Dissipative dynamics of a kink state in a Bose-condensed gas.

P.O. Fedichev1,2, A.E. Muryshkev2 and G.V. Shlyapnikov1−3
1 FOM Institute for Atomic and Molecular Physics, Kruislaan 407, 1098 SJ Amsterdam, The Netherlands.
2 Russian Research Center, Kurchatov Institute, Kurchatov Square, 123182 Moscow, Russia.
3 Laboratoire Kastler Brossel**, 24, Rue Lhomond, F-75231, Paris Cedex 05, France

We develop a theory of dissipative dynamics of a kink state in a finite-temperature Bose-condensed gas. We find that due to the interaction with the thermal cloud the kink state accelerates towards the velocity of sound and continuously transforms to the ground-state condensate. We calculate the lifetime of a kink state in a trapped gas and discuss possible experimental implications.

I. INTRODUCTION

The recent successful experiments on Bose-Einstein condensation (BEC) in trapped clouds of alkali atoms1−3 have boosted an interest in the physics of ultracold gases.4−6 One of the challenging goals is to create and study macroscopically (topologically) excited Bose-condensed states, such as 2D or 3D vortices and their 1D analogies, the so-called kinks. These excited states attract a wide attention because they behave as particle-like objects (solitons) and thus bring in the analogies with high-energy physics, where elementary particles are suggested to be solitons of fundamental fields.

The simplest example of an excited condensate state is a kink-wise state in a cylindrical harmonic trap.7,8 This state has a macroscopic wavefunction with one nodal plane perpendicular to the symmetry axis and represents a non-linear wave of matter. The investigation of these waves is especially interesting for understanding to what extent they are similar to light waves in non-linear optics and what is the difference related, in particular, to the interparticle interaction in a matter wave.

Fundamental limitations for creating and studying kink states concern their stability. First of all, a kink state is always thermodynamically unstable, since its energy is larger than the energy of the ground-state Bose condensate. In contrast to the well-known kink solution in 1D,6 in three dimensions the kink state is also characterized by the presence of a transverse dynamical instability, i.e. the instability of small-amplitude normal modes of the motion parallel to the nodal plane. In order to suppress this instability in a trap, one should strongly confine the radial (transverse) motion and turn to an effectively 1D case by making the radial frequency larger than the mean-field interparticle interaction.6 It is this condition that makes a kink-wise condensate stable in the limit of zero temperature.

The thermodynamic instability will lead to decay of the kink state only in the presence of dissipative processes. Those are related, for example, to the interaction of a kink with a thermal cloud. In this Letter we analyze the dissipative dynamics of a kink state in a Bose condensate at finite temperatures. We show that the thermal excitations scattered by a kink stimulate its diffusive motion, with an increasing mean velocity, and induce the dissipative force proportional to the velocity of the kink. After a short time the dissipative force takes over the diffusion and the kink steadily accelerates towards the velocity of sound. In the course of accelerating the kink state continuously transforms to the ground-state condensate.

Importantly, this process is insensitive to the size of the condensate in the direction perpendicular to the nodal plane. The key features remain the same as in an infinitely long condensate. This is fundamentally different from the dissipative dynamics of a vortex state,9 although in some sense the interaction with the thermal excitations is similar. The vortex state has a topological charge (circulation) and in an infinitely large system can not continuously transform to the ground-state condensate. The vortex can loose the charge and, hence, decay only when approaching the border of the condensate.

II. MEAN-FIELD DYNAMICS OF A KINK STATE

We first briefly outline the mean-field (frictionless) dynamics of a kink state as a solution of the 1D Gross-Pitaevskii equation (GPE) in a homogeneous condensate of density \( n_0 \).

\[
\begin{align*}
\imath \hbar \frac{\partial \Psi_0}{\partial t} &= \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + g |\Psi_0|^2 - \mu \right) \Psi_0, \\
\end{align*}
\]

where \( g = 4\pi \hbar^2 a/m \), \( a \) is the s-wave scattering length, \( m \) the atom mass, and \( \mu = g n_0 \) is the chemical potential. The condensate wavefunction \( \Psi_0 \) is characterized by a single distance scale, the correlation length \( l_0 = \hbar/\sqrt{mn_0g} \). The validity of Eq. (1) for describing spatial variations of \( \Psi_0 \) requires \( l_0 \) much larger than the mean interparticle separation along the x axis. The corresponding criterion can be written as

\[
n_0 Sl_0 \gg 1,
\]
where $S$ is the transverse cross section of the condensate. For the kink state with the $y,z$ nodal plane moving with a constant velocity $\dot{q}$ along the $x$ axis, Eq. (3) gives

$$\Psi_0(x) = \sqrt{n_0} \left( \frac{\dot{q}}{c_S} + \sqrt{1 - \frac{\dot{q}^2}{c_S^2}} \tanh \sqrt{1 - \frac{\dot{q}^2}{c_S^2}} \frac{x-q}{l_0} \right). \quad (3)$$

Here $q$ is the position of the nodal plane, $c_S = \sqrt{n_0 \hbar}/m$ is the speed of sound, and it is assumed that $|\dot{q}| < c_S$.

The energy of the kink state (3) can be represented in the form (see [2])

$$H(\dot{q}) = \frac{1}{3} M c_S^2 \left( 1 - \frac{\dot{q}^2}{c_S^2} \right)^{3/2}; \quad M = 4(n_0 S \hbar_0)m. \quad (4)$$

For the velocity of the nodal plane, $|\dot{q}| \ll c_S$, we find

$$H(\dot{q}) = \frac{1}{3} M c_S^2 - \frac{1}{2} \dot{M} \dot{q}^2. \quad (5)$$

On the basis of Eq. (3), the quantity $-M$ can be treated as a negative mass of the kink. The criterion (2) ensures that $M \gg m$, i.e., the kink as a particle-like object is much heavier than a gas particle. The kinetic energy term in Eq. (3) is negative, and the energy of the kink decreases with increasing velocity $\dot{q}$. This means that if the dissipation decreases the kink energy, then the kink accelerates, gradually reaching the velocity of sound. Once this happens, the wavefunction (3) becomes nothing else than the wavefunction of a motionless ground-state condensate. Accordingly, the time scale on which the kink accelerates towards the velocity of sound can be regarded as a life-time of a kink-wise state.

The equations of motion for the kink as a particle-like object follow from the Hamiltonian $H$ (3) expressed in terms of momentum $p$. The relation between $p$ and $\dot{q}$ is obtained straightforwardly from the Hamiltonian equation $p = \partial H/\partial \dot{q}$. For slow kinks we have $p = -M \dot{q}$ and $H(p) = M c_S^2/3 - p^2/2M$. Eq. (4) demonstrates an important kinematics property of a kink state. As the time derivative $\dot{p} = -M (1 - \dot{q}^2/c_S^2)^{1/2} \dot{\dot{q}}$, the quantity $-M (1 - \dot{q}^2/c_S^2)^{1/2}$ can be viewed as a kinematic mass of the kink. This kinematic mass decreases when $\dot{q}$ approaches the sound velocity $c_S$, and it becomes easier to accelerate the kink. This is opposite to the case of a relativistic particle, where the kinematic mass increases with $\dot{q}$, and an infinite force is required to accelerate the particle beyond the velocity of light. Here, the kink can ultimately reach the velocity of sound and disappear.

**III. KINETIC EQUATION FOR DISSIPATIVE EVOLUTION**

We now turn to the analysis of dissipation in the presence of a thermal cloud and treat the kink as a particle-like object. The motion of the kink in a dissipative environment is not deterministic and is characterized by the probability density $F(t,p,q)$ of having the momentum $p$ and coordinate $q$ at time $t$. The distribution function $F$ satisfies the kinetic equation (see, e.g. [2])

$$\frac{\partial F}{\partial t} - \frac{\partial}{\partial p} \left( \frac{\partial H}{\partial q} F \right) + \frac{\partial}{\partial q} \left( \frac{\partial H}{\partial p} F \right) = I[F]. \quad (6)$$

The collisional integral $I[F]$ in the r.h.s. of Eq. (6) describes the interaction of the kink with thermal excitations and, hence, is responsible for the dissipation. In the absence of dissipation ($I[F] = 0$) the l.h.s. describes the motion of the kink, governed by the Hamiltonian equations $\dot{p} = -\partial H/\partial q$, $\dot{q} = \partial H/\partial p$.

The collisional integral accounts for the change of the kink momentum and energy due to scattering of thermal excitations from a moving kink. As the kink is much heavier than the gas particles ($M \gg m$), one can expect that in each scattering event the relative change of the kink momentum $p$ is small. Then, similarly to the case of motion of a heavy particle in a gas of light particles, the collisional integral takes the Fokker-Plank form [2]

$$I[F] = \frac{\partial}{\partial p} \left( AF + \frac{\partial}{\partial q} (BF) \right). \quad (7)$$

The first term in the l.h.s. of Eq. (7) describes the mobility in the momentum space. The transport coefficient $A$ represents the mean momentum transferred per unit time from the thermal excitations to the moving kink, and hence is equal to the friction force acting on the kink. The second term is responsible for diffusion in the momentum space, with $B$ being the diffusion coefficient.

We will assume that the characteristic energies of transverse excited modes greatly exceed the gas temperature $T$. Accordingly, only longitudinal (and in this sense 1D) excitations are present in the system and interact with the kink. In the reference frame, where the kink is at rest, the reflection of an excitation from the kink predominantly changes the excitation wavevector $k$ to $-k$. This automatically follows from the condition $M \gg m$. Then, as in each scattering event the momentum transferred from the excitations to the kink is equal to $2\hbar k$, the equations for the transport coefficients read

$$\begin{pmatrix} A \\ B \end{pmatrix} = \int_{-\infty}^{\infty} \left( \frac{-2\hbar k}{2(hk)^2} \right) \left\{ \frac{dk}{2\pi} R(k) \frac{\partial \epsilon_k}{\hbar dk} N(\epsilon_k - \hbar kq) \right\},$$

where $\epsilon_k = (E_k^2 + 2\mu E_k)^{1/2}$ is the Bogolyubov energy of the excitations, $E_k = \hbar^2 k^2/2m$, $R(k)$ is the reflection coefficient of an excitation with momentum $\hbar k$, and $N(\epsilon_k)$ are equilibrium occupation numbers for the excitations.

The quantity in curly brackets represents the scattering rate for the excitations with wavevectors in the interval from $k$ to $k + dk$. For subsonic kinks ($|\dot{q}| \ll c_S$), we have $\hbar k q \ll \epsilon_k$. Then, in the first integral the occupation numbers $N(\epsilon_k - \hbar kq)$ have to be expanded in powers of $\dot{q}$ up to the linear term, which gives the mobility coefficient $A = -\alpha \dot{q}$. In the second integral one can put $\dot{q} = 0$, thus...
omitting an inessential dependence of the diffusion coefficient $B$ on $\dot{q}$. As a result, for the (positive) transport coefficients $\alpha$ and $B$ we can write

$$\alpha = -2\hbar \frac{d}{dk} \frac{R(k)}{2\pi} R(k) \frac{\partial k^2 \partial N}{\partial k^2}$$

(8)

$$B = 2\hbar \frac{d}{dk} \frac{R(k)}{2\pi} R(k) \frac{\partial k^2 \partial N}{\partial k}.$$

(9)

The ratio $B/\alpha = \epsilon_*$, where $\epsilon_* \sim \min\{T, \mu\}$ is a characteristic energy of the excitations which give the main contribution to the transport coefficients (see below).

**IV. DIFFUSION- AND MOBILITY-DOMINATED REGIMES**

We first analyze qualitatively the kinetic equation (3). For an axially homogeneous condensate $\partial F/\partial \dot{q} = 0$, and using Eqs. (3), (4) we obtain the equation describing the evolution of a subsonic kink ($p = -M\dot{q}$):

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial \dot{q}} \left( \alpha \dot{q} F + \frac{B}{M} \frac{\partial F}{\partial \dot{q}} \right).$$

(10)

At small times $t$ the diffusion dominates over the mobility and leads to increasing mean square velocity: $\langle \dot{q}^2 \rangle \sim Bt/M$. Accordingly, the ratio of the mobility to diffusion term in Eq. (10) also increases. Once $\langle \dot{q}^2 \rangle$ reaches a critical value $\dot{q}_0^2 \sim B/M\alpha \sim \epsilon_*/M$, the mobility takes over the diffusion. This happens on a time scale

$$t \sim \tau_D = M \alpha^{-1}.$$

(11)

Owing to the condition $M \gg m$, the velocity $\dot{q}_0 \ll c_S$, i.e. the kink remains subsonic when the mobility starts to dominate over the diffusion. The condition $M \gg m$ also justifies the use of the Fokker-Plank approach: The kink momentum $M\dot{q}_0$ can be written as $(M\mu/m_*)^{1/2} \epsilon_*/c_S$ and it greatly exceeds the momentum $2\hbar k \sim \epsilon_*/c_S$ transferred to the kink in a single scattering event.

At much larger times, $t \gg \tau_D$, the diffusion spreading of the kink velocity is negligible compared to the mobility-induced increase of $\dot{q}$. As the force $A$ acting on the kink is proportional to the kink velocity $\dot{q}$, the latter grows exponentially with time. Omitting the diffusion term, Eq. (10) is reduced to

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial \dot{q}} \left[ (\dot{p} - A) F \right] = 0$$

and coincides with the Liouville equation for particles moving in accordance with the law $\dot{p} = A$. Then, with $A = -\alpha \dot{q}$ and $p = -M\dot{q}$, we find $\dot{q} = \dot{q}(0) \exp(\exp(t/\tau_D))$ or $t = \tau_D \log(\dot{q}/\dot{q}(0))$. From Eq. (3), one can see that

$$\frac{dH}{dt} = -\alpha \dot{q}^2.$$

(12)

Eq. (12) clearly shows that the decrease of the kink energy is equal to the energy losses due to the friction.

The logarithmic dependence of $t$ on $\dot{q}$ ensures that the dissipative dynamics of a kink state has a bottleneck at $\dot{q}$ significantly smaller than $c_S$. With logarithmic accuracy, this result is also valid at $\dot{q} \sim c_S$, unless $\dot{q}$ is very close to $c_S$ in which case the kink state is practically indistinguishable from the ground-state condensate. Therefore, a characteristic time $\tau$ at which the kink acquires the sound velocity and disappears can be found from the obtained dependence $t(\dot{q})$, with $\dot{q} = c_S$:

$$\tau \approx \tau_D \log(c_S/\dot{q}(0)).$$

(13)

The life-time $\tau$ of the kink is much larger than $\tau_D$ and is insensitive to a precise value of the initial velocity. This allows one to put $\dot{q}(0) \sim \dot{q}_0 \sim (\epsilon_*/M)^{1/2}$.

**V. REFLECTION OF EXCITATIONS FROM A KINK**

For finding the time $\tau_D$ and velocity $\dot{q}_0$ (energy $\epsilon_*$) one should calculate the transport coefficients $\alpha$ and $B$. This requires us to solve a 1D scattering problem and find the reflection coefficient $R(k)$ of an excitation from a static kink. Following the Bogolyubov approach, we represent the non-condensed part of the field operator of atoms as $\hat{\Psi} = \sum_k (u_k \hat{b}_k - v_k \hat{b}_k^\dagger)$, where $\hat{b}_k, \hat{b}_k^\dagger$ are the annihilation/creation operators of elementary excitations. The Bogolyubov-de Gennes equations

$$\epsilon_k \hat{f}_k^\pm = -\hbar^2 \frac{d^2}{dx^2} \hat{f}_k^\pm + g|\Psi_0(x)|^2 \hat{f}_k^\pm - \mu \hat{f}_k^\mp,$$

(14)

$$\epsilon_k \hat{f}_k^\pm = -\hbar^2 \frac{d^2}{dx^2} \hat{f}_k^\mp + 3g|\Psi_0(x)|^2 \hat{f}_k^\pm - \mu \hat{f}_k^\pm.$$  

(15)

The condensate wavefunction $\Psi_0$ is given by Eq. (3) with $\dot{q} = 0$, and $|\Psi_0|^2$ is characterized by the presence of a density dip of spatial size $l_0$, associated with the kink.

For the phonon branch of the excitation spectrum ($\epsilon_\pm \ll \mu$) we have solved Eqs. (14),(15), relying on the presence of the small parameter $\epsilon_\pm/\mu$. The method is based on matching the wavefunctions of free motion of Bogolyubov excitations in the range of distances $x$ from the kink, where $l_0 \ll |x| \ll k^{-1}$, with the excitation wavefunctions obtained from Eqs. (14),(15) by the perturbation expansion in powers of $\epsilon_\pm/\mu$.

The Bogolyubov-de Gennes equations (14),(15) have 4 fundamental solutions corresponding to $\epsilon_\pm = 0, k = 0$:

$$f_1^+ = \tanh(x/l_0); f_1^- = 0,$$

$$f_2^+ = 0; f_2^- = \cosh^{-1}(x/l_0),$$

$$f_3^+ = (x/l_0) \tanh(x/l_0) - 1; f_3^- = 0,$$

$$f_4^+ = 0; f_4^- = \sinh(2x/l_0).$$

(16)

The modes $f_1^+, f_4^\pm$ are odd, and the modes $f_2^\pm, f_3^\pm$ are even with respect to changing the sign of $x$. For finite $\epsilon_k$
satisfying the condition \( \epsilon_k \ll \mu \) (\( kl_0 \ll 1 \)), at \( |x| \ll k^{-1} \) each particular solution of \( \epsilon_k/\mu \) around one of the fundamental modes. The coefficients of the expansion are obtained directly from Eqs. (14), (17). For finding the solution tending to \( f^+_k \) (or to \( f^-_k \)) as \( \epsilon_k \to 0 \), we substitute \( f^+_k \) (or \( f^-_k \)) into Eq. (13) and, performing the integration, obtain the contribution to \( f^-_k \), proportional to \( \epsilon_k/\mu \). The function \( f^+_k \) is then substituted into Eq. (14), which allows us to find the contribution to \( f^-_k \), proportional to \( (\epsilon_k/\mu)^2 \), and so on. A similar procedure is used for obtaining the solutions tending to \( f^+_k \) and \( f^-_k \) as \( \epsilon_k \to 0 \). For the solutions \( f^+_1 \) and \( f^-_3 \), which in the limit of zero \( \epsilon_k \) correspond to \( f^+_1 \) and \( f^-_3 \), it is sufficient to keep the terms independent of and linear in \( \epsilon_k/\mu \). Then we find

\[
\begin{align*}
  f^+_{1k} &= f^+_1; \quad f^-_{1k} = (\epsilon_k/2\mu) \left[ f^+_1 + (x/l_0) f^-_2 \right] \quad (17) \\
  f^+_{3k} &= f^+_3; \quad f^-_{3k} = (\epsilon_k/4\mu) \left[ \cos^2(\epsilon_k/\mu) f^+_1 + 2f^+_3 + 1 \right] \quad (18) \\
  &+ \left( x/l_0^2 + C_3 \right) f^-_2, \\
\end{align*}
\]

where the fundamental modes \( f^+_k \) are given by Eqs. (16), and \( C_3 \) is the integration constant. The fundamental mode \( f^-_2 \) decays exponentially at large \( |x| \). Hence, for the particular solution going into \( f^-_2 \) as \( \epsilon_k \to 0 \) we should also retain quadratic terms in the expansion. This gives

\[
\begin{align*}
  f^+_{2k} &= (\epsilon_k/\mu) \left[ (C_{1k} - 2) f^+_1 - 1 \right]; \\
  f^+_{2k} &= f^+_2 + (\epsilon_k/2\mu)^2 \left\{ C_{1k} \cosh^2(\epsilon_k/\mu) f^+_1 + (C_{1k} - 2) (2f^+_3 + 1 + \left( x/l_0^2 + C_2 \right) f^-_2) \right\},
\end{align*}
\]

with \( C_{1k} \) and \( C_2 \) being the integration constants.

A general solution of Eqs. (14), (17) can be written as

\[
  f^+_k = \sum_{i=1}^{\infty} A_i f^+_i.
\]

The odd fundamental mode \( f^+_3 \) grows exponentially at \( |x| \gg l_0 \), and so does the particular solution \( f^+_3 \). This growth cannot be canceled in Eq. (20) by the terms of the odd particular solution \( f^+_3 \). Therefore, one should put \( A_4 = 0 \). The coefficients \( C_{1k}, A_{2k} \) and \( A_{3k} \) should be chosen such that the terms of even particular solutions \( f^+_2, f^+_4, f^+_6, \ldots \), proportional to \( \cosh^2(\epsilon_k/\mu) f^+_1 \) and growing exponentially at large \( x \), cancel each other. This gives \( C_{1k} = -\mu A_{3k}/\epsilon_k A_{2k} \).

At distances from the kink, \( |x| \gg l_0 \), the fundamental modes \( f^+_{1k} \) and \( f^+_{3k} \) are \( f^+ \) on \( |x|/l_0 \), and one can put \( f^+_{2k} = 0 \). The ratio \( f^+_{1k}/f^+_{3k} = E_k/\epsilon_k \), and one can only deal with the function \( f^+_{1k} \). Then, in the spatial region \( l_0 \ll |x|/k^{-1} \), from Eqs. (14), (17) we obtain

\[
\begin{align*}
  f^+_k &= A_{1k} + \frac{\epsilon_k}{\mu} A_{2k} \left( 1 - \frac{2x}{l_0} \right), \quad x > 0 \\
  f^+_k &= -A_{1k} + \frac{\epsilon_k}{\mu} A_{2k} \left( 1 + \frac{2x}{l_0} \right), \quad x < 0.
\end{align*}
\]

For finding the reflection coefficient we assume that at \( x \to +\infty \) the excitation wavefunction is a superposition of incident and reflected waves:

\[
  f^+_k = \exp(ikx) + G \exp(-ix),
\]

and at \( x \to -\infty \) there is only a transmitted wave

\[
  f^+_k = D \exp(ikx).
\]

The reflection coefficient \( R = |G|^2 \), and the quantity \( |D|^2 = 1 - R \) is the transmission coefficient. The asymptotic wavefunctions \( f^+_k (x \to \pm \infty) \) retain their form in the range of distances, \( l_0 \ll |x| \ll k^{-1} \), where they become

\[
\begin{align*}
  f^+_1 &= 1 + i(1 - G)kx; \quad x > 0, \\
  f^+_1 &= D(1 + ikx); \quad x > 0,
\end{align*}
\]

The wavefunction (22) should coincide with \( f^+_k (21) \), which immediately gives the reflection coefficient

\[
  R = \epsilon_k^2/4\mu^2.
\]

The reflection coefficient increases with \( \epsilon_k \) and becomes of order unity at \( \epsilon_k \sim \mu \). At larger energies \( R(k) \) strongly decreases. Note that for single particles the kink-wise density dip is absolutely transparent \( R = 0 \), see [13].

**VI. TRANSPORT COEFFICIENTS AND LIFE-TIME OF A KINK STATE**

We can now calculate the transport coefficients and estimate the life-time of a kink state. At low temperatures \( T \ll \mu \) the main contribution to the transport coefficients comes from the scattering of excitations with energies \( \epsilon_k \sim T \). The calculation from Eqs. (6), (14), with \( R(k) (23) \), gives

\[
\begin{align*}
  \alpha &= \frac{m}{\hbar} \frac{12\zeta(4)}{\pi^3} T^4, \\
  B &= \frac{m}{\hbar} \frac{12\zeta(5)}{\pi^3} T^5.
\end{align*}
\]

The diffusion time \( \tau_D (11) \) and the life-time (13) depend on the mass \( M \) of the kink and, hence, on the transverse size of the condensate. For a harmonic transverse confinement with frequency \( \omega_\perp \), the transverse cross section \( S = \hbar/m\omega_\perp \), and from Eqs. (6), (13) and (24) we obtain

\[
\begin{align*}
  \tau_D^{-1} &= 24\zeta(4) \omega_\perp (\pi n_0 a^2)^{1/2} \left( \frac{T}{\mu} \right)^4; \\
  \tau^{-1} &= 2\tau_D^{-1} \log^{-1} [M\mu/mT],
\end{align*}
\]

where the Riemann \( \zeta \)-function \( \zeta(4) \approx 1.08 \).

In the opposite limiting case, where \( T \gg \mu \), the transport coefficients are determined by the scattering of excitations with energies \( \epsilon_k \sim \mu \). For these excitations the reflection coefficient \( R \sim 1 \), and we find \( \alpha \sim mT/\hbar \) and \( B \sim m\mu T/\hbar \). This gives
\[ \tau_D^{-1} \sim \omega_{\perp} (n_0 a^3)^{1/2} \frac{T}{\mu}, \]  
\[ \tau^{-1} = 2\tau_D^{-1} \log^{-1} (M/m). \]  

In the presence of harmonic confining potential in the axial direction the Hamiltonian (23) of a subsonic kink acquires an extra term \(-M\omega^2 q^2/4\), where \(\omega\) is the axial trap frequency. This means that in the absence of dissipation the kink undergoes oscillations with frequency \(\omega/\sqrt{2}\), studied in [6,7]. Importantly, at finite temperatures the discrete structure of the spectrum of thermal excitations does not manifest itself in their scattering from the kink, since the level spacing for quasiclassical longitudinal (axial) excitations is very close to \(h\omega/\sqrt{2}\). Hence, the transport coefficients and the diffusion time \(\tau_D\) will be the same as in the absence of axial confinement. In the regime of friction-induced acceleration \((t > \tau_D)\) the zero-temperature equation of motion for the kink [8] contains an extra term \(-\dot{q}/\tau_D:\)

\[ \ddot{q} - \frac{\ddot{q}}{\tau_D} + \frac{\omega^2}{2} q = 0. \]  

The character of evolution of the kink in a trap depends on the parameter \(\omega \tau_D\). For \(\omega \tau_D \gg 1\) the solution of Eq. (29) represents harmonic oscillations with frequency \(\omega/\sqrt{2}\) and slowly increasing amplitude: \(q(t) = q(0) \exp(t/2\tau_D) \sin(\omega t/\sqrt{2} + \phi)\). Hence, the kink reaches the border of the condensate and disappears on a time scale \(\tau = 2\tau_D \log(l_c/q(0))\), where \(q(0)\) is the initial amplitude of the oscillations, and \(l_c = (2\mu/m\omega^2)^{1/2}\) the axial size of the condensate. The amplitude \(q(0)\) can be found from the condition that in the end of the diffusion stage of the evolution the kinetic energy of the kink \(M\dot{q}^2/2 \sim M\omega^2 q(0)^2/4\). This gives \(q(0) \sim (\epsilon_c/M\omega^2)^{1/2}\). Accordingly, the life-time \(\tau\) of the kink state practically coincides with that in a homogeneous condensate and follows from Eqs. (29), (23). This means that near the border of the condensate the kink velocity is comparable with the velocity of sound.

For \(\omega \tau_D \ll 1\), from Eq. (29) we obtain \(q(t) = q(0) \exp(t/\tau_D)\), and the kink acquires the sound velocity and disappears before reaching the border of the condensate. Thus, the dissipative evolution will be the same as in the case of an axially homogeneous condensate.

For Na condensates at densities \(n_0 \approx 3 \times 10^{14} \text{ cm}^{-3}\) the chemical potential \(\mu \approx 200\) nK. Then, according to Eqs. (25), (26), for the radial confinement with \(\omega_{\perp} \approx 2\) kHz (satisfying the criterion of dynamical stability for the axially Thomas-Fermi condensate, \(\mu < 2/\hbar\omega_{\perp}\)) one should have \(T \lesssim 50\) nK in order to reach the life-time of the kink state \(\tau \gtrsim 1\) s. For Rb condensate at the same \(n_0\) and \(\omega_{\perp}\) the life-time \(\tau \gtrsim 1\) s requires temperatures \(T \lesssim 15\) nK. One can think of achieving these conditions in an optically confined cigar-shaped condensate, similar to the one in the MIT experiment [4].

\section{VII. CONCLUSIONS}

In conclusion, we have developed a theory of dissipative dynamics of a kink state in a finite-temperature Bose-condensed gas and found that the dynamics is fundamentally different from that of vortices [7]. As the kink has a negative mass, due to friction-induced energy losses it accelerates towards the velocity of sound, and the kink state continuously transforms to the ground-state condensate. This makes the kink dynamics fast and insensitive to the longitudinal (axial) size of the condensate. The fast dissipative dynamics of kinks is important for understanding the picture of relaxation under a rapid quench of strongly elongated condensates, where one expects the formation of a number of kinks (see, e.g. in [13]). Importantly, by decreasing temperature well below the chemical potential one can make \(\tau\) sufficiently large for studying analogies between the standing waves of matter and light.

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