Properties of Baryons Containing a Heavy Quark in the Skyrme Model

Zachary Guralnik, Michael Luke and Aneesh V. Manohar

Department of Physics 0319
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0319

Abstract

The properties of baryons containing one heavy quark are studied in the Skyrme model, where they are treated as bound states of a heavy meson with an $SU(2)$ chiral soliton. In the large $N_c$ limit, the baryon spectrum is an infinite tower of degenerate states with isospin and spin of the light degrees of freedom $I = s_\ell = 0, 1, 2, \ldots$, and total spin $s = |s_\ell \pm 1/2|$. Exotic states with no quark model analogues in the large $N_c$ limit are found to be unbound. The $\Sigma_c - \Lambda_c$ mass difference is computed and found to be in good agreement with experiment. The $\Sigma_c \Sigma_c \pi$ and $\Sigma_c \Lambda_c \pi$ coupling constants are calculated to leading order in $N_c$ in terms of $g_A$, the axial coupling constant of the nucleon. We discuss the extension of our results to $SU(3)$ chiral solitons.
1. Introduction

Skyrme originally suggested that baryons are solitons in the non-linear chiral lagrangian used to describe the self-interactions of the Goldstone pions [1]. The solitons of the chiral lagrangian have the correct quantum numbers to be the baryons [2] provided one includes the Wess-Zumino term [3]. The model of QCD baryons as solitons can be used to compute many of their properties [4]. The soliton model was originally considered in Ref. [4] for the case of two light flavours, but can be generalised to three light flavours in two different ways. The first method is to treat the s quark as light, and consider the solitons of the SU(3)$_L \times$ SU(3)$_R$ chiral lagrangian [5][6]. The second method is to treat the strange quark as heavy, and consider baryons containing a s quark as soliton-K meson bound states [7], where the solitons are those of an SU(2)$_L \times$ SU(2)$_R$ chiral lagrangian. Heavy quarks such as the c and b quarks have masses which are not small compared with the scale of chiral symmetry breaking. Thus baryons containing a heavy quark can only be treated by the second method, as bound states of solitons with heavy D and B mesons [8][9]. Callan and Klebanov considered a theory with an SU(2)$_L \times$ SU(2)$_R$ chiral lagrangian as well as K mesons by expanding an SU(3)$_L \times$ SU(3)$_R$ chiral lagrangian, so that the K meson couplings to the pions were determined by chiral SU(3)$_L \times$ SU(3)$_R$ symmetry. They also neglected the K* meson because it is much heavier than the K meson. While these approximations are reasonable for the s quark, they are not valid for the c and b quarks. A better starting point for heavy quarks is to expand about the limit $m_Q \to \infty$. In this limit, the pseudoscalar and vector mesons, such as the D and D* (or B and B*) are degenerate, and must both be included in the chiral lagrangian as matter fields. In addition, the couplings of the heavy mesons are no longer related to the pion couplings by chiral symmetry transformations involving the heavy quark. The analysis of baryons containing a heavy quark in Ref. [8] was done without including D* and B* mesons, and hence does not respect the heavy quark symmetries.

We first review the formalism for heavy-meson–soliton bound states of Ref. [9] that is necessary for the computations discussed in this paper. We then compute the spectrum of baryons containing a heavy quark in the case of two light flavours, using the soliton solution of the SU(2)$_L \times$ SU(2)$_R$ chiral lagrangian. In Sec. 4, the $\Sigma_c - \Lambda_c$ mass difference is computed in terms of the $\Delta - N$ mass difference. The result agrees well with experiment. In Sec. 5, we discuss the quantum number K which can be used to label the soliton–heavy-meson bound states. The energies of the bound states depend only on K in the large $N_c$.
limit, which explains the degeneracy of the spectrum computed in Sec. 3. In Sec. 6, we compute the pion-baryon coupling constants in terms of the $D$-meson–pion coupling constant and $g_A$, the axial vector coupling of the nucleon. In Sec. 7, we generalise the results of the previous sections to the $SU(3)$ case.

2. The Formalism for Studying Soliton-Meson Bound States

The basic formalism for studying properties of baryons containing a heavy quark in the soliton framework was described in detail in [9]. In the limit $m_Q \to \infty$, QCD has a heavy quark spin-flavour symmetry that determines many of the properties of hadrons containing a heavy quark [10] [11]. In the heavy quark limit, the total angular momentum of the light degrees of freedom (light quarks and gluons) $\vec{S}_\ell$ is conserved, where

$$\vec{S}_\ell = \vec{S} - \vec{S}_Q,$$

(2.1)

$\vec{S}$ is the total spin, and $\vec{S}_Q$ is the spin of the heavy quark. The lowest lying mesons containing a heavy quark with $Q\bar{q}_a$ ($q_1 = u$, $q_2 = d$, $q_3 = s$) flavour quantum numbers have $s_\ell$, the eigenvalue of $S_\ell^2 = s_\ell(s_\ell + 1)$ equal to 1/2, and come in a degenerate doublet containing a spin zero pseudoscalar and a spin one vector. For $Q = c$, these are the $D$ and $D^*$ mesons. The heavy meson field for the ground state $Q\bar{q}_a$ mesons is written as a $4 \times 4$ bispinor matrix,

$$H_a = \left(1 + \frac{\not{v}}{2}\right) \left[P_{a\mu}^* \gamma^\mu - P_a \gamma_5\right],$$

(2.2)

where $v^\mu$ is the heavy quark four-velocity, $v^2 = 1$. The fields $P_a$ and $P_{a\mu}^*$ destroy the heavy pseudoscalar and vector particles that comprise the ground state $s_\ell = 1/2$ doublet, and satisfy the constraint $P_{a\mu}^* v^\mu = 0$. The transformation rule for the $H$ field under $SU(2)_Q$ heavy quark spin symmetry is

$$H_a \to S H_a,$$

(2.3)

where $S \in SU(2)_Q$, and the transformation rule under chiral $SU(3)_L \times SU(3)_R$ is

$$H_a \to (HR^\dagger)_a,$$

(2.4)

where we will use the primed basis for the $H$ fields defined in Ref. [9]. It is also convenient to define

$$\overline{H}^a = \gamma^0 H^\dagger_a \gamma^0 = \left[P_{a\mu}^* \gamma^\mu + P_a \gamma_5\right] \frac{1 + \not{\gamma}}{2}.$$
The Goldstone boson fields have the $SU(3)_L \times SU(3)_R$ transformation law

$$\Sigma(x) \to L \Sigma(x) R^\dagger. \quad (2.6)$$

The $\Sigma$ field can be written in terms of the pion fields as

$$\Sigma(x) = e^{2iM/f}, \quad (2.7)$$

where $M$ is the Goldstone-boson matrix

$$M = \begin{bmatrix}
\pi^0/\sqrt{2} + \eta^0/\sqrt{6} & \pi^+ & K^+
\pi^- & -\pi^0/\sqrt{2} + \eta^0/\sqrt{6} & K^0
K^- & K^0 & -2\eta^0/\sqrt{6}
\end{bmatrix}, \quad (2.8)$$

and the pion decay constant $f \approx 132$ MeV. The transformation rules under parity are

$$H_a(x^0, \vec{x}) \to \gamma^0 H_b(x^0, -\vec{x}) \gamma^0 \Sigma^\dagger_a(x^0, -\vec{x}), \quad (2.9)$$

and

$$\Sigma(x^0, \vec{x}) \to \Sigma^\dagger(x^0, -\vec{x}). \quad (2.10)$$

The chiral lagrangian density for heavy-meson–pion interactions is

$$\mathcal{L} = -i \text{Tr} \bar{H} v_\mu \partial^\mu H + \frac{i}{2} \text{Tr} \bar{H} H v^\mu (\Sigma^\dagger \partial_\mu \Sigma) + \frac{ig}{2} \text{Tr} \bar{H} H \gamma^\nu \gamma^5 (\Sigma^\dagger \partial_\nu \Sigma) + \ldots, \quad (2.11)$$

where the ellipsis denotes terms with more derivatives and the trace is over spinor and flavour indices. Eq. (2.11) is the most general lagrangian density invariant under chiral $SU(3)_L \times SU(3)_R$, heavy quark spin symmetry and parity. It is easy to generalise this lagrangian density to include explicit $SU(3)_L \times SU(3)_R$ symmetry breaking from $u, d,$ and $s$ quark masses and explicit $SU(2)_Q$ symmetry breaking from $\Lambda_{QCD}/m_Q$ effects. The coupling $g$ determines the $D^* \to D \pi$ decay width,

$$\Gamma(D^* \to D^0 \pi^+) = \frac{f^2}{6\pi} \frac{g^2}{f^2} |\vec{p}_\pi|^3. \quad (2.12)$$

The present experimental limit \[ \Gamma(D^* \to D^0 \pi^+) \lesssim 72 \text{ keV}, \] implies that $g^2 \lesssim 0.4$. 

4
3. Masses of Baryons containing a Heavy Quark: The SU(2) Case

The masses of baryons containing a heavy quark are computed in this section for the case of two light flavours. The algebra is considerably simpler than for the SU(3) case, which is studied in Sec. 7. We will use the fields and interaction lagrangian of the previous section, with the light quark index restricted to the values 1,2.

The soliton solution of the SU(2)\(_L\) × SU(2)\(_R\) chiral lagrangian is

\[ \Sigma = A \Sigma_0 A^{-1}, \]  

where

\[ \Sigma_0 = \exp \left[ iF(\|\vec{x}\|) \vec{x} \cdot \vec{\tau} \right], \]  

and \( A \in SU(2) \) is the collective coordinate associated with isospin transformations of the soliton solution \( \Sigma_0 \). The radial shape function \( F(r) \) satisfies \( F(0) = -\pi \) and \( F(\infty) = 0 \) for a soliton with baryon number one. The soliton shape \( F(r) \) depends on the details of higher derivative terms in the chiral lagrangian. In the quantum theory, baryons have wavefunctions that are functions of the matrix \( A \in SU(2) \). The wavefunctions are

\[ \psi_{Ram}(A) = (-1)^{R+m} \sqrt{\text{dim} R} D_{a-m}^{(R)}(A), \]  

for a state with isospin \( I = R, I_3 = a, J = R, \) and \( J_3 = m \). The matrices \( D^{(R)} \) are the representation matrices for \( SU(2) \), and we have normalised the measure on the \( SU(2) \) group so that

\[ \int_{SU(2)} dA = 1. \]  

In QCD, the only soliton states in the theory are those with \( 2I = 2J = \text{odd} \).

In the large \( N_c \) limit, the soliton is very heavy, and the semiclassical approximation is valid. In this limit, time derivatives can be neglected to leading order, so that the interaction hamiltonian is

\[ H_I = -\frac{ig}{2} \int d^3 \vec{x} \ Tr (\bar{\Sigma} \gamma_5 [\Sigma] \partial_j \gamma_j) + \ldots, \]  

\[ \text{1 The baryon number current in Ref. } 4 \text{ should have the opposite sign, which changes the sign of } F(0) \text{ for a baryon relative to that used in } 4. \]
with $\Sigma$ given by Eq. (3.1). In the limit that the $H$ field is very heavy, the interaction energy is determined by the value of Eq. (3.5) with the $H$ field at the origin [9]. Using the expansion

$$F(r) = F(0) + rF'(0) + \ldots = -\pi + rF'(0) + \ldots,$$

in Eq. (3.2) gives

$$\Sigma_0 = -1 - i\vec{r} \cdot \vec{x} F'(0) + \ldots,$$

so that the interaction hamiltonian is

$$H_I = \frac{gF'(0)}{2} \int d^3 \vec{x} \, \text{Tr} \bar{H} H \gamma^j \gamma_5 \tau^j A^{-1}$$

$$= \frac{gF'(0)}{4} \int d^3 \vec{x} \, \text{Tr} \bar{H} H \gamma^j \gamma_5 \tau^k \text{Tr} A \tau^j A^{-1} \tau^k.$$

The isospin operator on the $H$ field is

$$I^k_H \, H = -H \frac{\tau^k}{2},$$

and the spin operator of the light degrees of freedom acting on $H$ is

$$S^k_{\ell H} \, H = -H \frac{\sigma^k}{2}.$$

Thus the interaction hamiltonian of Eq. (3.8) can be written in the form

$$H_I = gF'(0) \, I^k_H \, S^j_{\ell H} \, \text{Tr} A \tau^j A^{-1} \tau^k,$$

using the fact that $H \gamma^j \gamma^5 = -H \sigma^j$ in the rest frame of the $H$ field, and noting that $-\text{Tr} \bar{H} H$ creates $H$ particles with probability +1. The interaction hamiltonian gives a binding energy which is of order $N^0_c$. The total energy of the bound state is the interaction energy plus the mass of soliton (which is of order $N_c$) and the mass of the $D$ meson (which is of order $N^0_c$). Note that the interaction hamiltonian also produces a distortion in the shape function $F$ in the presence of an $H$ particle. However, this is an effect of order $1/N_c$, since the coupling constant $g$ is of order one, whereas the terms in the lagrangian with no $H$ field are of order $N_c$.

The problem of determining the energy of heavy-meson–soliton bound states is reduced to the problem of determining the spectrum of the interaction hamiltonian Eq. (3.11). The

---

2 This was also noted by J. Hughes.
spectrum will respect the heavy quark symmetry, since $\bar{S}_Q$ commutes with $H_I$. In Ref. [3], the soliton states were restricted to the nucleon subspace. In this case, the operator $\text{Tr} A^{\tau_j} A^{-1} \tau^k$ can be replaced by $-8 I^j_{\Sigma} S^{j}_{\Sigma}/3$, where $I_{\Sigma}$ and $S_{\Sigma}$ are the isospin and spin of the light degrees of freedom acting on the soliton state [4]. The spin operator in the soliton sector is the same as the operator for the spin of the light degrees of freedom in the soliton sector, since the soliton does not contain any heavy quarks. Thus, in this case we find that
\[
H_I = -\frac{8gF'(0)}{3} I^k_{H} S^j_{\ell H} I^k_{\Sigma} S^j_{\Sigma}
= -\frac{2gF'(0)}{3} (I^2 - I_{H}^2 - I_{\Sigma}^2) (S^2_{\ell} - S^2_{\ell H} - S^2_{\Sigma})
\] (3.12)
using $I = I_{\Sigma} + I_{H}$, and $S_{\ell} = S_{\Sigma} + S_{\ell H}$. This is the result given in [3].

In the large $N_c$ limit, all the collective excitations of the soliton are degenerate, and one cannot restrict the soliton states to the nucleon subspace. The interaction hamiltonian will mix different isospin excitations of the soliton. Since the interaction hamiltonian Eq. (3.11) involves only the light degrees of freedom of the $H$ field, it is convenient to treat the $H$ field as having $s_{\ell} = 1/2$, and combine the spin of the heavy quark with the spin of the light degrees of freedom at the end of the calculation. The interaction hamiltonian commutes with total isospin and total spin, so it is useful to combine soliton states with the light degrees of freedom in $H$ into states which are eigenstates of $I$ and $S_{\ell}$, \[
|I a \ s_{\ell} m; R \ I_{H} \ s_{\ell H} p\rangle = |R b n\rangle \ |I_{H} c \ s_{\ell H} p\rangle \left(\begin{array}{c} R \\
 b \\
 c \\
 a \\
 n \\
 p \\
 m \\
\end{array}\right) \left(\begin{array}{c} I_{H} \\
 I \\
 I_{\ell H} \\
 I_{\ell} \\
 s_{\ell H} \\
 s_{\ell} \\
 s_{\ell \ell H} \\
 s_{\ell \ell} \\
\end{array}\right).
\] (3.13)
where $|R b n\rangle$ denotes a soliton with $I = s_{\ell} = R$, $I_3 = b$, $s_{\ell 3} = n$, and $|I_{H} c \ s_{\ell H} p\rangle$ denotes the light degrees of freedom of $H$ with $I = I_{H}$, $I_3 = c$, $s_{\ell} = s_{\ell H}$ and $s_{\ell 3} = p$. We only need to consider the case where $I_{H} = s_{\ell H} = 1/2$. The soliton wavefunctions for the states $|R a m\rangle$ are given explicitly in Eq. (3.3). Throughout this paper, soliton states will be denoted by $|\rangle$, the light degrees of freedom of the heavy meson field $H$ will be denoted by $|\rangle$, and the bound state of the two will be denoted by $|\rangle$.

We can now compute the matrix elements of (3.11) between the states given in Eq. (3.13), \[
\langle I' a' \ s_{\ell}' m'; R' \ I_{H} \ s_{\ell H} | H_I | I a \ s_{\ell} m; R \ I_{H} \ s_{\ell H} \rangle = gF'(0)
\times \left(\begin{array}{c} R \\
 b \\
 c \\
 a \\
 n \\
 p \\
 m \\
\end{array}\right) \left(\begin{array}{c} I_{H} \\
 I \\
 I_{\ell H} \\
 I_{\ell} \\
 s_{\ell H} \\
 s_{\ell} \\
 s_{\ell \ell H} \\
 s_{\ell \ell} \\
\end{array}\right) \left(\begin{array}{c} R' \\
 b' \\
 c' \\
 a' \\
 n' \\
 p' \\
 m' \\
\end{array}\right) \left(\begin{array}{c} I' \\
 R' \\
 I_{H} \\
 I_{\ell H} \\
 s_{\ell H} \\
 s_{\ell} \\
 s_{\ell \ell H} \\
 s_{\ell \ell} \\
\end{array}\right) \left(\begin{array}{c} R' \\
 b' \\
 n' \\
 p' \\
 m' \\
\end{array}\right) \left(\begin{array}{c} I_{H} c \ s_{\ell H} p' \rangle \ I_{H}^k \ S_{\ell H}^j \ |I_{H} c \ s_{\ell H} p\rangle \right).
\] (3.14)
The matrix elements in the $H$ sector are the matrix elements of the generators of $SU(2)$ in the given irreducible representation,

$$\{ I_H c' s_{\ell H} p' | I_H^k S^j_{\ell H} | I_H c s_{\ell H} p \} = T^{k(I_H)}_{c'c} T^{j(s_{\ell H})}_{p'p},$$

(3.15)

where $T^{k(R)}$ is the generator in the irreducible representation $R$. The generators can be written in terms of Clebsch-Gordan coefficients using the Wigner-Eckart theorem,

$$T^{k(R)}_{ba} = \sqrt{R(R+1)} \left( \begin{array}{c} R \\ a \\ R \\ b \end{array} \right).$$

(3.16)

The matrix elements in the soliton sector can be evaluated in terms of the representation matrices of the adjoint representation using the identity

$$\text{Tr} A \tau^j A^{-1} \tau^k = 2 D^{(1)}_{kj}(A).$$

(3.17)

The product of two representation matrices can be written in terms of a single representation matrix,

$$D^{(R)}_{ab}(A) D^{(S)}_{cd}(A) = \left( \begin{array}{c} R \\ a \\ S \\ e \\ T \\ c \\ e \\ f \end{array} \right) \left( \begin{array}{c} R \\ b \\ S \\ d \\ T \\ b \end{array} \right) D^{(T)}_{ef}(A),$$

(3.18)

and the integral over the $SU(2)$ group of the product of three $D$ matrices can be evaluated by using Eq. (3.18) and the orthogonality relation

$$\int_{SU(2)} D^{(R)}_{ab}(A) D^{(S)\ast}_{cd}(A) = \frac{1}{\dim R} \delta_{RS} \delta_{ac} \delta_{bd}.$$ 

(3.19)

The soliton matrix element needed for Eq. (3.14) is then

$$(R' b' n'| \text{Tr} A \tau^j A^{-1} \tau^k | R b n) = 2 (-1)^{R+n+R'+n'} \sqrt{\dim R \dim R'} \int_{SU(2)} D^{(R')}_{b'-n'}(A) D^{(R)}_{b-n}(A) D^{(1)}_{kj}(A)$$

$$= 2 \sqrt{\frac{\dim R}{\dim R'}} \left( \begin{array}{c} R \\ b \\ k \\ b' \end{array} \right) \left( \begin{array}{c} R \\ n \\ 1 \\ -n' \end{array} \right),$$

(3.20)

using the fact that the adjoint representation is real. The matrix element of the interaction hamiltonian has thus been evaluated in terms of sums of products of eight Clebsch-Gordan
coefficients by combining Eqs. (3.14)–(3.20). These can be evaluated explicitly in terms of $6j$-symbols to give

$$
\langle I' a' s'_s m'; R' I_H s_{\ell H} | H_I | I a s_\ell m; R I_H s_{\ell H} \rangle = 
-2gF'(0)(-1)^{I+I_H+s_\ell+s_{\ell H}+2R} \times \sqrt{\dim R \dim R'} \dim I_H \dim s_{\ell H} \dim I_H(I_H+1)s_{\ell H}(s_{\ell H}+1)
$$

$$
\times \left\{ \begin{array}{ccc}
R & 1 & R' \\
I_H & I & I_H
\end{array} \right\} \left\{ \begin{array}{ccc}
R & 1 & R' \\
s_{\ell H} & s_\ell & s_{\ell H}
\end{array} \right\} \delta_{I'I'} \delta_{aa'} \delta_{s's_{\ell}} \delta_{mm'}.
$$

(3.21)

The interaction Hamiltonian is diagonal in isospin and in the spin of the light degrees of freedom.

It is now straightforward to determine the energies of the lightest heavy quark baryon states. The light degrees of freedom have the spin and flavour quantum numbers of an antiquark ($I_H = 1/2, s_{\ell H} = 1/2$), and will be denoted by $\vec{q}$. Since $I_H$ and $s_{\ell H}$ in Eq. (3.21) are both fixed to be $1/2$ and the Hamiltonian is diagonal in $a$ and $m$, we will drop those labels from now on, and denote the states $| I a s_\ell m; R I_H = 1/2 s_{\ell H} = 1/2 \rangle$ by $| I s_{\ell}; R \rangle$. States which are linear combinations of $| I s_{\ell}; R \rangle$ for different values of $R$ will be denoted by $| I s_{\ell} \rangle$. The nucleon $N$ has $I = 1/2$, $s_\ell = 1/2$ so the $N \otimes \vec{q}$ states have $I = 0, 1$ and $s_\ell = 0, 1$. Similarly, the $\Delta \otimes \vec{q}$ states have $I = 1, 2$ and $s_\ell = 1, 2$, etc. The interaction Hamiltonian mixes the $I = 1$, $s_\ell = 1$ states in the nucleon and delta sectors, but the other states in the $N\vec{q}$ sector do not mix with any other soliton-$\vec{q}$ states. The energies of the $| 0 0; \frac{1}{2} \rangle$, $| 1 0; \frac{1}{2} \rangle$, and $| 0 1; \frac{1}{2} \rangle$ states are obtained from (3.21) as $-3gF'(0)/2$, $gF'(0)/2$ and $gF'(0)/2$ respectively, which are the binding energies given in Ref. [9]. In the $I = 1$, $s_\ell = 1$ channel, we have the interaction Hamiltonian in the $| 1 1; \frac{1}{2} \rangle$ basis

$$
H_I = -\frac{gF'(0)}{6} \left( \begin{array}{cc}
1 & 4\sqrt{2} \\
4\sqrt{2} & 5
\end{array} \right).
$$

(3.22)

In Ref. [9], the $\Delta$ states were omitted, so that the energy of the $| 1 1 \rangle$ state was the $1\overline{1}$ matrix element in Eq. (3.22), $-gF'(0)/6$. In the large $N_\ell$ limit, where the $N$ and $\Delta$ are degenerate, we have two $| 1 1 \rangle$ states which are the eigenstates of $H_I$ in Eq. (3.22),

$$
| 1 1 \rangle_0 = \sqrt{\frac{3}{5}} \left| 1 1; \frac{1}{2} \right| + \sqrt{\frac{2}{5}} \left| 1 1; \frac{3}{2} \right|,
$$

$$
| 1 1 \rangle_1 = \sqrt{\frac{2}{3}} \left| 1 1; \frac{1}{2} \right| - \sqrt{\frac{1}{3}} \left| 1 1; \frac{3}{2} \right|,
$$

(3.23)

with energies $-3gF'(0)/2$ and $gF'(0)/2$ respectively. The choice of subscripts for the $| 1 1 \rangle$ states will be explained in Sec. 5. In the large $N_\ell$ limit, we see that the states $| 0 0; \frac{1}{2} \rangle$
and $|1\ 1\rangle_0$ are degenerate. The state $|0\ 0; \frac{1}{2}\rangle$ when combined with the heavy quark is the spin-1/2 $\Lambda_c$ baryon, and the state $|1\ 1\rangle_0$ when combined with the the heavy quark is the degenerate multiplet of the spin-1/2 $\Sigma_c$ and the spin-3/2 $\Sigma^*_c$. We see that in the large $N_c$ limit, the $\Lambda_c$ and $\Sigma_c$ are degenerate. The $\Sigma_c - \Lambda_c$ mass splitting is studied in the next section. We have also found that the $|0\ 1; \frac{1}{2}\rangle$, $|1\ 0; \frac{1}{2}\rangle$ and $|1\ 1\rangle_1$ states are degenerate. These are exotic baryons with no quark model analogues; it is reassuring to see that, for positive $g$, they are unbound. One can use Eq. (3.21) to compute the interaction hamiltonian in the different $I$, $s$ sectors. All states in the spectrum have an energy of either $-3gF'(0)/2$ or $gF'(0)/2$. The reason for this degeneracy will be explained in Sec. 5.

4. The $\Sigma_c - \Lambda_c$ Mass Splitting

The $\Sigma_c$ and $\Lambda_c$ are degenerate at leading order in $1/N_c$, but at subleading order there are two terms which break the degeneracy. The first is the rotational kinetic energy of the soliton which splits the nucleon-delta degeneracy. This term has a coefficient of order $N_c$ in the lagrangian, and has two time derivatives. Since each time derivative brings a factor of $1/N_c$ suppression, this term produces an energy splitting of order $1/N_c$. The degeneracy is also broken by the second term in Eq. (2.11). Its coefficient is of order $N^0_c$ and it has one time derivative, so it too produces an energy splitting of order $1/N_c$. However, for the $SU(2)$ soliton

$$\Sigma^+ \frac{d}{dt} \Sigma$$

vanishes at the origin where the $H$ particle is bound. Thus to leading order in the derivative expansion the interaction term can be neglected, and only the $N - \Delta$ splitting contributes. The interaction hamiltonian in the $|1\ 1; \frac{1}{2}\rangle$, $|1\ 1; \frac{3}{2}\rangle$ sector can be written as

$$H_I = -\frac{gF'(0)}{6} \left( \begin{array}{cc} 1 & 4\sqrt{2} \\ 4\sqrt{2} & 5 \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ 0 & \Delta M \end{array} \right),$$

where $\Delta M$ is the $\Delta - N$ mass difference.

We will determine the energies of the lowest heavy quark baryons using the experimental value for the $\Delta - N$ mass difference and adjusting the unknown value of $gF'(0)$ to fit the observed value of the $\Lambda_c$ mass,

$$m_{\Lambda_c} = 2285 \text{ MeV} = m_N + m_H - 3gF'(0)/2,$$
which implies $gF'(0) = 419 \text{ MeV}$ using the weighted average for the $D - D^*$ multiplet of $m_H = (3D^* + D)/4 = 1975 \text{ MeV}$. Thus we find that the energies of the $|1 0; \frac{1}{2} \rangle$ and $|0 1; \frac{1}{2} \rangle$ states are $3124 \text{ MeV}$. These states are exotic states, and the soliton model predicts that they are unbound. The eigenvalues of (4.1) are, to first order in $\Delta M$,

$$E_- = -\frac{3gF'(0)}{2} + \frac{2}{3} \Delta M$$

$$E_+ = \frac{gF'(0)}{2} + \frac{1}{3} \Delta M.$$  

Thus one of the $|1 1 \rangle$ states is unbound, and we have the relation\(^3\)

$$\overline{m}_{\Sigma_c} - m_{\Lambda_c} = \frac{2}{3} \Delta M,$$  

where we have defined $\overline{m}_{\Sigma_c}$ as the $\Sigma_c - \Sigma_c^*$ multiplet average mass $(2m_{\Sigma_c^*} + m_{\Sigma_c})/3$. This gives the parameter free prediction

$$\overline{m}_{\Sigma_c} = 2480 \text{ MeV}.$$  

The $\Sigma_c^*$ mass has not been measured, but the experimental value of the $\Sigma_c$ mass is 2453 MeV, so this agrees well with experiment.

We have computed the energies of baryons containing a heavy quark by expanding the effective action in derivatives. In the semiclassical expansion, time derivatives are considered to be small, but space derivatives are not necessarily small. Terms in the interaction hamiltonian with no time derivatives but an arbitrary number of space derivatives can produce only two kinds of interaction terms, which can be written in the form

$$H_I = V_0(F) + V_1(F) \text{ Tr } A^j A^{-1} A^k I_H^j S_{\mu H},$$  

where $V_0$ and $V_1$ are functionals of the soliton shape function $F$ and are of order $N_c^0$. The leading contribution to $V_1$ is first order in derivatives, $V_1 = gF'(0)$, as we determined in Eq. (3.11). The leading contribution to $V_0$ is second order in space derivatives, e.g. from a term of the form $\text{ Tr } \overline{H} H \text{ Tr } \partial^\mu \Sigma \partial_\mu \Sigma^\dagger$. Terms with one time derivative and two $H$ fields are of order $1/N_c$, and can be written in the form

$$H_I = V_2(F) I_H^k I_{\Sigma_c}^k,$$  

\(^3\) This result is true in the constituent quark model, and was also obtained in the Skyrme model in Ref. [7]. However, Callan and Klebanov found a $\Sigma_c^* - \Sigma_c$ mass difference in the heavy quark limit because they did not include $D^*$ mesons.
where $V_2$ is of order $1/N_c$. The leading term with one time derivative and no space derivatives from Eq. (2.11) vanishes, so $V_2(F)$ starts at second order in space derivatives. Terms with two time derivatives and no $H$ fields have the form

$$\lambda(F) \frac{I^2_S}{2},$$

where the functional $1/2\lambda$ is the moment of inertia of the soliton, starts off at zero space derivatives, and is of order $N_c$. The value of $\lambda$ can be chosen to fit the observed $N - \Delta$ mass difference. That leaves three parameters $V_0$, $V_1$ and $V_2$ to fit the observed $\Lambda_c$ and $\Sigma_c$ masses, so there is no predictive power. If we assume that we also have a systematic expansion in space derivatives, then we can neglect $V_0$ and $V_2$ relative to $V_1$, and we get the predictions discussed in the preceding paragraph. Finally, one can use the value $V_1 = gF'(0)$ combined with the slope of the shape function $F'(0) \simeq 700$ MeV used in Ref. [4] to get $g^2 \simeq 0.36$ compared with the experimental limit $g^2 < 0.4$.

5. Masses in the Large $N_c$ Limit: The Quantum Number $K$

In the previous section, we noticed that the $|0 0; \frac{1}{2}\rangle$ and $|1 1\rangle_0$ states were degenerate, as were the $|1 0; \frac{1}{2}\rangle$, $|0 1; \frac{1}{2}\rangle$ and $|1 1\rangle_1$ states. We will derive this result by a different method, which is useful in generalising the results to $SU(3)$. The interaction hamiltonian we consider has the form

$$H_I = V_1(F) I^k_H S^j_{\ell H} \text{ Tr } A \tau^j A^{-1} \tau^k,$$

(5.1)

with $V_1$ an arbitrary functional of $F$. In the previous section, we diagonalised this hamiltonian by considering states $|I s_\ell; R\rangle$ which were eigenstates of total isospin and total light-spin built from products of soliton and $\bar{q}$ states. In this section, we consider instead the states

$$|\Sigma_0\rangle |a m\rangle$$

(5.2)

which are the tensor product of a soliton with $A = 1\bar{1}$ and a $\bar{q}$ state with $I_3 = a$ and $s_{\ell 3} = m$. Acting on this state with $H_I$ gives

$$H_I |\Sigma_0\rangle |a m\rangle = V_1 (2 I_H \cdot S_{\ell H}) |\Sigma_0\rangle |a m\rangle.$$

(5.3)

\footnote{i.e. the wave function of the collective coordinate is $\psi(A) = \delta(A = 1)$}
It is convenient to define $\vec{K} = \vec{I} + \vec{S}_\ell$, and use $\vec{q}$ states which are representations of $K$\footnote{Many of the Skyrme model predictions follow from invariance of $|\Sigma_0\rangle$ under $K$. This invariance was used in Ref. \cite{9} to show that the quark model and Skyrme model are equivalent in the large $N_c$ limit. It has also been used to obtain relations between pion-nucleon scattering amplitudes \cite{10}. $K$ is equal to $I + S_\ell$ only for solitons $|\Sigma_0\rangle$ which have $A = 1$. For the generic soliton configuration $|A\Sigma_0A^{-1}\rangle$ with collective coordinate $A$, $K$ is equal to $\mathcal{R}(A)I\mathcal{R}(A^{-1}) + S_\ell$ where $\mathcal{R}(A)$ is an isospin rotation by $A$. Terms in the chiral lagrangian with no time derivative are invariant under $K$.}. Since $\vec{q}$ has $I_H = 1/2$ and $s_{\ell H} = 1/2$, we have states with $K = 0, 1$, which will be labeled $|Kk\rangle$. This allows us to diagonalise the interaction hamiltonian,

$$H_I |\Sigma_0\rangle |Kk\rangle = V_1 (2 I_H \cdot S_\ell) |\Sigma_0\rangle |Kk\rangle = V_1 (K^2_H - I^2_H - S_\ell^2) |\Sigma_0\rangle |Kk\rangle.$$ \hfill (5.4)

We see that the states with $K = 1$ have energy $V_1/2$ and that states with $K = 0$ have energy $-3V_1/2$, which are precisely the allowed energies found in the previous section. We can then find states with definite values of total isospin and light-spin by applying isospin and light-spin projection operators to $|\Sigma_0\rangle |Kk\rangle$.

$$|Ia s_\ell m; K k b m\rangle \propto P^R_{ab} (I) P^S_{mn} (S_\ell) |\Sigma_0\rangle |Kk\rangle$$ \hfill (5.5)

where the $SU(2)$ projection operators are defined by

$$P^R_{ab}(X) = \int_{SU(2)} dg D^{(R)}_{ab}(g) \hat{U}_X(g),$$ \hfill (5.6)

where $\hat{U}_X(g)$ is the group transformation operator with generator $X$. The isospin projector $\hat{U}_I$ is the exponential of the isospin generators, and the spin projector $\hat{U}_{S_\ell}$ is the exponential of the spin generators of the light degrees of freedom. We will also need $\hat{U}_K$ which is the exponential of $K = I + S_\ell$. The analysis of the allowed states is now a generalisation of the results of \cite{3}. The soliton $|\Sigma_0\rangle$ is invariant under the action of $I + S_\ell$, which places constraints on which states can be projected out in Eq. (5.5) because of the identity

$$\hat{U}_I(g) \hat{U}_{S_\ell(h)} |\Sigma_0\rangle = \hat{U}_I(gh^{-1}) |\Sigma_0\rangle.$$ \hfill (5.7)
The isospin operator can be written as the product of the isospin operators on the soliton and \( \overline{q} \), and similarly for the spin operators, so that

\[
| I a \ s_\ell \ m; K \ k \ b \ n \rangle = \int_{\text{SU}(2)} dg \ D_{ab}^{(R)}(g) \int_{\text{SU}(2)} dh \ D_{mn}^{(s_\ell)}(h) \\
\times \hat{U}_I(g) \hat{U}_{S_\ell}(h) |\Sigma_0\rangle \hat{U}_I(g) \hat{U}_{S_\ell}(h) |K \ k\rangle
\]

\[
= \int_{\text{SU}(2)} dg \ D_{ab}^{(R)}(g) \int_{\text{SU}(2)} dh \ D_{mn}^{(s_\ell)}(h) \\
\times \hat{U}_I(gh^{-1}) |\Sigma_0\rangle \hat{U}_I(g) \hat{U}_{S_\ell}(h) |K \ k\rangle,
\]

using Eq. (5.7). Replacing the dummy group element \( g \) by \( gh \), and using the group property of the representation matrices and the \( \hat{U} \) operators, we get

\[
| I a \ s_\ell \ m; K \ k \ b \ n \rangle = \int_{\text{SU}(2)} dg \int_{\text{SU}(2)} dh \ D_{ac}^{(R)}(g) D_{cb}^{(R)}(h) D_{mn}^{(s_\ell)}(h) \\
\times \hat{U}_I(g) |\Sigma_0\rangle \hat{U}_I(g) \hat{U}_{S_\ell}(h) |K \ k\rangle.
\]

We also have the identity

\[
U_I(h) U_{S_\ell}(h) |K \ k\rangle = U_K(h) |K \ k\rangle = |K \ k'\rangle D_{k'k}^{(K)}(h),
\]

since \( K = I + S_\ell \), and \( |K \ k\rangle \) is an irreducible representation under \( K \). Thus Eq. (5.9) can be simplified to

\[
| I a \ s_\ell \ m; K \ k \ b \ n \rangle = \int_{\text{SU}(2)} dg \int_{\text{SU}(2)} dh \ D_{ac}^{(R)}(g) D_{cb}^{(R)}(h) D_{mn}^{(s_\ell)}(h) \\
\times \hat{U}_I(g) |\Sigma_0\rangle \hat{U}_I(g) |K \ k'\rangle
\]

\[
= \frac{1}{\dim K} \left( \begin{array}{c} I \\ c \\ m \end{array} \right | K \left( \begin{array}{c} I \\ s_\ell \\ k \end{array} \right) \left( \begin{array}{c} I \\ b \\ n \end{array} \right | K \left( \begin{array}{c} I \\ s_\ell \\ k \end{array} \right) \int_{\text{SU}(2)} dg \ D_{ac}^{(R)}(g) \\
\times \hat{U}_I(g) |\Sigma_0\rangle \hat{U}_I(g) |K \ k'\rangle.
\]

Multiplying both sides of the equation by \( \left( \begin{array}{c} I \\ b \\ n \end{array} \right | K \left( \begin{array}{c} I \\ s_\ell \\ k \end{array} \right) \), we find

\[
| I a \ s_\ell \ m \rangle_K = \lambda \left( \begin{array}{c} I \\ b \\ n \end{array} \right | K \left( \begin{array}{c} I \\ s_\ell \\ k \end{array} \right) | I a \ s_\ell \ m; K \ k \ b \ n \rangle
\]

\[
= \lambda \int_{\text{SU}(2)} dg \ D_{ac}^{(I)}(g) \hat{U}_I(g) |\Sigma_0\rangle \hat{U}_I(g) |K \ k\rangle \left( \begin{array}{c} I \\ c \\ m \end{array} \right | K \left( \begin{array}{c} I \\ s_\ell \\ k \end{array} \right). \quad (5.11)
\]

14
which defines the state \(|I \, a \, s_\ell \, m_\ell\rangle_K\). \(\lambda\) is a normalisation constant which has been inserted so that the state has unit norm. To determine \(\lambda\), consider the overlap

\[
K' \langle I' \, a' \, s'_\ell \, m' \mid I \, a \, s_\ell \, m_\ell \rangle_K = \lambda' \lambda \int_{SU(2)} dg \int_{SU(2)} dg' \, D_{ac}^{(I)}(g) \, D_{ac'}^{(I')}(g')
\times (\Sigma_0 |\hat{U}_I(g')^{-1}\hat{U}_I(g)| \Sigma_0) \{K' \, k'\mid |K\, k\}
\times \left( I \mid s_\ell \mid K \right) \left( I' \mid s'_{\ell'} \mid K' \right).
\]

The orthogonality of soliton states with different rotations reduces the double integral to a single integral,

\[
K' \langle I' \, a' \, s'_\ell \, m' \mid I \, a \, s_\ell \, m_\ell \rangle_K = \lambda' \lambda \int_{SU(2)} dg \, D_{ac}^{(I)}(g) \, D_{ac'}^{(I')}(g)
\times \left( I \mid s_\ell \mid K \right) \left( I' \mid s'_{\ell'} \mid K' \right) \delta_{KK'} \delta_{kk'}
\]

\[
= \frac{\lambda' \lambda}{\text{dim} I} \delta_{II'} \delta_{aa'} \delta_{cc'} \delta_{KK'} \delta_{kk'}
\times \left( I \mid s_\ell \mid K \right) \left( I' \mid s'_{\ell'} \mid K' \right)
\]

\[
= \frac{\lambda' \lambda}{\text{dim} I \, \text{dim} s_\ell} \delta_{II'} \delta_{aa'} \delta_{cc'} \delta_{KK'} \delta_{kk'}.
\]

so that the normalisation factor \(\lambda\) for \(|I \, s_\ell\rangle_K\) is

\[
\lambda = \sqrt{\frac{\text{dim} I \, \text{dim} s_\ell}{\text{dim} K}}.
\]

The only states that one can construct are those for which the Clebsch-Gordan coefficient in Eq. (5.11) does not vanish, \(K \subset I \otimes s_\ell\), and have \(2I = \text{even}\). The fermionic nature of \(\vec{q}\) changes the quantisation condition from \(2I = \text{odd}\) for the soliton to \(2I = \text{even}\) for the soliton-\(\vec{q}\) composite state. The subscript on the states \(|1 \, 1\rangle\) in Eq. (3.23) is the value of \(K\)). For \(K = 0\), we have an infinite tower of states with \(I = s_\ell = 0, 1, 2, \ldots\) with energy \(-3V_1/2\). For \(K = 1\), we have three degenerate infinite towers, \(I = s_\ell + 1 = 1, 2, 3, \ldots\), \(s_\ell = I + 1 = 1, 2, 3, \ldots\), and \(I = s_\ell = 1, 2, 3 \ldots\) with energy \(gV_1/2\). The states \(|1 \, 0; \frac{1}{2}\rangle\) and \(|0 \, 1; \frac{1}{2}\rangle\) and \(|1 \, 1\rangle_1\) of the previous section are from the \(K = 1\) series, and the states \(|0 \, 0; \frac{1}{2}\rangle\) and \(|1 \, 1\rangle_0\) are from the \(K = 0\) series. The usual quantisation condition for the soliton (i.e. without any heavy mesons) is a special case of the above construction. We can recover the results by omitting the \(\vec{q}\) quantum numbers, so that \(K = 0\), and \(2I = \text{odd}\). This gives the usual tower \(I = s_\ell = 1/2, 3/2, 5/2, \ldots\).
6. Pion-Baryon Coupling Constants

The soliton model can also be used to determine the pion coupling constants of the baryons in terms of the pion coupling constant of the heavy meson, \( g \). These can be evaluated using the matrix element of the axial vector current in the soliton heavy meson bound state. We will evaluate the matrix element of the axial current \( j^{\mu A} \) at zero momentum transfer, with \( \mu = 3 \) and \( A = + \). There are two independent coupling constants that determine all the pion couplings. We will follow the notation of Cho \[17\] who denoted the couplings by \( g_2 \) and \( g_3 \). (\( g_1 \) as defined by Cho is equal to the meson pion coupling constant \( g \) used in this paper.) To determine \( g_2 \) and \( g_3 \) we need to compute two independent matrix elements of \( j^{3+} \). This axial current can be written in terms of baryon fields as

\[
j^{3+} = -\frac{\sqrt{2}}{3} g_2 \Sigma^{++}_c \sigma_3 \Sigma^+_c - \frac{1}{\sqrt{3}} g_3 \Lambda^+_c \sigma_3 \Sigma^0_c + \ldots \tag{6.1}
\]

where we have used the lagrangian given in Ref. [17] and retained only the terms we will use in the calculation.

The axial current in the chiral lagrangian is

\[
j^{3+} = \frac{\pi D}{2e^2} \text{Tr} A \tau^3 A^{-1} \tau^+ - g \text{Tr} \overline{H} H \gamma^3 \gamma_5 \left[ \frac{\Sigma^+_c \tau^+ \Sigma^+ + \tau^+}{2} \right] + \ldots , \tag{6.2}
\]

where the ellipsis denotes higher derivative terms in the chiral lagrangian. The first term in the axial current is from the purely pionic sector of the chiral lagrangian, and is identical to that studied by Adkins, Nappi and Witten \[4\], and the constants \( D \) and \( e \) are defined in their paper.\[4\] The second term is the axial current of the \( H \) field from Eq. (2.11). In the heavy quark limit, \( \Sigma \) can be replaced by its value at the origin, \(-1 \).

The two independent matrix elements that we will compute are

\[
\langle \Sigma^{++}_c \uparrow | j^{3+} | \Sigma^+_c \uparrow \rangle = -\frac{\sqrt{2}}{3} g_2 ,
\]

\[
\langle \Lambda^+_c \uparrow | j^{3+} | \Sigma^0_c \uparrow \rangle = -\frac{1}{\sqrt{3}} g_3 ,
\]

\[\text{We have included the factor of } 3/2 \text{ mentioned before Eq. (20) of Ref. [4] into the definition of the axial current. There is a correction to the expression for } D \text{ given in Ref. [4] because of the distortion of the shape function } F \text{ in the presence of the } H \text{ field. However, as mentioned in Sec. 3, this is a } 1/N_c \text{ effect proportional to } g, \text{ and we will neglect it. The distortion of } F \text{ also changes the long distance behaviour of the soliton solution so that the Goldberger-Treiman relation is satisfied for the bound state.}\]
using Eq. (6.1). The same matrix elements computed using soliton heavy meson bound states will determine \( g_2 \) and \( g_3 \). The states we have constructed so far have omitted the spin of the heavy quark. To obtain the \( \Lambda_c \) state, we need to combine the heavy quark with the state \( |0 0\rangle \), to obtain

\[
|\Lambda_c \uparrow\rangle = |0 0\rangle \uparrow\rangle_Q,
\]

where \( \uparrow\rangle_Q \) denotes the heavy quark spin state. Similarly, the \( \Sigma_c \) states are obtained by combining the heavy quark spin with the state \( |1 1\rangle \),

\[
|\Sigma_c \uparrow\rangle = \sqrt{\frac{2}{3}} (|1 1 s_{\ell 3} = 1\rangle \downarrow\rangle_Q - \sqrt{\frac{1}{3}} (|1 1 s_{\ell 3} = 0\rangle |\uparrow\rangle_Q.
\]

The state \( \Sigma_c \) has been chosen to be the state \( |1 1\rangle \), which is the leading contribution in the large \( N_c \) limit.

The matrix element of the first term in Eq. (6.2) uses the identity

\[
A \hat{U}_I(g) |\Sigma_0\rangle = g \hat{U}_I(g) |\Sigma_0\rangle,
\]

where \( A \) is the soliton collective coordinate, and Eq. (3.17). It is convenient to label the \( \tau \) matrices by the angular momentum indices \( \pm 1, 0 \), instead of the cartesian labels \( x, y, z \). The relation between the two bases is

\[
\tau_{\pm 1} = \mp \frac{\tau_x \pm i \tau_y}{\sqrt{2}} \quad \tau_0 = \tau_3.
\]

In the angular momentum basis, Eq. (3.17) becomes

\[
\text{Tr} A \tau^b A^{-1} \tau^a = 2 (-1)^b D^{(1)}_{a-b}(A).
\]

The expression for the \( |\rangle \rangle_K \) states, Eq. (5.11) can be simplified for the case \( K = 0 \),

\[
|I a s_\ell m\rangle_0 = \sqrt{\text{dim} I} (-1)^{s_\ell+m} \int_{\text{SU}(2)} dg \, D_{a-m}^{(1)}(g) \hat{U}_I(g) |\Sigma_0\rangle \hat{U}_I(g) |K = 0\rangle,
\]

where we must have \( I = s_\ell \) since \( K = 0 \). Using this simplified expression for the states gives the matrix element

\[
0 \langle\langle I' a' s'_{\ell'} m' | \text{Tr} A \tau^r A^{-1} \tau^s |I a s_\ell m\rangle_0 = 2 (-1)^{r + s_\ell + m - s'_{\ell'} - m'} \times \sqrt{\text{dim} I} \int_{\text{SU}(2)} dg \, D_{a-m}^{(1)}(g) D_{s-r}^{(1)}(g) D_{a'-m'}^{(1)}(g)
\]

\[
= (-2)^{s_\ell + m} \sqrt{\text{dim} I} \int_{\text{dim} I'} \left( I a 1 I' \right) \left( I m 1 m' \right) \left( I' \right).
\]
Thus we obtain the matrix element of the first term in Eq. (6.2),
\[
\langle \Sigma_c^{++} \uparrow | j^{3+} | \Sigma_c^{+} \uparrow \rangle = -\frac{\pi D}{3\sqrt{2}e^2},
\]
\[
\langle \Lambda_c^+ \uparrow | j^{3+} | \Sigma_c^{0} \uparrow \rangle = \frac{\pi D}{3\sqrt{2}e^2}.
\]
(6.8)

The second part of the axial current can be written in the form
\[
-g \text{Tr} \overline{H}H \gamma^r \gamma_5 \tau^a = -2g S^r_{\ell H} I^a_H,
\]
(6.9)
so that we need to evaluate the matrix element:
\[
0 \langle \langle I' a' s_{\ell'} m' | S^r_{\ell H} I^a_H | I a s_{\ell} m \rangle \rangle_0 = \sqrt{\dim I \dim I'} (-1)^{I+m-I'-m'}
\times \int_{SU(2)} dg \, D^*(I)_{a-m} D(I')_{a'-m}, \quad \{ K = 0 | \hat{U}_I(g^{-1}) S^r I^a \hat{U}_I(g) | K = 0 \}
= \sqrt{\dim I \dim I'} (-1)^{I+m-I'-m'} \{ K = 0 | S^r_{\ell H} I^a_H | K = 0 \}
\times \left( \begin{array}{cc} I & 1 \\ a & a' \end{array} \right) \left( \begin{array}{cc} I & 1 \\ -m & p \end{array} \right),
\]
(6.10)
since
\[
\hat{U}_I(g^{-1}) S^r_{\ell H} I^a_H \hat{U}_I(g) = S^r I^a D^*(I)(g)_{sp}.
\]
The operators $S^r$ and $I^a$ are irreducible tensor operators under $K$ with $K = 1$, so their product can have $K = 0, 1, 2$. Only the $K = 0$ part of $S^r I^a$ has a non-zero matrix element between $K = 0$ states, so we can make the replacement
\[
\{ K = 0 | S^r_{\ell H} I^a_H | K = 0 \} \rightarrow \frac{1}{3} (-1)^r \delta_{p-r} \{ K = 0 | S_{\ell H} \cdot I_{H} | K = 0 \}
= -\frac{1}{4} (-1)^r \delta_{p-r}
\]
(6.11)
since $2 S_{\ell H} \cdot I_{H} = K_{H}^2 - I_{H}^2 - S^2_{\ell H} = -3/2$. This allows the sum in Eq. (6.10) to be evaluated,
\[
0 \langle \langle I' a' s_{\ell'} m' | S^r I^a | I a s_{\ell} m \rangle \rangle_0 = \frac{1}{4} \sqrt{\dim I \dim I'} \left( \begin{array}{cc} I & 1 \\ a & a' \end{array} \right) \left( \begin{array}{cc} I & 1 \\ m & r \end{array} \right).
\]
(6.12)
The matrix element of $j^{3+}$ is obtained by setting $r = 0$ and $s = +$ in Eq. (6.12) and multiplying by $(-2g)(-\sqrt{2})$, since $I^+ = -\sqrt{2}I^1$. This gives the matrix elements of the second term of Eq. (6.2),
\[
\langle \Sigma_c^{++} \uparrow | j^{3+} | \Sigma_c^{+} \uparrow \rangle = -\frac{g}{3\sqrt{2}},
\]
\[
\langle \Lambda_c^+ \uparrow | j^{3+} | \Sigma_c^{0} \uparrow \rangle = \frac{g}{3\sqrt{2}}.
\]
(6.13)
Combining Eqs. (6.8) and (6.11), and comparing with Eq. (6.1), we get
\[ g_2 = \frac{\pi D}{2e^2} + \frac{g}{2} = -\frac{3}{2}g_A + \frac{g}{2}, \] (6.14)
\[ g_3 = -\frac{\pi D}{\sqrt{6}e^2} - \frac{g}{\sqrt{6}} = \sqrt{\frac{3}{2}}g_A - \frac{g}{\sqrt{6}}, \]

since the axial coupling constant of the nucleon is
\[ g_A = -\frac{\pi D}{3e^2}. \] (6.15)

Using \( g_A = 1.25 \), and setting \( g \) equal to its experimental upper bound [15] of 0.63, we get
\[ g_2 = -1.6, \quad g_3 = 1.3. \] (6.16)

To all orders in the derivative expansion, the \( g_A \) term has the form given in Eq. (6.2), where \( D \) is some functional of the shape function \( F \). The value of \( D \) is normalised using the formula for \( g_A \) in Eq. (6.15). The \( g \) term gets corrections from higher derivative terms in the effective action. These higher derivative terms retain the flavour structure of the leading term \( \text{Tr} \overline{H}H\sigma^3\tau^+, \) but renormalise the coefficient so that it is no longer \( g \). This renormalisation of \( g \) cancels in the ratio, so that we have the large \( N_c \) prediction
\[ \frac{g_2}{g_3} = -\sqrt{\frac{3}{2}}. \] (6.17)

In addition, since \( g \) is formally of order \( N_c^0 \) whereas \( g_A \) is of order \( N_c \), the leading \( N_c \) predictions for the values of \( g_2 \) and \( g_3 \) are given by retaining only the \( g_A \) term in Eq. (6.14), which leads to the model independent large \( N_c \) prediction
\[ g_2 = -\frac{3}{2}g_A = -1.9, \quad g_3 = \sqrt{\frac{3}{2}}g_A = 1.5. \] (6.18)

The correction to Eq. (6.18) is of order \( 1/N_c \) relative to the leading term, and the correction to Eq. (6.17) is of order \( 1/N_c^2 \) relative to the leading term. Using the value (6.16) for \( g_3 \) along with the measured baryon masses gives the prediction
\[ \Gamma(\Sigma_c^{++} \rightarrow \Lambda_c^{+}\pi^+) = \frac{g_3^2 |\vec{p}_\pi|^3}{6\pi f^2} \simeq 3.7 \text{ MeV}. \] (6.19)

The couplings \( g_2 \) and \( g_3 \) have also been computed in the constituent quark model, and the prediction for the decay width Eq. (6.19) is between the values found in Ref. [14] for the naive quark model, and for the chiral quark model [18].
7. Baryons containing a Heavy Quark: The $SU(3)$ Case

It is straightforward to generalise the results of the previous section to the case of $SU(3)$. The soliton in $SU(3)$ has the form Eq. (3.2) where the isospin matrices form a subgroup of $SU(3)$. The soliton is still invariant under $K = I + s_\ell$. In addition, the Wess-Zumino term requires that the soliton have a definite hypercharge \cite{5} [6],

$$3Y |\Sigma_0\rangle = N_c |\Sigma_0\rangle,$$

where we are interested in the case where the number of colours $N_c = 3$. The hypercharge generator is

$$Y = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix}. $$

The soliton-$\bar{q}$ bound states can be obtained using the methods of the previous section. The $\bar{q}$ state transforms as a $\bar{3}$ under $SU(3)$ and as a doublet under the spin of the light degrees of freedom. Under the $SU(2) \times U(1)$ subgroup, it decomposes into an isospin doublet with hypercharge $-1$, and an isospin singlet, with hypercharge 2. Thus we have $\bar{q}$ states with \{ $K = 0, 1; 3Y = -1$\}, and \{ $K = 1/2; 3Y = 2$\}. The energies are still given by Eq. (5.4), so that we have a tower of states with energy $V_1/2$ from the $K = 1$ states, energy $-3V_1/2$ from the $K = 0$ states, and energy zero from the $K = 1/2$ states. The bound states can be computed by a procedure similar to that used in Sec. 5. The details are uninteresting, and the final result is that the allowed states are given by

$$|Ra s_\ell m\rangle_K = \frac{\dim R \dim s_\ell}{\dim K} \int_{SU(3)} dg D_{ac}^{(R)}(g)$$

$$\times \hat{U}_I(g) |\Sigma_0\rangle \hat{U}_J(g) |K k'\rangle \begin{pmatrix} I_c & s_\ell & m \\ K \end{pmatrix},$$

with the additional constraint that

$$Y_c = Y_\Sigma + Y_K = 1 + Y_K,$$

where $I_c$ and $Y_c$ are the isospin and hypercharge of the $SU(3)$ state $c$ of the representation $R$, and $Y_K$ is the hypercharge of the $\bar{q}$ state with isospin $K$. Thus the allowed $SU(3)$ representations for a spin $s_\ell$ state are those which contain an element with isospin $I$ and hypercharge $Y$ such that

$$(a) \ I = s_\ell, \ 3Y = 2,$$

$$(b) \ 1 \subset I \otimes s_\ell, \ 3Y = 2,$$

$$(c) \ \frac{1}{2} \subset I \otimes s_\ell, \ 3Y = 5,$$

(7.4)
so that the Clebsch-Gordan coefficient in Eq. (7.2) does not vanish, and the hypercharge constraint is satisfied. Cases (a)–(c) are from the \( \{ K = 0, 3Y_K = -1 \} \), \( \{ K = 1, 3Y_K = -1 \} \), and \( \{ K = 1/2, 3Y_K = 2 \} \) states of \( q \) respectively. The usual soliton quantisation condition for \( SU(3) \) (i.e. with no heavy meson fields) is obtained by omitting the \( q \) state, i.e. using \( \{ K = 0, Y_K = 0 \} \), and is that \( R \) must contain a state with \( I = s \ell \) and with \( Y = 1 \).

To classify the allowed states, we need to find the decomposition of irreducible tensors of \( SU(3) \) into the \( SU(2) \times U(1) \) subgroup generated by isospin and hypercharge. An irreducible \( SU(3) \) representation \( (m, n) \) is a traceless tensor which is completely symmetric in \( m \) upper and \( n \) lower indices, and has dimension

\[
\dim (m, n) = \frac{(m + 1)(n + 1)(m + n + 2)}{2}
\]  

(7.5)

If \( [m, n] \) denotes the (reducible) tensor which is completely symmetric in its upper and lower indices, then the representations in \( (m, n) \) are those in \( [m, n] \oplus [m - 1, n - 1] \), because the trace of \( [m, n] \) is the tensor \( [m - 1, n - 1] \). It is much simpler to compute the decomposition of \( [m, n] \) because we do not have the constraint that the tensor has zero trace. The tensor \( [m, n] \) under the \( SU(2) \times U(1) \) subgroup decomposes into the sum

\[
\sum_{k=0}^{m} \sum_{r=0}^{n} [m - k, n - r]_{3Y = m - n - 3k + 3r},
\]  

(7.6)

where the term \( [m - k, n - r] \) is obtained by setting \( r \) upper and \( k \) lower indices equal to 3. The remaining \( m - k \) upper and \( n - r \) lower indices can have values 1,2. The tensor \( [m - k, n - r] \) is symmetric on \( m - k \) upper and \( n - r \) lower indices, and is the reducible representation given by the tensor product of the representations \( 2I = m - k \) and \( 2I = n - r \) of \( SU(2) \). By subtracting the decomposition of \( [m - 1, n - 1] \) from \( [m, n] \), one obtains the decomposition

\[
(m, n) \overset{SU(2)}{\longrightarrow} \sum_{r=0}^{n-1} \begin{cases} 
\frac{|m-r|}{2}, \frac{|m-r|}{2} + 1, \ldots, \frac{m+r}{2} \\
Y = m+2n-3r
\end{cases} \]

\[
\oplus \sum_{r=0}^{m-1} \begin{cases} 
\frac{|n-r|}{2}, \frac{|n-r|}{2} + 1, \ldots, \frac{n+r}{2} \\
Y = -n-2m+3r
\end{cases} \]

\[
\oplus \begin{cases} 
\frac{|m-n|}{2}, \frac{|m-n|}{2} + 1, \ldots, \frac{m+n}{2} \\
Y = m-n
\end{cases}
\]  

(7.7)

21
The soliton heavy meson bound states are now constructed in two different ways. We first construct the states by tensoring the soliton states with the $\bar{q}$ states. These states are the generalisation of the $|I s_\ell; R\rangle$ states of Sec. 3. We then construct the bound states directly using Eq. (7.2). The states obtained by this method are the generalisation of the $|I s_\ell\rangle_K$ states of Sec. 5. The soliton states have $I = s_\ell$ and $Y = 1$. The lowest representations are $8 = (1, 1)$ with $s_\ell = 1/2$, $10 = (3, 0)$ with $s_\ell = 3/2$, $27 = (2, 2)$ with $s_\ell = 1/2, 3/2$, $35 = (4, 1)$ with $s_\ell = 3/2, 5/2$, $\bar{10} = (0, 3)$ with $s_\ell = 1/2$, etc. The tensor product of the soliton with $\bar{q}$ can be computed using the formula

$$(m, n) \otimes (0, 1) = (m, n + 1) \oplus (m + 1, n - 1) \oplus (m - 1, n).$$

Thus the lowest soliton heavy meson bound states are: $8 \otimes 3 = (1, 1) \otimes (0, 1) = (1, 2) \oplus (2, 0) \oplus (0, 1) = \bar{15} \oplus 6 \oplus 3$, and have light-spin $s_\ell = 0, 1$; $10 \otimes 3 = (3, 0) \otimes (0, 1) = (3, 1) \oplus (2, 0) = 24 \oplus 6$ and have light-spin $s_\ell = 1, 2$, etc. The quark model states are the $3 = (0, 3)$ with $s_\ell = 0$ and the $6 = (2, 0)$ with $s_\ell = 1$. The other states are exotics. When combined with the heavy quark $Q = c$, the $3$ gives the multiplet $\{\Xi_c^+, \Xi_c^0, \Lambda_c^+\}$ with spin-1/2, and the $6$ gives the multiplets $\{\Sigma_c^{++}, \Sigma_c^+, \Sigma_c^0, \Xi_c^0, \Omega_c^0, \Omega_c^0\}$ with spin-1/2 and $\{\Sigma_c^{*++}, \Sigma_c^{*+}, \Sigma_c^{*0}, \Xi_c^{*0}, \Xi_c^{*0} \}$ with spin-3/2.

To determine the energies of the states, we need to classify them according to their values of $K$ using the allowed states given in Eq. (7.4). We will discuss only the $3 = (0, 1)$ and $6 = (2, 0)$ states here. The state $3 = (0, 1)$ with $s_\ell = 0$ arises from using rule $(a)$ for the $K = 0$, $3Y = 2$ state in $\bar{q}$ and has energy $-3V_1/2$, and the state $3 = (0, 1)$ with $s_\ell = 1$ arises from using rule $(b)$ for the $K = 1$, $3Y = 2$ state in $\bar{q}$ and has energy $V_1/2$. These states are unique, and so must correspond to the states obtained by tensoring the $8 = (1, 1)$ soliton states with $\bar{q}$, i.e.

$$|\bar{3} 0; 8\rangle = |\bar{3} 0\rangle_0, \quad |\bar{3} 1; 8\rangle = |\bar{3} 1\rangle_1.$$  

The states $6 = (2, 0)$ with $s_\ell = 2, 1, 0$ arises from using rule $(b)$ for the $K = 1$, $3Y = 2$ state in $\bar{q}$ and have energy $V_1/2$. A state $6 = (2, 0)$ with $s_\ell = 1$ arises from using rule $(a)$ for the $K = 0$, $3Y = 2$ state in $\bar{q}$ and has energy $-3V_1/2$. The $6 = (2, 0)$ states with $s_\ell = 1$, $|61\rangle_{0,1}$ must be linear combinations of the states $|6 1; 8\rangle$ and $|6 1; 10\rangle$ obtained by tensoring $8 = (1, 1)$ and $10 = (3, 0)$ with $\bar{q}$ respectively. We need to find the transformation relating the two sets of basis states (analogous to Eq. (3.23)) to determine the energies of the $6$ states including the $10 - 8$ mass splitting.
The straightforward way to determine the linear combinations is to compute the interaction hamiltonian for $SU(3)$. The result is essentially identical to Eq. (3.21) with the $SU(2)$ $6j$-symbol for isospin replaced by the corresponding $SU(3)$ $6j$-symbol, and with the adjoint representation 1 of $SU(2)$ replaced by the adjoint representation 8 of $SU(3)$. The labels $R, R'$ in the $SU(2)$ $6j$-symbol for spin refer to the isospin of the states in the $SU(3)$ representation that satisfy the hypercharge constraint. The $SU(3)$ $6j$-symbols are complicated, and have four additional labels because there can be more than one singlet in the tensor product of three $SU(3)$ representations.

Instead of doing this, we will determine the linear combinations for the 6 by a different method. Let us consider the state $|6 1\rangle_0$ which can be written as the linear combination $|6 1\rangle_0 = \alpha |6 1; 8\rangle + \beta |6 1; 10\rangle$. The ratio $\alpha/\beta$ is determined by making the right hand side an eigenstate of $K$ with eigenvalue zero. The algebra is relegated to Appendix A. The result is that we get the same linear combination for $SU(3)$ as we did for $SU(2)$ in Eq. (3.23). Thus the hamiltonian Eq. (4.1) is the hamiltonian for the 6 of $SU(3)$, where $\Delta M$ is now the $10−8$ mass difference, and the mass formula Eq. (4.4) now gives the $6−3$ mass difference.

The baryon-meson couplings in $SU(3)$ are easily determined from the results of Sec. 6; the only difference is that we replace Eqs. (6.7) and (6.12) by the corresponding $SU(3)$ expressions

$$0 \langle R' a' s_{\ell'} m' | Tr A^r A^{-1} r^s | R a s_{\ell} m \rangle_0 = (-2) \sqrt{\frac{\text{dim } R}{\text{dim } R'}} \left( \begin{array}{c|c} R & 8 \\ a & s \end{array} \right) \left( \begin{array}{c|c} R' & 8 \\ a' & s' \end{array} \right),$$

and

$$0 \langle R' a' s_{\ell'} m' | S^r I^s | I a s_{\ell} m \rangle_0 = \frac{1}{4} \sqrt{\frac{\text{dim } R}{\text{dim } R'}} \left( \begin{array}{c|c} R & 8 \\ a & s \end{array} \right) \left( \begin{array}{c|c} R' & 8 \\ a' & s' \end{array} \right) \left( \begin{array}{c|c} R & 8 \\ b & r \end{array} \right) \left( \begin{array}{c|c} R' & 8 \\ b' & r' \end{array} \right),$$

where the states $b$ and $b'$ have $I = s_{\ell}$ ($s'_{\ell}$), $I_3 = m$ ($m'$), and $3Y = 2$, the factors on the right are $SU(3)$ Clebsch-Gordan coefficients, and the dimensions are those of the $SU(3)$ representations. The Clebsch-Gordan coefficients are evaluated in terms of $SU(2)$ Clebsch-Gordan coefficients and the appropriate isoscalar factors. Using the tensor methods discussed in Appendix A, we find the isoscalar factors

$$\left( \begin{array}{c|c} 6 & 8 \\ \Sigma_c & \pi \end{array} \right) = \frac{\sqrt{3}}{5},$$

$$\left( \begin{array}{c|c} 6 & 8 \\ \Sigma_c & \Lambda_c \end{array} \right) = \frac{\sqrt{3}}{2}.$$
This gives
\[ g_2 = -\frac{9}{7} g_A + \frac{3}{10} g, \]  
\[ g_3 = \frac{15}{14} g_A - \frac{1}{4} g, \]  
(7.14)
compared to the $SU(2)$ prediction (6.14). The ratio of the couplings is
\[ \frac{g_2}{g_3} = -\frac{6}{5}, \]  
(7.15)
for $SU(3)$, and differs from the $SU(2)$ prediction of Eq. (6.17) by the factor $\sqrt{24/25} = 0.98$.

The Skyrme model predictions for the couplings are different for $SU(2)$ and $SU(3)$. This is a generic feature which is a peculiarity of the Skyrme model. It arises because the $SU(3)$ Clebsch-Gordan coefficients used are those for the baryon representations for $N_c = 3$ rather than those for the true large $N_c$ baryon representations which are Young tableaux with $N_c$ boxes \[6\]. If we had used the true large $N_c$ Clebsch-Gordan coefficients (and retained only the leading term as $N_c \to \infty$), we would have obtained the same results for $SU(2)$ and for $SU(3)$. A simple way to see this is to note that the quark model results in the large $N_c$ limit have the same group theoretic structure as the Skyrme model \[6\], and that the quark model results for baryons with no strange quark are obviously the same for $SU(2)$ and $SU(3)$.

Acknowledgements

We would like to thank E. Jenkins and M. B. Wise for useful discussions. A.M. would like to thank the Fermilab theory group for hospitality while this paper was being written. This work was supported in part by DOE grant #DOE-FG03-90ER40546, and by a NSF Presidential Young Investigator award PHY-8958081.

Appendix A. Determining $K$ Eigenstates for $SU(3)$

The state
\[ |6 \ 1 \rangle_0 = \alpha |6 \ 1; 8 \rangle + \beta |6 \ 1; 10 \rangle, \]  
(A.1)
discussed in Eq. (7.10) was defined to be an eigenstate of $K$ with eigenvalue 0. In this appendix, we determine the ratio of $\alpha/\beta$ for which this is true. The state can be written as
\[
|6c1m\rangle_0 = \alpha \left| \begin{array}{c} 8 \\ a \\ b \\ 3 \\ c \end{array} \right| \left( \begin{array}{c} \frac{1}{2} \\ n \\ p \\ 1 \\ m \end{array} \right) |8a\frac{1}{2}n\rangle |bp\rangle
+ \beta \left| \begin{array}{c} 10 \\ a' \\ b' \\ 3 \\ c \end{array} \right| \left( \begin{array}{c} \frac{3}{2} \\ n' \\ p' \\ 1 \\ m \end{array} \right) |10a'\frac{3}{2}n'\rangle |b'p'\rangle
\]
(A.2)
in terms of the soliton states $|$ and the $\bar{q}$ states $\}$. The normalised soliton states are
\[
|R a \ s_{\ell} \ m\rangle = \sqrt{\dim \mathbb{R}} \ (-1)^{s_{\ell}+m} \int_{SU(3)} dg \ D_{ad}^{*}(g) \ U(g) |\Sigma_0\rangle
\]
where the index $d$ is a state with $I = s_{\ell}, I_3 = -m$ and $Y = 1$. The eigenvalues of $K$ are defined for the $\bar{q}$ state when the soliton is in the state $|\Sigma_0\rangle$. Thus we only need to consider the soliton wavefunction Eq. (A.3) when $g = 1$, so that $D_{ad}^{*}(g) = \delta_{ad}$. With this substitution, one obtains
\[
|6 1\rangle_0 \rightarrow \alpha\sqrt{8} \ (-1)^{1/2+n} \left| \begin{array}{c} 8 \\ K \\ -n \\ b \\ c \end{array} \right| \left( \begin{array}{c} \frac{1}{2} \\ n \\ p \\ 1 \\ m \end{array} \right) |\Sigma_0\rangle |b\rangle
+ \beta\sqrt{10} \ (-1)^{3/2+n'} \left| \begin{array}{c} 10 \\ \Delta \\ -n' \\ c \end{array} \right| \left( \begin{array}{c} \frac{3}{2} \\ n' \\ p' \\ 1 \\ m \end{array} \right) |\Sigma_0\rangle |b'p'\rangle
\]
(A.4)
where $\delta_{ad}$ has restricted the sum over $a$ to states with $Y = 1$ and $I = 1/2$, and the sum over $a'$ to states with $Y = 1$ and $I = 3/2$. The $3$ decomposes under $SU(2)$ into a doublet and a singlet, which we will call $\bar{q}$ and $\bar{q}$ respectively. The $6$ decomposes into the representations $I = 1, 1/2, 0$ which will be called $qq, qs$ and $ss$ respectively. The $SU(3)$ Clebsch-Gordan coefficients in Eq. (A.4) can be written as the product of $SU(3)$ iso-scalar factors times $SU(2)$ Clebsch-Gordan coefficients. The non-zero iso-scalar factors are $K \otimes \bar{q} \rightarrow qq$ and $\Delta \otimes \bar{q} \rightarrow qq$. This restricts the index $b, b'$ to $I = 1/2, 3Y = -1$. Since the index $b, b'$ cannot have $I = 0$, we see that there is no amplitude for the right hand side of Eq. (A.4) to be in a $K = 1/2$ sector. We only need to ensure that it is orthogonal to the $K = 1$ sector. Rewriting the $SU(3)$ Clebsch-Gordan coefficients in terms of iso-scalar factors, we get
\[
|6 1\rangle_0 \rightarrow \alpha\sqrt{8} \ (-1)^{1/2+n} \left| \begin{array}{c} 8 \\ K \\ qq \end{array} \right| \left( \begin{array}{c} \frac{1}{2} \\ -n \\ r \\ c \end{array} \right)
\times \left( \begin{array}{c} \frac{1}{2} \\ n \\ p \\ 1 \\ m \end{array} \right) |\Sigma_0\rangle |r p\rangle
+ \beta\sqrt{10} \ (-1)^{3/2+n'} \left| \begin{array}{c} 10 \\ \Delta \\ qq \end{array} \right| \left( \begin{array}{c} \frac{3}{2} \\ -n' \\ r' \end{array} \right)
\times \left( \begin{array}{c} \frac{1}{2} \\ n' \\ p' \\ 1 \\ m \end{array} \right) |\Sigma_0\rangle |r'p'\rangle
\]
(A.5)
where \( r \) and \( r' \) are summed over 1,2. We rewrite the \( H \) states as linear combinations of \( K \) states,
\[
| r p \rangle = \left( \begin{array}{c} \frac{1}{2} \\ r \\ \frac{1}{2} \end{array} \right) \left( \begin{array}{c} K \\ p \\ k \end{array} \right) | K k \rangle .
\]
(A.6)

To ensure that the right hand side of Eq. (A.5) is orthogonal to \( K = 1 \), the coefficient of \( | K = 1 k \rangle \) must vanish. This requires that
\[
0 = \alpha \sqrt{8} \left( -1 \right)^{1/2+n} \left( \begin{array}{c} 8 \\ 3 \\ 6 \end{array} \right) \left( \begin{array}{c} K \\ q \\ qq \end{array} \right) \left( \begin{array}{c} -n \\ r \\ c \end{array} \right) \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ m \end{array} \right) \times | \Sigma_0 \rangle \left( \begin{array}{c} \frac{1}{2} \\ p \\ 1 \end{array} \right) + \beta \sqrt{10} \left( -1 \right)^{3/2+n'} \left( \begin{array}{c} 10 \\ 3 \\ 6 \end{array} \right) \left( \begin{array}{c} \Delta \\ \delta \\ qq \end{array} \right) \left( \begin{array}{c} -n' \\ r' \\ c \end{array} \right) \left( \begin{array}{c} \frac{3}{2} \\ \frac{1}{2} \\ m \end{array} \right) \times | \Sigma_0 \rangle \left( \begin{array}{c} \frac{1}{2} \\ p' \\ 1 \end{array} \right).
\]
(A.7)

The \( SU(2) \) Clebsch-Gordan coefficients on the right hand side can be evaluated in terms of \( 6j \)-symbols. A simpler procedure is to pick the particular values \( k = 1, m = 1 \) and \( c = 0 \), for which the sums are trivial, and gives
\[
0 = \alpha \sqrt{8} \left( -1 \right)^{1/2+n} \left( \begin{array}{c} 8 \\ 3 \end{array} \right) \left( \begin{array}{c} 6 \end{array} \right) \left( \begin{array}{c} -n \\ r \\ c \end{array} \right) \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ m \end{array} \right) \times | \Sigma_0 \rangle \left( \begin{array}{c} \frac{1}{2} \\ p \\ 1 \end{array} \right) - \frac{1}{2} \beta \sqrt{10} \left( -1 \right)^{3/2+n'} \left( \begin{array}{c} 10 \\ 3 \end{array} \right) \left( \begin{array}{c} 6 \end{array} \right) \left( \begin{array}{c} -n' \\ r' \\ c \end{array} \right) \left( \begin{array}{c} \frac{3}{2} \\ \frac{1}{2} \\ m \end{array} \right) \times | \Sigma_0 \rangle \left( \begin{array}{c} \frac{1}{2} \\ p' \\ 1 \end{array} \right).
\]
(A.8)

The isoscalar factors can be easily evaluated by tensor methods. The Clebsch-Gordan coefficients for \( 8 \otimes 3 \rightarrow 6 \) can be written as the \( SU(3) \) invariant combination
\[
\chi \, 8^{k}_{j} \, \bar{3}^{m}_{l} \, 6^{l}_{kl} \, \epsilon^{jml}
\]
(A.9)
in an obvious notation. To determine the normalisation constant \( \chi \), we can analyze the Clebsch-Gordan coefficients for \( 6^{11}_{11} \), for which Eq. (A.8) gives
\[
\chi \left( 8^{1}_{2} \, \bar{3}^{1}_{3} \, 8^{1}_{3} \, \bar{3}^{1}_{2} \right) \, 6^{11}_{11},
\]
(A.10)
which determines \( \chi = 1/\sqrt{2} \) for an amplitude normalised to unity. The \( SU(3) \) Clebsch-Gordan coefficient
\[
\left( \begin{array}{c} 8 \\ K^{+} \\ \bar{d} \end{array} \right) \left( \begin{array}{c} 6 \\ uu \end{array} \right) = \left( \begin{array}{c} 8 \\ \bar{3} \\ \bar{d} \end{array} \right) \left( \begin{array}{c} 6 \\ qq \end{array} \right) \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{array} \right) \left( \begin{array}{c} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{array} \right) = -\frac{1}{\sqrt{2}},
\]
(A.11)
so that the isoscalar factor is

\[
\left( \begin{array}{c|c} 8 & \bar{3} \\ \hline K & q \\ \hline \bar{q} & qq \\ \end{array} \right) = -\frac{1}{\sqrt{2}}. \tag{A.12}
\]

Similarly, one determines the isoscalar factor

\[
\left( \begin{array}{c|c} 10 & \bar{3} \\ \hline \Delta & q \\ \hline \bar{q} & qq \\ \end{array} \right) = \frac{2}{\sqrt{5}}, \tag{A.13}
\]

using the normalised invariant combination

\[
\sqrt{\frac{3}{5}} \, 10^{ijk} \, \bar{3}_i \, 6^\dagger_{jk}. \tag{A.14}
\]

Substituting in Eq. (A.8), we find that \( \beta = \sqrt{2}\alpha \), which is precisely the linear combination obtained in eq. (3.23) for the state with \( K = 0 \). The orthogonal linear combination with \( \alpha = -\sqrt{2}\beta \) must be the \( K = 1 \) state, since that is the only other \( |6, 1\rangle_K \) state in the spectrum.
References

[1] T.H.R. Skyrme, Proc. Roy. Soc. A260 (1961) 127
[2] E. Witten, Nucl. Phys. B223 (1983) 433
[3] J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95
[4] G.S. Adkins, C.R. Nappi and E. Witten, Nucl. Phys. B228 (1983) 552
[5] E. Guadagnini, Nucl. Phys. B236 (1984) 35
[6] A.V. Manohar, Nucl. Phys. B248 (1984) 19
[7] C. Callan and I. Klebanov, Nucl. Phys. B262 (1985) 365, Phys. Lett. 202B (1988) 260
[8] M. Rho, D.O. Riska, and N.N. Scoccola, Phys. Lett. 251B (1990) 597, Z. Phys. A341 (1992) 343; Y. Oh, D. Min, M. Rho, and N. Scoccola, Nucl. Phys. A534 (1991) 493
[9] E. Jenkins, A.V. Manohar and M.B. Wise, Caltech Preprint CALT-68-1783 (1992)
[10] N. Isgur and M.B. Wise, Phys. Lett. 232B (1989) 113, Phys. Lett. 237B (1990) 527
[11] H. Georgi, Phys. Lett. 240B (1990) 447
[12] M.B. Wise, Phys. Rev. D45 (1992) 2118
[13] G. Burdman and J.F. Donoghue, Phys. Lett. 280B (1992) 287
[14] Tung-Mow Yan, Hai-Yang Cheng, Chi-Yee Cheung, Guey-Lin Lin, Y.C. Lin and Hoi-Lai Yu, CLNS 92/1138 (1992)
[15] The ACCMOR Collaboration (S. Barlag, et al.), Phys. Lett. 278B (1992) 480
[16] M. Mattis and M. Mukerjee, Phys. Rev. Lett. 61 (1988) 1344
[17] P. Cho, Harvard Preprint HUTP-92/A014 (1992)
[18] A.V. Manohar and H. Georgi, Nucl. Phys. B234 (1984) 189