A Sampling-Aware Interpretation of Linear Logic: Syntax and Categorical Semantics

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Abstract

The usual resource interpretation of linear logic says that variables have to be used exactly once. However, there are models of linear logic where this interpretation is too restrictive.

In this work we show how in probabilistic models of linear logic the correct resource interpretation should be sampling, i.e. the linear arrow should be read as "the output may only sample once from its input".

We accommodate this new interpretation by defining a multilanguage syntax and its categorical semantics that bridges the Markov kernel and linear logic interpretations of probabilistic programs.

Keywords Linear Logic, Probabilistic Programming, Categorical Semantics.

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1 Introduction

Probabilistic primitives have been a standard feature of programming languages since the 70s. At first, randomness was mostly used to program so called random algorithms, i.e. algorithms that require access to a source of randomness. Recently, however, with the rise of computational statistics and machine learning, randomness is also used to program statistical models and do inference on them.

Programming languages researchers have seen this rise in interest as an opportunity to further study the interaction of probability and programming languages, establishing it as an active subfield within the PL community.

One of its main challenges is giving semantics to programming languages that are both expressive in the regular PL sense as well as in its abilities to program with randomness. One particular difficulty is that the mathematical machineries used for probability theory, such as measure theory, do not interact well with higher-order functions.

In this work we focus on two classes of models of probabilistic programming — in its broad sense — that have found numerous applications: models based in linear logic and models based on Markov kernels. Out of the linear-logic-based semantics there are two that we highlight:

• Ehrhard et. al [3, 5, 6] have defined models of linear logic with probabilistic primitives and have used the translation of intuitionistic logic into linear logic $A \to B = !A \multimap B$, where $!A$ is the exponential modality, to give semantics to a stochastic $\lambda$-calculus.

• Dahlqvist and Kozen [2] have defined an imperative, higher-order, linear language and added the type constructor $!A$ to accommodate non-linear programs.

Though their semantical models are very similar, their calculi lie on quite different foundations. By using the intuitionistic logic translation into linear logic, Ehrhard et al. get a call-by-name (CBN) semantics for a functional language [12]. This semantics is not well-suited for practical purposes as it makes it impossible to reuse a value sampled from a distribution. Consider, for instance, the following program written in a language that has a fair coin primitive $\sample{\text{coin}} : \mathbb{N}$:

\[
\text{let } x = \text{coin in } (x, \neg x)
\]

Under a call-by-name semantics the program above is equivalent to $(\text{coin}, \text{coin})$, since it samples from coin each time $x$ is used. To get around this limitation the authors have introduced a call-by-value (CBV) let operator, which would make the example program be interpreted as $\frac{1}{2}\delta_{(0,1)} + \frac{1}{2}\delta_{(1,0)}$, where $\delta_a$ is the point mass distribution at $a$. Unfortunately, from a semantical point of view, the existence of said operator relies on subtle mathematical constructions which complicate the semantics.

On the other hand, the calculus proposed by Dahlqvist and Kozen adopts a linear typing discipline and extends their type system with the exponential modality $!A$. Their language provides a conceptually elegant presentation of soft-conditioning based on the Kothe dual of vector spaces [1].

While these linear-logic-based semantics have been proposed, semantics based on Markov kernels have also been shown to properly handle conditioning and higher-order
functions [10, 19]. In particular, the calculus interpreted using \( \omega \)-quasi Borel spaces [19] can soundly interpret recursive types, soft-conditioning, higher-order functions and provides a monadic interface for its probabilistic fragment.

Though both styles of semantics provide insights into how to interpret probabilistic programming languages (PPL), it is still too early to claim that we have a “correct” semantics which subsumes all of the existing ones. The approaches mentioned above both have their advantages and their drawbacks. A major advantage of the models based on linear logic mentioned above is that its morphisms are linear operators of norm \( \leq 1 \) – a formalism that has been extensively used to reason about stochastic processes. Indeed, Dahlqvist and Kozen has only been recently discovered, there is still much work to be done when it comes to proving useful lemmas required for reasoning about programs. Even though it is a conservative extension of regular measure theory, it is still unclear which theorems hold in the higher-order setting.

That being said, there are also drawbacks to the linear logic approach. The syntactic linearity restriction imposes severe restrictions on the programming model and, as we mentioned above, making the exponential capable of reusing samples requires significant mathematical effort.

Therefore, it is essential to understand how these two families of semantics relate to each other. In this work we shed some light into the matter by showing how we can use both styles of semantics to interpret a linear calculus that does not suffer from some of the drawbacks mentioned above, while maintaining its rich reasoning principles. We believe that in the context of probabilistic programming, linear logic departs from its usual Computer Science ethos of enforcing syntactic invariants, and instead it provides a natural mathematical formalism in which to express ideas from probability theory, such as the soft-conditioning construction mentioned above.

We bridge the gap of these semantics by noting that the regular resource interpretation of linear logic, i.e. \( A \rightarrow B \) being equivalent to “by using one copy of \( A \) I get one copy of \( B \)” is too restrictive an interpretation for probabilistic programming. Instead, we should think of usage as being equivalent to sampling. Therefore the linear arrow \( A \rightarrow B \) should be thought of as “by sampling from \( A \) once I get \( B \)”.

From a programming languages perspective it is not completely straightforward to realize this new interpretation. The linear arrow expects its argument to be something that can be sampled from (e.g. a measure), but when we abstract over the argument, we want to think of it as if it were a value. This suggests that there are two different interpretations of variables.

We avoid this problem with a multilanguage approach: we have one language where variables roughly correspond to measures, one language where variables are values, and syntax that transports programs from the former language into the latter.

With this new interpretation of linearity, composition already corresponds to the CBV let used by Ehrhard et al while avoiding complicating the model. Furthermore, we get a more expressive language than that of Dahlqvist and Kozen, since we can reuse variables while still maintaining a resource interpretation of randomness.

Besides, since linear logic is at the center of our approach, we can use it to handle alternative computational interpretations of linear logic, such as session types. Indeed, there has been work done on using probabilistic session types to reason about channel communication in the presence of probabilities [4, 11]. In this context we should use the usual resource interpretation of linear logic when handling the communication of processes, but the probabilistic resource interpretation when computing within a single process.

Our contributions are:

- We define a multi-language syntax that allows us to realize our “sampling as resource” interpretation of linear logic.
- We define its categorical semantics and prove certain interesting equations satisfied by it.
- We show that our construction is already present in existing models for probabilistic programming.

### 2 Mathematical Preliminaries

We are assuming that the reader is familiar with basic notions of category theory such as categories and functors.

Transition matrices are one of the simplest abstractions used when modeling stochastic processes. Given two countable sets \( A \) and \( B \), the entry \((a, b)\) of a transition matrix is the probability of ending up in state \( b \in B \) whenever you start from initial state \( a \in A \).

**Definition 2.1.** The category \( \text{CountStoch} \) has countable sets as objects and transition matrices as morphisms. The identity morphism is the identity matrix and composition is given by matrix multiplication.

This category can only model discrete probabilistic processes. In order to generalize it we must use Markov kernels between measurable sets.

**Definition 2.2.** A measurable set is a pair \((A, \Sigma_A)\), where \( A \) is a set and \( \Sigma_A \subseteq \mathcal{P}(A) \) such that it contains the empty set, it is closed under complements and countable unions.

**Definition 2.3.** A function \( f : (A, \Sigma_A) \rightarrow (B, \Sigma_B) \) is called measurable if for every \( B \in \Sigma_B \), \( f^{-1}(B) \in \Sigma_A \).

**Definition 2.4.** Let \((A, \Sigma_A)\) be a measurable space. A probability measure \((A, \Sigma_A) \rightarrow [0, 1] \) is a function \( \mu : \Sigma_A \rightarrow [0, 1] \) such that \( \mu(\emptyset) = 0 \), \( \mu(A) = 1 \) and \( \mu(\bigcup_{i \in I} A_i) = \sum_{i \in I} \mu(A_i) \).
The intuition behind Definition 2.5 is that it duplicates the input once” probabilistic processes. The connection to linearity in the algebraic sense comes when noting that every Markov kernel $A \rightarrow MB$ can, by integration, be seen as a linear function $MA \rightarrow MB$.

We want a syntax that reflects the fact that linearity corresponds to sampling, not variable usage. We achieve this by making use of a multi-language semantics that enables the programmer to transport programs defined in a Markov kernel-centric language to a linear, higher-order, language.

This approach contrasts with existing multi-language semantics where one of the languages is linear and is used to increase the expressivity of a non-linear language by, for instance, allowing it to have access to mutable state [15]. Our language flips this idea on its head: we define syntax so that we may increase the expressivity of a linear calculus.

Since many probabilistic programming constructs, such as soft-conditioning and Markov kernels, can be naturally interpreted in linear logic terms, we believe that our calculus allows the user to benefit from the insights linearity provides to PPL while unburdening them from worrying about syntactic restrictions.

We use standard notation from the literature: $\Gamma \vdash t : \tau$ means that program $t$ has type $\tau$ under context $\Gamma$, $t(x/u)$ means substitution of $u$ for $x$ in $t$ and $t(\overline{x}/\overline{u})$ is the simultaneous substitution of the term list $\overline{x}$ for a variable list $\overline{u}$ in $t$.

We are going to use orange to represent MK programs and purple to represent LL programs. Both languages will be defined in the following sections.
(F(X) ⊗D F(Y)) ⊗D F(Z) \xrightarrow{\mu \circ \text{id}} F(X ⊗C Y) ⊗D F(Z) \xrightarrow{\mu} F((X ⊗C Y) ⊗C Z) \xrightarrow{F\alpha} F(X ⊗C (Y ⊗C Z))

![Lax monoidal diagrams](image)

**Figure 1.** Lax monoidal diagrams

\[ \tau := 1 \quad \tau \times \tau \]
\[ t, u := x \quad \text{unit} \quad \text{let } x = t \text{ in } u \quad (t, u) \quad \pi_1 t \quad \pi_2 u \quad f(M) \]
\[ \Gamma := \cdot \quad x : \tau, \Gamma \]

**3.1 A Markov Kernel Language**

We need a language which can program Markov kernels. Since we are aiming at generality, we are assuming the least amount of structure possible. As such we will be working with the internal language of Markov categories, as presented in Figure 2 and Figure 3. Note that we are implicitly assuming a set of primitives for the functions \( f \).

Due to obvious reasons, every Markov category can interpret this language, as we show in Figure 8. However, as it stands, this language is not very expressive, as it does not have any probabilistic primitives nor does it have any interesting types since \( 1 \times 1 \equiv 1 \).

When working with concrete models (c.f. Section 5) we can extend it with more expressive types as well as with concrete probabilistic primitives. For instance, in the context of continuous probabilities we could add a \( \mathbb{R} \) datatype and a \( \cdot \) uniform : \( \mathbb{R} \) uniform distribution primitive.

**3.2 A Linear Language**

Our second language is a linear simply-typed \( \lambda \)-calculus. Its main appeal is that it has higher-order functions.

The type system depicted in Figure 5 follows the standard presentation of linear \( \lambda \)-calculi.

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variables of ground type are measures. In order to bridge these language we must use the observation that Markov kernels — i.e. open MK terms — have a natural interpretation as linear maps between measure spaces — i.e. open LL term. The combined syntax for the language is depicted in Figure 6.

We now have a language design problem: we want to capture the fact that every open MK program is, semantically, also an open LL term. The naive typing rule is:

$$\forall x : \tau_1, \ldots, x_n : \tau_n \vdash_{MK} M : \tau$$

$$\vdash_{LL} \Gamma_{\tau_1, \ldots, \tau_n} \vdash_{LL} MK(M) : \Gamma_{\tau}$$

The problem with this rule is that it breaks substitution: the variables in the premise are MK variables whereas the ones in the conclusion are LL variables. One way around this would be defining a dual-context calculus for the LL language. In this case the context would be split between LL variables and MK variables and the rule would look like:

$$\forall x : \tau_1, \ldots, x_n : \tau_n \vdash_{MK} M : \tau$$

$$\vdash_{LL} \Gamma_{\tau_1, \ldots, \tau_n} \vdash_{LL} MK(M) : \Gamma_{\tau}$$

This fixes the issue of the previous version of the rule, we just have to state the substitution theorem a bit differently; variables in the LL context can only be substituted for by LL programs and the variables of the MK context can only be substituted for by MK programs. That being said, we want our syntax to reflect a common idiom of PPL’s: we compute a distribution (an element of $\Gamma_{\tau}$), we sample from it and then use the result in a non-linear continuation. This is captured by the following syntax:

Note that we are sampling from LL programs $t_i$ (possibly an empty list), outputting the results to MK variables $x_i$ and binding them to an MK program $M$. When clear from the context we simply use sample $t_i$ as $x_i$ in $M$. Its corresponding typing rule is:

As it is the case of linear $\lambda$-calculi, every symmetric monoidal closed category gives semantics to this language. Once again, we are aiming at generality instead of expressivity. In a concrete setting it would be fairly easy to extend the calculus with a datatype $\mathbb{N}$ for natural numbers and probabilistic primitives such as $\cdot \vdash \text{coin} : \mathbb{N}$.

### 3.3 Combining Languages

The main drawback of the linear calculus above is that the syntactic linearity restriction makes it harder to program in it, while the main drawback of the Markov language is that it does not allow for higher-order constructs. In this section we will show how we can combine both language so that we get a calculus with looser linearity restrictions while still being higher-order.

When looking at concrete models for these languages it is easy to see that the semantical interpretations of variables in both languages are completely different: in the MK language variables should be thought of as values, as the result of sampling from a distribution, whereas in the LL language,
them, bind the samples to the variables \{x_i\} in the MK language \(M\) where there are no linearity restrictions. Note that the rule above looks very similar to a monadic composition, though they are semantically different (cf. Section 4).

With this new syntax we can finally program in accordance with our new resource interpretation of linear logic, allowing us to write the program

\[
sample\ \text{coin}\ \text{as}\ \text{x}\ \text{in}\ (x = x),
\]

which flips a coin once and tests the result for equality with itself, making it equivalent to true.

This combined calculus enjoys the expected syntactic properties\(^2\).

**Theorem 3.1.** Let \(\Gamma, x : t_1 \vdash t : \tau\) and \(\Delta \vdash u : t_2\) be well-typed terms, then \(\Gamma, \Delta \vdash \{x/u\} : \tau\).

**Proof:** To be more formal we would have to prove such a theorem for both the MK and LL type systems. However, since the MK calculus has been shown to satisfy these metatheoretic properties, we only focus on the LL case, in which case the proof follows by structural induction on the typing derivation \(\Gamma, x : t_1 \vdash t : \tau\):

- **Axiom:** Since \(t = x\) then \(\{x/u\} = u\) and \(t_1 = \tau\).
- **Abstraction:** By hypothesis, \(\Gamma, x : t_1, y : t_2 \vdash t : \tau_3\). Since we can assume wlog that \(x \neq y\) and that \(y \notin \Gamma\), let \(y.\ \{x/u\} = \lambda y.\ \{x/u\} \vdash \lambda y.\ \{x/u\}.\) Therefore we can show that \(\Gamma, \Delta \vdash \lambda y.\ \{x/u\} : \tau_2 \o \tau_3\) by applying the rule Abstraction and by the induction hypothesis.
- **Application:** \(t_1 \cdot t_2\) \(\{x/u\} = t_1\{x/u\} \cdot t_2\{x/u\}\). Since the language LL is linear, only one of \(t_1\) or \(t_2\) will have \(x\) as a free variable. By symmetry we can assume that \(t_1\) has \(x\) as a free variable and we can prove \(\Gamma, \Delta \vdash t_1\{x/u\} \cdot t_2 : \gamma\) by applying the rule Application and by the induction hypothesis.
- **Sample:** It is easy to prove that \((\text{sample } t\ \text{as} \ y\ \text{in} M\{x/u\}) = \text{sample } (t\{x/u\})\) as \(y\) in \(M\).

\[\square\]

The following example illustrates how we can use the MK language to duplicate and discard linear variables.

**Example 3.2.** The program which samples from a distribution \(t\) and then returns a perfectly correlated pair is given by:

\[
sample\ t\ \text{as}\ x\ \text{in}\ (x, x)
\]

Similarly, the program that samples from a distribution \(t\) and does not use its sampled value is represented by the term

\[
sample\ t\ \text{as}\ x\ \text{in}\ \text{unit}
\]

\(^2\)To avoid visually polluting the proofs we will drop the color code in Theorem 3.1 and Theorem 4.4.

**Example 3.3.** Suppose that we have a Markov kernel given by an open MK term \(x : \mathbb{N} \vdash M : \mathbb{N}\). If we want to encapsulate it as a linear program of type \(M(\mathbb{N} \o M\mathbb{N})\) we can write:

\[
\lambda\ \text{meas.}\ \text{(sample} \text{meas as } x\ \text{in } M\text{)}
\]

**Example 3.4.** The language defined by Dahlgqvist and Kozen has types \(\mathbb{R}^n\) for every \(n \in \mathbb{N}\) and interprets them as the set of measures over \(\mathbb{R}^n\), making their syntax not well-equipped to manipulate elements of type \(\mathbb{R}^n\) as if they were tuples. To do that it is necessary to add constants to their syntax corresponding to projection combinators. In contrast, in our combined language we can write the program sample \(t\) as \(x\) in \(\pi_1\ x\) which samples from a distribution over pairs and returns only the first component.

Unfortunately there are some aspects of this language that still are restrictive. For instance, imagine that we want to write an LL program that receives two Markov kernels and a distribution over \(\mathbb{R}\) as inputs, samples from the input measure, feeds the result to the Markov kernels, samples from them and adds the results. Its type would be

\[
(M(\mathbb{N} \o \mathbb{N})) \o (M(\mathbb{N} \o M\mathbb{N})) \o M\mathbb{N} \o M\mathbb{N}
\]

Even though the algorithm only requires you to sample once from each measure, it is still not possible to write it in the linear language.

We will show in Section 4 how the type constructor \(M\) actually corresponds to an applicative functor [13]. Therefore, the limitation above is actually a particular case of a fundamental difference between programming with applicative functors and programming with monads.

In applicative style there are restrictions on the kinds of dependencies you can create between distributions since the continuation of the sample construction needs to be an MK program, making it impossible to typecheck the program:

\[
sample\ t\ \text{as}\ x\ \text{in}\ \text{(sample } t\ \text{as } y\ \text{in } (f\ x)\text{ as } y\ \text{in } (g\ x)\text{)}
\]

That being said, if you have two open MK programs \(x : \mathbb{N} \vdash M_1 : \mathbb{N}\) and \(x : \mathbb{N} \vdash M_2 : \mathbb{N}\) you can write the following open MK program:

\[
let\ y_1 = M_1\ in\ (let\ y_2 = M_2\ in\ (y_1 + y_2))
\]

**Remark 3.1.** We now have two languages that can interpret probabilistic primitives such as coin. However, every primitive \(M\) in the MK language can be easily transported to an LL program by sample \(\text{as } \text{in } M\). Therefore it makes sense to only add these primitives to the MK language.

**4 Categorical Semantics**

As it is the case with categorical interpretations of languages/logics, types and contexts are interpreted as objects in a category and every well-typed program/proof gives rise to a morphism in the category.
\[
\begin{align*}
[1] &= 1_c \\
[\Pi_1 \otimes \Pi_2] &= [\Pi_1] \otimes [\Pi_2] \\
[\Pi_1 \circ \Pi_2] &= [\Pi_1] \circ [\Pi_2] \\
[\Pi] &= 1_c \\
[\Pi : r, \Gamma] &= [\Pi] \otimes [\Gamma]
\end{align*}
\]

**Axiom**

\[
\tau \xrightarrow{id} \tau
\]

**Tensor**

\[
\begin{pmatrix}
\Gamma_1 & \rightarrow & \Gamma_1 \\
\Gamma_2 & \rightarrow & \Gamma_2
\end{pmatrix}
\]

**LetTensor**

\[
\begin{pmatrix}
\Gamma_1 & \rightarrow & \Gamma_1 \\
\Gamma_2 & \rightarrow & \Gamma_2
\end{pmatrix}
\]

**Application**

\[
\begin{pmatrix}
\Gamma & \rightarrow & \Gamma \\
\Gamma_1 & \rightarrow & \Gamma_1 \\
\Gamma_2 & \rightarrow & \Gamma_2
\end{pmatrix}
\]

**Abstraction**

\[
\begin{pmatrix}
\Gamma & \rightarrow & \Gamma \\
\Gamma & \rightarrow & \Gamma \\
\Gamma_1 & \rightarrow & \Gamma_1 \\
\Gamma_2 & \rightarrow & \Gamma_2
\end{pmatrix}
\]

**Figure 7.** Denotational semantics for LL

In our case, MK types \(\tau\) are interpreted as objects \([\tau]\) in a Markov category \((M, \times)\) and well-typed programs \(\Gamma \vdash_{MK} M : \tau\) are interpreted as an \(M\) morphism \([\Gamma] \rightarrow [\tau]\), as shown in Figure 8. Similarly, LL types \(\tau\) are interpreted as objects \([\tau]\) in a model of linear logic \((C, \otimes, \rightarrow)\) and well-typed programs \(\Gamma \vdash_{LL} t : \tau\) are interpreted as a \(C\) morphism \([\Gamma] \rightarrow [\tau]\), as shown in Figure 7.

To give semantics to the combined language is not as straightforward. The sample rule allows the programmer to run LL programs, bind the results to MK variables and use said variables in an MK continuation. The implication of this rule in our formalism is that our semantics should provide a way of translating MK programs into LL programs. In category theory this is usually achieved by a functor \(M\).

However, we can easily see that functors are not enough to interpret the sample rule. Consider what happens when you apply \(M\) to an MK program \(N\) with two free variables:

\[
M[N] : M(\Pi_1 \otimes \Pi_2) \rightarrow M\tau
\]

To precompose it with two LL programs outputting \(M\Pi_1\) and \(M\Pi_2\) we need a morphism \(\mu_{\Pi_1, \Pi_2} : M\Pi_1 \circ M\Pi_2 \rightarrow M(\Pi_1 \otimes \Pi_2)\). Furthermore, if \(N\) has three or more free variables, there would be several ways of applying \(\mu\). Since from a programming standpoint it should not matter how the LL programs are associated, we require that \(\mu_{\Pi_1, \Pi_2}\) makes the lax monoidality diagrams to commute. Under this assumption we may interpret the sample rule:

**Sample**

\[
\begin{pmatrix}
\Pi_1 \times \cdots \times \Pi_n & \rightarrow & \Pi \\
\Pi_1 & \rightarrow & M\Pi_1 \\
\Pi_n & \rightarrow & M\Pi_n
\end{pmatrix}
\]

**Figure 8.** Denotational semantics for MK

\[
\Gamma \vdash_{MK} M : \Pi, \Gamma \\
\Gamma \vdash_{LL} t : \Pi, \Gamma
\]

In case it only has one MK variable, the semantics is given by \([\tau] ; M[N]\) and in case it does not have any free variables the semantics is \(\varepsilon ; M[N]\).

The equational theories of the MK and LL languages are well-known. Something which is not obvious is understanding it follows the two program equivalences:

**Theorem 4.1.** Let \(t, M\) and \(N\) be well-typed programs,

\[
[(\lambda y. \text{sample } y \text{ as } z \text{ in } N) \text{ (sample } t \text{ as } x \text{ in } M)] =
[(\text{sample } t \text{ as } x \text{ in } (\text{let } y = M \text{ in } N))]
\]
Lemma 4.2. Let $t$ be a well-typed program,

$$
[sample\ t\ as\ x\ in\ x] = [t]
$$

Proof. The proof follows by induction on the typing derivation of $t$.

- **Axiom:** Since $t = x$ then $t(x/t_0) = t_0$ and $[t(x/t_0)] = [t_0] = [t_0] ; id = [t_0] ; [x]$.
- **Unit:** Since $t = x$ then $t(x/t_0) = t_0$ and $[t(x/t_0)] = [t_0] = [t_0] ; id = [t_0] ; [x]$.
- **Tensor:** We know that $t = t_1 \otimes t_2$. Furthermore, from linearity we know that each free variable appears either in $t_1$ or in $t_2$. Without loss of generality we can assume that

$$
(t_1 \otimes t_2)(x_1, \cdots, x_n/u_1, \cdots, u_k) =
(t_1(x_1, \cdots, x_k/u_1, \cdots, u_k) \otimes (t_2(x_{k+1}, \cdots, x_n/u_{k+1}, \cdots, u_n))
$$

We can conclude this case from the induction hypothesis and linearity of $\otimes$.

- **LetTensor:** This case follows from the functionality of $\otimes$ and the induction hypothesis.

Theorem 4.2. Let $t$ be a well-typed program,

$$
[sample\ t\ as\ x\ in\ x] = [t]
$$

Proof. The proof follows by induction on the typing derivation of $t$.

- **Axiom:** Since $t = x$ then $t(x/t_0) = t_0$ and $[t(x/t_0)] = [t_0] = [t_0] ; id = [t_0] ; [x]$.
- **Unit:** Since $t = x$ then $t(x/t_0) = t_0$ and $[t(x/t_0)] = [t_0] = [t_0] ; id = [t_0] ; [x]$.
- **Tensor:** We know that $t = t_1 \otimes t_2$. Furthermore, from linearity we know that each free variable appears either in $t_1$ or in $t_2$. Without loss of generality we can assume that

$$
(t_1 \otimes t_2)(x_1, \cdots, x_n/u_1, \cdots, u_k) =
(t_1(x_1, \cdots, x_k/u_1, \cdots, u_k) \otimes (t_2(x_{k+1}, \cdots, x_n/u_{k+1}, \cdots, u_n))
$$

We can conclude this case from the induction hypothesis and linearity of $\otimes$.

- **LetTensor:** This case follows from the functionality of $\otimes$ and the induction hypothesis.

- **Abstraction:** This case follows from unfolding the definitions, using the induction hypothesis and by naturality of cur.
- **Application and Sample:** Analogous to the Tensor case.

From this theorem we can conclude:

**Corollary 4.5.** The rule shown in Figure 9 is sound with respect to the categorical semantics.

Lax monoidal functors, under the name applicative functors, are widely used in programming languages research. They are often used to define embedded domain-specific languages (eDSL) within a host language. This suggests that the Markov kernel language can be thought of as an eDSL inside a linear language.

We have just shown that $\mathcal{M}$ being lax monoidal is sufficient to give semantics to our combined language, but what would happen if it had even more structure? If it were also full it would be possible to add a reification command:\(^3\)

$$
\mathcal{M}\Gamma \vdash t : \mathcal{M}\tau
$$

where $M\Gamma$ is notation for every variable in $\Gamma$ being of the form $M\tau'$, for some $\tau'$. The semantics for the rule would be taking the inverse image of $M$. As we will show in the next section, there are some concrete models where $\mathcal{M}$ is full and some other models where $\mathcal{M}$ is not full.

A property which is easier to satisfy is asking for $\mathcal{M}$ to be faithful. In this case the translation of the MK language into the LL language would be fully-abstract in the following sense:

**Theorem 4.6.** Let $x : \tau_1 \vdash M : \tau_2$ and $x : \tau_1 \vdash N : \tau_2$ be two well-typed $\mathcal{M}$ programs. If $M$ is faithful then $[sample\ x\ as\ x\ in\ M] = [sample\ x\ as\ x\ in\ N]$ implies $[M] = [N]$.\(^\Box\)

Proof. The proof follows by induction on the typing derivation of $t$.

- **Axiom:** Since $t = x$ then $t(x/t_0) = t_0$ and $[t(x/t_0)] = [t_0] = [t_0] ; id = [t_0] ; [x]$.\(^\Box\)
- **Unit:** Since $t = x$ then $t(x/t_0) = t_0$ and $[t(x/t_0)] = [t_0] = [t_0] ; id = [t_0] ; [x]$.\(^\Box\)
- **Tensor:** We know that $t = t_1 \otimes t_2$. Furthermore, from linearity we know that each free variable appears either in $t_1$ or in $t_2$. Without loss of generality we can assume that

$$
(t_1 \otimes t_2)(x_1, \cdots, x_n/u_1, \cdots, u_k) =
(t_1(x_1, \cdots, x_k/u_1, \cdots, u_k) \otimes (t_2(x_{k+1}, \cdots, x_n/u_{k+1}, \cdots, u_n))
$$

We can conclude this case from the induction hypothesis and linearity of $\otimes$.

- **LetTensor:** This case follows from the functionality of $\otimes$ and the induction hypothesis.

5 Concrete Models

In this section we show how existing models for both discrete as well as continuous probabilities fit within our formalism.

5.1 Discrete Probability

For the sake of simplicity we will denote the monoidal product of $\texttt{CountStoch}$ as $\times$.

The probabilistic coherence space model of linear logic has been extensively studied in the context of semantics of discrete probabilistic languages\(^[3]\).

**Definition 5.1 (Probabilistic Coherence Spaces \([3]\)).** A probabilistic coherence space (PCS) is a pair $(|X|, \mathcal{P}(X))$ where $|X|$ is a countable set and $\mathcal{P}(X) \subseteq |X| \rightarrow \mathbb{R}^+$ is a set, called the web, such that:

- $\forall a \in X \exists e_a > 0 \cdot e_a \cdot \delta_a \in \mathcal{P}(X)$, where $\delta_a(a') = 1$ iff $a = a'$ and 0 otherwise;\(^3\)

\(^3\)The proposed rule breaks the substitution theorem, but it is possible to define a variant for it where this is not the case.
Theorem 5.8. \( \forall a \in X \exists \lambda_a \forall x \in \mathcal{P}(X) x_a \leq \lambda_a; \)
\( \mathcal{P}(X)^{1+} = \mathcal{P}(X), \) where \( \mathcal{P}(X)^{1+} = \{ x \in X \to \mathbb{R}^+ | \forall v \in \mathcal{P}(X) \sum_{x \in X} x_a v_a \leq 1 \}. \)

We can define a category \( \text{PCoh} \) where objects are probabilistic coherence spaces and morphisms \( X \to Y \) are matrices \( f : [X] \times [Y] \to \mathbb{R}^+ \) such that for every \( v \in \mathcal{P}(X), (f v)_b = \sum_{a \in [A]} f(a,b) v_a. \)

Definition 5.2. Let \( \mathcal{P}_Y \) be a countable set, the pair \( X, \{ \mu : X \to \mathbb{R}^+ | \sum_{x \in X} \mu(x) \leq 1 \} \) is a PCS. The lax monoidal structure is given by
\[ \otimes : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \otimes Y), \]
where \( (x \otimes y)(a,b) = x(a)y(b) \)

Lemma 5.3. Let \( X \) be a countable set, the pair \( (X, \{ \mu : X \to \mathbb{R}^+ | \sum_{x \in X} \mu(x) \leq 1 \}) \) is a PCS.

Proof. The first two points are obvious, as the Dirac measure is a subprobability measure and every subprobability measure is bounded above by the constant function \( \mu_1(x) = 1. \)

To prove the last point we use the — easy to prove — fact that \( \mathcal{P} \subseteq \mathcal{P}^{1+}. \) Therefore we must only prove the other direction. First, observe that, if \( \mu \in \{ \mu : X \to \mathbb{R}^+ | \sum_{x \in X} \mu(x) \leq 1 \}, \) then we have \( \sum_{x \in X} \mu(x) \leq 1, \)
\[ \mu(x) \leq 1. \]

Let \( \tilde{\mu} \in \{ \mu : X \to \mathbb{R}^+ | \sum_{x \in X} \mu(x) \leq 1 \}^{1+}. \) By definition,
\[ \sum_{x \in X} \mu(x) \leq 1, \] and, therefore, the third point holds.

Lemma 5.4. Let \( X \to Y \) be a CountStoch morphism. It is also a PCoh morphism.

Theorem 5.5. There is a lax monoidal functor \( \mathcal{M} : \text{CountStoch} \to \text{P coh}. \)

Proof. The functor is defined using the lemmas above. Functoriality holds due to the functor being the identity on arrows. The lax monoidal structure is given by \( \epsilon = id_1 \) and \( \mu_{X,Y} = id_{X \otimes Y} \)

Lemma 5.6. If \( \mu \in \{ x \otimes y | x \in \mathcal{M}(X), y \in \mathcal{M}(Y) \}^{1+} \) then for every \( x \in X \) and \( y \in Y, \mu(x,y) \leq 1. \)

Proof. If there were such indices such that \( \mu(x_1,y_1) > 1 \)
\[ \sum_{x \in X} \mu(x,y)(\delta_{x_1} \otimes \delta_{y_1})(x,y) > \mu(x_1,y_1)(\delta_{x_1} \otimes \delta_{y_1})(x_1,y_1) \]
\( = \mu(x_1,y_1) > 1, \)
which is a contradiction.

Lemma 5.7. Let \( X \) and \( Y \) be two countable sets, then
\[ \mathcal{M}(X \times Y) = \left( \left\{ \mu : X \times Y \to \mathbb{R}^+ | \sum_{(x,y) \in X \times Y} \mu(x,y) \leq 1 \right\} \right) \]
\( \mathcal{M}(X \times Y). \)

Proof. By the lemma above it follows that if we have a joint probability distribution \( \tilde{\mu} \) over \( X \times Y \) and an element \( \mu \in \{ x \otimes y | x \in \mathcal{M}(X), y \in \mathcal{M}(Y) \}^{1+} \) then \( \sum_{x \in X} \sum_{y \in Y} \mu(x,y) \leq 1, \)
\( \mu(x,y) \leq 1. \)

Theorem 5.9. The functor \( \mathcal{M} \) is full.

Proof. Since \( \epsilon \) is the identity morphism, it is trivially an isomorphism. The morphisms \( \mu_{X,Y} \) being an isomorphism is a direct consequence of the lemmas above.

5.2 Continuous Probability

Definition 5.10 (\([2]\)). The category \( \text{RoBan} \) has regular ordered Banach spaces as objects and regular linear functions as morphisms.

Theorem 5.11. There is a lax monoidal functor \( \mathcal{M} : \text{BorelStoch} \to \text{RoBan}. \)

Proof. The functor acts on objects by sending a measurable space to the set of signed measures over it, which can be equipped with a \( \text{RoBan} \) structure. On morphisms it sends a Markov kernel \( f \) to the linear function \( \mathcal{M}(f)(\mu) = \int f d\mu. \)

The monoidal structure of \( \text{RoBan} \) satisfies the universal property of tensor products and, therefore, we can define the natural transformation \( \mu_{X,Y} : \mathcal{M}(X) \otimes \mathcal{M}(Y) \to \mathcal{M}(X \times Y) \) as the function generated by the bilinear function \( \mathcal{M}(X) \times \mathcal{M}(Y) \) which maps a pair of distributions to its product measure. The map \( \epsilon \) is, once again, equal to the identity function.

The commutativity of the lax monoidal diagrams follows from the universal property of the tensor product: it suffices to verify it for elements \( \mu_A \otimes \mu_B \otimes \mu_C. \)

Even though \( \mathcal{M} \) looks very similar to the discrete case, the functor is not strong monoidal, which has a few consequences in how we write programs in the combined language, as we may only project out of a pair by using the MK language, as the program below illustrates:

\[
\text{sample } t \text{ as } y \text{ in } (\pi_1 y)
\]

This lack of isomorphism is a well-known theorem from functional analysis. In practice it means that there are joint probability distributions (elements of \( \mathcal{M}(AXB) \)) that cannot
be represented as an element of the tensor $M(A) \otimes M(B)$, which is usually defined as a closure of product probability distributions, as illustrated by [2].

6 Beyond Probability

There are other models of linear logic where syntactic usage is too restrictive to capture linear morphisms. One such example is non-determinism, where linear morphisms $A \rightarrow B$ should be seen as “by observing the non-deterministic value of A once I get B”.

It is possible to adapt the syntax and categorical semantics presented in this paper to give semantics to a language for non-deterministic programming. For instance, consider the powerset $\mathcal{P}$ monad over the category $\mathbf{Set}$. Non-deterministic functions can be modeled by functions $A \rightarrow \mathcal{P}(B)$. Linear morphisms in the Kleisli category for the powerset monad.

Furthermore the powerset monad is isomorphic to the category $\text{Rel}$, which is a model of linear logic. In such a scenario there is an obvious lax monoidal functor $\text{id}_{\mathcal{Rel}} : \text{Rel} \rightarrow \text{Rel}$ and we could add the syntax observe $t$ as $x$ in $u$ to the language as the non-deterministic variant of the sample syntax. In this particular case the non-deterministic calculus does not suffer from any of the drawbacks mentioned in the probabilistic case.

7 Related Work

Session types are an alternative computational interpretation for linear logic. It is used to model distributed computation and has been recently extended to program with randomness [4, 11]. They can be used to program finite state Markov chains. Unfortunately, the linearity syntactic restrictions reduces the expressivity of the calculus. One way they work around it is by extending the type system with a new family of additive connectives $A \oplus_p B$, where $p \in [0, 1]$. This approach, however, is not enough, as they can only model finite state systems and the probability transitions are static. Our semantics could solve these issues by defining a functional-message passing language [18] where the functional core would deal with the probabilistic aspects of the system and the message passing core would deal with the communication between agents.

Ehrhard et al. [5, 6] have defined a model of linear logic which can be used to interpret a higher-order probabilistic programming language. They have used the CBN translation of intuitionistic logic into linear logic $A \rightarrow B = !A \rightarrow B$ to give semantics to their language. One consequence of such a semantics is that it is not possible to store a sampled valued, meaning that every use of a variable corresponds to a different sampling from a random variable, making it unfit for writing many useful programs. In order to mitigate this problem the authors extend their language with a CBN "let" syntax which allows to record a sampled valued through out the execution of the program. In order to give semantics to this new language they must modify their semantical domain. Our approach differs from theirs because we do not make use of the exponential in order to interpret seemingly non-linear programs, meaning that we can use their original, simpler, model to interpret our language.

Dahlqvist and Kozen [2] have defined a category of partially ordered Banach spaces and shown that it is a model of intuitionistic linear logic. An important difference from their approach and the one mentioned above is that they embrace linearity as part of their syntax. As we argued in this paper, we believe that the syntactic restriction of linearity they have used is not adequate for the purposes of probabilistic programming. Therefore, by using the techniques presented in this paper we can give semantics to a more expressive version of their language.

Quasi Borel spaces [10] are a conservative extension of $\text{Meas}$ that are Cartesian closed category equipped with a commutative probability monad. The drawback of this model is that it is still not as well understood as their measure theoretic counterpart. A practical consequence of this fact is there are probability theorems that are useful when proving properties about stochastic processes and programs that may not be true in $\text{QBS}$.

Recently, Geoffroy [8] has made progress in connecting linear logic and quasi Borel Spaces by showing that a certain subcategory of the Eilenberg-Moore category for the Giry monad in $\text{QBS}$ is a model of classical linear logic.

This result has interesting implications to our work as we can instantiate our categorical semantics with their concrete model and have an MK language with higher-order features and allow for the sampling of higher-order functions in the LL language. It is an interesting research question to understand if the characterizations of soft-conditioning as linear duality could be used in this model as well.

The idea of having two distinct languages that are connected by a functorial layer is reminiscent of Call-by-Push-Value (CBPV), which has a type system for values and a type system for computations that are connected by an adjunction. There has been recent work on using a CBPV semantics to interpret probabilistic computation. In the first one, Ehrhard and Tasson [17] use the monoidal adjunction generated by the linear logic exponential ! in order to define a calculus that can interpret lazy and eager probabilistic computation. Goubault-Larrecq [9] has defined a CBPV domain semantics that mixes probability and non-determinism.

There are a couple of important differences between these calculi and our language. Namely, the adjunction manifests itself in CBPV by requiring a way of moving from the value computation and vice-versa. Our language only requires translations in one direction. As a consequence, our semantics require less structure than the CBPV one. Indeed, neither of the models presented in Section 5 are CBPV models. Furthermore, in CBPV calculi variables are interpreted the same in
both languages. In our combined language this is not the case.

8 Conclusion and Future Work

In this paper we have presented a new syntax that is better suited for our “sampling as a resource” interpretation of linear logic. We have also defined its categorical semantics, shown how the equational theories of the languages interact and presented concrete models for it that can handle discrete as well as continuous probabilities.

Due to the modularity of the semantics, it is fairly standard to provide extensions to the language by extending each of the languages separately. For instance, recent work by Stein and Staton [16] have shown how to extend Markov categories so that they may also handle probabilistic conditioning. On the linear logic side of things, Dahlqvist et al. have shown how Bayesian inference can be defined using dual vector spaces.

An interesting question is understanding whether we can adopt their models to our language and understand how they would interact. Models of linear logic such as PCoh have also been used to give semantics to more expressive features of programming languages, such as recursive types.

Finally, this new resource interpretation of linear logic has deep connections to probabilistic independence. Under this new interpretation, the Tensor rule of linear logic says that in order to produce an element of type $\Gamma_1 \otimes \Gamma_2$ you have to sample from distinct sources of randomness (the elements of the disjoint contexts $\Gamma_1$ and $\Gamma_2$), guaranteeing probabilistic independence. For future work we would to further explore the connections between our new resource interpretation of linear logic and probabilistic independence.

Going a different direction, it would also be interesting to extend this approach to other effects. Syntactically it is easy to generalize our syntax to general effects, the main difficulty arises when considering models for these new calculi. We conjecture that game semantics would provide a natural setting to construct said models.

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