Abstract

In this work a non-conservative balance law formulation is considered that encompasses the rotating, compressible Euler equations for dry atmospheric flows. We develop a semi-discretely entropy stable discontinuous Galerkin method on curvilinear meshes using a generalization of flux differencing for numerical fluxes in fluctuation form. The method uses the skew-hybridized formulation of the element operators to ensure that, even in the presence of under-integration on curvilinear meshes, the resulting discretization is entropy stable. Several atmospheric flow test cases in one, two, and three dimensions confirm the theoretical entropy stability results as well as show the high-order accuracy and robustness of the method.

Keywords: Balance laws, Entropy stable, Discontinuous Galerkin, Geophysical flow

1. Introduction

Discontinuous Galerkin (DG) schemes for hyperbolic equations have many favorable properties, such as high-order accuracy, compact stencils, geometric flexibility, and the ability to add physical dissipation through upwind-biased numerical fluxes. DG belongs to the class of high-order element-based schemes [26, 29, 33] that can harness the fine-grained GPU parallelism needed for efficient exascale discretizations [1, 2, 32]. Due to their compact nature, DG schemes have demonstrated excellent scalability [41]. Provably stable DG discretizations can be constructed for linear systems [35, 60], and often work well for nonlinear systems with smooth and reasonably well-resolved solutions. However, in the presence of discontinuities and under-resolved phenomena, DG schemes often require additional stabilization. Classical stabilization techniques include dealiasing through over-integration [40], spectral filters [21], limiters [36, 42], and artificial viscosity [59, 63]. While these techniques enabled DG to perform complex simulations [5], they are not without drawbacks such as the possible loss of accuracy, hand-tuning of multiple parameters, and lack of a solid theoretical foundation.

Recently, entropy-stable DG schemes have emerged as a way to construct high-order discretizations with rigorous nonlinear stability estimates in a parameter-free fashion [24, 48]. Namely, semi-discrete entropy stability estimates are guaranteed by using specially constructed numerical fluxes [53] and flux-differencing [22]. Importantly, the schemes are computationally affordable because they do not rely on exact integration but instead exploit the summation-by-parts property of polynomial differentiation operators [23]. Following the pioneering work in Carpenter et al. [9], Gassner et al. [25] for the compressible Euler equations, the class of entropy-stable DG schemes has been extended to general systems of nonlinear conservation laws by Chen and Shu [16]. Chan [13, 14] generalized this to “modal” DG formulations which do not satisfy summation-by-parts. Most entropy stable DG schemes are constructed assuming exact integration in time, and Ranocha et al. [45] achieved fully discrete entropy-stability using relaxation Runge–Kutta time integrators.
Entropy-stable DG schemes have been constructed for many systems of nonlinear conservation and balance laws, such as the shallow water equations [61], the compressible multi-component Euler equations [47], special relativistic ideal magneto-hydrodynamics [20], and the ten-moment Gaussian closure equations [6]. However, many equations of physical interest do not conform to the standard balance law form and require extensions of the basic entropy-stability framework. Entropy-stable treatment of non-conservative source terms appearing in certain formulations of magneto-hydrodynamics has been presented in Liu et al. [37] for the ideal equations and in Bohm et al. [8] for the resistive equations. Renac [46] constructed an entropy-stable DG spectral element scheme for a one-dimensional, nonlinear hyperbolic systems in non-conservative form and applied it to two-phase flow. Coquel et al. [17] then used this framework for the Baer–Nunziato two-phase flow model.

The motivation for this paper is the construction of entropy stable schemes for the Euler equations with gravity. These equations are of fundamental importance in simulations of astrophysical phenomena, atmospheric large-eddy simulations, numerical weather prediction, and climate modeling. Due to their excellent properties, DG schemes are already emerging in computational astrophysics [31, 49] as well as in small- and large-scale atmospheric modeling [3, 28, 39, 56]. In contrast to engineering applications, many atmospheric models formulate the equations in term of an entropy-type variable known as potential temperature, which has favorable properties for simulating atmospheric flows [32]. However, in climate modeling applications, conservation of total energy and balanced energy transfers are also of great interest. This can be achieved, for example, by using mimetic methods [54]. An alternative, that we propose in this paper, is to guarantee conservation of energy by using the total energy formulation, and use the entropy-stable approach for robustness and consistency with the second law of thermodynamics. Crucially, this requires an extension of the existing entropy-stable framework, since in the total energy formulation of the Euler equations the gravity force enters as a non-conservative and spatially varying source term.

To do this we recast the Euler equations with gravity in a more general, non-conservative balance law formulation. We then construct arbitrary order flux differencing DG schemes for this balance law formulation and give conditions that the numerical fluxes must satisfy to ensure entropy stability. We present, in explicit form, an entropy-conserving numerical flux for the Euler equations with gravity and show how to modify it to make it entropy-stable. A variation on the classical one-dimensional Sod shock tube benchmark, put into a gravitational field, is used to show the versatility of the scheme in terms of choice of basis and quadrature rule. Two-dimensional atmospheric test cases of buoyant convection and gravity wave propagation verify, respectively, the entropy-conservation and high-order accuracy of the scheme. Finally, a baroclinic wave benchmark on the sphere demonstrates the robustness of the scheme for long-time simulations of idealized weather-like phenomena in a curvilinear three-dimensional geometry.

2. Balance law formulation

The rotating, compressible Euler equations for dry atmospheric flows in $d$ dimensions ($d = 1, 2, 3$) are

\[
\frac{\partial \rho}{\partial t} + \sum_{k=1}^{d} \frac{\partial \rho u_k}{\partial x_k} = 0, \quad \text{(1a)}
\]

\[
\frac{\partial \rho u_i}{\partial t} + \sum_{k=1}^{d} \frac{\partial \rho u_i u_k}{\partial x_k} + \frac{\partial p}{\partial x_i} + \rho \frac{\partial \phi}{\partial x_i} = -2 \sum_{j,k=1}^{d} \varepsilon_{ijk} \omega_j u_k, \quad i = 1, \ldots, d, \quad \text{(1b)}
\]

\[
\frac{\partial \rho e}{\partial t} + \sum_{k=1}^{d} \frac{\partial u_k (\rho e + p)}{\partial x_k} = 0. \quad \text{(1c)}
\]

Here the prognostic variables are density $\rho$, momentum $\rho u_i$ for $i = 1, \ldots, d$, and total energy $\rho e$. The pressure $p$ is

\[
p = (\gamma - 1) \left( \rho e - \rho \phi - \frac{1}{2} \sum_{j=1}^{d} \rho u_j^2 \right), \quad \text{(2)}
\]

where $\phi$ is a spatially varying geopotential function and the constant $\gamma = \frac{c_p}{c_v}$ is the specific heat ratio given in terms of the specific heats for constant pressure and volume, $c_p$ and $c_v$, respectively; we take $\gamma = 7/5$. The components of a
(potentially spatially varying) planetary rotation vector are $\omega_i$ for $i = 1, \ldots, d$. The Coriolis source term is given as twice the cross product of the rotation vector with the fluid velocity, expressed using the Levi–Civita symbol

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } ijk \text{ is 123, 312, or 231}, \\ -1, & \text{if } ijk \text{ is 312, 132, or 213}, \\ 0, & \text{otherwise}. \end{cases} \quad (3)$$

Governing equations (1) can be rewritten compactly as a balance law of the form

$$\frac{\partial q(x,t)}{\partial t} + \sum_{i=1}^{d} A_i(q(x,t),x) \frac{\partial h_i(q(x,t),x)}{\partial x_i} = g(q(x,t),x). \quad (4)$$

Here, $q(x,t)$ is the solution vector of length $N_c$, the total number of prognostic variables. Additionally, $h_i(q,x) \in \mathbb{R}^{N_d}$ is a vector-valued function and $A_i(q,x) \in \mathbb{R}^{N_c \times N_d}$ is a matrix-valued function, where in order to allow modeling flexibility $N_d$ is allowed to be different from $N_c$; for conservation laws $N_d = N_c$ and $A_i = I$ with $h_i$ being the flux function. Balance law formulation (4) allows for source terms to be included in $A_i \frac{\partial h_i}{\partial t}$ or the vector-valued function $g(q,x) \in \mathbb{R}^{N_c}$.

With $d = 3$ the atmospheric equations (1) can be written in balance law form (4) by defining the solution vector

$$q = \begin{bmatrix} \rho & \rho u_1 & \rho u_2 & \rho u_3 & \rho e \end{bmatrix}^T. \quad (5)$$

The matrix function $A_k$ and vector function $h_k$ for $k = 1, 2, 3$ have $N_d = 6$ and are defined as

$$A_k(q,x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad h_k(q,x) = \begin{bmatrix} \rho u_k \\ \rho u_1 u_k + \delta_{1k} \rho \\ \rho u_2 u_k + \delta_{2k} \rho \\ \rho u_3 u_k + \delta_{3k} \rho \\ u_k(\rho e + p) \end{bmatrix}, \quad (6)$$

with $\delta_{1k}$ being the Kronecker delta. The source term is

$$g(q,x) = \sum_{j,k=1}^{d} \begin{bmatrix} 0 \\ -2\epsilon_{1jk} \omega_j u_k \\ -2\epsilon_{2jk} \omega_j u_k \\ -2\epsilon_{3jk} \omega_j u_k \\ 0 \end{bmatrix}; \quad (7)$$

note that only the Coriolis term is included in the source and the geopotential is included in the differential term.

3. Entropy analysis

Here we summarize continuous entropy results needed for the discrete analysis that follows; for a more complete overview of the theory see, for example, Dafermos [18, Sections 1.4, 1.5, 3.1, and 3.2].

3.1. Companion balance law

We assume that balance law (4) is endowed with a nontrivial companion balance law

$$\frac{\partial \eta(q(x,t),x)}{\partial t} + \sum_{j=1}^{d} \frac{\partial \zeta_j(q(x,t),x)}{\partial x_j} = \Pi(q(x,t),x), \quad (8)$$

where $\eta$ is a scalar convex entropy function and $\zeta_j$ are associated entropy fluxes. This assumption is valid for many systems of balance laws from continuum physics which typically have a natural companion balance law related to the
second law of thermodynamics, e.g., Dafermos [18, Section 3.3]. Specifically, companion balance law (8) provides an entropy balance constraint on the solution of balance law (4). The vector of entropy variables $\mathbf{\beta}$ is obtained by differentiating the entropy function with respect to the state $\mathbf{q}$. That is, $\mathbf{\beta} = \frac{\partial \eta}{\partial \mathbf{q}}$, with component $\alpha = 1, \ldots, N_e$ given by

$$
\beta_\alpha(q, x) = \frac{\partial \eta(q, x)}{\partial q_\alpha}.
$$

To obtain an entropy balance condition, we first contract the balance law (4) with the entropy variables (9),

$$
\frac{\partial \eta(q, x)}{\partial t} + \sum_{i=1}^{d} \beta^T(q, x) A_i(q, x) \frac{\partial h_i(q, x)}{\partial x_i} = \beta^T(q, x) g(q, x).
$$

Comparing (10) with the companion balance law (8) gives

$$
\frac{\partial \zeta_j(q, x)}{\partial q_\alpha} = \beta^T(q, x) A_j(q, x) \frac{\partial h_j(q, x)}{\partial q_\alpha},
$$

when the entropy production function is taken as

$$
\Pi(q, x) = \beta(q, x)^T g(q, x) + \sum_{i=1}^{d} \left( \frac{\partial \zeta_i(q, x)}{\partial x_i} - \beta^T(q, x) A_i(q, x) \frac{\partial h_i(q, x)}{\partial x_i} \right).
$$

We will restrict ourselves to equations for which the summation term is exactly zero so that $
Pi(q, x) = \beta(q, x)^T g(q, x)$.

Integrating (8) over the domain $\Omega$ with boundary $\partial \Omega$ and outward unit normal $\mathbf{n}$ gives the continuous entropy balance equality

$$
\int_\Omega \frac{\partial \eta(q, x)}{\partial t} + \sum_{j=1}^{d} \int_{\partial \Omega} n_j \zeta_j(q, x) = \int_\Omega \Pi(q, x).
$$

For sufficiently regular (smooth) solutions $\mathbf{q}$, entropy balance (13) is satisfied and entropy conservation holds. For non-smooth solutions, an admissible weak solution to the governing equations should satisfy an entropy dissipation inequality:

$$
\int_\Omega \frac{\partial \eta(q, x)}{\partial t} + \sum_{j=1}^{d} \int_{\partial \Omega} n_j \zeta_j(q, x) \leq \int_\Omega \Pi(q, x).
$$

The primary objective of semi-discrete entropy analysis is to ensure a discrete analogue of entropy balance (13) and dissipation (14).

### 3.2. Entropy for the atmospheric Euler equations

For the atmospheric Euler equations (1), we define the convex mathematical entropy function

$$
\eta = -\frac{\rho s}{\gamma - 1},
$$

where $s = \log(p/\rho^\gamma)$ is the specific (physical) entropy. The components of the entropy flux (11) are taken to be

$$
\zeta_j = u_j \eta.
$$

The entropy variables (9) are obtained with $\eta$ defined by (15) resulting in

$$
\beta_1 = \frac{\gamma - s}{\gamma - 1} \left( u_1^2 + u_2^2 + u_3^2 - 2 \phi \right) b, \quad \beta_2 = 2bu_1, \quad \beta_3 = 2bu_2, \quad \beta_4 = 2bu_2, \quad \beta_5 = -2b.
$$
with \( b = \rho / 2p \) is the inverse temperature. Recognizing that \( b = -\beta \rho \), the prognostic state can be recovered from the entropy variables using

\[
\begin{align*}
  u_1 &= \frac{\beta_2}{2b}, \\
  u_2 &= \frac{\beta_3}{2b}, \\
  u_3 &= \frac{\beta_4}{2b}, \\
  e &= \frac{1}{(\gamma - 1)2b} + \frac{u_1^2 + u_2^2 + u_3^2}{2} + \phi, \\
  \rho &= \left( 2b \exp \left( (\gamma - 1) \left( -\beta + \left( 2\phi - \left( u_1^2 + u_2^2 + u_3^2 \right) \right) b \right) \right) \right)^{-1/(\gamma - 1)}.
\end{align*}
\]  

For the mapping between entropy and conserved variables to be well-defined, pressure and density must be positive.

The entropy production for the atmospheric Euler equations (1) is zero, i.e., \( \Pi = 0 \).

4. Flux differencing discontinuous Galerkin methods

In this section we establish our notation and a flux differencing formulation for balance law (4); our notation is motivated by Castro et al. [11], Chan [13, 14], Renac [46].

4.1. Notation

Let the domain \( \Omega \) be tessellated by a mesh of disjoint elements \( K \subset \mathbb{R}^d \) with boundaries \( \partial K \), where each polygonal element \( K \) is mapped to a reference element \( \hat{K} \) with boundary \( \partial \hat{K} \). Points on the physical and reference elements are denoted as \( x \) and \( \xi \), respectively. The outward unit normal in the physical coordinate system is \( n \) with \( \hat{n} \) denoting the unit normal to \( \partial \hat{K} \).

For each element \( K \), we assume there is a suitably smooth (differentiable) transformation between the physical and reference element. For a given function \( u(x) \) defined on a physical element \( K \), we denote its mapping to the reference element \( \hat{K} \) as \( \hat{u}(\xi) \). We let \( J^v(\xi) \) be the Jacobian determinant of the volume transformation and \( J^f(\xi) \) the Jacobian determinant of the surface mapping. Using these Jacobian factors, integrals over physical elements in terms of integrals on the reference element are

\[
\begin{align*}
  \int_K u(x) &= \int_{\hat{K}} J^v(\xi) \hat{u}(\xi), \\
  \int_{\partial K} u(x) &= \int_{\partial \hat{K}} J^f(\xi) \hat{u}(\xi).
\end{align*}
\]

The reference and physical outward unit normals are related by

\[
J^f n_i = \sum_{k=1}^d J^v \frac{\partial \xi_i}{\partial x_k} \hat{n}_k.
\]

On the reference element we assume that functions are approximated in some polynomial space \( \mathbb{K}^N = \mathbb{K}^N(\hat{K}) \), where \( N \) is the degree of approximation. In the numerical results we use tensor product (interval, quadrilateral, and hexahedral) elements to represent the computational domain, and \( \mathbb{K}^N \) is selected to be the space of \( d \)-dimensional tensor product polynomial of total degree \( N \). However, the analysis that follows is more general and can be applied to other elements types, e.g., simplices; see Chan [14] for a fuller discussion of other element types. Specifically, in the definition of the operators there is no assumption that the interpolation and quadrature points are collocated. Additionally, we do not assume that the surface quadrature points are a subset of the volume quadrature points. That said, when Legendre–Gauss–Lobatto points are used for interpolation and quadrature, e.g., a spectral element method, many of the operators simplify; see Remark 1 below.
We let \( \{ \phi_i(\xi) \}_{i=1}^{N_p} \) denote a basis for \( \mathbb{R}^N \), where \( N_p \) is the dimensionality of the space. Any function \( q^N(\xi) \in \mathbb{R}^N \) can therefore be represented as

\[
q^N(\xi) = \Phi^T(\xi) q^N,
\]

where \( \Phi(\xi) \) is a vector of basis functions and \( q^N \) is a vector of expansion coefficients.

4.2. Skew-hybridized summation-by-parts operators

Following Chan [14], we construct skew-hybridized operators involving both volume and surface quadrature nodes. Here we highlight the notation needed in this paper; further details and discussion on the operators accuracy requirements can be found in Chan [14]. Throughout we use the notation \([\cdot]_n\) and \([\cdot]_{n,m}\) to denote the elements of a vector and matrix, respectively.

The volume and surface quadrature rules on the reference element \( K \) are \( \{ [\xi^v], \, [w^v] \}_{i=1}^{N_q^v} \) and \( \{ [\xi^f], \, [w^f] \}_{i=1}^{N_q^f} \), with \( N_q^v \) and \( N_q^f \) being the total number of volume and surface quadrature points, respectively. Here \( \xi^v \) and \( \xi^f \) denote vectors of quadrature points, while \( w^v \) and \( w^f \) are the weights. Diagonal matrices of the quadrature weights are

\[
W^v = \text{Diagonal}(w^v), \quad W^f = \text{Diagonal}(w^f).
\]

Similarly, we define diagonal matrices of the volume and surface Jacobians evaluated at the quadrature points,

\[
J^v = \text{Diagonal}(J^v(\xi^v)), \quad J^f = \text{Diagonal}(J^f(\xi^f)).
\]

As discussed later, \( J^v \) and \( J^f \) are often not exact but approximations in \( \mathbb{R}^N \).

Vandermonde matrices for the volume \( V^v \in \mathbb{R}^{N_q^v \times N_p} \) and surface \( V^f \in \mathbb{R}^{N_q^f \times N_p} \) that interpolate the expansion coefficients to the quadrature points have elements

\[
[V^v]_{n,m} = \phi_m([\xi^v]_n), \quad [V^f]_{n,m} = \phi_m([\xi^f]_n).
\]

The combined quadrature Vandermonde matrix

\[
V = \begin{bmatrix} V^v \\ V^f \end{bmatrix},
\]

interpolates from the expansion coefficients to the volume and surface quadrature points. With the above notation the volume mass matrix

\[
M^v = (V^v)^T W^v J^v V^v,
\]

is defined to approximate inner products of functions in \( \mathbb{R}^N \) on the reference element in physical space.

In order to project functions defined on the volume quadrature points to polynomials in \( \mathbb{R}^N \) we define a quadrature-based \( L^2 \) projection operator

\[
P^v = (M^v)^{-1}(V^v)^T J^v W^v.
\]

We observe that going from the expansion coefficients to the quadrature points and back is exact:

\[
P^v V^v = I.
\]

The polynomial differentiation matrix with respect to the \( i \)-th reference coordinate is \( D_i^N \in \mathbb{R}^{N_p \times N_p} \). Namely, if \( q \) are the expansion coefficients of \( q^N \in \mathbb{R}^N \), then \( D_i^N q^N \) are the expansion coefficients for \( \frac{\partial q^N}{\partial \xi^i} \in \mathbb{R}^N \). The quadrature-based volume integrated differentiation operator with respect to the \( i \)-th reference coordinate is

\[
Q^v_i = W^v V^v D_i^N P^v.
\]

With this, we follow Chan [14] and define the skew-hybridized summation-by-parts (SBP) operator

\[
Q_i = \frac{1}{2} \begin{bmatrix} Q^v_i - (Q^v_i)^T (E^v)^T B^f_i \\ -B_i^f E^v \end{bmatrix}.
\]
Here the matrix $E^v = V^f P^v$ projects between the volume quadrature to the face quadrature and $B_i^f = W^f \hat{N}_i^f J^f$ is the reference normal-weighted surface quadrature with $\hat{N}_i^f$ being the diagonal matrix of component $i$ of $\hat{n}$ at the surface quadrature points. Since $Q_i \in \mathbb{R}^{(N_v^r + N_f^r) \times (N_v^r + N_f^r)}$ it operates on functions defined on the combine volume and surface quadrature.

The skew-hybridized SBP operator is almost skew-symmetric, as characterized by the following lemma from Chan [14, Lemma 1].

**Lemma 1 (The skew-hybridized SBP operator).** The skew-hybridized SBP operator $Q_i$ satisfies the generalized summation-by-parts (SBP) property:

$$Q_i + Q_i^T = \begin{bmatrix} 0 & B_i^f \end{bmatrix}.$$

(32)

The reason for $Q_i$ being referred to as an SBP operator is because the discrete identity

$$\left( p^N \right)^T V^T \left( Q_i + Q_i^T \right) V q^N = \left( V^f p^N \right)^T W^f \hat{N}_i^f J_i^f V^f q^N,$$

(33)

is an approximation of the continuous integration-by-parts identity

$$\int_{\mathbb{R}} \left( \frac{\partial p}{\partial \xi} q + p \frac{\partial q}{\partial \xi} \right) = \int_{\partial \mathbb{R}} \hat{n}_i p q.$$

(34)

The advantage of using $Q_i$ is that one is guaranteed to satisfy the SBP property, no matter the strength of the volume and surface quadrature rules. This allows for the construction of SBP operators in spite of aliasing errors due to inexact integration [14, 23, 44]. Later we will assume consistency for the skew-hybridized SBP operator; see Chan [14] for required accuracy conditions.

**Assumption 1.** The skew-hybridized SBP operator is assumed to be exact for constants:

$$Q_i 1 = 0.$$

(35)

The approximation of the metric terms $J^v \frac{\partial \xi_i}{\partial x_k}$ at the combined volume and surface quadrature points is denoted with the diagonal matrix $G_{jk}$. As noted above, these are often approximated using the discrete operators. One benefit of this is that the volume $G_{jk}^v$ and surface $G_{jk}^f$ can be defined in a consistent manner, e.g., $G_{jk}^f = \text{Diagonal} \left( E^v \text{ diag} \left( G_{jk}^v \right) \right)$ where diag extracts the diagonal of a matrix as a vector. In the discrete entropy stability analysis we will require that the metrics discretely satisfy the geometric conservation law [34, 55]

$$\sum_{k=1}^{d} Q_k G_{jk} 1 = 0,$$

(36)

which is a discrete statement of the continuous metric identity

$$\sum_{k=1}^{d} \frac{\partial}{\partial \xi_k} \left( f \frac{\partial \xi_k}{\partial x_j} \right) = 0.$$

(37)

4.3. Flux differencing in fluctuation form

Motivated by Castro et al. [11] and Renac [46] we define a numerical flux function in fluctuation form as

$$\mathcal{D}_i \left( p, y, q, x \right) = \mathcal{A}_i \left( p, y, q, x \right) \mathcal{H}_i \left( p, y, q, x \right) + \mathcal{A}_i \left( q, x, p, y \right) \mathcal{H}_i \left( q, x, p, y \right) - \left( \mathcal{A}_i \left( p, y, q, x \right) + \mathcal{A}_i \left( q, x, p, y \right) \right) h_i \left( p, y \right).$$

(38)
The matrix-valued function \( \mathcal{A}_i(p, y, q, x) \) and vector-valued function \( \mathcal{H}_i(p, y, q, x) \) are assumed to satisfy the consistency conditions

\[
\begin{align*}
2\mathcal{A}_i(q, x, q, x) &= \mathcal{A}_i(q, x), \\
\mathcal{H}_i(q, x, q, x) &= \mathcal{H}_i(q, x).
\end{align*}
\tag{39}
\tag{40}
\]

In the special case of balance law (4) being a conservation law, i.e., \( \mathcal{I}_i \) and let \( \mathcal{F}_i(p, y, q, x) \) be a symmetric and consistent numerical flux so that

\[
\mathcal{D}_i^{\text{cons}}(p, y, q, x) = \mathcal{F}_i(p, y, q, x) - f_i(p, y).
\tag{41}
\]

The following lemma relates the derivative of the numerical flux function \( \mathcal{D}_i \) to the spatial derivatives in balance law (4), i.e., a flux differencing form of the derivative.

**Lemma 2.** If numerical flux \( \mathcal{D}_i \), defined by (38) satisfies consistency conditions (39) and (40) then

\[
2 \frac{\partial \mathcal{D}_i(q(y), y, q(x), x)}{\partial x_j} \bigg|_{y=x} = A_i(q(x), x) \frac{\partial h_i(q(x), x)}{\partial x_j}
\tag{42}
\]

and

\[
p(x)^T A_j(q(x), x) \frac{\partial h(q(x), x)}{\partial x_j}
= \frac{\partial}{\partial x_j} \left( p(y)^T D_i(q(y), y, q(x), x) - p(x)^T D_i(q(x), x, q(y), y) \right) \bigg|_{y=x}.
\tag{43}
\]

**Proof.** Identity (42) follows from the consistency conditions:

\[
\begin{align*}
& A_i(q(x), x) \frac{\partial h_i(q(x), x)}{\partial x_j} \\
& = \frac{\partial A_i(q(x), x) h_i(q(x), x)}{\partial x_j} - \frac{\partial A_i(q(x), x)}{\partial x_j} h_i(q(x), x) \\
& = \frac{\partial}{\partial x_j} \left( A_i(q(x), x) (h_i(q(x), x) - h_i(q(y), y)) \right) \bigg|_{y=x} \\
& = 2 \frac{\partial}{\partial x_j} \left( \mathcal{A}_i(q(x), x, q(x), x) (\mathcal{H}_i(q(x), x, q(x), x) - h_i(q(y), y)) \right) \bigg|_{y=x} \\
& = 2 \frac{\partial}{\partial x_j} \left( \mathcal{A}_i(q(y), y, q(x), x) (\mathcal{H}_i(q(y), y, q(x), x) - h_i(q(y), y)) \right) \bigg|_{y=x} \\
& \quad + 2 \frac{\partial}{\partial x_j} \left( \mathcal{A}_i(q(x), x, q(y), y) (\mathcal{H}_i(q(x), x, q(y), y) - h_i(q(y), y)) \right) \bigg|_{y=x} \\
& = 2 \frac{\partial}{\partial x_j} \mathcal{D}_i(q(y), y, q(x), x) \bigg|_{y=x}.
\end{align*}
\]
Consistency implies that \( D_i(q, x, q, x) = 0 \), which when combined with (42) implies (43):
\[
p(x)^T A_j(q(x), x) \frac{\partial h(q(x), x)}{\partial x_j} = 2 p(x)^T \frac{\partial D_i(q(y), y, q(x), x)}{\partial x_j} \bigg|_{y=x} = \left( 2 p(y)^T \frac{\partial D_i(q(y), y, q(x), x)}{\partial x_j} \right) \bigg|_{y=x} - \frac{\partial p(x)^T D_i(q(x), x, q(x), x)}{\partial x_i} \bigg|_{y=x} \]
\[
= \frac{\partial}{\partial x_j} \left( p(y)^T D_i(q(y), y, q(x), x) - p(x)^T D_i(q(x), x, q(y), y) \right) \bigg|_{y=x} .
\]

For conservation laws we have the following corollary showing the equivalence to the conservation law flux differencing [13].

**Corollary 3.** If the numerical flux (41) is used then
\[
\frac{\partial D_i^{\text{cons}}(q(y), y, q(x), x)}{\partial x_j} \bigg|_{y=x} = \frac{\partial F_i(q(y), y, q(x), x)}{\partial x_j} \bigg|_{y=x} .
\]

In Renac [46] the case of \( h_i(q, x) = q \) was considered, and the above formulation is equivalent when \( F_i^{\text{Renac}}(p, y, q, x) = q \) in which case the numerical flux reduces to
\[
D_i^{\text{Renac}}(p, y, q, x) = R_i^{\text{Renac}}(p, y, q, x) (q - p) .
\]

**4.4. Discontinuous Galerkin methods with flux differencing in fluctuation form**

To derive a flux differencing scheme we begin with a “strong derivative” DG scheme for balance law (4) for a single element \( K \),
\[
\int_K p(x)^T \frac{\partial q(x,t)}{\partial t} + \sum_{j=1}^d \int_K p(x)^T A_j(q(x), x) \frac{\partial h(q(x), x)}{\partial x_j} \]
\[
= \int_K p(x)^T g(q(x,t), x) - \sum_{j=1}^d \int_{\partial K} p(x)^T D_j^*(q^-(x,t), x^-, q^+(x,t), x^+) n_j ,
\]

Here \( p \) and \( q \) are the test and trial functions, and unless needed for clarity no accent is added to denote that all functions are defined over the element \( K \). The numerical flux on the face is \( D_j^* \) with the superscripts \( - \) and \( + \) denoting values on the two-sides of the surface, with \( - \) being the inside value and \( + \) the outside value; the unit normal \( n \) is outward to element \( K \).

Using the flux differencing identity (43), the DG scheme (48) can be rewritten as
\[
\int_K p(x)^T \frac{\partial q(x,t)}{\partial t} + \sum_{j=1}^d \int_K \frac{\partial}{\partial x_j} \left( p(y)^T D_j(q(y,t), y, q(x), x) - p(x)^T D_j(q(x), x, q(y), y) \right) \bigg|_{y=x} \]
\[
= \int_K p(x)^T g(q(x,t), x) - \sum_{j=1}^d \int_{\partial K} p(x)^T D_j^*(q^-(x,t), x^-, q^+(x,t), x^+) n_j ,
\]
where the volume and surface numerical fluxes can differ. Changing from the physical element $K$ to the reference element $\hat{K}$ gives the final variational form

\[
\int_{\hat{K}} J^v(\xi)p(\xi)^T \frac{\partial q(\xi,t)}{\partial t} + \sum_{j,k=1}^{d} \int_{\hat{K}} G_{jk}(\xi) \frac{\partial}{\partial \xi_k} \left( p(\eta)^T \mathcal{D}_i(q(\eta,t), x(\eta), q(\xi,t), x(\xi)) \right) \bigg|_{\eta=\xi} - \frac{d}{\partial t} \int_{\hat{K}} G_{jk}(\xi) \frac{\partial}{\partial \xi_k} \left( p(\xi)^T \mathcal{D}_i(q(\xi,t), x(\xi), q(\eta,t), x(\eta)) \right) \bigg|_{\eta=\xi} = \int_{\hat{K}} J^v(\xi)p(\xi)^T g(q(\xi,t), x(\xi), t)
\]

with $G_{jk} = J^v \frac{\partial \xi_k}{\partial x_k}$.

A matrix form of the DG scheme (50) for solution component $\alpha$ is

\[
M^\nu \frac{d q^\nu}{d t} + V^T \sum_{j,k=1}^{d} \left( G_{jk} (Q_k \circ \mathcal{D}_{j,\alpha}) - \left( \mathcal{D}_{j,\alpha} \circ Q_k^T \right) G_{jk} \right) 1 = (V^\nu)^T J^v W^\nu \nu^\nu - \sum_{j,k=1}^{d} \left( V^j \right)^T G_{jk}^{\nu} B_{k}^{j} \mathcal{D}^{\nu}_{j,\alpha},
\]

where $\circ$ denotes the Hadamard (element-wise) product. Here the matrix $\mathcal{D}_{j,\alpha}$ denotes component $\alpha$ of the numerical flux $\mathcal{D}_j$ evaluated pairwise at the quadrature nodes with components

\[
[\mathcal{D}_{j,\alpha}]_{n,m} = \left[ \mathcal{D}_j \left[ (\hat{q})_n, [x]_n, [\hat{q}]_m, [x]_m \right] \right]_{\alpha},
\]

and the vector $\nu^\nu$ is the evaluation of the source at the volume quadrature

\[
\left[ \nu^\nu \right]_n = \left[ \nu \left[ (\hat{q}^\nu)_n, [x^\nu]_n \right] \right]_{\alpha}.
\]

The vector $\hat{q}$ is defined on the combined volume and surface quadrature grid from the expansion coefficients $q$. A natural and consistent choice is $\hat{q}_\alpha = V q^\nu$, but as will be seen in the next section, for general element types entropy stability requires that $\hat{q}$ be defined through an entropy-projection. With the flux matrix notation (52) we have the approximation

\[
\int_{\hat{K}} G_{jk}(\xi) \frac{\partial}{\partial \xi_k} \left( p_\alpha(\eta) \left[ \mathcal{D}_j(q(\eta), x(\eta), q(\xi), x(\xi)) \right] \right) \bigg|_{\eta=\xi} \approx \left( p^\nu_\alpha \right)^T V^T G_{jk} (Q_k \circ \mathcal{D}_{j,\alpha}) 1,
\]

and

\[
\int_{\hat{K}} G_{jk}(\xi) \frac{\partial}{\partial \xi_k} \left( p_\alpha(\xi) \left[ \mathcal{D}_j(q(\xi), x(\xi), q(\eta), x(\eta)) \right] \right) \bigg|_{\eta=\xi} \approx \left( p^\nu_\alpha \right)^T V^T \left( \mathcal{D}_{j,\alpha} \circ Q_k^T \right) G_{jk} 1.
\]

**Remark 1.** When the quadrature and interpolation points are both Legendre–Gauss–Lobatto, i.e., the DG spectral element method, $Q_j$ can be replaced with $Q_j^\nu$, the mass matrix $M^\nu$ is diagonal, $P^\nu = V^\nu = I$, and $V^j$ selects points from the volume quadrature that are on the boundary.
4.5. Entropy stable DG with flux differencing in fluctuation form

By construction, DG scheme \((51)\) is consistent, and here we discuss what is required to ensure entropy stability. Following Castro et al. [11], we assume the volume numerical flux satisfies the entropy conservation property\(^3\),

\[
\beta^T (q, x) \mathcal{D}_j (q, x, p, y) - \beta^T (p, y) \mathcal{D}_j (p, y, q, x) = \zeta_i (p, y) - \zeta_i (q, x), \tag{56a}
\]

and the surface flux the entropy dissipation property

\[
\beta^T (q, x) \mathcal{D}_j^s (q, x, p, y) - \beta^T (p, y) \mathcal{D}_j^s (p, y, q, x) \leq \zeta_i (p, y) - \zeta_i (q, x); \tag{56b}
\]

we refer to relations \((56a)\) and \((56b)\) as conservative and dissipative entropy shuffle relations. For entropy stability it is critical that the numerical fluxes, source terms, and entropy variables be evaluated with a \(\tilde{q} \) that properly accounts for variational crimes. To do this we define the auxiliary quantities

\[
q^\alpha = V^\alpha q^N, \quad q_f^\alpha = V_f^\alpha q^N, \quad q_a = V q^N, \quad \alpha = 1, \ldots, N_c, \tag{57a}
\]

\[
x^\alpha = V^\alpha x^N, \quad x_f^\alpha = V_f^\alpha x^N, \quad x_i = V x^N, \quad i = 1, \ldots, 3, \tag{57b}
\]

\[
with q^\alpha, q^f, x^\alpha, x_f^\alpha, and x being the solution and coordinate vectors with all the components. The expansion coefficients of the entropy variables are defined using the quadrature \(L^2\) projection operator \((28)\),

\[
\beta_a^N = P^\alpha \beta_a (q^\alpha, x^\alpha), \tag{58}
\]

with \(\beta^N\) being the vectors of all components; we similarly define \(\beta^V\), \(\beta_f\), \(\beta_a\), \(\beta^V\), \(\beta^f\), and \(\beta\) as above. With this, the entropy-projected solution vectors are

\[
\tilde{q}^\alpha = \beta_a^{-1} (\beta_a^V, x^\alpha), \tag{59a}
\]

\[
\tilde{q}_f^\alpha = \beta_a^{-1} (\beta_f^V, x_f^\alpha), \tag{59b}
\]

\[
\tilde{q}_a = \beta_a^{-1} (\beta_a, x) = \begin{bmatrix} \tilde{q}^V \\ \tilde{q}^f \end{bmatrix}, \tag{59c}
\]

with \(\tilde{q}^V, \tilde{q}^f\), and \(\tilde{q}\) being the combined entropy-projected solution vectors.

**Theorem 4.** If numerical flux \(\mathcal{D}_j\) satisfies conservative entropy shuffle \((56)\) and \(\mathcal{D}_{j,a}\) is evaluated with \(\tilde{q}\) as defined in \((59)\) then DG scheme \((51)\) on a single element satisfies the entropy relationship

\[
\sum_{\alpha=1}^{N_c} T J^\alpha W^\alpha \frac{d}{dt} (q^\alpha) \leq T J^\alpha W^\alpha \Pi (\tilde{q}^V, x^\alpha) - \sum_{j,f,k=1}^d \sum_{\alpha=1}^{N_c} \left( T G_{j,k}^f B_k^f \xi_j^f + \sum_{\alpha=1}^{N_c} \left( \beta_a^{-1} (\beta_a^V, x^\alpha) \right)^T G_{j,k}^f B_k^f \mathcal{D}_{j,a}^\alpha \right), \tag{60}
\]

**Proof.** On a single element, assuming exactness in time, it follows that

\[
T J^\alpha W^\alpha \frac{d}{dt} q^\alpha = \sum_{\alpha=1}^{N_c} \left( \beta_a (q^\alpha))^T J^\alpha W^\alpha dq^\alpha \right), \tag{61}
\]

\[
= \sum_{\alpha=1}^{N_c} \left( \beta_a (q^\alpha))^T (P^\alpha)^T (V^\alpha)^T J^\alpha W^\alpha V^\alpha dq^N \right), \tag{61}
\]

\[
= \sum_{\alpha=1}^{N_c} \left( \beta_a^N \right)^T M^\alpha dq^N \frac{d}{dt}. \tag{61}
\]

\(^3\)For conservation laws, using the numerical flux in fluctuation form \((41)\) it is straightforward to show that \((56)\) reduces to the standard conservative property: \([\beta^T (q, x) - \beta^T (p, y)] F_j (q, x, p, y) = \psi_i (q, x) - \psi_i (p, y); \) see Castro et al. [11].
Using DG scheme (51) then gives
\[
\sum_{a=1}^{N_c} (\beta_{\alpha}^V)^T M^V \frac{dq_{\alpha}^V}{dt} = -\sum_{a=1}^{N_c} \sum_{j,k=1}^{d} \beta_{\alpha}^V \left( G_{jk} \left( Q_k \circ D_{j,a} \right) - \left( D_{j,a} \circ Q_k^T \right) G_{jk} \right) 1
+ \sum_{a=1}^{N_c} (\beta_{\alpha}^V)^T J^V W^V \alpha^V - \sum_{a=1}^{N_c} \sum_{j,k=1}^{d} \left( \beta_{\alpha}^f \right)^T G_{jk}^f B_k^f \mathcal{D}_{j,a}. \tag{62}
\]

Considering the source term, we have that
\[
\sum_{a=1}^{N_c} (\beta_{\alpha}^V)^T J^V W^V \alpha^V = \sum_{a=1}^{N_c} (\beta_{\alpha}^V)^T J^V W^V \alpha^V (\bar{q}^V, x^V) = 1^T J^V W^V \Pi (\bar{q}^V, x^V). \tag{63}
\]

The volume flux term gives
\[
\sum_{a=1}^{N_c} \sum_{j,k=1}^{d} \beta_{\alpha}^V \left( G_{jk} \left( Q_k \circ D_{j,a} \right) - \left( D_{j,a} \circ Q_k^T \right) G_{jk} \right) 1
= \sum_{a=1}^{N_c} \sum_{n,m=1}^{N_a} \sum_{j,k=1}^{d} \left[ \beta_{\alpha}^V \right]_n \left[ D_{j,a} \right]_m \left( \left[ G_{jk} \right]_n \left[ Q_k \right]_m - \left[ Q_k \right]_m \left[ G_{jk} \right]_n \right)
= \sum_{n,m=1}^{N_a} \sum_{j,k=1}^{d} \left[ G_{jk} \right]_n \left[ Q_k \right]_m \left( \left[ \beta_{\alpha}^V \right]_n \left[ D_{j,a} \right]_m - \left[ D_{j,a} \right]_m \left[ \beta_{\alpha}^V \right]_n \right)
= \sum_{n,m=1}^{N_a} \sum_{j,k=1}^{d} \left[ \xi_j \right]_m \left( \left[ G_{jk} \right]_n \left[ Q_k \right]_m - \left[ Q_k \right]_m \left[ G_{jk} \right]_n \right)
= \sum_{j,k=1}^{d} \left( 1^T G_{jk} Q_k \xi_j - \xi_j^T G_{jk} Q_k 1 \right)
= \sum_{j,k=1}^{d} 1^T G_{jk} Q_k \xi_j,
\]

where we have used the entropy conservative shuffle (56a) and exactness of the derivative matrix for constants (35).

Using SBP property (32) and geometric conservation law assumption (36) gives
\[
\sum_{j,k=1}^{d} 1^T G_{jk} Q_k \xi_j = \sum_{j,k=1}^{d} \left( 1^T G_{jk}^f B_k^f \xi_j^f - 1^T G_{jk} Q_k^T \xi_j \right) = \sum_{j,k=1}^{d} 1^T G_{jk}^{-f} B_k^{-f} \xi_j^{-f}. \tag{65}
\]

Now considering the face term yields
\[
\sum_{a=1}^{N_c} \sum_{j,k=1}^{d} \left( \beta_{\alpha}^f \right)^T G_{jk}^f B_k^f \mathcal{D}_{j,a}
= \sum_{a=1}^{N_c} \sum_{j,k=1}^{d} \left( \beta_{\alpha}^f \right)^T G_{jk}^f B_k^f \mathcal{D}_{j,a} \tag{66}
\leq -\sum_{a=1}^{N_c} \sum_{j,k=1}^{d} \left( \beta_{\alpha}^f \right)^T G_{jk}^f B_k^f \mathcal{D}_{j,a}^{*s} + 1^T G_{jk}^{-f} B_k^{-f} \xi_j^{-f} + 1^T G_{jk}^f B_k^f \xi_j^f,
\]

where we have used that \( \sum_{k=1}^{d} G_{jk}^f B_k^f = -\sum_{k=1}^{d} G_{jk}^{-f} B_k^{-f} \) along with dissipative entropy shuffle (56b).

Putting the source (63), volume (65), and face (66) contributions together gives the desired result.
**Corollary 5.** If the conditions of Theorem 4 are satisfied, then DG scheme (51) on a periodic mesh satisfies the entropy stability relationship

\[
\sum_{k \in T} \sum_{a=1}^{N_e} T^T J^{v \cdot K} W^{v \cdot K} \frac{d}{dt} \left( q^{v \cdot K} \right) \leq \sum_{k \in T} T^T J^{v \cdot K} W^{v \cdot K} \Pi \left( q^{v \cdot K}, u^{v \cdot K} \right),
\]

(67)

where \( T \) is the set of all elements.

**Proof.** The result follows directly from the entropy shuffle (56) applied to neighboring face terms of the single element entropy relation (60) of Theorem 4.

### 4.6. Entropy conservative flux for the atmospheric Euler equations

For the atmospheric Euler equations (1), we construct numerical fluxes \( D_k \) such that the conservative entropy shuffle (56) is satisfied. Here we use a construction based on the two-point entropy conserving flux of Chandrashekar [15] and Renac [46]. First, we define the following auxiliary quantities:

\[
\begin{align*}
(\rho u_1) &= \langle \rho \rangle \log \langle u_1 \rangle, \\
p^* &= \frac{\langle \rho \rangle}{2 \langle b \rangle}, \\
e^* &= \frac{1}{2(\gamma - 1)} \langle \phi(x) \rangle + \| \langle u \rangle \|^2_2 - \frac{\| \langle u \rangle \|^2_2}{2}, \\
\hat{\rho}^- &= \frac{\langle b \rangle \langle \rho \rangle}{b^-},
\end{align*}
\]

(68a, b, c, d)

where the average and log-average operators are defined as follows

\[
\begin{align*}
\langle a \rangle &= a^+ + a^-, \\
\langle a \rangle \log &= \frac{a^+ - a^-}{\log (a^+) - \log (a^-)}.
\end{align*}
\]

(69a, b)

To evaluate \( \langle \cdot \rangle \log \) in a numerically stable manner we follow the approach outlined by Ismail and Roe [30, Appendix B]. With these definitions, an entropy conservative numerical flux for \( k = 1, 2, 3 \) is

\[
\begin{align*}
\mathcal{A}_k(q^-, x^-, q^+, x^+) &= \frac{1}{2} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \delta_{1k} \hat{\rho}^- & 0 \\
0 & 0 & 1 & 0 & \delta_{2k} \hat{\rho}^- & 0 \\
0 & 0 & 0 & 1 & \delta_{3k} \hat{\rho}^- & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix} \\
\mathcal{H}_k(q^-, x^-, q^+, x^+) &= \begin{bmatrix}
(\rho u_k)^* \langle u_1 \rangle + \delta_{1k} p^* \\
(\rho u_k)^* \langle u_2 \rangle + \delta_{2k} p^* \\
(\rho u_k)^* \langle u_3 \rangle + \delta_{3k} p^* \\
e^* (\rho u_k)^* + \langle u_k \rangle p^* \\
\phi(x^+)
\end{bmatrix}.
\end{align*}
\]

(70a, b)

Using these expressions in the definition of the numerical flux (38) gives

\[
\begin{align*}
D_k(q^-, x^-, q^+, x^+) &= \begin{bmatrix}
(\rho u_k)^* \\
(\rho u_k)^* \langle u_1 \rangle + \delta_{1k} p^* + \frac{1}{2} \hat{\rho}^- \langle \phi \rangle \\
(\rho u_k)^* \langle u_2 \rangle + \delta_{2k} p^* + \frac{1}{2} \hat{\rho}^- \langle \phi \rangle \\
(\rho u_k)^* \langle u_3 \rangle + \delta_{3k} p^* + \frac{1}{2} \hat{\rho}^- \langle \phi \rangle \\
e^* (\rho u_k)^* + \langle u_k \rangle p^*
\end{bmatrix} - \begin{bmatrix}
\rho^- u_k^- \\
\rho^- u_k^- u_1^- + \delta_{1k} p^- \\
\rho^- u_k^- u_2^- + \delta_{2k} p^- \\
\rho^- u_k^- u_3^- + \delta_{3k} p^- \\
\rho^- e^- u_k^- + \hat{u}_k^- p^-
\end{bmatrix},
\end{align*}
\]

(71)
where $[\phi] = \phi(x^+) - \phi(x^-)$.

Since the second term of (71) is independent of $q^+$, it need not be included in either the volume or surface numerical fluxes. The reason for this is that in the first derivative term of DG scheme (51) the derivative of this term is zero since the derivative is with respect to the final arguments of the flux. The reason that it can be neglected in the second derivative term and the surface term is we assume a discrete SBP property and consistency of the surface and volume metric terms.

Entropy-stable fluxes can be constructed by adding entropy dissipation terms to the entropy conservative flux in a number of ways, the simplest being local Lax-Friedrichs (Rusanov) dissipation \[43, Section 6.1\]. In Appendix A an entropy dissipative flux using a matrix dissipation term is described following the methodology presented in Winters et al. \[62\]. The matrix dissipation term improves upon scalar dissipation by distinguishing between advective and acoustic waves using the flux Jacobian eigendecomposition.

5. Numerical results

In this section, numerical results verifying correctness and highlighting the benefits of the novel entropy-stable scheme are presented. A one-dimensional Sod problem under gravity showcases the flexibility of the method in terms of different possible choices of quadrature points, while demonstrating stability in the presence of shocks. A classical atmospheric benchmark of a thermal bubble convection is used to numerically validate entropy conservation and highlight the benefits of the high-order DG scheme with numerical flux dissipation for a large eddy simulation scale flow with sharp gradients and small-scale features. A gravity wave test case in a channel, having an exact linear solution, verifies convergence and the high-order accuracy of the scheme. Lastly, a baroclinic wave benchmark demonstrates the robustness of high-order entropy stable numerics for an archetype of global weather simulation on anisotropic spherical grids. Unless otherwise noted, the surface flux used in the test problems is the entropy dissipative flux from Appendix A; in all cases the entropy conservative numerical flux (38) is used in the volume. In all examples, the Courant–Friedrichs–Lewy (CFL) number is defined with respect to the minimum distance between interpolation points and an estimate of the acoustic wave speed based on the test case initial condition.

5.0.1. Sod shock tube with gravity

Luo et al. \[38\] introduced a classical Sod shock tube placed in a gravitational field. The domain is $0 - 1 = \Phi = x$, and the initial condition is defined as follows

$$ (\rho, u, p) = \begin{cases} (1, 0, 0) & x < 0.5, \\ (0.125, 0, 0.1) & x \geq 0.5. \end{cases} \quad (72) $$

No-flux boundary conditions are used on both sides of the domain. The domain is resolved using 32 elements with polynomial order 4. Time integration is performed using the fourth-order five-stage low storage Runge–Kutta (LSRK54) method from Carpenter and Kennedy \[10,(5,4)\] with CFL = 0.2. To assess the impact of quadrature rule accuracy, simulations using two sets of quadrature points are considered: $N + 1$ Legendre–Gauss–Lobatto (LGL) points and a more accurate $N + 2$ point Gauss quadrature. Figure 1 and Figure 2 show the density, velocity, total energy, and pressure fields at time $t = 0.2$ for the LGL and Gauss point simulations, respectively, along with a reference simulation that used a $5^{th}$ order WENO scheme with 2000 cells. The results suggest that for this test problem, there is no significant improvement with more accurate Gauss quadrature.

5.0.2. Rising thermal bubble

To verify the entropy conservation and entropy stability of the discretization we consider the classic atmospheric benchmark of thermal convection. Our compressible setup closely follows the anelastic experiments presented in Smolarkiewicz and Pudykiewicz \[51\]. A buoyant bubble is placed in a neutrally stratified environment with potential temperature 300 K. The domain $(-L, L) \times (0, H)$ with $(L, H) = (2 \text{ km}, 2 \text{ km})$ is periodic in the horizontal with rigid-lid boundaries at the top and bottom. The initial perturbation is a thermal bubble of radius 250 m with its center 260 m above the bottom and a potential temperature constant value of 0.5 K. The domain is resolved using $K \times K$ elements with polynomial order $N = 4$. Here, and in the subsequent examples, only the $N + 1$ point LGL quadrature rule is
Figure 1: Density, velocity, total energy, and pressure at time $t = 0.2$ for the Sod shock tube under gravity. The solution is obtained on a grid with 32 elements with polynomial order $N = 4$ using an $N + 1$ point LGL quadrature rule. The reference results were obtained using a $5^{th}$ order WENO scheme with 2000 cells.

Figure 2: Density, velocity, total energy, and pressure at time $t = 0.2$ for the Sod shock tube under gravity. The solution is obtained on a grid with 32 elements with polynomial order $N = 4$ using an $N + 2$ point Gauss quadrature rule. The reference results were obtained using a $5^{th}$ order WENO scheme with 2000 cells.
Figure 3: Potential temperature perturbation for the rising thermal bubble test case using the matrix dissipation flux at time 1000 s on a fine mesh.

Figure 4: Time evolution of normalized entropy change for the rising thermal bubble simulations using entropy conservative and entropy stable fluxes on a coarse warped mesh. With the entropy conservative flux the maximum value of entropy change is less than $2.6 \times 10^{-15}$.

considered. Since the aim of this test is to verify the entropy conservation and stability properties of the scheme, we augment the LSRK54 scheme with the relaxation technique from Ranocha et al. [45]. The CFL number is 0.4.

Figure 3 shows the potential temperature perturbation at time 1000 s computed using the entropy stable flux on a fine mesh with $K = 40$ elements. Since the scheme has minimal dissipation through the matrix flux, the solution is rich in small-scale structures a feature that would not be possible with other stabilization mechanisms at this resolution.

Figure 4 shows the normalized entropy change over time for the rising thermal bubble using both the entropy conservative and entropy dissipative surface fluxes. To challenge the scheme, a coarser mesh with $K = 10$ elements is used and the mesh is warped by the transformation

$$
\tilde{x}_1 = x_1 + \frac{L}{3} \sin \left( \frac{\pi (x_1 + \frac{L}{2})}{L} \right) \sin \frac{2\pi x_2}{H},
$$

$$
\tilde{x}_2 = x_2 - \frac{H}{3} \sin \left( \frac{2\pi (x_1 + \frac{L}{2})}{L} \right) \frac{\pi x_2}{H}.
$$

As expected, the simulation using the entropy conservative flux conserves entropy to machine precision, while the
we refer the reader to Baldauf and Brdar [4] for the details of the setup and the analytical “small-scale setup” solution. To verify the convergence of models solving the full nonlinear equations, provided the initial perturbation is small enough; simulation with the entropy stable flux shows strict entropy decay. Also shown in Figure 4 is a line denoting where the vanilla DG spectral element simulation with a Rusanov flux and no other stabilization method crashes with a NaN error.

5.0.3. Gravity wave in a channel

To verify the high-order convergence of the entropy-stable scheme, a gravity wave test case from Baldauf and Brdar [4] was adopted. The two-dimensional setup specifies a channel of size $L \times H$ with $L = 300$ km and $H = 10$ km. An isothermal background state with temperature $T_0 = 250$ K is perturbed by a warm bubble with a maximum temperature perturbation $\Delta T$ triggering the evolution of gravity and acoustic waves. A uniform background flow of speed $20$ m/s and no Coriolis force is assumed. An exact solution of the linearized problem is available, which can also be used to verify the convergence of models solving the full nonlinear equations, provided the initial perturbation is small enough; we refer the reader to Baldauf and Brdar [4] for the details of the setup and the analytical “small-scale setup” solution. Experience from Blaise et al. [7] and Baldauf [3], which used this test case to evaluate high-order DG solvers, shows that for high orders of accuracy $\Delta T$ has to be very small to avoid error saturation due to nonlinear effects. Following Baldauf [3], we deviate from the canonical setup in Baldauf and Brdar [4] by decreasing $\Delta T$ to $10^{-3}$ K.

The channel is discretized by a uniform grid with $K_H \times K_V$ elements. In all simulations the ratio between the horizontal element size $\Delta x = \frac{L}{K_H}$ and the vertical element size $\Delta z = \frac{H}{K_V}$ is held constant $\frac{\Delta x}{\Delta z} = 3$. Time integration is done using the LSRK54 time stepper with CFL = 0.1.

Figure 5 presents temperature and vertical velocity perturbations at the final simulation time 30 min using polynomial order $N = 3$ for coarse and fine grids with $\Delta x = 6$ km and $\Delta x = 3$ km, respectively. The contours present the analytical results whereas shading corresponds to the numerical solution. Small differences between the numerical and analytical solutions can be observed on the coarse grid, whereas the two are nearly indistinguishable on the fine grid. This is further corroborated by Figure 6 which shows the numerical and analytical vertical velocity perturbation at $z = 5$ km for the two grids. The vertical velocity perturbation for $\Delta x = 3$ km in Figure 5 can be directly compared to Figure 2 in
without any explicit dissipation. To resolve the sphere, we use an equiangular cubed-sphere mesh with which is explained by its well-balanced property. Similarly as Baldauf [3], we see a small influence of nonlinear effects well-balanced schemes are better at approximating small perturbations around balanced states [12], which is exactly the situation in this test problem. Figure 7 shows convergence in the normalized $L_2$ error norm for the temperature and vertical velocity. The $L_2$ error norm is computed as

$$L_2(q) = \frac{\int (q - q_{\text{exact}})^2}{LH}.$$  

(75)

Overall, we observe convergence rates close to the expected $N + 1$ for approximation with polynomial order $N$. When compared to Figure 3 in Baldauf [3] we see much smaller errors of the entropy-stable scheme at coarse resolutions, which is explained by its well-balanced property. Similarly as Baldauf [3], we see a small influence of nonlinear effects for the highest polynomial order $N = 4$, leading to suboptimal convergence near the finest resolution.

5.0.4. Baroclinic instability

Here, we set up a standard benchmark for atmospheric models in a spherical configuration, used to compare dynamical cores within the 2016 edition of DCMIP (Dynamical Core Intercomparison Project) [57], following the prescription from Ullrich et al. [58]. The test case is designed to idealize global weather evolution in the midlatitudes. The model is initialized with a deep-atmosphere, balanced steady-state solution that is axisymmetric about the Earth’s rotation axis, using analytical formulae from Ullrich et al. [58, Section 3]. The planetary rotation vector is $\Omega = (0, 0, 2\Omega)$ with $\Omega = 7.29212 \times 10^{-5}$ $\text{s}^{-1}$. A localized Gaussian hill perturbation of the zonal wind field is introduced in the northern midlatitudes, which triggers the evolution of a baroclinic wave over the course of several days. In the initial 7-day period, the wave dynamics is linear, after which the dynamics transitions to a nonlinear regime with a steepening and breaking of the wave. To compare with other models, in our presentation the focus is on the first 15 days, and breaking of the wave. To compare with other models, in our presentation the focus is on the first 15 days, and particularly day 8 (before the wave breaking) and day 10 (after the wave breaking). However, to properly evaluate the robustness of the entropy-stable scheme, we confirmed that all our simulations successfully run for over 100 days without any explicit dissipation. To resolve the sphere, we use an equiangular cubed-sphere mesh with $K_h$ elements per horizontal side on each of the six panels and $K_v$ elements in the vertical. The model top is 30 km and rigid lid boundary conditions are assumed at the top and bottom. We perform two simulations with polynomial orders 3 and 7. The simulation with polynomial order 3 uses $K_h = 30$ and $K_v = 8$, while the simulation using polynomial order 7 uses $K_h = 15$ and $K_v = 4$. These choices ensure that the total number of degree of freedom is the same for both simulations, corresponding to about 100 km average horizontal resolution and 32 vertical levels. Time stepping was performed using the second order additive Runge Kutta scheme from Giraldo et al. [27] with CFL = 3.0. Figure 8 and Figure 9 show plots of surface pressure, 850 hPa temperature and relative vorticity at days 8 and 10 for polynomial orders 3 and 7, respectively. These results can be directly compared to results from the MCore and ENDGame models [58,
Figure 8: Results for the baroclinic wave benchmark using polynomial order 3 at day 8 (left) and 10 (right). Top to bottom: surface pressure, temperature at 850 hPa, and relative vorticity at 850 hPa.

Figure 9: Results for the baroclinic wave benchmark using polynomial order 7 at day 8 (left) and 10 (right). Top to bottom: surface pressure, temperature at 850 hPa, and relative vorticity at 850 hPa.
Figures 4 and 5] and the MPAS model [50, Figure 2]. The pressure and temperature fields in their overall structure look smooth and compare very well with other models, showing only small amounts of grid-scale oscillations due to minimal numerical diffusion of our method. More noise can be seen in the relative vorticity fields, especially with polynomial order 7. This is consistent with Skamarock et al. [50], which found that the relative vorticity field is more sensitive to numerical dissipation than the pressure and temperature fields.

Figure 10 shows the time evolution of the minimum surface pressure and maximum horizontal velocity for both polynomial orders, and can be compared to results from the MPAS model in Skamarock et al. [50, Figures 3 and 4]. The surface pressure evolution is close for both polynomial orders up until day 12 and compares well to other models. The maximum horizontal speed shows consistently higher maxima for polynomial order 7. This further confirms that the higher order simulation introduces less numerical dissipation. The presented results successfully demonstrate that the new entropy-stable scheme is robust and provides high-quality solutions using only numerical flux dissipation for very high orders of accuracy in atmospheric simulations. For realistic applications, e.g., climate simulations and weather prediction, it might be beneficial to add more dissipation to damp grid-scale oscillations; an advantage of the entropy-stable scheme is that any additional regularization can be guided solely by the needs of the model physics since the dynamics is provably stable.

6. Concluding Remarks

We have extended the work of Renac [46] to develop a flux differencing formulation of the DG method for non-conservative balance laws on elements that have operators that can be skew-hybridized [14]. By specifying the volume and surface numerical fluxes so that they satisfy an additional entropy conservation and dissipation relation, the scheme was proved to be entropy stable. Based on the new framework, an entropy-stable DG discretization of the Euler equations with gravity has been constructed. Entropy-conservative and entropy-stable numerical fluxes for the Euler equations with gravity have been given in explicit form. The entropy stability and high-order accuracy of the novel scheme have been verified on standard atmospheric benchmark problems. As the atmospheric test problems show, the scheme is quite robust even without any explicit stabilization techniques beyond the choice of numerical fluxes.

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Appendix A. Entropy stable flux with matrix dissipation

To construct an entropy stable flux the normal entropy conservative flux $\mathcal{D}_n = n_1 \mathcal{D}_1 + n_2 \mathcal{D}_2 + n_3 \mathcal{D}_3$ is combined with a dissipation operator $\mathcal{H}_n$ acting on the jump in entropy variables, leading to the normal entropy stable flux

$$\mathcal{D}^*_n = \mathcal{D}_n - \frac{1}{2} \mathcal{H}_n \left[\beta\right]. \quad (A.1)$$
To ensure entropy dissipation and stability, the operator $H_n$ has to be constructed carefully [19]. Here, motivated by Winters et al. [62], the dissipation operator is constructed in the so-called “matrix” form

$$H_n = R_n^T |A_n| T_n R_n,$$

where $R_n$ is the matrix of right eigenvectors of the normal flux Jacobian, $|A_n|$ is the corresponding matrix of absolute eigenvalues, and $T_n$ is a scaling matrix. These matrices are given explicitly as

$$R_n = \begin{bmatrix}
1 & 1 & 1 & 1 \\
\{u_1\} - c^* n_1 & \{u_2\} - c^* n_2 & \{u_3\} - c^* n_3 & h^* - u_n c^* \\
\{u_1\} & \{u_2\} & \{u_3\} & \frac{m}{\gamma} + \{|\phi\|} \\
m_1 & m_2 & m_3 & m \\
\ell_1 & \ell_2 & \ell_3 & \ell \\
\{u_1\} + c^* n_1 & \{u_2\} + c^* n_2 & \{u_3\} + c^* n_3 & h^* + u_n c^*
\end{bmatrix},$$

(A.2)

where

$$|A_n| = \text{diag} (|u_n - c^*|, |u_n|, |u_n|, |u_n + c^|),$$

(A.3)

and

$$T_n = \text{diag} \left( \frac{|\rho|^\text{log} (\gamma - 1)}{2\gamma}, \frac{p^*}{\gamma}, \frac{p^*}{\gamma}, \frac{|\rho|^\text{log} (\gamma - 1)}{2\gamma} \right),$$

(A.4)

where $n, m, \ell$ are orthonormal vectors. The action of the dissipation operator on the jump in entropy variables can be written without having to specify non-uniquely defined tangent vectors $m$ and $\ell$ as

$$(H_n [\mathbf{\beta}])_\rho = w_1 + w_2 + w_3,$$

(A.5)

$$(H_n [\mathbf{\beta}])_{\rho u} = (w_1 (|u_n - c^*|) + w_2 (|u_n|) + w_3 (|u_n + c^*|)$$

(A.6)

and

$$(H_n [\mathbf{\beta}])_{\rho e} = \left( w_1 (h^* - u_n c^*) + w_2 \left( \frac{u_n^2}{2} + |\phi| \right) \right) + w_3 (h^* + u_n c^*),$$

(A.7)

where

$$T = I - n \otimes n,$$

(A.8)

and

$$w_1 = |u_n - c^*| \frac{|\rho|^\text{log} (\gamma - 1)}{2\gamma} \left( |\mathbf{\beta}|_\rho + (|\mathbf{\beta}|_\rho - |\mathbf{\beta}|_{pu} + (h^* - c^* u_n) |\mathbf{\beta}|_{pe} \right),$$

(A.9)

$$w_2 = |u_n| \frac{(\gamma - 1)}{\gamma} \frac{|\rho|^\text{log} (\gamma - 1)}{2\gamma} \left( |\mathbf{\beta}|_\rho + (|\mathbf{\beta}|_\rho - |\mathbf{\beta}|_{pu} + (h^* + c^* u_n) |\mathbf{\beta}|_{pe} \right),$$

(A.10)

and

$$w_3 = |u_n + c^*| \frac{|\rho|^\text{log} (\gamma - 1)}{2\gamma} \left( |\mathbf{\beta}|_\rho + (|\mathbf{\beta}|_\rho - |\mathbf{\beta}|_{pu} + (h^* + c^* u_n) |\mathbf{\beta}|_{pe} \right),$$

(A.11)
and $[[\beta]]_{p}$, $[[\beta]]_{pu}$, $[[\beta]]_{rv}$ denotes $[[\beta]]_{1}$, $[[\beta]]_{2}$, $[[\beta]]_{3}$, $[[\beta]]_{4}$, $[[\beta]]_{5}$, respectively.

Appendix B. Well-balanced property for isothermal atmosphere

Here we prove that the new entropy-stable scheme is well-balanced for an isothermal atmosphere for an unwarped “box” domain. That is we assume that the domain is $\Omega$ is such that all the elements are aligned with $\xi_3$ being the direction that gravity acts and $\xi_1$ and $\xi_2$ are orthogonal. This is possible on, for instance, a perfect spherical shell domain with gravity acting radially or an unwarped hexahedral domain with gravity acting downward. In both cases we let $r$ be the direction that gravity acts so that the geopotential is $\phi = gr$ where $g$ is a constant. In each element we assume that $r \equiv r(\xi_3)$ so that $\phi$ is also only a function of $\xi_3$. To demonstrate hydrostatic balance we show that the entropy conservative flux (71) is zero when the initial conditions are set to the analytic hydrostatic state for an isothermal atmosphere.

A hydrostatically balanced state is a state which is independent of time with a zero velocity and depends only on $r$. After assuming a stationary solution which only depends on $r$ with $u_k = 0$, the atmospheric equations (1) reduce to the hydrostatic balance equation

$$\frac{\partial p}{\partial r} = -\rho g.$$  \hspace{1cm} (B.1)

Under the isothermal assumption of the ideal gas law $p = pR_dT_0$ with $R_d$ and $T_0$ being the gas constant and constant reference temperature, respectively, we then have an exponential solution for pressure $p$ and density $\rho$

$$p = p_0 \exp\left(-\frac{gr}{R_dT_0}\right),$$  \hspace{1cm} (B.2a)

$$\rho = \frac{p_0}{R_dT_0} \exp\left(-\frac{gr}{R_dT_0}\right),$$  \hspace{1cm} (B.2b)

where $p_0$ is the pressure at $r = 0$.

Since the solution only varies in $r$ and thus is constant with respect to $\xi_1$ and $\xi_2$ in each element, consistency of a numerical flux in fluctuation form (38) implies that $D_1$ and $D_2$ are zero. The third component of the entropy conservative numerical flux (71) simplifies to

$$D_3(q^-,x^-,q^+,x^+) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \rho^+ + \frac{1}{2} \hat{\rho}^- [[\phi]] \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \rho^- \end{bmatrix}. $$  \hspace{1cm} (B.3)

Since in the balance solution (B.2) pressure is proportional to density the inverse temperature is constant, $b = \rho/2p = 1/R_dT_0$, and the expressions for the auxiliary variables (68b) and (68d) simplify to

$$\rho^+ = \frac{R_dT_0}{2} \langle [\rho] \rangle,$$  \hspace{1cm} (B.4a)

$$\hat{\rho}^- = \langle [\rho] \rangle_{\log}.$$  \hspace{1cm} (B.4b)

Evaluating the density logarithmic average with the balanced density solution (B.2b) gives

$$\hat{\rho}^- = \rho^+ - \rho^- \frac{\log \rho^+ - \log \rho^-}{g \rho^+ - g \rho^-} = -R_dT_0 \frac{[\rho]}{[\phi]}.$$  \hspace{1cm} (B.5)

Using the auxiliary expressions (B.4a) and (B.5) in the non-zero term of the numerical flux (B.3) leads to

$$\left(\rho^+ + \frac{1}{2} \hat{\rho}^- [[\phi]] \right) - \rho^- = \frac{R_dT_0}{2} \langle [\rho] \rangle - \frac{R_dT_0}{2} \langle [\rho] \rangle - p^- = R_dT_0 \rho^- - p^- = p^- - p^- = 0.$$  \hspace{1cm} (B.6)

This implies that $D_3$ is zero and thus the solution is well-balanced.
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