Decay Properties of the Connectivity for Mixed Long Range Percolation Models on $\mathbb{Z}^d$

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Abstract

In this short note we consider mixed short-long range independent bond percolation models on $\mathbb{Z}^{k+d}$. Let $p_{uv}$ be the probability that the edge $(u, v)$ will be open. Allowing a $x, y$-dependent length scale and using a multi-scale analysis due to Aizenman and Newman, we show that the long distance behavior of the connectivity $\tau_{xy}$ is governed by the probability $p_{xy}$. The result holds up to the critical point.

1 Introduction

In this short note we consider a long range percolation model on $L = (\mathbb{Z}^{k+d}, \mathcal{B})$, where $u \in \mathbb{Z}^{k+d}$ is of the form $u = (\vec{u}_0, \vec{u}_1)$, with $\vec{u}_0 \in \mathbb{Z}^k$ and $\vec{u}_1 \in \mathbb{Z}^d$ and $\mathcal{B}$ is the set of edges (unordered pairs) $(u, v), u \neq v \in \mathbb{Z}^{k+d}$. To each edge $(u, v)$ we associate a Bernoulli random variable $\omega_b$ which is open $(\omega_b = 1)$ with probability

$$p_{uv} = p_{uv}(\beta) \equiv \beta J_{uv}, \quad u, v \in \mathbb{Z}^{k+d}$$

where $\beta \in [0, 1]$ and, for $\epsilon > 0$, $J_{uv}$ is

$$J_{uv} = \begin{cases} 2(1 + \|\vec{u}_1 - \vec{v}_1\|^{d+\epsilon}^{-1})^{-1} & \text{if } \vec{u}_0 = \vec{v}_0 \text{ and } \vec{u}_1 \neq \vec{v}_1; \\ 1 & \text{if } \vec{u}_1 = \vec{v}_1 \text{ and } \|\vec{u}_0 - \vec{v}_0\| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

(2)
We denote the event \(\{\omega \in \Omega : \text{there is an open path connecting } x \text{ to } y\}\) by \(\{x \leftrightarrow y\}\) and define the connectivity function by \(\tau_{xy} \equiv P\{x \leftrightarrow y\}\). Let \(|x| = |x_1| + \cdots + |x_d|\) be the \(L^1\) norm on \(\mathbb{Z}^d\) and \(\beta_c = \sup\{\beta \in [0, 1] : \chi(\beta) < \infty\}\). Our aim is to show the following result

**Theorem.** Suppose \(\beta < \beta_c\) and consider the long range percolation model with \(p_{uv}\) given by (1) and \(J_{uv}\) given by (2). Then there exist positive constants \(C = C(\beta)\) and \(m = m(\beta)\) such that

\[
\tau_{xy} \leq C \frac{e^{-m||\vec{x}_0 - \vec{y}_0||}}{1 + ||\vec{x}_1 - \vec{y}_1||^{d+\varepsilon}}
\]

for all \(x, y \in \mathbb{Z}^{k+d}\).

The above result says that the probability \(p_{uv}\) dictates the long distance behavior of the connectivity function in the subcritical regime (similar lower bounds are easily obtained from the FKG inequality). For the one dimensional \(\mathbb{Z}^{0+1}\) percolation model, the above result is known to hold, see [1], the same being true for one dimensional \(\mathbb{Z}^{0+1}\) \(O(N)\) spin models, \(1 \leq N \leq 4\), see [2]. The result is expected to hold in the \(d\)-dimensional \(\mathbb{Z}^{0+d}\) lattice but it is not clear how to prove it if \(\beta < \beta_c\), although one can see it holds if \(\beta \approx 0\). Our upper bound (3) holds if \(\beta < \beta_c\) and for \((k+d)\)-dimensional lattices, \(k \geq 0\) and \(d \geq 1\). For lattice spin models, the upper bound (3) is known at the high temperature regime, see Ref. [3] for bounded spin models and Ref. [4] for unbounded (and discrete) ones. Ref. [5] extends some of the results of [3, 4] to a general class of continuous spin systems, with \(J_{uv}\) given by (1) and \(u, v \in \mathbb{Z}^{0+d}\), while [6] considers the more general mixed decay model. In both cases, the polymer expansion (see [7] and references therein) is used and the results hold only in the perturbative regime.

The Hammersley-Simon-Lieb inequality ([8, 9, 10]) is a key ingredient in [1] and [2] and here we also adopt this “correlation inequality” point of view. For completeness, we state this inequality in the form we will use, see [11]. For each set \(S \subset \mathbb{Z}^d\), let \(\tau_{xy}^S \equiv P\{x \leftrightarrow y \text{ inside } S\}\). Then

**Hammersley-Simon-Lieb Inequality (HSL)** Given \(x, y \in \mathbb{Z}^{k+d}\), if \(S \subset \mathbb{Z}^{k+d}\) is such that \(x \in S\) and \(y \in S^c\), then

\[
\tau_{xy} \leq \sum_{\{u \in S, v \in S^c\}} \tau_{xu}^S p_{uv} \tau_{vy}.
\]

We now recall some known facts about the long range percolation model defined by (1) and (2). Let \(\theta(\beta, \varepsilon) = P_{\beta, \varepsilon}\{0 \leftrightarrow \infty\}\) be the probability that the origin will be connected to infinity. If \(k + d \geq 2\) is the space dimension then, by comparing with the nearest neighbor model and for any positive \(\varepsilon\), there exists \(\beta_c = \hat{\beta}(d, \varepsilon)\) such that \(\theta(\beta, \varepsilon) = 0\) if \(\beta < \beta_c\) and \(\theta(\beta, \varepsilon) > 0\) if \(\beta > \beta_c\). For \(k = 0\) and \(d = 1\), it is known that the existence of \(\beta_c\) depends upon \(\varepsilon\), if \(\varepsilon > 1\) then there is no phase transition [12] while it shows up
if $0 \leq \varepsilon \leq 1$, see [13]. A phase transition can also be measured in terms of $\chi$, the mean cluster size, given by $\chi = \sum_x \tau_0 x$. Let $\pi_c(d, \varepsilon) = \sup \{ \beta : \chi(\beta, \varepsilon) < \infty \}$. Then, it comes from the FKG Inequality [14] that $\pi_c(d, \varepsilon) = \beta_c(d, \varepsilon)$. The equality $\pi_c(d, \varepsilon) = \beta_c(d, \varepsilon)$ holds for the class of models we are dealing with and it was proved independently by Aizenman and Barsky in [15] and Menshikov in [16]. We will use the condition $\chi < \infty$ to characterize the subcritical region.

The remaining of this note is divided as follows: in the next section we prove Theorem 1 and in Section 3 we make some concluding remarks regarding the validity of our results to ferromagnetic spin models.

## 2 Proof of the Theorem

Let $x = (\bar{x}_0, \bar{x}_1) \in \mathbb{Z}^{k+d}$. We first observe that

$$\chi = \sum_{n \geq 0} \sum_{\|\bar{x}_0\| = n} \sum_{\bar{x}_1 \in \mathbb{Z}^d} \tau_{0x}.$$ 

Since $\chi < \infty$, given $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have

$$\sum_{\|\bar{x}_0\| = n} \sum_{\bar{x}_1 \in \mathbb{Z}^d} \tau_{0x} < \lambda.$$

Consider now $x = (\bar{x}_0, \bar{x}_1)$ with $\|\bar{x}_0\| > n_0$. Using the translation invariance of the model and applying iteratively the HSL Inequality with $S = \{ x \in \mathbb{Z}^{k+d} ; \|\bar{x}_0\| \leq n_0 \}$, we obtain

$$\sum_{\bar{x}_1 \in \mathbb{Z}^d} \tau_{0x} \leq \lambda^{\|\bar{x}_0\|/n_0} \leq C_1 \exp(-(m+\delta)\|\bar{x}_0\|),$$

where $\lfloor r \rfloor$ denotes the integer part of $r$ and $\delta > 0$ is given and $m$ is defined by $e^{-(m+\delta)} = \lambda^{1/n_0}$.

Next we show that a HSL type inequality holds for the modified connectivity function $T_m(x, y) \equiv e^m\|\bar{x}_0 - \bar{y}_0\|\tau_{xy}$ and for the set $S = \mathcal{C}_r(x) \equiv \{ z \in \mathbb{Z}^{k+d} ; \|\bar{x}_1 - \bar{z}_1\| \leq r \}$. Applying the HSL Inequality with the above specified $S$, we have

$$\tau_{xy} \leq \sum_{u \in \mathcal{C}_r(x)} \sum_{v \in \mathcal{C}_r^L(x)} \tau_{ux} p_{uv} \tau_{vy}.$$ 

Then, for $y \in \mathcal{C}_r(x)$ for some $L > 1$, we obtain

$$T_m(x, y) = e^m\|\bar{x}_0 - \bar{y}_0\|\tau_{xy} \leq e^m\|\bar{x}_0 - \bar{y}_0\| \sum_{u \in \mathcal{C}_r(x)} \sum_{v \in \mathcal{C}_r^L(x)} \tau_{ux} p_{uv} \tau_{vy}.$$

$$\leq \sum_{u \in \mathcal{C}_r(x)} \sum_{v \in \mathcal{C}_r^L(x)} e^m\|\bar{x}_0 - \bar{u}_0\|\tau_{ux} p_{uv} e^m\|\bar{y}_0 - \bar{v}_0\|\tau_{vy}.$$
since we necessarily have that \( \bar{u}_0 = \bar{v}_0 \). It then follows that
\[
T_m(x, y) \leq \sum_{u \in C_L(x)} \sum_{v \in C_L^c(x)} T_m(x, u)p_{uv}T_m(v, y).
\]

We remark that \( \chi_m \equiv \sum_{x \in \mathbb{Z}^{k+d}} T_m(0, x) < \infty \) is finite if \( \beta < \beta_c \) since
\[
\sum_{x \in \mathbb{Z}^{k+d}} T_m(0, x) = \sum_{x \in \mathbb{Z}^{k+d}} e^m\|\bar{x}_0\|\tau_0x \leq \sum_{k \geq 0} \sum_{\|\bar{x}_0\| = k} e^m\|\bar{x}_0\| \sum_{\bar{x}_1 \in \mathbb{Z}^d} \tau_0x \\
\leq \sum_{k \geq 0} 2dk^{d-1}e^{-\delta k} < \infty.
\]

From now on we closely follow Section 3 of [1] and prove the polynomial decay of \( T_m \) up to the critical point. For fixed \( x, y \in \mathbb{Z}^{k+d} \) and \( L \equiv \|\bar{x}_1 - \bar{y}_1\|/4 \), we know that
\[
T_m(x, y) \leq \sum_{u \in C_L(x)} \sum_{v \in C_L^c(x)} T_m(x, u)p_{uv}T_m(v, y)
\]
\[
\leq \sum_{u \in C_L(x)} \sum_{v \in C_L^c(x)} T_m(u, x)p_{uv}T_m(v, y) + \sum_{u \in C_L(x)} \sum_{v \in C_L^c(x) \cap C_{3L}(x)} T_m(x, u)p_{uv}T_m(v, y). \tag{4}
\]

Let
\[
T_m(L) \equiv \sup\{T_m(0, u); u \in C_L^c(0)\} \quad \text{and} \quad \gamma_L \equiv \sum_{u \in C_L(x)} \sum_{v \in C_L^c(x)} T_m(x, u)p_{uv}.
\]

Then, the first term on the r.h.s. of (4) is bounded above by \( T_m(L/2)\gamma_L \) while the second one is bounded by
\[
\frac{\beta \chi_m^2}{1 + \|\bar{x}_1 - \bar{y}_1\|^{d+\varepsilon}},
\]
leading to
\[
T_m(x, y) \leq \frac{\beta \chi_m^2}{1 + \|\bar{x}_1 - \bar{y}_1\|^{d+\varepsilon}} + \gamma_L T_m \left( \frac{L}{2} \right).
\]

Now, since \( \chi_m < \infty \) for \( \beta < \beta_c \) and since \( \sum_u p_{0u} < \infty \), we have that \( \gamma_L \to 0 \) as \( L \to \infty \). For \( \alpha \in (0, 2^{-d+\varepsilon}) \), there exists \( L_0 > 0 \) such that \( \gamma_L < \alpha \) for all \( L \geq L_0 \). Considering \( L > L_0 \), it follows that
\[
T_m(L) \leq \frac{\beta \chi_m^2}{1 + \|\bar{x}_1 - \bar{y}_1\|^{d+\varepsilon}} + \alpha T_m \left( \frac{L}{2} \right). \tag{5}
\]

Iterating (5) \( n \) times, with \( n \) the smallest integer for which \( L2^{-n} \leq L_0 \), we have for all \( L > L_0 \)
\[
T_m(L) \leq \frac{2 \beta \chi_m^2}{1 + \|\bar{x}_1 - \bar{y}_1\|^{d+\varepsilon}} + \alpha^n T_m \left( \frac{L}{2^n} \right).
\]

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Noting that $T_m(x, y) \leq T_m(L)$, that $T_m(L) \leq 1$ for any $L > 0$ and that

$$\alpha^n \leq 2^{-(d+\varepsilon)n} = \frac{1}{(1 + L^{d+\varepsilon})} \frac{(1 + L^{d+\varepsilon})}{2^{(d+\varepsilon)n}} \leq \frac{2 \cdot 2^{d+\varepsilon}}{1 + \|x_1 - y_1\|^{d+\varepsilon} \left( \frac{L}{2^n} \right)^{d+\varepsilon}} \leq \frac{2(2L_0)^{d+\varepsilon}}{1 + \|x_1 - y_1\|^{d+\varepsilon}},$$

we can conclude that, for $\beta < \beta_c$,

$$\tau_{xy} \leq \frac{C \, e^{-m\|x_0 - y_0\|}}{1 + \|x_1 - y_1\|^{d+\varepsilon}}$$

and the bound (3) holds.

\[ \square \]

3 Concluding Remarks

The strategy used to prove the main Theorem can also be applied to $\mathbb{Z}^d$ ferromagnetic models with free boundary conditions and pair interaction $J_{uv}$ given by (2). For this class of models the Griffiths inequalities [17, 18] are valid, guaranteeing the positivity of spin-spin correlations $\langle \sigma_x \sigma_y \rangle$. Simon-Lieb Inequality also holds in this case (see [7] and references therein), with $p_{uv}$ replaced by $\beta J_{uv}$, where $\beta$ is the inverse of the temperature, and with $\tau_{xy}$ replaced by $\langle \sigma_x \sigma_y \rangle$. Finally, since the uniqueness of the critical point is guaranteed in [19], the results of Section 2 are also valid for these models.

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