Autonomous emergence of connectivity assemblies via spike triplet interactions

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\textbf{Fourier transform of covariance}

We calculate the Fourier transform of the covariance $C_{ij}$ (Eq. 29 in the Methods section) as

$$\tilde{C}_{ij}(\omega) = \int_{-\infty}^{\infty} \left( \sum_{k=1}^{N} r_k \int_{-\infty}^{\infty} R_{ik}(u)R_{jk}(u-\tau)du \right) e^{-j\omega\tau}d\tau$$

$$= \sum_{k=1}^{N} r_k \int_{-\infty}^{\infty} R_{ik}(u) \left( \int_{-\infty}^{\infty} R_{jk}(u-\tau)e^{-j\omega\tau}d\tau \right)du.$$  \hspace{1cm} (S1.1)

Multiplying the right side of the equation by $e^{\pm j\omega u}$ and changing the variables as

$$\begin{cases} 
\tau = x + y \\
u = y 
\end{cases}$$  \hspace{1cm} (S1.2)

Then, since the Jacobian of the transformation is 1, we obtain Eq. 33 as

$$\tilde{C}_{ij}(\omega) = \sum_{k=1}^{N} r_k \int_{-\infty}^{\infty} R_{ik}(y)e^{-j\omega y} \left( \int_{-\infty}^{\infty} R_{jk}(x)e^{-jx\omega}dx \right)dy$$

$$= \sum_{k=1}^{N} r_k \tilde{R}_{jk}(-\omega) \int_{-\infty}^{\infty} R_{ik}(y)e^{-j\omega y}dy.$$  \hspace{1cm} (S1.3)

$$= \sum_{k=1}^{N} r_k \tilde{R}_{ik}(\omega)\tilde{R}_{jk}(-\omega).$$
Fourier transform of third order cumulant

In the same manner, we calculate the Fourier transform of the third order cumulant $K_{ij}$ (Eq. 35 in the Methods section) as

$$K_{ij}(\omega_1, \omega_2) = \int \int d\tau_1 d\tau_2 e^{-j(\omega_1 \tau_1 + \omega_2 \tau_2)} \left( \sum_{k=1}^{N} r_k \int \int_{-\infty}^{\infty} R_{ik}(u) R_{jk}(u - \tau_1) R_{lk}(u - \tau_2) du \right)$$

$$+ \sum_{k,l=1}^{N} r_k \int \int_{-\infty}^{\infty} R_{ik}(u) R_{jl}(v - \tau_1) R_{il}(v - \tau_2) \Psi_{ik}(v - u) dv du$$

$$+ \sum_{k,l=1}^{N} r_k \int \int_{-\infty}^{\infty} R_{jk}(u - \tau_1) R_{il}(v - \tau_2) \Psi_{ik}(v - u) dv du$$

$$+ \sum_{k,l=1}^{N} r_k \int \int_{-\infty}^{\infty} R_{ik}(u - \tau_2) R_{il}(v) R_{jl}(v - \tau_1) \Psi_{ik}(v - u) dv du. \quad (S1.4)$$

Although Eq. S1.4 consists on four different terms, since the Fourier transform is a linear operation we can calculate each term independently. The first term is given by

$$1 = \sum_{k=1}^{N} r_k \int \int_{-\infty}^{\infty} R_{ik}(u) R_{jk}(u - \tau_1) R_{ik}(u - \tau_2) e^{-j(\omega_1 \tau_1 + \omega_2 \tau_2)} d\tau_1 d\tau_2 du \quad (S1.5)$$

$$= \sum_{k=1}^{N} r_k \int_{-\infty}^{\infty} R_{ik}(u) \int_{-\infty}^{\infty} R_{jk}(u - \tau_1) e^{-j\omega_1 \tau_1} d\tau_1 \int_{-\infty}^{\infty} R_{ik}(u - \tau_2) e^{-j\omega_2 \tau_2} d\tau_2 du.$$

Multiplying the right side of the equation by $e^{\pm j\omega_1 u}$ and changing the variables as

$$\begin{align*}
\tau_1 &= x + y \\
\tau_2 &= x' + y \\
u &= y
\end{align*} \quad (S1.6)$$

Again, since the Jacobian of the transformation is 1, we obtain

$$1 = \sum_{k=1}^{N} r_k \int_{-\infty}^{\infty} R_{ik}(y)e^{-j(\omega_1 + \omega_2)y}dy \int_{-\infty}^{\infty} R_{jk}(-x)e^{-j\omega_1 x}dx \int_{-\infty}^{\infty} R_{ik}(-x')e^{-j\omega_2 x'}dx' \quad (S1.7)$$

$$= \sum_{k=1}^{N} r_k \tilde{R}_{ik}(\omega_1 + \omega_2) \tilde{R}_{jk}(-\omega_1) \tilde{R}_{ik}(-\omega_2).$$

The second term of Eq. S1.4 is

$$2 = \sum_{k,l=1}^{N} r_k \int \int_{-\infty}^{\infty} R_{ik}(u) R_{jl}(v - \tau_1) R_{il}(v - \tau_2) \Psi_{ik}(v - u) e^{-j(\omega_1 \tau_1 + \omega_2 \tau_2)} dv du d\tau_1 d\tau_2 \quad (S1.8)$$

$$= \sum_{k,l=1}^{N} r_k \int_{-\infty}^{\infty} R_{ik}(u) \Psi_{ik}(v - u) \int_{-\infty}^{\infty} R_{jl}(v - \tau_1) e^{-j\omega_1 \tau_1} d\tau_1 \int_{-\infty}^{\infty} R_{il}(v - \tau_2) e^{-j\omega_2 \tau_2} d\tau_2 dv du.$$
First, we multiply the right side of the equation by $e^{\pm j\omega_1 y}$ and $e^{\pm j\omega_2 v}$ and change the variables as

$$\begin{align*}
\tau_1 &= x+y' \\
\tau_2 &= x'+y' \\
u &= y \\
v &= y'
\end{align*}$$  \hspace{1cm} (S1.9)

Again, since the Jacobian of the transformation is 1, we obtain

$$\begin{align*}
(2) &= \sum_{k,l=1}^{N} r_k \int_{-\infty}^{\infty} R_{ik}(y) \Psi_{lk}(y' - y) e^{-j(\omega_1 + \omega_2)y'} dy' dy \int_{-\infty}^{\infty} R_{jl}(-x)e^{-j\omega_1 x} dx \int_{-\infty}^{\infty} R_{il}(-x')e^{-j\omega_2 x'} dx' \tau_2 \\
 &= \sum_{k,l=1}^{N} r_k \tilde{R}_{jl}(-\omega_1) \tilde{R}_{il}(-\omega_2) \int_{-\infty}^{\infty} R_{ik}(y) \Psi_{lk}(y' - y) e^{-j(\omega_1 + \omega_2)y'} dy' dy.
\end{align*}$$  \hspace{1cm} (S1.10)

Second, we multiply the right side of the equation by $e^{\pm j(\omega_1 + \omega_2)y}$ and change the variables as

$$\begin{align*}
y' &= x'' + y'' \\
y &= y''
\end{align*}$$  \hspace{1cm} (S1.11)

Next, the second term now is calculated as

$$\begin{align*}
(2) &= \sum_{k,l=1}^{N} r_k \tilde{R}_{il}(-\omega_1 + \omega_2) \tilde{R}_{jl}(-\omega_1) \tilde{R}_{il}(-\omega_2) \tilde{\Psi}_{lk}(\omega_1 + \omega_2) x'' dx''
\end{align*}$$  \hspace{1cm} (S1.12)

The last two terms (3) and (4) of Eq. S1.4 are solved in the same way as done with the second term, and we obtain

$$\begin{align*}
(3) &= \sum_{k,l=1}^{N} r_k \tilde{R}_{il}(-\omega_1 + \omega_2) \tilde{R}_{jl}(-\omega_1) \tilde{R}_{il}(-\omega_2) \tilde{\Psi}_{lk}(\omega_1) \\
(4) &= \sum_{k,l=1}^{N} r_k \tilde{R}_{il}(-\omega_1 + \omega_2) \tilde{R}_{jl}(-\omega_1) \tilde{R}_{il}(-\omega_2) \tilde{\Psi}_{lk}(\omega_2).
\end{align*}$$  \hspace{1cm} (S1.13, S1.14)

Finally, we derive Eq. 38 by summing all four terms in the Fourier domain.