Well-posedness for a class of dissipative stochastic evolution equations with Wiener and Poisson noise

Carlo Marinelli*
12 October 2011

Abstract

We prove existence and uniqueness of mild and generalized solutions for a class of stochastic semilinear evolution equations driven by additive Wiener and Poisson noise. The non-linear drift term is supposed to be the evaluation operator associated to a continuous monotone function satisfying a polynomial growth condition. The results are extensions to the jump-diffusion case of the corresponding ones proved in [4] for equations driven by purely discontinuous noise.

1 Introduction

The purpose of this note is to show that stochastic evolution equations of the type

\[
\frac{du(t)}{dt} + Au(t) \, dt + f(u(t)) \, dt = B(t) \, dW(t) + \int_Z G(z,t) \, \bar{\mu}(dz, dt), \quad u(0) = u_0, \tag{1}
\]

where \( A \) is a linear \( m \)-accretive operator on a Hilbert space \( H \), \( f : \mathbb{R} \to \mathbb{R} \) a monotone increasing function of polynomial growth, \( W \) is a cylindrical Wiener noise on \( H \), and \( \bar{\mu} \) is a compensated Poisson random measure, admit a unique mild solution. Precise assumptions on the space on which the equation is considered and on the data of the problem are given in the next section.

Global well-posedness of (1) in the case of purely discontinuous noise (i.e. with \( B \equiv 0 \)) has been proved in [4] showing that solutions to regularized equations converge to a process which solves the original equation. This is achieved proving a priori estimates for the approximating processes by rewriting the regularized stochastic equations as deterministic evolution equations with random coefficients and using monotonicity arguments. These a priori estimates essentially rely, in turn, on a maximal inequality of Bichteler-Jacod type for stochastic convolutions on \( L_p \) spaces with respect to compensated Poisson random measures, also proved in [4].

The well-posedness results of [4] will be here extended to the more general class of equation (1). We shall adapt the method used in [4], but instead of rewriting the

*Facoltà di Economia, Libera Università di Bolzano, I-39100 Bolzano, Italy.
regularized (stochastic) equations as deterministic equations with random coefficients, we shall rewrite them as stochastic equations driven just by Wiener noise (we might say that, in a sense, we “hide the jumps”), the solutions of which will be shown to satisfy suitable a priori estimates allowing to pass to the limit in the regularized equations.

The result might be interesting even in the case of equations driven only by a Wiener process (i.e. with $G \equiv 0$). In fact, the usual approach to establish well-posedness for such equations (cf. e.g. [1,2]) is to rewrite them as deterministic equations with random coefficients and to consider them on a Banach space of continuous functions. This approach requires the stochastic convolution to have paths in such a space of continuous functions. The latter condition is not needed in our setting.

Let us conclude this introductory section with some words about notation used throughout the paper: $a \lesssim b$ stands for $a \leq Nb$ for some constant $N$ (if the constant $N$ depends on parameters $p_1, \ldots, p_n$ we shall write $N(p_1, \ldots, p_n)$ and $a \lesssim_{p_1,\ldots,p_n}$, respectively). For any $p \geq 0$, we set $p^* := p^2/2$. Given two (metric) spaces $E, F$, we shall denote the space of Lipschitz continuous functions from $E$ to $F$ by $C^{0,1}(E \to F)$. The duality mapping of a Banach space $E$ with dual $E'$ and duality form $\langle \cdot, \cdot \rangle$ is the (multivalued) map $J : E \to 2^{E'}$, $J : x \mapsto \{x^* \in E' : \langle x^*, x \rangle = \|x\|_E^2 \}$.

## 2 Main result

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, with $T > 0$ fixed, be a filtered probability space satisfying the usual conditions, and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. All stochastic elements will be defined on this stochastic basis, and any equality or inequality between random quantities will be meant to hold $\mathbb{P}$-almost surely. Let $(Z, \mathcal{Z}, m)$ be a measure space, $\mu$ a Poisson measure on $[0,T] \times Z$ with compensator $\text{Leb} \otimes m$, where $\text{Leb}$ stands for Lebesgue measure. Let $D$ be an open bounded subset of $\mathbb{R}^d$ with smooth boundary. All Lebesgue spaces on $D$ will be denote without explicit mention of the domain, e.g. $L_p := L_p(D)$. We shall sometimes denote $L_2$ by $H$. Given $q \geq 1$ and a Banach space $E$, we shall denote the set of all $E$-valued random variables $\xi$ such that $\mathbb{E} |\xi|^q < \infty$ by $\mathbb{L}_q(E)$. We call $\mathbb{H}_p(E)$ the set of all adapted $E$-valued processes such that

$$\|u\|_{\mathbb{H}_p(E)} := \left(\mathbb{E} \sup_{t \leq T} \|u(t)\|_E^p \right)^{1/p} < \infty.$$

For compactness of notation, we shall also write $\mathbb{L}_q$ in place of $\mathbb{L}_q(L_q)$. and $\mathbb{H}_q$ in place of $\mathbb{H}_q(L_q)$. We shall denote by $W$ a cylindrical Wiener process on $L_2(D)$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $f(0) = 0$, for which there exists $p \geq 2$ such that $|f(r)| \leq 1 + |r|^{p/2}$ for all $r \in \mathbb{R}$.

Let $A$ be a linear (unbounded) $m$-accretive operator in the spaces $H = L_2$, $E := L_p$ and $L_{p^*}$, and assume that $S$, the strongly continuous semigroup generated by $-A$ on $E$, is analytic. We shall not distinguish notionally the different realizations of $A$ and $S$ on the above spaces.
Denoting by $\gamma(H \to E)$ the space of $\gamma$-radonifying operators from $H$ to $E$, for any $q \geq 1$, the class of adapted processes $B : [0,T] \to L(H \to E)$ such that
\[
\|B\|_{L^q}^q := \mathbb{E} \int_0^T \|B(t)\|_{\gamma(H \to E)}^q dt < \infty
\]
will be denoted by $L^q_{\gamma}(H \to E)$. Similarly, denoting the predictable $\sigma$-algebra by $\mathcal{P}$ and the Borel $\sigma$-algebra of $\mathbb{R}^d$ by $\mathcal{B}(\mathbb{R}^d)$, the space of $\mathcal{P} \otimes Z \otimes \mathcal{B}(\mathbb{R}^d)$-measurable processes $g : [0,T] \times Z \times D \to \mathbb{R}$ such that
\[
\|g\|_{L^q}^q := \mathbb{E} \int_0^T \int_Z \|g(t,z)\|_{L^q}^q m(dz) dt + \mathbb{E} \int_0^T \left( \int_Z \|g(t,z)\|_{L^q}^2 m(dz) \right)^{q/2} dt < \infty
\]
will be denoted by $L^q_{m}$. It was proved in [4] that, for any strongly continuous semigroup of positive contractions $R$ on $L^q$, $q \in [2, \infty[$, one has the maximal inequality
\[
\mathbb{E} \sup_{t \leq T} \left\| R(t-s)g(s) d\mu(ds,dz) \right\|_{L^q}^q \lesssim \|g\|_{L^q}^q. \tag{2}
\]

Let us now define mild and generalized solutions of (1).

**Definition 1.** Let $u_0$ be an $H$-valued $\mathcal{F}_0$-measurable random variable. A (strongly) measurable adapted $H$-valued process $u$ is a mild solution to (1) if, for all $t \in [0,T],
\[
\begin{align*}
    u(t) + \int_0^t S(t-s) f(u(s)) ds &= S(t)u_0 + \int_0^t S(t-s)B(s) dW(s) + \int_0^t \int_Z S(t-s)G(s,z) \mu(dz,ds)
\end{align*}
\]
and all integrals are well-defined.

As is well known, the stochastic convolution with respect to $W$ is well-defined if the operator $Q_t$ is nuclear for all $t \in [0,T]$, where
\[
Q_t := \int_0^t S(t-s)B(s)B^*(s)S^*(t-s) ds.
\]
This condition is verified, for instance, if $B \in L^2_\gamma$, i.e. if
\[
\mathbb{E} \int_0^T \|B(s)\|_{\gamma(H \to H)}^2 ds < \infty
\]
(recall that $\gamma(H \to H)$ is just the space of Hilbert-Schmidt operators from $H$ to itself). Similarly, the stochastic convolution with respect to $\mu$ is well-defined if $G \in L^2_{m}$, i.e. if
\[
\mathbb{E} \int_0^T \int_Z \|G(s,z)\|_{L^2_m}^2 m(dz) ds < \infty.
\]
The deterministic convolution term is well-defined if $f(u) \in L^1([0,T] \to H)$, or if $u \in L^p((0,T] \to L^p)$.
**Definition 2.** A process \( u \in \mathbb{H}_2(T) \) is a generalized solution to (1) if there exist \( \{u_{0n}\}_n \subset L_p, \{B_n\}_n \subset \mathcal{L}^\gamma_p, \{G_n\}_n \subset \mathcal{L}^m_p, \) and \( \{u_n\}_n \subset \mathbb{H}_2(T) \) such that \( u_{0n} \to u_0 \) in \( L_2, B_n \to B \) in \( \mathcal{L}^\gamma_2 \), \( G_n \to G \) in \( \mathcal{L}^m_2 \) and \( u_n \to u \) in \( \mathbb{H}_2(T) \) as \( n \to \infty \), where \( u_n \) is the (unique) mild solution of

\[
du_n(t) + Au_n(t) \, dt + f(u_n(t)) \, dt = B_n(s) \, dW(t) + \int_Z G_n(z) \, \mu(dt,dz), \quad u_n(0) = u_{0n}.
\]

Here are the results, which will be proved in the next sections.

**Theorem 3.** Assume that \( u_0 \in L_p, B \in \mathcal{L}^\gamma_p \) and \( G \in \mathcal{L}^m_p \). Then there exists a unique càdlàg mild solution \( u \in \mathbb{H}_2 \) to equation (1) such that \( f(u) \in L^1[0,T] \to H \).

**Theorem 4.** Assume that \( u_0 \in L_2, B \in \mathcal{L}^\gamma_2 \), \( G \in \mathcal{L}^m_2 \). Then there exists a unique generalized solution to equation (1).

**Remark 5.** By inspection of the corresponding proof in [4], it is clear that the same argument applies to Theorem 3 if one assumes \( B \in \mathcal{L}^\gamma_p \), i.e.

\[
\mathbb{E} \sup_{t \leq T} \| W_A(t) \|_{L^p}^p < \infty.
\]

In the proof of Theorem 3 below we show that \( B \in \mathcal{L}^\gamma_p \) is too strong an assumption, and that \( B \in \mathcal{L}^\gamma_2 \) is enough. It is natural to conjecture that also \( G \in \mathcal{L}^m_p \) is too strong, and it should suffice to assume \( G \in \mathcal{L}^m_2 \). Unfortunately, thus far we have not been able to replace the exponent \( p^* \) by \( p \) in the hypotheses on \( G \) of Theorem 3.

### 3 Proofs

#### 3.1 Proofs of Theorem 3

Let \( f_\lambda := \lambda^{-1}(I - (I + \lambda f)^{-1}) \), \( \lambda > 0 \), be the Yosida approximation of \( f \), and recall that \( f_\lambda \in \dot{C}^{0,1}(\mathbb{R}) \), with \( \|f_\lambda\|_{C^{0,1}} \leq 2/\lambda \). Let us consider the regularized equation

\[
du_\lambda(t) + Au_\lambda(t) \, dt + f_\lambda(u_\lambda(t)) \, dt = B(t) \, dW(t) + \int_Z G(z,t) \, \mu(dz,dt), \quad u_\lambda(0) = u_0. \tag{3}
\]

Assuming that \( B \in \mathcal{L}^\gamma_p \) and \( G \in \mathcal{L}^m_p \), one could prove by a fixed point argument that (3) admits a unique càdlàg mild \( E \)-solution (by which we mean, here and in the following, a mild solution with values in \( E \)). However, we prefer to proceed in a less direct way, for reasons that will become apparent later. In particular, we “hide the jumps” in (3) writing an equation for the difference between \( u_\lambda \) and the stochastic convolution with respect to the Poisson random measure as follows: setting, for notational compactness,

\[
W_A(t) := \int_0^t S(t-s)B(s) \, dW(s), \quad G_A(t) := \int_0^t \int_Z S(t-s)G(s,z) \, \mu(ds,dz),
\]

4
the integral form of (3) reads
\[ u_\lambda(t) + \int_0^t S(t-s)f_\lambda(u_\lambda(s)) \, ds = S(t)u_0 + W_A(t) + G_A(t), \] (4)
which can be equivalently written as
\[ u_\lambda(t) - G_A(t) + \int_0^t S(t-s)f_\lambda(u_\lambda(s) - G_A(s) + G_A(s)) \, ds = S(t)u_0 + W_A(t), \]

hence also, setting \( v_\lambda := u_\lambda - G_A \) and \( \tilde{f}_\lambda(t,y) := f_\lambda(y + G_A(t)) \), for \( y \in \mathbb{R} \) and \( t \geq 0 \), as
\[ v_\lambda(t) + \int_0^t S(t-s)\tilde{f}_\lambda(v_\lambda(s)) \, ds = S(t)u_0 + W_A(t), \]
which is the mild form of
\[ dv_\lambda(t) + Av_\lambda(t) \, dt + \tilde{f}_\lambda(t,v_\lambda(t)) \, dt = B(t) \, dW(t), \quad v_\lambda(0) = u_0. \] (5)

It is clear that \( v_\lambda \) is a mild \( E \)-solution of (5) if and only if \( v_\lambda + G_A \) is a mild \( E \)-solution of (3).

In the next Proposition we show that (5) admits a unique mild \( E \)-solution \( v_\lambda \), hence identifying also the unique \( E \)-mild solution of (3).

**Proposition 6.** If \( u_0 \in L_p \), \( B \in \mathcal{L}_p^{\gamma} \) and \( G \in \mathcal{L}_p^m \), then equation (4) admits a unique càdlàg mild \( E \)-solution \( v_\lambda \in \mathbb{H}_p \). Therefore equation (5) admits a unique càdlàg mild \( E \)-solution \( u_\lambda \in \mathbb{H}_p \), and \( u_\lambda = v_\lambda + G_A \).

**Proof.** We use a fixed point argument on the space \( \mathbb{H}_p \). Let us consider the operator
\[ \tilde{\Phi} : \mathbb{H}_p \ni \phi \mapsto \left( t \mapsto S(t)u_0 - \int_0^t S(t-s)\tilde{f}_\lambda(s,\phi(s)) \, ds + W_A(t) \right). \]

We shall prove that \( \tilde{\Phi} \) is a contraction on \( \mathbb{H}_p \), if \( T \) is small enough. Since \( u_0 \in L_p \) and \( S \) is strongly continuous on \( L_p \), it is clear that we can (and will) assume, without loss of generality, that \( u_0 = 0 \). Then
\[ \| \tilde{\Phi}(\phi) \|_{\mathbb{H}_p} \leq \| S * \tilde{f}_\lambda(\cdot,\phi) \|_{\mathbb{H}_p} + \| W_A \|_{\mathbb{H}_p}. \]

By a maximal inequality for stochastic convolutions we have
\[ \| W_A \|_{\mathbb{H}_p} \leq \mathbb{E} \int_0^T \| B(t) \|_{\gamma(H \to E)}^p \, dt, \]
where the right-hand side is finite by assumption. Moreover, Jensen’s inequality and strong continuity of \( S \) on \( L_p \) yield
\[ \mathbb{E} \sup_{t \leq T} \left\| \int_0^t S(t-s)\tilde{f}_\lambda(s,\phi(s)) \, ds \right\|_E^p \leq_T \mathbb{E} \sup_{t \leq T} \| \tilde{f}_\lambda(t,\phi(t)) \|_E^p. \]
Since $\|f_\lambda\|_{C^{0,1}} \leq 2/\lambda$, we have
\[
|\tilde{f}_\lambda(t,x) - \tilde{f}_\lambda(t,y)| = |f_\lambda(x + G_A(t)) - f_\lambda(y + G_A(t))| \leq \frac{2}{\lambda}|x - y|,
\]
hence
\[
|\tilde{f}_\lambda(t,x)| \leq |\tilde{f}_\lambda(t,x) - \tilde{f}_\lambda(t,0)| + |\tilde{f}_\lambda(t,0)| \leq \frac{2}{\lambda}|x| + |f_\lambda(G_A(t))| \leq \frac{2}{\lambda}|x| + \frac{2}{\lambda}|G_A(t)|,
\]
thus also
\[
E \sup_{t \leq T} \|\tilde{f}_\lambda(t,\phi(t))\|_E^p \lesssim \lambda E \sup_{t \leq T} \|\phi(t)\|_E^p + E \sup_{t \leq T} \|G_A(t)\|_{E^p}^p,
\]
where the right-hand side is finite because of (2) and because $G \in L^m_p$ by hypothesis. We have thus proved that $\tilde{g}(\mathbb{H}_\mu) \subseteq \mathbb{H}_p$. Since $x \mapsto \tilde{f}_\lambda(t,x,\omega)$ is Lipschitz continuous, uniformly over $t \in [0,T]$ and $\omega \in \Omega$, analogous computations show that $\tilde{g}$ is Lipschitz on $\mathbb{H}_p$, with a Lipschitz constant that depends continuously on $T$. Choosing $T = T_0$, for a small enough $T_0$ such that $\tilde{g}$ is a contraction, and then covering the interval $[0,T]$ by intervals of length $T_0$, one obtains the desired existence and uniqueness of a fixed point of $\tilde{g}$ in a standard way. □

Remark 7. (i) Note that we have assumed the more natural condition $G \in L^m_p$ for the well-posedness of the regularized equation (3) rather than $G \in L_p^{m^*}$. Let us show that the latter condition also ensures that $\|G_A\|_{E^p}$ is finite: since $D$ has finite Lebesgue measure and $p^* = p^2/2 \geq p$, Hölder’s inequality implies
\[
E \sup_{t \leq T} \|G_A(t)\|_{E^p}^p \lesssim D E \sup_{t \leq T} \|G_A(t)\|_{E^p}^p \leq \left(E \sup_{t \leq T} \|G_A(t)\|_{E^p}^p\right)^{2/p} < \infty.
\]
(ii) The previous existence and uniqueness result also follows by an adaptation of [6, Thm. 6.2], which is a more general and more precise result about well-posedness for equations with Wiener noise and Lipschitz coefficients. In [6] the nonlinearity in the drift is Lipschitz continuous and satisfies a linear growth condition with a constant that does not depend on $t$ and $\omega$, hence it does not apply directly to our situation. A reasoning completely analogous to the above one permits however to circumvent this problem.

We shall need the following a priori estimate for the solution to the regularized equation (3).

Lemma 8. Assume that $u_0 \in L_p$, $B \in L_p^*$ and $G \in L^{m^*}_p$. Then there exists a constant $N$, independent of $\lambda$, such that
\[
E \sup_{t \leq T} \|u_\lambda(t)\|_E^p \leq N(1 + E\|u_0\|_E^p).
\]
Proof. Let \( v_\lambda \) be the mild \( E \)-solution to (5). For \( \varepsilon > 0 \), set

\[
\begin{align*}
  u_0^\varepsilon := (I + \varepsilon A)^{-1}u_0, \\
  B^\varepsilon(t) := (I + \varepsilon A)^{-1}B(t), \\
  g_\lambda(t) := f_\lambda(t, v_\lambda), \\
  g_\lambda^\varepsilon(t) := (I + \varepsilon A)^{-1}g_\lambda(t),
\end{align*}
\]

and let \( w_\lambda^\varepsilon \) be the mild \( E \)-solution to

\[
dw_\lambda^\varepsilon + Aw_\lambda dt + g_\lambda^\varepsilon dt = B^\varepsilon dW, \quad w_\lambda^\varepsilon(0) = u_0^\varepsilon,
\]

that is

\[
w_\lambda^\varepsilon(t) = S(t)u_0^\varepsilon - \int_0^t S(t-s)g_\lambda^\varepsilon(s) ds + \int_0^t S(t-s)B^\varepsilon(s) ds
\]

for all \( t \in [0, T] \). It is easily seen that \( w_\lambda^\varepsilon \) is a strong solution, i.e. that one has

\[
w_\lambda^\varepsilon(t) + \int_0^t (Aw^\varepsilon_\lambda(s) + g^\varepsilon_\lambda(s)) ds = u_0^\varepsilon + \int_0^t B(s) dW(s)
\]

for all \( t \in [0, T] \), and that \( w_\lambda^\varepsilon = (I + \varepsilon A)^{-1}v_\lambda \rightarrow v_\lambda \) in \( H_p \) as \( \varepsilon \rightarrow 0 \). We are going to apply Itô’s formula (in particular we shall use the version in [5] Thm. 3.1) to obtain estimates for \( \| w_\lambda^\varepsilon \|_{E}^p \). To this purpose, we have to check that

\[
\mathbb{E}\left( \int_0^T \| b(t) \|_E dt \right)^p < \infty,
\]

where \( b := Aw^\varepsilon_\lambda + g^\varepsilon_\lambda \). One has

\[
\mathbb{E}\left( \int_0^T \| b(t) \|_E dt \right)^p \lesssim \mathbb{E}\int_0^T \| b(t) \|_{E}^p dt \lesssim \mathbb{E}\int_0^T \| Aw^\varepsilon_\lambda \|_{E}^p dt + \mathbb{E}\int_0^T \| g^\varepsilon_\lambda \|_{E}^p dt,
\]

where

\[
\| Aw^\varepsilon_\lambda \|_E = \| A(I + \varepsilon A)^{-1}v_\lambda \|_E \lesssim \| v_\lambda \|_E
\]

and

\[
\| g^\varepsilon_\lambda \|_E = \| (I + \varepsilon A)^{-1}f_\lambda(v_\lambda + G_A) \|_E \leq \| f_\lambda(v_\lambda + G_A) \|_E \lesssim_\lambda \| v_\lambda \|_E + \| G_A \|_E,
\]

hence

\[
\mathbb{E}\left( \int_0^T \| b(t) \|_E dt \right)^p \lesssim_{\lambda, \varepsilon, T} \| v_\lambda \|_{H_p}^p + \| G_A \|_{H_p}^p < \infty,
\]

which justifies applying Itô’s formula. Setting \( \psi(x) := \| x \|_{E}^p \), we have

\[
\psi(w_\lambda^\varepsilon) + \int_0^t (Aw^\varepsilon_\lambda + g^\varepsilon_\lambda, \psi'(w^\varepsilon_\lambda)) ds = \int_0^t \psi'(w^\varepsilon_\lambda)B(s) dW(s) + R(t),
\]

where \( R \) is a “remainder” term, the precise definition of which is given in [5]. Note that \( \psi(u) = (\| u \|_{E}^2)^{p/2} \) and \( \psi'(u) = p\| u \|_{E}^{p-2}u = p\| u \|_{E}^{p-2}J(u) \), where \( J \) is the duality mapping of \( E \),

\[
J : u \mapsto u\| u \|_{E}^{p-2},
\]

\[
7
\]
i.e. \( J \) is the Gâteaux (and Fréchet) derivative of \( \| \cdot \|_E^2/2 \). Since \( A \) is \( m \)-accretive on \( E \), it holds

\[
(\mathbb{A} w^\varepsilon, \psi'(w^\varepsilon)) = p\|w^\varepsilon\|_{E}^{p-2}(\mathbb{A} w^\varepsilon, J(w^\varepsilon)) \geq 0.
\]

Moreover, there exists \( \delta > 0 \) and \( N = N(\delta) > 0 \) such that (cf. \([5]\))

\[
\mathbb{E}\sup_{t \leq T}|R(t)| \leq \delta \mathbb{E}\sup_{t \leq T}\|w^\varepsilon(t)\|_{E}^p + N\mathbb{E}\left( \int_{0}^{T}\|B(s)\|_{\gamma(H \rightarrow E)}^2\,ds \right)^{p/2}
\]

and, by some calculations based on Young’s and Burkholder’s inequalities,

\[
\mathbb{E}\sup_{t \leq T}\left| \int_{0}^{t}\psi'(w^\varepsilon(s))B(s)\,dW(s) \right| \lesssim \delta \mathbb{E}\sup_{t \leq T}\|w^\varepsilon(t)\|_{E}^p + N\mathbb{E}\left( \int_{0}^{T}\|B(s)\|_{\gamma(H \rightarrow E)}^2\,ds \right)^{p/2}.
\]

We thus arrive at the estimate

\[
\mathbb{E}\sup_{t \leq T}\|w^\varepsilon\|_{E}^p \lesssim \delta \mathbb{E}\sup_{t \leq T}\|w^\varepsilon\|_{E}^p + \|B\|_{L^p}^p + \mathbb{E}\sup_{t \leq T}\int_{0}^{T}\langle -g^\varepsilon, w^\varepsilon \rangle \|w^\varepsilon\|_{E}^{p-2}\,ds.
\]

Letting \( \varepsilon \to 0 \), we are left with

\[
\mathbb{E}\sup_{t \leq T}\|v_\lambda\|_{E}^p \lesssim \delta \mathbb{E}\sup_{t \leq T}\|v_\lambda\|_{E}^p + \|B\|_{L^p}^p + \mathbb{E}\sup_{t \leq T}\int_{0}^{T}\langle -\tilde{f}_\lambda(s, v_\lambda(s)), \psi'(v_\lambda(s)) \rangle \,ds.
\]

Note that we have

\[
\langle \tilde{f}_\lambda(t, v_\lambda), \psi'(v_\lambda) \rangle = p\|v_\lambda\|_{E}^{p-2}\langle \tilde{f}_\lambda(t, v_\lambda), J(v_\lambda) \rangle = p\|v_\lambda\|_{E}^{p-2}\langle f_\lambda(G_\lambda + v_\lambda), J(v_\lambda) \rangle,
\]

where, by accretivity of \( f_\lambda \),

\[
\langle f_\lambda(G_\lambda + v_\lambda), J(v_\lambda) \rangle = \langle f_\lambda(G_\lambda + v_\lambda) - f(G_\lambda), J(G_\lambda + v_\lambda - G_\lambda) \rangle + \langle f_\lambda(G_\lambda), J(v_\lambda) \rangle \\
\geq \langle f_\lambda(G_\lambda), J(v_\lambda) \rangle,
\]

hence, recalling that \( \psi'(u) = pu|u|^{p-2} \),

\[
\langle \tilde{f}_\lambda(t, v_\lambda), \psi'(v_\lambda) \rangle \geq p(f_\lambda(G_\lambda), v_\lambda|v_\lambda|^{p-2}),
\]

and, by Young’s inequality with conjugate exponents \( p \) and \( p' = p/(p-1) \),

\[
\langle f_\lambda(G_\lambda), v_\lambda|v_\lambda|^{p-2} \rangle \lesssim N\|f_\lambda(G_\lambda)\|_{L^p}^p + \|v_\lambda|v_\lambda|^{p-2}\|_{L^{p'}}^p = N\|f_\lambda(G_\lambda)\|_{L^p}^p + \|v_\lambda\|_{L^p}^p,
\]

so that

\[
\mathbb{E}\sup_{t \leq T}\left| \int_{0}^{T}\langle \tilde{f}_\lambda(t, v_\lambda), \psi'(v_\lambda) \rangle \,ds \right| \lesssim \delta \mathbb{E}\sup_{t \leq T}\|v_\lambda(t)\|_{E}^p + N\mathbb{E}\sup_{t \leq T}\|f_\lambda(G_\lambda(t))\|_{E}^p \\
\lesssim 1 + \delta \mathbb{E}\sup_{t \leq T}\|v_\lambda(t)\|_{E}^p + N\|G\|_{L^{p'}}^p.
\]
where the last constant does not depend on \( \lambda \).

Combining the above estimates and choosing \( \delta \) small enough, we are left with

\[
\mathbb{E} \sup_{t \leq T} \|v_\lambda\|^p_{L^2} \lesssim 1 + \mathbb{E}\|u_0\|^p_{L^2} + \|G\|^p_{L^{p^*}} + \|B\|^p_{L^{p^*}},
\]

with implicit constant independent of \( \lambda \).

Thanks to the a priori estimate just established, we are now going to show that \( \{u_\lambda\}_\lambda \) is a Cauchy sequence in \( \mathbb{H}_2 \), hence that there exists \( u \in \mathbb{H}_2 \) such that \( u_\lambda \to u \) in \( \mathbb{H}_2 \) as \( \lambda \to 0 \). In particular, we have

\[
d(u_\lambda - u_\mu) + A(u_\lambda - u_\mu) \, dt + (f_\lambda(u_\lambda) - f_\mu(u_\mu)) \, dt = 0,
\]

from which we obtain, using the same argument as in \cite{4} pp. 1539-1540,

\[
\mathbb{E} \sup_{t \leq T} \|u_\lambda - u_\mu\|^2_{L^2} \lesssim_T (\lambda + \mu) \left( \mathbb{E} \sup_{t \leq T} \|f_\lambda(u_\lambda(t))\|^2_{L^2} + \mathbb{E} \sup_{t \leq T} \|f_\mu(u_\mu(t))\|^2_{L^2} \right)
\]

\[
\lesssim_T (\lambda + \mu) \left( 1 + \mathbb{E} \sup_{t \leq T} \|u_\lambda(t)\|^p_{L^p} + \mathbb{E} \sup_{t \leq T} \|u_\mu(t)\|^p_{L^p} \right).
\]

Since

\[
\|u_\lambda\|_{\mathbb{H}_p} \leq \|v_\lambda\|_{\mathbb{H}_p} + \|G\|_{\mathbb{H}_p}
\]

and \( \|G\|_{\mathbb{H}_p} \) is finite because \( G \in L^{p^*}_\mu \), we conclude that \( \mathbb{E} \sup_{t \leq T} \|u_\lambda(t)\|^p_{L^p} \) is bounded uniformly over \( \lambda \), hence that there exists \( u \in \mathbb{H}_2 \) such that \( u_\lambda \to u \) in \( \mathbb{H}_2 \) as \( \lambda \to 0 \).

As in \cite{4}, one can now pass to the limit as \( \lambda \to 0 \) in \cite{4}, concluding that \( u \) is indeed a mild solution of (\( \Pi \)). Since \( \mathbb{E} \sup_{t \leq T} \|u\|^p_{L^p} < \infty \), one also gets that \( f(u) \in L_1([0, T] \to H) \), hence, by the uniqueness results in \cite{3}, \( u \) is the unique càdlàg mild solution belonging to \( \mathbb{H}_2 \).

### 3.2 Proof of Theorem \[4\]

We need the following lemma, whose proof is completely analogous to the proof of \cite{4} Lemma 9, hence omitted.

**Lemma 9.** Assume that \( u_{01}, u_{02} \in \mathbb{L}_p \); \( B_1, B_2 \in L^\gamma_p \); \( G_1, G_2 \in L^{\gamma^*}_p \), and denote the unique càdlàg mild solutions of

\[
du + Au \, dt + f(u) \, dt = B_1 \, dW + \int Z \, G_1 \, d\mu,
\]

and

\[
du + Au \, dt + f(u) \, dt = B_2 \, dW + \int Z \, G_2 \, d\mu,
\]

by \( u_1 \) and \( u_2 \), respectively. Then one has

\[
\mathbb{E} \sup_{t \leq T} \|u_1(t) - u_2(t)\|^2_{H} \lesssim_T \mathbb{E} \|u_{01} - u_{02}\|^2_{H}
\]

\[
+ \mathbb{E} \int_0^T \|B_1(t) - B_2(t)\|^2_{\gamma(H \to H)} \, dt + \mathbb{E} \int_0^T \|G_1(t, z) - G_2(t, z)\|^2_{H} \, m(dz) \, dt.
\]

(6)
Let us consider sequences \( \{u_0^n\}_n \subset L^p \), \( \{B_n\}_n \subset L^\gamma_p \) and \( \{G_n\}_n \subset L^m_{2,p} \) such that \( u_0^n \to u_0 \) in \( L^2 \), \( B_n \to B \) in \( L^\gamma_2 \) and \( G_n \to G \) in \( L^m_2 \) as \( n \to \infty \). Denoting by \( u_n \) the unique mild solution in \( H^2 \) of
\[
du_n + Au_n \, dt + f(u_n) \, dt = B_n \, dW + \int_Z G_n \, d\bar{\mu}, \quad u_n(0) = u_0^n,\]
the previous lemma yields
\[
\mathbb{E} \sup_{t \leq T} \|u_n(t) - u_m(t)\|_H^2 \lesssim_T \mathbb{E} \|u_0^n - u_0^m\|_H^2
\]
\[+ \mathbb{E} \int_0^T \|B_n(t) - B_m(t)\|_{\gamma(H \to H)}^2 \, dt + \mathbb{E} \int_0^T \int_Z \|G_n(t,z) - G_m(t,z)\|_H^2 \, m(dz) \, dt,\]
i.e. \( \{u_n\}_n \) is a Cauchy sequence in \( H^2 \). This implies that \( u_n \to u \) in \( H^2 \) as \( n \to \infty \), and \( u \) is a generalized solution of (1). Since the limit does not depend on the choice of \( u_0^n, B_n \) and \( G_n \), the generalized solution is unique.

References

[1] S. Cerrai, *Second order PDE’s in finite and infinite dimension*, Lecture Notes in Mathematics, vol. 1762, Springer-Verlag, Berlin, 2001. MR 2002j:35327

[2] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge, 1992. MR MR1207136 (95g:60073)

[3] C. Marinelli and M. Röckner, *On uniqueness of mild solutions for dissipative stochastic evolution equations*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 13 (2010), no. 3, 363–376. MR 2729590 (2011k:60220)

[4] ———, *Well-posedness and asymptotic behavior for stochastic reaction-diffusion equations with multiplicative Poisson noise*, Electron. J. Probab. 15 (2010), no. 49, 1528–1555. MR 2727320

[5] J. van Neerven and J. Zhu, *A maximal inequality for stochastic convolutions in 2-smooth Banach spaces*, arXiv:1105.4720v1.

[6] J. M. A. M. van Neerven, M. C. Veraar, and L. Weis, *Stochastic evolution equations in UMD Banach spaces*, J. Funct. Anal. 255 (2008), no. 4, 940–993. MR 2433958 (2009h:35465)