Ground-state energy shift of $n$ pions and $m$ kaons in a finite volume

Brian Smigielski and Joseph Wasem
Department of Physics, University of Washington
Box 351560, Seattle, WA 98195, USA

The ground state energy of a collection of $n$ pions and $m$ kaons with short range interactions is calculated for a volume with finite spatial extent $L$ and periodic boundary conditions. This calculation is accomplished to order $L^{-6}$ in the large volume expansion. With this result one can extract the various two- and three-body interactions between pions and kaons from lattice QCD data.

Recently, efforts have been made using lattice QCD to extract both the $\pi^+\pi^+$ and the $K^+K^+$ scattering lengths as well as the three-body interactions $\pi^+\pi^+\pi^+$ and $K^+K^+K^+$ [1, 2, 3, 4]. These efforts have made steps toward using lattice QCD to rigorously determine the nuclear equation of state (NEOS) directly from the underlying theory of QCD. A knowledge of the NEOS would be of great importance to many research areas, including helping to predict the evolution of supernovae. In addition, an understanding of multi-pion and multi-kaon systems provide insight into strongly interacting boson gases. References [1, 2, 3, 4] have made use of a method involving the volume dependence of the energy spectrum (below inelastic thresholds) of two hadrons as a function of their scattering length [5, 6], where lattice QCD is used to calculate the energy of multi-hadron states and this result is then applied to the calculated dependence of the finite volume ground state energy shift to determine interaction strengths. The ground state energy of $n$ identical bosons with short range interactions is calculated in the large volume expansion to $O(L^{-7})$ in Ref. [7], providing a way to extract the three-body interaction strength between identical bosons which enter at $O(L^{-6})$, and building upon the earlier work of Refs. [8, 9, 10, 11, 12]. In this work we extend the calculation to determine the ground state energy shift in finite volume of $n$ pions and $m$ kaons to $O(L^{-6})$ in the large volume expansion using a multispecies extension of the techniques used in Refs. [7, 8]. This result will allow for the systematic extraction of $\pi K$ scattering lengths and the three-body interactions $\pi\pi K$ and $\pi K K$, as well as provide a different way to extract $\pi\pi$ and $KK$ scattering lengths by using mixed pion-kaon systems in lattice calculations.

To calculate the ground state energy there are multiple (and entirely equivalent) methods that one can use. One such method uses a resolvent function of the system Hamiltonian defined by [3]:

$$F(z) = \langle 0 | \frac{1}{z - \mathcal{H}} | 0 \rangle$$
$$= \frac{1}{z - E_0 - r(z)}$$

(1)
with
\[ r(z) = \langle 0 | \hat{V} \sum_{n=0}^{\infty} \left( \frac{\hat{Q}_0}{z - \mathcal{H}_0} \right)^n | 0 \rangle \]  
(2)

where \(|0\rangle\) is the ground-state of the system and \(\hat{Q}_0 = 1 - |0\rangle \langle 0|\) is a projection operator discussed below. Here \(\mathcal{H}\) is the full Hamiltonian of the system while \(\mathcal{H}_0\) is the free Hamiltonian. In eqn. (1), when \(F(z)\) takes the form found in the first line, a perturbative expansion leads to a pole near \(z = E_0\). However, projecting out that portion of the propagator containing the pole using the operator \(\hat{Q}_0\) leads to the function \(r(z)\) in the second line which is smooth in the neighborhood of \(E_0\) and which can be perturbatively expanded in powers of the potential \(\hat{V}\). To find the ground state energy shift one needs to expand the function \(r(z)\) around the point \(z = E_0\). The leading terms in this expansion are:

\[ r_j(z) = \frac{1}{j!} \frac{\partial^j}{\partial z^j} r(z) \big|_{z=E_0} \]

\[ \Delta E = r_0(E_0) + r_0(E_0)r_1(E_0) + r_0(E_0)r_2^2(E_0) + r_0^3(E_0)r_2(E_0) + \ldots . \]

(3)

The method of pseudo-potentials can be employed to determine \(r(z)\) above, or one can use the different but equivalent approach of nonrelativistic time-independent perturbation theory (NRPT). The potential used in this calculation is:

\[ \hat{V} = \sum_{i<j \in \mathcal{P}_\pi} \chi_{\pi\pi} \delta^{(3)}(\vec{r}_i - \vec{r}_j) + \sum_{i \in \mathcal{P}_\pi, j \in \mathcal{P}_K} \chi_{\pi K} \delta^{(3)}(\vec{r}_i - \vec{r}_j) + \sum_{i \in \mathcal{P}_K, j \in \mathcal{P}_K} \chi_{KK} \delta^{(3)}(\vec{r}_i - \vec{r}_j) + \]

\[ \sum_{i<j<k \in \mathcal{P}_\pi} \eta_{3,\pi\pi} \delta^{(3)}(\vec{r}_i - \vec{r}_j) \delta^{(3)}(\vec{r}_i - \vec{r}_k) + \sum_{i \in \mathcal{P}_\pi, j \in \mathcal{P}_K, k \in \mathcal{P}_K} \eta_{3,\pi K} \delta^{(3)}(\vec{r}_i - \vec{r}_j) \delta^{(3)}(\vec{r}_i - \vec{r}_k) + \]

\[ \sum_{i \in \mathcal{P}_\pi, j \in \mathcal{P}_K, k \in \mathcal{P}_K} \eta_{3,PK} \delta^{(3)}(\vec{r}_i - \vec{r}_j) \delta^{(3)}(\vec{r}_i - \vec{r}_k) \]

(4)

where the first line is the two-body interaction potential while the remaining lines constitute the various three-body potentials. The sets \(\mathcal{P}_\pi\) and \(\mathcal{P}_K\) refer to the set of all pions and kaons in the system, respectively. Due to the requirement that bound states do not form, the interactions between pions and kaons studied are restricted to be repulsive. Hence all \(\pi\) and \(K\) fields referenced are shorthand for \(\pi^+\) and \(K^+\). Because the particles are nonrelativistic, interactions such as \(KK \rightarrow \pi\pi\) or \(\pi\pi \rightarrow KK\) are not considered.

The \(\chi\)’s can be related to the scattering amplitude and for small external momentum be expanded using effective range theory according to \(\chi = -\frac{4\pi}{M} p^{-1} \tan \delta(p) \rightarrow -\frac{4\pi}{M} \left( -\frac{1}{a} + \frac{1}{4} r_0 p^2 + \cdots \right)^{-1}\) where \(a\) is the s-wave scattering length and \(r_0\) is the effective range. The modern language and description of this approach is pionless effective field theory or EFT(\#)\[13, 14, 15\]. In this framework, the two-body interaction \(\chi\) corresponds to the expansion of the \(2 \rightarrow 2\) contact interaction in the Lagrangian\[14\]. The language of EFT(\#) possesses a more natural way of dealing with issues of renormalization and power counting as the choice of a particular subtraction scheme leads to order-by-order renormalization of loop divergences.

The contact potential from eqn. (4) can be reexpressed in terms of a Hamiltonian in NRPT, as described in ref. 4. Using this method, each term in the large volume expansion of the function \(r(z)\) can be expressed as a sum of
diagrams in the perturbation theory. Vertices in these diagrams are given by the χ and η shown above, while the propagators are given by the expectation of the free theory resolvent \( \frac{1}{z-H_0} \) over intermediate states. The presence of the projection operator \( \hat{Q}_0 \) removes from consideration any diagrams where the intermediate state expectation gives \( \langle \mathcal{H}_0 \rangle = 0 \). Loop divergences are regulated with dimensional regularization in our calculation. The divergences arising from two-loop diagrams will necessitate the inclusion of three-body interaction strengths as a function of the renormalization scale \( \mu \). In finite volume the integrals normally associated with the continuum expression for each diagram will become sums because the momenta will be restricted to the possible values \( p = 2\pi n/L \) for \( n_j \in \mathbb{Z} \).

It is simple to show that generating all of the diagrams and summing their contributions leads to the following result for the ground state energy shift:

\[
\Delta E_0(n, m, L) = E_0(n, m, L) - n m_\pi - m m_K \\
= E_\pi(n, L) + E_K(m, L) + E_{\pi K}(n, m, L)
\]  

(5)

with \((m_{\pi K} = m_\pi m_K/(m_\pi + m_K))\)

\[
E_i(n, L) = \frac{4\pi \bar{a}_i}{m_i L^3} \left( \frac{n}{2} \right) \left[ 1 - \left( \frac{\bar{a}_i}{\pi L} \right) I + \left( \frac{\bar{a}_i}{\pi L} \right)^2 \left( I^2 + (2n - 5)J \right) \right. \\
- \left. \left( \frac{\bar{a}_i}{\pi L} \right)^3 \left( I^3 + (2n - 7)IJ + (5n^2 - 41n + 63)K \right) \right] + \left( \frac{n}{3} \right) \frac{\bar{g}_{3,i}(\mu)}{L^3} + O(L^{-7})
\]

(6)

\[
E_{\pi K}(n, m, L) = \frac{2\pi \bar{a}_{\pi K} mn}{m_{\pi K} L^3} \left[ 1 - \left( \frac{\bar{a}_{\pi K}}{\pi L} \right) I \right. \\
+ \left( \frac{\bar{a}_{\pi K}}{\pi L} \right)^2 \left[ I^2 + J \left[ -1 + \frac{\bar{a}_{\pi K}}{\bar{a}_{\pi K}}(n - 1) \left\{ \frac{1}{m_{\pi K}} + 2 \right\} + \bar{a}_{\pi K}(m - 1) \left\{ \frac{1}{m_{\pi K}} + \frac{2}{m_K} \right\} \right] \right] \\
+ \left( \frac{\bar{a}_{\pi K}}{\pi L} \right)^3 \left[ -I^3 + J K + \sum_{i=0}^{\infty} \sum_{p=\pi,K} \left( f_{ij}^{\pi K} J + f_{ij}^{\pi K} K \right) \left( \frac{\bar{g}_{p}}{\bar{a}_{\pi K}} \right) \right] \right] \\
+ \frac{nm(n - 1)\bar{g}_{3,\pi K}(L)}{2L^6} + \frac{nm(m - 1)\bar{g}_{3,\pi K K}(L)}{2L^6} + O(L^{-7})
\]

(7)

with the scattering lengths given in terms of the parameters \( \bar{a}_i = \frac{M_{\chi_i}}{4\pi} \) and the effective ranges \( r_i \) by

\[
a^{j=2}_\pi = \bar{a}_\pi - \frac{2\pi a^{j=2}_\pi r^{j=2}_\pi}{L^3} \\
a^{j=1}_K = \bar{a}_K - \frac{2\pi a^{j=1}_K r^{j=1}_K}{L^3} \\
a^{j=3/2}_\pi = \bar{a}_\pi K - \frac{2\pi a^{j=3/2}_K r^{j=3/2}_\pi}{L^3}.
\]  

(8)

The four volume dependent (but renormalization scale independent) quantities from the three-body interactions are
defined by \((y = m_\pi/m_K)\):

\[
\bar{\eta}_{3,i}(L) = \eta_{3,i}(\mu) + \frac{64\pi a_4^4}{m_i} (3\sqrt{3} - 4\pi) \log(\mu L) - \frac{96a_4^4}{\pi^2 m_i} (2q[1, 1] + r[1, 1])
\]

\[
\bar{\eta}_{3,\pi\pi K}(L, y) = \eta_{3,\pi\pi K}(\mu, y) - \frac{4a_{\pi K}^4}{\pi^2 m_{\pi K}} \sum_{i=0}^{2} \sum_{p=\pi, K} \sum_{N=1}^{N_1} \left( \frac{a_p}{a_{\pi K}} \right)^i f_i^{N,p} N
\]

\[
\bar{\eta}_{3,\pi K K}(L, y) = \eta_{3,\pi K K}(\mu, y) - \frac{4a_{\pi K}^4}{\pi^2 m_{\pi K}} \sum_{i=0}^{2} \sum_{p=\pi, K} \sum_{N=1}^{N_2} \left( \frac{a_p}{a_{\pi K}} \right)^i f_i^{N,p} N
\]  

(9)

and

\[
N_1 = \left\{ \hat{Q}(1, y), \hat{Q}(y, 1), \hat{R}(y, 1), \hat{R}(1/y, 1/y) \right\}
\]

\[
N_2 = \left\{ \hat{Q}(1, 1/y), \hat{Q}(y, y), \hat{R}(y, y), \hat{R}(1, 1/y) \right\}
\]  

(10)

The functions \(\hat{Q}, \hat{R}, q,\) and \(r\) are defined in the appendix along with the coefficients \(f_i\). The finite parts of \(\hat{Q}(a, b)\) and \(\hat{R}(a, b)\) are scheme dependent quantities where changes in the value will be absorbed by \(\eta_3(\mu)\). The numerical values for the MS scheme are given in the appendix. However, the \(\bar{\eta}_3\) are not scheme dependent and this is the quantity that will be determined during a lattice calculation. Furthermore, note that the three-body interactions in the \(\pi\pi K\) and \(\pi K K\) cases depend on the \(\pi K\) mass ratio. One can immediately see that this is necessary if one takes the limiting case where either the pion or the kaon become infinitely heavy, where the heavy particle decouples from the theory and hence all cross species interactions must go to zero. Finally, in the limit of \(n \to 0\) or \(m \to 0\) this result simplifies to the previously determined \(n\)-boson case while the limit \(m \to n\) with \(m_K \to m_\pi\) and all interactions set to be equal simplifies to the \(2n\)-boson case. The result given agrees with previously calculated results in the single species limit \([7, 8, 16]\).

In this work we have calculated the ground state energy shift of the mixed species system of \(n\) pions and \(m\) kaons. Using this result a rigorous connection can be made between Euclidean-space lattice QCD calculations of mixed pion-kaon systems and Minkowski-space multibody interaction strengths. In Ref. \([3]\) the isospin and strangeness chemical potentials are analyzed with lattice QCD at values relevant to dense nuclear matter for purely kaon condensates. With the results presented in this paper such an analysis can be done for mixed pion-kaon systems, allowing lattice calculations of the \(\pi K\) scattering length and three-body interaction strengths \(\pi\pi K\) and \(\pi K K\). Such extractions are necessary for examinations of the nuclear equation of state for values of the isospin and strangeness chemical potentials where it is not energetically favorable to form either pure pion or pure kaon condensates, but where possibly mixed systems of pions and kaons exist.

Acknowledgments

B.S. and J.W. would like to thank Martin Savage and William Detmold for their time and many useful discussions.
APPENDIX

The coefficients $f_i$ are given by (where $y = m_\pi/m_K$ and $m_\pi K = m_\pi m_K/(m_\pi + m_K)$):

\begin{align}
    f_0^{I,\pi} &= \frac{1}{2} (1 + m + n) \\
    f_1^{I,\pi} &= 2 \left( 1 + \frac{2m_\pi K}{m_\pi} \right) (1 - n) \\
    f_2^{I,\pi} &= \frac{m_\pi}{m_\pi K} \frac{m_\pi - m_K}{m_\pi m_K} (1 - n) \\
    f_i^{I,\pi} &= f_i^{I,\pi} (n \leftrightarrow m, m_\pi \leftrightarrow m_K) \\

    f_0^{K,\pi} &= \frac{1}{2} \left( 4n + 4m - 3mn - 6 + \frac{m_\pi^2 + m_K^2}{m_\pi m_K} (m + n - mn - 1) \right) \\
    f_1^{K,\pi} &= -8(n - 1)(m - 1) \left( 1 + \frac{m_\pi K}{m_\pi m_K} \right) \\
    f_1^{K,\pi} &= \frac{2m_\pi K (m_\pi^2 + 9m_\pi^2 + 4m_\pi m_K)}{m_\pi m_K^2} (n - 1) \\
    f_2^{K,\pi} &= \frac{m_\pi^2}{m_\pi m_K^2} (1 - n) \left[ m_K^3 (13n - 45) - m_\pi^3 + m_\pi m_K^2 (14n - 39) + 5m_\pi^2 m_K (n - 3) \right] \\
    f_i^{K,\pi} &= f_i^{K,\pi} (n \leftrightarrow m, m_\pi \leftrightarrow m_K) \\

    f_0^{\hat{Q}(1,y),\pi} &= f_0^{\hat{R}(1,y),\pi} = \frac{m_\pi}{m_\pi K} \\
    f_1^{\hat{Q}(1,y),\pi} &= f_1^{\hat{R}(1,y),\pi} = \frac{2m_\pi}{m_\pi K} \\
    f_2^{\hat{Q}(1,y),\pi} &= f_2^{\hat{R}(1,y),\pi} = \frac{m_\pi}{m_\pi K} \\
    f_1^{\hat{Q}(1,1/y),\pi} &= f_1^{\hat{R}(1,1/y),\pi} = f_1^{\hat{Q}(1,y),\pi} (m_\pi \leftrightarrow m_K) = f_i^{\hat{Q}(1,y),\pi} = f_i^{\hat{R}(1,1/y),\pi} = 0 \\
    f_i^{\hat{Q}(1,y),\pi} &= f_i^{\hat{R}(1,y),\pi} = f_i^{\hat{Q}(1,1/y),\pi} = f_i^{\hat{R}(1,1/y),\pi} = 0 \\

    f_0^{\hat{Q}(y,1),\pi} &= f_0^{\hat{Q}(y,1),\pi} = f_1^{\hat{Q}(y,1),\pi} = f_i^{\hat{Q}(y,1),\pi} = 0 \\
    f_1^{\hat{Q}(y,1),\pi} &= 8 \\
    f_2^{\hat{Q}(y,1),\pi} &= 8 \\
    f_i^{\hat{Q}(y,1),\pi} &= \frac{m_\pi}{m_K} f_i^{\hat{Q}(y,1),\pi} \\

    f_0^{\hat{R}(y,1),\pi} &= f_1^{\hat{R}(y,1),\pi} = f_2^{\hat{R}(y,1),\pi} = f_0^{\hat{R}(y,1),\pi} = f_1^{\hat{R}(y,1),\pi} = f_i^{\hat{R}(y,1),\pi} = 0 \\
    f_i^{\hat{R}(y,1),\pi} &= f_i^{\hat{R}(y,1),\pi} = f_i^{\hat{R}(y,1),\pi} = f_i^{\hat{R}(1/y,1/y),\pi} = 0 \\
    f_i^{\hat{R}(1/y,1/y),\pi} &= f_i^{\hat{R}(y,1),\pi} (m_\pi \leftrightarrow m_K) \\

    f_0^{\hat{R}(y),\pi} &= f_1^{\hat{R}(y),\pi} = f_2^{\hat{R}(y),\pi} = f_0^{\hat{R}(y),\pi} = f_1^{\hat{R}(y),\pi} = f_i^{\hat{R}(y),\pi} = 0 \\
    f_2^{\hat{R}(y),\pi} &= \frac{8m_\pi m_\pi K}{m_K^2} \\
    f_i^{\hat{R}(1/y,1/y),\pi} &= f_i^{\hat{R}(y),\pi} (m_\pi \leftrightarrow m_K) \\

\end{align}
The integer sums that contribute to $O(L^{-6})$ include:

$$\mathcal{I} = \lim_{\Lambda_j \to \infty} \left( \sum_{\vec{n} \neq 0} \frac{1}{|\vec{n}|^2} - 4\pi \Lambda_j \right) = -8.9136329 \quad (A-22)$$

$$\mathcal{J} = \sum_{\vec{n} \neq 0} \frac{1}{|\vec{n}|^4} = 16.532316 \quad (A-23)$$

$$\mathcal{K} = \sum_{\vec{n} \neq 0} \frac{1}{|\vec{n}|^6} = 8.4019240 \quad (A-24)$$

$$Q(a,b) = \sum_{\vec{n} \neq 0} \sum_{\vec{m} \neq 0} \frac{1}{|\vec{n}|^2 |\vec{m}|^2} \frac{1}{|\vec{n}|^2 + a|\vec{n}|^2 + b(\vec{n} + \vec{m})^2} \quad (A-25)$$

$$R(a,b) = \sum_{\vec{n} \neq 0} \left[ \sum_{\vec{m} \neq 0} \frac{1}{|\vec{n}|^4 |\vec{m}|^2} \frac{1}{|\vec{n}|^2 + a|\vec{n}|^2 + b(\vec{n} + \vec{m})^2} - \frac{1}{a+b} \int d^d n \frac{1}{|\vec{n}|^2} \right] \quad (A-26)$$

where the $Q$ and $R$ sums need to be regulated (defined) using dimensional regulation and in the $R$ sum the subtracted integral removes a nested divergence that would otherwise affect the one-loop scattering amplitude. One can calculate

$$Q(a,b) = Q_{\Sigma}(a,b) + Q_{\Lambda}(a,b) - \frac{8\pi^3}{b} \sin^{-1} \left( \frac{b}{\sqrt{(a+b)(1+b)}} \right) \frac{1}{\epsilon} \quad (A-27)$$

$$R(a,b) = R_{\Sigma}(a,b) + R_{\Lambda}(a,b) - \frac{8\pi^3 \sqrt{a+b+ab}}{(a+b)^2} \frac{1}{\epsilon} \quad (A-28)$$

where the cutoff dependencies of the sums $Q_{\Sigma}$ and $R_{\Sigma}$ are canceled by the cutoff dependence of the $Q_{\Lambda}$ and $R_{\Lambda}$ integrals. The dimensionally regularized integrals in turn counter the remaining portions of the cutoff integrals, leaving one with the functions $\hat{Q}$ and $\hat{R}$ found in eqn. (7). These are given by

$$\hat{Q}(a,b,\mu) = q(a,b) + \frac{8\pi^3}{b} \sin^{-1} \left( \frac{b}{\sqrt{(a+b)(1+b)}} \right) \log(\mu L) \quad (A-29)$$

$$\hat{R}(a,b,\mu) = r(a,b) - \frac{8\pi^3 \sqrt{a+b+ab}}{(a+b)^2} \log(\mu L) \quad (A-30)$$

where in Tables I and II the numerical portions of $q(a,b)$ and $r(a,b)$ are explicitly evaluated for values of the arguments $a$ and $b$ relevant to the calculation.

**TABLE I: Numerical evaluation of the function $q(a,b)$.**

| $y = \frac{a+b}{\mu L}$ | $q(1,y)$ | $q(1.1/y)$ | $q(y,1)$ | $q(y,y)$ |
|-------------------------|----------|------------|----------|----------|
| 0.2                     | -167.96763 | -34.22227 | -94.42134 | -113.71723 |
| 0.2827                  | -157.34173 | -45.55930 | -95.75651 | -127.15818 |
| 0.4                     | -144.45515 | -59.06500 | -97.36366 | -129.53719 |
| 0.6                     | -126.85940 | -77.19563 | -99.49508 | -121.84065 |
| 0.8                     | -113.15457 | -91.07490 | -101.03837 | -111.68392 |
| 1                       | -102.15598 | -102.15598 | -102.15598 | -102.15598 |
TABLE II: Numerical evaluation of the function \( r(a,b) \).

|   | \( r(y,1) \) | \( r(1,1/y) \) | \( r(1/y,1/y) \) | \( r(y,y) \) |
|---|----------|----------|----------------|-----------|
| 0.2 | 49.695247 | -0.358353 | -0.141373 | 517.146558 |
| 0.2827 | 44.958613 | 0.993223 | 0.391640 | 265.289351 |
| 0.4 | 38.951624 | 3.803394 | 1.738871 | 133.442877 |
| 0.6 | 30.564875 | 9.279473 | 5.616888 | 58.127093 |
| 0.8 | 24.132625 | 14.530094 | 11.449288 | 31.463511 |
| 1 | 19.186903 | 19.186903 | 19.186903 | 19.186903 |

[1] S. R. Beane et al., Phys. Rev. Lett. 100, 082004 (2008), 0710.1827.
[2] W. Detmold et al. (2008), 0803.2728.
[3] W. Detmold, K. Orginos, M. J. Savage, and A. Walker-Loud (2008), 0807.1856.
[4] W. Detmold (2008), 0810.1079.
[5] M. Luscher, Commun. Math. Phys. 105, 153 (1986).
[6] M. Luscher, Nucl. Phys. B354, 531 (1991).
[7] W. Detmold and M. J. Savage, Phys. Rev. D77, 057502 (2008), 0801.0763.
[8] S. R. Beane, W. Detmold, and M. J. Savage (2007), arXiv:0707.1670 [hep-lat].
[9] K. Huang and C. N. Yang, Phys. Rev. 105, 767 (1957).
[10] T. T. Wu, Phys. Rev. 115, 1390 (1959).
[11] N. M. Hugenholtz and D. Pines, Phys. Rev. 116, 489 (1959).
[12] K. Sawada, Phys. Rev. 116, 1344 (1959).
[13] D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. B424, 390 (1998), nucl-th/9801034.
[14] U. van Kolck, Nucl. Phys. A645, 273 (1999), nucl-th/9808007.
[15] J.-W. Chen, G. Rupak, and M. J. Savage, Nucl. Phys. A653, 386 (1999), nucl-th/9902056.
[16] S. Tan (2007), 0709.2530.