Abstract

We analyze the quantum cosmology of one-loop string effective models which exhibit an $O(d,d)$ symmetry. It is shown that due to the large symmetry of these models the Wheeler–de Witt equation can completely be solved. As a result, we find a basis of solutions with well defined transformation properties under $O(d,d)$ and under scale factor duality in particular. The general results are explicitly applied to 2-dimensional target spaces while some aspects of higher dimensional cases are also discussed. Moreover, a semiclassical wave function for the 2-dimensional black hole is constructed as a superposition of our basis.
1 Introduction

String theory as a candidate for the fundamental theory of nature is expected to predict not just the low energy effective particle theory but the initial status of our universe [1]–[6]. In any case the prediction crucially depends on which of the perturbatively degenerated vacua of string theory is chosen. Nonperturbative effects should remove this degeneracy and pick out the true vacuum. So far, however, the only information available comes from nonperturbative effects like gaugino condensation [7] which can be treated in the context of a low energy effective model. Therefore, string cosmology at present cannot fully predict the initial state of the universe. At most, the various promising candidate vacua can be checked with respect to their cosmological acceptability.

Unfortunately, it has turned out that early universe cosmologies based on gaugino condensate models have rather generic problems [8, 9]. This has motivated the construction of alternative string cosmologies e. g. based on the idea of topology change [10] or the use of stringy symmetries like duality [3]. The latter, so called pre–big–bang cosmologies relate the superinflationary (pre–big–bang) and ordinary phase to each other by a duality transformation [4], also named scale factor duality. The pre–big–bang models can be conveniently analyzed in the framework of an $O(d,d)$–invariant one–loop string effective action [5] with scale factor duality as a discrete subgroup. For our purpose, namely to make contact between string– and quantum cosmology we will concentrate on this interesting class of models.

Quantum Cosmology [11] provides a completely different approach towards a theory of initial conditions of the universe. Typically, the Wheeler–de Witt equation is solved in minisuperspace i. e. for a finite number of degrees of freedom. This is done, adopting a proposal for the boundary conditions several of which have been advocated so far, e. g. [12, 13]. A consistent interpretation of the resulting wave function can be given in the semiclassical region and certain predictions can be extracted in this limit.

Most probably, in the context of string theory such an approach does not have any fundamental meaning though attempts have been made in this direction [14]. Nevertheless one might ask the following questions : What is the “effective” meaning of the quantum cosmological approach in string theory and – more pragmatically – can the “stringy” properties of string effective models be helpful to understand their quantum cosmology? Certainly, the first of these questions will not be answered here though our results might provide some hints for further study in this direction. The answer to the second question will be definitely “yes”.

Our study along these questions will be concentrated on the aforementioned models with $O(d,d)$ symmetry for two reasons : First of all, as discussed before, cosmologically interesting models are included in this class of string vacua so that relations between string– and quantum cosmology can be discussed. This has been done in ref. [15] for the first time. Second, the large symmetry of these models turns out to be very useful in solving their quantum cosmology. In particular, due to the symmetry the system consists of a finite number of degrees of freedom only so that we are dealing with a min-
isuperspace model. We will not attempt to impose certain boundary conditions on our solutions following one of the proposals in the literature. Instead, we will give a complete classification of wave functions solving the Wheeler–de Witt equation in terms of $O(d, d)$ quantum numbers and construct some interesting semiclassical examples as appropriate superpositions.

The plan of the paper is as follows: In section 2 we present a short review of the $O(d, d)$–invariant effective action, its properties and the Hamiltonian formulation. Section 3 deals with the operator realization of $O(d, d)$ which is used to calculate the quadratic $O(d, d)$ Casimir. These results are applied in section 4 to find the general solution of the Wheeler–de Witt equation for arbitrary dimension. The 2–dimensional case is discussed in section 5 where some semiclassical wave functions are also constructed using our basis of solutions. Furthermore, we consider some aspects of higher dimensional cases, in particular the 3– and 4–dimensional ones. Relevant $O(d, d)$-group theory has been summarized in the appendix.

2 Setup of general formalism

The bosonic part of the one–loop string effective action \[16\] reads

$$S = \int d^D x \sqrt{-g} e^{-\phi} \left[ \Lambda - R - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} h_{\mu\nu\rho} h^{\mu\nu\rho} \right],$$

(1)

where $\Lambda$ is the cosmological constant, $\phi$ the dilaton, $g_{\mu\nu}$ the $\sigma$–model metric and the torsion $h_{\mu\nu\rho}$ is given in terms of the antisymmetric field $b_{\mu\nu}$ as

$$h_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu}.$$

(2)

If $g$ and $b$ do not depend on $d$ of the $D$ space–time coordinates the above action turns out to have an $O(d, d)$ symmetry \[17\]. For cosmological considerations we assume that all fields in eq. (1) are time dependent only ($d = D - 1$). By a suitable coordinate and gauge transformation $g$ and $b$ can be brought to the block diagonal form

$$g = \begin{pmatrix} -N^2(t) & 0 \\ 0 & G(t) \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 0 & B(t) \end{pmatrix},$$

(3)

where $G, B$ are $d \times d$ matrices and we have also introduced a lapse function $N$. Using the above form of the metric and the antisymmetric tensor, the action (1) turns out to be

$$S = \int dt N \sqrt{\det(G)} e^{-\phi} \left[ \Lambda + \frac{1}{N^2} \left( \dot{\phi}^2 - \dot{\phi} \text{tr}(G^{-1} \dot{G}) + \frac{1}{4} (\text{tr}(G^{-1} \dot{G}))^2 \right) + \frac{1}{4N^2} \text{tr} \left( (G^{-1} \dot{B})^2 - (G^{-1} \dot{G})^2 \right) \right].$$

(4)
The $O(d, d)$ invariance of eq. (4) can be made explicit by introducing the matrix

$$M = \begin{pmatrix} \frac{1}{2} (G^{-1} - G^{-1} B) \\ \frac{1}{2} (B G^{-1} - G^{-1} B) \end{pmatrix}$$

(5)

which transforms as

$$M \rightarrow \Omega^T M \Omega$$

(6)

under the action of an $O(d, d)$–element $\Omega$ which, by definition, satisfies

$$\Omega^T \eta \Omega = \eta , \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(7)

In particular, the $O(d, d)$–element $\Omega = \eta$ generates $T$–duality transformations which are direct generalizations of the $R \rightarrow 1/R$ symmetry of the toroidally compactified string. Moreover, the matrix $M$ is an $O(d, d)$ element itself which is restricted to be symmetric, i. e.,

$$M^T \eta M = \eta , \quad M = M^T.$$

(8)

We may express the action (4) in a manifest $O(d, d)$–invariant form as

$$S = \int dt \, Ne^{-\Phi} \left[ \Lambda + \frac{1}{N^2} \dot{\Phi}^2 + \frac{1}{8N^2} \text{tr} \left( \dot{M}^2 \eta \right) \right],$$

(9)

by using the $O(d, d)$–invariant dilaton

$$\Phi = \phi - \ln \sqrt{\det(G)}$$

(10)

and the relation

$$\text{tr} \left( (\dot{M} \eta)^2 \right) = 2 \text{tr} \left[ (G^{-1} \dot{B})^2 - (G^{-1} \dot{G})^2 \right].$$

(11)

The classical solutions of the action (9), classified in terms of a “$\beta$–function” which determines the running of $e^{\Phi}$, may completely be expressed in terms of quadratures (6).

Turning to the quantum theory now we may employ the Hamiltonian formulation. The action (4) defines a constrained system with the constraints given by eq. (8). Its quantization would therefore lead to unnecessary complications which can be avoided by using the fields $G, B$ instead of the $O(d, d)$–covariant object $M$. Nevertheless, the invariant combination $\Phi$ turns out to be useful in the following. Consequently, we have chosen, as the starting point, the field basis $\{\Phi, G, B\}$ and the corresponding Lagrangian

$$\mathcal{L} = Ne^{-\Phi} \left[ \Lambda + \frac{1}{N^2} \dot{\Phi}^2 + \frac{1}{4N^2} \text{tr} \left( (G^{-1} \dot{B})^2 - (G^{-1} \dot{G})^2 \right) \right].$$

(12)

The conjugate momenta are

$$\pi_\Phi = \frac{2}{N} e^{-\Phi} \dot{\Phi},$$

$$\pi_G = -\frac{1}{N} e^{-\Phi} G^{-1} \dot{G} G^{-1},$$

$$\pi_B = -\frac{1}{N} e^{-\Phi} G^{-1} \dot{B} G^{-1},$$

(13)
the Hamiltonian turns out to be
\[ H = N \left[ \frac{e^{\Phi}}{4} \left( \pi_\Phi^2 + \text{tr} \left( (G\pi_B)^2 - (G\pi_G)^2 \right) \right) - e^{-\Phi} \Lambda \right] , \] (14)
and the canonical quantization relations are
\[ [\Phi, \pi_\Phi] = i \quad [G_{ij}, \pi_{G_{kl}}] = i(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad [B_{ij}, \pi_{B_{kl}}] = i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) . \] (15)
The wave function \( \Psi = \Psi(\Phi, G, B) \) is subject to the Wheeler–de Witt equation
\[ \hat{H}\Psi = 0 \] (16)
with the operator version \( \hat{H} \) of the Hamiltonian (14). The above equation is a zero energy Schrödinger equation which – due to the restriction of \( O(d, d) \) invariance – contains a finite number of degrees of freedom only. In the language of Quantum Cosmology it constitutes a minisuperspace model \[ [11] \]. Turning the Hamiltonian (14) into \( \hat{H} \) we encounter the familiar operator–ordering ambiguity which appears in the term \( \text{tr}(G\pi_G)^2 \) \[ [18] \]. Later on we will comment on this ambiguity and we will show how to partially fix it.

3 Operator realization of the \( O(d, d) \) symmetry

Classically, to any \( O(d, d) \) generator \( T \) corresponds a conserved current which turns out to be
\[ J_T = e^{-\Phi} \text{tr} \left( T\eta M\eta M \right) . \] (17)
Working in the standard basis \( [A4] \) of matrices \( (T_{AB}) \), these currents take the form \[ [1] \]
\[ (J_{AB}) = \begin{pmatrix} S & C \\ D & R \end{pmatrix} \] (18)
with
\[ S = \pi_B \]
\[ C = \pi_G G - \pi_B B \]
\[ D = -(G\pi_G - B\pi_B) \]
\[ R = G\pi_G B - B\pi_B B + B\pi_G G - G\pi_B G . \] (19)
It can be easily seen that \( S \) generates the shifts of the antisymmetric tensor field whereas \( C \) generates coordinate transformations. They correspond to the explicit transformations given in eq. \( [A3] \).
To promote these currents to their operator version we have to carefully handle the operator–ordering problem. Let us define the operators

\[
\hat{S} = S \\
\hat{C} = C + ic1 \\
\hat{R} = R + irB
\]

with an operator ordering in \(S, C, R\) defined by the order of matrices as in eq. (19), while the real quantities \(c, r\) parameterize the operator–ordering ambiguity. In the standard basis we can write

\[
\left( \hat{J}_{AB} \right) = \begin{pmatrix} \hat{S} & \hat{C} \\ -\hat{C}^T & \hat{R} \end{pmatrix} .
\]

By an explicit calculation we find that these operators satisfy the \(O(d, d)\) algebra

\[
[\hat{S}_{ij}, \hat{S}_{kl}] = 0 \\
[\hat{S}_{ij}, \hat{C}_{kl}] = i(\delta_{ij}\hat{S}_{jk} - \delta_{jl}\hat{S}_{ik}) \\
[\hat{S}_{ij}, \hat{R}_{kl}] = i(\delta_{jk}\hat{C}_{il} + \delta_{il}\hat{C}_{jk} - \delta_{ik}\hat{C}_{jl} - \delta_{jl}\hat{C}_{ik}) \\
[\hat{C}_{ij}, \hat{C}_{kl}] = i(\delta_{jk}\hat{C}_{il} - \delta_{il}\hat{C}_{kj}) \\
[\hat{C}_{ij}, \hat{R}_{kl}] = i(\delta_{il}\hat{R}_{jk} - \delta_{jk}\hat{R}_{il}) \\
[\hat{R}_{ij}, \hat{R}_{kl}] = 0 ,
\]

if and only if

\[
2c - r = 2d .
\]

If the condition (23) is violated, however, unwanted central terms appear. Next we want to calculate the quadratic Casimir.

\[
C = -\frac{1}{16} f_{AB)(EF)(GH)f_{(CD)(GH)(EF)}J_{AB}J_{CD}
\]

in terms of the operator representation \(J_{AB}\). With the structure constants to be read off from eq. (A.3) we arrive at

\[
C = -2(d - 1)\text{tr} \left[ (J_\eta)^2 \right] .
\]

A straightforward calculation using the explicit expressions for \(J\) and eq. (23) leads to

\[
C = 4(d - 1) \left[ \text{tr} \left( (G\pi_B)^2 - (G\pi_G)^2 \right) + i(3d - 2c + 1)\text{tr} (\pi_G G) + dc(c - d + 2) \right] .
\]

\(^2\)Lower case letters \(i, j, k, \cdots = 1, \cdots, d\) refer to one of the \(O(d)\) subgroups or a spatial coordinate whereas upper case letters \(A, B, C, \cdots = 1, \cdots, 2d\) denote \(O(d, d)\) group indices.

\(^3\)If not specified explicitly, we refer to operators from now on and we drop the hat notation.
The parameter $c$ is still arbitrary and can e. g. be chosen such that one of the two extra terms in $C$ disappears. We observe that the first term exactly coincides with the $G, B$-dependent part of the classical Hamiltonian (14). To make $\mathcal{H}$ and $C$ fully compatible an appropriate operator ordering prescription for the Hamiltonian can be chosen and in the following we assume that this has been done. Then the ambiguity in promoting the classical Hamiltonian to its quantum version is partially fixed. We stress that such a choice does not exist if the condition (23) was violated or, equivalently, $O(d, d)$ symmetry was broken. In that case the expression for $C$ contains a term proportional to $\text{tr}(\pi B B)$ which never can appear in the Hamiltonian.

4 Solving the Wheeler–de Witt equation

Now we are ready to solve the Wheeler–de Witt equation exploiting the $O(d, d)$ symmetry. According to the above remarks, the $G, B$-dependent part of the Hamiltonian can be chosen to coincide with the Casimir so that

$$\mathcal{H} = N \left[ \frac{e^\Phi}{4} \left( \pi_\Phi^2 + \frac{1}{4(d-1)} C \right) - e^{-\Phi} \Lambda \right], \quad d > 1. \quad (27)$$

We emphasize that $\mathcal{H}$ can be entirely expressed in terms of the quadratic Casimir. No higher Casimir operators or “radial” parts are present. Obviously, the above expression does not hold for the abelian case $d = 1$ since the Casimir vanishes. It has to be replaced by the $O(1, 1)$–charge $Q = iG\pi_G$ leading to the Hamiltonian

$$\mathcal{H} = N \left[ \frac{e^\Phi}{4} \left( \pi_\Phi^2 + Q^2 \right) - e^{-\Phi} \Lambda \right], \quad d = 1. \quad (28)$$

Note that the definition of $Q$ fixes the operator ordering for $d = 1$.

Via the ansatz $\Psi(\Phi, G, B) = \xi(\Phi)\chi(G, B)$, the Wheeler–de Witt equation can be separated into a $G, B$–dependent part

$$E\chi = \begin{cases} \frac{C}{4(d-1)}\chi, & d > 1 \\ \frac{Q^2}{\chi}, & d = 1 \end{cases} \quad (29)$$

and a dilaton part

$$\left( \pi_\Phi^2 - 4e^{-2\Phi} \Lambda + E \right) \xi = 0. \quad (30)$$

The first equation (29) specifies the spectral parameter $E$ in terms of the Casimir $C$. We will concentrate on the finite dimensional representations of $O(d, d)$ which can be directly obtained from the corresponding $O(2d)$ representations by multiplying the noncompact generators with $i$ (Weyl’s trick). Referring to these representations by Dynkin labels $[20]$ $(\mathbf{r}) = (a_1, \cdots, a_d)$, the Casimir is

$$C(\mathbf{r}) = (\mathbf{r}, \mathbf{r} + \delta), \quad \delta = (2, \cdots, 2) \quad (31)$$
where the product \((...,..)\) is calculated with the metric tensor of \(SO(2d)\). The solutions \(\chi\) are harmonic functions on the moduli space \(O(d,d)/O(d) \times O(d)\). To find these harmonics we have to concentrate on those representations \((r)\) of \(O(2d)\) which in their decomposition under \(O(d) \times O(d)\) contain a singlet. Let us denote weights in a certain representation \((r)\) by \((m) = (m_1, \cdots, m_d)\) and the representation matrices by \(D^{(r)}_{(m)(m')}\). Then the \(G, B\)–dependent part of the wave function takes the form

\[
\chi(G, B) \sim D^{(r)}_{(0\zeta)(m)}(M(G, B)) \tag{32}
\]

where \((0\zeta)\) denotes the \(O(d) \times O(d)\) singlet directions. For representations which can be obtained as tensor products of the fundamental representation of \(O(d,d)\) there is a simple way to get explicit expressions for \(\chi\). They can be written as the following products of matrix elements \(M_{AB}\):

\[
\chi \sim (M_+M_+^T)_{A_1B_1} \cdots (M_+M_+^T)_{A_mB_m}(M_-M_-^T)_{C_1D_1} \cdots (M_-M_-^T)_{C_mD_n} \tag{33}
\]

Here \(M_\pm\) denote the matrices consisting of the \(d\) upper or lower rows of \(M\), respectively. The indices \(A_k, B_k, C_k, D_k\) have to be symmetrizes appropriately in accordance with the desired \(O(d,d)\) representation.

The equation (30) for \(\Phi\) becomes simple for a vanishing cosmological constant. For \(\Lambda\) nonvanishing and by using the substitution

\[
\rho = 2\sqrt{|\Lambda|}e^{-\Phi},
\]

it can be rewritten as a Bessel differential equation. The solution may then be expressed as

\[
\xi = \begin{cases} 
    ae^{\sqrt{E\Phi}} + be^{-\sqrt{E\Phi}}, & \text{\(\Lambda = 0\)} \\
    aJ_{\sqrt{E}(\rho)} + bN_{\sqrt{E}(\rho)}, & \text{\(\Lambda > 0\)} \\
    aJ_{\sqrt{E}(\rho)} + bN_{\sqrt{E}(\rho)}, & \text{\(\Lambda < 0\)}
\end{cases} \tag{35}
\]

in terms of the Bessel– and von Neumann–functions \(J_\nu, N_\nu\) and arbitrary coefficients \(a, b\). The expressions (32) and (35) with the Casimir (31), taken for all \(O(2d)\) representations which contain \(O(d) \times O(d)\) singlets, form a complete set of solutions of the Wheeler–de Witt equation. Explicit decompositions of certain semiclassical wave functions in terms of this basis will be given in the next section.

## 5 Application to 2–dimensional models

In this section we apply our results to 2–dimensional models in order to find explicit expressions for our solutions. Furthermore, we show how to construct semiclassical wave functions by appropriate superpositions and discuss some properties of the higher dimensional cases.
Clearly, for \( d = 1 \) the situation simplifies a lot. The matrix \( M \) is of the form
\[
M = \begin{pmatrix}
G^{-1} & 0 \\
0 & G
\end{pmatrix},
\]
where \( G \) is a single degree of freedom from the metric and obviously no antisymmetric tensor field exists. Of course the Wheeler–de Witt equation can be integrated immediately in that case. To illustrate the methods presented above, however, we follow the group theoretical approach here.

The relevant group \( O(1, 1) \) can be parameterized as
\[
\Omega(\alpha) = \begin{pmatrix}
\cosh \frac{\alpha}{2} & \sinh \frac{\alpha}{2} \\
\sinh \frac{\alpha}{2} & \cosh \frac{\alpha}{2}
\end{pmatrix}
\]
and its representations can be labeled by a charge \( q \):
\[
D^{(q)}(\Omega(\alpha)) = e^{q\alpha}.
\]
The above parameterization refers to the basis where \( \eta \) is diagonal. In this basis the matrix \( M \) takes the form
\[
M = \frac{1}{2} \begin{pmatrix}
G + G^{-1} & G - G^{-1} \\
G - G^{-1} & G + G^{-1}
\end{pmatrix}
\]
and by comparison with eq. (37) one reads off that \( G = e^{\alpha/2} \). Therefore \( \chi \sim G^q/2 \) and the complete wave function can be written as
\[
\Psi^{(q)}(G, \Phi) = G^{q/2} \begin{cases}
ac_G |q\rangle |\Phi\rangle + be^{-|q\rangle |\Phi\rangle} & , \ \Lambda = 0 \\
aj_{|q\rangle} \left(2\Lambda e^{-\Phi}\right) + bN_{|q\rangle} \left(2\Lambda e^{-\Phi}\right) & , \ \Lambda > 0 \\
aj_{|q\rangle} \left(i2\Lambda e^{-\Phi}\right) + bN_{|q\rangle} \left(i2\Lambda e^{-\Phi}\right) & , \ \Lambda < 0
\end{cases}
\]
As we will see below, duality acts on the representations by complex conjugation, i.e. by \( q \to -q \) in the case under consideration. Indeed, eq. (40) shows that this transformation is equivalent to \( G \to G^{-1} \).

Next, we construct semiclassical wave functions as superpositions of the solutions (40), i.e. wave functions of the form
\[
\Psi = Ce^{iS}
\]
with a classical action \( S \) and a (real) prefactor \( C \). Following the interpretation that a peak in the wave function is considered as a prediction \([22]\) it can be shown \([23]\) that such a semiclassical wave function selects a subset of classical trajectories specified by
\[
\pi_G = \frac{\partial S}{\partial G}, \ \quad \pi_\Phi = \frac{\partial S}{\partial \Phi}.
\]
Here $\pi_G$ and $\pi_\Phi$ denote the classical expressions for the conjugate momenta. On this subset the square of the wave function $|\Psi|^2 = C^2$ can be consistently interpreted as a probability $\Pi$.

Let us start with the case of vanishing cosmological constant $\Lambda = 0$. For a given charge $q$ we have the solution $(G^{\pm 1/2} e^{-\Phi})^q$ where the sign is specified by the sign of $q$. This implies that we can easily construct a semiclassical wave function with

$$S_\pm \sim G^{\pm 1/2} e^{-\Phi}$$

which by inspections of eqs. (42) corresponds to a Milne universe and its dual [2, 24], i.e. $G \sim t^{\mp 2}$ and $\Phi \sim \ln t$.

For positive cosmological constant an approximate wave function in the weak coupling region $e^{-\Phi} \gg 1$ can be obtained from the asymptotic expansion of the Bessel function. For a fixed charge $q$ we find

$$S \simeq 2\sqrt{\Lambda} e^{-\Phi}, \quad C \simeq \sqrt{\frac{1}{\pi\sqrt{\Lambda}}} G^{\pi/2} e^{\Phi/2}.$$ (44)

This implies a set of classical solutions specified by $\dot{\Phi} = -\sqrt{\Lambda}$ and $\dot{G} = 0$ which describe a static universe with a linearly moving dilaton [2]. It also corresponds to the linear dilaton solution [25] in 2–d black hole physics. Depending on the sign of $q$ the probability $C^2$ shows a preference for large or small static universes and both cases are related to each other by $R \to 1/R$ duality.

In addition to the linear dilaton solution we should also be able to find a semiclassical black hole wave function. The corresponding classical solution reads [21, 26]

$$G = \tanh^{\pm 2} \left( \sqrt{\Lambda}(T - t)/2 \right), \quad e^\Phi = \frac{2\sqrt{\Lambda}}{\kappa \sinh \left( \sqrt{\Lambda}(T - t) \right)},$$ (45)

with arbitrary constants $\kappa, T$. Its classical action can be computed from

$$S = 2\Lambda \int dt \, e^{-\Phi}$$ (46)

and leads to the semiclassical wave function

$$\Psi = \exp \left[ i\sqrt{\Lambda} e^{-\Phi} \left( \sqrt{G} + \frac{1}{\sqrt{G}} \right) \right],$$ (47)

which shows invariance under duality. Let us concentrate on the particular solution $\Psi^{(q)} = G^{q/2} J_q(2\sqrt{\Lambda} e^{-\Phi})$ taken from eq. (10). We can use the Laurent series which generates Bessel functions [26] to arrive at the result

$$\Psi = \Psi^{(0)} + \sum_{q=1}^{\infty} i^q \left( \Psi^{(q)} + \Psi^{(-q)} \right).$$ (48)
The absence of any prefactor in addition to eq. (47) shows that $\Psi$—though computed semiclassically—is an exact solution of the Wheeler–de Witt equation. Of course this can also be confirmed by direct calculation if the operator ordering fixed by eq. (40) is used.

An advantage of our choice of basis is clearly its well defined transformation property under $O(d, d)$. Though explicit expressions are more difficult to find in higher dimensions this allows to discuss some properties of our solution using group theoretical methods. For $d = 2$, e. g. we have to consider representations of $SO(4) \simeq SU_L(2) \times SU_R(2)$ which are specified by $(r) = (a_L, a_R) = 2(j_L, j_R)$. The Casimir is given by

$$C_{SO(4)}((j_L, j_R)) = 2(j_L(j_L + 1) + j_R(j_R + 1))$$

Locally, the moduli space can be written as $SU_L(2) \times SU_R(2)/U_L(1) \times U_R(1)$ and the representations which contain a $U_L(1) \times U_R(1)$ singlet are just the vector representations $(j_L, j_R)$ with $j_L, j_R$ integer.

For $d = 3$ the representations of $SO(6) \simeq SU(4)$ are labeled by $(r) = (a_1, a_2, a_3)$. With the metric tensor

$$G_m(SU(4)) = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

we get for the Casimir

$$C_{SU(4)}((a_1, a_2, a_3)) = \frac{3}{4}a_1^2 + 3a_1 + a_1a_2 + 4a_2 + \frac{1}{2}a_1a_3 + 3a_3 + a_2^2 + a_2a_3 + \frac{3}{4}a_3^2.$$  

(51)

For a given representation $(r) = (a_1, a_2, a_3)$ the complex conjugate representation is specified by the label $(r^*) = (a_3, a_2, a_1)$. These are exactly the representations which are mapped into each other by duality. From the local form of the moduli space $SU(4)/SU(2) \times SU(2)$ and the weight projection matrix

$$P(SU(4) \to SU(2) \times SU(2)) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

(52)

it can be seen that the representations containing $SU(2) \times SU(2)$ singlets are just the one with $a_1 + a_3$ even.

Finally, let us denote how duality acts on our solutions. One may see from eq. (A.11) that a state of weight $(m)$ in a given representation $(r)$ of $O(d, d)$ is mapped to a state of weight $(-m)$. Correspondingly, the representation $(r)$ is mapped into its complex conjugate $(r^*)$. This implies for our wave functions that

$$\chi(G, B) \sim D_{(m)(0)}^{(r)}(M(G, B)) \quad \to \quad D_{-(m)(0)}^{(r^*)}(M(G, B))$$

$$\xi(\Phi) \to \xi(\Phi).$$

(53)
6 Conclusions

Above, we have discussed quantum cosmological aspects of the one-loop string effective action. Although, string theory is supposed to be a consistent theory for quantum gravity, we believe that it is meaningful to discuss quantum cosmology in the context of the effective theory. Especially, as long as the information about nonperturbative string theory is very limited, such an effective approach may provide us with some hints about the correct vacuum of the theory.

In our case, we have considered cosmological backgrounds where all fields, namely, the metric, antisymmetric tensor and the dilaton are present. In particular, we have assumed homogeneity but not isotropy of the D-dimensional space-time. Moreover, the surfaces of simultaneity are assumed to be flat so that we are actually discussing here the quantum cosmology of the rolling–moduli solution [2]. The central point is the existence of an $O(d, d)$ symmetry of the one-loop beta-function equations. This symmetry allowed us to separate and finally solve the Wheeler–de Witt equation in minisuperspace. Although the general solution is quit complicated, it is possible to discuss explicitly the lower-dimensional (D=2) case. Here, we have found all the known 2-dimensional solutions, namely, the Milne universe, the linear dilaton solution as well as the 2-dimensional black hole one, depending on the value of the cosmological constant. It should be noted that these solutions are the only ones which have a CFT description, at least for specific (discrete) values of $\Lambda$.

Since we are dealing with noncompact space–times we have not attempted to impose one of the boundary proposals of quantum cosmology on our solutions. Therefore, we are not able to specify e. g. the ground state wave function selected by these proposals. However, one might speculate about another way to choose a ground state, namely by demanding that it is invariant under (scale factor) duality. In the 2–dimensional case this implies a vanishing $O(1, 1)$ charge $q$ and an essentially unique wave function which depends on the invariant dilaton $\Phi$ only.

As a final comment let us note that one may consider higher–loop corrections to the one-loop effective action as perturbations in the Wheeler–de Witt equation and use standard perturbation theory to deal with them.

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Appendix
A  Some useful properties of $O(d,d)$

Our final goal is to find the solutions of the Wheeler–de Witt equation (16) classified according to $O(d,d)$ quantum numbers. As a preparation, we will now present a brief summary of the $O(d,d)$ algebra and discuss its operator realization [19].

By means of eq. (7) a generator $T$ corresponding to the infinitesimal transformation
\[ \Omega = 1 + iT \]
is subject to the constraint
\[ T^T \eta + \eta T = 0 \quad \text{(A.1)} \]
Therefore the Lie algebra is given by matrices $T$ of the form
\[ T = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix} \quad \text{(A.2)} \]
with $d \times d$ matrices $a, b, c$. Here $a$ is arbitrary and $b, c$ are antisymmetric. The above form implies that $T \eta$ is a generally antisymmetric matrix. Infinitesimal shifts of the $B$-field $T_\theta$ and coordinate transformations $T_a$ are generated by
\[ T_\theta = \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix}, \quad T_a = \begin{pmatrix} -a^T & 0 \\ 0 & a \end{pmatrix} \quad \text{(A.3)} \]
respectively. As a standard basis for the Lie algebra we choose the matrices
\[ (T_{AB})^C_D = i (\delta^C_A \eta_{BD} - \delta^C_B \eta_{AD}) \quad \text{(A.4)} \]
which satisfy the well–known commutation relations
\[ [T_{AB}, T_{CD}] = i (-\eta_{AC} T_{BD} - \eta_{BD} T_{AC} + \eta_{AD} T_{BC} + \eta_{BC} T_{AD}) \quad \text{(A.5)} \]
Let us also explicitly perform the Cartan decomposition of $O(d,d)$. Here we switch to a different basis with
\[ \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{(A.6)} \]
Defining the elements of the Cartan subalgebra as
\[ H_i = T_{i+d,i} \quad \text{(A.7)} \]
and raising and lowering operators as
\[ E_{ij}^{++} = \frac{1}{2} (T_{ij} + T_{i+d,j+d} + T_{i,j+d} + T_{i+d,j}) \]
\[ E_{ij}^{--} = -\frac{1}{2} (T_{ij} + T_{i+d,j+d} - T_{i,j+d} - T_{i+d,j}) \quad \text{(A.8)} \]
\[ E_{ij}^{+-} = \frac{1}{2} (T_{ij} - T_{i+d,j+d} - T_{i,j+d} + T_{i+d,j}) \]
\[ E_{ij}^{-+} = -\frac{1}{2} (T_{ij} - T_{i+d,j+d} + T_{i,j+d} - T_{i+d,j}) \]
we indeed find

\[
[H_i, H_j] = 0
\]
\[
[H_i, E^{(\epsilon_1 \epsilon_2)}_{kl}] = \alpha_i^{(\epsilon_1 \epsilon_2,kl)} E^{\epsilon_1 \epsilon_2}_{kl}
\]
\[
[E^{(++)}_{ij}, E^{(--)}_{ij}] = \sum_k \alpha_k^{(++,ij)} H_k
\]
\[
[E^{(+--)}_{ij}, E^{(--+)}_{ij}] = \sum_k \alpha_k^{(+--ij)} H_k
\]

with the \(SO(2d)\) roots

\[
\alpha_i^{(\epsilon_1 \epsilon_2,kl)} = \epsilon_1 \delta_i^k + \epsilon_2 \delta_i^l
\]

and \(\epsilon_1, \epsilon_2 = \pm\). It is interesting to work out how duality acts on these generators. From the transformation property \(T_{AB} \rightarrow \eta T_{AB} \eta\) of \(T_{AB}\) and the definitions (A.7) and (A.8) we find

\[
H_i \rightarrow -H_i
\]
\[
E^{(++)}_{ij} \rightarrow -E^{(--)}_{ij}
\]
\[
E^{(+--)}_{ij} \rightarrow -E^{(--+)}_{ij}
\]

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