Fuzzy Aggregates in Fuzzy Answer Set Programming

Emad Saad
emsaad@gmail.com

Abstract
Fuzzy answer set programming [Saad, 2010; Saad, 2009; Subrahmanian, 1994] is a declarative framework for representing and reasoning about knowledge in fuzzy environments. However, the unavailability of fuzzy aggregates in disjunctive fuzzy logic programs, DFLP, with fuzzy answer set semantics [Saad, 2010] prohibits the natural and concise representation of many interesting problems. In this paper, we extend DFLP to allow arbitrary fuzzy aggregates. We define fuzzy answer set semantics for DFLP with arbitrary fuzzy aggregates including monotone, antimonotone, and nonmonotone fuzzy aggregates. We show that the proposed fuzzy answer set semantics subsumes both the original fuzzy answer set semantics of DFLP [Saad, 2010] and the classical answer set semantics of classical disjunctive logic programs with classical aggregates [Faber et al., 2010], and consequently subsumes the classical answer set semantics of classical disjunctive logic programs [Gelfond and Lifschitz, 1991]. We show that the proposed fuzzy answer sets of DFLP with fuzzy aggregates are minimal fuzzy models and hence incomparable, which is an important property for nonmonotonic fuzzy reasoning.

1 Introduction
Fuzzy answer set programming [Saad, 2010; Saad, 2009; Subrahmanian, 1994] is a declarative programming framework that has been shown effective for knowledge representation and reasoning in fuzzy environments. These include representing and reasoning about actions with fuzzy effects and fuzzy planning [Saad, 2009; Saad et al., 2009] as well as representing and reasoning about fuzzy preferences [Saad, 2010]. The fuzzy answer set programming framework includes disjunctive fuzzy logic programs [Saad, 2010], extended fuzzy logic programs [Saad, 2009], and normal fuzzy logic programs [Subrahmanian, 1994] with fuzzy answer set semantics. However, the unavailability of fuzzy aggregates in fuzzy answer set programming [Saad, 2010; Saad, 2009; Subrahmanian, 1994] disallows the natural and concise representation of many new interesting problems.

Example 1 Consider the same company control problem described in [Faber et al., 2010]. Assume that a company $C_3$ owns $P\%$ of a company $C_2$ shares, represented by the predicate $\text{ownsStk}(C_1, C_2, P)$. If the company $C_1$ owns a total sum of more than $50\%$ of shares of the company $C_2$ directly (through $C_1$ itself) or indirectly (through another company $C_3$ controlled by $C_1$), then we say that company $C_1$ controls company $C_2$. Let $\text{controls}(C_1, C_2)$ denote that company $C_1$ controls company $C_2$. Let $\text{controlStk}(C_1, C_2, C_3, P)$ denote that company $C_1$ controls $P\%$ of company $C_3$ shares through company $C_2$, since $C_1$ controls $C_2$ and $C_2$ owns $P\%$ of $C_3$ shares. Assume information about companies shares are represented as facts as described below. This company control problem is represented as a classical disjunctive logic program with classical aggregates, described below, whose answer set describes the intuitive and correct solution to the problem as illustrated in [Faber et al., 2010] as:

- $\text{ownsStk}(a, b, 40) \leftarrow$
- $\text{ownsStk}(c, b, 20) \leftarrow$
- $\text{ownsStk}(a, c, 40) \leftarrow$
- $\text{ownsStk}(b, c, 20) \leftarrow$
- $\text{controlStk}(C_1, C_1, C_2, C_3, P) \leftarrow \text{ownsStk}(C_1, C_2, P)$.
- $\text{controlStk}(C_1, C_2, C_3, P) \leftarrow \text{controls}(C_1, C_2)$.
- $\text{ownsStk}(C_2, C_3, P)$.

$\text{controls}(C_1, C_3) \leftarrow \text{sum}(P, C_2, \text{controlStk}(C_1, C_2, C_3, P)) > 50.$

The above representation of the company control problem as a classical disjunctive logic program with classical aggregates is entirely correct if our knowledge regarding the companies shares are perfect. However this is not always the case. Consider our knowledge regarding the company shares is not perfect. Thus, we cannot absolutely assert that some company $C_1$ controls another company $C_2$ as in the above representation. Instead, we can assert that a company $C_1$ controls another company $C_2$ with a certain degree of beliefs. In the presence of such uncertainties, the above company control problem need to be redefined to deal with imperfect knowledge about companies shares (namely fuzzy company control problem), where the imperfect knowledge about the companies shares are represented as a fuzzy set over companies shares. Consequently, a logical framework different from classical disjunctive logic programs with classical aggregates.
is needed for representing and reasoning about such fuzzy reasoning problems.

Consider that the fuzzy set over companies shares, presented in Example (1), is described as; company \(a\) owns 40\% of company \(b\) with grade membership 0.7; company \(c\) owns 20\% of company \(b\) with grade membership 0.6; company \(a\) owns 40\% of company \(c\) with grade membership 0.9; and company \(b\) owns 20\% of company \(c\) with grade membership 0.8. Consider also that the same company control strategy as in Example (1) is employed. Thus, this fuzzy company control problem cannot be represented as a classical disjunctive logic program with classical aggregates, since classical disjunctive logic programs with classical aggregates do not allow neither representing and reasoning in the presence of fuzzy uncertainty nor allow aggregation over fuzzy sets. Moreover, this fuzzy company control problem cannot be represented as a disjunctive fuzzy logic program with fuzzy answer set semantics either, since disjunctive fuzzy logic programs with fuzzy answer set semantics do not allow aggregations over fuzzy sets by means of fuzzy aggregates for intuitive and concise representation of the problem.

Therefore, we propose to extend disjunctive fuzzy logic programs with fuzzy answer set semantics [Saad, 2010], denoted by DFLP, with arbitrary fuzzy aggregates to allow intuitive and concise representation of many real-world problems. To the best of our knowledge, this development is the first that defines semantics for fuzzy aggregates in a fuzzy answer set programming framework.

The contributions of this paper are as follows. We extend the original language of DFLP to allow arbitrary fuzzy annotation function including monotone, antimonotone, and nonmonotone annotation functions. We define the notions of fuzzy aggregates and fuzzy aggregate atoms in DFLP. We develop the fuzzy answer set semantics of DFLP with arbitrary fuzzy aggregates, denoted by DFLP\(^{FA}\), including monotone, antimonotone, and nonmonotone fuzzy aggregates. We show that the presented fuzzy answer set semantics of DFLP\(^{FA}\) subsumes and generalizes both the original fuzzy answer set semantics of DFLP [Saad, 2010] and the classical answer set semantics of the classical disjunctive logic programs with classical aggregates, denoted by DLP\(^A\) [Faber et al., 2010], and consequently subsumes the classical answer set semantics of classical disjunctive logic programs, denoted by DLP [Gelfond and Lifschitz, 1991]. We show that the fuzzy answer sets of DFLP\(^{FA}\) are minimal fuzzy models and hence incomparable, which is an important property for nonmonotonic fuzzy reasoning.

The choice of DFLP for extension with fuzzy aggregates is interesting for many reasons. First, DFLP is very expressive form of fuzzy answer set programming that allows disjunctions to appear in the heads of rules. It has been shown in [Saad, 2010] that: (1) DFLP is capable of representing and reasoning with both fuzzy uncertainty and qualitative uncertainty in which fuzzy uncertainly need to be defined over qualitative uncertainty; (2) DFLP is shown to be sophisticated logical framework for representing and reasoning about fuzzy preferences; (3) DFLP is a natural extension to DLP and its fuzzy answer set semantics subsumes the classical answer set semantics of DLP [Gelfond and Lifschitz, 1991]; (4) DFLP with fuzzy answer set semantics subsumes the fuzzy answer set programming framework of [Subrahmanian, 1994], which are DFLP programs with only an atom appearing in heads of rules.

2 DFLP\(^{FA}\) : Fuzzy Aggregates Disjunctive Fuzzy Logic Programs

In this section we present the basic language of DFLP\(^{FA}\), the notions of fuzzy aggregates and fuzzy aggregate atoms, and the syntax of DFLP\(^{FA}\) programs.

2.1 The Basic Language of DFLP\(^{FA}\)

Let \(L\) denotes an arbitrary first-order language with finitely many predicate symbols, function symbols, constants, and infinitely many variables. A term is a constant, a variable or a function. An atom, \(a\), is a predicate in \(L\), where \(B_L\) is the Herbrand base of \(L\). The Herbrand universe of \(L\) is denoted by \(U_L\). Non-monotonic negation or the negation as failure is denoted by \(\neg\). In fuzzy aggregates disjunctive fuzzy logic programs, DFLP\(^{FA}\), the grade membership values are assigned to atoms in \(B_L\) as values from \([0, 1]\). The set \([0, 1]\) and the relation \(\leq\) form a complete lattice, where the join (\(\oplus\)) operation is defined as \(\alpha_1 \oplus \alpha_2 = \max(\alpha_1, \alpha_2)\) and the meet (\(\otimes\)) is defined as \(\alpha_1 \otimes \alpha_2 = \min(\alpha_1, \alpha_2)\).

A fuzzy annotation, \(\mu\), is either a constant in \([0, 1]\) (called fuzzy annotation constant), a variable ranging over \([0, 1]\) (called fuzzy annotation variable), or \(f(\alpha_1, \ldots, \alpha_n)\) (called fuzzy annotation function) where \(f\) is a representation of a monotone, antimonotone, or nonmonotone total or partial function \(f : ([0, 1])^n \rightarrow [0, 1]\) and \(\alpha_1, \ldots, \alpha_n\) are fuzzy annotations. If \(a\) is an atom and \(\mu\) is a fuzzy annotation then \(a : \mu\) is called a fuzzy annotated atom.

2.2 Fuzzy Aggregate Atoms

A symbolic fuzzy set is an expression of the form \(\{X : \mu \mid C\}\), where \(X\) is a variable or a function term and \(U\) is fuzzy annotation variable or fuzzy annotation function, and \(C\) is a conjunction of fuzzy annotated atoms. A ground fuzzy set is a set of pairs of the form \((X^g : U^g \mid C^g)\) such that \(X^g\) is a constant term and \(U^g\) is fuzzy annotation constant, and \(C^g\) is a ground conjunction of fuzzy annotated atoms. A symbolic fuzzy set or ground fuzzy set is called a fuzzy set term. Let \(f\) be a fuzzy aggregate function symbol and \(S\) be a fuzzy set term, then \(f(S)\) is said a fuzzy aggregate, where \(f \in \{\text{sum}_F, \text{times}_F, \text{min}_F, \text{max}_F, \text{count}_F\}\). If \(f(S)\) is a fuzzy aggregate and \(T\) is a constant, a variable or a function term, called guard, then we say \(f(S) \times T\) is a fuzzy aggregate atom, where \(<\epsilon \in \{=, \neq, <, \geq, \leq, \}angle\).

Example 2 The following are examples for fuzzy aggregate atoms representation in DFLP\(^{FA}\) language.

\[
\text{max}_F\{X : U \mid \text{benefit}(X) : U\} > 99
\]

\[
\text{sum}_F\{(2 : 0.4 \mid a(1, 2) : 0.4), (5 : 0.7 \mid a(1, 5) : 0.7)\} \leq 11
\]

Definition 1 Let \(f(S)\) be a fuzzy aggregate. A variable, \(X\), is a local variable to \(f(S)\) if and only if \(X\) appears in \(S\) and \(X\) does not appear in the DFLP\(^{FA}\) rule that contains \(f(S)\).
Definition 1 characterizes the local variables for a fuzzy aggregate function. For example, for the first fuzzy aggregate atom in Example 2, the variables X and U are local variables to the fuzzy aggregate $\max_F$.

Definition 2 A global variable is a variable that is not a local variable.

2.3 DFLP$^{FA}$ Program Syntax

This section defines the syntax of rules and programs in the language of DFLP$^{FA}$.

Definition 3 A DFLP$^{FA}$ rule is an expression of the form

$$a_1: \mu_1 \lor \ldots \lor a_k: \mu_k \rightarrow a_{k+1}: \mu_{k+1}, \ldots, a_m: \mu_m,$$

where $\forall (1 \leq i \leq k) a_i$ are atoms, $\forall (k + 1 \leq i \leq n) a_i$ are atoms or fuzzy aggregate atoms, and $\forall (1 \leq i \leq n) \mu_i$ are fuzzy annotations.

A DFLP$^{FA}$ rule means that if for each $a_i: \mu_i$, where $k + 1 \leq i \leq m$, it is believable that the grade membership value of $a_i$ is at least $\mu_i$ w.r.t. $\leq$ and for each not $a_j: \mu_j$, where $m + 1 \leq j \leq n$, it is not believable that the grade membership value of $a_i$ is at least $\mu_j$ w.r.t. $\leq$, then there exists at least $a_i$, where $1 \leq i \leq k$, such that the grade membership value of $a_i$ is at least $\mu_i$.

Definition 4 A DFLP$^{FA}$ program, $\Pi$, is a set of DFLP$^{FA}$ rules.

For the simplicity of the presentation, atoms that appear in DFLP$^{FA}$ programs without fuzzy annotations are assumed to be associated with the fuzzy annotation constant 1.

Example 3 The fuzzy company control problem described in Example 7 can be concisely and intuitively represented as a DFLP$^{FA}$ program, $\Pi$, that consists of the DFLP$^{FA}$ rules:

$$\text{ownsStk}(a, b, 40) : 0.7 \leftarrow$$
$$\text{ownsStk}(b, c, 20) : 0.6 \leftarrow$$
$$\text{ownsStk}(a, c, 40) : 0.9 \leftarrow$$
$$\text{ownsStk}(b, c, 20) : 0.8 \leftarrow$$

$$\text{controlStk}(C_1, C_1, C_2, P) : V \leftarrow$$
$$\text{controlStk}(C_1, C_2, C_3, P) : V \leftarrow$$
$$\text{controls}(C_1, C_2) : 0.55 \leftarrow$$
$$\text{controls}(C_1, C_3) : 0.55 \leftarrow$$

$$\text{sum}_F\{P : V \mid \text{controlStk}(C_1, C_2, C_3, P) : V\} > 50 : 0.6$$

The last DFLP$^{FA}$ rule in, $\Pi$, says that if it is at least 0.6 grade membership value believable that company $C_1$ owns a total sum of more than 50% of shares of the company $C_3$ directly (through $C_1$ itself) or indirectly (through another company $C_2$ controlled by $C_1$), then it is 0.55 grade membership value believable that company $C_1$ controls company $C_3$.

Definition 5 The ground instantiation of a symbolic fuzzy set $S = \{X : U \mid C\}$ is the set of all ground pairs of the form $\langle \theta(X) : \theta(U) \mid \theta(C) \rangle$, where $\theta$ is a substitution of every local variable appearing in $S$ to a constant from $U_C$.

3 Fuzzy Aggregates Semantics

A fuzzy aggregate is an aggregation over a fuzzy set that returns the evaluation of a classical aggregate and the grade membership value of the evaluation of that classical aggregate over a given fuzzy set. The fuzzy aggregates that we consider are $\text{sum}_F$, $\text{times}_F$, $\text{min}_F$, $\text{max}_F$, and $\text{count}_F$ that find the evaluation of the classical aggregates $\text{sum}$, $\text{times}$, $\text{min}$, $\text{max}$, and $\text{count}$ respectively along with the grade membership value of their evaluations. The application of fuzzy aggregates is on ground fuzzy sets which are sets of constants terms along with their associated grade membership values.

3.1 Mappings

Let $X$ denotes a set of objects. Then, we use $2^X$ to denote the set of all multisets over elements in $X$. Let $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{N}$ denotes the set of all natural numbers, and $U_C$ denotes the Herbrand universe. Let $\bot$ be a symbol that does not occur in $L$. Therefore, the mappings of the fuzzy aggregates are given by:

- $\text{sum}_F : 2^{\mathbb{R} \times [0,1]} \rightarrow \mathbb{R} \times [0,1].$
- $\text{times}_F : 2^{\mathbb{R} \times [0,1]} \rightarrow \mathbb{R} \times [0,1].$
- $\text{min}_F : (2^{\mathbb{R} \times [0,1]} - \emptyset) \rightarrow \mathbb{R} \times [0,1].$
- $\text{max}_F : (2^{\mathbb{R} \times [0,1]} - \emptyset) \rightarrow \mathbb{R} \times [0,1].$
- $\text{count}_F : 2^{U_C \times [0,1]} \rightarrow \mathbb{N} \times [0,1].$

The application of $\text{sum}_F$ and $\text{times}_F$ on the empty multiset return $(0, 1)$ and $(1, 1)$ respectively. The application of $\text{count}_F$ on the empty multiset returns $(0, 1)$. However, the application of $\text{max}_F$ and $\text{min}_F$ on the empty multiset is undefined.

Definition 7 A fuzzy interpretation of a DFLP$^{FA}$ program, $\Pi$, is a mapping $I : B_C \rightarrow [0, 1]$. 
3.2 Semantics of Fuzzy Aggregates

The semantics of fuzzy aggregates is defined with respect to a fuzzy interpretation, which is a representation of fuzzy sets. A fuzzy annotated atom, \( a : \mu \), is true (satisfied) with respect to a fuzzy interpretation, \( I \), if and only if \( \mu \leq I(a) \). The negation of a fuzzy annotated atom, \( \neg a : \mu \), is true (satisfied) with respect to \( I \) if and only if \( \mu \not\leq I(a) \). The evaluation of a fuzzy aggregate, and hence the truth valuation of a fuzzy aggregate atom, are established with respect to a given fuzzy interpretation, \( I \), as presented in the following definitions.

**Definition 8** Let \( f(S) \) be a ground fuzzy aggregate and \( I \) be a fuzzy interpretation. Then, we define \( S_I \) to be the multiset constructed from elements in \( S \), where \( S_I = \{X^a : U^a \mid X^a : U^a \in S, C^a \subseteq S \land C^a \text{ is true w.r.t. } I\} \).

**Definition 9** Let \( f(S) \) be a ground fuzzy aggregate and \( I \) be a fuzzy interpretation. Then, the evaluation of \( f(S) \) with respect to \( I \) is, \( f(S_I) \), the result of the application of \( f \) to \( S \), where \( f(S_I) = \top \) if \( S_I \) is not in the domain of \( f \) and

- \( \sum_F(S_I) = \langle \sum_{X^a : U^a \in S_I} X^a, \min_{X^a : U^a \in S_I} U^a \rangle \)
- \( \times_F(S_I) = \langle \prod_{X^a : U^a \in S_I} X^a, \min_{X^a : U^a \in S_I} U^a \rangle \)
- \( \min_F(S_I) = \langle \min_{X^a : U^a \in S_I} X^a, \min_{X^a : U^a \in S_I} U^a \rangle \)
- \( \max_F(S_I) = \langle \max_{X^a : U^a \in S_I} X^a, \min_{X^a : U^a \in S_I} U^a \rangle \)
- \( \text{count}_F(S_I) = \langle \sum_{X^a : U^a \in S_I} X^a, \min_{X^a : U^a \in S_I} U^a \rangle \)

4 Fuzzy Answer Set Semantics of DFLP\(^{F,A}\)

In this section we define the satisfaction, fuzzy models, and the fuzzy answer set semantics of fuzzy aggregates disjunctive fuzzy logic programs, DFLP\(^{F,A}\). Let \( r \) be a DFLP\(^{F,A}\) rule and \( \text{head}(r) = a_1 : \mu_1 \lor \ldots \lor a_k : \mu_k \) and

\( \text{body}(r) = a_{k+1} : \mu_{k+1}, \ldots, a_m : \mu_m, \text{not } a_{m+1} : \mu_{m+1}, \ldots, \text{not } a_n : \mu_n \).

**Definition 10** Let \( \Pi \) be a ground DFLP\(^{F,A}\) program, \( r \) be a DFLP\(^{F,A}\) rule in \( \Pi \), \( I \) be a fuzzy interpretation for \( \Pi \), and \( f \in \{\sum_F, \times_F, \min_F, \max_F, \text{count}_F\} \). Then,

1. \( I \) satisfies \( a_1 : \mu_1 \) in \( \text{head}(r) \) iff \( \mu_1 \leq I(a_1) \).
2. \( I \) satisfies \( f(S) \ll T : \mu \) in \( \text{body}(r) \) iff \( f(S_I) = (x, \nu) \neq \top \) and \( x \ll T \) and \( \mu \leq \nu \).
3. \( I \) satisfies \( \neg f(S) \ll T : \mu \) in \( \text{body}(r) \) iff \( f(S_I) = \top \) or \( f(S_I) = (x, \nu) \neq \top \) and \( x \not\ll T \) or \( \mu \not\leq \nu \).
4. \( I \) satisfies \( a_i : \mu_i \) in \( \text{body}(r) \) iff \( \mu_i \leq I(a_i) \).
5. \( I \) satisfies \( \neg a_j : \mu_j \) in \( \text{body}(r) \) iff \( \mu_j \not\leq I(a_j) \).
6. \( I \) satisfies \( \text{body}(r) \) iff \( \forall (k+1 \leq i \leq m), I \) satisfies \( a_i : \mu_i \) and \( \forall (m+1 \leq j \leq n), I \) satisfies not \( a_j : \mu_j \).
7. \( I \) satisfies \( \text{head}(r) \) iff \( \exists i \) (\( 1 \leq i \leq k \)) such that \( I \) satisfies \( a_i : \mu_i \).
8. \( I \) satisfies \( r \) iff \( I \) satisfies \( \text{head}(r) \) whenever \( I \) satisfies \( \text{body}(r) \) or \( I \) does not satisfy \( \text{body}(r) \).
9. \( I \) satisfies \( \Pi \) iff \( I \) satisfies every DFLP\(^{F,A}\) rule in \( \Pi \) and

\( \max\{\mu_i \mid \text{head}(r) \iff \text{body}(r) \in \Pi\} \leq I(a_i) \) such that \( I \) satisfies \( \text{body}(r) \) and \( I \) satisfies \( a_i : \mu_i \) in \( \text{head}(r) \).

**Example 5** Let \( \Pi \) be a DFLP\(^{F,A}\) program that consists of the DFLP\(^{F,A}\) rules:

\[
\begin{align*}
(a(1), 1) & : 0.8 \lor (a(1), 2) : 0.4 \iff \\
(a(2), 1) & : 0.3 \lor (a(2), 2) : 0.9 \iff \\
\end{align*}
\]

The ground instantiation of \( r \) is given by:

\[
\begin{align*}
\text{head}(r) & \iff \text{body}(r) \in \{1 : 0.8 | a(1), 1) : 0.8), (2 : 0.4 | a(1), 2) : 0.4), \\
& \quad (1 : 0.3 | a(2), 1) : 0.3), (2 : 0.9 | a(2), 2) : 0.9) \}
\end{align*}
\]

\( r \) satisfies \( \Pi \) and assigns \( 0.9 \) to the remaining atoms in \( B_\Pi \).

Thus the evaluation of the fuzzy aggregate atom, \( \min_F(S) \leq 1 \) in \( r \) w.r.t. \( I \) is given as follows, where

\[
S = \{(1 : 0.8 | a(1), 1) : 0.8), (2 : 0.4 | a(1), 2) : 0.4), \\
\quad (1 : 0.3 | a(2), 1) : 0.3), (2 : 0.9 | a(2), 2) : 0.9) \}
\]

\( S_I = \{2 : 0.4, 2 : 0.9\} \). Therefore, \( \min_F\{2 : 0.4, 2 : 0.9\} = (2 : 0.4) \) and consequently, the fuzzy annotated fuzzy annotated aggregate atom \( \min_F(S) \leq 1 \) is not satisfied by \( I \). This is because \( \min_F\{2 : 0.4, 2 : 0.9\} = (2 : 0.4) \neq \bot \) and \( 2 \not\leq 1 \) although \( 0.4 \leq 0.4 \).

**Definition 11** A fuzzy model for a DFLP\(^{F,A}\) program, \( \Pi \), is a fuzzy interpretation for \( \Pi \) that satisfies \( \Pi \). A fuzzy model \( I \) for \( \Pi \) is \( \leq\)-minimal if there does not exist a fuzzy model \( I' \) for \( \Pi \) such that \( I' < I \).

**Example 6** It can easily verified that the fuzzy interpretation, \( I, \) for DFLP\(^{F,A}\) program, \( \Pi, \) described in Example 3, is a minimal fuzzy model for \( \Pi \). However, the fuzzy interpretation, \( I', \) for \( \Pi, \) described in Example 3, is not a fuzzy model for \( \Pi \).

**Definition 12** Let \( \Pi \) be a ground DFLP\(^{F,A}\) program, \( r \) be a DFLP\(^{F,A}\) rule in \( \Pi \), and \( I \) be a fuzzy interpretation for \( \Pi \).

Let \( I \models \text{body}(r) \) denotes \( I \) satisfies \( \text{body}(r) \). Then, the fuzzy
reduct, \( \Pi' \), of \( \Pi \) w.r.t. \( I \) is a ground \( \text{DFLP}^{F,A} \) program \( \Pi' \) where

\[
\Pi' = \{ \text{head}(r) \leftarrow \text{body}(r) \mid r \in \Pi \land I \models \text{body}(r) \}
\]

**Definition 13** A fuzzy interpretation, \( I \), of a ground \( \text{DFLP}^{F,A} \) program, \( \Pi \), is a fuzzy answer set for \( \Pi \) if \( I \) is \( \leq \)-minimal fuzzy model for \( \Pi' \).

Observe that the definitions of the fuzzy reduct and the fuzzy answer sets for \( \text{DFLP}^{F,A} \) programs are generalizations of the fuzzy reduct and the fuzzy answer sets of the original \( \text{DFLP} \) programs described in [Saad, 2010].

**Example 7** It can be easily verified that the \( \text{DFLP}^{F,A} \) program described in Example (5) has three fuzzy answer sets \( I_1 \), \( I_2 \), and \( I_3 \) presented below, where atoms in \( B_C \) that are not appearing in \( I_1 \), \( I_2 \), and \( I_3 \) are assumed to be assigned the fuzzy annotation 0.0.

\[
\begin{align*}
I_1 &= \{ a(1,1) : 0.8, a(2,1) : 0.3 \} \\
I_2 &= \{ a(1,2) : 0.4, a(2,1) : 0.3 \} \\
I_3 &= \{ a(1,2) : 0.4, a(2,2) : 0.9 \}
\end{align*}
\]

**Example 8** The \( \text{DFLP}^{F,A} \) program representation of the fuzzy company control problem, \( \Pi \), described in Example 3 has one fuzzy answer set, \( I \), which, after omitting the facts and assuming atoms in \( B_C \) that do not appear in \( I \) are assigned the annotation 0.0, is

\[
I = \{ \\
\text{controlStk}(a, a, b, 40) : 0.7, \\
\text{controlStk}(a, a, c, 40) : 0.9, \\
\text{controlStk}(b, b, c, 20) : 0.8, \\
\text{controlStk}(c, c, b, 20) : 0.6 \}.
\]

The fuzzy answer set, \( I \), implies that no company fuzzy controls another company.

5 \( \text{DFLP}^{F,A} \) Semantics Properties

In this section we study the semantics properties of \( \text{DFLP}^{F,A} \) programs and its relationship to the original fuzzy answer set semantics of disjunctive fuzzy logic programs, denoted by \( \text{DFLP} \) [Saad, 2010]; the classical answer set semantics of classical disjunctive logic programs with classical aggregates, denoted by \( \text{DLP}^{A} \) [Faber et al., 2010]; and the original classical answer set semantics of classical disjunctive logic programs, denoted by \( \text{DLP} \) [Gelfond and Lifschitz, 1991].

**Theorem 1** Let \( \Pi \) be a \( \text{DFLP}^{F,A} \) program. The fuzzy answer sets for \( \Pi \) are \( \leq \)-minimal fuzzy models for \( \Pi \).

The following theorem shows that the fuzzy answer set semantics of \( \text{DFLP}^{F,A} \) subsumes and generalizes the fuzzy answer set semantics of \( \text{DFLP} \) [Saad, 2010], which are \( \text{DFLP}^{F,A} \) programs without fuzzy aggregates atoms and with only monotone fuzzy annotation functions.

**Theorem 2** Let \( \Pi \) be a \( \text{DFLP} \) program and \( I \) be a fuzzy interpretation. Then, \( I \) is a fuzzy answer set for \( \Pi \) iff \( I \) is a fuzzy answer set for \( \Pi \) w.r.t. the fuzzy answer set semantics of [Saad, 2010].

Now we show that the fuzzy answer set semantics of \( \text{DFLP}^{F,A} \) programs naturally subsumes and generalizes the classical answer set semantics of the classical disjunctive logic programs with the classical aggregates, \( \text{DLP}^{A} \) [Faber et al., 2010], which consequently naturally subsumes the classical answer set semantics of the original classical disjunctive logic programs, \( \text{DLP} \) [Gelfond and Lifschitz, 1991].

Any \( \text{DLP}^{A} \) program, \( \Pi \), is represented as a \( \text{DFLP}^{F,A} \) program, \( \Pi' \), where each \( \text{DLP}^{A} \) rule in \( \Pi \) of the form

\[
a_1 \lor \ldots \lor a_k \leftarrow a_{k+1}, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n
\]

is represented, in \( \Pi' \), as a \( \text{DFLP}^{F,A} \) rule of the form

\[
a_1 : 1 \lor \ldots \lor a_k : 1 \leftarrow a_{k+1} : 1, \ldots, a_m : 1, \text{not } a_{m+1} : 1, \ldots, \text{not } a_n : 1
\]

where \( a_1, \ldots, a_k \) are atoms and \( a_{k+1}, \ldots, a_n \) are atoms or fuzzy aggregate atoms whose fuzzy aggregates contain fuzzy sets that involve conjunctions of fuzzy annotated atoms with the fuzzy annotation 1, where 1 represents the truth value \textit{true}. We call this class of \( \text{DFLP}^{F,A} \) programs as \( \text{DFLP}^{F,A} \). Any \( \text{DLP} \) program is represented as a \( \text{DFLP}^{F,A} \) program by the same way as \( \text{DLP}^{A} \) except that \( \text{DLP} \) disallows classical aggregate atoms. The following results show that \( \text{DFLP}^{F,A} \) programs subsume both \( \text{DLP}^{A} \) and \( \text{DLP} \) programs.

**Theorem 3** Let \( \Pi' \) be a \( \text{DFLP}^{F,A} \) program equivalent to a \( \text{DLP}^{A} \) program \( \Pi \). Then, \( I \) is a fuzzy answer set for \( \Pi' \) iff \( I \) is a classical answer set for \( \Pi \), where \( I'(a) = 1 \) iff \( a \in I \) and \( I'(b) = 0 \) iff \( b \in B_C - I \).

**Proposition 1** Let \( \Pi' \) be a \( \text{DFLP}^{F,A} \) program equivalent to a \( \text{DLP} \) program \( \Pi \). Then, \( I \) is a fuzzy answer set for \( \Pi' \) iff \( I \) is a classical answer set for \( \Pi \), where \( I'(a) = 1 \) iff \( a \in I \) and \( I'(b) = 0 \) iff \( b \in B_C - I \).

6 Conclusions and Related Work

We presented the syntax and semantics of the fuzzy aggregates disjunctive fuzzy logic programs, \( \text{DFLP}^{F,A} \), that extends the original disjunctive fuzzy logic programs, \( \text{DFLP} \) [Saad, 2010], with arbitrary fuzzy annotation functions and with arbitrary fuzzy aggregates. We introduced the fuzzy answer set semantics of \( \text{DFLP}^{F,A} \) programs with arbitrary fuzzy aggregates including monotone, antimonotone, and nonmonotone fuzzy aggregates. We have shown that the fuzzy answer set semantics of \( \text{DFLP}^{F,A} \) subsumes and generalizes the fuzzy answer set semantics of the original \( \text{DFLP} \) [Saad, 2010]. In addition, we proved that the fuzzy answer sets of \( \text{DFLP}^{F,A} \) are minimal fuzzy models and consequently incomparable, which is an important property for nonmonotonic fuzzy reasoning. We have shown that the fuzzy answer set semantics of \( \text{DFLP}^{F,A} \) subsumes and generalizes the classical answer set semantics of both the classical aggregates classical disjunctive logic programs and the original classical disjunctive logic programs. To the best of our knowledge, this development is the first to consider fuzzy aggregates in fuzzy logical reasoning in general and in fuzzy answer set programming in particular. However, classical aggregates were extensively investigated in classical answer set programming [Faber et al., 2010], [Niemelä and Simons, 2001].
Pelov et al., 2007; Pelov and Truszczynski, 2004; Ferraris and Lifschitz, 2005; Ferraris and Lifschitz, 2010; Pelov, 2004. A comprehensive comparisons among these approaches to classical aggregates in classical answer set programming [Faber et al., 2010; Niemelä and Simons, 2001; Pelov et al., 2007; Pelov and Truszczynski, 2004; Ferraris and Lifschitz, 2005; Ferraris and Lifschitz, 2010; Pelov, 2004] in general and between these approaches and DLP$^A$ in particular is found in [Faber et al., 2010].

References

[Faber et al., 2010] W. Faber, N. Leone, and G. Pfeifer. Semantics and complexity of recursive aggregates in answer set programming. Artificial Intelligence, 2010.

[Ferraris and Lifschitz, 2005] P. Ferraris and V. Lifschitz. Weight constraints as nested expressions. TPLP, 5:45–74, 2005.

[Ferraris and Lifschitz, 2010] P. Ferraris and V. Lifschitz. On the stable model semantics of first-order formulas with aggregates. In Nonmonotonic Reasoning, 2010.

[Gelfond and Lifschitz, 1991] M. Gelfond and V. Lifschitz. Classical negation in logic programs and disjunctive databases. New Generation Computing, 9(3-4):363–385, 1991.

[Niemelä and Simons, 2001] I. Niemelä and P. Simons. Extending the smodels system with cardinality and weight constraints. In Logic-Based AI, 2001.

[Pelov and Truszczynski, 2004] N. Pelov and M. Truszczynski. Semantics of disjunctive programs with monotone aggregates - an operator-based approach. In NMR, 2004.

[Pelov et al., 2007] N. Pelov, M. Denecker, and M. Bruynooghe. Well-founded and stable semantics of logic programs with aggregates. TPLP, 7:355–375, 2007.

[Pelov, 2004] N. Pelov. Semantics of logic programs with aggregates. PhD thesis, Katholieke Universiteit Leuven, Leuven, Belgium, 2004.

[Saad et al., 2009] E. Saad, S. Elmorsy, M. Gaber, and Y. Hassan. Reasoning about actions in fuzzy environment. In World Congress of the International Fuzzy Systems Association/European society for Fuzzy Logic and Technology (IFSA/EUSFLAT-09), 2009.

[Saad, 2009] E. Saad. Extended fuzzy logic programs with fuzzy answer set semantics. In 3rd International Conference on Scalable Uncertainty Management, 2009.

[Saad, 2010] E. Saad. Disjunctive fuzzy logic programs with fuzzy answer set semantics. In 4rd International Conference on Scalable Uncertainty Management, 2010.

[Subrahmanian, 1994] V.S. Subrahmanian. Amalgamating knowledge bases. ACM TDS, 19(2):291–331, 1994.