See second panel (page 2?) for title and abstract.

**Poster construction**

There are twelve panels, arranged in four rows of three panels in each row to be placed on a background with panel ratio 4:3. Final size will be close to the paper size + 5% in both dimensions, but limiting space is 1.6 meters tall and .9 meters wide. This page has meta information and is not part of the eventual display.

The assembly will be folded and unfolded, so each panel will have some holes cut out of the corners to ease folding. There will be an alternating pattern of circle and diamond shapes, with a circle at the top left between the upper left and upper middle panel, and a diamond at the top right between the upper middle and upper right panel. The holes will have a radius about 5% of the height of the panel. (Yes, you heard right: the presentation has holes in it.)

The paper will be centered on each panel, and likely have the same aspect ratio. At present the paper size is 14 inches by 10.5 inches.

In addition to the text, there will be a watermark pattern underlying the text. (At present, I cannot use the ArXiv system to display the characters. I apologize for the inconvenience.) There will be a Korean word for logic (nonli ??), one for mathematics (suhag ??) and one for welcome (hwan-yeong ??), as well as ICM and 2014 in some arrangement. The current arrangement is to have ICM vertically on the right most upper three panels, 2014 on the bottom three panels, and the remaining panels have welcome on the top row, logic on the second row, and mathematics on the third row. (At this time, the watermark/panel arrangement is hwan- yeong I
non li C
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2 01 4)

I thank Joy Song for her consulting time on the arrangement and choice of Korean words to use.

Note: to fix margins, I have placed what is geometrically the first (hwan-) panel second in this sequence. In assembling, be sure to arrange the hwan- and yeong panels so that the title (yeong panel) is in the middle.
On Two Problems From "Hyperidentities and Clones"

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Abstract

A hyperidentity $E$ can be viewed as a statement in second order logic. When combined with a similarity type $\tau$, it can also be considered as a set of first order statements. Based on examples from [5], which included hyperassociativity and $\tau = < 2 >$, it was conjectured that each first order theory so produced was finitely axiomatizable. Part of the analysis suggested further investigating the relatively free 2-generated semigroup satisfying one or both of the equations $xxyyz = xxyyx$ and $zyxx = zyxxyx$.

At ICM 1994, the conjecture above was refuted, and a finite basis problem arose: Is it decidable which pairs $< E, \tau >$ give rise to finitely axiomatizable theories? This problem will be examined, and its connections to other fields (e.g. symbolic dynamics) will be reviewed. In doing so, we give partial solutions to problems 27 and 28 from [2].

In 1993, we came across the question: "How small can an equational basis for hyperassociativity be?" Hyperassociativity then meant an infinite set of equations, one for each term $t$ of arity two formed in the language $L$ of one binary operation symbol, that said that $t$ obeyed the associative law. If $\Omega_2$ is the collection of these terms, one can write informally

$$F(x,F(y,z)) \approx F(F(x,y),z)$$

or the hyperidentity given by $F(x,F(y,z)) \approx F(F(x,y),z)$ is represented or realized (in $L$) by the infinite set of identities. Any algebra with a binary operation that satisfied this infinite set would be hyperassociative: its operation and all binary derived operation would be associative. Thus, such an algebra would be a semigroup that obeyed the semigroup laws $x^2 = x^4$ (corresponding to $t(x,y) = x^2$), $xyxx = xyzyx$, $x^2y^2z = x^2yx^2yz$, and $xy^2z^2 = xyz^2y^2z$ (corresponding to $t(x,y)$ being $xyx$ or $xxy$ or $xyy$ respectively).

Denecke and Koppitz had investigated the question in [11] and found a finite basis of roughly 1000 equations for this infinite set. Independently (and with different methods) Polak [3] and we [5] showed that there was a basis of five equations: $(xy)z = x(yz)$ and the four parentheses-saving semigroup laws above. This led to some further investigations in structure [4] and language [6] [7].

These studies led to the Main Question below, which we call the Finite Basis Problem: informally, is there a computer program that will tell us if there is a finitely based equational theory equivalent to the one represented by the input pair $< E, \tau >$?

A more general version of this question appears as Question 27 in p.290 of "Hyperidentities and Clones" [2].

This presentation will be online with a URL like http://arxiv.org/abs/1408.XXXX.
What Are Hyperidentities?

Hyperidentities are statements in second-order logic which can be viewed as an extension of equational logic. Informally, they are universally quantified equations with the outer quantifiers over a domain of functions of certain arities, and inner quantifiers over a domain of individuals. The following example is written with subscripts to emphasize that a symbol for a function variable such as $F_2$ and $G_3$ must have the same arity in all occurrences.

$$\forall F_2 \forall G_3 \forall xyz F_2(G_3(x, y, z), G_3(x, y, z)) \approx G_3(F_2(x, x), F_2(y, y), F_2(z, z))$$

We will use a more casual style, suppressing quantifiers and subscripts. We write the above as:

$$F(G(x, y, z), G(x, y, z)) \approx G(F(x, x), F(y, y), F(z, z))$$

What does this mean? Different things depending on the domain of interpretation. There will be one set $U$ of individuals and varying sets of functions $\Omega$:

- Students of V. Belousov and Y. Movsisyan use the convention (e.g. in [11], [12]) that $\Omega$ is a set of fundamental operations. This is in the context of studying structures with finitely many operations $\cdot, \circ, \ldots$. Some papers have these as binary quasigroup operations on the same set, and the hyperidentity is a relation involving only these members of $\Omega$.

- Students of W. Neumann and W. Taylor (as in [13]) will fix a similarity type $\tau$, say $\tau = < 2, 2, 3, 1 >$ has symbols for two binary operations and a symbol for a unary and another ternary operation. $\Omega$ is then the collection of all terms formed from these fundamental operations, including the projection operations which appear as single variables. This presentation will often use this interpretation.

- Students of K. Denecke, D. Schweigert, et.al. have a notion of hypersubstitution which is covered in [2]; an effect is that certain varieties of hyperidentities can be seen as having $\Omega$ range over certain subsets of the set of terms. In particular, prehyperidentities will exclude the projection terms, and M-hyperidentities are a subset which are derived from a monoid M of hypersubstitutions. Parts of this presentation apply to this interpretation.

In all of these interpretations, arity is strictly preserved. One informally can substitute a projection function $p_i(x_1, \ldots, x_n) = x_i$ in interpreting an $n$-ary function symbol $F_n(x_1, \ldots, x_n)$, but technically one has to select only functions of the right arity $n$ from $\Omega$ for interpreting $F_n$.

A concrete clone is collection of functions over an algebra which contains the projection functions and is closed under composition. The hyperidentity is interpreted as an equation between terms formed from clone members of certain arities. Note that only Taylor’s use of hyperidentity corresponds to a clone identity. The other uses correspond to the equation holding among certain pairs of members of the clone, and not to all pairs of given arities belonging to the clone.

Technical Note: To avoid complications in presentation, we assume no constant symbols or functions of arity 0, and we assume the type has finitely many symbols of finite arity. We call such similarity types nice.

Given a nice type $\tau$ and a hyperidentity $E$, we can form the set of equations in the language of $\tau$-structures which come from all possible substitutions (sometimes restricted by terms which realize only functions in $\Omega$ for a given algebra) of a term of arity $n$ for the function symbol of same arity in $E$. Given this set, it makes sense to ask if it is logically equivalent (in equational logic) to a finite set of equations in the same language.
A small basis for various hyperidentities

We had been given the intuition that a small basis for hyperassociativity did exist. We started by looking at consequences of small terms satisfying the associative law. We would use that to reduce the number of binary terms to be considered. $xy$ was a natural term to use, and let us use semigroup notation and avoid many parentheses. The starting term was $xy$.

Ralph McKenzie pointed out to us that the semigroup variety given by the equation $xyzzxyx = xyzxy$ was locally finite. In particular, the 2-generated free semigroup of this variety was finite and we found it had 94 elements. One could also use the associative law for the term $xx (x^2 = x^4)$ to reduce the number of terms to examine.

We then undertook to show that the remaining laws were a consequence of two more laws $xxyyz = xxyxyy$ and $xyzz = yxzyzz$. Fortunately this broke down into a few cases, based on the number of alternations of $x$ and $y$ in the term $t (x^a y^b, x^a y^b x^c, x^a y^b x^c y^d, \text{ and } xyaxy)$, the last of which took advantage of analysis of smaller terms. We did the derivation, which was later used in Kunc’s thesis \[9\].

After getting this result, we did some more explorations and found several other examples of pairs of hyperidentities and nice types which gave finitely based equational theories.

Hyperidempotency:

$$F(x) \approx x,$$

finitely based for all types

Hypermediality:

$$F(G(x, y), G(z, w)) \approx G(F(x, z), F(y, w)),$$

finitely based for all types containing function symbols of arity at most 2,

A version of entropic identity:

$$F(G(x_1, \ldots, x_{1n}), \ldots, G(x_{n1}, \ldots, x_{nn})) \approx G(F(x_1, \ldots, x_{1n}), \ldots, F(x_{n1}, \ldots, x_{nn})),$$

finitely based for types with symbols having arity at most $n$.

HyperCommutativity:

$$F(x, y) \approx F(y, x),$$

finitely based for all types, using the Taylor interpretation. This included projections, yielding the trivial base $x = y$. For other interpretations which excluded projections, work by Dudek and Kiesilewicz showed in \[14\] that relative to the variety of semigroups, ”totally commutative” semigroups belonged to one of four finitely based semigroup varieties. In unpublished work, McKenzie and Paseman showed that ”total commutativity” was not finitely based relative to the variety of all magmas i.e. all universal algebras of type $<2>$. 

In the face of these examples, we conjectured that every hyperidentity had a finitely based representation in a nice type. We were wrong.
The Finite Basis Problem For Different Hyperidentities

Note that the finite basis problem is trivial when $\Omega$ is finite. The study of small bases can still be important, especially for optimization problems.

When $\Omega$ is the full term of clones, we have some degree of uniformity. We can use a model theory argument (sketched below) to show that representing a finitely based hyperidentity is preserved across reducts and a certain type of arity reduction.

When one uses monoids of hypersubstitutions, one must take more care. The sets of first order equations depends on the hypersubstitution, and is usually restricted to within a particular type. (There is also some conflation between types used in the first order language and the type of the language used to express the hyperidentity. This is necessary because the goal is to study things “in a first order way”: hypersubstitutions take place inside a monoid of terms of a given type, and not as a substitution of algebraic operations for logical function symbols.) However, if the set of terms is ”rich enough”, one can show an infinite basis in one type extends to an infinite basis in a larger type using the same model theoretic argument.

The basic argument involves infinite sequences of models. Given a sequence of $\tau$-models related to an equational theory derived from a given hyperidentity $E$, we suppose this sequence witnesses the fact that the theory is not finitely based, say by the $k$th model satisfying only the first $k$ equations of the theory. We create a new sequence of $\sigma$-models, where $\sigma$ is like $\tau$, except that $\sigma$ either has an extra function symbol, or $\sigma$ differs from $\tau$ in that one function symbol has arity one more in $\sigma$ than in $\tau$. It is then routine to construct a new sequence of $\sigma$-models which witnesses the fact that the equational theory in the language $\sigma$ derived from $E$ is also not finitely based, by preserving as much of the functionality of the given sequence of models. We called this a ”convexity theorem”; the set of types which might not give rise to a finitely based representation of a hyperidentity looked like a convex set in the partial order of all nice types induced by the preserving relation.
Going Back Twenty Years

We presented some of this work in [6] in Zurich ICM 1994. Specifically:

Hyperassociativity is finitely based for type < 2 > (and many unary types)

A list of various other <hyperidentity,type> pairs yield finitely based equational theories.

\[ F(F(x)) \approx F(F(F(x))) \]

is not finitely based for the type < 1, 1, 1 >. Our earlier conjecture that all such pairs were finitely based was wrong.

(The proof is based on the Thue-Morse word on three letters which is square free. We can choose certain long subwords of this and make large unary terms \( f(x) \) to substitute in the above hyperidentity. The resulting term \( f(f(x)) \) has no "small" subterms to use in a substitution, and so no "small" equations can be used to derive \( f(f(x)) = f(f(f(x))) \). Thus there is no "small" (finite) basis for the above theory. We are indebted to Stuart Margolis for this suggestion.)

There is a computable partial order \( \prec \) on types which respects the property of being finitely based: if \( \sigma \prec \tau \), then for any hyperidentity \( E \), \( < E, \sigma > \) not finitely based implies \( < E, \tau > \) is also not finitely based. (Conversely, \( < E, \tau > \) is f.b. implies \( < E, \sigma > \) is f.b.. This comes from the "convexity theorem" mentioned in another panel.)

The relation \( \sigma \prec \tau \) includes and is generated by the relation \( \sigma \) has one less function symbol than \( \tau \), and the relation \( \sigma \) is the same as \( \tau \), except one symbol of \( \tau \) has arity one greater than the corresponding symbol of \( \sigma \).

Let there be a computable encoding of the countable set

\[ A = \{ < E, \tau > | \ E \text{ is a hyperidentity and } \tau \text{ is a nice type } \} \]

Let \( B \) be that subset of \( A \) for which there is a finite equational basis which is logically equivalent to the first-order equational theory. The relation \( \prec \) being computable implies that for fixed \( E \), the set \( B_E = \{ < E, \tau > \in B \} \) is recursive. Also, there is an algorithm to determine if the equivalent theory is trivial, i.e. has the basis \( x = y \).

Main Question: Is \( B \) recursive?
Hyperassociativity is Not Finitely Based for $<2, 2>$

This was one of the key results in [7]. The preliminary example showing $F^3(x) \approx F^2(x)$ is not finitely based for the type $<1, 1, 1>$ relied on finding an infinite sequence of unary terms (in this case an infinite word), which would not allow for appropriate substitutions to reduce large instances of the hyperidentity to smaller instances.

Similarly, we needed an "infinite term" from which we could use certain subterms to show that large instances of hyperassociativity did not follow from smaller instances. In this case the term was built up from alternating composition of one binary operation $\cdot$ with the other $\circ$. The family of terms were used, along with a "dual" map from terms which switched the binary operation symbols:

$$t_{00} = (x \cdot x)$$
$$t_{01} = (x \cdot y)$$
$$t_{02} = (y \cdot x)$$
$$t_{03} = (y \cdot y)$$

along with their $*$-counterparts, e.g. $t_{02}^* = (y \circ x)$. Then larger terms were built using arrangements of the terms, as so:

$$t_{10} = t_{00} \cdot (t_{01} \cdot t_{02}^*)$$
$$t_{11} = t_{00} \cdot (t_{01} \cdot t_{03}^*)$$
$$t_{12} = t_{00} \cdot (t_{02} \cdot t_{03}^*)$$
$$t_{13} = t_{01} \cdot (t_{02} \cdot t_{03}^*)$$

along with their counterparts, e.g $t_{11}^* = (x \cdot x) \circ ((x \cdot y) \circ (y \cdot y))$.

The general pattern is given (for $n \geq 0$) by:

$$t_{(n+1)0} = t_{n0} \cdot (t_{n1} \cdot t_{n2}^*)$$
$$t_{(n+1)1} = t_{n0} \cdot (t_{n1} \cdot t_{n3}^*)$$
$$t_{(n+1)2} = t_{n0} \cdot (t_{n2} \cdot t_{n3}^*)$$
$$t_{(n+1)3} = t_{n1} \cdot (t_{n2} \cdot t_{n3}^*)$$
$$t_{(n+1)0}^* = t_{n0} \circ (t_{n1} \circ t_{n2})$$
$$t_{(n+1)1}^* = t_{n0} \circ (t_{n1} \circ t_{n3})$$
$$t_{(n+1)2}^* = t_{n0} \circ (t_{n2} \circ t_{n3})$$
$$t_{(n+1)3}^* = t_{n1} \circ (t_{n2} \circ t_{n3})$$

Then $t = t_{(2k+1)0}(x, t_{(2k+1)0}(y, z))$ has no “small” substitution instances appearing inside it. One would need such instances if there were a finite basis for hyperassociativity to transform that term to $u = t_{(2k+1)0}(t_{(2k+1)0}(x, y), z)$, but the only such allowed terms have more than $k$ binary operation symbols, which means one needs large instances of hyperassociativity to prove this instance $t = u$ in equational logic.

Similar arguments were used to show hyperassociativity is not finitely based for the types $<3>$ and for $<2, 1>$. This establishes the finite basis character of hyperassociativity for all nice types.
Aside: Hypersubstitutions and model theory

This portion diverges from considering the two problems, but is of related interest.

[1] provided the inspiration for the original problem on hyperassociativity and some results in [5]. However, the desire to understand the model theoretic side spurred the development arc in [5], [6], [7], and [8].

Corresponding to the linguistic side of hyperidentities is the model theoretic side of hypervarieties and solid varieties. Hypervarieties according to [13] are "varieties of varieties", or classes of varieties which are closed under certain operations on varieties such as varietal product and reduct. While of some interest, the study of clone identities for abstract and concrete clones has appeared more than studies focussed on hypervarieties.

Solid varieties are the by-product of the research direction using hypersubstitutions. Briefly, one considers maps on the term algebra of a type $\tau$ to itself, and one can form a monoid of such maps which stand for replacing a basic function symbol $\tau$ of arity $n$ by a term from $\tau$ of the same arity. One then sees a hyperidentity as relation among hypersubstitutions, and one can use this idea to form derived algebras, which are algebras enhanced with term operations arising from these hypersubstitutions. A solid variety is now a variety containing these derived algebras, and the collection of solid varieties of $\tau$-algebras forms a complete sub-lattice of the lattice of varieties of all $\tau$-algebras. As Polak showed in [4] even for semigroups there is some richness to this sublattice.

The Finite Basis Problem should relate to locally finite solid varieties, and hopefully reveal their relation to other solid varieties. In particular it should help talk about doing computations (or not) in the lattice of solid varieties, perhaps determine whether such a variety represents a join-irreducible element, or reveal other information.

The Finite Basis Problem should also play a role in seeing how complex the lattice can be in the case the hyperidentity is not finitely based with respect to a given type. Further, the preorder $\prec$ should give rise to a categorical mapping between solid varieties of $\prec$-related types, possibly providing embeddings of one lattice of solid varieties into another.
A Theory of Small Trees, With Applications

In [8], we reported some of the progress in finding non-finitely based pairs, particularly those involving hyperassociativity and the types $\langle 2, 2 \rangle$, $\langle 2, 1 \rangle$, and $\langle 3 \rangle$.

Also, Ralph McKenzie and George Bergman had shown us similar families of terms witnessing that hypermediality was not finitely based for some types.

Finally, as the first example of a nfb representation came from symbolic dynamics, we suggested in [8] the notion that symbolic dynamics should be extended to solve the Finite Basis problem.

In particular, instead of a tree representation for a term, consider a class of decorated directed sets, or infinite trees. (every two nodes belong to some common infinite sub term, each node decorated with a function symbol whose arity corresponded to the number of children of that node.). Such an infinite tree could be used to derive or otherwise represent an infinite sequence of terms, with earlier terms being subterms of later terms. One now has the concept of which trees map to their own shifts, much as occurs in symbolic dynamics. Indeed, standard symbolic dynamics on $\tau^\mathbb{Z}$ (where $\tau$ is playing a role as both a type of unary functions and a symbol alphabet) is a special case of this, the infinite term of composed unary functions corresponding to a trajectory.

One can study collections of such trees to see how they act under generalized shifts. In particular, where certain subterms correspond to one side of a hyperidentity, one might find rigid trees which have few or no such shifts that would correspond to substitutions in a derivation in equational logic. Such a "rigid" tree would serve a role analogous to the infinite sequence of models in witnessing the lack of a possibility for a finite basis for the hyperidentity.

In addition to finding collections of "rigid" trees, one needs to determine which aspects of these collections lend to computability results. If it turns out that "rigid" trees form a non-recursive set, that may lead to a negative answer of the Finite Basis Problem.

At the time of this presentation, we were becoming aware of the work following Denecke’s school, but did not see how the concept of hypersubstitution was changing the face of the research on hyperidentities. We think that some computability issues raised in [8] could inspire algorithmic complexity issues for hyperidentities.
The case $xxyyz = xxyxxyz$

If we consider a variation of the argument for the hyperidentity $F(F(x)) \approx F(F(F(x)))$, we should find in the semigroup variety satisfying $xxyyz = xxyxxyz$ relatively-free 3 or 4-generated free algebras which are not finite. This is because we should find semigroup words in 3 or 4 variables which are squarefree, and will be distinct in the free algebra of this variety.

Even though we do not have the local finiteness of semigroups in the case $xyxxyx = xyzyx$, it is enough to have the relatively free 2-generated semigroup be finite. This amounts to looking for repeated squares in words on an alphabet of two letters. Fortunately there are enough repeated squares that we can claim a similar finiteness result.

Note that we have $x^2y^2z^2 = x^2y^4z$, and $x^2y^b z = (x^2y)^b z$. Further, let $W(x,y)$ and $Z(x,y)$ be words in $x$ and $y$, with $Z(x,y)$ not the empty word. By the two consequences above, any word in $x$ and $y$ which is of the form $W(x,y)x^2y^3Z(x,y)$ or $W(x,y)x^2y^2x^2Z(x,y)$ has a logical equivalent where $Z(x,y)$ is replaced by $Z'(x,y)$. $Z'(x,y)$ has all but the last power reduced modulo 2, e.g. if $Z(x,y)$ is $x^{a(1)}y^{a(2)}\ldots x^{a(2n-1)}y^{a(2n)}$, and $b(i) = a(i) \mod 2$, and one writes the empty string as $x^0$ or $y^0$, then $Z'(x,y)$ is $x^{b(1)}y^{b(2)}\ldots x^{b(2n-1)}y^{a(2n)}$. This combined with $x^2y^3z = x^2y^5z$ gives finitely many alternatives for $Z'(x,z)$.

Similarly, one can reduce large patterns strictly containing $xyxyZ(x,y)^2$. One then finds only finitely many inequivalent alternatives for $W(x,y)$. This results in a finite set of words, allowing another route to showing that hyperassociativity is finitely based for $\langle 2 \rangle$.

Of course, one can perform a similar analysis with the subvariety which also satisfies $xyzz = xyyzzz$. This addresses problem 28 from [2].
Future Directions and Applications

In unpublished work, we found that an analogue to Burnside’s problem was not finitely based. This means that $F^n(x) \approx F(x)$ was not finitely based for the type $< 2 >$, and so for "larger" types.

One unexplored avenue regards categorical interpretations of the problem. The clones correspond to algebraic theories of Lawvere, which have a substantial literature. We are unsure how the Finite Basis problem is realized in this context.

At the moment, the connection to symbolic dynamics seems intriguing. We hope to extend the notion of dynamics on $S^Z$ to dynamics on directed sets, sections of which represent "infinite terms". We hope this extension of dynamics will allow the analysis and solution of the Main Question.

We also note work of Melkonian [10] in circuit optimization, who used hyperidentities and hyper-gates, to analyze and solve a k-out-of-n circuits problem. We hope to see more applications to circuit optimization.
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