How is an optimized path of classical mechanics affected by random noise?

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Abstract. The variational principle is one of important guiding principles in physics. Classical equations of motion of particle can be formulated so as to give the optimized path of an action. However, when there exist uncontrollable degrees of freedom such as noise, the optimized path is affected and the original classical equations of motion may not correspond to the optimized path. The stochastic variational method (SVM) is a framework to calculate the modified optimized path by the effect of noise. This method has been developed to show that the Schrödinger equation can be derived from the classical action which leads to Newton’s equation of motion by taking into account the modification of the optimized path due to noise. In this work, we will extend this idea to the case of the continuum media and show that the Euler equation of the ideal fluid is converted to the Navier-Stokes equation or the Gross-Pitaevskii equation in SVM.

1. Introduction
As is well-known, the classical dynamics is described by Newton’s equation of motion, and the path described by this equation corresponds to the optimized path of an action. However, it is also possible to consider the situation where this optimized path is not necessarily realized.

For example, let us consider to walk on a deck board on a luxury liner. When the ship lays at anchor, it is easy to go wherever we want to. However, when it is on the ocean and the sea runs high, it is difficult to reach there by the most direct way.

Similarly, if there exist uncontrollable degrees of freedom such as random noise, the optimized dynamics should be affected. Then is it possible to estimate the effect? The stochastic variational method (SVM) is a promising approach to discuss optimization under the Gaussian white noise[1].

So far, SVM has been exclusively applied to particle systems and used to discuss the relation between classical dynamics and quantum mechanics. In this work, we briefly summarize the application of SVM to the system of particle and report the extension of the method to the case of continuum media [2].

2. Stochastic Variational Method
In this section, we briefly summarize SVM. See also Ref. [2] and references therein for more details.
2.1. Definition of velocity
In general, there will exist various possibilities for the introduction of the effect of noise in the variational method. In SVM, this effect is taken into account through the definition of velocity as is the case of the Brownian motion.

First, we introduce the forward stochastic differential equation (SDE),
\[ d\mathbf{r}(t) = \mathbf{u}(\mathbf{r}(t), t)dt + \sqrt{2\nu}d\mathbf{W}(t), \quad (dt > 0), \]
where \( \nu \) is the strength of the noise and \( \mathbf{W}(t) \) is the Wiener process,
\[
\begin{align*}
E[W_j(s) - W_j(u)] &= 0, \\
E[(W_j(s) - W_j(u))(W_k(s) - W_k(u))] &= \delta^{jk}|s - u|, \\
E[(W_j(s) - W_j(u))W_k(t)] &= 0.
\end{align*}
\]

The trajectory \( \mathbf{r}(t) \) is not smooth and hence we cannot define the time derivative at \( t \) uniquely, as is shown in Fig. 1. Consequently, we further introduce backward SDE which describes the time reversed process of Eq. (1),
\[ d\mathbf{r}(t) = \tilde{\mathbf{u}}(\mathbf{r}(t), t)dt + \sqrt{2\nu}d\tilde{\mathbf{W}}(t), \quad (dt < 0) \]
The noise \( d\tilde{\mathbf{W}} \) satisfies the same correlation properties as \( d\mathbf{W} \) and there is no correlation between \( d\mathbf{W} \) and \( d\tilde{\mathbf{W}} \).

The backward SDE must describe the time reversed process of Eq. (1). To satisfy this condition, the two Fokker-Planck equations which are derived from Eqs. (1) and (5) should be equivalent. From this condition, \( \tilde{\mathbf{u}} \) is related to \( \mathbf{u} \), as
\[ \mathbf{u} = \tilde{\mathbf{u}} + 2\nu\nabla \ln \rho, \]
where \( \rho \) is the particle density which is given by the solution of the Fokker-Planck equation,
\[ \partial_t \rho = \nabla(-\mathbf{u} + \nu \nabla)\rho = \nabla(-\tilde{\mathbf{u}} - \nu \nabla)\rho. \]

Correspondingly to the two SDEs, we introduce the two different definitions of the time derivative: one is the mean forward derivative,
\[ D\mathbf{r} = \mathbf{u}, \]
and the other the mean backward derivative,
\[ \tilde{D}\mathbf{r} = \tilde{\mathbf{u}}. \]
More precise definitions of these two derivatives, see Ref. [3]. The time derivative terms which appear in the kinetic term of the Lagrangian are replaced by using these \( D\mathbf{r} \) and \( \tilde{D}\mathbf{r} \).

2.2. Various formulas
The most part of the variational procedures in SVM is same as those in the classical variational method. However we cannot use the Taylor expansion and the partial integration formula. These are modified in SVM as follows.
(i) Ito formula
For an arbitrary differentiable function \( F \) which is a function of the stochastic variable \( \mathbf{r} \) satisfying Eq. (1), there is the following expansion,
\[ dF(\mathbf{r}, t) = (\partial_t F + \mathbf{u} \cdot \nabla F + \nu \nabla^2 F)dt + O(dt^{3/2}). \]

(ii) stochastic partial integration formula [4]
For the stochastic variables \( \mathbf{X} \) and \( \mathbf{Y} \), the partial integration formula is extended as follows,
\[ \int_{t_a}^{t_b} dtE[D\mathbf{X}(t) \cdot \dot{\mathbf{Y}}(t) + \mathbf{X}(t) \cdot \tilde{D}\mathbf{Y}(t)] = E[\mathbf{X}(t_b)\dot{\mathbf{Y}}(t_b) - \mathbf{X}(t_a)\dot{\mathbf{Y}}(t_a)]. \]
3. Application to Systems of Particle and Continuum Media

Let us apply SVM to an action which leads to Newton’s equation of motion,

\[ I(r) = \int_{t_a}^{t_b} dt \left[ \frac{m}{2} \left( \frac{dr}{dt} \right)^2 - V(r) \right], \]

(12)

where \( t_a \) and \( t_b \) are the initial time and the final time of the evolution of dynamics, respectively. As was discussed, there are two time derivatives in SVM and it is not unique to express the kinetic term in terms of the stochastic variable. In this section, we consider the following replacement,

\[ I(r) = \int_{t_a}^{t_b} dt E \left[ \frac{m}{2} (Dr)^2 + \left( \tilde{Dr} \right)^2 / 2 - V(r) \right], \]

(13)

By considering the variation of the stochastic trajectory \( r \rightarrow r + \delta r \), we obtain

\[ (\partial_t + u_m \cdot \nabla)u^i_m - 2\nu^2 \nabla^i (\rho^{-1/2} \nabla^2 \sqrt{\rho}) = -\frac{1}{m} \nabla^i V, \]

(14)

where \( u_m = (u + \tilde{u})/2 \). As was mentioned, \( \rho \) is the solution of the Fokker-Planck equation.

This result can be cast into another form. The phase variable \( \theta \) can be introduced by \( \nabla \theta = u_m / (2\nu) \). Then the new variable \( \phi = \sqrt{\rho} e^{i\theta} \) satisfies the following equation,

\[ i\partial_t \phi = \left[ -\nu \nabla^2 + \frac{1}{2\nu m} V \right] \phi. \]

(15)
This equation is reduced to the Schrödinger equation by choosing $\nu = \hbar/(2m)$. That is, the Schrödinger equation can be derived from Newton’s equation of motion by considering the fluctuation by random noise. It is also easy to see that the particle density is given by $\rho = |\phi|^2$ without introducing the interpretation of the wave function $\phi$.

For the case of continuum media, we consider the classical action given by the following equation,

$$ I(r) = \int_{t_1}^{t_2} dt \int d^3R \left[ \frac{\rho^{\alpha}(R)}{2} \left( \frac{dR(R,t)}{dt} \right)^2 - J\varepsilon \right], $$

where $\varepsilon$ is an internal energy density, $J$ is the Jacobian between $r$ and $R$, and the mass density is defined by $\rho_m = m\rho$. Because this action is expressed in the Lagrangian coordinate, the spatial integral is done for the initial distribution of the mass density $\rho^\alpha_m$, which is a function of $R$. When we apply the classical variational method to this action, the Euler equation is obtained.

The corresponding stochastic Lagrangian density is given by

$$ \mathcal{L} = \frac{\rho_0^m}{2} \left[ \left( \frac{1}{2} + \alpha_2 \right) \left( \frac{1}{2} + \alpha_1 \right) (D\nu(t))^2 + \left( \frac{1}{2} - \alpha_1 \right) (D\nu(t))^2 \right] - J\varepsilon. $$

Here $\alpha_1$ and $\alpha_2$ are arbitrary constants, come from the ambiguity for the stochastic representation of the kinetic term. One can easily check that, in the vanishing noise limit, $\nu \to 0$, the difference between $D\nu$ and $\dot{D}\nu$ disappears, and hence the above Lagrangian density always reproduces the classical one, independently of $\alpha_1$ and $\alpha_2$.

After applying SVM, we obtain

$$ \rho^m (\partial_t + u_m \cdot \nabla) u^i_m - \sum_j \partial_j (\mu \rho_m e^m_{ij}) - 2\kappa \rho^m \sum_j \partial_i (\sqrt{\rho^m}^{-1} \partial_j \sqrt{\rho^m}) = -\nabla P, $$

where $\eta = \alpha_1(1 + 2\alpha_2)\nu \rho^m$, $\kappa = 2\alpha_2 \nu^2$ and $e^m_{ij} = \partial_j u^i_m + \partial_i u^j_m$. The pressure $P$ is defined by $(\rho^m)^2 d(\varepsilon/\rho_m)/d\rho_m$.

One can see that the second term on the left hand side corresponds to the viscosity and this equation is reduced to the Navier-Stokes (NS) equation when we set $\alpha_2 = 0$. When $\alpha_2 \neq 0$, the third term is finite and corresponds to the correction term to the NS equation. It is worth mentioning that the same correction term was, later, re-derived from a different phenomenological argument [5].

The above NS equation is reduced to the Gross-Pitaevskii (GP) equation in a special limit. Let us set $(\alpha_1, \alpha_2) = (0, 1/2)$ and $\nu = \hbar/(2m)$. As for the internal energy density, we choose $\varepsilon = V(x)\rho_m/m + U_0/(\rho_m/m)^2$ where $V$ is a trapped potential and $U_0$ is the interaction strength. Then the equation for the wave function which is defined as is the case with the previous section is given by the GP equation.

4. Concluding remarks

We discussed the optimization under random noise in the framework of SVM. We showed the application to the systems of particle and continuum media, and derived the Schrödinger, NS and GP equations. SVM further predicted that the NS equation can be modified.

This method has not yet been established. For example, the applicability to field theory and relativistic systems is still open question. The derivation of the uncertainty relation in SVM is now under investigation.

\footnote{Exactly speaking, the second coefficient of viscosity vanishes in this equation. To obtain a finite value, we have to consider the variation of entropy. See Ref. [2]}
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