No resource theory for non-Markovian channels?

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Finite-time Markovian channels, unlike their infinitesimal counterparts, do not form a
convex set, calling into question the possibility of a resource theory of non-Markovianity for
channels. As a particular instance of this observation, we consider the problem of mixing the
three Pauli channels, conservatively assumed to be quantum dynamical semigroups, and fully
characterize the resulting “Pauli simplex”. We show that neither the set of non-Markovian
(CP-indivisible) nor Markovian channels is convex in the Pauli simplex, and the measure of
non-Markovian channels is about 0.87. All channels in the Pauli simplex are P-divisible.

I. INTRODUCTION

Memory effects in open quantum systems [1, 2] have become a potential experimental area
of research in quantum information science. At the same time, the theoretical study of non-
Markovian dynamics of open quantum systems continues to gain wide interest and poses newer
conceptual challenges and surprises [3]. The dynamics of open quantum systems are described
by time dependent completely positive trace preserving (CPTP) channels, referred to as quantum
channels or dynamical maps [4, 5]. For understanding quantum non-Markovianity, the two widely
used approaches are based on a hierarchical divisibility criterion [6, 7] and the criterion based on
the distinguishability of states [8].

Recently, there has been an interest on convex combinations of quantum channels. An example
of mixing two Markovian evolutions to create a non-Markovian one was reported in [9]. The idea
in a more rudimentary form (before the divisibility and distinguishability criteria were developed)
appears in an earlier paper [10]. In [11], the counter-intuitive behaviour was explained in terms of
correlations and the information flow between the system and environment. An example of a convex
combination of two non-Markovian channels leading to a memoryless evolution was discussed in [12].
In [13], it was also shown that a master equation with an always negative decay rate as arising from
a mixture of Markovian semi-groups. An experimental realization was recently reported in [14].

Here, we generalize the idea of mixing channels to the case of three-way mixing, and discuss
the convexity of sets of quantum (non-)Markovian dynamical maps. For simplicity, we restrict to
the case of Pauli channels, whose geometrical structure is well understood [15]. We discuss the

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geometry of the sets and point out how some earlier results arise as special cases. An important consequence of our result is that unlike the set of generators of Markovian evolutions, the set of Markovian channels is not convex and thus cannot form ‘free states’ from the perspective of a resource theory. By contrast, it has been shown that Markovian evolutions over an infinitesimal time interval are closed under convex combinations, leading to the concept of free states in the resource theoretical approach to non-Markovianity [16, 17].

The paper is organized as follows. Sec. II and Sec. III discusses the convex combination of two and three Pauli channels respectively. We show that for finite mixing of two channels, the resultant channel is non-Markovian throughout, whereas for the case of mixing three channels, a more complex picture emerges, which we fully characterize. We discuss the implications of our result and give a measure of the sets of Markovian and non-Markovian channels in Sec. IV. We then conclude in Sec. V.

II. CONVEX COMBINATION OF TWO CHANNELS

Consider the channel $\mathcal{E}$ acting on a qubit, represented by the density matrix

$$\rho = \frac{1}{2}(1 + a_i \sigma_i) = \frac{1}{2} \begin{pmatrix} 1 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & 1 - a_3 \end{pmatrix}. \tag{1}$$

The vector $a = (a_1, a_2, a_3)$, with $|a| \leq 1$, is the Bloch vector. Here, we consider Pauli channels which are unital, as defined by $\mathcal{E}(\sigma_I) = \sigma_I$, and $\mathcal{E}(\sigma_i) = x_i \sigma_i$, where $\sigma_I = 1$ and $\sigma_i$’s are the Pauli matrices.

Consider the two Pauli channels $\mathcal{E}_y^p(\rho) = (1 - p)\rho + p\sigma_y \rho \sigma_y$ and $\mathcal{E}_z^p(\rho) = (1 - p)\rho + p\sigma_z \rho \sigma_z$, where $p$ is a decoherence parameter, with $p \in [0, \frac{1}{2})$, which in general is time-dependent. Conservatively, we choose $p$ to be $1 - \exp(-rt)/2$, where $r$ is a constant. This corresponds to a quantum dynamical semigroup (QDS), with a time-independent Lindblad generator. More generally, we may allow $p$ to be any function that monotonically rises from 0 to 0.5. Note that we use the same $p$ for each of the Pauli channels, making our study and analysis interesting.

Consider the convex combination of two channels as

$$\mathcal{E}_s(p) = a \mathcal{E}_z^p + (1 - a) \mathcal{E}_y^p. \tag{2}$$

Recasting this as $(1 - p)I + ap\sigma_z \rho \sigma_z + (1 - a)p\sigma_y \rho \sigma_y$ shows that $\mathcal{E}_s(p)$ to be a generalized Pauli channel, where the $y$ and $z$ fractions in the mixture remain constant under the evolution. To check the non-Markovianity of the channel $\mathcal{E}_s$ according to the RHP criterion [6], we consider the intermediate map $\mathcal{E}_s(q,p)$ defined by $\mathcal{E}_s(q) = \mathcal{E}_s(q,p)\mathcal{E}_s(p)$, with $p \leq q < \frac{1}{2}$ and $a \in (0, 1)$. For this, we use the $A$-matrix representation of the map following [4, 5]. The $A$-matrix acts on the density matrix expressed as a column vector. The $A$-matrix for the intermediate map is therefore readily
obtained by $A_\ast(a,q,p) = A_\ast(a,q)A_\ast(a,p)^{-1}$. By re-arranging the entries of $A_\ast(a,q,p)$, one obtains the dynamical (or, Choi [18]) matrix

$$
\mathfrak{B}_\ast(a,q,p) = \frac{1}{2} \begin{pmatrix}
1 + x_3 & 0 & 0 & x_1 + x_2 \\
0 & 1 - x_3 & x_1 - x_2 & 0 \\
0 & x_1 - x_2 & 1 - x_3 & 0 \\
x_1 + x_2 & 0 & 0 & 1 + x_3
\end{pmatrix},
$$

(3)

with $x_1 = \frac{1 - 2q}{1 - 2p}$, $x_2 = \frac{1 - 2a}{1 - 2ap}$ and $x_3 = \frac{2a - 2q + 1}{2ap - 2p + 1}$. For complete positivity of the dynamical matrix Eq. (3), and hence the Markovianity of $\mathcal{E}_\ast$, all eigenvalues of $\mathfrak{B}_\ast(a,q,p)$ must be positive. The conditions for that can be evaluated to be $|1 \pm x_3| \geq |x_1 \pm x_2|$. For $\mathcal{E}_\ast(p)$, the corresponding dynamical matrix is of the form of Eq. (3), with $x_1 = 1 - 2p$, $x_2 = 1 - 2ap$ and $x_3 = 2ap - 2p + 1$. One can easily see that the map $\mathcal{E}_\ast(p)$ is CP irrespective of $a$. However, the intermediate map is not-completely positive (NCP) (indicative of a negative eigenvalue for $\mathfrak{B}_\ast(a,q,p)$) and hence non-Markovian for all $a \in (0,1)$. For instance, consider $\mathfrak{B}_\ast(0.1,0.45,0.4)$, which has a negative eigenvalue, $\approx -0.0839$, indicative of the NCP nature of the intermediate map and hence non-Markovianity. Thus, any non-zero mixing of two Pauli channels produces non-Markovianity, giving the following result, which generalizes the result of [12] (corresponding to $a = \frac{1}{2}$).

**Theorem 1.** Any finite degree of mixing of two Markovian (QDS) Pauli channels results in non-Markovianity.

For the case of $a = 1$, the channel is Markovian corresponding to the two positive eigenvalues $1 \pm \nu$ where $\nu = \frac{1 - 2q}{1 - 2p}$. Theorem 1 can be proven by an argument based on generators, as discussed below.

**Proof.** The time-local generator $L$ of a channel $\mathcal{E}$, is defined by $\dot{\mathcal{E}} = LE$. For the channel, Eq. (2), the time-local generator is found to be

$$
L = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{2p}{2p - 1} & 0 & 0 \\
0 & 0 & \frac{2ap}{2ap - 1} & 0 \\
0 & 0 & 0 & \frac{2(a-1)p}{2(a-1)p+1}
\end{pmatrix},
$$

(4)

Using the time-local generator, Eq. (4), the differential form of the channel can be written as

$$
L(\rho) = \sum_{k=X,Y,Z} \gamma_k (\sigma_k \rho \sigma_k - \rho),
$$

(5)

where $\gamma_k$’s are the decay rates. Note that we work in the Pauli basis $\{1, \sigma_i\}$. The decay rate, $\gamma_X$ turns out to be

$$
\gamma_X = -\left[\frac{(1-a)a(1-p)p}{(1-2p)(1-2(1-a)p)(1-2ap)}\right] \dot{\rho}.
$$

(6)
For any choice of $a \in (0, 1)$, it is clear that $\gamma_X$ is always negative, since expression with the square brackets and $\dot{p}$ in the RHS of this equation are positive. This implies that the mixing of any two Markovian Pauli channels produces a channel which is non-Markovian. A similar result follows for the other convex combinations of any other pair of Pauli channels as well.

III. CONVEX COMBINATION OF THREE CHANNELS

Here, we consider the simplex obtained by arbitrary convex combinations of the three Pauli channels, which are assumed to have a QDS form and the same decay rate $c$. A general three-way mixture is described by

$$\tilde{E}_x(p) = aE^p_x + bE^p_y + cE^p_z,$$

with $a + b + c = 1$ and $E^p_x(\rho) = (1 - p)\rho + p\sigma_x\rho\sigma_x$. We shall call this the Pauli simplex. This is an equilateral triangle, whose vertices are QDS Pauli channels having a constant decay rate.

The time-local generator $L$ is given by $L = \tilde{E}E^{-1}$. $L$ is straightforwardly evaluated to be

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{2(b+c)\dot{p}}{1-2(b+c)p} & 0 & 0 \\
0 & 0 & -\frac{2(a+c)\dot{p}}{1-2(a+c)p} & 0 \\
0 & 0 & 0 & -\frac{2(a+b)\dot{p}}{1-2(a+b)p}
\end{pmatrix}.
$$

The differential form of the channel follows to be of the same form as in Eq. (5), with the decay rates being

$$
\gamma_X = \left(\frac{1 - b}{1 - 2(1 - b)p} + \frac{1 - c}{1 - 2(1 - c)p} - \frac{1 - a}{1 - 2(1 - a)p}\right)\dot{p}
$$

$$
\gamma_Y = \left(\frac{1 - a}{1 - 2(1 - a)p} + \frac{1 - c}{1 - 2(1 - c)p} - \frac{1 - b}{1 - 2(1 - b)p}\right)\dot{p}
$$

$$
\gamma_Z = \left(\frac{1 - a}{1 - 2(1 - a)p} + \frac{1 - b}{1 - 2(1 - b)p} - \frac{1 - c}{1 - 2(1 - c)p}\right)\dot{p}.
$$

For arbitrary choices of $a, b$ and $c$, it can be seen that in general one or more of the decay rates can become negative, implying the CP-indivisibility (and thus non-Markovian nature) of the channel. The following result holds.

**Theorem 2.** In the Pauli simplex, (a) neither the set of Markovian channels nor that of non-Markovian (CP-indivisible) maps is convex; (b) all channels are P-divisible, i.e. Markovian according to the distinguishability criterion.

**Proof.** The proof of (a) follows readily from Figure 1, derived in the following section. Here, the Markovian set is seen to constitute a curved-edge (horn) triangle within the Pauli simplex.
and having its vertex angles of 0 degrees. To prove (b), we note that in Eq. (9), the decay rate expressions have the form

\[
\begin{align*}
\gamma_X(a,b,c) &= -f(a,p) + f(b,p) + f(c,p) \\
\gamma_Y(a,b,c) &= f(a,p) - f(b,p) + f(c,p) \\
\gamma_Z(a,b,c) &= f(a,p) + f(b,p) - f(c,p),
\end{align*}
\]

where \( f(\alpha,p) \geq 0 \) for all \( p \in [0, \frac{1}{2}] \) and \( \alpha \in \{a,b,c\} \). The sum \( \gamma_i + \gamma_j, i, j = X, Y, Z, i \neq j \) is always positive, even though an individual rate may be negative. For example \( \gamma_X + \gamma_Y = 2f(c,p) \geq 0 \). This implies that the dynamics obtained by the mixing is \( P \)-divisible and in the qubit context, is equivalent to Markovianity according to the BLP distinguishability criterion [9].

Below, we determine the measure or area of (non-)Markovian maps in the Pauli simplex.

**IV. MEASURE OF (NON-)MARKOVIAN MAPS IN THE PAULI TRIANGLE**

The inherent 3-way symmetry in the problem helps simplify the analysis.

**Lemma 1.** If a channel obtained as a mixture of the three Pauli (QDS) channels is non-Markovian, then precisely one of the three rates \( \gamma_j \) is negative.

**Proof.** It follows from Eq. (10) at most one of the three decay rates can be negative. Next, note that \( \frac{df(\alpha,p)}{dp} = \frac{2(1-\alpha)^2}{(1-2(1-\alpha)p)^2} > 0 \) for all \( p, \alpha \). Thus, in a given rate \( \gamma_j \ (j \in \{X,Y,Z\}) \), two of the terms will produce a monotonic increase in rate whereas the negative term will produce a monotonic decrease. Further, all rate components \( \gamma_j \) start at a positive value. For example, \( \gamma_Y(a,b,1-a-b,p=0) = 2b \).

Now, suppose \( \gamma_X \) is negative in some region, it follows from the considerations of the preceding paragraph that there is a \( p_0 \) such that \( \gamma_X \) is negative for \( p \geq p_0 \) and positive otherwise. That is, for \( p > p_0 \), we have \( f(b,p) > f(a,p) + f(c,p) \), and it remains so for \( p \in [p_0, \frac{1}{2}] \). But, by that token, notice that \( \gamma_Y \) and \( \gamma_Z \) will always be positive throughout.

It follows from Lemma 1 that the regions \( R_X, R_Y \) and \( R_Z \) in the \( (a,b,c) \) parameter space, where \( \gamma_X, \gamma_Y \) and \( \gamma_Z \) turn negative, will be non-overlapping. In Figure 1, the (convex) regions of negative rates are indicated by the corresponding symbol.

**Theorem 3.** The measure of non-Markovian maps in the Pauli simplex is about 0.867.

**Proof.** As noted, by virtue of the monotony of \( f(\alpha,p) \), if a given rate (say) \( \gamma_Y(a,b,c,p) \) turns negative at \( p = p_0 \), then it remains negative throughout the remaining range of \( p \), and in particular at \( p = \frac{1}{2} \).
Thus, $\gamma_Y(a, b, p)$ will turn negative if and only if
\[
\gamma_Y(a, b, \frac{1}{2}) = \frac{a + b}{1 - a - b} + \frac{1 - a}{a} - \frac{1 - b}{b} \equiv \Gamma_Y(a, b) < 0. \tag{11}
\]
We wish to determine the set of all points $(a, b)$ that yield negative $\gamma_Y$ at $p = \frac{1}{2}$. To this end, we solve $\gamma_Y(a, b, \frac{1}{2} - x) = 0$ for $a$ in terms of $b$, which yields
\[
a_Y^\pm(b, x) = \frac{1}{2} \left( \pm \sqrt{\frac{(b - \frac{2x - 1}{2x - 1})(b - \frac{2x + 1}{2x - 1})}{4(b - 1)x^2 - 4bx + b + 1} - b + 1} \right). \tag{12}
\]
From this, one finds that for $b \in [0, \beta(x)]$, where $\beta(x) \equiv \frac{2 - \sqrt{4x^2 - 4x + 5}}{2x - 1}$ is the region for which $\gamma_Y(a, b, 0.5 - x) = 0$. In this range, points $(a, b)$ such that $a \in (a_Y^+(b, x), a_Y^-(b, x))$ (resp., $[0, a_Y^+(b, x)] \cup [a_Y^+(b, x), 1]$) represent those for which $\gamma_Y(a, b, 0.5 - x)$ is negative (resp. positive). Regions of $b > \beta(x)$ are those for which $\gamma_Y(a, b)$ is still positive for $p = 0.5 - x$. Thus, we fully determine $R_Y$, by setting $x := 0$. Accordingly:
\[
|R_Y| = 2 \times \int_{b=0}^{\beta(0)} \left( a_+(b, 0) - a_-(b, 0) \right) db = 2 \times \int_{b=0}^{2+\sqrt{5}} \frac{\sqrt{b^4 + 4b^2 - 2b^2 - 4b + 1}}{b + 1} db \approx 0.2898, \tag{13}
\]
where the pre-factor comes from the normalization $\int_{a=0}^{1} \int_{b=0}^{1-a} da \, db = \frac{1}{2}$.

It follows from the fact that the three non-Markovian regions are non-overlapping that the measure of all non-Markovian channels in the Pauli simplex is $3|R_Y|$, is 0.867. Thus, the measure of Markovian maps in the Pauli triangle is about 0.133.

Eq. (12) is the equation of $\gamma_Y$ in the representation where the coordinates $(a, b)$ are used. The diagrammatic depiction of the Pauli simplex in the $(a, b)$ representation, and analogously in the $(a, c)$ or $(b, c)$ representation, is a right angle triangle. With a suitable linear transformation, one can shift to the “Pauli neutral” representation, which depicts the Pauli simplex as an equilateral triangle, as shown in Fig. 1.

The curved-edge (horn) triangle in Figure 1, colored in blue, encloses the non-convex Markovian region. The vertex angle for this triangle are all 0°. The convex region $\mathcal{R}_Z$, of area about 28.98% that of the Pauli simplex, represents the set of maps where $\gamma_Z$ alone turns negative after sufficiently long time; similarly for $\mathcal{R}_X$ and $\mathcal{R}_Z$. However, the union of these three non-Markovian regions is clearly not convex. Also, whilst the Markovian region is a connected region, the non-Markovian region is not, being the union of three disjoint regions.
FIG. 1. (Color online) The Pauli simplex (outer triangle), whose vertices are the Pauli QDS channels, and represented in convex coordinates in the Pauli-neutral representation (see text). The curved-edge (horn) triangle, with its interior colored blue, represents the Markovian region, tapering to 0° at each vertex. The three convex regions marked $\mathcal{R}_j, (j \in \{X,Y,Z\})$ correspond to non-Markovian (CP-indivisible). The corresponding edges of the horn triangle are described by the equations $\gamma_X(p = \frac{1}{2}) = 0$, $\gamma_Y(p = \frac{1}{2}) = 0$ and $\gamma_Z(p = \frac{1}{2}) = 0$. The area of the horn triangle is about 0.87 of the Pauli simplex.

V. CONCLUSIONS

We have discussed the convex combination of two and three Markovian Pauli channels, taken to be quantum dynamical semigroups. For finite mixing of two channels, the resultant channel is non-Markovian (CP-indivisible) throughout. However, the case of mixing the three Markovian Pauli channels may produce a Markovian or non-Markovian channel. We characterize the Pauli simplex, which is the set of all possible mixtures of the three Pauli channels. Neither the Markovian nor non-Markovian regions is convex. The measure of the non-Markovian region in the Pauli simplex is 0.87. This means that if the three channels are mixed in a random proportion, then the probability that the resulting channel will be non-Markovian is about 0.87.

In any resource theory, the set of free states by definition forms a convex set. In a recently formulated resource theory of quantum non-Markovianity [16, 17], the set of generators of CP-divisible channels (represented as the Choi matrices of the corresponding infinitesimal map) act as free states in the resource theory. Our result above implies that at the level of finite-time channels, the set of Markovian (CP-divisible) channels do not form a convex set. Thus one cannot have a resource theory of quantum non-Markovian channels where finite-time Markovian channels correspond to free states.
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[1] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, 2002).
[2] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Rev. Mod. Phys. 88, 021002 (2016).
[3] L. Li, M. J. W. Hall, and H. M. Wiseman, Phys. Rep. 759, 1 (2018).
[4] E. C. G. Sudarshan, P. M. Mathews, and J. Rau, Phys. Rev. 121, 920 (1961).
[5] V. Jagadish and F. Petruccione, Quanta 7, 54 (2018).
[6] A. Rivas, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. 105, 050403 (2010).
[7] M. J. W. Hall, J. D. Cresser, L. Li, and E. Andersson, Phys. Rev. A 89, 042120 (2014).
[8] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett. 103, 210401 (2009).
[9] D. Chruściński and F. A. Wudarski, Phys. Rev. A 91, 012104 (2015).
[10] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, Phys. Rev. Lett. 101, 150402 (2008).
[11] H.-P. Breuer, G. Amato, and B. Vacchini, New J. Phys. 20, 043007 (2018).
[12] F. A. Wudarski and D. Chruściński, Phys. Rev. A 93, 042120 (2016).
[13] N. Megier, D. Chruściński, J. Piilo, and W. T. Strunz, Sci. Rep. 7, 1 (2017).
[14] S. A. Uriri, F. Wudarski, I. Sinayskiy, F. Petruccione, and M. S. Tame, arXiv:1908.08085 [quant-ph] (2019).
[15] I. Bengtsson and K. Życzkowski, Geometry of Quantum States an Introduction to Quantum Entanglement, 2nd ed. (Cambridge University Press, 2017).
[16] S. Bhattacharya, B. Bhattacharya, and A. S. Majumdar, arXiv:1803.06881 [quant-ph] (2018).
[17] N. Anand and T. A. Brun, arXiv:1903.03880 [quant-ph] (2019).
[18] M.-D. Choi, Linear Algebra Appl. 10, 285 (1975).