The small black hole illusion

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Small black holes in string theory are characterized by a classically singular horizon with vanishing Bekenstein-Hawking entropy. It has been argued that higher-curvature corrections resolve the horizon and that the associated Wald entropy is in agreement with the microscopic degeneracy. In this note we study the heterotic small black hole and question this result, which we claim is caused by a misidentification of the fundamental constituents of the system studied when higher-curvature interactions are present. In particular, we argue that the resolution of the heterotic small black hole reported in the literature involves the introduction of solitonic 5-branes, whose asymptotic charge vanishes due to a screening effect induced by the higher-derivative interactions.

Consider a fundamental heterotic string carrying winding number \( w \) and momentum \( n \) along a circle \( S^1 \) [1, 2], which forms part of the compact space \( T^4 \times S^1 \times S^1 \). For large values of \( n \) and \( w \), the entropy associated to the degeneracy of these states is

\[
S \approx 4\pi \sqrt{nw}, \quad n, w \gg 1.
\]

It was soon suggested that this system, among others, could be also represented as a black hole [3, 4]. In this case the black hole would be small, because the event horizon scale would be of the order of the string length.

Working with the heterotic effective action at lowest order in the perturbative expansion, a solution to the equations of motion carrying the same two charges as this configuration was found in [5]. That solution is characterized by a singular horizon of vanishing area and entropy, so higher-curvature terms in the effective expansion cannot be ignored in the near-horizon region. This just means that the effective description fails to give a good approximation of the system. In a seminal article [6] it was then conjectured that the higher-curvature corrections might somehow render the horizon regular, and that the Wald entropy of such black hole would match the microscopic value (1).

Always assuming the existence of a regular horizon and making use of the attractor mechanism, a precise matching of the macroscopic and microscopic computations of the entropy was later reported in [7] — see also [8]. This matching was interpreted as a proof of the resolution of the horizon previously conjectured. There are, however, certain points in this approach that need clarification.

In the first place, the techniques employed in those articles only allow for the study of the near-horizon regime, but the analytic construction of a full black hole solution interpolating between those and asymptotic Minkowski space is missing. According to numerical studies [9], the solution exists and its causal structure is identical to that of four-charge regular black holes [10]. Nevertheless, in order to fully understand the system studied an analytic solution is needed, specially if we take into consideration that higher-curvature corrections introduce significant global interactions [11].

In the second place, it is certainly surprising that the resolution of the small black hole described in [7–10] is achieved with the inclusion of only curvature squared terms. Since the departing system is singular, there is no reason to expect that further higher-derivative corrections can be disregarded for that purpose.

And in the third place, the resolution of similar singular systems, like a Type II string with winding and momentum charge, has not been observed. The different behavior of small black holes in diverse theories raises a puzzle whose resolution has remained unclear so far.

In this note we argue that the resolution of the heterotic small black hole via higher-curvature corrections does not actually occur. To do so, we apply the results of [12, 13], where the analytic construction of general four-charge, supersymmetric black holes including curvature square terms has been performed.

We claim that the resolution of the horizon previously reported is an illusion; the higher-curvature corrections modify the physical meaning of the parameters in the uncorrected solution. The corrections do not resolve the singularity of [5], but they substitute the system under consideration by a special four-charge black hole which is already regular at zeroth-order and whose entropy is precisely given by \( 4\pi \sqrt{nw} \).

Although this might seem to represent a step back in the understanding of small black holes, we actually believe that our result helps to clarify the situation. It puts every small black hole at the same qualitative level; the system is non-perturbative and cannot be properly described incorporating a subgroup of the higher-curvature corrections. Moreover, we recall that the heterotic small black hole can be resolved in the type IIB frame using the uncorrected effective action [14, 15], via smooth geometries whose degeneracy agrees with (1), an observation that gave rise to the fuzzball proposal [16, 17].

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The singular black hole: Let us start revisiting the small black hole singular solution. We first review four-charge regular black holes and then particularize to the two-charge case. We work directly in the ten dimensional formulation of low energy heterotic string theory. The bosonic field content consists of the metric \( g_{\mu
u} \), the dilaton \( e^\phi \) and the Kalb-Ramond 2-form \( B_{\mu
u} \), with field strength \( H_{\mu
u} \). The action and equations of motion, including all relevant terms at quadratic order in curvature, are contained in appendix A. In our first approach we neglect these higher-curvature corrections, which will be recovered afterwards. The four-charge solution is,
\[
\frac{d\sigma^2}{4} = 2\frac{Z}{Z_0} [dv - \frac{1}{2}Z \dd u] - Z_0 d\sigma^2_{(4)} - dy^2,
\]
\[
e^{-2\phi} = \frac{1}{g_s^2} \frac{Z}{Z_0},
\]
\[
H = dz^{-1} \wedge du \wedge dv + *_{(3)} d\sigma_{(3)},
\]
where the Hodge dual in the last equation is associated to the four-dimensional metric \( ds^2_{(4)} \), which is a Gibbons-Hawking (GH) space:
\[
ds^2_{(4)} = V^{-1} (dz + \chi)^2 + \mathcal{V} d\mathcal{z}_{(3)}^2, \quad d\mathcal{V} = \ast_{(3)} d\chi. \tag{3}
\]
Six of the coordinates are compact: the coordinates \( y^i \) parametrize a four-torus \( T^4 \) with no dynamics, while \( u \) and \( z \) have respective periods \( 2\pi R_u \) and \( 2\pi R_z \) and they parametrize two circles that we denote \( S^1 \) and \( \tilde{S}^1 \).

The functions \( Z_{0,\pm} \) and \( \mathcal{V} \) characterize the solution, and are given by
\[
Z_{0,\pm} = 1 + \frac{q_{0,\pm}}{r}, \quad \mathcal{V} = 1 + \frac{q_u}{r} \tag{4}
\]
where \( r \) is the radial coordinate of the Euclidean space \( dx^2_{(3)} \). Notice that all these functions are harmonic in this space. This solution represents the superposition of:
- a string wrapping the circle \( S^1 \) with winding number \( w \) and momentum \( n \),
- a stack of \( N \) solitonic 5-branes (S5) wrapped on \( T^4 \times S^1 \),
- and a Kaluza-Klein monopole (KK) of charge \( W \) associated with \( \tilde{S}^1 \).

The charge parameters \( q_i \) are given in terms of the integer numbers \( n \), \( w \), \( N \) and \( W \) according to
\[
q_+ = \frac{\alpha'^2 g_s^2 N}{2 R_u R_z}, \quad q_- = \frac{\alpha' g_s^2 w}{2 R_z}, \quad q_0 = \frac{\alpha' N}{2 R_z}, \quad q_v = \frac{WR_z}{2}. \tag{5}
\]
After compactification in \( T^4 \times S^1 \times \tilde{S}^1 \), the lower dimensional spacetime metric in the Einstein frame is [13]
\[
ds^2_{(4)} = (Z_+ Z_- Z_0 \mathcal{V})^{-\frac{1}{2}} dt^2 - (Z_+ Z_- Z_0 \mathcal{V})^{\frac{1}{2}} d\mathcal{z}_{(3)}^2. \tag{6}
\]
For non-vanishing charges this geometry represents an extremal black hole whose horizon is placed at \( r = 0 \) and its area is \( A \propto \sqrt{nwNW} \).

The small black hole described in the introduction is that without KK monopole and S5 brane: \( N = W = 0 \). In that case, at \( r = 0 \) there is still a horizon because \( g_{tt} \) vanishes. However, its area vanishes and, even worse, its curvature diverges. Hence, classically, these solutions have singular horizon and vanishing Bekenstein-Hawking entropy.\(^1\) The dilaton \( e^\phi \) vanishes at the horizon, so loop corrections can be neglected in this region, but the singularity in curvature signals that the tree-level supergravity description of this system is not valid for small values of \( r \). When trying to describe the physics near the horizon, one is forced to include the tower of higher-curvature corrections to the heterotic effective action [18]. For quite some time, it has been believed that their inclusion render the horizon regular and make the value of the Wald entropy of the solution coincide with that of (1).

Let us discuss how the first set of these corrections, which are quadratic in the curvature, alter relevant aspects of the solutions. These appear at first-order in the \( \alpha' \) perturbative expansion of the effective field theory.

\( \alpha' \) corrections and S5 charge: A first set of corrections comes from the inclusion of Lorentz and gauge Chern-Simons terms in the Kalb-Ramond field strength, such that its Bianchi identity becomes
\[
dH = \frac{\alpha'}{4} (F^A \wedge F^A + R_{(-)}^a b \wedge R_{(-)}^b a). \tag{7}
\]
In the small black hole literature the gauge fields are set to zero, so we simply take \( F^A = 0 \) from now on. On the other hand, \( R_{(-)}^a b \) is the 2-form curvature of the torsionful spin connection, defined as \( \Omega_{(-)}^a b = \omega^a b - \frac{1}{2} R_{\mu\nu}^a b dx^\mu \), where \( \omega^a b \) is the spin connection.

The recursive definition of \( H \) introduces a tower of infinite corrections in the perturbative expansion in \( \alpha' \) that breaks the supersymmetry of the action, which has to be recovered order by order. At the order we are working, the term \( -\frac{\alpha'}{4} R_{(-)}^a b R_{(-)}^b a \) in the action \( (A1) \) includes all the corrections quadratic in curvature to the heterotic effective action\(^2\). Notice that, although we are working at first-order in \( \alpha' \) in the action, the Bianchi identity (7) is exact in this expansion.

The analysis of the higher-curvature corrections to black hole solutions in the literature has been mostly limited to near-horizon geometries. Only very recently, the first-order in \( \alpha' \) corrections to the solution (2)-(4) have been determined [12, 13]. These are summarized in Appendix B. An important property of the corrections is that they introduce delocalized sources in a way that

\(^1\) This statement holds when any of the four charges vanishes.
\(^2\) Other corrections unrelated to the supersymmetrization of the Chern-Simons term appear first at fourth-order in curvature.
The asymptotic charges and the near-horizon charges are different. These charges are effectively defined as the coefficient of the $1/r$ term in the functions $Z_{0,\pm}$, $V$ when $r \to \infty$ and when $r \to 0$, respectively. Of course, this poses the question of which of those counts the number of the corresponding stringy objects. It is particularly relevant for our discussion the case of S5 branes, codified by $Z_{0}$, whose $\alpha'$-corrected expression is given by

$$Z_{0} = 1 + \frac{q_0}{r} - \alpha' \left( F(r;q_0) + F(r;q_v) \right) + \mathcal{O}(\alpha'^2), \quad (8)$$

where

$$F(r;k) := \frac{(r+q_v)(r+2k)+k^2}{4q_v(r+q_v)(r+k)^2}. \quad (9)$$

In the limits $r \to 0$, $\infty$, this function behaves as

$$\lim_{r \to 0} Z_{0} = \frac{q_0}{r}, \quad \lim_{r \to \infty} Z_{0} = 1 + \frac{q_0 - \alpha'/(2q_v)}{r}, \quad (10)$$

so that near-horizon and asymptotic charges do not coincide. In the language of [19], these are respectively the “Page charge” and “Maxwell charge”.

The correct way to determine the relation between those and the number of solitonic 5-branes is to couple the theory to a stack of $N$ of these branes. This can be done by dualizing the Kalb-Ramond 3-form into the NSNS 7-form $\tilde{H} = dB = *e^{-2\phi}H$ and coupling the 6-form $B$ to the worldvolume action of $N$ solitonic 5-branes by means of Wess-Zumino terms, as reviewed in Appendix C. The net effect is a localized source at the right-hand-side of the Bianchi identity (7). Thus, the number of S5-branes in our solution may be computed according to

$$N = \frac{1}{4\pi^2\alpha'} \int_{\mathbb{R}^5 \times \mathbb{S}_5^{\infty}} H - \frac{1}{16\pi^2} \int R_{(-)\ a}^a \wedge R_{(-)\ b}^b. \quad (11)$$

In the first term we used Stokes’ theorem and the integral is taken on the boundary of GH space (3), while in the second the integral is taken over the full GH space. The result of the integration coincides with the identification in (5) — see Appendix C — and therefore it is the near-horizon charge $q_0$ the one that counts the number of S5-branes.

On the other hand, the asymptotic charge does have a physical meaning by itself and, moreover, gives the contribution to the mass of the black hole. The negative subtraction in (10) is telling us that the higher-curvature terms introduce a screening mechanism such that the charge and mass detected at infinity are smaller than the local charge produced by the 5-branes. The effective number of S5-branes detected at infinity is

$$N^{\text{eff}} = N - \frac{2}{W}. \quad (12)$$

The origin of the negative shift can be identified with precision. It is caused by the presence of two self-dual gravitational instantons in the four-dimensional space parametrized by $(z, \vec{x}(3))$, one for each $\mathfrak{so}(3)$ factor in the decomposition of the group of local Lorentz transformations $\mathfrak{so}(4) \cong \mathfrak{so}_{L}(3) \times \mathfrak{so}_{R}(3)$. The two instantons are sourced by the KK monopole and by the stack of S5 branes respectively, and each one contributes to the asymptotic charge with a factor of $-1/W$.

We obtained this result from the first-order corrected solution (8), which is expected to receive other corrections in the $\alpha'$ expansion. However, one can see that (12) is actually exact and receives no further corrections. The way to prove it is to note that the effective number of S5 branes observed at infinity is given by

$$N^{\text{eff}} = \frac{1}{4\pi^2\alpha'} \int_{\mathbb{R}^5 \times \mathbb{S}_5^{\infty}} H - \frac{1}{16\pi^2} \int R_{(-)\ a}^a \wedge R_{(-)\ b}^b, \quad (13)$$

where in the second line we used (11). But now, the integral in the second line is actually a topological invariant: it is not modified at all by continuous deformations of the connection $\Omega_{(-)\ a}^a$, such as the ones introduced by $\alpha'$ corrections in perturbative solutions. Hence, the value of that integral is always $-32\pi^2/W$, and the S5-brane charge measured at infinity is exactly given by (12).

A very important consequence of this result is that the asymptotic S5 charge vanishes for configurations with $NW = 2$.

**Fake resolution of small black holes:** We are now ready to present a fake resolution of the singular small black hole presented before. Let us describe a four-charge black hole of the form (2) as a solution of the heterotic effective theory that includes all relevant terms of quadratic order in curvature. The functions $Z_-$ and $V$ remain uncorrected as in (4), while $Z_0$ is given by (8), while $Z_+$ receives a correction that behaves as a delocalized source of $+2w/(NW)$ units of momentum charge, whose form is not particularly illuminating for our discussion. 3

The solution has a regular event horizon at $r = 0$, with area $A_h \propto \sqrt{nwNW}$. The near-horizon geometry is $AdS_3 \times S^3/Z_w \times T^4$, and the Wald entropy is $4$

$$S = 2\pi \sqrt{nw(NW+2)}. \quad (14)$$

Here we observe a difference with the usual expression for the entropy that has appeared in the literature be-

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3 See Appendix B for details. Contrary to the contribution to solitonic 5-brane charge, this delocalized momentum does not seem to be protected under higher order corrections.

4 If the Wald entropy is computed to first-order in $\alpha'$ one obtains $S = 2\pi \sqrt{nwNW} \left[ 1 + 1/(NW) \right]$. The expression (14) is exact in $\alpha'$ and can be obtained using the entropy function method, see [20, 21].
fore — see for instance [22] and references therein — which is written in terms of the solitonic 5-brane asymptotic charge $N_{\text{eff}} = N - 2/W$, thus replacing the correction factor of $+2$ by a $+4$. The crucial point is that the shift between asymptotic charge and number of branes remained unnoticed, so in the preceding literature $N_{\text{eff}}$ was incorrectly identified with $N$. There, the parameters of the near-horizon solution are identified in terms of the asymptotic charges using the zeroth-order solution. However, as we just saw, the relation between the near-horizon parameters and the asymptotic charges is altered when the higher-curvature corrections are included. It is for this reason that setting $N_{\text{eff}} = 0$ does not automatically imply the absence of solitonic 5-branes.

The fake small black hole is a very special four-charge solution with $NW = 2$ and arbitrary $n, w$. Its asymptotic solitonic 5-brane charge vanishes, it has a regular horizon and its entropy just happens to match the value (1), although its microscopic origin is clearly different. Had we not taken into account the effect of the delocalized sources, it would seem that the near-horizon solution was already regular in the zeroth-order approximation.

Discussion: At this point we wonder if the singular horizon can be cured without adding other fundamental objects of string theory. As we discussed, this is what happens when harmonic poles are introduced in the functions $V$ and $Z_0$, so in order to avoid it both functions should be set to 1. Moreover, we do not expect the structure of the fields in (2) to be modified, as that configuration describes general spherically symmetric, four-charge, asymptotically flat supersymmetric solutions. The only possibility, thus, is that the functions $Z_\pm$ are modified by the higher-curvature terms just in a precise manner to render the horizon regular. As we already mentioned, the function $Z_-$ is not affected by the inclusion of terms quadratic in curvature. It is determined from the 2-form equation (A8), which remains unmodified for this family of solutions. On the other hand, the function $Z_+$ does get corrected. In this case the treatment is a bit subtle, because the general form of the function given in (B1) is not valid in the particular case in which $q_\epsilon = q_0 = 0$, and we have to use (B7) instead. By looking at that expression, we see that for the two-charge system $\lim_{r \to 0} Z_+ \sim 1/r^3$, which is just the right behavior to obtain a horizon with non-vanishing area in (6). However, this does not fix the singularity problem, since the curvature of the full ten-dimensional solution is still divergent at $r = 0$. Moreover, this even implies that the expression (B7) cannot be trusted, since in its derivation it is assumed that the ten-dimensional curvature is regular at several stages. Our conclusion is that corrections quadratic in curvature are not sufficient to resolve the singular small black hole.

From these observations we doubt there exists a true resolution of the heterotic small black hole singularity, as it does not seem that we can modify the structure of the fields, and a finite sized horizon built with only two functions $Z_\pm$ is headed toward a singularity in curvature.

The resolution of this system reported in [7–10] considered regular near-horizon solutions in the dual frame of Type IIA on $K3 \times T^2$, using a $N = 2$ four-dimensional effective description. This phenomenon was also observed directly in the heterotic string on $T^4 \times S^1 \times S^5$ in [21]. In all the cases, the $S5$ asymptotic charge $N_{\text{eff}}$ is set to zero under the assumption that this implies the absence of $S5$ branes. As we just saw, that premise is not true. This is a crucial fact that, if dismissed, can produce the illusion of a stretched small horizon. On the other hand, in those works it is also stated that $W = 0$ but, as we just argued, we found this fact to be incompatible with the assumption of a regular horizon. Then, from all angles, it seems the solution described there corresponds to the fake small black hole described above.

Our conclusions can be straightforwardly extended to five-dimensional small black holes by using $\mathbb{R}^4$ for the Gibbons-Hawking space $d^2$. This case is simpler because there is no KK monopole and we get $N_{\text{eff}} = N - 1$ for the screening effect. A fake resolution of the five-dimensional small black hole is then straightforward. In this case there is just one solitonic 5-brane, whose asymptotic charge and mass vanish. The entropy is then given by $S = 2\pi \sqrt{nw} (N + 2)$, which no longer happens to coincide with (1) for $N = 1$. Notice that the resolution of five-dimensional small black holes had so far remained uncertain — see the discussion in [21, 23].

Notice that our results do not apply to the qualitatively different class of solutions that come into existence only after the corrections are included — see [24, 25] for an explicit example. The understanding of these black holes in the context of string theory is still an open problem.

Before closing the article, we can very briefly consider the small black hole made of a Type II string carrying winding and momentum charges. We recall that in this theory the Bianchi identity does not receive corrections, so the charges are the same at the horizon and asymptotically. This means that one cannot design a fake resolution of the singularity in the terms we just described, which according to our proposal clarifies why no cure for their singularity has been observed.

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Appendices

Appendix A: The Heterotic Superstring effective action at first-order in $\alpha'$

The effective action of the Heterotic Superstring at first-order in $\alpha'$ is given by \[ (A1) \]

\[ S = \frac{g_s^2}{16\pi G_N(10)} \int d^{10}x \sqrt{|g|} e^{-2\phi} \left\{ R - \frac{4}{3} (\partial \phi)^2 + \frac{1}{2} \nabla^2 \phi - \frac{\alpha'}{4} \left( F^A_{\mu\nu} F^{A\mu\nu} + R(-)_{\mu\nu}^a b R(-)_{\mu\nu}^b a \right) \right\} . \]

From now on we will set to zero the Yang-Mills, $F^A = 0$, following the standard treatment in the small black hole literature. Here $R_{(-)}^a b$ is the curvature of the torsionful spin connection $\Omega_{(-)}^a b = \omega_a b - \frac{i}{2} H_a b dx^\mu$, namely \[ (A2) \]

\[ R_{(-)}^a b = d\Omega_{(-)}^a b - \Omega_{(-)}^a c \wedge \Omega_{(-)}^c b . \]

The field strength $H$ of the Kalb-Ramond 2-form $B$ includes the Lorentz Chern-Simons term \[ (A3) \]

\[ H = dB + \frac{\alpha'}{4} \omega^L_{(-)} , \]

where \[ (A4) \]

\[ \omega^L_{(-)} = d\Omega_{(-)}^a b \wedge \Omega_{(-)}^b a - \frac{2}{3} \Omega_{(-)}^a b \wedge \Omega_{(-)}^b c \wedge \Omega_{(-)}^c a . \]

The Bianchi identity reads \[ (A5) \]

\[ dH - \frac{\alpha'}{4} R_{(-)}^a b \wedge R_{(-)}^b a = 0 . \]

The $\alpha'$-corrected equations of motion are \[ (A6) \]

\[ R_{\mu\nu} - 2\nabla_\mu \partial_\nu \phi + \frac{1}{4} H_{\mu\nu\rho} H^{\rho\sigma} - \frac{\alpha'}{4} R_{(-)\mu\nu}^a b R_{(-)\nu}^b a = 0 , \]

\[ (\partial \phi)^2 - \frac{1}{2} \nabla^2 \phi - \frac{1}{3!} H^2 + \alpha' \frac{R_{(-)\mu\nu}^a b R_{(-)\mu\nu}^b a}{3} = 0 , \]

\[ d(e^{-2\phi} \star H) = 0 . \]

Appendix B: First-order $\alpha'$- corrections to 4-charge black holes

Here we explicitly write down the first-order in $\alpha'$ corrections to the functions $Z_{0,+,-}$ and $\mathcal{V}$ considered in the text when no Yang-Mills fields are present. This result, including the possibility of adding non-trivial Yang-Mills instanton fields, was originally derived in [12, 13]. The functions are given by

\[ (B1) \]

\[ Z_+ = 1 + \frac{q_+}{r} + \frac{\alpha' q_+ r^2 + r(q_0 + q_- + q_v) + q_v q_0 + q_v q_- + q_0 q_-}{(r + q_v)(r + q_0)(r + q_-)} + \mathcal{O}(\alpha'^2) , \]

\[ (B2) \]

\[ Z_- = 1 + \frac{q_-}{r} + \mathcal{O}(\alpha'^2) , \]

\[ (B3) \]

\[ Z_0 = 1 + \frac{q_0}{r} - \alpha' [F(r; q_0) + F(r; q_v)] + \mathcal{O}(\alpha'^2) , \]

\[ (B4) \]

\[ \mathcal{V} = 1 + \frac{q_v}{r} + \mathcal{O}(\alpha'^2) , \]
where
\[ F(r; k) := \frac{(r + q_v)(r + 2k) + k^2}{4q(r + q_v)(r + k)^2}. \] (B5)

The mass of the solution is
\[ M = \frac{R_u}{g_s^2 \ell_s^2} \left( N - \frac{2}{W} \right) + \frac{R_u}{\ell_s^2} n + \frac{w}{R_u} \left( 1 + \frac{2}{NW} \right) + W \frac{R^2 R_u}{g_s^2 \ell_s^4}, \] (B6)
where we have used (5) to write the mass in terms of the integer numbers \( n, w, N \) and \( W \).

On the other hand, when \( q_v = q_0 = 0 \), the solution is qualitatively different [13] and is given by
\[ Z_+ = 1 + \frac{q_+}{r} - \frac{\alpha' q_+ q_-}{2r^3(r + q_-)} + O(\alpha'^2), \] (B7)
\[ Z_- = 1 + \frac{q_-}{r} + O(\alpha'^2), \] (B8)
\[ Z_0 = 1 + O(\alpha'^2), \] (B9)
\[ V = 1 + O(\alpha'^2). \] (B10)

However, as mentioned in the text, these functions (B7)-(B10) yield a singular ten-dimensional geometry and should not be trusted.

### Appendix C: Wess-Zumino term for S5-branes

Since solitonic 5-branes couple electrically to the dual 6-form \( \tilde{B} \), let us start writing an action equivalent to (A1), but adapted to \( \tilde{B} \). In a differential form language, such action reads
\[ S = \frac{g_s^2}{16\pi G_N^{(10)}} \int \left\{ e^{-2\phi} \left[ \star R - 4d\phi \wedge \star d\phi + \frac{\alpha'}{4} R_{(-)}^{a} \wedge \star R_{(-)}^{b} \right] + \frac{1}{2} e^{2\phi} \tilde{H} \wedge \star \tilde{H} + \frac{\alpha'}{4} \tilde{B} \wedge R_{(-)}^{a} \wedge R_{(-)}^{b} \right\}, \] (C1)
where \( \star e^{-2\phi} H = \tilde{H} \equiv d\tilde{B} \).

In order to account for the coupling of solitonic five branes to the 6-form \( \tilde{B} \), we must add to the action (C1) a Wess-Zumino term of the form
\[ S_{WZ} = g_s^2 T_{S5} N \int \tilde{B}, \] where \( T_{S5} = \frac{1}{(2\pi \ell_s)^5 \ell_s g_s^2} \), (C2)
\[ d\left( \star e^{2\phi} \tilde{H} \right) - \frac{\alpha'}{4} R_{(-)}^{a} \wedge R_{(-)}^{b} = 4\pi^2 \alpha' N \star_{(4)} \delta^{(4)}(r), \] (C3)
with \( \int_{M_4} \star_{(4)} \delta^{(4)}(r) = 1 \), where \( M_4 \) is the four-dimensional space transverse to the S5-branes, which in the case at hands corresponds to the Gibbons-Hawking space (3). Integrating this equation we obtain expression (11).

Then, in order to obtain a relation between \( q_0 \) and the number of solitonic 5 branes, let us integrate (11). The first term gives
\[
\int_{\mathring{S}^1_{\infty} \times \mathring{S}^2_{\infty}} H = \int_{\mathring{S}^1_{\infty} \times \mathring{S}^2_{\infty}} \ast_4 dZ_0 = 8\pi^2 R q_0 - \frac{8\pi^2 \alpha'}{W},
\]  
(C4)

where \(\mathring{S}^1_{\infty} \times \mathring{S}^2_{\infty}\) is the boundary of the four-dimensional Gibbons-Hawking space. The computation of the second term gives the following contribution

\[
\int_{\mathcal{M}_4} R_{(-)}^a \wedge R_{(-)}^b = -\int_{\mathcal{M}_4} d \ast_4 dF(r; q_v) - \int_{\mathcal{M}_4} d \ast_4 dF(r; q_0),
\]  
(C5)

where \(F(r; k)\) is the function defined in (9). The value of each term above is actually the same, since the integral is independent of the parameter \(k\), i.e.

\[
\int_{\mathcal{M}_4} d \ast_4 dF(r; k) = \frac{8\pi^2 R z}{q_v} = \frac{16\pi^2}{W}.
\]  
(C6)

Then, we obtain

\[
\int_{\mathcal{M}_4} R_{(-)}^a \wedge R_{(-)}^b = -\frac{32\pi^2}{W},
\]  
(C7)

and, putting everything together, we get

\[
q_0 = \frac{\alpha' N}{2R^2}.
\]  
(C8)

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