ON A COHOMOLOGICAL GENERALIZATION OF THE SHAFAREVICH CONJECTURE FOR K3 SURFACES

TEPPEI TAKAMATSU

Abstract. The Shafarevich conjecture for K3 surfaces states the finiteness of isomorphism classes of K3 surfaces over a fixed number field admitting good reduction away from a fixed finite set of finite places. André proved this conjecture for polarized K3 surfaces of fixed degree, and recently She proved it for polarized K3 surfaces of unspecified degree. In this paper, we prove a certain generalization of their results, which is stated by the unramifiedness of ℓ-adic étale cohomology groups for K3 surfaces over finitely generated fields of characteristic 0. As a corollary, we get the original Shafarevich conjecture for K3 surfaces without assuming the extendability of polarization, which is stronger than the results of André and She. We also give an additional result which states an ℓ-independence about the unramifiedness. Moreover, as an application, we get the finiteness of twists of K3 surfaces via a finite extension of characteristic 0 fields.

1. Introduction

The Shafarevich conjecture for abelian varieties is a remarkable result which states the finiteness of isomorphism classes of abelian varieties of a fixed dimension over a fixed number field admitting good reduction away from a fixed finite set of finite places. This theorem was proved by Faltings (in the polarized case, see [Fal83]) and Zarhin (in the unpolarized case, see [Zar85]).

In this paper, we shall prove an analogue of this theorem for K3 surfaces. For any discrete valuation field $K$ and a K3 surface $X$ over $K$, we say $X$ has good reduction if $X$ admits a smooth proper model over the valuation ring of $K$, as an algebraic space (see [LM18, Section 1]). Remark that it is natural to admit an integral model being an algebraic space rather than a scheme in the case of K3 surfaces (see [Mat15, Section 5.2]). Then one can formulate the analogue of the Shafarevich conjecture for K3 surfaces. Previously, this conjecture was studied by André ([And96]) and She ([She17]) for polarized K3 surfaces. The goal of this paper is to generalize their results in terms of the unramifiedness of ℓ-adic étale cohomology groups. Our main theorem is the following (for more generalized form, see Theorem 6.1.1).

**Theorem 1.0.1** (compare with Theorem 6.1.1). Let $F$ be a finitely generated field over $\mathbb{Q}$, $\ell$ be a prime number, and $R$ be a finite type algebra over $\mathbb{Z}$ which is a normal domain with the fraction field $F$. Then, the set

$$
\text{Shaf}(F, R) := \left\{ X : \begin{array}{l}
X: \text{K3 surface over } F, \\
\text{for any height 1 prime ideal } p \in \text{Spec } R, \\
\text{there exists a prime number } \ell \notin p \\
\text{such that } H^2_{\text{ét}}(X, \mathbb{Q}_{\ell}) \text{ is unramified at } p \\
\end{array} \right\} / F\text{-isom}
$$
is finite.

As a corollary, we have the original Shafarevich conjecture for K3 surfaces over finitely generated fields of characteristic 0.

**Corollary 1.0.2** (Corollary 6.1.4). Let $F$ be a finitely generated field over $\mathbb{Q}$, and $R$ be a finite type algebra over $\mathbb{Z}$ whose fraction field is $F$. Then, the set

$$\left\{ X : \text{K3 surface over } F, X \text{ has good reduction at any height 1 prime ideal } p \in \text{Spec } R \right\} / F\text{-isom}$$

is finite.

Note that our results are stronger than results of André and She (see Remark 1.0.5 for details). Moreover, as an application of our cohomological generalization, we get the following corollary, which states the finiteness of twists of a K3 surface via a finite extensions of characteristic 0 fields.

**Corollary 1.0.3** (Corollary 6.2.1). Let $F$ be a field of characteristic 0, $E/F$ be a finite extension, and $X$ be a K3 surface over $F$. Then, the set

$$\text{Tw}_{E/F}(X) := \left\{ Y : \text{K3 surface over } F \mid Y_E \simeq E X_E \right\} / F\text{-isom}$$

is finite. Here $Y_E \simeq E X_E$ means the K3 surfaces $Y_E := Y \otimes_F E$ and $X_E := X \otimes_F E$ are isomorphic over $E$.

We note that our cohomological generalization is necessary for this application, i.e. the original statement of the Shafarevich conjecture (Corollary 1.0.2) is not enough to show Corollary 1.0.3.

Additionally, we also give the following $\ell$-independence result to prove the main theorem. The following result is well-known if the residue field $k$ is finite (see also Remark 5.0.2), but we consider more general situations to apply to our settings in Theorem 1.0.1.

**Theorem 1.0.4** (Theorem 5.0.1). Let $K$ be a Henselian discrete valuation field, $k$ be the residue field of $K$, $p$ be the characteristic of $k$. Assume that the characteristic of $K$ is different from 2. Then, for any K3 surface $X$ over $K$, the following are equivalent.

(a) The $\text{Gal}(\overline{K}/K)$-representation on $H^2_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for some $\ell \neq p$.
(b) The $\text{Gal}(\overline{K}/K)$-representation on $H^2_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for all $\ell \neq p$.

Moreover, if $K$ is a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field $k$, then (a) $\iff (b)$ is equivalent to the following.

(c) The $\text{Gal}(\overline{K}/K)$-representation on $H^2_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline.

Let us give some comments on the statement of Theorem 1.0.1. Theorem 1.0.1 is motivated by the good reduction criterion for K3 surfaces given by Liedtke and Matsumoto ([LMT18]). For K3 surfaces over a Henselian discrete valuation field satisfying some assumptions, they showed the equivalence between the unramifiedness of $\ell$-adic étale cohomology groups and admitting good reduction after a finite unramified extension ([LMT18 Theorem 1.3]). Remark that the latter condition cannot be replaced by ‘admitting good reduction’ (see [LMT18 Theorem 1.6]), so our cohomological generalization is stronger than the original Shafarevich conjecture. Moreover, we deal with finitely generated fields of characteristic 0 rather than number fields, motivated by the application to Corollary 1.0.3. In fact, André also proved the Shafarevich conjecture
for polarized K3 surfaces in this nature (see [And96, Theorem 9.1.1], and see also the following Remark 1.0.5).

**Remark 1.0.5.** Our results are stronger than previous results obtained by André and She. To explain this, we briefly recall their results. André proved the Shafarevich conjecture for polarized K3 surfaces ([And96, Theorem 9.1.1]), i.e. the finiteness of isomorphism classes of polarized K3 surfaces of fixed degree over a fixed number field which admit good reduction away from a fixed finite set of finite places (actually, as stated above, André dealt with finitely generated fields of characteristic 0). Here, André said that a polarized K3 surface $(X, L)$ admits good reduction if there exists a smooth proper model $X$ of $X$ as a scheme such that the ample line bundle $L$ extends to an ample line bundle on $X$. Recently She proved it for polarized K3 surfaces of unspecified degree ([She17, Theorem 1.1.5]). More correctly, She proved the finiteness of K3 surfaces over a fixed number field which admit good reduction as polarized K3 surfaces (without fixing polarization degree) away from a fixed finite set of finite places. Here, we remark that She’s result does not cover K3 surfaces admitting a smooth proper model only as an algebraic space. Moreover, there exists an example of a K3 surface admitting good reduction such that any smooth proper model does not have a polarization (therefore this K3 surface does not admit good reduction as polarized K3 surfaces) (see [Mat15, Section 5.2]). Therefore, Corollary 1.0.2 is also stronger than previous results, even in the number field case.

The strategy of the proof of Theorem 1.0.1 is as follows. We basically take the approach of André and She. We first show the polarized version of Theorem 1.0.1 before dealing with the unpolarized case. Our main tool is the uniform Kuga-Satake construction introduced by She. We use it to study K3 surfaces of all degrees simultaneously. Our proof is slightly different from She’s proof, and here we will sketch the differences. In She’s paper ([She17]), it is crucial to show that K3 surfaces admitting good reduction are sent to abelian varieties admitting good reduction via the uniform Kuga-Satake map. She proves this using integral canonical models of certain Shimura varieties (the argument like ‘$O$-valued points go to $O$-valued points’). However, in our case, we do not assume that each K3 surface admits a smooth proper model, so instead of She’s method, we use the Néron-Ogg-Shafarevich criterion for abelian varieties. For this purpose, we study She’s uniform Kuga-Satake construction in detail in Section 3. Note that our proof does not require the theory of integral canonical models of Shimura varieties. Because of the bad behavior of $SO$ and $GSpin$, we need the unramifiedness of 2-adic representations. Hence to complete the proof of the main theorem, we need the $\ell$-independence result (Theorem 1.0.4). In Section 5, we prove it using the same technique as above, i.e. using the Kuga-Satake construction and the Néron-Ogg-Shafarevich criterion for abelian varieties.

The outline of this paper is as follows. In Section 2, we will recall the basic results on K3 surfaces, and define the moduli space of K3 surfaces introduced by Rizov and Madapusi Pera. In Section 3, we will define several algebraic groups to introduce the uniform Kuga-Satake abelian varieties, and study their basic properties. In Section 4, we will prove the main theorem in a little weaker form (i.e. only considering 2-adic cohomology) by using the results of Section 3 and the arguments given by André and She. In Section 5, we will show $\ell$-independence results on the unramifiedness by using Ochiai’s $\ell$-independence results and the Kuga-Satake abelian varieties. In Section 6, we will complete the proof of the main theorem combining the results in Section 4 and
Section 5, and prove the finiteness of twists via a finite extension of characteristic 0 fields.

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2. K3 surfaces and their moduli

2.1. Basic definitions for K3 surfaces. In this subsection, we give definitions and basic notations about K3 surfaces.

Definition 2.1.1.

1. For any field \( k \), a \( K3 \) surface over \( k \) is a smooth proper surface \( X \) over \( k \) with \( \Omega^2_{X/k} \cong \mathcal{O}_{X/k} \) and \( H^1(X, \mathcal{O}_{X/k}) = 0 \).

2. For any scheme \( S \), a \( K3 \) family over \( S \) is a smooth proper algebraic space \( X \) over \( S \) whose geometric fibers are K3 surfaces.

Remark 2.1.2. For any field \( k \), a K3 family over \( k \) is automatically a K3 surface over \( k \) since smooth proper algebraic spaces of dimension 2 over a field are schemes.

Definition 2.1.3 (Riz06 Definition 3.2.2, Definition 3.2.3).

1. A polarization on a K3 family \( \pi: X \to S \) is an element \( \lambda \in \text{Pic}_X(S) \) whose pullback by any geometric point of \( S \) is an ample line bundle. Here \( \text{Pic}_X(S) \) is the relative Picard functor.

2. A polarization \( \lambda \) is primitive if its pullback by any geometric point of \( S \) is primitive, i.e. not divisible by an integer greater than 1.

3. A polarization \( \lambda \) is of degree \( 2d \) if its pullback by any geometric point of \( S \) has degree \( 2d \), i.e. its self intersection number is \( 2d \).

Remark 2.1.4. Let \( F \) be a subfield of \( \mathbb{C} \). For a K3 surface \( X \) over \( F \), the relative Picard functor \( \text{Pic}_{X/F} \) is represented by a scheme, thus \( \text{Pic}_{X/F}(F) = \text{Pic}(X)_{\text{Gal}(\overline{F}/F)} \) is a primitive sublattice in \( H^2(X(\mathbb{C}), \mathbb{Z}(1)) \) via the Chern class map as in [She17, Lemma 2.2.3]. Hence there exists a primitive polarization for each \( X \) (by dividing a polarization by an integer greater than 1 if necessary). Remark that the inclusion \( \text{Pic}(X) \subset \text{Pic}_{X/F}(F) \) may be proper in general, though it always has a finite cokernel (see [Huy16, Chapter 17, Section 2.2]).

Definition 2.1.5. (1) A K3 lattice \( \mathcal{L}_{K3} \) is a unimodular lattice of signature (19, 3) which is defined as

\[
\mathcal{L}_{K3} := E_8^8 \oplus \mathbb{H}^3,
\]

where \( E_8 \) is the (positive signature) \( E_8 \)-lattice as in [Huy16, Chapter 14, Example 0.3], and \( \mathbb{H} \) is the hyperbolic plane.

(2) Consider the last component \( \mathbb{H} \subset \mathcal{L}_{K3} \), and take \( e, f \in \mathbb{H} \subset \mathcal{L}_{K3} \) satisfying

\[
(e,f) = (f,e) = 1, \quad (e,e) = (f,f) = 0.
\]
Let $v_d := e - df$. Then the degree $2d$ primitive part of $L_{K3}$ is defined as
\[ L_d := v_d^\perp \cong \mathbb{E}_8^\perp \oplus \mathbb{H}^2 \oplus \langle 2d \rangle. \]

The lattice $L_d$ is a primitive sublattice of $L_{K3}$, and $\text{disc}(L_d) = 2d$.

**Remark 2.1.6** ([Riz06] Remark 2.3.2). For a K3 surface $X$ over $\mathbb{C}$ and its primitive polarization $L$ of degree $2d$, there exists an isomorphism
\[ (H^2(X(\mathbb{C}), \mathbb{Z}(1)), -\cup) \cong L_{K3} \]
which sends $\text{ch}_2(L)$ to $v_d$. Here $-\cup$ denotes the minus of the cup product. Therefore, for a primitively polarized K3 surface $(X, L)$ of degree $2d$ over a field $F$ which is contained in $\mathbb{C}$, we sometimes identify $H^2(X(\mathbb{C}), \mathbb{Z}(1))$ with $L_{K3}$, and $P^2((X(\mathbb{C}), L, \mathbb{Z}(1)))$ with $L_{K3,2} := L_{K3,2} \cong \mathbb{Z}^2$, $P^2((X, L, \mathbb{Z}(1)))$ with $L_d$, and $P^2((X, L, \mathbb{Z}(1)))$ with $L_{d,2} := L_d \otimes \mathbb{Z}$. Here, we denote the primitive parts of the singular cohomology group and the étale cohomology group by
\[ P^2((X(\mathbb{C}), L, \mathbb{Z}(1))) := \text{ch}(L) \perp \subset H^2(X(\mathbb{C}), \mathbb{Z}), \]
\[ P^2((X, L, \mathbb{Z}(1))) := \text{ch}(L) \perp \subset H^2(X, \mathbb{Z}). \]
To simplify the notation, in the following of this paper, we omit the pairing. We denote $(H^2(X(\mathbb{C}), \mathbb{Z}(1)), -\cup)$ by $H^2(X(\mathbb{C}), \mathbb{Z}(1))$, and same with others.

**Definition 2.1.7.** The discriminant kernel of $L_d$ is
\[ D_d := \{ g \in \text{SO}(L_{d,2}) \mid g \text{ acts trivially on } L_{d,2}/L_{d,2} \}. \]
Remark that $D_d$ is a compact open subgroup of $\text{SO}(L_{d,2})$. For any prime number $\ell$, we denote its $\mathbb{Z}_\ell$-component by $(D_d)_{\ell}$.

**Proposition 2.1.8** ([MP16] Lemma 2.6]). There is a natural identification
\[ D_d = \{ \tilde{g} \in \text{SO}(L_{K3,2}) \mid \tilde{g}(v_d) = v_d \}. \]

**Proof.** This is proved in [MP16] Lemma 2.6]. We include its proof because we need to recall the identification explicitly. Let $\ell$ be any prime number, and we will verify this claim for each $\mathbb{Z}_\ell$-component. First, we will define a map from the left-hand side to the right-hand side. For
\[ g_{\ell} \in (D_d)_{\ell} = \{ g \in \text{SO}(L_{d,2}) \mid g \text{ acts trivially on } L_{d,2}/L_{d,2} \}, \]
define $\tilde{g}_{\ell}$ as the image of $g_{\ell}$ via the composition of the following
\[ \text{SO}(L_{d,2}) \hookrightarrow \text{SO}(L_{d,2}) \hookrightarrow \text{SO}(L_{K3,2}). \]
Then we have $\tilde{g}_{\ell}v_d = v_d$. We will show that $\tilde{g}_{\ell}L_{K3,2} = L_{K3,2}$. Consider the morphisms
\[ L_{K3,2} \cong L_{K3,2} \hookrightarrow L_{d,2} \oplus \langle v_d \rangle \cong L_{d,2}. \]
For any $v \in L_{K3,2}$, denote its image in $L_{d,2} \oplus \langle v_d \rangle$ by $u_1 + u_2$. Then we have
\[ \tilde{g}_{\ell}(u_1 + u_2) = g_{\ell}(u_1) + u_2 = (g_{\ell}(u_1) - u_1) + (u_1 + u_2) \in L_{K3,2}, \]
because $g_{\ell}$ acts trivially on $L_{d,2}$. Hence $\tilde{g}_{\ell}^{\pm 1}L_{K3,2} \subset L_{K3,2}$, thus $\tilde{g}_{\ell}L_{K3,2} = L_{K3,2}$, and we can define the desired map.

Next, we will define a map from the right-hand side to the left-hand side. For $h_{\ell} \in \text{SO}(L_{K3,2})$ such that $h_{\ell}v_d = v_d$, we can associate $h_{\ell} \in \text{SO}(L_{d,2})$ as the restriction of $h_{\ell}$. We can show that $h_{\ell}$ acts trivially on $L_{d,2}/L_{d,2}$. Indeed, because the embedding
\( \mathcal{L}_d \hookrightarrow \mathcal{L}_{K3} \) is primitive, the composition of (1) is surjective, so for any \( u_1 \in \mathcal{L}'_{d, \mathbb{Z}_p} \), there exists \( u_2 \in (v_d)^{\vee} \) such that \( u_1 + u_2 \in \mathcal{L}_{K3, \mathbb{Z}_p} \). Thus we have

\[
h_e(u_1) - u_1 = \hat{h}_e(u_1 + u_2) - (u_1 + u_2) \in \mathcal{L}_{K3, \mathbb{Z}_p} \cap \mathcal{L}_{d, \mathbb{Q}_p} = \mathcal{L}_{d, \mathbb{Z}_p}.
\]

Clearly, the above maps are inverses of each other, so it finishes the proof. \( \square \)

### 2.2. Moduli spaces of K3 surfaces and the Torelli theorem.

In this subsection, we recall the Torelli theorem in terms of moduli spaces. First, we recall the definition of the moduli space of K3 surfaces with oriented level structures. See [Riz06, Section 6], [MP15, Section 3], [IIK18, Section 5] for details.

We define a groupoid-valued moduli functor \( M^o_{2d, \mathbb{Q}} \) by

\[
M^o_{2d, \mathbb{Q}}(S) := \left\{ (\pi: \mathcal{X} \to S, \lambda \in \text{Pic}_{\mathcal{X}/S}(S)) \mid \pi: \text{K3 family over } S, \lambda: \text{primitive polarization of degree } 2d \right\}
\]

for any \( \mathbb{Q} \)-scheme \( S \). Let \( \tilde{M}^o_{2d, \mathbb{Q}} \) be the twofold finite étale cover constructed by Madapusi Pera ([MP15, Section 5]) which parameterizes orientations. Then, for any \( S \to \tilde{M}^o_{2d, \mathbb{Q}} \) we get \( (\pi, \lambda, \nu) \) where \( (\pi, \lambda) \) is as above, and \( \nu \) is an isometry of \( \hat{\mathbb{Z}} \)-local systems

\[
\nu: \det \mathcal{L}_{d, \hat{\mathbb{Z}}} \cong \det P^2\pi_*\hat{\mathbb{Z}}
\]

such that for any \( s \in S(\mathbb{C}) \), the isometry \( \nu \) restricts to an isometry

\[
\nu_s: \det \mathcal{L}_d \cong \det P^2(\mathcal{X}_s, \mathbb{Z}).
\]

Here, we put

\[
P^2\pi_*\hat{\mathbb{Z}}(1) := \text{ch}_{\hat{\mathbb{Z}}}(\lambda)^{\perp} \subset R^2\pi_*\hat{\mathbb{Z}}(1),
\]

where \( \text{ch}_{\hat{\mathbb{Z}}}(\lambda) \) is the Chern class of \( \lambda \) ([MP15 3.10]). Let \( \mathbb{K} \subset D_d \) be a compact open subgroup. For any scheme \( S \) over \( \tilde{M}^o_{2d, \mathbb{Q}} \), one can define the étale sheaf \( I \) by

\[
I(T) := \left\{ g: \mathcal{L}_{K3, \hat{\mathbb{Z}}} \to R^2\pi|_T\hat{\mathbb{Z}}(1) \mid g: \text{isometry}, g(v_d) = \text{ch}_{\hat{\mathbb{Z}}}(\lambda), \det g \text{ induces } \nu|_T \right\},
\]

for any étale morphism \( T \to S \). A \( \mathbb{K} \)-level structure on \( S \to \tilde{M}^o_{2d, \mathbb{Q}} \) is a section \( \alpha \in H^0(S, I/\mathbb{K}) \), where \( \mathbb{K} \) acts on \( I \) through \( \mathcal{L}_{K3, \hat{\mathbb{Z}}} \). Then, one can define the moduli functor \( M^o_{2d, \mathbb{K}, \mathbb{Q}} \) over \( \tilde{M}^o_{2d, \mathbb{Q}} \) which parameterizes \( \mathbb{K} \)-level structures. For simplicity, we write an each element of \( M^o_{2d, \mathbb{K}, \mathbb{Q}}(S) \) as \( (\mathcal{X}, \lambda, \nu, \alpha) \). Moreover, for any field \( F \) of characteristic 0, we denote the base change by \( M^o_{2d, \mathbb{K}, F} \).

**Definition 2.2.1.**

1. \( \text{SO}_{\mathcal{L}_d} \) is an algebraic group over \( \mathbb{Q} \) whose \( R \)-valued points are given by

\[
\text{SO}_{\mathcal{L}_d}(R) := \{ g \in \text{SL}(\mathcal{L}_{d, R}) \mid (gv, gw) = (v, w), \text{ for any } v, w \in \mathcal{L}_{d, R} \}.
\]

2. We put

\[
\Omega_{\text{SO}_{\mathcal{L}_d}}^\pm := \{ \text{oriented negative definite planes in } \mathcal{L}_{d, \mathbb{R}} \}.
\]

Then \( \Omega_{\text{SO}_{\mathcal{L}_d}}^\pm \) is naturally identified with \( X_{\text{SO}_{\mathcal{L}_d}} \) which gives the Shimura datum \( (\text{SO}_{\mathcal{L}_d}, X_{\text{SO}_{\mathcal{L}_d}}) \) with a reflex field \( \mathbb{Q} \). More precisely, \( X_{\text{SO}_{\mathcal{L}_d}} \) is the image of \( X_{\text{GSpin}_{\mathcal{L}_d}} \) which is defined as in Definition 3.2.3.

Here, we quickly state the moduli interpretation of the Torelli theorem over \( \mathbb{Q} \).
Proposition 2.2.2 (The Torelli theorem, [MP15, Corollary 5.4, Theorem 5.8]). Let \( \mathbb{K} \subset D_d \) be a compact open subgroup. Moreover, assume that \( \mathbb{K} \) is contained in the principal level \( n \) congruence subgroup of \( \text{SO}(L_d, \hat{\mathbb{Z}}) \) with \( n \geq 3 \). Then \( M_{2d, \mathbb{K}, \mathbb{Q}} \) is representable by a scheme, and moreover there is the period map which is an \( \acute{e}tale \) morphism between \( \mathbb{Q} \)-schemes

\[
j: M_{2d, \mathbb{K}, \mathbb{Q}} \to \text{Sh}_K(\text{SO}_{L_d}, X_{\text{SO}_{L_d}}).
\]

Here \( \text{Sh}_K(\text{SO}_{L_d}, X_{\text{SO}_{L_d}}) \) is the canonical model of the Shimura variety over \( \mathbb{Q} \).

In Proposition 3.3.3, we will use the more detailed properties of the period map \( j \).

3. The uniform Kuga-Satake construction

In this section, we recall the definition and properties of the Kuga-Satake construction. In this section, we use only the uniform Kuga-Satake construction introduced by She. In fact, the classical Kuga-Satake construction is enough for proving the polarized case (Theorem 4.1.3), but we need She’s methods to prove the unpolarized case (Theorem 4.1.4). Hence we omit the classical Kuga-Satake construction for avoiding some repetitions.

3.1. Preparation I. In this and next subsection, we will define several algebraic groups and their adelic subgroups which play an important role in the Kuga-Satake construction. In this subsection we discuss objects related with the lattice \( L_d \).

For any algebra \( R \) and any lattice \( N \) over \( R \), we denote the Clifford algebra (resp. even Clifford algebra) of \( N \) by \( C(N) \) (resp. \( C^+(N) \)). Here, a lattice over \( R \) means a finite free module with a symmetric bilinear form.

Definition 3.1.1. \( \text{GSpin}_{L_d} \) is an algebraic group over \( \mathbb{Q} \), whose \( R \)-valued points are given by

\[
\text{GSpin}_{L_d}(R) := \{ z \in C^+(L_{d,R})^\times \mid zL_{d,R}z^{-1} = L_{d,R} \}.
\]

Remark 3.1.2. (1) There exists the following natural homomorphism of algebraic groups over \( \mathbb{Q} \)

\[
f_d: \text{GSpin}_{L_d} \to \text{SO}_{L_d}; g \mapsto (l \mapsto glg^{-1}).
\]

(2) For any \( \mathbb{Z} \)-algebra \( R \), we put

\[
\text{GSpin}(L_{d,R}) := \{ z \in C^+(L_{d,R})^\times \mid zL_{d,R}z^{-1} = L_{d,R} \}.
\]

Then, for any prime number \( \ell \), we can define

\[
f_d: \text{GSpin}(L_{d,\mathbb{Z}_\ell}) \to \text{SO}(L_{d,\mathbb{Z}_\ell})
\]

by the conjugation. Moreover, it is easy to confirm the following identity

\[
\text{GSpin}(L_{d,\mathbb{Z}_\ell}) = \text{GSpin}(L_{d,\mathbb{Q}_\ell}) \cap C^+(L_{d,\mathbb{Z}_\ell})^\times.
\]

(3) For any \( \mathbb{Z} \)-algebra \( R \), we will use the notation \( \text{GSpin}(L_{K3,R}) \) in the similar sense as in (2). Moreover, for any prime number \( \ell \), we denote the conjugation map \( \text{GSpin}(L_{K3,\mathbb{Z}_\ell}) \to \text{SO}(L_{K3,\mathbb{Z}_\ell}) \) by \( f_{K3} \). As in (2), it follows that

\[
\text{GSpin}(L_{K3,\mathbb{Z}_\ell}) = \text{GSpin}(L_{K3,\mathbb{Q}_\ell}) \cap C^+(L_{K3,\mathbb{Z}_\ell})^\times.
\]
Lemma 3.1.3 ([MP16, (2.6.1)]). Let \( \ell \) be any prime number. Through the natural inclusion \( C^+(L_{d,Z}) \subset C^+(L_{K3,Z}) \), we have
\[
C^+(L_{d,Z}) = \{ z \in C^+(L_{K3,Z}) \mid v_dz = zv_d \}.
\]
Moreover, the above inclusion induces an embedding
\[
\text{GSpin}(L_{d,Z}) \subset \text{GSpin}(L_{K3,Z}).
\]

Proof. The first claim is essentially proved in [MP16, (2.6.1)]. For the sake of completeness, we recall the proof. For the first claim, both sides of the desired identity are primitive \( Z_d \)-modules in \( C^+(L_{K3,Z}) \). Thus, it is enough to show that
\[
C^+(L_{d,Q_\ell}) = \{ z \in C^+(L_{K3,Q_\ell}) \mid v_dz = zv_d \}.
\]
It can be easily verified by using a basis of \( L_{d,Q_\ell} \) and \( v_d \). For the second claim, by Remark 3.1.2 (2) and (3), we can reduce the problem to the obvious inclusion \( \text{GSpin}(L_{d,Q_\ell}) \subset \text{GSpin}(L_{K3,Q_\ell}) \).

Definition 3.1.4 ([And96, Section 4.4], [Riz06, Example 5.1.4]). For any positive integer \( n \), we define a compact open subgroup \( K_{d,n}^{\text{sp}} \subset \text{GSpin}_d(A_f) \) by
\[
K_{d,n}^{\text{sp}} := \{ g \in \text{GSpin}(L_{d,\mathbb{Z}}) \mid g = 1 \text{ in } C^+(L_{d,\mathbb{Z}/(n\mathbb{Z})}) \}.
\]

Proposition 3.1.5 (cf. [And96, Section 4.4], [MP15, Section 4.4]).
\[
D_d(n) := f_d(K_{d,n}^{\text{sp}}) \subset \text{SO}(L_{d,\mathbb{Z}})
\]
is a compact open subgroup of \( D_d \).

Proof. First, we shall show that \( D_d(n) \) is contained in \( D_d \). Lemma 3.1.3 shows that
\[
f_{K3}(\text{GSpin}(L_{d,Z})) \subset \{ g \in \text{SO}(L_{K3,Z}) \mid g v_d = v_d \},
\]
thus the desired inclusion follows from Proposition 2.1.8.

For the openness, it is enough to show that for any \( \ell \) not dividing \( 2dn \), the \( Z_d \)-component of \( D_d(n) \) is equal to \( \text{SO}(L_{d,Z}) \). It follows from [And96, Section 4.4].

The following proposition gives more information about \( D_d(n) \).

Proposition 3.1.6. For any odd prime number \( \ell \neq 2 \), we have
\[
f_d(\text{GSpin}(L_{d,Z})) = (D_d)_\ell.
\]
If \( \ell = 2 \), as a subset of \( \text{SO}(L_{K3,Z}) \), we have
\[
f_d(\text{GSpin}(L_{d,Z})) = (D_d)_2 \cap f_{K3}(\text{GSpin}(L_{K3,Z})).
\]

Proof. If \( \ell \) does not divide \( 2d \), these results are essentially shown in the proof of Proposition 3.1.5.

First, for any prime number \( \ell \), we have \( f_d(\text{GSpin}(L_{d,Z})) \subset (D_d)_\ell \) as in the proof of Proposition 3.1.5. We assume \( \ell \neq 2 \). For any \( g \in (D_d)_\ell \subset \text{SO}(L_{K3,Z}) \), by the same argument as in [And96, Section 4.4] (here we use \( \ell \neq 2 \)), there exists \( z \in \text{GSpin}(L_{K3,Z}) \) such that \( f_{K3}(z) = g \). Proposition 2.1.8 implies \( zv_dz^{-1} = v_d \), and so in fact, \( z \in C^+(L_{d,Z}) \) via conjugation, thus \( z \in \text{GSpin}(L_{d,Z}) \) and it finishes the proof of the first claim. If \( \ell = 2 \), the second claim follows by the same arguments.
Remark 3.1.7. Unfortunately, if $l = 2$, we have $f_d(GSpin(\mathcal{L}_{d,\mathbb{Z_2}})) \neq (D_d)_2$. Indeed, there exists $g_2 \in (D_d)_2$ which is non-trivial in $SO(\mathcal{L}_{d,\mathbb{Z_2}/2\mathbb{Z}})$ (for example, permutation of two components $\mathbb{H}_{\mathbb{Z_2}} \subset \mathcal{L}_{d,\mathbb{Z_2}}$), though any element in the image of $f_d$ is trivial there.

Corollary 3.1.8. Let $(D_d(n))_\ell$ be the $\mathbb{Z_\ell}$-component of $D_d(n)$, and $n_\ell$ be the $\ell$-part of $n$. Then, for any prime number $\ell \neq 2$, we have

$$[(D_d)_\ell : (D_d(n))_\ell] \leq n_\ell^{(20)}.$$  

Moreover, there exists a positive integer $N$ which is independent of $d$ and $n$ such that

$$[(D_d)_2 : (D_d(n))_2] \leq N \cdot n_2^{(20)}.$$  

Proof. Assume $\ell \neq 2$. We have the following commutative diagram.

$$\begin{array}{ccc}
GSpin(\mathcal{L}_{d,\mathbb{Z_\ell}}) & \rightarrow & (D_d)_\ell \\
\downarrow & & \downarrow \\
(\mathbb{K}_{d,n}^{sp})_\ell & \rightarrow & (D_d(n))_\ell
\end{array}$$

Here we have

$$(\mathbb{K}_{d,n}^{sp})_\ell = \{ g \in GSpin(\mathcal{L}_{d,\mathbb{Z_\ell}}) \mid g = 1 \text{ in } C^+(\mathcal{L}_{d,\mathbb{Z_\ell}/n_i\mathbb{Z_\ell}}) \}.$$  

Since

$$\#(C^+(\mathcal{L}_{d,\mathbb{Z_\ell}/n_i\mathbb{Z_\ell}})^*) \leq \#(C^+(\mathcal{L}_{d,\mathbb{Z_\ell}/n_i\mathbb{Z_\ell}})) = n_\ell^{(20)},$$

the index of $(\mathbb{K}_{d,n}^{sp})_\ell$ in $GSpin(\mathcal{L}_{d,\mathbb{Z_\ell}})$ is bounded by $n_\ell^{(20)}$, and it finishes the proof of the first claim.

For the second claim, we put

$$N := [SO(\mathcal{L}_{K3,\mathbb{Z_2}}) : f_{K3}(GSpin(\mathcal{L}_{K3,\mathbb{Z_2}}))].$$

Then, by the second claim of Proposition 3.1.6 and the above arguments, we have

$$[(D_d)_2 : D_d(n)_2] \leq [(D_d)_2 : (D_d)_2 \cap f_{K3}(GSpin(\mathcal{L}_{K3,\mathbb{Z_2}}))] \cdot n_2^{(20)} \leq N \cdot n_2^{(20)}.$$  

\[\Box\]

3.2. Preparation II. Here, we will introduce a unimodular lattice $\mathcal{L}$ of signature $(23,2)$ which contains all $\mathcal{L}_d$. Then we will define related objects as in the previous subsection.

Proposition 3.2.1. We put

$$\mathcal{L} := \mathbb{H}_8^2 \oplus \mathbb{H}_2^2 \oplus \langle 1 \rangle^5.$$  

For any positive integer $d$, there exists a primitive embedding of lattices

$$i_d : \mathcal{L}_d \hookrightarrow \mathcal{L}.$$  

Proof. See [She17, Lemma 3.3.1].  

Remark 3.2.2. By the definition, the lattice $\mathcal{L}$ is unimodular. Hence $SO(\mathcal{L}_2)$ is the discriminant kernel of $\mathcal{L}$, which is defined as in Definition 2.1.7.

Next, we will define related algebraic groups and Shimura data for $\mathcal{L}$ as in Definition 2.2.1 and Definition 3.1.1.
Definition 3.2.3. (1) $GSpin_L$ is the algebraic group over $\mathbb{Q}$ whose $R$-valued points are given by

$$GSpin_L(R) := \{ z \in C^+(\mathcal{L}_R) \times | z\mathcal{L}_Rz^{-1} = \mathcal{L}_R \}.$$ 

(2) Take a 2-dimensional negative definite subspace of $\mathcal{L}_\mathbb{Q}$, and let $e_1, e_2$ be its orthogonal basis. Let $e_1', e_2'$ be an orthonormal basis over $\mathbb{R}$ which are given by constant multiples of $e_1, e_2$, and $J := e_1'e_2' \in C^+(\mathcal{L}_\mathbb{R})$. Let $\psi$ be the following map

$$\psi: \mathbb{R} \rightarrow GSpin_{L,\mathbb{R}}; \alpha + \beta i \mapsto \alpha + \beta J,$$

and $X_{GSpin_L}$ be a $GSpin_L(\mathbb{R})$-conjugacy class containing $\psi$.

(3) $SO_L$ is the algebraic group over $\mathbb{Q}$ whose $R$-valued points are given by

$$SO_L(R) := \{ g \in SL(\mathcal{L}_R) | (gv, gw) = (v, w) \text{ for any } v, w \in \mathcal{L}_R \}.$$ 

(4) $X_{SO_L}$ is the image of $X_{GSpin_L}$ via the natural map $GSpin_L \rightarrow SO_L$.

(5) For $V := C(\mathcal{L})$ and a fixed $\alpha \in V$ which is a constant multiple of $e_1e_2$, define $\phi_\alpha: V \times V \rightarrow \mathbb{Z}$ as $\phi_\alpha(x, y) := \text{tr}_{V/\mathbb{Q}}(xay^*)$. Here $\text{tr}_{V/\mathbb{Q}}(x)$ means the trace of a left multiplication map by $x$ as in [Huy16, Chapter 4, Section 2.2], and $*$ denotes the natural anti-automorphism on the Clifford algebra. Then $\phi_\alpha$ is a non-degenerate alternative form. We denote its degree by $r$. Let $GSp_{V,\alpha}$ be the algebraic group over $\mathbb{Q}$ whose $R$-valued points are given by

$$GSp_{V,\alpha}(R) := \left\{ g \in GL(V_R) \left| \begin{array}{l}
\text{there exists } c \in R^x \text{ such that } \\
\phi_\alpha(gx, gy) = c\phi_\alpha(x, y) \text{ for any } x, y \in V_R \end{array} \right. \right\}.$$ 

Let $(GSp_{V,\alpha}, X_{GSp_{V,\alpha}})$ be the Shimura datum associated with $(V, \phi_\alpha)$.

Remark 3.2.4. (1) As in Remark 3.1.2 (1), we can define a homomorphism

$$f: GSpin_L \rightarrow SO_L; g \mapsto (l \mapsto glg^{-1}).$$

Moreover, it induces a morphism of Shimura data

$$(GSpin_L, X_{GSpin_L}) \rightarrow (SO_L, X_{SO_L}).$$

(2) We can define a homomorphisms

$$h: GSpin_L \rightarrow GSp_{V,\alpha}; g \mapsto (v \mapsto gv).$$

Moreover, it induces an embedding of Shimura data

$$(GSpin_L, X_{GSpin_L}) \rightarrow (GSp_{V,\alpha}, X_{GSp_{V,\alpha}})$$

by our definition of $a$ (see [Huy16, Chapter 4, Section 2.2]).

(3) We will use a similar notation as in Remark 3.1.2 (2), (3) for $\mathcal{L}$.

Definition 3.2.5. For any positive integer $n$, we define compact open subgroups $K_n^{sp} \subset GSpin_L(A_f)$ and $K_n \subset GSp_{V,\alpha}(A_f)$ by

$$K_n^{sp} := \{ g \in GSpin_L(\mathcal{L}_\mathbb{A}) | g = 1 \text{ in } C^+(\mathcal{L}_{\mathbb{A}/n\mathbb{A}}) \},$$

$$K_n := \{ g \in GSp_{V,\alpha}(A_f) | gV_\mathbb{A} = V_\mathbb{A}, g \text{ acts trivial on } V_{\mathbb{A}/n\mathbb{A}} \}.$$ 

Remark 3.2.6. (1) One can show that $h(K_n^{sp}) \subset K_n$ and $h^{-1}(K_n) = K_n^{sp}$. Moreover, our definition of $K_n$ coincides with $\Lambda_n$ in Rizov’s paper [Riz10, Section 5.5]. Therefore, as in [Riz10, Section 5.5], we have an embedding

$$\text{Sh}_{K_n}(GSp_{V,\alpha}, X_{GSp_{V,\alpha}}) \hookrightarrow \mathcal{A}_{g, \sqrt{n}, \mathbb{Q}}.$$
Here, we put $g := 2^{24}$, and $\mathcal{A}_{g, Y_n, \mathcal{Q}}$ is the moduli space of $g$-dimensional degree $r$ polarized abelian schemes with level $n$-structure.

(2) The lattice embedding $i_d : \mathcal{L}_d \hookrightarrow \mathcal{L}$ induces a morphism of algebraic groups $i_d : SO_{\mathcal{L}_d} \to SO_{\mathcal{L}}$. It induces an embedding of Shimura data

$$(SO_{\mathcal{L}_d}, X_{SO_{\mathcal{L}_d}}) \to (SO_{\mathcal{L}}, X_{SO_{\mathcal{L}}}).$$

(3) One can show that $D(n) := f(\mathbb{F}_p^{\text{cn}})$ is a compact open subgroup of $SO(\mathcal{L}_d)$ similarly as in Proposition [3.1.2]. Moreover, it is clear that $i_d(D(n)) \subset D(n)$ because we have $GSpin(\mathcal{L}_d, \mathbb{Z}) \subset GSpin(\mathcal{L}, \mathbb{Z})$ as in Lemma [3.1.3].

3.3. The uniform Kuga-Satake construction. In this subsection, we assume that a positive integer $n$ is sufficiently large (in our application, $n$ would be a sufficiently large power of 2). Previous two subsections imply that there exists the following diagram of schemes over $\mathbb{Q}$.

$$\begin{array}{ccc}
M_{2d, D_d(n), \mathbb{Q}} & \xrightarrow{j} & \text{Sh}_{d(n)}(SO_{\mathcal{L}_d}, X_{SO_{\mathcal{L}_d}}) & \xrightarrow{i_d} & \text{Sh}_d(n)(SO_{\mathcal{L}}, X_{SO_{\mathcal{L}}}) & \xrightarrow{h} & \text{Sh}_n(GSp_{V,a}, X_{GSp_{V,a}}) \\
\downarrow{f} & & & & & & \\
\text{Sh}_{\mathbb{Q}}(GSpin_{\mathbb{L}}, X_{GSpin_{\mathbb{L}}}) & & & & & & \\
\end{array}$$

Here $\text{Sh}_n(G,X)$ means the canonical model of a Shimura variety of level $\mathbb{K}$ associated with $(G,X)$ over $\mathbb{Q}$, which is the reflex field of $(G,X)$. Then, by the arguments in [Riz10, Section 5.5], we can find $\delta$ which is a section of $f$ over a certain number field $E_n$. Indeed, as in [Riz10, Section 5.5], our definition of $D(n)$ guarantees that $f$ in the above diagram induces isomorphisms between geometric connected components of the above Shimura varieties. Hence we can find a section of $f$ over a number field on which all geometric connected components are defined.

In the following of this subsection, we fix a field $F$ containing $E_n$. We consider the base change from $\mathbb{Q}$ to $F$ of the above diagram.

$$\begin{array}{ccc}
M_{2d, D_d(n), F} & \xrightarrow{j} & \text{Sh}_{d(n)}(SO_{\mathcal{L}_d}) & \xrightarrow{i_d} & \text{Sh}_d(n)(SO_{\mathcal{L}}) & \xrightarrow{h} & \text{Sh}_n(GSp_{V,a}) \\
\downarrow{\delta} & & & & & & \\
\text{Sh}_{\mathbb{Q}}(GSpin_{\mathbb{L}}) & & & & & & \\
\end{array}$$

Here, and in the following of this paper, for simplicity, we denote $(\text{Sh}_n(G,X))_F$ by $\text{Sh}_n(G)$. Moreover, we denote the composition $h \circ \delta \circ i_d \circ j$ by $\Delta_d$.

Our Remark [3.2.6] implies that there exists the universal abelian scheme $\mathcal{A}$ over $\text{Sh}_{\mathbb{Q}}(GSp_{V,a})$ possessing the degree $r$ polarization and the level $n$-structure. Then, for $(X, L, v, \alpha) \in M_{2d, D_d(n), F}(F)$ which corresponds to a morphism $t : \text{Spec } F \to M_{2d, D_d(n), F}$, we can associate an abelian variety $A_{(X, L, \alpha)}$ by pulling back $\mathcal{A}$ via $\Delta_d \circ t$. We will quickly recall the properties of $A_{(X, L, \alpha)}$.

**Definition 3.3.1.** Let $\ell$ be any prime number.

1. Let $S$ be any (scheme) connected component of $\text{Sh}_{d(n)}(SO_{\mathcal{L}})$, and $\overline{S} \to S$ be a geometric point. Then, as in [Mil90, III, Remark 6.1], we can show that

$$\lim_{K} \text{Sh}_{K}(SO_{\mathcal{L}}) \to \text{Sh}_{d(n)}(SO_{\mathcal{L}})$$
is a Galois covering with a Galois group $D(n)$, and so we can associate the representation

$$\pi_1(S, \overline{s}) \to (D(n))_\ell \to \text{SO}(L_{\mathbb{Z}_\ell}).$$

We define $L^\text{shf}_{\mathbb{Z}_\ell}$ as the corresponding $\mathbb{Z}_\ell$-sheaf on $\text{Sh}_{D(n)}(\text{SO}_d)$, which have a symmetric pairing structure.

(2) Similarly, we define $L^\text{shf}_{d, \mathbb{Z}_\ell}$ as the $\mathbb{Z}_\ell$-sheaf on $\text{Sh}_{D_d(n)}(\text{SO}_d)$ corresponding to the representation $(D_d(n))_\ell \to \text{SO}(L_{d, \mathbb{Z}_\ell})$. The sheaf $L^\text{shf}_{d, \mathbb{Z}_\ell}$ have a symmetric pairing structure.

(3) Similarly, we define $V^\text{shf}_{d, \mathbb{Z}_\ell}$ as the $\mathbb{Z}_\ell$-sheaf on $\text{Sh}_{\mathbb{K}_a}(\text{GSp}_d)$ corresponding to the representation $(\mathbb{K}_n)_\ell \to \text{GSp}(V_{\mathbb{Z}_\ell}, \phi_a)$. The sheaf $V^\text{shf}_{d, \mathbb{Z}_\ell}$ have a symplectic pairing structure.

**Lemma 3.3.2.**

1. There exists the natural injection of étale sheaves $L^\text{shf}_{d, \mathbb{Z}_\ell} \to i^*_d L^\text{shf}_{d, \overline{s}}$ preserving the pairing. Moreover, $(L^\text{shf}_{d, \overline{s}})^\perp$ is trivial as a $\mathbb{Z}_\ell$-sheaf.

2. There exists the natural injection of étale sheaves $f^* L^\text{shf}_{\mathbb{Z}_\ell} \to \text{End}(h^*(V^\text{shf}_{d, \mathbb{Z}_\ell}))$, which induces a ‘left multiplication’ on a stalk.

**Proof.** For (1), it is enough to show that $i_d: L_{d, \mathbb{Z}_\ell} \to L_{\mathbb{Z}_\ell}$ is $\pi_1(S, \overline{s})$-equivariant, and $(L_{d, \mathbb{Z}_\ell})^\perp$ is a trivial $\pi_1(S, \overline{s})$-module. Here $S$ is any connected component of $\text{Sh}_{D_d(n)}(\text{SO}_d)$, $\overline{s}$ is a geometric point of $S$, and $\pi_1(S, \overline{s})$-module structure on $L_{d, \mathbb{Z}_\ell}$, $L_{\mathbb{Z}_\ell}$ correspond to $L_{d, \mathbb{Z}_\ell}$, $i^*_d L^\text{shf}_{d, \overline{s}}$. In regard to a $\mathbb{Z}_\ell$-sheaf given by a representation of adelic subgroup, a pullback of a $\mathbb{Z}_\ell$-sheaf corresponds to a pullback of a representation. Thus $\pi_1(S, \overline{s})$-module structure on $L_{\mathbb{Z}_\ell}$ is given by

$$\pi_1(S, \overline{s}) \to (D_d(n))_\ell \hookrightarrow (D(n))_\ell \hookrightarrow \text{SO}(L_{\mathbb{Z}_\ell}).$$

Hence the desired claim is clear.

For (2), it is enough to show that the morphism

$$L_{\mathbb{Z}_\ell} \to \text{End}(V_{\mathbb{Z}_\ell}); v \mapsto (z \mapsto vz)$$

is $\pi_1(S, \overline{s})$-equivariant, where $S$ is any connected component of $\text{Sh}_{\mathbb{K}_a}^\text{sp}(\text{GSpin}_d)$, and $\pi_1(S, \overline{s})$-module structure on $L_{\mathbb{Z}_\ell}$, $V_{\mathbb{Z}_\ell}$ correspond to $f^*(L^\text{shf}_{d, \mathbb{Z}_\ell})$, $h^*(V^\text{shf}_{d, \mathbb{Z}_\ell})$. By the same reason as (1), these structure are given by

$$\pi_1(S, \overline{s}) \to (\mathbb{K}_{d, n}^\text{sp})_\ell \xrightarrow{j} (D(n))_\ell \hookrightarrow \text{SO}(L_{\mathbb{Z}_\ell}),$$

$$\pi_1(S, \overline{s}) \to (\mathbb{K}_{d, n}^\text{sp})_\ell \xrightarrow{h} (\mathbb{K}_n)_\ell \to \text{GSp}(V_{\mathbb{Z}_\ell}, \phi_a).$$

Hence if we denote the first arrows of the both by $\sigma$, these actions are described as

$$\gamma(v) = \sigma(\gamma)v\sigma(\gamma)^{-1}, \quad \gamma(z) = \sigma(\gamma)(z),$$

for $\gamma \in \pi_1(S, \overline{s}), v \in L_{\mathbb{Z}_\ell}$, and $z \in V_{\mathbb{Z}_\ell}$. Thus the desired equivariantness is clear. \qed

**Proposition 3.3.3.** Let $\ell$ be any prime number, $t: \text{Spec } F \to M^\circ_{2d, D_d(n), F}$ be the point corresponding to $(X, L, \nu, \alpha) \in M^\circ_{2d, D_d(n), F}(F)$, $A^{(X, L, \alpha)}$ be the abelian variety given by $(\Delta_d \circ t)^*(A)$, and $L_{\mathbb{Z}_\ell}(X, L, \alpha)$ be the $\text{Gal}(\overline{\mathbb{F}}/F)$-lattice identified with $(i_d \circ j \circ t)^*(L^\text{shf}_{d, \mathbb{Z}_\ell})$. Then, the following hold.

1. There exists a Galois equivariant lattice embedding

$$P^2_{\text{ét}}((X_{\mathbb{Z}_\ell}, L_{\mathbb{Z}_\ell}), \mathbb{Z}_\ell(1)) \subset L_{\mathbb{Z}_\ell}(X, L, \alpha)$$
such that \( \text{Gal}(\overline{F}/F) \) acts trivially on the orthogonal complement
\[
P^2_\ell((X_\mathcal{T}, L_\mathcal{T}), \mathbb{Z}_\ell(1)) \subset \mathcal{L}_{\mathbb{Z}_\ell}(X, L, \alpha),
\]

(2) The abelian variety \( A^{(X, L, \alpha)} \) has a level \( n \)-structure defined over \( F \). Thus each \( n \)-torsion point of \( A^{(X, L, \alpha)} \) is \( F \)-rational.

(3) The abelian variety \( A^{(X, L, \alpha)} \) admits a left \( C(\mathcal{L}) \)-action over \( F \), and moreover there exists a isomorphism of \( \mathbb{Z}_\ell \)-modules
\[
H^1_{\text{ét}}(A^{(X, L, \alpha)}_F, \mathbb{Z}_\ell) \simeq C(\mathcal{L}_{\mathbb{Z}_\ell}(X, L, \alpha))
\]
which identifies the algebra
\[
C(\mathcal{L}_{\mathbb{Z}_\ell})^{\text{op}} \subset \text{End}(H^1_{\text{ét}}(A^{(X, L, \alpha)}_F, \mathbb{Z}_\ell))
\]
with
\[
C(\mathcal{L}_{\mathbb{Z}_\ell}(X, L, \alpha))^{\text{op}} \subset \text{End}(C(\mathcal{L}_{\mathbb{Z}_\ell}(X, L, \alpha))).
\]
Here, the former inclusion of algebras is induced by the above \( C(\mathcal{L}) \)-action, and the latter is induced by the right multiplication.

(4) The left multiplication by \( C(\mathcal{L}_{\mathbb{Z}_\ell}(X, L, \alpha)) \) on the right-hand side of the isomorphism in (3) induces a Galois equivariant isomorphism
\[
C(\mathcal{L}_{\mathbb{Z}_\ell}(X, L, \alpha)) \simeq \text{End}(C(\mathcal{L}_{\mathbb{Z}_\ell})^{\text{op}}(H^1_{\text{ét}}(A^{(X, L, \alpha)}_F, \mathbb{Z}_\ell))).
\]
Here, the (left) \( C(\mathcal{L}_{\mathbb{Z}_\ell})^{\text{op}} \)-module structure is induced by the left \( C(\mathcal{L}_{\mathbb{Z}_\ell})^{\text{op}} \)-action on \( A^{(X, L, \alpha)} \) as in (3).

Proof. These results are essentially proved in [She17, Proposition 3.5.8].

(1) follows from Lemma 3.3.2 (1) and the fact
\[
(j \circ t)^*(\mathcal{L}_{\mathbb{Z}_\ell}) \simeq P^2_\ell((X_\mathcal{T}, L_\mathcal{T}), \mathbb{Z}_\ell(1))
\]
(see [MP15 Proposition 5.6 (1)]).

(2) is clear because the universal family \( \mathcal{A} \) admits a level \( n \)-structure.

Before proving (3) and (4), we note that for the universal abelian scheme \( u: \mathcal{A} \to \text{Sh}_{\mathbb{Q}}(\text{GSp}_{V, \alpha}) \), we have
\[
R^1u_*\mathbb{Z}_\ell \simeq V_{Z^\text{sh}}^\text{sh}.
\]

For (3), as in [MP16, Section 3.10], \( h^*(\mathcal{A})_{\text{Sh}_{\mathbb{C}}(\text{GSpin}_L)^c} \) admits a \( C(\mathcal{L}) \)-action which corresponds to a right multiplication on the cohomology, since our definition of \( h \) guarantees that the right multiplication preserves the Hodge structure. This action descends to \( F \) by [MP16 Proposition 3.11] and induces a \( C(\mathcal{L}) \)-action on \( A^{(X, L, \alpha)} \) with desired properties.

For (4), the statement (3) of this proposition and Lemma 3.3.2 (2) implies the well-definedness and the Galois equivariantness of our morphism, and it is clearly bijective.

\[\square\]

4. Proof of the Main Theorems

4.1. Statements.

Lemma 4.1.1. Let \( F \) be a finitely generated field over \( \mathbb{Q} \), \( R \) be a smooth algebra over \( \mathbb{Z} \) which is an integral domain with the fraction field \( F' \), and \( \overline{s} \) be a geometric point corresponding to an algebraic closure \( \overline{F} \) over \( F \). For any \( \pi_1(\text{Spec } F, \overline{s}) \)-module \( M \) such that \( \text{Ker}(\pi_1(\text{Spec } F, \overline{s}) \to \text{Aut}(M)) \) is closed, the following are equivalent.

(1) The \( \pi_1(\text{Spec } F, \overline{s}) \)-action on \( M \) descends to the \( \pi_1(\text{Spec } R, \overline{s}) \)-action on \( M \).
(2) For any height 1 prime ideal $p \in \text{Spec } R$, the $\pi_1(\text{Spec } F, \overline{s})$-action on $M$ descends to the $\pi_1(\text{Spec } R_p, \overline{s})$-action on $M$.

(3) For any height 1 prime ideal $p \in \text{Spec } R$, $M$ is unramified at $p$, i.e. if we take $\overline{s}$ which is an extension of valuation $p$ to $\overline{F}$, the inertia group $I_{\overline{s}}$ acts trivially on $M$.

When $M$ satisfies the above equivalent conditions, we say $M$ is unramified over $\text{Spec } R$.

**Proof.** First, we recall that $\pi_1(\text{Spec } F, \overline{s}) \to \pi_1(\text{Spec } R, \overline{s})$ is surjective whose kernel is identified with $\text{Gal}(F/F_{\text{ur}})$, where $F_{\text{ur}}$ is the composite of finite extensions $E/F$ which are unramified over $\text{Spec } R$ ([Fu15, Proposition 3.3.6]). Here, we say $E/F$ is unramified over $\text{Spec } R$ if the normalization of $\text{Spec } R$ in $E$ is unramified over $\text{Spec } R$.

Same results hold for $\text{Spec } R_p$.

(1) $\iff$ (2) By the assumption on $M$, it suffices to show that $\text{Ker}(\pi_1(\text{Spec } F, \overline{s}) \to \pi_1(\text{Spec } R, \overline{s}))$ as a topological group. By the above remark, it is enough to show that $F_{\text{ur}} = \bigcap_{ht(p)=1} F_{\text{ur}}^{R_p}$. The inclusion $F_{\text{ur}}^{R_p} \subset \bigcap_{ht(p)=1} F_{\text{ur}}^{R_p}$ is obvious, and another direction follows from the Zariski-Nagata purity.

(2) $\iff$ (3) By the assumption on $M$, it suffices to show that $\text{Ker}(\pi_1(\text{Spec } F, \overline{s}) \to \pi_1(\text{Spec } R_p, \overline{s}))$ is generated by $(I_{\overline{s}})_{\overline{s}}$ over $p$ as a topological group, but it follows from the above remark. $\square$

**Remark 4.1.2.** The condition ‘$M$ is unramified at $p$’ does not depend on a choice of $\overline{s}$. Indeed, for each $p$, the inertia group $I_{\overline{s}}$ is determined by $p$ up to conjugation in $\text{Gal}(F/F)$.

The following are the statements of results of this section (for more generalized statements, see Theorem [6.1.1]).

**Theorem 4.1.3.** Let $F$ be a finitely generated field over $\mathbb{Q}$, $R$ be a smooth algebra over $\mathbb{Z}$ which is an integral domain with the fraction field $F$, and $d$ be a positive integer. Then, the set

$$\text{Shaf}(F, R, d) := \left\{ (X, L) \middle| \begin{array}{l} X: \text{K3 surface over } F, \\
L \in \text{Pic}_{X/F}(F): \text{primitive ample,} \\
H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_2): \text{unramified over } \text{Spec } R, \\
\deg L = 2d \end{array} \right\} /F\text{-isom.}$$

is finite.

**Theorem 4.1.4.** Let $F$ be a finitely generated field over $\mathbb{Q}$, and $R$ be a smooth algebra over $\mathbb{Z}$ which is an integral domain with the fraction field $F$. Then, the set

$$\text{Shaf}(F, R) := \left\{ X \middle| \begin{array}{l} X: \text{K3 surface over } F, \\
H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_2): \text{unramified over } \text{Spec } R \end{array} \right\} /F\text{-isom.}$$

is finite.

**Remark 4.1.5.** For a non-empty open subscheme $\text{Spec } (R') \subset \text{Spec } (R)$, the finiteness of $\text{Shaf}(F, R', d)$ (resp. $\text{Shaf}(F, R')$) clearly implies the finiteness of $\text{Shaf}(F, R, d)$ (resp. $\text{Shaf}(F, R)$). Thus, to prove Theorem 4.1.3 and Theorem 4.1.4, we may assume $1/2 \in$
Remark that it is equivalent to say that the residual characteristic at any point of \( \text{Spec } R \) is different from 2.

### 4.2. Proof of Theorem 4.1.3

In this subsection, we use the same notation as Theorem 4.1.3 unless otherwise noted. First, for using the Kuga-Satake construction, we will replace \( F \) by an appropriate finite extension of it to provide a level structure on \((X, L) \in \text{Shaf}(F, R, d)\). The following lemma is essential for justifying this replacement.

**Lemma 4.2.1.** Let \( E/F \) be a finite extension, \( X_0 \) be a K3 surface over \( F \), and \( L_0 \in \text{Pic}_{X_0/F}(F) \) be a polarization. Then, the set

\[
\left\{ (X, L) \mid X \text{ is a K3 surface over } F, \quad L \in \text{Pic}_{X/F}(F) : \text{ample}, \quad (X_E, L_E) \simeq_E (X_0, L_0) \right\} /F\text{-isom}
\]

is finite.

**Proof.** Taking the Galois closure of \( E \) in \( \overline{F} \), we may assume that \( E/F \) is a Galois extension. Then we can identify this set with the Galois cohomology group \( H^1(\text{Gal}(E/F), \text{Aut}_E(X_0, L_0)) \). The finiteness of this set follows from [Huy16, Chapter 5, Proposition 3.3]. □

**Lemma 4.2.2** (cf. [And96, Lemma 8.4.1]). Let \( X \) be a K3 surface over \( F \), \( L \in \text{Pic}_{X/F}(F) \) be a primitive polarization of degree \( 2d \) on \( X \) over \( F \), and \( n \) be a positive integer. We put

\[
W_{\hat{Z}} = P^2_a((X_{\hat{F}}, L_{\hat{F}}), \hat{Z}(1)).
\]

Let

\[
\rho: \text{Gal}(\overline{F}/F) \to O(W_{\hat{Z}})
\]

be the natural Galois representation. Fix an isometry

\[
i_{(X,L)}: \mathcal{L}_{K3,\hat{Z}} \simeq H^2_a(X_{\hat{F}}, \hat{Z}(1)),
\]

which restricts to an isometry \( \mathcal{L}_{K3} \simeq H^2(X(\mathbb{C}), \mathbb{Z}(1)) \), and which sends \( v_d \) to \( \text{ch}_{\hat{Z}}(L) \) (see Remark 2.1.6). Using \( i_{(X,L)} \), we identify \( D_d(n) \) with a compact open subgroup of \( \text{SO}(W_{\hat{Z}}) \). Then, for any finite extension \( E/F \), we have

\[
\rho(\text{Gal}(\overline{F}/E)) \subset D_d(n) \iff \rho_\ell(\text{Gal}(\overline{F}/E)) \subset (D_d(n))_\ell \text{ for any } \ell \mid 2n.
\]

**Proof.** In the following, we identify \( \text{SO}(\mathcal{L}_{d,\hat{Z}}) \) with \( \text{SO}(W_{\hat{Z}}) \) via \( i_{(X,L)} \). This lemma is essentially shown in [And96, Lemma 8.4.1]. André shows the following claim in the proof of [And96, Lemma 8.4.1], using specialization arguments and the Weil conjecture.

**Claim.** If there exists a prime number \( \ell \) such that \( \rho_\ell(\text{Gal}(\overline{F}/E)) \subset \text{SO}(W_{\hat{Z}}) \), then \( \rho(\text{Gal}(\overline{F}/E)) \subset \text{SO}(W_{\hat{Z}}) \).

André states that the above claim implies the following result.

\[
\rho(\text{Gal}(\overline{F}/E)) \subset D_d(n) \iff \rho_\ell(\text{Gal}(\overline{F}/E)) \subset (D_d(n))_\ell \text{ for any } \ell \mid 2dn.
\]

Indeed, for \( \ell \nmid 2dn \), we have \((D_d(n))_\ell = \text{SO}(\mathcal{L}_{d,\hat{Z}})\).

More generally, for \( \ell \mid 2n \), we have \((D_d(n))_\ell = (D_d)_\ell\). (See Corollary 5.1.8.) Therefore, to generalize André’s result to our lemma, it is enough to show that if
\[ \rho_t(\text{Gal}(\overline{F}/E)) \subset \text{SO}(W_{\mathbb{Z}}), \] then \( \rho_t(\text{Gal}(\overline{F}/E)) \subset (D_d)_{\ell}. \) However, since \( \text{Gal}(\overline{F}/F) \) stabilizes \( ch_{Z}(L) \), it follows from our description of the discriminant kernel

\[ (D_d)_{\ell} = \{ \tilde{g}_{\ell} \in \text{SO}(H^2_d(X_{\mathbb{T}}, \mathbb{Z} \ell)) \mid \tilde{g}_{\ell}(ch_{Z}(L)) = ch_{Z}(L) \}, \]

which follows from Proposition 2.1.8.

In the rest of this section, fix a positive integer \( n \) which is a sufficiently large power of 2.

**Proposition 4.2.3.** To prove Theorem 4.1.3, it is enough to show that

\[ \text{Shaf}'(F, R, d) := \left\{ (X, L) \mid \begin{array}{l}
X: \text{K3 surface over } F, \\
L \in \text{Pic}_{X/F}(F): \text{primitive ample}, \\
H^2_d(X_{\mathbb{T}}, \mathbb{Q}_2): \text{unramified over } \text{Spec } R, \\
\deg L = 2d. \\
(X, L) \text{ admits a } D_d(n)\text{-level structure} \\
\end{array} \right\} /F\text{-isom} \]

is a finite set for any \( F, R, d \) as in Theorem 4.1.3. Moreover, if we fix a number field \( F' \), it suffices to show only in the case where \( F \supseteq F' \) and \( 1/2 \in F \).

Here, ‘\( (X, L) \text{ admits a } D_d(n)\text{-level structure} \)’ means that there exists an element \( (X, L, \rho(X,L), (O(X,L)) \in M^2_{d,d}(F,F'). \)

**Proof.** We should prove the finiteness of \( \text{Shaf}(F, R, d) \). By Remark 1.1.3, we may assume \( 1/2 \in R \) (so the Tate twist \( \otimes \mathbb{Z}_2(1) \) does not effect on the unramifiedness over \( \text{Spec } R \), see Lemma 1.2.2).

First, we will show that there exists a finite extension \( E/F \) such that for any \( (X, L) \in \text{Shaf}(F, R, d) \), the pair \( (X_E, L_E) \) admits a \( D_d(n)\text{-level structure} \). We fix \( (X, L) \in \text{Shaf}(F, R, d) \) and \( i(X,L) \), moreover we use the same identification as in Lemma 1.2.2.

Let

\[ \overline{\rho}_2 := \rho_{(X,L),2}: \pi_1(\text{Spec } R, \overline{s}) \to O(P^2_{\text{et}}((X_{\mathbb{T}}, L_{\mathbb{T}}), \mathbb{Z}_2(1))) \]

be the representation induced by

\[ \rho := \rho_{(X,L)}: \text{Gal}(\overline{F}/F) \to O(P^2_{\text{et}}((X_{\mathbb{T}}, L_{\mathbb{T}}), \hat{\mathbb{Z}}(1))). \]

The inverse image \( \overline{\rho}_2^{-1}((D_d(n))_2) \) is a finite index subgroup, so we can associate a pointed finite étale cover \( \text{Spec } \tilde{R} \to \text{Spec } R \). Then we have \( \overline{\rho}_2(\pi_1(\text{Spec } \tilde{R}, \overline{s})) \subset (D_d(n))_2 \). The former is equal to \( \rho_2(\text{Gal}(\overline{F}/\text{Frac}(\tilde{R}))) \), and by Lemma 1.2.2 we can get the \( D_d(n)\text{-level structure} \) on \( (X_{\text{Frac}(\tilde{R})}, L_{\text{Frac}(\tilde{R})}) \) by \( i(X,L) \).

Here, note that

\[ \pi_1(\text{Spec } R, \overline{s}) : \overline{\rho}_2^{-1}((D_d(n))_2) \leq C_d := [O(L_{d,\mathbb{Z}_2}) : (D_d(n))_2], \]

where \( C_d \) is independent of \( (X, L) \) and \( i(X,L). \) By the analogue of the Hermite-Minkowski theorem [HH09, Proposition 2.3, Theorem 2.9], the family of subsets

\[ C := \{ H \subset \pi_1(\text{Spec } R, \overline{s}) : \text{open subgroup } \mid [\pi_1(\text{Spec } R, \overline{s}) : H] \leq C_d \} \]

is finite, therefore

\[ H_0 := \bigcap_{H \in C} H \]

is an open subgroup. Let \( \text{Spec } \tilde{R}_0 \to \text{Spec } \tilde{R} \) be the corresponding pointed finite étale covering, then by the above argument, we can get a \( D_d(n)\text{-level structure} \) on \( (X_{\text{Frac}(\tilde{R}_0)}, L_{\text{Frac}(\tilde{R}_0)}) \). Hence we now get a desired finite extension \( E := \text{Frac}(\tilde{R}_0). \)
Thus, by using the assumption for Shaf′(E, R, d) and Lemma 4.2.1 we can show the finiteness of Shaf(F, R, d). Remark that the latter statement is clear by Lemma 4.2.1.

**Proposition 4.2.4.** Assume F ⊇ E_n, where E_n is as in 3.3. For (X, L, ν, α) ∈ M_{2d,D}^o,n,F(F), let A^{(X, L, α)} be the Kuga-Satake abelian variety as in Proposition 3.3.3. Let R be a smooth algebra over Z which is an integral domain with the fraction field F, and assume 1/2 ∈ R. Assume that H^2_a((X, L), \mathbb{Q}_2) is unramified over Spec R (its Tate twists are unramified too, because 1/2 ∈ R). Then, for any height 1 prime ideal \mathfrak{p} ∈ Spec R, the abelian variety A^{(X, L, α)} has good reduction at \mathfrak{p}.

**Proof.** By the Néron-Ogg-Shafarevich criterion for abelian varieties (it is true whether a residue field is perfect or not), it is enough to show that H^1_{et}(A^{(X, L, α)}, \mathbb{Z}_2) is unramified at \mathfrak{p} (here we use 1/2 ∈ R). Let \overline{\mathfrak{p}} be an extension on \overline{F} of the valuation \mathfrak{p}, and \varphi: I_{\overline{\mathfrak{p}}} \to \text{Aut}(H^1_{et}(A^{(X, L, α)}, \mathbb{Z}_2)) be a restriction of the Galois representation. Since C(L)-action on A^{(X, L, α)} is defined over F, for any γ ∈ I_{\overline{\mathfrak{p}}}, we have

\[ \varphi(\gamma) \in \text{End}_C(\mathcal{L}_{\mathfrak{p}})\varphi(H^1_{et}(A^{(X, L, α)}, \mathbb{Z}_2)) \simeq C(\mathcal{L}_{\mathfrak{p}}(\mathbb{Z}, X, L, α)) \]

(see Proposition 3.3.3 (3), (4)). Thus we also denote its image by \varphi(\gamma) ∈ C(\mathcal{L}_{\mathfrak{p}}(\mathbb{Z}, X, L, α)).

On the other hand, I_{\overline{\mathfrak{p}}} acts trivially on P^2_{et}((X, L), \mathbb{Z}_2(1)) by our assumptions (see Lemma 4.1.1), and moreover acts trivially on P^2_{et}((X, L), \mathbb{Z}_2(1))^+ ⊂ L_{\mathfrak{p}}(\mathbb{Z}, X, L, α) by Proposition 3.3.3 (1), thus γ(c) = c for any γ ∈ I_{\overline{\mathfrak{p}}} and c ∈ C(\mathcal{L}_{\mathfrak{p}}(\mathbb{Z}, X, L, α)). By Proposition 3.3.3 (4), we have γ(z → cz) = (z → cz) in End_C(\mathcal{L}_{\mathfrak{p}})\varphi(H^1_{et}(A^{(X, L, α)}, \mathbb{Z}_2)), where the left-hand side is (z → \varphi(γ)c\varphi(γ)^{-1}z). This implies \varphi(γ) is contained in a center of C(\mathcal{L}_{\mathfrak{p}}(\mathbb{Z}, X, L, α)), which is a reduced algebra.

The Raynaud semi-abelian reduction criterion [GRR72, Exposé IX, Proposition 4.7] and Proposition 3.3.3 (2) imply that A^{(X, L, α)} has semi-abelian reduction at \mathfrak{p} (i.e. A^{(X, L, α)} extends to a semi-abelian scheme over Spec R_b). Here, we use that n ≥ 3 is a power of 2, and the residual characteristic of \mathfrak{p} is not 2. Thus for any γ ∈ I_{\overline{\mathfrak{p}}}, \varphi(γ) is a unipotent element of a reduced algebra, it is identity. Hence it finishes the proof.

We now complete the proof of Theorem 4.1.3. By Proposition 4.2.3 it is enough to show the finiteness of Shaf′(F, R, d) when F ⊇ E_n and 1/2 ∈ R. Here, we take E_n as in 3.3. In the following, we identify (X, L) ∈ Shaf′(F, R, d) with (X, L, ν, α) ∈ M_{2d,D}^o,n,F(F) by choosing a level structure. Hence for (X, L, ν, α) ∈ Shaf′(F, R, d), we can associate A^{(X, L, α)}, and since in the diagram (⋆) of 3.3, each fiber of i_d is finite and h is injective (because they are induced by an embedding of Shimura data), it suffices to show the finiteness of Δ_d(Shaf′(F, R, d)). The image Δ_d((X, L, ν, α)) corresponds to A^{(X, L, α)} with their degree r polarization and level n-structure. However, by Proposition 4.2.4 the abelian variety A^{(X, L, α)} has good reduction at any height 1 prime of Spec R, so this set is finite by [Fal83, Satz 6] (for finitely generated fields of characteristic 0, see [FWG+92, VI, §1, Theorem 2])

4.3. **Proof of Theorem 4.1.4.** In this subsection, we use the same notation as in Theorem 4.1.4 unless otherwise noted. The strategy is the same as [She17], i.e. we use Theorem 4.1.3 for reducing the problem to the finiteness of Picard lattices, and use the uniform Kuga-Satake maps for associating Shaf(F, R) with a finite set of abelian varieties.
Lemma 4.3.1 (cf. [She17, Corollary 4.1.3]). For any \( X_0 \in \text{Shaf}(F, R) \), there exist only finitely many \( X \in \text{Shaf}(F, R) \) whose Picard lattice \( \text{Pic}_{X/F}(F) \) is isometric to the Picard lattice \( \text{Pic}_{X_0/F}(F) \).

Proof. As in [She17, Proposition 4.1.2], a K3 surface \( X \) over \( F \) admits a primitive polarization whose degree bounded by constant depend only on an isometry class of \( \text{Pic}_{X/F}(F) \). Hence this lemma follows from Theorem 4.1.3. \( \square \)

Lemma 4.3.2 (cf. [She17, Lemma 4.1.4]). Let \( E/F \) be a finite extension. For any \( X_0 \in \text{Shaf}(F, R) \), the set

\[ \{ X \in \text{Shaf}(F, R) \mid X_E \simeq_E X_0,E \} \]

is finite.

Proof. Taking a Galois closure, we may assume \( E/F \) is a Galois extension. By Lemma 4.3.1 it suffices to show the finiteness of isometry classes of Picard lattices \( \text{Pic}_{X/F}(F) \) associated with the considering set. Remark that \( \text{Pic}_{X/F}(E) \) isomorphic to \( \text{Pic}_{X/F}(F) \) and \( \text{Pic}_{X/F}(E) \) is isometric to \( \text{Pic}_{X_0/F}(E) \). Since the conjugacy classes of subgroups of \( \text{O}(\text{Pic}_{X_0/F}(E)) \) with the order \( [E:F] \) is finite by [Bor63, Section 5, (a)], the desired finiteness follows. \( \square \)

Proposition 4.3.3. Recall that we fixed a positive integer \( n \) which is a power of 2. To show Theorem 4.1.7, it is enough to show that

\[
\text{Shaf}'(F, R) := \left\{ X \mid \begin{array}{l}
X: \text{K3 surface over } F, \\
H^2_{ad}(X_{\overline{F}}, \mathbb{Q}_2): \text{unramified over } \text{Spec } R, \\
\text{there exists } d_X, L_X, \nu_X, \alpha_X \text{ such that } (X, L_X, \nu_X, \alpha_X) \in M_{2d_X, D_{d_X}(n)}(F)
\end{array} \right\} / F\text{-isom}
\]

is a finite set for any \( (F, R) \) as in Theorem 4.1.4. Moreover, if we fix a number field \( F' \), it suffices to show only in the case where \( F \supset F' \) and \( 1/2 \in R \).

Proof. The proof is similar to Proposition 4.2.3, but we need more precise evaluation since we should discuss all degrees simultaneously.

As in the proof of Proposition 4.2.3 it is enough to show the finiteness of \( \text{Shaf}(F, R) \) with \( 1/2 \in R \). First, remark that every K3 surface over \( F \) admits some primitive polarization over \( F \). Therefore, for any \( X \in \text{Shaf}(F, R) \), we can associate a primitive polarization \( L_X \). Let \( 2d_X \) be the degree of \( L_X \). We will show that there exists a finite extension \( E/F \) such that for any \( X \in \text{Shaf}(F, R) \), the pair \( (X_E, L_{X,E}) \) admits a \( D_{d_X}(n) \)-level structure. For each \( X \in \text{Shaf}(F, R) \), we fix \( i_{(X,L_X)} \) as in Lemma 4.2.2 and we use the notation \( \overline{\rho}_2 := \overline{\rho}_{(X,L_X),2} \) in the same sense as in Proposition 4.2.3. To get a desired extension, we should replace the bound \( C_{d_X} \) in the proof of Proposition 4.2.3 by a bound which is independent of \( X \). For \( \Gamma := \overline{\rho}_2(\pi_1(\text{Spec } R, \pi)) \), we have

\[ [\Gamma : \Gamma \cap (D_{d_X}(n))_2] = [\Gamma : \Gamma \cap \text{SO}(\mathcal{L}_{d_X}(n))] \cdot [\Gamma \cap \text{SO}(\mathcal{L}_{d_X}(n)) : \Gamma \cap (D_{d_X}(n))_2] \]

\[ \leq 2N \cdot n^{(2m)}. \]

Here, we use \( \Gamma \cap \text{SO}(\mathcal{L}_{d_X}) \subset \Gamma \cap (D_{d})_2 \) (follows from Proposition 2.1.8 see the proof of Lemma 4.2.2, and Corollary 3.1.8. We note that this bound is independent of \( X, L_X \), and \( i_{(X,L_X)} \). Hence replacing \( C_d \) by \( 2N \cdot n^{(2m)} \) in the arguments in the proof of Proposition 4.2.3 we get a pointed finite étale covering \( \text{Spec } \overline{R_0} \to \text{Spec } R \).
whose fraction field $E$ satisfies the desired property. Thus, by using the assumption for $\text{Shaf}'(E, \hat{R}_0)$ and Lemma 4.3.2 we get the finiteness of $\text{Shaf}(F, R)$. The latter statement is clear by Lemma 4.3.2.

**Definition 4.3.4** ([She17 Definition 4.1.10]). Let $F$ be a subfield of $\mathbb{C}$, $X$ be a K3 surface over $F$, and $\ell$ be any prime number. We define (relative) *transcendental lattices* by

$$
T(X) := \text{Pic}_{X/F}(F)^\perp \subset H^2(X(\mathbb{C}), \mathbb{Z}(1)),
$$
$$
T(X)_{\mathbb{Z}_\ell} := \text{Pic}_{X/F}(F)^\perp \subset H^2_{\text{et}}(X^\wedge, \mathbb{Z}_\ell(1)),
$$
$$
T(X)_{\mathbb{Z}} := \text{Pic}_{X/F}(F)^\perp \subset H^2_{\text{et}}(X^\wedge, \hat{\mathbb{Z}}(1)).
$$

Here we omit the Chern class map. Clearly this notation is compatible with a base change.

**Remark 4.3.5** (cf. [She17 Corollary 4.1.13]). Recall that $M := H^2(X(\mathbb{C}), \mathbb{Z}(1)) \simeq \mathcal{L}_{K3}$ is unimodular, and $N := \text{Pic}_{X/F}(F)$ is a primitive sublattice. In this situation, one can verify a canonical isomorphisms

$$N^\vee / N \simeq M / (N + N^\perp) \simeq (N^\perp)^\vee / N^\perp.
$$

Thus we get $\text{disc}(\text{Pic}_{X/F}(F)) = \text{disc}(T(X))$.

**Lemma 4.3.6** (cf. [She17 Proposition 4.1.11]). For $(X, L, \nu, \alpha) \in M^0_{2d,d(1),F}(F)$ and any prime number $\ell$, we have

$$
(L^\text{Gal}(\overline{F}/F))_{\mathbb{Z}_\ell}(X, L, \alpha) = T(X)_{\mathbb{Z}_\ell}.
$$

Here, the orthogonal complement of the left-hand side is taken in $L_{\mathbb{Z}_\ell}(X, L, \alpha)$, and the above equality is as a sublattice of $P^2_{\text{et}}((X^\wedge, L^\wedge), \mathbb{Z}_\ell)$.

**Proof.** First, we can show that

$$T(X)_{\mathbb{Z}_\ell} = (P^2_{\text{et}}((X^\wedge, L^\wedge), \mathbb{Z}_\ell(1)))^\text{Gal}^\perp
$$

(the orthogonal complement of the right-hand side is taken in $P^2_{\text{et}}((X^\wedge, L^\wedge), \mathbb{Z}_\ell(1))$).

Indeed, since the both sides of this equality is primitive in $P^2_{\text{et}}((X^\wedge, L^\wedge), \mathbb{Z}_\ell(1))$, it suffices to show this equality after inverting $\ell$, which follows directly from the Tate conjecture over $F$ ([Tat94 Theorem 5.6 (a)])

Hence we have to show that

$$
(P^2_{\text{et}}((X^\wedge, L^\wedge), \mathbb{Z}_\ell(1)))^\text{Gal}^\perp = (L^\text{Gal}(\overline{F}/F))_{\mathbb{Z}_\ell}(X, L, \alpha)^\perp
$$

(remark that the $\perp$ in both sides have different meaning). However, since the both sides are primitive in $L_{\mathbb{Z}_\ell}(X, L, \alpha)$, we may invert $\ell$ for showing this equality, so it follows obviously from Proposition 3.3.3 (1).

Let us complete the proof of Theorem 4.1.1. As the previous subsection, by Proposition 4.1.3 it suffices to show the finiteness of $\text{Shaf}'(F, R)$ when $F \supset E_n$ and $1/2 \in R$. By Lemma 4.3.3 and the fact [Cas82 ch. 9, Theorem 1.1] which states the finiteness of isometry classes of lattices with bounded rank and discriminant, it is enough to show that $\text{disc}(\text{Pic}_{X/F}(F))$ ($X \in \text{Shaf}'(F, R)$) is bounded. Using Remark 4.3.5 we can reduce the problem to the finiteness of $\{T(X)_{\mathbb{Z}} \mid X \in \text{Shaf}'(F, R)/\text{isometry}\}$. 

□
For $X \in \text{Shaf}(F,R)$, we choose an element

$$(X, L_X, \nu_X, \alpha_X) \in M^2_{\Delta x, D_{dX}(n), F}(F).$$

Then, by Proposition 4.2.4 and [Zar85, Theorem 1] (for finitely generated fields of characteristic 0, see [FWG+92, VI, §1, Theorem 2]), the subset

$$\{\Delta_{dX}(X, L_X, \nu_X, \alpha_X) \mid X \in \text{Shaf}(F, R)\} \subset \text{Sh}_{\Delta_n}(\text{GSp}_{\nu(a)})(F)$$

is finite. We denote them by $t_1, \ldots, t_m$, and we put

$$\text{Shaf}(F, R)_i := \{X \in \text{Shaf}(F, R) \mid \Delta_{dX}(X, L_X, \nu_X, \alpha_X) = t_i\}.$$ 

Thus, the desired finiteness follows from the following lemma.

**Lemma 4.3.7.** The $\hat{\mathbb{Z}}$-lattices $T(X)_i (X \in \text{Shaf}(F, R)_i)$ are isometric to each other.

*Proof.* By Lemma 4.3.6 it suffices to show that $\mathcal{L}_{\mathbb{Z},(X, L_X, \alpha_X)} (X \in \text{Shaf}(F, R)_i)$ is unique up to a $\text{Gal}(F/F)$-equivariant isometry, for any $\ell$. We denote the lift of $t_i$ on $\text{Sh}_{D(n)}(\text{SO}_2)$ via $h \circ \delta$ (it exists by the definition of $t_i$, and it is unique because $h \circ \delta$ is injective) by $\tilde{t}_i$. Recall that we have the étale sheaf $\mathcal{L}_{\mathbb{Z}, \ell}^{\text{shf}}$, which have a symmetric pairing structure, so we get the $\text{Gal}(F/F)$-lattice $\tilde{t}_i^*(\mathcal{L}_{\mathbb{Z}, \ell}^{\text{shf}})$, which depends only on $t_i$. By our construction of $\mathcal{L}_{\mathbb{Z},(X, L_X, \alpha_X)}$ in Proposition 3.3.3, for any $X \in \text{Shaf}(F, R)_i$, the Gal($F/F$)-lattice $\mathcal{L}_{\mathbb{Z},(X, L_X, \alpha_X)}$ is no other than $\tilde{t}_i^*(\mathcal{L}_{\mathbb{Z}, \ell}^{\text{shf}})$ and it finishes the proof. \qed

5. $\ell$-independence

In this section, we give some $\ell$-independence results for completing the proof of the main theorem. The following result is well-known if the residue field is finite.

**Theorem 5.0.1.** Let $K$ be a Henselian discrete valuation field, $k$ be the residue field of $K$, $p$ be the characteristic of $k$. Assume that the characteristic of $K$ is different from 2. Then, for any K3 surface $X$ over $K$, the following are equivalent.

(a) The $\text{Gal}(\overline{K}/K)$-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for some $\ell \neq p$.

(b) The $\text{Gal}(\overline{K}/K)$-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for all $\ell \neq p$.

Moreover, if $K$ is a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field $k$, then (a) $(\Leftrightarrow) (b)$ is equivalent to the following.

(c) The $\text{Gal}(\overline{K}/K)$-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline.

**Remark 5.0.2.** If we assume that $X$ admits a Kulikov model after a finite extension of $K$, then these results are already known as a corollary of a good reduction criterion for K3 surfaces (see [CLL17, Theorem 1.1] for example). Moreover, because the weight monodromy conjecture is true for surfaces, there is another approach especially when $k$ is a finite field (see [Sai03, Corollary 0.4] for example).

5.1. $\ell$ versus $\ell'$ part. In this subsection, we prove the equivalence (a) $(\Leftrightarrow) (b)$ in Theorem 5.0.1. Let $K$ be a Henselian discrete valuation field with characteristic different from 2, $k$ be the residue field of $K$, $p$ be the characteristic of $k$, and $X$ be a K3 surface over $K$ in this subsection.

First, we recall the definition of the monodromy operator.

**Definition 5.1.1.** Let $\ell$ be a prime number different from $p$. Consider the representation

$$\rho_\ell: \text{Gal}(\overline{K}/K) \to \text{GL}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)).$$
By Grothendieck’s monodromy theorem, there exists an open subgroup of the inertia subgroup $J \subset I_K$ and the nilpotent operator

$$N_t : H^2_{\acute{e}t}(X_{\mathbb{R}}, \mathbb{Q}_t)(1) \to H^2_{\acute{e}t}(X_{\mathbb{R}}, \mathbb{Q}_t)$$

such that for all $\sigma \in J$, we have $\rho_\ell(\sigma) = \exp(t_\ell(\sigma)N_t)$, where $t_\ell : I_K \to \mathbb{Z}_\ell(1)$ is a natural projection. By fixing an isomorphism $\mathbb{Q}_t(1) \simeq \mathbb{Q}_t$, we regard $N_t$ as a linear endomorphism of $H^2_{\acute{e}t}(X_{\mathbb{R}}, \mathbb{Q}_t)$, which is called the monodromy operator.

**Remark 5.1.2.** By the definition, $N_\ell$ does not change if we replace $K$ by a finite extension of it.

The following lemma is an elementary fact about $\ell$-adic representations.

**Lemma 5.1.3.** The following are equivalent.

1. The $\ell$-adic representation $\rho_\ell$ is unramified.
2. $N_\ell = 0$ and $\text{tr}(\rho_\ell(\sigma)) = 22$ for any $\sigma \in I_K$.

**Proof.** (1) $\Rightarrow$ (2) is trivial. Therefore we prove the opposite direction. By the definition of the monodromy operator, we have $\rho_\ell(g) = 1$ for any $g \in J$, where $J$ is an open subgroup of $I_K$. Hence for any $\sigma \in I_K$, we get $\rho_\ell(\sigma)$ is of finite order, and the trace condition implies that $\rho_\ell(\sigma) = 1$. \qed

To prove Theorem 5.0.1 we need the Kuga-Satake construction. For simplicity, here we quickly recall the classical Kuga-Satake construction, rather than the uniform one. The following result was obtained by Madapusi Pera.

**Proposition 5.1.4** (see [MP15 Theorem 5.17]). Fix the primitive polarization $L$ of $X$ over $K$, and put $(L, L) = 2d$. Let $W_\ell$ be the primitive cohomology $P^2_{\acute{e}t}((X_{\mathbb{R}}, L_\mathbb{R}), \mathbb{Q}_\ell)$, and $\mathcal{L}_d$ be as in Definition 2.1.3. Then, there exists a finite separable extension $K'/K$ and an abelian variety $A$ over $K'$ with the following properties.

1. The abelian variety $A$ has semi-abelian reduction.
2. The abelian variety $A$ admits a left $C(\mathcal{L}_d)$-action over $K'$. Moreover, for any prime number $\ell$ which is different from the characteristic of $K$, there exists an isomorphism of $\mathbb{Q}_\ell$-modules

$$V_\ell := H^1_{\acute{e}t}(A_{\mathbb{R}}, \mathbb{Q}_\ell) \simeq C(W_\ell)$$

identifying the algebra $C(\mathcal{L}_d, \mathbb{Q}_\ell)^{op} \subset \text{End}(V_\ell)$ with $C(W_\ell)^{op} \subset \text{End}(C(W_\ell))$. Here, the former inclusion of algebras is induced by the above $C(\mathcal{L}_d)$-action, and the latter is induced by the right multiplication.

3. The left multiplication by $C(W_\ell)$ on the right-hand side of the isomorphism in (2) induces a Galois equivariant isomorphism

$$C(W_\ell) \simeq \text{End}_{C(\mathcal{L}_d, \mathbb{Q}_\ell)^{op}}(V_\ell).$$

**Remark 5.1.5.** As in Proposition 3.3.3, these results hold for cohomology groups with $\mathbb{Z}_\ell$-coefficients. However, we only need $\mathbb{Q}_\ell$-coefficients in this section.

**Lemma 5.1.6.** Let $K$ and $X$ be as in Theorem 5.0.1. Let $\ell, \ell' \neq p$ be prime numbers. If $N_\ell = 0$, then $N_{\ell'} = 0$.

**Proof.** The proof is same as in Proposition 4.2.3, but for the comparison to the $p$-adic case, we recall the argument. By the definition of the monodromy operators, we can replace $K$ by a finite extension for showing this claim. Therefore, by fixing
primitive polarization \(L\), we can associate the Kuga-Satake abelian variety \(A\) over \(K\) (see Proposition 5.1.4). Moreover, we may assume that \(H^2_{\text{et}}(X_K, \mathbb{Q}_p)\) is an unramified \(\text{Gal}(\overline{K}/K)\)-representation. Let \(\varphi: I_K \to \text{Aut}(V_\ell)\) be a restriction of the natural Galois representation. Then for any \(\gamma \in I_K\), we denote the image of

\[
\varphi(\gamma) \in \text{End}_{C(L,d_{a,q})^\text{op}}(V_\ell) \simeq C(W_\ell)
\]

by \(\varphi(\gamma)\) too (see Proposition 5.1.4 (3)). Because of the unramifiedness of \(C(W_\ell)\), for any \(c \in C(W_\ell), \gamma(z \mapsto cz) = (z \mapsto cz)\) in \(\text{End}_{C(L,d_{a,q})^\text{op}}(V_\ell)\). The former is \((z \mapsto \varphi(\gamma)c\varphi(\gamma)^{-1}z)\), therefore \(\varphi(\gamma)\) lies in the center of \(C(W_\ell)\), which is a reduced algebra. By Proposition 5.1.4 (1), we get \(\varphi(\gamma) = 1\), i.e. \(V_\ell\) is an unramified representation. By the Néron-Ogg-Shafarevich criterion for abelian varieties, \(A\) admits good reduction and therefore \(V_\ell\) is an unramified representation. Thus \(C(W_\ell) \simeq \text{End}_{C(L,d_{a,q})^\text{op}}(V_\ell)\) is unramified, and we get \(N_\ell = 0\).

**The proof of \((a) \iff (b)\) in the Theorem 5.0.1.** Taking a completion, we may assume that \(K\) is complete. We shall prove \((a) \Rightarrow (b)\). Take prime numbers \(\ell, \ell' \neq p\). By [Och99] Theorem 2.4 (for imperfect residue fields, see [Yr04, Proposition 4.2]), we get \(\text{tr}(\rho_p(\sigma)) = \text{tr}(\rho_{\ell'}(\sigma))\) for any \(\sigma \in I_K\). Therefore, by Lemma 5.1.3 and Lemma 5.1.6 we get the desired implication.

5.2. \(\ell\) versus \(p\) part. In this subsection, we prove the remaining part of Theorem 5.0.1. As in Theorem 5.0.1 let \(K\) be a complete discrete valuation field of mixed characteristic \((0, p)\) with a perfect residue field \(k\) in this subsection. First, we recall some generality of the \(p\)-adic representations.

**Definition 5.2.1.** Let \(\rho_p: \text{Gal}(\overline{K}/K) \to \text{GL}(V)\) be a \(p\)-adic representation (i.e. \(V\) is a finite dimensional \(\mathbb{Q}_p\)-vector space, and \(\rho_p\) is a continuous homomorphism). Define \(D_{\text{pst}}(V), D_{\text{st}}(V), D_{\text{crys}}(V)\) by

\[
\begin{align*}
D_{\text{pst}}(V) &\coloneqq \lim_{K' / K} (V \otimes_{\mathbb{Q}_p} B_{\text{st}})^{\text{Gal}(\overline{K}/K')} , \\
D_{\text{st}}(V) &\coloneqq (V \otimes_{\mathbb{Q}_p} B_{\text{st}})^{\text{Gal}(\overline{K}/K)} , \\
D_{\text{crys}}(V) &\coloneqq (V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{\text{Gal}(\overline{K}/K)} .
\end{align*}
\]

Here \(B_{\text{st}}, B_{\text{crys}}\) are the \(p\)-adic period rings introduced by Fontaine, and \(K'\) runs over finite extensions over \(K\). Let \(K_0\) be the ring of Witt vectors \(W(k)\), and \(K_0^\text{nr}\) be the maximal unramified extension of \(K_0\). Then \(D_{\text{pst}}(V)\) is a finite dimensional \(K_0^\text{nr}\)-vector space with the semi-linear Frobenius \(\varphi\) and the semi-linear action \(\widehat{\rho}_p\) of \(\text{Gal}(\overline{K}/K)\). Similarly, we can show that \(D_{\text{st}}(V)\) and \(D_{\text{crys}}(V)\) are finite-dimensional \(K_0\)-vector spaces with the semi-linear Frobenius \(\varphi\). Moreover, \(D_{\text{pst}}(V)\) (resp. \(D_{\text{st}}(V)\)) admits a \(K_0^\text{nr}\)-linear (resp. \(K_0\)-linear) endomorphism \(N_p\), which is called the monodromy operator.

**Lemma 5.2.2.** Let \(\rho_p: \text{Gal}(\overline{K}/K) \to \text{GL}(V)\) be a potentially semi-stable \(p\)-adic representation. Then, the following are equivalent.

1. The \(p\)-adic representation \(\rho_p\) is crystalline.
2. The monodromy operator \(N_p\) on \(D_{\text{pst}}(V)\) is a zero map, and \(\text{tr}(\widehat{\rho}_p(\sigma)) = \dim_{\mathbb{Q}_p} V\) for any \(\sigma \in I_K\).
Proof. (1) → (2) is trivial by the definition of the crystalline representation. Assume (2). Then, we have $V$ is a potentially crystalline representation, i.e. there exists a finite extension $K'$ over $K$ such that $\rho_p|_{\text{Gal}((K')/K)}$ is crystalline. Then, we have

$$D_{\text{pst}}(V) = D^{(K')}_{\text{crys}}(V) \otimes_{K'_0} K'_0,$$

where $D^{(K')}_{\text{crys}}$ is $D_{\text{crys}}$ for $\text{Gal}(K'/K)$-representations. Because of the trace assumption, $I_K$ acts trivially on $D_{\text{pst}}(V)$. Take $L$ which is a maximal unramified extension over $K$ in $K'$. Then, we have

$$\dim_{K'_0}(V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{\text{Gal}((\mathbb{R})/L)} = \dim_{K'_0}(V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{\text{Gal}((\mathbb{R})/K')} = \dim_{\mathbb{Q}_p} V,$$

where the first equality follows from the triviality of $I_K$ action. Thus we get $\rho_p|_{\text{Gal}((\mathbb{R})/L)}$ is crystalline, so $\rho_p$ is crystalline.

Lemma 5.2.3. Let $X$ be a K3 surface over $K$, and consider the $p$-adic representation $H^2_{\text{et}}(X_{\mathbb{R}}, \mathbb{Q}_p)$. Let $N_p$ be the monodromy operator on $D_{\text{pst}}(H^2_{\text{et}}(X_{\mathbb{R}}, \mathbb{Q}_p))$. Moreover, as in the previous subsection, let $\ell$ be a prime number different from $p$, and let $N_{\ell}$ be the monodromy operator on the $\ell$-adic representation $H^2_{\text{et}}(X_{\mathbb{R}}, \mathbb{Q}_\ell)$. Then, we have $N_p = 0 \iff N_{\ell} = 0$.

Proof. As in Lemma 5.1.6, we can extend $K$ to get the Kuga-Satake abelian variety $A$ over $K$. First, we assume $N_{\ell} = 0$. Extending $K$ if necessary, we may assume the $\text{Gal}(K'/K)$-representation $H^2_{\text{et}}(X_{\mathbb{R}}, \mathbb{Q}_\ell)$ is unramified. Then, by the arguments in Lemma 5.1.6, we get that $A$ admits good reduction. Hence $V_p = H^1_{\text{et}}(A_{\mathbb{R}}, \mathbb{Q}_p)$ is crystalline, so by Proposition 5.1.4 (3), we get $W_p = P_{\text{crys}}^p((X_{\mathbb{R}}, L_{\mathbb{R}}), \mathbb{Q}_p)$ is crystalline. Therefore $H^2_{\text{et}}(X_{\mathbb{R}}, \mathbb{Q}_p)$ is crystalline and so $N_p = 0$.

Next, we will show the opposite direction. As before, we may assume $H^2_{\text{et}}(X_{\mathbb{R}}, \mathbb{Q}_p)$ is crystalline. We will prove that $V_p$ is crystalline, by the analogous argument to Lemma 5.1.6. In the following, we write $\text{End}^{(K_0)}$ when we consider a $K_0$-linear endomorphism algebra. First, remark that

$$D_{\text{st}}(\text{End}_{C(\mathcal{L}_d, \mathbb{Q}_p)}^{(p)}(V_p)) \simeq \text{End}^{(K_0)}_{C(\mathcal{L}_d, \mathbb{Q}_p)}(D_{\text{st}}(V)).$$

Indeed, if we take the $\mathbb{Q}_p$-basis $c_i$ $(i = 1, \ldots, 2^{21})$ of $C(\mathcal{L}_d, \mathbb{Q}_p)$, we can consider the exact sequence

$$0 \to \text{End}_{C(\mathcal{L}_d, \mathbb{Q}_p)}^{(p)}(V_p) \to \text{End}(V_p) \to \text{End}(V_p)^{\otimes 221},$$

where the right arrow is $f \mapsto (f \cdot c_i - c_i \cdot f)_{1 \leq i \leq 221}$. Then, it is the exact sequence of $\text{Gal}(K'/K)$-module because the $C(\mathcal{L}_d)$-action on $A$ is defined over $K$. Applying $D_{\text{st}}$, we get the desired isomorphism. Here, we use the isomorphism

$$D_{\text{st}}(\text{End}(V_p)) \simeq \text{End}^{(K_0)}(D_{\text{st}}(V_p)).$$

Consider the monodromy operator

$$N_{V_p} \in \text{End}^{(K_0)}_{C(\mathcal{L}_d, \mathbb{Q}_p)}(D_{\text{st}}(V_p)) \simeq D_{\text{st}}(\text{End}_{C(\mathcal{L}_d, \mathbb{Q}_p)}^{(p)}(V_p)) \simeq D_{\text{st}}(C(W_p)),$$

and we denote its image in $D_{\text{st}}(C(W_p))$ by $N_{V_p}$. To prove $N_{V_p} = 0$, it is enough to show that $\gamma := \exp(N_{V_p}) = 1$. Since $W_p$ is crystalline, we get $\exp(N_{W_p})(c) = c$ for any $c \in D_{\text{st}}(C(W_p))$. Via the isomorphisms

$$D_{\text{st}}(C(W_p)) \simeq D_{\text{st}}(\text{End}_{C(\mathcal{L}_d, \mathbb{Q}_p)}^{(p)}(V_p)) \simeq \text{End}^{(K_0)}_{C(\mathcal{L}_d, \mathbb{Q}_p)}(D_{\text{st}}(V_p)),$$
the above equality implies $\gamma c^{-1} = c$, since the latter isomorphism comes from the natural isomorphism $D_{st}(\text{End}(V_p)) \simeq \text{End}^{(K)}(D_{st}(V_p))$. Therefore $\gamma$ lies in the center of $D_{st}(C(W_p))$ which is included in the center of $D_{st}(C(W_p)) \otimes_{K_B} B_{st} \simeq C(W_p) \otimes_{Q_p} B_{st}$, which is a reduced algebra. Since $\gamma - 1$ is nilpotent, we get $\gamma = 1$, i.e. $V_p$ is crystalline. By a crystalline analogue of the Néron-Ogg-Shafarevich criterion for abelian varieties [Cf99, Theorem 1], the abelian variety $A$ admits good reduction, and therefore we get $V_\ell$ is unramified, so $W_\ell$ is unramified. Finally we get $N_\ell = 0$. 

The proof of $(a) \iff (c)$ in Theorem 5.0.1. By [Och99, Theorem 3.1], we get $\text{tr}(\rho_\ell(\sigma)) = \text{tr}(\hat{\rho}_p(\sigma))$ for any $\sigma \in I_K$, where $\hat{\rho}_p$ means the semi-linear $\text{Gal}(\overline{K}/K)$-action on $D_{st}(H^2_{\text{et}}(X_{\overline{\kappa}}, Q_\ell))$. Therefore, the desired implications follow from Lemma 5.1.3, Lemma 5.2.2 and Lemma 5.2.3.

6. Corollaries

6.1. Some remarks. First, combining Theorem 4.1.4 with Theorem 5.0.1, we obtain the main theorem in more generalized form.

Theorem 6.1.1. Let $F$ be a finitely generated field over $\mathbb{Q}$, $R$ be a finite type algebra over $\mathbb{Z}$ which is a normal domain with the fraction field $F$, and $d$ be a positive integer. Then, the set

$$S(F, R) := \{X \mid X : \text{K3 surface over } F \text{ satisfying the condition (C)}\}/F\text{-isom}$$

is finite. Here, the condition (C) is the following.

- (C) For any height 1 prime $p \in \text{Spec}(R)$, take a discrete valuation field $E_p$ such that $E_p$ is an algebraic extension of discrete valuation fields over $F = \text{Frac}(R_p)$, the residue field of $E_p$ is the perfection of the residue field of $R_p$, and a uniformizer of $R_p$ is also a uniformizer of $E_p$. Then, there exists a prime number $\ell$ different from the residual characteristic of $p$ such that $H^2_{\text{et}}(X_{\overline{\kappa}}, \mathbb{Q}_\ell)$ is an unramified $\text{Gal}(\overline{F}/E_p)$-representation.

Remark 6.1.2. (1) The field extension $E_p$ in the condition (C) always exists by [Mat89, Theorem 29.1].

(2) By Theorem 5.0.1 the unramifiedness assumption in the condition (C) is independent of $\ell$. If the residual characteristic of $p$ is positive, replacing $E_p$ by the completion of it, we can replace this condition in terms of crystalline representations.

Proof. Shrinking Spec $R$ if necessary, we may assume that $R$ is smooth over $\mathbb{Z}$ since the generic fiber $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is generically smooth over $\mathbb{Q}$. Let $M$ be the order of $\text{GL}_{22}(\mathbb{F}_2)$. Shrinking Spec $R$ again, we may assume that $1/M \in R$. Consider a height 1 prime $p \in \text{Spec}(R)$, and denote its residual characteristic by $p \geq 0$. Take an extension of valuation $p$ to $\overline{p}$, and denote it by $p$. We denote the inertia subgroups by $I_p \subset \text{Gal}(\overline{F}/F)$, $I_p' \subset \text{Gal}(\overline{F}/E_p)$. We denote the $\text{Gal}(\overline{F}/F)$-representation $H^2_{\text{et}}(X_{\overline{\kappa}}, \mathbb{Z}_2)$ by $\rho$. Then, by Remark 6.1.2 (2), we get $\rho(I_p') = 1$. If $p = 0$, we have $\rho(I_p) = \rho(I_p') = 1$. If $p > 0$, for any finite index open normal subgroup $H$ of $\rho(I_p)$, we get $\{\rho(I_p) : H\}$ is a $p$-group by using the fundamental equality of discrete valuation fields. Therefore we get $\rho(I_p) \cap (1 + 2 \cdot \text{Mat}_{22}(\mathbb{Z}_2)) = 1$. 


since the former is pro-$p$ and the latter is pro-2. Moreover, the image of $\rho(I_v)$ in $\text{GL}_{22}(\mathbb{F}_2)$ via the reduction map is trivial because $p$ does not divide $M$. Therefore, we get $\rho(I_v) = 1$ even if $p > 0$. Thus we have $S(F, R) \subset \text{Shaf}(F, R)$, so $S(F, R)$ is a finite set.

Next, as an immediate consequence of Theorem 6.1.1, we obtain the unpolarized Shafarevich conjecture for K3 surfaces over finitely generated fields of characteristic 0.

**Definition 6.1.3.** Let $R_p$ be a discrete valuation ring with maximal ideal $p$, and $F$ be the fraction field of $R_p$. For a K3 surface $X$ over $F$, we say $X$ has good reduction at $p$ if there exists a smooth proper algebraic space over $R_p$ whose generic fiber is isomorphic to $X$. Remark that such model would be automatically a K3 family over $\text{Spec} R_p$ (see Definition 2.1.1 (2)).

**Corollary 6.1.4.** Let $F$ be a finitely generated field over $\mathbb{Q}$, and $R$ be a finite type algebra over $\mathbb{Z}$ which is a normal domain with the fraction field $F$. Then, the set

$$\left\{ X \mid X: \text{K3 surface over } F, \right. \right.$$

$$X \text{ has good reduction at any height 1 prime ideal } p \in \text{Spec } R \left. \right\} / F\text{-isom}$$

is finite.

6.2. The finiteness of twists. Here, we give the finiteness result of twists of K3 surfaces via a finite extension of characteristic 0 fields.

**Corollary 6.2.1.** Let $F$ be a field of characteristic 0, $E/F$ be a finite extension, and $X$ be a K3 surface over $F$. Then, the set

$$\text{Tw}_{E/F}(X) := \left\{ Y: \text{K3 surface over } F \mid Y_E \simeq_E X_E \right\} / F\text{-isom}$$

is finite.

**Proof.** Clearly, we may assume $E/F$ is a finite Galois extension. First, we will reduce the problem to the case of finitely generated fields. Since $\text{Aut}(X_F)$ is a finitely generated group ([Ste85, Proposition 2.2]), extending $E$ if necessary, we may assume $\text{Aut}(X_E) = \text{Aut}(X_F)$. We can take a finitely generated field $E' \subset E$ on which $X$ and any elements of $\text{Aut}(X_E)$ are defined. Moreover, by extending $E'$ if necessary, we may assume $E'$ is $\text{Gal}(E/F)$-stable and $\text{Gal}(E/F) \rightarrow \text{Aut}(E')$ is injective. Let $F'$ be the fixed subfield $E'^{\text{Gal}(E/F)}$. Then, the description of twists

$$\text{Tw}_{E/F}(X) \simeq H^1(\text{Gal}(E/F), \text{Aut}(X_E))$$

$$\simeq H^1(\text{Gal}(E'/F'), \text{Aut}(X_{E'})),$$}

implies that the desired finiteness is reduced to the case of $E'/F'$.

Thus, in the following of this proof, we assume $F$ is a finitely generated field and $E/F$ is a finite Galois extension. One can take a smooth proper morphism of schemes $X \rightarrow \text{Spec } R$ whose generic fiber is $X$, where $R$ is a smooth algebra over $\mathbb{Z}$ which is an integral domain with the fraction field $F$ and $1/2 \in R$. Then, via a monodromy action, we get $H^2_d(X_F, \mathbb{Z}_2)$ is unramified over $\text{Spec } R$. Let $\tilde{R}$ be the normalization of $R$ in $E$. Shrinking $\text{Spec } R$ if necessary, we may assume $\text{Spec } \tilde{R} \rightarrow \text{Spec } R$ is a finite étale covering. Since $E$ is unramified over $\text{Spec } R$, by [Fu15, Proposition 3.3.6], we have

$$\ker(\pi_1(\text{Spec } E, \mathfrak{m}) \rightarrow \pi_1(\text{Spec } \tilde{R}, \mathfrak{m})) = \ker(\pi_1(\text{Spec } F, \mathfrak{m}) \rightarrow \pi_1(\text{Spec } R, \mathfrak{m})).$$
For any $Y \in \text{Tw}_{E/F}(X)$, the isomorphism $Y_E \simeq E_X$ implies that the $\text{Gal}((\overline{F}/E)$-action on $H^2_{\acute{e}t}(Y_{\overline{F}}, \mathbb{Z}_2)$ descends to a $\pi_1(\text{Spec}\overline{R}, \overline{\mathfrak{r}})$-action. Moreover, because of the above equality, the $\text{Gal}((\overline{F}/F)$-action on $H^2_{\acute{e}t}(Y_F, \mathbb{Z}_2)$ also descends to a $\pi_1(\text{Spec} R, \mathfrak{r})$-action. Hence we get a natural inclusion $\text{Tw}_{E/F}(X) \hookrightarrow \text{Shaf}(F, R)$, and thus the desired finiteness follows from Theorem 4.1.4.

□

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Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Tokyo, 153-8914, Japan

E-mail address: teppei@ms.u-tokyo.ac.jp