On integrable deformations of the spherical top.

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The motion on the sphere $S^2$ in potential $V = (x_1 x_2 x_3)^{-2/3}$ is considered. The Lax representation and the linearisation procedure for this two-dimensional integrable system are discussed.
1 Description of the model

The system under consideration is a special case of the following mechanical system in the nine-dimensional space \( \mathbb{R}^9 \)

\[
F^*(t) \frac{d^2 F(t)}{dt^2} + | \det F(t) |^{1-\gamma} G = 0. \tag{1.1}
\]

Here \( F(t) \) and \( G \) are 3x3 matrices, \( F^* \) means transpose matrix and \( \gamma \) is an polytropic index. The components \( F_{jk} \) of the matrix \( F \) are coordinates on the configuration space \( \mathbb{R}^9 \). These equations of motion have been studied many authors, see [12, 8, 4, 4].

According to [12], we shall consider symmetric constant matrices \( G = G^* \) only. In this case, by using canonical transformation of variables \( F' = UFU^* \), we can reduce the constant matrix \( G \) to the diagonal matrix with the following diagonal elements \( \pm 1 \) or 0. Moreover, with physical point of view we can put \( G = I \) without loss of generality [12].

The Newton equations (1.1) arise in solution of the hydrodynamical equations representing dynamics of a cloud of compressible gas expanding freely in an otherwise empty space. This model has a rich history related with the names of Dirichlet, Dedekind and Riemann. For an extensive discussion of the model we refer to [4], a book which should be taken as a general reference on the subject.

At \( G = G^* \) the invariance of the problem under rotation and under internal motion of the gas leads to conservation of the angular momentum and operator of vorticity

\[
J = F(t) F^*(t) - F^*(t) F(t), \quad K = F^*(t) F(t) - F^*(t) F(t).
\]

Each of \( J, K \) is an antisymmetric matrix with 3 independent components. Thus, equations (1.1) possess an enlarged symmetry group \( SO(4) \cong SO(3) \times SO(3) \). There is also a discrete symmetry, which allows the vorticity and the angular momentum to be interchanged. This discrete symmetry is identical with the duality principle of Dedekind [8].

For the perfect monatomic gas, at \( \gamma = 5/3 \) the system of equations (1.1) possesses one more integral of motion [2]

\[
r^2 = tr \left( F^*(t) F(t) \right), \quad \text{at} \quad \gamma = \frac{5}{3}. \tag{1.2}
\]

The qualitative behavior of solutions at the different values of the adiabatic index \( \gamma \) and some partial cases of motion are discussed in [3].

It is easy to separate the diagonal and non-diagonal components of equations (1.1) [12, 8, 4]. The non-diagonal components give six kinematical equations involving only the inertial properties of the gas-cloud but not the pressure force. The diagonal components give three dynamical equations determining the rate of expansion of the gas-cloud under the influence of the pressure force.

Below at \( \gamma = 5/3 \) we shall consider a free expansion of an ellipsoidal gas cloud with the fixed orientation, having zero angular momentum and zero vorticity \( K = J = 0 \). In this case matrix \( F(t) \) is diagonal for all \( t \)

\[
F(t) = diag( x_1, x_2, x_3 ) (t)
\]

and the constant matrix \( G = I \) is equal to unit. The corresponding three equations of motion are given by

\[
x_1 \ddot{x}_1 = x_2 \ddot{x}_2 = x_3 \ddot{x}_3 = \frac{const}{(x_1 x_2 x_3)^{2/3}}, \quad \text{at} \quad \gamma = \frac{5}{3}. \tag{1.3}
\]
Additional integral of motion (1.2) is equal to the radius of sphere \( r^2 = \sum x_k^2 \). By using this integral our system on \( \mathbb{R}^3 \) may be reduced to the system on the sphere \( S^2 \). At \( \gamma = 5/3 \) the third independent integral of motion was founded in [9].

The reduced system has the configuration space diffeomorphic to the Euclidean motion group \( E(3) = SO(3) \times \mathbb{R}^3 \). It allows to identify the phase space of this system on \( T^*S^2 \) with the cotangent bundle \( T^*E(3) \). The kinetic energy is a left-invariant Riemannian metric on \( E(3) \). It is determined by some quadratic form on the dual space \( e^*(3) \) [3].

By using the Killing form the dual space \( e^*(3) \) may be identified with algebra \( e(3) = so(3) \oplus \mathbb{R}^3 \), the semi-direct sum of \( so(3) \) and the abelian space \( \mathbb{R}^3 \). Let two vectors \( J \in so(3) \simeq \mathbb{R}^3 \) and \( x \in \mathbb{R}^3 \) be coordinates in the dual space \( e^*(3) \) equipped with a natural Lie-Poisson brackets

\[
\{ J_i , J_j \} = \varepsilon_{ijk} J_k , \quad \{ J_i , x_j \} = \varepsilon_{ijk} x_k , \quad \{ x_i , x_j \} = 0 , \quad i,j,k = 1,2,3 .
\] (1.4)

Here \( \varepsilon_{ijk} \) is the standard totally skew-symmetric tensor. The generic coadjoint orbits of \( E(3) \) in \( e^*(3) \) are four dimensional symplectic leaves specified by the two Casimir elements

\[
C_1 = (x,x) = x_i x_i ; \quad C_2 = (J,x) = J_i x_i .
\] (1.5)

Here \((x,y)\) means inner product in \( \mathbb{R}^3 \). Thus, the dual space \( e^*(3) \) decomposes into the coadjoint orbits

\[
O_{c_1,c_2} = \left\{ \{ J,x \} \in \mathbb{R}^6 : C_1 = c_1 , C_2 = c_2 \right\}
\] (1.6)

which are invariant with respect to the usual Euler-Poisson equations in \( e^*(3) \) [3].

Let us introduce a complex analog of the Lie algebra \( e(3) \), as a semi-direct sum of \( so(3) \otimes \mathbb{C} \) and the complex space \( \mathbb{C}^3 \). This algebra \( e(3,\mathbb{C}) = so(3,\mathbb{C}) \oplus \mathbb{C}^3 \) is equipped with the same Lie-Poisson brackets (1.4) and the Casimir operators (1.3). In contrast with the usual \( e(3) \) algebra, it allows us to consider non-trivial representations at zero value \( c_1 = 0 \) of the first Casimir operator \( C_1 \).

The condition \( c_1 = 0 \) has no obvious physical or geometric meaning. Of course, we can not consider real sphere of zero radius, but with mathematical point of view \( c_1 \) is an arbitrary value of the Casimir element. Note, that ”non-physical” representations of the algebra \( sl(2) \) with the zero spin \( s = 0 \) are useful in physics [14] as well.

**Proposition 1** At zero value \( c_1 = 0 \) of the first Casimir operator \( C_1 \) (1.3) the following transformation in \( so^*(3,\mathbb{C}) \subset e^*(3,\mathbb{C}) \)

\[
J \rightarrow \tilde{J} = J + \frac{ia}{(x_1 x_2 x_3)^{1/3}} x , \quad a \in \mathbb{C}
\] (1.7)

is an outer automorphism of the representation of \( e(3,\mathbb{C}) \).

By using embedding \( e(3) \subset e(3,\mathbb{C}) \) let us consider known integrable tops on this complex algebra \( e^*(3,\mathbb{C}) \). Applying transformation (1.7) one can get integrable deformations of these top on the one-parameter set of orbits \( O_1 \) \((c_1 = 0,c_2 = const)\). Sometimes outer automorphism (1.7) allows us to get much more.
As an example, let us consider spherical top with the standard Hamilton function

$$H = (\tilde{J}, \tilde{J}) = \tilde{J}_1^2 + \tilde{J}_2^2 + \tilde{J}_3^2 \quad (1.8)$$

and with the non-standard second integral of motion

$$K = \tilde{J}_1 \tilde{J}_2 \tilde{J}_3 \quad (1.9)$$

defined on the subalgebra $so(3)$ only. Of course, substituting vector $\tilde{J}$ (1.7) one gets integrable deformation of this symmetric top at $c_1 = 0$. However, we can prove the following

**Proposition 2** Outer automorphism (1.4) maps Hamiltonian (1.8) of the spherical top into the following Hamiltonian

$$H = \sum_{k=1}^{3} J_k^2 + 2ia \frac{c_2}{(x_1 x_2 x_3)^{1/3}} - a^2 \frac{c_1}{(x_1 x_2 x_3)^{2/3}}. \quad (1.10)$$

The proposed deformation of the spherical top is completely integrable on the both one-parameter sets of orbits

$$O_1 : \ (c_1 = 0, \ c_2 = \text{const}) \quad \text{and} \quad O_2 : \ (c_1 = \text{const}, \ c_2 = 0).$$

Moreover, to consider the second orbits we can return to the usual real Lie algebra $e^*(3)$.

At $c_1 = 0$ the second integral is the image of known integral $K$ (1.9)

$$K = J_1 J_2 J_3 - a^2 \left( \frac{J_1}{x_1} + \frac{J_2}{x_2} + \frac{J_3}{x_3} \right) (x_1 x_2 x_3)^{1/3}$$

$$+ \frac{2ia}{(x_1 x_2 x_3)^{1/3}} (J_1 J_2 x_3 + J_1 x_2 J_3 + x_1 J_2 J_3) - ia^3. \quad (1.11)$$

At $c_2 = 0$ the Hamilton function (1.11) is in involution with the following second integral of motion

$$K = J_1 J_2 J_3 + a^2 \left( \frac{J_1}{y_1} + \frac{J_2}{y_2} + \frac{J_3}{y_3} \right) (x_1 x_2 x_3)^{1/3}. \quad (1.12)$$

In contrast with (1.11), here we removed imaginary terms and changed the sign before the rest potential term. We do not know origin of such additional transformation as yet.

In the natural variables

$$y = \frac{x}{(x_1 x_2 x_3)^{1/3}}, \quad y_1 y_2 y_3 = 1$$

transformation (1.7) becomes a shift $\tilde{J} = J + i a y$ and the integrals of motion (1.10) and (1.12) are given by

$$H = J_1^2 + J_2^2 + J_3^2 - a^2 (y_1^2 + y_2^2 + y_3^2), \quad (1.13)$$

$$K = J_1 J_2 J_3 + a^2 \left( \frac{J_1}{y_1} + \frac{J_2}{y_2} + \frac{J_3}{y_3} \right).$$
The Euler-Poisson equations on $e^*(3)$ generated by (1.10) are given by
\[
\frac{d}{dt} J = 2a^2 \left( y, y \right) y \times y^{-1}, \quad y^{-1} = (y_1^{-1}, y_2^{-1}, y_3^{-1}),
\]
(1.14)
\[
\frac{d}{dt} y = -\frac{2}{3} \left( y, y \right) J \times y^{-1}, \quad \left( y, y \right) = y_1^2 + y_2^2 + y_3^2,
\]
where $x \times y$ means standard vector product in $\mathbb{R}^3$. Thus, we rewrite initial very symmetric equations of motion (1.3) defined on the configuration space $\mathbb{R}^3$ as the Euler-Poisson equations (1.14) defined on the phase space $e^*(3)$.

2 Lax representation

The main purpose of this note is to rewrite equations of motion (1.14) in the Lax form
\[
\frac{d}{dt} L = [L, M]. \tag{2.15}
\]

Let us briefly recall construction of the Lax pair for the Neumann system. The Neumann system is an integrable system on the sphere with quadratic potential (see (1.13)). Its phase space may be modelled on the dual space $e^*(3)$ at $c_2 = 0$. The corresponding Euler-Poisson equations are equal to
\[
\frac{d}{dt} J = x \times z, \quad \frac{d}{dt} x = -J \times x, \quad z = -\text{diag}(a_1, a_2, a_3) x. \tag{2.16}
\]
Here $a_j$ be arbitrary parameters.

The Neumann system possesses the necessary number of the quadratic integrals of motion. Nevertheless, the Lax pair can not be constructed in framework of the algebra $e(3) = so(3) \oplus \mathbb{R}^3$. Namely, for the Neumann system and some others system, we have to use the Cartan-type decomposition of the Lie algebra $gl(3, \mathbb{R}) = so(3) + Symm(3) \ [13]$.

Let us introduce antisymmetric matrix of angular momentum $J \in so(3)$ and symmetric matrix of coordinates $X \in Symm(3)$
\[
J \in so(3) \simeq \mathbb{R}^3 : \quad J_{ij} = \varepsilon_{ijk} J_k, \]
\[
X \in Symm(3) : \quad X_{ij} = x_i x_j.
\]
Then the Lax representation for the Neumann system are given by
\[
L = \text{diag}(a_1, a_2, a_3) \lambda + J + \lambda^{-1} X, \quad M = -\lambda^{-1} X. \tag{2.17}
\]
Let us present these Lax matrices explicitly
\[
L = \begin{pmatrix}
a_1 \lambda & 0 & 0 \\
0 & a_2 \lambda & 0 \\
0 & 0 & a_3 \lambda
\end{pmatrix}
+ \begin{pmatrix}
0 & -J_3 & J_2 \\
J_3 & 0 & -J_1 \\
-J_2 & J_1 & 0
\end{pmatrix}
+ \frac{1}{\lambda} \begin{pmatrix}
x_1^2 & x_1 x_2 & x_1 x_3 \\
x_1 x_2 & x_2^2 & 2 x_2 x_3 \\
x_1 x_3 & 2 x_2 x_3 & x_3^2
\end{pmatrix},
\]
Now let us turn to the deformation of the completely symmetric top \((1.10)\). The Lie algebras \(\mathbb{R}^3\) with vector product and \(so(3)\) with usual commutator may be identified by using the Lie algebras isomorphism

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \rightarrow X = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix} \in so(3)
\]

In \((1.7)\) the element \(J \in so(3)\) has been added with the vector \(x \in \mathbb{R}^3\). Thus, defining outer automorphism \((1.7)\), we implicitly used this property of the three dimensional Euclidean space. Below we shall use the same property to construct the Lax representation.

The main recipe is to rearrange items in decomposition \(gl(3, \mathbb{R}) = so(3) + Symm(3)\). Let us introduce symmetric matrix of angular momentum \(J \in Symm(3)\) and antisymmetric matrix of coordinates \(X \in so(3)\)

\[
J \in Symm(3): \quad J_{ij} = |\epsilon_{ijk}| J_k,
\]

\[
X \in so(3) \simeq \mathbb{R}^3: \quad X_{ij} = \epsilon_{ijk} y_k,
\]

where \( |\epsilon_{ijk}| \) means absolute value of \(\epsilon_{ijk}\).

**Proposition 3** At \(c_2 = 0\) the equations of motion \((1.14)\) on the sphere \(S^2\) generated by the Hamilton function \((1.13)\) can be written in the Lax form \((2.15)\) with the following matrices

\[
L = \lambda I + J + a X, \quad M_{ij} = \frac{2a}{3} |\epsilon_{ijk}| x^{-1}_k.
\]

More explicitly, the first matrix is

\[
L = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & J_3 & J_2 \\ J_3 & 0 & J_1 \\ J_2 & J_1 & 0 \end{pmatrix} + a \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix},
\]

and the second matrix is given by

\[
M = \frac{2a c_1^{1/2}}{3 \sqrt{(y, y)}} \begin{pmatrix} 0 & y_3^{-1} & y_2^{-1} \\ y_3 & 0 & y_1^{-1} \\ y_2 & y_1 & 0 \end{pmatrix}
\]

The spectral invariants of \(L(\lambda)\) give rise to both integrals of motion in involution \((1.13)\)

\[
\det L(\lambda) = \lambda^3 - H\lambda + 2K.
\]
The proposed Lax matrix $L(\lambda)$ has a trivial dependence on spectral parameter $\lambda$, which is similar to the Lax matrix for the Kowalewski top due by Perelomov [13]. Therefore, we can not construct a suitable spectral curve and can not directly integrate equations of motion. Recall, in [13] the Perelomov matrices have been embedded into the general Lax matrices with the spectral parameter. It forces us to consider the Lax representation (2.18) as a first attempt to build an adequate Lax pair. We believe the desire Lax pair explains the peculiar geometry and the origin of integrability of the considered motion on the sphere $S^2$.

3 Linearisation procedure

In closing this note we briefly discuss results obtained in [10] within the modern theory of linearisation of the two-dimensional integrable systems [1, 4, 16]. The proposed in [10] procedure of integration has an unusual form, which is closely related with the concrete system of equations. This construction is technically closed to the Chaplygin approach [7] to the Kirchhoff equations at $c_2 = 0$. On the other hand, modern theory of linearisation allows us to consider different integrable systems such as the Neumann problem, Henon-Hailes system, Toda lattice, Kowalewski top, Goryachev-Chaplygin top and many other [1, 4, 16].

However, in this common powerful method it is necessary to rewrite equations of motion at some suitable variables. These variables have to satisfy the special conditions [1, 4, 16]. As an example, to integrate the Toda lattice we have to introduce so-called Flaschka variables. For other system such variables may be introduced by using the Kowalewski-Painlevé analysis or the algebro-geometric tools. Nevertheless, if we might have introduced such variables, the Adler and van Moerbeke methods [1, 4, 16] enable us to integrate a given mechanical system.

The aim of this Section is to introduce an analog of the Flaschka variables for the deformations of the spherical top. At these variables we may directly apply the Adler and van Moerbeke methods to a given integrable system. These results will be published at the consequent publications.

The three body Toda lattice is the Hamiltonian system defined as

$$H = \frac{1}{2} \sum_{j=1}^{3} p_j^2 + e^{q_1 - q_2} + e^{q_2 - q_3} + e^{q_3 - q_1}.$$  

Here $\{p_j, q_j\}_{j=1}^3$ be pairs of the canonical physical variables. According to [1, 4, 16], in the Flaschka variables

$$z_1 = e^{q_1 - q_2}, \quad z_2 = e^{q_2 - q_3}, \quad z_3 = e^{q_3 - q_1}, \quad z_1 z_2 z_3 = 1$$

$$z_4 = -p_1, \quad z_5 = -p_2, \quad z_6 = -p_3$$

the corresponding equation of motion have the following form

$$\frac{d}{dt} z_1 = z_1 (z_5 - z_4), \quad \frac{d}{dt} z_4 = z_1 - z_3,$$

$$\frac{d}{dt} z_2 = z_2 (z_6 - z_5), \quad \frac{d}{dt} z_5 = z_2 - z_1,$$

$$\frac{d}{dt} z_3 = z_3 (z_4 - z_6), \quad \frac{d}{dt} z_6 = z_3 - z_2.$$  

(3.19)
The Toda flow has the following four constants of motion

\[
\begin{align*}
Z_1 &= z_1 z_2 z_3 = 1, \\
Z_2 &= z_4 + z_5 + z_6 = d_1 = 0, \\
Z_3 &= \frac{1}{2} (z_4^2 + z_5^2 + z_6^2) + z_1 + z_2 + z_3 = a_1, \\
Z_4 &= z_4 z_5 z_6 - z_1 z_6 - z_2 z_4 - z_3 z_5 = b_1 .
\end{align*}
\] (3.20)

At \( d_1 \neq 0 \) the variable \( Q = q_1 + q_2 + q_3 \) can not be restored from variables \( \{z\}_j \). Really, we have to add to variables \( \{z\}_j \) some other variables with the trivial dynamics \([4]\). Below we shall introduce analog of the Flaschka variables for the integrable deformations of the spherical top.

There is a discrete permutation group acting on the vectors \( q \) and \( p \) simultaneously

\[
q \mapsto \mathcal{D} q, \quad p \mapsto \mathcal{D} p, \quad \mathcal{D} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \quad \mathcal{D}^3 = 1.
\] (3.21)

According this point symmetry, the invariant manifold defined by (3.20) has the third order automorphism given by

\[
(z_1, z_2, z_3, z_4, z_5, z_6) \mapsto (z_2, z_3, z_1, z_5, z_6, z_4).
\] (3.22)

This automorphism simplifies the Adler and van Moerbeke analysis \([1, 4]\), which applied to this system, gives linearisation to the Toda flow.

Recall, the Kowalewski-Painlevé analysis enables us to integrate equations of motion for the Goryachev-Chaplygin top \([4, 16]\) as well. It is another integrable system on \( e^*_3 \) at \( c_2 = 0 \). In this case we have to introduce some seven-dimensional system with the five constants of motion. Then this seven-dimensional system may be reduced to the Toda system \([4]\). The similar relations between the Toda flow and the integrable system on \( e^*_3 \) is discussed in \([14]\). Now we want to compare the Toda flow with another integrable system on \( e^*_3 \).

Let us turn to the deformation of the completely symmetric top \([1.10]\). In \([3]\), the following transformation of the independent time variable was proposed

\[
t \to u : \quad \frac{d}{du} = \frac{4}{3} (y, y) \frac{d}{dt},
\] (3.23)

because of the "weak" Kowalewski-Painlevé criterion. In the new time variable the initial Euler-Poisson equations \((1.14)\) are

\[
\frac{d}{du} J = \frac{a^2}{2} y \times y^{-1}, \quad \frac{d}{du} y = -\frac{1}{2} J \times y^{-1}.
\] (3.24)

Namely these equations were integrated in hyperelliptic quadratures in \([10]\).

If we want to compare integrable system on the sphere \( S^2 \) with the Toda system, note that equations \((3.19)\) are invariant by

\[
z_j \mapsto \frac{1}{z_j}, \quad z_{j+3} \mapsto -z_{j+3}, \quad j = 1, 2, 3.
\] (3.25)
The first equation in (3.24) has the similar property by
\[ y_j \mapsto \frac{1}{y_j}, \quad J_j \mapsto -J_j, \quad j = 1, 2, 3. \]

Using this observations we introduce analog of the Flaschka variable
\[ s_1 = y_1^{-2}, \quad s_2 = y_3^{-2}, \quad s_3 = y_2^{-2}, \quad s_1 s_2 s_3 = 1, \]
\[ s_6 = y_1 J_1, \quad s_4 = y_3 J_3, \quad s_5 = y_2 J_2. \]
satisfying the following equations
\[
\begin{align*}
\frac{d}{du} s_1 &= s_1 (s_5 - s_4), \\
\frac{d}{du} s_2 &= s_2 (s_6 - s_5), \\
\frac{d}{du} s_3 &= s_3 (s_4 - s_6), \\
\frac{d}{du} s_4 &= \frac{a^2}{2} (s_1 - s_3) + \frac{s_4}{2} (s_5 - s_6), \\
\frac{d}{du} s_5 &= \frac{a^2}{2} (s_2 - s_1) + \frac{s_5}{2} (s_6 - s_5), \\
\frac{d}{du} s_6 &= \frac{a^2}{2} (s_3 - s_2) + \frac{s_6}{2} (s_4 - s_5).
\end{align*}
\]

The first column of equations is completely coincided with the corresponding Toda equations (3.19). The second columns differ on the at most quadratic polynomials. Thus, for these equations we can directly apply the linearisation procedure due by Adler and van Moerbeke [1, 4]. The freedom in definition of the variables \( s_j \) may be used to make the Laurent solutions a bit easier.

The constants of motion for the flow (3.26) are given by
\[
\begin{align*}
S_1 &= s_1 s_2 s_3 = 1, \\
S_2 &= s_4 + s_5 + s_6 = c_2 = 0, \\
S_3 &= (s_4^2 s_2 + s_5^2 s_3 + s_6^2 s_1) - a^2 (s_1 s_2 + s_1 s_3 + s_2 s_3) = a_1, \\
S_4 &= s_4 s_5 s_6 + a^2 (s_1 s_6 + s_2 s_4 + s_3 s_5) = b_1.
\end{align*}
\]
The three constants \( S_1, \ S_2 \) and \( S_4 \) coincides with the corresponding Toda constants. The Hamiltonian \( S_3 \) is a qubic polynomials now.

There is the permutation invariance similar to the Toda lattice (3.21). Thus, the invariant manifold defined by (3.27) possesses the third order automorphism (3.22), which simplifies the linearisation procedure.

It may of course be quite difficult to find variables similar to \( \{z_j\} \) or \( \{s_j\} \) associated with a given mechanical system. These variables have to satisfy some special conditions [1, 16]. For instance, the corresponding equations of motion have to include at most second order polynomials only, see (3.19,3.26). However, if we can introduce such variables, the Adler and van Moerbeke methods [1, 4, 16] enable us to integrate a given mechanical system.

Thus, for a motion on sphere (1.13) one has to embed the affine invariant surface defined by (3.27) into the projective space, whose closure is a principally polarised Abelian surface. It enables one to define the system to linearising variables. Then we have to prove that the vector field corresponding to \( S_4 (3.27) \) gives the highest flow with respect to the same hyper-elliptic curve of genus two. It will complete the linearisation of the integrable deformation of the spherical top. Of course, this general machinery leads to the particular results obtained in [10].
4 Conclusion

In the algebro-geometric approach due to Adler and van Moerbeke [1, 4, 16], the algebraic curve may be constructed without any Lax pair representation. For the considered motion on sphere (1.10), by substituting the Laurent solutions in the invariants (3.27) one gets the hyperelliptic curve [10]. Starting with this curve and the linearising variables [10] the $2 \times 2$ Lax pair may be obtained (see [16] for a review).

In this note we tried to construct the Lax pair in framework of the group-theoretical approach to integrable system [13, 5]. Applying the method of finite-band integration to the adequate Lax matrix, we hope to get solutions, which may be more simpler than the original formulae [10], as for the Kowalewski top [3].

Further properties of the integrable deformation of the completely symmetric top, like action-angles variables, Poisson structures of the seven-dimensional system and separation of variables are under study. The results and the more detailed geometric description will be published elsewhere.

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References

[1] M. Adler and P. van Moerbeke. Algebraic completely integrable systems: a systematic approach. Academic Press, 1993

[2] S.I. Anisimov and I.I. Lysikov. J.Appl. Math. Mech., v.34, p.926, 1970.

[3] V.I. Arnold. Mathematical methods of classical mechanics. Springer, 1989.

[4] C. Bechlivanidis and P. van Moerbeke. Comm.Math.Phys, v.110, p.317, 1987.

[5] A. I. Bobenko, A.G. Reyman, and M.A. Semenov-Tian-Shansky. Commun.Math.Phys., v.122, p.321, 1989.

[6] O.I. Bogoyavlensky. Methods of qualitative theory of dynamical systems in astrophysics and gas dynamics. M.: Nauka, 1980.

[7] S.A. Chaplygin. Collected works, v.1, Gostekhizdat., Moscow, 1948.

[8] F.J. Dyson. J. Math. Mech., v.18, p.91, 1968.

[9] B. Gaffet. J.Fluid.Mech., v.325, p.113, 1996.

[10] B. Gaffet. J.Phys. A, v.31, p.1581 and p.8341, 1998.

[11] L.D. Faddeev and G.P. Korchemsky. Phys. Lett. B, v.342, p.311, 1995.

[12] L.V. Ovsyannikov Dokl. Akad. Nauk SSSR, v.111, pp.47, 1956.
[13] A.G. Reyman and M.A. Semenov-Tian Shansky. In V.I. Arnold and S.P. Novikov, editors, *Dynamical systems VII*, volume EMS 16, Berlin, Springer (1993).

[14] V.B. Kuznetsov and A.V. Tsiganov. *J.Phys.*, v.22, p.L73, 1989.

[15] A.V. Tsiganov. *Phys. Lett. A*, v.251, p.354, 1999.

[16] P. Vanhaecke. *Math.Z.*, v.211, p.265, 1992.