The Partition Function Of Argyres-Douglas Theory On A Four-Manifold

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ABSTRACT: Using the $u$-plane integral as a tool, we derive a formula for the partition function of the simplest nontrivial (topologically twisted) Argyres-Douglas theory on compact, oriented, simply connected, four-manifolds without boundary and with $b_2^+ > 0$. The result can be expressed in terms of classical cohomological invariants and Seiberg-Witten invariants. Our results hint at the existence of standard four-manifolds that are not of Seiberg-Witten simple type.

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1. Introduction And Conclusion

One of the great moments in the history of Physical Mathematics is Witten’s formulation \[39\] of the Seiberg-Witten invariants of four-manifolds together with his proposal for how to express the Donaldson invariants of four-manifolds in terms of the Seiberg-Witten invariants - a result known to four-manifold theorists as “the Witten conjecture.” The introduction of Seiberg-Witten invariants led to rapid progress in the theory of four-manifolds. See Donaldson’s review for a masterful account \[11\]. Nevertheless, many interesting questions in the field remain open \[14, 30, 35\].

Given Witten’s remarkable application of the basic Seiberg-Witten solution of pure \(SU(2)\) \(N_f = 2\) supersymmetric Yang-Mills theory \[31\] to four-manifold theory, one naturally asks whether topological twisting of other supersymmetric quantum field theories will lead to other new four-manifold invariants. One evident hunting ground in the search for such new invariants is the set of topologically twisted four-dimensional \(N = 2\) theories. This was, in fact, the main motivation for works such as \[24\]. While far from definitive,
the main conclusion of [24] was that, for Lagrangian theories, the partition functions of the
topologically twisted theories, while intricate and interesting, will nevertheless be ex-
pressible in terms of the classical topological invariants and Seiberg-Witten invariants of
a four-manifold. This narrows the search for new invariants to non-Lagrangian supercon-
formal theories. Again, this was the motivation for [23, 24]. Those papers again failed
to discover new invariants, but they did manage to show that the very existence of super-
conformal theories is related to nontrivial sum rules on the Seiberg-Witten invariants, now
known as the “superconformal simple type condition.”

The superconformal theory used in [25, 26] is the simplest nontrivial Argyres-Douglas
theory and is denoted here as AD3 (it is sometimes also denoted as the $(A_1, A_2)$ theory).
It arises in the Coulomb branch of pure $SU(3)$ SYM [4] and in the Coulomb branch of
$SU(2)$ SYM coupled to a single hypermultiplet in the fundamental representation [8].
The present paper completes the story of [25, 26] by giving an explicit formula for the partition
function of the topologically twisted AD3 theory on compact, oriented, four-manifolds
without boundary, henceforth denoted by $X$, in the case that $b_1(X) = 0$ and $b_2^+(X) > 0$.
Four-manifolds of this type that satisfy the further condition that $b_2^+(X) > 1$ are typically
referred to as standard four-manifolds. We will argue that the partition function of the
twisted AD3 theory on standard four-manifolds can, once again, be expressed using the
classical topological invariants and the Seiberg-Witten invariants. Our proposal for this
partition function on standard four-manifolds is given by equation (6.28). While this
partition function does not provide new four-manifold invariants it might nevertheless be
useful. For example, when $X$ is of Seiberg-Witten simple type ¹ the formula simplifies
dramatically to equation (6.30). (See Section 6.3 for a discussion of this simplification.)
This brings into sharp focus the distinction between those manifolds of Seiberg-Witten
simple type and those hypothetical manifolds that are not of Seiberg-Witten simple type.
In particular, equation (6.30) has the remarkable property that the 0-observable is a “null-
vector” in the sense that insertions of this operator always lead to zero correlation function.
That property is not true of the more general expression (6.28). Strangely enough, all
known standard four-manifolds are of Seiberg-Witten simple type. Why this should be so
is mysterious, at present. There ought to be a good physical reason why the zero observable
is a null vector. In the absence of such a reason one must suspect that, actually, there do
exist standard four-manifolds that are not of Seiberg-Witten simple type.

As explained in Section 7, on manifolds with $b_2^+(X) = 1$ the topologically twisted AD3
partition function is, in fact, not a diffeomorphism invariant, but varies continuously
with the metric. This is probably a general feature of twisted superconformal field theories.
The basic reason that lies behind this failure of the general expectations of topological field
theory was first noted for the $SU(2)$, $N_f = 4$ theory in [27].

Our result does not imply that other topologically twisted $N=2$ theories won’t lead
to new four-manifold invariants, although it might dim the ardor of those in pursuit of
such new invariants. Whether or not other theories lead to new invariants, the computa-

¹A standard four-manifold $X$ is said to be of Seiberg-Witten simple type if the Seiberg-Witten invariant
associated to a spin-c structure is only nonvanishing when the moduli space of solutions to the Seiberg-
Witten equations is of dimension zero. For mathematical discussions see [11, 29].
tion of these partition functions remains an interesting challenge for the future. Among other things, it would be of interest to apply our methods to the other theories with one-dimensional Coulomb branches described in [1, 3, 7].

Our basic line of reasoning is the following. From reference [8] we know that in the $SU(2)N_f = 1$ theory at a special value of the quark mass, $m = m^*$, the IR physics near a special vacuum $v = v^*$ is described by the AD3 theory. Moreover there is no noncompact Higgs branch for the $SU(2)N_f = 1$ theory so if we take the $m \to m^*$ limit of the partition function we should be able to extract the AD3 partition function. In section 5 below we will make this reasoning a little more precise, and Appendix B carries out the procedure in great detail. Our derivation makes use of a very general relation of $u$-plane integrands to total derivatives. Motivated by a recent paper of Korpas and Manschot [19] we make some comments on the extent to which one can write the $u$-plane integrand as a total derivative in Appendix A.

While we find the physical argument we will give compelling our proposal is nevertheless conjectural. We can only offer some fairly limited evidence that it is correct: First, we remind the reader that - as already observed long ago in [25, 26] - the very existence of the twisted partition function in the limit $m \to m^*$ is rather nontrivial. The most striking new piece of evidence is that the correlation functions satisfy the $U(1)_R$-charge selection rules expected for the topologically twisted $AD3$ theory on $X$ [17, 24, 33]. However, since the background charge was in fact determined from the behavior of the $u$-plane measure in the first place this is not really a very strong piece of evidence. In section 7 we give very explicit formulae for the continuous metric dependence when $b_2^+ (X) = 1$. If it could be checked more directly that would be very helpful. Indeed, any direct checks of (or counterarguments against!) our proposal would be most welcome.

Even within the extremely limited context of the correlators for twisted AD3 theory we have left many unanswered questions. The extension to manifolds with $b_1 \neq 0$ is of some interest for two reasons. First, in this case the 3-form descendent of the 0-observable has negative ghost number and hence the ghost number selection rule admits the possibility that there is an infinite number of nonzero correlation functions, in strong contrast to the simply-connected case. Moreover, non-simply connected manifolds are probably best suited for comparison with the approach to computing topologically twisted $d=4 N=2$ partition functions suggested in [17]. The extension of our computations (and indeed of the original computations in [27]) to the case $b_2^+ = 0$ should be quite interesting. We must note that there is some tension between our conjecture and some remarks in [3] (located between their equations (5.14) and (5.16)) so further study of the $b_2^+ = 0$ case is called for.

Finally, the microscopic interpretation of our partition function in terms of moduli spaces

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As we review in section 8 below, the discussion of Shapere and Tachikawa [33] expresses the $U(1)_R$ background charge in terms of the conformal anomaly coefficients $a, c$. These authors then derived the values of $a, c$ for the AD3 theory from the anomalous behavior of the $u$-plane measure. Although we do not directly use the $U(1)_R$ anomaly in our derivation of the AD3 correlators, our selection rule can be traced to the $U(1)_R$ dependence of the $u$-plane measure, so it is hardly surprising that if we input the Shapere-Tachikawa values of $a, c$ into the general formula for the $U(1)_R$ background charge we rederive our selection rule! Fortunately, there are some other discussions of these coefficients [1, 24, 31] which are asserted to be logically independent of the computation of Shapere and Tachikawa.
of traditional partial differential equations is an interesting open problem. In principle one should be able to translate our definition (5.5) into some subtle aspect of intersection theory on the moduli space of the nonabelian monopole equations, but we suspect there is a more compelling formulation.

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2. SU(2) $N_f = 1$ and AD3

Here we review some well-known facts. See [36] for a more extensive discussion.

The Seiberg-Witten curve $\Sigma$ for $SU(2)$ theory coupled to a single hypermultiplet in the fundamental was first presented in [32]. The class S presentation is [15]:

$$\lambda_1^2 = \left( \frac{\Lambda_1^2}{2} + 3u + 2\Lambda_1 mz + \Lambda_1^2 z^2 \right) \left( \frac{dz}{z} \right)^2$$

(2.1)

where $\Lambda_1$ is the UV scale, $m$ is the mass of the hypermultiplet, $u$ is a coordinate on the Coulomb branch, and $z \in C \cong \mathbb{C}^*$ is a coordinate on the UV curve. The Seiberg-Witten curve is a subset of $T^*C$ where the restriction of the canonical Liouville one-form on $T^*C$ to $\Sigma$ is the canonical Seiberg-Witten differential.

As observed in [8] when $m = \frac{3}{2}\omega\Lambda_1$, with $\omega$ a third root of unity, three branch points of the curve collide and the discriminant of the curve has a multiple zero. For definiteness we consider the limiting behavior as $m \to m_* := \frac{3}{2}\Lambda_1$, so the discriminant has a double zero at $u = u_* := \Lambda_1^2$ where two roots $u_{\pm}(m)$ collide. To define a scaling limit we change variables:

$$m = \frac{3}{2}\Lambda_1 + \delta m \quad u = \Lambda_1^2 + \delta u \quad z = -1 + \tilde{z}$$

(2.2)

define

$$\tilde{z} = -\epsilon z_{AD}$$

$$4\Lambda_1 \delta m - 3\delta u = 3\epsilon^2 \Lambda_1^2$$

$$2\Lambda_1 \delta m - 3\delta u = -\epsilon^3 u_{AD}$$

$$\lambda = \epsilon^{5/2} \lambda_{AD}$$

(2.3)

and take $\epsilon \to 0$ holding all quantities with subscript $AD$ fixed. The result is the AD3 family of curves as used in [15]:

$$\lambda^2 = (z^3 - 3\Lambda^2 z + u)(dz)^2 .$$

(2.4)
Note that in the above equation we have dropped the subscript $AD$ on $\lambda, z, \Lambda, u$ to avoid clutter. We will continue to do this in what follows, and we trust which $u$-plane is meant will be clear from context. We sometimes write $\Lambda_{AD}$ when we wish to emphasize that it is the mass deformation in the $AD3$ theory but often we simply write this as $\Lambda$ to avoid clutter. The superconformal point is described by the limit $\Lambda_{AD} \to 0$ at the origin of the $AD3$ Coulomb branch $u = 0$.

A key point made in [8] is that at the points $u_\pm(m)$ of the $SU(2)$, $N_f = 1$ family, hypermultiplets with mutually nonlocal charges become massless. Therefore, when $m \to m_*$ and $u \to u_*$ there are massless nonlocally related particles and the low energy effective theory cannot be a Lagrangian field theory. It is, in fact, the $AD3$ theory weakly coupled to other degrees of freedom in the $SU(2), N_f = 1$ theory.

We remark that the $AD3$ theory was first discovered at a point in the Coulomb branch of pure $SU(3) \to 2$ SYM [4]. However, in that case the $U(1)$ flavor symmetry associated with the mass parameter is gauged and therefore integrated over. For our purposes it is much better to keep it as a free parameter.

3. $u$-Plane Integrals

A systematic derivation of the Witten conjecture of four-manifold theory (equation (2.14) of [39]) relating the Donaldson and Seiberg-Witten invariants was presented in [27]. It involves an integral over the Coulomb branch of the $SU(2) N_f = 0$ theory. $^3$ It is referred to informally as the “$u$-plane integral.” For additional background and discussion of the $u$-plane integral see [19, 22, 28]. The original discussion of [27] applied just to $SU(2)$ Yang-Mills coupled to $N_f \leq 4$ fundamental hypermultiplets or one adjoint hypermultiplet, but in fact the measure makes sense for any one-dimensional Coulomb branch. $^4$ Although the integral is, conceptually, best written as an integral over the $u$-plane, the path integral derivation leads more naturally to an integral over a special coordinate $a$ so that it becomes

$$Z_u = K_u e^{2\pi i \lambda_0^2} \int da d\bar{a} A^\lambda B^\sigma e^{2pu + S^T(a)} \Psi$$

(3.1)

$$\Psi := \sum_{\lambda \in \lambda_0, \infty + \Gamma} e^{2\pi i (\lambda - \lambda_0, \infty) \cdot \xi_\infty} N_\lambda$$

(3.2)

$$N_\lambda := \frac{S^2}{d\bar{a}} e^{\frac{S^2}{2\pi y} \left( \frac{du}{da} \right)^2} e^{-i \pi \sigma^2 - i \pi \tau^2 - i \frac{du}{da} S \cdot \lambda_+} \left[ \lambda_+ + \frac{i}{4\pi y} \frac{du}{da} S_+ \right]$$

(3.3)

Our notation is the following:

1. $p$ is a fugacity conjugate to the insertion of the 0-observable $\mathcal{O} = u$ in the twisted partition function. $S \in H_2(X; \mathbb{Z})$ is a homology class and determines a canonical

$^3$Mathematically rigorous proofs of the Witten conjecture have been given in [14] for complex algebraic manifolds and in [13] for all standard four-manifolds of Seiberg-Witten simple type.

$^4$What is far less obvious is whether the measure is single-valued on the $u$-plane and whether the integral over the $u$-plane is well-defined for other families of Seiberg-Witten curve and differential.
2-observable $\mathcal{O}(S) := \int_S K^2 u$, via the descent formalism [27]. Here $K$ is a one-form supercharge such that $[K, \mathcal{Q}] = d$. The expression $Z_u$ should be viewed as a formal power series in $p, S$ and it is the contribution of the Coulomb branch to the correlation function

$$\langle e^{p\mathcal{O} + \mathcal{O}(S)} \rangle$$

in the twisted theory on $X$.

2. The measure factors are

$$A := \alpha \left( \frac{du}{da} \right)^{1/2} \quad B := \beta \Delta^{1/8}$$

and correspond to the terms in the low energy effective action on the Coulomb branch describing the coupling of the $U(1)$ vectormultiplet to the Euler character $\chi$ and the signature $\sigma$. Here $\Delta = \prod_s (u - u_s)$ is a holomorphic function with first order zeroes at the discriminant locus $\{u_s\}$ where a hypermultiplet becomes massless. The factors $\alpha, \beta$ are independent of $u$ but can depend on the theory, the scale $\Lambda$, and the masses. In principle they could also vary nontrivially on the the conformal manifold in the superconformal case. For example, detailed analysis of the mass-deformed $N = 2^*$ theory strongly suggests that they depend on $\tau_0$ in that case [21].

It would be very interesting to clarify this last point and understand their dependence on the conformal manifold in general class S theories.

3. $a$ is a special coordinate suitable to a duality frame at $u \to \infty$. It is the period of the Seiberg-Witten differential on a cycle that is invariant (up to a sign) under the path $u \to e^{2\pi i u}$ at large $|u|$. Once we choose a B-cycle we have $\tau = x + iy$, decomposed in terms of real and imaginary parts. In the case of $SU(2)$ with $N_f < 4$ this is a frame in which $y = \text{Im} \tau \to \infty$ as $u \to \infty$.

4. The lattice $\Gamma := \text{H}^2(X; \mathbb{Z})/\text{Tors}$, where Tors is the torsion subgroup of $\text{H}^2(X; \mathbb{Z})$. The sum $\Psi$ is, essentially, the classical partition function of the $U(1)$ gauge field on the four-manifold. We think of $\Gamma$ as embedded in the quadratic vector space $\text{H}^2(X; \mathbb{R})$. We have introduced a shift $\lambda_{0,\infty}$ and a phase $\xi_{\infty}$. In the case of $SU(2)$, $N_f = 0$ we have $\lambda_{0,\infty} = \frac{1}{2} w_2(P)$ where $P$ is a principal $SO(3)$ bundle, and $\xi_{\infty} = \frac{1}{2} w_2(X)$ and the overline denotes an integral lift. When $\tau \to \infty$ as $u \to \infty$, $2\xi$ must be a characteristic vector on $\Gamma$ for the measure to be well-defined. In this case we can write $\lambda = v + \lambda_{0,\infty}$, $v \in \Gamma$ and the phase

$$e^{2\pi i (\lambda - \lambda_{0,\infty}) \cdot \xi} = (-1)^v w_2(X)$$

In the $SU(2)$ $N_f > 0$ case we must take $w_2(P) = w_2(X)$ so we should take $2\lambda_{0,\infty} = 2\xi_{0,\infty}$ to be an integral lift of $w_2(X)$.

5As pointed out in [33] one should, in general, distinguish the “physical discriminant” from the “mathematical discriminant.” For our main example the two will be equal up to a constant.

6Note that general vectors in the torsor $\lambda_{0,\infty} + \Gamma$ are denoted by $\lambda$. There is an unfortunate clash of notation with the standard notation for the Seiberg-Witten differential. Which one is meant should be clear from context. We will mostly be using $\lambda$ to denote a vector in the cohomology torsor from now on.
5. The \( u \)-plane integrand depends on a choice of Riemannian metric on \( X \), but the dependence only enters through the cohomology class of a self-dual two-form \( \omega \in H^2(X, \mathbb{R}) \), so \( \omega = \ast \omega \). We can normalize it such that \( \int_X \omega^2 = 1 \). When \( b_2^+ = 1 \) there is a Lorentzian signature on the quadratic space \( H^2(X, \mathbb{R}) \), and we must choose a component of the lightcone, which we can call the “forward light cone,” in order to specify \( \omega \) uniquely. Such an \( \omega \) is sometimes called a \textit{period point}. In the path integral derivation of the \( u \)-plane integral one must integrate over the fermion zeromodes and this requires a choice of orientation of the vector space \( H^0(X; \mathbb{R}) \oplus H^1(X; \mathbb{R}) \oplus H^{2,+(X; \mathbb{R})} \). For \( SU(2), N_f = 0 \) this corresponds nicely to Donaldson’s discussion of orientations of instanton moduli space \([10]\). For \( H^1 = 0 \) and \( b_2^+ = 1 \) such an orientation amounts to a choice of “forward” component of the light-cone. Finally, we define \( \lambda_+ := \lambda \cdot \omega \) and \( \lambda_- := \lambda - \lambda_+ \omega \).

6. \( T(a) \) depends on the choice of duality frame and is known as a “contact term.” It is given by

\[
T(a) = -\frac{1}{24} E_2(\tau) \left( \frac{du}{da} \right)^2 + H(u)
\]

where it is claimed in \([27]\) that \( H(u) = u/3 \) for all \( N_f < 4 \). For systematic treatments of such contact terms in twisted four-dimensional \( N=2 \) theories see \([12, 22, 23]\). Several arguments show that for the \( AD3 \) theory \( H(u) = 0 \).

7. All anti-holomorphic dependence of the integrand on \( \bar{u} \), and all metric dependence of the integrand is subsumed in the expression \( N_\lambda \).

8. \( K_u \) is a numerical normalization. In TFT path integrals carry a canonical normalization: They count solutions to equations. However, the correct normalization factor for the \( u \)-plane integral is not obvious. Note, for example that there is an extra factor of \( e^{2\pi i \lambda_{b,\infty}} \) which could have been absorbed into \( K_u \), but we leave it this way because a shift of \( \lambda_{b,\infty} \) corresponds to a change of sign of Donaldson polynomials familiar from Donaldson theory. Note that for \( b_2^+ = 1 \) and \( b_1 = 0 \) we have \( \chi + \sigma = 4 \). Therefore there is some ambiguity in how we normalize \( K_u, \alpha, \beta \). Since

\[
K_u \alpha^\chi \beta^\sigma = (K_u \kappa^{-4}) (\kappa \alpha)^\chi (\kappa \beta)^\sigma
\]

for any nonzero constant \( \kappa \). Thus we must regard the normalization constants

\[
(K_u, \alpha, \beta) \sim (K_u \kappa^{-4}, \kappa \alpha, \kappa \beta)
\]

as equivalent.

9. The \( u \)-plane integral is quite subtle and requires careful definition. The integrand is typically quite singular at points \( u = u_s \) in the \( u \)-plane corresponding to zeroes of the discriminant. The procedure for defining the integral, outlined in \([27]\), is to cut out a small disk around \( u_s \), perform the angular integral and then take the radius to zero.
10. For some four-manifolds $X$ it is possible to write the integrand of the $u$-plane integral as a total derivative on the Coulomb branch and evaluate the integral as a sum of contours around the singular points. This kind of representation will be important to our extraction of the AD3 contributions to the $u$-plane integral. For details see Appendix A.

11. We have written the $u$-plane integral in a form that applies to any one-dimensional Coulomb branch. In this paper we will apply it to the $SU(2), N_f = 1$ family (2.1) and the AD3 family (2.4), in which case we will write $Z_{SU(2), N_f = 1}^u$ and $Z_{AD3 \text{ family}}^u$, respectively. As mentioned above, it would be quite interesting to investigate the integral for the other one-dimensional Coulomb branches described in [5, 6, 7].

4. The Topological Partition Function

The full partition function of the topologically twisted theory on $X$:

$$Z = \langle e^{pu + O(S)} \rangle,$$  \hspace{1cm} (4.1)

where $O(S) = \int_S K^2 u$ is the canonical 2-observable associated to $S$, is a sum of the $u$-plane integral together with contributions that guarantee that the contribution of the vacua near $u \cong u_s$ gives a topologically invariant answer - up to known metric dependence from the region $u \to \infty$. It is logically possible that in theories other than $SU(2)$ coupled to matter, new four manifold invariants other than the Seiberg-Witten invariants (but with exactly the same wall-crossing behavior when $b^+_2(X) = 1$) can be used to achieve this topological invariance. However, especially for theories such as the AD3 theory which appear in the IR limit of Lagrangian theories, we find this exceedingly unlikely. In any case, proceeding using the basic logic of [27] the Seiberg-Witten invariants are sufficient to do the job. The full partition function can therefore be written as:

$$Z = Z_u + Z_{SW} \tag{4.2}$$

where

$$Z_{SW} = \sum_s Z(u_s) \tag{4.3}$$

and the sum over $s$ is a sum over the discriminant locus of the family of Seiberg-Witten curves. When the family of elliptic curves in a neighborhood of $u_s$ is of Kodaira type $I_1$ (i.e., the discriminant has a first order zero at $u = u_s$ while the Weierstrass invariants $g_2, g_3$ are nonzero at $u_s$) the method used in [27] can be applied to derive

$$Z(u_s) = \left( \sqrt{32\pi K_u^2} \beta^{\sigma} \alpha^\chi \right) \cdot \left( e^{2\pi i (\lambda_{0,s} - 2i \xi_s)} \eta_s \right) \sum_{\lambda \in \lambda_{0,s} + \Gamma} e^{2\pi i \lambda \cdot \xi_s} (-1)^{n(\lambda)} SW(\lambda). \tag{4.4}$$

\[
\left[ \left( \frac{a_s}{q_s} \right)^{1/4} \frac{du}{dq_s} \left( \frac{\Delta}{q_s} \right)^{\sigma/8} \left( \frac{da_s}{du} \right)^{1-\chi/2} \right] e^{2pu + S^2 T_s(a_s) + \frac{\beta}{2\pi} \frac{du}{dq_s}} \frac{1}{q_s} \right] q_s^{2}\]
Here $\lambda_{0,s}$ and $\xi_s$ are the theta characteristics resulting from the duality transformation applied to $\Psi$ in the neighborhood of $u_s$. Similarly, $\eta_s$ is a root of unity arising from the multiplier system in that duality transformation. The expression only makes sense for $\lambda_{0,s} = \frac{1}{2}w_2(X)$ so that the sum on $\lambda$ can be interpreted as a sum over the characteristic class of spin-c structures on $X$. Then $\text{SW}(\lambda)$ is the corresponding Seiberg-Witten invariant associated with the Seiberg-Witten moduli space of real dimension $2n(\lambda)$ where

$$n(\lambda) = \frac{1}{2}\lambda^2 - \frac{\sigma}{8} - \chi_h$$

and $\chi_h := (\chi + \sigma)/4$. (For a complex surface $\chi_h$ is the holomorphic Euler characteristic.)

The special coordinate $a_s$ vanishes at $u_s$ and the coordinate $q_s = e^{2\pi i \tau_s} \to 0$ as $u \to u_s$.

The contact term $T_s(a_s)$ is obtained from $T(a)$ by duality transformation.

5. Deriving The AD3 Partition Function

In order to extract the partition function of the AD3 theory from that of the $SU(2), N_f = 1$ theory we use the following principles:

1. The limit of $Z^{SU(2),N_f=1}$ as $m \to m_*$ must exist since there are no noncompact Higgs branches. (Noncompact Higgs branches are the only source of IR divergences given that $X$ is compact and the contribution from $u \to \infty$ is finite.)

2. The resulting path integral must be an integral over all $Q$-invariant field configurations.

3. According to [8] those $Q$-invariant configurations include the supersymmetric “states” of the AD theory, perhaps coupled to other degrees of freedom in the $SU(2)$ theory. However, at $m = m_*$ those couplings should be arbitrarily weak in the scaling region of the $u$-plane near $u_*$.

4. We can therefore isolate the AD configurations by focusing on the contribution from an infinitesimally small neighborhood of the colliding singularities $u_{\pm}(m)$ plus the SW contributions associated with just those points.

When $m = m_*$ the family (2.1) has a singularity at $u = u_*$, where two singularities $u_{\pm}(m)$ have collided, and another singularity $u_0$ far away from the scaling region. Since the definition of the integral requires a subtle regularization over the noncompact regions it turns out that:

$$Z^{SU(2),N_f=1}_u = \int dudu \lim_{m \to m_*} \left( \frac{d}{du} A^* B^e e^{2\mu u} + S^2 T(u)\Psi \right)$$

has a nonzero Laurent expansion in powers of $\mu^{1/4}$ around $\mu = 0$ where $\mu := (m - m_*)/\Lambda_1$.

Here the integral of the $m \to m_*$ limit of the integrand of $Z^{SU(2),N_f=1}_u$ is defined by cutting

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7 $(2\lambda)$ is the characteristic class of the spin-c structure, modulo torsion.
out disks around \( u_0 \) and \( u_* \) and taking the limit as the disks shrink. The singular terms in the expansion \((5.1)\) will cancel against similar singular terms from \( Z_{SW} \). The constant term (i.e. the coefficient of \( \mu^0 \)) is in general nonzero and does not cancel against the constant term from \( Z_{SW} \).

The quantity \((5.1)\) comes from the integration around an infinitesimal region near \( u = u_* \). Indeed, for any \( \epsilon > 0 \) let \( B(\epsilon; u_*) \) be a disk around \( u_* \) with \( |u - u_*| < \epsilon \). When \( m \) is sufficiently close to \( m_* \) the two colliding singularities \( u_\pm(m) \) will be inside this disk.

Therefore, for any fixed \( \epsilon > 0 \):

\[
\lim_{m \to m_*} \int_{C-B(\epsilon; u_*)} d\bar{u} d\bar{u} \left( \frac{da}{du} \right)^2 A^\lambda B^\sigma e^{2\mu u + S^2 T(u)} \Psi = \\
\int_{C-B(\epsilon; u_*)} d\bar{u} d\bar{u} \lim_{m \to m_*} \left( \frac{da}{du} \right)^2 A^\lambda B^\sigma e^{2\mu u + S^2 T(u)} \Psi
\]

(5.2)

Therefore, in view of the limiting behavior reviewed in section 2 we should attribute the difference \((5.1)\) to the contribution of the AD partition function on four-manifolds with \( b_2^+ = 1 \).

In the \( SU(2) \) \( N_f = 1 \) family when \( m \to m_* \) and \( u \to u_* \) the AD3 theory is still weakly coupled to other degrees of freedom in the original gauge theory. The detailed considerations of Appendix A and Appendix B show that we should extract a factor \( \exp[2p(u_* + \frac{2}{3} \mu) + S^2 T_*] \) to account for these couplings. Here and henceforth we will choose units so that \( \Lambda_1 = 1 \). The peculiar shift by \( 2\mu/3 \) in the coefficient of \( p \) is due to the linear combinations (2.4). Thus we consider the constant term in the Laurent expansion around \( \mu = 0 \):

\[
\left[ e^{-2p(u_* + \frac{2}{3} \mu) - S^2 T_*} \left( Z_{SU(2), N_f = 1} - \int d\bar{u} d\bar{u} \lim_{m \to m_*} \left( A^\lambda B^\sigma e^{2\mu u + S^2 T(u)} \Psi \right) \right) \right]^{\mu^0} \quad (5.3)
\]

Again, the detailed considerations of Appendix A and Appendix B strongly motivate the following conjectures:

1. The constant term in \((5.3)\) is in fact a polynomial in \( p \) and \( S \), in striking contrast to the partition functions of Donaldson-Witten theory. We will denote it by \( P_1(p, S) \).

2. Furthermore, if we define a grading of the polynomial \( P_1(p, S) \) by “\( R \) charge” with \( R[p] = 6 \) and \( R[S] = 1 \) then the highest degree is given by \( 6\ell + r = 28 := -\frac{1}{4}(7\chi + 11\sigma) \).

3. If one considers the \( u \)-plane integral for the \( AD3 \) family \((2.4)\) it has a similar expansion in powers of \( \Lambda_{AD}^{1/2} \) around \( \Lambda_{AD} = 0 \) and the constant term \( P_{AD}(p, S) \) is also a polynomial in \( p \) and \( S \).

4. Finally, defining \( P_{1, top}^1(p, S) \) be the sum of terms with maximal \( R \)-charge we have:

\[
P_{1, top}^1(p, S) = N P_{AD}(n_0 p, n_2 S)
\]

(5.4)

for suitable constants \( N, n_0, n_2 \).

\footnote{We interpret the terms of lower \( R \)-charge in the polynomial \( P_1(p, S) \) as effects arising from the coupling of the AD3 theory to other degrees of freedom in the \( SU(2) \) \( N_f = 1 \) theory. It would certainly be useful to understand the physics of the lower order terms better.}
The results of Appendix A and Appendix B are enough to prove all the above claims for the difference of u-plane integrals for any two choices of metric. Moreover, they establish the above claims absolutely when $X$ has a homotopy type so that the $u$-plane integral has a vanishing chamber in the sense explained in section 5 of [27].

These considerations motivate our central formula for how to extract the physics of the AD3 theory from the expansion around $\mu = 0$ of the $SU(2)$, $N_f = 1$ partition function:

$$Z_{AD} := \left[ e^{-2p(u_++\frac{2}{3}\mu)} - S^2T \left( Z_u^{SU(2),N_f=1} - \int d\bar{u} \lim_{m \to m^*} A^x B^x e^{2p\bar{u} + S^2T(u)} \right) \right]_{\mu^0}^\text{top}$$

$$+ \left[ e^{-2p(u_++\frac{2}{3}\mu)} - S^2T \left( Z_{SW}^{SU(2),N_f=1}(u_+(m)) + Z_{SW}^{SU(2),N_f=1}(u_-(m)) \right) \right]_{\mu^0}^\text{top}$$  (5.5)

On the other hand, a very natural way to define the partition function of the AD3 theory is to use directly the family of curves (2.4) and define:

$$Z_{AD} := \lim_{\Lambda_{AD} \to 0} \left[ Z_u^{AD_{\text{family}}} + Z_{SW}^{AD_{\text{family}}} \right]$$  (5.6)

We conjecture that, up to an overall constant and a renormalization of $p$ and $S$ as in (5.4), we have $Z_{AD} = Z_{AD}$. Again, a full proof of this statement follows from the considerations of Appendix A and Appendix B, if we consider the difference of partition functions for two metrics, or if we consider a homotopy type of $X$ admitting a vanishing chamber. Moreover, if $b^+_2 > 1$ then the statement is an easy consequence of the relationship of the two curves described in section 4.

Our main conjecture is that

$$Z_{AD} = \langle e^{p\mathcal{O}(S)} \rangle$$  (5.7)

for the topologically twisted AD3 theory on four-manifolds $X$ with $b^+_2 > 0$.

6. The SW Contribution To $Z_{AD}$

When $X$ has $b^+_2 > 1$ only $Z_{SW}$ contributes to the partition function. In this section we will evaluate it fairly explicitly for the AD3 family (2.4) in the limit $\Lambda_{AD} \to 0$. Thus we are starting from the definition (5.6).

6.1 A General Simplification Of $Z(u_s)$

To begin we put (4.4) in a form which is more suitable for explicit evaluation. In fact our derivation of the result (6.15) below applies to any family of elliptic Seiberg-Witten curves with a simple zero of the discriminant at $u = u_s$ such that the Weierstrass invariants $g_2, g_3$ are nonzero at $u = u_s$. (This is Kodaira type $I_1$.) We also assume $\lambda_{0,s} = \xi_s = \frac{1}{2}w_2(X)$. This holds for the $SU(2)$, $N_f = 1$ family and therefore for the AD3 family. Moreover, the duality transformations needed to transform from the duality frame at $u = \infty$ to $u$ near $u_s$
are all, according to equation (11.17) of [32], conjugate to $T$. It turns out that the measure of the $u$-plane transforms by a character under $S$ and $T$. Therefore the root of unity $\eta_s$ is independent of $s$ and we will just denote it by $\eta$.

Now we can replace the sum over $\lambda$ by the average over $\lambda$ and $-\lambda$. Because $\lambda_{0,s} = \xi_s = \frac{1}{2} w_2(X)$ we have

$$e^{-4\pi i \lambda \xi_s} = e^{-2\pi i (\nu + \frac{1}{2} w_2) w_2} = e^{-i\pi w_2^2} = (-1)^{\sigma}.$$  

Moreover it is a standard result of Seiberg-Witten theory that

$$\text{SW}(-\lambda) = (-1)^{\chi_h} \text{SW}(\lambda)$$

so in the sum over $\lambda$ in (6.4) we can freely make the replacement:

$$\text{SW}(\lambda)e^{2\pi i \lambda \nu_0} e^{-i\lambda \cdot S \frac{da}{du}} \to \frac{1}{2} \text{SW}(\lambda)e^{2\pi i \lambda \nu_0} \left( e^{-i(\frac{da}{du}) S \lambda} + (-1)^{\chi_h + \sigma} e^{i(\frac{da}{du}) S \lambda} \right)$$  

The reason this is useful is that the expansion in $S \cdot \lambda$ only involves powers of $(\frac{da}{du})$ of a definite parity independent of $\lambda$. That will be important since, as we will see below, we can readily determine the $q_s$-expansion of $(\frac{da}{du})^2$ near $u_s$, but taking the square-root could be tricky. Equation (6.3) motivates us to define:

$$\frac{1}{2} \left( e^{-i(\frac{da}{du}) S \lambda} + (-1)^{\chi_h + \sigma} e^{i(\frac{da}{du}) S \lambda} \right) := \sum_{n \geq 0} \hat{c}_n^{\chi_h + \sigma} (S) \left( \frac{da}{du} \right)^{-n}$$

with

$$\hat{c}_n^{\chi_h + \sigma} (S) = \begin{cases} e^{-i\pi n/2} (S \cdot \lambda)^n & n = (\chi_h + \sigma) \mod 2 \\ 0 & n \neq (\chi_h + \sigma) \mod 2 \end{cases}$$

Now suppose we have a SW curve presented in the form:

$$y^2 = x^3 + A_2 x^2 + A_4 x + A_6$$

and there is a special coordinate $a_s$ so that $a_s \to 0$ but

$$\frac{da_s}{du} = \frac{\rho}{\pi} \omega_1$$

is nonvanishing as $q_s = e^{2\pi i \tau_s} \to 0$.  

In order to evaluate (6.4) we need to know the expansions

$$u = u_s + \mu_1 q_s + \mu_2 q_s^2 + \cdots$$

$^9$Here $\rho$ is a relative normalization between the standard periods $\omega$ of the elliptic curve and $\frac{da_s}{du}$. Its value depends on the conventions used to normalize the central charge. In the conventions of [33], the central charge is $Z(\gamma) = \pi^{-1} \int L_1 \lambda$, so $\frac{da}{du} = \pi^{-1} \int L_1 \frac{ds}{dz}$. Next, for an elliptic curve presented in the form (6.4), the canonically normalized holomorphic differential is $\sqrt{2} \frac{dz}{\omega_1}$. Finally, we note that for the AD3 family (2.4) we have $\frac{da}{du} = \frac{1}{2} \frac{ds}{dy}$. We thus conclude that for natural conventions for class $S$ we have $\rho = 1/\sqrt{8}$. However, we leave $\rho$ undetermined above since it is different if one uses other conventions such as those of [22] or [27]. The results for different choices of $\rho$ are simply related by a renormalization of $S$. 

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\[ a_s = \kappa_1 q_s + \kappa_2 q_s^2 + \cdots \] (6.9)

We now show how to extract these expansions - in principle - from the SW curve.

From \( A_2, A_4, A_6 \) we can construct the standard Weierstrass invariants \( g_2, g_3 \). For \( SU(2) \) theories and the AD3 family these will be polynomials in \( u \). In general we have

\[ (12)^3 \frac{g_2^3}{g_2^4 - 27g_3^2} = j(\tau_s) = q_s^{-1} + 744 + 196884q_s + 21493760q_s^2 + \cdots \] (6.10)

Actually, for our purposes, this equation is more usefully written as

\[ (27) \frac{g_2^3}{g_2^4} = \frac{E_6^2}{E_4^3} \] (6.11)

Plugging (6.8) into either version gives a triangular system of equations from which we can extract the coefficients \( \mu_n \).

Next, if we have chosen a basis so that \( \tau_s = \omega_2/\omega_1 \) then the period \( \omega_1 \) is expressed in terms of coefficients of the elliptic curve and \( \tau_s \) by

\[ \omega_1^2 = 2 \left( \frac{\pi}{3} \right)^2 \frac{E_6(\tau)}{E_4(\tau)} \cdot \frac{g_2}{g_3} \] (6.12)

and hence

\[ \left( \frac{da_s}{du} \right)^2 = 2 \left( \frac{\rho}{3} \right)^2 \frac{E_6(\tau)}{E_4(\tau)} \cdot \frac{g_2}{g_3} \] (6.13)

Now we use the standard expansions of \( E_4, E_6 \) in terms of \( q_s \) and we expand the polynomials \( g_2, g_3 \) of \( u \) around \( u_s \) and use (6.8). This gives \( \kappa_1^2 \) and all the \( \kappa_n/\kappa_1 \) for \( n > 1 \).

We also write \( \Delta = N^{\mathfrak{u}}_{\text{math}} \Delta^{\text{math}} \) where \( \Delta^{\text{math}} \) is the mathematical discriminant of the elliptic curve,

\[ \Delta^{\text{math}} = (e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2 = 4(4g_2^3 - 27g_3^2) = 2^{-22} \left( \frac{da_s}{du} \right)^{-12} \eta(\tau_s)^{24} \] (6.14)

where \( e_i, i = 1, 2, 3 \) are the roots of the cubic. Putting all these things together we can write (4.4) in the form

\[ Z_{SW}^s = \left( \sqrt{32\pi} K_0^\beta \alpha^\lambda 2^{-11\sigma/4} (N^{\mathfrak{u}}_{\text{math}})^{\sigma/8} \eta \right) \sum_{n \geq 0} \sum_{\lambda \in \mathbb{Z}_{\omega_2 + \Gamma}} e^{\pi i \lambda \cdot w_2} (-1)^n(\lambda) \cdot \hat{c}^{\chi_{h+\sigma}}(S) \cdot \left[ \frac{du}{dq_s} \left( \frac{\eta(\tau_s)^{24}}{q_s} \right)^{\sigma/8} \left( \frac{a_s}{q_s} \right)^{\chi_h - 1} \left( \frac{da_s}{du} \right)^{1-2\chi_h - \sigma - n} e^{2p_u + S^2 T(s)(a_s) q_s^{-n}(\lambda)} \right] \] (6.15)

This result is a slight generalization of, and improvement upon, equation (11.28) of \textbf{[27]}.

### 6.2 Specializing To The AD3 Family

We now specialize (6.15) to the AD3 family of curves.
For the $AD3$ family we have $g_2 = 3\Lambda^2$ and $g_3 = -u/2$, $\Delta$ is quadratic in $u$ and so there are just two singularities $u_{\pm}$. Near each of them we have the expansion in $q_s$:

$$u = u_s \frac{E_6}{E_4^{3/2}}$$

(6.16)

with $u_s = 2\zeta_s \Lambda^3$. Here $\zeta_s = \pm 1$ at the two singularities and $E_6, E_4$ are power series in $q_s$ beginning at 1. Fractional powers of Eisenstein series are to be interpreted as power series in $q_s$. Note that

$$\frac{du}{dq} = -\zeta_s (12\Lambda)^3 (q^{-1} \eta^{24}) \cdot E_4^{-5/2}$$

(6.17)

From (6.13) we obtain $\kappa_1^2 = -\frac{1}{2} (12\Lambda)^3 \zeta_s$ and

$$\tilde{E}_1(q) := \frac{a_s/\kappa_1}{q} := 1 + \sum_{n \geq 2} \frac{\kappa_n}{\kappa_1} q^{n-1}$$

(6.18)

is independent of $s$ and satisfies the equation:

$$q \frac{d}{dq} (q \tilde{E}_1(q)) = \eta^{24} E_4^{-9/4} = (12)^{-3} (E_4^3 - E_6^2) E_4^{-9/4}$$

(6.19)

from which one may generate its $q$-series. There does not appear to be any simple expression for $\tilde{E}_1$ in terms of $E_2$, $E_4$ and $E_6$ and we will, regrettably, take the above as its definition.

Using these formulae and (6.15) one can derive

$$Z_{SW}^{AD3 family} = C_1 \sum_{r_1 \geq 0} \sum_{\lambda} \sum_{\zeta_s = \pm 1} \Lambda^{\lambda/2} \left( r_1 - (\chi_s - c_s^2) \right) e^{i \pi \lambda - w_2 (1)^n \lambda \cdot \text{SW}(\lambda)} \frac{1}{(4\rho)^{r_1} r_1!} \left( \zeta_s^{4(r_1 - \chi_h + \sigma + r_1)} E_4^{\chi_h - 24} \right)$$

$$\cdot E_4^{2(\chi_h + \sigma + 9 + r_1)} \left( \frac{E_4^{2\chi_h + 9 + r_1}}{E_4^{3/2} + (\zeta_s \Lambda^3)^2 E_4^{-3/2}} \right)^{24} \left( \frac{E_4^3 - E_6^2}{E_4^{-9/4}} \right)$$

(6.20)

where $\delta_{r_1}$ enforces the constraint $r_1 = (\chi_h + \sigma) \mod 2$ and

$$C_1 = \left( \sqrt{32\pi} K_u \cdot \beta^* \cdot \alpha^h \eta \right) \left( -i2^{11/2} \gamma^{3/2} \rho^{-3/2} \right)^{2\chi_h} \left( 2^{13/4} \beta^* \eta \cdot 1^{3/2} \rho^{-3/2} \right)^{\alpha \cdot \text{math}}$$

(6.21)

Next, we expand the terms with $p$ and $S^2$ in the exponential. We find that the terms proportional to $(S \cdot \lambda)^{r_1} (S^2)^{r_2} p^\ell$ come with the power $\Lambda^{U/2}$ where

$$U := r_1 + 2r_2 + 6\ell - 2\mathfrak{B} \quad \mathfrak{B} := \chi_h - c_1^2 = -\frac{7\chi + 11\sigma}{4}$$

(6.22)

This is, of course, a reflection of the emergent $U(1)_R$ symmetry at the superconformal point.

---

10In this section we write $\Lambda$ instead of $\Lambda_{AD}$.

11The quantity $\mathfrak{B}$ is very natural in this subject. The quantity $2\mathfrak{B}$ provides a lower bound for the number of Seiberg-Witten basic classes of $X$. \cite{22}. 
Next, the entire dependence of the expression (6.20) on the two values \( s = + \) and \( s = - \) is summarized by the power

\[
\zeta_s^{(\sigma + r_1 - \chi_h) + \ell + r_2},
\]

so the sum over \( \zeta \) imposes the selection rule \( U = 0 \mod 4 \). (This selection rule implies that \( r_1 = (\chi_h + \sigma) \mod 2 \) so we can now drop that constraint.) The result of these considerations is that:

\[
Z_{SW}^{AD3family} = 2C_1 \sum_{U=0 \mod 4} \sum_\lambda \Lambda^{\frac{1}{2}}U e^{i\pi \lambda - w_2 (-1)^{n(\lambda)} \text{SW}(\lambda)} \left( \frac{\sqrt{34S \cdot \lambda}}{4} \right)^{r_1} \frac{(S^2)^{r_2}}{(4\lambda)^{2r_2}} \frac{1}{\ell!}
\]

\[
\left[ \tilde{E}_1^{\chi_h - 1}(q^{-1} \eta^{24})^{1 + \frac{\sigma}{2}} E_2 E_4^{-\frac{1}{2}(9 + U - 5 \chi_h)} E_6^\ell \right] q^{n(\lambda)}
\]

(6.24)

where the first sum is over all integers \( r_1, r_2, \ell \geq 0 \) such that \( U = 0 \mod 4 \).

Now we wish to take the \( \Lambda \to 0 \) limit. We can organize the sum by the degree \( U \). Note that there are potentially negative powers of \( \Lambda \) if \( \mathfrak{B} > 0 \). Nevertheless, the correlators should be finite in the \( \Lambda \to 0 \) limit. This was the original argument of \([25]\) used to derive sum rules on Seiberg-Witten invariants. However, unlike \([23]\), here we are not assuming that \( X \) is of Seiberg-Witten simple type. \(^{12}\)

For any given \( X \) there will be a finite number of sum rules, one for each nonnegative integer \( k \) such that \( k - \mathfrak{B} < 0 \) and \( k = \mathfrak{B} \mod 4 \). For each such \( k \) the sum, for fixed degree \( U = k - \mathfrak{B} \) must vanish. To be concrete:

1. Suppose \( \chi_h - c_1^2 > 0 \) and \( \chi_h - c_1^2 = 0 \mod 4 \). Then

\[
0 = \sum_\lambda e^{i\pi \lambda - w_2 (-1)^{n(\lambda)} \text{SW}(\lambda)} \left[ \tilde{E}_1^{\chi_h - 1}(q^{-1} \eta^{24})^{1 + \frac{\sigma}{2}} E_2 E_4^{-\frac{1}{2}(10 + c_1^2 - 6 \chi_h)} \right] q^{n(\lambda)}
\]

(6.25)

2. Suppose \( \chi_h - c_1^2 > 1 \) and \( \chi_h - c_1^2 = 1 \mod 4 \). Then the \( U = 1 - (\chi_h - c_1^2) \) only gets a contribution from \( r_1 = 1, r_2 = \ell = 0 \) and hence

\[
0 = \sum_\lambda e^{i\pi \lambda - w_2 (-1)^{n(\lambda)} \text{SW}(\lambda)} (S \cdot \lambda) \left[ \tilde{E}_1^{\chi_h - 1}(q^{-1} \eta^{24})^{1 + \frac{\sigma}{2}} E_2 E_4^{-\frac{1}{2}(10 + c_1^2 - 6 \chi_h)} \right] q^{n(\lambda)}
\]

(6.26)

3. Suppose \( \chi_h - c_1^2 > 2 \) and \( \chi_h - c_1^2 = 2 \mod 4 \). Then the \( U = 2 - (\chi_h - c_1^2) \) gets a contribution from \( r_1 = 2, r_2 = \ell = 0 \) and \( r_1 = 0, r_2 = 1, \ell = 0 \) hence

\[
0 = \sum_\lambda e^{i\pi \lambda - w_2 (-1)^{n(\lambda)} \text{SW}(\lambda)}
\]

\[
\left\{ S^2 \left[ \tilde{E}_1^{\chi_h - 1}(q^{-1} \eta^{24})^{1 + \frac{\sigma}{2}} E_2 E_4^{-\frac{1}{2}(11 + c_1^2 - 6 \chi_h)} \right] q^{n(\lambda)} \right. \]

\[
\left. - \frac{1}{2} (-24)^{\chi_h + \sigma + 1} (S \cdot \lambda)^2 \left[ \tilde{E}_1^{\chi_h - 1}(q^{-1} \eta^{24})^{1 + \frac{\sigma}{2}} E_2 E_4^{-\frac{1}{2}(11 + c_1^2 - 6 \chi_h)} \right] q^{n(\lambda)} \right\}
\]

(6.27)

\(^{12}\)“Seiberg-Witten simple type” is often given the acronym SWST below.
4. And so on: We get rather complicated polynomials in $S^2$, and $S \cdot \lambda$ which must vanish. If we assume SWST then only the spin-c structures with $n(\lambda) = 0$ contribute and we get the criteria of [23]. In this case the formulae simplify a lot because all the factors of the form $[E_1^{\chi h - 1} \cdots ]_{q^n(\lambda)}$ can be put equal to 1.

Now we consider the actual value at $\Lambda = 0$. According to our conjecture above, this should give the partition function of topologically twisted $AD3$ theory on standard four-manifolds. Technically, we simply keep the terms above with $U = 0$ so our formula is

$$\langle e^{pO+O(S)} \rangle_{AD3} = 2C_1 \sum_{U=0} \sum_{\lambda} e^{i\pi \lambda \cdot w^2} (-1)^{n(\lambda)} SW(\lambda) \frac{\sqrt{24S \cdot \lambda}}{(4\rho)^{r_1}} \frac{(S^2)^{r_2} (4\rho)^{\ell} \ell!}{(4\rho)^{2r_2 r_2}!} \left[ E_1^{\chi h - 1}(q^{-1}\eta^2)^{1+\frac{3}{2}} E_2^{r_2} E_4^{\frac{1}{2}(9-5\chi h)} E_6^\ell \right]_{q^n(\lambda)}$$

(6.28)

This is the generator of correlation functions of the twisted $AD3$ theory on four-manifolds $X$ with $b_1 = 0$ and $b_2^+ > 1$. It is only nonvanishing for $B = \chi_h - c_1^2 > 0$.

We now assume that $X$ has Seiberg-Witten simple type (SWST) so that only spin-c structures with $n(\lambda) = 0$ contribute. Moreover, we will also assume that $X$ is of superconformal simple type (SCST) with $B \geq 4$. According to [23, 26] this means that

$$\sum_{\lambda} e^{i\pi \lambda \cdot w^2} SW(\lambda)(\lambda \cdot S)^k = 0 \quad 0 \leq k \leq B - 4$$

(6.29)

Therefore, given the constraint $U = 0$ the only terms that can contribute are $r_1 = \chi_h - c_1^2 - 2 = B - 2$, $r_2 = 1$, $\ell = 0$, and $r_1 = \chi_h - c_1^2 = B$, $r_2 = \ell = 0$, and our partition function simplifies to

$$\langle e^{pO+O(S)} \rangle_{AD3} = C_2 \sum_{\lambda} e^{i\pi \lambda \cdot w^2} SW(\lambda) \left[ \frac{B(B - 1)}{24} S^2 (S \cdot \lambda)^B - (S \cdot \lambda)^B \right]$$

(6.30)

and to get the constant we observe that $\Delta^{\text{math}} = -27(u^2 - 2\Lambda^2)$ so $N^{u}_{\text{math}} = -1/27$. After some computation we find:

$$C_2 = \frac{\sqrt{128}\pi \eta'}{12!} K_u \left( \frac{32\beta \rho^{3/2} \sqrt{3}/8} \right)^\sigma \left( 2^{9/4} \alpha \rho^{3/2} \right)^\chi$$

(6.31)

where $\eta'$ is an eighth root of unity we have not determined. (One could probably use the fact that $SU(2) N_f = 1$ theory is time-reversal invariant for $\Lambda$ real to constrain this phase.)

6.3 Discussion: Seiberg-Witten Simple Type vs. Superconformal Simple Type

A striking property of (6.30) is that it does not depend at all on $p$. This comes about because when the condition $U = 0$ is combined with the SCST condition, the only solutions have $\ell = 0$. This means the 0-observable is a “null vector.” That is, insertions of $O$ into correlators always vanish for such four-manifolds. Although it is certainly true that $O_{\text{classical}} = 0$ it is not obvious why this should be true in the quantum theory and this leads us to take seriously the possibility that there might be standard four-manifolds that are
not of SWST. Indeed, if we drop the SCST condition \((6.29)\) there are many more solutions to \(U = 0\), i.e. \(r_1 + 2r_2 + 6\ell = \mathfrak{B}\) which will contribute to \((6.28)\). Some will include \(\ell \neq 0\), and we cannot use the necessary conditions \((6.25), (6.26), (6.27)\), et. seq. to eliminate the \(p\)-dependence.

At this point it is important to recall that reference \[25\] derived necessary conditions for the finiteness of the \(\Lambda \to 0\) limit (these are the conditions \((6.25), (6.26), (6.27)\), et. seq. above in the special case of SWST). These conditions are quite complicated so the authors of \[25\] also formulated the SCST condition, namely, that either \(\mathfrak{B} \leq 3\) or \((6.29)\) holds. The SCST condition is a sufficient condition for finiteness of the \(\Lambda \to 0\) limit. The authors of \[25\] then checked that all known (as of 1998) standard four-manifolds satisfy the SCST condition and they conjectured that all standard four-manifolds are of SCST. The work of \[16\] gave a different argument that complex algebraic manifolds are of SCST. The work \[13\] shows - subject to an unproven hypothesis - that for all standard four-manifolds, SWST implies SCST. Therefore, (accepting the work of \[13\]), all standard four-manifolds of SWST have the property that the topological correlators are given by \((6.30)\), and, in particular, the 0-observable is a “null-vector.” We reiterate that in the absence of any compelling reason for \(\mathcal{O}\) to be a null-vector, one must suspect that there are in fact standard four-manifolds that are not of Seiberg-Witten simple type.

Witten has pointed out \[40\] that the null-vector property of \(\mathcal{O}\) has an interesting similarity with the appearance of the Newstead-Ramanan conjecture in the framework of two-dimensional nonabelian gauge theory, as described in \[38\] (see section 4.3, especially equation (4.51) of that paper).

7. The \(u\)-plane Contribution To \(Z_{AD}\)

We now turn to the \(u\)-plane integral \(Z_{u}^{AD3Family}\). We will find that, once again, the coefficient of \(\Lambda_0^{AD}\) in the expansion around \(\Lambda_{AD} \to 0\) is a polynomial with terms satisfying the selection rule \(U = 0\). (In particular, it vanishes for manifolds such as \(S^2 \times S^2\) and \(\mathbb{C}P^2\), cases where the corresponding integrals in Donaldson theory are quite interesting.)

As discussed in Appendix A we do not know how to give a general contour integral expression for the result of the \(u\)-plane integral, but one key feature can be immediately noticed: In the AD3 family the \(\tau\)-parameter approaches a finite value \(\tau_*\) as \(u \to \infty\). Just as in case of the \(SU(2), N_f = 4\) theory studied in \[27\] this results in continuous metric dependence: The general arguments for invariance of the topological partition function fail utterly. We expect this to be a generic feature of topologically twisted superconformal partition functions on four-manifolds of \(b_2^+ = 1\).

Note that for the AD3 family, even when \(\Lambda_{AD} \neq 0\) for \(u \to \infty\) we have, in any duality frame

\[
a \to \kappa u^{5/6} + \cdots
\]

\[
a_D \to \kappa \tau_* u^{5/6} + \cdots
\]

where \(\kappa\) is a nonzero constant and \(\tau_*\) is in the \(PSL(2,\mathbb{Z})\) orbit of \(e^{i\pi/3}\). For concreteness, we will choose a frame so that \(\tau_* = e^{i\pi/3}\). This means that \(da/du \sim u^{-1/6} + \cdots\) is not
single-valued on the $u$-plane. It is thus quite nontrivial, and somewhat remarkable, that the $u$-plane measure is in fact well-defined at $u \to \infty$. Nevertheless, one can indeed check that it is well defined by directly making the modular transformation of the integrand by $(TS)^{-1}$. From the physical viewpoint it is quite important that the measure be well-defined on the $u$-plane and not just on some cover.

As explained in Appendix A, it is possible to write the $u$-plane integral as a sum of contour integrals when we consider the difference of integrals for two period points $\omega$ and $\omega_0$. The continuous metric dependence for the AD3 family comes from the contour at $u \to \infty$ and, as explained in Appendix A, this difference can be written as $G_\omega^\infty - G_{\omega_0}^\infty$ where $G_\omega^\infty$ is a contour integral depending only on $\omega$ and not both $\omega, \omega_0$. Using the expansions in (B.17) et. seq. we can be quite explicit. Up to an overall normalization factor we have:

$$G_\omega^\infty = -\int_{\gamma_\infty} \frac{du}{u} u^{-\mathfrak{B}/6} e^{-\frac{w^2}{24} E_2(\tau_*)} \sum_\lambda \left( \int_0^\infty e^{-2\pi t^2} dt \right) e^{-i\pi \tau_* \lambda^2 - iw \cdot \lambda (-1)^{\lambda - \lambda_0} w_2} + 2\pi i \sum_{n=1}^\infty \frac{(i w \cdot \omega y_{\tau_*})^n}{n!} \sum_\lambda H_{n-1}(2\pi \sqrt{y_{\tau_*} \lambda \cdot \omega}) e^{-i\pi \tau_* \lambda^2 - i\pi \tau_* \lambda^2 - iw \cdot \lambda (-1)^{\lambda - \lambda_0} w_2}$$

(7.2)

where $w = \kappa_2 u^{1/6} S$, the constant $\kappa_2$ is given in equation (B.21), and $H_n$ are standard Hermite polynomials.

In particular, if $\sigma = -7$ so $\mathfrak{B} = 0$ then we have a nonzero constant:

$$G_\omega^\infty = -2\pi i \sum_\lambda \left( \int_0^\infty e^{-2\pi t^2} dt \right) e^{-i\pi \tau_* \lambda^2 - iw \cdot \lambda (-1)^{\lambda - \lambda_0} w_2}$$

(7.3)

and if $\sigma = -8$ so $\mathfrak{B} = 1$ then we have a linear function of $S$:

$$G_\omega^\infty = -2\pi \kappa_2 \sum_\lambda \left( \int_0^\infty e^{-2\pi t^2} dt \right) (S \cdot \lambda) e^{-i\pi \tau_* \lambda^2 - iw \cdot \lambda (-1)^{\lambda - \lambda_0} w_2}$$

$$+ \frac{\pi}{\sqrt{y_{\tau_*}}} S \cdot \omega \sum_\lambda e^{-i\pi \tau_* \lambda_i^2 - i\pi \tau_* \lambda_i^2} (-1)^{\lambda - \lambda_0} w_2$$

(7.4)

and so on. Clearly, these expressions depend continuously on the metric and do not vanish as $\omega$ approaches any boundary of the light cone.

8. The $U(1)_R$ Charge Anomaly

There is a simple conceptual reason for the selection rule $U = 0$ we have found: It is the selection rule enforced by the the $U(1)_R$ symmetry of a superconformal theory. As mentioned in the introduction, this is not surprising given the work of [33].

It was pointed out that such $U(1)_R$ selection rules would apply to twisted superconformal correlators in [24] although the background charge for the AD3 theory deduced
from the measure of the $SU(3)$ Coulomb branch was incorrectly stated in that paper to be $-\chi/10$. The correct determination from the measure, expressed in terms of the conformal anomalies $a$ and $c$, was given in [33]. We briefly recall the derivation here.

When an $N = 2$ theory is coupled to external fields the anomaly for the $U(1)_R$ current can be deduced via the descent formalism from an index density in six dimensions. We introduce a $U(1)_R$ symmetry line bundle $\mathcal{R}$ with connection. Let $F_1$ be the fieldstrength of that connection. Similarly we introduce a principal $SU(2)_R$ symmetry bundle $P_R$. Let $E$ denote the associated bundle in the spinor representation. It has a connection with fieldstrength $F_2$. In a Lagrangian theory we can write the relevant index density as:

$$I_6 = \left[ (\text{Tr} \frac{F_1 \otimes 1 + 1 \otimes F_2}{2\pi}) \mathcal{A} \right]_6.$$  

(8.1)

where the trace is taken over the fermionic fields in the $N = 2$ field multiplets with $F_1$ and $F_2$ in the corresponding representation. Expanding this out we get:

$$I_6 = \text{Tr}(T^3_{U(1)}) \frac{c_1(\mathcal{R})^3}{3!} + \text{Tr}(T_{U(1)}T^2_{SU(2)}c_1(\mathcal{R})\text{ch}_2(E)) - \text{Tr}(T_{U(1)})c_1(\mathcal{R})\frac{p_1}{24}$$

(8.2)

where $T_{U(1)}$ is the generator of $U(1)_R$ symmetry and $T_{SU(2)}$ is any generator of the $SU(2)_R$ symmetry.

Now we use the relation between the $U(1)_R$ symmetry anomaly and the $a$ and $c$ coefficients of the stress-tensor correlators, as derived in [2, 3, 20]. These results are based on the structure of superconformal multiplets. (See [18] for a useful discussion.) The result is that

$$\text{Tr}(T_{U(1)})^3 = 48(a - c)$$

(8.3)

$$\text{Tr}\left(T_{U(1)}T^a_{SU(2)}T^b_{SU(2)}\right) = \delta_{ab}(4a - 2c)$$

(8.4)

Substitution into the anomaly polynomial then expresses it in terms of the conformal anomalies $a, c$:

$$I_6 = 2(a - c)\left(4c_1(\mathcal{R})^3 - c_1(\mathcal{R})p_1\right) + 2(2a - c)c_1(\mathcal{R})\text{ch}_2(E)$$

(8.5)

and the corresponding background charge computed via the descent formalism is:

$$\Delta T_{U(1)} = (a - c) \int_X \left(12c_1(\mathcal{R})^2 - p_1\right) + 2(2a - c) \int_X \text{ch}_2(E).$$

(8.6)

Now, all three quantities $a, c$ and $I_6$ make sense in all $N = 2$ theories, and in particular in non-Lagrangian theories. It is therefore natural to postulate that the expression for the anomaly polynomial (8.5) holds universally for all $N = 2, d = 4$ theories. We will adopt this hypothesis. The computation in this paper can be viewed as a nontrivial check that the hypothesis is correct.

\footnote{If $w_2(P_R)$ is nonzero we can make appropriate modifications by working in the adjoint representation. But this normalization is the most convenient.}
Now in a twisted $N = 2$ theory we have an isomorphism $E \cong S^+$, but

$$\int_X \text{ch}_2(S^\pm) = \frac{3\sigma \pm 2\chi}{2}$$  \hspace{1cm} (8.7)

and on any oriented four-manifold $\int_X p_1 = 3\sigma$. Putting these facts together we recover the result of [33] that in a topologically twisted theory:  \hspace{1cm} 14

$$\Delta T_{U(1)} = (2a - c)\chi + \frac{3}{2}c\sigma$$  \hspace{1cm} (8.8)

Plugging in the values [1, 33, 41] $a = 43/120$ and $c = 11/30$ leads to the specific result:

$$\Delta T_{U(1)} = \frac{7\chi + 11\sigma}{20}.$$  \hspace{1cm} (8.9)

The sum of this value with the $R$-charges of the observables must vanish. The $U(1)_R$ charge of the canonical 0-observable is $6/5$ and hence that of the 2-observable $O(S)$ is $1/5$. \hspace{1cm} 15 Therefore, dividing the selection rule $U = 0$, as found in our computations above, by 5 gives the expected $U(1)_R$ symmetry selection rule:

$$\frac{6}{5}n_0 + \frac{1}{5}n_1 + \frac{2}{5}n_2 - \frac{3}{5}n_3 = \frac{\chi h - c^2}{5} = -\frac{1}{20}(7\chi + 11\sigma)$$  \hspace{1cm} (8.10)

in perfect harmony with (8.9).

Remark: When $b_1$ is nonzero we can also introduce 1- and 3-observables $O(\gamma) = \int_{\gamma} Ku$ and $O(\Sigma) = \int_{\Sigma} K^3 u$, for $\gamma \in H_1(X; \mathbb{Z})$ and $\Sigma \in H_3(X; \mathbb{Z})$, respectively. The selection rule now becomes

$$\frac{12}{10}n_0 + \frac{7}{10}n_1 + \frac{2}{10}n_2 - \frac{3}{10}n_3 = \frac{\chi h - c^2}{5} = -\frac{1}{20}(7\chi + 11\sigma)$$  \hspace{1cm} (8.11)

where $n_k$ is the number of insertions of the $k$-observable. The notable feature here is that the relative minus sign in the sum on the left-hand side allows the possibility of infinitely many nontrivial correlation functions.

A. The $u$-plane Integrand And Total Derivatives

In this appendix we will show that if we consider the difference of two $u$-plane measures at different period points $\omega$ and $\omega_0$ then the measure can naturally be written as a total derivative of a well-defined one-form on the $u$-plane. Our approach here was strongly influenced by the recent paper of Korpas and Manschot [19]. The wall-crossing formula (A.20) below is equivalent to that derived in [27].

Up to an overall constant the measure on the $u$-plane can be written as

$$d\mu_{\text{Coulomb}}^\omega = du\bar{u}\hat{H}\hat{\psi}$$  \hspace{1cm} (A.1)

\hspace{1cm} 14Our normalization of the $U(1)_R$ charge differs by a factor of 2 from that of [33].

\hspace{1cm} 15To see this note that $\lambda$ and $Z$ have $U(1)_R$ charge $+1$, the supersymmetry operator $K$ has charge $-\frac{1}{2}$. 

\hspace{1cm}
where (using $\chi + \sigma = 4$ for $b^+_2 = 1$)

$$\hat{H} = \left( \frac{du}{da} \right)^{1-\sigma/2} \Delta^\sigma/8 e^{2\pi u + S^2 T}$$  \hspace{1cm} (A.2)

is purely holomorphic, $\hat{\Psi} = \frac{da}{du} \Psi$, and the period point $\omega$ is explicitly written in the notation in equation (A.1) because the dependence of the measure on $\omega$ will be important in what follows. We can rewrite $\hat{\Psi}$ in a useful way as follows. Define

$$\rho^\omega_\lambda := \sqrt{\gamma} \lambda - \frac{i}{4\pi} \sqrt{y} S^\lambda \frac{du}{da}$$  \hspace{1cm} (A.3)

so that

$$N_\lambda = -4i \left( \frac{d\rho^\omega_\lambda}{da} e^{-2\pi(\rho^\omega_\lambda)^2} \right) e^{-i\pi \tau \lambda^2 - i S \cdot \lambda} \frac{du}{da}$$  \hspace{1cm} (A.4)

Now define the entire function of $r$ and $b$:

$$E(r; b) := \int_r^b e^{-2\pi t^2} dt$$  \hspace{1cm} (A.5)

We will also denote $E(r) := E(r; 0)$. It follows that we can write

$$\hat{\Psi} = -4i \sum_{\lambda} \left( \frac{d}{du} E(\rho^\omega_\lambda; b_\lambda) \right) e^{-i\pi \tau \lambda^2 - i S \cdot \lambda} \frac{du}{da} (-1)^{(\lambda - \lambda_0, \infty) \cdot \xi}$$  \hspace{1cm} (A.6)

where the lower bounds $b_\lambda$ in the contour integral (A.5) are fairly arbitrary. They can depend on $\lambda$ and $u$, but not on $\bar{u}$.

Given the expression (A.6) and the fact that $\hat{H}$ is holomorphic one is strongly tempted to write the $u$-plane integrand as a total derivative

$$d\mu^\omega_\text{Coulomb} = d\Omega$$  \hspace{1cm} (A.7)

where $\Omega$ is a $(1, 0)$ form:

$$\Omega = -d\bar{u} \hat{H} \tilde{\Theta}$$  \hspace{1cm} (A.8)

In this expression we introduced an indefinite theta function:

$$\tilde{\Theta} = \tilde{\Theta}(\xi, \lambda_0; \tau, z; \{b_\lambda\}) := \sum_{\lambda \in \lambda_0 + \Gamma} E(\rho^\omega_\lambda(z); b_\lambda) e^{-i\pi \tau \lambda^2 - 2\pi i z \cdot \lambda} e^{2\pi i (\lambda - \lambda_0) \cdot \xi}$$  \hspace{1cm} (A.9)

where

$$\rho^\omega_\lambda(z) := \sqrt{y} \left( \lambda - \frac{iz}{2y} \right) \cdot \omega$$  \hspace{1cm} (A.10)

and the lower bounds $\{b_\lambda\}$ should be chosen so that the summation is absolutely convergent. Note that the factor

$$e^{-i\pi \tau \lambda^2} = e^{\pi y \lambda^2} e^{-i\pi y \lambda^2}$$  \hspace{1cm} (A.11)

can potentially lead to an exponential divergence from an infinite sum of vectors $\lambda$ with $\lambda^2 \to +\infty$, so the constants $\{b_\lambda\}$ must be chosen so that the error function decays fast enough to overwhelm this potential divergence.
The one-form in (A.8) uses the function $\tilde{\Theta}$ with $\Gamma = \tilde{H}^2(X) = H^2(X)/\text{Tors}$ and

$$z = \frac{1}{2\pi} S \frac{du}{da}. \quad (A.12)$$

The problem with the expression (A.9) is that there is a conflict between absolute convergence and single-valuedness of $\Omega$ on the $u$-plane. There are choices of $\{b_\lambda\}$, e.g. $b_\lambda = 0$ for which a formal application of the Poisson summation formula would prove that $\Omega$ is single-valued, but such a choice leads to divergences since if we take $b_\lambda = 0$ then

$$E(\rho_{\chi}^\xi) \to \frac{1}{\sqrt{8}} \text{sign}(\lambda_+) \quad (A.13)$$

for $\lambda_+ \to \infty$. In fact, if there were a one form $\Omega = -du\tilde{H}$ such that (A.7) holds and such that $\Omega$ is single-valued on the $u$-plane then we would have

$$0 = \oint d\bar{u} \frac{d}{d\bar{u}} (\tilde{H}F) = \oint d\bar{u} \tilde{H} \Psi \quad (A.14)$$

around any closed path in the $u$-plane. In particular this includes paths along which the monodromy of the local system of electro-magnetic charges is nontrivial. It is easy to check that in general the relevant periods are nonzero.

The situation is quite different if we consider instead the difference of $u$-plane measures for two different metrics with period points $\omega$ and $\omega_0$. Then we can indeed write

$$d\mu_{\text{Coulomb}}^\omega - d\mu_{\text{Coulomb}}^{\omega_0} = d\Omega_{\omega,\omega_0} \quad (A.15)$$

where $\Omega_{\omega,\omega_0}$ is a $(1,0)$ form:

$$\Omega_{\omega,\omega_0} = -du\tilde{H}\tilde{\Theta}_{\omega,\omega_0} \quad (A.16)$$

In this expression we make use of the general indefinite theta series

$$\tilde{\Theta}_{\omega,\omega_0}(\xi, \lambda_0; \tau, z) := \sum_{\lambda \in \lambda_0 + \Gamma} \mathcal{E}(\rho_{\lambda}^\omega(z), \rho_{\lambda_0}^{\omega_0}(z)) e^{-i\pi \tau \lambda^2 - 2\pi i z \cdot \lambda} e^{2\pi i (\lambda - \lambda_0) \cdot \xi} \quad (A.17)$$

It is both absolutely convergent and satisfies the modular transformation properties

$$\tilde{\Theta}_{\omega,\omega_0}(\xi, \lambda_0; \tau + 1, z) = e^{-i\pi \lambda_0^2} \tilde{\Theta}_{\omega,\omega_0}(\xi - \frac{1}{2}(w_2 + 2\lambda_0), \lambda_0; \tau, z) \quad (A.18)$$

$$\tilde{\Theta}_{\omega,\omega_0}(\xi, \lambda_0; -\frac{1}{\tau}, \frac{z}{\tau}) = e^{i\pi/2}(-i\tau)^{d/2} e^{-2\pi i \lambda_0 \xi} e^{-i\pi z^2/\tau} \tilde{\Theta}_{\omega,\omega_0}(\lambda_0, -\xi; \tau, z) \quad (A.19)$$

where $d$ is the rank of $\Gamma$.

Now, in (A.16) we use the function (A.17) with $z$ given in (A.12). It is straightforward to show that $\Omega_{\omega,\omega_0}$ is single-valued on the $u$-plane.

\[\text{We remark that the function (A.17) is similar to, but different from that discussed in [13, 22, 27]. The difference is that in the error function we do not take the imaginary part of } z.\]
It now follows that the difference of \(u\)-plane integrals can be written as:

\[
Z_u^\omega - Z_u^{\omega_0} = 4i e^{2\pi i \lambda_0^2} K_u \int_{u\text{-plane}} d\left( u \hat{H} \Theta^{u,\omega_0} (\xi, \lambda_0; \tau, z) \right)
\]

\[
= 4i e^{2\pi i \lambda_0^2} K_u \lim_{\epsilon \to 0} \left[ \oint_{|u|=1/\epsilon} d\hat{H} \Theta^{u,\omega_0} - \sum_s \oint_{|u-u_s|=\epsilon} d\hat{H} \Theta^{u,\omega_0} \right]
\]  \hspace{1cm} (A.20)

The contour integrals are all oriented counterclockwise. This is a somewhat better way of phrasing the wall-crossing formula presented in [27].

We now use the contour integral representation (A.20) to show that there is a formal series in \(p, S\) expressed as a contour integral, and denoted \(G^{\omega}(p, S)\), or just \(G^{\omega}\), such that

\[
Z_u^\omega - Z_u^{\omega_0} = G^{\omega} - G^{\omega_0} .
\]  \hspace{1cm} (A.21)

The point here is that \(G^{\omega}\) only depends on a single period point, and yet it is expressed as a contour integral. Before deriving (A.21) let us draw from it some useful consequences.

First, (A.21) implies that \(Z_u^\omega = G^{\omega} + \text{constant}\) where the “constant” does not depend on \(\omega\) but can be a power series in \(p\) and \(S\). As we will see in the derivation of (A.21) the formula is only valid when \(\omega\) and \(\omega_0\) are in the same component of the light cone in \(H^2(X, \mathbb{R})\). On the other hand, \(Z_u^\omega\) is defined for \(\omega\) in either component and moreover \(Z_u^{-\omega} = -Z_u^\omega\). Therefore we can conclude that

\[
Z_u^\omega = G^{\omega} + C(p, S)\text{sign}(\omega^t)
\]  \hspace{1cm} (A.22)

where \(C(p, S)\) is independent of \(\omega\) and \(\omega^t\) is the “time component” of \(\omega\).

It is, unfortunately, difficult to give useful explicit expressions for \(C(p, S)\). However, there is one case in which we can be more definitive. For \(X\) of a suitable homotopy type there are vanishing chambers for \(Z_u^\omega\) in the sense explained in sections 5 and 6 of [27]. That is, for any monomial \(p^\ell S^r\) in the power series there is a region \(V_{\ell, r}\) near the boundary of the light cone so that the contribution of \(Z_u^\omega\) to that monomial vanishes for \(\omega \in V_{\ell, r}\). Moreover the regions form an inverse system: There is an ordering so for \((\ell', r') > (\ell, r)\) \(V_{\ell', r'} \subset V_{\ell, r}\). Let \(V\) be the inverse limit of these vanishing chambers so we can say that \(Z_u^\omega = 0\) for \(\omega_0 \in V\). This simply means that the coefficient of any monomial \(p^\ell S^r\) in \(Z_u^{\omega_0}\) vanishes for \(\omega_0 \in V_{\ell', r'}\) for \((\ell', r') > (\ell, r)\). In this sense it follows from (A.21) that

\[
Z_u^\omega = G^{\omega} - G^{\omega_0} \quad \omega_0 \in V .
\]  \hspace{1cm} (A.23)

Therefore for such \(X\) we can express \(Z_u^\omega\) as a sum of contour integrals around the singular points. The class of homotopy types for which this applies is rather broad. It includes rational surfaces and blow-ups of surfaces for suitable choices of \(\lambda_{0, \infty}\). In particular it applies to such manifolds for the main example of this paper, where \(\lambda_{0, \infty} = \xi_{\infty} = \lambda_0 = \frac{1}{2} w_2(X)\).

It remains to prove (A.21). We begin with the contribution of a finite point \(u_s\) and assume that \(\text{Im} \tau_s \to \infty\) and \(da_s/du\) is a finite period as \(u \to u_s\). Then, provided the metric
is generic so that there is no $\lambda$ with $\lambda_+ = 0$, equation (A.17) simplifies and we can replace the difference of error functions by
\[
\frac{1}{\sqrt{8}} \left( \text{sign}(\lambda \cdot \omega) - \text{sign}(\lambda \cdot \omega_0) \right) \tag{A.24}
\]
But now we note that if $\omega$ and $\omega_0$ are in the same component of the lightcone then their time components $\omega^t$ and $\omega_0^t$ have the same sign and hence
\[
\text{sign}(\lambda \cdot \omega) - \text{sign}(\lambda \cdot \omega_0) = 0 \tag{A.25}
\]
when $\lambda^2 \geq 0$. Therefore, in evaluating the residue integral $\Delta Z_{u,s}^{\omega_0}$ around $u_s$ in (A.20) we can replace $\tilde{\Theta}^{\omega,\omega_0}$ by $F_s^{\omega} - F_s^{\omega_0}$ where we define:
\[
F_s^{\omega} := \frac{1}{\sqrt{8}} \sum_{\lambda : \lambda^2 < 0} \text{sign}(\lambda \cdot \omega) e^{-i\pi \tau \lambda^2 - 2\pi i z \cdot \lambda} e^{2\pi i (\lambda - \lambda_0) \cdot \xi} \tag{A.26}
\]
and $z$ is defined as in (A.12). Note carefully that because of the restriction $\lambda^2 < 0$ this sum converges absolutely. Moreover it is a function purely of $\omega$ and not of $\omega_0$. Let $G_s^{\omega}$ be the corresponding contour integral
\[
G_s^{\omega} := \oint_{u_s} du \mathcal{H} F_s^{\omega}. \tag{A.27}
\]
We would like to do something similar to write $\Delta Z_{u,s}^{\omega,\omega_0} = F_s^{\omega} - F_s^{\omega_0}$ but here we have two complications:

1. For $SU(2)$ $N_f < 4$ we have $du/da \to \infty$ as $u \to \infty$.

2. For the conformal theories of interest we have $\tau \to \tau_s$ as $u \to \infty$ and $\text{Im}(\tau)$ does not go to infinity so we cannot replace the error functions by differences of sign functions.

To deal with these complications we note that the $u$-plane integral really only has meaning as a formal power series in $p$ and $S$. Therefore, we should use the expansion of the error function
\[
\mathcal{E}(r + a) = \mathcal{E}(r) - 2\pi e^{-2\pi r^2} \sum_{n=1}^{\infty} \frac{(-2\pi a)^n}{n!} H_{n-1}(2\pi r) \tag{A.28}
\]
where $H_n(x)$ are the standard Hermite polynomials. We apply (A.28) with $r = \sqrt{y} \lambda \cdot \omega$ and $a = -\frac{i}{4\pi \sqrt{y}} \frac{du}{da} S \cdot \omega$. This gives:
\[
\tilde{\Theta}^{\omega,\omega_0}(\xi, \lambda_0; \tau, z) := \sum_{\lambda \in \mathbb{R}^4 + \Gamma} \left[ \mathcal{E}(\sqrt{y} \lambda \cdot \omega) - \mathcal{E}(\sqrt{y} \lambda \cdot \omega_0) \right] e^{-i\pi \tau \lambda^2 - 2\pi i z \cdot \lambda} e^{2\pi i (\lambda - \lambda_0) \cdot \xi} + \sum_{n=1}^{\infty} (\Theta_n^{\omega} - \Theta_n^{\omega_0}) \tag{A.29}
\]
where the $\Theta_n^{\omega}$ come from the $n^{th}$ term in the sum in (A.28) and are absolutely convergent sums on $\lambda$. For a fixed monomial $p^t S^r$ only a finite number of such terms will contribute
so we do not need to worry about the convergence of the sum on \( n \) in \( \sum_n \Theta_n \). Now, since we are considering a contour on a circle whose radius goes to infinity, if \( y \to \infty \) we can replace this expression by \( F_\infty^\omega - F_\infty^{\omega_0} \) where

\[
F_\infty^\omega := \sum_{\lambda \in \lambda_0 + \Gamma, \lambda^2 < 0} \text{sign}(\lambda : \omega) e^{-i \pi \lambda^2 - 2 \pi i \lambda \cdot \lambda_0} e^{2 \pi i (\lambda - \lambda_0) \cdot \xi} + \sum_{n=1}^{\infty} \Theta_n
\]  

(A.30)

is a well-defined function of a single period point \( \omega \).

In the conformal case where \( y \to y_* \) has a finite limit as \( u \to \infty \) we write

\[
E(\sqrt{y} \lambda : \omega) - E(\sqrt{y} \lambda : \omega_0) = E(\sqrt{y} \lambda ; \infty) - E(\sqrt{y} \lambda ; \omega_0; \infty)
\]  

(A.31)

Now we can separate terms and obtain a well-defined function \( F_\infty^\omega \).

Finally, let \( G_\infty^\omega \) denote the contour integral of \( du \hat{H} F_\infty^\omega \) around the circle at infinity and let

\[
G_\infty^\omega := G_\infty^\omega + \sum_s G_s^\omega.
\]  

(A.32)

This completes the proof of (A.21).

B. Detailed Derivation Of The Relation Of \( SU(2), N_f = 1 \) And \( AD3 \) \( u \)-plane Integrals

In this appendix we prove the crucial claims made between equations (5.2) and (5.7) for the difference of \( u \)-plane integrals for different period points \( \omega, \omega_0 \). Using the results of Appendix A we see that if we take the difference of the quantity in equation (5.1) for two period points then it can be written as a sum of contour integrals.

We consider a small disk \( B(u_*; \epsilon) \) of radius \( \epsilon \) around the critical point \( u_* \). Let \( \gamma_\epsilon \) be the counterclockwise oriented boundary. Set \( \Lambda_1 = 1 \) so that \( u_* = 1 \) and define the deviation from the critical mass by \( m = \frac{3}{2} + \mu \). Then we cut out disks of radius \( \delta \), with \( \delta \ll \epsilon \) around the colliding points \( u_\pm \) in the discriminant locus and let \( \gamma_\pm \) be the ccw oriented boundaries of these disks. We are going to prove that

\[
P_1(p, S) := \left[ e^{-2p(u_* + \frac{3}{2} \mu - T_* S^2)} \left( \oint_{\gamma_\epsilon} \Omega - \oint_{\gamma_+} \Omega - \oint_{\gamma_-} \Omega \right) \right] \mu^0
\]  

(B.1)

is a polynomial in \( p \) and \( S \). Here it is understood that we take \( \delta \to 0 \) then \( \epsilon \to 0 \). As mentioned above the quantity in square brackets might have divergent terms for \( \mu \to 0 \). It has a Laurent expansion in \( \mu^{1/4} \) around \( \mu = 0 \). The singular terms will cancel against terms coming from the Seiberg-Witten contribution to the partition function. In any case, our main focus here is on the constant term, i.e. the coefficient of \( \mu^0 \).

Moreover, we will compare the polynomial \( P_1(p, S) \) to the \( u \)-plane contribution for the AD3 theory

\[
P_{AD}(p, S) := \left[ \left( \oint_{\gamma_\infty} \Omega - \oint_{\gamma_+} \Omega - \oint_{\gamma_-} \Omega \right) \right] \Lambda_{AD}^0
\]  

(B.2)
where now $\gamma_{\pm}^{AD}$ are small contours of radius $\epsilon$ around the two points in the AD3 discriminant locus $u_{\pm} = \pm 2A_{AD}^3$. We will show that $P_{AD}(p, S)$ is also a polynomial in $p$ and $S$. Furthermore, if we define a grading of the polynomial $P_1$ by “$R$ charge” with $R[p] = 6$ and $R[S] = 1$ then we will show that the highest degree is given by $6\ell + r = 2B = \frac{1}{4}(7\chi + 11\sigma)$. Finally, defining $P_1^{top}(p, S)$ be be the sum of terms with maximal $R$-charge we will show that

$$P_1^{top}(p, S) = NP_{AD}(n_0p, n_2S)$$

(B.3)

for suitable constants $N, n_0, n_2$.

In the proof it is useful to note that for $b^+_2 = 1$ we have $2B = -\frac{7}{4} - \sigma$ and $1 - \chi/2 = \sigma/2 - 1$ and we recall that, up to an overall normalization we have (A.13) with

$$\Omega = du \left( \frac{du}{da} \right)^{1-\sigma/2} \Delta^{\sigma/8} e^{2pu + S^2T} \bar{\Theta}^{\omega} \omega_0(\lambda_0, \lambda_0; \tau, z)$$

(B.4)

where $\lambda_0 = \frac{1}{2}w_2(X)$. It will be crucial to compare expressions for $du/da$ and $u$ in the relevant expansions in the $N_f = 1$ and AD3 contour integrals.

We begin with the expression in the $N_f = 1$ theory

$$\left[ e^{-2p(u^* + \frac{3}{2} \mu) - T\cdot S^2} \int_{\gamma^+} \Omega \right]_{\mu^0}$$

(B.5)

Here we can set $\mu^0 = 0$ in the expressions for $\Omega$ so that the two points $u_{\pm}$ collide at $u = u^*$. In evaluating this integral we expand the integrand in power $s$ of $(u - u^*)$ and perform the contour integral. When $\mu = 0$ we find that $\tau(u)$ approaches $\tau^* = e^{i\pi/3}$ as $u \to u^*$ and indeed

$$\tau = \tau^* + PS((u - u^*)^{1/3})$$

(B.6)

where $PS(x)$ means a power series in positive powers of $x$ that vanishes at $x = 0$. Similarly:

$$\left( \frac{du}{da} \right) = \kappa_1 (u - u^*)^{1/6} \left( 1 + PS((u - u^*)^{1/3}) \right)$$

(B.7)

with

$$\kappa_1 = \left( \frac{1}{4} \frac{3}{\rho} \right)^{1/3} \left( \frac{4}{9} \right)^{1/3} \left( E_6(\tau^*) \right)^{-1/3}$$

(B.8)

Similarly,

$$du \left( \frac{du}{da} \right)^{1-\sigma/2} \Delta^{\sigma/8} = \Lambda_\infty^1 \frac{d(u - u^*)}{(u - u^*)} (u - u^*)^{-3/6} (1 + PS((u - u^*)^{1/3}))$$

(B.9)

with

$$\Lambda_\infty^1 = \kappa_1^{1-\sigma/2} (u^* - u_0)^{\sigma/8}$$

(B.10)

and finally, $T^* = u^*/3$ and

$$T - T^* = -\frac{\kappa_2^2}{24} E_2(\tau^*)(u - u^*)^{1/3} \left( 1 + PS((u - u^*)^{1/3}) \right)$$

(B.11)
The integral over the phase of $u - u_*$ will kill all terms in the power series except those proportional to
\[
\frac{d(u - u_*)}{(u - u_*)} |(u - u_*)^{1/3}|^n 
\] (B.12)
for some integer $n$, and in our expressions $n$ is always nonnegative. However, since we also take the $\epsilon \to 0$ limit, only the terms with $n = 0$ will contribute. We thus concentrate on the Laurent expansion in $(u - u_*)^{1/3}$ working to zeroth order in the power series expansion in $(\bar{u} - \bar{u}_*)^{1/3}$.

Now, since $(\tau - \tau_*)$ and $du/da$ are expansions in positive powers of $(u - u_*)^{1/3}$ the resulting contour integral is a polynomial in $p$ and $S$. Moreover, $S$ always multiplies $du/da$, so by (3.7) if we assign charge +1 to $S$ and +6 to $p$ then the leading powers of $(u - u_*)^{1/3}$ are governed by the natural grading $6\ell + r$. The higher order terms in the expansions in $(u - u_*)^{1/3}$ above will contribute to lower degree terms in the polynomial $P_1$. So the contribution to $P_{1,\text{top}}$ only comes from the leading order terms in the above expansions giving the contribution to the polynomial:
\[
P_{1,\text{top}}(p, S) = N_1 \int d(u - u_*) (u - u_*)^{-2/6} e^{2p(u - u_*)} F_\infty(\kappa_1 (u - u_*)^{1/6} S) \] (B.13)

where
\[
F_\infty(w) = e^{-\frac{w^2}{2\pi} E_2(\tau_*)} \sum_{\lambda \in \Gamma + \lambda_0} (-1)^{w_2(\lambda - \lambda_0)} E(\rho^w_\lambda(w); \rho^{\lambda_0}_\lambda(w)) e^{-i\pi \tau_\lambda^2 - iw \cdot \lambda} \] (B.14)
and here
\[
\rho^w_\lambda(w) := -\sqrt{y_+^{\lambda}} - \frac{i}{4\pi \sqrt{y_+}} w_+ \] (B.15)

Let us compare the above contribution to $P_{1,\text{top}}$ with the corresponding expression in the AD3 theory
\[
\left[ \oint_{\gamma_\infty} \Omega^{\text{AD}} \right]_{\Lambda_{\text{AD}}} \] (B.16)

Since we are after the constant term we consider the AD3 family with $\Lambda_{\text{AD}} \to 0$. Equation (6.11) can be written as:
\[
\left( \frac{E_4(\tau)}{E_6(\tau)} \right)^3 = 4 \left( \frac{\Lambda_3}{u} \right)^2 \] (B.17)
and (6.13) can be written as
\[
\left( \frac{du}{da} \right)^2 = -\frac{1}{6} \left( \frac{3}{\rho} \right)^2 \frac{E_4(\tau)}{E_6(\tau)} u \] \left( \frac{\Lambda_3}{u} \right)^2 \] (B.18)

Now, as $u \to \infty$,
\[
\tau - \tau_* = 2^{2/3} \frac{(E_6(\tau_*))^{2/3}}{E_4(\tau_*)} \left( \frac{\Lambda_3}{u} \right)^{2/3} \left( 1 + PS\left( \left( \frac{\Lambda_3}{u} \right)^{2/3} \right) \right) \] (B.19)
and
\[
\left( \frac{du}{da} \right) = \kappa_2 u^{1/6} \left( 1 + PS\left( \frac{\Lambda^3}{u} \right)^{2/3} \right) \tag{B.20}
\]
\[\kappa_2 = \left( -\frac{1}{12} \left( \frac{3}{p} \right)^2 \frac{2^{2/3}}{(E(\tau^*_s))^{1/3}} \right)^{1/2} \tag{B.21}\]

Similarly,
\[
\frac{du}{da} \left( \frac{1}{1 - \sigma/2} \Delta^{\sigma/8} = N_{AD}^{\infty} du \frac{u^{-\sigma/6}}{u} \right) (1 + PS\left( \frac{\Lambda^3}{u} \right)^{2/3}) \tag{B.22}
\]
with
\[
N_{AD}^{\infty} = \kappa_2^{1-\sigma/2} \tag{B.23}
\]

Once again, since we are taking the contour to infinity, we can focus on the holomorphic expansion in $u^{1/6}$. All the higher order terms in the power series have positive powers of $\Lambda_{AD}$ and hence, again, we need only consider the leading order terms to get the contribution at $\Lambda_{AD}^0$. We have
\[
P_{AD,\infty}(p, S) = N_{AD}^{\infty} \oint_{\infty} du \frac{u^{-\sigma/6}}{u} e^{2p u} F_{\infty}(\kappa_2 u^{1/6} S) \tag{B.24}
\]
with the same function $F_{\infty}$ defined in (B.14).

Comparing the two expressions we will find an equality of the kind (B.3), for this contribution to the polynomial, provided
\[
N_{1}^{\infty}(2p)^\ell (\kappa_1 S)^r = N_{AD}^{\infty}(2n_0 p)^\ell (n_2 \kappa_2 S)^r \tag{B.25}
\]
for $r + 6\ell = B$. We solve for $r$ in terms of $\ell$ and $B$ and then since different powers of $\ell$ appear in the polynomial we must have
\[
N_{1}^{\infty} \kappa_1^B = N_{AD}^{\infty}(n_2 \kappa_2)^B \tag{B.26}
\]
\[
\frac{(\kappa_2}{\kappa_1}^6 = \frac{n_0}{n_2}^B \tag{B.27}
\]

Now we consider an analogous computation for the contributions from $\gamma_\pm$. First we consider
\[
\left[ e^{-2p(u_+ + \frac{2}{3} \mu)} - T, S^2 \left( \oint_{\gamma_\pm} d\phi \Omega \right) \right]_{\mu^0} \tag{B.28}
\]
in the $N_f = 1$ theory. Here we will be writing the integrand as a power series in the local duality frame variable $q_\pm$.

For small $\mu$ the two points in the discriminant locus have an expansion
\[
u_+ = 1 + \frac{2}{3} \mu + \left( \frac{2}{3} \right)^{5/2} \mu^{3/2} + \cdots \tag{B.29}
\]
\[
u_- = 1 + \frac{2}{3} \mu - \left( \frac{2}{3} \right)^{5/2} \mu^{3/2} + \cdots
\]
A subtle point is that if we take the limit as \( u \to u_* \) with \( \mu \) held fixed then the expansions for \( u \) and \( du/da \) involve an infinite series of increasingly divergent terms in \( \mu \). The correct scaling limit \(^{17}\) is to define
\[
u = u_* + \mu^{3/2} v \tag{B.30}
\]
and take the limit \( \mu \to 0 \) holding \( v \) fixed. With this understood we have
\[
\exp(2p \nu) = \exp(2p(u_* + \frac{2}{3} \mu)) \exp(2p \mu^{3/2} (2/3)^{5/2} E_6 / E_4^{3/2} (1 + O(\mu^{1/2}))) \tag{B.31}
\]
where the Eisenstein series are expansions in \( q \) in the standard way. Next we can write
\[
\frac{du}{da} = \kappa_3 E_4^{-1/4} \mu^{1/4} \left( 1 + PS(\mu^{1/2}) \right) \tag{B.32}
\]
and similarly
\[
du \left( \frac{du}{da} \right)^{1-\sigma/2} \Delta^{\sigma/8} = N_\pm^1 \mu^{-\sigma/4} \left( \frac{dq}{q} H(q) \left( 1 + PS(\mu^{1/2}) \right) \right) \tag{B.34}
\]
where the power series in \( \mu^{1/2} \) has coefficients which are themselves power series in \( q \).

Here
\[
H(q) := \left( \frac{d}{dq} \left( \frac{E_6}{E_4^{3/2}} \right) \right) E_4^{-(\sigma+1)/4} (E_6^2 - E_4^3)^{\sigma/8} \tag{B.35}
\]
\[
N_\pm^1 = \pm \left( \frac{2}{3} \right)^{5/2(1+\sigma/4)} \kappa_3^{1-\sigma/2} (u_* - u_0)^{\sigma/8} \tag{B.36}
\]

Now, the expansion in \( p^\ell S^r \) comes with a power \( \mu^{(r+6\ell)/4} \) so the \( \mu^0 \) term satisfies the selection rule and the higher powers in the \( \mu \) expansion contribute lower order terms. Thus, the contribution to the polynomial from these two singularities is the sum over + and − of
\[
P^\text{top}_{1,\pm}(p, S) = \eta N_\pm^1 \left[ \mu^{-\sigma/4} \oint \frac{dq}{q} H(q) \exp(2p \mu^{3/2} (2/3)^{5/2} E_6 / E_4^{3/2} F_\pm(\kappa_3 \mu^{1/4} E_4^{-1/4} S)) \right]_{\mu^0} \tag{B.37}
\]
where
\[
F_\pm(w) = \frac{1}{\sqrt{8}} e^{-\frac{w^2}{2(\tau)}} \sum_{\lambda \in \Gamma + \lambda_0} (-1)^w (\lambda - \lambda_0) (\text{sign}(\lambda \cdot \omega) - \text{sign}(\lambda \cdot \omega_0)) e^{-i\pi \tau^2 \lambda^2 - i\omega \lambda} \tag{B.38}
\]

Finally we come to the contributions
\[
\left[ \oint_{\gamma_\pm^D} \Omega^{AD} \right]_{\Lambda_0^D} \tag{B.39}
\]

\(^{17}\) This is a consequence of the linear combinations we found in equation \([2.3]\) above.
in the AD3 theory.

In the AD3 theory we have the exact formulae for the expansions in $q_\pm$ near $u_\pm$:

$$u = \pm 2\Lambda^3_\text{AD} \frac{E_6}{E_4^{3/2}}$$  \hspace{1cm} (B.40)

$$\frac{du}{da} = \kappa_4 E_4^{-1/4} \Lambda^{1/2}_\text{AD}$$  \hspace{1cm} (B.41)

$$\kappa_4 = \left(-\zeta \frac{1}{6} \left(\frac{3}{\rho}\right)^2\right)^{1/2}$$  \hspace{1cm} (B.42)

and we compute:

$$du \left(\frac{du}{da}\right)^{1-\sigma/2} \Delta^{\sigma/8} = \mathcal{N}_\pm^{\text{AD}} \Lambda^{-3/2}_\text{AD} \frac{dq_\pm}{q_\pm} H(q_\pm)$$  \hspace{1cm} (B.43)

$$\mathcal{N}_\pm^{\text{AD}} = \pm 2^{1+\sigma/4} \kappa_4^{1-\sigma/2}$$  \hspace{1cm} (B.44)

So these terms contribute to the polynomial

$$P_{\text{AD},\pm} = \eta \mathcal{N}_\pm^{\text{AD}} \left[ \Lambda^{-3/2}_\text{AD} \int \frac{dq_\pm}{q_\pm} H(q_\pm) e^{\pm 4p\Lambda^3_\text{AD} E_4^{3/2}} F_\pm (\kappa_4 \Lambda^{1/2}_\text{AD} E_4^{1/4} S) \right]$$  \hspace{1cm} (B.45)

Now to match these using the rescalings (B.3) we have the conditions

$$\mathcal{N}_1^{\text{AD}} \left(2p \left(\frac{2}{3}\right)^{5/2}\right)^{\ell} (\kappa_3 S)^r = N \mathcal{N}_\pm^{\text{AD}} (4n_0 p)^\ell (\kappa_4 n_2 S)^r$$  \hspace{1cm} (B.46)

when $6\ell + r = 9$. In a way similar to (B.26) and (B.27) we obtain:

$$\mathcal{N}_1^{\text{AD}} \kappa_3^{3/2} = N \mathcal{N}_\pm^{\text{AD}} (n_2 \kappa_4)^{3/2}$$  \hspace{1cm} (B.47)

$$\left(\frac{\kappa_4}{\kappa_3}\right)^6 = 2^{3/2} \left(\frac{3}{2}\right)^{5/2} \frac{n_0}{n_2^4}$$  \hspace{1cm} (B.48)

We now ask if there are constants $N, n_0, n_2$ that allow us to solve the four conditions (B.26) (B.27) (B.47) (B.48). The conditions are not all independent, and in fact, there are such constants iff we have

$$\left(\frac{\kappa_1 \kappa_4}{\kappa_2 \kappa_3}\right)^6 = 2^{-3/2} \left(\frac{3}{2}\right)^{5/2}$$  \hspace{1cm} (B.49)

$$\frac{\mathcal{N}_\pm^{\text{AD}}}{\mathcal{N}_1^{\text{AD}}} \left(\frac{\kappa_1}{\kappa_3}\right)^{3/2} = \frac{\mathcal{N}_\pm^{\text{AD}}}{\mathcal{N}_1^{\text{AD}}} \left(\frac{\kappa_2}{\kappa_4}\right)^{3/2}$$  \hspace{1cm} (B.50)

Plugging in the above values we can confirm that these conditions are indeed satisfied.
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