From String Backgrounds to Topological Field Theories

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Abstract: The BRST formalism has played a fundamental role in the construction of bosonic closed string backgrounds, i.e. the stringy analogs of classical solutions to the field equations of general relativity. The concept of a string background has been extended to the notion of $W$-strings, where the BRST symmetry is still largely conjectural. More recently, the BRST formalism has entered the construction of two dimensional topological conformal quantum field theories, such as those that arise from Calabi-Yau varieties.

In this lecture, we focus on common features of the BRST cohomology algebras of string backgrounds and topological field theories. In this context, we present some new evidence for a remarkable relationship that transports us from bosonic and $W$-string backgrounds to the B-model topological conformal field theories associated to certain noncompact Calabi-Yau varieties. This paper will appear in the proceedings of the Symposium on BRS Symmetry held at RIMS, September 18-22, 1995.

1 Introduction

The BRST formalism is a widely, if not universally, recognized approach to the imposition of the Virasoro constraints in string theory (for some early works, see [24][16][51][13][14];

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see also the paper by Kato in this volume.). Over the last dozen years physicists and mathematicians alike have pondered the BRST-structure of string backgrounds, both abstract and concrete. During the same period, conformal field theory techniques have played an ever increasing role in the theory of string backgrounds (see [3] [1]). For a long time, the general theory has been dominated by the standard construction of ghost number one BRST invariant fields from dimension one primary matter fields (see for example [16]). It is well known that such standard invariant fields form a Lie algebra, at least modulo exact fields.

A number of research groups have understood that the operator product expansion for the background chiral algebra leads to a much richer algebraic structure in the full BRST cohomology, where all ghost numbers are on an equal footing [19] [7] [71] [74] [71] [23]. In [34], the authors have recognized that the full BRST cohomology of a conformal string background has the structure of a Batalin-Vilkovisky algebra, which is a special type of a Gerstenhaber algebra, or G-algebra. A G-algebra is a generalization of a Poisson algebra, which incorporates simultaneously the structure of a commutative algebra and a Lie algebra [17] [18] [19].

The particular G-algebra of the 2D string background is especially complex and has probably made no previous appearance in pure mathematics. In particular, in ghost number one BRST cohomology we obtain a (noncentral) extension of the Lie algebra of vector fields in the plane by an infinite dimensional abelian Lie algebra [34]. In this sense, string theory has enriched our understanding of algebraic structures. At the same time, there is a connection between the G-algebra for 2D strings and the anti-bracket formalism. The latter appears in both the work of Schouten-Nijenhuis [12] on the tensor calculus as well as the work of Batalin-Vilkovisky [2] [46] on the quantization of constrained field theories. Thus, string theory has also illuminated our understanding of geometrical structures.

As a biproduct of our structural analysis of the 2D string background, we obtain a new and very simple description of an explicit basis for the (chiral) BRST cohomology. This basis appears already in the work of Witten and Zwiebach [53]. Our enumeration of the basis exploits the full strength of the G-algebra structure, as well as a beautiful representation of the group $SL(2, \mathbb{C})$ in the cohomology.

In the last few years, many physicists have worked on the BRST formalism in the context of topological conformal field theory (see the papers of Kanno and Bonora in this volume). It’s also been shown that many conformal string backgrounds are equivalent to TCFTs (see [9]). Moreover TCFTs arise naturally in supersymmetric sigma models based on compact Calabi-Yau manifolds. Recently Ghoshal and Vafa have presented remarkable evidence for the strong relationship between certain 2D string backgrounds and TCFTs associated with certain noncompact Calabi-Yau threefolds [21] [43] [11]. This
relationship provides a strong motivation for extending the theory of sigma models to noncompact, possibly singular, varieties. Such an extension ought to involve a prominent role for both Gerstenhaber and Batalin-Vilkovisky algebras. Furthermore, the mirror symmetry between the so-called A and B models in the compact case should now be extended to the noncompact case as well as the case of a degenerate Kähler structure.

In [36], we raised the question of extending our results about a 2D string background to the case of $W$-string backgrounds (see for example [3][5][8]). In the meantime, the relationship in [21] suggested to us a connection between $W$-string backgrounds and certain higher dimensional noncompact Calabi-Yau varieties [47].

In this paper, we begin with a brief review of Gerstenhaber and Batalin-Vilkovisky algebras. We also review the BRST formalism in the context of chiral algebras of conformal string backgrounds. A particular 2D string background will be a fundamental example throughout our review. Using the elementary notions such as the normal ordered product and the descent operation, we show how to construct some fundamental algebraic structures on the BRST cohomology. In the case of the 2D string background, these structures can be described to a large extent in terms of the geometric $G$-algebra of the affine plane $\mathbb{C}^2$. We also describe an explicit basis for the BRST cohomology using the $SL(2, \mathbb{C})$ group action on $\mathbb{C}^2$. The first five sections of this paper are meant to be expository and contain no new results beyond [36].

In section 6, we review the notion of a topological chiral algebra (TCA) [34] as well as the BV structure on the BRST cohomology of a TCA. We then comment on a few examples of topological conformal field theories (TCFTs). Furthermore, we discuss the A and B model TCFTs naturally associated to the sigma model attached to a Calabi-Yau variety. We speculate on the extension of this theory to the case of Calabi-Yau varieties which are possibly noncompact, or possibly with degenerate Kähler structure. Relevant to this discussion are the two classical cohomology theories: the Kodaira-Spencer cohomology and the Poisson cohomology. In section 7, we discuss briefly the connection motivated by [21]. In section 8, we discuss a generalization of this connection to $W_n$-string backgrounds. We state some conjectures on the relationship between $W_n$-string backgrounds and certain complex varieties directly associated to the group $SL(n, \mathbb{C})$.

## 2 G-algebras and BV-algebras

Let $M$ be a smooth manifold. Then the space $V^*(M)$ of polyvector fields (ie. antisymmetric contravariant tensor fields) on $M$ admits the structure of a G-algebra, known
as the Schouten algebra of $M$. The dot product of a $p$-vector field $P$ and a $q$-vector $Q$ is given by
\[(P \cdot Q)^{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_q} = P^{[\nu_1 \cdots \nu_p} Q^{\mu_1 \cdots \mu_q]}.
\] (2.1)

Now let $P, Q$ be elements of $V^{p+1}(M), V^{q+1}(M)$ respectively. The Schouten bracket can be described as follows: in local coordinates the Schouten bracket $[P, Q]_S$ is given by
\[[P, Q]_S^{\nu_1 \cdots \nu_p \lambda \mu_1 \cdots \mu_q} = (p + 1) P^{\rho \nu_1 \cdots \nu_p} \partial_\rho Q^{\lambda \mu_1 \cdots \mu_q]} - (q + 1) Q^{\rho \mu_1 \cdots \mu_q} \partial_\rho P^{\lambda \nu_1 \cdots \nu_p}].
\] (2.2)

Note that if $p = q = 0$, then $[P, Q]_S$ is the ordinary Lie bracket of the vector fields $P, Q$.

**Definition 2.1** A $G$-algebra is an integrally graded vector space $A^*$ equipped with two bilinear products, denoted by $u \cdot v$ and $\{u, v\}$ respectively, satisfying the following assumptions:

(i) If $u, v$ are homogeneous elements of degree $|u|, |v|$ respectively, then $u \cdot v$ and $\{u, v\}$ are of degrees $|u| + |v|$ and $|u| + |v| - 1$ respectively.

(ii) Let $u, v, t$ be elements in $A$ of degrees $|u|, |v|, |t|$ respectively. Then we have
   
   (a) $u \cdot v = (-1)^{|u||v|} v \cdot u$
   
   (b) $(u \cdot v) \cdot t = u \cdot (v \cdot t)$
   
   (c) $\{u, v\} = -(-1)^{|u|(|v| - 1)} \{v, u\}$
   
   (d) $(-1)^{|u|(|v| - 1)} \{u, \{v, t\}\} + (-1)^{|t|(|v| - 1)} \{t, \{u, v\}\} + (-1)^{|v|(|u| - 1)} \{v, \{t, u\}\} = 0$
   
   (e) $\{u, v \cdot t\} = \{u, v\} \cdot t + (-1)^{|u||v|} v \cdot \{u, t\}$

For references on the mathematical theory of $G$-algebras, see [17] [18] [19] [34] (appendix B) [27].

**Definition 2.2** A $BV$ algebra is a $G$-algebra, $A^*$, equipped with a linear operation $\Delta$ satisfying

(f) $(-1)^{|u|} \{u, v\} = \Delta(u \cdot v) - (\Delta u) \cdot v - (-1)^{|u|} u \cdot \Delta v$

(g) $|\Delta| = -1$ and $\Delta^2 = 0$.

For references on the mathematical theory of $BV$ algebras, see [28] [27] [16] [50] [4].

Let $A^*(\mathbb{C}^2)$ be the holomorphic polyvector fields on $\mathbb{C}^2$ with polynomial coefficients. As a linear space, $A^*(\mathbb{C}^2)$ is the super polynomial algebra $\mathbb{C}[x, y, \partial_x, \partial_y]$, where $x, y$
have degree zero and $\partial_x, \partial_y$ have degree one. Introduce the notations $x^* = \partial_x$, $y^* = \partial_y$.

Define the linear operation on $A^*(C^2)$: $D = \frac{\partial}{\partial x} \frac{\partial}{\partial x^*} + \frac{\partial}{\partial y} \frac{\partial}{\partial y^*}$.

**Proposition 2.3** $A^*(C^2)$ equipped with $D$ is a BV algebra.

(See [34][50].)

### 3 BRST Formalism for String Backgrounds

The chiral algebra of a conformal string background is of the form

$$\mathcal{A} = A^{\text{ghost}} \otimes A^{\text{matter}}$$

where $A^{\text{ghost}}, A^{\text{matter}}$ are respectively the chiral algebras of the ghost and the matter sectors. Thus explicitly an element $O$ of $\mathcal{A}$ is a finite sum of holomorphic fields of the form $P(b, \partial b, \cdots, c, \partial c, \cdots)\Phi$, where $P$ is a normal ordered differential polynomial of the ghost fields $b, c$; $\Phi$ is a field in the matter sector. The respective (holomorphic part of the) stress energy tensors of the two sectors are $L^{\text{ghost}}, L^{\text{matter}}$ whose central charges are respectively $c = 26$, and $c = -26$.

The BRST current is

$$J = c(L^{\text{matter}} + \frac{1}{2}L^{\text{ghost}}),$$

and the ghost number current is

$$F = cb.$$  

Let’s recall a few basics of the BRST formalism. The BRST charge

$$Q = \oint_{C_0} J(z) dz$$

has the property that $[Q, Q] = 2Q^2 = 0$, where square brackets $[,]$ denotes the super commutator. Here $C_0$ is a small contour around 0. A BRST invariant field $O$ is one which satisfies $[Q, O] = 0$; and a BRST exact field is one which is of the form $O = [Q, O']$ for some field $O'$. The BRST cohomology of the string background $\mathcal{A}$ is the quotient space

$$H^*(\mathcal{A}) = \{Q\text{-invariant fields}\}/\{Q\text{-exact fields}\}$$
which is graded by the ghost number $\ast$.

A standard way to obtain BRST invariants is as follows. Let $V$ be a primary field of dimension one in $A^{\text{matter}}$. Then both $cV$ and $c\partial cV$ are BRST invariants of ghost number 1,2 respectively. The fields $O = 1$ and $O = c\partial c\partial^2 c$ are two universal BRST invariants known from the bosonic string theory.

3.1 An example of a string background

The chiral algebra $A_{2D}$ of a particular 2D string background is generated by the fields $b, c, \partial X, \partial \phi, e^{\pm (\pm iX-\phi)/\sqrt{2}}$, where $X, \phi$ are free bosons with the usual OPEs. Their corresponding stress energy tensors are:

$$L^X = -\frac{1}{2} (\partial X)^2$$

$$L^\phi = -\frac{1}{2} (\partial \phi)^2 + \sqrt{2} \partial^2 \phi$$

whose respective central charges are $c = 1, c = 25$.

We now describe some BRST invariants of the chiral algebra $A_{2D}$. Since $e^{(\pm iX+\phi)/\sqrt{2}}$ are matter primary fields of dimension one, we immediately obtain some standard BRST invariants $-ce^{(\pm iX+\phi)/\sqrt{2}}$. We denote them by $Y_{1/2,\pm 1/2}$. There are also exotic BRST invariants which cannot be obtained in the above standard way. For example, in ghost number zero we have

$$O_{1/2,\pm 1/2} = \left( cb + \frac{i}{\sqrt{2}} (\pm \partial X - i \partial \phi) \right) e^{(\pm iX-\phi)/\sqrt{2}}. \quad (3.9)$$

These BRST invariants were identified in [35] (see also [5]), and their explicit formulas were given in [49][53]. Explicit formulas for infinitely many BRST invariants of the 2D string background are also known. See [5][39][54][53].

4 Fundamental Operations

Using the antighost field $b$ and contour integral, one can define a number of useful operations on the fields $O$ in a string background. Given a dimension $h$ field $O$, we
attach to it a field of dimension \( h + 1 \)

\[
\mathcal{O}^{(1)}(w) = \oint_{C_w} b(z) \mathcal{O}(w) \, dz
\]  

(4.10)

where \( C_w \) is a small contour around \( w \). We call this linear operation, which reduces ghost number by one, the descent operation. Similarly, to \( \mathcal{O} \) we can attach a field of dimension \( h \)

\[
\Delta \mathcal{O}(w) = \oint_{C_w} b(z) \mathcal{O}(w)(z - w) \, dz.
\]  

(4.11)

This linear operation is called the Delta operation.

There are two important bilinear operations defined as follows. The first one is the dot product (a.k.a. the normal ordered product). Given two fields \( \mathcal{O}_1, \mathcal{O}_2 \), we define

\[
\mathcal{O}_1(w) \cdot \mathcal{O}_2(w) = \oint_{C_w} \mathcal{O}_1(z) \mathcal{O}_2(w)(z - w)^{-1} \, dz.
\]  

(4.12)

Under the dot product, both the conformal dimension and the ghost number are additive.

We define also the bracket operation

\[
\{ \mathcal{O}_1(w), \mathcal{O}_2(w) \} = \oint_{C_w} \mathcal{O}_1(z) \mathcal{O}_2(w) \, dz.
\]  

(4.13)

Note that conformal dimension is additive, while ghost number is shifted by -1 under the bracket operation. This operation was implicit in [49][53], and was studied in general in [34].

Let

\[
Q* \mathcal{O}(w) = \oint_{C(w)} J(z) \mathcal{O}(w) \, dz = [Q, \mathcal{O}(w)]
\]

\[
\Sigma* \mathcal{O}(w) = \oint_{C(w)} L_{\text{total}}(z) \mathcal{O}(w)(z - w) \, dz.
\]  

(4.14)

Then we have the following algebra of operations:

\[
\partial \mathcal{O} = [Q, \mathcal{O}^{(1)}]
\]

\[
\Delta^2 = 0
\]

\[
(Q*)^2 = 0
\]

\[
[Q*, \Delta] = \Sigma*
\]

\[
\Sigma* \mathcal{O} = n\mathcal{O} \text{ iff the conformal dimension of } \mathcal{O} \text{ is } n.
\]  

(4.15)

Moreover, we have the following identities [34]:

(a) \( Q* (\mathcal{O}_1 \cdot \mathcal{O}_2) = (Q* \mathcal{O}_1) \cdot \mathcal{O}_2 + (-1)^{\mathcal{O}_1} \mathcal{O}_1 \cdot (Q* \mathcal{O}_2) \)

(b) \( Q* \{\mathcal{O}_1, \mathcal{O}_2\} = \{Q* \mathcal{O}_1, \mathcal{O}_2\} + (-1)^{\mathcal{O}_1} \mathcal{O}_1 \cdot \{Q* \mathcal{O}_2\} \)

(c) \( (-1)^{\mathcal{O}_1} \{\mathcal{O}_1, \mathcal{O}_2\} = \Delta(\mathcal{O}_1 \cdot \mathcal{O}_2) - (\Delta \mathcal{O}_1) \cdot \mathcal{O}_2 - (-1)^{\mathcal{O}_1} \mathcal{O}_1 \cdot \Delta \mathcal{O}_2 \)

(4.16)
where $|\mathcal{O}|$ denotes the ghost number of $\mathcal{O}$. Note that the first equation in (4.13) is known as the descent equation [53].

### 4.1 Induced algebraic structures in BRST cohomology

Let $[\mathcal{O}], [\mathcal{O}_1], [\mathcal{O}_2]$ be cohomology classes in $H^*(\mathcal{A})$. By virtue of the identities (4.13) and (4.16), the above operations induce the following well-defined operations on cohomology:

\[
\Delta[\mathcal{O}] = [\Delta\mathcal{O}]_0 \\
{[\mathcal{O}_1] \cdot [\mathcal{O}_2]} = [{\mathcal{O}_1} \cdot {\mathcal{O}_2}] \\
\{[\mathcal{O}_1], [\mathcal{O}_2]\} = \{[\mathcal{O}_1, \mathcal{O}_2]\}.
\] (4.17)

Here $\mathcal{O}_0$ is the projection of $\mathcal{O}$ onto the subspace of $\mathcal{A}$ consisting of fields of zero conformal dimension. For example, given the standard BRST cohomology classes $\mathcal{O}_i = cV_i$, $i = 1, 2$, where the $V_i$ are matter primary fields of conformal dimension one, we have

\[
\{[\mathcal{O}_1], [\mathcal{O}_2]\} = [\mathcal{O}_3]
\] (4.18)

where

\[
\mathcal{O}_3(w) = c(w) \oint_{C_w} V_1(z)V_2(w)dz.
\] (4.19)

### 4.2 Chiral ground ring

Since ghost number is additive under dot product in cohomology, it follows that $H^0(\mathcal{A})$ is closed under this product.

**Theorem 4.1** [49] *The dot product in $H^0(\mathcal{A})$ is commutative and associative.*

The commutative associative algebra $H^0(\mathcal{A})$ is called the chiral ground ring of $\mathcal{A}$. For example it is shown in [49] [44] that in the case $\mathcal{A} = \mathcal{A}_{2D}$, the ground ring is the polynomial algebra generated by the classes $[\mathcal{O}_{1/2,1/2}]$ and $[\mathcal{O}_{1/2,-1/2}]$.

### 4.3 Chiral cohomology ring

**Theorem 4.2** [49] *$H^*(\mathcal{A})$ is a BV-algebra with BV operator $\Delta$ and with the dot and bracket products defined above.*
For related papers on algebraic structure in BRST cohomology, see [14][20][23][25][26][33][38][44][54][1].

5 A 2D String Background

An important example of a G-algebra is the one given by the BRST cohomology of the 2D string background discussed in section 3.1:

Theorem 5.1 As a G-algebra, \( H^*(A_{2D}) \) is generated by the four classes \([O_{1/2, \pm 1/2}], [Y^+_{1/2, \pm 1/2}]\). Moreover, \( H^p(A_{2D}) \) vanishes except for \( p = 0, 1, 2, 3 \).

Note that \( \{[Y^+_{1/2, -1/2}], [Y^+_{1/2, 1/2}]\} = [ce^{\sqrt{2}\phi}] \neq 0 \). (See [34].)

Theorem 5.2 The assignment \([O_{1/2, 1/2}] \mapsto x, [O_{1/2, -1/2}] \mapsto y, [Y^+_{1/2, -1/2}] \mapsto \partial_x, [Y^+_{1/2, 1/2}] \mapsto \partial_y\) extends to a G-algebra homomorphism \( \psi \) of \( H^*(A_{2D}) \) onto \( A^*(C^2) \).

For details on this result, see [34].

Proposition 5.3 For \( u \) in \( H^*(A_{2D}) \), \( \psi(\Delta u) = -D(\psi u) \).

(See [34][53].)

Introduce the notations \( \tilde{x} = [O_{1/2, 1/2}], \tilde{y} = [O_{1/2, -1/2}], \tilde{\partial}_x = [Y^+_{1/2, -1/2}], \tilde{\partial}_y = [Y^+_{1/2, 1/2}], \tilde{J}_+ = \tilde{x}\tilde{\partial}_y, \tilde{J}_0 = \tilde{x}\tilde{\partial}_x - \tilde{y}\tilde{\partial}_y, \tilde{J}_- = \tilde{y}\tilde{\partial}_x \). Similarly let \( J_+, J_0, J_- \in A^*(C^2) \) be defined by analogous formulas but without the tildes.

Proposition 5.4 \( \text{Span}\{\tilde{J}_+, \tilde{J}_0, \tilde{J}_-\}, \text{Span}\{J_+, J_0, J_-\} \) are both closed under the bracket \( \{\cdot, \cdot\} \) and are isomorphic to the Lie algebra \( sl(2, C) \).

(See [49][53][34][5].)

Now introduce an \( sl(2, C) \)-action on \( H^*(A_{2D}) \) by \( \tilde{J}_a \cdot O = \{\tilde{J}_a, O\} \) where \( a = +, 0, - \), \( O \in H^*(A_{2D}) \). Similarly, introduce an \( sl(2, C) \)-action on \( A^*(C^2) \) by \( J_a \cdot X = \{J_a, X\} \), \( X \in A^*(C^2) \). We make the observation (see [34][5]) that \( \psi : H^*(A_{2D}) \to A^*(C^2) \) intertwines actions of \( sl(2, C) \).
Proposition 5.5 \[\ker \psi\] is a $G$-ideal with vanishing dot product and bracket product.

That is, for $\mathcal{O} \in H^*(A_{2D})$, $\mathcal{O}', \mathcal{O}'' \in \ker \psi$, we have $\mathcal{O} \cdot \mathcal{O}'$, $\{\mathcal{O}, \mathcal{O}'\} \in \ker \psi$, and $\mathcal{O}' \cdot \mathcal{O}'' = \{\mathcal{O}', \mathcal{O}''\} = 0$.

Proposition 5.6 \[\ker \psi\] as a $G$-ideal is generated by the class $[ce\sqrt{2}\phi]$.

This means that $[ce\sqrt{2}\phi] \in \ker \psi$, and the smallest subspace containing $[ce\sqrt{2}\phi]$ and stable under the action of $H^*(A_{2D})$ by both the dot product and the bracket product is the whole $\ker \psi$ itself.

Let’s give an explicit basis for $\ker \psi$. Introduce the following linear operations on cohomology classes $[\mathcal{O}]$:

\[
\begin{align*}
A[\mathcal{O}] &= \{\tilde{\partial}_x, [\mathcal{O}]\} \\
B[\mathcal{O}] &= \{\tilde{\partial}_y, [\mathcal{O}]\} \\
C[\mathcal{O}] &= \tilde{\partial}_x \cdot [\mathcal{O}] \\
D[\mathcal{O}] &= \tilde{\partial}_y \cdot [\mathcal{O}].
\end{align*}
\]

Fix $\mathcal{K} = ce\sqrt{2}\phi$. Then we have

Proposition 5.7 $\ker \psi$ as a vector space has a basis consisting of the following classes:

ghost no. 1 : $A^{s-n}B^{s+n}[\mathcal{K}]$
ghost no. 2 : $(s - n)A^{s-n-1}B^{s+n}C[\mathcal{K}] + (s + n)A^{s-n}B^{s+n-1}D[\mathcal{K}]$
ghost no. 2 : $A^{s-n+1}B^{s+n}D[\mathcal{K}] - A^{s-n}B^{s+n+1}C[\mathcal{K}]$
ghost no. 3 : $A^{s-n}B^{s+n}CD[\mathcal{K}]$

where $s = 0, \frac{1}{2}, 1, ..., n = -s, -s + 1, .., s$.

We observe that the above basis vectors are in fact weight vectors for the $sl(2, \mathbb{C})$ action which we have previously described. Here $s, n$ are respectively the total spin and axial spin quantum numbers of the $sl(2, \mathbb{C})$ representation.

For completeness, we list here a basis for the algebra $A^*(\mathbb{C}^2) \cong H^*(A_{2D})/\ker \psi$:

ghost no. 0 : $x^{s-n} \cdot y^{s+n}$
ghost no. 1 : $\partial_x(x^{s-n} \cdot y^{s+n}) \cdot \partial_y - \partial_y(x^{s-n} \cdot y^{s+n}) \cdot \partial_x$
ghost no. 1 : $x^{s-n} \cdot y^{s+n} \cdot (\partial_x + y \cdot \partial_y)$
ghost no. 2 : $x^{s-n} \cdot y^{s+n} \cdot \partial_x \cdot \partial_y$

where $s = 0, \frac{1}{2}, 1, ..., n = -s, -s + 1, .., s$. 

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6 BRST Formalism for Topological Conformal Field Theories

Definition 6.1 [34] A topological chiral algebra (TCA) consists of the following data: a chiral (super)algebra $C^*$, a weight one even current $F(z)$ whose charge $F_0$ is the fermion number operator, a weight one odd primary field $J(z)$ having fermion number one and having a square zero charge $Q$, and a weight two odd primary field $G(z)$ having fermion number $-1$ and satisfying $[Q,G(z)] = L(z)$, where $L(z)$ is the stress-energy field. We denote the cohomology of the complex $(C^*,Q)$ by $H^*(C)$. We call the charge $Q$ the BRST operator of the TCA $C^*$.

Theorem 6.2 [34] Given a TCA $C^*$, $H^*(C)$ is a BV-algebra with BV operator $G_0$ and with dot product induced by the Wick product.

Given a particular conformal field theory [3], we can begin with the full operator algebra and then extract the chiral algebra [41] as the subalgebra of purely holomorphic operators. Since the spin of any operator in the full operator algebra is an integer, the dimension of any holomorphic operator is again an integer. In the same way, we can begin with the full operator (super)algebra of a topological conformal field theory–TCFT–and extract the topological chiral algebra–TCA–of purely holomorphic operators [12][10]. Just as a TCA has a cohomology ring which is a BV algebra, the full TCFT operator algebra has a cohomology ring with is also a BV algebra [20][14][29], and which contains the cohomology ring of the TCA.

For example, let

$$\mathcal{B} = \mathcal{A}_{2D} \otimes \bar{\mathcal{A}}_{2D}$$

(6.23)

where $\bar{\mathcal{A}}_{2D}$ is the antiholomorphic counterpart of $\mathcal{A}_{2D}$. We can regard $\mathcal{B}$ as the “bilateral” operator algebra of a particular TCFT with the BRST operator $Q + \bar{Q}$. The associated TCA is just $\mathcal{A}_{2D}$ itself. The BRST cohomology of $\mathcal{B}$ is given by $H^*(\mathcal{A}_{2D}) \otimes H^*(\bar{\mathcal{A}}_{2D})$ as a BV-algebra with the BV operator $\Delta + \bar{\Delta}$.

For a more interesting example, we can take $\mathcal{B}'$ be the subalgebra of $\mathcal{B}$ consisting of operators with zero charge relative to current $\partial \phi - \bar{\partial} \phi$. In this case $Q + \bar{Q}$ is still the BRST operator, but the cohomology is a proper subalgebra of $H^*(\mathcal{A}_{2D}) \otimes H^*(\bar{\mathcal{A}}_{2D})$. The
TCA associated with $B'$ is a subTCA of $A_{2D}$ generated by the holomorphic operators $b, c, e^{\pm i\sqrt{2}X}, \partial \phi$. The ground ring $\mathcal{H}$ of the string background $B'$ is the degree zero BRST cohomology of $B'$. This is a commutative algebra generated by

$$x_1 = [\mathcal{O}_{1/2,1/2}, \mathcal{O}_{1/2,1/2}]$$
$$x_2 = [\mathcal{O}_{1/2,1/2}, \mathcal{O}_{1/2,-1/2}]$$
$$x_3 = [\mathcal{O}_{1/2,-1/2}, \mathcal{O}_{1/2,2}]$$
$$x_4 = [\mathcal{O}_{-1/2,-1/2}, \mathcal{O}_{-1/2,-1/2}],$$

with one relation: $x_4x_1 - x_2x_3 = 0$. In other words,

$$H^0(B') \cong C[x_1, \ldots, x_4]/(x_4x_1 - x_2x_3)$$

the right hand side being the coordinate ring of the quadric cone in $C^4$.

### 6.1 TCFTs and Calabi-Yau varieties

Because a fermion number current $F(z)$ appears as a special element of a TCA, a TCFT will have both a bosonic and fermionic sector, unlike a typical CFT, which is purely bosonic. Thus, a TCFT is a cousin of a supersymmetric conformal field theory. In fact, there is a standard construction that in two ways twists an $N = 2$ super CFT to produce two TCFTs, the so-called A and B models. Moreover, many TCFTs are known to arise as twisted $N = 2$ theories\[12\]. In the same way, many TCAs arise from twisting an $N = 2$ chiral superalgebra. The TCA $A_{2D}$ analyzed previously is a good example of a twisted $N = 2$ chiral superalgebra. Vafa and Mukhi [40] have made a study of this particular twisting.

In the following, we will discuss the recent and exciting interaction between the theory of TCFTs and the theory of Calabi-Yau varieties.

1) Cohomology rings of mirror manifolds (See for example [48] [52] [10]): Physicists hypothesize the existence of an $N = 2$ supersymmetric CFT (“the supersymmetric sigma model”) associated to a given CY variety. Although the full state space and operator algebra of this $N = 2$ theory are not in general describable in closed form, the so-called topological sectors of the A and B twisted theories are at least partially understood. In particular, the A and B model cohomology rings have been identified with possibly deformed versions of classical cohomology rings of the CY variety. A remarkable consequence of this identification is the famous mirror variety hypothesis. We describe below some features of the A and B models:

The A-model: For a smooth, compact and Kähler CY variety, the cohomology ring of the A model is in general a nontrivial “quantum” deformation of its classical counterpart,
the deRham cohomology. The only natural BV operator in de Rham cohomology is the zero operator. In fact, the natural BV operator on the cohomology of the A-model TCFT is also the zero operator.

The B-model: The B model cohomology ring is precisely isomorphic to its classical counterpart, the Kodaira-Spencer cohomology ring:

**Definition 6.3** Let $X$ be a complex manifold, let $\mathcal{O}$ be the sheaf of holomorphic functions, and let $\mathcal{T}$ be the sheaf of holomorphic vector fields. The Kodaira-Spencer cohomology ring $[10]$ is the sheaf cohomology $H^\ast(X, \wedge^\ast \mathcal{O} \mathcal{T})$.

One can regard $H^\ast(X, \wedge^\ast \mathcal{O} \mathcal{T})$ as the Dolbeault $\bar{\partial}$-cohomology of $X$ with coefficients in holomorphic polyvector fields.

If the $n$-dimensional complex manifold $X$ has trivial canonical bundle, the holomorphic volume form leads to an isomorphism of graded vector spaces:

$$H^\ast(X, \wedge^n \mathcal{O} \mathcal{T}) \rightarrow H^\ast(X, \wedge^n \mathcal{O} \mathcal{\bar{\partial}})$$

(6.26)

Here $\Omega$ is the sheaf of holomorphic 1-forms. Thus $H^\ast(X, \wedge^n \mathcal{O} \mathcal{T})$ can be regarded as the $\bar{\partial}$-cohomology of $X$ with coefficients in holomorphic forms. The operator $\partial$ on $\Omega$ induces an operator on the range of the above isomorphism, and in turn a natural BV operator $\Delta$ on the Kodaira-Spencer ring itself. If $X$ is a smooth, compact and Kähler CY variety, then $\Delta$ vanishes identically.

**Comparison of the models:** Recall that for a smooth projective CY variety, only the de Rham cohomology is quantum-deformed. This observation is of course the key to the successful applications of the mirror symmetry hypothesis. However, one can ask whether this asymmetry between the A and B model continues to hold for more general complex manifolds.

2) Generalizations of cohomology rings: We also see from the above that for a smooth projective CY variety, both of the classical cohomology rings have trivial BV structure. At the level of topological conformal field theory, the respective BV operators are likewise trivial in both the A and B model cohomology rings attached to $X$. Thus, only the graded commutative (dot) product is nontrivial; the graded Lie bracket product vanishes identically. Thus there are two distinguished asymmetries: one between the A and the B models of $X$, and one between the dot and the bracket products of the BV algebras associated to $X$. We have been puzzled by these asymmetries in the case of smooth projective CY varieties. We have therefore been studying varieties of a more general type. It will be easier to consider the B model first.
From the purely mathematical definition of the Kodaira-Spencer cohomology, we have observed the following:

i) For any complex manifold, there is a natural bracket product which makes the Kodaira-Spencer cohomology a $G$-algebra. This bracket is induced by the Schouten bracket on holomorphic polyvector fields.

ii) For a complex manifold with trivial canonical bundle (a “zero canonical manifold”), the natural BV operator on the Kodaira-Spencer ring induces the natural bracket in i).

iii) For noncompact complex manifolds, the natural bracket can be nontrivial; likewise for noncompact zero canonical manifolds, the BV operator can be nontrivial. Simple examples arise as open dense submanifolds of compact complex manifolds (see the next section).

In principle, the physicists’ theory of smooth projective CY varieties ought to extend to some class of noncompact complex varieties. In particular, such an extension should include the open subvariety of smooth points (the ”smooth locus”) on a singular CY variety. In this way, the BV-structure of TCFT theory should come to play an important role. We therefore propose to continue our study of the Kodaira-Spencer cohomology of complex varieties.

The A model analog: Unfortunately, the de Rham cohomology of a smooth manifold (compact or noncompact) has no natural nontrivial BV algebra structure. However, the A model involves in an essential way the space of Kähler forms on the CY variety. A Kähler form gives rise to a Poisson tensor, which is of course nondegenerate, and completely determines the Kähler form itself.

**Definition 6.4** Let $Y$ be a smooth manifold and let $V^*(Y)$ be the smooth sections of the Grassman algebra bundle generated by the tangent bundle of $Y$. $V^*(Y)$ is a $G$-algebra relative to the wedge and Schouten products. Let $P$ in $V^2(Y)$ be a Poisson bivector field on $Y$. We call $P$ a Poisson tensor if the Schouten bracket $[P,P] = 0$.

It is well-known that the inverse of a Kähler (in fact even symplectic) form is a Poisson tensor. For arbitrary Poisson tensors, there is a geometrical invariant known as the Poisson cohomology, which was introduced by Lichnerowicz [29] and later studied by Koszul [28]. For more details, see these two references.

**Definition 6.5** [29] The operator $\delta_P = [P,-]$ is square zero and is a derivation of
degree one of the $G$-algebra structure on $V^*(Y)$. Finally, the Poisson cohomology ring of $Y$, $H^*_P(Y)$, is the cohomology of $\delta_P$ in $V^*(Y)$.

If the Poisson tensor is nondegenerate (invertible), the Poisson cohomology is naturally isomorphic to the de Rham cohomology. The Poisson cohomology is, for any Poisson tensor, a Gerstenhaber algebra; in some circumstances it is even a BV algebra. We have found simple examples with nontrivial $G$-algebra or BV-algebra structure. We propose to study a generalization of the $N = 2$ supersymmetric sigma model to the case where the Kähler form is replaced by a degenerate Poisson tensor [45]. For such target spaces, the A model cohomology should in some cases be a nontrivial BV algebra.

3) Degenerations of complex and Kähler structure: We hope to eventually generalize the mirror variety hypothesis beyond the context of smooth projective CY varieties. The new context should at least include Poisson varieties that arise by degeneration of Kähler structure, as well as the smooth loci of singular varieties that arise by degeneration of the complex structure. We have found simple examples to illustrate the notion of a degeneration of Kähler structure on a complex manifold. Of course, degeneration of complex structure is a well studied topic in the theory of Calabi-Yau varieties (see for example [11]).

The generalized mirror variety hypothesis should include a description of how the BV algebra structure of the A model produces a quantum deformation of the BV algebra structure of the Poisson cohomology. Likewise, there must be a description of how the B-model BV algebra produces a quantum deformation of the Kodaira-Spencer cohomology.

7 From The 2D String to Quadric Threefolds

We have discovered a potential example of a B-model deformation:

Consider the quadric cone in $\mathbb{C}^4$. This cone has a nodal singularity at the vertex. Suppose that $X$ is the cone minus the vertex. $X$ is a zero canonical manifold, and we have calculated its Kodaira-Spencer cohomology as a BV algebra. Following a striking suggestion by Ghoshal and Vafa [21], we have compared the Kodaira-Spencer cohomology of $X$ with the bilateral BRST cohomology of the TCFT $B'$, which arises in the theory the $c = 1$ string (see [6], [49], [53], [15]). The two cohomologies are nearly isomorphic as graded commutative algebras, but the G-bracket and BV operator of the $c = 1$ string TCFT are nontrivial deformations of the corresponding bracket and operator on the Kodaira-Spencer cohomology.
We expect that there is a generalization of the B-model to the case of a noncompact zero canonical variety (such as the cone $X$). We conjecture that the B-model cohomology of the cone $X$ is isomorphic to the BRST cohomology of the $c = 1$ string TCFT. In principle, there should even be an equivalence at the level of TCFTs. There is remarkable evidence in the Ghoshal-Vafa paper [21] for an equivalence between the TCFT of the smooth quadric and the TCFT obtained by perturbing the $c = 1$ string theory by an appropriate marginal operator [49]. Whether or not this evidence is relevant to the singular cone is still not clear.

We propose to continue our study of particular complex varieties which may be noncompact rather than projective and/or Poisson rather than Kähler. Our main goal is the precise formulation of a generalized mirror symmetry hypothesis, as well as the application of such a hypothesis to some specific families of CY varieties.

8 From the $W_n$-String Background to a Degeneration of $SL(n, \mathbb{C})$

The $c = 1$ string TCFT is the first in a series of TCFTs called W-string backgrounds [3]. Such backgrounds are associated with simply-laced simple Lie algebras over the complex numbers. The $c = 1$ string theory itself is connected with $sl(2, \mathbb{C})$. The so-called $W_n$-string background is connected with $sl(n, \mathbb{C})$.

**Conjecture 8.1** For arbitrary $n > 1$, there is a canonical BRST current for a $W_n$-string background. The cohomology of the corresponding TCA admits a natural filtration that respects the natural BV algebra structure. The associated graded BV algebra is isomorphic to the Kodaira-Spencer BV algebra of the noncompact complex variety (the “base affine space”)

$$X_n = SL(n, \mathbb{C})/R_n$$

where $R_n$ is the group of $n \times n$ upper triangular unipotent matrices. This isomorphism respects the natural action of $SL(n, \mathbb{C})$ on the $W_n$ BV algebra and the Kodaira-Spencer BV algebra of $X_n$, respectively.

A critical step in the development of the above conjecture was an earlier conjecture by Bouwknegt, McCarthy and Pilch: the algebra of holomorphic polyvector fields on
the base affine space is isomorphic as a graded commutative algebra to a distinguished subalgebra of the chiral cohomology of the \( W_n \)-string background \cite{9} \cite{6} \cite{7} \cite{8}. This statement has been verified for the case of \( n = 2 \) \cite{34} and \( n = 3 \) \cite{6}.

Let \( D_n \) be the group of diagonal matrices in \( SL(n, \mathbb{C}) \). Define an action of \( D_n \) on the Cartesian product \( X_n \times X_n \) by the formula

\[
h \cdot (gR_n, g'R_n) = (ghR_n, g'h^{-1}R_n).
\]

(8.27)

Let \( Z_n \) be the quotient of \( X_n \times X_n \) by the above action of \( D_n \); this action is free and \( Z_n \) is a smooth zero canonical noncompact complex manifold. In fact, \( Z_n \) is the smooth locus of a particular algebraic degeneration of the group variety, \( SL(n, \mathbb{C}) \). For \( n = 2 \) we obtain the quadric cone discussed in the previous section. By a general theorem of Tian and Yau \cite{47}, \( Z_n \) supports Calabi-Yau metrics.

For each \( n \) there is an appropriate TCFT \( B'_n \) generalizing the case of \( B'_2 := B' \) (see previous section).

**Conjecture 8.2** For arbitrary \( n > 1 \), the cohomology of the corresponding TCFT \( B'_n \) is nearly isomorphic as a graded commutative algebra to the Kodaira-Spencer algebra of the CY variety \( Z_n \).

We propose to compute by Kodaira-Spencer cohomology of the homogeneous complex manifold \( X_n \) by means of a standard reduction \cite{22} to a Lie algebra cohomology of the Lie algebra of \( R_n \). A similar reduction applies to the homogeneous complex manifold \( Z_n \). We have recently learned that Bouwknegt, McCarthy and Pilch have also observed that the \( R_n \)-Lie algebra cohomology theory plays a crucial role in the cohomology theory of \( W_n \)-string backgrounds. See reference \cite{8} for an early hint of this role. We look forward to future progress on the above conjectures.

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