Burghelea-Haller analytic torsion for twisted de Rham complexes

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Abstract

In this paper, we extend the Burghelea-Haller analytic torsion to the twisted de Rham complexes. We also compare it with the twisted refined analytic torsion defined by Huang.

1 Introduction

Let $E$ be a unitary flat vector bundle on a closed Riemannian manifold $M$. In [21], Ray and Singer defined an analytic torsion associated to $(M, E)$ and proved that it does not depend on the Riemannian metric on $M$. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on $M$ (cf. [14]). This conjecture was later proved in the celebrated papers of Cheeger [9] and Müller [15]. Müller generalized this result in [10] to the case when $E$ is a unimodular flat vector bundle on $M$. In [1], inspired by the considerations of Quillen [20], Bismut and Zhang reformulated the above Cheeger-Müller theorem as an equality between the Reidemeister and Ray-Singer metrics defined on the determinant of cohomology, and proved an extension of it to the case of general flat vector bundle over $M$. The method used in [1] is different from those of Cheeger and Müller in that it makes use of a deformation by Morse functions introduced by Witten [24] on the de Rham complex.

Braverman and Kappeler [2, 3, 4] defined the refined analytic torsion for flat vector bundle over odd dimensional manifolds, and show that it equals to the Turaev torsion (cf. [11, 23]) up to a multiplication by a complex number of absolute value one. Burghelea and Haller [6, 7], following a suggestion of Müller, defined a generalized analytic torsion associated to a non-degenerate symmetric bilinear form on a flat vector bundle over an arbitrary dimensional manifold and make an explicit conjecture between this generalized analytic torsion and the Turaev torsion. This conjecture was proved up to sign by Burghelea-Haller [8] and in full generality by Su-Zhang [22]. Cappell and Miller [10] used non-self-adjoint Laplace operators to define another complex valued analytic torsion and used the method in [22] to prove an extension of the Cheeger-Müller theorem.

In [13, 19], Mathai and Wu generalized the classical Ray-Singer analytic torsion to the twisted de Rham complex with an odd degree closed differential form $H$. Recently, Huang [13] generalized the refined analytic torsion [2, 3, 4] to the twisted de Rham complex, got a duality theorem and compared it with the twisted Ray-Singer metric which also was defined in [13].

In this paper, suppose there exists a non-degenerate symmetric bilinear form on the flat vector bundle $E$, we generalize the Burghelea-Haller analytic torsion to the twisted de Rham complex. For the odd dimensional manifold, we also compare it with the twisted refined analytic torsion and the twisted Ray-Singer metric.

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The rest of this paper is organized as follows. In Section 2, suppose there exists a $\mathbb{Z}_2$-graded non-degenerate symmetric bilinear form on a $\mathbb{Z}_2$-graded finite dimensional complex, we define a symmetric bilinear torsion on it. In Section 3, we generalize the Burghelea-Haller analytic torsion to the twisted de Rham complex. In Section 4, when the dimension of the manifold is odd, we show that the twisted Burghelea-Haller analytic torsion is independent of the Riemannian metric $g$, the symmetric bilinear form $b$ and the representative $H$ in the cohomology class $[H]$. In Section 5, we compare it with the twisted refined analytic torsion. In Section 6, we briefly discuss the Cappell-Miller analytic torsion on the twisted de Rham complex.

2 Symmetric bilinear torsion on a finite dimensional $\mathbb{Z}_2$-graded complex

Let

$$0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n \longrightarrow 0$$

be a cochain complex of finite dimensional complex vector space. Set

$$C^k = \bigoplus_{i=k \mod 2} C^i, \quad d^k = \sum_{i=k \mod 2} d_i, \quad k = 0, 1.$$

Then we get a $\mathbb{Z}_2$-graded cochain complex

$$(2.1) \quad \cdots \xrightarrow{d_{i+1}} C^{i+1} \xrightarrow{d_i} C^i \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_0} C^0 \longrightarrow 0$$

Denote its cohomology by $H^k$, $k = 0, 1$. Set

$$\text{det}(C^\bullet, d) = \text{det}C^0 \otimes \left(\text{det}C^1\right)^{-1}, \quad \text{det}(H^\bullet, d) = \text{det}H^0 \otimes \left(\text{det}H^1\right)^{-1}.$$

Then we have a canonical isomorphism between the determinant lines

$$(2.2) \quad \phi : \text{det}(C^\bullet, d) \longrightarrow \text{det}(H^\bullet, d).$$

Suppose that there is a non-degenerate symmetric bilinear form on $C^k$, $k = 0, 1$. Then it induces a non-degenerate symmetric bilinear form $b_{\det H^\bullet(C^\bullet, d)}$ on the determinant line $\text{det}(H^\bullet, d)$ via the isomorphism (2.2). Let $d_k^\# \otimes \lambda$ be the adjoint of $d_k$ with respect to the non-degenerate symmetric bilinear form and define $\Delta_{b,k} = d_k^\# \otimes \lambda$ + $d_{k+1}^\# \otimes \lambda$. Let us write $C^k_b(\lambda)$ for the generalized $\lambda$-eigen space of $\Delta_{b,k}$. Then we have a $b$-orthogonal decomposition

$$(2.3) \quad C^k = \bigoplus_{\lambda} C^k_b(\lambda)$$

and the inclusion $C^k_b(0) \hookrightarrow C^k$ induces an isomorphism in cohomology. Particularly, we obtain a canonical isomorphism

$$(2.4) \quad \text{det}H^\bullet(C^\bullet(0)) \cong \text{det}H^\bullet(C^\bullet).$$

**Proposition 2.1.** The following identity holds,

$$(2.5) \quad b_{\det H^\bullet(C^\bullet, d)} = b_{\det H^\bullet(C^\bullet(0), d)} \cdot \det \left( \begin{vmatrix} d_k^\# & d_k(0) \\ C^k_b(0) & \otimes \text{im}d^\#_k \end{vmatrix} \right)^{-1} \cdot \det \left( \begin{vmatrix} d_1^\# & d_1(0) \\ C^1_b(0) & \otimes \text{im}d_1^\# \end{vmatrix} \right),$$

where $C^k_b(0) = \bigoplus_{\lambda \neq 0} C^k_b(\lambda)$, $k = 0, 1$. 

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Proof. The proof is the same as [7, Lemma 3.3]. Suppose \((C^\bullet_1, b_1)\) and \((C^\bullet_2, b_2)\) are finite-dimensional \(\mathbb{Z}_2\)-graded complexes equipped with \(\mathbb{Z}_2\)-graded non-degenerate symmetric bilinear forms. Clearly, \(H^\bullet(C^\bullet_1 \oplus C^\bullet_2) = H^\bullet(C^\bullet_1) \oplus H^\bullet(C^\bullet_2)\) and we obtain a canonical isomorphism of determinant lines

\[
\det H^\bullet(C^\bullet_1 \oplus C^\bullet_2) = \det H^\bullet(C^\bullet_1) \otimes \det H^\bullet(C^\bullet_2).
\]

Then we have

\[
b_{\det H^\bullet(C^\bullet_1 \oplus C^\bullet_2)} = b_{\det H^\bullet(C^\bullet_1)} \otimes b_{\det H^\bullet(C^\bullet_2)}.
\]

In view of the \(b\)-orthogonal decomposition \((2.3)\) we may therefore without loss of generality assume \(\ker \Delta^k_{b, \bar{k}} = 0, \quad k = 0, 1\). Then by [7, Lemma 3.3] we have

\[
C^k = \text{im} d^k_{\bar{k-1}} \oplus \text{im} d^k_{\bar{k}}.
\]

This decomposition is \(b\)-orthogonal and invariant under \(\Delta_b\). Then we have the following two exact complexes

\[
0 \rightarrow C^0 \cap \text{im} d^k_{\bar{k}} \xrightarrow{d_0} C^1 \cap \text{im} d^k_{\bar{k}} \rightarrow 0
\]

and

\[
0 \rightarrow C^1 \cap \text{im} d^k_{\bar{k}} \xrightarrow{d_1} C^0 \cap \text{im} d^k_{\bar{k}} \rightarrow 0.
\]

Then from [7, Example 3.2], we get the proposition. \(\square\)

3 Symmetric bilinear torsion on the twisted de Rham complexes

In this section, we suppose that there is a fiber-wise non-degenerate symmetric bilinear form on \(E\). Then we define a symmetric bilinear torsion on the determinant line of the twisted de Rham complex.

3.1 Twisted de Rham complexes

In this subsection, we review the twisted de Rham complexes from [18].

Let \(M\) be a closed Riemannian manifold and \(E \rightarrow M\) be a complex flat vector bundle with flat connection \(\nabla\). Let \(H\) be an odd-degree closed differential form on \(M\). We set \(\Omega^0 = \Omega^{\text{even}}(M, E), \Omega^1 = \Omega^{\text{odd}}(M, E)\) and \(\nabla^H = \nabla + H \wedge\). We define the twisted de Rham cohomology groups as

\[
H^k(M, E, H) = \frac{\ker \left(\nabla^H : \Omega^k(M, E) \rightarrow \Omega^{k+1}(M, E)\right)}{\text{im} \left(\nabla^H : \Omega^{k+1}(M, E) \rightarrow \Omega^k(M, E)\right)}, \quad k = 0, 1.
\]

Suppose \(H\) is replaced by \(H' = H - dB\) for some \(B \in \Omega^0(M)\), then there is an isomorphism \(\varepsilon_B = e^B \wedge : \Omega^\bullet(M, E) \rightarrow \Omega^\bullet(M, E)\) satisfying

\[
\varepsilon_B \circ \nabla^H = \nabla^{H'} \circ \varepsilon_B.
\]

Therefore \(\varepsilon_B\) induces an isomorphism

\[
\varepsilon_B : H^\bullet(M, E, H) \rightarrow H^\bullet(M, E, H')
\]

on the twisted de Rham cohomology.
3.2 The construction of the symmetric bilinear torsion

Suppose that there exists a non-degenerate symmetric bilinear form on $E$. To simplify notation, let $C^k = \Omega^k(X,E)$ and let $d_k = d^H_k$ be the operator $\nabla^H$ acting on $C^k$ ($k=0,1$). Then $d_1d_0 = d_0d_1 = 0$ and we have a complex

$$\cdots \xrightarrow{d_1} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^0 \xrightarrow{d_0} \cdots.$$  \hspace{1cm} (3.1)

The metric $g^M$ and the symmetric bilinear form $b$ determine together a symmetric bilinear form on $\Omega^* (M,E)$ such that if $u = \alpha f$, $v = \beta g \in \Omega^* (M,E)$, then

$$\beta_{g,b} (u, v) = \int_M (\alpha \wedge \ast \beta) b(f, g),$$

where $\ast$ is the Hodge star operator. Denote by $d_k^\#$ the adjoint of $d_k$ with respect to the non-degenerate symmetric bilinear form (3.2). Then the Laplacians

$$\Delta_{b,k} = d_k^\# d_k + d_{k+1}^\# d_k^\# \quad k = 0,1$$

If $\lambda$ is in the spectrum of $\Delta_{b,k}$, then the image of the associated spectral projection is finite dimensional and contains smooth forms only. We refer to this image as the (generalized) $\lambda$-eigen space of $\Delta_{b,k}$ and denote it by $\Omega^k (\lambda) (M,E)$ and there exists $N_{\lambda} \in \mathbb{N}$ such that

$$(\Delta_{b,k} - \lambda)^{N_{\lambda}} |_{\Omega^k (\lambda) (M,E)} = 0.$$  \hspace{1cm} (3.2)

Then for different generalized eigenvalues $\lambda, \mu$, the spaces $\Omega^k (\lambda) (M,E)$ and $\Omega^k (\mu) (M,E)$ are $\beta_{g,b}$-orthogonal.

For any $a \geq 0$, set

$$\Omega^k_{[0,a]} (M,E) = \bigoplus_{0 \leq |\lambda| \leq a} \Omega^k (\lambda) (M,E).$$

Then $\Omega^k_{[0,a]} (M,E)$ is of finite dimensional and one gets a non-degenerate symmetric bilinear form $b_{\det H^* (\Omega^k_{[0,a]})}$ on $\det H^* (\Omega^k_{[0,a]})$. Let $\Omega^k_{(a, +\infty)} (M,E)$ denote the $\beta_{g,b}$-orthogonal complement to $\Omega^k_{[0,a]} (M,E)$.

For the subcomplexes $(\Omega^k_{(a, +\infty)} (M,E), d)$, since the operators $d_k d_k^\#$ and $\Delta_{b,k}$ are equal and invertible on $\text{im}(d_k) \cap \Omega^k_{(a, +\infty)} (M,E)$, we have

$$P_k := d_k^\# (d_k d_k^\#)^{-1} d_k = d_k^\# \left( \Delta_{b,k+1} \right)^{-1} d_k$$

is a pseudodifferential operator of order 0 and satisfies

$$P_k^2 = P_k.$$  \hspace{1cm} (3.3)

By definition we have

$$\zeta \left( s, d_k^\# d_k |_{\text{im}(d_k) \cap \Omega^k_{(a, +\infty)} (M,E)} \right) \big|_{M} = \text{Tr} \left( \Delta_{b,k}^{-\frac{s}{2}} P_k |_{\Omega^k_{(a, +\infty)} (M,E)} \right)$$

$$= \text{Tr} \left( P_k \Delta_{b,k}^{-\frac{s}{2}} |_{\Omega^k_{(a, +\infty)} (M,E)} \right).$$

Then $\zeta \left( s, d_k^\# d_k |_{\text{im}(d_k) \cap \Omega^k_{(a, +\infty)} (M,E)} \right)$ has a meromorphic extension to the whole complex plane and, by [23, Section 7], it is regular at 0. Then by [12, 25], we have the following result which is an analogue of [18, Theorem 2.1].
Theorem 3.1. For \( k = 0, 1 \), \( \zeta \left( s, d^*_k d_k |_{\Omega^\bullet_{(a,+\infty)}(M,E)} \right) \) is holomorphic in the half plane for \( \text{Re}(s) > n/2 \) and extends meromorphically to \( \mathbb{C} \) with possible poles at \( \{ \frac{n}{2l}, l = 0, 1, 2, \ldots \} \) only, and is holomorphic at \( s = 0 \).

Then for \( k = 0, 1 \) and any \( a \geq 0 \), the following regularized zeta determinant is well defined:

\[
(3.5) \quad \det' \left( d_k^* d_k \big|_{\Omega^\bullet_{(a,+\infty)}(M,E)} \right) := \exp \left( -\zeta' \left( 0, d_k^* d_k \big|_{\Omega^\bullet_{(a,+\infty)}(M,E)} \right) \right).
\]

Proposition 3.2. The symmetric bilinear form on \( \det H^\bullet(\Omega^\bullet(M,E,H), d) \) defined by

\[
(3.6) \quad b_{\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E),d)} \cdot \det' \left( d_k^* d_k \big|_{\Omega^\bullet_{[0,a]}(M,E)} \right)^{-1} \cdot \left( \det' \left( d_k^* d_k \big|_{\Omega^\bullet_{(a,+\infty)}(M,E)} \right) \right)
\]

is independent of the choice of \( a \geq 0 \).

Proof. Let \( 0 \leq a < c < \infty \). We have

\[
(3.7) \quad \left( \Omega^\bullet_{[0,c]}(M,E), d_k \right) = \left( \Omega^\bullet_{[0,a]}(M,E), d_k \right) \bigoplus \left( \Omega^\bullet_{(a,c]}(M,E), d_k \right)
\]

and

\[
\left( \Omega^\bullet_{(a,+\infty)}(M,E), d_k \right) = \left( \Omega^\bullet_{[a,c]}(M,E), d_k \right) \bigoplus \left( \Omega^\bullet_{(c,+\infty)}(M,E), d_k \right).
\]

Then by definition of the determinant, we get

\[
(3.8) \quad \det' \left( d_k^* d_k \big|_{\Omega^\bullet_{(a,+\infty)}(M,E)} \right) = \det' \left( d_k^* d_k \big|_{\Omega^\bullet_{[a,c]}(M,E)} \right) \cdot \det' \left( d_k^* d_k \big|_{\Omega^\bullet_{(c,+\infty)}(M,E)} \right).
\]

Applying Proposition 2.1 to (3.7), we get

\[
b_{\det H^\bullet(\Omega^\bullet_{[0,c]}(M,E))} = b_{\det H^\bullet(\Omega^\bullet_{[0,a]}(M,E))} \cdot \det' \left( d_k^* d_k \big|_{\Omega^\bullet_{[a,c]}(M,E)} \right)^{-1} \cdot \left( \det' \left( d_k^* d_k \big|_{\Omega^\bullet_{(c,+\infty)}(M,E)} \right) \right).
\]

Then we get the proposition. \( \square \)

Definition 3.3. The symmetric bilinear form defined by (3.6) is called the Ray-Singer symmetric bilinear torsion on \( \det H^\bullet(\Omega^\bullet(M,E,H), d) \) and is denoted by \( \tau_{b,\nabla,H} \).

4 Symmetric bilinear torsion under metric and flux deformations

In this section, we will use the methods in [18] to study the dependence of the torsion on the metric \( g \), the symmetric bilinear form \( b \) and the flux \( H \).

4.1 Variation of the torsion with respect to the metric and symmetric bilinear form

We assume that \( M \) is a closed compact oriented manifold of odd dimension. Suppose the pair \((g_u, b_u)\) is deformed smoothly along a one-parameter family with parameter \( u \in \mathbb{R} \). Let \( Q_k \) be the spectral projection onto \( \Omega^k_{[0,a]}(M,E) \) and \( \Pi_k = 1 - Q_k \) be the spectral projection onto \( \Omega^k_{(a,+\infty)}(M,E) \). Let

\[
\alpha = \star_u^{-1} \frac{\partial \star u}{\partial u} + b_u^{-1} \frac{\partial b_u}{\partial u}.
\]
Lemma 4.1. Under the above assumptions,

\begin{equation}
(4.1) \quad \frac{\partial}{\partial u} \log \left[ \det' \left( d_k^2 \right) \right]_0^1 (M, E) = \left( \det' \left( d_k^2 \right) \right)_0^1 (M, E) = - \sum_{k=0,1} (-1)^k \text{Tr}(\alpha Q_k).
\end{equation}

**Proof.** While \( d_k \) is independent of \( u \), we have

\[
\frac{\partial d_k^2}{\partial u} = - \left[ \alpha, d_k^2 \right].
\]

Using \( P_k d_k^2 = d_k^2, d_k P_k = d_k \) and \( P_k^2 = P_k \), we get \( d_k^2 d_k P_k = P_k d_k^2 d_k = d_k^2 d_k \) and

\[
\frac{\partial P_k}{\partial u} = \frac{\partial P_k}{\partial d_k} P_k, \quad P_k \frac{\partial P_k}{\partial u} = 0.
\]

Following the \( \mathbb{Z} \)-graded case, we set

\begin{equation}
(4.2) \quad f(s, u) = \sum_{k=0,1} (-1)^k \int_0^1 t^{s-1} \text{Tr} \left( e^{-td_k^2t_k} P_k \right) dt
= \Gamma(s) \sum_{k=0,1} (-1)^k \zeta \left( s, d_k^2 d_k \right).
\end{equation}

Using the above identities on \( P_k \), the trace property and by an application of Duhamel’s principal, we get

\begin{equation}
(4.3) \quad \frac{\partial f}{\partial u} = \sum_{k=0,1} (-1)^k \int_0^{1} t^{s-1} \text{Tr} \left( t ^{k} \left[ \alpha, d_k^2 \right] d_k e^{-td_k^2 t_k} \Pi_k + e^{-td_k^2 t_k} d_k \frac{\partial P_k}{\partial u} \right) dt
= \sum_{k=0,1} (-1)^k \int_0^{1} t^{s-1} \text{Tr} \left( \alpha \left( e^{-td_k^2 t_k} d_k^2 d_k - e^{-td_k^2 t_k} d_k d_k^2 \right) \Pi_k + e^{-td_k^2 t_k} d_k \frac{\partial P_k}{\partial u} \right) dt
= \sum_{k=0,1} (-1)^k \int_0^{1} t^s \text{Tr} \left( \alpha e^{-t \Delta_k \Pi_k} \right) dt
= - \sum_{k=0,1} (-1)^k \int_0^{1} t^s \left. \frac{\partial}{\partial t} \text{Tr} \left( \alpha e^{-t \Delta_k \Pi_k} \right) \right|_0^{1} dt.
\end{equation}

Integrating by parts, we have

\begin{equation}
(4.4) \quad \frac{\partial f}{\partial u} = s \sum_{k=0,1} (-1)^k \int_0^{1} t^{s-1} \text{Tr} \left( \alpha e^{-t \Delta_k \Pi_k} \right) dt
= s \sum_{k=0,1} (-1)^k \left( \int_0^{1} + \int_1^{+\infty} \right) t^{s-1} \text{Tr} \left( \alpha e^{-t \Delta_k \Pi_k} \right) dt.
\end{equation}
Since $\alpha$ is a smooth tensor and $n$ is odd, the asymptotic expansion as $t \downarrow 0$ for $\mathrm{Tr}(\alpha e^{-t\Delta_{b,k}})$ does not contain a constant term. Therefore

$$\int_0^1 t^{s-1} \mathrm{Tr}(\alpha e^{-t\Delta_{b,k}}) \, dt$$

does not have a pole at $s = 0$. On the other hand, because of the exponential decay of $\mathrm{Tr}(\alpha e^{-t\Delta_{b,k}} \Pi_k)$ for large $t$,

$$\int_{1}^{+\infty} t^{s-1} \mathrm{Tr}(\alpha e^{-t\Delta_{b,k}} \Pi_k)$$

is an entire function in $s$. So

$$(4.5) \quad \frac{\partial f}{\partial u} \bigg|_{s=0} = -s \sum_{k=0,1} (-1)^k \int_0^1 t^{s-1} \mathrm{Tr}(\alpha Q_k) dt \bigg|_{s=0} = - \sum_{k=0,1} (-1)^k \mathrm{Tr}(\alpha Q_k)$$

and hence

$$(4.6) \quad \frac{\partial}{\partial u} \sum_{k=0,1} (-1)^k \zeta \left(0, d^b_k \big|_{\Omega^k_{(u,\infty)}(M,E)} \right) = 0.$$ 

Finally, the result follows from (4.3), (4.6), and

$$(4.7) \quad \log \left[ \det' \left( d^b_0 \big|_{\Omega^0_{(u,\infty)}(M,E)} \right)^{-1} \right] \cdot \left( \det' \left( d^b_1 \big|_{\Omega^1_{(u,\infty)}(M,E)} \right) \right)$$

$$= \lim_{s \to 0} \left[ f(s,u) - \frac{1}{s} \sum_{k=0,1} (-1)^k \zeta \left(0, d^b_k \big|_{\Omega^k_{(u,\infty)}(M,E)} \right) \right].$$

\[\square\]

**Lemma 4.2.** Under the same assumptions, along any one-parameter deformation of $(g_u, b_u)$, we have

$$(4.8) \quad \frac{\partial}{\partial u} \left( \frac{b_{w,\det H^*} \left( \Omega^0_{(0,a)}(M,E), d \right)}{b_{u,\det H^*} \left( \Omega^0_{(0,a)}(M,E), d \right)} \right) = \sum_{k=0,1} (-1)^k \mathrm{Tr}(\alpha Q_k).$$

**Proof.** For sufficiently small $w - u$, the restriction of the spectral projection

$$Q_k \big|_{\Omega^k_{0,0,a}}(M,E) : \Omega^k_{0,0,a}(M,E) \rightarrow \Omega^k_{0,0,a}(M,E)$$

is an isomorphism of complexes. Then for sufficiently small $w - u$, we have

$$(4.9) \quad \frac{b_{w,\det H^*} \left( \Omega^0_{0,0,a}(M,E), d \right)}{b_{u,\det H^*} \left( \Omega^0_{0,0,a}(M,E), d \right)} = \det \left( \left( \beta_{g_u,b_u} \big|_{\Omega^0_{0,0,a}}(M,E) \right)^{-1} \left( Q_0 \big|_{\Omega^0_{0,0,a}}(M,E) \right) \right)^* \cdot \det \left( \left( \beta_{g_u,b_u} \big|_{\Omega^1_{0,0,a}}(M,E) \right)^{-1} \left( Q_1 \big|_{\Omega^1_{0,0,a}}(M,E) \right) \right)^* \cdot \left( \beta_{g_u,b_u} \big|_{\Omega^1_{0,0,a}}(M,E) \right)^{-1}.$$ 

Then similarly as in [7], we get (4.8).

\[\square\]

**Theorem 4.3.** Let $M$ be a closed, compact manifold of odd dimension, $E$ be a flat vector bundle over $M$, and $H$ be a closed differential form on $M$ of odd degree. Then the symmetric bilinear torsion $\tau_{b_\nabla,H}$ on the twisted de Rham complex does not depend on the choices of the Riemannian metric on $M$ and the symmetric bilinear form $b$ in a same homotopy class of non-degenerate symmetric bilinear forms on $E$. 

\[\square\]
4.2 Variation of analytic torsion with respect to the flux in a cohomology class

We continue to assume that dim\(M\) is odd and use the same notation as above. Suppose the (real) flux form \(H\) is deformed smoothly along a one-parameter family with parameter \(v \in \mathbb{R}\) in such a way that the cohomology class \([H] \in H^1(M, \mathbb{R})\) is fixed. Then \(\frac{\partial H}{\partial v} = -dB\) for some form \(B \in \Omega^0(M)\) that depends smoothly on \(v\); let

\[
\beta = B \wedge .
\]

**Lemma 4.4.** Under the above assumptions,

\[
\frac{\partial}{\partial v} \log \left[ \det^t \left( d_0^# d_0[\Omega^0_{(a,+,\infty)}(M,E)] \right)^{-1} \right] \left( \det^t \left( d_1^# d_1[\Omega^1_{(a,+,\infty)}(M,E)] \right) \right) = -2 \sum_{k=0,1} (-1)^k \text{Tr}(\beta Q_k).
\]

**Proof.** As in the proof of Lemma 4.1 we set

\[
f(s,v) = \sum_{k=0,1} (-1)^k \int_0^{+\infty} t^{s-1} \text{Tr} \left( e^{-td_k^# d_k} P_k[\Omega^k_{(a,+,\infty)}(M,E)] \right) dt.
\]

We note that \(B\), hence \(\beta\) is real. Using

\[
\frac{\partial d_k}{\partial v} = [\beta, d_k], \quad \frac{\partial d_k^#}{\partial v} = - \left[ \beta^#, d_k^# \right],
\]

\[
P_k^2 = P_k = P_k^#, \quad P_k \frac{\partial P_k}{\partial v} P_k = 0
\]

and by Dumahel’s principle, similarly as [18, Lemma 3.5], we get

\[
\frac{\partial f}{\partial v} = -2 \sum_{k=0,1} (-1)^k \int_0^{+\infty} t^{s} \frac{\partial}{\partial t} \text{Tr} \left( e^{-t\Delta_{v,k}H_k} \right) dt.
\]

The rest is similar to the proof of Lemma 4.1.

**Lemma 4.5.** Under the same assumptions, along any one-parameter deformation of \(H\) that fixes the cohomology class \([H]\), then we have

\[
\frac{\partial}{\partial v} \left( \frac{b_{\det H^*[\Omega^0_{[a]}(M,E,H^\nu),d)]}{b_{\det H^*[\Omega^0_{[a]}(M,E,H^\mu),d]}(M,E,H^\nu)} \right) = \sum_{k=0,1} (-1)^k \text{Tr}(\beta Q_k),
\]

where we identify \(\det H^*(M,E,H)\) along the deformation.

**Proof.** For sufficiently small \(w - v\), we have

\[
Q_{k,v} : \Omega^k_{[0,a]}(M,E,H^\nu) \rightarrow \Omega^k_{[0,a]}(M,E,H^\mu)
\]

is an isomorphism of complexes and the induced symmetric bilinear form on the determinant line \(\det H^* \left( \Omega^0_{[a]}(M,E,H^\nu), d \right)\) is

\[
\left( \left( \det (Q_{k,v})^* b_{\det H^*[\Omega^0_{[a]}(M,E,H^\nu),d]} \right) \right) \left( \cdot, \cdot \right) = b_{\det H^*[\Omega^0_{[a]}(M,E,H^\mu),d]} \left( \det (Q_{k,v}) \cdot, \det (Q_{k,v}) \cdot \right),
\]

where

\[
\det (Q_{k,v}) : \det H^* \left( \Omega^0_{[a]}(M,E,H^\nu) \right) \rightarrow \det H^* \left( \Omega^0_{[a]}(M,E,H^\mu) \right)
\]

is the induced isomorphism on the determinant lines. Then we can compare it with \(b_{\det H^*[\Omega^0_{[a]}(M,E,H^\nu),d]}\), similarly as [18, Lemma 3.7], we get (4.12).

\[\square\]
Combining Lemma 4.4 and Lemma 4.5, we have

**Theorem 4.6.** Let $M$ be a closed, compact manifold of odd dimension, $E$ be a flat vector bundle over $M$. Suppose $H$ and $H'$ are closed differential forms on $M$ of odd degrees representing the same de Rham cohomology class, and let $B$ be an even form so that $H' = H - dB$. Then the symmetric bilinear torsion $(\det B)^* \tau_{b,\nabla,H} = \tau_{b,\nabla,H}$.

5 Compare with the refined analytic torsion

In this section, we will compare the symmetric bilinear torsion $\tau_{b,\nabla,H}$ with the refined analytic torsion $\rho_{\text{an}}(\nabla^H)$ defined in [13]. The main theorem of this section is the following.

**Theorem 5.1.** Let $M$ be a closed odd dimensional manifold, $E$ be a complex vector bundle over $M$ with connection $\nabla$. $H$ be a closed odd-degree differential form on $M$. Suppose there exists a non-degenerate symmetric bilinear form on $E$. Then we have

$$
\tau_{b,\nabla,H} \left( \rho_{\text{an}} \left( \nabla^H \right) \right) = \pm e^{-\pi i (\eta(\nabla^H) - \text{rank}_E \cdot \eta_{\text{trivial}})} ,
$$

where $\eta(\nabla^H)$ and $\eta_{\text{trivial}}$ are defined in [13].

We will use the method in [5] to prove the theorem and the proof will be given later.

Let $h$ be a Hermitian metric on $E$. Then one can construct the Ray-Singer analytic torsion as an inner product on $\det H^*(M, E, H)$, or equivalently as a metric on the determinant line (cf. [13 (6.13)]). We denote the resulting inner product by $\tau_{b,\nabla,H}$. Then by Theorem 5.1 and [13, Theorem 6.2], we get

**Corollary 5.2.** If $\dim M$ is odd, then the following identity holds:

$$
\left| \frac{\tau_{b,\nabla,H}}{\tau_{h,\nabla,H}} \right| = 1.
$$

5.1 The dual connection

Let $M$ be an odd dimensional closed manifold and $E$ be a flat vector bundle over $M$, with flat connection $\nabla$. Assume that there exists a non-degenerate symmetric bilinear form $b$ on $E$. The dual connection $\nabla'$ to $\nabla$ on $E$ with respect to the form $b$ is defined by the formula

$$
db(u, v) = b(\nabla u, v) + b(u, \nabla' v), \quad u, v \in \Gamma(M, E).
$$

We denote by $E'$ the flat vector bundle $(E, \nabla')$.

5.2 Choices of the metric and the spectral cut

Till the end of this section we fix a Riemannian metric $g$ on $M$ and set $B^H = B(\nabla^H, g) = \Gamma \nabla^H + \nabla^H \Gamma$ and $B'^H = B'(\nabla'^H, g) = \Gamma \nabla'^H + \nabla'^H \Gamma$, where $\Gamma : \Omega^*(M, E) \to \Omega^*(M, E)$ is the chirality operator defined by

$$
\Gamma \omega = i^{\frac{n+1}{2}} (\pm (-1) \frac{n+1}{2}) \omega, \quad \omega \in \Omega^q(M, E).
$$

We also fix $\theta \in (-\pi/2, 0)$ such that both $\theta$ and $\theta + \pi$ are Agmon angles for the odd signature operator $B^H$. One easily checks that

$$
(\nabla^H)^\# = \Gamma \nabla^H \Gamma, \quad (\nabla'^H)^\# = \Gamma \nabla'^H \Gamma, \quad \text{and} \quad (B^H)^\# = B'^H.
$$

As $B^H$ and $(B^H)^\#$ have the same spectrum it then follows that

$$
\eta \left( B'^H \right) = \eta \left( B^H \right) \quad \text{and} \quad \text{Det}_{gr, \theta} (B^H) = \text{Det}_{gr, \theta} (B'^H).$$
5.3 A proof of Theorem 5.1

The symmetric bilinear form $\beta_{g,h}$ induces a non-degenerate symmetric bilinear form

$$H^j(M, E') \otimes H^{n-j}(M, E) \rightarrow \mathbb{C}, \quad j = 0, \ldots, n,$$

and, hence, identifies $H^j(M, E')$ with the dual space of $H^{n-j}(M, E)$. Using the construction of [13, Section 5.1] (with $\tau : \mathbb{C} \rightarrow \mathbb{C}$ be the identity map) we obtain a linear isomorphism

$$\alpha : \det H^\bullet(M, E, H) \rightarrow \det H^\bullet(M, E', H). \tag{5.4}$$

**Lemma 5.3.** Let $E \rightarrow M$ be a complex vector bundle over a closed oriented odd-dimensional manifold $M$ endowed with a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ and will be omitted. Let $H$ be a closed odd-degree differential form on $M$. Then

$$\alpha (\rho_{an} (\nabla^H)) = \rho_{an} (\nabla^H). \tag{5.5}$$

The proof is the same as the proof of [13, Theorem 5.3] (actually, it is simple, since $B^H$ and $B^H$ have the same spectrum and, hence, there is no complex conjugation involved) and will be omitted.

For simplicity, we denote $\det (d^H_0 \cdots d^H_{(a, +\infty)}(M, E))^{-1} \cdot (\det (d^H_0 \cdots d^H_{(a, +\infty)}(M, E)))$ by $\tau_{b, \nabla, H, (a, +\infty)}$. Let $\Delta^H = (\nabla^H)\otimes \nabla^H + \nabla^H (\nabla^H)\otimes$, then we have

$$\Delta^H = \Gamma \Delta^H \Gamma.$$

**Lemma 5.4.** The following identity holds,

$$\tau_{b, \nabla, H, (a, +\infty)} = \tau_{b, \nabla', H, (a, +\infty)}. \tag{5.6}$$

**Proof.** Applying (5.2) and using the fact that

$$\nabla^H : \Omega^{(a, +\infty)}(M, E, H) \cap \im (\nabla^H)\otimes \rightarrow \Omega^{k+1, (a, +\infty)}(M, E, H) \cap \im \nabla^H$$

is an isomorphism, we get

$$\tau_{b, \nabla, H, (a, +\infty)} = \prod_{k=0}^{\infty} \det' \left( \left( (\nabla^H)\otimes \nabla^H |_{\Omega^{k+1, (a, +\infty)}(M, E, H)} \right) \right)^{(-1)^{k+1}}$$

$$= \prod_{k=0}^{\infty} \det' \left( \Gamma \nabla^H (\nabla^H)\otimes \Gamma |_{\Omega^{k+1, (a, +\infty)}(M, E, H)} \right)^{(-1)^{k+1}}$$

$$= \prod_{k=0}^{\infty} \det' \left( \nabla^H (\nabla^H)\otimes |_{\Omega^{k+1, (a, +\infty)}(M, E, H)} \right)^{(-1)^k}$$

$$= \prod_{k=0}^{\infty} \det' \left( (\nabla^H)\otimes \nabla^H |_{\Omega^{k+1, (a, +\infty)}(M, E, H)} \right)^{(-1)^{k+1}} = \tau_{b, \nabla', H}. \tag{5.7}$$

Then for any $h \in \det H^\bullet(M, E, H)$, we have

$$\tau_{b, \nabla, H}(h) = \tau_{b, \nabla', H}(\alpha(h)). \tag{5.8}$$
Then form (5.5) and (5.8) we get
\[ \tau_b, \nabla, H (\rho_{\text{an}} (\nabla H)) = \tau_b, \nabla', H (\rho_{\text{an}} (\nabla'H)). \] (5.9)

Let
\[ \tilde{\nabla} = \begin{pmatrix} \nabla & 0 \\ 0 & \nabla' \end{pmatrix} \]
and
\[ \tilde{\nabla}'H = \begin{pmatrix} \nabla'H & 0 \\ 0 & \nabla'H \end{pmatrix}. \]

Then for any \( a \geq 0 \), we have
\[ \tau_b, \tilde{\nabla}, H, (a, +\infty) = \tau_b, \nabla, H, (a, +\infty) \cdot \nabla, H, (a, +\infty). \]

Then combining the latter equality with (5.9), we get
\[ \tau_b, \tilde{\nabla}, H (\rho_{\text{an}} (\tilde{\nabla}H)) = \tau_b, \nabla, H (\rho_{\text{an}} (\nabla H)) \cdot \tau_b, \nabla', H (\rho_{\text{an}} (\nabla'H)). \]

Hence, \( \rho_{\text{an}} (\tilde{\nabla}'H) \) is equivalent to the equality
\[ \tau_b, \tilde{\nabla}, H (\rho_{\text{an}} (\tilde{\nabla}H)) = e^{-4\pi i(\eta(\nabla H) - \text{rank}E \cdot \eta\text{trivial})}. \] (5.10)

By a slight modification of the deformation argument in \([5, \text{Section 4.7}]\), where the untwisted case was treated, we can obtain (5.10). Hence, we finish the proof of Theorem 5.1.

6 On the Cappell-Miller analytic torsion

In this section, we briefly discuss the extension of the Cappell-Miller analytic torsion to the twisted de Rham complexes. Let \( \dim M \) be odd.

Using notations above, we have the twisted de Rham complex \( \nabla^H : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E) \) and the chirality operator \( \Gamma : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E), \ k = 0, 1. \)

Define
\[ d_k = \Gamma d_k \Gamma : \Omega^k(M, E) \to \Omega^{k+1}(M, E). \]

Then consider the non-self-adjoint Laplacian
\[ \Delta_k^l = \left( d_k + d_k^* \right)^2 : \Omega^l(M, E) \to \Omega^l(M, E). \]

For any \( a \geq 0 \), let \( \Omega^{\bullet}_l(M, E) \) denote the span in \( \Omega^\bullet(M, E) \) of the generalized eigensolutions of \( \Delta_k^l \) with generalized eigenvalues with absolute value in \( [0, a] \). Then we have the decomposition of the complex
\[ (\Omega^\bullet(M, E), d) = \left( \Omega^{\bullet}_l(M, E), d \right) \oplus \left( \Omega^{\bullet}_l(M, E), d \right). \]

The subcomplex \( (\Omega^{\bullet}_l(M, E), d) \) is a \( \mathbb{Z}_2 \)-graded finite dimensional complex. Then we can define the torsion element \( \rho^{\bullet}_{l, [0, a]} \otimes \rho^{\bullet}_{l, [0, a]} \in \text{det}H^\bullet(\Omega^{\bullet}_l(M, E), d)^2 \cong \text{det}H^\bullet(M, E, H)^2 \), where \( \rho^{\bullet}_{l, [0, a]} \) defined by \([13, (2.22)]\). On the other hand, for
the subcomplex $(Ω^{\bullet}_{(a,++\infty)}(M, E), d)$, the following zeta-regularized determinant is well defined (cf. (3.5))

\[
\det'(d^k d_k|_{Ω^{\bullet}_{(a,++\infty)}(M, E)}) := \exp\left(-\zeta'(0, d^k d_k|_{im(0)\cap Ω^{\bullet}_{(a,++\infty)}(M, E)})\right).
\]

Consider the square of the graded determinant defined in [13, (2.38)], we find that for $\mathbb{Z}_2$-graded finite dimensional complex $Ω^{\bullet}_{(a,c)}(M, E)$, $0 \leq a < c < \infty$,

\[
\det'(d^0 d_0|_{Ω^{0}_{(a,c)}(M, E)}) \cdot \det'(d^1 d_1|_{Ω^{1}_{(a,c)}(M, E)})^{-1} = \left(\text{Det}_{\text{gr}}(B_0|_{Ω^{\bullet}_{(a,c)}(M, E)})\right)^2.
\]

Then by [13, Proposition 2.7], we easily get

**Proposition 6.1.** The torsion element defined by

\[
(6.1) \quad \rho^{\bullet}_{(0,0]} \otimes \rho^{\bullet}_{(0,0]} \cdot \prod_{k=0,1} \left(\det'(d^k d_k|_{Ω^{\bullet}_{(a,++\infty)}(M, E)})\right)^{(-1)^k} \in \det H^\bullet(M, E, H)^2
\]

is independent of the choice of $a \geq 0$.

**Definition 6.2.** The torsion element in $\det H^\bullet(M, E, H)$ defined by (6.2) is called the twisted Cappell-Miller analytic torsion for the twisted de Rham complex and is denoted by $\tau_{\nabla, H}$.

Next we study the torsion $\tau_{\nabla, H}$ under metric and flux deformations. Since the methods are the same as the cases in the twisted refined analytic torsion [13] and the twisted Burghelea-Haller analytic torsion above, we only briefly outline the results.

**Theorem 6.3.** (metric independence) Let $M$ be a closed odd dimensional manifold, $E$ be a complex vector bundle over $M$ with flat connection $\nabla$ and $H$ be a closed odd-degree differential form on $M$. Then the torsion $\tau_{\nabla, H}$ is independent of the choice of the Riemannian metric $g$.

**Proof.** By the definition of $\tau_{\nabla, H}$ and the observation on the determinants, this theorem follows easily from [13, Proposition 2.4], [13, (3.18)] and [13, (4.14)].

**Theorem 6.4.** (flux representative independence) Let $M$ be a closed odd dimensional manifold, $E$ be a complex vector bundle over $M$ with flat connection $\nabla$. Suppose $H$ and $H'$ are closed differential forms on $M$ of odd degrees representing the same de Rham cohomology class, and let $B$ be an even form so that $H' = H - dB$. Then we have $\tau_{\nabla, H'} = \det(\varepsilon_B)\tau_{\nabla, H}$.

**Proof.** From the above observation, this follows easily from [13, Lemma 4.6] and [13, Lemma 4.7].

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