Interaction Effects of Magnetic Monopoles and Kinks in a Doped Dipolar Superlattice Gas

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Abstract—Quantum spin systems possess various topological excitations, such as the magnetic kink and spinon, the skyrmion, the Majorana mode, as well as the magnetic monopole. These quasiparticles possess rich magnetic properties and endow the spin systems with potential applications, such as functional spintronic devices. The interaction between different quasiparticles is of particular importance, since it not only enriches the dynamical properties of the spin systems, but also provides efficient manipulation tools for corresponding applications. The coexistence and interaction effects of these quasiparticles in the pseudospin chain of the atomic dipolar superlattice gas. In this chain, the magnetic kink is the intrinsic quasiparticle, and the particle/hole defect takes over the role of the north/south magnetic monopole, exerting monopolar magnetic fields to neighboring spins. A confinement effect between the monopole and kink is revealed, which renormalizes the dispersion of the kink. The corresponding dynamical deconfinement process is observed and arises due to the kink-antikink annihilation. The rich interaction effects of the two quasiparticles could stimulate corresponding investigations in bulk spin systems.

Introduction.—Quantum spin systems possess various topological excitations, such as the magnetic kink \cite{1,2}, the spinon \cite{3,4}, the skyrmion \cite{5}, the Majorana mode \cite{6}, as well as the magnetic monopole \cite{7,8,9}. These quasiparticles possess rich magnetic properties and endow the spin systems with potential applications, such as functional spintronic devices \cite{10,11,12}. The interaction between different quasiparticles is of particular importance, since it not only enriches the dynamical properties of the spin systems, but also provides efficient manipulation tools for corresponding applications. The coexistence and interaction effects of these quasiparticles in the pseudospin chain of the atomic dipolar superlattice gas. In this chain, the magnetic kink is the intrinsic quasiparticle, and the particle/hole defect takes over the role of the north/south magnetic monopole, exerting monopolar magnetic fields to neighboring spins. A confinement effect between the monopole and kink is revealed, which renormalizes the dispersion of the kink. The corresponding dynamical deconfinement process is observed and arises due to the kink-antikink annihilation. The rich interaction effects of the two quasiparticles could stimulate corresponding investigations in bulk spin systems.

Ultracold atomic gases have become one of the major platforms for quantum simulation \cite{23,24,25,26,27,28}, thanks to the high degrees of freedom of the atomic gas to construct the target Hilbert space and the tunability to engineer the demanded Hamiltonians. Concerning quantum simulation of spin systems, the target spin degree of freedom can be modeled by the atomic species \cite{29,30,31,32,33,34,35}, the atomic internal states \cite{29,36,37,38,39}, as well as the spatial modes of lattice atoms, such as the occupation states in tilted lattices \cite{40,41,42} or superlattices \cite{43,44,45,46,47}. The effective interaction such as spin-spin interactions and the spin-orbital couplings \cite{48,49,50,51} have been extensively investigated, whereas the coexistence condition and the interaction properties between the magnetic kink and the monopole are still open questions. It is then of potential interest to address this open question in real spin systems and/or in pseudospin systems emulated with e.g. ultracold atomic gases.

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The current quantum simulation schemes mostly embed the monopole in the superfluid phase. In this letter we propose an alternative simulation scheme, which places the monopole in the spin chain and allows the investigation of the coexistence and interactions of monopoles with other quasiparticles, e.g. magnetic kinks. Our simulation scheme adapts and generalizes the pseudospin mapping of ultracold atoms in the double-well superlattice, which has been exploited to simulate the spin-orbital coupling \cite{52} and the corresponding supersolid-like phase \cite{53}, as well as the magnetic phase transitions \cite{54} and quasiparticles \cite{55}. This scheme shares a close similarity with that based on spin ices \cite{7,8}. In both setups, monopoles are manifested as defects surrounded by monopolar polarized spins, and the Dirac string is modelled by the spin flipping along the hopping path of monopoles. Comparing to the monopole in spin ices, the present scheme possesses some unique features: On the one hand, the monopoles can be not only excited by the pair production, but also by particle/hole doping, which provides more direct access to a single monopole. On the other hand, the monopole in our scheme is generated on top of a two-domain ferromagnetic background with an intrinsic kink, which enables the investigation of the coexistence and interaction effects between these quasiparticles.

Setup and pseudospin mapping.—We consider the dipolar superlattice gas (DSG) of spin polarized fermions confined in the one-dimensional double-well superlattice, in which the fermions interact with each other through the repulsive dipole-dipole interaction (DDI) \cite{56}. The DSG system can be described by the following Fermi-Hubbard Hamiltonian:

$$H_{\text{FH}} = -J \sum_{i=1}^{M} (\hat{f}_{i}^{\dagger} \hat{f}_{i+1}^{\dagger} + \text{H.c.}) - J_{1} \sum_{i=1}^{M-1} (\hat{f}_{i}^{\dagger} \hat{f}_{i+1}^{\dagger} + \text{H.c.}) + \sum_{i<j \in [1,2M]} V_{d}(j-i) \hat{n}_{i} \hat{n}_{j}$$

where \(\hat{f}_{i}^{\dagger} \hat{f}_{i}^{\dagger}\) and \(\hat{f}_{2i-1}^{\dagger} \hat{f}_{2i}^{\dagger}\) is the fermionic creation (annihilation) operator at the left/right site of the \(i\)-th supercell, and the operator \(\hat{n}_{i} = \hat{f}_{i}^{\dagger} \hat{f}_{i}\) counts the number of fermions at site \(i\). The
FIG. 1. (color online). (a) The pseudospin chain based on the DSG system. The NM and SM are sketched with the dark and light blue balls, respectively. The transparent arrows of orange, dark and light blue color refer to the ABM and the monopolar magnetic fields around the NM and SM, respectively. (b) The polarization of neighboring spins around a localized NM, in terms of \( \langle \hat{\sigma}_z \rangle \) (solid lines) and \( \langle \hat{\sigma}_x \rangle \) (dashed lines). The spin polarization is also explicitly shown at the bottom. (c) The dynamical process of the pair excitation, the tunneling of the NM and SM, as well as the spin flipping along the tunneling are shown.

The pseudospin mapping transfers the DSG system to an effective spin chain, with the Hamiltonian \( \hat{H}_{\text{spin}} = \hat{H}_0 + \hat{H}_{\text{SD}} \):

\[
\hat{H}_0 = -J \sum_{\alpha} \hat{\sigma}_z^\alpha - \frac{d}{4} \sum_{a=1}^{M-1} \hat{\sigma}_z^\alpha \hat{\sigma}_z^{\alpha+1} + \frac{d}{2} \sum_{\alpha} \hat{\sigma}_z^\alpha
\]

\[
+ \frac{d}{4} \left( \hat{\sigma}_z^1 - \hat{\sigma}_z^M \right) + \frac{d}{4} \left( \hat{\sigma}_z^M - \hat{\sigma}_z^1 \right),
\]

\[
\hat{H}_{\text{SD}} = \frac{d}{4} \sum_{\alpha} \hat{\sigma}_z^\alpha \left( \hat{\sigma}_z^{\alpha-1} - \hat{\sigma}_z^{\alpha+1} \right)
\]

\[
- J_1 \sum_{\alpha} \left[ \left( \hat{\sigma}_z^\alpha \hat{H}_{\text{SD}} + H.c. \right) + \left( \hat{\sigma}_z^\alpha \hat{P}_{\text{SM}} \hat{\sigma}_z^{\alpha+1} + H.c. \right) \right]
\]

In \( \hat{H}_0 \), \( \hat{\sigma}_x^\alpha \) and \( \hat{\sigma}_z^\alpha \) are the Pauli operators exerted to the pseudospins, and \( \hat{\sigma}_x^\alpha = |P_{\alpha} \rangle \langle P| - |H_{\alpha} \rangle \langle H| \) is the effective Pauli operator exerted to the defects. To describe the counter-propagation of the defect and pseudospin, the exchanging operator is introduced in \( \hat{H}_{\text{SD}} \), defined as \( \hat{\sigma}_z^\alpha \equiv |\Lambda \rangle_\alpha \langle \sigma| \), with \( \Lambda \in \{P, H\} \) and \( \sigma \in \{\to, \leftarrow\} \). In the following, we term the effective spin chain as the DSG pseudospin chain.

The DSG pseudospin chain is manifested as a transverse Ising spin chain, of which the pseudospins interact by the Ising-type spin-spin interaction and are subjected to a transverse magnetic field, as indicated by the first two terms of \( \hat{H}_0 \). The fourth term of \( \hat{H}_0 \) refers to the antiparallel boundary magnetic (ABM) field localized at the two edges of the chain. The ABM field has been recognized as an efficient way to excite kink since the original studies on the magnetic kinks \([1, 48, 49]\) and its gives rises to an intrinsic kink in the DSG pseudospin chain. More interestingly, as indicated by the last term of \( \hat{H}_0 \), the ABM fields also exert an attractive/repelling potential to the hole/particle defect along the direction of the field, which mimics the response of the magnetic south/north monopole to the external magnetic field.

Besides the response to the magnetic field, further fingerprints of the magnetic monopole, i.e. the monopolar magnetic field and the Dirac string are also reproduced by the particle and hole defects, which enables the particle and hole defects well resemble the north monopole (NM) and south monopole (SM), respectively. The monopolar magnetic field is normally evidenced by the spin texture around the monopole, and as indicated by the first term of \( \hat{H}_{\text{SD}} \), the particle (hole) defect polarizes the neighbor spins away from (towards) the defect, which resembles the monopolar magnetic field surrounding the NM (SM). The second term of \( \hat{H}_{\text{SD}} \) further demonstrates that the hopping of the defects is accompanied by the flipping of the counter-propagating spin, which has been recognized as a signature of the Dirac string for the monopoles \([7, 10]\). The last term in \( \hat{H}_{\text{SD}} \) refers to the pair production of an NM and a SM, which manifests as the main excitation channel of the monopoles in spin ices.
the 14-sites pseudospin chain with \(d\) = 40\(J\). The frequency interval of \(\omega \in (0.5, 36)\) is removed, where the gap between the first two bands lies. The representative basis states contributed to each band are shown to the right of the structure factor figure, in which different ferromagnetic domains are emphasized with solid-line steps and the (anti)kinks locate at the edges of the steps. (b) The NM-kink correlation for the ground state. (c) The three-body correlation \(C_3\) (blue) and four-body correlation \(C_4\) (yellow) of the second band.

In Fig. 1(a) the DSG pseudospin chain is sketched, in which the ABM and the monopolar magnetic field around the NM and SM is schematically shown. Figure 1(b) shows the expectation values of \(\langle \hat{\sigma}_x \rangle\) and \(\langle \hat{\sigma}_z \rangle\) of the pseudospins around a localized NM in the paramagnetic phase, in which the transverse magnetic field aligns the pseudospins to the \(y\)-direction. It can be seen that, the pseudospins far away from the NM aligns along the transverse direction, while the neighboring spins to the NM are polarized away from the NM, as indicated by the spin texture at the bottom of the figure. Figure 1(c) summarizes the pair production and spin-flipping effects of \(\hat{H}_SD\) with a dynamical process that initially a pair of monopoles are excited and then hop away from each other, accompanied by the spin flipping. The detailed sketch of the pseudospin mapping and the comparison between the spin polarization around the NM, SM and a normal magnetic defect in the pseudospin chain are given in the supplemental material.

Interaction effects between NM and kink.—In the strong interaction regime, the DSG pseudospin chain sustains the coexistence of the magnetic monopole and kinks, and provides an ideal platform to investigate the interplay of the two quasiparticles. Here, we focus on the doping of a single NM defect to the DSG chain, and the results can be straightforwardly generalized to the SM doping. We define the tail-to-tail and head-to-head kink as the kink and antikink, respectively. The Hilbert space is truncated to the subspace spanned by the basis states \(|n\rangle_{NM} \otimes |\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots , \tilde{\alpha}_{2N}\rangle\), in which \(|n\rangle_{NM}\) denotes the position of the monopole and \(|\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots , \tilde{\alpha}_{2N}\rangle = |\cdots \leftarrow \tilde{\alpha}_0 \rightarrow \cdots \tilde{\alpha}_1 \rightarrow \cdots \leftarrow \tilde{\alpha}_{2N} \rightarrow \cdots \rangle\) indicates the location of the (anti)kinks in the squeezed space where the monopole site is removed \([26, 43]\). (more details of the definition of the squeezed space is given in the supplemental material). Accordingly, the Hamiltonian can be spanned in the monopole-kink subspace as: \(\hat{H}_{\text{doped-spin}} = \hat{H}_K + \hat{H}_{\text{NM-K}}\), in which

\[
\hat{H}_K = \sum_{\alpha=1}^{M} \hat{\rho}_A^\dagger \hat{\rho}_A - J \sum_{\alpha=1}^{M-1} \left( \hat{S}_\alpha^+ + \hat{S}_\alpha^- \right) \left( \hat{S}_{\alpha+1}^+ + \hat{S}_{\alpha+1}^- \right),
\]

(4)

\[
\hat{H}_{\text{NM-K}} = -d \sum_{\alpha=\tilde{\alpha}}^{M} \hat{\rho}_K^\dagger \hat{\rho}_K - J_1 \sum_{\alpha=\tilde{\alpha}}^{M-1} \left( \hat{\rho}_N^\dagger \hat{\rho}_N \hat{S}_\alpha^+ \hat{S}_{\alpha+1}^- + H.c. \right),
\]

(5)

In \(\hat{H}_K\), \(\hat{\rho}_A^\dagger\) refers to the number of antikinks between sites \(\tilde{\alpha}\) and \(\tilde{\alpha} + 1\) in the squeezed space, and \(\hat{S}_\alpha^+ = (\hat{\sigma}_A^x - \hat{\sigma}_A^z) / (\hat{\sigma}_A^+ + \hat{\sigma}_A^-)\) combines the creation of a kink and the annihilation of a antikink. In \(\hat{H}_{\text{NM-K}}\), \(\hat{\rho}_K^\dagger (\hat{\rho}_K)\) denotes the creation (annihilation) of a NM on the \(\alpha\)-th site, with \(\hat{\rho}_N^\dagger = \hat{\rho}_N^\dagger \hat{\rho}_K^\dagger \hat{\rho}_N\). \(\hat{H}_{\text{NM-K}}\) then describes the interaction between the monopole and kinks, which includes the attractive interaction between a NM and a kink, and the effect of monopole hopping to the (anti)kink.

The interaction effects between the NM and the (anti)kinks can be captured by the dynamical structure factor \(S(k, \omega)\) \([1, 2]\), and the \(S(k, \omega)\) of the DSG pseudospin chain is shown in Fig. 2(a), which is calculated using the multi-layer multi-configuration time-dependent Hartree method for arbitrary bosonic (fermionic) mixtures \([44-48]\). (for more details see supplemental material). In Fig. 2(a), a single-mode branch appears in the first band, and gives a strong hint that that the doped NM and the intrinsic kink are bound and behave as a single composite quasiparticle. The emergence of the NM-kink bound state can be confirmed by the NM-kink correlation \(\langle \psi | \hat{\rho}_K^\dagger \hat{\rho}_K^\dagger \hat{\rho}_N^\dagger \hat{\rho}_N^\dagger | \psi \rangle \) running through all eigenstates in the first band. Fig. 2(b) shows the NM-kink correlation for an arbitrary eigenstate in the first band, and it clearly demonstrate that the NM and the intrinsic kink always occupy the same site. The NM-kink correlations for the other eigenstates in the first band all present the same bound behavior, though not shown here.

The second band in Fig. 2(a) presents a broad spectrum, leading to a continuum band in the infinite-long chain limit. It is known that in the absence of the NM, the antikink-kink pair excitation dominates the excitation from the ground to higher bands, which leads to continuum excited bands in the infinite-chain limit. An analysis via the multi-particle correlations, however, reveals that the presence of a NM not only preserves the excitation channel of the antikink-kink pair excitation, but also brings in a new channel contributing to the second
The multi-particle correlations have become powerful and experimentally accessible tools to identify quasiparticle excitation in the ultracold-atom simulated pseudospin chains [51], and here we determine the four-body correlation \( C_4 = \sum_{\alpha, \beta, \gamma} \langle \hat{\eta}_{\alpha}^{\alpha} \hat{\eta}_{\beta}^{\beta} \hat{\eta}_{\gamma}^{\gamma} \rangle \) and the three-body correlation \( C_3 = \sum_{\alpha, \beta} \left( 1 - \hat{\eta}_{\alpha}^{\alpha} \hat{\eta}_{\beta}^{\beta} \right) \hat{\eta}_{\alpha}^{\alpha} \hat{\eta}_{\beta}^{\beta} \hat{\eta}_{\alpha}^{\alpha} \) to identify the excitation channels from the first to the second band, where \( \alpha, \beta \) and \( \gamma \) run over all sites in the (squeezed) chain with \( \alpha \neq \beta \). \( C_4 \) and \( C_3 \) signify the antikink-kink pair excitation in the presence of the bound NM-kink, and the deconfinement of the bound NM-kink, respectively. The non-vanishing correlations \( C_4 \) and \( C_3 \) for the eigenstates in the second band, shown in Fig. 2(c), demonstrate that both excitation channels contribute to the second band, and also suggest that the excitation of an antikink-kink pair can be transformed to the deconfinement state of the NM-kink bond, which has potential applications for manipulation of the monopoles and kinks. In Fig. 2, the boxes to the right of Fig. 2(a) sketch the dominant contributions to the first two bands, and from bottom up they are the NM-kink bound state, the free pair of an NM and a kink and the coexistence of the bound NM-kink with the antikink-kink pair.

It is well known that composite quasiparticles composed of two types of particles, such as polarons [49, 50], can renormalize the dispersion and mobility of the bare particles and provides a unique control tool. The NM-kink bound state also shares this renormalization effect to the bare kink. As shown in Fig. 3, the dispersion of the bound state is significantly changed from that of the bare kink and can be tuned by the mobility of the NM. Furthermore, to verify the manipulation of the bound NM-kink by the antikink-kink pair excitation, we determine the dynamical process with the initial state that a bound NM-kink and antikink-kink pair are located at the left and right edges of the pseudospin chain. The temporal evolutions of \( C_3(t) \) and \( C_4(t) \) as shown in Fig. 4(a) indicate that, in the beginning of the dynamics the system is dominated by the coexistence of the bound NM-kink and the antikink-kink pairs, whereas for later times as marked by the grey vertical lines in the figure, the deconfinement of the bound NM-kink takes place, accompanied with the disappearance of the coexistence of the NM-kink and antikink-kink pairs. This confirms that the antikink-kink pair excitation can induce the deconfinement through the annihilation of the antikink with the kink bound to the NM. In Fig. 4(b)-(d), we also show the spatial densities of the NM, kink and antikink at the beginning and later times marked in Fig. 4(a). The densities clearly show that in the beginning the NM-kink and antikink-kink pairs are separately located on opposite edges, and at later times the NM and kink become deconfined with the antikink almost vanished. The spatial densities further verify the deconfinement of the bound NM-kink by the antikink-kink annihilation process. The renormalization effect and the manipulation of the NM-kink bound state with the antikink suggests rich interaction effects between the NM and the (anti)kinks, and provides a potential control manner of the kink by monopoles.

**Summary and discussion.**—Dipolar quantum gases, composed of ultracold polar atoms [52, 53]. Rydberg atoms [54, 55] or polar molecules [56, 57] have become an important platform for quantum simulation. In particular, rich spin models can be realized and simulated based on this platform. In this letter, we propose a new quantum simulation scheme for magnetic monopoles employing ultracold dipolar atoms confined in the one-dimensional superlattice, and this scheme allows for the investigation of interaction effects between the magnetic monopole and kinks. Our scheme can be straightforwardly generalized to simulate monopoles in more complex spin lattices, such as the square [58] and triangular lattice [59, 60], by applying two-dimensional double-well superlattices. It is also expected that the simulation based on ultracold quantum gases would stimulate related investigations in (artificial) spin lattices [61, 62].

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SUPPLEMENTARY MATERIAL

S-I. THE PSEUDOSPIN MAPPING

In this section, a more visualizable demonstration of the pseudospin mapping is provided in Fig. S1. As introduced in the main text, there are four single-cell occupation states \{0, 1\}, \{0, 1\}, \{0, 1\}, \{0, 0\}, each of which is mapped to a spin state and/or defect state. The mapping of the four occupation states to the corresponding spin and/or defect states is now given in Fig. S1(a). Following the pseudospin mapping, the whole superlattice loaded with atoms can be mapped to a spin chain with doped NM and SM, and the mapping between the superlattice of a particular atom filling configuration and the corresponding doped spin chain is shown in the Fig. S1(b).

![Pseudospin Mapping](image)

**FIG. S1.** Illustration of the pseudospin mapping. (a) The occupation states of arbitrary cell are mapped to the pseudospin/defect states. (b) The original DSG system (upper panel) and the effective doped pseudospin chain (lower panel).

S-II. THE POLARIZATION EFFECT OF THE MONOPOLE

Here, we provide more calculation results on the spin polarization effect of the NM and SM, which are compared to the spin polarization induced by a normal magnetic defect. To accomplish this, we consider a transverse Ising spin chain doped with a defect localized at the middle of the chain. Without the doping, the spins in the chain are all aligned to the x-direction by the transverse magnetic field, and the spin chain resides in the paramagnetic phase. The doped defect can interact with its neighboring spins, and polarize these spins to a new direction. Defects of different types can result in very different spin textures of the neighboring spins. Here we separately consider three types of defects, namely the NM, SM and a normal magnetic defect, and compare the spin textures from these defects. We model the normal magnetic defect as a particle of 1/2 spin, and the spin state of the defect is fixed to |↑⟩_{z}.

In our study, the Hamiltonian is taken as \( \hat{H} = \hat{H}_0 + \hat{H}_{\text{defect}} \), in which \( \hat{H}_0 \) refers to that of the transverse Ising spin chain, as introduced in the main text. \( \hat{H}_{\text{defect}} \) corresponds to the spin-defect interaction, with \( a = \text{NM, SM and normal} \), denoting the NM, SM and normal magnetic defects. The different spin-defect interactions read:

\[
\hat{H}_{\text{defect}}^{\text{NM}} = \frac{d}{4} \sum_{\alpha} \hat{h}_\alpha^{\text{NM}} (\hat{\sigma}^{-1}_{\alpha} - \hat{\sigma}^{1+}_{\alpha}), \tag{S1a}
\]

\[
\hat{H}_{\text{defect}}^{\text{SM}} = \frac{d}{4} \sum_{\alpha} \hat{h}_\alpha^{\text{SM}} (-\hat{\sigma}^{-1}_{\alpha} + \hat{\sigma}^{1+}_{\alpha}), \tag{S1b}
\]

\[
\hat{H}_{\text{defect}}^{\text{normal}} = \frac{d}{4} \sum_{\alpha} \hat{h}_\alpha^{\text{normal}} (-\hat{\sigma}^{-1}_{\alpha} - \hat{\sigma}^{1+}_{\alpha}). \tag{S1c}
\]

\( \hat{H}_{\text{defect}}^{\text{normal}} \) is taken from the situation of, for instance, doping magnetic atoms to electron gases, where the doped atom plays the role of the normal magnetic defect. The results are shown in Fig. S2, and Fig. S2(a) shows the spin polarization of the undoped DSG spin chain in the weak interaction regime, which correspond to the paramagnetic phase. In Fig. S2 (b) and (c), we present the results for the NM and SM defects. Here we observe that the neighboring spins are pointing away from (towards) the NM (SM) defect. Figure S2(d) provides a comparison to the normal magnetic defect, which polarizes the neighbor spins on its left- and right-hand side to the same direction.

S-III. THE SQUEEZED SPACE

In this section, we use Fig. S3 to give a more explicit demonstration of the basis defined in the squeezed space. Fig S3(a) shows one of the basis states of the pseudospin chain with nine spins labeled by \( \alpha = 1 \sim 9 \). There are two kinks and an antikink locate between site (2, 3), (8, 9) and (6, 7). The basis with a doped particle-defect at the 5-th site is shown in Fig. S3(b). Subsequently, we remove the particle-defect from the pseudospin chain and the left spins (Fig. S3(c)) form the squeezed space. The spins on the right side of particle-defect are squeezed forward. The left spins in squeezed space are relabeled by ˜\( \alpha \) = 1 \sim 8, where \( \tilde{\alpha} = \alpha \) and \( \tilde{\alpha} = \alpha + 1 \) for \( \alpha < 5 \) and \( \alpha > 5 \), respectively. In the squeezed space, the kinks and antikink are located between sites (2, 3), (7, 8) and (5, 6). The falling and rising edges of the purple solid line in Fig. S3(d) indicate the positions of kinks and antikink, respectively.

S-IV. THE ARPES CALCULATION

In this section, we present two approaches to calculate the dynamical structure factor. The first approach is based on [1], in which \( S(k, \omega) \) is determined using the Fourier transformation of the time-dependent correlation function. The second approach, following the proposal in [2], couples the DSG system to a particle bath and applies a periodic variation of the coupling strength. Then \( S(k, \omega) \) is extracted from the dynamical process under the periodic driving, which resembles the
Angle-Resolved Photo Emission Spectroscopy (ARPES) signal. The two approaches give qualitatively the same results, and the main difference arises from the quantitative weights of each mode. Given the comparison between the two approaches, we show the results of the first approach in Fig. 2(a) of the main text.

Firstly, we provide the definition of the single-particle spectral function, which is the imaginary part of the single-particle retarded Green’s function. The single-particle retarded Green’s function $G^R(k, t)$ in Lehmann representation at zero temperature reads [3]:

$$G^R(k, \omega) = \sum_{n, \sigma} \left\{ \frac{|\langle \psi_n | \hat{a}^\dagger_{k, \sigma} | \phi_0 \rangle|^2}{\omega + \epsilon_n - \epsilon_0 + i\eta} + \frac{|\langle \psi_\sigma | \hat{a}_{k, \sigma} | \phi_0 \rangle|^2}{\omega - \epsilon_n + \epsilon_0 + i\eta} \right\},$$  

(S2)

where $\hat{a}^\dagger_{k, \sigma} = \sqrt{2/((M+1)\sum_{j=1}^M \sin|k \cdot j/(M+1)|} \hat{a}^\dagger_{k, \sigma}$ creates a fermion with momentum $k$ and spin $\sigma$. $|\psi_0\rangle$ is the ground state of the system without doping and $|\psi_n\rangle$ is the $n$-th eigenstate of the system with single particle doping, while $\epsilon_0$ and $\epsilon_n$ are their energy. $\eta \in \mathbb{R}^+$ and we consider the limit $\eta \rightarrow 0^+$. We focus on the single-particle excitations, and we therefore have $|\langle \psi_n | \hat{a}^\dagger_{k, \sigma} \phi_0 \rangle|^2 = 0$. The single-particle spectral function $S^G(k, \omega)$ takes on the following appearance:

$$S^G(k, \omega) = -\frac{1}{\pi} \text{Im} \{G^R(k, \omega)\} = \sum_{n, \sigma} \left| \langle \psi_n | \hat{a}^\dagger_{k, \sigma} | \phi_0 \rangle \right|^2 \delta(\omega + \epsilon_n - \epsilon_0),$$

(S3)

and the density of states (DOS) is defined as $\rho^G(\omega) = \int dk S^G(k, \omega)$.

### Approach-1

Following Eq. [A1] in Ref [1], the approach firstly determines the spin-spin time-dependent correlation function, which is defined as:

$$C(i, t) = \sum_{\sigma} \left\langle \phi_0 \left| e^{iHt} \hat{a}_{i, \sigma} e^{-iHt} \hat{a}^\dagger_{i, \sigma} \right| \phi_0 \right\rangle,$$

(S4)

where $\hat{a}_{i, \sigma} = \hat{a}^\dagger_{i, \sigma} \hat{a}_{i, \sigma}$ is the fermionic creation (annihilation) operator at the right/left site of the $i$-th supercell. Then the spatial Fourier transform is performed and gives rise to

$$A(k, t) = \sqrt{\frac{2}{M+1}} \sum_{i} \sin \left( \frac{i \cdot k}{M+1} \right) C(i, t).$$

(S5)

Finally, the dynamical structure factor $S^F(k, \omega)$ is obtained by the Fourier transformation in time:

$$S^F(k, \omega) = \frac{1}{2\pi} \int_{-\infty}^\infty dt A(k, t).$$

(S6)
and the DOS is given by \( \rho^F(\omega) = \int dk S^F(k, \omega) \)

**Approach-2**

The second approach involves the simulation of the ARPES process. In the simulation of ARPES, the undoped DSG system is firstly connected to a bath system, which allows particle hopping between the DSG chain and the bath. A periodic shaking is then applied to the coupling strength, with a particular shaking frequency. During the dynamical process under the shaking, particles can tunnel from the bath to the DSG chain, corresponding to the doping of a particle to the DSG system. The structure factor is then associated with the momentum distribution of the hole in the bath at the particular driving frequency. We illustrate our simulation setup of this ARPES process as follows: Firstly, we consider a (DSG+Bath) system as in Fig. S4, in which the bath is a lattice of atoms in the Mott state. The lattice of the bath system has the same period with the double-well superlattice but there is a single site per cell. We load the same spin polarized fermions into the DSG system and the bath system, while the DSG and bath system is half and unit filling, respectively.

The Hamiltonian of the (DSG+Bath) system reads:

\[
\hat{H}_{\text{DSG+bath}} = \hat{H}_{\text{DSG}} + \hat{H}_{\text{bath}} + \hat{H}_{\text{intra}},
\]

\[
\hat{H}_{\text{DSG}} = -J \sum_{i=1}^{M} \left( \hat{c}_{i}^{\dagger} \hat{c}_{i-1} + \text{H.c.} \right) - J_1 \sum_{i=1}^{M-1} \left( \hat{f}_{i}^{\dagger} \hat{f}_{i+1} + \text{H.c.} \right)
\]

\[
+ \sum_{i<j \in \{1,2,M\}} V_d(j-i) \hat{n}_i \hat{n}_j,
\]

\[
\hat{H}_{\text{bath}} = -J_0 \sum_{i=1}^{M} \left( \hat{c}_{i}^{\dagger} \hat{c}_{i-1} + \text{H.c.} \right) + \Delta \sum_{i=1}^{M-1} \hat{n}_i,
\]

\[
\hat{H}_{\text{inter}} = -J_{\text{inter}} \sin(\omega S T) \sum_{i=1}^{M} \left[ \hat{c}_{i}^{\dagger} (\hat{f}_{i+1}^{\dagger} + \hat{f}_{i+1}) + \text{H.c.} \right].
\]

where \( \hat{f}_{i}^{\dagger} \) and \( \hat{f}_{i} \) are the fermionic creation/annihilation operator at the right/left site of the \( i \)-th supercell (cell) in the DSG (bath) system, and the operator \( \hat{n}_i = \hat{f}_i^{\dagger} \hat{f}_i \) (\( \hat{n}_i = \hat{c}_i^{\dagger} \hat{c}_i \)) counts the number of fermions at site \( i \) in the DSG (bath) system. The first two terms in \( \hat{H}_{\text{DSG}} \) describe the intra- and inter-cell hopping, respectively, with the hopping amplitudes \( J \) and \( J_1 \). \( \Delta \) is the offset of the bath relative to the DSG and the hopping amplitude of the fermions in the bath is \( J_0 \). The dipole-dipole interaction (DDI) between two fermions located at the \( i \)- and \( j \)-th site is taken as \( V_d(j-i) = d|x_j - x_i|^{3} \), where \( x_j \) (\( x_i \)) is the local minimum of the corresponding site, and \( d \) denotes the DDI strength. In the bath system, the fermions are well-separated from each other and we therefore ignore the DDI among them. The lattice modulation can be described by \( \hat{H}_{\text{inter}} \), \( J_{\text{inter}} \approx J_1 \) is the perturbation term.

\( \hat{H}_{\text{inter}} \) induces hopping of atoms from the bath to the DSG system, and the energy change of the DSG system due to the doping is \( \hbar \omega = E_{M+1} - E_{M} \) for single atom hopping. For a lattice modulation frequency \( \omega_S \), this is determined by the energy conservation:

\[
\hbar \omega = \hbar \omega_S - E^B(k) - \Delta,
\]

where \( E^B(k) = -2J_0 \cos(k) \) is the energy of the hole in the bath system. The offset \( \Delta \) is taken as \( 8d \) and \( 9d \) when we detect the dynamical structure factor of the first and second band, corresponding to the energy of a NM and the energy of one NM and antikink, respectively. The momentum and energy resolution spectrum function are obtained by detecting the momentum distribution of the hole for different \( \omega_S \).

The structure factor \( S^M(k,\omega_S) \) is determined from the reduced density matrix of the hole in the bath system as:

\[
S^M(k,\omega_S) = \frac{\hbar}{2\pi J_0^2} \Gamma(k,\omega_S),
\]

where \( \Gamma(k,\omega_S) = \frac{1}{M+1} \sum_{i,j} \sin(ik) \sin(jk) R_{\omega_S}(i,j) \) is the probability of creating a particle with momentum \( k \) and energy \( \hbar \omega_S \). \( R_{\omega_S}(i,j) = \text{tr}_{\text{DSG}} \left[ \hat{\psi}(t_{\omega_S}) \langle \hat{\psi}(t_{\omega_S}) \rangle \right] \) is the reduced density matrix of the hole in the bath, while \( \hat{\psi}(t_{\omega_S}) \) is the wavefunction of the complete system at \( t_{\omega_S} \). At \( t = t_{\omega_S} \), the probability of \( \text{tr} \left[ \hat{R}_{\omega_S}(i,j) \right] \) hole takes a maximum for a given shaking with frequency \( \omega_S \). The DOS is naturally defined as \( \rho^M(\omega) = \int dk S^M(k,\omega) \).

In the strong interaction regime, which is of interest here, the gap between the adjacent bands is much larger than the strength of shaking. As a result, we only need to take the resonant states into account. For a certain \( \omega_S \), after transforming to the rotating frame, performing a rotating wave approxima-
tion, one obtains the Hamiltonian:

\[
\hat{H}^{\text{RWA}} = \hat{H}_{\text{DSG}} + \hat{H}_{\text{bath}}^{\text{RWA}} + \hat{H}_{\text{inter}}^{\text{RWA}},
\]

where

\[
\hat{H}_{\text{DSG}} = -J \sum_{i=1}^{M} \left( \hat{f}_{2i}^\dagger \hat{f}_{2i-1} + \text{H.c.} \right) - J_1 \sum_{i=1}^{M-1} \left( \hat{f}_{2i}^\dagger \hat{f}_{2i+1} + \text{H.c.} \right)
\]

\[
+ \sum_{i=1}^{M-1} V_d (j-i) \hat{n}_i \hat{n}_j,
\]

\[
\hat{H}_{\text{bath}}^{\text{RWA}} = -J_0 \sum_{i=1}^{M} \left( \hat{c}_i^\dagger \hat{c}_{i-1} + \text{H.c.} \right) + (\Delta - \omega_S) \sum_{i=1}^{M} \hat{n}_i,
\]

\[
\hat{H}_{\text{inter}}^{\text{RWA}} = -\frac{J_\text{inter}}{2} \sum_{i=1}^{M} \left[ \hat{c}_i^\dagger \left( \hat{f}_{2i}^\dagger + \hat{f}_{2i} \right) + \text{H.c.} \right].
\]

Throughout our numerical calculation, we set \( J = 1 \) as the unit. The other parameters \( J_0 = J_1 = 0.1, d = 40, J_{\text{inter}} = 0.01 \) and the evolution time \( T = 1000 \).

The structure factor computed with the first and second approach is given in Fig. S5(a) and (b) with \( S^F(k, \omega) \) and \( S^M(k, \omega) \), respectively. We omit the internal of \( \omega \in (0.5, 35) \) as \( S^{F/M}(k, \omega) \sim 0 \), which is the gap between the first two bands. \( S^X(k, \omega) \) is normalized to unity \( \int d\omega S^X(k, \omega) = 1 \), with \( X = F, M, G \). \( S^F(k, \omega) \) and \( S^M(k, \omega) \) are qualitatively the same, although it looks like that there are more details in Fig. S5(b). This is confirmed by the DOS shown in Fig. S5(c) and (d). Fig. S5(c) shows \( \rho^G(\omega) \) (purple), \( \rho^F(\omega) \) (blue) and \( \rho^M(\omega) \) (red) for \( \omega \in [-0.5, 0.5] \), and their peaks locate at the same position with similar amplitude. In Fig. S5(d), we show the DOS for \( \omega \in [-35, 45] \). The peaks are nearly matched, although \( \rho^F(\omega) \) is invisible for the higher excited states of the second band. This is due to \( \left| \langle \psi_n \mid \phi^\dagger_{k_\omega} \rangle \right|^2 \sim 0 \) for these higher excited states. The most direct way to improve the intensity of \( S^F(k, \omega) \) is taking the rest of the eigenstates of the undoped system into account.

The above analysis compares different approaches to obtain the dynamical structure factor, which give qualitatively the same spectrum, with the difference mainly arising in the quantitative amplitude of each mode. We then adapt the first approach since it is more setup-independent and not relying on the setting of e.g. the bath.

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FIG. S5. Dynamical structure factor of the 6-sites DSG pseudospin chain. (a) \( S^F(k, \omega) \) obtained by the Fourier transform of the time-dependent correlation function. (b) \( S^M(k, \omega) \) originating from the lattice modulation. (c) and (d) are the DOS. The purple, blue and red lines correspond to \( \rho^G \), \( \rho^F \) and \( \rho^M \), respectively.