Topological quantum field theory structure on symplectic cohomology

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Abstract
We construct the topological quantum field theory (TQFT) on symplectic cohomology and wrapped Floer cohomology, possibly twisted by a local system of coefficients, and we prove that Viterbo restriction preserves the TQFT. This yields new applications in symplectic topology relating to the Arnol’d chord conjecture and to exact contact embeddings. We prove that if a Liouville domain $M$ admits an exact embedding into an exact convex symplectic manifold $X$, and the boundary $\partial M$ is displaceable in $X$, then the symplectic cohomology of $M$ vanishes and the chord conjecture holds for any Lagrangianly fillable Legendrian in $\partial M$. The TQFT respects the isomorphism between the symplectic cohomology of a cotangent bundle and the homology of the free loop space, so it recovers the TQFT of string topology.

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Motivation and outline of the paper
Symplectic cohomology has become an important tool in symplectic topology, ever since its introduction in Viterbo’s [43] foundational paper. For example, it can be used to prove the existence of closed Hamiltonian orbits and Reeb chords, and it gives rise to obstructions to the existence of exact Lagrangian submanifolds and of exact contact hypersurfaces. It also has theoretical importance: the wrapped Fukaya category (a version of the Fukaya category
for non-compact symplectic manifolds introduced in \[23\]) is built using the wrapped Floer cohomologies, which are modules over the symplectic cohomology ring.

Symplectic cohomology is an invariant of exact symplectic manifolds \((M, d\theta)\) whose boundary \(\partial M\) is of contact type. Equivalently, after attaching a conical end, symplectic cohomology is an invariant of non-compact exact symplectic manifolds \((\overline{M}, d\theta)\) which are conical at infinity, meaning that outside a bounded domain, \(\overline{M}\) is symplectomorphic to a conical end \((\Sigma \times [1, \infty), d(R\alpha))\) where \((\Sigma, \alpha)\) is a contact manifold, and \(R\) is the coordinate on \([1, \infty)\).

For example, a disc cotangent bundle \(M = DT^*N\) of a closed Riemannian manifold \(N\) with the standard symplectic form \(\sum dp_j \wedge dq_j = d(\sum p_j dq_j)\) in local coordinates \((q, p)\), then \(\overline{M} = T^*N\) is the cotangent bundle and \(R = \vert p \vert\). In this example, Viterbo \[43\] proved that symplectic cohomology recovers the homology of the free loop space \(LN = C^\infty(S^1, N)\),

\[
SH^*(T^*N; \mathbb{Z}/2) \cong H_{n-*}(LN; \mathbb{Z}/2),
\]

and Abbondandolo and Schwarz \[2\] proved that \(SH^*(T^*N; \mathbb{Z}/2)\) has a product structure, called ‘pair-of-pants product’, which under the above isomorphism corresponds to the Chas–Sullivan loop product on \(H_*(LN; \mathbb{Z}/2)\) coming from string topology \[11\].

In his influential survey on symplectic cohomology \[39\], Seidel described how to define a pair-of-pants product for any \(M\), and more generally how to define topological quantum field theory (TQFT) operations

\[
\psi_S : SH^*(M)^{\otimes q} \rightarrow SH^*(M)^{\otimes p} \quad (p \geq 1, q \geq 0).
\]

The idea is to count maps \(u : S \rightarrow \overline{M}\) from a Riemann surface \(S\) as in Figure 1, carrying \(p\) negative punctures, \(q\) positive punctures, and satisfying a perturbed Cauchy–Riemann equation of the form \((du - X \otimes \beta)^{0,1} = 0\). Here, \(X\) is a Hamiltonian vector field, and \(\beta\) is a 1-form on \(S\) satisfying \(d\beta \leq 0\) such that near the punctures the equation turns into Floer’s equation \(\partial_t u + J(\partial_x u - cX) = 0\) for constant weights \(c > 0\) depending on the puncture.

For closed symplectic manifolds such a TQFT was constructed on Floer cohomology by Piunikhin, Salamon and Schwarz \[30\] and in detail by Schwarz \[38\], over fields of characteristic 2. In this closed setup, one can just require such \(u\) to be holomorphic away from the punctures, so \(du^{0,1} = 0\), and then one interpolates this equation with Floer’s equation near the punctures. Since symplectic cohomology is defined as a direct limit of Floer cohomologies for Hamiltonians \(H : M \rightarrow \mathbb{R}\) which are linear in the \(R\) coordinate at infinity, one might at first imagine that the construction of the TQFT easily carries over. Unfortunately, this is not the case: this construction would run into the danger that a sequence of curves may escape to infinity. So the use of \(\beta\) and the condition \(d\beta \leq 0\) are fundamental in preventing this problem, and this somewhat complicates the construction of the TQFT.

For cotangent bundles, Abbondandolo and Schwarz \[2\], Section 3.2] construct the product while circumventing the use of \(\beta\), by using a clever construction of a pair-of-pants surface \(S\) which has explicit time coordinates. For general \(M\), the product is discussed by Seidel \[39\], Section (8a)] and by McLean \[28\], Section 2.3], who proved in \[28\], Section 10] how the product behaves under Viterbo restriction and under boundary connected sums. A key ingredient, namely preventing that solutions of \((du - X \otimes \beta)^{0,1} = 0\) escape to infinity, is due to Abouzaid and Seidel \[4\], Lemma 7.2], and we recall this in Appendix D.
Applications in symplectic topology typically arise from proving a vanishing result for $SH^*(M)$. For example, Viterbo [43] proved that the vanishing of $SH^*(M)$ implies that there is a closed Reeb chord in $\partial M$, which is the Weinstein Conjecture. It is for this reason that the most important part of the TQFT is the unit and the product. Indeed $SH^*(M)$ vanishes if and only if the unit vanishes, since then $y = y \cdot 1 = y \cdot 0 = 0$ for any $y \in SH^*(M)$. Secondly, if $W \subset \overline{M}$ is a subdomain with contact type boundary then there is a unital ring homomorphism called Viterbo restriction,

$$\varphi : SH^*(M) \longrightarrow SH^*(W).$$

So if $SH^*(M) = 0$, then $1 = \varphi(1) = \varphi(0) = 0$ in $SH^*(W)$, and so $SH^*(W) = 0$. This circle of ideas is due to Viterbo, Seidel and McLean [43, 39, 28]. We will recall this in Sections 9 and 10.

We now briefly outline the structure of the paper. Since a lot of the analysis and applications are based on previous work, this will give us an opportunity to properly credit the existing literature. A detailed Introduction will follow in Section 1.

One of the goals of this paper is to put in written form a detailed construction of the TQFT structure on symplectic cohomology. Since a large part of this material was either known or expected within the circle of specialists in symplectic cohomology, we decided to isolate this part in Appendix A. For the convenience of the reader, we summarize this Appendix at the start of Section 6.

We will recall the basic definitions and conventions in Sections 2 and 5: the definition of exact symplectic manifolds conical at infinity (Section 2); the construction of symplectic cohomology $SH^*(M)$ and symplectic homology $SH_*(M)$ (Section 3); the relative analogue of $SH^*$ with Lagrangian boundary conditions, called the wrapped Floer cohomology $HW^*(L)$, for Lagrangians $L \subset M$ intersecting $\partial M$ in a Legendrian submanifold (Section 4); and the maps

$$c^* : H^*(M) \longrightarrow SH^*(M) \quad \text{and} \quad c^* : H^*(L) \longrightarrow HW^*(L)$$

from the ordinary cohomology to the symplectic and wrapped cohomologies (Section 5).

The initial motivation of this paper was to construct algebraic structures on a deformation of symplectic cohomology called twisted symplectic cohomology introduced by the author [31], with the purpose of continuing the study of exact contact hypersurfaces initiated by Cieliebak and Frauenfelder [13]. The twisted symplectic cohomology $SH^*(M)_\alpha$ is an invariant of $M$ associated to a class $\alpha \in H^1(\mathcal{L}M)$ on the free loop space, and for $\alpha = 0$ it recovers $SH^*(M)$. In fact it is the Novikov cohomology theory applied to $SH^*$, which involves using twisted coefficients in a bundle of Novikov rings defined in Subsection 7.1. This twisting yields very concrete applications in symplectic topology, for example, in [31], we used it to prove that there are no exact Klein bottles in $T^*S^2$.

There is no TQFT on $SH^*(M)_\alpha$ in general because there is no canonical way of viewing surfaces in $M$ (such as pairs of pants) as chains in $\mathcal{L}M$. Even when a TQFT can be defined on $SH^*(M)_\alpha$, it may not possess a unit for the same reasons that the Novikov cohomology of a manifold usually does not. However, in Section 12, we prove that if the form $\alpha \in H^1(\mathcal{L}M)$ is the transgression of a class $\eta \in H^2(M)$, then the TQFT structure exists and possesses a unit. In Section 9, we prove that the twisted Viterbo restriction maps $\varphi : SH^*(M)_\eta \rightarrow SH^*(W)_{\eta|W}$ from [31] are TQFT maps, in particular, they are unital ring homomorphisms. Our interest in the twisted theory stems from the fact that for $\pi_1(N) = 1$, we have a vanishing result [31]:

$$SH^*(T^*N; \mathbb{Z}/2)_\eta \cong H_{n-\eta}(\mathcal{L}N; \mathbb{Z}/2)_\eta = 0 \quad \text{for non-zero} \ \eta \in H^2(N).$$

The applications to exact contact hypersurfaces in Section 12 are based on this result.

In Section 11 we give a family of examples where the Arnol’d chord conjecture is satisfied, namely the existence of Reeb chords in $\partial M$ with ends on certain Legendrian submanifolds.
This application follows from the module structure of $HW^*(L)$ over $SH^*(M)$ (Subsection 6.14) and it relies on vanishing criteria for these cohomology groups (Section 10). Some of these examples, but not all, exploit the twisted theory.

The application of the TQFT to prove that $SH^*(M)$ vanishes if $\partial M$ has certain displaceability properties (Section 13) is independent of the twisted theory. Rather, it relies on foundational work due to Cieliebak and Frauenfelder [13] and Cieliebak, Frauenfelder and Oancea [14]. Namely Cieliebak and Frauenfelder [13] construct an invariant of $M$ called Rabinowitz Floer cohomology $RFH^*(M)$, and proves that it vanishes under certain displaceability properties of $\partial M$; and Cieliebak, Frauenfelder and Oancea [14] construct a long exact sequence relating $SH_*(M)$, $SH^*(M)$ and $RFH^*(M)$. Our result then follows from investigating the role played by the unit in the long exact sequence and then exploiting the vanishing criteria in Section 10. This argument is slightly trickier when one makes no simplifying assumption on the first Chern class, since then one loses the $\mathbb{Z}$ grading. In this case, we need to exploit the fact that $c^*: H^*(M) \to SH^*(M)$ respects the product structure, which is interesting in its own right. In fact, we prove in Section 15 that the $c^*$ maps are compatible with a TQFT structure on $H^*(M)$. Section 15 relies on generalizing the PSS-maps from [30] to the non-compact setting required by $SH^*$ by using the analytical machinery in the foundational work of Salamon and Zehnder [37], which we will briefly recall.

Section 14 proves that the isomorphism $SH^*(T^*N; \mathbb{Z}/2) \cong H_{n-*}(\mathcal{L}N; \mathbb{Z}/2)$ preserves the TQFT structure and the deformed TQFT structure. We include this section mainly for theoretical interest. The result is actually a rather formal consequence of the TQFT axioms which follows from the foundational work of Abbondandolo and Schwarz [2] which proved that the isomorphism preserves the pair-of-pants product. There is one technical aspect worth mentioning: in the work of Abbondandolo–Schwarz, symplectic cohomology is defined as the Floer cohomology of one Hamiltonian of quadratic growth in $R$ using a certain non-contact type almost complex structure $J$, whereas in our work we always use contact type $J$ and we use linear growth Hamiltonians and then take a direct limit of the resulting Floer cohomologies. We reconcile the two approaches for general $M$ in Appendix C.

Appendix B discusses orientation signs for $SH^*(M)$. Orientation signs were constructed for Floer cohomology by Floer and Hofer [20] (which immediately generalizes to $SH^*(M)$) but they have not yet been written up for the TQFT structure. However, for the TQFT signs one can mimic the construction of orientation signs used in Symplectic Field Theory, which was carried out by Eliashberg, Givental and Hofer [16] and by Bourgeois and Mohrke [9].

A word about cohomological conventions. We explain (1) why we preferred to use symplectic cohomology and (2) why we define $SH^*$ as the direct limit of Floer cohomologies (rather than the inverse limit, which, for example, is the convention of [14]).

(1) The symplectic cohomology $SH^*$ is much better behaved algebraically because it possesses a unit, which is crucial in applications. While $SH_*$ only has a counit. Moreover, $SH_*$ can be recovered from $SH^*$ by dualization, but not vice versa.

(2) The conventions we use to define Floer cohomology actually go back historically to Floer’s original papers. But this convention also happens to be consistent with the fact that there is a map $c^*: H^*(M) \to SH^*(M)$ which respects the unit and product structures, all of which naturally respect our grading conventions when $SH^*$ is $\mathbb{Z}$-graded. In this convention, Viterbo’s map $SH^*(M) \to SH^*(W)$ can be interpreted as a restriction map generalizing the ordinary restriction map $H^*(M) \to H^*(W)$ induced by the inclusion $W \subset M$ (Viterbo [43] originally interpreted it as a transfer map $FH^*(W) \to FH^*(M)$ because for disc cotangent bundles $W, M$ it defines a transfer map on the cohomology of the free loop spaces).

We point out however that the other conventions are also very reasonable, for example, the conventions of [14] are particularly well-suited for studying cotangent bundles since in their conventions one has $SH_*(T^*N; \mathbb{Z}/2) \cong H_*(\mathcal{L}N; \mathbb{Z}/2)$. 
1. Introduction

1.1. Topological quantum field theory

Formally speaking, a TQFT [6] is a tensor functor from the category of 2-dimensional orientable cobordisms between 1-manifolds to the category of vector spaces. We make this explicit in our setup: to a circle we associate the vector space $SH^*(M)_\eta$, the symplectic cohomology of $M$ computed over a field $\mathbb{K}$ of coefficients (when twistings are present, we keep track of $\eta \in H^2(M)$, and we work over the Novikov field $\Lambda$, which is a $\mathbb{K}$ algebra defined in Subsection 7.1). To the disjoint union of $p$ circles, we associate the tensor product $SH^*(M)^{\otimes p}_\eta$ of $p$ copies of $SH^*(M)_\eta$, and to the empty set we associate the base field $\mathbb{K}$. On morphisms, the functor is

$$(\text{Punctured Riemann surface } S) \mapsto (\text{Operation } \psi_S : SH^*(M)^{\otimes q}_\eta \rightarrow SH^*(M)^{\otimes p}_\eta),$$

where $S$ represents a cobordism between $p \geq 1$ and $q \geq 0$ circles (Figure 1). Functoriality means that $\psi_Z = \text{id}$ for cylinders $Z : p = q = 1$, and that compositions are respected: gluing surfaces along the cylindrical ends near the punctures yields compositions of $\psi_S$ maps. The QFT is topological since $\psi_S$ only depends on $p, q$ and the genus of $S$.

**Theorem** (Theorem 16.14 and Theorem 7.3). $SH^*(M)_\eta$ is a $(1+1)$-dimensional TQFT, except it does not possess operations for $p = 0$ (so no counit).

The $\psi_S$ are not defined for $p = 0$ due to a non-compactness issue. Indeed, if $p = 0$ were allowed, then the TQFT axioms would imply $SH^*(M)$ is finite-dimensional (see the Remark below), which is false for simply connected cotangent bundles. This differs from the Floer cohomology of a closed symplectic manifold $M : p = 0$ is legitimate, and $FH^*(M) \cong H^*(M)$ is finite-dimensional.

**Remark.** Suppose the $\psi_S$ were defined also for $p = 0$. Let $a_i$ be a basis for the $\mathbb{K}$ vector space $SH^*(M; \mathbb{K})$. Denote $S_{pq}$ the surface of genus $0$ with $p$ negative and $q$ positive punctures. Observe that $S_{20}$ arises from gluing $Q = S_{21}$ onto $C = S_{10}$. This determines a finite sum $\psi_{S_{20}}(1) = \sum k_{ij} a_i \otimes a_j$, where $k_{ij} \in \mathbb{K}$. Since $S = (Z \cup S_{02})#(S_{20} \sqcup Z)$ is actually a cylinder, it acts by the identity. So, for any $x \in SH^*$, $x = \psi_S(x) = \psi_{Z\cup S_{02}}(\sum k_{ij} a_i \otimes a_j \otimes x) = \sum (k_{ij} \psi_{S_{02}}(a_j, x)) a_i$. But this means $SH^*$ is spanned by just finitely many vectors $a_i$, so $SH^*$ is finite-dimensional.

The untwisted TQFT is summarized in Figure 2 (in Section 3, we will define $SH_*(M)$ and prove that it is canonically the dual of $SH^*(M)$, but not vice versa).

**Corollary** (Theorem 7.3). There is a graded-commutative associative unital ring structure on $SH^*(M)_\eta$ with product $\psi_P$ and unit $e = \psi_C(1)$ (this structure can also be defined if we replace $\mathbb{K}$ by $\mathbb{Z}$).

1.2. TQFT structure on ordinary cohomology

A TQFT also exists on $H^*(M) \cong H^*(\overline{M})$ if we use 1-dimensional oriented cobordisms (we allow cobordisms which are not strictly speaking 1-manifolds, but which are oriented graphs). The ordinary cohomology can be identified with the Morse cohomology $MH^*(f)$ of any Morse function $f : \overline{M} \rightarrow \mathbb{R}$ for which $-\nabla f$ points inwards along the conical end of $\overline{M}$. Replace surfaces $S$ by directed graphs $S'$. Consider, for example, the Y-shaped graph $P'$ in Figure 6. Assign a generic Morse function $f_i$ to each edge. The count of isolated negative gradient flow lines along
Figure 2. Summary of the TQFT structure.

the graph defines

$$\psi' : MH^*(f_2) \otimes MH^*(f_3) \rightarrow MH^*(f_1).$$

After identifications with ordinary cohomology, this is the cup product on $H^*(M)$. Indeed $H^*(M)$ is a $(0+1)$-dimensional TQFT: to a point associate $H_*(M)$, and to a cobordism between $p+q$ points represented by a directed graph $S'$ associate

$$\psi' : H^*(M)^{\otimes q} \rightarrow H^*(M)^{\otimes p}.$$ 

This TQFT is well-known [7, 22].

1.3. Wrapped Floer cohomology

Wrapped Floer cohomology $HW_*(L)$ is an invariant of an exact Lagrangian $L \subset (M, d\theta)$ having Legendrian intersection $\partial L = L \cap \partial M$ with $\partial M$. Special cases for $M = T^*N$ were introduced by Abbondandolo and Schwarz [1], the general definition arises in [23], and a detailed construction is in [4]. The construction in [4] is more complicated than ours because the authors’ aim was to construct an $A_\infty$ structure at the chain level.
The construction of $SH^*(M)$ involves closed Hamiltonian orbits whereas $HW^*(L)$ involves open Hamiltonian orbits with ends on $L$. This open-closed string theory analogy is a dictionary to pass from $SH^*$ to $HW^*$: we now use half the surface obtained after cutting Figure 1 with a vertical plane, so $S$ has boundary components. We count maps $u : S \to M$ satisfying the Lagrangian boundary condition $u(\partial S) \subset L$ (where one has extended $L$ to the conical end of $\overline{M}$), converging to open Hamiltonian orbits at the $p + q$ punctures, yielding:

$$\mathcal{W}_S : HW^*(L)^{\otimes q} \to HW^*(L)^{\otimes p}. \quad (p \geq 1, q \geq 0).$$

One can also define open-closed operations by making $\overline{M}$ to pass from $\overline{HW}^*$ to $\overline{SH}^*$. We will also deduce that:

**Theorem (Theorem 11.4).** If $SH^*(\overline{M})_{\overline{\eta}} = 0$ or $HW^*(\overline{L})_{\overline{\eta}} = 0$ then the chord conjecture holds and generically there are at least rank $H^*(\overline{L})$ chords.

This applies, for example, to subcritical Stein manifolds $M$ since $SH^*(M) = 0$ by [12]. For subcritical Stein $M$, the existence of one Reeb chord for any Legendrian $K \subset \partial M$ is due to Mohnke [29]. We will also deduce that:

**Corollary (Theorem 13.4).** If $M$ admits an exact embedding into an exact convex symplectic manifold $X$ (such as $\overline{M}$), and $\partial M$ is displaceable by a compactly supported Hamiltonian flow in $X$, then the chord conjecture holds for any $\partial L$, and generically there are rank $H^*(\overline{L})$ chords.

**Theorem (Theorem 11.3).** For $L \subset M = DT^*N$, with $N$ closed and simply connected, such that $H^2(T^*N) \to H^2(\overline{L})$ is not injective, the chord conjecture holds and generically there are at least rank $H^*(\overline{L})$ chords. It also holds after attaching subcritical handles to $DT^*N$.

An ALE space $\overline{M}$ is a simply connected hyperkähler 4-manifold which at infinity looks like $\mathbb{C}^2/G$ for a finite subgroup $G \subset SL(2,\mathbb{C})$. Such $\overline{M}$ arise by attaching a conical end to the plumbing $M$ of copies of $DT^*S^2$ according to ADE Dynkin diagrams [32].

**Theorem (Theorem 11.4).** For any ALE space the chord conjecture holds for any $\partial L$ and generically there are at least rank $H^*(\overline{L})$ Reeb chords.
1.5. Obstructions to exact contact embeddings

An embedding $j: \Sigma^{2n-1} \hookrightarrow (M^{2n}, d\theta)$ is an exact contact embedding if there is a contact form $\alpha$ on $\Sigma$ with $\alpha - j^*\theta = \text{exact}$. For example, if $L \subset M$ is a closed exact Lagrangian, then a Weinstein neighbourhood $W \cong DT^*L$ yields an exact contact hypersurface $ST^*L \cong \partial W \hookrightarrow M$. Using the deformed TQFT we prove the following theorem (stronger results are discussed in Subsections 12.4 and 12.5).

**Theorem (Theorem 12.9).** Let $L$ and $N$ be closed simply connected $n$-manifolds, $n \geq 4$. For any exact contact embedding $ST^*L \hookrightarrow T^*N$, the following hold:

1. $H^2(N) \rightarrow H^2(L)$ is injective;
2. $\pi_2(L) \rightarrow \pi_2(N)$ has finite cokernel;
3. if $H^2(N) \neq 0$, then $H_*(L) \cong H_*(W)$ for the filling $W$ of $ST^*L \subset T^*N$.

**Conjecture.** For simply connected $L$ and $N$ of dimension at least 4, all exact contact $ST^*L \hookrightarrow T^*N$ always arise as the boundary of a Weinstein neighbourhood of an exact Lagrangian $L \hookrightarrow T^*N$.

1.6. Displaceability of contact hypersurfaces

By exploiting recent literature on the Rabinowitz–Floer cohomology $RF^*(M)$ (see [13, 14]), and after proving that $RF^*(M) = 0$ if and only if $SH^*(M) = 0$, we will deduce the following corollary:

**Corollary (Theorem 13.4).** If $\partial M$ is displaceable by a compactly supported Hamiltonian flow in $\tilde{M}$, then $SH^*(M) = 0$, so there are no closed exact Lagrangians in $M$. This also holds if $M$ exactly embeds into an exact convex symplectic manifold $X$, and $\partial M \subset X$ is displaceable.

1.7. String topology

The Pontryagin product on the homology of the space $\Omega N$ of based loops in $N$ is defined by concatenating loops to form figure-8 loops (see, for example, [2]). The analogue on the homology of the space $LN$ of free loops in $N$ is called Chas–Sullivan loop product [11]; given two families of loops, one forms the family of all possible figure-8 loops obtained when the two base points happen to coincide.

Abbondandolo and Schwarz [2] proved that the products on $HW^*(T^*N; \mathbb{Z}/2)$ and $SH^*(T^*N; \mathbb{Z}/2)$ agree with the Pontryagin and Chas–Sullivan products on $H_{n-s}(\Omega N; \mathbb{Z}/2)$ and $H_{n-s}(LN; \mathbb{Z}/2)$ via the respective isomorphisms. We extend the result to the following theorem:

**Theorem (Theorem 14.3).** The isomorphisms $SH^*(T^*N; \mathbb{Z}/2)_\eta \cong H_{n-s}(LN; \mathbb{Z}/2)_\eta$ and $HW^*(T^*N; \mathbb{Z}/2)_\eta \cong H_{n-s}(\Omega N; \mathbb{Z}/2)_\eta$ respect the TQFT structures, and the units are $[N]$ and $[\text{base-point}]$ respectively.

2. Liouville domains

2.1. Reeb periods, the contact type condition, Hamiltonians

A Liouville domain $(M^{2n}, \omega = d\theta)$ is a compact exact symplectic manifold with boundary, such that the Liouville vector field $Z$ defined by $\omega(Z, \cdot) = \theta$ points strictly outwards along $\partial M$. The form $\alpha = \theta|_{\partial M}$ is a contact form on $\partial M$, and it defines the Reeb vector field $R$ on $\partial M$ by the
conditions

\[ \alpha(\mathcal{R}) = 1, \quad d\alpha(\mathcal{R}, \cdot) = 0. \]

The closed orbits of \( \mathcal{R} \) on \( \partial M \) are called Reeb orbits, and the (non-zero) periods of the orbits are called Reeb periods. For a generic choice of \( \alpha \) (for example, a generic choice of \( \theta \) subject to \( d\theta = \omega \)), the Reeb periods form a countable closed subset of \([0, \infty)\), which we will assume.

The symplectization \( \overline{M} = M \cup_{\partial M} [0, \infty) \times \partial M \) of \( M \) is obtained by gluing a conical collar onto \( \partial M \). The flow of \( Z \) for small time \( r \leq 0 \) parametrizes a neighbourhood \((-\varepsilon, 0] \times \partial M\) of \( \partial M \), so all the data naturally extend to \( \overline{M} \) by \( Z = \partial_r, \theta = \epsilon' \alpha, \omega = d\theta \). The flow of \( Z \) for time \( r \in [-\infty, \infty) \) starting from \( \partial M \) defines the coordinate \( R = e^r \in [0, \infty) \) on \( \overline{M} \) with \( \partial M = \{R = 1\} \). We will use \( R \) instead of \( r \) from now on, so

\[ \overline{M} = M \cup [1, \infty) \times \partial M. \]

Let \( J \) be an \( \omega \)-compatible almost complex structure on \( \overline{M} \). Let \( g = \omega(\cdot, J\cdot) \) denote the \( J \)-invariant metric on \( \overline{M} \). We always assume that \( J \) is of contact type for large \( R \):

\[ J^* \theta = dR \quad (\text{equivalently: } JZ = \mathcal{R}). \]

Since symplectic cohomology is invariant under deformations of such \( J \), the particular choice of \( J \) will not matter, because the space of such \( J \) is contractible \([26, \text{Proposition 2.51}]\).

For \( H \in C^\infty(\overline{M}, \mathbb{R}) \), define the Hamiltonian vector field \( X = X_H \) by \( \omega(\cdot, X) = dH \). We call \( 1 \)-orbits of \( H \) the 1-periodic Hamiltonian orbits \( x : S^1 \to \overline{M} \), so \( \dot{x}(t) = X(x(t)) \).

We always assume \( H \) is \( C^2 \) small and Morse inside \( M \). So the only 1-orbits inside \( M \) are constants: the critical points of \( H \). We assume that for large \( R \), \( H = h(R) \) depends only on \( R \). So \( X = h'(R)\mathcal{R} \), so the 1-orbits \( x \) of \( X \) on the collar have constant \( R = R(x) \) and they correspond to the Reeb orbits \( y \) of period \( T = h'(R) \) via \( y(t) = x(t/T) \). We assume that for large \( R \), \( h \) becomes linear in \( R \) with slope not equal to a Reeb period; hence in the region at infinity where \( h \) is linear there are no 1-orbits.

2.2. Action functional
Let \( \mathcal{L}\overline{M} = C^\infty(S^1, \overline{M}) \) be the space of free loops in \( \overline{M} \). Let \( \mathcal{A}_H \) denote the \( H \)-perturbed action functional for \( x \in \mathcal{L}\overline{M} \),

\[ \mathcal{A}_H(x) = -\int x^*\theta + \int_0^1 H(x(t)) \, dt. \]

The differential of \( \mathcal{A}_H \) at \( x \in \mathcal{L}\overline{M} \) is \( d\mathcal{A}_H \cdot \xi = -\int_0^1 \omega(\xi, \dot{x} - X) \, dt \) in the direction of \( \xi \in T_x \mathcal{L}\overline{M} = C^\infty(S^1, x^*T\overline{M}) \). So the critical points of \( \mathcal{A}_H \) are the 1-orbits of \( H \).

For a 1-orbit \( x \) on the collar in \( \{R\} \times \partial M \), the action is \( \mathcal{A}_H(x) = \mathcal{A}_h(R) = -Rh'(R) + h(R) \).

2.3. Floer trajectories
With respect to the \( L^2 \)-metric \( \int_0^1 g(\cdot, \cdot) \, dt \), the gradient is \( \nabla \mathcal{A}_H = J(\dot{x} - X) \). For \( u : \mathbb{R} \to \mathcal{L}\overline{M} \), or equivalently \( u : \mathbb{R} \times S^1 \to \overline{M} \), the negative \( L^2 \)-gradient flow equation \( \partial_s u = -\nabla \mathcal{A}_H(u) \) in the coordinates \( (s, t) \in \mathbb{R} \times S^1 \) is

\[ \partial_s u + J(\partial_t u - X) = 0 \quad (\text{Floer’s equation}). \]

Let \( \hat{M}(x_-, x_+) \) denote the solutions \( u \) converging to 1-orbits \( x_\pm \) of \( H \) at the ends \( s \to \pm \infty \).
Then \( \mathcal{M}(x_-, x_+) = \hat{M}(x_-, x_+)/\mathbb{R} \) denotes the moduli space of Floer trajectories, where we identify \( u(\cdot, \cdot) \sim u(\cdot + \text{constant}, \cdot) \) (the \( \mathbb{R} \) reparametrization freedom).
2.4. Energy

The energy $E(u) = \int |\partial_s u|^2 ds \wedge dt$ of a Floer trajectory $u \in \mathcal{M}(x_-, x_+)$ satisfies

$$E(u) = \int d\theta (\partial_s u, \partial_t u - X) ds \wedge dt$$

$$= \int u^* d\theta - dH(\partial_s u) ds \wedge dt$$

$$= \int_0^1 (\theta(\dot{x}_+) - \theta(\dot{x}_-) - H(x_+) + H(x_-)) dt$$

$$= \mathcal{A}_H(x_-) - \mathcal{A}_H(x_+).$$

Thus, we have an a priori energy estimate (in terms only of the ends $x_\pm$ not of $u$).

2.5. Transversality and compactness

Standard Floer theory methods (see [35] and [21, Theorem 5.1]) show that for a generic time-dependent perturbation $(H_t, J_t)$ of $(H, J)$ the 1-orbits are non-degenerate and the moduli spaces $\mathcal{M}(x_-, x_+)$ are smooth manifolds. Write $\mathcal{M}_k(x_-, x_+)$ for the $k$-dimensional part of $\mathcal{M}(x_-, x_+)$.

Convention: We write $(H, J)$ even though one actually uses a perturbed $(H_t, J_t)$.

Technical Remarks. A 1-orbit of $H_t$ is non-degenerate if 1 is not an eigenvalue of the linearization of $\varphi_{H_t}^1$ (the time-1 flow of $X_{H_t}$). If the 1-orbits are non-degenerate, then they are isolated and $\mathcal{M}(x_-, x_+)$ is the zero set of a Fredholm map. For time-independent $H$ the 1-orbits are non-degenerate if and only if $H$ is Morse and the 1-orbits are critical points of $H$. So given $H$ or $H_t$, one typically needs to make a time-dependent perturbation to ensure non-degeneracy.

Suppose $H_t$ satisfies this non-degeneracy. Then, by [21, Theorem 5.1], we can ensure that all $\mathcal{M}(x_-, x_+)$ are smooth after either a generic time-dependent perturbation $H_t$ of $H$, or a generic time-dependent perturbation $J_t$ of $J$ (or perturbing both). We do not need to perturb $(H, J)$ in the region $R > 0$ where $H$ is linear since there are no 1-orbits in this region (recall Subsection 2.1) and since no Floer trajectories enter this region by the following lemma.

**Lemma 2.1.** Solutions of $\partial_s u + J(\partial_t u - X) = 0$ converging to $x_\pm$ at the ends are entirely contained in the region $R \leq \max\{R(\pm), R_0\}$ when $J$ is of contact type for $R \geq R_0$.

Lemma 2.1 is a consequence of a maximum principle (Lemma D.1). It ensures that all $u \in \mathcal{M}(x_-, x_+)$ stay in a compact region of $\overline{\mathcal{M}}$. So, we reduce to checking whether the compactness proofs that hold for closed symplectic manifolds [35] are applicable. Indeed, we have the two sufficient requirements: an a priori energy estimate and a reason to exclude the bubbling-off of $J$-holomorphic spheres (there are no non-constant $J$-holomorphic spheres by Stokes’ theorem since $\omega = d\theta$ is exact). Thus, the $\mathcal{M}(x_-, x_+)$ have natural compactifications, whose boundaries are described by broken Floer trajectories (see Figure A.2). In particular, $\mathcal{M}_0(x_-, x_+)$ is already compact, so it is a finite set of points called isolated solutions.

3. Symplectic cohomology and symplectic homology

3.1. Symplectic chain complex

Pick a base field $\mathbb{K}$. Let $SC^*(H)$ denote the $\mathbb{K}$ vector space generated by the 1-orbits of $H$,

$$SC^*(H) = \bigoplus \{ \mathbb{K} \cdot x : x \in \mathcal{L}\overline{\mathcal{M}}, \dot{x}(t) = X(x(t)) \}.$$
The differential \( d \) on \( SC^*(H) \) is the \( \mathbb{K} \) linear map, which on a generator \( y \) is defined by counting incoming isolated Floer trajectories,

\[
dy = \sum_{u \in M_0(x,y)} \epsilon_u x
\]

where \( \epsilon_u \in \{ \pm 1 \} \) are orientation signs (see Appendix B). By convention, the constant solution \( u(s,t) = y(t) \in M_0(y,y) \) is not counted. A standard argument [35] shows that \( d \circ d = 0 \) (see Appendix B.6), so we can define \( SH^*(H) = H^*(SC^*(H); d) \).

When gradings are defined (Subsection 3.6), \( \dim M(x,y) = |x| - |y| - 1 \), so \( |x| = |y| + 1 \) in the above.

### 3.2. Continuation maps

Let \((H_\pm, J_\pm)\) be two choices of data for which we have defined \( SH^* \) (the data may depend on \( t \in S^1 \)). Let \((H_z, J_z)_z=2(t)\in \mathbb{R} \times S^1 \) be an interpolation of the data such that for large \( |s| \), \( H_z = H_\pm \) and \( J_z = J_\pm \). Assume \( J_z \) is of contact type for \( R \gg 0 \).

The moduli space \( M_H^\pm(x_-, x_+) \) of Floer continuation solutions are the \( v: \mathbb{R} \times S^1 \rightarrow \mathbb{M} \) solving \( \partial_s v + J_z(\partial_t v - X_{H_z}) = 0 \), converging to 1-orbits \( x_\pm \) of \( H_\pm \) as \( s \rightarrow \pm \infty \). We make no identifications of solutions (there is no \( \mathbb{R} \) reparametrization freedom since \( J_z \) depends on \( s \)).

The maximum principle (Lemma D.1) holds for such \( v \) if \((H_z, J_z)\) is a monotone homotopy:

> for large \( R \), we assume: \( H_z = h_z(R), \partial_s h'_z \leq 0, J'_z d\theta = dR \) (contact type condition).

In particular, the slopes \( m_\pm \) of the \( H_z \) at infinity must satisfy \( m_+ \leq m_- \).

As usual, we often conceal the \( t \)-dependence from the notation, so we talk about a homotopy \((H_s, J_s)\) from \((H_, J_-)\) to \((H_+, J_+),\) instead of writing \((H_z, J_z)\).

Say the boxed assumption holds for \( R \geq R_0 \), then all \( v \in M_H^s(x_-, x_+) \) are contained in the compact set \( C \) defined by \( R \leq \max(R(x_\pm), R_0) \). Suppose that \( H_s \) and \( J_s \) are \( s \)-dependent only for \( a \leq s \leq b \), then we get an a priori energy estimate for all \( v \in M_H^s(x_-, x_+) \),

\[
E(v) = \int |\partial_s v|_{\mathfrak{g}_s}^2 ds \wedge dt
\]

\[
= \mathcal{A}_{H_-}(x_-) - \mathcal{A}_{H_+}(x_+) + \left( \partial_s H_s \right)(v) ds \wedge dt
\]

\[
\leq \mathcal{A}_{H_-}(x_-) - \mathcal{A}_{H_+}(x_+) + (b-a) \cdot \sup_{m \in C} |\partial_s H_s(m)|.
\]

For generic \((H_z, J_z)\), the \( M_{H_z}(x_-, x_+) \) are smooth manifolds with compactifications by broken solutions (Floer trajectories for \( H_\pm \) may break off at the respective ends of a continuation solution, see Figure A.2). More precisely, as in the Technical Remarks in Subsection 2.5, given \((H_z, J_z)\) for which \((H_\pm, J_\pm)\) satisfy non-degeneracy, it suffices to make a \( C^2 \) small \( z \)-dependent perturbation of \( H_z \) or \( J_z \) (or both) away from the ends to ensure that \( M_{H_z}(x_-, x_+) \) is smooth.

For monotone \( H_s \), counting incoming isolated Floer continuation solutions defines a continuation map \( \varphi: SC^*(H_+) \rightarrow SC^*(H_-) \). On generators

\[
\varphi(x_+) = \sum_{v \in M_0^H(x_-, x_+)} \epsilon_v x_-,
\]

where \( \epsilon_v \in \{ \pm 1 \} \) are orientation signs (see Appendix B). Then extend \( \varphi \) linearly. A standard argument [35] shows that \( \varphi \) is a chain map (see Appendix B.7).

When gradings are defined (Subsection 3.6), \( \dim M_{H_z}(x_-, x_+) = |x_-| - |x_+| \), so \( |x_-| = |x_+| \) in the above.
Lemma 3.1 (see, for example, [31]).

(1) Changing the monotone homotopy \((H_s, J_s)\) changes \(\varphi\) by a chain homotopy. So on cohomology \(\varphi : SH^*(H_s) \to SH^*(H_t)\) is independent of the choice of \((H_s, J_s)\).

(2) \(\varphi\) equals any composite \(SH^*(H_s) \to SH^*(K) \to SH^*(H_t)\) induced by continuation maps for monotone homotopies from \(H_-\) to \(K\) and from \(K\) to \(H_+\).

(3) The constant homotopy \(H_s = H\) induces the identity on \(SH^*(H)\).

(4) For \(H_{\pm}\) linear at infinity of the same slope, \(\varphi\) is an isomorphism in cohomology.

3.3. Hamiltonians \(H^m\) of slope \(m\) at infinity

Let \(H^m\) be any Hamiltonian equal to \(mR + \text{constant}\) for large \(R\) (for generic \(m > 0\) so that \(m\) is not a Reeb period). By Lemma 3.1, \(SH^*(H^m)\) only depends on \(m\) and continuations \(SH^*(H^m) \to SH^*(H^{m'})\) exist for \(m \leq m'\).

On cohomology \(SH^*(H^m)\) only depends on the slope \(m\) by Lemma 3.1(4), so if one wanted, one could always take \(H^m = mH\) for a fixed Hamiltonian \(H : \overline{M} \to \mathbb{R}\) of slope 1.

3.4. Symplectic cohomology

Define \(SH^*(M)\) by taking the direct limit over the above continuation maps between the \(SH^*(H)\) groups for Hamiltonians linear at infinity,

\[
SH^*(M) = SH^* \left(\overline{M}\right) = \lim_{\to} SH^*(H).
\]

So \(SH^*(M) \cong \lim SH^*(H^m_k)\) for any \(H^m_k\) as in Subsection 3.3 with slopes \(m_k \to \infty\) as \(k \to \infty\).

In Theorem 6.8, we prove that the continuation maps \(SH^*(H^m) \to SH^*(H^{m'})\) are the pair-of-pants product by a special element \(e_{H^m} \in SH^0(H^m)\) for \(m = m' - m \geq 0\).

In Appendix C, we prove that \(SH^*(M)\) can be defined as \(SH^*(Q)\) for one Hamiltonian \(Q : \overline{M} \to \mathbb{R}\) growing faster than linearly at infinity. This makes many arguments unnecessarily complicated (see Appendix C.2) so for now we assume our Hamiltonians to be linear at infinity.

3.5. Invariance under symplectomorphisms of contact type

For a proof of the invariance of \(SH^*(M)\) we refer to [12, 39, 43]. The following formulation is written in detail in [32, Theorem 8], which is based on Seidel’s exposition [39, Section (3e)].

Let \(M\) and \(N\) be Liouville domains. A symplectomorphism \(\varphi : \overline{M} \to \overline{N}\) is of contact type if

\[
\varphi^* \theta_N = \theta_M + d(\text{compactly supported function}).
\]

It follows that at infinity \(\varphi(e^r, y) = (e^r - f(y), \psi(y))\), where \(y \in \partial M\), for a smooth \(f : \partial M \to \mathbb{R}\) and a contact isomorphism \(\psi : \partial M \to \partial N\) with \(\psi^* \alpha_N = e^f \alpha_M\).

Under such a map \(\varphi : \overline{M} \to \overline{N}\), the Floer solutions on \(\overline{N}\) for \((H, d\theta_N, J_N)\) correspond precisely to the Floer solutions on \(\overline{M}\) for \((\varphi^* H, d\theta_M, \varphi^* J_N)\). However, for \(H\) on \(\overline{N}\) linear at infinity, \(\varphi^* H(e^r, y) = h(e^r - f(y))\) is not linear at infinity.

So \(SH^*(N)\) is isomorphic via \(\varphi^*\) to \(SH^*_f(M) = \lim SH^*(H_f)\) calculated for Hamiltonians of the form \(H_f = h(R_f)\), with \(R_f = e^r - f(y)\), using almost complex structures \(J\) satisfying \(J^* d\theta = dR_f\). It still turns out that \(SH^*(M) \cong SH^*_f(M)\). This is proved by a continuation argument by homotopying \(f\) to zero and proving a maximum principle for \(R_f \circ u\) for such homotopies (for example, see [32, Lemma 7]).

Lemma 3.2. A contact type \(\varphi : \overline{M} \cong \overline{N}\) induces \(\varphi_* : SH^*(M) \cong SH^*(N)\).
3.6. Grading of symplectic cohomology

We refer to [34, 35] for a detailed exposition on the Maslov index and gradings in Floer theory. We now explain that if $c_1(M) = c_1(TM, J)$ is zero, then $SH^*(H)$ is $\mathbb{Z}$-graded. In Appendix B.16, we discuss dimension counts for Floer moduli spaces and we discuss the case $c_1(M)\tau_2M = 0$.

Since $c_1(M) = 0$, we can pick a trivialization of the canonical bundle $K = \Lambda^{n,0}T^*M$ (Explanation: we can view $T^*M$ as a complex vector bundle of rank $n = (\dim_{\mathbb{R}} M)/2$ since we chose an almost complex structure $J$ on $M$. Then $K$ is a complex line bundle with $c_1(K) = -c_1(M)$. If $H^1(M) = 0$, then maps $M \to U(1)$ are contractible, so there is only one homotopy class of trivializations for $K$, and so the above choice of trivialization will not matter.)

Over any 1-orbit $x$ for $H$, trivialize $x^*TM$ so that it induces an isomorphic trivialization of $K$. Let $\phi_t$ denote the linearization $D\phi_{X_H}(x(0))$ of the time $t$ flow for $X_H$ written in the trivializing frame for $x^*TM$. Let $\text{sign}(t)$ denote the signature of the quadratic form $\omega(\cdot, \partial_t\phi_t\cdot) : \ker(\phi_t - \text{id}) \to \mathbb{R}$, assuming that we perturbed $\phi_t$ relative endpoints to make the quadratic form non-degenerate and to make $\ker(\phi_t - \text{id}) = 0$ except at finitely many $t$.

The Maslov index $\mu(x)$ of $x$ is $\mu(x) = \frac{1}{2}\text{sign}(0) + \sum_{0 < t < 1} \text{sign}(t) + \frac{1}{2}\text{sign}(1)$.

The Maslov index is invariant under homotopy relative endpoints, and it is additive with respect to concatenations. If $\phi_t$ is a loop of unitary transformations, then its Maslov index is the winding number of the determinant, $\det(\phi_t) : K \to K$.

For example, $\phi_t = e^{2\pi it} \in U(1)$ for $t \in [0,1]$ has Maslov index 1.

So when $c_1(M) = 0$, we can define the following $\mathbb{Z}$-grading $\lfloor \cdot \rceil$ on generators of $SC^*(H)$:

$$|x| = \frac{\dim(M)}{2} - \mu(x).$$

We call this the Conley–Zehnder index of $x$ (there are various conventions, we explain in Remark 3.3 why we chose this convention). As mentioned in Subsection 3.2, the continuation maps preserve this $\mathbb{Z}$-grading, so also $SH^*(M)$ is $\mathbb{Z}$-graded.

When $c_1(M) \neq 0$ this still defines a $\mathbb{Z}/2\mathbb{Z}$ grading since the Conley–Zehnder indices change by an even integer in $2c_1(M)(\tau_2(M))$ when we change the trivialization over a Hamiltonian orbit. In particular, Koszul signs $(-1)^{|x|}$ are well-defined (we will need this in Appendix B.15).

Remark 3.3. Our convention ensures that $|x|$ equals the Morse index of $x$ when $x$ is a critical point of a $C^2$ small Morse $H : \mathcal{M} \to \mathbb{R}$. This ensures that: the unit of $SH^*(M)$ lies in degree 0; the product is additive on degrees; and the canonical map $c^* : H^*(M) \to SH^*(M)$ defined in Section 5 is degree-preserving. This differs from Schwarz’s convention [38], $\|x\| = n - \text{ind}_{\text{Morse}}(x)$, which explains why the index of Theorem A.4 is $-\sum \|x_a\| + \sum \|y_b\| + n \cdot \chi(S)$ in [38], with $\chi(S) = 2 - 2g - p - q$. Our convention actually agrees with Salamon’s grading $\mu_H$ (see [35, Exercise 2.8]), since we use an opposite sign convention for $H$ in the action functional.

3.7. Symplectic homology

Using the notation from Subsections 3.1 and 3.2, define:

$$SC_s(H) = \prod \{ \mathbb{K}x : x \in \mathcal{L}M, \dot{x}(t) = X(x(t)) \},$$
$$\delta : SC_s(H) \to SC_{s+1}(H), \quad \delta x = \sum_{u \in \mathcal{M}_0(x,y)} \epsilon_u y \quad (\text{cf. Subsection 3.1}),$$
\[ \varphi_* : SC_*(H_-) \to SC_*(H_+) \], \quad \varphi_*(x_-) = \sum_{v \in M_0^{H_*}(x_-, x_+)} \epsilon_v x_+ \quad (\text{cf. Subsection 3.2}), \]

\[ SH_*(H) = H_*(SC_*(H); \delta) \quad (\text{cf. Subsection 3.1}), \]

\[ SH_*(M) = \lim SH_*(H) \quad (\text{taking the inverse limit over the } \varphi_*, \text{ cf. Subsection 3.4}). \]

Note that we count solutions flowing out of a generator not into. This reverses all maps, so the symplectic homology is the inverse limit over continuation maps.

**Technical Remark.** \( SC_*(H) \) and \( SC_*(H) \) are canonically identifiable as graded vector spaces, since our choice of \( H \) ensures that there are only finitely many generators, so there is no actual difference between \( \bigoplus \) and \( \prod \). Of course, the real difference comes from the fact that the differentials and continuation maps are reversed. If we had used a Hamiltonian of quadratic growth (see Appendix C), then it is crucial to use \( \prod \) since \( \delta x \) may involve infinitely many generators (unlike \( dx \) for \( SC_*(H) \) which is a finite sum for energy/action reasons).

### 3.8. Symplectic homology is the dual of symplectic cohomology

Observe that

\[ SC_*(H) = \text{Hom}(SC_*(H), \mathbb{K}) \]

canonically and that the differentials \( \delta \) and \( d \) are dual to each other, indeed on generators: 
\[ \delta x = \sum A_{x,y} y \text{ and } dy = \sum A_{x,y} x \] where \( A_{x,y} = \#M_0(x, y) \) (count the elements \( u \) with signs \( \epsilon_u \)). Thus, \( \delta \) and \( d \) are represented by the matrices \( (A_{y,x}) \) and \( (A_{x,y}) \), so one is the transpose of the other. Thus, \( SC_*(H) \) is canonically the dual of \( SC_*(H) \), and so by universal coefficients \( SH_*(H) \) is the dual of \( SH_*(H) \) and comes with a canonical isomorphism

\[ SH_*(H) \to \text{Hom}(SH_*(H), \mathbb{K}). \]

Similarly, \( \varphi_* : SC_*(H_-) \to SC_*(H_+) \) is dual to the \( \varphi^* = \varphi : SC_*(H_+) \to SC_*(H_-) \) from Subsection 3.2.

**Theorem 3.4.** Symplectic homology is canonically dual to symplectic cohomology:

\[ SH_*(M) \cong SH^*(M)^\vee. \]

**Proof.** Using the above canonical isomorphisms, we get the commutative diagram

\[ SH_*(H_+) \cong \text{Hom}(SH_*(H_+), \mathbb{K}), \]

\[ \varphi^* \]

\[ SH_*(H_-) \cong \text{Hom}(SH^*(H_-), \mathbb{K}). \]

By category theory, \( \text{Hom}(\lim M_i, N) \cong \lim \text{Hom}(M_i, N) \), for any module \( N \) and any directed system of modules \( M_i \). Using \( SH^*(H), \mathbb{K} \) in place of \( M_i \) and \( N \), we deduce

\[ SH^*(M)^\vee \equiv \text{Hom}(SH^*(M), \mathbb{K}) \cong \lim \text{Hom}(SH_*(H), \mathbb{K}) \cong \lim SH_*(H) = SH_*(M) \]

using the commutative diagram to obtain the third identification. \( \square \)

**Corollary 3.5.** \( SH^*(M) = 0 \) if and only if \( SH_*(M) = 0. \)
4. Wrapped Floer cohomology

4.1. Lagrangians inside Liouville domains

Observe Figure 11 (ignore W). Let \((M^{2n}, d\theta)\) be a Liouville domain. Let \(L^n \subset M\) be an exact Lagrangian submanifold with Legendrian boundary \(\partial L = L \cap \partial M\) such that this intersection is transverse. \(\text{Exact Lagrangian means the pull-back } \theta|_L = \text{exact. Legendrian means } T\partial L \subset \ker \alpha,\) or equivalently \(\theta|_{\partial L} = 0.\)

We strengthen the last condition: the pull-back \(\theta|_L = df\) vanishes near \(\partial L\). This stronger condition can always be achieved for the new data obtained after deforming \(L\) by a Hamiltonian isotopy of \(M\) relative to \(\partial M\) (see [4, Lemma 4.1]).

This condition ensures that near \(\partial M, L\) has the form \((\text{interval}) \times \partial L\) in the coordinates \((0, \infty) \times \partial M \subset \overline{M}\) of Subsection 2.1. We extend \(L\) to the non-compact exact Lagrangian

\[L = L \cup ([1, \infty) \times \partial L) \subset \overline{M},\]

with \(\theta|_{\overline{L}} = df\), and \(f\) locally constant on \(\overline{L} \setminus L\) since \(\theta|_{\overline{L} \setminus L} = 0.\)

**Example 4.1.** The fibre \(\bar{L} = T_{q}^*N\) in \(\overline{M} = T^*N\), where \(M = DT^*N\). More generally, the conormal bundle \(\bar{L} = N^*K = \{(q, p) : q \in K, p|T_qK = 0\} \subset T^*N\) of a proper submanifold \(K \subset N\).

4.2. Hamiltonian chord and Reeb chords

A **Hamiltonian chord** of \(H : \overline{M} \to \mathbb{R}\) is a map

\[x : [0, 1] \to \overline{M}\]

with \(x(t) = X(x(t))\) and ends \(x(0), x(1) \in \bar{L}\),

where we recall that \(X = X_H\) is defined by \(\omega(\cdot, X) = dH\). A **Reeb chord** of period \(T\) is a map

\[y : [0, T] \to \partial M\]

with \(y(t) = R(y(t))\) and ends \(y(0), y(1) \in \partial L\).

Let \(\varphi_H^t\) denote the time \(t\) flow of \(X\). The Hamiltonian chords correspond to the intersections \(\varphi_H^1(L) \cap L\). We choose \(H\) such that on \(R < 1, H\) is \(C^2\) small and Morse, and on \(R \geq 1, H = h(R), h' > 0,\) and \(h' = m\) for \(R \gg 0.\) It follows that:

**Lemma 4.2.** The Hamiltonian chords \(x\) in the collar have constant \(H(x) = h(R)\) and correspond to Reeb chords \(y(t) = x(t/T)\) of period \(T \leq m\) where \(T = h'(R)\).

**Lemma 4.3.** For generic \(m, H\) and \(\bar{L}\), there are only finitely many Hamiltonian chords.

**Proof.** Unlike the \(SH^*(H)\) construction, we will not need to make time-dependent perturbations of \(H\), but we need to allow perturbations \(\psi(L)\) of \(L\) by a compactly supported Hamiltonian flow \(\psi\) (one can alternatively view this as keeping \(L\) fixed but perturbing \(H\) to \(H \circ \psi^{-1}\), at the cost of losing Lemma 4.2). For generic \(m > 0\), there are no Reeb chords of period \(m\) (by a Sard’s Lemma argument). After a small generic (time-independent) compactly supported Hamiltonian perturbation of \(H\) and \(\bar{L}\) one can ensure the Hamiltonian chords are non-degenerate, that is, \(\varphi_H^1(L)\) is transverse to \(L\) (see [4, Lemma 8.1]). So the Hamiltonian chords are isolated. \(\square\)

4.3. Action functional

Consider the space of smooth paths with ends in \(\bar{L}:

\[\Omega(\overline{M}, \bar{L}) = \{x \in C^\infty([0, 1], \overline{M}) : x(0), x(1) \in \bar{L}\}.\]
Define $\mathcal{H}_H : \Omega(\overline{M}, L) \to \mathbb{R}$ analogously to Subsection 2.2,

$$\mathcal{H}_H(x) = f(x(1)) - f(x(0)) - \int x' \theta + \int_0^1 H(x(t)) \, dt.$$  

The motivation for the first three terms is that they would arise from $-\int u^* \omega$ by Stokes’ theorem if $u$ were a disc with boundary in $\overline{L} \cup \text{image}(x)$. This ensures that $d\mathcal{H}_H : \xi = -\int_0^1 \omega(\xi, \dot{x} - X) \, dt$, where $\xi \in T_x \Omega(\overline{M}, L) = \{ \xi \in C^\infty([0, 1], x^* T\overline{M}) : \xi(0), \xi(1) \in T\overline{L} \}$. Therefore, the critical points of $\mathcal{H}_H$ are the Hamiltonian chords.

For Hamiltonian chords $x$ on the collar, $\mathcal{H}_H(x) = f(x(1)) - f(x(0)) - Rh'(R) + h(R)$.

4.4. Wrapped trajectories

Pick $J$ and $g$ as in Subsection 2.1. The solutions $u : \mathbb{R} \to \Omega(\overline{M}, L)$ of $\partial_s u = -\nabla \mathcal{H}_H$ are the solutions $u : \mathbb{R} \times [0, 1] \to \overline{M}$ of

$$\partial_s u + J(\partial_t u - X) = 0,$$

with Lagrangian boundary conditions $u(\cdot, 0), u(\cdot, 1) \in \overline{L}$. Let $\mathcal{W}(x_-, x_+)$ denote the solutions converging to $x_\pm$ at the ends $s \to \pm \infty$. Let $\mathcal{W}(x_-, x_+) = \mathcal{W}(x_-, x_+)/\mathbb{R}$ denote the moduli space of wrapped trajectories, identifying $u(\cdot, \cdot) \sim u(\cdot + \text{constant}, \cdot)$.

**Remark 4.4.** These moduli spaces define the Lagrangian Floer cohomology $HF^*(\varphi^1_H, (\overline{L}, \overline{L}))$. Indeed wrapped trajectories $u(s, t)$ correspond precisely to pseudo-holomorphic strips with boundaries in $\varphi^1_H(\overline{L})$ and $\overline{L}$, that is, solutions of $\partial_s v + \overline{J}_t \partial_t v = 0$, converging to $v(\pm \infty, \cdot) = x_\pm \in \varphi^1_H(\overline{L}) \cap \overline{L}$, where $J_t = d\varphi^1_H^{-1} \circ J \circ d\varphi^1_H^{-1}$.

**Proof.** Let $v(s, t) = \varphi^1_H^{-1}(u(s, t))$.

4.5. Energy

The energy of $u \in \mathcal{W}(x_-, x_+)$ is defined as $E(u) = \int |\partial_s u|^2 \, ds \wedge dt$. The same calculation as in Subsection 2.4, using $\theta|_\overline{L} = df$, yields $E(u) = \mathcal{H}_H(x_-) - \mathcal{H}_H(x_+)$. 

4.6. Transversality and compactness

Just as in the Floer case, a generic time-dependent perturbation of $J$ ensures that the $\mathcal{W}(x_-, x_+)$ are smooth manifolds. By Lemma D.2 we get the following lemma:

**Lemma 4.5.** All $u \in \mathcal{W}(x_-, x_+)$ lie in $R \leq \max(R(x_\pm), R_0)$ for $J$ of contact type on $R \geq R_0$.

The energy estimate in Subsection 4.5 implies that the $\mathcal{W}(x_-, x_+)$ have compactifications by broken wrapped trajectories. Indeed the analysis for $\mathcal{W}(x_-, x_+)$ reduces to the known case of Lagrangian Floer cohomology by Remark 4.4.

**Technical Remark.** In general, in addition to breaking, limits of families of pseudo-holomorphic strips (in Remark 4.4) may also carry sphere bubbles and disc bubbles which bound $\overline{L}$ or $\varphi^1_H(\overline{L})$. However, since in our setup $\omega$ is exact and $\overline{L}$ is exact, these bubbling phenomena are ruled out by Stokes’ theorem. The bubbling-off analysis in the Lagrangian case is due to Floer [18].
4.7. Wrapped Floer cohomology

Pick a base field $\mathbb{K}$. The wrapped Floer complex $CW^*(L; H)$ is the $\mathbb{K}$ vector space generated by the Hamiltonian chords of $H$,

$$CW^*(L; H) = \bigoplus \{ \mathbb{K} x : x \in \Omega(M, L), \dot{x}(t) = X(x(t)) \},$$

whose differential $d$ on generators counts isolated wrapped trajectories,

$$dy = \sum_{u \in \mathcal{W}_0(x,y)} \epsilon_u x,$$

where $\epsilon_u \in \{ \pm 1 \}$ are orientation signs (see Remark 4.6). The constant solution $u(s, t) = y(t) \in \mathcal{W}_0(y, y)$ is not counted. That $d^2 = 0$ is a standard consequence of Subsection 4.6. We call $HW^*(L; H) = H^*(CW^*(L; H); d)$ the wrapped Floer cohomology for $H$.

**Remark 4.6.** For fields $\mathbb{K}$ of characteristic 2, the orientation signs are not necessary. In general, if $L$ is spin (the Stiefel–Whitney class $w_2(L) = 0 \in H^2(L; \mathbb{Z}/2)$), then ([40, Section (12b)]) orientation signs can be defined. If the relative Chern class $2c_1(M, L) = 0 \in H^2(M, L; \mathbb{Z})$, then a grading on Hamiltonian chords can be defined and $\dim \mathcal{W}(x_-, x_+) = |x_-| - |x_+| - 1$. For a detailed discussion and further references we refer the reader to [4, Section 9].

**Remark 4.7.** Via Remark 4.4, $HW^*(L; H) \cong HF^*(\varphi_L^1(L), \bar{L})$. Due to the non-compactness of $\bar{M}$, this heavily depends on $H$: as the slope $m \to \infty$, $\varphi_L^1(L)$ wraps more and more times around $\bar{L}$ giving rise to more chain generators (namely intersections $\varphi_L^1(L) \cap \bar{L}$).

As in Subsection 3.2, a monotone homotopy $H_s$ defines a wrapped continuation $\varphi : HW^*(H_+; L) \to HW^*(H_-; L)$ by counting wrapped continuation solutions. These are solutions $v : \mathbb{R} \times [0, 1] \to \bar{M}$ of $\partial_s v + J_s(\partial_t v - X_{H_s}) = 0$ with Lagrangian boundary conditions $v(\cdot, 0), v(\cdot, 1) \in \bar{L}$, converging to Hamiltonian chords on the ends $s \to \pm \infty$. The a priori energy estimate for wrapped continuation solutions is the same as in Subsection 3.2 and the maximum principle holds by Lemma D.2. This implies the smoothness and compactifiability of the moduli spaces and the analogue of Lemma 3.1. Thus, $HW^*(L; H)$ only depends on the slope $m$ of $H$ at infinity, and as in Subsection 3.3 for $m \leq m'$ there is a wrapped continuation $HW^*(L; H^m) \to HW^*(L; H^{m'})$.

**Definition.** Define the wrapped Floer cohomology by

$$HW^*(L) = \lim_{k \to \infty} HW^*(L; H),$$

taking the direct limit over wrapped continuation maps between Hamiltonians linear at infinity (Subsection 3.3). So $HW^*(L) \cong \lim_{k \to \infty} HW^*(L; H^{m_k})$ for any sequence of slopes $m_k \to \infty$ as $k \to \infty$.

In Theorem 6.16, we prove that the continuation maps $HW^*(L; H^m) \to HW^*(L; H^{m'})$ are the open pair-of-pants product by a special element $e_{H^\ell} \in HW^0(L; H^\ell)$ for $\ell = m' - m \geq 0$.

The $HW^*(L)$ are invariant under symplectomorphisms of contact type, arguing as in Section 3.5. The $HW^*(L; H)$ are invariant under compactly supported Hamiltonian isotopies of $L$ (this follows from Remark 4.7 and invariance of Lagrangian Floer cohomology). So, $HW^*(L; H)$ is invariant under compactly supported isotopies of $\bar{L}$ through exact Lagrangians (since these are automatically Hamiltonian isotopies by the Weinstein neighbourhood theorem and the fact that graphs of 1-forms $\mu$ on $\bar{L}$ are exact in $T^*\bar{L}$ precisely when $\mu$ is exact). So, $HW^*(L)$ is also invariant under compactly supported isotopies of $\bar{L}$ through exact Lagrangians.
5. Canonical $c^*$ maps from ordinary cohomology

For $\delta > 0$ smaller than all Reeb periods, consider $H^{\delta}$ as in Subsection 3.3. If $H^{\delta}$ is time-independent, Morse and $C^2$ small on $M$, then $SC^*(H^{\delta})$ reduces to the Morse complex for $H^{\delta}$, generated by Crit($H^{\delta}$) and whose differential counts $-\nabla H^{\delta}$ trajectories (for a proof, see Subsection 15.3). The Morse cohomology is isomorphic to the ordinary cohomology, so $SH^*(H^{\delta}) \cong H^*(M; \mathbb{K})$. Since $H^{\delta}$ is part of the direct limit construction, we automatically obtain a map

$$c^*: H^*(M; \mathbb{K}) \to SH^*(M) \quad \text{(and dually } c_*: SH_*(M) \to H_*(M; \mathbb{K})).$$

We often write $H^*(M)$ instead of $H^*(M; \mathbb{K})$. The analogue for the wrapped case is

$$c^*: H^*(L; \mathbb{K}) \to HW^*(L).$$

This arises from $c^*: HW^*(L; H^{\delta}) \to HW^*(L)$, where $H^{\delta}$ is as above so that all intersections $\varphi_{H^{\delta}}(L) \cap L$ lie inside $M$. By Lemma 4.5, all wrapped trajectories lie in $M$, so we reduce to a compact setup with $\omega = dt$. By Remark 4.7, $HW^*(L; H^{\delta}) \cong HF^*(\varphi_{H^{\delta}}^1(L), L)$, and by [18] this is isomorphic to the Morse cohomology $MH^*(L; H^{\delta})$. The idea is $\varphi_{H^{\delta}}^1(L)$ lies in a Weinstein neighbourhood $W \cong DT^*L$ of $L$ so it is the graph($dt) \subset DT^*L$ of a function $\ell \in C^\infty(L)$. The intersections $\varphi_{H^{\delta}}^1(L) \cap L$ correspond to Crit($\ell$), and by an implicit function theorem argument, if $H^{\delta}$ is $C^2$ small in $M$, then the evaluation $u \mapsto u(s, 0)$ sets up a bijection between pseudo-holomorphic strips $M$ with boundary in $L$ and $-\nabla \ell$ trajectories in $L$.

6. TQFT structure on $SH^*(M)$, $H^*(M)$, $HW^*(L)$

6.1. Summary of the TQFT on $SH^*(M)$

In Appendix A, we carry out the detailed construction of the TQFT structure on $SH^*(M)$, which we summarize here. Suppose that we are given the following.

(1) A Riemann surface $(S, j)$ with $p + q$ punctures, with fixed complex structure $j$.
(2) $\text{Ends}$: a cylindrical parametrization $s + it$ near each puncture, with $j\partial_s = \partial_t$.
(3) $p \geq 1$ of the punctures are negative (so $s \to -\infty$), they are indexed by $a = 1, \ldots, p$.
(4) $q \geq 0$ of the punctures are positive (so $s \to +\infty$), they are indexed by $b = 1, \ldots, q$.
(5) Weights: constants $A_a, B_b > 0$ satisfying $\sum A_a - \sum B_b \geq 0$.
(6) A $1$-form $\beta$ on $S$ with $d\beta \leq 0$, and on the ends $\beta = A_a dt$, $\beta = B_b dt$ for large $|s|$.

REMARKS. Negative/positive parametrizations are modelled on $(-\infty, 0] \times S^1$ and $[0, \infty) \times S^1$. In (6), $d\beta \leq 0$ means $d\beta(v, jv) \leq 0$ for all $v \in TS$. By Stokes, $\sum A_a - \sum B_b = -\int_S d\beta \geq 0$. This forces $p \geq 1$ and (5). Subject to this inequality, such $\beta$ exist (Lemma A.1).

Fix a Hamiltonian $H : \overline{M} \to \mathbb{R}$ linear at infinity (Subsection 3.3) with $H \geq 0$ (required in Appendix A.3), this defines $X = X_H$ (Subsection 2.1). Fix an almost complex structure $J$ on $M$ of contact type at infinity.

The moduli space $\mathcal{M}(x_a; y_b; S, \beta)$ of Floer solutions consists of smooth maps $u : S \to \overline{M}$ such that $du - X \otimes \beta$ is $(J, J)$-holomorphic, and $u$ converges on the ends to $1$-orbits $x_a, y_b$ of $A_a H$ and $B_b H$ which we call the asymptotics. After a small generic $S$-dependent perturbation $J_\epsilon$ of $J$, $\mathcal{M}(x_a; y_b; S, \beta)$ is a smooth manifold. One can ensure that on the ends $J_\epsilon$ does not depend on $z = s + it \in S$ for $|s| > 0$. Just as for Floer continuations (Subsection 3.2), a maximum principle and an a priori energy estimate $E(u) = \sum A_{A_a H}(x_a) - \sum B_{B_b H}(y_b)$ hold, so the $\mathcal{M}(x_a; y_b; S, \beta)$ have compactifications by broken Floer solutions: Floer trajectories for $A_a H$ and $B_b H$ can break off at the respective ends (Figure A.1). When gradings are defined
(Subsection 3.6),
\[ \dim \mathcal{M}(x_a; y_b; S, \beta) = \sum |x_a| - \sum |y_b| + 2n(1 - g - p). \]

Define \( \psi_S : \bigotimes_{b=1}^{q} SC^*(B_b H) \to \bigotimes_{a=1}^{p} SC^*(A_a H) \) on generators by counting isolated Floer solutions
\[ \psi_S(y_1 \otimes \ldots \otimes y_q) = \sum_{u \in \mathcal{M}_0(x_a; y_b; S, \beta)} \epsilon_u x_1 \otimes \ldots \otimes x_p, \]
where \( \epsilon_u \in \{ \pm 1 \} \) are orientation signs (Appendix B). Then extend \( \psi_S \) linearly.

**Remark.** We use cohomological conventions: the operation \( \psi_S \) receives inputs \( y_b \) at the positive punctures of \( S \), and emits outputs \( x_a \) at the negative punctures. So the operation goes ‘from right to left’ if we draw the surface \( S \) as in Figure 3.

The \( \psi_S \) are chain maps. On cohomology, \( \psi_S : \bigotimes_{b=1}^{q} SH^*(B_b H) \to \bigotimes_{a=1}^{p} SH^*(A_a H) \) is independent of the choices \( (\beta, j, J) \) relative to the ends. Taking direct limits:
\[ \psi_S : SH^*(M)^{\otimes q} \to SH^*(M)^{\otimes p} \quad (p \geq 1, q \geq 0). \]

So, \( SH^*(M) \) has a unit \( \psi_C(1) \), product \( \psi_P \), coproduct \( \psi_Q \), but no counit since \( p \geq 1 \).

For \( SH_* \) all arrows are reversed: \( \psi^S : \bigotimes_{a=1}^{p} SC_* (A_a H) \to \bigotimes_{b=1}^{q} SC_* (B_b H) \), on generators
\[ \psi^S(x_1 \otimes \ldots \otimes x_p) = \sum \epsilon_u y_1 \otimes \ldots \otimes y_q \text{ summing over } u \in \mathcal{M}_0(x_a; y_b; S, \beta). \]
Take inverse limits:
\[ \psi^S : SH_* (M)^{\otimes p} \to SH_* (M)^{\otimes q} \quad (p \geq 1, q \geq 0). \]

So, \( SH_* (M) \) has a counit \( \psi^C \), product \( \psi^P \), coproduct \( \psi^Q \), but no unit since \( p \geq 1 \).

6.2. **The product**

The pair-of-pants surface \( P \) (Figure 1) defines the product
\[ \psi_P : SH^i(M) \otimes SH^j(M) \to SH^{i+j}(M), \quad x \cdot y = \psi_P(x, y), \]
which is graded-commutative and associative.

Commutativity follows from the construction of the orientation signs in Theorem B.3 (this uses only the \( \mathbb{Z}/2\mathbb{Z} \)-grading on \( SH^* \), Appendix B.15). At a deeper level, this commutativity is really a consequence of the fact that there is no preferred order for three points on a sphere (unlike three points on the boundary of a disc, which is why the product on \( HW^*(L) \) in Theorem 6.13 is typically not graded-commutative).

Associativity is proved by gluing: glue onto one positive end of \( P \) the negative end of another copy of \( P \). This yields a surface \( S' \) with \( p = 1 \) and \( q = 3 \), independent of the positive end we chose. So, \( \psi_P(\psi_P(x, y), z) = \psi_S(x, y, z) = \psi_P(x, \psi_P(y, z)) \) for \( x, y, z \in SH^*(M) \).
6.3. The unit

Let \( C = C \) with \( p = 1 \) and \( q = 0 \). The end is parametrized by \((-\infty, 0] \times S^1\) via \( s + it \mapsto e^{-2\pi(s+it)}\). On this end, \( \beta = f(s)dt \) with \( f'(s) \leq 0 \), with \( f(s) = 1 \) for \( s \leq -2 \) and \( f(s) = 0 \) for \( s \leq -1 \). Extend by \( \beta = 0 \) away from the end. Thus, \( \psi_C : \mathbb{K} \rightarrow SH^*(H) \) (Figure 4).

DEFINITION. Let \( e_H = \psi_C(1) \in SH^0(H) \). Define \( e = \lim_{s \to 0} e_H \in SH^0(M) \).

THEOREM 6.1. \( e \) is the unit for the multiplication on \( SH^*(M) \).

Proof. By the gluing illustrated in the picture below, \( \psi_P(e, \cdot) = \psi_P#C(\cdot) = \psi_Z(\cdot) = \text{id} \). □

REMARK. For ‘gluing = compositions’ results, see Theorems A.10, A.12, A.14. Before taking direct limits, the above is the continuation map \( SH^*(H) \xrightarrow{e_H \otimes} SH^*(H) \otimes^2 \psi_{\beta} : SH^*(2H) \).

LEMMA 6.2. \( e_H \) is a count of the isolated finite energy Floer continuation solutions \( u : \mathbb{R} \times S^1 \to \overline{M} \) for the homotopy \( f(s)H \) from \( H \) to \( 0 \).

Proof. As \( X_H \otimes \beta = X_{f(s)H} \otimes dt \), \( (du - X \otimes \beta)^{0,1} = 0 \) becomes \( \partial_s u + J(\partial_t u - X_{f(s)H}) = 0 \). Near \( s = \infty \) such \( u \) are \( J \)-holomorphic, \( \partial_s u + J\partial_t u = 0 \). The finite energy condition implies that \( u \) converges to an orbit of \( X_H \) at \( s = -\infty \) by Theorem A.3 and that the singularity at \( s = \infty \) is removable. So, \( u \) extends to a \( J \)-holomorphic map over the puncture at \( s = \infty \) (see [25, Section 4.2]) recovering the solution \( C \to \overline{M} \) counted by \( \psi_C \). □

LEMMA 6.3. For \( H = H^6 \) as in Section 5, \( e_H = \text{sum of the local minima of } H \).

We also consider more general surfaces \( S \) obtained from the above model surfaces via the following two operations: (1) taking disjoint unions; (2) gluing surfaces along several ends (provided \( p \geq 1 \) after gluing). Therefore the surface \( S \) may contain holes.

Proof. Pick \( J \) so that \( g = \omega(\cdot, J\cdot) \) is Morse–Smale for \( H \). Rescale \( H \) by a small constant so that all Floer continuation solutions for \( f(s)H \) are time-independent (this is possible by Claim 5 in Subsection 15.3: the Floer solutions for \( C \) stay inside \( M \) by the maximum principle, and we have an a priori energy estimate). Define \( \sigma(s) = -\int_s^0 f(\tilde{s}) \tilde{s} ds \), so \( \sigma'(s) = f(s) \). Suppose \( v \) is a \(-\nabla H \) flow line: \( v'(s) = -\nabla H \). Define \( u(s) = v(\sigma(s)) \) so \( u'(s) = -f(s)\nabla H = -\nabla(f(s)H) \). Thus, \(-\nabla(f(s)H) \) flow lines out of \( x \in \text{Crit}(H) \) contain a subset parametrized by \( W^u(x; H) \), so these are never isolated unless \( x \) is a local minimum, in which case the \(-\nabla(f(s)H) \) flow...
line is necessarily constant. The second claim follows because the Floer solutions counted by \( \psi_C : \mathbb{K} \to SH^*(H) \) have energy \( E(u) = k_H(x) \geq 0 \) by Appendix A.3.

**Theorem 6.4.** \( e = \lim \mathfrak{e}_H \) is the image of 1 under \( c^* : H^*(M) \to SH^*(M) \), and \( \mathfrak{e}_H = c_H^*(1) \), where \( c_H^* : H^*(M) \cong SH^*(H^\delta) \to SH^*(H) \) is the continuation map.

**Proof.** By Lemma 6.3, \( e_{H^\delta} = c_0^*(1) \) via \( c_0^* : H^*(M) \cong MH^*(H^\delta) \equiv SH^*(H^\delta) \). Via the continuation \( \varphi : SH^*(H^\delta) \to SH^*(H) \), \( e_H = \varphi(e_{H^\delta}) = \varphi(c_0^*1) = c_H^*(1) \) (by Theorem A.14). Now take direct limits. The claim also follows by Theorem 6.6.

**Remark.** By Section 8, one can always construct a Hamiltonian \( H = H^\ell \) of slope \( \ell \) so that \( c_H^* : MH^*(H^\delta) \equiv SH^*(H^\delta) \to SH^*(H) \) is the inclusion of a subcomplex.

### 6.4. A minimal set of generating surfaces under gluing

Write \( S_{pq} \) for the surface of genus zero with \( p \) negative and \( q \) positive punctures, and write \( S_{pqg} \) when it has genus \( g \). For example, the surfaces \( C = S_{10} \), \( Z = S_{11} \), \( P = S_{12} \), \( Q = S_{23} \) are shown in Figure 2, and \( S_{20} = Q \# C \), \( S_{101} = P \# S_{20} \), \( S_{111} = P \# Q \) are shown in Figure 5.

**Theorem 6.5.** Every operation \( \psi_S : SH^*(M)^\otimes q \to SH^*(M)^\otimes p \) is a composition of operations, chosen among the four basic ones: the unit \( \psi_C : \mathbb{K} \to SH^*(M) \), the identity \( \psi_Z = \text{id} \), the product \( \psi_P \) and the coproduct \( \psi_Q : SH^*(M) \to SH^*(M)^\otimes 2 \).

**Proof.** We already showed that \( C, Z, P \) and \( Q \) generate \( S_{20}, S_{101} \) and \( S_{111} \). As an illustration, consider \( S = S_{223} \) in Figure 5: \( S \) is isotopic to the last picture, so \( S = Z \# Q \# P \# S_{11} \# S_{11} \# S_{101} \). For general \( S \), we first isotope all punctures into a small disc \( D \subset S \); then the complement \( S \setminus D \) is the form \( S_{111} \# \ldots \# S_{11} \# S_{101} \) (with genus \( S \) – 1 copies of \( S_{111} \)). Since \( S = D \# (S \setminus D) \), we reduce to generating genus zero surfaces such as \( D \) (viewing \( \partial D \) as a positive puncture, for example, in the illustration \( D = S_{23} \)). By ordering the punctures from left to right as in the last picture in Figure 5 (negative punctures on the left), we observe that for \( q \geq 1 \), \( S_{pq} = Z \# Q \# \ldots \# Q \# P \# \ldots \# P \) (with \( p - 1 \) copies of \( Q \) and \( q - 1 \) copies of \( P \)) where we always only glue on the first puncture. Similarly, for \( p \geq 2 \), \( S_{p0} = Z \# Q \# \ldots \# Q \# S_{20} \) (with \( p - 2 \) copies of \( Q \)) always gluing only on the first puncture.

**Remark.** This is essentially the statement that \((1+1)\)-TQFTs are Frobenius algebras \([6]\), the only subtlety is that we must avoid using surfaces with \( p = 0 \).
For $p \geq 1$ and $q \geq 1$, the proof also shows that only $Z, P$ and $Q$ are needed to generate (for genus at least 1) move the last + puncture into the $S_{101}$ region to turn it into $S_{111} Z$).

6.5. The TQFT on $SH^*(M)$ is compatible with the filtration by $H_1(M)$

We can filter $SC^*(H) = \bigoplus \gamma SC^*_\gamma(H)$ by the homology classes $\gamma \in H_1(M)$ of the generators. The Floer differential preserves the filtration, and so do Floer operations on a cylinder and a cap. For the product, $\psi_S : SH^*_\gamma(M) \otimes SH^*_\tau(M) \to SH^*_{\gamma + \tau}(M)$, and for the coproduct, $\psi_Q : SH^*_\gamma(M) \to \bigoplus SH^*_\gamma(M) \otimes SH^*_\tau(M)$ summing over $\gamma + \tau = \gamma \in H_1(M)$. For general $S$, decompose $S$ as in Section 6.4. Thus,

$$\psi_S : \bigotimes_b SH^*_\gamma(M) \to \bigotimes_a SH^*_\gamma(M) \text{ is zero unless } \sum \gamma_a = \sum \gamma_b \in H_1(M).$$

We can also filter $SH^*(M) = \bigoplus \gamma SH^*_\gamma(M)$ by the free homotopy classes $\gamma \in [S^1, M]$ of the generators. The TQFT operations for genus zero surfaces are compatible with the filtration (the equation above holds after replacing $\sum$ by concatenation of free loops).

Let $SH_0^*(M)$ denote the summand corresponding to the contractible loops. Considering only contractible loops determine a TQFT with operations $\psi_S : SH_0^*(M)^{\otimes q} \to SH_0^*(M)^{\otimes p}$ ($p \geq 1, q \geq 0$). Also $c^* : H^*(M) \to SH_0^*(M) \subset SH^*(M)$ naturally lands in $SH_0^*(M)$.

6.6. Unital ring structure on $SH^*(M)$ over $\mathbb{Z}$

Suppose that we worked with a (commutative) ring $\mathbb{K}$ instead of a field $\mathbb{K}$ (in the twisted case in Section 7 this implies that $\Lambda$ is a ring, instead of a field). We used the assumption that $\mathbb{K}$ was a field in the proof of Theorem A.10 by using the Künneth theorem. For a ring $\mathbb{K}$, we can nevertheless define as in Subsection 6.9 the operations

$$\psi_S : SH^* \bigotimes_a B_a H \to H^* \bigotimes_a SC^*(B_a H) \to H^* \bigotimes_a SC^*(A_a H),$$

where the first map is explicitly $\bigotimes a x_a \mapsto \bigotimes a x_a$ over a field, this first map is Künneth’s isomorphism, but over principal ideal domains this map is typically only injective so we cannot in general define a reverse map $H^* \bigotimes_a SC^*(A_a H) \to \bigotimes_a SH^*(A_a H)$.

For surfaces $S$ with $p = 1$, we have $H^* \bigotimes_a SC^*(A_a H) = SH^*(A_1 H)$, so we do get all $\psi_S : \bigotimes_b SH^* \bigotimes_a B_a H \to SH^*(A_1 H)$. This part of the TQFT includes the unital ring structure.

6.7. Associated graphs

Given a decomposition as in Section 6.4, associate a directed graph $S'$ to the decomposition by replacing $C, Z, P$ and $Q$ by the corresponding graphs $C', Z', P'$ and $Q'$ (Figure 6) and gluing them according to the decomposition of $S$.

6.8. TQFT on $H^*(M)$ via Morse operations

Consider an oriented graph $S'$ associated to $S$ in Subsection 6.7. The oriented edges of $S'$ are parametrized by $(-\infty, 0], [0, \infty)$ or $[0, \ell]$ (for finite $\ell \geq 0$). We call them, respectively, negative ends $e_a$, positive ends $e_b$ and internal edges $e_c$ of length $\ell$. For each edge $e_i$, pick a Morse

\[ C' \quad \quad Z' \quad \quad P' \quad \quad Q' \]

**Figure 6.** Graphs $C', Z', P'$ and $Q'$. Dark dots are ‘internal’ vertices. Crosses are ‘external’ vertices, namely punctures where the edge is infinite.
function \( f_i : M \to \mathbb{R} \) with \(-\nabla f_i\) inward-pointing along the collar (this condition ensures that the Morse cohomology is isomorphic to \( H^*(M) \)).

For critical points \( x_a \in \text{Crit}(f_a), y_b \in \text{Crit}(f_b) \) define the moduli space of Morse solutions, \( M(x_a; y_b; S', f_i) \), consisting of the continuous maps \( u : S' \to M \) which are \(-\nabla f_i\) gradient flows along \( e_i \) and which converge to \( x_a \) and \( y_b \) at infinity on the edges \( e_a \) and \( e_b \). The value of \( u \) at vertices where edges meet corresponds to points lying in the intersection of stable/unstable manifolds of the \( f_i \). The smoothness of \( M(x_a; y_b; S', f_i) \) is guaranteed by making these intersections transverse by choosing the \( f_i \) generically.

After identifications with ordinary cohomology, counting isolated Morse solutions defines

\[
\psi_{S'} : H^*(M)^{\otimes q} \to H^*(M)^{\otimes p} \quad (p \geq 1, q \geq 0).
\]

This map does not depend on the choice of \( S' \) associated to \( S \), or the choices \( \ell_i, f_a \) and \( f_b \) by continuation and gluing arguments analogous to the \( SH^*(M) \) case. One could also define the operations for \( p = 0 \) using appropriate oriented graphs, so one actually gets the full TQFT.

These TQFT operations are well-known for closed manifolds \([7, 22]\). The cup product on \( H^*(M) \) corresponds to the Morse product operation \( \psi_{P^*} \) (Figure 6).

For \( C' \), we are counting isolated flow lines \( v : (-\infty, 0] \to M \) with \( v'(s) = -\nabla f \), but these are of course never isolated unless \( v(-\infty) \) is a local minimum of \( f \). So, \( \psi_{C'}(1) \) is the sum of local minima of \( f \), which is the element \( 1 \in H^0(M) \) (compare this with Lemma 6.3, indeed the same argument would hold if we defined \( \psi_{C'} \) as a time-independent analogue of the construction in Subsection 6.3: so, using a homotopy \( f_s \) from \( f \) to \( 0 \) on the negative end of \( C' \) and counting isolated finite-energy \(-\nabla f_s\) flow lines along \( C' \), where the energy is \( E(v) = \int |\partial_s v|^2 \, ds \).

6.9. \( c^* : H^*(M) \to SH^*(M) \) respects the TQFT and \( SH^*(M) \) is an \( H^*(M) \)-module

By Subsection 6.8, \( H^*(M) \) and dually \( H_*(M) \) have a TQFT structure via Morse theory analogous to the TQFT structures on \( SH^*(M) \) and \( SH_*(M) \). We postpone the following result to Section 15.

**Theorem 6.6.** \( c^* : H^*(M) \to SH^*(M) \) and \( c_* : SH_*(M) \to H_*(M) \) are TQFT maps. So they are unital ring maps using cup product on \( H^*(M) \) and intersection product on \( H_*(M) \).

Recall that \( H^m \) denotes a Hamiltonian linear at infinity of generic slope \( m \) (see Subsection 3.3). The proof of Theorem 6.6 in fact shows that \( c^*_{H^m} : H^*(M) \cong SH^*(H^m) \) is a TQFT map, for small \( \delta > 0 \). So we obtain an \( H^*(M) \)-module structure on \( SH^*(H^m) \) as follows:

\[
H^*(M) \otimes SH^*(H) \to SH^*(H), \quad \alpha \otimes x \mapsto \psi_p(c^*_{H^m}(\alpha), x).
\]

**Remark.** This product actually arises as \( SH^*(H^\delta) \otimes SH^*(H^m) \to SH^*(H^{\delta+m}) \), but for small \( \delta \) no Reeb periods lie in \([m, \delta + m]\) so a monotone homotopy in the region where \( H^m \) is linear forces the continuation \( SC^*(H^\delta) \to SC^*(H^{\delta+m}) \) to be the identity (no new 1-orbits appear and, by Lemma D.1, isolated continuation solutions lie in the region where the homotopy is \( s \)-independent, so by \( \mathbb{R} \)-symmetry they cannot be isolated unless they are constant).

Taking the direct limit, we get a module structure \( H^*(M) \otimes SH^*(M) \to SH^*(M) \):

**Corollary 6.7.** \( SH^*(M) \) is an \( H^*(M) \)-module via \( \alpha \otimes x \mapsto \psi_p(c^*(\alpha), x) \).

**Theorem 6.8.** The continuation map \( \psi_{C_\ell} : SH^*(H^m) \to SH^*(H^m') \) equals the product by the element \( e_{H^\ell} \in SH^0(H^\delta) \) for \( \ell = m' - m \geq 0 \) (defined in Subsection 6.3 and Theorem 6.4).
Proof. Let $e_{H^f} = \psi_C(1)$ where $\psi_C : \mathbb{K} \to SH^*(H^f)$ (see Subsection 6.3 for details). Now consider the picture in the proof of Theorem 6.1 using weights $(m'; l, m)$ for $P$. Since gluing corresponds to compositions (Theorems A.12 and A.10), we deduce: $\psi_P(e_{H^f}, \cdot') = \psi_P(\#(C \cup Z))(\cdot') = \psi_Z(\cdot')$. For cylinders, $\psi_Z$ is the continuation map (Example A.2, Theorem A.10).

6.10. The coproduct

Denote $\psi_Q : H^*(M) \to H^*(M) \otimes H^*(M)$ the classical coproduct (see Subsection 6.8). Let $f_2$ and $f_3$ be the Morse functions we use on the negative ends of $Q'$, and use an $f$ on the positive end which has a unique local minimum. By definition $\psi_Q(1)$ is a count of intersection numbers $W^u(x; f_2) \cdot W^u(y; f_3)$ for $x, y$ of complementary Morse indices (for generic $f$, the intersection point at the internal vertex of $Q'$ will always have a unique $-\nabla f$ flow line to min $f$). Since intersection numbers are homotopy invariants, we can in fact write

$$\psi_Q(1) = \sum ([W^u(x; f)] \cdot [W^u(y; f)]), \quad x \otimes y \in MH^{2n}(f).$$

Example 6.9. For $M = T^*N$, the coproduct $\psi_Q$ is determined by the Euler characteristic: $\psi_Q(1) = ([N] \cdot [N]) \operatorname{vol}_N \otimes \operatorname{vol}_N = \pm \chi(N)(\operatorname{vol}_N)^{\otimes 2}$ (working over $\mathbb{Z}/2$ if $N$ is not oriented).

Theorem 6.10. The image of $\psi_Q : SH^*(M) \to SH^*(M) \otimes SH^*(M)$ lies in the image of $(c^*)^{\otimes 2} : H^*(M) \otimes SH^*(M) \otimes SH^*(M)$. Moreover, $\psi_Q$ is determined by $\psi_Q(1)$ and the $H^*(M)$-module structure of $SH^*(M)$ defined in Subsection 6.9.

Proof. The fact that $\psi_Q$ lands in $c^*(H^*(M)) \otimes SH^*(M)$ follows from the factorization

$$\psi_Q = \psi_{(Z \cup Z) \# Q} : SH^*(H) \overset{\psi_Q \circ id}{\to} SH^*(H) \otimes SH^*(H) \to SH^*(H) \otimes SH^*(H),$$

where $\varphi : SH^*(\delta H) \to SH^*(H)$ is the continuation, using small enough $\delta > 0$ so that $SH^*(\delta H) \cong H^*(M)$. A similar factorization shows $\operatorname{im}(\psi_Q) \subset SH^*(M) \otimes c^*(H^*(M))$.

We can also factorize $\psi_Q : H^*(M) \to SH^*(2H)$ as

$$\psi_Q = \psi_{(Z \cup P) \# ((Q \cap C) \cup Z)}(1 \otimes \cdot) : SH^*(H) \overset{\psi_Q \circ id}{\to} SH^*(H) \otimes SH^*(H) \overset{\varphi \otimes \varphi}{\to} SH^*(2H) \otimes SH^*(2H),$$

where $\varphi : SH^*(\delta H) \to SH^*(2H)$ is the continuation.

For small $\delta > 0$, $\psi_Q \circ id(1) \in SH^*(\delta H) \otimes SH^*(\delta H)$ equals $(c^*)^{\otimes 2}(\psi_Q(1))$ (since $c^*_{\delta H}$ is compatible with TQFT by the proof of Theorem 6.6). Say $(c^*)^{\otimes 2}(\psi_Q(1)) = \sum k_{ij} x_i \otimes x_j \in SH^*(\delta H) \otimes SH^*(\delta H)$, then by the factorization: $\psi_Q(y) = \sum k_{ij} \varphi(x_i) \otimes \psi_P(x_j, y)$. Finally, in the direct limit, $\psi_P(x_j, y) \in SH^*(M)$ is the $H^*(M)$-module structure of $SH^*(M)$ since $x_j \in \operatorname{im}(c^*)$.

Observe that we proved that if $\psi_Q(1) = \sum k_{ij} x_i \otimes x_j$, then $\operatorname{im}(\psi_Q) \subset \operatorname{span}\{c^*(x_i)\} \otimes \operatorname{span}\{c^*(x_j)\}$. In particular, if $\psi_Q(1) = 0$, then $\psi_Q \equiv 0$.

If the surface $S$ has non-zero genus or has $p \geq 2$, then by Theorem 6.5 the decomposition of $S$ into generators $C, Z, P$ and $Q$ contains at least one $Q$. So Theorem 6.10 implies:

Corollary 6.11. If $\psi_Q(1) = 0$, then all TQFT operations on $SH^*(M)$ for surfaces of non-zero genus and for surfaces with $p \geq 2$ will vanish.
Example 6.12. For $\mathcal{M} = T^*N$, the only possible non-zero contribution to $\psi_Q$ is a map $SH^0(M) \to \mathbb{K} (c^* \text{vol}_N)^{\otimes 2}$ for degree reasons, and the Corollary applies whenever $\chi(N) = 0$.

6.11. Open model surfaces $S$

Let $(S, j)$ be a Riemann surface isomorphic to the closed unit disc with $p + q$ boundary points removed. These punctures are marked in the oriented order as 1, ..., $p$, 1, ..., $q$, the first $p \geq 1$ are called negative, the other $q \geq 0$ positive. Fix parametrizations as strip-like ends $(-\infty, 0] \times [0, 1]$ and $[0, \infty) \times [0, 1]$ near negative and positive punctures, respectively, such that $j$ becomes standard: $j\partial_x = \partial_t$ (Figure 7).

Fix a one-form $\beta$ on $S$ satisfying: $d\beta \leq 0$; on each strip-like end $\beta$ is a positive constant multiple of $dt$, these constants $A_a, B_b > 0$ are called weights and are generic (not periods of Reeb chords); $\beta$ is exact near $\partial S$; and the pull-back $\beta|_{\partial S} = 0$ (so $\beta|_{\partial S}(T\partial S) = 0$).

By Stokes’ theorem, $\sum A_a - \sum B_b > -\int_S d\beta \geq 0$. Arguing as in Lemma A.1, one shows that this condition on the weights $A_a$ and $B_b$ is the only obstruction to constructing such a form $\beta$, and one can also ensure that $d\beta = 0$ everywhere except on a chosen small disc in the interior of $S$. At the cohomology level, the particular choice of $\beta$ will not matter.

Given two such sets of data for $S$ and $S'$, then as in Appendix A.10 we can glue a positive end of $S$ with a negative end of $S'$ provided they carry the same weight. The resulting surface $S \# S'$ carries glued data $j \# j'$ and $\beta \# \beta'$ satisfying the above conditions.

We also consider more general surfaces $S$ obtained from the above model surfaces via the following two operations: (1) taking disjoint unions; (2) gluing surfaces along several ends (provided $p \geq 1$ after gluing). Therefore the surface $S$ may contain holes.

6.12. TQFT structure on $HW^*(L)$

Fix a Hamiltonian $H : \mathcal{M} \to \mathbb{R}$ as in the definition of $HW^*(L; H)$ with $H \geq 0$ and (possibly non-generic) slope 1 at infinity. Write $X = X_H$.

In analogy with Appendix A.2, define the moduli space of wrapped solutions $W(x_a; y_b; S, \beta)$ consisting of smooth $u : S \to \mathcal{M}$ with $u(\partial S) \subset L$, solving $(du - X \otimes \beta)^{0,1} = 0$, converging on the ends to the Hamiltonian chords $x_a$ and $y_b$ for $A_a H$ and $B_b H$.

The maximum principle holds for these moduli spaces by Lemma D.2, and the a priori energy estimate $E(u) \leq \sum_{a=1}^p A_{A_a H}(x_a) - \sum_{b=1}^q A_{B_b H}(y_b)$ holds by Appendix D.2. We now make a Technical Remark about transversality, which proves that for generic $J$ the wrapped moduli spaces are smooth manifolds.

Technical Remark. For transversality, proceed as in Appendix A.5, so Appendix A.6 can be applied, except that we need to discuss trivial solutions $du - X \otimes \beta = 0$. Let $\gamma$ be a boundary path in $\partial S$ connecting two consecutive punctures $z$ and $z'$ with weights $c$ and $c'$. Since the pull-back of $\beta$ to $\partial S$ vanishes and $\beta$ is exact near $\partial S$, we can define a complex coordinate $s + it$ near $\gamma$ such that: $\partial_z = \gamma$ and $\beta = (\ell(t)) dt$ along $\gamma$, for some function $\ell(t)$ interpolating $c$ and $c'$. Since $du - X \otimes \beta = 0$, $du(\gamma) = 0$ so $u$ is constant along $\gamma$. Thus, $u(z)$ and $u(z')$ are
two (possibly equal) Hamiltonian chords such that the initial point of one is the end point of the other. This gives rise to a triple intersection in \( L \cap \varphi(L) \cap \varphi^c(L) \). As in Subsection 4.2, after perturbing \( H \) and \( L \), one can \cite[Lemma 8.2]{4} rule out these intersections provided \( \dim M \geq 4 \), which we tacitly assume (for \( \dim M = 2 \) there is a work-around by passing to \( M \times DT^*S^2 \) (see \cite[Section 5b]{4})).

By the maximum principle and the energy estimate, \( W(x_a; y_b; S, \beta) \) is compact up to breaking at the ends (as mentioned in Subsection 4.6, no sphere or disc bubbling can occur since \( \overline{M} \) and \( L \) are exact). On the ends, the equation turns into Floer’s equation, so the breaking analysis is the same as for strips. The construction of the TQFT is now analogous to the \( SH^*(M) \) case by counting wrapped solutions. The analogue of Theorem A.14 is as follows:

**Theorem 6.13.** There are TQFT maps \( W_S : \bigotimes S HW^*(L; B_bH) \rightarrow \bigotimes L HW^*(L; A_H^\ell) \) which in the direct limit become TQFT operations \( W_S^\ell : HW^*(L) \rightarrow HW^*(L) \), where \( P \) is a disc with \( 1 + 2 \) punctures and \( C \) is a disc with \( 1 + 0 \) punctures. As in Subsection 6.2, the product is associative. It is not graded-commutative in general (see Subsection 6.2).

6.13. \( c^* : H^*(L) \rightarrow HW^*(L) \) respects the TQFT and \( HW^*(L) \) is an \( H^*(L) \)-module

We postpone the following result to Subsection 15.6.

**Theorem 6.14.** The maps \( c^* : H^*(L) \rightarrow HW^*(L) \) from Section 5 preserve the TQFT.

Proceeding as in Subsection 6.9, we deduce the following two results:

**Corollary 6.15.** \( HW^*(L) \) is an \( H^*(L) \)-module via

\[
H^*(L) \otimes HW^*(L) \rightarrow HW^*(L), \quad \alpha \otimes x \mapsto W_{P}(c^*(\alpha), x).
\]

**Theorem 6.16.** The continuation map \( W_Z = W_C(\epsilon_H, \cdot) : HW^*(L; H^\ell; L, H^m) \rightarrow HW^*(L; H^\ell) \) equals the product by the element \( \epsilon_H^\ell = W_C(1) \in HW^0(L; H^\ell) \), where \( \ell = m' - m \geq 0 \).

Just as in Subsection 6.9 and Theorem 6.4, the elements \( \epsilon_H^\ell \in HW^0(L; H^\ell) \) are the image of 1 under the continuation map \( c^*_H : H^*(L) \cong HW^*(L; H^\ell) \rightarrow HW^*(L; H^\ell) \) (recall Section 5), and by Section 8 one can construct \( H^\ell \) so that \( c^*_H \) is the inclusion of a subcomplex.

6.14. \( HW^*(L) \) is a module over \( SH^*(M) \)

Pick a surface \( S \) as in Subsection 6.11. Introduce \( p_{\text{int}} \) negative and \( q_{\text{int}} \) positive punctures in the interior of \( S \). Fix cylindrical parametrizations \( (-\infty, 0] \times S^1 \) and \([0, \infty) \times S^1 \), respectively, near those punctures, with \( j\partial_s = \partial_t \). Let \( \beta \) be a 1-form on the resulting surface \( S \) with: \( d\beta \leq 0 \), \( \beta \) being a positive multiple of \( dt \) near each puncture, \( \beta \) is exact near \( \partial S \), the pull-back \( \beta|_{\partial S} = 0 \). Consider the moduli space of solutions \( u : S \rightarrow \overline{M} \) of \((du - X \otimes \beta)^{0:1} = 0\), with \( u(\partial S) \subset L \), converging at boundary punctures to Hamiltonian chords and at interior punctures to 1-periodic Hamiltonian orbits. The maximum principle and energy estimate still hold, so
transversality and compactness up to breaking on the ends are proved as before. The count of these moduli spaces defines operations

\[ \text{SH}^*(M)^{\otimes q_{\text{int}}} \otimes \text{HW}^*(L)^{\otimes p} \rightarrow \text{SH}^*(M)^{\otimes p_{\text{int}}} \otimes \text{HW}^*(L)^{\otimes p} \quad (p + p_{\text{int}} \geq 1, q + q_{\text{int}} \geq 0). \]

Consider a punctured disc \( D \) with \((p, q; p_{\text{int}}, q_{\text{int}}) = (1, 1; 0, 1)\). At the chain level a solution as in Figure 8 contributes \( \pm w' \) to \( W_D(x \otimes w) \), where

\[ W_D : \text{SH}^*(M) \otimes \text{HW}^*(L) \rightarrow \text{HW}^*(L). \]

We deduce the following result, which was known by specialists in symplectic cohomology:

**Theorem 6.17.** \( W_D \) defines a module structure of \( \text{HW}^*(L) \) over \( \text{SH}^*(M) \). \( \text{HW}^*(L) \) over \( \text{SH}^*(M) \).

**Proof.** By construction, \( W_D \) is linear in \( \text{SH}^*(M) \) and \( \text{HW}^*(L) \). Next, we check that \( W_D(e, \cdot) = \text{id} \) for the unit \( e \in \text{SH}^*(M) \). Given \( H \) and weights, consider Figure 9: capping off the closed orbit in Figure 8 yields a strip, which we can homotope to the standard continuation strip \((\mathbb{R} \times [0, 1], \beta = f(s) \, dt, j \partial_s = \partial_t)\), where \( f'(s) \leq 0 \) and \( f \) interpolates the weights. So, \( W_D(e_H, \cdot) \) is chain homotopic to a continuation map. So, in cohomology, taking direct limits, \( W_D(e, \cdot) \) is the identity, as required.

Finally, we need \( W_D \) to be compatible with the product on \( \text{SH}^*(M) \), namely \( W_D \circ (\text{id} \otimes W_D) = W_D \circ (\psi_P \otimes \text{id}) \). It suffices to check that for given \( H \) and weights, at the chain level \( W_D(x, W_D(x', w)) \) is chain homotopic to \( W_D(\psi_P(x, x'), w) \).

Consider the last two configurations in Figure 9: one is the gluing of two copies of Figure 8 corresponding to \( W_D(x, W_D(x', w)) \), the other is obtained by an isotopy of this one and it is the gluing corresponding to \( W_D(\psi_P(x, x'), w) \). In the last figure, we first pull-back the form \( \beta \) via the isotopy, then we deform it (by linear interpolation) to make it equal to the \( \beta \) obtained from the gluing of the punctured strip with the pair of pants.

Now run the same argument as in the proof of Theorem A.12: combining a stretching argument with the a priori energy estimate shows that each of the two configurations arises precisely from gluing broken solutions. Since the two configurations are homotopic, \( W_D(x, W_D(x', w)) \) and \( W_D(\psi_P(x, x'), w) \) are chain homotopic. \( \square \)
7. Twisted theory

7.1. Novikov bundles

The Novikov field $\Lambda$ in the formal variable $t$ is the $\mathbb{K}$-algebra

$$
\Lambda = \left\{ \sum_{j=0}^{\infty} k_j t^{a_j} \text{ for any } k_j \in \mathbb{K}, \; a_j \in \mathbb{R}, \text{ with } \lim_{j \to \infty} a_j = \infty \right\}.
$$

Recall that $\overline{LM} = C^\infty(S^1, \overline{M})$ is the free loop space. Let $\alpha \in C^1_{\text{sing}}(\overline{LM}; \mathbb{R})$ be an $\mathbb{R}$-valued singular cocycle representing a class $a \in H^1(\overline{LM}; \mathbb{R})$. The Novikov bundle $\Lambda_\alpha$ is a local system of coefficients on $\overline{LM}$; over a loop $x \in \overline{LM}$ its fibre is a copy $\Lambda_x$ of $\Lambda$, and the parallel translation over a path $u$ in $\overline{LM}$ from $x$ to $y$ is the multiplication isomorphism

$$
\rho^\alpha[u] : \Lambda_y \longrightarrow \Lambda_x,
$$

where $\alpha[.] : C^0_{\text{sing}}(\overline{LM}; \mathbb{R}) \to \mathbb{R}$ denotes the evaluation of $\alpha$ on singular one-chains.

Up to isomorphism, this local system only depends on the class $a$, so by abuse of notation, we write $\Lambda_a$ and $\alpha[u]$. (Proof. Changing the representative $\alpha$ to $\alpha + df$, for $f \in C^0_{\text{sing}}(\overline{LM}; \mathbb{R})$, corresponds to an isomorphism of the local systems given by $\lambda_x \mapsto t^{-f(x)}\lambda_x$ on the $\Lambda_x$ fibres.)

7.2. Transgressions

Let $ev : \overline{LM} \times S^1 \to \overline{M}$ be the evaluation map. Define

$$
\tau = \pi \circ ev^* : H^2(\overline{M}; \mathbb{R}) \xrightarrow{ev^*} H^2(\overline{LM} \times S^1; \mathbb{R}) \xrightarrow{\pi} H^1(\overline{LM}; \mathbb{R}),
$$

where $\pi$ is projection to the Künneth summand. This is an isomorphism when $\overline{M}$ is simply connected. Explicitly, pick a closed de Rham 2-form $\eta$ to represent the given $H^2(\overline{M}; \mathbb{R})$ class, then, for any path $u \subset \overline{LM}$, define $\tau \eta \in C^1_{\text{sing}}(\overline{LM}; \mathbb{R})$ by

$$
\tau \eta[u] = \int \eta(\partial_s u, \partial_t u) \, ds \wedge dt.
$$

Remark. The path $u : [0, 1] \to \overline{LM}$ defines a cylinder $u : [0, 1] \times S^1 \to \overline{M}$, and we can homotope $u$ relative to the ends to make it smooth, the above integral is then just $\int u^* \eta$. By Stokes’ theorem, this does not depend on the choice of homotopy, since $\eta$ is a closed form.

Observe that $\tau \eta$ vanishes on time-independent paths in $\overline{LM}$, so $c^* \Lambda_{\tau \eta}$ is trivial when we pull-back by the inclusion of constant loops $c : \overline{M} \to \overline{LM}$.

More generally, $\tau \eta$ evaluated on a Floer solution $u : S \to \overline{M}$ is defined by

$$
\tau \eta[u] = \int_S u^* \eta.
$$

This integral is finite despite the non-compactness of $S$ because of the exponential convergence of $u$ to the asymptotics (Theorem A.3). Indeed, on an end, $|\partial_s u| \leq c e^{-\delta|s|}$, $\partial_s u + J(\partial_t u - CX) = 0$ and $|\eta(u)| \leq C'$ for some constants $c, \delta, C$ and $C'$. Thus,

$$
\int_{\text{end}} \eta(\partial_s u, \partial_t u) \, ds \wedge dt \leq C' \cdot \int_{\text{end}} |\partial_s u| \cdot |\partial_t u| \leq C' \cdot \int_{\text{end}} c e^{-\delta|s|}(c e^{-\delta|s|} + C|X|) < \infty.
$$
7.3. Twisted symplectic (co)homology

Now introduce weights corresponding to $\alpha \in H^1(\mathcal{L}M; \mathbb{R}) \cong H^1(\mathcal{L}\mathcal{M}; \mathbb{R})$ in the construction of Subsection 3.1 (for details, see [31]):

$$SC^*(H)_\alpha = SC^*(H; \Delta_{\alpha}) = \bigoplus \{Ax : x \in \mathcal{L}\mathcal{M}, \dot{x}(t) = X_H(x(t))\},$$

$$dy = \sum_{u \in \mathcal{M}_0(x,y)} \epsilon_u t^{\alpha[u]}x$$ (see Subsection 3.1),

$$SH^*(H)_\alpha = SH^*(H; \Delta_{\alpha}) = H^*(SC^*(H; \Delta_{\alpha}); d),$$

$$\varphi : SC^*(H_+) \alpha \longrightarrow SC^*(H_-) \alpha, \varphi(x_+) = \sum_{v \in \mathcal{M}_{0}^H(x_-, x_+)} \epsilon_v t^{\alpha[v]}x_-$$ (see Subsection 3.2).

**Remark.** These maps are compatible with the isomorphism of local systems described in Subsection 7.1, since changing the representative $\alpha$ to $\alpha + df$ corresponds to an isomorphism $SC^*(H; \Delta_{\alpha}) \longrightarrow SC^*(H; \Delta_{\alpha} + df)$ given by $x \mapsto t^{-f(x)}x$ on generators.

For $(M, d\theta; \alpha)$, define $SH^*(M)_\alpha = SH^*(M; \Delta_{\alpha}) = \lim SH^*(H)_{\alpha}$, taking the direct limit over the twisted continuation maps $\varphi$ above (cf. Subsection 3.4).

By inserting the local system of coefficients $\epsilon^*\mathcal{A}_{\alpha}$ in $MH^*(H^3)$ in Section 5, we obtain $\epsilon^* : H^*(M; \epsilon^*\mathcal{A}_{\alpha}) \longrightarrow SH^*(M; \Delta_{\alpha})$. For $\alpha = \tau \eta$, $\epsilon^*\mathcal{A}_{\alpha}$ is trivial (Subsection 7.2), so

$$\epsilon^* : H^*(M) \otimes \Lambda \longrightarrow SH^*(M; \Delta_{\tau \eta}).$$

We sometimes abbreviate $H^*(M; \mathcal{A}_{\alpha})$ by $H^*(M)_{\alpha}$, and $H^*(M) \otimes \Lambda$ by $H^*(M)$.

The construction of $SH_{\tau}(M; \mathcal{A}_{\alpha})$ is dual to the above so, in the notation of Section 3.7, we define $\delta x = \sum_{u \in \mathcal{A}_{\alpha}^H(x_-, x_+)} \epsilon_u t^{-\alpha[u]}y$ and $\varphi_s(x_+) = \sum_{u \in \mathcal{A}_{\alpha}^H(x_-, x_+)} \epsilon_u t^{-\alpha[u]}x_+$. The $c_{\ast}$ map is $c_{\ast} : SH_{\tau}(M; \Delta_{\alpha}) \longrightarrow H_{\tau}(M; \epsilon^*\mathcal{A}_{\alpha})$.

The dualization result in Subsection 3.8 now reads: $\text{Hom}_{\Lambda}(SH^*(M)_{\alpha}, \Lambda) \cong SH_{\tau}(M; -\alpha)$.

7.4. Twisted TQFT

From now on, use $\alpha = \tau \eta$ (Subsection 7.2). In Subsection 7.5, we mention other options. Abbreviate $SC^*(H)_{\eta} = SC^*(H; \Delta_{\tau \eta})$, $SH^*(H)_{\eta} = SH^*(H; \Delta_{\tau \eta})$. Define

$$\psi_S : \bigotimes_{b=1}^{q} SC^*(B_b H)_{\eta} \longrightarrow \bigotimes_{a=1}^{p} SC^*(A_a H)_{\eta},$$

$$\psi_S(y_1 \otimes \ldots \otimes y_q) = \sum_{u \in \mathcal{M}_0(x_a; y_b; S, \beta)} \epsilon_u t^{\tau \eta[u]}x_1 \otimes \ldots \otimes x_p$$ (compare Subsection 6.9).

**Remark** (Compare Subsection 7.3). These maps are compatible with the isomorphism of local systems described in Subsection 7.1. Indeed, changing the representative $\eta$ to $\eta + d\mu$ for $\mu \in C^1_\text{derRham}(\mathcal{M}; \mathbb{R})$ determines the change of basis isomorphism $SC^*(H; \Delta_{\tau \eta}) \longrightarrow SC^*(H; \Delta_{\tau(\eta + d\mu)})$ given by $x \mapsto t^{-\int_{\gamma} x^*\mu}x$ on generators. If $u \in \mathcal{M}(x_a; y_b; S, \beta)$ contributes $t^{\int_{\gamma} u^*\eta\mu}x_1 \otimes \ldots \otimes x_p$ to $\psi_S(y_1 \otimes \ldots \otimes y_q)$ when twisting by $\eta$, then by Stokes’ theorem it will contribute $t^{\int_{\gamma} u^*\eta\mu - \int_{\gamma} \int_{\gamma} x_1^* \mu \Sigma \int_{\gamma} \int_{\gamma} \mu \mu \mu x_1 \otimes \ldots \otimes x_p}$ when twisting by $\eta + d\mu$. So the twisted TQFT structure respects the change of basis.

**Theorem 7.1.** (1) The weights $\tau \eta[u]$ are locally constant on $\mathcal{M}(x_a; y_b; S, \beta)$.

(2) The weights $\tau \eta[u]$ are additive under the breaking of Floer solutions: if $u_{\lambda}$ converge to a broken solution $u \# v$, then $\tau \eta[u_{\lambda}] = \tau \eta[u] + \tau \eta[v]$.

(3) $\tau \eta[v]$ is constant on components of the compactification of $\mathcal{M}(x_a; y_b; S, \beta)$.
Proof. Suppose $w(\cdot, \cdot, \lambda) = u_\lambda(\cdot, \cdot)$ is a smooth path of solutions in $M(x_\alpha; y_\beta; S, \beta), \lambda \in [0, 1]$. Now $d\eta = 0$, since $\eta$ is a closed form, so (1) follows by Stokes:

$$0 = \int_{S \times [0, 1]} w^* d\eta = \int_S u_\lambda^* \eta - \int_S u_0^* \eta = \tau \eta[u_1] - \tau \eta[u_0].$$

To prove (2), consider a smooth family $w(\cdot, \cdot, \lambda) = u_\lambda \in M(x_\alpha; y_\beta; S, \beta)$ which is parametrized by $\lambda \in [0, 1)$, such that $u_\lambda \to u \neq v$ as $\lambda \to 1$.

Fix $\epsilon > 0$. Pick a large $c > 0$ so that the restrictions $u', v'$ of $u, v$ to the complement of the neighbourhoods $(-\infty, -c) \times S^1$, $(c, \infty) \times S^1$ of all the ends of $u$ and $v$ satisfy

$$|\tau \eta[u'] + \tau \eta[v'] - \tau \eta[u'_\lambda] - \tau \eta[v'_\lambda]| < \epsilon.$$

This is possible by the calculation of Subsection 7.2, which shows that the integrals evaluated near the ends are arbitrarily small because $u$ and $v$ converge exponentially fast.

Denote by $u'_\lambda$ the analogous restriction of $u_\lambda$. The domain of $u'_\lambda$ is compact so, by Lemma A.9, $u'_\lambda$ converges $C^2$-uniformly to $u'$. Moreover, there exist $s_\lambda \in \mathbb{R}$ such that $v'_\lambda(s, t) = u_\lambda(s + s_\lambda, t)|_{s \in [-c, c]}$ converges $C^2$-uniformly to $v'$. So, for large $c$, and $\lambda$ close to 1,

$$|\tau \eta[u'_\lambda] + \tau \eta[v'_\lambda] - \tau \eta[u'_\lambda] - \tau \eta[v'_\lambda]| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, to conclude the proof of (2), we show that for $c \gg 0$ and $\lambda \approx 1$,

$$|\tau \eta[u'_\lambda] + \tau \eta[v'_\lambda] - \tau \eta[u_\lambda]| < \epsilon.$$

By Theorem A.3, near the ends $|\partial_s u| < C e^{-\delta|s|}$ and $|\partial_s v| < C e^{-\delta|s|}$ for some $\delta, C$. For large $c$, these hold at $|s| = c$, which are the boundaries of the domains for $u'$ and $v'$.

Thus, on an end of $u$, say $K = (-\infty, -c) \times S^1$, apply Stokes’ theorem to $K \times [1, \lambda]$ as in (1) to deduce that $|\int_K u'_\lambda \eta - \int_K u^\alpha \eta| = |\int_{\mu \in \lambda, 1, s = -c} u'_\lambda | \leq C e^{-\delta c}$ for $\mu \approx 1$. A similar argument holds for the other ends of $u$ and $v$.

Finally, consider what happens near the breaking. On the end, where $u_\lambda$ breaks, define $w_\lambda(s, t) = u_\lambda(s, t)|_{s \in [c, -c + s_\lambda]}$. At the boundaries $s = c$ and $s = -c + s_\lambda$ the $u_\lambda$ converge $C^2$-uniformly to $u(c, t)$ and $v(-c, t)$, respectively, so we can assume $\partial_s w = \lambda$ is bounded by a constant multiple of $e^{-\delta c}$ for $\lambda \approx 1$. Apply Stokes to $\bigcup_{\mu \in [1, 1]} (c, -c + s_\mu) \times S^1 \times \{\mu\}$ as in (1) to deduce:

$$\left|\int_{s > c} u'_\lambda - \int_{s < -c} v'_\lambda\right| = \left|\int_{\mu \in \lambda, 1, s = -c} u'_\lambda - \int_{\mu \in \lambda, 1, s = -c + s_\mu} v'_\lambda\right|$$

which again is bounded by a constant multiple of $e^{-\delta c}$, and so is arbitrarily small for large $c$. Thus, (2) follows, and (3) follows from (1) and (2).

\[\Box\]

**Theorem 7.2.** The twisted TQFT map $\psi_S$ is a chain map, so it induces a map $\psi_S : \bigotimes_b SH^*(B_b, H)_{\eta} \to \bigotimes_a SH^*(A_a, H)_{\eta}$ independent of the choice of data $(\beta, j, J)$ (Theorem A.10). For cylinders, the $\psi_S$ are twisted continuation maps (Example A.2). Gluing surfaces correspond to compositions (Theorem A.12). The $\psi_S$ maps are compatible with the twisted continuation maps (Theorem A.14). In the direct limit, we get operations independent of $H$

$$\psi_S : SH^*(M)_{\eta}^{\otimes q} \to SH^*(M)_{\eta}^{\otimes p} (p \geq 1, q \geq 0).$$

**Proof.** The claims involve the same moduli spaces used to prove the untwisted results. Theorem 7.1 implies that we count the moduli spaces with correct weights: broken solutions on the boundaries $\partial M$ of a component $M$ of a moduli space are counted with equal weights. \[\Box\]
Theorem 7.3. $SH^*(M)_{\eta}$ carries a twisted TQFT structure, in particular, it is a unital ring. The map $c^*: H^*(M) \otimes \Lambda \to SH^*(M)_{\eta}$ of Subsection 7.3 respects the TQFT, using Morse operations on $H^*(M)$. In particular, the unit is $e = c^*(1) \in SH^0(M)_{\eta}$.

Proof. The proof is analogous to that of Theorems 6.1 and 6.6, which are proved via Lemma 15.11. We only need to insert appropriate weights. In the notation of Definition 15.7, define the twisted versions of the maps $\phi$ and $\psi$ on generators as follows:

$$\phi: SC^*(H^\delta)_{\eta} \to MC^*(f) \otimes \Lambda, \quad \psi: MC^*(f) \otimes \Lambda \to SC^*(H^\delta)_{\eta},$$

$$\phi(y) = \sum_{v \# u \in M^0_\eta(x,y)} \epsilon_{v \# u} t^{\tau[\eta[\text{cov}^n] + \tau[\eta]_x} y,$$

$$\psi(x) = \sum_{u \# v \in M^0_\eta(y,x)} \epsilon_{u \# v} t^{\tau[\eta][\text{cov}^n] y}.$$

The weights for Morse solutions $v$ are redundant: $t^{\tau[\eta[\text{cov}^n]} = 1$ since $c \circ v$ only depends on one variable (see Subsection 7.2 on why $c^*\Lambda_{\eta}$ is trivial). However, inserting these trivial weights makes it clear that the proof of Theorem 7.1 can be applied to the above moduli spaces: the weights are locally constant on the compactifications of $M^0(x,y)$ and $M^0(y,x)$.

Theorem 7.4. A $\varphi: \overline{M} \cong \tilde{N}$ of contact type induces $\varphi_*: SH^*(M)_{\alpha} \cong SH^*(N)_{\varphi_* \alpha}$. For $\alpha = \tau\eta$, $\varphi_*$ respects the twisted TQFT structure: $\varphi_*^\otimes \circ \psi_{S,M} = \psi_{S,N} \circ \varphi_*^{\otimes \eta}$.

Proof. In Lemma 3.2 and Theorem A.15 insert weights $t^{\tau[\nu]}$ in the definition of the continuation maps defining $\varphi_*: SH^*(M) \cong SH^*(N)$, and apply Theorem 7.1.

7.5. TQFT structure for other twistings

For general $\alpha \in H^1(\mathcal{L}M)$, it does not seem possible to define the TQFT structure on $SH^*(M)_{\alpha}$. Naively one might divide $S$ into cylinders, the restrictions of $u$ yield 1-chains in $\mathcal{L}M$, then evaluate $\alpha$ and add up. This depends heavily on choices and will not yield an invariant TQFT. We will now show that it is possible to define TQFT operations in two special cases. We restrict ourselves, in this discussion, only to the TQFT defined by surfaces $S$ of genus zero.

1. In Subsection 7.6, we show how to twist by $\alpha = \tau\tilde{\eta}$ for $\tilde{\eta} \in H^2(\tilde{M})$ taken from the universal cover $\tilde{M}$. If we restrict to contractible loops, then we obtain a TQFT with unit.

2. In Subsection 7.7, we twist by pull-backs $\alpha = ev^*\mu$, for $\mu \in H^1(M)$, where $ev: \mathcal{L}M \to M$, $x \mapsto x(0)$. Then we obtain a TQFT with no unit.

If we restrict to contractible loops, all $\alpha \in H^1(\mathcal{L}_0M)$ arise from combining (1) and (2) (Remark 7.8). For simply connected $M$, all twistings in $H^1(\mathcal{L}M)$ can be achieved using (1).

Technical Remark. For simply connected $M$, in (1) we can also allow surfaces of non-zero genus. This is because for a loop $\gamma$ in $S$ wrapping around a hole of $S$, we can fill the image $u(\gamma)$ in the target space $\overline{M}$ of the map $u: S \to \overline{M}$ by a disc $D$. So in the construction of Theorem 7.5, one can still form a sphere by capping off $u(S)$ at the ends and adding and subtracting such discs.

7.6. Twisting by closed 2-forms from the universal cover

Let $\tilde{M}$ be the universal cover of $M$. Then the transgression $\tau: H^2(\tilde{M}) \to H^1(\mathcal{L}\tilde{M})$ is an isomorphism, and

$$H^1(\mathcal{L}\tilde{M}) \cong \text{Hom}(H_1(\mathcal{L}\tilde{M}), \mathbb{Z}) \cong \text{Hom}(\pi_1(\mathcal{L}\tilde{M}), \mathbb{Z}) \cong \text{Hom}(\pi_2(M), \mathbb{Z})$$
(via evaluation of $\tau x$ on 1-chains). For the component $L_0M$ of contractible loops:

$$H^1(L_0M) \cong \text{Hom}(\pi_1(L_0M), \mathbb{Z}) \cong \text{Hom}(\pi_1 M \times \pi_2 M, \mathbb{Z}) \cong \text{Hom}(H_1 M \times H_2 M, \mathbb{Z}).$$

Via these, $H^1(\tilde{L}M) \subset H^1(L_0M)$, so we can define $SH^*(M)\tilde{=} = SH^*(M; \Delta_{x, \tilde{y}})$, which comes with $c^* : H^*(M) \otimes \Lambda \to SH^*_0(M)\tilde{=} \subset SH^*(M)$ (see Subsection 6.5 for the $SH^*_0$ notation).

**Remark.** We will use integral singular 1-cocycles $\alpha$ below, but one could also use real 1-cocycles in $\tau H^2(M; \mathbb{R}) \subset H^1(L_0M; \mathbb{R})$ via $H^1(\tilde{L}M; \mathbb{R}) \cong \text{Hom}(\pi_2 M, \mathbb{R})$.

**Theorem 7.5.** Restrict to $L_0M$ (see Subsection 6.5) and consider only the TQFT defined by genus 0 surfaces. Then $SH^*_0(M)\tilde{=} \subset H^1(L_0M; \mathbb{R})$ is a TQFT with unit $\psi_C(1) \in SH^0_0(M)\tilde{=} \subset H^1(L_0M; \mathbb{R})$. The analogues of Theorems 9.5, 10.1 and 14.3 hold.

**Proof.** Let $\alpha$ denote a singular 1-cocycle representing the image of $\tau y$ in $H^1(L_0M)$. Pick as a base point for $L_0M$ the constant loop $m$ at a point in $M$.

Let $u : S \to \tilde{M}$ be a Floer solution whose ends $z_i$ are contractible Hamiltonian orbits. Pick maps $C \to \tilde{M}$, whose images are caps $C_i$ which contract $z_i$ down to $m$. An appropriate gluing $\sigma = (\cup C_i) \cup u(S)$ defines a sphere $\sigma : S^2 \to \tilde{M}$. Now $\alpha$ corresponds to a homomorphism $\pi_2(M) \to \mathbb{Z}$ under the identification $H^1(L_0M) \cong \text{Hom}(\pi_1 M \times \pi_2 M, \mathbb{Z})$. Thus, $\alpha[\sigma]$ is defined (using that $L_0M \simeq L_0\tilde{M}$ are homotopy equivalent).

Parametrizing $C$ by polar coordinates, the caps $C_i$ can be viewed as paths in $L_0M$ from $z_i$ to $m$. So $C_i$ is a singular 1-chain in $L_0M$ and $\alpha[C_i]$ is defined. Let

$$\tilde{\eta}[u] = \alpha[\sigma] - \sum \pm \alpha[C_i],$$

where $\pm$ is the sign of the end $z_i$ of $u$.

We claim $\tilde{\eta}[u]$ is independent of the choices of caps. Suppose that we change $C$ to $C'$, so the sphere $\sigma$ changes to a new sphere $\sigma'$. The gluing $\sigma'' = -(C') \# C$ is a sphere in $M$ but also a sum of two 1-chains in $L_0M$, and under our identifications,

$$\alpha[\sigma''] = -\alpha[C'] + \alpha[C].$$

In $\pi_2(M)$, the equality $\sigma'' + \sigma = \sigma'$ holds, so evaluating $\alpha$ we deduce

$$-\alpha[C'] + \alpha[C] + \alpha[\sigma] = \alpha[\sigma'].$$

So, $\alpha[\sigma] + \alpha[C]$ does not change when we replace $\sigma, C$ with $\sigma', C'$. Similarly, $\alpha[\sigma] - \alpha[C]$ does not change when we replace positive caps $C$. So $\tilde{\eta}[u]$ only depends on $u$, not on the caps.

The construction of the TQFT for $SH^*_0(M)\tilde{=} \subset H^1(L_0M; \mathbb{R})$ is now routine using weights $\tilde{\eta}[u]$.

Just as in Subsection 7.4, we also need to show that when we change the representative cocycle $\alpha$ to $\alpha + df$, with $f \in C^0_{\text{sing}}(L_0M)$, the TQFT will change in a way compatible with an isomorphism of the $SH^*_0$ groups induced by an isomorphism of the corresponding local systems on $L_0M$. On a positive cap $C$, viewed as a path from $y$ to $m$ we have by definition:

$$df[C] = f(m) - f(y).$$

So in the above definition, $\tilde{\eta}[u]$ changes by

$$df[\sigma] - \sum \pm df[C_i] = 0 - \sum \pm (f(m) - f(z_i)) = \sum f(y_k) - \sum f(x_a) + (p - q)f(m),$$

where $x_a$ and $y_k$ are, respectively, the $p$ negative and the $q$ positive asymptotics of $u$. So the TQFT changes compatibly with the isomorphism $\Lambda_x \mapsto \Lambda_x$, $\Lambda_x \mapsto t^{-f(x)}f(m)\lambda_x$ of the local systems, which induces $SH^*_0(M; \Delta_\alpha) \cong SH^*_0(M; \Delta_{\alpha+df})$ given by $x \mapsto t^{-f(x)}f(m)\lambda_x$ on generators.
7.7. Twisting by closed 1-forms from the base

Let \( ev : \mathcal{LM} \to M, x \mapsto x(0) \) be the evaluation at zero, and fix a closed 1-form \( \mu \) on \( M \). Let \( u : S \to \overline{M} \) be a Floer solution.

For cylinders \( S \), \( ev^*\mu[u] = \int_S ev(u)^*\mu \) evaluates \( \mu \) on the 1-chain in \( M \) given by \( s \mapsto u(s,0) \).

For a cap \( S \), \( ev^*\mu[u] \) cannot be defined consistently (if we puncture the cap and parametrize the punctured cap as a cylinder, then the value \( ev^*\mu[u] \) would depend on the choice of puncture; in the approach that we will explain below, we would need to embed a suitable graph in the cap, but no choice of graph guarantees that caps glue compatibly with the TQFT structure.)

For other \( S \), let \( S' \) be an associated graph (Subsection 6.7). Pick any continuous map \( S' \to S \) (not necessarily an embedding) so that the ends of \( S' \) (‘external vertices’) converge to \( t = 0 \in S^1 \) in the asymptotic circles in \( S \). So, to clarify, for \( u \in \mathcal{M}(x_a; y_b; S, \beta) \), the external vertices map to \( x_a(0), y_b(0) \) via \( u \).

Associate to each edge \( e_i \) a constant \( k_i \in \mathbb{R} \). These define a closed 1-form \( \mu_i = k_i\mu \) for each edge \( e_i \). Denote by \( \mu_a = k_a\mu \) and \( \mu_b = k_b\mu \) the forms defined in this way on those edges which connect to the external vertices. We now fix these constants \( k_a \geq 0 \) and \( k_b \geq 0 \), but we allow the other constants \( k_i \) to be arbitrary subject only to the zero-sum condition: at any internal vertex \( v \) of the graph we require that the signed sum of the constants vanishes,

\[
\sum_{\text{edges } e_i \text{ coming into } v} k_i - \sum_{\text{edges } e_i \text{ leaving } v} k_i = 0.
\]

Integrating \( \mu_i \) along the image under \( u \) of the edge \( e_i \), and adding up, defines

\[
\mu[u] = \sum_{e_i} \mu_i[u|_{e_i}].
\]

This construction also defines \( \mu[u] \) when we deal with Morse solutions \( u : S' \to \overline{M} \) (see Subsection 6.8).

**Example.** \( P' \) (Figure 6): using \( 2\mu \) on the incoming edge, and \( \mu \) on each outgoing edge.

**Remark.** The zero-sum condition implies the steady-state condition \( \sum k_a = \sum k_b \), and conversely the steady-state condition guarantees that there is a choice of constants \( k_i \) as above.

Counting Floer solutions with weight \( \mu^a[u] \) defines

\[
\psi_S : \bigotimes_{b=1}^q SH^*(M)_{ev^*\mu_b} \to \bigotimes_{a=1}^p SH^*(M)_{ev^*\mu_a} \quad (p \geq 1, q \geq 0, (p, q) \neq (1, 0)).
\]

These \( \psi_S \) are well-defined but depend on \( S' \) and the choice of map \( S' \to S \).

**Example.** Take \( S = Z \) and \( S' = Z' \) as in Subsection 6.7. If we map \( S' \) inside the line \( t = 0 \) in \( Z \), we get \( \mu[u] = ev^*\mu[u] \), but if we map \( S' \) onto a curve in \( Z \) which winds once around the \( S^1 \) factor, then \( \mu[u] = ev^*\mu[u] + \mu[\gamma] \), where \( \gamma \in H_1(M) \) is the \( H_1 \)-class of the asymptotics of \( u \).

So to obtain TQFT operations which glue consistently (assuming that the forms \( \mu_i \) agree on the gluing ends), we need to make choices as follows. We now assume \( S \) has genus zero and \( p \geq 1, q \geq 0, (p, q) \neq (1, 0) \). Such \( S \) are generated by \( Z, P, Q \) and \( S_{20} \) (see Section 6.4). Cut the \( Z, P, Q \) and \( S_{20} \) in half by planes (this produces open model surfaces as in Subsection 6.11). We can make this cut so that there are halves \( Z_{up} \subset Z, P_{up} \subset P, Q_{up} \subset Q \) and \( (S_{20})_{up} \subset S_{20} \) whose asymptotics consist of the arcs in \( S^1 \) parametrized by \( -\frac{1}{4} \leq t \leq \frac{1}{4} \). We embed the \( Z', P', Q' \) and \( S'_{20} \) of Subsection 6.7 into \( Z_{up}, P_{up}, Q_{up} \) and \( (S_{20})_{up} \) (where \( S'_{20} \) is the graph with two
Lemma 7.6. The $\mu[u]$ constructed for $S' \subset S_{up}$ only depend on the choices of constants $k_a$ and $k_b$ at the ends (and on the connected component of $u \in M(x_a; y_b; S, \beta)$).

Proof. Observe that, if we ignored weights, such an $S'$ will be homotopic in $S_{up}$ to the standard graph consisting of $p$ negative ends and $q$ positive ends which meet in one common internal vertex. This relies on the fact that $S$ has genus zero, so $S'$ is contractible.

Consider first a homotopy of $S'$ inside $S_{up}$ which does not change the combinatorial type of the graph. Each edge $e_1$ will sweep out via $u$ a 2-simplex in $M$, and these 2-simplices are joined along the 1-simplices swept out by the internal vertices (the external vertices are fixed). Evaluating $d\mu = 0$ on these 2-simplices shows that $\mu[u]$ before and after the homotopy has changed by an amount equal to the sum of the $\mu_i$ evaluated on those 1-simplices. Taking into account the orientations, this amount vanishes by the zero-sum condition.

Now, we show how to change the combinatorial type of the graph. Adding or removing internal vertices which touch only two edges will not affect $\mu[u]$, assuming that the zero-sum condition holds. Next, suppose we add to $S'$ an oriented triangle whose edges are assigned the same form $-k\mu$, with at least one vertex being part of the vertices of $S'$. Since the triangle is contractible after mapping it into $S_{up}$ (since $S$ has genus 0), and since $d\mu = 0$, this triangle does not affect $\mu[u]$. Secondly, we can remove two equally oriented edges which carry opposite weights, since these give cancelling contributions to $\mu[u]$.

These two moves allow us to slide an edge along another: given edges $e' : v_1 \rightarrow v_2$, $e : v_2 \rightarrow v_3$ of weight $k'$ and $k$, adding such a triangle with vertices $v_1, v_2$ and $v_3$ allows us to replace $e'$ and $e$ by edges: $v_1 \rightarrow v_2$ and $v_1 \rightarrow v_3$ of weight $k' - k$ and $k$.

These moves allow us to turn $S'$ into a standard graph. But all standard graphs are homotopic through a homotopy which does not affect the combinatorial type of the graph, so $\mu[u]$ is independent of choices.

Remark 7.8. Restricting to contractible loops, $\alpha = ev^*\mu + \tau\eta$ yields all $\alpha \in H^1(L_0 M)$ since $\pi_1(L_0 M) \cong \pi_1(M) \times \pi_2(M)$ so $H^1(L_0 M) \cong ev^*H^1(M) \oplus \text{Hom}(\pi_2(M), \mathbb{Z})$. 

Theorem 7.7. $NSH^*(M)_{\eta}$ is a TQFT (using genus zero $S$, $p \geq 1$) with unit $\psi_C(1) \in SH_{0, \eta}.$

Abbreviate $SH_{\mu} = SH^*(M)_{ev^*\mu}$. Arguing as in Theorem A.14, we obtain operations $\psi_S : \bigotimes_a SH_{\mu_a} \rightarrow \bigotimes_a SH_{\mu_a}$ for genus zero surfaces $S$ with $\nu \geq 0, q \geq 0, (p, q) \neq (1, 0)$ which compose correctly. Changing $\mu$ to $\mu + df$ corresponds to an isomorphism of the local systems $x \mapsto t^{-f(x)}x$, just as in Subsection 7.4.

One could identify $SH_{ev^*\mu} \cong SH_{\mu}$ via the isomorphism $t \mapsto t^{1/c}$ of local systems, for any $c > 0$, so that appropriately rescaled operations give a TQFT on $SH^*(M)_{ev^*\mu}$ without unit.

A more natural approach is to take $NSH^*(M) = \bigoplus_{\mu} SH_{\mu}$ summing over all $\mu \in H^1(M).$ One can also twist by $\eta \in H^2(M); NSH^*(M)_{\eta} = \bigoplus_{\mu} SH_{\mu, \eta}$ where $SH_{\mu, \eta} = SH^*(M)_{ev^*\mu + \tau \eta}.$ So, $NSH^*(M) = NSH^*(M)_{0}.$ For example, the product is $SH_{\mu_1, \eta} \otimes SH_{\mu_2, \eta} \rightarrow SH_{\mu_1 + \mu_2, \eta},$ the canonical map is $e^* : NH^*(M) = \bigoplus_{\mu} H^*(M)_{\mu} \rightarrow NSH^*(M)_{\eta}$ (the Novikov cohomology $H^*(M)_{\mu}$ is the ordinary cohomology with coefficients in the local system defined by $\mu$).

Theorem 7.7. $NSH^*(M)_{\eta}$ is a TQFT (using genus zero $S$, $p \geq 1$) with unit $\psi_C(1) \in SH_{0, \eta}.$
So $\oplus_\alpha SH^0_\alpha(M) = \oplus_\eta NSH^0_\eta(M)$ is a TQFT with unit (for genus zero, $p \geq 1$), $c^*: NH^*(M) \to \oplus_\alpha SH^0_\alpha(M)_\alpha$.

7.8. Twisted wrapped theory

Let $\alpha \in H^1(\Omega(M, L)) \cong H^1(\Omega(\bar{M}, \bar{L}))$ (Subsection 4.3). As in Subsection 7.1, $\alpha$ defines a local system of coefficients $\Omega_\alpha$ on $\Omega(\bar{M}, \bar{L})$. Define $CW^*(L; H)$ over the Novikov ring $\Lambda$ and insert weights $t^{|u|}$ in the definitions of the differential and the continuation maps. This yields $\Lambda$-modules $HW^*(L; H; \Omega_\alpha)$ and their direct limit defines $HW^*(L) = HW^*(L; \Omega_\alpha)$.

These come with a canonical map $c^*: H^*(L; c^*\Omega_\alpha) \to HW^*(L; \Omega_\alpha)$, where $c: \Omega(\bar{L}, \bar{L}) \to \Omega(\bar{M}, \bar{L})$ is the inclusion of paths lying in $\bar{L}$ (observe that $\Omega(\bar{L}, \bar{L})$ has the homotopy type of $L$).

Let $\eta \in H^2(\bar{M}, L; \mathbb{R}) \cong H^2(\bar{M}, L; \mathbb{R}) \cong H^2(M/L; \mathbb{R})$. One can represent $\eta$ by a closed de Rham 2-form which is supported away from $\bar{L}$ (see [24, Section 11.1]). Evaluating by $\int_C \sigma^* \eta$ on smooth 1-chains $\sigma: \mathcal{C} \to \bar{M}$ defines a transgression cycle $\tau = \frac{H^*(\Omega(M, L)) \cong \text{Hom}(\pi_2(M/L), \mathbb{R})}$.

This is an isomorphism when $\pi_2(M/L) = 1$ (for example, if $\pi_2(M) = 1$). For simply connected $L$ and $M$, $\tau$ is the Hurewicz isomorphism $H^2(M/L; \mathbb{R}) \cong \text{Hom}(\pi_2(M/L), \mathbb{R})$.

For a wrapped solution $u: S \to \bar{M}$ define $\tau = \int_S u^* \eta$. These weights are locally constant on the moduli spaces of wrapped solutions since $\tau = u^* \eta$ remains constant if we homotope $u$ relative to the ends while keeping $u(\partial S) \subset L$. Inserting weights in Subsections 6.12–6.14, we deduce:

**Theorem 7.9.** $HW^*(L; \Omega_\eta)$ has a TQFT structure and $HW^*(L; \Omega_\eta)$ is a module over $SH^*(M; \Lambda_\eta)$, where $\eta \in H^2(M; \mathbb{R})$ is the image of $\eta$ via $H^2(M/L; \mathbb{R}) \to H^2(M; \mathbb{R})$.

The twisted version of the map $c^*$ from Section 5 is $c^*: H^*(L) \otimes \Lambda \to HW^*(L; \Omega_\eta)$, and this is a TQFT map by Theorem 6.14 (after inserting Novikov weights as in Theorem 7.3).

8. The action filtration, and the $SH^*_+$, $HW^*_+$ groups

Let $H: \bar{M} \to \mathbb{R}$ be linear at infinity (Subsection 3.3). By Subsection 2.1, we can ensure that the only 1-orbits of $H$ inside $M$ are critical points of $H$ and, picking $|H| < \delta$ small on $M$, these have action close to zero: $\kappa_H \in (-\delta, \delta)$. As explained in the following Remark, we can ensure $H = h(R)$ for $R \geq 1$ with $h'$ growing so slowly that the 1-orbits of $H$ on the collar have arbitrarily negative action, in particular $\kappa_H < -\delta$.

**Remark.** Recall the action formula $\kappa_h(R) = -R w''(R) + h(R)$ of Subsection 2.2. If $h$ is convex, then $\partial_R \kappa_h = -R h''(R) \leq 0$ so the action decreases with $R$. Pick $h$ convex for $R \geq R_0$. For $R \in (1, R_0)$ pick $h' \leq T_0/2$, where $T_0$ is the smallest Reeb period (so there are no 1-orbits since they have $h' \geq T_0$), with $h(R_0) = 2\delta$ and $h'(R_0) = T_0/2$. Then 1-orbits on the collar have action less than $\kappa_h(R_0) = -R_0 T_0/2 + 2\delta$. Finally, by picking $R_0 \gg 0$, we can ensure that the action is arbitrarily negative.

Write $SC^*(A, \infty)_\alpha$ for the chain subcomplex of $SC^*(H)_\alpha$ generated only by those generators with action in the interval $(A, \infty)$ (for generic $A \in \mathbb{R} \cup \{\pm \infty\}$, so not Reeb periods, and where
we keep track of the twisting by $\alpha \in H^1(\mathcal{L}M)$ when twistings are present). Note that this is a subcomplex because the differential counts incoming Floer trajectories, and the action decreases along trajectories. Define the quotient complex

$$SC^*(A, B)_\alpha = SC^*(A, \infty)_\alpha / SC^*(B, \infty)_\alpha.$$ 

Increasing $A$ gives a subcomplex: $SC^*(A', B)_\alpha \hookrightarrow SC^*(A, B)_\alpha$ for $A < A' < B$. Decreasing $B$ gives quotient maps: $SC^*(A, B)_\alpha \to SC^*(A, B')_\alpha$ for $A < B' < B$.

**Example.** $SC^*(-\delta, \infty) \equiv SC^*(-\delta, \delta)$ is a subcomplex of $SC^*(H)$ and it computes precisely the cohomology $SH^*(H^3)$ from Section 5, which one can identify with $H^*(M)$.

**Example.** Define $SC^+_\alpha(H)_\alpha = SC^*(-\infty, -\delta)_\alpha$. Define $SH^*_+(M)_\alpha$ to be the direct limit of the cohomology groups $SH^*_+(H)_\alpha$ as we increase the slope of $H$ at infinity. In the untwisted setup, the $SH^*_+(M)$ groups are well-known [43, 10].

These two examples fit naturally into a short exact sequence

$$0 \to SC^*(-\delta, \delta)_\alpha \xrightarrow{c^*} SC^*(-\infty, \delta)_\alpha \to SC^*(-\infty, -\delta)_\alpha \to 0.$$ 

(More generally, for $s < m < \ell$, one gets $0 \to SC^*(m, \ell)_\alpha \hookrightarrow SC^*(s, \ell)_\alpha \to SC^*(s, m)_\alpha \to 0$.) Taking the associated long exact sequence in cohomology, and taking direct limits, yields:

**Lemma 8.1.** There is a long exact sequence induced by action–restriction maps

$$\ldots \to H^*(M)_\alpha \xrightarrow{c^*} SH^*(M)_\alpha \to SH^*_+(M)_\alpha \to H^{*+1}(M)_\alpha \to \ldots.$$ 

**Example.** $\overline{M} = T^*N$: we assume $\mathbb{K}$ has characteristic 2 (we explain why in Section 14). By [1] (and the twisted analogue [31, Theorem 3]), the isomorphism $SH^*(T^*N)_\alpha \cong H_{n-\delta}(\mathcal{L}N)_\alpha$ respects certain action filtrations (see Section 14). In this case:

**Corollary 8.2.** For $\overline{M} = T^*N$, the long exact sequence is that of the pair $(\mathcal{L}N, N)$, viewing $c : N \hookrightarrow \mathcal{L}N$ as the subspace of constant loops,

$$\ldots \to H^*(T^*N)_\alpha \xrightarrow{c^*} SH^*(T^*N)_\alpha \to SH^*_+(T^*N)_\alpha \to \ldots$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\ldots \to H_{n-\delta}(N)_\alpha \xrightarrow{c_{n-\delta}} H_{n-\delta}(\mathcal{L}N)_\alpha \to H_{n-\delta}(\mathcal{L}N, N)_\alpha \to \ldots.$$ 

Now consider the wrapped analogue of the construction. The action contains the additional term $f(x(1)) - f(x(0))$ (see Subsection 4.3). This will not affect the $h_R < -\delta$ estimate on the collar, since we can make the $-Rh^*(R) + h(R)$ contribution arbitrarily small, whereas $f$ is locally constant on $\mathcal{L} \setminus L$ and hence bounded. So, we can define the subcomplex $CW^*(A, \infty)_\alpha \subset CW^*(L; H)_\alpha$ for $\alpha \in H^1(\Omega(M, L))$: the quotient complex $CW^*(A, B)_\alpha$; and $HW^*_+(L) = HW^*(-\infty, -\delta)_\alpha$.

**Lemma 8.3.** There is a long exact sequence for wrapped Floer cohomology,

$$\ldots \to H^*(L)_\alpha \xrightarrow{c^*} HW^*(L)_\alpha \to HW^*_+(L)_\alpha \to H^{*+1}(L)_\alpha \to \ldots.$$
9. Viterbo functoriality

9.1. Liouville subdomains

Observe Figure 11 (ignore $L$). A Liouville subdomain $i : (W, \theta_W) \hookrightarrow (\mathcal{M}, \theta)$ is the embedding of a Liouville domain $(W, \theta_W)$ of the same dimension as $M$ such that $i^* \theta = e^\rho \theta_W$ + exact form, for some $\rho \in \mathbb{R}$.

**Example 9.1.** Suppose $L \subset T^*N$ is a closed exact Lagrangian submanifold ($\theta|_L$ = exact, and dim $L = \dim N$). This induces a Weinstein embedding $i : DT^*L \hookrightarrow T^*N$ of a small disc cotangent bundle of $L$ onto a neighbourhood of $L$, and $i$ is a Liouville subdomain.

**Lemma 9.2.** For the purposes of symplectic cohomology, one can always assume a Liouville embedding is an inclusion $W \subset \mathcal{M}$, using the same $\theta, Z$ for $W$ as for $\mathcal{M}$.

**Proof.** We can always assume $\rho = 1$ by redefining $\theta_W$ to be $e^\rho \theta_W$. This does not affect the symplectic cohomology theory for $W$ (it rescales all Hamiltonians by $e^\rho$).

Identify $W$ and $i(W)$. We are given $\theta_W = \theta - df$, with $f : W \to \mathbb{R}$. Extend $f$ smoothly to $\mathcal{M}$, so that it vanishes outside of a compact neighbourhood of $W$. Then extend $\theta_W, Z_W$ to all of $\mathcal{M}$ by $\theta_W = \theta - df$ and $\omega(Z_W, \cdot) = \theta_W$. Since $d\theta_W = d\theta$ everywhere and $\theta_W = \theta$ for large $R$ on the collar of $\mathcal{M}$, the Hamiltonian vector fields agree and we can use the same almost complex structure. So the Floer theory of $\mathcal{M}$ is not affected if we replace $\theta$ by $\theta_W$.

**Remark.** If $W$ is not entirely contained in $M$, then the change in $\theta$ does affect the definition of the coordinate $R$ and thus of the class of linear Hamiltonians. But in that case, the identity $(\mathcal{M}, \theta) \to (\mathcal{M}, \theta_W)$ is a symplectomorphism of contact type so $SH^*$ is invariant by Section 3.5.

**Remark 9.3.** Liouville embeddings $i : (W, \theta) \hookrightarrow (\mathcal{M}, \theta)$ can be extended to $i : (\bar{W}, \theta) \hookrightarrow (\mathcal{M}, \theta)$: define $i$ on the collar to be the Liouville flow of $Z_W$ in $\mathcal{M}$ (so $[1, \infty) \times \partial W \ni (R = e^r, y) \mapsto \varphi_{Z_W}^r(y) \in \mathcal{M}$). Since $Z_W$ is outward pointing along $\partial W$, this flow will not reintersect $W$.

By Lemma 9.2, we always assume $W \subset \mathcal{M}$. Let $C_W = [1, 1 + \varepsilon) \times \partial W$ be a collar of $\partial W \subset \mathcal{M}$. Choose $J$ on $\mathcal{M}$ of contact type near $R = e^r = 1 + \varepsilon$ (using the $R$ coordinate of the $C_W$ collar). Choose $H : \mathcal{M} \to \mathbb{R}$ with $H = mR$ near $R = 1 + \varepsilon$ and $H \geq 0$ on $\mathcal{M} \setminus (W \cup C_W)$.

**Corollary 9.4.** Any Floer solution with asymptotics in $W \cup C_W$ cannot escape $W$.

**Proof.** Let $V = \mathcal{M} \setminus (\bar{W} \cup [1, R) \times \partial W)$, with $R$ close to $1 + \varepsilon$ chosen generically so that $\partial V = \{R\} \times \partial W$ intersects the Floer solution $v$ transversely. Apply Lemma D.3 to the restriction $u$ of $v$ to the preimage of each component of $\text{im} \, v \cap V$. Hence, $\text{im} \, v \subset W \cup C_W$.

9.2. Viterbo functoriality

For Liouville subdomains $W \subset \mathcal{M}$, Viterbo [43] constructed a restriction map $SH^*(M) \to SH^*(W)$ and McLean [28, Lemma 10.2] proved that it is a ring homomorphism. We now prove a stronger statement:
Theorem 9.5. Let \( i : (W, \theta_W) \hookrightarrow (M, \theta) \) be a Liouville subdomain. Then there exists a restriction map, \( SH^*(i) : SH^*(M)_{\eta} \rightarrow SH^*(W)_{i, \eta} \), which is a TQFT map which fits into a commutative diagram which respects TQFT structures:

\[
\begin{array}{cccc}
SH^*(W) & \xrightarrow{SH^*(i)} & SH^*(M) \\
\downarrow c^* & & \downarrow c^* \\
H^*(W) & \xleftarrow{i^*} & H^*(M)
\end{array}
\]

In particular, all maps are unital ring homomorphisms.

Proof. We may assume \( W \subset \overline{M} \) and \( \theta_W = \theta|_W \) by Lemma 9.2.

Pick a (smoothing of the) step-shaped Hamiltonian \( H \geq 0 \) in Figure 10. More precisely, the Liouville flow parametrizes a tubular neighbourhood \((0, 1 + \varepsilon) \times \partial W \) of \( \partial W \subset \overline{M} \) (cf. Subsection 2.1). This determines an \( R \) coordinate on that neighbourhood with \( \partial W = \{ R = 1 \} \) (which in general is not related to the \( R \) coordinate that we use on the collar of \( \overline{M} \)). Using these \( R \) coordinates on the two collars, define \( H \) to be linear of slope \( m \) for \( R \gg 0 \) on the collar of \( \overline{M} \) and linear of slope \( w \gg m \) on \((c, 1 + \varepsilon) \times \partial W \). Make \( H \) constant elsewhere (rounding off the corners in the graph). Here, \( w \) and \( m \) are generic (so not Reeb periods of \( \partial W \) and \( \partial M \)). In the regions where the graph is flat we make a time-independent positive Morse perturbation of \( H \) so the 1-orbits there are critical points of \( H \) and their action is the value of \( H \) \( \geq 0 \).

Even if \( W \) is not contained in \( M \), we can fix a large enough \( R_0 \) so that the region \( R \geq R_0 \) on the collar of \( \partial M \) where we make \( H \) have slope \( m \) is away from \( W \).

We now show how to separate the action values of the 1-orbits lying in \( W \) and those outside \( W \) (we mimic the discussion of Ritter [31, Section 4.3] which is based on Viterbo’s [43] original argument). By construction, the only non-constant 1-orbits of \( H \) arise near the regions where the graph of Figure 10 has corners. By Subsection 2.1, the action of these orbits is \( \mathcal{A}(R) = -Rh'(R) + h(R) \) (in the appropriate \( R \) coordinate), which is the value where the tangent line to the graph of \( H \) intersects the vertical axis.

So \( w \gg m \) ensures that the smallest values of \( \mathcal{A}(R) \) outside \( W \) arise near \( R = 1 + \varepsilon \), since the 1-orbits outside the corner \( R = 1 + \varepsilon \) will have large positive action if we make \( w \) large.

Fix \( 0 < \delta_w \leq 1 \) such that no Reeb periods are in \([w - \delta_w, w + \delta_w]\) (the Reeb periods are a discrete subset of \([0, \infty)\) by Subsection 2.1). By Subsection 2.1, the slopes of \( h \) at 1-orbits are Reeb periods, so the 1-orbits near \( R = 1 + \varepsilon \) have \( \mathcal{A}(R) \geq -(1 + \varepsilon)(w - \delta_w) + h(1 + \varepsilon) \geq (1 + \varepsilon)\delta_w - wc \), using \( h(1 + \varepsilon) = w(1 + \varepsilon - c) \). So, for \( c \ll 1/w \), the 1-orbits outside \( W \) have action in \((\delta_w, \infty)\).

The 1-orbits in \( W \) near \( R = c \) have action in \([-wc, 0]\) and the critical points of \( H \) in \( W \) have arbitrarily small positive action. So given \( w \), for \( c \ll 1/w \), the 1-orbits in \( W \) have action in \((-\delta_w/G_c, \delta_w/G_c)\) for a large number \( G_c \geq 1 \), such that \( G_c \to \infty \) as we let \( c \to 0 \).

Figure 10. A step-shaped Hamiltonian with slopes \( w \) and \( m \). The dashed line is the action function \( \mathcal{A}(R) = -Rh'(R) + h(R) \). See also Figure 11.
Let $H'$ be another step-shaped Hamiltonian for $w' \geq w, m' \geq m$ with $w' \gg m'$ so that the 1-orbits outside the corner $R = 1 + \varepsilon$ have large action (action at least 1 suffices). We can modify $H$ by decreasing $c$ so that $\delta_w / G_c < \delta_{w'}$. We then modify $H'$ by decreasing $c'$ so that we have the following.

1. $c' < c$, so $H' > H$, so there is a monotone homotopy $H_s$ from $H'$ to $H$ with $\partial_s H_s \leq 0$.
2. $\delta_{w'}/G_{c'} < \delta_w/G_c < \delta_{w'}$.

Sub-claim: there is a commutative diagram:

\[
\begin{array}{ccc}
SC^*(W, i^* H')_{i^*\eta} & \xrightarrow{\eta} & SC^*(M, H'; \mathcal{A}_H < \delta_w/G_c)_{\eta} \\
\downarrow \text{cont.} & & \downarrow \text{cont.} \\
SC^*(W, i^* H)_{i^*\eta} & \xrightarrow{\eta} & SC^*(M, H; \mathcal{A}_H < \delta_w/G_c)_{\eta}
\end{array}
\]

where the vertical maps are the (twisted) continuation maps defined by $H_s$; and the horizontal maps in the second square are action-restriction quotient maps (see Section 8).

We now prove the Sub-claim. The second square commutes because the action decreases along Floer continuation solutions for $H_s$ (using $\partial_s H_s \leq 0$ and the energy estimate in Subsection 3.2). In the first square of the above diagram, the only 1-orbits being considered lie in $W$ because of the action constraint. By Corollary 9.4, Floer trajectories with ends in $W$ do not escape $W$. So the $i^{-1}$ maps are well-defined identifications of chain complexes. The first square commutes because, by Corollary 9.4 and Remark D.4(2), also Floer continuation solutions with ends in $W$ do not escape $W$.

We now prove that the above diagram respects the TQFT structure on cohomology, in the sense of Theorem A.14 (for $A_a = A'_a$, $B_b = B'_b$). For the first square, this follows because the Floer solutions in $\overline{M}$ with asymptotics in $W$ and the Floer solutions in $W$ with asymptotics in $W$ must both lie in $W$ by Corollary 9.4, and so they coincide via $i^{-1}$. For the second square, we first need to consider the behaviour of the action-restriction map along TQFT operations.

Consider $\psi_S : \bigotimes_b SC^*(B_b H) \rightarrow \bigotimes_a SC^*(A_a H)$ on $\overline{M}$. By Theorem A.14, on cohomology $\psi_S$ does not change if we increase $w$ and decrease $c$: it only depends on the slope $m$ at infinity. By the above estimates, for $w \gg m, c \ll 1/w$, we can ensure that: $G_c > p + q - 1$; the 1-orbits of $A_a H$ and $B_b H$ in $W$ have action in $(-\delta_w/G_c, \delta_w/G_c)$; the 1-orbits of $A_a H$ and $B_b H$ outside $W$ have action more than $\delta_w < \delta_w/G_c$. Therefore, in particular, all 1-orbits of $B_b H$ have $\mathcal{A}_{B_b H} \gg -\delta_w/G_c$.

For action-restrictions to commute with $\psi_S$, we need to ensure that no Floer solution $u : S \rightarrow M$, which contributes a non-zero multiple of $\bigotimes_a x_a$ to $\psi_S(\bigotimes_b y_b)$ has some $\mathcal{A}_{B_b H}(y_b) > \delta_w$ but all $\mathcal{A}_{A_a H}(x_a) < \delta_w/G_c$. (The opposite problem would be if both $\mathcal{A}_{B_b H}$ and $\mathcal{A}_{A_a H}$ were zero, and the action-restriction would get $\bigotimes_a x_a$ zero.) Suppose by contradiction that this was the case. Then by the energy estimate in Subsection 3.2 and using $G_c > p + q - 1$ and $\mathcal{A}_{B_b H}(y_b) \gg -\delta_w/G_c$,

\[
E(u) = \sum_a \mathcal{A}_{A_a H}(x_a) - \sum_b \mathcal{A}_{B_b H}(y_b) < \frac{\delta_w}{G_c} + (q - 1) \frac{\delta_w}{G_c} - \delta_w < 0
\]

contradicting $E(u) \geq 0$. So the only contributions to the TQFT operation $\psi_S$ which survive under action-restriction involve Floer solutions $u$ all of whose asymptotics lie in $W$. A similar argument holds for $H'$ in place of $H$. This, combined with Theorem A.14, implies that the second square respects the TQFT operation $\psi_S$ on cohomology. This proves the Sub-claim.

Call $i_H : SH^*(W, i^* H)_{i^*\eta} \rightarrow SH^*(M, H)_{\eta}$ the composite on cohomology of the horizontal maps in the Sub-claim. By the Sub-claim, the direct limit of the $i_H$ as we let $w' \gg m' \rightarrow \infty$ is defined. This limit defines the map $SH^*(i)_{\eta}$ in the statement of Theorem 9.5. By construction, $SH^*(i)_{\eta}$ fits into a commutative diagram with $i_H$. To obtain the diagram in the Theorem,
define $H$ by taking $w \ll 1$ and then $m \ll w$, so that no Reeb periods for $\partial W$ and $\partial M$ lie in $[0, w]$ and $[0, m]$. So the only 1-orbits of $H$ are critical points and, by Section 5 and Theorem 7.3, $i_H$ can be identified (in a TQFT-preserving way) with the restriction on Morse cohomology $MH^*(W, i^*H) \otimes \Lambda \leftarrow MH^*(M, H) \otimes \Lambda$, which indeed is the pull-back on ordinary cohomology.

\begin{theorem}
(1) The restriction map $SH^*(i)_\eta : SH^*(M)_\eta \rightarrow SH^*(W)_{i^*\eta}$ is invariant under isotopies of $i : W \hookrightarrow \overline{M}$ among embeddings satisfying $i^*\theta_M - e^{\text{constant}}\theta_W = \text{exact}$.

(2) Functoriality: given nested Liouville subdomains $W' \hookrightarrow W \hookrightarrow \overline{M}$, then $SH^*(M)_\eta \rightarrow SH^*(W)_{i^*\eta} \rightarrow SH^*(W')_{i^*\eta}$ equals $SH^*(M)_\eta \rightarrow SH^*(W')_{i^*\eta}$.

(3) $SH^*(i)_\eta = \text{id}$ for the following maps: $i = \text{id} : M \subset \overline{M}$; the Liouville flow $i = \varphi^\tau_Z : M \hookrightarrow \overline{M}$ for time $\tau \in \mathbb{R}$; and the inclusions $i : M \cup [1, R_0] \times \partial M \hookrightarrow \overline{M}$.
\end{theorem}

\begin{proof}
(1) Consider a family $i_\lambda$, $0 \leq \lambda \leq 1$, of such embeddings. By the proof of Lemma 9.2 (and the Remark contained therein), this reduces to considering a fixed $W \subset \text{int}(M)$ with Liouville form $\theta_\lambda = \theta_0 + d\lambda f_\lambda$ on $\overline{M}$, such that $f_\lambda$ is supported in a neighbourhood of $W$ in $\text{int}(M)$ and $\theta_\lambda|_W = \theta_W$, where $W_\varepsilon = W \cup (1, 1 + \varepsilon) \times \partial W$ (we extend $\theta_W$ to the collar just as for $W$). Note that the $R$ coordinate on the collar of $\overline{M}$ will be independent of $\lambda$.

Choose the step-shaped Hamiltonian $H$ with slopes $w$ and $m$ so that its constant step includes the bounded region outside $W_\varepsilon$, where $\theta_\lambda \neq \theta_0$ (in the two regions where we make $H$ linear the two $R$ coordinates do not depend on $\lambda$ since $\theta_\lambda = \theta_0$ there). Now consider the computation of the actions in the proof of Theorem 9.5: near the regions where $H$ has slopes $w$ and $m$ the form $\theta_\lambda$ equals $\theta_W$ and $\theta_0$, respectively, and in the region where $H$ is constant the action estimates do not involve $\theta_\lambda$ (the generators there, after a small perturbation, are critical points of $H$, and these have actions equal to the value of $H$). As explained in the proof of Theorem 9.5, the Floer solutions joining orbits in $W$ stay entirely in $W$, so the Floer theory there depends only on the form $\theta_W$. Finally, the restriction map for each $\lambda$ can be constructed by using action–restrictions for such Hamiltonians $H$, so we have shown that the restriction maps do not depend on whether we use $\theta_0$ or $\theta_1$ on $\overline{M}$.

(2) Using an appropriate step-shaped Hamiltonian with two steps instead of one, one can separate the action values of the generators in $W'$, $W \setminus W'$ and $\overline{M} \setminus W$. So claim (2) follows because the composite of action–restriction maps is an action–restriction map.

(3) For $i = \text{id}$, the step-shaped Hamiltonian induces the continuation $SH^*(H^m) \rightarrow SH^*(H^w)$ in the notation of Subsection 3.3. So the direct limit as $w \gg m \rightarrow \infty$ is the identity.

For $i = \varphi^\tau_Z : M \hookrightarrow \overline{M}$, $i^*\theta_M = e^\tau \theta_M$, and since $i$ is isotopic to the identity, $SH^*(i)_\eta = \text{id}$ by (1). The inclusions $i$ in (3) are isotopic to the identity via $(\varphi^\tau_Z)_{\tau \in [0, \log R_0]}$, so $SH^*(i)_\eta = \text{id}$ by (1).
\end{proof}

9.3. Wrapped solutions with asymptotics in $W$ do not escape $W$

Observe Figure 11. Let $W \hookrightarrow M$ be a Liouville subdomain, and $L$ an exact Lagrangian with transverse Legendrian intersections with $\partial W$ and $\partial M$ (by Lemma 9.2, we can assume $W \subset M$, $\theta_W = \theta|_W$). So, explicitly, $\theta|_L = df$, for $f : L \rightarrow \mathbb{R}$, and $df = 0$ on $L \cap (\partial W \cup \partial M)$.

In preparation for the construction of the wrapped Viterbo restriction map in Subsection 9.4, we make an additional assumption (the necessity of which is explained in [4, Example 4.2]):

One can extend $f : L \rightarrow \mathbb{R}$ to $f : M \rightarrow \mathbb{R}$ so that $f$ is locally constant on $\partial W \cup \partial M$.

The condition we really want is that $f|_L = 0$ near $\partial W \cup \partial M$ in order to apply Lemma D.6. The somewhat weaker assumption above actually guarantees that this condition can be achieved
after deforming $L$ by a Hamiltonian isotopy of $M$ relative to $\partial W \cup \partial M$ (see [4, Lemma 4.1]). In particular, $L$ then has the form $(\text{interval}) \times (L \cap \partial W)$ near $\partial W$ and similarly near $\partial M$, and we can extend $f$ to $\overline{M}$ by defining it to be locally constant on the collar of $\overline{M}$.

By the no escape Lemma D.6, Corollary 9.4 generalizes to the following corollary:

**Corollary 9.7.** Any wrapped Floer solution with asymptotics in $W \cup C_W$ cannot escape $W$.

### 9.4. Wrapped Viterbo restriction

Continuing with the notation and assumptions introduced in Corollary 9.7, we now prove the wrapped analogue of Theorem 9.5. The construction of the Viterbo restriction for wrapped cohomology is due to Abouzaid and Seidel [4]: they prove that it respects the $A_\infty$ operations on a chain-level telescope model for $H^W(L)$.

**Theorem 9.8.** There is a restriction map $H^W(L \subset M) \to H^W(L \cap W \subset W)$, commuting with $c^*$, and compatible with the TQFT and the $SH^*$-module structure.

More generally, there is a twisted restriction $H^W(L; \Omega_\alpha) \to H^W(L \cap W; \Omega_\alpha|W)$, which for $\alpha = \tau \eta$ respects the structures of Theorem 7.9.

**Proof.** Observe that if we can separate the action values as in the proof of Theorem 9.5, then the proof is identical after replacing the groups $SC^*(L; H)$, $SC^*(W; H)$ by $CW^*(L; H)$, $CW^*(L \cap W; H)$, and using the no escape Corollary 9.7 instead of Corollary 9.4.

We now show how to separate the action values. In the regions where $f = 0$, the action of Hamiltonian chords is $-R h'(R) + h(R)$ (see Subsection 4.3). Thus, in those regions, the same estimates that we made in the proof of Theorem 9.5 for the step-shaped Hamiltonian $H$ of slopes $w$ and $m$ will also hold in the wrapped setup. The non-constant Hamiltonian chords lie in the regions where we smoothen the corners of the graph of $H$ drawn in Figure 10: near $\partial M$ and $\partial W$, we indeed have $f = 0$, but the region near $R = c$ inside $W$ is problematic since it is far away from $\partial W$ as we let $c \to 0$, so $f = 0$ may fail there.

Let $\varphi = \varphi_2^c$ be the Liouville flow for time $\log c < 0$. Push $L \cap W$ into $\varphi(L \cap W)$. Since $L$ has the form $(\text{interval}) \times (L \cap \partial W)$ near $\partial W$, the flow rescales the interval by $c$, so we can smoothly join $\varphi(L \cap W)$ to $\overline{L} \setminus W$ by some $(\text{interval}) \times (L \cap \partial W)$. Call $P$ the resulting Lagrangian. The part of $P$ contained in the region $c \leq R \leq 1 + \varepsilon$ of $W \cup C_W$ is conical: $[c, 1 + \varepsilon] \times (L \cap \partial W)$. So, $\theta|_P = 0$ on that part, and so globally $\theta|_P = df_P$ for a function $f_P : \overline{M} \to \mathbb{R}$ which is zero also in the problematic region near $R = c$. Thus, for $(P, H, W, M, \theta)$, the action values can be separated. So we have a restriction

$$H^W(P; H) \xrightarrow{\text{restrict}} H^W(P; H, \delta_H < \delta_w/G_c) \cong H^W(W \cap P; H|_W).$$
We claim that this naturally induces a restriction \( HW^*(L; H^m) \to HW^*(W \cap L; H^w) \) for some \( H^m \) and \( H^w \) with slopes \( m \) and \( w \) at infinity on \( \overline{M} \) and \( \overline{W} \). This claim implies the Theorem after taking direct limits \( w \gg m \to \infty \).

Since the above construction gives a family of exact Lagrangians \( P_\lambda \) interpolating \( P_0 = \overline{L} \) and \( P_1 = \overline{P} \), there is a Hamiltonian isometry \( \psi \) of \( \overline{M} \) supported in \( W \) which maps \( \overline{L} \) onto \( P \). This induces a natural isomorphism \( \psi^* : HW^*(P; H) \cong HW^*(L; H^m) \) by pulling back the Floer data, which determines the Hamiltonian \( H^m = \psi^*H \) with slope \( m \) at infinity.

Secondly, the above Liouville map \( \varphi \) also induces a natural pull-back isomorphism

\[
\varphi^* : HW^*(W \cap P; H|_W; \theta) \cong HW^*(W \cap L; \varphi^*(H|_W); c\theta),
\]

where \( \varphi^*(H|_W) = H|_W \circ \varphi \) has slope \( wc \) at infinity on \( \overline{W} \). Rescaling \( c\theta \) and \( \varphi^*(H|_W) \) by \( 1/c \) preserves the Hamiltonian vector field and thus the whole Floer theory:

\[
HW^*(W \cap L; \varphi^*(H|_W); c\theta) \equiv HW^*(W \cap L; H^w; \theta),
\]

where \( H^w = \varphi^*(H|_W)/c \) has slope \( w \) at infinity, as required.

Both these isomorphisms are compatible with the algebraic structures (the \( e^* \) map, the TQFT and the \( SH^* \)-module structure) since analogous isomorphisms can be defined for the algebraic structures again simply by pulling back the relevant Floer data by global maps (in the first case, a Hamiltonian isometry \( \psi \), and in the second case, the Liouville map \( \varphi \)). We briefly illustrate how this works in the case of the \( e^* \) maps, but we omit a discussion of the other algebraic structures for sake of brevity.

Recall that \( e^* : H^*(L) \to HW^*(L) \) arises at the Floer level as a direct limit of continuation maps \( \varphi^* : HW^*(L; H^\delta) \to HW^*(L; H) \) which increase the slope of \( H \) at infinity (see Section 5). Let \( H \) be the step-shaped Hamiltonian from above (with slopes \( w \gg m \gg 1 \)) and let \( H^\delta \) be a step-shaped Hamiltonian with slopes \( \delta_1 \) and \( \delta_2 \) where \( 1 \gg \delta_1 \gg \delta_2 > 0 \). These conditions are chosen so as to ensure that restriction maps can be constructed for \( H \) and for \( H^\delta \), and that the slopes \( \delta_1 \) and \( \delta_2 \) are appropriate for defining the \( e^* \) maps for \( L \cap W \) and \( L \). Let \( H_s \) be a monotone homotopy interpolating \( H \) with \( H^\delta \) with \( \partial_s H_s \leq 0 \). By our previous discussion, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
HW^*(W \cap L; H^w) & \xrightarrow{\varphi^*} & HW^*(W \cap P; H|_W) \\
\xrightarrow{\text{cont.}} & & \xrightarrow{\text{cont.}} \\
HW^*(W \cap L; H^\delta_1) & \xrightarrow{\text{restr.}} & HW^*(W \cap P; H^\delta_1|_W) \\
\end{array}
\]

where the first square arises from pulling back via \( \varphi \) the data \( H^\delta|_W, H|_W, H_s|_W, J|_W \); the second square arises from action–restrictions since for the Lagrangian \( P \) we have action estimates analogous to those in the proof of Theorem 9.5; and the last square arises from pulling back via \( \psi \) the data \( H^\delta, H, H_s \) and \( J \).

10. Vanishing criteria

This section relies on Theorem 7.3 and Subsection 6.3. We recall that by construction the map \( e^* \) factors via \( e_H : H^*(M) \to SH^*(H)_\eta \) as \( e^* : H^*(M) \to SH^*(H)_\eta \to SH^*(M)_\eta \) for any Hamiltonian \( H \) linear at infinity. Denote by \( 1 \mapsto e_H \mapsto e_M \) the image of the unit via \( e^* \).

The results in this section also hold in the untwisted setup (simply ignore \( \eta \)).

**Theorem 10.1.** (1) \( SH^*(M)_\eta = 0 \) if and only if the unit \( e_M = 0 \in SH^0(M)_\eta \).

(2) For a Liouville subdomain \( i : (W, \theta_W) \hookrightarrow (\overline{M}, \theta) \), if \( SH^*(M)_\eta = 0 \), then \( SH^*(W)_*, \eta = 0 \).
Proof. (1) If \( e_M = 0 \), then for all \( y \in SH^*(M)_\eta \), \( y = e_M \cdot y = 0 \cdot y = 0 \), so \( SH^*(M)_\eta = 0 \).

(2) Suppose \( SH^*(M)_\eta = 0 \), so \( e_M = 0 \). The restriction map \( \varphi : SH^*(M)_\eta \to SH^*(W)_{\eta'}_\eta \) respects the TQFT by Theorem 9.5. So it sends unit to unit. Thus, \( \epsilon_W = \varphi(\epsilon_M) = \varphi(0) = 0 \).

But by (1) applied to \( W \), \( \epsilon_W = 0 \) forces \( SH^*(W)_{\eta'}_\eta = 0 \).

\[ \text{THEOREM 10.2.} \quad \text{The following conditions are equivalent.} \]

1. \( SH^*(M)_\eta = 0 \).
2. \( \epsilon_H = 0 \) for some Hamiltonian \( H = H^\ell \) linear of slope \( \ell > 0 \) at infinity.
3. \( c^*_H : H^*(M) \to SH^*(H)_\eta \) vanishes for some \( H \).
4. \( c^* : H^*(M) \to SH^*(M)_\eta \) vanishes.
5. For some \( \ell > 0 \), the continuation \( SH^*(H^m)_\eta \to SH^*(H^{\ell+m})_\eta \) is zero for all slopes \( m \).

\[ \text{Proof.} \quad \text{By definition, } \epsilon_M = \lim \epsilon_H = 0 \iff \text{some } \epsilon_H = 0. \text{ So (1) } \Rightarrow (2) \text{ by Theorem 10.1.} \]

If \( \epsilon_H = 0 \), then \( 0 = \psi_P(\epsilon_H, y) = c^*_H y \in SH^*(H)_\eta \) for all \( y \in H^*(M) \) using the product \( \psi_P : SH^*(H)_\eta \otimes H^*(M) \to SH^*(H)_\eta \). So (2) \Rightarrow (3). For (3) \Rightarrow (2) recall that \( \epsilon_H = c^*_H(1) \).

The factorization \( c^* : H^*(M) \to SH^*(H^m)_\eta \to SH^*(H^{\ell+m})_\eta \to SH^*(M)_\eta \), for slopes \( 0 < m \leq m' \), shows that the rank of the image \( c^*_H(H^*(M)) \subset SH^*(H)_\eta \) decreases as the slope of \( H \) increases. So the rank must stabilize for large enough slopes. So (3) \Rightarrow (4).

Finally (5) \Rightarrow (3) by taking \( m \) to be very small, and (2) \Rightarrow (5) by Theorem 6.8.

\[ \text{THEOREM 10.3.} \quad \text{The following are equivalent.} \]

1. \( SH_*(M)_\eta = 0 \).
2. The counit \( \psi^C : SH_*(M)_\eta \to \mathbb{K} \) vanishes.
3. \( c^H : SH_*(H)_\eta \to H_*(M) \) vanishes for some \( H \).
4. \( c_* : SH_*(M)_\eta \to H_*(M) \) vanishes.

\[ \text{Proof.} \quad \text{This can be proved either directly as for Theorem 10.2, or by dualizing the statement of Theorem 10.2 using the dualization results of Subsection 3.8.} \]

\[ \text{THEOREM 10.4.} \quad SH^*(M)_\eta = 0 \text{ if and only if } SH_*(M)_-\eta = 0. \]

\[ \text{Proof.} \quad \text{This follows by Theorems 10.2 and 10.3 since } c^* \text{ and } c_* \text{ are dual to each other (Subsection 3.8).} \]

\[ \text{THEOREM 10.5.} \quad \text{The following are equivalent.} \]

1. \( HW^*(L)_\eta = 0 \).
2. The unit \( W_C(1) = c^*(1) \in HW_0^0(L)_\eta \) vanishes.
3. \( c^*_H(1) \in 0 \in HW^0_0(L; H^\ell)_\eta \) for some slope \( \ell > 0 \).
4. \( c^*_H : H^*(L) \to HW^*(L; H)_\eta \) vanishes for some \( H \).
5. \( c^* : H^*(L) \to HW^*(L)_\eta \) vanishes.
6. For some \( \ell > 0 \), the continuation \( HW^*(L; H^m)_\eta \to HW^*(L; H^{\ell+m})_\eta \) is zero for all \( m \).

\[ \text{Proof.} \quad \text{This follows by mimicking the proof of Theorem 10.2 in the wrapped setup.} \]
Theorem 10.6.  (1) If $SH^*(M) = 0$, then $HW^*(L) = 0$.
(2) If $SH^*(M) \eta = 0$, then $HW^*(L) \eta = 0$ (in the notation of Theorem 7.9).

Proof. This follows from the fact that $HW^*$ is a module over $SH^*$ (Theorems 6.17 and 7.9): if $e = 0$ in $SH^*$, then $y = e \cdot y = 0 \cdot y = 0$ for all $y \in HW^*$.

11. Arnol’d chord conjecture

The chord conjecture states that a contact manifold containing a Legendrian submanifold has a Reeb chord with ends on the Legendrian. This was originally stated for Legendrian knots in the standard 3-sphere by Arnol’d [5], and we recommend the work of Cieliebak [12] for a modern Floer-theoretic approach to the general statement. Our setup in Subsection 4.1 involves the Legendrian $\partial L = L \cap \partial M$ inside the contact manifold $\partial M \subset \overline{M}$.

Theorem 11.1. If $HW^*(L) = 0$, then the Arnol’d chord conjecture holds for $\partial L \subset \partial M$ (existence of a Reeb chord). For a generic contact form $\alpha$ there are at least rank $H^*(L)$ chords. The same holds if $HW^*(L) \eta = 0$ for $\eta \in H^2(M, L; \mathbb{R})$.

Proof. Suppose that there are no Reeb chords. Then we do not need to perturb $\alpha$ to avoid transversality issues caused by degenerate Reeb chords. By Lemma 4.2, there are no Hamiltonian chords on the collar of $\overline{M}$, and by the maximum principle in Appendix D.4 all wrapped trajectories lie in $M$ where we can ensure that $H^0(M) = H^0_0(M)$ for any Hamiltonian $H$ linear at infinity. So, by Section 5, $c^*: H^*(L) \cong HW^*(L; H^0) \cong HW^*(L)$, so $HW^*(L) \neq 0$.

Now suppose $HW^*(L) = 0$, $\alpha$ generic. By Lemma 8.3, if $HW^*(L) = 0$, then $HW^+_{\alpha}(L) \cong H^{*+1}(L)$, so there are at least rank $H^*(L)$ distinct Hamiltonian chords on the collar (see the Technical Remarks in Subsection 6.12), and the same holds for Reeb chords by Lemma 4.2. The twisted case is analogous, since $HW^*(L) \eta = 0$ implies $HW^+_{\alpha}(L) \eta \cong H^{*+1}(L) \otimes \Lambda$.

Remark. The proof is similar to Viterbo’s [43] applications to the Weinstein conjecture (existence of a closed Reeb orbit): if there are no closed Reeb orbits, then $c^*: H^*(M) \rightarrow SH^*(M)$ is an isomorphism. So the Weinstein conjecture holds if $SH^*(M) = 0$ or $SH^*(M) \eta = 0$.

Remark. In general, rank $H^*(L) \geq \frac{1}{2}$rank $H^*(\partial L)$ (use the long exact sequence for the pair $(L, \partial L)$, and rank $H^*(L) = rank H^*(L, \partial L)$ by Poincaré duality and universal coefficients).

Example 11.2. Let $M$ be a subcritical Stein manifold. Then Theorem 11.1 applies since $SH^*(M) = 0$ by Cieliebak [12] so $HW^*(L) = 0$ by Theorem 10.6.

Recent work of Bourgeois, Ekholm and Eliashberg [8, Remark 6.2] can tackle the chord conjecture for Legendrian spheres in the boundary of Stein domains with vanishing $SH^*$.

Let $N$ be a simply connected closed manifold, and $L \subset DT^*N$ an exact Lagrangian with transverse Legendrian intersection $\partial L = L \cap ST^*N$.

Theorem 11.3. If $H^2(T^*N) \rightarrow H^2(L)$ is not injective, then the chord conjecture holds and generically there are at least rank $H^*(L)$ Reeb chords.
Proof. The non-injectivity implies the existence of an \( \eta \in H^2(T^*N, L) \) with \( \bar{\eta} \neq 0 \in H^2(T^*N) \). Combining Theorem 10.6 and the vanishing \( SH^*(T^*N; \mathbb{Z}/2) \bar{\eta} = 0 \) (see Subsection 14.6), we deduce that \( HW^*(L; \mathbb{Z}/2) \eta = 0 \). Thus the claim follows by Theorem 11.1.

Theorem 11.3 applies, for example, for \( L \) a conormal bundle to a submanifold \( K \subset N \) such that \( H^2(N) \to H^2(K) \) has kernel (Example 4.1).

Theorem 11.3 also holds after attaching subcritical handles to \( DT^*N \), since Cieliebak [12] proved that \( SH^* \) does not change for action reasons (so this also holds for twisted \( SH^* \)).

**Theorem 11.4.** For any ALE space (see Section 1.4), the chord conjecture holds for any \( \partial L \) and generically there are at least rank \( H^*(L) \) Reeb chords.

Proof. By [32], \( SH^*(M) \bar{\eta} = 0 \) for generic \( \bar{\eta} \in H^2(M) \). Now \( L \) is a 2-manifold with boundary so \( H^2(L) = 0 \), so \( \bar{\eta} \) lifts to an \( \eta \in H^2(M, L) \). Now see the proof of Theorem 11.3.

12. Exact contact hypersurfaces

12.1. Exact contact hypersurfaces

The definition and study of exact contact hypersurfaces first appeared in [13]. We briefly recall the definition.

**Definition 12.1.** An exact contact hypersurface is an embedding \( i: (\Sigma^{2n-1}, \xi) \hookrightarrow (\overline{M}^{2n}, d\theta) \) of a closed contact manifold \( (\Sigma, \xi) \) such that for some 1-form \( \alpha \in \Omega^1(\Sigma) \), \( \xi = \ker \alpha \) and \( \alpha - i^*\theta \) is exact. Moreover, we will always assume that \( i(\Sigma) \) separates \( \overline{M} \) into a compact connected submanifold \( W \subset \overline{M} \) and an unbounded component \( \overline{M} \setminus W \).

**Remark 12.2.** The separating assumption is automatic if \( H_{2n-1}(M) = 0 \). Indeed, if \( \overline{M} \setminus \Sigma \) is connected, then there is a loop \( \gamma \) in \( \overline{M} \) cutting \( \Sigma \) once transversely, so the intersection number \( [\gamma] \cdot [\Sigma] \) is non-zero, so \( [\Sigma] \in H_{2n-1}(\overline{M}) \cong H_{2n-1}(M) \) is non-trivial, a contradiction.

When \( H^1(\Sigma) = 0 \), the condition \( \alpha - i^*\theta \) is exact is equivalent to \( d\alpha = i^*\omega \).

**Example 12.3.** Sphere bundles \( ST^*N \hookrightarrow T^*N \) and the boundaries \( ST^*L \hookrightarrow M \) of the Liouville domains in Example 9.1 (assume \( n \geq 2 \) for connectedness). The separating condition for \( \Sigma \hookrightarrow T^*N \) is automatic if \( \dim(N) = n \geq 2 \), since \( H_{2n-1}(T^*N) \cong H_{2n-1}(N) = 0 \).

**Lemma 12.4.** For the purposes of symplectic cohomology, we can always assume that an exact contact hypersurface \( \Sigma \) bounds a Liouville subdomain \( W \) of \( M \).

Proof. Assume \( i(\Sigma) \subset M \) by redefining \( M \) to be \( M \cup [1, R_0] \times \partial M \) (Theorem 9.6). Identify \( i(\Sigma) = \Sigma \). Use a bump function to extend \( \theta \) to a neighbourhood of \( \Sigma \subset \overline{M} \). Near \( \Sigma \), \( \alpha - \theta = df \) for a function \( f \) supported near \( \Sigma \). Then replace \( \theta \) by \( \theta + df: SH^*(M) \) has not changed (cf. Lemma 9.2). Thus, we can assume \( W \subset M \) with \( \alpha = \theta|_{\Sigma} \) a contact form. Define \( Z \) by \( \omega(Z, \cdot) = \theta \), so \( \omega(Z, \cdot)|_{\Sigma} = \alpha \). Thus, \( \omega^n(Z, \ldots)|_{\Sigma} = \alpha \land (d\alpha)^{n-1} \neq 0 \) pointwise (contact condition) and since the flow of \( Z \) expands volumes, \( Z \) must be pointing strictly outwards along \( \Sigma \). So \( W \subset M \) is Liouville. 

\( \square \)
12.2. **Stretching-of-the-neck argument (Bourgeois and Oancea [10, Section 5.2])**

Consider an isolated Floer trajectory in \( \overline{M} \) joining orbits lying in the collar, assuming \( J \) is of contact type on the collar. Consider what happens to this Floer trajectory as you stretch a neighbourhood of \( \partial M \) (more precisely: you insert a collar \([R_0, 1] \times \partial M\) in between \( M \) and \([1, \infty) \times \partial M\), and you rescale \( \omega, H \) on \( M \) by \(1/R_0\) so that the data glue correctly). By a compactness argument, this 1-family of Floer trajectories will converge in the limit to a Floer cylinder with punctures

\[
(\mathbb{R} \times S^1) \setminus \{\text{punctures}\} \longrightarrow \partial M \times \mathbb{R}
\]

and the map rescaled near the punctures converges to Reeb orbits, each of which is ‘capped off’ by an isolated holomorphic plane \( C \rightarrow \overline{M} \) (converging to the Reeb orbit at infinity). The proof that the only holomorphic curves capping off the Reeb orbits in \( \overline{M} \) are planes (and not holomorphic cylinders, for example) is a consequence of a subtle action argument [10, Section 5.2, Proof of Proposition 5, Step 1] which shows that the limit curve must have a connected component containing both Hamiltonian orbits (the fixed ends of the Floer trajectory). A dimension count and the fact that cylinders have genus 0 then proves that the above are the only limits.

**Remark 12.5 (Transversality).** In general, this argument would require knowing transversality for symplectic field theory [10, Remark 9, Section 3.1]. In our applications this is not necessary, as explained to us by Oancea: we will always assume that the virtual dimension \((2n - 3) - |x|\) of the moduli space of holomorphic planes \( C \rightarrow \overline{M} \) converging to a Reeb orbit \( x \) at infinity is at least 1, therefore the above punctured Floer cylinders do not have punctures. Indeed, if they had punctures, then since the virtual dimension of the planes is at least 1, the virtual dimension of the component containing the two Hamiltonian orbits (modulo the \( \mathbb{R} \) translation action) would be negative. But for that component, transversality is guaranteed by a time-dependent perturbation of the almost complex structure just like for non-punctured Floer cylinders. Thus that component does not exist, which is a contradiction.

12.3. **Independence of the filling**

**Theorem 12.6 (Bourgeois and Oancea [10], Cieliebak–Frauenfelder–Oancea [14, Theorem 1.14]).** Let \( W \) be a Liouville domain with \( c_1(W)|_{\pi_2(W)} = 0 \) such that all closed Reeb orbits \( x \) in \( \partial W \) are contractible in \( W \) have \( |x| < 2n - 3 \). Then for small \( C > 0 \), all isolated Floer trajectories in \( W \) connecting orbits in the collar lie in \( R \geq C \).

For a disc cotangent bundle \( W = DT^*L \) these conditions are satisfied if \( \dim L \geq 4 \).

**Proof.** If no such \( C \) existed, the Subsection 12.2 implies the existence of an isolated Floer cylinder capped off by at least one holomorphic plane in \( W \). But \((2n - 3) - |x| \geq 1\), so Remark 12.5 implies that the cylinder cannot have punctures, which is a contradiction.

For \( W^{2n} = DT^*L \), \( c_1(W) = 0 \). Our grading convention is \( SH^*(T^*L; \mathbb{Z}/2) \cong H_{n-*}(\mathcal{LL}; \mathbb{Z}/2) \), so \( |x| = n - \text{index}(\gamma_x) \) where \( \gamma_x \) is the closed geodesic (Reeb orbit) corresponding to \( x \). So \((2n - 3) - |x| = \text{index}(\gamma_x) + \dim L - 3 \). The claim follows since \( \text{index}(\gamma_x) \geq 0 \).

**Definition 12.7 (Boundary symplectic cohomology).** Let \((\Sigma, \alpha)\) be a contact manifold admitting a filling by a Liouville domain \( W \) with \( c_1(W)|_{\pi_2(W)} = 0 \) such that all closed Reeb orbits \( x \) in \( \Sigma \) which are contractible in \( W \) have \( |x| < 2n - 3 \). ('Filling' here means that there
is an exact contact hypersurface $\Sigma \hookrightarrow W$ mapping onto $\partial W$. For $\eta \in H^2(W)$, define

$$BSH^*(\Sigma)_{\eta|_{\Sigma}} = SH^*_+(W)_{\eta} \quad \text{(constructed in Section 8)}.$$  

**Corollary 12.8.**  
(1) $BSH^*(\Sigma)_{\eta|_{\Sigma}}$ is independent of the choice of filling $W$ in the above Definition. In particular, it only depends on the restriction of $\eta$ to $H^2(\partial W)$. 

(2) Let $i: \Sigma \hookrightarrow \overline{M}$ be an exact contact hypersurface with $c_1(M) = 0$ such that all closed Reeb orbits $x$ in $\Sigma$ have $|x| < 2n - 3$ in $\overline{M}$. Then $BSH^*(\Sigma)_{\eta|_{\Sigma}} \cong SH^*_+(W)_{\eta}$ for the Liouville subdomain $W$ bounding $i(\Sigma)$ in $\overline{M}$ (Lemma 12.4).

**Proof.** (1) We can choose the same Hamiltonian and almost complex structure on $R \geq C$ for two such fillings, so the Floer equations are identical. Twisting is determined by $\eta$ restricted to $R \geq C$, but this region is homotopy equivalent to $\partial W$.

(2) $c_1(M) = 0$, so $c_1(W) = 0$ by naturality, so gradings for $W, M$ can both be calculated with respect to the trivialization of the canonical bundle of $\overline{M}$ (Subsection 3.6). Now apply (1). □

12.4. Obstructions to exact $ST^*L \hookrightarrow T^*N$, simply connected case

**Convention:** In Subsections 12.4–12.7, ordinary (co)homology is computed over $\mathbb{Z}$ coefficients.

**Theorem 12.9.** Let $L$ and $N$ be closed simply connected $n$-manifolds, $n \geq 4$. For any exact contact hypersurface $ST^*L \hookrightarrow T^*N$, the following hold:

1. $H^2(N) \rightarrow H^2(L)$ is injective;
2. $\pi_2(L) \rightarrow \pi_2(N)$ has finite cokernel;
3. if $H^2(N) \neq 0$, then $H_*(L) \cong H_*(W)$ (where $W$ is the filling of $ST^*L \subset T^*N$).

**Technical Remark:** $SH^*(T^*N)_{\eta} \cong H_{n-1}(\Sigma L)_{\eta}$ is known to hold only for char $\mathbb{K} = 2$ (see Section 14). This does not affect (1) and (2): those arise from the integral cohomology classes which define the parallel transport maps of the local systems, so it does not matter if we take $\mathbb{K} = \mathbb{Z}/2$ in the definition of $\lambda$. However, (3) heavily involves $\lambda$ in the last lines of the proof. So, strictly speaking, our argument only proves $H_*(L; \mathbb{K}) \cong H_*(W; \mathbb{K})$ for char $\mathbb{K} = 2$. However, assuming the predictions explained at the start of Section 14, the argument holds for any $\mathbb{K}$.

**Remark 12.10.** The maps in (1), (2), (3) are defined by the composites

$$\pi_2(L) \cong \pi_2(ST^*L) \rightarrow \pi_2(W) \rightarrow \pi_2(T^*N) \cong \pi_2(N),$$

$$H^2(N) \cong H^2(T^*N) \rightarrow H^2(W) \rightarrow H^2(ST^*L) \cong H^2(L).$$

Using dim($L$) $\geq 4$, we identify $\pi_1(L), \pi_2(L), H_1(L), H^2(L)$ with the analogous groups for $\partial W = ST^*L$ via the long exact sequence in homotopy for $S^{n-1} \rightarrow ST^*L \rightarrow L$ and the Gysin sequence $H^*(ST^*L) \rightarrow H^{*-0}(L) \rightarrow H^{*-1}(L) \rightarrow H^{*+1}(ST^*L)$.

**Proof of Theorem 12.9.** Suppose by contradiction that $\eta \neq 0 \in H^2(N)$ vanishes in $H^2(L) \cong H^2(\partial W)$. By Corollary 14.5, $SH^*(T^*N)_{\eta} = 0$. By Theorem 10.1, $SH^*(W)_{\eta|W} = 0$. By Lemma 8.1, $SH^*_+(W)_{\eta|W} \cong H^{*+1}(W) \otimes \Lambda$ has finite rank. By Corollary 12.8, $SH^*_+(W)_{\eta|W} = \ldots$
$\text{BSH}^*(ST^*L)_{\eta|\partial W}$. Using $\eta|\partial W = 0 \in H^2(\partial W)$ and Corollary 8.2,

$$\text{BSH}^*(ST^*L)_{\eta|\partial W} \cong \text{BSH}^*(ST^*L) \otimes \Lambda \cong H_{n-*}(L, L) \otimes \Lambda.$$ This has infinite rank as $H_*(LL)$ does $[42]$, using $L$ is closed with $\pi_1(L) = 1$. So $\text{SH}^*(W)_{\eta|W}$ has both finite and infinite rank. This is a contradiction, proving (1).

We have in fact just proved more, since the isomorphism class of the local system used to twist $\text{SH}^*$ actually depends on the transgressed classes $\tau\eta$ and $\tau(\eta|\partial W)$ (recall Subsection 7.2). So we actually proved that if $\tau\eta \neq 0$ in $H^1(L\mathcal{L})$, then $\tau(\eta|\partial W) \neq 0$ in $H^1(L\partial W) \cong H^1(LL)$.

In general, for simply connected manifolds $X$, the transgression $\tau : H^2(X) \rightarrow H^1(LX)$ is an isomorphism and one can identify the groups $H^1(LX) \cong \text{Hom}(H_1(LX), \mathbb{Z}) \cong \text{Hom}(\pi_2(X), \mathbb{Z})$. So, using this general fact for $X = N$ and $X = L$, the above actually proved that

$$\text{Hom}(\pi_2(N), \mathbb{Z}) \rightarrow \text{Hom}(\pi_2(L), \mathbb{Z}), \quad \tau\eta \rightarrow \tau(\eta|\partial W)$$

must be injective. Dualizing this statement yields (2).

If $H^2(N) \neq 0$ then pick an $\eta \neq 0 \in H^2(N)$. By Corollary 14.5, $\text{SH}^*(T^*N)_{\eta} = 0$ so $\text{SH}^*(W)_{\eta|W} = 0$. By (1) it restricts to $\eta|\partial W \neq 0 \in H^2(ST^*L) \cong H^2(T^*L)$. By Corollary 14.5, $\text{SH}^*(T^*L)_{\eta|\partial W} = 0$. Using Lemma 8.1 and Corollary 12.8,

$$H^{*+1}(W) \otimes \Lambda \cong \text{SH}^*_+(W)_{\eta|W} \cong \text{BSH}^*(ST^*L)_{\eta|\partial W} \cong \text{SH}^*_+(DT^*L)_{\eta|\partial W} \cong H^{*+1}(L) \otimes \Lambda.$$ 

Passing to homology by universal coefficients, proves (3) over $\mathbb{K}$ coefficients. In fact it also holds over $\mathbb{Z}$ coefficients by Subsection 6.6 (defining $\Lambda$ over $\mathbb{Z}$ instead of over $\mathbb{K}$). 

12.5. Obstructions to exact $ST^*L \rightarrow T^*N$, general case

The key to Theorem 12.9 was that $H_*(LL)$ had infinite rank. This holds for closed $L$ with $\pi_1(L) = 1$, since Sullivan $[42]$ showed $H_*(LL; \mathbb{Q})$ is infinite dimensional via rational homotopy theory. Sullivan’s proof also works for nilpotent spaces. We do not know when the result holds for finite $\pi_1(L)$ (we remark that $L$ need not be nilpotent: $\mathbb{R}P^2$ has $\pi_1 = \mathbb{Z}/2$ but does not act nilpotently on $\pi_2 = \mathbb{Z}$). When $\pi_1(L)$ has infinitely many conjugacy classes (for example infinite abelian $\pi_1(L)$) the result holds because the connected components of $LL$ are indexed by these conjugacy classes, so rank $H_0(LL) = \infty$.

Convention: In Subsections 12.4–12.7, ordinary (co)homology is computed over $\mathbb{Z}$ coefficients.

Notation: Denote $L_0L \subset LL$ is the subspace of contractible loops in $L$.

**Theorem 12.11.** Let $L, N$ be closed $n$-manifolds, $n \geq 4$, with $N$ of finite type (Subsection 14.6). Let $ST^*L \rightarrow T^*N$ be an exact contact hypersurface. If rank $H_*(L_0L) = \infty$ then:

1. $\pi_2(L) \rightarrow \pi_2(N)$ has finite cokernel.
2. if $N$ is simply connected, $H^2(N) \rightarrow H^2(L)$ is injective;
3. $H^2(\tilde{N}) \rightarrow H^2(\tilde{L})$ and $H^2(N) \setminus \ker \pi^* \rightarrow H^2(L)$ are injective; (Here $\pi : \tilde{N} \rightarrow N$ is the universal cover, inducing $\pi^* : H^2(N) \rightarrow H^2(N)$.)
4. if $L$ has finite type and rank $(\pi_2(N)) \neq 0$, then (see the Technical Remark below Theorem 12.9.) $H_*(L) \cong H_*(W)$ where $W$ is the filling of $ST^*L \subset T^*N$.

If rank $H_*(LL) = \infty$ then: (1), (2), (3) hold after replacing $\pi_2(L)$, $H^2(\tilde{L})$ by $\pi_2(W)$, $H^2(W)$ respectively. Also if any one of (1)–(4) failed, then Corollary 12.13(1)–(4) all hold and $\ker \pi^* = H^2(N)$ (so the image of the Hurewicz homomorphism $\pi_2(M) \rightarrow H_2(M)$ is torsion).
Remark 12.12. The distinction (2), (3) is because we need \( \tau \eta \neq 0 \in H^1(\mathcal{L}_0, N) \) in Corollary 14.5: \( \tau : H^2(N) \to H^1(\mathcal{L}_0, N) \) is an isomorphism if \( \pi_1(N) = 1 \), but otherwise has kernel \( \ker \pi^* \). We identify \( H^2(N) \cong \text{Hom}(\pi_2(N), \mathbb{Z}) \) with \( \text{im}(\pi) \subset H^1(\mathcal{L}_0, N) \) (Subsection 7.6). The second part of (3) and (2) can be proved as in Theorem 12.9 using \( \text{rank} H_\ast(\mathcal{L}L) = \infty \). Note: (1) implies (2), (3). The first part of (3) is dual to (1), the second part follows by the diagram

\[
H^2(N) \xrightarrow{\tau} \text{Hom}(\pi_2(N), \mathbb{Z}) \subset H^1(\mathcal{L}_0, N) \\
\downarrow \quad \downarrow \\
H^2(L) \xrightarrow{\tau} \text{Hom}(\pi_2(L), \mathbb{Z}) \subset H^1(\mathcal{L}_0, L).
\]

Proof. To prove (1) we twist by 2-forms \( \tilde{\eta} \neq 0 \in H^2(\tilde{N}) \) on the universal cover of \( N \) as explained in Subsection 7.6. By transgression, these give rise to 1-forms on the space \( \mathcal{L}_0, N \) of contractible loops in \( N \). By Theorem 7.5 there is a TQFT structure on the subgroup \( SH_0^\ast(M)_{\tilde{\eta}} \) of \( SH^\ast(M)_{\tilde{\eta}} \) generated by the contractible orbits.

Pass to universal covers and use transgressions (Subsection 7.6) to identify the two rows:

\[
\begin{array}{ccc}
H^2(N) & \xrightarrow{\tau} & H^2(W) \\
\| & & \| \\
\text{Hom}(\pi_2 N, \mathbb{Z}) & \xrightarrow{\tau} & \text{Hom}(\pi_2 W, \mathbb{Z}) \\
& & \| \\
& & \text{Hom}(\pi_2 L, \mathbb{Z}).
\end{array}
\]

Case 1: rank \( H_\ast(\mathcal{L}_0, L) = \infty \). Suppose by contradiction that \( \tilde{\eta} \) vanishes under this composition: \( \tilde{\eta}|_L = 0 \in H^2(\tilde{L}) \). By [31], \( SH_0^\ast(T^\ast N)_{\tilde{\eta}} \cong H_{n-s}(\mathcal{L}_0, N)_{\tilde{\eta}} = 0 \) since \( N \) has finite type. As before, \( SH_0^\ast(W)_{\tilde{\eta}} = 0 \), so \( SH^\ast_{+, 0}(W)_{\tilde{\eta}} = H^{s+1}(W) \otimes \Lambda \) has finite rank. But, by independence of the filling, this group is isomorphic to

\[
SH^\ast_{+, 0}(DT^\ast L)_{\tilde{\eta}|L} = SH^\ast_{+, 0}(DT^\ast L)_0 \cong H_{n-s}(\mathcal{L}_0, L) \otimes \Lambda,
\]

which has infinite rank, a contradiction. So the composition is injective, so dualizing gives (1). By Remark 12.12 it remains to prove (4). If rank \( \pi_2(N) \neq 0 \), pick \( \tau \tilde{\eta} \neq 0 \). By (2), also \( \tau \eta|_L \neq 0 \), so since \( L \) has finite type, \( SH_0^\ast(T^\ast L)_{\tilde{\eta}|L} = 0 \). Thus

\[
H^{s+1}(W) \otimes \Lambda \cong SH^\ast_{+, 0}(W)_{\tilde{\eta}} \cong BSH_0^\ast(ST^\ast L)_{\tilde{\eta}} \cong SH^\ast_{+, 0}(DT^\ast L)_{\tilde{\eta}} \cong H^{s+1}(L) \otimes \Lambda.
\]

Passing to homology by universal coefficients and using Subsection 6.6, proves (4).

Case 2: rank \( H_\ast(\mathcal{L}L) = \infty \). Assume by contradiction that \( \tilde{\eta}|_W = 0 \in H^2(W) \). As before, \( SH_0^\ast(W)_{\tilde{\eta}} = 0 \). Since \( \tilde{\eta}|_W = 0 \), this implies \( SH_0^\ast(W) = 0 \). By Theorem 10.1, \( SH^\ast(W) = 0 \) since the unit lies in \( SH_0^\ast(W) = 0 \). By Lemma 8.1, \( SH^\ast_+(W) \cong H^{s+1}(W) \) has finite rank. But by Corollaries 12.8 and 8.2,

\[
SH^\ast_+(W) \cong BSH^\ast(ST^\ast L) \cong H_{n-s}(\mathcal{L}L, L),
\]

which has infinite rank since \( H_\ast(\mathcal{L}L) \) does. This is a contradiction. So \( H^2(\tilde{N}) \to H^2(\tilde{W}) \) is injective and dually \( \pi_2(W) \to \pi_2(N) \) has finite cokernel.

12.6. Pathological \( L \)

Write \([S^1, L]\) for the set of free homotopy classes of maps \( S^1 \to L \), equivalently: it is the set of conjugacy classes of \( \pi_1(L) \).

Call \( L \) pathological if \([S^1, L] \neq 1 \) is a finite set and \( H_\ast(\mathcal{L}L) \) has finite rank. This is the only case when Theorem 12.11 may not apply.

Corollary 12.13. Let \( L, N \) be closed \( n \)-manifolds, \( n \geq 4 \), with \( N \) of finite type and \( L \) pathological. Let \( ST^\ast L \hookrightarrow T^\ast N \) be an exact contact hypersurface. Then either the results of
Theorem 12.11 all hold, or else the following must hold:

1. $H_1(L) \to H_1(W)$ and $[S^1, L] \to [S^1, W]$ both vanish;
2. rank $H^{n+1}(W) = \# [S^1, L] - 1 \geq 1$;
3. if $L$ is of finite type then $H^2(\bar{N}) \to H^2(\bar{L})$, $H^2(W) \setminus \ker \pi^* \to H^2(L)$ vanish;
4. if $L$ is of finite type then the image of $\pi_2(L) \to \pi_2(W)$ is torsion.

In particular, if $W$ is Stein then the results of Theorem 12.11 hold.

Proof. For the last claim: $H^{n+1}(W) \neq 0$ by (2), but $H^{n+1}(W) = 0$ for Stein $W^{2n}$.

The connected components $\mathcal{L}_c \cup \mathcal{L}_L$ are indexed by $c \in [S^1, L]$. Filter $SH^*(DT^*L)$ by $[S^1, L]$ as in Subsection 6.3, and consider $i : [S^1, L] \cong [S^1, ST^*L] \to [S^1, W]$.

Suppose Theorem 12.11 fails. By Remark 12.12 some $\tilde{\eta} \neq 0 \in H^2(\bar{N})$ has $\tilde{\eta}|_L = 0 \in H^2(\bar{L})$.

By the proof of Theorem 12.11 $SH^*(W) = 0$ and, using Lemma 8.1,

$$H_0(\mathcal{L}_L, L) \cong SH^+_n(\mathcal{L}_c, L) \cong SH^+_n(W) \cong \bigoplus_{\eta \in [S^1, W]} SH^+_n, w(\eta) \cong H^{n+1}(W),$$

$$H_0(\mathcal{L}_c, L) \cong \bigoplus_{\eta \in [S^1, W]} H_0(\mathcal{L}_c, L) \cong \bigoplus_{\eta \in [S^1, W]} \bigoplus_{\eta \in [S^1, W]} SH^+_{n,c}(\mathcal{L}_c, L) \cong \bigoplus_{\eta \in [S^1, W]} \bigoplus_{\eta \in [S^1, W]} SH^+_{n,c}(\mathcal{L}_c, L).$$

Now $H_0(\mathcal{L}_c, L) = \mathbb{Z}$, so the two lines imply rank $H^{n+1}(W) = \# [S^1, L] - 1$. Also, $SH^+_n, w(\eta) \cong SH^+_n(W) = 0$ for $\eta \neq 0 \in [S^1, W]$. So, since the second line respects summands, $\eta$ must vanish (and similarly if we filter by $H_1$, Subsection 6.5), proving (1), (2).

Suppose $\tilde{\eta} \in H^2(\bar{N})$ has $\tilde{\eta}|_L \neq 0 \in H^2(\bar{L})$. Then $SH^+_n(T^*N)\tilde{\eta} = 0$, $SH^+_n(W)\tilde{\eta} = 0$,

$$H^+ + 1(W) \otimes \mathcal{L} \cong SH^+_{+,0}(W)\tilde{\eta} \cong BSH^*_0(ST^*L)\tilde{\eta} \cong SH^+_{+,0}(DT^*L)\tilde{\eta} \cong H_{n-\varepsilon}(\mathcal{L}_0, L)\tilde{\eta}.$$ But for finite type $L$, $H_{n-s}(\mathcal{L}_0, L)\tilde{\eta} = 0$ since $\tilde{\eta} \neq 0 \in H^2(\bar{L})$. Thus, $H_{n-\varepsilon}(\mathcal{L}_0, L)\tilde{\eta} \cong H_{n-\varepsilon}(L) \cong 0$ by Corollary 8.2. So $H^{n+1}(W) \cong H_{-1}(L) = 0$ contradicts (2). Thus $H^2(\bar{N}) \to H^2(\bar{L})$ vanishes, proving the first part of (3). Taking transgressions and dualizing implies (4), which implies the second part of (3) (Remark 12.12).

12.7. Obstructions to exact contact embeddings

The only input about $\overline{\mathcal{M}} = T^*N$ that we used in the proofs of Theorem 12.11 and Corollary 12.13 was the vanishing theorem $SH^*(M)\tilde{\eta} = 0$ from Corollary 14.5. Thus, the same proofs show more generally the following:

**Theorem 12.14.** Let $M^{2n}$ be a Liouville domain with $c_1(M) = 0$, $n \geq 4$. Let $L^n$ be a non-pathological closed manifold admitting an exact contact embedding $i : ST^*L \to \overline{\mathcal{M}}$. Then

1. if $SH^*(M)\tilde{\eta} = 0$ then $i^*\tilde{\eta} \neq 0 \in H^2(L)$ and $\tau(i^*\tilde{\eta}) \neq 0 \in H^1(L_0)$;
2. if $SH^*(M)\tilde{\eta} = 0$ then $i^*\tilde{\eta} \neq 0 \in H^2(L)$;
3. if $L$ is of finite type and $SH^*(M)\tilde{\eta} = 0$, then (see the Technical Remark below Theorem 12.9) $H_s(L) \cong H_s(L)$ where $W$ is the filling of $ST^*L \subset \overline{\mathcal{M}}$.

For pathological $L$: if (1) or (2) fails, then Corollary 12.13(1,2) hold and $W$ is not Stein.

For finite type pathological $L$: $SH^*(M)\tilde{\eta} = 0$ implies $\tau(i^*\tilde{\eta}) = 0$, $SH^*(M)\tilde{\eta} = 0$ implies $i^*\tilde{\eta} = 0$.

**Corollary 12.15.** In the setup of Theorem 12.14, if $SH^*(M) = 0$ or $SH^*(W) = 0$, then:

(i) $L$ must be pathological and $W$ is not Stein;
(ii) $H_1(L) \to H_1(W)$ and $[S^1, L] \to [S^1, W]$ both vanish;
(iii) rank $H^{n+1}(W) = \# [S^1, L] - 1 \geq 1$. 

13. Displaceability of \( \partial M \) and Rabinowitz Floer theory

13.1. Rabinowitz Floer cohomology

The Lagrange multiplier analogue of \( SH^* \) is Rabinowitz Floer cohomology, due to Cieliebak and Frauenfelder [13]. It involves changing the action \( \lambda_H \) by rescaling \( H \) by a Lagrange multiplier variable \( \lambda \in \mathbb{R} \):

\[
\mathbb{L}_H : \mathcal{LM} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathbb{L}_H(x, \lambda) = -\int_0^1 x^*\theta + \lambda \int_0^1 H(x(t)) \, dt.
\]

This tweak has a considerable effect: it forces all critical points of the new action functional \( \mathbb{L}_H \) to lie in \( H^{-1}(0) \). So take \( H : \mathcal{M} \rightarrow \mathbb{R} \) such that \( H^{-1}(0) = \partial M \), with \( H = h(R) \) near \( \partial M = \{ R = 1 \} \) with \( h'(1) = 1 \). Then the critical points of \( \mathbb{L}_H \) are the solutions \( x : S^1 \rightarrow \partial M \) of \( \dot{x} = \lambda \mathcal{L}(x) \) with action \( \mathbb{L}_H = -\lambda \). So, after reparametrizing \( y(t) = x(t/|t|) \), these critical points are precisely: the Reeb orbits of period \( \lambda > 0 \), the constant loops in \( \partial M \) (\( \lambda = 0 \)), and the negative Reeb orbits of period \( -\lambda > 0 \). The chain complex generated by the critical points of \( \mathbb{L}_H \) has a differential defined by counting negative gradient flow lines of \( \mathbb{L}_H \). The resulting cohomology is called Rabinowitz Floer cohomology \( RFH^*(M) \) (see [13] for details).

**Theorem 13.1** (Cieliebak, Frauenfelder and Oancea [14]). There is a long exact sequence

\[
\ldots \longrightarrow SH_{2n-\ast}(M) \longrightarrow SH^*(M) \longrightarrow RFH^*(M) \longrightarrow SH_{2n-\ast-1}(M) \longrightarrow \ldots
\]

where the map \( SH_{2n-\ast}(M) \rightarrow SH^*(M) \) is the composite

\[
SH_{2n-\ast}(M) \xrightarrow{c_*} H_{2n-\ast}(M) \xrightarrow{\text{Poincaré}} H^*(M, \partial M) \xrightarrow{\text{inclusion}^*} H^*(M) \xrightarrow{c^*} SH^*(M).
\]

**Remark 13.2.** The Theorem is written in our conventions. Comparison: conventions in [14] dictate \( SH_*(T^*N; \mathbb{Z}/2) \cong H_*(\mathcal{L}N; \mathbb{Z}/2) \) instead of our \( SH^{n-\ast}(T^*N; \mathbb{Z}/2) \cong H_*(\mathcal{L}N; \mathbb{Z}/2) \).

13.2. Vanishing theorem

**Theorem 13.3.** \( RFH^*(M) = 0 \) iff \( SH^*(M) = 0 \) iff \( SH_*(M) = 0 \). \( RFH^*(M) = 0 \) if and only if \( SH^*(M) = 0 \) if and only if \( SH_*(M) = 0 \).

If one among \( SH_0, SH^0 \) and \( RFH^0 \) is 0 (even just on the component \( \mathcal{L}_0 \mathcal{M} \) of contractible loops, Subsection 6.5), then all \( SH_*, SH^* \) and \( RFH^* \) are 0. Here we used \( \mathbb{Z} \)-gradings if \( c_1 = 0 \) (or \( c_1|_{\tau_2 M} = 0 \) if we restrict to \( \mathcal{L}_0 \mathcal{M} \)), otherwise use \( \mathbb{Z}/2\mathbb{Z} \)-gradings (see Subsection 3.6).

**Proof.** By Theorem 10.4, \( SH^*(M) = 0 \) if and only if \( SH_*(M) = 0 \) if and only if one of them vanishes in degree 0 (where the unit/ counit lie). By Theorem 13.1, if \( SH^*(M) = SH_*(M) = 0 \), then \( RFH^*(M) = 0 \). Conversely, suppose \( RFH^0(M) = 0 \) (even restricted to contractible loops, since the long exact sequence of Theorem 13.1 respects the filtrations of Subsection 6.5). By Theorems 10.1 and 10.4, it suffices to show that the unit \( e \in SH^0(M) \) vanishes.

Case 1. Suppose that we have a \( \mathbb{Z} \)-grading. Theorem 13.1 yields the exact sequence \( SH_{2n}(M) \rightarrow SH^0(M) \rightarrow RFH^0(M) \), whose first map is the composite

\[
SH_{2n}(M) \longrightarrow H_{2n}(M) \longrightarrow H^0(M, \partial M) \longrightarrow H^0(M) \longrightarrow SH^0(M).
\]
Now $M$ is a $2n$-manifold with non-empty boundary so $H_{2n}(M) = 0$. Thus the above composite is zero, so in the above exact sequence, $SH^0(M) \hookrightarrow \text{RFH}^0(M)$ is injective. Thus $\text{RFH}^0(M) = 0$ implies $SH^0(M) = 0$ so $e = 0$.

Case 2. Suppose that we have a $\mathbb{Z}/2\mathbb{Z}$-grading. By Theorem 13.1, $e$ is in the image of $\text{SH}^{\text{even}}(M) \to \text{SH}^{\text{even}}(M)$ since it maps to zero via $\text{SH}^{\text{even}}(M) \to \text{RFH}^{\text{even}}(M) = 0$. The $H^0(M)$ summands of the image of $\text{SH}^{\text{even}}(M) \to H^{\text{even}}(M) \to H^{\text{even}}(M, \partial M) \to H^{\text{even}}(M)$ must vanish since they factor through $H_{2n}(M) = 0$. So since $e$ is in the image of $\text{SH}^{\text{even}}(M) \to \text{SH}^{\text{even}}(M)$, $e = c^*(z)$ for some $z \in H^*(M)$ with no $H^0(M)$-summand. But $c^*$ is a TQFT map by Theorem 6.6 and $z^{n+1} = 0$ for degree reasons, so $0 = c^*(z^n) = c^*(z)^{n+1} = e^{n+1} = e$.

### 13.3. Displaceability of $\partial M$ implies vanishing of $\text{SH}^*(M)$

Call an exact symplectic manifold $(X^{2n}, d\theta)$ convex, if it is connected without boundary; it admits an exhaustion $X = \bigcup_k X_k$ by compact sets $X_k \subset X_{k+1}$ for which the Liouville vector field $Z$ defined by $i_Z d\theta = \theta$ points strictly outwards along $\partial X_k$; $Z$ is complete, meaning that its flow is defined for all time; and $Z \not= 0$ outside of some $X_k$. For example, $X = \overline{M}$ for a Liouville domain $M$.

**Theorem 13.4.** If $M$ admits an exact embedding into an exact convex symplectic manifold $X$, such that $M$ is displaceable by a compactly supported Hamiltonian flow inside $X$, then $\text{SH}^*(M) = 0$, there are no closed exact Lagrangians in $M$, and the Arnol’d chord conjecture holds for any Legendrian arising as in Subsection 4.1.

**Proof.** We work over $\mathbb{K} = \mathbb{Z}/2$ throughout the proof. If $\partial M \subset X$ is displaceable, then $\text{RFH}^*(\partial M \subset X) = 0$ by [13]. But $\text{RFH}^*$ only depends on the filling $M$ of $\partial M$, so $\text{RFH}^*(M) = 0$. By Theorem 13.3, $\text{SH}^*(M) = 0$.

If $L \subset M$ was a closed exact Lagrangian, then for $W \cong DT^*L$ as in Example 9.1, the restriction $0 = \text{SH}^*(M) \to \text{SH}^*(W) \cong H_{n-s}(\mathcal{L}L)$ implies $H_s(\mathcal{L}L) = 0$, by Theorem 10.1, which is absurd since $H_0(\mathcal{L}L) \neq 0$. For the last claim use Theorems 10.6 and 11.1.

**Example.** If $ST^*L \hookrightarrow X$ is an exact contact hypersurface such that $ST^*L$ is displaceable, with $c_1(X) = 0$ and $n \geq 4$, then $\text{SH}^*(W) = 0$ so Corollary 12.15 holds. So for non-pathological $L$, $ST^*L$ is never displaceable (this mildly strengthens [14, Theorem 1.17]).

### 14. String topology

For closed $n$-manifolds $N$, Viterbo [43] proved that $\text{SH}^*(T^*N; \mathbb{Z}/2) \cong H_{n-s}(\mathcal{L}N; \mathbb{Z}/2)$, and there are now alternative proofs by Abbondandolo and Schwarz [1] and Salamon and Weber [36]. We use the approach of Abbondandolo and Schwarz [1] as we used it in [31] to prove $\text{SH}^*(T^*N; \mathbb{Z}/2)_\eta \cong H_{n-s}(\mathcal{L}N; \mathbb{Z}/2)_\eta$. Our goal is to show that these isomorphisms respect the TQFT.

It was initially believed that the isomorphism $\text{SH}^*(T^*N) \cong H_{n-s}(\mathcal{L}N)$ should hold as written for any choice of coefficients when $N^n$ is an orientable manifold. However, recent observations by Seidel [41] show that in fact this is not quite correct due to an ambiguity in the choice of orientation signs for the relevant moduli spaces. The prediction, according to Seidel [41], is that the isomorphism holds as written over any coefficients if the orientable manifold $N$ is spin (meaning $w_2(TN) = 0$). For a general closed manifold $N$, the prediction [41] is

$$\text{SH}^*(T^*N)_\eta \cong H_{n-s}(\mathcal{L}N; \mathbb{Z}e_i^*(w_1(TN))) \bigotimes \mathbb{Z}_{e_i^*(w_1(TN))} \bigotimes \mathbb{Z}_{\tau_7},$$
where we now explain the local system of coefficients used to compute the homology of $\mathcal{L}N$.

1. $\tau_2 : H^2(N;\mathbb{Z}/2) \to H^1(\mathcal{L}N;\mathbb{Z}/2)$ is the transgression map over $\mathbb{Z}/2$ (see Subsection 7.2).
2. The map $ev : \mathcal{L}N \to N$ is the evaluation $x \mapsto x(0)$.
3. $\mathbb{Z}_\alpha$ for $\alpha \in H^1(\mathcal{L}N;\mathbb{Z}/2)$ is a local system defined analogously to the local system $\Delta_\alpha$ of Subsection 7.1: the fibre $\mathbb{Z}_x$ over $x \in \mathcal{L}N$ is $\mathbb{Z}$, and the parallel transport map over a path $u$ in $\mathcal{L}N$ connecting $x$ to $y$ is the multiplication isomorphism $\pm 1 \ni \alpha[u] : \mathbb{Z}_y \to \mathbb{Z}_x$.
4. $w_1(TN) \in H^1(N;\mathbb{Z}/2)$ and $w_2(TN) \in H^2(N;\mathbb{Z}/2)$ are the Stiefel–Whitney classes of $N$, so, in particular, $\mathbb{Z}_{ev^*}(w_1(TN))$ is the pull-back via $ev$ of the orientation sheaf of $N$.

To avoid these complications, we assume throughout this section that the field $\mathbb{K}$ has characteristic 2, so that orientation signs are irrelevant and we assume that only $N$ is a closed manifold. We write $H_*(\mathcal{L}N)_\eta$ to keep track of twistings by $\eta \in H^2(N)$ when present (in which case it means $H_*(\mathcal{L}N;\Delta_\tau\eta)$) using coefficients in the local system defined in Subsection 7.1).

14.1. The Abbondandolo–Schwarz construction [1] We first identify $H_*(\mathcal{L}N)_\eta$ with the Morse homology $MH_*(\mathcal{L}N)_\eta$ of the Lagrangian action functional

$$\mathcal{E}(\gamma) = \int_0^1 L(\gamma, \dot{\gamma}) \, dt,$$

for a suitable function $L : TN \to \mathbb{R}$ which is quadratic in the fibres, for example, the Lagrangian functions $L(q, v) = \frac{1}{2}|v|^2 - U(q)$ from classical mechanics. One actually replaces $\mathcal{L}N = C^\infty(S^1, N)$ with the homotopy equivalent space $W^{1,2}(S^1, N)$, and one proves that Morse homology for $\mathcal{E}$ on this space is well-defined for most $L$ (the quadratic growth of $L$ ensures that $\mathcal{E}$ is bounded below and that it has finite-dimensional unstable manifolds).

The choice of $L$ determines the choice of a quadratic Hamiltonian on $T^*N$,

$$H(q, p) = \max_{v \in T_qN} (p \cdot v - L(q, v)).$$

For example: $L = \frac{1}{2}|v|^2$ gives $H = \frac{1}{2}|p|^2$. The convexity of $L$ implies that the maximum is achieved at precisely one point $p$, which equals the vertical differential of $L$ at $(q, v)$. This defines a fibre-preserving diffeomorphism, called Legendre transform:

$$\mathcal{L} : TN \to T^*N, \quad (q, v) \mapsto (q, dL(q, v)|_{T^*_qN} \equiv T_q(N)).$$

This relates the dynamics of $H$ on $T^*N$ with that of $\mathcal{E}$ on $\mathcal{L}N$: $\gamma \in \text{Crit}(\mathcal{E})$ is a critical point of $\mathcal{E}$ in $\mathcal{L}N$ if and only if $x = \mathcal{L}(\gamma, \dot{\gamma}) \in \text{Crit}(\mathcal{A}_H)$ is a 1-periodic orbit of $X_H$ in $T^*N$. One can even relate the indices and the action values of $\mathcal{E}$ and $\mathcal{A}_H$.

The isomorphism $H_*(\mathcal{L}N)_\eta \to SH^{n-*}(T^*N)_\eta$ is induced by the chain isomorphism

$$\varphi : MC_*(\mathcal{E})_\eta \to SC^{n-*}(H)_\eta, \quad \varphi(\gamma) = \sum_{v \neq u \in \mathcal{M}^+_\eta(\gamma,x)} \epsilon_{v \neq u} t^{-\tau u[v]+\tau u[\tau u]} x$$

where $\epsilon_{v \neq u} \in \{\pm 1\}$ are orientation signs and where $\mathcal{M}^+_\eta(\gamma,x)$ consists of pairs $v \neq u$, with $v : (-\infty,0] \to \mathcal{L}N$ a $-\mathcal{E}$ flow line converging to $\gamma \in \text{Crit}(\mathcal{E})$ as $s \to -\infty$, and $u : (-\infty,0] \times S^1 \to T^*N$ solves Floer’s equation and converges to $x \in \text{Crit}(\mathcal{A}_H)$ as $s \to -\infty$, with Lagrangian boundary condition $\pi \circ u(0,t) = v(0,t)$, where $\pi : T^*N \to N$ is the projection.

As usual, one actually makes a $C^2$-small time-dependent perturbation of $L$ (and hence also of $H$), but we keep the notation simple. Appendix C explains why we can use a quadratic
Hamiltonian instead of taking a direct limit over linear Hamiltonians, and in Appendix C.4 we explain why the use of non-contact type $J$ in Abbondandolo–Schwarz is also not problematic.

Finally, $MH_* (\mathcal{L} N; \mathcal{E})_\eta \cong H_* (\mathcal{L} N)_\eta$ is essentially obtained by mapping a critical point $\gamma$ of $\mathcal{E}$ to the pseudo-cycle given by the unstable manifold $W^u (\gamma; \mathcal{E})$.

**Remark 14.1.** In our conventions [31, Section 3], any $w : [0, 1] \to T^* N$ has $\mathcal{E} (\pi w) \geq -\hbar_H (w)$. Since $\mathcal{E}$ decreases along $v$ and $-\hbar_H$ decreases along $s \mapsto u(\cdot - s, \cdot)$, the isomorphism $\varphi$ respects action–filtrations: $\varphi_{\leq c} : MH_* (\mathcal{E} < c) \cong SH^{n-\ast} (\hbar_H > -c)$ for all $c \in \mathbb{R}$ (see Section 8).

For small $c > 0$, this filtered isomorphism becomes $MH_* (N; L) \cong MH^{n-\ast} (T^* N; H)$, which is the Poincaré duality $H_* (N) \cong H^n - \ast (T^* N) \cong H^{n-\ast} (N)$. So, via $\varphi$, the map $c^* : H^* (T^* N) \to SH^* (T^* N)_\eta$ becomes the inclusion of constant loops $c_* : H_* (N) \to H_* (\mathcal{L} N)_\eta$. In particular, $\varphi^{-1} (e) = c_* [N]$ since $[N]$ is the Poincaré dual of 1 and $e = c^* (1)$ (Theorem 6.4).

14.2. TQFT structure on $H_* (\mathcal{L} N)$

Given a graph $S'$ as in Subsection 6.8, use the Morse function $\mathcal{E}_i (\gamma) = \int_0^1 L_i (\gamma, \dot{\gamma}) \, dt$ for edge $e_i$ for a generic $L = L_i$ as above. This yields

$$\psi_{S'} : MH_* (\mathcal{L} N)^{\otimes p} \longrightarrow MH_* (\mathcal{L} N)^{\otimes q} \quad (p \geq 1, q \geq 0).$$

Note that we use homological conventions (see Subsection 6.1), so the operation goes ‘from left to right’ in Figure 6. For example, $\psi_Q : MH_* (\mathcal{L} N)^{\otimes 2} \to MH_* (\mathcal{L} N)_\eta$ is a product.

**Remark.** $p \geq 1$ is needed since only the unstable manifolds of the $\mathcal{E}_i$ are finite-dimensional, the stable ones are infinite-dimensional. In particular, we cannot construct a unit via Morse theory since $\mathcal{E}$ does not have a maximum (but one can construct a counit, compare Subsection 6.8). Nevertheless, there is a unit $c_* [N]$ (see Remark 14.1 and the remarks in Subsections 14.3 and 14.4).

Identifying $MH_* (\mathcal{L} N)_\eta \cong H_* (\mathcal{L} N)_\eta$, these Morse operations define a TQFT on $H_* (\mathcal{L} N)_\eta$. A string topology focused description of these can be found in [15].

14.3. The Chas–Sullivan loop product

Let $ev : \mathcal{L} N \to N$ be evaluation at 0. Let $\sigma : \Delta^a \to \mathcal{L} N$ and $\tau : \Delta^b \to \mathcal{L} N$ be two singular chains, thought of as $a$- and $b$-dimensional families of loops. When two loops from those two families happen to have the same base point, we can form a ‘figure-8’ loop. Let $E = \{(s, t) \in \Delta^a \times \Delta^b : ev (\sigma (s)) = ev (\tau (t))\}$ parametrize those match-ups and let $j_E : E \to \mathcal{L} N$ output those figure-8 loops. Now $E = (ev \times ev)^{-1} (\text{diagonal})$ via $ev \times ev : \Delta^a \times \Delta^b \to N \times N$. So $E$ is a manifold when the two families of base points, $ev (\sigma)$ and $ev (\tau)$, intersect transversely in $N$, and then $\dim (E) = a + b - n$. The loop product $\sigma \cdot \tau$ in [11] is then defined to be the cycle $(j_E)_* [E] \in H_{a+b-n} (\mathcal{L} N)$ (see [2] for details).

**Theorem 14.2 (Abbondandolo and Schwarz [2]).** The map $\varphi : MH_* (\mathcal{L} N) \to SH^{n-\ast} (T^* N)$ preserves the product structure, so $\varphi \circ \psi_Q = \psi_P \circ \varphi^{\otimes 2}$. Moreover, the product $\psi_Q$ on $MH_* (\mathcal{L} N)$ corresponds to the loop product on $H_* (\mathcal{L} N)$.

**Proof (Sketch).** Observe Figure 12. Consider the 1-dimensional family of solutions corresponding to the configurations in the middle column ($\ell$ is a free parameter). On the right are the limiting configurations as $\ell \to 0$ or $\infty$. The middle two pictures in the last column
Figure 12. Dark circles are critical points of $E$, dark rectangles are Hamiltonian orbits. POP are pairs of pants, Floer are Floer trajectories. The configurations in the left column are counted by $\varphi, \psi_{Q'}, \psi_P$. are the same. So the maps counting the top and bottom configurations in the last column are chain homotopic. These are, respectively, $\varphi \circ \psi_{Q'}$ and $\psi_P \circ \varphi \otimes^2$.

Remark. Theorem 14.2 and Remark 14.1 prove that the unit for the loop product is $c_s[N]$.

14.4. $\varphi$ is a TQFT isomorphism

To $S$, we associated a graph $S'$ (Subsection 6.7). Let $\tilde{S}'$ be the graph obtained from $S'$ by switching the orientations of the edges ($p$ and $q$ get interchanged). So for $S$ with $p + q$ punctures, $\psi_{\tilde{S}'} : MH_*(\mathcal{L}N)^{\otimes q} \to MH_*(\mathcal{L}N)^{\otimes p}$.

Example. For $P$ and $Q$ of Section 6.4, we get the $P'$ and $Q'$ of Subsection 6.7 and $\tilde{P}' = Q'$ and $\tilde{Q}' = P'$.

Theorem 14.3. The isomorphism $\varphi : H_*(\mathcal{L}N)_\eta \cong MH_*(\mathcal{L}N)_\eta \to SH^{n*-}(T^*N)_\eta$ preserves the (possibly twisted) TQFT structures: $\varphi^{\otimes p} \circ \psi_{\tilde{S}'} = \psi_S \circ \varphi^{\otimes q}$, where $p \geq 1$ and $q \geq 1$.

Proof. For general $S$ with $p \geq 1$ and $q \geq 1$, decompose $S$ into copies of $Z, P$ and $Q$ as in Theorem 6.5 (this can be done without using $C$ by the remark in Section 6.4). So, $\psi_S$ is a composition of operations $\psi_Z, \psi_P$ and $\psi_Q$. Thus $\psi_{\tilde{S}'}$ is the corresponding composition of operations obtained by replacing $Z, P$ and $Q$ by $\tilde{Z}', \tilde{P}'$ and $\tilde{Q}'$. So the general relation follows if we can prove it for $Z, P$ and $Q$.

For $S = Z, \psi_Z$ and $\psi_{\tilde{Z}'}$ are the identity, so there is nothing to prove.

For $S = P$, the relation $\varphi \circ \psi_{Q'} = \psi_P \circ \varphi^{\otimes 2}$ is Theorem 14.2.

For $S = Q$, we need to check $\varphi^{\otimes 2} \circ \psi_{S'} = \psi_Q \circ \varphi$. One can either prove this by an argument similar to Figure 12 using the techniques of Abbondandolo and Schwarz [2], or one can use the second factorization in the proof of Theorem 6.10 as follows. The coproduct $Q$ factors as a gluing of $(P \sqcup Z)\#(Z \sqcup (Q\#C))$, where one can arrange the weights so that $Q\#C$ uses $H^3$ on each output (recall $H^3$ from Section 5). Since Theorem 14.3 holds for $P$ and $Z$,
we reduce to checking that \( \psi_Q(1) = \psi_Q(\varphi([N])) = \varphi^{S_2}(\psi_{P'}[N]) \) (recall \( \varphi(c_*[N]) = c^*(1) \) by Remark 14.1, and that using \( H^8 \) instead of \( H \) is equivalent to restricting to action values close to zero by Section 8). By Theorem 6.6, we can identify \( SH^*(H^8) \) with the Morse cohomology, and \( \psi_Q(1) = \psi_Q(1) = \pm \chi(N) \vol_{S_2} \) by Example 6.9. One then checks that by restricting to \( E \)-actions close to zero (see Remark 14.1), the TQFT solutions for \( LN \) become time-independent (this would involve mimicking Subsection 15.3, which we will not carry out). We now compute \( \psi_{P'}[N] \). Since \([N] \) is represented by the maximum of \( L|_N \), the first edge of \( P' \) will sweep out \([N] \). We need to perturb \( E = \int L \, dt \) on one of the two outgoing edges of \( P' \) to achieve transversality. We do this by using \( L' = L \circ \varphi^1_K : TN \to \mathbb{R} \) instead of \( L \) on that edge, where \( \varphi^1_K : TN \to TN \) is a time-1 Hamiltonian flow so that \( \varphi^1_K(N) \) are transverse in \( TN \). Thus, the only non-zero contribution to \( \psi_{P'}[N] \) is the count of \( w \in N \cap \varphi^1_K(N) \) which admit a configuration consisting of a semi-infinite \( -\nabla L|_N \) flow line from \( w \) to the minimum of \( L|_N \) and a semi-infinite \( -\nabla L'|_N \) flow line from \( w \) to the minimum of \( L'|_N \). So \( \psi_{P'}([N]) = \pm \chi(N)\vert pt\vert^{S_2} \in H_0(N)^{S_2} \). So, \( \psi_Q(1) = \varphi^{S_2}(\psi_{P'}[N]) \) since \( \varphi([pt]) = \vol_N \) by Remark 14.1.

For the twisted case, \( \varphi \) was constructed in [31]. For the argument above to hold in the twisted case, it suffices that the weights are locally constant on the TQFT moduli spaces and the \( M^+_\delta(\gamma, x) \) moduli spaces. The former is Theorem 7.1, the latter is proved similarly and was done in detail in [31].

\[ \square \]

**Remark.** Action-restricting the above result as in Remark 14.1, one obtains that \( c_* : H_*(N) \to H_*(LN)_\eta \) is a TQFT map, with unit \([N] \to c_*[N] \), using the (twisted) loop product on \( H_*(LN)_\eta \), and the intersection product on \( H_*(N) \) (the Poincaré dual of the cup product).

**Corollary 14.4.** For a closed exact Lagrangian submanifold \( L \subset T^*N \), Theorem 9.5 applied to Example 9.1 yields via Theorem 14.3 the commutative TQFT diagram of transfer maps

\[
\begin{array}{ccc}
H_*(LL)_\eta|_L & \to & H_*(LN)_\eta \\
\uparrow c_* & & \uparrow c_* \\
H_*(L) \otimes \Lambda & \to & H_*(N) \otimes \Lambda.
\end{array}
\]

**Remark.** The untwisted diagram (without TQFT) is due to Viterbo [44].

14.5. The based loop space

The wrapped analogue of \( SH^*(T^*N) \cong H_{n-\ast}(LN) \) for a fibre \( Tq_0N \subset T^*N \) is Abbondandolo–Schwarz’s ring isomorphism [2]:

\[
HW^*(Tq_0N) \cong H_{n-\ast}(\Omega N),
\]

using the Pontryagin product on \( H_*(\Omega N) \) induced by concatenation of based loops. As in Theorem 14.3, this respects the TQFT and the twisted analogue holds.

14.6. Vanishing of the Novikov homology of the free loop space

**Convention.** Recall, as mentioned at the start of Section 14, that we assume \( \text{char}(\mathbb{K}) = 2 \).

**Definition.** A space \( N \) has finite type if \( \pi_m(N) \) is finitely generated for each \( m \geq 2 \).

**Examples.** (1) Simply connected closed manifolds and (2) closed manifolds with trivial \( \pi_1 \) action on higher homotopy groups.
For closed $N$ of finite type, the Novikov homology $H_*(\mathcal{L}_0 N; \Lambda_{\tau \eta}) = 0$ for $\tau \eta \neq 0 \in H^1(\mathcal{L}_0 N)$ by [31], where $\mathcal{L}_0 N$ is the component of contractible loops. We can now extend it to $\mathcal{L} N$:

**Corollary 14.5.** For closed manifolds $N$ of finite type, and any $\tau \eta \neq 0 \in H^1(\mathcal{L}_0 N)$,

$$SH^*(T^* N)_{\eta} \cong H_{n-s}(\mathcal{L} N; \Lambda_{\tau \eta}) = 0.$$

**Proof.** By Theorem 14.3, $SH^*(T^* N)_{\eta} \cong H_{n-s}(\mathcal{L} N)_{\eta}$ is a ring with unit $c_*(N) \in H_*(\mathcal{L}_0 N)_{\eta}$. But $H_*(\mathcal{L}_0 N)_{\eta} = 0$ by [31]. So the claim follows by Theorem 10.1. \qed

**Remark.** $\tau \eta \neq 0 \in H^1(\mathcal{L}_0 N)$ iff $\pi^* \eta \neq 0 \in H^2(\tilde{N})$ for the universal cover $\pi : \tilde{N} \to N$.

15. The $c^*$ maps preserve the TQFT structure

15.1. PSS map

Theorem 6.6 is the analogue of the Piunikhin, Salamon and Schwarz [30] ring isomorphism $FH^*(M, H) \to QH^*(M)$, which holds for weakly monotone closed symplectic manifolds $(M, \omega)$. It turns the pair-of-pants product on Floer cohomology into the quantum cup product on quantum cohomology. We will briefly survey their proof.

Pick a generic Morse function $f$ and a Hamiltonian $H$ on $M$. Pick a homotopy $H_s$ interpolating 0 and $H$. The PSS map $\phi : FC^*(H) \to MC^*(f)$ on generators is $\phi(y) = \sum N_{x, y} x$, where $N_{x, y}$ is the oriented count of isolated spiked discs converging to $x$ and $y$ (we omit a detail here: one actually has to work over a Novikov ring since $\omega$ is not exact [30, Section 4]). Spiked discs are maps $u : \Sigma \to M$ such that $u(e^{2\pi i (s+it)})$ satisfies Floer’s continuation equation for $H_s$, converging at $s = \infty$ to a 1-orbit $y$ for $H$ and at $s = -\infty$ to a point $u(0) \in W^u(x, f)$ in the unstable manifold of $f$. The ‘spike’ is the $-\nabla f$ flow line connecting $x$ to $u(0)$. An inverse $\psi : MC^*(f) \to FC^*(H)$ (up to chain homotopy) is defined by counting isolated spiked discs flowing in the reverse direction: $u(e^{-2\pi i (s+it)})$ is an Floer continuation solution for $H_{-s}$ converging at $s = -\infty$ to a 1-orbit $y$ for $H$ and at $s = \infty$ to a point $u(0) \in W^u(x, f)$.

To show that $\phi$ and $\psi$ are inverses up to chain homotopy, consider $\phi \circ \psi$ and $\psi \circ \phi$. After gluing solutions, these count configurations on the left in Figure 13. Up to chain homotopy, the maps do not change if we homotope, respectively, $H$ and $\ell$ to zero. So we count the configurations on the right. In the first case, for dimensional reasons the $J$-holomorphic sphere in the figure is constant for generic $J$ (this uses the weak monotonicity of $M$). So we count isolated Morse continuation solutions for a constant homotopy $f$, which are constants (otherwise there is a 1-family of reparametrized solutions via $s \mapsto s + \text{constant}$). So $\phi \circ \psi \simeq \text{id}$ are chain homotopic. In the second case, we used a gluing theorem for $J$-holomorphic curves to obtain a cylinder (for such gluing arguments, we recommend the appendix in [25]). Up to chain homotopy, we can homotope $H_s$ to an $s$-independent $H$. So we count isolated Floer continuation solutions for a constant homotopy $H$, which again must be constants. So $\psi \circ \phi \simeq \text{id}$.

To prove that $\phi$ and $\psi$ respect the ring structure, observe Figure 14. We glue spiked discs to cap off, respectively, Floer and quantum-Morse product solutions (the quantum product replaces the vertex of the Morse-product graph $P'$ of Subsection 6.8 with a $J$-holomorphic
sphere). Up to chain homotopy, we can homotope, respectively, $H$ and $\ell_1, \ell_2, \ell_3$ to zero, reducing to the configurations on the right. In the second case, we used the gluing theorem for $J$-holomorphic curves to get a pair-of-pants Floer solution. So up to chain homotopy, we obtain the products on $QH^*$ and $FH^*$ respectively. A similar argument works for all (quantum) TQFT operations.

15.2. Proof for Liouville domains

For Liouville domains $M$, on the one hand, the above construction simplifies because the $J$-holomorphic spheres must be constant by Stokes' theorem (so only the non-quantum Morse operations defined in Subsection 6.8 contribute), on the other hand, the construction complicates because of the non-compactness of $M$. Indeed, the compactness of the above moduli spaces must fail, for otherwise $\phi$ and $\psi$ would always induce an isomorphism $SH^*(M) \cong H^*(M)$ which is false in general. Compactness in fact fails for $\phi$ since it uses a non-monotone homotopy $H_s$. The other map, $\psi : MC^*(f) \to SC^*(H)$, is actually well-defined by Lemma D.1, since we can pick $H_{-s}$ to be monotone, and we ensure that $-\nabla f$ is pointing inwards on the collar of $M$ so that $MH^*(f)$ computes $H^*(M)$.

**Lemma 15.1.** For $H = H^\delta$ as in Section 5, $\psi : MH^*(f) \to SH^*(H^\delta)$ is equal to $c_{H_s}^* : H^*(M) \cong SH^*(H^\delta)$ (see Theorem 6.4).

**Proof.** $H^\delta$ is $C^2$ small and Morse so the Floer solutions counted by $\psi$ are time-independent. Now $\psi$ changes by a chain homotopy if we homotope $f, H_s$ to $H, H$. So we end up counting
isolated Morse continuation solutions for the constant homotopy \( H \), which must be constant solutions. So \( \psi \simeq c_H \) are chain homotopic.

Goal: To prove Theorem 6.6 by the methods of Subsection 15.1, it remains to construct a well-defined map \( \phi : SH^*(H^\infty) \to H^*(M) \) inverse to \( \psi \). We will do this in Subsection 15.4.

15.3. Floer trajectories converging to broken Morse trajectories

The analytical machinery that Subsection 15.4 is based upon is due to Salamon and Zehnder [37]. For the convenience of the reader, we briefly review how they applied this machinery to prove \( FH^*(H) \cong H^*(M) \) for \( C^2 \) small \( H \) on closed symplectic manifolds \((M, \omega)\), assuming \( M \) is symplectically aspherical: \( \omega|_{\pi_2(M)} = 0 \) (this ensures, in particular, that \( J \)-holomorphic spheres are constant).

Let \( S^1_T = \mathbb{R}/T\mathbb{Z} \). We will be considering \( T \)-periodic Floer theory, as opposed to the 1-periodic theory, and we will study the limit \( T \to 0 \). Consider solutions \( u : \mathbb{R} \times S^1_T \to \mathbb{R}^{2n} \) to Floer’s local equation \( F_T(u) = 0 \), where \( F_T : W^{1,2}(\mathbb{R} \times S^1_T; \mathbb{R}^{2n}) \to L^2(\mathbb{R} \times S^1_T; \mathbb{R}^{2n}) \) is the local Floer operator defined by

\[
F_T(u) = \partial_s u + J\partial_t u + (S + A)u,
\]

where \( J \) is standard, \( S = S(s) \) and \( A = A(s) \) are continuous matrix-valued functions on \( \mathbb{R} \) which are, respectively, symmetric and antisymmetric, and which converge to \( S_\pm \) and \( A_\pm = 0 \) as \( s \to \pm \infty \) (the time-independence of \( S \) and \( A \) is the local equivalent of the assumption that the Hamiltonian \( H \) is time-independent).

Claim 1. The solutions \( u \) of \( F_T(u) = 0 \) are time-independent whenever the \( L^2 \)-norms satisfy \( \|S\| + \|A\| \leq c < 1/T \) for some constant \( c \).

Proof (Sketch). [37, Proposition 4.2] The averaged solution \( \bar{u}(s) = (\int_0^T u(s,t) \, dt)/T \) is a solution \( F_0(\bar{u}) = 0 \) of the local Morse operator \( F_0 : W^{1,2}(\mathbb{R}, \mathbb{R}^{2n}) \to L^2(\mathbb{R}, \mathbb{R}^{2n}) \),

\[
F_0(\bar{u}) = \partial_s \bar{u} + (S + A)\bar{u},
\]

and then one checks that \( u(s,t) \equiv \bar{u}(s) \) when \( S \) and \( A \) are small in the above sense. We remark that this result can also be proved using Fourier series.

Assume the hypothesis of Claim 1, and assume in addition that \( S_\pm \) are non-singular (which is the local equivalent of the non-degeneracy condition of orbits in Floer theory).

Claim 2. \( F_T \) is onto if and only if \( F_0 \) is onto.

Proof (Sketch). [37, Corollary 4.3] This follows because they have the same Fredholm index, and by Claim 1 they have isomorphic kernels.

The local result can now be applied globally. Pick an \( \omega \)-compatible almost complex structure on the closed symplectic manifold \((M, \omega)\) such that \( \omega(\cdot, J\cdot) \) is a Morse–Smale metric for a given Morse function \( H : M \to \mathbb{R} \).

Claim 3. For small \( T > 0 \), every isolated finite energy Floer trajectory \( u : \mathbb{R} \times S^1_T \to M \) is time-independent and regular (the linearization of the Floer operator at \( u \) is onto).
Proof (Sketch). [37, Theorem 7.3] When \( u \) is a time-independent Floer trajectory, that is, a \( -\nabla H \) flow, we reduce to the above local case by picking an orthonormal frame: \( F_0 \) is onto by the Morse–Smale condition, so \( F_T \) is onto, so \( u \) is regular.

To prove that \( u \) must be time-independent, we need a Gromov compactness trick. By contradiction suppose there are isolated finite energy Floer solutions \( u_n : \mathbb{R} \times S^1_{T_n} \to M \) with periods \( T_n \to 0 \), which are not time-independent, connecting two given critical points \( x, y \) of \( H \) (we assume \( T_n \) is so small that there are no \( T_n \)-periodic Hamiltonian orbits, this uses the fact that \( H \) is Morse). The energy

\[
E(u_n) = \frac{1}{T_n} \int_{-\infty}^{\infty} \int_0^{T_n} |\partial_s u_n|^2 \, dt \, ds = H(x) - H(y)
\]

is bounded, therefore, by Gromov compactness, a subsequence of the \( u_n \) must converge to some broken Morse trajectory \( v_1 \# \ldots \# v_m \) for \( H \), where \( m \) is bounded by the difference of the Morse indices of \( x \) and \( y \). Note that this uses the assumption \( \omega|_{x}(\mathcal{M}) = 0 \) to exclude sphere bubbling (sphere bubbles can otherwise naturally appear and in general one should expect the Gromov limits to consist of Morse trajectories intermediated by trees of \( J \)-holomorphic spheres, an example of this phenomenon for the non-exact total space of \( \mathcal{O}_{p_1}(-1) \) is explained in [33]). Since the \( u_n \) are isolated (up to \( \mathbb{R} \) reparametrization), the index of the Fredholm problem is 1, so also the broken Morse trajectory must have total index 1. Therefore, \( m = 1 \) and \( v = v_1 \) is an isolated Morse trajectory, that is, a time-independent isolated Floer trajectory.

We claim that for large \( n \), \( u_n(s + s_n, t) = v(s) \) coincide for some \( s_n \in \mathbb{R} \). This would contradict the assumption that \( u_n \) is not time-independent. So suppose by contradiction that \( u_n \) and \( v \) never coincide in this way. If there are integers \( C_n \) with \( C_nT_n = T \), with \( T \) so small that \( F_T \) is onto at \( v \) by the first part of the proof, then \( F_T(u_n) = 0 \) and \( u_n \to v \) contradicts the fact that \( v \) is an isolated solution of \( F_T \). In general, pick integers \( C_n \) with \( C_nT_n \to T \) and apply the same argument uniformly in an interval around \( T \).

Claim 4. Given a Morse function \( H : M \to \mathbb{R} \) and a small enough \( T > 0 \), all isolated Floer trajectories for \( T \cdot H \) are time-independent and regular.

Proof. The \( T \)-periodic and 1-periodic Floer theories are isomorphic, by sending a \( T \)-periodic Floer trajectory \( u(s, t) \) for \( H \) to the 1-periodic Floer trajectory \( u(sT, tT) \) for \( T \cdot H \). By Claim 3, for \( T \ll 1 \), the \( T \)-periodic Floer trajectories for \( H \) are time-independent and regular, so the same is true for the 1-periodic ones for \( T \cdot H \).

We can apply the same arguments to Floer continuation solutions \( u : \mathbb{R} \times S^1_{T} \to M \) and Morse continuation solutions for homotopies \( H_s : M \to \mathbb{R} \) connecting two Morse functions and picking \( J_s \) so that \( g_s = \omega(\cdot, J_s \cdot) \) is Morse–Smale for \( H_s \). Thus, we have the following claim:

Claim 5. For small \( T > 0 \), all isolated Floer continuation solutions for \( T \cdot H_s \) are time-independent and regular.

We now show that the arguments of Claims 3–5 also hold for non-isolated Floer solutions. This result is stated in [37, p. 1343]: Salamon and Zehnder mention that this generalization can be proved by a parameter-valued gluing argument. Strictly speaking, we only need this generalization in the proof of Lemma 15.8, and we mention in Remark 15.9 a work-around.
Theorem 15.2. For a time-independent $H$ which is Morse–Smale for the metric induced by an almost complex structure $J$, and for small enough $T > 0$, the finite energy Floer trajectories for $T \cdot H$ are time-independent and regular. Similarly for Floer continuation solutions of $T \cdot H$.

Proof. We take a closer look at the proof of Claim 3. Let $\mathcal{M}_T$ be the moduli space of Floer trajectories $u : \mathbb{R} \times S^1_T \to M$ joining two critical points $x$ and $y$ of $H$ (for small $T$ the 1-orbits of $T \cdot H$ are $\text{Crit}(H)$). The proof is by induction on the (finitely many possible values of the) Morse index difference between $x$ and $y$. To keep the notation under control, we assume the index difference between $x$ and $y$ is 2, so the inductive hypothesis is the result for isolated Floer trajectories. Our proof of the inductive step from 1 to 2 easily generalizes to the general case.

Let $\mathcal{M}_0$ be the moduli space of Morse trajectories from $x$ to $y$. By the Morse–Smale assumption, this moduli space is a smooth, typically non-compact, manifold of the expected dimension 1 (the Morse index difference minus the $\mathbb{R}$ reparametrization freedom), and it has a compactification $\bar{\mathcal{M}}_0$ consisting of all possibly broken Morse trajectories from $x$ to $y$. So, in particular, $\mathcal{M}_0$ contains all Gromov limits that arise from sequences $u_{T_n} \in \mathcal{M}_{T_n}$ as $T_n \to 0$ (as emphasized in the proof of Claim 3, this uses that $J$-holomorphic spheres in $M$ are constant).

Let us abbreviate by $F_0$ and $F_T$ the linearizations of the relevant Fredholm maps which define the moduli spaces $\mathcal{M}_0$ and $\mathcal{M}_T$. Since Fredholm indices are locally constant and $F_0$ has index 2, also $F_T$ must have index 2 for small enough $T$ (the maps in $\mathcal{M}_T$ must get close to the maps in $\mathcal{M}_0$, otherwise thanks to the energy estimate $E = H(x) - H(y)$ a Gromov compactness argument as $T \to 0$ would produce a broken Morse trajectory which is not in $\mathcal{M}_0$). So $\mathcal{M}_T$ has virtual dimension 1. For small enough $T$, by Claim 2, the operators $F_0$ and $F_T$ have isomorphic kernels. But since their Fredholm indices equal, also their cokernels have the same rank. Regularity of $F_0$ means that this cokernel vanishes, so also $F_T$ is regular. Therefore, $\mathcal{M}_T$ is a smooth 1-dimensional manifold, and it has a compactification $\bar{\mathcal{M}}_T$ by broken Floer trajectories. The boundary $\partial \mathcal{M}_T$ consists of broken Floer trajectories, which by the inductive hypothesis must be time-independent for all small enough $T$, so $\partial \mathcal{M}_T \subset \partial \mathcal{M}_0$.

Observe that $\mathcal{M}_0 \subset \mathcal{M}_T$ viewing Morse trajectories for $H$ as time-independent Floer trajectories. But both $\mathcal{M}_0$ and $\mathcal{M}_T$ are compact 1-manifolds so the inclusion $\mathcal{M}_0 \subset \mathcal{M}_T$ identifies those connected components which contain some possibly broken time-independent trajectory. So for these particular connected components of $\mathcal{M}_T$ the Floer trajectories are time-independent.

Now run the final argument of Claim 3: we can assume there are integers $C_n$ with $C_nT_n = T$ with $T$ small so that $F_T$ is onto (otherwise apply the argument in a small interval around $T$ and use $C_nT_n \to T$). Suppose by contradiction that for $T_n \to 0$, there are $u_n \in \mathcal{M}_{T_n}$ lying in a connected component of $\mathcal{M}_{T_n}$ which does not contain any possibly broken time-independent trajectory. As $n \to \infty$, $u_n$ converges to a possibly broken time-independent trajectory $u \in \bar{\mathcal{M}}_0$. Since $F_T(u_n) = F_{C_nT_n}(u_n) = 0$, both $u_n, u \in \mathcal{M}_T$. But $\mathcal{M}_T$ is a compact 1-manifold, so $u_n$ converging to $u$ implies that for large $n$ all the $u_n$ lie in the same connected component of $\mathcal{M}_T$ as $u$, but this contradicts the choice of $u_n$ since $u$ is time-independent. \hfill \Box

15.4. Non-monotone homotopies of small Hamiltonians

We now deal with the ‘Goal’ of Subsection 15.2. From now on, $M$ is a Liouville domain, $H = H^\delta$ is as in Section 5 so the 1-orbits of $H$ are the critical points of $H$. Pick a Morse function $f : \overline{M} \to \mathbb{R}$ and a (non-monotone) homotopy $H_s$ from 0 to $H$ of the form $f = f(R)$, $H_s = h_s(R)$ on the collar, with $f'(R) > 0$ and $h'_s(R) > 0$ (unless $H_s \equiv 0$). This ensures that $-\nabla f$ and $-\nabla H_s$ are inward-pointing on the collar, so in particular the Morse cohomologies of $f$ and $H$ are well defined and are isomorphic to $H^*(M)$. 

**Definition 15.3.** A \( \phi \) solution \( v \# u \) from \( x \in \text{Crit}(f) \) to \( y(t) \equiv y \in \text{Crit}(H) \) consists of the following.

1. \( v : (-\infty, 0] \to \overline{M} \), which is a \( -\nabla f \) flow line from \( x \) to \( u(0) \).
2. \( u : \mathbb{C} \to \overline{M} \), such that \( u(e^{2\pi i (s+it)}) \) is a Floer continuation solution for \( H_s \) converging to \( y(t) \) as \( s \to \infty \).

Write \( \mathcal{M}^\phi(x, y) \) for the moduli space of \( \phi \) solutions. We call it a \( T \)-periodic \( \phi \) solution if we instead use \( u(e^{2\pi i (s+it)/T}) \) and parametrize \( t \) by \( \mathbb{R}/T\mathbb{Z} \) instead of \( \mathbb{R}/\mathbb{Z} \).

We choose the relevant almost complex structures \( J_s, J_e \) so that they induce Morse–Smale metrics for \( H \) and \( H_s \), and we choose \( f \) generically so that the time-independent \( \phi \) solutions are regular solutions of the defining Fredholm problem.

**Lemma 15.4.** Fix a compact \( M' \subset \overline{M} \) containing \( M \). For small enough \( T > 0 \) all \( \phi \) solutions for \( T \cdot H_s \) lying in \( M' \) are time-independent, regular and lie in \( M \).

**Proof.** We discuss the case of isolated solutions, the general case is proved by mimicking the proof of Theorem 15.2.

Apply the Gromov compactness trick of Subsection 15.3 Claim 3: suppose by contradiction that there are \( T_n \)-periodic isolated \( \phi \) solutions \( v_n \# u_n \in \mathcal{M}^\phi(x, y) \) lying in \( M' \), with \( T_n \to 0 \), such that \( u_n \) is not time-independent. The \( v_n \# u_n \) have bounded energy

\[
E(v_n) + E(u_n) = (f(x) - f(u(0))) + (H(u(0)) - H(y)) \\
\leq f(x) - H(y) + \max(f|_M) + \max(H|_M)
\]

because \( u(0) \) must lie in \( M \) (since \( -\nabla f \) is inward-pointing on the collar). So by Gromov compactness, we can extract a subsequence converging to a time-independent broken \( \phi \) solution. But since \( v_n \# u_n \) are isolated, the index of the Fredholm problem is 0, so the broken \( \phi \) solution also has total index 0, so in fact it is not broken. So this time-independent \( \phi \) solution \( v \# u \) is isolated, and it is regular by the comments above the statement of the lemma.

We claim that for large \( n \), \( u_n(s, t) \equiv u(s) \) coincide, contradicting the time-dependence of \( u_n \). If \( u_n \) and \( u \) never coincide in this way, then a time-rescaling argument as in the proof of Claim 3 of Subsection 15.3 would contradict that \( v \# u \) is an isolated \( \phi \) solution. Thus, for small enough \( T > 0 \), all \( T \)-periodic \( \phi \) solutions lying in \( M' \) are time-independent and regular.

For time-independent \( \phi \) solutions, since the ends lie in \( M \), the flow line cannot exit \( M \), otherwise \( -\nabla f \) or \( -\nabla H_s \) would point outward at some point on the collar. The result follows by a rescaling trick like Claim 4 of Subsection 15.3. \( \square \)

**Lemma 15.5.** Let \( \mathcal{M} \) be a connected component of \( \mathcal{M}^\phi(x, y) \). Then \( \mathcal{M} \) consists either entirely of solutions lying in \( M \) or entirely of solutions which escape \( M' \).

**Proof.** By Lemma 15.4, \( \phi \) solutions lying in \( M' \) must lie in \( M \), so they cannot be \( C^0 \)-close to a \( \phi \) solution which escapes \( M' \). \( \square \)

Similarly define \( \psi \) solutions, \( \mathcal{M}^\psi(y, x) \) for \( f, H_{-s} \) (see Subsection 15.1), with \( H_{-s} \) monotone. Recall that for these the maximum principle holds, so:

**Lemma 15.6.** All \( \psi \) solutions are contained in \( M \). Moreover, for small \( T > 0 \), all \( \psi \) solutions for \( T \cdot H_{-s} \) are time-independent, regular and contained in \( M \).
**Definition 15.7.** Replace $H_s$ by $T \cdot H_s$ so that the previous two lemmas hold. Define $\phi : SC^*(H) \to MC^*(f)$ and $\psi : MC^*(f) \to SC^*(H)$ by counting only the isolated $\phi$ and $\psi$ solutions which lie in $M$ (so we ignore the $\phi$ solutions which exit $M'$).

**Lemma 15.8.** $\phi$ and $\psi$ are chain maps.

**Proof.** Consider a 1-dimensional connected component $M \subset M^\phi_1(y, x)$. The boundary consists of broken $\psi$ solutions, where a Floer or a Morse trajectory breaks off at one of the two ends. The count of these broken solutions proves that $d_{\text{Floer}} \circ \psi - \psi \circ d_{\text{Morse}} = 0$. The same proof works for $\phi$, by Lemma 15.5, since we can ignore those $M \subset M^\phi_1(x, y)$ which contain a $\phi$ solution which escapes $M'$.

**Remark 15.9.** One can prove Lemma 15.8 while only using the result of Lemma 15.4 for isolated solutions (so we avoid appealing to Theorem 15.2). Indeed, suppose by contradiction that there exist $T_n$-periodic $\phi$ solutions $v_n \# u_n$ of index 1 lying in $M'$ but which exit $M$, where $T_n \to 0$. After passing to a subsequence, $v_n \# u_n$ converges as $T_n \to \infty$ either to a genuine time-independent $\phi$ solution of index 1 or to a broken time-independent $\phi$ solution. But both those limits are contained in the interior of $M$, so also $v_n \# u_n$ lies in $M$, a contradiction.

**Theorem 15.10.** $\psi$ and $\phi$ are inverse to each other up to chain homotopy. In particular, $\phi : SH^*(H^\delta) \to H^*(M)$ is an inverse for $\psi = c^* : H^*(M) \to SH^*(H^\delta)$.

**Proof.** Run the argument of Subsection 15.1, using that all $J$-holomorphic spheres are constant since the symplectic form is exact. The proof of Figure 13 involves the connected 1-dimensional components $M$ of the moduli space $\bigcup_{c \in [0, 1]} M^{\phi \circ \psi}(x, x', f, c \cdot H_s)$ of glued solutions for $\phi \circ \psi$ for $c \cdot H_s$. The count of all $\partial M$ proves the existence of the chain homotopy mentioned in Figure 13 like in the proof of Theorem A.10. We need to justify why we can ignore those $M$ which contain a solution which escapes $M'$. This is proved like Lemma 15.5, since the glued solutions for $c \cdot H_s$ either lie in $M$ or escape $M'$, by combining Lemmas 15.4 and 15.6.

**Technical Remark.** More precisely, this last argument is as follows. The glued solutions depend on a gluing parameter $\lambda > 0$ and as $\lambda \to \infty$, they converge to broken solutions, which by assumption lie in $M$. So, for $\lambda \gg 0$, we claim that the glued solutions in $M$ must also lie in $M$. Suppose not. Then there is a sequence of glued solutions $w^\phi_n \# w^\psi_n \in M$ lying in $M'$ which exit $M$ with $\lambda_n \to \infty$ as $n \to \infty$. By passing to a subsequence, we can assume that they arise from gluings for $c_n \cdot H_s$ with $c_n \to c_\infty \in [0, 1]$. In the limit, this converges to a broken solution in $M'$ consisting of a $\phi$ solution and a $\psi$ solution (defined using $c_\infty H$) such that at least one of them does not lie in the interior of $M$. So this contradicts Lemma 15.4 or 15.6.

So $\phi \circ \psi$ is chain homotopic to the map we get at $c = 0$, which is the identity since it counts isolated Morse continuation solutions for a constant homotopy (Subsection 15.1). Similarly, $\psi \circ \phi \simeq \text{id}$.

**15.5. Proof of Theorem 6.6**

Let $H = H^\delta, H_s, f$ be as above. Let $S'$ be a model surface with weights $A_a, B_b \leq 1$ (see Appendix A.1). Denote by

\[ \phi_a : SH^*(A_a H^\delta) \to MH^*(f_a) \quad \text{and} \quad \psi_b : MH^*(f_b) \to SH^*(B_b H^\delta) \]

the $\phi$ and $\psi$ maps obtained for $A_a \cdot H_s$ and $B_b \cdot H_{-s}$ and Morse functions $f_a$ and $f_b$ obtained by generically perturbing $f$. Let $S'$ be the graph associated to $S$ (Subsections 6.7 and 6.8).
Then
\[ \psi_S : \bigotimes_b \mathcal{SH}^*(B_bH^\delta) \to \bigotimes_a \mathcal{SH}^*(A_aH^\delta) \quad \text{and} \quad \psi_{S'} : \bigotimes_b \mathcal{MH}^*(f_b) \to \bigotimes_a \mathcal{MH}^*(f_a). \]

**Lemma 15.11.** \( \bigotimes_a \phi_a \circ \psi_S \circ \bigotimes_b \psi_b \) is chain homotopic to \( \psi_{S'} \) (compare Figure 14).

**Proof.** The proof follows like Theorem 15.10 using the fact that glued solutions of \( \bigotimes_a \phi_a \circ \psi_S \circ \bigotimes_b \psi_b \) either lie in \( M \) or escape \( M' \), combining Lemmas D.1, 15.4 and 15.6.

Similarly, \( \bigotimes_a \psi_a \circ \psi_{S'} \circ \bigotimes_b \psi_b \) is chain homotopic to \( \psi_S \), in the obvious notation. Theorem 6.6 then follows upon taking direct limits and using Theorems A.14 and 15.10.

15.6. The maps \( c^* : H^*(L) \to HW^*(L) \) preserve the TQFT (Theorem 6.14)

Recall from Section 5 that in the wrapped case there is a map \( c^* : H^*(L) \cong HW^*(L; H^\delta) \to HW^*(L) \). To prove that this map is a TQFT map one needs to show that continuation maps \( HW^*(L; H^\delta) \to HW^*(L; H^\delta) \) are compatible with the TQFT, and that the isomorphism \( H^*(L) \otimes \Lambda \to HW^*(L; H^\delta) \) of Section 5 is compatible with the TQFT. The proof of the former is analogous to the proof for the \( SH^* \) groups (Theorem A.14), but the proof of the latter requires a PSS-description of the isomorphism \( H^*(L) \otimes \Lambda \to HW^*(L; H^\delta) \) (cf. Subsection 15.1). We will not pursue this in great detail, but we will mention how this can be done by changing the setup in Subsection 15.1 as follows.

We assume \( \omega|_{\pi_2(M,L)} = 0 \) so \( J \)-holomorphic discs bounding \( L \) are constant (this will hold by Stokes’ theorem in the Liouville setup when mimicking Subsection 15.2 since \( L \subset \bar{M} \) is exact). Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \} \). The spiked disc \( u : \mathbb{C} \to M \) of Subsection 15.1 which intersected the \( -\nabla f \) spike at \( 0 \in \mathbb{C} \) gets replaced by a punctured disc \( u : \mathbb{D} \setminus \{+1\} \to M \) which intersects the spike at \( z = -1 \in \partial \mathbb{D} \). We pick a holomorphic strip-like end parametrization \([0, \infty) \times [0, 1]\) for \( \mathbb{D} \setminus \{+1\} \) near \( +1 \in \partial \mathbb{D} \), and on this strip we define the interpolation \( H_s \) from \( H_s = 0 \) for \( s \leq 1 \) to \( H_s = H^\delta \) for \( s \geq 2 \). We require that the map \( u : \mathbb{D} \setminus \{+1\} \to M \) is \( J \)-holomorphic away from the strip-like end; it satisfies Floer’s equation for \( H_s \) on the strip-like end; and it satisfies the Lagrangian boundary condition \( u(\partial \mathbb{D}) \subset L \). The discussion in Subsection 15.1 can now be carried out analogously. For example, in Figure 13, instead of a sphere which we homotope to a holomorphic sphere, we have a disc which arises from gluing two punctured discs; we then homotope the Hamiltonian to zero so the disc becomes a \( J \)-holomorphic disc \( v \) bounding \( L \), which must be constant since its energy \( \int v^* \omega \) is zero by \( \omega|_{\pi_2(M,L)} = 0 \).

Appendix A. TQFT structure on \( SH^*(M) \)

A.1. Model Riemann surfaces

See Figure 3. Let \((S, j)\) be a Riemann surface with \( p \geq 1 \) negative punctures and \( q \geq 0 \) positive punctures, with a fixed choice of complex structure \( j \) and a fixed choice of parametrization \((-\infty, 0] \times S^1 \) and \([0, \infty) \times S^1 \), respectively, near the negative and positive punctures so that \( j \partial_s = \partial_t \). We call these parametrizations the *cylindrical ends*. We assume \( S \) has no boundary and that away from the ends \( S \) is compact.

There is a contractible set of choices of complex structures \( j \) on \( S \) extending the \( j \) on the ends \((j \partial_s = \partial_t)\), see [38, 2.2.1]. We emphasize that we keep \( j \) fixed (see the remark in Appendix A.7).

We also fix a 1-form \( \beta \) on \( S \) satisfying \( d\beta \leq 0 \), such that on each cylindrical end \( \beta \) is some positive constant multiple of \( dt \). These constants \( A_1, \ldots, A_p, B_1, \ldots, B_q > 0 \) will be called
weights. The reason for using $\beta$ is that we do not have a global form $dt$ for a general surface $S$, unless $S$ is a cylinder. The condition $d\beta \equiv 0$ will be crucial to prove a priori energy estimate and a maximum principle (Appendices D.1 and D.3).

By Stokes’ theorem $\sum A_a - \sum B_b = - \int_S d\beta \geq 0$ (observe this forces $p \geq 1$). Conversely, we now show that given $A_a, B_b > 0$, this inequality is the only obstruction to constructing $\beta$.

**Lemma A.1.** If $c = \sum A_a - \sum B_b \geq 0$, then such forms $\beta$ exist. If $c = 0$, then $d\beta = 0$. If $c > 0$, then one can ensure that $d\beta = 0$ except on an arbitrarily small disc $D$ embedded in $S$.

**Proof.** Suppose that $A_a$ and $B_b$ satisfy $c = \sum A_a - \sum B_b = 0$. Consider the exact sequence $H^1(S) \xrightarrow{f} H^1(\partial S) \xrightarrow{\partial} H^2(S, \partial S) \to 0$ for the pair $(S, \partial S)$. On de Rham forms, $f[\beta] = [\beta|_{\partial S}]$ is the pull-back and $g[\alpha] = [d\alpha]$ for any extension $\alpha$ of $\alpha$ to $S$. We can identify $H^2(S, \partial S) \cong \mathbb{R}$ by integration $[\alpha] \to \int_S \alpha$. Let $a$ be the 1-form on $\partial S$ given by the data $A_a dt$ and $B_b dt$. Under that identification, the map $g$ becomes $g[\alpha] = - \sum A_a + \sum B_b = 0$. So by exactness there is a 1-form $\beta$ on $S$ with $d\beta = 0$, and $[\beta|_{\partial S}] = [\alpha]$. So by adding to $\beta$ an exact form supported near the ends we can ensure that $[\beta|_{\partial S}]$ equals the data $A_a dt$ and $B_b dt$ on the ends.

Now suppose $c = \sum A_a - \sum B_b > 0$. Let $\tilde{S}$ be the Riemann surface $S$ with an additional positive puncture at a chosen point of $S$, and fix a cylindrical end parametrization near this new puncture. The above procedure applied to $\tilde{S}$ yields a 1-form $\tilde{\beta}$ on $\tilde{S}$ with $d\tilde{\beta} = 0$, such that $\tilde{\beta}$ equals $A_a dt$ and $B_b dt$ near the original ends and equals $c dt$ on the new end. Finally, extend $\tilde{\beta}$ to a form $\beta$ on $S$ by defining $\beta = h(s) dt$ on the new cylindrical end $[0, \infty) \times S^1$ with $h = c$ for $s \leq 1$, $h = 0$ for $s \geq 2$, and $h' \leq 0$ everywhere. So $d\beta = 0$ on $S$ except in the small region near the new puncture where $h' \neq 0$, and there $d\beta = h'(s) ds \wedge dt \leq 0$. □

### A.2. Floer solutions

Let $H : \overline{M} \to \mathbb{R}$ be a Hamiltonian linear at infinity (Subsection 3.3) and assume $H \geq 0$ (required in Appendix A.3). A Floer solution is a smooth map $u : S \to \overline{M}$ converging to Hamiltonian orbits as $s \to \pm \infty$ on the ends, such that $du - X \otimes \beta$ is $(j, J)$-holomorphic:

$$(du - X \otimes \beta)^{0,1} \equiv \frac{1}{2} \{(du - X \otimes \beta) + J \circ (du - X \otimes \beta) \circ J\} = 0.$$

On a cylindrical end with weight $c$ this is Floer’s equation $\partial_s u + J(\partial_t u - cX) = 0$ for the Hamiltonian $cH$. So $u$ converges to 1-orbits of $A_a H$ and $B_b H$.

Let $\mathcal{M}(x_1, \ldots, x_p; y_1, \ldots, y_q; S, \beta)$ denote the moduli space of Floer solutions which converge to Hamiltonian orbits $x_a$ at the negative ends and $y_b$ at the positive ends. We abbreviate it by $\mathcal{M}(x_a; y_b; S, \beta)$ and we call $x_a$ and $y_b$ the asymptotics. We write $\mathcal{M}_k(x_a; y_b; S, \beta)$ for the $k$-dimensional part of $\mathcal{M}(x_a; y_b; S, \beta)$.

**Example A.2** (Continuation cylinders). Let $S$ be the cylinder $Z = \mathbb{R} \times S^1$ with $p = q = 1$; weights $A_1 = m'$, $B_1 = m$ with $m' \geq m$; $\beta = f(s) dt$ with $f'(s) \leq 0$. Then $u \in \mathcal{M}(x; y; Z, \beta)$ solves $\partial_s u + J(\partial_t u - f(s)X) = 0$, so it is a Floer continuation solution for the homotopy $H \Rightarrow f(s)H$ from $m'H$ to $mH$. The monotonicity condition $\partial_s h' = f'(s)h' \leq 0$ of Subsection 3.2 is equivalent to the condition $d\beta = f'(s) ds \wedge dt \leq 0$ of Appendix A.1. Thus, the count of $\mathcal{M}_0(x; y; Z, \beta)$ is the continuation map $SH^*(mH) \to SH^*(m'H)$.

### A.3. Energy

A solution $u : S \to \overline{M}$ of $(du - X \otimes \beta)^{0,1} = 0$ has energy defined by

$$E(u) = \frac{1}{2} \int_S \|du - X \otimes \beta\|^2 \text{vol}_S = \int_S u^* \omega - u^*(dH) \wedge \beta.$$
In Appendix D.1, we show that, since $H \geq 0$, $\omega = d\theta$ and $d\beta \leq 0$, the energy is determined a priori from the asymptotics:

$$E(u) \leq \int_S u^* \omega - d(u^* H \beta) = \sum_{\text{negative ends } a} \mathcal{A}_{A_a H}(x_a) - \sum_{\text{positive ends } b} \mathcal{A}_{B_b H}(y_b).$$

### A.4. Fredholm theory

Lemma D.1 proves that all $u \in \mathcal{M}(x_a; y_b; S, \beta)$ lie in the compact set $R \leq \max(R(x_a), R(y_b), R_0)$ in $\overline{M}$ for $J$ of contact type for $R \geq R_0$. So the following Theorems do not notice that $\overline{M}$ is non-compact, so the proofs for closed manifolds apply. After a small generic time-dependent perturbation of $(H, J)$ on the ends, the following hold:

**Theorem A.3 (38, 2.5.7).** Solutions of $(du - X \otimes \beta)^{0,1} = 0$ are smooth and those of finite energy converge exponentially fast to Hamiltonian orbits at the ends, that is near each end $|\partial_s u| \leq ce^{-\delta|s|}$ for some constants $c, \delta > 0$.

**Theorem A.4 ([38, 3.1.31, 3.3.11] or [9, Proposition 4]).** Let $u$ be a Floer solution. Let $D_u$ be the linearization at $u$ of the operator defining $(du - X \otimes \beta)^{0,1} = 0$ (see Appendix A.6), then $D_u$ is Fredholm with index $D_u = \sum |x_a| - \sum |y_b| + 2n(1 - g - p)$, where $g$ is the genus of $S$ (after filling in the punctures). See Remark 3.3 for the grading conventions. See Appendix B.16 for the index calculation.

### A.5. Smoothness of the Moduli spaces

For closed symplectic manifolds $M$, the condition $d\beta \leq 0$ was not necessary. So the natural approach [38] was to pick any Hamiltonians $H_a$ and $H_b$ on the ends and any tensor $\kappa = X \otimes \beta$ which equals $X_{H_a}$ and $X_{H_b}$ on the ends, with $\kappa = 0$ away from the ends (so $u$ is $J$-holomorphic there: $du^{0,1} = 0$). Using this approach, for closed $M$, Schwarz [38, 4.2.17, 4.2.20] proved smoothness of $\mathcal{M}(x_a; y_b; S, \beta)$ for generic $\kappa$ close to 0 away from the ends. Unfortunately, for Liouville domains $M$ we cannot use this approach, since the maximum principle and the a priori energy estimate fail without $d\beta \leq 0$. So we need to reprove the smoothness result.

Just as in the Technical Remarks in Subsection 2.5, we need the 1-orbits of $A_a H$ and $B_b H$ to be non-degenerate so that they are isolated and so that $\mathcal{M}(x_a; y_b; S, \beta)$ is the zero set of a Fredholm map, so one typically needs to make a time-dependent perturbation $H_t$ of $H$. Just as for Floer continuation solutions in Subsection 3.2, we typically need to make a $C^2$ small perturbation $J_z$ of $J$ depending on $z \in S$ and supported away from the ends of $S$ to ensure the smoothness of $\mathcal{M}(x_a; y_b; S, \beta)$. In Appendix A.6, we explain this perturbation. Such perturbed $J$ are called regular (we discuss this below). So the precise dependence of $H$ and $J$ on parameters is as follows.

1. $J \in C^\infty(S \times \overline{M}, T\overline{M})$ is $\omega$-compatible, and $H \in C^\infty(S \times \overline{M}, \mathbb{R})$ with $H \geq 0$.
2. $H_z = H(z, \cdot)$ does not depend on $z \in S$ except possibly on the ends of $S$.
3. For large $R$, $J_z = J(z, \cdot)$ is of contact type $(dR = J_z^* \theta)$ for each $z \in S$.
4. For large $R$, $H_z = h_z(R)$ only depends on $R$, and $h_z'(R) \geq 0$.
5. On the ends of $S$: for large $|s|$ both $H$ and $J$ can depend on $t$ but not on $s$; for large $R$, $\partial_s h_z'(R) \leq 0$ on the ends of $S$.

**Remarks.** The conditions ensure that Lemma D.1(2) applies. Since we can construct $SH^*$ using Hamiltonians which for $R \gg 0$ are linear of generic slope and time-independent, we can further simplify the above by taking $h_z = h$ and $J_z = J$ to be independent of $z$ for $R \gg 0$. 


Indeed, for $R \gg 0$ there are no 1-orbits of $A_uH$ and $B_bH$ (see Subsection 2.1) and Lemma D.1 ensures that Floer solutions do not enter the region $R \gg 0$, so we do not need to perturb there.

The equation $(du - X \otimes \beta)^{0,1} = 0$ corresponds to $\overline{\partial}u = 0$ for the operator
\[ \overline{\partial} = \frac{1}{2} \{(d - X \otimes \beta) + J \circ (d - X \otimes \beta) \circ j \}. \]

So the moduli space $\mathcal{M}(x_a; y_b; S, \beta)$ is the intersection of the section $\overline{\partial}$ and the zero section of a suitable Banach bundle (for details on this bundle see [38, 2.2.4]):
\[ \mathcal{M}(x_a; y_b; S, \beta) = \overline{\partial}^{-1}(0). \]

Regularity of $J$ means that $\overline{\partial}$ is transverse to the zero section and so $\mathcal{M}(x_a; y_b; S, \beta)$ is a smooth manifold by the implicit function theorem. Regularity of $J$ is equivalent to the surjectivity of the linearizations $D_u$ of $\overline{\partial}$ at Floer solutions $u$, in particular, the index of $D_u$ will then be the dimension of the kernel of $D_u$. This kernel is the tangent space of $\overline{\partial}^{-1}(0)$.

So, when gradings are defined (Subsection 3.6), by Theorem A.4, we have:
\[ \dim \mathcal{M}(x_a; y_b; S, \beta) = \sum |x_a| - \sum |y_b| + 2n(1 - g - p). \]

A.6. Existence of regular $J$

We claim that for generic (in the sense of Baire) $J$, which depend on $S$, regularity holds: the linearization $D_u$ of $\overline{\partial}$ is surjective at Floer solutions $u$.

Let $(S, j, \beta)$ be the data described in Appendix A.1 and $J = J_z, H = H_z$ the data described in Appendix A.5 (we assume that near the ends $H = H_j$ has been perturbed time-dependently to ensure that the asymptotics are non-degenerate).

By Lemma D.1, all Floer solutions lie in a compact $K \subset \overline{M}$ determined by the asymptotics. We will achieve transversality by perturbing $J$ on $K$, keeping $J$ of contact type outside of $K$ (so the maximum principle still holds outside $K$).

Let $D_u : W^{1,p}(u^*TM) \to L^p(\text{Hom}^{0,1}(TS, u^*TM))$ denote the linearization of $\overline{\partial}$ at $u \in \mathcal{M}(x_a; y_b; S, \beta)$ (where $p > 2$). We claim that after a generic perturbation of $J$, all $D_u$ are surjective (equivalently: $\overline{\partial}$ is transverse to the zero section).

Consider $\overline{\partial}$ for all choices of $J$ simultaneously, where $J$ can depend on $z \in S$. Let $\mathcal{J}$ denote the collection of all $J$, appropriately topologized as a Banach bundle.

Technical Remarks. The precise conditions on $J$ are listed in (1), (3), (5) of Appendix A.5. Call $J_0$ a fixed such $J$, and fix a large $s_\infty > 0$. Define $\mathcal{J}$ to be the set of such $J$ which agree with $J_0$ for $|s| \geq s_\infty$ on the ends (this is to ensure that a generic perturbation of $J_0$ within $\mathcal{J}$ will still satisfy those conditions). What regularity we demand from $J$ depends on how we want to topologize $\mathcal{J}$. To topologize, one first identifies $\mathcal{J}$ with a suitable vector subbundle of $\text{End}(TM)$-valued sections over $S$ (see [38, 4.2.10]). To make $\mathcal{J}$ into a Banach bundle, one can use Floer’s $C^\infty$ norm [38, 4.2.6–4.2.9 and 4.2.11], alternatively one can, as in [25, Section 3.4], define $\mathcal{J} = \mathcal{J}^\ell$ using the $C^\ell$ norm (for large $\ell \in \mathbb{N}$ so that the Sard–Smale theorem applies for a given index of the Fredholm map) and afterwards one proves that smooth regular $J \in \mathcal{J}^\ell$ are generic also in the (non-Banach) $C^\infty$-topology. The approach in [40, Section (8h)] and [4, Section (3b)] allows the $J \in \mathcal{J}$ to differ from $J_0$ also on the ends: it only requires the $J$ to converge to $J_0$ faster than exponentially on the ends. In this case, the perturbed $J$ may not satisfy the first condition in (5) of Appendix A.5 but, by appealing to Lemma D.1(3), it turns out that this is not problematic.

Linearizing at a solution $\overline{\partial}_J(u) = 0$:
\[ F : T_J \mathcal{J} \oplus W^{1,p}(u^*TM) \to L^p(\text{Hom}^{0,1}(TS, u^*TM)), \]
\[ F(Y, \xi) = \frac{1}{2} Y_{z,u(z)} \circ (du - X \otimes \beta)_{z,u(z)} \circ j_z + D_u \xi, \]
where $T_J \mathcal{J}$ is the tangent space (for example, see the approach in [25, Section 3.4]).
**Lemma A.5.** If $F$ is surjective, then there is a Baire second category subset $\mathcal{J}_{reg}(S, \beta) \subset \mathcal{J}$ of regular $J$ (those for which $D_u$ is surjective for all $u \in \mathcal{M}(x_a; y_b; S, \beta, J)$).

**Proof (Sketch).** This is a standard trick, for example, see [27, 3.1.5, p. 51]. One considers the universal moduli space $\mathcal{M}$ of all Floer solutions $u \in \mathcal{M}(x_a; y_b; S, \beta, J)$ for all $J \in \mathcal{J}$. Let $\pi : \mathcal{M} \to \mathcal{J}$ denote the projection. Using the surjectivity of $F$, one checks that the kernel and cokernel of $d\pi$ are isomorphic to those of $D_u$. Thus, $d\pi$ is Fredholm since $D_u$ is Fredholm; and $d\pi$ is onto if and only if $D_u$ is onto. Thus, the regular values of $\pi$ correspond to the regular $J$. By the Sard–Smale theorem the regular values of $\pi$ in $\mathcal{J}$ are of Baire second category.

**Lemma A.6.** If $du - X \otimes \beta \neq 0$ on a non-empty open subset of $S$, then $F$ is surjective.

**Proof.** This is a standard argument [38, 4.2.18]. By Theorem A.4, $D_u$ is Fredholm so the image has a finite codimension. Since $\text{Im} \, D_u \subset \text{Im} \, F$, the same is true for the image of $F$. Thus, it suffices to show that the orthogonal complement to the image $\text{Im} \, F$ is zero. Suppose by contradiction that there is an $\eta \neq 0$ in the dual space, $L^2(\text{Hom}^{0,1}(TS, u^*TM))$ for $1/q + 1/p = 1$, which is $L^2$-perpendicular to $\text{Im} \, F$.

Then $\eta$ is perpendicular to $\text{Im} \, D_u$; $\langle D_u \xi, \eta \rangle_{L^2} = 0$ for all $\xi$. So $\eta$ is a weak solution of $D^*_u \eta = 0$, and by elliptic regularity [38, 2.5.3] $\eta$ is smooth. By the Carleman similarity principle [21, Corollary 2.3], $\eta$ can vanish at most on a discrete set of points in $S$. Since $\eta$ is perpendicular to $\text{Im} \, F$ and $\langle D_u \xi, \eta \rangle_{L^2} = 0$

$$0 = \langle F(Y, \xi), \eta \rangle_{L^2} = \frac{1}{2} \langle Y(du - X \otimes \beta) \circ \cdot j, \eta \rangle_{L^2}$$

for all $Y, \xi$.

Pick $z_0 \in S$ in the open subset where $du - X \otimes \beta \neq 0$ such that $\eta_{z_0} \neq 0$. Then, by [37, 8.1], one can pick a vector $Y$ at $(z_0, u(z_0)) \in S \times M$ such that $g(Y(du - X \otimes \beta)j, \eta) > 0$. Extend $Y$ locally and multiply it by a cut-off function which depends on $z \in S$ to make $Y$ globally defined so that $g(Y(du - X \otimes \beta)j, \eta) \geq 0$ everywhere and $> 0$ near $z_0$. This contradicts the vanishing of the above $L^2$ inner product.

**Technical Remark.** In the previous Technical Remarks, we imposed that the $J \in \mathcal{J}$ all agree for $|s| \geq s_\infty$ on the ends, so we cannot pick $Y \neq 0$ there. So in the above argument, one needs to show that one can find a $z_0$ in the complement of the $|s| \geq s_\infty$ regions such that $(du - X \otimes \beta)_{z_0} \neq 0$. By contradiction, suppose $du - X \otimes C \partial t = 0$ for $s_\infty - 1 < |s| < s_\infty$ (there $\beta = C \partial t$ for some $C > 0$) and that no such $z_0$ existed in $|s| \geq s_\infty$. This equation is equivalent to $\partial_s u = 0$ and $\partial_t u - X_{\partial t} = 0$, and it is equivalent to $v = 0$ where $v(s, t) = \varphi^{-1}_{C, H} u(s, t))$. But $v$ is a pseudo-holomorphic strip (cf. Remark 4.4), so it has isolated zeros unless it is identically zero. So $v = 0$ on $s_\infty - 1 < |s| < s_\infty$ implies $v = 0$ on $|s| \geq s_\infty$, so $du - X \otimes \beta = 0$ on $|s| \geq s_\infty$, a contradiction, see [38, 4.2.13, 4.2.15] for details on this argument.

**Lemma A.7.** Combining the previous two lemmas, we deduce that transversality holds unless $du - X \otimes \beta = 0$ on all of $S$ (since being non-zero is an open condition). By making a generic time-dependent perturbation of $H$ at each end of $S$, we can ensure that $du - X \otimes \beta = 0$ is impossible for all Floer solutions $u$.

**Proof.** After a generic time-dependent perturbation of $H$ at each end of $S$, the asymptotics of $u$ can never be constant orbits (since Crit($H_t$) will change in time, generically). Suppose $du - X \otimes \beta \equiv 0$. Then $du(TS) = X \otimes \beta(TS) \subset \text{span}(X)$, so the image of $u$ lands inside a Hamiltonian orbit $v$ of $H$. When $S$ is a disc $(p = 1, q = 0)$, $u(S) \subset v(S^1)$ means that $v$ is contractible within $v(S^1)$ and hence constant, which we ruled out. In the general case, since
Figure A.1. Floer solutions $u_n \in \mathcal{M}(x_1, x_2, x_3; y_1, y_2; S, \beta)$ converging to a broken solution $v \# u \in \mathcal{M}(x_2, z) \times \mathcal{M}(x_1, z, x_3; y_1, y_2; S, \beta)$.

Figure A.2. Floer solutions $u_\lambda$ converging to a broken solution $u \# v$.

we choose the time-dependent perturbations to be different on each end, we can ensure that $v$ is not a common orbit of the perturbed Hamiltonians even up to rescaling.

Remark A.8. If the $A_a$ and $B_b$ of $S$ are $\mathbb{Q}$-independent, we can make the same perturbation of $H$ at each end (in the proof: $v$ has periods $A_a$ and $B_b$ so it would be constant).

A.7. Compactness of the Moduli spaces

We claim that the $\mathcal{M}(x_a; y_b; S, \beta)$ have compactifications whose boundaries consist of broken Floer solutions (Figures A.1 and A.2). Since the complex surface $(S, j)$ is fixed, away from the ends the energy estimate forces $C^\infty$ local hence uniform convergence to a solution, thus breaking of Floer solutions can only occur at the cylindrical ends. On those ends, the equation turns into Floer’s equation so the breaking is analytically identical to the breaking of Floer trajectories.

Remark. We emphasize that we keep the complex structure $j$ fixed on $S$. One could in fact allow $(S, j)$ to vary in certain families [39, Section (8a)], and one could also allow the surface to degenerate to a node in regions where $\beta = 0$ (so the gluing theorem for $J$-holomorphic curves applies), but in this paper we will avoid these additional complications.

Details of the construction of compactifications for these moduli spaces can be found in [38, 4.3.21, 4.4.1]. Schwarz’s work for closed symplectic manifolds $M$ applies here because of the following conditions.

1. All $u \in \mathcal{M}(x_a; y_b; S, \beta)$ lie in the compact subset $R \subseteq \max(R(x_a), R(y_b), R_0)$ of $\overline{M}$, by Appendix D.3.
2. They satisfy an a priori energy estimate by Appendix A.3.
3. No bubbling off of $J$-holomorphic spheres occurs since $\omega = d\theta$ is exact.
Lemma A.9. Breaking: Suppose $u_n \in \mathcal{M}_1(x_n; y_n; S, \beta)$ has no $C^\infty$-convergent subsequence. Then, passing to a subsequence, $u_n$ converges $C^\infty$ locally to an isolated Floer solution $u$ defined on the same $(S, \beta)$ but with one of the asymptotics $w$ among $x_n$ and $y_n$ changed to a new asymptotic $z$ at an end. On that end of $S$, there are reparametrizations $u_n(\cdot + s_n \cdot) \to v$ converging $C^\infty$ locally to an isolated Floer trajectory for $C \cdot H$ as $s_n \to \pm \infty$ with asymptotics $w$ and $z$ (where $\pm$ is the sign of that end of $S$ and $C$ is the weight for that end).

Gluing theorem: given such isolated solutions $v$ and $u$, for large $\lambda \in \mathbb{R}$, there is a smooth 1-parameter family $u_\lambda \in \mathcal{M}_1(x_n; y_n; S, \beta)$ converging to the broken trajectory $v \# u$ or $u \# v$ in the above sense (respectively, for breakings at negative or positive ends).

Index: the Fredholm index of solutions is additive with respect to gluing, so the local dimension of moduli spaces is additive (for Floer trajectories count dim $\mathcal{M}$, see Subsection 2.3).

A.8. Parametrized moduli spaces

Given two sets of regular data $(S, j_0, \beta_0, J_0)$ and $(S, j_1, \beta_1, J_1)$ interpolating between them (while keeping the same data $j, \beta$ and $J$ near the ends). This is always possible since the choices of $j, \beta$ and $J$ form a contractible set. Again one proves that for a generic choice of the interpolation $J_\lambda$ the relevant operator $\bar{\partial}$ is transverse to the zero section and thus the moduli space $\bigcup \mathcal{M}(x_n; y_n; S, \beta_\lambda, J_\lambda)$ is smooth. The compactness results of Appendix A.7 also carry over to this setup \cite{38, 5.2.3}. Indeed, near the ends we are not varying $j, \beta$ and $J$ so the same breaking of solutions applies, and away from the ends the energy estimate forces $C^\infty$-local hence uniform convergence.

A.9. TQFT operations

Define

$$\psi_S : SC^*(B_1 H) \otimes \ldots \otimes SC^*(B_q H) \to SC^*(A_1 H) \otimes \ldots \otimes SC^*(A_p H)$$

on the generators by counting isolated Floer solutions

$$\psi_S(y_1 \otimes \ldots \otimes y_q) = \sum_{u \in \mathcal{M}_0(x_1, \ldots, x_p; y_1, \ldots, y_q; S, \beta)} \epsilon_u x_1 \otimes \ldots \otimes x_p,$$

where $\epsilon_u \in \{ \pm 1 \}$ are orientation signs (Appendix B). Then extend $\psi_S$ linearly. When gradings are defined, degree$(\psi_S) = \sum |x_n| - \sum |y_n| = -2n(1 - g - p)$ by Appendix A.5.

Theorem A.10. The $\psi_S$ are chain maps and, using the fact that we work over a field (see Subsection 6.6), they yield maps $\psi_S : \bigotimes_{b=1}^t SH^*(B_b H) \to \bigotimes_{a=1}^p SH^*(A_n H)$ which are independent of the choice of the data $(\beta, j, J)$ relative to the ends.

Proof. To prove that $\psi_S$ is a chain map consider the 1-dimensional $\mathcal{M}_1(x_n; y_n; S, \beta)$. A sequence of Floer solutions approaching the boundary will break giving rise to an isolated Floer trajectory at one of the ends (Appendix A.7, Figure A.1). This is precisely the definition of the Floer differential $\bar{\partial}$ on the tensor products (Appendix B.15). Since the oriented count of the boundary components of a compact 1-dimensional manifold is zero, and after checking orientation signs in Appendix B.15, we conclude that $\psi_S \circ \bar{\partial} = \bar{\partial} \circ \psi_S$. Thus $\psi_S$ descends to cohomology: $H^*(\bigotimes_{b} SC^*(B_b H)) \to H^*(\bigotimes_{a} SC^*(A_n H))$. Since we work over a field, we can apply the Künneth theorem to obtain $\psi_S : \bigotimes_{b} SH^*(B_b H) \to \bigotimes_{a} SH^*(A_n H)$.

To prove independence of the auxiliary data $(\beta, j, J)$ consider the 1-dimensional part of $\bigcup_{0 \leq \lambda \leq 1} \mathcal{M}(x_n; y_n; S, \lambda)$ as in Appendix A.6, where $S, \lambda = (S, \beta_\lambda, J_\lambda, J_\lambda)_{0 \leq \lambda \leq 1}$ is a regular homotopy of the data $(\beta, j, J)$ relative ends (in particular, the $A_n, B_b$ are fixed). We show
that \( \psi_{S_0} \) and \( \psi_{S_1} \) are chain homotopic, so the claim follows. We run the usual invariance proof of Floer homology [35, Lemma 3.12]. The boundaries of the moduli spaces are of two types: either \( \lambda \to 0 \) or 1, which, respectively, yield contributions to \( -\psi_{S_0} \) and \( \psi_{S_1} \), or \( \lambda \to \lambda_0 \in (0, 1) \).

In the latter case, the boundary is a broken solution consisting of a Floer trajectory and a Floer solution in \( \mathcal{M}_{-1} \), where \( \mathcal{M}_{-1} = \mathcal{M}_{-1}(x_{a}; y_{b}; S_{\lambda}) \) has asymptotics \( x_{a} \) and \( y_{b} \) equal to the \( x_{a} \) and \( y_{b} \) except for one asymptotic which has changed (because the Floer trajectory broke off).

Generically \( \mathcal{M}_{-1} \) is empty since it has virtual dimension \( -1 \), however at finitely many \( \lambda \), there may actually exist a solution: call these the unexpected solutions.

Let \( K \) denote the oriented count of the unexpected solutions. Then the broken solutions described above contribute to \( K \circ \partial \) or \( \partial \circ K \) depending on whether the end where the breaking occurs is a positive or a negative end respectively (and where \( \partial \) is acting on \( \bigotimes_{a} SC^{*}(B_{b}H) \) or \( \bigotimes_{a} SC^{*}(A_{a}H) \) respectively, see Appendix B.15). The oriented count of the boundaries of a compact 1-manifold is zero, so \( -\psi_{S_0} + \psi_{S_1} + K \circ \partial + \partial \circ K = 0 \). So \( K \) is the required chain homotopy.

\[ \square \]

**A.10. Gluing surfaces yields compositions of operations**

Two surfaces \( S \) and \( S' \) as in Appendix A.1 can be glued along opposite ends carrying equal weights. Pick a gluing parameter \( \lambda \gg 0 \). Suppose a positive end \( [0, \infty) \times S^{1} \) of \( S \) and a negative end \( (-\infty, 0] \times S^{1} \) of \( S' \) carry the same weight. Cut off \( (\lambda, \infty) \times S^{1} \) from \( S \) and \( (-\infty, -\lambda) \times S^{1} \) from \( S' \). Glue what remains of the ends \( [0, \lambda] \times S^{1} \cup [-\lambda, 0] \times S^{1} \) via \( (\lambda, t) \sim (-\lambda, t) \). As the weights agree, for \( \lambda \gg 0 \), we get a 1-form \( \beta_{\lambda} \) on the glued surface \( S_{\lambda} \) from the \( \beta \)-forms on the \( S \) and \( S' \) parts of \( S_{\lambda} \). The result is a 1-family of surfaces \( S_{\lambda} = S \#_{\lambda} S' \). Similarly, one can glue several pairs of ends with matching weights.

**Lemma A.11.** Let \( \mathcal{M} \subset \mathcal{M}_{1}(x_{a}; y_{b}; S, \beta) \) be a connected component. Near \( \partial \mathcal{M} \), \( \mathcal{M} \) is parametrized by Floer solutions \( u_{\lambda} \) defined on a gluing \( S_{\lambda} = S \#_{\lambda} S' \) (infinite cylinder, \( \beta = C dt \)). As \( \lambda \to \infty \), \( u_{\lambda} \) converges to a broken solution, broken at the gluing end.

**Proof.** Observe Figure A.2. We will deal with the case of a breaking \( u \# v \) on a positive end as \( \lambda \to \infty \), the other case is analogous. By Lemma A.9, the Floer solutions in \( \mathcal{M} \) near \( u \# v \) are parametrized by a gluing parameter \( \lambda \gg 0 \). Indeed, by an implicit function theorem argument, they can be described by a gluing construction which ‘glues’ \( u(s, t)|_{s \leq \lambda} \) with \( v(s - 2\lambda, t)|_{s \geq \lambda} \). In particular, the glued solution \( u_{\lambda} \) will converge \( C^{\infty} \) locally to \( u \), and on the breaking end the rescaled map \( u_{\lambda}(s + 2\lambda, t) \) will converge \( C^{\infty} \) locally to \( v(s, t) \). The Hamiltonian orbit \( v(-\infty, t) \), where the breaking occurs is the limit of \( u_{\lambda}(\lambda, t) \) (dotted line in Figure A.2). Let \( S_{\lambda} = S \#_{\lambda}(\mathbb{R} \times S^{1}) \) glued at the end where the breaking occurs, extending \( \beta = C dt \) to the cylinder. ‘Extend’ \( u_{\lambda} \) to \( S_{\lambda} \) so that it equals \( u_{\lambda} \) on the \( S \)-part and it equals \( u_{\lambda}(s + 2\lambda, t) \) at \( (s, t) \in [-\lambda, \infty) \times S^{1} \).

This is well defined since we identified \( (\lambda, t) \sim (-\lambda, t) \).

\[ \square \]

**Theorem A.12.** Let \( S_{\lambda} = S \#_{\lambda} S' \) with all positive ends of \( S \) glued to all negative ends of \( S' \) in the given order. Then on the chain level \( \psi_{S_{\lambda}} = \psi_{S} \circ \psi_{S'} \) for \( \lambda \gg 0 \).

**Remark A.13.** If we do not glue all ends, the composite \( \psi_{S} \circ \psi_{S'} \) is ill-defined. The remedy is to take disjoint unions with cylinders \( Z = (\mathbb{R} \times S^{1}, \beta = C dt) \) (which induce the identity map by Lemma 3.1(3)) so that the resulting surfaces can be fully glued.

**Proof.** \( \psi_{S} \circ \psi_{S'} \) counts isolated broken Floer solutions \( u \# u' \) which broke at the ends where \( S \) and \( S' \) get glued. Arguing as in Lemmas A.9 and A.11, there is a 1-parameter family \( u_{\lambda} \) of Floer solutions on \( S_{\lambda} \) parametrizing the Floer solutions close to \( u \# u' \).
Conversely, as \( \lambda \to \infty \) a 1-parameter family of isolated Floer solutions \( u_\lambda \) on \( S_\lambda \) must converge to a broken solution counted by \( \psi_S \circ \psi_{S'} \). Indeed, the \( u_\lambda \) have a priori bounded energy depending on the fixed asymptotics (see Appendix A.3), so, by Theorem A.3, they must break at the stretched ends. Since the \( u_\lambda \) are isolated, for dimensional reasons they can only break once on each stretched end, and they cannot break at the non-stretched ends.

Since the Hamiltonian is linear at infinity, the relevant chain complexes are finitely generated and the union of the moduli spaces defining the operations \( \psi_{S_\lambda} \), \( \psi_S \) and \( \psi_{S'} \) is finite. So, for large \( \lambda \), there is a bijection between the moduli spaces counted by \( \psi_{S_\lambda} \) and those counted by \( \psi_S \circ \psi_{S'} \). We prove in Appendix B.14 that the solutions are counted with the same orientation signs. Thus, \( \psi_{S_\lambda} = \psi_S \circ \psi_{S'} \) for large \( \lambda \).

\( \square \)

A.11. TQFT operations on \( SH^*(M) \)

**Theorem A.14.** There is a commutative diagram,

\[
SH^*(B_1 H) \otimes \ldots \otimes SH^*(B_q H) \xrightarrow{\psi_S} SH^*(A_1 H) \otimes \ldots \otimes SH^*(A_p H)
\]

\[
\xrightarrow{\text{continuation}} \quad \varphi
\]

\[
SH^*(B'_1 H') \otimes \ldots \otimes SH^*(B'_{q'} H') \xrightarrow{\psi_{S'}} SH^*(A'_1 H') \otimes \ldots \otimes SH^*(A'_{p'} H')
\]

\[
\xrightarrow{\text{continuation}} \quad \varphi'
\]

where the surfaces \( S', S \) are equal but \( \beta' \) and \( \beta \) may differ; for \( \psi_{S'} \), we use \( H' \) instead of \( H \); the vertical maps are tensors of monotone continuations (so \( B_m \leq B'_m, A_m \leq A'_m \) where \( m \) and \( m' \) are the slopes of \( H \) and \( H' \) at infinity; and \( \sum A_a \geq \sum B_b, \sum A'_a \geq \sum B'_b \) by Lemma A.1).

**Proof.** Suppose first \( H' = H \). Then \( \varphi = \psi_{\bigcup_a Z_a}, \varphi' = \psi_{\bigcup_a Z_a} \), where \( \bigcup_b Z_b, \bigcup_a Z_a \) are disjoint unions of continuation cylinders (Example A.2). By Theorem A.12, \( \varphi' \circ \psi_S = \psi_{\bigcup_a Z_a} \# S \) and \( \psi_{S'} \circ \varphi = \psi_{S'} \# (\bigcup_b Z_b) \) (using large glueing parameters). But the gluings \( (\bigcup_a Z_a) \# S \) and \( S' \# (\bigcup_b Z_b) \) consist of the same topological surface, namely \( S \), and although their data (\( \beta, j \)) differ, the data agree on the ends (the weights are \( A'_a, B'_b \)). Therefore, by Theorem A.10, the two maps are chain homotopic, and hence \( \varphi' \circ \psi_S = \psi_{S'} \circ \varphi \) on cohomology.

Now suppose \( H' \neq H \). The difficulty is that a gluing argument as above would end up with completely different Hamiltonians on the glued surface \( S \). So the argument is more subtle.

Observe that by rescaling \( \beta, A_a, B_b \) by \( m \) and \( H \) by \( 1/m \), we can assume that \( H \) has slope 1, and similarly for \( \beta', A'_a, B'_b, H' \). So \( H \) and \( H' \) now have the same slope. Consider the diagram

\[
\begin{array}{ccc}
\bigotimes_b SH^*(B_b H) & \xrightarrow{\psi_S} & \bigotimes_a SH^*(A_a H) \\
\downarrow \varphi & & \downarrow \varphi' \\
\bigotimes_b SH^*(B_b H') & \xrightarrow{\psi_{S'}} & \bigotimes_a SH^*(A'_a H')
\end{array}
\]

By Lemma 3.1(2), the two outer triangles of continuation maps commute, and the two diagonal continuations are isomorphisms by Lemma 3.1(4) since the slopes agree. By the proof in the case \( H = H' \), the composites of the horizontal maps are again TQFT operations \( \psi_S \) and \( \psi_{S'} \) (after modifying \( \beta \) and \( \beta' \)). Therefore, to prove the Theorem, we may assume that \( A_a = A'_a, B_b = B'_b \) and that \( \varphi \) and \( \varphi' \) are compatible. So it suffices to prove that \( \psi_S = (\varphi')^{-1} \circ \psi_{S'} \circ \varphi \).

By Theorem A.12, \( (\varphi')^{-1} \circ \psi_{S'} \circ \varphi = \psi_{S''} \) for the gluing \( S'' = (\bigcup_a Z_a) \# S' \# (\bigcup_b Z_b) \), where on the cylinders \( Z_a \) and \( Z_b \) we use \( \beta = A_a dt \) and \( \beta = B_b dt \) and we use monotone Hamiltonians \( H_z \) and \( H_{-z} \) which depend on the cylinders’ coordinate \( z \) and which interpolate \( H, H' \) and \( H', H \) respectively. Note that since \( H \) and \( H' \) have the same slopes, we can ensure \( \partial_z H_z = 0, \partial_z H_{-z} = 0 \) for \( R \gg 0 \), so by Lemma D.1(2) the maximum principle holds for \( S'' \).
Now observe that $S''$ and $S$ involve the same data near the ends (namely weights $A_a$ and $B_b$) and also involve the same Hamiltonian $H$ near the ends. Therefore, a parametrized moduli space argument as in Appendix A.8, and mimicking the proof of Theorem A.10, shows that $\psi_{S''}$ and $\psi_S$ are chain homotopic. Thus, on cohomology, $\psi_S = (\varphi')^{-1} \circ \psi_{S'} \circ \varphi$. \hfill \Box

The diagram in Theorem A.14 for $H' = H$ proves that the direct limit of the maps $\bigotimes B SH^*(B_bH) \to \bigotimes A SH^*(A_aH)$ as $A_a \to \infty$ is defined, yielding $\bigotimes SH^*(B_bH) \to SH^*(M)^{\otimes p}$. The resulting diagram, after letting $A_a = A_a' \to \infty$, shows that the direct limit of the maps $\bigotimes B SH^*(B_bH) \to SH^*(M)^{\otimes p}$ as $B_b \to \infty$ is defined. This defines the map $\psi_S: SH^*(M)^{\otimes q} \to SH^*(M)^{\otimes p}$ ($p \geq 1, q \geq 0$).

Theorem A.10 proves that $\psi_S$ depends on only $p$ and $q$ and the genus of $S$ (for $S$ connected), and not on $\beta, j, J$. Theorem A.14 proves that $\psi_S$ does not depend on the choice of $H$. Theorem A.12 proves that the $\psi_S$ satisfy the TQFT axioms, see Subsection 1.1 (the axiom $\psi_Z = \text{id}$ follows by category theory since the $\psi_Z: SH^*(A_aH) \to SH^*(A_{a'})H$ define $SH^*(M) = \lim \limits_{A_a \to \infty} SH^*(A_aH))$.

A.12. Invariance of the TQFT structure on $SH^*(M)$

**Theorem A.15.** For a symplectomorphism $\varphi: \mathcal{M} \cong \mathcal{N}$ of contact type (see Section 3.5),

$$\varphi^{\otimes p} \circ \psi_{S,M} = \psi_{S,N} \circ \varphi^{\otimes q},$$

where $\psi_{S,M}$ and $\psi_{S,N}$ are the $\psi_S$ operations for $SH^*(M)$ and $SH^*(N)$ respectively.

So the isomorphism $\varphi_* : SH^*(M) \to SH^*(N)$ respects the TQFT.

**Proof.** Recall from Section 3.5 that instead of working with two manifolds $M$ and $N$, we can just work with $M$ and two choices of data $H, J$ and $\varphi^* H, \varphi^* J$. We abbreviate $H_f = \varphi^* H$ and $J_f = \varphi^* J$. One needs some caution because $H_f$ is not linear for large $R$: $H_f(R) = h(e^{-f(y)}R)$ for $f: \partial M \to \mathbb{R}$, using collar coordinates $(R, y) \in [1, \infty) \times \partial M$. We write $SH^*_f$ instead of $SH^*$ to remind ourselves that we use Hamiltonians of that form, and that we use almost complex structures for which the contact type condition $J_f^{\prime\prime} d\theta = dR_f$ for $R_f = e^{-f(y)} R$.

By [32, Lemma 7], after fixing a homotopy $(f_s)_{s \in \mathbb{R}}$ from $0$ to $f$ (independent of $s$ for large $|s|$), there is a constant $K > 0$ (depending only on a $C^2$-bound for the homotopy $f_s$) such that if the weights $B', B > 0$ satisfy $B' \geq KB$, then one can define the continuation map $\varphi_* : SH^*(BH) \to SH^*_f(B'H_f)$. The proof in fact shows that there is a homotopy $(H_s, J_s)$ from $(B'H_f, J_f)$ to $(BH, J)$ for which the maximum principle holds for continuation solutions. Similarly, for a homotopy $f_s$ from $0$ to $f$, and $A \geq KA'$, we get $\psi_* : SH^*_f(A'H_f) \to SH^*(AH)$.

We will call special continuation cylinders the domains and the auxiliary data used to construct these two continuation maps.

By [32, Theorem 8], a direct limit over these special continuations defines mutually inverse isomorphisms $SH^*(M) \to SH^*_f(M)$ and $SH^*_f(M) \to SH^*(M)$ (and recall that $SH^*_f(M)$ is identifiable with $SH^*(N)$ by pull-back via the symplectomorphism $\varphi$).

We now want to construct a commutative diagram of the form

$$
\begin{array}{ccc}
\bigotimes B SH^*(B_bH) & \xrightarrow{\psi_S} & \bigotimes A SH^*(A_aH) \\
\downarrow \varphi^{\otimes q} & & \uparrow \varphi^{\otimes p} \\
\bigotimes B SH^*_f(B'_bH_f) & \xrightarrow{\psi_{S,f}} & \bigotimes A SH^*_f(A'_aH_f).
\end{array}
$$

In the diagram, the vertical maps are special continuations which are defined provided that $A_a \geq KA_a'$ and $B'_b \geq KB_b$, and the horizontal maps are defined provided that $\sum A_a \geq \sum B_b$ and $\sum A'_a \geq \sum B'_b$ (Lemma A.1). The surfaces $S$ and $S_f$ are the same (but the $\beta$ forms may
orientations were first constructed by Floer and Hofer. We emphasize that coherent orientations are not unique, since the construction involves choices.

The composite \( \psi_S \circ \varphi_S \circ \varphi_s \) corresponds to gluing special continuation cylinders onto the ends of the surface \( S_f \). The gluing determines auxiliary data \((S, \beta, H, J)\) and \((S, \beta, H_f, J_f)\). We choose this interpolation so that on the collar of \( \partial \overline{M} \) and away from the ends of \( S \) we have: \( H_\lambda(R) = h(R_{\lambda f}) \) and \( J_\lambda d\theta = dR_{\lambda f} \), where \( R_{\lambda f} = e^{-\lambda f(y)} R \). Observe that only near the ends of \( S \) the function \( f_s \) depends on the coordinate \( z = s + it \) of \( S \), namely where the special continuation cylinders got glued. So on the ends of \( S \), we ensure that those same formulas for \( H_\lambda \) and \( J_\lambda \) hold on the collar of \( \partial \overline{M} \) with \( f \) replaced by \( f_s \).

Technical Remark. This last observation is important because it means that we do not need to prove a new maximum principle for Floer solutions for \( S_\lambda \) (for fixed \( \lambda \)): away from the ends, the maximum principle holds since \( f \) does not depend on \( z \in S \), and on the ends the maximum principle holds because it holds for special continuation cylinders defined using \( \lambda f_s \).

The 1-parameter family argument in the proof of Theorem A.10 then defines a chain homotopy \( K \) such that \(-\psi_S + \psi_S^\# \circ \psi_S \circ \varphi_s^\# + K \circ \partial + \partial \circ K = 0\) at the chain level, and so \( \psi_S = \psi_S^\# \circ \psi_S \circ \varphi_s^\# \) on cohomology. Thus, the above diagram commutes, as required. \( \Box \)

Appendix B. Coherent orientations

B.1. Coherent orientations for Floer trajectories

A coherent orientation for the moduli spaces \( \mathcal{M}(x, y) \) of Subsection 2.3 is a continuous map associating to \( u \in \mathcal{M}(x, y) \) an orientation \( \sigma(u) \) of the determinant line bundle of the operator \( D_u \) (the linearization of Floer’s equation),

\[
\text{Det } D_u = \Lambda^{\text{max}} \ker D_u = \Lambda^{\text{max}} T_u \mathcal{M}(x, y),
\]

such that the orientations glue correctly: there is an associative gluing operation \( \# \) such that \( \sigma(u) \# \sigma(v) = \sigma(u \# v) \) for \( \lambda > 0 \) (here \( u \# v \) is any family of Floer trajectories depending on a gluing parameter \( \lambda \in (\lambda_0, \infty) \) converging to the broken trajectory \( u \# v \) as \( \lambda \to \infty \)). Coherent orientations were first constructed by Floer and Hofer [20]. We recall the construction below. We emphasize that coherent orientations are not unique, since the construction involves choices.

B.2. Fredholm operators on trivial bundles over a cylinder

One first constructs coherent orientations for families of Fredholm operators of the form

\[
L : W^{1,p}(\text{triv}) \to L^p(\text{triv}), \quad Lu = \partial_s u + J_{s,t} \partial_t u + A_{s,t} u
\]

on the trivial vector bundle \( Z = \mathbb{C}^n \to Z \) over the cylinder \( Z = \mathbb{R} \times S^1 \) (compactified appropriately by adding two circles \( S^1_\pm \) at infinity), such that \( A_{s,t} \in \text{End}(\text{triv}) \) converges to self-adjoint isomorphisms \( A_{s,\infty,t} \) and \( L_\pm = J_{s,\infty,t} \partial_t + A_{s,\infty,t} \) as \( s \to \infty, \) and \( p > 2 \).

Once the asymptotic operators \( L^\pm \) at the ends are fixed, the family \( \mathcal{O}_{\text{triv}}(L^-, L^+) \) of such operators is a contractible (indeed convex) set [20, Proposition 7]. Therefore, the real line bundle \( \text{Det}(\mathcal{O}_{\text{triv}}(L^-, L^+)) \to \mathcal{O}_{\text{triv}}(L^-, L^+) \) of all determinants

\[
\text{Det}(L) = (\Lambda^{\text{max}} \text{coker } L)^* \otimes \Lambda^{\text{max}} \ker L \quad \text{(over } \mathbb{R})
\]

is trivial, and an orientation \( \sigma(L) \) is a choice of trivialization.
B.3. Gluing the operators and coherent orientations

Operators $L$ and $K$ which become constant, respectively, for $s \gg 0$ and $s \ll 0$, with $L^+ = K^-$, can be glued $L\#K$ to yield a natural gluing isomorphism $\det L\#K \cong \det L \times \det K$ (see [20, Proposition 9]), which we now explain.

**Explanation.** Suppose $L$ and $K$ are surjective (in general, one uses a stabilization trick to reduce to this situation [20, above Proposition 9]). Then it is enough to construct a natural linear isomorphism $\psi : \ker L \oplus \ker K \to \ker(L\#K)$ and to show that $L\#K$ is surjective, since then the top exterior power of $\psi$ gives the required isomorphism of the determinants (note that the cokernels vanish). The gluing $L\#K$ actually depends on a parameter $\lambda > 0$:

$$(L\#K)(s) = L(s + 2\lambda)$$

for $s \leq -\lambda$ and $$(L\#K)(s) = K(s - 2\lambda)$$

for $s \geq \lambda$, so, for large $\lambda$, this patches correctly with $L\#K$ since then the top exterior power of $\psi$ gives the required isomorphism of the determinants.

Finally, we explain the construction of $\psi$. From $(u, v) \in \ker L \oplus \ker K$, one constructs the shifted sum $(u_{2,\lambda} \# v_{-2,\lambda})(s) = u(s + 2\lambda) + v(s - 2\lambda)$. Then $\psi(u, v) \in \ker L\#K$ is defined by orthogonally projecting $u_{2,\lambda} \# v_{-2,\lambda}$ onto $\ker L\#K$. The proof that $L\#K$ is injective on $(\det \psi)^+_{\lambda}$ for large $\lambda$ is an argument by contradiction which requires some analysis [20, Proposition 9].

Define $\sigma(L\#K)$ by gluing two given orientations $\sigma(L)\#\sigma(K)$ via the gluing map

$$O_{\text{triv}}(L^+, L^+) \times O_{\text{triv}}(L^+, K^+) \to O_{\text{triv}}(L^-, K^+).$$

Now construct a coherent orientation as follows. Fix an asymptotic operator $L_0$ of the type that arises at $s = -\infty$. Choose the orientation $1^\vee \otimes 1 \in \mathbb{R}^\vee \otimes \mathbb{R}$ for the isomorphism $L = \partial_s + L_0$. So, we get an orientation on $O_{\text{triv}}(L_0, L_0)$. Pick any orientation of $O_{\text{triv}}(L_0, L^+)$ for all $L^+ \neq L_0$. Coherence requires the orientations on $O_{\text{triv}}(L_0, L^+)$ and $O_{\text{triv}}(L_0^+, L_0)$ to determine that on $O_{\text{triv}}(L_0, L_0)$ via gluing, so we deduce an orientation for $O_{\text{triv}}(L^+, L_0)$. Similarly orientations on $O_{\text{triv}}(L_0^-, L^-)$, $O_{\text{triv}}(L^-, L^+)$ and $O_{\text{triv}}(L^+, L_0)$ determine that on $O_{\text{triv}}(L_0, L_0)$, so we deduce an orientation for $O_{\text{triv}}(L^-, L^+)$. The construction is well-defined since gluing $1^\vee \otimes 1$ with itself yields $1^\vee \otimes 1$.

B.4. Fredholm operators for a bundle $E \to \overline{M}$

Let $E \to \overline{M}$ be any symplectic vector bundle. Consider all operators $L$ on $u^*E$ for smooth maps $u : Z \to \overline{M}$ such that $L$ is an operator of the type above in some (and hence any) symplectic trivialization of $u^*E$. The bundles $u^*E$ have the homotopy type of $S^1$ and $u^*E$ is trivial since $\pi_0(Sp(2n)) = 0$, but two choices of trivialization may be non-homotopic as $\pi_1(Sp(2n)) = \mathbb{Z}$. To overcome this issue, one classifies the operators into equivalence classes $[L, u]$ determined by the following data: the asymptotics $\hat{x}^\pm(t) = u(\pm \infty, t)$, the asymptotic $L^\pm$, and the Fredholm index $\text{Ind}(L)$ (when deciding whether two pairs belong to the same equivalence class, we compare the $L^\pm$ in a common symplectic trivialization of $(x^\pm)^*E$). We now explain how a choice of orientation for $L$ naturally determines an orientation for all operators in the class $[L, u]$ (see [20, Lemmas 13/15]).

**Explanation.** Given equivalent $(L^u, u)$ and $(L^v, v)$ (so they share the same $L^\pm$ over the common $x^\pm$ and have the same Fredholm index), we can pick trivializations $\phi : u^*E \cong Z \times \mathbb{C}^n$ and $\phi : v^*E \cong Z \times \mathbb{C}^n$ which agree over $x^\pm$. In the trivializations, the $L^u$ and $L^v$ determine the data $J_{s,t}^u, A_{s,t}^u$ and $J_{s,t}^v, A_{s,t}^v$ in $\text{End}(\mathbb{C}^n)$, and these data determine symplectic paths $\gamma^u_{\pm \infty}, \gamma^v_{\pm \infty}$ in $\text{Sp}(\mathbb{C}^n)$ by integrating the 1-dimensional equation system $\dot{\gamma}^u_{\pm}(t) = J_{s,\pm \infty} A_{\pm \infty, t} A_{\pm \infty, t}^\dagger \gamma^u_{\pm}(t)$, $\gamma_{\pm}(0) = \text{Id}$. The **Conley–Zehnder index** [37] is an integer-valued map defined on paths in
Sp(\mathbb{C}^n) joining the identity to an element which does not have eigenvalue 1. We recall three key properties:

1. two such paths are homotopic if and only if the indices equal;
2. under the multiplication action of \psi \in \pi_1(\text{Sp}(\mathbb{C}^n), \text{Id}) on such paths, the index changes by subtracting twice the Maslov index of \psi (using our conventions, see Remark 3.3);
3. the Fredholm index of an operator \( L \) as in Appendix B.2 is the difference of the Conley–Zehnder indices of the asymptotics: \(|\gamma^-_{\infty} - |\gamma^+_{\infty}|\).

In our situation: \( \gamma^-_{\infty} \equiv \gamma^+_{\infty} \) and the Fredholm indices of \( L^u \) and \( L^v \) equal. So the Conley–Zehnder indices of the paths \( \gamma_u \) and \( \gamma_v \) are equal, so \( \gamma^-_{\infty} \) and \( \gamma^+_{\infty} \) are homotopic, so \( \phi_{\infty,t}^{-1}(\phi_{\infty,t})^{-1} \) has zero Maslov index, so \( \phi_u^\infty \) and \( \phi_v^\infty \) are homotopic. So, by homotopying \( \phi^v \), we can assume that \( \phi^u \) and \( \phi^v \) agree over both ends \( x^\pm \), and since the asymptotics \( L^\pm \) agree, also the asymptotics \( L_{Cn}^\pm \) in the common trivializations over \( x^\pm \). Since \( O_{\text{triv}}(L_{Cn}, L_{Cn}^\pm) \) is contractible, a choice of orientation for \( L^u \) induces via \( \phi^u \) and \( \phi^v \) an orientation for \( L^v \) (and a similar homotopy argument shows that the induced orientation on \( L^v \) does not depend on the choices of \( \phi^u \) and \( \phi^v \)).

Now, we construct coherent orientations \( \sigma([L, u]) \) for all classes \([L, u]\). For each homotopy class \([S^1, M]\), pick a representative loop \( x_0 \) and an asymptotic operator \( L_0 \). Define \( \sigma(L_0, x_0) = 1^V \otimes 1 \) (viewing \( x_0 \) as an \( s \)-independent cylinder). Consider all \((L, u)\) such that both ends of \( u \) are \( x_0 \) and \( L^\pm = L_0 \). Then the index \( \text{Ind}(L) \in 2c_1(TM) / (\pi_2(M)) \subset \mathbb{Z} \) by the Riemann–Roch theorem [27, Appendix C]. If there is such an \((L, u)\) with non-zero index, then all such classes \([L, u]\) can be related by gluings using a class \([L_{\min}, u_{\min}]\) with minimal positive index, so it is enough to pick an orientation for \([L_{\min}, u_{\min}]\) to determine orientations for all such \([L, u]\). The remaining part of the construction of coherent orientations now proceeds as in Appendix B.3.

B.5. Definition of orientation signs for \( SH^*(M) \)

Apply Appendix B.4 to \( T\overline{M} \rightarrow M \) to get a coherent orientation \( \sigma \). Then \( \sigma \) is defined on all linearizations \( D_u \) of Floer’s equation along a Floer trajectory \( u \). An isolated Floer trajectory \( u \) has a natural orientation \( \partial_s u \) determined by its flow. Define the \( \epsilon_u \in \{\pm 1\} \) of Subsection 3.1 by

\[
\sigma(u) = \epsilon_u \cdot \partial_s u.
\]

For an isolated Floer continuation solution \( v \) define the \( \epsilon_v \in \{\pm 1\} \) of Subsection 3.2 by

\[
\sigma(v) = \epsilon_v \cdot (1^V \otimes 1).
\]

B.6. Using orientation signs to prove \( d \circ d = 0 \)

Let \( u \neq u' \in \mathcal{M}_0(x, y) \# \mathcal{M}_0(y, z) \) be a broken Floer trajectory lying at the boundary of a connected component \( \mathcal{M} \subset \mathcal{M}_1(x, z) \). Floer’s gluing map parametrizes a neighbourhood of this boundary by a gluing parameter \( \lambda \gg 0 \) as follows. It interpolates the two maps (gluing data)

\[
(u_\lambda = u(s + \lambda, t))|_{s \leq -1} \quad \text{and} \quad (u'_\lambda = u'(s - \lambda, t))|_{s \geq 1}
\]
to obtain an approximate solution to Floer’s equation, and via the implicit function theorem the approximate solution uniquely determines a genuine Floer solution \( u \# \lambda u' \in \mathcal{M} \). As \( \lambda \rightarrow \infty \), the 1-family \( u \# \lambda u' \) converges to the broken trajectory \( u \# u' \).

In general, given 1-dimensional vector spaces \( \mathbb{R}e \) and \( \mathbb{R}f \) oriented by the vectors \( e \) and \( f \), we canonically orient \( \mathbb{R}e \times \mathbb{R}f \) by \( (e, 0) \wedge (0, f) \), or equivalently by \( (e, -f) \wedge (e, f) \). So, using the notation \( \mathcal{M} \) of Subsection 2.3, \( \mathcal{M}(x, y) \times \mathcal{M}(y, z) \) is canonically oriented at \((u, u')\) by the pair of basis vectors

\[
((\partial_s u, -\partial_s u'), (\partial_s u, \partial_s u')).
\]
This agrees with the natural orientation of the domain \((\lambda_0, \infty) \times \mathbb{R}\) of \((\lambda, s)\) which parametrizes the gluing data \((u_\lambda', u_\lambda')\), since differentiating the data in \(\lambda\) and \(s\) yields the two vectors \((\partial_\lambda u_\lambda, -\partial_\lambda u_\lambda')\) and \((\partial_s u_\lambda, \partial_s u_\lambda')\), which agree with the orientation defined by the above pair.

Therefore, \((\partial_\lambda(u\#u'), \partial_s(u\#u'))\) is the orientation of \(\tilde{\mathcal{M}}(x, z)\) induced by the gluing map. Quotienting by the \(s\) translations, \(\partial_\lambda(u\#u')\) is the orientation on \(\mathcal{M}\) \((\cong\) interval\) induced by the gluing. Now, \(u\#u'\) approaches the boundary as \(\lambda\) increases, so the gluing map induces the orientation which points outward along the boundary of the 1-manifold \(\mathcal{M}\).

Therefore, if \(u\#u'\) and \(\tilde{u}\#\tilde{u}'\) are the two boundaries of \(\mathcal{M}\), then the gluing map has assigned opposite orientations to them, and we abbreviate this fact by \(\partial_s u\#\partial_s u' = -\partial_s \tilde{u}\#\partial_s \tilde{u}'\).

Now consider the \(\sigma\)-orientations. The \(u\#u'\) and \(\tilde{u}\#\tilde{u}'\) belong to the same component \(\mathcal{M}\), so they carry the same asymptotic operators and they have the same Fredholm index, so the \(\sigma\) orientations of \(u\#u'\) and \(\tilde{u}\#\tilde{u}'\) are identical. Using orientation signs to prove \(\phi\) says it is a chain map. Similarly, \(\partial_\lambda(u\#u')\) and \(\partial_\lambda(v\#v')\) agree with the orientation defined by the data in \(\phi\).

\[\text{B.7. Using orientation signs to prove } d \circ \varphi = \varphi \circ d\]

Here, \(\varphi\) is a continuation map (see Subsection 3.2) and \(d \circ \varphi = \varphi \circ d\) says it is a chain map. This equation arises from an oriented count of the breakings of the 1-dimensional moduli spaces \(\mathcal{M}^1_{\text{Floer}}(x, z)\) of Floer continuations. We will now prove that this equation indeed holds for the choices of signs \(\epsilon_u\) and \(\epsilon_v\) defined in Appendix B.5.

Let \(u\#v\) and \(v'\#u'\) be broken continuation solutions, where \(v\) and \(v'\) are Floer continuation solutions, \(u\) and \(u'\) are Floer trajectories. The orientations of \(u\#v\) and \(v'\#u'\) are defined by the orientations \(\partial_s \tilde{u}\) and \(\partial_s \tilde{u}'\) respectively. However, the gluing data are now

\[(u_{2\lambda}|_{s \leq -\lambda-1}, v|_{s \geq -\lambda+1}) \quad \text{and} \quad (v'|_{s \leq \lambda-1}, u'|_{s \geq \lambda+1}).\]

Differentiating in \(\lambda\): \((2\partial_\lambda u_{2\lambda}, 0), (0, -2\partial_\lambda u'_{2\lambda})\). So, for \(\lambda > 0\), the glued orientations are \(\partial_\lambda(u\#v)\) (outward-pointing near the boundary) and \(-\partial_\lambda(v'\#u')\) (inward-pointing).

Let \(\mathcal{M} \subset \mathcal{M}^1_{\text{Floer}}(x, z)\) be a 1-dimensional component whose boundaries are the broken solutions \(u\#v\) and \(\tilde{u}\#\tilde{v}\). According to the above gluing, they are both oriented in the outward direction, so they have opposite orientations, so we abusively write \(\partial_s u = -\partial_s \tilde{u}\).

The \(\sigma\)-orientations are coherent and constant on \(\mathcal{M}\) so \(\sigma(u)\#\sigma(v) = \sigma(u\#v) = \sigma(\tilde{u}\#\tilde{v}) = \sigma(\tilde{u})\#\sigma(\tilde{v})\) by definition. \(\sigma(u)\#\sigma(v) = \epsilon_u \epsilon_v \partial_s u, \sigma(\tilde{u})\#\sigma(\tilde{v}) = \epsilon_{\tilde{u}} \epsilon_{\tilde{v}} \partial_s \tilde{u}\). Using \(\partial_s u = -\partial_s \tilde{u}\), we get \(\epsilon_u \epsilon_v = -\epsilon_{\tilde{u}} \epsilon_{\tilde{v}}\). So these two breakings contribute cancelling contributions to \(d \circ \varphi\).

Similarly, if the boundaries of \(\mathcal{M}\) are \(v\#u\) and \(\tilde{v}\#\tilde{u}\), we get cancelling contributions to \(\varphi \circ d\). Finally, if the boundaries of \(\mathcal{M}\) are of different types, say \(u\#v\) and \(\tilde{v}\#\tilde{u}\), then the gluing map assigns equal orientations to them, so proceeding as above we get \(\epsilon_u \epsilon_v = \epsilon_{\tilde{u}} \epsilon_{\tilde{v}}\). So these breakings give equal contributions to \(d \circ \varphi\) and \(\varphi \circ d\). This completes the proof of \(d \circ \varphi = \varphi \circ d\).

\[\text{B.8. Coherent orientations for Floer solutions}\]

For TQFT operations, we need coherent orientations using smooth maps \(u : S \to \overline{M}\) on a punctured Riemann surface \(S\) with prescribed parametrizations on the cylindrical ends (appropriately compactified with asymptotic circles \(S^1_a\) and \(S^1_b\) at the ends). We mimic the construction of coherent orientations for symplectic field theory due to Bourgeois–Mohnke [9] which builds upon [16, 1.8].
B.9. Fredholm operators over punctured surfaces

Let $E \to S$ be a complex vector bundle with prescribed trivializations on the asymptotic circles. Consider operators

$$ L: W^{1,p}(E) \to L^p(\text{Hom}^{0,1}(TS, E)), \quad Lu \cdot Z = \nabla_Z u + J\nabla_J Z u + A_z(u) \cdot Z $$

which restrict on the cylindrical ends to an operator of the type in Appendix B.2. Here, $Z \in TS$; $A_z \in \text{Hom}(E, \text{Hom}^{0,1}(TS, E))$ depending on $z \in S$; and $J \in \text{End}(E)$ is the complex structure of the vector bundle $E$. Denote the space of such operators by

$$ \mathcal{O}_E(L_a; L_b) = \mathcal{O}_E(L_1, \ldots, L_p; L_1, \ldots, L_q), $$

where $L_a$ and $L_b$ are the asymptotic operators over the asymptotic circles, respectively, at the negative and positive ends. Just as in Appendix B.2, $\mathcal{O}_E(L_a; L_b)$ is a contractible space so the determinant bundle over $\mathcal{O}_E(L_a; L_b)$ is trivial and an orientation is a choice of trivialization.

B.10. Gluings and disjoint unions

Just as in Appendix B.3 there is a gluing operation: given $E \to S$ and $E' \to S'$ with matching trivializations over the respective punctures $y_c$ and $x'_c$, where we glue, we obtain a glued bundle $E'' \to S''$ and a gluing map

$$ \#: \mathcal{O}_E(L_a; L_b, L_{-c}) \times \mathcal{O}_{E'}(L_c, L_{a'}; L_{b'}) \to \mathcal{O}_{E''}(L_a, L_{a'}; L_b, L_{b'}), $$

where we always use the convention that $L_{-c}$ is an abbreviation for the reversed ordering $(\ldots, L_2, L_1)$ of the operators $(L_1, L_2, \ldots)$, so inductively in $c$ we are gluing on the $c$-end of the pair of asymptotics $L_{-c} = L_c$.

The disjoint union of bundles determines a natural isomorphism

$$ \text{Det} L \otimes \text{Det} L' \to \text{Det}(L \cup L'). $$

**Lemma B.1.** The two natural isomorphisms $\text{Det} L \otimes \text{Det} L' \to \text{Det}(L \cup L')$ and $\text{Det} L' \otimes \text{Det} L \to \text{Det}(L \cup L')$ differ by the sign $(-1)^{\text{ind} L \cdot \text{ind} L'}$.

**Proof.** We first illustrate this when $L$ and $L'$ are surjective with 1-dimensional kernels. Suppose orientations have been chosen for their determinants: $\text{Det} L = \ker L = \mathbb{R} e$, $\text{Det} L' = \ker L' = \mathbb{R} e'$. Here $e$ and $e'$ are sections of two bundles $E$ and $E'$ over some surfaces $S$ and $S'$, and we can naturally view them as elements in $\ker(L \cup L')$ by extending them to sections of $E \cup E' \to S \cup S'$ by defining them to be zero, respectively, over $S'$ and $S$. Then $\ker(L \cup L') = \mathbb{R} e + \mathbb{R} e'$ and $\text{Det}(L \cup L') = \Lambda^2(\ker L \cup L') = \mathbb{R} e \wedge e'$. The two isomorphisms in the claim are, respectively, induced by $e \otimes e' \mapsto e \wedge e'$ and $e' \otimes e \mapsto e' \wedge e = (-1)^{1-1} e \wedge e'$, so the sign is as predicted.

In general, we first stipulate more precisely what the isomorphism preceding the claim is. Abbreviate by $k, k', c$ and $c'$ the dimensions of the kernels and cokernels of $L$ and $L'$. Given an orientation $(f_1^c \wedge \ldots \wedge f^c_1) \otimes (e_1 \wedge \ldots \wedge e_k)$ for $\text{Det}(L)$, and similarly (using $f', e', c', k'$) an orientation for $\text{Det}(L')$, we first extend the $f, f', e$ and $e'$ sections over $S \cup S'$ as in the example above, and then we declare $(-1)^d(f_1^c \wedge \ldots \wedge f^c_1 \wedge f'^{c'}_1 \wedge \ldots \wedge f'^{c'}_1) \otimes (e_1 \wedge \ldots \wedge e_k \wedge e'_1 \wedge \ldots \wedge e'_{k'})$ to be the orientation of $\text{Det}(L \cup L')$, where $d = c' + k$ follows Koszul sign rules (cf. [40, Section (11a)] for a similar discussion). So the difference between the two isomorphisms in the claim is the Koszul sign arising from switching the order of the two brackets in

$$ (\Lambda^\text{max} \ker L \otimes \Lambda^\text{max} \ker L) \otimes (\Lambda^\text{max} \ker L' \otimes \Lambda^\text{max} \ker L'), $$

So the sign is: $(-1)^{(c+k)(c'+k')} = (-1)^{(k-c)(k'-c')} = (-1)^{\text{ind} L \cdot \text{ind} L'}$. \qed
Denote by \( \sigma(L) \cup \sigma(L') \) the orientation for \( L \cup L' \) induced from the orientations \( \sigma(L), \sigma(L') \) via the above natural isomorphism.

If a gluing between \( K \) and \( L \cup L' \) only involves gluing \( K \) with \( L \), then distributivity holds:

\[
\sigma(K) \# (\sigma(L) \cup \sigma(L')) = (\sigma(K) \# \sigma(L)) \cup \sigma(L').
\]

### B.11. Axiomatic construction of coherent orientations

If \( S \) has no punctures, then \( \mathcal{O}_E(\emptyset; \emptyset) \) contains a Cauchy–Riemann operator \( \bar{\partial} \), which is \( \mathbb{C} \)-linear. So \( \ker \bar{\partial} \) and \( \operatorname{coker} \bar{\partial} \) are complex, so they are canonically oriented. So we obtain a canonical orientation on \( \mathcal{O}_E(\emptyset; \emptyset) \).

Now consider the standard trivial bundles \( \text{triv} \to \mathbb{C} \). Denote the operators \( L \) on these bundles by \( D^\pm \) depending on whether infinity is a positive or negative puncture for \( \mathbb{C} \). Define an orientation on each \( \mathcal{O}_{\text{triv} \to \mathbb{C}}(L^-; \emptyset) \) by fixing an operator \( D^- \) and picking an orientation for it. This determines an orientation for any \( D^+ \) by coherence: gluing an appropriate \( D^- \) forces \( \sigma(D^+) \# \sigma(D^-) \) to be the canonical orientation.

Since we established orientations, for any \( D^-_i \in \mathcal{O}_{\text{triv} \to \mathbb{C}}(L^-_i; \emptyset) \), we can define

\[
\sigma(D^-_1 \cup \ldots \cup D^-_k) = \sigma(D^-_1) \cup \ldots \cup \sigma(D^-_k),
\]

and similarly for \( D^+_i \in \mathcal{O}_{\text{triv} \to \mathbb{C}}(\emptyset; L^+_i) \).

**Definition B.2.** The orientation \( \sigma(K) \) for \( K \in \mathcal{O}_E(L_a; L_b) \) is defined by capping off the punctures and requiring that we obtain the canonical orientation:

\[
\sigma(\cup D^+_a) \# \sigma(K) \# \sigma(\cup D^-_b) = \text{canonical},
\]

with the convention that \( D^-_a \) is the reverse ordering \((\ldots, D_2, D_1)\) of the \( D_a \).

### B.12. Orientation signs arising from gluing

We now want to find out whether a glued orientation \( \sigma(L) \# \sigma(L') \) agrees with \( \sigma(L \# L') \) or not.

**Theorem B.3 ([9, Theorem 2]).** If we exchange the position of two consecutive \( b \)-labels \( k \) and \( k+1 \), the coherent orientation of \( \mathcal{O}_E(L_a; L_b) \) changes by \((-1)^{\text{ind } D_b \cdot \text{ind } D_{k+1}} \), where the \( D_b \in \mathcal{O}_{\text{triv} \to \mathbb{C}}(L_b; \emptyset) \) cap off the \( b \)-ends. Similarly for \( a \)-labels using the \( D_a \in \mathcal{O}_{\text{triv} \to \mathbb{C}}(\emptyset; L_a) \). For a general permutation of the ends, iterate this result.

**Proof.** For simplicity, we consider the pair-of-pants case. Observe the following diagram:

We want to compare \( L \in \mathcal{O}_E(L_1; L_2, L_3) \) and \( L' \in \mathcal{O}_E(L_1; L_3, L_2) \). We cap off the positive ends 2 and 3 so that both operators now lie in \( \mathcal{O}_{E'}(L_1; \emptyset) \) so they are equally oriented. Thus, \( \sigma(L \# (D_3 \cup D_2)) = \sigma(L' \# (D_2 \cup D_3)) \). So, by our axiomatic construction,

\[
\sigma(L) \# (\sigma(D_3) \cup \sigma(D_2)) = \sigma(L') \# (\sigma(D_2) \cup \sigma(D_3)).
\]
Thus, $\sigma(L)$ and $\sigma(L')$ are the same if and only if $\sigma(D_2) \cup \sigma(D_2)$ and $\sigma(D_2) \cup \sigma(D_3)$ are the same. By Lemma B.1, these are the same if and only if $(-1)^{\text{Ind}(D_2) - \text{Ind}(D_3)} = +1$.

**Theorem B.4** ([9, Propositions 8, 10]). $\sigma(L\#L')$ agrees with $\sigma(L)\#\sigma(L')$ when all ends are glued, so gluings of type $\mathcal{O}_E(L_a; L_{-c}) \times \mathcal{O}_E(L_c; L_{b'}) \to \mathcal{O}_E(L_a; L_{b'})$. For partial gluings, $\mathcal{O}_E(L_a; L_b; L_{-c}) \times \mathcal{O}_E(L_c; L_{b'}; L_{b''}) \to \mathcal{O}_E(L_a; L_{b''}'; L_{b'}; L_{b''})$, the orientations $\sigma(L\#L')$ and $\sigma(L)\#\sigma(L')$ differ by $(-1)^{\sum_{\text{ind } D_b} \sum_{\text{ind } D_{b'}}}$. (Mnemonically: we reorder $L_{b''}'; L_{a'}$ to $L_{a'}; L_{b''}$.)

**Proof (Sketch).** The key behind the first claim is that for complex linear operators, gluings preserve the complex orientations. Now for general $L$ and $L'$, we can assume the ends labelled $a$ and $b'$ are capped off since they do not matter. Then attach cylinders $Z_{-c}$ and $Z'_c$ at the c-ends of $S$ and $S'$ and extend $E$ and $E'$ so that the new asymptotic operators are complex linear. The resulting operators $L \cup Z_{-c}$ and $Z'_c \cup L'$ are now homotopic to complex linear ones, so they glue well: $\sigma((L \cup Z_{-c})\#(Z'_c \cup L')) = \sigma(L \cup Z_{-c})\#\sigma(Z'_c \cup L')$. Finally, by the associativity of $\#$, we can move the $Z'_c$ over to the $Z_{-c}$ and ‘cancel them off’ in pairs, to conclude the first claim: $\sigma(L \cup L') = \sigma(L)\#\sigma(L')$.

For the second claim, write $\sigma_a = \bigcup_a \sigma(D_a)$. Using distributivity and associativity of $\#$,

$$(\sigma_{-a'} \cup \sigma_{-a}) \# (\sigma(L)\#\sigma(L')) \# (\sigma_{-b'} \cup \sigma_{-b}) = [\sigma_{-a'} \cup (\sigma_{-a} \# \sigma(L))] \# [(\sigma(L')\#\sigma_{-b'}) \cup \sigma_{-b}].$$

The latter is a gluing of all ends, so we can apply the first claim provided that the ends are correctly ordered. Apply Theorem B.3 iteratively to correctly reorder the (negative) ends of the second square bracket: this gives rise to the sign in the second claim. Except for this sign, the result of the gluing must be the canonical complex orientation just like for $(\sigma_{-a'} \cup \sigma_{-a}) \# (\sigma(L)\#L') \# (\sigma_{-b'} \cup \sigma_{-b})$. Removing the caps that we added in both cases shows $\sigma(L)\#\sigma(L')$, $\sigma(L\#L')$ differ by that sign.

**B.13. Orientation signs for the TQFT**

We now show how the above axiomatic coherent orientation induces coherent orientations for the moduli spaces $\mathcal{M}(x; y; S, \beta)$. Pick once and for all a choice of trivialization of $x^*T\overline{M}$ over each Hamiltonian orbit $x$ (we discuss this further in Appendix B.16). For $u \in \mathcal{M}(x; y; S, \beta, J)$, the linearization $L = D_u$ of $\partial$ from Appendix A.5 is an operator as in Appendix B.9 for the complex vector bundle $E = u^*T\overline{M} \to S$ (using $J$) with asymptotic operators $(L_1^-, \ldots, L_p^-, L_1^+, \ldots, L_q^+)$ in the chosen trivializations over the asymptotics $x_a$ and $y_b$. Define the orientation sign $\epsilon_u \in \{\pm1\}$ arising for isolated $u$ in Appendix A.9 by

$$\epsilon_u \cdot (1^y \otimes 1) = \sigma(u) = \sigma(O_{x^*T\overline{M}-S}(L_1^-, \ldots, L_p^-, L_1^+, \ldots, L_q^+)).$$

Note that we reversed the order of the positive punctures. Observe that the definition involves the bundle $u^*T\overline{M}$, not just the asymptotics $L^\pm$. Also note that the capping off in Appendix B.12 is an abstract construction: it does not require the asymptotics $x_a$ and $y_b$ of $u$ to be contractible in $\overline{M}$.

**B.14. Using orientation signs to prove TQFT maps compose correctly**

In Theorem A.12, we claimed that $\psi_S \circ \psi_{S'} = \psi_{S \# S'}$ for any large enough gluing parameter $\lambda$. Recall that the proof produces a unique family of glued solutions $u\#_\lambda v$ which converges to the broken solution $u\#v$ as $\lambda \to \infty$. The broken solution is counted with sign $\epsilon_u \epsilon_v$ by $\psi_S \circ \psi_{S'}$, whereas $u\#_\lambda v$ is counted with sign $\epsilon_u \epsilon_{\lambda,v}$ by $\psi_{S \# S'}$. To complete the proof we still need to show that $\epsilon_u \epsilon_v = \epsilon_u \epsilon_{\lambda,v}$. To keep the notation under control, we illustrate the proof in the case
where $S = P$ and $S' = Q$ (see Figure 2). So the gluing operation is
\[
O_{u^\ast T\overline{\mathbb{P}} - S} (L^\ast_1; L^+_2, L^+_1) \otimes O_{v^\ast T\overline{\mathbb{P}} - S'} (L^-_1, L^-_2; L^+_1) \rightarrow O_{(u^\#_\lambda v)^\ast T\overline{\mathbb{P}} - S^\#_\lambda S} (L^\ast_1; L^+_1)
\]
with $L^+_1 = L^-_1$, $L^+_2 = L^-_2$. By Theorem B.4 this gluing is orientation-preserving for $\lambda \gg 0$ (the Explanation in Appendix B.3 clarifies the role of $\lambda$). We deduce $\epsilon_u \epsilon_v = \epsilon_{u^\#_\lambda v}$ as required.

B.15. Using orientation signs to prove TQFT maps are chain maps

If $SH^*$ is $\mathbb{Z}$-graded (Subsection 3.6), the differential $\partial$ on $SC^\ast (H_1) \otimes \cdots \otimes SC^\ast (H_k)$ is defined by
\[
\partial(a_1 \otimes \cdots \otimes a_k) = d(a_1) \otimes a_2 \otimes \cdots \otimes a_k + (-1)^{|a_1|} a_1 \otimes d(a_2) \otimes a_3 \otimes \cdots \otimes a_k + \cdots + (-1)^{|a_1|+\cdots+|a_{k-1}|} a_1 \otimes \cdots \otimes a_{k-1} \otimes d(a_k),
\]
this is the Koszul sign convention with $d$ in degree 1. In general, $SH^*$ is only $\mathbb{Z}/2$-graded, but this $\partial$ makes sense since only the parity of $|a_1|, |a_2|, \ldots$ matter.

In Theorem A.10, we claimed $\psi_S : \bigotimes_k SC^\ast (B_k H) \rightarrow \bigotimes_k SC^\ast (A_k H)$ is a chain map: $\partial \circ \psi_S = \psi_S \circ \partial$. This equation arose from an oriented count of the broken solutions arising at the boundary of the 1-dimensional moduli spaces of Floer solutions, but we still need to check the orientation signs. This involves two steps: (1) we need to explain how the Koszul signs in the above definition of $\partial$ arise; (2) we need to explain the minus sign in $\partial \circ \psi_S - \psi_S \circ \partial = 0$.

The proof of (2) is identical to the proof in Appendix B.7: replace $M^H_1 (x, z)$ by $M_1 (x; z_j; S, \beta)$, replace continuation solutions $v$ and $v'$ by Floer solutions $v$ and $v'$ (and the $s$-coordinate for $v$ and $v'$ now refers to the $s$-coordinate on the ends of $v$ and $v'$, where the Floer trajectories $u$ and $u'$ broke off). The argument in Appendix B.7 then shows that if $\sigma$-orientations are respected (meaning $\sigma(u \#_\lambda v) = \sigma(u) \#_\lambda \sigma(v)$ and $\sigma(v' \#_\lambda u') = \sigma(v') \#_\lambda \sigma(u')$), then the gluing map sends the orientations $\partial_x u, \partial_x u'$ of $u \#_\lambda v, v' \#_\lambda u'$ to $\partial_x (u \#_\lambda v), -\partial_x (v' \#_\lambda u')$, which are, respectively, outward and inward pointing near the boundary of $M$ (since they approach the boundary as $\lambda \rightarrow \infty$).

Proof of (1): Suppose $u \#_\lambda v$ is a broken Floer solution, where an isolated Floer trajectory $u$ broke off at the first negative end of $v$. The gluing of the linearizations $D_u \# D_v$ is
\[
O(D^-_1; D^+_1) \times O(D^+_u, L^+_2, \ldots, L^+_p; L^-_q, \ldots, L^-_1) \rightarrow O(D^+_u, L^-_2, \ldots, L^-_p; L^+_q, \ldots, L^+_1),
\]
so by Theorem B.4, $\sigma(u \#_\lambda v) = \sigma(u) \#_\lambda \sigma(v)$. Similarly, for broken Floer solutions $v' \#_\lambda u'$ when $u'$ broke off at the first positive end of $v'$, we get $\sigma(v' \#_\lambda u') = \sigma(v') \#_\lambda \sigma(u')$.

Via step (2) this shows that these two breakings contribute, respectively, to $(d \otimes 1 \otimes \cdots \otimes 1) \circ \psi_S$ and $\psi_S \circ (d \otimes 1 \otimes \cdots \otimes 1)$ in the equation $\partial \circ \psi_S = \psi_S \circ \partial$ (here 1 are identity maps). To prove that the breakings at the other ends also contribute correctly to $\partial \circ \psi_S = \psi_S \circ \partial$, we reduce to the previous two cases by reordering the ends twice using Theorem B.3, as follows.

Suppose $u$ broke off at the $k$th negative end of $v$, so $L^-_k = D^+_u$. Then Theorem B.3 allows us to move the $k$th end into the first position at the cost of introducing the sign $(-1)^{|\text{Ind}(D_1) + \cdots + |\text{Ind}(D_{k-1})|}$ $|\text{Ind}(D_k)|$ where $D_i \in O_{\text{triv} - \epsilon} (\emptyset; L^-_1)$ cap off the negative ends. After this reordering, we are in the case discussed previously where $u$ broke off at the first negative end. So gluing the cylinder with asymptotics $(D^-_u; D^+_u)$ onto the first negative end respects $\sigma$-orientations. By Theorem B.3, we can now move the first end back to its original $k$th position, at the cost of the sign $(-1)^{|\text{Ind}(D_k)|}$ $(-1)^{|\text{Ind}(D_{k-1})|}$ (here we used that $u$ is an isolated Floer trajectory, so the Fredholm indices of the caps attached to $L^-_k = D^+_u$ and to $D^-_u$ differ by 1). This final position is the actual gluing without reorderings that we are interested in. Call $a_i$ the asymptotic orbits at the negative ends of the Floer solution $v$. Then by definition $(-1)^{|a_i|} = (-1)^{|\text{Ind}(D_i)|}$ (By (2)–(3) in the Technical Remark in Appendix B.16,
the Conley–Zehnder grading $|L^\pm| \equiv \text{ind } D^\pm \pmod{2}$.) So the total sign caused by the reorderings is

$$(1)^{[\text{ind}(D_1) + \ldots + \text{ind}(D_{k-1})]} \text{ind}(D_{k}) (1)^{\left[\text{ind}(D_k) - 1\right]} \text{ind}(D_{k+1}) + \ldots + \text{ind}(D_{k+1})] = (-1)^{|a_1| + \ldots + |a_{k-1}|},$$

which is precisely the Koszul sign for the $k$th term in the definition of $\partial$ above. The discussion when $u'$ breaks off at a positive end of a Floer solution $\nu'$ is analogous. This proves (1).

B.16. The choice of trivializations over the Hamiltonian orbits, the role of the canonical bundle $K$ and dimension counts

Recall that we chose trivializations of $x^*TM$ over all possible asymptotics $x$ in Appendix B.13. By construction, we actually only need to choose a homotopy class of trivializations, so we only need to choose an orientation of $x^*TM$.

For example, suppose $c_1(M) \equiv c_1(TM, J) = 0$. Then the canonical bundle $K = \Lambda_c^{\text{max}}x^*TM$ is trivial (we encountered $K$ in Subsection 3.6). A choice of trivialization of the complex line bundle $K$ naturally determines a trivialization of the complex line bundles $\Lambda_c^{\text{max}}(x^*TM) = x^*(K^\vee)$. So explicitly, we require that the trivialization of $x^*TM$ induces the trivialization up to homotopy (this condition is an obstruction lying in $\pi_1(U(n)) \cong \mathbb{Z}$). Equivalently, pick a complex volume form $\eta$, namely a non-vanishing section of $K$, and require that the trivialization $x^*K \cong S^1 \times \mathbb{C}^n$ sends $\eta$ to the standard volume form of $\mathbb{C}^n$ (since $\pi_1(SU(n)) = 0$, this determines the trivialization up to homotopy).

Recall from Subsection 3.6 that the condition $c_1(M) = 0$ induces a Conley–Zehnder grading on the orbits. We now explain how the grading determines the Fredholm indices by the Riemann–Roch theorem [27, Theorem C.1.10].

Technical Remark: The linearization $D_a$ arising in Floer theory is an elliptic first-order partial differential operator which equals a complex linear operator up to a lower-order term which can be ignored when computing Fredholm indices [25, Section 3.1]. The Riemann–Roch theorem is then applied to the complex linear part, which is a Cauchy–Riemann operator which induces a holomorphic structure on $u^*TM$.

(1) For $L \in \mathcal{O}_{u^*TM \to Z}(L^{-}; L^+)$, where $u : Z \to \overline{M}$ has asymptotics $x^\pm$, we can prescribe the trivialization of $u^*TM$ in Appendix B.3 to be the one which induces the given trivialization of $u^*K$. The Conley–Zehnder grading described in the Explanation in Appendices B.3 and B.4 agrees with the grading from Subsection 3.6, so:

$$\text{Ind}(L) = |x^-| - |x^+|.$$

(2) For $D^+ \in \mathcal{O}_{E_{-C}}(\emptyset; L^+)$, a Riemann–Roch argument [27, Appendix C.4] shows that $\text{Ind}(D^+) = n + \mu(L^+) = 2n - |L^+|$, where $n = \text{rank}_C E$, $\mu(L^+)$ is a Maslov index, and $|L^+|$ is the Conley–Zehnder index (see Subsection 3.6 and the explanation in Appendix B.3). If $E = u^*TM$ for some $u : C \to \overline{M}$ with asymptotic $x^+$, and $c_1(TM)|_{\pi_2(M)} = 0$, then we deduce:

$$\text{Ind}(D^+) = 2n - |x^+|.$$

(3) For $D^- \in \mathcal{O}_{E_{-C}^{-}}(L^-; \emptyset)$, where $C'$ is $C$ but viewing infinity as a negative puncture, glue an appropriate $D^+$ from (2). Then additivity of indices and Riemann–Roch determines $\text{Ind}(D^-)$ via: $\text{Ind}(D^+) + \text{Ind}(D^-) = \text{Ind}(D^+ \# D^-) = 2n + 2c_1(E \# E')[C' \# C']$. If $E = u^*TM$ for $u : C' \to \overline{M}$ with asymptotic $x^-$, and $c_1(TM)|_{\pi_2(M)} = 0$, then we deduce:

$$\text{Ind}(D^-) = |x^-|.$$

(4) For $L \in \mathcal{O}_{E_{-S}}(L^{-}_a; L^+)$ with $S$ of genus $g$ with no punctures, by Riemann–Roch $\text{Ind}(L) = 2(1 - g)n + 2c_1(E)[S].$

(5) For $L \in \mathcal{O}_{E_{-S}}(L^{-}_a; L^+)$ with $S$ of genus $g$ with $p$ negative and $q$ positive punctures, cap off the ends using $D^+_a, D^-_b$ of type (2),(3) respectively. Then by (4), $\text{Ind}(L) + \sum \text{Ind}(D^+_a) + \sum \text{Ind}(D^-_b) = 2(1 - g)n + 2c_1(\bigcup_a D^+_a \# E \# \bigcup_b D^-_b)[\bigcup_a C \# S \# \bigcup_b C'].$
(6) Suppose \( c_1(M) = 0 \). Fix a trivialization of \( K \). If \( E = u^*\overline{T\overline{M}} \) for \( u : S \to \overline{M} \), for an \( S \) as in (5), then pick a trivialization of \( E \) inducing the given trivialization of \( u^*K \). Choose trivial bundles for the caps \( D^+_a, D^-_b \) agreeing with the trivializations on the asymptotics of \( x_a \) and \( y_b \) of \( u \). Then the \( c_1 \) term in (5) vanishes, so by (2) and (3), we obtain

\[
\operatorname{Ind}(L) + 2np - \sum |x_a| + \sum |y_b| = 2(1 - g - p).
\]

We deduce the formula in Theorem A.4:

\[
\operatorname{Ind}(L) = \sum |x_a| - \sum |y_b| + 2n(1 - g - p).
\]

If we restrict the TQFT to genus 0 surfaces and we restrict to contractible orbits (so we use \( SH^*_0 \) as in Subsection 6.5), the above discussion shows that we can weaken \( c_1(TM)|_{\pi_2(M)} = 0 \) to just \( c_1(TM)|_{\pi_2(M)} = 0 \). In this case, instead of using \( K \), one chooses the trivialization of \( x^*T\overline{M} \) to be induced from a trivialization of \( \overline{x}^*T\overline{M} \), where \( \overline{x} \) is a disc in \( \overline{M} \) bounding \( x \). In (6) one uses such discs to cap off \( u : S \to \overline{M} \) to obtain a sphere in \( \overline{M} \). The choices of trivialization of \( \overline{x}^*T\overline{M} \) do not affect the index, since two choices are related by the gluing of a bundle over a sphere, and by Riemann–Roch, gluing \( E' \to CP^1 \) onto \( E \to S \) can only change Fredholm indices by a value in \( 2c_1(TM)|_{\pi_2(M)} = 0 \).

In conclusion, when \( c_1(TM)|_{\pi_2(M)} = 0 \), \( SH^*_0(M) \) is a \( \mathbb{Z} \)-graded unital ring.

For cotangent bundles \( \overline{M} = T^*N \) of closed oriented manifolds \( N \), there is a preferred homotopy class of trivializations for \( K = \Lambda^*_C \max TT^*N = \Lambda^*_\mathbb{R} \max T^*N \otimes_\mathbb{R} \mathbb{C} \), since the orientation for \( N \) determines a trivialization of \( \Lambda^*_\mathbb{R} T^*N \).

B.17. Avoiding choices

The system of coherent orientations constructed axiomatically in Appendix B.11 depends on the choices of orientations for \( \mathcal{O}_{\text{triv} - \mathbb{C}}(L^-; \emptyset) \). Different choices will give different orientation signs. To avoid making these choices, one can incorporate all choices into the symplectic chain complex, which is a construction due to Seidel [40, Section (12f)].

For a 1-dimensional \( \mathbb{R} \)-vector space \( V \), define the orientation space \( \mathcal{O}(V) \) to be the 1-dimensional \( \mathbb{K} \)-vector space generated by the two possible orientations of \( V \), subject to the relation that the two generators sum to zero. This is functorial: any \( \mathbb{R} \)-linear map \( V \to W \) defines a natural \( \mathbb{K} \)-linear map \( \mathcal{O}(V) \to \mathcal{O}(W) \), and similarly for multi-linear maps.

Fix a homotopy class of trivializations for \( x^*T\overline{M} \) for any Hamiltonian orbit \( x \) (for example by Appendix B.16, when \( c_1(M) = 0 \) a set of choices can be described in terms of a trivialization of \( K \)). The linearization of Floer’s equation \( \partial_a u + J(\partial_t u - X_H) = 0 \) determines an asymptotic operator \( L^+_x \) of the form described in Appendix B.2 in such a trivialization of \( x^*T\overline{M} \). This in turn determines, up to homotopy, an operator \( D^+_x : W^{1,p}(\mathbb{C}, \mathbb{C}^n) \to L^p(\mathbb{C}, \mathbb{C}^n) \) in \( \mathcal{O}_{\text{triv} - \mathbb{C}}(\emptyset; L^+_x) \) as in Appendix B.9 defined on the trivial bundle over \( \mathbb{C} \) with asymptotic operator \( L^+_x \).

Denote by \( \mathcal{O}(x) = \mathcal{O}(V) \) the orientation space of \( V = \operatorname{Det}(D^+_x) \). Then define

\[
SC^*(H) = \bigoplus \{ \mathcal{O}(x) : x \in \mathcal{L}\overline{M}, \dot{x}(t) = X_H(x(t)) \}.
\]

Suppose we are given a choice of generators \( X_a \in \mathcal{O}(x_a), Y_b \in \mathcal{O}(y_b) \). Let \( L = D_u \) be any linearization arising from a Floer solution \( u : S \to \overline{M} \) with asymptotics \( x_a \) and \( y_b \). This determines asymptotic operators \( L^-_a \) and \( L^+_b \) over \( x_a \) and \( y_b \) in the given trivializations of \( x^*_a T\overline{M} \) and \( y^*_b T\overline{M} \). The caps \( D^+_a \in \mathcal{O}_{\text{triv} - \mathbb{C}}(\emptyset; L^-_a) \) and \( D^-_b \in \mathcal{O}_{\text{triv} - \mathbb{C}}(L^+_b; \emptyset) \) are oriented, respectively, by \( X_a \) and by the condition \( Y_b \# \operatorname{Det}(D^-_b) = \text{canonical} \) (cf. Appendix B.11). Call \( Y'_b \) the orientation that \( Y_b \) determines for \( \operatorname{Det}(D^-_b) \) in this way. Then \( X_a \) and \( Y'_b \) uniquely determine an orientation for \( L \) via the condition \( \bigcup_a X_{-a} \# \operatorname{Det}(L) \# \bigcup_b Y'_{-b} = \text{canonical} \) (mimicking Definition B.2).

This proves that the \( \mathcal{M}(x_a; y_b; S, \beta) \) are canonically oriented relative to the ends.
For isolated \( u \in \mathcal{M}_0(x_a; y_b; S, \beta) \), the \( X_a \) and \( Y_b \) determine an orientation sign \( \epsilon_u(x_a; y_b) \in \{ \pm 1 \} \) via \( \text{Det}(D_u) = \epsilon_u(x_a; y_b) \cdot (1^\vee \otimes 1) \). So in this formalism the \( \epsilon_u \) become isomorphisms

\[
\epsilon_u : \bigotimes_b \mathcal{O}(y_b) \rightarrow \bigotimes_a \mathcal{O}(x_a),
\]

extending \( Y_1 \otimes \ldots \otimes Y_q \mapsto \epsilon_u(x_a; y_b) X_1 \otimes \ldots \otimes X_p \) multi-linearly. So \( \epsilon_u \) no longer depends on the choice of a system of coherent orientations, since all choices are taken into account at the same time. Then, summing over all \( u \in \mathcal{M}_0(x_a; y_b; S, \beta) \), we define the TQFT operation by

\[
\psi_S : \bigotimes_b \mathcal{O}(y_b) \rightarrow \bigotimes_a \mathcal{O}(x_a), \quad \psi_S = \sum \epsilon_u.
\]

Similarly, we orient the moduli spaces of Floer trajectories, so for instance \( u \in \mathcal{M}_0(x, y) \) defines an isomorphism \( \epsilon_u : \mathcal{O}(y) \rightarrow \mathcal{O}(x) \), which, if we made explicit choices of orientation generators, would be multiplication by the sign \( \epsilon_u \) of Subsection 3.1. Then define the Floer differential \( d : \mathcal{O}(y) \rightarrow \mathcal{O}(x) \) by the linear map \( d = \sum \epsilon_u \), summing over \( u \in \mathcal{M}_0(x, y) \).

For twisted symplectic cohomology, we replace the field \( \mathbb{K} \) by the Novikov algebra \( \Lambda \), and we insert the weights \( \alpha^{[\alpha]} \) in front of the \( \epsilon_u \) maps.

**Appendix C.** \( SH^*(M) \) defined using a nonlinear Hamiltonian

This section is required in Subsection 14.1, since it uses Hamiltonians of quadratic growth.

**C.1. An alternative definition of \( SH^*(M) \)**

Let \( Q : \overline{M} \rightarrow \mathbb{R} \) be equal to a convex function \( q(R) \) for \( R \geq 1 \), with \( q'(R) \rightarrow \infty \) as \( R \rightarrow \infty \) (example: \( Q = \frac{1}{2} R^2 \)). Define the action function

\[
\mathbb{H}_q(R) = -Rq'(R) + q(R).
\]

**Lemma C.1.** There is a natural identification \( SH^*(Q) \cong SH^*(M) \), where \( SH^*(Q) \) is defined as in Subsection 3.1.

**Proof.** By convexity of \( q \), \( \partial_R \mathbb{H}_q = -Rq''(R) \leq 0 \), so the action \( \mathbb{H}_q \) decreases on the collar.

This convexity together with the condition \( q'(R) \rightarrow \infty \) ensures that \( \mathbb{H}_q(R) \rightarrow -\infty \) as \( R \rightarrow \infty \) (we explain this in the following Remark).

**Remark.** If \( q'(R_a) = a \) and \( q'(R_b) = b \geq a \), then \( \mathbb{H}_q(R_b) = -R_bq + q(R_b) \leq -R_ab + q(R_a) \) since \( \partial_R(-Rb + q(R)) = -b + q'(R) \leq 0 \) for \( R \in [R_a, R_b] \) since \( q'' \geq 0 \). Now observe \( -R_ab + q(R_a) \rightarrow -\infty \) as \( b \rightarrow \infty \).

So, using the action–restrictions from Section 8, we have a chain complex

\[
SC^*(Q; \mathbb{H}_Q \geq \mathbb{H}_Q(R_m)) \cong SC^*(H^m),
\]

where \( q'(R_m) = m \) and where \( H^m \) is obtained from \( Q \) by changing it to be linear of slope \( m \) in the region \( R \geq R_m \) where \( q' \geq m \). By Section 8, \( SC^*(H^m) \subset SC^*(H^{m'}) \subset SC^*(Q) \) for \( m \leq m' \) are inclusions of subcomplexes. Since \( SC^*(Q) \) consists of finite linear combinations of generators, we deduce \( SH^*(Q) = \bigcup_m SH^*(H^m) \cong \lim SH^*(H) = SH^*(M) \). Strictly speaking, to define \( SH^*(Q) \) we need to make a subtle time-dependent perturbation of \( Q \) to ensure the non-degeneracy of the orbits: we do this in the Technical Remarks below. \( \square \)

By Lemma C.1, one could define \( SH^*(M) = SH^*(Q) \). Call nonlinear Hamiltonians the functions \( Q \) described above. One could drop the convexity assumption, and only assume \( q'(R) \rightarrow \infty \) and \( \mathbb{H}_q(R) \rightarrow -\infty \) as \( R \rightarrow \infty \) then Lemma C.1 can still be proved by a similar argument as follows:
Remark. Define $H^m$ as before: so $H^m$ is linear of slope $m$ at infinity, but now, due to the non-convexity of $q$, there may be some points where $H^m$ has slope $>m$. Given $m > 0$, all 1-orbits of $H^m$ have $c_{H^m} > c$ for $c \ll -m$, so for a monotone homotopy $H_s$ from $Q$ to $H^m$ with $\partial_s H_s \leq 0$ (so actions decrease along continuation solutions by Subsection 3.2) and depending on $s$ only on $V_m = \{Q \neq H^m\}$, the continuation map sends $SH^s(H^m) \to SH^s(Q; \delta Q > c)$. Given $c < 0$, for $m' \gg -c$ there are inclusions $SC^s(Q; \delta Q > c) \to SC^s(H^m)$ since the 1-orbits of $Q$ on $V_{m'}$ will have $\delta Q \ll c$. Then observe: (1) for $m' \gg -c \gg m$, the composite $SH^s(H^m) \to SH^s(Q; \delta Q > c) \to SH^s(H^{m'})$ is the continuation $SH^s(H^m) \to SH^s(H^m)$ (using Lemma 3.1(2)); and (2) for $-c' \gg m' \gg -c$, the composite $SC^s(Q; \delta Q > c) \to SC^s(H^m) \to SC^s(Q; \delta Q > c')$ is the inclusion $SC^s(Q; \delta Q > c) \to SC^s(Q; \delta Q > c')$. The claim now follows by (1), (2), $SH^s(M) = \lim_{m' \to \infty} SH^s(H^{m'})$ and $SH^s(Q) = \lim_{c' \to -\infty} SH^s(Q; \delta Q > c_j)$. 

Technical Remarks. As mentioned in Subsection 2.5, we always need the 1-orbits to be non-degenerate. In the nonlinear setup, this becomes a problem: to make the orbits non-degenerate (in particular, to remove the $S^1$-symmetry), one needs to perturb $Q : \overline{M} \to \mathbb{R}$ by adding a small time-dependent function $P : S^1 \times \overline{M} \to \mathbb{R}$, and it is not possible to make $Q + P$ depend only on $t$ and $R$ near these orbits, so the maximum principle in Lemma D.1 may fail. We mimic Abouzaid’s approach [3, Appendix B]: we avoid using the maximum principle, and we rather use the no escape lemma, the version that we need here is proved in Lemma D.5.

In our setup, $q$ is convex so $\partial_R \delta_q \leq 0$. So to guarantee that Lemma D.5 applies, we ensure $\delta_q \leq 0$ for all $R \geq 1$ by requiring that $-q'(1) + q(1) \leq 0$.

To make the 1-orbits non-degenerate it suffices to make a generic $C^2$-small perturbation of $Q$ supported near the orbits. So for Floer trajectories of $Q$, we choose $P : S^1 \times \overline{M} \to \mathbb{R}$ to be generic $C^2$-small and supported near the 1-orbits of $Q$, in particular, we ensure that $P = 0$ in certain regions where there are no 1-orbits of $Q$: fix a sequence $R_m \to \infty$ such that $q'(R_m)$ are not Reeb periods and ensure $P = 0$ for $R$ close to $R_m$. This guarantees that Lemma D.5 applies to $H = Q + P$ on any $V \subset \overline{M}$ defined by $R \geq R_m$ (assuming that $J$ is of contact type along $R = R_m$). So if a Floer trajectory $v : \mathbb{R} \times S^1 \to \overline{M}$ has asymptotics $x, y$ with $R(x) \leq R_m, R(y) \leq R_m$; then $v$ cannot enter the region $R > R_m$ since otherwise $u = v|_{v_1(V)}$ would violate Lemma D.5.

Now consider Floer solutions for $Q$ on a model surface $S$ (a similar discussion holds for Floer continuations for $Q_s$, using $P : \mathbb{R} \times S^1 \times \overline{M} \to \mathbb{R}$). We choose $P : S \times \overline{M} \to \mathbb{R}$ to be: zero away from the ends of $S$; generic $C^2$ small on the ends of $S$ with $\partial_x P \leq 0$ and $P$ only depending on $t$ for $|s| \gg 0$; and $P$ depends near the 1-orbits of $Q$. As before, we pick a sequence $R_m \to \infty$ such that $A_\alpha q'(R_m)$ and $B_\beta q'(R_m)$ are not Reeb periods, where $A_\alpha$ and $B_\beta$ are the weights for $S$, and we ensure $P = 0$ for $R$ near $R_m$. So if a Floer solution $v$ has asymptotics $x_a, y_b$ with $R(x_a) \leq R_m, R(y_b) \leq R_m$; then $v$ cannot enter the region $R > R_m$ since otherwise $u = v|_{v_1(V)}$ violates Lemma D.5 for $H = Q + P$.

In conclusion, once the data $S, A_\alpha$ and $B_\beta$ are fixed, a generic perturbation $Q + P$ of $Q$ as above ensures that both the orbits are non-degenerate and that the Floer moduli spaces are contained in a compact subset of $\overline{M}$ determined by the $R$ values of the asymptotics.

C.2. Contact type $J$ makes transversality fail for $SH^s(Q)$

For Hamiltonians $H$ of linear growth, we always arranged that the 1-orbits of $H$ were in a compact set $R \leq R_0$, so if $J$ is of contact type for $R > R_0$, then no Floer trajectories enter the region $R > R_0$ and so $J$ does not need to be perturbed there to achieve transversality for the Floer moduli spaces. For nonlinear Hamiltonians $Q$ as above, 1-orbits keep appearing for larger and larger $R$, so $J$ will typically need to be perturbed everywhere on $\overline{M}$ and in all directions to achieve transversality. So we cannot impose the contact type condition $J\partial_t = R$ which
leaves no freedom to perturb $J$ in the plane $\text{span}(\partial_r, \mathcal{R})$ (recall the notation of Subsection 2.1). Without the contact type condition, the maximum principle in Lemma D.1 (or the no escape Lemma D.3) may fail, so the Floer moduli spaces may not have compactifications by broken solutions. Thus, we must allow non-contact type $J$, and we need to reprove a version of the maximum principle.

We found two approaches to solve this issue which we now describe. From now on, when we say ‘Floer moduli spaces’ or Floer solutions we refer generally to the moduli spaces of Floer trajectories, Floer continuation solutions or Floer solutions. Without the contact type condition, the maximum principle in Lemma D.1 (or the no escape Lemma D.3) may fail, so the Floer moduli spaces may not have compactifications by broken trajectories, Floer continuation solutions or Floer solutions.

(1) First approach: we allow $J$ to be of non-contact type, provided that there is a sequence $R_m \to \infty$ such that $J$ satisfies the contact type condition along the hypersurfaces $\{ R = R_m \}$. We assume that $Q = q(R)$ only depends on $R$ near $R = R_m$ (see the Technical Remarks in Appendix C.1) and the slopes $q'(R_m)$ are not Reeb periods (for Floer solutions, we require this for $A_0q'(R_m), B_0q'(R_m)$, where $A_0$ and $B_0$ are the weights). This ensures that the asymptotics for the Floer moduli spaces do not lie in $\{ R = R_m \}$.

(2) Second approach: we allow a non-contact type almost complex structure $J_\infty$, but we require that it decays sufficiently rapidly in $R$ to a fixed $J = J_0$ which is of contact type for large $R$ (and we assume $Q = q(R)$ only depends on $R$ for $R \gg 0$ up to perturbations as in the Technical Remarks of Appendix C.1). More precisely, we fix an exhausting sequence of compact sets $\overline{M} = \bigcup_m K_m$, with $K_m \subset \text{int}(K_{m+1})$, for example, $K_m = \{ R \leq m \}$. Then we prove that there are reals $\delta_m > 0$ such that any $\omega$-compatible almost complex structure $J_\infty$ satisfying $|J_\infty - J| < \delta_m$ on $K_m$ will satisfy a maximum principle. We also show that this leaves enough room to make generic perturbations of $J_\infty$ on each $K_m$ to ensure the regularity of Floer solutions.

**Remark 1.** The second approach is trickier, but it is of independent interest since the assumptions can be significantly weakened as follows. Fix any time-dependent Hamiltonian $Q : S^1 \times \overline{M} \to \mathbb{R}$ for which in any given compact subset of $\overline{M}$ there are only finitely many 1-orbits of $Q$ (and the orbits are non-degenerate), in the case of Floer solutions we assume this for $A_0Q$ and $B_0Q$. Then we need that only $J = J_0$ is an $\omega$-compatible almost complex structure for which a no escape Lemma of the following form holds: given any compact subset $K \subset \overline{M}$, and any $\omega$-compatible almost complex structure $J'$ which equals $J$ outside of $K$, all $J'$-Floer solutions automatically lie in a compact subset of $\overline{M}$ determined by $K$ and the asymptotics (and not depending on $J'$). This generalization is useful when $\overline{M}$ does not have contact hypersurfaces at infinity (for example: negative vector bundles of rank $\mathbb{C} \geq 2$ (see [33]).

**Lemma C.2.** In the first approach, after a generic perturbation of $J$ which preserves the contact type condition along the hypersurfaces $\{ R = R_m \}$, we can ensure compactness and transversality for Floer moduli spaces.

**Proof.** By the no escape Lemma D.3, using that $J$ is of contact type on $R = R_m$, a Floer solution $u$ with asymptotics $x_a$ and $y_b$ must lie in $\{ R \leq R_m \}$ for any $m$ satisfying $R_m > \max\{ R(x_a), R(y_b) \}$. So compactness results for the Floer moduli spaces are not problematic.

For transversality, we run the proof of Lemma A.6. Recall this relied on the construction of a vector $Y$ at $(z_0, u(z_0)) \in S \times \overline{M}$ such that $g(Y(du - X \otimes \beta)j, \eta) > 0$ at $z_0$. Since $Y$ arises from differentiating $J$, and since we impose that $J\partial_r = \mathcal{R}$ on $R = R_m$, we are forced to choose $Y(\text{span}(\partial_r, \mathcal{R})) = 0$ on $R = R_m$ (this is a closed condition, so we have restricted $Y$ to a Banach subbundle). So the proof of Lemma A.6 fails only if two conditions hold: (1) $u(z_0) \in \{ R = R_m \}$ and (2) $\text{im}(du - X \otimes \beta) \subset \text{span}(\partial_r, \mathcal{R})$ at $z_0$. Now $X = q'(R)\mathcal{R}$ near $R = R_m$, since $Q = q(R)$ there, so (2) implies $\partial_s u, \partial_t u \in \text{span}(\partial_r, \mathcal{R})$ at $z_0$. Since $u$ is a Floer solution, $u$ satisfies...
equation (D.1) in Appendix D.3, so \( \partial_t u - X\beta_t = J(\partial_s u - X\beta_s) \). This equation implies that if \( \partial_t u \) and \( \partial_s u \) both lie in \( \text{span}(R) \) at \( z_0 \), then in fact \( \partial_t u = X\beta_t \) and \( \partial_s u = X\beta_s \), contradicting the assumption that \( du - X \otimes \beta \neq 0 \) at \( z_0 \). On the other hand, if \( \partial_s u \) or \( \partial_t u \) has a non-zero \( \partial_t \)-component at \( z_0 \), then we can pick \( z_0' \) arbitrarily close to \( z_0 \) such that \( u(z_0') \notin \{ R = R_m \} \), so we can run the proof of Lemma A.6 for \( z_0' \) instead of \( z_0 \). This proves transversality (and for Floer continuation solutions).

For Floer trajectories \( u : Z \to \overline{M} \) joining \( z^\pm(t) \), the proof of transversality is slightly different since \( J \) can be perturbed \( t \)-dependently but not \( s \)-dependently: the construction of the above \( Y \) is done in the same way as before [37, pp. 1346–1347], but the problem is that the cut-off function that is used to extend \( Y \) locally in the proof of Lemma A.6 must be chosen to be independent of \( s \). Define the set of regular points \( \mathcal{R}(u) = \{ z_0 = (s_0, t_0) \in Z : u(z_0) \neq x^\pm(t_0), \partial_s u(z_0) \neq 0, u(s, t_0) \neq u(z_0) \text{ for all } s \in \mathbb{R} \} \). By [21, Theorem 4.3], \( \mathcal{R}(u) \subset Z \) is open and dense unless \( \partial_s u \equiv 0 \) (if \( \partial_s u \equiv 0 \), then \( u \) does not contribute to the Floer complex, so we ignore it). If \( z_0 \in \mathcal{R}(u) \), one can check that there is a \( t \)-dependent cut-off function \( \phi_t : \overline{M} \to [0,1] \) supported near \( u(z_0) \) and near \( t = t_0 \) such that \( \phi \circ u : Z \to \mathbb{R} \) is a cut-off function supported near \( z_0 \) of the type required in the proof of Lemma A.6. So we can use \( \phi \) to extend \( Y \) locally, so the proof of Lemma A.6 works in our setup provided \( z_0 \in \mathcal{R}(u) \) and \( z_0 \notin \{ R = R_m \} \) (since then \( Y \) has no constraints). Finally observe that we can always choose \( z_0 \in \mathcal{R}(u) \) with \( u(z_0) \notin \{ R = R_m \} \); otherwise \( u(\mathcal{R}(u)) \subset \{ R = R_m \} \), but then by the density of \( \mathcal{R}(u) \subset Z \) we have \( u(Z) \subset \{ R = R_m \} \), contradicting \( x^\pm \notin \{ R = R_m \} \).

The second approach. By Appendix A.6, a generic arbitrarily small perturbation of \( J \) achieves transversality for Floer solutions lying in a given compact subset \( W \subset \overline{M} \). Unfortunately one cannot naively perturb \( J \) inductively on larger and larger compact sets because new Floer solutions for fixed asymptotics may suddenly appear reaching into the regions where we perturbed \( J \). So a more subtle argument is needed. We first introduce some notation.

A Floer moduli space is described by a finite data set \( \mathcal{A} \) consisting of: a model surface \((S, j, \beta)\), weights \( A_a \) and \( B_b \), and asymptotics \( x_a \) and \( y_b \) (1-orbits of \( A_aQ \) and \( B_bQ \)). For example, for Floer trajectories the data set \( \mathcal{A} \) is the cylinder \( Z \) and asymptotics \( x^\pm \). We call such data sets \( \mathcal{A} \) the auxiliary data. We also allow an auxiliary data set \( \mathcal{A} \) to consist of a finite collection of such data sets, and we say ‘\( J' \)-Floer solution for \( \mathcal{A} \)’ to mean an element of the Floer moduli space defined using \( J' \) and one of the finitely many data sets in \( \mathcal{A} \). Then define

\[
R_{\mathcal{A}}(J') = \max\{ \max R(u) : u \text{ is a possibly broken } J' \text{-Floer solution for } \mathcal{A} \},
\]

\[
K_{\mathcal{A}}(J') = \{ R \leq R_{\mathcal{A}}(J') \} \subset \overline{M}.
\]

The assumptions on \( J \) in (2) at the start of Appendix C.2 ensure by the no escape Lemma that for any compact \( K \subset \overline{M} \) and any \( J' \) which equals \( J \) outside \( K \), the \( R_{\mathcal{A}}(J') \) are bounded by constants which depend on \( J, \mathcal{A} \) and \( K \) but not on \( J' \) (a similar statement holds for \( J \) as in Remark 1).

Let \( \mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \ldots \) be a sequence of auxiliary data sets. The goal is to obtain almost complex structures \( J_\infty \) which are regular for \( J_\infty \)-Floer solutions for any data set in \( \bigcup_m \mathcal{A}_m \) (so we ensure regularity simultaneously for any countable collection of data sets). For example, for Floer trajectories for \( Q \), let \( x_1, x_2, \ldots \) be a listing of the (non-degenerate) 1-orbits of \( Q \), and let \( \mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \) be determined by a listing of all possible choices of pairs \( x^\pm \) taken from the \( x_1, x_2, \ldots \) with each \( \mathcal{A}_j \) being only a finite subset of such pairs.

Remark 2. Observe that a ‘no escape Lemma’ for \( J' \)-Floer solutions for \( \mathcal{A} \) corresponds to giving a bound for \( R_{\mathcal{A}}(J') \). In the more general settings alluded to in Remark 1 above, one may not have a \( R \)-coordinate, so one defines \( K_{\mathcal{A}}(J') \) to be the union of all \( J' \)-Floer solutions for \( \mathcal{A} \), and a ‘no escape Lemma’ would mean giving a bound on the diameter of
In this general setting, there may not be a priori bounds for the energy or for the Fredholm index for the Floer solution in terms of the asymptotics. In this case, part of the auxiliary data $A_j$ will be a choice of bounds on the index and the energy of the solutions considered.

We need a brief comment about the analysis, since we will use Floer’s norm [19, p. 807]. To give the reader a gist of this norm, we consider for simplicity smooth functions $[0, 1] \to \mathbb{R}$, and we refer to [37, p. 1345] which explains how to adapt this norm to the space of $\omega$-compatible almost complex structures. For smooth functions, Floer’s norm is defined as $|f| = \sum c_k |f|_{C^k}$, where $| \cdot |_{C^k}$ is the $C^k$-norm and where $c_k > 0$ are constants. The subspace of $C^\infty$-functions of bounded Floer norm is a separable Banach space $B$. If the $c_k$ decrease sufficiently rapidly in $k$, then $B$ is dense in $L^2$.

**Remark.** This is because $B$ contains many cut-off functions [19, Lemma 5.1]. Let $\phi : [0, 1] \to [0, 1]$ be a cut-off function which equals 1 near 0 and 0 near 1, and extend $\phi$ constantly to $\mathbb{R}$. Then, by taking $c_k \leq 1/(k^k|\phi|_{C^k})$ for $k > 1$, the cut-off functions $b(x) = \phi(||x - x_0||^2/\delta)$ are in $B$ for any $x_0 \in [0, 1]$ and any $\delta > 0$.

This topology is stronger than the $C^\infty$-topology; it is dense in $C^\infty$ by the Stone–Weierstrass theorem; and it has a useful property:

**Lemma C.3.** A sequence $b_i \in B$ of bounded Floer norm $|b_i| \leq \alpha < \infty$ has a subsequence which converges in the $C^\infty$-topology (but not necessarily in the $B$-topology) to a smooth $b$ with Floer norm $|b| \leq \alpha$, so $b \in B$.

**Proof.** For fixed $k$, the $C^k$-norms of the $b_i$ are bounded. Since $[0, 1]$ is compact, by the Arzelà–Ascoli theorem a subsequence of the $b_i$ converges in $C^{k-1}$. By a diagonal argument, a subsequence of the $b_i$ converges in $C^k$ for each $k$, so it converges in $C^\infty$ to some $b$. So $\sum_{k=1}^K c_k |b|_{C^k} = \lim_{i \to \infty} \sum_{k=1}^K c_k |b_i|_{C^k} \leq \alpha$ for any finite $K$. Hence, $\sum_{k=1}^\infty c_k |b|_{C^k} \leq \alpha$. \hfill $\Box$

We will write $|J'|$ for Floer’s norm, and we write $|J'|_W$ for the Floer norm of the restriction $J'|_W$ to the compactification of a bounded set $W \subset \overline{M}$, and we observe that the useful property mentioned above holds for Floer’s norm provided we work on a compact subset of $\overline{M}$.

**Theorem C.4.** Given a strictly increasing sequence $\varepsilon_m > 0$, there exist small constants $d_m > 0$ and large constants $c_m > 0$, such that: for any $\omega$-compatible almost complex structures $J_m$ which satisfy

1. $J_m = J$ on $\overline{M} \setminus K_m$,
2. $|J_m - J|_{K_k \setminus K_{k-1}} \leq d_k$ for all $k \in \{1, \ldots, m\}$ (defining $K_0 = \emptyset$),
we can guarantee that

3. $R_{A_k}(J_m) \leq R_{A_k}(J) + \varepsilon_m + \sum_{j=1}^k c_j$, for $k \in \{1, \ldots, m\}$, and all $m \in \mathbb{N}$.

We emphasize that these $J_m$ do not need to be related amongst each other.

**Proof.** The proof is by induction on $m$. For $m = 0$, there is nothing to prove. Suppose by contradiction that the induction fails at step $m$, so no $d_m > 0$ can be found. Then there is a
sequence $J'_i$ such that $J'_i = J$ on $\overline{M} \setminus K_m$ satisfying
\[
|J'_i - J|_{K_k \setminus K_{k-1}} \leq d_k \quad \text{for } k \in \{1, \ldots, m-1\}; \\
|J'_i - J|_{K_m \setminus K_{m-1}} \leq \delta_i \quad \text{with } \delta_i \to 0 \text{ as } i \to \infty; \\
R_{A_k}(J'_i) > R_{A_k}(J) + D_k \quad \text{for some } k \in \{1, \ldots, m\},
\]
where $D_k = \varepsilon_m + \sum_{j=1}^k c_j$ (we will define $c_m$ later). By passing to a subsequence, we can assume the latter inequality holds for the same $k$ and for the same choice of auxiliary data $\mathcal{A} \in \mathcal{A}_k$ for all $i$. Thus, there is a sequence of $J'_i$-Floer solutions $u_i$ for the same auxiliary data $\mathcal{A}$ such that $u_i$ escapes $K_{A_k}(J)$ by a distance of at least $D_k$.

The $J'_i$ only differ amongst each other on the compact set $K_m$, so the above useful property (Lemma C.3) applies. Thus, after passing to a subsequence, the $J'_i$ converge in the $C^\infty$-topology to an $\omega$-compatible complex structure $J'$ with $J' = J$ on $\overline{M} \setminus K_m$ and whose Floer norms satisfy the same bounds as the $J'_i$: $|J' - J|_{K_k \setminus K_{k-1}} \leq d_k$ for $k \in \{1, \ldots, m-1\}$. So $J'$ satisfies the conditions required for $J_{m-1}$ in the claim. Therefore, by the inductive hypothesis, $R_{A_k}(J') \leq R_{A_k}(J) + D_k - (\varepsilon_m - \varepsilon_{m-1})$ for $k \in \{1, \ldots, m-1\}$. So $J'$-Floer solutions for $A_k$ cannot escape $K_{A_k}(J)$ by distance $D_k$ for $k \in \{1, \ldots, m-1\}$ since $\varepsilon_m > \varepsilon_{m-1}$.

Since the $J'_i$ differ from $J$ only on the compact set $K_m$, the $u_i$ lie in a compact subset of $\overline{M}$ independent of $i$ (this follows from the no escape Lemma D.5, or from the assumptions on $J$ in Remark 1). Since $J'_i \to J'$ in the $C^\infty$-topology and since the $u_i$ all lie in a fixed compact subset, the $u_i$ converge by Gromov compactness to a possibly broken $J'$-Floer solution $u$. But then by continuity the $J'$-Floer solution $u$ escapes $K_{A_k}(J)$ by a distance of at least $D_k$ since the $u_i$ do: this contradicts the previous paragraph if $k \in \{1, \ldots, m-1\}$.

So finally suppose $k = m$, so $\mathcal{A} \in \mathcal{A}_m$. Since $J'$ differs from $J$ only on the compact set $K_{m-1}$, all $J'$-Floer solutions for $A_m$ lie in some compact subset $C_m \subset \overline{M}$ depending only on $J$, $A_m$, $K_{m-1}$ and not depending on $J'$ (again by the no escape Lemma or by the assumptions on $J$ in Remark 1). Define $c_m > 0$ large enough so that $\{R < R_{A_m}(J) + D_m\}$ contains $C_m$. We emphasize $c_m$ depends only on $J, A_m$ and $K_{m-1}$ and not on $J'$. By construction of $c_m$, all $J'$-Floer solutions for $A_m$ lie in $R < R_{A_m}(J) + D_m$. But the above argument produced a $J'$-Floer solution $u$ which contradicts this inequality. So, we obtained a contradiction. \qed

From now on, choose $\varepsilon_m - \varepsilon < \infty$ in Theorem C.4, and note that we may assume that $d_m \to 0$ as $m \to \infty$. Also, fix small neighbourhoods $W_m \subset K_m \setminus K_{m-1}$ of $\partial K_m$.

The difficulty in constructing $J_m$ as above is that it is not clear that one can interpolate a given $J_m|_{K_m \setminus W_m}$ with $J|_{\overline{M} \setminus K_m}$ without destroying the bound $|J_m - J|_{K_m \setminus K_{m-1}} \leq d_m$.

By the Remark above Lemma C.3, there is a cut-off function
\[
\phi_m : \overline{M} \to \mathbb{R}
\]
of bounded Floer norm which on $K_m \setminus W_m$ equals 1 and which equals 0 near $\partial K_m$ and outside $K_m$. One can choose the constants $c_k$ which define the Floer norm so that in general $|\phi_m \cdot J'| \leq |\phi_m||J'|$.

**Remark.** We prove that we can ensure $|fg| \leq |f| |g|$ for functions $[0,1] \to \mathbb{R}$ (this generalizes to almost complex structures). It suffices to ensure $|fg| \leq C|f| |g|$ for $C > 0$ independent of $f, g$ since we can then redefine $|\cdot|$ by rescaling it by $C$. By the Leibniz formula, $c_k|fg|_{C^k} = c_k \sum (\ell_j) |\partial^{i'} f|_{C^0} |\partial^{j'} g|_{C^0}$ summing over $\ell \leq k$, $i' + j' = \ell$. It suffices to bound this by $C \sum_{i+j=k} c_k \sum (\ell_j) |\partial^{i'} f|_{C^0} |\partial^{j'} g|_{C^0}$. Decomposing $|\cdot|_{C^j} = \sum_{j' \leq k} |\partial^{j'}|_{C^{j'}}$ the latter sum contains $C c_k |\partial^{i'} f|_{C^0} |\partial^{j'} g|_{C^0}$ for $i' + j' = \ell - k - i = j$. So it suffices to ensure $(\ell_j) c_k \leq C c_k e_{k-i}$, since $(\ell_j) \leq \binom{k}{i}$. So let $C = 1/c_0$ and inductively ensure $c_k \leq C c_k e_{k-i}/(\ell_j)$ for $1 \leq i \leq k - 1$. 

So if \( J_m \) satisfies \( |J_m - J|_{K_m \setminus K_m^{-1}} \leq d_m/|\phi_m| \) (instead of \( d_m \)), then the interpolation \( J' = (1 - \phi_m)J + \phi_m J_m \) satisfies the required bound:

\[
|J' - J|_{W_m} = |\phi_m (J_m - J)|_{W_m} \leq |\phi_m||J_m - J|_{W_m} \leq d_m
\]

(the linear combination which defines \( J' \) is understood to be computed in the Banach vector bundle of almost complex structures).

Using this interpolation trick, we can inductively construct \( J_m \) to satisfy conditions (1) and (2) of Theorem C.4 in such a way that \( J_i = J_m \) on \( K_m \setminus W_m \) for all \( i \geq m \). Since \( d_m \to 0 \) as \( m \to \infty \), these \( J_m \) manifestly converge to a \( J_{\infty} \) satisfying \( J_{\infty}|_{K_m^{-1}} = J_m|_{K_m^{-1}} \).

**Corollary C.5.** Let \( J_{\infty} \) be any \( \omega \)-compatible complex structure \( J_{\infty} \) on \( \overline{M} \) satisfying

\[
|J_{\infty} - J|_{K_k \setminus K_{k-1}} \leq d_k/|\phi_k| \quad \text{for all } k \geq 1.
\]

Then a ‘no escape Lemma’ holds: \( R_{A_k}(J_{\infty}) \leq R_{A_k}(J) + \varepsilon + \sum_{j=1}^{k} c_j \) for all \( k \geq 1 \).

**Proof.** Suppose by contradiction that a \( J_{\infty} \)-Floer solution \( u \) for \( A_k \) escapes \( K_{A_k}(J) \) by at least distance \( D = \varepsilon + \sum_{j=1}^{k} c_j \). The asymptotics of \( u \) are fixed, so the image of \( u \) lies in some \( K_{m-1} \) for \( m \gg k \). By the hypothesis, \( J_{\infty} \) arises via the above interpolation construction: let \( J_m = J_{\infty} \) on \( K_m \setminus W_m \) and then interpolate with \( J|_{M \setminus K_m} \). So \( u \) is a \( J_m \)-Floer solution for \( A_k \) since \( J_m|_{K_m^{-1}} = J_{\infty}|_{K_m^{-1}} \). But \( J_m \) satisfies the conditions in Theorem C.4, so \( u \) cannot escape \( K_{A_k}(J) \) by distance \( D \) since \( \varepsilon > \varepsilon_m \). This is a contradiction.

Let \( B \) denote the Banach space of \( \omega \)-compatible almost complex structures defined on some compact subset \( W \subset \overline{M} \) which have finite Floer norm. When we speak of making a generic perturbation of \( J_{\infty} \) on \( W \), we mean perturbing \( J_{\infty} \) on \( W \) by an element of a Baire subset of \( B \). We proved regularity statements for generic perturbations in Appendix A.6 using the \( C^\ell \)-norm, from which one bootstraps to the \( C^\infty \)-topology. Those results can also be proved using the Floer norm [37, p. 1345]; however, a Baire subset for the \( B \)-topology, although dense in \( B \) and so dense in \( C^\infty \), may not actually be \( C^\infty \)-Baire. However, for genericity statements, we really only care that countable intersections are still dense, and this is still true: a countable intersection of Baire subsets for the \( B \)-topology is Baire in \( B \) and hence dense in \( C^\infty \).

**Theorem C.6.** After a generic perturbation on each \( K_m \) of the \( J_{\infty} \) of Corollary C.5, the \( J_{\infty} \) will be regular for \( J_{\infty} \)-Floer solutions for \( A_k \), for all \( k \geq 1 \), and it will still satisfy the no escape Lemma \( R_{A_k}(J_{\infty}) \leq R_{A_k}(J) + \varepsilon + \sum_{j=1}^{k} c_j \) for all \( k \geq 1 \).

**Proof.** By the above interpolation procedure, \( J_{\infty} \) is a limit of \( J_m \) satisfying Theorem C.4 with \( J_{\infty}|_{K_m^{-1}} = J_m|_{K_m^{-1}} \). So a generic perturbation of \( J_{\infty} \) on \( K_m \) is precisely a generic perturbation of \( J_{m+1} \) on \( K_m \). By Appendix A.6, a generic perturbation of \( J_{m+1} \) on \( K_m \) ensures regularity for \( J_{m+1} \)-Floer solutions for \( A_k \) lying in \( K_m \). A countable intersection of Baire subsets is Baire, so we can ensure this for all \( k \geq 1 \) simultaneously (with \( m \) fixed). As we increase \( m \), we typically need to perturb \( J_{m+2}, J_{m+3}, \ldots \) also on \( K_m \) so that \( J_{m+1} \) on \( K_m \) has no longer equal \( J_{m+1}|_{K_m} \). However, these perturbations on \( K_m \) again involve a countable intersection of Baire subsets, so in fact we have a Baire subset worth of choices for the value of \( J_{m+1}|_{K_m} \) while still satisfying Theorem C.4. Any \( J_{\infty} \)-Floer solution \( u \) for \( A_k \) lands in some \( K_m \), so \( u \) is also a \( J_{m+1} \)-Floer solution, and so \( u \) is regular since \( J_{m+1} \) is regular for solutions lying in \( K_m \) for any \( A_k \). 

\[ \square \]
C.7. Transversality and compactness can both be achieved for the moduli spaces of Floer solutions for nonlinear Hamiltonians, after a generic (non-contact type) perturbation $J_{\infty}$ of a given $J$ of contact type at infinity, provided the perturbation decays to zero sufficiently fast as $R \to \infty$ (we require $|J_{\infty} - J|_{K_m \setminus K_{m-1}} \leq \delta_m$ for certain $\delta_m > 0$ depending on $J, K_m$).

C.3. TQFT structure using nonlinear Hamiltonians

By the previous section, we can define TQFT operations $\psi_S : \bigotimes_b SH^* (B_b Q) \to \bigotimes_a SH^* (A_a Q)$ for $\sum A_a \geq \sum B_b$.

By Lemma C.1, all these $SH^*(cQ)$ can be identified with $SH^*(Q)$. So in fact, if one defined $SH^*(M) = SH^*(Q)$, one would define the TQFT operations

$$\psi_S : SH^*(Q)^{\otimes q} \to SH^*(Q)^{\otimes p} \quad (p \geq 1, q \geq 0),$$

by first applying $\psi_S : SH^*(Q)^{\otimes q} \to SH^*(cQ)^{\otimes p}$, for any $c > 0$ such that $q \leq cp$, and then identifying $SH^*(cQ) \cong SH^*(Q)$ via the direct limit of the continuation isomorphisms $SH^*(cH^m) \cong SH^*(H^{cm})$, using the notation $H^m$ defined in the proof of Lemma C.1.

Theorem C.8. The $\psi_S : SH^*(Q)^{\otimes q} \to SH^*(Q)^{\otimes p}$ are independent of $c > 0$ and they define a TQFT on $SH^*(Q)$ which, via the identification $SH^*(M) = \lim SH^*(H) \cong SH^*(Q)$ of Lemma C.1, agrees with the TQFT in Appendix A.11 constructed using linear Hamiltonians.

Proof. For $q \leq cp$, $m \leq m'$, $c \leq c'$ there is a diagram

$$SH^*(H^m)^{\otimes q} \xrightarrow{\text{TF} \psi_{m',c}} SH^*(cH^m)^{\otimes p} \xrightarrow{\text{cont.}} SH^*(c'H^m)^{\otimes p} \xrightarrow{\text{cont.}} SH^*(c'H^{cm})^{\otimes p} \xrightarrow{\text{cont.}} SH^*(H^{cm})^{\otimes p} \xrightarrow{\text{cont.}} SH^*(H^m)^{\otimes p},$$

where $\psi_{m,c}$ and $\psi_{m',c}$ are the $\psi_S$-maps defined using the same $(S, \beta)$, but different $H^m$ and $H^{cm}$.

The 1st square commutes by Theorem A.14, the other squares commute by Lemma 3.1(2). The horizontal continuations in the 2nd and 4th squares are isomorphisms by Lemma 3.1(4), and induce $SH^*(cQ) \cong SH^*(Q)$, $SH^*(c'Q) \cong SH^*(Q)$.

The 1-orbits of $H^m$ and $cH^m$ lie in $K_m = \{ R \leq R_m \}$ (the complement of the region where $H^m$ has slope $m$). By the no escape Lemma D.3 the Floer solutions for $\psi_{m,c}$ lie in $K_m$. By the no escape Lemma D.5, the Floer solutions with asymptotics in $K_m$ involved in $\psi_S : SC^*(Q)^{\otimes q} \to SC^*(cQ)^{\otimes p}$ must also lie in $K_m$. By construction, $Q|_{K_m} = H^m|_{K_m}, cQ|_{K_m} = cH^m|_{K_m}$, so those two collections of Floer solutions are exactly the same.

By definition $SC^*(Q), SC^*(cQ)$ involve only finite linear combinations of 1-orbits, and by the energy estimate Appendix D.1 there are only finitely many Floer solutions involved in $\psi(\bigotimes y_b)$ for given 1-orbits $y_b$ of $cQ$. So, in the direct limit $m \to \infty$, the 1st and 2nd squares yield

$$SH^*(Q)^{\otimes q} \to SH^*(cQ)^{\otimes p} \to SH^*(Q)^{\otimes p},$$

where the first map is $\psi_S = \lim \psi_{m,c}$. We now prove that this composite is independent of the choice of $c$. By Lemma 3.1(2) the composite of the horizontal maps of the 2nd and 3rd squares give the continuation maps $SH^*(cH^m) \to SH^*(c'H^m)$ and $SH^*(cH^m) \to SH^*(c'H^m)$, and by Lemma 3.1(1) these are equal to the continuations induced by a continuation cylinder $(Z, \beta_Z)$ with weights $c'$ and $c$ at the ends (see Example A.2). So composing the 1st square with these continuations corresponds to attaching $(Z, \beta_Z)$ to the negative ends of $(S, \beta)$. So by
Theorem A.14, the composite of the first three squares gives the TQFT operations $\psi_{m',c'}$ and $\psi_{m,c'}$, and these together with the 4th square gives $SH^*(Q)^{\otimes q} \to SH^*(c'Q)^{\otimes p} \to SH^*(Q)^{\otimes p}$ in the limit. So to prove independence of $c$, it remains to show that composing the 3rd and 4th square induces the identity on $SH^*(Q)$ in the limit. But again by Lemma 3.1(s) the composite of the 3rd and 4th squares gives a square of continuation maps involving only the Hamiltonians $H^{c_m}, H^{c_m'}, H^{c_m}$, and $H^{c_m'}$, and they induce the identity on $SH^*(Q)$ in the direct limit since all these maps are in fact inclusions by construction (this is assuming $q$ is convex, but a ladder argument (see the Remark following the definition of nonlinear Hamiltonians in Section C.1) proves this also if we only assume $q'(R) \to \infty$ and $\partial_q(R) \to -\infty$).

This very last argument also proves that a cylinder induces the identity operation on $SH^*(Q)$. Independence of the TQFT operations on the choices of $S, \beta$ and $J$ and invariance follow from the above construction and the analogous properties for linear Hamiltonians proved in Section A. It remains to prove that gluings of surfaces $S_2 \# S_1$ correspond to compositions.

Consider the 1st and 2nd squares above for two choices of data $(S_1, \beta_1, c_1)$ and $(S_2, \beta_2, c_2)$ with $p_1 = q_2$. Composing these four squares together gives horizontal maps

$$SH^*(H^m)^{\otimes q_1} \xrightarrow{\psi_{S_1}} SH^*(c_1 H^m)^{\otimes p_1} \xrightarrow{\text{cont}} SH^*(H^{c_1 m})^{\otimes p_1} \xrightarrow{\psi_{S_2}} SH^*(c_2 H^{c_1 m})^{\otimes p_2} \xrightarrow{\text{cont}} SH^*(H^{c_2 c_1 m})^{\otimes p_2}.$$ 

Here, $\psi_{S_2}$ is defined using the data $(S_2, \beta_2, H^{c_1 m})$ with weights $c_2$ and 1, respectively, at negative and positive ends. Without changing the moduli spaces, we can instead use the data $(S_2, c_2 \beta_2, (1/c_1) H^{c_1 m})$ using weights $c_2 c_1$ and $c_1$ respectively (this will ensure that $\beta_1$ and $\beta_2$ have matching weights when we later glue). The 2nd map involves cylinders $(Z, \beta_2)$ with $\beta_2 = c_1 dt$ which use a Hamiltonian $H_z$ depending on $z \in Z$ which equals $(1/c_1) H^{c_1 m}$ and $H^m$ near the two ends (these Hamiltonians both have slope $m$ at infinity, so we can pick $H_z$ independent of $z$ for $R \gg 0$). Let $\psi_0$ denote the composite of the first three maps above (the no escape Lemma still holds by Remark D.4(2) since $\partial_z H_z = 0$ for $R \gg 0$). By Theorem A.12, this corresponds to gluing $S_2 \# (\bigcup_{p_1} Z') \# S_1$. Now consider a different gluing: $(\bigcup_{p_2} Z') \# S_2 \# S_1$ obtained by using the Hamiltonian $H^m$ on both $S_2$ and $S_1$ but on the cylinder $Z'$ we use $H_z$ and $\beta_z = c_1 c_2 dt$. These two gluings $S_2 \# (\bigcup_{p_1} Z') \# S_1$ and $(\bigcup_{p_2} Z') \# S_2 \# S_1$ share the same asymptotic data (in particular, the Hamiltonians agree at the asymptotics). The same proof as in Theorem A.10, using parametrized moduli spaces, shows that the two gluings induce chain homotopic maps. By Theorem A.12, $\psi_{(\bigcup Z')} \# S_2 \# S_1 \circ \psi_{S_2} \circ \psi_{S_2} \# S_1$. Thus the above horizontal map equals

$$SH^*(H^m)^{\otimes q_1} \xrightarrow{\psi_{S_2} \# S_1} SH^*(c_2 c_1 H^m)^{\otimes p_1} \xrightarrow{\text{cont}} SH^*(c_2 H^{c_1 m})^{\otimes p_2} \xrightarrow{\psi_{S_2} \# S_1} SH^*(H^{c_2 c_1 m})^{\otimes p_2}.$$ 

By Lemma 3.1(2), the composite of the last two maps is the continuation $SH^*(c_2 c_1 H^m)^{\otimes p_2} \to SH^*(H^{c_2 c_1 m})^{\otimes p_2}$. As before, this argument is compatible with the continuations which change $H^m$ to $H'$, so in the direct limit this proves that $\psi_{S_2} \circ \psi_{S_1} = \psi_{S_2 \# S_1}$ on $SH^*(Q)$.

**Remark.** Similarly, in the wrapped case, $HW^*(L) \cong HW^*(L; Q)$ respects the TQFT.

C.4. The TQFT for $T^*N$ using Levi-Civita $J$ instead of contact type $J$

For $\overline{M} = T^*N$, Abbondandolo-Schwarz [1, 2] use an $\omega$-compatible almost complex structure $J_{LC}$ on $\overline{M} = T^*N$ which is induced by a Levi-Civita connection for $N$ ([1, Section 1.5]), and these are not of contact type at infinity. We now prove that their Floer cohomology $SH^*(Q; J_{LC})$ agrees with our $SH^*(Q; J)$ for $Q$ with $q(R)$ of quadratic growth in $R$, and that the products agree.

**Claim 1.** There is an isomorphism $SH^*(Q; J_{LC}) \cong SH^*(Q; J)$ respecting action-filtrations.
Proof. The Floer complex admits a filtration by action (Section 8), and recall $\mathcal{A}_Q$ is bounded above since $\mathcal{A}_Q$ is bounded above as $R \to \infty$. By Lemma C.1, it suffices to do the following:

1. build filtered-isomorphisms $\varphi_c : \text{SH}^*(Q; J_{LC}; \mathcal{A}_Q \geq c) \cong \text{SH}^*(Q; J; \mathcal{A}_Q \geq c)$;
2. show that the $\varphi_c$ are compatible with decreasing the action bound $c \in \mathbb{R}$.

Our $\text{SH}^*(Q; J; \mathcal{A}_Q \geq c)$ are invariant under deforming $J$ on a compact subset, provided $J$ is of contact type at infinity. So we can pick $J$ to equal $J_{LC}$ on any large compact $K \subset \overline{M}$. Abbondandolo and Schwarz [1, Theorem 1.14] proved $L^\infty$ estimates which imply that $J_{LC}$-Floer trajectories whose asymptotics satisfy $\mathcal{A}_Q \geq c$ do not have enough energy to escape a compact subset of $\overline{M}$ determined by $c$, and we pick $K$ so that $K$ contains this subset. Thus, the same statement holds for $J$-Floer trajectories since $J|_K = J_{LC}|_K$. So $\varphi_c$ in (1) is an identification for this $J$. Then (2) follows because the inclusions $\text{SC}^*(Q; J; \mathcal{A}_Q \geq c) \to \text{SC}^*(Q; J; \mathcal{A}_Q \geq c')$, for $c \geq c'$, are invariant on cohomology under deforming $J$ on compact subsets. □

Claim 2. The isomorphism $\text{SH}^*(Q; J_{LC}) \cong \text{SH}^*(Q; J)$ respects the product structures.

Proof. This follows by the same argument, since Abbondandolo–Schwarz proved $L^\infty$ estimates also for $J_{LC}$-Floer solutions defined on a suitable pair-of-pants surface [2, Proposition 6.2]. □

Claim 3. Theorem 14.2 holds for $\text{SH}^*(Q; J)$.

Proof. By [1, Theorem 3.1] $\text{SH}^*(Q; J_{LC}) \cong H_{n-*}(\mathcal{L}N)$ respects the action-filtrations (Remark 14.1). By Claim 1, $\text{SH}^*(Q; J_{LC}) \cong \text{SH}^*(Q; J)$ respects the filtration. □

Appendix D. Energy, maximum principle and no escape Lemma

D.1. Energy of Floer solutions

Define the energy of a map $u : S \to \overline{M}$ by

$$E(u) = \frac{1}{2} \int_S \|du - X \otimes \beta\|^2 \text{vol}_S.$$  

Explanation. Let $Y \in \text{Hom}(TS, u^*T\overline{M})$. In a local holomorphic coordinate $s + it$ for $(S, j)$:

$$\text{vol}_S = ds \land dt; \quad Y = Y_s ds + Y_t dt \quad \text{where} \quad Y_s = Y(\partial_s), \quad Y_t = Y(\partial_t); \quad \text{and} \quad \|Y\|^2 = |Y_s|^2 + |Y_t|^2$$

where $|\cdot|^2 = g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$ on $T\overline{M}$. Fact: $\|Y\|^2 \text{vol}_S$ is independent of the choice of $s + it$.

Example. For a cylinder $S = Z$ with $\beta = dt$, $E(u) = \frac{1}{2} \int (|\partial_s u|^2 + |\partial_t u - X|^2) \, ds \land dt$, so for a Floer trajectory $u$ we get the usual energy $E(u) = \int |\partial_s u|^2 \, ds \land dt$.

Recall $Y^{0,1} = \frac{1}{2} (Y + j \circ Y \circ j)$, so $Y^{0,1} = 0$ implies $Y \circ j = J \circ Y$, so locally $Y_t = JY_s$ since $j\partial_s = \partial_t$, therefore $\frac{1}{2} \|Y\|^2 \text{vol}_S = \omega(Y_s, Y_t) \, ds \land dt$.

For $Y = du - X \otimes \beta$, decompose $\beta = \beta_s ds + \beta_t dt$, then $(du - X \otimes \beta)^{0,1} = 0$ implies

$$\frac{1}{2} \|du - X \otimes \beta\|^2 \text{vol}_S = \omega(\partial_s u - X \beta_s, \partial_t u - X \beta_t) \, ds \land dt$$

$$= [\omega(\partial_s u, \partial_t u) + dH(\partial_t u) \beta_s - dH(\partial_s u) \beta_t] \, ds \land dt$$

$$= u^* \omega - u^* (dH) \land \beta.$$
We can rewrite \(-u^*(dH) \wedge \beta = -d(u^*(H)\beta) + u^*(H) \, d\beta\). Assuming \(u\) is a Floer solution for weights \(A_a\) and \(B_b\), with \(H \geq 0\), \(d\beta \leq 0\), by Stokes’ theorem we obtain the energy estimate:

\[
E(u) = \int_S u^*\omega - u^*(dH) \wedge \beta 
\leq \int_S u^*\omega - d(u^*(H)\beta) \quad \text{(since } u^*(H) \, d\beta \leq 0) \\
= \int_S d(u^*\theta - u^*(H) \beta) \quad \text{(since } \omega = d\theta) \\
= \sum_{\text{negative ends } a} k_{A_aH}(x_a) - \sum_{\text{positive ends } b} k_{B_bH}(y_b).
\]

\[D.2. \quad \text{Energy of wrapped solutions}\]

Recall from Subsection 6.12 that wrapped solutions \(u : S \to \overline{M}\) solve \((du - X \otimes \beta)^{0,1} = 0\) with \(u(\partial S) \subset \bar{L}\), where \(\beta|_{\partial S} = 0\), \(d\beta \leq 0\) and \(\theta|_{\bar{L}} = df\). The same proof as in Appendix D.1 shows that wrapped solutions with asymptotic chords \(x_a, y_b\) satisfy the a priori estimate \(E(u) \leq \sum k_{A_aH}(x_a) - \sum k_{B_bH}(y_b)\) (recall \(k_H\), defined in Subsection 4.3, involves \(f\)).

\[D.3. \quad \text{Maximum principle for Floer solutions}\]

By the discussion in Appendix D.1, the equation \(Y^{0,1} = 0\) for \(Y = du - X \otimes \beta\) in a local holomorphic coordinate \(s + it\) for \(S\) corresponds to

\[
\begin{align*}
\partial_t u &= X\beta_t + J\partial_s u - JX\beta_s, \\
\partial_s u &= X\beta_s - J\partial_t u + JX\beta_t.
\end{align*}
\]

(\text{D.1) }

\[\text{Lemma D.1. For } R \geq R_0 \text{ assume: } J \text{ is of contact type and } H = h(R) \text{ only depends on } R \text{ with } h' > 0. \text{ Then for any local Floer solution } u \text{ landing in } R \geq R_0, \text{ the coordinate } R \circ u \text{ cannot have local maxima unless it is constant. So global Floer solutions with asymptotics } x_a, y_b \text{ must lie in the region } R \leq \max\{R(x_a), R(y_b), R_0\}. \text{ The result also holds in the following situations:}\]

1. \(\text{if } H \text{ is time-dependent near the ends (where } \beta \text{ is a multiple of } dt;\)
2. \(\text{if } h = h_z(R) \text{ depends on } z = s + it \text{ near the ends (where } \beta \text{ is a multiple of } dt), \text{ provided we assume the monotonicity condition: } \partial_h h'_z \leq 0 \text{ for } R \geq R_0;\)
3. \(\text{if } J = J_z \text{ depends on } z \in S, \text{ assuming the contact condition } dR = J_z^*\theta \text{ for } R \geq R_0.\)

Proof. Let \(\rho(s,t)\) denote the \(R\) coordinate of \(u(s,t)\). Using the contact condition \(\theta = -dR \circ J\),

\[
\begin{align*}
\partial_s \rho &= dR(\partial_s u) = dR(X\beta_s - J\partial_t u + JX\beta_t) \\
&= \theta(\partial_t u) - \theta(X)\beta_t, \\
\partial_t \rho &= dR(\partial_t u) = dR(X\beta_t + J\partial_s u - JX\beta_s) \\
&= -\theta(\partial_s u) + \theta(X)\beta_s, \\
d^c \rho &= d\rho \circ j = (\partial_t \rho) \, ds - (\partial_s \rho) \, dt \\
&= -\theta(\partial_s u) \, ds - \theta(\partial_t u) \, dt + \theta(X)\beta_s \, ds + \theta(X)\beta_t \, dt \\
&= -u^*\theta + \theta(X)\beta, \\
-\ddbar \rho &= u^*\omega - d(\theta(X)\beta) \\
&= \frac{1}{2} \|du - X \otimes \beta\|^2 \, ds \wedge dt + u^*(dH) \wedge \beta - d(\theta(X)\beta).
\end{align*}
\]
By Subsection 2.1, $X = h'(R)\mathcal{R}$ where $\mathcal{R}$ is the Reeb vector field, so $\theta(X) = Rh'(R)$. Thus, 

$$u^*(dH) \wedge \beta - d(\theta(X)\beta) = h'(\rho)d\rho \wedge \beta - d(\rho h'(\rho)\beta) = -\rho h''(\rho)d\rho \wedge \beta - \rho h'(\rho)d\beta.$$ 

Thus, $\Delta \rho ds \wedge dt = -ddc\rho \geq \rho h''(\rho)d\rho \wedge \beta - \rho h'(\rho)d\beta$, so 

$$\Delta \rho + \text{(first-order terms in } \rho) \geq -\rho h'(\rho)d\beta/ds \wedge dt.$$ 

Since $d\beta \leq 0$, $h' \geq 0$, $\rho \geq 0$, the right-hand side is at least 0. The claim follows by the maximum principle [17, Section 6.4] for the elliptic operator $L = \Delta + Rh''(R)\beta_{\partial_S} - Rh''(R)\beta_{\partial_t}$ since $Lu \geq 0$.

1. If $H = h_t$ on an end where $\beta = C dt$ ($C > 0$), then the extra term $-\rho(\partial_t h'_t)dt \wedge \beta = 0$.
2. If $H = h_z$, then the extra term $-\rho(\partial_z h'_z)C \geq 0$ provided $\partial_z h'_z \leq 0$.
3. A $z$-dependence for $J = J_z$ does not affect the proof.

D.4. Maximum principle for the Lagrangian setting

Recall from Subsection 6.12 that wrapped solutions $u : S \to \overline{\mathcal{M}}$ solve $(du - X \otimes \beta)^{0,1} = 0$ with $u(\partial S) \subset \overline{L}$, where $\beta|_{\partial S} = 0$ and $d\beta \leq 0$.

**Lemma D.2.** Under the assumptions of Lemma D.1, also any local wrapped solution $u$ landing in $R \geq R_0 \geq 1$ cannot admit a local maximum of $R \circ u$ unless $R \circ u$ is constant, so global wrapped solutions with asymptotics $x_a$ and $y_b$ lie in the region $R \leq \max \{R(x_a), R(y_b), R_0\}$.

**Proof.** The proof is identical to Lemma D.1, except maxima might occur on $\partial S$. But then by Hopf’s Lemma [17, Section 6.4.2] the derivative $\partial_t \rho$ of $\rho = R \circ u$ in the outward normal direction $\partial_t$ on $\partial S$ would be strictly positive at those maxima, which is false:

$$\partial_t \rho = dR(\partial_t u) = dR(X\beta_t + J\partial_s u - JX\beta_s) = -\theta(\partial_s u) = 0$$

using $dR(X) = 0$, $dR \circ J = -\theta$, the pull-back $\beta|_{\partial S} = 0$ so $\beta_s = 0$ on $\partial S$, and finally the pull-back $\theta|_{\partial S} = 0$ on the collar and $\partial_s u \in TL$ on $\partial S$ (since $\partial_s = -j\partial_t$ is tangent to $\partial S$).

D.5. No escape Lemma

In Corollary 9.4, we need the following lemma. This result is analogous to Abouzaid and Seidel [4, Lemma 7.2], adapted to our setup.

Let $(V, d\theta)$ be a non-compact symplectic manifold with boundary $\partial V$ of negative contact type: the Liouville field $Z$, defined by $d\theta(Z, \cdot) = \theta$, points strictly inwards.

Near $\partial V$ define the coordinate $R = R_0 e^r$ parametrizing the time $r$ flow of $Z$ starting from $\partial V = \{R = R_0\}$ (cf. Subsection 2.1).

Assume $J$ is of contact type along $\partial V$, meaning $J^* \theta = dR$ holds at points of $\partial V$. Suppose $H : V \to [0, \infty)$ only depends on $R$ near $\partial V$, say $H = h(R)$. Define $X$ by $d\theta(\cdot, X) = dH$ and define $R = JZ$. Then by the contact condition one easily obtains that $X = h'(R)\mathcal{R}$ near $\partial V$.

Suppose $h(R) = mR$ is linear near $\partial V$ (see Lemma D.5 for the general case). Let $S$ be a compact Riemann surface with boundary and let $\beta$ be a 1-form on $S$ with $d\beta \leq 0$.

**Lemma D.3.** Any solution $u : S \to V$ of $(du - X \otimes \beta)^{0,1} = 0$, with $u(\partial S) \subset \partial V$, must map entirely into $\partial V$ and must solve $du = X \otimes \beta$. 

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Proof. We use a trick we learnt from Mohammed Abouzaid: we show \( E(u) \leq 0 \), so \( E(u) = 0 \), so \( du = X \otimes \beta \), so \( du \) lands in the span of \( X = h'\mathcal{R} \subset T\partial V \), thus \( u(S) \subset \partial V \) as required.

\[
E(u) = \int_S u^* d\theta - u^* (dH) \wedge \beta \quad \text{(Subsection D.1 using } \omega = d\theta) \\
\leq \int_S d(u^* \theta) - d(u^* H \beta) \quad (d\beta \leq 0 \text{ and } H \geq 0) \\
= \int_S u^* \theta - (u^* H) \beta \quad \text{(Stokes' Theorem)} \\
= \int_S u^* \theta - \theta(X) \beta \quad \text{(on } u(\partial S) \subset \partial V : H = mR = \theta(h'(R)\mathcal{R}) = \theta(X)) \\
= \int_S \theta(du - X \otimes \beta) \\
= - \theta J(du - X \otimes \beta) \quad \text{(since } (du - X \otimes \beta)^{0,1} = 0) \\
= \int_{\partial S} -dR(du - X \otimes \beta) \quad \text{ } (J \text{ is of contact type along } u(\partial S) \subset \partial V) \\
= \int_{\partial S} -dR(du) \quad \text{ } (dR \text{ vanishes on } X = h'(R)\mathcal{R} \text{ on } u(\partial S) \subset \partial V)
\]

Let \( \mathbf{n} \) be outward normal direction along \( \partial S \subset S \). Then \( (\mathbf{n}, j\mathbf{n}) \) is an oriented frame, so \( \partial S \) is oriented by \( j\mathbf{n} \). Now \( dR(du) j(j\mathbf{n}) = d(R \circ u) \cdot (j\mathbf{n}) \geq 0 \) since in the inward direction \( -\mathbf{n}, R \circ u \) can only increase since \( \partial V \) minimizes \( R \). So \( E(u) \leq 0 \).

Remark D.4. (1) One can allow \( u \) to have negative/positive punctures, provided that the asymptotics \( x_a \) and \( y_b \) satisfy \( \sum A_{a,H}(x_a) - \sum A_{b,H}(y_b) \leq 0 \) (this is the new contribution in the above proof to the integral \( \int_{\partial S} u^* \theta - (u^* H) \beta \)).

(2) If \( H = H_z \) depends on \( z \in S \) on a subset of \( S \) parametrized by a holomorphic coordinate \( z = s + it \) for a subset of values of \( (s, t) \in \mathbb{R} \times S^1 \) where \( \beta = c dt \) for some \( c > 0 \), then on that subset we require that \( \partial_s H \leq 0 \) and near \( \partial V \) we require \( H = m_a R \) for slopes \( m_a > 0 \) depending on \( s \). So in the above proof, \( u^*(dH) \wedge \beta = d(u^* H) \wedge \beta - c \partial_s (u^* H) ds \wedge dt \), and this last term has the correct sign needed to prove \( E(u) \leq 0 \). Also \( H = m_a R = \theta(X_{H_z}) \) on \( \partial V \) and hence on \( u(\partial S) \subset \partial V \).

In the setting of Lemma D.5, if \( H = H_z \) on a subset as above, then we require \( \partial_s H \leq 0 \) and near \( \partial V \) we require \( H = q_z(R) \) with \( A_{q_z}(R_0) \leq 0 \) and \( \partial_s A_{q_z}(R_0) \geq 0 \). This ensures that \( -d(A_{q_z}(R_0)) = -c \partial_s A_{q_z}(R_0) ds \wedge dt \leq 0 \) on that subset, so in the last line of the proof of Lemma D.5 we still obtain \( -\int_S d(A_{q_z}(R_0) \beta) \leq 0 \).

(3) If \( J \) is of contact type on all of \( V \) then, by the proof of Lemma D.1, \( \Delta R ds \wedge dt = -d\sqrt{R} = u^* d\theta - d(\theta(X) \beta) \). So the second half of the above proof is Green’s formula \cite{[17], Appendix C.2} for \( S \): \( \int_{\partial S} u^* \theta - \theta(X) \beta = \int_S \Delta R ds \wedge dt = \int_{\partial S} (\partial R/\partial \mathbf{n}) dS \leq 0 \).

D.6. No escape Lemma for nonlinear \( H \)

We need the following lemma in the Technical Remarks in Appendix C.1. For \( q : \mathbb{R} \to \mathbb{R} \), define the action function \( A_q(R) = -Rq'(R) + q(R) \).

Lemma D.5. In Lemma D.3, suppose \( H : V \to [0, \infty) \) has the form \( H = q(R) \) near \( \partial V = \{ R = R_0 \} \) instead of \( H = mR \). If \( A_q(R_0) \leq 0 \) then the conclusion of the lemma holds.
Proof. It suffices to show \( \int_{\partial S} u^{*}\theta - (u^{*}H)\beta \leq \int_{\partial S} u^{*}\theta - \theta(X)\beta \) since this was the only equality in the proof of Lemma D.3 that used the assumption \( H = mR \) near \( \partial V \).

So we need \( \int_{\partial S} (\theta(X) - u^{*}H)\beta \leq 0 \). On \( u(\partial S) \subset \partial V \), \( \theta(X) = \theta(q'(R)R) = Rq'(R_0) \) and \( u^{*}H = q(R_0) \). So, using Stokes' theorem, \( d\beta \leq 0 \) and \( \mathcal{A}_{q}(R_0) \leq 0 \):

\[
\int_{\partial S} (\theta(X) - u^{*}H)\beta = \int_{\partial S} (Rq'(R_0) - q(R_0))\beta = -\mathcal{A}_{q}(R_0) \int_{\partial S} \beta \\
= -\mathcal{A}_{q}(R_0) \int_{S} d\beta \leq 0.
\]

\( \square \)

D.7. No escape Lemma for the Lagrangian setting

To study the Viterbo restriction in the wrapped setup in Subsection 9.3, we need the analogue of Lemma D.3 when the map \( u : S \to V \) has Lagrangian boundary conditions, which is due to Abouzaid and Seidel [4, Lemma 7.2].

We use the same notation as in Appendix D.5: \( (V, d\theta) \), \( R, \partial V = \{ R = R_0 \} \), \( H, J, S, \beta \). The novelty is that the Riemann surface \( S \) has corners, and the corners divide the boundary of \( S \) into pieces which are closed intervals or circles. These pieces are labelled by the two letters \( b, l \) (because later our maps \( u : S \to V \) will be required to land in the boundary \( \partial V \) on the \( b \) pieces and in a Lagrangian \( L \) on the \( l \) pieces), and two pieces intersecting at a corner must carry different labels. Abbreviate the decomposition by \( \partial S = \partial_{b}S \cup \partial_{l}S \). We require \( \partial_{b}S \neq \emptyset \).

Let \( L \subset V \) be an exact Lagrangian, say \( \theta|_L = df \), such that \( \theta|_L \) vanishes to infinite order on the boundary \( \partial L = L \cap \partial V \). In addition to \( d\beta \leq 0 \), we also require that the pull-back \( \beta|_{\partial_{l}S} = 0 \).

Finally, we need to strengthen the exactness assumption on \( L : \{ f|_{\partial L} = 0 \} \).

**Lemma D.6.** Any solution \( u : S \to V \) of \( (du - X \otimes \beta)^{0,1} = 0 \), with \( u(\partial_{b}S) \subset \partial V \) and \( u(\partial_{l}S) \subset L \), must map entirely into \( \partial V \) and must solve \( du = X \otimes \beta \).

**Proof.** We run the proof of Lemma D.3. Thus, \( E(u) \leq \int_{\partial S} u^{*}\theta - (u^{*}H)\beta \). The new contributions arise along the pieces \( \gamma \subset \partial_{l}S \). Since \( (u^{*}H)\beta|_{\gamma} = 0 \), only the \( u^{*}\theta|_{L} = u^{*}df \) integrand contributes. By Stokes' theorem, \( \int_{\gamma} u^{*}\theta = 0 \) for circles \( \gamma \). For intervals \( \gamma \), \( \int_{\gamma} u^{*}\theta = f(z) - f(z') \) for corners \( z, z' \in \partial_{l}S \cap \partial_{b}S \) so this also vanishes by the assumption \( f|_{\partial L} = 0 \).

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