RESTRICTION THEOREMS FOR HOMOGENEOUS BUNDLES

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Abstract. We prove that for an irreducible representation \( \tau : GL(n) \to GL(W) \), the associated homogeneous \( \mathbb{P}^n_k \)-vector bundle \( W_\tau \) is strongly semistable when restricted to any smooth quadric or to any smooth cubic in \( \mathbb{P}^n_k \), where \( k \) is an algebraically closed field of characteristic \( \neq 2, 3 \) respectively. In particular \( W_\tau \) is semistable when restricted to general hypersurfaces of degree \( \geq 2 \) and is strongly semistable when restricted to the \( k \)-generic hypersurface of degree \( \geq 2 \).

1. Introduction

In this paper we study the semistable restriction theorem for the homogeneous vector bundles on \( \mathbb{P}^n_k \) which come from irreducible \( GL(n) \)-representations.

In general suppose \( G \) is a reductive algebraic group over an algebraically closed field \( k \) and \( P \subset G \) is a parabolic group. Then there is an equivalence between the category of homogeneous \( G \)-bundles over \( G/P \) and the category of \( P \)-representations, where a \( P \)-representation \( \rho : P \to GL(V) \) on a \( k \)-vector space \( V \) induces a homogeneous \( G \)-bundle \( \nabla_\rho \) on \( G/P \) given by

\[
\nabla_\rho = \frac{G \times V}{P} = \frac{G \times V}{\{(g, v) \cong (gh, h^{-1}v) \mid g \in G, v \in V, h \in P\}}.
\]

Now for the rest of the paper we fix the following

**Notation 1.1.** The field \( k \) is an algebraically closed field and \( G = SL(n+1, k) \), and \( P \) is the maximal parabolic subgroup of \( G \) given by

\[
P = \left\{ \begin{bmatrix} g_{11} & * \\ 0 & A \end{bmatrix} \in SL(n+1), \text{ where } A \in GL(n) \right\}
\]

and \( G/P \cong \mathbb{P}^n_k \) is a canonical isomorphism.

Now, if \( \sigma : GL(n) \to GL(V) \) is an irreducible \( GL(n) \)-representation then it induces an irreducible \( P \)-representation \( \rho : P \to GL(V) \) given by

\[
(1.1) \quad \begin{bmatrix} g_{11} & * \\ 0 & A \end{bmatrix} \mapsto \sigma(A),
\]

which gives a \( G \)-homogeneous bundle on \( G/P = \mathbb{P}^n_k \). Conversely, any \( G \)-homogeneous bundle \( \nabla \), given by an irreducible \( P \)-representation \( \rho : P \to GL(V) \), is in fact induced by an irreducible \( GL(n) \)-representation (upto tensoring by \( \mathcal{O}_{\mathbb{P}^n_k}(r) \), for some \( r \)).

In this paper we prove the following

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Theorem 1.2. Let \( \tau : GL(n) \to GL(W) \) be an irreducible \( GL(n) \)-representation, where \( W \) is a \( k \)-vector space. Let \( \mathbb{W}_\tau \) be the associated \( G \)-homogeneous bundle on \( G/P = \mathbb{P}_k^n \). Let

1. \( X = \) smooth quadric, if \( \text{char } k \neq 2 \), or
2. \( X = \) smooth cubic, if \( \text{char } k \neq 3 \).

Then the bundle \( \mathbb{W}_\tau \mid_X \) is strongly semistable.

We note that Theorem 1.2 implies \( \mathbb{W}_\tau \) itself is semistable on \( \mathbb{P}_k^n \). However this result, in much more general form, has been proved in [R], [U], [MR1] and [B].

Theorem 1.2 implies (see Corollary 5.4) that, provided \( \text{char } k \neq 2, 3, \) the bundle \( \mathbb{W}_\tau \mid_H \) is semistable, for a general hypersurface \( H \) of degree \( \geq 2 \) in \( \mathbb{P}_k^n \), and \( \mathbb{W}_\tau \mid_{H_0} \) is strongly semistable for generic hypersurface \( H_0 \) of degree \( d \geq 2 \). This is equivalent to the statement that, given \( s \geq 0 \), the \( s^{th} \) Frobenius pull back \( F^s \mathbb{W}_\tau \mid_H \) is semistable for a general hypersurface \( H \) of degree \( d \geq 2 \) in \( \mathbb{P}_k^n \). Moreover when the bundle \( \mathbb{W}_\tau \) comes from the standard representation, i.e., \( \mathbb{W}_\tau \) is the tangent bundle (up to a twist by a line bundle) of \( \mathbb{P}_k^n \), where \( n \geq 4 \), then we can prove a stronger statement, by replacing the word ‘semistable’ by ‘stable’ everywhere in Theorem 1.2 and Corollary 5.4.

In this context we recall that, Mehta-Ramanathan [MR2] have proved that if \( E \) is a semistable sheaf on a smooth projective variety (over a field of arbitrary characteristic) then \( E \) restricted to a general hypersurface of degree \( a \) (where \( a \) is any sufficiently large integer) is semistable. On the other hand, Flenner [F] proved this assertion, where the degree \( a \) of the hypersurface depends only on the rank of \( E \) and degree of the variety \( X \), provided the characteristic is 0.

The paper is organised as follows: In Section 2, we recall some general facts about smooth quadrics. Then we discuss the vector bundle \( \mathbb{V}_\sigma = T_{\mathbb{P}_k^n}(-1) \) associated to the standard representation \( \sigma : GL(n) \to GL(V) \) and its restriction to smooth quadrics. In particular, for a smooth quadric \( Q \subset \mathbb{P}_k^n \), we show that \( \mathbb{V}_\sigma \mid_Q \) has a unique \( SO(n+1) \)-homogeneous proper subbundle, if \( n \geq 4 \), (see remark 5.4 for details).

In Section 3, we prove that if \( \text{char } k \neq 2 \) then \( T_{\mathbb{P}_k^n} \mid_Q \) is strongly stable if \( n \geq 3 \), and is strongly semistable if \( n = 2 \). Moreover the tangent bundle \( T_{\mathbb{P}_k^n} \) of \( Q \) is semistable and is of positive slope.

In Section 4 we prove that, if \( \text{char } k \neq 3 \) and \( X \subset \mathbb{P}_k^n \) is an arbitrary smooth cubic hypersurface then \( T_{\mathbb{P}_k^n} \mid_X \) is strongly stable if \( n \geq 4 \) and strongly semistable if \( n = 2 \) or \( n = 3 \). Moreover the tangent bundle \( T_X \) of \( X \) is either stable if \( n \neq 3 \), or \( \mu_{\min}(T_X) \geq 0 \) if \( n = 3 \). In fact, we show that the statement given in [PW], to prove stability of \( T_X \), for a smooth hypersurface of degree \( d \geq 3 \), \( n \geq 4 \) and \( k = \mathbb{C} \), can be modified so as to work over any algebraically closed field of characteristic coprime to \( d \) (this hypothesis is needed so that the cup product with \( c_1(O_{\mathbb{P}_k^n}(d)) \) is an injective map).

Finally in Section 5, we show (see Theorem 1.2) that, if \( \mathbb{V}_\sigma \mid_X \) is semistable and \( \mu_{\min}(\mathbb{V}_\sigma \mid_X) \geq 0 \), where \( X \) is a smooth hypersurface in \( \mathbb{P}_k^n \) then the bundle \( \mathbb{W}_\tau \mid_X \) is strongly semistable for any irreducible representation \( \tau : GL(n) \to GL(W) \).
2. Some general facts about quadrics

2.1. Embedding of quadrics in $\mathbb{P}^n_k$. Let $V$ be a vector-space of dimension $n + 1$ over $k$ (characteristic $k \neq 2$). Let us choose a basis $\{e_1, \ldots, e_{n+1}\}$ of $V$. Represent a point $v \in V$ by

\[
v = (x_1, \ldots, x_{n/2}, z, y_1, \ldots, y_{n/2}), \quad \text{if } n \text{ is even},
\]
\[
v = (x_1, \ldots, x_{(n+1)/2}, y_1, \ldots, y_{(n+1)/2}), \quad \text{if } n \text{ is odd},
\]

with respect to the basis $\{e_1, \ldots, e_{n+1}\}$. Without loss of generality, one can assume that any fixed smooth quadric $Q \subset \mathbb{P}^n_k$ is given by the quadratic form

\[
\tilde{Q}(v) = z^2 + 2(x_1 y_{n/2} + \cdots + x_{n/2} y_1), \quad \text{if } n \text{ is even and}
\]
\[
\tilde{Q}(v) = x_1 y_{(n+1)/2} + \cdots + x_{(n+1)/2} y_1), \quad \text{if } n \text{ is odd}.
\]

Let

\[
SO(n + 1) = \{ A \in SL(n + 1) \mid \tilde{Q}(Av) = \tilde{Q}(v) \text{ for all } v \in V \}
\]

where

\[
J = \begin{bmatrix}
0 & \cdots & 1 \\
0 & \ddots & 0 \\
1 & \cdots & 0
\end{bmatrix} \in GL(n + 1).
\]

Notation 2.1. Let $P_1 = P \cap SO(n + 1)$ denote the maximal parabolic group in $SO(n + 1)$ such that

\[
\left\{ \begin{bmatrix}
a_{11} & 0 & 0 \\
0 & A & 0 \\
0 & 0 & a_{11}^{-1}
\end{bmatrix}, \quad \text{where } A \in SO(n - 1), a_{11} \in k^* \right\} \subseteq P_1, \quad \text{and}
\]

\[
P_1 \subseteq \left\{ \begin{bmatrix}
a_{11} & * & * \\
0 & A & * \\
0 & 0 & a_{11}^{-1}
\end{bmatrix}, \quad \text{where } A \in SO(n - 1), a_{11} \in k^* \right\}.
\]

Then we have the canonical identification

\[
\mathbb{P}^n_k \cong SL(n + 1)/P \quad \uparrow \quad \tilde{Q} \cong SO(n + 1)/P_1.
\]
2.2. Standard representation of $GL(n)$. Consider the canonical short exact sequence of sheaves of $\mathcal{O}_{\mathbb{P}^n_k}$-modules

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n_k}(1) \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) \otimes \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(1) \longrightarrow 0.$$  

The dual sequence is

$$(2.1) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1) \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) \otimes \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{T}_{\mathbb{P}^n_k}(-1) \longrightarrow 0,$$

where $\mathcal{T}_{\mathbb{P}^n_k}$ is the tangent sheaf of $\mathbb{P}^n_k$. Now this sequence is also a short exact sequence of $G$-homogeneous bundles on $G/P = \mathbb{P}^n_k$ (see [1.1]). Hence there exists a corresponding short exact sequence of $P$-modules

$$0 \longrightarrow V_2 \xrightarrow{f} V_1 \xrightarrow{\eta} V \longrightarrow 0,$$

where the $P$-module structure is given as follows.

Let $V_1$, $V$ and $V_2$ be $n+1$, $n$ and 1 dimensional $k$-vector spaces respectively, with fixed bases. Let $f : (c) \mapsto (c,0,\ldots,0)$ and let $\eta : (a_1,\ldots,a_{n+1}) \mapsto (0,a_2,\ldots,a_{n+1})$.

Now representing the elements of the vector spaces as column vectors and expressing any $g \in P$ as

$$g = \begin{bmatrix} g_{11} & \ast \\ 0 & B \end{bmatrix}, \quad \text{where } B \in GL(n),$$

we define the representations as follows:

The representation $\rho_1 : P \longrightarrow GL(V_1)$ is given by

$$\rho_1(g) \begin{bmatrix} a_1 \\ \vdots \\ a_{n+1} \end{bmatrix} = [g] \begin{bmatrix} a_1 \\ \vdots \\ a_{n+1} \end{bmatrix}.$$  

The representation $\rho_2 : P \longrightarrow GL(V_2)$ is given by

$$\rho_2(g)[c] = [g_{11}][c]$$

and the representation $\sigma : P \longrightarrow GL(V)$ is given by

$$\sigma(g) \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = [B] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

which is the standard representation $\sigma : GL(n) \longrightarrow GL(V)$. Thus

$$\mathcal{T}_{\mathbb{P}^n_k}(-1) = \mathbb{V}_\sigma$$

is the homogeneous bundle on $G/P$ associated to the standard representation $\sigma$. One can easily check that the maps $f$ and $\eta$ are compatible with the $P$-module structure of $V_2$, $V_1$ and $V$.

We write the sequence (2.1) as

$$0 \longrightarrow \mathbb{V}_{\rho_2} \longrightarrow \mathbb{V}_{\rho_1} \longrightarrow \mathbb{V}_\sigma \longrightarrow 0.$$
2.3. **Restriction of \( V_\sigma \) to the quadric** \( Q \subset P^n_k \). The bundle \( V_\sigma = T_{P^n_k}(-1) \), when restricted to \( Q \), fits into an extension

\[
0 \longrightarrow T_Q(-1) \longrightarrow T_{P^n_k}(-1) \otimes_{\mathcal{O}_{P^n_k}} \mathcal{O}_Q \longrightarrow N_{Q/P^n_k}(-1) \longrightarrow 0,
\]

where \( T_Q \) and \( N_{Q/P^n_k} \) denote the tangent sheaf and the normal sheaf of \( Q \subset P^n_k \). Note that this is also a short exact sequence of \( SO(n + 1) \)-homogeneous bundles on \( Q = SO(n + 1)/P_1 \) (see 2.1), hence there exists the corresponding short exact sequence of \( P_1 \)-modules

\[
0 \longrightarrow U_1 \xrightarrow{\tilde{f}} V \xrightarrow{\tilde{g}} U_3 \longrightarrow 0,
\]

where \( U_1 \) and \( U_3 \) are \( k \)-vector spaces of dimensions \( n - 1 \) and 1 respectively. We define

\[\tilde{f}: (b_1, \ldots, b_{n-1}) \rightarrow (b_1, \ldots, b_{n-1}, 0)\]

and

\[\tilde{g}: (a_1, \ldots, a_n) \rightarrow (a_n)\]

Now any \( g \in P_1 \) can be written as

\[
g = \begin{bmatrix}
a_{11} & * & * \\
0 & A & * \\
0 & 0 & a_{11}^{-1}
\end{bmatrix}
\]

where \( A \in SO(n - 1) \) and \( a_{11} \in k \setminus \{0\} \). The representation \( \tilde{\sigma}: P_1 \rightarrow GL(V) \) is given by

\[
(2.4) \quad \tilde{\sigma}(g) \begin{bmatrix}
b_1 \\
\vdots \\
b_n
\end{bmatrix} = \begin{bmatrix}
A & * \\
0 & a_{11}^{-1}
\end{bmatrix} \begin{bmatrix}
b_1 \\
\vdots \\
b_n
\end{bmatrix}
\]

The representation \( \rho_3: P_1 \rightarrow GL(U_3) \) is given by

\[
\rho_3(g)[x] = [a_{11}^{-1}][x]
\]

and the representation \( \sigma_1: P_1 \rightarrow GL(U_1) \) is given by

\[
(2.5) \quad \sigma_1(g) \begin{bmatrix}
c_1 \\
\vdots \\
c_{n-1}
\end{bmatrix} = [A] \begin{bmatrix}
c_1 \\
\vdots \\
c_{n-1}
\end{bmatrix}
\]

We write the sequence (2.3) as

\[
0 \longrightarrow \mathbb{U}_1 \longrightarrow \mathbb{V}_\sigma \longrightarrow \mathbb{U}_3 \longrightarrow 0.
\]

**Remark 2.2.** Note that \( \sigma_1: P_1 \rightarrow GL(U_1) \) factors through the standard representation \( \tilde{\sigma}_1: SO(n - 1) \rightarrow GL(U_1) \) and hence is irreducible, for \( n \neq 3 \). This implies that the tangent bundle \( T_Q \) is semistable. For \( n = 3 \), the representation \( \sigma_1 \) is not irreducible and \( U_1 \) is a direct sum of two \( P_1 \)-submodules, namely \( k(1,0,0) \subset V \) and \( k(0,1,0) \subset V \) respectively. In fact one can check easily that the only \( P_1 \)-submodules of \( V \) are given by \( k(1,0,0) \), \( k(0,1,0) \), \( U_1 \) and \( V \) itself. In particular, all the homogeneous subbundles of \( \mathbb{V}_\sigma \) are given by these four \( P_1 \)-submodules.
A smooth quadric $Q \subset P^3_k$ is isomorphic to $P^1_k \times P^1_k$ and therefore the tangent bundle $T_Q$ is a direct sum of line bundles of same degree. Hence the tangent bundle $T_Q$ is always a semistable vector bundle for a smooth quadric $Q$. Moreover, by (2.2), one can compute that $\mu(T_Q) > 0$, if $n \geq 2$.

3. Stablity of $T_{P^3_k}$ | smooth quadric

Proposition 3.1. Let $\sigma : GL(n) \rightarrow GL(V)$ be the standard representation (i.e., $\sigma(g) = g$). Let $\mathcal{V}_\sigma$ be the associated $G$-homogeneous bundle on $G/P = P^n_k$. Then for characteristic $k \neq 2$, the restriction of the bundle $\mathcal{V}_\sigma = T_{P^n_k}(-1)$ to any smooth quadric $Q \subset P^n_k$ is semistable.

Remark This result in characteristic 0 is proved by [F]. In fact later we prove a stronger version of the above proposition (see Proposition 3.6).

For the proof of the proposition we need the following two lemmas.

Lemma 3.2. Let $U_1$ and $V_\tilde{g}$ denote the $SO(n+1)$-homogeneous bundles, associated to the $\sigma_1$ and $\tilde{\sigma}$ respectively (as given in Section 2), on $Q = SO(n+1)/P_1$. Then

$$\mu(U_1) < \mu(V_\tilde{g})$$

Proof. We are given that

$$\mathcal{V}_\tilde{g} = \mathcal{V}_\sigma |_Q = T_{P^n_k}(-1) \otimes \mathcal{O}_{P^n_k} \mathcal{O}_Q$$

and $U_1 = T_Q(-1)$. Now

$$\text{deg } T_{P^n_k}(-1) \otimes \mathcal{O}_{P^n_k} \mathcal{O}_Q = 2 \text{ deg } T_{P^n_k}(-1) = 2(\text{ deg } H^0(P^n_k, \mathcal{O}_{P^n_k}) \otimes \mathcal{O}_{P^n_k} - \text{ deg } \mathcal{O}_{P^n_k}(-1)) = 2$$

where the second last equality follows from (2.1). As

$$\mathcal{N}_{Q/P^n_k} \simeq (\mathcal{I}/\mathcal{I}^2)^{\mathcal{V}} = \mathcal{O}_{P^n_k}(-2)^{\mathcal{V}} |_Q = \mathcal{O}_{P^n_k}(2) |_Q,$$

where $\mathcal{I}$ is the ideal sheaf of $Q \subset P^n_k$, we have

$$\text{deg } \mathcal{N}_{Q/P^n_k}(-1) = \text{ deg } \mathcal{O}_{P^n_k}(1) |_Q = 2.$$

Therefore

$$\text{deg } U_1 = \text{deg } T_Q(-1) = \text{deg } T_{P^n_k}(-1) - \text{deg } \mathcal{N}_{Q/P^n_k}(-1) = 0.$$

Hence $\mu(U_1) = 0 < \mu(V_\tilde{g}) = 2/n$. This proves the lemma. \hfill \Box

Lemma 3.3. The sequence (2.3)

$$0 \rightarrow U_1 \xrightarrow{f} V \xrightarrow{\tilde{g}} U_3 \rightarrow 0,$$

defined as above, of $P_1$-representations does not split.

Proof. It is enough to prove that the short exact sequence (2.2) does not split as sheaves of $\mathcal{O}_Q$-modules. Suppose it does, then so does

$$0 \rightarrow T_Q(-2) \rightarrow T_{P^n_k}(-2) \otimes \mathcal{O}_{P^n_k} \mathcal{O}_Q \rightarrow \mathcal{N}_{Q/P^n_k}(-2) \rightarrow 0,$$
where we know that \( N_{Q/P^n_k(-2)} \simeq \mathcal{O}_Q \). This implies that \( H^0(Q, \mathcal{T}_{P^n_k}(-2) \otimes \mathcal{O}_{P^n_k} \mathcal{O}_Q) \neq 0 \). However we have

\[
0 \rightarrow \mathcal{T}_{P^n_k}(-4) \rightarrow \mathcal{T}_{P^n_k}(-2) \rightarrow \mathcal{T}_{P^n_k}(-2) \otimes \mathcal{O}_{P^n_k} \mathcal{O}_Q \rightarrow 0,
\]

where the first map is multiplication by the quadratic equation defining \( Q \subset P^n_k \).

If we assume the following

Claim. \( H^0(P^n_k, \mathcal{T}_{P^n_k}(-2)) = 0 = H^1(P^n_k, \mathcal{T}_{P^n_k}(-4)) \).

Then (3.1) implies that \( H^0(Q, \mathcal{T}_{P^n_k}(-2) \otimes \mathcal{O}_{P^n_k} \mathcal{O}_Q) = 0 \), which contradicts the hypothesis. Now we give the Proof of the claim. Consider the following short exact sequence (which is derived from (2.1))

\[
0 \rightarrow \mathcal{O}_{P^n_k}(-2) \rightarrow \mathcal{O}_{P^n_k}(-1)^{n+1} \rightarrow \mathcal{T}_{P^n_k}(-2) \rightarrow 0.
\]

As \( n \geq 2 \), we have \( H^1(P^n_k, \mathcal{O}_{P^n_k}(-2)) = H^0(P^n_k, \mathcal{O}_{P^n_k}(-1)) = 0 \), which implies \( H^0(P^n_k, \mathcal{T}_{P^n_k}(-2)) = 0 \). The above sequence also gives the long exact sequence

\[
\oplus^{n+1}H^1(P^n_k, \mathcal{O}_{P^n_k}(-3)) \rightarrow H^1(P^n_k, \mathcal{T}_{P^n_k}(-4)) \rightarrow H^2(P^n_k, \mathcal{O}_{P^n_k}(-4)) \rightarrow H^3(P^n_k, \mathcal{O}_{P^n_k}(-3)) \rightarrow \]

(1) If \( n \geq 3 \) then \( H^1(P^n_k, \mathcal{O}_{P^n_k}(-3)) = H^2(P^n_k, \mathcal{O}_{P^n_k}(-4)) = 0 \), which implies \( H^1(P^n_k, \mathcal{T}_{P^n_k}(-4)) = 0 \).

(2) If \( n = 2 \) then \( H^1(P^n_k, \mathcal{O}_{P^n_k}(-3)) = 0 \). Moreover the map

\[
H^2(P^n_k, \mathcal{O}_{P^n_k}(-4)) \rightarrow \oplus^3 H^2(P^n_k, \mathcal{O}_{P^n_k}(-3))
\]

is dual to

\[
\oplus^3 H^0(P^n_k, \mathcal{O}_{P^n_k}^2) \rightarrow H^0(P^n_k, \mathcal{O}_{P^n_k}^2(1))
\]

which is an isomorphism as it comes from the evaluation map

\[
H^0(P^n_k, \mathcal{O}_{P^n_k}^2(1)) \otimes \mathcal{O}_{P^n_k} \rightarrow \mathcal{O}_{P^n_k}(1).
\]

This implies \( H^1(P^n_k, \mathcal{T}_{P^n_k}(-4)) = 0 \).

This proves the claim and hence the lemma.

Proof of Proposition 3.1 Now suppose the \( SO(n+1) \)-homogeneous bundle \( V_\sigma \) on \( Q \) is not semistable. Then it has a Harder-Narasimhan filtration

\[
0 \subset V_1 \subset \cdots \subset V_k = V_\sigma
\]

where \( \mu(V_1) > \mu(V_\sigma) \). Now the uniqueness of the HN filtration implies that \( V_1 \) is a \( SO(n+1) \)-homogeneous subbundle of \( V_\sigma \). Therefore there exists a corresponding \( P_1 \)-representation, say, \( \rho_4 : P_1 \rightarrow GL(\tilde{V}_1) \) and an inclusion of \( P_1 \)-modules \( \tilde{V}_1 \hookrightarrow V \) corresponding to the inclusion \( V_1 \hookrightarrow V_\sigma \).

Claim. \( U_1 \subset \tilde{V}_1 \), where \( \sigma_1 : P_1 \rightarrow GL(U_1) \) is the \( P_1 \)-representation as defined in (2.5).

We assume the claim for the moment. Since \( V/U_1 \) is an irreducible \( P_1 \)-module, we have either \( \tilde{V}_1 = U_1 \) or \( \tilde{V}_1 = V \), i.e., \( V_1 = U_1 \) or \( V_1 = V_\sigma \). By Lemma 3.2
in both the cases \( \mu(\mathbb{V}_1) \leq \mu(\mathbb{V}_\sigma) \), which contradicts the fact that \( \mathbb{V}_1 \) is a term of the HN filtration of \( \mathbb{V}_\sigma \). Hence we conclude that the \( \mathbb{V}_\sigma \) is semistable.

Now we give

**Proof of the claim.** Suppose \( \tilde{V}_1 \cap U_1 = 0 \). Then the composition map

\[
\tilde{V}_1 = \frac{\tilde{V}_1}{\tilde{V}_1 \cap U_1} \hookrightarrow \frac{V}{U_1} \hookrightarrow U_3,
\]

gives an isomorphism \( \tilde{V}_1 \to U_3 \), which implies that \((2.3)\) splits as a sequence of \( P_1 \)-modules; by Lemma \( \text{[3.3]} \) this is a contradiction.

Hence \( \tilde{V}_1 \cap U_1 \neq 0 \). If \( n \neq 3 \) then \( U_1 \) is an irreducible \( P_1 \)-module (see Remark \( \text{[2.2]} \)), which implies that \( U_1 \subset \tilde{V}_1 \). Let \( n = 3 \) and \( U_1 \not\subset \tilde{V}_1 \). Then Remark \( \text{[2.2]} \) implies that \( V_1 \subset U_1 \) as a \( P_1 \)-submodule of rank 1 and therefore \( \mu(\mathbb{V}_1) = \mu(\mathbb{U}_1) < \mu(\mathbb{V}_\sigma) \), which is a contradiction. Therefore \( U_1 \subset \tilde{V}_1 \). Hence the claim. This proves the proposition. \( \square \)

**Remark 3.4.** The argument in the above proposition implies that the only \( SO(n + 1) \)-homogeneous subbundle of \( \mathcal{T}_{\mathbb{P}^n_k}(-1) \mid_Q = \mathbb{V}_\sigma \) is either \( \mathbb{U}_1 \) or \( \mathbb{V}_\sigma \) itself, if \( n \neq 3 \). If \( n = 3 \) then the homogeneous subbundle of \( \mathbb{V}_\sigma \) is one of the two homogeneous line subbundles of \( \mathbb{U}_1 \) (as given in Remark \( \text{[2.2]} \)) or \( \mathbb{U}_1 \) or \( \mathbb{V}_\sigma \) itself.

**Remark 3.5.** For \( n = 3 \), we can give another proof of the stability of \( \mathbb{V}_\sigma \) by reversing the role of cubic and quadric in the proof of Lemma \( \text{[1.5]} \).

Now we can strengthen Proposition \( \text{[3.1]} \) as follows.

**Proposition 3.6.** With the notations as in Proposition \( \text{[3.1]} \), for \( n \geq 3 \), the restriction of the \( \mathbb{P}^n_k \)-bundle, \( \mathbb{V}_\sigma \) to any smooth quadric \( Q \subset \mathbb{P}^n_k \) is stable. If \( n = 2 \) then \( \mathbb{V}_\sigma \mid_Q \) is a direct sum of two copies of a line bundle on \( Q \).

Before coming to the proof of this proposition we need the following lemma (which, perhaps, is already known to the experts). For this we recall some general facts. Let \( H \) be a reductive algebraic group over \( k \) and \( P' \subset H \) be a parabolic group. Let \( \mathbb{V}_\rho \) be a homogeneous \( H \)-bundle on \( X = H/P' \) induced by a \( P' \)-representation \( \rho : P' \to GL(V) \) on a \( k \)-vector space \( V \). Let the \( H \) action on \( \mathbb{V}_\rho \) be given by the map \( L : H \times \mathbb{V}_\rho \to \mathbb{V}_\rho \), where we write \( L(g, v) = L_g(v) \), for \( g \in H \) and \( v \in \mathbb{V}_\rho \). This induces the canonical \( H \)-action on the dual of \( \mathbb{V}_\rho \), which makes \( \mathbb{V}_\rho ^\vee \) and \( \mathbb{V}_\rho \otimes \mathbb{V}_\rho ^\vee \) into \( H \)-homogeneous bundles such that the map

\[
\mathcal{E}nd_{\mathcal{O}_X}(\mathbb{V}_\rho) \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathbb{V}_\rho) \to \mathcal{E}nd_{\mathcal{O}_X}(\mathbb{V}_\rho),
\]

\[
(\mathbb{V}_\rho \otimes_{\mathcal{O}_X} \mathbb{V}_\rho ^\vee) \otimes_{\mathcal{O}_X} (\mathbb{V}_\rho \otimes_{\mathcal{O}_X} \mathbb{V}_\rho ^\vee) \to (\mathbb{V}_\rho \otimes_{\mathcal{O}_X} \mathbb{V}_\rho ^\vee),
\]

given by

\[
(v_1 \otimes \phi_1) \otimes (v_2 \otimes \phi_2) \mapsto \phi_1(v_1)(v_2) \otimes \phi_2.
\]

is \( H \)-equivariant. Hence \( \text{End}_{\mathcal{O}_X}(\mathbb{V}_\rho) = H^0(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathbb{V}_\rho)) \) is a \( H \)-module such that \( H \) respects the algebra structure on it. This gives the homomorphism

\[
\tilde{L} : H \to \text{Aut}(\text{End}_{\mathcal{O}_X}(\mathbb{V}_\rho)),
\]
given by \( \bar{L}(g)(\phi) = L_g \cdot \phi \cdot L_g^{-1} \), where

\[
\text{Aut}(\text{End}_{\mathcal{O}_X}(V)) = \text{the set of ring automorphism on End}_{\mathcal{O}_X}(V).
\]

**Lemma 3.7.** With the above notations, assume that the map \( \bar{L} \), defined as above, is the trivial map. Then any subbundle of \( V \) on \( X \), which is also a direct summand of \( V \), is \( H \)-homogeneous vector subbundle.

**Proof.** Now let \( V = U_1 \oplus U_2 \) be the direct sum of subbundles \( U_1 \) and \( U_2 \). Let \( \phi \in \text{End}_{\mathcal{O}_X}(V) \) be given by

\[
\phi|_{U_1} = \text{Id} \text{ and } \phi|_{U_2} = 0.
\]

Now, since \( \bar{L} \) is trivial, we have

\[
\bar{L}(g)(\phi) = \phi \text{ for all } g \in G.
\]

i.e.,

\[
(3.2) \quad L_g \cdot \phi \cdot L_g^{-1} = \phi.
\]

Let \( (V)_x \) be the fiber of \( V \) over \( x \in X \). Then, by (3.2), we have the following commutative diagram

\[
\begin{array}{ccc}
(V)_x & \xrightarrow{L_g^{-1}} & (V)_{g^{-1}x} \\
\downarrow \phi & & \downarrow \phi_{g^{-1}} \\
(V)_x & \xrightarrow{L_g^{-1}} & (V)_{g^{-1}x},
\end{array}
\]

for each \( x \in X \). This may be written as

\[
\begin{array}{ccc}
U_1^x \oplus U_2^x & \xrightarrow{L_g^{-1}} & U_1^{g^{-1}x} \oplus U_2^{g^{-1}x} \\
\downarrow \phi_x & & \downarrow \phi_{g^{-1}x} \\
U_1^x \oplus U_2^x & \xrightarrow{L_g^{-1}} & U_1^{g^{-1}x} \oplus U_2^{g^{-1}x}.
\end{array}
\]

Now

\[
U_2^x \subseteq \ker \phi_x \implies U_2^x \subseteq \ker(L_g \cdot \phi_{g^{-1}x} \cdot L_g^{-1}) = \ker(\phi_{g^{-1}x} \cdot L_g^{-1}).
\]

This implies

\[
L_g^{-1}(U_2^x) \subseteq \ker \phi_{g^{-1}x} = U_2^{g^{-1}x}.
\]

Hence \( L_g^{-1}(U_2^x) \subseteq U_2^x \), i.e., \( U_2^x \) is a \( H \)-homogeneous subbundle of \( V \). This proves the lemma. \( \square \)

**Proof of Proposition 3.6** By Proposition 3.1 for a quadric \( Q \subseteq \mathbf{P}_k^n \), the bundle \( V_\sigma|_Q \simeq V_\tilde{\sigma} \) is semistable. Hence there exists a nontrivial socle \( F \subseteq V_\tilde{\sigma} \) such that \( \mu(F) = \mu(V_\tilde{\sigma}) \) and \( F \) is the maximal polystable subsheaf. Hence, by the uniqueness of maximal polystable sheaf, it follows that it is an \( SO(n+1) \)-homogeneous subbundle of \( V_\tilde{\sigma} \). Therefore, by Remark 3.4 either \( F = U_1 \) or \( F = V_\tilde{\sigma} \). But \( \mu(F) = \mu(V_\tilde{\sigma}) \geq \mu(U_1) \), which implies \( F = V_\tilde{\sigma} \). Therefore we can write

\[
V_\tilde{\sigma} = F_1 \oplus F_2 \oplus \cdots \oplus F_r,
\]

where \( F_i \) is a direct sum of isomorphic stable sheaves, and the stable summands of distinct \( F_i \) are non-isomorphic. But each \( F_i \) is an \( SO(n+1) \)-homogeneous
subbundle of \( V_\tilde{a} \) and is of the same slope as of \( V_\tilde{a} \). Hence \( r = 1 \) and \( V_\tilde{a} \) is a direct sum of isomorphic stable sub-bundles, \( i.e. \)
\[
V_\tilde{a} = \oplus' U, \quad \text{where } \mu(U) = \mu(V_\tilde{a}).
\]

By Equation (2.1), we have
\[
2 = \deg V_\tilde{a} = t \cdot \deg U.
\]
Hence \( t = 1 \) or \( t = 2 \).

Suppose \( n = 2 \). Then \( Q \cong P^1_k \), hence \( V_\tilde{a} \) being rank 2 vector bundle on \( Q \) splits as a direct sum of two line bundles. Therefore in this case \( t = 2 \).

Suppose \( n \geq 3 \). If \( t = 1 \) then we are done. Let \( t = 2 \). Let
\[
\tilde{L} : SO(n + 1) \longrightarrow \text{Aut}(H^0(Q, \text{End}(V_\tilde{a})))
\]
be the induced map. We are given that \( V_\tilde{a} = U \oplus U \), where \( U \) is a stable bundle on \( Q \). But \( \text{End}_Q(U) \) consists of scalars, and so
\[
\text{End}_Q(V_\tilde{a}) \cong M(2, k) \text{ is the algebra of } 2 \times 2 \text{ matrices.}
\]
Hence \( \text{Aut}(H^0(Q, \text{End}(V_\tilde{a}))) \cong SO(3) \). So, we have the map
\[
\tilde{L} : SO(n + 1) \longrightarrow SO(3).
\]
But \( SO(n + 1) \) is an almost simple group, which implies, that

either \( \dim \text{Im } \tilde{L} = 0 \) or \( \dim SO(n + 1) = \dim \text{Im } \tilde{L} \leq \dim SO(3) \).

Hence, for \( n \geq 3 \), \( \dim \text{Im } \tilde{L} = 0 \), which means \( \tilde{L} \) is trivial. Therefore, by Lemma 3.7 the bundle \( U \) is homogeneous.

However, by Remark 3.4 and Lemma 3.2 the only \( G \)-homogeneous subbundle of \( V_\tilde{a} \), of the same slope as \( V_\tilde{a} \), is \( V_\tilde{a} \) itself. Hence we conclude that \( V_\tilde{a} = U \) is stable, if \( n \geq 3 \). This proves the proposition.

\[ \square \]

**Corollary 3.8.** If \( Q \subset P^n_k \) is a smooth quadric such that \( k \) is an algebraically closed field of char \( \neq 2 \) then

1. \( \Omega_{P^n_k} |_Q \) is strongly semistable if \( n = 2 \) and
2. \( \Omega_{P^n_k} |_Q \) is strongly stable if \( n \geq 3 \).

**Proof.** If \( n = 2 \) then the corollary follows from Proposition 3.6. Suppose \( n \geq 3 \). Then, by Proposition 3.6, the bundle \( \Omega_{P^n_k} |_Q \) is stable. Moreover, by Remark 2.2 the tangent bundle \( T_Q \) of \( Q \) is semistable and \( \mu(T_Q) > 0 \). Hence, by Theorem 2.1 of [MR1], the bundle \( \Omega_{P^n_k} |_Q \) is strongly stable. This proves the corollary.

\[ \square \]

### 4. Stability of \( T_{P^n_k}|_{\text{smooth cubic}} \)

We recall the Bott vanishing theorem for \((P^n_k, \Omega_{P^n_k}(t))\), where \( k \) an arbitrary field of arbitrary characteristic.

\[
\begin{align*}
H^0(P^n_k, \Omega_{P^n_k}^q(t)) & \neq 0, \text{ if } 0 \leq q \leq n, \text{ and } t > q \\
H^n(P^n_k, \Omega_{P^n_k}^q(t)) & \neq 0 \text{ if } 0 \leq q \leq n, \text{ and } t < q - n \\
H^p(P^n_k, \Omega_{P^n_k}^q) & = k, \text{ if } 0 \leq p \leq n \\
H^p(P^n_k, \Omega_{P^n_k}^q(t)) & = 0 \text{ otherwise.}
\end{align*}
\]
Now throughout this section we fix a smooth hypersurface $X$ of degree $d \geq 3$ in $Y = \mathbb{P}^n$, $(d, \text{char } k) = 1$. We have the following short exact sequences

(4.1) \[ 0 \rightarrow \Omega_X^p(t) \rightarrow \Omega_X^p(t + d) \rightarrow \Omega_X^p(t + d) |_X \rightarrow 0 \]

(4.2) \[ 0 \rightarrow \Omega_X^q(t) \rightarrow \Omega_Y^{q+1}(t + d) |_X \rightarrow \Omega_Y^{q+1}(t + d) \rightarrow 0 \]

1. If $p + q < \dim X$ and $p, q \geq 0$ then from Bott vanishing and the short exact sequences (4.1) and (4.2), it follows that $H^p(X, \Omega_X^q(t)) = 0$ for $t < 0$.

2. If $p + q < \dim X$ then

\[ H^p(X, \Omega_X^q) \simeq H^p(Y, \Omega_Y^q). \]

3. Consider the following commutative diagram of natural maps

\[
\begin{array}{ccc}
H^p(Y, \Omega_Y^q) & \longrightarrow & H^{p+1}(Y, \Omega_Y^{q+1}) \\
\downarrow & & \downarrow \\
H^p(X, \Omega_X^q) & \longrightarrow & H^{p+1}(X, \Omega_X^{q+1}),
\end{array}
\]

where the horizontal maps are given by the cup product with $c_1(\mathcal{O}_Y(d)) = d \cdot c_1(\mathcal{O}_Y(1))$ and $c_1(\mathcal{O}_X(d))$ respectively. Since $(\text{char } k, d) = 1$, the map $H^p(Y, \Omega_Y^q) \longrightarrow H^{p+1}(Y, \Omega_Y^{q+1})$ is an isomorphism for every $p, q$ with $p, q \geq 0$ and $p + 1 \leq \dim Y$. In particular, the induced composite map

(4.3) \[ \eta_{p,q} : H^p(X, \Omega_X^q) \longrightarrow H^{p+1}(Y, \Omega_Y^{q+1}) \]

is an isomorphism if $p, q \geq 0$ and $p + q < \dim X$.

We prove the following Lemma 4.1 and Corollary 4.2 along the same line of arguments, as given for the case $k = \mathbb{C}$, in [PW].

**Lemma 4.1.** Let $X \subseteq \mathbb{P}^n_k$ be a hypersurface of deg $d \geq 3$. Let $n \geq 2$ and $(\text{char } k, d) = 1$. If $p, q \geq 0$ and $p + q < \dim X$ and $t \leq q(n+1-d)/(n-1)$ then

1. $H^p(X, \Omega_X^q(t)) = 0$, if $t \neq 0$ and
2. $H^p(X, \Omega_X^q) \simeq H^p(Y, \Omega_Y^q)$.

**Proof.** As discussed above, (a) for $t < 0$, the statement (1) holds, i.e., for $t < 0$, we have $H^p(X, \Omega_X^q(t)) = 0$, and (b) the statement (2) always holds.

Suppose $t = d$. In particular $q \geq 2$. Now (4.2) gives the long exact sequence

\[ H^p(\Omega_X^{q-1}) \xrightarrow{f_{p,q-1}} H^p(\Omega_X^q(d) |_X) \longrightarrow H^p(\Omega_Y^q(d)) \longrightarrow H^{p+1}(\Omega_X^{q-1}) \xrightarrow{f_{p+1,q-1}} H^{p+1}(\Omega_Y^q(d) |_X). \]

Hence to prove that $H^p(X, \Omega_X^q(d)) = 0$, it is enough to prove the following Claim: The map $f_{p,q}$ is an isomorphism, if $p, q \geq 0$ and $p + q < \dim X$.

**Proof of the claim.** Note that we have the following commutative diagram

\[
\begin{array}{ccc}
H^p(X, \Omega_X^q) & \xrightarrow{f_{p,q}} & H^p(Y, \Omega_Y^{q+1}(d) |_X) \\
\downarrow \eta_{p,q} & & \downarrow g_{p,q+1} \\
H^{p+1}(Y, \Omega_Y^{q+1}) & & \end{array}
\]

where, by (4.3) the map $\eta_{p,q}$ is an isomorphism. Hence the map $g_{p,q+1}$ is surjective, in this case. Moreover, by (4.1) we also have the exact sequence
\[ H^p(Y, \Omega_Y^{q+1}(d)) \rightarrow H^p(X, \Omega_Y^{q+1}(d) |_{X}) \xrightarrow{g_{p,q+1}} H^{p+1}(Y, \Omega_Y^{q+1}) , \]

where \( H^p(Y, \Omega_Y^{q+1}(d)) = 0 \), by Bott vanishing. Therefore the map \( g_{p,q+1} \) is an isomorphism. This implies that \( f_{p,q} \) is an isomorphism. This proves the claim. Hence \( H^p(X, \Omega_X^q(d)) = 0 \) if \( p, q \geq 0 \) and \( p + q < \dim X \).

By induction on \( t \), we can assume that for \( m < t \) and \( m \neq 0 \), we have

\[ H^i(X, \Omega_X^j(m)) = 0, \] where \( i, j \geq 0 \), \( i + j < \dim X \) and \( m \leq \frac{j(n+1-d)}{n-1} \).

Now, to prove the proposition, it remains to show that,

\[ t \leq \frac{q(n+1-d)}{(n-1)}, \ t \notin \{0,d\}, \ p,q \geq 0, \ p + q < \dim X \implies H^p(X, \Omega_X^q(t)) = 0. \]

Note that the hypothesis that \( t \leq \frac{q(n+1-d)}{(n-1)} \implies t \leq q. \)

Consider the following long exact sequence (obtained from (4.2))

\[ H^p(X, \Omega_Y^t |_{X}) \rightarrow H^p(X, \Omega_X^q(t)) \rightarrow H^{p+1}(X, \Omega_X^{q-1}(t - d)) \]

If \( q - 1 < 0 \) then the last term is 0. If \( q - 1 \geq 0 \) then as

\[ t \leq \frac{q(n+1-d)}{n-1} \implies t - d \leq \frac{(q-1)(n+1-d)}{n-1}, \]

by induction hypothesis on \( t \), the last term of the sequence is 0. Consider the exact sequence (obtained from (4.1))

\[ H^p(Y, \Omega_Y^t) \rightarrow H^p(X, \Omega_X^q(t) |_{X}) \rightarrow H^{p+1}(Y, \Omega_Y^q(t - d)) \]

then, by Bott vanishing, the first and the last term of the sequence are 0. This implies that \( H^p(X, \Omega_X^q(t) |_{X}) = 0. \) Hence \( H^p(X, \Omega_X^q(t)) = 0. \) This completes the proof of the proposition. \( \square \)

**Corollary 4.2.** Let \( X \subset \mathbb{P}_k^n \) be a smooth hypersurface of degree \( d \geq 3 \). Let \( n \geq 4 \) and \( g.c.d. (\text{char} k, d) = 1. \) Then \( \Omega_X \) is stable.

**Proof.** Suppose \( \Omega_X \) is not stable then there exists a subbundle \( W \subset \Omega_X \) of rank \( q \leq n-2 \), such that \( \mu(W) \geq \mu(\Omega_X) \). Then \( \wedge^q W \hookrightarrow \wedge^q \Omega_X \). Since \( \wedge^q W \in \text{Pic}(X) \), we have \( \wedge^q W = \mathcal{O}_{\mathbb{P}_k^n}(-t) |_{X} \), as \( n \geq 4 \) implies that the map \( \text{Pic}(\mathbb{P}_k^n) \rightarrow \text{Pic}(X) \) is an isomorphism. This implies that \( H^0(X, \Omega_X(t)) \neq 0. \) Hence to prove that the bundle \( \Omega_X \) is stable, it is enough to prove that

\[ H^0(X, \Omega_X^q(t)) = 0, \] for \( t \leq \frac{q(n+1-d)}{n-1} \),

which immediately follows by Lemma 4.1. Hence \( \Omega_X \) is stable. \( \square \)

**Lemma 4.3.** Let \( X \subset \mathbb{P}_k^3 \) be a smooth hypersurface of degree \( d = 3 \). Then \( \mu_{\min} (T_X) \geq 0. \)
Proof. Let \( H \subset \mathbb{P}^3_k \) be a general hyperplane such that \( C = X \cap H \) is a nonsingular complete intersection on \( \mathbb{P}^3_k \). In particular \( C \) is an elliptic curve. This gives the canonical short exact sequence
\[
0 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{T}_X \mid_C \longrightarrow \mathcal{N}_{C/X} \longrightarrow 0,
\]
which is equivalent to
\[
0 \longrightarrow \mathcal{O}_C \xrightarrow{f_1} \mathcal{T}_X \mid_C \xrightarrow{f_2} \mathcal{O}_C(1) \longrightarrow 0.
\]
If \( T_X \) is semistable then \( \mu_{\min}(T_X) = \mu(T_X) = 1/2 > 0 \). We can assume that \( T_X \) is not semistable. Let \( \mathcal{L} \subset T_X \) be the Harder-Narasimhan filtration of \( T_X \), which gives a short exact sequence of coherent sheaves (where \( \mathcal{L} \) is a line bundle on \( X \)),
\[
0 \longrightarrow \mathcal{L} \xrightarrow{g_1} T_X \xrightarrow{g_2} \mathcal{M} \longrightarrow 0.
\]
By definition, \( \mu_{\min}(T_X) = \deg \mathcal{M} \), therefore it is enough to prove that \( \deg \mathcal{M} > 0 \), which is same as to prove that \( \deg \mathcal{M} \mid_C = \mathcal{M} \cdot H > 0 \). Consider the composite map
\[
\mathcal{O}_C \xrightarrow{f_1} \mathcal{T}_X \mid_C \xrightarrow{g_2} \mathcal{M} \mid_C.
\]

Case 1. If \( g_2 \mid_C \circ f_1 = 0 \) then the induced map \( \mathcal{O}_C(1) \longrightarrow \mathcal{M} \mid_C \) is surjective. This implies that \( \deg \mathcal{M} \mid_C > 0 \). Case 2. If \( g_2 \mid_C \circ f_1 \neq 0 \) then there exists a nonzero map \( \mathcal{O}_C \longrightarrow \mathcal{M} \mid_C \), which implies that \( \deg \mathcal{M} \mid_C \geq 0 \). This proves the lemma.

Lemma 4.4. Let \( X \subset \mathbb{P}^n_k \) be a smooth hypersurface of degree \( d \geq 3 \). Let \( n \geq 4 \) and \( g.c.d.(\text{char } k, d) = 1 \). Then \( \Omega^n_{\mathbb{P}^n_k} \mid_X \) is stable.

Proof. As argued in Corollary 4.2 it is enough to prove that
\[
H^0(X, \Omega^n_{\mathbb{P}^n_k}(t) \mid_X) = 0, \quad \text{for } t \leq q(n+1)/n \text{ and } 1 \leq q \leq n - 1.
\]
Now, consider
\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_X \longrightarrow 0,
\]
which gives
\[
0 \longrightarrow \Omega^n_{\mathbb{P}^n_k}(t-d) \longrightarrow \Omega^n_{\mathbb{P}^n_k}(t) \longrightarrow \Omega^n_{\mathbb{P}^n_k}(t) \mid_X \longrightarrow 0.
\]
Since \( t \leq q(n+1)/n \Rightarrow t \leq q \), by Bott vanishing we have
\[
H^0(\mathbb{P}^n_k, \Omega^n_{\mathbb{P}^n_k}(t)) = 0, \quad \text{for } t \leq q(n+1)/n,
\]
and
\[
H^1(\mathbb{P}^n_k, \Omega^n_{\mathbb{P}^n_k}(t-d)) = 0, \quad \text{if } t \neq d \text{ or } q \neq 1.
\]
Therefore the exact sequence
\[
H^0(\mathbb{P}^n_k, \Omega^n_{\mathbb{P}^n_k}(t)) \longrightarrow H^0(\mathbb{P}^n_k, \Omega^n_{\mathbb{P}^n_k}(t) \mid_X) \longrightarrow H^1(\mathbb{P}^n_k, \Omega^n_{\mathbb{P}^n_k}(t-d))
\]
implies that for \( t \leq q(n+1)/n \)
\[
H^0(\mathbb{P}^n_k, \Omega^n_{\mathbb{P}^n_k}(t) \mid_X) = 0, \quad \text{if } t \neq d \text{ or } q \neq 1.
\]
However the case, when \( t = d \) and \( q = 1 \) and \( t \leq q(n+1)/n \) does not arise, as these conditions imply that \( d = t \leq 1 + (1/n) < 2 \). Hence we conclude that
\[
H^0(\mathbb{P}^n_k, \Omega^n_{\mathbb{P}^n_k}(t) \mid_X) = 0 \text{ if } t \leq q(n+1)/n. \] This proves the lemma. \( \square \)
Lemma 4.5. Let $X \subset \mathbf{P}^n_k$ be a smooth cubic hypersurface such that $n = 2$ or $n = 3$. Then $\Omega_{\mathbf{P}^n_k} |_X$ is strongly semistable.

Proof. Suppose $n = 2$, then $X$ is an elliptic curve. Hence $\Omega_{\mathbf{P}^n_k} |_X$ is an indecomposable rank 2 vector bundle on $X$ (see the proof of Theorem 3.16 of [NT]) and is of negative degree. Hence strong semistability follows from the facts that a vector bundle of negative degree has no sections and a semistable bundle is strongly semistable on an elliptic curve.

Suppose $n = 3$. Let $Q \subset \mathbf{P}^3_k$ be a general smooth quadric such that $C = Q \cap X$ is a smooth complete intersection nonsingular curve in $\mathbf{P}^3_k$. Then $C$ is curve of genus $= 4$ such that $\mathcal{O}_{\mathbf{P}^3_k}(1) |_C = \omega_C$ and the restriction of the short exact sequence

$$0 \to \Omega_{\mathbf{P}^3_k}(1) \to H^0(\mathbf{P}^3_k, \mathcal{O}_{\mathbf{P}^3_k}(1)) \otimes \mathcal{O}_{\mathbf{P}^3_k} \to \mathcal{O}_{\mathbf{P}^3_k}(1) \to 0,$$

to $C$, is

$$0 \to \Omega_{\mathbf{P}^3_k}(1) |_C \to H^0(C, \omega_C) \otimes \mathcal{O}_C \to \omega_C \to 0.$$

Note that $C$ is a non-hyperelliptic curve, hence by Corollory 3.5 of [PR] (the proof given there for $k = \mathbb{C}$ works for any algebraically closed field $k$ of arbitrary characteristic), the bundle $\Omega_{\mathbf{P}^3_k}(1) |_C$ is stable. By Lemma 4.3, we have $\mu_{\min}(T_X) \geq 0$. Therefore Theorem 2.1 of [MR1] implies that $\Omega_{\mathbf{P}^3_k}(1) |_X$ is strongly semistable, for general curve $C \subset X$, of degree 3. Hence $\Omega_{\mathbf{P}^3_k}(1) |_X$ is strongly semistable. Hence the lemma.

Corollary 4.6. If $X \subset \mathbf{P}^n_k$ is a smooth cubic such that $k$ is an algebraically closed field of characteristic $\neq 3$, then

1. $\Omega_{\mathbf{P}^n_k} |_X$ is strongly semistable, if $n = 2$ or $n = 3$ and
2. $\Omega_{\mathbf{P}^n_k} |_X$ is strongly stable, if $n \geq 4$

Proof. The cases $n = 2$ and $n = 3$ follow from Lemma 4.5. Hence it is enough to prove the corollary for $n \geq 4$. Now, by Corollory 4.2, the tangent bundle $T_X = \Omega^X_X$ of $X$ is semistable and is of positive slope. By Lemma 4.4, the bundle $\Omega_{\mathbf{P}^n_k} |_X$ is stable. Hence, again, by Theorem 2.1 of [MR1], we deduce that $\Omega_{\mathbf{P}^n_k} |_X$ is strongly stable. Hence the corollary.

5. Main results

Notation 5.1. We recall the notion of ‘generic’ and ‘general’ as given in Section 1 of [MR2]. Let $k$ be an algebraically closed field of arbitrary characteristic. Let $S_d = \text{Proj}(H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}))$. Then we have

$$\mathbf{P}^n_k \times S_d \supset Z_d \to S_d \quad \text{via} \quad p_d \quad \text{and} \quad q_d$$

wher $Z_d = \{(x, s) \in \mathbf{P}^n_k \times S_d \mid s(x) = 0\}$ and $p_d, q_d$ are projections. The fiber of $q_d$ over $s \in S_d$ is the embedding in $\mathbf{P}^n_k$ via $p_d$ as the hypersurface of $\mathbf{P}^n_k$ defined
by the ideal generated by \( s \). Let \( K_d \) be the function field of \( S_d \). Let \( Y_d \) be the generic fiber of \( q_d \) given by the fiber product

\[
\begin{align*}
Z_d & \rightarrow S_d \\
\uparrow^{q_d} & \uparrow \\
Y_d & \rightarrow \text{Spec } K_d,
\end{align*}
\]

where \( Y_d \) is an absolutely irreducible, nonsingular hypersurface, and there is a nonempty open subset of \( S_d \) over which the geometric fibres of \( q_d \) are irreducible.

We call \( Y_d \) the \textit{generic hypersurface} of degree \( d \). Whenever a property holds for \( q_d^{-1}(s) \) for \( s \) in a nonempty Zariski open subset of \( S_d \), then we say it holds for a \textit{general} \( s \).

**Remark 5.2.** For a torsion free sheaf \( V \) on a smooth projective variety (which is \( \mathbb{P}^n_k \) in our case), the restriction of \( V \) to the generic hypersurface \( Y_d \) is semistable (geometrically stable) if and only if the restriction of \( V \) to a general hypersurface of degree \( d \) is semistable (geometrically stable): because, for any coherent torsion free sheaf \( F \) of \( X \), the sheaf \( p_d^* F \) forms a flat family over a nonempty open subset of \( S_d \) (see Proposition 1.5 of [MR2]), and the property of coherent sheaves being semistable (geometrically stable) is open in flat families.

**Remark 5.3.** If

1. \( X = \text{smooth quadric}, \) if \( \text{char } k \neq 2 \), or \( k \neq 3 \)
2. \( X = \text{smooth cubic}, \) if \( \text{char } k \neq 3 \)

then, by Corollary 3.8 and Corollary 4.6, the bundle \( \Omega_{\mathbb{P}^n_k | X} \) is strongly semistable. Moreover, by Remark 2.2, Corollary 4.2 and Lemma 4.3, we have \( \mu_{\min}(T_X) \geq 0 \). In particular, by Theorem 2.1 of [MR1] and Theorem 3.23 of [RR], any semistable bundle on \( X \) remains semistable after applying the functors like Frobenius pull backs, tensor powers, symmetric powers, and exterior powers on \( X \).

**Proof of Theorem 1.2.** By Remark 5.3, it is enough to prove that \( W_\tau \) is semistable on \( X \). By Proposition 2.4 of [J], given an irreducible representation

\[
\tau : GL(n) \rightarrow GL(W),
\]

there exists \( \lambda \in \chi(T) \) (for a fixed torus \( T \) of \( GL(n) \)) such that

\[
W = L(\lambda),
\]

where following the notation of [J], the \( GL(n) \)-module \( L(\lambda) = \text{socle of } H^0(\lambda) \). Moreover, by corollary 2.5 of [J], the module dual to \( L(\lambda) \) is

\[
L(\lambda)^\vee = L(-w_0 \lambda).
\]

Let \( \epsilon_i \in \chi(T) \) be given by \( \epsilon_i(t_1, t_2, \ldots, t_n) = t_i \) and let \( \omega_i = \epsilon_1 + \cdots + \epsilon_i \). Then any \( \nu \in \chi(T) \) can be written as

\[
\nu = \sum_i a_i \omega_i = \sum_i \nu_i \epsilon_i,
\]

where \( \nu_i \in \mathbb{Z} \) and \( \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \).

Let \( \mathcal{H}^0(L_\nu) \) be the vector bundle on \( G/P = \mathbb{P}^n_k \) corresponding to the \( GL(n) \)-representation \( H^0(L_\nu) \).
Claim. The bundle $\mathbb{H}^0(L_\nu) |_X$ is semistable on $X \subset \mathbb{P}_k^n$ and

$$
\mu(\mathbb{H}^0(L_\nu) |_X) = (\sum_i \nu_i)(\mu(\mathbb{V}_\sigma |_X)),
$$

Proof of the claim: Let us denote

$$
S(a_1, \ldots, a_n, V) = S^{a_1}(V) \otimes S^{a_2}(\wedge^2 V) \otimes \cdots \otimes S^{a_n}(\wedge^n V),
$$

for a vector space $V$, and let us denote

$$
S(a_1, \ldots, a_n, \mathbb{V}) = S^{a_1}(\mathbb{V}) \otimes S^{a_2}(\wedge^2 \mathbb{V}) \otimes \cdots \otimes S^{a_n}(\wedge^n \mathbb{V}),
$$

for a vector bundle $\mathbb{V}$. By definition of $H^0(L_\nu)$, we have a surjection of $GL(n)$-modules

$$
S(a_1, \ldots, a_n, V) \longrightarrow H^0(L_\nu),
$$

where $\sigma : GL(n) \longrightarrow GL(n) = GL(V)$ is the standard representation. Hence we have the surjection of $G$-homogeneous bundles on $\mathbb{P}_k^n$

$$
S(a_1, \ldots, a_n, \mathbb{V}_\sigma) \longrightarrow \mathbb{H}^0(L_\nu),
$$

where we recall that $\mathbb{V}_\sigma = \mathcal{T}_{\mathbb{P}_k^n}(-1) = (\Omega_{\mathbb{P}_k^n}(1))^\vee$ is the vector bundle associated to the representation $\sigma$. Therefore we have the surjection of bundles on $X$

$$
S(a_1, \ldots, a_n, \mathbb{V}_\sigma |_X) \longrightarrow \mathbb{H}^0(L_\nu) |_X.
$$

By Theorem 1.1 (and Cor. 1.3), exposé XXV, Schémas en groupes III, [SGA-3], $GL(n)/B$ ($B$ is a Borel group of $GL(n)$) can be lifted to characteristic zero. Therefore the degree and rank of these vector bundles are independent of the characteristic of the field. Now over a field of characteristic 0, sequence (5.1) split, which implies that sequence (5.2) splits as bundles on $\mathbb{P}_k^n$, defined over field of characteristic 0. Now since $S(a_1, \ldots, a_n, \mathbb{V}_\sigma)$ is semistable vector bundle, we have

$$
\mu(\mathbb{H}^0(L_\nu)) = \mu(S(a_1, \ldots, a_n, \mathbb{V}_\sigma)) = (a_1 + 2a_2 + \cdots + na_n)\mu(\mathbb{V}_\sigma) = (\sum_i \nu_i)\mu(\mathbb{V}_\sigma),
$$

where the last inequality follows as $\nu_i = a_i + \cdots a_n$. Hence

$$
(5.4) \quad \mu(\mathbb{H}^0(L_\nu) |_X) = (\sum_i \nu_i)(\mu(\mathbb{V}_\sigma |_X)).
$$

By Remark 5.3, the bundle $S(a_1, \ldots, a_n, \mathbb{V}_\sigma |_X)$ is semistable. Therefore, by (5.3) and (5.4), the bundle $\mathbb{H}^0(L_\nu) |_X$ is semistable. Hence the claim.

Now, coming back to $W = L(\lambda)$, let

$$
\lambda = \sum_i a_i \omega_i = \sum_i \lambda_i \epsilon_i.
$$

Then, as $w_0(\epsilon_i) = \epsilon_{n+1-i}$, we have

$$
-w_0 \lambda = a_{n-1} \omega_1 + \cdots + a_1 \omega_{n-1} + (-a_1 + \cdots - a_n) \omega_n = -\sum_i (\lambda_{n+1-i}) \epsilon_i.
$$
This implies that \( \mu(\mathbb{H}^0(L_{-w_0\lambda})) = -\mu(\mathbb{H}^0(L_\lambda)) \), therefore

\[(5.5) \quad \mu(\mathbb{H}^0(L_{-w_0\lambda}) \mid X) = -\mu(\mathbb{H}^0(L_\lambda) \mid X).\]

Moreover there exists the surjective map of vector bundles on \( X \)

\[(5.6) \quad S(a_1, \ldots, a_n, \mathbb{V}_\sigma \mid X) \otimes S(a_{n-1}, \ldots, a_1, -(a_1+\cdots+a_n), \mathbb{V}_\sigma \mid X) \rightarrow (\mathbb{H}^0(L_\lambda) \otimes \mathbb{H}^0(L_{-w_0\lambda})) \mid X,\]

where the L.H.S. is a semistable vector bundle of slope \( = 0 \). Moreover, by \((5.5)\), the slope of R.H.S. is also \( = 0 \). Hence \( \mathbb{H}^0(L_\lambda) \mid X \otimes \mathbb{H}^0(L_{-w_0\lambda}) \mid X \) is semistable of slope \( 0 \). Now, consider the injective map

\[(5.7) \quad \mathbb{W}_\tau \otimes \mathbb{W}_\tau^\vee \rightarrow \mathbb{H}^0(L_\lambda) \otimes \mathbb{H}^0(L_{-w_0\lambda}),\]

which give the injective map

\[\mathbb{W}_\tau \mid X \otimes \mathbb{W}_\tau^\vee \mid X \rightarrow \mathbb{H}^0(L_\lambda) \mid X \otimes \mathbb{H}^0(L_{-w_0\lambda}) \mid X\]

is injective, where the slope of L.H.S is \( = 0 \), which is same as the slope of R.H.S.. Hence \( \mathbb{W}_\tau \mid X \otimes \mathbb{W}_\tau^\vee \mid X \) is semistable. This implies that \( \mathbb{W}_\tau \mid X \) is semistable, which proves the theorem. \( \square \)

**Corollary 5.4.** Let \( \mathbb{W}_\tau \) be the homogeneous bundle on \( \mathbb{P}_k^n \) associated to an irreducible representation \( \tau : GL(n) \rightarrow GL(W) \). Let \( k \) be an algebraically closed field of characteristic \( \neq 2,3 \). Then

1. for \( s \geq 0 \), the \( s^{\text{th}} \) Frobenius power \( F^{s*}\mathbb{W}_\tau \mid_H \) is semistable, for general hypersurface \( H \) of degree \( d \geq 2 \) in \( \mathbb{P}_k^n \). In particular
2. \( \mathbb{W}_\tau \mid_{H_0} \) is strongly semistable, where \( H_0 \subset \mathbb{P}_k^n \) is the \( k \)-generic hypersurface of degree \( d \geq 2 \).

Moreover, if \( \mathbb{W}_\tau \) is the tangent bundle on \( \mathbb{P}_k^n \) and \( n \geq 4 \) then we can replace the word ‘semistable’ by ‘stable’ everywhere in the above statement.

**Proof.** By Theorem \[1.2\] the bundle \( \mathbb{W}_\tau \mid X \) is strongly semistable, where \( X \) is a smooth quadric or a smooth cubic in \( \mathbb{P}_k^n \). In other words, for \( s \geq 0 \) and for the \( s^{\text{th}} \) iterated Frobenius pull back, \( F^{s*}\mathbb{W}_\tau \) of \( \mathbb{W}_\tau \), the bundle \( F^{s*}\mathbb{W}_\tau \mid_X \) is semistable, where \( X \) is a smooth quadric or a smooth cubic. Hence, by the proof of the restriction theorem of [MR2], it follows that \( F^{s*}\mathbb{W}_\tau \mid_H \) is semistable when restricted to a general hypersurface \( H \subset \mathbb{P}_k^n \) of degree \( \geq 2 \) (see also the modified proof of the above mentioned restriction theorem given in [HL]). This proves part (1) of the corollory.

Moreover this implies that, for any \( s \geq 0 \) and for generic hypersurface \( H_0 \) of degree \( \geq 2 \), the bundle \( F^{s*}\mathbb{W}_\tau \mid_{H_0} \) is semistable (see Remark \[5.2\]). In particular, the bundle \( \mathbb{W}_\tau \mid_{H_0} \) is strongly semistable. This proves the part (2) of the corollory.

Note that, for \( n \geq 4 \), by Corollories \[3.8\] and \[4.6\] the bundle \( \mathcal{T}_{\mathbb{P}_k^n} \mid_X \) is strongly stable and hence geometrically strongly stable (as the underlying field \( k \) is algebraically closed). Now the similar arguments, as above, applied to the tangent bundle \( \mathcal{T}_{\mathbb{P}_k^n} \), prove the rest of the corollory. \( \square \)
Remark 5.5. By Proposition 3.6, the bundle $\mathcal{T}_{P^n_k}|_Q$ is stable for a smooth quadric $Q \subset P^n_k$, for $n \geq 3$. One may ask the following: If $\tau : GL(n) \rightarrow GL(W)$ is an irreducible representation, then is the associated bundle $\mathbb{W}_\tau$ stable on $Q$? More generally if $\tau : GL(n) \rightarrow H$ is any irreducible representation, with $H$ semisimple, then is the induced $H$ bundle semistable on $Q$?

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