ROBUST RESIDUAL-BASED A POSTERIORI ERROR ESTIMATORS FOR MIXED FINITE ELEMENT METHODS FOR FOURTH ORDER ELLIPTIC SINGULARLY PERTURBED PROBLEMS

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Abstract. We consider mixed finite element approximation of a singularly perturbed fourth-order elliptic problem with two different boundary conditions, and present a new measure of the error, whose components are balanced with respect to the perturbation parameter. Robust residual-based a posteriori estimators for the new measure are obtained, which are achieved via a novel analytical technique based on an approximation result. Numerical examples are presented to validate our theory.

Key words. fourth order elliptic singularly perturbed problems, mixed finite element methods, a new measure of the error, robust residual-based a posteriori error estimators

AMS subject classifications. 65N15, 65N30, 65J15

1. Introduction. Let $\Omega$ be a bounded polygonal or polyhedral domain with Lipschitz boundary $\Gamma = \partial \Omega$ in $\mathbb{R}^d$, $d = 2$ or 3. Consider the following fourth-order singularly perturbed elliptic equation

$$\varepsilon^2 \Delta^2 u - \Delta u = f \quad \text{in } \Omega$$

with boundary conditions

$$u = \Delta u = 0 \quad \text{on } \Gamma$$

or

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma,$$

where $f \in L^2(\Omega)$, $\Delta$ is the standard Laplace operator, and $\frac{\partial u}{\partial n}$ denotes the outer normal derivative on $\Gamma$. In two dimensional cases, the boundary value problems (1.1)-(1.2) and (1.1)-(1.3) arise in the context of linear elasticity of thin bucking plate with $u$ representing the displacement of the plate. The dimensionless positive parameter $\varepsilon$, assumed to be small (i.e., $\varepsilon \ll 1$), is defined by

$$\varepsilon = \frac{t^3 E}{12(1-\nu^2)t^2T},$$

where, $t$ is the thickness of the plate, $E$ is the Young modulus of the elastic material, $\nu$ is the Poisson ratio, $l$ is the characteristic diameter of the plate, and $T$ is the absolute value of the density of the isotropic stretching force applied at the end of the plate [20]. In three dimensions, problems (1.1)-(1.2) and (1.1)-(1.3) can be a gross simplification of the stationary Cahn-Hilliard equations with $\varepsilon$ being the length of the transition region of phase separation.

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Conforming, nonconforming, and mixed finite element methods for fourth order problem have been extensively studied \cite{2, 3, 7, 8, 15, 19, 28, 29, 31, 32, 34, 35, 36}. However, its \textit{a posteriori} error estimation is a much less explored topic. Even for the Kirchhoff plate bending problem, the finite element \textit{a posteriori} error analysis is still in its infancy. In 2007, Beirão et al. \cite{5} developed an estimator for the Morley element approximation using the standard technique for nonconforming element. Later, Hu et al. \cite{25} improved the methods of \cite{5, 37, 38} by dropping two edge jump terms in both the energy norm of the error and the estimator, and by dropping the normal component in the estimators of \cite{5, 37}. Therefore, a naive extension of the estimators in \cite{5, 37, 38} to the current problem may probably not be robust in the parameter \(\varepsilon\).

Designing robust \textit{a posteriori} estimators is challenging, especially for singularly perturbed problems, since constants occurring in estimators usually depend on the small perturbation parameter \(\varepsilon\). This motivates us to think about the question: What method and norm are suitable for the singularly perturbed fourth-order elliptic problem? In the literature, \((\varepsilon^2 \|\Delta u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)})^{1/2}\) is a widely used measure for the primal weak formulation. We recall \textit{a priori} estimates in \cite{23} for boundary condition (1.3) and convex domain \(\Omega\):

\[
\|u\|_{H^s(\Omega)} \leq C \varepsilon^{\frac{s}{2}} \|f\|_{L^2(\Omega)}, \quad \text{for } s = 2, 3. \tag{1.4}
\]

Hereafter, we use \(C > 0\) to denote a generic constant independent of \(\varepsilon\) with different value at different occurrence. This leads to

\[
\varepsilon \|\Delta u\|_{L^2(\Omega)} \leq C \|u\|_{H^2(\Omega)} \leq C \varepsilon^{1/2} \|f\|_{L^2(\Omega)}.
\]

Multiply both sides of (1.1) by \(u\), and then integrate over \(\Omega\). Using integration by parts and boundary condition (1.2) or (1.3), we have from the Poincaré inequality that

\[
\varepsilon^2 (\Delta u, \Delta u) + (\nabla u, \nabla u) = (f, u) \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)}.
\]

As a consequence, \(\|\nabla u\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}\). This suggests that the two components of \(\varepsilon \|\Delta u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\) are unbalanced with respect to \(\varepsilon\) if \(f \in L^2(\Omega)\).

Furthermore, if we set \(\psi = -\Delta u\), then problem (1.1) is written as

\[-\varepsilon^2 \Delta \psi + \psi = f.\]

Note that \(\psi\) has boundary layer, but \(u\) usually does not have one. Thus,

\[(\varepsilon^2 \|\Delta u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2)^{1/2} = (\varepsilon^2 \|\psi\|_{L^2(\Omega)}^2 + |u|_{H^1(\Omega)}^2)^{1/2}\]

approaches to \(|u|_{H^1(\Omega)}\) as \(\varepsilon \to 0^+\), which fails to describe the layer of \(\psi\).

An observation of the two decoupled equations \(-\varepsilon^2 \Delta \psi + \psi = f\) and \(-\Delta u = \psi\) suggests that the two measures \(\varepsilon^2 \|\nabla \psi\|_{L^2(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 + |u|_{H^1(\Omega)}^2\) and \((\varepsilon^2 \|\nabla \psi\|_{L^2(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2)^{1/2}\) can portray the layer of \(\psi\) and the first and second derivatives of \(u\). From (1.4), we have

\[
\varepsilon \|\nabla \psi\|_{L^2(\Omega)} \leq C \varepsilon \|u\|_{H^1(\Omega)} \leq C \varepsilon^{-1/2} \|f\|_{L^2(\Omega)}.
\]

Notice that

\[
\|\psi\|_{L^2(\Omega)} \leq C \|u\|_{H^2(\Omega)} \leq C \varepsilon^{-1/2} \|f\|_{L^2(\Omega)}.
\]
If \( f \in L^2(\Omega) \), then \( \varepsilon \| \nabla \psi \|_{L^2(\Omega)} \) and \( \| \psi \|_{L^2(\Omega)} \) are balanced with respect to \( \varepsilon \) for the boundary condition \( (1.2) \). These inspire us to think about the mixed finite element method for the problem \( (1.1) \) and the two aforementioned measures.

However, the mixed finite element method for the problem \( (1.1) \) is a much less explored topic, since there exist some special problems such as the fourth order problem, where attempts at using the results of Brezzi and Babuška were not entirely successful since not all of the stability conditions were satisfied, cf. [3] and the reference therein. To overcome this difficulty, Falk et al. developed abstract results from which optimal error estimates for these (biharmonic equation) and other problems could be derived ([19, 8]). However, it is not easy to extend the results of [19] to the problem \( (1.1) \), because of the existence of an extra term of the stability conditions were satisfied, cf. [3] and the reference therein. To overcome this difficulty, Falk et al. developed abstract results from which optimal error estimates for these problems were derived in [15]. We refer to [10, 22] about the \( a \) posteriori estimation of Ciarlet-Raviart methods for the biharmonic equation.

In this work, our goal is to develop robust residual-type \( a \) posteriori estimators for a mixed finite element method for the problem \( (1.1) \) in the two aforementioned measures. The main difficulty lies in the fact that the boundary condition \( (1.3) \) does not include any information on the immediate variable \( \psi \). In order to overcome this difficulty, we develop a novel technique to analyze residual-based \( a \) posteriori error estimator. The key idea is to replace a function \( v \in H^1(\Omega) \) (such that \( -\varepsilon^2 \Delta v + v \neq 0 \) without boundary restriction) by a function \( \tilde{v} \in H^1_0(\Omega) \) with boundary restriction, which catches at least \( \gamma \) times of \( v \) in the \( \varepsilon \)-weighted energy norm (see Lemma 3.3 below). Combining this novel design with standard tools, we develop uniformly robust residual-type \( a \) posteriori estimators with respect to the singularly perturbed parameter \( \varepsilon \). Recently, for a fourth order reaction diffusion equation, the error estimates of its mixed finite element method was derived in [15]. We refer to [10, 22] about the \( a \) posteriori estimation of Ciarlet-Raviart methods for the biharmonic equation.

The rest of this paper is organized as follows: In Section 2, we introduce mixed weak formulations and some notations, and prove an equivalent relation between the primal weak solution and the weak solution determined by its mixed formulation. Some preliminary results are provided in Section 3. Residual-type \( a \) posteriori estimators are developed and proven to be reliable in Section 4. An efficient lower bound is proved in Section 5. In Section 6, numerical tests are provided to support our theory.

2. The mixed weak formulations. Setting \( \psi = -\Delta u \), and employing the boundary condition \( (1.2) \), we attain the Ciarlet-Raviart mixed problem \( P_1 \):

\[
\begin{align*}
-\varepsilon^2 \Delta \psi + \psi &= f & \text{in } \Omega \\
-\Delta u &= \psi & \text{in } \Omega \\
u &= 0 & \text{on } \Gamma.
\end{align*}
\]  

Similarly, using the boundary condition \( (1.3) \), we arrive at the Ciarlet-Raviart mixed formulation \( P_2 \):

\[
\begin{align*}
-\varepsilon^2 \Delta \psi + \psi &= f & \text{in } \Omega \\
-\Delta u &= \psi & \text{in } \Omega \\
u &= 0 & \text{on } \Gamma.
\end{align*}
\]  

For any bounded open subset \( \omega \) of \( \Omega \) with Lipschitz boundary \( \gamma \), let \( L^2(\gamma) \) and \( H^m(\omega) \) be the standard Lebesgue and Sobolev spaces equipped with standard norms \( \| \cdot \|_{L^2(\gamma)} \) and \( \| \cdot \|_{H^m(\omega)} \), \( m \in \mathbb{N} \) (see [11] for details). Note that \( H^0(\omega) = L^2(\omega) \). We denote \( \| \cdot \|_{m, \omega} \) the semi-norm in \( H^m(\omega) \). Similarly, denote \( (\cdot, \cdot)_\gamma \) and \( (\cdot, \cdot)_\omega \) the \( L^2 \) inner products on \( \gamma \) and \( \omega \), respectively. We shall omit the symbol \( \Omega \) in the notations above if \( \omega = \Omega \).
The weak formulation of problem $P_1$ reads: find $(\psi, u) \in H^1_0(\Omega) \times H^1_0(\Omega)$ such that
\begin{equation}
\begin{cases}
(\varepsilon^2 \nabla \psi, \nabla \varphi) + (\psi, \varphi) = (f, \varphi) & \forall \varphi \in H^1_0(\Omega), \\
(\nabla u, \nabla v) = (\psi, v) & \forall v \in H^1_0(\Omega).
\end{cases}
\tag{2.3}
\end{equation}

The weak formulation of the problem $P_2$ reads: find $(\psi, u) \in H^1(\Omega) \times H^1_0(\Omega)$ such that
\begin{equation}
\begin{cases}
(\varepsilon^2 \nabla \psi, \nabla \varphi) + (\psi, \varphi) = (f, \varphi) & \forall \varphi \in H^1_0(\Omega), \\
(\nabla u, \nabla v) = (\psi, v) & \forall v \in H^1(\Omega).
\end{cases}
\tag{2.4}
\end{equation}

Note that, by the Lax-Milgram lemma, both systems (2.3) and (2.4) have a unique solution. In fact, by regularity theory for elliptic problems [21], if $\Omega$ is convex and $f \in H^{-1}(\Omega)$, then $u \in H^3(\Omega)$ and $\psi \in H^2_0(\Omega)$. Thus (2.3) has solution, which is unique since its homogeneous system has only one solution satisfying $(\psi, u) = 0$. Similar conclusion can be drawn for the system (2.4).

It is well known that the primal weak formulation of (1.1)-(1.2) is: find $\tilde{u} \in H^2(\Omega) \cap H^1_0(\Omega)$ such that
\begin{equation}
(\varepsilon^2 \Delta \tilde{u}, \Delta v) + (\nabla \tilde{u}, \nabla v) = (f, v), \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega),
\tag{2.5}
\end{equation}
and that the one of (1.1)-(1.3) is: find $\tilde{u} \in H^2_0(\Omega)$ such that
\begin{equation}
(\varepsilon^2 \Delta \tilde{u}, \Delta v) + (\nabla \tilde{u}, \nabla v) = (f, v), \quad \forall v \in H^2_0(\Omega).
\tag{2.6}
\end{equation}

The classical results of PDEs imply that (2.5) and (2.6) have unique solutions (see [21]). A natural question is whether the $u$ determined by (2.3) (or (2.4)) is the solution of (2.5) (or (2.6)). In [39], for biharmonic equation on a reentrant corners polygon, a counterexample is shown. The following theorems answer this question.

**Theorem 2.1.** The solution $\tilde{u}$ of (2.5) and the $u$ determined by (2.3) are identical if and only if $u \in H^2(\Omega)$.

**Proof.** The necessity is trivial. If the solution of (2.3) is such that $u \in H^2(\Omega) \cap H^1_0(\Omega)$, then we have from the second equation
\begin{equation}
(-\Delta u, w) = (\psi, w), \quad \forall w \in H^1_0(\Omega).
\end{equation}
Notice that $H^1_0(\Omega)$ is dense in $L^2(\Omega)$. It follows that
\begin{equation}
(-\Delta u, w) = (\psi, w), \quad \forall w \in L^2(\Omega).
\end{equation}

Integration by parts yields
\begin{equation}
(-\Delta u, \Delta \varphi) = (\psi, \Delta \varphi) = -(\nabla \psi, \nabla \varphi), \quad \forall \varphi \in H^2(\Omega) \cap H^1_0(\Omega),
\end{equation}
which implies
\begin{equation}
(\varepsilon^2 \Delta u, \Delta \varphi) = (\varepsilon^2 \nabla \psi, \nabla \varphi), \quad \forall \varphi \in H^2(\Omega) \cap H^1_0(\Omega).
\tag{2.7}
\end{equation}
We obtain from (2.7) and the second equation of (2.3) that
\begin{equation}
(\varepsilon^2 \Delta u, \Delta \varphi) + (\nabla u, \nabla \varphi) = (\varepsilon^2 \nabla \psi, \nabla \varphi) + (\psi, \varphi), \quad \forall \varphi \in H^2(\Omega) \cap H^1_0(\Omega).
\end{equation}
In terms of (2.5), we proved that $u$ is the solution of (2.5). \qed

**Theorem 2.2.** The solution $\tilde{u}$ of (2.6) and the $u$ determined by (2.4) are identical if and only if $u \in H^2_0(\Omega)$. \hfill \Box
Proof. The necessity is trivial. From the second equation of (2.4), integration by parts, and variational principle, we know that the Neumann boundary condition $\partial u / \partial n = 0$ on $\Gamma$ is automatically satisfied. Following the proof of Theorem 2.1, we know that if the solution $u$ of (2.4) is in $H^2_0(\Omega)$, then $u$ is the solution of (2.5).

Let $T_h$ be a shape regular partition of $\Omega$ into triangles (tetrahedra for $d = 3$) or parallelograms (parallelepiped for $d = 3$) satisfying the angle condition [13], i.e., there exists a constant $C_0$ such that

$$C_0^{-1} h_K^d \leq |K| \leq C_0 h_K^d \quad \forall K \in T_h,$$

(2.8)

where $h_K := \text{diam}(K)$. Let $P_k(K)$ be the space of polynomials of total degree at most $k$ if $K$ is a simplex, or the space of polynomials with degree at most $k$ for each variable if $K$ is a parallelogram/parallelepiped. Define the finite element spaces $V_h$ and $V_h^0$ by

$$V_h := \{ v_h \in C(\bar{\Omega}) : v_h|_K \in P_k(K), \forall K \in T_h \}$$

and

$$V_h^0 := \{ v_h \in V_h : v_h|_\Gamma = 0 \},$$

respectively.

We introduce the mixed finite element method for problem $P_1$: find $(\psi_h, u_h) \in V_h^0 \times V_h^0$ such that

$$\begin{cases} 
(\varepsilon^2 \nabla \psi_h, \nabla \varphi_h) + (\psi_h, \varphi_h) = (f, \varphi_h) & \forall \varphi_h \in V_h^0, \\
(\nabla u_h, \nabla v_h) = (\psi_h, v_h) & \forall v_h \in V_h^0.
\end{cases}$$

(2.9)

For problem $P_2$, the mixed problem reads: find $(\psi_h, u_h) \in V_h \times V_h^0$ such that

$$\begin{cases} 
(\varepsilon^2 \nabla \psi_h, \nabla \varphi_h) + (\psi_h, \varphi_h) = (f, \varphi_h) & \forall \varphi_h \in V_h^0, \\
(\nabla u_h, \nabla v_h) = (\psi_h, v_h) & \forall v_h \in V_h^0.
\end{cases}$$

(2.10)

By standard arguments, problem (2.9) possesses a unique solution provided there exist functions $\psi_h$ and $u_h$ satisfying

$$\begin{cases} 
(\varepsilon^2 \nabla \psi_h, \nabla \varphi_h) + (\psi_h, \varphi_h) = 0 & \forall \varphi_h \in V_h^0, \\
(\nabla u_h, \nabla v_h) = (\psi_h, v_h) & \forall v_h \in V_h^0,
\end{cases}$$

(2.11)

then $(\psi_h, u_h)$ is the trivial solution to the system. In fact, taking $\varphi_h = \psi_h$ in the first equation of (2.11), one gets $\psi_h = 0$. Setting $v_h = u_h$ in the second equation of (2.11), one obtains $u_h = 0$. Similarly, it is verified that problem (2.10) has also a unique solution.

We define a measure of the error between the exact solution $(\psi, u)$ and the numerical solution $(\psi_h, u_h)$ by

$$|| (\psi - \psi_h, u - u_h) ||^2 := || \psi - \psi_h ||^2_\varepsilon + || u - u_h ||^2_1,$$

where

$$|| \psi - \psi_h ||_\varepsilon = (\varepsilon^2 || \psi - \psi_h ||^2_1 + || \psi - \psi_h ||^2)^{1/2},$$

is the standard energy norm of the numerical error $\psi - \psi_h$. In this paper, we aim at robust a posterior error estimators for the numerical errors $|| \psi - \psi_h ||_\varepsilon$, $|| u - u_h ||_1$, and $|| (\psi - \psi_h, u - u_h) ||$. 

Robust Residual-Based a Posteriori Error Estimators

5
We next introduce some notations that will be used later. We denote $\mathcal{E}_h$ the set of interior sides (if $d = 2$) or faces (if $d = 3$) in $T_h$, $\mathcal{E}_T$ the set of sides or faces of $T \in T_h$, and $\mathcal{E}_0$ the union of all elements in $T_h$ sharing at least one point with $T$. For a side or face $E$ in $\mathcal{E}_h$, which is the set of element sides or faces in $T_h$, let $h_E$ be the diameter of $E$, and $\omega_E$ be the union of all elements in $T_h$ sharing $E$. For a function $v$ in the "broken Sobolev space" $H^1(\bigcup T_h)$, we define $[v]|_E := (v|_{T_+})_E - (v|_{T_-})_E$ as the jump of $v$ across an interior side or face $E$, where $T_+$ and $T_-$ are the two neighboring elements such that $E = T_+ \cap T_-$. Throughout of this paper, we denote by $C_Q$ a constant depending only on $Q$, and denote by $C_i (i = 0, 1, \cdots)$ constants depending on the mesh shape regularity and $d$. In what follows we use the notation $A \lesssim F$ to represent $A \leq CF$ with a generic constant $C > 0$ independent of mesh size. In addition, $A \approx F$ abbreviates $A \lesssim F \lesssim A$.

3. Preliminary results. For problem $P_2$, $\psi$ and $u$ are decoupled. However, $\psi$ does not obtain any information directly from boundary conditions. It will be difficult to develop residual-based $a$ posteriori error estimates if the residual on the boundary is not clear. To overcome this difficulty, we shall develop a novel analytical technique (see Section 4), which is based on the following approximation result.

**Lemma 3.1.** Let $v \in H^1(\Omega)/H^1_0(\Omega)$ which satisfies $-\varepsilon^2 \Delta v + v = 0$ (operator to be understood in weak sense). Then it holds

$$\inf_{w \in H^1_0(\Omega)} \frac{\varepsilon^2 |v - w|^2 + ||v - w||^2}{\varepsilon^2 |v|^2 + ||v||^2} = 1.$$  

**Proof.** Consider the functional

$$J(w) = \varepsilon^2 |v - w|^2 + ||v - w||^2, \quad \forall \ w \in H^1_0(\Omega).$$

Minimization of such functional in $H^1_0(\Omega)$ immediately leads to the following variational problem: Find $\tilde{v} \in H^1_0(\Omega)$ such that

$$\int_{\Omega} (\varepsilon^2 \nabla (v - \tilde{v}) \cdot \nabla \phi + (v - \tilde{v}) \phi) dx = 0, \quad \forall \ \phi \in H^1_0(\Omega).$$

(3.1)

Integrating by parts, we arrive at

$$\int_{\Omega} (-\varepsilon^2 \Delta (v - \tilde{v}) + v - \tilde{v}) \phi dx = 0, \quad \forall \ \phi \in H^1_0(\Omega),$$

which implies

$$-\varepsilon^2 \Delta \tilde{v} + \tilde{v} = -\varepsilon^2 \Delta v + v = 0.$$  

So $\tilde{v}$ is the solution to the following problem:

$$\begin{cases}
-\varepsilon^2 \Delta \tilde{v} + \tilde{v} = 0 & \text{in } \Omega, \\
\tilde{v} = 0 & \text{on } \Gamma.
\end{cases}$$

(3.2)

Since the problem (3.2) has only trivial solution, we have

$$\inf_{w \in H^1_0(\Omega)} \frac{\varepsilon^2 |v - w|^2 + ||v - w||^2}{\varepsilon^2 |v|^2 + ||v||^2} = \inf_{w \in H^1_0(\omega)} \frac{J(w)}{\varepsilon^2 |v|^2 + ||v||^2} = \frac{\varepsilon^2 |v - \tilde{v}|^2 + ||v - \tilde{v}||^2}{\varepsilon^2 |v|^2 + ||v||^2} =\frac{\varepsilon^2 |v|^2 + ||v||^2}{\varepsilon^2 |v|^2 + ||v||^2} = 1,$$
which completes the proof.  

**Lemma 3.2.** If there holds the following relation

$$ \inf_{w \in H^1_0(\Omega)} \frac{\varepsilon^2 |v - w|^2_1 + \|v - w\|^2}{\varepsilon^2 |v|^2_1 + \|v\|^2} = 1, $$

then \( v \) satisfies \( -\varepsilon^2 \Delta v + v = 0 \).

**Proof.** Since \( \tilde{v} = 0 \in H^1_0(\Omega) \), the condition

$$ 1 = \inf_{w \in H^1_0(\Omega)} \frac{\varepsilon^2 |v - w|^2_1 + \|v - w\|^2}{\varepsilon^2 |v|^2_1 + \|v\|^2} $$

can be satisfied when \( w = \tilde{v} \). On the other hand, \( \tilde{v} \) is the solution to the variational problem: Find \( \tilde{v} \in H^1_0(\Omega) \) such that

$$ \int_{\Omega} (\varepsilon^2 \nabla (v - \tilde{v}) \cdot \nabla \phi + (v - \tilde{v})\phi) \, dx = 0, \ \forall \phi \in H^1_0(\Omega). $$

By integrating by parts, we obtain

$$ \int_{\Omega} (\varepsilon^2 \Delta (v - \tilde{v}) + (v - \tilde{v})) \phi \, dx = 0, \ \forall \phi \in H^1_0(\Omega), $$

which leads to \( -\varepsilon^2 \Delta (v - \tilde{v}) + (v - \tilde{v}) = 0 \), this means

$$ -\varepsilon^2 \Delta v + v = -\varepsilon^2 \Delta \tilde{v} + \tilde{v} = 0. $$

**Lemma 3.3.** Let \( v \in H^1(\Omega) \) such that \( -\varepsilon^2 \Delta v + v \neq 0 \). Then there exists \( \gamma \in (0, 1) \) such that

$$ \inf_{w \in H^1_0(\Omega)} \frac{\varepsilon^2 |v - w|^2_1 + \|v - w\|^2}{\varepsilon^2 |v|^2_1 + \|v\|^2} \leq \gamma. $$

**Proof.** We only prove the case \( v \in H^1(\Omega)/H^1_0(\Omega) \) using proof by contradiction, since \( v \in H^1_0(\Omega) \) is obvious. Assume that there does not exist \( \gamma \in (0, 1) \) such that

$$ \inf_{w \in H^1_0(\omega)} (\varepsilon^2 |v - w|^2_1 + \|v - w\|^2) \leq \gamma (\varepsilon^2 |v|^2_1 + \|v\|^2), $$

which means

$$ \inf_{w \in H^1_0(\omega)} (\varepsilon^2 |v - w|^2_1 + \|v - w\|^2) > \gamma (\varepsilon^2 |v|^2_1 + \|v\|^2) $$

for all \( \gamma \in (0, 1) \). From the proof of Lemma 3.1, there exists \( \tilde{v} \in H^1_0(\Omega) \) such that

$$ \varepsilon^2 |v|^2_1 + \|v\|^2 \geq \inf_{w \in H^1_0(\Omega)} (\varepsilon^2 |v - w|^2_1 + \|v - w\|^2) $$

$$ = \varepsilon^2 |v - \tilde{v}|^2_1 + \|v - \tilde{v}\|^2 $$

$$ \geq \gamma (\varepsilon^2 |v|^2_1 + \|v\|^2). $$

In particular, for any \( n \in \mathbb{N} \), let \( \gamma = \frac{n-1}{n} \), it holds that

$$ \varepsilon^2 |v|^2_1 + \|v\|^2 \geq \varepsilon^2 |v - \tilde{v}|^2_1 + \|v - \tilde{v}\|^2 > \frac{n-1}{n} (\varepsilon^2 |v|^2_1 + \|v\|^2). $$
Since $\varepsilon^2|v - \tilde{v}|_1^2 + ||v - \tilde{v}||^2$ is the upper bound for $\begin{split}
frac{\varepsilon^2}{h^2}(\varepsilon^2|v|_1^2 + ||v||^2)
\end{split}$ with respect to the positive integer number $n$, and $\varepsilon^2|v|_1^2 + ||v||^2$ is its supremum. Therefore,

\[\varepsilon^2|v|_1^2 + ||v||^2 \leq \varepsilon^2|v - \tilde{v}|_1^2 + ||v - \tilde{v}||^2 \leq \varepsilon^2|v|_1^2 + ||v||^2,\]

which means

\[\varepsilon^2|v - \tilde{v}|_1^2 + ||v - \tilde{v}||^2 = \varepsilon^2|v|_1^2 + ||v||^2.\]

This leads to

\[\inf_{w \in H^1_0(\Omega)} \frac{\varepsilon^2|v - w|_1^2 + ||v - w||^2}{\varepsilon^2|v|_1^2 + ||v||^2} = \frac{\varepsilon^2|v - \tilde{v}|_1^2 + ||v - \tilde{v}||^2}{\varepsilon^2|v|_1^2 + ||v||^2} = 1.\]

From Lemma 3.2 we have $-\varepsilon^2\Delta v + v = 0$, this leads to a contradiction. We complete the proof. \[\square\]

**Remark 3.1.** Note that we cannot prove that \{ $v \in H^1(\Omega)$ : $-\Delta v + v \neq 0$ \} is dense in $H^1_0(\Omega)$ by recursion by using Lemma 3.3, because of $-\Delta(v - \tilde{v}) + (v - \tilde{v}) = 0$.

Denote by $I_h : L^2(\Omega) \rightarrow V_h^0$ the quasi-interpolation operator of Clément (cf. \cite{13, 14, 33}).

**Lemma 3.4.** For all $T \in T_h, E \subset \partial T$, define $\alpha_T$ and $\alpha_E$ the weighted factors by

\[\alpha_T := \min\{h_T\varepsilon^{-1}, 1\}\] and \[\alpha_E := \varepsilon^{-1/2}\min\{h_T\varepsilon^{-1}, 1\},\]

respectively. Then the following local error estimates hold for $v \in H^1(\tilde{\omega}_T)$:

\[\|v - I_h v\|_T \lesssim \alpha_T\|v\|_{\tilde{\omega}_T}\] (3.3)

and

\[\|v - I_h v\|_E \lesssim \alpha_E\|v\|_{\tilde{\omega}_T}.\] (3.4)

**Proof.** Following the line of the proof of Lemma 3.2 in \cite{33}, we obtain the desired estimates \cite{33} and \cite{34}. \[\square\]

For $\theta \in (0, 1), T \in T_h, E \in \mathcal{E}_h$, denote $\psi_T$ and $\psi_{E, \theta}$ the two bubble functions defined in \cite{33}, and $P_E$ a continuation operator introduced in \cite{33} by

\[P_E : L^\infty(E) \rightarrow L^\infty(\omega_E),\]

which maps polynomials onto piecewise polynomials of the same degree.

**Lemma 3.5.** The following estimates hold for all $v \in \mathcal{P}_h$ (the set of polynomials of degree at most $k$) and $T \in T_h$

\[\|v\|_T^2 \lesssim (v, \psi_T v)_T,\] (3.5)

\[\|v\psi_T\|_T \leq \|v\|_T,\] (3.6)

\[\|v\psi_T\|_{\tilde{\omega}, T} \lesssim \alpha^{-1}_T\|v\|_T.\] (3.7)
Furthermore, for $E \in \mathcal{E}_h$, set $\theta_E := \min \{ \epsilon h_E^{-1}, 1 \}$. Then there hold the following estimates for all $E \in \mathcal{E}_h$ and $\sigma \in \mathcal{P}_k|_E$.

$$
\| \sigma \|_E^2 \lesssim (\sigma, \psi_{E,\theta_E} P_E \sigma)_E,
$$

(3.8)

$$
\| \psi_{E,\theta_E} P_E \sigma \|_{\omega_E} \lesssim \epsilon^{1/2} \min \{ h_E^{-1}, 1 \}^{1/2} \| \sigma \|_E,
$$

(3.9)

$$
\| \psi_{E,\theta_E} P_E \sigma \|_{E,\omega_E} \lesssim \epsilon^{1/2} \min \{ h_E^{-1}, 1 \}^{-1/2} \| \sigma \|_E.
$$

(3.10)

**Proof.** Following the line of the proof of Lemma 3.3 in [33], we attain (3.5)-(3.10).

4. **A reliable upper bound.** For all $T \in \mathcal{T}_h$, define $\eta_{\psi,T}$ and $\eta_{u,T}$ the elementwise indicators of $\psi$ and $u$, respectively, by

$$
\eta_{\psi,T} := \left\{ \frac{\alpha_T^2}{2} \| f + \epsilon^2 \Delta \psi_h - \psi_h \|_T^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_h^0} \alpha_E^2 \left[ \epsilon^2 \frac{\partial \psi_h}{\partial n} \right]_E \right\}^{1/2},
$$

and

$$
\eta_{u,T} := \left\{ \frac{h_T^2}{2} \| \Delta u_h + \psi_h \|_T^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_h^0} h_E \left[ \frac{\partial u_h}{\partial n} \right]_E \right\}^{1/2}.
$$

**THEOREM 4.1.** Let $(\psi, u) \in H^1_0(\Omega) \times H^1_0(\Omega)$ and $(\psi_h, u_h) \in V^0_h \times V^0_h$ be the solutions to (2.3) and (2.9), respectively. Then there exist positive constants $C_1, C_2$, and $C_3$, independent of the mesh-size function $h$ and $\epsilon$, such that

$$
\| \psi - \psi_h \|_E \leq C_1 \left\{ \sum_{T \in \mathcal{T}_h} \eta_{\psi,T}^2 \right\}^{1/2},
$$

(4.1)

$$
|u - u_h|_1 \leq C_2 \left\{ \sum_{T \in \mathcal{T}_h} \eta_{\psi,T}^2 + \eta_{u,T}^2 \right\}^{1/2},
$$

(4.2)

$$
\| (\psi - \psi_h, u - u_h) \| \leq C_3 \left\{ \sum_{T \in \mathcal{T}_h} \eta_{\psi,T}^2 + \eta_{u,T}^2 \right\}^{1/2}.
$$

(4.3)

**Proof.** From the definition of the measure $\| (\psi - \psi_h, u - u_h) \|$, (4.3) follows from (4.1) and (4.2). We need to prove (4.1) and (4.2). We have from the first equations of (2.3) and (2.9) that

$$
(\epsilon^2 \nabla (\psi - \psi_h), \nabla \varphi_h) + (\psi - \psi_h, \varphi_h) = 0, \ \forall \varphi_h \in V^0_h.
$$

(4.4)
For any $\varphi \in H^1_0(\Omega)$, let $\varphi_h$ be the Clemént interpolation of $\varphi$ in $V^0_h$, i.e., $\varphi_h = I_h\varphi$. Applying integration by parts and (4.4), we get

\begin{align*}
&\langle \varepsilon^2\nabla(\psi - \psi_h), \nabla\varphi \rangle + (\psi - \psi_h, \varphi) \\
= &\sum_{T \in T_h} \int_T (-\varepsilon^2 \Delta \psi + \varepsilon^2 \Delta \psi_h + \psi - \psi_h)(\varphi - \varphi_h) + \int_{\partial T} \varepsilon^2 \frac{\partial(\psi - \psi_h)}{\partial n} (\varphi - \varphi_h) \\
= &\sum_{T \in T_h} \int_T (f + \varepsilon^2 \Delta \psi_h - \psi_h)(\varphi - \varphi_h) - \int_{\partial T} \varepsilon^2 \frac{\partial \psi_h}{\partial n} (\varphi - \varphi_h) \\
\leq &\sum_{T \in T_h} \left\{ \|f + \varepsilon^2 \Delta \psi_h - \psi_h\|_T \|\varphi - \varphi_h\|_T + \frac{1}{2} \sum_{E \in \mathcal{E}_h \cap \mathcal{E}_h^0} \left\| \varepsilon^2 \frac{\partial \psi_h}{\partial n} \left\|_E \right\| \|\varphi - \varphi_h\|_E \right\}. \quad (4.5)
\end{align*}

Notice that

\begin{align*}
\|\psi - \psi_h\|_E &= \frac{\langle \varepsilon^2\nabla(\psi - \psi_h), \nabla(\psi - \psi_h) \rangle + (\psi - \psi_h, \psi - \psi_h)}{\|\psi - \psi_h\|_E} \\
&\leq \sup_{0 \neq \varphi \in H^1_0(\Omega)} \frac{\langle \varepsilon^2\nabla(\psi - \psi_h), \nabla\varphi \rangle + (\psi - \psi_h, \varphi)}{\|\varphi\|_E}. \quad (4.6)
\end{align*}

The first estimate (4.1) follows from a combination of (4.6), (4.5), and (3.3)-(3.4).

We next prove (4.2). From the second equation of (2.3) and (2.9), we get

\begin{align*}
(\nabla(u - u_h), \nabla v_h) = (\psi - \psi_h, v_h), \quad \forall v_h \in V^0_h. \quad (4.7)
\end{align*}

Similarly, we have, for any $v \in H^1_0(\Omega)$ and $v_h = I_h v$,

\begin{align*}
&(\nabla(u - u_h), \nabla v) = (\nabla(u - u_h), \nabla(v - v_h)) + (\nabla(u - u_h), \nabla v_h) \\
= &\sum_{T \in T_h} \int_T (-\Delta u + \Delta u_h)(v - v_h) + \int_{\partial T} \frac{\partial(u - u_h)}{\partial n} (v - v_h) + (\psi - \psi_h, v_h) \\
= &\sum_{T \in T_h} \int_T (-\Delta u + \Delta u_h - \psi + \psi_h)(v - v_h) + \int_{\partial T} \frac{\partial(u - u_h)}{\partial n} (v - v_h) + (\psi - \psi_h, v) \\
= &\sum_{T \in T_h} \int_T (\Delta u_h + \psi_h)(v - v_h) - \int_{\partial T} \frac{\partial u_h}{\partial n} (v - v_h) + (\psi - \psi_h, v) \\
\leq &\sum_{T \in T_h} \left\{ \|\Delta u_h + \psi_h\|_T \|v - v_h\|_T + \frac{1}{2} \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_h^0} \left\| \frac{\partial u_h}{\partial n} \right\|_E \|v - v_h\|_E \right\} \\
&+ \|\psi - \psi_h\|_E \|v\|. \quad (4.8)
\end{align*}

Recall the following estimates on Clemént interpolation (cf. [14]):

\begin{align*}
\|v - I_h v\|_T \lesssim h_T \|v\|_{1,\omega_T} \quad \text{for all } T \in T_h, \ v \in H^1(\tilde{\omega}_T) \quad (4.9)
\end{align*}

and

\begin{align*}
\|v - I_h v\|_E \lesssim h_{E,T}^{1/2} \|v\|_{1,\omega_T} \quad \text{for all } E \in \mathcal{E}_h, E \subset \partial T, \ v \in H^1(\tilde{\omega}_T). \quad (4.10)
\end{align*}

For any $v \in H^1_0(\Omega)$, the Poincaré inequality implies

\begin{align*}
\|v\| \leq \|v\|_1 \lesssim |v|_1. \quad (4.11)
\end{align*}
A combination of (4.8) and (4.9)-(4.11) yields
\[
\frac{1}{2} \sum_{E \in E_T \cap E_h^T} h_T \left( \left\| \frac{\partial u_h}{\partial n} \right\|_E \right)^2 + \| \psi - \psi_h \|_E \right) \right] |v|_1. \tag{4.12}
\]

Notice that \[|u - u_h|_1 \leq \sup_{0 \neq v \in H^1_0(\Omega)} \frac{(\nabla (u - u_h), \nabla v)}{|v|_1}. \tag{4.13}\]

A combination of (4.13), (4.12), and (4.1) yields
\[|u - u_h|_1 \lesssim \left\{ \sum_{T \in T_h} \left( \eta_{\psi,T}^2 + \eta_{u,T}^2 \right) \right\}^{1/2}. \tag{4.14}\]

This completes the proof of (4.2). \( \square \)

**Theorem 4.2.** Let \((\psi, u) \in H^1(\Omega) \times H^1_0(\Omega)\) and \((\psi_h, u_h) \in V_h \times V_h^0\) be the solutions to (4.13) and (4.10), respectively. If \(-\varepsilon^2 \Delta (\psi - \psi_h) + (\psi - \psi_h) \neq 0\), then there exist positive constants \(C_4, C_5, \) and \(C_6\), independent of the mesh-size function \(h\) and \(\varepsilon\), such that
\[
\| \psi - \psi_h \|_E \leq C_4 \left\{ \sum_{T \in T_h} \eta_{\psi,T}^2 \right\}^{1/2}, \tag{4.15}\]
\[
|u - u_h|_1 \leq C_5 \left\{ \sum_{T \in T_h} \eta_{\psi,T}^2 + \eta_{u,T}^2 \right\}^{1/2}, \tag{4.16}\]
\[
\| (\psi - \psi_h, u - u_h) \| \leq C_6 \left\{ \sum_{T \in T_h} \eta_{\psi,T}^2 + \eta_{u,T}^2 \right\}^{1/2}. \tag{4.17}\]

**Proof.** We have from \(-\varepsilon^2 \Delta (\psi - \psi_h) + (\psi - \psi_h) \neq 0\)
\[
\| \psi - \psi_h \|_E \leq \sup_{v \in H^1(\Omega), -\varepsilon^2 \Delta v + v \neq 0} \frac{(\varepsilon^2 \nabla (\psi - \psi_h), \nabla v) + (\psi - \psi_h, v)}{|v|_E}. \tag{4.18}\]

For \(v \in H^1(\Omega)\) satisfying \(-\varepsilon^2 \Delta v + v \neq 0\), from the proofs of Lemmas 3.1 and 3.3 there exist \(\tilde{v} \in H^1_0(\Omega)\) and \(\gamma \in (0, 1)\), such that
\[
\varepsilon^2 |v - \tilde{v}|_1^2 + |v - \tilde{v}|^2 \leq \inf_{w \in H^1_0(\Omega)} (\varepsilon^2 |v - w|_1^2 + |v - w|^2) \leq \gamma (\varepsilon^2 |v|_1^2 + |v|^2) = \gamma |v|_E^2. \tag{4.19}\]

Let \(\tilde{v}_h = I_h \tilde{v}\) be the Clemént interpolation of \(\tilde{v}\) in \(V_h^0\). From the first equation of (2.4) and (2.10), we have
\[
(\varepsilon^2 \nabla (\psi - \psi_h), \nabla \tilde{v}_h) + (\psi - \psi_h, \tilde{v}_h) = 0. \tag{4.20}\]
From (4.19), we have

\[
(\varepsilon^2 \nabla (\psi - \psi_h), \nabla v) + (\psi - \psi_h, v) = (\varepsilon^2 \nabla (\psi - \psi_h), \nabla (\tilde{v} - \tilde{v}_h)) + (\psi - \psi_h, v - \tilde{v}) \\
+ (\varepsilon^2 \nabla (\psi - \psi_h), \nabla \tilde{v}) + (\psi - \psi_h, \tilde{v})
\]

\[
\leq \|\psi - \psi_h\|_{\mathcal{E}} \|v - \tilde{v}\|_{\mathcal{E}} + (\varepsilon^2 \nabla (\psi - \psi_h), \nabla (\tilde{v} - \tilde{v}_h)) + (\psi - \psi_h, \tilde{v} - \tilde{v}_h).
\]

(4.20)

Repeating the proof of (4.5), and applying (3.3)-(3.4), we have

\[
(\varepsilon^2 \nabla (\psi - \psi_h), \nabla (\tilde{v} - \tilde{v}_h)) + (\psi - \psi_h, \tilde{v} - \tilde{v}_h)
\]

\[
\leq \sum_{T \in \mathcal{T}_h} \left\{ \|f + \varepsilon^2 \Delta \psi_h - \psi_h\|_{\mathcal{T}} \|\tilde{v} - \tilde{v}_h\|_{\mathcal{T}} + \frac{1}{2} \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_{h}^T} \left\| \varepsilon \frac{\partial \psi_h}{\partial n} \right\|_{E} \|\tilde{v} - \tilde{v}_h\|_{E} \right\}
\]

\[
\leq C \left\{ \sum_{T \in \mathcal{T}_h} \eta_{\psi,T}^2 \right\}^{1/2} \|\tilde{v}\|_{\mathcal{E}}.
\]

(4.21)

Using the triangle inequality and (4.19), we have

\[
\|\tilde{v}\|_{\mathcal{E}} \lesssim \|v - \tilde{v}\|_{\mathcal{E}}^2 + \|v\|_{\mathcal{E}}^2 \lesssim \|v\|_{\mathcal{E}}.
\]

(4.22)

A combination of (4.20), (4.18), (4.21), and (4.22) yields

\[
(\varepsilon^2 \nabla (\psi - \psi_h), \nabla v) + (\psi - \psi_h, v) \leq \left\{ \sqrt{\gamma} \|\psi - \psi_h\|_{\mathcal{E}} + C \left( \sum_{T \in \mathcal{T}_h} \eta_{\psi,T}^2 \right)^{1/2} \right\} \|v\|_{\mathcal{E}}.
\]

(4.23)

From (4.17) and (4.23), we obtain

\[
\|\psi - \psi_h\|_{\mathcal{E}} \leq \sqrt{\gamma} \|\psi - \psi_h\|_{\mathcal{E}} + C \left( \sum_{T \in \mathcal{T}_h} \eta_{\psi,T}^2 \right)^{1/2},
\]

which leads to the desired estimate (4.14). Repeating the proof of (4.2) and (4.3), we obtain (4.15) and (4.16). \( \square \)

**Remark 4.1.** The condition \(-\varepsilon^2 \Delta (\psi - \psi_h) + (\psi - \psi_h) \neq 0\) is usually satisfied, since

\[
-\varepsilon^2 \Delta (\psi - \psi_h) + (\psi - \psi_h) = f - (-\varepsilon^2 \Delta \psi_h + \psi_h)
\]

is the residual, which doesn’t vanish in usual. Here \(\Delta \psi_h\) is the piecewise Laplacian of \(\psi_h\).

### 5. The analysis of the efficiency on the estimators.

In this section, we analyze the efficiency of the *a posteriori* error estimates developed in Section 4. To avoid the appearance of high order term, we assume that \(f\) is a piecewise polynomial.

**Lemma 5.1.** For all \(T \in \mathcal{T}_h\), there hold

\[
\alpha_T \|f + \varepsilon^2 \Delta \psi_h - \psi_h\|_{\mathcal{T}} \lesssim \|\psi - \psi_h\|_{\mathcal{E},T}
\]

and

\[
h_T \|\Delta u_h + \psi_h\|_{T} \lesssim |u - u_h|_{1,T} + h_T \|\psi - \psi_h\|_{T}.
\]

(5.1)

(5.2)
Proof. We first prove (5.1). To this end, let \( v = f + \varepsilon^2 \Delta \psi_h - \psi_h \). Recall the bubble function \( \psi_T \) introduced in Section 3. From (3.5), integration by parts, and (3.7), we have

\[
\|v\|_T^2 \lesssim (\psi_T v, v)_T \\
= (-\varepsilon^2 \Delta (\psi - \psi_h), \psi_T v)_T + (\psi - \psi_h, \psi_T v)_T \\
= \varepsilon^2 (\nabla (\psi - \psi_h), \nabla (\psi_T v))_T + (\psi - \psi_h, \psi_T v)_T \\
\leq \|\psi - \psi_h\|_{\mathcal{E},T} \|\psi_T v\|_{\mathcal{E},T} \\
\lesssim \|\psi - \psi_h\|_{\mathcal{E},T} \alpha_T^{-1} \|v\|_T.
\]

The desired estimate (5.1) follows.

We next prove (5.2). For convenience, denote \( v = \Delta u_h + \psi_h \). Similarly, we have from \( \psi = -\Delta u \) that

\[
\|v\|_T^2 \lesssim (\psi_T v, v)_T \\
= (\Delta u_h - \Delta u + \Delta u + \psi_h, \psi_T v)_T \\
= - (\Delta(u - u_h), \psi_T v)_T - (\psi - \psi_h, \psi_T v)_T \\
= (\nabla (u - u_h), \nabla (\psi_T v))_T - (\psi - \psi_h, \psi_T v)_T \\
\leq |u - u_h|_{1,T} |\psi_T v|_{1,T} + \|\psi - \psi_h\|_{T} \|\psi_T v\|_{T}.
\]

Applying inverse estimate and (3.6), we have

\[
\|v\|_T^2 \lesssim (h_T^{-1} |u - u_h|_{1,T} + \|\psi - \psi_h\|_{T}) \|v\|_T.
\]

The estimate (5.2) follows immediately. \( \square \)

Lemma 5.2. For all \( E \in \mathcal{E}_h^0 \), there hold

\[
\alpha_E \bigg\| \varepsilon^2 \frac{\partial \psi_h}{\partial n} \bigg\|_E \lesssim \|\psi - \psi_h\|_{\mathcal{E},\omega_E} \tag{5.3}
\]

and

\[
h_E^{1/2} \bigg\| \frac{\partial u_h}{\partial n} \bigg\|_E \lesssim |u - u_h|_{1,\omega_E} + h_E \|\psi - \psi_h\|_{\omega_E}. \tag{5.4}
\]

Proof. We first prove (5.3). To this end, let \( \sigma = [\varepsilon^2 \frac{\partial \psi_h}{\partial n}] \). Recall the bubble function \( \psi_{E,\theta_E} \) and the extension operator \( P_E \) introduced in Section 3. Let \( v_E = \psi_{E,\theta_E} P_E \sigma \). An application of integration by parts leads to

\[
(\varepsilon^2 \nabla (\psi - \psi_h), \nabla v_E)_{\omega_E} + (\psi - \psi_h, v_E)_{\omega_E} = (f + \varepsilon^2 \Delta_h \psi_h - \psi_h, v_E)_{\omega_E} + \left(- [\varepsilon^2 \frac{\partial \psi_h}{\partial n}], v_E \right)_{\omega_E},
\]
where \( \Delta_h \) is the elementwise Laplace operator. A combination of the above equality and (3.8)-(3.10) leads to

\[
\| \sigma \|_E^2 \lesssim \left( [\varepsilon^2 \frac{\partial \psi_h}{\partial n}, v_E]_E \right)
= (f + \varepsilon^2 \Delta_h \psi_h - \psi_h, v_E)_{\omega_E} - (\varepsilon^2 \nabla (\psi - \psi_h), \nabla v_E)_{\omega_E} - (\psi - \psi_h, v_E)_{\omega_E}
\lesssim \| f + \varepsilon^2 \Delta_h \psi_h - \psi_h \|_{\omega_E} \| v_E \|_{\omega_E} + \| \psi - \psi_h \|_{\omega_E} \| v_E \|_{\omega_E}
\lesssim \varepsilon^{1/2} \min \{ 1, h_E \varepsilon^{-1} \}^{1/2} \| f + \varepsilon^2 \Delta_h \psi_h - \psi_h \|_{\omega_E} + \| \psi - \psi_h \|_{\omega_E}
\lesssim \| \psi - \psi_h \|_{\omega_E}.
\]

By the definition of \( \alpha_E \) for \( E \subset \partial T \) and the local shape regularity of the mesh, we obtain

\[
\alpha_E \| \sigma \|_E \lesssim \varepsilon^{1/2} \min \{ 1, h_E \varepsilon^{-1} \}^{1/2} \varepsilon^{-1/2} \min \{ 1, h_E \varepsilon^{-1} \}^{1/2}\| f + \varepsilon^2 \Delta_h \psi_h - \psi_h \|_{\omega_E}
+ \| \psi - \psi_h \|_{\omega_E} \varepsilon^{1/2} \min \{ 1, h_E \varepsilon^{-1} \}^{-1/2} \varepsilon^{-1/2} \min \{ 1, h_E \varepsilon^{-1} \}^{1/2}
\lesssim \min \{ 1, h_E \varepsilon^{-1} \} \| f + \varepsilon^2 \Delta_h \psi_h - \psi_h \|_{\omega_E} + \| \psi - \psi_h \|_{\omega_E}
\lesssim \| \psi - \psi_h \|_{\omega_E}.
\]

In the last step, estimate (5.1) is used. We complete the proof of (5.3).

We next prove (5.4). For convenience, denote \( \sigma = \left( \frac{\partial \psi_h}{\partial n} \right) \) and \( v_E = \psi_E P_E \sigma \), where \( \psi_E = \psi_E, \theta_E \) for \( \theta_E = 1 \). Similarly, we have

\[
(\nabla (u - u_h), \nabla v_E)_{\omega_E} = (-\Delta u + \Delta_h u_h, v_E)_{\omega_E} - \left( \left( \frac{\partial \psi_h}{\partial n} \right), v_E \right)_E,
\]

which leads to the following estimate:

\[
\| \sigma \|_E^2 \lesssim \left( \frac{\partial \psi_h}{\partial n}, v_E \right)
= (\psi + \Delta_h u_h, v_E)_{\omega_E} - (\nabla (u - u_h), \nabla v_E)_{\omega_E}
= (\psi - \psi_h, v_E)_{\omega_E} + (\Delta_h u_h + \psi_h, v_E)_{\omega_E} - (\nabla (u - u_h), \nabla v_E)_{\omega_E}
\lesssim \| \psi - \psi_h \|_{\omega_E} h_E^{1/2} \| \sigma \|_E + h_E^{1/2} \| \Delta_h u_h + \psi_h \|_{\omega_E} \| \sigma \|_E
+ |u - u_h|_{1, \omega_E} h_E^{-1/2} \| \sigma \|_E.
\]

We obtain from the above inequality that

\[
h_E^{1/2} \| \sigma \|_E \lesssim h_E \| \psi - \psi_h \|_{\omega_E} + h_E \| \Delta_h u_h + \psi_h \|_{\omega_E} + |u - u_h|_{1, \omega_E}
\lesssim h_E \| \psi - \psi_h \|_{\omega_E} + |u - u_h|_{1, \omega_E}.
\]

In the last step above, we employ the estimate (5.2). We complete the proof of (5.4). \( \square \)

**THEOREM 5.3.** Let \( (\psi, u) \in H^1_0(\Omega) \times H^1_0(\Omega) \) and \( (\psi_h, u_h) \in V_h^0 \times V_h^0 \) be the solutions to (2.3) and (2.4), respectively. Then there exist positive constants \( C_7 \) and \( C_8 \), independent of the mesh-size function \( h \) and \( \varepsilon \), such that

\[
C_7 \left( \sum_{T \in Th} \eta_{h,T}^2 \right)^{1/2} \leq \| \psi - \psi_h \|_E
\]
It is observed that the function \(\psi(x, y)\) has a boundary layer, which varies significantly near \(x = 0\). The function \(u\) has a boundary layer, which varies significantly near \(x = 0\). We suppose the exact solution of this model has the form

\[ u(x, y) = 256(x^2 + \varepsilon^2(1 - \exp(-x/\varepsilon))^2)(x - 1)^2y^2(y - 1)^2. \]

The function \(u\) has a boundary layer, which varies significantly near \(x = 0\). Our initial mesh consists of eight isosceles right triangles. We employ Dörfler marking strategy to obtain an admissible mesh. Plots in Figure 6.1 depict the estimators of \(\|\psi - \psi_h\|_C = (\varepsilon^2\|\nabla(\psi - \psi_h)\|^2 + \|\psi - \psi_h\|^2)^{1/2} \) (upper and middle), and \((\|\psi - \psi_h, u - u_h\|) = (\|\psi - \psi_h\|^2 + \|\nabla(u - u_h)\|^2)^{1/2} \) (lower), respectively. We observe that strong mesh refinement near the line \(x = 0\), which indicates the estimators of the errors \(\|\psi - \psi_h\|_C\) and \((\|\psi - \psi_h, u - u_h\|)\) are asymptotically exact even for very small \(\varepsilon\). We also observe that the estimated convergence curve is parallel to the curve \((\|\psi - \psi_h, u - u_h\|)\) independent of \(\varepsilon = 10^{-6}\), and both curves decrease in optimal rates. Note that the study of convergence and
FIG. 6.1. Upper: The mesh after 10 iterations with 214 triangles (left) and the mesh after 12 iterations with 402 triangles (right). Middle: The mesh after 14 iterations with 727 triangles (left) and the mesh after 16 iterations with 4470 triangles (right). These four plots depict the elementwise indicator $\eta_{\psi,T}$. Lower: The mesh after 10 iterations with 478 triangles (left) and the mesh after 12 iterations with 1010 triangles (right), generated by the elementwise indicator $\eta_{\psi,T} + \eta_{u,T}$. Here $\varepsilon = 10^{-6}$ and $\theta = 0.3$ for all plots.

The optimality of adaptive algorithms is still in its infancy, and has been carried out mainly for standard adaptive finite element method for general second order elliptic problems; see, e.g., [4, 6, 9, 11, 12, 16, 18, 27].

The two lower plots of Figure 6.3 depict error curves for $\|\psi - \psi_h\|_{\mathcal{E}}$ (left) and $\|(\psi - \psi_h, u - u_h)\|$ (right), respectively. It is observed that the convergence curves for $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{-6}$ are consistent, which indicates that the errors reduce uniformly with respect to $\varepsilon$. In addition, we include in Figure 6.3 an optimal theoretical convergence line with slope $-1/2$. The plots indicate that $\|\psi - \psi_h\|_{\mathcal{E}}$ and $\|(\psi - \psi_h, u - u_h)\|$ decrease in the optimal convergence rates.

Tables 6.1 and 6.2 show some results of the actual errors $\|\psi - \psi_h\|_{\mathcal{E}}$ and $\|(\psi - \psi_h, u - u_h)\|$, the a posteriori indicators $\eta_{\psi}$ and $\eta_{\psi} + \eta_{u}$, and the effectivity indexes eff-index$_{\psi}$ for $\psi$ and eff-index$_{\psi+u}$ for $(\psi, u)$ for Example 1, where eff-index$_{\psi} = \eta_{\psi}/\|\psi - \psi_h\|_{\mathcal{E}}$, eff-index$_{\psi+u} = (\eta_{\psi} + \eta_{u})/\|(\psi - \psi_h, u - u_h)\|$. It is observed that the effectivity indices of the
Fig. 6.2. Approximations for $u$ (left) and $\psi = -\Delta u$ (right) on an adaptively refined mesh with 6094 triangles, which are generated by the elementwise indicator $\eta_{\psi,T}$. Here $\varepsilon = 10^{-6}$ and $\theta = 0.3$.

Fig. 6.3. Upper: Estimated and exact errors of $\|\psi - \psi_h\|_E$ (left) and $\|(\psi - \psi_h, u - u_h)\|$ (right) against the number of elements in adaptively refined meshes for $\varepsilon = 10^{-6}$. Lower: Exact errors of $\|\psi - \psi_h\|_E$ (left) and $\|(\psi - \psi_h, u - u_h)\|$ (right) against the number of elements in adaptively refined meshes for $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{-6}$. Here the marking parameter $\theta = 0.3$.

Error $\|\psi - \psi_h\|_E$ are close to 1, and that the effectivity indices of the error $\|\psi - \psi_h\|_E$ are about 1.5. This suggests that our estimators are robust with respect to $\varepsilon$.

6.2. Example two. This model is taken from [24]. Consider (1.1)-(1.2) on the unit square $\Omega = (0,1) \times (0,1)$ with the source term

$$f(x,y) = 2\pi^2(1 - \cos 2\pi x \cos 2\pi y).$$
Table 6.1

Example 1: $k$ – number of iterations; $\eta_\psi$ – numerical result of estimated error for $\|\psi - \psi_h\|_E$; eff-index $\psi$ – the corresponding effectivity index for $\psi$ (the ratio of estimated and exact errors). Here $\varepsilon = 10^{-5}$, $\theta = 0.5$.

| $k$ | 2  | 4  | 6  | 8  | 10 | 12 | 14 | 16 |
|-----|----|----|----|----|----|----|----|----|
| $\|\psi - \psi_h\|_E$ | 10.170 | 5.1837 | 3.6752 | 2.1023 | 1.1593 | 0.6290 | 0.3374 | 0.1911 |
| $\eta_\psi$ | 10.400 | 5.2507 | 3.7688 | 2.1295 | 1.1557 | 0.6286 | 0.3336 | 0.2158 |
| eff-index $\psi$ | 1.0226 | 1.0129 | 1.0249 | 1.0130 | 0.9969 | 0.9914 | 0.9887 | 1.1290 |

Table 6.2

Example 1: $\eta_\psi + \eta_u$ – numerical result of estimated error for $\|\psi(u) - \psi_h(u)\|$ (is denoted by $\text{err}_{\psi + u}$); eff-index $\psi + u$ – the corresponding effectivity index for $(\psi, u)$. Here $\varepsilon = 10^{-5}$, $\theta = 0.5$.

| $k$ | 1  | 3  | 5  | 7  | 9  | 11 | 13 | 15 |
|-----|----|----|----|----|----|----|----|----|
| $\text{err}_{\psi + u}$ | 17.559 | 7.3066 | 3.6330 | 1.9267 | 0.9408 | 0.5028 | 0.3040 | 0.1808 |
| $\eta_\psi + \eta_u$ | 24.481 | 11.2702 | 5.9308 | 2.9577 | 1.5630 | 0.8571 | 0.4748 | 0.2999 |
| eff-index $\psi + u$ | 1.3942 | 1.5425 | 1.6325 | 1.5351 | 1.6613 | 1.7045 | 1.5617 | 1.6583 |

Although the exact solution of this model problem is unknown, we know that the exact solution $u$ has four sharp boundary layers near the boundary.

We choose the same initial mesh as in Example one and set the marking parameter $\theta = 0.3$. The upper and middle four plots of Figure 6.4 show the mesh generated by the estimator of $\|\psi - \psi_h\|_E$ after 10, 12, 14, and 16 iterations, and the lower two plots show the mesh by the estimator of $\|\psi - \psi_h\|_E$ and $\|\psi(u) - \psi_h(u)\|$ capture the layers well, and that the refinement concentrates around four sharp boundary layers. This indicates that our estimators recognize the behavior of the solution well, even when the singularly perturbed parameter is very small.

Figure 6.5 reports the finite element approximation to $u$ (left) and $\psi = -\Delta u$ (right), respectively. Notice that the immediate variable $\psi$ has four sharp boundary layers, and that the primal variable $u$ does not have layer.

Table 6.3 reports the given tolerance TOL, the number of iterations $k$, the estimated error ($\eta_k$) for $\|\psi - \psi_h\|_E$, the degrees of freedom DOF, the smallest mesh size $h_{\text{min}}(\varepsilon)$ for example 2, which show that the required DOF depends on both TOL and $\varepsilon$, and that the layer is gradually resolved, because the smallest mesh size $h_{\text{min}}(\varepsilon)$ has arrived at the magnitude of $\varepsilon$ after 22 iterations.

Figure 6.6 shows the estimated errors of $\|\psi - \psi_h\|_E$ (left) and $\|\psi(u) - \psi_h(u)\|$ (or $|u - u_h|_1$) (right), respectively. We observe again that the estimated errors reduce uniformly with respect to $\varepsilon$ in both norms with almost optimal rate $-1/2$.

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Robust Residual-Based *a Posteriori* Error Estimators

**Fig. 6.4.** Upper: The mesh after 10 iterations with 344 triangles (left) and the mesh after 12 iterations with 611 triangles (right), generated by the elementwise indicator $\eta_{\psi, T}$. Middle: The mesh after 14 iterations with 1136 triangles (left) and the mesh after 16 iterations with 2109 triangles (right), generated by the elementwise indicator $\eta_{\psi, T}$. Lower: The mesh after 8 iterations with 418 triangles (left) and the mesh after 12 iterations with 1936 triangles (right), generated by the elementwise indicator $\eta_{\psi, T} + \eta_{u, T}$. Here $\varepsilon = 10^{-7}$, $\theta = 0.3$.

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Fig. 6.5. Approximations for $u$ (left) and $\psi = -\Delta u$ (right), respectively, on an adaptively refined mesh with 9355 triangles, which are generated by the elementwise indicator $\eta_{\psi,T}$. Here $\epsilon = 10^{-7}$ and $\theta = 0.3$.

Fig. 6.6. $\|\psi - \psi_h\|_E$ (left) and $\| (\psi - \psi_h, u - u_h) \|$ (or $|u - u_h|_1$) (right) against the number of elements in adaptively refined meshes for $\epsilon$ from $\epsilon = 10^{-4}$ to $\epsilon = 10^{-7}$, where the marking parameter $\theta = 0.3$.
Table 6.3

Example 2: TOL – given tolerance, k – number of iterations; $\eta_k$ – numerical result of estimated error for $\|\psi - \psi_h\|_{\varepsilon}$, DOF – degrees of freedom, $h_{\text{min}}(\varepsilon)$ – smallest mesh size. Here $\varepsilon = 10^{-5}$ and $\theta = 0.4$.

| TOL  | 20  | 10  | 5   | 2.5 | 1.25 | 0.625 | 0.3125 |
|------|-----|-----|-----|-----|-----|-------|--------|
| k    | 1   | 6   | 9   | 12  | 15  | 19    | 22     |
| $\eta_k$ | 19.200 | 9.991 | 4.0486 | 2.2030 | 1.1698 | 0.5093 | 0.2754 |
| DOF  | 9   | 56  | 278 | 1041 | 5243 | 19062 | 67485  |
| $h_{\text{min}}(\varepsilon)$ | 0.500 | 0.1250 | 1.56e-02 | 3.91e-03 | 1.38e-03 | 2.44e -04 | 6.10e -05 |

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