Ext Groups between Irreducible $\text{GL}_n(q)$-modules in Cross Characteristic

Veronica Shalotenko
vvs9cc@virginia.edu
Department of Mathematics, University of Virginia, Charlottesville, VA 22903

Abstract

Let $G = \text{GL}_n(q)$ be the general linear group over the finite field $\mathbb{F}_q$ of $q$ elements, and let $k$ be an algebraically closed field of characteristic $r > 0$ such that $r \nmid q(q-1)$. In 1999, Cline, Parshall, and Scott showed that under these assumptions, cohomology calculations for $G$ may be translated to $\text{Ext}^i$ calculations over a $q$-Schur algebra. The aim of this paper is to extend the results of Cline, Parshall, and Scott and show that $\text{Ext}^i$ calculations for $\text{GL}_n(q)$ may also be translated to $\text{Ext}^i$ calculations over an appropriate $q$-Schur algebra (both for $i = 1$ and $i > 1$). To that end, we establish formulas relating certain $\text{Ext}$ groups for $\text{GL}_n(q)$ to $\text{Ext}$ groups for the $q$-Schur algebra $S_q(n, n)_k$. As a consequence, we show that there are no non-split self-extensions of irreducible $kG$-modules belonging to the unipotent principal Harish-Chandra series. As an application in higher degree, we describe a method which yields vanishing results for higher $\text{Ext}$ groups between irreducible $kG$-modules and demonstrate this method in a series of examples.

1 Introduction

Let $q$ be a power of a prime $p$, let $G = \text{GL}_n(q)$ be the general linear group over the finite field $\mathbb{F}_q$ of $q$ elements, and let $k$ be an algebraically closed field of characteristic $r > 0$ such that $r \nmid q(q-1)$. We will work with right $kG$-modules, and all $kG$-modules will be assumed to be finite-dimensional over $k$. Let $B$ be the subgroup of $G$ consisting of invertible upper triangular $n \times n$ matrices; let $U$ be the subgroup of $B$ consisting of invertible upper triangular matrices with 1’s along the main diagonal, and let $T$ be the subgroup of $B$ consisting of invertible diagonal matrices. In this case, $|U| = q^{\frac{n^2-n}{2}}$, $|T| = (q-1)^n$, and $B = U \rtimes T$. The assumption $r \nmid q(q-1)$ ensures that $r \nmid |B|$, so that $k|_B^G$ is a projective $kG$-module.

In this paper, we study the relationship between $\text{Ext}$ groups for $G$ and $\text{Ext}$ groups for the $q$-Schur algebra $S_q(n, n)_k$ in cross-characteristic ($S_q(n, n)_k$ is defined in Section 2.2). Since the category $\text{mod-}S_q(n, n)_k$ of right $S_q(n, n)_k$-modules is a highest weight category, many $\text{Ext}$ groups for $S_q(n, n)_k$ are known (see, for instance, [Do98 App. A2]). Thus, it is advantageous to translate $\text{Ext}$ calculations over $G$ to $\text{Ext}$ calculations over $S_q(n, n)_k$: certain $\text{Ext}$ groups
between $kG$-modules may be easier to compute in mod-$S_q(n,n)_k$.

To carry out the Ext calculations presented in this paper, we will require an indexing of the irreducible $kG$-modules. We will, for the most part, work with the parameterization of Cline, Parshall, and Scott [CPS99]. In the CPS indexing, the irreducible constituents of the permutation module $k|_{B^G}^G$ may be labeled by partitions of $\lambda$ of $n$; given a partition $\lambda \vdash n$, the corresponding irreducible $kG$-module $D(1,\lambda)$ occurs at least once as a composition factor of $k|_{B^G}^G$ [S98, Thm. 2.1]. (See Section 3.1 for a detailed description of the CPS indexing on the irreducible $kG$-modules.) In this paper, the indexing of the irreducible $kG$-modules in the unipotent principal Harish-Chandra series $\text{Irr}_k(G|B)$ will play a particularly important role. The unipotent principal series $\text{Irr}_k(G|B)$ consists of the irreducible $kG$-modules which can be found in the head (and, equivalently, the socle) of the permutation module $k|_{B^G}^G$. (We refer the reader to [GJ11, Sec. 4.2] for a discussion of Harish-Chandra series.) Let $|q \pmod{r}|$ denote the multiplicative order of $q$ modulo $r$, and define $l \in \mathbb{Z}^+$ by

$$l = \begin{cases} r & \text{if } |q \pmod{r}| = 1 \\ |q \pmod{r}| & \text{if } |q \pmod{r}| > 1. \end{cases}$$

In the CPS indexing, the irreducible $kG$-modules belonging to $\text{Irr}_k(G|B)$ correspond to $l$-restricted partitions $\lambda$ of $n$, and we have

$$\text{Irr}_k(G|B) = \{D(1,\lambda)|\lambda \vdash n, \lambda \text{ is } l\text{-restricted}\}.$$ 

In Section 4 of this paper, we show that for an $l$-restricted partition $\lambda \vdash n$ and an arbitrary partition $\mu \vdash n$,

$$\text{Ext}^1_{kG}(D(1,\lambda),D(1,\mu)) \cong \text{Ext}^1_{S_q(n,n)_k}(L^k(\lambda),L^k(\mu)),$$

where $S_q(n,n)_k$ is the $q$-Schur algebra of bidegree $(n,n)$ over $k$ and for any $\lambda \vdash n$, $L^k(\lambda)$ denotes the corresponding irreducible $S_q(n,n)_k$-module. In Section 5, we extend this result to show that certain higher Ext groups for $\text{GL}_n(q)$ in cross-characteristic are isomorphic to Ext groups over a $q$-Schur algebra.

This work was inspired by the cohomology computations of [CPS99], where the authors connect $H^i$ calculations for $G$ in cross characteristic to Ext$^i$ calculations over a $q$-Schur algebra. In degree one, Cline, Parshall, and Scott show that

$$H^1(G,D(1,\mu)) \cong \text{Ext}^1_{S_q(n,n)_k}(L^k(1^n),L^k(\mu))$$

for any partition $\mu \vdash n$ [CPS99, Thm. 10.1]. This result is extended to higher degree cohomology groups in [CPS99] Thm. 12.4] (under certain additional assumptions on the 

1Furthermore, Ext$^i$ calculations over a $q$-Schur algebra can be translated to Ext$^i$ calculations in the category of integrable modules for the quantum enveloping algebra $\tilde{U}_q(\mathfrak{gl}_n)$. So, the connection between Ext$^i$ calculations for $\text{GL}_n(q)$ and Ext$^i$ calculations for a $q$-Schur algebra opens the possibility of using the theory of quantum groups to learn more about the structure of Ext groups for $\text{GL}_n(q)$.

2A partition if $\lambda \vdash n$ is called $l$-restricted if its dual $\lambda'$ is $l$-regular, meaning that every part of $\lambda'$ occurs less than $l$ times.
characteristic of $k$). In this paper, we generalize [CPS99] Thm. 10.1 and [CPS99] Thm. 12.4 to obtain new Ext computations for $G$ in cross characteristic.

Our generalization of [CPS99] Thm. 10.1 (given in Theorem 4.1) follows readily once we observe that the trivial irreducible $kG$-module $k$ shares certain properties with all irreducible $kG$-modules belonging to $\text{Irr}_k(G|B)$. As a consequence, we prove that there are no non-split self-extensions of irreducible $kG$-modules belonging to the unipotent principal series. In other words, given any irreducible $kG$-module $Y$ belonging to the unipotent principal series $\text{Irr}_k(G|B)$, $\text{Ext}^1_{kG}(Y, Y) = 0$.\footnote{Guralnick and Tiep [GT11] found bounds on the dimension of $H^1$ for finite groups of Lie type in cross characteristic, which were extended to $\text{Ext}^1$ by the author [Sh20]. Using the methods of Guralnick and Tiep, it is only possible to show that $\dim \text{Ext}^1_{kG}(Y, Y) \leq |W| = n!$ (where $W = S_n$ is the Weyl group of $\text{GL}_n(q)$). Thus, the result of this paper significantly improves our understanding of $\text{Ext}^1$ groups for $\text{GL}_n(q)$ in cross characteristic.}

This generalization requires much more background machinery; the key is the existence of a resolution with certain desirable properties, which we construct in Lemma 5.7. In Section 6, we give examples to illustrate some applications of the theory developed in Section 5; in particular, this theory allows us to obtain higher Ext vanishing results for $\text{GL}_n(q)$. The main theorems of Sections 4 and 5 of this paper were originally proved in the author’s thesis [Sh18].

2 Preliminaries

2.1 The Hecke algebra

Let $\mathcal{Z} = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in an indeterminate $t$. Let $(W, S)$ be a finite Coxeter system with length function $l$, and let $\widetilde{H} = \widetilde{H}(W, \mathcal{Z})$ denote the associated generic Hecke algebra over $\mathcal{Z}$. The Hecke algebra $\widetilde{H}$ is free over $\mathcal{Z}$; it has basis $\{\tau_w\}_{w \in W}$, satisfying the relations

$$\tau_s \tau_w = \begin{cases} 
\tau_{sw} & \text{if } l(sw) > l(w) \\
t\tau_{sw} + (t-1)\tau_w & \text{otherwise}
\end{cases}$$

for $s \in S$, $w \in W$.

There is a $\mathcal{Z}$-involution $\Phi : \widetilde{H} \to \widetilde{H}$ with $\Phi(\tau_w) = (-t)^{l(w)}\tau_{w^{-1}}^{-1}$ for all $w \in W$ [PS05 (2.0.3)]. Given a right $\widetilde{H}$-module $\widetilde{M}$, let $\widetilde{M}^\Phi$ denote the right $\widetilde{H}$-module obtained by twisting the action of $\widetilde{H}$ on $\widetilde{M}$ by $\Phi$. The assignment $\widetilde{M} \mapsto \widetilde{M}^\Phi$ defines an exact functor $\Phi : \text{mod-}\widetilde{H} \to \text{mod-}\widetilde{H}$ (where $\text{mod-}\widetilde{H}$ denotes the category of right $\widetilde{H}$-modules).

There is also a $\mathcal{Z}$-anti-involution $\iota : \widetilde{H} \to \widetilde{H}$ given by $\iota(\tau_w) = \tau_{w^{-1}}$ for any $w \in W$ [PS05 (2.0.5)]. Given a right $\widetilde{H}$-module $\widetilde{M}$, we let $\widetilde{M}^\iota$ denote the left $\widetilde{H}$-module obtained by twisting the action of $\widetilde{H}$ on $\widetilde{M}$ by $\iota$. The anti-automorphism $\iota$ is used primarily as a means to convert the left action of $\widetilde{H}$ on the dual of a right $\widetilde{H}$-module to a right action. Let $(-)^* = \text{Hom}_\mathcal{Z}(-, \mathcal{Z})$ denote the linear dual on the category $\text{mod-}\widetilde{H}$ of right $\widetilde{H}$-modules.
Then, given a right \( \widetilde{H} \)-module \( \tilde{M} \in \text{mod-}\widetilde{H} \), the dual \( \tilde{M}^* \) of \( \tilde{M} \) is a left \( \widetilde{H} \)-module. Thus, twisting the action of \( \widetilde{H} \) on \( \tilde{M}^* \) by \( \iota \) yields a right \( \widetilde{H} \)-module. For any \( \tilde{M} \in \text{mod-}\widetilde{H} \), we will write \( \tilde{M}^{D_\iota} := \tilde{M}^{\ast *\iota} \). The functor \( D_\iota : \text{mod-}\widetilde{H} \to \text{mod-}\widetilde{H} \), \( \tilde{M} \mapsto \tilde{M}^{D_\iota} \) is a contravariant duality functor. Given any \( Z \)-free right \( \widetilde{H} \)-module \( \tilde{M} \), we have \( \tilde{M}^{D_\iota} \cong \tilde{M} \).

For a subset \( J \subseteq S \), let \( W_J = \langle s \rangle_{s \in J} \) be the parabolic subgroup of \( W \) corresponding to \( J \). We can similarly define a parabolic subalgebra \( \widetilde{H}_J \) of \( \widetilde{H} \) by setting \( \widetilde{H}_J = \langle \tau_s \rangle_{s \in J} \). Also, to any \( J \subseteq S \), we can associate elements \( x_J = \sum_{w \in W_J} \tau_w \in \widetilde{H} \) and \( y_J = \sum_{w \in W_J} (-t)^{\lambda(w)} \tau_w \in \widetilde{H} \).

(The right \( \widetilde{H} \)-modules \( x_J \widetilde{H} \) and \( y_J \widetilde{H} \) are induced from the parabolic subalgebra \( \widetilde{H}_J \) of \( \widetilde{H} \) [DPS2, Lem. 1.1].) We will use the elements \( x_J \) and \( y_J \) in Section 3 in order to describe a Morita equivalence of Cline, Parshall, and Scott [CPS99], which yields an indexing of the irreducible \( k\text{GL}_n(q) \)-modules.

### 2.2 The \( q \)-Schur algebra

Let \( Z = \mathbb{Z}[t, t^{-1}] \) and let \( (W, S) = (\mathfrak{S}_m, S) \) be the Coxeter system in which \( \mathfrak{S}_m \) is the symmetric group on \( m \) letters and \( S = \{(1, 2), (2, 3), \ldots, (m-1, m)\} \) is the set of fundamental reflections. Let \( \widetilde{H} = \widetilde{H}(\mathfrak{S}_m, Z) \) be the corresponding generic Hecke algebra.

Let \( V \) be a free \( Z \)-module of rank \( n > 0 \) and let \( \{v_1, \ldots, v_n\} \) be an ordered basis of \( V \). Given a sequence \( J = (j_1, \ldots, j_m) \) of integers with \( 1 \leq j_i \leq n \), let \( v_J = v_{j_1} \otimes \cdots \otimes v_{j_m} \). The elements \( v_J \) give a basis of \( V^\otimes m \). For \( \sigma \in \mathfrak{S}_m \), let \( J\sigma = (j_{\sigma^{-1}(1)}, \ldots, j_{\sigma^{-1}(m)}) \). Then, for any \( s \in S \), we can define an action of the generator \( \tau_s \) of \( \widetilde{H} \) on \( V^\otimes m \) by

\[
v_J \tau_s = \begin{cases} tv_{Js} & \text{if } j_i \leq j_{i+1} \\ v_{Js} + (t-1)v_J & \text{otherwise.} \end{cases}
\]

This action extends to give a right action of \( \widetilde{H} \) on \( V^\otimes m \), and the endomorphism algebra \( S_t(n, m) := \operatorname{End}_\widetilde{H}(V^\otimes m) \) is the \( t \)-Schur algebra of bidegree \((n, m)\) over \( Z \) (this is the definition given in [CPS99, (8.2)])).

There is another description of the tensor product \( V^\otimes m \), involving partitions of \( m \). Let \( \Lambda(n, m) \) denote the set of compositions of \( m \) with at most \( n \) non-zero parts, and let \( \Lambda^+(n, m) \) denote the set of partitions of \( m \) with at most \( n \) non-zero parts. By rearranging parts, every composition \( \lambda \) corresponds to a unique partition \( \lambda^+ \in \Lambda^+ \). As a right \( \widetilde{H} \)-module, \( V^\otimes m \cong \bigoplus_{\lambda \in \Lambda(n, m)} x_\lambda \widetilde{H} \). To every composition \( \lambda \in \Lambda(n, m) \), we can associate a subset \( J(\lambda) \) of \( S \), where \( J(\lambda) \) consists of those \( s \in S \) which stabilize the rows of the standard tableau of shape \( \lambda \) [CPS99, Sec. 8]. Now, \( x_\lambda \widetilde{H} \cong x_\mu \widetilde{H} \) when \( W_J(\lambda) \) and \( W_J(\mu) \) are conjugate in \( W \), which occurs if and only if \( \lambda^+ = \mu^+ \). So, for every \( \lambda \in \Lambda(n, m) \), we have \( x_\lambda \widetilde{H} \cong x_{\lambda^+} \widetilde{H} \). It follows that

\[
V^\otimes m \cong \bigoplus_{\lambda \in \Lambda^+(n, m)} x_\lambda \widetilde{H}^\otimes k_{\lambda(n, m)}
\] (1)
For a commutative ring $R$ and a homomorphism $\mathbb{Z} \to R$, $t \mapsto q$, let $S_t(n,m) \otimes_{\mathbb{Z}} R = S_q(n,m)_R$. If $R = k$ is a field and there is an algebra homomorphism $\mathbb{Z} \to k$, $t \mapsto q$, then the $q$-Schur algebra $S_q(n,m)$ of bidegree $(n,m)$ is quasi-hereditary. Thus, the category $\text{mod}-S_q(n,m)_k$ of right $S_q(n,m)_k$-modules is a highest weight category with poset $\Lambda^+(n,m)$ [DPS1 Thm. 2.8]. The poset structure $\preceq$ on the set $\Lambda^+(n,m)$ is the dominance order, defined by $\lambda = (\lambda_1, \lambda_2, \ldots) \preceq \mu = (\mu_1, \mu_2, \ldots)$ if and only if $\lambda_1 \leq \mu_1, \lambda_1 + \lambda_2 \leq \mu_1 + \mu_2, \ldots$.

For every $\lambda \in \Lambda^+(n,m)$, there exists a right $S_t(n,m)$-module $\widetilde{\Delta}(\lambda)$ such that $\Delta(\lambda)_k = \Delta(\lambda)$ is the standard object in $\text{mod}-S_q(n,m)_k$ corresponding to the partition $\lambda$. Similarly, for every $\lambda \in \Lambda^+(n,m)$, there exists a right $S_t(n,m)$-module $\widetilde{\nabla}(\lambda)$ such that $\nabla(\lambda)_k = \nabla(\lambda)$ is the costandard object in $\text{mod}-S_q(n,m)_k$ corresponding to the partition $\lambda$. The irreducible $S_q(n,m)_k$-modules are indexed by $\Lambda^+(n,m)$. We will denote the irreducible $S_q(n,m)_k$-module corresponding to a partition $\lambda \in \Lambda^+(n,m)$ by $L^k(\lambda)$.

The category $\text{mod}-S_q(n,m)_k$ is also a highest weight category with poset $\Lambda^+(n,m)$. We will denote the standard object of $\text{mod}-S_q(n,m)_k$ corresponding to $\lambda \in \Lambda^+(n,m)$ by $\Delta^\text{left}(\lambda)$, and we will denote the costandard object of $\text{mod}-S_q(n,m)_k$ corresponding to $\lambda \in \Lambda^+(n,m)$ by $\nabla^\text{left}(\lambda)$. The standard objects of $\text{mod}-S_q(n,m)_k$ correspond to the costandard objects of $\text{mod}-S_q(n,m)_k$ via the linear dual; for any $\lambda \in \Lambda^+(n,m)$, $\nabla(\lambda)_k = \Delta^\text{left}(\lambda)^*$. Similarly, the costandard objects of $\text{mod}-S_q(n,m)_k$ correspond to the standard objects of $\text{mod}-S_q(n,m)_k$ via duality: for any $\lambda \in \Lambda^+(n,m)$, $\Delta(\lambda) = \nabla^\text{left}(\lambda)^*$. [DDPW08, pg. 707].

### 3 The Indexing of the Irreducible $k\text{GL}_n(q)$-Modules

In this paper, it will be necessary to use results of Cline, Parshall, and Scott [CPS99] together with those of Dipper and James [DJ89] and Dipper and Du [DD97]. The indexing of the irreducible $kG$-modules is not consistent across these papers; in particular, the indexing used by CPS differs from the two indexings used by Dipper and James and Dipper and Du. Since the methods used by CPS [CPS99] are most relevant to the proofs appearing in Sections 4 and 5 of this paper, we will describe the indexing of CPS in greater detail than the indexings of DJ and DD.

#### 3.1 The CPS Indexing of the Irreducible $k\text{GL}_n(q)$-Modules

In this section, we will describe how the Morita equivalence constructed in [CPS99 Sec. 9] leads to an indexing of the irreducible right $k\text{GL}_n(q)$-modules. To establish the results of Sections 4 and 5 of this paper, we will need to assume that the characteristic $r > 0$ of our algebraically closed field $k$ does not divide $q(q-1)$. However, for the developments of Section 3, it is sufficient to assume that $r > 0$ and $r \nmid q$. Here, we will work with an $r$-modular system $(\mathcal{O}, K, k)$ (where $\mathcal{O}$ is a discrete valuation ring, $K$ is the quotient field of $\mathcal{O}$, and $k$ is the residue field). We will assume that the quotient field $K$ of $\mathcal{O}$ is large enough so that it is a
splitting field for $\text{GL}_n(q)$.

3.1.1 A decomposition of $\mathcal{O}_{\text{GL}_n(q)}$ into sums of blocks

Let $\mathcal{C}$ denote a fixed set of representatives of the conjugacy classes in $\text{GL}_n(q)$. The finite group $\text{GL}_n(q)$ may be obtained as the set of fixed points of the algebraic group $\text{GL}_n$ (over the algebraic closure of $\mathbb{F}_q$) under a Frobenius morphism. Let $\mathcal{G}_{ss}$ denote the set of semisimple elements in the algebraic group $\text{GL}_n$, and let $G_{ss} = \mathcal{G}_{ss} \cap \text{GL}_n(q)$. We set $\mathcal{C}_{ss} = G_{ss} \cap \mathcal{C}$ (so, $\mathcal{C}_{ss}$ consists of the representatives of the conjugacy classes in $\text{GL}_n(q)$ which are semisimple when viewed as elements of the algebraic group $\text{GL}_n$). Let $\mathcal{C}_{ss,r'}$ denote the set of elements in $\mathcal{C}_{ss}$ of order prime to $r$.

Following [M95, Ch. IV, Sec. 2], we will describe the structure of the centralizer $Z_G(s)$ of an element $s \in G_{ss}$. Let $\mathbb{F}_q[t]$ denote the polynomial ring over $\mathbb{F}_q$ in an indeterminate $t$, and let $V$ be the $n$-dimensional $\mathbb{F}_q$-vector space $V = (\mathbb{F}_q)^n$. Let $V_s$ be the $\mathbb{F}_q[t]$-module with $V_s = V$ as an $\mathbb{F}_q$-vector space and $t$-action given by $t \cdot v = s \cdot v$ for all $v \in V$. Since $\mathbb{F}_q[t]$ is a principal ideal domain, the $\mathbb{F}_q[t]$ module $V_s$ has a direct sum decomposition with summands of the form $\mathbb{F}_q[t]/(f)^m$, where $f \in \mathbb{F}_q[t]$ is an irreducible monic polynomial and $m_f \geq 1$ is an integer. In fact, since $s$ is semisimple, $m_f = 1$ for all elementary divisors $f$ of $V_s$. Suppose that $V_s$ has $m(s)$ distinct elementary divisors $f_1, f_2, \ldots, f_{m(s)}$ (where $f_1, f_2, \ldots, f_{m(s)} \in \mathbb{F}_q[t]$ are monic irreducible polynomials), and let $n_i(s)$ denote the number of times a summand $\mathbb{F}_q[t]/(f_i)$ occurs in the direct sum decomposition of $V_s$. We can write

$$V_s \cong \bigoplus_{i=1}^{m(s)} (\mathbb{F}_q[t]/(f_i))^{\oplus n_i(s)}.$$ 

For $1 \leq i \leq m(s)$, let $a_i(s)$ denote the degree of the polynomial $f_i$. Then, $\mathbb{F}_q[t]/(f_i) \cong \mathbb{F}_{q^{a_i(s)}}$ for each $i$ and $V_s \cong \bigoplus_{i=1}^{m(s)} (\mathbb{F}_{q^{a_i(s)}})^{n_i(s)}$. Since $V$ is $n$-dimensional over $\mathbb{F}_q$, we have $m(s) \sum_{i=1}^{m(s)} a_i(s) n_i(s) = n$. The centralizer $Z_G(s)$ of $s$ in $G$ can now be identified with the set of automorphisms of $V_s$. For any $1 \leq i \leq m(s)$, $\text{Aut}(\mathbb{F}_{q^{a_i(s)}}^{n_i(s)}) \cong \text{GL}_{n_i(s)}(q^{a_i(s)})$, and it follows that

$$Z_G(s) \cong \text{Aut}(V_s) \cong \prod_{i=1}^{m(s)} \text{GL}_{n_i(s)}(q^{a_i(s)}),$$

with $\sum_{i=1}^{m(s)} a_i(s) n_i(s) = n$.

For any $s \in G_{ss}$, we let $\underline{n}(s) = (n_1(s), \ldots, n_{m(s)}(s))$ be the partition of $n$ corresponding to this direct product decomposition of $Z_G(s)$. Let $\Lambda^+(\underline{n}(s))$ denote the set of multipartitions of $\underline{n}(s)$. An element $\lambda \in \Lambda^+(\underline{n}(s))$ is of the form $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m(s))})$, where $\lambda^{(1)} \vdash n_1(s)$, $\ldots$, $\lambda^{(m(s))} \vdash n_{m(s)}(s)$. The ordinary irreducible characters of $G$ (which were first parameterized by Green) may be indexed by pairs $(s, \lambda)$, where $s \in \mathcal{C}_{ss}$ and $\lambda \in \Lambda^+(\underline{n}(s))$. [CPS99]
We will denote the irreducible character of $G$ corresponding to the pair $(s, \lambda)$ by $\chi_{s,\lambda}$.

The group algebra $O_{GL_n}(q)$ decomposes as a direct sum of two-sided ideals called blocks, with every irreducible $O_{GL_n}(q)$-module belonging to a unique block. Given an element $s \in C_{ss,r}$, let $B_{s,G}$ be the sum of the blocks of $O_{GL_n}(q)$ which contain a character of the form $\chi_{st,\lambda}$, where $t \in Z_{GL_n}(q)(s)$ is an $r$-element. Then, $O_{GL_n}(q)$ has the direct sum decomposition

$$O_{GL_n}(q) = \bigoplus_{s \in C_{ss,r}} B_{s,G}.$$ 

### 3.1.2 An $O_{GL_n}(q)$-module whose endomorphisms are described by $q$-Schur algebras

Given an element $s \in C_{ss,r}$ with centralizer $Z_G(s) \cong \prod_{i=1}^{m(s)} GL_{a_i(s)}(q^{n_i(s)})$ (where $\sum_{i=1}^{m(s)} a_i(s)n_i(s) = n$), let $L_s(q)$ be the subgroup of $GL_n(q)$ defined by

$$L_s(q) = \prod_{i=1}^{m(s)} GL_{a_i(s)}(q^{n_i(s)}).$$

Let $H_s(q)$ be the subgroup of $L_s(q)$ defined by

$$H_s(q) = \prod_{i=1}^{m(s)} GL_{a_i(s)}(q)^{n_i(s)}.$$ 

(So, $H_s(q)$ consists of block-diagonal matrices in $L_s(q)$.)

$L_s(q)$ is a Levi subgroup of a parabolic subgroup of $G$, and $H_s(q)$ is a Levi subgroup of a parabolic subgroup of $L_s(q)$. Therefore, we have Harish-Chandra induction functors

$$R_{H_s(q)}^{L_s(q)} : \text{mod-}O_{H_s(q)} \rightarrow \text{mod-}O_{L_s(q)},$$

$$R_{GL_n(q)}^{L_s(q)} : \text{mod-}O_{L_s(q)} \rightarrow \text{mod-}O_{GL_n(q)}.$$ 

To any element $s \in C_{ss,r}$, we associate a $KH_s(q)$-module $C_K(s)$, which is a tensor product of certain irreducible cuspidal modules defined by Dipper and James (a more precise description of the module $C_K(s)$ can be found in [CPS99, 9.6]). The representation $C_K(s)$ contains an $O_{H_s(q)}$-lattice $C_O(s)$, and we can define a right $O_{L_s(q)}$-module associated to the element $s \in C_{ss,r}$ by

$$M_{s,L_s(q),O} = R_{H_s(q)}^{L_s(q)} C_O(s).$$

By [DJ89 (2.17)], the endomorphism algebra of the $O_{L_s(q)}$-module $M_{s,L_s(q),O}$ is a tensor product of Hecke algebras. Specifically,

$$\text{End}_{O_{L_s(q)}}(M_{s,L_s(q),O}) \cong \bigotimes_{i=1}^{m(s)} H(\mathfrak{S}_{n_i(s)}, O, q^{a_i(s)}).$$

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Now, given a multipartition $\lambda \vdash \underline{n}(s)$, let

$$y_\lambda = y_{\lambda(1)} \otimes \cdots \otimes y_{\lambda(m(s))} \in \bigotimes_{i=1}^{m(s)} H(\mathfrak{S}_{n_i(s)}, O, q^{a_i(s)})$$

(the elements $y_{\lambda(i)} \in H(\mathfrak{S}_{n_i(s)}, O, q^{a_i(s)})$ are defined in Section 2.1). Since

$$\text{End}_{O_L(s)}(M_{s,L_s(q),O}) \cong \bigotimes_{i=1}^{m(s)} H(\mathfrak{S}_{n_i(s)}, O, q^{a_i(s)}),$$

it follows that $y_\lambda \in \bigotimes_{i=1}^{m(s)} H(\mathfrak{S}_{n_i(s)}, O, q^{a_i(s)})$ acts on $M_{s,L_s(q),O}$ as an $O_L(s)$-module endomorphism. In particular, $y_\lambda M_{s,L_s(q),O}$ is an $O$-submodule of $M_{s,L_s(q),O}$. Let $\sqrt{y_\lambda M_{s,L_s(q),O}}$ denote the smallest $O$-submodule of $M_{s,L_s(q),O}$ such that $y_\lambda M_{s,L_s(q),O} \subseteq \sqrt{y_\lambda M_{s,L_s(q),O}}$ and $M_{s,L_s(q),O}/\sqrt{y_\lambda M_{s,L_s(q),O}}$ is $O$-torsion free.

We can now define an $O_L(s)$-module $\widehat{M}_{s,L_s(q),O}$ whose endomorphism algebra is a tensor product of $q^{a_i(s)}$-Schur algebras (where the integers $a_i(s)$ are determined by the structure of the centralizer $Z_G(s)$ of $s$ in $G$). First, for every $1 \leq i \leq m(s)$ and $\mu \vdash n_i(s)$, we write

$$\bigoplus_{\mu \in \Lambda(n_i(s), n_i(s))} x_{\mu} H \cong \bigoplus_{\mu \in \Lambda^+(n_i(s), n_i(s))} x_{\mu} H_{k_{\mu}(n_i(s), n_i(s))},$$

with the positive integers $k_{\mu}(n_i(s), n_i(s))$ defined as in Section 2.2, \(\square\). Given a multipartition $\lambda = (\lambda(1), \ldots, \lambda(m(s))) \vdash \underline{n}(s)$, let $m_{\lambda} = \prod_{i=1}^{m(s)} k_{\lambda(i)}(n_i(s), n_i(s))$. We define the $O_L(s)$-module $\widehat{M}_{s,L_s(q),O}$ by

$$\widehat{M}_{s,L_s(q),O} = \bigoplus_{\lambda \vdash \underline{n}(s)} y_{\lambda} M_{s,L_s(q),O} \cong \bigoplus_{\lambda \vdash \underline{n}(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_O.$$  

By \[CPS99\] Lem. 9.11,

$$\text{End}_{O_L(s)}(\widehat{M}_{s,L_s(q),O}) \cong \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_O.$$  

We can now apply the Harish-Chandra induction functor $R_{L_s(q)}^{GL_n(q)}$ to the $O_L(s)$-module $\widehat{M}_{s,L_s(q),O}$ to obtain an $OGL_n(q)$-module module whose endomorphisms are described by $q^{a_i(s)}$-Schur algebras. Given $s \in \mathcal{C}_{s,t}$, let

$$\widehat{M}_{s,GL_n(q),O} = R_{L_s(q)}^{GL_n(q)} \widehat{M}_{s,L_s(q),O}. \quad (2)$$

(To simplify the notation, we will denote $\widehat{M}_{s,GL_n(q),O}$ simply by $\widehat{M}_{s,G,O}$.) Then, $\widehat{M}_{s,GL_n(q),O}$ is the desired $OGL_n(q)$-module satisfying

$$\text{End}_{OGL_n(q)}(\widehat{M}_{s,G,O}) \cong \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_O$$

\[CPS99\] Cor. 9.14.]
3.1.3 A Morita Equivalence for $\text{GL}_n(q)$.

Viewing $\hat{M}_{s,G,\mathcal{O}}$ as a module for the subalgebra $B_{s,G}$ of $\mathcal{O}\text{GL}_n(q)$, let $J_s = \text{Ann}_{B_{s,G}}(\hat{M}_{s,G,\mathcal{O}})$. Then, by [CPS99, Lem. 9.1] and [CPS99, Thm. 9.2], the functor

$$F_s(-) = \text{Hom}_{B_{s,G} / J_s}(\hat{M}_{s,G,\mathcal{O}}, -) : \text{mod-}B_{s,G} / J_s \to \text{mod-}\text{End}_{B_{s,G}}(\hat{M}_{s,G,\mathcal{O}})$$

gives a Morita equivalence.

In fact, since $\text{End}_{B_{s,G}}(\hat{M}_{s,G,\mathcal{O}}) \cong \text{End}_{\mathcal{O}}(\hat{M}_{s,G,\mathcal{O}}) \cong \bigotimes_{i=1}^{m(s)} S_{q^{n_i}(s)}(n_i(s), n_i(s))_\mathcal{O}$, the functor $F_s$ gives a Morita equivalence

$$\text{mod-}B_{s,G} / J_s(q) \cong \text{mod-} \bigotimes_{i=1}^{m(s)} S_{q^{n_i}(s)}(n_i(s), n_i(s))_\mathcal{O}. \quad (3)$$

Let $J = \sum_{s \in \mathcal{C}_{ss,r'}} J_s$. Since the group algebra $\mathcal{O}\text{GL}_n(q)$ may be decomposed as

$$\mathcal{O}\text{GL}_n(q) = \bigoplus_{s \in \mathcal{C}_{ss,r'}} B_{s,G},$$

it follows that $\mathcal{O}\text{GL}_n(q) / J \cong \bigoplus_{s \in \mathcal{C}_{ss,r'}} B_{s,G} / J_s$. Therefore, taking direct sums over $s \in \mathcal{C}_{ss,r'}$ in (3), we have a Morita equivalence

$$F(-) = \text{Hom}_{\mathcal{O}\text{GL}_n(q) / J}(\bigoplus_{s \in \mathcal{C}_{ss,r'}} \hat{M}_{s,G,\mathcal{O}}, -) :$$

$$\text{mod-}\mathcal{O}\text{GL}_n(q) / J \to \text{mod-} \bigoplus_{s \in \mathcal{C}_{ss,r'}} \bigotimes_{i=1}^{m(s)} S_{q^{n_i}(s)}(n_i(s), n_i(s))_\mathcal{O}.$$

By [CPS99, Thm. 9.17], this Morita equivalence remains valid upon base change to $k$; so, the category of right $k\text{GL}_n(q) / J_k$-modules is Morita equivalent to the category of right

$$\bigoplus_{s \in \mathcal{C}_{ss,r'}} \bigotimes_{i=1}^{m(s)} S_{q^{n_i}(s)}(n_i(s), n_i(s))_k$$-modules. More precisely, there is a Morita equivalence

$$\tilde{F}(-) = \text{Hom}_{k\text{GL}_n(q) / J_k}(\bigoplus_{s \in \mathcal{C}_{ss,r'}} \hat{M}_{s,G,k}, -) :$$

$$\text{mod-}k\text{GL}_n(q) / J_k \to \text{mod-} \bigoplus_{s \in \mathcal{C}_{ss,r'}} \bigotimes_{i=1}^{m(s)} S_{q^{n_i}(s)}(n_i(s), n_i(s))_k$$

(where $\hat{M}_{s,G,k} = \hat{M}_{s,G,\mathcal{O}} \otimes \mathcal{O} k$).

3.1.4 An Indexing of the Irreducible $k\text{GL}_n(q)$-modules (CPS)

By [CPS99, Thm. 9.17], the algebras $k\text{GL}_n(q)$ and $k\text{GL}_n(q) / J_k$ have the same irreducible modules. Thus, the Morita equivalence

$$\tilde{F}(-) : \text{mod-}k\text{GL}_n(q) / J_k \to \text{mod-} \bigoplus_{s \in \mathcal{C}_{ss,r'}} \bigotimes_{i=1}^{m(s)} S_{q^{n_i}(s)}(n_i(s), n_i(s))_k$$
may be used to index the irreducible $kGL_n(q)$-modules [CPS99, pg. 35]. Since the irreducible $S_{q_i^e(s)}(n_i(s), n_i(s))_k$-modules are indexed by the set $\Lambda^+(n_i(s))$ of partitions of $n_i(s)$, the irreducible $\otimes_{i=1}^{m(s)} S_{q_i^e(s)}(n_i(s), n_i(s))_k$-modules are indexed by the set of multipartitions $\Lambda^+(n(s))$. Therefore, the irreducible $\oplus_{s \in \mathcal{C}_{ss,r'}} \otimes_{i=1}^{m(s)} S_{q_i^e(s)}(n_i(s), n_i(s))_k$-modules (and, consequently, the irreducible $kGL_n(q)$-modules) are indexed by pairs $(s, \lambda)$, where $s \in \mathcal{C}_{ss,r'}$ and $\lambda$ is a multipartition of $n(s)$. We will denote the irreducible $kGL_n(q)$-module corresponding to the pair $(s, \lambda)$ by $D(s, \lambda)$.

A special class of irreducible $kG$-modules is obtained by taking $s = 1$ above. Since $Z_{GL_n(q)}(1) = GL_n(q)$, $n(1) = (n)$. Therefore, the irreducible $kG$-modules corresponding to the element $s = 1$ may be labeled as $D(1, \lambda)$, with $\lambda$ a partition of $n$. By [S98, Thm. 2.1], the irreducible $kG$-modules $D(1, \lambda)$, $\lambda \vdash n$, are precisely the composition factors of the permutation module $k[G]_B$. (An irreducible module $D(1, \lambda)$ may appear more than once as a composition factor of $k[G]_B$. By [CPS99, Rem. 9.18(b)], the trivial module $k$ may be parameterized as $D(1, (1^n))$, where $(1^n)$ is the partition of $n$ with each part equal to 1.

3.2 The DD and CPS Indexings of Unipotent Principle Series Representations

In [DD97], Dipper and Du use two indexings of the irreducible $kGL_n(q)$-modules, both based on work of Dipper and James [DJ91] and Dipper [D98]. The first indexing, which is presented in [DD97, (4.2.3)], relies on the Hecke functors between the category mod-$kG$ of right $kG$-modules and the category of right modules for an appropriate Hecke algebra. The second indexing, which is presented in [DD97, (4.2.11)]], relies on the $q$-Schur functors between mod-$kG$ and the category of right modules for an appropriate $q$-Schur algebra. By [DD97, (4.2.11, (3))], these indexings agree for the irreducible $kG$-modules belonging to the unipotent principal series $\text{Irr}_k(G|B)$.

As in the introduction, let

$$l = \begin{cases} r & \text{if } |q \mod r| = 1 \\ \frac{|q \mod r|}{|q \mod r|} & \text{if } |q \mod r| > 1. \end{cases}$$

A partition $\lambda \vdash n$ is called $l$-regular if every part of $\lambda$ occurs less than $l$ times. By [DJ86, Thm. 7.6], both DD indexings yield $\text{Irr}_k(G|B) = \{D'(1, \lambda)|\lambda \vdash n, \lambda \text{ is } l\text{-regular}\}^4$

**Lemma 3.1.** Let $\lambda$ be an $l$-regular partition of $n$, and let $\lambda'$ be the dual partition. (So, the Young diagram of shape $\lambda'$ is the transpose of the Young diagram of shape $\lambda$.) If $D'(1, \lambda)$ is the irreducible $kG$-module corresponding to $\lambda$ in the DD indexing, then $D'(1, \lambda) \cong D(1, \lambda')$, where $D(1, \lambda')$ is the irreducible $kG$-module corresponding to $\lambda'$ in the CPS indexing. Thus, in the CPS indexing, $\text{Irr}_k(G|B) = \{D(1, \lambda)|\lambda \vdash n, \lambda \text{ is } l\text{-restricted}\}$.

---

4In [DD97], the modules $D'(1, \lambda)$, $\lambda$ $l$-regular, are denoted by $D(1, \lambda)$. Here, we have changed the notation to distinguish between the CPS and DD indexings.
Proof. We fix an $l$-regular partition $\lambda \vdash n$ and apply the CPS functor $\overset{\sim}{F}$ (defined in Section 3.1.4) to the irreducible $kG$-module $D'(1, \lambda)$. Since $D'(1, \lambda) \in \text{Irr}_k(G|B)$, $D'(1, \lambda)$ is in the head of $k|_B^G$, which means that $\text{Hom}_{kG}(k|_B^G, D'(1, \lambda)) \neq 0$. Since $k|_B^G$ is a direct summand of the module $\tilde{M}_{l,G,k}$ by [CPS99] Rem. 9.18 (b)] and $\overset{\sim}{F}(D'(1, \lambda))$ is irreducible, we can compute

$$\overset{\sim}{F}(D'(1, \lambda)) = \text{Hom}_{kG/J_k}(\bigoplus_{s \in \mathcal{E}_{ss,r'}} \tilde{M}_{s,G,k}, D'(1, \lambda))$$

$$\cong \text{Hom}_{kG}(\bigoplus_{s \in \mathcal{E}_{ss,r'}} \tilde{M}_{s,G,k}, D'(1, \lambda))$$

$$\cong \text{Hom}_{kG}(k|_B^G, D'(1, \lambda)).$$

In [DD97] (4.2.11], Dipper and Du describe $q$-Schur functors $S_1 : \text{mod-}kG \to \text{mod-}S_q(n,n)_k$ and $\hat{S}_1 : \text{mod-}S_q(n,n)_k \to \text{mod-}kG$, where $\hat{S}_1$ is a right inverse of $S_1$. Tracing through the definition of $S_1 : \text{mod-}kG \to \text{mod-}S_q(n,n)_k$ given in [D98] Sec. 6] (and using the fact that $S_1(D'(1, \lambda))$ must be an irreducible $S_q(n,n)_k$-module), we find that $\text{Hom}_{kG}(k|_B^G, D'(1, \lambda)) \cong S_1(D'(1, \lambda))$. Now, $D'(1, \lambda) = \hat{S}_1(L^k(\lambda'))$ by [DD97] 4.2.11 (3). So, using the fact that the functor $\hat{S}_1$ is a right inverse of $S_1$, we have $S_1(D'(1, \lambda)) \cong S_1(\hat{S}_1(L^k(\lambda'))) \cong L^k(\lambda')$. Therefore, $\overset{\sim}{F}(D'(1, \lambda)) \cong L^k(\lambda')$, and it follows that $D'(1, \lambda) = D(1, \lambda')$ in the indexing of CPS when $\lambda$ is an $l$-regular partition of $n$. 

\section{Self-Extensions of Unipotent Principal Series Representations}

Let $G = G = \text{GL}_n(q)$ (where $q$ is a power of a prime $p$), and let $B, U,$ and $T$ be the subgroups of $G$ described in the introduction. Let $k$ be an algebraically closed field of characteristic $r > 0$ such that $r \nmid q(q - 1)$. Let $Y$ be an irreducible $kG$-module belonging to the unipotent principal series $\text{Irr}_k(G|B)$ and $V$ be an arbitrary irreducible $kG$-module. In Theorem 4.1 we will show that $\text{Ext}_{kG}^1(Y, V)$ is isomorphic to an $\text{Ext}_{kG}^1$ group over a $q$-Schur algebra. The proof of this result relies on techniques and methods of [CPS99].

Let $J_k \leq kG$ be the ideal described in Section 3 such that there is a Morita equivalence

$$\overset{\sim}{F} : \text{mod-}kG/J_k \to \text{mod-} \bigoplus_{s \in \mathcal{E}_{ss,r'}} q^{m(s)} \otimes S_{q^m(s)}(n_i(s), n_i(s))_k.$$ 

Since $kG$ and $kG/J_k$ have the same irreducible $kG$-modules [CPS99, 9.17], the Morita equivalence $\overset{\sim}{F}$ yields an indexing of the irreducible $kG$-modules. In this indexing, the full set of irreducible $kG$-modules is given by

$$\{D(s, \lambda) \mid s \in \mathcal{E}_{ss,r'}, \lambda \vdash n(s)\}$$

(see Section 3.1.4). The irreducible constituents of the permutation module $k|_B^G$ are indexed by $D(1, \lambda)$, $\lambda \vdash n$, and the irreducible $kG$-modules belonging to the unipotent principal series $\text{Irr}_k(G|B)$ are indexed by $D(1, \lambda)$, where $\lambda \vdash n$ is $l$-restricted.

The $q$-Schur algebra $S_q(n,n)_k$ (defined in Section 2.2) has weight poset $\Lambda^+(n) = \{\lambda | \lambda \vdash n\}$, with the poset structure on $\Lambda^+(n)$ given by the dominance order. Therefore, the irreducible $S_q(n,n)_k$-modules are indexed by partitions $\lambda$ of $n$. Given a partition $\lambda \vdash n$, let $L^k(\lambda)$
denote the corresponding irreducible $S_q(n,n)_k$-module. By construction, $\bar{F}(D(1,\lambda)) = L^k(\lambda)$ for any $\lambda \vdash n$.

In [CPS99], CPS establish a connection between $H^1$ calculations for $G = \GL_n(q)$ and $\Ext^1$ calculations for the $q$-Schur algebra $S_q(n,n)_k$. In particular, when $\ch(k) = r \nmid q(q-1)$,

$$H^1(G, D(s, \mu)) \cong \begin{cases} \Ext^1_{S_q(n,n)_k}(L^k((1^n)), L^k(\mu)) & \text{if } s = 1 \\ 0 & \text{if } s \neq 1 \end{cases}$$

for any $s \in \mathcal{C}_{ss,r'}$ and any multipartition $\mu \vdash n(s)$ [CPS99 Thm. 10.1]. Since $k = D(1,(1^n))$, the result of [CPS99 Thm. 10.1] may be restated as follows: if $\ch(k) = r \nmid q(q-1)$,

$$\Ext^1_{kG}(D(1,(1^n)), D(s, \mu)) \cong \begin{cases} \Ext^1_{S_q(n,n)_k}(L^k((1^n)), L^k(\mu)) & \text{if } s = 1 \\ 0 & \text{if } s \neq 1 \end{cases}$$

for any $s \in \mathcal{C}_{ss,r'}$ and any multipartition $\mu \vdash n(s)$.

The key component of the proof of [CPS99 Thm. 10.1] is the observation that $k$ is in the head of the permutation module $k|_B^G$. But, $k$ is not the only irreducible $kG$-module contained in $\head(k|_B^G)$: any irreducible $kG$-module belonging to the unipotent principal series $\Irr_k(G|B)$ occurs as a composition factor of $\head(k|_B^G)$ [GJ11, 4.2.6]. In particular, by Lemma 3.1, any $kG$-module of the form $D(1,\lambda)$, where $\lambda$ is an $l$-restricted partition of $n$, is contained in $\head(k|_B^G)$. Hence, the proof of [CPS99 Thm. 10.1] holds if we replace $k = D(1,(1^n))$ by an irreducible $kG$-module $D(1,\lambda)$ where $\lambda \vdash n$ is $l$-restricted.

**Theorem 4.1.** Suppose that $r \nmid q(q-1)$ and that $\lambda$ is an $l$-restricted partition of $n$. Then, for any $s \in \mathcal{C}_{ss,r'}$ and any multipartition $\mu \vdash \underline{n}(s)$,

$$\Ext^1_{kG}(D(1,\lambda), D(s, \mu)) \cong \begin{cases} \Ext^1_{S_q(n,n)_k}(L^k(\lambda), L^k(\mu)) & \text{if } s = 1 \\ 0 & \text{if } s \neq 1. \end{cases}$$

**Proof.** (The following proof is due to CPS [CPS99 10.1] – no modifications are necessary when $k$ is replaced by an irreducible $kG$-module $D(1,\lambda)$ with $\lambda \vdash n$ $l$-restricted.) Let $s \in \mathcal{C}_{ss,r'}$, let $\mu \vdash \underline{n}(s)$, and let $D(s, \mu)$ be the corresponding irreducible $kG$-module.

Since $\lambda \vdash n$ is $l$-restricted, $D(1,\lambda) \in \Irr_k(G|B)$, which means that $D(1,\lambda) \subseteq \head(k|_B^G)$. Therefore, there exists a short exact sequence of $kG$-modules

$$0 \to \mathcal{L} \to k|_B^G \to D(1,\lambda) \to 0$$

(4)

for some submodule $\mathcal{L}$ of $k|_B^G$. Now, since $r \nmid q(q-1)$, we have $r \nmid |B|$, which means that every $kB$-module is projective. Since induction from $B$ to $G$ is exact, $k|_B^G$ is a projective $kG$-module, so that $\Ext^1_{kG}(k|_B^G, D(s, \mu)) = 0$. Therefore, the short exact sequence (4) induces the exact sequence

$$\Hom_{kG}(k|_B^G, D(s, \mu)) \to \Hom_{kG}(\mathcal{L}, D(s, \mu)) \to \Ext^1_{kG}(D(1,\lambda), D(s, \mu)) \to 0.$$

(5)
Now, by [CPS99, Thm. 9.17], any irreducible \(kG\)-module \(D(s, \tau) (\tau \vdash \underline{\mu}(s))\) is also an irreducible \(kG/J_k\)-module. So, \(kG|_B\) is a \(kG/J_k\)-direct summand of a projective \(kG/J_k\)-module and thus is itself a projective \(kG/J_k\)-module [CPS99, Rem. 9.18(c)]. It follows that (4) is a short exact sequence of \(kG/J_k\)-modules in which \(kG|_B\) is a projective \(kG/J_k\)-module. Therefore, (4) also induces the exact sequence

\[
\text{Hom}_{kG/J_k}(kG|_B, D(s, \mu)) \rightarrow \text{Hom}_{kG/J_k}(L, D(s, \mu)) \rightarrow \text{Ext}^1_{kG/J_k}(D(1, \lambda), D(s, \mu)) \rightarrow 0. \tag{6}
\]

The exact sequences (5) and (6) give rise to a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{kG/J_k}(kG|_B, D(s, \mu)) & \rightarrow & \text{Hom}_{kG}(L, D(s, \mu)) \\
\downarrow & & \downarrow \\
\text{Hom}_{kG/J_k}(kG|_B, D(s, \mu)) & \rightarrow & \text{Hom}_{kG/J_k}(L, D(s, \mu)) \\
\end{array}
\]

which has exact rows. The two left vertical arrows of the commutative diagram above are isomorphisms, and it follows that the third vertical arrow is also an isomorphism. Therefore, \(\text{Ext}^1_{kG/J_k}(D(1, \lambda), D(s, \mu)) \cong \text{Ext}^1_{kG/J_k}(D(1, \lambda), D(s, \mu)).\) Since \(\bar{F}(D(1, \lambda)) = L^k(\lambda),\) it follows by [CPS99, Thm. 9.17] that \(\text{Ext}^1_{kG/J_k}(D(1, \lambda), D(s, \mu)) \cong \text{Ext}^1_{S_q(n, n)_k}(L^k(\lambda), \bar{F}(D(s, \mu)))\). So,

\[
\text{Ext}^1_{kG/J_k}(D(1, \lambda), D(s, \mu)) \cong \text{Ext}^1_{S_q(n, n)_k}(L^k(\lambda), \bar{F}(D(s, \mu))). \tag{7}
\]

If \(s \neq 1,\) then \(D(1, \lambda)\) and \(D(s, \mu)\) belong to different blocks, which means that

\[
\text{Ext}^1_{kG/J_k}(D(1, \lambda), D(s, \mu)) = 0.
\]

If \(s = 1,\) then \(\mu\) is a partition of \(n,\) \(\bar{F}(D(s, \mu)) = \bar{F}(D(1, \mu)) = L^k(\mu),\) and the isomorphism (7) gives \(\text{Ext}^1_{kG/J_k}(D(1, \lambda), D(s, \mu)) \cong \text{Ext}^1_{S_q(n, n)_k}(L^k(\lambda), L^k(\mu)).\)

\[\square\]

As an application of Theorem 4.1, we will show that there are no non-split self-extensions of irreducible \(kG\)-modules belonging to the unipotent principal series \(\text{Irr}_k(G|B).\) It is known that there are no non-split self-extensions of irreducible modules for the \(q\)-Schur algebra \(S_q(n, n)_k.\)

**Proposition 4.2.** If \(\lambda\) is a partition of \(n\) and \(L^k(\lambda)\) is the corresponding irreducible \(S_q(n, n)_k\)-module, then \(\text{Ext}^1_{S_q(n, n)_k}(L^k(\lambda), L^k(\lambda)) = 0.\)

5The irreducible \(S_q(n, n)_k\)-module \(L^k(\lambda)\) has the structure of a \(\bigoplus_{s \in E_{ss'}} \oplus m(s) S_{q_s} (n_i(s), n_i(s))_k\)-module via the natural quotient map \(\bigoplus_{s \in E_{ss'}} \oplus m(s) S_{q_s} (n_i(s), n_i(s))_k \rightarrow S_q(n, n)_k\) (where \(S_q(n, n)_k\) is the summand of \(\bigoplus_{s \in E_{ss'}} \oplus m(s) S_{q_s} (n_i(s), n_i(s))_k\) corresponding to \(s = 1).\) So, for \(s \neq 1,\) any element of the tensor product \(\bigoplus_{s \in E_{ss'}} \oplus m(s) S_{q_s} (n_i(s), n_i(s))_k\) acts as the zero map on \(L^k(\lambda),\) and it follows that \(\text{Ext}^1_{kG/J_q}(D(1, \lambda), D(s, \mu)) \cong \text{Ext}^1_{S_q(n, n)_k}(L^k(\lambda), \bar{F}(D(s, \mu))).\)
(See [CPS88, Lem. 3.2(b)] for a proof of Proposition 4.2.)

Combining the result of Theorem 4.1 with that of Proposition 4.2 yields the next corollary.

**Corollary 4.3.** If \( \lambda \) is an \( l \)-restricted partition of \( n \) (so that \( D(1, \lambda) \in \text{Irr}_k(G|B) \)), then \( \text{Ext}_{kG}^1(D(1, \lambda), D(1, \lambda)) = 0 \).

5 Calculations of Higher \( \text{Ext} \) Groups for \( \text{GL}_n(q) \) in Cross Characteristic

As above, let \( G = \text{GL}_n(q) \), and let \( k \) be an algebraically closed field of characteristic \( r > 0 \), \( r \nmid q(q-1) \). Let \((\mathcal{O}, K, k)\) be an \( r \)-modular system, where the quotient field \( K \) of \( \mathcal{O} \) is large enough so that it is a splitting field for \( G \). In this section, we will generalize the result of [CPS99, Thm. 12.4], which states that if \( r \nmid q^m + 1 \prod_{j=1}^{m+1} (q^j - 1) \) for some integer \( m \geq 0 \) and \( V \) is a right \( kG \)-module with \( J_k \subseteq \text{Ann}_{kG}(V) \),

\[
H^i(G, V) \cong \text{Ext}_{S_q(n,n)_k}^i(L^k((1^n)), \tilde{F}(V))
\]

for \( 0 \leq i \leq m + 1 \).

Let \( S_n \) denote the symmetric group on \( n \) letters, and let \( S \) be the generating set of fundamental reflections in \( S_n \). Let \( H = \tilde{H}(S_n, S) \) be the generic Hecke algebra over the Laurent polynomial ring \( Z = \mathbb{Z}[t, t^{-1}] \) corresponding to the pair \((S_n, S)\) (defined in Section 2.1). Let \( \tilde{H}_O \) denote the \( \mathcal{O} \)-algebra obtained by base change to \( \mathcal{O} \), and let \( \tilde{H}_k \) denote the \( k \)-algebra obtained by base change to \( k \). As in [PS05], we will denote \( \tilde{H}_k \) by \( H_k \). For consistency, we will also denote the \( q \)-Schur algebra \( S_q(n,n)_k \) by \( S_q(n,n) \) for the remainder of this paper.

By [GJ11 4.3.1], \( H \cong \text{End}_{kG}(k^G_B) \). Therefore, there is a “Hecke functor”

\[
\mathfrak{F}_k : = \text{Hom}_{kG}(k^G_B, -) : \text{mod}-kG \to \text{mod}-H.
\]

The functor \( \mathfrak{F}_k \) has a right inverse \( \mathfrak{G}_k : \text{mod}-H \to \text{mod}-kG \), which is given by \( \mathfrak{G}_k(E) = E \otimes_H k^G_B \) for any right \( H \)-module \( E \). (The right \( kG \)-module \( k^G_B \) is naturally a left module for the endomorphism algebra \( H = \text{End}_{kG}(k^G_B) \).) We have adopted the notation of Geck and Jacon [GJ11] here. Since \( k^G_B \) is projective, \( \mathfrak{F}_k = H_1 \) and \( \mathfrak{G}_k = H_1 \) in Dipper and Du’s notation [DD97 Sec. 4.1].

**Remark 5.1.** By [GJ11 4.1.4], we can say more about the relationship between the functors \( \mathfrak{F}_k \) and \( \mathfrak{G}_k \) under the current assumptions on the characteristic \( r \) of \( k \). Since \( r \nmid q \) and \( k^G_B \) is projective, \( \mathfrak{G}_k \) is a two-sided inverse of \( \mathfrak{F}_k \) on the full subcategory of \( \text{mod}-kG \) consisting of all \( V \in \text{mod}-kG \) such that

1. every non-zero submodule of \( V \) has a composition factor in \( \text{Irr}_k(G|B) \), and
2. every non-zero quotient module of \( V \) has a composition factor in \( \text{Irr}_k(G|B) \)
(As stated in [GJ11 4.1.4], this result is based on work of Green [G80] and Brundan, Dipper, and Kleschev [BDK01].)

**Lemma 5.2.** If $V$ is a $kG$-module in the image of the functor $\mathfrak{G}_k : \text{mod-}H \rightarrow \text{mod-}kG$, then $V$ is annihilated by $J_k$.

**Proof.** Let $E$ be a right $H$-module such that $V = \mathfrak{G}_k(E) = E \otimes_{kG} k|_B^G$. Given $\alpha \in kG$ and $e \otimes x \in E \otimes_{kG} k|_B^G$ (where $e \in E$ and $x \in k|_B^G$), we have $(e \otimes x)\alpha = e \otimes (x\alpha)$. But, since $k|_B^G$ is a direct summand of the right $kG/J_k$-module $\widehat{M}_{1,G,k}$ [CPS99, Rem. 9.18 (b)], $k|_B^G$ is annihilated by $J_k$. Thus, $e \otimes (x\alpha) = e \otimes 0 = 0$. It follows that $E \otimes_{kG} k|_B^G$ is annihilated by $J_k$, which means that the same statement holds for $V$. 

Before we proceed, we must define a certain $S_q(n,n)-H$ bimodule which is used in [PS05] to link the representation theory of $H$ to the representation theory of $S_q(n,n)$. As in Section 2.2, let $V$ be a free $Z = \mathbb{Z}[t,t^{-1}]$-module of rank $n$. Given $\lambda \vdash n$, let $k_\lambda \in \mathbb{Z}$ be such that $V^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda^+(n)} (x\lambda \tilde{H})^{k_\lambda}$ (see (1)). We define a right $\tilde{H}$-module

$$\widetilde{T} = \bigoplus_{\lambda \in \Lambda^+(n)} (x\lambda \tilde{H})^{k_\lambda}.$$

Since $S_t(n,n) = \text{End}_H(V^{\otimes n}) \cong \text{End}_H(\widetilde{T})$, there is also a natural left action of $S_t(n,n)$ on $\widetilde{T}$. Let $T = \widetilde{T}_k$ be the $S_q(n,n)-H$ bimodule obtained by reduction of $\widetilde{T}$ to $k$.

Since $k|_B^G$ is a right $kG$-module, $k|_B^G$ is a left module for $H = \text{End}_{kG}(k|_B^G)$, which means that the dual $(k|_B^G)^*$ has the structure of a right $H$-module. (The module $k|_B^G$ is self-dual; however, we will continue to write $(k|_B^G)^*$ for the right $H$-module described here to distinguish it from $k|_B^G$ as a right $kG$-module.)

**Lemma 5.3.** Suppose that $r \mid q(q-1)$, and let $\widehat{M}_{1,G,k} \otimes_{kG}(k|_B^G)^*$ be the right $H$-module defined via the right action of $H$ on $(k|_B^G)^*$. There is an isomorphism $\widehat{M}_{1,G,k} \otimes_{kG}(k|_B^G)^* \cong T^\Phi$ of right $H$-modules (where $T^\Phi$ denotes the right $H$-module obtained by twisting the action of $H$ on $T$ by the involution $\Phi$ of $H$ defined in Section 2.1).

**Proof.** Under the given assumptions on $r$, $k|_B^G$ is projective. Thus, there is an isomorphism of right $H$-modules

$$\widehat{M}_{1,G,k} \otimes_{kG}(k|_B^G)^* \cong \text{Hom}_{kG}(k|_B^G, \widehat{M}_{1,G,k})$$

[D90] Rem. 2.3, with the right action of $H$ on $\text{Hom}_{kG}(k|_B^G, \widehat{M}_{1,G,k})$ induced by the left action of $H$ on $k|_B^G$. We also have an isomorphism

$$\text{Hom}_{kG}(k|_B^G, \widetilde{M}_{1,G,k}) \cong \text{Hom}_{\text{OG}}(\mathcal{O}|_B^G, \widetilde{M}_{1,G,k})$$

\(^6\text{See (2) of Section 3.1.2 for the definition of } \widetilde{M}_{1,G,k}.$$
since $k|G_B$ is projective. By definition, $\widetilde{M}_{1,G,O} = \oplus_{\lambda \vdash n} \sqrt{y_{\lambda} O_B^{G_B}}^{k_{\lambda}}$ (see Section 3.1.2); so,

$$\text{Hom}_{OG}(O_B^{G_B}, \widetilde{M}_{1,G,O}) \simeq \oplus_{\lambda \vdash n} \text{Hom}_{OG}(O_B^{G_B}, \sqrt{y_{\lambda} O_B^{G_B}}^{k_{\lambda}}).$$

By [DJ89] Lem. 2.22, 2.15 (ii), and 2.8], $\text{Hom}_{OG}(O_B^{G_B}, \sqrt{y_{\lambda} O_B^{G_B}}) \simeq y_{\lambda}H_O^{G_B}$. Thus,

$$\text{Hom}_{kG}(k|G_B, \widetilde{M}_{1,G,O}) \simeq \oplus_{\lambda \vdash n} (y_{\lambda}H_O)^{k_{\lambda}} \simeq (\oplus_{\lambda \vdash n} (y_{\lambda}H_O)^{k_{\lambda}}) \simeq \widetilde{T}^{\Phi} \simeq T^{\Phi},$$

where the isomorphism $\oplus_{\lambda \vdash n} (y_{\lambda}H_O)^{k_{\lambda}} \simeq \widetilde{T}^{\Phi}$ holds by [DPS2] Lem. 1.1 (c)].

Before we proceed to the next result, we must compare the indexing of Specht modules for the Hecke algebra used by Du, Parshall, and Scott [DPS2] and Parshall and Scott [PS05] with that used by Dipper and James [DJ89]. In the work of DPS and PS, the Specht module in mod-$\tilde{H}_O$ corresponding to a partition $\lambda \vdash n$ is denoted by $\tilde{S}_\lambda$. First, we will define the Specht module $\tilde{S}_\lambda$ following Parshall and Scott [PS05 Sec. 2.2]. If $\lambda$ is a partition of $n$, then $\text{Hom}_{\tilde{H}_O}(y_{\lambda} \tilde{H}_O, x_{\lambda} \tilde{H}_O) \simeq O$. Therefore, there exist indecomposable summands $\tilde{Y}_\lambda$ of $y_{\lambda} \tilde{H}_O$ and $\tilde{Y}_\lambda$ of $x_{\lambda} \tilde{H}_O$ such that $\text{Hom}_{\tilde{H}_O}(\tilde{Y}_\lambda, \tilde{Y}_\lambda) \simeq O$. (For $\lambda \vdash n$, $\tilde{Y}_\lambda$ is called a twisted Young module, and $\tilde{Y}_\lambda$ is called a Young module.) Let $\zeta : \tilde{Y}_\lambda \rightarrow \tilde{Y}_\lambda$ be a generator of $\text{Hom}_{\tilde{H}_O}(\tilde{Y}_\lambda, \tilde{Y}_\lambda)$. Then, the Specht module $\tilde{S}_\lambda$ corresponding to $\lambda$ is defined to be the image of $\zeta$.

In the work of Dipper and James [DJ86], [DJ89] (and, in the work of Dipper and Du [DD97]), the Specht module corresponding to the partition $\lambda \vdash n$ is denoted by $\tilde{S}_\lambda$ and defined as the right ideal $\tilde{S}_\lambda = x_{\lambda} \tilde{H}_O y_{\lambda} \tilde{H}_O$ of $\tilde{H}_O$, where $w_{0\lambda}$ is the longest element of the parabolic subgroup $W_\lambda$ of $W$ (see [DJ86 Sec. 4]).

**Lemma 5.4.** The DPS labeling of the Specht modules for $H_O$ is consistent with the DJ labeling. That is, for any $\lambda \vdash n$, $\tilde{S}_\lambda \simeq \tilde{S}_\lambda$ as right $\tilde{H}_O$-modules.

**Proof.** This result holds since both $\tilde{S}_\lambda$ and $\tilde{S}_\lambda$ are isomorphic to $x_{\lambda} \tilde{H}_O y_{\lambda} \tilde{H}_O$. The isomorphism $\tilde{S}_\lambda \simeq x_{\lambda} \tilde{H}_O y_{\lambda} \tilde{H}_O$ follows by [DPS2] Lem. 1.1 (hf) and (f), and the isomorphism $\tilde{S}_\lambda = x_{\lambda} \tilde{H}_O y_{\lambda} \tilde{H}_O$ follows by [DJ86] Cor. 4.2].

7In the notation of Dipper and James, $O_B^{G_B} = M$ and the lemmas of [DJ89] apply as follows. Let $\lambda \vdash n$. By [DJ89] Lem. 2.22, $\sqrt{y_{\lambda} O_B^{G_B}} = \sqrt{y_{\lambda} M} \simeq M_{y_{\lambda}} = \{u \in M \mid l(y_{\lambda} H_O)u = 0\}$, where $l(y_{\lambda} H_O)$ is the left annihilator of $y_{\lambda} H_O$ in $H_O$. By [DJ89] Lem. 2.15 (ii), $\text{Hom}_{OG}(O_B^{G_B}, M_{y_{\lambda}}) = rl(y_{\lambda} H_O)$, where $rl(y_{\lambda} H_O)$ denotes the right annihilator of $l(y_{\lambda} H_O)$ in $H_O$. Finally, by [DJ89] Lem. 2.8], $rl(y_{\lambda} H_O) = y_{\lambda} H_O$. Combining these results, we see that $\text{Hom}_{OG}(O_B^{G_B}, \sqrt{y_{\lambda} O_B^{G_B}}) \simeq y_{\lambda} H_O$. Combining
The PS indexing of the irreducible $H$-modules is also consistent with the DJ indexing. Let $S_\lambda = \tilde{S}_\lambda_k$ denote the Specht module for $H$ obtained by base change to $k$. As above, let

$$l = \begin{cases} r & \text{if } |q \mod r| = 1 \\ |q \mod r| & \text{if } |q \mod r| > 1. \end{cases}$$

When $\lambda$ is an $l$-regular partition of $n$, $D_\lambda := S_\lambda/\text{rad}(S_\lambda)$ is an irreducible $H$-module. A full set of irreducible $H$-modules is given by $\{D_\lambda \mid \lambda \text{ is } l-\text{regular}\}$. In the work of Dipper and James, the irreducible $H$-module corresponding to an $l$-regular partition $\lambda$ of $n$ is denoted by $D^\lambda$. By Lemma 5.4, we have $D^\lambda \cong D_\lambda$ for all $l$-regular partitions $\lambda$ of $n$.

Suppose that $\lambda$ is an $l$-regular partition of $n$. Dipper and James [DJ89, (3.1)] associate to $\lambda$ a certain indecomposable right $kG$-module $S(1, \lambda)$ with the property that $\text{head}(S(1, \lambda)) = D'(1, \lambda)$ (where $D'(1, \lambda)$ is indexed following Dipper and Du [DD97 (4.2.3)]). The $kG$-module $S(1, \lambda)$ is closely connected to the Specht module $S_\lambda$ for the Hecke algebra $kG$ (this is the reason for the similarity in notation); the Hecke functor $\text{mod}-kG \to \text{mod}-H$ maps $S(1, \lambda)$ to $S_\lambda$ [DJ89 (3.1)]. By [J84, Def. 11.11], $S(1, \lambda)$ is a submodule of the permutation module $k_P^{G_{P_{\lambda}}}$, where $P_{\lambda}$ is the parabolic subgroup of $G$ corresponding to $\lambda$. It follows that $S(1, \lambda)$ is also a submodule of $k_{P_{\lambda}}^G$ (since $k \subseteq k_{P_{\lambda}}^G$ and induction from $P_{\lambda}$ to $G$ is exact, we have $k_{P_{\lambda}}^G \subseteq k_{P_{\lambda}}^G|_{P_{\lambda}} \cong k_{P_{\lambda}}^G$). Since every irreducible $kG$-module in the socle of $k_{P_{\lambda}}^G$ belongs to the unipotent principal series $\text{Irr}_k(G|B)$, the same is true of $S(1, \lambda)$. When $\lambda$ is $l$-regular, $\text{head}(S(1, \lambda)) = D'(1, \lambda)$ also belongs to $\text{Irr}_k(G|B)$. These observations justify the next result.

**Lemma 5.5.** Let $\lambda$ be an $l$-regular partition of $n$. Then, the $kG$-module $S(1, \lambda)$ satisfies the assumptions of Remark 5.7 (i.e., every non-zero submodule and quotient module of $S(1, \lambda)$ has a composition factor in $\text{Irr}_k(G|B)$). In particular, $S(1, \lambda) \cong \mathfrak{S}_k(\mathfrak{S}_k(S(1, \lambda)))$.

Suppose now that $\lambda$ is an $l$-restricted partition of $n$. Then, $\lambda'$ is $l$-regular and Lemma 5.5 tells us that $S(1, \lambda')$ is in the image of the Hecke functor $\mathfrak{S}_k : \text{mod}-H \to \text{mod}-kG$. So, by Lemma 5.2, $S(1, \lambda')$ is annihilated by the ideal $J_k$ of $kG$, which means that we may apply the CPS functor $\hat{F}$ (which takes as inputs $kG/J_k$-modules) to $S(1, \lambda')$. In Proposition 5.6 we will show that $\hat{F}$ maps $S(1, \lambda')$ to the standard module $\Delta(\lambda)$ for the $q$-Schur algebra $S_q(n, n)$. The apparent mismatch in labeling is due to the differences between the CPS and DJ indexings of the irreducible $kG$-modules. In the indexing of Dipper and James (and Dipper and Du), $\text{head}(S(1, \lambda')) = D'(1, \lambda')$. By Lemma 3.1, $D'(1, \lambda')$ is isomorphic to $D(1, \lambda)$ in the indexing of Cline, Parshall, and Scott. It follows that when $\lambda \vdash n$ is $l$-restricted, the indecomposable right $kG$-module $S(1, \lambda')$ of Dipper and James has the irreducible $D(1, \lambda)$ (in the CPS indexing) as its head.

**Proposition 5.6.** If $l > 2$, $\lambda$ is an $l$-restricted partition of $n$, and $r \nmid q(q - 1)$, then $\hat{F}(S(1, \lambda')) \cong \Delta(\lambda)$ (where $\Delta(\lambda)$ is the standard object corresponding to $\lambda$ in the category of right $S_q(n, n)$-modules).
Proof. Since $\lambda'$ is $l$-regular, it follows from the discussion preceding Lemma 5.5 that all composition factors of $S(1, \lambda')$ belong to $B_{1,G}$, where $B_{1,G}$ is the sum of the unipotent blocks of $G$. But, as shown in the proof of [CPS99] Thm. 9.17, all composition factors of the head of the $kG/J_k\hat{\phi}^{-}$ have irreducible constituents belonging to $B_{1,G}$. So, viewing $\hat{M}_{s,G,k}$ as a $kG$-module via the natural quotient map $kG \twoheadrightarrow kG/J_k$, we have $\text{Hom}_{kG}(\hat{M}_{s,G,k}, S(1, \lambda')) = 0$ when $s \neq 1$. Thus,

$$F(S(1, \lambda')) = \text{Hom}_{kG/J_k}(\bigoplus_{s \in \mathcal{E}_{s,G,k}} \hat{M}_{s,G,k}, S(1, \lambda'))$$

$$\cong \text{Hom}_{kG}(\bigoplus_{s \in \mathcal{E}_{s,G,k}} \hat{M}_{s,G,k}, S(1, \lambda'))$$

$$\cong \text{Hom}_{kG}(\hat{M}_{1,G,k}, S(1, \lambda')),$$

where $\text{Hom}_{kG}(\hat{M}_{1,G,k}, S(1, \lambda'))$ is viewed as a right $S_q(n, n)_k\hat{\phi}$-module via the natural left action of $S_q(n, n)_k \cong \text{End}_{kG}(\hat{M}_{1,G,k})$ on the right $kG$-module $\hat{M}_{1,G,k}$.

According to Lemma 5.5, we can write $S(1, \lambda') \cong \mathfrak{S}_k(\mathfrak{S}_k(S(1, \lambda')))$. By [DD97] (4.2.3, (1)), the Hecke functor $\mathfrak{S}_k$ maps $S(1, \lambda')$ to the Specht module $S_{\lambda'}$, which means that $S(1, \lambda') \cong \mathfrak{S}_k(S_{\lambda'}) = S_{\lambda'} \otimes_H k[G]_B^\ast$ (here, we have used Lemma 5.4 to identify the Dipper-James Specht module $S_{\lambda'}$ with the CPS Specht module $S_{\lambda'}$). We claim that there is an isomorphism $S_{\lambda'} \otimes_H k[G]_B^\ast \cong \text{Hom}_H((k[G]_B^\ast)^*, S_{\lambda'})$ of right $kG$-modules. (The right action of $kG$ on $\text{Hom}_H((k[G]_B^\ast)^*, S_{\lambda'})$ is given by $(\phi \cdot \alpha)(f) = \phi(\alpha \cdot f)$ for any $\alpha \in kG$, $\phi \in \text{Hom}_H((k[G]_B^\ast)^*, S_{\lambda'})$, and $f \in (k[G]_B^\ast)^*$.) It is known that there is a vector space isomorphism $\Omega : S_{\lambda'} \otimes_H k[G]_B^\ast \to \text{Hom}_H((k[G]_B^\ast)^*, S_{\lambda'})$, given by $\Omega(s \otimes x) = (f \mapsto f(x)s)$ for any $s \in S_{\lambda'}$ and $x \in k[G]_B^\ast$. So, to prove the claim, it suffices to show that $\Omega$ respects the right action of $kG$. But, given any $\alpha \in kG$, $s \in S_{\lambda'}$, $x \in k[G]_B^\ast$, and $f \in (k[G]_B^\ast)^*$, $\Omega((s \otimes x) \cdot \alpha)(f) = \Omega(s \otimes x \cdot \alpha)(f) = f(x \cdot \alpha)(s) = (\alpha \cdot f)(x)s = \Omega(s \otimes x)(\alpha \cdot f) = (\alpha \cdot f)(x)s = \Omega(s \otimes x)(\alpha \cdot f) = (\alpha \cdot f)(x)s$.

Since $S_{\lambda'} \otimes_H k[G]_B^\ast \cong \text{End}_{kG}((k[G]_B^\ast)^*, S_{\lambda'})$ (as right $kG$-modules),

$$\text{Hom}_{kG}(\hat{M}_{1,G,k}, S(1, \lambda')) \cong \text{Hom}_{kG}(\hat{M}_{1,G,k}, \text{Hom}_{kG}((k[G]_B^\ast)^*, S_{\lambda'})) \cong \text{Hom}_{kG}(\hat{M}_{1,G,k} \otimes_{kG} (k[G]_B^\ast)^*, S_{\lambda'})$$

as right $S_q(n, n)$-modules (the third isomorphism in the chain of isomorphisms above follows by tensor-hom adjunction, which preserves the right $S_q(n, n)$-module structure of $\text{Hom}_{kG}(\hat{M}_{1,G,k}, \text{Hom}_{kG}((k[G]_B^\ast)^*, S_{\lambda'}))$).

By Lemma 5.3, $\hat{M}_{1,G,k} \otimes_{kG} (k[G]_B^\ast)^* \cong T^\Phi$ as right $H$-modules. Thus, tracing through the calculations above, we have

$$F(S(1, \lambda')) \cong \text{Hom}_{H}(T^\Phi, S_{\lambda'}),$$

with the right action of $S_q(n, n)$ on $\text{Hom}_H(T^\Phi, S_{\lambda'})$ defined via the left action of $S_q(n, n)$ on $T^\Phi$. Now, it follows from the proof of [DPS2] Thm. 7.7 that the right $S_q(n, n)_\mathcal{O}$-module
Hom_{H_O}(\tilde{T}^\Phi, \tilde{S}_\lambda) identifies with \(\tilde{\Delta}^{\left(\tilde{\Delta}(\lambda)\right)}\), where \(\tilde{\Delta}(\lambda)\) is the standard object corresponding to the partition \(\lambda\) in the category of left \(S_q(n, n)_{\sigma}\)-modules and \(\tilde{\Delta}^{\left(\tilde{\Delta}(\lambda)\right)}\) is the right \(S_q(n, n)_{\sigma}\)-module obtained by converting the left action of \(S_q(n, n)_{\sigma}\) on \(\tilde{\Delta}(\lambda)\) to a right action via the anti-automorphism \(\beta\) defined in [DPS2, Lem. 2.2]. But, since

\[
(\tilde{\Delta}(\lambda))^{\beta} = (\tilde{\Delta}(\lambda))^{D_{S_q(n, n)_{\sigma}}(n, s)} \cong \tilde{\Delta}(\lambda)
\]

(where \(D_{S_q(n, n)_{\sigma}}\) is the duality on \(\text{mod-}S_q(n, n)_{\sigma}\)), we have \(\tilde{\Delta}(\lambda)^{\beta} \cong \tilde{\Delta}(\lambda)^*\), and

\[
\text{Hom}_{H_O}(\tilde{T}^\Phi, \tilde{S}_\lambda) \cong \tilde{\Delta}(\lambda)^{\beta} \cong \tilde{\Delta}(\lambda)^*.
\]

Since \(\tilde{\Delta}(\lambda)^* \cong \tilde{\Delta}(\lambda)\) (where \(\tilde{\Delta}(\lambda)\) is the standard object corresponding to \(\lambda\) in the category of right \(S_q(n, n)_{\sigma}\)-modules), it follows that

\[
\text{Hom}_{H_O}(\tilde{T}^\Phi, \tilde{S}_\lambda) \cong \tilde{\Delta}(\lambda).
\]

Finally, when \(l > 2\), an argument analogous to that given in the second part of [PS05, Lem. 2.4] shows that the isomorphism \(\text{Hom}_{H_O}(\tilde{T}^\Phi, \tilde{S}_\lambda) \cong \tilde{\Delta}(\lambda)\) holds upon base change to \(k\). Therefore, when \(l > 2\),

\[
\tilde{F}(S(1, \lambda')) \cong \text{Hom}_H(T^\Phi, S_\lambda) \cong \Delta(\lambda).
\]

As above, let \(\tilde{H}\) denote the generic Hecke algebra corresponding to the pair \((\mathfrak{S}_n, S)\), where \(\mathfrak{S}_n\) is the symmetric group on \(n\) letters, and \(S\) is the generating set of fundamental reflections in \(\mathfrak{S}_n\). Let \(\tilde{H}_O\) denote the \(O\)-algebra obtained by base change to \(O\). In order to generalize [CPS99] Thm. 12.4], we must construct a suitable resolution of \(S(1, \lambda')\) when \(\lambda\) is \(l\)-restricted. To construct this resolution, we will use the functors \(\mathfrak{N}_i : \text{mod-}\tilde{H}_O \rightarrow \text{mod-}\tilde{H}_O\) \((0 \leq i \leq n - 1)\) defined by Parshall and Scott in [PS05, Sec. 3.1]. Given a subset \(J \subseteq S\), there is a restriction functor \(\text{Res}_{\tilde{H}_O(J)}^{\tilde{H}_O}\) from \(\text{mod-}\tilde{H}_O\) (the category of right \(\tilde{H}_O\)-modules) to \(\text{mod-}(\tilde{H}_O)_J\) (the category of right modules for the parabolic subalgebra \((\tilde{H}_O)_J\) of \(\tilde{H}_O\)). There is also an induction functor \(\text{Ind}_{\tilde{H}_O(J)}^{\tilde{H}_O}\) is a left adjoint of \(\text{Res}_{\tilde{H}_O(J)}^{\tilde{H}_O}\). Since \(|S| = n - 1\), we may define \(\mathfrak{N}_i\) for \(0 \leq i \leq n - 1\) by

\[
\mathfrak{N}_i = \prod_{J \subseteq S, |J| = i} \text{Ind}_{\tilde{H}_O(J)}^{\tilde{H}_O} \circ \text{Res}_{\tilde{H}_O(J)}^{\tilde{H}_O} : \text{mod-}\tilde{H}_O \rightarrow \text{mod-}\tilde{H}_O.
\]

**Lemma 5.7.** Suppose that \(r \not| q(q - 1)\) and \(\lambda\) is an \(l\)-restricted partition of \(n\). Then, there exists an exact sequence of right \(kG\)-modules of the form \(0 \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow S(1, \lambda') \rightarrow 0\) in which every module is annihilated by \(J_k\) and \(M_i\) is projective as both a \(kG\) and a \(kG/J_k\)-module for \(0 \leq i \leq l - 2\).
Proof. By [PS05 Thm. 3.4], there exists an exact sequence of right $\tilde{H}_G$-modules of the form

$$0 \rightarrow S^\phi_\lambda \rightarrow N_0(S_\lambda) \rightarrow N_1(S_\lambda) \rightarrow \cdots \rightarrow N_{n-1}(S_\lambda) \rightarrow 0.$$ 

By [PS05 Rem. 3.5], this sequence remains exact upon base change to $k$, yielding the exact sequence

$$0 \rightarrow S^\phi_\lambda \rightarrow N_0(S_\lambda) \rightarrow N_1(S_\lambda) \rightarrow \cdots \rightarrow N_{n-1}(S_\lambda) \rightarrow 0$$

of right $H$-modules. Now, by [PS05 Rem. 3.10], $N_i(S_\lambda)$ is a projective right $H$-module for $i \leq l - 2$ (this follows because $H_J$ is semisimple for any $J \subseteq S$ with $|J| = i$ and $N_i$ is exact). Applying the contravariant duality functor $D_H : \text{mod-}H \rightarrow \text{mod-}H$ to the exact sequence above, we obtain the exact sequence

$$0 \rightarrow N_{n-1}(S_\lambda)^{D_H} \rightarrow \cdots \rightarrow N_1(S_\lambda)^{D_H} \rightarrow N_0(S_\lambda)^{D_H} \rightarrow (S^\phi_\lambda)^{D_H} \rightarrow 0$$

of right $H$-modules, in which $N_i(S_\lambda)^{D_H}$ is projective for $i \leq l - 2$. But, by [DPS2 Prop. 7.3], $(S^\phi_\lambda)^{D_H} \cong S_\lambda$; therefore, the exact sequence above can be re-written as

$$0 \rightarrow N_{n-1}(S_\lambda)^{D_H} \rightarrow \cdots \rightarrow N_1(S_\lambda)^{D_H} \rightarrow N_0(S_\lambda)^{D_H} \rightarrow S_\lambda \rightarrow 0.$$ 

Since $r \not| q(q-1)$ and $k|G_B$ is a projective right $kG$-module, the functor $G_k(-) = - \otimes_H k|G_B$ is exact. Applying $G_k$ to the exact sequence above and using the isomorphism $G_k(S_{\lambda'}) \cong S(1, \lambda')$ (valid since $S(1, \lambda')$ satisfies the assumptions of Remark 5.1), we obtain the exact sequence

$$0 \rightarrow G_k(N_{n-1}(S_\lambda)^{D_H}) \rightarrow \cdots \rightarrow G_k(N_1(S_\lambda)^{D_H}) \rightarrow G_k(N_0(S_\lambda)^{D_H}) \rightarrow S(1, \lambda') \rightarrow 0 \quad (8)$$

of right $kG$-modules.

By Lemma 5.2, each of the right $kG$-modules in the exact sequence [8] is annihilated by $J_k$, which means that [8] is also an exact sequence of right $kG/J_k$-modules. For $0 \leq i \leq n-1$, let $M_i = G_k(N_i(S_\lambda))^{D_H}$. Since $N_i(S_\lambda)^{D_H}$ is a projective right $H$-module for $i \leq l - 2$ and $G_k$ is exact, $M_i$ is a projective right $kG$-module for $i \leq l - 2$. It follows from these observations that $M_i$ is also a projective $kG/J_k$-module for $i \leq l - 2$.

We are now ready to prove our generalization of [CPS99 Thm. 12.4].

Theorem 5.8. Suppose that $r \not| q(q-1)$, $l > 2$, and $\lambda$ is an $l$-restricted partition of $n$. If $V$ is a right $kG$-module with $J_k \subseteq \text{Ann}_{kG}(V)$, then $\text{Ext}^{i}_{kG}(S(1, \lambda'), V) \cong \text{Ext}^{i}_{S(q,n,n)}(\Delta(\lambda), F(V))$ for $0 \leq i \leq l - 1$.

\footnote{Since $k|G_B$ is a projective $kG$-module, the functors $G_k$ and $G_k$ form an equivalence of abelian categories between the full subcategory of mod-$kG$ consisting of modules satisfying the conditions of Remark 5.1 and mod-$H$ [GJTTh 4.1.4]. It follows that the functor $G_k$ is exact.}
Proof. Given the results of Proposition 5.6 and Lemma 5.7, the proof of [CPS99, Thm. 12.4] goes through with virtually no change.

Let \( 0 \to M_{n-1} \to \cdots \to M_1 \to M_0 \to S(1, \lambda') \to 0 \) be the exact sequence obtained in Lemma 5.7. This sequence is exact in both the category of right \( kG \)-modules and the category of right \( kG/J_k \)-modules. For \( 0 \leq i \leq l-2 \), \( M_i \) is projective for both \( kG \) and \( kG/J_k \). Let \( R = kG \) or \( kG/J_k \), and let \( C_i \) denote the double complex obtained by applying the functor \( \text{Hom}_R(-, V) \) to a Cartan-Eilenberg resolution of the complex \( M \). Filtering \( C_i \) by columns \( C_i \) leads to the spectral sequence

\[
\text{Ext}_R^i(M_i, V) \Rightarrow \text{Ext}_R^{i+l}(S(1, \lambda'), V).
\]

But, \( M_i \) is projective for \( 0 \leq i \leq l-2 \), so \( \text{Ext}_R^i(M_i, V) = 0 \) for \( t > 0 \) and \( 0 \leq i \leq l-2 \). Since \( \text{Hom}_{kG}(-, -) \) and \( \text{Hom}_{kG/J_k}(-, -) \) are equivalent bifunctors on \( \text{mod-}kG/J_k \), we have

\[
\text{Ext}_{kG}^i(S(1, \lambda'), V) \cong \text{Ext}_{kG/J_k}^i(S(1, \lambda'), V) \text{ for } 0 \leq i \leq l-1.
\]

Finally, since \( \tilde{F}(S(1, \lambda')) \cong \Delta(\lambda) \) when \( l > 2 \) (by Proposition 5.6) and \( \tilde{F} \) is a Morita equivalence, we have \( \text{Ext}_{kG/J_k}^i(S(1, \lambda'), V) \cong \text{Ext}_{S_q(n,n)}^i(\Delta(\lambda), \tilde{F}(V)) \) for all \( i \). Combining this isomorphism with the isomorphism of (9), we have \( \text{Ext}_{kG}^i(S(1, \lambda'), V) \cong \text{Ext}_{S_q(n,n)}^i(\Delta(\lambda), \tilde{F}(V)) \) for \( 0 \leq i \leq l-1 \).

Remark 5.9. When \( \lambda = (1^n) \), \( \lambda' = (n) \) and \( S(1, \lambda') = S(1, (n)) = D'(1, (n)) = D(1, (1^n)) = k \). So, in this case, Theorem 5.8 yields \( H^i(G, V) \cong \text{Ext}_{kG}^i(k, V) \cong \text{Ext}_{S_q(n,n)}^i(L^k((1^n)), \tilde{F}(V)) \) for \( 0 \leq i \leq l-1 \).

Remark 5.10. The spectral sequences described in the proof of Theorem 5.8 also yield a version of [CPS99] (12.4.2). Thus, if \( r \nmid q(q-1) \), \( l > 2 \), \( \lambda \) is an \( l \)-restricted partition of \( n \), and \( V \) is a right \( kG \)-module with \( J_k \subseteq \text{Ann}_{kG}(V) \), then there is an injection \( \text{Ext}_{S_q(n,n)}^i(\Delta(\lambda), \tilde{F}(V)) \hookrightarrow \text{Ext}_{kG}^i(S(1, \lambda'), V) \).

Theorem 5.8 allows us to use known Ext vanishing results for \( S_q(n,n) \)-modules to obtain new Ext vanishing results for \( kG \)-modules. For example, we can use the fact that \( \text{mod-}S_q(n,n) \) is a highest weight category to prove the following corollary.

Corollary 5.11. Suppose that \( r \nmid q(q-1) \), \( l > 2 \), and \( \lambda \) is an \( l \)-restricted partition of \( n \).

(a) If \( \mu \vdash n \) is such that \( \mu \subseteq \lambda \), then \( \text{Ext}_{kG}^i(S(1, \lambda'), D(1, \mu)) = 0 \) for \( 1 \leq i \leq l-1 \).

(b) If \( \mu \vdash n \) is \( l \)-restricted and \( \mu \subseteq \lambda \), then \( \text{Ext}_{kG}^i(S(1, \lambda'), S(1, \mu')) = 0 \) for \( 1 \leq i \leq l-1 \).

Proof. (a) By Theorem 5.8

\[
\text{Ext}_{kG}^i(S(1, \lambda'), D(1, \mu)) \cong \text{Ext}_{S_q(n,n)}^i(\Delta(\lambda), \tilde{F}(D(1, \mu))) = \text{Ext}_{S_q(n,n)}^i(\Delta(\lambda), L^k(\mu))
\]
for $1 \leq i \leq l - 1$. Since $\mu \preceq \lambda$, $\text{Ext}^i_{S_q(n,n)}(\Delta(\lambda), L^k(\mu)) = 0$ by [DDPW08 Prop. C.13 (2)].

(b) By Proposition 5.6, $\overline{\bar{F}}(S(1, \mu')) \cong \Delta(\mu)$. Thus, Theorem 5.8 yields

$$\text{Ext}^i_{kG}(S(1, \lambda'), S(1, \mu')) \cong \text{Ext}^i_{S_q(n,n)}(\Delta(\lambda), \Delta(\mu))$$

for $1 \leq i \leq l - 1$. Since $\mu \preceq \lambda$, $\text{Ext}^i_{S_q(n,n)}(\Delta(\lambda), \Delta(\mu)) = 0$ by [DDPW08 Prop. C.13 (2)].

6 An Application of Theorem 5.8: Ext Groups between Irreducible $k\text{GL}_n(q)$-modules

The $k\text{GL}_n(q)$-modules $S(1, \lambda')$ appearing in Section 5 are closely connected to Specht modules for the Hecke algebra and, consequently, play a key role in the representation theory of $\text{GL}_n(q)$ in non-defining characteristic. In particular, we can apply our Ext results for the modules $S(1, \lambda')$ to study many other Ext groups for $\text{GL}_n(q)$. In this section, we will demonstrate one such application. First, we will describe an algorithm of James [J84, Ch. 20], which determines whether an $kG$-module $S(1, \lambda')$ corresponding to a two-part partition $\lambda'$ is irreducible. We will then use James’ algorithm together with the result of Corollary 5.11(a) in several examples to obtain vanishing results for higher Ext groups between irreducible $k\text{GL}_n(q)$-modules.

6.1 An Irreducibility Criterion for $S(1, \lambda)$

For the remainder of this paper, we let $G = \text{GL}_n(q)$ and $k$ be an algebraically closed field of characteristic $r > 0$, $r \nmid q(q - 1)$. As above, let

$$l = \begin{cases} 
    r & \text{if } |q \pmod{r}| = 1 \\
    |q \pmod{r}| & \text{if } |q \pmod{r}| > 1.
\end{cases}$$

Given a partition $\lambda \vdash n$, let $S(1, \lambda)$ denote the associated indecomposable $kG$-module which maps to a Specht module under an appropriate Hecke functor [DJ89 (3.1)]. In [J84 Ch. 24], James presents an algorithm which may be used to determine whether $S(1, \lambda)$ is irreducible in the case that $\lambda$ has two non-zero parts. We demonstrate this algorithm in the next example.

Example 6.1. Let $n = 6$ and $q = 3$, so that $G = \text{GL}_6(3)$. Suppose that the characteristic of the field $k$ is $r = 13$. The smallest positive integer $i$ such that $13 \mid (3^i - 1)$ is $i = 3$. Thus, in this case, $l = |3 \pmod{13}| = 3$. We will use James’s algorithm to show that $S(1, (3, 3)) = S(1, (3^2))$ is not an irreducible $kG$-module.

---

9James conjectures that an analogous irreducibility criterion holds for partitions $\lambda \vdash n$ having more than two non-zero parts [J84 Conj. 20.5].
Following [JS4, Def. 20.1], we construct the hook graph for the two-part partition \((3^2)\) of 6. We start with the diagram of shape \((3^2)\) (denoted by \([3^2]\)), which has three nodes in the first row and three nodes in the second row.

\[
[3^2] = \begin{array}{ccc}
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\end{array}
\]

We obtain the hook graph of \((3^2)\), we replace each node \((i, j)\) of \([3^2]\) with the hook length

\[
h_{ij} = (3^2)_i + (3^2)'_j + 1 - i - j,
\]

where \((3^2)_i\) denotes the \(i\)th part of \((3^2)\) and \((3^2)'_j\) denotes the \(j\)th part of the dual partition \((3^2)\)' (or, equivalently, the number of entries in the \(j\)th column of \([3^2]\)). Thus, we replace the node \((1, 1)\) in the diagram \([3^2]\) with \(h_{11} = (3^2)_1 + (3^2)'_1 + 1 - 1 - 1 = 3 + 1 - 1 = 4\). We replace the node \((1, 2)\) with \(h_{12} = 3 + 2 + 1 - 1 - 2 = 3\). Continuing in this manner, we obtain the hook graph of \((3^2)\):

\[
\begin{array}{ccc}
4 & 3 & 2 \\
3 & 2 & 1 \\
\end{array}
\]

Next, we use the hook graph of \((3^2)\) to construct a new array \([3^2]\)_{13, 3}. Given a node \((i, j)\) of the diagram \([3^2]\), we check whether \(h_{ij}\) is divisible by \(l = 3\). If \(l \mid h_{ij}\), we replace \(h_{ij}\) with the largest integer \(m\) such that \(r^m\) divides \(h_{ij}\). If \(l \nmid h_{ij}\), we replace \(h_{ij}\) with \(\infty\). In this example, \(h_{12}\) and \(h_{21}\) are the only entries of the hook graph which are divisible by \(l = 3\). The largest integer \(m\) such that \(13^m\) divides \(h_{12} = 3\) is \(i = 0\); thus, we replace \(h_{12}\) by 0. Similarly, we replace \(h_{21}\) by 0. All of the other entries of the hook graph are replaced by \(\infty\).

\[
[3^2]_{13, 3} = \begin{array}{ccc}
\infty & 0 & \infty \\
0 & \infty & \infty \\
\end{array}
\]

By [JS4, Thm. 20.3], \(S(1, (3^2))\) is an irreducible \(kG\)-module if and only if the symbols in any given column of \([3^2]_{13, 3}\) are the same. Since the first and second columns of \([3^2]_{13, 3}\) each contain two different symbols, we conclude that \(S(1, (3^2))\) is not irreducible.

### 6.2 Some Ext Computations for \(\text{GL}_n(q)\)

Corollary 5.11(a) shows that if \(l > 2\) and \(\lambda\) is an \(l\)-restricted partition of \(n\), then

\[
\text{Ext}^i_{kG}(S(1, \lambda'), D(1, \mu)) = 0
\]

for any \(\mu \vdash n\) such that \(\mu \leq \lambda\) and \(1 \leq i \leq l - 1\) (where \(D(1, \lambda)\) is an irreducible unipotent \(kG\)-module in the CPS indexing). Assume, additionally, that \(\lambda\) is a partition of \(n\) with the property that \(S(1, \lambda')\) is irreducible. In this case, \(S(1, \lambda') = D(1, \lambda)\) (where \(D(1, \lambda)\) is, again, indexed following CPS) and Corollary 5.11(a) yields the following Ext vanishing result.

**Proposition 6.2.** Suppose that \(r \nmid q(q-1)\), \(l > 2\), and \(\lambda\) is an \(l\)-restricted partition of \(n\) with the property that \(S(1, \lambda')\) is an irreducible \(kG\)-module. Then, \(\text{Ext}^i_{kG}(D(1, \lambda), D(1, \mu)) = 0\) for all \(\mu \vdash n\) such that \(\mu \leq \lambda\) and all \(i\) such that \(1 \leq i \leq l - 1\).
In the next two examples, we will use Proposition 6.2 along with James’s irreducibility criterion to show that certain Ext groups between irreducible \( kG \)-modules vanish.

**Example 6.3.** Let \( n = 4 \) and \( q = 7 \), so that \( G = \text{GL}_4(7) \). Let \( \text{char}(k) = r = 5 \). The order of \( G \) is \( |G| = (7^4 - 1)(7^4 - 7)(7^4 - 7^2)(7^4 - 7^3) \). Since \( r = 5 \) divides \( (7^4 - 1) \), we have \( r \mid |G| \); therefore, the homological algebra of \( G \) over \( k \) is non-trivial. The smallest integer \( i \) such that \( 5 \mid (7^i - 1) \) is \( i = 4 \); thus, \( \ell = \left( \frac{7}{5} \right) \equiv 4 \mod 5 \). Since \( \ell > 2 \) and \( r \not\mid q(q-1) \), we may apply Proposition 6.2.

The are five partitions of 4: (4), (3,1), (2,2), (2,1,1), and (1,1,1,1). The 4-restricted partitions are (3,1), (2,2), (2,1,1), and (1,1,1,1). To apply the result of Proposition 6.2, we must identify 4-restricted partitions \( \lambda \vdash 4 \) for which \( S(1, \lambda') \) is irreducible. By [J84, Ex. 11.17 (i)], \( S(1, (1^4)') = S(1, (4)) = k \); thus, \( (1^4) \) is one such partition.

Since \((2^2)' = (2^2)\) and \((2,1^2)' = (3,1)\) are two-part partitions, we can apply James’s irreducibility criterion to determine whether \( S(1, (2^2)') = S(1, (2^2)) \) and \( S(1, (2,1^2)') = S(1, (3,1)) \) are irreducible \( kG \)-modules.

\( S(1, (2^2)) \):

The hook graph of \((2^2)\) is shown below.

\[
\begin{array}{ccc}
3 & 2 \\
2 & 1 \\
\end{array}
\]

Since \( l = 4 \) does not divide any entry of the hook graph, the array \([ (2^2)_r, l = [(2^2)]_{5,4} \) contains only the symbol \( \infty \).

\[
[(2^2)]_{5,4} = \infty \infty \infty \infty
\]

James’s irreducibility criterion for two-part partitions indicates that \( S(1, (2^2)) \) is irreducible. Thus, \( S(1, (2^2)) = S(1, (2^2)') = D(1, (2^2)) \) (in the CPS indexing).

\( S(1, (3,1)) \):

In this case, the hook graph is \[
\begin{array}{ccc}
4 & 2 & 1 \\
1 & & \\
\end{array}
\]

and James’s algorithm yields the following array.

\[
[(3,1)]_{5,4} = 0 \infty \infty \infty \infty
\]

Since the first column of \([ (3,1)]_{5,4} \) contains two different symbols, \( S(1, (3,1)) \) is not irreducible.

We cannot use James’s irreducibility criterion to check whether \( S(1, \lambda') \) is irreducible for \( \lambda = (4) \) (\( \lambda' = (1^4) \) has four non-zero parts) and \( \lambda = (3,1) \) (\( \lambda' = (2,1^2) \) has three non-zero parts. To apply Proposition 6.2 with \( \lambda = (1^4) \) and \( \lambda = (2^2) \), we must identify (for each \( \lambda \)) partitions \( \mu \vdash 4 \) such that \( \mu \leq \lambda \). The poset structure \( \leq \) on the set of partitions of 4 is the
dominance order, defined in Section 2.2. If $\lambda = (1^4)$, the only partition $\mu$ with the property $\mu \leq (1^4)$ is $(1^4)$ itself. If $\lambda = (2^2)$, we have $(2^2) \leq (2^2), (2, 1^2) \leq (2^2)$, and $(1^4) \leq (2^2)$. Since $D(1,(1^4)) = k$, Proposition 6.2 yields the following Ext vanishing results:

1. $\text{Ext}^i_{kG}(k, k) = 0$ for $1 \leq i \leq 3$,
2. $\text{Ext}^i_{kG}(D(1,(2^2)), D(1,(2^2))) = 0$ for $1 \leq i \leq 3$,
3. $\text{Ext}^i_{kG}(D(1,(2^2)), D(1,(2,1^2))) = 0$ for $1 \leq i \leq 3$, and
4. $\text{Ext}^i_{kG}(D(1,(2^2)), k) = 0$ for $1 \leq i \leq 3$.

**Example 6.4.** Let $G = \text{GL}_q(3)$ and $r = 13$. (Again, we have chosen $r$ such that $r \mid |G|$ to avoid triviality.) In Example 6.1, we found that $l = 3$.

The partitions of 6 are: $(6), (5,1), (4,2), (4,1^2), (3,2,1), (3,1^2), (2^3), (2^2,1^2), (2,1^4)$, and $(1^6)$. The 3-restricted partitions of 6 are: $(3,2,1), (3,1^2), (2^3), (2^2,1^2), (2,1^4)$, and $(1^6)$. The 3-restricted partitions $\lambda \vdash 6$ such that the dual partition $\lambda'$ has two parts are: $(2^2), (2^2,1^1)$, and $(2,1^4)$. For these partitions $\lambda$, James’s algorithm yields the following irreducibility results.

| $\lambda$      | $\lambda'$      | $S(1,\lambda')$ | Result of algorithm |
|-----------------|-----------------|------------------|---------------------|
| $(1^4)$         | $(3^2)$         | $S(1,(3^2))$    | Not irreducible     |
| $(2^2,1^2)$     | $(4,2)$         | $S(1,(4,2))$    | Irreducible         |
| $(2,1^4)$       | $(5,1)$         | $S(1,(5,1))$    | Not irreducible     |

Thus, James’s irreducibility criterion shows that $S(1,(4,2)) = D(1,(2^2,1^2))$ is irreducible.

We also know that $S(1,(6))$ is irreducible since $S(1,(6)) = D(1,(1^6)) = k$. The only partition $\mu \vdash 6$ such that $\mu \leq (1^6)$ is $\mu = (1^6)$. The partitions $\mu \vdash 6$ such that $\mu \leq (2^2,1^2)$ are $\mu = (2^2,1^2), (2,1^4)$, and $(1^6)$. Thus, Proposition 6.2 yields the following results:

1. $\text{Ext}^i_{kG}(k, k) = 0$ for $1 \leq i \leq 2$,
2. $\text{Ext}^i_{kG}(D(2^2,1^2), D(2^2,1^2)) = 0$ for $1 \leq i \leq 2$,
3. $\text{Ext}^i_{kG}(D(2^2,1^2), D(2,1^4)) = 0$ for $1 \leq i \leq 2$, and
4. $\text{Ext}^i_{kG}(D(2^2,1^2), k) = 0$ for $1 \leq i \leq 2$.

**Remark 6.5.** In this paper, we used the connection between $k\text{GL}_n(q)$-modules and $S_q(n,n)$-modules to obtain a variety of Ext vanishing results. However, there is also potential to use the $q$-Schur algebra to compute non-zero Ext groups for $\text{GL}_n(q)$; we believe that further analysis of the action of the CPS functor $F$ on $kG$-modules will yield new Ext calculations for $\text{GL}_n(q)$.

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References

[BDK01] J. Brundan, R. Dipper, and A. Kleshchev. Quantum linear groups and representations of $\text{GL}_n(\mathbb{F}_q)$. Mem. Amer. Math. Soc. 149 (2001), no. 706.

[CPS88] E. Cline, B. Parshall, and L. Scott. Finite dimensional algebras and highest weight categories. J. reine angew. Math. 391 (1988), 85–99.

[CPS99] E. Cline, B. Parshall, and L. Scott. Generic and q-Rational Representation Theory. Publications of the Research Institute for Mathematical Sciences Kyoto 35 (1999), 31–90.

[DDPW08] B. Deng, J. Du, B. Parshall, and J. Wang. Finite Dimensional Algebras and Quantum Groups. Amer. Math. Soc., Providence, 2008.

[D90] R. Dipper. On Quotients of Hom-Functors and Representations of Finite General Linear Groups, I. J. Algebra 130 (1990), 235–259.

[D98] R. Dipper. On Quotients of Hom-Functors and Representations of Finite General Linear Groups, II. J. Algebra 209 (1998), 199–269.

[DD97] R. Dipper and J. Du. Harish-Chandra vertices and Steinberg’s tensor product theorems for finite general linear groups. Proc. London Math. Soc. 75 (1997), 559–599.

[DJ91] R. Dipper and G. D. James. q-tensor space and q-Weyl modules. Trans. Amer. Math. Soc. 327 (1991) 251–282.

[DJ86] R. Dipper and G. D. James. Representations of Hecke algebras of general linear groups. Proc. London Math. Soc. (3), 52 (1986), 20–52.

[DJ89] R. Dipper and G. D. James. The q-Schur algebra. Proc. London Math Soc. (3) 59 (1989), 23–50.

[Do98] S. Donkin. The q-Schur Algebra. Cambridge University Press, Cambridge, 1998.

[DPS1] J. Du, B. Parshall, and L. Scott. Cells and q-Schur algebras. J. Transformation Groups (3) (1998), 33–44.
[DPS2] J. Du, B. Parshall, and L. Scott. *Quantum Weyl reciprocity and tilting modules*. Commun. Math. Phys. 195 (1998), 321–352.

[GJ11] M. Geck and N. Jacon. *Representations of Hecke Algebras at Roots of Unity*. Algebra and Applications, vol. 15, Springer-Verlag, 2011.

[G80] J. A. Green. *Polynomial representations of GL_n*. Lectures Notes in Math. 830, Springer-Verlag, Berlin, 1980.

[GT11] R. Guralnick and P. H. Tiep. *First cohomology groups of Chevalley groups in cross characteristic*. Ann. of Math. 174 (2011), 543–559.

[J84] G. D. James *Representations of General Linear Groups*. London Math. Soc. Lecture Note Series, Cambridge University press, Cambridge, 1984.

[M95] I. G. Macdonald. *Symmetric Functions and Hall Polynomials, 2nd Edition*. Oxford University Press Inc., New York, 1995.

[PS05] B. Parshall, and L. Scott. *Quantum Weyl reciprocity for cohomology*. Proc. London Math. Soc. (3) 90 (2005), 655–688.

[S98] L. Scott. *Linear and nonlinear group Actions, and the Newton Institute Program*. Algebraic groups and their representations, Proceedings of the 1997 Newton Institute conference on representations of finite and related algebraic groups, edited by R. Carter and J. Saxl, Kluwer Scientific (1998) 1–23.

[Sh20] V. Shalotenko. *Bounds on the dimension of Ext for finite groups of Lie type*. J. Algebra 550 (2020), 266–289.

[Sh18] V. Shalotenko. *In Search of Bounds on the Dimension of Ext between Irreducible Modules for Finite Groups of Lie Type*. PhD dissertation. University of Virginia, Charlottesville, VA, 2018.