On Equivalence of Matrices

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Abstract

Motivated by semi-tensor product (STP) of matrices, certain equivalences of matrices are proposed, which allow the matrix operators being more general, beyond the dimension restrictions. Precisely speaking, it allows the matrix product being available for two arbitrary matrices, and matrix addition being available for certain matrices of different dimensions. In addition, the characteristic function, eigenvalue/eigenvector etc. are also defined for a large class of non-square matrices. All these extensions are the generalization of classical ones. That is, when the matrices meet the dimension requirement for the classical definitions, the extended definitions coincide with the classical ones.

To this end, a new matrix product, called the STP, is first reviewed, which extends the classical matrix product to two arbitrary matrices. Under STP the set of matrices becomes a monoid (semi-group with identity). Some related structures and properties are investigated. Then the generalized matrix addition is also presented, which extends the classical matrix addition to a class of two matrices with different dimensions.

Two kinds of equivalences are introduced, which are matrix equivalence and vector equivalence. The lattice structure has been established for each equivalence. Under each equivalence, the corresponding quotient space becomes a vector space. Under matrix equivalence, many algebraic, geometric, and analytic structures have been posed to the quotient space, which include (i) lattice structure; (ii) inner product and norm (distance); (iii) topology; (iv) a fiber bundle structure, called the discrete bundle; (v) bundled differential manifold; (vi) bundled Lie group and Lie algebra. Under vector space equivalence, vectors of different dimensions form a vector space \( V \), and a matrix \( A \) of arbitrary dimension is considered as an operator (linear mapping) on \( V \). When \( A \) is a bounded operator (not necessarily square but includes square matrices as a special case), the generalized characteristic function, eigenvalue and eigenvector etc. are defined.

In one word, this new matrix theory overcomes the dimensional barrier in certain sense. It provides much more freedom for using matrix approach to practical problems.

Key words: Semi-tensor product/addition (STA/STA), vector product/addition (VP/VA), matrix/vector equivalence (ME/VE), lattice, topology, fiber bundle, bundled manifold/Lie algebra/Lie group (BM/BLA/BLG).

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1.2 Introduction

Matrix theory and Calculus are two classical and fundamental mathematical tools in modern science and technology. There are two mostly used operators on the set of matrices: conventional matrix product and matrix addition. Roughly speaking, the object of matrix theory is $(\mathcal{M}, +, \times)$, where $\mathcal{M}$ is the set of all matrices. Unfortunately, unlike the arithmetic system $(\mathbb{R}, +, \times)$, in matrix theory both “+” and “×” are restricted by the matrix dimensions. Precisely speaking: consider two matrices $A \in \mathcal{M}_{m \times n}$ and $B \in \mathcal{M}_{p \times q}$. Then the “product”, $A \times B$, is well posed, if and only if, $n = p$; the “addi-
tion” $A + B$, is defined, if and only if, $m = p$ and $n = q$. Though there are some other matrix products such as Kronecker product, Hadamard product etc., but they are of different stories [23].

The main purpose of this paper is to develop a new matrix theory, which intendes to overcome the dimension barrier by extending the matrix product and matrix addition to two matrices which do not meet the classical di-
mension requirement. As generalizations of the classical ones, they should be consistent with the classical definitions. That is, when the dimension requirements of two argument matrices in classical matrix theory are satis-
fied, the newly defined operators should coincide with the original one.

Because of the extension of the two fundamental operators, many related concepts can be extended. For in-
stance, the characteristic functions, the eigenvalues and eigenvectors of a square matrix can be extended to cer-
tain non-square matrices; Lei algebraic structure can also be extended to dimension-varying square matrices. All these extensions should be consistent with the classical ones. In one word, we are developing the classical matrix theory but not violating any of the original matrix theory.

When we were developing the extended operators we meet a serious problem: Though the extended operators are applicable to certain sets of matrices with different dimensions, they fail to be vector space anymore. This drawback is not acceptable, because it blocked the way to extend many nice algebraic or geometric structures in matrix theory, such as Lie algebraic structure, manifold structure etc., to the enlarged set, which includes matrices of different dimensions. To overcome this ob-
stacle, we introduced certain equivalence relations. Then
the quotient spaces, called the equivalence spaces, become vector spaces. Two equivalence relations have been proposed. They are matrix equivalence (ME) and vector equivalence (VE).

Then many nice algebraic and geometric structures have been developed on the matrix equivalence spaces. They are briefly introduced as follows:

- **Lattice structure**: The elements in each equivalent class form a lattice. The class of spaces with different dimensions also form a lattice. The former and the latter are homomorphic. The lattices obtained for ME and VE are isomorphic.
- **Topological structure**: A topological structure is proposed to the equivalence space, which is built using topological sub-base. It is then proved that under this topology the equivalence space is Hausdorff ($T_2$) space and is second countable.
- **Inner product structure**: An inner product is proposed on the equivalence space. The norm (distance) is also obtained. It is proved that the topology induced by this norm is the same as the topology produced by using topological sub-base.
- **Fiber bundle structure**: A fiber bundle structure is proposed for the set of matrices (as total space) and the equivalent classes (as base space). The bundle structure is named the discrete bundle, because each fiber has discrete topology.
- **Bundle manifold structure**: A (dimension-varying) manifold structure is proposed for the equivalence space. Its coordinate charts are constructed via the discrete bundle. Hence it is called a bundled manifold.
- **Bundled Lie algebraic structure**: A Lie algebra structure is proposed for the equivalence space. The Lie algebra is of infinite dimensional, but almost all the properties of finite dimensional Lie algebras remain available.
- **Bundled Lie group structure**: For the equivalence classes of square nonsingular matrices a group structure is proposed. It has also the dimension-varying manifold structure. Both algebraic and geometric structures are consistent and hence it becomes a Lie group. The relationship of this Lie group with the bundled Lie algebra is also investigated.

Under vector equivalence, all the vectors form a vector space $V$, and any matrix $A$ can be considered as an linear operator on $V$. A very important class of $A$, called the bounded operator, is investigated in details. For a bounded operator $A$, which could be non-square, its characteristic function is proposed. Its eigenvalue and the corresponding eigenvector are obtained. $A$-invariant subspace has been discussed in detail.

This work is motivated by the semi-tensor product (STP). The STP of matrices was proposed firstly and formally in 2001 [3]. Then it has been used to some Newton differential dynamic systems and their control problems [4], [46], [36]. A basic summarization was given in [5].

Since 2008, STP has been used to formulate and analyze Boolean networks as well as general logical dynamic systems, and to solve control problems for those systems. It has then been developed rapidly. This is witnessed by hundreds of research papers. A multitude of applications of STP include (i) logical dynamic systems [6], [17], [30]; (ii) systems biology [49], [19]; (iii) graph theory and formation control [40], [50]; (iv) circuit design and failure detection [31], [32], [9]; (v) game theory [20], [10], [11]; (vi) finite automata and symbolic dynamics [45], [52], [22]; (vii) coding and cryptography [53], [51]; (viii) fuzzy control [8], [16]; (ix) some engineering applications [44], [33]; and many other topics [7], [35], [47], [54], [55], [34]; just to name a few.

As a generalization of conventional matrix product, STP is applicable to two matrices of arbitrary dimensions. In addition, this generalization keeps all fundamental properties of conventional matrix product available. Therefore, it becomes a very conventional tool for investigating many matrix computation related problems.

Up to this time, the main effort has been put on its applications. Now when we start to explore the mathematical foundation of STP, we found that the most significant characteristic of STP is that it overcomes the dimension barrier. After serious thought, it can be seen that in fact STP is defined on equivalent classes. Following this thought of train, the matrix theory on equivalence space emerges. The outline of this new matrix theory is presented in this manuscript.

### 1.3 Symbols

For statement ease, we first give some notations:

1. $\mathbb{N}$: Set of natural numbers (i.e., $\mathbb{N} = \{1, 2, \cdots \}$);
2. $\mathbb{Z}$: set of integers;
(3) \( \mathbb{Q} \): Set of rational numbers, \((\mathbb{Q}_+ : \text{Set of positive rational numbers})\);
(4) \( \mathbb{R} \): Field of real numbers;
(5) \( \mathbb{C} \): Field of complex numbers;
(6) \( F \): certain field of characteristic number 0 (Particularly, we can understand \( F = \mathbb{R} \) or \( F = \mathbb{C} \)).

(7) \( \mathcal{M}^F_{m \times n} \): set of \( m \times n \) dimensional matrices over field \( F \). When the field \( F \) is obvious or does not affect the discussion, the superscript \( F \) can be omitted, and as a default: \( F = \mathbb{R} \) can be assumed.

(8) \( \text{Col}(A) \) (Row(A)): the set of columns (rows) of \( A \); \( \text{Col}_i(A) \) (Row\(_i\)(A)): the \( i \)-th column (row) of \( A \).

(9) \( D_k = \{1, 2, \cdots, k\}, D := D_2 \);
(10) \( \delta_i^j \): the \( i \)-th column of the identity matrix \( I_n \);
(11) \( \Delta_n = \{\delta_i^i | i = 1, 2, \cdots, n\}; \)
(12) \( L \in \mathcal{M}_{m \times r} \) is a logical matrix, if \( \text{Col}(L) \subset \Delta_m \).

The set of \( m \times r \) logical matrices is denoted as \( \mathcal{L}_{m \times r} \);

(13) Assume \( A \in \mathcal{L}_{m \times r} \). Then \( L = [\delta_1^m, \delta_2^m, \cdots, \delta_n^m] \). It is briefly denoted as

\[
L = \delta_m[i_1, i_2, \cdots, i_r].
\]

(14) Let \( A = (a_{i,j}) \in \mathcal{M}_{m \times n} \), and \( a_{i,j} \in \{0, 1\}, \forall i, j \).

Then \( A \) is called a Boolean matrix. Denote the set of \( m \times n \) dimensional Boolean matrices by \( \mathcal{B}_{m \times n} \).

(15) Set of probabilistic vectors:

\[
\Upsilon_k := \left\{(r_1, r_2, \cdots, r_k)^T \mid r_i \geq 0, \sum_{i=1}^k r_i = 1\right\}.
\]

(16) Set of probabilistic matrices:

\[
\Upsilon_{m \times n} := \left\{M \in \mathcal{M}_{m \times n} \mid \text{Col}(M) \subset \Upsilon_m \right\}.
\]

(17) \( a | b \): \( a \) divides \( b \).
(18) \( m \wedge n = \gcd(m, n) \): The greatest common divisor of \( m, n \).
(19) \( m \vee n = \mathsf{lcm}(m, n) \): The least common multiple of \( m, n \).

(20) \( \times \) (\( \times \)): The left (right) semi-tensor product of matrices.
(21) \( \times \times \) (\( \times \times \)): The left (right) vector product of matrices.
(22) \( \sim \) (\( \sim \)): The matrix equivalence ((left, right) matrix equivalence).
(23) \( \leftrightarrow \) (\( \leftrightarrow \)): The vector equivalence ((left, right) vector equivalence).

(24) The set of all matrices:

\[
\mathcal{M} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \mathcal{M}_{i \times j}.
\]

(25) \( \mathcal{M}_{\times q} := \{ A \in \mathcal{M} \mid \text{column number of } A \text{ is } q \} \).

(26) The set of matrices:

\[
\mathcal{M}_\mu := \{ A \in \mathcal{M}_{m \times n} | m/n = \mu \}
\]

(27) Lattice homomorphism: \( \cong \)
(28) Lattice isomorphism: \( \approx \)
(29) Vector order: \( \leq \)
(30) Vector space order: \( < \)
(31) Matrix order: \( \prec \)
(32) Matrix space order: \( \preceq \)
(33) The overall matrix quotient space:

\[
\Sigma_{\mathcal{M}} = \mathcal{M} / \sim.
\]

(34) The \( \mu \)-matrix quotient space:

\[
\Sigma_{\mu} = \mathcal{M}_\mu / \sim.
\]

(35) The \( q \)-vector quotient space:

\[
\Omega_q := \mathcal{M}_{\times q} / \leftrightarrow.
\]

2 Matrix Equivalence and Lattice Structure

2.1 STP of Matrices

This section gives a brief review on STP. We refer to [5], [6], [7] for details.

**Definition 1** Let \( A \in \mathcal{M}_{m \times n}, B \in \mathcal{M}_{p \times q} \), and \( t = \mathsf{lcm}(n, p) \) be the least common multiple of \( n \) and \( p \). Then

\[
(1) \text{ the left STP of } A \text{ and } B, \text{ denoted by } A \times B, \text{ is defined as}
\]

\[
A \times B := (A \otimes I_{t/n}) (B \otimes I_{t/p}),
\]

where the \( \otimes \) is the Kronecker product [23];

\[
(2) \text{ the right STP of } A \text{ and } B \text{ is defined as}
\]

\[
A \times B := (I_{t/n} \otimes A) (I_{t/p} \otimes B).
\]
In the following we mainly discuss the left STP, and briefly call the left STP as STP. Most of the properties of left STP have their corresponding ones for right STP. Please also refer to [5] or [6] for their major differences.

**Remark 2** If $n = p$, both left and right STP defined in Definition 1 degenerate to the conventional matrix product. That is, STP is a generalization of the conventional matrix product. Hence, as a default, in most cases the symbol $\times$ can be omitted (but not $\odot$).

The following proposition shows that this generalization not only keeps the main properties of conventional matrix product available, but also adds some new properties such as certain commutativity.

Associativity and distribution are two fundamental properties of conventional matrix product. When the product is generalized to STP, these two properties remain available.

**Proposition 3** 1. (Distributive Law)

\[
\begin{align*}
F \odot (aG \pm bH) &= aF \odot G \pm bF \odot H, \\
(aF \pm bG) \odot H &= aF \odot H \pm bG \odot H, & a, b \in F.
\end{align*}
\]

(3)

2. (Associative Law)

\[
(F \odot G) \odot H = F \odot (G \odot H).
\]

(4)

The following proposition is inherited from the conventional matrix product.

**Proposition 4** 1.

\[
(A \odot B)^T = B^T \odot A^T.
\]

(5)

2. Assume $A$ and $B$ are invertible, then

\[
(A \odot B)^{-1} = B^{-1} \odot A^{-1}.
\]

(6)

The following proposition shows that the STP has certain commutative property.

**Proposition 5** Given $A \in \mathcal{M}_{m \times n}$.

1. Let $Z \in \mathbb{R}^t$ be a column vector. Then

\[
ZA = (I_t \odot A)Z.
\]

(7)

2. Let $Z \in \mathbb{R}^t$ be a row vector. Then

\[
AZ = Z(I_t \otimes A).
\]

(8)

To explore further commutating properties, we introduce a swap matrix.

**Definition 6** [23] A swap matrix $W_{[m,n]} \in \mathcal{M}_{mn \times mn}$ is defined as follows:

\[
W_{[m,n]} = \delta_{mn}[1, m + 1, \cdots, (n - 1)m + 1; 2, m + 2, \cdots, (n - 1)m + 2; \cdots; m, 2m, \cdots, nm].
\]

(9)

The following proposition shows that the swap matrix is orthogonal.

**Proposition 7**

\[
W_{[m,n]}^T = W_{[m,n]}^{-1} = W_{[n,m]}.
\]

(10)

Its fundamental function is to swap two factors.

**Proposition 8** 1. Let $X \in \mathbb{R}^m$, $Y \in \mathbb{R}^n$ be two column vectors. Then

\[
W_{[m,n]} \odot X \odot Y = Y \odot X.
\]

(11)

2. Let $X \in \mathbb{R}^m$, $Y \in \mathbb{R}^n$ be two row vectors. Then

\[
X \odot Y \odot W_{[m,n]} = Y \odot X.
\]

(12)

The swap matrix can also swap two factor matrices of the Kronecker product [5], [42].

**Proposition 9** Let $A \in \mathcal{M}_{m \times n}$ and $B \in \mathcal{M}_{p \times q}$. Then

\[
W_{[m,p]}(A \otimes B)W_{[q,n]} = B \otimes A.
\]

(13)

**Remark 10** Assume $A \in \mathcal{M}_{m \times n}$ and $B \in \mathcal{M}_{p \times p}$ are square matrices. Then (13) becomes

\[
W_{[n,p]}(A \oplus B)W_{[p,n]} = B \oplus A.
\]

(14)

As a consequence, $A \oplus B$ and $B \oplus A$ are similar.

The following example is an application of Proposition 9.
Example 11  Prove
\[ e^{A \otimes I_k} = e^A \otimes I_k. \] (15)

Assume \( A \in M_{n \times n} \) is a square matrix and \( B = A \otimes I_k \).

Note that
\[ W(A \otimes I_k)W^{-1} = I_k \otimes A = \text{diag}(A, A, \cdots, A), \]
where \( W = W_{[n,k]} \). Then
\[ e^B = e^{W^{-1}(I_k \otimes A)W} = W^{-1} e^\text{diag}(A, A, \cdots, A)W = W^{-1} \text{diag}(e^A, e^A, \cdots, e^A)W = W^{-1} [I_k \otimes e^A] W = e^A \otimes I_k. \]

Remark 12  Comparing the product of numbers with the product of matrices, two major differences are (i) matrix production has dimension restriction while the scalar product has no restriction; (ii) the matrix product is not commutative while the scalar product is. When the conventional matrix product is extended to STP, these two weaknesses have been eliminated in certain degree. First, the dimension restriction has been removed. Second, in addition to Proposition 5, which shows certain commutativity, the use of swap matrix also provides certain commutativity property. All these improvements make the STP more convenient in use than the conventional matrix product.

2.2 Matrix Equivalence of Matrices

The set of all matrices (over certain field \( \mathbb{F} \)) is denoted by \( M \), that is
\[ M = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} M_{m \times n}. \]

It is obvious that STP is an operator defined as \( \otimes ( \text{or } \times) : M \times M \rightarrow M \). Observing the definition of STP carefully, it is not difficult to find that when we use STP to multiply \( A \) with \( B \), instead of \( A \) itself, we modify \( A \) by different sizes of identity to multiply different \( B \)'s. In fact, STP multiplies an equivalent class of \( A \), precisely, \( (A) = \{ A, A \otimes I_2, A \otimes I_3, \cdots \} \), with an equivalent class of \( B \), i.e., \( (B) = \{ B, B \otimes I_2, B \otimes I_3, \cdots \} \).

Motivated by this fact, we first propose an equivalence over set of matrices, called the identity equivalence \( \sim_I \) or \( \sim_r \). Then STP can be considered as an operator over the equivalent classes. We give a rigorous definition for the equivalence.

Definition 13  Let \( A, B \in M \) be two matrices.

(1) \( A \) and \( B \) are said to be left matrix equivalent (LME), denoted by \( A \sim_I B \), if there exist two identity matrices \( I_s, I_t, s, t \in \mathbb{N} \), such that
\[ A \otimes I_s = B \otimes I_t. \]

(2) \( A \) and \( B \) are said to be right matrix equivalent (RME), denoted by \( A \sim_r B \), if there exist two identity matrices \( I_s, I_t, s, t \in \mathbb{N} \), such that
\[ I_s \otimes A = I_t \otimes B. \]

Remark 14  It is easy to verify that the LME \( \sim_I \) (similarly, RME \( \sim_r \)) is an equivalence relation. That is, it is (i) self-reflexive \((A \sim_I A)\); (ii) symmetric \((A \sim_I B, \text{then } B \sim_I A)\); and (iii) transitive \((A \sim_I B, \text{and } B \sim_I C, \text{then } A \sim_I C)\).

Definition 15  Given \( A \in M \).

(1) The left equivalent class of \( A \) is denoted by
\[ \langle A \rangle_I := \{ B \mid B \sim_I A \} ; \]

(2) The right equivalent class of \( A \) is denoted by
\[ \langle A \rangle_r := \{ B \mid B \sim_r A \} . \]

(3) \( A \) is left (right) reducible, if there is an \( I_s, s \geq 2, \) and a matrix \( B, \text{such that } A = B \otimes I_s \) (correspondingly, \( A = I_s \otimes B \)). Otherwise, \( A \) is left (right) irreducible.

Lemma 16  Assume \( A \in M_{\beta \times \beta} \) and \( B \in M_{\alpha \times \alpha}, \) where \( \alpha, \beta \in \mathbb{N}, \alpha \text{ and } \beta \text{ are co-prime, and} \)
\[ A \otimes I_\alpha = B \otimes I_\beta. \] (16)

Then there exists a \( \lambda \in \mathbb{F} \) such that
\[ A = \lambda I_\beta, \quad B = \lambda I_\alpha. \] (17)
Proof. Split $A \otimes I_\alpha$ into equal size blocks as

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1\beta} \\ \vdots \\ A_{\beta 1} & \cdots & A_{\beta \beta} \end{bmatrix}$$

where $A_{ij} \in \mathcal{M}_{\alpha \times \alpha}, i, j = 1, \cdots, \beta$. Then we have

$$A_{i,j} = b_{i,j}I_\beta. \quad (18)$$

Note that $\alpha$ and $\beta$ are co-prime. Comparing the entries of both sides of (18), it is clear that (i) the diagonal elements of all $A_{ii}$ are the same; (ii) all other elements are zero. Hence $A = b_{11}I_\beta$. Similarly, we have $B = a_{11}I_\alpha$.

But (16) requires $a_{11} = b_{11}$, which is the required $\lambda$. The conclusion follows.

Theorem 17 (1) If $A \sim \ell B$, then there exists a matrix $\Lambda$ such that

$$A = \Lambda \otimes I_\beta, \quad B = \Lambda \otimes I_\alpha. \quad (19)$$

(2) In each class $\langle A \rangle_\ell$, there exists a unique $A_1 \in \langle A \rangle_\ell$, such that $A_1$ is left irreducible.

Proof.

(1) Assume $A \sim \ell B$, that is, there exist $I_\alpha$ and $I_\beta$ such that

$$A \otimes I_\alpha = B \otimes I_\beta. \quad (20)$$

Without loss of generality, we assume $\alpha$ and $\beta$ are co-prime. Otherwise, assume their greatest common divisor is $r = \gcd(\alpha, \beta)$, the $\alpha$ and $\beta$ in (20) can be replaced by $\alpha/r$ and $\beta/r$ respectively.

Assume $A \in \mathcal{M}_{m \times n}$ and $B \in \mathcal{M}_{p \times q}$. Then

$$m\alpha = p\beta, \quad n\alpha = q\beta.$$ 

Since $\alpha$ and $\beta$ are co-prime, we have

$$m = s\beta, \quad n = t\beta, \quad p = sa, \quad q = ta.$$ 

Split $A$ and $B$ into block forms as

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1t} \\ \vdots \\ A_{s1} & \cdots & A_{st} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & \cdots & B_{1t} \\ \vdots \\ B_{s1} & \cdots & B_{st} \end{bmatrix}$$

where $A_{i,j} \in \mathcal{M}_{\beta \times \beta}$ and $B_{i,j} \in \mathcal{M}_{\alpha \times \alpha}, i = 1, \cdots, s, j = 1, \cdots, t$. Now (20) is equivalent to

$$A_{i,j} \otimes I_\alpha = B_{i,j} \otimes I_\beta, \quad \forall i, j. \quad (21)$$

According to Lemma 16, we have $A_{i,j} = \lambda_{ij}I_\beta$ and $B_{i,j} = \lambda_{ij}I_\alpha$. Define

$$\Lambda := \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1t} \\ \vdots \\ \lambda_{s1} & \cdots & \lambda_{st} \end{bmatrix}$$

equation (19) follows.

(2) For each $A \in \langle A \rangle$, we can find $A_0$ irreducible such that $A = A_0 \otimes I_\alpha$. To prove it is unique, let $B \in \langle A \rangle$ and $B_0$ is irreducible and $B = B_0 \otimes I_\alpha$. We claim that $A_0 = B_0$. Since $A_0 \sim \ell B_0$, there exists $\Gamma$ such that

$$A_0 = \Gamma \otimes I_p, \quad B_0 = \Gamma \otimes I_q.$$ 

Since both $A_0$ and $B_0$ are irreducible, we have $p = q = 1$, which proves the claim.

Remark 18 Theorem 17 is also true for $\sim_r$ with obvious modification.

Remark 19 For statement ease, we propose the following terminologies:

(1) If $A = B \otimes I_\alpha$, then $B$ is called a divisor of $A$ and $A$ is called a multiple of $B$.

(2) If (20) holds and $\alpha, \beta$ are co-prime, then the $\Lambda$ satisfying (19) is called the greatest common divisor of $A$ and $B$. Moreover, $\Lambda = \gcd(A, B)$ is unique.

(3) If (20) holds and $\alpha, \beta$ are co-prime, then

$$\Theta := A \otimes I_\alpha = B \otimes I_\beta \quad (22)$$

is called the least common multiple of $A$ and $B$. Moreover, $\Theta = \text{lcm}(A, B)$ is unique. (Refer to Fig.1)

(4) Consider an equivalent class $\langle A \rangle$, denote the unique irreducible element by $A_1$, which is called the root element. All the elements in $\langle A \rangle$ can be expressed as

$$A_i = A_1 \otimes I_i, \quad i = 1, 2, \cdots. \quad (23)$$
\[ A_i \text{ is called the } i\text{-th element of } \langle A \rangle. \text{ Hence, an equivalent class } \langle A \rangle \text{ is a well ordered sequence as:} \]
\[ \langle A \rangle = \{ A_1, A_2, A_3, \ldots \} . \]

![Diagram](image)

**Fig. 1.** \( \Theta = \text{lcm}(A, B) \) and \( \Lambda = \text{gcd}(A, B) \)

Next, we modify some classical matrix functions to make them available for the equivalence class.

**Definition 20** (1) Let \( A \in \mathcal{M}_{n \times n} \). Then a modified determinant is defined as
\[
\text{Dt}(A) = [\det(A)]^{1/n}. \tag{24}
\]

(2) Consider an equivalence of square matrices \( \langle A \rangle \), the “determinant” of \( \langle A \rangle \) is defined as
\[
\text{Dt}(\langle A \rangle) = \text{Dt}(A), \quad A \in \langle A \rangle . \tag{25}
\]

**Proposition 21** (25) is well defined, i.e., it is independent of the choice of the representative \( A \).

**Proof.** To see (25) is well defined, we need to check that \( A \sim B \) implies \( \text{Dt}(A) = \text{Dt}(B) \). Now assume \( A = \Lambda \otimes I_\beta, B = \Lambda \otimes I_\alpha \) and \( \Lambda \in \mathcal{M}_{k \times k} \), then
\[
\text{Dt}(A) = [\det(\Lambda \otimes I_\beta)]^{1/k\beta} = [\det(\Lambda)]^{1/k},
\]
\[
\text{Dt}(B) = [\det(\Lambda \otimes I_\alpha)]^{1/k\alpha} = [\det(\Lambda)]^{1/k}.
\]

It follows that (25) is well defined. \( \square \)

**Definition 22** (1) Let \( A \in \mathcal{M}_{n \times n} \). Then a modified trace is defined as
\[
\text{Tr}(A) = \frac{1}{n} \text{trace}(A). \tag{26}
\]

(2) Consider an equivalence of square matrices \( \langle A \rangle \), the “trace” of \( \langle A \rangle \) is defined as
\[
\text{Tr}(\langle A \rangle) = \text{Tr}(A), \quad A \in \langle A \rangle . \tag{27}
\]

Similar to Definition 20, we need and can easily prove (27) is well defined. These two functions will be used frequently in the sequel.

Next, we show that \( \langle A \rangle = \{ A_1, A_2, \ldots \} \) has a lattice structure.

**Definition 23** [2] A poset \( L \) is a lattice if and only if for every pair \( a, b \in L \) both \( \text{sup} \{ a, b \} \) and \( \text{inf} \{ a, b \} \) exist.

Let \( A, B \in \langle A \rangle \). If \( B \) is a divisor (multiple) of \( A \), then \( B \) is said to be proceeding (succeeding) \( A \) and denoted by \( B \prec A \ (B \succ A) \). Then \( \prec \) is a partial order for \( \langle A \rangle \).

**Theorem 24** \( \langle (A), \prec \rangle \) is a lattice.

**Proof.** Assume \( A, B \in \langle A \rangle \). It is enough to prove that the \( \Lambda = \text{gcd}(A, B) \) defined in (19) is the \( \text{inf}(A, B) \), and the \( \Theta = \text{lcm}(A, B) \) defined in (22) is the \( \text{sup}(A, B) \).

To prove \( \Lambda = \text{inf}(A, B) \) we assume \( C \prec A \) and \( C \prec B \), then we need only to prove that \( C \prec \Lambda \). Since \( C \prec A \) and \( C \prec B \), there exist \( I_p \) and \( I_q \) such that \( C \otimes I_p = A \) and \( C \otimes I_q = B \). Now
\[
C \otimes I_p = A = \Lambda \otimes I_\beta,
\]
\[
C \otimes I_q = B = \Lambda \otimes I_\alpha.
\]
Hence
\[
C \otimes I_p \otimes I_q = \Lambda \otimes I_\beta \otimes I_q = \Lambda \otimes I_\alpha \otimes I_p.
\]

It follows that
\[
\beta q = \alpha p.
\]
Since \( \alpha \) and \( \beta \) are co-prime, we have \( p = m\beta \) and \( q = n\alpha \), where \( m, n \in \mathbb{N} \). Then we have
\[
C \otimes I_p = C \otimes I_m \otimes I_\beta = \Lambda \otimes I_\beta.
\]
That is
\[
C \otimes I_m = \Lambda.
\]
Hence, \( C \prec \Lambda \).

To prove \( \Theta = \text{sup}(A, B) \) assume \( D \succ A \) and \( D \succ B \). Then we can prove \( D \succ \Theta \) in a similar way. \( \square \)

**Definition 25** \( \langle A \rangle \) is said to possess a property, if every \( A \in \langle A \rangle \) possesses this property. The property is also said to be consistent with the equivalence relation.
In the following some easily verifiable consistent properties are collected.

**Proposition 26** (1) Assume \( A \in \mathcal{M} \) is a square matrix. The following properties are consistent with the matrix equivalence (\( \sim_t \) or \( \sim_r \)):  
- \( A \) is orthogonal, that is \( A^{-1} = A^T \);  
- \( \det(A) = 1 \);  
- \( \text{trace}(A) = 0 \);  
- \( A \) is upper (lower) triangle;  
- \( A \) is strictly upper (lower) triangle;  
- \( A \) is symmetric (skew-symmetric);  
- \( A \) is diagonal;

(2) Assume \( A \in \mathcal{M}_{2n \times 2n}, n = 1, 2, \ldots, \) and  
\[
J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{28}
\]

The following property is consistent with the matrix equivalence:  
\[
J \ll A + A^T \ll J = 0. \tag{29}
\]

**Remark 27** As long as a property is consistent with an equivalence, then we can say if the equivalent class has the property or not. For instance, because of Proposition 26 we can say (A) is orthogonal, \( \det((A)) = 1 \), etc.

### 2.3 Lattice Structure on \( \mathcal{M}_\mu \)

Denote by  
\[
\mathcal{M}_\mu := \{ A \in \mathcal{M}_{m \times n} \mid m/n = \mu \}. \tag{30}
\]

Then it is clear that we have a partition as  
\[
\mathcal{M} = \bigcup_{\mu \in \mathbb{Q}_+} \mathcal{M}_\mu, \tag{31}
\]

where \( \mathbb{Q}_+ \) is the set of positive rational numbers.

**Definition 28** (1) Let \( \mu \in \mathbb{Q}_+, p \) and \( q \) are co-prime and \( p/q = \mu \). Then we denote by \( \mu_y = p \) and \( \mu_x = q \) as \( y \) and \( x \) components of \( \mu \).  
(2) Denote the spaces of various dimensions in \( \mathcal{M}_\mu \) as  
\[
\mathcal{M}_\mu^i := \mathcal{M}_{\mu_y \times \mu_x}, \quad i = 1, 2, \ldots
\]

Assume \( A_\alpha \in \mathcal{M}_\mu^\alpha, A_\beta \in \mathcal{M}_\mu^\beta, A_\alpha \sim A_\beta, \) and \( \alpha \mid \beta \), then \( A_\alpha \otimes I_k = A_\beta, \) where \( k = \beta/\alpha \). One sees easily that we can define an embedding mapping \( bd_k : \mathcal{M}_\alpha \rightarrow \mathcal{M}_\beta \) as  
\[
bd_k(A) := A \otimes I_k. \tag{32}
\]

In this way, \( \mathcal{M}_\mu^\alpha \) can be considered as a subspace of \( \mathcal{M}_\mu^\beta \).  
The order determined by this space-subspace relation is denoted as  
\[
\mathcal{M}_\mu^\alpha \sqsubseteq \mathcal{M}_\mu^\beta. \tag{33}
\]

If (33) holds, \( \mathcal{M}_\mu^\alpha \) is called a divisor of \( \mathcal{M}_\mu^\beta \), and \( \mathcal{M}_\mu^\beta \) is called a multiple of \( \mathcal{M}_\mu^\alpha \).

Denote by \( i \wedge j = \gcd(i, j) \) and \( i \vee j = \text{lcm}(i, j) \). Using the order of (33), \( \mathcal{M}_\mu \) has the following structure.

**Theorem 29** (1) Given \( \mathcal{M}_\mu^i \) and \( \mathcal{M}_\mu^j \). The greatest common divisor is \( \mathcal{M}_\mu^{i \wedge j} \), and the least common multiple is \( \mathcal{M}_\mu^{i \vee j} \). (Please refer to Fig. 2)  
(2) Assume \( A \sim B, A \in \mathcal{M}_\mu^i \) and \( B \in \mathcal{M}_\mu^j \). Then their greatest common divisor \( \Lambda = \gcd(A, B) \in \mathcal{M}_\mu^{i \wedge j} \), and their least common multiple \( \Theta = \text{lcm}(A, B) \in \mathcal{M}_\mu^{i \vee j} \).

![Fig. 2. Lattice Structure of \( \mathcal{M}_\mu \)](image_url)

Next, we define  
\[
\mathcal{M}_\mu^i \wedge \mathcal{M}_\mu^j = \mathcal{M}_\mu^{i \wedge j} \tag{34}
\]
\[
\mathcal{M}_\mu^i \vee \mathcal{M}_\mu^j = \mathcal{M}_\mu^{i \vee j}
\]

From above discussion, the following result is obvious:
Proposition 30 Consider \( M_\mu \). The followings are equivalent:

1. \( M_\mu^\alpha \) is a subspace of \( M_\mu^\beta \).
2. \( \alpha \) is a factor of \( \beta \), i.e., \( \alpha|\beta \).
3. \( M_\mu^\alpha \wedge M_\mu^\beta = M_\mu^\alpha \).
4. \( M_\mu^\alpha \lor M_\mu^\beta = M_\mu^\beta \).

Using the order \( \sqsubseteq \) defined by (33), it is clear that all the fixed dimension vector spaces \( M_i^\mu \), \( i = 1, 2, \cdots \) form a lattice.

Proposition 31 \( (M_\mu, \sqsubseteq) \) is a lattice with

\[
\begin{align*}
\sup (M_\mu^\alpha, M_\mu^\beta) &= M_\mu^{\alpha \lor \beta} \\
\inf (M_\mu^\alpha, M_\mu^\beta) &= M_\mu^{\alpha \land \beta}.
\end{align*}
\]

(35)

The following properties are easily verifiable.

Proposition 32 Consider the lattice \( (M_\mu, \sqsubseteq) \).

1. It has a smallest (root) subspace \( M_\mu^1 \), where \( p, q \) are co-prime and \( p/q = \mu \). That is,

\[
M_\mu^1 \cap M_\mu^1 = M_\mu^1
\]

\[
M_\mu^1 \lor M_\mu^1 = M_\mu^1.
\]

But there is no largest element.

2. The lattice is distributive, i.e.,

\[
M_\mu^1 \wedge (M_\mu^2 \lor M_\mu^3) = (M_\mu^1 \wedge M_\mu^2) \lor (M_\mu^1 \wedge M_\mu^3)
\]

\[
M_\mu^1 \lor (M_\mu^2 \wedge M_\mu^3) = (M_\mu^1 \lor M_\mu^2) \land (M_\mu^1 \lor M_\mu^3).
\]

3. For any finite set of spaces \( M_i^\mu \), \( s = 1, 2, \cdots, r \).

There exists a smallest numerator \( u = \bigvee_{s=1}^r M_i^\mu \), such that

- \( M_i^\mu \cap M_i^\alpha, s = 1, 2, \cdots, r\);
- \( M_i^\mu \cap M_i^\beta, s = 1, 2, \cdots, r\).

then

\[
M_i^\mu \subseteq M_i^\mu.
\]

Definition 33 Let \( (L, \prec) \) and \( (M, \sqsubseteq) \) be two lattices. \( (L, \prec) \) and \( (M, \sqsubseteq) \) are said to be lattice homomorphic, denoted by \( (L, \prec) \cong (M, \sqsubseteq) \), if there exists a mapping \( \varphi : L \to M \) satisfying the following condition:

\[
\ell_1 \prec \ell_2 \Rightarrow \varphi(\ell_1) \sqsubseteq \varphi(\ell_2). \tag{36}
\]

\( \varphi \) is called the homomorphism of \( (L, \prec) \) and \( (M, \sqsubseteq) \).

Moreover, if \( \varphi \) is one to one and onto, \( (L, \prec) \) and \( (M, \sqsubseteq) \) are said to be isomorphic and \( \varphi \) is an isomorphism, denoted by \( (L, \prec) \cong (M, \sqsubseteq) \).

Assume \( A \in M_i^\mu \) is irreducible, define \( \varphi : (A) \to M_\mu \) as

\[
\varphi(A_j) := M_i^\mu, \quad j = 1, 2, \cdots, \tag{37}
\]

then it is easy to verify the following result.

Proposition 34 The mapping \( \varphi : (A) \to M_\mu \) defined in (37) is a lattice homomorphism from \( (A, \prec) \) to \( (M_\mu, \sqsubseteq) \).

Next, we consider the \( M_\mu \) for different \( \mu \)'s. It is also easy to verify the following result.

Proposition 35 Define a mapping \( \varphi : M_\mu \to M_\lambda \) as

\[
\varphi(M_i^\mu) := M_i^\lambda.
\]

The mapping \( \varphi : (M_\mu, \sqsubseteq) \to (M_\lambda, \sqsubseteq) \) is a lattice isomorphism.

Definition 36 Let \( (L, \prec) \) be a lattice and \( S \subseteq L \). If \( (S, \prec) \) is also a lattice, it is called a sub-lattice of \( (L, \prec) \).

Remark 37 Let \( \varphi : (H, \prec) \to (M, \sqsubseteq) \) be an injective (i.e., one-to-one) lattice homomorphism. Then \( \varphi : H \to \varphi(H) \) is a lattice isomorphism. Hence \( \varphi(H) \) is a sub-lattice of \( (M, \sqsubseteq) \). If we identify \( H \) with \( \varphi(H) \), we can simply say that \( H \) is a sub-lattice of \( M \).

Definition 38 Let \( (L, \prec) \) and \( (M, \sqsubseteq) \) be two lattices. The product order \( \sqsubseteq := \times \sqsubseteq \) defined on the product set

\[
L \times M := \{ (\ell, m) \mid \ell \in L, m \in M \}
\]

is: \((\ell_1, m_1) \subseteq (\ell_2, m_2)\) if and only if \( \ell_1 \prec \ell_2 \) and \( m_1 \sqsubseteq m_2 \).
**Theorem 39** Let \((L, \prec)\) and \((M, \sqsubseteq)\) be two lattices. Then \((L \times M, \prec \times \sqsubseteq)\) is also a lattice, called the product lattice of \((L, \prec)\) and \((M, \sqsubseteq)\).

**Proof.** Let \((\ell_1, m_1)\) and \((\ell_2, m_2)\) be two elements in \(L \times M\). Denote by \(\ell_s = \sup(\ell_1, \ell_2)\) and \(m_s = \sup(m_1, m_2)\). Then \((\ell_s, m_s) \supseteq (\ell_i, m_i), i = 1, 2\). To see \((\ell_s, m_s) = \sup((\ell_1, m_1), (\ell_2, m_2))\) let \((\ell, m) \supseteq (\ell_i, m_i), i = 1, 2\). Then \(\ell \succ \ell_i\) and \(m \sqsupset m_i, i = 1, 2\). It follows that \(\ell \succ \ell_s\) and \(m \sqsupset m_s\). That is, \((\ell, m) \supseteq (\ell_s, m_s)\). We conclude that

\[
(\ell_s, m_s) = \sup((\ell_1, m_1), (\ell_2, m_2)).
\]

Similarly, we set \(\ell_i = \inf(\ell_1, \ell_2)\) and \(m_i = \inf(m_1, m_2)\), then we can prove that

\[
(\ell_i, m_i) = \inf((\ell_1, m_1), (\ell_2, m_2)).
\]

Finally, we have the following result.

**Example 40** Consider the product of two lattices \((M_\mu, \sqsubseteq)\) and \((M_\lambda, \sqsubseteq)\). Define a mapping

\[
\varphi : (M_\mu, \sqsubseteq) \times (M_\lambda, \sqsubseteq) \rightarrow (M_{\mu \lambda}, \sqsubseteq)
\]

as

\[
\varphi (M_\mu^p \times M_\lambda^q) := M_{\mu \lambda}^{pq}.
\]

Assume \(M_\mu^i \sqsubseteq M_\mu^j\) and \(M_\lambda^k \sqsubseteq M_\lambda^l\), then \(i|j\) and \(s|t\), and by the definition of product lattice, we have

\[
M_\mu^i \times M_\lambda^k \sqsubseteq \sqsubseteq \sqsubseteq M_\mu^j \times M_\lambda^l.
\]

Since \(i|j\), we have

\[
\varphi (M_\mu^i \times M_\lambda^k) = M_{\mu \lambda}^{is}
\]

\[
\sqsubseteq M_{\mu \lambda}^{jt} = \varphi (M_\mu^j \times M_\lambda^l).
\]

That is, \(\varphi\) is a lattice homomorphism.

It is obvious that \(\varphi\) is not injective. \((M_\mu \times M_\lambda, \sqsubseteq \times \sqsubseteq)\) can not be considered as a sub-lattice of \(M_{\mu \lambda}\).

### 2.4 Monoid and Quotient Monoid

A monoid is a semigroup with identity. We refer readers to [27], [24], [18] for concepts and properties.

Recall that

\[
M := \bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} M_{m \times n}.
\]

We have the following algebraic structure.

**Proposition 41** The algebraic system \((M, \times)\) is a monoid.

**Proof.** The associativity comes from the property of \(\times\) (refer to (4)). The identity element is 1.

One sees easily that this monoid covers numbers, vectors, and matrices of arbitrary dimensions.

In the following some of its useful sub-monoids are presented:

- \(M(k)\):
  \[
  M(k) := \bigcup_{\alpha \in \mathbb{N} \beta \in \mathbb{N}} M_{k^\alpha \times k^\beta},
  \]
  where \(k \in \mathbb{N}\) and \(k > 1\).

  It is obvious that \(M(k) \prec M\). (In this section \(A \prec B\) means \(A\) is a submonoid of \(B\)). This sub-monoid is useful for calculating the product of tensors over \(k\) dimensional vector space [1]. It is particularly useful for \(k\)-valued logical dynamic systems [6], [7]. When \(k = 2\), it is used for Boolean dynamic systems.

  In this sub-monoid the STP can be defined as follows:

**Definition 42** (1) Let \(X \in \mathbb{F}^n\) be a column vector, \(Y \in \mathbb{F}^m\) a row vector.

- Assume \(n = pm\) (denoted by \(X \succ_p Y\)): Split \(X\) into \(m\) equal blocks as

  \[
  X = [X_1^T, X_2^T, \cdots, X_m^T]^T,
  \]

  where \(X_i \in \mathbb{F}^p, \forall i\). Define

  \[
  X \times Y := \sum_{s=1}^m X_s y_s \in \mathbb{F}^p.
  \]

- Assume \(np = m\) (denoted by \(X \prec_p Y\)): Split \(Y\) into \(n\) equal blocks as

  \[
  Y = [Y_1, Y_2, \cdots, Y_n],
  \]
where $Y_i \in \mathbb{F}^p$, $\forall i$. Define

$$X \times Y := \sum_{s=1}^{m} x_s Y_s \in \mathbb{F}^p.$$  

(2) Assume $A \in \mathcal{M}_{m \times n}$, $B \in \mathcal{M}_{p \times q}$, where $n = tp$ (denoted by $A \succ_t B$), or $nt = p$ (denoted by $A \prec_t B$). Then

$$A \times B := C = (c_{i,j}),$$

where

$$c_{i,j} = \text{Row}_i(A) \times \text{Col}_j(B).$$

Remark 43 (1) It is easy to prove that when $A \prec_t B$ or $B \prec_t A$ for some $t \in \mathbb{N}$, this definition of left STP coincides with Definition 1. Though this definition is not general, it has clear physical meaning. Particularly, so far this definition covers almost all the applications.

(2) Unfortunately, this definition is not suitable for right STP. This is a big difference between left and right STPs.

- **$\mathcal{V}$:**
  
  $$\mathcal{V} := \bigcup_{k \in \mathbb{N}} \mathcal{M}_{k \times 1}.$$  

  It is obvious that $\mathcal{V} < \mathcal{M}$. This sub-monoid consists of column vectors. In this sub-monoid the STP is degenerated to Kronecker product.

  We denote by $\mathcal{V}^T$ the sub-monoid of row vectors. It is also clear that $\mathcal{V}^T < \mathcal{M}$.

- **$\mathcal{L}$:**
  
  $$\mathcal{L} := \{ A \in \mathcal{M} \mid \text{Col}(A) \subset \Delta_s, s \in \mathbb{N} \}.$$  

  It is obvious that $\mathcal{L} < \mathcal{M}$. This sub-monoid consists of all logical matrices. It is used to express the product of logical mappings.

- **$\mathcal{P}$:**
  
  $$\mathcal{P} := \{ A \in \mathcal{M} \mid \text{Col}(A) \subset T_s, \text{ for some } s \in \mathbb{N} \}.$$  

  It is obvious that $\mathcal{P} < \mathcal{M}$. This monoid is useful for probabilistic logical mappings.

- **$\mathcal{L}(k)$:**
  
  $$\mathcal{L}(k) := \mathcal{L} \cap \mathcal{M}(k).$$  

  It is obvious that $\mathcal{L}(k) < \mathcal{L} < \mathcal{M}$. We use it for $k$-valued logical mappings.

Next, we define the set of “short” matrices as

$$S := \{ A \in \mathcal{M}_{m \times n} \mid m \leq n \},$$

and its subset

$$S^r := \{ A \in \mathcal{M} \mid A \text{ is of full row rank} \}.$$

Then we have the following result.

**Proposition 44**

$$S^r < S < \mathcal{M}. \quad (39)$$

**Proof.** Assume $A \in \mathcal{M}_{m \times n}$, $B \in \mathcal{M}_{p \times q}$ and $A, B \in S$, then $m \leq n$ and $p \leq q$. Let $t = \text{lcm}(n, p)$. Then $AB \in \mathcal{M}_{\frac{mt}{n} \times \frac{pq}{t}}$. It is easy to see that $\frac{mt}{n} \leq \frac{mt}{p}$, so $AB \in S$. The second part is proved.

As for the first part, Assume $A, B \in S^r$. Then

$$\text{rank}(AB) = \text{rank} \left[ (A \otimes I_{t/n}) (B \otimes I_{t/p}) \right] \geq \text{rank} \left[ (A \otimes I_{t/n}) (I_p \otimes I_{t/p}) \right] = \text{rank} \left( A \otimes I_{t/n} \right) = mt/n.$$

Hence, $AB \in S^r$. \qed

Similarly, we can define the set of “tall” matrices $\mathcal{H}$ and the set of matrices with full column rank $\mathcal{H}^c$. We can also prove that

$$\mathcal{H}^c < \mathcal{H} < \mathcal{M}. \quad (40)$$

Next, we consider the quotient space

$$\Sigma_M := \mathcal{M} / \sim.$$

**Definition 45** [37]

(1) A nonempty set $S$ with a binary operation $\sigma : S \times S \to S$ is called an algebraic system.
(2) Assume $\sim$ is an equivalence relation on an algebraic system $(S, \sigma)$. The equivalence relation is a congruence relation, if for any $A, B, C, D \in S$, $A \sim C$ and $B \sim D$, we have

$$A\sigma B \sim C\sigma D.$$  \hfill (41)

**Proposition 46** Consider the algebraic system $(M, \star)$ with the equivalence relation $\sim = \sim_\ell$. The equivalence relation $\sim$ is congruence.

**Proof.** Let $A \sim \tilde{A}$ and $B \sim \tilde{B}$. According to Theorem 17, there exist $U \in \mathcal{M}_{m \times n}$ and $V \in \mathcal{M}_{p \times q}$ such that

$$A = U \otimes I_\alpha, \quad \tilde{A} = U \otimes I_\beta;$$

$$B = V \otimes I_\alpha, \quad \tilde{B} = V \otimes I_\beta.$$  

Denote

$$\text{lcm}(n, p) = r, \quad \text{lcm}(ns, op) = r\xi, \quad \text{lcm}(nt, \beta p) = r\eta.$$  

Then

$$A \star B = (U \otimes I_\alpha \otimes I_{r\xi/m\alpha}) (V \otimes I_\beta \otimes I_{r\xi/\alpha p}) = (U \otimes I_{r/m}) (V \otimes I_{r/p}) \otimes I_\xi.$$  

Similarly, we have

$$\tilde{A} \star \tilde{B} = (U \otimes I_{r/m}) (V \otimes I_{r/p}) \otimes I_\eta.$$  

Hence we have $A \star B \sim \tilde{A} \star \tilde{B}$.  \hfill $\square$  

According to Proposition 46, we know that $\star$ is well defined on the quotient space $\Sigma_M$. Moreover, the following result is obvious:

**Proposition 47** (1) $(\Sigma_M, \star)$ is a monoid.

(2) Let $S < M$ be a sub-monoid. Then $S/ \sim$ is a sub-monoid of $\Sigma_M$, that is,

$$S/ \sim < \Sigma_M.$$  

Since the $S$ in Proposition 47 could be any sub-monoid of $M$. All the aforementioned sub-monoids have their corresponding quotient sub-monoids, which are the sub-monoids of $\Sigma_M$. For instance, $V/ \sim$, $L/ \sim$, etc. are the sub-monoids of $\Sigma_M$.

### 2.5 Group Structure on $\mathcal{M}^\mu$

**Proposition 48** Assume $A \in \mathcal{M}_{\mu_1}$ and $B \in \mathcal{M}_{\mu_2}$ then $A \times B \in \mathcal{M}_{\mu_1\mu_2}$. That is, the operation $\times$ is a mapping

$$\times : \Sigma_{\mu_1} \times \Sigma_{\mu_2} \to \Sigma_{\mu_1\mu_2}.$$  

**Proof.** Assume $A \in \mathcal{M}_{m \times n}$ and $B \in \mathcal{M}_{p \times q}$, where $\mu_1 = m/n$ and $\mu_2 = p/q$, and $t = \text{lcm}(n, p)$. Then

$$A \times B = (A \otimes I_{t/n}) (B \otimes I_{t/p}) \in \mathcal{M}_{mt/n \times qt/p} \subset \mathcal{M}_{\mu_1\mu_2}.$$  

$\square$

**Definition 49** (1) Define

$$\mathcal{M}^\mu := \bigcup_{z \in \mathbb{Z}} \mathcal{M}_{zu}.$$  

Then $\mathcal{M}^\mu$ is closed under operator $\times$.

(2) Define

$$\Sigma^\mu := \bigcup_{z \in \mathbb{Z}} \Sigma_{zu}.$$  

Then $\Sigma^\mu$ is also closed under operator $\times$.

(3) $\langle A \rangle, \langle B \rangle \in \Sigma^\mu$ is said to be power equivalent, denoted by $\langle A \rangle \sim_p \langle B \rangle$, if there exists an integer $z \in \mathbb{Z}$ such that both $\langle A \rangle, \langle B \rangle \in \Sigma_{zu}$. Denote

$$\langle \langle A \rangle \rangle := \{ \langle B \rangle \mid \langle B \rangle \sim_p \langle A \rangle \}$$  \hfill (42)

**Remark 50** It is obvious that $\times$ is consistent with $\sim_p$.

Hence $\times$ is well defined on the set of equivalent classes as

$$\langle \langle A \rangle \rangle \times \langle \langle B \rangle \rangle := \langle \langle A \times B \rangle \rangle.$$  \hfill (43)

Then we have the following group structure.

**Theorem 51** $(\Sigma^\mu/ \sim_p, \times)$ is a group, which is isomorphic to $(\mathbb{Z}, +)$. Precisely, assume $A \in \mathcal{M}_{\mu z}$ then $\Psi : \Sigma^\mu/ \sim_p \to \mathbb{Z}$ is defined as

$$\Psi (\langle \langle A \rangle \rangle) := z,$$

which is a group isomorphism.
2.6 Semi-tensor Addition and Vector Space Structure of $\Sigma_\mu$

Definition 52 Let $A$, $B \in M_\mu$. Precisely, $A \in M_{m \times n}$, $B \in M_{p \times q}$, and $m/n = p/q = \mu$. Set $t = \mathrm{lcm}\{m, p\}$. Then

(1) the left STA of $A$ and $B$, denote by $\langle \cdot \rangle$, is defined as

$$A \leftarrow B := (A \otimes I_{\frac{t}{m}}) + (B \otimes I_{\frac{t}{p}}).$$

(44)

We also denote the left semi-tensor subtraction as

$$A - B := A \leftarrow (-B).$$

(45)

(2) The right STA of $A$ and $B$, denote by $\langle \cdot \rangle$, is defined as

$$A \rightarrow B := (I_{\frac{t}{m}} \otimes A) + (I_{\frac{t}{p}} \otimes B).$$

(46)

We also denote the right semi-tensor subtraction as

$$A \rightarrow B := A \rightarrow (-B).$$

(47)

Remark 53 Let $\sigma \in \{\langle \cdot \rangle, \leftarrow, \rightarrow, \langle \cdot \rangle\}$ be one of the four binary operators. Then it is easy to verify that

(1) if $A$ and $B$ are in $M_\mu$, the $A \sigma B$ is $M_\mu$;

(2) if $A$ and $B$ are as in Definition 52, then $A \sigma B \in M_{t \times s}$;

(3) Set $s = \mathrm{lcm}(n, q)$, then $s/n = t/m$ and $s/q = t/p$. So $\sigma$ can also be defined by using column numbers respectively, e.g.,

$$A \leftarrow B := (A \otimes I_{\frac{s}{n}}) + (B \otimes I_{\frac{s}{q}}),$$

etc.

Theorem 54 Consider the algebraic system $(M_\mu, \sigma)$, where $\sigma \in \{\langle \cdot \rangle, \leftarrow\}$ and $\sim = \sim_\ell$ (or $\sigma \in \{\langle \cdot \rangle, \rightarrow\}$ and $\sim = \sim_r$). Then the equivalence relation $\sim$ is a congruence relation with respect to $\sigma$.

Proof. We prove one case, where $\sigma = \leftarrow$ and $\sim = \sim_\ell$. Proofs for other cases are similar.

Assume $\tilde{A} \sim_\ell A$ and $\tilde{B} \sim_\ell B$. Set $P = \gcd(\tilde{A}, A)$ and $Q = \gcd(\tilde{B}, B)$, then

$$\tilde{A} = P \otimes I_\beta, \quad A = P \otimes I_\alpha;$$

(48)

$$\tilde{B} = Q \otimes I_\gamma, \quad B = Q \otimes I_\delta,$$

(49)

where $P \in M_{x \times x}$, $Q \in M_{y \times y}$, $x$, $y \in \mathbb{N}$ are certain numbers.

Now consider $\tilde{A} \leftarrow \tilde{B}$. Assume $\eta = \mathrm{lcm}(x, y)$, $t = \mathrm{lcm}(x\beta, y\gamma) = \eta\xi$, $s = \mathrm{lcm}(x\alpha, y\delta) = \eta\zeta$. Then we have

$$\tilde{A} \leftarrow \tilde{B} = P \otimes I_\beta \otimes I_{t/x\beta}$$

$$+ Q \otimes I_\gamma \otimes I_{t/y\gamma}$$

(50)

$$= [(P \otimes I_{\eta/x}) + (Q \otimes I_{\eta/y})] \otimes I_\xi.$$  

(51)

Similarly, we have

$$A \leftarrow B = [(P \otimes I_{\eta/x}) + (Q \otimes I_{\eta/y})] \otimes I_\xi.$$  

(52)

(50) and (51) imply that $\tilde{A} \leftarrow \tilde{B} \sim A \leftarrow B$. □

Define the left and right quotient spaces $\Sigma_\mu^\ell$ and $\Sigma_\mu^r$ respectively as

$$\Sigma_\mu^\ell := M_\mu / \sim_\ell;$$

(52)

$$\Sigma_\mu^r := M_\mu / \sim_r.$$  

(53)

According to Theorem 54, the operation $\leftarrow$ (or $\rightarrow$) can be extended to $\Sigma_\mu^\ell$ as

$$\langle A \rangle_\ell \leftarrow \langle B \rangle_\ell := < A \leftarrow B >_\ell;$$

(54)

$$\langle A \rangle_\ell \rightarrow \langle B \rangle_\ell := < A \rightarrow B >_\ell, \quad \langle A \rangle_\ell, \langle B \rangle_\ell \in \Sigma_\mu^\ell.$$  

Similarly, we can define $\leftarrow$ (or $\rightarrow$) on the quotient space $\Sigma_\mu^r$ as

$$\langle A \rangle_r \leftarrow \langle B \rangle_r := < A \leftarrow B >_r;$$

(55)

$$\langle A \rangle_r \rightarrow \langle B \rangle_r := < A \rightarrow B >_r, \quad \langle A \rangle_r, \langle B \rangle_r \in \Sigma_\mu^r.$$  

The following result is important, and the verification is straightforward.

Theorem 55 Using the definitions in (54) (correspondingly, (55)), the quotient space $(\Sigma_\mu^\ell, \leftarrow)$ (correspondingly, $(\Sigma_\mu^r, \leftarrow)$) is a vector space.
**Remark 56** As a consequence, \((\Sigma^\ell_\mu, \oplus)\) (or \((\Sigma^r_\mu, \oplus)\)) is an Abelian group.

**Remark 57** Recall Example 11, it shows that the exponential function \(\exp\) is well defined on the quotient space \(\Sigma := \Sigma_1\).

Since each \(\langle A \rangle \in \Sigma\) has a unique left (or right) identity irreducible element \(A_1\) (or \(B_1\)) such that \(A \sim_\ell A_1\) (or \(A \sim_r B_1\)), in general, we can use the irreducible element, which is also called the root element of an equivalent class, as the representation of this class. But this is not compulsory.

For notational and statement ease, hereafter we consider \(\Sigma^\ell_\mu\) only unless elsewhere stated. As a convention, the omitted script ("f" or "r") means \(\ell\). For instance, \(\Sigma = \Sigma^\ell_\mu, \sim = \sim_\ell, \langle A \rangle = \langle A \rangle_\ell\) etc.

### 3 Topology on Equivalence Space

#### 3.1 Topology via Sub-basis

This subsection builds step by step a topology on quotient space \(\Sigma_\mu\) using a sub-basis.

First, we consider the partition (31), it is natural to assume that each \(M_\mu\) is a clopen subset in \(M\), because distinct \(\mu\)'s correspond to distinct shapes of matrices. Now inside each \(M_\mu\) we assume \(\mu_y, \mu_x \in \mathbb{N}\) are co-prime and \(\mu_y/\mu_x = \mu\). Then

\[ M_\mu = \bigcup_{i=1}^{\infty} M^i_\mu, \]

where

\[ M^i_\mu = M_{i\mu_y \times i\mu_x}, \quad i = 1, 2, \cdots. \]

Because of the similar reason, we also assume each \(M^i_\mu\) is clopen.

Overall, we have a set structure on \(M\) as

\[ M = \bigcup_{\mu \in \mathbb{Q}_+} \bigcup_{i=1}^{\infty} M^i_\mu. \quad (56) \]

**Definition 58** A natural topology on \(M\), denoted by \(\mathcal{T}_M\), consists of

1. a partition of countable clopen subsets \(M^i_\mu, \mu \in \mathbb{Q}_+, i \in \mathbb{N}\);
2. the conventional Euclidean \(\mathbb{R}^{2\mu_y\mu_x}\) topology for \(M^i_\mu\).

Next, we consider the quotient space

\[ \Sigma_M := M/ \sim. \]

It is clear that

\[ \Sigma_M = \bigcup_{\mu \in \mathbb{Q}_+} \Sigma_\mu. \quad (57) \]

Moreover, (57) is also a partition. Hence each \(\Sigma_\mu\) can be considered as a clopen set in \(\Sigma_M\). We are, therefore, interested only in constructing a topology on each \(\Sigma_\mu\).

**Definition 59**

1. Consider \(M^i_\mu\) as an Euclidean space \(\mathbb{R}^{2\mu_y\mu_x}\) with conventional Euclidean topology. Assume \(\alpha_i \neq \emptyset\) is an open set. Define a subset \(s_i(\alpha_i) \subset \Sigma_\mu\) as follows:

\[ \langle A \rangle \in s_i(\alpha_i) \iff \langle A \rangle \cap \alpha_i \neq \emptyset. \quad (58) \]

2. Let

\[ O_i = \{ \alpha_i \mid \alpha_i \text{ is an open ball in } M^i_\mu \}

with rational center and rational radius \(\}\).

3. Using \(O_i\), we construct a set of subsets \(S_i \subset 2^{\Sigma_\mu}\) as

\[ S_i := \{ s_i \mid s_i = s_i(\alpha_i) \text{ for some } \alpha_i \in O_i \}, \quad i = 1, 2, \cdots. \]

Taking \(S = \bigcup_{i=1}^{\infty} S_i\) as a topological sub-basis, the topology generated by \(S\) is denoted by \(\mathcal{T}\), which makes

\[ (\Sigma_\mu, \mathcal{T}) \]

a topological space. (We refer to [28] for a topology produced from a sub-basis.)

Note that the topological basis consists of the set of finite intersections of \(s_i \in S_i\).

**Remark 60**

1. It is clear that \(\mathcal{T}\) makes \((\Sigma_\mu, \mathcal{T})\) a topological space.
2. The topological basis is

\[ \mathcal{B} := \{ s_{i_1} \cap s_{i_2} \cap \cdots \cap s_{i_r} \mid s_{i_j} \in S_{i_j}; \quad j = 1, \cdots, r; \quad r < \infty \}. \quad (59) \]
If we consider $T$ as a product topological space, then $\prod M_i^\mu$ is the product topology of the standard product topology on the product topology.)

Since $A_o \cap A_o = \emptyset$, we can find two open discs with rational center and rational radius. Then $s_1(o_1)$ and $s_2(o_2)$ are two elements in the sub-basis, and

$$s_1 \cap s_2 = \{ (A) \mid (A) \cap o_i \neq \emptyset, i = 1, 2 \}$$

is an element in the basis.

**Theorem 61** The topological space $(\Sigma, T)$ is a second countable, Hausdorff (or $T_2$) space.

**Proof.** To see $(\Sigma, T)$ is second countable, It is easy to see that $O_i$ is countable. Then $\{O_i|i=1, 2, \cdots\}$, as countable union of countable set, is countable. Finally, $B$, as the finite subset of a countable set, is countable.

Next, consider $(A) \neq (B) \in \Sigma$. Let $A_1 \in (A)$ and $B_1 \in (B)$ be their irreducible elements respectively. If $A_1, B_1 \in M_i^\mu$ for the same $i$, then we can find two open sets $\emptyset \neq o_a, o_b \subset M_i^\mu$, $o_a \cap o_b = \emptyset$, such that $A_1 \in o_a$ and $B_1 \in o_b$. Then by definition, $s_a(\emptyset) \cap s_b(\emptyset) = \emptyset$ and $(A) \in s_a, (B) \in s_b$.

Finally, assume $A_1 \in M_i^\mu$, $B_1 \in M_j^\mu$ and $i \neq j$. Let $t = \text{lcm}(i, j)$. Then

$$A_{ij} = A_1 \otimes I_{ij} \in M_i^\mu, \quad B_{ij} = B_1 \otimes I_{ij} \in M_j^\mu.$$ 

Since $A_{ij} \neq B_{ij}$, we can find $o_a, o_b \subset M_i^\mu$, $o_a \cap o_b = \emptyset$ and $A_{ij} \in o_a$ and $B_{ij} \in o_b$. That is, $s_a(o_a)$ and $s_b(o_b)$ separate $(A)$ and $(B)$.

If we consider

$$\mathcal{M} := \prod_{i=1}^\infty \prod_{j=1}^\infty M_{i,j} \quad (60)$$

as a product topological space, then $\mathcal{T}$ is the quotient topology of the standard product topology on the product space $\mathcal{M}$ defined by (60). (We refer to [43] for product topology.)

### 3.2 Bundle Structure on $\mathcal{M}_\mu$

**Definition 62** [26] A bundle is a triple $(E, p, B)$, where $E$ and $B$ are two topological spaces and $p : E \rightarrow B$ is a continuous map. $E$ and $B$ are called the total space and base space respectively. For each $b \in B$, $p^{-1}(b)$ is called the fibre of the bundle at $b \in B$.

Observing the two topologies $T_{\mathcal{M}}$ and $T$ constructed in previous subsection, the following result is obvious:

**Proposition 63** $(\mathcal{M}_\mu, Pr, \Sigma)$ is a bundle, where $Pr$ is the natural projection, i.e.,

$$Pr(A) = (A).$$

![Fig. 4. Fiber Bundle Structure](image)

**Remark 64** (1) Of course, $(\mathcal{M}, Pr, \Sigma)$ is also a bundle. But it is of less interest because it is a discrete union of $(\mathcal{M}_\mu, Pr, \Sigma\mu), \mu \in \mathbb{Q}_+$. 

(2) Consider an equivalent class $(A) = \{A_1, A_2, \cdots\} \in \Sigma_\mu$, where $A_1$ is irreducible. Then the fibre over $(A)$ is a discrete set:

$$Pr^{-1}(\{A\}) = \{A_1, A_2, A_3, \cdots\}.$$

Hence this fiber bundle is named discrete bundle.

(3) Fig 4 illustrates the fiber bundle structure of $(\mathcal{M}_\mu, Pr, \Sigma\mu)$. Here $A_1 \in (A)$ and $B_1 \in (B)$ are irreducible $A_1 \in M_1^\mu$ and $B_1 \in M_2^\mu$. Their fibers are depicted in the Fig 4.

We can define a set of cross sections [26] $c_i : \Sigma \rightarrow \mathcal{M}_\mu$ as:

$$c_i((A)) := A_i, \quad i = 1, 2, \cdots \quad (61)$$

It is clear that $Pr \circ c_i = 1_{\Sigma \mu}$, where $1_{\Sigma \mu}$ is the identity mapping on $\Sigma \mu$. 

![Diagram showing $s_1 \cap s_2$](image)
Next, we consider some truncated sub-bundles of \((\mathcal{M}_\mu, Pr, \Sigma_\mu)\).

**Definition 65** Assume \(k \in \mathbb{N}\).

1. Set
   \[
   \mathcal{M}^{[k]}_\mu := \{M_{m \times n} | M_{m \times n} \in \mathcal{M}_\mu, \text{ and } m|k\mu_y\}. 
   \]

   Then \(\mathcal{M}^{[k]}_\mu\) is called the \(k\)-upper bounded subspace of \(\mathcal{M}_\mu\).

2. \(\Sigma^{[k]}_\mu := \mathcal{M}^{[k]}_\mu / \sim\) is called the \(k\)-upper bounded subspace of \(\Sigma_\mu\).

The natural projection \(Pr : \mathcal{M}^{[k]}_\mu \rightarrow \Sigma^{[k]}_\mu\) is obviously defined. Then we have the following bundle structure.

**Proposition 66** \((\mathcal{M}^{[k]}_\mu, Pr, \Sigma^{[k]}_\mu)\) is a sub-bundle of \((\mathcal{M}_\mu, Pr, \Sigma_\mu)\). Precisely speaking, the following graph (63) is commutative, where \(\pi\) and \(\pi'\) are including mappings. (63) is also called the bundle morphism.

\[
\begin{array}{ccc}
\mathcal{M}^{[k]}_\mu & \xrightarrow{\pi} & \mathcal{M}_\mu \\
Pr \downarrow & & \downarrow Pr \\
\Sigma^{[k]}_\mu & \xrightarrow{\pi'} & \Sigma_\mu
\end{array}
\]

(3) Assume \(\alpha|\beta\). Define

\[
\mathcal{M}^{[\alpha,\beta]}_\mu := \mathcal{M}_\mu^{[\alpha]} \cap \mathcal{M}_\mu^{[\beta]},
\]

which is called the \([\alpha, \beta]\)-bounded subspace of \(\mathcal{M}_\mu\).

(4) Define the quotient space

\[
\Sigma^{[\alpha,\beta]}_\mu := \mathcal{M}^{[\alpha,\beta]}_\mu / \sim,
\]

which is called the \([\alpha, \beta]\)-bounded subspace of \(\Sigma_\mu\).

**Remark 69** Proposition 66 is also true for the two other truncated forms: \((\mathcal{M}^{[k]}_\mu, Pr, \Sigma^{[k]}_\mu)\) and \((\mathcal{M}^{[\alpha,\beta]}_\mu, Pr, \Sigma^{[\alpha,\beta]}_\mu)\) respectively. Precisely speaking, in (63) if both \(\mathcal{M}^{[k]}_\mu\) and \(\Sigma^{[k]}_\mu\) are replaced by \(\mathcal{M}^{[k]}_\mu\) and \(\Sigma^{[k]}_\mu\) respectively, or by \(\mathcal{M}^{[\alpha,\beta]}_\mu\) and \(\Sigma^{[\alpha,\beta]}_\mu\) respectively, (63) remains commutative.

### 3.3 Coordinate Frame on \(\Sigma_\mu\)

It is obvious that \((\Sigma_\mu, +)\) is an infinite dimensional vector space. Since each \(\langle A \rangle \in \Sigma_\mu\) has finite coordinate expression, we may try to avoid using a basis with infinite elements. To this end, we construct a set of “consistent” coordinate frames as \(\{B_1, B_2, \cdots \}\). Then \(\langle A \rangle\) can be expressed by \(A_1 \in \Span\{B_i\}, A_2 \in \Span\{B_{i+1}\}\), and so on. Moreover, \(B_i \subset B_{i+1}\) is a subset (or \(B_i\) is part of coordinate elements in \(B_{i+1}\)). Then \(\langle A \rangle\) can always be expressed in \(B_i\) no matter which representative is chosen. The purpose of this section is to build such a set of consistent coordinate sub-frames, which form an overall coordinate frame.

Assume \(A_\alpha \in \mathcal{M}^\alpha_\mu, A_\beta \in \mathcal{M}^\beta_\mu, A_\alpha \sim A_\beta\), and \(\alpha|\beta\), then \(A_\alpha \otimes I_k = A_\beta\), where \(k = \beta/\alpha\). The order determined by this space-subspace relation is denoted as

\[
\mathcal{M}^\alpha_\mu \subset \mathcal{M}^\beta_\mu.
\]

One sees easily that we can define an embedding mapping \(bd_k : \mathcal{M}_\alpha \rightarrow \mathcal{M}_\beta\) as

\[
bd_k(A) := A \otimes I_k.
\]

In this way, \(\mathcal{M}^\alpha_\mu\) can be considered as a subspace of \(\mathcal{M}^\beta_\mu\). Our purpose is to make \(\mathcal{M}^\alpha_\mu\) a coordinate subspace, which means it is generated by part of coordinate variables of \(\mathcal{M}^\beta_\mu\). To this end, we build a set of orthonormal basis on \(\mathcal{M}^\beta_\mu\) as follows:
Assume \( p = \mu_y \) and \( q = \mu_x \). Splitting \( C \in \mathcal{M}_\mu^\beta \) into \( ap \times aq \) blocks, where each block is of dimension \( k \times k \), yields

\[
C = \begin{bmatrix}
C^{1,1} & C^{1,2} & \ldots & C^{1,aq} \\
\vdots & \ddots & \ddots & \vdots \\
C^{ap,1} & C^{ap,2} & \ldots & C^{ap,aq}
\end{bmatrix}.
\]

Then for each \( C^{I,J} \in \mathcal{M}_{k \times k} \) we construct a basis, which consists of three classes:

- **Class 1:**
  \[
  \Delta_{i,j}^{I,J} = (b_{u,v}) \in \mathcal{M}_{k \times k}, \quad i \neq j,
  \]
  where
  \[
  b_{u,v} = \begin{cases} 
  1, & u = i, v = j \\
  0, & \text{otherwise}. 
  \end{cases}
  \]
  That is, for \((I, J)\)-th block, at each non-diagonal position \((i, j)\), set it to be 1, and all other entries to be 0.

- **Class 2:**
  \[
  D_{t,J}^{I,J} := \frac{1}{\sqrt{k}} I_{k}^{I,J}.
  \]
  That is, at each \((I, J)\)-th block, set \( D_{t,J}^{I,J} = \frac{1}{\sqrt{k}} I_k \) as a basis element.

- **Class 3:**
  \[
  E_t^{I,J} = \frac{1}{\sqrt{t(t-1)}} \text{diag} \left( \frac{1}{t-1}, \ldots, 1, -(t-1), 0, \ldots, 0 \right),
  \]
  \[
  t = 2, \ldots, k.
  \]
  That is, set \( E_t^{I,J} \) as the rest of basis elements of the diagonal subspace of \((I, J)\)-th block, which are orthogonal to \( D_{t,J}^{I,J} \).

Let \( A, B \in \mathcal{M}_{m \times n} \). The Frobenius inner product is defined as

\[
(A|B)_F := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} b_{i,j}.
\]

Correspondingly, the Frobenius norm is defined as

\[
\|A\|_F := \sqrt{(A|A)_F}.
\]

Using Frobenius inner product, it is easy to verify the following result.

**Proposition 70**

1. Set

\[
B^{I,J} := \{ \Delta_{i,j}^{I,J}, 1 \leq i \neq j \leq k; D_{t,J}^{I,J}; E_t^{I,J}, t = 2, \ldots, k \}.
\]

Then \( B^{I,J} \) is an orthonormal basis for \((I, J)\)-th block.

2. Set

\[
B := \{ B^{I,J} \mid I = 1, 2, \ldots, ap; J = 1, 2, \ldots, aq \}.
\]

Then \( B \) is an orthonormal basis for \( \mathcal{M}_\mu^\beta \).

**Example 71**

Consider \( \mathcal{M}_{1/2}^2 \subset \mathcal{M}_{1/2}^4 \). For any \( A \in \mathcal{M}_{1/2}^4 \), we split \( A \) as

\[
A = \begin{bmatrix}
A^{1,1} & A^{1,2} & A^{1,3} & A^{1,4} \\
A^{2,1} & A^{2,2} & A^{2,3} & A^{2,4}
\end{bmatrix}.
\]

Then we build the orthonormal basis block-wise as

\[
B^{I,J} := \begin{cases} 
\Delta_{1,2}^{I,J}, & \Delta_{2,1}^{I,J} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
D_{1,J}^{i,J}, & E_{2,J}^{i,J} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{cases},
\]

The orthonormal basis as proposed above is

\[
B = \{ B^{I,J} \mid I = 1, 2; \quad J = 1, 2, 3, 4 \}.
\]

Assume \( A \in \mathcal{M}_\mu^\beta \) and \( C \in \mathcal{M}_\mu^\alpha \) and \( \alpha k = \beta \). Using (71), matrix \( A \) can be expressed as

\[
A = \sum_i \sum_j \left( \sum_{i \neq j} c_{i,j}^{I,J} \Delta_{i,j}^{I,J} + d_{I,J} D_{I,J}^{I,J} + \sum_{t=2}^{k} e_t^{I,J} E_t^{I,J} \right).
\]

When \( \mathcal{M}_\mu^\alpha \) is merged into \( \mathcal{M}_\mu^\beta \) as a subspace, matrix \( C \) can be expressed as

\[
C = \sum_i \sum_j d_{I,J} D_{I,J}^{I,J}.
\]
The Frobenius norm is
\[ \|A\|_F = \sqrt{k_{I,J} D_{I,J}}, \]
where \( I = 1, \ldots, \alpha \mu_y; J = 1, \ldots, \alpha \mu_x. \)

To be consistent with this, we define the projection as follows:

**Definition 73** Assume \( A = (A^I,J) \in \mathcal{M}_\mu^\beta, \) where \( \beta = k\alpha \) and \( A^I,J = \left(A^I_{i,j}\right) \in \mathcal{M}_{k \times k}. \) The projection \( pr_k : \mathcal{M}_\mu^\beta \rightarrow \mathcal{M}_\mu^\alpha \) is defined as

\[ pr_k(A) = C = (c_{I,J}) \in \mathcal{M}_\mu^\alpha, \]

where
\[ c_{I,J} = \frac{1}{k} \sum_{d=1}^k A^I_{i,j} I_{d,d}. \]

According to the above construction, it is easy to verify the following:

**Proposition 74** The composed mapping \( pr_k \circ bd_k \) is an identity mapping.

### 3.4 Inner Product on \( \Sigma_\mu \)

Let \( A = (a_{i,j}), B = (b_{i,j}) \in \mathcal{M}_{m \times n}. \) It is well known that the Frobenius inner product of \( A \) and \( B \) is

\[ (A \mid B)_F := \sum_{i=1}^m \sum_{j=1}^n a_{i,j} b_{i,j}. \]

The Frobenius norm is
\[ \|A\|_F := \sqrt{(A \mid A)_F}. \]

The following lemma comes from a straightforward computation.

**Lemma 75** Let \( A, B \in \mathcal{M}_{m \times n}. \) Then
\[ (A \otimes I_k \mid B \otimes I_k)_F = k(A \mid B)_F. \]

**Definition 76** Let \( A, B \in \mathcal{M}_\mu, \) where \( A \in \mathcal{M}_\mu^\alpha \) and \( B \in \mathcal{M}_\mu^\beta. \) Then the weighted inner product of \( A, B \) is defined as
\[ (A \mid B)_W := \frac{1}{t} \left( A \otimes I_{\alpha} \mid B \otimes I_{\beta}\right)_F, \]
where \( t = \text{lcm}(\alpha, \beta) \) is the least common multiple of \( \alpha \) and \( \beta. \)

Using Lemma 75 and Definition 76, we have the following property.

**Proposition 77** Let \( A, B \in \mathcal{M}_\mu, \) if \( A \) and \( B \) are orthogonal, i.e., \( (A \mid B) = 0, \) then \( A \otimes I_{\xi} \) and \( B \otimes I_{\xi} \) are also orthogonal.

Now we are ready to define the inner product on \( \Sigma_\mu. \)

**Definition 78** Let \( (A), (B) \in \Sigma_\mu. \) Their inner product is defined as
\[ ( (A) \mid (B) ) := (A \mid B)_W. \]

The following proposition shows that (78) is well defined.

**Proposition 79** Definition 78 is well defined. That is, (78) is independent of the choice of the representatives \( A \) and \( B. \)

**Proof.** Assume \( A_1 \in (A) \) and \( B_1 \in (B) \) are irreducible. Then it is enough to prove that
\[ (A \mid B)_W = (A_1 \mid B_1)_W, \quad A \in (A), B \in (B). \]

Assume \( A_1 \in \mathcal{M}_\mu^\alpha \) and \( B_1 \in \mathcal{M}_\mu^\beta. \) Let
\[ A = A_1 \otimes I_{\xi} \in \mathcal{M}_\mu^\alpha_{\xi}, \quad B = B_1 \otimes I_{\eta} \in \mathcal{M}_\mu^\beta_{\eta}. \]

Denote by \( t = \text{lcm}(\alpha, \beta), s = \text{lcm}(\alpha \xi, \beta \eta), \) and \( s = t \ell. \)

Using (77), we have
\[ (A \mid B)_W = \frac{1}{s} \left( A \otimes I_{s \xi} \mid B \otimes I_{s \eta}\right)_F = \frac{1}{s} \left( A_1 \otimes I_{\xi} \mid B_1 \otimes I_{\eta} \right)_F = \frac{1}{t \ell} \left( A_1 \otimes I_{\xi} \mid B_1 \otimes I_{\eta} \right)_F = (A_1 \mid B_1)_W. \]
Theorem 82 Assume $\langle A \rangle, \langle B \rangle \in \Sigma_\mu$. Then we have the following

\begin{enumerate}
  \item (Schwarz Inequality)
  \[ \| \langle A \rangle \| \cdot \| \langle B \rangle \| \leq \langle A \rangle \| \langle B \rangle \|; \]  
  \item (Triangular Inequality)
  \[ \| \langle A \rangle + \langle B \rangle \| \leq \| \langle A \rangle \| + \| \langle B \rangle \|; \]  
  \item (Parallelogram Law)
  \[ \| \langle A \rangle + \langle B \rangle \|^2 + \| \langle A \rangle - \langle B \rangle \|^2 = 2 \| \langle A \rangle \|^2 + 2 \| \langle B \rangle \|^2. \]
\end{enumerate}

Note that the above properties show that $\Sigma_\mu$ is a normed space.

Finally, we present the generalized Pythagorean theorem:

\[ \| \langle A_1 \rangle + \langle A_2 \rangle + \cdots + \langle A_n \rangle \|^2 = \| \langle A_1 \rangle \|^2 + \| \langle A_2 \rangle \|^2 + \cdots + \| \langle A_n \rangle \|^2. \]  

**Theorem 83** Let $\langle A_i \rangle \in \Sigma_\mu$, $i = 1, 2, \cdots, n$ be an orthogonal set. Then

\[ \| \langle A_1 \rangle + \langle A_2 \rangle + \cdots + \langle A_n \rangle \|^2 = \| \langle A_1 \rangle \|^2 + \| \langle A_2 \rangle \|^2 + \cdots + \| \langle A_n \rangle \|^2. \]  

A natural question is that “is $\Sigma_\mu$ a Hilbert space?” Unfortunately, this is not true. This fact is shown in the following counter-example.

**Example 84** Define a sequence of elements, denoted as $\{ \langle A_k \rangle \mid k = 1, 2, \cdots \}$, as follows: $A_1 \in \mathcal{M}_{\mu}^2$ is arbitrary. Define $A_k$ inductively as

\[ A_{k+1} = A_k \otimes I_2 + E_{k+1} \in \mathcal{M}_{\mu}^{2^k}, \quad k = 1, 2, \cdots, \]

where $E_{k+1} = (e_{i,j}^{k+1}) \in \mathcal{M}_{\mu}^{2^k}$ is defined as

\[ e_{i,j}^{k+1} = \begin{cases} \frac{1}{2}, & i = 1, j = 2 \\ 0, & \text{Otherwise.} \end{cases} \]

First, we claim that $\{ \langle A_k \rangle \mid k = 1, 2, \cdots \}$ is a Cauchy sequence. Let $n > m$. Then

\[ \| \langle A_m \rangle - \langle A_n \rangle \| \leq \| \langle A_m \rangle - \langle A_{m+1} \rangle \| + \cdots + \| \langle A_{n-1} \rangle - \langle A_n \rangle \| \]

\[ \leq \frac{1}{2^m} + \cdots + \frac{1}{2^n} \leq \frac{1}{2^m}. \]

Then we prove by contradiction that it does not converge to any element. Assume it converges to $\langle A_0 \rangle$, it is enough to consider the following three cases:

- **Case 1**, assume $A_0 \in \mathcal{M}_{\mu}^{2^s}$ and $A_0 = A_{s+1}$. Then
  \[ \| \langle A_0 \rangle - \langle A_{s+2} \rangle \| = \| \langle A_{s+1} \rangle - \langle A_{s+2} \rangle \| = \frac{1}{2^{s+2}}. \]
  Similar to (85) we can prove that
  \[ \| \langle A_0 \rangle - \langle A_t \rangle \| > \frac{1}{2^s}, \quad t > s + 2. \]
  Hence the sequence cannot converge to $\langle A_0 \rangle$. 

\[ \Box \]
• Case 2, assume $A_0 \in \mathcal{M}_\mu^{2*}$ and $A_0 \neq A_{s+1}$. Note that $A_0 - A_{s+1}$ is orthogonal to $E_{s+2}$, then it is clear that
$$\|\langle A_0 - A_{s+2} \rangle\| > \|\langle A_0 - A_{s+1} \rangle\|.$$ Similar to (85), we can prove that
$$\|\langle A_0 - A_{t} \rangle\| > \|\langle A_0 - A_{s+1} \rangle\|, \quad t > s + 2.$$ Hence the sequence cannot converge to $\langle A_0 \rangle$.

• Case 3, $A_0 \in \mathcal{M}_\mu^{2*}$, where $\xi > 1$ is odd. Corresponding to Case 1, we assume $A_0 = A_{s+1} \otimes I_\xi$. Then we have
$$\|\langle A_0 - A_{s+2} \rangle\| = \|\langle A_{s+1} \otimes I_\xi - (A_{s+1} \otimes I_2 + E_{s+2}) \otimes I_\xi \rangle\| = \|\langle E_{s+2} \otimes I_\xi \rangle\| = \frac{1}{2^{s+2}}.$$ and
$$\|\langle A_0 - A_{t} \rangle\| > \frac{1}{2^{s+2}}, \quad t > s + 2.$$ So the sequence cannot converge to $\langle A_0 \rangle$.

Corresponding to Case 2, assume $A_0 \neq A_{s+1} \otimes I_\xi$. Using Proposition 77, a similar argument shows that the sequence cannot converge to $\langle A_0 \rangle$ too.

3.5 $\Sigma_\mu$ as a Matric Space

Using the norm defined in previous section one sees easily that $\Sigma_\mu$ is a matric space:

**Theorem 85** $\Sigma_\mu$ with distance

$$d(\langle A \rangle, \langle B \rangle) := \|\langle A \rangle - \langle B \rangle\|, \quad \langle A \rangle, \langle B \rangle \in \Sigma_\mu \quad (86)$$

is a matric space.

**Theorem 86** Consider $\Sigma_\mu$. The topology deduced by the distance $d$, denoted by $T_\mu$ is exactly the same as the topology $T$ defined in Definition 59.

**Proof.** Assume $U \in T_\mu$ and $p \in U$. Then there exists a ball $B_\epsilon(p)$ such that $B_\epsilon(p) \subset U$, where $\epsilon > 0$. Assume $p = \langle A_0 \rangle$ and $A_0 \in \mathcal{M}_\mu^\mu = \mathcal{M}_{\mu \times \mu \times \mu}$.

Now we can construct a ball $B_\delta(A_0) \subset \mathcal{M}_\mu^\mu$, where $\delta > 0$. Note that $B_\delta(A_0)$ is a sub-basis element of $T$, and hence is an open set in $(\Sigma_\mu, T)$. By continuity, as $\delta > 0$ small enough, $q \in B_\delta(A_0)$ implies $d(q, A_0) < \epsilon$. That is,
$$B_\delta(\langle A_0 \rangle) \subset B_\epsilon(p) \subset U,$$

which means $U \in T$. Hence, $T_\mu \subset T$.

Conversely, assume $q \in V \in T$. Then there exists a basic open set $s_1 \cap \cdots \cap s_r \in T$ such that $q \in s_1 \cap \cdots \cap s_r \in T$. Express
$$q = \{ A_1, A_2, \cdots, A_r \},$$

where $A_i \in s_i \subset \mathcal{M}_\mu^\mu$, $i = 1, \cdots, r$. For each $A_i$, we can find $B_\epsilon^i(\langle A_i \rangle) \subset s_i$, where $\epsilon > 0$, $i = 1, \cdots, r$. Choosing $\delta_i > 0$ small enough such that
$$B_\delta^i(\langle A_i \rangle) \subset B_\epsilon^i(\langle A_i \rangle) \subset s_i, \quad i = 1, \cdots, r.$$ Then we have
$$q \in \bigcap_{i=1}^r B_\delta_i(\langle A_i \rangle) \in T_\mu.$$ That is, $V \in T_\mu$. Hence, $T \subset T_\mu$.

We conclude that $T_\mu = T$. \hfill $\square$

**Definition 87** [15]

(1) A topological space is regular (or $T_3$) if for each closed set $X$ and $x \notin X$ there exist open neighborhoods $U_x$ of $x$ and $U_X$ of $X$, such that $U_x \cap U_X = \emptyset$.

(2) A topological space is normal (or $T_4$) if for each pair of closed sets $X$ and $Y$ there exist open neighborhoods $U_X$ of $X$ and $U_Y$ of $Y$, such that $U_X \cap U_Y = \emptyset$.

Since a matric space is regular and normal, as a corollary of Theorem 86, we have the following result.

**Corollary 88** The topological space $(\Sigma_\mu, T)$, defined in Definition 59, is both regular and normal.

Note that
$$T_4 \Rightarrow T_3 \Rightarrow T_2.$$ Finally, we show some properties of $\Sigma_\mu$.

**Proposition 89** $\Sigma_\mu$ is convex. Hence it is arcwise connected.
Proposition 91 \( \langle A \rangle, \langle B \rangle \in \Sigma_{\mu} \). Then it is clear that 
\[ \lambda \langle A \rangle \| (1 - \lambda) \langle B \rangle = \langle (\lambda A) \| (1 - \lambda) B \rangle \in \Sigma_{\mu}, \quad \lambda \in [0, 1]. \]

So \( \Sigma_{\mu} \) is convex. Let \( \lambda \) go from 1 to 0, we have a path connecting \( \langle A \rangle \) and \( \langle B \rangle \).

\[ \square \]

Proposition 90 \( \Sigma_{\mu} \) and \( \Sigma_{\mu/\mu} \) are isometric spaces.

Proof. Consider the transpose:
\[ \langle A \rangle \mapsto \langle A^T \rangle. \]

Then it is obvious that 
\[ d(\langle A \rangle, \langle B \rangle) = d(\langle A^T \rangle, \langle B^T \rangle). \]

Hence the transpose is an isometry. Moreover, its inverse is itself.

\[ \square \]

3.6 Subspaces of \( \Sigma_{\mu} \)

Consider the \( k \)-upper bounded subspace \( \Sigma_{\mu}^{[k]} \). We have

Proposition 91 \( \Sigma_{\mu}^{[k]} \) is a Hilbert space.

Proof. Since \( \Sigma_{\mu}^{[k]} \) is a finite dimensional vector space and any finite dimensional inner product space is Hilbert [14].

\[ \square \]

Proposition 92 [14] Let \( E \) be an inner product space, \( \{0\} \neq F \subset E \) be a Hilbert subspace.

1. For each \( x \in E \) there exists a unique \( y := P_F(x) \in F \), called the projection of \( x \) on \( F \), such that
\[ \|x - y\| = \min_{z \in F} \|x - z\|. \quad (87) \]

2. \[ F^\perp := P_F^{-1}(0) \quad (88) \]

is the subspace orthogonal to \( F \).

3. \[ E = F \oplus F^\perp, \quad (89) \]

where \( \oplus \) stands for orthogonal sum.

Using above proposition, we consider the projection: \( P_F : \Sigma_{\mu} \to \Sigma_{\mu}^{[k]} \). Let \( \langle A \rangle \in \Sigma_{\mu}^{[k]} \). Assume \( \langle X \rangle \in \Sigma_{\mu}^{\alpha} \), \( t = \text{lcm}(\alpha, \beta) \). Then the norm of \( \langle A \rangle \vdash \langle X \rangle \) is:
\[ \|\langle A \rangle \vdash \langle X \rangle\| = \frac{1}{\sqrt{t}} \|A \otimes I_{t/\beta} - X \otimes I_{t/\alpha}\|_F. \quad (90) \]

Set \( p = \mu_q \) \( q = \mu_x \), and \( k := t/\alpha \). We split \( A \) as
\[ A \otimes I_{t/\beta} = \begin{bmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,q_\alpha} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,q_\alpha} \\
\vdots & & & \\
A_{p_\alpha,1} & A_{p_\alpha,2} & \cdots & A_{p_\alpha,q_\alpha}
\end{bmatrix}, \]

where \( A_{i,j} \in M_{k \times k}, i = 1, \ldots, p_\alpha; j = 1, \ldots, q_\alpha. \) set
\[ C := \arg \min_{x \in \Sigma_{\mu}^{[k]}} A \otimes I_{t/\beta} - X \otimes I_{t/\alpha} \|_F. \quad (91) \]

Then it is easy to calculate that
\[ c_{i,j} = \frac{1}{k} \text{trace}(A_{i,j}), \quad i = 1, \ldots, p_\alpha; j = 1, \ldots, q_\alpha. \quad (92) \]

We conclude that

Proposition 93 Let \( \langle A \rangle \in \Sigma_{\mu}, P_F : \Sigma_{\mu} \to \Sigma_{\mu}^{[k]} \). Precisely speaking, \( \langle A \rangle \in \Sigma_{\mu}^{[k]} \). Using the above notations, the projection of \( \langle A \rangle \) is
\[ P_F(\langle A \rangle) = C, \quad (93) \]

where \( C \) is defined in (92).

We give an example to depict this.

Example 94 Given 
\[ A = \begin{bmatrix}
1 & 2 & -3 & 0 & 2 & 1 \\
2 & 1 & -2 & -1 & 1 & 0 \\
0 & -1 & -1 & 3 & 1 & -2
\end{bmatrix} \in \Sigma_{\mu}^{[3,2]} \]

We consider its projection to \( \Sigma_{\mu}^{[2,2]} \). Denote by \( t = \text{lcm}(2, 3) = 6 \). Using formulas (92)-(93), we have
\[ P_F(\langle A \rangle) = \begin{bmatrix}
1 & 0 & 1/3 & 0 \\
0 & -1/3 & 0 & -1
\end{bmatrix}.\]
Then we have

$$\langle E \rangle = \langle A \rangle - P_F(\langle A \rangle),$$

where

$$E = \begin{bmatrix}
0 & 2 & 0 & -3 & 0 & -1/3 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 2 & 0 & -3 & 0 & -1/3 & 0 & 2 & 0 & 1 \\
2 & 0 & 0 & -2 & 0 & -1 & 0 & 2/3 & 0 & 0 & 0 \\
0 & 2 & 1/3 & 0 & -2 & 0 & -1 & 0 & 2 & 0 & 0 \\
0 & 0 & -1 & 0 & -2/3 & 0 & 3 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & -2/3 & 0 & 3 & 0 & 1 & 0 & -1
\end{bmatrix}.$$ 

It is easy to verify that $\langle E \rangle$ and $\langle A \rangle$ are mutually orthogonal.

We also have $\Sigma_{\mu}^{[k, k]}$ and $\Sigma_{\mu}^{[\alpha, \beta]}$ (where $\alpha|\beta$) as metric subspaces of $\Sigma_{\mu}$.

Finally, we would like to point out that since $\Sigma_{\mu}$ is an infinity dimensional vector space, it is possible that $\Sigma_{\mu}$ is isometric to its proper subspace. For instance, consider the following example.

**Example 95** Consider a mapping $\varphi : \Sigma_{\mu} \to \Sigma_{\mu}^{[k, k]}$ defined by $\langle A \rangle \mapsto \langle A \otimes I_k \rangle$. It is clear that this mapping satisfies

$$\| \langle A \rangle - \langle B \rangle \| = \| \varphi(\langle A \rangle) - \varphi(\langle B \rangle) \|, \quad \langle A \rangle, \langle B \rangle \in \Sigma_{\mu}.$$ 

That is, $M_{\mu}$ can be isometrically embedded into its proper subspace.

## 4 Differential Structure on Equivalence Space

### 4.1 Bundled Manifold

Unlike the conventional manifolds, which have fixed dimensions, this section explores a new kinds of manifolds, called the bundled manifold. Intuitively speaking, it is a fiber bundle, which has fibers belonging to manifolds of different dimensions. To begin with, the following definition is proposed, which is a mimic to the definition of an $n$ dimension manifold [1].

**Definition 96** Let $\{M, T\}$ be a topological space.

1. An open set $U \neq \emptyset$ is said to be a simple coordinate chart, if there is an open set $\Theta \subset \mathbb{R}^s$ and a homeomorphism $\phi : U \to \Theta \subset \mathbb{R}^s$. The integer $s$ is said to be the dimension of $U$.
2. An open set $U \neq \emptyset$ is said to be a bundled coordinate chart, if there exist finite simple coordinate charts $U_i$ with homeomorphisms $\phi_i : U_i \to \Theta_i \subset \mathbb{R}^s$, $i = 1, \cdots, k, k < \infty$ is set $U$ depending, such that

   $$U = \bigcap_{i=1}^{k} U_i.$$ 

3. Let $U, V$ be two bundled coordinate charts. $U = \cap_{i=1}^{k} U_i$, $V = \cap_{j=1}^{k} V_j$. $U$ and $V$ are said to be $C^r$ comparable if for any $U_i$ and $V_j$, as long as their dimensions are equal, they are $C^r$ comparable. (Where $r$ could be $\infty$, that is they are $C^\infty$ comparable; or $\omega$, which means they are analytically comparable.)

**Remark 97** In Definition 96 for a bundled coordinate chart $U = \bigcap_{i=1}^{k} U_i$, we can, without loss of generality, assume $\dim(U_i), i = 1, \cdots, k$ are distinct. Because simple coordinate charts of same dimension can be put together by set union $\cup$. Hereafter, this is assumed.

![Fig. 5. Multiple Coordinate Charts](image-url)
only if the two mappings
\[
\psi \circ \phi^{-1} : \phi(U_2 \cap V_1) \to \psi(U_2 \cap V_1) \quad \text{and} \\
\phi \circ \psi^{-1} : \psi(U_2 \cap V_1) \to \phi(U_2 \cap V_1)
\]
are \( C^r \). As a convention, we assume \( \dim(U_1) \neq \dim(U_2) \), \( \dim(V_1) \neq \dim(V_2) \).

**Definition 98** A topological space \( M \) is a bundled \( C^r \) (or \( C^\infty \), or analytic, denoted by \( C^\omega \)) manifold, if the following conditions are satisfied.

1. \( M \) is second countable and Hausdorff.
2. There exists an open cover of \( M \), described as
   \[ C = \{ U_\lambda \mid \lambda \in \Lambda \}, \]
   where each \( U_\lambda \) is a bundled coordinate chart. Moreover, any two bundled coordinate charts in \( C \) are \( C^r \) comparable.
3. If a bundled coordinate chart \( W \) is comparable with \( U_\lambda, \forall \lambda \in \Lambda \), then \( W \in C \).

It is obvious that the topological structure of \( \Sigma_\mu \) with natural \( (\mathbb{R}^2^{\mu_y \times \mu_z}) \) coordinates on each cross section (or leaf) meets the above requirements for a bundled manifold. Hence, we have the following result.

**Theorem 99** \( \Sigma_\mu \) is a bundled analytic manifold.

**Proof.** Condition 1 has been proved in Theorem 61. For condition 2, set \( p = \mu_y \), \( q = \mu_z \) and
\[ O_k := \mathcal{M}_\mu^k, \quad k = 1, 2, \ldots . \]
Choosing any finite open subset \( \alpha_{i_s} \subset O_{i_s}, s = 1, 2, \ldots, t \), \( t < \infty \), and constructing corresponding \( s(o_{i_1}), \ldots, s(o_{i_s}) \). Set \( U_I := s(o_{i_1}) \cap \cdots \cap s(o_{i_s}) \), where \( I = \{i_1, i_2, \ldots, i_t\} \). Define
\[ W := \{ U_I \mid I \text{ is a finite subset of } \mathbb{N} \} \]
Then \( W \) is an open cover of \( M \). Identity mappings from \( s(o_i) \to \mathcal{M}_\mu^i \simeq \mathbb{R}^{\mu_y \times \mu_z} \) makes any two \( U_I \) and \( U_J \) being \( C^\omega \) comparable. As for condition 3, just add all bundled coordinate charts which are comparable with \( W \) into \( W \), the condition is satisfied.

Next, we consider the lattice-related coordinates on \( \Sigma_\mu \).

Consider \( \Sigma_\mu \) and assume \( p = \mu_y \) and \( q = \mu_z \). Then \( \Sigma_\mu \) has leaves
\[ \Sigma_\mu = \{ \mathcal{M}_\mu^i \mid i = 1, 2, \ldots \}, \]
where \( \mathcal{M}_\mu^i = \mathcal{M}_{ip \times iq} \).

Consider an element \( \langle x \rangle \in \Sigma_\mu \), then there exists a unique irreducible \( x_1 \in \langle x \rangle \) such that \( \langle x \rangle = \{ x_j = x_1 \otimes I_j \mid j = 1, 2, \ldots \} \). Now assume \( x_1 \in \mathcal{M}_\mu^1 \). As defined above, \( \mathcal{M}_\mu^1 \) is the root leaf of \( \langle x \rangle \).

It is obvious that \( \langle x \rangle \) has different coordinate representations on different leaves. But because of the subspace lattice structure, they must be consistent. Fig. 6 shows the lattice-related subspaces. Any geometric objects defined on its root leaf must be consistent with its representations on all embedded spaces and projected spaces.

![Fig. 6. Lattice-related Coordinates](image_url)

As shown in Fig. 6 the following subspaces are related:

- **Class 1 (Embedded Elements):** Starting from \( x^1 \in \mathcal{M}_\mu^1 \), we have
  \[ x^1 < x^2 < x^3 < \cdots . \]

- **Class 2 (Projected Elements):** Let \( x^{(t)} \in \mathcal{M}_\mu^t \), where \( t \mid s \). Then
  \[ x^{(t)} < x^1 . \]
  Particularly, \( x^{(1)} \in \mathcal{M}_\mu^1 \) satisfies
  \[ x^{(1)} < x^1 . \]

- **Class 3 (Embedded Elements from Projected Elements):** Starting from any \( x^{(t)}, t \mid s \), we have
  \[ x^{(t)} < x^{(2t)} < x^{(3t)} < \cdots . \]
Remark 100  (1) Classes 1 - 3 are the set of coordinates, which are related to a given irreducible element $x_1$.
(2) Elements in Class 1 are particularly important. Say, we may firstly define an root element on $M^n_\mu$, such as $A_1 \in M^n_\mu$. Then we use it to get an equivalent class, such as $\langle A \rangle$, and use the elements in this class to perform certain calculation, such as STP. All the elements in the equivalent class, such as $\langle A \rangle$, are of Class 1.
(3) The elements in subspace and their equivalent classes are less important. Sometimes we may concern only the elements of Class 1, say for STP etc.
(4) The subspace elements obtained by project mapping may not be “unified” with the object obtained from the real subspaces of the original space. More discussion will be seen in the sequel.
(5) Because of the above argument, sometimes we may consider only the equivalent classes which have their root elements defined on their root leaf. Therefore, the objects may only be defined on the multiple of the root leaf (root space).

4.2 $C^\tau$ Functions on $\Sigma_\mu$

Definition 101  Let $M$ be a bundled manifold, $f : M \to \mathbb{R}$ is called a $C^\tau$ function, if for each simple coordinate chart $U \subset M^n_\mu$, $f|_U := f_s$ is $C^\tau$. The set of $C^\tau$ functions on $M$ is denoted by $C^\tau(M)$.

Assume $f \in C^\tau(\Sigma_\mu)$, $A$, $B \in M_\mu$ and $A \sim B$. Then $f$ is well defined on $\Sigma_\mu$, means it is defined on different leaves consistently, and hence on leaves corresponding to $A$ and $B$ we have $f(A) = f(B)$. To this end, the $f$ can be constructed as follows:

Definition 102  Assume $f$ is firstly defined on root leaf $M^n_\mu$ as $f_s(x)$. Then we extend it to other leaves as:

- Step 1. Let $Q = \{ t \in \mathbb{N} \mid t|s \}$. Then
\[
f_t(y) := f_s(x = bd_k(y)), \quad (94)
\]
where $k = \frac{t}{s} \in \mathbb{N}$.
- Step 2. Assume $\text{gcd}(\ell,s) = t$. If $\ell = t$, $f_t = f_1$ has already been defined in Step 1. So we assume $\ell = kt$. Then
\[
f_\ell(y) := f_t(x = pr_k(y)). \quad (95)
\]

Note that in Step 2, $t = s$ is allowed.

Then it is easy to verify the following:

Proposition 103  The function $f$ defined in Definition 102 is consistent with the equivalence $\sim$. Hence it is well defined on $\Sigma_\mu$.

Example 104  Consider $\Sigma_2$, and assume $f$ is defined on its root leaf $M_2^n$ firstly as
\[
f_2 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} := a_{11}a_{22} - a_{32}a_{41}.
\]

Then we can determine the other expressions of $f$ as follows:

- Consider $f_1$:
\[
f_1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = f_2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes I_2 = a_1^2.
\]

- Consider $f_3$. Let $A = (a_{i,j}) \in M_3^n$. Then
\[
f_3(A) = f_1(pr_3(A)) = f_1 \begin{pmatrix} \frac{1}{3}(a_{11} + a_{22} + a_{33}) \\ \frac{1}{3}(a_{41} + a_{52} + a_{63}) \end{pmatrix}
\]
\[
= \frac{1}{3}(a_{11} + a_{22} + a_{33})^2.
\]

Similarly, for any $n = 2k - 1$ we have
\[
f_n(A) = \frac{1}{n^2} (a_{11} + a_{22} + \cdots + a_{nn})^2.
\]

- Consider $f_4$. Let $A = (a_{i,j}) \in M_4^n$. Then
\[
f_4(A) = f_2(pr_2(A))
\]
\[
= f_2 \begin{pmatrix} \frac{1}{2}(a_{11} + a_{22}) & \frac{1}{2}(a_{13} + a_{24}) \\ \frac{1}{2}(a_{31} + a_{42}) & \frac{1}{2}(a_{33} + a_{44}) \\ \frac{1}{2}(a_{51} + a_{62}) & \frac{1}{2}(a_{53} + a_{64}) \\ \frac{1}{2}(a_{71} + a_{82}) & \frac{1}{2}(a_{73} + a_{84}) \end{pmatrix}
\]
\[
= \frac{1}{4} \left( (a_{11} + a_{22})(a_{33} + a_{44}) - (a_{53} + a_{64})(a_{71} + a_{82}) \right)
\]
Similarly, for \( n = 2k \) \((k \geq 2\) we have

\[
f_n(A) = \frac{1}{k^2} \left[ (a_{11} + a_{22} + \cdots + a_{kk}) \right.
\]
\[
\left. (a_{k+1,k+1} + a_{k+2,k+2} + \cdots + a_{2k,2k}) \right.
\]
\[
\left. - (a_{2k+1,k+1} + a_{2k+2,k+2} + \cdots + a_{3k,2k}) \right.
\]
\[
\left. (a_{3k+1,1} + a_{3k+2,2} + \cdots + a_{4k,k}) \right].
\]

Remark 105 For a smooth function \( f \) defined firstly on \( \mathcal{M}_\mu^p \), its extensions to both \( \mathcal{M}_\mu^{[a,\infty]} \) and \( \mathcal{M}_\mu^{[\infty,a]} \) are consistently defined. Hence, \( f \) is completely well posed on \( \Sigma_\mu \).

4.3 Generalized Inner Products

We define the generalized Frobenius inner product as follows.

Definition 106 Given \( A \in \mathcal{M}_{m \times n} \) and \( B \in \mathcal{M}_{p \times q} \).

Case 1 (Special Case): Assume \( p = rm \) and \( q = sn \). Split \( B \) into equal blocks as

\[
B = \begin{bmatrix}
B_{1,1} & B_{1,2} & \cdots & B_{1,s} \\
B_{2,1} & B_{2,2} & \cdots & B_{2,s} \\
& \vdots & & \\
B_{r,1} & B_{r,2} & \cdots & B_{r,s}
\end{bmatrix}
\]

where \( B_{i,j} \in \mathcal{M}_{m \times n}, i = 1, \cdots, r; j = 1, \cdots, s \). Then the generalized Frobenius inner product of \( A \) and \( B \) is defined as

\[
(A | B)_F := \begin{bmatrix}
(A|B_{1,1}F)F & (A|B_{1,2}F)F & \cdots & (A|B_{1,s}F)F \\
(A|B_{2,1}F)F & (A|B_{2,2}F)F & \cdots & (A|B_{2,s}F)F \\
& \vdots & & \\
(A|B_{r,1}F)F & (A|B_{r,2}F)F & \cdots & (A|B_{r,s}F)F
\end{bmatrix}
\]

(96)

Note that here \((A|B_{i,j})F\) is the standard Frobenius inner product defined in (74).

Case 2 (General Case): Assume \( A \in \mathcal{M}_{m \times n} \) and \( B \in \mathcal{M}_{p \times q} \) and let the great common divisor of \( m \), \( p \) be \( \alpha = \gcd(m,p) \), and the great common divisor of \( n \), \( q \) be \( \beta = \gcd(n,q) \). Denote by \( \xi = m/\alpha \) and \( \eta = n/\beta \), \( r = p/\alpha \) and \( s = q/\beta \). Then we split \( A \) into \( \xi \times \eta \) blocks as

\[
A = \begin{bmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,\eta} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,\eta} \\
& \vdots & & \\
A_{\xi,1} & A_{\xi,2} & \cdots & A_{\xi,\eta}
\end{bmatrix}
\]

(97)

where \( A_{i,j} \in \mathcal{M}_{m/\alpha \times n/\beta} \), \( i = 1, \cdots, \xi; j = 1, \cdots, \eta \). Then the generalized Frobenius inner product of \( A \) and \( B \) is defined as

\[
(A | B)_F := \begin{bmatrix}
(A_{1,1}|B_{1,1}F)F & (A_{1,2}|B_{1,2}F)F & \cdots & (A_{1,\eta}|B_{1,\eta}F)F \\
(A_{2,1}|B_{2,1}F)F & (A_{2,2}|B_{2,2}F)F & \cdots & (A_{2,\eta}|B_{2,\eta}F)F \\
& \vdots & & \\
(A_{\xi,1}|B_{\xi,1}F)F & (A_{\xi,2}|B_{\xi,2}F)F & \cdots & (A_{\xi,\eta}|B_{\xi,\eta}F)F
\end{bmatrix}
\]

Note that here \((A_{i,j}|B_{i,j})F\) is the (Case 1) generalized Frobenius inner product defined in (96).

Example 107 Let

\[
A = \begin{bmatrix}
1 & -1 & 1 & 0 \\
1 & 2 & 0 & 1
\end{bmatrix} \in \mathcal{M}^2_{2,5},
\]

and

\[
B = \begin{bmatrix}
1 & 0 \\
-1 & 2 \\
-1 & 0 \\
1 & -1
\end{bmatrix} \in \mathcal{M}^4_{3,2}.
\]

Note that \( m = 2, n = 4, p = 4, q = 2 \), \( \alpha = \gcd(m,p) = 2, \beta = \gcd(n,q) = 2 \). Then we split \( A \) and \( B \) as follows

\[
A = \begin{bmatrix}
A_{1,1} & A_{1,2} \\
& \\
& B_{2,1}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
B_{1,1} \\
B_{2,1} \\\nB_{3,1} \\
B_{4,1}
\end{bmatrix}
\]

were \( A_{i,j} \), \( B_{k,\ell} \in \mathcal{M}_{2 \times 2}, i = 1,2; j = 1,2; k = 1,2; \ell = 1 \).

Finally, we have

\[
(A | B)_F = \begin{bmatrix}
(A_{1,1}|B_{1,1})F & (A_{1,2}|B_{1,1})F \\
(A_{1,1}|B_{2,1})F & (A_{1,2}|B_{2,1})F
\end{bmatrix} = \begin{bmatrix}
4 & 3 \\
-2 & -2
\end{bmatrix}.
\]
Definition 108 Assume $A \in \mathcal{M}_\sigma^\alpha$ and $B \in \mathcal{M}_\lambda^\beta$. $\gcd(\mu_y, \lambda_y) = s$, $\gcd(\mu_x, \lambda_x) = t$, $\frac{\mu}{s} = m$, $\frac{\mu}{n} = n$, $\frac{\lambda}{t} = p$, $\frac{\lambda}{q} = q$. Since $s$, $t$ are co-prime, denote $\sigma = s/t$, then $\sigma_y = s$ and $\sigma_x = t$.

Split $A$ as

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{bmatrix}$$

where $A_{i,j} \in \mathcal{M}_\sigma^\alpha$, and split $B$ as

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,q} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,q} \\ \vdots \\ B_{p,1} & B_{p,2} & \cdots & B_{p,q} \end{bmatrix}$$

where $B_{i,j} \in \mathcal{M}_\sigma^\beta$. Then the generalized weighted inner product is defined as

$$(A \mid B)_W := \begin{vmatrix} (A_{1,1}|B_{1,1})_W & \cdots & (A_{1,1}|B_{1,q})_W & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ (A_{m,1}|B_{1,1})_W & \cdots & (A_{m,1}|B_{1,q})_W & \cdots \end{vmatrix} \quad (98)$$

$$(A_{1,n}|B_{1,1})_W \cdots (A_{1,n}|B_{1,q})_W$$

$$(A_{2,n}|B_{1,1})_W \cdots (A_{2,n}|B_{1,q})_W$$

$$\vdots$$

$$(A_{m,n}|B_{1,1})_W \cdots (A_{m,n}|B_{1,q})_W$$

where $(A_{i,j}|B_{r,s})$ are defined in (77).

Definition 109 Assume $(A) \in \Sigma_\mu$ and $(B) \in \Sigma_\lambda$. Then the generalized inner product of $(A)$ and $(B)$, denoted by $((A) \mid (B))$, is defined as

$$( (A) \mid (B) ) := (A \mid B)_W. \quad (99)$$

Of course, we need to prove that (99) is independent of the choice of representatives $A$ and $B$. This is verified by a straightforward computation.

Next, we would like to define another “inner product” called the $\delta$-inner product, where $\delta \in \mathbb{Q}_+$. First we introduce a new notation:

Definition 110 Let $\mu, \delta \in \mathbb{Q}_+$, $\mu$ is said to be superior to $\delta$, denoted by

$$\mu \gg \delta,$$

if $\delta_y|\mu_y$ and $\delta_x|\mu_x$.

The $\delta$ inner product is a mapping $(\cdot|\cdot) : \bigcup_{\mu \gg \delta} \Sigma_\mu \times \bigcup_{\mu \gg \delta} \Sigma_\mu \rightarrow \Sigma_\delta$.

Definition 111 Assume $A \in \mathcal{M}_\mu^\alpha$, $B \in \mathcal{M}_\lambda^\beta$ and $\mu \gg \delta$, $\lambda \gg \delta$. Denote $\mu_y/\delta_y = \xi$, $\mu_x/\delta_x = \eta$, $\lambda_y/\delta_y = \zeta$, and $\lambda_x/\delta_x = \ell$, then the $\delta$-inner product of $A$ and $B$ is defined as follows: Split

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots \\ A_{\xi,1} & A_{\xi,2} & \cdots & A_{\xi,\eta} \end{bmatrix},$$

$$(A_{1,n}|B_{1,1})_W \cdots (A_{1,n}|B_{1,q})_W$$

$$(A_{2,n}|B_{1,1})_W \cdots (A_{2,n}|B_{1,q})_W$$

$$\vdots$$

$$(A_{m,n}|B_{1,1})_W \cdots (A_{m,n}|B_{1,q})_W$$

where $A_{i,j} \in \mathcal{M}_\delta^\alpha$, and

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,\ell} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,\ell} \\ \vdots \\ B_{\zeta,1} & B_{\zeta,2} & \cdots & B_{\zeta,\ell} \end{bmatrix}$$

where $B_{i,j} \in \mathcal{M}_\delta^\beta$. Then the $\delta$-inner product of $A$ and $B$ is defined as

$$(A|B)_\delta := \begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,\eta} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,\eta} \\ \vdots \\ C_{\xi,1} & C_{\xi,2} & \cdots & C_{\xi,\eta} \end{bmatrix} \quad (100)$$
We express a vector field in a matrix form. That is, let

\[
C_{i,j} := \begin{pmatrix}
(A_{i,j}|B_{1,1})_W & (A_{i,j}|B_{1,2})_W & \cdots & (A_{i,j}|B_{1,t})_W \\
(A_{i,j}|B_{2,1})_W & (A_{i,j}|B_{2,2})_W & \cdots & (A_{i,j}|B_{2,t})_W \\
\vdots & \vdots & \ddots & \vdots \\
(A_{i,j}|B_{\xi,1})_W & (A_{i,j}|B_{\xi,2})_W & \cdots & (A_{i,j}|B_{\xi,t})_W
\end{pmatrix}
\]  

(101)

**Definition 112** Assume \( \langle A \rangle \in \Sigma_\mu \) and \( \langle B \rangle \in \Sigma_\lambda \), where \( \mu \gg \delta \) and \( \lambda \gg \delta \). Then the \( \delta \)-inner product of \( \langle A \rangle \) and \( \langle B \rangle \) is defined as

\[
(\langle A \rangle \mid \langle B \rangle)_\delta := \langle A \mid B \rangle_\delta, \quad A \in \langle A \rangle, \quad B \in \langle B \rangle.
\]  

(102)

**Remark 113**

1. It is easy to verify that (102) is independent of the choice of \( A \) and \( B \). Hence the \( \delta \)-inner product is well defined.
2. Definition 108 (or Definition 109) is a special case of Definition 111 (correspondingly, Definition 112).
3. Definition 106 cannot be extended to the equivalence space, because it depends on the choice of representatives.
4. Unlike the generalized inner product defined in Definition 108 (as well as Definition 109), the \( \delta \)-inner product is defined on a subset of \( M \) (or \( \Sigma \)).

Using \( \delta \)-inner product, we have the following.

**Proposition 114** Assume \( \varphi : \Sigma_\mu \to \Sigma_\lambda \) is a linear mapping. Then there exists a matrix \( \Lambda \in M_{r_\mu \times r_\lambda} \), called the structure matrix of \( \varphi \), such that

\[
\varphi(\langle A \rangle) = \langle A \mid \Lambda \rangle_\mu.
\]  

(103)

4.4 Vector Fields

**Definition 115** Let \( M \) be a bundled manifold and \( T(M) \) the tangent space of \( M \). \( V : M \to T(M) \) is called \( C^r \) vector field, if for each simple coordinate chart \( U \), \( V|_U \) is \( C^r \). The set of \( C^r \) vector fields on \( M \) is denoted by \( V^r(M) \).

We express a vector field in a matrix form. That is, let \( X \in V^r \). Then

\[
X = \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j}(x) \frac{\partial}{\partial x_{i,j}} := [f_{i,j}(x)] \in M_{m \times n}.
\]

Similar to smooth functions, the vector fields on \( M_\mu \) can be defined as follows:

**Definition 116** Assume \( \langle X \rangle \) is firstly defined on \( T(M_\mu) \) as \( X_s(x) \), i.e., \( M_\mu \) is the root leaf of \( \langle X \rangle \). Then we extend it to other leaves as:

- **Step 1.** Let \( Q = \{ t \in \mathbb{N} \mid t \mid s \} \). Then for

\[
X_t(y) := (pr_k)_* (X_s)(x = bd_k(y)),
\]  

(104)

where \( k = \frac{s}{t} \in \mathbb{N} \).

- **Step 2.** Assume \( gcd(t, s) = t \). If \( \ell = t \), \( X_\ell = X_t \) has already been defined in Step 1. So we assume \( \ell = kt \).

Then

\[
X_\ell(y) := (bd_k)_* (X_t)(x = pr_k(y)) \otimes I_k.
\]  

(105)

Next, we consider the computation of the related expressions of a vector field, which is originally defined on its root leaf.

- To calculate (104) we first set

\[
x = y \otimes I_k
\]  

(106)

to get \( X_s(x(y)) := X_s(y) \). Then we split \( X_s(y) \) into \( tp \times tq \) blocks as \( X_s = [X_s^{i,j}] \), where each \( X_s^{i,j} \in M_{k \times k} \).

Then \( X_t = [X_t^{i,j}] \in T_x (M_{tp \times tq}) \), and

\[
X_t^{i,j} = \text{Tr} \left( [X_s^{i,j}] \right).
\]  

(107)

- To calculate (105) we split \( y \) into \( tp \times tq \) blocks as \( y = [y^{i,j}] \), where each \( y^{i,j} \in M_{k \times k} \).

Then \( X_t(y) \) is obtained by replacing \( x \) by \( x = [x_{i,j}] \in M_{tp \times tq} \) as

\[
x_{i,j} = \text{Tr} \left( [y^{i,j}] \right).
\]  

(108)

It follows that

\[
X_t(y) = X_t(y) \otimes I_k.
\]  

(109)

Then it is easy to verify the following:

**Proposition 117** The vector field \( X \) defined in Definition 116 is consistent with the equivalence \( \sim \left[ M_\mu^{s,i,j} \right] \).

Hence the equivalent class \( \langle X \rangle \) is well defined on \( T(\Sigma_\mu^{s,i,j}) \).
**Remark 118** In fact, Proposition 117 only claim that the representations on embedded supper spaces are consistent, which is obviously weaker than Proposition 103. Please refer to Remark 100 and the latter Remark 123 for the extension of \((X)\) to the projected subspace.

**Example 119** Consider \(\Sigma_{1/2}\). Assume \(X\) is firstly defined on \(T\left(M_{1/2}^2\right)\) as

\[
X_2(x) = F(x) = \begin{bmatrix} F^{11}(x) & F^{12}(x) \end{bmatrix},
\]

where \(x = (x_{i,j}) \in M_{1/2}^2\) and

\[
F^{1,1}(x) = \begin{bmatrix} x_{1,1} & 0 \\ 0 & x_{1,3} \end{bmatrix};
\]

\[
F^{1,2}(x) = \begin{bmatrix} x_{2,2} & 0 \\ x_{1,1} & 0 \end{bmatrix}.
\]

Then we consider the expression of \(X\) on the other cross sections:

- **Consider** \(X_1 \in T\left(M_{1/2}^1\right)\). Set

\[
X_1(y) = \begin{bmatrix} f_1(y) & f_2(y) \end{bmatrix},
\]

where \(y = (y_1, y_2) \in M_{1/2}^1\). According to (104),

\[
f_1(y) = \frac{1}{2} \left( F^{1,1}_{1,1}(bd_2(y)) + F^{1,1}_{2,2}(bd_2(y)) \right) = \frac{y_1 + y_2}{2};
\]

and

\[
f_2(y) = \frac{1}{2} \left( F^{1,2}_{1,1}(bd_2(y)) + F^{1,2}_{2,2}(bd_2(y)) \right) = \frac{1}{2} y_1.
\]

- **Consider** \(X_3 \in T\left(M_{1/2}^3\right)\). Set

\[
X_3(z) = (g_{i,j}(z)) \in M_{1/2}^3,
\]

where \(z = (z_{i,j}) \in M_{1/2}^3\). Consider the projection \(pr_3 : M_{1/2}^3 \to M_{1/2}^1\):

\[
pr_3(z) = \begin{bmatrix} z_{1,1} + z_{2,2} + z_{3,3}, & z_{1,4} + z_{2,3} + z_{3,6}, & z_{1,4} + z_{2,3} + z_{3,6} \end{bmatrix}.
\]

According to (105),

\[
X_3(z) = [G_1(z), G_2(z)] \otimes I_3,
\]

where

\[
G_1(z) = \frac{1}{2} \left( z_{1,1} + z_{2,2} + z_{3,3} + z_{1,4} + z_{2,3} + z_{3,6} \right)
\]

\[
G_2(z) = \frac{1}{2} \left( z_{1,1} + z_{2,2} + z_{3,3} \right).
\]

Similarly, for \(n = 2k - 1\) we have

\[
X_n(z) = [G_1(z), G_2(z)] \otimes I_n \in T\left(M_{1/2}^n\right),
\]

where

\[
G_1(z) = \frac{1}{2n} \left( z_{1,1} + z_{2,2} + \cdots + z_{n,n} \right)
\]

\[
G_2(z) = \frac{1}{2n} \left( z_{1,1} + z_{2,2} + \cdots + z_{n,n} \right).
\]

- **Consider** \(X_4 \in M_{1/2}^4\). Set

\[
X_4(z) = (g_{i,j}(z)) \in T\left(M_{1/2}^4\right),
\]

where \(z = (z_{i,j}) \in M_{1/2}^4\). Consider the projection \(pr_2 : M_{1/2}^4 \to M_{1/2}^2\):

\[
pr_2(z) = \begin{bmatrix} z_{1,1} + z_{2,2}, & z_{3,3} + z_{4,4}, & z_{1,1} + z_{2,2}, & z_{3,3} + z_{4,4}, & z_{1,1} + z_{2,2}, & z_{3,3} + z_{4,4} \end{bmatrix}.
\]

According to (105),

\[
X_4(z) = \begin{bmatrix} G_{1,1}(z) & G_{1,2}(z) & G_{1,3}(z) & G_{1,4}(z) \\ G_{2,1}(z) & G_{2,2}(z) & G_{2,3}(z) & G_{2,4}(z) \end{bmatrix} \otimes I_2,
\]

where

\[
G_{1,1}(z) = \frac{z_{1,1} + z_{2,2}}{2}, \quad G_{1,2}(z) = 0,
\]

\[
G_{1,3}(z) = \frac{z_{3,3} + z_{4,4}}{2}, \quad G_{1,4}(z) = 0,
\]

\[
G_{2,1}(z) = 0, \quad G_{2,2}(z) = \frac{z_{1,1} + z_{2,2}}{2}
\]

\[
G_{2,3}(z) = \frac{z_{1,1} + z_{2,2}}{2}, \quad G_{2,4}(z) = 0.
\]
Similarly, for \( n = 2k \) we have \( X_n \in T \left( \mathcal{M}_{1/2}^n \right) \) as

\[
X_n(z) = \begin{bmatrix} G_{1,1}(z) & G_{1,2}(z) & G_{1,3}(z) & G_{1,4}(z) \\ G_{2,1}(z) & G_{2,2}(z) & G_{2,3}(z) & G_{2,4}(z) \end{bmatrix} \otimes I_k,
\]

where

\[
\begin{align*}
G_{1,1}(z) &= \frac{z_{1,1} + z_{2,1} + \cdots + z_{k,1}}{k}, \\
G_{1,2}(z) &= 0, \\
G_{1,3}(z) &= \frac{z_{1,k+1,2} + z_{2,k+2,2} + \cdots + z_{k,k,2}}{k}, \\
G_{1,4}(z) &= 0, \\
G_{2,1}(z) &= 0, \\
G_{2,2}(z) &= \frac{z_{1,2,k+1,2} + z_{2,2,k+2,2} + \cdots + z_{k,k,2}}{k}, \\
G_{2,3}(z) &= \frac{z_{1,1} + z_{2,2} + \cdots + z_{k,k}}{k}, \\
G_{2,4}(z) &= 0.
\end{align*}
\]

### 4.5 Integral Curves

**Definition 120** Let \( \langle \xi \rangle \in V^r(\Sigma_\mu) \) be a vector field. Then for each \( \langle A \rangle \in \Sigma_\mu \), there exists a curve \( \langle X(t) \rangle \) such that \( \langle X(0) \rangle = \langle A \rangle \) and

\[
\left\langle \dot{X}(t) \right\rangle = \langle \xi \rangle \left( \left\langle X(t) \right\rangle \right),
\]

which is called the integral curve of \( \langle \xi \rangle \), starting from \( \langle A \rangle \).

In fact, the integral curve of \( \langle \xi \rangle \) is a bundled integral curve. Assume \( \langle A \rangle = \langle A_0 \rangle \), where \( A_0 \in \mathcal{M}_{m \times n} \) is irreducible. Here \( m, n \) may not be co-prime but \( m/n = \mu \). So we denote \( A_0 \in \mathcal{M}_\mu^n \). Then on this root leaf we denote

\[
\xi^s = \langle \xi \rangle \cap T(\mathcal{M}_{m \times n}),
\]

and the integral curve of \( \xi^s \), starting from \( A_0 \) is a standard one, which is the cross section of the bundled integral curve \( \langle X \rangle \) passing through \( A_0 \). That is, it is the solution of

\[
\begin{align*}
\dot{X}_s(t) &= \xi^s(X_s) \\
X_s(0) &= A_0.
\end{align*}
\]

We may denote the solution as

\[
X_s(t) = \Phi_t^{\xi^s}(A_0).
\]

Next, we consider the other cross sections of the integral curve, which correspond to the cross sections of \( \langle \xi \rangle \) respectively.

Recall Definition 116, we can get the cross sections of the bundled integral curve on each leaf by using the cross sections of \( \langle \xi \rangle \) on corresponding leafs. The following result is then obvious:

**Theorem 121** Assume \( s|\tau \). The corresponding cross section of the bundled integral curve is the integral curve, denoted by

\[
X_\tau(t) = \Phi_t^{\xi^s}(bd_k(A_0)), \quad k = \tau/s,
\]

satisfying

\[
\begin{align*}
\dot{X}_\tau(t) &= \xi^s(X_\tau) \\
X_\tau(0) &= bd_k(A_0),
\end{align*}
\]

where \( \xi^s \) is defined by (105).

Fig. 7 shows the integral curve and its projections on each leaf.

![Fig. 7. Integral Curve of Vector Field](image)

The following result comes from the construction directly.

**Theorem 122** Assume \( A_\alpha = \langle A \rangle \cap \mathcal{M}_\mu^n, A_\beta = \langle A \rangle \cap \mathcal{M}_\mu^n \), \( \xi^\alpha = \langle \xi \rangle \cap T(\mathcal{M}_\mu^n), \xi^\beta = \langle \xi \rangle \cap T(\mathcal{M}_\mu^n), \) and \( \alpha = k\beta, k \geq 2 \). \( \xi \) is defined firstly on \( T(\mathcal{M}_\mu^n) \), where \( s|\beta \). Then there exists a one-to-one correspondence between the two cross sections (or corresponding integral curves).
Precisely,

\[
\Phi^\xi_i(A_\alpha) = pr_k \left[ \Phi^\xi_i(pr_k(A_\alpha)) \right]
\]

\[
\Phi^\xi_i(A_\beta) = bd_k \left[ \Phi^\xi_i(bd_k(A_\beta)) \right].
\]  

(114)

Particularly, assume the vector field is a linear vector field and \( X_0 \in M^1_1 \), then we have

\[
\Phi^\xi_{i,j,k} = (X_{i,j} | X_0)_{\delta}, \quad i = 1, \ldots, m; j = 1, \ldots, n.
\]  

(115)

Moreover, on \( M_{s\times n} \), the cross section of the integral curve can be expressed by modifying (115) as

\[
\Phi^\xi_{i,j} = \Phi^\xi_{i,j} \otimes I_s, \quad i = 1, \ldots, m; j = 1, \ldots, n.
\]  

(116)

Remark 123  • Note that if \( \xi \) is firstly defined on \( T (M^1_\mu \otimes I_k) \), \( t < s \) and \( k = s/t \in \mathbb{N} \). Then \( \xi^{(t)} \) is obtained through the following two steps: (i) Restrict \( \xi^s \) on \( M^1_\mu \otimes I_k \) as

\[
\xi^s(y = bd(x)) = \xi^s|_{M^1_\mu \otimes I_k}.
\]

(ii) Project \( \xi^s(y = bd(x)) \) onto the tangent space of the subspace \( T (M^1_\mu \otimes I_k) \). Because of (ii), \( \xi^{(t)}(A) \) does not correspond to \( \xi^s(A \otimes I_k) \). Hence, the relationship demonstrated in Theorems 121 and 122 are not available for the integral curves of \( \xi^{(t)} \) and \( \xi^s \) respectively.

* The projections of \( \xi \) are useful in some other problems. For instance, assume \( \xi \) is a constant vector field. Using notations in Theorem 122, we set

\[
D := \text{Span} \left\{ D^{i,j} | I = 1, \ldots, m; J = 1, \ldots, n \right\},
\]

where \( D^{i,j} \) is defined in (67). Then the projection of \( \xi^{i,j} \) on \( D \), denoted by \( \xi^{i,j} \), is well defined. Moreover, (114) becomes

\[
\Phi^\xi_i(A_\alpha) = pr_k \left[ \Phi^\xi_i(pr_k(A_\alpha)) \right]
\]

\[
\Phi^\xi_i(A_\beta) = bd_k \left[ \Phi^\xi_i(bd_k(A_\beta)) \right].
\]  

(117)

Example 124  Recall Example 104. Since it is a linear vector field, it is easy to calculate that the cross section on

\[
M_{2\times 4} \text{ can be expressed as in (115), where } X_0 \in M_{2\times 4}
\]

and

\[
X^2_{1,1} = \begin{bmatrix}
e^t & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}; \quad X^2_{1,2} = \begin{bmatrix} 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix};
\]

\[
X^2_{1,3} = \begin{bmatrix} 0 & 0 & 0 \\
0 & \frac{e^t + e^{-t}}{2} & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}; \quad X^2_{1,4} = \begin{bmatrix} 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix};
\]

\[
X^2_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}; \quad X^2_{2,2} = \begin{bmatrix} 0 & 0 & 0 \\
0 & \frac{e^t + e^{-t}}{2} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix};
\]

\[
X^2_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}; \quad X^2_{2,4} = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Consider the cross section on \( M_{2k \times 4k} \) with \( X^k_0 = X_0 \otimes I_k \in (X_0) \). Then

\[
X^2_{i,j} = X^2_{i,j} \otimes I_k, \quad i = 1, 2; \ j = 1, 2, 3, 4.
\]

An integral manifold of an involutive distribution on \( T(\Sigma_\mu) \) can be defined and calculated in a similar way.

4.6 Forms

Definition 125  Let \( M \) be a bundled manifold, \( \omega : M \to T^*(M) \) is called a \( C^r \) co-vector field (or one form), if for each simple coordinate chart \( U \omega|_U \) is a \( C^r \) co-vector field. The set of \( C^r \) co-vector fields on \( M \) is denoted by \( V^*(M) \).

We express a co-vector field in matrix form. That is, let \( \omega \in V^*(M_{m \times n}) \). Then

\[
\omega = \sum_{i=1}^m \sum_{j=1}^n \omega_{i,j}(x) dx_{i,j} := [\omega_{i,j}(x)] \in M_{m \times n}.
\]

Similar to the construction of vector fields, the co-vector fields on \( \Sigma_\mu \) can be established as follows:

Definition 126  Assume \( \langle \omega \rangle \) is firstly defined on \( T^*(M^1_\mu) \) as \( \omega_\alpha(x) \), where \( M^1_\mu \) is the root leaf of \( \langle \omega \rangle \). Then we extend it to other leafs as:

* Step 1. Let \( Q = \{ t \in \mathbb{N} | t|s \} \).

\[
\omega_t(y) := k \left[ (bd_k)^\ast(\omega_\alpha)(x = bd_k(y)) \right],
\]

(118)
where $k = \frac{p}{q} \in \mathbb{N}$.

- Step 2. Assume $\gcd(\ell, s) = t$. If $\ell = t$, $\omega_t = \omega_t$ has already been defined in Step 1. So we assume $\ell = kt$, $k \geq 2$. Then

$$
\omega_t(y) := \frac{1}{k} \left[ (pr_k) \otimes (\omega_t)(x = pr_k(y)) \otimes I_k \right]. \quad (119)
$$

Similar to the calculation of vector fields, to calculate (118) we first set

$$
x = y \otimes I_k \quad (120)
$$

to get $\omega_t(x(y)) := \omega_t(y)$. Then we split $\omega_s(y)$ into $tp \times tq$ blocks as $\omega_s = [\omega_{s;i,j}]$, where each $\omega_{s;i,j} \in \mathcal{M}_{k \times k}$. Then $\omega_t = [\omega_{t;i,j}] \in T^s_x(\mathcal{M}_{tp \times tq})$, where the entries are

$$
\omega_{i,j} = \text{Tr} \left( [\omega_{s;i,j}] \right), \quad i = 1, \ldots, tp; j = 1, \ldots, tq. \quad (121)
$$

To calculate (119) we split $y$ into $tp \times tq$ blocks as $y = [y^{i,j}]$, where each $y^{i,j} \in \mathcal{M}_{k \times k}$. Then $\omega_t(y)$ is obtained by replacing $x$ by $x = [x_{i,j}] \in \mathcal{M}_{tp \times tq}$ as

$$
x_{i,j} = \text{Tr} \left( [y^{i,j}] \right). \quad (122)
$$

It follows that

$$
\omega_t(y) = \frac{1}{k} \left( \omega_t(y) \otimes I_k \right). \quad (123)
$$

Then it is easy to verify the following:

**Proposition 127** The co-vector field $\langle \omega \rangle$ defined in Definition 126 is consistent with the equivalence $\sim [\mathcal{M}_s^{[k,s]}]$. Hence it is well defined on $\Sigma_{\mu}^{[k,s]}$.

**Definition 128** Let $\langle \omega \rangle \in V^*(\Sigma_{\mu})$ and $\langle X \rangle \in V(\Sigma_{\mu})$. Then the action of $\langle \omega \rangle$ on $\langle X \rangle$ is defined as

$$
\langle \omega \rangle \left( \langle X \rangle \right) := \langle \langle \omega \rangle \mid \langle X \rangle \rangle_W. \quad (124)
$$

Similar to vector field case, if the co-vector field is firstly defined on $\mathcal{M}_s^a$, then we can assume $\langle \omega \rangle$ is only defined on $\mathcal{M}_s^a$ satisfying

$$
\mathcal{M}_s^a \sqsupset \mathcal{M}_s^\beta.
$$

**4.7 Tensor Fields**

The set of tensor fields on $\Sigma_{\mu}$ of covariant order $\alpha$ and contravariant order $\beta$ is denoted by $T_{\beta}^{\alpha}(\Sigma_{\mu})$. To avoid complexity, we consider only the covariant tensor, $\langle t \rangle \in T^{\alpha}(\Sigma_{\mu})$.

**Definition 129** A covariant tensor field $\langle t \rangle \in T^{\alpha}(\Sigma_{\mu})$ is a multi-linear mapping

$$
\langle t \rangle : V^\alpha(\Sigma_{\mu}) \times \cdots \times V^\alpha(\Sigma_{\mu}) \to C(\Sigma_{\mu}).
$$

Assume $\langle t \rangle$ is defined on root leaf at $x \in \mathcal{M}_s^a$ as $t_s \in T(\mathcal{M}_s^a)$, $p$, $q$ are co-prime and $p/q = \mu$. The calculation is performed as follows: Construct the structure matrix of $t_s$ as

$$
M_s(x) :=
\begin{bmatrix}
  t_{1,1} \cdots t_{1,1}
  & t_{1,2} \cdots t_{1,2}
  & \cdots
  & t_{1,sp} \cdots t_{1,sp}

  t_{2,1} \cdots t_{2,1}
  & t_{2,2} \cdots t_{2,2}
  & \cdots
  & t_{2,sp} \cdots t_{2,sp}

  \vdots

  t_{sp,1} \cdots t_{sp,1}
  & t_{sp,2} \cdots t_{sp,2}
  & \cdots
  & t_{sp,sp}
\end{bmatrix}
\in \mathcal{M}_{(sp)^a \times (sq)^a},
$$

where

$$
t_{ij} = t_0 \left( \frac{\partial}{\partial x_{i,j}^1} \cdots \frac{\partial}{\partial x_{i,j}^d} \right) \bigg| x_{id} = 1, \ldots, sp; j_d = 1, \ldots, sq; d = 1, \ldots, \alpha;
\quad x \in \mathcal{M}_{sp \times sq}.
$$

Consider $\langle X^k \rangle \in V(\Sigma_{\mu})$ with its irreducible element $X_s^k \in V(\mathcal{M}_s^a)$, where $\mathcal{M}_s^a$ is the root leaf of $\langle X^k \rangle$. $X_s^k$ is expressed in matrix form as

$$
X_s^k := \sum_{i=1}^{sp} \sum_{j=1}^{sq} \left[ \frac{\partial}{\partial x_{i,j}^k} \right] = \left[ \partial_{x_{i,j}^k} \right], \quad k = 1, \ldots, \alpha.
$$

Then we have
Algorithm 1

Assume \( (t) \) is firstly defined on \( T^\alpha (\mathcal{M}_{\mu}^s) \) as \( t_s(x) \). Then we extend it to other leaves as:

1. **Step 1.** Denote \( Q := \{ r \mid r | s \} \). Let \( \tau \in Q \) and \( k = \frac{\tau}{r} \in N \). Then

\[
t_\tau(y) = (bd_k)^* (t_s(bd_k(y)) \tag{127}
\]

can be calculated by constructing \( M_\tau(y) \) (\( y \in T^\alpha (\mathcal{M}_{\tau p \times \tau q}) \)) as follows: First, split \( M_s(bd(y)) \) into \((\tau p)^\alpha \times (\tau q)^\alpha\) blocks

\[
M_s(bd_\tau(y)) = \begin{bmatrix}
T_{1,\ldots,1}^{1,\ldots,1}(y) & T_{1,\ldots,2}^{1,\ldots,1}(y) & \cdots & T_{q,\ldots,q}^{1,\ldots,1}(y) \\
T_{1,\ldots,1}^{1,\ldots,2}(y) & T_{1,\ldots,2}^{1,\ldots,2}(y) & \cdots & T_{q,\ldots,q}^{1,\ldots,2}(y) \\
& \cdots & \ddots & \cdots \\
T_{1,\ldots,1}^{p,\ldots,p}(y) & T_{1,\ldots,2}^{p,\ldots,p}(y) & \cdots & T_{q,\ldots,q}^{p,\ldots,p}(y)
\end{bmatrix},
\]

where each block \( T_{j_1,\ldots,j_\alpha}^{i_1,\ldots,i_\alpha} \in \mathcal{M}_{k \times k} \). Then we set

\[
M_\tau(y) := k^\alpha [\xi_{j_1,\ldots,j_\alpha}^{i_1,\ldots,i_\alpha}(y)] \in \mathcal{M}_{(\tau p)^\alpha \times (\tau q)^\alpha}, \tag{128}
\]

where

\[
\xi_{j_1,\ldots,j_\alpha}^{i_1,\ldots,i_\alpha}(y) = \text{Tr} \left( T_{j_1,\ldots,j_\alpha}^{i_1,\ldots,i_\alpha} \right).
\]

2. **Step 2.** Assume \( \gcd(\ell, s) = \tau \). If \( \ell = \tau \), \( t_\ell = t_\tau \) has already been defined in Step 1. So we assume \( \ell = k\tau \), where \( k \geq 2 \). Then

\[
t_\ell(z) = (pr_k)^* (t_\tau(pr_k(y)) \tag{129}
\]

can be calculated by constructing

\[
M_\ell(z) := \frac{1}{k^\alpha} [M_\tau(y = pr_k(z)) \otimes I_{k^\alpha}], \quad z \in \mathcal{M}_{\ell p \times \ell q}. \tag{130}
\]

Example 131

Consider a covariant tensor field \( (t) \in T^2 (\Sigma_\mu) \), where \( \mu = \frac{\tau}{r} \). Moreover, \( (t) \) is firstly defined at \( x = (x_{i,j}) \in \mathcal{M}_{2 \times 3} \) with its structure matrix as

\[
M_1(x) = \begin{bmatrix}
1 & x_{12} & 0 & 0 & 0 & 0 & 0 & x_{22} \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & x_{22} & 0 & 0 \\
0 & 0 & x_{21} & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}. \tag{131}
\]

(1) Let \( X(x), Y(x) \in V (\Sigma_\mu) \) be defined at \( x \in \mathcal{M}_{2 \times 3} \) as

\[
X(x) = \begin{bmatrix}
x_{13} & 0 & 0 \\
0 & 0 & x_{21}
\end{bmatrix}, \quad Y(x) = \begin{bmatrix}
1 & 0 & 1 \\
0 & x_{11} & 0
\end{bmatrix}.
\]

Evaluate \( (t) (X, Y) \).

First,

\[
t_1(X, \cdot) = (M_1(x) \mid X) \begin{bmatrix}
x_{13} + x_{21} & x_{12}x_{13} - x_{21} & 0 \\
0 & x_{13} & x_{21}x_{23}
\end{bmatrix}.
\]

Then

\[
t_1(X, Y) = x_{13} (1 + x_{11}^2) + x_{21}.
\]

(2) Expressing \( (t) \) on leaf \( \mathcal{M}_{4 \times 6} \): Note that

\[
x = pr_2(y) = \begin{bmatrix}
w_1 + w_2 & \frac{w_1 + w_2}{2} & \frac{w_1 + w_2}{2} & \frac{w_1 + w_2}{2} \\
\frac{w_1 + w_2}{2} & \frac{w_1 + w_2}{2} & w_3 + w_4 & \frac{w_3 + w_4}{2} \\
\frac{w_1 + w_2}{2} & w_3 + w_4 & \frac{w_3 + w_4}{2} & \frac{w_3 + w_4}{2}
\end{bmatrix},
\]

where \( y = [y_{i,j}] \in \mathcal{M}_{4 \times 6} \).

Using (130), we have

\[
M_2(y) = \frac{1}{2^7} \begin{bmatrix}
t_{11} & t_{12} & t_{13} \\
t_{21} & t_{22} & t_{23}
\end{bmatrix},
\]
where

\[
\begin{align*}
t^{11} &= \begin{bmatrix}
\frac{y_{13} + y_{24}}{2} & 0 \\
0 & 1 & 0
\end{bmatrix} \otimes I_2; \\
t^{12} &= \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix} \otimes I_2; \\
t^{13} &= \begin{bmatrix}
0 & 0 & \frac{y_{13} + y_{24}}{2} \\
0 & 0 & 0
\end{bmatrix} \otimes I_2; \\
t^{21} &= \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix} \otimes I_2; \\
t^{22} &= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \frac{y_{13} + y_{24}}{2}
\end{bmatrix} \otimes I_2; \\
t^{23} &= \begin{bmatrix}
1 & -1 & 0 \\
0 & 0 & \frac{y_{13} + y_{24}}{2}
\end{bmatrix} \otimes I_2.
\end{align*}
\]

5 Lie Algebra on Square Equivalence Space

5.1 Ring Structure on \( \Sigma \)

Consider the vector space of the equivalent classes of square matrices \( \Sigma := \Sigma_1 \). Since \( \Sigma \) is closed under the STP, more algebraic structures may be posed on it. First, polynomials; Second, Lie algebra structure.

Then we extend some fundamental concepts of matrices to their equivalent classes.

Definition 132 (1) \( \langle A \rangle \) is nonsingular (symmetric, skew symmetric, positive/negative (semi)-definite, upper/lower (strictly) triangular, diagonal, etc.) if its irreducible element \( A_1 \) is (equivalently, every \( A_i \) is).

(2) \( \langle A \rangle \) and \( \langle B \rangle \) are similar, denoted by \( \langle A \rangle \sim \langle B \rangle \), if there exists a nonsingular \( (P) \) such that

\[
\langle P^{-1} \rangle \langle A \rangle \langle P \rangle = \langle B \rangle .
\] (132)

(3) \( \langle A \rangle \) and \( \langle B \rangle \) are congruent, denoted by \( \langle A \rangle \cong \langle B \rangle \), if there exists a nonsingular \( (P) \) such that

\[
\langle P^T \rangle \langle A \rangle \langle P \rangle = \langle B \rangle .
\] (133)

(4) \( \langle J \rangle \) is called the Jordan normal form of \( \langle A \rangle \), if the irreducible element \( J_1 \in \langle J \rangle \) is the Jordan normal form of \( A_1 \).

Definition 133 [29] A set \( R \) with two operators \(+, \times\) is a ring. If the followings hold:

(1) \( (R, +) \) is an Abelian group;
(2) \( (R, \times) \) is a monoid;
(3) (Distributive Rule)

\[
(a + b) \times c = a \times c + b \times c,
\]

\[
c \times (a + b) = c \times a + c \times b,
\]

\( a, b, c \in R \).

Observing \( \mathcal{M}_1 \), which consists of all square matrices, both \( \vdash \) (including \( \vdash \)) and \( \times \) are well defined. Unfortunately, \( (\mathcal{M}_1, \vdash) \) is not a group because there is no unit element. Since both \( \vdash \) and \( \times \) are consistent with the equivalence \( \sim \), we consider \( \Sigma := \mathcal{M}_1 / \sim \). Then it is easy to verify the following:

Proposition 134 \( (\Sigma, \vdash, \times) \) is a ring.

Consider a polynomial on a ring \( R, p : R \to R \), as

\[
p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,
\]

\[
a_i \in R, \ i = 0, 1, \cdots, n.
\] (134)

It is obvious that this is well defined. Set \( R = (\Sigma, \vdash, \times) \), then \( p(x) \) defined in (134) is also well defined on \( R \). Particularly, the coefficients \( a_i \) can be chosen as \( \langle a_i \rangle \) for \( a_i \in F, i = 1, \cdots, n \). Then \( p(x) \) is as a “standard” polynomial. Unless elsewhere is stated, in this paper only such standard polynomials are considered.

For any \( \langle A \rangle \in \Sigma \), the polynomial \( p(\langle A \rangle) \) is well defined, and it is clear that

\[
p(\langle A \rangle) = \langle p(A) \rangle , \ \text{for any } A \in \langle A \rangle .
\] (135)

Using Taylor expansion, we can consider general matrix functions. For instance, we have the following result:

Theorem 135 Let \( f(x) \) be an analytic function. Then \( f(\langle A \rangle) \) is well defined provided \( f(A) \) is well defined. Moreover,

\[
f(\langle A \rangle) = \langle f(A) \rangle .
\] (136)

In fact, the above result can be extended to multi-variable case.
\textbf{Definition 136} Let $F(x_1, \cdots, x_k)$ be a $k$-variable analytic function. Then $F(\langle A \rangle_1, \cdots, \langle A \rangle_k)$ is a well posed expression, where $\langle A \rangle_i \in \Sigma$, $i = 1, \cdots, k$. Assume $A_i \in \langle A \rangle_i$, $i = 1, \cdots, k$, then $F(A_1, \cdots, A_k)$ is a realization of $F(\langle A \rangle_1, \cdots, \langle A \rangle_k)$. Particularly, if $A_i \in \mathcal{M}_{r \times r}$, $\forall i$, we call $F(A_1, \cdots, A_k)$ a realization on $r$-th leaf.

Similar to (136), we also have

$$F(\langle A \rangle_1, \cdots, \langle A \rangle_k) = F(A_1, \cdots, A_k),$$

provided $F(A_1, \cdots, A_k)$ is well defined.

In the following we consider some fundamental matrix functions for $\Sigma$. We refer to [13] for the definitions and basic properties of some fundamental matrix functions. Using these acknowledges, the following result is obvious:

\textbf{Theorem 137} Let $\langle A \rangle$, $\langle B \rangle$ in $\Sigma$ (i.e., $A, B$ are square matrices). Then the following hold:

\begin{enumerate}
\item $e^{\langle A \rangle} \otimes e^{\langle B \rangle} = e^{\langle A \otimes B \rangle}.$
\item If $\langle A \rangle$ is real skew symmetric, then $e^{\langle A \rangle}$ is orthogonal.
\item Assume $B$ is invertible, we denote $\langle B \rangle^{-1} = \langle B^{-1} \rangle$. Then
\begin{equation}
 e^{\langle B \rangle^{-1}} \langle A \rangle \langle B \rangle = \langle B \rangle^{-1} \otimes e^{\langle A \rangle} \otimes \langle B \rangle.
\end{equation}
\item Let $A, B$ be closed enough to identity so that $\log(\langle A \rangle) \otimes \langle B \rangle$ and $\log(\langle B \rangle)$ are defined. Then
\begin{equation}
 \log(\langle A \rangle \otimes \langle B \rangle) = \log(\langle A \rangle) + \log(\langle B \rangle).
\end{equation}
\end{enumerate}

Many known results for matrix functions can be extended to $\Sigma$. For instance, it is easy to prove the following Euler formula:

\textbf{Proposition 138} Consider $\mathbb{F} = \mathbb{R}$ and let $\langle A \rangle \in \Sigma$. Then the Euler formula holds. That is,

$$e^{\langle A \rangle} = \cos(\langle A \rangle) + i \sin(\langle A \rangle).$$

Recall the modification of trace and determinant in Definitions 20 and 22. The following proposition shows that for the modifications the relationship between trace($A$) and $\det(A)$ [13] remain available.

\textbf{Proposition 139} Assume $\mathbb{F} = \mathbb{R}$ (or $\mathbb{F} = \mathbb{C}$), $\langle A \rangle \in \Sigma$, then

$$e^{\text{Tr}(\langle A \rangle)} = \det \left( e^{\langle A \rangle} \right).$$

\textbf{Proof.} Let $A_0 \in \langle A \rangle$ and $A_0 \in \mathcal{M}_{n \times n}$. Then

$$e^{\text{Tr}(\langle A \rangle)} = \left( e^{\frac{1}{2} \text{trace}(A_0)} \right) = \left( e^{\text{trace}(A_0)} \right)^{\frac{1}{2}} = \left( \text{det}(e^{A_0}) \right) = \text{det} \left( e^{\langle A \rangle} \right).$$

\hfill \qed

Next, we consider the characteristic polynomial of an equivalent class, they comes from standard matrix theory [23].

\textbf{Definition 140} Let $\langle A \rangle \in \Sigma$. $A_1 \in \langle A \rangle$ is its irreducible element. Then

$$p_{\langle A \rangle}(\lambda) := \det(\lambda \vdash A_1)$$

is called the characteristic polynomial of $\langle A \rangle$.

The following result is an immediate consequence of the definition.

\textbf{Theorem 141} (Cayley-Hamilton) Let $p_{\langle A \rangle}$ be the characteristic polynomial of $\langle A \rangle$. Then

$$p_{\langle A \rangle}(\langle A \rangle) = 0.$$

\textbf{Remark 142} (1) If we choose $A_k = A_1 \otimes I_k$ and calculate the characteristic polynomial of $p_{A_k}$, then $p_{A_k}(\lambda) = (p_{A_1}(\lambda))^k$. So $p_{\langle A \rangle}(\langle A \rangle) = 0$ is equivalent to $p_{A_1}(\langle A \rangle) = 0$.

(2) Choosing any $A_i \in \langle A \rangle$, the corresponding minimal polynomials $q_{A_i}(\lambda)$ are the same. So we have unique minimal polynomial as $q_{\langle A \rangle}(\lambda) = q_{A_i}(\lambda)$.

\subsection{Bundled Lie Algebra}

Consider the vector space of the equivalent classes of square matrices $\Sigma := \Sigma_1$, this section gives a Lie algebraic structure to it.
Definition 143 [21] A Lie algebra is a vector space $g$ over some field $\mathbb{F}$ with a binary operation $[-,-] : g \times g \to g$, satisfying

1. (bi-linearity)
$$[\alpha A + \beta B, C] = \alpha [A, C] + \beta [B, C];$$
$$[C, \alpha A + \beta B] = \alpha [C, A] + \beta [C, B];$$  \hspace{1cm} (145)

where $\alpha, \beta \in \mathbb{F}$.

2. (skew-symmetry)
$$[A, B] = -[B, A];$$  \hspace{1cm} (146)

3. (Jacobi Identity)
$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \hspace{1cm} \forall A, B, C \in g.$$  \hspace{1cm} (147)

Definition 144 Let $(E, Pr, B)$ be a discrete bundle. If

1. $(B, \oplus, \otimes)$ is a Lie algebra;
2. $(E_i, +, \times)$ is a Lie algebra, $i = 1, 2, \cdots$;
3. The restriction $Pr|_{E_i} : (E_i, +, \times) \to (B, \oplus, \otimes)$ is a Lie algebra homomorphism, $i = 1, 2, \cdots$,

then $(B, \oplus, \otimes)$ is called a bundled Lie algebra.

(We refer to Definition 62 and Remark 65 for the concept of discrete bundle.)

On vector space $\Sigma$ we define an operation $[-,-] : \Sigma \times \Sigma \to \Sigma$ as

$$[(A), (B)] := \langle A \rangle \times \langle B \rangle + \langle B \rangle \times \langle A \rangle.$$  \hspace{1cm} (148)

Then we have the following Lie algebra:

Theorem 145 The vector space $\Sigma$ with Lie bracket $[-,-]$ defined in (148), is a bundled Lie algebra, denoted by $gl(\mathbb{F})$.  \hspace{1cm} $\Box$

Proof. Let $M = \bigcup_{i=1}^\infty M^i$, where $M^i = M^i_1$. Then it is clear that $(M, Pr, \Sigma)$ is a discrete bundle.

Next, we prove $(\Sigma, \cdot, [-,-])$ is a Lie algebra. Equations (145) and (146) are obvious. We prove (147) only.

Assume $A_1 \in \langle A \rangle$, $B_1 \in \langle B \rangle$ and $C_1 \in \langle C \rangle$ are irreducible, and $A_1 \in M_{n \times m}$, $B_1 \in M_{m \times n}$, and $C_1 \in M_{r \times r}$. Let $t = \text{lcm}(n, m, r)$. Then it is easy to verify that

$$[(A), [B], (C)] =$$
$$\langle [A_1 \otimes I_t/m], [(B_1 \otimes I_t/n), (C_1 \otimes I_t/r)] \rangle.$$  \hspace{1cm} (149)

Similarly, we have

$$[(B), [C], (A)] =$$
$$\langle [(B_1 \otimes I_t/n), (C_1 \otimes I_t/r), (A_1 \otimes I_t/m)] \rangle.$$  \hspace{1cm} (150)

$$[(C), [A], (B)] =$$
$$\langle [(C_1 \otimes I_t/r), ([A_1 \otimes I_t/m], (B_1 \otimes I_t/n)] \rangle.$$  \hspace{1cm} (151)

Since (147) is true for any $A, B, C \in gl(t, \mathbb{F})$, it is true for $A = A_1 \otimes I_t/m$, $B = B_1 \otimes I_t/n$, and $C = C_1 \otimes I_t/r$. Using this fact and equations (149)-(151), we have

$$\langle [A], [(B), (C)] \rangle + \langle [B], [(C), (A)] \rangle + \langle [C], [(A), (B)] \rangle =$$
$$\langle ([A_1 \otimes I_t/m], [(B_1 \otimes I_t/n), (C_1 \otimes I_t/r)] \rangle +$$
$$\langle [(B_1 \otimes I_t/n), (C_1 \otimes I_t/r), (A_1 \otimes I_t/m)] \rangle +$$
$$\langle [(C_1 \otimes I_t/r), ([A_1 \otimes I_t/m], (B_1 \otimes I_t/n)] \rangle \rangle = 0 = 0.$$  \hspace{1cm} (152)

Let the Lie algebraic structure on $M^n$ be $gl(n, \mathbb{F})$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$). It follows from the consistence of $\parallel$ and $\cdot$ with the equivalence that $Pr : gl(n, \mathbb{F}) \to gl(\mathbb{F})$ is a Lie algebra homomorphism.  \hspace{1cm} $\Box$

5.3 Bundled Lie Sub-algebra

This section considers some useful Lie sub-algebras of Lie algebra $gl(\mathbb{F})$. Assume $g$ is a Lie algebra, $h \subset g$ is a vector subspace. Then $h$ is called a Lie sub-algebra if and only if $[h, h] \subset h$.

Definition 146 Let $(E, Pr, B)$ be a bundled Lie algebra. If

1. $(H, \oplus, \otimes)$ is a Lie sub-algebra of $(B, \oplus, \otimes)$;
2. $(F_i, +, \times)$ is a Lie sub-algebra of $(E_i, +, \times)$, $i = 1, 2, \cdots$;
3. The restriction $Pr|_{E_i} : (F_i, +, \times) \to (H, \oplus, \otimes)$ is a Lie algebra homomorphism, $i = 1, 2, \cdots$,  \hspace{1cm} (153)

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then \((F, Pr, H)\) is called a bundled Lie sub-algebra of \((E, Pr, B)\).

It is well known that there are some useful Lie sub-algebras of Lie algebra \(gl(n, F)\). When \(gl(n, F)\), \(\forall n\), are generalized to the bundled Lie algebra \(gl(F)\) (over \(\Sigma\)), the corresponding Lie sub-algebras are investigated one-by-one in this section.

- **Bundled orthogonal Lie sub-algebra**

\[\text{Definition 147} \quad \langle A \rangle \in \Sigma \text{ is said to be symmetric (skew symmetric)} \text{ if } A^T = A \text{ (} A^T = -A \text{), } \forall A \in \langle A \rangle.\]

The symmetric (skew symmetric) \(\langle A \rangle\) is well defined because if \(A \sim B\) and \(A^T = A\) (or \(A^T = -A\)), then so is \(B\). It is also easy to verify the following:

\[\text{Proposition 148} \quad \text{Assume } \langle A \rangle \text{ and } \langle B \rangle \text{ are skew symmetric, then so is } [[\langle A \rangle, \langle B \rangle]].\]

We, therefore, can define the following bundled Lie sub-algebra.

\[\text{Definition 149} \quad o(F) := \left\{ \langle A \rangle \in gl(F) \mid \langle A \rangle^T = -\langle A \rangle \right\}\]

is called the bundled orthogonal algebra.

- **Bundled special linear algebra**

\[\text{Definition 150} \quad sl(F) := \{ \langle A \rangle \in gl(F) \mid \text{Tr}(\langle A \rangle) = 0 \}\]

is called the bundled special linear algebra.

Similar to the case of orthogonal algebra, it is easy to verify that \((F, o)\) is a Lie sub-algebra of \(gl(F)\).

- **Bundled upper triangular algebra**

\[\text{Definition 151} \quad t(F) := \{ \langle A \rangle \in gl(F) \mid \langle A \rangle \text{ is upper triangular} \}\]

is called the bundled upper triangular algebra.

Similarly, we can define bundled lower triangular algebras.

- **Bundled strictly upper triangular algebra**

\[\text{Definition 152} \quad n(F) := \{ \langle A \rangle \in gl(F) \mid \langle A \rangle \text{ is strictly upper triangular} \}\]

is called the bundled strictly upper triangular algebra.

- **Bundled diagonal algebra**

\[\text{Definition 153} \quad d(F) := \{ \langle A \rangle \in gl(F) \mid \langle A \rangle \text{ is diagonal} \}\]

is called the bundled diagonal algebra.

- **Bundled symplectic algebra**

\[\text{Definition 154} \quad sp(F) := \{ \langle A \rangle \in gl(F) \mid \langle A \rangle \text{ satisfies (152)} \}
\]

\[\text{and } A_1 \in \mathcal{M}_{2n \times 2n}, \ n \in N \},\]

is called the bundled symplectic algebra.

\[\langle J \rangle \times \langle A \rangle \rtimes \langle A \rangle^T \times \langle J \rangle = 0, \quad (152)\]

where

\[J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.\]

\[\text{Definition 155} \quad \text{A Lie sub-algebra } J \subset G \text{ is called an idea, if}\]

\[[g, J] \in J. \quad (153)\]

**Example 156** \(sl(F)\) is an idea of \(gl(F)\). Because

\[\text{Tr}[g, h] = \text{Tr}(g \rtimes h \rtimes h \rtimes g) = 0, \quad \forall g \in gl(F), \forall h \in (F).\]

**Hence**, \([gl(F), F] \subset (F)\).

Many properties of the sub-algebra of \(gl(n, F)\) can be extended to the sub-algebra of \(gl(F)\). The following is an example.

**Proposition 157**

\[gl(F) = sl(F) + r \{1\}, \quad r \in F. \quad (154)\]

**Proof.** It is obvious that

\[\langle A \rangle = (\langle A \rangle \rtimes \text{Tr}(\langle A \rangle)) \rtimes \text{Tr}(\langle A \rangle) \{1\}.\]

Since \(\text{Tr}(\langle A \rangle) = 0\), which means

\[\langle A \rangle \rtimes \text{Tr}(\langle A \rangle) \in \langle F \rangle.\]

The conclusion follows.
Example 158  (1) Denote by

\[ gl(\langle A \rangle, F) := \{ \langle X \rangle \mid \langle X \rangle \langle A \rangle \langle A \rangle \langle X \rangle = 0 \} \tag{155} \]

It is obvious that \( gl(\langle A \rangle, F) \) is a vector sub-space of \( gl(F) \). Let \( \langle X \rangle, \langle Y \rangle \in gl(\langle A \rangle, F) \). Then

\[
\begin{align*}
\langle [X, Y] \rangle (A) &+ \langle [X, Y] \rangle (X) (A) \\
& + \langle \langle X^T \rangle \langle X \rangle (A) \\
& + \langle \langle Y \rangle (A) \rangle \langle X \rangle (A) \rangle = 0.
\end{align*}
\]

Hence, \( gl(\langle A \rangle, F) \subset gl(F) \) is a bundled Lie sub-algebra.

(2) Assume \( \langle A \rangle \) and \( \langle B \rangle \) are congruent. That is, there exists a non-singular \( (P) \) such that \( \langle A \rangle = \langle P^T \rangle \langle B \rangle (P) \). Then it is easy to verify that \( \pi : gl(\langle A \rangle, F) \rightarrow gl(\langle B \rangle, F) \) is an isomorphism, where

\[ \pi(\langle X \rangle) = \langle P^{-T} \rangle \langle X \rangle (P) . \]

5.4 Further Properties of \( gl(F) \)

Definition 159 Let \( p \in \mathbb{N} \).

(1) The \( p \)-truncated equivalent class is defined as

\[ (A)^{[\cdot,p]} := \{ A_i \in \langle A \rangle \mid i|p \} . \tag{156} \]

(2) The \( p \)-truncated square matrices is defined as

\[ M^{[\cdot,p]} := \{ A \in M_{n \times n} \mid n|p \} . \tag{157} \]

(3) The \( p \)-truncated equivalence space is defined as

\[ \Sigma^{[\cdot,p]} := M^{[\cdot,p]} / \sim . \tag{158} \]

Remark 160 (1) If \( A_1 \in \langle A \rangle \) is irreducible, \( A_1 \in M_{n \times n} \) and \( n \) is not a divisor of \( p \), then \( (A)^{[\cdot,p]} = \emptyset \).

(2) It is obvious that \( (M^{[\cdot,p]}, Pr, \Sigma^{[\cdot,p]}) \) is a discrete bundle, which is a sub-bundle of \( (M, Pr, \Sigma) \). That is, the following (159) is commutative:

\[ \begin{array}{ccc}
M^{[\cdot,k]} & \xrightarrow{\pi} & M \\
\downarrow Pr & & \downarrow Pr \\
\Sigma^{[\cdot,k]} & \xrightarrow{\pi'} & \Sigma
\end{array} \tag{159} \]

Where \( \pi \) and \( \pi' \) are including mappings.

(3) It is ready to verify that \( \Sigma^{[\cdot,p]} \) is closed with \( \langle \cdot, \cdot \rangle \), defined in (148), hence the including mapping \( \pi' : \Sigma^{[\cdot,p]} \rightarrow \Sigma \) is a Lie algebra homomorphism. Identifying \( \langle A \rangle^{[\cdot,p]} \) with its image \( \langle A \rangle = \pi' \left( \langle A \rangle^{[\cdot,p]} \right) \), then \( \Sigma^{[\cdot,p]} \) becomes a Lie sub-algebra of \( gl(F) \). We denote this as

\[ gl^{[\cdot,p]}(F) := \left( \Sigma^{[\cdot,p]}, \langle \cdot, \cdot \rangle \right) . \tag{160} \]

and call \( gl^{[\cdot,p]}(F) \) the \( p \)-truncated Lie sub-algebra of \( gl(F) \).

(4) Let \( \Gamma \subset gl(F) \) be a Lie sub-algebra. Then its \( p \)-truncated sub-algebra \( \Gamma^{[\cdot,p]} \) is defined in a similar way as for \( gl^{[\cdot,p]} \). Alternatively, it can be considered as

\[ \Gamma^{[\cdot,p]} := \Gamma \cap \Sigma^{[\cdot,p]} . \tag{161} \]

Definition 161 [25] Let \( g \) be a Lie algebra.

(1) Denote the derived serious as \( D(g) := [g, g] \), and

\[ D^{(k+1)}(g) := D \left( D^k(g) \right) , \quad k = 1, 2, \cdots . \]

Then \( g \) is solvable, if there exists an \( n \in \mathbb{N} \) such that \( D^{(n)}(g) = \{ 0 \} \).

(2) Denote the descending central series as \( C(g) := [g, g] \), and

\[ C^{(k+1)}(g) := [g, C^k(g)] , \quad k = 1, 2, \cdots . \]

The \( g \) is nilpotent, if there exists an \( n \in \mathbb{N} \) such that \( C^{(n)}(g) = \{ 0 \} \).

Definition 162 Let \( \Gamma \subset gl(F) \) be a sub-algebra of \( gl(F) \).

(1) \( \Gamma \) is solvable, if for any \( p \in \mathbb{N} \), the truncated sub-algebra \( \Gamma^{[\cdot,p]} \) is solvable.

(2) \( \Gamma \) is nilpotent, if for any \( p \in \mathbb{N} \), the truncated sub-algebra \( \Gamma^{[\cdot,p]} \) is nilpotent.

Definition 163 [25] Let \( g \) be a Lie algebra.
(1) $g$ is simple if it has no non-trivial idea (the only ideas are \{0\} and $g$ itself).
(2) $g$ is semi-simple if it has no solvable idea except \{0\}.

Though the following proposition is simple, it is also fundamental.

**Proposition 164** The Lie algebra $gl^{[\cdot, \cdot]}(\mathbb{F})$ is isomorphic to the classical linear algebra $gl(p, \mathbb{F})$.

**Proof.** First we construct a mapping $\pi : gl^{[\cdot, \cdot]}(\mathbb{F}) \to gl(p, \mathbb{F})$ as follows: Assume $\langle A \rangle \in gl^{[\cdot, \cdot]}(\mathbb{F})$ and $A_1 \in \langle A \rangle$ is irreducible. Say, $A_1 \in M_{n \times n}$, then by definition, $n | p$.

Denote by $s = p/n$, then define

$$\pi(\langle A \rangle) := A_1 \otimes I_s \in gl(p, \mathbb{F}).$$

Set $\pi' : gl(p, \mathbb{F}) \to gl^{[\cdot, \cdot]}(\mathbb{F})$ as

$$\pi'(B) := \langle B \rangle \in gl^{[\cdot, \cdot]}(\mathbb{F}),$$

then it is ready to verify that $\pi$ is a bijective mapping and $\pi^{-1} = \pi'$.

By the definitions of $\oplus$ and $[\cdot, \cdot]$, it is obvious that $\pi$ is a Lie algebra isomorphism. \hfill $\Box$

The following properties are available for classical $gl(n, \mathbb{F})$ [25], [41]. Using Proposition 164, it is easy to verify that they are also available for $gl(\mathbb{F})$.

**Proposition 165** Let $g \subset gl(\mathbb{F})$ be a Lie sub-algebra.

(1) If $g$ is nilpotent then it is solvable.
(2) If $g$ is solvable (or nilpotent) then so is its sub-algebra, its homomorphic image.
(3) If $h \subset g$ is an idea of $g$ and $h$ and $g/h$ are solvable, then $g$ is also solvable.

**Definition 166** Let $\langle A \rangle \in gl(\mathbb{F})$. The adjoint representation $ad_{\langle A \rangle} : gl(\mathbb{F}) \to gl(\mathbb{F})$ is defined as

$$ad_{\langle A \rangle} \langle B \rangle = [\langle A \rangle , \langle B \rangle].$$

(162)

To see (162) is well defined, we have to prove that

$$ad_{\langle A \rangle} \langle B \rangle = \langle ad_A B \rangle, \quad A \in \langle A \rangle, \quad \langle B \rangle \in \langle B \rangle.$$ (163)

It follows from the consistence of $\otimes$ and $\oplus$ (\(\sim\)) with $\sim_{\ell}$ immediately.

**Example 167** Consider $\langle A \rangle \in \Sigma$. Assume $\langle A \rangle$ is nilpotent, that is, there is a $k > 0$ such that $\langle A \rangle^k = 0$. Then $ad_{\langle A \rangle}$ is also nilpotent.

Note that $A^k = 0$ if and only if $(A \otimes I_s)^k = 0$. Similarly, $ad_A^k = 0$ if and only if $ad_{A \otimes I_s}^k = 0$. Hence, we need only to show that $ad_A$ is nilpotent, where $A \in \langle A \rangle$ and $A \in M_{n \times n}$. It is easy to verify that

$$ad_A^m B = \sum_{i=0}^{m} (-1)^i \binom{m}{i} A^{m-i} BA^i.$$ As $m = 2k - 1$, it is clear that $ad_A^m B = 0, \forall B \in M_{n \times n}$. It follows that $ad_A^{2k-1} = 0$.

**Definition 168** (1) Let $A, B \in gl(n, \mathbb{F})$. Then the Killing form $\langle \cdot, \cdot \rangle_K : gl(n, \mathbb{F}) \times gl(n, \mathbb{F}) \to \mathbb{F}$ is defined as (We refer to [25] for original definition. The following definition is with a mild modification.)

$$\langle A, B \rangle_K := Tr(ad_A ad_B).$$ (164)

(2) Assume $\langle A \rangle, \langle B \rangle \in gl(\mathbb{F})$. The Killing form $\langle \cdot, \cdot \rangle : gl(\mathbb{F}) \times gl(\mathbb{F}) \to \mathbb{F}$ is defined as

$$\langle \langle A \rangle, \langle B \rangle \rangle_K := Tr (ad_{\langle A \rangle} \times ad_{\langle B \rangle}).$$ (165)

To see the killing form is well defined, we also need to prove

$$\langle \langle A \rangle, \langle B \rangle \rangle_K = \langle A, B \rangle_K, \quad A \in \langle A \rangle, \quad B \in \langle B \rangle.$$ (166)

Similar to (163), it can be verified by a straightforward calculation.

Because of the equations (163) and (166), the following properties of finite dimensional Lie algebras [41] can easily be extended to $gl(\mathbb{F})$:

**Proposition 169** Consider $g = gl(\mathbb{F})$. Let $\langle A \rangle, \langle A \rangle_1, \langle A \rangle_2, \langle B \rangle, \langle E \rangle \in g$, $c_1, c_2 \in \mathbb{F}$. Then

$$\langle \langle A \rangle, \langle B \rangle \rangle_K = \langle \langle B \rangle, \langle A \rangle \rangle_K.$$ (167)
The Engel theorem can easily be extended to $gl(F)$:

**Theorem 170** Let $\{0\} \neq g \subset gl(F)$ be a bundled Lie sub-algebra. Assume each $\langle A \rangle \in g$ is nilpotent, \textit{i.e.}, for each $\langle A \rangle \in g$ there exists a $k > 0$ such that $\langle A \rangle^k = \langle A^k \rangle = 0$.

1. If $g$ is finitely generated, then there exists a vector $X \neq 0$ (of suitable dimension) such that $G \times X = 0$, $\forall G \in g$.
2. $g$ is nilpotent.

**Definition 171** [25] Let $V$ be an $n$ dimensional vector space. A flag in $V$ is a chain of subspaces $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V,$ with $\text{dim}(V_i) = i.$ Let $A \in \text{End}(V)$ be an endomorphism of $V.$ $A$ is said to stabilize this flag if $A V_i \subset V_i, \text{ } i = 1, \cdots, n.$

Lie theory can be extended to $gl(F)$ as follows.

**Theorem 172** Assume $g \subset gl(F)$ is a solvable Lie sub-algebra. Then for any $p > 0$ there is a flag of ideals $0 = I_0 \subset I_1 \subset \cdots \subset I_p$, such that the truncated $g^{[-p]}$ stabilizes the flag.

---

Corollary 173 Assume $g \subset gl(F)$ is a Lie sub-algebra. $g$ is solvable if and only if, $D(g)$ is nilpotent.

Example 174 Consider the bundled Lie sub-algebras $t(F)$ and $n(F)$. It is easy to verify the following:

1. $t(F)$ is solvable;
2. $n(F)$ is nilpotent.

Even though $gl(F)$ is an infinite dimensional Lie algebra, it has almost all the properties of finite dimensional Lie algebras. This claim can be verified one by one easily. The reason is: $gl(F)$ is a union of finite dimensional Lie algebras.

6 Lie Group on Nonsingular Equivalence Space

6.1 Bundled Lie Group

Consider $\Sigma := \Sigma_1$, we define a subset

$$GL(F) := \{ \langle A \rangle \in \Sigma \mid \text{Dt}(\langle A \rangle) \neq 0 \}.$$ (171)

We emphasize the fact that $GL(F)$ is an open subset of $\Sigma$. For an open subset of bundled manifolds we have the following result.

**Proposition 175** Let $M$ be a bundled manifold, and $N$ an open subset of $M$. Then $N$ is also a bundled manifold.

Proof. It is enough to construct an open cover of $N$. Starting from the open cover of $M$, which is denoted as $\mathcal{C} = \{ U_\lambda \mid \lambda \in \Lambda \}$, we construct $\mathcal{C}_N := \{ U_\lambda \cap N \mid U_\lambda \in \mathcal{C}, \lambda \in \Lambda \}$

Then we can prove that it is $C^\infty$ ($C^\omega$ or $C^\omega$) comparable as long as $\mathcal{C}$ is. Verifying other conditions is trivial. □

**Corollary 176** $GL(F)$ is a bundled manifold.

Note that, Proposition 46 shows that $\ltimes : GL(F) \times GL(F) \rightarrow GL(F)$ is well define.

**Definition 177** A topological space $G$ is a bundled Lie group, if
(1) it is a bundled analytic manifold;
(2) it is a group;
(3) the product \( A \times B \rightarrow AB \) and the inverse mapping \( A \rightarrow A^{-1} \) are analytic.

The following result is an immediate consequence of the definition.

**Theorem 178** \( GL(\mathbb{F}) \) is a bundled Lie group.

**Proof.** We already known then \( GL(\mathbb{F}) \) is a bundled analytic manifold. We first prove \( GL(\mathbb{F}) \) is a group. It is ready to verify that (1) is the identity. Moreover, \( (A)^{-1} = (A^{-1}) \). The conclusion follows.

Using a simple coordinate chart, it is obvious that the inverse and product are two analytic mappings. \( \Box \)

6.2 Relationship with \( gl(\mathbb{F}) \)

Denote by

\[
W := \{ A \in \mathcal{M}_1 \mid \det(A) \neq 0 \},
\]

and

\[
W_s := W \cap \mathcal{M}_{s \times s}, \quad s = 1, 2, \ldots.
\]

Consider the bundle: \( (\mathcal{M}_1, Pr, \Sigma) \), where the map \( Pr : A \mapsto A \) is the national projection. It has a natural sub-bundle: \( (W, Pr, GL(\mathbb{F})) \) via the following bundle morphism as

\[
\begin{array}{ccc}
W & \xrightarrow{\pi} & \mathcal{M}_1 \\
Pr \downarrow & & Pr \downarrow \\
GL(\mathbb{F}) & \xrightarrow{\pi'} & \Sigma
\end{array}
\]

In fact, the projection leads to Lie group homomorphism.

**Theorem 179** (1) With natural group and differential structures, \( W_s = GL(s, \mathbb{F}) \) is a Lie group.

(2) Consider the projection \( Pr \). Restrict it to each leaf yields

\[
Pr|_{W_s} : GL(s, \mathbb{F}) \rightarrow GL(\mathbb{F}).
\]

Then \( Pr|_{W_s} \) is a Lie group homomorphism.

(3) Set the image set as \( Pr(GL(s, \mathbb{F})) := \Psi_s \). Then \( \Psi_s < GL(\mathbb{F}) \) is a Lie sub-group. Moreover,

\[
Pr|_{W_s} : GL(s, \mathbb{F}) \rightarrow \Psi_s
\]

is a Lie group isomorphism.

**Definition 180** A vector field \( \langle \xi \rangle \in V(GL(\mathbb{F})) \), (where for each \( P \in GL(\mathbb{F}), \langle \xi(P) \rangle \in T_P (GL(\mathbb{F})) \), is called a left-invariant vector field, if for any \( \langle A \rangle \in GL(\mathbb{F}) \)

\[
(L(A))_* (\langle \xi \rangle (P)) := (\langle LA \rangle)_* (\langle \xi \rangle (P)) = \langle \xi (AP) \rangle = \langle \xi (AP) \rangle.
\]

Then it is easy to verify the following relationship between \( GL(\mathbb{F}) \) and \( gl(\mathbb{F}) \):

**Theorem 181** The corresponding Lie algebra of the bundled Lie group \( GL(\mathbb{F}) \) is \( gl(\mathbb{F}) \) in the following natural sense:

\[
\begin{array}{ccc}
\mathfrak{gl}(\mathbb{F}) & \xrightarrow{\exp} & GL(n, \mathbb{F}) \\
Pr \downarrow & & Pr \downarrow \\
\mathfrak{gl}(\mathbb{F}) & \xrightarrow{\exp} & GL(n, \mathbb{F})
\end{array}
\]

That is, \( \mathfrak{gl}(\mathbb{F}) \) is a Lie algebra isomorphic to the Lie algebra consists of the vectors on the tangent space of \( GL(\mathbb{F}) \) at identity. Then these vectors generate the left-invariant vector fields which form the tangent space at any \( \langle A \rangle \in GL(\mathbb{F}) \).

Let \( Pr : \mathcal{M}_{n \times n} \rightarrow \Sigma \) be the natural mapping \( A \mapsto (A) \). Then we have the following commutative picture:

\[
\begin{array}{ccc}
\mathfrak{gl}(n, \mathbb{F}) & \xrightarrow{\exp} & GL(n, \mathbb{F}) \\
Pr \downarrow & & Pr \downarrow \\
\mathfrak{gl}(\mathbb{F}) & \xrightarrow{\exp} & GL(\mathbb{F})
\end{array}
\]

where \( n = 1, 2, \ldots \). Recall (15), we know that the exponential mapping \exp is well defined \( \forall \langle X \rangle \in \mathfrak{gl}(\mathbb{F}) \).

Graph (175) also shows the relationship between \( gl(\mathbb{F}) \) and \( GL(\mathbb{F}) \), which is a generalization of the relationship between \( gl(n, \mathbb{F}) \) and \( GL(n, \mathbb{F}) \).

6.3 Lie Subgroups of \( GL(\mathbb{F}) \)

It has been discussed that \( gl(\mathbb{F}) \) has some useful Lie sub-algebras. It is obvious that \( GL(\mathbb{F}) \) has some Lie subgroups, corresponding to those sub-algebras of \( gl(\mathbb{F}) \). They are briefly discussed as follows.
• Bundled orthogonal Lie sub-group

**Definition 182** \( (A) \in GL(\mathbb{F}) \) is said to be orthogonal, if \( A^T = A^{-1} \).

It is also easy to verify the following:

**Proposition 183** Assume \( (A) \) and \( (B) \) are orthogonal, then so is \( (A) \times (B) \).

We, therefore, can define the following bundled Lie sub-group of \( GL(\mathbb{F}) \) as follows.

**Definition 184**

\[
O(\mathbb{F}) := \left\{(A) \in GL(\mathbb{F}) \mid (A)^T = (A)^{-1}\right\}
\]

is called the bundled orthogonal group.

It is easy to verify the following proposition:

**Proposition 185** Consider the bundled orthogonal group.

1. \( O(\mathbb{F}) \) is a Lie sub-group of \( GL(\mathbb{F}) \), i.e., \( O(\mathbb{F}) < GL(\mathbb{F}) \).
2. \( SO(\mathbb{F}) := \left\{(A) \in O(\mathbb{F}) \mid Dt((A)) = 1\right\} \).

Then \( SO(\mathbb{F}) < O(\mathbb{F}) < GL(\mathbb{F}) \).

3. The Lie algebra for both \( O(\mathbb{F}) \) and \( SO(\mathbb{F}) \) is \( o(\mathbb{F}) \).

• Bundled special linear group

**Definition 186**

\[
SL(\mathbb{F}) := \left\{(A) \in GL(\mathbb{F}) \mid Dt((A)) = 1\right\}
\]

is called the bundled special linear group.

Similar to the case of orthogonal algebra, it is easy to verify the following:

**Proposition 187** Consider the bundled special linear group.

1. \( SL(\mathbb{F}) \) is a Lie sub-group of \( GL(\mathbb{F}) \), i.e., \( SL(\mathbb{F}) < GL(\mathbb{F}) \).
2. The Lie algebra of \( SL(\mathbb{F}) \) is \( sl(\mathbb{F}) \).

• Bundled upper triangular group

**Definition 188**

\[
T(\mathbb{F}) := \left\{(A) \in GL(\mathbb{F}) \mid (A) \text{ is upper triangular}\right\}
\]

is called the bundled upper triangular group.

**Proposition 189** Consider the bundled upper triangular group.

1. \( T(\mathbb{F}) \) is a Lie sub-group of \( GL(\mathbb{F}) \), i.e., \( T(\mathbb{F}) < GL(\mathbb{F}) \).
2. The Lie algebra of \( T(\mathbb{F}) \) is \( t(\mathbb{F}) \).

• Bundled special upper triangular group

**Definition 190**

\[
N(\mathbb{F}) := \left\{(A) \in T(\mathbb{F}) \mid Dt((A)) = 1\right\}
\]

is called the bundled special upper triangular group.

**Proposition 191** Consider the bundled special upper triangular group.

1. \( N(\mathbb{F}) \) is a Lie sub-group of \( T(\mathbb{F}) \), i.e., \( N(\mathbb{F}) < T(\mathbb{F}) < GL(\mathbb{F}) \).
2. The Lie algebra of \( N(\mathbb{F}) \) is \( n(\mathbb{F}) \).

• Bundled symplectic group

**Definition 192**

\[
SP(\mathbb{F}) := \left\{(A) \in GL(\mathbb{F}) \mid A_1 \in M_{2n}, \text{satisfies (176)}, n \in \mathbb{N}\right\},
\]

is called the bundled symplectic group.

\[
\langle A \rangle^T \langle J \rangle \langle A \rangle = \langle J \rangle,
\]

where \( J \) is defined in (28).

**Proposition 193** Consider the bundled symplectic group.

1. \( SP(\mathbb{F}) \) is a Lie sub-group of \( GL(\mathbb{F}) \), i.e., \( SP(\mathbb{F}) < GL(\mathbb{F}) \).
2. The Lie algebra of \( SP(\mathbb{F}) \) is \( sp(\mathbb{F}) \).

6.4 Symmetric Group

Let \( S_k \) be the \( k \)-th order symmetric group. Denote

\[
S := \bigcup_{k=1}^{\infty} S_k.
\]

**Definition 194** A matrix \( A \in M_k \) is called a permutation matrix, if \( \text{Col}(A) \subset \Delta_k \) and \( \text{Col}(A^T) \subset \Delta_k \). The set of \( k \times k \) permutation matrices is denoted by \( P_k \).

**Proposition 195** Consider the set of permutation matrix.

1. If \( P \in P_k \), then

\[
P^T = P^{-1}.
\]

2. \( P_k < O(k, \mathbb{R}) < GL(k, \mathbb{R}) \).
Let $\sigma \in S_k$. Define a permutation matrix $M_\sigma \in P_k$ as $M_\sigma := [m_{i,j}]$, where

$$m_{i,j} = \begin{cases} 1, & \sigma(j) = i \\ 0, & \text{otherwise.} \end{cases} \quad (178)$$

The following proposition is easily verifiable.

**Proposition 196** Define $\pi : S_k \to P_k$, where $\pi(\sigma) := M_\sigma$ is constructed by (178). Then $\pi$ is an isomorphism.

Assume $\sigma, \lambda \in S_k$. Then Proposition 196 leads to

$$M_{\sigma \circ \lambda} = M_\sigma M_\lambda. \quad (179)$$

Next, assume $\sigma \in S_m, \lambda \in S_n$, we try to generalize (179).

**Definition 197** Assume $\sigma \in S_m, \lambda \in S_n$. The (left) STP of $\sigma$ and $\lambda$ is defined by

$$M_{\sigma \circ \lambda} = M_\sigma \times M_\lambda \in P_t, \quad (180)$$

where $t = \text{lcm}(m, n)$. That is,

$$\sigma \circ \lambda := \pi^{-1}(M_\sigma \times M_\lambda) \in S_t. \quad (181)$$

Similarly, we can define the right STP of $\sigma$ and $\lambda$.

Now, it is clear that $(S, \circ) < (M_2, \times)$ is a sub-monoid.

To get a bundled Lie subgroup structure, we consider the quotient space

$$\mathcal{D} := (S, \circ) / \sim_t.$$ 

Then we have the following:

**Theorem 198** $\mathcal{D}$ is a discrete bundled sub-Lie group of $GL(\mathbb{F})$.

$\mathcal{D}$ may be used to investigate the permutation of uncertain number of elements.

## 7 Vector Equivalence

### 7.1 Equivalence of Vectors of Different Dimensions

Consider the set of vectors on field $\mathbb{F}$. Denote it as

$$\mathcal{V} := \bigcup_{i=1}^{\infty} \mathcal{V}_i, \quad (182)$$

where $\mathcal{V}_i$ is the $i$ dimensional vector subspace of $\mathcal{V}$.

Our purpose is to build a vector space structure on $\mathcal{V}$. To this end, we first propose an equivalence relation.

**Definition 199** (1) Let $X, Y \in \mathcal{V}$. $X$ and $Y$ are said to be vector equivalent, denoted by $X \leftrightarrow Y$, if there exist two one-vectors $1_s$ and $1_t$ such that

$$X \otimes 1_s = Y \otimes 1_t. \quad (183)$$

(2) The equivalent class is denoted as

$$[X] := \{Y \mid Y \leftrightarrow X\}.$$ 

(3) In an equivalent class $[X]$ a partial order ($\leq$) is defined as: $X \leq Y$, if there exists a one-vector $1_s$ such that $X \otimes 1_s = Y$. $X_1 \in [X]$ is irreducible, if there are no $Y$ and $1_s, s > 1$, such that $X_1 = Y \otimes 1_s$.

**Remark 200** (1) The equivalence defined above can be seen as the left equivalence. Formally, we set $\leftrightarrow_1 := \leftrightarrow_1$.

(2) The right equivalence can be defined as follows: Let $X, Y \in \mathcal{V}$. $X$ and $Y$ are said to be right equivalent, denoted by $X \leftrightarrow_r Y$, if there exist two one-vectors $1_s$ and $1_t$ such that

$$1_s \otimes X = 1_t \otimes Y. \quad (184)$$

(3) The right equivalent class of $X$ is denoted as $[X]_r$. Of course, we have $[X]_1 = [X]$.

(4) Hereafter, the left equivalence is considered as the defaulted one. That is, we always assume $\leftrightarrow := \leftrightarrow_1$.

But with some obvious modifications one sees easily that the arguments/results in the sequel are also valid for right equivalence.

The following properties of the matrix equivalence are also true for vector equivalence. The proofs are also similar to the matrix equivalence. Therefore, they are omitted.
Theorem 201  (1) If $X \leftrightarrow Y$, then there exists a vector $\Gamma$ such that

$$X = \Gamma \otimes 1_\beta, \quad Y = \Gamma \otimes 1_\alpha.$$  \hspace{1cm} (185)

(2) In each class $[X]$ there exists a unique $X_1 \in [X]$, such that $X_1$ is irreducible.

Remark 202  (1) If $X = Y \otimes 1_s$, then $Y$ is called a divisor of $X$ and $X$ is called a multiple of $Y$. This relation determined the order $Y \leq X$.

(2) If (185) holds and $\alpha, \beta$ are co-prime, then the $\Gamma$ satisfying (185) is called the greatest common divisor of $X$ and $Y$. Moreover, $\Sigma = \gcd(X, Y)$ is unique.

(3) If (183) holds and $s, t$ are co-prime, then

$$\Xi := X \otimes 1_s = Y \otimes 1_t$$  \hspace{1cm} (186)

is called the least common multiple of $X$ and $Y$. Moreover, $\Xi = \lcm(X, Y)$ is unique.

(4) Consider an equivalent class $[X]$, denote the unique irreducible element by $X_1$, which is called the root element. All the elements in $[X]$ can be expressed as

$$X_i = X_1 \otimes 1_i, \quad i = 1, 2, \cdots.$$  \hspace{1cm} (187)

$X_i$ is called the $i$-th element of $[X]$. Hence, an equivalent class $[X]$ is a well ordered sequence as:

$$[X] = \{X_1, X_2, X_3, \cdots\}.$$  

We also have a lattice structure on $[X] = \{X_1, X_2, \cdots\}$:

Proposition 203  $([X], \leq)$ is a lattice.

Proof. It is easy to verify that for $U, V \in [X]$,

$$\sup(U, V) = \lcm(U, V); \quad \inf(U, V) = \gcd(U, V).$$

The conclusion follows. \square

Proposition 204 Let $A_1 \in \mathcal{M}$ and $X_1 \in \mathcal{V}$ be both irreducible. Then the two lattices $(A)$ and $[X]$ generated by $A_1$ and $X_1$ are isomorphic. Precisely,

$$([A], \prec) \cong ([X], \leq).$$  \hspace{1cm} (188)

The isomorphism is: $\varphi : A_s = A_1 \otimes I_s \mapsto X_s = X_1 \otimes 1_s$.

Next, we consider the lattice structure of

$$\mathcal{V} = \bigcup_{i=1}^\infty \mathcal{V}_i,$$

which is obviously a partition. Observing that if $q|p$, say $p = qs$, Then $\mathcal{V}_q \otimes 1_s$ is a subspace of $\mathcal{V}_p$. We give an order as $\mathcal{V}_q \leq \mathcal{V}_p$. Under this partial order we have a lattice structure on $\mathcal{V}$ as follows:

Proposition 205 $([V], \leq)$ is a lattice with

$$\sup(V_\alpha, V_\beta) = V_{\alpha \lor \beta} \quad \text{and} \quad \inf(V_\alpha, V_\beta) = V_{\alpha \land \beta}$$  \hspace{1cm} (189)

We also have the following isomorphic relation:

Proposition 206 Define $\varphi : \mathcal{V} \rightarrow \mathcal{M}_\mu$ as:

$$\varphi(V_i) := \mathcal{M}_\mu^i.$$  \hspace{1cm} (190)

Then $\varphi$ is a lattice isomorphism.

Definition 207  (1) Let $p \in \mathbb{N}$. The $p$-lower truncated vector space is defined as

$$\mathcal{V}^{[p:]} := \bigcup_{(s \mid p|s)} \mathcal{V}_s.$$  \hspace{1cm} (191)

(2) The quotient space of $\mathcal{V}/\leftrightarrow$ is denoted as

$$\Omega_\mathcal{V} := \{[X] \mid X \in \mathcal{V}\}.$$  \hspace{1cm} (192)

(3) The subspace

$$\Omega_i := \mathcal{V}^{[i:]} / \leftrightarrow \quad \text{and} \quad \{[X] \mid X_1 \in \mathcal{V}_i\}.$$  \hspace{1cm} (193)

7.2 Vector Space Structure on Vector Equivalence Space

Next, we define an addition between vectors of different dimensions.

Definition 208 Let $X \in \mathcal{V}_p$ and $Y \in \mathcal{V}_q$. $t = \lcm(p, q)$. Then

$$X \oplus Y = X \otimes 1_q.$$
(1) the vector addition $\leftrightarrow: \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ of $X$ and $Y$ is defined as

$$X \leftrightarrow Y := (X \otimes 1_{t/p}) + (Y \otimes 1_{t/q}); \quad (194)$$

(2) the subtraction $\leftarrow$ is defined as

$$X \leftarrow Y := X \leftrightarrow (-Y). \quad (195)$$

**Remark 209** Let $X \in \mathcal{V}^T_p$ and $Y \in \mathcal{V}^T_q$ be tow row vectors. Then we define

(1) the vector addition as

$$X \leftrightarrow Y := \left( X \otimes 1_{t/p} \right) + \left( Y \otimes 1_{t/q} \right); \quad (196)$$

(2) the subtraction as

$$X \leftarrow Y := X \leftrightarrow (-Y). \quad (197)$$

**Proposition 210** The vector addition $\leftrightarrow$ is consistent with the equivalence $\leftrightarrow$. That is, if $X \leftrightarrow \tilde{X}$ and $Y \leftrightarrow \tilde{Y}$, then $X \leftrightarrow Y \leftrightarrow \tilde{X} \leftrightarrow \tilde{Y}$.

**Proof.** Since $X \leftrightarrow \tilde{X}$, according to Theorem 201, there exists $\Gamma$, say, $\Gamma \in \mathcal{V}_p$, such that

$$X = \Gamma \otimes 1_s, \quad \tilde{X} = \Gamma \otimes 1_{j}. \quad (198)$$

Similarly, there exists $\Pi$, say, $\Pi \in \mathcal{V}_q$, such that

$$Y = \Pi \otimes 1_s, \quad \tilde{Y} = \Pi \otimes 1_{\ell}. \quad (199)$$

Let $\xi = \text{lcm}(p, q), \eta = \text{lcm}(sp, sq)$, and $n = \xi \ell$. Then

$$X \leftrightarrow Y = (\Gamma \otimes 1_n) \leftrightarrow (\Pi \otimes 1_n)$$

$$= \left[ (\Gamma \otimes 1_n) \otimes 1_{\xi/\eta} \right] + \left[ (\Pi \otimes 1_n) \otimes 1_{\eta/(sp)} \right]$$

$$= [\Gamma \otimes 1_{n/\eta}] + [\Pi \otimes 1_{n/\eta}]$$

$$= (\Gamma \otimes 1_{n/\eta/p}) + (\Pi \otimes 1_{n/\eta/q})$$

$$= \left( (\Gamma \otimes 1_{n/\eta/p}) \otimes 1_{\xi/\eta} \right) \otimes 1_{\ell}$$

$$= (\Gamma \leftrightarrow \Pi) \otimes 1_{\ell}. \quad (200)$$

Hence $X \leftrightarrow Y \leftrightarrow \Gamma \leftrightarrow \Pi$. Similarly, we can show that $\tilde{X} \leftrightarrow \tilde{Y} \leftrightarrow \Gamma \leftrightarrow \Pi$. The conclusion follows. \hfill \Box

**Corollary 211** The vector addition $\leftrightarrow$ (or subtraction $\leftarrow$) is well defined on the quotient space $\Omega$, as well as $\Omega_i, i = 1, 2, \ldots$. That is,

$$[X] \leftrightarrow [Y] := [X \leftrightarrow Y], \quad (201)$$

$$X, Y \in \mathcal{V}, (or [X], [Y] \in \Omega \cup \Omega_i). \quad (202)$$

Let $[X] \in \Omega$ (or $[X] \in \Omega_i$). Then we define a scale product

$$a[X] := [aX], \quad a \in \mathbb{F}. \quad (203)$$

Using (198) and (199), one sees easily that $\Omega$ becomes a vector space:

**Theorem 212** Using the vector addition defined in (198) and the scale product defined in (199), we have

(1) $\Omega$ is a vector space over $\mathbb{F}$;

(2) $\Omega_i, i = 1, 2, \ldots$ are the subspaces of $\Omega$;

(3) if $i|j$, then $\Omega_i$ is a subspace of $\Omega_j$.

Let $E \subset \mathcal{V}$ be a set of vectors. Then

$$[E] := \{ [X] \mid X \in E \}. \quad (204)$$

The following proposition shows that the vector equivalence keeps the space-subspace relationship unchanged.

**Proposition 213** Assume $E \subset \mathcal{V}_i$ is a subspace of $\mathcal{V}_i$. Then $[E] \subset \Omega_i$ is a subspace of $\Omega_i$.

Definition 208 can be translated to its corresponding right one as follows:

**Definition 214** Let $X \in \mathcal{V}_p$ and $Y \in \mathcal{V}_q, t = \text{lcm}(p, q)$. Then

(1) the (right) vector addition $\Rightarrow: \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ of $X$ and $Y$ is defined as

$$X \Rightarrow Y := (1_{t/p} \otimes X) + (1_{t/q} \otimes Y); \quad (205)$$

(2) the subtraction $\rightarrow$ is defined as

$$X \rightarrow Y := X \Rightarrow (-Y). \quad (206)$$

Then all the arguments in this subsection and the following several subsections can be stated in a parallel way for right addition and corresponding linear spaces.
7.3 Inner Product and Linear Mappings

Definition 215 Let \( X \in V_m \subset V, Y \in V_n \subset V, \) and \( t = \text{lcm}(m, n) \). Then the weighted inner product is defied as

\[
(X \mid Y)_W := \frac{1}{t} (X \otimes I_{t/m} \mid Y \otimes I_{t/n}), \quad (203)
\]

where \((X \mid Y)\) is the conventional inner product. Say, if \( F = \mathbb{R} \), \((X \mid Y) = X^T Y \), and if \( F = \mathbb{C} \), \((X \mid Y) = X^T \bar{Y} \).

Definition 216 Let \( [X], [Y] \in \Omega \). Their inner product is defined as

\[
([X] \mid [Y]) := (X \mid Y)_W, \quad X \in [X], Y \in [Y].
\]

(204)

It is easy to verify the following proposition, which assures the Definition 216 is reasonable.

Proposition 217 Equation (204) is well defined. That is, it is independent of the choice of \( X \) and \( Y \).

Since \( \Omega \) is a vector space, using (204) as an inner product on \( \Omega \), then we have the follows:

Proposition 218 The \( \Omega \) with inner product defined by (204) is an inner product space. It is not a Hilbert space.

Proof. The first part is obvious. As for the second part, we construct a sequence as

\[
\begin{cases}
X_1 = a \in F \\
X_{i+1} = X_i \otimes I_{2+i} \left( \delta_{2+i} - \delta_{2+i+1} \right), \quad i = 1, 2, \ldots.
\end{cases}
\]

Then we can prove that \( \{X_i\} \) is a Cauchy sequence and it does not converge to any \( X \in V \).

Given a vector \( X \in V \), then it determined a linear mapping \( \varphi_X : V \rightarrow F \) via inner product as

\[ \varphi_X : Y \mapsto (X \mid Y)_W. \]

Similarly, \([X] \in \Sigma\) can also determine a linear mapping \( \varphi_{[X]} : \Sigma \rightarrow F \) as

\[ \varphi_{[X]} : [Y] \mapsto ([X] \mid [Y]). \]

Unfortunately, the inverse is not true. Because \( \Sigma \) is an infinite dimensional vector space but any vector in \( \Sigma \) has only finite dimensional representatives.

Next, we consider the linear mappings on \( V \). It is well known that assume \( A \in M_{m \times n} \) and \( X \in V_n \). Then the product \( \times : M_{m \times n} \times V_n \rightarrow V_m \), defined as \((A, X) \mapsto AX\), for given \( A \) is a linear mapping. We intend to generalize such a linear mapping to arbitrary matrix and arbitrary vector.

Definition 219 Let \( A \in M_{m \times n}, X \in V_p, \) and \( t = \text{lcm}(n, p) \). Then the vector product, denoted by \( \times \), is defined as

\[ A \times X := (A \otimes I_{n/t}) (X \otimes I_{p/t}). \quad (205) \]

Remark 220 (1) The vector product defined in (205) is the left vector product. Of course, we can define right vector product, denoted by \( \triangleright \), defined as

\[ A \triangleright X := (I_{n/t} \otimes A) (1/p \otimes X). \quad (206) \]

(2) In fact, \( \times \) is a combination of \( \sim \) of matrices and \( \leftrightarrow \) of vectors, and \( \triangleright \) is a combination of \( \sim r \) of matrices and \( \leftrightarrow r \) of vectors. Of course, we may define two more vector products by combinations of \( \sim l \) with \( \leftrightarrow r \) and \( \sim r \) with \( \leftrightarrow l \) respectively.

(3) Note that when \( n = P, A \times X = AX \). That is, the linear mapping defined in (205) is a generalization of conventional linear mapping. It is also true for other vector products.

(4) To avoid similar but tedious arguments, hereafter the default vector product is \( \times \).

The following proposition is easily verifiable.

Proposition 221 Consider the vector product

\[ \times : M \times V \rightarrow V. \]

(1) It is linear with respect to the second variable, precisely,

\[ A \times(aX \triangleright bY) = A \times X \triangleright B \times Y, \quad a, b \in F. \quad (207) \]

(2) Assume both \( A, B \in M_P \), then the vector product is also linear with respect to the first variable, precisely,

\[ (A+B) \times X = A \times X \triangleright B \times X. \quad (208) \]
The following proposition shows that the vector product is consistent with both matrix equivalence and vector equivalence.

**Proposition 222** Assume \( A \sim B \) and \( X \leftrightarrow Y \). Then

\[
A \sim X \leftrightarrow B \sim Y. \quad (209)
\]

**Proof.** Assume \( A = \Lambda \otimes I_s, B = \Lambda \otimes I_t \); \( X = \Gamma \otimes 1_t \), \( Y = \Gamma \otimes 1_s \), where \( \Lambda \in \mathcal{M}_{n \times p} \) and \( \Gamma \in \mathcal{V}_q \). Denote \( \xi = \text{lcm}(p, q), \eta = \text{lcm}(ps, qt) \), and \( \eta = k\xi \). Then we have

\[
A \sim X = (\Lambda \otimes I_s) (\Gamma \otimes 1_t)
\]

\[
= (\Lambda \otimes I_s \otimes I_{\eta/pn}) (\Gamma \otimes 1_t \otimes 1_{\eta/qr})
\]

\[
= (\Lambda \otimes I_{\xi/p} \otimes I_k) (\Gamma \otimes 1_{\xi/q} \otimes 1_k)
\]

\[
= [(\Lambda \otimes I_{\xi/p}) (\Gamma \otimes 1_{\xi/q})] \otimes [I_k 1_k]
\]

\[
= (A \sim \Gamma) \otimes 1_k.
\]

Hence

\[
A \sim X \leftrightarrow \Lambda \sim \Gamma.
\]

Similarly, we have

\[
B \sim Y \leftrightarrow \Lambda \sim \Gamma.
\]

Equation (209) follows. \( \square \)

The above propositions have an immediate consequence as follows:

**Corollary 223** The vector product \( \sim : \mathcal{M} \times \mathcal{V} \to \mathcal{V} \) can be extended to \( \sim : \Sigma_M \times \Omega \to \Omega \). Particularly, each \( (\Lambda) \in \Sigma_M \) determines a linear mapping on the vector space \( \Omega \).

Next, we extend the vector equivalence to matrices.

**Definition 224** (1) Assume \( V, W \in \mathcal{M}_{x n} \). \( V \) and \( W \) are said to be vector equivalent, denoted by \( V \leftrightarrow W \), if there exist \( 1_s \) and \( 1_t \) such that

\[
V \otimes 1_s = W \otimes 1_t. \quad (210)
\]

(2) The equivalence class of \( V \) is denoted as

\[
[V] := \{W \mid W \leftrightarrow V\}. \quad (211)
\]

**Remark 225** (1) It is clear the equivalence defined by (210) is left vector equivalence \( \leftrightarrow \). The right vector equivalence \( \leftrightarrow \), can be defined similarly. Moreover, the right vector equivalence class is denoted by \([·]_r\).

(2) A matrix can be considered as a linear mapping, and correspondingly the matrix equivalence is considered. A matrix can also be considered as a vector subspace generated by its columns. In this way its vector equivalence, defined by (210)-(211), makes sense. Denote the equivalence class as

\[
\Omega_M := \mathcal{M}/ \leftrightarrow.
\]

All the results about vector space \( \mathcal{V} \) can be extended to \( \mathcal{M}_{x n} \) for \( n \in \mathbb{N} \). They are briefly summarized as follows:

**Proposition 226** Given \( n \in \mathbb{N} \).

Assume \( V \in \mathcal{M}_{r \times n}, W \in \mathcal{M}_{s \times n}, V \leftrightarrow W \) and \( r \mid s \). Then we said \( V \) is a divisor of \( W \) or \( W \) is a multiple of \( V \). Moreover, an order is determined by this as \( V \leq W \).

(2) \( \{\mathcal{V}, \leq\} \) is a lattice.

(3) Assume \( r \mid s \), then an order is given in \( \mathcal{M}_{x n} \) as

\[
\mathcal{M}_{r \times n} \subset \mathcal{M}_{s \times n}.
\]

(4) \( (\mathcal{M}_{Xn}, \leq) \) is a lattice.

(5) Assume \( V \in \mathcal{M}_{r \times n} \) is irreducible. Then \( \varphi : [V] \to \mathcal{M}_{x n} \) defined as \( V_i \mapsto M_{ir \otimes n} \) is a lattice homomorphism.

The operators \( \leftrightarrow \) and \( \sim \) can also be defined in a similar way as for vectors:

**Definition 227** (1) Assume \( V \in \mathcal{M}_{p \times n} \) and \( W \in \mathcal{M}_{q \times n}, t = \text{lcm}(p, q) \). The vector addition is defined as

\[
V \leftrightarrow W := (V \otimes 1_{t/p}) + (W \otimes 1_{t/q}).
\]

(2) Let \( A \in \mathcal{M}_{m \times n}, V \in \mathcal{M}_{p \times q} \), and \( t = \text{lcm}(n, p) \). Then the vector product of \( A \) and \( V \) is defined as

\[
A \leftrightarrow V := (A \otimes I_{t/n}) (V \otimes 1_{t/p}). \quad (212)
\]

We can denote the equivalence class of \( \mathcal{M}_{x n} \) as

\[
\Sigma^n_M := \mathcal{M}_{x n}/ \leftrightarrow.
\]
For vector product and vector addition we also have the distributive law with respect to these two operators for matrix case.

**Proposition 228** \( \ll : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \) is distributive. Precisely, Let \( A, B \in \mathcal{M}_\mu \) and \( C \in \mathcal{M}_{n \times \beta}, \ D \in \mathcal{M}_{p \times \beta}. \) Then

\[
(aA+bB) \ll C = aA \ll C \ll bB \ll C, \quad a, b \in \mathbb{F}.
\]

(213)

\[
A \ll (aC + bD) = aA \ll C \ll bA \ll D, \quad a, b \in \mathbb{F}.
\]

(214)

**Proof.** We prove (213). The proof of (214) is similar. Denote \( \text{lcm}(m, n) = s, \ \text{lcm}(m, p) = r, \ \text{lcm}(n, p) = t, \) and \( \text{lcm}(m, n, p) = \xi. \) Then for (213) we have

\[
\text{LHS} = (aA \otimes I_{s/m} + bB \otimes I_{s/n}) \ll C
\]

\[
= [(aA \otimes I_{s/m} + bB \otimes I_{s/n}) \otimes I_{\xi/s}] \otimes (C \otimes 1_{\xi/p})
\]

\[
= a (A \otimes I_{\xi/m}) (C \otimes 1_{\xi/p}) + b (B \otimes I_{\xi/n}) (C \otimes 1_{\xi/p})
\]

\[
= a [(A \otimes I_{s/m}) (C \otimes 1_{s/p})] \otimes 1_{\xi/s}
\]

\[
+ b [(B \otimes I_{t/n}) (C \otimes 1_{t/p})] \otimes 1_{\xi/t}
\]

\[
= a [(A \otimes I_{s/m}) (C \otimes 1_{s/p})]
\]

\[
\ll b [(B \otimes I_{t/n}) (C \otimes 1_{t/p})]
\]

\[
= \text{RHS}.
\]

\[\square\]

Finally, we can extend the distributive law to equivalence spaces as follows.

**Corollary 229** \( \ll : \Sigma_{\mathcal{M}} \times \Sigma_{\mathcal{M}} \to \Sigma_{\mathcal{M}} \) is distributive. Precisely, Let \( \langle A \rangle, \ \langle B \rangle \in \Sigma_{\mu} \) and \( [C], [D] \in \Sigma_{\beta}. \) Then

\[
(a \langle A \rangle + b \langle B \rangle) \ll [C] = a \langle A \rangle \ll [C] \ll b \langle B \rangle \ll [C],
\]

(215)

\[
A \ll (a[C] + b[D]) = a \langle A \rangle \ll [C] \ll b \langle A \rangle \ll [D],
\]

(216)

**7.4 Type-1 Invariant Subspace**

Given \( A \in \mathcal{M}_\mu, \) we seek a subspace \( S \subset \mathcal{V} \) which is \( A \) invariant.

**Definition 230** Let \( S \subset \mathcal{V} \) be a vector subspace. If \( A < \times S \subset S, \) \( S \) is called an \( A \)-invariant subspace. Moreover, if \( S \subset \mathcal{V}_t, \) it is called the type 1 invariant subspace; Otherwise, it is called the type 2 invariant subspace.

This subsection considers the type 1 invariant subspace only.

**Proposition 231** Let \( A \in \mathcal{M}_\mu \) and \( S = \mathcal{V}_t. \) Then \( S \) is \( A \)-invariant, if and only if,

(i)

\[
\mu_y = 1;
\]

(217)

(ii) \( A \in \mathcal{M}_\mu^i, \) where \( i \) satisfies

\[
\text{lcm}(i\mu_x, t) = t\mu_x.
\]

(218)

**Proof.** (Necessity) Assume \( \xi = \text{lcm}(i\mu_x, t), \) by definition we have

\[
A \ll X = (A \otimes I_{\xi/i\mu_x}) (X \otimes 1_{\xi/t})
\]

\[
\in \mathcal{V}_t, \quad X \in \mathcal{V}_t.
\]

Hence we have

\[
(i\mu_y) \left( \frac{\xi}{i\mu_x} \right) = t.
\]

(219)

It follows from (219) that

\[
\frac{\xi}{t} = \frac{\mu_x}{\mu_y}.
\]

(220)

Since \( \mu_x \) and \( \mu_y \) are co-prime and the left hand side of (220) is an integer, we have \( \mu_y = 1. \) It follows that

\[
\xi = t\mu_x.
\]

(221)

A \ll X \in \mathcal{V}_t, \quad \text{when } X \in \mathcal{V}_t.

\[\square\]
Assume \( A \in \mathcal{M}_\mu \) and \( S = \mathcal{V}_t \), a natural question is: Can we find \( A \) such that \( S \) is \( A \)-invariant? According to Proposition 231, we know that it is necessary that \( \mu_y = 1 \). Let \( k_1, \ldots, k_t \) be the prime divisors of \( \gcd(\mu_x, t) \), then we have

\[
\mu_x = k_1^{\alpha_1} \cdots k_t^{\alpha_t} p; \ t = k_1^{\beta_1} \cdots k_t^{\beta_t} q,
\]

(221)

where \( p, q \) are co-prime and \( k_i \mid p, k_i \mid q, \forall i \).

Now it is obvious that to meet (218) it is necessary and sufficient that

\[
i = k_1^{\beta_1} \cdots k_t^{\beta_t} \lambda,
\]

(222)

where \( \lambda \mid q \).

Summarizing the above argument we have the following result.

**Proposition 232** Assume \( A \in \mathcal{M}_\mu \), \( S = \mathcal{V}_t \). \( S \) is \( A \)-invariant, if and only if, (i) \( \mu_y = 1 \), and (ii) \( A \in \mathcal{M}_\mu \) where \( i \) satisfies (222).

**Remark 233** Assume \( A \in \mathcal{M}_1 \), \( S = \mathcal{V}_t \). Using Proposition 232, it is clear that \( S \) is \( A \)-invariant, if and only if, \( A \in \mathcal{M}_1^t \) with

\[
i \in \{ \ell \mid \ell \mid t \}.
\]

Particularly, when \( i = 1 \) then \( A = a \) is a number. So it is a scale product of \( S \), i.e., for \( V \in S \) we have a \( \times V = aV \in S \). when \( i = t \) then \( A \in \mathcal{M}_{t \times t} \). So we have \( A \preceq V = AV \in S \). This is the classical linear mapping on \( \mathcal{V}_t \). We call these two linear mapping the standard linear mapping. The following example shows that there are lots of non-standard linear mappings.

**Example 234** (1) Assume \( \mu = 0.5, t = 6 \). Then \( \mu_y = 1 \), \( \mu_x = 2 \). \( \gcd(\mu_x, t) = 2 \). Using (221), \( \mu_x = 2 \times 1 \) and \( t = 2 \times 3 \). According to (222) \( i = 2^{\beta_1} \lambda \), where \( \beta_1 = 1 \), \( \lambda = 1 \) or \( \lambda = 3 \). According to Proposition 232, \( V_6 \) is \( A \)-invariant, if and only if, \( A \in \mathcal{M}_{6 \times 6}^2 \) or \( A \in \mathcal{M}_{6 \times 6}^3 \).

(2) Next, we give a numerical example: Assume

\[
A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{M}_{6 \times 6}^2.
\]

(223)

Then \( \mathbb{R}^6 \) is \( A \)-invariant space. For instance, let

\[
X = \begin{bmatrix} 1 + i & 2 & 1 & -i & 0 & 0 \end{bmatrix}^T \in \mathbb{C}^6.
\]

(224)

Then

\[
A \preceq X = iX.
\]

Motivated by (225), we give the following definition.

**Definition 235** Assume \( A \in \mathcal{M} \) and \( X \in \mathcal{V} \). If

\[
A \preceq X = \alpha X, \quad \alpha \in \mathbb{F}, X \neq 0,
\]

(226)

then \( \alpha \) is called an eigenvalue of \( A \), and \( X \) is called an eigenvector of \( A \) with respect to \( \alpha \).

**Example 236** Recall Example 234. In fact it is easy to verify that matrix \( A \) in (223), as a linear mapping on \( \mathbb{R}^6 \), has 6 eigenvalues: Precisely,

\[
\sigma(A) = \{ i, -i, 0, 0, 0, 1 \}.
\]

Correspondingly, the first eigenvector is \( X \), defined in (224). The other 5 eigenvectors are \((1-i, 2, 1+i, 0, 0, 0)^T, (1, 1, 1, 0, 0, 0)^T, (0, 0, 0, 0, 0, 0)^T \) (this is a root vector), \((0, 0, 0, 0, 0, 1)^T \), and \((0, 0, 0, 1, 1, 1)^T \), respectively.

Assume \( S = \mathcal{V}_t \) is \( A \)-invariant with respect to \( \preceq \), that is,

\[
A \preceq S \subset S.
\]

(227)

Then the restriction \( A|_S \) is a linear mapping on \( S \). It follows that there exists a matrix, denoted by \( A|_t \in \mathcal{M}_{t \times t} \), such that \( A|_S \) is equivalent to \( A|_t \). We state it as a proposition.

**Proposition 237** Assume \( A \in \mathcal{M} \) and \( S = \mathcal{V}_t \) is \( A \)-invariant. Then there exists a unique matrix \( A|_t \in \mathcal{M}_{t \times t} \), such that \( A|_S \) is equivalent to \( A|_t \). Precisely,

\[
A \preceq X = A|_t X, \quad \forall X \in S.
\]

(228)

\( A|_t \) is called the realization of \( A \) on \( S = \mathcal{V}_t \).

**Remark 238** (1) To calculate \( A|_t \) from \( A \) is easy. In fact, it is clear that

\[
\text{Col}_i (A|_t) = A \preceq \delta_i, \quad i = 1, \ldots, t.
\]

(229)
(2) Consider Example 234 again.

\[
\text{Col}_1(A|_6) = A \preceq \delta^3_6
\]

\[
= \left[\begin{array}{cccc}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right] \otimes I_3
\]

\[
= (1, 1, 0, 0, 0, 0)^T.
\]

Similarly, we can calculate all other columns. Finally, we have

\[
A|_6 = \left[\begin{array}{cccccc}
1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\]

Finally, given a matrix \( A \in M^i_{\mu} \), we would like to know whether it has (at least one) type-1 invariant subspace \( S = V_t \)?

Then as a corollary of Proposition 231, we can prove the following result.

**Corollary 239** Assume \( A \in M^i_{\mu} \), then \( A \) has (at least one) type 1 invariant subspace \( S = V_t \), if and only if,

\[
\mu_y = 1; \quad (230)
\]

**Proof.** According to Proposition 231, \( \mu_y = 1 \) is obvious necessary. We prove it is also sufficient. Assume \( i \) is factorized into its prime factors as

\[
i = \prod_{j=1}^{n} i_j^{k_j}, \quad (231)
\]

and correspondingly, \( \mu_x \) is factorized as

\[
\mu_x = \prod_{j=1}^{n} i_j^{t_j} p, \quad (232)
\]

where \( p \) is co-prime with \( i \); \( t \) is factorized as

\[
t = \prod_{j=1}^{n} i_j^{t_j} q, \quad (233)
\]

where \( q \) is co-prime with \( i \).

Using Proposition 231 again, we have only to prove that there exists at least one \( t \) satisfying (218). Calculate that

\[
(i\mu_x, t) = \prod_{j=1}^{n} i_j^{\text{max}(t_j+k_j, r_j)} \text{lcm}(p, q);
\]

\[
\mu_x t = \prod_{j=1}^{n} i_j^{t_j+r_j} pq.
\]

To meet (218) a necessary condition is: \( p \) and \( q \) are co-prime. Next, fix \( j \), we consider two cases: (i) \( r_j > k_j + t_j \): Then on the LHS (left hand side) of (218) we have factor \( \prod_{j=1}^{n} i_j^{t_j} \) and on the RHS of (218) we have factor \( i_j^{t_j+k_j+r_j} \). Hence, as long as \( t_j = 0 \), we can choose \( r_j > k_j \) to meet (218). (ii) \( r_j < k_j + t_j \): Then on the LHS we have factor \( i_j^{k_j+t_j} \) and on the RHS we have factor \( i_j^{t_j+k_j+r_j} \). Hence, as long as \( r_j = k_j, \) (218) is satisfied. \( \square \)

Using above notations, we also have the following result

**Corollary 240** Assume \( \mu_y = 1 \). Then \( V_t \) is \( A \)-invariant, if and only if, (i) for \( t_j = 0 \), the corresponding \( r_j \geq k_j \); (ii) for \( t_j > 0 \), the corresponding \( r_j = k_j \).

**Example 241** Recall Example 234 again.

(1) Since the matrix \( A \) defined in (223) is in \( M^2_{0.5} \). We have \( i = 2, \mu_y = 1, \mu_x = 2 = ip \) and hence \( p = 1 \). According to Corollary 240, \( S = V_t \) is \( A \)-invariant, if and only if, \( t = iq = 2q \) and \( q \) is co-prime with \( i = 2 \). Hence

\[
V_{2(2n+1)}, \quad n = 0, 1, 2, \ldots ,
\]

are type-1 invariant subspaces of \( A \).
(2) Assume \( q = 5 \). Then the restriction is

\[
A|_{\mathbb{R}^{10}} = A|_{[10]} = \begin{bmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

The eigenvalues are

\[\sigma(A|_{\mathbb{R}^{10}}) = \{-1, i, -i, 0, 1, 0, 0, 0, 1\}.\]

The corresponding eigenvectors are:

\[
E_1 = (0, 1, 0, 1, 2, 0, 0, 0, 0, 0)^T \\
E_2 = (0.3162 + 0.1054i, 0.5270, 0.4216 - 0.2108i, 0.3162 - 0.4216i, 0.1054 - 0.3162i, 0, 0, 0, 0, 0)^T \\
E_3 = (0.3162 - 0.1054i, 0.5270, 0.4216 + 0.2108i, 0.3162 + 0.4216i, 0.1054 + 0.3162i, 0, 0, 0, 0, 0)^T \\
E_4 = (1, 1, 1, 1, 1, 0, 0, 0, 0, 0)^T \\
E_5 = (2, 1, 0, 1, 0, 0, 0, 0, 0, 0)^T \\
E_6 = (0, 0, 0, 0, 0, 0, 1, 0, 0, 0)^T \\
E_7 = (0, 0, 0, 0, 0, 0, 1, 1, 0, 0)^T \\
E_8 = (0, 0, 0, 0, 0, 0, 0, 0, 1, 0)^T \\
E_9 = (0, 0, 0, 0, 0, 0, 0, 1, 0, 0)^T \\
E_{10} = (0, 0, 0, 0, 0, 1, 1, 1, 1, 1)^T.
\]

Note that \( R_6, R_7 \) are two root vectors. That is,

\[A \preceq R_6 = R_7; \quad A \preceq R_7 = E_8.\]

Remark 242 In fact, the set of type-1 \( A \)-invariant subspace depends only on the shape of \( A \). Hence we define

\[\mathcal{I}_\mu^i := \{V_i \mid V_i \text{ is } A \in \mathcal{M}_\mu^i \text{ invariant}\}.
\]

We can also briefly call \( \mathcal{I}_\mu^i \) the set of \( \mathcal{M}_\mu^i \)-invariant subspaces.

Before ending this subsection we consider a general type-1 invariant subspace \( S \subset \mathcal{V}_i \). The following proposition is obvious.

**Proposition 243** If \( S \subset \mathcal{V}_i \) is an \( A \)-invariant subspace, then \( \mathcal{V}_i \) is also an \( A \)-invariant subspace.

Because of Proposition 243 searching \( S \) becomes a classical problem. Because we can first find a matrix \( P \in \mathcal{M}_{1 \times \ell} \), which is equivalent to \( A|_{\mathcal{V}_i} \). Then \( S \) must be a classical invariant subspace of \( P \).

7.5 Type-2 Invariant Subspace

Denote the set of Type-1 \( A \)-invariant subspaces as

\[\mathcal{I}_A := \{V_s \mid V_s \text{ is } A - \text{invariant}\}.
\]

Assume \( A \in \mathcal{M}_\mu^i \). Then \( \mathcal{I}_A = \mathcal{I}_\mu^i \).

To assure \( \mathcal{I}_A \neq \emptyset \), through this subsection we assume \( \mu_y = 1 \). We give a definition for this.

**Definition 244** (1) Assume \( A \in \mathcal{M}_\mu \). A is said to be a bounded operator (or briefly, A is bounded,) if \( \mu_y = 1 \).

(2) A sequence \( \{V_i \mid i = 1, 2, \ldots\} \), is called an \( A \) generated sequence if

\[A \preceq V_i \subset V_{i+1}, \quad i = 1, 2, \ldots.
\]

(3) A finite sequence \( \{V_i \mid i = 1, 2, \ldots, p\} \), is called an \( A \) generated loop if \( V_p = V_1 \).

**Lemma 245** Assume there is an \( A \) generated loop \( V_p, V_{q_1}, \ldots, V_{q_r}, V_p \), as depicted in (234).

\[
V_p \xrightarrow{A} V_{q_1} \xrightarrow{A} \cdots \xrightarrow{A} V_{q_r} \xrightarrow{A} V_p \quad (234)
\]

Then

\[q_j = p, \quad j = 1, \ldots, r. \quad (235)\]
Proof. Assume $A \in \mathcal{M}_s$. According to Definition 219, we have the following dimension relationship:

\begin{align}
    s_0 &:= \text{lcm}(i\mu_x, p) \quad \Rightarrow \quad q_1 = \mu s_0; \\
    s_1 &:= \text{lcm}(i\mu_x, q_1) \quad \Rightarrow \quad q_2 = \mu s_1; \\
    &\vdots \\
    s_{r-1} &:= \text{lcm}(i\mu_x, q_{r-1}) \quad \Rightarrow \quad q_r = \mu s_{r-1}; \\
    s_r &:= \text{lcm}(i\mu_x, q_r) \quad \Rightarrow \quad p = \mu s_r.
\end{align}

Next, we

\begin{align}
    \text{set } s_0 := t_0p &\quad \text{then } q_1 = \mu t_0p; \\
    \text{set } s_1 := t_1q_1 &\quad \text{then } q_2 = \mu^2t_10p; \\
    &\vdots \\
    \text{set } s_{r-1} := t_{r-1}q_{r-1} &\quad \text{then } q_r = \mu^rt_{r-1}\cdots t_00p; \\
    \text{set } s_r := t_rq_r &\quad \text{then } p = \mu^{r+1}t_r\cdots t_00p.
\end{align}

We conclude that

$$
\mu^{r+1}t_r\cdots t_0 = 1. \quad (237)
$$

Equivalently, we have

$$
\frac{\mu^r}{\mu^1} = t_rt_{r-1}\cdots t_0
$$

It follows that

$$
\mu^1 = 1.
$$

Define

$$
s_r = \text{lcm}(i\mu_x, q_r) := i\mu_x\xi,
$$

where $\xi \in \mathbb{N}$. Then from the last equation of (236) we have

$$
p = i\xi. \quad (238)
$$

That is,

$$
\mu \text{lcm}(i\mu_x, q_r) = i\xi.
$$

Using (237), and the expression

$$
q_r = \mu^rt_{r-1}\cdots t_00p,
$$

we have

$$
\text{lcm} \left( \mu_x, \frac{\mu_x\xi}{t_r} \right) = \xi\mu_x.
$$

From above it is clear that

$$
t_r|\mu_x \quad \text{gcd}(\mu_x, \xi) = 1. \quad (239)
$$

Next, using last two equation in (236), we have

$$
\mu \text{lcm}(i\mu_x, \mu \text{lcm}(i\mu_x, q_{r-1})) = p = i\xi.
$$

Similar to the above argument, we have

$$
\xi\mu_x \mid \text{lcm} \left( \mu_x, \mu \text{lcm}(\mu, \frac{q_{r-1}}{t_r}) \right).
$$

Hence

$$
\xi\mu_x \mid \text{lcm} \left( \mu_x, \mu \text{lcm}(\mu, \frac{\mu_x^2}{t_r}) \right).
$$

To meet this requirement, it is necessary that

$$
t_r | t_{r-1} \mu_x^2.
$$

Continuing this process, finally we have

$$
t_r t_{r-1}\cdots t_s \mu_x^{r-s+1}, \quad s = r-1, r-2, \cdots, 0. \quad (240)
$$

Combining (237) with (240) yields that

$$
t_s = \mu_x, \quad s = 0, 1, \cdots, r.
$$

That is, $q_1 = q_2 = \cdots = q_r = p$. \hfill \Box

**Theorem 246** A finite dimensional subspace $S \subset \mathcal{V}$ is $A$-invariant, if and only if, $S$ has the following structure:

$$
S = \oplus_{i=1}^t S^i, \quad (241)
$$

where

$$
A \ll S^i \subset S^{i+1}, \quad i = 1, \cdots, \ell - 1.
$$

Proof. Since $S$ is of finite dimension, there are only finite $\mathcal{V}_i$, such that

$$
S^i := S \cap \mathcal{V}_i \neq \{0\}.
$$

Now for each $0 \neq X_0 \in S^j \subset \mathcal{V}_i$, we construct $X_1 := A \ll X \in \mathcal{V}_j$ for certain $t_r$. Note that $S$ is $A$-invariant, if for all $X_0 \in S^j$, we have $t_r = t_j$, then this $S^j = S^\ell$.\hfill \Box
is the end element in the sequence. Otherwise, we can find a successor \( S' = S \cap V_s \). Note that since there are only finite \( S' \), according to Lemma 245, starting from \( X_0 \in S \) there are only finite sequence of different \( S' \) till it reach an \( A \)-invariant \( S' \) (equivalently, \( A \)-invariant \( V_s \)). The claim follows.

7.6 Higher Order Linear Mapping

**Definition 247** Let \( A \in M_{A_t}^I \), \( V_t \) is \( A \)-invariant subspace of \( V \). That is, \( A : V_t \to V_t \) is a linear mapping. The higher order linear mapping of \( A \), is defined as

\[
\begin{aligned}
A^{[1]} &\ll X := A \ll X, \quad X \in V_t \\
A^{[k+1]} &\ll X := A \ll (A^{[k]} \ll X), \quad k \geq 1.
\end{aligned}
\]

**Definition 248** Let \( X \in V \). The \( A \)-sequence of \( X \) is the sequence \( \{X_i\} \), where

\[
\begin{aligned}
X_0 &= X \\
X_{i+1} &= A \ll X_i, \quad i = 0, 1, 2, \ldots .
\end{aligned}
\]

Using notations (231)-(233), we have the following result.

**Lemma 249** Assume \( A \) is bounded. Then \( A \ll X \in V_s \in I_A \), if and only if, for each \( 0 < j < n \), one of the following is true:

\[ r_j = 0; \]  
\[ \text{or} \]

\[ t_j \leq k_j + r_j. \]

**Proof.**

\[
\text{lcm}(i_{X_t}, t) = \text{lcm} \left( \prod_{j=1}^{n} \frac{\max(k_j + r_j, t_j)}{\prod_{j=1}^{n} t_j}, \prod_{j=1}^{n} t_j \right)
\]

and

\[
\hat{t} = \text{lcm}(i_{X_t}, t) \mu
\]

\[
= \prod_{j=1}^{n} \frac{\max(k_j + r_j, t_j) - r_j \text{lcm}(p, q)}{\text{lcm}(p, q)/p}.
\]

Then

\[
\text{lcm}(i_{X_t}, \hat{t}) = \prod_{j=1}^{n} t_j^{\max(k_j + r_j, \max(k_j + r_j, t_j) - r_j)} \text{lcm}(p, q),
\]

and

\[
\mu_{X_t} \hat{t} = \prod_{j=1}^{n} t_j^{\max(k_j + r_j, t_j)} \text{lcm}(p, q).
\]

Now \( A \ll X \in V_s \in I_A \) is assured by (218), which leads to

\[
\text{max}(k_j + r_j, \text{max}(k_j + r_j, t_j) - r_j) = \text{max}(k_j + r_j, t_j).
\]

Case 1 : \( t_j > k_j + 2r_j \), (245) leads to \( r_j = 0 \). Hence, we have

\[
r_j = 0 \text{ and } t_j > k_j.
\]

Case 2 : \( k_j + r_j \leq t_j \leq k_j + 2r_j \), which leads to \( k_j + r_j = t_j \).

Case 3 : \( t_j < k_j + r_j \), which assures (245). Combining Case 2 and Case 3 yields

\[
t_j \leq k_j + r_j.
\]

Note that when \( r_j = 0 \), if \( t_j \leq k_j \) we have (247). Hence \( t_j \leq k_j \) is not necessary. The conclusion follows.

The following result is important.

**Theorem 250** Let \( A \in M_{I}^I \) be bounded. Then for any \( X \in V \) the \( A \)-sequence of \( X \) will enter a \( V_s \in I_A \) at finite steps.

**Proof.** Assume \( A \ll X \in V_t \). Using notations (231)-(233), it is easy to calculate that after one step the \( j \)-th index of \( t \) becomes

\[
t_j^1 = \text{max}(k_j + r_j, t_j^0) - r_j.
\]

Assume \( r_j = 0 \), this component already meets the requirement of Lemma 249. Assume for some \( j, r_j > 0 \) and \( t_j^0 > k_j + r_j \), then we have

\[
t_j^1 = t_j^0 - r_j < t_j^0.
\]

Hence after finite times, say \( k \), the \( j \)-th index of \( X_k \), denoted by \( t_j^k \), satisfies

\[
t_j^k = t_j^0 - kr_j.
\]

will satisfy (244), and as long as (244) holds, \( t_j^s = t_j^k \) \( \forall s > k \). Hence, after finite steps either (243) or (244) (or both) is satisfied. Then at the next step the sequence enters into \( V_t \in I_A \).
**Definition 251** Given a polynomial

\[ p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0, \]  

(249)
a matrix \( A \in \mathcal{M} \) and a vector \( X \in \mathcal{V} \).

1. \( p(x) \) is called an \( A \)-annihilator of \( X \), if

\[ p(A)X := A^n X \preceq c_{n-1} A \preceq \cdots c_1 A X \preceq c_0 = 0. \]  

(250)

2. Assume \( q(x) \) is the \( A \)-annihilator of \( X \) with minimum degree, then \( q(x) \) is called the minimum \( A \)-annihilator of \( X \).

The following result is obvious:

**Proposition 252** The minimum \( A \)-annihilator of \( X \) divides any \( A \)-annihilator of \( X \).

The following result is an immediate consequence of Theorem 250.

**Corollary 253** Assume \( A \) is bounded, then for any \( X \in \mathcal{V} \) there exists at least one \( A \)-annihilator of \( X \).

**Proof.** According to Theorem 250, there is a finite \( k \) such that \( A^k X \in \mathcal{V}_s \) with \( \mathcal{V}_s \) being \( A \)-invariant. Now in \( \mathcal{V}_s \) assume the minimum annihilator polynomial for \( A^k X \) is \( q(x) \), then \( p(x) = x^k q(x) \) is an \( A \)-annihilator of \( X \). \( \square \)

**Example 254** (1) Assume \( A \in \mathcal{M}_{2,3} \). Since \( \mu_g = 2 \neq 1 \), we know any \( X \) does not have its \( A \)-annihilator.

Now assume \( X_0 \in \mathcal{V}_k \), where \( k = 3^p p \), and \( 3, p \) are co-prime. Then it is easy to see that the \( A \)-sequence of \( X_0 \) has the dimensions, \( \dim (X_i) := d_i \), where \( d_1 = 2 \times 3^p \), \( d_2 = 2^2 \times 3^{p-2} \), \( \ldots \), \( d_s = 2^s p \), \( d_s+1 = 2^{s+1} p \), \( d_{s+2} = 2^{s+2} p \). It can not reach a \( \mathcal{V}_t \in \mathcal{I}_A \).

(2) Given

\[
A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

We try to find the minimum \( A \)-annihilator of \( X \). Set \( X_0 = X \). It is easy to see that

\[ X_1 = A \preceq X_0 \in \mathcal{V}_6 \in \mathcal{I}_A. \]

Hence, we can find the annihilator of \( X \) in the space of \( \mathbb{R}^6 \). Calculating

\[
X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad X_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad X_6 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},
\]

it is easy to verify that \( X_1, X_2, X_3, X_4, X_5 \) are linearly independent. Moreover,

\[ X_6 = X_1 + X_2 + X_3 + X_4 - X_5. \]

The minimum \( A \)-annihilator of \( X = X_0 \) follows as

\[ p(x) = x^6 + x^5 - x^4 - x^3 - x^2 - x. \]

7.7 Invariant Subspace on Equivalence Space

Denote

\[ \Omega_M := \mathcal{M}/\leftrightarrow. \]

We extend the vector product to the equivalence spaces.

**Definition 255** Let \( A \in \mathcal{M} \) and \( B \in \mathcal{M}_{x,q} \). Then we define \( \preceq : \Sigma_M \times \Omega_M^q \to \Omega_M^q \) as

\[ \langle A \rangle \preceq \langle B \rangle := [A \preceq B]. \]  

(251)

The following proposition shows that (251) is well defined.

**Proposition 256** (251) is independent of the choice of \( A \) and \( B \).
Proof. Assume \( A_1 \in \langle A \rangle \) is irreducible and \( A_1 \in M_{m \times n} \); \( B_1 \in \langle B \rangle \) is also irreducible and \( B_1 \in M_{p \times q}, A_1 = A_1 \otimes I_i \) and \( B_1 = B_1 \otimes 1_j \) Set \( s = \text{lcm}(n, p), t = \text{lcm}(ni, pj) \), and \( s\xi = t \). Then

\[
\begin{align*}
(A_i &\preccurlyeq B_j) \\
&= (A_i \otimes I_{ni/pj}) (B_j \otimes 1_{t/pj}) \\
&= (A_i \otimes I_i \otimes I_{ni/p}) (B_1 \otimes 1_{t/pj}) \\
&= (A_i \otimes I_i \otimes 1_{t/p}) \\
&= (A_i \preccurlyeq B_1) \otimes 1_\xi \leftrightarrow (A_1 \preccurlyeq B_1).
\end{align*}
\]

\( \square \)

Precisely speaking, because \( V \) is not a vector space, “invariant subspace” is not a rigorous subspace. But \( \Omega_V := V/\leftrightarrow \) is a vector space. It is easy to see that the results about \( V \), and \( M_{\times n} \) can be extended to \( \Omega_V \) and \( \Omega_M \). For instance, \( \langle A \rangle \in \Sigma_n \) is a bounded operator on \( \Omega_V \) if and only if, \( \mu_y = 1 \).

8 Conclusion

Matrix theory is one of the most fundamental and useful tools in modern science and technology. But one of the major weaknesses is its dimension restriction. To overcome this barrier, the purpose of this paper is to set up a framework for an almost dimension-free matrix theory.

First, we review the STP (\( \kappa \)), which extends the conventional matrix product to overall matrices \( M \). The related monoid structure for \( (M, \kappa) \) is obtained. The matrix equivalence \( \sim \) is proposed. A lattice structure over each equivalence class is obtained. The equivalence space, as the quotient space \( M/\sim \) is discussed.

Second, the set of overall matrices is partitioned into subspaces \( M = \bigcup_{\mu \in \mathbb{Q}_+} M_\mu \). The STA (\( \oplus \)) is proposed. Under this addition the quotient spaces \( \Sigma_n = M_\mu / \sim \) become vector spaces. Certain geometric and algebraic structures are revealed. Including topological structure, inner product structure, differential manifold structure, etc.

Particularly, when \( \mu = 1 \), (corresponding to square matrices) we have extended the Lie algebra and Lie group theory to \( M_1 \). A fiber bundle structure, called the discrete bundle, is proposed for \( M_\mu \) and the extended Lie group and Lie algebra.

Finally, the set of overall vectors \( V \) are considered as a universal vector space, based on the vector equivalence \( \leftrightarrow \). A matrix \( A \) of any dimension can be considered either a linear mapping on \( V \), or a subspace generated by its columns. The \( A \)-invariant subspace is discussed in details. Many key concepts such as eigenvalue/eigenvector, characteristic polynomial of a matrix have been extended from square matrices to no-square matrices.

It was said by Asimov that “Only in mathematics is there no significant correction - only existence. Once the Greeks had developed the deductive method, they were correct in what they did, correct for all the time.” [38] All extensions we did in this paper consist with the classical ones. That is, when the dimension restrictions required by the classical matrix theory are satisfied the new operators proposed in this paper coincide with the classical ones.

There are many questions remain for further discussion. For instance, is it possible to construct an equivalence over \( M \), which is consistent with certain matrix product, such that the quotient space becomes a vector space?

The followings are some possible equivalences on \( M \).

(1) Equivalence 1:

**Definition 257** Let \( A, B \in M \). \( A \) is said to be equivalent to \( B \), if there exist \( I_i, I_j, 1_s, 1_t \) such that

\[
1^T_A \otimes A \otimes I_i = 1^T_B \otimes B \otimes I_j \tag{252}
\]

It is easy to verify that \( \simeq \) is an equivalence relation. Moreover, similar to matrix equivalence or vector equivalence, we have the following result:

**Theorem 258** Assume \( A \simeq B \), then there exists a \( \Lambda \), such that

\[
A = 1^T_p \otimes \Lambda \otimes I_s \quad B = 1^T_q \otimes \Lambda \otimes I_t. \tag{253}
\]

Hence the lattice structure similar to matrix equivalence exists.

It may be considered as a combination of matrix and vector equivalences. Unfortunately, (i) it is not consistent with STP (\( \kappa \)); (ii) the quotient space is not a vector space.

(2) Equivalence 2:
Definition 259 Let $A$, $B \in \mathcal{M}$. $A$ is said to be equivalent to $B$, if there exist $1_i$, $1_j$, $1_s$, $1_t$ such that

$$1^T_\alpha \otimes A \otimes 1_i = 1^T_\beta \otimes B \otimes 1_j \quad (254)$$

The lattice structure can also be determined in a similar way. Moreover, the quotient space is a vector space. Unfortunately, a proper product, which is consistent with the equivalence, is unknown.

Further geometric/algebraic structures may be investigated.

(1) More geometric structure on equivalence space could be interesting. For instance, a Riemannian geometric structure or a Symplectic geometric structure may be posed on the equivalence space.

(2) Under the STP and $\mp$ ($\pm$), for any $A \in \mathcal{M}_\mu$ and $B \in \mathcal{M}_\lambda$,

$$[A, B] = A \ltimes B \mp B \ltimes A \quad (255)$$

is well defined. Moreover, the three requirements (145)-(147) in Definition 143 can also be satisfied (under obvious modification). Exploring the properties of this generalized Lie algebra is challenging and interesting.

One may be more interested in its applications. For instance, can we use the extended structure proposed in this paper to the analysis and control of certain dynamic systems? Particularly, we may consider the following special cases:

(1) Consider a dynamic system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n,$$

where $A(t)$ satisfies

$$\dot{A}_t = V(x),$$

where $V(x)$ is a vector field on $\mathcal{M}_{n \times n}$. What can we say about this system? Is it possible to extend this system to the equivalent space $\Sigma$?

(2) A dimension-varying dynamic control system as

$$\begin{cases} x(t + 1) = A \triangleleft x(t) \triangleleft B \triangleleft u(t) \\ y(t) = C \triangleleft x(t), \end{cases}$$

where $x(t) \in \mathcal{V}$.

What can we say about this, say, controllability? Observability etc.?

In one word, this paper could be the beginning of investigating dimension-free matrix theory and its applications.

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