1. INTRODUCTION

Over the last several years, considerable effort has gone into the construction of large wide-field low-frequency radio interferometers. One of the many challenges that these new instruments must overcome is that of phase and amplitude fluctuations due to the ionosphere (Hewish 1951, 1952). This is particularly important at meter wavelengths where these instruments operate (see, e.g., Cohen & Röttgering 2009) and the refractive index of the ionized medium can be particularly large.1 Moreover, low-frequency radio telescopes often have wide fields of view (FOV; i.e., up to tens of degrees), making them not only susceptible to the combined electric field from large parts of the sky, but also to its modifications by the full three-dimensional electron-density structure of the ionosphere.

For high-frequency interferometers—having dish-antennae that probe a small area of the sky (a few degrees or less)—the three-dimensional ionosphere can be collapsed along the line of sight and be well approximated by a two-dimensional thin phase screen without losing substantial information (Salpeter 1967). This approximation, however, does not hold if the FOV becomes large and the widely separated antennae observe different sources, under large angles, through similar parts of the ionosphere (e.g., Lonsdale 2005). In the case of wide-field low-frequency interferometers, a full three-dimensional model of the ionosphere is required for self-consistent ionospheric calibration. Whereas considerable work has recently gone into developing (multi-layer) two-dimensional wide-field ionospheric models (e.g., Intema et al. 2009; Matejek & Morales 2009), no physically intuitive and self-consistent description seems readily available for a full-scale three-dimensional wide-field modeling of the ionosphere.

In this paper, a step toward such a complete physical description is presented based on the tomographic theory of electric-field scattering by weakly inhomogeneous media (Wolf 1969), as applied to the ionosphere. The main result is that the ionosphere, over a wide FOV, acts as a scatterer with a spatially varying point-spread function (PSF). The instantaneous two-dimensional PSF at a position \(s\) (a directional unit vector) around a point source at \(s_0\) is identical to the instantaneous three-dimensional power spectrum of the ionospheric electron density sampled from points \((s - s_0)\) on the surface of an “Ewald sphere of reflection” (Ewald 1969), i.e., a spherical surface with unit radius centered in the direction of the source. Having many point or very compact sources over a larger FOV allows one to sample the three-dimensional Fourier structure of the ionosphere at many distinct points. These points can be used to constrain a three-dimensional model of the power spectrum of the ionosphere. This model can subsequently be used to “deconvolve” the image, also at points where there are no strong sources, and remove the dominant phase errors due the three-dimensional ionosphere. We term this “tomographic self-calibration,” because it involves a three-dimensional ionosphere and not a single phase screen. This calibration cannot be done in the classical way through self-calibration (see, e.g., Pearson & Readhead 1984) because the scattering PSF is not spatially invariant as is implicit in that method. It either requires the solution of a matrix (measurement) equation (e.g., Hamaker et al.

1 The ratio of the plasma over the observing frequency at 150 MHz is \(\sim 0.03\); hence, intensity scattering is at the level of \(\sim 0.001\) and scales with \(\lambda^2\).
for a wavelength $\lambda$ (r), we take the case in radio astronomy (i.e., $\nu$), where the refractive index of the medium is near unity, which is often dependent, each component satisfies the same solution and we drop the explicit frequency dependence of the electric field, but the electric field can be conveniently rewritten as

$$E(r) = E^{(1)}(r) + \int_V \Phi(r') E(r') \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d^3 \mathbf{r'},$$

(3)

where $E(r) = E^{(1)}(r) + E^{(1)}(r)$. The first term is the incident (plane) wave and the second term is the scattered wave. The latter is equal to the integral term above and carried out over the entire volume $V$ in which $n \neq 1$. The last factor in the integrand indicates that $E^{(1)}(r)$ is a spherically outgoing wave, assuming that the extent of the scattering potential is small compared to the distance between the scatterer and scattering medium.

2.1. Weak Scattering of Multiple Point Sources

We now extend the single plane-wave description of Wolf (1969) to an incident electric field that results from the sum of $N$ point sources that satisfy the solution of the Helmholtz equation in free space. Later in the paper, we further extend this to a continuous intensity field. We also express all physical distances in units of $\lambda$, i.e., $\mathbf{u} \equiv \mathbf{r}/\lambda = (u, v, w)$ in the remainder of the paper. Its Fourier equivalent is $\mathbf{s} = (s_u, s_v, s_w)$. The incident electric field in this case becomes

$$E^{(1)}(\mathbf{u}) = \sum_s \sqrt{S_u} e^{2\pi i (s u u + s v v + s w w)} E^{(1)}(\mathbf{u})$$

(4)

where $S_u$ is the flux density of the point source $n = 1 \ldots N$ and $s_u, s_v, s_w$ are unit vectors that point in the directions of the point sources. These point sources are (for now) assumed to dominate the electric field and can be compared to the phase calibrators in radio interferometry. In the weak scattering approximation, the scattered wave has a relatively low amplitude compared to the incident wave. We can then replace $E(\mathbf{u})$ with $E^{(1)}(\mathbf{u})$ in the first-order (Born) approximation, finding

$$E_1^{(1)}(\mathbf{u}) = \sum_s \sqrt{S_u} \int_V \Phi(u') e^{2\pi i (s u u + s v v + s w w)} d^3 \mathbf{u},$$

(5)

where the subscript indicates that the scattered field is that to the first-order Born approximation and $\Phi(\mathbf{u}) \equiv [n^2 - 1]$. We note here that the second order can be obtained by substituting $E^{(1)}(\mathbf{u}) + E_1^{(1)}(\mathbf{u})$ into the original integral equation to obtain a solution to the second order, etc. This iterative scheme only works for weak scattering, where the values of $\Phi(\mathbf{u})$ do not exceed unity.

We now use the fact that the spherical outgoing wave can be written as (Weyl 1919)

$$e^{2\pi i (u u u + v v v + w w w)} d^3 \mathbf{u} = i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{S_u} e^{2\pi i (s u u + s v v + s w w)} ds_u ds_v,$$

(6)

3 We note here that even though scattering can be “weak” in terms of a small deviation of the refractive index from unity everywhere, the integral over the line of sight can still lead to substantial phase changes in the electric field, leading to both diffractive and refractive effects and strong scintillation (e.g., Narayan 1992). In the current discussion, “weak scattering” means both a refractive index very close to unity and phase fluctuations much less than unity. The latter is the nominal mode of the ionosphere and data obtained with interferometers during strong scintillation (e.g., during solar bursts or sun rise/set) occur only rarely and are often discarded in analyses.
with $s_u^2 = 1 - s_v^2 + s_w^2$ for $s_v^2 + s_w^2 \leq 1$ or $s_u^2 = -(s_v^2 + s_w^2) - 1$ otherwise. Complex values of $s_u$ lead to exponentially decaying (evanescent) electric fields that are typically not measurable far from the scatterer. The other (homogeneous) waves are those measured by a distant observer. If one further uses the Fourier transform of the scattering potential, $\tilde{\Phi}(s) = \iint \Phi(u) e^{-2\pi i su} u du$, one can write the scattered electric field as

$$E_1^{(v)}(u) = i \sum_n S_n \int \frac{1}{s_w} \tilde{\Phi}(s - s_{0,n}) e^{2\pi i su} ds_w ds_v,$$

where we assume a geometry where $w = 0$ is the ground plane below the ionosphere where $n = 1$, and that $w > 0$ is in the direction of the zenith or the phase-reference center (see below). The interferometer is placed in a plane defined at a constant $w_{\text{ant}} = z_{\text{ant}}/\lambda$. Typically, one can assume $w_{\text{ant}} = 0$.

Hence, one finds a relation between the Fourier transform of the observed electric field in the plane of the interferometer at $w_{\text{ant}}$ and the Fourier transform of the scattering potential

$$\tilde{E}^{(v)}(s_u, s_v) = \frac{i}{s_w} e^{2\pi i s_w w_{\text{ant}}} \sum_n S_n \tilde{\Phi}(s - s_{0,n}),$$

with $\tilde{E}^{(v)}(s_u, s_v) = \int E_1^{(v)}(u, v, w_{\text{ant}}) e^{-2\pi i (s_u u + s_v v)} du dv$. This can be regarded as the Fourier transform of a two-dimensional slice through a three-dimensional scattered electric field. In this paper, we do not treat the case of an interferometer with varying $w_{\text{ant}}$. A planar array is a reasonable assumption for relatively compact (i.e., km scale) interferometers, but breaks down on large scales where the curvature of the Earth cannot be neglected (see, e.g., Carozzi & Woan 2009). For a planar array, however, the $w$-term due to the array can be neglected for small integration times (i.e., instantaneous sampling of the electric field in a plane), in contrast to visibilities from very different integration times (i.e., DCTs). This equation is exact for phase-coherent point sources to the first-order Born approximation. However, the sky is an incoherent emitter (see Mandel & Wolf 1965 for an exposure on the coherence properties of electric fields). Hence, the cross-terms with $n \neq m$ depend on the electric field coming from incoherent point sources and vanish, such that we are left with

$$\delta I^{(v)}(s_u, s_v) = \frac{1}{s_w^2} \sum_n S_n |\tilde{\Phi}(s - s_{0,n})|^2,$$

where we dropped the subscript. This equation forms the basis for further discussions in the paper. The above equation is only correct for an interferometer and an electric field measured in a plane. In three dimensions, one would no longer be able to use simple Fourier transforms (see below), because $s_{0,n}$ depends explicitly on $s_u$ and $s_v$.

To understand the physical interpretation of the above equation, one might suppose a point source in the zenith (or equivalently in the phase center) emitting a plane wave in the absence of the ionosphere. Because the phase of the electric field is the same at each antenna (by construction), its Fourier transform yields a complex delta function in the zenith with a time-varying amplitude. Multiplied with its complex conjugate, this recovers the complete scattered intensity

$$\delta I^{(v)}(s_u, s_v) = \left(\tilde{E}^{(v)}(s_u, s_v) \tilde{E}^{(v)*}(s_u, s_v)\right)_1,$$

where the dependence on $w_{\text{ant}}$ disappears. Using Equation (8), we find the following result:

$$\delta I^{(v)}(s_u, s_v) = \frac{1}{s_w^2} \sum_n \sum_m \sqrt{S_n S_m} \tilde{\Phi}^*(s - s_{0,n}) \tilde{\Phi}(s - s_{0,m}).$$

3. THE EFFECTS OF WEAK IONOSPHERIC SCATTERING ON INTERFEROMETRIC IMAGES

In radio interferometry one does not analyze the electric field itself. In that case, Equation (8) would directly yield the three-dimensional structure of the ionosphere (per integration time) because the phase information of the Fourier transform of the electron density of the ionosphere is fully retained in the phase information of the scattered electric field. In reality, only the cross-correlations of the electric field, measured at different antennae pairs, are stored (i.e., the complex visibilities) and the phase information of the ionospheric density fluctuations is lost. In the following, we assume that the total electric field from the entire sky (i.e., the antenna sensitivity is directionally independent) is measured over the infinite interferometer plane with $w = w_{\text{ant}}$.

Visibilities are sampled from the cross-correlation of the electric field $E(u) = E^{(c)}(u) + E^{(v)}(u)$ with its complex conjugate, i.e., $V(b) \equiv (E(u) E^*(u + b))$, with $b$ being the baseline between two points (antennae) in plane of the interferometer. The averaging is assumed to be over time. The Fourier transform of the visibilities forms the incident intensity from the sky, as follows from the van Cittert–Zernike theorem (e.g., Carozzi & Woan 2009). The same intensity is also the product of the Fourier transform of the electric field with its complex conjugate. A bit of algebra shows that the cross-correlation between the incident and scattered fields depends on the imaginary part of the zero mode, $\Phi(0)$, of the ionosphere, and consequently is equal to zero. The multiplication of the Fourier transform of the scattered electric field with its complex conjugate therefore provides the complete scattered intensity

$$\delta I^{(v)}(s_u, s_v) = \left(\tilde{E}^{(v)}(s_u, s_v) \tilde{E}^{(v)*}(s_u, s_v)\right)_1,$$
(a speckle), offset by the projected phase frequency onto the array and with an intensity proportional to the amplitude of the wave mode squared. The sum of all speckles create a halo of scattered emission around the point source, when not corrected for through phase calibration.

3.1. Speckle and Speckle Noise

Because the phase fluctuations of the electric field are measured in the plane of the array and are typically drawn from a Gaussian random field realization from some (ensemble average) power spectrum, taking the Fourier transform of this field and multiplying it with its complex conjugate yields that each point source exhibits a diffuse “halo” of scattered intensity (e.g., Salpeter 1967; Cronyn 1972). Its Fourier transform is related to the usual phase structure function \( D_{\phi}; \) see below). This pattern is referred to as “speckle” in optical (laser) interferometry. We define the instantaneous ionospheric scattering PSF (ISP hereafter) as the sum of a delta function plus its scattered speckle pattern, re-normalized to a flux density of unity such that flux is conserved in a convolution process. The ISP is related to the Fourier transform of optical transfer function (Goodman 1985), determined by phase fluctuations induced by the ionosphere. Equation (11) shows that the point-source intensity is convolved with an ISP that reflects a curved surface in the Fourier transform of the instantaneous ionospheric electron-density fluctuations. Disregarding a geometrically determined distortion that depends only on the angle of the observed point on the sky away from the zenith, a three-dimensional ionosphere causes a spatially varying convolution. In contrast, a thin two-dimensional ionospheric screen causes a spatially invariant convolution. This sets our analysis apart from most studies up to the present that have focused on scattering by ionized media (see, e.g., Narayan 1992, for an excellent review) where wide-field effects and extended (thick) screens can be neglected in nearly all circumstances.

For ionospheric electron-density fluctuations set by Kolmogorov turbulence, one expects that \( |\Phi(\Delta s)|^2 \propto |\Delta s|^{-\beta} \) with \( \beta = 11/3 \) over a scale of meters to tens of kilometers (e.g., Thompson et al. 2001). This implies that in the weak-scattering regime, point sources on average exhibit a speckle pattern which rapidly decreases in intensity with distance from the source (Goodman 1985). The speckle pattern, at a given moment in time, is a Gaussian random realization from a power spectrum with expectation value \( |\Phi(\Delta s)|^2 \). Scattered flux is spread over scales from arcseconds to degrees, corresponding to tens of km to tens of meter-scale ionospheric electron-density fluctuations, respectively. It is interesting to note that ionospheric modes of a given physical scale give rise to image distortions that are visible only on baselines equal or larger to that same physical scale. The properties of large isoplanatic patches over a given scale are therefore easier to determine if one can observe the sky with an array that exceeds that scale (see also, e.g., Jacobson & Erickson 1992). Long baselines thus significantly help in calibrating shorter baselines, even if the science of interest is obtained by the short baselines.⁴

The presence of a speckle pattern can have consequences. Once the ISP reaches an intensity level comparable to the average intensity of the surrounding sources, the confusion noise, or the noise itself, one can no longer distinguish it from these contaminants and calibration is typically limited to a maximum \( k \)-mode. This could place a limit on the calibratability of the instrument, especially at very low frequencies near the ionospheric cutoff, because the signal-to-noise ratio of an image snapshot, taken over the maximum timescale of substantial ionospheric changes, might not be large enough to correct for low signal-to-noise speckles further away from the sources. This leaves residual speckles which are a function of wavelength (scaling with \( \lambda^2 \) in strength and with \( \lambda \) in scale).

A detailed calculation of the residual speckle intensity and its associated noise goes beyond the scope of the current paper. We note that these speckles do not average away over time but ultimately form a halo (i.e., “seeing”) around each source. Whereas a bright source might be very useful in determining the ionospheric scattering, it might also leave a residual speckle pattern in the image due to uncalibrated phase fluctuations from ionospheric scale below the smallest baseline. It could therefore be advantageous to place bright sources near the half-power of the FOV such that they still probe the same ionosphere to the first order, but that their residual speckle patterns do not contaminate the field of interest.

Speckle noise (i.e., the expected standard deviation from the expectation value of the speckle intensity at a given point), however, does average away. To see this, we note that a convolution of the intensity implies a multiplication of the visibilities with the Fourier transform of the ISP. In the case of weak scattering and assuming a thin phase screen and small FOV, this multiplication function to the first order is \( 1 - D_{\phi}(\theta) \), where the latter term is the phase structure function. Cronyn (1972) showed that the variance in the scattered part of the intensity is set by the amplitude of \( D_{\phi}(\theta) \) and to the first order is equal to its expectation value. Hence, speckle noise around point sources average away as one over the square root of the number of independent realizations of the ionosphere. Obviously, calibration can remove much of the speckle pattern, but not that caused by the structure below the smallest and above the longest baselines or below the noise level. This leftover halo of speckle can be compared to that in optical observations (after AO correction) in the search for faint companions around nearby stars (e.g., Racine et al. 1999). Speckle and speckle noise ultimately set the detection threshold and a similar effect can play a role in very high dynamic range imaging at low radio frequencies. We finally note that a convolution implies a suppression of visibilities on long baselines. In particular, if no calibration is applied for ionospheric effects, phases on long baselines become less correlated and average out stronger (Bramley 1954) and a “seeing” halo develops around the point sources.

4. VARIATION IN THE IONOSPHERIC SCATTERING PSF OVER THE FIELD OF VIEW

In the case of a single source in the phase center and narrow FOV over which \( \epsilon_2 \) changes little from zero, the ISP around a point source (assuming a perfectly sampled \( \nu \)-plane or an electric field) represents the instantaneous power spectrum of the electron-density fluctuations on a two-dimensional surface in three-dimensional Fourier space. In the case of a narrow FOV and a compact speckle pattern with \( \mu \rightarrow 0 \), this pattern is nearly equal to the instantaneous two-dimensional power spectrum of the medium integrated along the line of sight, which has thus far motivated the use of a phase screen or a thin ionospheric model (e.g., Intema et al. 2009).

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⁴ To correct for all modes in the FOV (\( \theta_{\text{FOV}} \)), the longest baseline should be \( b \gtrsim 25 \text{ km} \times \left( \frac{H_{\text{ion}}}{300 \text{ km}} \right) \cdot \left( \frac{\theta_{\text{FOV}}}{5'} \right) \), where \( H_{\text{ion}} \) is the typical height of the ionosphere.
To assess how the ISP is modified when displaced from the phase-reference center (see Thompson et al. 2001; usually the center of the antenna beam), we assume that the phase-reference center is in the zenith, such that the $w$-axis points exactly upward and the $uv$-plane is perpendicular to it. The planar array lies at constant $w$ as assumed thus far. It can be shown, however, that for any planar array, the coordinate system can be rotated to any phase-reference center. By correcting the phases of each of the antennae such that the fringe rate is zero for a chosen phase-reference center, one effectively re-orient the planar array such that it acts as being perpendicular to the line connecting the array and the phase-reference center. Hence, the analysis performed in the coordinate system with the phase-reference center in the zenith is in fact valid for any chosen phase-reference center for planar arrays by defining the $w$-axis in that direction.

We now assume two point sources, one in the phase-reference center at $s_{\text{zenith}} = (0, 0, 1)$ and one slightly offset from the zenith at a unit vector $s_{\text{src}}$, such that $|s_{\text{src}} - s_{\text{zenith}}| \ll 1$. One can then Taylor expand the ISP to the first order. To do this properly, however, one needs to compare two points that are offset from the two sources (one in the zenith and one offset from it) by an identical two-dimensional vector $\delta s_{\text{2D}} = (\delta s_u, \delta s_v)$. In that case, one compares the ratio of their aligned speckle patterns in the $(s_u, s_v)$ plane. Furthermore, because the first two coordinates of $s - s_{\text{src}} = (\delta s_u, \delta s_v - \delta s_{\text{src}}) = (1 - |\delta s_{\text{2D}}|^2)^{1/2} - (1 - |\delta s_{\text{2D}}|^2)^{1/2}$ are the same by construction for both sources, any difference occurs because of the dependence of the ISP on $\delta s_{\text{2D}}$ and hence one only needs to Taylor expand Equation (11) for a single point source with respect to changes in $\delta s_{\text{2D}}$, which is the offset of the source from the phase center.

After a bit of algebra one finds that the fractional intensity difference between the offset ISP divided by the one in the phase center is

$$f_j^{(s)}(\delta s_{\text{2D}}) \approx \frac{2 \, \delta s_{\text{2D}} \cdot \delta s_{\text{2D}}}{1 - |\delta s_{\text{2D}}|^2} \left[ 1 - \frac{\sqrt{1 - |\delta s_{\text{2D}}|^2}}{2 \Phi(t)} \frac{\partial \Phi(t)}{\partial t_w} \right].$$

(12)

with the derivative of the power spectrum being evaluated at the point $t = (\delta s_{\text{2D}}, \sqrt{1 - |\delta s_{\text{2D}}|^2} - 1)$. The length of $\delta s_{\text{2D}}$ does not necessarily have to be much smaller than unity because so far we only expanded in $\delta s_{\text{2D}}$. If we also further assume that $|\delta s_{\text{2D}}| \ll 1$, then the equation further simplifies to

$$f_j^{(s)}(\delta s_{\text{2D}}) \approx \frac{2 \, \delta s_{\text{2D}} \cdot \delta s_{\text{2D}}}{1 - |\delta s_{\text{2D}}|^2} \left[ 1 - \frac{\partial \ln \Phi(t)}{2 \partial t_w} \right].$$

(13)

The ISP is therefore no longer spatially invariant but starts to depend on the power spectrum of the ionosphere in the $w$-direction and on the offset of the source from the phase-reference center. Whereas to the first order the ISP remains identical to that of a thin phase screen, the large-scale structure (i.e., with $t_w \approx 0$) of the ionosphere in the $w$-direction becomes important because $\partial \ln \Phi(t)/\partial t_w \neq \text{constant}$ for a thick ionosphere. Because this additional term depends strongest on the large-scale $w$-modes, we expect this term to be rather insensitive to relative small changes in $s_{\text{2D}}$ and show slower diurnal and seasonal variations.

If, instead of working with the image $\Delta J^{(s)}(s_u, s_v)$, we work with $\delta J^{(s)}(s_u, s_v) = s_{\text{2D}}^2 \Delta J^{(s)}(s_u, s_v)$ (see also Section 5.2), the fractional difference is

$$f_j^{(s)}(\delta s_{\text{2D}}) \approx - \left( \delta s_{\text{2D}} \cdot \delta s_{\text{2D}} \right) \left[ \frac{\sqrt{1 - |\delta s_{\text{2D}}|^2}}{|\Phi(t)|} \frac{\partial |\Phi(t)|^2}{\partial t_w} \right].$$

(14)

A physical interpretation of this is the following: when observing away from the phase center, one becomes sensitive to large-scale modes in the $w$-direction. This causes a gradient in the electron density over a scale that corresponds to depth of the ionosphere along the line of sight to the source projected onto the array. For small angles away from the phase center, this projected baseline is small and interference between the opposite parts of the ionosphere is not symmetric. This results in large-scale interference patterns on the plane of the array. These patterns correlate strongest on long baselines and thus show up as small-scale structures in the ISP, i.e., near the point sources. We note that in the case of a thin ionosphere, which has $\partial \ln |\Phi(t)|^2/\partial t_w \equiv 0$, the ISP is spatially invariant to all orders.

The interpretation of Equation (14) is that offsets, $s_{\text{2D}}$, perpendicular to $s_{\text{2D}}$ lead to a zero fractional difference, whereas parallel offsets lead to maximal differences (either positive or negative). This implies that the lowest order the ISP is invariant in the tangential direction when moved from the phase center on radial spokes and is stretched or squeezed in the radial direction further away from the phase center, depending on the gradient of the ionospheric power spectrum in the $w$-direction. Equation (14) can thus be used to determine the ISP over a wider FOV, taking the first-order effect of the thickness of the ionosphere into account. By having several calibrators spread over the FOV, one can determine the nearly constant value of $\partial \ln |\Phi(t)|^2/\partial t_w$. Once known, the ISP can be determined to the lowest order for the rest of the FOV by simple rescaling.

To estimate the effect of the thickness of the ionosphere let us assume it has a thickness $d_w$ in units of wavelength and a uniform electron density in the $w$-direction. The normalized power spectrum in the $w$-direction has the functional form $|\Phi(t_w)|^2 = (d_w/\pi) \times \sin^2(d_w t_w)$. Inserting this in Equation (14) and Taylor expanding to the lowest order yields, along radial spokes, $f_j^{(s)} \approx (\pi/3) d_w |\Phi(t_w)|^2 |s_{\text{2D}}|^2$. We note that even though the fractional changes in the ISP are smaller near the sources, the absolute changes are larger because of the steep increase of the power spectrum, i.e., $|\Phi(t_w)|^2 \propto |s_{\text{2D}}|^2$. For $\beta = 11/3$ i.e., a Kolmogorov spectrum. Note that this equation breaks down near the outer scale (i.e., whose effect shows up closest to the images) of the ionosphere; otherwise, it would not be bounded.

5. THREE-DIMENSIONAL IONOSPHERIC POWER-SPECTRUM TOMOGRAPHY

After having derived an equation in the previous section that describes the scattering of point sources due to the three-dimensional ionosphere, we introduce here the equivalent

5 Numerically, we find $f_j^{(s)} \sim 10^{-2} (d_w/10^9) \, |s_{\text{2D}}|^2 / |\Phi(t_w)|^2$. As an example, for an ionosphere with a 200 km thickness and observed at 2 m wavelength, the ISP about 10 away from sources near the edge of a 10 FOV changes by ~1% of the ISP maximum. This might seem small, but could still be considerably larger than the noise or confusion level for bright sources in the field. Hence, in high dynamic range and wide FOV imaging experiments the third dimension of the ionosphere becomes very important.
expression for a continuous intensity field and then discuss the consequences of these results for ionospheric calibration.

5.1. Scattering of a Continuous Intensity Field

One can simply extend the point-source equation for scattering (Equation (11)) to a convolution-type operation on a continuous intensity field by relating the flux density with intensity times the area. In that case, one readily sees that

\[ \delta I^{(i)}(s_u, s_v) = \frac{1}{s_w^2} \int \int \delta I^{(s)}(s_{0,u}, s_{0,v}) |\hat{\Phi}(s - s_0)|^2 ds_{0,u} ds_{0,v}. \]  

(15)

We note that this is a two-dimensional convolution with a three-dimensional kernel. This makes it more difficult to deconvolve using simple two-dimensional Fourier techniques. For a small FOV, one can simply set \( s_w = s_{0,w} = 0 \) and perform a two-dimensional convolution through fast Fourier transform methods. For wider FOVs, this cannot be done. However, one can rewrite the equation as a three-dimensional convolution as follows:

\[ \delta I^{(i)}(s_x, s_y) = \frac{1}{s_w^2} \int \int \int \delta I^{(s)}(s_{0,x}, s_{0,y}) \delta_s(s_{w,u}) |\hat{\Phi}(s - s_0)|^2 ds_0, \]  

(16)

where \( \delta_s(s_{w,u}) = s_{0,w} - (1 - s_{w,u}^2 - s_{0,v}^2)^{1/2} \) and \( \delta_k \) is a Kronecker delta function. Note that in this equation \( s_0 \) should no longer be treated as a unit vector but that \( (1 - s_{0,u}^2 - s_{0,v}^2) \geq 0 \) still. The intensity \( I^{(s)}(s_0) \) is a cylinder of radius unity that has the same value as \( I^{(s)}(s_{0,u}, s_{0,v}) \) for each value of \( s_w \). Writing the equation like this makes it a three-dimensional convolution which can be performed using Fourier transform techniques, but at the cost of more memory and computational effort.

5.2. Determination of the Three-dimensional Ionospheric Power Spectrum

Here, we derive an expression for the full ionospheric power spectrum in terms of the scattered intensity field and the incident field. To do this, first we define the rescaled version of the scattered intensity as

\[ \delta J^{(i)}(s_u, s_v) \equiv s_w^2 \delta I^{(i)}(s_u, s_v), \]  

(17)

and

\[ \delta \tilde{J}^{(i)}(u, v) = \int \int \delta J^{(i)}(s_u, s_v) e^{2\pi i (s_u u + s_v v)} ds_u ds_v \]  

as its Fourier transform. We note that the scattered intensity is zero if \( s_u^2 + s_v^2 > 1 \). If we now define (see, e.g., Sramek & Schaw 1989; Cornwell & Perley 1992; for a similar approach for non-coplanar arrays\(^6\))

\[ J^{(i)}_{3D}(s_0) \equiv I^{(i)}_{3D}(s_0) \cdot \delta \left( s_{0,w} - \sqrt{1 - s_{0,u}^2 - s_{0,v}^2} \right), \]  

(18)

we obtain

\[ \delta J^{(i)}(s_u, s_v) = \int \int J^{(i)}_{3D}(s_0) |\hat{\Phi}(s - s_0)|^2 ds_0. \]  

(19)

We note that this equation can be integrated over infinity, as long as the intensities are zero (as they are) when \( |s_0| > 1 \). Using the relation between convolution and Fourier transforms, we can now write this as

\[ \delta \tilde{J}^{(i)}(u, v) = \tilde{J}^{(i)}_{3D}(u) \cdot F(\hat{\Phi}(s)^2)(u). \]  

(20)

with \( \tilde{J}^{(i)}_{3D}(u) = \int \int J^{(i)}_{3D}(s_0) e^{2\pi i s_0 u} ds_0 \) and \( F(\hat{\Phi}(s)^2) \) being the autocorrelation function of the ionospheric scattering function. This remarkable equation shows that the two-dimensional field \( \delta \tilde{J}^{(i)}(u, v) \) contains information about the full three-dimensional structure of the ionosphere if a reference field \( \tilde{J}^{(i)}_{3D}(u) \) is available; this is closely related to the technique of “holography.” We can now go one step further and show after a little algebra that

\[ \tilde{J}^{(i)}_{3D}(u) = \tilde{I}^{(i)}(u, v) \otimes H(u, v; w), \]  

(21)

being a two-dimensional convolution, with \( w \) as control parameter. The function

\[ H(u, v; w) = 2\pi \int_0^1 e^{-2\pi i w \sqrt{1-x^2}} J_0(s \cdot u_{2D}) s \, ds, \]  

(22)

with \( u_{2D} = \sqrt{u^2 + v^2} \) is a Hankel transform of \( e^{-2\pi i w \sqrt{1-x^2-x'^2}} \). Putting this all together, we find the final result

\[ |\hat{\Phi}(s)|^2 = F^{-1} \left[ \frac{\delta \tilde{J}^{(i)}(u, v)}{\tilde{I}^{(i)}(u, v) \otimes H(u, v; w)} \right]. \]  

(23)

Hence, the three-dimensional power spectrum of the ionosphere or its autocorrelation function can be reconstructed from the ratio between the Fourier transform of the rescaled scattered intensity field and the Fourier transform of the incident radiation field convolved with a Hankel function.

So what does Equation (23) mean? Looking carefully at the equation, we see that \( H(u, v; w) \) is the Fourier transform of a unit phasor with a phase that is determined by the distance (in wavelength) from a half-sphere of radius \( w \) to the \( u,v \)-plane along a line perpendicular to the latter. In other words, \( H(u, v; w) \) acts as the Fourier transform of a complex optical transfer function in the pupil plane (see, e.g., Goodman 1985), which in this case is the full sky and not (as usual) the interferometer plane itself. Equivalently, \( H(u, v; w) \) acts as a complex PSF in \( u,v \)-space (i.e., convolving \( I^{(i)}(u, v) \)). Turning this around, a convolution in \( u,v \)-space is a multiplication of the sky intensity with the reciprocal of the convolution kernel. Hence, \( H(u, v; w) \) causes a complex beam of unit amplitude on the sky equal to the phasor in Equation (22). This exactly extracts the information from \( \delta \tilde{J}^{(i)}(u, v) \) on a particular \( w \)-slice cut through the three-dimensional autocorrelation function of the ionosphere. Hence, by simply multiplying the sky model \( I^{(i)}(s_u, s_v) \) with the complex phasor in Equation (22) one obtains a complex sky-intensity cube. Fourier transforming this back slice by slice provides the denominator in Equation (23).

To illustrate this further, imagine that the sky contains only one point source of unit flux density at \((s_{0,u}, s_{0,v})\). In that case,

\[ F(|\hat{\Phi}(s)|^2) = \left[ e^{-2\pi i (s_{0,u} u + s_{0,v} v) \sqrt{1-s_u^2-s_v^2}} \right]. \]  

(24)

Substituting Equation (16) back into this equation shows that the left- and right-hand sides of the equation are identical as
required. By bringing the denominator to the left side of the equation, we find for a point source
\[
|\tilde{\Phi}(s - s_0)|^2 = \mathcal{F}^{-1}[\delta \tilde{J}^{(s)}(u, v)].
\] (25)

This is the inversion of Equation (11) for a single point source. Hence, every point source in the sky probes the ionospheric power spectrum on an Ewald sphere of reflection. The full sky probes the power spectrum of the ionosphere, as encoded in power spectrum on an Ewald sphere of reflection. The full sky which the ISP will change is its effect on the scattering in the image. One also finds that the scale over which the ionosphere typically is densest, is encoded in the image on scales ten times below the resolution limit. Hence, baselines well beyond 20–30 km are required to fully extract the three-dimensional structure of the ionosphere and that light rays travel on a straight line through the medium.

6. CONNECTION TO THE PHASE SCREEN

While in this paper we started our discussion from the Born approximation and derived the scattered field, this is only a valid approach in the weak scattering regime. Despite this approximation, it is remarkably accurate up to intensity fluctuations very close to unity. However, there are other ways to solve for the scattered electric field (e.g., Kravtsov 1992 for a detailed review) in the weak scattering regime that are closely connected to the formalism as introduced above and to the phase-screen approach. Instead of looking at every point of the medium as a source of a single scattering (as is done in the Born approximation), one can also assume that the medium does not modify the amplitude of the incident wave to the first order and that light rays travel on a straight line through the medium. In that case, since the medium does not absorb or amplify the wave, only a phase shift occurs between different points in the medium when a plane wave enters the ionosphere.

We can again describe this in terms of the refractive index or electron density as follows. A phase shift of \(\delta \psi \approx k \cdot \delta n(r) ds\) is introduced between a wave traveling in a medium with refractive index \(1 + \delta n(r)\) and unity, respectively, where the integral is carried out along a straight line through the medium (see, e.g., Lutomirski & Yura 1971). A plane wave of a source with unit amplitude entering medium exits as a wave with a “wrinkled” phase-front due to varying refractive indices along different paths (Cronyn 1972). The auto-correlation of this “phase screen” is the function with which the visibilities of that source are multiplied in the \(uv\)-plane and its Fourier transform is the ISP as we discussed it before. This allows us to directly connect the previous analysis with that of the phase screen through the so-called Radon transform, which is related to the Fourier projection-slice theorem.

The Fourier projection-slice theorem (Bracewell 1986), in our context, states that the two-dimensional Fourier transform of the Radon projection (Deans 1983), along straight lines, of a three-dimensional medium equals a two-dimensional slice (perpendicular to the projection direction) through the three-dimensional Fourier transform of the object. We now note that the phase screen, up to a constant, is in fact a two-dimensional projection of the refractive index \(\delta n(r)\) in three dimensions. Hence, the Fourier transform of the phase screen is simply a two-dimensional slice through the three-dimensional Fourier transform of the refractive index \(\delta n(r)\) in the medium. The autocorrelation of the phase screen as measured in the \(uv\)-plane is then a slice through the three-dimensional power spectrum of the refractive index, hence that of the electron-density distribution. We immediately see the connection to the discussion in Section 5 and how this connects to Equation (11).

Since point-sources in different directions project the three-dimensional power spectrum differently on the \(uv\)-plane, they probe different slices through the power spectrum. Disregarding the geometric curvature terms, this slice is the tangent plane to the Ewald sphere of reflection at the point \(s - s_0\) which for small vectorial differences are all slices through \(s - s_0 = (0, 0, 0)\), which is exactly the Fourier projection-slice theorem.

Observing over a wide FOV allows one to build a three-dimensional electron-density power spectrum from these different slices. Obviously, the measured auto-correlation of the
phases are the result of many point sources and need to be disentangled. This was discussed in Section 5 in detail and is identical in the current situation. Thence, the phase-screen approach extended to three dimensions is completely identical to the approach taken in this paper.

7. RESULTS AND CONCLUSIONS

A tomographic method has been introduced that allows us to quantify the three-dimensional power spectrum of the ionospheric electron-density fluctuations based on radio-interferometric observations by a two-dimensional planar array. The goal has been to provide a more complete and physically intuitive description of the effects of the full three-dimensional ionosphere over a wide FOV on radio-interferometric images, without any approximations about either a small FOV and/or a thin ionospheric slab (i.e., phase screens). Neither is expected to be sufficiently accurate in upcoming high dynamic range observations with low frequency arrays. The description is valid to the first-order Born approximation, which holds well for frequencies well above the plasma frequency of the ionosphere. Second-order corrections are typically several orders of magnitude below the first-order corrections at frequencies $\gtrsim 100$ MHz. However, we stress that the method is only valid for weak scattering and that the higher order corrections are needed if the phase fluctuations approach unity. The main results and conclusions are as follows.

1. When modeling the ionosphere, it should not be the ionospheric electron-density distribution that is the primary structure to model, but its autocorrelation function or equivalently its power spectrum. Any information about the phase structure of the ionospheric electron-density distribution is lost in the cross-correlation of the electric field when obtaining the visibilities.

2. A three-dimensional ionosphere causes a spatially varying convolution of the sky (at the second-order level), whereas a two-dimensional phase screen or a thin ionosphere results in a spatially invariant convolution. Ionospheric structure in the $w$-direction causes, to the lowest order, radial stretching or squeezing of the ionospheric scattering PSF but leaves its tangential structure invariant. Correcting for the thickness of the ionosphere can thus be reduced (to the lowest order) to determining a single number, i.e., the level of stretching or squeezing.

3. Residual speckle, which cannot be corrected in short time integrations, causes a diffuse intensity halo around bright sources beyond a certain distance from the source. Whereas longer integrations probe more of the total scattered flux of the ISP, because of its very steep intensity decline away from the source, these longer integrations also cause smearing of the instantaneous speckle pattern due to variations of the ionosphere. This halo (“seeing”) and related speckle noise might therefore pose a fundamental limitation on the ability to reach the thermal noise level in interferometers at very low frequencies after long total integrations.

4. Long baselines substantially help in correcting for the effects of the largest wave modes of the ionosphere, as seen inside the FOV of typical interferometers, as well as for its three-dimensional structure in the $w$-direction. The reason is that these large-scale modes cause a small-scale diffraction pattern and thus show up in the image on small angular scales that can only be resolved on long baselines (i.e., of a size equal to or larger than the projected scale of the wave mode).

5. An exact mathematical expression is derived that provides the power spectrum of the ionospheric electron-density fluctuations from a rescaled scattered intensity field and an incident intensity field convolved with a complex unit phasor defined on the full sky pupil plane. This is related to a holographic principle. In the limit of a small FOV, the method reduces to the usual thin phase screen approximation. It is also shown, through the application of a Radon projection and the Fourier projection-slice theorem that the extension of the phase-screen approach to three dimensions is identical to the introduced tomographic method.

Whereas in this paper no direct implementation of an algorithm is given on how to calibrate the ionosphere, a more physically intuitive picture was presented that goes beyond the single (or multiple) phase-screen models of the ionosphere and is valid in the presence of a full three-dimensional ionosphere and full-sky FOV. A mathematical expression was presented that shows that the ionosphere causes a spatially varying PSF (at the second order level) over the FOV, determined by the instantaneous three-dimensional power spectrum of the ionospheric electron-density fluctuations. Subsequently, this expression was inverted, showing that through a “holographic” principle one can extract information about the three-dimensional structure of the ionosphere from only electric-field measurements on a two-dimensional plane. In a forthcoming work, it remains to be investigated whether these results can be implemented in a practical algorithm in the context of the measurement equation (see, e.g., Matejeck & Morales 2009) in order to calibrate visibilities without assumptions about thin ionospheric phase screens, and by using instead the full three-dimensional power spectrum of the ionosphere (i.e., “tomographic self-calibration”).

L.K. is supported through an NWO-VIDI program subsidy. The author also acknowledges Saleem Zaroubi, Rajat Thomas, and Ger de Bruyn for very useful and clarifying discussions.

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