Consider Lagrangian tracer particles that emanate from a point source in a turbulent fluid. If \( t_R \) is the time at which a tracer, initially at the origin of a sphere of radius \( R \), crosses the surface of the sphere for the first time, what is the probability distribution function (PDF) \( \mathcal{P}(R, t_R) \)? The answer to this question is of central importance in both fundamental nonequilibrium statistical mechanics and in understanding the dispersal of tracers by a turbulent flow, a problem whose significance cannot be overemphasized, for it is relevant to the advection of pollutants in the atmosphere. First-passage-time problems have been studied extensively \([2–5]\) and they have found applications in a variety of areas in physics and astronomy, chemistry \([6]\), biology \([7]\), economics \([8]\), and finance \([9]\) or various statistical measures of two-particle dispersion including exit-time statistics for such dispersions in two- and three-dimensional (2D and 3D) turbulent flows \([10, 11]\). In contrast to these earlier studies (e.g., Refs. \([10, 12]\) ), the first-passage-time problem we pose considers one tracer in a turbulent flow that is statistically homogeneous and isotropic. For such a particle we show, via extensive direct numerical simulations (DNSs), that \( \mathcal{P}(R, t_R) \) displays a crossover between two qualitatively different behaviors: (a) for \( R \ll L_1 \), \( \mathcal{P}(R, t_R) \sim t_R^{-\alpha} \), with \( \alpha = 4 \); (b) for \( L_1 \ll R \), \( \mathcal{P}(R, t_R) \) has an exponentially decaying tail (Fig. 1). We develop models that allow us to obtain these asymptotic behaviors analytically.

The 3D incompressible, Navier-Stokes equation is

\[
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \tag{1a}
\]

and

\[
\nabla \cdot \mathbf{u} = 0. \tag{1b}
\]

Here, \( \mathbf{u}(\mathbf{x}, t) \) is the Eulerian velocity at position \( \mathbf{x} \) at time \( t \), \( p(\mathbf{x}, t) \) is the pressure field, and \( \nu \) is the kinematic viscosity of the fluid; the constant density is chosen to be unity. Our direct numerical simulation (DNS) uses the pseudo-spectral method \([13]\), with the 2/3 rule for dealiasing, in a triply periodic cubical domain with \( N^3 \) collocation points; we employ the second-order, exponential, Adams-Bashforth scheme for time stepping \([14]\). We obtain a nonequilibrium, statistically stationary turbulent state via a forcing term \( \mathbf{f} \) which imposes a constant rate of energy injection \([15, 16]\) in wave-number shells \( k = 1 \) and \( k = 2 \) in Fourier space; this turbulent state is statistically homogeneous and isotropic.

To obtain the statistical properties of Lagrangian tracers that are advected by this turbulent flow, we seed the flow with \( N_0 \) independent, identical tracer particles. If the Lagrangian displacement of a tracer, which was at position \( \mathbf{r}_0 \) at time \( t_0 \), is \( \mathbf{r}(t \mid \mathbf{r}_0, t_0) \), then its temporal evolution is given by

\[
\frac{d}{dt} \mathbf{r} = \mathbf{v}(t \mid \mathbf{r}_0, t_0) = \mathbf{u}(\mathbf{r}, t), \tag{2}
\]

where \( \mathbf{v} \) is its Lagrangian velocity. In Eq. (2), we need the Eulerian flow velocity at off-grid points; we obtain this by tri-linear interpolation; and we use the first-order Euler method for time marching (see, e.g., Ref. \([14]\) ).

Clearly, \( t_R \) is the first time at which \( |\mathbf{r}| \) becomes equal to \( R \). Instead of computing the PDF (or histogram) of \( t_R \) numerically, we calculate the complementary cumulative probability distribution function (CPDF) \( Q(t_R) \), by using the rank-order method \([17]\), to circumvent binning errors. In Fig. 1 we present log-log and semi-log plots of \( Q(t_R) \) versus \( t_R/T_{\text{edd}} \), for various values of \( R \). From Fig. 1(a) we conclude that, for \( R \ll L_1 \), \( Q(t_R/T_{\text{edd}}) \sim (t_R/T_{\text{edd}})^{\alpha-1} \), for large \( t_R/T_{\text{edd}} \); with \( \alpha \approx 4 \); note that, in this power-law scaling regime, the complementary CPDFs for different values of \( R/L_1 \) collapse onto a universal scaling form, if we plot \( Q(t_R/T_{\text{edd}}) \). In contrast, Fig. 1(b) shows that, for \( L_1 \ll R \), the tail of \( Q(t_R/T_{\text{edd}}) \) decays exponentially. For the first-passage-time problem for tracers in homogeneous and isotropic fluid turbulence.

A first-passage-time problem for tracers in homogeneous and isotropic fluid turbulence

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We define a new first-passage-time problem for Lagrangian tracers that are advected by a statistically stationary, homogeneous, and isotropic turbulent flow: By direct numerical simulations of the three-dimensional (3D) incompressible, Navier-Stokes equation, we obtain the time \( t_R \) at which a tracer, initially at the origin of a sphere of radius \( R \), crosses the surface of the sphere for the first time. We obtain the probability distribution function \( \mathcal{P}(R, t_R) \) and show that it displays two qualitatively different behaviors: (a) for \( R \ll L_1 \), \( \mathcal{P}(t_R) \) has a power-law tail \( \sim t_R^{-\alpha} \), with the exponent \( \alpha = 4 \) and \( L_1 \) the integral scale; (b) for \( L_1 \lesssim R \), the tail of \( \mathcal{P}(R, t_R) \) decays exponentially. We develop models that allow us to obtain these asymptotic behaviors analytically.
time PDF, these results imply that
\[ \mathcal{P}(t_R/T_{\text{eddy}}) \sim \begin{cases} \left( R/t_R/T_{\text{eddy}} \right)^{-4} & \text{for } R \ll L_1; \\ \exp\left(-t_R/T_{\text{eddy}}\right) & \text{for } L_1 \lesssim R. \end{cases} \]  
(3)

We now develop models that allow us to understand these two asymptotic behaviors analytically.

To understand the power-law behavior of \( \mathcal{P} \), in the range \( R \ll L_1 \), we construct the following, natural, ballistic model: Tracer particles emanate from the origin with (a) a velocity whose magnitude \( v \) is a random variable with a PDF \( p(v) \); and (b) when it starts out from the origin, the tracer’s velocity vector points in a random direction. Tracers move ballistically, for short times. Therefore, for \( R \ll L_1 \), the first-passage time \( t_R = R/v \); and the first-passage PDF is
\[ \mathcal{P}(R, t_R) = \int \delta(t_R - R/v)p(v)dv. \]  
(4)

In statistically homogeneous and isotropic and incompressible-fluid turbulence, each component of the Eulerian velocity has a PDF that is very close to Gaussian \[18\], so \( p(v) \) has the Maxwellian \[19\] form
\[ p(v) = C_d v^{d-1} \exp(-v^2/\sigma^2), \]  
(5)

where \( C_d \) depends on the spatial dimension \( d \) and \( \sigma = \langle v^2 \rangle \). We substitute Eq. (5) in Eq. (4); then, by integrating over \( v \), we obtain
\[ \mathcal{P}(R, t_R) = C_d t_R^3 \exp\left(-R^2/(t_R^2 \sigma^2)\right). \]  
(6)

Therefore, in the limit of small \( R \) and large \( \tau \), the first-passage-time probability is
\[ \mathcal{P}(R, t_R) \sim R^3/t_R^4, \quad \text{for } d = 3; \]  
(7)

this power-law exponent is the same as the one we have obtained from our DNSs above (Table I and Fig. I).

We can obtain the tail \( \mathcal{P}(t_R/T_{\text{eddy}}) \sim \exp\left(-t_R/T_{\text{eddy}}\right) \) for \( L_1 \lesssim R \) as follows. At times that are larger than the typical auto-correlation time of velocities in the Lagrangian description, we follow Taylor \[20\] and assume that the motion of a tracer particle is diffusive. Therefore, we consider a Brownian particle in three dimensions (3D). To calculate the first-passage-time PDF, we must first obtain the survival probability \( S(t, R(0)) \), i.e., the probability that the particle has not reached the surface of the sphere of radius \( R \) up to time \( t \), if it has started from the origin of this sphere.
fluctuations-dissipation theorem (FDT) holds. Note that
boundary condition
obtain the following solution:

\[
P(r, t) = \frac{1}{2R^2} \sum_{n=0}^{\infty} \frac{n!}{r} \sin \left( \frac{n \pi r}{R} \right) \exp \left( -K n^2 \pi^2 t/R^2 \right),
\]
whence we get

\[
S(R, t_R) = \int_0^R P(r, t) 4 \pi r^2 dr = 2 \sum_{n=0}^{\infty} (-1)^{n+1} \exp(-K n^2 \pi^2 t_R/R^2),
\]
where, in the last step, we have used Eq. (9). The first-passage-time probability is

\[
\mathcal{P}(R, t_R) = -\frac{\partial}{\partial t_R} S(R, t_R) = \frac{2K n^2}{R^2} \sum_{n=0}^{\infty} (-1)^{n+1} n^2 \exp(-K n^2 \pi^2 t_R/R^2).
\]

At large times, the first term \((n = 1)\) is the dominant one; therefore,

\[
\mathcal{P}(R, t_R) \sim (1/R^2) \exp(-K \pi^2 t_R/R^2),
\]
the exponential form that we have obtained from our DNS (Fig. 1(b)); the \(1/R^2\) pre-factor cannot be extracted reliably from our DNS data, because this requires much longer runs than are possible with our computational resources.

We now show that both the small- and large-\(R/L_1\) behaviors of \(\mathcal{P}\) in Eq. (11) can be obtained from one stochastic model for the motion of a particle. The simplest such model uses a particle that obeys the following Ornstein-Uhlenbeck (OU) model:

\[
\frac{dx_i}{dt} = v_i, \tag{13a}
\]
\[
\frac{dv_i}{dt} = -\gamma v_i + \frac{\sqrt{\Gamma}}{m} \zeta_i. \tag{13b}
\]

Here, \(\gamma\) and \(\Gamma\) are positive constants; \(x_i\) and \(v_i\) are the Cartesian components of the position and velocity of the particle; in three dimensions, \(i = 1, 2,\) and 3; \(\zeta(t)\) is a zero-mean Gaussian white noise with \((\zeta_i) = 0\) and \((\zeta_i(t)\zeta_j(t')) = \delta_{ij} \delta(t - t');\) this noise is such that the fluctuations-dissipation theorem (FDT) holds. Note that there is no FDT for turbulence. However, for the one particle statistics we consider, the simple OU model is adequate. We use \(N_p = 50,000\) particles; for each particle, the initial-position components \(x_i(t = 0)\) are distributed randomly and uniformly on the interval \([0, 2\pi]\); and the velocity components \(v_i(t = 0)\) are chosen from a Gaussian distribution. For each particle, we obtain, numerically, the time \(t_R\) at which it reaches a distance \(R\) from the origin for the first time. We then obtain the first-passage-time complementary PDF \(\mathcal{Q}(t_R)\), which we plot in Fig. 2 for \(R \ll L\) and \(L \lesssim R\), where \(L = \sqrt{\gamma/m}\); the natural length scale for Eq. (13), plays the role of \(L_1\) in our DNSs above (Table I and Fig. 1). We find

\[
\mathcal{P}(R, t_R) \sim \left[ \frac{t_R \gamma}{(R/L)^4} \right]^{-4}, \text{for } R \ll L;
\]

\[
\mathcal{P}(R, t_R) \sim \exp\left(-\frac{t_R \gamma}{(R/L)^2}\right), \text{for } L \lesssim R; \tag{14}
\]
these are the OU-model analogs of our DNS results Eq. (3). We have carried out two OU-model simulations: (a) we have designed the first, with \(\gamma = 0.01\), to explore the form of \(\mathcal{P}\) in the ballistic regime \(R \ll L\); (b) the second, with \(\gamma = 30\), allows us to uncover the form of \(\mathcal{P}\) in the diffuse regime \(L \lesssim R\). (From a numerical perspective, it is expensive to obtain the precise form of \(\mathcal{P}\) in both ballistic and diffusive regimes, with one value of \(\gamma\).) We now explore in detail the forms of \(\mathcal{P}\) in these two regimes. In Fig. 2(a), we present log-log plots of the complementary CPDFs of the scaled first-passage time \(t_R/R\), for \(R \ll L\) and \(\gamma = 0.01\). The complementary CPDFs of \(t_R/R\), for \(R/L = 0.0002, R/L = 0.00035, \) and \(R/L = 0.0005\), collapse onto one curve; i.e., in this regime, \(t_R\) scales a \(L\), which is a clear manifestation of ballistic motion. In Fig. 2(b), we present semi-log plots of the complementary CPDFs of the scaled first-passage time \(t_R/R^2\), for \(L \lesssim R\) and \(\gamma = 30\). The complementary CPDFs of \(t_R/R^2\), for \(R/L = 10, R/L = 14, R/L = 18,\) and \(R/L = 20\), collapse onto one curve; from this we conclude that, in this regime, \(t_R\) scales as \(R^2\), which is a clear signature of diffusive motion.

We have defined and studied a new first-passage-time problem for Lagrangian tracers that are advected by a 3D turbulent flow that is statistically steady, homogeneous and isotropic. Our work shows that the first-passage-time PDF \(\mathcal{P}(t_R)\) has tails that cross over from a power-law form to an exponentially decaying form as we move from the regime \(R \ll L_1\) to \(L_1 \lesssim R\) (Eq. 3). We develop ballistic-transport and diffusive models, for which we can obtain these limiting asymptotic behaviors of \(\mathcal{P}\) analytically. We also demonstrate that an OU model, with Gaussian white noise, which mimics the effects of turbulence, suffices to obtain the crossover between these limiting forms. Of course, such a simple stochastic model cannot be used for more complicated multifractal properties of turbulent flows \([11, 18, 22]\).
Earlier studies have concentrated on two-particle relative dispersion by using doubling-time statistics, in 2D fluid turbulence; in particular, they have shown that the PDF of this doubling time has an exponential tail. Studies of velocity zero crossings, in a turbulent fluid turbulence; in particular, they have shown that PDFs of the zero-crossing times have exponential tails.

The single-particle first-passage-time statistics that we study have not been explored so far. We hope that our work will encourage experimental groups to measure first-passage times and verify the asymptotic behaviors that we have elucidated above.

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FIG. 2. (a) Log-log plots of the complementary CPDFs $Q(t_{R/L})$ of the scaled first-passage time $t_{R/L}$, for $R \ll L$ and $\gamma = 0.01$; the complementary CPDFs, for $R/L = 0.0002$ (green), $R/L = 0.00035$ (blue), and $R/L = 0.0005$ (orange), collapse onto one curve; (b) semi-log plots of the complementary CPDFs of the scaled first-passage time $t_{R/L}$, for $L \ll R$ and $\gamma = 30$. The complementary CPDF of $t_{R/L}$, for $R/L = 10$ (purple), $R/L = 14$ (green), $R/L = 18$ (blue), and $R/L = 20$ (orange), collapse onto one curve. Plots of the complementary CPDFs $Q(t_{R/L})$ versus $t_{R/L}$ are shown in the insets.

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