Quantum Group Covariant Noncommutative Geometry. *

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Abstract

The algebraic formulation of the quantum group covariant noncommutative geometry in the framework of the $R$-matrix approach to the theory of quantum groups is given. We consider structure groups taking values in the quantum groups and introduce the notion of the noncommutative connections and curvatures transformed as comodules under the "local" coaction of the structure group which is exterior extension of $GL_q(N)$. These noncommutative connections and curvatures generate $GL_q(N)$-covariant quantum algebras. For such algebras we find combinations of the generators which are invariants under the coaction of the "local" quantum group and one can formally consider these invariants as the noncommutative images of the Lagrangians for the topological Chern-Simons models, non-abelian gauge theories and the Einstein gravity. We present also an explicit realization of such covariant quantum algebras via the investigation the coset construction $GL_q(N + 1)/(GL_q(N) \otimes GL(1))$.

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1 Introduction.

Noncommutative geometry [1] has started to play a significant role in the mathematical physics for last few years. One of the nontrivial examples of the noncommutative geometry is given by the quantum groups [2, 3]. After the paper [4], the differential geometric aspects of the theory of quantum groups have been intensively investigated recently (see e.g. [3]-[8]). Using these investigations various approaches to formulate quantum group gauge theories have been developed [9]-[13].

In this paper we prolong the researches of a quantum group covariant noncommutative geometry proposed in [9, 14]. In the Sect.2 we describe how it is possible to revise the usual commutative geometry (related to the geometry of the principal fibre bundle) and introduce differentials covariant under the special quantum group co-transformation interpreted as a local (structure) transformation. Here the special quantum group is an exterior extension of $GL_q(N)$. Then we define corresponding geometrical objects such as noncommutative 1-form connections and curvature 2-forms. We show that these noncommutative geometrical objects generate $GL_q(N)$-covariant quantum algebras. In the Sect.3 we discuss the noncommutative geometry related to the coset space $GL_q(N + 1)/(GL_q(N) \otimes GL(1))$. This geometry yields the nontrivial explicit example of the algebraical constructions considered in the Sect.2. Then, in the Sect.4, we compose from the generators of the $GL_q(N)$-covariant quantum algebras the set of $GL_q(N)$-local invariants, which could be considered as the noncommutative images of the well known gauge invariant Lagrangians (e.g. discrete gauge theories and Einstein gravity). Some of these invariants are nothing but noncommutative analogs of the Chern characters. We would like to stress, however, that this analogy with the conventional Lagrangians is rather formal and, strictly speaking, it may not lead to the $q$-deformations of the corresponding field theories.

We use the notation and methods of the paper [2] in which $R$-matrix formulation of the quantum groups have been elaborated. Some further development [15] of the $R$-matrix notation considerably simplifying the calculations is also employed. According to the results obtained in [13] one can reformulate our algebraical construction of the noncommutative geometry for the case of the unitary structure groups $U_q(N)$. Moreover we believe that using Brzezinski theorem [16] (and a generalization of it on the braided case [17]) about exterior Hopf algebras one can apply our construction to the case of any quasitriangular Hopf algebra with bicovariant first order differential calculus. In the Conclusion we briefly discuss this possibility and make some other remarks.

2 $GL_q(N)$-covariant derivatives, noncommutative connections and curvature.

Let us consider a $Z_2$-graded finite dimensional Zamolodchikov algebra (denoted by $\Omega_Z$) generated by the operators $\{e^i, (de)^j\}$, $(i, j = 1, 2, \ldots, N)$ with the following commutation relations:

\[
Ree' = cee', \quad (\pm)cR(de)e' = e(de)', \quad R(de)(de)' = -\frac{1}{c}(de)(de)', \quad (2.1)
\]
where \( e = e_1 \) is a \( q \)-vector in the first space, \( e' = e_2 \) is a \( q \)-vector in the second space, \( R = P_{12} R_{12} \) is a matrix which acts in the first and second spaces simultaneously, 
\[
P_{12} = \delta_{j_2}^{i_2} \delta_{j_1}^{i_1} \]
is the permutation matrix and
\[
R_{12} = R_{j_1,i_2}^{i_1,j_2} = \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} (1 + (q - 1) \delta_{j_1}^{i_1} \delta_{j_2}^{i_2}) + (q - q^{-1}) \delta_{j_2}^{i_2} \Theta_{i_1 i_2},
\]
(2.2)
is the \( GL_q(N) \) \( R \)-matrix satisfying the Hecke relation \((\lambda = q - q^{-1})\). 
\[
R^2 = \lambda R + 1.
\]
(2.3)
Here \( 1 \) is \((N^2 \times N^2)\) unit matrix. We imply the wedge product in the multiplication of the differential forms in the formulas (2.2) (we also omit \( \wedge \) in all formulas below).
One can recognize in the relations (2.1) \((\pm) = +1\) the Wess-Zumino formulas of the covariant differential calculus on the bosonic \((c = q)\) and fermionic \((c = -1/q)\) quantum hyperplanes [18] where \( e^i \) are the coordinates of the quantum hyperplane while \((de)^i\) are the associated differentials. The choice \((\pm) = -1\) corresponds to the case when \( e^i \) are bosonic \((c = -1/q)\) and fermionic \((c = q)\) vielbein 1-forms.
Note, that there is the second version of the algebra (2.1) obtaining by means of the replacement \( R \to R^{-1}, \, c \to c^{-1} \). Below, we concentrate only on the consideration of the algebra [18] (the other type can be treated analogously).
It has been suggested in [20, 19, 9] to consider the algebra \( \Omega_Z (2.1) \) as a comodule with respect to the coaction of the \( Z_2 \)-graded quantum group \( \Omega_{GL_q(N)} \) with the \( GL_q(N) \)-generators \( \{ T_j \} \) and additional generators \( \{(dT)^{ij}_l \} \) \((i, j, k, l = 1, 2, \ldots, N)\) which are the basis of the differential 1-forms on the quantum group \( GL_q(N) \). This coaction \( \Omega_Z \overset{g}{\to} \Omega_{GL_q(N)} \otimes \Omega_Z \) conserves the grading and can be written down as a homomorphism:
\[
e^i \overset{g}{\to} e^i = T_j^i \otimes e^j,
\]
(2.4)
\[
(dT)^i \overset{g}{\to} (dT)^i = (dT)^i_j \otimes e^j + T_j^i \otimes (dT)^i_j.
\]
(2.5)
Here \( \otimes \) denotes the graded tensor product: \( a \otimes b = (-1)^{\delta(a \otimes b)} (a \otimes 1) \), where \( \delta = \text{deg} (f) \) and \( a \in \Omega^{(a)}_{GL_q(N)}, \, b \in \Omega^{(b)}_Z \). We recall that the algebra \( \Omega_Z \) with the generators (2.1) has the following expansion \( \Omega_Z = \bigoplus_{n=0}^{\infty} \Omega^{(n)}_Z \), where \( \Omega^{(n)}_Z \) denotes the subspace of the differential \( n \)-forms and there exists a similar expansion for the \( Z_2 \)-graded quantum group \( \Omega_{GL_q(N)} = \bigoplus_{n=0}^{\infty} \Omega^{(n)}_{GL_q(N)} \). Substituting transformed algebra \( \{ e^i, (dT)^i \} \) into the commutation relations (2.1) we obtain the following equations for the generators \( \{ T_j^i, (dT)^i \} \)
\[
(R - c)TT'(R + c^{-1}) = 0 , \quad (R(dT)T' - T(dT)'R^{-1})(R + c^{-1}) = 0 ,
\]
(2.6)
\[
(R + c^{-1})(dT)(dT)'(R + c^{-1}) = 0 , \quad (R + c^{-1})(dT)T'R - R^{-1}T(dT)' = 0 ,
\]
(2.7)
where \( T = T_1 = T \otimes I \) while \( T' = T_2 = I \otimes T \) and \( I \) is a \((N \times N)\) unit matrix. The relations (2.6), (2.7) have to be fulfilled as for \( c = q \) as for \( c = -q^{-1} \) therefore
\[
\]
we deduce from them the following $q$-commutation relations for the bicovariant differential complex on $GL_q(N)$ (see [19, 4, 8]):

$$RTT' = TT'R, \quad (2.8)$$
$$R(dT)T' = T(dT)'R^{-1}, \quad (2.9)$$
$$R(dT)(dT)' = -(dT)(dT)'R^{-1}. \quad (2.10)$$

We stress that (2.10) follows from (2.9) if the differential $d$ is nilpotent $d^2 = 0$ and obeys the graded Leibnitz rule $d(fg) = d(f)g + (-1)^i f d(g)$. It is interesting to note (see [7]) that the algebra $\Omega_{GL_q(N)} (2.8)-(2.10)$ is the Hopf algebra. The comultiplication $\Delta$, the counit $\epsilon$ and the antipode $S$ are defined by

$$\Delta(T) = T \otimes T, \quad \epsilon(T) = I, \quad S(T) = T^{-1},$$
$$\Delta(dT) =dT \otimes T + T \otimes dT, \quad \epsilon(dT) = 0, \quad S(dT) = -T^{-1}dTT^{-1}, \quad (2.11)$$

and satisfy to the all axioms of the Hopf algebra. Thus, the algebra $\Omega_{GL_q(N)}$ yields the special example of the general exterior Hopf algebras discussed in [16]. We stress that the example of $GL_q(N)$-exterior Hopf algebra proposed in [16] has slightly different comultiplication comparing with the Hopf algebra $\Omega_{GL_q(N)} (2.8)-(2.11)$ independently introduced in [3]. One can show that it is possible to extend the action of the differential $d$ over the tensoring and apply $d$ to the algebra $\Omega_{GL_q(N)} \otimes \Omega_Z$ in such a way that: $d(g \otimes \Omega_Z) = d(g) \otimes \Omega_Z + (-1)^k g \otimes d(\Omega_Z)$, where $g \in \Omega^{(k)}_{GL_q(N)}$ and $d^2 = 0$.

Now we would like to interpret the formulas (2.4) and (2.5) as a local (structure) quantum group transformation of the comodule $e^i$. Here the matrix $T^i_j$ is understood as a noncommutative analog of a local (structure) group element. In view of this, it is natural to consider the appearing of the additional term $(dT)_j^i \otimes e^i$ in (2.5) as a noncovariance of the comodule $(de)^i$ under the "gauge" rotation (2.4) (or as an indication that the differentials $(d e)^i$ describe "nonparallel transporting" of the vector $e^i$). To restore the covariance let us introduce a covariant differential $\nabla$ in such a way that the transformations (2.4), (2.5) are rewritten in the form

$$e^i \xrightarrow{g} \bar{e}^i = T^i_j \otimes e^j, \quad (2.12)$$
$$\nabla (e)^i \xrightarrow{g} (\nabla e)^i = T^i_j \otimes (\nabla e)^j. \quad (2.13)$$

In general $(\nabla e)^i \notin \Omega_Z$ and, hence, the action of the operator $\nabla$ enlarges the algebra $\Omega_Z$ up to some new algebra $\Omega_{\bar{Z}}$. The operator $d$ can be induced (as a differential) onto the whole algebra $\Omega_{\bar{Z}}$ and this algebra is naturally decomposed as $\Omega_{\bar{Z}} = \bigoplus_{n=0}^{\infty} \Omega^{(n)}_{\bar{Z}}$, where $\Omega^{(n)}_{\bar{Z}}$ is the subspace of $n$-forms. Then we postulate that the elements $(\nabla e)^i \in \Omega^{(1)}_{\bar{Z}}$ are expanded over the generators $\{e^i, (de)^j\}$ of $\Omega_{\bar{Z}}$ in the following way:

$$(\nabla e)^i = (de)^i - A^i_j e^j, \quad (2.14)$$

It is clear that the coefficients $A^i_j$ belong to the subspace $\Omega^{(1)}_{\bar{Z}}$ and it is natural to consider them as noncommutative analogs of the connection 1-forms. Under the transformations (2.12) and (2.13) 1-forms $A^i_j$ are transformed as:

$$A^i_k \xrightarrow{g} \bar{A}^i_k = T^i_j (T^{-1})^j_k \otimes A^j_l + dT^i_j (T^{-1})^j_k \otimes 1 \equiv (TAT^{-1})^i_k + (dTT^{-1})^i_k, \quad (2.15)$$
\[ \nabla(\nabla e) = -(d(A) - A^2) e = -Fe. \] (2.16)

The quantum co-transformation (2.15) for the curvature 2-forms \( F^i_j \) is represented as the adjoint coaction:
\[
F^i_j \rightarrow \tilde{F}^i_j = T^i_k(T^{-1})^j_l \otimes F^k_l \equiv T^i_k F^k_l (T^{-1})^j_l .
\] (2.17)

The curvature tensor \( F^i_j \) is a reducible adjoint representation of \( GL_q(N) \) and it is possible to decompose it into the scalar \( F^0 = Tr_q(F) \) and the \( q \)-traceless tensor:
\[
\tilde{F}^i_j = F^i_j - \delta^i_j Tr_q(F)/Tr_q(I).
\]

Below we need the feature of invariance of \( q \)-trace:
\[
Tr_q(TET^{-1}) = Tr_q(E)
\] (2.19)

where \([T_{ij}, E_{kl}] = 0 \) and \( T^i_j \in GL_q(N) \). In particular, we have
\[
Tr_{q2}(RE\overline{R}^{-1}) = Tr_{q2}(\overline{R}^{-1}ER) = Tr_q(E)
\] (2.20)

Here \( Tr_{q2}(.) \) denotes quantum trace over the second space. We also use the relations:
\[
Tr_q(R^{\pm 1}) = q^{\pm N}, \quad Tr_q(I) = \frac{q^N - q^{-N}}{q - q^{-1}} \equiv [N]_q
\] (2.21)

The next action of the covariant derivative on the formula (2.14) yields the Bianchi identities that are represented in the classical form:
\[
d(F) = [A, F].
\]

To complete the definition of the algebra \( \Omega_{\overline{Z}} \) we have to deduce the commutation relations of the new generators \( \{A^i_j, F^i_j, \ldots\} \) and their cross-commutation relations with the generators \( \{e^i, (de)^j\} \). First of all, let us note that the choice of the connection in the pure gauge form (see (2.15))
\[
A^i_j = dT^i_k(T^{-1})^k_j \otimes 1 ,
\] (2.22)

leads to the conclusion that the generators \( A^i_j \) could satisfy the following \( q \)-deformed anticommutation relations:
\[
RARA + ARAR^{-1} = 0 ,
\] (2.23)
where $A = A_1 = A \otimes I$. These relations for the noncommutative 1-form connections (gauge fields) have been postulated in [3, 12]. Note, however, that in the right hand side of Eq. (2.23) one may add the arbitrary linear combination of the curvature 2-forms $F = dA - A^2$ which is vanished on the solution (2.22). Thus, the general covariant commutation relations for $A_i^j$ are

$$RARA + ARAR^{-1} = a(R)(FR + R^{-1}F) + \kappa(R)F^0 \equiv \Delta(F), \quad (2.24)$$

where $F = F_1 = F \otimes I$, $a(R) = a_1 + a_2R$ and for convenience we choose the parameter $\kappa(R)$ in the form: $\kappa(R) = (\kappa_1 + \kappa_2R)(R + R^{-1})$.

The special form of the right hand side of Eq. (2.24) is dictated by the symmetry properties of the $q$-anticommutator appeared in the left hand side of this equation ($c = \pm q^\pm$):

$$(R - c)(RARA + ARAR^{-1})(R + c^{-1}) = 0.$$ 

We stress that the anticommutation relations (2.24) are covariant under the transformations (2.15) and (2.17). Moreover one can extract from the relations (2.24) subsets of covariant relations using the methods proposed in [13]. Namely, applying $Tr_{q(2)}(\ldots)$ and $Tr_{q(2)}(\ldots R)$ to (2.24) and using (2.21) we obtain two sets of relations transformed as adjoint comodules:

$$\lambda q^N A^2 + \{A^0, A\} = [a_1(q^N + q^{-N}) + a_2([N]_q + \lambda q^N)]F + a_2F^0 +$$

$$\quad + \{\kappa_1(q^N + q^{-N}) + \kappa_2(2[N]_q + \lambda q^N)]F^0,$$

$$q^N A^2 + (A * A) = [a_1([N]_q + \lambda q^N) + a_2q^2(q^2 + q^{-2})]F + (a_1 + \lambda a_2)F^0 +$$

$$\quad + \left[\kappa_1(2[N]_q + \lambda q^N) + \kappa_2(q^2 + q^{-2}) + \lambda[N]_q\right]F^0,$$

where $(A * A) = Tr_{q(2)}(RARA), \quad F^0 = Tr_q(F), \quad A^0 = Tr_q(A)$. Then, applying $Tr_{q(1)}(\ldots)$ to (2.23) and (2.26) we obtain two scalar relations ($q^2 \neq -1$)

$$Tr_q(A^2) = \left[(a_1 + \kappa_1)q^{-N}[N]_q + (a_2 + \kappa_2)\right]F^0,$$

$$(A^0)^2 = [(a_1 + \kappa_1)q^{-N} + (a_2 + \kappa_2)[N]_q]F^0. \quad (2.27)$$

We see that in the noncommutative case Eqs. (2.27)-(2.28) give additional relations of 1-form connections $A$ and 2-form curvatures $F \equiv dA - A^2$.

Arbitrary parameters $a_i, \quad \kappa_i$ introduced in Eq. (2.24) depend on the choice of the noncommutative geometry and have to be fixed partially by the consistency conditions (with respect to the two ways of ordering of any cubic monomial) for the algebra $\Omega_q$. It is amusing to note that the additional nonzero term included into the right-hand side of (2.24) looks similar to the quantum anomaly terms arising in the (anti)commutators of fields (or currents) in certain conventional quantum field theories.

In order to find commutation relations $A^i_j$ with the generators $\{e^i, (de)^i\}$ we postulate that the coordinates of the comodule (2.14) commute in the same way as the components of 1-forms $(de)^i$ (see (2.1))

$$R(\nabla e)(\nabla e)' = -\frac{1}{c}(\nabla e)(\nabla e)' \quad (2.29)$$
\[ (\pm)(c - b)R(\nabla e)e' = e(\nabla e)' . \]  
(2.30)

where \( b \) is a constant which is fixed below. Let us stress that Eqs. (2.29), (2.30) are not the general covariant relations of that kind. For example one can add to (2.29) the terms of the type \((Fe)e'\). We however prefer to consider here the simplest case of the relations (2.29), (2.30). From (2.1) and (2.30) we deduce covariant commutation relations of \( A \) and \( e \):

\[ (\pm)eA' = RAR e + bR(\nabla e) \]  
(2.31)

Considering the consistency condition for the reordering (in two different ways) the monomial \( ee'\mathcal{A}'' \equiv e_1e_2A_3 \) we obtain only two solutions for the parameter \( b \):

\[ A.) \quad b = 0 \quad B.) \quad b = \lambda . \]  
(2.32)

Thus, we have two variants for the Eq. (2.31)

\[ A.) \quad (\pm)eA' = RAR e , \quad B.) \quad (\pm)eA' = RAR^{-1}e + \lambda R(de) . \]  
(2.33)

Note, that in the paper [9] we have considered only the first case A.) : \( b = 0 \). Taking into account (2.29) one can obtain the corresponding commutation relations for \((de)\) and \( A \)

\[ (\pm)(de)A' = -R^{-1}AR(de) + (b - \lambda)AR(\nabla e) + \tilde{a}(R)\Delta(F)e , \]  
(2.34)

where

\[ \tilde{a}(R) = \frac{1 + \gamma(R - c)}{1 + c^2} \]  
(2.35)

and \( \gamma \) is a new arbitrary constant to be fixed below. Type A.) and type B.) commutation relations (2.31), (2.34) are covariant under the gauge coactions (2.4), (2.5) and (2.15) and both cases lead to the same covariant commutation relation for \((\nabla e)\) and \( A \):

\[ (\pm)(\nabla e)A' = -RAR(\nabla e) + (\tilde{a}(R) - 1)\Delta(F)e , \]  
(2.36)

Differentiating (2.31) and, then, using (2.34) one can derive

\[ eF' = RF(R - b)e + \tilde{a}(R)\Delta(F)e = \]  
(2.37)

\[ = (R + \tilde{a}(R)a(R))FR e + (\tilde{a}(R)a(R)R^{-1} - bR)F e + \tilde{a}(R)\kappa(R)F^0 e \]

where we define

\[ \tilde{a}(R) = -(1 + bR)\tilde{a}(R) + (b - \lambda)R . \]  
(2.38)

Considering the reordering of the monomials \( ee'\mathcal{F}' \) in two possible ways and comparing the results we obtain for both types A.) \( b = 0 \) and B.) \( b = \lambda \) the restrictions

\[ 1.) \quad \tilde{a}(R)a(R) = 0 , \quad \tilde{a}(R)\kappa(R) = 0 , \]  
(2.39)

which leads to the commutation relation:

\[ eF' = RF(R - b)e . \]  
(2.40)
Note, that for the type A.) $(b = 0)$ we have an additional solution

$$2.) \tilde{a}(R)a(R) = -\lambda, \quad \tilde{a}(R)\kappa(R) = 0$$

equivalent to the relation: $eF' = R^{-1}FR^{-1}e$. This relation, however, is consistent with the algebra (2.24), (2.31) and (2.36) only if some additional relations on the generators of $\Omega_Z$ will be fixed. One can prove this by considering two different ways of reordering of the monomials $eR'A'R'A'$ where $R' = P_{23}R_{23}$.

Taking into account the conditions (2.39) we obtain from the definitions (2.38) and (2.35) the following solutions for the parameters $a(R)$ and $\gamma$

$$1.) \ a(R) = 0, \ \kappa(R) = 0 \Rightarrow \Delta(F) = 0,$$

$$2.) \ a(R) = a_0(R - c), \ \kappa(R) = \kappa_0(R - c), \ \gamma = \frac{1}{c + c^{-1}} + (b - \lambda) \Rightarrow \quad (2.41)$$

$$(R - c)\tilde{a}(R) = \frac{(\lambda - b)}{c}(R - c), \quad (R - c)\tilde{a}(R) = \frac{b(\lambda - b)}{c^2}(R - c) \equiv 0.$$  

Here $a_0 \neq 0, \ \kappa_0 \neq 0$ are constants.

Now, we deduce the covariant commutation relations for the generators $F^i_j$ postulating the following natural quantum hyperplane condition

$$\quad (R - c)(Fe)(F'e') = 0. \quad (2.42)$$

Using (2.40) one can obtain from (2.42) the following relations

$$\quad (R - c)FRF(R + c^{-1}) = 0. \quad (2.43)$$

The commutation relations for the curvature 2-form $F^i_j$ have to be independent of the class of the comodule $\{e^i\}$ and therefore of the choice of the parameter $c = \pm q^{\pm 1}$. So, we deduce from Eqs.(2.43) the commutation relations

$$RFRF = FRFR. \quad (2.44)$$

These relations are known, first, as reflection equations [22], second, as the commutation relations for invariant vector fields on $GL_q(N)$ [7, 8] and, third, as the defining relations for the braided algebras [23].

To complete the definition of the algebra $\Omega_Z$ one can deduce the following cross-commutation relation for $F$ and $A$:

$$FRAR = RARF. \quad (2.45)$$

This is the simplest relation covariant under the coactions presented in (2.13) and (2.17) and allowing one to push the operators $F$ through the operators $A$.

Thus, leaving aside the commutation relations with generators $\{e, de\}$, we come to the following algebra with generators $A$ (1-form connection) and $F = dA - A^2$ (2-form curvature):

$$FRAR = RARF, \quad RFRF = FRFR, \quad RARA + ARAR^{-1} = a(R)(FR + R^{-1}F) + \kappa(R)F^0, \quad (2.46)$$
where \( \alpha (\mathbf{R}) = (\mathbf{R} - c) a_0 \) and \( \kappa (\mathbf{R}) = (\mathbf{R} - c) \kappa_0 \) (see Eqs. (2.41)). Note, that for the case \( a_0 \neq 0 \) and \( \kappa_0 \neq 0 \) the consistence conditions for the whole covariant algebra \( \Omega_Z \) give some additional constraints on the generators of this algebra. In particular, one can deduce

\[
(\mathbf{R} - c) \tilde{F} \mathbf{R} e = 0,
\]

(2.47)

where \( \tilde{F} = F - \frac{\kappa_0}{a_0 (c + c^{-1})} F^0 \).

3 \( GL_q(N+1)/(GL_q(N) \otimes GL(1)) \) noncommutative geometry.

In this Section we present an explicit realization of such covariant algebra \( \Omega_Z \) where parameters \( a_0, \kappa_0 \) and additional relations (of the type (2.47)) on the generators will be fixed. We consider the differential geometry on the group \( GL_q(N+1) \) \([19, 20, 7, 8]\) and interpret it as the noncommutative geometry on the total space of the principal fibre bundle with the base space \( GL_q(N+1)/(GL_q(N) \otimes GL(1)) \) and the structure group being \( GL_q(N) \otimes GL(1) \).

Let us introduce \( Z_2 \)-graded extension of the \( GL_q(N+1) \) quantum group (exterior Hopf algebra) with the generators \( \{ T^I_J, dT^I_J \} \) \((I, J = 0, 1, \ldots N)\) satisfying the commutation relations (2.8)-(2.10) where \( GL_q(N+1) \) \( R \)-matrix acts in the space \( \text{Mat}(N+1) \times \text{Mat}(N+1) \). Then, we consider the following left coaction of the group \( GL_q(N) \otimes GL(1) \) on the group \( GL_q(N+1) \):

\[
T^I_J \rightarrow \left( \begin{array}{cc} t & 0 \\ 0 & T^i_k \end{array} \right) \otimes \left( \begin{array}{cc} T^0_0 & T^0_J \\ T^k_0 & T^k_J \end{array} \right)
\]

(3.1)

where as usual \( i, j, k = 1, 2, \ldots N \) and \( t \) \(([t, T^i_j] = 0)\) is a dilaton generator of \( GL(1) \). It is evident (from the commutation relations for the \( GL_q(N+1) \)-generators) that the elements \( T^i_j \) generate the quantum group \( GL_q(N) \). The noncommutative coordinates for the "base space" \( GL_q(N+1)/(GL_q(N) \otimes GL(1)) \) could be related with the generators \( T^0_i \) and \( T^j_0 \). For the Cartan 1-forms on the \( GL_q(N+1) \)-group:

\[
\Omega^I_J = dT^K_J(T^{-1})^K_I = \left( \begin{array}{cc} \omega & \Omega^0_j = < \bar{e}|_j \\ \Omega^i_0 = |e >^i & A^i_j \end{array} \right)
\]

(3.2)

the coaction (3.1) is represented in the form:

\[
\left( \begin{array}{c} \omega < \bar{e}| \\ |e > A \end{array} \right) \rightarrow \left( \begin{array}{c} \omega + dtT^{-1} < \bar{e}|T^{-1}t \\ t^{-1}T|e > TAT^{-1} + dTT^{-1} \end{array} \right)
\]

(3.3)

where the short notation have been used (see e.g. (2.15)). Comparing these transformations with the transformations (2.12) and (2.13) it becomes clear that the Cartan 1-forms \( |e > A, \omega \) can be interpreted as veilein 1-forms and connection 1-forms respectively. Then, the generators \( < \bar{e}| \) are nothing but contragradient
The Maurer-Cartan equation $d\Omega^I_J = \Omega^K_I \Omega^K_J$ leads to the following constraints on the noncommutative differential 1-forms $\Omega^I_J$:

$$
\begin{pmatrix}
 d\omega - \omega^2 - <\bar{e}|e> & d<\bar{e}|A-e <\bar{e}|e> \\
 d|e>-|e| > e > -|e| > e > & dA - A^2 - |e| <\bar{e}|e>
\end{pmatrix} = 0 .
$$

(3.4)

The $q$-deformed commutators for the noncommutative Cartan 1-forms (3.2) is deduced from the $N+1$-dimensional analog of the relations presented in (2.23). Taking into account Maurer-Cartan equations (3.4) and using the notation (3.2) we rewrite these relations in the form:

$$
\text{RARA} + \text{ARAR}^{-1} = -\lambda (\text{RF} + \text{FR}^{-1})
$$

(3.5)

$$
-e\omega' = \text{RAR} e + \lambda R(d-e-A e) , \quad -A'\bar{e} = \bar{e} \text{RAR} + \lambda (d\bar{e} - \bar{e}A)R
$$

(3.6)

$$
\bar{e}R e = -q e'\bar{e}' , \quad R e e' = -q^{-1} e e' , \quad e' \bar{e} R = -q^{-1} \bar{e} e ,
$$

(3.7)

$$
\omega^2 = 0 , \quad [\omega, e]_+ = [\omega, \bar{e}]_+ = 0 , \quad [A, \omega]_+ = q \lambda |e| <\bar{e}| = q \lambda F.
$$

(3.8)

Here we have also introduced the notation for the curvature 2-form:

$$
F = dA - A^2 = |e| <\bar{e}| = -q^{-1} <\bar{e}|1R|e|_1 .
$$

(3.9)

The last two equalities follow from Eqs.(3.4) and (3.7) and reveals the dependence of the curvature 2-forms and the veilbein 1-forms. Note, that for the such form (3.9) of the curvature one can directly prove the identity (2.47) (for $\kappa_0 = 0$) using the relations (3.7). Now, we find (applying the commutation relations (3.5)-(3.8) and Eq.(3.9)), that the following relations for $F$ and $A$ are hold:

$$
\text{RFRF} = \text{FRFR} , \quad \text{RARF} = \text{FRAR} + \lambda (\text{RF} \omega - \text{F} \omega R)
$$

(3.10)

We would like to compare these relations with the relations (2.46) but on this stage we can not do it in view of the appearing in (3.10) the additional scalar generator $\omega$ which is nothing but $GL(1)$-connection 1-form (see (3.3)). To exclude from the considerations these scalar connection 1-form we introduce new total $GL_q(N) \otimes GL(1)$ connection:

$$
A_t = A - \omega I ,
$$

(3.11)

for which we have

$$
\nabla_t e = \nabla_t \bar{e} = 0 ,
$$

(3.12)

(see (3.4)) and the corresponding curvature 2-form:

$$
F_t = q^2 F - <\bar{e}|e> = q^2 F + q^{1-N} . F^0
$$

(3.13)

satisfy the conditions

$$
F_t|e >= <\bar{e}|F_t = 0 .
$$

(3.14)
The scalar 2-form $F^0 = \text{Tr}_q(F)$ in (3.13) is defined by Eq. (2.18) and is invariant under the adjoint coaction (2.17). Finally, we find from Eqs. (3.6)-(3.8) and (3.10) that the elements $\{e, A_t, F\}$ generate the following closed algebra:

\[
\begin{align*}
R^FRF &= FRFR, \quad RA_tRF = FRA_tR, \\
RA_tA_t + A_tA_tA_t^{-1} &= a_0(FR^{-1} + RF)(R - c), \\
eA_t' &= RA_tRe, \quad eF' = RFRe
\end{align*}
\]

where $a_0 = 1 - q^2$ and $c = -q^{-1}$.

Comparing the commutation relations (3.7) and (3.15) with the relations (2.1), (2.31) and (2.46) one can infer that we have explicitly realized the defining relations for the covariant quantum algebra $\bar{\Omega}_Z$ of the type $A_1$ (2.32), (2.33) in terms of the algebraic objects related to the $GL_q(N + 1)/GL_q(N) \otimes GL(1)$-geometry. To be precise we have to consider the algebra of the type (3.15) with substitution $F \leftrightarrow F_t$.

The corresponding defining relations are

\[
\begin{align*}
R^FRF_t &= F_tRF_R, \quad RA_tRF_t = F_tRA_tR, \\
RA_tA_t + A_tA_tA_t^{-1} &= a_0(F_tR^{-1} + RF_t)(R - c) + \frac{a_0(c + q^{-1})(R - c)}{q^{1[N]_q}} F^0_t, \\
eA_t' &= RA_tRe, \quad eF'_t = q^{-2}R^{-1}F_tRe
\end{align*}
\]

One can note that such kind algebras in view of the last relation of (3.16) were not presented in general consideration of Sect.2. The explanation of this fact is that in the Sect.2 we essentially use the conditions $\nabla e \neq 0, F_t|e > \neq 0$ which are not fulfilled here (see (3.12) and (3.14)). That is why we have not received in the Sect.2 the cross-commutation relations for $F$ and $e$ presented in (3.16).

4 \quad GL_q(N)$-local co-invariants and Chern characters.

Our final aim is to define composite elements $\mathcal{L}$ for the extended algebra $\bar{\Omega}_Z$ which are co-invariant $\mathcal{L} \to 1 \otimes \mathcal{L}$ under the $GL_q(N)$ local transformations (2.3), (2.4), (2.13) and (2.17). We would like to interpret these elements $\mathcal{L}$ as noncommutative Lagrangians. However, we stress that this interpretation is rather formal because the elements $\mathcal{L}$ are not the usual Lagrangians for certain field theories. To write down such noncommutative Lagrangians we further extend the algebra $\bar{\Omega}_Z$ described in the Sect.2 by virtue of introducing $Z_2$-graded contragradient comodule $(\bar{e}_i, d\bar{e}_j)$ with the following commutation relations:

\[
\begin{align*}
\bar{e}' \cdot \bar{e}R &= c\bar{e}'\bar{e}, \quad (d\bar{e})'(d\bar{e}) = (\pm)c\bar{e}'(d\bar{e})R, \\
(d\bar{e})'(d\bar{e})R &= -\frac{1}{c}(d\bar{e})'(d\bar{e}).
\end{align*}
\]

Note that contragradient $q$-vectors have naturally appeared in the context of the explicit example of the $GL_q(N)$-covariant noncommutative geometry considered in the Sect.3. The quantum group local (structure) transformation of the vector $(\bar{e}_i, d\bar{e}_j)$
is expressed as the following homomorphism of the algebra (4.1):

\[
(\bar{e}, d\bar{e}) \xrightarrow{\theta} \left( (T^{-1})^i_k \otimes \bar{e}_k, \ d(T^{-1})^i_j \otimes \bar{e}_k + (T^{-1})^i_j \otimes d\bar{e}_k \right) \equiv \\
\equiv (\bar{e}, d\bar{e}) \cdot \left( \begin{array}{c} T^{-1}, \\
0, \\
T^{-1}dTT^{-1} \end{array} \right),
\]

(4.2)

where in the last equality of (4.2) we have used the short notation (see (2.13), (2.17)) and the operators \( T^i_j \) and \( dT_k^i \) are the same as in Eqs.(2.8)-(2.10). The commutation relations for the coordinates of the contragradient \( \bar{q} \)-vectors \( \{\bar{e}_i, d\bar{e}_j\} \) with the former generators of \( \Omega_{\bar{Z}} \) can be found using covariance of these relations under the gauge co-actions (2.4), (2.5), (2.15), (2.17) and (4.2). For example, one can assume the relations of the type appeared in the explicit construction of the Sect.3:

\[
\begin{align*}
\bar{e}' \bar{e}' &= c\bar{e}R \bar{e}, \\
(\pm)(de)' \bar{e}' &= c(\bar{e}R(de) + \lambda \bar{e}RAR\bar{e}), \\
A'\bar{e} &= (\pm)\bar{e}RAR, \\
F' \bar{e} &= \bar{e}RFR.
\end{align*}
\]

(4.3)

(4.4)

These relations are not unique covariant relations for the generators \( \{e, \bar{e}, A, F, \ldots\} \). There are another choices corresponding to the another noncommutative geometry. For example in our paper [9] we have proposed the noncommutative geometry with different relations (4.3).

Now one can define the co-invariant elements of \( \Omega_{\bar{Z}} \) transformed under the local co-transformations as \( L \rightarrow 1 \otimes L \). For example, using the noncommutative generators \( e^i, \bar{e}_i \) and \( A^i_j \) we construct the co-invariant

\[
L = \bar{e}_i \left( de^i - A^i_j e^j \right).
\]

(4.5)

We call these composite elements of the algebra \( \Omega_{\bar{Z}} \) the noncommutative (algebraical) Lagrangians bearing in mind the formal similarity of (4.5) to the Lagrangians for the one dimensional discrete gauge models (see e.g. [24]).

In order to write down other local quantum group co-invariants, it is convenient to use the curvature 2-form \( F \) transformed as the adjoint comodule (2.17). As an example we present the noncommutative analogs of Chern characters. For this, let us consider the special case of the closed algebra (2.43) with the generators \( A \) and \( F \) where the parameters \( a(R) = 0 \) and \( \kappa(R) = 0 \). Here, as we have explained above, \( A^i_j \) are noncommutative analogs of connection 1-forms, while \( F^i_j \) are interpreted as curvature 2-forms. In analogy with the classical case (see e.g. [25]), we consider as invariant characters the following expressions:

\[
C_k = Tr_q(F^k) = D^i_j F^j_{i1} \cdots F^j_{ik-1},
\]

(4.6)

where we have used the \( q \)-deformed trace defined in (2.18). Using (2.19) we immediately obtain that 2k-forms \( C_k \) (1.6) are invariant under the adjoint coaction (2.17). Moreover, \( C_k \) are the closed 2k-forms. Indeed, from the Bianchi identities \( dF = [A, F] \) we deduce

\[
dC_k = Tr_q(AF^k - F^k A) = 0,
\]

(4.7)
where we have taken into account (see Eqs. (2.46), (2.20) and (2.21))

\[ Tr_q(AF^k) = q^{-N} Tr_q1(Tr_q2(R^{-1}RARF^k)) = \]

\[ q^{-N} Tr_q1(Tr_q2(F^kRA)) = Tr_q(F^kA). \]

We believe that \( C_k \) have to be presented as the exact form \( C_k = dL^{(k)}_{CS} \), where the Chern-Simons \((2k - 1)\)-forms \( L^{(k)}_{CS} \) are represented as

\[ L^{(k)}_{CS} = Tr_q \{ A(dA)^{k-1} + \frac{1}{h_1^{(k)}} A^3(dA)^{k-2} + \ldots + \frac{1}{h_{(k)}} A^{2k-1} \} \] (4.8)

and the constants \( h_i^{(k)} \) depend on the deformation parameter \( q \). We do not have explicit formulas for all parameters \( h_i^{(k)} \) (in the classical case \( q = 1 \) these formulas are known [26]), but for the case \( k = 2 \) one can obtain a noncommutative analog of the three-dimensional Chern-Simons term in the form:

\[ L^{(2)}_{CS} = Tr_q \{ AdA - \frac{1}{h_1^{(2)}} A^3 \}, \quad h_1^{(2)} = 1 + \frac{1}{q^2 + q^{-2}}. \] (4.9)

We would like to note that it is extremely interesting to write the Chern characters for the general case of the algebra (2.46) when the parameters \( a(R) \neq 0 \) and \( \kappa(R) \neq 0 \).

At the end of this Section we propose the way how to find the algebraical Lagrangian corresponding to the field theoretical Lagrangian for the Einstein gravity. First, we take the four generators of the underlying Zamolodchikov algebra (2.1) in the form of \( 2 \times 2 \) matrix \( e_{ij} \) (\( i, j = 1, 2; \quad e^\dagger = e \)) interpreted as the spinorial representation for the 4-dimensional veilbein 1-forms. The differential complex \( \Omega_Z \) for this algebra is the anticommuting version \((\pm) = +1\) of the differential complex for the \( q \)-Minkowski space [28, 29]

\[ R e R e + e R e R^{-1} = 0, \] (4.10)

\[ R de R e - (\pm)e R de R = 0, \] (4.11)

\[ R de R de - de R de R = 0. \] (4.12)

Note that there is another consistent differential complex with the choice of eq. (4.11) in the form \( R e R de = (\pm)de R e R \). Here we do not consider this possibility which is absolutely parallel. The factor \( (\pm) = -1 \) corresponds to the fermionic version of the \( q \)-Minkowski space. The algebra (4.10)-(4.12) is covariant under the \( q \)-Lorentz global transformations

\[ e \rightarrow Te\tilde{T}^{-1}, \] (4.13)

\[ de \rightarrow Tde\tilde{T}^{-1}. \] (4.14)

where \( \{ e, de \} \) commute with \( \{ T, \tilde{T} \} \) and elements of matrices \( T \) and \( \tilde{T} = (T^\dagger)^{-1} \) are the generators of the two \( SL_q(2) \)-groups with the following crossing-commutation relations

\[ RT\tilde{T}' = \tilde{T}'T R, \] (4.15)
This formulation of the $q$-Lorentz group have been proposed and investigated in [27]-[29]. Using the $q$-trace (2.18) one can construct from the generators $e^{ij}$ the contragradient veilbein 1-forms $\bar{e}_{ij}$:

$$\bar{e}_{ij} = e^{ij} - q^{-1}Tr_q(\epsilon)\delta_{ij}. \quad (4.16)$$

The co-transformation (4.13),(4.14) for $\bar{e}$ reads

$$\bar{e} \to \bar{T}eT^{-1}, \quad d\bar{e} \to \bar{T}deT^{-1}. \quad (4.17)$$

Further we need the differential calculus on $SL_q(2)$. Up to now we do not have the appropriate calculus on $SL_q(N)$ (see however [30]). Therefore we will consider the case of extended Lorentz symmetry generated by $\Omega_{GL_q(2)}$. In this case one can consider the local version of the transformation (4.14)

$$de \to dT e T^{-1} + T de T^{-1} - (\pm)Ted\bar{T}^{-1}, \quad (4.18)$$

where $\{T, dT\}$ and $\{\bar{T}, d\bar{T}\}$ are two isomorphic $GL_q(2)$-exterior algebras (2.8)-(2.10) with the cross-commutation relations defined by eq.(4.15) and

$$RT d\bar{T}' = d\bar{T} T' R, \quad RdT' \bar{T}' = \bar{T} dT' R, \quad RdT d\bar{T}' = -d\bar{T} dT' R, \quad (4.19)$$

Note that the formulas (2.8)-(2.10),(4.15) and (4.19) for the $GL_q(N)$ $R$-matrix define the differential complex on $GL_q(N,C)$. Then one can introduce the covariant derivative

$$(\nabla e) = de - Ae - e\tilde{A} \quad (4.20)$$

where connection 1-forms $A$ and $\tilde{A}$ are transformed as

$$A \to TAT^{-1} + dTT^{-1}, \quad \tilde{A} \to \bar{T}\tilde{A}\bar{T}^{-1} + d\bar{T}\bar{T}^{-1}. \quad (4.21)$$

For the consistence we demand that $\tilde{A} = A^\dagger$. The corresponding curvature 2-forms $F$ and $\bar{F}$ are defined as usual

$$F = dA - A^2, \quad \bar{F} = d\tilde{A} - \tilde{A}^2. \quad (4.22)$$

We assume that 2-forms $F$ and $\bar{F}$ admit the expansion over the basis of the veilbein 1-forms (cf. with (3.3))

$$F_1 = Tr_q(\epsilon_2 F_{12} e_2) \to \bar{F}_1 = Tr_q(\epsilon_2 \bar{F}_{12} e_2). \quad (4.23)$$

The noncommutative scalar curvature could be obtained as a real combination of the coefficients $F_{12}, \bar{F}_{12}$:

$$\mathcal{F} = Tr_q F_{12} + \bar{F}_{12}, \quad (4.24)$$

and the corresponding algebraical version of the Einstein Lagrangian reads

$$\mathcal{L} = \mu(\epsilon^{ij}) \cdot \mathcal{F}$$

where the invariant 4-dimensional real measure $\mu$ can be chosen in the form:

$$\mu = i (Tr_q(\epsilon \epsilon \epsilon \bar{e}) - Tr_q(\epsilon \bar{e} \epsilon \epsilon)). \quad (4.25)$$

Here $\bar{e}_i$ are contragradient veilbein 1-forms transformed as in (4.17).
5 Discussion and Conclusion

To conclude the paper we would like to make some remarks and comments.

1. We note that there is the realization of the differential complex \( (2.8)-(2.10) \) with the usual differential \( d = dz\partial_z + d\bar{z}\partial_{\bar{z}} \) over the classical 2-dimensional space \( \{z, \bar{z}\} \). Indeed, let us consider the algebra

\[
\begin{align*}
R TT' &= TT'R, \\
TM' &= RMR T, \\
RM'R = MRM'R, \\
\bar{M}^2 = 0,
\end{align*}
\]

where as usual \( M = M_1 \) and \( M' = M_2 \) etc. Then one can prove that the operators

\[
T(z, \bar{z}) = \exp(zM)T\exp(\bar{z}\bar{M}),
\]

\[
dT(z, \bar{z}) = dz(\partial_zT) + d\bar{z}(\partial_{\bar{z}}T) = dzMT + d\bar{z}\bar{M}T
\]

satisfy the commutation relations \( (2.8)-(2.10) \). The generators \( \{e^i, (de)^i\} \) of the exterior algebra \( \Omega_z (2.1) \) for \( c = q \) can be realized now as columns of the quantum matrices \( T_j^i(z, \bar{z}) \) and \( dT_j^i(z, \bar{z}) \). In this sense we indeed can consider Eqs.\( (2.7), (2.3) \) as a local co-transformations where \( \{z, \bar{z}\} \) are coordinates of the space-time. We stress also that Eqs.(5.1) and (5.2) remind the formulas appeared in the framework of the Hamiltonian quantizing of the WZWN models (see e.g. [31] and references therein) and related toy model [32].

2. Another attractive possibility is the choice of the noncommutative space-time isomorphic to the space of the quantum group e.g. \( GL_q(N) \). In this case it is tempting to explore monopole-instanton type gauge potential 1-forms

\[
A^i_j = dT^i_kM^k_j(Z)(T^{-1})^i_j = dT^i_k(T^{-1})^i_jM^k_j(Z),
\]

where \( Z = det_qT \) and (\( [M(Z), T] = 0, M(Z)dT = dT M(q^2 Z) \)). Substituting (5.3) in the anticommutation relations \( (2.23) \) we obtain that \( M \) satisfy reflection equation:

\[
M(q^2 Z)R^{-1}M(Z)R^{-1} - R^{-1}M(q^2 Z)R^{-1}M(Z) = 0.
\]

3. For arbitrary invertible Yang-Baxter \( R \)-matrix satisfying the characteristic equation (generalization of (2.3))

\[
(R - \lambda_1)(R - \lambda_2) \cdots (R - \lambda_m) = 0, \quad (\lambda_i \neq \lambda_j \text{ if } i \neq j)
\]

one can introduce [33] the set of the quantum hyperplanes and covariant differential calculi on them. Namely, for each eigenvalue \( \lambda_k \) we define the exterior algebra \( \{e, (de)\} \) with the commutation relations [33]

\[
\prod_{j \neq k} \frac{R - \lambda_j}{(\lambda_k - \lambda_j)} ee' \equiv P_{k}ee' = 0,
\]

\[
R(de)e' = -\lambda_k e(de)',
\]

\[
R(de)(de)' = \lambda_k(de)(de)',
\]

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We choose two variants of the hyperplanes related to the eigenvalues $\lambda_k$ and $\lambda_i$ for which projectors $P_k$ and $P_i$ are $q$-analogs of a symmetrizer and an antisymmetrizer (fermionic and bosonic hyperplanes). Then we deduce the commutation relations for $T$ and $dT$ substituting the transformations (2.4), (2.5) into these two variants of the relations (5.5). Surprisingly these relations coincide with the relations (2.8)-(2.10) for $\lambda_k \lambda_i = 1$ and, as it can be easily shown, such differential complex is not consistent for $m > 2$, e.g. for the quantum groups such as $SO_q(N)$ and $SP_q(2N)$ for which $m = 3$. Our conjecture is that the consistent differential complex for quantum groups with general $R$-matrices satisfying (5.4) can be represented in the form (cf. with formulas presented in [6])

$$R TT' = TT'R , \quad (5.6)$$

$$T(dT)' = \sum_{k,j=1}^{m} \alpha_{kj} P_k (dT) T' P_j - (dT) T' , \quad (5.7)$$

$$\sum_{k,j=1}^{m} \alpha_{kj} P_k (dT) (dT)' P_j = 0 . \quad (5.8)$$

Here the differential $d$ satisfies the undeformed graded Leibnitz rule, the coefficients $\alpha_{kj} = 0, 1 \ (k \neq j)$ and $\alpha_k \equiv \alpha_{kk}$ have to be fixed from the diamond condition (the unique lexicographic ordering of cubic monomials) for the algebra (5.0)-(5.8). In particular one can deduce the following condition on $\alpha_k$

$$[X(\Omega), \ R] = 0$$

where $X(\Omega) = (1 - \sum_k \alpha_k P_k') \Omega_1 + \sum_{k,l} \alpha_k \alpha_l P_k P_l \Omega_1 P_l P_k'$. Note, that the algebra (5.0)-(5.8) is an exterior Hopf algebra with the structure maps defined in (2.11).

4. Now we make some notes about Brzezinski theorem [16] and its application to the construction of the quantum group covariant noncommutative geometry.

Let $(A, \Delta, S, e)$ be a Hopf algebra and $(\Gamma, d)$ - first order differential calculus on $A$, where $\Gamma$ is a space of 1-forms on $A$, while $d$ is a differential mapping which is nilpotent $d^2 = 0$ and satisfies graded Leibnitz rule. Denote the basic elements of $A$ (including unity) as $\{t_i, t_0 = 1\}$ and define

$$t_i t_j = m_{ij}^k t_k , \quad (5.9)$$

$$\Delta(t_i) = \Delta_i^{kj} t_k \otimes t_j , \quad (5.10)$$

$$S(t_i) = S_j^i t_j . \quad (5.11)$$

The comultiplication $\Delta$ is a homomorphistic mapping for the algebra (5.3) and therefore we have the following condition on the structure constants

$$\Delta_i^{kn} \Delta_j^{ql} m_{kj}^p m_{nl}^r = m_{ij}^k \Delta_k^{pr} . \quad (5.12)$$

Let us choose in $\Gamma$ the basis of independent 1-forms $\{\omega_\alpha\}$ defined by the relations

$$dt_i = (\chi^\alpha)_i^j \omega_\alpha t_j . \quad (5.13)$$
where \( \chi^a \) are some numerical matrices. Each element in \( \Gamma \) can be uniquely represented in the form \( \sum a_\alpha \omega_\alpha \) or \( \sum \omega_\alpha b_\alpha \), \( (a_\alpha, b_\alpha \in A) \) and therefore we have to be able to commute \( \{ t_m \} \) and \( \{ \omega_\alpha \} \):

\[
t_n \omega_\beta = (F^\alpha_\beta)^k_n \omega_\alpha t_k ,
\]

(5.14)

where

\[
(F^\alpha_\beta)^k_n = \eta_{\gamma\beta} \left( (\chi^\alpha)^j_\gamma m^r_\gamma (\chi^\gamma)^j_0 T r (\chi^\alpha)^k_n \right);
\]

(\( \eta_{\alpha\beta} \eta^{\beta\gamma} = \delta^\gamma_\alpha \) and \( \eta^{\alpha\beta} = T r (\chi^\alpha \chi^\beta) \))

are again some invertible numerical matrices. The corresponding commutation relations for the basis of 1-forms (in other words the definition of the exterior product \( \omega \wedge \omega \)) can be easily deduced by the differentiation of Eq.(5.14)

\[
\left[ \chi^\alpha \omega_\alpha , F^\beta_\gamma \omega_\gamma \right] = \left( F^\alpha_\beta f^\gamma_\delta - f^\alpha_\delta F^\gamma_\beta \right) \omega_\gamma \omega_\delta .
\]

(5.15)

One can guarantee that there are no other quadratic relations on \( \omega_\alpha \) since we choose these 1-forms as independent. We imply in Eq.(5.15) the exterior products of the differential forms and introduce structure constants \( f^\alpha_\beta \) appeared in the Maurer-Cartan equation

\[
d\omega_\alpha = f^\beta_\gamma \omega_\beta \wedge \omega_\gamma .
\]

(5.16)

Comparing this relation with the differential of Eq.(5.13) one can express \( f^\beta_\gamma \) in terms of the matrices \( \chi^\gamma \).

The relations (5.9), (5.14) and (5.15) are defining relations for the exterior algebra \( \Omega = \bigoplus \Omega^{(n)} \) of \( A \). Here \( \Omega^{(0)} = A \), \( \Omega^{(1)} = \Gamma \) and \( \Omega^{(n)} \) denotes the space of n-forms. Now let us consider the mapping \( \Delta' : \Omega \to \Omega \otimes \Omega \) where \( \otimes \) is a graded tensor product and \( \Delta' (A) \equiv \Delta (A) \). Define the action of \( d \) on \( \Omega \otimes \Omega \) as

\[
d(\Omega^{(n)} \otimes \Omega^{(k)}) = d\Omega^{(n)} \otimes \Omega^{(k)} + (-1)^n \Omega^{(n)} \otimes d\Omega^{(k)}.
\]

Our proposition is that if the mapping \( \Delta' \) (co-action) commutes with \( d \):

\[
d\Delta' = \Delta' d
\]

(5.17)

and the relations (5.14) are covariant under the co-action \( \Delta' \), then the differential complex (5.9), (5.14) and (5.13) defines the exterior Hopf algebra of \( A \).

**Proof:** First, we note that from the condition (5.17) we obtain the explicit definition of \( \Delta' \):

\[
\Delta'(t_i) = \Delta(t_i)
\]

\[
\Delta'(dt_i) = d\Delta'(t_i) = \Delta'^{ij} (dt_k \otimes t_j + t_k \otimes dt_j) .
\]

(5.18)

The co-action on the higher differential forms \( \Omega^{(n)} \) can be derived from (5.18). From the covariance of the relations (5.14) it is not hard to show (applying
Leibnitz rule and the condition (5.17)) that the relations (5.15) are also covariant under the co-action (5.18). The coassociativity of \( \Delta^k_j \Delta^l_n = \Delta^k_j \Delta^l_n \) leads to the coassociativity of \( \Delta' \) (5.18). Thus, \( \Delta' \) is a coproduct for \( A \oplus \Gamma \) and therefore for \( \Omega \). Finally, we define the extended versions of the antipode \( S' \) and the counite \( \epsilon' \) for the exterior algebra \( \Omega \) by means of the relations

\[
S'(t_i) = S(t_i), \quad S'(dt_i) = dS(t_i), \quad \epsilon'(t_i) = \epsilon(t_i), \quad \epsilon'(dt_i) = d\epsilon(t_i) = 0.
\]  

(5.19)

All axioms for \( S' \) and \( \epsilon' \) follow from the corresponding axioms for \( S \) and \( \epsilon \).

This proposition immediately implies Brzezinski theorem [16] since the bico-variance for \( (\Gamma, d) \) is nothing but the covariance of the relations (5.9), (5.14) and (5.15) with respect to the left \( \Phi_L \) and right \( \Phi_R \) coactions on \( A \oplus \Gamma \)

\[
\Phi_L(t_i) = \Delta(t_i), \quad \Phi_L(dt_i) = \Delta^k_j t_k \otimes dt_j, \quad \Phi_R(dt_i) = \Delta^k_j dt_k \otimes t_j,
\]  

(5.20)

and therefore relations (5.14) are also covariant under the coaction (5.18).

Now we consider the left coaction of the exterior Hopf algebra \( \Omega \) on a left comodule represented by some exterior algebra \( \Omega_Z \):

\[
x_\alpha \rightarrow (C^i)_\alpha^\beta t_i \otimes x_\beta,
\]

\[
dx_\alpha \rightarrow (C^i)_\alpha^\beta (dt_i \otimes x_\beta + t_i \otimes dx_\beta).
\]  

(5.21)

Here \( \{x_\alpha, dx_\alpha\} \) are generators of \( \Omega_Z \) and matrices \( C^i \) represent the dual object: \( (C^i)_\alpha^\beta (C^j)_\beta^\gamma = \Delta^j_k (C^k)_\alpha^\gamma \). If we extend the algebra \( \Omega_Z \rightarrow \Omega_{\bar{Z}} \) by adding new generators \( A^\beta_\alpha \) such that \( A^\beta_\alpha \in \Omega^{(1)}_{\bar{Z}} \) and introduce a new differential \( \nabla x_\alpha = dx_\alpha - A^\beta_\alpha x_\beta \) transformed covariantly under (5.21)

\[
\nabla x_\alpha \rightarrow (C^i)_\alpha^\beta t_i \otimes \nabla x_\beta,
\]  

(5.22)

then we interpret \( A^\beta_\alpha \) as connection 1-forms. The definition of the curvature 2-forms is evident. One can try to construct the cross-product of the algebras \( \Omega \) and \( \Omega_{\bar{Z}} \) and obtain a new exterior Hopf algebra \( G \) for which \( \Omega \) will be a Hopf subalgebra. In this case \( A^\beta_\alpha \) could be realized as right-covariant 1-forms on \( G \). Just this realization have been done in the Sect.3 where \( \Omega \equiv \Omega_{GL_q(N)} \) and \( G \equiv \Omega_{GL_q(N+1)} \). So we see that in principal the algebraical constructions of the Sections 2 and 3 could be adapt with the case of the arbitrary exterior Hopf algebra.

5. Finally, we would like to stress that there are many variants of the quantum group covariant commutation relations for connections, curvatures, veilbeins etc. For each variant (and for the same quantum group of covariance) one can obtain different noncommutative geometries. Therefore the structure cogroup (the group of the covariance) is not define the noncommutative geometry
uniquely. Indeed, we can embed the structure quantum group $\Omega$ in various large algebras $G$ and correspondingly to obtain various geometrical structures. For example, one can consider the embedding of the structure group $\Omega = \Omega_{GL_q(N)}$ in the arbitrary group $\Omega_{GL_q(M)}$ for $M > N + 1$. Obviously this will be the generalization of the noncommutative geometry for $M = (N+1)$ presented in Sect.3.

It seems that all these ideas very closely related to the concept of the noncommutative geometry on the quantum principal fibre bundles [11]. However, we stress that we have not done the sequential analyses of these relations. It would be very interesting to interpret the quantum group covariant noncommutative geometries as geometries on noncommutative principal fibre bundles.

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