Faster Least Squares Optimization

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Abstract

We investigate randomized methods for solving overdetermined linear least-squares problems, where the Hessian is approximated based on a random projection of the data matrix. We consider a random subspace embedding which is either drawn at the beginning of the algorithm and fixed throughout, or, refreshed at each iteration. We provide an exact finite-time analysis of the refreshed embeddings method for a broad class of random matrices including the Gaussian ensemble, an exact asymptotic analysis of the fixed embedding method with a Gaussian matrix, and a non-asymptotic analysis of the fixed embedding method for Gaussian and Subsampled Randomized Hadamard Transforms (SRHT) matrices, with and without a Heavy-ball momentum acceleration. Surprisingly, we show that, for Gaussian matrices, the refreshed sketching method with no momentum yields the same asymptotic rate of convergence as the fixed embedding method accelerated with momentum. Furthermore, we characterize optimal step sizes and prove that, for a broad class of random matrices including the Gaussian ensemble, momentum does not accelerate the refreshed embeddings method. Hence, among the class of randomized algorithms we consider, a fixed subspace embedding with momentum yields the fastest rate of convergence, along with the lowest computational complexity.

Then, picking the accelerated, fixed embedding method as the algorithm of choice among the randomized methods we consider, we obtain a faster algorithm by optimizing over the choice of the sketching dimension. We show that our choice of the sketch size yields an algorithm, for solving overdetermined least-squares problem, with a lower computational complexity compared to current state-of-the-art least-squares iterative methods based on randomized pre-conditioners. In particular, given the sketched data matrix, as the sample size grows, the resulting computational complexity becomes sub-linear in the problem dimensions. We validate numerically our guarantees on large sample datasets, both for Gaussian and SRHT embeddings.

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1 Least-Squares, Preconditioning and Randomized Newton Methods

We consider an overdetermined least-squares optimization problem

\[
\min_{x \in \mathbb{R}^d} \left\{ f(x) := \frac{1}{2} \|Ax - b\|_2^2 \right\},
\]

where \( A \in \mathbb{R}^{n \times d} \) is a data matrix with \( n \geq d \) and \( b \in \mathbb{R}^n \) is a vector of observations. Let \( x^* := A^*b \) be the optimal least-squares solution, where \( A^* \) denotes the pseudo-inverse of \( A \). In general, computing the solution \( x^* \) – by, first, forming the matrix \( A^TA \), and then, inverting either a QR decomposition or a Cholesky factorization of \( A^TA \) – takes \( O(nd^2) \) computational time, which can be prohibitive for a large sample size \( n \).

Sketching-based methods involve using a random matrix \( S \in \mathbb{R}^{m \times n} \) to project the data matrix \( A \) and/or the data vector \( b \) to a lower dimensional space \( (m \ll n) \), and then solving the approximated least-squares problem to obtain an approximate solution \( \tilde{x} \). There are many choices of random sketching matrices, that we discuss in Section 1.1. Given some choice of random embedding \( S \), a standard form of sketching least-squares is based on solving the approximated objective \( \tilde{O} := \min \|SAx - Sb\|_2^2 \) in which the data matrix-vector pair \((A,b)\) is approximated by its sketched version \((SA,Sb)\). A number of works (e.g., [22] [7] [14] [18]) have investigated the properties of the sketched solution \( \tilde{x} \), and, accordingly, we refer to it as the classical sketched solution. In the overdetermined regime \( n \gg d \), this approach can lead to substantial computational savings provided than \( m \ll n \). Indeed, in general, the resulting computational cost of solving the classical sketch least-squares problem is \( O(md^2 + C_{SA}) \), where \( C_{SA} \) is the cost of forming \((SA,Sb)\). Under appropriate choice of the sketching matrix \( S \) or in parallel computing architectures, the cost \( C_{SA} \) can be much lower than the classical cost of least-squares, \( O(nd^2) \).

There are various ways in which the quality of an approximate solution \( \tilde{x} \) can be assessed. A standard one is in terms of the prediction (semi)-norm \( \|A(\tilde{x} - x^*)\|_2 \). It is natural to wonder whether or not, for a reasonable sketch size, the resulting guarantees are satisfactory. For instance, using arguments from [7] or [18], it can be shown that, given a precision \( \varepsilon \), the classical sketched solution \( \tilde{x} \) using Gaussian subspace embeddings satisfies, with high probability,

\[
\|A(\tilde{x} - x^*)\|_2^2 \leq (1 + \varepsilon) \|Ax^* - b\|_2^2,
\]

provided that the sketch dimension \( m \) is selected so that \( m \gtrsim \varepsilon^{-1}d \). However, in many applications, such as statistical estimation, the precision \( \varepsilon \) needs to be as small as \( \varepsilon \approx \frac{d}{n} \), and for the classical sketched solution \( x^c \), it results in a sketching size \( m \gtrsim n \). This scaling is undesirable in the regime \( n \gg d \), as the whole point of sketching is to have the sketch dimension \( m \) much lower than \( n \).

This apparent inefficiency of the classical sketched solution \( \tilde{x} \) has been investigated in [19]. Their analysis reveals that part of the sub-optimality is in fact due to sketching the observations vector \( b \rightarrow Sb \). Instead, they introduced an iterative method, namely, the iterative Hessian sketch, where only the data matrix \( A \) is sketched, as opposed to the data pair \((A,b)\). Concretely, starting from an initial point \( x_0 \in \mathbb{R}^d \) and considering a sequence of independent sketching matrices \( S_0, \ldots, S_{T-1} \), the iterative Hessian sketch performs the following update at time \( t \geq 0 \),

\[
x_{t+1} = x_t + \left( A^T S_t^T S_t A \right)^\dagger \left( A^T (b - Ax_t) \right),
\]

and returns the approximate solution \( \tilde{x}^H = x_T \). For a precision \( \varepsilon > 0 \), using a sketch size \( m = \alpha d \) with \( \alpha > 1 \), the approximate solution \( \tilde{x}^H \) satisfies \( \|A(\tilde{x}^H - x^*)\|_2^2 \leq \varepsilon \|A(x_0 - x^*)\|_2^2 \) provided
that $T \gtrsim \frac{\log(1/\varepsilon)}{\log \alpha}$, resulting in a total computational cost $\mathcal{O}\left((C_{\text{sketch}} \lor nd)\frac{\log(n/d)}{\log \alpha}\right)$, where $C_{\text{sketch}}$ is the cost of forming $S_tA$. Under appropriate choice of the sketching matrix $S_t$ or in parallel computational architectures, the cost $C_{\text{sketch}}$ can be reduced to $\mathcal{O}(nd)$, up to logarithmic factors. For statistical estimation, with $\varepsilon \asymp \frac{a}{d}$ and $\alpha = \mathcal{O}(1)$, computing the iterative Hessian sketch solution $\tilde{x}^H$ then requires no more than $\mathcal{O}(\log(1/\varepsilon)nd)$ operations, with a sketching size $m = \Omega(d)$, thus offering significant computational savings compared to direct computation of the solution $x^*$ or an $\varepsilon$-approximate solution $\tilde{x}^H$.

The iterative Hessian sketch belongs to the class of randomized quasi-Newton methods, that is, a method which starts from an initial guess $x_0 \in \mathbb{R}^d$ and then, repeats an update of the form

$$x_{t+1} = x_t - \mu_t H_t^{-1}g(x_t),$$

where $\mu_t$ is a step size, $g(x_t) = A^\top(Ax_t - b)$ is the gradient of the least-squares objective $f$ evaluated at the current iterate $x_t$, and $H_t$ is an approximation of the Hessian $A^\top A$ of $f$, based on a random sketch $S_tA$.

Instead of refreshing the approximation $H_t$ at each iteration, one can compute an approximate $H$ at the beginning of the iterative procedure, and then use $H_t = H$ for all $t \geq 0$ in the update (4). Hence, the matrix $H$ acts as a pre-conditioner to improve on the condition number of $A$, resulting in more efficient gradient descent updates. On the other hand, randomized right pre-conditioning methods – that is, methods which first compute a matrix $P$ based on a random sketch $SA$ and such that the matrix $AP^{-1}$ is well-conditioned, and then employ a first-order method such as gradient descent, Chebychev method, or, the Conjugate Gradient method to the pre-conditioned least-squares objective (1) – have been the subject of many works, which we discuss in more details in Section 2. Roughly, the number of iterations of these methods takes, in general, the form $\mathcal{O}(\kappa/\varepsilon)$ or $\mathcal{O}(\kappa \log(1/\varepsilon))$, where $\kappa$ is the condition number of the pre-conditioned matrix $AP^{-1}$, and $\kappa$ is usually a decreasing function of the sketch size.

In this paper, we provide an exact analysis of the iterative Hessian sketch, where, in each step, we compute the gradient $g(x_t) = A^\top(Ax_t - b)$, we draw a randomized approximate Hessian $H_t = A^\top S_t^\top S_tA$, and we perform the update (4). We will consider this algorithm with sketching matrices $S_t$ that are either refreshed (independently drawn at each iteration), or, fixed ($S_t = S_0$ for all $t \geq 0$). Further, we also provide an exact analysis of the iterative Hessian sketch with momentum acceleration, which, starting from initial points $x_0, x_1 \in \mathbb{R}^d$, at each iteration $t \geq 1$, performs the update

$$x_{t+1} = x_t - \mu_t H_t^{-1}g(x_t) + \beta(x_t - x_{t-1}).$$

### 1.1 Different types of randomized sketches

Various types of randomized sketches are possible, and we describe the few of them that we consider in our algorithmic procedures.

#### 1.1.1 Gaussian sketches

A classical sketch is based on a random matrix $S \in \mathbb{R}^{m \times n}$ with i.i.d. standard Gaussian entries $\mathcal{N}(0, \frac{1}{m})$. Many distributional properties of such a matrix $S$ are tightly characterized [S 24], and hence, the sketch $SA$ is the most straightforward to control from a probabilistic perspective – resulting in sharp choices of the algorithm’s parameters such as the sketch or step sizes. However, on the computational side, a disadvantage of Gaussian sketches is that they require a matrix multiplication $S \cdot A$ which, under no further assumption, is unstructured. That is, given a generic
data matrix $A \in \mathbb{R}^{n \times d}$, computing the sketch $SA$ requires $O(mnd)$ basic operations (using classical matrix multiplication). Under parallel computation, such a matrix multiplication can be efficiently accelerated, resulting in a computational time $O(nd)$.

1.1.2 Subsampled randomized Hadamard transforms

The second type of randomized sketch we consider is the subsampled randomized Hadamard transform (SRHT), for which matrix multiplication can be performed much more efficiently. SRHT matrices belong to the more general class of randomized orthonormal systems (ROS). Although our guarantees will apply, up to different universal constants, to other ROS sketches such as the subsampled randomized Fourier transform (SRFT), we restrict our attention to SRHT matrices for concreteness.

For an integer $n = 2^p$, with $p \geq 1$, we introduce the (normalized) Walsh-Hadamard transform, which is defined recursively as

$$H_n = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n/2} & H_{n/2} \\ H_{n/2} & -H_{n/2} \end{bmatrix},$$

with $H_1 = 1$. For a sketch size $m \geq 1$, the SRHT matrix is then defined as $S := \sqrt{\frac{m}{n}} RH_n D$, where $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix whose diagonal entries are i.i.d. random variables uniformly distributed on $\{-1, +1\}$, and, the matrix $R$ is a row-subsampling matrix, i.e., $R = [e_i, \ldots, e_{im}]^\top \in \mathbb{R}^{m \times n}$, where the indices $i_1, \ldots, i_m$ are selected uniformly at random (without replacement) from $\{1, \ldots, n\}$, and, $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$.

Due to the sparsity of the matrices $R$ and $D$, and, to the recursive structure of the matrix $H_n$, the sketched matrix $SA$ can be formed in $O(nd \log m)$ time (see, for instance, [3, 1, 7]).

1.1.3 Sparse Johnson Lindenstrauss transforms

For sparse data matrices, the sketching operation can be done faster if the sketching matrix is chosen from a distribution over sparse matrices. Several works developed sparse J.L. embeddings [2, 11, 12] and sparse subspace embeddings [16]. Here we describe a construction given by [16, 12]. Given an integer $s$, each column of the embedding matrix $S \in \mathbb{R}^{m \times n}$ is chosen to have exactly $s$ non-zero entries in random locations, each equal to $\pm 1$. Equivalently, we write $a_t \preceq b_t$ if $\lim_{t \to \infty} \frac{a_t}{b_t} < +\infty$. We write $a_t \succeq b_t$ if $b_t \preceq a_t$. We write $a_t \asymp b_t$ if $a_t \preceq b_t$ and $a_t \succeq b_t$. We write $a_t \sim b_t$ if $\lim_{t \to \infty} \frac{a_t}{b_t} = 1$.

For a vector $z \in \mathbb{R}^p$, the notation $\|z\|_2$ refers to the Euclidean norm of $z$, i.e., $\|z\|_2 = \sqrt{\sum_{i=1}^{d} z_i^2}$. For a matrix $M \in \mathbb{R}^{p \times q}$, we denote by $M^\dagger$ the pseudo-inverse of $M$. The condition number $\kappa(M)$ of $M$ is the ratio between its largest and smallest non-zero singular values. The operator norm of $M$, denoted by $\|M\|_2$, is the largest singular value of $M$. The spectral radius $\rho(M)$ of $M$ is defined as the largest module of its eigenvalues. In particular, for a symmetric matrix, it holds that $\rho(M) = \|M\|$. 

1.2 Notations and assumptions

For real-valued, positive sequences $\{a_t\}_{t \geq 0}$ and $\{b_t\}_{t \geq 0}$, we use the notation $a_t \lesssim b_t$ if $\limsup_{t \to \infty} \frac{a_t}{b_t} < +\infty$. Equivalently, we write $a_t \gtrsim b_t$ if $b_t \preceq a_t$. We write $a_t \asymp b_t$ if $a_t \preceq b_t$ and $a_t \succeq b_t$. We write $a_t \sim b_t$ if $\lim_{t \to \infty} \frac{a_t}{b_t} = 1$.
We denote by \( k \in \mathbb{N} \) the rank of \( A \). The optimal solution of the least-squares problem (1) is denoted by \( x^* \in \mathbb{R}^d \). A standard result states that \( x^* = A^\dagger b \). We write a singular value decomposition of \( A = U \Sigma V^\top \), where \( U \in \mathbb{R}^{n \times k} \) has orthonormal columns, \( V \in \mathbb{R}^{d \times k} \) has orthonormal columns and \( \Sigma \in \mathbb{R}^{k \times k} \) is a diagonal with positive entries. We denote by \( \kappa \) the condition number of \( A \), i.e., \( \kappa := \kappa(A) \).

### 2 Prior work and Our Contributions

Classical algorithms for solving the least-squares problem (1), are based on solving the normal equation \( A^\top A x = A^\top b \) by, first, forming the matrix \( A^\top A \), then computing a QR or Cholesky factorization of \( A^\top A \) and finally, inverting the latter factorization. In the large sample regime \( n \gg d \), the cost of forming \( A^\top A \) dominates the computational cost of these classical procedures, and is equal to \( \mathcal{O}(n d^2) \).

First-order methods such as gradient descent (G.D.) or the conjugate gradient method (C.G.) applied to (1) have computational cost at least \( \mathcal{O}(\sqrt{\kappa} \log(1/\varepsilon) n d) \), where \( \kappa \) is the condition number of the matrix \( A \), the term \( \varepsilon \) is the desired precision of the approximate solution, and \( \mathcal{O}(n d) \) is the cost of each G.D. (or C.G.) iteration. When the matrix \( A \) is ill-conditioned (\( \kappa \gg 1 \)), the resulting computational cost becomes prohibitively large.

Randomized pre-conditioning methods [21, 4] address the latter issue. For instance, instead of applying a first-order method to the least-squares problem (1), one first computes a right pre-conditioner \( P \in \mathbb{R}^{d \times d} \) such that the matrix \( P^\top A \) has a much lower condition number than \( A \). Then, starting from \( y_0 \in \mathbb{R}^d \), one solves iteratively the following least-squares problem

\[
\min_{y \in \mathbb{R}^d} \| P^\top y - b \|_2^2,
\]

up to a relative error \( \varepsilon \), that is, \( \| P^\top y_\varepsilon - A x^* \|_2^2 \leq \varepsilon \| P^\top y_0 - A x^* \|_2^2 \). Finally, one sets \( \tilde{x} := P^\top y_\varepsilon \).

In [21], the authors propose the following pre-conditioned C.G. method. For a sketch size \( m \), the algorithm computes a sketch \( S \) where \( S \) is a SRFT matrix, which takes \( \mathcal{O}(n d \log m) \) time. Then, a pivoted QR decomposition of \( S A \) is computed, that is, \( S A = QR \Pi \), where \( Q \in \mathbb{R}^{n \times d} \) has orthonormal columns, \( R \in \mathbb{R}^{d \times d} \) is upper-triangular and \( \Pi \in \mathbb{R}^{d \times d} \) is a permutation matrix. This factorization procedure takes \( \mathcal{O}(md^2) \) time. Finally, the algorithm solves problem (7) with the pre-conditioning matrix \( P = R \Pi \). Provided that the sketch size satisfies \( m \geq 4d^2 \), they show that the condition number of \( A P^\top \) is of order \( \mathcal{O}(1) \), resulting in a total computational cost of order

\[
nd \log(d) + d^4 + nd \log(1/\varepsilon).
\]

which is dominated by the term \( \mathcal{O}(n d \log(1/\varepsilon)) \), for large sample sizes \( n \gg d \) and small enough relative error \( \varepsilon \). To our knowledge, the computational complexity [4] is the best known complexity to achieve an \( \varepsilon \)-relative error solution in highly overdetermined least-squares problems.

In a related vein, sketching methods have also been used to restrict the optimization variable to a lower-dimensional subspace. In [2], the authors propose a randomized iterative method with linear convergence rate, which, at each iteration, performs the proximal update \( x_{t+1} = \arg\min_{x \in T} \| x - x_t \|_2^2 \), where the next iterate \( x_{t+1} \) is restricted to lie within an affine subspace \( T = x_t + \text{range}(A^\top S) \), and \( S \) is a \( n \times m \) dimension-reduction matrix with \( m \leq \min\{n, d\} \). In the context of kernel ridge regression, the authors of [26] propose to approximate the \( n \)-dimensional kernel matrix by sketching its columns to a lower \( m \)-dimensional subspace, chosen uniformly at random. From the low dimensional kernel ridge solution \( \alpha^* \in \mathbb{R}^m \), they show how to reconstruct an approximation \( \tilde{x} \in \mathbb{R}^n \) of the high dimensional solution \( x^* \in \mathbb{R}^n \). Provided that the sketching
dimension \( m \) is large enough, the estimate \( \bar{x} \) retains some statistical properties of \( x^* \), e.g., minimaxity. More generally, randomized approximate Newton methods, which involves left-sketching the data matrix, and random subspace optimization, which involves right-sketching the latter, have been extended to convex optimization problems (e.g., [20, 13]).

### 2.1 Our contributions

In this work, we analyze different versions of the randomized algorithm which uses the update

\[
x_{t+1} = x_t + \mu_t (A^T S_t^T S_t A)^\dagger A^T (b - Ax_t),
\]

as well as its accelerated version,

\[
x_{t+1} = x_t + \mu_t (A^T S_t^T S_t A)^\dagger A^T (b - Ax_t) + \beta (x_t - x_{t-1}).
\]

Our algorithm improves the time complexity over the previous algorithms by optimizing a trade-off between sketching accuracy and number of iterations. Our contributions are the following.

1. We provide an exact, closed-form formula of the expected squared error norm \( E[\|A(x_t - x^*)\|^2] \), and, the optimal sequence of step sizes \( \{\mu_t\}_{t \geq 0} \), when using refreshed sketching matrices \( \{S_t\}_{t \geq 0} \). Our analysis holds for a broad class of random embeddings, which includes in particular the Gaussian ensemble. Leveraging our analysis, we present an optimal step-size refreshed sketches algorithm (see Algorithm 1).

2. We consider the accelerated version (see Algorithm 2) of the refreshed sketches method (Alg. 1), which uses the update formula (10). Surprisingly, we show that Heavy-ball momentum does not improve the rate of convergence, that is, the optimal momentum parameter \( \beta^* \) is equal to 0.

3. Leveraging standard concentration results on the extremal eigenvalues of certain random matrices, we provide a high-probability convergence analysis of the error \( \|A(x_t - x^*)\|^2 \), when using a fixed random embedding \( S \) in the update formula (9). Using our probabilistic bounds to derive a step size \( \mu \), we present a fixed sketch algorithm (see Algorithm 3) that uses a fixed embedding \( S \), and its accelerated version (see Algorithm 4) based on the update formula (10). Then, we provide a detailed comparison of Algorithms 1, 2, 3 and 4. We show that the accelerated fixed sketched method (Alg. 4) converges faster than its unaccelerated version (Alg. 3), and, surprisingly, approximately as fast as the refreshed sketches strategy (Alg. 1). Thus, we pick the accelerated fixed sketch (Alg. 4) as the algorithm of choice among iterative Hessian sketching methods.

4. By picking the sketch size which trades-off optimally between the different computational costs involved in iterative subspace embedding methods, we show that the accelerated fixed sketch method (Alg. 4) achieves \( \varepsilon \)-error in time \( O(nd \sqrt{\log(1/\varepsilon)} + nd \log(d)) \) for not too small errors \( \varepsilon \), and, in time \( O\left( n d \frac{\log(1/\varepsilon)}{\log(n/d^2)} \right) \) for arbitrarily small values of \( \varepsilon \). Further, if the sketched matrix computation \( SA \) can be parallelized over the rows of \( S \), then the optimal time complexity can be reduced to \( O\left( n d \frac{\log(1/\varepsilon)}{\log(n/d^2)} \right) \). We validate numerically our computational trade-off analysis, and show significant time improvements over the current state-of-the-art pre-conditioned C.G. method ([21]).
5. We derive the exact rate of convergence of the fixed Gaussian sketch method (Alg. 3), in the asymptotic regime
\[ \frac{m}{d} \to \alpha, \quad \text{with } \alpha \in (1, +\infty). \]

Our analysis employs the limiting eigenvalue density of the Gaussian ensemble, namely, the Marcenko-Pastur density [15]. Consequently, we are able to characterize the optimal step size \( \mu^* \), as the solution of a polynomial, convex optimization problem. As opposed to the refreshed sketches method (Alg. 1), the optimal sketch size depends on the hindsight computational horizon, which opens a new range of theoretical and empirical research directions for randomized quadratic optimization solvers.

### 3 Refreshed Sketching Matrices – an Exact Error Formula

Given a sketch size \( m \geq 1 \), a sequence of independent matrices \( \{S_t\}_{t \geq 0} \) of size \( m \times n \), an initial point \( x_0 \in \mathbb{R}^d \), and a sequence of step sizes \( \{\mu_t\}_{t \geq 0} \), we consider the sequence of iterates \( \{x_t\}_{t \geq 0} \) produced by the update (9), that is, by the following algorithm.

**Algorithm 1: Refreshed Sketches.**

**Input:** Data matrix \( A \in \mathbb{R}^{n \times d} \), sketch size \( m \), initial point \( x_0 \in \mathbb{R}^d \), sequence of step sizes \( \{\mu_t\}_{t \geq 0} \).

1. for \( t = 0, 1, \ldots, T - 1 \) do
   2. Sample a sketching matrix \( S_t \in \mathbb{R}^{m \times n} \), independently of \( S_0, \ldots, S_{t-1} \).
   3. Compute the sketched matrix \( S_t A = S_t S_t^\top A \).
   4. Compute a factorization of the approximate Hessian \( H_t = S_t^\top S_t A \).
   5. Compute the gradient \( g_t = A^\top (Ax_t - b) \), and perform the update \( x_{t+1} = x_t - \mu_t H_t^\top g_t \).
1. end

Return the last iterate \( x_T \).

Intuitively, one can hope that re-drawing the randomized Hessian approximation \( A^\top S_t^\top S_t A \) provides a better convergence rate by decoupling randomness between different iterations.

We recall that we denote by \( U \in \mathbb{R}^{n \times k} \) the matrix of left singular vectors of \( A \), where \( k \) is the rank of \( A \). We define the error vector \( \Delta_t := A(x_t - x^*) \), where \( x^* \) is the least-squares solution of (1), and we aim to characterize the expected squared error norm \( \mathbb{E} \left[ \|\Delta_t\|_2^2 \right] \). First, we provide a general error formula for sketching matrices \( S_t \overset{i.i.d.}{\sim} P_S \), where \( P_S \) is a distribution over \( \mathbb{R}^{m \times n} \), which satisfies some bounded moment conditions, for which the Gaussian ensemble will be a special case.

#### 3.1 An exact error formula for bounded moments sketching matrices

Given a sketch size \( m \geq k \), let \( P_S \) be a distribution over \( \mathbb{R}^{m \times n} \). For \( S \sim P_S \), we denote by \( \sigma_1, \sigma_2, \ldots, \sigma_k \) the singular values of the matrix \( SU \), and we define the inverse moments
\[
\theta_1 := \mathbb{E} \left[ \frac{1}{k} \sum_{i=1}^{k} \frac{1}{\sigma_i^2} \right], \quad \text{(11)}
\]
\[
\theta_2 := \mathbb{E} \left[ \frac{1}{k} \sum_{i=1}^{k} \frac{1}{\sigma_i^4} \right]. \quad \text{(12)}
\]
Assumption 3.1. For $S \sim P_S$, we assume the following. The matrix $SU$ is full-column rank almost surely. The inverse moments $\theta_1$ and $\theta_2$ satisfy $\theta_1, \theta_2 < +\infty$. The right singular vectors $w_1, \ldots, w_k$ of the matrix $SU$ are isotropic, i.e., $\mathbb{E}[w_i w_i^\top] = \frac{1}{k}I_k$. Further, each $w_i$ is independent of its associated singular value $\sigma_i$.

An important example of such a class of embeddings is the Gaussian ensemble. Indeed, if $S$ is a random matrix of size $m \times n$ with independent entries $\mathcal{N}(0, \frac{1}{m})$, then the matrix $SU$ is a matrix of size $m \times k$, also with independent entries $\mathcal{N}(0, \frac{1}{m})$. It follows that the right singular vectors $\{w_i\}_{i=1}^k$ of $SU$ are uniformly distributed on the unit sphere $S^{k-1}$, and each $w_i$ is independent of its corresponding singular value $\sigma_i$. Further, according to standard trace calculations for Wishart matrices (see Lemma 2.3 in [10]), provided that $m \geq k + 4$, it holds that

$$
\theta_1 = \frac{m}{m - k - 1},
$$

$$
\theta_2 = \frac{m^2(m - 1)}{(m - k)(m - k - 1)(m - k - 3)}.
$$

Lemma 1. Let $m \geq k$ and $P_S$ be a distribution over $\mathbb{R}^{m \times n}$ satisfying Assumption 3.1. Then, it holds that

$$
\mathbb{E}\left[\left(U^\top S^\top SU\right)^{-1}\right] = \theta_1 I_k, \quad \mathbb{E}\left[\left(U^\top S^\top SU\right)^{-2}\right] = \theta_2 I_k.
$$

It follows, in particular, that $\theta_2^2 \leq \theta_2$.

Proof. Denote the matrix of eigenvectors of $U^\top S^\top SU$ by $W := [w_1, \ldots, w_k] \in \mathbb{R}^{k \times k}$, and let $\Sigma := \text{diag}(\sigma_1^2, \ldots, \sigma_k^2)$ be the matrix of associated eigenvalues. Let $j \in \{1, 2\}$. Then,

$$
\mathbb{E}\left[U^\top S^\top SU\right]^{-j} = \mathbb{E}[W \Sigma^{-2j} W^\top] = \sum_{i=1}^k \mathbb{E}\left[\sigma_i^{-j} w_i w_i^\top\right] = \sum_{i=1}^k \mathbb{E}\left[\sigma_i^{-2j}\right] \mathbb{E}[w_i w_i^\top],
$$

where the last equality holds by independence of $\sigma_i$ and $w_i$. By assumption, we have $\mathbb{E}[w_i w_i^\top] = k^{-1}I_k$. Therefore, we obtain

$$
\mathbb{E}\left[U^\top S^\top SU\right]^{-j} = \mathbb{E}\left[\frac{1}{k} \sum_{i=1}^k \sigma_i^{-2j}\right].
$$

It remains to show that $\theta_2^2 \leq \theta_2$, or equivalently, that $\theta_1 \leq \theta_2^{\frac{1}{2}}$. Let $z \in \mathbb{R}^k$ such that $\|z\|_2 = 1$. Then,

$$
\theta_1 = \theta_1 \|z\|_2^2 = \mathbb{E}\left[z^\top \left(U^\top S^\top SU\right)^{-1} z\right] \leq \mathbb{E}\left[z^\top \left(U^\top S^\top SU\right)^{-1} z \left(U^\top S^\top SU\right)^{-1} z\right]^\frac{1}{2} \tag{15}
$$

$$
\leq \mathbb{E}\left[z^\top \left(U^\top S^\top SU\right)^{-2} z\right]^\frac{1}{2} \tag{16}
$$

$$
= \theta_2^{\frac{1}{2}}. \tag{17}
$$

Inequality (i) is a Cauchy-Schwarz inequality. For inequality (ii), we used the fact that $zz^\top \preceq I_k$.

Theorem 1. Let $m \geq k$, and $P_S$ be a distribution over $\mathbb{R}^{m \times n}$ satisfying Assumption 3.1. Then, for any $T \geq 0$, we have the exact error formula

$$
\mathbb{E}\|\Delta_T\|_2^2 = \prod_{t=0}^{T-1} \left[\left(\frac{\theta_1}{\sqrt{\theta_2}} - \mu_t \sqrt{\theta_2}\right)^2 + 1 - \frac{\theta_2^2}{\theta_2}\right] \|\Delta_0\|_2^2.
$$

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Proof. The proof of Theorem 1 is deferred to Appendix 7.1.

**Corollary 1.** Let \( m \geq k \), and \( P_S \) be a distribution over \( \mathbb{R}^{m \times n} \) satisfying Assumption 3.1. Then, for any \( T \geq 0 \), the minimum error \( \mathbb{E} \| \Delta_T \|_2^2 \) is obtained by choosing \( \mu_t = \frac{\theta_1}{\theta_2} \) for all \( 0 \leq t < T \). Consequently,

\[
\left(1 - \frac{\theta_1^2}{\theta_2^2}\right)^T \| \Delta_0 \|_2^2 = \min_{\mu_0, \ldots, \mu_{T-1}} \mathbb{E} \| \Delta_T \|_2^2.
\]

**Proof.** Picking \( \mu_t = \frac{\theta_1}{\theta_2} \) for each \( 0 \leq t < T \) minimizes the right-hand-side of (18). It immediately yields the error formula \( \mathbb{E} \| \Delta_T \|_2^2 = \left(1 - \frac{\theta_1^2}{\theta_2^2}\right)^T \| \Delta_0 \|_2^2 \).

An important aspect of the results above is that the expected error formula is exact, and it holds universally for every input data \( A \) and \( b \) with equality. In particular, one cannot hope to do better by adjusting the step-sizes as long as they are independent of the randomness in the sketches. Note that, for Gaussian sketches, the above result holds for any sketch dimension \( m \) greater than \( k + 4 \). This is in contrast to earlier works on preconditioning, where \( S \) has to be a subspace embedding for convergence results to hold. An exception is [5], where the authors obtain upper-bounds for leverage score based on row sampling by directly analyzing the expectation of the error iteration.

### 3.2 Application to the Gaussian random ensemble

Leveraging the result of Corollary 1, we now specialize Algorithm 1 to the Gaussian random ensemble. Provided that \( m \geq k + 4 \), plugging-in the formulas (13) and (14) into the expression \( \theta_1/\theta_2 \), we get that the optimal step sizes are given by

\[
\mu_t = \frac{(m-k)(m-k-3)}{m(m-1)}.
\]

**Corollary 2.** For \( m \geq k + 4 \), if the sequence \( \{S_t\}_{t \geq 0} \) is taken from the Gaussian ensemble, then Algorithm 1 has expected squared error given by the exact formula

\[
\mathbb{E} \| \Delta_T \|_2^2 = \rho^* T \| \Delta_0 \|_2^2,
\]

where

\[
\rho^* = \frac{k + 1}{m - 1} + \frac{2}{(m - 1)(m - k - 1)}.
\]

**Proof.** Using the fact that \( \mathbb{E} \| \Delta_T \|_2^2 = \left(1 - \frac{\theta_1^2}{\theta_2^2}\right)^T \| \Delta_0 \|_2^2 \), and plugging-in the values \( \theta_1 \) and \( \theta_2 \) given in (13) and (14), we get \( 1 - \frac{\theta_1^2}{\theta_2^2} = \frac{k+1}{m-1} + \frac{2}{(m-1)(m-k-1)} = \rho^* \).

### 3.3 Momentum does not accelerate the refreshed Hessian sketch

A natural question is whether the rate of convergence of Algorithm 1 can be improved by adding a momentum term to the update formula (9). Surprisingly, in contrast to standard first-order methods, the answer is negative.

Let us precise the accelerated method we consider. Let \( m \geq k \) be a sketch size and \( P_S \) be a distribution over \( \mathbb{R}^{m \times n} \), such that Assumption 3.1 is satisfied. For initial points \( x_0, x_1 \in \mathbb{R}^d \), we
consider the heavy-ball update [10] with i.i.d. sketching matrices $S_0, \ldots, S_t, \ldots \sim P_S$, that is, the following algorithm.

**Algorithm 2: Accelerated Refreshed Sketches, with Heavy-ball Momentum.**

**Input:** Data matrix $A \in \mathbb{R}^{n \times d}$, sketch size $m$, initial points $x_0, x_1 \in \mathbb{R}^d$, step size $\mu \geq 0$ and momentum parameter $\beta \geq 0$.

1. for $t = 1, 2, \ldots, T - 1$ do
2. Sample a sketching matrix $S_t \in \mathbb{R}^{m \times n}$, independently of $S_0, \ldots, S_{t-1}$.
3. Compute the sketched matrix $S_A = S_t A$.
4. Compute a factorization of the approximate Hessian $H_t = S_A^T S_A$.
5. Compute the gradient $g_t = A^T (A x_t - b)$, and perform the update
   $$x_{t+1} = x_t - \mu H_t^t g_t + \beta (x_t - x_{t-1})$$
6. end
7. Return the last iterate $x_T$.

We recall that we define the error vector at time $t \geq 0$ as $\Delta_t := A(x_t - x^*)$. For a given momentum parameter $\beta \geq 0$ and step size $\mu \geq 0$, we define the upper and lower asymptotic rates of convergence as

$$\rho^+ (\beta, \mu) := \limsup_{t \to \infty} \frac{\mathbb{E}\left[\|\Delta_{t+1}\|_2^2\right]}{\mathbb{E}\left[\|\Delta_t\|_2^2\right]}, \quad \text{and} \quad \rho^- (\beta, \mu) := \liminf_{t \to \infty} \frac{\mathbb{E}\left[\|\Delta_{t+1}\|_2^2\right]}{\mathbb{E}\left[\|\Delta_t\|_2^2\right]}.$$  \hspace{1cm} (23)

According to Corollary 1, we already know that for $\beta = 0$ and $\mu = \frac{\theta_1}{\theta_2}$, the asymptotic rates of convergence are both equal to $\rho^* = 1 - \frac{\theta_1^2}{\theta_2^2}$ (provided that the initial error is non-zero, i.e., $\|A(x_0 - x^*)\|_2 \neq 0$). Given some initial points $x_0, x_1 \in \mathbb{R}^d$, we aim to characterize the optimal momentum parameter $\beta^*$, and the corresponding optimal step size $\mu^*$. As the next result states, surprisingly, a momentum correction does not improve the asymptotic rate of convergence, i.e., the optimal parameter $\beta^*$ is equal to 0, and the corresponding optimal step size is $\mu^* = \frac{\theta_1}{\theta_2}$.

**Theorem 2.** Let $x_0, x_1 \in \mathbb{R}^d$. Then, provided that $(A x_0, A x_1) \neq (A x^*, A x^*)$, it holds that

$$\rho^* = \inf_{\beta, \mu \geq 0} \rho^+ (\beta, \mu) = \inf_{\beta, \mu \geq 0} \rho^- (\beta, \mu),$$

where $\rho^* := 1 - \frac{\theta_1^2}{\theta_2^2}$. Further, both infima are attained at $\beta^* = 0$ and $\mu^* = \frac{\theta_1}{\theta_2}$.

**Proof.** The proof of Theorem 2 relies on analyzing the roots of the characteristic polynomial of the linear dynamics that the expected squared error $\mathbb{E}\left[\|\Delta_t\|_2^2\right]$ satisfies. We show in particular that, for any $\beta, \mu \geq 0$, the largest module of the roots is always greater than $\rho^*$. We defer details of the proof to Section 7.2. □

Let us illustrate numerically the result of Theorem 2. We set $n = 10^5$, $d = 200$ and we generate a data matrix $A \in \mathbb{R}^{n \times d}$ such that $A = U \Sigma V^T$, where $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{d \times d}$ have orthonormal columns, generated uniformly at random, and, $\Sigma = [\Sigma, 0]^T \in \mathbb{R}^{n \times d}$, where $\Sigma \in \mathbb{R}^{d \times d}$ is a diagonal matrix, with entries $\Sigma_{ii} = 0.97^i$, so that the condition number of $A$ is approximately equal to 400. We sample a planted vector $x \in \mathbb{R}^d$, with i.i.d. Gaussian entries $N(0, 1/d)$, and, given $x$, we sample a response vector $b \sim N(A x, I_n)$. The least-squares solution $x^*$ is then given by $x^* = \frac{A^T A}{\|A^T A\|}$.

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\((A^T A)^{-1} A^T b\). We run Algorithm 2 with Gaussian embeddings \(S_t \in \mathbb{R}^{m \times n}\) where \(m = 4d\), and we use the step size \(\mu = \frac{\theta_t}{\theta_t^2}\) and different values of the momentum parameter \(\beta\). Figure 1 shows the (scaled) prediction error \(\frac{1}{n} \|A(x_t - x^*)\|^2\) versus the number of iterations, for each value of \(\beta \in \{0, 0.03, 0.06, 0.12, 0.25, 0.5, 1.0\}\). As predicted by Theorem 2, momentum does not accelerate the refreshed sketches method, and the value \(\beta = 0\) yields a smaller error. Further, we see on Figure 1(c) that the exact rate of convergence \(\rho^* = 1 - \frac{\theta_t^2}{\theta_t^2}\) given by Theorem 1 is observed numerically, that is, the predicted line with (log-scale) slope \(-\log(\rho^*)\) interpolates the points \(n^{-1} \|A(x_t - x^*)\|^2\) when \(\beta = 0\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Error \(n^{-1} \|A(x_t - x^*)\|^2\) versus number of iterations for Algorithm 2 with refreshed Gaussian embeddings. We used \(n = 100000, d = 200\), and a sketch size \(m = 4d\). Results are averaged over 50 trials. We choose the optimal step size \(\mu^* = \frac{\theta_t}{\theta_t^2}\), and \(\beta \in \{0, 0.03, 0.06, 0.12, 0.25, 0.5, 1.0\}\).}
\end{figure}

4 Fixed Sketching Matrix – Exact Asymptotic Analysis

We consider the update [5] with a fixed step size \(\mu\) and a fixed subspace embedding \(S\), which is computed once at the beginning and fixed throughout.

We recall the following notations. The thin singular value decomposition of the rank \(k\) matrix \(A\) is denoted by \(A = U \Sigma V^T\), where \(U \in \mathbb{R}^{n \times k}\) has orthogonal columns, \(V \in \mathbb{R}^{d \times k}\) has orthogonal columns and \(\Sigma\) is a \(k \times k\) diagonal matrix with positive diagonal entries. Again, we define the error vector as \(\Delta_t := A(x_t - x^*)\).

4.1 An exact error formula for delocalized sketching matrices

Given a sketch size \(m \geq k\), let \(P_S\) be a distribution over \(\mathbb{R}^{m \times n}\). For \(S \sim P_S\), we denote by \(w_1, \ldots, w_k\) the eigenvectors of the matrix \(U^T S^T S U\), and \(\lambda_1, \ldots, \lambda_k\) their associated eigenvalues. Similarly to Assumption 3.1, we make the following assumptions about the distribution \(P_S\).

Assumption 4.1. Let \(S \sim P_S\). The matrix \(SU\) is full-column rank almost surely. Further, the eigenvectors \(w_1, \ldots, w_k\) of the matrix \(U^T S^T S U\) are isotropic, and each \(w_i\) is independent of its associated eigenvalue \(\lambda_i\).

As already shown in the previous section, the Gaussian ensemble satisfies the above assumptions. Our next result characterizes exactly the expected squared error, \(\mathbb{E} \|\Delta_T\|^2\).

**Theorem 3.** Let \(m \geq k\) and consider a sketching distribution \(P_S\) over \(\mathbb{R}^{m \times n}\), which satisfies Assumption 4.1. Then, for any horizon \(T \geq 0\) and step size \(\mu \geq 0\), it holds that

\[
\mathbb{E} \|\Delta_T\|^2 = \mathbb{E} \left[ \frac{1}{k} \sum_{i=1}^{k} \left( 1 - \frac{\mu}{\lambda_i} \right)^{2T} \right] \|\Delta_0\|^2.
\]
Proof. The proof is deferred to Section 7.3.

Hence, for a finite \( T \geq 0 \), we get, from Theorem 3, that the optimal step size \( \mu^* \) is a minimizer of the convex function \( \Gamma_{m,k}^{m,k} \), defined as

\[
\Gamma_{m,k}^{m,k}(\mu) := \mathbb{E}\left[ \frac{1}{k} \sum_{i=1}^{k} \left( 1 - \frac{\mu}{\lambda_i} \right)^{2T} \right].
\]

4.2 Application to the Gaussian ensemble – asymptotic analysis based on the Marcenko-Pastur distribution

We recall the definition of the Marcenko-Pastur distribution (see [15]) with parameter \((\alpha, \sigma) \in (1, +\infty) \times (0, +\infty)\), which we denote by \( \text{MP}(\alpha, \sigma) \), and which has density with respect to the Lebesgue measure on \( \mathbb{R} \) given by

\[
\nu(\lambda) = \frac{\alpha}{2\pi\sigma^2} \sqrt{\frac{1}{\lambda - \lambda_+}} \frac{1}{\lambda} 1_{[\lambda_-, \lambda_+]}(\lambda),
\]

where \( \lambda_+ = \sigma^2 \left( 1 - \sqrt{\frac{1}{\alpha}} \right)^2 \) and \( \lambda_- = \sigma^2 \left( 1 + \sqrt{\frac{1}{\alpha}} \right)^2 \).

Let \( S \) be an \( m \times n \) subspace embedding, with i.i.d. Gaussian entries \( \mathcal{N}(0, m^{-1}) \). By rotational invariance of the Gaussian distribution, the matrix \( SU \) is also a matrix with i.i.d. Gaussian entries \( \mathcal{N}(0, m^{-1}) \). Given a horizon \( t \geq 1 \), we aim to characterize the step size \( \mu_t^* \) which minimizes the function \( \Gamma_{m,k}^{m,k}(\mu) \) in an asymptotic regime. That is, we assume that \( m, k \to \infty \), such that \( m/k \to \alpha \in (1, +\infty) \). According to a classical result ([15]), it holds that the empirical distribution of the eigenvalues \( \lambda_1, \ldots, \lambda_k \) of the matrix \( U^\top SSU \) converges to the distribution \( \text{MP}(\alpha, 1) \). It follows that the function \( \Gamma_{m,k}^{m,k} \) converges pointwise to the function

\[
\Gamma_t^\alpha(\mu) := \mathbb{E}_{\lambda \sim \text{MP}(\alpha, 1)} \left[ \left( 1 - \frac{\mu}{\lambda} \right)^{2t} \right] = \frac{\alpha}{2\pi} \int_{\lambda_-}^{\lambda_+} \left( 1 - \frac{\mu}{\lambda} \right)^{2t} \sqrt{\frac{\lambda - \lambda_+}{\lambda - \lambda_-}} d\lambda.
\]

In particular, the function \( \Gamma_t^\alpha \) is strongly convex and goes to \( +\infty \) as \( |\mu| \to +\infty \), so that it admits a unique minimizer \( \mu_t^* \). Our next result provides an equivalent optimization problem, which relates finding the optimal step size \( \mu_t^* \) to finding the shift of a Marcenko-Pastur distributed random variable, which minimizes its moment of order \( 2t \).

Lemma 2. Let us introduce the parameters \( \nu = \left( \frac{\sqrt{\lambda_+ + \lambda_-}}{\sqrt{\lambda_+ - \sqrt{\lambda_-}}} \right)^2 \) and \( \sigma = \frac{\sqrt{\lambda_+ - \sqrt{\lambda_-}}}{2} \). Let \( t \geq 1 \), and define

\[
\beta_t^* = \arg\min_{\beta \in \mathbb{R}} \mathbb{E}_{X \sim \text{MP}(\nu, \sigma)} \left[ (\beta - X)^{2t} \right].
\]

Then, the minimizer \( \mu_t^* \) of \( \Gamma_t^\alpha \) satisfies

\[
\mu_t^* = \frac{1}{\beta_t^*}.
\]
Proof. This reduction follows from a simple change of variables, and we defer details of the calculation to Section 8.4.

The function $P(\beta) := \mathbb{E}_{X \sim \text{MP}(\nu, \sigma)} [(\beta - X)^{2t}]$ is a convex polynomial of degree $2t$, whose coefficients are expressed in terms of the known moment expressions of the distribution $\text{MP}(\nu, \sigma)$. Precisely, $P(\beta) = \sum_{k=0}^{2t} c_k \beta^k$, where $c_k = \binom{2t}{k} (-1)^{2t-k} m_{2t-k}$, and $m_j = \mathbb{E}_{X \sim \text{MP}(\nu, \sigma)} [X^j] = \sigma^2 \sum_{i=0}^{j-1} \binom{\nu^{-1}}{i} \binom{j-1}{i}$. Thus, one could efficiently find $\beta^*_1$ by minimizing $P$ using, for instance, a bisection method. However, to our knowledge, finding a closed-form expression or an analytic expansion of $\beta^*_1$ – and then a closed-form expression of the rate of convergence $\Gamma^*_t(1/\beta^*_1)$ – for a general value $t \geq 1$ is an open problem. For $t = 1$, the answer is the mean of the distribution $\text{MP}(\nu, \sigma)$, given by $\beta^*_1 = \sigma^2 = (1 - \frac{1}{\alpha})^{-2}$, so that $\mu^*_1 = (1 - \frac{1}{\alpha})^2$. For $t \to \infty$, we provide next a closed-form expression for the optimal step size, and the resulting rate of convergence. That is, we are interested in finding a step size $\mu^*$, if any, which satisfies the following property,

$$\liminf_{t \to +\infty} \frac{\Gamma^*_t(\mu)}{\Gamma^*_t(\mu^*)} \geq 1, \quad \text{for all } \mu \in \mathbb{R}.$$ 

Further, we aim to characterize the asymptotic rates of convergence $\rho^+_\infty$ and $\rho^-\infty$, defined as

$$\rho^+\infty := \limsup_{t \to +\infty} \frac{\Gamma^*_{t+1}(\mu^*)}{\Gamma^*_t(\mu^*)}, \quad \text{and,} \quad \rho^-\infty := \liminf_{t \to +\infty} \frac{\Gamma^*_{t+1}(\mu^*)}{\Gamma^*_t(\mu^*)}$$

Our next result gives the values of $\mu^*$, $\rho^+\infty$ and $\rho^-\infty$.

Theorem 4. Let $\mu^* = \frac{(1 - \frac{1}{\alpha})^2}{1 + \frac{1}{\alpha}}$. Then, for any $\mu \neq \mu^*$, it holds that

$$\lim_{t \to +\infty} \frac{\Gamma^*_t(\mu)}{\Gamma^*_t(\mu^*)} = +\infty, \quad \text{and} \quad \rho^\infty := 4\alpha (1 + \alpha)^2 = \rho^+_\infty = \rho^-\infty.$$

Proof. The proof is based on standard asymptotic methods for integral calculations ([?] ), and we defer the details to Section 7.4.

A natural question is how the term $\rho^\infty$ compares with the convergence rate $\rho^*$ obtained for the refreshed Gaussian sketches method (Alg. [4]), that is, $\rho^* = \frac{k+1}{m-1} + \frac{2}{(m-1)(m-k-1)}$. When $m/k \to \alpha$, we have $\rho^* \to \frac{4}{\alpha}$. Hence,

$$\frac{\rho^\infty}{\rho^*} \sim \frac{4}{(1 + \frac{1}{\alpha})^2}.$$ 

Since $\alpha \geq 1$, we have $4 \geq (1 + \frac{1}{\alpha})^2$, i.e., $\rho^\infty \geq \rho^*$. Thus, in such an asymptotic regime, the refreshed sketches method always converges faster, and up to four times faster when $\alpha \gg 1$.

5 Fixed Sketching Matrix – Non-asymptotic Analysis

Let $S \in \mathbb{R}^{m \times n}$ be a fixed sketching matrix and $\mu \geq 0$ a fixed step size. We recall that we denote by $U \in \mathbb{R}^{n \times k}$ the matrix of left-singular vectors of $A$, where $k$ is the rank of the data matrix $A$. Given a sequence of estimates $\{x_t\}_{t \geq 0}$, we define the error vector $\Delta_t$ at time $t \geq 0$ as $\Delta_t := A(x_t - x^*)$. Let $\lambda, \Lambda > 0$, and define the ($S$-measurable) event

$$\mathcal{E} := \left\{ \lambda \leq \lambda_{\min} \left( U^T S^T SU \right) \right\} \cap \left\{ \lambda_{\max} \left( U^T S^T SU \right) \leq \Lambda \right\}.$$
In Table 1, we recall classical results on high-probability bounds over the extremal eigenvalues of $U^\top S^\top SU$, for Gaussian, SRHT and sparse J.L. matrices, for which proofs and references are provided in Section 7.6.

Table 1. Extremal eigenvalues of $U^\top S^\top SU$, where $U \in \mathbb{R}^{n \times k}$ has orthonormal columns, and $S \in \mathbb{R}^{m \times n}$ a random embedding. We consider the event $E = \{\lambda \leq \lambda_{\min}(U^\top S^\top SU) \leq \lambda_{\max}(U^\top S^\top SU) \leq \Lambda\}$, and let $\eta > 0$ be a probability such that $P[E] \geq 1 - \eta$. The term $\alpha$ must satisfy $\alpha > 1$. For Gaussian embeddings, we leave a generic term $c > 0$ as a user’s choice constant, e.g., $c \sim 1/\sqrt{k}$. For SRHT embeddings, we define the quantity $\delta_{k,n} = (1 + \sqrt{8k - 1})/2$. For sparse J.L. matrices, the column sparsity parameter satisfies $s = \Theta(\sqrt{\alpha \log (k^2)})$ (we refer the reader to [16] for details).

| Embedding    | Sketch size $m$ | Lower bound $\lambda$ | Upper bound $\Lambda$ | $\eta$ |
|--------------|-----------------|------------------------|------------------------|-------|
| Gaussian, $\mathcal{N}(0, m^{-1})$ | $(1 + \sqrt{2c})^2\alpha k$ | $\left(1 - \frac{1}{\sqrt{\alpha}}\right)^2$ | $\left(1 + \frac{1}{\sqrt{\alpha}}\right)^2$ | $2e^{-ck}$ |
| SRHT         | $\frac{8}{5}ak\delta_{k,n}\log(2k^2)$ | $1 - \frac{1}{\sqrt{\alpha}}$ | $1 + \frac{1}{\sqrt{\alpha}}$ | $\frac{1}{\kappa}$ |
| Sparse J.L.  | $\Omega(ak\log^8(k^2))$ | $\left(1 - \frac{1}{\sqrt{\alpha}}\right)^2$ | $\left(1 + \frac{1}{\sqrt{\alpha}}\right)^2$ | $\frac{1}{\kappa}$ |

5.1 Optimal Step Size Convergence

We consider the update (9) for a fixed sketch $S$, which yields Algorithm 3.

Algorithm 3: Fixed Sketch.

**Input:** Data matrix $A \in \mathbb{R}^{n \times d}$, sketch size $m \geq k$, initial point $x_0 \in \mathbb{R}^d$, and step size $\mu \geq 0$.

1. Sample a sketching matrix $S \in \mathbb{R}^{m \times n}$.
2. Compute the sketched matrix $S_A = SA$.
3. Compute and cache a factorization of the approximate Hessian matrix $H = S_A^\top S_A$.
4. for $t = 0, 1, \ldots, T - 1$
5. Compute the gradient $g_t = A^\top (Ax_t - b)$.
6. Perform the update $x_{t+1} = x_t - \mu H^t g_t$.
7. end
8. Return the last iterate $x_T$.

**Lemma 3.** Consider the step size $\mu^* = \frac{2}{\lambda_{1+\Lambda}^{-1} + \Lambda}$. Conditional on the event $E$ is true, then, for any $T \geq 0$, the output of Algorithm 3 satisfies

$$\|\Delta_T\|_2 \leq \left(\frac{\kappa - 1}{\kappa + 1}\right)^T \|\Delta_0\|_2,$$

where $\kappa := \frac{3}{\Lambda}$. Thus, the asymptotic rate of convergence satisfies

$$\limsup_{t \to +\infty} \frac{\|\Delta_{t+1}\|_2^2}{\|\Delta_t\|_2^2} \leq \left(\frac{\kappa - 1}{\kappa + 1}\right)^2.$$

**Proof.** The proof is deferred to Section 7.5.

Hence, given a random embedding $S \in \mathbb{R}^{m \times n}$, if we know high-probability bounds $\lambda$ and $\Lambda$ on the extremal eigenvalues of the matrix $U^\top S^\top SU$, then the result of Lemma 3 is immediately actionable. Indeed, running Algorithm 3 with $\mu^* = \frac{2}{\lambda_{1+\Lambda}^{-1}}$ over a horizon $T \geq 0$, we obtain a solution $x_T$,
such that $\|A(x_T - x^*)\|_2 \leq \left(\frac{\kappa-1}{\kappa+1}\right)^T \|A(x_0 - x^*)\|_2$, with high-probability, where $\kappa = \frac{A}{\lambda}$. Our next result is an immediate consequence of Lemma 3.

**Corollary 3.** Let $\varepsilon > 0$ be a target relative error. Suppose that $\mathbb{P}[E] \geq 1 - \eta$, for some $\eta \in [0, 1]$. Then, using $\mu^* = \frac{2}{\lambda + \alpha - 1}$, it holds with probability at least $1 - \eta$,

$$\frac{\|\Delta T\|^2_2}{\|\Delta 0\|^2_2} \leq \varepsilon,$$

provided that $T \geq \frac{\log(1/\varepsilon)}{\log(1/\rho)}$, where $\rho := \left(\frac{A-\lambda}{\lambda+\lambda}\right)^{2}$.

In Table 2, we provide convergence rates and horizon’s lower bounds in order to achieve $\varepsilon$-relative error, for Gaussian, SRHT and sparse J.L. embeddings. These results are immediate consequences of Corollary 3 and the high-probability bounds on extremal eigenvalues presented in Table 1.

| Embedding     | Sketch size $m$ | Step size $\mu$ | Rate $\rho$ | Horizon $T$ | $\eta$ |
|---------------|-----------------|-----------------|-------------|-------------|-------|
| Gaussian      | $(1 + \sqrt{2c})^2 \alpha k$ | $\frac{(1 - \frac{1}{\alpha})^2}{1 + \frac{1}{\alpha}}$ | $\frac{4\alpha}{(1 + \alpha)^2}$ | $\frac{\log(1/\varepsilon)}{\log(1 + \alpha^2)}$ | $2e^{-ck}$ |
| SRHT          | $\frac{8\alpha k \delta_{k,n} \log(2k^2)}{\beta}$ | $1 - \frac{1}{\alpha}$ | $\frac{1}{\alpha}$ | $\frac{\log(1/\varepsilon)}{\log(\alpha)}$ | $\frac{1}{k}$ |
| Sparse J.L.   | $\Omega(\alpha k \log^8(k^2))$ | $\frac{(1 - \frac{1}{\alpha})^2}{1 + \frac{1}{\alpha}}$ | $\frac{4\alpha}{(1 + \alpha)^2}$ | $\frac{\log(1/\varepsilon)}{\log(1 + \alpha^2)}$ | $\frac{1}{k}$ |

5.2 **Acceleration through the Heavy-ball method**

Given a momentum parameter $\beta \geq 0$, we consider the update (10) for a fixed subspace embedding $S$, which yields Algorithm 4.

**Algorithm 4:** Accelerated Fixed Sketch, with Heavy-ball Momentum.

- **Input:** Data matrix $A \in \mathbb{R}^{n \times d}$, sketch size $m \geq k$, initial points $x_0, x_1 \in \mathbb{R}^d$, step size $\mu \geq 0$ and momentum parameter $\beta \geq 0$.
- 1. Sample a sketching matrix $S \in \mathbb{R}^{m \times n}$.
- 2. Compute the sketched matrix $S_A = SA$.
- 3. Compute and cache a factorization of the approximate Hessian matrix $H = S_A^T S_A$.
- 4. for $t = 1, 2, \ldots, T - 1$ do
- 5. Compute the gradient $g_t = A^T (Ax_t - b)$.
- 6. Perform the update $x_{t+1} = x_t - \mu_t H^t g_t + \beta(x_t - x_{t-1})$.
- 7. end
- 8. Return the last iterate $x_T$. 

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Lemma 4. Consider the step size $\mu^* = \frac{4}{(\sqrt{\lambda^{-1}} + \sqrt{\lambda^{-1}})^2}$ and the momentum parameter $\beta^* = \left(\frac{\lambda^{-\frac{1}{2}} - \Lambda^{-\frac{1}{2}}}{\lambda^{-\frac{1}{2}} + \Lambda^{-\frac{1}{2}}}\right)^2$. Conditional on the event $\mathcal{E}$ is true, it holds that for any $T \geq 0$, the output of Algorithm 4 satisfies
\[
\left\| \begin{bmatrix} \Delta_{T+1} \\ \Delta_T \end{bmatrix} \right\|_2 \leq \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} + a_T\right)^T \left\| \begin{bmatrix} \Delta_1 \\ \Delta_0 \end{bmatrix} \right\|_2,
\]
where $\kappa := \frac{1}{\lambda}$, and $\{a_k\}_{k \geq 0}$ is a ($\mathcal{S}$-measurable) sequence which converges almost surely to 0. Thus, the asymptotic rate of convergence satisfies
\[
\limsup_{t \to +\infty} \left( \frac{\left\| \Delta_{t+1} \right\|_2^2 + \left\| \Delta_t \right\|_2^2}{\left\| \Delta_t \right\|_2^2 + \left\| \Delta_{t-1} \right\|_2^2} \right) \leq \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^2.
\]

Proof. The proof is deferred to Section 7.7. \hfill \square

Corollary 4. Let $\varepsilon > 0$ be a target relative error. Suppose that $\mathbb{P}[\mathcal{E}] \geq 1 - \eta$, for some $\eta \in [0, 1]$. Then, using $\mu^* = \frac{4}{(\sqrt{\lambda^{-1}} + \sqrt{\lambda^{-1}})^2}$ and $\beta^* = \left(\frac{\lambda^{-\frac{1}{2}} - \Lambda^{-\frac{1}{2}}}{\lambda^{-\frac{1}{2}} + \Lambda^{-\frac{1}{2}}}\right)^2$, it holds with probability at least $1 - \eta$,
\[
\frac{\left\| \Delta_T \right\|_2^2}{\left\| \Delta_0 \right\|_2^2} \leq \varepsilon,
\]
provided that $T \geq \frac{\log(1/\varepsilon)}{\log((\sqrt{\sigma + \alpha})^2 - \sqrt{\sigma})}$, where $\rho := \left(\frac{\sqrt{\lambda} - \sqrt{\Lambda}}{\sqrt{\lambda} + \sqrt{\Lambda}}\right)^2$.

In Table 3, we provide convergence rates and horizon’s lower bounds in order to achieve an $\varepsilon$-accurate solution, for Gaussian, SRHT and sparse J.L. embeddings. These results are immediate consequences of Corollary 4 and the high-probability bounds on extremal eigenvalues presented in Table 1.

**Table 3.** Rate of convergence $\rho$ and horizon’s lower bounds $T$ to achieve $\varepsilon$-relative error. We consider the event $\mathcal{E} = \{\lambda \leq \lambda_{\min}(U^\top S^1 U) \leq \lambda_{\max}(U^\top S^1 U) \leq \Lambda\}$, and a probability $\eta > 0$ such that $\mathbb{P}[\mathcal{E}] \geq 1 - \eta$. For Gaussian embeddings, we specify the user’s choice constant $c > 0$ (introduced in Table 1) to $c = \frac{1}{\sqrt{k}}$, which will be convenient for next discussions in the paper. We define the oversampling function $\varphi(k) = (1 + \sqrt{\frac{\alpha}{\lambda}})^2$ for Gaussian embeddings, $\varphi(k) = \frac{8}{\sqrt{\lambda}} \delta_k \log(2\delta_k^2)$ for SRHT matrices, where $\delta_k = (1 + \sqrt{8\alpha^{-1} \log(2\delta_k^2)})^2$, and $\varphi(k) = \log^6(k^2)$ for sparse J.L. matrices. For readability, we discard the error term $a_T$ which converges to 0 as $T \to +\infty$ in the reported lower bounds on the horizon $T$. For further clarity, for SRHT matrices, we report a slightly smaller lower bound on $T$, which will not affect the next discussions in the paper (the latter lower bound on the horizon could be refined to $\log(1/\varepsilon) / \log\left(\frac{\sqrt{\varphi + \sqrt{\alpha}/\varphi} - \sqrt{\alpha}}{\sqrt{\varphi + \sqrt{\alpha}/\varphi} - \sqrt{\alpha}}\right)$).

| Embedding     | Sketch size $m$ | Step size $\mu$ | Momentum parameter $\beta$ | Convergence rate $\rho$ | Horizon $T$ | $\eta$ |
|---------------|----------------|----------------|----------------------------|------------------------|-------------|-------|
| Gaussian      | $\varphi(k)ak$ | $(1 - \frac{1}{a})^2$ | $\frac{1}{a}$ | $\frac{1}{a}$ | $\log(1/\varepsilon) / \log(\alpha)$ | $2e^{-\sqrt{\kappa}}$ |
| SRHT          | $\frac{2(\alpha - 1)}{\alpha + \sqrt{\alpha} - \alpha}$ | $\frac{\sqrt{\alpha} - \sqrt{\alpha - 1}}{\sqrt{\alpha} + \sqrt{\alpha - 1}}$ | $\frac{\sqrt{\alpha} - \sqrt{\alpha - 1}}{\sqrt{\alpha} + \sqrt{\alpha - 1}}$ | $\log(1/\varepsilon) / \log(\alpha)$ | $\frac{1}{k}$ |
| Sparse J.L.   | $\varphi(k)ak$ | $(1 - \frac{1}{a})^2$ | $\frac{1}{a}$ | $\frac{1}{a}$ | $\log(1/\varepsilon) / \log(\alpha)$ | $\frac{1}{k}$ |
5.3 Comparison of Convergence Rates

5.3.1 Gaussian Embeddings

Let \( m \geq 1 \) be a sketch size. We consider Gaussian embeddings \( S \in \mathbb{R}^{m \times n} \), where \( S \) has independent Gaussian entries with mean zero and variance \( \frac{1}{m} \). We would like to compare the performance of Algorithms 1, 2, 3 and 4 through their respective asymptotic rate of convergence, defined as

\[
\rho := \limsup_{t \to \infty} \left( \frac{\mathbb{E} \| A(x_{t+1} - x^*) \|^2_2}{\mathbb{E} \| A(x_t - x^*) \|^2_2} \right).
\]

(32)

For the sake of clarity, we assume that \( m, k \to +\infty \) and \( \frac{m}{k} \to \alpha \), for some \( \alpha > 1 \). Collecting the results from Corollary 2, Theorem 2, Table 2 and Table 3, we obtain the following comparison (see also Table 4 for a summary). The fixed sketch method (Alg. 3) has rate \( \rho^f = \frac{4\alpha}{(1+\alpha)^2} \), whereas the accelerated fixed sketch (Alg. 4) has rate \( \rho^f_m = \frac{1}{\alpha} \). It holds that \( \rho^f \geq \rho^f_m \) for any \( \alpha > 1 \), and as \( \alpha \to +\infty \), we get \( \rho^f_m \sim 4 \). As previously discussed, the refreshed sketches method (Alg. 1) has exact rate \( \rho^* = 1 - \frac{1}{\alpha} \left( 1 - \frac{1}{\alpha} \right)^2 \), and as \( k \to +\infty \), we get \( \rho^* \sim \frac{1}{\alpha} \). Further, according to Theorem 2, momentum does not accelerate the refreshed sketches strategy. In summary, we have the following theoretical comparison.

- Fixed Sketch (Alg. 3) \( \ll \) Refreshed Sketches (Alg. 1)
- \( \sim \) Refreshed Sketches, with Momentum (Alg. 2)
- \( \sim \) Fixed Sketch, with Momentum (Alg. 4).

Table 4. Let \( S \in \mathbb{R}^{m \times n} \) be a subspace embedding with i.i.d. Gaussian entries \( \mathcal{N}(0, \frac{1}{m}) \). For each algorithm, we consider its asymptotic rate of convergence \( \rho \), defined in (32). We let \( m, k \to +\infty \), with \( \frac{m}{k} \to \alpha \), and \( \alpha > 1 \).

| Algorithm | Rate \( \rho \) | Optimal \( \mu \) | Optimal \( \beta \) |
|-----------|----------------|-----------------|-----------------|
| Refreshed, no momentum (Alg. 1) | \( \frac{1}{\alpha} \) | \( (1 - \frac{1}{\alpha})^2 \) | - |
| Refreshed, momentum (Alg. 2) | \( \frac{1}{\alpha} \) | \( (1 - \frac{1}{\alpha})^2 \) | 0 |
| Fixed, no momentum (Alg. 3) | \( \frac{4\alpha}{(1+\alpha)^2} \) | \( \frac{(1-\frac{1}{\alpha})^2}{1+\frac{1}{\alpha}} \) | - |
| Fixed, momentum (Alg. 4) | \( \frac{1}{\alpha} \) | \( (1 - \frac{1}{\alpha})^2 \) | \( \frac{1}{\alpha} \) |

Remarkably, Algorithms 1, 2 and 4 – with their respective optimal parameters – have same asymptotic rates, and should exhibit a similar performance when dealing with large dimensions. That is, a fixed sketch with momentum strategy is equivalent to refreshing sketching matrices, for which momentum does not provide any advantage in our setting. Of natural interest is the practical performance of each algorithm, and if the high-probability upper bounds used to analyze Algorithms 3 and 4 provide a tight description of their empirical behavior.

In Figure 2, we observe that the empirical curves of the refreshed sketches method and the fixed sketch with momentum, and, their predicted curves, are all nearly superposed, which confirms our analysis (Table 4). On the other hand, the empirical performance of the fixed sketch (Alg. 3) is slightly better than its predicted curve. As the sketch size increases, the empirical and theoretical curves get closer to each other, which tends to confirm our infinite dimensional analysis (see
Figure 2. Error $n^{-1}\|Ax - x^*\|_2^2$ versus number of iterations, for refreshed Gaussian embeddings (Refreshed), fixed Gaussian embedding (Fixed), fixed Gaussian embedding with momentum (Fixed Mom.), the Conjugate Gradient algorithm pre-conditioned with a Gaussian embedding (p.C.G.), and, the Conjugate Gradient algorithm pre-conditioned with a Gaussian embedding using the sketch size prescribed in [21] (Pre-p.C.G.). We used $n = 100000$, $d = 500$, and the same sketch size $m = 4d$ for all algorithms (except Pre-p.C.G., for which we used $m = 4d$), with (a) $\alpha = 4$, (b) $\alpha = 8$, (c) $\alpha = 16$ and (d) $\alpha = 32$. The (red) dashed line represents the predicted error of (Fixed), which has (log-scale) slope $\alpha/1 + \alpha^2$. The (black) dotted line represents the predicted error of (Refreshed) with (log-scale) slope $\approx 1/\alpha$. Results are averaged over 50 trials.

Theorem [4]. Additionally, we compare our algorithms to the pre-conditioned CG method [21], with a Gaussian pre-conditioning matrix. We run the pre-conditioned CG method first, with the same sketch size as the other algorithms and, then, with the sketch size prescribed in [21], that is, $m = 4d$. We observe that the pre-conditioned CG method with same sketch size has the same empirical rate of convergence as Algorithms [1] and [4]. However, the pre-conditioned CG with sketch size $m = 4d$ performs worse compared to the other algorithms with larger sketch sizes. Hence, we get the following empirical comparison,

Pre-Conditioned C.G., Fixed Sketch (Alg. [3]) $\ll$ Refreshed Sketches (Alg. [1])

$(m = 4d)$

$\sim$ Refreshed Sketches, with Momentum (Alg. [2])

$\sim$ Fixed Sketch, with Momentum (Alg. [4])

$\sim$ Pre-Conditioned C.G.

5.3.2 SRHT Embeddings

Let $m \geq 1$, and $S \in \mathbb{R}^{m \times n}$ be a SRHT matrix. We assume that $m, k \to +\infty$, and $\frac{m}{k} \to \alpha$, for some $\alpha > 1$. According to our results provided in Tables [2] and [3], the fixed sketch method (Alg. [3]) has asymptotic convergence rate $\rho_f = \frac{1}{\alpha}$, and, with momentum (Alg. [4]), this rate becomes $\rho_{fm} = \frac{\sqrt{\alpha} - \sqrt{1 - \alpha}}{\sqrt{\alpha} + \sqrt{1 - \alpha}}$. It holds that $\rho_f \geq \rho_{fm}$ for any $\alpha > 1$, and as $\alpha \to +\infty$, we get that $\rho_{fm} \sim 1 - 1 + \frac{1}{\alpha}$, i.e., $\rho_{fm} \sim \frac{1}{\alpha^2}$, and hence, $\frac{\rho_f}{\rho_{fm}} \sim 4$.

We compare the numerical performance of the fixed sketch method, with and without momentum. Although our analysis leverages the tightest known bounds on the extremal singular values of SRHT embeddings, we observed that using the corresponding optimal parameters, as described in Tables [2] and [3] does not yield a tight prediction of the practical behavior of these algorithms. The observed gap is significant, and the empirical performance is stronger than its theoretical prediction. We hypothesize that this better practical performance might simply be due to the fact that the bounds we leverage can be improved, which results in pessimistic theoretical predictions.
We leave such investigations for future work. However, we further experimented our different algorithms with the parameters we prescribe for Gaussian embeddings. Interestingly, our empirical observations (see Figure 3) are the same as for Gaussian embeddings, regarding the fixed sketch with and without momentum, the pre-conditioned CG algorithm using the same sketch size, and the pre-conditioned CG method using the sketch size prescribed in [21]. However, regarding the refreshed sketches method, the parameters we prescribe for Gaussian embeddings did not yield a satisfying performance with SRHT embeddings, that is, the error diverged in all our experiments.

![Figure 3](image-url)

Figure 3. Error $n^{-1}\|A(x_t - x^*)\|^2_2$ versus number of iterations for fixed SRHT embedding (Fixed), fixed SRHT embedding with momentum (Fixed Mom.), the Conjugate Gradient algorithm pre-conditioned with a SRHT embedding (p.C.G.), and, the Conjugate Gradient algorithm pre-conditioned with a SRHT embedding using the sketch size prescribed in [21] (Pre-p.C.G.). We used $n = 100000$, $d = 500$, and the same sketch size $m = \alpha d$ for all algorithms (except Pre-p.C.G., for which we used $m = 4d$), with (a) $\alpha = 4$, (b) $\alpha = 8$, (c) $\alpha = 16$ and (d) $\alpha = 32$. The (red) dashed line represents the predicted error of (Fixed), which has (log-scale) slope $\frac{4}{1+\alpha^2}$. The (black) dotted line represents the predicted error of (Refreshed) with (log-scale) slope $\approx \frac{1}{\alpha}$. Results are averaged over 50 trials.

In summary, we observe a gap between the empirical and theoretical performance of our algorithm with SRHT embeddings, which motivates a refined analysis of the extremal eigenvalues of SRHT embeddings. However, treating SRHT embeddings as Gaussian ones yields the expected behavior for the fixed sketch with and without momentum. The performance of the two latter algorithms is equivalent to the pre-conditioned CG method with SRHT embeddings and using the same sketch size. However, the pre-conditioned CG method with the sketch size prescribed in [21] ($m = 4d$) has much weaker performance.

6 A Faster Least-Squares Algorithm: Optimal Sketch Size and Computational Complexity

We present a least-squares algorithm which has a better computational complexity than current state-of-the-art iterative least-squares solvers, in order to achieve a given relative error $\varepsilon$. Our method is based on choosing an optimal sketch size, that is, the sketch size which trades-off optimally between the different costs of the algorithmic procedure. But first, let us discuss the actual choice of the algorithm. According to our comparison of Alg. 1, 2, 3 and 4 and the pre-conditioned CG method [21], it holds that the fixed sketch with momentum (Alg. 4) exhibits the best rate of convergence, equally well – both theoretically and empirically – as the refreshed sketches strategy (Alg. 1), and – empirically – as the pre-conditioned CG method [21]. Thus, we consider as candidate the latter three algorithms. Since the refreshed sketches strategy involves re-computing a sketch matrix at each iteration, its computational cost is at least as large as the cost of the fixed sketch.
with momentum. Consequently, we prefer here the fixed sketch with momentum over the refreshed sketches strategy. Then, let us compare the fixed sketch with momentum and the pre-conditioned CG method, from a computational standpoint. As already mentioned, our empirical comparison of these two algorithms, in terms of accuracy versus number of iterations, suggests that they have the same convergence rate. Although we have provided a detailed mathematical analysis of the convergence rate of the accelerated fixed sketch in terms of the sketch size, we are not aware of any such analysis for the pre-conditioned CG method, and we cannot confirm formally these empirical observations. Further, recent empirical comparisons of these two algorithms in terms of accuracy versus number of floating-point operations (see, for instance, [17]) suggest that the fixed sketch with momentum has better computational properties. Consequently, we pick the latter as the algorithm of choice for our study of the optimal sketch size.

Given a sketch size of the form $m \equiv m(\alpha) = \alpha \varphi(d)d$ where $\varphi(d)$ is given in Table [3] and depends on the embedding choice – and a relative error $\varepsilon > 0$ (the ratio between the initial error and the target error), the computational cost of Alg. [4] is given by

$$C(\alpha) = C_S(\alpha, n, d) + md^2 + nd \frac{\log(\frac{1}{\varepsilon})}{\log \alpha},$$

where $C_S(\alpha, n, d)$ is the cost of forming the sketched matrix $SA$, the term $md^2$ corresponds to the factorization cost, that is, the cost of forming and factoring the approximate Hessian $(A^T S^T S A)$, and, the term $nd \frac{\log(\frac{1}{\varepsilon})}{\log \alpha}$ is the per-iteration cost times the number of iterations to achieve an $\varepsilon$-relative error solution. It should be noted that we do not consider in our discussions the numerical constants in front of each term. For instance, the approximate Hessian could be factorized through several different procedures, some of them improving by significant constants the factorization cost (e.g., Cholesky or QR factorization versus SVD). However, discarding these constants will not affect the scaling of the optimal sketch size in terms of the problem’s dimensions and the relative error $\varepsilon$.

Before providing our detailed results, let us distinguish two settings, that is, one has access to a parallel computational architecture and can fastly compute the sketched matrix, or, one does not. In the latter situation, when $S$ is an unstructured, dense matrix, e.g., a Gaussian embedding, then, computing the sketched matrix $S_A := SA$ takes, in general, $O(mnd)$ floating point operations. However, in a parallelized computational architecture, the sketching operation can be easily reduced to $O(nd)$. Therefore, we assume that, for a Gaussian embedding $S$, the matrix multiplication $S_A := SA$ can be performed in parallel, i.e., if $s_i^T$ is the $i$-th row of $S$, then the matrix-vector multiplications $s_i^T A$ can be split across $m$ different workers. Hence, as detailed in Section 6.2, the optimal sketch size $m^*$ trades-off between the factorization cost $md^2$ and the total iterations’ cost. Regarding SRHT embeddings, with no parallelization, the sketching operation can be performed in time $O(nd \log m)$ (see, for instance, [3]), so that having access to a parallelized environment is less critical, and we will not make such an assumption.

For the sake of clarity of the results to come, we assume that the matrix $A$ is full-column rank, i.e., rank $A = d$, and the least-squares problem [1] is highly overparameterized, that is, there exists a constant $\omega > 0$ such that $n = \varphi(d)d^{2+\omega}$. It should be noted that if such a scaling does not hold, i.e., $n \leq \varphi(d)d^2$, then the factorization cost dominates the total cost $C(\alpha)$, and the resulting optimal sketch size simply scales as $m \approx d$. Consequently, our discussion will be relevant for highly overparameterized settings.

Finally, we make the convenient assumptions that $\omega$ is not too small relatively to $d$, that is, $\omega \log d \geq 1$, and, that $\varepsilon$ is not too small relatively to $d$ and $\omega$, that is, $\log(1/\varepsilon) \leq d\omega \log d$. Note that for the latter assumption to hold, it suffices that $\varepsilon \geq e^{-d}$. Thus, for any $d \geq 22$, the latter assumption only excludes values of $\varepsilon$ which are far below 32-bits machine precision.

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6.1 Optimized Complexity for SRHT Embeddings

Given a relative precision $\varepsilon > 0$ and a sketch size $m = \alpha \varphi(d) d$ (see Table 3 for the expression of $\varphi(d)$), it holds with probability at least $(1 - \frac{1}{d})$ that a sufficient number of iterations $T$ for Algorithm 4 to achieve an $\varepsilon$-relative error solution $x_T$, that is, $\|A(x_T - x^*)\|_2^2 \leq \varepsilon$, is given by $T = \frac{\log(1/\varepsilon)}{\log(\alpha)}$. As previously detailed, the corresponding complexity $C(\alpha)$ is equal to $C(\alpha) = nd \log m + \frac{m}{d} + nd \frac{\log(1/\varepsilon)}{\log(\alpha)}$. We aim to find a closed-form approximation of the value $\alpha$ which minimizes the above cost. Due to the scaling $n = \varphi(d) d^{2+\omega}$, we introduce the convenient parameterization $\alpha = d^\gamma$, with $\gamma > 0$, which will ease the exposition of the results. Naturally, we require the sketch size $m$ to be smaller than the sample size $n$, which results in the constraint $\gamma < 1 + \omega$.

**Lemma 5.** Let $m = \alpha \varphi(d)$ with $\alpha = d^\gamma$ for some $\gamma \in (0, 1 + \omega)$. Let $S \in \mathbb{R}^{m \times n}$ be a SRHT matrix, and $\varepsilon > 0$ be a target relative error.

- If $\sqrt{\frac{\log(1/\varepsilon)}{\log(d)}} \leq \omega$, then we pick $\gamma^* = \sqrt{\frac{\log(1/\varepsilon)}{\log(d)}}$, which yields
  $$m^* = \frac{1}{d^\omega} \frac{n}{\omega}, \quad T \asymp \sqrt{\log(1/\varepsilon)}, \quad C^* \asymp nd \left( \log d + 2 \sqrt{\log(1/\varepsilon)} \right). \quad (33)$$

- Otherwise, if $\sqrt{\frac{\log(1/\varepsilon)}{\log(d)}} > \omega$, then we pick $\gamma^* = \omega + \frac{1}{\log d} \log \left( \frac{\log(1/\varepsilon)}{\omega \log d} \right)$, which yields
  $$m^* = \frac{n}{d^\omega} \frac{\log(1/\varepsilon)}{\omega \log d}, \quad T \asymp \frac{\log(1/\varepsilon)}{\omega \log d}, \quad C^* \asymp nd \left( \log d + 3 \frac{\log(1/\varepsilon)}{\omega \log d} \right). \quad (34)$$

In particular, for the standard statistical error $\varepsilon = \frac{d}{n}$, we obtain the following.

- If $\sqrt{\frac{\omega + 1}{\log d}} \leq \omega$, then we pick $\gamma^* = \sqrt{\frac{\omega + 1}{\log d}}$, which yields
  $$m^* = \frac{1}{d^\omega} \frac{n}{\sqrt{\frac{\omega + 1}{\log d}}}, \quad T \asymp (\omega + 1) \log d, \quad C^* \asymp nd \left( \log d + 2 \sqrt{(\omega + 1) \log d} \right). \quad (35)$$

- Otherwise, if $\sqrt{\frac{\omega + 1}{\log d}} > \omega$, then we pick $\gamma^* = \omega + \frac{1}{\log d} \log \left( \frac{\omega + 1}{\omega} \right)$, which yields
  $$m^* = \frac{(\omega + 1)}{\omega} n, \quad T \asymp \frac{(\omega + 1)}{\omega}, \quad C^* \asymp nd \left( \log d + 3 \frac{(\omega + 1)}{\omega} \right). \quad (36)$$

**Proof.** We defer the proof to Section 7.8. Essentially, when $\sqrt{\frac{\log(1/\varepsilon)}{\log(d)}} \leq \omega$, i.e., then the iterations’ cost remains relatively small. Thus, one can pick a small sketch size for which the factorization cost is negligible, and the (approximate) optimal sketch size trades-off between the sketching and iterations’ cost. On the other hand, when the required precision is large enough so that $\sqrt{\frac{\log(1/\varepsilon)}{\log(d)}} > \omega$, then one must take a larger sketch size to decrease the iterations’ cost, which results in a trade-off between the sketching, factorization and iterations’ cost. \( \square \)

Let us compare the above optimized computational costs to a state-of-the-art iterative solver. Consider the statistical relative error $\varepsilon = \frac{d}{n}$. As prescribed in [21], the cost of the pre-conditioned CG method scales as $C_{CG} \asymp nd (\omega + 2) \log d$. According to our previous analysis, for $\omega \geq \sqrt{\frac{\omega + 1}{\log d}}$,
we have \( C^* / C_{CG} \lesssim 1 / \omega^2 + 2 \sqrt{\omega + 1} / (\omega + 2)^{3 / \omega} \log d \), and, for \( \omega < \sqrt{\frac{1 + 3}{\omega^2}} \), we have \( C^* / C_{CG} \lesssim 1 / \omega^2 + 3 / \omega \log d \). In any case, optimizing the sketch size offers computational improvements, which get more and more significant as the dimensions \( \omega \) and \( d \) increase. For instance, when \( n = 10^7 \) and \( d = 50 \), the optimized sketch size algorithm is almost twice as fast than a state-of-the-art solver. We confirm our theoretical predictions through numerical simulations, reported in Figure 4.

\[
\begin{align*}
\text{(a) } & \epsilon = 10^{-4}, n = 5 \times 10^6, d = 20 \quad \text{(b) } \epsilon = 10^{-4}, n = 5 \times 10^6, d = 20 \\
\text{(c) } & \epsilon = 10^{-4}, n = 5 \times 10^6, d = 20 \quad \text{(d) } \epsilon = 10^{-16}, n = 5 \times 10^6, d = 20 \\
\text{(e) } & \epsilon = 10^{-4}, n = 5 \times 10^6, d = 40 \quad \text{(f) } \epsilon = 10^{-4}, n = 5 \times 10^6, d = 40 \\
\text{(g) } & \epsilon = 10^{-4}, n = 5 \times 10^6, d = 40 \quad \text{(h) } \epsilon = 10^{-16}, n = 5 \times 10^6, d = 40 \\
\text{(i) } & \epsilon = 10^{-4}, n = 10^7, d = 100 \quad \text{(j) } \epsilon = 10^{-4}, n = 10^7, d = 100 \\
\text{(k) } & \epsilon = 10^{-4}, n = 10^7, d = 100 \quad \text{(l) } \epsilon = 10^{-16}, n = 10^7, d = 100 \\
\end{align*}
\]

**Figure 4.** Computational performance of Algorithm 4 with SRHT embeddings. Are shown the sketching, factorization and total iterations’ times (seconds) to reach a given relative error \( \epsilon \), as a function of the sketch size, for values of \( \epsilon \in \{10^{-4}, 10^{-8}, 10^{-12}, 10^{-16}\} \), and for different dimensions \( n \) and \( d \) reported on each plot. The optimal predicted value \( m^* \) is shown on the x-axis. The (red) dashed line shows the computational time of the pre-conditioned CG method [21], using the prescribed sketch size \( m = 4d \).

### 6.2 Optimized Complexity for Gaussian Embeddings

Given a Gaussian embedding \( S \in \mathbb{R}^{m \times n} \), we assume that the sketched matrix \( S_A := SA \) can be formed in \( O(nd) \) computational time, through parallelization. Then, the sketching cost is negligible compared to the factor and iterations’ cost. Given a relative precision \( \epsilon > 0 \) and a sketch size \( m = \alpha \varphi(d) d \) (see Table 3 for the expression of \( \varphi(d) \)), it holds with probability at least \( 1 - e^{-\sqrt{d}} \) that a sufficient number of iterations \( T \) for Algorithm 4 to achieve an \( \epsilon \)-relative error solution \( x_T \), is given by \( T = \frac{\log(1/\epsilon)}{\log \alpha} \). Discarding the sketching cost, the total cost is then \( C(\alpha) = nd \left( \alpha d^{-\omega} + \frac{\log(1/\epsilon)}{\log \alpha} \right) \).

Naturally, we require the sketch size \( m \) to be smaller than \( n \), which enforces the constraint \( \alpha < d^{1+\omega} \).

**Lemma 6.** Let \( m = \alpha d \varphi(d) \), with \( \alpha \in (1, d^{1+\omega}) \). Let \( S \in \mathbb{R}^{m \times n} \) be a Gaussian embedding, and \( \epsilon > 0 \) be a target relative error. Then, picking \( \alpha = d^{2 \log(1/\epsilon) / \omega \log d} \) yields

\[
m^* = \frac{n \log(1/\epsilon)}{d \omega \log d}, \quad T \asymp \frac{\log(1/\epsilon)}{\omega \log d}, \quad C^* \asymp 2nd \frac{\log(1/\epsilon)}{\omega \log d}.
\]
In particular, for the standard statistical precision \( \varepsilon = \frac{4}{n} \), we obtain

\[
m^* = \frac{n}{d} \left( \frac{\omega + 1}{\omega} \right), \quad T \asymp \left( \frac{\omega + 1}{\omega} \right), \quad C^* \asymp 2nd \frac{\omega + 1}{\omega}.
\]  

(38)

For an arbitrary precision \( \varepsilon > 0 \), we have the following comparison ratio between the above optimized cost and the cost of the pre-conditioned CG method, \( \frac{C^*}{C_{CG}} \sim \frac{2}{\omega \log d} \approx \frac{2}{\log \left( \frac{m}{n^2} \right)} \). Thus, optimizing the sketch size yields cost improvements, which become more and more significant as the sample size \( n \) increases. For instance, when \( n = 10^5 \) and \( d = 50 \), then choosing our prescribed sketch size yields an algorithm which is almost 5 times faster than a state-of-the-art iterative solver. We confirm our theoretical predictions through numerical simulations, reported in Figure 5.

\[\begin{array}{cccc}
(a) & (b) & (c) & (d) \\
(e) & (f) & (g) & (h) \\
(i) & (j) & (k) & (l) \\
\end{array}\]

Figure 5. Computational performance of Algorithm 4 with Gaussian embeddings. Are shown the factorization and total iterations’ times (seconds) to reach a given relative error \( \varepsilon \), as a function of the sketch size, for values of \( \varepsilon \in \{10^{-4}, 10^{-8}, 10^{-12}, 10^{-16}\} \), and for different dimensions \( n \) and \( d \) reported on each plot. The optimal predicted value \( m^* \) is shown on the x-axis. The (red) dashed line shows the computational time of the pre-conditioned CG method [21], using the prescribed sketch size \( m = 4d \).

6.3 Complexity for Sparse J.L. Embeddings

Let \( S \in \mathbb{R}^{m \times n} \) be a sparse J.L. embedding matrix, with \( m = \alpha \varphi(d) \), and a column sparsity parameter \( s = \sqrt{n} \log^3(d^2) \). Assuming that the data matrix \( A \) is sparse with \( \|A\|_0 \) non-zero entries and \( \|A\|_0 \geq d \), the cost of forming the sketched matrix \( SA \) is \( O(s \|A\|_0) \). Given \( \varepsilon > 0 \), we need \( T = \log(1/\varepsilon) \) iterations of Algorithm 4 in order to achieve an \( \varepsilon \)-relative error solution. Each iteration requires \( O(s \|A\|_0) \) operations to compute the gradient \( g_t \), and, \( O(d) \) operations to compute the momentum term. Instead of computing and caching the approximate Hessian \( H = A^T S^T SA \) at the beginning of Algorithm 4 one can solve at each iteration the linear system \( Hz = g_t \). Using the conjugate gradient algorithm, it requires at most \( d \) iterations of C.G. (assuming exact
arithmetic), and each C.G. iteration requires $\mathcal{O}(s\|A\|_0)$ operations. Then, the total cost becomes

$$C(\alpha) = \|A\|_0 \log^3(d^2) \sqrt{\alpha} \left( 1 + \frac{\log(1/\varepsilon)}{\log \alpha} \right),$$

which is minimized for $\alpha$ close to 1. Thus, no significant trade-off appears for sparse data matrices using sparse J.L. embeddings.

### 6.4 Summary Tables for Optimized Costs

| Algorithm | Embedding | Computational complexity $C^*$ | Prob. failure $\eta$ | Sketch size $m^\ast$ | Horizon $T$ |
|-----------|-----------|-------------------------------|--------------------|-----------------|-------------|
| Alg. 4 $\gamma \leq \omega$ | Gaussian | $2nd \frac{\log(1/\varepsilon)}{\omega \log d}$ | $2e^{-\sqrt{d}}$ | $\frac{n \log(1/\varepsilon)}{\omega \log d}$ | $\frac{\log(1/\varepsilon)}{\omega \log d}$ |
| Alg. 4 $\gamma > \omega$ | SRHT | $nd \left( \log d + 2 \sqrt{\log(1/\varepsilon)} \right)$ | $d^{-1}$ | $\frac{1}{d_0} \frac{n \log(1/\varepsilon)}{\omega \log d}$ | $\sqrt{\log(1/\varepsilon)}$ |
| p.C.G | SRFT | $nd (\log d + (1/\varepsilon))$ | - | $4d$ | $\log(1/\varepsilon)$ |
| Standard C.G. | - | $nd \sqrt{\kappa} \log(1/\varepsilon)$ | - | - | $\sqrt{\kappa} \log(1/\varepsilon)$ |

| Algorithm | Embedding | Computational complexity $C^*$ | Prob. failure $\eta$ | Sketch size $m^\ast$ | Horizon $T$ |
|-----------|-----------|-------------------------------|--------------------|-----------------|-------------|
| Alg. 4 $\gamma \leq \omega$ | Gaussian | $2nd \frac{\omega+1}{\omega}$ | $2e^{-\sqrt{d}}$ | $\frac{n \omega+1}{\omega}$ | $\frac{\omega+1}{\omega}$ |
| Alg. 4 $\gamma > \omega$ | SRHT | $nd \left( \log d + 2 \sqrt{(\omega + 1) \log d} \right)$ | $d^{-1}$ | $\frac{1}{d_0} \frac{n \omega+1}{\omega}$ | $\sqrt{(\omega + 1) \log(d)}$ |
| p.C.G | SRFT | $nd (\omega + 2) \log d$ | - | $4d$ | $(\omega + 1) \log d)$ |
| Standard C.G. | - | $nd \sqrt{\kappa} \log(1/\varepsilon)$ | - | - | $\sqrt{\kappa} \log(1/\varepsilon)$ |

Table 5. Let $\varepsilon > 0$ be a target relative error. We assume that $\text{rank}(A) = d$. For Gaussian embeddings, we set $\varphi(d) = \left(1 + \sqrt{\frac{3}{2}}\right)^2$, and, for SRHT embeddings, $\varphi(d) = \frac{8}{3} \delta_{d,n} \log(2d^2)$, where $
_{\delta_{d,n}} := (1 + \sqrt{8d^{-1} \log(2dn)})^2$. We assume that there exists $\omega > 0$ such that $n = \varphi(d)d^2 + \omega$. We make the convenient assumptions that $\omega \log d \geq 1$ and $\omega d \log d \geq \log(1/\varepsilon)$. We define $\gamma^* = \frac{\log(1/\varepsilon)}{\log(d)}$. Are shown below values of the sketch size $m^\ast$, horizon $T$ and corresponding computational complexity $C^\ast$, which yields a solution with relative error less than $\varepsilon$, with failure probability less than $\eta$. p.C.G stands for the pre-conditioned CG method, as prescribed in [21]. We also report the cost of the standard CG method, with no pre-conditioning, and we denote by $\kappa$ the condition number of the data matrix $A$.

Table 6. We use the same assumptions and notations as in Table 5, but we set $\varepsilon$ to the standard minimax error $\frac{d}{\pi}$, for least-squares regression under zero-mean unit-variance Gaussian additive noise.
7 Proofs of Main Results

7.1 Refreshed Embeddings (Algorithm 1) – Proof of Theorem 1

Let $S \sim P_S$. We denote the thin singular value decomposition of the rank $k$ matrix $A$ by $A = UDV^\top$. It follows that $A^\top S^\top SA = V^\top DU^\top S^\top SUDV^\top$. By assumption, the matrix $SU$ is almost surely full-column rank, and we can write $(A^\top S^\top SA)^\dagger = VD^{-1}(U^\top S^\top SU)^{-1}D^{-1}V^\top$. Then,

$$A \left( A^\top S^\top SA \right)^{-1} A^\top = UDV^\top VD^{-1}(U^\top S^\top SU)^{-1}D^{-1}V^\top VDU^\top = U(U^\top S^\top SU)^{-1}U^\top.$$ 

Multiplying both sides of the update formula (9) by $A$, substracting by $Ax^*$ and using the normal equation $A^\top Ax^* = A^\top b$, we obtain

$$A(x_{t+1} - x^*) = \left(I_n - \mu_tU(U^\top S^\top_t S_tU)^{-1}U^\top\right)A(x_t - x^*).$$

That is, we have the equality $\Delta_{t+1} = (I_n - \mu_tU(U^\top S^\top_t S_tU)^{-1}U^\top)\Delta_t$, which, after multiplication of both sides by $U^\top$, yields

$$U^\top \Delta_{t+1} = U^\top \left(I_n - \mu_t(U^\top S^\top_t S_tU)^{-1}U^\top\right)\Delta_t$$

$$= \left(U^\top - \mu_t(U^\top S^\top_t S_tU)^{-1}U^\top\right)\Delta_t$$

$$= \left(I_k - \mu_t(U^\top S^\top_t S_tU)^{-1}\right)U^\top \Delta_t,$$

and then, taking the squared norm,

$$\|U^\top \Delta_{t+1}\|_2^2 = \left(U^\top \Delta_t\right)^\top \left(I_k - \mu_t(U^\top S^\top_t S_tU)^{-1}\right)^2 \left(U^\top \Delta_t\right).$$

Taking the expectation with respect to $S_t$ and using the fact that $S_t$ is independent of $S_0, \ldots, S_{t-1}$, and hence, independent of $\Delta_t$, we have

$$\mathbb{E}_{S_t} \left[\|U^\top \Delta_{t+1}\|_2^2\right] = \left(U^\top \Delta_t\right)^\top \mathbb{E} \left[\left(I - \mu_t(U^\top S^\top_t S_tU)^{-1}\right)^2\right] U^\top \Delta_t$$

$$= \left(U^\top \Delta_t\right)^\top \left(1 - 2\mu t \theta_1 + \mu_t^2 \theta_2 \right) I_k \left(U^\top \Delta_t\right)$$

$$= \left(1 - 2\mu t \theta_1 + \mu_t^2 \theta_2 \right) \|U^\top \Delta_t\|_2^2.$$

Taking the expectation with respect to $S_0, S_1, \ldots, S_{t-1}$, we obtain

$$\mathbb{E} \left[\|U^\top \Delta_{t+1}\|_2^2\right] = \left(1 - 2\mu t \theta_1 + \mu_t^2 \theta_2 \right) \mathbb{E} \left[\|U^\top \Delta_t\|_2^2\right].$$

By induction, it immediately follows that for any $T \geq 0$,

$$\mathbb{E} \left[\|U^\top \Delta_T\|_2^2\right] = \prod_{t=0}^{T-1} \left(1 - 2\mu t \theta_1 + \mu_t^2 \theta_2 \right) \|U^\top \Delta_0\|_2^2$$

$$= \prod_{t=0}^{T-1} \left[\left(\frac{\theta_1}{\sqrt{\theta_2}} - \mu_t \sqrt{\theta_2}\right)^2 + 1 - \frac{\theta_1^2}{\theta_2}\right] \|U^\top \Delta_0\|_2^2.$$
Finally, we observe that for any \( t \geq 0 \),
\[
\|UD_t\|_2 = \|UD_tDU^T(x_t - x^*)\|_2 = \|DV^T(x_t - x^*)\|_2 = \|UDV^T(x_t - x^*)\|_2 = A(x_t - x^*)\|_2 = \|\Delta_t\|_2,
\]
where equality \((i)\) follows from the fact that for any \( y \in \mathbb{R}^k \), \( \|Uy\|_2 = (\|y\|_2 + E)^2 = (y^TUy)^2 = \|y\|_2 \). Therefore, we get the desired equality,
\[
\mathbb{E}[\|\Delta_T\|_2^2] = \mathbb{E}[\|U^\top \Delta_T\|_2^2] = \prod_{t=0}^{T-1} \left( \left( \frac{\theta_t}{\theta_2} - \mu_t \sqrt{\theta_2} \right)^2 + 1 - \frac{\theta_t^2}{\theta_2} \right) \|U^\top \Delta_0\|_2^2 \]
\[
= \prod_{t=0}^{T-1} \left( \left( \frac{\theta_t}{\theta_2} - \mu_t \sqrt{\theta_2} \right)^2 + 1 - \frac{\theta_t^2}{\theta_2} \right) \|\Delta_0\|_2^2.
\]

**7.2 Momentum does not Accelerate the Refreshed Iterative Hessian Sketch – Proof of Theorem 2**

Let us introduce some preliminary notations and results. For \( t \geq 1 \), multiplying both sides of \((40)\) by \( A \) and then subtracting \( Ax^* \), we obtain the linear dynamics
\[
\Delta_{t+1} = \Delta_t - \mu U \left( U^\top S_t^\top S_t U \right)^{-1} U^\top \Delta_t + \beta (\Delta_t - \Delta_{t-1}). \tag{40}
\]
We denote \( \eta \equiv \eta(\beta, \mu) := (1 + \beta)^2 - 2\mu \theta_1 (1 + \beta) + \mu^2 \theta_2 \) and \( \gamma \equiv \gamma(\beta, \mu) := 1 + \beta - \mu \theta_1 \). Further, we denote \( \eta_0 \equiv \eta_0(\mu) := \eta(0, \mu) \) and \( \gamma_0 \equiv \gamma_0(\mu) := \gamma(0, \mu) \). It holds that \( \eta = \eta_0 + 2\gamma_0 \beta + \beta^2 \), and, \( \gamma = \gamma_0 + \beta \), for any \( \beta, \mu \geq 0 \).

Multiplying both sides of \((40)\) by \( \Delta_{t+1}^\top \), taking the expectation and using the independence of the sketching matrices, it follows that
\[
\mathbb{E}[\|\Delta_{t+1}\|_2^2] = \eta \mathbb{E}[\|\Delta_t\|_2^2] - 2\beta \mathbb{E}[\Delta_t^\top \Delta_t] + \beta^2 \mathbb{E}[\|\Delta_{t-1}\|_2^2]. \tag{41}
\]
Similarly, multiplying both sides of \((40)\) by \( \Delta_t^\top \) and taking the expectation, we get that the cross-term \( \mathbb{E}[\Delta_{t-1}^\top \Delta_t] \) satisfies the dynamics
\[
\mathbb{E}[\Delta_{t+1}^\top \Delta_t] = \gamma \mathbb{E}[\|\Delta_t\|_2^2] - \beta \mathbb{E}[\Delta_t^\top \Delta_{t-1}]. \tag{42}
\]
For \( t \geq 0 \), setting \( a_t := \mathbb{E}[\|\Delta_{t+1}\|_2^2] \), \( b_t := -\beta \mathbb{E}[\Delta_{t+1}^\top \Delta_{t+1}] \) and \( c_t := \beta^2 \mathbb{E}[\|\Delta_t\|_2^2] \), we obtain the three-dimensional linear dynamical system
\[
\begin{bmatrix}
a_{t+1} \\
b_{t+1} \\
c_{t+1}
\end{bmatrix} =
\begin{bmatrix}
\eta & 2\gamma & 1 \\
-\beta \gamma & -\beta & 0 \\
\beta^2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a_t \\
b_t \\
c_t
\end{bmatrix}, \tag{43}
\]
which characteristic polynomial is given by
\[
\chi_{\mu, \beta}(\lambda) = \lambda^3 + \left( -\beta^2 + (1 - 2\gamma_0)\beta - \eta_0 \right) \lambda^2 + \beta \left[ \beta^2 - (1 - 2\gamma_0)\beta + 2\gamma_0^2 - \eta_0 \right] \lambda - \beta^3.
\]
For a polynomial $P \in \mathbb{C}[X]$ with degree $\deg(P) \geq 1$, we recall that its root radius $\Lambda(P)$ is the largest module of its roots, i.e.,

$$
\Lambda(P) := \max \{ |\lambda| \mid \lambda \in \mathbb{C}, \; P(\lambda) = 0 \}.
$$

(44)

For $\mu, \beta \in \mathbb{R}$, we denote by $\Lambda(\mu, \beta)$ the root radius of the characteristic polynomial $\chi_{\mu, \beta}$, i.e., $\Lambda(\mu, \beta) := \Lambda(\chi_{\mu, \beta})$.

**Theorem 7.1** (Minimal root radius). It holds that

$$
\rho^* = \inf \left\{ \Lambda(\mu, \beta) \mid \mu \geq 0, \beta \geq 0 \right\},
$$

(45)

where $\rho^* := 1 - \frac{\theta_1^2}{\theta_2^2}$. Further, the infimum is attained at $\beta^* = 0$ and $\mu^* = \frac{\theta_1}{\theta_2}$.

**Proof.** The proof of Theorem 7.1 is deferred to Section 8.4.

For $\beta = 0$, we already know from Theorem 1 that the optimal step size is $\mu = \frac{\theta_1}{\theta_2}$, and the corresponding rate of convergence is $\rho^*$. Therefore, let us consider a positive momentum parameter, $\beta > 0$. We fix a step size $\mu \geq 0$, and we denote by $\lambda_1, \lambda_2, \lambda_3$ the complex roots of the polynomial $\chi_{\mu, \beta}$. By definition, we have $\Lambda(\mu, \beta) = \max\{ |\lambda_1|, |\lambda_2|, |\lambda_3| \}$. Given an initial point $(\Delta_0, \Delta_1) \in \text{range}(A)^2$ for the dynamics (40), we know – from fundamental results in linear dynamical systems – that there exists $\nu_1, \nu_2, \nu_3 \in \mathbb{C}$ and $i_1, i_2, i_3 \in \{0, 1, 2\}$ such that

$$
E[\|\Delta_t\|^2] = \nu_1 t^{i_1} \lambda_1^t + \nu_2 t^{i_2} \lambda_2^t + \nu_3 t^{i_3} \lambda_3^t,
$$

(46)

and the elements $(\lambda_j, i_j)$ – for $j = 1, 2, 3$ – are pairwise distinct.

**Definition 1.** A point $(\Delta_0, \Delta_1) \in \text{range}(A)^2$ is said to be singular if the resulting sequence $E[\|\Delta_t\|^2]$ satisfies

$$
\liminf_{t \to \infty} \left\{ \Lambda(\mu, \beta)^{-t} E[\|\Delta_t\|^2] \right\} = 0.
$$

We denote by $\Gamma$ the set of singular points.

Clearly, according to the decomposition (46), an initial point $(\Delta_0, \Delta_1)$ is singular if and only if $\nu_j = 0$ for all $j \in \{1, 2, 3\}$ such that $|\lambda_j| = \Lambda(\mu, \beta)$. Further, it is immediate to observe that the initial point $(\Delta_0, \Delta_1) = (0, 0)$ yields a sequence $\{\Delta_t\}$ which is constant and equal to 0. Therefore, the point $(0, 0)$ is singular.

We claim that $\Gamma = \{(0, 0)\}$. Let us first understand the implications of the latter statement. Suppose it is true. Then, for $(\Delta_0, \Delta_1) \neq (0, 0)$, according to the decomposition (46), it implies the existence of an index $j \in \{1, 2, 3\}$ and an exponent $i_j \in \{0, 1, 2\}$ such that $\nu_j \neq 0$, $|\lambda_j| = \Lambda(\mu, \beta)$, and, $E[\|\Delta_t\|^2] \sim \nu_j t^{i_j} |\lambda_j|^t$. Since the term $E[\|\Delta_t\|^2]$ is non-negative, the latter asymptotic scaling is equivalent to $E[\|\Delta_t\|^2] \sim |\nu_j| t^{i_j} |\lambda_j|^t$. Consequently, we have the asymptotic rate of convergence

$$
\lim_{t \to \infty} \frac{E[\|\Delta_{t+1}\|^2]}{E[\|\Delta_t\|^2]} = \Lambda(\mu, \beta).
$$

But, by Theorem 7.1 we have the inequality $\Lambda(\mu, \beta) \geq \rho^*$, which implies the claim of Theorem 2.

In conclusion, in order to prove Theorem 2 it suffices to show that $\Gamma = \{(0, 0)\}$.
Lemma 7. The subset of non-singular points

\[ \Gamma^c := \left\{ (\Delta_0, \Delta_1) \in \text{range}(A)^2 \mid \liminf_{t \to \infty} \left\{ \Lambda(\mu, \beta) - t\mathbb{E}[\|\Delta_t\|^2] \right\} > 0 \right\} \]

is not empty. In particular, the sets

\[ E := \left\{ (\Delta_0, 0) \mid \Delta_0 \in \text{range}(A), \Delta_0 \neq 0 \right\} , \]
\[ F := \left\{ (0, \Delta_1) \mid \Delta_1 \in \text{range}(A), \Delta_1 \neq 0 \right\} \]

are subsets of \( \Gamma^c \).

Proof. The proof of Lemma 7 is deferred to Appendix 8.3.

Lemma 8. The set of singular points \( \Gamma \) is a subspace of range(\( A \))^2, and \( \text{dim}(\Gamma) \leq \text{rank}(A) \).

Proof. First, we show that \( \Gamma \) is a subspace of range(\( A \))^2. The latter could be proved by leveraging standard results on spectral decomposition, but we provide a proof based on basic arguments. Since \( (0,0) \in \Gamma \), it suffices to show that \( \Gamma \) is stable by linear combination. Let \( (\Delta_0, \Delta_1), (\Delta'_0, \Delta'_1) \in \Gamma \), and let \( \alpha, \alpha' \in \mathbb{R} \). We define \( (\Delta_0, \Delta_1) = \alpha(\Delta_0, \Delta_1) + \alpha'(\Delta'_0, \Delta'_1) \). Given a sequence of sketching matrices \( (S_t) \), by linearity of the dynamics (40), the resulting sequence \( (\Delta_t) \) satisfies

\[ \Delta_t = \alpha \Delta_t + \alpha' \Delta'_t, \quad \forall t \geq 0. \] (49)

By triangular inequality, we get \( \|\Delta_t\|_2 \leq |\alpha|\|\Delta_t\|_2 + |\alpha'|\|\Delta'_t\|_2 \). Squaring the previous inequality and using the fact that \( (a + b)^2 \leq 2(a^2 + b^2) \) for any \( a, b \in \mathbb{R} \), we obtain that \( \|\Delta_t\|_2^2 \leq 2\alpha^2\|\Delta_t\|_2^2 + 2\alpha'^2\|\Delta'_t\|_2^2 \). Taking the expectation, it follows that

\[ \mathbb{E} \left[ \|\Delta_t\|_2^2 \right] \leq 2\alpha^2 \mathbb{E} \left[ \|\Delta_t\|_2^2 \right] + 2\alpha'^2 \mathbb{E} \left[ \|\Delta'_t\|_2^2 \right]. \]

and consequently,

\[ \limsup_{t \to \infty} \left\{ \Lambda(\mu, \beta) - t\mathbb{E}[\|\Delta_t\|^2] \right\} \leq 2\alpha^2 \limsup_{t \to \infty} \left\{ \Lambda(\mu, \beta) - t\mathbb{E}[\|\Delta_t\|^2] \right\} + 2\alpha'^2 \limsup_{t \to \infty} \left\{ \Lambda(\mu, \beta) - t\mathbb{E}[\|\Delta'_t\|^2] \right\} = 0. \]

Thus, \( \Gamma \) is stable by linear combinations and is a subspace of range(\( A \))^2. The fact that \( \Gamma \) is a strict subspace with dimension no greater than \( \text{rank}(A) \) is a consequence of Lemma 7. Indeed, suppose by contradiction that \( \text{dim} \Gamma \geq \text{rank} A + 1 \). Consider the set of non-singular points \( F \) defined in (48). Then, \( F := F \cup \{(0,0)\} \) is a subspace of range(\( A \))^2, with \( \text{dim} F = \text{rank} A \). Since both \( F \) and \( \Gamma \) are subspaces of range(\( A \))^2, which dimension is equal to \( 2 \text{rank}(A) \), it follows that the intersection \( F \cap \Gamma \) is a subspace with dimension at least 1, which is a contradiction.

Lemma 9. Let \( (\Delta_0, \Delta_1) \in \Gamma \). Then, it holds that \( (\Delta_1, \Delta_2) \in \Gamma \) almost surely.

Proof. Let \( S \equiv S_1 \) be the first sketching matrix. Then,

\[ \Delta_2 = \Delta_1 - \mu U(U^T S^T SU)^{-1} U^T \Delta_1 + \beta(\Delta_1 - \Delta_0). \]
Suppose, by contradiction, that the event $\mathcal{E} := \{(\Delta_1, \Delta_2) \notin \Gamma\}$ holds with probability $\eta > 0$. Let $\tilde{\Delta}_t$ be the sequence satisfying the dynamics \[^{[40]}\] started at the initial point $(\tilde{\Delta}_0, \tilde{\Delta}_1) = (\Delta_1, \Delta_2)$. Define

$$\nu(S) := \liminf_{t \to \infty} \Lambda(\mu, \beta)^{-t} \mathbb{E}[\|\tilde{\Delta}_t\|_2^2 \mid S]$$

Conditional on $\mathcal{E}$, it holds, by definition of the event $\mathcal{E}$, that $\nu(S) > 0$. Taking the expectation with respect to $S$, it follows that

$$\mathbb{E}_S \left[ \liminf_{t \to \infty} \Lambda(\mu, \beta)^{-t} \mathbb{E}[\|\tilde{\Delta}_t\|_2^2 \mid S] \right] = \mathbb{E}_S [\nu(S)] > \eta \mathbb{E}_S [\nu(S) \mid \mathcal{E}] > 0.$$

Further, using Fatou’s lemma and the tower rule of expectation, we obtain

$$\liminf_{t \to \infty} \left\{ \Lambda(\mu, \beta)^{-t} \mathbb{E}[\|\tilde{\Delta}_t\|_2^2] \right\} > 0.$$

Equivalently,

$$\liminf_{t \to \infty} \left\{ \Lambda(\mu, \beta)^{-t} \mathbb{E}[\|\Delta_t\|_2^2] \right\} > 0,$$

which contradicts the fact that $(\Delta_0, \Delta_1) \in \Gamma$. Therefore, we must have $(\Delta_1, \Delta_2) \in \Gamma$ almost surely, which concludes the proof.

We are ready to show that $\Gamma$ is reduced to the singleton $\{(0, 0)\}$. By contradiction, we assume that there exists a singular point $(\Delta_0, \Delta_1) \in \Gamma$ such that $(\Delta_0, \Delta_1) \neq (0, 0)$. From Lemma \[^{[7]}\] we know that $\Delta_1$ cannot be equal to 0. From Lemma \[^{[9]}\] we know that the point $(\Delta_1, \Delta_2)$ belongs to $\Gamma$ almost surely. Denoting $S \equiv S_1$ the first sketching matrix, we have

\[
(1 + \beta) \begin{bmatrix} \Delta_0 \\ \Delta_1 \\ \Delta_2 \end{bmatrix} - \begin{bmatrix} \Delta_1 \\ \Delta_1 \end{bmatrix} = \begin{bmatrix} (1 + \beta) \Delta_0 - \Delta_1 \\ \beta \Delta_0 + \mu U (U^T S^T S U)^{-1} U^T \Delta_1 \end{bmatrix}
\]

Since $\Gamma$ is stable by linear combinations, the point $\tilde{X}_S := (1 + \beta)[\Delta_0, \Delta_1]^T - [\Delta_1, \Delta_2]^T$ belongs to $\Gamma$. Therefore, we must have, almost surely, that

$$\tilde{X}_S = \begin{bmatrix} (1 + \beta) \Delta_0 - \Delta_1 \\ \beta \Delta_0 + \mu U (U^T S^T S U)^{-1} U^T \Delta_1 \end{bmatrix} \in \Gamma.$$}

Let $\Gamma'$ (resp. $\tilde{X}_S'$) be the projection of $\Gamma$ (resp. $\tilde{X}_S$) onto $(e_{n+1}, \ldots, e_{2n})$, where $(e_1, \ldots, e_{2n})$ is the canonical basis of $\mathbb{R}^{2n}$. We claim that we must have $\Gamma' = \text{range}(A)$. Indeed, observe, first, that the matrix $(U^T S^T S U)^{-1}$ follows an invert Wishart distribution. Since $\Delta_1 \neq 0$, the vector $v := (U^T S^T S U)^{-1} U^T \Delta_1$ has a density with respect to the Lebesgue measure on $\mathbb{R}^{\text{rank}(A)}$, which further implies that $Uv$, and thus, $\tilde{X}_S$, have a density with respect to the Lebesgue measure on $\text{range}(A)$. If $\Gamma'$ was a strict subspace of $\text{range}(A)$, then the measure of $\Gamma'$ would be equal to 0, and it would further imply $\mathbb{P}(\tilde{X}_S' \in \Gamma') = 0$, contradicting the fact that $\tilde{X}_S \in \Gamma$ almost surely. Consequently, we must have $\Gamma' = \text{range}(A)$. But, according to Lemma \[^{[8]}\] $\dim \Gamma \leq \text{rank}(A)$. Hence, we must have $\Gamma = \mathcal{F}$, where $\mathcal{F} = \{(0, \Delta_1) \mid \Delta_1 \in \text{range}(A)\}$. But according to Lemma \[^{[7]}\] the set $\overline{\mathcal{F}} - \{(0, 0)\}$ contains only non-singular points. The latter contradiction concludes the proof of Theorem \[^{[2]}\].

\[\square\]
7.3 Fixed Embedding – Exact Analysis of Algorithm 3, Proof of Theorem 3

Fix \( t \geq 0 \). After subtracting \( x^* \) from the update formula (9), and then left-multiplying both sides with \( U^\top A \), we obtain

\[
U^\top \Delta_{t+1} = \left( I_k - \mu (U^\top S^\top SU)^{-1} \right) U^\top \Delta_t.
\]

It follows that after \( T \) iterations,

\[
U^\top \Delta_T = \left( I_k - \mu (U^\top S^\top SU)^{-1} \right)^T U^\top \Delta_0.
\]

Using the eigenvalue decomposition \( U^\top S^\top SU = W \Lambda W^\top \) where \( W = [w_1, \ldots, w_k] \in \mathbb{R}^{k \times k} \) is orthogonal and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_k) \), we have

\[
(I_k - \mu (U^\top S^\top SU)^{-1})^T = W (I_k - \mu \Lambda^{-1})^T W^\top,
\]

from which we obtain

\[
\|U^\top \Delta_T\|_2^2 = \|(I_k - \mu \Lambda^{-1})^T W^\top U^\top \Delta_0\|_2^2 = \sum_{i=1}^k (1 - \mu \lambda_i^{-1})^{2T} \left( w_i^\top \Sigma V^T (x_0 - x^*) \right)^2.
\]

Then, taking expectations, using the fact that \( \mathbb{E} \left[ w_i w_i^\top \right] = \frac{1}{k} I_k \) for \( i = 1, \ldots, k \), and using the independence of \( w_i \) and \( \lambda_i \), it follows that

\[
\mathbb{E} \left[ \|\Delta_T\|_2^2 \right] = \mathbb{E} \left[ \frac{1}{k} \sum_{i=1}^k (1 - \mu \lambda_i^{-1})^{2T} \|\Sigma V^T (x_0 - x^*)\|_2^2 \right]
\]

\[
= \mathbb{E} \left[ \frac{1}{k} \sum_{i=1}^k (1 - \mu \lambda_i^{-1})^{2T} \|U \Sigma V^T (x_0 - x^*)\|_2^2 \right]
\]

\[
= \mathbb{E} \left[ \frac{1}{k} \sum_{i=1}^k (1 - \mu \lambda_i^{-1})^{2T} \|\Delta_0\|_2^2 \right].
\]

\( \square \)

7.4 Fixed Gaussian Embedding – Exact Asymptotic Analysis of Algorithm 3, Proof of Theorem 4

By using the change of variables \( x = \frac{1}{\lambda} \), we have that

\[
\Gamma_t^a(\mu) = \frac{\alpha}{2\pi \sqrt{ab}} \int_a^b (1 - x \mu)^{2T} \sqrt{(x - a)(b - x)} \frac{dx}{x^2}.
\]

where \( a := \lambda_+^{-1} \) and \( b := \lambda_-^{-1} \). Further, defining

\[
\varphi_t(\beta) := \frac{1}{\beta^{2T}} \int_a^b (\beta - x)^{2T} \sqrt{(x - a)(b - x)} \frac{dx}{x^2},
\]

we have the identity

\[
\Gamma_t^a(\mu) = \frac{\alpha}{2\pi \sqrt{ab}} \varphi_t(1/\mu). \tag{51}
\]

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We set $\gamma = \frac{b-a}{2a}$, and we make a change a variable in the integral defining $\varphi_t$, by setting $z = \frac{x-a}{b-a}$, so that we obtain
\[
\varphi_t \left( \frac{b-a}{2} \gamma + \frac{a+b}{2} \right) = \frac{1}{(\gamma + \frac{a+b}{b-a})^{2t}} \int_{-1}^{1} \left( \gamma - z \right)^{2t} \sqrt{1 - z^2} \, dz.
\]  
(52)

For convenience, we define $\tilde{\varphi}_t(\gamma) := \varphi_t \left( \frac{b-a}{2} \gamma + \frac{a+b}{2} \right)$, $\kappa := \frac{b+a}{b-a}$ and $h(z) := \frac{1}{(z+\kappa)^2}$. Hence, we get the more compact expression
\[
\tilde{\varphi}_t(\gamma) = \frac{1}{(\gamma + \kappa)^{2t}} \int_{-1}^{1} (z - \gamma)^{2t} \sqrt{1 - z^2} h(z) \, dz.
\]  
(53)

**Lemma 10.** Define the function $\psi_t(\gamma) := \int_{-1}^{1} (z - \gamma)^{2t} \sqrt{1 - z^2} h(z) \, dz$. Let $\gamma \in (-1, 1)$ such that $\gamma \neq 0$. Then, we have the following asymptotic expansions as $t \to +\infty$,
\[
\psi_t(\gamma) = \sqrt{\frac{\pi}{8e}} \frac{(1 + |\gamma|)^{2t+3/2}}{(\gamma^2 + \kappa)^{t^2}} (1 + o(1)), \quad \text{and}, \quad \psi_t(0) = \sqrt{\frac{\pi}{2e}} \frac{1 + \kappa^2}{(\kappa^2 - 1)^{t^2}} (1 + o(1)).
\]

**Proof.** The proof is deferred to Appendix 8.2. It relies on standard techniques for asymptotic approximations of integrals, namely, Laplace approximations [6].

**Corollary 5.** For $\gamma \in (-1, 1)$ such that $\gamma \neq 0$, it holds that
\[
\lim_{t \to +\infty} \tilde{\varphi}_t(0) = +\infty, \quad \text{and}, \quad \lim_{t \to +\infty} \tilde{\varphi}_t(0) = \left( \frac{b-a}{b+a} \right)^2.
\]

**Proof.** For any $\gamma \in (-1, 1)$, we have the identity $\tilde{\varphi}_t(\gamma) = \frac{1}{(\gamma + \kappa)^{2t}} \psi_t(\gamma)$. Let $\gamma \in (-1, 1)$ such that $\gamma \neq 0$. Using Lemma 10, it follows that
\[
\tilde{\varphi}_t(\gamma) = \frac{1}{(-\gamma/|\gamma| + \kappa)^2} \sqrt{\frac{\pi}{8e}} \frac{(1 + |\gamma|)^{2t+3/2}}{(\gamma^2 + \kappa)^{t^2}} (1 + o(1)),
\]
\[
\tilde{\varphi}_t(0) = \sqrt{\frac{\pi}{2e}} \frac{1 + \kappa^2}{(\kappa^2 - 1)^{t^2}} \frac{1}{\kappa^2} (1 + o(1)).
\]

Thus, we get that
\[
\frac{\tilde{\varphi}_t(\gamma)}{\tilde{\varphi}_t(0)} \sim \left( \frac{1 + |\gamma|}{\frac{2}{\kappa} + 1} \right)^{2t}.
\]

It holds that $1 > \frac{1}{\kappa} > 0$. Multiplying by $|\gamma|$, we get that $|\gamma| \geq \frac{|\gamma|}{\kappa} \geq \frac{2}{\kappa}$, which further implies that $\frac{1 + |\gamma|}{1 + \frac{2}{\kappa}} > 1$, and
\[
\lim_{t \to +\infty} \left( \frac{1 + |\gamma|}{\frac{2}{\kappa} + 1} \right)^{2t} = +\infty.
\]

Thus, we have the claimed divergence result, that is, $\lim_{t \to +\infty} \frac{\tilde{\varphi}_t(\gamma)}{\tilde{\varphi}_t(0)} = +\infty$. In order to conclude, we observe that
\[
\frac{\tilde{\varphi}_{t+1}(0)}{\tilde{\varphi}_t(0)} \sim \frac{t^2}{(t+1)^2} \kappa^{-2},
\]

so that $\lim_{t \to +\infty} \frac{\tilde{\varphi}_{t+1}(0)}{\tilde{\varphi}_t(0)} = \kappa^{-2} = \left( \frac{b-a}{b+a} \right)^2$.  

\[\square\]
We are ready to prove Theorem 4. We set \( \beta^* := \frac{a+b}{2} \). Using the identity \( \varphi_t \left( \frac{\beta - \frac{a+b}{2}}{b-a} \right) = \varphi_t (\beta) \) and the fact that \( \frac{\beta - \frac{a+b}{2}}{b-a} \in (-1, 1) \) if and only if \( \beta \in (a, b) \), it immediately follows from Corollary 5 that for any \( \beta \in (a, b) \) such that \( \beta \neq \beta^* \), we have

\[
\lim_{t \to +\infty} \varphi_t (\beta) = +\infty ,
\]

and

\[
\lim_{t \to +\infty} \varphi_{t+1} (\beta^*) = \left( \frac{b-a}{b+a} \right)^2 .
\]

Similarly, using the identity (51) and the fact that \( \mu^* = 1/\beta^* \), we get that for any \( \mu \in (b^{-1}, a^{-1}) \) such that \( \mu \neq \mu^* \),

\[
\lim_{t \to +\infty} \Gamma_t^\alpha (\mu) = +\infty ,
\]

and

\[
\lim_{t \to +\infty} \frac{\Gamma_{t+1}^\alpha (\mu^*)}{\Gamma_t^\alpha (\mu^*)} = \left( \frac{b-a}{b+a} \right)^2 .
\]

It remains to show that the divergence result holds for any \( \mu \in \mathbb{R} \). We leverage the convexity of the function \( \Gamma_t^\alpha \). Let \( \mu \in \mathbb{R} \) be an arbitrary step size. Let \( \varepsilon > 0 \) be sufficiently small such that \( \mu\varepsilon = \varepsilon \mu + (1-\varepsilon)\mu^* \in (b^{-1}, a^{-1}) \). Then, by convexity of \( \Gamma_t^\alpha \),

\[
\Gamma_t^\alpha (\mu\varepsilon) \leq \varepsilon \Gamma_t^\alpha (\mu) + (1-\varepsilon) \Gamma_t^\alpha (\mu^*) .
\]

Dividing both sides of the latter inequality by \( \Gamma_t^\alpha (\mu^*) \), we have that

\[
\frac{\Gamma_t^\alpha (\mu\varepsilon)}{\Gamma_t^\alpha (\mu^*)} \leq \varepsilon \frac{\Gamma_t^\alpha (\mu)}{\Gamma_t^\alpha (\mu^*)} + (1-\varepsilon) .
\]

Taking \( t \to +\infty \), it follows that

\[
\liminf_{t \to +\infty} \frac{\Gamma_t^\alpha (\mu\varepsilon)}{\Gamma_t^\alpha (\mu^*)} \leq \varepsilon \liminf_{t \to +\infty} \frac{\Gamma_t^\alpha (\mu)}{\Gamma_t^\alpha (\mu^*)} + (1-\varepsilon) .
\]

Therefore, we necessarily have

\[
\liminf_{t \to +\infty} \frac{\Gamma_t^\alpha (\mu)}{\Gamma_t^\alpha (\mu^*)} = +\infty ,
\]

Finally, a simple calculation gives that

\[
\left( \frac{b-a}{b+a} \right)^2 = \frac{4\alpha}{(1+\alpha)^2} = \rho_\infty ,
\]

which concludes the proof of Theorem 4. \( \square \)
7.5 Fixed Embedding (Algorithm 3) – Proof of Lemma 3

Suppose that the event $E$ is true, and let $\mu \geq 0$ be any step size. After left-multiplying both sides of the update formula (59) by $U^T A$, and subtracting $U^T A x^*$ to each side, we obtain the recursion formula

$$U^T \Delta_{t+1} = \left( I - \mu \left( U^T S^T S U \right)^{-1} \right) U^T \Delta_t .$$

Since $\Delta_t \in \text{range}(U)$, it holds that $\|U^T \Delta_t\|_2 = \|\Delta_t\|_2$. Hence, we can bound the norm of the error $\Delta_t$ using the operator norm of the matrix $I - \mu(U^T S^T S U)^{-1}$ as follows,

$$\|\Delta_{t+1}\|_2 \leq \left\| I - \mu \left( U^T S^T S U \right)^{-1} \right\|_2 \|\Delta_t\|_2$$

$$= \max \left\{ \left\| 1 - \mu \lambda^{-1}_\text{min} \left( U^T S^T S U \right) \right\|, \left\| 1 - \mu \lambda^{-1}_\text{max} \left( U^T S^T S U \right) \right\| \right\} \|\Delta_t\|_2$$

$$\leq \max \left\{ \left\| 1 - \mu \lambda^{-1} \right\|, \left\| 1 - \mu \lambda^{-1} \right\| \right\} \|\Delta_t\|_2 .$$

By recursion, it follows that for any $T \geq 0$,

$$\|\Delta_T\|_2 \leq \|\Delta_0\|_2 \max \left\{ \left\| 1 - \mu \lambda^{-1} \right\|, \left\| 1 - \mu \lambda^{-1} \right\| \right\} .$$

(60)

The step size $\mu$ which minimizes the right hand side of inequality (60) is the solution of the equation $1 - \mu \lambda^{-1} = \mu \lambda^{-1} - 1$, which is exactly $\mu^* = \frac{2}{\lambda + 1 + \lambda^{-1}}$. Plugging-in the value of $\mu^*$ in inequality (60), we finally obtain that $\|\Delta_T\|_2 \leq \left( \frac{\kappa - 1}{\kappa + 1} \right)^T \|\Delta_0\|_2 .$

7.6 Extremal Eigenvalues of Random Embeddings – Proofs and References for Table 3

7.6.1 Gaussian Ensemble

Lemma 11. Let $U \in \mathbb{R}^{n \times k}$ be a matrix with orthonormal columns, and $S \in \mathbb{R}^{m \times n}$ a random matrix with i.i.d. Gaussian entries $\mathcal{N}(0, \frac{1}{m})$. Then, for any $c > 0$, it holds that

$$\left( 1 - \left( 1 + \sqrt{2c} \right) \sqrt{\frac{k}{m}} \right)^2 \leq \lambda_{\text{min}}(U^T S^T S U) \leq \lambda_{\text{max}}(U^T S^T S U) \leq \left( 1 + \left( 1 + \sqrt{2c} \right) \sqrt{\frac{k}{m}} \right)^2 ,$$

(61)

with probability at least $1 - 2e^{-ck}$.

Thus, if $m = (1 + \sqrt{2c})\alpha k$ where $\alpha > 1$, we have with probability at least $1 - 2e^{-ck}$ that

$$\left( 1 - \frac{1}{\sqrt{\alpha}} \right)^2 \leq \lambda_{\text{min}}(U^T S^T S U) \leq \lambda_{\text{max}}(U^T S^T S U) \leq \left( 1 + \frac{1}{\sqrt{\alpha}} \right)^2$$

(62)

Proof. Using Corollary 7.3.3 from [24], for $t > 0$, it holds with probability at least $1 - 2e^{-\frac{t^2}{2}}$ that

$$\left( 1 - \sqrt{\frac{t}{\sqrt{m}}} \right)^2 \leq \lambda_{\text{min}}(U^T S^T S U) \leq \lambda_{\text{max}}(U^T S^T S U) \leq \left( 1 + \sqrt{\frac{t}{\sqrt{m}}} \right)^2 .$$

Picking $t = \sqrt{2ck}$ yields the first claim (61). The second claim immediately follows. 

□
7.6.2 SRHT Matrices

**Lemma 12.** Let $U \in \mathbb{R}^{n \times k}$ be a matrix with orthonormal columns, and $\alpha > 1$. Define $\delta_{k,n} := (1 + \sqrt{8k^{-1} \log(2kn)})^2$. Then, for any sketch size $m \geq \frac{8}{3} \alpha k \delta_{k,n} \log(2k^2)$, it holds with probability at least $1 - \frac{1}{k}$ that $1 - \frac{1}{\sqrt{\alpha}} \leq \lambda_{\min}(U^T S^T S U) \leq \lambda_{\max}(U^T S^T S U) \leq 1 + \frac{1}{\sqrt{\alpha}}$.

**Proof.** From Lemma 12 in [25] – which is a simplified statement of results from [24] – it holds with probability at least $1 - \eta$ that any eigenvalue $\lambda_i$ of $U^T S^T S U$ satisfies $|\lambda_i - 1| \leq \varepsilon$ with $\varepsilon \in (0, 1)$, provided that $m \geq \frac{8}{3} \frac{k}{\varepsilon^2} \left(1 + \sqrt{8k^{-1} \log(2\eta^{-1} n)}\right)^{\frac{1}{2}} \log \left(\frac{2k}{\eta}\right)$. Picking $\varepsilon = \frac{1}{\sqrt{\alpha}}$, $\eta = \frac{1}{k}$ and $m \geq \frac{8}{3} \alpha k \delta_{k,n} \log(2k^2)$, we obtain the claimed result.

\[ \left(1 - \frac{1}{\sqrt{\alpha}}\right)^2 \leq \lambda_{\min}(U^T S^T S U) \leq \lambda_{\max}(U^T S^T S U) \leq \left(1 + \frac{1}{\sqrt{\alpha}}\right)^2. \]

**Proof.** The first part of the claim is exactly Theorem 9 in [16]. The second part follows immediately from the stated change of variables.

7.6.3 Sparse J.L. matrices

**Lemma 13.** Let $U \in \mathbb{R}^{n \times k}$ be a matrix with orthonormal columns. For $S \in \mathbb{R}^{m \times n}$ a sparse J.L. matrix with $s = \Theta((k/\delta)/\varepsilon)$ and $\varepsilon \in (0, 1)$, with probability at least $1 - \delta$, all singular values of $SU$ are $1 \pm \varepsilon$, as long as $m = \Omega(k \log^2(k/\delta)/\varepsilon^2)$ (and under additional technical conditions – see [16] for details). Consequently, picking $\delta = 1/k$ and $\varepsilon = 1/\sqrt{\alpha}$ for some $\alpha > 1$, with probability at least $1 - \frac{1}{k}$,

\[ \left(1 - \frac{1}{\sqrt{\alpha}}\right)^2 \leq \lambda_{\min}(U^T S^T S U) \leq \lambda_{\max}(U^T S^T S U) \leq \left(1 + \frac{1}{\sqrt{\alpha}}\right)^2. \]

Let $S \in \mathbb{R}^{m \times n}$ be a random embedding. Given two reals $\lambda, \Lambda > 0$, consider the event

\[ \mathcal{E} := \left\{ \lambda \leq \lambda_{\min}(U^T S^T S U) \leq \lambda_{\max}(U^T S^T S U) \leq \Lambda \right\}, \quad (63) \]

and suppose that the event $\mathcal{E}$ is true.

Let $\mu, \beta \geq 0$. Multiplying the update formula (10) by $U^T A$, and subtracting $U^T A x^*$, we obtain the error recursion formula

\[ U^T \Delta_{t+1} = U^T \Delta_t - \mu(U^T S^T S U)^{-1} U^T \Delta_t + \beta(U^T \Delta_t - U^T \Delta_{t-1}) \]

\[ = ((1 + \beta)I - \mu(U^T S^T S U)^{-1}) U^T \Delta_t - \beta U^T \Delta_{t-1}. \]  

\[ (64) \]

Then, the vector $[U^T \Delta_{t+1}, U^T \Delta_t]$ satisfies the two-dimensional linear dynamics

\[ \begin{bmatrix} U^T \Delta_{t+1} \\ U^T \Delta_t \end{bmatrix} = \begin{bmatrix} (1 + \beta)I - \mu(U^T S^T S U)^{-1} & -\beta I \\ I & 0 \end{bmatrix} \begin{bmatrix} U^T \Delta_t \\ U^T \Delta_{t-1} \end{bmatrix}, \]

\[ := M \]

which further implies, by induction, that

\[ \left\| \begin{bmatrix} U^T \Delta_{T+1} \\ U^T \Delta_T \end{bmatrix} \right\|_2 \leq \left\| M^T \right\|_2 \left\| \begin{bmatrix} U^T \Delta_1 \\ U^T \Delta_0 \end{bmatrix} \right\|_2. \]  

\[ (66) \]

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Using the fact that $\|U^T \Delta_t\|_2 = \|\Delta_t\|_2$ for any $t \geq 0$, we get
\[
\left\| \begin{bmatrix} \Delta_{T+1} \\ \Delta_T \end{bmatrix} \right\|_2 \leq \|M^T\|_2 \left\| \begin{bmatrix} \Delta_1 \\ \Delta_0 \end{bmatrix} \right\|_2.
\] (67)

Denote by $\Lambda(M)$ the spectral norm of the matrix $M$, that is, $\Lambda(M)$ is equal to the largest module of its eigenvalues. A standard linear algebra result gives that there exists a sequence $\{a_t\}_{t \geq 0}$ such that, for any $t \geq 0$,
\[
\|M^t\|_2 \leq (\rho(M) + a_t)^t, \quad \text{and}, \quad \lim_{t \to \infty} a_t = 0.
\] (68)

From a standard Heavy-ball method analysis, we get for $\beta \geq \max \{1 - \sqrt{\mu}, 1 - \sqrt{\mu}\}$, then $\rho(M) \leq \sqrt{\beta}$. Choosing $\mu = \frac{4}{(\sqrt{\frac{\beta}{\lambda} + \sqrt{\frac{\beta}{\lambda}}} - 1)^2}$ and $\beta = \left(\frac{\sqrt{\frac{\beta}{\lambda}} - 1}{\sqrt{\frac{\beta}{\lambda}} + 1} + a_T\right)^T$ yields the desired inequality,
\[
\left\| \begin{bmatrix} \Delta_{T+1} \\ \Delta_T \end{bmatrix} \right\|_2 \leq \left(\frac{\sqrt{\frac{\beta}{\lambda}} - 1}{\sqrt{\frac{\beta}{\lambda}} + 1} + a_T\right)^T \left\| \begin{bmatrix} \Delta_1 \\ \Delta_0 \end{bmatrix} \right\|_2,
\] (69)

where $\kappa = \frac{\lambda}{\chi}$.

\section{Optimal Sketch Size for SRHT Embeddings — Proof of Lemma 5}

Recall that we make the following two assumptions,
\[
\omega \log d \geq 1, \quad \text{and}, \quad \log(1/\varepsilon) \leq d \omega \log d.
\] (70)

Let $m = \varphi(d)d^\alpha$, where $\varphi(d)$ is the oversampling factor for SRHT embeddings given in Table 3. Suppose that $\alpha = d^\gamma$ for some $\gamma \in (0, 1 + \omega]$, and express the computational complexity in terms of $\gamma$,
\[
C(\gamma) \equiv C(d^\gamma) = \varphi(d)d^3\left(d^\gamma(1 + \gamma) \log d + d^\gamma + d^\gamma \frac{\log(1/\varepsilon)}{\gamma \log d}\right).
\] (71)

We introduce the value $\bar{\gamma} = \sqrt{\log(1/\varepsilon)} / \log d$.

- Suppose that $\bar{\gamma} < \omega$. We define $\tilde{C}(\gamma) = nd\left(\log d + \gamma \log d + \frac{\log(1/\varepsilon)}{\gamma \log d}\right)$, which approximates $C(\gamma)$, by discarding the factorization cost $d^\gamma$. Over the range $\gamma \in (0, \omega]$, we have $C(\gamma) \leq (1 + \frac{1}{\log d}) \tilde{C}(\gamma)$. The function $\tilde{C}$ is minimized at $\gamma^* = \bar{\gamma}$, which, by assumption, belongs to the interval $(0, \omega)$. Hence, we have $C(\gamma^*) \leq (1 + \frac{1}{\log d}) nd\left(\log d + 2 \sqrt{\log(1/\varepsilon)}\right)$, which up to the bounded factor $\left(1 + \frac{1}{\log d}\right) -$ is the claimed result. Plugging-in the value of $\gamma^*$ into the expression of the sketch size $m$ and the horizon $T$ yields the rest of the claim.

- Suppose that $\bar{\gamma} \geq \omega$. We define $\tilde{C}(\gamma) = \varphi(d)d^3\left(d^\gamma + d^{\frac{\log(1/\varepsilon)}{\bar{\gamma} \log d}}\right)$, which approximate $C(\gamma)$ by discarding the sketching cost. Picking $\gamma^* = \omega + \frac{1}{\log d} \log\left(\frac{\log(1/\varepsilon)}{\omega \log d}\right)$, we get that $d^\gamma^* = d^{\omega \frac{\log(1/\varepsilon)}{\omega \log d}}$, and
\[
\tilde{C}(\gamma^*) = nd\left(\frac{\log(1/\varepsilon)}{\omega \log d} + \frac{\log(1/\varepsilon)}{\omega \log d + \log\left(\frac{\log(1/\varepsilon)}{\omega \log d}\right)}\right).
\] (72)
By assumption, we have $\gamma \geq \omega$ and $\omega \log d \geq 1$, which together imply that $\log(1/\varepsilon) \geq \omega \log(d)$. Consequently, $\tilde{C}(\gamma^*) \leq 2nd\log(1/\varepsilon) / \omega \log(d)$. The sketching cost at the value $\gamma^*$ is given by $nd\log m = nd \log d + nd\omega \log(d) + nd \log \left( \frac{\log(1/\varepsilon)}{\omega \log(d)} \right)$. But, by assumption, $\omega \log d \leq \frac{\log(1/\varepsilon)}{\omega \log(d)}$, and therefore, $nd\log m \leq nd \log d + nd\frac{\log(1/\varepsilon)}{\omega \log(d)}$. Hence, we have $C(\gamma) \leq nd \log d + 3nd\frac{\log(1/\varepsilon)}{\omega \log(d)}$. It remains to verify that $\gamma^* \leq \omega + 1$. We have $\gamma^* = \omega + \frac{1}{\log d} \log \left( \frac{\log(1/\varepsilon)}{\omega \log(d)} \right)$, and the inequality $\gamma^* \leq \omega + 1$ is equivalent to $\log(1/\varepsilon) \leq d\omega \log d$, which we assumed to be true.

Plugging-in $\varepsilon = \frac{d}{n}$ into the expressions of $\gamma^*$, $T$, $m^*$ and $C^*$ yields the rest of Lemma 5.

8 Proof of Technical Results

8.1 Proof of Lemma 2

By first-order optimality conditions, the step size $\mu^*_t$ which minimizes $\Gamma^o_t$ must satisfy

$$E_{\lambda \sim MP(\alpha, \gamma)} \left[ \frac{1}{\lambda} \left( 1 - \frac{\mu^*_t}{\lambda} \right)^{2t-1} \right] = 0.$$ 

Equivalently,

$$\int_{\lambda_-}^{\lambda_+} \left( 1 - \frac{\mu^*_t}{\lambda} \right) \frac{2t-1}{\lambda^2} \sqrt{\frac{(\lambda_+ - \lambda)(\lambda - \lambda_-)}{\lambda_-}} d\lambda = 0.$$ 

Using the change of variable $x = \frac{1}{\lambda}$, we have the following set of equalities,

$$\int_{\lambda_-}^{\lambda_+} \left( 1 - \frac{\mu^*_t}{\lambda} \right) \frac{2t-1}{\lambda^2} \sqrt{\frac{(\lambda_+ - \lambda)(\lambda - \lambda_-)}{\lambda_-}} d\lambda = \int_{1/\lambda_+}^{1/\lambda_-} (1 - \mu^*_t x)^{2t-1} \sqrt{(\lambda_+ - x^{-1})(x^{-1} - \lambda_-)} dx$$

$$= \mu^*_t^{2t-1} \sqrt{\lambda_- \lambda_+} \int_{1/\lambda_+}^{1/\lambda_-} \left( \frac{1}{\mu^*_t} - x \right)^{2t-1} \sqrt{\frac{(x - \lambda_+^{-1})(\lambda_-^{-1} - x)}{x}} dx.$$ 

Using the change of variable $\beta^*_t = \mu^*_t^{-1}$, it follows that finding $\mu^*_t$ which minimizes $\Gamma^o_t(\mu)$ is equivalent to finding $\beta^*_t$ such that

$$\int_{1/\lambda_+}^{1/\lambda_-} (\beta^*_t - x)^{2t-1} \sqrt{\frac{(x - \lambda_+^{-1})(\lambda_-^{-1} - x)}{x}} dx = 0. \tag{73}$$

We introduce the following function in $\beta$,

$$\gamma^o_t(\beta) := \int_{1/\lambda_+}^{1/\lambda_-} (\beta - x)^{2t} \sqrt{\frac{(x - \lambda_+^{-1})(\lambda_-^{-1} - x)}{x}} dx. \tag{74}$$

The function $\gamma^o_t$ is strongly convex, and goes to $+\infty$ as $|\beta| \to +\infty$. Thus, it admits a unique minimizer $\beta_t$, which is characterized by first-order optimality conditions,

$$\int_{1/\lambda_+}^{1/\lambda_-} (\beta_t - x)^{2t-1} \sqrt{\frac{(x - \lambda_+^{-1})(\lambda_-^{-1} - x)}{x}} dx = 0.$$ 

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We recognize the equation (73), and therefore, we have that $\beta_t^* = \tilde{\beta}_t$. Setting $\nu = \left(\frac{\sqrt{\lambda_+} + \sqrt{\lambda_-}}{\sqrt{\lambda_+} - \sqrt{\lambda_-}}\right)^2$ and $\sigma = \frac{\sqrt{\lambda_+} + \sqrt{\lambda_-}}{2}$, we get that, up to a positive constant independent of $\beta$, the function $\gamma_t^0$ is equal to

$$E_{X \sim \text{MP}(\nu, \sigma)} [(X - \beta)^2],$$  

which concludes the proof. \hfill \Box

### 8.2 Proof of Lemma 10

Let $\gamma \in (-1, 1)$. We distinguish three cases, that is, $\gamma < 0$, $\gamma > 0$ and $\gamma = 0$.

**Case 1: $-1 < \gamma < 0$.**

Using the fact that, for any $z \in [-1, \gamma]$, we have $|z - \gamma| \leq |\gamma + 1| < 1$, it follows, by dominated convergence, that the integral

$$\int_{-1}^{\gamma} (z - \gamma)^2t \sqrt{1 - z^2} h(z) \, dz$$

converges to 0 as $t \to +\infty$. Thus, we study the asymptotics of

$$\int_{\gamma}^{1} (z - \gamma)^2t \sqrt{1 - z^2} h(z) \, dz = \int_{\gamma}^{1} e^{tg_t(z)} h(z) \, dz,$$  

where we introduced the function $g_t(z) := 2 \log(z - \gamma) + \frac{1}{2t} \log(1 - z^2)$. We have the three following facts. First, the function $g_t$ is continuous over $(\gamma, 1)$. Second, $\lim_{z \to 1^-} g_t(z) = -\infty$. Third, $\lim_{z \to \gamma^+} g_t(z) = -\infty$. Therefore, the function $g_t$ admits a maximizer $z^* \in (\gamma, 1)$. Let us characterize $z^*$. By first-order optimality conditions, we have $g_t'(z^*) = 0$, i.e.,

$$\frac{2}{z^* - \gamma} - \frac{z^*}{t(1 - z^*^2)} = 0.$$  

(77)

After re-arranging the above equation, we get that $z^*$ must satisfy the second-order equation

$$(1 + 2t)z^*^2 - \gamma z^* - 2t = 0,$$  

(78)

whose two solutions are equal to $z_{\pm} = \frac{\gamma}{2(1 + 2t)} \pm \sqrt{\frac{2t}{2t + 1} + \frac{\gamma^2}{4(1 + 2t)^2}}$. The solution $z^*$ corresponds to the positive branch. Indeed, when $t \to +\infty$, we have that $z_- \to -1$, whereas we must have $z^* > \gamma > -1$, hence, ruling out the equality $z^* = z_-$. Thus, the maximizer $z^*$ is unique, equal to $z_+$, and, by Laplace approximation of integrals [4], we have

$$\psi_t(\gamma) = e^{tg_t(z^*)} h(z^*) \sqrt{\frac{2\pi}{t |g_t'(z^*)|}} \left(1 + O\left(\frac{1}{t}\right)\right).$$  

(79)

Let us expand $z^*$ at first-order in terms of $1/t$. We have,

$$z^* = \frac{\gamma}{2(1 + 2t)} + \sqrt{\frac{2t}{2t + 1} + \frac{\gamma^2}{4(1 + 2t)^2}} = 1 - \frac{1 - \gamma}{2(1 + 2t)} + o\left(\frac{1}{t}\right),$$  

(80)

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which further implies that
\[(z^* - \gamma)^{2t} = (1 - \gamma)^{2t} \left(1 - \frac{1}{2(1 + 2t)} + o(1/t)\right)^{2t} = (1 - \gamma)^{2t}(1/\sqrt{e} + o(1))\]
and
\[\sqrt{1 - z^*^2} = \sqrt{\frac{1 - \gamma}{1 + 2t}(1 + o(1))}.\]
Combining the two previous expansions, we obtain that
\[e^{\gamma t}(z^*) = (z^* - \gamma)^{2t}\sqrt{1 - z^*^2} = \frac{1}{\sqrt{1 + 2t}}(1 - \gamma)^{2t + \frac{1}{2}}(1/\sqrt{e} + o(1)).\]
Further, we have \[g''(z^*) = -\frac{1}{(z^* - \gamma)^2} - \frac{z^* + 1}{2t(1 - z^*^2)^2}.\] Using the expansions
\[
\begin{align*}
\frac{1}{(z^* - \gamma)^2} &= \frac{1}{(1 - \gamma)^2} \left(1 + \frac{1}{2t} + o(1/t)\right), \\
z^*^2 + 1 &= 2 - \frac{1 - \gamma}{2t} + o(1/t), \\
\frac{1}{(1 - z^*^2)^2} &= \frac{(1 + 2t)^2}{(1 - \gamma)^2} + o(1) \quad (1 + o(1)),
\end{align*}
\]
we obtain \[g''(z^*) = \frac{8t}{(1 - \gamma)^2}(1 + o(1))\] and, thus,
\[\sqrt{\frac{2\pi}{t|g''(z^*)|}} = \sqrt{\frac{\pi 1 - \gamma}{4t}}(1 + o(1)).\]

It is immediate to verify that \[h(z^*) = \frac{1}{(1 + \kappa)^2}(1 + o(1)).\] Hence, using the Laplace approximation formula [79], we finally obtain
\[\psi_t(\gamma) = \sqrt{\frac{\pi}{8e}} \frac{(1 - \gamma)^{2t + 3/2}}{(1 + \kappa)^{2t^2/2}}(1 + o(1)).\]

**Case 2: 0 < \gamma < 1.**

By using the change of variable \(z' = -z\), and setting \(\gamma' = -\gamma\), this case study follows exactly the same lines as the previous one, and we obtain
\[\psi_t(\gamma) = \sqrt{\frac{\pi}{8e}} \frac{(1 + \gamma)^{2t + 3/2}}{(-1 + \kappa)^{2t^2/2}}(1 + o(1)).\]

**Case 3: \gamma = 0.**

Separating the integral defining \(\psi_t(0)\) at \(z = 0\), we need to study separately the asymptotics of the two following integrals,
\[
\int_{-1}^{0} z^{2t}\sqrt{1 - z^2}h(z) \, dz, \quad \text{and}, \quad \int_{0}^{1} z^{2t}\sqrt{1 - z^2}h(z) \, dz.
\]
Following similar steps as in the first case-study, we obtain that
\[
\int_{-1}^{0} z^2 \sqrt{1 - z^2} h(z) \, dz = \sqrt{\frac{\pi}{8e}} \frac{1}{t^{\frac{3}{2}}} (1 + o(1)),
\]
\[
\int_{0}^{1} z^2 \sqrt{1 - z^2} h(z) \, dz = \sqrt{\frac{\pi}{8e}} \frac{1}{t^{\frac{3}{2}}} (1 + o(1)).
\]

By summing the two above expansions, we obtain
\[
\int_{-1}^{1} z^2 \sqrt{1 - z^2} h(z) \, dz = \sqrt{\frac{\pi}{8e}} \frac{1}{t^{\frac{3}{2}}} \left( \frac{1}{(1 + \kappa)^2} + \frac{1}{(-1 + \kappa)^2} \right) (1 + o(1))
\]
\[
= \sqrt{\frac{\pi}{2e} \frac{1 + \kappa^2}{t^{\frac{3}{2}} (\kappa^2 - 1)^2}} (1 + o(1)),
\]
which is the desired result.

8.3 Proof of Lemma 7

Let \( \beta > 0 \) and \( \mu \geq 0 \). We aim to show that if
\[
(x_0, x_1) \in \{(x_0, x_1) \mid Ax_0 = Ax^*, Ax_1 \neq Ax^*\} \cup \{(x_0, x_1) \mid Ax_0 \neq Ax^*, Ax_1 = Ax^*\},
\]
then, the sequence \( E[\|\Delta_t\|^2] \) satisfies the asymptotic scaling \( E[\|\Delta_t\|^2] \gtrsim \Lambda(\mu, \beta)^t \). Recall the definition of the sequence \( c_t = \beta^2 E[\|\Delta_t\|^2] \). Since we have \( \beta > 0 \), the statement we aim to show is equivalent to \( c_t \gtrsim \Lambda(\mu, \beta)^t \).

We distinguish three cases, that is, the roots of the characteristic polynomial \( \chi_{\mu, \beta} \) are pairwise distinct (Case 1), two roots exactly are equal (Case 2), and, the roots are all equal (Case 3).

**Case 1: the roots are pairwise distinct**

Assume that the roots \( \lambda_1, \lambda_2, \lambda_3 \) of the characteristic polynomial \( \chi_{\mu, \beta} \) are pairwise distinct. Then, according to fundamental results in linear dynamical systems, there exist \( \nu_1, \nu_2, \nu_3 \in \mathbb{C} \) such that for any \( t \geq 0 \),
\[
c_t = \nu_1 \lambda_1^t + \nu_2 \lambda_2^t + \nu_3 \lambda_3^t.
\]

**Lemma 14.** Assume that \( Ax_0 = Ax^* \) and \( Ax_1 \neq Ax^* \), i.e., \( \Delta_0 = 0 \) and \( \Delta_1 \neq 0 \). Then, one of the following statements must be true,

(a) \( \nu_1 \neq 0 \), and \( |\lambda_1| = \Lambda(\mu, \beta) \),

(b) \( \nu_2 \neq 0 \), and \( |\lambda_2| = \Lambda(\mu, \beta) \),

(c) \( \nu_3 \neq 0 \) and \( |\lambda_3| = \Lambda(\mu, \beta) \).

**Proof.** Without loss of generality, we assume that the root \( \lambda_1 \) satisfies \( \Lambda(\mu, \beta) = |\lambda_1| \geq \max(|\lambda_2|, |\lambda_3|) \). If \( \nu_1 \neq 0 \), then the result is proved. Hence, let us assume that \( \nu_1 = 0 \). By assumption, we also have
\[ \| \Delta_0 \|_2^2 = 0 \text{ and } \| \Delta_1 \|_2^2 \neq 0. \] From the linear dynamics (43) for \( t = 0, 1, 2 \), we obtain that \( c_0 = 0, c_1 = \beta^2 \| \Delta_1 \|_2^2 \) and \( c_2 = \beta^2 \eta \| \Delta_1 \|_2^2. \) Further, using the formula (81) for \( t = 0, 1, 2 \), we then obtain
\[
\begin{align*}
\nu_2 + \nu_3 &= 0 \\
\nu_2 \lambda_2 + \nu_3 \lambda_3 &= \beta^2 \| \Delta_1 \|_2^2 \\
\nu_2 \lambda_2^2 + \nu_3 \lambda_3^2 &= \beta^2 \eta \| \Delta_1 \|_2^2.
\end{align*}
\] (82)

Since \( \beta^2 \| \Delta_1 \|_2^2 \neq 0 \), we get from the system of equations (82) that \( \lambda_2 + \lambda_3 = \eta \). On the other hand, the (negative) sum of the roots \(- (\lambda_1 + \lambda_2 + \lambda_3)\) is equal to the second coefficient of \( \chi_{\mu, \beta} \), i.e., \( \lambda_1 + \lambda_2 + \lambda_3 = \eta - \beta \). Thus, we obtain \( \lambda_1 = -\beta \). Further, the product of the roots is equal to (minus) the last coefficient of \( \chi_{\mu, \beta} \), i.e., \( \lambda_1 \lambda_2 \lambda_3 = \beta^3 \). By assumption, \( |\lambda_1| \geq |\lambda_2|, |\lambda_3| \). Hence, we must have \( |\lambda_1| = |\lambda_2| = |\lambda_3| \). Since \( \nu_2 \nu_3 \neq 0 \) (otherwise, the second equation of the system (82) has a null left-hand side and a non-null right-hand side), it follows that either \( |\lambda_2| = \Lambda(\mu, \beta) \) and \( \nu_2 \neq 0 \), or, \( |\lambda_3| = \Lambda(\mu, \beta) \) and \( \nu_3 \neq 0 \), which concludes the proof.

\[ \text{Lemma 15. Assume that } Ax_0 \neq Ax^* \text{ and } Ax_1 = Ax^*, \text{ i.e., } \Delta_0 \neq 0 \text{ and } \Delta_1 = 0. \text{ Then, one of the following statements must be true,} \]

(a) \( \nu_1 \neq 0 \), and \( |\lambda_1| = \Lambda(\mu, \beta) \),

(b) \( \nu_2 \neq 0 \) and \( |\lambda_2| = \Lambda(\mu, \beta) \),

(c) \( \nu_3 \neq 0 \) and \( |\lambda_3| = \Lambda(\mu, \beta) \).

\[ \text{Proof. The proof follows similar lines to that of Lemma 14. Without loss of generality, we assume that} \]
\[ \text{the root } \lambda_1 \text{ satisfies } \Lambda(\mu, \beta) = |\lambda_1| \geq \max(|\lambda_2|, |\lambda_3|). \text{ If } \nu_1 \neq 0, \text{ then the result is proved.} \]
\[ \text{Hence, let us assume that } \nu_1 = 0. \text{ By assumption, we also have } || \Delta_1 ||_2^2 = 0 \text{ and } || \Delta_0 ||_2^2 \neq 0. \]
\[ \text{From the linear dynamics (43) for } t = 0, 1, 2, \text{ we obtain that } c_0 = \beta^2 || \Delta_0 ||_2^2, c_1 = 0 \text{ and } c_2 = \beta^4 || \Delta_0 ||_2^2. \]
\[ \text{Further, using the formula (81) for } t = 0, 1, 2, \text{ we then obtain} \]
\[
\begin{align*}
\nu_2 + \nu_3 &= \beta^2 || \Delta_0 ||_2^2 \\
\nu_2 \lambda_2 + \nu_3 \lambda_3 &= 0 \\
\nu_2 \lambda_2^2 + \nu_3 \lambda_3^2 &= \beta^4 || \Delta_0 ||_2^2.
\end{align*}
\] (83)

Since \( \beta^2 || \Delta_0 ||_2^2 \neq 0 \), we get from the system of equations (83) that \( - \lambda_3 \lambda_2 = \beta^2 \). Using the facts that \( \lambda_1 \lambda_2 \lambda_3 = \beta^3 \), \( |\lambda_1| \geq \max(|\lambda_2|, |\lambda_3|) \) and \( \beta \neq 0 \), it follows that \( \lambda_1 = -\beta \), and thus, \( |\lambda_1| = |\lambda_2| = |\lambda_3| \). Finally, we must have \( \nu_2 \neq 0 \) or \( \nu_3 \neq 0 \) (otherwise, the first equation of the system (83) has a null left-hand side and a non-null right-hand side). It follows that either \( |\lambda_2| = \Lambda(\mu, \beta) \) and \( \nu_2 \neq 0 \), or, \( |\lambda_3| = \Lambda(\mu, \beta) \) and \( \nu_3 \neq 0 \), which concludes the proof. \]

\[ \text{Case 2: two roots are equal} \]

Assume that \( \lambda_1 = \lambda_2 \), and \( \lambda_1 \neq \lambda_3 \). Then, there exist \( \nu_1, \nu_2, \nu_3 \in \mathbb{C} \) such that for any \( t \geq 0 \),
\[ c_t = \nu_1 \lambda_1^t + \nu_2 t \lambda_1^t + \nu_3 \lambda_3^t. \] (84)

\[ \text{Lemma 16. Assume that } Ax_0 = Ax^* \text{ and } Ax_1 \neq Ax^*, \text{ i.e., } \Delta_0 = 0 \text{ and } \Delta_1 \neq 0. \text{ Then, one of the} \]
\[ \text{following statements must be true,} \]

(a) \( \nu_1 \neq 0 \) or \( \nu_2 \neq 0 \), and, \( |\lambda_1| = \Lambda(\mu, \beta) \),
(b) \( \nu_3 \neq 0 \) and \( |\lambda_3| = \Lambda(\mu, \beta) \).

**Proof.** First, we assume that \( \Lambda(\mu, \beta) = |\lambda_1| \geq |\lambda_3| \). If \( \nu_1 \neq 0 \) or \( \nu_2 \neq 0 \), then the result is proved. Hence, let us assume that \( \nu_1 = \nu_2 = 0 \). By assumption, we also have \( \|\Delta_0\|^2 = 0 \) and \( \|\Delta_1\|^2 \neq 0 \). From the linear dynamics (143) for \( t = 0, 1, 2 \), we obtain that \( c_0 = 0, c_1 = \beta^2\|\Delta_1\|^2_2 \) and \( c_2 = \beta^2 \eta_1 \|\Delta_1\|^2_2 \). Using the formula (84) for \( t = 0 \), we then obtain \( \nu_3 = 0 \), and hence, \( c_1 = 0 \) for all \( t \geq 0 \). But this is in contradiction with the equality \( c_1 = \beta^2\|\Delta_1\|^2_2 \), which is positive. Therefore, we must have \( \nu_1 \neq 0 \) or \( \nu_2 \neq 0 \).

Now, we assume that \( \Lambda(\mu, \beta) = |\lambda_3| > |\lambda_1| \). If \( \nu_3 \neq 0 \), then the result is proved. Hence, let us assume that \( \nu_3 = 0 \). Combining the identities \( c_0 = 0, c_1 = \beta^2\|\Delta_1\|^2_2 \) and \( c_2 = \beta^2 \eta_1 \|\Delta_1\|^2_2 \) with the formula (84) for \( t = 0, 1, 2 \), we then obtain

\[
\begin{align*}
\nu_1 &= 0 \\
\nu_1\lambda_1 + \nu_2\lambda_1 &= \beta^2\|\Delta_1\|^2_2 \\
\nu_1\lambda_1^2 + 2\nu_2\lambda_1^2 &= \beta^2 \eta_1 \|\Delta_1\|^2_2,
\end{align*}
\]

from which, after a few manipulations, we obtain \( 2\lambda_1 = \eta \). On the other hand, the sum of the roots satisfies the equality \( \lambda_3 + 2\lambda_1 = \eta - \beta \). Thus, we get \( \lambda_3 = -\beta \). Since the product of the roots satisfies \( \lambda_1^2 \lambda_3 = \beta^3 \), \( \beta \neq 0 \) and \( |\lambda_3| \geq |\lambda_1| \), we get \( |\lambda_1| = |\lambda_3| \), which is in contradiction with the assumption that \( |\lambda_3| > |\lambda_1| \). Therefore, we must have \( |\lambda_3| = \Lambda(\mu, \beta) \) and \( \nu_3 \neq 0 \), which concludes the proof. \( \square \)

**Lemma 17.** Assume that \( Ax_0 \neq Ax^* \) and \( Ax_1 = Ax^* \), i.e., \( \Delta_0 \neq 0 \) and \( \Delta_1 = 0 \). Then, one of the following statements must be true,

(a) \( \nu_1 \neq 0 \) or \( \nu_2 \neq 0 \), and \( |\lambda_1| = \Lambda(\mu, \beta) \),

(b) \( \nu_3 \neq 0 \) and \( |\lambda_3| = \Lambda(\mu, \beta) \).

**Proof.** The proof follows very similar lines to the proofs of Lemma 15 and Lemma 16 and we skip it for the sake of conciseness. \( \square \)

**Case 3: the roots are all equal**

Assume that \( \lambda_1 = \lambda_2 = \lambda_3 \). Then, there exist \( \nu_1, \nu_2, \nu_3 \in \mathbb{C} \) such that for any \( t \geq 0 \),

\[
c_t = \nu_1\lambda_1^t + \nu_2 t\lambda_1^t + \nu_3 t^2\lambda_1^t.
\]

If the system is initialized at a point \( (x_0, x_1) \) such that \( (Ax_0, Ax_1) \neq (Ax^*, Ax^*) \), then the next result immediately follows from the fact that the sequence \( \{c_t\} \) cannot be constant and equal to 0.

**Lemma 18.** Assume that \( Ax_0 = Ax^* \) and \( Ax_1 \neq Ax^* \), or, \( Ax_0 \neq Ax^* \) and \( Ax_1 = Ax^* \). Then, one of the following statements must be true,

(a) \( \nu_1 \neq 0 \), and \( |\lambda_1| = \Lambda(\mu, \beta) \),

(b) \( \nu_2 \neq 0 \) and \( |\lambda_2| = \Lambda(\mu, \beta) \),

(c) \( \nu_3 \neq 0 \) and \( |\lambda_3| = \Lambda(\mu, \beta) \).
Concluding the proof of Lemma 7

Proof. Let $\lambda_1, \lambda_2, \lambda_3$ be the three roots of $\chi_{\mu,\beta}$. Combining the case-study results of Lemma 14, Lemma 15, Lemma 16, Lemma 17 and Lemma 18, we know that there exist $i \in \{1, 2, 3\}$ and $\nu_i \neq 0$ such that $|\lambda_i| = \Lambda(\mu, \beta)$, and, $c_t \sim \nu_i \lambda_i^t$, or, $c_t \sim \nu_i^2 \lambda_i^t$. Using the fact that $c_t$ is non-negative and setting $\nu = |\nu_i|$, it follows that $c_t \sim \nu \Lambda(\mu, \beta)^t$, or, $c_t \sim \nu^2 \Lambda(\mu, \beta)^t$. Consequently,

$$\liminf_{t \to \infty} \{\Lambda(\mu, \beta)^{-t} c_t\} > 0,$$

which concludes the proof of Lemma 7.

8.4 Proof of Theorem 7.1

First, we show that $(\mu^*, \beta^*) = (\theta_2^{-1} \theta_1, 0)$ is a local minimum of the function $(\mu, \beta) \mapsto \Lambda(\mu, \beta)$ over $\mathbb{R}^2$.

Lemma 19 (Local optimality). The root radius function

$$\begin{align*}
(\mu, \beta) &\mapsto \Lambda(\mu, \beta) \\
\mathbb{R}^2 &\to \mathbb{R}_+
\end{align*}$$

has a strong local minimum at $(\mu^*, \beta^*) = (\theta_2^{-1} \theta_1, 0)$, which is equal to $\rho^* = 1 - \frac{\theta_2^2}{\theta_1^2}$.

Proof. We defer the proof to Section 8.5

We turn to the proof of global optimality of $(\mu^*, \beta^*)$ over $\mathbb{R}^2_+$. For convenience of notations, we use the linear change of variable $\alpha := \frac{\theta_2}{\theta_1} \mu$, and re-parameterize the characteristic polynomial $\chi_{\mu,\beta}$ as $\chi_{\alpha,\beta}$. Thus, the point $(\alpha^*, \beta^*) = (1, 0)$ is a strong local minimizer of the root radius $\Lambda(\alpha, \beta)$ of the polynomial $\chi_{\alpha,\beta}$. Before developing formally the results, we describe the main steps of the proof, which proceeds by contradiction.

1. First, we assume that there exist some parameters $\hat{\beta} > 0$ and $\hat{\alpha} \geq 0$ such that $\Lambda(\hat{\alpha}, \hat{\beta}) < \rho^*$. We show, in Lemma 20, that the latter assumption implies the existence of parameters $\beta', \alpha' \in \mathbb{R}$ such that

$$\begin{align*}
\alpha' &\geq 0, \\
\beta' &\in (0, \rho^*), \\
\chi_{\alpha', \beta'}(\rho^*) &= 0.
\end{align*}$$

The proof of Lemma 20 relies on the strong local optimality of $(\alpha^*, \beta^*)$, and, the continuity of the roots of the polynomial $\chi_{\alpha,\beta}$ in the variable $(\alpha, \beta)$.

2. Then, we seek a contradiction with the system (87). We introduce, for a fixed parameter $\alpha \geq 0$, the function $P_\alpha : \beta \mapsto \chi_{\alpha,\beta}(\rho^*)$, which is a polynomial in $\beta$. In Lemma 21, we show that for any $\alpha \geq 0$, provided that $\rho^* \neq \frac{1}{2}$, the polynomial $P_\alpha$ has no real root within the interval $(0, \rho^*)$. This will be a contradiction since, according to (87), for $\alpha = \alpha'$, we have $P_{\alpha'}(\beta') = 0$, with $\beta' \in (0, \rho^*)$. The proof of Lemma 21 reduces the problem to an analysis of the roots of the degree-two polynomial $\frac{dP_{\alpha'}}{d\beta'}$, which enjoys simpler analytical properties. Details are developed in Lemma 22, Corollary 7, Lemma 23, Lemma 24 and Lemma 25.
3. By continuity arguments, we conclude that the result holds for any \( \rho^* \in (0, 1) \) (see Corollary 6).

**Lemma 20.** Suppose that for some parameters \( \hat{\beta} > 0 \) and \( \hat{\alpha} \geq 0 \), the root radius \( \Lambda(\hat{\alpha}, \hat{\beta}) \) satisfies \( \Lambda(\hat{\alpha}, \hat{\beta}) < \rho^* \). Then, there exist parameters \( \beta' \in (0, \rho^*) \) and \( \alpha' \geq 0 \), such that \( \chi_{\alpha', \beta'}(\rho^*) = 0 \).

**Proof.** For fixed parameters \( \alpha, \beta \geq 0 \), the root radius \( \Lambda(\alpha, \beta) \) satisfies \( \Lambda(\alpha, \beta) \geq \beta \). Indeed, we have \( \lambda_1, \lambda_2, \lambda_3 = \beta^3 \), where \( \lambda_1, \lambda_2, \lambda_3 \) are the roots of \( \chi_{\alpha, \beta} \). Thus, \( |\lambda_1\lambda_2\lambda_3| = \beta^3 \), which further implies that \( \max(|\lambda_1|, |\lambda_2|, |\lambda_3|) \geq \beta \), i.e., \( \Lambda(\alpha, \beta) \geq \beta \). In particular, we have that \( \hat{\beta} \leq \Lambda(\hat{\alpha}, \hat{\beta}) \). Along with the assumption \( \Lambda(\hat{\alpha}, \hat{\beta}) < \rho^* \), it follows that

\[
\hat{\beta} < \rho^*. \tag{88}
\]

We know that \( \Lambda(1, 0) = \rho^* \). Let \((\alpha(t), \beta(t))_{t \in [0, 1]}\) be a continuous, injective path in the rectangle \([1, \hat{\alpha}] \times [0, \hat{\beta}]\) such that \((\alpha(0), \beta(0)) = (1, 0), (\alpha(1), \beta(1)) = (\hat{\alpha}, \hat{\beta})\) and \(\beta(t) > 0\) for \(t > 0\). Using (88), we have

\[
\beta(t) \in (0, \rho^*), \quad \text{for } t \in (0, 1].
\]

Denote \( \Lambda(t) = \Lambda(\alpha(t), \beta(t)) \). We introduce continuous parameterizations of the roots \( \lambda_1(t), \lambda_2(t), \lambda_3(t) \) of \( \chi_{\alpha(t), \beta(t)} \). Then, it suffices to show that one of the roots \( \lambda_1(t), \lambda_2(t) \) or \( \lambda_3(t) \) takes the value \( \rho^* \) for some \( t > 0 \). Indeed, by setting \( \alpha' = \alpha(t) \) and \( \beta' = \beta(t) \), it will imply the claim, since \( \beta(t) \in (0, \rho^*) \).

We study the dynamics of \( \Lambda(t) \), which goes continuously from \( \Lambda(0) \) to \( \Lambda(1) \), and the corresponding dynamics of the continuous curves \( \lambda_i(t) \) for \( i = 1, 2, 3 \).

We have that \( \Lambda(0) = \rho^* \), and, \( \Lambda(1) = \Lambda(\hat{\alpha}, \hat{\beta}) < \rho^* \). By continuity of \( \Lambda(t) \) and using the fact that \( \Lambda(t) \) has a strong local minimum at \( t = 0 \), it follows that there exists \( t_0 > 0 \) such that \( \Lambda(t_0) = \rho^* \) and \( \Lambda(t) > \rho^* \) for \( t \in (0, t_0) \).

Without loss of generality, we choose an indexing of the roots such that \( \lambda_1(0) = \rho^* \) and \( \lambda_2(0) = \lambda_3(0) = 0 \). For \( t \) close to 0, by continuity, the root \( \lambda_1(t) \) is not the conjugate of \( \lambda_2(t) \) and \( \lambda_3(t) \). Therefore, for \( t \) close to 0, \( \lambda_1(t) \) is real, equal to \( \Lambda(t) \) and thus, strictly greater than \( \rho^* \). Since \( |\lambda_1(t)| \leq \rho^* \), by continuity of the root, there must exist \( t_1 \in (0, t_0] \) such that \( |\lambda_1(t_1)| = \rho^* \). Either \( \lambda_1(t_1) = \rho^* \), which concludes the proof. Or, \( \lambda_1(t_1) \) is strictly complex or equal to \( -\rho^* \). In both cases, by continuity, there must exist \( t \in (0, t_0) \) such that \( \lambda_1(t) \) is strictly complex. Denote by \( t_2 \) the infimum time at which \( \lambda_1(t) \) becomes strictly complex. It holds that \( t_2 > 0 \), since \( \lambda_1(0) \) has single multiplicity. For \( t \in (0, t_2) \), the root \( \lambda_1(t) \) is real. Either there exists \( t \in (0, t_2) \) such that \( \lambda_1(t) = \rho^* \), which concludes the proof. Or, \( \lambda_1(t) > \rho^* \) for all \( t \in (0, t_2) \). By conjugacy of the complex roots, we must have \( \lambda_1(t_2) = \lambda_i(t_2) > \rho^* \) for some \( i \in \{2, 3\} \) (without loss of generality, say \( i = 2 \)). Since \( \lambda_2(0) = 0 \), by continuity of the root, there must exist \( t_3 \in (0, t_2) \) such that \( |\lambda_2(t_3)| = \rho^* \).

Either the root \( \lambda_2(t) \) crosses the (complex) circle of radius \( \rho^* \) along the real axis, at the point \( \rho^* \), which concludes the proof.

Or, the root \( \lambda_3(t) \) crosses the circle of radius \( \rho^* \) in the (strictly) complex plane or at \( -\rho^* \) and then it hits the real axis \([\rho^*, +\infty)\). Denote \( t_4 \) the first time at which \( \lambda_2(t) \) hits the real axis \([\rho^*, +\infty)\). Then, by conjugacy of \( \lambda_2(t_4) \) and \( \lambda_3(t_4) \) (since, right before \( t_4 \), \( \lambda_1(t) \) is real and \( \lambda_2(t) \) must be complex and hence, conjugate to \( \lambda_3(t) \)), we must have \( \lambda_2(t_4) = \lambda_3(t_4) \). Hence, \( \lambda_1(t_4)\lambda_2(t_4)\lambda_3(t_4) > \rho^3 \), which yields that \( \beta(t_4)^3 < \rho^3 < \lambda_1(t_4)\lambda_2(t_4)\lambda_3(t_4) = \beta(t_4)^3 \). The latter set of inequalities yields a contradiction, and thus the claim.
According to the previous result, we seek a contradiction by studying the equation $\chi_{\alpha,\beta}(\rho^*) = 0$. Let $\alpha, \beta \geq 0$ be some parameters such that $\chi_{\alpha,\beta}(\rho^*) = 0$. A simple calculation gives

$$\frac{\chi_{\alpha,\beta}(\rho^*)}{(1 - \rho^*)} = -\beta^3 + \rho^*(1 - 2\alpha)\beta^2 + \rho^*(1 - \alpha)(1 + \alpha(2\rho^* - 1))\beta - \rho^2(1 - \alpha)^2.$$ 

Thus, it holds that $\chi_{\alpha,\beta}(\rho^*) = 0$ if and only if $P_{\alpha}(\beta) = 0$, where

$$P_{\alpha}(\beta) := -\beta^3 + \rho^*(1 - 2\alpha)\beta^2 + \rho^*(1 - \alpha)(1 + \alpha(2\rho^* - 1))\beta - \rho^2(1 - \alpha)^2.$$ 

**Lemma 21.** Suppose that $\rho^* \neq \frac{1}{2}$. Then, for any $\alpha \geq 0$, the polynomial $P_{\alpha}$ has no root within the interval $(0, \rho^*)$.

**Proof.** We defer the proof to Section 8.6.

**Corollary 6.** For any value of $\rho^* \in (0, 1)$, it holds that

$$\inf_{\alpha \geq 0, \beta \geq 0} \Lambda(\alpha, \beta) = \rho^*,$$

and the infimum is uniquely attained at $(\alpha^*, \beta^*) = (1, 0)$.

**Proof.** Suppose first that $\rho^* \neq \frac{1}{2}$. Assume that for some parameters $\hat{\beta} \geq 0$ and $\hat{\alpha} \geq 0$, the root radius $\Lambda(\hat{\alpha}, \hat{\beta})$ satisfies

$$\Lambda(\hat{\alpha}, \hat{\beta}) < \rho^*.$$ 

Then, we must have $\hat{\beta} > 0$. Indeed, we have already shown in Corollary 1 that

$$\inf_{\alpha \geq 0} \Lambda(\alpha, 0) = \rho^*.$$ 

From Lemma 21, we know that there must exist some parameters $0 < \beta' < \rho^*$ and $\alpha' \geq 0$ such that $\chi_{\alpha',\beta'}(\rho^*) = 0$. That is, the polynomial $P_{\alpha'}$ has a root $\beta' \in (0, \rho^*)$. But this is contradiction with Lemma 21.

We turn to the value $\frac{1}{2}$. For any $\rho^* \in (0, 1)$, we denote $\Lambda(\alpha, \beta, \rho^*) \equiv \Lambda(\alpha, \beta)$. Note that, as $\alpha \to +\infty$ or $\beta \to +\infty$, then $\Lambda(\alpha, \beta, \rho^*) \to +\infty$. Therefore, for $\rho^* = \frac{1}{2}$, we can restrict the range of $(\alpha, \beta)$ to a rectangle $[0, R]^2$ for $R \geq 1$ large enough. Therefore, for any $\rho^* \in (0, 1)$,

$$\inf_{\alpha,\beta \geq 0} \Lambda(\alpha, \beta, \rho^*) = \inf_{0 \leq \alpha, \beta \leq R} \Lambda(\alpha, \beta, \rho^*). \quad (89)$$

We introduce the functions $f(\alpha, \beta, \rho^*) = -\rho^* + \Lambda(\alpha, \beta, \rho^*)$ and $F_R(\rho^*) = \inf_{0 \leq \alpha, \beta \leq R} f(\alpha, \beta, \rho^*)$. The function $f$ is continuous. Partial minimization of a continuous function over a compact preserves continuity with respect to the other variables. Therefore, $F_R$ is continuous in $\rho^*$. Further, we know that for $\rho^* \neq 1/2$, $F(\rho^*) = 0$. Hence, $F \equiv 0$ everywhere, which implies

$$\inf_{0 \leq \alpha, \beta \leq R} \Lambda(\alpha, \beta, \frac{1}{2}) = \frac{1}{2},$$

and concludes the proof of Theorem 7.1.
8.5 Proof of Lemma 19

For $\mu, \beta \in \mathbb{R}$, we define

$$
\begin{align*}
\begin{cases}
a_2 = -\beta^2 + (1 - 2\gamma_0)\beta - \eta_0, \\
a_1 = \beta (3\beta^2 - (1 - 2\gamma_0)\beta + 2\gamma_0^2 - \gamma_0), \\
a_0 = -\beta^3,
\end{cases}
\end{align*}
$$

so that $\chi_{\mu,\beta}(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$. Further, denote $\beta^* = 0$ and $\mu^* = \frac{\theta_2}{\rho_2}$. The roots of $\chi_{\mu^*,\beta^*}$ are equal to $\rho^* := 1 - \frac{\theta_2}{\rho_2}$ (with single multiplicity), and, 0 (with double multiplicity).

We introduce continuous parameterizations of the roots of $\chi_{\mu,\beta}$, that is, $\lambda_1 \equiv \lambda_1(\mu, \beta)$, $\lambda_2 \equiv \lambda_2(\mu, \beta)$ and $\lambda_3 \equiv \lambda_3(\mu, \beta)$ are the three roots of $\chi_{\mu,\beta}$, and continuous functions in $(\mu, \beta)$. Without loss of generality, we assume that $\lambda_1(\mu^*, \beta^*) = \rho^*$, $\lambda_2(\mu^*, \beta^*) = 0$, and, $\lambda_3(\mu^*, \beta^*) = 0$.

For $(\mu, \beta)$ in a neighborhood of $(\mu^*, \beta^*)$, by continuity of $(\lambda_1, \lambda_2, \lambda_3)$, we have that $\lambda_2$ and $\lambda_3$ are close to 0 and $\lambda_1$ close to $\rho^*$, and hence, $\lambda_1$ is necessarily a real root of the polynomial $\chi_{\beta,\mu}$, with single multiplicity (otherwise, $\lambda_1$ would be the complex conjugate of $\lambda_2$ or $\lambda_3$, but $\lambda_1$ is far apart from $\lambda_2$ and $\lambda_3$). Since $\lambda_1$ is real and has single multiplicity, it follows that $\lambda_1$ is differentiable in a neighborhood of $(\mu^*, \beta^*)$.

The function $\beta \mapsto \chi_{\mu,\beta}(\lambda_1(\mu, \beta))$ is constant and equal to 0. Denoting $a_1 = a_1(\mu, \beta)$, $a_2 = a_2(\mu, \beta)$ and $a_3 = a_3(\mu, \beta)$, we get, for $(\mu, \beta)$ close to $(\mu^*, \beta^*)$, that

$$
0 = \frac{d\chi_{\mu,\beta}(\lambda_1)}{d\beta} = \frac{d\lambda_1}{d\beta} (3\lambda_1^2 + 2\lambda_1 a_2 + a_1) + \lambda_1 \frac{da_2}{d\beta} + \lambda_1 \frac{da_1}{d\beta} + \frac{da_0}{d\beta}.
$$

At $(\mu, \beta) = (\mu^*, \beta^*)$, we have

$$
\begin{align*}
\begin{cases}
a_2 = -\eta_0 \\
a_1 = 0 \\
a_0 = 0
\end{cases}
\end{align*}
\begin{align*}
\begin{cases}
\frac{da_2}{d\beta} = 1 - 2\gamma_0 \\
\frac{da_1}{d\beta} = 2\gamma_0^2 - \eta_0 \\
\frac{da_0}{d\beta} = 0
\end{cases}
\begin{cases}
\gamma_0 = \rho^* \\
\eta_0 = \rho^*
\end{cases}
\end{align*}
$$

Hence, $3\rho^* + 2\rho^* a_2 + a_1 = \rho^2 \neq 0$. By continuity over a neighborhood of $(\mu^*, \beta^*)$, the term $3\lambda_1^2 + 2\lambda_1 a_2 + a_1$ is non-zero, and, from (90), we obtain $\frac{d\lambda_1}{d\beta} = -\frac{\lambda_1^2 a_2 + \lambda_1 a_1 + a_0}{3\lambda_1^2 + 2\lambda_1 a_2 + a_1}$. In particular, it implies that $\lambda_1$ is infinitely differentiable with respect to $\beta$, around $\beta^*$. Evaluating the latter derivative at $(\mu^*, \beta^*)$, we find that $\frac{d\lambda_1}{d\beta}(\mu^*, \beta^*) = 0$. Thus, $\beta^*$ is a stationary point of $\beta \mapsto \lambda_1(\mu^*, \beta)$. Further, differentiating again (90) with respect to $\beta$, evaluating at $(\mu^*, \beta^*)$ and using the fact that $\frac{d\lambda_1}{d\beta}(\mu^*, \beta^*) = 0$, we get

$$
0 = \frac{d^2\lambda_1}{d\beta^2} (3\rho^* + 2\rho^* a_2 + a_1) + \rho^* \frac{d^2 a_2}{d\beta^2} + \rho^* \frac{d^2 a_1}{d\beta^2} + \frac{d^2 a_0}{d\beta^2},
$$

so that $\frac{d^2\lambda_1}{d\beta^2} = -\frac{\rho^* \frac{d^2 a_2}{d\beta^2} + \rho^* \frac{d^2 a_1}{d\beta^2} + \frac{d^2 a_0}{d\beta^2}}{3\rho^2 + 2\rho^* a_2 + a_1}$. At $(\mu^*, \beta^*)$, we find

$$
\frac{d^2 a_0}{d\beta^2} = 0, \quad \frac{d^2 a_1}{d\beta^2} = 4\rho^* - 2, \quad \frac{d^2 a_2}{d\beta^2} = -2.
$$
Thus,
\[
\frac{d^2 \lambda_1}{d\beta^2}(\mu^*, \beta^*) = 2 \frac{1 - \rho^*}{\rho^*}.
\]

Differentiating \[90\] with respect to \( \mu \), evaluating at \((\mu^*, \beta^*)\) and using the fact that \( \frac{d\lambda_1}{d\mu}(\mu^*, \beta^*) = 0 \), we get
\[
0 = \frac{d^2 \lambda_1}{d\beta d\mu}\left(3\rho^* + 2\rho^* a_2 + a_1\right) + \rho^* \frac{d^2 a_2}{d\beta d\mu} + \rho^* \frac{d^2 a_1}{d\beta d\mu} + \frac{d^2 a_0}{d\beta d\mu},
\]
At \((\mu^*, \beta^*)\), we have
\[
\frac{d^2 a_2}{d\beta d\mu} = 2\theta_1, \quad \frac{d^2 a_1}{d\beta d\mu} = 4\theta_1 \rho^*, \quad \frac{d^2 a_0}{d\beta d\mu} = 0,
\]
from which we obtain
\[
\frac{d^2 \lambda_1}{d\beta d\mu}(\mu^*, \beta^*) = 2\theta_1.
\]
Using the same type of derivations, we obtain that around \((\mu^*, \beta^*)\),
\[
0 = \frac{d\chi_{\mu, \beta}(\lambda_1)}{d\mu} = \frac{d\lambda_1}{d\mu}\left(3\lambda_1^2 + 2\lambda_1 a_2 + a_1\right) + \lambda_1 \frac{d a_2}{d\mu} + \lambda_1 \frac{d a_1}{d\mu} + \frac{d a_0}{d\mu},
\]
from which we get that
\[
\frac{d\lambda_1}{d\mu} = -\frac{\lambda_1^2 \frac{d a_2}{d\mu} + \frac{d a_1}{d\mu} + \frac{d a_0}{d\mu}}{3\lambda_1^2 + 2\lambda_1 a_2 + a_1}.
\]
At \((\mu^*, \beta^*)\), we have \( \frac{d a_2}{d\mu} = \frac{d a_1}{d\mu} = \frac{d a_0}{d\mu} = 0 \), and thus, \( \frac{d\lambda_1}{d\mu}(\mu^*, \beta^*) = 0 \). Differentiating \[91\] with respect to \( \mu \), evaluating at \((\mu^*, \beta^*)\) and using the fact that \( \frac{d\lambda_1}{d\mu}(\mu^*, \beta^*) = 0 \), we get
\[
0 = \frac{d^2 \lambda_1}{d\mu^2}\left(3\rho^* + 2\rho^* a_2 + a_1\right) + \rho^* \frac{d^2 a_2}{d\mu^2} + \rho^* \frac{d^2 a_1}{d\mu^2} + \frac{d^2 a_0}{d\mu^2}.
\]
Further, at \((\mu^*, \beta^*)\), we find
\[
\frac{d^2 a_0}{d\mu^2} = 0, \quad \frac{d^2 a_1}{d\mu^2} = 0, \quad \frac{d^2 a_2}{d\mu^2} = -2\theta_2,
\]
from which we get
\[
\frac{d^2 \lambda_1}{d\mu^2}(\mu^*, \beta^*) = 2\theta_2.
\]
Thus, the Hessian of \( \lambda_1 \) at \((\mu^*, \beta^*)\) is \( M = \left[ \begin{array}{cc} \frac{\sigma^2}{\theta_2} & 2\theta_1 \\ 2\theta_1 & 2\theta_2 \end{array} \right] \). We set \( x := \frac{\sigma^2}{\theta_2} \). Since \( x \in (0, 1) \), we find that \( \text{Trace}(M) = 2\frac{x^2}{1 - x} + 2\theta_2 \), which is positive. Further, \( \det(M) = 4\theta_2 x^2 \) is also positive. Therefore, the matrix \( M \) is positive definite and \((\mu^*, \beta^*)\) is a strong local minimum.
8.6 Proof of Lemma 21

We recall the definition of the polynomial $P_\alpha$,

$$P_\alpha(\beta) := -\beta^3 + \rho^*(1 - 2\alpha)\beta^2 + \rho^*(1 - \alpha)(1 + \alpha(2\rho^* - 1))\beta - \rho^{*2}(1 - \alpha)^2.$$ 

We denote the coefficients of the polynomial $P_\alpha$ as

$$a := -1, \quad b \equiv b(\alpha, \rho^*) := \rho^*(1 - 2\alpha), \quad c \equiv c(\alpha, \rho^*) := \rho^*(1 - \alpha)(1 + \alpha(2\rho^* - 1)), \quad d \equiv d(\alpha, \rho^*) := -\rho^{*2}(1 - \alpha)^2.$$ 

In order to prove Lemma 21, we rely on simple variational properties of the degree three polynomial $P_\alpha$ and its derivative $P'_\alpha$. In the next result, we enumerate several properties satisfied by $P_\alpha$ and $P'_\alpha$, which results from simple calculations, and will be used throughout the rest of the proof.

**Lemma 22.** Suppose that $\rho^* \in (0, 1)$ and let $\alpha \in \mathbb{R}$. Then, the following statements are true.

(a) $P_\alpha(0) = -\rho^{*2}(1 - \alpha)^2 \leq 0$, with equality if and only if $\alpha = 1$.

(b) Suppose that $\alpha \geq 0$. It holds that $P_\alpha(\rho^*) = -2\alpha\rho^3$. Thus, $P(\rho^*) \leq 0$, with equality if and only if $\alpha = 0$. Further, $P_\alpha(\beta) = -\rho^*(1 - \alpha)(\beta - \sqrt{\rho^*})(\beta + \sqrt{\rho^*})$.

(c) The discriminant of the degree two polynomial $P'_\alpha$ is equal to $b^2 + 3c$, which satisfies

$$b^2 + 3c = -\frac{\rho^*}{3 - 2\rho^*} \left[ (3 - 2\rho^*)\alpha + \rho^* - 3 \right] + 3\rho^*(1 - \rho^*).$$

Consequently, we have $b^2 + 3c > 0$. Hence, the degree two polynomial $P'_\alpha$ has two distinct real roots, and its maximal root is given by

$$\beta_+(\alpha) \equiv \beta_+(\alpha, \rho^*) = \frac{1}{3} \left[ b + \sqrt{b^2 + 3c} \right].$$

Further, we have that

$$P_\alpha(\beta_+(\alpha)) = \frac{1}{27} \left[ 27d + 9bc + 2b^3 + 2(b^2 + 3c)^{3/2} \right].$$

**Corollary 7.** Suppose that $\rho^* \neq \frac{1}{2}$ and $\alpha \in \mathbb{R}$. Then, the following statements are true.

(a) $P_\alpha(\beta) \to +\infty$ as $\alpha \to -\infty$ and $P_\alpha(\beta_+(\alpha)) \to +\infty$ as $|\alpha| \to +\infty$.

(b) If $\rho^* < \frac{1}{2}$, then $\beta_+(\alpha) \to +\infty$ as $|\alpha| \to +\infty$. If $\rho^* > \frac{1}{2}$, then $\beta_+(\alpha) \to +\infty$ as $\alpha \to +\infty$.

(c) If $\alpha \neq 1$, the polynomial $P_\alpha$ has a negative root. If $\alpha = 1$, then $P_\alpha(0) = 0$.

(d) Let $\alpha \geq 0$, and suppose that the polynomial $P_\alpha$ has a root within $(0, \rho^*)$. Then, it holds that $\alpha > 0$ and $\alpha \neq 1$. Further, it holds that, either $P_\alpha$ has two distinct real roots $\beta_1(\alpha), \beta_2(\alpha) \in (0, \rho^*)$ such that $\beta_1(\alpha) < \beta_+(\alpha) < \beta_2(\alpha)$, or, $P_\alpha$ has one real root $\beta(\alpha)$ with multiplicity two within $(0, \rho^*)$ and $\beta(\alpha) = \beta_+(\alpha)$.

**Proof.** (a) The function $P$ is a degree three polynomial and its dominant coefficient is negative. Thus, $P_\alpha(\beta) \to +\infty$ as $\alpha \to +\infty$.

We have $P_\alpha(\beta_+(\alpha)) = \frac{2}{27} \left[ (\rho^* (3 - 2\rho^*))^{3/2} |\alpha|^3 + (10\rho^* - 9\rho^{*2}) |\alpha|^3 \right] + O(|\alpha|^3)$, as $|\alpha| \to +\infty$. As $\alpha \to +\infty$, we have $P_\alpha(\beta_+(\alpha)) \to +\infty$ if and only if $(\rho^* (3 - 2\rho^*))^{3/2} + (10\rho^* - 9\rho^{*2}) > 0$. The
latter inequality is equivalent to $10 - \frac{9}{\rho^3} + \left(\frac{3}{\rho^2} - 2\right)^{\frac{3}{2}} > 0$. Set $f(x) = 10 - 9x + (3x - 2)^{\frac{3}{2}}$, for $x > 1$. Then $f(x) \to +\infty$ as $x \to +\infty$ and $f'(x) = 9 \left(-1 + \frac{1}{2}(3x - 2)^{\frac{1}{2}}\right) = 0$ if and only if $x = 2$. Further, we have $f(2) = 0$. Therefore, the minimal value of $f(x)$ is 0, which is strictly attained at $x = 2$. Provided that $\rho^* \neq \frac{1}{2}$, it follows that $P_\alpha(\beta_+(\alpha)) \to +\infty$ when $\alpha \to +\infty$. On the other hand, when $\alpha \to -\infty$, we have $P_\alpha(\beta_+(\alpha)) = \frac{2\alpha^3}{\rho^2} \left((\rho^* (3 - 2\rho^*))^3 + 9\rho^* - 10\rho^* + O(\alpha^3)\right)$.

Further, $(\rho^* (3 - 2\rho^*))^3 + 9\rho^* - 10\rho^* > 0$ if and only if $\frac{9}{\rho^2} - 10 + \left(\frac{3}{\rho^2} - 2\right)^{\frac{3}{2}} > 0$. Setting $g(x) = 9x - 10 + (3x - 2)^{\frac{3}{2}}$, for $x \geq 1$, we have that $g'(x) = 9 + \frac{9}{2}\sqrt{3x - 2} > 0$, and $g(1) = 0$. Therefore, $P_\alpha(\beta_+(\alpha)) \to +\infty$ when $\alpha \to -\infty$.

(b) We have

$$\beta_+(\alpha) = \frac{1}{3} \left[\rho^*(1 - 2\alpha) + \sqrt{\frac{\rho^*}{3 - 2\rho^*}} \sqrt{((3 - 2\rho^*)\alpha + \rho^* - 3)^2 + 3\rho^*(1 - \rho^*)}\right].$$

Thus, it holds that $\beta_+(\alpha) = \frac{1}{3} \left(\sqrt{\rho^*(3 - 2\rho^*)} |\alpha| - 2\rho^* \alpha\right) + O(1)$ as $|\alpha| \to +\infty$. Hence, the asymptotic limits of $\beta_+(\alpha)$ immediately follow from the fact that the inequality $\sqrt{\rho^*(3 - 2\rho^*)} > 2\rho^*$ is equivalent to $\rho^* < \frac{1}{2}$.

(c) If $\alpha \neq 1$, then the zero-order coefficient of $P_\alpha$, which is equal to $-\rho^2(1 - \alpha)^2$, is negative. Since the degree of $P_\alpha$ is odd and its dominant coefficient is negative, it follows that $P_\alpha$ must have a negative root. If $\alpha = 1$, the zero-order coefficient is equal to 0, i.e., $P_1(0) = 0$.

(d) Suppose that the polynomial $P_\alpha$ has a root $\tilde{\beta} \in (0, \rho^*)$. From Lemma 22b, we have $P_\alpha(\tilde{\beta}) = -\rho^*(\tilde{\beta} - \rho^*) = -\rho^*(\tilde{\beta} - \sqrt{\rho^*}) = -\rho^*(\tilde{\beta} + \sqrt{\rho^*})$, which cannot be equal to 0 since $\sqrt{\rho^*} < \rho^* < \sqrt{\rho^*}$. On the other hand, we find that $P_1(\beta) = -\beta^3 - \rho^* \beta^2$, which roots are exactly 0 with multiplicity two and $-\rho^*$. Hence, $P_1$ has no root within $(0, \rho^*)$. Therefore, we must have $\alpha > 0$ and $\alpha \neq 1$. We claim that, either $P_\alpha$ has a second root within $(0, \rho^*)$, or, its root $\tilde{\beta}$ has multiplicity two. From Lemma 22 we have $P_\alpha(0) < 0$ and $P_\alpha(\rho^*) < 0$. First, assume that $P_\alpha(\beta) \leq 0$ in a neighborhood $\beta$, i.e., the root $\beta$ is a local maximum. In that case, $P_\alpha(\beta) = 0$, which implies that $\beta$ is a real root of $P_\alpha$ with multiplicity at least two. Since $P_\alpha(-\infty) = +\infty$ and $P_\alpha(0) < 0$, we must have $\beta = \beta_+(\alpha)$. On the other hand, if $\beta$ is not a local maximum, then $P_\alpha$ takes both negative and positive values within $(0, \rho^*)$. Since $P_\alpha(0) < 0$ and $P_\alpha(\rho^*) < 0$, we obtain that $P_\alpha$ must cross the x-axis at least twice, at $\beta_1(\alpha)$ and $\beta_2(\alpha)$, and that $P_\alpha(\beta) > 0$ for $\beta \in (\beta_1(\alpha), \beta_2(\alpha))$, which further implies that $\beta_1(\alpha) < \beta_+(\alpha) < \beta_2(\alpha)$.

Lemma 23. Let $\rho^* \in (0, 1)$ and $\rho^* \neq \frac{1}{2}$. Then the following statements are true.

(a) Suppose that $\rho^* < \frac{1}{2}$. Then, there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $0 < \alpha_1 < 1 < \alpha_2$ and, for any $\alpha \geq 0$, $\beta_+(\alpha) < \rho^*$ if and only if $\alpha \in (\alpha_1, \alpha_2)$. Further, $\beta_+(\alpha_1) = \beta_+(\alpha_2) = \rho^*$.

(b) Suppose that $\rho^* > \frac{1}{2}$. Then, there exists $\overline{\alpha} \in (0, 1)$ such that for any $\alpha \geq 0$, $0 < \beta_+(\alpha) < \rho^*$ if and only if $\alpha \in (\overline{\alpha}, 1)$. Further, $\beta_+(\overline{\alpha}) = \rho^*$ and $\beta_+(1) = 0$.

Proof. Fix $\alpha \in \mathbb{R}$. Using the expression of $\beta_+(\alpha)$ given in Lemma 22 we have that $\beta_+(\alpha) < \rho^*$ if and only if

$$\frac{1}{3} \left[\rho^*(1 - 2\alpha) + \sqrt{\frac{\rho^*}{3 - 2\rho^*} \left[((3 - 2\rho^*)\alpha + \rho^* - 3)^2 + 3\rho^*(1 - \rho^*)\right]}\right] < \rho^*,$$
which is equivalent to
\[
\sqrt{\frac{\rho^*}{3 - 2\rho^*}} [(3 - 2\rho^*)\alpha + \rho^* - 3) + 3\rho^*(1 - \rho^*)] < 2\rho^*(1 + \alpha) .
\] (92)

Inequality (92) can only be true for \( \alpha > -1 \), which we assume from now on. Squaring both sides and after a few manipulations, we find that inequality (92) is equivalent to
\[
Q(\alpha) := \alpha^2(-1 + 2\rho^*) + 2\alpha(1 + \rho^*) + (-1 + \rho^*) > 0 .
\]

The polynomial \( Q \) has two distinct real roots \( \alpha_1, \alpha_2 \), which are given by
\[
\alpha_1 = \frac{\rho^* + 1 - \sqrt{\rho^*(5 - \rho^*)}}{1 - 2\rho^*}, \quad \alpha_2 = \frac{\rho^* + 1 + \sqrt{\rho^*(5 - \rho^*)}}{1 - 2\rho^*} .
\]

If \( \rho^* < \frac{1}{2} \), the dominant coefficient of \( Q \) is negative, and \( Q \) takes positive values between its two roots. Therefore, \( \beta_+(\alpha) < \rho^* \) if and only if \( \alpha \in (\alpha_1, \alpha_2) \cap (-1, +\infty) \). Further, it holds that \( \alpha_1 > 0, \alpha_1 < 1 \) and \( \alpha_2 > 1 \). Hence, \( \beta_+(\alpha) < \rho^* \) if and only if \( \alpha \in (\alpha_1, \alpha_2) \).

If \( \rho^* > \frac{1}{2} \), the dominant coefficient of \( Q \) is positive. Hence, \( \beta_+(\alpha) < \rho^* \) if and only if \( \alpha \in (1, +\infty) \cup (-\infty, \alpha_2) \) and \( \alpha > -1 \). It holds that \( \alpha_1 \in (0, 1) \) and \( \alpha_2 < -1 \). Thus, setting \( \alpha_2 = \alpha_1 \), we have \( \beta_+(\alpha) < \rho^* \) if and only if \( \alpha > \alpha_2 \). On the other hand, a calculation yields that \( \beta_+(\alpha) > 0 \) if and only if \( \rho^*(1 - \alpha)(1 + \alpha(2\rho^* - 1)) > 0 \). Since \( \rho^* > \frac{1}{2} \), it follows that \( \rho^*(1 + \alpha(2\rho^* - 1)) > 0 \), and thus, \( \alpha \) must be less than 1. Hence, \( \beta_+(\alpha) \in (0, \rho^*) \) if and only if \( \alpha \in (\alpha_2, 1) \).

**Lemma 24.** Let \( \rho^* \in (0, 1) \) such that \( \rho^* \neq \frac{1}{2} \). Then, the following statements are true.

(a) Suppose that \( \rho^* < \frac{1}{2} \). Then, the function \( \alpha \mapsto P_\alpha(\beta_+(\alpha)) \) over \( [\alpha_1, \alpha_2] \) attains uniquely its maximum at \( \alpha = 1 \), and the maximal value is equal to 0. Consequently, for any \( \alpha \geq 0 \) such that \( \beta_+(\alpha) \in (0, \rho^*) \), we have \( P_\alpha(\beta_+(\alpha)) < 0 \).

(b) Suppose that \( \rho^* > \frac{1}{2} \). Then, the function \( \alpha \mapsto P_\alpha(\beta_+(\alpha)) \) over \( [\alpha_1, 1] \) attains uniquely its maximum at \( \alpha = 1 \), and the maximal value is equal to 0. Consequently, for any \( \alpha \geq 0 \) such that \( \beta_+(\alpha) \in (0, \rho^*) \), we have \( P_\alpha(\beta_+(\alpha)) < 0 \).

**Proof.** Using the expression of \( P_\alpha(\beta_+(\alpha)) \) given in Lemma 22, a simple calculation yields that
\[
\frac{d}{d\alpha} P_\alpha(\beta_+(\alpha))_{\alpha = 1} = 0
\]
\[
\frac{d^2}{d\alpha^2} P_\alpha(\beta_+(\alpha))_{\alpha = 1} = -2(1 - \rho^*)\rho^{*2} .
\]

Therefore, \( \alpha = 1 \) is a strict local maximum of \( \alpha \mapsto P_\alpha(\beta_+(\alpha)) \), and we find through further calculation that \( P_1(\beta_+(1)) = 0 \).

Suppose that \( \rho^* < \frac{1}{2} \), and that the function \( P_\alpha(\beta_+(\alpha)) \) has a unique local maximum at \( \alpha = 1 \). From Lemma 23, we have \( \beta_+(\alpha_i) = \rho^* \), for \( i = 1, 2 \). Consequently, \( P_{\alpha_i}(\beta_+(\alpha_i)) = P_{\alpha_1}(\rho^*) \), for \( i = 1, 2 \). From Lemma 22 and the fact that \( \alpha_1, \alpha_2 > 0 \), we know that \( P_{\alpha_2}(\rho^*) < 0 \), and hence, \( P_{\alpha_i}(\beta_+(\alpha_i)) < 0 = P_1(\beta_+(1)) \) for \( i = 1, 2 \). Further, \( 1 \in (\alpha_1, \alpha_2) \). Therefore, the maximum of \( \alpha \mapsto P_\alpha(\beta_+(\alpha)) \) over \( [\alpha_1, \alpha_2] \) is uniquely attained at \( \alpha = 1 \). Hence, for any \( \alpha \in (\alpha_1, \alpha_2) \) and \( \alpha \neq 1 \), we get that \( P_{\alpha_2}(\beta_+(\alpha)) < P_1(\beta_+(1)) \), i.e., \( P_{\alpha_2}(\beta_+(\alpha)) < 0 \). According to Lemma 23, the latter statement implies that for any \( \alpha \geq 0 \) such that \( \beta_+(\alpha) \in (0, \rho^*) \), we have \( P_{\alpha_2}(\beta_+(\alpha)) < 0 \).
Suppose that $\rho^* > \frac{1}{2}$, and that the function $P_\alpha(\beta_+(\alpha))$ has a unique local maximum at $\alpha = 1$. From Lemma 23, we have $\beta_+(\overline{\alpha}) = \rho^*$, and $\overline{\alpha} > 0$. From Lemma 22 and the fact that $\overline{\alpha} > 0$, we get that $P_\overline{\alpha}(\rho^*) < 0$. Thus, $P_\overline{\alpha}(\beta_+(\overline{\alpha})) = P_\overline{\alpha}(\rho^*) < 0 = P_1(\beta_+(1))$. Hence, for $\alpha \in (\overline{\alpha}, 1)$, by unicity of the local maximum, we deduce that $P_\alpha(\beta_+(\alpha)) < 0$. According to Lemma 23, the latter statement implies that for any $\alpha \geq 0$ such that $\beta_+(\alpha) \in (0, \rho^*)$, we have $P_\alpha(\beta_+(\alpha)) < 0$.

Hence, it remains to show that $\alpha \mapsto P_\alpha(\beta_+(\alpha))$ has a unique local maximum at $\alpha = 1$. It holds that

$$P_\alpha(\beta_+(\alpha)) = Q(\alpha) + \frac{2}{27}(b^2 + 3c)^3,$$

where $Q(\alpha) := \frac{1}{27}(27d + 9bc + 2b^3)$. Then,

$$\frac{d}{d\alpha}P_\alpha(\beta_+(\alpha)) = Q'(\alpha) + R(\alpha),$$

where $R(\alpha) := \frac{1}{9}(2bb' + 3c')(b^2 + 3c)\frac{1}{2}$. By definition, any critical point of $\alpha \mapsto P_\alpha(\beta_+(\alpha))$ is a solution of the equation $Q'(\alpha) = -R(\alpha)$. Squaring both sides of the latter equation, we get that any critical point must satisfy

$$S(\alpha) := Q'(\alpha)^2 - R(\alpha)^2 = 0.$$

Through a simple calculation, we find that the function $S(\alpha)$ is a polynomial of degree less than four. Thus, the function $\alpha \mapsto P_\alpha(\beta_+(\alpha))$ has at most four critical points. Suppose by contradiction that there exist at least two local maxima. From Corollary 7, we know that $P_\alpha(\beta_+(\alpha)) \rightarrow +\infty$ as $|\alpha| \rightarrow +\infty$. The latter fact along with the existence of (at least) two local maxima implies that there must exist (at least) three local minima. Thus, there exist at least five critical points, which is a contradiction, and concludes the proof. 

**Lemma 25.** Let $\rho^* \in (0, 1)$ such that $\rho^* \neq \frac{1}{2}$. Suppose that, for some $\overline{\alpha} \geq 0$, the polynomial $P_{\overline{\alpha}}$ has a real root within the interval $(0, \rho^*)$. Then, there must exist some $\widehat{\alpha} > 0$ such that

$$\widehat{\alpha} \neq 1, \quad \beta_+(\widehat{\alpha}) = \beta(\widehat{\alpha}) \in (0, \rho^*), \quad P_{\widehat{\alpha}}(\beta(\widehat{\alpha})) = 0.$$

**Proof.** According to Corollary 7, we must have $\overline{\alpha} > 0$ and $\overline{\alpha} \neq 1$. Further, $P_{\overline{\alpha}}$ must either have one real root $\beta$ with multiplicity two within $(0, \rho^*)$ and $\beta = \beta_+(\overline{\alpha})$ – in which case the result holds by setting $\widehat{\alpha} = \overline{\alpha}$ –, or, $P_{\overline{\alpha}}$ must have two distinct real roots $\beta_1, \beta_2 \in (0, \rho^*)$ such that $\beta_1 < \beta_+(\overline{\alpha}) < \beta_2$. Thus, let us assume that $P_{\overline{\alpha}}$ has two distinct real roots $\beta_1, \beta_2 \in (0, \rho^*)$ such that $\beta_1 < \beta_+(\overline{\alpha}) < \beta_2$. We introduce $\alpha \mapsto \beta_1(\alpha)$ and $\alpha \mapsto \beta_2(\alpha)$ continuous parameterizations over $[0, +\infty)$ of two of the roots of the polynomial $P_\alpha$, such that $\beta_1(\overline{\alpha}) = \beta_1$ and $\beta_2(\overline{\alpha}) = \beta_2$. Let us distinguish two cases.

**Case 1: $\overline{\alpha} < 1$.**

Assume, by contradiction, that for any $\alpha \in [0, \overline{\alpha}]$, $\beta_1(\alpha) \neq \beta_2(\alpha)$. For any $\alpha \in [0, \overline{\alpha}]$, $P_\alpha(0) = -\rho^{\alpha^2}(1 - \alpha)^2 \neq 0$. Therefore, as $\alpha$ decreases from $\overline{\alpha}$ to 0, the roots $\beta_1(\alpha)$ and $\beta_2(\alpha)$ cannot become equal to 0, nor can the third root of $P_\alpha$ – which, we know from Corollary 7 is negative at $\overline{\alpha}$. Then, there are two cases.

- Either, $\beta_1(\alpha)$ and $\beta_2(\alpha)$ remain both real and, thus, positive. Then, we must have $\beta_1(\alpha) < \beta_2(\alpha)$ for any $\alpha \in [0, \overline{\alpha}]$, and the third root of $P_\alpha$ remains negative. Then, $\beta_2(\alpha)$ is the maximal root of $P_\alpha$ for any $\alpha \in [0, \overline{\alpha}]$. But, it holds that $\beta_2(\overline{\alpha}) < \rho^* < \sqrt{\rho^*}$, and $P_0$ has roots $\pm \sqrt{\rho^*}$ and $\rho^*$. It implies that $\beta_2(0) = \sqrt{\rho^*}$, so that, by continuity, $\beta_2(\alpha)$ must be equal to $\rho^*$ for some $\alpha \in (0, \overline{\alpha})$. But $P_\alpha(\rho^*) = 0$ implies that $\alpha = 0$, which is a contradiction.
Hence, $\beta_1(\alpha)$ or $\beta_2(\alpha)$ (say $\beta_1$) must become strictly complex for some $\alpha \in (0, \bar{\alpha})$. Let $\hat{\alpha} := \sup\{\alpha < \bar{\alpha} \mid \beta_1(\alpha) \in \mathbb{C} - \mathbb{R}\}$. By continuity of $\beta_1(\alpha)$ and the fact that $\beta_1(\bar{\alpha}) \in \mathbb{R}$, we must have $\beta_1(\hat{\alpha}) \in \mathbb{R}$. Since $\beta_1(\alpha)$ cannot cross the point 0 nor the point $\rho^*$ for $\alpha \in (\hat{\alpha}, \bar{\alpha})$, it holds that $\beta_1(\hat{\alpha}) \in (0, \rho^*)$. By continuity and conjugacy, we must have that $\beta_1(\hat{\alpha}) = \beta_2(\hat{\alpha})$, which is a contradiction.

**Case 2:** $\bar{\alpha} > 1$.

Assume by contradiction that for any $\alpha \geq \bar{\alpha}$, $\beta_1(\alpha) \neq \beta_2(\alpha)$. Then, there are two cases.

- Either $\beta_1(\alpha)$ and $\beta_2(\alpha)$ remain both real. Then, we must have $\beta_1(\alpha) < \beta_2(\alpha)$ for any $\alpha \geq \bar{\alpha}$. We have the inequality $\beta_1(\alpha) < \beta_2(\alpha)$. If $\rho^* < \frac{1}{2}$, then, according to Corollary 7, $\beta_+(-\alpha) \to +\infty$ as $\alpha \to +\infty$, which implies that $\beta_2(\alpha) \to +\infty$ and $\beta_2(\alpha)$ must be equal to $\rho^*$ for some $\alpha > \bar{\alpha}$. But $P_\alpha(\rho^*) = 0$ implies that $\alpha = 0$, which is a contradiction. If $\rho^* > \frac{1}{2}$, then $\beta_+(\alpha) \to -\infty$ as $\alpha \to +\infty$, so that $\beta_1(\alpha) \to -\infty$ and $\beta_1(\alpha)$ must be equal to 0 for some $\alpha > \bar{\alpha}$. But $P_\alpha(0) = -\rho^2(1 - \alpha)^2$, which cannot be equal to 0 for $\alpha \geq \bar{\alpha} > 1$, and is thus a contradiction.

- Hence, $\beta_1(\alpha)$ or $\beta_2(\alpha)$ (say $\beta_1$) must become strictly complex for some $\alpha \geq \bar{\alpha}$. Let $\hat{\alpha} := \inf\{\alpha > \bar{\alpha} \mid \beta_1(\alpha) \in \mathbb{C} - \mathbb{R}\}$. By continuity of $\beta_1(\alpha)$ and the fact that $\beta_1(\bar{\alpha}) \in \mathbb{R}$, we must have $\beta_1(\hat{\alpha}) \in \mathbb{R}$. Since $\beta_1(\alpha)$ cannot cross the point 0 nor the point $\rho^*$ for $\alpha \in (\bar{\alpha}, \hat{\alpha})$, it holds that $\beta_1(\hat{\alpha}) \in (0, \rho^*)$. By continuity and conjugacy, we must have that $\beta_1(\hat{\alpha}) = \beta_2(\hat{\alpha})$, which is a contradiction.

**Proof of Lemma 21.** Assume, by contradiction, that the polynomial $P_\alpha$ has a real root within the interval $(0, \rho^*)$. By Lemma 25, we know that there exists some $\hat{\alpha} > 0$ and $\hat{\alpha} \neq 1$ such that $\beta(\hat{\alpha})$ is a root of $P_\alpha$ and $\beta(\hat{\alpha}) = \beta_+(\hat{\alpha}) \in (0, \rho^*)$. By Lemma 23, since $\beta_+(\hat{\alpha}) \in (0, \rho^*)$, we obtain that $P_\alpha(\beta_+(\hat{\alpha})) < 0$, i.e., $P_\alpha(\beta(\hat{\alpha})) < 0$, which is a contradiction. 

\[ \square \]
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