Are Higher Order Membranes stable in Black Hole Spacetimes?

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Abstract

We continue the study of the existence and stability of static spherical membrane configurations in curved spacetimes. We first consider higher order membranes described by a Lagrangian which, besides the Dirac term, includes a term proportional to the scalar curvature of the world–volume $\nabla^2 R$. Notably, in this case, the equations of motion can be reduced to second order ones and an effective potential analysis can be made. The conditions for stability are then explicitly derived. We find a self–consistent static spherical membrane, determining the spacetime generated by the membrane itself. In this case we find, however, that the total energy of the membrane has to be negative, and no stable equilibrium can be achieved. We then generalize the discussion to a membrane described by a Lagrangian including all possible second derivative terms. We conclude the paper with some discussion on the generality of the results obtained.

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I. INTRODUCTION

In Ref. [1] we have started the investigation of the plausibility of having a stable spherical membrane in the curved background of a (spherically symmetric) black hole, with regards to the possibility of the membrane being able to represent, in an effective way, the quantum degrees of freedom of the event horizon [2]. We have found that, contrary to what happens in flat spacetime for bosonic Dirac membranes [3], static equilibrium solutions are possible. In fact, as an example we have given explicitly the equilibrium radii $r_m$ for the Schwarzschild–de Sitter background metric. Although equilibrium is reached, it is not stable against perturbations. We have identified the mode $l = 0$ (and under certain conditions also $l = 1$) as the responsible for the instability. We have then considered higher order membranes [4–6] (coupling to the extrinsic curvature), which can be thought of as finite thickness membranes, as opposite to Dirac membranes which have zero thickness. Also in this case it is known that no static spherical equilibrium solutions are possible in flat spacetime [7]. We have analyzed the stability of the higher order membranes in the Schwarzschild background, finding that equilibrium solutions actually do exist there, but they are unstable. However, this by no means proves in general the non–existence of stable equilibrium solutions in curved spacetimes.

Two possibilities are now open: One could try to prove some kind of “no hair” theorem for the case of membranes coupled only to the gravitational field, i.e., try to prove that it is actually impossible to have stable equilibrium membranes around a black hole. A second possibility, and this is the one we will follow in this paper, is to look at the back-reaction problem, i.e., to solve self-consistently the equations of motion representing a membrane plus black hole system in equilibrium, and try to provide a counter–example to the eventual “no hair” theorem for membranes.

The paper is organized as follows: In Sec. II we discuss the higher order membrane Lagrangian in general terms. Sec. III deals with the higher order membrane Lagrangian including only the term proportional to the scalar curvature of the world–volume. In this case an effective potential analysis can be made and the conditions for stability are then explicitly derived. In Sec. IV we find a self–consistent static spherical membrane coupled to Einsteinian gravity, that is to say, we determine the spacetime generated by the membrane instead of fixing it a priori. This self-consistent membrane is however also unstable. In Sec. V we then generalize the discussions of Secs. III and IV to the more complicated case of the membrane Lagrangian including all the possible second derivative terms. We finally end the paper with a discussion on the generality of the unstable behavior of membranes in black hole backgrounds.

II. HIGHER ORDER MEMBRANE IN A SPHERICALLY SYMMETRIC CURVED SPACETIME

Up to second derivatives in the membrane world-volume coordinates, the most general action can be written as [4–6]

$$S_m = \int d\tau d\rho d\sigma \sqrt{-\gamma} \left[ -T + A \left( \gamma^{ij} \Omega_{ij} \right)^2 + B \Omega^{ij} \Omega_{ij} + C \left( ^{(3)}R \right) \right], \quad (2.1)$$
where $\gamma_{ij}$ is the induced metric on the world-volume
\[ \gamma_{ij} = g_{\mu\nu} X_i^\mu X_j^\nu, \quad (2.2) \]
$\gamma$ is its determinant and $(3) R$ is its scalar curvature; $g_{\mu\nu}$ is the spacetime metric, while
\[ \Omega_{ij} = g_{\mu\nu} n^\mu X_i^\nu \nabla_\rho X_j^\nu, \quad (2.3) \]
is the second fundamental form (extrinsic curvature), where the normal vector $n^\mu$ is defined by
\[ g_{\mu\nu} n^\mu X^\nu, i = 0, \quad g_{\mu\nu} n^\mu n^\nu = 1, \quad (2.4) \]
and it fulfills the completeness relation
\[ g^\mu^\nu = n^\mu n^\nu + \gamma_{ij} X^\mu, i X^\nu, j. \quad (2.5) \]
Notice that the tension $T$ has dimension of $\text{length}^{-3}$ while the arbitrary constants $A$, $B$, and $C$ carry dimension of $\text{length}^{-1}$.

In Ricci–flat spacetimes, the scalar curvature of the world–volume is related to the two second fundamental form terms, via the Gauß–Codazzi equation
\[ (3) R = \left( \gamma^{ij} \Omega_{ij} \right)^2 - \Omega^{ij} \Omega_{ij}, \quad (2.6) \]
and therefore only two of the last three terms in Eq. (2.1) are independent. However, in this paper we will also be interested in studying non–Ricci–flat spacetimes, where all terms should be included.

We shall consider static and spherically symmetric backgrounds
\[ ds^2 = -a(r) dt^2 + b(r)^{-1} dr^2 + r^2 d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2. \quad (2.7) \]
A spherical membrane with time-dependent radius can be conveniently described by the following spherically symmetric rest gauge choice
\[ t = \tau, \quad r = r(\tau), \quad \theta = \rho, \quad \varphi = \sigma, \quad (2.8) \]
so that the induced metric on the world–volume becomes
\[ \gamma_{\tau\tau} = -a + r^2/b, \quad \gamma_{\rho\rho} = r^2, \quad \gamma_{\sigma\sigma} = r^2 \sin^2 \rho, \quad (2.9) \]
\[ \sqrt{-\gamma} = r \sin \rho \sqrt{a - \dot{r}^2/b}, \]
where a dot denotes derivative with respect to $\tau$.

From $\gamma_{ij}$ we can now compute the non–vanishing components of the (world–volume) Ricci tensor
\[ (3) R_{\tau\tau} = -\frac{\dot{r}^2}{r(a - \dot{r}^2/b)} \left( -a' + 2\dot{r}b^{-1} - \dot{r}^2 b' b^{-2} \right) - 2\ddot{r} \frac{\dot{r}}{r}, \quad (2.10) \]
\[ (3) R_{\rho\rho} = 1 + \frac{\dot{r}^2 + r\ddot{r}}{a - \dot{r}^2/b} + \frac{r\ddot{r}^2}{2(a - \dot{r}^2/b)} \left( -a' + 2\dot{r}b^{-1} - \dot{r}^2 b' b^{-2} \right), \quad (2.11) \]
\[ (3) R_{\sigma\sigma} = \sin^2 \rho \quad (3) R_{\rho\rho}, \quad (2.12) \]
where a prime denotes derivative with respect to \( r \). From these expressions, it is easy to compute \( ^{(3)}R \)

\[
^{(3)}R = \frac{2a}{r \left( a - \dot{r}^2/b \right)^2} (2\ddot{r} - a'b' - b'a) + \frac{2(a'b' + 2b'a)}{r \left( a - \dot{r}^2/b \right)} + \frac{2ab}{r^2 \left( a - \dot{r}^2/b \right)} + \frac{2(1 - b) - 2b'}{r^2}.
\]

(2.13)

We shall also need the components of the second fundamental form. The normal vector introduced in Eq. (2.4) is given by:

\[
n^\mu = \frac{\sqrt{b/a}}{\sqrt{a - \dot{r}^2/b}} (\dot{r}/b, a, 0, 0).
\]

(2.14)

and it is then straightforward to obtain explicit expressions for \( \Omega_{ij} \)

\[
\Omega_{\tau\tau} = \frac{a\sqrt{b/a}}{2b\sqrt{a - \dot{r}^2/b}} (2\ddot{r} - a'b' - b'a) + \frac{\sqrt{a - \dot{r}^2/b}}{a\sqrt{b/a}} (a'b' + b'a/2),
\]

(2.15)

\[
\Omega_{\rho\rho} = -ar\sqrt{b/a}.
\]

(2.16)

\[
\Omega_{\sigma\sigma} = \sin^2 \rho \Omega_{\rho\rho}
\]

(2.17)

It follows that

\[
(\gamma^{ij}\Omega_{ij})^2 = \frac{a}{4b(a - \dot{r}^2/b)^3} (2\ddot{r} - a'b' - b'a)^2 + \frac{(2ab + r(a'b' + b'a/2))^2}{r^2 ab(a - \dot{r}^2/b)} + \frac{(2ab + r(a'b' + b'a/2))}{rb(a - \dot{r}^2/b)^2} (2\ddot{r} - a'b' - b'a),
\]

(2.18)

while

\[
\Omega_{ij}\Omega^{ij} = \frac{a}{4b(a - \dot{r}^2/b)^3} (2\ddot{r} - a'b' - b'a)^2 + \frac{(a'b' + b'a/2)^2}{ab(a - \dot{r}^2/b)} + \frac{2ab}{b(a - \dot{r}^2/b)^2} (2\ddot{r} - a'b' - b'a) + \frac{2ab}{r^2(a - \dot{r}^2/b)}.\]

(2.19)

Comparing with Eq. (2.13) we see that

\[
^{(3)}R - \left( \gamma^{ij}\Omega_{ij} \right)^2 + \Omega^{ij}\Omega_{ij} = \frac{2(b/a - a'b)}{r \left( a - \dot{r}^2/b \right)} + \frac{2(1 - b) - 2b'}{r^2} \equiv ^{(4)}R - 2 \frac{4}{\mu\nu} n^\mu n^\nu,
\]

(2.20)

in agreement with the Gauß–Codazzi equation for non-Ricci-flat spacetimes of the form (2.7).
For the sake of simplicity we will consider in this section the reduced model of Eq. (2.1) with \( A = 0 = B \). This resulting action is essentially the 3-dimensional Einstein action with a cosmological constant. However, here, we must take the functional variation with respect to \( X^\mu(\tau, \sigma, \rho) \) instead of \( \gamma_{ij} \). This makes the generic equations of motion rather complicated. In this paper we are though interested in spherical membranes and, therefore, it is easier to derive the equations of motion from the effective Lagrangian

\[
L = -4\pi Tr^2 \sqrt{a - \dot{r}^2/b} + 8\pi C \left\{ \frac{ar(2\dot{r} - a'b - b'a)}{(a - \dot{r}^2/b)^{3/2}} + \frac{r(a'b + 2b'a) + ab}{\sqrt{a - \dot{r}^2/b}} + (1 - b - rb') \sqrt{a - \dot{r}^2/b} \right\},
\]

as obtained from equations (2.1), (2.10) and (2.13). Note that the Lagrangian depends on \( r, \dot{r} \) and \( \ddot{r} \). The standard way to deal with such situations is as follows:

We build up the conjugate momenta as

\[
P_1 = \frac{\delta L}{\delta \dot{r}} - \partial_r \left( \frac{\delta L}{\delta \ddot{r}} \right), \quad P_2 = \frac{\delta L}{\delta \ddot{r}}.
\]

The equations of motion then read

\[
\dot{P}_1 - \frac{\delta L}{\delta r} = 0
\]

and the Hamiltonian

\[
\mathcal{H} = P_1 \dot{r} + P_2 \ddot{r} - L.
\]

Note that this Hamiltonian is conserved in the usual sense: \( \dot{\mathcal{H}} = 0 \), i.e. \( \mathcal{H} = E = constant \) [this can be explicitly checked by the use of the generalized Euler–Lagrange equations (3.3)].

In our case, the Hamiltonian is explicitly given by

\[
\mathcal{H} = 4\pi T \frac{ar^2}{\sqrt{a - \dot{r}^2/b}} - 8\pi C \left[ \frac{a(1 - b)}{\sqrt{a - \dot{r}^2/b}} + \frac{a^2b}{(a - \dot{r}^2/b)^{3/2}} \right].
\]

Note that the higher derivatives of \( r \) have canceled out! This allows us to define an effective potential to analyze the stability of the membrane. In fact, \( \mathcal{H} = E \) and Eq. (3.5) leads to

\[
E^2 \dot{r}^2 = E^2 - V(r)^2,
\]

where \( E \) has dimension of length\(^{-1} = mass \) in units where \( c = \hbar = 1 \) but \( G \) is kept explicitly, and \( V(r) \) is the effective potential that can be explicitly obtained from (3.5) by inversion of a cubic equation.

The conditions for the existence of a static equilibrium solution at \( r = r_m \), i.e., \( \dot{r} = 0, \ V^2 = E^2 \) and \( (V^2)' = 0 \) for \( r = r_m \), are
\[ E = 4\pi \sqrt{a} \left( T r^2 - 2C \right) \bigg|_{r_m} \]

\[ \frac{T}{C} = \frac{2a'}{(r^2 a' + 4a r)} \bigg|_{r_m}. \]  

(3.7)

(3.8)

These two equations can be seen as determining the values of \( E \) and \( r_m \) for the equilibrium membrane for given values of \( T \) and \( C \). Alternatively, for a given value of \( r_m \), equation (3.8) determines the value of \( T/C \) necessary to support such a membrane.

This equilibrium solution will be stable when \((V^2)'' > 0\) for \( r = r_m\), i.e.

\[ T \left[ \frac{r^2 a''}{2\sqrt{a}} - \frac{x^2 a'^2}{4a^{3/2}} + 2 \frac{a'}{\sqrt{a}} + 2 \sqrt{a} \right] - C \left[ \frac{a''}{\sqrt{a}} - \frac{a'^2}{2a^{3/2}} \right] \right|_{r_m} > 0. \]  

(3.9)

[Note that both the equilibrium condition (3.8) and the stability one (3.9) are invariant under the constant scaling \( a(r) \to \Lambda a(r) \), although the reduced Einstein equations are not.]

It is easy to check that condition (3.8) cannot be fulfilled by a membrane in the Minkowski spacetime (as had been already shown in Ref. [7]). The case is however completely different when the background is curved. Consider, for example, a Schwarzschild black hole. Then \( a(r) = b(r) = 1 - 2M/r \) (taking now \( G = 1 \)). In this case Eq. (3.8) leads to

\[ \frac{T}{C} = \frac{2M}{(2r_m^3 - 3Mr_m^2)}, \]  

(3.10)

which can be fulfilled for any value of \( r_m \) (outside the event horizon) by a suitable choice of \( T/C \). This membrane, however, is in an unstable equilibrium since the inequality (3.3) is always violated for \( a(r) = b(r) = 1 - 2M/r \). It is however easy to construct by hand “black hole” metrics \( a(r), b(r) \) such that Eqs. (3.8), (3.3) can be fulfilled somewhere outside the horizon, see Section VI.

### IV. SELF-CONSISTENT STATIC MEMBRANE

So far we have considered the membrane propagating in an arbitrary curved background. In this section we will consider the self-consistent problem of computing the spacetime generated by the membrane itself. In general this is a very complicated system of non-linear equations. We shall restrict the study to the spherically symmetric case and where the only matter field is represented by the membrane of the previous section.

The total action will then consist of the following two pieces

\[ S_G = \frac{1}{16\pi} \int \left( R \sqrt{-g} \right) d^4x, \]  

(4.1)

\[ S_m = \int d\tau d\rho d\sigma \sqrt{-\gamma} (-T + C^{(3)} R). \]  

(4.2)

The usual functional variation with respect to the metric \( g_{\mu\nu} \) gives the Einstein equations

\[ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}, \]  

(4.3)
where the stress–energy–momentum tensor of the membrane is given by
\[
\sqrt{-g} T^{\mu\nu} = -2 \int d\tau d\rho d\sigma \sqrt{-\gamma} \delta \left( x^\lambda - X^\lambda(\tau, \rho, \sigma) \right) \left\{ \frac{T}{2} \gamma^{ij} + C \left( (3) R^{ij} - \frac{1}{2} (3) R \gamma^{ij} \right) \right\} X^\mu_i X^\nu_j,
\]
(4.4)

which for a static spherical membrane at \( r = r_m \) takes the following explicit form (where we have performed the integrals in (4.4))
\[
\begin{align*}
T_{tt} &= \frac{a \sqrt{b}}{r^2} (Tr^2 - 2C) \delta(r - r_m), \\
T_{rr} &= 0, \\
T_{\theta\theta} &= -T \sqrt{br^2} \delta(r - r_m), \\
T_{\phi\phi} &= \sin^2 \theta T_{\theta\theta}.
\end{align*}
\]
(4.5)

Note that there is no radial pressure and that the scalar curvature term only contributes to the energy density. The total energy of the membrane is given by
\[
\text{Energy} = -\int \sqrt{-g} T^t_t d^3 \vec{x} = 4\pi \sqrt{a(r_m)} \left( Tr_m^2 - 2C \right) = E,
\]
(4.6)

which coincides with the energy defined through the Hamiltonian (3.5), see equation (3.7).

The Einstein equations (4.3) are solved \([9]\) by
\[
\begin{align*}
b(r) &= 1 - \frac{2M}{r} - \frac{2E}{r} \Theta(r - r_m), \\
a(r) &= b(r) \exp \left\{ -E \left( 1 - 2\Theta(r - r_m) \right) \right\},
\end{align*}
\]
(4.7) (4.8)

where we have chosen integration constants such that \( a(r_m) = b(r_m) \), and where
\[
E = 4\pi \sqrt{1 - \frac{2M}{r_m} - \frac{E}{r_m} (Tr_m^2 - 2C)},
\]
(4.9)
\[
-16\pi T = \frac{E}{r_m^3} (2M + E) \left( 1 - \frac{2M}{r_m} - \frac{E}{r_m} \right)^{-3/2},
\]
(4.10)

and our conventions are such that \( \Theta(0) = 1/2 \). Note that equations (4.3)–(4.10) as obtained from the Einstein equations are exactly equivalent to equations (3.7)–(3.8) as obtained from the stability conditions, i.e. from the equations of motion for the membrane. We thus conclude that the static spherical membrane at \( r_m \) is a self–consistent solution in the spacetime (4.7)–(4.8). The string tension \( T \) is positive and we shall also assume that the integration constant \( M \) is positive. It follows that inside the spherical membrane, the spacetime is Schwarzschild corresponding to mass \( M \) while outside it is Schwarzschild corresponding to mass \( M + E \), as expected since \( E \) is the energy of the membrane. However, for Eq. (4.10) to be fulfilled, \( E \) must be negative (since we assumed \( T \) and \( M \) positives). In addition the inequality (3.9) is not satisfied, and we conclude that the static self-consistent membrane is in unstable equilibrium.
V. MEMBRANE WITH GENERIC SECOND DERIVATIVE TERMS

In this section we shall discuss the possibility of generalizing the analysis of sections III and IV to the generic case of second order membranes, as described by the action (2.1).

The effective Lagrangian is obtained from Eqs. (2.10), (2.13), (2.18) and (2.19), and we can then construct the Hamiltonian via the generalized Legendre transform as in Eqs. (3.2)–(3.4). This Hamiltonian will now depend on \( r, \partial_r r, \partial^2_r r \) and \( \partial^3_r r \), so we will not have a simple description of the dynamics in terms of an effective potential, in the form of an equation like (3.6). The conserved Hamiltonian “energy” is however still obtained from \( H = E = \text{constant} \), and the condition for having an equilibrium configuration can be obtained from the Euler-Lagrange equation (3.3) by setting \( \partial_r r = \partial^2_r r = \partial^3_r r = \partial^4_r r = 0 \). The two equations generalizing (3.7)–(3.8) thus become

\[
E = 4\pi \sqrt{a} \left\{ (Tr^2 - 2C) - (A + B) \left( 2b + \frac{br^2a'^2}{4a^2} \right) - A \left( 2b + \frac{2rba'}{a} \right) \right\},
\]

(5.1)

\[
T(r^2a' + 4ar) = 2Ca' + A \left( 4b'a + 6ba' - \frac{2rba'^2}{a} + 4rba'' + 4ra'b' \right)
+ (A + B) \left( 2ba' + 4ab' - \frac{3br^2a'^3}{4a^2} + \frac{r^2b'a'^2}{2a} + \frac{bra'^2}{a} + \frac{br^2a''}{a} \right),
\]

(5.2)

and \( r \) has to be evaluated everywhere at the equilibrium position \( r_m \). Again we notice that these equations cannot be fulfilled in Minkowski space \([7]\). They can however be easily fulfilled in most curved spacetimes, for instance the Schwarzschild spacetime, as was shown by the present authors in \([1]\).

Next we have to consider the question of stability of the equilibrium configurations. Since we do not, in this case, have a potential \( V(r) \) as in Eq. (3.6), we proceed as in Ref. \([1]\). Introduce the function \( \phi(r) \)

\[
r = r_m + \phi(\tau),
\]

(5.3)

and expand the Euler-Lagrange equation (3.3) to first order in \( \phi \). After some algebra, the resulting differential equation determining the radial fluctuations takes the general form

\[
\frac{d^4\phi}{d\tau^4} + F(r_m)\frac{d^2\phi}{d\tau^2} + G(r_m)\phi = 0,
\]

(5.4)

where \( F(r_m) \) and \( G(r_m) \) are complicated functions carrying the information about the static zeroth order solution and of the curved spacetime.

In the most general (non-degenerate) case, this fluctuation equation is solved by

\[
\phi(\tau) = c_1e^{d_1\tau} + c_2e^{d_2\tau} + c_3e^{d_3\tau} + c_4e^{d_4\tau},
\]

(5.5)

where \( (c_1, c_2, c_3, c_4) \) are arbitrary constants, and

\[
d_{(1,2,3,4)} = \pm \left( \frac{-F(r_m) \pm \sqrt{F^2(r_m) - 4G(r_m)}}{2} \right)^{1/2}.
\]

(5.6)
The necessary and sufficient condition for stability is that \(d_{1,2,3,4}\) are all purely imaginary, corresponding to \(\phi(\tau)\) being oscillatory.

Here we shall not give the (rather complicated) general expressions for the functions \(F(r_m), G(r_m)\). They were given in Ref. [1] for the case of the Schwarzschild spacetime (for which the constant \(C\) can be set equal to zero without loss of generality, c.f. Eq. (2.6)), and were shown to lead to the conclusion that a static spherical equilibrium membrane in the Schwarzschild spacetime is always unstable. This is not, however, the case in a general curved spacetime; see the next section.

We now briefly discuss the remaining question of the self-consistency of the equilibrium solutions for the generic second-derivative membranes. First one has to compute the contributions to the stress-energy-momentum tensor coming from the terms proportional to \(A\) and \(B\) in the action (2.1). For this purpose, it is convenient to eliminate the normal-vectors using the completeness-relation and the Gauß-Weingarten equation,

\[
\gamma^{ij}\Omega_{ij}^2 = g_{\mu\nu}(\Box x^\mu + \gamma^{ij}\Gamma^\mu_{\rho\sigma} x^\rho_i x^\sigma_j)(\Box x^\nu + \gamma^{ij}\Gamma^\nu_{\rho\sigma} x^\rho_i x^\sigma_j),
\]

\[
\Omega^{ij}\Omega_{ij} = \gamma^{ik}\gamma^{jl}(g_{\mu\rho}x^\lambda_i \nabla_\lambda x^\rho_j)(g_{\nu\sigma}x^\lambda_k \nabla_\lambda x^\sigma_l)(g^{\mu\nu} - \gamma^{mn}x^m_i x^n_j).
\]

Now it is straightforward (although tedious) to perform the functional variation with respect to the metric \(g_{\mu\nu}\). Here we will only give the results for the static spherical membrane at \(r = r_m\). From the \((\gamma^{ij}\Omega_{ij})^2\) term we get

\[
\Delta_AT_{tt} = 2\sqrt{b}\left(\frac{-2ab}{r^2} + \frac{3ba^2}{8a} + \frac{a'b}{2r}\right)\delta(r-r_m) - 2\sqrt{b}\left(\frac{2ab}{r} + \frac{a'b}{2}\right)\delta'(r-r_m),
\]

\[
\Delta_AT_{rr} = -2\sqrt{b}\left(\frac{4}{r^2} + \frac{a^2}{4a^2} + \frac{2a'}{ar}\right)\delta(r-r_m),
\]

\[
\Delta_AT_{\theta\theta} = 2\sqrt{b}\left(-2b + \frac{br^2a^2}{8a^2}\right)\delta(r-r_m) + 2\sqrt{b}\left(\frac{2br + \frac{r^2a'b}{2a}}{2}\right)\delta'(r-r_m),
\]

\[
\Delta_AT_{\phi\phi} = \sin^2\theta T_{\theta\theta}.
\]

The \(\Omega^{ij}\Omega_{ij}\) term gives rise to

\[
\Delta_BT_{tt} = 2\sqrt{b}\left(\frac{-ab}{r^2} + \frac{3ba^2}{8a}\right)\delta(r-r_m) - \sqrt{b}a'b \delta'(r-r_m),
\]

\[
\Delta_BT_{rr} = -2\sqrt{b}\left(\frac{2}{r^2} + \frac{a^2}{4a^2}\right)\delta(r-r_m),
\]

\[
\Delta_BT_{\theta\theta} = 2\sqrt{b}\left(-b + \frac{br^2a^2}{8a^2}\right)\delta(r-r_m) + 2\sqrt{b}br \delta'(r-r_m),
\]

\[
\Delta_BT_{\phi\phi} = \sin^2\theta T_{\theta\theta}.
\]

Note that the expressions for \(T_{tt}\) and \(T_{rr}\) can also (and more easily) be obtained directly from the effective Lagrangian by taking the functional variations with respect to \(-a(r)\) and \(1/b(r)\), respectively.

Together with equations (1.3), we now have the complete expressions for the stress-energy-momentum tensor for a static spherical membrane described by the action (2.1).
However, the presence of the $\delta'(r - r_m)$-terms is somewhat problematic. It is still possible to define the total energy of the membrane along the lines of (4.6), and to get a well-defined result. The $(tt)$ and $(rr)$ components of the Einstein equations (4.3) can also be easily integrated \[9\], at least formally. However, due to the $\delta'(r - r_m)$-terms in $T_{tt}$, we obtain ill-defined expressions for the functions $a(r)$ and $b(r)$ involving products of delta-functions and exponentials of delta-functions. Presently it is thus not clear how to interpret the results physically, at least not without some kind of regularisation of the singular functions. We therefore leave this problem for further study elsewhere.

VI. DISCUSSION

We have extended the analysis of the stability of spherical membranes in curved spacetimes, in terms of an effective potential, to the case of higher order membranes with a Lagrangian dependence proportional to $^{(3)}R$. We have also considered the problem of stability for the higher order membrane Lagrangian including all possible second derivative terms. We have found static solutions in fixed background spacetimes, and we have also been able to find self–consistent static solutions. However, these solutions are unstable against small radial perturbations. This does not mean, however, that we are ready to infer that no membranes can be kept in stable equilibrium outside a black hole. For that, one should prove a “no–hair” theorem for membranes including also other types of matter on the right hand side of the Einstein equations. At this point it is not clear whether it is possible or not to prove such theorem.

On the other hand it is relatively easy to construct, by hand, arbitrary functions $a(r)$ and $b(r)$ describing a black hole (plus a source with a positive mass density contribution) such that the stability Eqs. (3.8)–(3.9) are satisfied for some $r_m$ outside the horizon. As an example, consider

$$a(r) = b(r) = \frac{2}{3} - \frac{2M}{3r} + \frac{1}{3} \tanh \left( \frac{16r}{M} - 40 \right), \quad (6.1)$$

for some $r < R$ and then smoothly matched to the Schwarzschild metric. This functional form of the line–element allows for static stable membranes just outside the event horizon as can be easily verified. Such a spacetime can however not be supported by only the membrane itself, as follows from our analysis, but at this point it is yet not clear whether inclusion of some other kinds of ”normal” matter could help on that.

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