Open manifolds with non-homeomorphic positively curved souls

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Abstract

We extend two known existence results to simply connected manifolds with positive sectional curvature: we show that there exist pairs of simply connected positively-curved manifolds that are tangentially homotopy equivalent but not homeomorphic, and we deduce that an open manifold may admit a pair of non-homeomorphic simply connected and positively-curved souls. Examples of such pairs are given by explicit pairs of Eschenburg spaces. To deduce the second statement from the first, we extend our earlier work on the stable converse soul question and show that it has a positive answer for a class of spaces that includes all Eschenburg spaces.

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1. Introduction

The Soul Theorem [4] determines the structure of an open manifold $N$ endowed with a metric $g$ of non-negative sectional curvature: there exists a closed totally convex submanifold $S$, called the soul, such that $N$ is diffeomorphic to the normal bundle of $S$. This soul may not be unique, but for a given metric $g$ any two souls are isometric. Our work is motivated then by the following question: if $N$ admits different non-negatively curved metrics $g_1$, $g_2$, what can be said about the corresponding souls $S_1$, $S_2$? For convenience we will say that $S$ is a soul of $N$ if $S$ is a soul of $(N, g)$ in the usual sense for some metric $g$ of non-negative sectional curvature.

Open manifolds with different souls can be constructed in the following ways. It is well known that there exist 3-dimensional lens spaces $L_1$, $L_2$ that are homotopy equivalent but not homeomorphic, and such that their products with $\mathbb{R}^3$ are diffeomorphic [28, section 2]. Thus, the obvious product metrics on $L_1 \times \mathbb{R}^3 \cong L_2 \times \mathbb{R}^3$ have two non-homeomorphic souls. In a similar vein, all of the fourteen exotic 7-dimensional spheres $\Sigma^7$ (i.e. manifolds which are homeomorphic but not diffeomorphic to the standard sphere $S^7$) admit non-negatively curved

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metrics (see [15] and the recent preprint [12]), and they all become diffeomorphic after taking the product with $\mathbb{R}^3$. Thus, the obvious product metrics yield fifteen non-diffeomorphic souls of $S^7 \times \mathbb{R}^3$.

In a more elaborate construction, Belegradek showed that $S^3 \times S^4 \times \mathbb{R}^5$ admits infinitely many souls that are pairwise non-homeomorphic [1]. In [19] the same statement was shown over $S^2 \times S^3 \times S^1 \times S^3 \times \mathbb{R}^4$ for any $k > 10$, where the souls satisfy certain curvature-diameter properties. Finally, in [2] further examples in the same vein were constructed with the additional property that the souls have codimension four.

Our main interest in this note is the existence of souls with positive sectional curvature. For example, the lens spaces described above have metrics with constant positive sectional curvature. Unpublished work by Petersen–Wilhelm [30] announces a positively curved metric on one of the exotic spheres $\Sigma^7$: this would yield two non-diffeomorphic souls with positive curvature on $S^7 \times \mathbb{R}^3$. It also follows from [3] that there exist open manifolds with pairs of non-diffeomorphic homeomorphic souls with positive curvature: see Theorem 16 below for the precise statement and its proof. In all of the above examples, however, the pairs of souls satisfy at most two of the following three properties: they are simply connected, they are non-homeomorphic, they have positive sectional curvature. The situation is summarised in Figure 1. Here, we present open manifolds with pairs of souls that satisfy all three properties simultaneously:

**THEOREM A.** There exist simply connected open manifolds with a pair of non-homeomorphic souls of positive sectional curvature.

In combination with results of [2, 19], Theorem A yields some consequences on the topology of the moduli space of Riemannian metrics with non-negative sectional curvature on the corresponding spaces. This is explained in Section 6.
Theorem A will be proved in the following more explicit form:

**THEOREM A’.** There exist Eschenburg spaces $M$ with the following property: the total space of every real vector bundle over $M$ of rank $\geq 8$ admits a pair of non-homeomorphic souls of positive sectional curvature.

Of course, one of the souls is the given Eschenburg space $M$; the other soul is a homotopy equivalent but non-homeomorphic Eschenburg space $M'$. Recall that Eschenburg spaces [9] form an infinite family of 7-dimensional quotients of $SU(3)$ under certain circle actions. They inherit non-negatively curved metrics from $SU(3)$ which in many cases have positive sectional curvature (see Section 4 for details). The only known examples of pairs of simply connected manifolds with positive curvature which are homotopy equivalent but non-homeomorphic occur among these Eschenburg spaces [5, 32]. On the other hand, there are only finitely many homeomorphism classes of Eschenburg spaces in each homotopy type [5, proposition 1.7], so our strategy behind proving Theorem A’ cannot yield infinite families of non-homeomorphic souls.

This strategy is as follows. We use the classical fact that the total spaces of a vector bundle of high rank and its pull-back under a tangential homotopy equivalence are diffeomorphic. Here, two manifolds $M_1$, $M_2$ of the same dimension are called tangentially homotopy equivalent if there exists a homotopy equivalence $f: M_1 \to M_2$ such that the tangent bundle $TM_1$ and $f^*TM_2$ are stably isomorphic, i.e. such that $TM_1 \times \mathbb{R}^k$ and $f^*TM_2 \times \mathbb{R}^k$ are isomorphic as bundles over $M_1$ for some integer $k \geq 0$. Thus, Theorem A’ is a consequence of the two following results, in which each Eschenburg space is understood to come equipped with some metric which descends from a circle invariant non-negatively curved metric on $SU(3)$.

**THEOREM B.** There exist pairs of positively curved Eschenburg spaces which are tangentially homotopy equivalent but not homeomorphic.

**THEOREM C.** Let $M$ be an Eschenburg space. The total space of every real vector bundle over $M$ of rank $\geq 8$ admits a metric with non-negative sectional curvature whose soul is isometric to $M$.

Explicit pairs of Eschenburg spaces as in Theorem B are listed in Table I below. They constitute the first known examples of simply connected positively curved non-homeomorphic spaces that are tangentially homotopy equivalent. On the other hand, any two homeomorphic Eschenburg spaces are in particular tangentially homotopy equivalent. (This implication holds for many closed manifolds of dimension at most 7; see Corollary 3.) Pairs of simply connected non-negatively curved manifolds that are tangentially homotopy equivalent but not homeomorphic are already known: Crowley exhibited an explicit such pair of $S^3$-bundles over $S^4$ [7, p. 114], which carry metrics of non-negative sectional curvature by the work of Grove and Ziller [15].

Theorem C should be seen in the context of the converse soul question: does every vector bundle over a manifold with non-negative sectional curvature itself admit a metric of non-negative sectional curvature? While this is known to be false for general base manifolds, very little is known about this question for simply connected bases. Every vector bundle over a sphere $S^n$ with $2 \leq n \leq 5$ admits such a metric [15], and there exist partial positive
results over cohomogeneity-one four-manifolds \cite{16}. A stable version of the question is known to have an affirmative answer for all spheres \cite{31}, and also for many other families of homogeneous spaces including almost all the positively curved ones \cite{13, 14}. On the other hand, there is not a single known example of a vector bundle over a simply connected non-negatively curved closed manifold whose total space admits no metric of non-negative sectional curvature. Using the same techniques as in the proof of Theorem C, we can further extend the list of examples in which the converse soul question has a positive answer, at least after some form of stabilisation:

**THEOREM C'.** Let \( M \) be any of the closed manifolds listed below. The total space of every real vector bundle over \( M \) of rank \( \geq r \) admits a metric of non-negative sectional curvature, where \( r \) depends on \( M \) as listed:

1. generalised Witten spaces \( M \) with \( H^4(M) \) of odd order \( (r = 8) \);
2. generalised Witten spaces \( M \) with \( H^4(M) \) of even order \( (r = 18) \);
3. products of spheres \( S^2 \times S^m \) with \( m \equiv 3, 5 \mod 8 \) \( (r = m + 3) \);
4. the total space of the unique non-trivial linear \( S^m \)-bundle over \( S^2 \) where either \( m = 3 \) or \( m \equiv 5 \mod 8 \) (in any case \( r = m + 3 \)).

The generalised Witten spaces appearing here are a family of manifolds \( M_{k,l_1,l_2} \) defined as quotients of \( S^5 \times S^3 \) under the circle action

\[
\begin{aligned}
S^1 \times S^5 \times S^3 &\rightarrow S^5 \times S^3 \\
(z, (u_1, u_2, u_3), (v_1, v_2)) &\mapsto ((z^k u_1, z^k u_2, z^k u_3), (z^{l_1} v_1, z^{l_2} v_2)),
\end{aligned}
\]

where \( S^5 \subset C^3 \), \( S^1 \subset C^2 \), and \( k, l_1, l_2 \) are nonzero integers such that \( k, l_j \) are coprime for \( j = 1, 2 \); for such a space \( H^4(M_{k,l_1,l_2}) = \mathbb{Z}_{l_1 l_2} \). We refer to \cite{11} for details.

The unifying feature of the examples appearing in Theorem C is that the base manifolds come equipped with a principal \( S^1 \)-bundle that carries an invariant metric of non-negative sectional curvature, and whose associated complex line bundle generates the Picard group of the base manifold. The idea is then to show that any real vector bundle is stably equivalent to a sum of at most \( r/2 \) complex line bundles. See Proposition 8 below for a general form of Theorems C and C'.

Note that there are infinitely many manifolds in Theorems C and C' that are not diffeomorphic to homogeneous spaces. Indeed, there are infinitely many spaces among Eschenburg and generalised Witten spaces that are not even homotopy equivalent to any homogeneous space \cite{11, 32}.

**Outline**

The paper is organised as follows. All theorems above follow from a study of stable equivalence classes of real vector bundles over manifolds of dimension at most seven, with which we begin in Section 2. Theorems C and C' are deduced in Section 3. In Section 4, we use the results on stable equivalence classes to refine the homotopy classification of Eschenburg spaces due to Kruggel, Kreck and Stolz to a classification up to tangential homotopy equivalence. A search for pairs as in Theorem B can then easily be implemented as a computer program. The code we use is briefly discussed at the end of Section 4; we have made it freely available \cite{40}. Theorem A is finally proved in Section 5. We close in Section 6 with a brief discussion of implications for moduli spaces.
Notation

We write $H^*(-)$ to denote (singular) cohomology with integral coefficients, i.e. $H^*(X) := H^*(X, \mathbb{Z})$.

2. Vector bundles over seven-manifolds

Two real vector bundles $F$ and $F'$ over a common base $X$ are stably equivalent if $F \oplus \mathbb{R}^k \cong F' \oplus \mathbb{R}^{k'}$ for certain integers $k$ and $k'$. The main result of this section is that, over certain classes of 7-manifolds, any real vector bundle is stably equivalent to a sum of complex line bundles. See Proposition 4 for the precise statement and Remark 6 for slight generalisations.

Our calculations will make use of the Spin characteristic class $q_1$ constructed by Thomas [35]. Assume for the following brief discussion that our base $X$ is a finite-dimensional connected CW complex. A Spin bundle $F$ over $X$ is a real vector bundle whose first two Stiefel–Whitney classes $w_1 F$ and $w_2 F$ vanish. Equivalently, a real vector bundle $F$ is a Spin bundle if and only if its classifying map $f_F : X \to BO$ lifts to a map $\hat{f}_F : X \to BSpin$. The Spin characteristic class $q_1 F \in H^4(X)$ of such a Spin bundle is defined as the pullback under $\hat{f}_F$ of a distinguished generator of $H^4(BSpin)$. We will make frequent use of the following properties of the Spin characteristic class and its relation to the first Pontryagin class $p_1$ and the Chern classes $c_1$ and $c_2$.

**Proposition 1.** Let $F$ and $F'$ be two Spin bundles over $X$, let $E$ be a complex vector bundle over $X$, and let $r E$ be the underlying real vector bundle.

(i) $q_1(F) = 0$ if $F$ is a trivial vector bundle.

(ii) $q_1(F \oplus F') = q_1 F + q_1 F'$.

(iii) $2 q_1(F) = p_1(F)$ — “The Spin class is half the Pontryagin class.”

(iv) $p_1(r E) = (c_1 E)^2 - 2 c_2 E$.

(v) $q_1(r E) = - c_2 E$ if $c_1 E = 0$.

For the last identity, note the $r E$ is a Spin bundle if and only if the mod-2-reduction of $c_1 E$ in $H^2(X, \mathbb{Z}_2)$ vanishes. In particular, the stated stronger condition $c_1 E = 0$ implies that $r E$ is a Spin bundle.

**Proof.** The first claim is clear from the definition. For (ii) and (iii), see equations 1-10 and 1-5 in theorem 1-2 of [35]. Claim (iv) is a direct consequence of the definition of Pontryagin classes. Claim (v) is immediate from (iii) and (iv) when $H^4(X)$ contains no 2-torsion, an assumption we will frequently make below. To see that (v) also holds in general, note that stable equivalence classes of bundles with vanishing first Chern class are classified by $BSU$. So $q_1 \circ r$ defines a natural transformation $[X, BSU] \to H^4(X)$ and hence corresponds to an element of $H^4(BSU) = \mathbb{Z} c_2$. To see which element it is, we can evaluate, say, on $X = S^4$ and then use (iv).

**Proposition 2.** Suppose $X$ is a connected CW complex of dimension $\leq 7$. Then two Spin bundles $F$, $F'$ over $X$ are stably equivalent if and only if their Spin characteristic classes agree.

Suppose in addition that $H^4(X)$ contains no 2-torsion. Then two real bundles $F$, $F'$ over $X$ are stably equivalent if and only if their Stiefel–Whitney classes $w_1$ and $w_2$ and their first Pontryagin classes $p_1$ agree.
The distinction of cases here is necessary because, in contrast to \( w_1, w_2 \) and \( p_1 \), the Spin characteristic class \( q_1 \) is not defined for arbitrary real vector bundles.

**Proof.** Let \( \widetilde{KO}(X) \) denote the reduced real \( K \)-group of \( X \), i.e. the group of stable equivalence classes of real vector bundles over \( X \). (For background, see for example [18].) Let \( KSpin(X) \) denote the subgroup of stable equivalence classes of Spin bundles. Points (i) and (ii) of the previous proposition show that \( q_1 \) defines a homomorphism \( q_1 : KSpin(X) \to H^4(X) \). By [26, corollary 1], this homomorphism is an isomorphism for \( X \) of dimension at most seven, so the claim follows.

In general, \( KSpin(X) \) and \( \widetilde{KO}(X) \) fit into a short exact sequence as follows [26, remark 2]:

\[
0 \longrightarrow KSpin(X) \longrightarrow \widetilde{KO}(X) \xrightarrow{(w_1, w_2)} H^1(X, \mathbb{Z}_2) \times H^2(X, \mathbb{Z}_2) \longrightarrow 0
\]

Here, the group structure on \( H^1(X, \mathbb{Z}_2) \times H^2(X, \mathbb{Z}_2) \) is defined such that the map \((w_1, w_2)\) is a homomorphism. Given two real vector bundles \( F \) and \( F' \) whose Stiefel–Whitney classes \( w_1 \) and \( w_2 \) agree, we obtain an element \( F - F' \in \widetilde{KO}(X) \) that lies in the kernel of \((w_1, w_2)\) and hence in \( KSpin(X) \). If furthermore \( p_1(F) = p_1(F') \), we find that \( p_1(F - F') = 0 \) because the Whitney sum formula holds for Pontryagin classes up to 2-torsion [29, theorem 15.3] and because we have assumed that \( H^3(X) \) does not contain any such torsion. Using Proposition 1 (iii) and the same assumption on \( H^4(X) \), we deduce that \( q_1(F - F') = 0 \). As we saw in the first part of the proof, this implies that \( F - F' = 0 \) in \( KSpin(X) \). So \( F \) and \( F' \) are stably equivalent.

As \( q_1 \) is a homeomorphism invariant [6, 1.1/remark 2.1], and as Stiefel–Whitney classes are even homotopy invariants, the above proposition implies:

**Corollary 3.** Any two homeomorphic closed Spin manifolds of dimension \( \leq 7 \) are tangentially homotopy equivalent. Similarly, any two homeomorphic closed manifolds of dimension \( \leq 7 \) for which \( H^4(-) \) contains no 2-torsion are tangentially homotopy equivalent.

We introduce the following notation for a CW complex \( X \) with \( H^4(X) \) finite:

\[
\sigma_3(X) := \begin{cases} 
1 & \text{if } H^4(X) = 0, \\
4 & \text{if } |H^4(X)| \text{ is odd}, \\
9 & \text{if } |H^4(X)| \text{ is even and non-zero}.
\end{cases}
\] (2.1)

**Proposition 4.** Let \( X \) be a connected CW complex of dimension \( \leq 7 \) such that \( H^1(X, \mathbb{Z}_2) = 0 \), \( H^2(X) \) is (non-zero) cyclic, \( H^3(X) \) contains no 2-torsion, and \( H^4(X) \) is finite cyclic and generated by the square of a generator of \( H^2(X) \). Then any real vector bundle over \( X \) is stably equivalent to (the underlying real bundle of) a Whitney sum of \( \sigma_3(X) \) complex line bundles.

**Proof.** Under our assumptions, the Bockstein sequence shows that the reduction map \( H^2(X) \to H^2(X, \mathbb{Z}_2) \) is surjective, and that either \( H^2(X, \mathbb{Z}_2) = 0 \) or \( H^2(X, \mathbb{Z}_2) \cong \mathbb{Z}_2 \). We identify \( H^4(X) \) with \( \mathbb{Z}_2 \) for some positive integer \( s \). We will not distinguish between integers and their images in any of these residue groups notationally. Given an integer \( a \), we write \( L_a \) for the complex line bundle with \( c_1(L_a) = a \in H^2(X) \). More generally, a sum of such line bundles will be denoted \( L_{a_1} \oplus \cdots \oplus L_{a_s} \).
Proposition 4. Any real vector bundle over $X$ of rank $r_L a_1, \ldots, a_4$ such that

$$w_2(rL_{a_1, \ldots, a_4}) = w_2(F)$$  \hspace{1cm} (i)

$$p_1(rL_{a_1, \ldots, a_4}) = p_1(F).$$  \hspace{1cm} (ii)

If $H^2(X, \mathbb{Z}_2) = 0$, we can ignore the first condition; otherwise, $w_2(rL_{a_1, \ldots, a_4}) = a_1 + a_2 + a_3 + a_4 \mod 2$. For the Pontryagin class, part (iv) of Proposition 1 implies that

$$p_1(rL_{a_1, \ldots, a_4}) = a_1^2 + a_2^2 + a_3^2 + a_4^2 \in H^4(X).$$

So we can find integers $a_i$ satisfying condition (ii) by appealing to Lagrange’s Four Square Theorem: any positive integer can be written as a sum of a most four squares. In case these integers do not already satisfy condition (i), we can replace $a_1$ by $a_1 + s$: as $a_1 + s = a_1 + 1 \mod 2$ and $(a_1 + s)^2 = a_1^2 \mod s$, the new set of integers will then satisfy both conditions.

Finally, for arbitrary $s$, we can argue as follows. Let $F$ again be some given real vector bundle over $X$, but assume to begin with that $F$ is a Spin bundle. Then in view of Proposition 2 it suffices to show that there exists a Whitney sum of (at most nine) complex line bundles $L_{a_1, \ldots, a_4}$ such that $rL_{a_1, \ldots, a_4}$ is a Spin bundle with the same Spin characteristic class as $F$. As the first Chern class of such a sum is given by

$$c_1(rL_{a_1, \ldots, a_4}) = a_1 + \cdots + a_k,$$

$rL_{a_1, \ldots, a_4}$ is certainly a Spin bundle whenever $a_1 + \cdots + a_k \equiv 0 \mod 2$. Moreover, part (v) of Proposition 1 applies whenever $a_1 + \cdots + a_k = 0$ in $\mathbb{Z}$. In particular, we find that

$$q_1(rL_{a_1, \ldots, a_4}) = a_1^2,$$

and more generally that

$$q_1(rL_{a_1, a_2, a_3, a_4}) = a_1^2 + a_2^2 + a_3^2 + a_4^2 \in H^4(X).$$

So, again by Lagrange’s Four Square Theorem, we can find integers $a_1, a_2, a_3, a_4$ such that $q_1(rL_{a_1, a_2, a_3, a_4}) = q_1 F$, whatever the given value of $q_1 F$. So our Spin bundle $F$ is stably equivalent to a Whitney sum of eight complex line bundles.

When $F$ is an arbitrary real vector bundle, we can pick a complex line bundle $L_b$ such that $w_2(rL_b) = w_2(F)$. Then $F - rL_b$ is a stable equivalence class in $K\text{Spin}(X)$, the previous argument shows that $F - rL_b = rL_{a_1, -a_1, \ldots, a_4, -a_4}$ in $K\text{Spin}$, and hence $F$ is stably equivalent to the Whitney sum of nine complex line bundles $rL_{a_1, -a_1, \ldots, a_4, -a_4}$.

**Corollary 5.** Let $X$ be a connected CW complex satisfying the assumptions of Proposition 4. Any real vector bundle over $X$ of rank $\geq \max\{2\sigma_3(X), \text{dim}(X) + 1\}$ is isomorphic to a Whitney sum of $\sigma_3(X)$ complex line bundles and a trivial bundle.

**Proof.** This is immediate from Proposition 4 and the general fact that the notions of stable equivalence and isomorphism agree for bundles of sufficiently high rank: if two real vector bundles of the same rank $F$ and $F'$ over an $n$-dimensional CW complex are stably equivalent, and if the common rank of these bundles is greater than $n$, then $F$ and $F'$ are isomorphic (e.g. [18, chapter 9, proposition 1.1]).
Remark 6. We have deliberately refrained from stating Propositions 2 and 4 and Corollary 5 with minimal assumptions. In Proposition 2, the condition that $X$ is a connected CW complex of dimension $\leq 7$ could easily be replaced with the following weaker assumptions:

(i) $X$ is a connected finite-dimensional CW complex, and
(ii) the inclusion of the seven-skeleton $X^7$ induces an isomorphism $\widetilde{KO}(X^7) \cong \widetilde{KO}(X)$.

The Atiyah–Hirzebruch spectral sequence shows that a sufficient criterion for this to be the case is that all non-vanishing integral cohomology groups $H^i(X)$ in degrees $i \geq 5$ are torsion-free and concentrated in degrees $i$ with $(i \mod 8) \in \{3, 5, 6, 7\}$.

The additional assumptions needed in Proposition 4 and Corollary 5 are that $H^1(X, \mathbb{Z}_2)$, $H^2(X)$, $H^3(X)$ and $H^4(X)$ have the properties stated in Proposition 4.

3. Non-negative curvature

In this section we review a common construction of non-negatively curved metrics on vector bundles and prove Theorems C and C’, which give partial positive answers to the converse soul question for Eschenburg spaces and a few other spaces.

Let $G$ be a Lie group and let $P \to M$ be a principal $G$-bundle. Given a representation $\rho: G \to \mathbb{R}^m$, there exists a natural diagonal action on the product $P \times \mathbb{R}^m$ whose quotient space $E_\rho = P \times_G \mathbb{R}^m$ is the total space of a real vector bundle over $M$. This construction yields a natural semiring homomorphism:

$$\text{Rep}(G) \to \text{Vect}(M).$$

Suppose now that $P$ admits a $G$-invariant metric $g_P$ with non-negative sectional curvature. By the Gray–O’Neill formula for Riemannian submersions, $M$ inherits a metric $\bar{g}_P$ with non-negative sectional curvature. Now suppose that $\rho: G \to \mathbb{R}^m$ is an orthogonal representation with respect to the usual Euclidian metric $g_0$ on $\mathbb{R}^m$. Equip $P \times \mathbb{R}^m$ with the product metric $g_P \times g_0$. Then $P \times \mathbb{R}^m$ also has non-negative sectional curvature and the diagonal $G$-action on $P \times \mathbb{R}^m$ is by isometries. So, again by the Gray–O’Neill formula, $E_\rho$ inherits a metric with non-negative sectional curvature for which the zero-section $(P \times_G \{0\}, \bar{g}_P) = (M, \bar{g}_P)$ is a soul.

At the present time, this is the only known construction of open manifolds with non-negative sectional curvature, up to a change of metric (see [39, section 3.1]). It is natural to ask which vector bundles over $M$ can be constructed in this way, a purely topological question that is discussed at length in [14] for the case when $P \to M$ is the canonical $G$-bundle over a homogeneous space $G'/G$. Here, we consider circle bundles, i.e. the case $G = S^1$.

Proposition 7. Let $P \to M$ be a principal circle bundle over a closed manifold $M$. Assume that $P$ is 2-connected and that it admits an invariant metric $g_P$ of non-negative sectional curvature. Then the total space of any Whitney sum of complex line bundles over $M$ admits a metric of non-negative sectional curvature and with soul isometric to $(M, \bar{g}_P)$, where $\bar{g}_P$ denotes the quotient metric inherited from $g_P$.

Proof. As explained in [3, section 12], the fact that $P$ is 2-connected implies that $H^2(M) = \mathbb{Z}$ and that the first Chern class of the bundle is a generator of $H^2(M)$. It follows that any
complex line bundle over $M$ has the form $E_\rho = P \times S^1 \mathbb{C}$ for some character $\rho$ of $S^1$, and more generally that any Whitney sum of complex line bundles has the form $E_\rho = P \times S^1 \mathbb{C}^k$ for some direct sum of characters $\rho \in \text{Rep}(S^1)$. So the claim follows immediately from the discussion above.

Conditions for a circle bundle to admit invariant metrics with non-negative sectional curvature are given in [33].

Theorems C and C' of the introduction are particular cases of the following more general statement. Recall from equation (2·1) in Section 2 our notation $\sigma_4(M)$ for a space with $H^4(M)$ finite.

**Proposition 8.** Let $P \to M$ be a principal circle bundle over a closed manifold $M$. Assume that:

1. $P$ is 2-connected (so that $H^1(M) = 0$ and $H^2(M) = \mathbb{Z}$) and that it admits an invariant metric $g_P$ of non-negative sectional curvature, and that
2. $H^3(M)$ contains no 2-torsion, $H^4(M)$ is finite cyclic and generated by the square of a generator of $H^2(M)$, and all non-vanishing integral cohomology groups $H^i(M)$ in degrees $i \geq 5$ are torsion-free and concentrated in degrees $i$ with $(i \mod 8) \in \{3, 5, 6, 7\}$.

Then the total space of every real vector bundle of rank $\geq\max\{2\sigma_4(M), \dim(M) + 1\}$ over $M$ admits a metric with non-negative sectional curvature and soul isometric to $(M, \bar{g}_P)$, where $\bar{g}_P$ is the induced quotient metric on $M$.

**Proof.** Corollary 5 and Remark 6 show that any real vector bundle $F$ over $M$ of rank $\geq\max\{2\sigma_4(M), \dim(M) + 1\}$ is isomorphic to a Whitney sum of complex line bundles and a trivial vector bundle. The Whitney sum of complex line bundles admits a metric of non-negative sectional curvature by Proposition 7, and thus the product metric of this metric with the flat metric on the trivial summand yields a metric on $F$ with the desired properties.

To prove Theorems C and C’, it now suffices to check that the spaces in question satisfy the assumptions of Proposition 8.

**Proof of Theorems C and C’.** The cohomology of Eschenburg and generalised Witten spaces is well known [9, 11]: they are manifolds of type $r$ (see Definition 10 below). For Eschenburg spaces $|H^4(M)|$ is odd, while for generalized Witten spaces it can be either odd or even so both $\sigma_4(M) = 4$ and $\sigma_4(M) = 9$ occur. The total spaces of the corresponding principal bundles are $SU(3)$ and $S^3 \times S^5$, respectively, which clearly satisfy the topological assumptions of Proposition 8. The corresponding metrics on $SU(3)$ were constructed by Eschenburg [9], see Section 4 below. As for the generalised Witten spaces, the circle actions are by isometries with respect to the standard product metric on $S^3 \times S^5$ (see [11]).

The products $S^2 \times S^m$ and the unique non-trivial $S^m$-bundle over $S^2$ with $m \geq 2$ have the same cohomology ring, which clearly satisfies the topological assumptions when $m \equiv 3, 5 \mod 8$. The products $S^2 \times S^m$ are just quotients of $S^3 \times S^m$ via the Hopf fibration over the first factor. The unique non-trivial linear $S^m$-bundle over $S^2$ with $m = 3$ or $m \equiv 5 \mod 8$ can be described as a circle quotient of $S^3 \times S^m$ as well. Moreover, the corresponding action
is by isometries with respect to the standard product metric on $\mathbb{S}^3 \times \mathbb{S}^m$: see [8] for the case $m = 3$ and [38, item (b) above corollary 4] for the cases $m \equiv 5 \mod 8$.

4. Eschenburg spaces

Eschenburg spaces, first introduced and studied in [9], generalise the homogeneous 7-manifolds known as Aloff–Wallach spaces. Each Eschenburg space is a quotient of $SU(3)$ by a free action of $S^1$ of the following form:

$$S^1 \times SU(3) \rightarrow SU(3)$$

$$(z, A) \mapsto \text{diag}(z^{k_1}, z^{k_2}, z^{k_3}) \cdot A \cdot \text{diag}(z^{-l_1}, z^{-l_2}, z^{-l_3})$$

Following [5], we specify the action of $S^1$ and the resulting Eschenburg space $M = M(k, l)$ by the six-tuple of integer parameters $(k, l) = (k_1, k_2, k_3, l_1, l_2, l_3)$. We refer to this six-tuple as the parameter vector of $M$. The parameters need to satisfy $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$, as well as some further conditions that ensure that the $S^1$-action is free, see [5, (1.1)]. The Aloff–Wallach spaces are the Eschenburg spaces $M(k, l)$ whose parameter vector satisfies the following condition:

$$k_1 \geq k_2 > l_1 \geq l_2 \geq l_3 = 0. \quad (†)$$

In fact, as explained in [5, lemma 1-4], each of the Eschenburg spaces on which Eschenburg constructed a positively curved submersion metric is diffeomorphic to one of the spaces $M(k, l)$ satisfying $(†)$. Positively curved Eschenburg spaces display interesting phenomena that are not visible when studying the Aloff–Wallach subfamily alone. The following proposition is one example of this. Part (b) was already stated as Theorem B of the introduction.

**PROPOSITION 9.** For Aloff–Wallach spaces, the notions of homotopy equivalence, tangential homotopy equivalence and homeomorphism coincide. In contrast, for general positively curved Eschenburg spaces, these notions differ:

(a) there exist pairs of positively curved Eschenburg spaces which are homotopy equivalent to each other but not tangentially homotopy equivalent.

(b) there exist pairs of positively curved Eschenburg spaces which are tangentially homotopy equivalent but not homeomorphic.

Examples of both phenomena are displayed in Table 1.
homeomorphic but not diffeomorphic. The situation is illustrated in Figure 2.

Homotopy equivalent but not tangentially homotopy equivalent:

| k₁, k₂, k₃, l₁, l₂, l₃ | r | s | s₂ | s³ |
|-------------------------|---|---|----|----|
| (8, 7, −5, 6, 4, 0)    | 43 | −21 | 1 | 13 | 1/6 | −59/516 |
| (21, 21, −2, 20, 20, 0) | 43 | −21 | 1 | 26 | 1/6 | 55/516 |
| (12, 10, −8, 9, 5, 0)  | 101 | −50 | −1 | 21 | 1/6 | 565/1212 |
| (50, 50, −2, 49, 49, 0) | 101 | −50 | −1 | 55 | 1/6 | −125/1212 |
| (19, 17, −7, 16, 13, 0) | 137 | −68 | −1 | 23 | 1/6 | −743/1644 |
| (68, 68, −2, 67, 67, 0) | 137 | −68 | −1 | 73 | 1/6 | 241/1644 |
| (30, 26, −6, 25, 25, 0) | 181 | −26 | −1 | 164 | 1/6 | −193/2192 |
| (16, 16, −10, 13, 9, 0) | 181 | 26 | 1 | 85 | 1/6 | −443/2192 |
| (15, 14, −11, 12, 6, 0) | 181 | −43 | 0 | 35 | 0 | −55/181 |
| (45, 43, −4, 42, 42, 0) | 181 | −43 | 0 | 89 | 0 | 36/181 |
| (16, 13, −11, 12, 6, 0) | 183 | −91 | 0 | 33 | −1/6 | −99/2196 |
| (91, 91, −2, 90, 90, 0) | 183 | −91 | 0 | 96 | −1/6 | 413/2196 |

Tangentially homotopy equivalent but not homeomorphic:

| k₁, k₂, k₃, l₁, l₂, l₃ | r | s | s₂ | s³ |
|-------------------------|---|---|----|----|
| (58, 54, −34, 39, 39, 0) | 2197 | 1032 | 0 | 845 | 1/2 | 11147/8788 |
| (45, 41, −47, 39, 0, 0)  | 2197 | 1032 | 0 | 845 | 1/2 | −3247/8788 |
| (81, 69, −84, 56, 10, 0)  | 7571 | 74 | 0 | 5352 | 1/2 | −9219/30284 |
| (108, 63, −69, 56, 46, 0) | 7571 | 74 | 0 | 5352 | 1/2 | 5923/30284 |
| (88, 61, −107, 30, 12, 0) | 10935 | −5179 | 0 | 1368 | −1/6 | 55529/131220 |
| (77, 77, −106, 30, 18, 0) | 10935 | −5179 | 0 | 1368 | 1/6 | −11789/131220 |
| (79, 58, −131, 6, 0, 0)  | 13365 | −1183 | 0 | 72 | 1/3 | −3794/4019 |
| (92, 47, −127, 6, 6, 0)  | 13365 | 1183 | 0 | 72 | −1/3 | −1552/4019 |
| (115, 79, −116, 72, 6, 0) | 13851 | 1184 | 0 | 9576 | −1/6 | −77167/166212 |
| (128, 107, −97, 72, 66, 0) | 13851 | −1184 | 0 | 9576 | 1/6 | −61343/166212 |
| (1112, 1111, −13, 1110, 1100, 0) | 14467 | 2246 | −1 | 11744 | −1/6 | 68945/173604 |
| (127, 103, −106, 88, 36, 0) | 14467 | −2246 | 1 | 11744 | 1/6 | 17857/173604 |
| (188, 176, −82, 145, 137, 0) | 16625 | 3341 | 0 | 6608 | 1/2 | −25007/66500 |
| (176, 164, −94, 163, 83, 0) | 16625 | 3341 | 0 | 6608 | 1/2 | 8243/66500 |

For Aloff–Wallach spaces, the equivalence of the notions of homotopy equivalence and homeomorphism is due to Dickinson and Shankar [32]. A slightly weaker version of the statements for positively curved Eschenburg spaces, namely the existence of pairs of positively curved Eschenburg spaces which are homotopy equivalent but not homeomorphic, is known by [5, 32]. Also, there are known pairs of positively curved Eschenburg spaces [5, table 2] and even of Aloff–Wallach spaces [21, corollary on p. 467] which are homeomorphic but not diffeomorphic. The situation is illustrated in Figure 2.
Given the concrete examples in Table I, Proposition 9 can be treated as an application of the classification of Eschenburg spaces. We will first discuss this classification and then say a few words about how the examples were obtained.

Classifications of Eschenburg spaces are known up to various notions of equivalence. Most relevant for us are the classifications up to homotopy and homeomorphism due to Kruggel [23, 24, 25]. The simplest homotopy invariant used in these classifications is obtained via cohomology. Namely, all Eschenburg spaces are type-$r$-manifolds in the following sense [9, proposition 36]:

Definition 10 ([23]). A type-$r$-manifold is a simply connected closed 7-manifold $M$ whose cohomology has the following structure:

- $H^2(M) \cong \mathbb{Z}$, generated by some class $u$;
- $H^4(M) \cong \mathbb{Z}_r$, generated by $u^2$, for some finite integer $r \geq 1$;
- $H^5(M) \cong \mathbb{Z}$, generated by some class $v$;
- $H^7(M) \cong \mathbb{Z}$, generated by $uv$;
- $H^d(M) = 0$ in all other degrees $d > 0$.

In particular, the order $r$ of the fourth cohomology group is a homotopy invariant of Eschenburg spaces. A homeomorphism invariant used in Kruggel’s classification is the first Pontryagin class $p_1 \in H^4(M)$. Note that we can canonically identify $H^4(M)$ with $\mathbb{Z}_r$ as the generator $u^2$ does not depend on any (sign) choices. The additional invariants used by Kruggel are the linking number and certain invariants $s_i$ developed by Kreck and Stolz for arbitrary type-$r$-manifolds [20]. Closed expressions for the Kreck–Stolz invariants of Eschenburg spaces $M(k, l)$ are known only for spaces whose parameter vector $(k, l)$ satisfies a certain numerical “condition (C)” [5, section 2]. However, spaces violating this condition are relatively rare, see Examples 13 below. One last homotopy invariant of positively curved Eschenburg spaces worth mentioning is the value of $\Sigma := k_1 + k_2 + k_3 \mod 3$ [27, 32, proposition 12]. This invariant is not used in Kruggel’s classification, but it can still be useful when looking for the kind of phenomena we are studying here.

Table II attempts to give an overview over the different invariants, while Table III summarises the classification results. Note that the displayed classification of Eschenburg spaces...
Table II. Some invariants of an Eschenburg space $M(k, l)$. Our notation mostly follows the notation used in [5]. In the explicit formulae for the invariants, $\sigma_i$ denotes the $i$th elementary symmetric polynomial, i.e., $\sigma_1(k) = k_1 + k_2 + k_3$, $\sigma_2(k) = k_1k_2 + k_2k_3 + k_1k_3$ and $\sigma_3(k) = k_1k_2k_3$. The oriented invariants ("or.") change signs under a change of orientation.

| Invariant | Definition | Interpretation | Invariance |
|-----------|------------|----------------|------------|
| $r$       | $|\sigma_2(k) - \sigma_2(l)|$ | $\in \mathbb{Z}$ | order of $H^4(M(k, l))$ | homotopy |
| $s$       | $\sigma_3(k) - \sigma_3(l)$ | $\in \mathbb{Z}_r^\times$ | $-\sigma_3(k)/\sigma_3(l) \in \mathbb{Q}/\mathbb{Z}$ | or. homotopy |
| $\Sigma$  | $\sigma_1(l)$ | $\in \mathbb{Z}_3$ | – | or. homotopy |
| $p_1$     | $2\sigma_1(l)^2 - 6\sigma_2(l)$ | $\in \mathbb{Z}_r$ | first Pontryagin class | tangential homotopy |
| $s_{22}$  | $(2r s_2)$ | $\in \mathbb{Q}/\mathbb{Z}$ | – | or. homotopy |
| $s_2$     | (non-polynomial) | $\in \mathbb{Q}/\mathbb{Z}$ | (Kreck-Stolz invariant) | or. homeomorphism |

Table III. Classification of Eschenburg spaces satisfying Krüggen’s condition (C), up to various notions of equivalence. For example, the first line says that two such spaces are homotopy equivalent via an orientation preserving equivalence if and only if their invariants $r, s, s_{22}$ agree. For a more extensive and detailed summary, see [5, theorem 2.3].

| Invariants . . . agree | Spaces agree up to . . . | References |
|------------------------|--------------------------|------------|
| $r, s, s_{22}$         | oriented homotopy equivalence | [5, 24]    |
| $r, s, s_{22}, p_1$    | oriented tangential homotopy equivalence | Proposition 11 |
| $r, s, s_2, p_1$       | oriented homeomorphism     | [5, 25]    |

up to tangential homotopy equivalence is immediate from the classification up to homotopy equivalence:

**Proposition 11.** Two Eschenburg spaces are tangentially homotopy equivalent if and only if they are homotopy equivalent and their first Pontryagin classes agree.

**Proof.** The invariant $r$, the order of $H^4(M)$, is odd for any Eschenburg space $M$ [5, above proposition 1.7]. In particular, $H^4(M)$ contains no two-torsion, so that the claim follows directly from the second statement in Corollary 3.

**Proof of Proposition 9.** The classification results summarised in Table III and the examples in Table I immediately imply the claims concerning general positively curved Eschenburg spaces.

As for the statement concerning Aloff–Wallach spaces, the equivalence of the notions of homeomorphism and homotopy equivalence was proven in [32, proposition A-1]. Finally, the equivalence of the notions of homotopy equivalence and tangential homotopy equivalence follows from Proposition 11 since $p_1 = 0$ for Aloff–Wallach spaces (see Table II).

To find the examples listed in Table I, we followed the basic strategy outlined in [5]. That is, we employed a computer program that first generates all positively curved Eschenburg spaces satisfying (†) with $r$ bounded by some upper bound $R$, and then looks for families of spaces whose invariants agree. More precisely, given an upper bound $R \in \mathbb{N}$, the main steps of the program are:
(i) generate all parameter vectors \((k, l)\) satisfying \((\dagger)\) with \(r \leq R\);
(ii) among these parameter vectors, find all maximal families of two or more parameter vectors for which the invariants \(r, s\) and \(\Sigma\) agree, up to simultaneous sign changes of \(s\) and \(\Sigma\). (This intermediate step is necessary to avoid time-consuming computations of the invariant \(s_{22}\) for all generated parameter vectors.)
(iii) within those families, find all maximal (sub)families of two or more parameter vectors for which, in addition, the invariant \(s_{22}\) agrees, again up to simultaneous sign changes of \(s, \Sigma\) and \(s_{22}\). This results in a list of families of parameter vectors that describe homotopy equivalent positively curved Eschenburg spaces;
(iv) within the remaining families, find all maximal (sub)families of two or more parameter vectors for which, in addition, the first Pontryagin class agrees. This results in a list of families of parameter vectors that describe tangentially homotopy equivalent positively curved Eschenburg spaces;
(v) within the remaining families, find all maximal (sub)families of two or more parameter vectors for which, in addition, the invariant \(s_2\) agrees (up to simultaneous sign changes of \(s, \Sigma, s_{22}\), and \(s_2\)). This results in a list of families of parameter vectors that describe homeomorphic Eschenburg spaces.

The examples in Table I were obtained by comparing the different lists generated by the program. Unfortunately, the C-code referred to in \([5]\) seems to have been lost, so we reimplemented the whole program from scratch and added the additional functionality we needed (in particular steps (iii)-(v)). The new program, written completely in C++, is freely available \([40]\), and we encourage the reader to play around with it. Invariants of individual spaces can alternatively be computed using some Maple code that is still available from Wolfgang Ziller’s homepage.

The following empirical data obtained using the program is supplied purely for the reader’s amusement.

**Statistics** 12. Within the range of \(r \leq 100\,000\), there are

- 101,870,124 to 101,872,253 distinct homotopy classes,
- 103,602,166 distinct tangential homotopy classes, and
- 103,602,344 distinct homeomorphism classes

of positively curved Eschenburg spaces satisfying \((\dagger)\). We do not know the exact number of distinct homotopy classes due to the failure of Kruggel’s condition C in some cases.

**Examples** 13 (Condition C failures). Examples of positively curved Eschenburg spaces for which Kruggel’s condition C fails are discussed in \([5]\). An example of such a space with minimal value of \(r\) among those satisfying \((\dagger)\), taken from \([5, section 2]\), is displayed as space \(M_0\) in Table IV. The spaces \((M_1, M_2)\) in Table IV constitute a pair of positively curved Eschenburg spaces for which the invariants \(r, s, \Sigma\) and \(p_1\) agree, while we cannot compare the Kreck–Stolz invariants due to the failure of condition C for one of the spaces. The value \(r = 141,151\) is minimal among all such pairs of spaces satisfying \((\dagger)\).

**Example** 14 (Larger exotic families). The literature on Eschenburg spaces only studies pairs of exotic structures, for example pairs of homotopy equivalent spaces. However, there also seem to be lots of triples, quadruples, etc. of homotopy equivalent Eschenburg spaces.
Table IV. Some examples of positively curved Eschenburg spaces

| $M$     | $k_1$, | $k_2$, | $k_3$, | $l_1$, | $l_2$, | $l_3$ | $r$  | $s$  | $\Sigma$ | $p_1$ | $s_{22}$ | $s_2$ |
|---------|--------|--------|--------|--------|--------|-------|------|------|----------|-------|----------|-------|
| $M_0 := M(35, 21, -34, 12, 10, 0)$ | 1289   | 499    | 1      | 248    | [condition C fails] |
| $M_1 := M(440, 168, -320, 159, 129, 0)$ | 141151 | -58968 | 0      | 42822  | $-35047/141151$ |
| $M_2 := M(400, 168, -352, 165, 51, 0)$ | 141151 | -58968 | 0      | 42822  | [condition C fails] |
| $M_3 := M(410, 259, -457, 192, 20, 0)$ | 203383 | -79707 | -1     | 66848  | $-1/6$ | $614891/2440596$ |
| $M_4 := M(548, 497, -335, 374, 336, 0)$ | 203383 | -79707 | -1     | 50833  | $-1/6$ | $-621835/2440596$ |
| $M_5 := M(370, 287, -457, 126, 74, 0)$ | 203383 | -79707 | -1     | 24056  | $-1/6$ | $404657/2440596$ |
| $M_6 := M(610, 491, -325, 462, 314, 0)$ | 203383 | -79707 | -1     | 130561 | $-1/6$ | $123017/2440596$ |
| $M_7 := M(650, 491, -305, 432, 404, 0)$ | 203383 | -79707 | -1     | 147241 | $-1/6$ | $659411/2440596$ |
| $M_8 := M(548, 469, -355, 432, 230, 0)$ | 203383 | -79707 | -1     | 76945  | $-1/6$ | $-947995/2440596$ |
For example, the spaces $M_3, M_4, \ldots, M_8$ in Table IV constitute a six-tuple of homotopy equivalent, positively curved Eschenburg spaces, no two of which are tangentially homotopy equivalent. In contrast, we have not been able to find a single triple of tangentially homotopy equivalent but non-homeomorphic Eschenburg spaces. There appear to be no such triples of spaces satisfying (†) with $r \leq 300\,000$.

5. Proof of Theorem A

We are now ready to prove our main result. By Theorem B, there exist pairs of positively curved Eschenburg spaces $M_1, M_2$ that are tangentially homotopy equivalent but non-homeomorphic. Pick one such pair and a tangential homotopy equivalence $f: M_1 \to M_2$. We claim that $M := M_2$ has the property stated in Theorem A’. Indeed, let $E \to M_2$ be an arbitrary real vector bundle of rank $\geq 8$. Denote by $f^*E \to M_1$ its pullback along $f$. The induced map $h: f^*E \to E$ is still a tangential homotopy equivalence, see for example the proof of Proposition 1.3 in [14]. Now we need the following well-known corollary of a classical result of Siebenmann; it appears, for example, as [37, theorem 10.1.6], where it is dubbed “Work Horse Theorem”:

**THEOREM 15** (Siebenmann, Belegradek). Let $E_1 \to M_1$ and $E_2 \to M_2$ be two vector bundles of the same rank $l$ over two closed manifolds of the same dimension $n$. Suppose that $l \geq 3$ and $l > n$. Then any tangential homotopy equivalence $h: E_1 \to E_2$ is homotopic to a diffeomorphism.

**Proof sketch.** Note first that we might as well assume $M_1$ and $M_2$ to be connected, as we may argue one component at a time. For $n = 0$ or $n = 1$, the statement can be checked by elementary means. For $n \geq 2$, a proof is outlined in [1] below Proposition 5, as follows: first one observes that the total space $E$ of a vector bundle of rank $\geq 3$ over a closed connected manifold $M$ of dimension $\geq 2$ satisfies hypothesis (3) in [34, theorem 2.2]: it has one end, $\pi_1$ is essentially constant at $\infty$, and $\pi_1(\infty) \to \pi_1(E)$ is an isomorphism. Thus, if such a total space contains an embedded closed connected manifold $S$ such that the embedding $S \hookrightarrow E$ is a homotopy equivalence, then $E$ admits the structure of a vector bundle over $S$, with the given embedding as zero section. Slight generalizations of the arguments used in the proof of [34, theorem 2.3] then complete the proof: For $h: E_1 \to E_2$ as above and $s_1: M_1 \to E_1$ the zero section, the homotopy equivalence $h \circ s_1: M_1 \to E_2$ is homotopic to a smooth embedding $g: M_1 \to E_2$ by general position arguments [17, chapter 2, theorems 2.6 and 2.13]. It follows that $E_2$ has the structure of a vector bundle over $M_1$ and can be identified with the normal bundle $N_g$ of the embedding $g$. On the other hand, the assumption that $h$ is a tangential homotopy equivalence implies that the vector bundles $N_g$ and $E_1$ over $M_1$ are stably isomorphic, and since their rank $l$ is greater than $n$ it follows that $N_g \cong M_1$ (see the reference given in the proof of Corollary 5).

Returning to the proof of Theorem A, we find that the total spaces of our bundles $f^*E \to M_1$ and $E \to M_2$ are diffeomorphic. By Theorem C, they admit two metrics with non-negative sectional curvature, one with soul isometric to $M_1$ and the other with soul isometric to $M_2$. This completes the proof of Theorem A'/Theorem A.

The pairs of souls we have constructed have codimension $\geq 8$. This is probably not optimal. All we know is that any pair of souls as in Theorem A necessarily has codimension at least three: according to [2], any two codimension-two souls of a simply connected open
manifold are homeomorphic. There is, however, the following result on positively-curved codimension-two souls due to Belegradek, Kwasik and Schultz:

**Theorem 16 ([3]).** There exist Eschenburg spaces $M$ with the following property: the total space of every non-trivial complex line bundle over $M$ admits a pair of non-diffeomorphic, homeomorphic souls of positive sectional curvature.

Indeed, this is essentially the case $m = 0$ of [3, theorem 1·4]; the exact statement may easily be extracted from the proof of this theorem given there (see page 41). This result does not rely on the “Work Horse Theorem” stated as Theorem 15 above. Rather, the main topological tool that goes into it is [3, theorem 12·1]:

**Theorem.** Let $M_1, M_2$ be two closed simply connected manifolds of dimension $n \geq 5$ with $n \not\equiv 1 \pmod{4}$, such that $M_1$ is the connected sum of $M_2$ with a homotopy sphere that bounds a parallelisable manifold. Let $L_2 \to M_2$ be a non-trivial line bundle, and let $L_1 \to M_1$ be its pullback via the standard homeomorphism $M_1 \to M_2$. Then the total spaces $L_1$ and $L_2$ are diffeomorphic.

6. *Moduli spaces of Riemannian metrics*

Given a manifold $N$, denote by $\mathcal{R}(N)$ the space of all (complete) Riemannian metrics on $N$. We refer to [37, chapter 1] for basic properties of spaces of metrics. They can be topologized in different ways. Following [2], we consider:

- (u) the topology of uniform $C^\infty$-convergence;
- (c) the topology of uniform $C^\infty$-convergence on compact subsets.

The space of metrics equipped with one of these topologies will be denote $\mathcal{R}^u(N)$ and $\mathcal{R}^c(N)$, respectively. The diffeomorphism group $\text{Diff}(N)$ acts on $\mathcal{R}(N)$ by pulling back metrics. This action is continuous with respect to both topologies. The quotient spaces are called the *moduli spaces of metrics* and will be denoted by $\mathcal{M}^c(N)$ and $\mathcal{M}^u(N)$, respectively. While $\mathcal{M}^c(N)$ is always path-connected, $\mathcal{M}^u(N)$ can have uncountably many connected components if $N$ is non-compact.

For an open manifold $N$, we are interested in the subspace $\mathcal{R}_{K \geq 0}(N)$ of $\mathcal{R}(N)$ consisting of all metrics with non-negative sectional curvature. Pulling back metrics preserves curvature bounds, so we can consider the corresponding moduli spaces $\mathcal{M}^c_{K \geq 0}(N)$ and $\mathcal{M}^u_{K \geq 0}(N)$. Connectedness properties of these spaces have been the subject of much research; see [36] and [37, chapter 10] for recent surveys on this topic.

Our main result Theorem A suggests to also consider the subspace of those metrics with non-negative sectional curvature $K \geq 0$ whose souls $S$ have positive sectional curvature $K^3 > 0$. We will denote this subspace and the corresponding moduli space by $\mathcal{R}_{K \geq 0, K^3 > 0}(N)$ and $\mathcal{M}_{K \geq 0, K^3 > 0}(N)$, with the appropriate superscript again indicating the topology. Let us examine how the results above are reflected in the connectedness of these subspaces. We first consider the two topologies separately and then discuss the special case of codimension-one souls, for which both topologies coincide.

**Topology of uniform convergence**

The following result is an immediate consequence of [2, Theorem 1·5]:
THEOREM. Let \( g_1, g_2 \in \mathcal{R}^{u}_{K \geq 0}(N) \) with souls \( S_1, S_2 \). If \( S_1, S_2 \) are non-diffeomorphic, then the equivalence classes of \( g_1, g_2 \) lie in different path components of \( \mathcal{M}^{u}_{K \geq 0}(N) \).

So \( \mathcal{M}^{u}_{K \geq 0, K^*>0}(N) \) is not path-connected for any \( N \) as in Theorem A’ or Theorem 16.

Topology of uniform convergence on compact subsets

The following result is an immediate consequence of [19, Lemma 6-1]:

THEOREM. Let \( g_1, g_2 \in \mathcal{R}^{c}_{K \geq 0}(N) \) with souls \( S_1, S_2 \). Assume that the normal bundles of \( S_1 \) and \( S_2 \) in \( N \) both have non-trivial rational Euler class. If \( S_1, S_2 \) are non-diffeomorphic, then the equivalence classes of \( g_1, g_2 \) lie in different path components of \( \mathcal{M}^{c}_{K \geq 0}(N) \).

For dimensional reasons, the Euler classes of the vector bundles in Theorem A’ vanish. On the other hand, the Euler classes of the line bundles in Theorem 16 are non-zero by assumption. Thus, \( \mathcal{M}^{c}_{K \geq 0, K^*>0}(N) \) is not path-connected when \( N \) is the total space of a line bundle as in Theorem 16.

Codimension-one souls

In the special case where the souls have codimension one in \( N \) both topologies coincide. More precisely, the following result is [2, proposition 2-8]:

THEOREM. If \( N \) admits a metric with non-negative curvature and codimension-one soul, then the obvious map \( \mathcal{M}^{u}_{K \geq 0}(N) \to \mathcal{M}^{c}_{K \geq 0}(N) \) is a homeomorphism. Moreover, the natural map

\[
soul: \mathcal{M}^{u}_{K \geq 0}(N) \to \bigsqcup_i \mathcal{M}_{K \geq 0}(S_i)
\]

that assigns to each metric the metric of its soul is a homeomorphism as well, where \( \bigsqcup \) denotes disjoint union over are all possible diffeomorphism types \( S \) of souls of \( N \).

When \( N \) is simply-connected all codimension-one souls \( S \) are diffeomorphic, so that the map \( soul: \mathcal{M}^{u}_{K \geq 0}(N) \to \mathcal{M}_{K \geq 0}(S) \) is a homeomorphism. We can use this result to obtain further open manifolds \( N \) such that \( \mathcal{M}^{u}_{K \geq 0, K^*>0}(N) \) is not path-connected: Kreck and Stolz showed in [22] that there are Eschenburg spaces \( M \) for which the moduli space \( \mathcal{M}_{K \geq 0}(M) \) of metrics with positive sectional curvature is not path-connected. By considering Riemannian products with the real line we find that \( \mathcal{M}^{u}_{K \geq 0, K^*>0}(M \times \mathbb{R}) \) is not path-connected either.

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