Spherical Trigonometry of the Projected Baseline Angle

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(Dated: October 29, 2018)

PACS numbers: 95.10.Jk, 95.75.Kk

Keywords: projected baseline; parallactic angle; position angle; celestial sphere; spherical astronomy; stellar interferometry

I. SCOPE

The paper describes a standard to define the plane that contains the baseline of a stellar interferometer and a direction to a star. Two derived vectors are the delay vector, and the projected baseline vector in the plane of the wavefronts of the stellar light. The manuscript deals with the trigonometry of projecting the baseline further outwards onto the celestial sphere. The position angle of the projected baseline is defined, measured in a plane tangential to the celestial sphere, tangent point at the position of the star. This angle represents two orthogonal directions on the sky, differential star positions which are aligned with or orthogonal to the gradient of the delay recorded in the u − v plane. The North Celestial Pole is chosen as the reference direction of the projected baseline angle, adapted to the common definition of the “parallactic” angle.

II. SITE GEOMETRY

A. Telescope Positions

1. Spherical Approximation

In a geocentric coordinate system, the Cartesian coordinates of the two telescopes of a stellar interferometer are related to the geographic longitudes \( \lambda_i \), latitudes \( \phi_i \) and effective Earth radius \( \rho \) (sum of the Earth radius and altitude above sea level),

\[
T_i \equiv \rho \begin{pmatrix}
\cos \lambda_i \cos \phi_i \\
\sin \lambda_i \cos \phi_i \\
\sin \phi_i
\end{pmatrix}, \quad i = 1, 2. \tag{1}
\]

We add a “g” to the geocentric coordinates to set them apart from other Cartesian coordinates that will be used further down. A great circle of radius \( \rho \approx 6380 \) km, centered at the Earth center, connects \( T_1 \) and \( T_2 \): a vector perpendicular to this circle is \( \hat{J} \equiv \frac{1}{\rho \sin Z} T_1 \times T_2 \), where a baseline aperture angle \( Z \) has been defined under which the baseline vector

\[
b = T_2 - T_1 \tag{2}
\]

is seen from the Earth center:

\[
T_1 \cdot T_2 = |T_1| |T_2| \cos Z; \tag{3}
\]

\[
\cos Z = \cos \phi_1 \cos \phi_2 \cos (\lambda_2 - \lambda_1) + \sin \phi_1 \sin \phi_2. \tag{4}
\]

\( \hat{J} \) is the axis of rotation when \( T_1 \) is moved toward \( T_2 \). The nautical direction from \( T_1 \) to \( T_2 \) is computed as follows: define a tangent plane to the Earth at \( T_1 \). Two orthonormal vectors that span this plane are

\[
\hat{N}_1 \equiv \begin{pmatrix}
-\cos \lambda_1 \sin \phi_1 \\
-\sin \lambda_1 \sin \phi_1 \\
\cos \phi_1
\end{pmatrix} \sim \theta T_1 / \partial \phi_1; \quad |\hat{N}_1| = 1. \tag{5}
\]
to the North and
to the East. The unit vector from $\mathbf{T}_1$ to $\mathbf{T}_2$ along the great circle is $\mathbf{J} \times \frac{1}{\rho} \mathbf{T}_1$. The compass rose angle $\tau$ is computed by its decomposition $\mathbf{J} \times \frac{1}{\rho} \mathbf{T}_1 = \cos \tau \mathbf{N}_1 + \sin \tau \mathbf{E}_1$ within the tangent plane. It becomes zero if $\mathbf{T}_2$ is North of $\mathbf{T}_1$, and becomes $\pi/2$ if $\mathbf{T}_2$ is East of $\mathbf{T}_1$: Fig. 1.

(Recall that the tables in [5] use a different definition.) Straightforward algebra establishes $\tau$ from

$$\cos \tau = \frac{\cos \rho \sin \phi - \sin \phi_1 \cos \phi_2 \cos (\lambda_1 - \lambda_2)}{\sin Z},$$

$$\sin \tau = \frac{\cos \phi_2 \sin (\lambda_2 - \lambda_1)}{\sin Z}.$$  \hfill (7) \hfill (8)

2. Caveats

The angle $\tau$ of the previous section does not transform exactly into $\tau \pm \pi$ if the roles of the two telescopes are swapped, because the great circle, which has been used to define the direction, is not a loxodrome. This asymmetry within the definition indicates that a more generic framework to represent the geometry is useful.

If we consider (i) geodetic rather than geocentric representations of geographic latitudes or (ii) long baseline interferometers build into rugged landscapes, the 5 parameters above ($\rho, \lambda_i, \phi_i$) are too constrained to handle the 6 degrees of freedom of two earth-fixed telescope positions. Nevertheless, a telescope array “platform” is helpful to define a zenith, a horizon, and associated coordinates like the zenith distance or the star azimuth. This leads to the OIFITS concept, an array center $\mathbf{C}$, for example

$$\mathbf{C} = \begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix}.$$  \hfill (9)

plus local telescope coordinates

$$\mathbf{T}_i \equiv \mathbf{C} + \mathbf{t}_i, \quad i = 1, 2.$$  \hfill (10)

B. Sky coordinates

Geodetic longitude $\lambda$, geodetic latitude $\phi$ and altitude $H$ above the geoid are defined with the array center $\mathbf{C}$$2^1$$2^4$$2^5$, $\mathbf{C}$

$$\begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix} = \begin{pmatrix} (N + H) \cos \phi \cos \lambda \\ (N + H) \cos \phi \sin \lambda \\ N(1 - e^2) + H \sin \phi \end{pmatrix},$$  \hfill (11)

where $e$ is the eccentricity of the Earth ellipsoid, and $N \equiv \rho_e / \sqrt{1 - e^2} \sin^2 \phi$ the distance from the array center to the Earth axis measured along the local vertical.

In a plane tangential to the geoid at $\mathbf{C}$ we define a star azimuth $A$, a zenith angle $z$, and a star elevation $a = \pi/2 - z$,

$$\mathbf{s} = \begin{pmatrix} -\cos A \sin z \\ \sin A \sin z \\ \cos z \end{pmatrix} = \begin{pmatrix} -\cos a \cos \phi \\ \sin a \cos \phi \\ \sin a \sin \phi \end{pmatrix}.$$  \hfill (12)

We label this coordinate system $t$ as “topocentric,” with the first component North, the second component West and the third component up. The value of $A$ used in this script picks one of the (countably many) conventions: South means $A = 0$ and West means $A = +\pi/2$. This conventional definition of the horizontal means the star coordinates can be transformed to the equatorial parameters of hour angle $h = \ell - \alpha$, declination $\delta$, and right ascension $\alpha$ $10, (5.45) \ll 0, (2.13)$,

$$\cos a \sin A = \cos \delta \sin h;$$  \hfill (13)
$$\cos a \cos A = -\sin \delta \cos \phi + \cos \delta \cos h \sin \phi;$$  \hfill (14)
$$\sin a = \sin \delta \sin \phi + \cos \delta \cos h \cos \phi;$$  \hfill (15)
$$\cos \delta \cos h = \sin a \cos \phi + \cos a \cos A \sin \phi;$$  \hfill (16)
$$\sin \delta = \sin a \sin \phi - \cos a \cos A \cos \phi.$$  \hfill (17)

These convert (12) into

$$\mathbf{s} = \begin{pmatrix} \cos \phi \sin \delta - \sin \phi \cos \delta \cos h \\ \cos \delta \sin h \\ \sin \phi \sin \delta + \cos \phi \cos \delta \cos h \end{pmatrix}.$$  \hfill (18)

A star at hour angle $h = 0$ is on the Meridian,

$$\mathbf{s} = \begin{pmatrix} \sin(\delta - \phi) \\ 0 \\ \cos(\delta - \phi) \end{pmatrix}, \quad (h = 0).$$  \hfill (19)
north of the zenith at \( A = \pi \) if \( \delta - \phi > 0 \), south at \( A = 0 \) if \( \delta - \phi < 0 \). A reference point on the Celestial Sphere is the North Celestial Pole (NCP), given by insertion of \( \delta = \pi/2 \) into (18).

\[
\mathbf{s}_+ = \begin{pmatrix} \cos \phi \\ 0 \\ \sin \phi \end{pmatrix}. \tag{20}
\]

The angular distance between \( \mathbf{s} \) and \( \mathbf{s}_+ \) is

\[
\mathbf{s} \cdot \mathbf{s}_+ = \sin \delta. \tag{21}
\]

C. Baseline: Generic Position

The baseline vector coordinates of the geocentric OIFITS system

\[
\mathbf{b} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_g = \mathbf{T}_2 - \mathbf{T}_1 \tag{22}
\]

(defined to stretch from \( \mathbf{T}_1 \) to \( \mathbf{T}_2 \)—the opposite sign is also in use [17]) could be converted with

\[
\mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}_t = \begin{pmatrix} t_{2x} \\ t_{2y} \\ t_{2z} \end{pmatrix}_t - \begin{pmatrix} t_{1x} \\ t_{1y} \\ t_{1z} \end{pmatrix}_t = \mathbf{U}_{tg}(\phi, \lambda) \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_g \tag{23}
\]

to the topocentric system via the rotation matrix

\[
\mathbf{U}_{tg}(\phi, \lambda) = \begin{pmatrix} -\sin \phi \cos \lambda & -\sin \phi \sin \lambda & \cos \phi \\ \sin \lambda & -\cos \lambda & 0 \\ \cos \phi \cos \lambda & \cos \phi \sin \lambda & \sin \phi \end{pmatrix}. \tag{24}
\]

Just like the star direction (12), the unit vector \( \hat{\mathbf{b}} \) along the baseline direction defines a baseline azimuth \( A_b \) and a baseline elevation \( a_b \)

\[
\hat{\mathbf{b}} = \frac{\mathbf{b}}{b} = \begin{pmatrix} -\cos A_b \cos a_b \\ \sin A_b \cos a_b \\ \sin a_b \end{pmatrix}_t. \tag{25}
\]

The standard definition of the delay vector \( \mathbf{D} \) and projected baseline vector \( \mathbf{P} \) is

\[
\mathbf{b} = \mathbf{D} + \mathbf{P}, \quad \mathbf{D} \parallel \mathbf{s}; \quad \mathbf{P} \perp \mathbf{D}. \tag{26}
\]

\( \mathbf{P} \) and later its circle segment projected on the Celestial Sphere—inherits its direction from \( \mathbf{b} \) such that heads and tails of the vectors are associated with the same portion of the wavefront: Fig. 2. We use the term “projected baseline” both ways: for the straight vector (of length \( P \), units of meter) that connects the tails of \( \mathbf{D} \) and \( \mathbf{b} \), or the line segment (length \( \theta \), units of radian) on the Celestial Sphere.

The dot product of (12) by (25) is

\[
D = \mathbf{s} \cdot \mathbf{b} = b \cos \theta, \quad (0 \leq \theta \leq \pi), \tag{27}
\]

where the angular distance \( \theta \) between the baseline and star directions is introduced as

\[
\cos \theta = \cos a_b \cos a \cos(A_b - A) + \sin a_b \sin a. \tag{28}
\]

The star circles around the NCP in 24 hours, which changes the distance to the baseline in a centric periodic way (App. B).

III. PROJECTED BASELINE ANGLES

A. Definition

We define position angles of points on the Celestial Sphere relative to a star’s position as the bearing angle by which the NCP must be rotated around the star direction \( \mathbf{s} \) (the axis) until the NCP and the particular point of the object are aligned to the same direction (along a celestial circle) from the star. The sign convention is left-handed placing the head of \( \mathbf{s} \) at the center of the Celestial Sphere. This is equivalent to drawing a line between the star and the NCP, looking at it from inside the sphere, and defining position angles of points as the angles in polar coordinates in the mathematical sign convention, using this line as the abscissa and the star as the center. Another, fully equivalent definition uses a right-handed turn of the object around the star until it is North of the star. And finally, as an aid to memory, it is also the nautical course of a ship sailing the outer hull of the Celestial Sphere, which is currently poised at the star’s coordinates and is navigated toward the point.

This defines values modulo \( 2\pi \); whether these are finally represented as numbers in the interval \([0, 2\pi]\) or in the interval \((-\pi, +\pi]\)—as the \texttt{SLA Bear} and the \texttt{SLA PA} routines of the SLA library [23] do—is largely a matter of taste.

An overview on the relevant geometry is given in Figure 3 which looks at the celestial sphere from the outside. In Figure 3 the baseline has been infinitely extended straight outwards in both directions which defines telescope coordinates \( \mathbf{T}_1 \) and \( \mathbf{T}_2 \) also where the baseline penetrates the Celestial Sphere. Standing at the mid-point of the baseline, telescope 1 then is at azimuth \( \tau \), telescope 2 at azimuth \( A_b = \tau + \pi \), see Fig. 1. Let the object be at azimuth \( A \) and elevation \( a \).

The projection of the baseline occurs in a plane including the object and the baseline and thus defining the
great circle labeled \( \theta \) in Figure 3. If \( \theta \) is the angle on the sky between the object and the point on the horizon with azimuth \( A_b \), then the length \( P \) of the projected baseline is given by

\[
P = b \sin \theta,
\]  

(29)

where \( \theta \) is calculated from the relation \((28)\). With this auxiliary quantity, there are various ways of obtaining the position angle \( p_b \) of the projected baseline on the sky, some of which are detailed in Sections III B–III D. The common theme is that the baseline orientation \((A_b, a_b)\) in the local horizontal polar coordinate system is transferred to a rotated polar coordinate system with the star defining the new polar axis; the position angle is the difference of azimuths between \( T_2 \) and the NCP in this rotated polar coordinate system.

### B. In Horizontal Coordinates

We use a vector algebraic method to span a plane tangential to the celestial sphere; tangent point is the position \( s \) of the star. Within this orthographic zenithal projection we consider the two directions from the origin toward the NCP, \( s_+ \), on one hand and toward \( T_2 \), \( b \), on the other. The tangential plane is fixed by any two unit vectors perpendicular to the vector \( s \) of \((18)\). Rather arbitrarily they are chosen along \( \partial s/\partial A \) and \( \partial s/\partial a \), explicitly

\[
e_A = \begin{pmatrix} \sin A \\ \cos A \\ 0 \end{pmatrix}_t,
\]  

(30)

\[
e_a = \begin{pmatrix} \cos A \sin a \\ -\sin A \sin a \\ \cos a \end{pmatrix}_t.
\]  

(31)

(The equivalent exercise with a different choice of axes follows in Section III D.) These are orthonormal,

\[
s \cdot e_A = s \cdot e_a = e_A \cdot e_a = 0; \quad |e_A| = |e_a| = 1; \quad s \times e_a = e_A.
\]  

(32)

Decomposition of the direction from \( s \) to the NCP defines three projection coefficients \( c_{0,1,2} \),

\[
s_+ = c_0 s + c_1 e_A + c_2 e_a.
\]  

(33)

Building the square on both sides yields the familiar formula for the sum of squares of the projected cosines,

\[
c_0^2 + c_1^2 + c_2^2 = 1.
\]  

(34)

Two dot products of \((33)\) using \((20)\) and \((30)–(31)\) solve for two coefficients,

\[
c_1 = s_+ \cdot e_A = \cos \phi \sin A,
\]  

(35)

\[
c_2 = s_+ \cdot e_a = \cos \phi \cos A \sin a + \sin \phi \cos a.
\]  

(36)

![FIG. 3: Celestial sphere, seen from the outside. The north direction through the object is given by the great circle passing through the celestial poles and the object. The angle \( \theta \) is the vector \( P = b - D \) projected on the sphere.](image)

![FIG. 4: In the tangent plane spanned by the unit vectors \( e_A \) and \( s_+ \), the components of \( b \) and \( s_+ \) are \( c_1, c_1', c_2 \) and \( c_2' \), defined by \((30)\) and \((31)\). The view is from the outside onto the plane, so \( e_A \), the axis vector into the direction of increasing \( A \), points to the left.](image)
From (31), and since (21) equals $c_0$,
\[
\sqrt{c_1^2 + c_2^2} = \cos \delta. \tag{39}
\]

The equivalent splitting of $\hat{b}$ into components within and perpendicular to the tangent plane is:
\[
\hat{b} = c'_0 \mathbf{s} + c'_1 \mathbf{e}_A + c'_2 \mathbf{e}_a. \tag{40}
\]

Building the square on both sides yields
\[
c'_0^2 + c'_1^2 + c'_2^2 = 1. \tag{41}
\]

Dot products of (40) solve for the expansion coefficients, with (25) and (30)–(31):
\[
c'_1 = \hat{b} \cdot \mathbf{e}_A = -\cos A_b \cos a_b \sin A + \sin A_b \cos a_b \cos A = \cos a_b \sin(A_b - A), \tag{42}
\]
\[
c'_2 = \hat{b} \cdot \mathbf{e}_a = -\cos a_b \sin a \cos(A_b - A) + \sin a_b \cos a. \tag{43}
\]

This defines an angle $\varphi'$ in the polar coordinates of the tangent plane,
\[
\cos \varphi' = \frac{c'_1}{\sqrt{c'_1^2 + c'_2^2}}, \tag{44}
\]
\[
\sin \varphi' = \frac{c'_2}{\sqrt{c'_1^2 + c'_2^2}}. \tag{45}
\]

Since $c'_0$ equals $\hat{b} \cdot \mathbf{s} = \cos \theta$ in (28),
\[
\sqrt{c'_1^2 + c'_2^2} = \sin \theta. \tag{46}
\]

The baseline position angle is the difference between the two angles, as to redefine the reference direction from $\mathbf{e}_A$ to the direction of the NCP (see Fig. 4):
\[
p_b = \varphi' - \varphi \pmod{2\pi}. \tag{47}
\]

Sine and cosine of this imply [11, 4.3.16, 4.3.17]
\[
\sin p_b = \sin \varphi' \cos \varphi - \cos \varphi' \sin \varphi = \frac{c_1 c'_2 - c'_1 c_2}{\cos \delta \sin \theta}, \tag{48}
\]
\[
\cos p_b = \cos \varphi' \cos \varphi + \sin \varphi' \sin \varphi = \frac{c_1 c'_1 + c_2 c'_2}{\cos \delta \sin \theta}. \tag{49}
\]

where
\[
c_1 c'_2 - c'_1 c_2 = \cos \phi(\sin A \sin a_b \cos a - \cos a_b \sin a \sin A_b) - \sin \phi \cos a \cos a_b \sin(A_b - A), \tag{50}
\]
and
\[
c_1 c'_1 + c_2 c'_2 = \cos \phi(\cos A \cos a \cos \theta - \cos a_b \cos A_b) + c'_2 \sin \phi \cos a. \tag{51}
\]

The computational strategy is to build
\[
\tan p_b = \frac{c_1 c'_2 - c'_1 c_2}{c_1 c'_1 + c_2 c'_2}. \tag{52}
\]

The (positive) values of $\cos \delta$ and $\sin \theta$ in the denominators of (48) and (49) need not to be calculated. The branch ambiguity of the arctan is typically handled by use of the $\text{atan2}()$ functionality in the libraries of higher programming languages.

\section{C. Via Parallactic Angle}

If the position angle of the zenith $p$ is known by any other sources—see App. [A]—a quicker approach to the calculation of $p_b$ employs an auxiliary angle $\psi$ [11],
\[
p_b \equiv p + \pi + \psi \pmod{2\pi}. \tag{53}
\]

The calculation of $\psi$ is delegated to the calculation of its sines and cosines
\[
\cos \psi = -\cos p_b \cos p - \sin p_b \sin p; \tag{54}
\]
\[
\sin \psi = -\sin p_b \cos p + \cos p_b \sin p. \tag{55}
\]

In the right hand sides we insert (48), (49) and (51)–(52), and after some standard manipulations
\[
\sin \psi = \frac{\cos a_b \sin(A_b - A)}{\sin \theta}; \tag{56}
\]
\[
\cos \psi = \frac{\sin a \cos \theta - \sin a_b}{\sin \theta \cos a} = \frac{\cos a_b \sin a \cos(A_b - A) - \sin a_b \cos a}{\sin \theta}. \tag{57}
\]

These are $c'_1$ and $-c'_2$ of (42) and (43) divided by $\sin \theta$. In Fig. 4 $\psi$ can therefore be identified as the angle between $-\mathbf{e}_a$ and $\hat{b}$, and this is consistent with (53) which decomposes the rotation into the angle $p$, a rotation by the angle $\pi$ (which would join the end of $p$ with the start of $\psi$ in Fig. 4), and finally a rotation by $\psi$.

In summary, the disadvantage of this approach is that $p$ must be known by other means, and the advantage is that computation of $\psi$ via the arctan of (54) and (57) only needs $c'_1$ and $c'_2$, but not $c_1$ or $c_2$.

\section{D. In Equatorial Coordinates}

Section [11] provides $p_b$, given star coordinates $(A, a)$. Subsequently we derive the same value in terms of $\delta$ and $h$. The straightforward approach is the substitution of (13)–(17) in (50) and (51). A less exhaustive alternative works as follows [9]: the axes $\mathbf{e}_A$ and $\mathbf{e}_a$ in the tangential plane are replaced by two different orthonormal directions, better adapted to $\delta$ and $h$, namely the directions
III B: The axes are orthonormal.

The further calculation follows the scheme of Section III-B. The decomposition of \( s \) along \( \partial \) defines three expansion coefficients.

\[
\mathbf{e}_h = \begin{pmatrix} \sin \phi \sin h \\ \cos h \\ -\cos \phi \sin h \end{pmatrix},
\]

\[
\mathbf{e}_\delta = \begin{pmatrix} \cos \phi \cos \delta + \sin \phi \sin \delta \cos h \\ -\sin \delta \sin h \\ \sin \phi \cos \delta - \cos \phi \sin \delta \cos h \end{pmatrix}.
\]

The further calculation follows the scheme of Section III-B. The axes are orthonormal.

\[ \mathbf{s} \cdot \mathbf{e}_h = \mathbf{s} \cdot \mathbf{e}_\delta = \mathbf{e}_h \cdot \mathbf{e}_\delta = 0; \quad |\mathbf{e}_h| = |\mathbf{e}_\delta| = 1; \quad \mathbf{s} \times \mathbf{e}_\delta = \mathbf{e}_h. \]

The decomposition of \( s_+ \) in this system defines three expansion coefficients \( d_{0,1,2} \).

\[ s_+ = d_0 \mathbf{s} + d_1 \mathbf{e}_\delta + d_2 \mathbf{e}_h. \]

Multiplying this equation in turn with \( \mathbf{e}_h \) and \( \mathbf{e}_\delta \) yields

\[
d_2 = 0; \quad d_1 = s_+ \cdot \mathbf{e}_\delta = \cos \delta.
\]

The direction \( \hat{\mathbf{b}} \) to \( T_2 \) is projected into the same tangent plane, defining three expansion coefficients \( d'_{0,1,2} \).

\[ \hat{\mathbf{b}} = d'_0 \mathbf{s} + d'_1 \mathbf{e}_\delta + d'_2 \mathbf{e}_h. \]

FIG. 5: Fig. 4 after switching from the \(( \mathbf{e}_A, \mathbf{e}_a)\) axes to \(( \mathbf{e}_h, \mathbf{e}_\delta)\). Axis components of \( s_+ \) and \( \mathbf{b} \) are indicated.

Building the dot product of this equation with \( \mathbf{e}_h \) using

\[ d'_2 = -\cos A_b \cos a_b \sin \phi \sin h + \sin A_b \cos a_b \cos h - \sin a_b \cos \phi \sin h. \]

Building the dot product of (63) with \( \mathbf{e}_\delta \) yields

\[
d'_1 = -\cos A_b \cos a_b \cos \phi \cos \delta - \cos A_b \cos a_b \sin \phi \sin \delta \cos h - \sin A_b \cos a_b \sin \delta \sin h + \sin a_b \sin \phi \cos \delta - \sin a_b \cos \phi \sin \delta \cos h.
\]

Some simplification in this formula is obtained by trading the hour angle \( h \) for the angle \( \theta \), which might be more readily available since \( \theta \) is measured via the delay: In the second term we use (14) to substitute

\[ \sin \phi \cos h \rightarrow \frac{\cos \phi \sin \delta + \cos A \cos a}{\cos \delta}. \]

In the third term we use (13) to substitute

\[ \sin h \rightarrow \frac{\sin A \cos a}{\cos \delta}. \]

and in the last term we use (15) to substitute

\[ \cos \phi \cos h \rightarrow \frac{\sin a - \sin \delta \sin \phi}{\cos \delta}. \]

Further standard trigonometric identities \[1, 4.3.10, 4.3.17\] and \[28\] yield

\[
d'_1 = \frac{\sin a_b \sin \phi - \cos a_b \cos A_b \cos \phi - \sin \delta \cos \theta}{\cos \delta}. \]

In the numerator we recognize a baseline declination \( \delta_b \),

\[ s_+ \cdot \mathbf{b} \equiv \sin \delta_b = \sin a_b \sin \phi - \cos a_b \cos A_b \cos \phi. \]

\( p_b \) is the angle that rotates the projected vector \(( d_2, d_1) \) within the tangent plane into the direction \(( d'_2, d'_1) \) (Fig. 5).

\[
\cos p_b = \frac{d'_1}{\sin \theta} \quad (71)
\]

\[
\sin p_b = -\frac{d'_2}{\sin \theta} \quad (72)
\]

Again, the \( \sin \theta \) does not need actually to be calculated, but only

\[
\tan p_b = -\frac{d'_2}{d'_1}. \quad (73)
\]

and again, selection of the correct branch of the arctan is easy with \texttt{atan2()} functions if the negative sign is kept attached to \( d'_2 \).

Note also that the use of (69) is optional: \( d'_2 \) and \( d'_1 \) are completely defined in terms of the baseline direction \(( A_b, a_b) \), the geographic latitude \( \phi \) and the star coordinates \(( h, \delta) \) via (63) and (64). With these one can immediately proceed to (73).

OIFITS \[15, 40\] defines no associated angle in the plane perpendicular to the direction of the phase center.
The segment of the baseline in the tangent plane primarily defines an orientation in the \(u-v\) plane. However, modal decomposition of the amplitudes in products of radial and angular basis functions (akin to Zernike polynomials) means that the Fourier transform to the \(x-y\) plane preserves the angular basis functions (see the calculation for the 3D case, Spherical Harmonics, in \([13]\)). In this sense, the angle \(p_b\) is also "applicable" in the \(x-y\) plane.

**IV. DIFFERENTIALS IN THE FIELD-OF-VIEW**

**A. Sign Convention**

The provisions of the previous section, namely

- the OIFITS sign convention of the baseline vector, Eq. (2) and Fig. 2;

- the conventional formula (27)

fix the sign of the delay \(D\) as follows: if the wavefront hits \(T_2\) prior to \(T_1\), the values of \(D\) and \(\cos \theta\) are positive; if it hits \(T_1\) prior to \(T_2\), both values are negative.

The transitional case \(D = 0 (\theta = \pi/2)\) occurs if the star passes through the "baseline meridian" plane perpendicular to the baseline; the two lines to the Northwest and Southeast in Fig. 1 show (projections of) these directions. The baseline in most optical stellar interferometers is approximately horizontal, \(a_b \approx 0\). From (27) we see that this case is approximately equivalent to \(\cos(A_b - A) \approx 0\). Since \(\cos a > 0\), the sign of \(D\) coincides with the sign of \(\cos(A_b - A)\). This is probably the fastest way to obtain the sign of the Optical Path Difference from FITS [6] header keywords; the advantage of this recipe is that the formula is independent of which azimuth convention is actually in use.

The baseline length \(b\) follows immediately from the Euclidean distance of the two telescopes entries for column STAXYZ of the OIFITS table 01ARRAY [10].

**B. External Path Delay**

The total differential of (28) relates a direction \((\Delta A, \Delta a)\) away from the star to a change in the angular distance between the star and the baseline,

\[
- \sin \theta \Delta \theta = \Delta a \left[ - \cos a_b \sin a \cos (A_b - A) + \sin a_b \cos a \right] + \Delta A \cos a \cos a_b \sin (A_b - A). \tag{74}
\]

Equating the right hand side of the previous equation with zero, this means

\[
\frac{\Delta a}{\cos a \Delta A} = \tan \psi \quad (\Delta \theta = \Delta D = 0). \tag{76}
\]

For the direction of maximum change in \(\Delta D\), which runs perpendicular to (76),

\[
\frac{\Delta a}{\cos a \Delta A} = -\frac{1}{\tan \psi} \quad (p_b : \max \Delta D) \tag{77}
\]

which leads to

\[
\Delta D = -b \sin \theta \Delta \theta = b \sin \theta \frac{\cos a \Delta A}{\sin \psi} = -b \sin \theta \frac{\Delta a}{\cos \psi}. \quad (\max \Delta D) \tag{78}
\]

Example: a field of view of \(\Delta \theta = \pm 1''\) at a baseline of \(b = 100\) m at a "random" position \(\theta = 45^\circ\) scans \(\Delta D = \pm 340\) \(\mu\)m. The details of the optics determine how far this contributes to the instrumental visibility (loss).

Fig. 6 sketches this geometry: the change \(\Delta D\) when re-pointing from one star by \((\Delta A, \Delta a)\) to another star along some segment of the Celestial Sphere can be split into a change associated with the radial direction toward/away from \(T_2\) along a great circle, and a zero change moving along a circle of radius \(\cos \theta\) centered on the baseline azimuth.

![FIG. 6: Example of three concentric circles on the Celestial Sphere, centered at the baseline, which unite star directions of D = const. Two star positions are connected by the short diagonal dash. A quarter of each of the two projected baselines, which intersect at \((A_b, a_b)\) near the horizon, is also shown.](image)
This translates [3, App. C] to our variables. When \( \Delta \theta = 0 \), we have the directions of zero change in \( \Delta D \), perpendicular to the projected baseline, \[
\frac{\Delta \delta}{\cos \delta \Delta h} = \tan p_b, \quad (\Delta \theta = \Delta D = 0). \tag{83}
\]
The direction of maximum change is \[
\frac{\Delta \delta}{\cos \delta \Delta h} = -\frac{1}{\tan p_b}, \quad (p_b : \text{max } \Delta D), \tag{84}
\]
and along this gradient \[
\Delta D = -b \sin \theta \frac{\cos \delta \Delta h}{\sin p_b} = b \sin \theta \frac{\Delta \delta}{\cos p_b}, \quad (\text{max } \Delta D). \tag{85}
\]

V. SUMMARY

To standardize the nomenclature, we propose to choose the North Celestial Pole as the “reference” direction (direction of position angle zero) and a handedness to define the sign of the position angles (mathematically positive if looking at the Celestial Sphere from the inside).

The set of position angles discussed here includes

- the direction of the zenith, the “parallactic” angle,
- the direction toward the point where the baseline pinpoints the Celestial Sphere, the “projected baseline angle,”
- position angles of “secondary” stars in differential astrometry.

Computation of the angle proceeds via \([22]\) if the object coordinates are given in the local altitude-azimuth system, or via \([75]\) if they are given in the equatorial system.

The mathematics involved is applicable to observatories at the Northern and the Southern hemisphere: positions \((A, a)\) or \((\delta, h)\) are mapped onto a 2D zenithal coordinate system. The position angles play the role of the longitude (the North Celestial Pole the role of Greenwich or Aries). A distance between the object and the star (in the range from 0 to \(\pi\) along great circles through the Celestial Pole in an ARC projection) might take the role of the polar distance—although an orthographic SIN projection had been used for the calculations in this script.

APPENDIX A: PARALLACTIC ANGLE

We define the parallactic angle \( p \) as the position angle of the zenith. Since the formulas in Section [11] dealt with the general case of coordinates on the Celestial Sphere, one may just set \( a_b = \pi/2 \) in \([25]\), which induces \( c'_3 = 0 \), \( c'_2 = \cos a \). This also turns \([28]\) into \( \cos \theta = \sin a \). \([18, 19]\) become

\[
\sin p = \frac{\cos \phi \sin A}{\cos \delta}, \tag{A1}
\]
\[
\cos p = \frac{\cos \phi \cos A \sin a + \sin \phi \cos a}{\cos \delta}. \tag{A2}
\]

The ratio of these two equations is

\[
\tan p = \frac{\cos \phi \sin A}{\cos \phi \cos A \sin a + \sin \phi \cos a}. \tag{A3}
\]

This expression can be transformed from the horizontal to equatorial coordinates as follows: Multiply numerator and denominator by \( \cos a \) and use \([13]\) in the numerator,

\[
\tan p = \frac{\cos \phi \cos \delta \sin h}{\cos \phi \cos A \sin a \cos a + \sin \phi \cos^2 a}. \tag{A4}
\]

Insert \([17]\) in the denominator,

\[
\tan p = \frac{\cos \phi \cos \delta \sin h}{\sin a (\sin a \sin \phi - \sin \delta) + \sin \phi \cos^2 a}
= \frac{\cos \phi \cos \delta \sin h}{\sin \phi - \sin a \sin \delta}. \tag{A5}
\]

Replace \( \sin a \) with \([15]\) in the denominator, eventually divide numerator and denominator through \( \cos \phi \cos \delta \):

\[
\tan p = \frac{\cos \phi \cos \delta \sin h}{\sin \phi - (\sin \delta \sin \phi + \cos \delta \cos h \cos \phi) \sin \delta}
= \frac{\cos \phi \cos \delta \sin h}{\cos^2 \delta \sin \phi - \cos \delta \sin \delta \cos h \cos \phi}
= \frac{\sin h}{\cos \delta \tan \phi - \sin \delta \cos \phi}. \tag{A6}
\]

These manipulations involved only multiplications with positive factors like \( \cos \phi \), \( \cos \delta \) and \( \cos a \); therefore one can use \( \sin h \) and \( \cos \delta \tan \phi - \sin \delta \cos h \) separately as the two arguments of the \( \tan \) function to resolve the branch ambiguity of the inverse tangent.

This definition of the parallactic angle coincides with the dominant one in the literature \([2, 18, 19, 20, 22]\); others exist: Equation (1) in \([4]\) generates the 90°-complement of our definition, for example.

If we insert \([57-59]\) into \([75, 4.3.18]\) and compare with \([22]\), we find

\[
\frac{\Delta \delta}{\cos \delta \Delta h} = \frac{\Delta a}{\cos a \Delta A} + \tan p \frac{\sin a}{1 - \tan p \cos a \Delta A}, \tag{A7}
\]

which means the parallactic angle mediates between directions expressed as axis ratios in the two coordinate systems \([12]\).

APPENDIX B: DIURNAL MOTION

The diurnal motion of the external path difference is described by separating the terms \( \alpha \sin h \) and \( \alpha \cos h \).
We rewrite (27) in equatorial coordinates by multiplying (15) with (25), and collect terms with the aid of (70):

\[
D = s \cdot b = b[s \sin \delta \sin \delta_b + \cos \delta \cos \delta_b \cos(h - h_b)]. \quad (B1)
\]

A time-independent offset \(b \sin \delta \sin \delta_b\) and an amplitude \(b \cos \delta \cos \delta_b\), defined by products of the polar and equatorial components of star and baseline, respectively, constitute the regular motion of the delay. The phase lag \(h_b\) is determined by the arctan of

\[
\begin{align}
\cos \delta_b \sin h_b &= \sin A_b \cos a_b; \\
\cos \delta_b \cos h_b &= \cos A_b \cos a_b \sin \phi + \sin a_b \cos \phi. \quad (B3)
\end{align}
\]

and plays the role of an interferometric hour angle.

**APPENDIX C: CLOSURE RELATIONS**

An array \(T_1, T_2\) and \(T_3\) of telescopes forms a baseline triangle

\[
\mathbf{b}_{12} + \mathbf{b}_{23} + \mathbf{b}_{31} = 0. \quad (C1)
\]

If all point to a common direction \(s\), some phase closure sum rules result:

\[
\begin{align}
D_{12} + D_{23} + D_{31} &= 0; \\
D_{12} + D_{23} + D_{31} &= 0; \\
b_{12} \cos \theta_{12} + b_{23} \cos \theta_{23} + b_{31} \cos \theta_{31} &= 0. 
\end{align}
\]

The triangle of projected baselines in the tangent plane is

\[
\mathbf{P}_{12} + \mathbf{P}_{23} + \mathbf{P}_{31} = 0. \quad (C5)
\]

These vectors can be represented as numbers in a complex plane by introduction of three moduli \(P\) and three orientation angles \(p_b\):

\[
P_{12} e^{i \theta_{12}} + P_{23} e^{i \theta_{23}} + P_{31} e^{i \theta_{31}} = b_{12} \sin \theta_{12} e^{i \theta_{12}} + b_{23} \sin \theta_{23} e^{i \theta_{23}} + b_{31} \sin \theta_{31} e^{i \theta_{31}} = 0. \quad (C6)
\]

These equations remain correct if transformed by complex conjugation or multiplied by a complex phase factor, therefore these closure relations of the projected baselines angles are correct for (i) both senses of defining their orientation, and (ii) any reference direction of the zero angle.

**APPENDIX D: NOTATIONS**

| Notation | Description |
|----------|-------------|
| \(\alpha\) | right ascension |
| \(a, \Delta a\) | star elevation and its difference |
| \(A, \Delta A\) | star azimuth angle, and its difference |
| \(A_b\) | baseline azimuth |
| \(b, b\) | baseline vector and length |
| \(\delta\) | declination |
| \(D\) | signed external path difference |
| \(e_h, e_\delta, e_A, e_o\) | unit vector in tangent plane |
| \(e\) | eccentricity of geoid |
| \(\phi\) | geographic or geodetic latitude |
| \(h\) | hour angle |
| \(H\) | altitude above geoid |
| \(l\) | local sidereal time |
| \(\lambda\) | geographic longitude |
| \(N\) | distance to the Earth axis measured along local normal |
| \(p\) | position angle of the zenith, “parallactic angle” |
| \(p_b\) | position angle of the projected baseline |
| \(P\) | projected baseline length |
| \(\rho\) | Earth radius |
| \(\rho_e\) | Earth equatorial radius |
| \(s\) | unit vector into star direction |
| \(s_A\) | unit vector into NCP direction |
| \(\theta\) | angular separation between star and baseline vector |
| \(\psi\) | angle between projected baseline circle and star meridian |
| \(\psi\) | zenith angle |
| \(Z\) | baseline angle seen from the Earth center |

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