Conformal Moduli and $b - c$ Pictures for NSR Strings

Dimitri Polyakov†

Center for Advanced Mathematical Studies†
and Department of Physics
American University of Beirut
Beirut, Lebanon

Abstract

We explore the geometry of superconformal moduli of the NSR superstring theory in order to construct the consistent sigma-model for NSR strings, free of picture-changing complications. The sigma-model generating functional is constructed by the integration over the bosonic and fermionic moduli, corresponding to insertions of the vertex operators in scattering amplitudes. While the integration over the supermoduli leads to the standard picture-changing insertions, the integration over the bosonic moduli results in the appearance of picture-changing operators for the $b - c$ fermionic ghosts with the ghost number $-1$. Important example of the $b - c$ ghost pictures involves the vertex operators in integrated and unintegrated forms. We obtain the BRST-invariant expressions for the $b - c$ picture-changing operators for open and closed strings and study some of their properties. We also show that the superconformal moduli spaces of the NSR theory contain the global singularities, leading to the phenomenon of ghost-matter mixing and the appearance of non-perturbative D-brane creation operators. Keywords: PACS: 04.50.+h; 11.25.Mj.

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† dp02@aub.edu.lb
† associate member
1. Introduction

Despite the fact that the NSR (Neveu-Schwarz-Ramond) superstring theory appears to be the best explored, elegant and relatively simple model describing the superstring dynamics, it still suffers a number of serious complications. One, and perhaps the most important of them is related to the problem of picture-changing and the difficulties in constructing the generating functional for Ramond and Ramond-Ramond states, because of the picture-changing ambiguity \[1\]. Since the Ramond vertex operators have fractional ghost-numbers, one always has to consider their combinations at different ghost pictures in order to calculate their scattering amplitudes (except for the 4-point function where all the operators can be taken at the picture \(-1/2\)). As a result, constructing the generating functional for the Ramond scattering amplitudes becomes a confusing question. Actually, the problem of the picture-changing is not limited to the Ramond sector. Due to the ghost number anomaly cancellation condition, vertex operators of any correlation function in NSR string theory must have total superconformal ghost number \(-2\) and total \(b - c\) ghost number \(3\). This means that we cannot limit ourselves to just integrated and picture 0 vertices in the generating functional of the NSR model, since both integrated and unintegrated vertex operators, at various superconformal pictures must be involved in the sigma-model action. Therefore, due to this picture ambiguity, we face the problem building the generating functional for the consistent perturbation theory for NSR strings. In this paper we show how this problem can be resolved if one accurately takes into account the dependence of the superstring correlators on the superconformal moduli and explores the geometry of supermoduli spaces, related to the vertex operator insertions on the sphere or the disc. The paper is organized as follows. In the first section, we construct the basis for the supermoduli related to the vertex operator insertions and study the geometry of the related supermoduli spaces. In the second section we perform the integration over the supermoduli leading to the appearance of the bosonic and \(b - c\) ghost picture changing. Integrated and unintegrated vertex operators are shown to be the particular examples of different \(b - c\) pictures. The general BRST-invariant expression for the \(b - c\) picture-changing is derived. We show that if one takes all the vertex operators of the sigma-model unintegrated and at the picture \(-1\), the supermoduli integration insures the correct \(b - c\) and \(\beta - \gamma\) ghost number balance. In the third section we study the global singularities of the supermoduli spaces, leading to the phenomenon of the ghost-matter mixing \[2\], \[3\] and the appearance of the non-perturbative vertices carrying the RR-charges, which can be interpreted as the D-brane creation operators. We conclude by introducing the
general expression for the NSR sigma-model, free of picture-changing ambiguity, by using the relevant superconformal moduli structures. In the concluding section we discuss some possible implications of our results.

**NSR sigma-model and moduli integration**

Consider the string scattering amplitudes in the NSR formalism. To be certain, let us first consider the closed string case, the open strings can be treated similarly. The scattering amplitude on a sphere for \( N \) vertex operators in the NSR superstring theory is given by:

\[
<V_1(z_1, \bar{z}_1) ... V_N(z_N, \bar{z}_N) > = \int \prod_{i=1}^{M(N)} dm_i d\bar{m}_i \int \prod_{a=1}^{P(N)} d\theta_a d\bar{\theta}_a \int DX D\psi D\bar{\psi} [\text{ghosts}]
\]

\[
e^{-S_{NSR} + m_i \langle \xi^i | T_m + T^g_h > + \bar{m}_i \langle \bar{\xi}^i | T_m + T^g_h > + \theta_a \langle \chi^a | G_m + G^g_h > + \bar{\theta}_a \langle \bar{\chi}_a | G_m + G^g_h >}
\]

\[
\prod_{a=1}^{M(N)} \delta(\langle \chi^a | \beta >) \delta(\langle \bar{\chi}^a | \bar{\beta} >) \prod_{i=1}^{P(N)} <\xi^i | b > <\bar{\xi}^i | \bar{b} > V_1(z_1, \bar{z}_1) ... V_N(z_N, \bar{z}_N)
\]

(1)

Here \( z_1, ..., z_N \) are the points of the vertex operator insertions on the sphere and

\[
S_{NSR} \sim \int d^2z \{ \partial X_m \bar{\partial} X^m + \psi_m \bar{\partial} \psi^m + \bar{\psi}_m \partial \bar{\psi}^m + \bar{b} \partial c + b \bar{\partial} \bar{c} + \beta \partial \gamma + \bar{\beta} \bar{\partial} \bar{\gamma} \}
\]

\[
m = 0, ..., 9
\]

is the NSR superstring action in the superconformal gauge. Next, \( (m_i, \theta_a) \) are the holomorphic even and odd coordinates in the moduli superspace and \( (\xi^i, \chi^a) \) are their dual super Beltrami differentials (similarly for \( (\bar{m}_i, \bar{\theta}_a) \) and \( (\bar{\xi}^i, \bar{\chi}^a) \) ). The \( \langle ... > \) symbol stands for the scalar product in the Hilbert space and the delta-functions \( \delta(\langle \chi^a | \beta >) \) and \( \delta(\langle \xi^i | b >) = \langle \xi^i | b > \) are needed to insure that the basis in the moduli space is normal to variations along the superconformal gauge slices (similarly for the antiholomorphic counterparts). The dimensions of the moduli and supermoduli spaces are related to the number \( N \) of the vertex operators present and are different for the NS and Ramond sectors (see the discussion below). The standard bosonization relations for the \( b, c, \beta \) and \( \gamma \) ghosts are given by

\[
c(z) = e^\sigma(z); b(z) = e^{-\sigma(z)}
\]

\[
\gamma(z) = e^{\phi - \chi(z)}; \beta(z) = e^{\chi - \phi} \partial \chi(z)
\]

\[
\langle \sigma(z) \sigma(w) \rangle = \langle \chi(z) \chi(w) \rangle = - \langle \phi(z) \phi(w) \rangle = \log(z - w)
\]

(3)
The explanation is needed for the formula (1). In NSR superstring theory, while the partition function on the sphere does not have of course any modular dependence, the integration over the moduli and the supermoduli does appear for the sphere scattering amplitudes, for all the N-point correlators with $N \geq 3$. This is related to the important fact that in string theory the BRST and the local gauge invariances are not isomorphic to each other [4],[5]. Namely, the physical vertex operators while being BRST-invariant, are not necessarily invariant under the local supersymmetry. Moreover, while the integrated vertices are invariant under the local reparametrizations, the unintegrated ones are not. E.g. the infinitesimal conformal transformation of an unintegrated photon at zero momentum gives:

$$\delta_\epsilon V_{ph}(z) = \oint \frac{dw}{2i\pi} \epsilon(w) T(w)(c\partial X^m + \gamma \psi^m)(z)$$

$$= \oint \frac{dw}{2i\pi} \epsilon(w) \left\{ \frac{1}{z-w} \partial(c\partial X^m + \gamma \psi^m)(z) + O((z-w)^0) \right\} = \epsilon(z) \partial(c\partial X^m + \gamma \psi^m)(z)$$

(4)

More generally, under the infinitesimal conformal transformations $z \to z + \epsilon(z)$ any dimension 0 primary field transforms as

$$\Phi(z) \to \Phi(z + \epsilon(z)) = \Phi(z) + \epsilon \partial \Phi(z)$$

Similarly, applying the local supersymmetry generator $\oint \frac{dw}{2i\pi} \epsilon(w) G(w)$ to a photon at picture $-1$: $V_{ph} = ce^{-\phi} \psi^m$ with $G$ being a full matter+ghost supercurrent and $\epsilon$ now a fermionic parameter, it is easy to calculate

$$\delta_\epsilon V_{ph} = \epsilon(z) \left( -\frac{1}{2} ce^{-\phi} \partial X^m + c \partial ce^{\chi} \partial ce^{-2\phi} \psi^m \right) \neq 0.$$ 

Note, however, that the photon vertex operator at picture 0 is supersymmetric. The problem is that, due to the $b-c$ and $\beta-\gamma$ ghost number anomalies on the sphere we cannot limit ourself to just integrated vertices or those at the picture 0 but must always consider a combination of pictures in the amplitudes. This combination, however, is not arbitrary but must follow from the consistent superstring perturbation theory, or the expansion in the appropriate sigma-model terms. These terms are typically of the form $\lambda V$ where $V$ is a vertex operator of some physical state and $\lambda$ is the space-time field corresponding to this vertex operator. This term must be added to the original action as a string emits the $\lambda$-field. Then the string partition function must be expanded in $\lambda$ leading to the perturbation theory series and the effective action for $\lambda$, upon computing the correlators [6]. But the
important question is - which vertex operator should we take? Should it be integrated or unintegrated, which superconformal ghost picture should be chosen? If we choose the integrated and picture 0 vertices as the sigma-model terms, the local superconformal symmetry is preserved, but we face the problem of the ghost number anomaly cancellation. For this reason these vertex operators are not suitable as the sigma-model terms. If, on the other hand, we choose unintegrated vertices at the picture -1, then one faces the question of the gauge non-invariance - though these operators are BRST-invariant, they are not invariant under local superconformal gauge transformations. Therefore, in order to insure the gauge invariance of the scattering amplitudes, it is necessary to impose some restrictions on the parameter $\epsilon(z)$ of the superconformal transformations; namely, it must vanish at all the insertion points of the vertex operators. With such a restriction on the gauge parameter (effectively reducing the superconformal gauge group) the scattering amplitudes will be gauge invariant. Below we will prove that the consistent superstring perturbation theory must be defined as follows:

1) All the vertex operators must be taken unintegrated and at the picture $-1$.

2) The restrictions on the gauge parameter:

$$\epsilon(z_k) = 0; \ k = 1, ..., N$$

must be imposed at the insertion points of the vertex operators (where $\epsilon$ is either a bosonic infinitesimal reparametrization or a fermionic local supersymmetry transformation on the worldsheet).

The restrictions (5) on the gauge parameter effectively reduce the gauge group of superdiffeomorphisms on the worldsheet. As a result, it is in general no longer possible to choose the superconformal gauge, fully eliminating the functional integrals over the worldsheet metric $\gamma^{ab}$ and the worldsheet gravitino field $\chi^a_\alpha$ in the partition function. Instead these functional integrals are reduced to integration over the finite number of the conformal moduli $m_i; \ i = 1, ..., M(N)$ and the anticommuting moduli $\theta^a; \ a = 1, ..., P(N)$ of the gravitino while the gauge-fixed superstring action is modified by the supermoduli terms according to (1). The final remark about the general expression for the correlation function (1) concerns the dimensionalities of the odd and even moduli spaces as functions of the number N of the vertices. These dimensionalities are equal to the numbers of independent holomorphic super Beltrami differentials $\chi^a$ and $\xi^i$, dual to the basic vectors $\theta_a(z)$ and
\(m_i(z)\) in the supermoduli space. These numbers are different for the NS and Ramond sectors. Namely, the OPE’s of the stress-tensor and the supercurrent with the unintegrated perturbative vertex operators (e.g. a photon) are given by:

\[
T(z)V(w) \sim \frac{1}{z-w} \partial V(w) + O(z-w)^0
\]
\[
G(z)V(w) \sim \frac{1}{z-w} W(w) + O(z-w)^0
\]

(6)

where \(W\) is some normally ordered operator of conformal dimension \(1/2\). E.g. for a picture zero photon at zero momentum: \(V = c \partial X + \gamma \psi\) it is easy to check that \(W(w) = \frac{1}{2} \partial c\psi\)

Now since the induced worldsheet metric and the induced worldsheet gravitino field can be expressed as

\[
\gamma_{ab}(z) =: \partial_a X^m \partial_b X^m : (z) + \ldots \sim T_{ab}(z)
\]
\[
\chi_{aa}(z) =: \psi^m \partial_a X^m : (z) + \ldots \sim G_{aa}(z)
\]

(7)

\(a, \alpha = 1, 2\)

it is clear that the \(\chi_{aa}(z), \gamma_{aa}(z)\) and hence the corresponding superconformal moduli \(\theta_a(z)\) and \(m_i(z)\) behave as \((z - z_i)^{-1}; i = 1, \ldots, N\) when approaching a NS vertex point. For this reason the natural choice of the basis for \(m\) and \(\theta\), consistent with their pole structures and the holomorphic properties, is given by

\[
\theta_a(z) = z^a \prod_{j=1}^{N} (z - z_j)^{-1}
\]
\[
m_i(z) = z^i \prod_{j=1}^{N} (z - z_j)^{-1}
\]

(8)

\(a = 1, \ldots, M(N); i = 1, \ldots, P(N)\)

The holomorphy condition requires that at the infinity the \(\theta^a\) vectors must go to zero not slower than the supercurrent’s two-point function \(\lim_{z \to \infty} < G(0)G(z) > \sim \frac{1}{z^3}\) while the \(m^i(z)\) must decay as \(\lim_{z \to \infty} < T(0)T(z) > \sim \frac{1}{z^4}\) or faster. To see this, note that the expansions of \(T\) and \(G\) in terms of their normal modes are

\[
T(z) = \sum_n \frac{L_n}{z^{n+2}}
\]
\[
G(z) = \sum_n \frac{G_n}{z^{n+3/2}}
\]

(9)
therefore the normal ordering of these operators at zero point implies \( n \leq -2 \) for \( T \) and \( n \leq -\frac{3}{2} \) for \( G \). Under conformal transformation \( z \rightarrow u = \frac{1}{z} \) mapping the zero point to infinity, \( G \) and \( T \) transform as

\[
G(z) \rightarrow -i^3 G(u)
\]

\[
T(z) \rightarrow u^4 T(u) + ...
\]

(10)

therefore the normal ordering or the regularity at the infinity requires

\[
T(u) \sim \frac{1}{u^4} + O\left(\frac{1}{u^5}\right)
\]

\[
G(u) \sim \frac{1}{u^3} + O\left(\frac{1}{u^4}\right)
\]

(11)

implying the same asymptotic behaviour for \( \theta_a \) and \( m_i \), in the light of (7). This condition immediately implies that

\[
M(N) = N - 3
\]

\[
P(N) = N - 2
\]

(12)

for the scattering amplitudes in the NS-sector. For any \( N \) unintegrated Ramond operators \( V_R \) the OPE of \( T \) and \( V \) remains the same as in (6), therefore the basis and the number of the bosonic moduli do not change in the Ramond sector. However, the OPE of GSO-projected Ramond vertex operators with the supercurrent is given by

\[
G(z)V_R(w) \sim (z - w)^{-\frac{3}{2}} U(w) + ...
\]

(13)

where \( U(w) \) is some dimension 1 operator. For this reason the supermoduli of the gravitini, approaching the insertion points of \( V_R(z_i) \), behave as \( \chi^a(z) \sim (z - z_i)^{-\frac{3}{2}} \). For the scattering amplitudes on the sphere involving the \( N_1 \) NS vertices \( V(z_i), i = 1, \ldots, N_1 \) and \( N_2 \) Ramond operators \( V(w_j), j = 1, \ldots, N_2 \) the basis in the moduli space (involving the combinations of quadratic and 3/2-differentials) should be chosen as

\[
\theta^a(z) = z^a \prod_{i=1,j=1}^{N_1,N_2} \frac{1}{(z - z_i)\sqrt{z - w_j}}
\]

(14)

The holomorphy condition (11) for these differentials gives the total dimension of the supermoduli space for the NSR N-point scattering amplitudes:

\[
M(N) = N - 3
\]

\[
P(N) = N_1 + \frac{N_2}{2} - 2; N_1 + N_2 = N
\]

(15)
This concludes the explanation of the formula (1) for the NSR scattering amplitudes.

Performing the integration over $m_i$ and $\theta_a$ in (1) using (15) we obtain

$$< V_1^{NS-NS}(z_1, \bar{z}_1) ... V_{N_1}^{NS-NS}(z_{N_1}, \bar{z}_{N_1}) V_1^{RR}(w_1, \bar{w}_1) ... V_{N_2}^{RR}(w_{N_2}, \bar{w}_{N_2}) > = \int DXD\psi D\bar{\psi} e^{\frac{1}{2} N_2 - 2} \prod_{a=1}^{N_1+\frac{1}{2} N_2-2} |\delta(\langle \chi^a|\beta \rangle) - \chi^a|G_m + G_{gh} > |^2 \prod_{i=1}^{N_1+N_2-3} |< \xi^i|b \rangle \delta(\langle \xi^i|T \rangle)|^2$$

(16)

and similarly for the open string case. Each of the operators $\delta(\langle \chi^a|\beta \rangle) - \chi^a|G_m + G_{gh} >$ has the superconformal ghost number $+1$. These operators are the standard operators of picture-changing. Indeed, for the particular choice of $\chi^a = \delta(\langle z - z_a \rangle)$ we have

$$: \Gamma : (z_a) = \delta(\langle \chi^a|\beta \rangle) - \chi^a|G_m + G_{gh} > =: \delta(\beta)(G_m + G_{ghost})(z_a) =: e^\phi(G_m + G_{ghost}):(z_a)$$

(17)

i.e. the standard expression for the picture-changing operator and similarly for $\bar{\Gamma}(\bar{z}_a)$

The total ghost number of the left or right picture changing insertions following from the supermoduli integration is therefore equal to $N_1 + \frac{1}{2} N_2 - 2$. Thus if one takes all the NS-NS vertex operators, entering the correlator (16) or the sigma-model terms, at the canonical ($-1,-1$)-picture and all the RR operators at picture ($-\frac{1}{2},-\frac{1}{2}$), the integration over the supermoduli insures the correct ghost number of the correlation function to cancel the ghost number anomaly on the sphere. Similarly, the operator

$$Z =: < \xi^i|b \rangle \delta(\langle \xi^i|T \rangle) < \bar{\xi}^i|\bar{b} \rangle \delta(\langle \bar{\xi}^i|\bar{T} \rangle) :$$

(18)

resulting from the integration over the $m_i$-moduli has left and right fermionic ghost numbers $-1$. Since we found that the total number of these operators is equal to $N - 3$, one has to take all the vertex operators unintegrated (i.e. with the left and right $+1 b - c$ fermionic ghost number) so that the fermionic ghost number anomaly, equal to $-3$ on the sphere, is precisely cancelled by the moduli integration. The $Z$-operator (18) is a straightforward generalization of the picture-changing transformation (17) for the case of the fermionic $b - c$ pictures. In particular, the integrated and the unintegrated vertex operators in NSR string theory are simply two different $b - c$ ghost picture representations of the vertex
operator. The $Z$-operator must therefore reduce the $b - c$ ghost number of a vertex operator by one unit, at the same time transforming the local operators into the non-local (i.e. the dimension (1,1) operators integrated over the worldsheet). Namely, if $c\bar{c}V(z, \bar{z})$ is an unintegrated vertex of a conformal dimension zero, the $Z$-transformation should give

$$ : Z(c\bar{c}V) : \sim \int d^2z V(z, \bar{z}) + ... $$

i.e. the $c\bar{c}V$-operator is transformed into the the worldsheet integral of $V$ plus possibly some other terms insuring the BRST-invariance. The non-locality of the $Z$-operator follows from the non-locality of the delta-function of the full stress-energy tensor in the definition (18).

As previously one can choose the conformal coordinate patches on the Riemann surface excluding $N-3$ points corresponding to the basis $\xi^i(z) = \delta(z - z_i)$, so the $Z$-operator (18) becomes

$$ Z = : bb\delta(T)\delta(\bar{T}) : (z, \bar{z}) $$

The expression (20) for the $Z$-operator is still not quite convenient for practical calculations. Using the BRST invariance of $Z$ its OPE properties we shall try to derive a suitable representation for $\delta(T(z))$ (with the help of the arguments similar to those one uses to derive the exponential representations for $\delta(\gamma) = e^{-\phi}$ and $\delta(\beta) = e^{\phi}$) Since the $Z$ operator must have the conformal dimension 0, the operator $\delta(T)$ has conformal dimension $-2$. As the full matter+ghost stress tensor satisfies

$$ T(z)T(w) \sim 2(z - w)^{-2}T(w) + (z - w)^{-1}\partial T(w) + : TT : + ... , $$

the corresponding OPE for $\delta(T)$ must be given by

$$ : (z - w)^2\delta(T(z))\delta(T(w)) : \sim \delta(T(w)) + ... $$

In addition, the $\delta(T)$-operator must satisfy

$$ [Q_{brst}, Z] = : T\delta(T) : = 0 $$

since $Z$ is BRST-closed. Moreover, as one can formally write $: \delta(T(z)) : \sim (T(z) - i\epsilon)^{-1} : - : (T(z) + i\epsilon)^{-1} :$ where the “inverse” of $:T(z):$ can be represented as a dimension $-2$-operator

$$ T^{-1}(w) = \frac{2}{\alpha} \oint \frac{dz}{2i\pi} (w - z)^3 T(z) $$
(where $\alpha$ is a central charge) so that $[T^{-1}(z), T(z)] = 1$ we shall be looking for the representation of $Z$ in the form

$$ b\delta(T(z)) \sim : T^{-1} A : (z) $$

(24)

where $A$ is some dimension 2 operator chosen so that $Z$ satisfies (21), (22) With some effort, one finds

$$ A(z) = b - 4ce^{2x-2\phi}(T + b\partial c) $$

(25)

and hence

$$ : b\delta(T(w)) : = \int \frac{dz}{2i\pi} (z - w)^3 \{ bT - 4ce^{2x-2\phi}T(T - b\partial c) \} $$

$$ Z(w, \bar{w}) = \int d^2 z |z - w|^6 \{ (bT - 4ce^{2x-2\phi}T(T - b\partial c))(b\bar{T} - 4ce^{2\bar{x}-2\bar{\phi}}\bar{T}(\bar{T} - b\partial \bar{c})) \} $$

(26)

By simple calculation, using $\{ Q_{\text{brst}}, b \} = T$ and $\{ Q_{\text{brst}}, ce^{2x-2\phi} \} = \frac{1}{4} - \partial c e^{2x-2x}$ it is easy to check that the expression (26) for the fermionic ghost picture-changing operator is BRST-invariant. The operator $Z(w, \bar{w})$ particularly maps the unintegrated vertices into integrated ones. The rules of how $Z(w, \bar{w})$ acts on unintegrated vertices are as follows. Let $ceV(w, \bar{w})$ be an unintegrated vertex operator at $w$, where $V$ is the dimension $(1,1)$ operator. Writing

$$ Z(w, \bar{w}) \equiv \int d^2 z |z - w|^6 R(z) \bar{R}(\bar{z}) $$

(27)

where

$$ R(z) = bT(z) - 4ce^{2x-2\phi}T(z) - 4bc\partial ce^{2x-2\phi}T(z) $$

(28)

is defined according to (25), one has to calculate the OPE between $R(z)$ and $cV(w)$ around the $z$-point. For elementary perturbative vertices, such as a graviton, the relevant contributions (up to total derivatives and integrations by parts) will be of the order of $(z - w)^{-3}$, cancelling the factor of $(z - w)^3$ in (25) and removing any dependence on $w$. The operator $W(z)$ from the $(z-w)^{-3}$ of the OPE of the conformal dimension 1 will then be the integrand of the vertex in the integrated form. As for the possible more singular terms of the OPE, one can show that for the perturbative superstring vertices, such as a graviton, are generally the total derivatives; higher order terms will be either total derivatives or BRST trivial (being of the general form $TL$ where $L$ is BRST-closed). The $Z$-operator (26) is defined for the closed string vertices. Analogously, the operator of the $Z$-transformations in the open string case is given by

$$ Z_{\text{open}}(w) = \oint_C \frac{dz}{2i\pi} (z - w)^3 R(z) $$

(29)
In the latter case, the Z-transformation, applied to the unintegrated open string vertices, produces the open string operators, integrated over some contour $C$.

As a concrete illustration, let us consider the $Z$-transformation of the picture zero unintegrated graviton. For simplicity and brevity, let us consider the graviton at zero momentum. The $k \neq 0$ case can be treated analogously, even though the calculations would be a bit more cumbersome. The expression for the unintegrated vertex operator of the graviton is given by

$$ V(w)\bar{V}(\bar{w}) \sim (c\partial X^m + \gamma \psi^m)(\bar{c}\bar{\partial} \bar{X}^n + \bar{\gamma} \bar{\psi}^n) $$

Consider the expansion of $R(z)$ with $V$ around the $z$-point (the OPE of $\bar{R}$ with $\bar{V}$ can be evaluated similarly). Let’s start with the $Z$-transformation or the $c\partial X^m$ part of $V$. A simple calculation gives

$$ :bT:(z) : c\partial X^m : (w) \sim (z-w)^{-2}(\partial^2 X^m(z)+ : \partial \sigma \partial X^m : (z)) + O((z-w)^{-1}) \quad (30) $$

Next, consider the OPE of $V$ with the remaining two terms of $R(z)$ (28). The OPE calculation gives

$$ :4ce^{2\chi-2\phi}T \mathcal{T} - 4bce^{2\chi-2\phi} : (z)c\partial X^m(w) \sim 4(z-w)^{-2} : \partial \sigma c\partial ce^{2\chi-2\phi} \partial X^m : (z) + O((z-w)^{-1}) \quad (31) $$

It is easy to check that the operator of the $(z-w)^{-2}$ term of this OPE can be represented as

$$ : \partial \sigma c\partial ce^{2\chi-2\phi} \partial X^m : (z) = -\frac{1}{4} \partial \sigma \partial X^m(z) + \{Q_{\text{brst}}, : ce^{2\chi-2\phi} \partial \sigma \partial X^m : (z) \} \quad (32) $$

Next, since $\partial \sigma(z) = - : bc : (z)$ and since $\{Q_{\text{brst}}, c\partial X^m \} = 0$ due to the BRST invariance of the integrated vertex, we have

$$ [Q_{\text{brst}}, \partial \sigma \partial X^m] = -[Q_{\text{brst}}, bc\partial X^m] = -cT \partial X^m, \quad (33) $$

therefore

$$ \{Q_{\text{brst}}, ce^{2\chi-2\phi} \partial \sigma \partial X^m : (z) \} = \{Q_{\text{brst}}, ce^{2\chi-2\phi} \partial \sigma \partial X^m \} \quad (34) $$

since

$$ : ce^{2\chi-2\phi} [Q_{\text{brst}}, \partial \sigma \partial X^m] := cce^{2\chi-2\phi} T \partial X^m := 0 \quad (35) $$
as: $cc := 0$. For this reason

$$4\partial\sigma c\partial ce^2 x^{-2\phi} \partial X^m = -\partial\sigma \partial X^m + [Q_{\text{brst}}, ...]$$ (36)

and therefore the $(z - w)^{-2}$ order OPE term of (31) precisely cancels the corresponding $\partial\sigma \partial X^m$-term of the operator product (30) of $bT$ with $c\partial X^m$, up to the BRST-trivial piece.

With some more effort, one similarly can show that the $(z - w)^{-1}$ and other higher order terms of the OPE of $R(z)$ with $c\partial X^m$ are the exact BRST-commutators. Proceeding similarly with the antiholomorphic OPE's, we obtain

$$: Zc\bar{c}\partial X^m \bar{\partial} X^n : (w, \bar{w}) = \int d^2 z |z - w|^2 \partial^2 X^m \bar{\partial}^2 X^n (z, \bar{z}) + [Q_{\text{brst}}, ...]$$ (37)

Finally, integrating twice by parts we get

$$: Zc\bar{c}\partial X^m \bar{\partial} X^n := \int d^2 z \partial X^m \bar{\partial} X^n (z, \bar{z}) + [Q_{\text{brst}}, ...]$$ (38)

Next, consider the $\gamma \psi^m$-part of the graviton’s unintegrated vertex (left and right-moving alike). The calculation gives

$$: R : (z) \gamma \psi^m (w) \sim (z - w)^{-3} \partial (ce^x - \phi \psi^m)$$

$$+(z - w)^{-2}[Q_{\text{brst}}, : bc\beta \psi^m :] + [Q_{\text{brst}}, ...]$$ (39)

Performing the analogous calculation for the right-moving part and getting rid of the total derivatives we obtain

$$: Z \gamma \bar{\gamma} \psi^m \bar{\psi}^n := [Q_{\text{brst}}, ...]$$ (40)

Thus the full $Z$-transformation of the unintegrated graviton gives

$$: ZV_{\text{grav}}^{\text{unintegrate}} := \int d^2 z \partial X^m \bar{\partial} X^n (z, \bar{z}) + [Q_{\text{brst}}, ...]$$ (41)

Thus the result is given by the standard integrated vertex operator of the graviton, up to BRST-trivial terms. Note that if one naively applies the picture-changing operator $\Gamma$ (17) to the integrated photon $V_{ph}^{(-1)} = \oint \frac{dz}{2\pi i} e^{-\phi \psi^m}$, one gets $\oint \frac{dz}{2\pi i} (\partial X^m + c\beta \psi^m)$. The last term in this expression is not BRST-invariant, therefore at the first glance the picture-changing operation seems to violate the BRST-invariance for some integrated vertices. This contradiction is due to the fact that one is not allowed to straightforwardly apply the $\Gamma$-operator (which is the expression for the picture-changing consistent with the supermoduli integration) to the integrated vertices because, roughly speaking, the $\Gamma$-operation
does not “commute” with the worldsheet integration. More precisely, the contradiction arises because the “naive” application of the \( \Gamma \) picture-changing operator to the integrands ignores the OPE singularities between \( \Gamma \) and \( Z \) (the latter can be understood as the “operator of the worldsheet integration”). On the other hand, if one uses the “conventional” definition of the picture-changing operation given by \([Q_{brst}, e^\chi V]\) then one gets

\[
\{Q_{brst}, \oint \frac{dz}{2i\pi} e^{x-\phi\psi^m}\} = \oint \frac{dz}{2i\pi} (\partial X^m + \partial(ce^{x-\phi\psi^m}))
\]  

(42)

The second total derivative in this expression can be thrown out and we get the picture 0 photon. Now it is clear that the definition of the picture-changing as \([Q_{brst}, e^\chi V]\) is consistent with the supermoduli integration and the expression (17) for the local picture-changing operator, only if one accurately accounts for the singularities of the OPE between \( \Gamma \) and \( Z \).

We have shown that the \( Z \)-transformation (42) of the unintegrated graviton reproduces the full picture zero expression for the integrated vertex operator, up to BRST-trivial terms. Therefore the consistent procedure of the picture-changing implies that one always applies the picture-changing operator \( \Gamma \) to unintegrated vertices, with the subsequent \( Z \)-transformation, if necessary, i.e. the \( \beta\gamma \) picture-changing must be followed by the \( Z \)-transformation and not otherwise. Another useful expression is the \( Z \)-transformation of the picture-changing operator, or the integrated form of the picture-changing. Applying the \( Z \)-operator to \( :\Gamma\bar{\Gamma}: \) one obtains

\[
(\Gamma\bar{\Gamma})_{int}(w) =: Z\Gamma : w) = \int d^2z |z - w|^2 \{ : b\Gamma : -4 : c e^{2x-2\phi}(T - b\partial c)\Gamma : \} \times \text{c.c.}
\]

\[
= \int d^2z |z - w|^2 P(z)\bar{P}(\bar{z})
\]

(43)

Similarly, the expression for the single left or right integrated \( \Gamma_{int} \) can be written as

\[
\Gamma_{int} = \oint \frac{dz}{2i\pi} (z - w)P(z)
\]

(44)

The main advantage of using \( :\Gamma_{int}: \) is the absence of singularities in the OPE between \( \Gamma_{int}(w_1)\Gamma_{int}(w_2) \) which can be checked straightforwardly using the definition (44). The singularities in the \( \Gamma\Gamma \) operator products of the usual (unintegrated) picture-changing operators are well-known to result in complications and inconsistencies in the picture-changing procedure. As we saw, these complications and the appearance of the singularities.
are due to the fact that, strictly speaking, the picture-changing procedure is not well-defined without the appropriate $Z$-transformations.

3. Ghost-Matter Mixing and Moduli Space Singularities

The scattering amplitude (16) involves the insertion of picture-changing operators for bosonic and fermionic ghosts, which precise form depends on the choice of basis for super Beltrami quadratic and $3/2$--differentials. It has been shown [7], [8] that the scattering amplitudes are invariant under the small variations of the Beltrami basis, up to the total derivatives in the moduli space. In particular, if one chooses the delta-functional basis (17), (20) for $\xi^i$ and $\chi^a$, this symmetry implies the independence on the insertion points of picture-changing operators. The situation is more subtle, however, when the picture-changing insertions $z_a$ of (16) coincide with locations of the vertex operators (which precisely is the case for the amplitudes involving combinations of the vertex operators at different pictures) The equations (6), (7) and (13) imply the singular behavior of the supermoduli approaching the locations of the vertex operators. Namely, by simple conformal transformations it is easy to check that the singularities of (6), (7) and (13) at $z_i$ and $w_j$ correspond to orbifold points in the moduli space. As it has been pointed out in [7], if picture-changing operators are located at the orbifold points of the moduli space, the picture-changing gauge symmetry is reduced to the discrete automorphism group corresponding to all the possible permutations of the p.c. operators between these orbifold points. In particular, it’s easy to see that for $N_1$ Neveu-Schwarz and $N_2$ Ramond perturbative vertex operators having a a total superconformal ghost number $g$, the volume of this automorphism group is given by

$$\Xi_{N_1,N_2}(g) = (N_1 + N_2)^{N_1 + N_2 + g}$$  \hspace{1cm} (45)

The appearance of this discrete group is particularly a consequence of the polynomial property of picture-changing operators:

$$: \Gamma^m :: \Gamma :: n : \sim: \Gamma^{m+n} : + [Q_{BRST}, ...]$$  \hspace{1cm} (46)

which holds as long as the picture-changing transformations are applied to the perturbative string vertices, such as a graviton or a photon, which are equivalent at all the ghost pictures. However, apart from the usual massless states such as a graviton or a photon, the spectra of open and closed NSR strings also contain BRST-invariant and non-trivial vertex operators which cannot be interpreted in terms of emissions of point-like particles by a string. In
case of an open string, an example of such a vertex operator is an antisymmetric 5-form, given by

$$V_{\text{open}}^5(k) = H_{m_1...m_5}(k)e^{-3\phi}\psi_{m_1}\cdots\psi_{m_5}e^{ikX}$$

$$V_{\text{int}}^{5}(k) = H_{m_1...m_5}(k) \int \frac{dz}{2i\pi} e^{-3\phi}\psi_{m_1}\cdots\psi_{m_5}e^{ikX} + [Q_{\text{brst}},...]$$  \hspace{1cm} (47)$$

It has been shown that this vertex operator is physical, i.e. BRST-invariant and non-trivial. The BRST non-triviality of the operator (47) requires that the $H$ five-form is not closed:

$$k_{m_1}[H_{m_2...m_6}] \neq 0$$  \hspace{1cm} (48)$$

The vertex operator (47) exists only at nonzero ghost pictures below $-3$ and above $+1$, i.e. its coupling with the ghosts is more than just an artefact of a gauge and cannot be removed by picture-changing transformations. This situation is referred to as the ghost-matter mixing. In the closed string sector, an important example of the ghost-matter mixing vertex operator can be obtained by multiplying the five-form (47) by antiholomorphic photonic part:

$$V_{\text{open}}^5(-3) = H_{m_1...m_5m_6}(k)cce^{-3\phi}\bar{\psi}_{m_1}\cdots\bar{\psi}_{m_5}\psi_{m_6}e^{ikX}$$

$$V_{\text{int}}^{5}(-3) = H_{m_1...m_5m_6}(k) \int d^2z e^{-3\phi}\bar{\psi}_{m_1}\cdots\bar{\psi}_{m_5}\psi_{m_6}e^{ikX}(z,\bar{z}) + [Q_{\text{brst}},...]$$  \hspace{1cm} (49)$$

where the 6-tensor $H_{m_1...m_6}$ is antisymmetric in the first five indices. The BRST-invariance and non-triviality conditions for this vertex operator imply

$$k_{[m_7}H_{m_1...m_5]m_6}(k) \neq 0$$
$$k_{m_6}H_{m_1...m_5m_6}(k) = 0$$  \hspace{1cm} (50)$$

These constraints particularly entail the gauge transformations for the H-tensor

$$H_{m_1...m_5m_6}(k) \rightarrow H_{m_1...m_5m_6}(k) + k_{[m_1}R_{m_2...m_5]m_6}(k)$$  \hspace{1cm} (51)$$

where $R$ is a rank 5 tensor antisymmetric over the first 4 indices, satisfying

$$k_{m_6}R_{m_2...m_6} = 0$$

It is easy to check that the BRST constraints (50) including the related gauge transformations (51) eliminate 1260 out of 2520 independent components of the H-tensor. Therefore the total number of the degrees of freedom related to the closed string vertex operator (49)
is equal to 1260. Their physical meaning can be understood if we note that the BRST conditions (50) imply that for each particular polarization \( m_1...m_6 \) of the vertex operator (49) the momentum \( k \) must be normal to the directions of the polarization, i.e. confined to the four-dimensional subspace orthogonal to the \( m_1,...m_6 \) directions. The number of independent polarizations of \( V_5 \) in ten dimensions is equal to \( \frac{10!}{4!6!} = 210 \) therefore the total number of degrees of freedom per polarization is equal to 6. For instance consider \( m_1 = 4, ...m_6 = 9 \) so that \( k \) is polarized along the 0, 1, 2, 3 directions.

Then we can make a 4 + 6-split of the space-time indices \( m \to (a,t); a = 0,...,3; t = 4,...9; H_{m_1...m_6} \equiv H_{t_1...t_6} \). The tensor \( H_{t_1...t_6} \) is antisymmetric in the first five indices. Since the total number of independent degrees of freedom for this polarization is equal to 6, one can always choose it in the form \( t_6 \neq t_i, i = 1,...,5 \) by using suitable gauge transformations (51). This means in turn that one can choose the basis

\[
\lambda_t = H_{t_1...t_5 t},
\]

\[
t = 4,...9; t \neq t_1,...t_5
\]

with \( H \) antisymmetrized over \( t_1,...t_5 \). It is easy to see that the \( \lambda_t \) simply parametrize the 6 physical degrees of freedom for this particular polarization of \( V_5 \). To understand the meaning of \( \lambda_t \)-field one has to calculate its effective action. Computing the closed string 4-point correlation function \( \langle V_5(z_1, \bar{z}_1)...V_5(z_4, \bar{z}_4) \rangle \) and the three-point function \( \langle V_5V_5V_5 \rangle \) of two \( V_5 \)'s with the dilaton one can obtain that they reproduce the appropriate expansion terms of the DBI effective action for the D3-brane

\[
S_{eff}(\lambda) = \int d^4 xe^{-\phi} \sqrt{det(\eta_{ab} + \partial_a \lambda_t \partial_b \lambda^t)}
\]

(53)

where \( x \) is a Fourier transform of \( k \). The open string vertices (49) can also be shown to carry the Ramond-Ramond charges which can be demonstrated by calculating their disc correlation functions with the appropriate RR 5-form operator. That is, the disc correlation function \( \langle V_5^{(-3)}(k)V_5^{(+1)}(p)V_5^{(+1/2,-1/2)}(q) \rangle \), in which the RR vertex operator has to be taken at the \( (+1/2,-1/2) \)-picture, is linear in the momentum leading to the term \( \sim (dH)^2 A_{RR} \) in the effective action, where \( A \) is the Ramond-Ramond 4-form potential. This implies that the \( dH \)-field corresponds to the wavefunction of the RR-charge carrier, i.e. of the D-brane [9]. The BRST nontriviality condition (50) for the open string \( V_5 \)-vertices simply means that this wavefunction does not vanish. Thus the closed string \( V_5 \)-operators generate the kinetic term of the D-brane action while the open string \( V_5 \)-vertices account for its coupling with the RR-fields. The fact that the closed-string amplitudes lead to the
D-brane type dilaton coupling of the effective action is related to the non-perturbative nature of the $V_5$-vertices, which in turn is the consequence of the ghost-matter mixing, or the picture inequivalence of $V_5$. Let us explore this inequivalence in more details, from the point of view of the supermoduli geometry. The OPE of the full matter+ghost supercurrent $G(z) = -\frac{1}{2}\psi_m\partial X^m - \frac{1}{2}b\gamma + c\partial \beta + \frac{3}{2}\beta \partial e$ with $V_5(w)$ gives

$$G(z)V_5(w) \sim -\frac{1}{2(z-w)^3}H_{m_1...m_5}c\partial ce^{\chi}e^{ikX} + ...$$

and similarly for the OPE of $G$ with the left-moving part of the closed string $V_5$. This means that the supermoduli (8) of the gravitini behave as $\theta^a(z) \sim (z - z_k)^{-3}$ as they approach insertion points of $V_5$. Such a behavior of the supermoduli approaching the $V_5$ operators is much more singular than of those approaching the usual perturbative vertices (8). These singularities no longer correspond to the orbifold points of the moduli space. Instead they entirely overhaul the moduli space topology, effectively creating boundaries and global curvature singularities. To illustrate this consider the worldsheet metric

$$ds^2 = dzd\bar{z} + z^{-\alpha}(dz)^2 + \bar{z}^{-\alpha}(d\bar{z})^2$$

In terms of the $r, \varphi$ coordinates where $z = re^{i\varphi}, \bar{z} = re^{-i\varphi}$ this metric is given by

$$(1 + 2r^{-\alpha}\cos(\alpha - 2)\varphi)dr^2 + (1 - 2r^{-\alpha}\cos(\alpha - 2)\varphi)r^2d\varphi^2 - 4r^{1-\alpha}\sin(2 - \alpha)\varphi drd\varphi$$

The area of the disc of radius $\epsilon$ surrounding the origin point is given by

$$A = \int_0^\epsilon dr \int_0^{2\pi} d\varphi \sqrt{\gamma} = \int_0^{2\pi} d\varphi \int_0^\epsilon dr \sqrt{1 - 2r^{-2\alpha}}$$

This integral is of the order of $\epsilon^2$ for positive $\alpha$, $\epsilon^{2-\alpha}$ for $0 \leq \alpha < 2$ (this value reflects the deficit of the angle near the orbifold points) but it diverges if $\alpha \geq 2$ which means that for $\alpha = 3$ the disc is no longer compact but the origin point is blown up to become a global singularity. For this reason the scattering amplitudes in the presence of the $V_5$-vertices are no longer invariant under the discrete automorphism group (45). This means that the OPE’s involving the $V_5$-operators become picture-dependent. Therefore, in order to correctly describe the physical processes involving the $V_5$-operators one has to sum over all the previously equivalent gauges, i.e. all the admissible pictures (corresponding to the possible locations of the picture-changing operators (17) at the singularity points of the moduli space) and normalize by the volume (45) of the original gauge group of
automorphisms. The presence of the $V_5$-operators also modifies the suitable choice of the basis for the supermoduli and the number of independent $3/2$-differentials. For the amplitudes involving the total number $N_1$ of the $V_5$-insertions and $N_2$ of the standard perturbative vertices (without the ghost-matter mixing) the basis for $\theta^a$ is given by

$$
\theta_a(z) = z^a \prod_{i=1}^{N_1} (z - z_i)^{-3} \prod_{j=1}^{N_2} (z - w_j)^{-1} \tag{58}
$$

Accordingly the holomorphy condition (11) implies that in this case the number of the $3/2$-differentials (equal to the number of the picture-changing insertions) is equal to

$$
P(N) = 3N_1 + N_2 - 2 \tag{59}
$$

As previously, the supermoduli integration insures the correct total ghost number of the vertex operators in the amplitude provided that the perturbative vertices of the generating sigma-model are all taken at pictures $-1$ while the ghost-matter mixing $V_5$-vertices are at the picture $-3$.

When a string propagating in flat space-time emits particles or solitons, the background is perturbed by the appropriate vertex operators. These operators multiplied by the corresponding space-time fields must be added to the original NSR superstring action (2). The condition of conformal invariance then leads to the effective equations of motion for these space-time fields and to the corresponding effective field theory. We conclude this section by writing down the generating functional for the NSR string sigma-model, free of the picture-changing ambiguities. The partition function of superstring theory perturbed by the set $\{V_i\}$ of physical vertex operators is given by

$$
Z(\varphi_i) = \int DXD\psi D[ghosts]e^{-S_{NSR} + \varphi_i V^i} \rho(\Gamma; Z) \rho(\bar{\Gamma}; \bar{Z}) \tag{60}
$$

where $\rho(\Gamma; Z)$ is the picture-changing factor due to the appearance of the supermoduli when one expands in $\varphi_i$. Though this factor enters differently for each term of the expansion (as it is clear from (16)), it is straightforward to check that, due to the ghost number conservation, one can recast it in the invariant form (independent on the order of expansion)

$$
\rho(\Gamma; Z) = \sum_{m,n=0}^{\infty} \Xi^{-1}(n) \Xi^{-1}(m) \sum_{\{\xi^{(1)}...\xi^{(m)}, \chi^{(1)}...\chi^{(m)}\}} \delta(<\chi^{(1)}|\beta>) <\chi^{(1)}|G>... \delta(<\chi^{(m)}|\beta>) <\chi^{(m)}|G> <\xi^{(1)}|b> \delta(<\xi^{(1)}|T>)... <\xi^{(m)}|b> \delta(<\xi^{(m)}|T>) \tag{61}
$$
and accordingly for $\rho_{(\Gamma, Z)}$ where the sum over $\xi^{(i)}$ and $\chi^{(i)}$ implies the summation over all the basic vectors of the $(m, n)$-dimensional spaces of super Beltrami differentials. Note that due to the ghost number anomaly cancellation condition, for any N-point correlator appearing as a result of expansion in in $\lambda$ only the $m = N - 3$ and $n = N_1 + \frac{1}{2}N_2 - 2$ terms of $\rho_{(\Gamma, Z)}$ contribute to the sigma-model. $\Xi_{\xi}(m)$ and $\Xi_{\chi}(n)$ are the volumes of the symmetry groups related to the picture-changing gauge symmetry. (defined separately for the left and the right picture-changing). In the important case when the basic vectors are chosen at the orbifold points of the moduli spaces, the volumes are given by the relation (45). It’s easy to see that in the picture-independent case inserting the $\rho_{(\Gamma, Z)}$-factor in the partition function can be reduced to the trivial statement that if one sums over N equivalent amplitudes with ghost picture combinations of the vertices and then divides by N, one gets the value of the amplitude. In the ghost-matter mixing cases involving the global singularities in the moduli space, the situation is more complicated. Thus the generating functional (60), (61) of the NSR sigma-model is a straightforward consequence of the expression (16) for the scattering amplitudes derived from the supermoduli integration. As was already said above, the operators $V_i$ of the sigma-model action (60) are taken unintegrated at picture $-1$ (and those with the ghost-matter mixing are at the picture $-3$). These operators are generally BRST invariant but not invariant under superdiffeomorphisms which simply means that the gauge symmetry, related to global conformal transformations, is fixed from the very beginning in the model (60),(61). The choice of the insertion points of the vertex operators corresponds to the choice of the Koba-Niel sen’s measure in the correlation functions. Indeed, the $Z$-operators appearing as a result of the integration over the bosonic moduli, transform $N - 3$ out of $N$ vertex operators into the integrated ones, while the remaining 3 are left in the unintegrated form $\sim c\bar{c}V(z_i, \bar{z}_i); i = 1, 2, 3..$ Then the $c$ and $\bar{c}$-fields contribute the factor of $\prod_{i,j} |z_i - z_j|^2$ which precisely is the invariant Koba-Nielsen’s measure necessary for the calculations of the string scattering amplitudes.

Conclusions

In this paper we have constructed the sigma-model for NSR superstrings, leading to the consistent string perturbation theory, free of the picture-changing ambiguities. The important element of the construction is the appearance of the $b - c$-picture changing operators defined by (18),(20). The $b - c$ picture-changing operators particularly map unintegrated vertices into integrated ones, up to total derivatives and BRST-trivial terms. In case if the $Z$-operator (18),(20) acts on the integrated vertices, one obtains the “double-integrated” representations of the vertex operators, etc. Just like not all the physical
vertex operators can be represented at picture zero (due to the global singularities in the space of supermoduli), the global singularities in the spaces of the bosonic moduli lead to the $b-c$ picture inequivalence for some vertex operators. As a result there are the physical operators existing at some particular $b-c$ pictures with no equivalent version at other pictures, e.g. the operators that can be represented in the integrated pictures, but not in the unintegrated $cV$-form. In addition, the OPE of the $Z(w,\bar{w})$ operator defined by (18),(20), with unintegrated perturbative massless vertices (such as graviton) $cciW(w,\bar{w})$ gives an expression independent on $w$ (i.e. the integral of $W$ over the worldsheet). In general, however, it is possible that the global moduli space singularities may lead to the appearance of the terms of the form $\sim W(z) \sim \int d^2w |z-w|^{2n}U(w,\bar{w})$ in the expressions for physical vertex operators, where $U$ is some operator of conformal dimension $(n+1,n+1)$.

The important example of the $b-c$ picture inequivalence is the 5-form (49) at the picture $+1$. The naive picture $+1$-expression for the five-form (47) $\sim H_m^{\ldots 5} \oint e^{\phi} \psi_{m_1} \ldots \psi_{m_5} e^{ikX}$ is not BRST-invariant due to the non-commutation with the supercurrent part of $Q_{brst}$.

To insure the BRST-invariance of the five-form (47) at the $+1$-picture one has to add the double-integrated $b-c$ ghost terms so that the full BRST-invariant expression for the $V^{(+1)}_5$ is given by

$$V^{(+1)}_5 = H_{m_1 \ldots m_5}(k) \oint \frac{dz}{2i\pi} \{ e^\phi \psi_{m_1} \ldots \psi_{m_5} e^{ikX}$$

$$+ 2\hat{b}_3 c\partial c e^X \partial \chi (\psi_{m_1} \ldots \psi_{m_3} (\psi_m \partial X^m) + \psi_{[m_1} \ldots \psi_{m_4} (\partial X_{m_5]} (\partial \phi - \partial \chi) + \partial^2 X_{m_5]}))$$

$$+ i\psi_{m_1} \ldots \psi_{m_3} ((k\psi)(\partial \phi - \partial \chi) + (k\partial \psi)) e^{ikX}$$

$$+ \frac{1}{40} [[(\hat{T}^\chi)_6 (\partial \phi - \partial \chi) c\partial b e^{2\phi - X} \psi_{m_1} \ldots \psi_{m_5} (\psi \partial^2 X)] e^{ikX}])}$$

$$k_{[m_1} H_{m_2 \ldots m_6]} \neq 0$$

where the operators with the hat acting on an arbitrary operator $A(z)$ are defined as

$$\hat{b}_n A(z) = \oint \frac{dw}{2i\pi} (w-z)^{n+1} : b(w)A(w) :$$

$$(\hat{T}^\chi)_n A(z) = \oint \frac{dw}{2i\pi} (w-z)^{n+1} : T^\chi(z)A(z) :$$

$$T^\chi(z) = \partial \chi \partial \chi (z) - \partial^2 \chi (z)$$

It’s easy to show that the OPE of the full stress-tensor with $V^{(+1)}_5$ is given by

$$T(z) V^{(+1)}_5(w) \sim (z-w)^{-3}U(w) + \ldots$$

while the OPE of $T$ with usual integrated vertices contains no singularities at all (the last OPE can be represented as a worldsheet integral
of a total derivative). At the same time, the OPE of the supercurrent with the $V_5^{(+1)}$-operator contains only the regular $(z - w)^{-1}$ orbifold-type singularity (unlike the case of the five-form at the picture $-3$). Therefore the bosonic moduli behave as $(z - z_i)^{-3}$ approaching the $V_5^{(+1)}$ insertion points. Hence the $V_5^{(+1)}$-insertions correspond to the global singularities in the spaces of the bosonic moduli, while the anticommuting moduli have the usual orbifold singularities at these insertion points. Thus the picture $+1$ and $-3$ five-form operators (47) and (62), while representing the same massless physical state, are ontologically different from the point of view of the moduli space geometry: the first originates from the global singularities of the bosonic moduli space, while the second corresponds to those in the spaces of the anticommuting supermoduli (accordingly, they are related to the orbifold points in the fermionic and bosonic moduli spaces). As has been noted in this paper, the global singularities in the superconformal moduli spaces lead to the breaking of the picture-changing gauge symmetry. This results in the picture-dependence of the operator products involving the ghost-matter mixing vertex operators (related to the creation of the D brane-like solitons (49),(52),(53)). As a result, due to this picture-dependence, the conformal $\beta$-function equations involving these vertex operators are entirely different from the usual ones; namely, these equations become stochastic, having the form of the non-Markovian Langevin equations, where the operator of the stochastic noise is given by the worldsheet integral of $V_5$, cut at a certain scale $\Lambda$. The worldsheet cutoff $\Lambda$ corresponds to the stochastic time of the process, and the memory structuer of the noise is determined by the worldsheet correlators of $V_5$. In our next paper, currently in progress and close to the conclusion, we will show how the equations of the hydrodynamics (Navier-Stokes and the passive scalar equations) emerge for the dilaton and the D-brane’s U(1) field in the ghost-matter mixing backgrounds. These stochastic processes possibly play a vital role in the gauge-string correspondence $\square$, $\square$, in the formation of the space-time geometry of our world, and may hint at the existence of the deep connections between string theory and the physics of hydrodynamics, chaos and turbulence. We will discuss these connections in details in our next paper to come.

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