EXTRA-LARGE METRICS

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Abstract. We show that every two dimensional spherical cone metric with all cone angles greater than 2π and the lengths of all closed geodesics greater than 2π admits a triangulation whose 0-skeleton is precisely the set of cone points – this is, in fact, the Delaunay triangulation of the set of cone points.

1. Introduction

In this note we study extra large spherical cone manifolds in dimension 2 (though many of our results and techniques extend to higher dimensions.

A 2-dimensional spherical cone manifold is a metric space where all but finitely many points has a neighborhood isometric to a neighborhood of a point on the round sphere $S^2$. The exceptional points (cone points) have neighborhoods isometric to a spherical cone, the angle of which is the cone angle at that point.

If $M$ is a cone manifold, we define a geodesic to be a locally length minimizing curve on $M$. It is easy to see that such a curve is locally a great circle, except at the cone points. There, the geodesic must have the property that it subtends an angle no smaller than $\pi$ on either side. Consequently, no geodesics can pass through cone points, where the cone angles are smaller than 2π (such cone points are known as positively curved cone points, since the curvature of a cone point is defined as $2\pi$ less the cone angle at the point).

We say that a spherical cone manifold is extra large if

1. All the cone points are negatively curved.
2. All closed geodesics are longer than $2\pi$.

Such spaces are of considerable importance in geometry in general (due to work of A.D.Aleksandrov, M. Gromov, and then R. Charney and M. Davis [1]), and in three-dimensional hyperbolic geometry in particular, due in large part to the results of the author ([2],[3],[5],[4]).

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who showed that the polar duals of convex compact polyhedra in $\mathbb{H}^3$ are precisely the extra large spherical cone manifolds homeomorphic to $S^2$. In that work, the term *extra large* was not used – it was invented by Gromov, to describe the vertex links in negatively curved spaces.

The main objective of this paper is to show:

**Theorem 1.** An extra large spherical cone surface admits a cell decomposition whose 0-skeleton is precisely the set of cone points.

For the impatient reader, we first give the recipe for constructing the cell decomposition whose existence is postulated in Theorem 1.

First, recall that the Voronoi diagram $V_P$ of a metric space $M$ with respect to a point set $P = \{p_1, \ldots, p_n, \ldots\}$ is the decomposition of $M$ into Voronoi cells

$$V_i = \{x \in M | d(x, p_i) \leq d(x, p_j), \quad \forall j\}.$$  

Clearly,

$$\bigcup_i V_i = M,$$

and

$$\hat{V}_i \cap \hat{V}_j = \emptyset, \quad i \neq j.$$  

As will be shown below, each $V_i$ is a geodesic polygon, and the Voronoi diagram $V_P$ is a cell decomposition of $M$. The Delaunay tesselation $D_P$ of $M$ with respect to $P$ is the Poincaré dual of $V_P$: its edges correspond to pairs $p_i, p_j$ of sites whose Voronoi cells share an edge, while its faces correspond to points of $M$ equidistant from three or more elements of $P$. The cells of $D_P$ are convex, and so the tesselation $D_P$ can be completed to a triangulation of $M$.

To push the program above through, we will need a number of steps.

### 2. The injectivity radius

We define the *injectivity radius* of the space $M$ at a point $p$ as the radius of the largest disk in the tangent space of $p$ for which the exponential map is an embedding. The injectivity radius of $M$ is the infimum over all points $p$ of the injectivity radii of $M$ at $p$. In simpler terms, the injectivity radius of $M$ is the smallest $d$ such that there exist at least two distinct curves from $p$ to $q$ realizing $d(p, q) = d$.

Our first result is:

**Theorem 2.** A space $M$ is extra large if and only if the cone points of $M$ are negatively curved and the injectivity radius of $M$ is greater than $\pi$. 
Proof. Let \( p, q \) be the pair of points realizing the injectivity radius. This means that there are two shortest curves \( \gamma_1, \gamma_2 \) of length \( L = \ell(\gamma_1) = \ell(\gamma_2) \leq \pi \) connecting \( p \) to \( q \), and \( L \) is the smallest with this property.

Let \( \gamma = \gamma_1 \cup \gamma_2 \). If \( \gamma \) is geodesic, then \( L(\gamma) \leq 2\pi \), so we have a contradiction to extra-largeness. If not, suppose (without loss of generality) that \( \gamma \) has a “corner” at \( p \). That means that on one side, the angle \( \alpha \) subtended by \( \gamma \) at \( p \) is smaller than \( \pi \).

If \( \gamma_1 \) and \( \gamma_2 \) are both smooth, then take the bisector of \( \alpha \) and move \( p \) a very small distance \( \rho \) along the bisector. By elementary spherical geometry,

\[
\frac{d\ell(\gamma_1)}{d\rho} = \frac{d\ell(\gamma_2)}{d\rho} < 0,
\]

which contradicts the minimality of \( L \). If (without loss of generality) \( \gamma_1 \) is not smooth, while \( \gamma_2 \) is, let \( x \) be the cone point of \( \gamma_1 \) closest to \( p \). Note that if the angle subtended by \( \gamma_1 \) at \( x \) on the side of the corner at \( p \) equals \( \pi \), then \( x \) can be treated as a smooth point, so the correct definition of \( x \) is: the closest point of \( \gamma_1 \) to \( p \), where \( \gamma_1 \) is not smooth on the side of the corner (if such a point does not exist between \( p \) and \( q \), then we find ourselves back in the smooth case, which corresponds to \( x = q \)). Let \( L_x = d(p, x) \).

In any event, now, instead of the bisector of the angle \( \alpha \) at \( p \), we pick a direction, such that

\[
\frac{d\ell(\gamma_2)}{d\rho} = \frac{dL_x}{d\rho} < 0.
\]

Such a direction exists by the intermediate value theorem. The above argument adapts in the obvious way if both \( \gamma_1 \) and \( \gamma_2 \) are singular. \( \square \)

3. Voronoi diagrams

Lemma 1. For all \( x \in V_i \), \( d(x, p_i) < \pi/2 \).

Proof. Suppose that there exists an \( x \) contradicting the assertion of the Lemma. Then the distance from \( x \) to the cone locus of \( M \) is at least \( \pi/2 \), and so there is a smooth hemisphere around \( x \). The boundary of that hemisphere is a closed geodesic of length \( 2\pi \).

\( \square \)

Corollary 1. The diameter of the Voronoi cell \( V_i \) is less than \( \pi \).

We will need the following simple lemma from spherical geometry:

Theorem 3. The boundary of a Voronoi cell \( V_i \) is a convex polygonal curve.
Proof. Let \( x \in V_i \). Let \( r = d(x, p_i) \); we know that \( r < \pi/2 \). Consider the disk \( D_x(r) \) of radius \( r \) around \( x \).

There are the following possibilities:

Firstly, \( p_i \) might be the only cone point in \( D_x(r) \). In that case, a neighborhood of \( x \) is in \( V_i \), and so \( x \) is in the interior of \( V_i \).

Secondly, there may be exactly one other point \( p_j \) such that \( d(p_i, p_j) = r \). In that case, a small geodesic segment bisecting the angle \( p_i x p_j \) lies in \( V_i \cap V_j \).

Thirdly, there can be a number of points \( p_i, p_j, p_{j_1}, \ldots, p_{j_k} \) at distance \( r \) from \( x \). In that case a small part of the cone from \( x \) to the Voronoi region of \( p_i \) on the boundary of \( D_x(r) \) lies in \( V_i \) — note that this argument works in arbitrary dimension. \( \square \)

Theorem 4. A Voronoi cell \( V_i \) is star-shaped with respect to \( p_i \).

Proof. Let \( x \in V_i \), and let \( y \) be on the segment \( p_i x \). By the triangle inequality, we see that
\[
d(y, \partial D_x(r)) + d(y, x) > r.
\]
Since for any \( j \neq i \) we have that
\[
d(y, p_j) > d(y, \partial D_x(r)),
\]
the assertion of the Theorem follows. \( \square \)

Theorem 5. Every Voronoi cell \( V_i \) is convex.

Proof. By the preceding result, every Voronoi cell \( V_i \) is a starshaped subset of a cone of radius \( \pi/2 \) centered on \( p_i \), with geodesically convex boundary. Take two points \( p \) and \( q \) in \( V_i \). If one of the angles \( pp_iq \) does not exceed \( \pi \), the result follows from elementary spherical geometry. If both the angles \( pp_iq \) are at least \( \pi \), the broken line \( pp_iq \) is geodesic. \( \square \)

Theorem 6. Let \( V_i, V_j \) be two Voronoi cells, then \( V_i \cap V_j \) is connected.

Proof. Let \( p, q \in V_i \cap V_j \), here is a shortest geodesic \( \gamma_i \) from \( p \) to \( q \) in \( V_i \) and a shortest geodesic \( \gamma_j \) from \( p \) to \( q \) in \( V_j \). Since the diameters of \( V_i \) and \( V_j \) are smaller than \( \pi \), it follows that \( \gamma_i = \gamma_j \). Thus, \( \gamma = \gamma_i = \gamma_j \subseteq V_i \cap V_j \), hence \( V_i \cap V_j \) is path connected. In fact, the argument (together with the results above) easily shows that \( V_i \cap V_j \) is an edge of both. \( \square \)

The above results show that \( \mathcal{V}_P = \{V_1, \ldots, V_n, \ldots\} \) is a simplicial cell decomposition, and so its dual is a cellulation of \( M \) with vertices at \( p_1, \ldots, p_n, \ldots \). The cells of cellulation are convex (in fact inscribed in circles; The centers are precisely the corners of the boundaries of the cells \( V_i \)), and so the proof of Theorem 1 is complete.
4. Remarks

An identical argument with the appropriate modification of the extra-largeness hypothesis can be used to show an analogous result for a Riemannian surface with cone singularies. In particular, for Euclidean and Hyperbolic cone surfaces, it is sufficient to require the cone angles to be non-positively curved.

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