Efficient finite-dimensional solution of initial value problems in infinite-dimensional Banach spaces

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21.02.2018

Abstract

We deal with the approximate solution of initial value problems in infinite-dimensional Banach spaces with a Schauder basis. We only allow finite-dimensional algorithms acting in the spaces $\mathbb{R}^N$, with varying $N$. The error of such algorithms depends on two parameters: the truncation parameters $N$ and a discretization parameter $n$. For a class of $C^r$ right-hand side functions, we define an algorithm with varying $N$, based on possibly non-uniform mesh, and we analyse its error and cost. For constant $N$, we show a matching (up to a constant) lower bound on the error of any algorithm in terms of $N$ and $n$, as $N, n \to \infty$. We stress that in the standard error analysis the dimension $N$ is fixed, and the dependence on $N$ is usually hidden in error coefficient. For a certain model of cost, for many cases of interest, we show tight (up to a constant) upper and lower bounds on the minimal cost of computing an $\varepsilon$-approximation to the solution (the $\varepsilon$-complexity of the problem). The results are illustrated by an example of the initial value problem in the weighted $\ell_p$ space ($1 \leq p < \infty$).

Key words: initial value problems, infinite-dimensional Banach space, Schauder basis, Galerkin-type algorithms, finite-dimensional approximation, optimality, complexity

\textsuperscript{1}The author was partially supported by the Polish NCN grant - decision No. DEC-2017/25/B/ST1/00945 and by the Polish Ministry of Science and Higher Education
\textsuperscript{2}The author was partially supported by the Faculty of Applied Mathematics AGH UST dean grant No.15.11.420.038/1 for PhD students and young researchers within subsidy of Ministry of Science and Higher Education
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1 Introduction

Let \((E, \| \cdot \|)\) be an infinite-dimensional Banach space over \(\mathbb{R}\) with a Schauder basis. We study the solution of an initial value problem

\[
z'(t) = f(z(t)), \quad t \in [a, b], \quad z(a) = \eta, \tag{1}
\]

where \(a < b\), \(\eta \in E\) and \(f : E \to E\) is a Lipschitz function in \(E\). The Lipschitz condition implies the existence and uniqueness of a solution \(z : [a, b] \to E\), see e.g. \([4], [8]\) or \([16]\). We aim at approximating the solution \(z\) in \([a, b]\).

Infinite countable systems of the form \((1)\) have been investigated for many years. They often appear in various applications inspired by physical, chemical or mechanical problems, see, for example, \([1], [2], [8], [14]\) or \([19]\). Many basic results have already been surveyed in \([8]\). According to \([8]\), one can distinguish between two main approaches to countable systems: a direct approach, where we look for a sequence \(z(t)\) satisfying the sequence of equations \((1)\), and a Banach space approach, where \(f\) acts in a Banach space, and the solution \(z\) is a Banach space valued function.

In this paper we consider computational aspects of \((1)\). In contrast to theoretical properties of infinite systems, much less is known about efficient approximation of the solutions, see e.g. \([3], [8]\). The authors of most papers concentrate on basic Galerkin-type devices that allow us to truncate the infinite system to a finite-dimensional one. An extensive complexity analysis of problem \((1)\) in a Banach space in the deterministic and randomized settings is recently presented in \([9], [6]\) and \([7]\). The authors assume that computations in the underlying Banach space are allowed. This means, in particular, that the evaluations of the Banach space valued right-hand side function \(f\) and its partial derivatives can be performed with the unit cost. Complexity upper bounds are obtained by a complex multilevel projection algorithm.

It is well known that in the finite-dimensional case, for \(\mathbb{R}^N\)-valued functions \(f\) with finite and fixed \(N\), there is a vast literature devoted to optimal approximation of the solution of \((1)\), see, for example, \([5], [11], [12], [13]\), or many other papers.

Motivated by computer applications, we restrict ourselves in this paper to algorithms for \((1)\) that are only based on finite-dimensional computations. No operations performed in \(E\) are allowed. In particular, we do not admit computation of \(f(y)\) for \(y \in E\). Thus, we consider a different computational model compared to that in \([9]\). We wish to study the quality of such finite-dimensional solution of \((1)\).

Our approach is different from that in \([9], [6]\) and \([7]\). We assume that the space \(E\) has a Schauder basis. Since most important spaces appearing in applications, such as \(\ell_p\) or \(L_p\) for \(1 \leq p < \infty\), have Schauder bases, this is not a restrictive assumption in practice. The class of problems under consideration and the class of algorithms are defined in terms of that basis. Our main results are as follows:

- For a class of \(C^r\) functions \(f : E \to E\) we define an algorithm \(\phi_{n, N}^*\) based on possibly non-uniform mesh (with \(n + 1\) points) and restricted, in each time step, to finite-dimensional
computations with varying dimensions, represented by the vector $\vec{N}$. We show an upper bound on the error of $\phi^*_{n,\vec{N}}$ expressed in the terms of the truncation vector $\vec{N}$ and the step sizes. In contrast to the usual analysis in the finite-dimensional case, the parameter $N$ is now not a constant number, which may be hidden in error coefficient, but it tends to infinity. This requires somewhat different analysis including the tractability questions, see [17].

• For constant dimensions equal to $N$, we bound from below the error of any algorithm $\phi_{n,\vec{N}}$ for solving (1). The bound shows that the algorithm $\phi^*_{n,\vec{N}}$ is error optimal (up to a constant) as $n, N \to \infty$.

• Based on two-sided error bounds, for $\varepsilon > 0$ we discuss upper and lower bounds on the minimal cost of computing an $\varepsilon$-approximation to the solution of (1) (the $\varepsilon$-complexity of the problem). To consult a general notion of the $\varepsilon$-complexity, see [20].

• We illustrate the results by an example of a countable system of equations in $\ell_p$, $1 \leq p < \infty$, embedding it to the Banach space setting with a weighted $\ell_p$ space.

The paper is organized as follows. In Section 2 we present basic notions and definitions, and we define the model of computation. In Section 3 we define the algorithm $\phi^*_{n,\vec{N}}$ and prove an upper error bound in Theorem 1. Theorem 2 shows a lower error bound for any algorithm $\phi_{n,\vec{N}}$ based on constant truncation parameters. In Propositions 1 and 2 we discuss the resulting $\varepsilon$-complexity bounds for the problem (1). Section 4 contains an example of a countable system, to which we apply the results described in Section 3. We show how to select $N$ and $n$ to get the error at most $\varepsilon$, and we establish the cost of computing the $\varepsilon$-approximation. In Section 5 we recall, for convenience of the reader, basic facts used in the paper about integration, differentiation and interpolation in a Banach space with Schauder basis.

## 2 Preliminaries

Let $\{e_1, e_2, \ldots \}$ with $\|e_j\| = 1$ be a Schauder basis in $E$. Let $f(y) = \sum_{j=1}^{\infty} f^j(y)e_j$ for $y \in E$.

For $k \in \mathbb{N}$, let $P_k : E \to E$ be the projection operator, i.e., for $z \in E$, $z = \sum_{j=1}^{\infty} z^j e_j$ we have $P_k z := \sum_{j=1}^{k} z^j e_j$. The operator $P_k$ is linear and bounded, with $\sup_{k} \|P_k\| = P < \infty$, see [15] p. 1–2. The number $P$ is called the basis constant of $\{e_1, e_2, \ldots \}$. Since $|z^k| = \|(P_k - P_{k-1})z\|$ (for $P_0 = 0$), it holds $|z^k| \leq 2P\|z\|$, for $k = 1, 2, \ldots$.

**The class of problems**

Let $r$ be a nonnegative integer. Let $L, M, D$ be positive numbers, and $\Gamma = \{\gamma(k)\}_{k=1}^{\infty}$ and $\Delta = \{\delta(k)\}_{k=1}^{\infty}$ positive, nonincreasing, convergent to zero sequences. We shall consider a class $F_r = F_r(L, M, D, P, \Gamma, \Delta)$ of pairs $(f, \eta)$ defined by the following conditions (A1)–(A5).
(A1) \( \| \eta - P_k \eta \| \leq \gamma(k) \) for \( k \in \mathbb{N} \),
(A2) \( \| f(y) - f(\bar{y}) \| \leq L \| y - \bar{y} \| \), for \( y, \bar{y} \in E \),
(A3) \( \| f(\eta) \| \leq M \).

Let \( R = R(L, M, P, a, b) \) be a number, existence of which is shown in Lemma 1 below, and let \( K = K(\eta, R) = \{ y \in E : \| y - \eta \| \leq R \} \). In addition to (A1)–(A3), we assume that

(A4) \( f \in C^r(K) \) (where the derivatives are meant in the Fréchet sense) and
\[
\sup_{y \in K} \| f^{(l)}(y) \| \leq D, \quad l = 1, 2, \ldots, r.
\]

(In the last inequality, \( \| \cdot \| \) means the norm of a bounded \( l \)-linear operator in \( E^l \), defined by the norm in \( E \).) We note that for \( l = 0 \) we have from (A2) and (A3) the bound
\[
\sup_{y \in K} \| f(y) \| \leq M + LR.
\]

(A5) \( \sup_{y \in K} \| f(y) - P_k f(y) \| \leq \delta(k), \quad k \in \mathbb{N} \).

The parameter \( P \) of the space \( E \), as well as the parameters \( L, M, D, \{ \gamma(k) \}_{k=1}^\infty, \{ \delta(k) \}_{k=1}^\infty \) of the class \( F \), are unknown, and they cannot be used by an algorithm. The numbers \( a, b, r \) are known.

The class of algorithms

To approximate \( z \), we shall only allow Galerkin-type algorithms that base on finite-dimensional computations. Let \( n \in \mathbb{N} \) be a discretization parameter, and let \( \{ \alpha(n) \}_{n=1}^\infty \) be a nonincreasing sequence convergent to 0 as \( n \to \infty \). We shall consider a family of partitions of \( [a, b] \) given by points \( a = t_0^n < t_1^n < \ldots < t_{n}^n = b \) such that
\[
\max_{k=0,1,\ldots,n-1} (t_{k+1}^n - t_k^n) \leq \alpha(n), \quad n = 1, 2, \ldots.
\]

(Obviously, it must hold \( \alpha(n) \geq (b - a)/n \).) In what follows, we shall omit in the notation the superscript \( n \). Furthermore, to keep the notation legible, we will not indicate the dependence of information and an algorithm on \( \{ t_k \}_{k=0}^n \).

Let \( N_{-1} \in \mathbb{N} \) and \( N_k, M_k \in \mathbb{N}, k = 0, 1, \ldots, n - 1 \). For a given partition \( \{ t_k \}_{k=0}^n \) and given numbers \( \{ N_k \}_{k=-1}^{n-1} \) and \( \{ M_k \}_{k=0}^{n-1} \), we shall allow algorithms based on the (approximate) successive solution of finite-dimensional local problems in \( [t_k, t_{k+1}] \), \( k = 0, 1, \ldots, n - 1 \). Let \( f_k = P_{N_k} f \), and let \( \tilde{z}_k : [t_k, t_{k+1}] \to E \) be the solution of the local problem
\[
\tilde{z}'_k(t) = \tilde{f}_k(\tilde{z}_k(t)), \quad \tilde{z}_k(t_k) = P_{M_k} y_k, \quad t \in [t_k, t_{k+1}],
\]
where \( y_k \) is a given point in \( E \). This is a finite-dimensional problem defined by truncation parameters \( N_k \) (which describes the number of components of \( f \) that are considered) and \( M_k \) (which describes the number of components of the arguments taken into account). Note that \( \tilde{f}_k \) is a Lipschitz function in \( E \) with the (uniform) constant \( PL \). An algorithm
successively computes \( y_k \) and approximations \( l_k \) to \( \bar{z}_k \) in \( [t_k, t_{k+1}] \). The approximation to the solution \( z \) of (1) in \( [a, b] \) is a spline function \( l : [a, b] \to E \) composed of \( l_k \).

Consider information about \( f \) that is allowed in the computation of \( l \). The function \( f \) can only be accessed through the components \( \bar{f}_k \) of \( \bar{f}_k \), \( j = 1, 2, \ldots, N_k \). Available information is given by evaluations \( \bar{f}_k(P_{M_k}y) \), or evaluations of partial derivatives of \( \bar{f}_k(P_{M_k}y) \) at \( P_{M_k}y \) (up to order \( r \)), for some component \( j \), at some information points \( y \). We assume that the number of information points \( s \) is proportional to the number of subintervals, that is, there is \( \hat{K} \) such that

\[
s \leq \hat{K}n, \tag{4}
\]

\( n = 1, 2, \ldots \). For example, the standard explicit Runge-Kutta methods in \( \mathbb{R}^N \) of order \( p \) are based, at each time interval \( [t_k, t_{k+1}] \), on the constant number of \( p \) function evaluations.

For what concerns the initial condition \( \eta \), we assume to have access to \( P_{N_{-1}}\eta \) for any \( N_{-1} \in \mathbb{N} \). We assume that information is adaptive in the following sense. We allow successive adaptive selection of the information points, indices \( j \) of the components of \( \bar{f}_k \), and orders of partial derivatives to be evaluated. This means that these elements can be computed based on information computed so far. In this paper, the sequences \( \{t_k\}_{k=0}^n \), \( \{N_k\}_{k=0}^{n-1} \) and \( \{M_k\}_{k=0}^{n-1} \) are given in advance.

Infinite-dimensional ‘computation’ is not allowed; for example, computing \( f(y) \in E \) for \( y \in E \) is in general not possible.

Let \( \bar{N} = [N_{-1}, N_0, \ldots, N_{n-1}] \) and \( \bar{M} = [M_0, M_1, \ldots, M_{n-1}] \). Information computed as described above in the interval \( [a, b] \) for \( f \) and \( \eta \) will be denoted by \( \mathcal{N}_{n, \bar{N}, \bar{M}}(f, \eta) \). By an algorithm \( \phi_{n, \bar{N}, \bar{M}} \) we mean a mapping that assigns to the vector \( \mathcal{N}_{n, \bar{N}, \bar{M}}(f, \eta) \) the function \( l \) described above, \( l = \phi_{n, \bar{N}, \bar{M}}(\mathcal{N}_{n, \bar{N}, \bar{M}}(f, \eta)) \). The (worst case) error of an algorithm \( \phi_{n, \bar{N}, \bar{M}} \) with information \( \mathcal{N}_{n, \bar{N}, \bar{M}} \) in the class \( F_r \) is defined by

\[
e(\phi_{n, \bar{N}, \bar{M}}, \mathcal{N}_{n, \bar{N}, \bar{M}}, F_r) = \sup_{(f, \eta) \in F_r} \sup_{t \in [a, b]} \|z(t) - l(t)\|. \tag{5}
\]

Let us consider the cost of computing information \( \mathcal{N}_{n, \bar{N}, \bar{M}}(f, \eta) \). For each \( j \), we assume that the cost of computing the value of the function \( \bar{f}_k \) or its partial derivative at \( P_{M_k}y \) is \( c(M_k) \), where \( c \) is a nondecreasing function. That is, the cost of computing these real-valued functions is determined by the number of variables \( M_k \).

The number of such scalar evaluations at each time step \( [t_k, t_{k+1}] \) depends on particular information; we denote this number by \( \ell (N_k, M_k) \). For instance, if we only compute at each time step a single value \( \bar{f}_k(P_{M_k}y) \) for some \( y \), then \( \ell (N_k, M_k) = N_k \). If we compute at \( P_{M_k}y \) all partial derivatives up to order \( r \) of each component of \( \bar{f}_k \), then \( \ell (N_k, M_k) = \Theta (N_k M_k^r) \).

We assume to have access to \( P_{N_{-1}}\eta \) for any \( N_{-1} \) with no cost.

The total cost of computing information is thus

\[
\sum_{k=0}^{n-1} c(M_k)\ell (N_k, M_k).
\]

For a given \( \varepsilon > 0 \), by the \( \varepsilon \)-complexity \( \text{comp}(\varepsilon) \) of the problem (1), we mean the minimal
Cost of computing an $\varepsilon$-approximation. More precisely,

$$\text{comp}(\varepsilon) = \inf \left\{ \sum_{k=0}^{n-1} c(M_k)\ell(N_k,M_k) : n, \bar{N}, \bar{M} \text{ are such that } \exists N_{n,\bar{N},\bar{M}}, \phi_{n,\bar{N},\bar{M}}, \leq \varepsilon \right\},$$

with $e(\phi_{n,\bar{N},\bar{M}}, N_{n,\bar{N},\bar{M}}, F_r) \leq \varepsilon$.

(6)

The $\varepsilon$-complexity measures an intrinsic difficulty of solving the problem (1) by finite-dimensional computation. We shall establish in this paper bounds on $\text{comp}(\varepsilon)$.

Unless otherwise stated, all coefficients that appear in this paper will only depend on $L$, $M$, $P$, $D$, $r$, $a$ and $b$.

3 Upper error and complexity bounds

3.1 The variable dimension algorithm

Let $h_k = t_{k+1} - t_k$. We associate with each subinterval $[t_k, t_{k+1}]$ a number (dimension) $N_k$, $k = 0, 1, \ldots, n - 1$. For $\bar{N} = [N_{-1}, N_0, \ldots, N_{n-1}]$, we define an algorithm $\phi^*_{n,\bar{N}}$ for solving (1). Let $\bar{y}_0 = P_{N_{-1}} \eta$. Let $\bar{f}_k = P_{N_k} f$ and, for a given $\bar{y}_k \in E$, consider the local problem

$$\bar{z}_k(t) = \bar{f}_k(\bar{z}_k(t)), \quad \bar{z}_k(t_k) = \bar{y}_k, \quad t \in [t_k, t_{k+1}].$$

(7)

The following general idea of approximating the solution $z$ of (1) has been used several times in various contexts, see e.g. [3], [9], [13]. Let $r \geq 1$. We define a function $\bar{l}_{k,r}$ in $[t_k, t_{k+1}]$ as follows. Let $\bar{l}_{k,0}(t) \equiv \bar{y}_k$. For $s \geq 0$ and a given function $\bar{l}_{k,s}$, we define the Lagrange interpolation polynomial of degree at most $s$ by

$$\bar{q}_{k,s}(t) = \sum_{p=0}^{s} \prod_{l=0, l \neq p}^{s} \frac{t - \xi_{k,l}}{\xi_{k,p} - \xi_{k,l}} \bar{f}_k(\bar{l}_{k,s}(\xi_{k,p})), \quad \text{(8)}$$

where $\xi_{k,p} = t_k + ph / s, \quad p = 0, 1, \ldots, s$ (with $\prod_{l=0, l \neq p}^{s} = 1$). We define a polynomial

$$\bar{l}_{k,s+1}(t) = \bar{y}_k + \int_{t_k}^{t} \bar{q}_{k,s}(\xi) \, d\xi, \quad t \in [t_k, t_{k+1}]. \quad \text{(9)}$$

We repeat (8) and (9) for $s = 0, 1, \ldots, r - 1$ to get a final polynomial $\bar{l}_{k,r}$, and we set $\bar{y}_{k+1} = \bar{l}_{k,r}(t_{k+1})$. After passing through all the intervals $[t_k, t_{k+1}], \quad k = 0, 1, \ldots, n - 1$, we get an approximation to $z$ in $[a, b]$ defined as a piecewise polynomial continuous function $\bar{l}_n(t) : = \bar{l}_{k,r}(t)$, if $t \in [t_k, t_{k+1}]$.

For $r = 0$ we define $\bar{l}_n$ as we did above for $r = 1$, i.e.,

$$\bar{l}_n(t) = \bar{y}_k + (t - t_k) \bar{f}_k(\bar{y}_k), \quad \text{if } t \in [t_k, t_{k+1}],$$

and we set as above $\bar{y}_{k+1} = \bar{l}_n(t_{k+1})$.

Note that the computation of $\bar{l}_n$ only involves finite-dimensional operations. We see that
the computations are determined by the vector \( \vec{N} = [N_{-1}, N_0, \ldots, N_{n-1}] \), since the dimensions \( M_k \) are given by \( M_k = \max_{j=1, 0, \ldots, k} N_j \) for \( k = 0, 1, \ldots, n - 1 \).

We denote the information about \( f \) used above to construct \( t_n \) by \( N_n^*(f; \eta) \). It is easy to see that it consists of \( O(r^2 n) \) evaluations of finite-dimensional truncations of \( f \) at finite-dimensional truncations of some points in \( E \). We define an algorithm \( \phi_n^* \) that approximates \( z \) by

\[
\phi_{n, \vec{N}}^* \left( N_{n, \vec{N}}^*(f, \eta) \right)(t) = t_n(t), \quad t \in [a, b].
\]

### 3.2 Upper error bound

We show in this section an upper bound on the error of \( \phi_{n, \vec{N}}^* \). We start with a lemma that assures that the solutions of (1), (7) stay in a certain ball \( K(\eta, R) \) with radius \( R \) that only depends on the parameters appearing in assumptions (A1), (A2) and (A3).

**Lemma 1** There exist \( R = R(L, M, \gamma(1), P, a, b) \) such that:

(a) \( z(t) \in K(\eta, R) \) for \( t \in [a, b] \)

and

(b) for any \( \{a(n)\}^\infty_{n=1} \) there is \( \hat{n} \) such that for any \( n \geq \hat{n} \), any \( \{t_k\}_{k=0}^n \) satisfying (2), any \( \vec{N} \) and any \( f \) satisfying (A1), (A2) and (A3) it holds

\[
\bar{z}_k(t) \in K(\eta, R) \text{ for } t \in [t_k, t_{k+1}], \quad k = 0, 1, \ldots, n - 1.
\]

**Proof** Note that the boundedness of \( \|z(t) - \eta\| \leq \bar{C}_1 \) by some constant \( \bar{C}_1 = \bar{C}_1(L, M, P, a, b) \) immediately follows from the Gronwall inequality, (A2) and (A3). We show a bound on \( \|\bar{z}_k(t) - \eta\| \).

For \( r = 0 \) the algorithm is defined by the same expression as for \( r = 1 \), so that we can consider formulas (8) and (9) with \( r \geq 1 \). We have

\[
\bar{I}_{k,s+1}(t) - \bar{y}_k = \int_{t_k}^t \left( \sum_{p=0}^s \prod_{l=0, l \neq p}^s \frac{\xi - \xi_{k,l}}{\xi_{k,p} - \xi_{k,l}} \left( \bar{f}_k(\bar{I}_{k,s}(\xi_{k,p})) - \bar{f}_k(\bar{y}_k) \right) \right) d\xi.
\]

Since, \( \bar{f}_k \) is the Lipschitz function with the constant \( PL \), we have that

\[
\|\bar{I}_{k,s+1}(t) - \bar{y}_k\| \leq h_k PLC_r \sup_{t \in [t_k, t_{k+1}]} \|\bar{I}_{k,s}(t) - \bar{y}_k\| + h_k \|\bar{f}_k(\bar{y}_k)\|,
\]

(11)

\( t \in [t_k, t_{k+1}], \ s = 0, 1, \ldots, r - 1 \), where \( C_r \) is an upper bound (dependent only on \( r \)) on

\[
\sup_{\xi \in [t_k, t_{k+1}]} \sum_{p=0}^s \prod_{l=0, l \neq p}^s \frac{\xi - \xi_{k,l}}{\xi_{k,p} - \xi_{k,l}}.
\]

Taking the \( \sup \) in the left hand side of (11), and solving the resulting recurrence inequality, we get for \( n \) such that \( 2\alpha(n)PLC_r \leq 1 \) the bound

\[
\sup_{t \in [t_k, t_{k+1}]} \|\bar{I}_{k,s}(t) - \bar{y}_k\| \leq 2h_k \|\bar{f}_k(\bar{y}_k)\| \leq 2h_k P \|f(\bar{y}_k)\|,
\]

(12)
We now bound \( E_k = \|\tilde{y}_k - \bar{y}_0\| \) \((E_0 = 0)\). Since \( \|f(\bar{y}_0)\| \leq \|f(\tilde{y}_0)\| + LE_k\), we get from (13) that
\[
E_{k+1} \leq (1 + 2h_k PL)E_k + 2h_k P\|f(\bar{y}_0)\|, \quad k = 0, 1, \ldots, n-1.
\]
By solving the recurrence inequality, remembering that \( \sum_{j=0}^{k} h_j = t_{k+1} - t_0 \leq b - a \), we get
\[
\|\bar{y}_k - \bar{y}_0\| \leq C\|f(\bar{y}_0)\|, \quad k = 0, 1, \ldots, n-1,
\]
for some constant \( C \) only dependent on \( L, P, a, b \) and sufficiently large \( n \).

We now estimate \( \|\bar{z}_k(t) - \eta\|, \quad t \in [t_k, t_{k+1}] \). We have from (7) that
\[
\bar{z}_k(t) = \bar{y}_k + \int_{t_k}^{t} \bar{f}_k(\bar{z}_k(\xi)) d\xi.
\]
Subtracting from both sides \( \bar{y}_0 \) and applying the Lipschitz condition for \( \bar{f}_k \), we get
\[
\|\bar{z}_k(t) - \bar{y}_0\| \leq \|\bar{y}_k - \bar{y}_0\| + PL \int_{t_k}^{t} \|\bar{z}_k(\xi) - \bar{y}_0\| d\xi + h_k P\|f(\bar{y}_0)\|, \quad t \in [t_k, t_{k+1}].
\]
By Gronwall’s lemma, we get
\[
\|\bar{z}_k(t) - \bar{y}_0\| \leq \exp(h_k PL) (\|\bar{y}_k - \bar{y}_0\| + h_k P\|f(\bar{y}_0)\|),
\]
which yields according to (14) that
\[
\|\bar{z}_k(t) - \bar{y}_0\| \leq C\|f(\bar{y}_0)\|, \quad t \in [t_k, t_{k+1}], \quad k = 0, 1, \ldots, n-1,
\]
for some constant \( C \) (different from that in (14)) and sufficiently large \( n \).

Since \( \bar{y}_0 = P_{N-1} \eta \), it follows from (15) and (A1), (A2) and (A3) that
\[
\|\bar{z}_k(t) - \bar{y}_0\| \leq C (L\gamma(N_{-1}) + M) \leq C (L\gamma(1) + M).
\]
Finally, we get for \( n \) sufficiently large, \( t \in [t_k, t_{k+1}] \) and \( k = 0, 1, \ldots, n-1 \) that
\[
\|\bar{z}_k(t) - \eta\| \leq \|\bar{z}_k(t) - \bar{y}_0\| + \|P_{N-1} \eta - \eta\| \leq \tilde{C}_2,
\]
for some constant \( \tilde{C}_2 \) which only depends on the parameters appearing in assumptions (A1), (A2) and (A3), \( P, a \) and \( b \). To complete the proof, in the statement of the lemma we take \( R = \max\{\tilde{C}_1, \tilde{C}_2\} \).

The following theorem gives us an upper bound on the error of the algorithm \( \phi^*_{n,N} \). In the proof, we have to pay attention to the independence of constants appearing in the bounds of \( N_{-1}, N_0, \ldots, N_{n-1} \).
Theorem 1  There exist $C$ such that for any $\{\alpha(n)\}_{n=1}^{\infty}$ there is $\hat{n}$ such that for $n \geq \hat{n}$, any $\{t_k\}_{k=0}^{n}$ satisfying (2), any $\tilde{N} = [N_1, N_0, \ldots, N_{n-1}]$, any $(f, \eta) \in F_r$ it holds

$$
\sup_{t \in [t_k, t_{k+1}]} \|z(t) - \tilde{l}_n(t)\| \leq C \left( \gamma(N-1) + \sum_{j=0}^{k} h_j \delta(N_j) + \sum_{j=0}^{k} h_j^{\max\{r,1\}+1} \right), \quad (17)
$$

$k = 0, 1, \ldots, n-1$.

Consequently,

$$
e(\phi^*, N^*, \tilde{N}, F_r) \leq C \left( \gamma(N-1) + \sum_{j=0}^{n-1} h_j \delta(N_j) + \sum_{j=0}^{n-1} h_j^{\max\{r,1\}+1} \right). \quad (18)
$$

Note that the last term in (18) can be bounded by $\alpha(n)^{\max\{r,1\}}(b-a)$.

Proof  Let $(f, \eta) \in F_r$. We need to estimate for $t \in [a, b]$

$$\|z(t) - \phi^*_n(N^*(f, \eta))(t)\|.$$

We first show that for $r \geq 1$, $k = 0, 1, \ldots, n-1$

$$\sup_{\xi \in [t_k, t_{k+1}]} \|\tilde{z}_k(\xi) - \tilde{l}_{k,r}(\xi)\| \leq C_1 h_k^{r+1}, \quad (19)$$

for some constant $C_1$ only dependent on the parameters of the class $F_r$, and sufficiently large $n$.

For the knots $\xi_{k,p}$ given in (5), we let $\tilde{w}_{k,s}$ be the Lagrange interpolation polynomial for $\hat{z}_k$ defined in a similar way as $\tilde{q}_{k,s}$,

$$\tilde{w}_{k,s}(\xi) = \sum_{p=0}^{s} \prod_{l=0, l \neq p}^{s} \frac{\xi - \xi_{k,l}}{\xi_{k,p} - \xi_{k,l}} \hat{z}_k(\xi_{k,p}). \quad (20)$$

Note that $\tilde{w}_{k,0}(\xi) = \hat{z}_k(t_k) = \tilde{f}_k(\tilde{y}_k) = \tilde{q}_{k,0}(\xi)$. From (9) we have for $t \in [t_k, t_{k+1}]$ and $s = 0, 1, \ldots, r-1$ that

$$\tilde{z}_k(t) - \tilde{l}_{k,s+1}(t) = \int_{t_k}^{t} (\tilde{z}_k^*(\xi) - \tilde{w}_{k,s}(\xi)) d\xi + \int_{t_k}^{t} (\tilde{w}_{k,s}(\xi) - \tilde{q}_{k,s}(\xi)) d\xi,$$

which yields

$$\|\tilde{z}_k(t) - \tilde{l}_{k,s+1}(t)\| \leq \int_{t_k}^{t} \|\tilde{z}_k^*(\xi) - \tilde{w}_{k,s}(\xi)\| d\xi + \int_{t_k}^{t} \|\tilde{w}_{k,s}(\xi) - \tilde{q}_{k,s}(\xi)\| d\xi. \quad (21)$$

We now bound both terms in the right-hand side of (21). Using the integral form of the Lagrange interpolation remainder formula (60), we get for $\xi \in [t_k, t_{k+1}]$ and $s = 0, 1, \ldots, r-1$

$$\|\tilde{z}_k^*(\xi) - \tilde{w}_{k,s}(\xi)\| \leq h_k^{s+1}/(s+1)! \sup_{t \in [t_k, t_{k+1}]} \|\tilde{z}_k^{(s+2)}(t)\|. \quad (22)$$
We now estimate the global error in $z_{k}^{(s)}(t)$ for $s = 1, 2, \ldots, r + 1$ can be expressed in the well known way as a sum of multilinear expressions involving the Fréchet derivatives of $\bar{f}_k = P_{N_k} f$ of order $0, 1, \ldots, r$, evaluated at $\bar{z}_k(t)$. Since, by Lemma 1, $\bar{z}_k(t)$ lies in the ball $K$ for sufficiently large $n$, from the assumption (A4) we get for any $N_k$ that

$$
\|z_{k}^{(s)}(t)\| \leq C_2, \quad \text{for } t \in [t_k, t_{k+1}], \ k = 0, 1, \ldots, n - 1,
$$

for some number $C_2$ only dependent on the parameters of the class $F_r$ and $P$, and sufficiently large $n$. Hence,

$$
\int_{t_k}^{t} \|\bar{z}_k'(\xi) - \bar{w}_{k,s}(\xi)\| \, d\xi \leq \hat{C}_2 h_{k}^{s+2}, \quad t \in [t_k, t_{k+1}],
$$

for $s = 0, 1, \ldots, r - 1$. To bound the second term in (21), we estimate the difference between two Lagrange polynomials

$$
\|\bar{w}_{k,s}(\xi) - \bar{q}_{k,s}(\xi)\| = \left| \sum_{p=0}^{s} \prod_{l=0,l\neq p}^{s} \frac{\xi - \xi_{k,l}}{\xi_{k,p} - \xi_{k,l}} \left( \bar{f}_k(\bar{z}_k(\xi_{k,p})) - \bar{f}_k(\bar{t}_k(\xi_{k,p})) \right) \right|
\leq PL \sum_{p=0}^{s} \|\bar{z}_k(\xi_{k,p}) - \bar{t}_k(\xi_{k,p})\| \prod_{l=0,l\neq p}^{s} \left| \frac{\xi - \xi_{k,l}}{\xi_{k,p} - \xi_{k,l}} \right|
\leq PLC_r \sup_{\xi \in [t_k, t_{k+1}]} \|\bar{z}_k(\xi) - \bar{t}_k(\xi)\|,
$$

where $C_r$ is given in (11). From this we get for $t \in [t_k, t_{k+1}]

$$
\int_{t_k}^{t} \|\bar{w}_{k,s}(\xi) - \bar{q}_{k,s}(\xi)\| \, d\xi \leq h_{k} PLC_r \sup_{\xi \in [t_k, t_{k+1}]} \|\bar{z}_k(\xi) - \bar{t}_k(\xi)\|.
$$

We now come back to (21) to get from (23) and (25) that

$$
\sup_{t \in [t_k, t_{k+1}]} \|z_k(t) - \bar{t}_{k,s+1}(t)\| \leq \hat{C}_2 h_{k}^{s+2} + h_{k} PLC_r \sup_{t \in [t_k, t_{k+1}]} \|\bar{z}_k(t) - \bar{t}_{k,s}(t)\|,
$$

$s = 0, 1, \ldots, r - 1$, where $\sup_{t \in [t_k, t_{k+1}]} \|\bar{z}_k(t) - \bar{t}_{k,0}(t)\| \leq P(M + LR)h_{k}$ for sufficiently large $n$.

By solving the recurrence inequality with respect to $s$, we obtain (19).

We now estimate the global error in $[a, b]$. We have

$$
\|z(t_{k+1}) - \bar{y}_{k+1}\| \leq \|z(t_{k+1}) - \bar{z}_k(t_{k+1})\| + \|\bar{z}_k(t_{k+1}) - \bar{y}_{k+1}\|.
$$

Note that $z$ and $\bar{z}_k$ are solutions of the initial value problems in $[t_k, t_{k+1}]$ with right-hand sides $f$ nad $\bar{f}_k$, and initial conditions $z(t_k)$ and $\bar{y}_k$, respectively. By a standard use of Gronwall’s inequality, using (A5) and Lemma 1, we get for $t \in [t_k, t_{k+1}]$ and sufficiently large $n$

$$
\|z(t) - \bar{z}_k(t)\| \leq e^{L h_{k}} (\|z(t_k) - \bar{y}_k\| + h_{k} \delta(N_k)).
$$

Hence,

$$
\|z(t_{k+1}) - \bar{y}_{k+1}\| \leq e^{L h_{k}} (\|z(t_k) - \bar{y}_k\| + h_{k} \delta(N_k)) + C_1 h_{k}^{r+1},
$$
for \( k = 0, 1, \ldots, n - 1 \), where \( \| z(t_0) - \bar{y}_0 \| = \| \eta - P_{N-1} \eta \| \leq \gamma (N-1) \). By solving this recurrence inequality with respect to \( k \), we get that there is a number \( C_3 \) only dependent on the parameters of the class \( E_r \) and \( P \) such that

\[
\| z(t_k) - \bar{y}_k \| \leq C_3 \left( \gamma (N-1) + \sum_{j=0}^{k-1} h_j \delta (N_j) + \sum_{j=0}^{k-1} h_{r_j+1}^{r_j+1} \right),
\]

for \( k = 0, 1, \ldots, n - 1 \), for \( n \) sufficiently large (\( \sum_{j=0}^{k-1} = 0 \)). From (27), by slightly changing the constant \( C_3 \), we have for \( t \in [t_k, t_{k+1}] \)

\[
\| z(t) - \bar{z}_k(t) \| \leq C_3 \left( \gamma (N-1) + \sum_{j=0}^{k} h_j \delta (N_j) + \sum_{j=0}^{k-1} h_{r_j+1}^{r_j+1} \right),
\]

for \( k = 0, 1, \ldots, n - 1 \), and \( n \) sufficiently large.

By (19) and (30) we obtain for \( t \in [t_k, t_{k+1}] \) and \( k = 0, 1, \ldots, n - 1 \)

\[
\left\| z(t) - \phi^*_{n,N}(N^*, (f, \eta)) (t) \right\| \leq \| z(t) - \bar{z}_k(t) \| + \left\| \bar{z}_k(t) - \phi^*_{n,N}(N^*, (f, \eta)) (t) \right\|
\]

\[
\leq C_3 \left( \gamma (N-1) + \sum_{j=0}^{k} h_j \delta (N_j) + \sum_{j=0}^{k-1} h_{r_j+1}^{r_j+1} \right) + C_1 h_{r_k+1}^{r_k+1},
\]

for \( n \) sufficiently large. This implies the statement of the theorem in the case \( r \geq 1 \).

For \( r = 0 \), similarly as for \( r = 1 \), the algorithm \( \phi^*_{n,N} \) reduces to the Euler method, i.e., the final approximation is given by

\[
\bar{I}_n(t) = \bar{y}_k + (t - t_k) \bar{f}_k(\bar{y}_k), \quad \text{if} \quad t \in [t_k, t_{k+1}].
\]

It suffices to note that the error analysis in the case \( r = 1 \) only requires the Lipschitz condition for \( \bar{f}_k \). That is, it can be repeated for \( r = 0 \). This completes the proof of the theorem.

\[\blacksquare\]

**Remark 1** In the special case of uniform discretization of \([a,b]\) and constant truncation parameters, that is, when \( N_{-1} = N_0 = \cdots = N_{n-1} = N \), the estimate (18) can be derived from Lemma 1 above and Theorem 3.3 in [9]. One has to apply the random algorithm from [9], for a fixed random instant, to the pair \((P_N f, P_N \eta)\), use Lemma 1 and note that information needed for that input is \( N \)-dimensional. This observation was made by Stefan Heinrich in private communication.

### 3.3 Upper complexity bound

Let \( \varepsilon > 0 \). The cost of computing an \( \varepsilon \)-approximation using information \( N^*_{n,N} \) and the algorithm \( \phi^*_{n,N} \) provides an upper bound on the \( \varepsilon \)-complexity. The computations in \( \phi^*_{n,N} \) are determined by vectors \( \bar{N} = [N_{-1}, N_0, \ldots, N_{n-1}] \) and \( \bar{M} = [M_0, M_1, \ldots, M_{n-1}] \), where \( M_k \) is given by \( M_k = \max_{j=-1,0,\ldots,k} N_j \), \( k = 0, 1, \ldots, n - 1 \). The number \( s \) of information
points in $N_{n,N}^*$ is $O(r^2n)$, with an absolute constant in the 'O' notation. Neglecting the coefficient that only depends on $r$, the cost of $\phi_{n,N}^*$ is thus

$$
\sum_{k=0}^{n-1} c \left( \max_{j=-1,0,\ldots,k} N_j \right) N_k,
$$

(32)

where $c(N)$ is the cost function defined in Section 2. Given the mesh points $\{t_k\}$, let cost$^*(\varepsilon)$ be the minimal cost of computing an $\varepsilon$-approximation by this class of algorithms, the minimum taken with respect to the selection of $n$ and the dimensions $N_{-1}, N_0, \ldots, N_{n-1}$. Let

$$
U(\varepsilon) = \inf \left\{ \sum_{k=0}^{n-1} c \left( \max_{j=-1,0,\ldots,k} N_j \right) N_k : \gamma(N_{-1}) + \sum_{j=0}^{n-1} h_j \delta(N_j) + \sum_{j=0}^{n-1} h_j \max\{r,1\} + 1 \leq \varepsilon \right\},
$$

(33)

where the infimum is taken with respect to $n$, $\{h_j\}$ and $\tilde{N}$ satisfying the bound. For a given sequence $\{\alpha(n)\}_{n=1}^{\infty}$, given cost function $c$, and given functions $\gamma$ and $\delta$ defining the class of problems, $U(\varepsilon)$ can be computed.

Due to Theorem 1, the $\varepsilon$-complexity for sufficiently small $\varepsilon$ is bounded by

$$
\text{comp}(\varepsilon) \leq \text{cost}^*(\varepsilon) \leq U(\varepsilon/C),
$$

(34)

where $C$ is the constant from Theorem 1. An obvious choice is to take the truncation parameters constant in each interval $[t_k, t_{k+1}]$, i.e., $N_{-1} = N_0 = \ldots = N_{n-1} = N$. The minimization in this subclass gives us the value

$$
U^{eq-dim}(\varepsilon) = \inf \left\{ nc(N) N : \gamma(N) + (b-a)\delta(N) + \sum_{j=0}^{n-1} h_j \max\{r,1\} + 1 \leq \varepsilon \right\}.
$$

(35)

We have for sufficiently small $\varepsilon > 0$ that

$$
\text{comp}(\varepsilon) \leq U(\varepsilon/C) \leq U^{eq-dim}(\varepsilon/C).
$$

(36)

To further bound $U^{eq-dim}(\varepsilon/C)$ from above, it suffices to take the value of $nc(N) N$ with the minimal $n$ and $N$ such that

$$
C\gamma(N) \leq \varepsilon/3, \quad C(b-a)\delta(N) \leq \varepsilon/3, \quad C(b-a)\alpha(n)\max\{r,1\} \leq \varepsilon/3.
$$

We get

**Proposition 1** There exist positive numbers $C_1$ and $\varepsilon_0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ it holds

$$
\text{comp}(\varepsilon) \leq n(\varepsilon/C_1) c(N(\varepsilon/C_1)) N(\varepsilon/C_1),
$$

(37)

where

$$
n(\varepsilon) = \alpha^{-1} \left( \varepsilon^{1/\max\{r,1\}} \right), \quad N(\varepsilon) = \max \left\{ \gamma^{-1}(\varepsilon), \delta^{-1}(\varepsilon) \right\}.
$$

(For a nonincreasing function $g$ acting from $[1, \infty]$ onto $(0, p]$, $(p > 0)$, by $g^{-1}$ we mean a function on $(0, p)$ defined by $g^{-1}(\varepsilon) = \inf \{ x \in [1, \infty) : g(x) \leq \varepsilon \}.$)
4 Lower error and complexity bounds

In this section we discuss lower error and complexity bounds. We restrict ourselves to a special (but still interesting) case of constant truncation parameters, that is, we assume that \( N_{-1} = N_0 = \ldots = N_{n-1} = N \). Furthermore, we assume that the partitions of \([a, b]\) satisfy the following condition (which most often holds in practice): there exists \( \hat{K}_1 \) such that

\[
\alpha(n) \leq \hat{K}_1 n^{-1}, \tag{38}
\]

\( n = 1, 2, \ldots \). We shall denote information and an algorithm in this case by \( \mathcal{N}_{n, N} \) and \( \phi_{n, N} \), respectively. Theorem 1 assures the existence of \( C \) (dependent on \( \hat{K}_1 \)) such that for \( n \) sufficiently large

\[
e(\phi^*_{n, N}, \mathcal{N}^*_{n, N}, F_r) \leq C \left( \gamma(N) + \delta(N) + n^{-\max\{r, 1\}} \right). \tag{39}\]

Upper complexity bound of Proposition 1 now holds with

\[
n(\varepsilon) = (1/\varepsilon)^{1/\max\{r, 1\}}, \tag{40}\]

i.e.,

\[
\text{comp}(\varepsilon) \leq (C_1/\varepsilon)^{1/\max\{r, 1\}} e(N(\varepsilon/C_1)) N(\varepsilon/C_1). \tag{41}\]

4.1 Lower error bound

We now show a matching (up to a constant) lower bound, with respect to \( \varepsilon \), on the error of any algorithm \( \phi_{n, N} \) using any information \( \mathcal{N}_{n, N} \) from the considered class. The solution of \( \Phi \) for a right-hand side \( f \) and an initial vector \( \eta \) will be denoted by \( z_{f, \eta} \).

**Theorem 2** For any \( \hat{K} \) in \( [4] \) and \( \hat{K}_1 \) in \( [5] \) there exist positive \( \hat{C}, n_0 \) and \( \hat{N} \) such that for any \( n \geq n_0, N \geq \hat{N}, \) for any information \( \mathcal{N}_{n, N} \) and any algorithm \( \phi_{n, N} \) it holds

\[
e(\phi_{n, N}, \mathcal{N}_{n, N}, F_r) \geq \hat{C} \left( \gamma(N) + \delta(N) + n^{-\max\{r, 1\}} \right). \tag{42}\]

**Proof** Let \((f, \eta), (g, \kappa) \in F_r \) be such that \( \mathcal{N}_{n, N}(f, \eta) = \mathcal{N}_{n, N}(g, \kappa) \). By the triangle inequality we have

\[
e(\phi_{n, N}, F_r) \geq \frac{1}{2} \sup \| z_{f, \eta}(t) - z_{g, \kappa}(t) \|. \tag{43}\]

Using \( \Phi \), in a standard way we get that

\[
\sup_{\xi \in [a, b]} \| z_{f, \eta}(\xi) - z_{g, \kappa}(\xi) \| \geq \frac{1}{1 + L(t - a)} \left\| \eta - \kappa - \int_a^t H(z_{f, \eta}(\xi)) d\xi \right\|, \quad t \in [a, b], \tag{44}\]

where \( H = g - f \).

We now construct suitable pairs \((f, \eta) \) and \((g, \kappa) \).

**Case I.** Let \( f = g = 0, \eta = \gamma(N) e_{N+1} \) and \( \kappa = 0 \). Then \((f, \eta), (g, \kappa) \in F_r \) and \( \mathcal{N}_{n, N}(f, \eta) = \mathcal{N}_{n, N}(g, \kappa) \). Indeed, for example to show \((A1)\), we note that \( \| \eta - P_k \eta \| = \gamma(N) \leq \gamma(k) \) for \( k \leq N \), and \( \| \eta - P_k \eta \| = 0 \) for \( k \geq N + 1 \). We have from \( \Phi \) that

\[
\sup_{\xi \in [a, b]} \| z_{f, \eta}(\xi) - z_{g, \kappa}(\xi) \| \geq \gamma(N). \tag{45}\]
Case II. Let $f(y) = \delta(N)e_{N+1}$ for $y \in E$ and $N$ sufficiently large so that $\delta(N) \leq M$.  
Take $g = 0$ and $\eta = \kappa$, where $\eta$ satisfies (A1). Then $(f, \eta), (g, \kappa) \in F_r$ and $N_{n,N}(f, \eta) = N_{n,N}(g, \kappa)$. For instance, to see (A5), note that $\|f(y) - P_k f(y)\| = \delta(N) \leq \delta(k)$ for $k \leq N$, and $\|f(y) - P_k f(y)\| = 0$ for $k \geq N + 1$. From (44) we get

$$\sup_{\xi \in [a,b]} \|z_{f,\eta}(\xi) - z_{g,\kappa}(\xi)\| \geq (b-a)\delta(N). \tag{46}$$

Case III. Let $\eta$ satisfy (A1) and $\kappa = \eta$. We take $f(y) = Ae_1$, where $A > 0$. The solution $z_{f,\eta}$ is given by

$$z_{f,\eta}(t) = A(t-a)e_1 + \eta, \quad t \in [a,b].$$

Compute the adaptive information $N_{n,N}(f, \eta)$ for $f$ and $\eta$. By definition, $N_{n,N}(f, \eta)$ is based on evaluations of the components, or partial derivatives of the components, of the function $P_N f$, at some information points $\hat{y}$ such that $\hat{y} = P_N \hat{y}$. The number of these points is $O(n)$.

The function $g$ is defined as $g = f + H$, where $H$ will be given below. Note that the integral in (44) with $t = b$ has now the form

$$\int_a^b H(A(\xi - a)e_1 + \eta) d\xi.$$

Let $r \geq 1$ and $H^{scal} : \mathbb{R} \to \mathbb{R}$ be a nonnegative function such that:

$H^{scal} \in C^r(\mathbb{R})$, $(H^{scal})^{(j)}(\hat{y}^1) = 0$ for $j = 0,1,\ldots,r$, where $\hat{y}^1$ is the first component of any information point (the number of $\hat{y}^1$ is $O(n)$),

$H^{scal}$ is a Lipschitz function with a constant $L_1$,

$H^{scal}(y^1) \leq M_1$, $|(H^{scal})^{(j)}(y^1)| \leq D_1$, for $y^1 \in \mathbb{R}$, $j = 1,2,\ldots,r$, for some $M_1, D_1$, and

$$\int_a^b H^{scal}(A(\xi - a) + \eta^1) d\xi = \Omega(n^{-r}). \tag{47}$$

For $r = 0$ we take the same function $H^{scal}$ as for $r = 1$. The construction of such a (bump) function $H^{scal}$ is a standard tool when proving lower bounds, see for instance [12]. We now define for $r \geq 0$

$$H(y) = H^{scal}(y^1)e_1.$$

By taking sufficiently small $A$, $M_1$, $L_1$ and $D_1$, we assure that $(f, \eta), (g, \kappa) \in F_r$. Since the derivatives of $H^{scal}$ of order 0,1,\ldots,r vanish at first component of each information point, we have that $N_{n,N}(f, \eta) = N_{n,N}(g, \kappa)$. Hence, by (44) and (47) we get

$$\sup_{\xi \in [a,b]} \|z_{f,\eta}(\xi) - z_{g,\kappa}(\xi)\| = \Omega(n^{-\max(r,1)}). \tag{48}$$

The bounds obtained in the three cases above together with (43) lead to the statement of the theorem.
4.2 Lower complexity bound

In this section we discuss a lower \( \varepsilon \)-complexity bound for (1). Theorem 2 immediately leads to such a bound under certain condition, which we believe holds true under mild assumptions. The condition concerns the number of scalar evaluations \( \ell(M_k, N_k) \) in the definition of the complexity. In our case, we have that \( \ell(M_k, N_k) = \ell(N, N) \). The condition reads:

(C) for information used by an algorithm for solving (1) with a right-hand side \( P_N f \) and an initial condition \( P_N \eta \), in \( \Omega(n) \) time intervals it holds \( \ell(N, N) = \Omega(N) \), with coefficients in the '\( \Omega \)' notation only dependent on \( \hat{K} \) and \( \hat{K}_1 \) (and the parameters of the class \( F_r, a \) and \( b \)).

Under condition (C), the cost of any algorithm is \( \Omega(n c(N)N) \).

**Proposition 2** For any \( \hat{K} \) and \( \hat{K}_1 \), if the class of information satisfies (C), then there exist positive numbers \( C_2 \) and \( \varepsilon_0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) it holds

\[
\text{comp}(\varepsilon) \geq C_2 n(\varepsilon) c(N(\varepsilon/C_2)) N(\varepsilon/C_2),
\]

where \( n(\varepsilon) \) and \( N(\varepsilon) \) are given in (40) and Proposition 1, respectively.

**Proof** Consider an arbitrary algorithm \( \phi_{n,N} \) based on some information \( \mathcal{N}_{n,N} \) for which (C) holds. If \( \varepsilon(\phi_{n,N}, \mathcal{N}_{n,N}, F_r) \leq \varepsilon \), then, due to Theorem 2, we have

\[
\hat{C} \gamma(N) \leq \varepsilon, \quad \hat{C} \delta(N) \leq \varepsilon, \quad \hat{C} n^{-\max\{r,1\}} \leq \varepsilon.
\]

This yields that

\[
n \geq \hat{C}^{1/\max\{r,1\}} n(\varepsilon), \quad N \geq N(\varepsilon/\hat{C}).
\]

Since, by assumption (C), the cost of an algorithm is \( \Omega(nc(N)N) \), we get the desired lower bound. \( \blacksquare \)

Under condition (C), if \( \alpha(n) = O(n^{-1}) \), the lower bound in (49) matches, up to a constant, that in (41).

**Remark 2** Note that the Taylor algorithm can potentially be used to solve the finite-dimensional problem in \( \mathbb{R}^N \). However, the cost of computing the Taylor information, which can be as large as \( nN^{r+1}c(N) \), is much larger than the minimal cost as \( N \to \infty \) (unless function \( f \) is very special).

5 Illustration – weighted \( \ell_p \) spaces

Consider a countable system of equations
in the interval \([a, b] = [0, 1]\). We embed this problem into the Banach space setting. Note that often in applications components of an infinite sequence \(y = (y^1, y^2, \ldots)\) are not of the same importance. Some components may be crucial, while other may even be neglected. It seems reasonable to associate with the components certain positive weights \(w_j, j = 1, 2, \ldots\). For \(1 \leq p < \infty\), we assume that

\[
\sum_{j=1}^{\infty} w_j^p = W^p < \infty.
\]

Consider the Banach space of sequences

\[
E = \ell_p^w = \left\{ y = (y^1, y^2, \ldots) : \sum_{j=1}^{\infty} |y^j|^p w_j^p < \infty \right\},
\]

with the norm

\[
\|y\|_{\ell_p^w} = \left( \sum_{j=1}^{\infty} |y^j|^p w_j^p \right)^{1/p}.
\]

The normalized Schauder basis in \(\ell_p^w\) is given by

\[
e_j = (0, \ldots, 0, 1/w_j, 0, \ldots),
\]

where the \(j\)th position is nonzero, \(j = 1, 2, \ldots\). Note that the basis constant \(P\) equals 1. The \(j\)th coordinate of the sequence \(y = (y^1, y^2, \ldots)\) in that basis is given by \(w_j y^j\). We assume that \(\eta = (\eta^1, \eta^2, \ldots) \in \ell_p^w\). The components of the system of equations \(f^j, j = 1, 2, \ldots\), are now treated as functions

\[
f^j : \ell_p^w \to \mathbb{R}.
\]

We assume about the components \(f^j\) and \(\eta^j\) that

\[
|f^j(\eta)| \leq M_j, \quad |f^j(y) - f^j(\bar{y})| \leq L_j \|y - \bar{y}\|_{\ell_p^w}, \quad \text{and} \quad |\eta^j| \leq T_j
\]

for \(y, \bar{y} \in \ell_p^w\), for some nonnegative numbers \(M_j, L_j\) and \(T_j\).

For \(y \in \ell_p^w\), we define \(f(y)\) as a sequence

\[
f(y) := (f^1(y), f^2(y), \ldots).
\]

We now show, under certain assumptions on \(M_j, L_j\) and \(T_j\), that the conditions (A1)–(A5) hold with \(r = 0\) for the pair \((f, \eta)\).

**Proposition 3** Let

\[
W_1 := \left( \sum_{j=1}^{\infty} M_j^p w_j^p \right)^{1/p}, \quad W_2 := \left( \sum_{j=1}^{\infty} L_j^p w_j^p \right)^{1/p} \quad \text{and} \quad W_3 := \left( \sum_{j=1}^{\infty} T_j^p w_j^p \right)^{1/p}
\]
be finite numbers. Then
\[ f : \ell^w_p \to \ell^w_p, \]  
\[ \|f(\eta)\|_{\ell^w_p} \leq W_1, \]  
\[ \|f(y) - f(\bar{y})\|_{\ell^w_p} \leq W_2\|y - \bar{y}\|_{\ell^w_p}, \]  
\[ \|f(y) - P_kf(y)\|_{\ell^w_p} \leq 2^{1-1/p} \left( \sum_{j=k+1}^{\infty} M^j_{j+1} w^p_j + \|y - \eta\|_{\ell^w_p} \sum_{j=k+1}^{\infty} L_j^p w^p_j \right)^{1/p}, \]  
and
\[ \|\eta - P_k\eta\|_{\ell^w_p} \leq \left( \sum_{j=k+1}^{\infty} T^j_{j+1} w^p_j \right)^{1/p}. \]

**Proof** Note that
\[ |f^j(y)| \leq \left((|f^j(\eta)| + L_j\|y - \eta\|_{\ell^w_p})^p \right) \leq 2^{p-1} \left(|f^j(\eta)|^p + L_j^p\|y - \eta\|_{\ell^w_p}^p\right), \]
which gives
\[ \|f(y)\|_{\ell^w_p} \leq 2^{p-1} \left(W_1^p + W_2^p\|y - \eta\|_{\ell^w_p}^p\right). \]
Hence, (54) holds. The proofs of (55) and (56) are straightforward.

To show (57), we see that
\[ \|f(y) - P_kf(y)\|_{\ell^w_p} = \sum_{j=k+1}^{\infty} |f^j(y)|^p w^p_j, \]
and we use (59). We get
\[ \|f(y) - P_kf(y)\|_{\ell^w_p} \leq \sum_{j=k+1}^{\infty} 2^{p-1} \left(|f^j(\eta)|^p w^p_j + L_j^p\|y - \eta\|_{\ell^w_p}^p w^p_j\right), \]
which yields (57). The proof of (58) is similar.

Proposition 3 yields that the system that we started with has the form
\[ z'(t) = f(z(t)), \quad t \in [0, 1], \quad z(0) = \eta \]
in the Banach space \( E = \ell^w_p \), and the pair \((f, \eta)\) satisfies (A1)–(A5) with \( r = 0 \). The sequences \( \gamma(k) \) and \( \delta(k) \) are defined by the right-hand sides of (58) and (57), respectively, taking into account that \( y \in K(\eta, R) \).

There exists a unique solution \( z : [a, b] \to \ell^w_p \), which can be approximated using the truncated Euler algorithm described in Section 3.

We apply the Euler algorithm on the uniform mesh with \( h_k = (b - a)/n \) and \( N_{-1} = N_0 = \ldots = N_{n-1} = N \). The Euler approximation in \([a, b]\) to the solution \( z \), where
\[ z(t) = (z^1(t), z^2(t), \ldots, z^N(t), z^{N+1}(t), \ldots) \in \ell^w_p \]
is defined by
\[ \bar{z}_n(t) = (\bar{z}_n^1(t), \bar{z}_n^2(t), \ldots, \bar{z}_n^N(t), 0, 0, \ldots) \]
Consider the following example. Let \( p \in (1, \infty) \), \( w_j = 1/j \), \( M_j = L_j = T_j = 1 \). Then \( W_1 = W_2 = W_3 = W = \left( \sum_{j=1}^{\infty} (1/j)^p \right)^{1/p} \). Furthermore, \( M = L = W \) and \( R \) of Lemma 1 are known numbers. Since

\[
\sum_{j=k+1}^{\infty} (1/j)^p \leq \int_{k}^{\infty} (1/x)^p \, dx = (1/k)^{p-1}/(p-1),
\]

we can take \( \gamma(k) = \delta(k)/(2R) = (1/k)^{1-1/p}/(p-1)^{1/p} \). One can see that the error of the truncated Euler algorithm in the class \( F_0 \) is bounded by

\[
e(\phi_{n,N}^{*}, N_{n,N}^{*}, F_0) \leq A(1/N)^{1-1/p} + B(1/n),
\]

where

\[
A = (2R + 1) \exp(W)/(p-1)^{1/p}, \quad B = W(R + 1)(3 \exp(W) - 2)
\]

are known absolute constants. The cost of the truncated Euler algorithm is equal to \( nc(N)N \).

Let \( \varepsilon > 0 \). Take, for instance, the cost function \( c(N) = N^\beta, \beta \geq 0 \), and consider minimization of the cost of the algorithm with the error bounded by \( \varepsilon \):

\[
\text{minimize } nc(N)N = nN^{1+\beta} \quad \text{subject to } \quad A(1/N)^{1-1/p} + B(1/n) \leq \varepsilon.
\]

That is, we wish to find the best (in the framework of the example) discretization and truncation parameters \( n \) and \( N \). The solution is given by

\[
n = n(\varepsilon) = \left\lceil \frac{B(p(\beta + 2) - 1)}{(p-1)\varepsilon} \right\rceil
\]

and

\[
N = N(\varepsilon) = \left\lceil \frac{A(\beta + 2 - 1/p)}{(\beta + 1)\varepsilon} \right\rceil^{p/(p-1)}.
\]

The minimal cost is then equal to

\[
n(\varepsilon)(N(\varepsilon))^{1+\beta} = O\left( \frac{1}{\varepsilon} (p(\beta+2)-1)/(p-1) \right).
\]

For example, if \( p = 2 \) and \( \beta = 1 \), then the minimal cost is \( O((1/\varepsilon)^5) \). The constants in the ‘\( O \)’ notation are known absolute constants. For comparison, if we solve a finite-dimensional system of \( N \) equations with fixed \( N \), then the cost of the Euler algorithm is \( O(1/\varepsilon) \), and \( N \) enters the constant.

**Remark 3** Note that the weights \( w_j \) may enter the formulation of the problem through the norm in \( E \), or through the choice of a subclass of problems to be considered. Let, for example, \( E = \ell^w_p \) with the norm defined above. Consider the class of right-hand side functions \( f : \ell^w_p \to \ell^w_p \) and initial conditions defined by (52) with \( M_j = L_j = T_j = 1 \), and the problem

\[
z'(t) = f(z(t)), \quad z(0) = \eta, \quad t \in [0,1].
\]
Here the weights define the norm of the space, and they do not enter the definition of the class of right-hand side functions and initial conditions.

The problem can be reformulated as follows. For a sequence \( y = (y^1, y^2, \ldots) \in \ell_p \), let \( \tilde{f}(y) \) be a sequence \( (\tilde{f}^1(y), \tilde{f}^2(y), \ldots) \), where

\[
\tilde{f}^j(y) = w^j f^j(y^1/w_1, y^2/w_2, \ldots).
\]

Then \( \tilde{f} : \ell_p \to \ell_p \) with the standard norm (which does not depend on \( w_j \)). Consider the problem

\[
\tilde{z}'(t) = \tilde{f}(\tilde{z}(t)), \quad \tilde{z}(0) = \tilde{\eta}, \quad t \in [0,1],
\]

with \( \tilde{\eta} = (w_1\eta^1, w_2\eta^2, \ldots) \). Note that both initial value problems are equivalent, since

\[
\tilde{z}^j(t) = w_j z^j(t) \text{ for } j = 1, 2, \ldots.
\]

The restrictions (52) are equivalent to the following restrictions on \( \tilde{f}^j \) and \( \tilde{\eta}^j \)

\[
\frac{1}{w_j} |\tilde{f}^j(\tilde{\eta})| \leq 1, \quad \frac{1}{w_j} |\tilde{f}^j(y) - \tilde{f}^j(\bar{y})| \leq \|y - \bar{y}\|_p, \quad \frac{1}{w_j} |\tilde{\eta}^j| \leq 1.
\]

In the alternative formulation, the weights appear in the restrictions on \( \tilde{f} \) and \( \tilde{\eta} \), not in the norm of the space, see e.g. [8], p. 109.

6 Auxiliary facts

For convenience of the reader, we recall some well known facts that are used in this paper. Let \( \alpha : [a, b] \to E \). We define the Riemann integral

\[
\int_a^b \alpha(t) \, dt
\]

(an element of \( E \)) in the same way as we do for real functions as a limit of Riemann sums, see e.g. [16] or [18]. If \( \alpha(t) = \sum_{j=1}^{\infty} \alpha^j(t)e_j \) is Riemann integrable, then \( \alpha^j \) are also Riemann integrable real functions and

\[
\int_a^b \alpha(t) \, dt = \sum_{j=1}^{\infty} \left( \int_a^b \alpha^j(t) \, dt \right) e_j.
\]

We have that \( \alpha_1(t)e_1 = P_1 \alpha(t) \) and \( \alpha^j(t)e_j = (P_j - P_{j-1}) \alpha(t) \) for \( j \geq 2 \). If \( \alpha \) is a continuous function then \( \alpha^j \) are continuous, if \( \alpha \) is a Lipschitz function with a constant \( C \), then \( \alpha^j \) are Lipschitz functions with the constant \( 2PC \).

Let \( \alpha \) be \( k \) times Fréchet differentiable in \( [a, b] \). The derivative \( \alpha^{(k)}(t) \) can be identified with an element of \( E \), i.e., \( \alpha^{(k)} : [a, b] \to E \). Then \( \alpha^j \) are also \( k \) times differentiable functions, and

\[
\alpha^{(k)}(t) = \sum_{j=1}^{\infty} (\alpha^j)^{(k)}(t)e_j.
\]
Let \( a = t_0 < \ldots < t_p = b \). Let \( w : [a, b] \rightarrow E \) be an interpolation polynomial of degree at most \( p \) (i.e., a function of the form \( w(t) = \sum_{i=0}^{p} t^i a_i \) for some \( a_i \in E \)) such that

\[
    w(t_j) = \alpha(t_j), \quad j = 0, 1, \ldots, p.
\]

As in the case of real-valued functions, one can see that the interpolation conditions are satisfied for

\[
    w(t) = \sum_{i=0}^{p} \prod_{s=0, s \neq i}^{p} \frac{t - t_s}{t_i - t_s} \alpha(t_i).
\]

The coefficients \( w^j \) of \( w \) in the basis \( \{e_j\} \) are real-valued polynomials of degree at most \( p \).

Let \( \alpha \in C^{p+1}([a, b], E) \). In the same way as for real-valued functions, one can prove that the remainder \( R(t) = \alpha(t) - w(t) \) of the interpolation formula can be written in the integral form as

\[
    R(t) = \prod_{i=0}^{p} (t - t_i) G(t),
\]

where \( G(t) \in E \) is given by (see e.g. [10])

\[
    G(t) = \int_{0}^{1} \int_{0}^{\xi_0} \cdots \int_{0}^{\xi_{p-1}} \alpha^{(p+1)}(t + \xi_0(t_0 - t) + \ldots + \xi_p(t_p - t_{p-1})) \, d\xi_p \cdots d\xi_1 \, d\xi_0. \quad (60)
\]

Let \( D \subset E \) be an open nonempty subset, and \( g : D \rightarrow E \) be a \( k \) times Fréchet differentiable function in \( D \),

\[
    g(y) = \sum_{j=1}^{\infty} g^j(y)e_j.
\]

Since \( g^1(y)e_1 = P_1g(y) \) and \( g^j(y)e_j = (P_j - P_{j-1})g(y) \) for \( j \geq 2 \), by the definition of Fréchet derivative we see that \( g^j \) are also \( k \) times Fréchet differentiable functions. If \( \|g^{(k)}(y)\| \leq Z \) for some constant \( Z \), then \( \|g^{(k)}(y)\| \leq 2PZ \), where the first symbol \( \| \cdot \| \) means the norm of a \( k \)-linear operator in \( E^k \), while the second one means the norm of a \( k \)-linear functional.

7 Conclusions

We analyzed the finite-dimensional solution of initial value problems in infinite-dimensional Banach spaces. For \( r \)-smooth right-hand side functions, we showed an algorithm for solving such problems on a non-uniform mesh with variable dimensions. For a constant dimension \( N \), under additional assumptions, we proved its error and cost optimality (up to constants), as the truncation and discretization parameters \( N \) and \( n \) tend to infinity. The results were illustrated by a countable system in the weighted \( \ell_p \) space.

Acknowledgments We are indebted to Stefan Heinrich for reading the manuscript and giving his comments, see Remark 1.
References

[1] Bellman, R. (Ed.), (1973), Infinite Systems of Ordinary Differential Equations and Truncation – Chapter XV in Methods of Nonlinear Analysis, Math. in Sc. and Eng., vol. 61, Part 2, Pages iii-x, 1-261.

[2] Bellman, N.D., Adomian, G., (1985), Infinite Systems of Differential Equations – Chapter XIV in Partial Differential Equations, New Methods for Their Treatment and Solution, Springer Netherlands.

[3] Brzychczy, S., (2002), Existence and uniqueness of solutions of infinite systems of semilinear parabolic differential-functional equations in arbitrary domains in ordered Banach spaces, Math. and Comput. Model., 36, 1183–1192.

[4] Cartan, H., (1971), Differential Calculus, Hermann, Paris.

[5] Daun, T., (2011), On the randomized solution of initial value problems, J. Complexity, 27, 300–311.

[6] Daun, T., Heinrich, S., (2014) Complexity of parametric initial value problems in Banach spaces, J. Complexity, 30, 392–429.

[7] Daun, T., Heinrich, S., (2017) Complexity of parametric initial value problems for systems of ODEs, Mathematics and Computers in Simulation, 135, 72–85.

[8] Deimling, K., (1977), Ordinary Differential Equations in Banach Spaces, Springer-Verlag.

[9] Heinrich, S., (2013), Complexity of initial value problems in Banach spaces, J. Math. Phys. Anal. Geom., 9, 73–101.

[10] Jankowska, J., Jankowski, M., (1981), A Survey of Numerical Methods and Algorithms I, WNT, in Polish.

[11] Kacewicz, B., (1984), How to increase the order to get minimal-error algorithms for systems of ODE, Numer. Math., 45, 93–104.

[12] Kacewicz, B., (1988), Minimum asymptotic error of algorithms for solving ODE, J. Complexity, 4, 373–389.

[13] Kacewicz, B., Przybyłowicz, P., (2015), Complexity of the derivative-free solution of systems of IVPs with unknown singularity hypersurface, J. Complexity, 31, 75–97.

[14] Lewis, D. C., Jr, (1933), Infinite systems of ordinary differential equations with applications to certain second-order partial differential equations, Trans. of the American Math. Soc., 35, 792–823.
[15] LINDENSTRAUSS, J., TZAFRIRI, L., (1977), Classical Banach Spaces I, Springer-Verlag.

[16] LUSTERNIK, L. A., SOBOLEV, V. J., (1974), Elements of Functional Analysis, Hindustan Publishing Corporation (India).

[17] NOVAK, E., WOŹNIAKOWSKI, H., (2008), Tractability of Multivariate Problems, Vol. 1, Linear Information, European Math. Soc., Zürich.

[18] RALL, L. B., (1969), Computational Solution of Nonlinear Operator Equations, John Wiley and Sons.

[19] TEMAM, R., (1998), Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd Ed., Springer Science+Business Media New York.

[20] TRAUB, J.F., WASILKOWSKI G.W., WOŹNIAKOWSKI, H., (1988), Information-Based Complexity, Academic Press, New York.