No-Go Theorem for Energy-Momentum Conservation in Curved Spacetime

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Abstract

It is pointed out that in curved spacetime, one cannot define the sum of energy-momentum 4-vector over a space-like hypersurface. The difficulty in finding satisfactory gravitational energy-momentum complex stems from misunderstanding of this question. The law of conservation of energy-momentum holds valid only approximately when spacetime is not seriously curved.

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I. INTRODUCTION

The law of conservation of energy is the cornerstone of physics. As a matter of fact, far beyond physics, it governs all natural processes, and is considered the bedrock law of nature. In 1686, Leibniz proposed the concept of kinetic energy, when he noticed that the total kinetic energy remains unchanged in some processes, say, elastic collisions on a horizontal frictionless plane. Later, however, it was found that this was not true for some other processes. The concept of potential energy (elastic, gravitational, etc.) was then introduced, so that the total mechanic energy might remain unchanged. To keep the total energy unchanged, energy of heat, electromagnetic energy, chemical energy, etc., were introduced. People realized that energy can not be created nor be destroyed, it can only change its forms, and be transported from body to body. In a sense, the processes of development of physics since, can be regarded as the processes of discovery of new forms and new carriers of energy in order to keep the law of energy conservation alive. The discovery of neutrino in 1930’s to keep conservation of energy-momentum valid in $\beta$-decay, was a good example.

In 1905, more than two centuries after Leibniz proposed the concept of kinetic energy, Einstein presented his special theory of relativity, which changed revolutionarily the concepts of space and time. Space and time can no longer be separated absolutely. And energy is no longer a scalar, but the 0-component of the energy-momentum 4-vector. Conservation of energy and conservation of momentum either both hold valid, or both fail, because conservation of energy (momentum) is the consequence of homogeneity of time (space), and spacetime is inseparable. In special relativity (SR) spacetime is still flat, and we have conservation of energy-momentum:

$$\partial_\mu T^{\mu\nu}(x) = 0$$

$$\int_{\Omega} d^4x \partial_\mu T^{\mu\nu}(x) = \int_{\partial\Omega} ds_\mu T^{\mu\nu}(x) = 0$$

where $\Omega$ is a 4-dimensional spacetime region, $\partial\Omega$ its boundary and $(x^0, x^1, x^2, x^3)$ a Lorentzian coordinate system (coordinate system in which the metric tensor $g_{\alpha\beta}(x) \equiv \eta_{\alpha\beta}$, $[\eta] = diag(-1, 1, 1, 1)$). Special relativity also revealed that mass and energy are the same thing, $mc^2 = E$.

Ten years later, Einstein published his general theory of relativity. According to general
relativity (GR), our spacetime is no longer flat, but curved. It is no longer an affine space, but a 4-dimensional generalized Riemannian manifold with a Lorentzian signature metric. The law of energy-momentum conservation encounters difficulty in GR. In fact, from $\nabla_\mu j^\mu(x) = 0$, which is the generalization to curved spacetime of $\partial_\mu j^\mu(x) = 0$, the differential conservation law of some scalar in flat spacetime, we can get the integral conservation law of this scalar in curved spacetime:

$$\sqrt{-g(x)}\nabla_\mu j^\mu(x) = \partial_\mu(\sqrt{-g(x)}j^\mu(x)) = 0 \implies \int_{\partial\Omega} ds_\mu \sqrt{-g(x)}j^\mu(x) = 0 \quad (3)$$

But, from $\nabla_\mu T^{\mu\nu}(x) = 0$, the generalization of (1) to curved spacetime, we cannot get the integral conservation law of energy-momentum in curved spacetime. In fact,

$$\sqrt{-g(x)}\nabla_\mu T^{\mu\nu}(x) = \partial_\mu(\sqrt{-g(x)}T^{\mu\nu}(x)) + \sqrt{-g(x)}\Gamma^\nu_{\mu\lambda} T^{\mu\lambda}(x) = 0$$

$$\implies \int_{\partial\Omega} ds_\mu \sqrt{-g(x)}T^{\mu\nu}(x) = - \int_{\Omega} d^4x \sqrt{-g(x)}\Gamma^\nu_{\mu\lambda} T^{\mu\lambda}(x) \quad (4)$$

The rhs of eqn.(4) is not always equal to 0. Having realized this difficulty, Einstein rewrote $\nabla_\mu T^{\mu\nu}(x) = 0$ as[1],

$$\sqrt{-g(x)}\nabla_\mu T^{\mu\nu}(x) = \partial_\mu[\sqrt{-g(x)}(T^{\mu\nu} + t^{\mu\nu})] = 0 \quad (5)$$

and by integrating it over $\Omega$, he obtained

$$\int_{\Omega} d^4x \partial_\mu[\sqrt{-g(x)}(T^{\mu\nu} + t^{\mu\nu})] = \int_{\partial\Omega} ds_\mu[\sqrt{-g(x)}(T^{\mu\nu} + t^{\mu\nu})] = 0 \quad (6)$$

which he considered as the integral conservation law of the total energy-momentum with $t^{\mu\nu}$ being regarded as the energy-momentum tensor of gravity. However, $t^{\mu\nu}$ does not behave like a tensor and is not symmetric as expected. H. Bauer pointed out it leads to difficulty[2]. Following Einstein, Tolman[3], Landau, Lifshitz[4], Papapetrou[5], Möller[6], Weinberg[7], Bergmann and Thompson[8] proposed their gravitational energy-momentum complexes, coordinate dependent or coordinate free, symmetrical or asymmetrical, but all lacking in energy locality. Bondi argued that nonlocalizable energy is not allowed in GR[9]. Penrose and many others developed the concept of quasilocal energy[10]. So far, however, we still lack a generally accepted definition of energy-momentum of gravitational field. Energy-
momentum conservation in GR and energy-momentum density of gravity are still the focus of theoretical physicists[11].

The present paper consists of two parts. In the first part, by using variational principle approach, the field equation in GR is re-established, and Noether’s theorem is re-derived in a context more general than before, and the conservation laws of energy-momentum, angular momentum and electric charge are re-obtained. In the second part, a no-go theorem for energy-momentum conservation in curved spacetime is given. And finally, the physical implication is discussed.

II. VARIATIONAL PRINCIPLE APPROACH TO GENERAL RELATIVITY

In the present paper, we use Hilbert action for the metric field, and the total action is

\[ A = \int_{\Omega} d^4x \sqrt{-g(x)}[\frac{1}{16\pi G} R + \mathcal{L}_M(g(x), u(x), \nabla u(x))] = A_G + A_M \]  

where \( R \) is the Ricci scalar curvature, and \( \mathcal{L}_M \) the Lagrangian of matter field, which we restrict to scalar and vector fields without loss of generality for our purpose.

For the sake of generality, we use the following general action in our discussion.

\[ A = \int_{\Omega} d^4x L(\Phi(x), \partial\Phi(x), \partial^2\Phi(x)) \]  

A. Re-establishing Field Equation

The least action principle says for any 4-dimensional spacetime region \( \Omega \), the action on \( \Omega \) of real movement takes the stationary value among the actions on \( \Omega \) of all possible movements (all movements allowed by the constraints) with the same boundary condition:

\[ \delta \Phi_a(x)|_{\partial\Omega} = 0 \]  

for \( \Phi_a(x) \)'s whose second derivatives \( \partial^2\Phi_a(x) \) do not appear in \( L \).

\[ \delta \Phi_a(x)|_{\partial\Omega} = 0, \delta \partial_\mu \Phi_a(x)|_{\partial\Omega} = 0 \]  

for \( \Phi_a(x) \)'s whose second derivatives \( \partial^2\Phi_a(x) \) do appear in \( L \).
Using the standard procedure, we obtain

\[
\delta A = \int_{\Omega} d^4x \left( \frac{\partial L}{\partial \Phi_a(x)} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \Phi_a(x)} + \partial_\nu \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} \right) \delta \Phi_a(x)
\]

\[
+ \int_{\Omega} d^4x \partial_\mu \left( \frac{\partial L}{\partial \partial_\mu \Phi_a(x)} - \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} \right) \delta \Phi_a(x) + \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} \delta \partial_\nu \Phi_a(x)
\]

\[
= \int_{\Omega} d^4x \left( \frac{\partial L}{\partial \Phi_a(x)} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \Phi_a(x)} + \partial_\nu \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} \right) \delta \Phi_a(x)
\]

\[
+ \int_{\partial \Omega} d\sigma [\left( \frac{\partial L}{\partial \partial_\mu \Phi_a(x)} - \partial_\nu \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} \right) \delta \Phi_a(x) + \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} \delta \partial_\nu \Phi_a(x)]
\]

\[
= \int_{\Omega} d^4x \left( \frac{\partial L}{\partial \Phi_a(x)} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \Phi_a(x)} + \partial_\nu \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} \right) \delta \Phi_a(x) = 0
\]  \[11\]

Therefore the general field equation is

\[
\frac{\delta A}{\delta \Phi_a(x)} = \frac{\partial L}{\partial \Phi_a(x)} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \Phi_a(x)} + \partial_\nu \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} = 0
\]  \[12\]

For the case of general relativity

\[
L = \sqrt{-g(x)} \left[ \frac{1}{16\pi G} R + L_M(g(x), u(x), \nabla u(x)) \right] = L_G + L_M
\]  \[13\]

we have

\[
\frac{\delta A}{\delta \Phi_a(x)} = \frac{\delta A_G}{\delta \Phi_a(x)} + \frac{\delta A_M}{\delta \Phi_a(x)} = 0
\]  \[14\]

When \( u(x) \) in (13) is a scalar field,

\[
\frac{\partial L}{\partial \varphi(x)} = \sqrt{-g(x)} \frac{\partial L_M}{\partial \varphi(x)}
\]

\[
\partial_\mu \frac{\partial L}{\partial \partial_\mu \varphi(x)} = \partial_\mu \left[ \sqrt{-g(x)} \frac{\partial L_M}{\partial \partial_\mu \varphi(x)} \right] = \sqrt{-g(x)} \left[ \partial_\mu \frac{\partial L_M}{\partial \partial_\mu \varphi(x)} + \Gamma^\mu_{\mu\nu} \frac{\partial L_M}{\partial \partial_\nu \varphi(x)} \right]
\]

we get

\[
\frac{\delta A}{\delta \varphi(x)} = \sqrt{-g(x)} \left[ \frac{\partial L_M}{\partial \varphi(x)} - \nabla_\mu \frac{\partial L_M}{\partial \partial_\mu \varphi(x)} \right] = 0
\]  \[15\]

When \( u(x) \) in (13) is a vector field,

\[
\frac{\partial L}{\partial u_\alpha(x)} = \sqrt{-g(x)} \left[ \frac{\partial L_M}{\partial u_\alpha(x)} + \frac{\partial L_M}{\partial \nabla_\mu u_\beta(x)} (-\Gamma^\alpha_{\mu\beta}) \right]
\]
\[ \partial_{\mu} \frac{\partial L}{\partial \partial_{\mu} u_{\alpha}(x)} = \partial_{\mu} [\sqrt{-g(x)} \frac{\partial L_{M}}{\partial \partial_{\mu} u_{\alpha}(x)}] = \sqrt{-g(x)} [\partial_{\mu} \frac{\partial L_{M}}{\partial \partial_{\mu} u_{\alpha}(x)}] + \Gamma_{\mu}^{\nu} \frac{\partial L_{M}}{\partial \partial_{\nu} u_{\alpha}(x)} \]

we get

\[ \frac{\delta A}{\delta u_{\alpha}(x)} = \sqrt{-g(x)} \left[ \frac{\partial L_{M}}{\partial u_{\alpha}(x)} - \nabla_{\mu} \frac{\partial L_{M}}{\partial \partial_{\mu} u_{\alpha}(x)} \right] = 0 \quad (16) \]

For both cases, we have

\[ \frac{\delta A}{\delta g^{\alpha\beta}(x)} = \frac{\delta A_{G}}{\delta g^{\alpha\beta}(x)} + \frac{\delta A_{M}}{\delta g^{\alpha\beta}(x)} = 0 \quad (17) \]

\[ \frac{\partial L_{G}}{\partial g^{\alpha\beta}(x)} = \sqrt{-g(x)} \left[ \frac{1}{16\pi G} [\frac{1}{2} g_{\alpha\beta} R - \frac{1}{2} g_{\alpha\beta} R + \frac{\partial R}{\partial g^{\alpha\beta}(x)}] \right] \]

\[ \partial_{\mu} \frac{\partial L_{G}}{\partial \partial_{\mu} g^{\alpha\beta}(x)} = \partial_{\mu} \left[ \sqrt{-g(x)} \frac{1}{16\pi G} \frac{\partial R}{\partial \partial_{\mu} g^{\alpha\beta}(x)} \right] \]

\[ = \sqrt{-g(x)} \frac{1}{16\pi G} \left[ \frac{1}{4} g_{\rho\sigma} g_{\xi\eta} g_{\mu\nu} g_{\alpha\beta} \frac{\partial R}{\partial \partial_{\mu} g^{\alpha\beta}(x)} - g_{\mu\nu} g_{\rho\sigma} g_{\xi\eta} g_{\alpha\beta} \frac{\partial R}{\partial \partial_{\mu} g^{\alpha\beta}(x)} + \frac{1}{2} g_{\rho\sigma} \frac{\partial R}{\partial \partial_{\nu} g^{\alpha\beta}(x)} \right] \]

Substituting these equations into eqn.(17), we get

\[ \frac{\delta A}{\delta g^{\alpha\beta}(x)} = \sqrt{-g(x)} \frac{1}{16\pi G} [R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R - 8\pi G T_{\alpha\beta} + I] = 0 \quad (18) \]

where

\[ T_{\alpha\beta} \equiv \frac{-2}{\sqrt{-g(x)} \delta g^{\alpha\beta}(x)} \frac{\delta A_{M}}{\delta g^{\alpha\beta}(x)} \quad (19) \]

is the energy-momentum tensor of the matter field, and

\[ I \equiv -R_{\alpha\beta} + \frac{\partial R}{\partial g^{\alpha\beta}(x)} - \partial_{\mu} \frac{\partial R}{\partial \partial_{\mu} g^{\alpha\beta}(x)} + \partial_{\nu} \frac{\partial R}{\partial \partial_{\nu} g^{\alpha\beta}(x)} + \frac{1}{2} g_{\rho\sigma} \frac{\partial R}{\partial \partial_{\rho\sigma} g^{\alpha\beta}(x)} \]

\[ + \frac{1}{4} g_{\rho\sigma} g_{\xi\eta} g_{\mu\nu} g_{\alpha\beta} \frac{\partial R}{\partial \partial_{\mu\nu} g^{\alpha\beta}(x)} - g_{\mu\nu} g_{\rho\sigma} \frac{\partial R}{\partial \partial_{\rho\sigma} g^{\alpha\beta}(x)} + \frac{1}{2} g_{\rho\sigma} \frac{\partial R}{\partial \partial_{\nu} g^{\alpha\beta}(x)} \]
Identity (20) can be proved straightforwardly. Therefore, eqn.(18) is just Einstein’s field equation.

$$\frac{\delta A}{\delta g^{\alpha\beta}(x)} = \sqrt{-g(x)} \frac{1}{16\pi G} \left[ (R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R) - 8\pi G T_{\alpha\beta} \right] = 0$$  \hspace{1cm} (21)

Thus, eqns.(15)+(21) (eqns.(16)+(21)) are the field equations of scalar (vector) field plus metric field.

B. Re-deriving Noether’s Theorem

Noether’s theorem is re-derived in a more general context than before. And by using it, we get the conservation laws of ‘energy-momentum’ ‘angular momentum’ and electric charge in general relativity.

**Theorem 1** Noether’s theorem

*If the action of field system on any 4-dimensional spacetime region $\Omega$ remains unchanged under the $r$-parameter family of the following infinitesimal transformations of coordinates and fields*

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \delta x^\mu$$  \hspace{1cm} (22)

$$\Phi_a(x) \mapsto \tilde{\Phi}_a(\tilde{x}) = \Phi_a(x) + \delta \Phi_a(x)$$  \hspace{1cm} (23)

*then there exist $r$ conserved quantities.*

**Proof.** From equations (22) and (23), we have

$$\delta(d^4x) = (\partial_\sigma \delta x^\sigma)d^4x$$  \hspace{1cm} (24)

$$\delta \partial_\mu = - (\partial_\mu \delta x^\sigma) \partial_\sigma$$

$$\delta \partial_\mu \Phi_a(x) = \partial_\mu \delta \Phi_a(x) - \partial_\sigma \Phi_a(x)(\partial_\mu \delta x^\sigma)$$  \hspace{1cm} (25)

$$\delta \partial_\mu \partial_\nu \Phi_a(x) = \partial_\mu \partial_\nu \delta \Phi_a(x) - \partial_\sigma \partial_\nu \Phi_a(x) \partial_\mu \delta x^\sigma - \partial_\sigma \partial_\mu \Phi_a(x) \partial_\nu \delta x^\sigma - \partial_\sigma \Phi_a(x) \partial_\mu \partial_\nu \delta x^\sigma$$  \hspace{1cm} (26)

The variation of action is
\[ \delta A = \int_\Omega \! d^4x \left[ \frac{\partial L}{\partial \Phi_a(x)} \delta \Phi_a(x) + \frac{\partial L}{\partial \partial_\mu \Phi_a(x)}(\partial_\nu \delta \Phi_a(x) - \partial_\sigma \partial_\nu \Phi_a(x) \partial_\mu \delta x^\sigma + \partial_\sigma \partial_\mu \Phi_a(x) \partial_\nu \delta x^\sigma) - \partial_\sigma \Phi_a(x) \partial_\mu \partial_\nu \delta x^\sigma) + L \partial_\sigma \delta x^\sigma \right] + \int_\Omega \! d^4x \left[ \partial_\mu \left[ \delta_\mu \delta x^\sigma \right] - \partial_\nu \left( \frac{\partial L}{\partial \partial_\mu \Phi_a(x)} \right) \right] (\delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \delta x^\sigma) \]

\[ + \int_\Omega \! d^4x \delta x^\mu \delta x^\sigma \left( \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} \right) (\delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \delta x^\sigma) \]

\[ + \int_\Omega \! d^4x \delta x^\mu \delta x^\sigma \left( \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} \right) (\delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \delta x^\sigma) \] = 0 \tag{27} 

The first integral at rhs vanishes for real movement, hence the second integral at rhs does too. We get the following equation due to the arbitrariness of \( \Omega \).

\[ \partial_\mu \left[ \delta_\mu \delta x^\sigma \right] - \partial_\nu \left( \frac{\partial L}{\partial \partial_\mu \Phi_a(x)} \right) \right] (\delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \delta x^\sigma) \]

\[ + \int_\Omega \! d^4x \delta x^\mu \delta x^\sigma \left( \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} \right) (\delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \delta x^\sigma) \]

\[ = 0 \tag{28} \]

Noting that both \( \delta \Phi_a(x) \) and \( \delta x^\mu \) depend on \( r \) real parameters, we can consider equation (28) as the conservation laws of \( r \) quantities. \[ \blacksquare \]

**C. Conservation of ‘Energy-Momentum’**

Action (7) remains unchanged under the following coordinates shift.

\[ x^\mu \rightarrow x^\mu + \varepsilon^\mu, \quad \delta x^\mu = \varepsilon^\mu \tag{29} \]

\[ \delta u_a(x) = 0, \quad \delta g^{\alpha\beta}(x) = 0 \tag{30} \]

In this case, eqn.(28) reads

\[ \partial_\mu \left[ \delta_\mu L - \left( \frac{\partial L}{\partial \partial_\mu \Phi_a(x)} \right) \partial_\nu \Phi_a(x) - \frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi_a(x)} \partial_\sigma \partial_\nu \Phi_a(x) \right] = 0 \tag{31} \]
In order to find out the gravitational energy-momentum complex, for the sake of simplicity, we consider the complex scalar field in curved spacetime

$$\mathcal{L}_M = -g^{\sigma\rho} \nabla_\rho \varphi^* \nabla_\sigma \varphi - m^2 \varphi^* \varphi$$  \hspace{1cm} (32)

In this case, eqn.(31) reads

$$\partial_\mu \left\{ \sqrt{-g(x)} \left[ \delta_\mu^\nu \mathcal{L}_M - \frac{\partial \mathcal{L}_M}{\partial \nabla_\mu \varphi(x)} \nabla_\nu \varphi(x) - \frac{\partial \mathcal{L}_M}{\partial \nabla_\mu \varphi^*(x)} \nabla_\nu \varphi^*(x) \right] + \sqrt{-g(x)} \frac{1}{16\pi G} \left[ \delta_\mu^\nu R \right. \right.$$  

$$\left. \left. - \left( \frac{\partial R}{\partial \partial_\mu g^{a\beta}(x)} - \partial_\sigma \frac{\partial R}{\partial \partial_\mu \partial_\sigma g^{a\beta}(x)} \right) \partial_\nu g^{a\beta}(x) \right) \left] \right. \right.$$  

$$= 0, \ \forall \nu = 0, 1, 2, 3$$  \hspace{1cm} (33)

The expression in the first bracket is just the energy-momentum tensor of matter fields.

$$T_\mu^\nu = \delta_\mu^\nu \mathcal{L}_M - \frac{\partial \mathcal{L}_M}{\partial \nabla_\mu \varphi(x)} \nabla_\nu \varphi(x) - \frac{\partial \mathcal{L}_M}{\partial \nabla_\mu \varphi^*(x)} \nabla_\nu \varphi^*(x)$$  \hspace{1cm} (34)

Hence the second bracket with its coefficient

$$t_\mu^\nu = \frac{1}{16\pi G} \left[ \delta_\mu^\nu R - \left( \frac{\partial R}{\partial \partial_\mu g^{a\beta}(x)} - \partial_\sigma \frac{\partial R}{\partial \partial_\mu \partial_\sigma g^{a\beta}(x)} \right) \partial_\nu g^{a\beta}(x) \right. \right.$$  

$$\left. \left. + \frac{1}{2} g^{\xi\eta}(x) \partial_\sigma g^{\xi\eta}(x) \frac{\partial R}{\partial \partial_\mu \partial_\sigma g^{a\beta}(x)} \partial_\nu g^{a\beta}(x) - \frac{\partial R}{\partial \partial_\mu \partial_\sigma g^{a\beta}(x)} \partial_\nu \partial_\sigma g^{a\beta}(x) \right] \right.$$  \hspace{1cm} (35)

seems to be the gravitational counterpart. It is not difficult to show that eqn.(34) is consistent with eqn.(19) for Lagrangian (32).

\textbf{D. Conservation of ‘Angular Momentum’}

Action (7) remains unchanged under the following infinitesimal ‘Lorentz transformation’ (36) through (38). In this subsection, \(u(x)\) in (7) is supposed to be the complex scalar field for definiteness.

$$x^\mu \mapsto \tilde{x}^\mu = L_\nu^\mu x^\nu, \ L_\nu^\mu = \delta_\nu^\mu + \Lambda_\nu^\mu, \ |\Lambda_\nu^\mu| \ll 1, \ \eta_{\mu\lambda} \Lambda_\nu^\lambda \equiv \Lambda_{\mu\nu}, \ \Lambda_{\mu\lambda} = -\Lambda_{\lambda\mu}$$

9
\[
\begin{align*}
\delta x^\mu &= \Lambda_\nu^\mu x^\nu = \frac{1}{2}(\eta^{\mu\nu}x^\sigma - \eta^{\mu\sigma}x^\nu)\Lambda_{\rho\sigma} \\
\delta \varphi(x) &= 0, \quad \delta \varphi^*(x) = 0
\end{align*}
\] (36)

Substituting eqns.(36) through (38) into eqn.(28), we get

\[
\begin{align*}
\partial_\mu \left\{ \sqrt{-g(x)} \left[ T_\lambda^\mu (\eta^{\lambda\rho}x^\sigma - \eta^{\lambda\sigma}x^\rho) \right] + \sqrt{-g(x)}\left[ i\Lambda_\lambda (\eta^{\lambda\rho}x^\sigma - \eta^{\lambda\sigma}x^\rho) \right] \right. \\
+ & \left. \frac{1}{16\pi G} \left[ \partial_\mu \partial_\nu g^{\alpha\beta}(x) - \partial_\nu \partial_\mu g^{\alpha\beta}(x) \right] \right. \\
+ & \left. \frac{1}{2} g_{\lambda\eta} \partial_\nu g^{\lambda\beta} \partial_\eta \partial_\mu g^{\alpha\beta}(x) \right) (\eta^{\beta\rho}g^{\alpha\sigma} - \eta^{\beta\sigma}g^{\alpha\rho} + \eta^{\alpha\rho}g^{\sigma\beta} - \eta^{\alpha\sigma}g^{\rho\beta}) \\
+ & \frac{\partial R}{\partial g^{\alpha\beta}} \left[ \eta^{\beta\rho} \partial_\nu \eta^{\sigma\alpha} - \eta^{\beta\sigma} \partial_\nu \eta^{\alpha\rho} + \eta^{\alpha\rho} \partial_\nu \eta^{\sigma\beta} - \eta^{\alpha\sigma} \partial_\nu \eta^{\rho\beta} \right] \right. \\
- & \left. \frac{\partial R}{\partial g^{\alpha\beta}} \right] = 0
\end{align*}
\] (39)

The expression in the first bracket is just the angular momentum tensor of the matter fields. Hence the expression in the second bracket seems to be the gravitational orbital angular momentum, and the expression in the third bracket with its coefficient seems to be the gravitational spin angular momentum.

**E. Conservation of Electric Charge**

Action (7) for the complex scalar field in curved spacetime remains unchanged under the following infinitesimal transformation of field.

\[
\begin{align*}
\delta x^\mu = 0, \quad \delta \varphi(x) = i\epsilon \varphi(x), \quad \delta \varphi^*(x) = -i\epsilon \varphi^*(x), \quad \delta g^{\alpha\beta}(x) = 0
\end{align*}
\] (40)

Substituting eqns.(13), (32) and (40) into eqn.(28), we get electric charge conservation,

\[
\partial_\mu \left\{ \sqrt{-g(x)} J^\mu(x) \right\} = \sqrt{-g(x)} \nabla_\mu J^\mu(x) = 0
\] (41)

where
\[ J^\mu(x) = i e g^{\mu\nu}(x) [\varphi^* \nabla_\nu \varphi - (\nabla_\nu \varphi^*) \varphi] \] (42)

is the 4–vector of electric current density.

Now, we have obtained, by using Noether’s theorem, the conservation laws of ‘energy-momentum’, ‘angular momentum’, and electric charge. However, I am not going to investigate the obtained energy-momentum, angular momentum densities in detail here. Because I think only the last conservation law is geometrically and physically meaningful. The reason will be given in next section.

III. NO-GO THEOREM FOR CONSERVATION OF ENERGY-MOMENTUM IN CURVED SPACETIME

A. Some preliminaries from geometry

We will denote by \( \mathcal{M} \) the spacetime manifold. For any spacetime point (event) \( p \in \mathcal{M} \), denote by \( V_p \) the tangent space to \( \mathcal{M} \) at \( p \), denote by \( V_p^* \) the dual space of \( V_p \) and denote by \( V_p^{\otimes r} \) the tensor product

\[ V_p \otimes V_p \otimes \cdots \otimes V_p \equiv V_p^{\otimes r} \] (43)

There are \( r \) copies of \( V_p \) on the lhs.

Tangent vectors at different points \( (p, q, \ldots) \) of a (generalized) Riemannian manifold belong to different tangent spaces \( (V_p, V_q, \ldots) \). One can not add up vectors belonging to different linear spaces. In order to add them up, one has to parallelly transport them into one and the same tangent space first. Parallel transport of vectors is coordinate free. However, it depends on path, if the (generalized) Riemannian manifold is not flat. Therefore, one cannot add up tangent vectors at different points of a curved (generalized) Riemannian manifold in an objective way without any subjective factor.

But in an affine space, which can be regarded as a flat (generalized) Riemannian manifolds, parallel transport of tensors is coordinate free and independent of path. Adding up tangent vectors at different points always makes sense. Besides, in an affine coordinate system, the equation for parallel transport of vector \( \nu \) is trivial, \( \delta \nu^\gamma = 0 \), hence the \( \nu \)–component of sum vector is the sum of the \( \nu \)–components of addend vectors, which remain unchanged.
during parallel transport. In a curvilinear coordinate system, however, the value of \( \nu \)-component of sum vector is by no means the sum of \( \nu \)-components of addend vectors before being parallelly transported. The latter is meaningless. In particular, \( \int_{\partial \Omega} ds_\mu \sqrt{-g(x)} T^{\nu \mu}(x) \), the sum of \( ds_\mu \sqrt{-g(x)} T^{\nu \mu}(x) \) (\( \nu \)-components of addend vectors \( dP \) before being parallelly transported) is meaningless.

Scalars at different points \((p, q, \ldots)\) of a (generalized) Riemannian manifold belong to different linear spaces \((V_p^{\otimes 0}, V_q^{\otimes 0}, \ldots)\) too. Conceptually, before adding them up, one has to parallelly transport them to one and the same point of the manifold. Parallel transport of scalars is trivial \((\delta \phi = 0)\), coordinate free and path-independent. Therefore, adding up scalars at different points of a (generalized) Riemannian manifold is always meaningful, and the value of sum scalar is just the sum of values of addend scalars, independent of where the sum scalar lies. In particular, \( \int_{\partial \Omega} ds_\mu \sqrt{-g(x)} J^\mu(x) \), the sum of \( ds_\mu \sqrt{-g(x)} J^\mu(x) \) (values of addend scalars \( dQ \)) is meaningful.

The above argument shows

**Proposition 2** (No-Go Theorem) In a curved (generalized) Riemannian manifold, one cannot add up the tangent vectors at different points of the manifold in a way free of subjective options.

**Proposition 3** (No-Go Theorem) In a curvilinear coordinate system, sum of \( \nu \)-components of tangent vectors at different points of a (generalized) Riemannian manifold is meaningless, no matter the manifold is flat or curved.

These propositions are well-known to geometers. Unfortunately, however, neglect of them is the root of all the difficulties in energy-momentum conservation in GR.

Now, let us get to the conservation laws.

**B. Inadequate Expression for Conservation of a Vector in Curved Spacetime**

Let’s first consider a tensor field \( T \) of type \((2, 0)\) with null covariant divergence on Minkowski space. This tensor field \( T \) can be considered the density and current density of some vector, say, \( P \) (not necessarily energy-momentum 4-vector). In any Lorentzian coordinate system \((x^0, x^1, x^2, x^3)\), we have
\[
\sqrt{-g(x)}\nabla_\mu T^{\mu\nu}(x) = \partial_\mu T^{\mu\nu}(x) = 0 \tag{44}
\]

Integrating (44) over 4-dimensional spacetime region \(\Omega\), we get
\[
\int_\Omega d^4x \sqrt{-g(x)}\nabla_\mu T^{\mu\nu}(x) = \int_\Omega d^4x \partial_\mu T^{\mu\nu}(x) = \int_{\partial\Omega} ds_\mu T^{\mu\nu}(x) = 0 \tag{45}
\]

Choosing \(\Omega\) such that \(\partial\Omega\) consist of two pieces of space-like hypersurface, \(\Sigma_2\) on the top (future), and \(\Sigma_1\) on the bottom (past), and one piece of hypersurface \(\Sigma\) with Lorentzian signature reduced metric in between, we have
\[
\int_{\Sigma_2} ds_\mu T^{\mu\nu}(x) - \int_{\Sigma_1} ds_\mu T^{\mu\nu}(x) = -\int_{\Sigma} ds_\mu T^{\mu\nu}(x) \tag{46}
\]

Eqn.(46) is considered in SR by all physicists as the law of conservation of vector \(P^\nu\): The value of \(P^\nu\) on space-like hypersurface \(\Sigma_2\) is equal to the value of \(P^\nu\) on space-like hypersurface \(\Sigma_1\) plus the amount of \(P^\nu\) which flows in through the boundary during the corresponding evolution. Note that in this interpretation, \(\Sigma_1\) and \(\Sigma_2\) are not necessarily hyperplanes, while coordinate system \((x^0, x^1, x^2, x^3)\) must be Lorentzian.

Now let us switch to curvilinear coordinate system \((y^0, y^1, y^2, y^3)\), while keeping the tensor field and spacetime unchanged (the spacetime is still Minkowski space and the covariant divergence of \(T\) is still null). Eqn.(44) becomes
\[
\sqrt{-g(y)}\nabla_\mu T^{\mu\nu}(y) = \sqrt{-g(y)}[\partial_\mu T^{\mu\nu}(y) + \Gamma^\mu_{\mu\lambda}(y)T^{\lambda\nu}(y) + \Gamma^\nu_{\mu\lambda}(y)T^{\mu\lambda}(y)]
\]
\[
= \partial_\mu[\sqrt{-g(y)}\partial_\mu T^{\mu\nu}(y)] + \sqrt{-g(y)}\Gamma^\nu_{\mu\lambda}(y)T^{\mu\lambda}(y) = 0 \tag{47}
\]

Integrating \(\partial_\mu[\sqrt{-g(y)}\partial_\mu T^{\mu\nu}(y)]\) over 4-dimensional spacetime region \(\Omega\), we get
\[
\int_\Omega d^4y \partial_\mu[\sqrt{-g(y)}T^{\mu\nu}(y)] = \int_{\Sigma_2} ds_\mu \sqrt{-g(y)}T^{\mu\nu}(y) - \int_{\Sigma_1} ds_\mu \sqrt{-g(y)}T^{\mu\nu}(y)
\]
\[
+ \int_{\Sigma} ds_\mu \sqrt{-g(y)}T^{\mu\nu}(y) = -\int_\Omega d^4y[\sqrt{-g(y)}\Gamma^\nu_{\mu\lambda}(y)T^{\mu\lambda}(y)] \neq 0 \tag{48}
\]

Einstein and his followers on this subject consider \(\int_{\partial\Omega} ds_\mu \sqrt{-g(y)}T^{\mu\nu}(y) = 0\), or equivalently \(\partial_\mu[\sqrt{-g(y)}T^{\mu\nu}(y)] = 0\) as the integral or differential conservation law of vector \(P\). That is, they consider \(\int_{\Sigma_1} ds_\mu \sqrt{-g(y)}T^{\mu\nu}(y)\) as the total value of \(P^\nu\) over space-like hypersurface.
Σ₁. But, as has been shown above, this is right only in affine coordinate systems, and there
is no such coordinate system in curved spacetime. In curvilinear coordinate systems, the
ν-component of the sum of vectors \( dP \)'s over space-like hypersurface \( \Sigma₁ \) differs from the sum
of \( (dP)ν = ds_μ\sqrt{-g(y)T^\muν(y)} \) over space-like hypersurface \( \Sigma₁ \). The former is geometrically
meaningful, the latter is meaningless. \( \int_\Omega d^4y\partial_µ[\sqrt{-g(y)T^\muν(y)}] = 0 \) is an inadequate
expression for conservation of vector \( P \). It was just this misunderstanding that has
guided them in searching for gravitational energy-momentum tensor to keep the total energy-
momentum conserved. According to their point of view, inequality (48) would be read as
vector \( P \) is not conserved even in Minkowski space. Thus, originally conserved vector \( P \) in
Minkowski space is no longer conserved without any geometrical or physical change, just
due to switching to curvilinear coordinate system. This is obviously unreasonable, since the
objective law of nature should be coordinate free, and a proper generalization of a proposition
in SR to GR should retrieve it in the flatness limit.

C. Conclusion

Einstein and some others realized the difficulty of energy-momentum conservation
in GR. They take the law of conservation of energy-momentum for granted, consider
\( \int_\Omega d^4y\partial_µ[\sqrt{-g(y)T^\muν(y)}] = 0 \) as the adequate expression for conservation of energy-
momentum for matter field. In order to keep the total energy-momentum conservation
alive, they introduce the energy-momentum psuedotensor of gravity. After ninety years of
efforts, people realized there is no local solution to this question.

According to the no-go theorems proposed above, I think it is impossible to define the
total energy over a piece of space-like hypersurface, hence it is meaningless to talk about
energy-momentum conservation in curved spacetime. Gravitational field ( I’d rather call it
metric field) is different to matter fields. It shouldn’t carry energy.

The correctness of any proposition in natural science has to be tested by experiments.
Up to now, no evidence for gravity carrying energy has been tested experimentally yet. The
observed accelerated expansion of universe seems to support non-conservation of energy-
momentum.

The law of conservation of energy-momentum still works rather well in regions of space-
time which is approximately flat, like the part of universe nearby.
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[1] A. Einstein, Sber. preuss. 788(1915); 448(1918).
[2] H. Bauer, Phys. Z. 19, 163(1918).
[3] R.C. Tolman, Phys. Rev. 35, 875(1930).
[4] L.D. Landau and E.M. Lifshitz, The classical theory of fields, 4th ed., (Butterworth-Heinemann, Beijing, 1999) P.280.
[5] A. Papapetrou, Proc. R. Irish Acad. A52, 11(1948).
[6] C. Möller, Annals of Physics 4, 347(1958); Annals of Physics 12, 118(1961).
[7] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972), P.165.
[8] P.G. Bergmann and R. Thompson, Phys. Rev. D89, 400 (1953).
[9] H. Bondi, Proc. R. Soc. London A427, 249(1990).
[10] R. Penrose, Proc. R. Soc. London A388, 457(1982).
[11] M.B. Mensky, Phys. Lett. A328, 261(2004).