(x,t) = \overline{\sigma}(x,t) \gamma_5 u(x,t) \).
After contracting the quark field operators, the correlation function can be written as
\[ C_{\{n_{\pi}\pi^+, n_KK^+\}}(t) = n_{\pi}! n_K! \left\langle (\overline{\pi} A(t) \eta)^{n_{\pi}} (\overline{\pi} \kappa(t) \eta)^{n_K} \right\rangle \]
\[ \kappa(t) = \sum_x S(x,t;0,0) S^\dagger_x(x,t;0,0) , \] (18)
where \( S_s(x, t; 0, 0) \) is the strange quark propagator from the source located at \((0, 0)\) to the sink at \((x, t)\). The factor of \( n! \) that appears (instead of the \( n! \) in eq. (2)) corresponds to the number of ways of contracting both the anti-strange and anti-down light-quark field operators. Setting \( n = n_\pi + n_K \) in eq. (2) and eq. (3), making the replacement \( \lambda A \rightarrow \lambda A + \beta K \) and identifying terms that are of the same order in \( \lambda \beta^{n-j} \), we find that

\[
C_{\{n_\pi , n_K,K^+\}}(t) = (-)^{n_\pi + n_K} \frac{n_\pi! n_K!}{(N - n_\pi - n_K)!} \epsilon_{\alpha_1 \ldots \alpha_{N-n_\pi-n_K}} \epsilon_{\alpha'_{n_\pi} \ldots \alpha_{n_\pi+n_K}} \frac{[A(t)]_{\alpha_1 \ldots \alpha_{n_\pi}} [A(t)]_{\alpha'_{n_\pi+1} \ldots \alpha_{n_\pi+n_K}} \kappa(t)}{\alpha_{n_\pi+1} \ldots \alpha_{n_\pi+n_K}}
\]

\[
= (-)^{n_\pi + n_K} \frac{n_\pi! n_K!}{n_\pi+n_K C_{n_\pi}} \langle R_{\{n_\pi,n_K\}} \rangle ,
\]

where \( R_{\{n_\pi,n_K\}} \) is the generalization of \( R_n \) to the two-species system. By construction, we are restricted to systems with \( n_\pi + n_K \leq N \) for propagators from single sinks. As the recursion relation in eq. (19) is satisfied under the replacement \( \lambda A \rightarrow \lambda A + \beta K \), it is clear that the \( R_{\{n_\pi,n_K\}} \) satisfy a recursion relation

\[
R_{\{n_\pi,n_K\}} = \langle R_{\{n_\pi-1,n_K\}} \rangle A - (n_\pi + n_K - 1) \langle R_{\{n_\pi-1,n_K\}} \rangle A + \langle R_{\{n_\pi,n_K-1\}} \rangle \kappa - (n_\pi + n_K - 1) \langle R_{\{n_\pi,n_K-1\}} \rangle \kappa .
\]

(20)

The boundary conditions for the ascending recursion relations are

\[
R_{\{1,0\}} = A , \quad R_{\{0,1\}} = \kappa , \quad \langle R_{\{0,0\}} \rangle = 1 ,
\]

(21)

and \( R_{\{p,j\}} = 0 \) and \( R_{\{-j,p\}} = 0 \ \forall \ j > 0 \) and \( \forall \ p \).

The descending recursion relations are a little less obvious. Unlike the case of \( N \) \( \pi^+ \)'s for which there is a single system with \( n_\pi = N \), the mixed \( \pi^+-K^+ \) systems has a set of systems with \( n_\pi + n_K = N \). It remains the case that \( R_{j,N-j+1} = 0 \ \forall \ j \), and further, the single species results provide

\[
R_{\{N,0\}} = (N - 1)! \ det(A) I_N , \quad R_{\{0,N\}} = (N - 1)! \ det(\kappa) I_N .
\]

(22)

Using the replacement \( \lambda A \rightarrow \lambda A (1 + \beta A^{-1} \kappa) \) in eq. (6) we see that

\[
R_{\{N-j,j\}} = \frac{(N - 1)!}{j!} \ det(A) \langle R_{j}(A^{-1} \kappa) \rangle I_N
\]

\[
= \frac{(N - 1)!}{(N - j)!} \ det(\kappa) \langle R_{N-j}(\kappa^{-1} A) \rangle I_N ,
\]

(23)

which allows for the contractions of the systems with \( n_\pi + n_K = N \) to be related to each other.

To reduce the total number of mesons in the system to \( n_\pi + n_K < N \) requires use of eq. (20), which can be written as

\[
R_{\{N-p-j,j\}} = \frac{1}{N-p} \left[ \frac{1}{p} \left( \langle R_{\{N-p-j+1,j\}} A^{-1} \rangle - \langle R_{\{N-p-j+1,j-1\}} \rangle \kappa A^{-1} \right) + (N-p) \langle R_{\{N-p-j+1,j-1\}} \rangle \kappa A^{-1} \right] I_N
\]

\[
- \langle R_{\{N-p-j+1,j\}} A^{-1} \rangle + \langle R_{\{N-p-j+1,j-1\}} \rangle \kappa A^{-1} - (N-p) \langle R_{\{N-p-j+1,j-1\}} \rangle \kappa A^{-1} \right] ,
\]

(24)
from which the correlation function with \((N - p - j)\) \(\pi^+\)'s and \(j\) \(K^+\)'s can be determined from the correlation functions with \((N - p - j + 1)\) \(\pi^+\)'s and \(j\) \(K^+\)'s, and \((N - p - j + 1)\) \(\pi^+\)'s and \((j - 1)\) \(K^+\)'s. For instance, as we have expressions for the system with \(n_\pi \pi^+\)'s and 0 \(K^+\)'s, and also for the system with \(N - 1\) \(\pi^+\)'s and 1 \(K^+\)'s, the relation in eq. (25) can be used to obtain the correlation function for the system with \((N - 2)\) \(\pi^+\)'s and 1 \(K^+\)'s,

\[
R_{(N-2,1)} = \frac{1}{N-1} \left[ \langle \langle R_{(N-1,1)} A^{-1} \rangle - \langle R_{(N-1,0)} \rangle \langle \kappa A^{-1} \rangle + (N - 1) \langle R_{(N-1,0)} \kappa A^{-1} \rangle \rangle \right]
\]

Once this is known, it can be combined with the correlation function for \((N - 3)\) \(\pi^+\)'s and 2 \(K^+\)'s, to produce that for \((N - 3)\) \(\pi^+\)'s and \(2 K^+\)'s, and so forth, determining the correlation functions for all systems with \(n_\pi + n_K = N - 1\). This process can then be repeated to produce the correlation functions for all systems with \(n_\pi + n_K = N - 2\), \(n_\pi + n_K = N - 3\), and so forth. The fact that we have calculated the correlation functions for purely \(\pi^+\)-systems, purely \(K^+\) systems and mixed systems with a total of \(N\) \(\pi^+\)'s and \(K^+\)'s, allows for the correlation functions for all systems with \(n_\pi + n_K \leq N\) to be determined from descending recursion relations.

### IV. M- SPECIES MULTI-MESON SYSTEMS FROM ONE SOURCE

It is now possible to generalize the discussions of the previous sections, and arrive at the correlation functions for systems comprised of mesons of more than one species, generated with a single light-quark propagator, and multiple different light, strange or heavy quark propagators. This allows for the discussions of systems comprised of, for instance, \(n_\pi \pi^+\)'s, \(n_K K^+\)'s, \(n_D \overline{D}^0\)'s, and \(n_B B^+\)'s. A correlation function for a system composed of \(n_1\) mesons of type \(A_1\), \(n_2\) mesons of type \(A_2\), ..., \(n_m\) mesons of type \(A_m\), is of the form

\[
C_{\{n_1 A_1, ..., n_m A_m\}}(t) = \left\langle \left( \sum_x A_1(x, t) \right)^{n_1} ... \left( \sum_x A_m(x, t) \right)^{n_m} \right\rangle
\]

where the operator \(A_m(x, t)\) denotes a quark-level operator \(A_m(x, t) = \bar{q}_m(x, t) \gamma_5 u(x, t)\). After contracting the quark field operators, this can be written as

\[
A_j(t) = \sum_x S(x, t; 0, 0) S_j^+(x, t; 0, 0)
\]

and

\[
C_{\{n_1 A_1, ..., n_m A_m\}}(t) = n_1! ... n_m! \left\langle \left( \bar{q}_1 A_1(t) \eta \right)^{n_1} ... \left( \bar{q}_m A_m(t) \eta \right)^{n_m} \right\rangle,
\]

\[ (27) \]
where \( S_j(x, t; 0, 0) \) is the propagator of the \( j \)th quark flavor from the source located at \((0, 0)\) to the sink at \((x, t)\). Writing the total number of mesons in the system as \( \sum_i n_i = \mathcal{N} \), we have that

\[
C_{\{n_1A_1, \ldots, n_mA_m\}}(t) = (-)^\mathcal{N} \frac{\Pi_i n_i!}{(\mathcal{N} - \mathcal{N})!} \epsilon^{\alpha_1 \cdots \alpha_{\mathcal{N} - \mathcal{N}} \alpha_{\mathcal{N}} \cdots \alpha_{\mathcal{N}}} \epsilon_{\alpha_1 \cdots \alpha_{\mathcal{N} - \mathcal{N}} \beta_{\mathcal{N}} \cdots \beta_{\mathcal{N}}} \\
\left[ A_1(t) \right]^{\beta_1}_{\alpha_1} \cdots \left[ A_1(t) \right]^{\beta_{n_1}}_{\alpha_{n_1}} \left[ A_2(t) \right]^{\beta_{n_1 + 1}}_{\alpha_{n_1 + 1}} \cdots \left[ A_2(t) \right]^{\beta_{n_1 + n_2}}_{\alpha_{n_1 + n_2}} \cdots \left[ A_m(t) \right]^{\beta_{\mathcal{N} - n_m}}_{\alpha_{\mathcal{N} - n_m}} \cdots \left[ A_m(t) \right]^{\beta_{\mathcal{N}}}_{\alpha_{\mathcal{N}}}
= (-)^\mathcal{N} \frac{\Pi_i n_i!}{\mathcal{N}!} \langle R_{\{n_1, \ldots, n_m\}} \rangle ,
\]  

(28)

where \( R_{\{n_1, \ldots, n_m\}} \) is the generalization of \( R_n \) to the \( m \)-species system. The \( R_{\{n_1, \ldots, n_m\}} \) satisfy a set of recursion relations such as

\[
R_{\{n_1 + 1, n_2, \ldots, n_m\}} = \langle R_{\{n_1, n_2, \ldots, n_m\}} \rangle A_1 - \mathcal{N} R_{\{n_1, n_2, \ldots, n_m\}} A_1 \\
+ \langle R_{\{n_1 + 1, n_2 - 1, \ldots, n_m\}} \rangle A_2 - \mathcal{N} R_{\{n_1 + 1, n_2 - 1, \ldots, n_m\}} A_2 + \ldots \\
+ \langle R_{\{n_1 + 1, n_2, \ldots, n_k - 1, \ldots, n_m\}} \rangle A_k - \mathcal{N} R_{\{n_1 + 1, n_2, \ldots, n_k - 1, \ldots, n_m\}} A_k + \ldots \\
+ \langle R_{\{n_1 + 1, n_2, \ldots, n_m - 1\}} \rangle A_m - \mathcal{N} R_{\{n_1 + 1, n_2, \ldots, n_m - 1\}} A_m ,
\]  

(29)

which are an obvious generalization of eq. (20). These can be written more compactly as

\[
R_{\{n + 1\}} = \sum_{j=1}^{m} \langle R_{\{n + 1 - j\}} \rangle A_j - \mathcal{N} R_{\{n + 1 - j\}} A_j ,
\]  

(30)

As may be guessed from the complexity of the descending recursion relations in the two-species case, the descending recursion relations in the \( m \)-species case are quite unpleasant, and we do not present them.

V. SINGLE SPECIES MULTI-MESON SYSTEMS BEYOND \( n = 12 \)

Calculations using a single source for quark propagators are limited to systems involving \( n \leq 12 \pi^+ \)'s. Systems comprised of \( n > 12 \pi^+ \)'s can be studied by computing light-quark propagators produced from more than one source. For instance, systems with \( n \leq 24 \pi^+ \)'s can be studied by working with light-quark propagators produced from two different sources and, more generally, systems with \( n \leq 12 p \pi^+ \)'s can be studied by working with light-quark propagators produced from \( p \) different sources. It is then obvious that to study a system of 240 \( \pi^+ \)'s will require the calculation of light-quark propagators from 20 different sources.

A. Single Species Multi-Meson Systems from Two Sources

Instead of considering propagators from only a single source point at \((0, 0)\), we can consider propagators from two source points at \((y_1, 0)\) and \((y_2, 0)\). A correlation function for a system of \( n = n_1 + n_2 \pi^+ \)'s with \( n_1 \) emanating from \((y_1, 0)\) and \( n_2 \) emanating from \((y_2, 0)\) is

\[
C_{\{n_1\pi_1^+, n_2\pi_2^+\}}(t) = \left\langle \left( \sum_x \pi^+(x, t) \right)^{n_1+n_2} \left( \pi^-(y_1, 0) \right)^{n_1} \left( \pi^-(y_2, 0) \right)^{n_2} \right\rangle.
\]  

(31)
After contracting the quark field operators, the correlator can be written as

\[ C_{(n_1 \pi^+_L, n_2 \pi^-_L)}(t) = \frac{n!}{2 \pi} \langle (\bar{\pi} P_1(t) \eta)^{n_1} (\bar{\pi} P_2(t) \eta)^{n_2} \rangle, \]  

(32)

where \( \eta \) is now a 24-component Grassmann variable, corresponding to the 12-components of the up-quark field at position \((y_1, 0)\) and the 12-components of the up-quark field at position \((y_2, 0)\). The \(24 \times 24\) matrices

\[
\begin{align*}
P_1 &= \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ 0 & 0 \end{pmatrix}, & P_2 &= \begin{pmatrix} 0 & 0 \\ A_{21}(t) & A_{22}(t) \end{pmatrix},
\end{align*}
\]

(33)

are constructed from the \(12 \times 12\) matrices

\[
A_{ij} = \sum_x S(x, t; y_j, 0) S^\dagger(x, t; y_i, 0),
\]

(34)

with \(i, j\) denoting the propagator source locations. It is straightforward to show that

\[
C_{(n_1 \pi^+_L, n_2 \pi^-_L)}(t) = (-)^{\pi} \frac{n!}{(N - \pi)!} \epsilon^{a_1 \cdots a_n \pi a_{n+1} \cdots a_m} \epsilon_{a_1 \cdots a_m \pi b_{n+1} \cdots b_m} [P_1(t)]^{b_1}_{a_1} \cdots [P_1(t)]^{b_n}_{a_n} [P_2(t)]^{b_{n+1}}_{a_{n+1}} \cdots [P_2(t)]^{b_m}_{a_m}
\]

\[
= (-)^{\pi} \frac{n!}{n C_{n_1}} \langle Q_{(n_1, n_2)} \rangle,
\]

(35)

where we have defined \(N = 2N = 24\). It is obvious that if either \(n_1 > N\) or \(n_2 > N\), then the correlation function vanishes. The \(Q_{(n_1, n_2)}\) are \(N \times N\) matrices (that are time-dependent), and satisfy the recursion relation

\[
Q_{(n_1+1, n_2)} = \langle Q_{(n_1, n_2)} \rangle P_1 - (n_1 + n_2) Q_{(n_1, n_2)} P_1 + \langle Q_{(n_1+1, n_2-1)} \rangle P_2 - (n_1 + n_2) Q_{(n_1+1, n_2-1)} P_2,
\]

(36)

and a similar relation for \(Q_{(n_1, n_2+1)}\). The boundary condition for the recursion relation in eq. (36) is

\[
Q_{(1,0)} = P_1, \quad Q_{(0,1)} = P_2, \quad \langle Q_{(0,0)} \rangle = 1,
\]

(37)

with \(Q_{(j,k)} = 0\) if either \(j < 0\) or \(k < 0\). This recursion relation is somewhat less obvious than those that describe systems with a single light-quark propagator, and it is worth demonstrating its implementation. For the \(n_1 + n_2 = 2\) systems, the recursion relation gives

\[
Q_{(2,0)} = \langle Q_{(1,0)} \rangle P_1 - Q_{(1,0)} P_1
\]

\[
= \left( \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \right) - \left( \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \right)
\]

\[
= \left( A_{11}^2 - A_{11} \right) \left( A_{11}^2 - A_{11} \right),
\]

\[
\langle Q_{(2,0)} \rangle = \langle A_{11} \rangle^2 - \langle A_{11}^2 \rangle.
\]

(38)
\[ Q_{(0,2)} = \langle Q_{(0,1)} \rangle P_2 - Q_{(0,1)} \ P_2 \]
\[ = \left( \begin{array}{cc} 0 & 0 \\ A_{21} & A_{22} \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ A_{21} & A_{22} \end{array} \right) - \left( \begin{array}{cc} 0 & 0 \\ A_{21} & A_{22} \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ A_{21} & A_{22} \end{array} \right) \]
\[ = \left( \begin{array}{cc} 0 & 0 \\ A_{22} & A_{21} - A_{22} A_{21} \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ A_{22} & A_{22} - A_{22}^2 \end{array} \right) , \]
\[ \langle Q_{(0,2)} \rangle = \langle A_{22} \rangle^2 - \langle A_{22}^2 \rangle . \] (39)

\[ Q_{(1,1)} = \langle Q_{(0,1)} \rangle P_1 - Q_{(0,1)} \ P_1 + \langle Q_{(1,0)} \rangle P_2 - Q_{(1,0)} \ P_2 \]
\[ = \left( \begin{array}{cc} 0 & 0 \\ A_{21} & A_{22} \end{array} \right) \left( \begin{array}{cc} A_{11} & A_{12} \\ 0 & 0 \end{array} \right) - \left( \begin{array}{cc} 0 & 0 \\ A_{21} & A_{22} \end{array} \right) \left( \begin{array}{cc} A_{11} & A_{12} \\ 0 & 0 \end{array} \right) \]
\[ + \langle \left( \begin{array}{cc} A_{11} & A_{12} \\ 0 & 0 \end{array} \right) \right) \left( \begin{array}{cc} 0 & 0 \\ A_{21} & A_{22} \end{array} \right) - \left( \begin{array}{cc} 0 & 0 \\ A_{21} & A_{22} \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ A_{21} & A_{22} \end{array} \right) \]
\[ = \left( \begin{array}{cc} 0 & 0 \\ A_{22} & A_{21} - A_{12} A_{21} \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ A_{22} & A_{22} - A_{22} A_{22} \end{array} \right) , \]
\[ \langle Q_{(1,1)} \rangle = 2 \left[ \langle A_{22} \rangle \langle A_{11} \rangle - \langle A_{12} A_{21} \rangle \right] . \] (40)

The result for \langle Q_{(1,1)} \rangle obtained in eq. (40) exhibits the expected result when the second source is set to be identical to the first, reproducing the single source result multiplied by the combinatoric factor of \( 2\ C_1 \). Repeated application of the recursion relation generates all contractions possible from the two sources. In the case of three mesons, we find that
\[ \frac{1}{3C_2} \langle Q_{(2,1)} \rangle = [(A_{11})^2 \langle A_{22} \rangle - \langle A_{11}^2 \rangle \langle A_{22} \rangle + 2 \langle A_{11} A_{12} A_{21} \rangle - 2 \langle A_{12} A_{21} \rangle \langle A_{11} \rangle] , \] (41)
which recovers the single-source result when \( y_2 = y_1 \).

**B. Single Species Multi-Meson Systems from \( m \) Sources**

The extension of the two-source result in eq. (36), which provided a way to explore systems comprised of up to \( n \leq 24 \) \( \pi^+ \)'s, to systems generated with \( m \)-sources can be achieved with a similar construction. The correlation function for a system of \( \pi = \sum_i n_i \pi^+ \)'s is
\[ C_{(n_1 \pi^+_1, \ldots, n_m \pi^+_m)}(t) = \left( \sum \pi^+(x,t) \right) \left( \pi^-(y_1,0) \right)^{n_1} \ldots \left( \pi^-(y_m,0) \right)^{n_m} , \] (42)
which is equal to
\[ C_{(n_1 \pi^+_1, \ldots, n_m \pi^+_m)}(t) = \pi^! \{ \eta P_1 \eta \}^{n_1} \ldots \{ \pi^! P_m \eta \}^{n_m} , \] (43)
where the \( \eta \) are now \( m \times N \) component Grassmann variables and the
\[ P_k = \left( \begin{array}{cccc} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ A_{k1}(t) & A_{k2}(t) & \ldots & A_{km}(t) \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 \end{array} \right) , \] (44)
with the $A_{ij}(t)$ defined in eq. (34). This can be expressed as

$$C_{\{n_1 \pi_1^+, \ldots, n_m \pi_m^+\}}(t) = (-)^\pi \frac{\pi!}{(N - \pi)!} \varepsilon^{\alpha_1 \cdots \alpha_\pi \beta_1 \cdots \beta_{\pi - 1} \cdots \beta_{\pi - 2} \cdots \beta_1 \cdots \alpha_1} \epsilon_{\alpha_\pi \cdots \alpha_n \beta_{n+1} \cdots \beta_\pi}
$$

\[
\begin{align*}
[ P_1(t) ]_{\alpha_1} \cdots [ P_1(t) ]_{\alpha_n} [ P_2(t) ]_{\alpha_{n+1}} \cdots [ P_2(t) ]_{\alpha_{n+2}} \cdots [ P_m(t) ]_{\beta_1} \cdots [ P_m(t) ]_{\beta_\pi} \\
= (-)^\pi \left( \prod_i n_i! \right) \langle Q_{(n_1 \pi_1^+, \ldots, n_m \pi_m^+)} \rangle ,
\end{align*}
\]

(45)

where $N = m \cdot N$. The $Q_{(n_1 \pi_1^+, \ldots, n_m \pi_m^+)}$ satisfy the recursion relation

\[
Q_{(n_1 \pi_1^+, \ldots, n_m \pi_m^+)} = \langle Q_{(n_1 \pi_1^+, \ldots, n_m \pi_m^+)} \rangle P_1 - \pi Q_{(n_1 \pi_1^+, \ldots, n_m \pi_m^+)} P_1 \\
+ \langle Q_{(n_1 \pi_1^+, \ldots, n_k \pi_k^-, \ldots, n_m \pi_m^+)} \rangle P_k - \pi Q_{(n_1 \pi_1^+, \ldots, n_k \pi_k^-, \ldots, n_m \pi_m^+)} P_k \\
+ \langle Q_{(n_1 \pi_1^+, \ldots, n_m \pi_m^+ - 1)} \rangle P_m - \pi Q_{(n_1 \pi_1^+, \ldots, n_m \pi_m^+ - 1)} P_m ,
\]

(46)

which can be written in a more concise way as

$$Q_{(n_{1+k})} = \sum_{i=1}^{m} \langle Q_{(n_{1+k} - 1)} \rangle P_i - \pi Q_{(n_{1+k} - 1)} P_i ,
$$

(47)

where $(n) = (n_1, n_2, \ldots, n_m)$, and $(n + 1 \cdot k) = (n_1, n_2, \ldots, n_k + 1, \ldots, n_m)$, etc. The recursion relation in eq. (46) allows for the calculation of systems involving large numbers of $\pi^+$'s. As an example, the application of the recursion relation to the contraction for the $3\pi^+$ systems resulting from 3 different sources reproduces the correct result of

\[
\frac{1\!1\!1\!1\!1}{3!} Q_{(1,1,1)} = \left[ \langle A_{11} \rangle \left\langle A_{22} \right\rangle \left\langle A_{33} \right\rangle \\
- \langle A_{12} A_{21} \rangle \left\langle A_{33} \right\rangle - \langle A_{13} A_{31} \rangle \left\langle A_{22} \right\rangle - \langle A_{23} A_{32} \rangle \left\langle A_{11} \right\rangle \\
+ \langle A_{12} A_{23} A_{31} \rangle + \langle A_{13} A_{32} A_{21} \rangle \right] ,
\]

(48)

and recovers the single-source result when $y_3 = y_2 = y_1$.

VI. K SPECIES MULTI-MESON SYSTEMS FROM M SOURCES

In this section, we generalize the results of the previous sections to the correlation functions of systems composed of arbitrary numbers of species of mesons with the quantum numbers of $\bar{q}_i \gamma_5 u$ for $q_i \neq u$ (for instance, systems comprised of $\pi^+$'s, $K^+$'s, $D^0$'s, $B^+$'s) resulting from an arbitrary number of light-quark sources. A correlation function for a system composed of $n_{ij}$ mesons of the $i^{th}$ species from the $j^{th}$ source at $(y_j, 0)$, where $0 \leq i \leq k$ and $0 \leq j \leq m$, is of the form

\[
C_n(t) = \left\langle \left( \sum_x A_1(x, t) \right)^{N_1} \ldots \left( \sum_x A_k(x, t) \right)^{N_k} \left( A_{11}^\dagger(y_1, 0) \right)^{n_{11}} \ldots \left( A_{1m}^\dagger(y_m, 0) \right)^{n_{1m}} \ldots \left( A_{k1}^\dagger(y_1, 0) \right)^{n_{k1}} \ldots \left( A_{km}^\dagger(y_m, 0) \right)^{n_{km}} \right\rangle ,
\]

(49)
where \( N_i = \sum_j n_{ij} \) is the total number of mesons of species \( i \), and the subscript in \( C_n(t) \) labels the number of each species from each source,

\[
\mathbf{n} = \begin{pmatrix} n_{11} & n_{12} & \ldots & n_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ n_{k1} & n_{k2} & \ldots & n_{km} \end{pmatrix}.
\] (50)

The \( A_i(y, t) \) are defined immediately after eq. (26). It is straightforward to show that

\[
C_n(t) = \left( \prod_i \mathcal{N}_i! \right) \left\langle \prod_{i,j} (\bar{\eta} P_{ij} \eta)^{n_{ij}} \right\rangle,
\] (51)

where the \( \eta \) are \( m \times N \)-component Grassmann variables, and the \( P_{ij} \) are \( \bar{N} \times \bar{N} \) dimensional matrices, where \( \bar{N} = m \times N \), which are generalizations of the \( P_j \) defined in eq. (44) with an additional species index, \( i \). They are defined as

\[
P_{ij} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (A_i)_{j1}(t) & (A_i)_{j2}(t) & \ldots & (A_i)_{jm}(t) \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}.
\] (52)

These correlators can be expressed as

\[
C_n(t) = (-)^{\bar{N}} \frac{\prod_i \mathcal{N}_i!}{(\bar{N} - \bar{N})!} \epsilon_{\alpha_1 \ldots \alpha_{\bar{N}} \beta_1 \beta_{n_{11}} \beta_{n_{11}+1} \ldots \beta_{n_{1m}}} \epsilon_{\alpha_1 \ldots \alpha_{\bar{N}} \beta_1 \beta_{n_{11}} \beta_{n_{11}+1} \ldots \beta_{n_{k,m+1}}} [ P_{11}(t) ]_{\alpha_1}^{\beta_1} \ldots [ P_{11}(t) ]_{\alpha_{n_{11}}}^{\beta_{n_{11}}} \ldots [ P_{km}(t) ]_{\alpha_{n_{k,m+1}}}^{\beta_{n_{k,m+1}}} [ P_{km}(t) ]_{\alpha_1}^{\beta_1} = (-)^{\bar{N}} \frac{\prod_i \mathcal{N}_i! \left( \prod_{i,j} n_{ij}! \right)}{\bar{N}!} \langle T_n \rangle,
\] (54)

where \( \bar{N} = \sum_i \mathcal{N}_i \) is the total number of mesons in the system, with \( \bar{N} \leq \bar{N} \). The \( T_n \) satisfy the recursion relation

\[
T_{n+1,rs} = \sum_{i=1}^{k} \sum_{j=1}^{m} \langle T_{n+1,rs-1,ij} \rangle P_{ij} - \bar{N} T_{n+1,rs-1,ij} P_{ij},
\] (55)

where we have introduced the notation

\[
\mathbf{1}_{ij} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}.
\] (56)
where the non-zero value is in the \((i, j)\)th entry. Defining \(U_j = \sum_i n_{ij}\) to be the number of mesons from the \(j\)th source, it is clear that the correlation function vanishes when \(U_j > N\) for any source \(j\). Eq. (55) is the main result of this work. It allows the correlation functions of essentially arbitrary meson systems to be evaluated. The correlation function defined in eq. (49) can accommodate a total of \(12L^3\) mesons where \(L\) is the number of lattice sites in each spatial direction of the lattice. This filled system would correspond to a total meson density of \(1/b^3\) where \(b\) is the lattice spacing. To go to even higher densities, sources (and sinks) must be placed on multiple time-slices.

VII. DISCUSSION

In this work, we have developed recursion relations that enable the calculation of the correlation function of a system composed of arbitrary numbers of mesons of different species generated from quark propagators originating from different sources. These recursion relations will allow for Lattice QCD calculations of many-body systems that will elucidate the phase transitions (or cross-overs) that are expected to exist in QCD at finite meson density [7, 10], and will also allow for the exploration of systems at high density. Further, they will allow for fixed-density calculations in multiple lattice volumes, thereby providing a means to control the finite-volume systematic effects of such calculations.

The recursion relations scale to very large meson number and enable the calculation of the correlation functions of systems composed of (more precisely, with the quantum numbers of) large numbers of mesons of different species, which are presently not practical to evaluate. A further advantage of the recursive construction is that it significantly reduces the overall computational cost. Each application of the recursion requires only a single matrix multiplication for each type of meson (or source) involved and a few additional scalar operations. The memory requirements are also modest. In contrast, the expressions for the fully evaluated contractions (as displayed in Ref. [4] for the single species, single source case) contain a number of terms that grows exponentially in the number of mesons, with an exponent that rapidly increases with the complexity of the system (number of sources or species of meson). For the single source, single species case, the two methods are comparable, but for more complicated systems, the recursive approach requires fewer operations to evaluate the corresponding correlation functions.

The recursive method can also be applied to other types of meson systems such as those involving annihilation type diagrams (for example, multiple \(\pi^0\) systems). However, the construction of the equivalent of the \(A_{ij}(t)\) objects defined above is computationally expensive. In the case of baryons, or mixed meson-baryon systems, recursive relations exist, but are much more difficult to generalize. This is currently under investigation.

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