EXISTENCE OF GLOBAL SOLUTIONS AND BLOW-UP OF SOLUTIONS FOR COUPLED SYSTEMS OF FRACTIONAL DIFFUSION EQUATIONS

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ABSTRACT. We study the Cauchy problem for a system of semi-linear coupled fractional-diffusion equations with polynomial nonlinearities posed in $\mathbb{R}_+ \times \mathbb{R}^N$. Under appropriate conditions on the exponents and the orders of the fractional time derivatives, we present a critical value of the dimension $N$, for which global solutions with small data exist, otherwise solutions blow-up in finite time. Furthermore, the large time behavior of global solutions is discussed.

1. INTRODUCTION

We consider the system

\begin{align}
C D^{\gamma_1}_{0+} u - \Delta u &= f(v), \quad t > 0, \ x \in \mathbb{R}^N, \\
C D^{\gamma_2}_{0+} v - \Delta v &= g(u), \quad t > 0, \ x \in \mathbb{R}^N,
\end{align}

subject to the initial conditions

\begin{align}
u(0, x) &= u_0(x), \quad v(0, x) = v_0(x), \quad x \in \mathbb{R}^N, \ \ \ \ (1.2)
\end{align}

where $0 < \gamma_1, \gamma_2 < 1$, for $0 < \alpha < 1$, $C D_{0+}^\alpha u$ denotes the Caputo time fractional derivative defined, for an absolutely continuous function $u$, by

\begin{align}
(C D_{0+}^\alpha u)(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_t u(s, \cdot) \, ds, \quad 0 < \alpha < 1,
\end{align}

where $\Delta$ is the Laplace operator in $\mathbb{R}^N$. The functions $f(v)$ and $g(u)$ are the nonlinear source terms that will be determined later, and $u_0, v_0$ are given functions.

Before we present our results and comment on them, let us dwell on existing results concerning the limiting case $\gamma_1 = \gamma_2 = 1$. Escobedo and Herrero [7] studied the existence of global solutions, and blowing-up of solutions for the system

\begin{align}
\begin{cases}
u_t - \Delta u = v^p, \quad t > 0, \ x \in \mathbb{R}^N, \ v > 0, \\
v_t - \Delta v = u^q, \quad t > 0, \ x \in \mathbb{R}^N, \ u > 0.
\end{cases}
\end{align}

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They showed, in particular, that for
\[ pq > 1, \quad \frac{N}{2} \leq \frac{\max\{p, q\} + 1}{pq - 1}, \]
every nontrivial solution of (1.3) blows-up in a finite time \( T^* = T^*(\|u\|_{\infty}, \|v\|_{\infty}) \),
in the sense that
\[
\limsup_{t \to T^*} \|u(t)\|_{\infty} = \limsup_{t \to T^*} \|v(t)\|_{\infty} = +\infty.
\]
The work [7] has been followed by works of Escobedo and Herrero in a bounded domain, Escobedo and Levine [8] for more general nonlinear forcing terms, Lu [18], Lu and Sleeman [17], Mochizuki [22], Mochizuki and Huang [23], Takase and Sleeman [31, 32], Samarskii et al. [29], and many other authors; see the review papers [4, 1, 24].

Time-fractional differential equations/systems for global or blowing-up solutions have been studied, for example, in [5, 6, 12, 14, 20, 21, 28, 34, 39]. Kirane, Laskri and Tatar [14] studied the more general system
\[
\mathcal{C}D_{0,t}^{\gamma_1} u + (-\Delta)^{\beta/2} u = |v|^p, \quad t > 0, \quad x \in \mathbb{R}^N, \quad p > 1,
\]
\[
\mathcal{C}D_{0,t}^{\gamma_2} v + (-\Delta)^{\gamma/2} v = |u|^q, \quad t > 0, \quad x \in \mathbb{R}^N, \quad q > 1,
\]
(1.4)
(for the definition of \((-\Delta)^{\sigma/2}, 1 \leq \sigma \leq 2\) see [14]) with nonnegative initial data, and proved the non-existence of global solutions under the condition
\[ pq > 1, \quad N \leq \max \left\{ \frac{\gamma_2}{q} + \gamma_1 - \left(1 - \frac{1}{pq}\right), \frac{\gamma_1}{p} + \gamma_2 - \left(1 - \frac{1}{pq}\right) \right\}, \]
where \( p + p' = pp' \) and \( q + q' = qq' \).

Here, we consider problem (1.1)-(1.2) and will give conditions relating the space dimension \( N \) with parameters \( \gamma_1, \gamma_2, p, \) and \( q \) for which the solution of (1.1)-(1.2) exists globally in time and satisfies \( L^\infty \)-decay estimates. We also discuss blowing-up in finite time solutions with initial data having positive average. Our study of the existence of global solutions relies on the semigroup theory, while for the blow-up of solutions result, we use the test function approach due to Zhang [41] and developed by Mitidieri and Pohozaev [24], and used by several authors (see for example [14, 10, 39]). Our result on blowing-up solutions improves the one obtained in [14]. We should mention that to the best of our knowledge there are no global existence and large time behavior results for the time-fractional diffusion system with two different fractional powers. The paper of Zhang et al. [40] does not treat the case of different time fractional operators. Also in [40], the authors do not obtain the decay rate of the solution in the space \( L^\infty(\mathbb{R}^N) \).

The rest of this article is organized as follows. In section 2, we present some preliminary lemmas. In section 3, we present the main results of this paper. Finally, section 4 and section 5 are devoted to the proofs of small data global existence and blow-up in finite time of the solutions of problem (1.1)-(1.2).

Throughout this article, \( C \) will denote a positive constant. The space \( L^p(\mathbb{R}^N) \) \((1 \leq p < \infty)\) will be equipped with the usual norm \( \|u\|_{L^p(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u(x)|^pdx \). The space \( C_0(\mathbb{R}^N) \) denotes the set of all continuous functions decaying to zero at infinity, equipped with Chebychev’s norm \( \|u\|_{\infty} \).
2. Preliminaries

The left-sided and right-sided Riemann-Liouville integrals (see [30]), for \( \Psi \in L^1(0, T) \), \( 0 < \alpha < 1 \), are defined as

\[
(I^\alpha_{0, t} \Psi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\Psi(\sigma)}{(t - \sigma)^{1-\alpha}} d\sigma, \quad (I^\alpha_{t, T} \Psi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T \frac{\Psi(\sigma)}{(\sigma - t)^{1-\alpha}} d\sigma,
\]

respectively. \( \Gamma \) stands for the Euler gamma function.

The left-handed and right-handed Riemann-Liouville derivatives (see [30]), for \( \Psi \in AC^1([0, T]) \), \( 0 < \alpha < 1 \), are defined as

\[
(D^\alpha_{0, t} \Psi)(t) = \frac{d}{dt} \circ I^{1-\alpha}_{0, t} \Psi(t), \quad (D^\alpha_{t, T} \Psi)(t) = -\frac{d}{dt} \circ I^{1-\alpha}_{t, T} \Psi(t),
\]

respectively.

The Caputo fractional derivative for a function \( \Psi \in AC^1([0, T]) \) is defined by

\[
(^C D^\alpha_{0, t} \Psi)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\Psi'(\sigma)}{(t - \sigma)^\alpha} d\sigma,
\]

\[
(^C D^\alpha_{t, T} \Psi)(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{\Psi'(\sigma)}{(\sigma - t)^\alpha} d\sigma.
\]

For \( 0 < \alpha < 1 \) and \( \Psi \in AC^1([0, T]) \), we have

\[
(D^\alpha_{0, t} \Psi)(t) = \frac{1}{\Gamma(1-\alpha)} \left[ \Psi(0)\frac{t^\alpha}{t^\alpha} + \int_0^t \frac{\Psi'(\sigma)}{(t - \sigma)^\alpha} d\sigma \right],
\]

and

\[
(D^\alpha_{t, T} \Psi)(t) = \frac{1}{\Gamma(1-\alpha)} \left[ \Psi(T)\frac{1}{(T - t)^\alpha} - \int_t^T \frac{\Psi'(\sigma)}{(\sigma - t)^\alpha} d\sigma \right].
\]

The Caputo derivative is related to the Riemann-Liouville derivative by

\[
^C D^\alpha_{0, t} \Psi(t) = (D^\alpha_{0, t}) (\Psi(t) - \Psi(0)), \quad \text{for} \ \Psi \in AC^1([0, T]).
\]

Let \( 0 < \alpha < 1 \), \( f \in AC^1([0, T]) \) and \( g \in AC^1([0, T]) \). Then

\[
\int_0^T f(t) (D^\alpha_{0, t} g)(t) dt = \int_0^T g(t) (^C D^\alpha_{t, T} f)(t) dt + f(T) (I^{1-\alpha}_{0, t} g)(T).
\]

If \( f(T) = 0 \), then

\[
\int_0^T f(t) (D^\alpha_{0, t} g)(t) dt = \int_0^T g(t) (^C D^\alpha_{t, T} f)(t) dt.
\]

For later use, let

\[
\varphi(t) = \left( 1 - \frac{t}{T} \right)^l, \quad \text{for} \ t \geq 0, \ l \geq 2.
\]

By a direct calculation, we obtain

\[
^C D^\alpha_{0, t} \varphi(t) = \frac{\Gamma(l + 1)}{\Gamma(l + 1 - \alpha)} T^{-\alpha} \left( 1 - \frac{t}{T} \right)^l, \quad t \geq 0.
\]

Now, we present some properties of two special functions. The two parameter Mittag-Leffler function \([30]\) is defined for \( z \in \mathbb{C} \) as

\[
E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \ \Re(\alpha) > 0.
\]
It satisfies
\[ I_{0+}^{1-\alpha}(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)) = E_{\alpha,1}(\lambda t^\alpha) \quad \text{for } \lambda \in \mathbb{C}, 0 < \alpha < 1. \]

The Wright type function
\[ \phi_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k! \Gamma(-\alpha k + 1 - \alpha)}, \]
\[ = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-z)^k \Gamma(\alpha(k+1)) \sin(\pi(k+1)\alpha)}{k!}, \]
for \(0 < \alpha < 1\), is an entire function; it has the following properties:
(a) \( \phi_\alpha(\theta) \geq 0 \) for \( \theta \geq 0 \) and \( \int_0^{\infty} \phi_\alpha(\theta) d\theta = 1; \)
(b) \( \int_0^{+\infty} \phi_\alpha(\theta) \theta^r d\theta = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)} \) for \( r > -1; \)
(c) \( \int_0^{+\infty} \phi_\alpha(\theta) e^{-\theta} d\theta = E_{\alpha,1}(-z), z \in \mathbb{C}; \)
(d) \( \alpha \int_0^{+\infty} \theta \phi_\alpha(\theta) e^{-\theta} d\theta = E_{\alpha,\alpha}(-z), z \in \mathbb{C}. \)

The operator \( A = -\Delta \) with domain
\[ D(A) = \{ u \in C_0(\mathbb{R}^N): \Delta u \in C_0(\mathbb{R}^N) \}, \]
generates, on \( C_0(\mathbb{R}^N) \), a semigroup \( \{T(t)\}_{t \geq 0} \), where
\[ T(t)u_0(x) = \int_{\mathbb{R}^N} G(t,x,y)u_0(y)dy, \quad G(t,x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-|x|^2/4t}; \]
it is analytic and contractive on \( L^q(\mathbb{R}^N) \) \(^3\) and, for \( t > 0, x \in \mathbb{R}^N \), it satisfies
\[ \|T(t)u_0\|_{L^p(\mathbb{R}^N)} \leq (4\pi t)^{-\frac{N}{2}(1/q-1/p)} \|u_0\|_{L^q(\mathbb{R}^N)}, \quad (2.2) \]
for \( 1 \leq q \leq p \leq +\infty. \)

Let the operators \( P_\alpha(t) \) and \( S_\alpha(t) \) be defined by
\[ P_\alpha(t)u_0 = \int_0^{+\infty} \phi_\alpha(\theta)T(t^{\alpha}\theta)u_0 d\theta, \quad t \geq 0, \ u_0 \in C_0(\mathbb{R}^N), \quad (2.3) \]
\[ S_\alpha(t)u_0 = \alpha \int_0^{+\infty} \theta \phi_\alpha(\theta)T(t^{\alpha}\theta)u_0 d\theta, \quad t \geq 0, \ u_0 \in C_0(\mathbb{R}^N). \quad (2.4) \]

The operators \( P_\alpha(t) \) and \( S_\alpha(t) \) acting on the space \( C_0(\mathbb{R}^N) \) into itself, see \(^3\) Lemma 2.3, Lemma 2.4].

Consider the problem
\[ CD_0^{\alpha}u - \Delta u = f(t,x), \quad t > 0, \ x \in \mathbb{R}^N, \]
\[ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \quad (2.5) \]
where \( u_0 \in C_0(\mathbb{R}^N) \) and \( f \in L^1((0,T), C_0(\mathbb{R}^N)) \). If \( u \) is a solution of \((2.5)\), then by \(^3\), it satisfies
\[ u(t,x) = P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s,x) ds. \]

The following lemmas play an important role in obtaining the results of this paper; their proofs are obtained by combining smoothing effect of the heat semigroup property \((2.2)\) with formulas \((2.3)\) and \((2.4)\) (see \(^3\)).

**Lemma 2.1.** The operator \( \{P_\alpha(t)\}_{t \geq 0} \) has the following properties:
(a) If \( u_0 \geq 0, u_0 \neq 0 \), then \( P_\alpha(t)u_0 > 0 \) and \( \|P_\alpha(t)u_0\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)}; \)
Lemma 2.2. For the operator family \( \{S_\alpha(t)\}_{t>0} \), we have the following estimates:

(a) If \( u_0 \geq 0 \) and \( u_0 \neq 0 \), then \( S_\alpha(t)u_0 > 0 \) and

\[
\|S_\alpha(t)u_0\|_{L^1(\mathbb{R}^N)} = \frac{1}{\Gamma(\alpha)} \|u_0\|_{L^1(\mathbb{R}^N)};
\]

(b) If \( p \leq q \leq +\infty \) and \( 1/r = 1/p - 1/q \), \( 1/r < 2/N \), then

\[
\|S_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \leq \alpha(4\pi t^\alpha)^{-\frac{N}{2}} \frac{\Gamma(1-N/(2r))}{\Gamma(1-\alpha N/(2r))} \|u_0\|_{L^p(\mathbb{R}^N)}.
\]

Lemma 2.3. Let \( l \geq 1 \), and let the function \( f(t,x) \) satisfy

\[
\|f(t,\cdot)\|_l \leq \begin{cases} C_1, & 0 \leq t \leq 1, \\ C_2 t^{-\alpha}, & t > 1, \end{cases}
\]

for some positive constants \( C_1, C_2 \) and \( \alpha \). Then

\[
\|f(t,\cdot)\|_l \leq \max\{C_1, C_2\}(1+t)^{-\beta}, \quad \text{for all } 0 < \beta \leq \alpha \text{ and } t \geq 0.
\]

**Proof.** For \( 0 \leq t \leq 1 \), we have \( \|f(t,\cdot)\|_l \leq C_1 \leq C_1 2^\alpha(1+t)^{-\alpha} \), so

\[
\|f(t,\cdot)\|_l \leq K(1+t)^{-\beta},
\]

for some positive constant \( K > 0 \), and for all \( 0 < \beta \leq \alpha \).

When \( t \geq 1 \), it follows from \( \|f(t,\cdot)\|_l \leq C_2 t^{-\alpha} \) that there is a constant \( K' > 0 \), such that \( \|f(t,\cdot)\|_l \leq K'(1+t)^{-\alpha} \), and so for all \( 0 < \beta \leq \alpha \) and any \( t \geq 1 \), we have

\[
\|f(t,\cdot)\|_l \leq K'(1+t)^{-\beta}, \quad \text{for } 0 < \beta \leq \alpha.
\]

Therefore \( \|f(t,\cdot)\|_l \leq \max\{K, K'\}(1+t)^{-\beta} \), for all \( 0 < \beta \leq \alpha \) and \( t \geq 0 \). \( \square \)

3. Main results

In this section, we state our main result. First, we present the definition of a mild solution of problem (1.1)-(1.2).

**Definition 3.1.** Let \((u_0, v_0) \in C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N), 0 < \gamma_1, \gamma_2 < 1, p, q \geq 1 \) and \( T > 0 \). We say that \((u, v) \in C([0,T]; C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))\) a mild solution of system (1.1)-(1.2) if \((u, v)\) satisfies the integral equations

\[
\begin{align*}
\frac{d}{dt} u(t,x) &= P_{\gamma_1} \frac{d}{dt} u_0(t) + \int_0^t (t-\tau)\gamma_1^{-1} S_{\gamma_1}(t-\tau) f(v(\tau,\cdot)) \, d\tau, \\
\frac{d}{dt} v(t,x) &= P_{\gamma_2} \frac{d}{dt} v_0(t) + \int_0^t (t-\tau)\gamma_2^{-1} S_{\gamma_2}(t-\tau) g(u(\tau,\cdot)) \, d\tau.
\end{align*}
\]

Using the results in \[39\] Theorem 3.2 and \[40\] Theorem 3.2, the local solvability and uniqueness of (1.1)-(1.2) can be established.

**Proposition 3.2** (Existence of a local mild solution). Given \( u_0 \) and \( v_0 \) in \( C_0(\mathbb{R}^N) \), \( 0 < \gamma_1, \gamma_2 < 1, p, q \geq 1 \), there exist a maximal time \( T_{\max} > 0 \) and a unique mild solution \((u, v) \in C([0,T_{\max}); C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))\) to problem (1.1)-(1.2), such that either

(i) \( T_{\max} = \infty \) (the solution is global), or
(ii) $T_{\text{max}} < \infty$ and $\lim_{t \to T_{\text{max}}} (\|u(t)\|_{\infty} + \|v(t)\|_{\infty}) = \infty$ (the solution blows up in a finite time).

If, in addition, $u_0 \geq 0$, $v_0 \geq 0$, $u_0$, $v_0 \neq 0$, then $u(t) > 0$, $v(t) > 0$ and $u(t) \geq P_{\gamma_1}(t)u_0$, $v(t) \geq P_{\gamma_2}(t)v_0$ for $t \in (0, T_{\text{max}})$.
Moreover, if $(u_0, v_0) \in L^{1}(\mathbb{R}^{N}) \times L^{1}(\mathbb{R}^{N})$, then for all $s_1, s_2 \in (1, +\infty)$, $(u, v) \in C([0, T_{\text{max}}]; L^{s_1}(\mathbb{R}^{N}) \times L^{s_2}(\mathbb{R}^{N}))$.

Now, we state the first main result of this section concerning the existence of a global solution and large time behavior of solutions of (1.1)-(1.2).

**Theorem 3.3 (Existence of a global mild solution).** Let $N \geq 1$, let $q \geq p \geq 1$, be such that $pq > 1$, let $(f(v), g(u)) = (\pm |v|^{p-1}v, \pm |u|^{q-1}u)$, or $(\pm |v|^p, \pm |u|^q)$, and let $0 < \gamma_1 \leq \gamma_2 < 1$. If

$$\frac{N}{2} \geq \left( \frac{\gamma_2 - \gamma_1}{\gamma_1} \right) pq + q \gamma_2 + \gamma_1,$$

(3.2)

then, for

$$\|u_0\|_1 + \|u_0\|_{\infty} + \|v_0\|_1 + \|v_0\|_{\infty} \leq \varepsilon_0,$$

with some $\varepsilon_0 > 0$, there exist $s_1 > q$, $s_2 > p$ such that problem (1.1)-(1.2) admits a global mild solution with

$$u \in L^{\infty}([0, \infty), L^{\infty}(\mathbb{R}^{N})) \cap L^{\infty}([0, \infty), L^{s_1}(\mathbb{R}^{N})),
$$

$$v \in L^{\infty}([0, \infty), L^{\infty}(\mathbb{R}^{N})) \cap L^{\infty}([0, \infty), L^{s_2}(\mathbb{R}^{N})).$$

Furthermore, for all $\delta > 0$,

$$\max \left\{ 1 - \frac{(pq - 1)}{\gamma_2 q (p + 1)}, 1 - \frac{\gamma_1 (pq - 1)}{\gamma_2 (p + 1)}, 1 - \frac{(pq - 1)}{q + 1} \right\} < \delta \leq \min \left\{ 1, \frac{N (pq - 1)}{2 q (p + 1)} \right\},$$

$$\|u(t)\|_{s_1} \leq C(t + 1)^{-\frac{1}{\delta} - (\gamma_1 + \gamma_2)} \|u(t)\|_{s_2} \leq C(t + 1)^{-\frac{1}{\delta} - (\gamma_1 + \gamma_2)} \|v(t)\|_{s_2} \leq C(t + 1)^{-\frac{1}{\delta} - (\gamma_1 + \gamma_2)}, \quad t \geq 0.$$

If, in addition,

$$\frac{pN}{2 s_2} < 1 \quad \text{and} \quad \frac{qN}{2 s_1} < 1,$$

or

$$N > 2, \frac{pN}{2 s_2} < 1 \quad \text{and} \quad \frac{qN}{2 s_1} \geq 1,$$

or

$$N > 2, \quad \frac{qN}{2 s_1} \geq 1, \quad \frac{pN}{2 s_2} \geq 1 \quad \text{and} \quad q \geq p > 1$$

with

$$\max \left\{ \frac{q + 1}{pq (p + 1)}, \frac{pq - 1}{pq (p + 1)}, \frac{\gamma_2}{p} \sqrt{\frac{\gamma_2}{pq}} \right\} < \gamma_1 \leq \gamma_2 < 1,$$

then $u, v \in L^{\infty}([0, \infty), L^{\infty}(\mathbb{R}^{N}))$,

$$\|u(t)\|_{\infty} \leq C(t + 1)^{-\hat{\sigma}}, \quad \|v(t)\|_{\infty} \leq C(t + 1)^{-\hat{\sigma}}, \quad t \geq 0,$$

for some constants $\hat{\sigma} > 0$ and $\hat{\sigma} > 0$.

**Definition 3.4 (Weak solution).** Let $u_0, v_0 \in L^{1}_{\text{loc}}(\mathbb{R}^{N})$, $T > 0$. We say that

$$(u, v) \in L^{q}((0, T), L^{s_1}_{\text{loc}}(\mathbb{R}^{N})) \times L^{p}((0, T), L^{s_2}_{\text{loc}}(\mathbb{R}^{N}))$$

is a weak solution of (1.1)-(1.2) if

$$\int_{0}^{T} \int_{\mathbb{R}^{N}} (|v|^p \varphi + u_0 D_{(0, t)}^{\gamma_1} \varphi) \, dx \, dt = \int_{0}^{T} \int_{\mathbb{R}^{N}} u(-\Delta \varphi) \, dx \, dt + \int_{0}^{T} \int_{\mathbb{R}^{N}} u D_{(0, t)}^{\gamma_1} \varphi \, dx \, dt,$$
\[
\int_0^T \int_{\mathbb{R}^N} (|u|^q \varphi + v_0 D_{t_0}^{\gamma_2} \varphi) \, dx \, dt = \int_0^T \int_{\mathbb{R}^N} v(-\Delta \varphi) \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} v D_{t_0}^{\gamma_2} \varphi \, dx \, dt,
\]
for every \( \varphi \in C_c^{1,2}([0, T] \times \mathbb{R}^N) \) such that \( \operatorname{supp}_x \varphi \subset \mathbb{R}^N \) and \( \varphi(T, \cdot) = 0 \).

Similar to the proof in [39], we can easily obtain the following lemma asserting that the mild solution is the weak solution.

**Lemma 3.5.** Assume \( u_0, v_0 \in C_0(\mathbb{R}^N) \), and let \((u, v) \in C([0, T], C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))\) be a mild solution of \((1.1)-(1.2)\), then \((u, v)\) is a weak solution of \((1.1)-(1.2)\).

Our next result concerns the blowing-up of solutions of \((1.1)-(1.2)\).

**Theorem 3.6 (Blow-up of mild solutions).** Let \( N \geq 1, \ p > 1, \ q > 1, \ 0 < \gamma_1, \gamma_2 < 1, \) let \((f(v), g(u)) = (|v|^p, |u|^q)\), let \( u_0, \ v_0 \in C_0(\mathbb{R}^N) \), \( u_0 \geq 0, \ v_0 \geq 0, \ u_0 \neq 0 \) and \( v_0 \neq 0 \). If

\[
\frac{N}{2} < \min \left\{ \frac{(pq_2 + \gamma_2)}{\gamma_1(pq - 1)} \frac{\gamma_2(pq_2 + \gamma_1)}{\gamma_1(pq_1 + \gamma_2)} \frac{\gamma_1(pq_1 + \gamma_2)}{\gamma_1(pq - 1)} \right\} \left( \frac{p+1}{pq} \right) \quad \text{or}
\]

\[
\frac{N}{2} < \min \left\{ \frac{(pq_1 + \gamma_2)}{\gamma_2(pq - 1)} \frac{\gamma_2(pq_1 + \gamma_1)}{\gamma_2(pq_1 + \gamma_2)} \frac{\gamma_1(pq_1 + \gamma_2)}{\gamma_2(pq_1 - 1)} \right\} \left( \frac{p+1}{pq} \right),
\]

then the mild solution \((u, v)\) of \((1.1)-(1.2)\) blows up in a finite time.

Also if \( p = 1 \) and \( 1 < q < 1 + \frac{2}{N} \), or \( 1 < p < 1 + \frac{2}{N} \) and \( q = 1 \), then the solution blows-up in a finite time.

A result of blowing-up solutions can be obtained via differential inequalities. Let

\[
\chi(x) = \left( \int_{\mathbb{R}^N} e^{-\sqrt{N^2 + |x|^2}} \, dx \right)^{-1} e^{-\sqrt{N^2 + |x|^2}}, \quad x \in \mathbb{R}^N,
\]

which satisfies

\[
\int_{\mathbb{R}^N} \chi(x) \, dx = 1.
\]

In the next theorem, we take \( f(v) = |v|^p \) and \( g(u) = |u|^q \).

**Theorem 3.7.** Let \( \gamma_1 = \gamma_2 = \gamma \in (0, 1), \ u_0, \ v_0 \in C_0(\mathbb{R}^N) \) and \( u_0, \ v_0 \geq 0 \). Let \( p > 1, \ q > 1 \) such that \( p \leq q \) and let \((f(v), g(u)) = (|v|^p, |u|^q)\). If

\[
Z_0 := \int_{\mathbb{R}^N} (u_0(x) + v_0(x)) \chi(x) \, dx > 2 \frac{\pi^2}{N},
\]

then the solution of problem \((1.1)-(1.2)\) blows-up in a finite time. Moreover, we have estimate of the time blowing up \( t_{*} \leq \left[ \frac{\ln(1 - 2^{\gamma-\gamma_1})}{\Gamma(\gamma + 1)} \Gamma(\gamma + 1) \right]^{1/\gamma} \).

The next lemma plays an important role in establishing lower solution for Caputo fractional differential equation.

**Lemma 3.8 ([35, Lemma 3.1]).** Let \( u = u(t) \) is a solution of the ordinary differential equation

\[
\frac{du}{dt} = F(u), \quad u(0) = u_0,
\]

where \( F \) is a function of \( u \) such that \( F(0) \geq 0, \ F(u) > 0, \ F_u(u) \geq 0 \) for \( u \geq 0 \) then \( \varphi(t) = u(t) \) is a lower solution of a Caputo fractional differential equation

\[
C D_{0+}^{\alpha} u(t) = F(u), \quad u(0) = u_0,
\]
Clearly, we have
\[ C^\alpha_0(t) v(t) \leq F(v), \quad v(0) \leq u_0. \]

4. PROOFS OF MAIN RESULTS

Proof of Theorem 3.3. We proceed in three steps.

**Step 1:** Global existence for \((u, v)\) in \(L^p(H^N) \times L^q(H^N)\). Since \(q \geq p > 1, \ pq > 1, \ 0 < \gamma_1 \leq \gamma_2 < 1\) and
\[
\frac{N}{2} \geq (\gamma_2 - \gamma_1) pq + q \gamma_2 + \gamma_1 \quad \frac{\gamma_2 q(p+1) - \gamma_1 q(pq-1)}{\gamma_2 (pq-1)},
\]
we have
\[
\frac{N(pq-1)}{2q(p+1)} > \frac{\gamma_2(p+1) - \gamma_1(pq-1)}{\gamma_2(p+1)} = 1 - \frac{\gamma_1(pq-1)}{\gamma_2(p+1)}.
\]
Note that, from
\[
\frac{N}{2} \geq (\gamma_2 - \gamma_1) pq + q \gamma_2 + \gamma_1 \quad \frac{q + 1}{pq - 1},
\]
we obtain
\[
\frac{N(pq-1)}{2q(p+1)} = \frac{q + 1}{pq - 1} \frac{(pq-1)}{q(p+1)} > 1 - \frac{(pq-1)}{q+1},
\]
\[
\frac{N(pq-1)}{2q(p+1)} > \frac{q + 1}{q(p+1)} > 1 - \frac{(pq-1)}{q+1}.
\]
From these facts, we can choose \(\delta > 0\) such that
\[
\max \left\{ 1 - \frac{(pq-1)}{\gamma_2 q(p+1)}, 1 - \frac{\gamma_1(pq-1)}{\gamma_2(p+1)}, 1 - \frac{(pq-1)}{q+1} \right\}
\]
\[
< \delta < \min \left\{ 1, \frac{N(pq-1)}{2q(p+1)} \right\}.
\]

We set
\[
\begin{align*}
& r_1 = \frac{N \gamma_1 (pq-1)}{2 \gamma_1 (1+\delta p) + \gamma_2 p (1-\delta)}, \quad r_2 = \frac{N \gamma_2 (pq-1)}{2 \gamma_2 (1+\delta q) + \gamma_1 q (1-\delta)}, \\
& s_1 = \frac{2 \delta \ p + 1}{N \ pq - 1}, \quad s_2 = \frac{2 \delta \ q + 1}{N \ pq - 1}, \\
& \sigma_1 = \frac{(1-\delta)(\gamma_1 + \gamma_2 p)}{pq - 1}, \quad \sigma_2 = \frac{(1-\delta)(\gamma_2 + \gamma_1 q)}{pq - 1}.
\end{align*}
\]

Clearly, we have
\[
\begin{align*}
& \frac{1}{r_1} = \frac{2}{N \gamma_1} \frac{(1-\delta)(\gamma_1 + \gamma_2 p)}{pq - 1} + \frac{2 \delta \ (pq + 1)}{N \ pq - 1}, \\
& \frac{1}{r_2} = \frac{2}{N \gamma_2} \frac{(1-\delta)(\gamma_2 + \gamma_1 q)}{pq - 1} + \frac{2 \delta \ (pq + 1)}{N \ pq - 1}.
\end{align*}
\]

It is easy to check that \(s_1 > q, s_2 > p, ps_1 > s_2, q s_2 > s_1, \ s_1 > r_1 > 1, \ s_2 > r_2 > 1, \)
\[
\frac{N}{2} \gamma_1 \left( \frac{1}{r_1} - \frac{1}{s_1} \right) q < 1, \quad \frac{N}{2} \gamma_2 \left( \frac{1}{r_2} - \frac{1}{s_2} \right) p < 1, \quad \frac{N}{2} \left( \frac{p}{s_2} - \frac{1}{s_1} \right) = \frac{\delta}{2} \left( \frac{q}{s_1} - \frac{1}{s_2} \right),
\]
\(p \sigma_2 < 1, \) and \(q \sigma_1 < 1.\) From
\[
\frac{pq (\gamma_2-1) + 1 + q \gamma_1}{pq \gamma_2 + \gamma_1 q} = \frac{q(p \gamma_2 + \gamma_1) - (pq-1)}{pq \gamma_2 + \gamma_1 q} < 1 - \frac{(pq-1)}{\gamma_2 q(p+1)} < \delta
\]
we obtain $\delta > \frac{pq(\gamma_2-1) + q\gamma_1 + 1}{(\gamma_1+p\gamma_2)q}$ which is equivalent to

$$
\left(\gamma_1 - \frac{N}{2} \frac{p - \frac{1}{s_1}}{s_2} - p\sigma_2\right)q > -1.
$$

In fact, since $\delta = \frac{N}{2} \frac{p - \frac{1}{s_1}}{s_2}$, the above inequality gives

$$
(\gamma_1 - \delta \gamma_1 - p\sigma_2)q > -1,
$$

using definition of $\sigma_2 = \frac{(1-\delta)(\gamma_2 + \gamma_1 q)}{pq - 1}$, we obtain

$$
(\gamma_1 - \delta \gamma_1 - p\frac{(1-\delta)(\gamma_2 + \gamma_1 q)}{pq - 1})q > -1,
$$

so, we obtain

$$
(1-\delta)\left(\gamma_1 - p\frac{(\gamma_2 + \gamma_1 q)}{pq - 1}\right)q > -1.
$$

Therefore

$$
(1-\delta)\left(\frac{pq - 1}{pq - 1}\right)q > -1.
$$

By simplification, we obtain

$$
(\delta - 1)\left(\frac{\gamma_1 + pq\gamma_2}{pq - 1}\right)q > -1,
$$

or

$$
\delta\left(\frac{\gamma_1 + pq\gamma_2}{pq - 1}\right)q > -1.
$$

Thus

$$
\delta > 1 - \frac{pq - 1}{(\gamma_1 + pq\gamma_2)q} = \frac{pq(\gamma_2 - 1) + q\gamma_1 + 1}{(\gamma_1 + pq\gamma_2)q}.
$$

Similarly, we have

$$
\left(\gamma_2 - \frac{N}{2} \frac{q - \frac{1}{s_1}}{s_2} - q\sigma_1\right)p > -1.
$$

equivalent to $\delta > \frac{pq(\gamma_2-1) + p\gamma_2 + 1}{(\gamma_2+q\gamma_1)p}$.

Let $(u_0, v_0) \in C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N) \cap L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N)$, and let

$$(u, v) \in C([0, T_{\text{max}}); C_0(\mathbb{R}^N) \cap L^{s_1}(\mathbb{R}^N)) \times C([0, T_{\text{max}}); C_0(\mathbb{R}^N) \cap L^{s_2}(\mathbb{R}^N)).$$

For $t \in [0, T_{\text{max}})$, from (3.1), we have

$$
\|u(t)\|_{s_1} \leq \|P_{t}(t)u_0\|_{s_1} + \left\| \int_0^t (t - \tau)^{\gamma_1 - 1} S_{t}(t - \tau)|v(\tau)|^p d\tau \right\|_{s_1}
$$

$$
= \|P_{t}(t)u_0\|_{s_1} + \left[ \int_{\mathbb{R}^N} \left\| \int_0^t (t - \tau)^{\gamma_1 - 1} S_{t}(t - \tau)|v(\tau)|^p d\tau \right|^{s_1} dx \right]^{1/s_1}.
$$

We have

$$
\left[ \int_{\mathbb{R}^N} \left\| \int_0^t (t - \tau)^{\gamma_1 - 1} S_{t}(t - \tau)|v(\tau)|^p d\tau \right|^{s_1} dx \right]^{1/s_1}
$$

$$
\leq \left[ \int_{\mathbb{R}^N} \left( \int_0^t (t - \tau)^{\gamma_1 - 1} S_{t}(t - \tau)|v(\tau)|^p d\tau \right)^{s_1} dx \right]^{1/s_1}.
$$

Using Minkowski’s integral inequality, we obtain

$$
\left[ \int_{\mathbb{R}^N} \left( \int_0^t (t - \tau)^{\gamma_1 - 1} S_{t}(t - \tau)|v(\tau)|^p d\tau \right)^{s_1} dx \right]^{1/s_1}$$
Multiplying both sides of (4.7) by $t^{\gamma_1-1}$, we obtain
\[ \|u(t)\|_{s_1} \leq \|P_{\gamma_1}(t)u_0\|_{s_1} + t^{\gamma_1-1} \int_0^t \|S_{\gamma_1}(t-\tau)|v(\tau)|^{p}d\tau, \]
(4.3)
\[ \|v(t)\|_{s_2} \leq \|P_{\gamma_2}(t)v_0\|_{s_2} + t^{\gamma_2-1} \int_0^t \|S_{\gamma_2}(t-\tau)|u(\tau)|^{q}d\tau. \]
(4.4)

Applying lemmas 2.2 and 2.3, we obtain
\[ \|u(t)\|_{s_1} \leq \|u_0\|_{r_1} t^{-\sigma_1} + C \int_0^t \|S_{\gamma_1}(t-\tau)-N^{\gamma_1}(\frac{\gamma_1}{\gamma_2}-\frac{1}{r_1})|v(\tau)|^{p}d\tau, \]
(4.5)
\[ \|v(t)\|_{s_2} \leq \|v_0\|_{r_2} t^{-\sigma_2} + C \int_0^t \|S_{\gamma_2}(t-\tau)-N^{\gamma_2}(\frac{\gamma_1}{\gamma_2}-\frac{1}{r_2})|u(\tau)|^{q}d\tau. \]
(4.6)

By using (4.6) in (4.5), we obtain
\[ \|u(t)\|_{s_1} \leq \|u_0\|_{r_1} t^{-\sigma_1} + C \int_0^t (t-\tau)^{-\frac{\sigma_1}{\sigma_2}} \left( \|v_0\|_{r_2} t^{-\sigma_2} + C \int_0^t \|S_{\gamma_2}(t-\tau)|u(\tau)|^{q}d\tau \right)^p. \]
Hence
\[ \|u(t)\|_{s_1} \leq \|u_0\|_{r_1} t^{-\sigma_1} + C \int_0^t (t-\tau)^{-\frac{\sigma_1}{\sigma_2}} \left( \|v_0\|_{r_2} + C \int_0^t (t-\tau)^{-\frac{\sigma_2}{\gamma_2}} \|u(\tau)|^{q}d\tau \right)^p, \]
(4.7)

Multiplying both sides of (4.7) by $t^{\sigma_1}$, where $\sigma_1 = \frac{1-(\gamma_1+\gamma_2)p}{pq-1}$, we find that
\[ t^{\sigma_1} \|u(t)\|_{s_1} \leq \|u_0\|_{r_1} + Ct^{\sigma_1} \int_0^t (t-\tau)^{-\frac{\sigma_1}{\gamma_1}} \|v_0\|_{r_2}^{p} + C t^{\sigma_1} \int_0^t (t-\tau)^{-\frac{\sigma_1}{\gamma_2}} \|u(\tau)|^{q}d\tau \]
(4.8)
\[ \times \left( \tau^{\sigma_1} \|u(\tau)\|_{s_1} \right)^p. \]

Since $\gamma_1 - \frac{\gamma_2}{2} \gamma_1 + \frac{1}{s_2} > -1$, and $(\gamma_2 - \frac{\gamma_1}{2}) \gamma_2 \frac{1}{s_1} - \frac{1}{s_2} - q \sigma_1 > -1$, we have
\[ t^{\sigma_1} \|u(t)\|_{s_1} \leq \|u_0\|_{r_1} + Ct^{\sigma_1 + \gamma_1 - \frac{\gamma_2}{2} \gamma_1 - \frac{1}{s_2} - q \sigma_1} \|v_0\|_{r_2}^{p} \]
(4.9)
\[ + C t^{\sigma_1 + \gamma_2 - \frac{\gamma_1}{2} \gamma_2 \frac{1}{s_1} - \frac{1}{s_2} - q \sigma_1} \left( \sup_{0 \leq \tau < t} \tau^{\sigma_1} \|u(\tau)\|_{s_1} \right)^p. \]
Note that
\[
\sigma_1 = \frac{N}{2} \gamma_1 \left( \frac{1}{r_1} - \frac{1}{s_1} \right),
\]
\[
\sigma_1 + \gamma_1 - \frac{N}{2} \gamma_1 \left( \frac{p}{s_2} - \frac{1}{s_1} \right) - p \sigma_2 = 0,
\]
\[
\sigma_1 + \gamma_1 - \frac{N}{2} \gamma_1 \left( \frac{p}{s_2} - \frac{1}{s_1} \right) + \left( \gamma_2 - \frac{N}{2} \gamma_2 \left( \frac{q}{s_1} - \frac{1}{s_2} \right) - q \sigma_1 \right) p = 0,
\]
and
\[
\sigma_1 + \gamma_1 - \gamma_1 \delta + (\gamma_2 - \gamma_2 \delta - q \sigma_1) p = 0.
\]

Defining \( h(t) = \sup_{0 \leq \tau \leq t} \tau^{\gamma_1} \| u(\tau) \|_{s_1}, \) \( t \in [0, T_{\text{max}}] \), we deduce from (4.8) that
\[
h(t) \leq C(\| u_0 \|_{r_1} + \| v_0 \|_{r_2} + h(t)^{pq})
\]
for all \( t \in (0, T_{\text{max}}) \). Here \( C \) is independent of \( t \). Set
\[
A := \| u_0 \|_{r_1} + \| v_0 \|_{r_2}.
\]

Then, it follows by a continuity argument that for sufficiently small \( u_0 \) and \( v_0 \) such that \( A < (2C)^{\frac{1}{pq}} \), that
\[
h(t) \leq 2CA,
\]
for all \( t \in [0, T_{\text{max}}) \). (4.11)

Otherwise, there exists \( t_0 \in (0, T_{\text{max}}) \) such that \( h(t_0) > 2CA \); by the intermediate value theorem, since \( h \) is continuous and \( h(0) = 0 \), there exists \( t_1 \in (0, t_0) \) such that
\[
h(t_1) = 2CA.
\]

Using (4.10) in (4.11), we obtain
\[
h(t_1) \leq C(A + h(t_1)^{pq}) = \left( \frac{h(t_1)}{2} + Ch(t_1)^{pq} \right),
\]
from which, we infer
\[
\frac{h(t_1)}{2} \leq Ch(t_1)^{pq},
\]
using (4.10) in (4.13), we obtain \( CA \leq C(2CA)^{pq} \), so
\[
A \leq (2CA)^{pq} = (2C)^{pq} A^{pq},
\]
then it yields \( (2C)^{-pq} \leq A^{pq-1} \), which is equivalent to
\[
A \geq (2C)^{\frac{1}{pq}}.
\]

This contradicts the choice of \( A \). It then follows that \( h(t) \) remains bounded in all time \( t > 0 \) provided that \( \| u_0 \|_{r_1} \) and \( \| v_0 \|_{r_2} \) are small. Therefore
\[
t^{\sigma_1} \| u(t) \|_{s_1} \leq C, \quad \text{for all } t > 0.
\]
Similarly, we obtain
\[
t^{\sigma_2} \| v(t) \|_{s_2} \leq C, \quad \text{for all } t > 0.
\]

**Step 2:** \( L^\infty \)-global existence estimates of \( (u, v) \) in \( L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N) \). Let \( s_1, s_2 \) be the same as in (4.2). Since \( p \leq q \), we have
\[
\frac{Np}{2s_2} \leq \frac{Nq}{2s_1}.
\]

We further assume for some \( \xi > q \), \( w > p \), \( k_1 > 0 \), \( k_2 > 0 \) that \( u(t) \in L^w(\mathbb{R}^N) \), \( v(t) \in L^\xi(\mathbb{R}^N) \) and
\[
\| u(t) \|_w \leq C(1 + t^{k_1}), \quad \| v(t) \|_\xi \leq C(1 + t^{k_2}) \quad \text{for every } t \in [0, T_{\text{max}}). 
\]

(4.16)
Then, by applying Lemmas 2.1 and 2.2 again to (3.1), we obtain

\[ \|u(t)\|_\infty \leq \|P_{\gamma_1}(t)u_0\|_\infty + \int_0^t (t-\tau)^{\gamma_1-1} - \frac{Np_1}{2\xi} \|v(\tau)\|_\xi^p d\tau, \]  
\[ \|v(t)\|_\infty \leq \|P_{\gamma_2}(t)v_0\|_\infty + \int_0^t (t-\tau)^{\gamma_2-1} - \frac{Nq_2}{2\xi} \|u(\tau)\|_\xi^q d\tau, \]  
for all \( t \in [0, T_{\text{max}}] \). Now, if one can find \( \xi \) and \( w \) such that

\[ \frac{Np}{2\xi} < 1 \quad \text{or} \quad \frac{Nq}{2w} < 1, \]  
then the \( L^\infty \)-estimates of \((u, v)\) can be obtained. In fact, if \( \frac{Np}{2\xi} < 1 \), in view of (4.16), from (4.17) we have

\[ \|u(t)\|_\infty \leq \|P_{\gamma_1}(t)u_0\|_\infty + C \max_{\tau \in [0, t]} \|v(\tau)\|_{\xi}^{p/\gamma_1}, \]  
\[ \leq C(1 + t^{(1 - \frac{Np}{2\xi})\gamma_1 + pk_2}), \]  
and by taking \( w = \infty \) in (4.18), we obtain

\[ \|v(t)\|_\infty \leq \|P_{\gamma_2}(t)v_0\|_\infty + \int_0^t (t-\tau)^{\gamma_2-1} \|u(\tau)\|_{\xi}^{q} d\tau \]  
\[ \leq \|P_{\gamma_2}(t)v_0\|_\infty + \int_0^t (t-\tau)^{\gamma_2-1} \left(1 + t^{(1 - \frac{Nq}{2w})\gamma_2 + pk_2}\right)^q d\tau \]  
\[ \leq C \left(1 + t^{\gamma_2 + [(1 - \frac{Np}{2\xi})\gamma_1 + pk_2]q}\right). \]  
These estimates show that \( T_{\text{max}} = \infty \) and

\[ u, v \in L^\infty_\text{loc}([0, \infty); L^\infty(\mathbb{R}^N)). \]  
In a similar way, we can deal with the case \( \frac{Nq}{2w} < 1 \).

To find such appropriate \( \xi \) and \( w \), we note that if \( \frac{Nq}{2s_1} < 1 \) or \( \frac{Np}{2s_2} < 1 \), then (4.20) and (4.21) hold by taking \( \xi = s_1 \) or \( w = s_2 \). This is certainly the case if \( N \leq 2 \) as \( s_1 > q \) and \( s_2 > p \).

Thus it remains to deal with the case \( N > 2 \), \( \frac{Nq}{2s_1} \geq 1 \) and \( \frac{Np}{2s_2} \geq 1 \). We will do this via an iterative process. Define \( s_1' = s_1 \), \( s_2' = s_2 \). Since \( s_1' > q \) and \( s_2' > p \), using the Hölder inequality and lemmas 2.1, 2.2 we obtain from (4.3) and (4.4) that

\[ \|u(t)\|_{s_2'} \leq \|P_{\gamma_1}(t)u_0\|_{s_2'} + \int_0^t (t-\tau)^{\gamma_1-1} - \frac{Np_1}{2s_1'} \|v(\tau)\|_{s_1'}^p d\tau, \]  
\[ \|v(t)\|_{s_2'} \leq \|P_{\gamma_2}(t)v_0\|_{s_2'} + \int_0^t (t-\tau)^{\gamma_2-1} - \frac{Nq_2}{2s_2'} \|u(\tau)\|_{s_2'}^q d\tau, \]  
where \( s_2' \) and \( s_2'' \) are such that \( N \frac{p}{2s_1'} - \frac{1}{s_2'} < 1 \), \( N \frac{q}{2s_2' - \frac{1}{s_2''}} < 1 \). This can be verified by taking

\[ \frac{1}{s_2'} = \frac{p}{s_1'} - \frac{2}{N} + \eta, \quad \frac{1}{s_2'} = \frac{q}{s_2'} - \frac{2}{N} + \eta, \]
where \(0 < \eta < 2(1 - \delta)/N\). Observe that
\[
\frac{1}{s'_{i}} - \frac{1}{s''_{i}} = \frac{2}{N} (1 - \delta) - \eta > 0, \quad \frac{1}{s'_{i}} - \frac{1}{s''_{i}} = \frac{2}{N} (1 - \delta) - \eta > 0, \tag{4.23}
\]
and hence \(s''_{i} > s'_{i} > q\) and \(s''_{i} > s''_{i} > p\).

Next, we define the sequences \(\{s'_{i}\}_{i \geq 1}\) and \(\{s''_{i}\}_{i \geq 1}\) iteratively as follows
\[
\frac{1}{s'_{i}} = \frac{p}{s''_{i-1}} - \frac{2}{N} + \eta, \quad \frac{1}{s''_{i}} = \frac{q}{s'_{i-1}} - \frac{2}{N} + \eta, \quad i \geq 3. \tag{4.24}
\]
Then
\[
\frac{1}{s'_{i}} - \frac{1}{s'_{i+1}} = p\left(\frac{1}{s''_{i-1}} - \frac{1}{s''_{i}}\right) = pq\left(\frac{1}{s''_{i-2}} - \frac{1}{s''_{i-1}}\right)
\]
\[
\frac{1}{s''_{i}} - \frac{1}{s''_{i+1}} = q\left(\frac{1}{s'_{i-1}} - \frac{1}{s'_{i}}\right) = pq\left(\frac{1}{s'_{i-2}} - \frac{1}{s'_{i-1}}\right).
\]
Since \(pq > 1\), in view of (4.23), we obtain
\[
\frac{1}{s'_{i}} > \frac{1}{s'_{i+1}}, \quad \frac{1}{s''_{i}} > \frac{1}{s''_{i+1}}, \quad i \geq 1 \tag{4.25}
\]
\[
\lim_{i \to +\infty} \left(\frac{1}{s'_{i}} - \frac{1}{s'_{i+1}}\right) = \lim_{i \to +\infty} \left(\frac{1}{s''_{i}} - \frac{1}{s''_{i+1}}\right) = +\infty. \tag{4.26}
\]

Now, we ensure that there exists \(i_0\) such that
\[
\frac{p}{s''_{i_0}} < \frac{2}{N} \quad \text{or} \quad \frac{q}{s'_{i_0}} < \frac{2}{N}. \tag{4.27}
\]
In fact, if (4.27) is not true, that is \(\frac{p}{s''_{i_0}} \geq \frac{2}{N}\) and \(\frac{q}{s'_{i_0}} \geq \frac{2}{N}\) for all \(i \geq 1\).

Then, by (4.24), we see that \(s'_{i} > 0\), \(s''_{i} > 0\) for all \(i \geq 1\) and hence, by (4.25),
\[
q < s''_{i} < \cdots < s''_{i_0} < s'_{i_0} = \cdots < s'_{i} < \cdots .
\]
Therefore
\[
\left|\frac{1}{s'_{i}} - \frac{1}{s'_{i+1}}\right| = \frac{1}{s'_{i}} + \frac{1}{s'_{i+1}} < \frac{2}{q} < 2, \quad \text{for all } i \geq 1,
\]
which contradicts (4.26).

Let \(i_0\) be the smallest number that satisfies (4.27). We note that \(i_0 \geq 2\). Without loss of generality, we assume that
\[
\frac{p}{s''_{i_0}} < \frac{2}{N}, \quad \frac{p}{s''_{i_0}} \geq \frac{2}{N} \quad \text{for } 1 \leq i \leq i_0 - 1,
\]
\[
\frac{q}{s'_{i_0}} \geq \frac{2}{N} \quad \text{for } 1 \leq i \leq i_0. \tag{4.28}
\]
Thus (4.24) yields
\[
s'_{i} > 0 \quad \text{for } 1 \leq i \leq i_0, \quad s''_{i} > 0 \quad \text{for } 1 \leq i \leq i_0 + 1,
\]
which together with (4.25) leads to
\[
q < s''_{i_0-1} < s'_i < \cdots < s''_{i_0} < s'_i < \cdots .
\]
Now, from (4.24), for all \(i \geq 2\) we have
\[
\frac{N}{2} \left(\frac{p}{s''_{i-1}} - \frac{1}{s''_{i}}\right) = 1 - \frac{N}{2} \eta = \frac{N}{2} \left(\frac{q}{s'_{i-1}} - \frac{1}{s'_{i}}\right).
\]
Now, let us deal with the boundedness of \((u(t),v(t))\) in \(L'^{s'}(\mathbb{R}^{N}) \times L'^{s''}(\mathbb{R}^{N})\).
By using Lemmas 2.1 and 2.2, it follows from (4.1) inductively that, for all $2 \leq i \leq i_0$ and for all $t \in (0, T_{\text{max}})$,
\begin{align}
\|u(t)\|_{s_i'} \leq \|P_{\gamma_i}(t)u_0\|_{s_i'} + C \int_0^t (t-\tau)^{\gamma_i-1} \|v(\tau)\|_{P_{\gamma_i-1}}^p d\tau \\
\leq C\|u_0\|_{s_i'} + C \int_0^t (t-\tau)^{\gamma_i-1-\gamma_i(1-\frac{Np}{s_i'})} \|v(\tau)\|_{P_{\gamma_i-1}}^p d\tau,
\end{align}
(4.29)
and
\begin{align}
\|v(t)\|_{s_i'} \leq \|P_{\gamma_2}(t)v_0\|_{s_i'} + C \int_0^t (t-\tau)^{\gamma_i-1+\frac{Np}{s_i'}-\frac{1}{p}} \|u(\tau)\|_{P_{\gamma_i-1}}^q d\tau \\
\leq C\|v_0\|_{s_i'} + C \int_0^t (t-\tau)^{\gamma_i-1-\gamma_i(1-\frac{Np}{s_i'})} \|v(\tau)\|_{P_{\gamma_i-1}}^q d\tau,
\end{align}
(4.30)
for $2 \leq i \leq i_0 + 1$ and $t \in (0, T_{\text{max}})$.

From the Hölder inequality, we have
\begin{align}
\|P_{\gamma_i}(t)u_0\|_{s_i'} \leq \|u_0\|_{s_i'} \|u_0\|_{1-\frac{1}{s_i'}} < \infty,
\end{align}
(4.31)
\begin{align}
\|P_{\gamma_2}(t)v_0\|_{s_i'} \leq \|v_0\|_{s_i'} \|v_0\|_{1-\frac{1}{s_i'}} < \infty,
\end{align}
since $u_0 \in L^{s_1} \cap L^\infty$ and $v_0 \in L^{s_2} \cap L^\infty$. From (4.29), (4.30) and (4.31) it follows that
\begin{align}
u(t) \in L^{s_1}(\mathbb{R}^N), \quad \|u(t)\|_{s_i'} \leq C(1 + t^{a_i}), \quad 1 \leq i \leq i_0, \\
v(t) \in L^{s_2}(\mathbb{R}^N), \quad \|v(t)\|_{s_i'} \leq C(1 + t^{b_i}), \quad 1 \leq i \leq i_0 + 1,
\end{align}
(4.32)
for all $t \in (0, T_{\text{max}})$ and for some positive constants $a_i$, $b_i$. Since $\frac{Np}{2s_{i_0''}} < 1$, taking $s_2 = s_{i_0''}$, (4.19) holds; hence $T_{\text{max}} = +\infty$ and (4.22) holds.

**Step 3:** First, we show the following decay estimates
\begin{align}
\|u(t)\|_{s_1} \leq C(t+1)^{-\sigma_1}, \quad \|v(t)\|_{s_2} \leq C(t+1)^{-\sigma_2}, \quad \text{for } t \geq 0,
\end{align}
where $s_1$ and $s_2$ are given by (4.2).

According to Lemma 2.3, it suffices to prove that $\|u(t)\|_{s_1} \leq C, \|v(t)\|_{s_2} \leq C$, for all $t \in [0, 1]$.

To do this, we need to show that
\begin{align}
\|u(t)\|_{\infty} \leq C, \quad \|v(t)\|_{\infty} \leq C, \quad \text{for } t \in [0, 1].
\end{align}
(4.33)
In fact, by applying Lemmas 2.1 and 2.2 to (3.1) we see that
\begin{align}
\|u(s)\|_{\infty} \leq \|P_{\gamma_i}(s)u_0\|_{\infty} + \int_0^s (s-\tau)^{\gamma_i-1}\|v(\tau)\|_{P_{\gamma_i-1}}^p d\tau \\
\leq \|u_0\|_{\infty} + \int_0^s (s-\tau)^{\gamma_i-1}\|v(\tau)\|_{P_{\gamma_i-1}}^p d\tau,
\end{align}
\begin{align}
\|v(s)\|_{\infty} \leq \|P_{\gamma_2}(s)v_0\|_{\infty} + \int_0^s (s-\tau)^{\gamma_i-1}\|u(\tau)\|_{P_{\gamma_i-1}}^q d\tau \\
\leq \|v_0\|_{\infty} + \int_0^s (s-\tau)^{\gamma_i-1}\|u(\tau)\|_{P_{\gamma_i-1}}^q d\tau,
\end{align}
for $0 \leq s \leq t$. For $0 \leq s \leq t \leq 1$, the two inequalities above give

$$\sup_{0 \leq s \leq t} \|u(s)\|_\infty \leq \|u_0\|_\infty + \frac{1}{\gamma_1} \left( \sup_{0 \leq \tau \leq t} \|v(\tau)\|_\infty \right)^p t^{\gamma_1}$$

$$\leq \|u_0\|_\infty + \frac{1}{\gamma_1} \left( \sup_{0 \leq \tau \leq t} \|v(\tau)\|_\infty \right)^p,$$

$$\sup_{0 \leq s \leq t} \|v(s)\|_\infty \leq \|v_0\|_\infty + \frac{1}{\gamma_2} \left( \sup_{0 \leq \tau \leq t} \|u(\tau)\|_\infty \right)^q t^{\gamma_2}$$

$$\leq \|v_0\|_\infty + \frac{1}{\gamma_2} \left( \sup_{0 \leq \tau \leq t} \|u(\tau)\|_\infty \right)^q .$$

Using the second inequality into first inequality, it yields that

$$\sup_{0 \leq s \leq t} \|u(s)\|_\infty \leq \|u_0\|_\infty + \frac{1}{\gamma_1} \left( \|v_0\|_\infty + \frac{1}{\gamma_2} \left( \sup_{0 \leq \tau \leq t} \|u(\tau)\|_\infty \right)^q \right)^p$$

$$\leq C \left( \|u_0\|_\infty + \|v_0\|_\infty + \left( \sup_{0 \leq \tau \leq t} \|u(\tau)\|_\infty \right)^p \right) .$$

So, arguing as in the first step by setting $h(t) = \sup_{0 \leq s \leq t} \|u(s)\|_\infty$ and $A = \|u_0\|_\infty + \|v_0\|_\infty$, we obtain

$$h(t) \leq A + Ch^p(t), \text{ for all } t \leq 1,$$

which implies (4.33) for $A$ small since $pq > 1$.

We see from (4.33) and Lemmas 2.1, 2.2 that

$$\|u(t)\|_{s_1} \leq C \|u_0\|_{s_1} + C \int_0^t (t - \tau)^{\gamma_1 - 1} \|v(\tau)\|^p_{s_1} d\tau ,$$

where $s_1$ given explicitly by (4.2). Therefore

$$\|u(t)\|_{s_1} = C \|u_0\|_{s_1} + C \int_0^t (t - \tau)^{\gamma_1 - 1} \|v(\tau)\|^p_{s_1} d\tau .$$

By the interpolation inequality $\|v(\tau)\|^p_{s_1} \leq \|v(\tau)\|^{\frac{22}{22}}_{\infty} \|v(\tau)\|^{\frac{2q}{2q}}_{s_2}$, we obtain

$$\|u(t)\|_{s_1} = C \|u_0\|_{s_1} + C \sup_{\tau \in (0,t)} \|v(\tau)\|^{\frac{p(1 - \frac{22}{2q})}{\frac{22}{2q}}} \int_0^t (t - \tau)^{\gamma_1 - 1} \|v(\tau)\|^{\frac{2q}{2q}}_{s_2} d\tau . \quad (4.34)$$

Now, using (4.14) and (4.33) in (4.34), we obtain

$$\|u(t)\|_{s_1} \leq C \|u_0\|_{s_1} + C \int_0^t (t - \tau)^{\gamma_1 - 1 - \frac{22}{2q} \sigma_2} d\tau ,$$

provided that $\frac{s_1}{s_2} \sigma_2 < 1$. On the other hand, since $s_1$ and $s_2$ satisfy

$$\frac{s_1}{s_2} \sigma_2 = \frac{(1 - \delta)(p\gamma_2 + \gamma_1)s_1}{(pq - 1)s_2} \leq \gamma_2,$$ \quad

$$\frac{s_2}{s_1} \sigma_2 = \frac{(1 - \delta)(q\gamma_1 + \gamma_2)s_2}{(pq - 1)s_1} \leq \gamma_1 ,$$

we obtain $\gamma_1 - \frac{22}{2q} \sigma_2 \geq 0$ and consequently

$$\|u(t)\|_{s_1} \leq C \|u_0\|_{s_1} + C t^{\gamma_1 - \frac{22}{2q} \sigma_2} \leq C, \quad \text{for all } t \in [0, 1].$$

Analogously,

$$\|v(t)\|_{s_2} \leq C \quad \text{for all } t \in [0, 1].$$

From (4.14), (4.15), (4.33) and Lemma 2.3, we conclude that

$$\|u(t)\|_{s_1} \leq C(t + 1)^{-\frac{(1 - \delta)(p\gamma_2 + \gamma_1)}{pq - 1}}, \quad \|v(t)\|_{s_2} \leq C(t + 1)^{-\frac{(1 - \delta)(q\gamma_1 + \gamma_2)}{pq - 1}} . \quad (4.35)$$
for all $t \geq 0$.

Next, we derive $L^\infty$-decay estimates. Let

$$
\sigma_1 = \frac{(1 - \delta)(pq_2 + \gamma_1)}{pq - 1}, \quad \sigma_2 = \frac{(1 - \delta)(q\gamma_1 + \gamma_2)}{pq - 1}.
$$

If $\frac{pN}{s_2} < 1$, by taking $\xi = s_2$ in (4.18) and using (4.35), we obtain

$$
\|u(t)\|_\infty \leq Ct^{-\frac{N}{2}\gamma_1}\|u_0\|_1 + C\int_0^t (t - \tau)^{\gamma_1 - 1 - \frac{N\gamma_1}{2}s_2}\tau^{-p\sigma_2}d\tau
$$

(4.36)

and

$$
p\sigma_2 < 1, \quad \gamma_1 = \frac{N\gamma_1 p}{2} - p\sigma_2 = \frac{\lceil\gamma_1 + \gamma_1p\delta + (1 - \delta)p\gamma_2\rceil}{pq - 1}.
$$

On the other hand, we have

$$
\frac{\gamma_1 + \gamma_1p\delta + p\gamma_2(1 - \delta)}{pq - 1} < \frac{\gamma_1 + \gamma_2}{pq - 1} \leq \frac{N}{2}\gamma_1.
$$

(4.37)

Then, it follows from (4.36), (4.37) and lemma 2.3 that

$$
\|u(t)\|_\infty \leq Ct^{-\frac{N}{2}\gamma_1} + Ct^{-\frac{\gamma_1 + \gamma_1p\delta + (1 - \delta)p\gamma_2}{pq - 1}}.
$$

Thus

$$
\|u(t)\|_\infty \leq C(1 + t)^{-\frac{\gamma_1 + \gamma_1p\delta + (1 - \delta)p\gamma_2}{pq - 1}}.
$$

(4.38)

for all $t \geq 0$.

Similarly, if $\frac{qN}{s_1} < 1$, then one can find that

$$
\|v(t)\|_\infty \leq C(1 + t)^{-\frac{\gamma_2 + \gamma_2p\delta + (1 - \delta)\gamma_1}{pq - 1}}, \quad \text{for } t \geq 0.
$$

(4.39)

At the same time, (4.38) holds as $pN/(2s_2) \leq qN/(2s_1)$.

In particular, if $pq > q + 2$, and $\gamma_1q^2 > 2q + 1$, we can choose

$$
\delta > \max\left\{1 - \frac{(pq - 1)}{\gamma_2q(p + 1)}, 1 - \frac{\gamma_1(pq - 1)}{\gamma_2(p + 1)}\right\}
$$

and $\delta \approx \max\left\{1 - \frac{(pq - 1)}{\gamma_2q(p + 1)}, 1 - \frac{\gamma_1(pq - 1)}{\gamma_2(p + 1)}\right\}$ such that $qN/(2s_1) < 1$. Therefore, estimates (4.38) and (4.39) hold.

It is useful to note that $N \leq 2$ implies $pN/(2s_2) < 1$ and $qN/(2s_1) < 1$ implies $pq > 2 + q$.

It remains to consider the following two cases:

1. The case $N > 2$, $\frac{pN}{2s_2} < 1$ and $\frac{qN}{2s_1} \geq 1$. Let

$$
\sigma' = \frac{\gamma_1 + \gamma_1p\delta + (1 - \delta)p\gamma_2}{pq - 1}.
$$

For a positive $\mu$ such that $\mu < \min\{\sigma', \sigma_1\}$ and $q\mu < 1$, since $N > 2$ and $q > 1$, we can choose $k > 0$ such that $k > \frac{qN}{2}$ and $q\mu + \frac{2N\gamma_2}{2k} > \gamma_2$. Since $s_1 \leq qN/2$, we have $k > s_1$.

Using the interpolation inequality

$$
\|u(t)\|_k \leq \|u(t)\|_{(k-s_1)/k}^{(k-s_1)/k}\|u(t)\|_{s_1/k}^{s_1/k} \leq Ct^{-\sigma'(k-s_1)/k}t^{-k-s_1/k} \quad \text{for all } t > 0,
$$

it follows from (4.14) and (4.38) that

$$
\|u(t)\|_k \leq Ct^{-\mu} \quad \text{for all } t > 0.
$$
whereupon,

\[ \|v(t)\|_\infty \leq \|P_{\gamma_2}(t)v_0\|_\infty + C \int_0^t (t - \tau)^{\gamma_2 - 1 - \frac{N\gamma_2}{2k}} \|u(\tau)\|^q d\tau \]

\[ \leq Ct^{-\frac{N}{2} - \gamma_2} \|v_0\|_1 + C \int_0^t (t - \tau)^{\gamma_2 - 1 - \frac{N\gamma_2}{2k}} \tau^{-q\mu} d\tau \]

\[ \leq C(t^{-\frac{N}{2} - \gamma_2} + t^{\gamma_2 - \frac{N\gamma_2}{2k}}) \]

\[ \leq Ct^{-\alpha}, \]

for all \( t > 0 \), where \( \alpha = \min \{ \frac{N}{2}, \gamma_2 - \gamma_2 + \frac{N\gamma_2}{2k} + q\mu \} > 0 \). From (4.33) and (4.40), we infer that

\[ \|v(t)\|_\infty \leq C(1 + t)^{-\alpha} \quad \text{for all } t \geq 0. \]

We remark that, in the particular case \( p = 1, q > 3 \) and \( q^2 > \max \{ 4\gamma_2q + 1, \frac{4\gamma_2 + \gamma_1}{q_1} \} \), we can choose

\[ \delta > \max \left\{ 1 - \frac{(pq - 1)\gamma_2q}{\gamma_2q(p + 1)}, 1 - \frac{\gamma_1(pq - 1)}{\gamma_2(p + 1)}, 1 - \frac{(pq - 1)}{q + 1} \right\} \]

\[ = \max \left\{ 1 - \frac{(pq - 1)}{2\gamma_2q}, 1 - \frac{\gamma_1(q - 1)}{2\gamma_2}, 1 - \frac{(q - 1)}{q + 1} \right\} \]

and \( \delta \approx \max \{ 1 - \frac{q - 1}{2\gamma_2q}, 1 - \frac{\gamma_1(q - 1)}{2\gamma_2}, 1 - \frac{(q - 1)}{q + 1} \} \) such that \( N/(2s_2) < 1 \). Therefore, we have the estimate (4.33).

(2) The case: \( N > 2, qN/(2s_1) \geq 1, pN/(2s_2) \geq 1, q \geq p > 1, \) and \( \gamma_1 \leq \gamma_2 \).

It needs a careful handling and we need to restrict further the choice of \( \delta \). From \( \max \{ \frac{q + 1}{pq(p + 1)}, \frac{pq - 1}{pq(p + 1)}, \gamma_2/p, \sqrt{\frac{2\gamma_2}{pq}} \} < \gamma_1 \leq \gamma_2 < 1 \) and \( pq > 1 \), we obtain

\[ \max \left\{ 1 - \frac{(pq - 1)\gamma_2 q}{\gamma_2 q(p + 1)}, 1 - \frac{\gamma_1(pq - 1)}{\gamma_2(p + 1)}, 1 - \frac{(pq - 1)}{q + 1} \right\} \]

\[ < \min \left\{ 1 - \frac{(pq - 1)}{\gamma_1 pq(p + 1)}, \frac{N(pq - 1)}{2q(p + 1)} \right\}. \]

So, we select \( \delta > 0 \) such that

\[ \max \left\{ 1 - \frac{(pq - 1)\gamma_2 q}{\gamma_2 q(p + 1)}, 1 - \frac{\gamma_1(pq - 1)}{\gamma_2(p + 1)}, 1 - \frac{(pq - 1)}{q + 1} \right\} \]

\[ < \delta \leq \min \left\{ \frac{N(pq - 1)}{2q(p + 1)}, 1 - \frac{(pq - 1)}{\gamma_1 pq(p + 1)} \right\}. \]

We get immediately \( pq_2 \sigma > 1/q \) and \( q_2 > 1/p \). Further, we notice that there exist \( \varepsilon \in (0, 1) \) and \( \beta < 1 \) close to 1 such that

\[ pq_2 - \varepsilon > 1/q, \quad q_2 - \varepsilon > 1/p, \quad 1/p < \beta - \varepsilon, \quad 1/q < \beta - \varepsilon. \]  

(4.41)

By taking \( \eta = 2\varepsilon(1 - \delta)/N \), we find the integer \( i_0 \) as in the step 2, and, without loss of generality, we assume that (4.28) holds. We choose \( \beta \) in addition to (4.41) satisfying

\[ \gamma_1 \leq \gamma_1 \frac{pN}{2s_2} + \gamma_2 \beta, \quad \text{since } \gamma_2 \geq \gamma_1. \]

(4.42)

As

\[ \delta < \frac{N(pq - 1)}{2(p + 1)q} \leq \frac{N(pq - 1)}{2(q + 1)p}, \]
and $\beta < 1$, we have
\[
\beta + \frac{(p+1)q\delta}{(pq-1)} < 1 + \frac{N}{2}, \quad \beta + \frac{(q+1)p\delta}{(pq-1)} < 1 + \frac{N}{2} \tag{4.43}
\]
For $2 \leq i \leq i_0 - 1$, define $r'_{i+1}$ and $r''_{i+1}$ inductively as follows:
\[
\begin{align*}
\frac{1}{r'_{i+1}} &= \frac{1}{s_i'} + \frac{2}{N} \|p\sigma_2 - \varepsilon(1 - \delta)\|, \quad \frac{1}{r''_{i+1}} = \frac{1}{s_i''} + \frac{2}{N} [\beta - \varepsilon(1 - \delta)], \\
&= \frac{p}{s_i'} + \frac{2}{N} \varepsilon(1 - \delta) + \frac{2}{N} [\beta - \varepsilon(1 - \delta)], \\
&= \frac{2}{N} \left( p(q+1)\delta + \beta - 1 \right) < 1,
\end{align*}
\]
It is clear that $r'_i$, $r''_i > 0$ and $r'_i < s'_i$, $r''_i < s''_i$ for all $2 \leq i \leq i_0$. A simple calculation shows that $r'_i$, $r''_i > 1$. As $s'_i$ and $s''_i$ are increasing in $i$ for $1 \leq i \leq i_0$; we have
\[
\frac{1}{r'_{i+1}} < \frac{1}{s_i'} + \frac{2}{N} [\beta - \varepsilon(1 - \delta)]
\]
from (4.33); therefore $r'_{i+1} > 1$. Similarly, we can check that $r''_{i+1} > 1$.

From (4.22) and (4.32), we see that there exists a positive constant $C$ such that
\[
\|u(t)\|_\infty, \|v(t)\|_\infty, \|u(t)\|_{k_3}, \|v(t)\|_{k_3} \leq C \quad \text{for } 0 \leq t \leq 1 \tag{4.44}
\]
for all $s'_1 \leq k_1 \leq s''_{i_0}$, $s''_2 \leq k_2 \leq s''_{i_0}$. Furthermore, since $1 - \eta N/2 = 1 - \varepsilon(1 - \delta)$ and $p\sigma_2 < 1$, using (4.29), (4.30) with the help of (4.14) and (4.15), we arrive at the estimate
\[
\|u(t)\|_{s_2'} \leq \|P_{\gamma_1}(t)u_0\|_{s_2'} + C \int_0^t \tau^{1-\varepsilon(1-\delta)} \|u(\tau)\|_{s_2'} d\tau \leq C t^{-\frac{N}{2} \gamma_1 \left( \frac{1}{2} - \frac{1}{2} \right)} \|u_0\|_{s_2'} + C \int_0^t (1 - \varepsilon(1 - \delta)) \tau^{-p\sigma_2} d\tau \leq C t^{-\gamma_1 (p\sigma_2 - \varepsilon(1 - \delta))} \|u_0\|_{s_2'} + C \int_0^t (1 - \varepsilon(1 - \delta)) \tau^{-p\sigma_2} d\tau \leq C t^{-\gamma_1 (p\sigma_2 - \varepsilon(1 - \delta))} \quad \text{for all } t > 0.
\]
Similarly,
\[
\|v(t)\|_{s_2'} \leq C t^{-\gamma_2 (p\sigma_2 - \varepsilon(1 - \delta))} \quad \text{for all } t > 0.
\]
In view of (4.41) and $\beta < 1$, we conclude, thanks to lemma 2.3, that
\[
\|u(t)\|_{s'_o} \leq C t^{-\gamma_1 \beta/q}, \quad \|v(t)\|_{s''_o} \leq C t^{-\gamma_2 \beta/p} \quad \text{for all } t > 0.
\]
An iterative argument gives
\[
\|u(t)\|_{s'_o} \leq C t^{-\gamma_1 (\beta - \varepsilon(1 - \delta))} \leq C t^{-\gamma_1 \beta/q} \quad \text{for all } t > 0, \\
\|v(t)\|_{s''_o} \leq C t^{-\gamma_2 (\beta - \varepsilon(1 - \delta))} \leq C t^{-\gamma_2 \beta/p} \quad \text{for all } t > 0.
\]
Therefore, by (4.17) and (4.18), we have
\[
\|u(t)\|_\infty \leq C t^{-\frac{N}{2} \gamma_1} \|u_0\|_1 + C \int_0^t (1 - \varepsilon(1 - \delta)) \tau^{-p\sigma_2} \|v(\tau)\|_{s''_o} d\tau
\]
\[
\leq Ct^{-\frac{N}{2}\gamma_1}\|u_0\|_1 + C\int_0^t (t - \tau)^{\gamma_1 - 1 - \gamma_1 \frac{pN}{2s_0} \tau^{-\gamma_2}d\tau \\
\leq C\left(t^{-\frac{N}{2}\gamma_1} + (\frac{pN}{2s_0})^{\gamma_1 - \gamma_1^0 - \gamma_2^\beta}\right) \leq Ct^{-\sigma},
\]

where \(\sigma'' = \min\{\frac{N}{2}\gamma_1, \gamma_1 \frac{pN}{2s_0} - \gamma_1 + \gamma_2^\beta\} > 0\) from [4.42].

Since \(\frac{N\gamma_1}{2s_0} \geq 1\), using similar arguments as for the case \(\frac{N\gamma_1}{2s_0} < 1\) and \(\frac{N\gamma_1}{2s_0} \geq 1\), we obtain \(\|u(t)\|_\infty \leq Ct^{-\sigma''}\) for some \(\sigma'' > 0\) and for every \(t > 0\). This completes the proof. \(\square\)

**Remark 4.1.** In the particular case \(N > 2\), \(q\gamma_1/(2s_1) \geq 1\), \(p\gamma_1/(2s_2) \geq 1\), \(q > p = 1\) and \(q'' \leq 4\gamma_1q + 1\), using the above method, we obtain

\[
\|u(t)\|_\infty \leq Ct^{-\sigma''}, \quad t > 0,
\]

where \(\sigma'' = \min\{\frac{N}{2}\gamma_1, \frac{pN}{2s_0} - \gamma_1 + \gamma_2^\beta - \varepsilon(1-\delta)\}\). Here, \(\varepsilon > 0\) can be arbitrarily small, and \(\beta\) can be arbitrarily close to 1. However, since \(s_i^0\) depends on \(\varepsilon\) and \(s_i''^0\) is decreasing in \(\varepsilon\), it is not clear that \(\sigma''\) is positive.

**Proof of Theorem 3.6.**

**Case:** \(p > 1, q > 1\). The proof proceeds by contradiction. Suppose that \((u,v)\) is a nontrivial solution of (1.1) which exists globally in time. We make the judicious choice

\[
\varphi(t,x) = \varphi_1(t)\varphi_2(x) = \varphi_1\left(\frac{t}{T^\lambda}\right)\Phi\left(\left|\frac{x}{T^\lambda}\right|\right),
\]

where \(\Phi \in C_0^\infty(\mathbb{R})\), \(0 \leq \Phi(z) \leq 1\) is such that

\[
\Phi(z) = \begin{cases} 
1 & \text{if } |z| \leq 1, \\
0 & \text{if } |z| > 2,
\end{cases}
\]

\[
\varphi_1(t) = \begin{cases} 
(1 - \frac{t}{T^\lambda})^{-l} & \text{if } t \leq T^\lambda, \\
0 & \text{if } t > T^\lambda,
\end{cases}
\]

where \(l = \max\{1, \frac{q}{p} - \gamma_1 - 1, \frac{p}{p-1} \gamma_2 - 1\}\). We denote by \(Q_{T^\lambda} := \mathbb{R}^N \times [0, T^\lambda]\).

From Definition 3.4 of the weak solution, we have

\[
\int_{Q_{T^\lambda}} |u|^p \varphi_2(x) \varphi_1(t) dx dt + T^{\lambda(1-\gamma_1)} \int_{\mathbb{R}^N} u_0(x) \varphi_2(x) dx \\
= \int_{Q_{T^\lambda}} \varphi_2(x) u D_{t|T^\lambda} \varphi_1(t) dx dt - \int_{Q_{T^\lambda}} \Delta \varphi_2(x) \varphi_1(t) u dx dt,
\]

\[
= \int_{Q_{T^\lambda}} |u|^q \varphi_2(x) \varphi_1(t) dx dt + T^{\lambda(1-\gamma_2)} \int_{\mathbb{R}^N} v_0 \varphi_2(x) dx \\
= \int_{Q_{T^\lambda}} \varphi_2(x) u D_{t|T^\lambda} \varphi_1(t) dx dt - \int_{Q_{T^\lambda}} \varphi_1(t) \Delta \varphi_2(x) u dx dt.
\]

Using Hölder’s inequality with exponents \(q\) and \(q'\) (\(q + q' = qq'\)), to the right-hand sides of (4.45) and (4.46), we obtain

\[
\int_{Q_{T^\lambda}} u \varphi_2(x) D_{t|T^\lambda} \varphi_1(t) dx dt \\
= \int_{Q_{T^\lambda}} u |\varphi_1(t)|^{1/q} |\varphi_2(x)|^{\frac{1}{q'} + \frac{q}{2}} |\varphi_1(t)|^{-1/q} D_{t|T^\lambda} \varphi_1(t) dx dt
\]
Consequently,
\[
\left( \int_{Q_{T^\lambda}} |D_u^{\alpha} \varphi_1(t)|^{q'} |\varphi_1(t)|^{\frac{-q'}{q}} |\varphi_2(x)|^{(1-\frac{q}{q'})} q' \, dx \, dt \right)^{1/q'}
\times \left( \int_{Q_{T^\lambda}} |u|^q \varphi_2 \, dx \, dt \right)^{1/q},
\]
and
\[
\int_{Q_{T^\lambda}} u|\Delta \varphi_2(x)| \varphi_1(t) \, dx \, dt
\leq \left( \int_{R^N} |\Delta \varphi_2(x)|^{q'} \varphi_2(x) \varphi_1(t) d\chi \int_{0}^{T^\lambda} |\varphi_1(t)|^{(1-\frac{q}{q'})} \, dt \right)^{1/q'}
\times \left( \int_{Q_{T^\lambda}} |u|^q \varphi_2 \, dx \, dt \right)^{1/q}.
\]
Setting
\[
A(\sigma, \kappa, \kappa') = \left( \int_{Q_{T^\lambda}} |D_u^{\alpha} \varphi_1(t)|^{\kappa'} |\varphi_1(t)|^{\frac{-\kappa'}{\kappa}} |\varphi_2(x)|^{(1-\frac{1}{\kappa'})} \, dx \, dt \right)^{1/\kappa'},
\]
\[
B(\kappa, \kappa') = \left( \int_{Q_{T^\lambda}} |\Delta \varphi_2(x)|^{\kappa'} \varphi_2(x) \varphi_1(t) \, dx \, dt \right)^{1/\kappa'},
\]
and gathering the above estimates, we obtain
\[
\int_{Q_{T^\lambda}} |v|^p \varphi_1(t) \varphi_2(x) \, dx \, dt + T^{\lambda(1-\gamma_1)} \int_{R^N} u_0 \varphi_2(x) \, dx
\leq A(\gamma_1, q, q') \left( \int_{Q_{T^\lambda}} |u|^q \varphi_1(t) \varphi_2 \, dx \, dt \right)^{1/q} \tag{4.47}
\]
\[
+ B(q, q') \left( \int_{Q_{T^\lambda}} |u|^q \varphi_1(t) \varphi_2 \, dx \, dt \right)^{1/q}.
\]
Similarly, we obtain
\[
\int_{Q_{T^\lambda}} |u|^q \varphi_2(x) \varphi_1(t) \, dx \, dt + T^{\lambda(1-\gamma_2)} \int_{R^N} v_0 \varphi_2(x) \, dx
\leq A(\gamma_2, p, p') \left( \int_{Q_{T^\lambda}} |v|^p \varphi_1 \varphi_2 \, dx \, dt \right)^{1/p} + B(p, p') \left( \int_{Q_{T^\lambda}} |v|^p \varphi_1 \varphi_2 \, dx \, dt \right)^{1/p} \tag{4.48}
\]
Consequently,
\[
\int_{Q_{T^\lambda}} |v|^p \varphi_1(t) \varphi_2(x) \, dx \, dt + CT^{\lambda(1-\gamma_1)} \int_{R^N} u_0 \varphi_2(x) \, dx
\leq A \left( \int_{Q_{T^\lambda}} |u|^q \varphi_1 \varphi_2 \, dx \, dt \right)^{1/q},
\]
and
\[
\int_{Q_{T^\lambda}} |u|^q \varphi_1(t) \varphi_2(x) \, dx \, dt + CT^{\lambda(1-\gamma_2)} \int_{R^N} v_0 \varphi_2(x) \, dx
\leq B \left( \int_{Q_{T^\lambda}} |v|^p \varphi_1 \varphi_2 \, dx \, dt \right)^{1/p},
\]
where
\[ A = A(\gamma_1, q, q') + B(q, q'), \quad B = A(\gamma_2, p, p') + B(p, p'). \]

Using inequalities (4.47) and (4.48) in the last two inequalities, we obtain
\[
\begin{align*}
&\int_{Q_T^\lambda} |v|^p \varphi_1(t) \varphi_2(x) \, dx \, dt + C T^{\lambda(1-\gamma_1)} \int_{\mathbb{R}^N} u_0 \varphi_2(x) \, dx \\
&\leq A B^{1/q} \left( \int_{Q_T^\lambda} |v|^p \varphi_1 \varphi_2 \, dx \, dt \right)^{\frac{1}{pq}}, \\
&\int_{Q_T^\lambda} |u|^q \varphi_1(t) \varphi_2(x) \, dx \, dt + C T^{\lambda(1-\gamma_2)} \int_{\mathbb{R}^N} v_0 \varphi_2(x) \, dx \\
&\leq B A^{1/p} \left( \int_{Q_T^\lambda} |u|^q \varphi_1 \varphi_2 \, dx \, dt \right)^{\frac{1}{pq}}.
\end{align*}
\]

Now, applying Young’s inequality, we obtain
\[
(pq - 1) \int_0^T \int_{\mathbb{R}^N} |v|^p \varphi_2(x) \varphi_1(t) \, dx \, dt + C p q T^{\lambda(1-\gamma_1)} \int_{\mathbb{R}^N} u_0(x) \varphi_2(x) \, dx \\
\leq (pq - 1) (A B^{1/q})^{\frac{pq}{pq-1}},
\]
\[
(pq - 1) \int_0^T \int_{\mathbb{R}^N} |u|^q \varphi_2(x) \varphi_1(t) \, dx \, dt + C p q T^{\lambda(1-\gamma_2)} \int_{\mathbb{R}^N} v_0(x) \varphi_2(x) \, dx \\
\leq (pq - 1) (B A^{1/p})^{\frac{pq}{pq-1}}.
\]

At this stage, using the change of variables, \( x = T^2 y, \ t = T^\lambda \tau, \) with \( \lambda > 0 \) to be chosen later, we obtain
\[
\begin{align*}
&\int_0^T \int_{\mathbb{R}^N} |v|^p \varphi_2(x) \varphi_1(t) \, dx \, dt + T^{\lambda(1-\gamma_1)} \int_{\mathbb{R}^N} u_0 \varphi_2(x) \, dx \\
&\leq C \left( T^{-\lambda_1 + (\lambda + 2N) \frac{1}{p'}} + T^{-4+ (\lambda + 2N) \frac{1}{q'}} \right) \left( \int_{Q_T^\lambda} |u|^q \varphi_1(t) \varphi_2(x) \, dx \, dt \right)^{\frac{1}{q'}} \\
&\leq C T^{-\lambda_1 + (\lambda + 2N) \frac{1}{p'}} \left( \int_{Q_T^\lambda} |u|^q \varphi_1 \varphi_2 \, dx \, dt \right)^{\frac{1}{q'}}.
\end{align*}
\]

Analogously, we have
\[
\begin{align*}
&\int_0^T \int_{\mathbb{R}^N} |u|^q \varphi_2(x) \varphi_1(t) \, dx \, dt + C T^{\lambda(1-\gamma_2)} \int_{\mathbb{R}^N} v_0 \varphi_2(x) \, dx \\
&\leq C \left( T^{-\lambda_2 + (\lambda + 2N) \frac{1}{p'}} + T^{-4+ (\lambda + 2N) \frac{1}{q'}} \right) \left( \int_{Q_T^\lambda} |v|^p \varphi_1 \varphi_2 \, dx \, dt \right)^{\frac{1}{p}} \\
&= C T^{-\lambda_2 + (\lambda + 2N) \frac{1}{p'}} \left( \int_{Q_T^\lambda} |v|^p \varphi_1 \varphi_2 \, dx \, dt \right)^{\frac{1}{p}}.
\end{align*}
\]

Choosing \( \gamma_1 \lambda = 4, \) we have
\[
\begin{align*}
&\int_0^T \int_{\mathbb{R}^N} |v|^p \varphi_2(x) \varphi_1(t) \, dx \, dt + T^{\lambda(1-\gamma_1)} \int_{\mathbb{R}^N} u_0 \varphi_2(x) \, dx \\
&\leq C \left( T^{-\frac{1}{4} \gamma_2 + (\lambda + 2N) \frac{1}{p'}} + T^{-4+ (\lambda + 2N) \frac{1}{q'}} \right) \left( \int_{Q_T^\lambda} |v|^p \varphi_1 \varphi_2 \, dx \, dt \right)^{\frac{1}{p}}.
\end{align*}
\]
and

\[
\int_0^T \int_{\mathbb{R}^N} |u|^q \varphi_2(x) \varphi_1(t) \, dx \, dt + CT^{\lambda(1-\gamma_2)} \int_{\mathbb{R}^N} v_0 \varphi_2(x) \, dx \\
\leq C \left( T^{-\gamma_2 + (\lambda+2N)\frac{1}{p'}} + T^{-4 + (\lambda+2N)\frac{1}{p'}} \right) \left( \int_{Q_T^*} |v|^p \varphi_1 \varphi_2 \, dx \, dt \right)^{1/p} \\
\leq C \left( T^{-\gamma_2 + (\lambda+2N)\frac{1}{p'}} + T^{-4 + (\lambda+2N)\frac{1}{p'}} \right) T^{-4\frac{1}{p'} + \left( \frac{4}{q_1} + 2N \right)\frac{1}{p'}} + T^{-4 + \left( \frac{4}{q_1} + 2N \right)\frac{1}{p'}} \right) \\
\times \left( \int_{Q_T^*} |u|^q \varphi_1 \varphi_2 \, dx \, dt \right)^{\frac{1}{p'}}.
\]

Therefore, using the \( \varepsilon \)-Young inequality, we obtain

\[
\int_{\mathbb{R}^N} u_0(x) \varphi_2(x) \, dx \leq CT^{\delta_1}, \tag{4.49}
\]

\[
\int_{\mathbb{R}^N} v_0(x) \varphi_2(x) \, dx \leq CT^{\delta_2}, \tag{4.50}
\]

where

\[
\delta_1 = \max \left\{ \left( \frac{-4}{q_1} \gamma_2 + \left( \frac{4}{q_1} + 2N \right) \frac{1}{p'} q - 4 + \left( \frac{4}{q_1} + 2N \right) \frac{1}{q'} \right) \frac{pq}{pq - 1} + \frac{4}{q_1} (\gamma_1 - 1) \right\},
\]

\[
\left( \frac{-\frac{1}{q}}{q_1} + \left( \frac{4}{q_1} + 2N \right) \frac{1}{p' q} - 4 + \left( \frac{4}{q_1} + 2N \right) \frac{1}{q'} \right) \frac{pq}{pq - 1} + \frac{4}{q_1} (\gamma_1 - 1) \right\},
\]

and

\[
\delta_2 = \max \left\{ \left( \frac{-4}{q_1} + \left( \frac{4}{q_1} + 2N \right) \frac{1}{p'} - 4 + \left( \frac{4}{q_1} + 2N \right) \frac{1}{q'} \right) \frac{pq}{pq - 1} + \frac{4}{q_1} (\gamma_2 - 1) \right\},
\]

\[
\left( \frac{-4 + \left( \frac{4}{q_1} + 2N \right) \frac{1}{q'} - 4 + \left( \frac{4}{q_1} + 2N \right) \frac{1}{q'} \right) \frac{pq}{pq - 1} + \frac{4}{q_1} (\gamma_2 - 1) \right\}.
\]

The condition (3.2) leads to either \( \delta_1 < 0 \) or \( \delta_2 < 0 \). Then as \( T \to \infty \), the right-hand side of (4.49) (resp. (4.50)) tends to zero while the left-hand side tends to \( \int_{\mathbb{R}^N} u_0(x) \, dx \geq 0 \) (resp. \( \int_{\mathbb{R}^N} v_0(x) \, dx > 0 \)); a contradiction.

We repeat the same argument for \( \gamma_2 \lambda = 4 \) to conclude the proof of Theorem 3.6.

**Case** \( p = 1, q > 1 \) (the case \( p > 1, q = 1 \) is treated similarly). We still use the weak formulation of the solution and argue by contradiction. Let us set

\[
I = \int_0^T \int_\Omega \varphi \, dx \, dt, \quad J = \left( \int_0^T \int_\Omega u^q \varphi \, dx \, dt \right)^{1/q}.
\]

Then, applying Holder’s inequality as above, we obtain

\[
I + \int_0^T \int_\Omega u_0 D_t^{1/p} \varphi \, dx \, dt \leq J (A + B), \tag{4.51}
\]
where
\[ A = \left( \int_0^T \int_\Omega \varphi^{-\frac{q'}{q}} \Delta \varphi \, dx \, dt \right)^{\frac{1}{q'-q}}, \quad B = \left( \int_0^T \int_\Omega \varphi^{-\frac{q'}{q}} |D^{\gamma_1}_{\partial T} \varphi|^q \, dx \, dt \right)^{\frac{1}{q'}}; \]
and
\[ J^q + \int_0^T \int_\Omega v_0 D^{\gamma_1}_{\partial T} \varphi \, dx \, dt \leq \lambda \int_0^T \int_\Omega v \varphi \, dx \, dt + \int_0^T \int_\Omega v D^{\gamma_2}_{\partial T} \varphi \, dx \, dt \]
\[ \leq (\lambda + \varepsilon) T, \]  
(4.52)
thanks to the \( \varepsilon \)-Young inequality and where we have chosen \( \varphi \) as the first eigenfunction of the spectral problem
\[-\Delta \varphi = \lambda \varphi, \quad x \in B_T(0), \quad \varphi_{|_{\partial \Omega}} = 0,\]
where \( (\Omega = B_T(0) \subset \mathbb{R}^N \) is the ball centered in zero and of radius \( T \) and \( \partial \Omega \) is the boundary of \( \Omega \). Adding equation \( (4.52) \) to \( (\lambda + \varepsilon) \) times equation \( (4.51) \), we obtain
\[ J^q + (\lambda + \varepsilon) \int_0^T \int_\Omega u_0 D^{\gamma_1}_{\partial T} \varphi \, dx \, dt + \int_0^T \int_\Omega v_0 D^{\gamma_2}_{\partial T} \varphi \, dx \, dt \leq (\lambda + \varepsilon) J(A + B), \]
whereupon,
\[ J^{q-1} \leq A + B. \]
Replacing \( \varphi(x) \) by \( \varphi(\frac{x}{T}) \) and passing to the new variables \( y = T^{-1}x \) and \( \tau = T^{-1}t \), and then letting \( T \) go to infinity, we obtain a contradiction whenever \( q < 1 + \frac{2}{N} \). \( \Box \)

**Proof of Theorem 3.7.** Let \( u_0, v_0 \in C_0(\mathbb{R}^N) \) be nonnegative and \( (u, v) \) be the corresponding solution of \( (1.1)-(1.2) \). We proceed by contradiction. Assume that \( (u, v) \) exists globally in time, that is \( (u, v) \) exists in \( (0, t_*(u_0, v_0)) \), for all \( t_*(u_0, v_0) > 0 \).

Let \( T \in (0, t_*(u_0, v_0)) \) be arbitrarily fixed.

Taking \( \chi \) as test-function and setting
\[ X(t) := \int_{\mathbb{R}^N} u(t, x) \chi(x) \, dx, \quad Y(t) := \int_{\mathbb{R}^N} v(t, x) \chi(x) \, dx \]
\[ Z(t) = \int_\Omega \chi(u(t, x) + v(t, x)) \, dx, \quad Z_0 = \int_{\mathbb{R}^N} (u_0 + v_0) \chi(x) \, dx. \]
It follows from \( (1.1)-(1.2) \) that
\[ C D^{\gamma_1}_{(0, t)} \int_{\mathbb{R}^N} u(t, x) \chi(x) \, dx - \int_{\mathbb{R}^N} u \Delta \chi(x) \, dx = \int_{\mathbb{R}^N} |v(t, x)|^p \chi(x) \, dx, \quad t \in (0, T), \]
\[ C D^{\gamma_1}_{(0, t)} \int_{\mathbb{R}^N} v(t, x) \chi(x) \, dx - \int_{\mathbb{R}^N} v \Delta \chi(x) \, dx = \int_{\mathbb{R}^N} |u(t, x)|^q \chi(x) \, dx, \quad t \in (0, T), \]
(4.53)
supplemented with the initial conditions
\[ X(0) = \int_{\mathbb{R}^N} u_0(x) \chi(x) \, dx, \quad Y(0) = \int_{\mathbb{R}^N} v_0(x) \chi(x) \, dx. \]
(4.54)
From \( (4.53)-(4.54) \), we have
\[ D^{\gamma_1}_{(0, t)} ([Z - Z_0]) (t) = \int_{\mathbb{R}^N} (u(t, x) + v(t, x)) \Delta \chi(x) \, dx \]
\[ = \int_{\mathbb{R}^N} (|v(t, x)|^p + |u(t, x)|^q) \chi(x) \, dx, \quad t \in (0, T). \]
(4.55)
We observe that
\[ \int_{\mathbb{R}^N} v(x,t)\chi(x) \, dx = \int_{\mathbb{R}^N} v(x,t)\chi^{\frac{1}{\gamma}}(x)\chi^{\frac{1-\frac{1}{\gamma}}}{(x)} \, dx. \]

Since the function \( \chi \) satisfies \( \int_{\mathbb{R}^N} \chi(x) \, dx = 1 \), then it yields by Hölder’s inequality that
\[ \int_{\mathbb{R}^N} v(x,t)\chi(x) \, dx \leq \left( \int_{\mathbb{R}^N} |v(x,t)|^{p}\chi(x) \, dx \right)^{1/p}. \]

So
\[ \int_{\mathbb{R}^N} |v(x,t)|^{p}\chi(x) \, dx \geq \left( \int_{\mathbb{R}^N} v(x,t)\chi(x) \, dx \right)^{p} = Y^p(t). \quad (4.56) \]

Similarly, we obtain
\[ \int_{\mathbb{R}^N} |u(x,t)|^{q}\chi(x) \, dx \geq \left( \int_{\mathbb{R}^N} u(x,t)\chi(x) \, dx \right)^{q} = X^q(t). \quad (4.57) \]

Using estimates (4.56), (4.57) in (4.55), and the fact that the function \( \chi \) satisfies \( \Delta \chi \geq -\chi \), it yields
\[ D_{0|t}^{\gamma}([Z - Z_0]) + Z(t) \geq Y^p(t) + X^q(t), \quad t \in (0, T). \quad (4.58) \]

By adding \( Z(t) \) to the two members of (4.58), we obtain
\[ D_{0|t}^{\gamma}([Z(t) - Z_0]) + 2Z(t) \geq Y^p(t) + X^q(t) + X(t) + Y(t) \geq Y^p(t) + X^q(t) + X(t). \]

We assume that \( q \geq p \), by using the fact that \( X^q(t) + X(t) \geq X^p(t) \) and
\[ (a + b)^r \leq 2^{r-1}(a^{r} + b^{r}), \quad a, b > 0, \ r \geq 1, \]
we obtain
\[ D_{0|t}^{\gamma}([Z(t) - Z_0]) \geq 2^{1-p}Z(t)^p. \quad (4.59) \]

We put \( F(y) = 2^{1-p}y^p - 2y \), the function \( F \) is convex on \((0, \infty)\) (since \( F \in C^2(0, +\infty), F'' \geq 0 \)).

Writing \( \partial_t(k* [Z - Z_0])(t) \) instead of \( D_{0|t}^{\gamma}([Z(t) - Z_0]) \) with \( k(t) = t^{\frac{1-\gamma}{\gamma(1-\gamma)}} \) in (4.59), we obtain
\[ \partial_t(k* [Z - Z_0])(t) \geq F(Z(t)), \quad t \in (0, T). \quad (4.60) \]

It is clear that \( F(y) > 0 \) and \( F'(y) > 0 \) for all \( y > 2^{\frac{1}{p-1}} := \alpha_1 \).

Suppose now that \( Z_0 > \alpha_1 \). We claim that (4.60) implies that \( Z(t) > \alpha_1 \) for all \( t \in (0, T) \). In fact, for \( Z(0) = Z_0 > \alpha_1 \), we have by continuity of \( Z \), there exists \( \delta \in (0, T] \) such that \( Z(t) > \alpha_1 \) for all \( t \in (0, \delta) \). This implies that \( F(Z(t)) > 0 \) for all \( t \in (0, \delta) \).

By the comparison principle, it follows that \( Z(t) \geq Z_0 \) for all \( t \in (0, \delta) \). Setting \( \delta := \sup\{ s \in (0, T) : Z(t) \geq Z_0 \ t \in (0, s) \} \),
then \( \delta > 0 \). We want to show that \( \delta = T \).

Indeed, if \( \delta < T \), then by setting \( s = t - \delta \) for \( t \in (\delta, T) \) and \( \tilde{Z}(s) = Z(s + \delta) \), \( s \in (0, T - \delta_1) \), it follows from positivity of \( Z - Z_0 \) on \((0, \delta)\) and \( k \) being non-increasing that
\[ \partial_s(k* [\tilde{Z} - Z_0])(s) \geq \partial_s(k* [Z - Z_0])(s + \delta), \quad s \in (0, T - \delta_1). \quad (4.61) \]

From (4.60) and (4.61) we deduce that
\[ \partial_s(k* [\tilde{Z} - Z_0])(s) \geq F(\tilde{Z}(s)), \quad s \in (0, T - \delta_1). \]
This time-shifting property can be already found in [34]. So we may repeat the argument from above to see that there exists $\delta \in (0, T - \delta_1]$ such that $\tilde{Z}(s) \geq Z_0$ for all $s \in (0, \tilde{\delta})$. This leads to a contradiction with the definition of $\delta_1$.

Hence, the assumption $\delta_1 < T$ was not true. This proves the claim. Knowing that $Z(t) \geq Z_0 > C_1$ for all $t \in (0, T)$ it follows from (4.60) that

$$C D_0^\gamma Z(t) = \tilde{\partial}(k + \tilde{Z}(s))(t) \geq F(Z(t)) > 0, \quad \text{for all } t \in (0, T).$$

Therefore the function $Z(t)$ satisfying (4.62) is an upper solution of the problem

$$C D_0^\gamma y = F(y) = 2^{1-p} y^p - 2y, \quad y(0) = Z_0,$$

we have by comparison principle $Z(t) \geq y(t)$ (see [15, Theorem 2.3], [16, Theorem 4.10]).

On the other hand, since $F(0) > 0$, $F(y) > 0$ and $F'(y) > 0$, for all $y \geq Z_0 > 2^{p-1}$. It then follows from Lemma 3.8 that $v(t) = w(t) = 1/(1 + t)$ is a lower solution for (4.63) (which means

$$C D_0^\gamma u \leq F(v) = 2^{1-p} v^p - 2v, \quad v(0) = Z_0 \leq Z_0,$$

where $w(t)$ solves the ordinary differential equation

$$\frac{dw}{dt} = F(w) = 2^{1-p} w^p - 2w, \quad w(0) = Z_0.$$

By the comparison principle (see [15, Theorem 2.3], [16, Theorem 4.10]), we obtain $y(t) \geq v(t)$. So, by solving the Cauchy problem (4.64), which is equivalent to

$$\frac{d}{dt}(e^{2t} w) = 21-p e^{2(1-p)t}(e^{2t} w)^p, \quad w(0) = Z(0),$$

in which the explicit blow-up solution is

$$w(t) = \left(\frac{e^{2(1-p)t}}{2p} - 1 + Z_0 \right) e^{-2t},$$

which blows up in finite time $t_{**} = \frac{\ln(1-2^p Z_0^{1-p})}{2(1-p)}$. By the comparison principle (see [15, Theorem 2.3], [16, Theorem 4.10]), we conclude that

$$Z(t) \geq y(t) \geq v(t) = w\left(\frac{t^\gamma}{\Gamma(\gamma + 1)}\right) = \left(\frac{e^{2(1-p)t}}{2p} - 1 + Z_0 \right)^{1/\gamma} e^{-2t/\gamma}$$

which in turn leads to $Z(t)$ blows-up in finite time at $\tilde{t}_{**} \leq \left[\frac{\ln(1-2^p Z_0^{1-p})}{2(1-p)} \Gamma(\gamma + 1)\right]^{1/\gamma}$. Thus the same holds for the solution $(u, v)$ of (1.1)-(1.2), which in turn leads to a contradiction. □

Remark 4.2. Similar results were obtained in [40, Theorem 3.5] using another method, while the authors did not address the estimation of the time blow up.

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