Stochastic First-order Methods for Convex and Nonconvex Functional Constrained Optimization

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Abstract

Functional constrained optimization is becoming more and more important in machine learning and operations research. Such problems have potential applications in risk-averse machine learning, semisupervised learning and robust optimization among others. In this paper, we first present a novel Constraint Extrapolation (ConEx) method for solving convex functional constrained problems, which utilizes linear approximations of the constraint functions to define the extrapolation (or acceleration) step. We show that this method is a unified algorithm that achieves the best-known rate of convergence for solving different functional constrained convex composite problems, including convex or strongly convex, and smooth or nonsmooth problems with stochastic objective and/or stochastic constraints. Many of these rates of convergence were in fact obtained for the first time in the literature. In addition, ConEx is a single-loop algorithm that does not involve any penalty subproblems. Contrary to existing primal-dual methods, it does not require the projection of Lagrangian multipliers into a (possibly unknown) bounded set. Second, for nonconvex functional constrained problems, we introduce a new proximal point method which transforms the initial nonconvex problem into a sequence of convex problems by adding quadratic terms to both the objective and constraints. Under certain MFCQ-type assumption, we establish the convergence and rate of convergence of this method to KKT points when the convex subproblems are solved exactly or inexactly. For large-scale and stochastic problems, we present a more practical proximal point method in which the approximate solutions of the subproblems are computed by the aforementioned ConEx method. Under a strong feasibility assumption, we establish the total iteration complexity of ConEx required by this inexact proximal point method for a variety of problem settings, including nonconvex smooth or nonsmooth problems with stochastic objective and/or stochastic constraints. To the best of our knowledge, most of these convergence and complexity results for the proximal point method for nonconvex problems also seem to be new in the literature.

Keywords: functional constrained optimization, stochastic algorithms, convex and nonconvex optimization, acceleration.

AMS 2000 subject classification: 90C25, 90C06, 90C22, 49M37

1 Introduction

In this paper, we study the following composite optimization problem with functional constraints:

\[
\begin{align*}
\min_{x \in X} & \quad \psi_0(x) := f_0(x) + \chi_0(x) \\
\text{s.t.} & \quad \psi_i(x) := f_i(x) + \chi_i(x) \leq 0, \quad i = 1, \ldots, m.
\end{align*}
\]

(1.1)

Here, \(X \subseteq \mathbb{R}^n\) is a convex compact set, \(f_0 : X \to \mathbb{R}\) and \(f_i : X \to \mathbb{R}, \ i = 1, \ldots, m\) are continuous functions which are not necessarily convex, and \(\chi_i : X \to \mathbb{R}, \ i = 0, 1, \ldots, m\) are convex and continuous functions. Problem (1.1) covers different convex and nonconvex settings depending on the assumptions on \(f_i\) and \(\chi_i, \ i = 0, \ldots, m.\)

In the convex setting, we assume that \(f_i, \ i = 0, \ldots, m,\) are convex or strongly convex functions, which can be either smooth, nonsmooth or the sum of smooth and nonsmooth components. We also assume that \(\chi_i, \ i = 0, \ldots, m,\) are “simple” functions in the sense that, for any given vector \(v \in \mathbb{R}^n\) and non-negative weight vector \(w \in \mathbb{R}^m,\) a certain proximal operator associated

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with the function \( \chi_0(x) + \sum_{i=1}^{m} w_i \chi_i(x) + \langle v, x \rangle \) can be computed efficiently. For such problems, Lipschitz smoothness properties of \( \chi_i \)'s is of no consequence due to the simplicity of this proximal operator.

For the nonconvex case, we assume that \( f_i, i = 0, \ldots, m \), are smooth functions, which are not necessarily convex, but satisfying a certain lower curvature condition (c.f. (1.3)). However, we do not put the simplicity assumption about the proximal operator associated with convex functions \( \chi_i, i = 0, \ldots, m \), in order to cover a broad class of nonconvex problems, including those with non-differentiable objective functions or constraints.

Constrained optimization problems of the above form are prevalent in data science. One such example arises from risk averse machine learning. Let \( \ell(\cdot, \xi) : \mathbb{R}^n \times \Xi \rightarrow \mathbb{R} \) models the loss for a random data-point \( \xi \in \Xi \). Our goal is to minimize a certain risk measure \([42, 43]\), e.g., the so-called conditional value at risk that penalizes only the positive deviation of the loss function, subject to the constraint that the expected loss is less than a threshold value. Therefore, one can formulate this problem as

\[
\min_{x \in \mathcal{X}} \quad \text{CVaR}[\ell(x, \omega)] \\
\text{s.t.} \quad \mathbb{E}[\ell(x, \omega)] \leq c,
\]

(1.2)

where CVaR denotes conditional value at risk and \( c \) is the tolerance on the average loss that one can consider as acceptable. In many practical situations, the loss function \( \ell(x, \omega) \) is nonconvex w.r.t. \( x \). Other examples of problem (1.1) can also be found in semi-supervised learning, where one would like to minimize the loss function defined over the labeled samples, subject to certain proximity type constraints for the unlabeled samples.

There exists a variety of literature on solving convex functional constrained optimization problems (1.1). One research line focuses on primal methods, e.g., cooperative subgradient methods \([39, 27]\) and level-set methods \([28, 35, 30, 4, 29]\), which do not involve the Lagrange multipliers. One possible limitation of these methods is the difficulty to directly achieve accelerated rate of convergence when the objective or constraint functions are smooth. Constrained convex optimization problems can also be solved by reformulating them as saddle point problems which will then be solved by using primal-dual type algorithms (see \([34, 19]\)). The main hurdle for existing primal-dual methods exists in that they require the projection of dual multipliers onto a ball whose diameter is usually unknown. Other approaches for constrained convex problems include the classical exact penalty, quadratic penalty and augmented Lagrangian methods \([6, 23, 24, 46]\). These approaches however require the solutions of penalty subproblems and hence are more complicated than primal and primal-dual methods. Recently, research effort has also been directed to stochastic optimization problems with functional constraints \([27, 4]\). In spite of many interesting findings, existing methods for solving these problems are still limited: a) many primal methods solve only stochastic problems with deterministic constraints \([27]\), and the convergence for accelerated primal-dual methods \([34, 19]\) has not been studied for stochastic functional constrained problems; and b) a few algorithms for solving problems with expectation constraints require either a constraint evaluation step \([27]\), or stochastic lower bounds on the optimal value \([4]\), thus relying on a light-tail assumption for the stochastic noise and conservative sampling estimates based on Bernstein inequality. Some other algorithms require even more restrictive assumptions that the noise associated with stochastic constraints has to be bounded \([47]\).

The past few years has also seen a resurgence of interest in the design of efficient algorithms for nonconvex stochastic optimization, especially for stochastic and finite-sum problems due to their importance in machine learning. Most of these studies need to assume that the constraints are convex, and focus on the analysis of iteration complexity, i.e., the number of iterations required to find an approximate stationary point, as well as possible ways to accelerate convergence to such approximate solutions. If the nonconvex functional constraints do not appear, one approach for solving (1.1) is to directly generalize stochastic gradient descent type methods (see \([16, 17, 41, 1, 14, 48, 45, 36, 38, 21]\)) for solving problems with nonconvex objective functions. An alternative approach is to indirectly utilize convex optimization methods within the framework of proximal-point methods which transfer nonconvex optimization problems into a series of convex ones (see \([18, 7, 15, 11, 20, 25, 40, 37]\)). While direct methods are simpler and hence easier to implement, indirect methods may provide stronger theoretical performance guarantees under certain circumstances, e.g., when the problem has a large conditional number, many components and/or multiple blocks \([25]\). However, if nonconvex functional constraints \( \psi_i(x) \leq 0 \) do appear in (1.1), the study on its solution methods is scarce. While there is a large body of work on the asymptotic analysis and the optimality conditions of penalty-based approaches for general
constrained nonlinear programming (for example, see [6, 33, 3, 2, 12]), only a few works discussed the complexity of these methods for solving problems with nonconvex functional constraints [8, 44, 13]. However, these techniques are not applicable to our setting because they cannot guarantee the feasibility of the generated solutions, but certain local non-increasing properties for the constraint functions. On the other hand, the feasibility of the nonconvex functional constraints appear to be important in our problems of interest.

In this paper, we attempt to address some of the aforementioned significant issues associated with both convex and nonconvex functional constrained optimization. Our contributions mainly exist in the following several aspects.

First, for solving convex functional constrained problems, we present a novel primal-dual type method, referred to as the Constraint Extrapolation (ConEx) method. One distinctive feature of this method from existing primal-dual methods is that it utilizes linear approximations of the constraint functions to define the extrapolation (or acceleration/momentum) step. As a consequence, contrary to the well-known Nemirovski’s mirror-prox method [34] and an interesting primal-dual method recently developed by Hamedani and Aybat [19], ConEx does not require the projection of Lagrangian multipliers onto a (possibly unknown) bounded set. In addition, ConEx is a single-loop algorithm that does not involve any penalty subproblems. Due to the built-in acceleration step, this method can explore problem structures and hence achieve better rate of convergence than primal methods. In fact, we show that this method is a unified algorithm that achieves the best-known rate of convergence for solving different convex functional constrained problems, including convex or strongly convex, and smooth or non-smooth problems with stochastic objective and/or stochastic constraints.

Table 1: Different convergence rates of the ConEx method for strongly convex/convex, and smooth/nonsmooth objective and/or constraints. Deterministic means both objective and constraints are deterministic, semi-stochastic means objective is stochastic but constraints are deterministic, fully-stochastic means both objective and constraints are stochastic. (*) Results for the smooth case hold if \( B \geq \|y^*\|_2 + 1 \). Otherwise, the ConEx method converges at the rate of nonsmooth problems.

| Cases             | Strongly convex (1,1) | Convex (1,1) |
|-------------------|-----------------------|--------------|
|                   | Smooth* | Nonsmooth | Smooth* | Nonsmooth |
| Deterministic     | \( O(1/\sqrt{\varepsilon}) \) | \( O(1/\varepsilon) \) | \( O(1/\varepsilon) \) | \( O(1/\varepsilon^2) \) |
| Semi-stochastic   | \( O(1/\varepsilon) \) | \( O(1/\varepsilon) \) | \( O(1/\varepsilon^2) \) | \( O(1/\varepsilon^2) \) |
| Fully-stochastic  | \( O(1/\varepsilon^2) \) | \( O(1/\varepsilon^2) \) | \( O(1/\varepsilon^2) \) | \( O(1/\varepsilon^2) \) |

Table 1 provides a brief summary for the iteration complexity of the ConEx method for solving different functional constrained problems. For the strongly convex case, ConEx can obtain convergence to an \( \varepsilon \)-approximate solution (i.e., optimality gap and infeasibility are \( O(\varepsilon) \)) as well as convergence of the last iterate to the optimal solution. The complexity bounds provided in Table 1 for the strongly convex case hold for both types of convergence criterions. For semi- and fully-stochastic case, we use the notion of expected convergence instead of exact convergence used in the deterministic case. It should be noted that in Table 1, we ignore the impact of various Lipschitz constants and/or stochastic noises for the sake of simplicity. In fact, the ConEx method achieves quite a few new complexity results by reducing the impact of these Lipschitz constants. Moreover, to the best of our knowledge, it attains for the first time the optimal iteration and sampling complexity for solving general stochastic constrained problems without requiring the boundedness or light-tail assumptions on the stochastic subgradients (see Theorems 2.1 and 2.3 and discussions afterwards). Even though ConEx is a primal-dual type method, we can show its convergence irrespective of the knowledge of the optimal Lagrange multiplier \( y^* \), as it does not require the projection of multipliers onto the ball. In particular, convergence rates of the ConEx method for nonsmooth cases (either convex or strongly convex) in Table 1 holds irrespective of the knowledge of the optimal Lagrange multipliers. For smooth cases, if certain parameter \( B \) is not big enough, i.e., \( B < \|y^*\|_2 + 1 \), then it converges at the rates for nonsmooth problems of the respective case. As one can see from Table 1, such a change would cause a suboptimal convergence rate in terms of \( \varepsilon \) only for the deterministic case, but complexity will be the same for both semi- and fully-stochastic cases. It is worth mentioning that faster convergence rates for the smooth deterministic case can still be attained by incorporating certain line search procedures. To the best of our knowledge, this is the first time in the literature that a simple single-loop
algorithm was developed for solving all different types of convex functional constrained problems in an optimal manner.

Second, we extend the ConEx method for the nonconvex setting and present a new framework of proximal point method for solving the nonconvex functional constrained optimization problems, which otherwise seem to be difficult to solve by using direct approaches. The key component of our method is to exploit the structure of the nonconvex objective and constraints \( \psi_i, i = 0, \ldots, m \), thereby turning the original problem into a sequence of functional constrained subproblems with a strongly convex objective and strongly convex constraints. We show that if the proximal point method has a strictly feasible initial solution and its subproblem is exactly solved, then the whole generated sequence remains strictly feasible. Hence, Slater’s condition guarantees the existence of Lagrange multipliers and strong duality for each subproblem. Moreover, we show that the need of an exact optimal subproblem solution can be relaxed by a termination criterion based on the distance to the optimal solution, leading to a more general inexact proximal point method that still preserves the appealing property of strict feasibility. Under the Mangasarian-Fromovitz constraint qualification (MFCQ), we show that the inexact proximal point method converges asymptotically to the KKT points, and provide the first iteration complexity of such proximal point method. More specific, we show that this method requires \( O(1/\varepsilon) \) iterations to obtain an appropriately defined \( (\varepsilon, \varepsilon^2) \)-KKT point (see Theorem 3.10 and discussions afterwards).

For large-scale and stochastic optimization, we propose an inexact proximal point method for which the subproblems are approximately solved by first-order methods such as ConEx. However, due to the optimization challenge in this setting, it is in general difficult to obtain highly accurate solutions such that the whole sequence remains strictly feasible. To overcome this difficulty, we propose a new and verifiable strong feasibility assumption which alleviates the need of generating strict feasible solutions. We develop the inexact proximal point method using a termination criterion in terms of the functional optimality gap and the constraint violation, showing that this method requires \( O(\Delta/\varepsilon) \) iterations to obtain some \( (\varepsilon, \varepsilon) \)-KKT point solutions, where \( \Delta \) depends on the inexactness errors summed over the iterations. When the proximal point subproblems are solved by ConEx, we present the total iteration count of ConEx to achieve the approximate solutions under different smooth and nonsmooth settings (see Theorem 3.18, Corollary 3.20 and discussions afterwards).

Close to the completion of our paper, we notice that Ma et. al. [31] also worked independently on the analysis of the proximal-point methods for nonconvex functional constrained problems. In fact, the initial version of [31] was released almost at the same time as ours. In spite of some overlap, there exist a few essential differences between our work and [31]. First, we establish the convergence/complexity of the proximal point method under a variety of constraint qualification conditions, including MFCQ, a stronger notion of MFCQ (Assumption 3.2), and strong feasibility. Hence, our work covers a broader class of nonconvex problems, while [31] only consider problems that satisfy a uniform Slater’s condition. Strong feasibility condition is stronger than the uniform Slater’s condition but it is easier to verify. Second, [31] uses a different definition of subdifferential than ours and the definition of the KKT conditions in [31] comes from convex optimization problems. While it is unclear under what constraint qualification this KKT condition is necessary for local optimality, it is possible to put their problem into our composite framework in (1.1) and compute the subdifferential that provably yields our KKT condition under the aforementioned MFCQ. Third, for solving the convex subproblems we provide a unified algorithm, i.e., ConEx, that can achieve the best-known rate of convergence for solving different problem classes, including deterministic, semi- and fully-stochastic, smooth and nonsmooth problems. On the other hand, different methods were suggested for solving different types of problems in [31]. In particular, a variant of the switching subgradient method, which was firstly presented by Polyak in [39] for the general convex case, and later extended by [27] for the stochastic and strongly convex cases, was suggested for solving deterministic problems. For the stochastic case they directly apply the algorithm in [47] and hence require stochastic gradients to be bounded. These subgradient methods do not necessarily yield the best possible rate of convergence if the objective/constraint functions are smooth or contain certain smooth components.

**Outline** This paper is organized as follows. Section 1.1 describes notation and terminologies. Section 2 exclusively deals with the ConEx method for solving problem (1.1) in the convex setting. Section 2.1 states the main convergence results of the ConEx method and Section 2.2 shows the details of the convergence analysis. Section 3 presents exact and inexact proximal point methods for solving problem (1.1) in the nonconvex setting. Section 3.1 and 3.2 establishes their
convergence behavior and iteration complexity under the MFCQ type assumptions. Section 3.3 investigates the inexact proximal point method under the strong feasibility assumption, and shows an overall iteration complexity result when the subproblems are solved by the ConEx method.

1.1 Notation and terminologies

Throughout the paper, we use the following notations. Let \( \{ m \} := \{ 1, \ldots, m \} \), \( \psi(x) := [\psi_1(x), \ldots, \psi_m(x)]^T \), \( f(x) := [f_1(x), \ldots, f_m(x)]^T \) and \( \chi(x) := [\chi_1(x), \ldots, \chi_m(x)]^T \) and the constraints in (1.1) be expressed as \( \psi(x) \leq 0 \). Here bold \( 0 \) denotes the vector of elements 0. Size of the vector is left unspecified whenever it is clear from the context. \( \| \cdot \| \) denotes a general norm and \( \| \cdot \|_a \) denotes its dual norm defined as \( \| z \|_a := \sup \{ z^T x : \| x \| \leq 1 \} \). From this definition, we obtain the \( a^T b \leq \| a \| \| b \|_a \). Euclidean norm is denoted as \( \| \cdot \|_2 \) and standard inner product is denoted as \( \langle \cdot, \cdot \rangle \). Let \( B^r(r) := \{ x : \| x \|_2 \leq r \} \) be the Euclidean ball of radius \( r \) centered at origin. Non-negative orthonal of this ball is denoted as \( B^r_+(r) \). For a convex set \( X \), we denote the normal cone at \( x \in X \) as \( N_X(x) \) and its dual cone as \( N_X^*(x) \), interior as int \( X \) and relative interior as \( \text{rint} X \). For a scalar valued function \( f \) and a scalar \( t \), the notation \( \{ f \leq t \} \) stands for the set \( \{ x : f(x) \leq t \} \). The “ tighter” operation on sets denotes the Minkowski sum of the sets. We refer to the distance between two sets \( A, B \subset \mathbb{R}^n \) as \( d(A, B) := \inf_{a \in A, b \in B} \| a - b \| \).

\[ \lfloor x \rfloor_+ := \max\{x, 0\} \text{ for any } x \in \mathbb{R}. \]

For any vector \( x \in \mathbb{R}^k \), we define \( \lfloor x \rfloor_+ \) as element-wise application of the operator \( \lfloor . \rfloor_+ \). The \( i \)-th element of vector \( x \) is denoted as \( x^{(i)} \) unless otherwise explicitly specified a different notation for certain special vectors.

A function \( r(\cdot) \) is \( \lambda \)-Lipschitz smooth if the gradient \( \nabla r(x) \) is a \( \lambda \)-Lipschitz function, i.e. for some \( \lambda \geq 0 \)

\[ \| \nabla r(x) - \nabla r(y) \|_a \leq \lambda \| x - y \|, \quad \forall x, y \in \text{dom} r. \]

An equivalent form is:

\[ -\frac{\lambda}{2} \| x - y \|^2 \leq r(x) - r(y) - \langle \nabla r(y), x - y \rangle \leq \frac{\lambda}{2} \| x - y \|^2, \quad \forall x, y \in \text{dom} r. \]

Note that in the above relation, the upper and lower curvatures need not be same. A refined version of the above property differentiates between negative and positive curvatures.

\[ r(y) + \langle \nabla r(y), x - y \rangle - \frac{\lambda}{2} \| x - y \|^2 \leq r(x), \quad \forall x, y \in \text{dom} r. \]

Here, we say that \( r \) satisfies (1.3) with parameter \( \nu \) with respect to \( \| \cdot \| \). In many cases, it is possible that a convex function \( r \) is a combination of Lipschitz smooth and nonsmooth functions.

Let \( \omega : X \rightarrow \mathbb{R} \) be continuously differentiable with \( L_\omega \)-Lipschitz continuous gradient and \( 1 \)-strongly convex with respect to \( \| \cdot \| \). We define the prox-function associated with \( \omega(\cdot) \) as

\[ W(y, x) := \omega(y) - \omega(x) - \langle \nabla \omega(x), y - x \rangle, \quad \forall x, y \in X. \]

Based on the smoothness and strong convexity of \( \omega(x) \), we have the following relation

\[ W(y, x) \leq \frac{L_\omega}{2} \| x - y \|^2 \leq L_\omega W(y, x), \quad \forall x, y \in X. \]

We define the diameter of the compact set \( X \) by

\[ D_X := \max_{x, y \in X} \sqrt{2W(x, y)}. \]

It immediately follows from the strong convexity of \( \omega(\cdot) \) that \( \| x - y \| \leq D_X \) for any \( x, y \in X \). For any convex function \( h \), we denote the subdifferential as \( \partial h \) as follows: at a point \( x \) in the relative interior of \( X \), \( \partial h \) is comprised of all subgradients \( h' \) of \( h \) at \( x \) which are in the linear span of \( X \). For a point \( x \in X \backslash \text{rint} X \), the set \( \partial h(x) \) consists of all vectors \( h' \), if any, such that there exists \( x_t \in \text{rint} X \) and \( h'_t \in \partial h(x_t), i = 1, 2, \ldots \) with \( x = \lim_{t \to x_t} x_t \), \( h' = \lim_{t \to x_t} h'_t \). With this definition, it is well-known that, if a convex function \( h : X \rightarrow \mathbb{R} \) is Lipschitz continuous, with constant \( M \), with respect to a norm \( \| \cdot \| \), then the set \( \partial h(x) \) is nonempty for any \( x \in X \) and \( h' \in \partial h(x) \Rightarrow \| (h', d) \| \leq M \| d \|, \forall d \in \lim (X - X) \), which also implies

\[ h' \in \partial h(x) \Rightarrow \| h' \|_a \leq M, \]

where \( \| \cdot \|_a \) is the dual norm. See [5] for more details. We say that a function \( r(\cdot) \) is \( \beta \)-strongly convex with respect to \( W(\cdot, \cdot) \) if

\[ r(x) \geq r(y) + \langle r'(y), x - y \rangle + \beta W(x, y), \quad \forall x, y \in X. \]
2 Constraint Extrapolation for Convex Functional Constrained Optimization

In this section, we present a novel constraint extrapolation (ConEx) method for solving problem (1.1) in the convex setting. To motivate our proposed method, consider the equivalent Lagrangian saddle point problem:

\[
\min_{x \in X} \max_{y \geq 0} \left\{ \mathcal{L}(x, y) := \psi_0(x) + \sum_{i=1}^{m} y^{(i)} \psi_i(x) \right\} .
\] (2.1)

Let \((x^*, y^*)\) be a saddle point solution of (2.1), then it satisfies

\[
\mathcal{L}(x^*, y) \leq \mathcal{L}(x^*, y^*) \leq \mathcal{L}(x, y^*),
\] (2.2)

for all \(x \in X, y \geq 0\). Moreover, we have that \(x^*\) is the optimal solution of (1.1). Throughout this section, we assume the existence of \(y^*\) satisfying (2.2). Denote the optimal value \(\psi_0^* := \psi_0(x^*)\). Then, the following definition describes a widely used optimality measure for the convex problem (1.1).

**Definition 2.1.** A point \(\bar{x} \in X\) is called a \((\delta_0, \delta_c)\)-optimal solution of problem (1.1) if

\[
\psi_0(\bar{x}) - \psi_0^* \leq \delta_0 \quad \text{and} \quad \|\psi(\bar{x})\|_2 \leq \delta_c.
\]

A stochastic \((\delta_0, \delta_c)\)-approximately optimal solution satisfies

\[
\mathbb{E}[\psi_0(\bar{x}) - \psi_0^*] \leq \delta_0 \quad \text{and} \quad \mathbb{E}[\|\psi(\bar{x})\|_2] \leq \delta_c.
\]

As mentioned earlier, for the convex composite case, we assume that \(\chi_i, i = 0, \ldots, m\), are “simple” functions in the sense that, for any vector \(v \in \mathbb{R}^n\) and nonnegative \(w \in \mathbb{R}^m\), we can efficiently compute the following **prox** operator

\[
\text{prox}(w, v, \bar{x}, \eta) := \arg\min_{x \in X} \left\{ \chi_0(x) + \sum_{i=1}^{m} w_i \chi_i(x) + \langle v, x \rangle + \eta W(x, \bar{x}) \right\} .
\] (2.3)

The **prox** operator above uses Bregman divergence instead of Euclidean norm used in standard prox operator.

**2.1 The ConEx method**

ConEx is a single-loop primal-dual type method for functional constrained optimization. It evolves from the primal-dual methods for solving bilinear saddle point problems (e.g., [9, 10, 26, 22, 21]). Recently Hamedani and Aybat [19] show that these methods can also handle more general functional coupling term. However, as discussed earlier, existing primal-dual methods [34, 19] for general saddle point problems, when applied to functional constrained problems, require the projection of dual multipliers onto a possibly unknown bounded set in order to ensure the boundedness of the operators, as well as the proper selection of stepsizes. One distinctive feature of ConEx is to use value of linearized constraint functions in place of exact function values when defining the extrapolation/momentum step. With this modification, we show that the ConEx method still converges even though the feasible set of \(y\) in problem (2.1) is unbounded. In addition, we show that ConEx method is a unified algorithm for functional constrained optimization in the following sense. First, we establish an explicit rate of convergence of the ConEx method for solving functional constrained stochastic optimization problems where either the objective and/or constraints are given in the form of expectation. Second, we consider the composite constrained optimization problem in which the objective function \(f_0\) and/or constraints \(f_i, i = 1, \ldots, m\) can be nonsmooth. Third, we consider the two cases of convex or strongly convex objective, \(f_0\). For the strongly convex objective, we also establish the rate of convergence to the optimal solution \(x^*\).

Before proceeding to the algorithm, we introduce the problem setup in more details. First, we assume that \(f_0\) satisfies the following Lipschitz smoothness and nonsmoothness condition for some constants \(L_0, H_0 \geq 0\):

\[
f_0(x_1) - f_0(x_2) - \langle f'_0(x_2), x_1 - x_2 \rangle \leq \frac{L_0}{2} \|x_1 - x_2\|^2 + H_0 \|x_1 - x_2\|,
\] (2.4)

for all \(x_1, x_2 \in X\) and for all \(f'_0(x_2) \in \partial f_0(x_2)\). For constraints, we make a similar assumption as in (2.4). Moreover, we make an additional assumption that the constraint functions are Lipschitz continuous. In particular, we have

\[
f_i(x_1) - f_i(x_2) - \langle f'_i(x_2), x_1 - x_2 \rangle \leq \frac{L_i}{2} \|x_1 - x_2\|^2 + H_i \|x_1 - x_2\|,
\] (2.5)

for all \(x_1, x_2 \in X\) and for all \(f'_i(x_2) \in \partial f_i(x_2), i = 1, \ldots, m\), and

\[
f_i(x_1) - f_i(x_2) \leq M_{f_i} \|x_1 - x_2\|, \quad \forall x_1, x_2 \in X, i = 1, \ldots, m,
\]

\[
\chi_i(x_1) - \chi_i(x_2) \leq M_{\chi_i} \|x_1 - x_2\|, \quad \forall x_1, x_2 \in X, i = 1, \ldots, m.
\] (2.6)
Note that the Lipschitz-continuity assumption in (2.6) is common in the literature when \( f_i, i \in [m] \), are nonsmooth functions. If \( f_i, i \in [m] \) are Lipschitz smooth then their gradients are bounded due to the compactness of \( X \). Hence (2.6) is not a strong assumption for the given setting. Also note that due to definition of subgradient for convex function defined in Section 1.1, we have \( \| f'_i(x) \|_u \leq M_{f,i} \) which implies \( \| f'_i(x_2)^T(x_1 - x_2) \| \leq \| f'_i(x_2) \|_u \| x_1 - x_2 \| \leq M_{f,i} \| x_1 - x_2 \| \).

Using this relation, (2.5) and (2.6), we have the following four relations:

\[
\begin{align*}
\| f(x_1) - f(x_2) \|_2 & \leq M_f \| x_1 - x_2 \|, \\
\| x(x_1) - x(x_2) \|_2 & \leq M_N \| x_1 - x_2 \|, \\
\| f(x_1) - f(x_2) - f'(x_2)^T(x_1 - x_2) \| & \leq \frac{L_f}{2} \| x_1 - x_2 \|^2 + H_f \| x_1 - x_2 \|, \\
\| f'(x_2)^T(x_1 - x_2) \| & \leq M_{H_f} \| x_1 - x_2 \|,
\end{align*}
\]

(2.7)

for all \( x_1, x_2 \in X \). Here \( f'(\cdot) := [f'_i(\cdot), \ldots, f'_m(\cdot)] \in \mathbb{R}^{n \times m} \) and constants \( M_f, M_N, H_f \) and \( L_f \) are defined as

\[
\begin{align*}
M_f & := (\sum_{i=1}^m M_{f,i}^2)^{1/2}, \\
M_N & := (\sum_{i=1}^m M_{N,i}^2)^{1/2}, \\
H_f & := (\sum_{i=1}^m H_{f,i}^2)^{1/2}, \\
L_f & := (\sum_{i=1}^m L_{f,i}^2)^{1/2}.
\end{align*}
\]

(2.8)

We denote \( \alpha = (\alpha_1, \ldots, \alpha_m)^T \) as the vector of moduli of strong convexity for \( \chi_i, i \in [m] \), and \( \alpha_0 \) as the modulus of strong convexity for \( \chi_0 \). We say that convex problem (1.1) is a smooth composite if (2.5) is satisfied with \( H_i = 0 \) for all \( i = 1, \ldots, m \) and (2.4) is satisfied with \( H_0 = 0 \).

Otherwise, (1.1) is a nonsmooth problem. To be succinct, problem (1.1) is composite smooth if \( H_f = H_0 = 0 \), otherwise it is a nonsmooth problem.

We assume that we can access the first-order information of functions \( f_0, f_1 \) and zeroth-order information of function \( f_i \) using a stochastic oracle (SO). In particular, given \( x \in X \), SO outputs \( G_0(x, \xi), G_i(x, \xi), \) and \( F(x, \xi) \) such that

\[
\begin{align*}
\mathbb{E}[G_0(x, \xi)] & = f_0'(x), \\
\mathbb{E}[G_i(x, \xi)] & = f'_i(x), \quad i = 1, \ldots, m, \\
\mathbb{E}[F(x, \xi)] & = f(x), \\
\mathbb{E}[\| G_0(x, \xi) - f_0'(x) \|_u^2] & \leq \sigma_0^2, \\
\mathbb{E}[\| G_i(x, \xi) - f'_i(x) \|_u^2] & \leq \sigma_i^2, \quad i = 1, \ldots, m, \\
\mathbb{E}[\| F(x, \xi) - f(x) \|_u^2] & \leq \sigma_f^2,
\end{align*}
\]

(2.9)

where \( \xi \) is a random variable which models the source of uncertainty and is independent of the search point \( x \). Note that the last relation of (2.9) is satisfied if we have individual stochastic oracles \( F_i(x, \xi) \) such that \( \mathbb{E}[F_i(x, \xi) - f_i(x)]^2 \leq \sigma_i^2, \) in particular, we can set \( \sigma_i^2 = \sum_{j=1}^m \sigma_j^2, \) for all \( i \), respectively. We use stochastic subgradients \( G_i(x, \xi), i = 0, \ldots, m, \) at point \( x \), the t-th iteration of the ConEx method where \( \xi_t \) is a realization of random variable \( \xi \) which is independent of the search point \( x_t \). We denote by \( \sigma := [\sigma_1, \ldots, \sigma_m]^T \) a vector of standard deviation of stochastic subgradients of individual constraints.

We denote \( \ell^t_f(x_t) \) a linear approximation of \( f(\cdot) \) at point \( x_t \) with

\[
\ell^t_{f_{i-1}}(x_t) := f(x_{t-1}) + f'(x_{t-1})^T(x_t - x_{t-1}),
\]

where \( f'(x_{t-1}) := [f'_i(x_{t-1}), \ldots, f'_m(x_{t-1})] \) as defined earlier. For ease of notation, we denote \( \ell_{f,t-1}(x_t) \) as \( \ell_f(x_t) \). We can do this, since for all \( t \), we approximate \( f(x_t) \) with linear function approximation taken at \( x_{t-1} \). We use a stochastic version of \( \ell_f \) in our algorithm, which is denoted as \( \ell_{F,t} \). In particular, we have

\[
\ell_{F,t}(x_t) := F(x_{t-1}, \tilde{\xi}_{t-1}) + G(x_{t-1}, \tilde{\xi}_{t-1})^T(x_t - x_{t-1}),
\]

where \( G(x_{t-1}, \tilde{\xi}_{t-1}) := [G_i(x_{t-1}, \tilde{\xi}_{t-1}), \ldots, G_m(x_{t-1}, \tilde{\xi}_{t-1})] \in \mathbb{R}^{n \times m}. \) Here, we used \( \tilde{\xi}_t \) as an independent (of \( \xi_t \)) realization of random variable \( \xi_t \). In other words, \( G_i(x_t, \tilde{\xi}_t) \) and \( G_i(x_t, \xi_t) \) are conditionally independent estimates of \( f'_i(x_t) \) for \( i = 1, \ldots, m \) under the condition that \( x_t \) is fixed. As we show later, independent samples of \( \xi_t \) are required to show that \( \ell_f(x_t) \) is an unbiased estimator of \( \ell_f(x_t) \).

We are now ready to formally describe the constraint extrapolation method (see Algorithm 1).

As mentioned earlier, the \( \ell_f(x_t) \) term in Line 3 of Algorithm 1 is an unbiased estimator of \( \ell_f(x_t) \). Moreover, the term \( \chi(x_t) + \ell_f(x_t) \) is an approximation to \( \chi(x_t) + f(x_t) = \psi(x_t) \). Essentially, Line 3 represents a stochastic approximation for the term \( \psi(x_t) + \theta_i(\psi(x_t) - \psi(x_{t-1})) \) which is an extrapolation of the constraints, hence justifying the name of the algorithm. Line 4 is the standard \texttt{prox} operator which has a closed-form solution: \( y_{t+1} = [y_t + \frac{1}{\gamma_i} \xi_i]_+ \). Line 5 also uses a \texttt{prox} operator defined in (2.3). The final output of the algorithm in Line 7 is the weighted
average of all primal iterates generated. If we choose standard deviations \( \sigma_f = \sigma_0 = \sigma_i = 0 \) for \( i = 1, \ldots, m \) then we recover the deterministic gradients and function evaluation. Henceforth, we assume general non-negative values for all such standard deviations and provide a combined analysis for these settings. Later, we substitute appropriate values of standard deviations to finish the analysis for the following three different cases.

a) Deterministic setting where both the objective and constraints are deterministic. Here \( \sigma_0 = \sigma_1 = \sigma_f = 0 \) for all \( i \in [m] \).

b) Semi-stochastic setting where the constraints are deterministic but the objective is stochastic. Here, \( \sigma_f = \sigma_i = 0 \) for all \( i \in [m] \). However, \( \sigma_0 \geq 0 \) can take arbitrary values.

c) Fully-stochastic setting where both function and gradient evaluations are stochastic. Here, all \( \sigma_f, \sigma_0, \sigma_i \geq 0 \) can take arbitrary values.

Below, we specify a stepsize policy and state the convergence properties of Algorithm 1 for solving problem (1.1) in the strongly convex setting where \( \chi_0 \) is \( \alpha_0 \)-strongly convex. Note that in the stochastic case, stepsize \( \tau_t \) depends on the total number of iterations \( T \) which needs to be fixed beforehand. Similar policies are used in the stochastic approximation literature [21]. The proof of this theorem is involved and will be deferred to Section 2.2.

**Theorem 2.1.** Suppose (2.4), (2.5), (2.6) and (2.9) are satisfied. Let \( B \geq 1 \) be a constant, \( t_0 := \frac{4(L_\alpha + BL_d)}{\alpha_0^2} + 2, M := \max \{2M_f, M_h + M_f \} \), and \( \sigma_{X,f} := (\sigma_f^2 + D_X^2 \| \sigma \|^2) / 2 \). Set \( y_0 = 0 \), \( T \geq 1 \) and \( \{ \gamma_t, \tau_t, \eta_t, \gamma_t \} \) in Algorithm 1 according to the following:

\[
\gamma_t = t + t_0 + 2, \quad \tau_t = \frac{1}{t+1} \max \left\{ \frac{32M^2}{\alpha_0}, \frac{384\|c\|^2 T \sigma_{X,f} T^{3/2}}{B(t_0+2)^2} \right\}, \quad \eta_t = \frac{\alpha_0(t+t_0+1)}{2(t+t_0+2)}, \quad \theta_t = \frac{4(t+t_0)+2}{14(t+t_0+2)}. \tag{2.10}
\]

Then, we have

\[
\mathbb{E} [\psi(\bar{x}_T) - \psi(x^*)] \leq \frac{\alpha_0(t_0+1)(t_0+2)D_X^2}{T^2} + \frac{12B\sigma_{X,f}(t_0+1)(t_0+2)^{1/2}}{T^{3/2}} + \frac{16(\tilde{c}_0^2 + \tilde{H}_f^2)\|x^*\|^2}{\alpha_0 T} + \frac{8B(t_0+2)^{1/2}\sigma_{X,f}}{T^{1/2}}. \tag{2.11}
\]

and

\[
\mathbb{E} \left\| \frac{\psi(\bar{x}_T) - \psi(x^*)}{T} \right\|_2 \leq \frac{192(t_0+2)(\|y^*\|^2 + 1)^2M^2}{\alpha_0 T^2} + \frac{\alpha_0(t_0+1)(t_0+2)D_X^2}{T^{3/2}} + \frac{13B\sigma_{X,f}(t_0+1)(t_0+2)^{1/2}}{T^{1/2}}.
\]

\[
+ \frac{16(\tilde{c}_0^2 + \tilde{H}_f^2 + 144(t_0+2)(\|y^*\|^2 + 1)^2\|\sigma\|^2)\|H_f\|^2}{\alpha_0 T^2} + \frac{26B(t_0+2)^{1/2}\sigma_{X,f}}{3T^{1/2}}, \tag{2.12}
\]

where \( H_f := H_0 + (\|y^*\|^2 + 1)H_f + \frac{L_f D_X \|y^*\|^2 + 1 - B}{2} \),

\[
\zeta := 2\left\{ \left( \frac{c_0^2 + 12(t_0 + 3)}{2} \right) \|\sigma\|^2 \|y^*\|^2 + 96(t_0 + 2)B^2\|\sigma\|^2 + \frac{\hat{H}_f^2}{2} + \frac{3\alpha_0 B\sigma_{X,f}(t_0+2)^{1/2}}{2} \right) \right\}^{1/2}.
\]

Moreover, we obtain the last iterate convergence

\[
\mathbb{E} [W(x^*, X_T)] \leq \frac{192(t_0+2)(\|y^*\|^2 + 1)^2M^2}{\alpha_0^2 T^2} + \frac{(t_0+1)(t_0+2)D_X^2}{T^{3/2}} + \frac{12B\sigma_{X,f}(t_0+1)(t_0+2)^{1/2}}{\alpha_0 T^{1/2}}
\]

\[
+ \frac{16(\zeta^2 + \hat{H}_f^2 + 144(t_0+2)(\|y^*\|^2 + 1)^2\|\sigma\|^2)\|H_f\|^2}{\alpha_0^2 T^2} + \frac{26B(t_0+2)^{1/2}\|y^*\|^2\sigma_{X,f}}{3T^{1/2}} + \frac{8B(t_0+2)^{1/2}\sigma_{X,f}}{\alpha_0 T^{1/2}}. \tag{2.13}
\]

An immediate corollary of the above theorem is the following:
Corollary 2.2. We obtain an $(\varepsilon, \varepsilon)$-optimal solution of problem (1.1) in $T_\varepsilon$ iterations, where

$$T_\varepsilon = \max \left\{ \left( \frac{5m(2n+1)(m+2)}{c} + \frac{960(m+2)(n^{3/2}+1)^{2}M^{2}}{\alpha_{0}^{2}} \right)^{1/2}, \left( \frac{65B\sigma_{x_1}(m+2)^{3/2}}{\varepsilon_{0}} \right)^{2/3}, \frac{80(c^{2}+H_{b}^{2}+144(m+2)(n^{3/2}+1)^{2}\sigma_{\sigma_{1}}^{2})}{\alpha_{0}^{2}} \right\}$$

(2.14)

Moreover, we obtain $E[W(x^{*}, x_{p})] \leq \varepsilon$ in at most

$$\max \left\{ \left( \frac{10(m+2)(m+2)}{c} + \frac{960(m+2)(n^{3/2}+1)^{2}M^{2}}{\alpha_{0}^{2}} \right)^{1/2}, \left( \frac{60B\sigma_{x_1}(m+2)^{3/2}}{\varepsilon_{0}} \right)^{2/3}, \frac{80(c^{2}+H_{b}^{2}+144(m+2)(n^{3/2}+1)^{2}\sigma_{\sigma_{1}}^{2})}{\alpha_{0}^{2}} \right\}$$

iterations.

Proof. Using (2.12) and (2.14), we have $E[\|\psi(X_{T})\|_{2}] \leq \frac{\xi}{c} + \frac{\xi}{c} + \frac{\xi}{c} = \varepsilon$. Similarly, using (2.11) and (2.14), it is easy to observe that $E[\psi_{0}(X_{T}) - \psi_{0}(x^{*})] \leq \varepsilon$. Using (2.13) and (2.15), we have $E[W(x^{*}, x_{p})] \leq \frac{\xi}{c} + \frac{\xi}{c} + \frac{\xi}{c} = \varepsilon$. Hence we conclude the proof.

Theorem 2.1 and Corollary 2.2 provide unified iteration complexity bounds for solving strongly convex functional constrained optimization problems. These results will also be used later for solving subproblems arising from the proximal point method for nonconvex problems in Section 3.

Below we derive from (2.14) the convergence rate of Algorithm 1 for both nonsmooth problems, i.e., either $H_f$ or $H_0$ is strictly positive, and (composite) smooth problems, i.e., $H_f = 0$, $H_0 = 0$.

Let us start with nonsmooth problems for which (2.4) is satisfied with $H_0 > 0$ or (2.5) is satisfied with $H_1 > 0$ for at least one $i \in [m]$. In this case, we have

$$H_{a} = (\|y^{*}\| + 1)H_{f} + H_{0} + \left( \frac{L_{f}}{\lambda_{X}} \right) \|y^{*}\|_{2} + \left( \frac{L_{f}}{\lambda_{X}} \right) \frac{1}{\lambda_{X}} \|y^{*}\|_{2} > 0$$

irrespective of the value of $B$. Then, using (2.14), we obtain the iteration complexity of

$$O \left( \frac{1}{\sqrt{c}} (\frac{L_{0} + BL_{f}}{\sqrt{\alpha_{0}}})^{D_{X}} + \frac{\sqrt{L_{0} + BL_{f}BM}}{\alpha_{0}} + \frac{H_{a}}{\alpha_{0}} \right)$$

for the deterministic case. For the semi-stochastic case, the iteration complexity becomes

$$O \left( \frac{1}{\sqrt{c}} (\frac{L_{0} + BL_{f}}{\sqrt{\alpha_{0}}})^{D_{X}} + \frac{\sqrt{L_{0} + BL_{f}BM}}{\alpha_{0}} + \frac{H_{a}^{2} + \sigma_{2}^{2}}{\alpha_{0}^{2}} \right)$$

Similarly, for the fully-stochastic case, the iteration complexity is given by

$$O \left( \frac{1}{\sqrt{c}} (\frac{L_{0} + BL_{f}}{\sqrt{\alpha_{0}}})^{D_{X}} + \frac{\sqrt{L_{0} + BL_{f}BM}}{\alpha_{0}} + \frac{H_{a}^{2} + \sigma_{2}^{2}}{\alpha_{0}^{2}} + \frac{1}{\varepsilon_{c}} (\frac{B^{2}(L_{0} + BL_{f})(\sigma_{2}^{2} + \sigma_{2}^{2})}{\alpha_{0}} \|y^{*}\|_{2}^{3}) \right)$$

Observe that, due to the built-in acceleration scheme of the ConEx method, the Lipschitz constant $L_0$ will barely impact the convergence since it appears only in the $O(1/\sqrt{c})$ term. Similarly, the impact of the Lipschitz constant $L_f$ will be minimized for a large enough $B$ (i.e., $B \geq \|y^{*}\|_{2} + 1$).

To the best of our knowledge, these complexity results with separate impact of Lipschitz constants appear to be new for functional constrained optimization. Moreover, the iteration (and sampling) complexity for the fully-stochastic case, i.e., general stochastic constrained problems requiring only bounded second moments on noises, has not been obtained before in the literature.

Now let us consider smooth problems for which (2.4) and (2.5) are satisfied with $H_{0} = 0$ and $H_{1} = 0$ for all $i = 1, \ldots, m$, respectively. We distinguish two different scenarios depending on whether $B \geq \|y^{*}\|_{2} + 1$. First, if $B \geq \|y^{*}\|_{2} + 1$, then $H_{a} = H_{0} + H_{f}(\|y^{*}\|_{2} + 1) + L_{f}D_{X}(\|y^{*}\|_{2} + 1) = 0$ and the iteration complexity in (2.14) can be simplified as follows. For the deterministic case, the iteration complexity in (2.14) reduces to

$$O \left( \frac{1}{\sqrt{c}} (\frac{L_{0} + BL_{f}}{\sqrt{\alpha_{0}}})^{D_{X}} + \frac{\sqrt{L_{0} + BL_{f}BM}}{\alpha_{0}} \right)$$

(2.16)

Moreover, the complexity bounds for the semi- and fully-stochastic cases are given by

$$O \left( \frac{1}{\sqrt{c}} (\frac{L_{0} + BL_{f}}{\sqrt{\alpha_{0}}})^{D_{X}} + \frac{\sqrt{L_{0} + BL_{f}BM}}{\alpha_{0}} + \frac{\sigma_{2}^{2}}{\alpha_{0}^{2}} \right),$$

(2.17)

$$O \left( \frac{1}{\sqrt{c}} (\frac{L_{0} + BL_{f}}{\sqrt{\alpha_{0}}})^{D_{X}} + \frac{\sqrt{L_{0} + BL_{f}BM}}{\alpha_{0}} + \frac{\sigma_{2}^{2}}{\alpha_{0}^{2}} + \frac{B^{2}(L_{0} + BL_{f})(\sigma_{2}^{2} + \sigma_{2}^{2})}{\alpha_{0}} \|y^{*}\|_{2}^{3} \right)$$

(2.18)

respectively, where $\zeta^{2} = O(\sigma_{2}^{2} + B^{2}(L_{0} + BL_{f})\|y^{*}\|_{2}^{3}/\alpha_{0})$. It is worth noting that a similar bound to (2.16) has been obtained in [19] with a slightly different termination criterion. On the other hand, the complexity bounds in (2.17) and (2.18) for the semi-stochastic and fully-stochastic cases seem to be new in the literature.

Second, if $B < \|y^{*}\|_{2} + 1$ for the smooth case, then $H_{a} > 0$ and the ConEx method converges at the rate of nonsmooth problems in all these three settings described above. Hence, the ConEx method still converges albeit at a slower rate without knowing exact bound on $\|y^{*}\|_{2}$. On the other hand, existing primal-dual methods require correct upper-bound estimation on $\|y^{*}\|_{2}$ which is used as a bound on $y$ in order to define the projection operator and properly select stepsize. To
obtain a faster convergence rate, one can possibly perform a line search for the right value of $B$ when specifying the policy for $\{\gamma_t, \eta_t, \tau_t, \theta_t\}$ in the ConEx method, especially for the deterministic and semi-stochastic cases where the constraint violations $\|\psi(\cdot)\|_2$ can be measured precisely.

It is worth mentioning that for the complexity results discussed above, we do not require the constraints $\psi_i$, $i = 1, \ldots, m$, to be strongly convex. From (2.10), we can see that $\alpha_0 > 0$ is enough to ensure the selection of stepsize policy which yields accelerated convergence rates. In particular, if $\alpha_i = 0$ for all $i \in [m]$ (implying $\psi_i$’s are merely convex functions) then $\eta_i$ in relation (2.29) is required to satisfy the following more stringent relation: $\gamma_t\eta_t \leq \gamma_{t-1}(\eta_{t-1} + \alpha_0)$. Note that our stepsize policy already satisfies this relation. Hence Algorithm 1 exhibits accelerated convergence rates even if the constraints are merely convex.

Now we provide another theorem which states the stepsize policy and the resulting convergence guarantees of the ConEx method for solving problem (1.1) without any strong convexity assumptions. The proof of this result can be found in Section 2.2.

**Theorem 2.3.** Suppose (2.4), (2.5), (2.6) and (2.9) are satisfied. Let $B \geq 1$ be a given constant, $M$, $\sigma_{X,f}$ and $H_\alpha$ be defined as in Theorem 2.1. Set $y_0 = 0$ and $\{\gamma_t, \theta_t, \eta_t, \tau_t\}$ in Algorithm 1 according to the following:

$$
\begin{align*}
\gamma_t &= 1, \quad \eta_t = L_0 + B L_f + \eta_t, \\
\theta_t &= 1, \quad \tau_t = \tau,
\end{align*}
$$

(2.19)

where

$$
\begin{align*}
\eta &:= \sqrt{2T(D_X^2 + \max\{M, 4\sigma \|y\|_2\}) D_X^2 + \max\{M, 4\sigma \|y\|_2\} D_X^2}, \\
\tau &:= \max\{\sqrt{2T(M, \sigma \|y\|_2)}, 2D_X \max\{M, 4\sigma \|y\|_2\}\}.
\end{align*}
$$

Then, we have

$$
\begin{align*}
\mathbb{E}[\psi(\bar{x}_T) - \psi(\bar{x}_*)] &\leq \frac{(L_0 + B L_f) D_X^2 + \max\{M, 4\sigma \|y\|_2\} D_X}{T} + \frac{\sqrt{T} (\zeta + H_\alpha^2) D_X}{\sqrt{T} (H_\alpha^2 + \sigma^2 + 4 B^2 \|\sigma\|_2^2) + 3 BM} + \frac{\sqrt{T} \beta D_X \beta}{2 T} \\
\mathbb{E}[\|\psi(\bar{x}_T)\|_2^2] &\leq \frac{(L_0 + B L_f) D_X^2 + \max\{M, 4\sigma \|y\|_2\} D_X}{T} (B + \frac{\|y\|_2}{\sqrt{B}})^2 + \frac{\sqrt{T} D_X (\zeta + H_\alpha^2)}{\sqrt{T} (H_\alpha^2 + \sigma^2 + 4 B^2 \|\sigma\|_2^2) + 3 BM},
\end{align*}
$$

(2.20)

(2.21)

where

$$
\zeta := 2 \epsilon \left( \sigma^2 + \|\sigma\|_2^2 \left( 14 \|y\|_2 \|y\|_2 + 123 B^2 \right) + 2 \sqrt{3} \|\sigma\|_2 \left( 2 B H_\alpha + B \sigma_0 \right) \right)^{1/2}.
$$

As a consequence, the number of iterations performed by Algorithm 1 to find an $(\epsilon, \epsilon)$-optimal solution of problem (1.1) can be bounded by

$$
\begin{align*}
\max\left\{ \frac{3(L_0 + B L_f) D_X^2 + \max\{M, 4\sigma \|y\|_2\} (\|y\|_2 + 1) D_X}{\epsilon^2} \left( \frac{36 \sqrt{T} \|\sigma\|_2^2 + 13 B}{4 \sqrt{2}} \right)^2, \\
\frac{(L_0 + B L_f) D_X^2 + \max\{M, 4\sigma \|y\|_2\} D_X}{\epsilon^2} \left( \frac{36 \sqrt{T} \|\sigma\|_2^2 + 13 B}{4 \sqrt{2}} \right)^2 \right\}.
\end{align*}
$$

(2.22)

Theorem 2.3 provides unified iteration complexity bounds for solving convex functional constrained optimization problems. Below we derive from (2.22) the convergence rate of Algorithm 1 for solving both nonsmooth problems, i.e., either $H_f$ or $H_0$ is strictly positive, and (composite) smooth problems, i.e., $H_f = 0$, $H_0 = 0$.

Let us start with the more general nonsmooth problems. Since $H_i > 0$ for some $i = 0, \ldots, m$, we have $H_\alpha > 0$. Then, the complexity bound in (2.22) for the deterministic, semi-stochastic and fully-stochastic cases, respectively, will reduce to

$$
\begin{align*}
O\left( \frac{(L_0 + B D_X (L_i + X + M))}{\epsilon^2} + \frac{D_X^2 H_\alpha^2}{\epsilon^2} \right), \\
O\left( \frac{(L_0 + B D_X (L_i + X + M))}{\epsilon^2} + \frac{D_X^2 (H_\alpha^2 + \sigma_0^2)}{\epsilon^2} \right), \\
O\left( \frac{(L_0 + B D_X (L_i + X + M))}{\epsilon^2} + \frac{B \left( \sigma_0^2 + D_X^2 \|\sigma\|_2^2 \right) + D_X^2 \left( \sigma_0^2 + H_\alpha^2 \right)}{\epsilon^2} \right).
\end{align*}
$$

(2.23)

Similarly to the strongly convex case, the separate impact of the Lipschitz constants ($L_0$ and $L_f$) on these complexity bounds have not been obtained before. Moreover, the iteration (and sampling) complexity for the fully-stochastic case, i.e., general stochastic constrained problems requiring only bounded second moments on noises, appears to be new in the literature.
Now let us consider smooth problems for which $H_f = H_0 = 0$. We distinguish two different scenarios depending on whether $B \geq \|y^*\|_2 + 1$. First, if $B \geq \|y^*\|_2 + 1$, then $H_*$ = 0 and the complexity bound in (2.22) for the deterministic, semi-stochastic and fully-stochastic cases, respectively, will reduce to
\[
O\left(\frac{L_0 + BD_X(L_1 + D_X + M)}{\varepsilon}\right),
\]
(2.24)
and
\[
O\left(\frac{L_0 + BD_X(L_1 + D_X + M)}{\varepsilon \varepsilon} + \frac{\sigma_D^2}{\varepsilon \varepsilon}\right),
\]
(2.25)
where last bound is obtained from (2.22) by noting that $\zeta^2 = O(\sigma_D^2 + 48B^2\|\sigma\|_2^2)$ and replacing $\sigma_{X,f}^2 = \sigma_D^2 + D_X^2\|\sigma\|_2^2$. Note that similar bound as in (2.24) has been obtained before by using more complicated algorithms (e.g., penalty method) or different criterion. On the other hand the complexity bounds in (2.25) and (2.26) appear to be new in the literature. Second, if $B < \|y^*\|_2 + 1$, then $H_* > 0$ and as a result, the ConEx method still converges but at the rate of nonsmooth problems in all these three settings described above.

It should be noted that, different from the strongly convex case (c.f. (2.10)), the stepsize scheme in (2.19) depends on $H_*$, implying that we need to estimate whether $B > \|y^*\|_2 + 1$. However, we can replace $H_*$ in the definition of $\eta$ by $H_B := H_0 + BH_f$. In this way, similar complexity bounds will be obtained for most cases, including nonsmooth deterministic, nonsmooth semi-stochastic, nonsmooth fully-stochastic, as well as smooth semi-stochastic and smooth fully-stochastic problems. In particular, with this modification the last term in (2.22) will change to
\[
\frac{18}{\varepsilon^2} \left( D_X \sqrt{H_B^2 + \sigma_D^2 + 48B^2\|\sigma\|_2^2} + \frac{D_X(\zeta^2 + H_B^2)}{\sqrt{H_B^2 + \sigma_D^2 + 48B^2\|\sigma\|_2^2}} \right)^2.
\]
The only exception that this modification would not work is for smooth deterministic problems. In this case, since $H_B = 0$ but $H_* > 0$, the stepsize scheme (2.19) set according to replacing $H_*$ by $H_B$ does not yield convergence. In particular, the last term in the infeasibility bound (2.21) would change to $H_*^2/(\sqrt{TH_B + BM})$ which is a constant when $H_B = 0$. One possible solution for this is to artificially set $H_B > 0$ in the definition of $\eta$ to be some large positive number and forgo of the faster convergence of $O(1/\varepsilon)$. After this change, we would obtain a convergence rate of $O(1/\varepsilon^2)$. An alternative approach would be to design a line search procedure on $H_B$ for the right value of $H_*$, since there exists a verifiable condition based on the constraint violation $\|\psi(\cdot)\|_2$.

\section{Convergence analysis of the ConEx method}

In this section, we provide a combined analysis of Theorem 2.3 and Theorem 2.1. Note that Algorithm 1 is essentially a dual type method. In order to analyze this algorithm, we define a primal-dual gap function for the equivalent saddle point problem (2.1). In particular, given a pair of feasible solutions $z = (x, y)$ and $\tilde{z} = (\tilde{x}, \tilde{y})$ of (2.1), we define the primal-dual gap function $Q(z, \tilde{z})$ as
\[
Q(z, \tilde{z}) := \mathcal{L}(x, y) - \mathcal{L}(\tilde{x}, \tilde{y}).
\]
(2.27)
One can easily see from (2.2) that $Q(z, z^*) \geq 0$ and $Q(z^*, z) \leq 0$ for all feasible $z$. We use the gap function of the saddle point formulation (2.1) to bound the optimality and infeasibility of the convex problem (1.1) separately, in terms of Definition 2.1. We first develop an important upper-bound on the gap function in terms of primal, dual variables and randomness. This bound holds for all non-negative $\gamma_\tau, \eta_\tau$ and $\tau$. The precise statement is provided in Lemma 2.5.

The following technical result provides a simple form of the Three-point theorem (see, e.g., Lemma 3.5 of [21]) and will be used in the proof of Lemma 2.5.

**Lemma 2.4.** Assume that $g: S \rightarrow \mathbb{R}$ satisfies
\[
g(y) \geq g(x) + \langle g'(x), y - x \rangle + \mu W(y, x), \quad \forall x, y \in S
\]
for some $\mu \geq 0$, where $S$ is closed convex set in $\mathbb{R}^n$. If $x = \arg\min_{x \in S} \{g(x) + W(x, \bar{x})\}$, then
\[
g(x) + W(x, \bar{x}) + (\mu + 1)W(x, x) \leq g(x) + W(x, \bar{x}), \quad \forall x \in S.
\]
**Proof.** It follows from the definition of $W$ that $W(x, \bar{x}) = W(\bar{x}, \bar{x}) + \langle \nabla W(\bar{x}, \bar{x}), x - \bar{x} \rangle + W(x, \bar{x})$. Using this relation, (2.28) and the optimality condition for $\bar{x}$, we have
\[
g(x) + W(x, \bar{x}) = g(x) + [W(\bar{x}, \bar{x}) + \langle \nabla W(\bar{x}, \bar{x}), x - \bar{x} \rangle + W(x, \bar{x})]
\[ g(\bar{x}) + \langle \nabla g(x), x - \bar{x} \rangle + \mu W(x, \bar{x}) + [W(\bar{x}, \bar{x}) + \langle \nabla W(x, \bar{x}), x - \bar{x} \rangle + W(x, \bar{x})] \geq g(\bar{x}) + W(\bar{x}, \bar{x}) + (\mu + 1)W(x, \bar{x}). \]

Hence we conclude the proof.

**Lemma 2.5.** Suppose (2.4), (2.5), (2.6) and (2.9) are satisfied. Let \( B > 0 \) be a constant and assume that \( \{\gamma_t, \eta_t, \tau_t, \theta_t\} \) is a non-negative sequence satisfying for all \( t \geq 1, \)

\[
\begin{align*}
\gamma_t \theta_t &= \gamma_{t-1}, \\
\gamma_t \tau_t &\leq \gamma_{t-1} \tau_{t-1}, \\
\eta_t \theta_t &\leq \gamma_{t-1} \eta_{t-1} + \alpha_{0,t}, \\
\theta_t (M_f + M_x)^2 &\leq \frac{\tau_t (\eta_{t-1} - L_0 - BL_t)}{12}, \\
(M_f + M_x)^2 &\leq \frac{\tau_t (\eta_{t-1} - L_0 - BL_t)}{12}, \\
(2M_f)^2 &\leq \frac{\tau_t (\eta_{t-1} - L_0 - BL_t)}{12},
\end{align*}
\]

and, for all \( t \geq 2, \)

\[
(2M_f)^2 \frac{\theta_t}{\gamma_t} \leq \frac{\tau_t (\eta_{t-1} - L_0 - BL_t)}{12}.
\]

where \( \alpha_{0,t} := \alpha_0 + \alpha^T y_{t+1} \) and \( M_f, M_x, L_f \) are constants as defined in (2.8). Then, for all \( t \geq 0 \) and \( z \in \{(x, y) : x \in X, y \geq 0\}, \)

\[
\begin{align*}
\sum_{t=0}^{\gamma_t} &\gamma_t \partial Q(z_{t+1}, z) + \sum_{t=0}^{\gamma_t} \eta_t \langle \delta_{t}^2(x_t - x), x - x_t \rangle - \langle \ell_{t-1}^{(P)}, y_{t+1} - y \rangle \\
\leq &\gamma_0 \gamma_t W(x, x_0) - \gamma_t (\gamma_{t-1} + \alpha_{0,t}) W(x, x_{t+1}) + \frac{\gamma_{t+1}}{2} \|y - y_0\|^2 - \frac{\gamma_{t+1}}{2} \|y - y_{t+1}\|^2 \\
&+ \gamma_0 \gamma_t \sum_{t=0}^{\gamma_t-1} \frac{\gamma_{t+1}}{2} \left[ \|\delta_{t}^2\|^2 + (H_0 + H_0^f) \|y\|^2 + \frac{L_0^D}{12} \|y\|^2 - B \right] + \frac{3 \gamma_{t+1}}{2} \|y_t - y_{t+1}\|^2.
\end{align*}
\]

Here \( q_t := \ell_f(x_0) - \ell_f(x_{t-1}) + \chi(x_0) - \chi(x_{t-1}), \)

\[
\delta_t^2 := \ell_f(x_0) - \ell_f(x_t) \quad \text{and} \quad \delta_t^2 := \sum_{j=1}^{\gamma_t} G_j(x_t, t) q_{t}^{(j)} - f_0(x_t) - \sum_{j=1}^{\gamma_t} f_t(x_t, t) q_{t}^{(j)}.
\]

**Proof.** Note that \( y_{t+1} = \text{argmin} \{-s_t, y \} + \frac{\eta_t}{2} \|y - y_0\|^2. \) Hence, using Lemma 2.4, we have for all \( y \geq 0, \)

\[
-\langle s_t, y_{t+1} - y \rangle \leq \frac{\eta_t}{2} \|y - y_0\|^2 - \|y_{t+1} - y\|^2 - \|y - y_{t+1}\|^2.
\]

Let us denote \( v_t := \sum_{j=1}^{\gamma_t} f_t(x_{t+1}) q_{t+1}^{(j)} \) and \( V_t := G_0(x_t, \xi_t) + \sum_{j=1}^{\gamma_t} G_j(x_t, \xi_t) q_{t}^{(j)}. \) Then, due to the strong convexity of \( \chi_0 \) and \( \chi_j, j = 1, \ldots, m, \) the optimality of \( x_{t+1}, \) Lemma 2.4 and the definition of \( \alpha_{0,t}, \)

\[
\langle V_t, x - x_{t+1} \rangle + \chi_0(x_{t+1}) - \chi_0(x) + \sum_{j=1}^{\gamma_t} (\chi_j(x_{t+1}) - \chi_j(x)) y_{t}^{(j)} \\
\leq \eta_t [W(x, x_t) - W(x_{t+1}, x_t)] - (\eta_t \alpha_{0,t}) W(x, x_{t+1}).
\]

Due to the convexity of \( f_0 \) and \( f_t, \) (2.4), the definition of \( \ell_f \) and the fact that \( y_{t+1} \geq 0, \)

\[
\begin{align*}
\langle v_t, x_{t+1} - x \rangle &= \langle f_0(x_t) + \sum_{j=1}^{\gamma_t} f_t(x_t) q_{t+1}^{(j)}, x_{t+1} - x \rangle \\
&= \langle f_0(x_t), x_{t+1} - x \rangle + \langle f_t(x_t), y_{t+1} + x_{t+1} - x \rangle \\
&\geq f_0(x_t) - f_0(x) + f_0(x_{t+1}) - f_0(x_t) - \frac{L_0}{12} \|x_{t+1} - x_t\|^2 - H_0 \|x_{t+1} - x_t\| \\
&+ \eta_t \ell_f(x_{t+1}) - \ell_f(x_t) + \langle f(x_t), x_{t+1} - x \rangle \\
&= f_0(x_t) - f_0(x) - \langle \ell_f(x_{t+1}) - \ell_f(x_t), y_{t+1} \rangle - \left( \frac{L_0}{12} \|x_{t+1} - x_t\|^2 + H_0 \|x_{t+1} - x_t\| \right), \quad \text{as } x_{t+1} \to x_t,
\end{align*}
\]

where \( O_{t+1} := \frac{L_0}{12} \|x_{t+1} - x_t\|^2 + H_0 \|x_{t+1} - x_t\| \) is a ‘Lipschitz’-like term for the objective. Combining (2.34), (2.35), noting that \( \delta_t^2 = V_t - v_t \) and using \( \psi_0 = f_0 + \chi_0, \psi = f + \chi, \) we have

\[
\psi_0(x_{t+1}) - \psi_0(x) + \langle \ell_f(x_{t+1}), x_{t+1} \rangle - \chi(x_{t+1}) - \psi(x), y_{t+1} \rangle + \langle \delta_t^2, x_{t+1} - x \rangle \\
\leq \eta_t [W(x, x_t) - W(x_{t+1}, x_t)] - (\eta_t \alpha_{0,t}) W(x, x_{t+1}) + O_{t+1}.
\]

Noting the definition of \( Q(x, \cdot) \) in (2.27) and, adding (2.33) and (2.36), we obtain

\[
\begin{align*}
Q(z_{t+1}, z) - \langle \psi(x_{t+1}), y \rangle - \langle \ell_f(x_{t+1}), y_{t+1} \rangle - \langle s_t, y_{t+1} - y \rangle - \langle \delta_t^2, z_{t+1} - z \rangle \\
\leq \frac{\eta_t}{2} \|y - y_0\|^2 - \|y_{t+1} - y\|^2 - \|y - y_{t+1}\|^2 + \eta_t [W(x, x_t) - W(x_{t+1}, x_t)] - (\eta_t \alpha_{0,t}) W(x, x_{t+1}) + O_{t+1}.
\end{align*}
\]

In view of (2.5),

\[
f_t(x_{t+1}) - \ell_f(x_{t+1}) \leq \frac{L_0}{12} \|x_{t+1} - x_t\|^2 + H_0 \|x_{t+1} - x_t\|.
\]
Then, using Cauchy-Schwarz inequality and noting definitions of $L_f, H_f$, we have
\[
\langle y, f(x_{t+1}) - \ell_f(x_{t+1}) \rangle \leq \|y\|_2 \left[ \frac{L_f}{C_{t+1}} \|x_{t+1} - x_t\|^2 + H_f \|x_{t+1} - x_t\| \right],
\]
where $C_{t+1} := \frac{L_f}{2} \|x_{t+1} - x_t\|^2 + H_f \|x_{t+1} - x_t\|$ is a 'Lipschitz'-like term for the constraints. In view of the above relation and definitions of $\delta_i$ and $\delta^F_i$, we have
\[
\langle \ell_f(x_{t+1}) + \chi(x_{t+1}), y_t \rangle - \langle \psi(x_{t+1}), y \rangle - \langle s_t, y_t - y \rangle \\
\geq \langle \ell_f(x_{t+1}) + \chi(x_{t+1}), y_t \rangle - \langle \ell_f(x_{t+1}) + \chi(x_{t+1}), y \rangle - \langle s_t, y_t - y \rangle - \|y\|_2 \|C_{t+1} \\rangle \\
= \langle \ell_f(x_{t+1}) + \chi(x_{t+1}) - s_t, y_t - y \rangle - \|y\|_2 \|C_{t+1} \| \\
= \langle \ell_f(x_{t+1}) + \chi(x_{t+1}) - \ell_f(x_t) - \chi(x_t) - \delta_i, y_t - y \rangle - \|y\|_2 \|C_{t+1} \| \\
= \langle \delta^F_i, y_t - y \rangle - \delta_i \langle \theta_i, y_t - y \rangle - \delta_i \langle \theta_i, y_t - y \rangle - \delta_i \langle \delta^F_i, y_t - y \rangle - \|y\|_2 \|C_{t+1} \|.
\] (2.38)
Given the non-negative constant $B$ and using the definition of $C_{t+1}$, we have
\[
\|y\|_2 \|C_{t+1} \| = \frac{L_f}{2} \|y\|_2^2 - B \|x_{t+1} - x_t\|^2 + \frac{BL_f}{2} \|x_{t+1} - x_t\|^2 + \|y\|_2 H_f \|x_{t+1} - x_t\| \\
\leq \frac{L_f}{2} \|y\|_2^2 - B \|x_{t+1} - x_t\|^2 + \frac{BL_f}{2} \|x_{t+1} - x_t\|^2 + \|y\|_2 H_f \|x_{t+1} - x_t\| \\
\leq \frac{BL_f}{2} \|x_{t+1} - x_t\|^2 + \|y\|_2 H_f + \frac{L_f D_x}{2} \|y\|_2 \|y\|_2 - B \|x_{t+1} - x_t\|. 
\] (2.39)
Recall the definition of $D_x$ from (1.6). By (2.37), (2.38), and (2.39), noting the definition of $O_{t+1}$, using the relation $\frac{1}{2}(a - b)^2 \leq W(a, b)$ and replacing index $t$ with $i$, we have
\[
Q(\bar{q}_{i+1}, z) + \langle q_{i+1} - y \rangle - \theta_i(q_i, y) + \langle \delta^F_i, x_i - x \rangle - \delta_i \langle \theta_i, y_i - y \rangle \\
\leq \delta_i \langle \theta_i, y_{i+1} - y \rangle - \delta_i \langle \theta_i, y_{i+1} - x \rangle \\
+ \gamma_i W(x, x_i) - (\gamma_i + \alpha_i) W(x, x_{i+1}) + \frac{2}{\gamma_i} \|y - y_i\|^2 - \|y_{i+1} - y_i\|^2 - \|y - y_i\|^2 \\
- \gamma_i (L_0 - BL_f) W(x_{i+1}, x_{i+1}) + \gamma_i (H_0 + \|y\|_2 H_f + \frac{L_f D_x}{2} \|y\|_2 - B \|x_{i+1} - x_i\|.) 
\] (2.40)
Multiplying (2.40) by $\gamma_i$, summing them up from $i = 0$ to $t$ with $t \geq 0$ and noting that $q_0 = 0$, we obtain
\[
\sum_{i=0}^t \gamma_i Q(\bar{q}_{i+1}, z) + \sum_{i=0}^t \gamma_i \gamma_i \langle \bar{q}_{i+1}, y_{i+1} - y \rangle - \gamma_i \langle \bar{q}_{i+1}, y_i - y \rangle \\
+ \sum_{i=0}^t \gamma_i \delta^F_i \langle \theta_i, y_{i+1} - x \rangle - \delta^F_i \langle \theta_i, y_{i+1} - y \rangle \\
\leq \sum_{i=0}^t \gamma_i \langle \bar{q}_{i+1}, \bar{q}_{i+1}, y_{i+1} - y_i \rangle + \sum_{i=0}^t \gamma_i \langle \bar{q}_{i+1}, y_{i+1} - y_i \rangle + \sum_{i=0}^t \gamma_i \delta^F_i \langle \theta_i, x_i - x \rangle - \sum_{i=0}^t \gamma_i \delta^F_i \langle \theta_i, y_i - y \rangle \\
+ \sum_{i=0}^t \gamma_i \langle \gamma_i W(x, x_i) - (\gamma_i + \alpha_i) W(x, x_{i+1}) \|y - y_{i+1}\|^2 - \|y_{i+1} - y_i\|^2 - \|y - y_i\|^2 \\
+ \sum_{i=0}^t \gamma_i \langle \gamma_i (L_0 - BL_f) W(x_{i+1}, x_{i+1}) + \gamma_i (H_0 + \|y\|_2 H_f + \frac{L_f D_x}{2} \|y\|_2 - B \|x_{i+1} - x_i\|.) 
\] (2.41)
where $H(y, B) := H_0 + \|y\|_2 H_f + \frac{L_f D_x}{2} \|y\|_2 - B \|x_{i+1} - x_i\|$. Now we focus our attention to handle the inner product terms of (2.41). Noting the definition of $\bar{q}_i$, we have
\[
\|\bar{q}_i\|^2 = \langle \ell_f(x_i) - \ell_f(x_{i-1}) + \chi(x_i) - \chi(x_{i-1}) \| \\
\leq \langle f(x_{i-1}) + f'(x_{i-1}) T(x_{i-1} - x_{i-2}) - f(x_{i-2}) - f'(x_{i-2}) T(x_{i-1} - x_{i-2}) \|_2 + \| f'(x_{i-1}) - f'(x_{i-2}) \|_2 + \| f'(x_{i-1}) \|_2 + M_\chi \| x_{i-2} - x_{i-1} \| \\
\leq 2 M_\chi \| x_{i-2} - x_{i-1} \| + (M_f + M_\chi) \| x_{i-2} - x_{i-1} \|, 
\] (2.42)
where the last relation follows due to (2.7). Using the above relation, we have for all $i \geq 1$,
\[
\gamma_i \theta_i \langle \bar{q}_{i+1}, y_{i+1} - y_i \rangle - \frac{\gamma_i L_0}{2} \|y_{i+1} - y_i\|^2 - \frac{\gamma_i L_0}{2} \|y_{i+1} - y_i\|^2 \\
- \gamma_i (\gamma_i - L_0 - BL_f) W(x_{i-1}, x_{i-2}) \\
\leq \gamma_i \theta_i \|q_i\|_2 \|y_{i+1} - y_i\|^2 - \frac{2}{\gamma_i} \|y_{i+1} - y_i\|^2 - \|y_{i+1} - y_i\|^2 - \|y_{i+1} - y_i\|^2 \\
- \gamma_i (\gamma_i - L_0 - BL_f) W(x_{i-1}, x_{i-2}) - \gamma_i (\gamma_i - L_0 - BL_f) W(x_{i-1}, x_{i-2}) \\
\leq 2 \gamma_i \theta_i \|q_i\|_2 \|y_{i+1} - y_i\|^2 - \|y_{i+1} - y_i\|^2 - \|y_{i+1} - y_i\|^2 - \|y_{i+1} - y_i\|^2 \\
+ (M_f + M_\chi) \| q_i \|_2 \| y_{i+1} - y_i \|^2 - \gamma_i (\gamma_i - L_0 - BL_f) W(x_{i-1}, x_{i-2}) \\
\leq 0, 
\] (2.43)
where the last inequality follows by applying the relation $W(x, y) \geq \frac{1}{2} \|x - y\|^2$, Young’s inequality $(2ab \leq a^2 + b^2)$ applied twice, once with
\[
a = \frac{\gamma_i L_0}{6}, \quad b = \frac{\gamma_i L_0 - 2}{8} \|x_{i-1} - x_{i-2}\|^2, \quad \frac{1}{2} \|x_{i-1} - x_{i-2}\|^2.
\]
second with
\[ a = \left( \frac{2\tau_2}{6} \right)^{1/2} \| y_{t+1} - y_t \|, \quad b = \left( \frac{2\tau_2(\eta_{t-1} - L_0 - BL_L)}{8} \right)^{1/2} \| x_t - x_{t-1} \|, \]
and the fact that
\[ (2M_f)\gamma_t \theta_t \leq \left\{ \frac{2\tau_2(\eta_{t-1} - L_0 - BL_L)}{12} \right\}^{1/2} \quad \iff \quad (2M_f)^2 \theta_t \leq \frac{\tau_2(\eta_{t-1} - L_0 - BL_L)}{12}, \]
\[ (M_f + M_\gamma)\gamma_t \theta_t \leq \left\{ \frac{2\tau_2(\eta_{t-1} - L_0 - BL_L)}{12} \right\}^{1/2} \quad \iff \quad (M_f + M_\gamma)^2 \theta_t \leq \frac{\tau_2(\eta_{t-1} - L_0 - BL_L)}{12}, \]
where equivalences follow from (2.29a) and conditions follow from relations in (2.30a) and (2.31).

In particular, we require (2.30a) and (2.31) for \( t \geq 1 \) such that (2.43) is satisfied for \( i \geq 1 \). Moreover, \( \| x_{t-1} - x_{t-2} \| = 0 \) for \( i = 1 \). Hence, we require (2.31) for \( t \geq 2 \).

Using Young’s inequality, Cauchy-Schwarz inequality and the relation \( u^Tv \leq \| u \| \| v \| \), we have
\[ \gamma_t \theta_t \| x_t - y_t - \gamma_t(q_t - \bar{q}_t) \| \leq \frac{3\gamma_t^2}{2\tau_1} \| q_t - \bar{q}_t \|^2, \]
\[ \gamma_t \| x_{t+1} - x_t \| - \frac{\gamma_t(\eta_{t-1} - L_0 - BL_L)}{4} \| W(x_{t+1}, x_t) \| \leq \frac{\gamma_t}{\eta_{t-1} - L_0 - BL_L} \| \delta_t \|^2, \tag{2.44} \]
Using (2.43) and (2.44) for \( i = 1, \ldots, t \) inside (2.41) and noting (2.29), we have
\[ \sum_{i=0}^t \gamma_t Q(z_{t+1}, z) + \gamma_t \| q_{t+1} - y_t - y \| + \sum_{i=0}^t \gamma_t \left[ \delta_t \| x_t - x_i \| - \frac{\gamma_t(\eta_{t-1} - L_0 - BL_L)}{4} \| W(x_{t+1}, x_t) \| \right] \leq \gamma_t y_0 W(x_0, x_t) + \gamma_t \| y_t - y_0 \|^2 - \frac{\gamma_t^2}{\eta_{t-1} - L_0 - BL_L} \| y - y_t \|^2 - \frac{\gamma_t^2}{\eta_{t-1} - L_0 - BL_L} \| y_t - y \|^2 \]
\[ + \sum_{i=0}^t \gamma_t \| x_{t+1} - x_t \| - \gamma_t \| y_t - y \|^2 - \frac{\gamma_t^2}{\eta_{t-1} - L_0 - BL_L} \| y_t - y \|^2 \]
\[ - \gamma_t \| y_t - y \|^2 \leq \gamma_t H(y, B) \| x_{t+1} - x_t \| - \gamma_t \| y_t - y \|^2 \]
\[ \| y_t - y \|^2, \tag{2.45} \]
where on the left hand side of the above relation, we use the fact that \( q_0 = \ell_F(x_0) - \ell_F(x_{t-1}) + \chi(x_0) - \chi(x_{t-1}) = 0 \). Using (2.42), we have
\[ -\gamma_t \| q_{t+1} - y_t - y \| - \frac{\gamma_t^2}{\eta_{t-1} - L_0 - BL_L} \| y_t - y \|^2 \]
\[ - \frac{\gamma_t(\eta_{t-1} - L_0 - BL_L)}{4} \| W(x_{t+1}, x_t) \| \leq (M_f + M_\gamma)\gamma_t \| x_{t+1} - x_t \| \| y_t - y \|^2 - \frac{\gamma_t^2}{\eta_{t-1} - L_0 - BL_L} \| y_t - y \|^2 \]
\[ + 2M_f \| x_t - y_t - y \|^2 - \frac{\gamma_t^2}{\eta_{t-1} - L_0 - BL_L} \| y_t - y \|^2 \]
\[ - \frac{\gamma_t^2}{\eta_{t-1} - L_0 - BL_L} \| y_t - y \|^2 \]
\[ \leq - \gamma_t \| y_t - y \|^2, \tag{2.46} \]
where the last relation follows from (2.30b) and (2.30c), Young’s inequality and the fact that
\[ (2M_f)\gamma_t \leq \left\{ \frac{2\tau_2(\eta_{t-1} - L_0 - BL_L)}{12} \right\}^{1/2} \quad \iff \quad (2M_f)^2 \theta_t \leq \frac{\tau_2(\eta_{t-1} - L_0 - BL_L)}{12}, \]
\[ (M_f + M_\gamma)\gamma_t \leq \left\{ \frac{2\tau_2(\eta_{t-1} - L_0 - BL_L)}{12} \right\}^{1/2} \quad \iff \quad (M_f + M_\gamma)^2 \theta_t \leq \frac{\tau_2(\eta_{t-1} - L_0 - BL_L)}{12}. \]
Finally, using Young’s inequality and Cauchy-Schwarz inequality, we have
\[ -\gamma_t \| q_{t+1} - y_t - y \| - \frac{\gamma_t^2}{\eta_{t-1} - L_0 - BL_L} \| y_t - y \|^2 \leq \frac{\gamma_t}{\eta_{t-1} - L_0 - BL_L} \| q_{t+1} - y_t \|^2. \tag{2.47} \]
Using (2.46) and (2.47) in relation (2.45), noting that \( q_0 - \bar{q}_0 = 0 \) and replacing the definition of \( H(y, B) \), we obtain (2.32). Hence, we conclude the proof.

We now aim to convert the bound on the primal-dual gap function \( Q \) in Lemma 2.5 into bounds on the optimality and infeasibility of the ergodic solution \( x_T \) according to Definition 2.1.

For proving this lemma, we need one more simple result which is stated below.

**Lemma 2.6.** Let \( \rho_0, \ldots, \rho_T \) be a sequence of elements in \( \mathbb{R}^n \) and let \( S \) be a convex set in \( \mathbb{R}^n \).

Define the sequence \( v_t, t = 0, 1, \ldots, \) as follows: \( v_0 \in S \) and
\[ v_{t+1} = \text{argmin}_{x \in S} \langle \rho_t, x \rangle + \frac{1}{2} \| x - v_t \|^2. \]

Then for any \( x \in S \) and \( t \geq 0 \), the following inequalities hold
\[ \langle \rho_t, v_t - x \rangle \leq \frac{1}{2} \| x - v_t \|^2 - \frac{1}{2} \| x - v_{t+1} \|^2 + \frac{1}{2} \| \rho_t \|^2, \tag{2.48} \]
\[ \sum_{t=0}^T \langle \rho_t, v_t - x \rangle \leq \frac{1}{2} \| x - v_0 \|^2 + \frac{1}{2} \sum_{t=0}^T \| \rho_t \|^2. \tag{2.49} \]

**Proof.** Using Lemma 2.4 with \( g(x) = \langle \rho_t, x \rangle, W(y, x) = \frac{1}{2} \| y - x \|^2, \) \( x = v_t \) and \( \mu = 0 \), we have, due to the optimality of \( v_{t+1} \),
\[ \langle \rho_t, v_{t+1} - x \rangle + \frac{1}{2} \| v_{t+1} - v_t \|^2 + \frac{1}{2} \| x - v_{t+1} \|^2 \leq \frac{1}{2} \| x - v_t \|^2, \]

is satisfied for all \( x \in S \). The above relation and the fact
\[ \langle \rho_t, v_t - v_{t+1} \rangle - \frac{1}{2} \| v_{t+1} - v_t \|^2 \leq \frac{1}{2} \| \rho_t \|^2, \]

imply that
\[ \langle \rho_t, v_t - x \rangle \leq \frac{1}{2} \| x - v_t \|^2 - \frac{1}{2} \| x - v_{t+1} \|^2 + \frac{1}{2} \| \rho_t \|^2. \]
for all \( x \in S \). Summing up the above relations from \( t = 0 \) to \( j \) and noting the nonnegativity of \( \| \cdot \|^2 \), we obtain (2.49). Hence we conclude the proof.

Now we are ready to prove the lemma converting bound on the primal-dual gap to infeasibility and optimality gap.

**Lemma 2.7.** Suppose all assumptions in Lemma 2.5 are satisfied. Then, for \( T \geq 1 \), we have

\[
E[\psi_{t,T}(x_T) - \psi_0(x^*)] \leq \frac{1}{\tau_T} \left[ \gamma_T \theta_0 W(x^*, x_0) + \frac{2m_0}{\tau_T} \| y_0 \|^2 \right] \\
+ \sum_{t=0}^{T-1} \frac{2m_0}{\tau_{t+1} - \tau_t} E[\| \delta_t^G \|^2] + H_0^2 + \left( \sum_{t=1}^{T-1} \frac{12\gamma_0^2}{\tau_t} + \frac{12\gamma_{t-1}^2}{\tau_{t-1}} \right) (\sigma_t^2 + D_X^0 \| \sigma_t \|^2),
\]

(2.50)

\[
\gamma_T (\eta_{T-1} + \alpha_{T-1}) E[W(x^*, x_T)] \leq \frac{2m_0}{\tau_T} \| y^* - y_0 \|^2 + \gamma_T \theta_0 W(x^*, x_0) \\
+ \left( \sum_{t=1}^{T-1} \frac{12\gamma_0^2}{\tau_t} + \frac{12\gamma_{t-1}^2}{\tau_{t-1}} \right) (\sigma_t^2 + D_X^0 \| \sigma_t \|^2) \\
+ \sum_{t=0}^{T-1} \frac{2m_0}{\tau_{t+1} - \tau_t} E[\| \delta_t^G \|^2] + (H_0 + \| y^* \|^2) H_f + [\| y^* - B \|^2],
\]

(2.51)

and

\[
E[\| \psi(x_T) \|_2] \leq \frac{1}{\tau_T} \left[ \gamma_T \theta_0 \| y_0 \|^2 + 3(\| y^* \|^2 + 1)^2 \gamma_T \theta_0 + \gamma_T \theta_0 W(x^*, x_0) \\
+ \sum_{t=0}^{T-1} \frac{2m_0}{\tau_{t+1} - \tau_t} E[\| \delta_t^G \|^2] + (H_0 + \| y^* \|^2) H_f + \frac{L_D \theta_0 \| y^* \|^2 + \| B \|^2}{2\tau_T} \right] \\
+ \left( \sum_{t=1}^{T-1} \frac{12\gamma_0^2}{\tau_t} + \sum_{t=0}^{T-1} \frac{12\gamma_{t-1}^2}{\tau_t} \right) (\sigma_t^2 + D_X^0 \| \sigma_t \|^2),
\]

(2.52)

where \( \Gamma_T := \sum_{t=0}^{T-1} \gamma_t \).

**Proof.** Notice that conditional random variables \( G_i(x_t, \xi_t) [\xi_{t-1}, \xi_{t-2}] \), \( i = 0, \ldots, m \) satisfy properties of SO in (2.9) because \( x_0 \) is a constant when conditioned on random variables \( \xi_{t-1} := (\xi_0, \ldots, \xi_{t-1}) \) and \( \xi_{t-2} := (\xi_0, \ldots, \xi_{t-2}) \). Also, observe that, \( y_{t+1} \) is a constant when conditioned on random variables \( \xi_{t-1} \) and \( \xi_{t-1} \). In particular, using (2.9), we have

\[
E[\langle \delta_t^G, x_t - x \rangle] = E(E[\xi_{t-1}] [\xi_{t-1}] [\delta_t^G], x_t - x) = 0,
\]

(2.53)

for any non-random \( x \). This follows due to the following relation

\[
E[\xi_{t-1}] [\xi_{t-1}] [\delta_t^G] = E(E[\xi_{t-1}] [\xi_{t-1}] [\delta_t^F], x_t - x) = 0,
\]

(2.54)

Similarly, using (2.9), we have

\[
E[\langle \delta_t^F, y_{t+1} - y \rangle] = E(E[\xi_{t-1}] [\xi_{t-1}] [\delta_{t+1}^F], y_{t+1} - y) = 0,
\]

(2.55)

for any non-random \( y \). Here, we note that

\[
E[\xi_{t-1}] [\xi_{t-1}] [\delta_{t+1}^F] = E(E[\xi_{t-1}] [\xi_{t-1}] [F(x_t, \xi_t)], f(x_t)) \\
+ \langle E(E[\xi_{t-1}] [\xi_{t-1}] [G(x_t, \xi_t)], f'(x_t) \rangle \rangle_T (x_{t+1} - x_t) = 0,
\]

where the first term in RHS is 0 due to the third relation in (2.9) applied to \( \xi_t \), the second term is 0 due to the second relation of (2.9) applied to \( \xi_t \) and the common fact for both the terms that \( x_t, x_{t-1} \) are constants for given \( \xi_t, \xi_{t-1} \). We note that

\[
E[\| \delta_t^F \|^2] \leq 2E[\| \delta_t^F \|^2] + 2 \sum_{i=1}^m \left( \| G_i(x_t, \xi_t) - f_i(x_t) \|_2 \right)^2 \\
\leq 2\sigma_t^2 + 2 \sum_{i=1}^m \| G_i(x_t, \xi_t) - f_i(x_t) \|_2^2 \\
\leq 2\sigma_t^2 + 2 \sum_{i=1}^m \| G_i(x_t, \xi_t) - f_i(x_t) \|_2^2 \| x_t - x_t-1 \|_2^2 \\
\leq 2\sigma_t^2 + 2 \sum_{i=1}^m \| G_i(x_t, \xi_t) - f_i(x_t) \|_2^2 \| x_t - x_t-1 \|^2,
\]

(2.56)

Recall that \( \sigma = [\sigma_1, \ldots, \sigma_T]^T \). Then, in view of the above relation and definitions of \( q_t, \bar{q}_t \), we have

\[
E[\| q_t - \bar{q}_t \|^2] \leq 2E[\| \delta_t^F \|^2] + 2 \sum_{i=1}^m \| G_i(x_t, \xi_t) - f_i(x_t) \|_2^2 \leq 8(\sigma_t^2 + D_X^0 \| \sigma_t \|^2).
\]

(2.57)

In (2.32), choosing \( t = T - 1 \geq 0 \), taking expectation on both sides, and using relation (2.53), (2.54) and (2.57), we have for all non-random \( x \in \{ (x, y) : x \in X, y \geq 0 \} \),

\[
E[\sum_{t=0}^{T-1} \gamma_t Q(z_{t+1}, z_t)] \leq 2m_0 \| y - y_0 \|^2 + \gamma_T \theta_0 W(x, x_0) + \left( \sum_{t=1}^{T-1} \frac{12\gamma_0^2}{\tau_t} + \frac{12\gamma_{t-1}^2}{\tau_{t-1}} \right) (\sigma_t^2 + D_X^0 \| \sigma_t \|^2) \\
+ \sum_{t=0}^{T-1} \frac{2m_0}{\tau_{t+1} - \tau_t} E[\| \delta_t^G \|^2] + (H_0 + \| y^* \|^2) H_f + \frac{L_D \theta_0 \| y^* \|^2 + \| B \|^2}{2\tau_T} \right] \\
- \gamma_T (\eta_{T-1} + \alpha_{T-1}) E[W(x, x_T)],
\]

(2.58)
where we dropped $\|y - yT\|_2^2$. Using the convexity of $\psi_0(\cdot)$ and $\psi(\cdot)$, and noting the definition of $\Gamma_T$, we have for all non-random $y \geq 0$ and $x \in X$,

$$\Gamma_T E \left[ \psi_0(\tilde{y}_T) + \langle y, \psi(\tilde{y}_T) \rangle - \psi_0(x) - \langle \tilde{y}_T, \psi(x) \rangle \right] \leq E \left[ \sum_{t=0}^{T-1} \gamma_t Q(z_{t+1}, z) \right].$$

Combining (2.58) and (2.59), then choosing $x = x^*$, $y = 0$ (which are non-random) throughout the combined relation, observing that $[0 - B]_+ = 0$ for any $B \geq 0$, ignoring $W(x, x_T)$ term and noting that $\psi(x^*) \leq 0$ and $\tilde{y}_T \geq 0$ implies $\langle \tilde{y}_T, \psi(x^*) \rangle \leq 0$, we have (2.50).

Now, we prove a bound on $E[|W(x^*, x_T)|]$. Put $z = z^* := (x^*, y^*)$ in (2.58). Then we have that $Q(z_{t+1}, z^*) \geq 0$ for all $t = 0, \ldots, T - 1$. Hence, using $z = z^*$ in (2.58), dropping summation of $Q$-terms and taking expectation on both sides, we obtain (2.51).

Now, we focus our attention to the infeasibility bound. Let us define $R := \|y^*\|_2 + 1$ and an auxiliary sequence $\{y_t^1\}$ in the following way: $y_0^1 = y_1$ and for all $t \geq 1$, define

$$y_{t+1}^1 := \arg\min_{y \in \mathcal{B}_2^2 (R)} \frac{\gamma_t}{\gamma_{t-1}} (\delta^F_{t+1}, y) + \frac{1}{2} \|y - y_{t+1}^1\|_2^2,$$

where we recall that $\mathcal{B}_2^2 (R) = \{x \in \mathbb{R}^n : \|x\|_2 \leq R, x \geq 0\}$. Then in view of Lemma 2.6, in particular relation (2.48), for all $y \in \mathcal{B}_2^2 (R)$ we have

$$\frac{1}{\gamma_{t-1}} (\delta^F_{t+1}, y_{t+1}^1 - y) \leq \frac{1}{2} \|y - y_{t+1}^1\|_2^2 + \frac{1}{2} \|y - y_{t+1}^1\|_2^2 + \frac{3 \gamma_{t-1}}{\gamma_{t-1}} \|\delta^F_{t+1}\|_2^2.$$  

(2.60)

Multiplying by $\gamma_t$, taking a sum from $t = 0$ to $T - 1$ and noting (2.29b), we obtain

$$\sum_{t=0}^{T-1} \gamma_t (\delta^F_{t+1}, y_{t+1}^1 - y) \leq \frac{\gamma_m}{\gamma_{t-1}} \|y - y_{t+1}^1\|_2^2 + \sum_{t=0}^{T-1} \frac{\gamma_m}{\gamma_{t-1}} \|\delta^F_{t+1}\|_2^2.$$  

(2.61)

for all $y \in \mathcal{B}_2^2 (R)$. Replacing $i$ and $t$ in (2.32) by $t$ and $T - 1$ and summing with (2.61), we obtain

$$\sum_{t=0}^{T-1} \gamma_t Q(z_{t+1}, z) + \sum_{t=0}^{T-1} \gamma_t \|\delta^F_t\|_2^2 \|x_t - x\|_2 + (\delta^F_{t+1}, y_{t+1}^1 - y_{t+1}^1) \leq \frac{3 \gamma_{t-1}}{\gamma_{t-1}} (\|y - y_{t+1}^1\|_2^2 + \|y - y_{t+1}^1\|_2^2 + \gamma_t \|y_{t+1}^1 - y\|_2^2) \leq \frac{3 \gamma_{t-1}}{\gamma_{t-1}} \|y_{t+1}^1 - y\|_2^2 + \gamma_t \|y_{t+1}^1 - y\|_2^2.$$  

(2.62)

for all $z \in \{(x, y) : x \in X, y \in B_2^2 (R)\}$. Note that given $\xi_t$ and $\xi_{t-1}$, we have $y_{t+1}, y_{t+1}^1, x_{t+1}$ and $x_t$ are constants. Hence we have

$$E \left[ \delta^F_{t+1}, y_{t+1}^1 - y_{t+1}^1 \right] = E \left[ \|\delta^F_{t+1}, y_{t+1}^1 - y_{t+1}^1\|_2 \right] = 0,$$  

(2.63)

where equality directly follows from (2.55). Choosing $z = \tilde{z} := (x^*, \tilde{y})$ in (2.62) where $\tilde{y} := (\|y^*\|_2 + 1) \langle \psi(x^*) \rangle$, we have $\|\psi(x^*)\|_2 \leq 1$ in $\mathcal{B}_2^2 (R)$, taking expectation on both sides and noting (2.63), (2.56), (2.57), first relation in (2.53), we have

$$\sum_{t=0}^{T-1} \gamma_t Q(z_{t+1}, z) \leq \frac{3 \gamma_{t-1}}{\gamma_{t-1}} \frac{\gamma_t}{\gamma_{t-1}} \left( \|\delta^F_{t+1}\|_2^2 \right) + \left( H_0 + \|y^*\|_2 - 1 \right) H_1 + \frac{L_1 D_1 \|y^*\|_2^2 \|1-B\|_2^2}{2}.$$  

(2.64)

Noting the convexity of $Q$ in the first argument, we obtain

$$E \left[ Q(\tilde{z}, \tilde{z}) \right] \leq \frac{1}{\gamma_{t-1}} E \left[ \sum_{t=0}^{T-1} \gamma_t Q(z_{t+1}, z) \right].$$  

(2.65)

Now observe that

$$\mathcal{L}(\tilde{x}_T, y^*) - \mathcal{L}(x^*, y^*) \geq 0 \Rightarrow \psi_0(\tilde{x}_T) + \langle y^*, \psi(\tilde{x}_T) \rangle - \psi_0(x^*) \geq 0,$$

which in view of the relation

$$\langle y^*, \psi(\tilde{x}_T) \rangle \leq \langle y^*, [\psi(\tilde{x}_T)]_+ \rangle \leq \|y^*\|_2 \|\psi(\tilde{x}_T)\|_2 \|y^*\|_2,$$

implies that

$$\psi_0(\tilde{x}_T) + \|y^*\|_2 \|\psi(\tilde{x}_T)\|_2 - \psi_0(x^*) \geq 0.$$  

(2.66)

Moreover,

$$Q(\tilde{z}, \tilde{z}) = \mathcal{L}(\tilde{x}_T, \tilde{y}) - \mathcal{L}(x^*, \tilde{y}) \geq \mathcal{L}(\tilde{x}_T, \tilde{y}) - \mathcal{L}(x^*, \tilde{y}) = \psi_0(\tilde{x}_T) + (\|y^*\|_2 + 1) \|\psi(\tilde{x}_T)\|_2 - \psi_0(x^*),$$

along with (2.66) implies that

$$Q(\tilde{z}, \tilde{z}) \geq \|\psi(\tilde{x}_T)\|_2.$$  

The above relation (2.65) and (2.64) together yield

$$E \left[ \|\psi(\tilde{x}_T)\|_2 \right] \leq \frac{1}{\gamma_{t-1}} \left( \frac{3 \gamma_{t-1}}{\gamma_{t-1}} \frac{\gamma_t}{\gamma_{t-1}} \left( \|\delta^F_{t+1}\|_2^2 \right) + \left( H_0 + \|y^*\|_2 - 1 \right) H_1 + \frac{L_1 D_1 \|y^*\|_2^2 \|1-B\|_2^2}{2} \right)$$

$$+ \sum_{t=0}^{T-1} \frac{2 \gamma_t}{\gamma_{t-1}} \left( E \left[ \|\delta^F_{t+1}\|_2^2 \right] + \left( H_0 + \|y^*\|_2 - 1 \right) H_1 + \frac{L_1 D_1 \|y^*\|_2^2 \|1-B\|_2^2}{2} \right)$$

$$+ \sum_{t=0}^{T-1} \frac{2 \gamma_t}{\gamma_{t-1}} \left( \|\delta^F_{t+1}\|_2^2 \right).$$

(2.66)

Noting the bound $\|\tilde{y} - y_{t+1}^1\|_2 \leq 2R$ and $\|\tilde{y} - y_{t+1}^1\|_2 \leq \|y_{t+1}^1\|_2 + 2 \|\tilde{y}\|_2 \leq \|y_{t+1}^1\|_2 + 2R^2$ in the above relation and recalling that $R = \|y^*\|_2 + 1$, we obtain (2.52). Hence we conclude the proof. Note
that we still need to bound $\mathbb{E}[\|\delta^G_t\|_2^2]$. Below, we provide a simple lemma which is used to show such a bound.

**Lemma 2.8.** Let $\{a_t\}_{t \geq 0}$ be a nonnegative sequence, $m_1, m_2 \geq 0$ be constants such that $a_0 \leq m_1$ and the following relation holds for all $t \geq 1$:

$$a_t \leq m_1 + m_2 \sum_{k=0}^{t-1} a_k.$$

Then we have $a_t \leq m_1(1 + m_2)^t$.

**Proof.** We prove this lemma by induction. Clearly, it is true for $t = 0$. Suppose it is true for $a_t$. Then, using inductive hypothesis on $a_k$ for $k = 0, \ldots, t$, we have

$$a_{t+1} \leq m_1 + m_2 \sum_{k=0}^{t} a_k \leq m_1 \left[1 + m_2 \sum_{k=0}^{t-1} (1 + m_2)^k\right] \leq m_1(1 + m_2)^{t+1}.$$

Hence, we conclude the proof. Now, under some assumptions, we show a bound on $\mathbb{E}[\|\delta^G_t\|_2^2]$.

**Lemma 2.9.** Assume that $\{\gamma_t, \tau_t, \eta_t\}$ satisfy

$$\frac{96||\sigma^2||}{\gamma_t(\eta_t - \lambda_0 - BL_t^2)} < 1,$$

for all $t \leq T - 1$ and there exist constants $R_1$ and $R_2$ satisfying

$$R_1 \geq \left(1 - \frac{96||\sigma^2||}{\gamma_t(\eta_t - \lambda_0 - BL_t^2)}\right)^{-1} \frac{2\sigma_0^2 + \frac{48||\sigma^2||^2}{\gamma_t}}{\gamma_t} \left\{\gamma_0 y_0 W(x^*, x_0) + \frac{2\eta_0}{\gamma_t} \|y^* - y_0\|^2 + \frac{2\gamma_t}{\gamma_t} \|y^*\|^2 \right\}
+ \sum_{i=0}^{t} \frac{2\eta_i}{\eta_t - \lambda_0 - BL_t^2} \left(\frac{12\gamma_i}{\gamma_t} \sigma^2 \|\sigma\|^2 + 2 ||\eta-\lambda\|+\eta_t\right) \left(\frac{12\gamma_i}{\gamma_t} \sigma^2 \|\sigma\|^2 + \frac{2L_t D_t x^* \|y^* - y_t\|}{\gamma_t}\right)^2 + \sum_{t=0}^{T-1} \frac{12\gamma_t}{\gamma_t} \sigma^2 \|\sigma\|^2 + \frac{12\gamma_t}{\gamma_t} \sigma^2 \|\sigma\|^2 + \frac{D_t^2 \|\sigma\|^2}{\gamma_t},$$

for all $t \leq T - 1$ and

$$R_2 \geq \left(1 - \frac{96||\sigma^2||}{\gamma_t(\eta_t - \lambda_0 - BL_t^2)}\right)^{-1} \frac{96||\sigma^2||^2}{\gamma_t(\eta_t - \lambda_0 - BL_t^2)},$$

for all $t \leq T - 1$ and $i \leq t - 1$. Then, we have

$$\mathbb{E}[\|\delta^G_t\|_2^2] \leq R_1 (1 + R_2)^t,$$

for all $t \leq T - 1$. In particular, if $||\sigma||_2 = 0$, then we can set $R_1 = 2\sigma_0^2$ and $R_2 = 0$ implying $\mathbb{E}[\|\delta^G_t\|_2^2] \leq 2\sigma_0^2$.

**Proof.** Observe that $Q(z_{t+1}, z^*) \geq 0$ for all $t = 0, \ldots, T - 1$ where $z^* = (x^*, y^*)$. Choosing $z = z^*$ in (2.32) and taking expectation on both sides, using (2.53) with $x = x^*$ and (2.54) with $y = y^*$ and noting (2.57), we have

$$\frac{\gamma_t^2}{\gamma_t} \mathbb{E}[\|y^* - y_{t+1}\|_2^2] \leq \gamma_0 y_0 W(x^*, x_0) + \frac{2\eta_0}{\gamma_t} \|y^* - y_0\|^2 + \frac{2\gamma_t}{\gamma_t} \|y^*\|^2 + \sum_{i=0}^{t} \frac{2\eta_i}{\eta_t - \lambda_0 - BL_t^2} \left(\frac{12\gamma_i}{\gamma_t} \sigma^2 \|\sigma\|^2 + 2 ||\eta-\lambda\|+\eta_t\right) \left(\frac{12\gamma_i}{\gamma_t} \sigma^2 \|\sigma\|^2 + \frac{2L_t D_t x^* \|y^* - y_t\|}{\gamma_t}\right)^2 + \sum_{t=0}^{T-1} \frac{12\gamma_t}{\gamma_t} \sigma^2 \|\sigma\|^2 + \frac{D_t^2 \|\sigma\|^2}{\gamma_t},$$

Now, let us define $\delta^G_t := G_t(x_t, \xi_t) - f_t(x_t)$ for $i = 0, \ldots, m$. As a consequence, we have $\delta^G_t = \delta^G_{t,0} + \sum_{i=1}^{m} y_{t,i} \epsilon^G_{t,i}$. Then, we have

$$\mathbb{E}[\|\delta^G_t\|_2^2] = \mathbb{E}[\|\delta^G_{t,0}\|^2 + \sum_{i=1}^{m} \|y_{t,i}\|^2 \|\delta^G_{t,i}\|_2^2]
\leq 2\mathbb{E}[\|\delta^G_{t,0}\|^2] + 2\mathbb{E}[\sum_{i=1}^{m} \|y_{t,i}\|^2 \|\delta^G_{t,i}\|_2^2]
\leq 2\mathbb{E}[\|\delta^G_{t,0}\|^2] + 2\mathbb{E}[\sum_{i=1}^{m} \|y_{t+1}\|^2 \|\delta^G_{t+1}\|_2^2]
\leq 2\mathbb{E}[\|\delta^G_{t,0}\|^2] + 2\mathbb{E}[\|\delta^G_{t+1}\|_2^2]
\leq 2\mathbb{E}[\|\delta^G_{t,0}\|^2] + 2\mathbb{E}[\|\delta^G_{t+1}\|_2^2]
\leq 2\mathbb{E}[\|\delta^G_{t,0}\|^2] + 2\mathbb{E}[\|\delta^G_{t+1}\|_2^2]
\leq 2\mathbb{E}[\|\delta^G_{t,0}\|^2] + 2\mathbb{E}[\|\delta^G_{t+1}\|_2^2]
\leq 2\mathbb{E}[\|\delta^G_{t,0}\|^2] + 2\mathbb{E}[\|\delta^G_{t+1}\|_2^2]
\leq 2\mathbb{E}[\|\delta^G_{t,0}\|^2] + 2\mathbb{E}[\|\delta^G_{t+1}\|_2^2].$$

(2.71)

Here, relation (i) follows from the fact that $||a + b||_2^2 \leq (||a||_2 + ||b||_2)^2 \leq 2||a||_2^2 + 2||b||_2^2$, relation (ii) follows from Cauchy-Schwarz inequality, relation (iii) follows from the fact that $y_{t+1},$ conditioned on random variables $\xi_{t-1}, \xi_{t-1},$ is a constant and relation (iv) follows from fourth and fifth relation in (2.9) and the fact that $x_t$ is a constant when conditioned on random variables $\xi_{t-1}, \xi_{t-1}.$
Adding \( \frac{2\eta t^2}{\gamma_0^2 T} \| y^* \|^2 \) to both sides of (2.71), then multiplying the resulting relation by \( \frac{48\|x\|_2^2}{\gamma_0^2 T} \) and observing (2.72), we have

\[
\begin{align*}
\mathbb{E}[\|\delta_t\|^2] & \leq 2\alpha_0^2 + \frac{48\|x\|_2^2}{\gamma_0^2 T} \left\{ \gamma_0 \eta_0 W(x^*, x_0) + \frac{2\alpha_0}{\gamma_0} \| y^* - y_0 \|^2 + \frac{2\eta t^2}{\gamma_0^2} \| y^* \|^2 \right\} \\
& + \sum_{t=0}^T \frac{2\eta (t+1)^2}{\gamma_0^2 T} \left( H_0 + H_f \right) \left( \| y^* \|^2 + \frac{L_d x}{2} \right)
+ \left( \sum_{t=1}^T \frac{\eta (t+1)^2}{\gamma_0^2} \right) \left( \sigma_2^2 + D^2_2 \| \sigma_2^2 \| \right) + \sum_{t=0}^T \frac{96\|x\|_2^2}{\gamma_0^2} \left( \frac{H_0 + H_f}{2} \right) \mathbb{E}[\|\delta_t\|^2].
\end{align*}
\]

In view of (2.67), we have that the coefficient of the \( \delta_t^2 \) term on the right hand side of the above relation is strictly less than 1. Moving the \( \delta_t^2 \) term to the left hand side and noting the conditions imposed on constants \( R_1, R_2 \), we have

\[
\mathbb{E}[\|\delta_t\|^2] \leq R_1 + R_2 \sum_{t=0}^{T-1} \mathbb{E}[\|\delta_t\|^2],
\]

for all \( t \leq T - 1 \). Using Lemma 2.8 for the above relation, we have (2.70). Hence we conclude the proof.

**Remark 2.10.** Note that the bound in (2.70) is still a function of stepsize parameters since \( R_1 \) and \( R_2 \) need to satisfy relations (2.68) and (2.69), respectively. Now, we need to show that there exists a possible selection of stepsize parameters for which we can compute a uniform upper bound on \( \mathbb{E}[\|\delta_t\|^2] \) for all \( t \leq T - 1 \), in particular, we can obtain constants \( R_1 \) and \( R_2 \) satisfying (2.68) and (2.69), respectively. Moreover, the selected stepsize policy is meaningful in the sense that it yields convergence according (2.50) and (2.52). Below, we show that the stepsize policy in (2.10) of Theorem 2.1 and in (2.19) of Theorem 2.3 is specified in a way such that (2.29), (2.30), (2.31) and (2.67) are satisfied. Moreover, a uniform upper bound according to (2.70) for all \( t \leq T - 1 \) can be obtained and it also leads to the convergence according to (2.50) and (2.52). In particular, we show the proofs of Theorem 2.1 and Theorem 2.3 below.

First, we focus on the setting in which (1.1) is strongly convex, i.e., \( \alpha_0 > 0 \) and show the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Note that \( \{\gamma_t, \theta_t, \eta_t, \tau_t\} \) set according to (2.10) satisfy (2.29). It is easy to verify (2.29a) and (2.29b). To verify (2.29c), note that

\[
\gamma_{t-1}(\eta_t - 1) + \alpha_0 t \gamma_0 + \alpha_0 = (t + t + 1) \left( \frac{\alpha_0 (t + t + 1)}{2} + \alpha_0 \right) = \frac{2\alpha_0}{\gamma_0} (t + t + 1)(t + t + 2) = \gamma_t \eta_t.
\]

Note that all relations in (2.30) and (2.31) are satisfied if \( \frac{4}{3} \mathcal{M}^2 \leq \frac{\gamma_t (\eta_t - 1 - L_0 - B_L)}{12} \leq \frac{32 \mathcal{M}^2}{12 \gamma_0 (t + 1)} \left( \frac{\alpha_0 (t + t + 1)}{2} - \frac{\alpha_0 (t + t - 1)}{4} \right) = \frac{2(2t + 2) \mathcal{M}^2}{3(t + 1)} \geq 4 \mathcal{M}^2, \]

where the last inequality follows from \( t_0 \geq 2 \) by definition. Also note that

\[
\tau_t (\eta_t - L_0 - B_L) \geq \frac{384 \|x\|^2}{\alpha_0 (t + 1) T} \left( \frac{\alpha_0 (t + t + 1)}{2} - \frac{\alpha_0 (t + t - 1)}{4} \right) = \frac{96(2t + 2) \|y\|^2}{\alpha_0 (t + 1) T} \geq 192 \|y\|^2
\]

for all \( t \geq 0 \). Then the above relation implies that

\[
\frac{96 \|x\|^2}{\gamma_t (\eta_t - 1 - L_0 - B_L)} \leq \frac{1}{T},
\]

for all \( t \geq 0 \). Finally, we need to show the existence of constants \( R_1 \) and \( R_2 \) satisfying (2.68) and (2.69), respectively. Using the fact that \( \tau_t \geq \frac{384 \|x\|^2}{\alpha_0 (t + 1) T} \), we observe

\[
\frac{96 \|x\|^2}{\gamma_t (\eta_t - 1 - L_0 - B_L)} \leq \frac{384 \|x\|^2}{\alpha_0 (t + 1) T} \left( \frac{\alpha_0 (t + t + 2)}{2} + \frac{\alpha_0 (t + t + 1)}{4} \right) \leq \frac{1}{T}, \]

for all \( i \geq 0, t \geq 0 \). Noting (2.73), (2.74) and (2.69), we can set

\[
R_2 := \frac{2}{T}.
\]

Noting (2.68) along with definition of \( \mathcal{H}_q \) in the theorem statement, setting \( y_0 = 0 \), using (2.73),(2.57), and applying the following relations

\[
\gamma_t \tau_t \geq \max \left\{ \frac{384 \|x\|^2}{\alpha_0 (t + 1) T} \frac{\alpha_0}{2(t + t + 1)}, \right\}
\]

\[
\sum_{t=0}^{T-1} \frac{2\alpha_0^2}{\gamma_0} (t + 1)^2 \leq 4(t + 1) \alpha_0,
\]

\[
\sum_{t=1}^{T-1} \frac{2\eta (t+1)^2}{\gamma_0^2} \leq \frac{B(t+1)^2}{\sigma_\mathcal{M}^2} \left[ \frac{(t+1)^2}{3} + \frac{(t+1)^2(t+2)}{2} + \frac{(t+1)(t+2)(t+1)\alpha_0}{6} - (t + 1) \right],
\]

we have for all \( t \leq T - 1 \), RHS of (2.68) is at most

\[
2 \left\{ 2\alpha_0^2 + 48\|x\|^2 \right\} \left( \frac{(t + 1)^2}{2} + \frac{1}{12} \right) \| y^* \| \|y\|^2 + \frac{8T \mathcal{M}^2}{\alpha_0 (t + 1) T} \left[ \frac{\alpha_0}{2(t + t + 1)} + \frac{\alpha_0}{384 \sigma_\mathcal{M}^2 T} \left( \frac{T^2(t + t + 1)}{2} + \frac{T(t + t + 10)}{6} - (t + 1) \right) \right]
\]

\[
+ \frac{12 \eta}{\alpha_0} \mathcal{M} \left( \frac{(t + 1)^2}{2} + \frac{1}{12} \right) \| y^* \|^2 \|y\|^2 \left\{ 2\alpha_0^2 + 48\|x\|^2 \right\}.
\]
Then, noting \( \frac{1}{t} \leq 1 \) and ignoring \(- (t_0 + 1)\) term, we can set \( R_1 := \left[ 2 \left( 2 \sigma_{0}^2 + 24 (t_0 + 3) \| \sigma \|^2_2 \| y^* \|^2_2 + H_{\ast}^2 + 4 \times 48 (t_0 + 2) B^2 \| \sigma \|^2_2 + 3 \sigma_{0} B \sigma_{X,f} (t_0 + 2) \right)^{3/2} \right]. \) (2.76)

Then using Lemma 2.9 and noting (2.75), we have for all \( t \leq T - 1 \)
\[
E[\| \delta^G_i \|_2^2 ] \leq \begin{cases} 
2 \sigma_{0}^2 & \text{if } \| \sigma \|_2 = \sigma_{f} = 0; \\
R_1 (1 + \frac{2}{t})^{-1} & \text{otherwise}
\end{cases}
\]

Noting the above relation, (2.76) and the definition of \( \zeta \), we have
\[
E[\| \delta^G_i \|_2^2 ] \leq \zeta^2, \quad \forall t \leq T - 1.
\] (2.77)

So according to (2.50) with \( y_0 = 0 \) and using (2.77), we have
\[
E[\psi_{0}(\delta T) - \psi_{0}(y^*)] \leq 2 \left( \frac{1}{T + 2t_0 + 3} \right) \left\{ \frac{\alpha_{3} (t_0 + 1) (t_0 + 2) W(x^*, x_0) + 8 \left( \sigma_{0}^2 + B \| \delta_{X,f} \|_2^2 \right) }{\alpha_{0}^3} + \frac{2B (t_0 + 2)^{1/2} \sigma_{X,f} (t_1 + 1) (T + 2) (t_1 + 1)^{-1/2} }{3} \right\].
\]

Here we used the bound
\[
\sum_{t=1}^{T-1} \frac{2 \sigma_{t}^2}{\tau_{t}} + \frac{\tau_{t-1}}{\tau_{t}} \leq B (t_0 + 1) (T + 1) \quad \forall t \leq T - 1.
\] (2.78)

Noting the bound on \( W(x^*, x_0) \) in the earlier relation, we obtain (2.11). Using (2.52), (2.77) and the bounds in (2.78), we have
\[
E[\| \psi_{T} \|_2 ] \leq \frac{2}{\alpha_{0}} \left\{ \frac{3 (t_0 + 1) (t_0 + 2) W(x^*, x_0) + 13 B (t_0 + 2)^{1/2} \sigma_{X,f} \left( \frac{1}{2} \right) \left( t_1 + 1 \right)^{-1/2} }{2} \right\].
\]

Noting the bound on \( W(x^*, x_0) \) in (2.79), the definition of \( H_{\ast} \), using the fact that \( \frac{T + 2}{3} \leq T^{\gamma} \) and combining the \( T^{\\gamma} \) order terms, we obtain (2.12). From (2.51), we have
\[
E[W(x^*, x^*)] \leq \frac{2}{\alpha_{0}} \left\{ \frac{3 \left( t_0 + 1 \right)^{1/2} (t_0 + 2) W(x^*, x_0) + 12 B (t_0 + 2)^{1/2} \sigma_{X,f} \left( \frac{1}{2} \right) \left( t_1 + 1 \right)^{-1/2} }{2} \right\].
\]

With similar replacements in the above relation as in (2.79), we obtain (2.13). Hence we conclude the proof.

Now, we show proof of Theorem 2.3.

Proof of Theorem 2.3. It is easy to verify that \( \{ \gamma_{t}, \beta_{t}, \eta_{t}, \tau_{t} \} \) set according to (2.19) satisfy (2.29) with \( \alpha_{0} = 0 \). Note that all relations in (2.30) and (2.31) are satisfied if \( M^{2} \leq \frac{\tau_{1} (t_0 + 2) - L_{0} - B L_{f}}{t_0} \). This follows due to the fact that \( \{ \eta \} \) is an non-decreasing sequence, \( \{ \tau \} \) is a non-increasing sequence, \( \beta_{t} = 1 \) for all \( t \geq 0 \) and the definition of \( M \). Then we have
\[
\tau_{1} (t_0 + 2) - L_{0} - B L_{f} \geq \frac{6 M \beta_{t}}{2 M \beta_{t}^{2}} \times \frac{t_0}{t_0} = M^{2}.
\]

Also, since \( \frac{7 \beta_{t}}{2 M \beta_{t}^{2}} \geq \frac{7 B X}{2 M \beta_{t}^{2}} \), we have
\[
\tau_{1} (t_0 + 2) - L_{0} - B L_{f} \geq \frac{192 | \sigma |_{2}^{2}}{t_0} \quad \text{for all } t \geq 0.
\]

In view of the above relation, we have
\[
\frac{96 \| \sigma \|_{2}^{2}}{\gamma_{t}(t_0 + 2) - L_{0} - B L_{f}} \leq \frac{1}{4},
\] (2.80)

hence (2.67) is satisfied. We also need to show the existence of \( R_{1} \) and \( R_{2} \) satisfying (2.68) and (2.69), respectively. Using the fact that \( \gamma_{t}, \eta_{t}, \tau_{t} \) are constants for all \( t \geq 0 \), we have
\[
\tau_{1} \eta_{t} \leq \frac{8 M \beta_{t}}{B X} \quad \text{for all } t \geq 0.
\]

Using (2.80), we obtain
\[
\left( 1 - \frac{96 \| \sigma \|_{2}^{2}}{\gamma_{t}(t_0 + 2) - L_{0} - B L_{f}} \right) \leq \left( 1 - \frac{96 \| \sigma \|_{2}^{2}}{\gamma_{t}(t_0 + 2) - L_{0} - B L_{f}} \right) \leq \left( 1 - \frac{96 \| \sigma \|_{2}^{2}}{\gamma_{t}(t_0 + 2) - L_{0} - B L_{f}} \right) \leq \frac{1}{4},
\]

where in the last relation, we used the fact that \( \sigma_{X,f} \geq D X \| \sigma \|_{2} \). In view of the above relation and (2.69), we can set
\[
R_{2} = \frac{1}{4}.
\] (2.81)

Noting (2.68) along with the fact that \( H_{\ast} \geq H_{0} + H_{f} \| y^* \|_{2} + \frac{L_{f} D X \| y^* \|_{2}}{2} \), setting \( y_0 = 0 \), using (2.80), (2.57), \( \gamma_{t} \tau_{1} \tau_{t} = \tau \geq \sqrt{96 \| \sigma \|_{2}^{2} \tau_{0}} \), we have
\[
\sum_{t=1}^{T} \frac{\gamma_{t} \tau_{1} \tau_{t}}{\gamma_{t}} = \frac{t_0}{t_0} \leq \frac{1}{T} \quad \text{for all } t \leq T - 1,
\]

we can see that the RHS of (2.68) is at most
\[
2 \left( 2 \sigma_{0}^2 + 48 \| \sigma \|_{2}^{2} \| y^* \|_{2}^2 + \frac{8}{3} \delta_{f}^{2} + \frac{7 \| y^* \|_{2}^2 + B \delta_{X}^{2}}{96 \| \sigma \|_{2}^{2}} \right) \geq \frac{B}{\sqrt{H^{2} + 4 B^{2} \| \sigma \|_{2}^{2} + 96 \| \sigma \|_{2}^{2} \tau_{0} \delta_{X}^{2} \tau_{0}}} + 12 \sigma_{X,f} \frac{T}{\tau_{0}} \right]
\]
Our goal in this section is to extend the ConEx method to the nonconvex setting by leveraging the framework of proximal point methods for nonconvex functional constrained optimization. We first recall the assumptions mentioned briefly in Section 2. Note that the last term in the above sequence of relations is a constant satisfying the requirement in (2.82).

Then using Lemma 2.9 and noting (2.81), we have for all \( t \leq T - 1 \)

\[
\mathbb{E}[\|\delta^2 \|_2^2] \leq \begin{cases}
2\sigma_0^2 & \text{if } \|\sigma\|_2 = \sigma_f = 0; \\
R_1(1 + \frac{\nu}{\tau})^{T-1} \leq R_1 e^2 & \text{otherwise}.
\end{cases}
\]

Noting the above relation, (2.82) and the definition of \( \zeta \), we have

\[
\mathbb{E}[\|\delta^2 \|_2^2] \leq \zeta^2, \quad \forall t \leq T - 1.
\]

So according to (2.50) with \( y_0 = 0 \) and using (2.83), we have

\[
\mathbb{E}[\psi(x_T) - \psi_0(x^*)] \leq \frac{1}{\nu}[\eta L_0 + B L_f] W(x^*, x_0) + \frac{2(\zeta^2 + H^2)}{\eta} + 12\sigma_0^2 \frac{1}{\nu}.
\]

Using the bound \( W(x^*, x_0) \leq D_X \), we obtain (2.20). From (2.52) and (2.83), we have for \( T \geq 1 \)

\[
\mathbb{E}[\|\psi(x_T)\|_2^2] \leq \frac{1}{\nu}[\eta L_0 + B L_f] W(x^*, x_0) + \frac{2(\zeta^2 + H^2)}{\eta} + 12\sigma_0^2 \frac{1}{\nu}.
\]

Similarly, using (2.20) and (2.22), it is easy to observe that \( \mathbb{E}[\psi(x_T) - \psi(x^*)] \leq \frac{1}{2} \). Hence we conclude the proof.

3 Proximal Point Methods for Nonconvex Functional Constrained Problems

Our goal in this section is to extend the ConEx method to the nonconvex setting by leveraging the framework of proximal point methods. To achieve this goal, we first need to understand the convergence properties of the proximal point method for nonconvex functional constrained optimization. We first recall the assumptions mentioned briefly in Section 1 for the nonconvex case.

1. \( f_i : X \rightarrow \mathbb{R} \) are nonconvex and Lipschitz-smooth functions satisfying the lower curvature condition in (1.3) with parameters \( \mu_i(\geq 0), i = 0, \ldots, m \).

2. \( \chi_i : X \rightarrow \mathbb{R}, \ i = 0, \ldots, m \) are convex and continuous functions.

Let \( x^* \in X \) be a global optimal solution and \( \psi_0 = \psi_0(x^*) \) be the optimal value of problem (1.1). According to the above assumptions and compactness of \( X \), we have \( \psi_0^* > -\infty \).

It should be noted, however, that solving nonconvex problem (1.1) to the optimality condition in Definition 2.1 is generally difficult. Due to the hardness of the problem, we focus on the necessary condition for guaranteeing local optimality. For this purpose, we need to properly generalize the subdifferential for the objective function \( \psi_0 \) and constraints \( \psi_i \) because they are possibly nonconvex and nonsmooth. Let \( \partial \chi_0 \) and \( \partial \chi_i, i \in [m] \) be the subdifferentials of \( \chi_0 \) and \( \chi_i \). We define

\[
\partial \psi_i(x) = \{ \nabla f_i(x) \} + \partial \chi_i(x), \quad i = 0, \ldots, m.
\]

Note that \( \partial \psi_i = \{ \nabla f_i \} \) when \( \psi_i \) is a "purely" differentiable nonconvex function and \( \partial \psi_i = \partial \chi_i \) when \( \psi_i \) is a nonsmooth convex function.

Using these objects, we can define a Karush-Kuhn-Tucker (KKT) condition for the nonsmooth nonconvex problem (1.1) as follows.

**Definition 3.1.** We say that \( x^* \in X \) is a critical KKT point of (1.1) if \( \psi_i(x^*) \leq 0 \) and \( \exists y^* = [y^{*(1)}, \ldots, y^{*(m)}]^T \geq 0 \) s.t.

\[
y^{*(i)} \psi_i(x^*) = 0, \quad i \in [m],
\]

\[
d(\partial \psi_0(x^*) + \sum_{i=1}^m y^{*(i)} \partial \psi_i(x^*) + N_X(x^*), 0) = 0.
\]

20
The parameters \( \{y^{*\langle i\rangle}\}_{i \in [m]} \) are called Lagrange multipliers. For brevity, we use the notation \( y^* \) and \( \{y^{*\langle 1\rangle}, \ldots, y^{*\langle m\rangle}\}^T \) interchangeably.

To ensure that the KKT condition is necessary to achieve optimality, we need a constraint qualification (CQ). A well-known CQ for smooth nonlinear optimization is the classical Mangasarian-Fromovitz constraint qualification (MFCQ, see [32]). A slightly extended MFCQ to deal with nonsmooth functions is defined as follows.

**Definition 3.2 (MFCQ).** Let \( x \in X \) be a point such that \( \psi_i(x) \leq 0 \) for all \( i \in [m] \). We say that \( x \) satisfies MFCQ if there exists a direction \( z \in -N^*_X(x) \) such that
\[
\max_{v \in \psi_i(x)} v^T z < 0, \quad i \in \mathcal{A}(x),
\]
where \( \mathcal{A}(x) \) denotes the indicator set of all active constraints. Moreover, if \( \mathcal{A}(x) = \emptyset \) then we say that \( x \) satisfies MFCQ.

The following result ensures that the KKT condition in (3.1) is a first-order necessary optimality condition for the composite nonconvex optimization. The proof of this result is given in the appendix.

**Proposition 3.1.** Let \( x^* \) be a local optimal solution of the problem (1.1). If \( x^* \) satisfies MFCQ (Definition 3.2), then there exists \( y^{*\langle i\rangle} \geq 0, \ i \in [m] \) such that (3.1) holds.

In order to describe the stationary condition at the limit points of the solutions generated by the algorithm, we assume that MFCQ holds on an enlarged domain containing all the limit points of the algorithm.

**Assumption 3.2 (Strong MFCQ).** All the points in the feasible set of problem (1.1) satisfy MFCQ.

**Definition 3.3.** We say that a point \( x \in X \) is an \( \epsilon \)-KKT point for problem (1.1) if it is feasible, i.e., \( \psi(x) \leq 0 \), and there exists \( y \geq 0 \) such that
\[
\sum_{i=1}^{m} |y^{\langle i\rangle}\psi_i(x)| \leq \epsilon,
\]
\[
d(\bar{\psi}_0(x) + \sum_{i=1}^{m} y^{\langle i\rangle}\bar{\psi}_i(x) + N_X(x), 0)^2 \leq \epsilon.
\]  
Moreover, we call \( x \) a stochastic \( \epsilon \)-KKT point if \( x \) is a feasible random vector, and \( y \) is a random vector such that (3.3) is satisfied under expectation w.r.t. the random variables involved in the process generating \( x \).

Definition 3.3 describes a type of points which approximately satisfy the KKT condition to a specific accuracy. Particularly, a 0-KKT point satisfies the KKT condition (3.1) exactly and hence is a KKT point. However, as will be shown in our convergence analysis, often it is more convenient to describe convergence in terms of the following measure.

**Definition 3.4.** We say that a point \( \hat{x} \in X \) is an \( (\epsilon, \delta) \)-KKT point for problem (1.1) if there exists a \( \epsilon \)-KKT point \( x \) (\( \psi(x) \leq 0 \)) and
\[
||x - \hat{x}||^2 \leq \delta.
\]  
Moreover, \( \hat{x} \) is a stochastic \( (\epsilon, \delta) \)-KKT point if \( x \) is a stochastic \( \epsilon \)-KKT point and
\[
\mathbb{E}[||x - \hat{x}||^2] \leq \delta.
\]

### 3.1 Basic proximal point method

The main idea of the proximal point method (see Algorithm 2) is to transform the nonconvex problem into a sequence of convex subproblems by adding strongly convex terms to the objective and to the constraints. Specifically, each step of the proximal point algorithm involves a convex subproblem (3.7) with convex constraints. It can be observed that, by adding a strongly convex proximal term, \( \psi_0(x; x_{k-1}) \) is \( \mu_0 \)-strongly convex and \( \psi_1(x; x_{k-1}) \) is \( \mu_1 \)-strongly convex relative to \( W(\cdot, \cdot) \). Hence, if each subproblem is feasible, it must have a unique globally optimal solution.

One way to ensure the well-definedness of each subproblem is always keeping the solutions \( \{x_k\} \) feasible. Nevertheless, feasibility is insufficient to guarantee that the KKT condition holds for the original problem. For illustration, we consider the following example:

\[
\begin{align*}
\min_{x \in [−2,2]} & \quad x - x^2 \\
\text{s.t.} & \quad \frac{1}{2}(x - 1)^2 \leq 0.
\end{align*}
\]
Consider Algorithm 2 applied to the above problem with input $x_0 = 1$, $\mu_0 = 2, \mu_1 \geq 0$ and $W(x, y) = \frac{1}{2}(x - y)^2$. Clearly, $x_0$ is the only feasible solution and also the output of the algorithm. However, the main issue in this setting is that the KKT condition fails at $x_0$, since there does not exist a Lagrange multiplier $y \geq 0$ such that the stationarity condition

$$-1 + y(x - 1) = 0$$

holds at $x = 1$. Due to this issue, it is desired to generate a sequence of strictly feasible solutions $\{x_k\}$, which motivates the following strict feasibility assumption on the initial point.

**Assumption 3.3.** $x_0$ is a strictly feasible solution to the nonlinear functional constraints of problem (1.1), namely, $\psi(x_0) < 0$.

---

**Algorithm 2** Exact Constrained Proximal Point Algorithm

**Input:** Input $x_0$

1: for $k = 1, \ldots, K$ do
2: Set $\psi_i(x; x_{k-1}) := \psi_i(x) + 2\mu_i W(x, x_{k-1}), \quad i = 0, \ldots, m$.
3: Obtain $x_k$ as the optimal solution of the following problem

$$\min_{x \in X} \psi_0(x; x_{k-1})$$

s.t. $\psi_i(x; x_{k-1}) \leq 0, \quad i \in [m].$ \hspace{1cm} (3.7)

4: If $x_{k-1} = x_k$ then return $x_k$.
5: end for
6: return $x_K$

---

Strict feasibility of the generated solutions, as well as some other properties of Algorithm 2 are established by the following theorem.

**Theorem 3.4.** Suppose that Assumption 3.3 holds. Then,

a) all the generated points $x_1, x_2, \ldots, x_k, \ldots$ are strictly feasible for problem (1.1), and $\{\psi_0(x_k)\}$ is a monotonically decreasing sequence;

b) either there exists a $k$ such that $x_k = x_{k-1}$ and $x_k$ is a KKT point of (1.1), or $\{\psi_0(x_k)\}$ is a strictly decreasing sequence that has a limit point $\psi_0 > -\infty$, and we have $\lim_{k \to +\infty} \|x_k - x_{k-1}\| = 0$.

**Proof.** Part a). We prove the property of strict feasibility by induction. First, the strict feasibility of $x_0$ is by Assumption 3.3. Next, assume that our claim holds for $x_{k-1}$, namely $\psi_i(x_{k-1}) < 0$, then $x_{k-1}$ is strictly feasible for the $k$-th subproblem (3.7) with $\psi_0(x_{k-1})$ and $\psi(x_{k-1})$. If $x_k = x_{k-1}$, the claim holds by the induction assumption. Otherwise, by the feasibility of $x_k$ for (3.7), we have $\psi_i(x_k) < \psi_i(x_k; x_{k-1}) \leq 0$ for all $i \in [m]$.

To show the monotonicity of $\{\psi_0(x_k)\}$, we establish some sufficient descent property. Due to the optimality of $x_k$ for solving subproblem (3.7) and the strong convexity of objective $\psi_0(x_{k-1})$, we have for all feasible $x$ that

$$\psi_0(x; x_{k-1}) \geq \psi_0(x_k; x_{k-1}) + \mu_0 W(x, x_k) \geq \psi_0(x_k) + 2\mu_0 W(x_k, x_{k-1}) + \mu_0 W(x, x_k).$$

In above, taking $x = x_{k-1}$ and using the strong convexity of $\omega(x)$ we have

$$\|x_{k-1} - x_k\|^2 \leq \frac{\mu_0}{2\mu} [\psi_0(x_{k-1}) - \psi_0(x_k)],$$

which implies that $\{\psi_0(x_k)\}$ is a decreasing sequence.

Part b). Since we already show that $x_{k-1}$ is strictly feasible, if $x_k = x_{k-1}$, then we conclude from Slater’s condition that $x_k$ is a KKT point. If $x_k \neq x_{k-1}$ for all $k$, then (3.8) implies that $\{\psi_0(x_k)\}$ is strictly decreasing. Since this sequence is lower-bounded, we conclude that $\lim_{k \to +\infty} \psi_0(x_k) = \widetilde{\psi}_0$ for some $\widetilde{\psi}_0 \geq \psi_0^*$. Taking limit $k \to \infty$ in (3.8) we have $\lim_{k \to +\infty} \|x_k - x_{k-1}\| = 0$.

Strict feasibility established in Theorem 3.4 along with Slater’s condition guarantees that there exist Lagrange multipliers $\{y_k\}$ such that $(x_k, y_k)$ satisfies the KKT condition of subproblem (3.7) for each $k \geq 1$.

**Lemma 3.5.** Let $(x_k, y_k)$ be a KKT point of the subproblem (3.7). Then

$$\psi_0(x; x_{k-1}) - \psi_0(x_k; x_{k-1}) + \langle y_k, \psi(x; x_{k-1}) \rangle \geq (\mu_0 + \mu^T y_k) W(x, x_k), \quad x \in X.$$  

(3.9)
Proof. Due to the KKT condition, there exist \( \psi'_0(x_k, x_{k-1}) \in \partial \psi_0(x_k, x_{k-1}) \), \( \psi'_1(x_k, x_{k-1}) \in \partial \psi_1(x_k, x_{k-1}) \) and \( z^* \in N_X(x_k) \) satisfying the condition

\[
\psi'_0(x_k, x_{k-1}) + \sum_{i=1}^m y'_k (i) \psi'_1(x_k, x_{k-1}) + z^* = 0. \tag{3.10}
\]

According to the strong convexity of \( \psi_0() \), \( \psi_1() \), and the fact that \( y_k \geq 0 \), we have

\[
\psi_0(x; x_{k-1}) + \langle y_k, \psi(x; x_{k-1}) \rangle \geq \psi_0(x_k; x_{k-1}) + \langle \psi'_0(x_k; x_{k-1}), x - x_k \rangle + \mu_0 W(x, x_k)
\]

\[
+ \langle y_k, \psi(x_k; x_{k-1}) \rangle + \sum_{i=1}^m y'_k (i) \psi'_1(x_k; x_{k-1}), x - x_k \rangle + (\mu T y_k) W(x, x_k)
\]

\[
= \psi_0(x_k; x_{k-1}) + \langle \psi'_0(x_k; x_{k-1}), x - x_k \rangle + \sum_{i=1}^m y'_k (i) \psi'_1(x_k; x_{k-1}) \rangle, x - x_k \rangle
\]

\[
+ (\mu T y_k) W(x, x_k),
\]

where the last equality follows from the complementary slackness part of the KKT condition. Moreover, appealing to the definition of normal cone and property (3.10), we have that

\[
\langle \psi'_0(x_k; x_{k-1}) + \sum_{i=1}^m y'_k (i) \psi'_1(x_k; x_{k-1}), x - x_k \rangle = -\langle z^*, x - x_k \rangle \geq 0,
\]

Putting the above two inequalities together, we arrive at relation (3.9).

In view of Lemma 3.5 and Theorem 3.4, we develop a boundedness condition on the sequence \( \{y_k\} \).

Proposition 3.6. Suppose that Assumption 3.3 holds. Then, for all \( k \geq 1 \), there exists \( y_k = [y_k^{(1)}, \ldots, y_k^{(m)}] \) such that \( y_k \geq 0 \) and

\[
y_k^{(i)} \psi'_1(x_k; x_{k-1}) = 0, \quad i = 1, \ldots, m, \tag{3.11}
\]

and we have the following boundedness condition:

\[
\|y_k\| \leq \frac{\psi_0(x_{k-1}) - \psi_0(x_k)}{\min_{1 \leq i \leq m} \{-\psi_i(x_{k-1})\}}, \quad k = 1, 2, 3, \ldots \tag{3.12}
\]

Proof. Strict feasibility of \( x_0 \) along with Theorem 3.4a) imply that the subproblem (3.7) in Algorithm 2 satisfies Slater’s condition for all \( k \geq 1 \), which ensures that the KKT condition (3.11) holds at optimality. In particular, the first relation in (3.11) is a direct application of the KKT complementary slackness and the second relation is an application of the KKT stationarity. Similarly, applying Lemma 3.5 and setting \( x = x_{k-1} \) in (3.9) yield

\[
\psi_0(x_{k-1}) - \psi_0(x_k) \geq (\mu_0 + \mu T y_k) W(x_{k-1}, x_k) + 2 \mu_0 W(x_k, x_{k-1}) - \sum_{i=1}^m y_k^{(i)} \psi_i(x_{k-1})
\]

\[
\geq \|y_k^{(i)}\| \cdot \min_{1 \leq i \leq m} \{-\psi_i(x_{k-1})\}. \tag{3.12}
\]

Thus relation (3.12) immediately follows.

We have two comments about the above results. First, in order to show that the KKT solutions \( \{x_k, y_k\}_{k \geq 1} \) are well-defined, we need to ensure that Algorithm 2 generates a path of strictly feasible solutions \( \{x_k\}_{k \geq 0} \). However, to achieve this goal, Algorithm 2 requires an oracle that solves the convex subproblem (3.7) exactly. Since computation of the optimal solution can be impractical for large-scale or stochastic optimization, it is desirable to develop inexact variants of the proximal point method which can deal with approximate solutions for the subproblem (3.7).

Second, we note that the bound on the optimal dual solution \( \|y_k\| \), as provided in Proposition 3.6, depends on the algorithm. For some special cases, the bound is uniform for the whole sequence. For example, if the algorithm generates \( x_k = x_{k-1} \) for some \( k > 1 \), then the stationary point is interior to the inequality constraints and we have \( y_k = 0 \). However, (3.12) does not rule out the possibility that \( \|y_k\| \) tends to infinity when \( x_k \) converges to boundary points. Therefore, it is difficult to claim that the KKT conditions of problem (1.1) will be satisfied at the limit points of \( \{x_k\} \). Due to the above two concerns, we turn our attention to inexact proximal point methods which can deal with approximate solutions for the subproblem (3.7). In particular, we will discuss additional constraint qualifications to ensure that the optimal dual solutions have a uniform bound, and thus to allow us to establish the convergence of the inexact proximal point method to the KKT conditions.

3.2 Inexact proximal point method under MFCQ

In Algorithm 3, we present an inexact extension of the proximal point method, which generalizes Algorithm 2 by allowing the subproblems to be solved approximately. To distinguish exact solutions from approximate ones, we denote the exact solution as \( x^* \) and the corresponding dual solution as \( y_k^* \) hereafter. Since each subproblem (3.7) is solved inexactly, the sequence generated by Algorithm 3 can become infeasible with respect to the original problem. If \( x_{k-1} \) is infeasible
with respect to (1.1), then we can not guarantee feasibility of the subproblem (3.7) in general.
This also implies obtaining bounds on Lagrange multipliers is more challenging for inexact case.
However, we show that if the successive problems are solved accurately enough then both the
strict feasibility of the iterates and the boundedness of \( \{ y_k \} \) can be ensured.

**Algorithm 3 Inexact Constrained Proximal Point Algorithm**

1. Input \( x_0 \) is strictly feasible.
2. for \( k = 1, \ldots, K \) do
3. \( x_k \leftarrow \) an approximate solution of subproblem (3.7).
4. end for
5. \( \hat{k} = \text{argmin}_{1 \leq k \leq K} \psi_0(x_{k-1}) - \psi_0(x_k) \).
6. return \( x_{\hat{k}} \).

Throughout the rest of this subsection, we assume that \( \psi_i(x; x_{k-1}) \) is Lipschitz continuous
with constant \( M_i \), \( i = 0, \ldots, m \). Proposition 3.7 shows that the sequence \( \{ x_k \} \) is strictly feasible
if the subproblem (3.7) is solved accurately enough.

**Proposition 3.7.** In Algorithm 3, suppose that Assumption 3.3 holds and that the subproblem
(3.7), if solvable, is returned an approximate solution \( x_k \) satisfying

\[
\left\{ \frac{M_i}{\mu} \right\} \| x_k - x_k^* \| + \| x_k^* - x_k^0 \| \leq \frac{1}{2} \| x_{k-1} - x_k^0 \|, \quad i \in [m],
\]

(3.13)

where \( x_k^* \) (which depends on \( x_{k-1} \)) is the optimal solution, then the sequence \( \{ x_k \} \) generated
by Algorithm 3 is strictly feasible, i.e. \( \psi(x_k) < 0 \), \( k = 0, 1, 2, 3, \ldots \). Moreover, there exists
subproblem solutions \( \{ x_k^*, \mu_k \} \) satisfying the KKT condition for (3.7).

If we further assume that \( \{ x_k \} \) satisfies:

\[
\left\{ \frac{2M_i}{\mu_0} \| x_k - x_k^* \| + \| x_k^* - x_k^0 \| \leq \| x_{k-1} - x_k^0 \| \right\},
\]

(3.14)

then \( \{ \psi_0(x_k) \} \) is monotonically decreasing and converges to a limit point \( \tilde{\psi}_0 \), and

\[
\lim_{k \rightarrow \infty} W(x_k, x_{k-1}) = \lim_{k \rightarrow \infty} W(x_{k-1}, x_k^0) = 0.
\]

(3.15)

**Proof.** Note that the strict feasibility of \( x_{k-1} \) trivially implies that subproblem (3.7) is solvable.
We want to show that the strict feasibility of \( x_{k-1} \), along with condition (3.13), implies the strict
feasibility of \( x_k \). Therefore, using \( \psi(x_0) < 0 \), it is easy to prove the strict feasibility of the whole
sequence by induction.

Suppose that \( x_{k-1} \) is strictly feasible. From the definition of \( \psi_i(x; x_{k-1}) \) and feasibility of \( x_k^* \),
we have

\[
\psi(x_k) + 2\mu_i W(x_k, x_{k-1}) = \psi(x_k; x_{k-1}) \leq \psi(x_k; x_{k-1}) + M_i \| x_k - x_k^* \| = M_i \| x_k - x_k^* \|,
\]

where the first inequality follows from the Lipschitz continuity of \( \psi_i(x; x_{k-1}) \). Using the triangle
inequality and (3.13), we have

\[
\sqrt{2} \mu_i W(x_k, x_{k-1}) \geq \sqrt{M_i} \| x_k - x_{k-1} \| \geq \sqrt{M_i} \| x_{k-1} - x_k^* \| - \| x_k - x_k^* \|)
\]

\[
\geq 2 \sqrt{M_i} \| x_k - x_k^* \|,
\]

for \( i \in [m] \). Combining the above two results, we have \( \psi(x_k) + \frac{2\mu_i}{2} W(x_k, x_{k-1}) \leq 0 \). If \( x_k \neq x_{k-1} \),
then we have strict feasibility: \( \psi_0(x_k) < 0 \). Otherwise, \( x_k = x_{k-1} \) is strictly feasible. Using the induction
we complete the proof of strict feasibility of the whole sequence \( \{ x_k \} \). Moreover, the existence of \( (x_k^*, y_k) \)
immediately follows from Slater’s condition and the KKT condition.

Applying Lemma 3.5 with \( x = x_{k-1} \) and replacing the saddle point \( (x_k, y_k) \) therein by \( (x_k^*, y_k^*) \),
we deduce

\[
\psi_0(x_{k-1}) = \psi_0(x_{k-1}; x_{k-1})
\]

\[
\geq \psi_0(x_{k-1}; x_{k-1}) - \langle y_k^*, \psi(x_{k-1}; x_{k-1}) \rangle + \langle (\mu_0 + \mu^T y_k^*) W(x_{k-1}, x_k^*) \rangle
\]

\[
\geq \psi_0(x_{k-1}; x_{k-1}) - M_0 \| x_k - x_k^* \| + (\mu_0 + \mu^T y_k^*) W(x_{k-1}, x_k^*).
\]

\[
= \psi_0(x_k) + 2\mu_0 W(x_k, x_{k-1}) - M_0 \| x_k - x_k^* \| + (\mu_0 + \mu^T y_k^*) W(x_{k-1}, x_k^*),
\]

(3.16)

where the second inequality follows from the Lipschitz continuity of \( \psi_0(x_{k-1}; x_{k-1}) \) and the basic
property \( \langle y_k^*, \psi(x_{k-1}; x_{k-1}) \rangle = \langle y_k^*, \psi(x_{k-1}) \rangle \leq 0 \). Moreover, using (3.14) and similar argument
beforehand, we have

\[
\sqrt{2} \mu_0 W(x_k, x_{k-1}) \geq \sqrt{2M_0} \| x_k - x_k^* \|
\]
Putting this result in (3.16), we have
\[ \psi_0(x_k) + \mu_0 W(x_k, x_{k-1}) + (\mu_0 + \mu^T y_{k}^*) W(x_{k-1}, x_{k}^*) \leq \psi_0(x_{k-1}). \] (3.17)
We immediately observe that \( \psi_0(x_k) \) is decreasing. Since \( \psi_0 \) is bounded below, we have the convergence \( \lim_{k \to \infty} \psi_0(x_k) = \tilde{\psi}_0 \) for some \( \tilde{\psi}_0 > -\infty \). Summing up the above relation for \( k = 1, 2, \ldots \), we have
\[ \sum_{k=1}^{\infty} [\mu_0 W(x_k, x_{k-1}) + (\mu_0 + \mu^T y_{k}^*) W(x_{k-1}, x_{k}^*)] \leq \psi_0(x_0) - \tilde{\psi}_0 < +\infty. \] (3.18)
Therefore, the convergence results in (3.15) immediately follow.

**Remark 3.8.** It should be noted that the inexactness criteria include the subproblem optimality criteria for the exact proximal point method (Algorithm 2) as a special case. Specifically, if we set \( x_k = x_k^* \), then (3.13) and (3.14) hold trivially. Hence all our convergence analysis applies to the exact proximal point discussed in subsection 3.1.

The following theorem establishes the asymptotic convergence of the proposed inexact proximal point method under some mild constraint qualification.

**Theorem 3.9.** Suppose that all the assumptions in Proposition 3.7 hold, \( x^* \) is a limit point of the solution sequence and it satisfies MFCQ. If \( x_{j_k} \) is a subsequence converging to \( x^* \), then the dual sequence \( \{y_{j_k}^*\} \) is bounded. Moreover, if \( y^* \) is a limit point of \( \{y_{j_k}^*\} \), then \( (x^*, y^*) \) satisfies the KKT condition (3.1).

**Proof.** First, we establish the convergence of \( \{x_{j_k}^*\} \) to \( x^* \). It immediately follows from the assumption \( \lim_{k \to \infty} x_{j_k} = x^* \) and Proposition 3.7 that \( \lim_{k \to \infty} x_{j_k-1} = x^* \). Applying Proposition 3.7 and the triangle inequality, we have
\[ \lim_{k \to \infty} [x_{j_k}^* - x^*] = \lim_{k \to \infty} \|x_{j_k}^* - x_{j_k-1}\| + \|x_{j_k-1} - x^*\| = 0, \]
which implies that
\[ \lim_{k \to \infty} x_{j_k}^* = x^*. \] (3.19)
We prove the boundedness of the dual subsequence by contradiction. Suppose that \( \{y_{j_k}^*\} \) is unbounded. Passing to any subsequence if necessary, we have \( \lim_{k \to \infty} \|y_{j_k}^*\| = \infty \). In view of the KKT condition, we have
\[ \psi_0(x_{j_k}^*) + (y_{j_k}^*)^T \psi(x_{j_k}^*) \leq \psi_0(x) + (y_{j_k}^*)^T \psi(x) + 2(\mu_0 + \mu^T y_{j_k}^*) W(x, x_{j_k-1}) - W(x_{j_k}^*, x_{j_k-1}). \] (3.20)
for any \( x \in X \). Let \( v_{j_k} = y_{j_k}^*/\|y_{j_k}^*\| \), then \( \|v_{j_k}\| = 1 \), hence \( \{v_{j_k}\} \) must have a convergent subsequence. Without loss of generality, we assume \( \lim_{k \to \infty} v_{j_k} = v^* \). Let us divide both sides of (3.20) by \( \|y_{j_k}^*\| \) and take \( k \to \infty \). In view of (3.19) and \( \lim_{k \to \infty} 1/\|y_{j_k}^*\| = 0 \), we have
\[ (v^*)^T \psi(x^*) = \lim_{k \to \infty} (v_{j_k})^T \psi(x_{j_k}^*) \leq (v^*)^T \psi(x) + 2(\mu^T v^*) W(x, x_{j_k}^*), \quad \forall x \in X, \] (3.21)
which means that \( x^* \) is optimal for minimizing the right side of (3.21). Therefore, the first-order condition implies that
\[ d^T \sum_{i=1}^n \partial \psi_i(x^*) v^*(i) + N_X(x^*), 0 \] (3.22)
Let \( A(x^*) \) be the set of active constraints at \( x^* \). By this definition, for any \( i \notin A(x^*) \), we have \( \psi_i(x^*) < 0 \). Since \( \psi_i \) is continuous and \( \|x_{j_k}^* - x_{j_k-1}\|^2 \) converges to 0, there exists \( k_0 \) such that for all \( k > k_0 \), we have \( \psi_i(x_{j_k}^*; x_{j_k-1}) < 0 \). Hence, according to the KKT complementary slackness condition for the subproblem, \( y_{j_k}^*(i) = 0 \) for \( k > k_0 \). Taking \( k \to \infty \) we obtain \( v^*(i) = 0 \) for any \( i \notin A(x^*) \). So we can rewrite the equation (3.22) as
\[ d^T \sum_{i \in A(x^*)} \partial \psi_i(x^*) v^*(i) + N_X(x^*), 0 \]
Let \( \psi'_i(x^*) \in \partial \psi_i(x^*) \) for \( i \in [m] \) and \( u \in N_X(x^*) \) be such that \( u + \sum_{i \in A(x^*)} \psi'_i(x^*) v^*(i) = 0 \). Let \( z \) be the direction vector defined in MFCQ (3.2). Then,
\[ 0 = z^T u + \sum_{i \in A(x^*)} \psi'_i(x^*) z^T \psi'_i(x^*) \leq \sum_{i \in A(x^*)} \psi'_i(x^*) y^*(i) \max_{z \in \partial \psi_i(x^*)} z^T u < 0, \]
where the first inequality follows since \( z = -N_X(x^*) \) and \( u \in N_X(x^*) \) implies \( z^T u \leq 0 \), the second inequality follows due to the fact that \( v^*(i) \geq 0 \) and \( \psi'_i(x^*) \in \partial \psi_i(x^*) \) and the last strict inequality follows from MFCQ and \( v^*(i) > 0 \) for at least one \( i \in A(x^*) \). Hence we obtain a contradiction and conclude that \( \{y_{j_k}^*\} \) is a bounded sequence. Since \( \{y_{j_k}^*\} \) is bounded, it must have a limit point \( y^* \). Passing to any subsequence if necessary, we have \( \lim_{k \to \infty} y_{j_k}^* = y^* \).
To prove that \((x^*, y^*)\) satisfies the KKT condition, we first show that \((x^*, y^*)\) satisfies complementary slackness. Applying Lemma 3.5 with \(x = x_{k-1}\) and replacing \((x_k, y_k)\) by \((x_k^*, y_k^*)\), we have
\[
\psi_0(x_{k-1}) - \psi_0(x_k^*) \geq 2\mu_0 W(x_k^*, x_{k-1}) + (\mu_0 + \mu^T y_k^*) W(x_{k-1}, x_k^*). \tag{3.23}
\]
Moreover, the KKT condition for the subproblem (3.7) and the assumption (1.5) imply that
\[
-\sum_{i=1}^{m} y_k^{(i)} \psi_i(x_k^*) = 2(\mu^T y_k^*) W(x_k^*, x_{k-1}) \leq 2 L_w (\mu^T y_k^*) W(x_{k-1}, x_k^*). \tag{3.24}
\]
Combining (3.25) and (3.24) and taking the limit \(k \to \infty\), we have
\[
0 \leq - \lim_{k \to \infty} \sum_{i=1}^{m} y_k^{(i)} \psi_i(x_k^*)
\leq \lim_{k \to \infty} 2 L_w \left[ \psi_0(x_{k-1}) - \psi_0(x_k^*) \right]
\leq \lim_{k \to \infty} 2 L_w \left[ \psi_0(x_{k-1}, x_{k-1}) - \psi_0(x_k^*, x_{k-1}) + 2\mu_0 W(x_k^*, x_{k-1}) \right]
\leq \lim_{k \to \infty} 2 L_w \left[ M_0 \|x_{k-1} - x_k^*\| + 2\mu_0 W(x_k^*, x_{k-1}) \right]
\leq 0,
\]
where the last equality is implied by Proposition 3.7 and (1.5). Hence we have
\[
\lim_{k \to \infty} y_k^{(i)} \psi_i(x_k^*) = 0, \quad i = 1, 2, \ldots, m. \tag{3.25}
\]
Passing to the subsequence \((x_{j_k}, y_{j_k}^*)\) that converges to \((x^*, y^*)\), we conclude from (3.25) that
\[
y_k^{(i)} \psi_i(x_k^*) = 0, \quad i = 1, 2, \ldots, m. \tag{3.26}
\]
Next, we check the stationarity condition. Since \((x_{j_k}, y_{j_k}^*)\) converges to \((x^*, y^*)\), taking \(k \to \infty\) in (3.20) yields
\[
\psi_0(x^*) + (y^*)^T \psi(x^*) \leq \psi_0(x) + (y^*)^T \psi(x) + 2(\mu_0 + \mu^T y^*) W(x, x^*), \tag{3.27}
\]
implying that \(x^*\) minimizes the function \(\psi_0(x) + (y^*)^T \psi(x) + 2(\mu_0 + \mu^T y^*) W(x, x^*)\) over \(x \in X\). In other words, we have
\[
0 \in N_X(x^*) + \nabla \psi_0(x^*) + \sum_{i=1}^{m} y_k^{(i)} \nabla \psi_i(x^*). \tag{3.28}
\]
In view of (3.26) and (3.28), we complete the proof.

The following theorem shows the asymptotic convergence and the rate of convergence to stationarity of all the limit points of \(\{x_k\}\). An ingredient vital to the latter result is the uniform boundedness of the dual variables under Strong MFCQ.

**Theorem 3.10.** Suppose that all the assumptions in Proposition 3.7 and Assumption 3.2 hold. Then,

a) all the limit points of the sequence \(\{x_k\}\) are critical KKT points;

b) the whole sequence \(\{y_k^*\}\) is uniformly bounded, namely, there exists some constant \(B > 0\) that \(\|y_k^*\| \leq B\), for \(k = 1, 2, 3, \ldots\);

c) after \(K\) iterations, the solution sequence contains an \((\varepsilon_K, \bar{\varepsilon}_K)\)-KKT point with
\[
\varepsilon_K = \max \left\{ 2L_w, 8L_w (\mu_0 + \|\mu\|_B) \right\} \frac{\psi_0(x_0) - \psi_0^*}{K},
\tag{3.29}
\]
\[
\bar{\varepsilon}_K = \min \left\{ \frac{\psi_0(x_0) - \psi_0^*}{M^2 K^2}, \frac{1}{2\mu_0 K} \right\} \left[ \psi_0(x_0) - \psi_0^* \right],
\tag{3.30}
\]
where \(\bar{M} = \max_{0 \leq i \leq m} \frac{M_i}{\min X}\). Moreover, if Algorithm 3 generates exact solutions \(x_k = x_k^*\), i.e., it is reduced to Algorithm 2, then the solution sequence contains an \(\varepsilon_K\)-KKT point.

**Proof.** Part a): A natural consequence of Assumption 3.2 is that every limit point of \(\{x_k\}\) satisfies MFCQ (Definition 3.2). Then, applying Theorem 3.9, we immediately get part a).

Part b): We show the boundedness of \(\{y_k^*\}\) by contradiction. Suppose that the sequence is unbounded, then there exists a subsequence \(\{y_{j_k}^*\}\) such that
\[
\lim_{k \to \infty} \|y_{j_k}^*\| = \infty. \tag{3.31}
\]
Since \(X\) is compact and \(\{x_{j_k}\}\) is bounded, there exists a limit point \(x^*\) and a subsequence \(\{x_{j_k}\} \subseteq \{x_k\}\) that \(\lim_{k \to \infty} x_{j_k} = x^*\). Due to Assumption 3.2, we obtain that \(x^*\) satisfies MFCQ. However, according to Theorem 3.9, when \(x^*\) satisfies MFCQ, \(\|y_k^*\|\) must be bounded, thereby leading to a contradiction to (3.31). This completes the proof of the existence of a constant \(B > 0\) such that
\[
\|y_k^*\| \leq B \quad k = 1, 2, \ldots.
\]
Part c): Next we establish the rate of convergence to KKT condition. Due to the optimality of \(x_k^*\) in subproblem (3.7), we have
\[
d((\overline{c}\overline{\psi}_0(x_k^*; x_{k-1}) + \sum_{i=1}^{m} y_k^{*(i)} \overline{\psi}_i(x_k^*; x_{k-1}) + N_X(x_k^*), 0) \ni 0.
\]
Plugging the definition of \(\overline{\psi}_0(x_k^*; x_{k-1})\) and \(\overline{\psi}_i(x_k^*; x_{k-1}), i \in [m]\), into the above inequality yields
\[
d((\overline{c}\overline{\psi}_0(x_k^*; x_{k-1}) + \sum_{i=1}^{m} y_k^{*(i)} \overline{\psi}_i(x_k^*; x_{k-1}) + 2(\mu_0 + \mu^T y_k^*) (\nabla \omega(x_k^*) - \nabla \omega(x_{k-1})) + N_X(x_k^*), 0) = 0.
\] (3.32)
It follows that
\[
d((\overline{c}\overline{\psi}_0(x_k^*; x_{k-1}) + \sum_{i=1}^{m} y_k^{*(i)} \overline{\psi}_i(x_k^*; x_{k-1}) + N_X(x_k^*), 0)^2
\leq 4(\mu_0 + \mu^T y_k^*)^2 \|\nabla \omega(x_k^*) - \nabla \omega(x_{k-1})\|^2
\leq 8L^2_\omega(\mu_0 + \mu^T y_k^*)^2 W(x_k, x_{k-1})
\leq 8L^2_\omega(\mu_0 + \|\mu\|_B)[\psi_0(x_{k-1}) - \psi_0(x_k)],
\] (3.33)
where the second inequality uses the smoothness and strong convexity of \(\omega(x)\), and the last inequality follows from (3.17) and \(\mu_0 + \mu^T y_k^* \leq \mu_0 + |\mu|_B\). In addition, by complementary slackness and (3.17) we have
\[
\sum_{i=1}^{m} |y_k^{*(i)} \overline{\psi}_i(x_k^*)| = 2(\mu^T y_k^*) W(x_k^*, x_{k-1}) \leq 2L_\omega(\mu^T y_k^*) W(x_k, x_{k-1})
\leq 2L_\omega |\psi_0(x_{k-1}) - \psi_0(x_k)|.
\] (3.34)
Furthermore, by the distance contraction property (3.13), (3.14) and relation (3.17), we have
\[
|x_k - x_k^*|^2 \leq \frac{1}{4} |x_{k-1} - x_k^*|^2 \leq \frac{1}{4\mu_0}[\psi_0(x_{k-1}) - \psi_0(x_k)],
\] (3.35)
and
\[
|x_k - x_k^*|^2 \leq \min \left\{ \frac{\mu_0}{2M_0}, \min_{1 \leq i \leq m} \frac{\mu_i}{4M_i} \right\} |x_{k-1} - x_k^*|^2
\leq \min \left\{ \frac{1}{M_0}, \min_{1 \leq i \leq m} \frac{\mu_i}{2M_0M_i} \right\} [\psi_0(x_{k-1}) - \psi_0(x_k)]
\leq \frac{1}{M}[\psi_0(x_{k-1}) - \psi_0(x_k)].
\] (3.36)
Noting that \(\hat{k} = \arg\min_{1 \leq k \leq K} \psi_0(x_k) - \psi_0(x_k)\), we have
\[
\min_{1 \leq k \leq K} [\psi_0(x_{k-1}) - \psi_0(x_k)] \leq \frac{1}{K} \sum_{k=1}^{K} [\psi_0(x_{k-1}) - \psi_0(x_k)] \leq \frac{\psi_0(x_k) - \psi_0(x_k)}{K}.
\] (3.37)

Let \(x^k\) be the desired approximate KKT point. Combining (3.33), (3.34) and (3.37) we obtain (3.29), and combining (3.35), (3.36) and (3.37) gives (3.30). Finally, noting that an \((\epsilon, \delta)\)-KKT point is an \(\epsilon\)-KKT point when \(\delta = 0\), we immediately see that the exact proximal point method generates an \(\epsilon\)-KKT solution.

**Remark 3.11.** We leave several comments about the above convergence results. First, imposing MFCQ type assumption is quite common in the traditional nonlinear optimization for justifying the search of KKT solutions (see [6]). By means of MFCQ, we not only ensure asymptotic convergence to the KKT solution but also obtain a non-asymptotic convergence rate result. To the best of our knowledge, this appears to be the first complexity result and efficiency analysis of proximal point method under the MFCQ type assumption Second, while Assumption 3.2 requires MFCQ on the whole feasible domain, we only need every limit point of \(\{x_k\}\) to satisfy MFCQ throughout the proof. Strong MFCQ is an algorithm independent sufficient condition to achieve MFCQ on every limit point. Third, it should be noted that while Algorithm 3 generates approximate KKT points that are strictly feasible, feasibility is actually not essential for the underlying optimality measure in Definition 3.1. The next subsection will describe inexact proximal point method with possibly infeasible solution sequence. (Also see Remark 3.14).

**Remark 3.12.** All the results in this section can be easily extended to the case when \(\psi_i, i \in [m]\) are convex functions. In that case, we can replace \(\mu_i = 0\) for all \(i \in [m]\). As a result, the subproblem (3.7) of Algorithm 2 becomes
\[
\min_{x \in \mathcal{X}} \psi_0(x; x_{k-1})
\text{ s.t. } \psi_i(x) \leq 0, \quad i \in [m].
\] (3.38)
Hence, constraints are fixed for all iterations. This implies that we do not need (3.13) for ensuring the strict feasibility of iterates for every subproblem (3.38). It is given for free provided that there exists some (possibly unknown) strictly feasible solution for problem (3.38). However, we still need (3.14) for ensuring the convergence result in (3.15). After this modification in Proposition 3.7, it would be easy to obtain results similar to those in Theorem 3.9 and Theorem 3.10.
Remark 3.13. It is interesting to compare the complexity of proximal point iterations in Theorem 3.10 with that of proximal point for unconstrained nonconvex optimization (e.g. [25]). A quantitative difference is that for functional constrained problem the complexity bound has an unknown parameter $B$ which could possibly depend on the solution path, while for unconstrained problem, the parameters in the complexity bound usually globally depend on the initial solution, optimality gap or distance to the optimal solutions, and hence are easier to estimate. This distinction appears to indicate some substantial difficulty in developing complexity analysis for nonconvex optimization with nonconvex constraints.

Remark 3.14. The inexactness criteria (3.13) and (3.14) describe convergence to the optimal solution for each convex subproblem. These criteria can be satisfied eventually if we employ ConEx for the subproblems, thanks to the last iterate convergence (2.13). However, it is still difficult to estimate the total complexity, since ConEx does not guarantee convergence purely in terms of initial distance to the optimal solution. This difficulty holds for deterministic as well as stochastic cases. Due to these issues, it is desirable to exploit more practical scenarios where proximal point methods can be combined with first order methods (such as ConEx) for large-scale and stochastic optimization.

3.3 Inexact proximal point under the strong feasibility assumption

In this section, we present a variant of the inexact proximal point in which the subproblem is approximately solved by ConEx (see Algorithm 4). To understand our motivation, consider the case when the objective function is given in the form of $f(x) = E_\xi [F(x, \xi)]$, where $F(x, \xi)$ is a stochastic function on some random variable $\xi$ and is possibly nonconvex with respect to the parameter $x$. Consequently, the objective function in the subproblem (3.7) is given by $E_\xi [F(x, \xi)] + \mu_0 \|x - \bar{x}\|^2$. As discussed in the previous section, stochastic optimization algorithms (such as ConEx) for solving this type of problem will exhibit a sublinear rate of convergence in expectation, which does not fit the inexactness criterion raised in Proposition 3.7. To alleviate this issue, we propose a new assumption as follows.

**Assumption 3.15 (Strong feasibility).** There exists $\bar{x} \in X$ such that
\[
\psi_i(\bar{x}) \leq -2\mu_i D_X^2, \quad i = 1, \ldots, m,
\]
where $D_X$ is defined in (1.6).

The following proposition states that the KKT condition in (3.1) is a first-order necessary optimality condition when Assumption 3.15 holds. This result is similar to Proposition 3.1 where MFCQ is replaced by strong feasibility. We defer the proof of this result to the appendix.

**Proposition 3.16.** Let $x^*$ be a local optimal solution of the problem (1.1). If Assumption 3.15 is satisfied then there exists $y^{*(i)} \geq 0$, $i \in [m]$ such that (3.1) holds.

**Algorithm 4 Inexact Constrained Proximal Point Algorithm with ConEx**

1: Input $x_0$
2: for $k = 1, \ldots, K$ do
3: \hspace{1cm} $x_k \leftarrow$ a (stochastic) approximate solution of subproblem (3.7) by ConEx.
4: end for
5: Choose $\hat{k}$ uniformly at random from \{1, 2, ..., $K$\}.
6: return $x_{\hat{k}}$.

Note that (3.39) is a local and a verifiable condition. Moreover, we will show in Lemma 3.17 that it provides a computable uniform bound $B$ on the dual solutions. While it appears that (3.39) is quite distinct from MFCQ (Definition 3.2), we would like to point out certain similarities between these two conditions. To understand this connection better, let us assume that $\psi_i$ is smooth function. Then, for all $x \in X$, we have
\[
\psi_i(\bar{x}) \geq \psi_i(x) + \langle \nabla \psi_i(x), \bar{x} - x \rangle - \frac{\mu_i}{2} \|\bar{x} - x\|^2
\]
\[
\Rightarrow \langle \nabla \psi_i(x), x - \bar{x} \rangle \geq \psi_i(x) - \psi_i(\bar{x}) - \frac{\mu_i}{2} \|\bar{x} - x\|^2,
\]
which implies that
\[
\langle \nabla \psi_i(x), x - \bar{x} \rangle \geq 0, \quad \forall x \in X \cap \{\psi_i \geq -\frac{3}{2} \mu_i D_X^2\}.
\]

(3.40)
Recall that the existence of a Minty solution, \( \bar{x} \), for variational inequality problem on mapping \( \nabla \psi_k \), is the following condition
\[
\langle \nabla \psi_k(x), x - \bar{x} \rangle \geq 0, \quad \forall x \in \mathcal{X},
\]  
which is stronger than (3.40). Hence \( \psi \) satisfying (3.39) is not necessarily quasi-convex. However, existence of Minty solution, \( \bar{x} \), gives an ‘almost’ sufficient condition for ensuring (3.2) in the following way. Set \( x = x^* \) in (3.41). Then we obtain that \( z = \bar{x} - x^* \) satisfies (3.2) with strict inequality replaced by nonstrict inequality. Since there is no implication from (3.40) to (3.41) (in fact, the implication is in the opposite direction), a direct comparison for the weaker among the two conditions (3.39) and (3.2), can not be made as such.

Now, we state an important lemma which allows us to obtain dual boundedness under the strong feasibility assumption.

**Lemma 3.17.** Let \( \psi_i(x; \hat{x}) := \psi_i(x) + 2\mu_i W(x; \hat{x}), 0 \leq i \leq m \), where \( \hat{x} \in \mathcal{X} \) is an arbitrary proximal center. Then under Assumption 3.15, the convex problem:
\[
\min_{x \in \mathcal{X}} \psi_i(x; \hat{x}) \quad \text{s.t.} \quad \psi_i(x; \hat{x}) \leq 0, \quad i \in [m].
\]  
is always feasible, there exists a solution \( (x^+, y^+) \) satisfying the KKT condition, and the variable \( y^+ \) is uniformly bounded by:
\[
\|y^+\|_1 \leq B := \frac{\nu_0(\bar{x}) - \nu^*_p + \mu_D^2}{\mu_{\min} D^2},
\]  
where \( \mu_{\min} = \min_{1 \leq i \leq m} \mu_i \).

**Proof.** Based on Assumption 3.15, we have from subproblem (3.42) that
\[
\psi_i(\bar{x}, \hat{x}) \leq -2\mu_i D^2 + 2\mu_i W(\bar{x}, \hat{x}) \leq -\mu_i D^2_Y < 0.
\]  
Then the existence of KKT solution \((x^+, y^+)\) immediately follows from Slater’s condition.

Moreover, notice that problem (3.42) differs from problem (3.7) only at the choice of proximal center. Therefore, the same argument to prove Lemma 3.5 implies that
\[
\psi_0(x, \hat{x}) - \psi_0(x^+; \hat{x}) + \langle y^+, \psi(x; \hat{x}) \rangle \geq (\mu_0 + \mu^T y^+) W(x, x^+), \quad x \in \mathcal{X}.
\]  
Let us place \( x = \hat{x} \) in the above inequality. In view of the non-negativity of \((\mu_0 + \mu^T y^+)\) and the definition of \( \psi_0(x, \hat{x}) \), one has
\[
\psi_0(\bar{x}) + 2\mu_0 W(\bar{x}, \hat{x}) - \psi_0(x^+, \hat{x}) - 2\mu_0 W(x^+, \hat{x}) \geq \langle y^+, \psi(\bar{x}, \hat{x}) \rangle.
\]  
Combining the result (3.44) and (3.46) together, we deduce
\[
\mu_{\min} \|y^+\|_1 D^2_Y \leq (\mu^T y^+) D^2_Y \leq -\sum_{i=1}^m y^{+(i)} \psi_i(\bar{x}, \hat{x}) \leq \psi_0(\bar{x}) - \psi_0(x^+) + \mu_0 D^2_Y.
\]  
Finally, since \( x^+ \) is feasible to (3.42) and the feasible region of (3.42) is a subset of the feasible region of the original problem (1.1), we have \( \psi_0(x^+) \geq \psi^*_p \). The result (3.43) immediately follows.

Note that in view of Lemma 3.17, the strong feasibility assumption ensures two key requirements for the analysis of the convergence rate of the inexact proximal point: 1) feasibility of proximal point subproblems and 2) a uniform bound on the optimal dual variable \( y^+_k \). To see the second part, placing \( x^{k-1} = \hat{x} \) in (3.42), we immediately obtain the bound
\[
\|y^+_k\|_1 \leq B = \frac{\nu_0(\bar{x}) - \nu^*_p + \mu_D^2}{\mu_{\min} D^2}, \quad k = 1, 2, \ldots.
\]  
In this case, we only need to assume that \( x_k \) satisfies the functional optimality gap and constraint violation given in Definition 2.1 as compared to (3.13) and (3.14) in the previous section.

We develop the convergence result of Algorithm 4 in the following theorem.

**Theorem 3.18.** In Algorithm 4, suppose that Assumption 3.15 holds such that \( \|y^+_k\|_1 \leq B \) and \( B \) is given in Lemma 3.17. Moreover, assume that the definition of \( x_k \) in Algorithm 4 is given by
\[
x_k \leftarrow \text{a stochastic}(\delta_k, \delta_k)\text{-optimal solution (c.f. Definition 2.1) of (3.7).}
\]  
Then \( x_k \) is a stochastic \((\varepsilon_K, \bar{\varepsilon}_K)\)-KKT point of Problem (1.1) with
\[
\varepsilon_K = \max \{2L_{\min}, 8L_{\max}^2 \mu_0 + \mu_{\max} B \} \Gamma, \quad \text{and} \quad \bar{\varepsilon}_K = \frac{2}{\mu_{\max}} \Omega_K,
\]  
where \( \mu_{\max} := \max_{1 \leq i \leq m} \mu_i \), \( \Gamma := \Delta \psi_0 + B \Delta_0 + \Omega_K \), \( \Delta \psi_0 := \psi_0(\bar{x}) - \min_{x \in \mathcal{X}} \psi_0(x) \), \( \Delta_0 = \|\psi(x_0)\|_2 \) and \( \Omega_K = \sum_{k=1}^{K} \delta_k + B \sum_{k=1}^{K} \delta_k \).
Proof. Let $\Delta_k = \psi_0(x_k; x_{k-1}) - \psi_0(x_k^*; x_{k-1})$ and $\tilde{\Delta}_k = ||[\psi_0(x_k; x_{k-1})]_+||_2$. Using Definition 2.1 we have $E[|\Delta_k|] \leq \delta_k$ and $E[\tilde{\Delta}_k] \leq \tilde{\delta}_k$. In view of Lemma 3.5 and the strong convexity of $\psi_0(x_{k-1})$ and $\psi(x_{k-1})$, we have

$$\psi_0(x_k; x_{k-1}) + \sum_{i=1}^m y_k^{(i)} \psi_i(x_k; x_{k-1}) \geq \psi_0(x_k^*; x_{k-1}) + (\mu_0 + \mu^T y_k^*)_0 W(x_k; x_{k-1})$$

$$= \psi_0(x_k; x_{k-1}) - \Delta_k + (\mu_0 + \mu^T y_k^*)_0 W(x_k; x_{k-1})$$

$$= \psi_0(x_k) + 2\mu_0 W(x_k, x_{k-1}) - \Delta_k + (\mu_0 + \mu^T y_k^*)_0 W(x_k, x_{k-1}).$$

(3.50)

Setting $x = x_k$ in (3.50) yields

$$\psi_0(x_k; x_{k-1}) + \sum_{i=1}^m y_k^{(i)} \psi_i(x_k; x_{k-1}) \geq \psi_0(x_k^*; x_{k-1}) + (\mu_0 + \mu^T y_k^*)_0 W(x_k, x_{k-1}).$$

Setting $k = \tilde{k}$ in the above relation and taking expectation, we have

$$E[|x_k^* - x_{k-1}|^2] \leq E[W(x_{k-1}, x_k^-)] \leq \frac{2}{\mu_0} \sum_{k=1}^K E[\psi_0(x_k; x_{k-1}) - \psi_0(x_{k-1}) + \mu_0 + \mu^T y_k^*)_0 W(x_k, x_{k-1})]$$

$$\leq \frac{2}{\mu_0} \sum_{k=1}^K E[\psi_0(x_k; x_{k-1}) - \psi_0(x_{k-1}) + \sum_{i=1}^m y_k^{(i)} \psi_i(x_k; x_{k-1})]$$

$$\leq \frac{2}{\mu_0} \sum_{k=1}^K E[\Delta_k + B\tilde{\Delta}_k]$$

$$\leq \frac{2}{\mu_0} \sum_{k=1}^K (\tilde{\delta}_k + B\tilde{\delta}_k),$$

where the third inequality above is due to the Cauchy-Schwarz inequality and the boundedness of $\|y_k^*\|_2; \|y_k^*\|_2 \leq \|y_k^*\|_1 \leq 1 \leq B$.

Analogously, by setting $x = x_{k-1}$ in (3.50) and noticing $\psi_0(x_{k-1}; x_k - x_{k-1}) = \psi_0(x_{k-1})$ we have

$$\psi_0(x_k; x_{k-1}) + B\tilde{\Delta}_k \geq \psi_0(x_k; x_{k-1}) + ||y_k^*||_2 \tilde{\Delta}_k$$

$$\geq \psi_0(x_k; x_{k-1}) + \sum_{i=1}^m y_k^{(i)} \psi_i(x_k; x_{k-1})$$

$$\geq \psi_0(x_k; x_{k-1}) + (\mu_0 + \mu^T y_k^*)_0 W(x_k; x_{k-1})$$

$$\geq \psi_0(x_k) - \Delta_k + 2\mu_0 W(x_k, x_{k-1}) + (\mu_0 + \mu^T y_k^*)_0 W(x_k, x_{k-1}).$$

Here the second inequality use the following property: for $k > 1$, 

$$\sum_{i=1}^m y_k^{(i)} \psi_i(x_{k-1}; x_k - x_{k-1}) \leq \sum_{i=1}^m [y_k^{(i)} \psi_i(x_{k-1}; x_k - x_{k-1})]$$

and

$$\sum_{i=1}^m y_k^{(i)} \psi_i(x_0; x_0) \leq \sum_{i=1}^m y_k^{(i)} \psi_i(x_0) \leq \|y_k^*\|_2 \tilde{\Delta}_k.$$ 

Summing up the inequality (3.51) for $k = 1, \ldots, K$, we obtain

$$2\mu_0 \sum_{k=1}^K W(x_k, x_{k-1}) + \sum_{k=1}^K (\mu_0 + \mu^T y_k^*)_0 W(x_k; x_{k-1})$$

$$\leq \psi_0(x_0) - \psi_0(x_K) + \sum_{k=1}^K \Delta_k + B\sum_{k=1}^K \tilde{\Delta}_k$$

(3.53)

Furthermore, due to the KKT condition for (3.7), we have

$$d(\partial \psi_0(x_k; x_{k-1}) + \sum_{i=1}^m y_k^{(i)} \partial \psi_i(x_k; x_{k-1}) + [X(x_k^*), 0] = 0$$

Plugging the definition of $\partial \psi_0(x_k; x_{k-1})$ and $\partial \psi_i(x_k; x_{k-1}, i \in [m])$ into the above inequality yields

$$d(\partial \psi_0(x_k^*) + \sum_{i=1}^m y_k^{(i)} \partial \psi_i(x_k^*) + 2(\mu_0 + \mu^T y_k^*)_0 \nabla \omega(x_k^*) - \nabla \omega(x_{k-1})) + [X(x_k^*), 0] = 0$$

(3.54)

Let $\tilde{k}$ be the random index from $1, \ldots, K$. Then, in view of (3.54), (3.53) and bound on $\|y_k^*\|_1$, we have

$$E[d(\partial \psi_0(x_k^*) + \sum_{i=1}^m y_k^{(i)} \partial \psi_i(x_k^*) + [X(x_k^*), 0]^2]$$

$$\leq \frac{1}{\mu_0} E\left\{\sum_{k=1}^K (\mu_0^2 + \mu^2 y_k^*)^2 ||\nabla \omega(x_k^*) - \nabla \omega(x_{k-1})||^2\right\}$$

$$\leq \frac{2L^2(\mu_0^2 + \mu^2 B^2)}{\mu_0^2} E\left\{\sum_{k=1}^K (\mu_0 + \mu^T y_k^*)_0 W(x_k, x_{k-1})\right\}$$

$$\leq \frac{2L^2(\mu_0^2 + \mu^2 B^2)}{\mu_0^2} \Gamma_{\tilde{k}}$$

Moreover, using the complimentary slackness for the subproblem and the relation (3.53), we have

$$\sum_{k=1}^K \sum_{i=1}^m y_k^{(i)} \psi_i(x_k^*) = 2\sum_{k=1}^K (\mu^T y_k^*)_0 W(x_k^*, x_{k-1})$$

$$\leq 2L \sum_{k=1}^K (\mu^T y_k^*)_0 W(x_k, x_{k-1})$$

$$\leq 2L \omega \Delta_f + \sum_{k=1}^K \Delta_k + B\sum_{k=1}^K \tilde{\Delta}_k.$$
Therefore
\[ E\left[ \sum_{k=1}^{m} \left\| y_k^{(i)} \psi_i(x_k^i) \right\| \right] = \frac{1}{K} E\left[ \sum_{k=1}^{K} \sum_{i=1}^{m} \left\| y_k^{(i)} \psi_i(x_k^i) \right\| \right] \leq \frac{2\delta_k^\epsilon}{K} \Gamma K. \]
Hence we conclude the proof.

**Remark 3.19.** We should note that when \( \psi_i, i \in [m], \) are convex functions then we can obtain a variant of Algorithm 4 where \( x_k \) is a (stochastic) \( (\delta_k, \delta_k) \)-optimal solution of (3.38). For this variant of Algorithm 4, we can easily obtain the result of Theorem 3.18 under uniform boundedness of the Lagrange multiplier. Moreover, since constraints remain the same in (3.38) for all \( k \geq 1, \) we just need Slater’s condition to ensure the uniform boundedness of \( \{y_k^k\}. \)

In the following corollary, we state an immediate consequence of Theorem 3.18 as well as the final complexity when using the ConEx method as subroutine to solve subproblem 3.7. Before proceeding to the details of the corollary, we need to properly redefine \( B \) such that it satisfies \( B \geq \max\{\|y_k^k\|_1, \|y_k^k\|_2 + 1\}. \) This allows the use of \( B \) in the sense of Theorem 3.18 as well as in the stepsize policy for the ConEx method in (2.10).

**Corollary 3.20.** Under the assumptions of Theorem 3.18, suppose that in Algorithm 4, we set \( \delta_k = \delta \delta_k \) for some \( c > 0, \) and \( \delta_k = \epsilon/(2c_1c_2), \) where
\[ c_1 = \max\{2L_w, 8L^2_\mu (\mu_0 + \mu_{\max} B)\} \]
\[ c_2 = c + B \]
Then after running at most \( K = 2c_1(\Delta_f + B \Delta_0)/\epsilon \) iterations, we obtain an \( (\epsilon, \frac{2\epsilon}{\mu_{\max}}) \)-KKT point of Problem (1.1). In particular, if we run Algorithm 1 for subproblem (3.7), then we obtain an \( (\epsilon, \frac{2\epsilon}{\mu_{\max}}) \)-KKT point in \( O(\sqrt{T_\epsilon}) \) iterations, where \( T_\epsilon \) is defined in (2.14).

**Proof.** Suppose \( \delta_k \) and \( \delta_k \) are constants throughout Algorithm 4. Then, according to (3.49), we have \( \epsilon_K \leq c_1 \Gamma K/K. \) Choosing given values of \( \delta_k, \delta_k \) and \( K, \) we have
\[ \epsilon_K \leq c_1 \frac{\Gamma K}{K} = c_1 \frac{\Delta_f + B \Delta_0}{K} + (c + B) \delta_k = c_1 \frac{\Delta_f}{2c_1} + c_2 \frac{\epsilon}{2c_1c_2} = \epsilon. \]
Moreover, we have
\[ \epsilon_K = \frac{2\epsilon}{\mu_{\max}} \Omega K \leq \frac{2\epsilon}{\mu_{\max}} \Gamma K \leq \frac{2\epsilon}{\mu_{\max}}. \]
Now noting that \( \delta_k = \delta_k = O(\epsilon) \) is a constant and using Corollary 2.2, we obtain \( (\delta_k, \delta_k) \)-approximate solution of subproblem (3.7) in \( T_\epsilon \) iterations. Noting the definition \( K \) in the statement of the corollary, we conclude the proof.

In the above corollary, we assume that the subproblem (3.7) is solved using the ConEx method. Since \( \|y_k^k\|_2 \leq \|y_k^k\|_1, \) we have an upper bound \( B \) on \( \|y_k^k\|_2 \) in (3.47). Then, we can set the parameter \( B \) of ConEx method, (call it \( B_{\text{ConEx}} \) to avoid confusion with \( B \) in (3.47)) in Theorem 2.1 appropriately. In particular, we can set \( B_{\text{ConEx}} = B + 1 \) which gives accelerated convergence for smooth problems. Moreover, if \( \chi_i(x) \) is a simple function such that we can compute \( \text{prox} \) operator in (2.3) for functions \( \mu_i W(x, x_{k-1}) + \chi_i(x), \) \( i = 1, \ldots, m, \) efficiently, then we solve each subproblem in the smooth strongly convex setting, since \( f_i, i = 1, \ldots, m \) are smooth functions. Otherwise, we must include the nonsmooth convex function \( \chi_i(x) \) in totality (or part thereof) with \( f_i, \) and then we can assume \( \mu_i W(x, x_{k-1}) \) is a simple function. In this case, we solve the subproblems in a nonsmooth strongly convex setting. We can derive from Corollary 3.20 and the definition of \( T_\epsilon \) in (2.14) the final complexity bounds for different problem settings.

- **Smooth nonconvex case:** In this case, \( T_\epsilon \) can be bounded \( O(1/\epsilon^{1/2}) \) in the deterministic case, \( O(1/\epsilon) \) in the semi-stochastic case and \( O(1/\epsilon^2) \) in the fully-stochastic case. Hence, in view of Corollary 3.20, we can compute an \( (\epsilon, 2\epsilon/(\mu_{\max} C_1)) \)-KKT point of the nonconvex problem (1.1) in \( O(1/\epsilon^{3/2}) \), \( O(1/\epsilon^2) \), and \( O(1/\epsilon^3) \) iterations for the deterministic case, semi-stochastic case and fully-stochastic cases, respectively.

- **Nonsmooth nonconvex case:** In this case, \( T_\epsilon \) can be bounded by \( O(1/\epsilon) \) in the deterministic case, \( O(1/\epsilon) \) in the semi-stochastic case and \( O(1/\epsilon^2) \) in the fully-stochastic case. Hence, in view of Corollary 3.20, we can compute an \( (\epsilon, 2\epsilon/(\mu_{\max} C_1)) \)-KKT point of the nonconvex problem (1.1) in \( O(1/\epsilon^2) \) iterations for the deterministic and semi-stochastic cases, and \( O(1/\epsilon^3) \) iterations for the fully-stochastic case.

Note that the dependence of these complexity bounds on different problem parameters can be made more precise in view of the definition of \( T_\epsilon \) in (2.14).
4 Conclusion

This paper focuses on stochastic first-order methods for solving both convex and nonconvex functional constrained problems. For the convex case, we present a novel ConEx method which utilizes linear approximations of constraint functions to define the extrapolation step. This method is a simple and unified algorithm that can achieve the best-known convergence rates for solving different functional constrained convex composite problems. In particular, we show that ConEx attains a few new complexity results especially for the stochastic constrained setting and/or when the objective/constraint functions contains smooth components. For the nonconvex case, we present new proximal point methods which successively generate strictly feasible solutions from a sequence of strongly convex subproblems. Under some standard MFCQ type assumptions, we establish both asymptotic convergence to the KKT condition and iteration complexities to attain some approximate KKT points. Under a different strong feasibility assumption, we establish the convergence of inexact proximal point methods without requiring feasible solutions. This is particularly attractive for large-scale and stochastic optimization where high accuracy is unachievable. Efficiencies of the proximal point method which uses ConEx to solve the subproblems are established under different problem settings.

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A Proof of Proposition 3.1

Let us denote
\[ \tilde{\psi}_i(x) := \psi_i(x) + \mu_i W(x, x^*), \quad i = 0, \ldots, m. \]  
(A.1)
It is easy to see that \( \tilde{\psi}_0(x) \) and \( \tilde{\psi}_i(x) \), \( i \in [m] \), are convex functions. Moreover, their respective subdifferentials can be written as
\[ \partial \tilde{\psi}_i(x) = \{ \nabla f_j(x) + \mu_i \nabla W(x, x^*) \} + \partial \chi_i(x), \]
where \( \nabla W \) is the gradient in the first variable. Note that \( \nabla W(x^*, x^*) = 0 \). Consider the constrained convex optimization problem:
\[ \min_{x \in X} \tilde{\psi}_0(x) \quad \text{s.t.} \quad \tilde{\psi}_i(x) \leq 0, \quad i \in [m]. \]  
(A.2)
Note that \( x^* \) is a feasible solution of this problem. For sake of this proof, define \( \Psi_k(x) := \tilde{\psi}_0(x) + \frac{1}{2} \sum_{i=1}^m (\tilde{\psi}_i(x))^2 + \frac{1}{2} \| x - x^* \|_2^2 \). Let \( S = \{ x \in X : \| x - x^* \|_2 \leq \varepsilon \} \) for some \( \varepsilon > 0 \) such that any \( x \in S \) which is feasible for (A.2) satisfies \( \tilde{\psi}_0(x) \geq \tilde{\psi}_0(x^*) \). Let \( x_k := \arg\min_{x \in S} \Psi_k(x) \).

This is well-defined since \( \Psi_k \) is a strongly convex function. Note that
\[ \liminf_{k \to \infty} \Psi_k(x_k) \leq \limsup_{k \to \infty} \Psi_k(x_k) \leq \limsup_{k \to \infty} \Psi_k(x^*) = \tilde{\psi}_0(x^*) < \infty, \]  
(A.3)
where second inequality follows from the fact that \( x_k \) is optimal and \( x^* \in S \) is a feasible point. Note that as \( k \to \infty \), we have \( \limsup_{k \to \infty} \Psi_k(x_k) < \infty \Rightarrow \limsup_{k \to \infty} \tilde{\psi}_i(x_k) \leq 0 \).

Moreover, note that \( \text{dom}(\liminf_{k \to \infty} \Psi_k) \subseteq \{ x : \psi_i(x) \leq 0, i \in [m] \} \). Also note that \( \text{dom}(\liminf_{k \to \infty} \Psi_k) \cap \text{S} \neq \emptyset \) since both sets contain \( x^* \). Then, definition of set \( S \) implies \( \psi_0(x) \geq \psi_0(x^*) \) for all \( x \in \text{dom}(\liminf_{k \to \infty} \Psi_k) \cap \text{S} \). Hence, \( \liminf_{k \to \infty} \Psi_k(x_k) \geq \liminf_{k \to \infty} \tilde{\psi}_0(x_k) \geq \tilde{\psi}_0(x^*) \). This inequality with (A.3) implies that \( \liminf_{k \to \infty} \Psi_k(x_k) = \tilde{\psi}_0(x^*) \) and \( x_k \to x^* \). Hence, there exists \( k \) such that for all \( k > k_0, x_k \in \text{int}(S) \). So for such \( k \) we can write the following first-order criterion for convex optimization ([91] 2.3.6.2): \( 0 \in N_X(x_k) + \partial \tilde{\psi}_0(x_k) + k \partial \tilde{\psi}(x_k) + x_k - x^* \).

This implies that \( x_k \) is also the optimal solution of \( \min_{x \in X} \tilde{\psi}_0(x) + k \tilde{\psi}(x_k) + \frac{1}{2} \| x - x^* \|_2^2 \).

For simplicity, let us denote \( v_k = k [ \tilde{\psi}(x_k) ]_+ \). Due to the optimality of \( x_k \) of solving the above, we have
\[ \tilde{\psi}_0(x_k) + v_k^T \tilde{\psi}(x_k) + \frac{1}{2} \| x_k - x^* \|_2^2 \leq \tilde{\psi}_0(x) + v_k^T \tilde{\psi}(x) + \frac{1}{2} \| x - x^* \|_2^2, \quad \forall x \in X. \]  
(A.4)

We claim that \( \{ v_k \} \) is a bounded sequence. Indeed, if this is true, then we can find a convergent subsequence \( \{ v_{i_k} \} \) with \( \lim_{k \to \infty} v_{i_k} = v^* \). Taking \( k \to \infty \) in (A.4), we have
\[ \limsup_{k \to \infty} \psi_0(x_{i_k}) + v_{i_k}^T \psi(x_{i_k}) \leq \tilde{\psi}_0(x) + v^T \tilde{\psi}(x) + \frac{1}{2} \| x - x^* \|_2^2, \quad \forall x \in X. \]  
(A.5)

Placing \( x = x^* \), we have \( \tilde{\psi}_0(x^*) \geq \tilde{\psi}_0(x_{i_k}), \) thus \( \lim_{k \to \infty} \tilde{\psi}_0(x_{i_k}) = \tilde{\psi}_0(x^*) \) based on the lower semicontinuity of \( \tilde{\psi}_0 \). In view of this discussion, \( x^* \) optimizes the right side of (A.5). Thus, applying the first order criterion, we have
\[ 0 \in \partial \tilde{\psi}_0(x^*) + \sum_{i=1}^m v(i)^* \partial \tilde{\psi}(x^*) + N_X(x^*). \]

It remains to apply \( \partial \tilde{\psi}_0(x^*) = \partial \psi(x^*) \) and \( \partial \tilde{\psi}(x^*) = \partial \psi(x^*) \).

In addition, to prove complimentary slackness, it suffices to show when \( \tilde{\psi}_i(x^*) = \psi_i(x^*) < 0 \), we must have \( v(i)^* = 0 \). Since \( x_k \) converges to \( x^* \) and \( \tilde{\psi}_i \) is continuous, there exists some \( \exists k_0 > 0 \), such that \( \tilde{\psi}_i(x_{i_k}) < 0 \) when \( k > k_0 \). Hence \( v_{i_k}^* = 0 \) by its definition. Taking the limit, we have \( v(i)^* = 0 \).

It remains to show the missing piece, that \( \{ v_k \} \) is a bounded sequence. We will prove by contradiction. If this is not true, we may assume \( \lim_{k \to \infty} \| v_k \| = \infty \), passing to a subsequence if necessary. Moreover, define \( y_k = v_k / \| v_k \| \), since \( y_k \) is a unit vector, it has some limit point, let us assume \( \lim_{k \to \infty} y_k = y^* \) for a subsequence \( \{ j_k \} \). Dividing both sides of (A.4) by \( \| v_k \| \) and then passing it to the subsequence \( \{ j_k \} \), we have
\[ \tilde{\psi}_0(x_{j_k}) / \| v_{j_k} \| + y^T_{j_k} \tilde{\psi}(x_{j_k}) + \frac{1}{2} \| x_{j_k} - x^* \|_2^2 \leq \tilde{\psi}_0(x) / \| v_{j_k} \| + y^T_{j_k} \tilde{\psi}(x) + \frac{1}{2} \| x - x^* \|_2^2, \quad \forall x \in X. \]

Taking \( k \to \infty \), we have
\[ y^T \tilde{\psi}(x) \leq y^* T \tilde{\psi}(x), \quad \forall x \in X. \]

Since subsequence \( x_{j_k} \) converges to \( x^* \) and \( \tilde{\psi}_i \) is continuous, we see that \( \tilde{\psi}_i(x_{j_k}) < 0 \) for any \( i \notin A(x^*) \) for \( k \geq k_0 \). This implies \( y_{j_k} = j_k \tilde{\psi}(x_{j_k})_+ = 0 \) for all \( k \geq k_0 \) and for all
\[ i \notin A(x^*). \] So we must have \( 0 \in N_X(x^*) + \sum_{i \in A(x^*)} y^{(i)} \partial \psi_i(x^*). \) Here, we have used the fact that \( \nabla W(x^*, x^*) = 0, \) implying that \( \partial \psi_i(x^*) = \partial \psi_i(x^*) \) for all \( i = 0, \ldots, m. \) Let \( u \in N_X(x^*) \) and \( g_i(x^*) \in \partial \psi_i(x^*), i \in A(x^*) \) be such that
\[
u + \sum_{i \in A(x^*)} y^{(i)} g_i(x^*) = 0.
\]
Then we can derive a contradiction by using MFCQ (Definition 3.2). Assume that \( z \) satisfies MFCQ (3.2). Therefore, we have
\[
v = z^T u + \sum_{i \in A(x^*)} y^{(i)} z^T g_i(x^*) \leq \sum_{i \in A(x^*)} y^{(i)} z^T g_i(x^*) \leq \sum_{i \in A(x^*)} y^{(i)} \max_{\epsilon \in \partial \psi_i(x^*)} z^T \nu < 0,
\]
where first inequality follows since \( z \in -N_X(x^*) \) and \( u \in N_X(x^*) \) hence \( z^T u \leq 0, \) second inequality follows due to the fact that \( y^{(i)} \geq 0 \) and \( g_i(x^*) \in \partial \psi_i(x^*) \) and last strict inequality follows since (3.2) and \( y^{(i)} > 0 \) for at least one \( i \in A(x^*). \)

## B Proof of Proposition 3.16

Let us define \( \bar{\psi}_i, i = 0, \ldots, m \) as in (A.1) where \( x^* \) is a local solution of (1.1) then,
\[
\exists \varepsilon > 0 \text{ s.t. } \psi_0(x) \geq \psi_0(x^*) \text{ for all } x \in \{x \in X : \psi_i(x) \leq 0, i \in [m], \|x - x^*\| < \varepsilon\}
\]
\[
\Rightarrow \exists \varepsilon > 0 \text{ s.t. } \psi_0(x) \geq \psi_0(x^*) \text{ for all } x \in \{x \in X : \bar{\psi}_i(x) \leq 0, i \in [m], \|x - x^*\| < \varepsilon\}
\]
\[
\Rightarrow \exists \varepsilon > 0 \text{ s.t. } \bar{\psi}_0(x) \geq \bar{\psi}_0(x^*) = \bar{\psi}_0(x^*) \text{ for all } x \in \{x \in X : \bar{\psi}_i(x) \leq 0, i \in [m], \|x - x^*\| < \varepsilon\},
\]
where the first implication follows from the fact that \( \bar{\psi}_i(x) \geq \psi_i(x) \) for all \( i \in [m] \) or equivalently \( \{x \in X : \bar{\psi}_i(x) \leq 0, i \in [m], \|x - x^*\| < \varepsilon\} \subseteq \{x \in X : \psi_i(x) \leq 0, i \in [m], \|x - x^*\| < \varepsilon\}, \) and second implication follows from the fact that \( \bar{\psi}_i(x) \geq \psi_i(x). \)

The last statement implies that \( x^* \) is a local optimal solution for the convex problem (A.2). Hence, it is also a global optimal solution. Based on (3.39) from Assumption 3.15, we have,
\[
\bar{\psi}_i(x) = \psi_i(x) + \mu_i W(x, x^*) \leq -2\mu_i D^2 + \mu_i D^2 = -\mu_i D^2 < 0.
\]
Hence, by Slater condition, we have that there exists \( y^* \geq 0 \) such that \( (x^*, y^*) \) satisfy first order KKT-condition for the convex problem (A.2). Thus, we have
\[
\partial \bar{\psi}_0(x^*) + \sum_{i=0}^{m} y^{(i)} \partial \bar{\psi}_i(x^*) + N_X(x^*) \geq 0,
\]
\[
y^{(i)} \bar{\psi}_i(x^*) = 0, \quad i \in [m].
\]
It remains to apply \( \partial \bar{\psi}_i(x^*) = \partial \psi_i(x^*) \) and \( \bar{\psi}_i(x^*) = \psi_i(x^*) \) for all \( i = 0, \ldots, m. \) Hence, we conclude the proof.