Hyperbolic formulations of General Relativity with Hamiltonian structure

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With the aim of deriving symmetric hyperbolic free-evolution systems for GR that possess Hamiltonian structure and allow for the popular puncture gauge condition we analyze the hyperbolicity of Hamiltonian systems. We develop helpful tools which are applicable to either the first order in time, second order in space or the fully second order form of the equations of motion. For toy models we find that the Hamiltonian structure can simplify the proof of symmetric hyperbolicity. In GR we use a special structure of the principal part to prove symmetric hyperbolicity of a formulation that includes gauge conditions which are very similar to the puncture gauge.

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I. INTRODUCTION

In the build up to the solution of the binary black hole problem in general relativity, a lot of effort was spent examining the Einstein equations as a system of partial differential equations (PDEs). Continuous dependence of a solution on given data, or mathematical well-posedness, of the initial boundary value problem (IBVP) is an essential property both for meaningful physical models and for numerical applications. When GR is decomposed into and against spacelike hypersurfaces it becomes a system of ten coupled PDEs; six evolution equations and four constraint equations. The constraints must be satisfied at all times in the development of a spacetime. There are few statements about well-posedness for constrained systems [1, 2], but since the constraints are compatible with the evolution equations we may consider free-evolution schemes, in which for the purposes of analysis the constraints are not assumed; suitable initial and boundary data must be given to guarantee their satisfaction. Well-posedness of the free IBVP is determined by the character of the principal part of the system and the boundary conditions. In particular well-posedness of the IVP is guaranteed by strong hyperbolicity. For the IBVP the notion of symmetric hyperbolicity may be used to guarantee well-posedness [3, 4].

From the free-evolution PDEs point of view, the gauge freedom of GR has two aspects, the freedom to choose arbitrary equations of motion for the spacetime coordinates and the addition of the constraint equations to the evolution equations. Both aspects alter the principal part and therefore potentially the well-posedness of the IBVP. That the well-posedness of the IBVP depends upon the choice of coordinates is troublesome, since there may be gauge conditions of interest that do not give rise to a well-posed IBVP.

A coordinate choice that has been applied successfully in numerical GR is determined by the puncture gauge [5]. Here we want to derive Hamiltonian formulations that
allow for that gauge condition and possess a well-posed IBVP as a free-evolution system. Our methods are also applicable when other gauge conditions are desired.

The motivation is twofold. Firstly every Hamiltonian system has a conserved quantity. The existence of a conserved quantity is also part of the defining property of symmetric hyperbolic PDEs. Therefore one may expect a connection between the concepts. The relationship between symmetric hyperbolicity and Hamiltonian structure has not been previously studied by numerical relativists. After analyzing several symmetric hyperbolic Hamiltonian systems we find that the two energies are in general not related. However in some special cases the Hamiltonian energy can be used to simplify the analysis of symmetric hyperbolicity.

The second reason to consider Hamiltonian systems are their convenient properties. As discussed above Hamiltonian systems possess a conserved energy. Moreover it is guaranteed that the time translation map is symplectic. The literature contains numerical methods that preserve one of those properties exactly \[7, 8\], which makes Hamiltonian formulations of GR potentially interesting for numerical simulation. Previous studies of symplectic integrators in numerical GR show that indeed their application can have advantages over standard Runge-Kutta schemes \[9–11\].

In section II we describe the basic concepts of hyperbolicity for fully second order systems. We then introduce Hamiltonian systems in section III. In section IV we present several new tools that are useful in the analysis of hyperbolicity. Next those tools are applied to two toy models in section V. We then turn our attention to the Einstein equations. Section VI contains our Hyperbolicity analysis for a large class of Hamiltonian formulations of GR with live gauges. Finally we conclude in section VII.

II. HYPERBOLICITY OF SECOND ORDER SYSTEMS

We start by introducing strong and symmetric hyperbolicity for first order in time, second order in space as well as fully second order systems. The former case was treated comprehensively in \[12\]. For fully second order systems one can find material on strong hyperbolicity in \[13, 14\]. Here we also discuss symmetric hyperbolicity of fully second order systems.

A. Definitions

Well-posedness and hyperbolicity: Well-posedness is the requirement that an initial (boundary) value problem has a solution that is unique and depends continuously on the initial (and boundary) data. It can be shown \[8\] that the first order evolution system

\[
\partial_t u^\mu = A^\mu_\nu \partial_\nu u^\nu + S^\mu
\]

has a well-posed initial (boundary) value problem if it is strongly (symmetric) hyperbolic.

The system (1) is called strongly hyperbolic if the principal symbol, the contraction with a spatial vector \(s_i\) of the principal part \(A^\mu_\nu s_\nu\), has real eigenvalues and a complete set of eigenvectors which depend continuously on \(s_i\). The system is called symmetric hyperbolic if there exists a hermitian symmetrizer \(H\) such that (suppressing some matrix indices)

\[
HA^\mu = (HA^\rho)^\rho
\]

and where \(H\) is positive definite. A hermitian \(H\) that satisfies (2) but is not necessarily positive definite is called a candidate symmetrizer. Symmetric hyperbolicity is a more strict condition than strong hyperbolicity \[12, appendix C\]. It is equivalent to the existence of a conserved positive definite energy

\[
E = \int \epsilon dx, \quad \epsilon = u^i H u
\]

for the principal part of the system linearized around some background.

These definitions were used to define strong and symmetric hyperbolicity for first order in time, second order in space systems \[12\] of the form

\[
\partial_t v = A^1_1 \partial_1 v + A^1_2 v + A^2 w + a, \quad (4a)
\]

\[
\partial_t w = B^1_1 \partial_1 j v + B^1_2 \partial_2 j v + B^2_1 \partial_1 w + B^2_2 \partial_2 w + b. \quad (4b)
\]

In what follows we also consider fully second order systems of the form

\[
\partial_t^2 q = A^{ij} \partial_i j q + B^{ij} \partial_i \partial_j q + S. \quad (5)
\]

In fact the set of fully second order systems is a special case of the first order in time, second order in space systems. However in both forms of the equations certain calculations are simplified.

The system (3) is called strongly (symmetric) hyperbolic if there exists a reduction to first order that is strongly (symmetric) hyperbolic. We likewise call the system (5) strongly (symmetric) hyperbolic if there exists a fully first order reduction that is strongly (symmetric) hyperbolic.

With those definitions a system is strongly hyperbolic if and only if the principal symbol, \(P_a\), has a complete set of eigenvectors (with real eigenvalues) that depend continuously on a spatial vector \(s_i\). Strong hyperbolicity is equivalent to the existence of a complete set of characteristic variables with real speeds \[12, 14\] and appendix A. For first order in time, second order in space systems the principal symbol is

\[
P_a = \begin{pmatrix} A^1_1 s_i & A^2_2 \\ B^1_1 s_i s_j & B^2_2 s_i \end{pmatrix}
\]

(6)
and for fully second order systems it becomes
\[
P_2 = \begin{pmatrix} 0 & 1 \\ A^\dagger s_j s_i & B^\dagger s_i \end{pmatrix}.
\] (7)

The existence of a symmetric hyperbolic first order reduction is equivalent to the existence of a conserved positive definite energy, \( E = \int dx \) (again details can be found in \cite{12,14} and appendix \( \text{A} \) respectively). For first order systems, and for fully second order systems it is
\[
\epsilon_1 = u^*_i H^{ij}_1 \left( v \right) u_j
= \left( \begin{array}{c} \partial_i v \\ w \end{array} \right)^\dagger \left( \begin{array}{cc} H^{ij}_{11} & H^{ij}_{12} \\ H^{ij}_{12} & H^{ij}_{22} \end{array} \right) \left( \begin{array}{c} \partial_j v \\ w \end{array} \right),
\] (8)

and for fully second order systems it is
\[
\epsilon_2 = u^*_i H^{ij}_2 \left( q \right) u_j
= \left( \begin{array}{c} \partial_i q \\ \partial_j q \end{array} \right)^\dagger \left( \begin{array}{cc} H^{ij}_{11} & H^{ij}_{12} \\ H^{ij}_{12} & H^{ij}_{22} \end{array} \right) \left( \begin{array}{c} \partial_j q \\ \partial_i q \end{array} \right). \] (9)

In both cases we denote \( H^{ij}_1 \) a symmetrizer. Furthermore if there is a matrix \( H^{ij}_2 \) such that \( u^*_i H^{ij}_2 u_j \) is a conserved quantity, but not necessarily positive definite then we call \( H^{ij}_2 \) a candidate symmetrizer.

Since there is only one time coordinate, if a first order in time, second order in space system \( \mathcal{A} \) is obtained as the reduction of a fully second order system \( \mathcal{B} \), then the two systems will have the same level of hyperbolicity.

In App. \( \text{A} \) we demonstrate the statements about equivalence of the various flavors of hyperbolicity for fully second order systems.

We denote the matrices
\[
\mathcal{A}^p, = \begin{pmatrix} A^p_i & 0 \\ B^p_i & 0 \end{pmatrix}, \quad \mathcal{A}^p, = \begin{pmatrix} 0 & \delta^p_i \\ 0 & 0 \end{pmatrix}
\] (10)

the principal part matrices of the first order in time second order in space and the fully second order system respectively.

B. Symmetric hyperbolicity of fully second order systems

Since the structure of the matrices \( \mathcal{A}^p, \) in equation \( \text{(10)} \) is quite simple one can derive another general criterion for fully second order systems of the form \( \mathcal{B} \) to be symmetric hyperbolic:

**Lemma 1.** Symmetric hyperbolicity of the fully second order system \( \mathcal{B} \) is equivalent to the existence of a second order symmetrizer and fluxes \( (H_1, \phi^i, \phi^{ij}) \) satisfying
\[
\phi^{i\dagger} = \phi^i, \quad \phi^{ij} = \phi^{[ij]} = \phi^{[ij]}\dagger,
\] (11)

\[
s_i s_j s_k (\phi^i - B^{ij}_1 H_1) A^{jk} = A^{jk} (\phi^i - H_1 B^{ij}) s_i s_j s_k, \] (12)

for every spatial vector \( s_i \), and
\[
H^{ij} = \begin{pmatrix} H_1 A^{ij} + B^{i(\dagger} H_1 B^{j)} - B^{i(\dagger} \phi^{j)} + \phi^{i\dagger} \phi^j - B^{i\dagger} H_1 & \phi^{ij} \\ \phi^{ij} & H_1 \end{pmatrix}
\] (13)

is hermitian positive definite.

The lemma is proven in appendix \( \text{A} \). To simplify the criterion further we notice the following. If \( T^i \) is an invertible matrix then positivity of \( \mathcal{B} \) is equivalent to \( T^i H_1 T_j^{\dagger} \) being positive definite. We choose
\[
T^i = \begin{pmatrix} L^i & L^i B^i L - L^i \phi^j L L^j L \end{pmatrix}, \] (14)

with \( H_1 = L^{-1} L^{-1} \). This is possible because positivity of \( \mathcal{B} \) implies positivity of \( H_1 \). The invertible matrix \( L^{-1} \) can be chosen to be the Cholesky decomposition of \( H_1 \).

It follows that positivity of \( \mathcal{B} \) is equivalent to positivity of
\[
L^{-1} A^{ij} L + L^i \phi^j L L^j L - L^i \phi^j L L^j L \] (15)

If we redefine \( \phi^i \) and \( \phi^{ij} \) appropriately then we get the following

**Corollary 1.** The fully second order system \( \mathcal{B} \) is symmetric hyperbolic if and only if there exists an invertible matrix \( L \) and fluxes \( (\phi^i, \phi^{ij}) \) satisfying
\[
\phi^{i\dagger} = \phi^i, \quad \phi^{ij} = \phi^{[ij]} = \phi^{[ij]}\dagger
\] (16)

such that
\[
s_i s_j s_k (\phi^i - L^i B^i L L^{-1}) A^{jk} L = \] (17)

\[
L^{-1} A^{ij} L + \phi^{i\dagger} L L^j L - \phi^{i\dagger} \phi^j + \phi^{ij}\dagger
\] (18)

is hermitian positive definite.

The conditions of corollary \( \text{II} \) simplify significantly in the special case that \( B^i = 0 \). In that case symmetric hyperbolicity is equivalent to the existence of an invertible matrix \( L \) and fluxes \( \phi^{[ij]} \) such that
\[
H^{ij}_3 = L^{-1} A^{ij} L + \phi^{ij}\dagger
\] (19)

is hermitian positive definite. We can choose \( \phi^i = 0 \) in corollary \( \text{II} \) without affecting positivity, because \( \phi^{i\dagger} \phi^j \) is clearly positive semi-definite.

III. HAMILTONIAN SYSTEMS

In this article we aim to analyze the hyperbolicity of Hamiltonian systems. Here we give a short introduction to the basic notions.
A. Variational principle

In many cases the dynamics of a physical system can be described by a single functional, the action

$$S = \int L(q, \dot{q}) dt,$$

(20)

with the Lagrangian $L$, where $\dot{q}$ denotes the time derivative of $q$.

According to Hamilton’s principle classical systems behave such that the action is minimal. This leads to the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$  

(21)

The ADM equations [15] are of the form (21), if the extrinsic curvature, $K$, is interpreted as an abbreviation for the time derivative of the 3-metric, $\gamma$, using (73a). They are composed of the first order in time equations (74) and the second order equations (73b). The former being the constraints and the latter the dynamical ADM equations.

As mentioned in the previous section we are only interested in free evolution systems of the form (1) or (5).

For general relativity that means we consider systems whose solutions can be directly related to solutions of the ADM equations if the constraints are satisfied, but that also possess constraint violating solutions.

The term in the Euler-Lagrange equations (21) that contains the highest order time derivatives is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial^2 L}{\partial \dot{q} \partial \dot{q}} \dot{q} + \frac{\partial^2 L}{\partial \dot{q} \partial \ddot{q}} \ddot{q}.$$  

(22)

Hence, for a free evolution system the matrix

$$M := \frac{\partial^2 L}{\partial \ddot{q} \partial \dot{q}}$$

(23)

is invertible (the equation of motion of every variable may be solved its second time derivative). For this type of system one can simplify the structure of the Euler-Lagrange equations by writing them in their Hamiltonian formulation.

B. Hamiltonian structure

A Hamiltonian formulation is given by the specification of a Hamiltonian $\mathcal{H}$ for the system, constructed from so-called canonical positions and momenta $(q, p)$. It can be shown that the Hamiltonian is a conserved quantity of the system, the total energy.

Here the Hamiltonian can be expressed as the integral over space of a Hamiltonian density

$$\mathcal{H} = \int_{\Sigma} d^3 x \mathcal{H}_D(q, \partial_i q, p, \partial_i p),$$

(24)

where the Hamiltonian density is a local function of the canonical variables and their spatial derivatives. One obtains the following canonical equations of motion

$$\partial_t q = \frac{\partial \mathcal{H}_D}{\partial p} - \partial_t \left( \frac{\partial \mathcal{H}_D}{\partial (\partial_t q)} \right),$$

(25a)

$$\partial_t p = -\frac{\partial \mathcal{H}_D}{\partial q} + \partial_t \left( \frac{\partial \mathcal{H}_D}{\partial (\partial_t q)} \right).$$

(25b)

A system that can be written in the form (25) is said to have Hamiltonian structure.

If one treats the Hamiltonian formulation of a model as fundamental then also free evolution equations with singular matrix $M^{-1}$ exist. In some situations it may even be possible to reduce those systems to a fully second order form. We consider only formulations in which one can use the standard Legendre transformation to convert between (21) and (25).

C. Canonical equations of motion

For systems with Hamiltonian structure one may easily show that a candidate symmetrizer exists. We consider a Hamiltonian formulation of some quasilinear field theory in which the functional form of the Hamiltonian density is given by

$$\mathcal{H}_D = \frac{1}{2} u^\dagger H(q) u + S(q, \partial_q, p)$$

(26)

with $u^\dagger = (\partial_q q^\dagger, p^\dagger)$, where

$$H^{ij} = \begin{pmatrix} V^{ij} & F^i & M^{-1} \\ F^j & \bar{F} & \end{pmatrix}$$

(27)

and $S(q, \partial_q, p)$ does not contribute to the principal part of the system. Without loss of generality we can assume that

$$M^{-1} = M^{-\dagger}, \quad V^{ij} = V^{ji\dagger}, \quad V^{ij} = V^{ji}.$$  

(28)

The ansatz (26) might seem very restrictive, because e.g. the Hamiltonian density for the dynamical ADM equations contains curvature terms, i.e. second order spatial derivatives of the 3-metric. However using integration by parts, those terms can be transformed to match with the ansatz (26). Integration by parts introduces boundary terms, but the equations of motion are unaffected.

The canonical equations of motion are

$$\partial_t q = M^{-1} p + F^i \partial_i q + s_q,$$  

(28a)

$$\partial_t p = F^i \partial_i p + V^{ij} \partial_j \partial_i q + s_p,$$  

(28b)

where $s_p$ and $s_q$ denote terms that do not contribute to the principal part of the system.

If we identify $q$ and $p$ with $v$ and $w$ respectively then we see that (28) is a first order in time second order in space system of the form (1). A short calculation shows that the matrix $H$ from (27) is a candidate symmetrizer for (26), we denote it the canonical candidate. Thus, if $H$ is positive definite, like e.g. for the wave equation, then the system (26) is automatically symmetric hyperbolic.
IV. TOOLS TO ANALYZE HYPERBOLICITY

In this section we present several tools and important special cases that are helpful in the analysis of hyperbolicity.

A. Decoupled variables

The special case that simplifies our analysis the most is that of the principal part matrix having a special block structure.

An obvious special case is that of sets \((v_1, w_1)\) and \((v_2, w_2)\) of fields that decouple in the principal part of the system. It is easy to check that the system is strongly (symmetric) hyperbolic if and only if the two subsystems are. Hence one may analyze two disjoint systems in isolation. The most important example in the context of general relativity is that of matter and spacetime variables. Since only the metric can contract these indices the two groups of variables are usually not mixed.

For many systems that are relevant for numerical relativity is that of the principal part matrix having a special block structure.

Hence a system with a principal part matrix of the form (29) is symmetric hyperbolic if and only if it admits a symmetrizer of the form (31).

For fully second order formulations the principal part matrix is \(A^{ij} = \begin{pmatrix} A^{i1} & 0 \\ 0 & A^{j2} \end{pmatrix} \) defined in (10). In that matrix the upper left block always vanishes and the upper right block is the identity. Hence, those submatrices automatically have the desired block structure (29) and the argument reduces to conditions on the block structure of \(A^{ij}\) and \(B^i\). If they have the following form

\[
A^{ij} = \begin{pmatrix} A^{i1} & 0 \\ 0 & A^{j2} \end{pmatrix}, \quad B^i = \begin{pmatrix} 0 & B^i_2 \\ B^i_1 & 0 \end{pmatrix}
\]

then one can assume that the symmetrizer has the block structure (31).

These considerations about the block structure of symmetrizers can be very helpful when one tries to derive a symmetrizer by making an ansatz with certain parameters. A block structure like (31) halves the number of parameters in the ansatz, simplifying the calculations significantly.

B. Searching for symmetrizers in a family of matrices – the rank criterion

In this section we consider whether or not symmetrizers may be found in a set of matrices \(G\). We find a necessary condition for \(G\) to contain a symmetrizer. If \(G\) can be chosen big enough then one obtains a necessary condition for symmetric hyperbolicity. We refer to the condition as the rank criterion.

We find in applications that it is typically easier to check the rank criterion than the alternative, which is to take the set of candidate symmetrizers for a certain formulation and prove that the set contains no positive definite matrix.

Consider a family of formulations that is parametrized by \(l\) formulation parameters, \(c \in C^l \subset \mathbb{R}^l\). For a formulation with parameters \(c\) we denote the principal part matrix \(A^{i,j}(c) \in \mathbb{R}^{k \times k}\) (this can be the fully second order or the first order in time, second order in space principal part matrix (10)). Assume that \(A^{i,j}\) depends continuously on \(c\).

For the criterion to be applicable we also need to know that in every neighborhood \(U_c \subset C^l\) of each point \(c \in C^l\) there are formulations that do not possess a symmetrizer in the set of matrices, \(G\). For our examples we find this situation, e.g. when the set of strongly hyperbolic formulations inside \(C^l\) has lower dimensionality.

The idea is then to show that by continuity \(G\) can only contain symmetrizers for \(c\) if the space of candidates for \(c, G_c \subset G\), is bigger than the space of candidates for a generic \(\tilde{c} \in U_c\).

We assume that \(G\) is linearly parametrized, i.e. there exists a linear map \(G^{ij} : \mathbb{R}^n \to \mathbb{R}^{k \times k}\), that assigns hermitian \(k \times k\)-matrices to parameters, \(g \in \mathbb{R}^n\) such that
\( G = G^{ij}(\mathbb{R}^n) \). According to \([12]\) a matrix \( G^{ij}(g) \) is a candidate symmetrizer of a formulation \( c \in C^i \), if and only if the following matrix is hermitian for every spatial vector \( s \)

\[
S_i G^{ij}(g) A^j_p k(c) s_p S_k, \quad (33)
\]

with

\[
S_i = \begin{pmatrix} s_i & 0 \\ 0 & 1 \end{pmatrix}. \quad (34)
\]

Since \( G^{ij} \) is a linear map this defines linear equations for the parameters \( g \) of the form

\[
B(c) g = 0, \quad (35)
\]

where \( B : C^i \rightarrow \mathbb{R}^{m \times n} \) is a continuous map, because we assume that \( A^j_p \) is continuous.

In appendix \([13]\) we show the following. If for fixed \( N \in \mathbb{N} \) there is in every neighborhood of \( c \) a formulation \( \bar{c} \) with \( \text{rank}(B(\bar{c})) = N \) such that \( G \) does not contain symmetrizers for \( \bar{c} \) and \( \text{rank}(B(c)) \geq N \) then there is no symmetrizer for \( c \) in \( G \).

If \( G \) is the set of all hermitian \( k \times k \)-matrices and the rank of \( B \) is continuous at \( c \) this implies that \( c \) is not symmetric hyperbolic. Hence, it is not necessary to discuss positivity of matrices, one only needs to calculate the rank of \( B \).

C. Symmetric hyperbolicity of special Hamiltonian systems

In section \([11]\) we derived general criteria for a fully second order system to be symmetric hyperbolic. We now discuss simplifications that can be achieved when the system additionally has a reduction to first order in time, second order in space with Hamiltonian structure in the principal part.

Assume that there exist matrices \( M = M^i, F^i \) and \( V^{ij} = V^{(ij)\dagger} \) such that

\[
\begin{align*}
\partial_t^2 q &= M^{-1}[V^{ij} - F^{(i]} M F^{j)] \partial_i \partial_j q \\
&+ [F^a M^{-1} + M^{-1} F^{\dagger}] M \partial_i \partial_q + S,
\end{align*} \quad (36)
\]

where \( S \) denotes terms that do not contribute to the principal part.

The underlying Hamiltonian structure guarantees the system a canonical candidate symmetrizer

\[
\bar{H}^{ij} = M \begin{pmatrix} A^{ij} & 0 \\ 0 & 1 \end{pmatrix}, \quad (37)
\]

the matrices \( MB^i \) and \( MA^{ij} \) are hermitian for every \( i, j \):

\[
\begin{align*}
\bar{S}_2^i &= MB^i = MF^i + F^{i\dagger} M, \\
\bar{S}_3^{ij} &= MA^{ij} = V^{ij} - F^{(i\dagger} M F^{j)].
\end{align*} \quad (38)
\]

However one needs to be careful. Although the canonical candidate \( \bar{S}_3^{ij} \) is block diagonal one cannot assume that there exists a positive definite symmetrizer with this block structure. However if \( B^i \) vanishes and the system is symmetric hyperbolic then it is easy to check that there exists a block diagonal symmetrizer (this was discussed previously for general fully second order systems).

There are several special cases where the existence of the canonical candidate \( \bar{S}_3^{ij} \) simplifies the construction of a positive symmetrizer:

1. If \( M \) and \( \bar{S}_2^i \) are positive definite, the system is automatically symmetric hyperbolic.

The proof is trivial, because the canonical candidate is positive definite.

2. If \( A^{ij}, B^i \) and \( M \) commute for every \( i, j \), then the system is symmetric hyperbolic if \( A^{ij} \) has only positive eigenvalues.

\( M \) is hermitian and therefore diagonalizable. The matrices \( A^{ij}, B^i \) and \( M \) commute, therefore each \( A^{ij} \) and \( B^i \) is diagonal in the basis where \( M \) is diagonal. In this basis it is then obvious that \( \bar{M} \bar{H} \) is a candidate and if \( A^{ij} \) has only positive eigenvalues then \( \bar{M} \bar{H} \) is also positive definite.

3. The system \( (36) \) is symmetric hyperbolic if there exists a matrix \( S \) such that the two matrices

\[
\begin{align*}
SM_i, & \quad SM A^{ij}, \quad (39) \\
SMB^i
\end{align*}
\]

are hermitian positive definite and

\[
(40)
\]

is symmetric. The symmetrizer is then \( S \bar{H} \).

Positivity of \( S \bar{H} \) is obvious because of conditions \((39)\) and a straightforward calculation shows that \( S \bar{H} \) is also a candidate symmetrizer because of \((40)\).

V. TOY PROBLEMS

We now use the criteria of the previous sections to analyze several toy problems. Our aim is to demonstrate the methods without considering equations with a complicated structure.

A. Electromagnetism

The structure of the Maxwell equations is very similar to that of the Einstein equations. In particular they share a very similar gauge sector. To expand the vacuum Maxwell equations in flat-space

\[
\partial_i B^i = 0, \quad \partial_i E^i = 0, \quad (41a)
\]

\[
\partial_i B_i = -(\partial \times E)_i, \quad \partial_i E_i = (\partial \times B)_i. \quad (41b)
\]

with their largest gauge freedom we introduce the standard scalar and vector potentials \( \phi \) and \( A^i \),

\[
B_i = (\partial \times A)_i, \quad E_i = -\partial_t A_i - \partial_i \phi. \quad (42)
\]
The gauge freedom is in the time derivative of φ and the divergence of \( A_i \).

**Hamiltonian structure:** In the canonical variables \((A^i; \pi_i = -E_i)\) the dynamical part \( H \) of the system has Hamiltonian density

\[
H = \frac{1}{2} \left[ \pi^i \pi^i - (\partial_i A^i)(\partial_j A^j) + (\partial_i A^j)(\partial_j A^i) \right] + \phi C_m, \tag{43a}
\]

and in analogy with GR, the “momentum” constraint is

\[
C_m \equiv \partial_t \pi^i = 0. \tag{44}
\]

Here φ is considered a given field. Following the approach of \([16]\) with the Maxwell equations, we promote φ to the status of an evolved field by introducing a canonical momentum \( π \). In flat-space the dynamical vacuum Maxwell equations with live gauge conditions can be expanded and written with Hamiltonian density

\[
H_D = H_M + H_G, \tag{45a}
\]

\[
H_G = (c_1 - 1)\phi C_m - \frac{1}{2} c_2 \pi^2 - c_3 \pi \partial_i A^i. \tag{45b}
\]

where \( H_G \) denotes the Hamiltonian for the gauge sector of the theory. The canonical variables are \((A^i; \phi, \pi, π)\). In principle the \( c_i \) are arbitrary given scalar functions on the spacetime, however from the point of view of hyperbolicity analysis we will treat them as constants. Since we are in flat-space we do not denote the densitization of \( c_i \) in space. The principal symbol then decomposes into a strong hyperbolicity requires that \( \dot{c}_i \neq 0 \) as well.

Strong hyperbolicity: Following \([12]\) we investigate strong hyperbolicity by performing a \( 2 \times 1 \) decomposition in space. The principal symbol then decomposes into a scalar and a vector block. The characteristic speeds in the vector block are \( \pm 1 \) and this submatrix is always diagonalizable. In the scalar block of the expanded Maxwell equations the characteristic speeds are \( \pm \sqrt{c_1 c_3} \). Thus, strong hyperbolicity requires that \( c_1 \) and \( c_3 \) share a sign, or one of them vanishes. Further calculations reveal that the system is not strongly hyperbolic in the latter case.

Strong hyperbolicity additionally demands a complete set of characteristic variables. They exist only if \( c_1 c_2 = c_3 \neq 0 \), i.e. if in the fully second order system \( 19 \) the vector and scalar potentials are decoupled. The characteristic variables are given by

\[
U_{\pm v_1} = \dot{\partial}_i \phi \pm c_1 \sqrt{c_2} \partial_i \phi, \tag{51a}
\]

\[
U_{\pm v_2} = \dot{\partial}_i A^i \pm c_1 \sqrt{c_2} \partial_i A^i, \tag{51b}
\]

\[
U_{A \pm v_3} = \dot{\partial}_i A^i \pm \partial_i A^i. \tag{51c}
\]

The non-trivial characteristic speeds are \( v_1 = v_2 = \pm c_1 \sqrt{c_2} \) so strong hyperbolicity requires \( c_2 > 0 \) as well.

In the notation of \([27]\) we have \( u = (\partial_i A^i, \partial_i \phi, \pi, π) \)

\[
M^{-1} \dot{\eta}_k = \begin{pmatrix} \gamma_{jk} & 0 & 0 & c_2 \partial_i \phi \\ 0 & -c_2 & c_1 \eta^j & 0 \\ -c_3 \delta^j_k & 0 & 0 & c_2 \eta^j \\ 0 & -c_3 \delta^j_k & 0 & 0 \end{pmatrix}, \tag{48a}
\]

\[
E^{ij} \dot{\eta}_k = \begin{pmatrix} 0 & -c_1 \eta^j & 0 & 0 \\ -c_1 \eta^j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{48b}
\]

\[
V^{ij} \dot{\eta}_k = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{48c}
\]

Note that the structure of the gauge sector prevents the evolution equation of \( π \) from containing terms like \( \partial_\phi \phi \), since the evolution of the constraint subsystem must be closed. The index structure of the variables also prevents the evolution equation for \( A^i \) from containing terms like \( \partial_\phi \phi \). These two facts together guarantee the empty row and column of \( V^{ij} \). In fact any system with similar structure in the gauge have the same property. We will see this for the Einstein equations in the following sections.

The expanded Maxwell equations \([46]\) can be written in a fully second order form provided that \( c_2 \neq 0 \). One obtains

\[
\dot{\partial}_i A^i = \partial_j \partial^j A^i + \left( \frac{c_2}{c_1} - 1 \right) \partial^j \partial^i A^j - (c_1 c_2 - c_3) / c_2 \partial_\phi \phi, \tag{49a}
\]

\[
\dot{\partial}_i \phi = c_2 \partial_j \partial^j \phi + (c_1 c_2 - c_3) \partial_\phi \phi, \tag{49b}
\]

and we read off the principal part

\[
A^{ij} \dot{\eta}_k = \begin{pmatrix} \eta^{ij} \delta^j_k + (\frac{c_2}{c_1} - 1) \eta^{ij} \dot{\eta}_k & 0 & 0 \\ 0 & c_2 \eta^{ij} & 0 \end{pmatrix}, \tag{50a}
\]

\[
\dot{B}^{ij} \dot{\eta}_k = \begin{pmatrix} 0 & -c_1 (c_1 c_2 - c_3) \eta^{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{50b}
\]

A natural choice for the free parameters is the Lorentz gauge \( c_1 = c_2 = c_3 = 1 \). In this case equations \([19]\) are a first order in time second order in space formulation of decoupled wave equations. In what follows we will consider an arbitrary gauge choice \( c_i \) and derive conditions that the resulting PDE system be symmetric hyperbolic.
Symmetric hyperbolicity: Before we start the construction of positive symmetrizers we check whether the necessary conditions for the existence of positive candidates, i.e. strong hyperbolicity and the rank criterion, are satisfied. Thus, we immediately demand strong hyperbolicity so \( B^i \) vanishes, \( c_2 > 0, c_3 = c_1 c_2 \neq 0 \). Application of the rank criterion (see section \[ \text{V} \text{B} \]) results in a similar restriction. We find that the rank of the relevant matrix \( B(c) \) changes if and only if \( c_3 = c_1 c_2 \).

The remaining matrix \( A^{ij} \) is automatically symmetric. So, taking \( H_1 = 1, \phi^i = 0, \phi^{ij} = 0 \) in lemma \[ I \] the system is obviously symmetric hyperbolic if \( A^{ij} \) has positive eigenvalues. Positivity of the eigenvalues with \( c_1 c_2 = c_3 \) is equivalent to the condition \( \frac{1}{2} < c_1^2 c_2 < 3 \). In geometric units the characteristic speeds \( v \) of the system satisfy \( 1/\sqrt{2} < |v| < \sqrt{3} \).

Instead of the canonical candidate we may also consider the case where

\[
\phi^{ij} \mathbf{k} = \left( b_1 \eta^{i}(\delta j) \mathbf{k} 0 \right) = \phi^{ij} \mathbf{k},
\]

with some constant \( b_1 \). The eigenvalues of the resulting candidate symmetrizer are

\[
\lambda_1 = \frac{1 + c_1^2 c_2 - b_1}{2}, \quad \lambda_2 = \frac{3 - c_1^2 c_2 + b_1}{2},
\]
\[
\lambda_3 = -1 + 2 c_1^2 c_2 + b_1, \quad \lambda_4 = c_1^2 c_2, \quad \lambda_5 = 1.
\]

Hence, if \( c_2 > 0 \) we can always choose \( b_1 \) such that this candidate is positive definite. If \( c_1^2 c_2 \leq 4/3 \) then \( b_1 = 1 \) is a possible value and for \( c_1^2 c_2 > 4/3 \) one can take \( b_1 = c_1^2 c_2 \). Thus, every strongly hyperbolic formulation of Electromagnetism with Hamiltonian structure is symmetric hyperbolic.

B. The pure gauge system

In this section we consider scalar fields \( (T, X^i) \) with second order equations of motion. We determine the set of equations of motion with Hamiltonian structure which are strongly and symmetric hyperbolic. Finally we consider which equations of motion obtained for the lapse and shift when the fields are taken as coordinates.

Hamiltonian structure: We start by introducing canonical momenta \( (\Theta, \Pi_i) \) for the scalar fields \( (T, X^i) \) and write the most general ansatz for the Hamiltonian without introducing new geometric objects such that the equations of motion are linear and naturally written with \( \partial_0 = \frac{1}{\gamma} \partial_t - \beta^i \partial_i \) derivatives (where \( \alpha \) and \( \beta^i \) are lapse and shift of the background manifold slicing respectively)

\[
\mathcal{H}_D = \frac{\gamma^2}{2} \Theta^2 + \frac{c_2^2 \alpha}{2} \sqrt{7} D_i T D^i T + \frac{c_3 \alpha}{2} \Pi_i \Pi^i (54)
\]
\[
+ \frac{c_4^2 \alpha}{2} \sqrt{7} D_i X^j D^i X^j + \frac{c_5 \alpha}{2} \sqrt{7} D_i X^j D_j X^i
\]
\[
+ c_6 \alpha \Theta D_i X^i + c_7 \alpha \Pi_i D^i T + \Theta \beta^i D_i T + \Pi_i \beta^j D_j X^i.
\]

Terms like \( \Theta X^i D_i T \) would make the equations of motion nonlinear. We could also adjust the system by additional source terms. In the notation \( (\partial^1, X^i, \partial T, \Pi_i, \Theta) \) and

\[
M^{-1} k = \alpha \left( \begin{array}{cc} c_3 \gamma_{lk} & 0 \\ 0 & c_1 \end{array} \right),
\]

\[
F^{ij} = \alpha \left( \begin{array}{cc} c_4 \delta_{ij} & 0 \\ 0 & c_2 \end{array} \right),
\]

\[
V^{ij} = \alpha \left( \begin{array}{cc} c_4 \gamma_{ij} & c_5 \delta_{ij} \delta_{ik} \\ 0 & c_2 \delta_{ij} \right),
\]

The pure gauge system does not carry any constraints, so unlike in the Maxwell equations the \( V^{ij} \) matrix may contain terms on both diagonal components. Otherwise the block structure guarantees that the matrices \( \lambda \) look very similar to those of the Maxwell theory \[ \text{I} \]. In fact electromagnetism is a special case of the pure gauge system.

Fully second order system: The number of parameters now present in the system complicates the analysis of hyperbolicity. To simplify the calculations we immediately switch to the fully second order version of the system, in which the equations of motion have principal matrices

\[
A^{ij} = \alpha^2 A^{ij} - 2 \alpha \beta^{ij} B^0 - \beta^{ij} \beta^i, \quad (56a)
\]

\[
B^i = \alpha B^i + 2 \beta^i, \quad (56b)
\]

where the principal part in normal slicing \( (\alpha = 1, \beta^i = 0) \) becomes

\[
A^0^{ij} = \left( \begin{array}{cc} c_1 \gamma_{ij} & c_2 \gamma_{ij} \delta_{ik} \\ 0 & c_3 \gamma_{ij} \end{array} \right),
\]

\[
B_0^{ij} = \left( \begin{array}{cc} 0 & c_4 \delta_{ij} \\ c_5 \delta_{ij} & 0 \end{array} \right),
\]

and we write

\[
\bar{c}_1 = c_3 c_4, \quad \bar{c}_2 = \frac{c_4}{c_1} (c_1 c_3 - c_2^2),
\]

\[
\bar{c}_3 = \frac{c_4}{c_3} (c_2 c_3 - c_2^2), \quad \bar{c}_4 = \frac{1}{c_1} (c_1 c_7 + c_3 c_6),
\]

\[
\bar{c}_5 = \frac{1}{c_3} (c_1 c_7 + c_3 c_6),
\]

to simplify the matrices. In what follows we assume normal slicing, i.e. \( \alpha = 1, \beta^i = 0, A_0 = A \) and \( B_0 = B \).

Since we require \( M^{-1} \) invertible, \( c_1 \) and \( c_3 \) must be nonzero, which means that if one of \( \bar{c}_4 \) and \( \bar{c}_5 \) vanish, they both must.

Given the linear fully second order system \[ \text{I} \text{C} \] one can always derive a Hamiltonian formulation, provided \( \bar{c}_4 \not= 0 \neq \bar{c}_5 \), using

\[
c_1 = \frac{c_3 \bar{c}_5}{c_4}, \quad c_2 = \frac{c_3 c_4 + c_5 c_2^2}{c_3 \bar{c}_5},
\]

\[
c_4 = \frac{\bar{c}_3}{c_3}, \quad c_6 = \bar{c}_5 \left( 1 - \frac{\bar{c}_7}{\bar{c}_4} \right),
\]
and
\[ c_5 = \frac{1}{c_3 c_5} \left( c_2 \bar{c}_4 + \bar{c}_2^2 c_5 - 2 \bar{c}_4 c_5 c_7 + \bar{c}_5 c_2^2 \right), \]

together with \( [55] \) where \( c_3 \) and \( c_7 \) can be chosen freely.

**Strong hyperbolicity:** We assume a generic choice of the parameters \( c_1, \ldots, c_5 \), i.e., no degeneracies in the characteristic variables occur. Then we perform a \( 2 + 1 \) decomposition in space and introduce the auxiliary quantities
\[ \lambda_X = \sqrt{\bar{c}_1 + \bar{c}_2}, \quad \lambda_T = \sqrt{\bar{c}_3}, \]

the speeds of the decoupled wave equations (in the scalar sector) in normal slicing when \( B \) vanishes. In the vector sector the fully second order characteristic variables are
\[ U_{A,\pm \lambda} = \partial_t X_A \pm \bar{\lambda} v \partial_s X_A, \]

with speeds \( \lambda_V = \sqrt{c_1} \), so \( \bar{c}_1 > 0 \) is required. With non-vanishing \( B \) the characteristic speeds in the scalar sector \( \pm (v_T, v_X) \) are modified to
\[ 2v_T^2 = \lambda_T^2 + \lambda_X^2 + \bar{c}_4 \bar{c}_5 \]
\[ + \sqrt{(\lambda_T^2 - \lambda_X^2)^2 + 2 \bar{c}_4 \bar{c}_5 (\lambda_X^2 + \lambda_T^2)}, \]
\[ 2v_X^2 = \lambda_X^2 + \lambda_T^2 + \bar{c}_4 \bar{c}_5 \]
\[ - \sqrt{(\lambda_T^2 - \lambda_X^2)^2 + 2 \bar{c}_4 \bar{c}_5 (\lambda_X^2 + \lambda_T^2)}. \]

Note that if \( \bar{c}_4 \) or \( \bar{c}_5 \) vanish we recover the decoupled speeds. The characteristic variables in the scalar sector are
\[ U_{s,\pm v_X} = \partial_t X_s \pm v_X \partial_s X_s \]
\[ \pm \frac{\bar{c}_5 v_X}{v_X^2 - \lambda_T^2} (\partial_t T \pm v_X \partial_s T), \]
\[ U_{s,\pm v_T} = \partial_t T \pm v_T \partial_s T \]
\[ \pm \frac{\bar{c}_4 v_T}{v_T^2 - \lambda_X^2} (\partial_t X_s \pm v_T \partial_s X_s), \]

when \( v_X \neq \lambda_T \) and \( v_T \neq \lambda_X \). Further calculation using the fact that \( v_X^2 v_T^2 = \lambda_X^2 \lambda_T^2 \) reveals that \( v_X = \lambda_T \) if and only if \( v_T = \lambda_X \), which is only possible if \( \bar{c}_4 \) and \( \bar{c}_5 \) vanish. In this case the variables decouple and the only condition for strong hyperbolicity is that the speeds are real. In summary, the system is strongly hyperbolic provided that
\[ \lambda_V^2 = \bar{c}_1 > 0, \quad \lambda_X^2 = \bar{c}_1 + \bar{c}_2 \neq 0, \]
\[ \lambda_T^2 = \bar{c}_3 \neq 0, \quad v_T^2 > 0, \quad v_X^2 > 0. \]

To obtain the characteristic variables for the fully second order system in a general slicing from the expressions in normal slicing one can e.g. simply replace the \( \partial_s \) derivative in those expressions by a “\( \partial_s \)” derivative:
\[ \partial_s X \to \frac{1}{\alpha} (\partial_s X - \beta^s \partial_s X). \]

E.g. in the vector sector one obtains
\[ U_{A,\pm \lambda} = \frac{1}{\alpha} (\partial_t X_A - (\beta^s \mp \alpha \lambda) \partial_s X_A). \]

Given a characteristic speed \( v \) in normal slicing it becomes \( \alpha v + \beta^s \) for general slicings.

**Symmetric hyperbolicity:** As in the case of electromagnetism we first demand strong hyperbolicity. The second necessary condition, the rank criterion, cannot be applied here, because in the parameter space we do not find a dense set of formulations which are not symmetric hyperbolic.

We ask whether the modified canonical candidate
\[ \bar{H}^{ij}_{kl} = \left( \begin{array}{cc} M_{mn} A^{j[m} & \phi^{ij]}_{|kl|} \\ 0 & M_{kl} \end{array} \right), \]

with the matrix \( \phi^{ij}_{|kl|} \) defined in \([52] \) is positive. We immediately get that
\[ c_3 > 0, \quad c_4/c_5 > 0, \quad c_1 > 0 \]

must be satisfied, because the candidate is block diagonal with one diagonal block which has those elements.

The eigenvalues of the remaining upper left block are
\[ b_1 - \frac{\bar{c}_2 - \bar{c}_1}{2 c_3}, \quad \frac{2 c_1 + \bar{c}_2}{2 c_3}, \quad \frac{c_1 + 2 \bar{c}_2}{c_3} + 2 b_1. \]

Hence, we can choose \( b_1 \) such that the modified candidate becomes positive provided that
\[ \bar{c}_1 > 0, \quad \bar{c}_1 + \bar{c}_2 > 0. \]

Strong hyperbolicity of the system only implies that \( \bar{c}_1 > 0, \lambda_T^2 \lambda_X^2 > 0 \) and \( (\lambda_T^2 - \lambda_X^2)^2 + 2 \bar{c}_4 \bar{c}_5 (\lambda_X^2 + \lambda_T^2) > 0 \). Hence, as opposed to Electromagnetism, we get strongly hyperbolic formulations where the modified canonical candidate is not positive definite.

However, here the expressions are still simple enough to consider all matrices that can be constructed from the metric. We derive general candidate symmetrizers in that class and analyze their positivity. We find that for a generic choice of the parameters every strongly hyperbolic formulation is also symmetric hyperbolic.

**Discussion:** It is not clear by examining matrices \([57] \) exactly what restriction Hamiltonian structure puts upon the pure gauge system. The only condition is that when either of \( \bar{c}_4 \) or \( \bar{c}_5 \) vanish then they both do. One may analyze the level of hyperbolicity of the system even if Hamiltonian structure is abandoned. The two cases \( \bar{c}_4 = 0 \) and \( \bar{c}_5 = 0 \) are similar, so we describe only the first. We find that whenever \( \lambda_X \neq \lambda_T \) the system is both strongly and symmetric hyperbolic whenever \( \lambda_T, \lambda_X > 0 \).

With this example we demonstrate that there are strongly hyperbolic Hamiltonian formulations where apparently the existence of the canonical candidate does not simplify the symmetric hyperbolicity analysis. Moreover, in the fully second order form of the equations of motion Hamiltonian structure does not play a role. Thus, if general relativity is considered then we do not expect to find symmetrizers through a simple modification of the canonical candidate.
Evolution of the lapse and shift: To derive equations of motion for the lapse and shift from our pure gauge system we may simply use the relationship between the lapse and shift $\alpha, \beta^i$ and the local coordinates in time and space $(t, x^i)$

$$\alpha = -\frac{1}{n^a \partial_a t}, \quad \beta^i = -\alpha n^a \partial_a x^i, \quad (72)$$

with unit normal to the hypersurface $n_a$, and take $T = t, \ X^i = x^i$. The lapse and shift evolution equations do not inherit the same principal part as the pure gauge system from which they were derived. Furthermore, it is not possible to give the fully second order equations of motion for the lapse and shift until the gauge is coupled to particular equations of motion for the metric and extrinsic curvature.

VI. FORMULATIONS IN NUMERICAL GR

In this section we investigate formulations of GR with Hamiltonian structure. The aim is to derive a symmetric hyperbolic Hamiltonian formulation with properties that are believed to be important for stable numerical simulations and to relate this formulation to currently existing ones.

A. The ADM system

The starting point in the derivation of a new formulation is the ADM system [12]. It can be derived through a 3 + 1 decomposition of the vacuum Einstein equations [13]. If one writes the result in geometric variables $(\gamma_{ij}, K_{ij})$, where $\gamma_{ij}$ is the 3-metric and $K_{ij}$ is the extrinsic curvature induced in a spacelike slice, then one obtains equations of motion

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}, \quad (73a)$$
$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha [R_{ij} - 2K_{ik}K^k_j + K_{ij}K] + \mathcal{L}_\beta K_{ij}. \quad (73b)$$

Equation (73a) defines $K_{ij}$ and eqn. (73b) is the projection of the Einstein equations out of the slice. By contracting into the slice one also obtains the Hamiltonian $C$ and momentum $C_i$, constraints, which are given by

$$C = R + K^2 - K_{ij}K^{ij} = 0, \quad (74a)$$
$$C_i = D_j(K^{ij} - \gamma^{ij}K) = 0. \quad (74b)$$

The ADM equations can also be written in terms of canonical ADM variables, $(\gamma_{ij}; \pi^i)$, through the relation

$$\pi^i = \sqrt{\gamma}(K\gamma^{ij} - K^{ij}). \quad (75)$$

In these variables the dynamical ADM equations (73) have a Hamiltonian structure with Hamiltonian

$$\mathcal{H}_{ADM} = \int (-\alpha C + 2\beta^i C_i) \sqrt{\gamma} d^3 x, \quad (76)$$

where the constraints must be rewritten in terms of the canonical ADM variables.

The constraint evolution system is closed, so in the absence of a spatial boundary they need only be satisfied in one time-slice to guarantee their satisfaction later in time. This justifies analyzing the PDE properties of the system without assuming the constraints. When a spatial boundary is present suitable constraint preserving boundary conditions are also required.

In the ADM formulation the lapse $\alpha$ and shift $\beta^i$ are thought of as given fields. The formulation is weakly, but not strongly hyperbolic (the principal symbol has real eigenvalues). The initial value problem is not well-posed.

B. Gauge conditions

In the introduction we noted that from the free-evolution PDEs point of view, the gauge freedom of GR has two aspects, the first of which is the ability to choose arbitrary equations of motion for the lapse and shift. Choices in which either the lapse or shift evolve dynamically are known as live gauge conditions. In this section we describe two popular live gauge conditions.

Harmonic gauge: In terms of the lapse and shift the harmonic gauge condition $\Box X^\alpha = 0$ becomes

$$\partial_t \alpha = -\alpha^2 K + \beta^p \alpha_p, \quad (77a)$$
$$\partial_t \beta_i = \beta^p \partial_p \beta_i - \alpha \partial_t \alpha + \alpha^2 \gamma^{jk} \Gamma^i_{jk}. \quad (77b)$$

The use of this gauge was the key component in the first proofs that GR admits a well-posed initial value problem [12], since when it is appropriately coupled to the EE’s they become a particularly simple symmetric hyperbolic system of wave equations. When the gauge is modified with arbitrary source terms hyperbolicity of that formulation is not affected [21]. The modified gauge is known as generalized harmonic gauge or GHG.

Puncture gauge: We call the combination of the popular Bona-Massó lapse and gamma driver shift conditions

$$\partial_t \alpha = -\mu_L \alpha^2 K + \beta^p \alpha_p, \quad (78a)$$
$$\partial_t \beta_i = \mu_S \Gamma^i + \beta^p \partial_p \beta_i - \eta \beta. \quad (78b)$$

with

$$\Gamma^i = \gamma^{1/3} \gamma^{jk} \Gamma^i_{jk} + \frac{1}{3} \gamma^{1/3} \gamma_j^{\ i} \Gamma^k \Gamma^k_{ij} \quad (79)$$

the puncture gauge. In fact there are various choices of parameters inside the conditions which we do not consider here; for a review of the choices made by various groups see table 1 of [5]. Our gamma driver condition (78a) can be obtained by integrating the most commonly implemented condition once in time [6]. Stationary data for the lapse condition was discussed in [21, 22].
C. Hamiltonian formulations

Now we are interested in whether or not strongly or symmetric hyperbolic Hamiltonian formulations with a large class of live gauge conditions can be derived. Our Hamiltonian formulations are described in terms of the canonical variables \((\gamma_{ij}, \alpha, \beta^j, \pi^{ij}, \sigma, \rho_k)\). For solutions of the Einstein equations \(\sigma\) and \(\rho_k\) are constrained to vanish.

To get a Hamiltonian formulation in other variables, e.g. densitized ones, one can apply a symplectic transformation to obtain the correct Hamiltonian. This does not alter hyperbolicity. In general, if the desired position variables are \(f(q)\) then the corresponding canonical momenta are \(f'(q)^{-1}p_i\), where \(f'\) is the Jacobi matrix of \(f\) (\(f\) must not contain derivatives of \(q\)).

To derive formulations we follow the approach of Brown [16], i.e. we add appropriate gauge terms to the Super-Hamiltonian \(\mathcal{H}_{ADM}\):

\[
\mathcal{H} = \mathcal{H}_{ADM} + \mathcal{H}_{GHG} + \int d^3x \left( \Lambda \sigma + \Omega^i \rho_i \right),
\]

where \(\mathcal{H}_{ADM} + \mathcal{H}_{GHG}\) is the Hamiltonian for Brown’s generalized harmonic formulation [16]. The terms that appear in the principal part are

\[
\mathcal{H}_{GHG} = \beta_i \sigma D_i \alpha + \beta_j \rho_j \partial_i \beta^j + \alpha^2 \rho_i \Gamma^i_{jk} \gamma^{jk} - \alpha \rho^i D_i \alpha + \frac{1}{8\sqrt{\gamma}} \left(-4\alpha^2\gamma_{ij} \pi^{ij} \sigma + \alpha^3 \sigma^2 - 4\alpha^3 \rho_i \rho_j \gamma^{ij}\right).
\]

The terms in \(\Lambda\) and \(\Omega^i\) that affect the principal part of the equations of motion are linear in the canonical momenta and in the first spatial derivatives of the positions.

We restrict the possible terms further by considering those gauges for which the shift appears in the principal part only through its spatial derivatives and the obvious advection terms.

This is of course a restriction of the admissible gauges. But the live gauge conditions that are used in numerical GR are usually of this form. Moreover in those gauges the hyperbolicity analysis is simpler, because, as we will see, one may linearize around Minkowski space without loss of generality. Furthermore the discussion of Sect. [V.A] is applicable.

One also finds strongly hyperbolic formulations without this “\(\partial_0\)” restriction, but they are not considered here. We get the following expressions

\[
\Lambda = -C_1 \alpha^2 \gamma^{-1/2} \gamma_{ij} \pi^{ij} + C_4 \alpha^3 \gamma^{-1/2} \sigma + C_7 \left( \alpha D_i \beta^i - \frac{1}{2} \alpha \gamma_{ij} \beta^i \partial_i \gamma^{jk} \right),
\]

and

\[
\Omega^i = C_2 \alpha^2 \gamma_{jk}^i \gamma^{jk} + C_3 \alpha^3 \gamma_{ij} \gamma^{jk} - C_5 \alpha \gamma_{ij} D_j \alpha - C_6 \gamma^{-1/2} \alpha^3 \rho_j \gamma^{ij}.
\]

We denote \(C_0, \ldots, C_7\) formulation parameters and obtain Brown’s Hamiltonian [16] when \(C_7 = 0\) and we replace

\[
C_1 \rightarrow C_1 - 1/2, \quad C_2 \rightarrow C_2 - 1, \quad C_4 \rightarrow C_4 - 1/8, \quad C_5 \rightarrow C_5 - 1, \quad C_6 \rightarrow C_6 - 1/2.
\]

For those formulations we get in the notation of [27] that \(u=(\partial_i \gamma_{km}, \partial_i \alpha, \partial_i \beta^k, \pi^{km}, \sigma, \rho_k)^\dagger\) and

\[
M_{klmn}^{-1} = \alpha \gamma^{-1/2} \begin{pmatrix}
2(k(m \gamma_{ij}) & -\gamma_{kl} \gamma_{mn} & -(1/2 + C_1) \alpha \gamma_{kl} & 0 & 0 \\
-(1/2 + C_1) \alpha \gamma_{mn} & (1/4 + 2 C_4) \alpha^2 & 0 & -(1 + 2 C_6) \alpha^2 \gamma_{km} & 0 \\
0 & 0 & (1 + 2 C_6) \alpha^2 \gamma_{km} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
F_{jkl}^{mn} = \begin{pmatrix}
\beta_j (m \delta_k^m) & 0 & 0 & 0 \\
0 & \alpha \left((1 + C_2) \gamma_{j(m \gamma_{nk})} - (1/2 (C_2 - 3 + 1) \gamma_{jk} \gamma_{mn})\right) & 0 & 0 \\
0 & 0 & (1 + C_5) \alpha \gamma_{jk} & 0 \\
0 & 0 & 0 & \beta_j \delta_{mn}^k
\end{pmatrix},
\]

\[
V_{ijkl}^{mn} = \gamma^{1/2} \begin{pmatrix}
V_{ijkl}^{mn} & 0 & 0 & 0 \\
0 & \gamma^{1/2} \gamma_{ij} \gamma_{mn} & 0 & 0 \\
0 & 0 & \gamma^{1/2} \gamma_{ij} \gamma_{mn} & 0 \\
0 & 0 & 0 & \gamma^{1/2} \gamma_{ij} \gamma_{mn}
\end{pmatrix},
\]

where

\[
V_{ijkl}^{mn} = -\alpha (\gamma^{1/2} \gamma_{ij} \gamma_{kl} \gamma_{mn} + \gamma^{1/2} (k \gamma_{ij}) (m \gamma_{kl}) + \gamma^{1/2} (m \gamma_{ij}) (k \gamma_{kl}) + \gamma^{1/2} (l \gamma_{ij}) (m \gamma_{kl}) + \gamma^{1/2} (k \gamma_{ij}) (l \gamma_{mn}))/2.
\]

Various entries of \(V_{ijkl}^{mn}\) vanish for exactly the same reason as those in the Maxwell system [35].

To see that for the hyperbolicity analysis one may linearize around Minkowski space observe that when \(A^0 (\text{det}(\gamma_{ij}), \alpha, \beta^i)\) is the first order in time, second order in space principal part for an arbitrary background solution and

\[
T = \text{diag}(\alpha^{-1/2} \gamma^{-1/4}, \alpha^{-1/2} \gamma^{-1/4}, \alpha^{-1/2} \gamma^{-1/4}, \alpha^{-1/2} \gamma^{-1/4}, \alpha^{-3/2} \gamma^{-1/4}, \alpha^{-3/2} \gamma^{-1/4})
\]
then
\[ T^{-1}A^p(\det(\gamma_{ij}), \alpha, \beta^\nu)T = \alpha A^p(1, 1, 0) + \beta^\nu \mathbf{1}. \] (87)

Diagonalizability and eigenvalues are not altered by a similarity transformation, i.e., to analyze strong hyperbolicity one may consider the matrix \( s_p A^p(1, 1, 0) \) for an arbitrary spatial vector \( s \).

Concerning symmetric hyperbolicity a short calculation shows that when \( \mathcal{H} \) is a symmetrizer of \( \mathcal{A}^p(1, 1, 0) \) then \( \mathcal{T}^\dagger \mathcal{H} \mathcal{T} \) is a symmetrizer of \( \mathcal{A}^p(\det(\gamma_{ij}), \alpha, \beta^\nu) \) for an arbitrary background metric. In what follows we assume \( \alpha = 1 = \gamma \) and \( \beta^\nu = 0 \).

\textbf{Strong hyperbolicity:} Again we perform a 2 + 1 decomposition in space and, the principal symbol decomposes into three submatrices, a scalar, vector and trace-free tensor block. The characteristic speeds in the trace-free tensor and vector sector are
\[ \lambda_{TF}^2 = 1, \quad \lambda_V^2 = 1 + C_2 \] (88)
respectively. In the scalar block one finds
\[ 2v_+^2 = \lambda_+^2 + \lambda_-^2 - C_5 C_7 \] (89a)
\[ + \sqrt{\left(\lambda_+^2 - \lambda_-^2\right)^2 + C_3^2 C_7^2 - 2C_3C_7(\lambda_+^2 + \lambda_-^2)}, \]
\[ 2v_-^2 = \lambda_+^2 + \lambda_-^2 - C_5 C_7 \] (89b)
\[ - \sqrt{\left(\lambda_+^2 - \lambda_-^2\right)^2 + C_3^2 C_7^2 - 2C_3C_7(\lambda_+^2 + \lambda_-^2)}, \]
where
\[ 2\lambda_\pm^2 = (2C_1 + C_2 + C_3 - C_7 + 2) \]
\[ \pm (-2C_1 + C_2 + C_3 + C_7). \] (90)

Note that the speeds in the scalar sector are very similar to those of the pure gauge system, and simplify significantly in the special case \( C_5 = 0 \) or \( C_7 = 0 \) (or both). They simplify further when \( \lambda_+ = \lambda_- \).

Using the techniques described in App. \[\text{we can classify the formulations with respect to their strong hyperbolicity. We find three families } \mathcal{F}_{\alpha = 1,2,3} \text{ of strongly hyperbolic fully second order formulations.} \]

From the vector block of the principal symbol it is obvious that the condition
\[ C_0 = (\lambda_+^2 - 1)/2 \] (91)
must be satisfied in every case. Moreover it is clear that the characteristic speeds must be real, i.e.,
\[ \lambda_+^2 \geq 0, \quad v_+^2 \geq 0. \] (92)

The three families correspond to different solutions of the requirement that the scalar block is diagonalizable.

\textbf{Family } \mathcal{F}_1: \ The first family has four free parameters, \( (\lambda_+, \lambda_-, C_5, C_7) \), and one obtains strongly hyperbolic formulations if \( \lambda_- \neq 0, \ v_+ \neq v_- \)
\[ 0 \neq (1 + 2C_5)\lambda_-^2 - \lambda_+^2, \] (93a)
\[ 2\lambda_+^2 = 3\lambda_+^2 - 1 + \frac{C_5 \lambda_-^2 + 3C_7 \lambda_+^2}{(1 + 2C_5)\lambda_-^2 - \lambda_+^2}C_5, \] (93b)
\[ 8C_4 = \lambda_-^2 - 1 + \frac{C_5 \lambda_-^2 + 3C_7 \lambda_+^2}{(1 + 2C_5)\lambda_-^2 - \lambda_+^2}C_7. \] (93c)

where we have replaced \( C_1, C_2 \) and \( C_3 \) using the definitions \[\text{and } \text{of } \lambda_+ \text{ and } \lambda_-. \]

\textbf{Family } \mathcal{F}_2: \ If the inequality \[\text{is not satisfied, but } \lambda_- \neq 0, \ v_+ \neq v_- \] still hold, then the solution \[\text{becomes singular. One obtains another family with three free parameters, } (\lambda_-, \lambda_V, C_7) \):
\[ 12C_4 = 3\lambda_-^2 - 3 - 9C_7 - (1 + 6C_7)(\lambda_V^2 - 1), \] (94a)
\[ C_5 = -\frac{3C_7}{1 + 6C_7}, \quad \lambda_V^2 = \frac{\lambda_-^2}{1 + 6C_7}. \] (94b)

It must satisfy
\[ C_7 \neq 0, \quad C_7 \neq -1/6. \] (94c)

The first of these inequalities guarantees that there are no more repeated speeds and the second prevents the solution from becoming singular, both of which are special cases that contain no further strongly hyperbolic formulations. The characteristic speeds in the scalar sector become
\[ 2(1 + 6C_7)v_\pm^2 = 3C_7(2\lambda_-^2 + C_7) + 2\lambda_-^2 \]
\[ \pm C_7 \sqrt{9(2\lambda_-^2 + C_7)^2 + 12\lambda_-^2}. \] (95)

\textbf{Family } \mathcal{F}_3: \ If the scalar block has only one eigenvalue \( (v_+^2 = v_-^2) \) one obtains one parameter family of formulations with \( \lambda_+ > 0 \):
\[ \lambda_- = \lambda_+, \quad C_5 = 0, \quad C_7 = 0, \]
\[ 2\lambda_V^2 = 3\lambda_-^2 - 1, \quad C_4 = 1/(\lambda_-^2 - 1). \] (96)

This is the limit of \( \mathcal{F}_1 \) for \( \lambda_+ \to \lambda_- \) along curves with \( C_5 = 0 = C_7 \). But if one takes in \( \mathcal{F}_3 \) the limit \( v_+ \to v_- \), then the result is in general not strongly hyperbolic. To obtain \[\text{one may also take the limit } C_7 \to 0, \lambda_V^2 \to (3\lambda_-^2 - 1)/2 \text{ of } \mathcal{F}_2 \text{ (in any order). Again the case } C_7 = 0 \text{ is not strongly hyperbolic for general choices of } \lambda_V. \]

\textbf{Fully second order system:} The fully second order equations of motion can be constructed for each of three strongly hyperbolic systems \( \mathcal{F}_i \). We present only \( \mathcal{F}_3 \). In terms of variables \( u_{kl} = (\gamma_{kl}, \alpha, \beta_k)^\dagger \) the two parameter strongly hyperbolic system \( \mathcal{F}_3 \) has the fully second order principal part
\[ \partial_t^2 u_{kl} = A^{ij}_{kl} mn \partial_j u_{mn} + B^{ij}_{kl} mn \partial_i u_{mn}. \] (97)

The principal matrices \( A^{ij}_{kl} mn \) and \( B^{ij}_{kl} mn \) are
The parameters of $B^i$ may be constructed for each family $F_i$. The parameters in $A^{ij}_{kl}^m$ present them only for the subset $F_i$ where $F_i = 0$. The characteristic variables $A^{ij}_{kl}^m$ are given by

$$A^{ij}_{kl}^m = \begin{pmatrix} A^{ij}_{kl}^m & C_i \gamma^{ij} \delta_{kl} & 0 \\ 0 & C_0 \gamma^{ij} & 0 \\ 0 & 0 & C_2 \gamma^{ij} \delta_k^m \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ B_{i \delta_k^m} \end{pmatrix} \right), \quad (98a)$$

with $\lambda_V = \sqrt{1 + C_2}$. Finally the tensor sector has characteristic variables

$$U_{AB \pm 1}^{TF} = (\partial_t \pm \partial_s) \gamma_{AB}^{TF}, \quad (105)$$

where $\gamma_{AB}^{TF}$ denotes the transverse trace free part of the metric against $s_i$.

To find the characteristic variables of the system in an arbitrary background, one must transform the eigenvectors of the principal symbol through the similarity transformation defined by $T$ from $S_0^{\pm 1}$ and replace the time derivatives $\partial_t \to \partial_0$.

**Symmetric hyperbolicity:** As for the Maxwell equations we impose strong hyperbolicity and check where the rank criterion is satisfied. Interestingly we find again that the rank of the relevant matrix $H(c)$ from section [IV.B] is discontinuous (and drops) at every strongly hyperbolic formulation. Hence, this does not lead to immediate restrictions in the parameter space.

A complete analysis of symmetric hyperbolicity seems to be too hard in the general case, because expressions become very complicated. We want to discuss special cases where this problem can be avoided and conjecture about the general case.

In App. D we construct positive definite symmetrizers for formulations whose parameters satisfy

$$\lambda_V^2 = (3 \lambda_+^2 + 1)/4, \quad C_4 = (\lambda_+^2 - 1)/8, \quad C_5 = 0,$$

$$C_6 = 3(\lambda_+^2 - 1)/8, \quad \lambda_+^2 > 0, \quad \lambda_-^2 > 0. \quad (106)$$

Hence, we found a two parameter family of symmetric hyperbolic formulations. Using the same techniques we are also able to prove symmetric hyperbolicity of other two parameter families. With (106) they share that they belong to the four parameter family $F_1$ (93) and that $C_5 = 0$.

For the three parameter subfamily of $F_1$ that satisfies also $C_5 = 0$ we can construct candidates $H^{ij}$ such that the contractions $s_i H^{ij} s_j$ are positive definite for every spatial vector $s$. We think that the full matrices $H^{ij}$ can be chosen positive definite in this case, but are not able to prove it.

We also deal with the question whether also in GR every strongly hyperbolic formulation is symmetric hyperbolic, as it was the case for the toy problems in section [V]. We find that the (strongly hyperbolic) two parameter...
subfamily family of \(F_1\) with
\[
\lambda_T^2 = \frac{1}{4}(1 + C_3)(1 + 3\lambda^2), \quad \lambda_T^2 = (1 + C_5)\lambda^2., \quad C_4 = \frac{1}{8}(\lambda^2 - 1), \quad C_6 = \frac{1}{8}(C_5(3\lambda^2 + 1) + 3(\lambda^2 - 1)), \quad C_7 = 0, \quad \lambda^2 > 0, \quad C_5 > -1 \tag{107}
\]
and a generic choice of \(\lambda^2\) and \(C_5\) is not symmetric hyperbolic.

To prove that we first search for candidates in the set of matrices that can be written in terms of the metric. We find a six dimensional space, \(G = G_S \oplus G_A\), which decomposes into a four dimensional \(G_S\), the symmetric part \(H^{(s)}\), and a two dimensional \(G_A\), the antisymmetric \(\phi^{[ij]}\) \(\tag{11}\). We can show that there is no candidate with positive definite principal minor \(s_i H^{ij} s_j\) \((\text{where} s\text{ is any spatial vector})\). That means the four degrees of freedom in \(G_S\) are not sufficient to construct a positive symmetrizer for the principal symbol. Thus, since all principal minors of a positive definite matrix are positive as well, there is no symmetrizer which can be written in terms of the metric.

We then ask for the candidate symmetrizers in the most general set of matrices, i.e. the set which is only restricted by the appropriate block structure \(\tag{24}\). There we find a 76 dimensional space of candidates, \(\hat{G} = \hat{G}_S \oplus \hat{G}_A\), but again \(\hat{G}_S\) is four dimensional. The most general set of matrices of course contains all matrices that can be written in terms of the metric, i.e. \(\hat{G}_S = G_S\). Hence, \(\hat{G}\) does not contain a matrix with positive definite principal minor \(s_i H^{ij} s_j\), and therefore there is no positive definite symmetrizer for the formulations \(\tag{106}\).

This statement holds for a generic choice of the parameters in \(\tag{107}\). To deal with special cases we apply the rank criterion. It tells us that within \(\tag{107}\) one cannot find symmetrizers, except when \(G_S\) is at least five dimensional. This is the case for \(C_5 = 0\) only, which results in the (symmetric hyperbolic) one parameter family \(F_1\).

Thus, the obvious conjecture is that formulations where the dimension of \(G_S\) is at least five are symmetric hyperbolic, and that formulations where \(G_S\) is at most four dimensional are not.

To prove this one would need to analyze positivity of candidates for three and four parameter families. A problem that we could not solve so far.

We find that the dimension of \(G_S\) is at least five e.g. when one imposes \(C_5 = 0\) in the family \(F_1\) \(\tag{25}\), also for \(C_7 \neq 0\). But, as discussed above, for this three parameter family we can only show that certain sub blocks of a candidate are positive definite.

Discussion: We have also constructed the characteristic variables for the four \(F_1\) and three parameter \(F_2\) families of strongly hyperbolic Hamiltonian formulations of GR. It is interesting that in every fully second order strongly hyperbolic Hamiltonian formulation (Maxwell, pure gauge and GR) we find that the fully second order characteristic variables always take the simple form
\[
U_i = \sum_j a_{ij}(\partial_t \pm v \partial_\nu) u_j, \tag{108}
\]
with speeds \(\pm v\) given constants \(a_{ij}\) and primitive variables \(q_i\). There are strongly hyperbolic fully second order systems without this property, for example the Z4 formulation coupled to the puncture gauge \(\tag{24}\). At the moment it is not clear what causes the special form \(\tag{108}\).

We originally aimed to find strongly and symmetric hyperbolic Hamiltonian formulations with popular gauge conditions. Having analyzed a large class of gauges, we are now in a position to return to that question. The generalized harmonic gauge is straightforward and as discussed previously is recovered with \(C_i = 0, i = 1, \ldots, 7\). More interesting is the puncture gauge \(\tag{75}\).

The formulation parameters in the strongly and symmetric hyperbolic formulations can be used to adjust the characteristic speeds. If we regard them as scalar functions of the position variables \((\gamma_{ij}, \alpha, \beta^j)\) then the equations of motion will contain derivatives of the parameters, but in the principal part only the functions appear. Hence, hyperbolicity of the formulations is not altered.

Here we want to present the evolution equations for lapse and shift for a formulation that is very close to the puncture gauge. It belongs to the symmetric hyperbolic family \(\tag{106}\). We choose
\[
\lambda^2 = \mu L, \quad \lambda^2 = \frac{4\mu S \gamma^{1/3}}{3\sigma^2} - \frac{1}{3}, \quad \Theta = \frac{1}{2} \sigma \alpha, \quad Z_i = -\frac{\alpha \rho_i}{2 \sqrt{\gamma}} \tag{111}
\]
Thus, symmetric hyperbolicity requires
\[
\mu_L > 0, \quad 4^{1/3} \mu_S > \alpha^2, \tag{110}
\]
which reduces to \(0 < \alpha < \sqrt{3} \gamma^{1/6}\) for the popular choice \(\mu_L = 2/\alpha\), \(\mu_S = 3/4\). To make the comparison with other systems simpler we present the equations in terms of the \(Z^4\) variables. Instead of the canonical momenta \((\pi^i, \sigma, \rho_i)\) the \(Z^4\) formulation \(\tag{25}\) (which is not Hamiltonian) takes the extrinsic curvature and two fields called \((\Theta, Z_i)\) as evolved variables. The mapping to the canonical variables is defined through \(\tag{75}\) and
\[
\Theta = \frac{1}{2} \sigma \alpha, \quad Z_i = -\frac{\alpha \rho_i}{2 \sqrt{\gamma}} \tag{111}
\]
In these variables, restricting to Brown’s Hamiltonian results in a system that is as close to the \(Z^4\) formulation as possible. With the choice \(\tag{106}\) we find
\[
\partial_t \alpha = \beta^i \partial_j \alpha - \mu_L \alpha^2 K + \frac{1}{2} \mu_L \alpha^2 \Theta, \quad \partial_t \beta^i = \beta^j \partial_j \beta^i + \mu_S \gamma^{1/3} R_{j k}^i \gamma^{j k} + \frac{1}{3} (\mu_S \gamma^{1/3} - \alpha^2) \Gamma_{j k}^i \gamma^{j k} + 2 \mu_S \gamma^{1/3} Z^i - \alpha D^i \alpha. \tag{112}
\]
Hence, near the puncture (for \( \alpha \to 0 \)) we obtain the puncture gauge condition \( [78] \) when the constraints \( \Theta = 0 \) and \( Z_i = 0 \) are satisfied.

Brown finds in \([10] \) that a strongly hyperbolic Hamiltonian formulation with the puncture gauge does not exist. Here we have shown that with a slight modification of the \( \Gamma \)-driver shift condition even symmetric hyperbolicity can be achieved. The new terms are small near the puncture.

The final measure of how useful the formulations with these gauges are will be determined by numerical experiments. Some of the ingredients for successful numerical evolutions are now understood. In this work we have taken care of questions related to the continuum formulation. For long-term stable numerical evolutions of puncture data a convenient choice of variables is an important feature as well \([20] \). Furthermore, even if convenient variables can be taken on a suitably mathematically well-founded formulation, there is no guarantee that numerical simulations will be stable, either around flat-space or the puncture data we are interested in evolving. Numerical tests will be discussed in detail elsewhere.

VII. CONCLUSION

Motivated partially by the success of symplectic integrators in many numerical applications we have studied Hamiltonian formulations of GR with live gauges. An essential property for any numerical application is that the underlying PDE problem is well-posed. Thus we have concentrated on well-posedness of the I(B)VP and considered the level of hyperbolicity of the equations of motion when the gauge choice is made at the Hamiltonian level.

Several tools were developed to perform our analysis. We examined the relationship between the Hamiltonian energy and that required for symmetric hyperbolicity. In general we find that the Hamiltonian guarantees the existence of a candidate symmetrizer, which is typically not positive definite in applications. However there are important special cases in which Hamiltonian structure significantly simplifies hyperbolicity analysis. We have also translated the analysis in the literature on first order in time, second order in space systems to fully second order systems and now view the fully second order characteristic variables as the natural form whenever they can be constructed.

In our first applications we demonstrated that every strongly hyperbolic Hamiltonian formulation of electromagnetism is symmetric hyperbolic. We furthermore show that a generalization of electromagnetism, with Hamiltonian structure, the pure gauge system, is also always symmetric hyperbolic whenever it is strongly hyperbolic.

There are several interesting consequences of the Hamiltonian approach to gauge choice. Normally from the free-evolution PDEs point of view the gauge choice of GR (or electromagnetism) allows one to choose equations of motion for some quantities and the freedom to add combinations of the constraints to the equations of motion. But in the Hamiltonian approach in GR for example the constraint addition is determined completely by the choice of the lapse and shift. This restricts the choice of formulation. There are choices of evolution equations for the lapse and shift that allow for a strongly or symmetric hyperbolic system without Hamiltonian structure that are forbidden when Hamiltonian structure is imposed, including the popular puncture gauge. We find in every example that the fully second order characteristic variables of the Hamiltonian systems always have a special form. We do not know how generally this property holds, but there are certainly examples of strongly hyperbolic formulations without Hamiltonian structure that do not exhibit it.

In our analysis of GR we find several families of strongly hyperbolic formulations and are able to choose the gauge parameters so that we get very close to the puncture gauge choice. Analysis of symmetric hyperbolicity is rather more difficult, but for every strongly hyperbolic formulation we are at least able to obtain partial results. We analyze several cases completely and find a Hamiltonian formulation that is symmetric hyperbolic with a small modification of the puncture gauge.

The next practical step for applications is to investigate whether or not any of the formulations constructed here may be used successfully in numerical evolutions, either with or without symplectic integration. There are however delicate issues involved in the choice of evolved variables.

An interesting question about GR is: what are the set of gauge choices that admit a well-posed I(B)VP? This question may be asked whether or not one restricts to formulations with Hamiltonian structure, and has still not been answered to our satisfaction in the general case.

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Appendix A: Hyperbolicity of fully second order systems

In this section we discuss the various definitions of hyperbolicity for fully second order systems as summarized in Sect. [HA]
The key in the proofs is the following. If two matrices
\[ A^{ij} = M^{-1}[V^{ij} - G^i M F^j], \]
\[ B^i = [F^i M^{-1} + M^{-1} G^i] M \] (A1)
are given then the matrix
\[ Q^{i, j} = \begin{pmatrix} 0 & \delta^i_p \\ A_p^j & B^p \end{pmatrix} \] (A2)
is similar to the matrix \( A_p^j \):
\[ Q^{i, j} = T^{-1}_1 k A_p^j T_1^j, \] (A3)
where
\[ A_p^j = \begin{pmatrix} F^j & \delta^j \\ \delta^j_p & G_p \end{pmatrix} M^{-1} \delta^j, \] (A4)
\[ T^{-1}_1 k = \begin{pmatrix} \delta^j_k & 0 \\ F^j & M^{-1} \end{pmatrix}, \] (A5)
\[ T_1^j = \begin{pmatrix} \delta^j & 0 \\ -M F^j & M \end{pmatrix}. \] (A6)

This shows immediately that a fully second order system is strongly hyperbolic if and only if the fully second order principal symbol \( \Theta \) has a complete set of eigenvectors with real eigenvalues.

Moreover one may easily prove that the notions of strong and symmetric hyperbolicity for a first order in time, second order in space system and an equivalent fully second order system agree.

Now we show that symmetric hyperbolicity of the fully second order system is equivalent to the existence of a positive conserved quantity.

**Proof.** Fully second order system is symmetric hyperbolic \( \Rightarrow \) a positive conserved quantity exists: Without loss of generality we assume the following reduction to first order
\[ \partial_t q = M^{-1} w + F^i \partial_i q + S_q, \] (A7)
\[ \partial_i w = V^j \partial_i q + G \partial_i w + S_w, \]
which results in a first order in time, second order in space system with principal part matrix \( A_1 \). We notice that in the principal part one can write
\[ \partial_t \left( \frac{\partial_i q}{\partial_q} \right) = T^{-1}_1 k \partial_i \left( \frac{\partial_j q}{w} \right) = T^{-1}_1 k A_p^j T_1^j \partial_q \left( \frac{\partial_j q}{w} \right) = Q^{i, j} \partial_q \left( \frac{\partial_j q}{\partial_q} \right). \] (A8)

It is then straightforward to check that if \( H^{ij} \) is a symmetrizer of the first order in time second order in space system then \( T_1^i H^{ij} T_1^j \) defines a positive conserved quantity for the fully second order system. Positivity follows from Sylvester’s law of inertia. To show conservation of the corresponding energy we notice that
\[ S_i T_1^i H^{kl} T_1^j \partial_q^m s_p T_1^j = S_i T_1^i k H^{kl} A_p^j s_p T_1^j S_q^m \]
\[ = T_1^i s_p H^{kl} A_p^j s_p T_1^j \]
\[ = T_1^i s_p (s_k H^{kl} A_p^j s_p S_j)^j T_1^j \]
\[ = (S_i T_1^i k H^{kl} A_p^j s_p T_1^j S_q^m) I^j \]
\[ = (S_i T_1^i k H^{kl} T_1^j Q^{i, j} s_p S_q^m)^j, \] (A9)
where \( s_i \) is an arbitrary spatial unit vector and
\[ S_i = \begin{pmatrix} s_i & 0 \\ 0 & 1 \end{pmatrix}. \] (A10)

Using the techniques presented in \[ \[12\] \] one can then show that \( A_9 \) implies conservation of the energy
\[ \epsilon = \frac{1}{2} \left( \frac{\partial_j q^1}{\partial_t} \partial_j q^2 \right) T_1^i k H^{kl} T_1^j \left( \frac{\partial_j q^1}{\partial_t} \partial_j q^2 \right). \] (A11)

A positive conserved quantity exists \( \Rightarrow \) Fully second order system is symmetric hyperbolic: This direction is obvious if one chooses the natural first order reduction with \( \partial_i q = w \). The symmetrizer of the first order in time second order in space system is then the same as the symmetrizer of the fully second order system. 

**Lemma 1.** Symmetric hyperbolicity of the fully second order system \( D \) is equivalent to the existence of a second order symmetrizer and fluxes \( (H_1, \phi^i, \phi^{ij}) \) satisfying
\[ \phi^{i \dagger} = \phi^i, \]
\[ \phi^{ij} = \phi^{i(j)}, \]
\[ s_is_j s_k (\phi^i - B^{i\dagger}H_1) A_1^{jk} = \]
\[ \frac{1}{2} (\partial_j q^1 \partial_j q^2) T_1^i k H^{kl} T_1^j \left( \frac{\partial_j q^1}{\partial_t} \partial_j q^2 \right). \] (A12)
for every spatial vector \( s_i \), and
\[ (H_1 A^{ij} + B^{(i \dagger} H_1 B^{j)}) \phi^{ij} + \phi^{ij} \phi^{i - B^{i(} H_1) \phi^{i}} - H_1 \]
hermitian positive definite.

**Proof.** We proceed by demonstrating equivalence of \( A_{12} \)-\( A_{14} \) and the existence of a conserved positive quantity on the system.
\( A_{12} \)-\( A_{14} \) \( \Rightarrow \) conserved quantity exists: Consider the energy
\[ E = \int d^3 x \epsilon, \] (A15)
\[ \epsilon = \frac{1}{2} \left( \partial_j q^1 \partial_j q^2 \right) H^{ij} \left( \frac{\partial_j q^1}{\partial_t} \partial_j q^2 \right), \]
where \( H^{ij} \) is the matrix \( A_{14} \). The positivity of \( H^{ij} \) obviously implies positivity of \( E \). It remains to show that \( E \) is conserved.
Computing a time derivative of the energy density \( \epsilon \) we get
\[
2\partial_t \epsilon = \partial_p \left( \partial_q^T \phi^T \partial_q \right) \tag{A16}
\]
\[
+ \partial_q \left( q^T H^{[p]}_3 \partial_q \partial_q + \partial_q \partial_q^T H^{[p]}_3 q \right) \\
+ \partial_q \left( \partial_q A^{[p]} H_1 \partial_q + \partial_q \partial_q^T H_1 A^{[p]} \partial_q \right) \\
+ \partial_q \left( \partial_q H^{[p]}_3 A^{[p]} \partial_q + \partial_q \partial_q^T H^{[p]}_3 A^{[p]} \partial_q \right),
\]
where \( H^{[p]}_3 = \phi^T - H_1 B^T, H^{[p]}_3 = \phi^{[p]} + B^T \phi^{[p]} - B \phi^{[p]} H_1 B^T \).

The last two terms can be written as a divergence, too:
\[
\partial_t \partial_p q^T A^{[p]} H_2 \partial_k q + \partial_q \partial_q^T H_2 A^{[p]} \partial_k p q \\
= \partial_t \partial_p q^T A^{[p]} H_2 \partial_k q + \partial_q \partial_q^T H_2 A^{[p]} \partial_k p q \\
+ 2 \partial_q \partial_q q^T \left( A^{[p]} H_2 \partial_k - A^{k(p} H_2 \partial_k \right) \partial_k q \\
+ 2 \partial_q \partial_q q^T \left( H_2 A^{k(p} - H_2^{k(p} A \partial^{k)} \right) \partial_k p q \\
= \partial_p \left( \partial_q H_2 A^{[p]} \partial_k q \right) \\
+ 2 \partial_q \left( \partial_q q^T \left( A^{[p]} H_2 \partial_k - A^{k(p} H_2 \partial_k \right) \partial_k q \right) \\
+ 2 \partial_q \left( \partial_q q^T \left( H_2 A^{k(p} - H_2^{k(p} A \partial^{k)} \right) \partial_k p q \right), \tag{A17}
\]
where the last equality holds because of (A13) (for a tensor \( T_{ijk} \) the equation \( T_{ijk} = 0 \) is equivalent to the requirement \( T_{ijk} s_i s_j s_k = 0 \) for every covector \( s_i \)). Hence \( E \) is a conserved quantity.

Conserved quantity exists \( \Rightarrow \) (A12) \& (A14): Assume that we have the conserved quantity [A15] with
\[
H^{ij} = \begin{pmatrix} H^{ij}_2 & H^{ij}_3 \end{pmatrix}, \tag{A18}
\]
and parametrize the fluxes of the energy by
\[
\Phi^i = \begin{pmatrix} \phi_2^{ijk} & \phi_3^{ijk} \end{pmatrix}, \tag{A19}
\]
with \( \phi_2^{ijk} = \phi_1^i \) and \( \phi_3^{ijk} = \phi_3^{ijk} \). Computing and comparing \( \partial \epsilon \) and \( \partial \Phi \) reveals that energy conservation implies
\[
2 \phi_2^{ijk} = A^{[k} A^{j]} H_2^{i],} \tag{A20a}
2 \phi_2^{ij} = A^{ij} H_1, \tag{A20b}
2 \phi_2^{ij} = H_3^{ij} + H_2^{ij}, \tag{A20c}
2 \phi_2^i = B^{i} H_1 + H_2^i. \tag{A20d}
\]
The solution of (A20d) is obvious:
\[
H_3^{ij} = 2 \phi_2^i - H_2^i B^j. \tag{A21}
\]
From (A20d) and (A20c) we get
\[
H_3^{ij} = 2 \phi_2^j - H_2^j B^i \\
= A^{[i} H_1 - 2 \phi_2^{[i]} B^j - H_2^i B^j \\
= A^{[i} H_1 - 2 \phi_2^{[i]} B^j + B^{i} H_1 B^j. \tag{A22}
\]
Since \( H^{ij} \) defines a symmetrizer we also have \( H_3^{ij} = H_3^{ij} \). When we identify
\[
\phi_2^i = 2 \phi_1^i, \tag{A23}
\]
\[
\phi_3^{ij} = 2 \phi_2^{[ij]} + 2 \phi_2^{[i]} B^j - B^{[i} H_1 B^j \tag{A24}
\]
this implies that (A13) is hermitian, and (A12) is obvious.

Concerning (A20a) it was proven in [12 section IVB] that this equation together with \( \phi_3^{ijk} = \phi_3^{[ijk]} \) has a solution if and only if
\[
A^{ijk} H_2^{ij} = H_3^{ij} A^{ik}. \tag{A24}
\]
This equation is equivalent to (A13). Finally, the positivity of the symmetrizer \( H^{ij} \) implies that the matrix (A14) is positive definite.

**Appendix B: Proof of the rank criterion**

In section [IVB] we described the rank criterion, a feature that can be used as a necessary condition for symmetric hyperbolicity. Here we discuss the proof of this criterion.

We follow the notation of section [IVB]. Thus, \( C^l \subset \mathbb{R}^l \) is a set of formulation parameters, \( c \), and we consider \( k \times k \) principal part matrices \( A^p, i (c) \) that depend continuously on \( c \). We search for symmetrizers of \( A^p, i (c) \) in the set of hermitian \( k \times k \)-matrices, \( \mathcal{G} \), that is the image of a linear map \( G^{ij} : \mathbb{R}^n \rightarrow \mathbb{R}^{k \times k} \), \( g \rightarrow G^{ij} (g) \). The requirement that \( G^{ij} (g) \) is a candidate symmetrizer for the formulation \( c \) defines linear equations on \( g \) of the form
\[
B(c)g = 0, \tag{B1}
\]
where \( B(c) \in \mathbb{R}^{m \times n} \) depends continuously on \( c \).

We prove the following

**Lemma 2.** Let \( C^l \subset C^l \) be the set of formulations where the rank of \( B(c) \) is \( N \) and that do not possess symmetrizers in \( \mathcal{G} \). Let further \( c \) be a formulation such that the intersection of every neighborhood \( U_c \subset C^l \) of \( c \) with \( C^l \) is not empty, \( U_c \cap \tilde{C}^l \neq \emptyset \).

If there is a symmetrizer for \( c \) in \( \mathcal{G} \) then the rank of \( B(c) \) is smaller than \( N \).

For the proof we need the following

**Proposition 1.** Let \( B, \tilde{B} \in \mathbb{R}^{m \times n} \) with \( \| B - \tilde{B} \| < \varepsilon \) and \( g \in \ker B \). Let further be
\[
K(\tilde{B}) = \inf_{v \in (\ker B)^\perp \setminus \{0\}} \frac{\| \tilde{B} v \|}{\| v \|}. \tag{B2}
\]
where \( V^\perp = \{ v \in \mathbb{R}^n : \langle v, \tilde{v} \rangle = 0, \forall \tilde{v} \in V \} \). Theorem we get
\[
\inf_{h \in \ker \tilde{B}} \| g - h \| = \min_{h \in \ker \tilde{B}} \| g - h \| < \frac{\varepsilon}{K(\tilde{B})} \| g \|. \tag{B3}
\]
Proof. It is clear that there exists a $\tilde{g} \in \ker \tilde{B}$ such that
\[ \inf_{c \in \ker \tilde{B}} \| g - \tilde{h} \| = \| g - \tilde{g} \|. \]
It satisfies $g - \tilde{g} \in (\ker \tilde{B})^\perp$. If $g = \tilde{g}$ nothing needs to be shown. Hence, we assume that $g - \tilde{g} \in (\ker \tilde{B})^\perp \setminus \{0\}$. The definition of $K$ then implies
\[
K(\tilde{B}) \| g - \tilde{g} \| \leq \| \tilde{B}(g - \tilde{g}) \| = \| \tilde{B}g \| = \| (\tilde{B} - B)g \| \\
\leq \| \tilde{B} - B \| \| g \| < \varepsilon \| g \|. \tag{B4}
\]

Now we prove the above lemma. Since $B : \mathcal{C}^l \to \mathbb{R}^{m \times n}$ is continuous, there exists for every $\varepsilon > 0$ a $\delta > 0$ such that $\| B(c) - B(\tilde{c}) \| < \varepsilon$ for every $c$ with $\| c - \tilde{c} \| < \delta$. We assume that $g \in \mathbb{R}^n$ satisfies $B(c)g = 0$. Then, according to the proposition there is for every $\varepsilon > 0$ and every candidate $G^j(\tilde{g})$ of $c$ we find a candidate $G^j(\tilde{g})$ of $\tilde{c}$ that is not positive definite such that
\[
\| g - \tilde{g} \| < \frac{\varepsilon}{K(B(\tilde{c}))} \| g \|. \tag{B5}
\]

Since $U_c \cap \mathcal{C}^l \neq \emptyset$ for every neighborhood $U_c$ of $c$ we can choose $\tilde{c} \in \mathcal{C}^l$ such that $\| c - \tilde{c} \| < \delta$. Then we get for every $\tilde{g} \in \ker B(\tilde{c})$ that $G^j(\tilde{g})$ is not positive definite (because otherwise there is a symmetrizer for $\tilde{c} \in \mathcal{C}^l$).

Hence, for every $\varepsilon > 0$ and every candidate $G^j(g)$ of $c$ we find a candidate $G^j(\tilde{g})$ of $\tilde{c}$ that is not positive definite such that
\[
\| g - \tilde{g} \| < \frac{\varepsilon}{K(B(\tilde{c}))} \| g \|. \tag{B6}
\]

On the other hand, since we find a symmetrizer for $c$ in $g$ there exists a point $g_+ \in \ker B(c)$ such that $G^j(g_+)$ is positive definite.

We notice that $G^j$ is a continuous map (because it is linear) and that the map which assigns to a matrix its smallest eigenvalue is continuous, too. This implies that there is a neighborhood $U_+ \subset \mathcal{G}$ of $g_+$ such that the matrix $G^j(h)$ is positive definite for every $h \in U_+$.

We choose $g_+ \in \ker B(\tilde{c})$ such that
\[
\| g_+ - \tilde{g} \| < \frac{\varepsilon}{K(B(\tilde{c}))} \| g_+ \|. \tag{B7}
\]

We know that $\tilde{g}_+$ is not in $U_+$. Therefore there exists a $\rho > 0$ such that $\| g_+ - \tilde{g} \| \geq \rho$.

Equation $\text{(B7)}$ then implies that for every $\varepsilon > 0$ there is a formulation $\tilde{c} \in \mathcal{C}^l$ with
\[
K(B(\tilde{c})) < \Lambda \varepsilon, \tag{B8}
\]
with the constant $\Lambda = \| g_+ \| / \rho$.

To show this it implies a jump in the rank of $B$ we consider a sequence $\{ \tilde{c}_n \}_{n \in \mathbb{N}} \subset \mathcal{C}^l$ such that
\[
\lim_{n \to \infty} \tilde{c}_n = c. \tag{B9}
\]

We notice that $\lim_{n \to \infty} K(B(\tilde{c}_n)) = 0$.

Then, for every $c_n$, there exists a $g_n \in (\ker B(\tilde{c}_n))^\perp$ such that $\| g_n \| = 1$ and $K(B(c_n)) = \| B(c_n)g_n \|$ (the set $\{ \| g \| = 1 \}$ is compact). From the sequence $\{ g_n \}$ we choose a convergent subsequence $\{ g_{n_m} \}$ and get
\[
0 = \lim_{m \to \infty} K(B(c_{n_m})) h_{n_m} = \lim_{m \to \infty} B(c_{n_m}) h_{n_m} = \left( \lim_{m \to \infty} B(c_{n_m}) \right) \left( \lim_{m \to \infty} h_{n_m} \right) = B(c) h, \tag{B10}
\]
with $\| h \| = 1$. Analogously one can show that the limit of every convergent sequence $\{ g_n \in \ker B(c_{n_m}) \}$ is in the kernel of $B(c)$.

It follows that the dimension of the kernel of $B(c)$ is bigger than the dimension of the kernel of $B(c_n)$, i.e. the rank of $B(c)$ is smaller than $N$.

Appendix C: Derivation of conditions for strong hyperbolicity

In this appendix we discuss how conditions for strong hyperbolicity can be derived on the Hamiltonian formulations of GR introduced in section \textit{Appendix}. One finds that the special structure of the principal symbol which we assume in this article is very helpful for this purpose. We consider first order in time, second order in space formulations, but most steps can be directly carried forward to the fully second order case.

The key will be to write the symbol in a simple standard form where the conditions can be read off easily. This is meant to be a preliminary step for the construction of positive definite symmetrizers in the next appendix \textit{E}.

We are interested in Hamiltonian formulations of GR with the Hamiltonian $H$ given in \textit{Appendix}. Thus, we are free to choose the seven formulation parameters $C_1, \ldots, C_7$. Yet, we replace $C_1, C_2$ and $C_3$ by $\lambda^2_1$, $\lambda^2_2$ and $\lambda^2_3$ respectively, using \textit{Appendix} and \textit{Appendix}. The given matrix expressions are valid in the $Z_4$ variables \textit{Appendix}. But the analysis in the canonical variables is analogous. The only difference are factors of 2 at some places.

As discussed in section \textit{Appendix} for the hyperbolicity analysis one can assume without loss of generality $\alpha = 1$, $\beta^i = 0$. I.e. the principal part matrix depends on the 3-metric and the formulation parameters only.

After changing the order of variables to $(\gamma_{ij}, \alpha, Z_i, K_{ij}, \Theta, \beta^j)$ we find that the principal symbol, $P^a$, has the block structure
\[
P^a = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}, \tag{C1}
\]

To this matrix we apply similarity transformations to bring it to a form where the conditions for strong hyperbolicity can be read off easily. First we consider the case where $X$ is invertible. For our example that means $\lambda^2_2 \neq 0$. We get
\[
\tilde{P}^a = T_X^{-1} P^a T_X = \begin{pmatrix} 0 & 1 \\ XY & 0 \end{pmatrix}, \tag{C2}
\]
where

\[ T_X = \begin{pmatrix} 1 & 0 \\ 0 & X^{-1} \end{pmatrix}. \] (C3)

Since we assumed that the principal part matrix can be written in terms of the metric only one can write \( XY \) in block diagonal form. We define the orthogonal projector, \( q_i \), to the direction \( s \) and decompose the vectors and symmetric 2-tensors as

\[
V^i = q_i A V^A + s^i V^s, \\
T^{ij} = \left( q_i^j q^A_B - \frac{1}{2} q_i^j q_{AB} \right) T_{FF}^{AB} + \sqrt{2} q_i^{(s)j} T^{sA} + \frac{1}{\sqrt{2}} q_i^{(s)j} T^{sA} + s^i s^j T^{ss},
\]

where we use \( T^{sA} := s_i q_j A T^{ij} / \sqrt{2} = T^{sA} / \sqrt{2} \) and \( T^{qq} := \sqrt{2} q_i q_j T^{ij} \) instead of the usual decomposition with \( T^{sA} \) and \( T^{qq} \), because it makes the transformation map orthogonal. We apply this decomposition to \( P^s \) and get that it decomposes into a trace-free tensor, a vector and a scalar block, \( \hat{P}_T^s \), \( \hat{P}_V^s \) and \( \hat{P}_S^s \) respectively. The corresponding submatrices of \( XY \) are

\[
XY_{T^{ij} \ k^l} = q_i^j q^k_l - \frac{1}{\sqrt{2}} q_i^j q^{kl}, \\
XY_{Vi}^k = \frac{1}{\sqrt{2}} \left( \lambda_2^V \ -2 \sqrt{2} \lambda_2^V \ -1 - 2 C_6 \right), \\
XY_S = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix},
\]

where

\[
A = \begin{pmatrix} \lambda_2^V - (1 + C_5) C_7 / \sqrt{2} \\ \sqrt{2} \left( C_7^V - (1 + C_5)(C_7 + \lambda_2^V) \right) / C_7 + \lambda_2^V \end{pmatrix}, \\
C = \begin{pmatrix} \lambda_2^V - C_5 C_7 / 2 & C_7^V / \sqrt{2} \lambda_2^V \\ -2 C_5 & \lambda_2^V \end{pmatrix}, \\
B = \begin{pmatrix} B_{11} & \lambda_2^V - 1 - 8 C_6 C_7 + (2 \lambda_2^V - 4 \lambda_2^V - 1) \lambda_2^V + 1 \\ B_{12} & C_7^V + 2 \lambda_2^V + 4 \lambda_2^V + 1 \end{pmatrix},
\]

\[
2 B_{11} = 1 + C_5 + 8 C_4 (1 + C_5) + 8 C_6 C_7 - 4 \lambda_2^V - 4 (\lambda_2^V - 1) (C_7 + \lambda_2^V) + 3 (C_7 + \lambda_2^V) \lambda_2^V, \\
B_{12} = 8 C_6 - 4 (\lambda_2^V - 1) (1 + C_5) (C_7 + \lambda_2^V) + \lambda_2^V.
\]

We do not get the vanishing upper right block in \( XY_S \) when we start from the fully second order system. Therefore we prefer the first order in time system here.

We see that the trace-free tensor block is always diagonalizable, because \( q_i^j q^k_l - q_i^j q^{kl} / 2 \) is the identity. The vector block has the structure

\[
q_{ik} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_2^V & b_1 & 0 & 0 \\ 0 & \lambda_2^V & 0 & 0 \end{pmatrix}. 
\]

It is hence diagonalizable with real eigenvalues if and only if \( b_1 = 0 \), i.e. \( C_6 = (\lambda_2^V - 1)/2 \) and \( \lambda_2^V > 0 \).

Concerning the scalar part we use that the upper right block of \( XY_S \) vanishes. A necessary condition for its diagonalizability is hence that \( A \) and \( C \) are diagonalizable. One finds that the eigenvalues of \( A \) and \( C \) are the same, namely \( \lambda_2^V \). Hence, \( A \) and \( C \) are diagonalizable if \( (\lambda_2^V - \lambda_2^V)^2 + C_5^V C_7 (\lambda_2^V + \lambda_2^V) \neq 0 \).

When \( A \) and \( C \) have only one eigenvalue those matrices must be (as \( 2 \times 2 \)-matrices) already diagonal. This implies \( C_5 = 0 = C_7 \). Moreover, since there is only one eigenvalue, the matrix \( B \) must vanish. One obtains \( \hat{P}_S^s \).

The squared characteristic speeds are

\[
\lambda_2^V = 1/4(1 + 3 \lambda_2^V), \quad v_+^2 = \lambda_2^V + 1 = 2 C_1, 
\]

and \( P^s \) has real eigenvalues if \( \lambda_2^V \geq 0, \lambda_2^V \geq 0 \). However, the case \( \lambda_2^V = 0 \) is not allowed here, because \( X \) would be singular. Hence, we get \( \lambda_2^V > 0 \).

For \( v_+^2 \neq v_-^2 \), we can transform to a basis where \( A \) and \( C \) are diagonal. Let \( T_A \) and \( T_C \) diagonalize \( A \) and \( C \) respectively. We apply the transformation

\[
\hat{P}_S^s = T_S^{-1} \hat{P}_S^s T_S, 
\]

with

\[
T_S = \text{diag}(T_A, T_C, T_A, T_C), \quad T_S \text{ and } T_C \text{ diagonalize } A \text{ and } C \text{ respectively.}
\]

and get the scalar block \( \hat{P}_S^s \) with the following structure

\[
\hat{X} Y_S = \begin{pmatrix} v_+^2 & 0 & 0 & 0 \\ 0 & v_+^2 & 0 & 0 \\ B_{11} & B_{12} & v_+^2 & 0 \\ B_{21} & B_{22} & 0 & v_+^2 \end{pmatrix}. \]

Hence, to make \( \hat{P}_S^s \) diagonalizable one needs that \( B_{11} = 0 = B_{22} \). Those expressions are however very long so we do not present them. We discuss the solutions of the equations \( B_{11} = 0 = B_{22} \) in section \( \text{VTC} \). There we find three families of strongly hyperbolic formulations that we denote \( F_{1/2/3} \).

In the next section we prove symmetric hyperbolicity of the strongly hyperbolic formulations in the four parameter family \( F_1 \) \( \text{[15]} \) which also satisfy \( C_7 = 0 = C_5 \). In that case we get

\[
B_{11} = (\lambda_2^V - 1 - 8 C_4) / \sqrt{4}, \\
B_{22} = (3 \lambda_2^V - 8 \lambda_2^V + 8 C_6 + 5) / \sqrt{2}. 
\]

Hence strong hyperbolicity implies

\[
\lambda_2^V = (3 \lambda_2^V + 1)/4, \quad C_4 = (\lambda_2^V - 1)/8, \\
C_6 = 3(\lambda_2^V - 1)/8, \quad \lambda_2^V > 0, \quad \lambda_2^V \geq 0. \]

For \( \lambda_2^V = 0 \) the matrix \( XY_S \) has vanishing eigenvalues. Hence, \( \hat{P}_S^s \) is not diagonalizable. Thus, the case \( \lambda_2^V = 0 \) must be excluded.
Singular $X$: In the analysis above we needed the assumption that $X$ is invertible. If $X$ is not invertible ($\lambda^2 = 0$) then the diagonalizability of $XY$ is only a necessary condition for the diagonalizability of $P^s$ (when $P^s$ is diagonalizable then also $(P^s)^2$, and hence $XY$ and $YX$ are).

Indeed it turns out that the derived conditions are not sufficient in that case. However, having a complete set of eigenvectors for $(P^s)^2$ one obtains restrictions on the formulation parameters through the requirement that every eigenvector of $(P^s)^2$ must be an eigenvector of $P^s$ (details can be found e.g. in [11]). One finds a one parameter family of strongly hyperbolic formulations. But there the matrix $M$ [23] is not invertible, i.e. we cannot transform to a fully second order free evolution system. Therefore those formulations are not of interest here.

Appendix D: Construction of symmetrizers

In this appendix a possible procedure for the construction of a positive symmetrizer for formulations of GR is discussed. The difficulty is usually not the construction of general candidate symmetrizers but the proof of positivity. We use a method that simplifies the latter task. The approach works as follows.

First the principal symbol is transformed to a simple standard form using the calculations of the previous appendix [C]. It is then possible to construct the set of all positive definite matrices, $G$, that symmetrize this transformed symbol. The next step is to make an ansatz for the symbol, i.e. the matrices in $G$, and we were not able to reduce them.

Yet, for another two parameter family of strongly hyperbolic formulations, [107], we were able to show that it is not symmetric hyperbolic, which means that in GR there are Hamiltonian formulations that are strongly but not symmetric hyperbolic.

Decomposition of the ansatz candidate. In the construction of a symmetrizer we assume that we start from a reasonable ansatz candidate, $G$. The first step is a decomposition of its derivative indices into longitudinal and transverse part:

$$ G^{ij} = (s^i q^j)_A \begin{pmatrix} G^{ss} & G^{SB} \\ G^{As} & G^{AB} \end{pmatrix} s^j q^i, \quad (D1) $$

with $G^{ss+} = G^{ss}, G^{As+} = G^{As}, G^{AB+} = G^{BA}$. We can apply the same decomposition to the principal part matrix:

$$ s^j A^{ij} p = (s^i q^j)_A \begin{pmatrix} A^i & A^j \\ A^i & A^j \end{pmatrix} \begin{pmatrix} s^i \\ q^j \end{pmatrix}. \quad (D2) $$

Due to the structure of $A^{ij}$ we get $s^i q^j A^{ij} p = 0$, and it follows that $G^{ij}$ is a candidate symmetrizer of $A^{ij}$ if and only if for all spatial vectors $s$ the matrix $G^{ss} A^i s$ is hermitian.

Later we discuss the construction of a positive definite $G^{ss}$. But given this matrix it is clear that $G^{ij}$ is positive definite if and only if

$$ \begin{pmatrix} 1 & 0 \\ B^A & q^A C \end{pmatrix} \begin{pmatrix} G^{ss} & G^{ss} C \\ G^{ss} & G^{ss} D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B^A & q^A B \end{pmatrix} = \begin{pmatrix} G^{ss} & 0 \\ 0 & G^{AB} - G^{As}(G^{ss})^{-1}G^{sB} \end{pmatrix} > 0, \quad (D3) $$

is. Hence, given a positive definite $G^{ss}$ we need to prove positivity of

$$ G^{AB} - G^{As}(G^{ss})^{-1}G^{sB}. \quad (D4) $$

In our formulations of GR that means for the analysis of positivity the full candidate symmetrizer, a $40 \times 40$ matrix, is reduced to two $20 \times 20$ matrices.

Construction of a positive $G^{ss}$. Now we want to construct a matrix $G^{ss}$ that symmetrizes the principal symbol $P^s = A^i A^j$. One cannot expect that every symmetrizer of $P^s$ is part of a symmetrizer for the full principal part matrix, i.e. the constructed $G^{ss}$ must be sufficiently general to make it possible to adjust parameters in the construction of the full symmetrizer later on. But it must not contain too many parameters, because this complicate the positivity analysis of the matrix [125].

As a good compromise we found it reasonable to start from the ansatz that the full symmetrizer depends on the metric only, but is not restricted otherwise:

$$ G^{ijkl mn} = \begin{pmatrix} G^{ij \ kl mn} & G^{ij \ kl mn} & G^{ij \ kl mn} & G^{ij \ kl mn} \\ G^{ij \ kl mn} & G^{ij \ kl mn} & G^{ij \ kl mn} & G^{ij \ kl mn} \\ G^{ij \ kl mn} & G^{ij \ kl mn} & G^{ij \ kl mn} & G^{ij \ kl mn} \\ G^{ij \ kl mn} & G^{ij \ kl mn} & G^{ij \ kl mn} & G^{ij \ kl mn} \end{pmatrix} \quad (D6) $$

In this way it is possible to prove symmetric hyperbolicity for the two parameter family of strongly hyperbolic formulations [106] (it is the special case of the four parameter family $F_1$ [33] with $C_5 = 0 = C_7$).

In section [VI.C] we presented more strongly hyperbolic formulations, but we were not able to prove the existence of symmetrizers in the general case. The reason is that already the positivity conditions on the symmetrizers of the symbol, i.e. the matrices in $G$, become complicated and we were not able to reduce them.
with

\[
G^{ij} = G^{ij}_{11} = G^{ij}_{11} + 2G^{ij}_{12} + 2G^{ij}_{13} - 2G^{ij}_{14} + 2G^{ij}_{15} + 2G^{ij}_{16} - 2G^{ij}_{17} + 2G^{ij}_{18},
\]

\[
G^{ij} = G^{ij}_{12} = G^{ij}_{12} + 2G^{ij}_{13} + 2G^{ij}_{14} - 2G^{ij}_{15} + 2G^{ij}_{16} - 2G^{ij}_{17} + 2G^{ij}_{18},
\]

\[
G^{ij} = G^{ij}_{13} = G^{ij}_{13} + 2G^{ij}_{14} - 2G^{ij}_{15} + 2G^{ij}_{16} - 2G^{ij}_{17} + 2G^{ij}_{18},
\]

\[
G^{ij} = G^{ij}_{14} = G^{ij}_{14} - 2G^{ij}_{15} + 2G^{ij}_{16} - 2G^{ij}_{17} + 2G^{ij}_{18},
\]

\[
G^{ij} = G^{ij}_{15} = G^{ij}_{15} + 2G^{ij}_{16} - 2G^{ij}_{17} + 2G^{ij}_{18},
\]

\[
G^{ij} = G^{ij}_{16} = G^{ij}_{16} - 2G^{ij}_{17} + 2G^{ij}_{18},
\]

\[
G^{ij} = G^{ij}_{17} = G^{ij}_{17} + 2G^{ij}_{18},
\]

\[
G^{ij} = G^{ij}_{18} = G^{ij}_{18}.
\]

Now, from the previous appendix we know that for \(\lambda^2 \neq 0\) there exists a matrix \(T\) and a diagonal matrix \(\Lambda\) such that

\[
\tilde{P}^* = T^{-1} P^* T = \begin{pmatrix} 0 & 1 \\ \Lambda & 0 \end{pmatrix}.
\]  

We decompose the derivative indices of the ansatz candidate according to (D1) and apply a congruence transformation using the matrix \(T\):

\[
\begin{pmatrix} \tilde{G}^{ss} \\ \tilde{G}^{AB} \end{pmatrix} = \begin{pmatrix} T^{1} & 0 \\ 0 & qA_C \end{pmatrix} \begin{pmatrix} G^{ss} & G^{CD} \\ \tilde{G}^{DC} & \tilde{G}^{DD} \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & qD_B \end{pmatrix}
\]

The condition that \(G^{ij}\) is a candidate symmetrizer of the full problem then becomes

\[
\tilde{G}^{ss} \tilde{P}^* = (\tilde{G}^{ss} \tilde{P}^*)^T.
\]  

To simplify expressions we introduce new parameters:

\[
s_V := 2G^{ij}_{11} + \frac{G^{ij}_{10}}{2(3\lambda^2 + 1)},
\]

\[
s_{44} := \frac{G^{ij}_{10}}{2(3\lambda^2 + 1)},
\]

\[
s_{22} := \frac{4G^{ij}_{11} - 2G^{ij}_{33} + G^{ij}_{44} - G^{ij}_{10}(1 + 3\lambda^2)^{-1}}{\lambda^2},
\]

\[
s_{12} := \frac{4G^{ij}_{11} - 2G^{ij}_{33} + G^{ij}_{44} + 2G^{ij}_{45} - G^{ij}_{10}(1 + 3\lambda^2)^{-1}}{2\lambda^2},
\]

where \(s_V\) appears in the vector and \(s_{12}, s_{22}\) as well as \(s_{44}\) in the scalar block.

With these parameters and the observation that \(\tilde{G}^{ss}\) decomposes into \(2 \times 2\) blocks the positivity conditions on \(\tilde{G}^{ss}\) can be reduced using computer algebra. One finds that for solutions of (D10) \(\tilde{G}^{ss}\) is positive definite if and only if

\[
s_{44} < \frac{2}{9\lambda^2 s_{22}}(\lambda^2 s_{22}^2 - 2),
\]

\[
s_{22} > \frac{\lambda^2 s_{22}^2}{2(1 + 3\lambda^2)(2G^{ij}_{11} - s_V)},
\]

and

\[
s_V > 2G^{ij}_{11}, \quad s_{44} > \frac{2G^{ij}_{11}}{3}, \quad G^{ij}_{11} > 0, \quad \lambda^2 > 0, \quad \lambda^4 > 0.
\]

To simplify that condition further we define new parameters \(p_1, p_2, p_3\) as follows

\[
p_1 = \frac{1}{\lambda^2}(2s_{22}(s_V - 2G^{ij}_{11})(1 + 3\lambda^2) - \lambda^2 s_{12}),
\]

\[
p_2 = N_2 / D_2, \quad p_3 = \frac{3s_{44}}{2G^{ij}_{11}} - 1,
\]

\[
N_2 = s_{22}(2G^{ij}_{11} - s_V) \times (2G^{ij}_{11}(4 + 9\lambda^2) + 9\lambda^2 s_{44} - 4s_V(1 + 3\lambda^2))
\]

\[
+ (2G^{ij}_{11} - s_V)2\lambda^2 s_{12}
\]

\[
D_2 = 4G^{ij}_{11}(4 + 9\lambda^2)s_{22} - 9\lambda^2 s_{12} s_{44} s_{22} s_{44} s_{22} s_{44} s_{22} s_{44} s_{22} s_{44} s_{22} s_{44}
\]

\[
+ 2G^{ij}_{11}(2\lambda^2 s_{12} + 9\lambda^2 s_{12} s_{44} - (4 + 9\lambda^2) s_{22} s_{44})
\]

The positivity condition for \(\tilde{G}^{ss}\) then becomes just

\[
p_1 > 0, \quad p_2 > 0, \quad p_3 > 0, \quad G^{ij}_{11} > 0, \quad \lambda^2 > 0, \quad \lambda^4 > 0.
\]  

Positive of the transverse block. Now we consider the remaining transverse block (D5). It is easy to check that the transformation (D8) does not change this matrix.

We use the solution of (D9) to write (D5) in terms of \(p_1, p_2, p_3, G^{ij}_{11}, s_{12}, \lambda^2, \lambda^4, A_{11}^{ij}\) and \(A_{13}^{ij}\) and show that
the parameters can be chosen to make it positive definite for every $\lambda_1^2 > 0, \lambda_2^2 > 0$.

First we notice that (D5) has the following structure

$$G^{AB kl mn} = \begin{pmatrix} G^{AB kl mn} & 0 & 0 & 0 \\ 0 & G^{AB kl mn} & 0 & 0 \\ 0 & 0 & G^{AB kl mn} & 0 \\ 0 & 0 & 0 & G^{AB kl mn} \end{pmatrix}, \quad (D14)$$

where the component tensors can be written in terms of $q$ and $s$:

$$G^{AB kl mn}_{11} = g_{11}^{AB} q_{kl}^{mn} + 2 g_{12}^{AB} q_{kl}^{mn} q_{kl}^{mn} + 2 g_{13}^{AB} q_{kl}^{mn} q_{kl}^{mn} q_{kl}^{mn} + 2 g_{14}^{AB} q_{kl}^{mn} q_{kl}^{mn} q_{kl}^{mn}$$

When we assume that $G^{AB kl mn}_{22}$ is positive definite then we can transform this matrix without altering positivity to

$$\text{diag}(G^{AB kl mn}_{11}, G^{AB kl mn}_{22}, G^{AB kl mn}_{33}, G^{AB kl mn}_{44}), \quad (D16)$$

where

$$G^{AB kl mn}_{11} := G^{AB kl mn}_{11} - G^{AC kl}_{12} (G^{CD}_{22})^{-1} C D G^{BD}_{22}.$$

The structure of $G^{AB kl mn}_{11}$ is of course (D15a), too. Yet, we denote the scalar parameters $g_{11}$ instead of $g_{11}$.

**Positivity of $G^{AB kl mn}$.** Concerning the matrix $G^{AB kl mn}_{22}$ we find $g_{22}^2 = \lambda_1^2 p_1^2 / D$, with the denominator

$$D = 3 G^{11} q_{11}^2 (p_1 + s_{12}^2) p_9 (1 + p_2) + 4 p_1 p_2 (3 \lambda_2^2 + 1). \quad (D18)$$

We see that with the condition (D13) every term in $N$ and $D$ is positive. Hence, $G^{AB kl mn}_{22}$ is positive.

**Positivity of $G^{AB kl mn}_{33}$.** If we write the matrix $G^{AB kl mn}_{33}$ in an orthonormal basis that contains $s$ then we find the following eigenvalues

$$\{g_{33}^2, g_{33}^3, g_{33}^4 - g_{33}^3, g_{33}^5 + g_{33}^3, g_{33}^6 + 2 g_{33}^3 + g_{33}^5\}. \quad (D19)$$

Written in terms of $p_1, p_2, p_3, G^{11}_{11}, s_{12}, \lambda_2^2, \lambda_1^2, A_{11}^1$ and $A_{11}^3$ those eigenvalues have the form

$$g_{33}^2 = z_1(A_{11}^3 - z_2)(A_{11}^3 - z_3),$$

$$g_{33}^1 - g_{33}^4 = A_{11}^3 - z_2,$$

$$g_{33}^1 + g_{33}^4 = A_{11}^3 + z_3,$$

$$g_{33}^1 + 2 g_{33}^3 + g_{33}^5 = z_4(A_{11}^3 - z_5)(A_{11}^3 - z_3).$$

The expressions for $z_1, \ldots, z_5$ are quite complicated functions of the parameters. But using (D13) and computer algebra one can show that $z_1 < 0, z_4 < 0, z_2 < z_1$ and $z_5 < z_3$. Hence, we can choose $A_{11}^3$ such that $G^{AB kl mn}_{33}$ is positive definite, e.g. $A_{11}^3 = (\max(z_2, z_5) + z_3)/2$.

Since $A_{11}^3$ does not appear in $G^{AB}_{11}$ or $G^{AB kl mn}_{11}$ we impose conditions on $B$ that do not restrict the parameter choices in the rest of the matrix.

**Positivity of $G^{AB kl mn}_{11}$.** The hardest part in the positivity analysis is the matrix $G^{AB kl mn}_{11}$. Again the first step is to write it in an orthonormal basis that contains $s$. One finds that the resulting matrix decomposes into two $4 \times 4$ and two $2 \times 2$ blocks, where the $4 \times 4$ blocks are similar. The $2 \times 2$ blocks are

$$2 \left( \begin{array}{cc} g_{11}^1 & g_{11}^2 \\ g_{11}^2 & g_{11}^1 \end{array} \right),$$

Looking at the entries in the diagonal of the $4 \times 4$ blocks one finds complicated expressions only at two positions. Those entries are

$$2(g_{11}^1 + g_{11}^2 + 2 g_{11}^3 + g_{11}^4 + g_{11}^5) \quad \text{and} \quad g_{11}^1.$$

Using a congruence transformation we decompose the $4 \times 4$ blocks into $2 \times 2$ matrices such that the complicated expressions are in only one $2 \times 2$ matrix. The resulting matrices are

$$2 \left( \begin{array}{cc} g_{11}^1 + g_{11}^2 + g_{11}^3 & g_{11}^4 + g_{11}^5 \\ g_{11}^4 + g_{11}^5 & g_{11}^1 + g_{11}^2 \end{array} \right),$$

$$1 \left( \begin{array}{cc} m_{11} & m_{12} \\ m_{12} & m_{22} \end{array} \right), \quad (D22a)$$

where

$$m_{11} = (g_{11}^4)^2 + g_{11}^5 (g_{11}^1 + 2 g_{11}^2 + g_{11}^3 (2 g_{11}^1 + 3 g_{11}^2 + 2 g_{11}^3)$$

$$- 2((g_{11}^1)^2 + 2 g_{11}^3 + g_{11}^5 (g_{11}^4 + g_{11}^5)),$$

$$m_{12} = \sqrt{2}(2 g_{11}^1 g_{11}^5),$$

$$m_{22} = g_{11}^4 g_{11}^5.$$
We get the following eigenvalues

\[
\left\{ \frac{4(3G_{11}^2)\lambda_+^2 + 3G_{11}^2(\lambda_+^2 - 1)p_4 - 4p_4^2)}{(1 + 3\lambda_+^2)G_{11}^2}, \quad 3G_{11}^2 - 2p_4, 4p_4, 6p_4 - \frac{4p_4^2}{G_{11}^2}, 10p_4 - \frac{12p_4^2}{\lambda_+^2 G_{11}^2} \right\}. \tag{D25}
\]

Hence, together with \[\text{(D22a)}\], the three matrices \[\text{(D22b)}\] are positive definite if and only if the inequalities

\[
p_4 > 0, \quad p_4 < \frac{3}{2}G_{11}^2, \quad p_4 < \frac{5}{6}G_{11}^2 \lambda_+^2, \\
p_1 > 0, \quad p_2 > 0, \quad p_3 > 0, \\
s_{12} \in \mathbb{R}, \quad \lambda_+^2 > 0
\]

are satisfied.

Now, to prove positivity of the remaining block \[\text{(D22b)}\], we use the fact that the only condition on \(s_{12}\) is \(s_{12} \in \mathbb{R}\). We choose \(s_{12}\) such that \[\text{(D22b)}\] becomes diagonal, i.e., we solve the equation \(m_{12} = 0\) for \(s_{12}\). We get \(s_{12}^2 = -N_{12}/D_{12} - p_1\) with the numerator and denominator

\[
N_{12} = 4(1 + 3\lambda_+^2)p_1p_2x \\
\times \left[3(G_{11}^2)\lambda_+^2p_3 + 3(G_{11}^2)^2\lambda_+^2((\lambda_+^2 - 1)p_3 - 10)p_4 \\
+ 4G_{11}^2(9 - \lambda_+^4)(p_3 - 5))p_4^2 - 24p_4^2 \right], \\
D_{12} = 3\lambda_+^2p_3 \left[3(G_{11}^2)\lambda_+^2p_2p_3 \\
+ 3(G_{11}^2)^2\lambda_+^2((\lambda_+^2 - 1)p_2p_4 - 10(1 + p_2)p_4) \\
+ 4G_{11}^2(9 + 5\lambda_+^4)(1 + p_2) - \lambda_+^2 p_2 p_4) \right].
\]

The condition \(s_{12} \in \mathbb{R}\) can be written as \(s_{12}^2 \geq 0\). It restricts the allowed range of \(p_2\) and \(p_3\):

\[
p_{10} > \left( \frac{G_{11}^2 \lambda_+^2}{5G_{11}^2 - 6p_4 - 3G_{11}^2 - 2p_4} + \frac{G_{11}^2 \lambda_+^2}{p_4} \right)^{-1}, \\
p_2 \geq 6\lambda_+^2 p_3 (5G_{11}^2 \lambda_+^2 - 6p_4) (3G_{11}^2 - 2p_4)p_4 \\
(4 + 3\lambda_+^2(4 + p_1))D_2 \\
D_2 = 3(G_{11}^2)^3\lambda_+^2 p_3 + 3(G_{11}^2)^2\lambda_+^2 (\lambda_+^2 - 1)p_3 - 10)p_4 \\
+ 4G_{11}^2(9 - \lambda_+^4(p_3 - 5))p_4^2 - 24p_4^2,
\]

but if these parameters are chosen sufficiently big then the inequalities are satisfied.

If we replace \(s_{12}\) everywhere using \[\text{(D27)}\] then \[\text{(D22a)}\] has only one simple eigenvalue:

\[
\frac{2(5G_{11}^2 \lambda_+^2 - 6p_4)(3G_{11}^2 - 2p_4)p_4}{3(G_{11}^2)^3\lambda_+^2 + 3G_{11}^2(\lambda_+^2 - 1)p_4 - 4p_4^2}.
\tag{D29}
\]

One can check that numerator and denominator are positive if \[\text{(D25)}\] is satisfied.

Hence, indeed for every \(\lambda_+^2 > 0, \lambda_+^2 > 0\) the parameters can be chosen such that \(G_{11}^{AB kl mn}\) is positive definite.

The calculations presented here were performed using the computer algebra system \texttt{mathematica} \[\text{[27]}\] with the package \texttt{xtensor} by José-María Martín-García \[\text{[28]}\].

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