Blown-up p-Branes and the Cosmological Constant

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Abstract

We consider a blown-up 3-brane, with the resulting geometry $R^{(3,1)} \times S^{(N-1)}$, in an infinite-volume bulk with $N > 2$ extra dimensions. The action on the brane includes both an Einstein term and a cosmological constant. Similar setups have been proposed both to reproduce 4-d gravity on the brane, and to solve the cosmological constant problem. Here we obtain a singularity-free solution to Einstein’s equations everywhere in the bulk and on the brane, which allows us to address these question explicitly. One finds, however, that the proper volume of $S^{(N-1)}$ and the cosmological constant on the brane have to be fine-tuned relatively to each other, thus the cosmological constant problem is not solved. Moreover the scalar propagator on the brane behaves 4-dimensionally over a phenomenologically acceptable range only if the warp factor on the brane is huge, which aggravates the Weak Scale – Planck Scale hierarchy problem.
1 Introduction

Recently it has been claimed that the cosmological constant problem can be solved in brane worlds with $N > 2$ infinitely large extra dimensions with an extra Einstein term localized on the brane [1, 2]. Then the graviton propagator is soft (massive) beyond some scale $r \gtrsim r_c$, and consequently the gravitational field (the FRW scale factor $a(t)$) does not necessarily “react” to sources that are smooth at scales $\gtrsim r_c$ such as a cosmological constant. The success of the cosmological standard model, on the other hand, requires $r_c \gtrsim H_0^{-1}$ where $H_0$ is the Hubble constant today.

The setup in [1, 2] assumes a vanishing cosmological constant in the bulk, which can possibly be motivated by some unbroken supersymmetry (and $R$-parity) in the bulk. A 4-dimensional behaviour of gravity, hopefully over a large range of scales, is then due to an additional Einstein term on the brane [3-5]. The (classical) brane-to-brane graviton propagator actually suffers from a short distance singularity for $N > 2$, which requires an UV regularization in the form of “blowing up” the transverse size of the 3-brane, higher derivative terms in the bulk or a momentum cutoff [6-12].

A solution of the cosmological constant problem is achieved only if one includes, in addition, an arbitrary (not fine-tuned) cosmological constant on the brane. This induces a non-trivial gravitational field in the bulk surrounding the brane, which generically exhibits a naked singularity [13]. Again a regularization of this singularity requires a blowing-up of the brane [14, 10] (or higher derivative terms [12]).

In [1, 2] it has been argued that allowing the brane to inflate – with a phenomenologically acceptable small acceleration rate – removes the naked singularity. However, no explicit solution of the Einstein equations was given, and the proposed scenario supposedly requires at least fine-tuned initial conditions.

In the present paper we present a (singularity free) solution of Einstein’s equation, which corresponds to a blown-up 3-brane in $N > 2$ extra dimensions. In contrast to smoothed-out branes considered elsewhere Einstein’s equations are satisfied everywhere: Not only in the bulk for $r > r_b$ (where $r_b$ is the radius of the blown-up brane) but also for $r < r_b$. This is made possible through a particular choice for the profile function $f(r)$, which describes the distribution of the brane tension as a function of $r$ (which was concentrated at $r = 0$ originally): We assume
that the complete brane tension (i.e. the brane cosmological constant) is concentrated at the surface \( r = r_b \) of the blown-up brane, i.e. \( f(r) \sim \delta(r - r_b) \). The shape of the brane in the \( N \) extra dimensions is then the one of a sphere \( S^{(N-1)} \). Although the distribution of the brane tension is still \( \delta \)-function-like in the radial direction, this does not imply a singular gravitational field in the bulk (or on the brane) since one has just an effective co-dimension one problem.

A similar setup has already been considered in [8, 10]. In [10], however, the metric in the bulk “inside” the sphere (for \( r < r_b \)) was assumed to be constant, and no consistent solution to the junction conditions could be found.

Here we take instead, for \( r < r_b \), a copy of the known metric “outside” [13, 14] after an inversion \( r \to r_b^2/r \). This configuration has a \( \mathbb{Z}_2 \) symmetry: Changing the radial coordinate from \( r \) to \( y \) by \( r = r_b \exp(y/r_b) \) our setup is invariant under a reflection at \( y = 0 \); the surface of \( S^{(N-1)} \) is situated at \( y = 0 \). Now the notions “inside” and “outside” have actually lost their meaning, since the space “inside” is just a copy of the space “outside” and shares with it its infinite volume.

Nevertheless this corresponds to a regularization (“blowing-up”) of an infinitely thin 3-brane: Fields living on the brane with its world volume \( R^{(3,1)} \times S^{(N-1)} \) (where \( R^{(3,1)} \) is our 4-dimensional Minkowski space) just see the finite world volume of \( \mathcal{O}(r_b^{N-1}) \) of \( S^{(N-1)} \) independently from the infinite volumes “inside” and “outside”, and their Kaluza-Klein modes on \( S^{(N-1)} \) become infinitely heavy for \( r_b \to 0 \).

Since we have explicit singularity-free solutions of Einstein’s equations and junction conditions, we can ask whether the brane can support an arbitrary tension (cosmological constant) without inflation, i.e. without a naked singularity in the static case. It turns out, however, that the brane tension and the volume of the blown-up brane have to be relatively fine-tuned. This follows from the mere number of junction conditions to be satisfied, hence we have all reasons to believe that this result is generic and independent from the particular setup considered here (a profile function concentrated on the boundary). The Einstein term on the brane does not play a particular role for this result, it just modifies the required relation between the brane tension and its volume on \( S^{(N-1)} \).

Next we can ask whether the gravitational propagator on the brane behaves 4-
dimensionally over a phenomenologically acceptable range of scales or momenta $p^2$.

To this end we study the scalar propagator on the brane in the present gravitational background, with coefficients of the kinetic terms in the bulk and on the brane identical to the ones of the Einstein terms. This does not yet allow us to study the tensor structure of the graviton propagator, but a $p^{-2}$ behaviour of the scalar propagator over a large range of $p^2$ is already a necessary condition for an acceptable behaviour of gravity.

The result is that, one the one hand, a $p^{-2}$ behaviour of the scalar propagator over a phenomenologically acceptable range is possible, for a certain range of values of the gravitational constant $M_D$ in the $D$-dimensional bulk and the volume of $S^{(N-1)}$. However, for the corresponding values of these parameters the warp factor $A$ on the brane is huge:

$$A > 10^{15 \sqrt{\frac{N-1}{N+2}}}.$$  \hfill (1.1)

This behaviour of the warp factor is opposite to the one proposed in [15] as a solution to the Weak Scale – Planck Scale hierarchy problem. A scenario with many infinite extra dimensions, and 4-d gravity due to an (induced) Einstein term on the brane, is thus a priori difficult to realize.

On the other hand recall that, for a 4-dimensional behaviour of gravity over a phenomenologically acceptable range, the setup in [2] requires (for $N > 1$)

$$\left( \frac{M_{Pl}}{M_D} \right) > 10^{30}.$$  \hfill (1.2)

Here we find that the required hierarchy between the gravitational constants $M_D$ and $M_{Pl}$ is less dramatic, if one allows the proper volume of $S^{(N-1)}$ to be as small as $M_{Pl}^{1-N}$ (instead of $M_D^{1-N}$): Then a $p^{-2}$ behaviour of the propagator over a phenomenologically acceptable range requires only

$$\left( \frac{M_{Pl}}{M_D} \right)^{N+2} > 10^{30}$$  \hfill (1.3)

which, for $N$ large, is easier to realize than eq. (1.2).

In the next section we present our setup more explicitly, and solve the combined Einstein equations in the bulk and junction conditions on the blown-up brane. The
existence of static singularity-free (non-supersymmetric) \((D - 2)\)-brane configurations in \(D\)-dimensions, with \(N - 1\) dimensions of the brane compactified on \(S^{(N-1)}\), is quite remarkable in view of the naked singularities encountered generally \([13]\) and the negative result in \([10]\). As noted above, however, a relative fine-tuning of the input parameters is required, thus the cosmological constant problem is not solved.

In Section III we study the scalar propagator in this background. Although the full problem (the corresponding wave equation for all non-zero modes) cannot be solved explicitly \([14]\), the partial results in \([14]\) can be used here as well and allow to obtain the essential features of the \(p^2\)-dependence of the propagator. Then we derive the phenomenological constraints on the input parameters cited above and conclude.

2 Singularity Free Blown-up p-Branes

The original scenario consists in an infinitely thin 3-brane in a \(D = 4 + N\) dimensional bulk, i.e. with \(N\) extra dimensions. Our conventions for coordinates and indices are: \(x^\mu, \mu = 0 \ldots 3\), are the coordinates of flat Minkowski space \(R^{(3,1)}\) along the 3-brane, with the metric \(\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)\). \(y^i, i = 1 \ldots N\), denote the coordinates of the extra dimensions. Indices \(A, B = 0 \ldots D - 1\) run over all \(D = 4 + N\) coordinates.

The original action includes an Einstein term in the bulk, and an Einstein term plus a cosmological constant on the brane situated at \(y^i = 0\):

\[
S = \int d^4x \ d^N y \left\{ \sqrt{-g^{(D)}} \frac{1}{2\kappa_D} R^{(D)} + \sqrt{-g^{(4)}} \delta^N(y) \left[ \frac{1}{2\kappa_4} R^{(4)} + \Lambda_4 \right] \right\}. \tag{2.1}
\]

Our convention for the curvature scalar \(R^{(D)}\) is such that, in a weak field expansion, \(R^{(D)} \sim g^{A,B} - g^{AB}_{\ A, B}, i.e.\ the sign of the curvature is opposite to the sign of \(R\). Then \(\Lambda_4\) in (2.1) corresponds to a positive cosmological constant for \(\Lambda_4 > 0\).

Smoothing out the 3-brane corresponds to a replacement of \(\delta^N(y)\) in (2.1) by a profile function \(f(y)\). Let us introduce spherical coordinates in the extra dimensions: \(y^i = \{r, y^\alpha\}\) where \(y^\alpha, \alpha = 1 \ldots N - 1\), are angles on \(S^{(N-1)}\) (an infinitely thin brane
would be situated at \( r = 0 \). The profile function now depends on \( r \) only, \( f = f(r) \). \( f(r) \) is non-vanishing only for \( r \leq r_b \), where \( r_b \) is the width of the brane.

An explicit solution of Einstein’s equation everywhere, i.e. also for \( r \leq r_b \), is not possible for general profile functions \( f(r) \). Therefore, as stated in the introduction, we make the particularly simple choice \( f(r) \sim \delta(r-r_b) \). The blown-up brane is then “hollow”, all tension is located at the surface \( r = r_b \). It corresponds to an infinitely thin \( D-2 \) brane with a geometry \( R^{(3,1)} \times S^{(N-1)} \), where \( S^{(N-1)} \) is a sphere with radius \( r_b \). The fact that this \( D-2 \) brane is again infinitely thin does not necessarily induce singularities in its surrounding gravitational field, since its co-dimension is just 1 (the dimension of its world volume is \( D-1 \)).

Correspondingly we should also replace the Einstein term \( \sqrt{-g^{(4)}R^{(4)}} \) on the brane by \( \sqrt{-g^{(D-1)}R^{(D-1)}} \), since this will be the structure induced by radiative corrections [3]. The \( D-1 \) coordinates along the brane are \( x^\mu \) and the angles \( y^\alpha \).

Thus the action (2.1) is replaced by

\[
S = \int d^4x \ d^N y \left\{ \sqrt{-g^{(D)}} \frac{1}{2\kappa_D} R^{(D)} + \sqrt{-g^{(D-1)}} \delta(r-r_b) \left[ \frac{1}{2\kappa_{D-1}} R^{(D-1)} + \Lambda_{D-1} \right] \right\} .
\]

(2.2)

Subsequently we will be interested in Poincaré invariant (on \( R^{(3,1)} \)) and spherically symmetric (on \( S^{(N-1)} \)) background metrics with \( g_{r\mu} = g_{r\alpha} = 0 \); then we have \( \sqrt{-g^{(D)}} = \sqrt{-g^{(D-1)}} g^{rr} \).

The Einstein equations derived from (2.2) read

\[
\frac{1}{\kappa_D} G^{(D)}_{AB} = \frac{1}{\sqrt{g^{rr}}} \delta(r-r_b) \left[ -\frac{1}{\kappa_{D-1}} G^{(D-1)}_{AB} + g^{(D-1)}_{AB} \Lambda_{D-1} \right]
\]

(2.3)

where \( g^{(D-1)}_{AB} \) is the pull-back of \( g^{(D)}_{AB} \) onto the brane, and the Einstein tensor \( G^{(D-1)}_{AB} \) is constructed from \( g^{(D-1)}_{AB} \). For \( A = r \) (or \( B = r \)) we have \( G^{(D-1)}_{rB} = g^{(D-1)}_{rB} = 0 \). Up to now this is the same setup as considered in [10].

A static metric with the desired symmetries has the form

\[
ds^2 = A^2(r) \eta_{\mu\nu} \ dx^\mu \ dx^{\nu} + B^2(r) \ dr^2 + C^2(r) \ d\Omega_{N-1}^2 ,
\]

(2.4)

where \( d\Omega_{N-1}^2 \) is the metric on \( S^{(N-1)} \). Note that our convention for \( C(r) \) is such
that flat space corresponds to $C(r) = r$. The parameters $\kappa_{D-1}, \Lambda_{D-1}$ in (2.2) are then related to the original 4-dimensional parameters $\kappa_4, \Lambda_4$, as

$$\frac{1}{\kappa_{D-1}} = \frac{1}{\Omega_{N-1} C^{N-1}(r_b)} \frac{1}{\kappa_4},$$  \hspace{1cm} (2.5a)$$

$$\Lambda_{D-1} = \frac{1}{\Omega_{N-1} C^{N-1}(r_b)} \Lambda_4$$ \hspace{1cm} (2.5b)

where $\Omega_{N-1}$ is the volume of the unit sphere $S^{(N-1)}$. An asymptotically (for $r \to \infty$) flat solution of the empty space Einstein equations (for $r > r_b$) for $A(r), B(r)$ and $C(r)$ has been given in [13, 14]:

$$A(r) = f_o^{\frac{1}{N+2}}$$ \hspace{1cm} (2.6a)

$$B(r) = f_o^{\frac{1}{N+2}} \left( \sqrt{\frac{N}{N+2}} \frac{N+3}{2} \right)$$ \hspace{1cm} (2.6b)

$$C(r) = r f_o^{\frac{1}{N+2}} \left( \frac{1}{2} + \sqrt{\frac{N}{N+2}} \right)$$ \hspace{1cm} (2.6c)

with

$$f_o = 1 + \frac{\alpha}{r^{N-2}}.$$ \hspace{1cm} (2.7)

(The index $o$ attached to $f_o$ stands for “outside”.) The parameter $\alpha$ in (2.7) can also be written as $\alpha = \pm r_0^{N-2}$, and will be determined in terms of the parameters in the action (2.2) below.

Next we seek for a solution of the empty space Einstein equations for $r < r_b$. In [10] the metric for $r < r_b$ has been chosen flat (constant), and no singularity free solution to the junction conditions (see below) has been found.

Instead, we chose for the metric for $r < r_b$ a copy of the metric for $r > r_b$ after a coordinate transformation

$$r' = \frac{r_b^2}{r}$$ \hspace{1cm} (2.8)
and renaming $r' \to r$. The coefficient $r_b^2$ in (2.8) ensures that all components of the metric are continuous across $r = r_b$. Explicitly the metric for $r < r_b$ reads

$$A(r) = f_i^{- \frac{1}{2}} \sqrt{\frac{N-1}{N+2}}$$

(2.9a)

$$B(r) = \frac{r_b^2}{r^2} f_i^{\frac{1}{N-2}} \left( \sqrt{\frac{N-1}{N+2}} - \frac{N-1}{2} \right)$$

(2.9b)

$$C(r) = \frac{r_b^2}{r} f_i^{\frac{1}{N-2}} \left( \frac{1}{2} + \sqrt{\frac{N-1}{N+2}} \right)$$

(2.9c)

with

$$f_i = 1 + \alpha \left( \frac{r}{r_b^2} \right)^{N-2}$$

(2.10)

and hence $f_i(r_b) = f_o(r_b)$.

If one introduces an exponential radial coordinate $y$ through $r = r_b \exp(y/r_b)$ for $r > r_b$, one sees that eqs. (2.9) and (2.6) are related by a $\mathbb{Z}_2$-symmetry (reflection) around $y = 0$. For convenience (the junction conditions below) we stick to a single coordinate $r$ ranging from 0 to $\infty$; the price to pay are the coordinate singularities for $r \to 0$ in (2.9): The divergence in $B(r)$ indicates that the point $r = 0$ is at infinite proper distance from the “surface” at $r_b$ (since it is an image of $+\infty$), and the behaviour of $C(r)$ shows that the proper volume of spheres at radii $r < r_b$ blows up the same way as the one of spheres at radii $r > r_b$.

In spite of the fact that the volumes both “inside” and “outside” are infinite (mirrors of each other), the proper volume of the extra dimensional part of the brane on $S^{(N-1)}$ at $r = r_b$ is finite and given by $\Omega_{N-1} C^{N-1}(r_b)$.

Finally we have to show, however, that the metric (2.6) for $r > r_b$, and (2.9) for $r < r_b$, allows to satisfy the junction conditions at $r = r_b$, i.e. to match the coefficients of $\delta(r - r_b)$ in the Einstein equations (2.3).

Let us first discuss the coefficients of $\delta(r - r_b)$ on the right-hand side of eq. (2.3), which are non-vanishing only for \{A, B\} = \{\mu, \nu\} or \{\alpha, \beta\}. To this end we need the \{\mu, \nu\}, \{\alpha, \beta\}-components of the Einstein tensor $G_{AB}^{(D-1)}$ computed from
the pull-back metric \( g^{(D-1)}_{\alpha\beta} \) [10] (Subsequently \( A, B \) and \( C \) denote \( A(r), B(r), \) and \( C(r) \) evaluated at \( r = r_b \):

\[
G^{(D-1)}_{\mu\nu}(r_b) = \frac{A^2(N - 1)(N - 2)}{2C^2} \, \eta_{\mu\nu} , \tag{2.11a}
\]

\[
G^{(D-1)}_{\alpha\beta}(r_b) = \frac{1}{2} (N - 2)(N - 3) g^{(N-1)}_{\alpha\beta} , \tag{2.11b}
\]

where \( g^{(N-1)}_{\alpha\beta} \) is the metric on \( S^{(N-1)} \) with unit radius. Thus the non-vanishing right-hand sides of eq. (2.3) read

\[
\delta(r - r_b) \eta_{\mu\nu} \left( -\frac{A^2(N - 1)(N - 2)}{2\kappa D-1 BC^2} + \frac{A^2}{B} \Lambda_{D-1} \right) , \tag{2.12a}
\]

\[
\delta(r - r_b) g^{(N-1)}_{\alpha\beta} \left( -\frac{(N - 2)(N - 3)}{2\kappa D-1 B} + \frac{C^2}{B} \Lambda_{D-1} \right) . \tag{2.12b}
\]

Terms \( \sim \delta(r - r_b) \) appear on the left-hand side of eq. (2.3) due to the discontinuous first derivatives of the metric across \( r = r_b \). Generally one finds

\[
G^{(D)}_{\mu\nu} = \delta(r - r_b) \eta_{\mu\nu} \frac{A^2}{B^2} \left( -3 \left[ \frac{A''}{A} \right] - (N - 1) \left[ \frac{C''}{C} \right] \right) + \text{regular} , \tag{2.13a}
\]

\[
G^{(D)}_{\alpha\beta} = \delta(r - r_b) g^{(N-1)}_{\alpha\beta} \frac{C^2}{B^2} \left( -4 \left[ \frac{A''}{A} \right] - (N - 2) \left[ \frac{C''}{C} \right] \right) + \text{regular} \tag{2.13b}
\]

where

\[
\left[ \frac{A''}{A} \right] = \left( \frac{\partial_r A}{A} \bigg|_{r_b+\epsilon} - \frac{\partial_r A}{A} \bigg|_{r_b-\epsilon} \right)_{\epsilon \to 0} , \tag{2.14}
\]

and similarly for \( [C''/C] \). No terms \( \sim \delta(r - r_b) \) appear in \( G_{rA}^{(D)} \) for any \( A \), in agreement with the right-hand side of eq. (2.3). From \( A(r), C(r) \) in eqs. (2.6) and (2.9) one finds
\[
\begin{align*}
\left[ \frac{A'}{A} \right] &= \frac{N - 2}{2r_b} \left( 1 - \frac{1}{f} \right) \sqrt{\frac{N - 1}{N + 2}}, \\
\left[ \frac{C'}{C} \right] &= \frac{1}{r_b} \left( 1 - 2 \sqrt{\frac{N - 1}{N + 2}} + \frac{1}{f} \left( 1 + 2 \sqrt{\frac{N - 1}{N + 2}} \right) \right).
\end{align*}
\] (2.15a)

where \( f = f_o(r_b) = f_i(r_b) \). Thus the terms \( \sim \delta(r - r_b) \) in eq. (2.3) imply

\[
\begin{align*}
\frac{A^2}{\kappa_D B^2 r_b} \left( 1 - N + w + \frac{1}{f}(1 - N - w) \right) &= \frac{A^2}{B} \left( \frac{(N - 1)(N - 2)}{2\kappa_{D-1} C^2} + \Lambda_{D-1} \right), \\
- \frac{C^2}{\kappa_D B^2 r_b} (N - 2) \left( 1 + \frac{1}{f} \right) &= \frac{C^2}{B} \left( \frac{(N - 2)(N - 3)}{2\kappa_{D-1} C^2} + \Lambda_{D-1} \right),
\end{align*}
\] (2.16a)

where we have defined \( w = \sqrt{(N + 2)(N - 1)/2} \). These equations can be considerably simplified. Using \( r_b B = C/\sqrt{f} \) (cf. eqs. (2.6b) and (2.6c), or (2.9b) and (2.9c)) they can be brought into the form

\[
\begin{align*}
\sqrt{f}(1 - w) + \frac{1}{\sqrt{f}}(1 + w) &= \frac{(N - 2)\kappa_D}{\kappa_{D-1} C}, \\
\sqrt{f}(N - 1 + (N - 3)w) + \frac{1}{\sqrt{f}}(N - 1 - (N - 3)w) &= -2\kappa_D \Lambda_{D-1} C.
\end{align*}
\] (2.17a)

Let us recall the basic parameters in the action (2.2) describing the blown-up brane. These are

i) The \( D = 4 + N \)-dimensional gravitational constant \( \kappa_D \), which can be written in terms of a \( D \)-dimensional gravitational scale \( M_D \) as

\[
\frac{1}{\kappa_D} = M_D^{2+N}.
\] (2.18)
ii) The radial position of the $S^{(N-1)}$-part of the brane in the $N$ extra dimensions is given by $r_b$; however, only its proper volume $V_{S^{(N-1)}}$ is of physical significance, which is given by

$$V_{S^{(N-1)}} = \Omega_{N-1} C^{N-1}$$  \hspace{1cm} (2.19)$$

where $\Omega_{N-1}$ is the volume of the unit sphere. Eq. (2.19) explains the physical meaning of $C \equiv C(r_b)$.

iii) The gravitational constant $\kappa_{D-1}$ on the brane is related through (2.5a) to the effective 4-dimensional gravitational constant $\kappa_4$. Defining $\kappa_4 = M_{Pl}^{-2}$ we thus have

$$\frac{1}{\kappa_{D-1}} = \frac{1}{V_{S^{(N-1)}} \kappa_4} = \frac{M_{Pl}^2}{\Omega_{N-1} C^{N-1}}.$$  \hspace{1cm} (2.20)$$

iv) The cosmological constant $\Lambda_{D-1}$ (or tension) on the brane is related through (2.5a) to the 4-dimensional cosmological constant $\Lambda_4$:

$$\Lambda_{D-1} = \frac{\Lambda_4}{V_{S^{(N-1)}}} = \frac{\Lambda_4}{\Omega_{N-1} C^{N-1}}.$$  \hspace{1cm} (2.21)$$

v) The width of the resulting gravitational field in the radial direction is given by a parameter $\alpha$ in $f_i(r_b) = f_o(r_b) \equiv f$, with

$$f = 1 + \frac{\alpha}{r_b^{N-2}}.$$  \hspace{1cm} (2.22)$$

Instead of $\alpha$, $f$ can be considered as a parameter to be solved for. Then $r_b$ is given in terms of $C$ and $f$ through eq. (2.6c) at $r = r_b$, and $\alpha$ in terms of $f$ and $r_b$ through (2.22).

Now we have 2 equations (2.17) (originating from the $\{\mu, \nu\}$ and $\{\alpha, \beta\}$ components of the junction conditions) to solve. This implies immediately that the 4 input parameters i)-iv) cannot be arbitrary; one “fine-tuning” relation has to be satisfied. Note that we count $C$ (or $r_b$) as an input parameter: Its origin is the profile function in the radial direction of the “blown-up” 3-brane (localized at its surface), introduced originally as an UV regulator.

Furthermore some consistency conditions have to be satisfied: $C$ and notably $f$ have to be positive. If $f < 0$, the metric is generically complex, and there exists
always a naked singularity in the bulk [13] which is precisely what we want to avoid.

Eqs. (2.17) can be solved most easily with $\kappa_D$, $\kappa_{D-1}$ and $C$ (positive) as input: Then eq. (2.17a) determines $f$, for which luckily always a positive solution exists. Subsequently eq. (2.17b) fixes $\Lambda_{D-1}$ (or $\Lambda_4$ with $C$ given). One sees immediately that $\Lambda_4$ has to be fine-tuned.

Apart from this constraint on $\Lambda_4$ (or $\Lambda_{D-1}$), the existence of consistent (singularity-free) solutions is quite remarkable, given the general presence of naked singularities in the case of flat $p$-branes in $N$ codimensions [13].

Let us first consider the case without an Einstein term on the brane, i.e. $\kappa_{D-1} \to 0$. Then eq. (2.17a) fixes $f = (w + 1)/(w - 1)$, and eq. (2.17b) gives $\kappa_D \Lambda_{D-1} C = -2\sqrt{(N + 2)(N - 1)(N - 2)/(N + 3)}$. Thus, with eqs. (2.18) and (2.21),

$$M_4^4 \sim M_D^{2+N} C^{N-2}. \quad (2.23)$$

With $f \sim O(1)$ we have $r_0^{N-2} (\equiv \alpha) \sim r_b^{N-2} \sim C^{N-2}$, hence

$$r_0^{N-2} \sim M_4^4 M_D^{-N+2}. \quad (2.24)$$

This order of magnitude of $r_0$ coincides with the one in [13, 14].

Of phenomenological interest (cf. the next section) is, however, the opposite limit

$$\frac{\kappa_D}{\kappa_{D-1} C} \gg 1. \quad (2.25)$$

Now eq. (2.17a) gives

$$\left(\sqrt{f}\right)^{-1} \sim \frac{\kappa_D}{\kappa_{D-1} C}. \quad (2.26)$$

($f < 1$ implies that the parameter $\alpha$, introduced below eq. (2.7), is negative.) Now eq. (2.26) implies, from eq. (2.6a) or (2.9a) for $r = r_b$,

$$A \sim \left(\frac{\kappa_D}{\kappa_{D-1} C}\right)^\frac{1}{2} \sqrt{\frac{N-1}{N+2}} \quad (2.27)$$
As we will see in the next section, at the example of the scalar propagator, only in this limit the graviton propagator on the brane has the possibility to behave 4-dimensionally over a wide range of scales.

3 The Scalar Propagator

As a first step towards the momentum dependence of the brane-to-brane graviton propagator in the present gravitational background we will study the corresponding propagator of a (dimensionless) scalar field, whose kinetic terms in the bulk and on the brane have identical coefficients as the Einstein terms in the action (2.2). Although an exact expression for this propagator cannot be given, the essential features of its momentum dependence can be deduced using results from [14].

The differential equation for the general (bulk-to-bulk) scalar propagator has the form

\[
\left( \frac{1}{\kappa} \Box^{(D)} + \frac{1}{\kappa_{D-1}} \delta(r - r_b) \Box^{(D-1)} \right) G(x - x', y, y') = \frac{-1}{\sqrt{-g^{(D)}}} \delta^4(x - x') \delta^N(y - y') .
\]

(3.1)

We recall that \( A(r), B(r) \) and \( C(r) \) define the metric (2.4), and \( A, B \) and \( C \) denote these functions at \( r = r_b \). The \( N \) extra coordinates \( y^i \) are split into \( y^i = \{r, y^\alpha\} \) where \( y^\alpha \) are the angles on \( S^{(N-1)} \), and subsequently \( g_{\alpha\beta} \) will denote the (dimensionless) angular part of the metric on \( S^{(N-1)} \).

Then the Laplacians in (3.1) are

\[
\Box^{(D)} = A^{-2}(r) \partial_\mu \eta^{\mu\nu} \partial_\nu \sqrt{-g^{(D)}} B^{-2}(r) \partial_r + C^{-2}(r) \partial_\alpha g^{\alpha\beta} \partial_\beta ,
\]

(3.2)

and

\[
\Box^{(D-1)} = A^{-2} \partial_\mu \eta^{\mu\nu} \partial_\nu + C^{-2} \partial_\alpha g^{\alpha\beta} \partial_\beta .
\]

(3.3)

The determinant of the \( D \)-dimensional metric is

\[
\sqrt{-g^{(D)}} = A^4(r) B(r) C^{N-1}(r) \sqrt{\text{det}(g_{\alpha\beta})} .
\]

(3.4)
We will Fourier transform \( G(x - x', y, y') \) with respect to the 4 Minkowski coordinates \( x^\mu - x'^\mu \), whereupon \( \partial_\mu \eta^{\mu\nu} \partial_\nu \) is replaced by \( -p^2 \). Furthermore, following [14], we will decompose the angular part of the propagator into spherical harmonics \( Y^{m_a}_\ell (a = 1 \ldots N - 2) \) on \( S^{(N-1)} \):

\[
G(x - x', r, y^\alpha, r', y'^\alpha) = \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \Omega_{N-1} \sum_{\ell, m_a} Y^{m_a*}_\ell (y^\alpha) Y^{m_a}_\ell (y'^\alpha) \cdot G_\ell (p, r, r') \tag{3.5}
\]

where \( \Omega_{N-1} \) is the volume of the unit sphere, and the spherical harmonics satisfy

\[
\sum_{\ell, m_a} Y^{m_a*}_\ell (y^\alpha) Y^{m_a}_\ell (y'^\alpha) = \frac{1}{\sqrt{\det(g_{\alpha\beta})}} \delta^{N-1} (y^\alpha - y'^\alpha) , \tag{3.6}
\]

\[
\partial_\alpha g^{\alpha\beta} \partial_\beta Y^{m_a}_\ell (y^\alpha) = -\ell (\ell + N - 2) Y^{m_a}_\ell (y^\alpha) . \tag{3.7}
\]

Putting \( r' = r_b \), the equation for the bulk-to-brane propagator \( G_\ell (p, r, r_b) \) becomes

\[
\begin{align*}
&\left\{ \frac{1}{\kappa_D} \left[ A^{-2}(r)p^2 - \frac{1}{\sqrt{-g^{(D)}}} \partial_r \sqrt{-g^{(D)}B^{-2}(r)} \partial_r + C^{-2}(r)\ell (\ell + N - 2) \right] \\
&\quad + \frac{1}{\kappa_{D-1}B} \delta(r - r_b) \left[ A^{-2} p^2 + C^{-2}\ell (\ell + N - 2) \right] \right\} G_\ell (p, r, r_b) \\
&\quad = \frac{1}{A^4 BC^{N-1}\Omega_{N-1}} \delta(r - r_b) . \tag{3.8}
\end{align*}
\]

Following [5] the solution of eq. (3.8) can be written as

\[
G_\ell (p, r, r_b) = \frac{D_\ell (p, r, r_b)}{1 + \frac{A^4 C^{N-1}\Omega_{N-1}}{\kappa_{D-1}} (A^{-2} p^2 + C^{-2}\ell (\ell + N - 2)) D_\ell (p, r_b, r_b)} \tag{3.9}
\]

where \( D_\ell (p, r, r') \) is the propagator in the absence of field dependent terms on the brane and satisfies

\[
\begin{align*}
&\left\{ \frac{1}{\kappa_D} \left[ A^{-2}(r)p^2 - \frac{1}{\sqrt{-g^{(D)}}} \partial_r \sqrt{-g^{(D)}B^{-2}(r)} \partial_r + C^{-2}(r)\ell (\ell + N - 2) \right] \\
&\quad + \frac{1}{\kappa_{D-1}B} \delta(r - r') \left[ A^{-2} p^2 + C^{-2}\ell (\ell + N - 2) \right] \right\} D_\ell (p, r, r') \\
&\quad = \frac{1}{A^4 BC^{N-1}\Omega_{N-1}} \delta(r - r') . \tag{3.10}
\end{align*}
\]
Defining an operator \( \mathcal{O}_\ell(r) \) as (using (3.4) for \( \sqrt{-g^{(D)}} \))

\[
\mathcal{O}_\ell(r) = \frac{-1}{A^2(r)B(r)C^{N-1}(r)} \partial_r A^4(r)B^{-1}(r)C^{N-1}(r)\partial_r + \frac{A^2(r)}{C^2(r)} \ell(\ell + N - 2)
\]

(3.11)

eq. (3.10) can be rewritten as

\[
\left( \mathcal{O}_\ell(r) + p^2 \right) D_\ell(p, r, r') = k \delta(r - r')
\]

(3.12)

with

\[
k = \frac{\kappa D}{A^2 BC^{N-1}\Omega_{N-1}}.
\]

(3.13)

The wave equation corresponding to eq. (3.12) (with a vanishing right-hand side, and in the notation \( p^2 = -m^2 \)) has been studied in [14]. There the setup is slightly different than the one considered here: In [14] the brane (for the purpose of solving the wave equation) is located at \( r = 0 \), and the functions \( A(r) \), \( B(r) \) and \( C(r) \) in \( \mathcal{O}_\ell \) are of the form (2.6) for \( r > 0 \). The remarkable result in [14] is that inspite of the singular behaviour of these functions for \( r \to 0 \) the solutions of the wave equation, and hence the scalar propagator, are well defined.

In the present case the brane is located at \( r = r_b \), and the functions \( A(r) \), \( B(r) \) and \( C(r) \) in \( \mathcal{O}_\ell \) are of the form (2.6) only for \( r_b \leq r < \infty \), but of the form (2.9) for \( 0 < r \leq r_b \), hence \( Z_2 \)-symmetric under \( r \leftrightarrow r_b^2/r \). Asymptotically, both for \( r \to \infty \) and \( r \to 0 \), \( \mathcal{O}_\ell(r) \) tends to the (radial) Laplacian in flat space. Hence its spectrum is continuous, with eigenfunctions \( \phi(\ell, q, r) \) that behave as plane waves for \( r \to \infty \) and \( r \to 0 \). Here \( q \) denotes a continuum variable conjugate to \( r \). These eigenfunctions can be decomposed into even ones under \( Z_2 \) (with vanishing radial derivatives at \( r = r_b \)) and odd ones under \( Z_2 \) (which vanish at \( r = r_b \)).

It is possible to repeat the analysis in [14] of the operator \( \mathcal{O}_\ell(r) \) in the present setup (without singularities for \( r \to 0 \), except for coordinate singularities, but jumps in the first radial derivatives of \( A(r) \), \( B(r) \) and \( C(r) \) at \( r = r_b \), with the same result: \( \mathcal{O}_\ell(r) \) is self-adjoint and thus semi-positive. Hence the eigenfunctions \( \phi(\ell, q, r) \) satisfying

\[
\mathcal{O}_\ell \, \phi(\ell, q, r) = e^2(q, \ell) \, \phi(\ell, q, r)
\]

(3.14)

15
with \( e^2(q, \ell) \geq 0 \) represent a complete basis.

Although neither these functions nor their eigenvalues are not known explicitely (except for the zero modes [14]) they can be used to construct a formal solution for the bulk Green function \( D_\ell(p, r, r') \) satisfying eq. (3.12):

\[
D_\ell(p, r, r') = k \int dq \frac{\phi(\ell, q, r) \phi(\ell, q, r')}{p^2 + e^2(q, \ell)}
\]  

(3.15)

The rough behaviour of \( D_\ell(p, r, r') \) for \( p \to 0 \) can be obtained from eq. (3.15) from the behaviour of the eigenvalues \( e^2(q, \ell) \) for \( q \to 0 \): First, dimensional analysis dictates that we must have \( e^2(q, \ell) \sim q^2 \). No powers of \( r_b \) can appear here, since the problem is non-singular for \( r_b \to 0 \) [14]. This shows already that \( D_\ell(p \to 0, r, r') \sim p^{-1} \).

Below we will be interested in the brane-to-brane propagator (and in the \( \ell = 0 \) partial wave), and in this case we can be somewhat more precise on the corresponding coefficient. Expanding \( O_{\ell=0} \) around \( r = r_b \) it becomes simply \( -A^2 B^{-2} \partial_r^2 \), in which case (3.12) can be solved by simple Fourier transformation and the argument of the \( dq \)-integral in (3.15) becomes \( 1/(p^2 + A^2 B^{-2} q^2) \). Thus one finds

\[
D_{\ell=0}(p \to 0, r_b, r_b) \sim \frac{k B}{Ap}.
\]  

(3.16)

Note that if one would compactify the extra dimensions inside a finite volume of size \( \sim R^N \), the spectrum of \( O_\ell(r) \) would be discrete. Then the integral \( dq \) in (3.15) would be replaced by a sum over discrete modes \( q_i \) with spacing \( \sim R^{-1} \), and the \( p \to 0 \) behaviour of \( D_\ell \) would be entirely due to the zero mode: \( D_\ell(p \to 0, r, r') \sim k/(R p^2) \) in contrast to the \( p^{-1} \)-behaviour obtained for infinite extra dimensions.

The result (3.16) can be plugged into (3.9), together with \( k \) given in (3.13). Up to factors of \( O(1) \) one finds

\[
G_\ell(p, r_b, r_b) \sim \frac{\kappa_D}{A^3 C^{N-1} \Omega_{N-1}} \cdot \frac{1}{p + \frac{A \kappa_D}{\kappa_{D-1}} \left( A^{-2} p^2 + C^{-2} \ell (l + N - 2) \right)}.
\]  

(3.17)

Of particular importance is the denominator in the second factor in (3.17), and the question where and when it behaves \( \sim p^2 \) such that a phenomenologically acceptable \( p^2 \) dependence of the propagator is obtained.
First, to this end the first term has to be small with respect to the second one, i.e.

$$p \gg \frac{A \kappa_D}{\kappa_D}$$

which is true at scales $1/p \ll r_c$ with

$$r_c \sim \frac{\kappa_D}{A \kappa_D - 1}.$$  

Now recall that the second and third terms in the denominator originate from the kinetic term $\Box^{(D-1)}$ localized on the brane, whose extra dimensions ($N - 1$ of them) are compactified on $S^{(N-1)}$. Once this term dominates, the propagator sees the corresponding Kaluza-Klein states (partial waves with $\ell > 0$) with masses $M_{kk} \sim A/C$ in agreement with the metric (2.4).

Contributions of these states to the full propagator $G$ in (3.5) (the sum over $G_\ell$) would not be acceptable, since they would again modify its $p^2$-dependence and turn it into a $D = 4 + N$ dimensional propagator. Thus one has to require

$$p \lesssim M_{kk} \sim \frac{A}{C}$$

in which regime only the $\ell = 0$ mode contributes.

Once the inequalities (3.18) and (3.20) are satisfied, the $\ell = 0$ brane-to-brane propagator behaves as

$$G_0(p, r_b, r_b) \sim \frac{\kappa_D - 1}{C^{N-1} \Omega_{N-1} A^2 p^2} \sim \frac{K_4}{A^2 p^2},$$

where we have used (2.5a) in the last step. The coefficient coincides with the one obtained from a 4-dimensional action

$$S = \frac{1}{2k_4} \int d^4x \sqrt{-g^{(4)}} \partial_\mu \phi g^{\mu \nu} \partial_\nu \phi$$

with $g^{(4)}_{\mu \nu} = A^2 \eta_{\mu \nu}$, as it should.

The two inequalities (3.18) and (3.20) are compatible only if

$$\frac{\kappa_D}{\kappa_D - 1 C} \gg 1. \quad (3.23)$$
Let us consider the required orders of magnitude in more detail. For the scale $r_c$, beyond which gravity would change, one can only tolerate $r_c \gtrsim H_0^{-1}$ [2], where $H_0$ is the Hubble constant today. Hence, rewriting eq. (3.19),

$$ r_c \sim \frac{\kappa_D}{C \kappa_{D-1}} \frac{C}{A} \gtrsim H_0^{-1} . $$

(3.24)

On the other side, in order not to change gravity at distances down to $10^{-4} \text{m}$ through Kaluza-Klein states with masses $M_{kk}$ trapped on $S^{(N-1)}$ we need

$$ M_{kk} \sim \frac{A}{C} > 10^{-3} \text{eV} . $$

(3.25)

With $H_0 \sim 10^{-33} \text{eV}$ eqs. (3.24) and (3.25) imply

$$ \frac{\kappa_D}{C \kappa_{D-1}} > 10^{30} . $$

(3.26)

Now recall that, from the junction conditions derived in the previous chapter, the combination of parameters in (3.26) is related to the warp factor $A$ on the brane. From eqs. (2.26), (2.27) with (3.26) one finds

$$ A > 10^{15} \sqrt{\frac{N}{N-1}} . $$

(3.27)

A huge warp factor on the brane aggravates the hierarchy problem (i.e. the smallness of the Weak Scale as compared to the Planck Scale) instead of solving it in the sense of [15] via a tiny warp factor on the brane.

If one is willing to swallow this complication, one can translate the constraint (3.26), via eqs. (2.18) and (2.20), into a constraint on the $D$-dimensional gravitational scale $M_D$ and $C$, which is related to the size of the blown-up brane in the extra dimensions via (2.19). Then (3.26) gives

$$ \frac{M_D^2}{\Omega_{N-1} M_{Pl}^2 (M_D C)^N} > 10^{30} . $$

(3.28)

For $C \sim M_D^{-1}$ one obtains

$$ M_D < 10^{-15} M_{Pl} $$

(3.29)
as in the co-dimension 1 case in [2]. However, eq. (3.28) allows to generate a huge number more easily, if one allows for \( C \ll M_D^{-1} \). In the extreme case \( C \sim M_{Pl}^{-1} \) one obtains

\[
\left( \frac{M_{Pl}}{M_D} \right)^{N+2} > 10^{30}
\]

which, for \( N \) large, allows for much more modest ratios \( M_D/M_{Pl} \).

What would be the required tension (cosmological constant) on the brane? From eqs. (2.17) one finds that, in the limit (2.25) and (2.26), its right hand sides are of comparable magnitude. Using eqs. (2.20) and (2.21) this gives

\[
\Lambda_4 \sim \frac{M_{Pl}^2}{C^2}
\]

or \( \Lambda_4 \sim M_{Pl}^2 M_D^2 \) in the case (3.29), \( \Lambda_4 \sim M_{Pl}^4 \) in the case (3.30).

In order to see whether this scenario is phenomenologically viable, the complete spectrum of physical fluctuations would have to be studied.

The tensor structure of 4-dimensional gravity, in the presence of an additional Einstein term on the brane, has recently been the subject of numerous investigations [3, 9, 16-21]. Note that the problematic result (ghosts and tachyons for \( N \geq 2 \)) obtained in [21] cannot be applied here, since we have a non-trivial gravitational background and, notably, replaced \( R^{(4)} \) by \( R^{(D-1)} \) on the blown-up brane, hence we have an effective codimension one scenario. The recent results in the co-dimension 1 case, on the other hand, point towards a phenomenologically acceptable tensor structure of 4-dimensional gravity [16-20]. Nevertheless the details in the present case, as well as the spectrum of 4-dimesional scalars (moduli), remain to be studied.

Independently thereof we want to emphasize that the present brane configuration can also be studied in compact extra dimensions, in which case the correct structure of 4-dimensional gravity is guaranteed even without an additional Einstein term on the brane. Given the present negative result on the solution of the cosmological constant problem, and the required hierarchies of scales in the case of infinite-volume extra dimensions (notably the need for \( M_D \ll M_{Pl} \), although our eq. (3.30) reduces the required hierarchy), compact extra dimensions are perhaps somewhat more promising. Work in this direction is in progress.
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