Which weighted circulant networks have perfect state transfer? *

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Abstract
The question of perfect state transfer existence in quantum spin networks based on weighted graphs has been recently presented by many authors. We give a simple condition for characterizing weighted circulant graphs allowing perfect state transfer in terms of their eigenvalues. This is done by extending the results about quantum periodicity existence in the networks obtained by Saxena, Severini and Shparlinski and characterizing integral graphs among weighted circulant graphs. Finally, classes of weighted circulant graphs supporting perfect state transfer are found. These classes completely cover the class of circulant graphs having perfect state transfer in the unweighted case. In fact, we show that there exists an weighted integral circulant graph with $n$ vertices having perfect state transfer if and only if $n$ is even. Moreover we prove the non-existence of perfect state transfer for several other classes of weighted integral circulant graphs of even order.

Keywords: Circulant networks; Quantum systems; Perfect state transfer; Weighted graphs.

1 Introduction
The transfer of a quantum state from one location to another is a crucial ingredient for many quantum information processing protocols. There are various physical systems that can serve as quantum channels, one of them being a quantum spin network. These networks consist of $n$ qubits where some pairs of qubits are coupled via XY-interaction. The perfect transfer of quantum states from one qubit to another in such networks, was first considered in [9, 10]. There are two special qubits $A$ and $B$ representing the input and output qubit, respectively. The transfer is implemented by setting the qubit $A$ in a prescribed quantum state and by retrieving the state from the output qubit $B$ after some time. A transfer is called perfect state transfer (PST) (transfer with unit fidelity) if the initial state of the qubit $A$ and then final state of the qubit $B$ are equal due to the local phase rotation. If the previous condition holds for $A = B$, the network is periodic at $A$. A network is periodic if it is periodic at each qubit $A$. For such networks, periodicity is a necessary condition for the perfect state transfer existence.

Every quantum spin network with fixed nearest-neighbor couplings is uniquely described by an undirected graph $G$ on a vertex set $V(G) = \{1, 2, \ldots, n\}$. The edges of the graph $G$ specify which qubits are coupled. In other words, there is an edge between vertices $i$ and $j$ if $i$-th and $j$-th qubit are coupled.

In [10] a simple XY coupling is considered such that the Hamiltonian of the system has the form

$$H_G = \frac{1}{2} \sum_{(i,j) \in E(G)} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y.$$ 

and $\sigma_i^x$, $\sigma_i^y$ and $\sigma_i^z$ are Pauli matrices acting on $i$-th qubit. The standard basis chosen for an individual qubit is $\{|0\rangle, |1\rangle\}$ and it is assumed that all spins initially point down ($|0\rangle$) along the prescribed $z$ axis. In other words, the initial state of the network is $|0\rangle = |0_40\ldots0_B\rangle$. This is an eigenstate of Hamiltonian $H_G$ corresponding to zero energy. The Hilbert space $H_G$ associated to a network is spanned by the vectors $|e_1e_2\ldots e_n\rangle$ where $e_i \in \{0, 1\}$ and, therefore, its dimension is $2^n$.

The process of transmitting a quantum state from $A$ to $B$ begins with the creation of the initial state $\alpha|0_40\ldots0_B\rangle + \beta|1_40\ldots0_B\rangle$ of the network. Since $|0\rangle$ is a zero-energy eigenstate of $H_G$, the coefficient $\alpha$ will not change in time. Since the operator of total $z$ component of the spin $\sigma_z = \sum_{i=1}^n \sigma_i^z$ commutes with $H_G$, state $|1_40\ldots0_B\rangle$ must evolve into a superposition of the states $|i\rangle = |0\ldots010\ldots0\rangle$ for $i = 1, \ldots, n$. Denote by $S_G$ the subspace of $H_G$ spanned by the vectors $|i\rangle$, $i = 1, \ldots, n$. Hence, the initial state of network evolves in time $t$ into the state

$$\alpha|0\rangle + \sum_{i=1}^n \beta_i(t)|i\rangle \in S_G.$$ 

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The previous equation shows that system dynamics is completely determined by the evolution in $n$-dimensional space $S_G$. The restriction of the Hamiltonian $H_G$ to the subspace $S_G$ is an $n \times n$ matrix identical to the adjacency matrix $A_G$ of the graph $G$.

Thus, the time evolution operator can be written in the form $F(t) = \exp(iA_G t)$. The matrix exponential $\exp(M)$ is defined as usual

$$\exp(M) = \sum_{n=0}^{+\infty} \frac{1}{n!} M^n.$$  

Perfect state transfer (PST) between different vertices (qubits) $a$ and $b$ ($1 \leq a, b \leq n$) is obtained in time $\tau$, if $(a|F(\tau)|b) = |F(\tau)|_{ab} = 1$. Now formally, the graph (network) is periodic at $a$ if $|F(\tau)|_{aa} = 1$ for some $\tau$.

In a recent work of Saxena, Severini and Shparlinski [21], circulant graphs were proposed as potential candidates for modeling quantum spin networks enabling the perfect state transfer between antipodal sites in a network. It was shown that a quantum network whose Hamiltonian is identical to the adjacency matrix of a circulant graph is periodic if and only if all eigenvalues of the graph are integers (that is, the graph is integral). Therefore, circulant graphs having PST must be integral circulant graphs. Circulant graphs are also an important class of interconnection networks in parallel and distributed computing (see [15]).

Integral circulant graphs were first characterized by So [22]. Some properties of integral circulant graphs, including the bound of the number of vertices, diameter and bipartiteness were later studied by [5][7][21][23]. Moreover, integral circulant graphs are a generalization of the well-known class of unitary Cayley graphs. Various properties of unitary Cayley graphs were investigated in some recent papers as the diameter, clique number, chromatic number, eigenvalues and size of the longest induced cycles [3][19]. Integral circulant graphs have found important applications in molecular chemistry for modeling energy-like quantities [16].

Some research of the existence of PST over circulant topologies was already performed. In [3] authors gave a simple and general characterization of perfect state transfer existence in integral circulant graphs and in a recent paper [4], complete characterization of integral circulant graph having PST was given. The existence of PST for some other network topologies was also recently considered. For example, Christandl et al. [8][10] proved that PST occurs in the paths of length one and two between its end-vertices and in Cartesian powers of these graphs between vertices at maximal distance. In the recent paper [13], Godsil constructed a class of distance-regular graphs of diameter three, with PST. Some properties of quantum dynamics on circulant graphs were studied in [1]. In all cases perfect quantum communication distances (i.e., the distances between vertices where PST occurs) are considerably small compared to the order of the graph. These were further increased by considering networks with fixed but different couplings between qubits. These networks correspond to graphs with weighted adjacency matrices with a Hamiltonian

$$H_G := \frac{1}{2} \sum_{(i,j) \in E} d_{ij}(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y)$$

(1)

where $\sigma_i^x$, $\sigma_i^y$ and $\sigma_i^z$ are the standard Pauli matrices acting on qubit $i$ and $d_{ij} > 0$ are coupling constants [20].

It was shown that PST can be achieved over arbitrarily long distances in a weighted linear chain. Many recent papers proposed such approach [7][8][13]. The aim of our paper is to provide a general characterization of the PST existence in weighted circulant topologies. By considering weighted circulant topologies, rather than the unweighted, one might improve some relevant properties of circulant topologies. For example, some new classes of graphs could be found having PST or the perfect quantum communication distance could be enlarged.

The paper is organized as follows. Section 3 deals with quantum periodicity, since it represents a necessary condition for PST existence. We show that a necessary condition for PST existence in weighted circulant spin networks is that the ratio of differences of any two pairs of eigenvalues is rational. In addition, we complete and generalize Theorem 1 of [21]. It is proved that a weighted circulant graph is periodic if and only if it is integral. Furthermore, we characterize weighted integral circulant graphs with integer weights. In Section 4 we give a simple and general condition for weighted integral circulant graphs having PST in terms of their adjacency matrix eigenvalues. In Section 5 we present new classes of weighted circulant graphs supporting PST. These classes completely cover all integral circulant graphs having PST in the unweighted case. In fact, we show that there exists an integral weighted circulant graph with $n$ vertices having PST if and only if $n$ is even. In Theorem [13] we prove nonexistence of PST in those WICG($n;C$) for which exactly two entries of $C$ are positive and $n/4 = n/2 = 0$.

2 Circulant graphs

Circulant quantum spin networks of identical qubit couplings are described by circulant graphs. A circulant graph $G(n;S)$ is a graph on vertices $Z_n = \{0, 1, \ldots, n - 1\}$ such that vertices $i$ and $j$ are adjacent if and only if $i - j \equiv s$ (mod $n$) for some $s \in S$. A set $S$ is called the symbol of graph $G(n;S)$. As we will consider undirected graphs without loops, we assume that $S = n - S = \{n - s | s \in S\}$ and $0 \notin S$. Note that the degree of the graph $G(n;S)$ is $|S|$. The eigenvalues and eigenvectors of $G(n;S)$ are given by [21].
\[ \lambda_j = \sum_{x \in S} \omega_n^{jx}, \quad v_j = [1, \omega_n, \omega_n^2, \ldots, \omega_n^{(n-1)}]^T, \] (2)

where \( \omega_n = e^{2\pi i/n} \) is the \( n \)-th root of unity.

A weighted circulant digraph \( G(n; C) \) is a weighted digraph of order \( n \), the adjacency matrix of which is a circulant matrix with first row vector \( C = (c_0, \ldots, c_{n-1}) \in \mathbb{R}^n \). Recall that, each row vector of a circulant matrix is rotated one element to the right relative to the preceding row vector. The eigenvalues and eigenvectors of \( G(n; C) \) are given by

\[ \lambda_j = \sum_{i=0}^{n-1} c_i \omega_n^{ji}, \quad v_j = [1, \omega_n^j, \omega_n^{2j}, \ldots, \omega_n^{(n-1)j}]^T. \] (3)

If the adjacency matrix of the digraph \( G(n; C) \) is symmetric with zero main diagonal then we say that \( G(n; C) \) is a weighted circulant graph. In other words the weight vector \( C \) is specified as \( c_i = c_{n-i} \) for \( 1 \leq i \leq n-1 \). Circulant quantum spin networks of fixed but different couplings between the qubits are described by weighted circulant graphs. The elements of the row vector \( C \) represent the coupling strength between qubits in the network and thus we may assume that the elements of \( C \) are nonnegative. But the results given in the paper in most of cases do not require this condition. In fact, such networks correspond to the Hamiltonian \( \mathcal{H} \).

## 3 Quantum periodicity of weighted circulant graphs

Let \( \mathbb{H} \) be a Hilbert space associated to a quantum network. The dynamics of the system is periodic if for every state \( |\psi\rangle \in \mathbb{H} \), there exists \( t \in \mathbb{R}^+ \), for which \( |\psi e^{-itH}|\psi\rangle = 1 \), \( \mathbb{R}^+ \). The number \( t \) is the period of the system.

As is well known, the evolution of a system with the Hamiltonian \( \mathcal{H} \), can be expressed using matrix of incidence \( A_G \) of the graph \( G \), i.e.,

\[ |\psi(t)\rangle = e^{itA_G}|\psi(0)\rangle. \]

Using the fact that the matrix \( A_G \) is symmetric, we calculate easily

\[ |\psi(t)\rangle = \sum_{k=1}^{n} \alpha_k e^{i(\lambda_k t)}|\lambda_k\rangle, \quad |\psi(0)\rangle = \sum_{k=1}^{n} \alpha_k|\lambda_k\rangle, \]

where \( \lambda_k \in \mathbb{R} \), for \( 1 \leq k \leq n \), are the eigenvalues of the matrix \( A_G \) counting multiplicities and \( |\lambda_k\rangle \) the corresponding eigenvectors.

Assume there is PST between the states \( |\psi(t_1)\rangle \) and \( |\psi(t_2)\rangle \). Using the periodicity condition \( |\psi(t)\rangle = e^{it\lambda}|\psi(0)\rangle \) we have that

\[ \sum_{k=1}^{n} \alpha_k e^{it\lambda_k} |\lambda_j\rangle = |\psi(t_1)\rangle = e^{i\phi} |\psi(t_2)\rangle = \sum_{k=1}^{n} \alpha_k e^{i\phi + it\lambda_k} |\lambda_j\rangle. \]

According to linear independence of \( |\lambda_j\rangle \), for \( 1 \leq j \leq n \), we have

\[ e^{it(t_2-t_1)\lambda_j + \phi} = 1, \quad \text{i.e.} \quad (t_2 - t_1)\lambda_j + \phi = 2k_j \pi, \]

for some \( k_j \in \mathbb{Z} \), \( 1 \leq j \leq n \). Eliminating \( t_2 - t_1 \), \( \phi \) and \( k_j \in \mathbb{Z} \), \( j = 1, \ldots, n \), from the previous system, for every quadruple \( \lambda_k, \lambda_j, \lambda_m, \lambda_h \), (with \( \lambda_m \neq \lambda_h \)), we get

\[ \frac{\lambda_k - \lambda_j}{\lambda_m - \lambda_h} \in \mathbb{Q}. \] (4)

In the sequel, we use the terms graph and quantum network as being equivalent.

The result below extends Theorem 1 of [21] and at the same time simplifies considerably the proof given there in.

**Theorem 1** Let \( G \) be a weighted circulant digraph without loops and with integer first row vector \( C \) such that the sum of the entries of \( C \) is nonzero. Then \( G \) satisfies condition \( \mathcal{F} \), if and only if it is integral.

**Proof.** Suppose that \( G \) satisfies condition \( \mathcal{F} \). We prove that all the eigenvalues are rational numbers.

According to relation \( \mathcal{E} \) we have \( \lambda_0 = \sum_{i=0}^{n-1} c_i \in \mathbb{Z} \). Let \( \lambda_i \) be an arbitrary eigenvalue of \( G \). If \( \lambda_i = \lambda_0 \) then \( \lambda_i \in \mathbb{Z} \) also.

Suppose now \( \lambda_i \neq \lambda_0 \). Using \( \mathcal{F} \), we have \( \frac{\lambda_j - \lambda_0}{\lambda_i - \lambda_0} = a_j \in \mathbb{Q} \) for \( 1 \leq j \leq n - 1 \), implying that

\[ \lambda_j = a_j \lambda_i + (1 - a_j) \lambda_0. \] (5)

Since \( G \) has no loops thus \( c_0 = 0 \) and the sum of all the eigenvalues of \( G \) is given by:
\[
\sum_{j=0}^{n-1} \lambda_j = \sum_{j=0}^{n-1} \sum_{i=1}^{n-1} c_i \omega_n^{ji} = \sum_{i=1}^{n-1} \sum_{j=0}^{n-1} c_i \omega_n^{ji} = n \alpha_0 + \sum_{i=1}^{n-1} c_i \omega_n^{d_i n} - 1 = 0. \quad (6)
\]

Suppose that \(\sum_{j=1}^{n-1} a_j = 0\). Relation (5) yields
\[
\sum_{j=1}^{n-1} \lambda_j = \lambda_0 \sum_{j=1}^{n-1} (1 - a_j).
\]

Furthermore, using (5) the above relation reduces to \(-\lambda_0 = (n-1)\lambda_0\). Finally, the last statement is true if and only if \(\lambda_0 = 0\) which is a contradiction.

Thus, let \(\sum_{j=1}^{n-1} a_j \neq 0\). By relations (5) and (6) it holds that
\[
\sum_{j=0}^{n-1} \lambda_j = \lambda_0 \sum_{j=0}^{n-1} (1 - a_j) = 0,
\]
which implies that
\[
\lambda_i = \frac{\lambda_0 (n - \sum_{j=0}^{n-1} a_j)}{\sum_{j=0}^{n-1} a_j} \in \mathbb{Q},
\]
and thus by (5), \(\lambda_i \in \mathbb{Q}\) for all \(0 \leq j \leq n - 1\). Hence all the eigenvalues are rational and they are also algebraic integers, which further implies the desired result.

The converse trivially holds. \(\square\)

**Corollary 2** Let \(G = G(n, C)\) be a weighted circulant graph which corresponds to the Hamiltonian given by (1) with integer first row vector \(C\). Then \(G\) satisfies condition (3), if and only if it is integral.

The last statement leads us to the conclusion that if \(G = G(n, C)\) has PST, then it has to be integral. So, in the rest of the section we characterize weighted integral circulant graphs with integer weights. In addition, using Ramanujan sums, the spectra of these graphs are explicitly computed.

Let \(D_n\) be the set of all positive divisors of \(n\), less than \(n\). Denote by
\[
G_n(d) = \{ k : \gcd(k, n) = d, \ 1 \leq k \leq n - 1 \}.
\]
Recall that
\[
\Phi_n(x) = \prod_{0 < i < n, \ \gcd(i, n) = 1} (x - \omega_n^i),
\]
is the \(n\)-th cyclotomic polynomial [12].

**Theorem 3** A weighted circulant graph \(G(n; C)\) with integer weights is integral if and only if for each divisor \(d \in D_n\) the numbers \(c_i\) are equal for all \(i \in G_n(d)\).

**Proof.** Given a divisor \(d\) of \(n\), define a polynomial
\[
\Phi_{n, d}(x) = \prod_{i \in G_n(d)} (x - \omega_n^i).
\]
Notice that, the condition \(\gcd(i, n) = d\) is equivalent to \(\gcd(i/d, n/d) = 1\). According to the equality \(\omega_n^i = \omega_n^{i/d}\) for \(i \in G_n(d)\), we have that the monic polynomials \(\Phi_{n, d}(x)\) and \(\Phi_{n/d, d}(x)\) are identical.

(\(\Leftarrow\)) Assume that for some divisor \(d \in D_n\) the numbers \(c_i\) are equal for all \(i \in G_n(d)\). Note that the \(j\)-th eigenvalue of the graph \(G(n; C)\) can be written
\[
\lambda_j = \sum_{i=0}^{n-1} c_i \omega_n^{ji} = \sum_{d \in D_n} \sum_{i \in G_n(d)} c_i \omega_n^{ji} = \sum_{d \in D_n} c_d \sum_{i \in G_{n/d}(d)} \omega_n^{ji},
\]
where \(c_d = c_i\) for all \(i \in G_n(d)\). We have used the fact that \(Z_n = \cup_{d \in D_n} G_n(d)\).

Denote by
\[
\mu_{j, d} = \sum_{i \in G_{n/d}(1)} \omega_n^{ji} = \sum_{i \in G_{n/d}(1)} \omega_n^{ji/d}.
\]
We will prove that \( \mu_{j,d} \in \mathbb{Z} \) for \( 0 \leq j \leq n - 1 \).

Let \( G_{n/d}(1) = \{i_1, \ldots, i_{\varphi(n/d)}\} \). From Vieta’s formulas and the well-known property that all coefficients of cyclotomic polynomials are integers, we conclude that the coefficients of \( \Phi_{n/d}(x) \) is equal to

\[
s_j(\omega_{n/d}^{i_1}, \ldots, \omega_{n/d}^{i_{\varphi(n/d)}}) \in \mathbb{Z},
\]

where \( s_j \) is the \( j \)-th elementary symmetric polynomial and \( 1 \leq j \leq \varphi(n/d) \).

Furthermore, using Newton–Girard formulas we have the following identities

\[
\mu_{j,d} = (-1)^{j+1} j s_j(\omega_{n/d}^{i_1}, \ldots, \omega_{n/d}^{i_{\varphi(n/d)}}) - \sum_{k=1}^{j-1} (-1)^{k+j} \mu_{k,d} s_{j-k}(\omega_{n/d}^{i_1}, \ldots, \omega_{n/d}^{i_{\varphi(n/d)}}).
\]

Finally, using mathematical induction we have that \( \mu_{j,d} \in \mathbb{Z} \) for \( 1 \leq j \leq \varphi(n/d) \). Since for \( \varphi(n/d) < j \leq n/d \) the numbers \( \mu_{j,d} \) can be represented as polynomials with integer coefficients in \( \mu_{1,d}, \ldots, \mu_{\varphi(n/d),d} \) we conclude that \( \mu_{j,d} \in \mathbb{Z} \) for \( 0 \leq j \leq n/d \).

(\( \Rightarrow \)) Now, assume that all the eigenvalues of \( G(n; C) \) are integers,

\[
\lambda_j = \sum_{i=0}^{n-1} c_i \omega_n^{ji} \in \mathbb{Z}
\]

for \( 0 \leq j \leq n - 1 \). Since the eigenvalue \( \lambda_j \) represents the sum of the \( j \)-th powers of the roots \( \omega_n^i \), we also have that \( \lambda_j \in \mathbb{Z} \) for \( j \geq n \). According to Newton–Girard formulas we have the following identities

\[
(-1)^{j} j s_j(1, \ldots, 1, \omega_n, \ldots, \omega_n, \ldots, 1, \omega_n^{n-1}, \ldots, \omega_n^{n-1}) + \sum_{k=1}^{j-1} (-1)^{k+j} \lambda_k s_{j-k}(1, \ldots, 1, \omega_n, \ldots, \omega_n, \ldots, 1, \omega_n^{n-1}, \ldots, \omega_n^{n-1}) = 0
\]

for each \( 1 \leq j \leq n \). Using mathematical induction we obtain that

\[
s_j(1, \ldots, 1, \omega_n, \ldots, \omega_n, \ldots, 1, \omega_n^{n-1}, \ldots, \omega_n^{n-1}) = \prod_{i=0}^{n-1} (x - \omega_n^i) c_i \in \mathbb{Q}[x]
\]

since the coefficients of \( p(x) = \prod_{i=0}^{n-1} (x - \omega_n^i) c_i \) are integers, we also have that \( s_j(1, \ldots, 1, \omega_n, \ldots, \omega_n, \ldots, 1, \omega_n^{n-1}, \ldots, \omega_n^{n-1}) \) up to sign.

Let \( i \) be an arbitrary index \( 0 \leq i \leq n-1 \) such that \( c_i \neq 0 \). According to basic properties of cyclotomic polynomials \[12\], the minimal nonzero polynomial of \( \omega_n^i \) over \( \mathbb{Q} \) is \( \Phi_{n,i}(x) \). This in turn means that \( \Phi_{n,i}(x) \mid p(x) \) and the numbers \( c_i \) are mutually equal for all \( i \in G_n(d) \) and \( d \in D_n \).

In the unweighted case (\( c_i \in \{0,1\} \)), from Theorem\[9\] we see that \( G(n; C) \) is integral if and only if it holds that two vertices \( a \) and \( b \) are adjacent if \( a \sim b \) in \( G_n(d) \) for some \( d \in D \subseteq D_n \). This means that circulant graphs with integer eigenvalues are uniquely determined by the order \( n \) and the set of divisors \( D \subseteq D_n \). So, in the rest of the paper we denote them by \( ICG_n(D) \).

Denote by \( c(j,n) = \sum_{i \in G_n(d)} \omega_n^{ij} \). The expression \( c(j,n) \) is known as the Ramanujan sum \((\cite{13}, p. 55)\). The eigenvalues of \( G(n, C) \) can be expressed using the following formula for the Ramanujan sum \[14\]:

\[
c(j,n) = \mu(t_{n,j}) \frac{\varphi(n)}{\varphi(t_{n,j})}, \quad t_{n,j} = \frac{n}{\gcd(n,j)}
\]

where \( \mu \) denotes the Möbius function defined as

\[
\mu(n) = \begin{cases} 
1, & \text{if } n = 1 \\
0, & \text{if } n \text{ is not square-free} \\
(-1)^k, & \text{if } n \text{ is product of } k \text{ distinct prime numbers.}
\end{cases}
\]

According to the notation in the previous formula, the \( j \)-th eigenvalue of \( G(n; C) \) is given by:

\[
\lambda_j = \sum_{d \in D_n} c_d \sum_{i \in G_n(d)} \omega_n^{ij} = \sum_{d \in D_n} c_d c(j,n/d) \quad 0 \leq j \leq n - 1.
\]

Let us observe that the Ramanujan function has the following properties given below. These basic properties will be used in the rest of the paper.
Proposition 4 For any positive integers \(n, j\) and \(d\) such that \(d \mid n\), the following are satisfied

\[
\begin{align*}
c(0, n/d) &= \varphi(n/d), \quad (10) \\
c(1, n/d) &= \mu(n/d), \quad (11) \\
c(2, n/d) &= \begin{cases} 
\mu(n/d), & n/d \in 2\mathbb{N} + 1 \\
\mu(n/2d), & n/d \in 4\mathbb{N} + 2 \\
2\mu(n/2d), & n/d \in 4\mathbb{N} 
\end{cases}, \quad (12) \\
c(n/2, n/d) &= \begin{cases} 
\varphi(n/d), & d \in 2\mathbb{N} \\
-\varphi(n/d), & d \in 2\mathbb{N} + 1 
\end{cases}, \quad (13) \\
c(n/2 + 1, n/d) &= \begin{cases} 
-\mu(n/d), & n \in 4\mathbb{N} + 2, \ d \in 2\mathbb{N} + 1 \\
\mu(n/d), & \text{otherwise} 
\end{cases}, \quad (14) \\
c(j, 2) &= \begin{cases} 
0, & j \in 2\mathbb{N} + 1 \\
1, & j \in 2\mathbb{N} 
\end{cases}, \quad (15) \\
c(j, 4) &= \begin{cases} 
0, & j \in 2\mathbb{N} + 1 \\
2, & j \in 4\mathbb{N} 
\end{cases}. \quad (16)
\end{align*}
\]

Proof. These follow directly from relation (17). As an illustration, we prove the relation in line 15. For an arbitrary odd prime \(p \mid n/d\) it holds that \(p \mid n/2\) and thus \(p \nmid n/2 + 1\). Hence we conclude that \(gcd(n/2 + 1, n/d) \in \{1, 2\}\) and

\[
gcd(n/2 + 1, n/d) = \begin{cases} 
2, & n \in 4\mathbb{N} + 2, \ d \in 2\mathbb{N} + 1 \\
1, & \text{otherwise} 
\end{cases}, \quad (17)
\]

Finally we get

\[
c(n/2 + 1, n/d) = \begin{cases} 
\mu(n/2d) = -\mu(n/d), & n \in 4\mathbb{N} + 2, \ d \in 2\mathbb{N} + 1 \\
\mu(n/d), & \text{otherwise} 
\end{cases}. \quad (18)
\]

4 Perfect state transfer in weighted circulant graphs

In this section we provide a general condition of perfect state transfer existence in weighted circulant graph with integer weights. A weighted integral circulant graph of order \(n\) and with the set of integer weights \(C\) will be denoted by WICG\((n; C)\). According to the notation of Theorem 3 we index weights from \(C\) by the divisors \(d \in D_n\), i.e. \(C = \{c_d \mid d \in D_n\}\). Indeed, from Theorem 3 we have that \(c_i = c_d\), for all \(0 \leq i \leq n - 1\) such that \(gcd(i, n) = d\).

For a given graph \(G\) we say that there is perfect state transfer (PST) between the vertices \(a\) and \(b\) if there is a positive real number \(t\) such that

\[
|\langle a | e^{iAt} | b \rangle| = 1. \quad (19)
\]

For a weighted circulant graph \(G = G(n; C)\), let \(v_j = [1, \omega_n^j, \ldots, \omega_n^{j(n-1)}]^T\) be an eigenvector of \(G\) and \(v_j^* = [1, \omega_n^{-j}, \ldots, \omega_n^{-(j(n-1))}]\) the conjugate transpose of the eigenvector \(v_j\). Thus we have \(A = \frac{1}{n} \sum_{l=0}^{n-1} \lambda_l v_l v_l^*\) and \(e^{iAt} = \frac{1}{n} \sum_{l=0}^{n-1} e^{i\lambda_l t} v_l v_l^*\). Therefore,

\[
|\langle a | e^{iAt} | b \rangle| = 1 \Leftrightarrow \left| \frac{1}{n} \sum_{l=0}^{n-1} e^{i\lambda_l t} \omega_n^b \omega_n^{-l(b)} \right| = \left| \frac{1}{n} \sum_{l=0}^{n-1} e^{i\lambda_l t} \omega_n^{j(a-b)} \right| = 1. \quad (20)
\]

From the triangle inequality it is obviously that \(\langle a | e^{iAt} | b \rangle \leq 1\) holds, where the equality is satisfied if and only if all the summands in (20) have the same argument, i.e. are equal. In other words, there is PST in \(G\) if and only if

\[
e^{i\lambda_0 t} = e^{i\lambda_1 t} = \ldots = e^{i\lambda_{n-1} t + \frac{2\pi}{n}}(a-b) = \ldots = e^{i\lambda_{n-1} t + \frac{2(n-1)\pi}{n}}(a-b). \quad (21)
\]

The last expression is equivalent to

\[
\lambda_0 t \equiv_{2\pi} \lambda_1 t + \frac{2\pi}{n}(a-b) \equiv_{2\pi} \lambda_2 t + \frac{2\pi}{n}(a-b) \equiv_{2\pi} \lambda_{n-1} t + \frac{2(n-1)\pi}{n}(a-b).
\]

\[
\]
Here the relation $\equiv_{2\pi}$ is defined by $A \equiv_{2\pi} B$ if $\frac{(A-B)}{2\pi} \in \mathbb{Z}$. Notice that (21) depends on $a$ and $b$ as a function of $a - b$ only. Therefore without loss of generality, we can take $b = 0$. Upon subtracting the adjacent congruences in the previous equation and substituting $b = 0$ we obtain that (21) is equivalent to the following $n - 1$ conditions

$$(\lambda_{j+1} - \lambda_j)t_1 + \frac{a}{n} \in \mathbb{Z}, \quad j = 0, \ldots, n - 2,$$

where $t_1 = t/(2\pi)$. From the last expression we can conclude that if there is PST in $G$, then $t_1$ is rational, i.e., there exist integers $p$ and $q$ such that $t_1 = p/q$ and $\gcd(p, q) = 1$.

The discussion above leads to the following result.

**Theorem 5** There exists PST in a weighted circulant graph $G(n; C)$ between vertices $a$ and $0$ if and only if there are integers $p$ and $q$ such that $\gcd(p, q) = 1$ and

$$\frac{p}{q}(\lambda_{j+1} - \lambda_j) + \frac{a}{n} \in \mathbb{Z},$$

for all $j = 0, \ldots, n - 2$.

Note that if there is PST in $G(n; C)$, then by (22) the following equation holds

$$\frac{p}{q}(\lambda_{j+2} - \lambda_j) + \frac{2a}{n} \in \mathbb{Z}, \quad j = 0, 1, \ldots, n - 3$$

The next corollary is derived from Theorem 5 and will be used as the criterion for the nonexistence of PST.

**Corollary 6** If $\lambda_j = \lambda_{j+1}$ for some $0 \leq j \leq n - 2$ then there is no PST in $G(n; C)$ between any two vertices $a$ and $b$.

**Proof.** Without loss of generality we can take $b = 0$. Theorem 5 yields that $a/n \in \mathbb{Z}$, i.e., $n \mid a$. This is impossible because $0 < a < n$. \qed

Using the above results for the graphs $G(n; C)$ we will derive some other for weighted integral circulant graphs.

**Theorem 7** There is no PST in WICG($n; C$) if for every $d$ such that $c_d \neq 0$ the integer $n/d$ is odd. For $n$ even, if there exists PST in WICG($n; C$) between vertices $a$ and $0$ then $a = n/2$.

**Proof.** First suppose that $n/d$ is odd for every $d \in D_n$ such that $c_d \neq 0$.

Using the relation (12) and (13) of Proposition 4 it is easy to see that

$$\lambda_1 = \lambda_2 = \sum_{d \in D_n} c_d \mu(n/d).$$

According to Corollary 6 there is no PST in WICG($n; C$).

Suppose now that $n$ is even. Let us observe that $\gcd(n/2 + 1, n/d) = \gcd(n/2 - 1, n/d) \in \{1, 2\}$. Therefore there holds that $t_{n/d,n/2+1} = t_{n/d,n/2-1}$, i.e., $c(n/2 - 1, n/d) = c(n/2 + 1, n/d)$. Using the last expression we obtain

$$\lambda_{n/2-1} = \sum_{d \in D_n} c_d c(n/2 - 1, n/d) = \sum_{d \in D_n} c_d c(n/2 + 1, n/d) = \lambda_{n/2+1}$$

Again using (23) we have that $(2a)/n \in \mathbb{Z}$, which is possible only for $a = n/2$. \qed

From the proof of the preceding theorem we see that PST may exists in WICG($n; C$) only for $n$ even and between vertices $0$ and $a = n/2$ (i.e., between $b$ and $n/2 + b$). Therefore, in the rest of the paper we assume that $n$ is even and $a = n/2$.

We also avoid referring to the input and output vertices and simply say that there exists PST in WICG($n; C$). Relation (22) now becomes

$$\frac{p(\lambda_{j+1} - \lambda_j)}{q} + \frac{1}{2} \in \mathbb{Z},$$

(24)

For a given prime number $p$ and integer $n$, denote by $S_p(n)$ the maximal number $\alpha$ such that $p^\alpha \mid n$. Using the following theorem a criteria of PST existence in WICG($n; C$) can be established.

**Theorem 8** There exists PST in WICG($n; C$), if and only if there exists an integer $m \in \mathbb{N}_0$ such that for all $j = 0, 1, \ldots, n - 2$ there holds

$$S_2(\lambda_{j+1} - \lambda_j) = m.$$  

(25)
\textbf{Proof.} Let $\lambda_{j+1} - \lambda_j = 2^i m_j$ where $s_j = S_2(\lambda_{j+1} - \lambda_j) \geq 0$ and $m_j$ is odd for each $j = 0, 1, \ldots, n - 2$.

(\Rightarrow) Suppose that WICG$(n; C)$ has PST. According to Theorem 5 there exist relatively prime integers $p, q$ such that (24) holds. Rewrite relation (24) in the following form

$$\frac{2^{s_j} - p m_j + q}{2q} \in \mathbb{Z}. \quad (26)$$

From the last expression we can conclude that $q \mid 2^{s_j} - p m_j$ (because gcd$(p, q) = 1$) and $2 \mid q$. Furthermore there must exist non-negative integers $s_q$ and $m_q \in 2\mathbb{N} + 1$ such that $q = 2^{s_q} + m_q$ where $s_q \leq s_j$ and $m_q \mid m_j$ for each $j = 0, 1, \ldots, n - 2$. \(20\) now becomes

$$\frac{2^{s_j} - q m_j + 1}{2} \in \mathbb{Z},$$

which directly implies that $s_j = s_q = S_2(q) - 1$. Putting $m = S_2(q) - 1$ we obtain (25).

(\Leftarrow) Now suppose that (25) is valid. Put $q = 2^{m+1}$ and $p = 1$. Then it holds

$$\frac{p(\lambda_{j+1} - \lambda_j)}{q} + \frac{1}{2} = \frac{m_j + 1}{2} \in \mathbb{Z},$$

for every $j = 0, 1, \ldots, n - 2$. According to Theorem 5 there is PST in WICG$(n; C)$. \hfill \Box

PST may exist only in the case when a graph (network) is connected. A graph ICG$_n(D)$ is connected if and only if

$$\gcd(n, d_1, d_2, \ldots, d_l) = 1,$$

for $d_i \in D$ and $1 \leq i \leq l$. \[5\]. Hence in the rest of the paper we assume that a weighted circulant graph WICG$(n; C)$ is connected. This means that the corresponding unweighted graph ICG$_n(D)$, where $D = \{d \mid n \div cd \neq 0\}$, is also connected.

5 Classes of weighted integral circulant graphs either having or not having PST

Let $D \subseteq D_n$ be an arbitrary set of divisors. We define sets $D_i \subseteq D$ for $0 \leq i \leq l$, where $l = S_2(n)$ in the following way

$$D_i = \{d \in D \mid S_2(n/d) = i\}.$$

For simplicity of notation we also define sets $D_0 \subseteq D$ to be $D_0 = \bigcup_{i \geq 3} D_i$.

Let us also introduce the notation $kA$ for the set \{\(ka \mid a \in A\)\} for a positive integer $k$ and some set of integers $A$. The following result concerns unweighted circulant graphs having PST.

\textbf{Theorem 9} \([4]\) ICG$_n(D)$ has PST if and only if in $4\mathbb{N}$, $D_1^* = 2D_2^*$, $D_0 = 4D_2^*$ and either $n/4 \in D$ or $n/2 \in D$, where $D_2^* = D_2 \setminus \{n/4\}$ and $D_1^* = D_1 \setminus \{n/2\}$.

Let $k$ be an arbitrary positive integer. Notice that WICG$(n; C)$ has PST if and only if WICG$(n; 2kC)$ has PST. Indeed, let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ and $\mu_0, \mu_1, \ldots, \mu_{n-1}$ be the eigenvalues of WICG$(n; C)$ and WICG$(n; 2kC)$, respectively. Now, we have the following relation for $1 \leq j \leq n - 1$

$$\mu_j - \mu_{j-1} = \sum_{d \in D_n} 2^k \text{cd}(c(j, n/d) - c(j - 1, n/d)) = 2^k(\lambda_j - \lambda_{j-1}).$$

This implies that $S_2(\mu_j - \mu_{j-1}) = S_2(\lambda_j - \lambda_{j-1}) + k$ and according to Theorem 8 the assertion holds.

The last observation implies that for WICG$(n; C)$ having PST, we can assume that at least one of the weights in $C$ is odd. If not, i.e., if all $cd$ for $d \in D_n$ are even, we can divide them sufficient number of times by 2, and obtain that at least one $cd$ is odd and the graph with the new weights will still have PST. Therefore, in the rest of section we assume that $cd \in 2\mathbb{N} + 1$ for some $d \in D_n$.

\textbf{Theorem 10} There exists PST in WICG$(n; C)$ if for some $a \in \{1, 2\}$ both $c_{n/2^a}$ is odd and $c_d \in 4\mathbb{N}$ for all $d \in D_n \setminus \{n/2^a\}$.
Proof. For \( a \in \{1,2\} \) and \( 1 \leq j \leq n-1 \), the difference between the eigenvalues is given by
\[
\lambda_j - \lambda_{j-1} = \sum_{d \in D_n \setminus \{n/2^n\}} c_d(c(j,n/d) - c(j-1,n/d)) + c_{n/2^n}(c(j,2^n) - c(j-1,2^n)).
\]
If \( a = 1 \), according to the relation (16) of Proposition \( \square \) we conclude that \( |c(j,2) - c(j-1,2)| = 2 \), and hence \( c_{n/2}(c(j,2) - c(j-1,2)) \in 4\mathbb{N}+2 \). If \( a = 2 \) then in both of the cases \( j \in 4\mathbb{N}+2 \) and \( j \in 4\mathbb{N} \) we also have that \( |c(j,4) - c(j-1,4)| = 2 \), and hence \( c_{n/4}(c(j,4) - c(j-1,4)) \in 4\mathbb{N}+2 \).

Now using the assumption of the theorem we readily see that \( \sum_{d \in D_n \setminus \{n/2^n\}} c_d(c(j,n/d) - c(j-1,n/d)) \in 4\mathbb{N} \).

Finally, we conclude that \( \lambda_j - \lambda_{j-1} \in 4\mathbb{N}+2 \) for \( 1 \leq j \leq n-1 \), and consequently that there is PST in WICG(\( n; C \)) according to Theorem \( \square \) \( \square \)

Notice that the assertion still holds if \( S_2(c_d) \geq S_2(c_{n/2^n}) + 2 \) for \( d \in D_n \setminus \{n/2^n\} \) and \( a \in \{1,2\} \).

From the previous theorem we see that we can associate suitable weights to the edges of any ICG\( n(D) \) such that \( n/2 \in D \) or \( n/4 \in D \) and obtain PST. This result evidently generalizes Theorem \( \square \) since here \( n/4 \) and \( n/2 \) may both belong to \( D \), there are no restrictions concerning the remaining divisors of \( D \) and \( n \) is only required to be even. So, in the rest of the section we focus on searching those WICG(\( n; C \)) with PST such that \( c_{n/4} = c_{n/2} = 0 \). In fact, we will see that there is no WICG(\( n; C \)) having PST such that its weight integer vector \( C \) has exactly two positive entries. This means that there is no way in which we can associate weights to the edges of an unweighted graph ICG\( n(D) \) to obtain PST if \( n/2, n/4 \not\in D \) and \( |D| = 2 \). We prove that in the sequel by using the foregoing two important lemmas.

Notice also that in the case when for exactly one \( d \in D_n \), the entry \( c_d \in C \) is positive, then WICG(\( n; C \)) has PST if and only if ICG\( n(\{d\}) \) has PST. This can be easily seen from the fact that \( \mu_i = c_d \lambda_i \) for \( 0 \leq i \leq n-1 \), where \( \mu_i \) and \( \lambda_i \) are the eigenvalues of WICG(\( n; C \)) and ICG\( n(\{d\}) \), respectively. Thus, according to Theorem 10 of \( \square \) we conclude that there is no PST in WICG(\( n; C \)) except in the trivial cases for the hypercubes \( K_2 \) and \( C_4 \).

Lemma 11 For \( n \geq 2 \) it holds that \( c(j,n) \in 2\mathbb{N}+1 \) if and only if \( 4 \nmid n \) and \( j = p_1^{\alpha_1-1} \cdots p_k^{\alpha_k-1} \) for some integer \( J \) such that \( \gcd(J,n) \in \{1,2\} \).

Proof.

(\( \Rightarrow \)) Suppose that \( c(j,n) \) is an odd integer. Since \( c(j,n) = \mu(t_{n,j})\varphi(n)/\varphi(t_{n,j}) \), it holds that \( \mu(t_{n,j}) = \pm 1 \), i.e. \( t_{n,j} \) is square-free and \( \varphi(n)/\varphi(t_{n,j}) \) is an odd integer.

Suppose that for some odd \( p_i \) it holds that \( p_i \nmid t_{n,j} \). Let \( n' = n/p_i^\alpha \). Since \( t_{n,j} \mid n' \) and so \( \varphi(t_{n,j}) \mid \varphi(n') \) we obtain that
\[
c(j,n) = \pm \frac{\varphi(n)}{\varphi(t_{n,j})} = \pm \frac{\varphi(p_i^\alpha)}{\varphi(p_i^\alpha t_{n,j})} = \pm p_i^{\alpha-1}(p_i - 1) \frac{\varphi(n')}{\varphi(t_{n,j})}.
\]
The last equation implies that \( c(j,n) \) is even since \( p_i - 1 \) is even. This is a contradiction and we can conclude that \( p_i \nmid t_{n,j} \) for every \( 2 \leq i \leq k \).

Now we have that \( \varphi(t_{n,j}) = (p_1 - 1) \cdots (p_k - 1) \) and thus
\[
c(j,n) = 2^{\alpha_1-1}p_2^{\alpha_2-1} \cdots p_k^{\alpha_k-1}.
\]
Since \( c(j,n) \) is odd it holds that \( 0 \leq \alpha_i \leq 1 \) or equivalently \( 4 \nmid n \).

If \( n \in 2\mathbb{N}+1 \) it must hold that \( t_{n,j} = p_1 \cdots p_k \) since \( t_{n,j} \) is square-free. If \( n \in 4\mathbb{N}+2 \), we have two possibilities for \( t_{n,j} \): \( t_{n,j} = p_1 \cdots p_k \) or \( t_{n,j} = 2p_1 \cdots p_k \) depending on the parity of \( j \).

Furthermore, using \( n = \gcd(n,j)t_{n,j} \) we obtain that \( \gcd(n,j) = p_1^{\alpha_1-1} \cdots p_k^{\alpha_k-1} \) (\( t_{n,j} \) and \( n \) have the same parity) or \( \gcd(n,j) = 2p_1^{\alpha_1-1} \cdots p_k^{\alpha_k-1} \) (otherwise). This implication of the lemma is now straightforward.

\( \Leftarrow \) Since \( \gcd(n,j) = p_1^{\alpha_1-1} \cdots p_k^{\alpha_k-1} \gcd(J,n) \) and \( \gcd(J,n) \in \{1,2\} \), it holds that \( t_{n,j} = p_1 \cdots p_k \) or \( t_{n,j} = 2p_1 \cdots p_k \). In either case we have that \( \varphi(t_{n,j}) = (p_1 - 1) \cdots (p_k - 1) \). Now since
\[
c(j,n) = \mu(t_{n,j})\varphi(n)/\varphi(t_{n,j}) = \pm p_1^{\alpha_1-1} \cdots p_k^{\alpha_k-1},
\]
we conclude that \( c(j,n) \in 2\mathbb{N}+1 \). \( \square \)

Lemma 12 Let \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \) be the eigenvalues of the graph WICG(\( n; C \)). Then \( \lambda_2 - \lambda_1 \) must be even.

Proof.

According to the relation (12) of Proposition \( \square \) we have \( \lambda_1 = \sum_{d \in D} c_d \mu(n/d) \). Since \( 4 \mid n/d \) for \( d \in D_2 \cup D_3 \) we conclude that \( \mu(n/d) = 0 \) and therefore \( \lambda_1 = \sum_{d \in D_2 \cup D_3} c_d \mu(n/d) \). Using Proposition \( \square \) (13) once again we see
that \( \lambda_2 = \sum_{d \in D_0} c_d \mu(n/d) + \sum_{d \in D_1} c_d \mu(n/2d) + \sum_{d \in D_2} 2c_d \mu(n/2d) \). For \( d \in D_3 \) we have \( 4 \mid n/2d \), which yields \( \mu(n/2d) = 0 \) and
\[
\lambda_2 - \lambda_1 = \sum_{d \in D_1} c_d (\mu(n/2d) - \mu(n/d)) + 2 \sum_{d \in D_2} c_d \mu(n/2d) = 2 \sum_{d \in D_1 \cup D_2} c_d \mu(n/2d) \in 2\mathbb{N}. \tag{27}
\]

\[ \square \]

**Theorem 13** Let \( WICG(n; C) \) be a weighted integral circulant graph such that \( n \) is a square-free number and \( c_{n/4} = 0 \). If there exist \( d_1, d_2 \in D_n \) such that \( c_{d_1}, c_{d_2} > 0 \) and \( c_d = 0 \) for all \( d \in D_n \setminus \{d_1, d_2\} \) then there is no PST in \( WICG(n; C) \).

**Proof.** Suppose that \( WICG(n; C) \) has PST. According to Proposition 4 we have that
\[
\lambda_1 - \lambda_0 = c_{d_1} (\mu(n/d_1) - \varphi(n/d_1)) + c_{d_2} (\mu(n/d_2) - \varphi(n/d_2)). \tag{28}
\]
Since \( WICG(n; C) \) has PST, using Theorem 8 and Lemma 12 it holds that \( \lambda_1 - \lambda_0 \in 2\mathbb{N} \). Furthermore, both \( n/d_1 \) and \( n/d_2 \) are square-free and thus \( \mu(n/d_i) \in \{-1, 1\} \) for \( 1 \leq i \leq 2 \). On the other hand, since \( d_i \neq n/2 \) for \( 1 \leq i \leq 2 \), it follows that \( \varphi(n/d_i) \in 2\mathbb{N} \) for \( 1 \leq i \leq 2 \). Finally, it can be concluded that both terms \( \varphi(n/d_1) - \mu(n/d_1) \) and \( \varphi(n/d_2) - \mu(n/d_2) \) are odd and since one of the weights \( c_{d_1} \) and \( c_{d_2} \) is odd, then both of them must be odd in order for \( \lambda_1 - \lambda_2 \) to be even.

As \( WICG(n; C) \) is connected then \( \gcd(d_1, d_2) = 1 \) meaning that \( d_1 \) and \( d_2 \) cannot be both even. Now consider the case when \( d_1 \) and \( d_2 \) have different parity and suppose without loss of generality that \( d_1 \in 2\mathbb{N} \) and \( d_2 \in 2\mathbb{N} + 1 \). Since \( n \) is square-free, it means that \( d_1 \in D_0 \) and \( d_2 \in D_1 \). The relation (27) is now reduced to
\[
\lambda_2 - \lambda_1 = c_{d_1} (\mu(n/2d_2) - \mu(n/d_1)) = 2c_{d_2} (\mu(n/d_2) \in 4\mathbb{N} + 2).
\]
According to Theorem 8 we have that \( \lambda_i - \lambda_{i-1} \in 4\mathbb{N} + 2 \) for all \( 1 \leq i \leq n - 1 \). On the other hand, using the relations (14) and (15) of Proposition 3 we have that
\[
\lambda_{n/2+1} - \lambda_{n/2} = c_{d_1} (\mu(n/d_1) - \varphi(n/d_1)) + c_{d_2} (\mu(n/d_2) + \varphi(n/d_2)).
\]
We proceed by subtracting the difference \( \lambda_1 - \lambda_0 \) and \( \lambda_{n/2+1} - \lambda_{n/2} \)
\[
(\lambda_1 - \lambda_0) - (\lambda_{n/2+1} - \lambda_{n/2}) = 2c_{d_2} (\mu(n/d_2) - \varphi(n/d_2)).
\]
Since \( \lambda_1 - \lambda_0, \lambda_{n/2+1} - \lambda_{n/2} \in 4\mathbb{N} + 2 \), the left hand side of the above relation is divisible by four. But this is a contradiction as \( 2c_{d_2} (\mu(n/d_2) - \varphi(n/d_2)) \in 4\mathbb{N} + 2 \).

Having disposed of the previous case, we can now investigate the case where both divisors \( d_1 \) and \( d_2 \) are odd. Let \( r_1, r_2, \ldots, r_s \) be all the odd prime divisors of \( n \), not dividing \( d_1 \) and let \( q_1, q_2, \ldots, q_l \) be all the odd prime divisors of \( n \), not dividing \( d_2 \). Without loss of generality we may assume that \( d_1 > d_2 \) and thus there exists an odd prime number \( p \) such that \( p \mid d_1 \). Hence, we obtain that \( p \not\in \{r_1, r_2, \ldots, r_s\} \). Since \( \gcd(d_1, d_2) = 1 \) then \( p \nmid d_2 \) and thus \( p \in \{q_1, q_2, \ldots, q_l\} \). Now, we can choose \( 0 \leq j_0 \leq n - 1 \) such that
\[
\begin{align*}
    j_0 &\neq \{0, 1\} \pmod{r_i} \text{ for } 1 \leq i \leq s, \\
    j_0 &\equiv 0 \pmod{p}, \\
    j_0 &\neq 1 \pmod{q_i} \text{ for } 1 \leq i \leq l \text{ such that } q_i \neq p.
\end{align*}
\]
This is possible by the Chinese remainder Theorem if we consider a suitable system of congruences modulo \( n/2 \).

We conclude that \( \gcd(j_0, n/d_1) \in \{1, 2\} \) and \( p \nmid \gcd(j_0, n/d_2) \), and thus \( c(j_0, n/d_1) \in 2\mathbb{N} + 1 \) and \( c(j_0, n/d_2) \in 2\mathbb{N} \), according to Lemma 11. Now we have that
\[
\lambda_{j_0} = c_{d_1} c(j_0, n/d_1) + c_{d_2} c(j_0, n/d_2) \in 2\mathbb{N} + 1.
\]

Similarly, \( \gcd(j_0 - 1, n/d_1) \in \{1, 2\} \) and \( \gcd(j_0 - 1, n/d_2) \in \{1, 2\} \) which yields that \( c(j_0 - 1, n/d_1) \in 2\mathbb{N} + 1 \) and \( c(j_0 - 1, n/d_2) \in 2\mathbb{N} + 1 \). Therefore
\[
\lambda_{j_0 - 1} - c_{d_1} c(j_0 - 1, n/d_1) + c_{d_2} c(j_0 - 1, n/d_2) \in 2\mathbb{N},
\]
and \( \lambda_{j_0} - \lambda_{j_0 - 1} \in 2\mathbb{N} + 1 \), a contradiction.

\[ \square \]
Theorem 14 Let WICG\((n; C)\) be a weighted integral circulant graph such that \(n\) is a twice even square-free number and \(c_{n/4} = c_{n/2} = 0\). If there exist \(d_1, d_2 \in D_n\) such that \(c_{d_1}, c_{d_2} > 0\) and \(c_d = 0\) for all \(d \in D_n \setminus \{d_1, d_2\}\) then there is no PST in WICG\((n; C)\).

Proof. Suppose that WICG\((n; C)\) has PST. As in the proof of the previous theorem we distinguish two cases.

Case 1. The divisors \(d_1\) and \(d_2\) have different parity. Without loss of generality suppose that \(d_1 \in 2\mathbb{N}\) and \(d_2 \in 2\mathbb{N} + 1\). Using the relation (28) we get
\[
\lambda_1 - \lambda_0 = c_{d_1}(\mu(n/d_1) - \varphi(n/d_1)) - c_{d_2}\varphi(n/d_2).
\]
Since WICG\((n; C)\) has PST, by Theorem 8 and Lemma 12 it holds that \(\lambda_1 - \lambda_0 \in 2\mathbb{N}\) and that \(\lambda_1 - \lambda_{i-1} \in 2\mathbb{N}\) for all \(1 \leq i \leq n - 1\). Since \(\mu(n/d_1) - \varphi(n/d_1) \in 2\mathbb{N} + 1\) and \(\varphi(n/d_2) \in 2\mathbb{N}\) we have that \(c_{d_1} \in 2\mathbb{N}\) and \(c_{d_2} \in 2\mathbb{N} + 1\) (at least one of the entries of \(C\) must be odd).

From the parity of \(d_1\) and \(d_2\) it follows that \(d_1 \in D_1 \cup D_1\) and \(d_2 \in D_2\), and
\[
\lambda_2 - \lambda_1 = \begin{cases} 
 c_{d_1}(\mu(n/2d_1) - \mu(n/d_1)) + 2c_{d_2}\mu(n/2d_2), & d_1 \in D_1 \\
 2c_{d_2}\mu(n/2d_2), & d_1 \in D_0.
\end{cases}
\] (29)

In both cases we have that \(\lambda_2 - \lambda_1 \in 4\mathbb{N} + 2\), due to the facts that \(c_{d_1}(\mu(n/2d_1) - \mu(n/d_1)) \in 4\mathbb{N}\) and \(2c_{d_2}\mu(n/2d_2) \in 4\mathbb{N}\). Using Theorem 8 again we obtain that \(\lambda_1 - \lambda_0 \in 4\mathbb{N} + 2\). The last relation is true if and only if \(c_{d_1} \in 4\mathbb{N}\) and \(\varphi(n/d_2) \in 4\mathbb{N} + 2\) or else, if \(c_{d_1} \in 4\mathbb{N} + 2\) and \(\varphi(n/d_2) \in 4\mathbb{N}\).

If \(\varphi(n/d_2) \in 4\mathbb{N} + 2\) then it is easy to see that \(n/d_2 \in \{p^\alpha, 2p^\alpha\}\) for some odd prime number \(p\) and \(\alpha \geq 1\). But, we have that \(n/d_2 \in 4\mathbb{N}\) and hence that \(\varphi(n/d_2) \not\in 4\mathbb{N} + 2\). Having disposed of this case, we can now assume that \(c_{d_1} \in 4\mathbb{N} + 2\).

Assume that \(d_2 > 1\). Then there exists an odd prime \(p\) such that \(p \mid d_2\) and, since \(\gcd(d_1, d_2) = 1\), then \(p \nmid d_1\). Let \(r_1, r_2, \ldots, r_s\) be all the odd prime divisors of \(n\), not dividing \(d_1\) and let \(q_1, q_2, \ldots, q_l\) be all the odd prime divisors of \(n\), not dividing \(d_2\). Now, we can choose \(0 \leq j_0 \leq n - 1\) such that
\[
j_0 \equiv 0 \pmod{p}.
\]

Notice that \(p \not\in \{q_1, q_2, \ldots, q_l\}\) since \(p \mid d_2\). This is possible by the Chinese remainder Theorem if we consider a suitable system of congruences modulo \(n/4\). Here we can also assume that \(j_0 \in 4\mathbb{N} + 2\).

We conclude that \(p \mid \gcd(j_0, n/d_1)\) and \(\gcd(j_0 - 1, n/d_1) = 1\) and hence \(c_{j_0, n/d_1} \in 2\mathbb{N}\) and \(c_{j_0 - 1, n/d_1} \in 2\mathbb{N} + 1\) according to Lemma 11. As \(j_0 \in 4\mathbb{N} + 2\) we conclude that \(\gcd(j_0 - 1, n/d_2) = 2\mathbb{N} + 1\) and thus \(4 \mid t_{n/d_2,j_0-1}\) which yields \(c_{j_0 - 1, n/d_2} = \mu(n/d_2) = 1\). On the other hand, from to the above system of congruences we see that \(\gcd(j_0, n/d_2) = 2\). This implies that \(t_{n/d_2,j_0} = n/2d_2\) and \(c_{j_0, n/d_2} = 2\mu(n/2d_2) \in 4\mathbb{N} + 2\). Next it can be concluded that \(c_{j_0, n/d_1} - c_{j_0 - 1, n/d_1} \in 2\mathbb{N} + 1\) and \(c_{j_0, n/d_2} - c_{j_0 - 1, n/d_2} = c_{j_0, n/d_2} \in 4\mathbb{N} + 2\). Finally, we have
\[
\lambda_{j_0} - \lambda_{j_0 - 1} = c_{d_1}(c_{j_0, n/d_1} - c_{j_0 - 1, n/d_1}) + c_{d_2}(c_{j_0, n/d_2} - c_{j_0 - 1, n/d_2}) \in 4\mathbb{N}
\]
since both of the summands \(c_{d_1}(c_{j_0, n/d_1} - c_{j_0 - 1, n/d_1})\) and \(c_{d_2}(c_{j_0, n/d_2} - c_{j_0 - 1, n/d_2})\) are in \(4\mathbb{N} + 2\). This leads us to a contradiction.

Now, let \(d_2 = 1\). Since \(d_1 \not\in \{n/4, n/2\}\) and \(n\) is twice even square-free, there exists an odd prime number \(p\) such that \(p \nmid d_1\). Let \(p_1, p_2, \ldots, p_k\) be all the odd prime divisors of \(n\). Now, consider \(0 \leq j_0 \leq n - 1\) such that
\[
j_0 \equiv 1 \pmod{p}.
\]

This is possible by the Chinese remainder Theorem if we consider a suitable system of congruences modulo \(n/4\). Now, we can choose \(j_0 \in 4\mathbb{N} + 2\).

We see that \(\gcd(j_0, n/d_1) \in \{1, 2\}\) and \(p \mid \gcd(j_0, n/d_1)\), thus \(c_{j_0, n/d_1} \in 2\mathbb{N} + 1\) and \(c_{j_0 - 1, n/d_1} \in 2\mathbb{N}\) according to Lemma 11. As \(j_0 \in 4\mathbb{N} + 2\) we conclude that \(\gcd(j_0 - 1, n) = 2\mathbb{N} + 1\) and thus \(4 \mid t_{n,j_0-1}\) which yields \(c_{j_0 - 1, n} = \mu(n/j_0) = 1\). On the other hand, from the above system of congruences we see that \(\gcd(j_0, n) = 2\). This implies that \(t_{n,j_0} = n/2\) and \(c_{j_0, n} = 2\mu(n/2) \in 4\mathbb{N} + 2\). Also \(c_{j_0, n} \in 2\mathbb{N} + 1\) and \(c_{j_0, n} - c_{j_0 - 1, n} = c_{j_0, n} \in 4\mathbb{N} + 2\) so finally, we have that
\[
\lambda_{j_0} - \lambda_{j_0 - 1} = c_{d_1}(c_{j_0, n/d_1} - c_{j_0 - 1, n/d_1}) + c_{d_2}(c_{j_0, n} - c_{j_0 - 1, n}) \in 4\mathbb{N},
\]
since \( c_{d_1}(c(j_0, n/d_1) - c(j_0 - 1, n/d_1)), c_{d_2}(c(j_0, n) - c(j_0 - 1, n)) \in 4\mathbb{N} + 2 \). This is a contradiction.

**Case 2.** \( d_1, d_2 \) are both odd. It follows that \( d_1, d_2 \in D_2 \). According to (27) we obtain that

\[
\begin{align*}
\lambda_1 - \lambda_0 &= -c_{d_1}\varphi(n/d_1) - c_{d_2}\varphi(n/d_2) \\
\lambda_2 - \lambda_1 &= 2c_{d_1}\mu(n/2d_1) + 2c_{d_2}\mu(n/2d_2).
\end{align*}
\]

Since \( d_1 \) and \( d_2 \) are both different than \( n/4 \), then there are odd prime numbers \( p_1, p_2 \) dividing \( n/d_1 \) and \( n/d_2 \), respectively. This implies that \( \varphi(n/d_1), \varphi(n/d_2) \in 4\mathbb{N} \), which yields \( \lambda_1 - \lambda_0 \in 4\mathbb{N} \). If we assume that \( c_{d_1} \) and \( c_{d_2} \) have different parity we get that \( \lambda_2 - \lambda_1 \in 4\mathbb{N} + 2 \), which is a contradiction due to Theorem [S]. Thus it can be concluded that both of the weights \( c_{d_1} \) and \( c_{d_2} \) must be odd, since one of them certainly must be.

As both divisors \( d_1 \) and \( d_2 \) are odd, without loss of generality we can assume that \( d_1 > d_2 \). This means that there is an odd prime number \( p \) such that \( p \mid d_1 \). Since \( \gcd(d_1, d_2) = 1 \) it holds that \( p \nmid d_2 \). Choose \( 0 \leq j_0 \leq n - 1 \) such that \( j_0 = 2p \). Now, we have that \( \gcd(j_0, n/d_1) = 2 \), and hence \( t_{n/d_1,j_0} = n/2d_1 \), and finally that \( c(j_0, n/d_1) = 2\mu(n/2d_1) \in 4\mathbb{N} + 2 \). On the other hand, we conclude that \( \gcd(j_0 - 1, n/d_1) = 2n + 1 \), and hence \( t_{n/d_1,j_0-1} \in 4\mathbb{N} \). Finally we have \( c(j_0 - 1, n/d_1) = \mu(t_{n/d_1,j_0-1}) = 0 \). From the preceding analysis we can infer that

\[
c_{d_1}(c(j_0, n/d_1) - c(j_0 - 1, n/d_1)) = 2c_{d_1}\mu(n/2d_1) \in 4\mathbb{N} + 2.
\]

Furthermore, from \( \gcd(j_0, n/d_2) = 2p \) it follows that \( t_{n/d_2,j_0} = n/(2pd_2) \). This further implies that \( c(j_0, n/d_2) = 2(p-1)\mu(n/2pd_2) \in 4\mathbb{N} \). On the other hand, we conclude that \( \gcd(j_0 - 1, n/d_2) = 2n + 1 \), and consequently \( t_{n/d_2,j_0-1} \in 4\mathbb{N} \). Finally \( c(j_0 - 1, n/d_2) = \mu(t_{n/d_2,j_0-1}) = 0 \). From the preceding analysis we see that

\[
c_{d_2}(c(j_0, n/d_2) - c(j_0 - 1, n/d_2)) = 2c_{d_2}(p-1)\mu(n/2pd_2) \in 4\mathbb{N}.
\]

Finally, using (32) and (33) we obtain

\[
\lambda_{j_0} - \lambda_{j_0-1} \in 4\mathbb{N} + 2,
\]

which is a contradiction due to Theorem [S].

Now, we turn to the general case.

**Theorem 15** Let WICG\((n; C)\) be a weighted integral circulant graph such that \( c_{n/4} = c_{n/2} = 0 \). If there exist \( d_1, d_2 \in D_n \) such that \( c_{d_1}, c_{d_2} > 0 \) and \( c_d = 0 \) for all \( d \in D_n \setminus \{d_1, d_2\} \) then there is no PST in WICG\((n; C)\).

**Proof.** Suppose there are some odd prime divisors \( q_i \) of \( n/d_i \), \( 1 \leq i \leq 2 \) such that for each \( i \in \{1, 2\} \) at least one of the cases \( 8 \mid n/d_i \) or \( q_i^2 \mid n/d_i \) holds. This implies that \( \mu(n/d_i) = \mu(n/2d_i) = 0 \), for \( 1 \leq i \leq 2 \), and hence \( \lambda_1 = \lambda_2 = 0 \) according to Proposition [4] (relations (12) and (13)). Now, using Corollary [S] there is no PST in WICG\((n; C)\). Thus, it can be concluded that for at least one of the divisors \( d_i \), \( 1 \leq i \leq 2 \), the integer \( n/d_i \) must be square-free or twice even square-free. Without loss of generality assume that \( n/d_2 \) is square-free or twice even square-free, and distinguish two cases.

**Case 1.** \( n/d_2 \) is square-free. Suppose that WICG\((n; C)\) has PST. According to Proposition [4] we have that

\[
\lambda_1 - \lambda_0 = c_{d_1}(\mu(n/d_1) - \varphi(n/d_1)) + c_{d_2}(\pm 1 - \varphi(n/d_2)).
\]

Since WICG\((n; C)\) has PST, using Theorem [S] and Lemma [23] it holds that \( \lambda_1 - \lambda_0 \in 2\mathbb{N} \).

Assume that \( c_{d_2} \in 2\mathbb{N} + 1 \). Since \( d_2 \neq n/2 \) we see that \( \varphi(n/d_2) \in 2\mathbb{N} \). It follows that \( c_{d_2}(\pm 1 - \varphi(n/d_2)) \in 2\mathbb{N} + 1 \) and from the fact that \( \lambda_1 - \lambda_0 \in 2\mathbb{N} \) we see that \( c_{d_1}(\mu(n/d_1) - \varphi(n/d_1)) \in 2\mathbb{N} + 1 \). The last relation is true if and only if \( c_{d_1} \in 2\mathbb{N} + 1 \) and \( \mu(n/d_1) \in 2\mathbb{N} + 1 \) (\( \varphi(n/d_1) \) is even). This means that \( n/d_1 \) is square-free.

Assume that there is a prime number \( p \) such that \( p^2 \mid n \). This implies that \( p \mid d_1 \) and \( p \mid d_2 \), since both \( n/d_1 \) and \( n/d_2 \) are square-free. But WICG\((n; C)\) is connected and \( \gcd(d_1, d_2) = 1 \), which is a contradiction. This means that \( n \) is square-free and according to Theorem [13] there is no PST in WICG\((n; C)\).

Now assume that \( c_{d_2} \in 2\mathbb{N} \). Since one of the weights is odd then we have that \( c_{d_1} \in 2\mathbb{N} + 1 \). As \( \lambda_1 - \lambda_0 \in 2\mathbb{N} \), \( c_{d_2} \in 2\mathbb{N} \) and \( c_{d_1} \in 2\mathbb{N} + 1 \) we conclude that \( \mu(n/d_1) - \varphi(n/d_1) \in 2\mathbb{N} \) and thus \( \mu(n/d_1) \in 2\mathbb{N} \). This means that \( n/d_1 \) is not square-free and \( \mu(n/d_1) = 0 \). The relation [34] is now reduced to

\[
\lambda_1 - \lambda_0 = -c_{d_1}\varphi(n/d_1) + c_{d_2}(\pm 1 - \varphi(n/d_2)).
\]

Assume that \( n/2d_1 \) is square-free. Using the fact that \( n/d_1 \) is not square-free we conclude that \( n/d_1 \) is a twice even square-free number. Since \( n/d_1 \) is twice even square-free and \( n/d_2 \) is square free, we can easily conclude that \( n \) is a twice even square-free number. But according to Theorem [14] there is no PST in WICG\((n; C)\). So, \( n/2d_1 \) is not
square-free and thus \( \mu(n/2d) = 0 \). Also since \( n/d_2 \) is square-free we have \( d_2 \in D_0 \cup D_1 \). It follows now that the relation \( (27) \) can be reduced to

\[
\lambda_2 - \lambda_1 = \begin{cases} 
-2c_{d_2}\mu(n/d_2), & d_2 \in D_1 \\
0, & d_2 \in D_0
\end{cases}.
\]

(36)

If \( d_2 \in D_0 \) then \( \lambda_1 = \lambda_2 \). This is a contradiction by Corollary \( 9 \). Thus we assume that \( d_2 \in D_1 \). The last relation implies that \( S_2(\lambda_2 - \lambda_1) = S_2(c_{d_2}) + 1 \) and according to Theorem \( 5 \) we have \( S_2(\lambda_i - \lambda_{i-1}) = S_2(c_{d_i}) + 1 \) for \( 1 \leq i \leq n - 1 \). Since \( S_2(\lambda_1 - \lambda_0) = S_2(c_{d_2}) + 1 \) and \( S_2(c_{d_2}(\pm 1 - \varphi(n/d_2))) = S_2(c_{d_2}) \) we conclude that for the first summand in \( (35) \) it holds \( S_2(c_{d_1}\varphi(n/d_1)) = S_2(c_{d_1}) \). From the fact that \( c_{d_1} \in 2\mathbb{N} + 1 \) we finally have that

\[
S_2(\varphi(n/d_1)) = S_2(c_{d_1}).
\]

(37)

Let \( q_1, q_2, \ldots, q_l \) be all the odd prime divisors of \( n/d_2 \). Since \( d_2 \neq n/2 \) and \( n/d_2 \) is square-free it holds that \( n/d_2 > 2 \) and \( l \geq 1 \).

Let \( p \in \{q_1, q_2, \ldots, q_l\} \). Consider \( 0 \leq j_0 \leq n - 1 \) such that

\[
\begin{align*}
\lambda_0 &\equiv 1 \pmod{p} \\
\lambda_0 &\not\equiv 0 \pmod{q_i} \text{ for } 1 \leq i \leq l \text{ such that } q_i \neq p.
\end{align*}
\]

(38)

This is possible by the Chinese remainder Theorem if we consider a suitable system of congruences modulo \( n/d_2 \). Furthermore, since \( \gcd(j_0, n/d_2) \in \{1, 2\} \) and \( p \mid \gcd(j_0, n/d_2) \) we obtain that \( c(j_0, n/d_2) \in 2\mathbb{N} + 1 \) and \( c(j_0, n/d_2) \in 2\mathbb{N} + 1 \), according to Lemma \( 11 \).

Assume that there exists an odd prime \( r_0 \) such that \( r_0^2 \mid n/d_1 \). If \( r_0 \neq p \) we can suppose that \( j_0 \notin \{0, 1\} \) (mod \( r_0 \)) by adjoining that condition to the above congruence system \( (38) \) and then obtain that both \( \gcd(j_0, n/d_1) \) and \( \gcd(j_0 - 1, n/d_1) \) are not divisible by \( r_0 \). This further means that \( r_0^2 \mid t_{n/d_0,j_0} \) and \( r_0^2 \mid t_{n/d_1,j_0-1} \), which implies that \( c(j_0, n/d_1) = \mu(t_{n/d_0,j_0}) = 0 \) and \( c(j_0 - 1, n/d_1) = \mu(t_{n/d_1,j_0-1}) = 0 \). Finally, it follows that

\[
S_2(\lambda_j - \lambda_{j-1}) = S_2(c_{d_2}(c(j_0, n/d_2) - c(j_0 - 1, n/d_2))) = S_2(c_{d_2}) < S_2(c_{d_2}) + 1,
\]

which is a contradiction.

If \( r_0 = p \) we can find \( 0 \leq j_0 \leq n - 1 \) such that

\[
\begin{align*}
\lambda_0 &\equiv p + 1 \pmod{p^2} \\
\lambda_0 &\not\equiv 0 \pmod{q_i} \text{ for } 1 \leq i \leq l \text{ such that } q_i \neq p.
\end{align*}
\]

(39)

We see that \( j_0 \equiv 1 \pmod{p} \) and thus as in the previous case we conclude that \( c(j_0, n/d_2) \in 2\mathbb{N} + 1 \) and \( c(j_0 - 1, n/d_2) \in 2\mathbb{N} + 1 \), according to Lemma \( 11 \). \( p \mid \gcd(j_0, n/d_1) \) clearly implies \( p^2 \mid t_{n/d_0,j_0} \). It follows that \( c(j_0, n/d_1) = \mu(t_{n/d_0,j_0}) = 0 \). If \( t_{n/d_0,j_0} \) is not square-free we have that \( c(j_0 - 1, n/d_1) = 0 \) and in the previous case we obtain that \( S_2(\lambda_j - \lambda_{j-1}) < S_2(c_{d_2}) + 1 \).

Assume that \( t_{n/d_0,j_0} \) is square-free. Since \( S_p(j_0 - 1) = 1 \), we see that \( S_p(\gcd(j_0 - 1, n/d_1)) = 1 \). Using the fact that \( S_p(n/d_1) = \alpha, \alpha \geq 2 \), we get \( S_p(t_{n/d_0,j_0-1}) = \alpha - 1 \). This implies that \( p - 1 \nmid \varphi(n/d_1)/\varphi(t_{n/d_0,j_0-1}) \). Therefore since \( p - 1 \nmid \varphi(n/d_1) \) and \( p - 1 \nmid c(j_0 - 1, n/d_1) \) we conclude that \( S_2(c(j_0 - 1, n/d_1)) < S_2(\varphi(n/d_1)) \). According to the relation \( (37) \) we obtain \( S_2(c(j_0 - 1, n/d_1)) < S_2(\varphi(n/d_1)) \).

From the above discussion we conclude that \( S_2(c_{d_2}(c(j_0, n/d_2) - c(j_0 - 1, n/d_2))) = S_2(c_{d_2}) \). Furthermore, from \( S_2(c_{d_2}(c(j_0, n/d_2) - c(j_0 - 1, n/d_2))) < S_2(\varphi(n/d_1)) \) we finally obtain that

\[
S_2(\lambda_j - \lambda_{j-1}) < S_2(c_{d_2})
\]

which is a contradiction.

If there is no odd prime \( r_0 \) such that \( r_0^2 \mid n/d_1 \) then \( n \) is a product of \( 2^\alpha \) and an odd square-free number, for some \( \alpha \geq 3 \). Indeed, assume that the last conclusion is not true. This means that there is an odd prime \( p \) such that \( S_p(n) \geq 2 > 0 \) or \( S_p(n) \leq 1 \). Suppose first that \( p^2 \mid n \). Since \( p^2 \mid n/d_1 \) it holds that \( p \mid d_1 \). On the other hand, it follows that \( p \mid d_2 \) since \( n/d_2 \) is square-free. This further means that \( p \mid \gcd(d_1, d_2) \) which is a contradiction since \( \text{WICG}(n; C) \) is connected. Now, suppose that \( 0 \leq S_p(n) \leq 1 \). This means that \( n \) is square-free or twice square-free, but in none of these cases there is \( \text{PST} \) in \( \text{WICG}(n; C) \).

Consider \( 0 \leq j_0 \leq n - 1 \) such that \( j_0 \) satisfies the congruence system \( (38) \) and \( j_0 \equiv 2 \pmod{4} \). As in the previous case, we have that \( c(j_0, n/d_2) \in 2\mathbb{N} + 1 \) and \( c(j_0 - 1, n/d_2) \in 2\mathbb{N} \). Furthermore, since \( j_0 \in 2\mathbb{N} + 1 \) we have that \( \gcd(j_0 - 1, n/d_1) \in 2\mathbb{N} + 1 \), and hence \( 2^\alpha \mid t_{n/d_0,j_0-1} \). From \( \alpha \geq 3 \) we see that \( c(j_0 - 1, n/d_1) = \mu(t_{n/d_0,j_0-1}) = 0 \).
As \( j_0 \in 4N + 2 \) it can be concluded that \( \gcd(j_0, n/d_1) \in 4N + 2 \) and \( 2^{n-1} \mid t_{n/d_1,j_0} \). From \( \alpha \geq 3 \) it follows \( c(j_0, n/d_1) = \mu(t_{n/d_1,j_0}) = 0 \). Now, same as in the previous case we have that

\[
S_2(\lambda_{j_0} - \lambda_{j_0-1}) = S_2(c_{d_d}(c(j_0, n/d_2) - c(j_0-1, n/d_2))) = S_2(c_{d_d}) < S_2(c_{d_d}) + 1
\]

**Case 2.** \( n/d_2 \) is a twice even square-free number. Since \( \mu(n/d_2) = 0 \) then according to Proposition 4 we have

\[
\lambda_1 - \lambda_0 = c_{d_d}(\mu(n/d_1) - \varphi(n/d_1)) - c_{d_d}\varphi(n/d_2).
\]

If \( n/d_1 \) is square-free, then for any odd prime divisor \( p \) of \( n \) such that \( S_p(n) \geq 2 \) we have that \( p \) divides both \( d_1 \) and \( d_2 \) which is impossible since \( \gcd(d_1, d_2) = 1 \). Similarly, we conclude that \( S_2(n) = 2 \). Thus, \( n \) is twice even square-free and we have that there is no PST in \( \text{WICG}(n;C) \), according to Theorem 13. So, \( n/d_1 \) can not be a square-free number, which implies that

\[
\lambda_1 - \lambda_0 = -c_{d_d}\varphi(n/d_1) - c_{d_d}\varphi(n/d_2).
\]

Notice that \( d_2 \in D_2 \). If \( d_1 \in D_1 \) then, since \( n/d_1 \) is not square-free, there exists an odd prime number \( p \) such that \( p^2 \mid n/d_1 \). This means that \( \mu(n/d_1) = \mu(n/2d_1) = 0 \). If \( d_1 \in D_2 \) then \( n/2d_1 \) is square-free if and only if \( n/d_1 \) is a twice square-free number. But if \( n/d_1 \) is twice square-free, from \( \gcd(d_1, d_2) = 1 \) we conclude that \( n \) is twice square-free. According to Theorem 13 there is no PST in \( \text{WICG}(n;C) \). Thus, we may assume that \( n/2d_1 \) is not square-free and \( \mu(n/2d_1) = 0 \). From the preceding analysis it can be concluded that the relation (27) is reduced to

\[
\lambda_2 - \lambda_1 = 2c_{d_d}\mu(n/2d_2).
\]

Now, we have that \( S_2(\lambda_2 - \lambda_1) = S_2(c_{d_d}) + 1 \). Also according to Theorem 8 it holds that \( S_2(\lambda_j - \lambda_{j-1}) = S_2(c_{d_d}) + 1 \) for \( 1 \leq j \leq n - 1 \). Since \( n/d_2 \) is twice even square-free and \( d_2 \neq n/4 \) we have that there exists an odd prime number \( p \) such that \( p \mid n/d_2 \). It follows that \( \varphi(4p) \mid \varphi(n/d_2) \) and hence \( \varphi(n/d_2) \in 4N \). This further implies that \( S_2(c_{d_d}\varphi(n/d_2)) \geq S_2(c_{d_d}) + 2 \) and thus \( S_2(c_{d_d}\varphi(n/d_1)) = S_2(c_{d_d}) + 1 \). From \( d_1 \neq n/2 \) we conclude that \( \varphi(n/d_1) \in 2N \) which yields

\[
S_2(c_{d_d}) + 1 \leq S_2(c_{d_d}\varphi(n/d_1)) = S_2(c_{d_d}) + 1.
\]

Since one of the coefficients \( c_{d_d} \) or \( c_{d_d} \) is odd it holds that \( c_{d_d} \in 2N + 1 \). Furthermore, it follows that

\[
S_2(\varphi(n/d_1)) = S_2(c_{d_d}) + 1.
\]

Let \( r_1, r_2, \ldots, r_s \) be all the prime divisors of \( n/d_1 \) such that \( S_{r_i}(n/d_1) \geq 2 \). Notice that \( s \geq 1 \) and let \( \beta_i = S_{r_i}(n/d_1) \) for \( 1 \leq i \leq s \).

Choose \( 0 \leq j_0 \leq n - 1 \) such that \( j_0 = 2r_1^{\beta_1 - 1}r_2^{\beta_2 - 1} \cdots r_s^{\beta_s - 1} \). Since \( n/d_2 \in 4N \), we can use Lemma 11 to see that \( c(j_0, n/d_2) \in 2N \). On the other hand, from \( \gcd(j_0 - 1, n/d_2) \in 2N + 1 \) we have that \( t_{n/d_2,j_0-1} \in 4N \) and consequently \( c(j_0 - 1, n/d_2) = \mu(t_{n/d_2,j_0-1}) = 0 \). From the previous discussion we can conclude that

\[
S_2(c_{d_d}(c(j_0, n/d_2) - c(j_0 - 1, n/d_2))) \geq S_2(c_{d_d}) + 1.
\]

It is easy to see that \( r_i \nmid \gcd(j_0 - 1, n/d_1) \) for \( 1 \leq i \leq s \) and therefore we have that \( r_i^2 \mid t_{n/d_1, j_0} \). It follows that \( c(j_0 - 1, n/d_1) = \mu(t_{n/d_1, j_0}) \). Furthermore, from the fact that \( r_1^{\beta_1 - 1}r_2^{\beta_2 - 1} \cdots r_s^{\beta_s - 1} \mid \gcd(j_0, n/d_1) \) we conclude that \( t_{n/d_1, j_0} \) is square-free and \( r_1r_2 \cdots r_s \mid t_{n/d_1, j_0} \). This implies that

\[
S_2(c_{d_d}(c(j_0, n/d_1)) = S_2(\varphi(n/d_1)) - S_2(\varphi(t_{n/d_1, j_0})) < S_2(\varphi(n/d_1)).
\]

Now, according to (40) and (42) we have that

\[
S_2(c_{d_d}(c(j_0, n/d_1)) = S_2(c(j_0, n/d_1)) = S_2(\varphi(n/d_1)) < S_2(c_{d_d}) + 1.
\]

Finally, using (11) and (13) we obtain

\[
S_2(\lambda_{j_0} - \lambda_{j_0-1}) < S_2(c_{d_d}) + 1
\]

which is a contradiction.

\[\square\]

The last theorem implies that we can assign to the edges of a graph \( \text{ICG}_n(D) \), where \( D = \{d_1, d_2\} \) and \( d_1 < d_2 \), some weights \( c_{d_d} \) and \( c_{d_d} \) so as to create PST if and only if \( n \) is even and \( d_2 \in \{n/4, n/2\} \). This means that for a given
In this paper, we show that the evolution of a quantum system, whose Hamiltonian is identical to the adjacency matrix of a weighted circulant graph, is periodic if and only if the graph is integral. We prove that by finding the necessary condition for PST existence in such systems, which is equivalent to the fact that the graph is integral. Thus, the next natural step was proving Theorem 3 by which we characterize integral graphs in the class of all weighted circulant graphs (with integer weights). In addition, we give a simple and general condition in terms of eigenvalues for weighted integral circulant graphs (with integer weights) to have PST (Theorem 8). By Theorem 7 we prove that there is no PST in weighted integral circulant graphs of odd order. Combining the previous result with Theorem 10 we obtain one of our main results of this paper which can be formulated as follows: for an arbitrary $n \in N$, there is a weighted integral circulant graph of order $n$ having a PST if and only if $n$ is even. Moreover, this result evidently extends the corresponding result for unweighted graphs, given by Theorem 9. Indeed, in the weighted case $n/4$ and $n/2$ may both belong to $D$, there are no restrictions concerning the remaining divisors of $D$ and $n$ is only required to be even. We can also calculate the number of weighted circulant networks satisfying the conditions of our theorem (thus having PST). In general case when $n \in 4N$, this number is equal to the number of integral circulant graphs such that $n/4 \in D$ or $n/2 \in D$ and can easily be shown to be $3 \cdot 2^{\tau(n)-1}$. Since, there are at most $2^{\tau(n)-1}$ integral circulant graphs on $n$ vertices, we conclude that the number of integral circulant networks having PST is asymptotically equal to the number of integral circulant graphs of a given order $n$.

In the rest of Section 5, we use Theorem 15 to prove nonexistence of PST in those $WICG(n; C)$ for which exactly two entries of $C$ are positive and $c_{n/4} = c_{n/2} = 0$. The proof requires an extensive discussion and falls into a good many of distinct cases. Generally, the proofs presented in this paper are based on the connection between number theory, polynomial theory and graph theory. Attempts to generalize Theorem 15 by allowing more than two positive entries of $C$ such that $c_{n/4} = c_{n/2} = 0$ would be much more demanding and probably require considering a significantly greater number of cases and we leave it for future research. In fact, finding classes of $WICG(n; C)$ having PST such that $c_{n/4} = c_{n/2} = 0$ could increase the maximal perfect quantum communication distance in such networks.

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