Comments on $SO/Sp$ Gauge Theories from Brane Configurations with an O6 Plane

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Abstract

We use the M theory approach in the presence of an orientifold O6 plane to understand some aspects of the moduli space of vacua for $N = 1$ supersymmetric $SO(N_c)/Sp(N_c)$ gauge theories in four dimensions. By exploiting some general properties of the O6 orientifold, we reproduce some results obtained previously with an orientifold O4 plane when the flavor group arises from the worldvolume dynamics of D6 branes. By using semi-infinite D4 branes instead of D6 branes, we derive the most general form of the rotated curve describing the moduli space of vacua for $N = 1$ supersymmetric gauge theory.

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1 Introduction

Recently, important progress has been made in studying the strongly coupled dynamics of low energy supersymmetric gauge theories in various dimensions. The D-brane dynamics provides a powerful geometrical description for the gauge theories which are obtained on their world-volume. A detailed account of many aspects of the interrelation between D-brane dynamics and supersymmetric gauge theory in different dimensions can be found in the very interesting review of Giveon and Kutasov [1].

Many aspects of the strongly coupled dynamics of supersymmetric gauge theories have been explained by using string theory results. The mirror symmetry of the $N = 4$ gauge theory in 3 dimensions was described in [2] as being due to the nonperturbative S-duality of type IIB string theory (See also [3]). A stringy derivation of Seiberg’s duality for $N = 1$ supersymmetric $SU(N_c)$ gauge theory with flavors appeared in [4, 5]. This description was generalized to the brane configurations with orientifolds where the gauge group is either $SO(N_c)$ or $Sp(N_c)$ [6] (See also [7] for an equivalent geometrical approach).

In string theory (10 dimensions) there are singularities where the branes are touching each other. The singularities are removed in 11 dimensions where the brane configuration becomes smooth. Both the D4 branes and NS brane used in type IIA string theory become a unique M5 brane in 11 dimensions. The D6 branes are the Kaluza-Klein monopoles given by Taub-NUT space [8]. The world volume of the M5 brane was observed to be the product of the four dimensional spacetime and the Seiberg-Witten curve uniquely identified with the solutions for the Coulomb branch of the four dimensional gauge theories [9]. Further generalizations of this configuration were obtained by inserting an O4 orientifold [10]. The low energy description of $N = 1$ supersymmetric $SU(N_c)$ gauge theories with flavors in 4 dimensions have been found in [11, 12, 13] (See also [14, 15] for theories with an orientifold 4 plane).

Many results have been very recently obtained from brane configurations in the presence of an O6 plane [5, 16, 17, 18, 19, 20, 21]. The brane configuration is very similar with the one constructed with an orientifold O4 plane but it allows us to obtain new gauge theories on the world-volume of the D4 branes lying between two NS branes like theories with $SO \times SU \times SU...$ gauge groups [16] or theories with matter in symmetric and antisymmetric representations [17, 18, 19, 20, 21].

In this paper, we study the M5 brane with an orientifold O6 plane, along the lines of [11, 12, 14] in order to understand the moduli space of vacua of $N = 1$ supersymmetric $SO(N_c)/Sp(N_c)$ gauge theories in 4 dimensions. We will see how our previous results of [14, 15] are rederived in the presence of an O6 orientifold instead of an O4 orientifold.
We also consider the case of semi-infinite D4 branes instead of D6 branes and explicitly derive the form of the M-theory curve for the rotated NS branes configuration by using the method developed in [13]. We obtain as solution for the result constructed in [20].

In section 2 we briefly review the results obtained in [16] for the \( N = 2 \) configuration. In section 3 we consider the case when the flavor group is given by D6 branes and we rederive the results of [14, 15] in the presence of O6 orientifold. In section 4 we explicitly derive the M-theory curve for rotated configuration in the presence of semi-infinite D4 branes.

## 2 Brane Configuration for \( N = 2 \)

We would like to review the work of Landsteiner and Lopez [16] where they considered brane configurations of type IIA string theory giving \( N = 2 \) SO\((N_c)\) or Sp\((N_c)\) gauge theory. In order to study the orthogonal and symplectic groups, an O6 plane parallel to the D6 branes is added into the SU\((N_c)\) gauge theory brane configuration. The O6 plane does not break further supersymmetry and there exist two possible signs for the O6 plane charges. The brane configuration consists of NS5 branes, D4 branes and D6 branes together with an orientifold O6 plane. Working on the double covering of the orientifold, we denote their worldvolumes:

\[
\begin{align*}
\text{NS5} & : (x^0, x^1, x^2, x^3, x^4, x^5) \\
\text{D4} & : (x^0, x^1, x^2, x^3, x^6) \\
\text{D6/O6} & : (x^0, x^1, x^2, x^3, x^7, x^8, x^9).
\end{align*}
\]

Here the D4 branes suspended between two NS5 branes are finite in \( x^6 \) direction. O6 plane acts as a mirror in \((x^4, x^5, x^6)\) directions due to the spacetime reflection and the two NS5 branes are mirror images of each other under this orientifold projection. Every D4 branes which does pass through \( x^4 = x^5 = 0 \) should have its mirror image.

As usual we write two complex coordinates as \( v = x^4 + ix^5 \), \( s = (x^6 + ix^{10})/R, t = e^{-s} \) where \( x^{10} \) is the 11th coordinate of M theory which is compactified on a circle of radius \( R \).

• SO\((2N_c)\) Case

Let us first construct the M theory curve for the SO\((2N_c)\) gauge theory in the presence of O6 plane. The O6 plane in SO gauge theory carries a +4 D6 brane charge. Thus the singularity associated with the O6-plane can be expressed as a quotient of a
surface

\[ xy = v^4 \]  

(2.2)

in \( \mathbb{C}^3 \) by a \( \mathbb{Z}_2 \) symmetry \( x \leftrightarrow y \) and \( v \leftrightarrow -v \) which corresponds to a \( D_4 \) singularity as observed in [16, 22]. Moreover, in the presence of D6 branes, the surface (2.2) in \( \mathbb{C}^3 \) will be generalized to

\[ xy = v^4 \prod_{i=1}^{N_f} (v^2 - m_i^2) \]  

(2.3)

where the orientifold projection allows only the configuration invariant under the \( x \leftrightarrow y \) and \( v \leftrightarrow -v \). In summary, M-theory for \( SO \) gauge group in the presence of O6 plane and D6 branes should be described on the quotient space of \( xy = v^4 \prod_{i=1}^{N_f} (v^2 - m_i^2) \) by a \( \mathbb{Z}_2 \) symmetry \( x \leftrightarrow y \) and \( v \leftrightarrow -v \) replacing the ordinary flat spacetime \( \mathbb{R}^3 \times S^1 \) (the coordinates being \( x^4, x^5, x^6, x^{10} \)).

The directions corresponding to (2.3) are transversal to the D6 branes. The D6 branes are located at \( x = y = 0, v = \pm m_i \) and \( N_f \) is the number of “physical” D6 branes.

Let us describe how the above brane configuration appears in M theory as a a generically smooth single M5 brane whose worldvolume has, in addition to four spacetime dimensions, another 2 directions given by \( \Sigma \). Furthermore, we can identify \( \Sigma \) with the Seiberg-Witten curve [23] which determines the solution of the Coulomb branch of the gauge theory. Following [9], the Seiberg-Witten curve \( \Sigma \) can be described by

\[ xy = \Lambda_{N=2}^{4N_c-4-2N_f} v^4 \prod_{i=1}^{N_f} (v^2 - m_i^2) \]  

(2.4)

\[ x + y = B(v^2). \]  

(2.5)

where \( B \) is an even polynomial of degree \( 2N_c \) in \( v \). We introduced the power of \( \Lambda_{N=2} \) in order to match the dimensions. The projected image of this curve in \( (y, v) \)-space will be given by

\[ y^2 - B(v^2)y + \Lambda_{N=2}^{4N_c-4-2N_f} v^{4} \prod_{i=1}^{N_f} (v^2 - m_i^2) = 0. \]  

(2.6)

It is easy to see that by redefining \( \tilde{y} = y/v^2 \) this reproduces the form of the curve of [23].

- \( SO(2N_c + 1) \) Case
In this case we have a supplementary D4 brane which has no mirror and is stuck at $v = 0$. Then the Seiberg-Witten curves are:

$$xy = \Lambda_{N=2}^{4N_c-2-2N_f} v^4 \prod_{i=1}^{N_f} \left(v^2 - m_i^2\right) (2.7)$$

$$-x + y = vB(v^2) (2.8)$$

with the same orientifold projection as before. The Seiberg-Witten curve will be projected to

$$y^2 - vB(v^2)y - \Lambda_{N=2}^{4N_c-2-2N_f} v^4 \prod_{i=1}^{N_f} \left(v^2 - m_i^2\right) = 0. (2.9)$$

in $(y, v)$-space and one can check that this leads to the result of [23] by redefining $\tilde{y} = -y/v^3$.

**$Sp(N_c)$ Case**

In case of the symplectic gauge group, O6 plane carries a $-4$ D6 brane charge. The orientifold is described by

$$xy = v^{-4}. (2.10)$$

with the same orientifold projection as before. As before, the presence of D6 branes modify the previous equation to

$$xy = v^{-4} \prod_{i=1}^{N_f} \left(v^2 - m_i^2\right). (2.11)$$

with the same orientifold projection as before. The curve is then given by

$$xy = \Lambda_{N=2}^{4N_c+4-2N_f} v^{-4} \prod_{i=1}^{N_f} \left(v^2 - m_i^2\right) (2.12)$$

$$x + y = C(v^2) + \Lambda_{N=2}^{2N_c+2-2N_f} v^{-2}. (2.13)$$

with the same orientifold projection as before. Away from the orientifold, the curve is described by a single equation in $(y, v)$-space

$$y^2 - (C(v^2) + \Lambda_{N=2}^{2N_c+2-2N_f} v^{-2} \prod_{i=1}^{N_f} m_i)y + \Lambda_{N=2}^{4N_c+4-2N_f} v^{-4} \prod_{i=1}^{N_f} \left(v^2 - m_i^2\right) = 0 (2.14)$$

Note that $C(v^2)$ is a polynomial in $v$ of degree $2N_c$ and the existence of $v^{-2}$ due to the orientifold projection dominates for small $v$ while it can be ignored for large $v$. If we write $\tilde{y}$ as $v^2 y$ the previous relation becomes the one in [23].
3 Rotated Configuration with D6 Branes

In this section we are going to work only within the brane configuration in order to derive the moduli space of vacua for the corresponding \(N = 1\) theory. We are not going to derive the results in field theory which were obtained in detail in \([14, 15]\) (see also \([24]\)) and we will limit ourselves to some comments at specific points. We refer the interested reader to \([14, 15]\) for the derivation of the field theory results.

- \(SO(2N_c)\) Case

It is convenient to introduce a complex coordinate \(w = x^8 + i x^9\). Before breaking the \(N = 2\) supersymmetry, the two NS5 branes are located at \(w = 0\). Now we rotate the right NS5 brane towards \(w\) direction and this determines its mirror image, the left NS5 brane, to be rotated towards its mirror direction. This gives a mass to the adjoint chiral multiplet in the \(N = 2\) vector multiplet. In order to rotate these NS5 branes, all the D4 branes are placed together and the motion of the D4 branes along \(v\) direction is not possible. Here we are assuming that the \(N = 2\) curve is irreducible. The Higgsing is possible only when the D4 branes breaks on the D6 branes. (This is an interesting case.) For this reason, we will assume that the D6 branes passes through \(v = 0\) i.e. we turn off the bare mass \(m_i\). In short, we are rotating the following curve:

\[
 x + y = v^{2N_c}, \quad xy = \Lambda_{N=2}^{4N_c-4-2N_f} v^{2N_f+4}. \tag{3.1}
\]

The asymptotic behavior of the rotated right NS5 brane and left NS5 brane when \(v \to \infty\) imposes the following boundary conditions for \(SO(2N_c)\)

\[
 w \to \mu v \quad \text{as} \quad v \to \infty, \quad y \sim v^{2N_c}
\]

\[
 w \to -\mu v \quad \text{as} \quad v \to \infty, \quad y \sim \Lambda_{N=2}^{2(2N_c-2-N_f)} v^{2N_f-2N_c+4} \tag{3.2}
\]

in the threefold \(V\) defined by

\[
 xy = \Lambda_{N=2}^{4N_c-4-2N_f} v^{2N_f+4}. \tag{3.3}
\]

Thus the worldvolume of the M5 brane describing the \(N = 1\) \(SO(2N_c)\) gauge theory will be given by \(\mathbb{R}^{3,1} \times \Sigma\), where the algebraic curve \(\Sigma\) fits the boundary conditions \((3.2)\) and is embedded in the threefold \(V\). We will describe the projection of \(\Sigma\) into \((y, v, w)\)-space. First we can compactify the curve \(\Sigma\) by adding two points at infinity \(v \to \infty\). that the functions \(w_+ \equiv w + \mu v\) or \(w_- \equiv w - \mu v\) have a simple pole at one of the points at infinity. This implies that \(\Sigma\) is a rational curve which can be globally parameterized by either \(w_+\) or \(w_-\). Thus we can express the functions \(y\) and \(v\) on \(\Sigma\) in terms of \(w_+\) by rational function.

\[
 v = P(w_+), \quad y = Q(w_+). \tag{3.4}
\]
Since the orientifold projection sends $v \rightarrow -v$ and $y \rightarrow x$, the following equation must hold:

$$v = -P(w_-), \quad x = Q(w_-).$$

(3.5)

Since $v$ and $y$ are finite except at $w_+ = 0, \infty$, these rational functions are polynomials of $w_+$ up to a factor of some power of $w_+$: $P(w_+) = w_+^ap(w_+)$ and $Q(w) = w_+^bq(w_+)$ where $a$ and $b$ are some integers and $p(w_+)$ and $q(w_+)$ are polynomials of $w_+$ which we may assume nonvanishing at $w_+ = 0$. Near one of the points at $w_+ = \infty$, $v$ and $y$ behave as $v \sim \mu^{-1}w_+$ and $y \sim v^{2N_c}$ by (3.2). Thus the rational functions are of the form

$$P(w_+) = w_+^a(w_+^{1-a} + \cdots)/2\mu \quad \text{and} \quad Q(w_+) = \mu^{-2N_c}w_+^b(w_+^{2N_c-b} + \cdots)$$

(3.6)

Now around the other infinity $w_+ = 0$, the curve $\Sigma$ can be parameterized by $1/v$ which vanishes as $w_+ \rightarrow 0$ from the boundary condition. Since $w_+$ and $1/v$ are two coordinates around the neighborhood $w_+ = 0$ in the compactification of $\Sigma$ and vanish at the same point, they must be linearly proportional to each other

$$v \sim \mu^{-1}w_+$$

and

$$y \sim v^{2N_c}$$

Now by putting $w = 0$ in this equation, we obtain

$$2(w_1 + w_2)\mu v = 0.$$  

(3.9)

Since $w_+ \sim 1/v$ in the limit $w_+ \rightarrow 0$, the functions $v$ and $w$ can not vanish simultaneously on $\Sigma$. Hence we have $w_1 = -w_2$. We let $w_0 = w_1$. Now the equation (3.4) becomes

$$v = P(w_+) = \frac{(w_+^2 - w_0^2)\mu}{2w_+}.$$  

(3.10)

Since $y \sim v^{2N_f - 2N_c + 4}$ and $w_+ \sim 1/v$ as $w_+ \rightarrow 0$, we get $b = 2N_c - 4 - 2N_f$ and thus,

$$y = Q(w_+) = \mu^{-2N_c}w_+^{2N_c-4-2N_f}(w_+^{2N_f+4} + \cdots)$$  

(3.11)

By substituting (3.10) and (3.11) into (3.3), we conclude

$$y = Q(w_+) = \mu^{-2N_c}w_+^{2N_c-4-2N_f}(w_+^2 - w_0^2)^{N_f+2}.$$  

(3.12)

Note that the solutions (3.10) and (3.12) satisfy the conditions (3.3). In fact, we have

$$w_- = w_+ - 2\mu v = w_+ - 2\mu P(w_+) = w_0/w_+$$  

(3.13)
and
\[ P(w_0^2/w_+) = -P(w_+) \quad Q(w_0^2/w_+) = x \text{(up to a constant factor)} \quad (3.14) \]
as required by (3.3).

To find the values of \( w_0 \), we observe that the solutions \( v = P(w_+) \) and \( y = Q(w_+) \)
should satisfy (3.1) namely
\[ y^2 - v^{2N_c} y + \Lambda^{4N_c - 2N_f} v^{2N_f + 4} = 0. \quad (3.15) \]
in order to keep the \( U(1) \) symmetry in the \( w \) direction as pointed out in [12]. By plugging
the equations (3.10) and (3.12) in the previous equation this equation and equating the
lowest order terms in \((w_+ \pm w_0)\), we obtain :
\[ w_0 = (-1)^{2N_c - 2N_f} \mu \Lambda_{N=2} \quad \text{or} \quad w_0 = 0. \quad (3.16) \]

As described in [12], the values for \( w \) are just the expectation values for the me-
son. The above results are identical with the ones obtained in [15] both in field theory
(where the values for \( w \) are the expectation values for the meson field \( M \)) and in brane
configuration with an O4 plane. We refer to [15] for the physical interpretation of the
result.

Without matter, the curve describing the \( N = 2 \) Coulomb branch is given by
\[ y^2 - B(v^2)y + \Lambda^{4N_c - 4} v^4 = 0. \quad (3.17) \]
The genus of the curve is \( 2N_c - 1 \) provided \( B(v^2) \) is a general polynomial. By dividing
\( v^4 \) and renaming \( yv^{-2} \) as \( \tilde{y} \), this becomes
\[ \tilde{y}^2 - B(v^2)\tilde{y}v^{-2} + \Lambda^{4N_c - 4} = 0. \quad (3.18) \]
Then this curve is completely degenerate at \( 2N_c - 2 \) points on the Coulomb branch. At
one of these points, the curve has the following form
\[ v^2 = \tilde{y}^{2N_c - 2} + \Lambda^2_{N=2} \tilde{y}^{-\frac{1}{2N_c - 2}}. \quad (3.19) \]

• \( Sp(N_c) \) Case

For this case we can proceed in a very similar way. We just describe the facts without
much details. The boundary conditions from the rotations of two NS5 branes can be
read off easily by looking the behavior of \( y \) and \( v \) in the (2.14) as follows:
\[
\begin{align*}
    w & \to \mu v \quad \text{as} \quad v \to \infty, \quad y \sim v^{2N_c} \\
    w & \to -\mu v \quad \text{as} \quad v \to \infty, \quad y \sim \Lambda^{2(2N_c + 2 - N_f)}_{N=2} v^{2N_f - 2N_c - 4}.
\end{align*} \quad (3.20)
\]
Since the functions $y$ and $v$ on $\Sigma$ can be written in terms of $w_+$ as rational function similar to (3.4) and the extra conditions (3.3) arising from the orientifold action hold as well, by following the same procedure through (3.6) and (3.9) we will get

$$P(w_+) = \frac{(w_+^2 - w_0^2)}{2\mu w_+}. \quad (3.21)$$

Since $y \sim v^{2N_f - 2N_c - 4}$ and $w_+ \sim 1/v$ as $w_+ \to 0$, we get $b = 2N_c + 4 - 2N_f$. This leads to

$$y = Q(w_+) = \mu^{2N_c} w_+^{4-2N_f} (w_+^2 - w_0^2)^{N_f-2}. \quad (3.22)$$

The solutions $v = P(w_+)$ and $y = Q(w_+)$ should satisfy by

$$y^2 - v^{2N_c} y + \Lambda^{4N_c+4-2N_f} v^{2N_f-4} = 0. \quad (3.23)$$

By plugging (3.21) and (3.22) in this equation and taking the limit $w_+ \to \pm w_0$, we obtain

$$w_0 = (-1)^{N_f} \mu \Lambda^{N_c+2}. \quad (3.24)$$

This result is again in accordance with the results obtained in [14] in field theory and from brane configuration in the presence of an O4 plane. We refer to [14] for the physical interpretation of the results.

Without matter, the curve describing the $N = 2$ Coulomb branch is given by

$$y^2 - C(v^2) y + \Lambda^{4N_c+4} v^{-4} = 0. \quad (3.25)$$

The genus of the curve is $2N_c - 1$ provided $C(v^2)$ is a general polynomial. By multiplying $v^4$ and renaming $yv^2$ as $\tilde{y}$, this becomes

$$\tilde{y}^2 - C(v^2) \tilde{y} v^2 + \Lambda^{4N_c+4} = 0. \quad (3.26)$$

Then this curve is completely degenerate at $2N_c + 2$ points on the Coulomb branch. At one of these points the curve has the following form

$$v^2 = \tilde{y}^{2N_c+2} + \Lambda^{2N_c+2} \tilde{y}^{-2N_c-2}. \quad (3.27)$$

4 Rotated Configuration with Semi-infinite D4 Branes

- $SO(2N_c)$ Case.
Now we insert $N_f$ semi-infinite D4 branes extending to the left from the left NS5 brane, and their mirrors extending to the right from the right NS brane. The equation for the Sieberg-Witten curve is given by

$$xy = \Lambda_{N=2}^{4N_c-4-2N_f} v^4$$

(4.1)

$$w = 0$$

(4.2)

$$(-1)^{N_f} \prod_{i=1}^{N_f} (v + m_i)x + \prod_{i=1}^{N_f} (v - m_i)y = B(v^2)$$

(4.3)

with orientifold projection $x \leftrightarrow y$ and $v \leftrightarrow -v$. One observation is in order here: in the case of an O4 plane, the variable $v$ always appears in equations as $v^2$ because for each D4 brane at $v$ there is one at $-v$. The O6 plane introduces the symmetry $x^6 \leftrightarrow -x^6$ besides the $v \leftrightarrow -v$ symmetry. So the effect on semi-infinite D4 branes is the following: each semi-infinite D4 brane ending at $v$ on the left of the left NS brane has a mirror ending at $-v$ on the right of the right NS brane. For this reason in equation (4.3) we have terms with $v$ and not $v^2$.

Since for large $y$ with fixed or small $x$, $y$ corresponds to $t$ and for large $x$ with fixed or small $y$, $x$ corresponds to $t^{-1}$, the equation (4.3) becomes

$$\prod_{i=1}^{N_f} (v - m_i)y \sim B(v^2), \quad \text{for large } y$$

(4.4)

$$\prod_{i=1}^{N_f} (v + m_i)x \sim B(v^2), \quad \text{for large } x.$$  

(4.5)

This shows that the above equations describe the brane configuration under consideration. The Seiberg-Witten curve will be projected to a curve in $(y, v)$-plane given by

$$\prod_{i=1}^{N_f} (v - m_i)y^2 - B(v^2)y + (-1)^{N_f} \Lambda_{N=2}^{4N_c-4-2N_f} v^4 \prod_{i=1}^{N_f} (v + m_i) = 0.$$  

(4.6)

Since in a brane configuration of these models the Coulomb branch is not modified by a change of the singular locus, we may simplify the spacetime by resolving the singularities. We desingularize the surface $S$, $xy = \Lambda_{N=2}^{4N_c-4-2N_f} v^4$, by blowing up the ideal $(x, y, v^2)$. Then the resolved surface $S'$ can be described by

$$x'y' = \Lambda_{N=2}^{4N_c-4-2N_f}$$

(4.7)

and the new surface $S'$ will map onto the singular surface $S$ by $x = x'v^2$ and $y = y'v^2$. Now the Seiberg-Witten curve on this new space will be given by

$$xy' = \Lambda_{N=2}^{4N_c-4-2N_f}$$

(4.8)

$$w = 0$$

(4.9)

$$(-1)^{N_f} \prod_{i=1}^{N_f} (v + m_i)v^2x' + \prod_{i=1}^{N_f} (v - m_i)v^2y' = B(v^2)$$

(4.10)
with an orientifold projection $x' \leftrightarrow y'$ and $v \leftrightarrow -v$. In this new space, the Sieberg-Witten curve will be projected to a curve in $(y', v)$-plane given by

$$\prod_{i=1}^{N_f} (v - m_i) v^2 y'^2 - B(v^2) y' + (-1)^{N_f} \Lambda_{N_c=2}^{4N_c-4-2N_f} v^2 \prod_{i=1}^{N_f} (v + m_i) = 0. \quad (4.11)$$

In terms of brane geometry, the effect of resolving the singularity corresponds to pushing +4 charge of O6 through the NS5 branes to $\pm \infty$, creating new non-dynamical D4 branes à la Hanany-Witten as described by Uranga [21].

Now let us rotate the left NS5 brane towards $w$ which determines the rotation of its mirror right brane towards $-w$ direction. Thus the left NS5 brane is located at $w_+ = 0$ and the right NS5 brane is located at $w_- = 0$ where we have again:

$$w_\pm = w \pm \mu v. \quad (4.12)$$

We can make another choice of coordinates to describe the brane configuration but $w_\pm$ are the most efficient in our approach as we will see below. Also we are only dealing in this section with massive quarks having generic masses $m_1, \ldots, m_{N_f}$ and we realize this in the brane configuration by putting the semi-infinite D4 branes at $(v, w) = (-m_i/2\mu, m_i/2)$ on the left NS5 brane and their mirrors at $(v, w) = (m_i/2\mu, m_i/2)$ on the right NS5 brane. Let $\Sigma$ be the corresponding Sieberg-Witten curve. On $\Sigma$, the function $w_+$ goes to infinity only at one point and $w_+$ has only a single pole there, since there is only one NS5 brane i.e. the right NS5 brane. Thus we can identify $\Sigma$ with the punctured complex $w_+$-plane possibly after resolving the singularity at $x = y = v = 0$. Similarly, we can argue that $w_-$ has a single pole on $\Sigma$. Since these are two rational functions on a rational curve, they are related by a linear fractional transformation which after suitable constant shifts can be written as

$$w_+ w_- = \zeta \quad (4.13)$$

where $\zeta$ is a constant.

Now we project this curve to $(y, w_+)$-space to obtain:

$$\prod_{i=1}^{N_f} (w_+ - m_i) y - P_1(w_+) = 0, \quad (4.14)$$

where

$$P_1(w_+) = w_+^{2N_c} + p_1 w_+^{2N_c-1} + \cdots + p_{2N_c} \quad (4.15)$$

is some polynomial of degree $2N_c$ because there are $2N_c$ finite D4 branes between the two NS branes and the the number of finite D4 branes gives the degree of $P_1$. Similarly
if we project the curve to $(y, w)-$space, we get

\[ Q_1(w-) y - A \prod_{i=1}^{N_f} (w_- - m_i) = 0 \tag{4.16} \]

where

\[ Q_1(w-) = w_-^{2N_c} + q_1 w_-^{2N_c-1} + \cdots + q_2 N_c \tag{4.17} \]

and $A$ is a normalization constant. In writing the equations (4.14) and (4.16) we have used the fact that one of the functions $w_+, w_-$ vanishes at one of the NS branes so when we project the curve to one of the $(y, w_{\pm})$ spaces, its equation becomes linear in $y$. We have used the idea of [13] where the configuration had two NS branes with the left NS brane was extended in the $v$ direction at $w = 0$ and the right NS brane was extended in the $w$ direction at $v = 0$ so the brane equations were linear when projected to $(y, v)$ or $(y, w)$ spaces.

In order to have (4.13), (4.14) and (4.16) simultaneously, it is required that

\[ P_1(w_+) Q_1(\zeta/w_+) \equiv A \prod_{i=1}^{N_f} (w_+ - m_i)(\zeta/w_+ - m_i) \tag{4.18} \]

for all $w_+ \in \mathbb{C}$. The most general solution for the massive flavor case is:

\[ P_1(w_+) = w_+^{2N_c-N_f} \prod_{i=1}^{N_f} (w_+ - \zeta/m_i) \tag{4.19} \]
\[ Q_1(w_-) = w_-^{2N_c-N_f} \prod_{i=1}^{N_f} (w_- - \zeta/m_i). \tag{4.20} \]

Now we plug (4.19) into (4.14) to obtain a Seiberg-Witten curve:

\[ 16\mu^4 y = \Lambda^{4N_c-4-2N_f} (w_+ - w_-)^4 \tag{4.21} \]
\[ w_+ w_- = \zeta \tag{4.22} \]
\[ \left( \prod_{i=1}^{N_f} m_i \right) y \prod_{i=1}^{N_f} \left( \frac{w_+ - m_i}{w_- - m_i} \right) = (-1)^{N_f} w_+^{2N_c} \tag{4.23} \]

As before, we may push the charge of O6 plane through the NS5 branes to $\pm \infty$ by going to the resolved surface $S'$ given by

\[ x'y' = \Lambda^{4N_c-4-2N_f}. \tag{4.24} \]
Similar computations can be done on the resolved surface \( S' \) after replacing (4.14) and (4.16) by

\[
N_f \prod_{i=1}^{N_f} (w_+ - m_i) w_+^2 y' - P_1 (w_+ ) = 0. \tag{4.25}
\]

\[
Q_1(w_-) y' - A w_-^2 \prod_{i=1}^{N_f} (w_- - m_i) = 0. \tag{4.26}
\]

Then as the most general solution for massive flavor, we obtain:

\[
\left( \prod_{i=1}^{N_f} m_i \right) y' \prod_{i=1}^{N_f} \left( \frac{w_+ - m_i}{w_- - m_i} \right) = (-1)^{N_f} w_+^{2N_c - 2}. \tag{4.27}
\]

Moreover, from (4.26) we can identify \( A \) with \( \Lambda_{N=1}^{3(2N_c - 2) - 2N_f} \) by taking into account the geometric symmetry \( U(1)_{w_+} \otimes U(1)_{w_-} \) of the M theory associated with the complex rotations of the left and right NS branes respectively. Thus the value for \( \zeta \) can be obtained as

\[
\zeta = \left( \Lambda_{N=1}^{3(N_c-1)} \prod_{i=1}^{N_f} m_i \right)^{\frac{1}{N_c-1}} \tag{4.28}
\]

by comparing (4.25) and (4.26) as in (4.18). If we map this curve to \((x, y, w_+, w_-)\)-space via \( x = v^2 x', y = v^2 y' \), we will obtain a set of equations similar to (5.2), (5.3) and (5.4) of [20]:

\[
xy = \Lambda_{N=2}^{4N_c - 4 - 2N_f} v^4 \tag{4.29}
\]

\[
w_+ w_- = \zeta \tag{4.30}
\]

\[
\left( \prod_{i=1}^{N_f} m_i \right) y \prod_{i=1}^{N_f} \left( \frac{w_+ - m_i}{w_- - m_i} \right) = (-1)^{N_f} w_+^{2N_c - 2} v^2. \tag{4.31}
\]

We remark that the solution curve (4.27) is embedded in the smooth surface (4.24) while the solution curve (4.31) is embedded in the singular surface (4.29). We repeat that the effect of resolving the singularity corresponds to pushing +4 charge of O6 through the NS5 branes to \( \pm \infty \), creating new non-dynamical D4 branes à la Hanany-Witten and thus there are no differences in the Coulomb branches between these two models.

Notice that there is a difference in scaling constant between (4.29) and (5.2) of [20]. From consideration of the RG equation:

\[
\Lambda_{N=2}^{4N_c - 4 - 2N_f} \mu^{2N_c - 2} = \Lambda_{N=1}^{6N_c - 6 - 2N_f} \tag{4.32}
\]

we can write \( y \) as \( \Lambda_{N=1}^{3N_c - 3 - N_f} \mu^{1-N_c} \tilde{y} \) where \( \tilde{y} \) has dimension 2 as compared with the \( y \) in (4.31) which has dimension \( 2N_c - N_f \). The dimension of \( \tilde{y} \) is the same as the one of \( y \).
in the equation (5.4) of [20] and (4.31) becomes almost identical to (5.4) of [20] when is written in terms of \( \tilde{y} \).

The extension to \( SO(2N_c + 1) \) is trivial. In the odd case the equations (4.14) and (4.16) become:

\[
P_1(w_+) = w_+^{2N_c} + p_1 w_+^{2N_c-1} + \cdots + p_{2N_c} \\
Q_1(w_-) = w_-^{2N_c} + q_1 w_-^{2N_c-1} + \cdots + q_{2N_c}
\]

and the most general solution for the massive flavor case is of the form

\[
P_1(w_+) = w_+^{2N_c+1-N_f} \prod_{i=1}^{N_f} (w_+ - \zeta/m_i) \\
Q_1(w_-) = w_-^{2N_c+1-N_f} \prod_{i=1}^{N_f} (w_- - \zeta/m_i)
\]

which yields

\[
\left( \prod_{i=1}^{N_f} m_i \right) \left( \prod_{i=1}^{N_f} \frac{w_+ - m_i}{w_- - m_i} \right) = (-1)^{N_f} w_+^{2N_c+1} v^2.
\]

\* \( Sp(2N_c) \) Case

The situation is similar to \( SO \) case. Now the O6 plane is described by:

\[
xy = \Lambda^{4N_c+4-2N_f} v^{-4}.
\]

Again the general solution for massive flavors is given by

\[
xy = \Lambda_{N=2}^{4N_c+4-2N_f} 16 \mu^4 (w_+ - w_-)^{-4} \\
w_+ w_- = \zeta \\
\left( \prod_{i=1}^{N_f} m_i \right) \left( \prod_{i=1}^{N_f} \frac{w_+ - m_i}{w_- - m_i} \right) = (-1)^{N_f} w_+^{2N_c}.
\]
Let us simplify the spacetime again by resolving singularities at infinity. The resolved surface can be given by

\[ x'y' = \Lambda_{N=2}^{4N_c+4-2N_f} \]

and the new surface maps onto the old surface via the map \( x = x'v^{-2}, y = y'v^{-2} \). On this new surface, the solution can be described by

\[ x'y' = \Lambda_{N=2}^{4N_c+4-2N_f} \]

\[ w_+w_- = \zeta \]

\[ \left( \prod_{i=1}^{N_f} m_i \right) y' \prod_{i=1}^{N_f} \left( \frac{w_+ - m_i}{w_- - m_i} \right) = (-1)^{N_f} w_+^{2N_c+2}. \]

If we map this curve to the old surface, then the last equation becomes:

\[ \left( \prod_{i=1}^{N_f} m_i \right) y \prod_{i=1}^{N_f} \left( \frac{w_+ - m_i}{w_- - m_i} \right) = (-1)^{N_f} v^{-2} w_+^{2N_c+2} \]

which is the same as (5.27) of [20] after rescaling of variables.

5 Conclusion

We have studied brane configurations for \( N = 1 \) supersymmetric \( SO/Sp \) gauge theories and the flavor group was given by either infinite D6 branes or semi-infinite D4 branes. For the case of D6 branes we obtained the results of [14, 15] which were derived from a brane configuration with an orientifold O4 plane. For the case of semi-infinite D4 branes, we obtained the equations of the Seiberg-Witten curve as in [20] from consideration of the brane configuration mixed with simplification of the spacetime via resolution of the orientifold singularity. While these equations were presented with some evidence in [20], we derive them rigorously by fully applying the method of [13] and working on the resolved surface \( S' \). The simplification of the spacetime via resolution and other possible solutions deserve further study.

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