Gravitational Radiation Damping and the Three-Body Problem

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March 24, 2022

Abstract

A model of three-body motion is developed which includes the effects of gravitational radiation reaction. The radiation reaction due to the emission of gravitational waves is the only post-Newtonian effect that is included here. For simplicity, all of the motion is taken to be planar. Two of the masses are viewed as a binary system and the third mass, whose motion will be a fixed orbit around the center-of-mass of the binary system, is viewed as a perturbation. This model aims to describe the motion of a relativistic binary pulsar that is perturbed by a third mass. Numerical integration of this simplified model reveals that given the right initial conditions and parameters one can see resonances. These \((m, n)\) resonances are defined by the resonance condition, \(m\omega = 2n\Omega\), where \(m\) and \(n\) are relatively prime integers and \(\omega\) and \(\Omega\) are the angular frequencies of the binary orbit and third mass orbit (around the center-of-mass of the binary), respectively. The resonance condition consequently fixes a value for the semimajor axis of the binary orbit for the duration of the resonance; therefore, the binary energy remains constant on the average while its angular momentum changes during the resonance.

Key words: celestial mechanics, relativity, gravitational waves

1 INTRODUCTION

In 1975, Hulse and Taylor found through the timing observations of binary pulsar PSR B1913+16 that the semimajor axis decayed at nearly the rate predicted by general relativity for the emission of gravitational radiation. This has served as significant indirect evidence of gravitational radiation reaction force as the

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result of energy balance due to the fact that gravitational waves leave the system. It has also shown the importance of the pulsar in probing strong gravity. Because of the universal nature of gravity, a correct analysis of astrophysical systems must involve the gravitational interaction between all of the masses in the environment plus other non-Newtonian effects. One would expect that over time, evidence of neighboring masses would appear in the timing observations of binary pulsars.

This study puts forth a model of three bodies, two of which constitute a relativistic binary pulsar and the third is a perturbing mass. This investigation extends the work of Chicone, Mashhoon, and Retzloff (1996a, 1996b, 1997a, 1997b), which originally dealt with a binary system perturbed by normally incident gravitational waves. As in these previous models, the model derived here will be searched for resonances. In the following analysis, an \((m, n)\) resonance will occur when the relation \(m\omega = 2n\Omega\) is satisfied, where \(m\) and \(n\) are relatively prime integers, \(\omega\) is the frequency of the binary system, and \(\Omega\) is the frequency of the orbit of the third mass around the center-of-mass of the binary. Such resonances are physically noteworthy because they correspond to events where the collapse of the semimajor axis of the binary halts. That is, on average it stays fixed.

The N-body problem in general relativity theory is rather complicated; therefore, to capture the main effects of three bodies plus gravitational radiation reaction one can start with a classical three-body system with gravitational radiation damping as a linear perturbation. That is, in this work all post-Newtonian (relativistic) corrections will be neglected except for gravitational radiation reaction that will be treated in the quadrupole approximation. For the sake of simplicity, the internal structure of the masses will be neglected so that one is in effect dealing with three Newtonian point masses.

Two astrophysical systems that may demonstrate measurable behavior based on this model are the relativistic binary pulsar near a massive third mass and a planet in an orbit around a binary pulsar. While this study will focus on the former case the latter proves to be a timely point of interest. Astronomers have in the last six years found more than 50 stars which include at least one planet. The present number of stars that are currently being observed for planets is 1000 (Cameron, 2001). Pulsars have also proven to be detectable sources that show evidence of extrasolar planets. Finding a pulsar planet, where the pulsar is also part of a binary system is a possibility. Microlensing technique in its extrasolar planet search has turned up a possible planet with two suns (Bennet et al., 1999). Another possibility would be the future detection of a binary pulsar in a globular cluster. A binary pulsar system that is situated in an approximately spherical cluster would be effectively the same as a binary attracted to a third mass equal to the mass of the matter inside the binary’s orbit.

The timing of radio pulses from pulsars has opened up a rich and exciting branch of astronomy and astrophysics. Analysis of pulsar signals over time offers insight into the gravitational environment such as the possible presence of a companion mass. Hence, pulsar astronomy offers a fruitful method of detection for the astrophysical processes discussed in this paper. On the frontiers of
astronomy, interferometric gravitational wave detectors based on Earth such as LIGO, VIRGO, GEO, AND TAMA aim to detect gravitational waves from known inspiraling binary systems. The shortening orbital separation that would arise as the result of energy loss via gravitational wave emission would result in an increasing frequency in the orbit. The gravitational wave signal would have twice this frequency (Blanchet, 2002). Thus, the predicted effect of a binary system’s capture into resonance would leave its imprint in terms of the detected gravitational waves. Namely, the interval when the distance between the members of the binary stays fixed on average would give a gravitational wave signal that on the average would be of near constant frequency. Furthermore, the next generation of planned space-based detectors like LISA will broaden the scope of gravitational wave detection.

The initial approach to this problem in Section 2 involves setting up the equations of motion for the three masses with the influence of gravitational radiation reaction included. The radiation reaction force is understood to be very small, so it is viewed as a perturbation. The model is then simplified in Section 3 to make it more amenable to analysis. The results of the numerical integration of the simplified model are presented in Section 4. Next, section 5 expands upon the numerical results with a discussion about resonance. Section 6 contains concluding remarks.

The nonlinear disposition of the equations of motion for the relative motion of the binary system requires analysis that is suited for nonlinear behavior. Chicone, Mashhoon, and Retzloff (1997b) developed such an averaging method, that elucidates details about the structure of the orbit of a nonlinear system, especially when a resonance occurs. The system developed and studied here is suited for further analysis, in particular the application of the method mentioned above.

## 2 EQUATIONS OF MOTION

The general equations of motion for the three-body problem with gravitational radiation damping are:

\[
m_1 \frac{d^2 x_1^i}{dt^2} + \frac{G m_1 m_2 (x_1^i - x_2^i)}{|x_1 - x_2|^3} = -\frac{G m_1 m_3 (x_1^i - x_3^i)}{|x_1 - x_3|^3} - \frac{2G}{15c^5} m_1 \frac{d^5 D_{ij}}{dt^5} x_1^j, \tag{1}
\]

\[
m_2 \frac{d^2 x_2^i}{dt^2} + \frac{G m_1 m_2 (x_2^i - x_1^i)}{|x_1 - x_2|^3} = -\frac{G m_2 m_3 (x_2^i - x_3^i)}{|x_2 - x_3|^3} - \frac{2G}{15c^5} m_2 \frac{d^5 D_{ij}}{dt^5} x_2^j, \tag{2}
\]

\[
m_3 \frac{d^2 x_3^i}{dt^2} + \frac{G m_3 m_1 (x_3^i - x_1^i)}{|x_3 - x_1|^3} = -\frac{G m_2 m_3 (x_3^i - x_2^i)}{|x_3 - x_2|^3} - \frac{2G}{15c^5} m_3 \frac{d^5 D_{ij}}{dt^5} x_3^j, \tag{3}
\]

where the quadrupole moment tensor for the three-body system is:

\[
D_{ij} = m_1 (3x_1^i x_1^j - \delta_{ij} x_1^2) + m_2 (3x_2^i x_2^j - \delta_{ij} x_2^2) + m_3 (3x_3^i x_3^j - \delta_{ij} x_3^2). \tag{4}
\]
Each equation includes the familiar Newtonian gravitational interactions between the masses plus the Newtonian gravitational radiation reaction force. This damping force results from the emission of gravitational waves by the system of masses. There is a loss of energy in the system that results from energy carried away by the waves. According to general relativity, the energy of gravitational waves leaving the system is given in the quadrupole approximation by

\[
\frac{dE}{dt} = \frac{G}{45c^5} \frac{d^3D_{ij}}{dt^3} \frac{d^3D^{ij}}{dt^3},
\]

where \( E \) is the amount of gravitational radiation energy emitted by the system (Landau and Lifshitz, 1971).

Energy conservation requires that this loss be reflected in the equations of motion; therefore, one can also arrive at this rate of energy loss by way of mechanics. If one multiplies each equation of motion by its time derivative of position and adds the three equations, one arrives at the following expression:

\[
\frac{dE}{dt} = -\frac{G}{45c^5} \frac{d^3D_{ij}}{dt^3} \frac{d^3D^{ij}}{dt^3},
\]

where \( E = E_N + E_S \). Here \( E_N \) is the total Newtonian energy of the three-body system. That is,

\[
E_N = \sum_{i=1}^{3} \frac{1}{2} m_i v_i^2 - \frac{1}{2} \sum_{i \neq j} G m_i m_j |\mathbf{x}_i - \mathbf{x}_j|,
\]

and \( E_S \) is the gravitational “Schott” energy,

\[
E_S = \frac{G}{45c^5} \left( \frac{d^4D_{ij}}{dt^4} \frac{dD^{ij}}{dt} - \frac{d^3D_{ij}}{dt^3} \frac{d^2D^{ij}}{dt^2} \right),
\]

that is analogous to the Schott term in electrodynamics (Schott, 1912).

For quasi-periodic motions, the change in the Schott energy over a period turns out to be of higher order and can be neglected. Hence there is agreement between the two different methods as to what the loss of energy should be since the right-hand side of (5) is equal and opposite to the right-hand side of (6). Comparison of the general relativistic and classical methods of calculating rate of change of angular momentum yields agreement in a similar way. Thus in the quadrupole approximation for the emission of gravitational waves, the three-body system loses energy and angular momentum to the radiation field. Moreover, adding equations (1)-(3) and making the following definition

\[
Z = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + m_3\mathbf{x}_3}{m_1 + m_2 + m_3}
\]

for the center-of-mass of the system we find

\[
\frac{d^2Z^i}{dt^2} + \frac{2G}{15c^5} \frac{d^3D_{ij}}{dt^3} Z^j = 0.
\]
Let us note that \( Z = 0 \) is a solution of this equation, so that if the center-of-mass of the whole system is initially at rest at the origin of coordinates, it will remain so in the quadrupole approximation under consideration here. This is consistent with the fact that gravitational waves do not carry away linear momentum in the quadrupole approximation.

Solutions to the proposed three-body problem, equations (1)-(3), by way of numerical integration offer some insight into the behavior of the system that has no known solution in closed form. To integrate such a higher-order system numerically, it is not sufficient to specify the positions and velocities of the three masses at some initial instant of time. In fact, it is known that such systems suffer from the existence of runaway modes that inevitably will lead to divergent results. It is therefore necessary to replace the system (1)-(3) by second-order equations of motion via iterative reduction as explained in a recent paper by Chicone, Kopeikin, Mashhoon, and Retzloff (2001). This procedure involves the repeated substitution of the equations of motion (1)-(3) in the evaluation of \( \frac{d^5 D_{ij}}{dt^5} \) and the subsequent reduction of the resulting system to one of second-order equations for the motion of the system (by dropping higher-order terms). The resulting system would be appropriate for numerical integration; however, it would have a rather complicated form. Therefore, we resort to certain simplifications in this first treatment of the classical three-body problem that takes gravitational radiation reaction into account.

The main physical model which will be under investigation in this study involves two masses, which represent the binary, whose center-of-mass orbits a massive third body. Since the mass of the third body is assumed to be much larger than the mass of the binary, one can take \( x_3 \) to be the origin of the coordinate system. This is because the center-of-mass of the entire system is very close to the location of the third mass. To simplify matters, we set \( x_3 = 0 \); moreover, the distance of the binary from the third mass is taken to be much larger than the semimajor axis of the binary system.

This model could approximate a relativistic binary pulsar system that orbits the center-of-mass of a globular cluster. The outer shell of stars would remain 'unseen' by the binary since a globular cluster is nearly spherical, and the interior stars would serve as a third mass located at the center-of-mass of the cluster.

### 3 SIMPLIFIED MODEL

The following equations of motion result from the approximations made in the model:

\[
\frac{d^2 x_1^i}{dt^2} = -\frac{Gm_3m_1x_1^i}{|x_1|^3} + \frac{Gm_1m_2(x_2^i - x_1^i)}{|x_2 - x_1|^3} - \frac{2m_1G}{15c^5} \frac{d x_j}{dt} \frac{d^5 D_{ij}}{dt^5} x_1^i, \quad (11)
\]

\[
\frac{d^2 x_2^i}{dt^2} = -\frac{Gm_3m_2x_2^i}{|x_2|^3} - \frac{Gm_1m_2(x_2^i - x_1^i)}{|x_2 - x_1|^3} - \frac{2m_2G}{15c^5} \frac{d x_j}{dt} \frac{d^5 D_{ij}}{dt^5} x_2^i, \quad (12)
\]

Here \( m_1 \) and \( m_2 \) are masses of the members of the binary, \( m_3 \) is the mass of the perturbing third body, and \( D_{ij} = m_1(3x_1^ix_1^j - \delta_{ij}x_1^j) + m_2(3x_2^ix_2^j - \delta_{ij}x_2^j) \)
is the quadrupole moment tensor.

In order to elucidate the relevant dynamics of the system, it is helpful to change the coordinates of the system. The relative and center-of-mass coordinates of the binary can be defined as:

\[ r^i = x_2^i - x_1^i, \quad X^i = \frac{m_1 x_1^i + m_2 x_2^i}{M}, \]  

where \( M = m_1 + m_2 \). The equation of relative motion is then

\[ \frac{d^2 r^i}{dt^2} = -\frac{GM r^i}{|r|^3} - \frac{Gm_3}{|x_2|^3} \left( \frac{x_1^i}{|x_1|^3} - \frac{x_2^i}{|x_2|^3} \right) - \frac{2G}{15c^5} \frac{d^5 D_{ij} r^j}{dt^5}, \]  

while the equation of motion of the center-of-mass of the binary is

\[ \frac{d^2 X^i}{dt^2} = -\frac{Gm_3}{M} \left( \frac{m_1 x_1^i}{|x_1|^3} + \frac{m_2 x_2^i}{|x_2|^3} \right) - \frac{2G}{15c^5} \frac{d^5 D_{ij} X^j}{dt^5}. \]  

One can show that the quadrupole moment tensor can be written in terms of the relative and center-of-mass variables as

\[ D_{ij} = \mu \left( 3 \frac{r^i r^j}{|r|^3} - \delta_{ij} \right) + M \left( 3 \frac{X^i X^j}{|X|^3} - \delta_{ij} \right), \]

where \( \mu = m_1 m_2 / M \) is the reduced mass of the binary system, \( r = |r| \), and \( X = |X| \). Replacing \( x_1 \) and \( x_2 \) in (14)-(15) with \( x_1^i = X^i - \gamma_2 r^i \) and \( x_2^i = X^i + \gamma_1 r^i \) and expanding terms in powers of \( r/X \), one obtains

\[ \frac{x_1^i}{|x_1|^3} = \frac{X^i - \gamma_2 r^i}{|X - \gamma_2 r|^3} = \frac{X^i}{X^3} - \gamma_2 \left( r^i - \frac{3(X \cdot r) X^i}{X^2} \right) + \ldots, \]  

\[ \frac{x_2^i}{|x_2|^3} = \frac{X^i + \gamma_1 r^i}{|X + \gamma_1 r|^3} = \frac{X^i}{X^3} + \gamma_1 \left( r^i - \frac{3(X \cdot r) X^i}{X^2} \right) + \ldots, \]

where \( \gamma_1 = m_1 / M \) and \( \gamma_2 = m_2 / M \).

The following equations result when terms to linear order in \( r/X \) are kept in the above expansions:

\[ \frac{d^2 r^i}{dt^2} = -\frac{GM}{r^3} r^i - \frac{Gm_3}{X^3} K_{ij} r^j - \frac{2G}{15c^5} \frac{d^5 D_{ij} r^j}{dt^5}, \]  

\[ \frac{d^2 X^i}{dt^2} = -\frac{Gm_3}{X^3} X^i - \frac{2G}{15c^5} \frac{d^5 D_{ij} X^j}{dt^5}. \]

where

\[ K_{ij} = \left( \delta_{ij} - \frac{3X^i X^j}{X^2} \right) \]

is the reduced tidal matrix.

Thus the problem reduces to the coupled system (18)-(19) for the motion of the binary system. To simplify the equations further, it will be assumed in what follows that the center-of-mass motion is circular in the absence of radiation reaction. Substituting such a solution for (19) in the equation of relative
motion (18), one finds that to lowest order in the perturbations the center-of-
mass motion can be taken to be simply circular in (18). This circular motion
contributes to the radiation reaction term in (18); however, this contribution
can also be neglected if
\[
\left( \frac{X}{r} \right)^{11/2} \left( \frac{m_1}{m_3} \right) \left( \frac{m_2}{m_3} \right) \left( \frac{m_1 + m_2}{m_3} \right)^{1/2} \gg 1.
\]
(21)
Assuming this inequality, the equation of relative motion reduces to
\[
\frac{d^2 \hat{r}_i}{dt^2} = -\frac{GM}{r^3} \hat{r}_i - \frac{Gm_3}{X^3} \hat{K}_{ij} \hat{r}_j - \frac{2G\mu}{15c^5} \frac{d^5 \hat{D}_{ij}}{dt^5} \hat{r}_j,
\]
(22)
where \( \hat{D}_{ij} = 3r^i r^j - \delta_{ij} r^2 \). The rest of this paper is devoted to the study of this
equation that describes the relative motion of two bodies under mutual graviti-
tional attraction, tidal interaction with a large third mass, and gravitational
radiation damping. These are the combined effects of interest here. This model
can, in effect, represent a second scenario as well, i.e. a binary system orbited
by a distant small mass \( m_3 \) (see Appendix A).
It is convenient to transform equation (22) into dimensionless form. To
this end, let all lengths and temporal variables be measured in units of \( R_0 \)
and \( T_0 \), respectively. Moreover, one assumes that these units are related to
the unperturbed motion of the binary such that \( gMT_0^2 = R_0^3 \). Thus, letting
\( r^i \rightarrow R_0 \hat{r}_i, \ X^i \rightarrow R_0 \hat{X}_i, \) and \( t \rightarrow T_0 \hat{t} \) in equation (22), it reduces to the form
\[
\frac{d^2 \hat{r}_i}{d\hat{t}^2} = -\frac{\hat{r}_i}{\hat{r}_3} - \Omega^2 \hat{K}_{ij} \hat{r}_j - \delta \frac{d^5 \hat{D}_{ij}}{d\hat{t}^5} \hat{r}_j,
\]
(23)
where the hats will be dropped in what follows for the sake of simplicity. Here
\[
\Omega^2 = \frac{Gm_3T_0^2}{X^3} = \frac{m_3}{M} \left( \frac{R_0}{X} \right)^3,
\]
(24)
\[
\delta = \frac{2G\mu R_0^2}{15c^5 T_0^3},
\]
(25)
and the center-of-mass motion is taken to be a circle in the \((x, y)\)-plane such
that \((K_{ij})\) has the form
\[
(K_{ij}) = \begin{bmatrix}
-\frac{3}{2} - \frac{3}{2} \cos 2\Omega t & -\frac{3}{2} \sin 2\Omega t & 0 \\
-\frac{3}{2} \sin 2\Omega t & -\frac{3}{2} + \frac{3}{2} \cos 2\Omega t & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
(26)
If the length unit \( R_0 \) were chosen to be the semimajor axis of the binary,
the corresponding initial period would be \( 2\pi T_0 \). A relativistic binary pulsar
such as PSR B1913+16 has an orbital period of nearly eight hours (Lyne and
Graham-Smith, 1998). The data regarding this Hulse-Taylor binary pulsar when
applied to the dimensionless formula (25) give \( \delta \approx 10^{-16} \). This value of \( \delta \) is also
approximately valid for the relativistic binary pulsar PSR B1534+12 (Stairs et al., 1998). Moreover, the tidal perturbation term is also assumed to be very small, i.e. $\Omega^2 << 1$. It proves useful to replace $\Omega^2$ by a free parameter $\epsilon << 1$ in what follows. Therefore, equation (22) will be replaced by

$$\frac{d^2 r^i}{dt^2} = -\frac{r^i}{r^3} - \epsilon K_{ij} r^j - \delta \frac{d^2 \hat{D}_{ij}}{dt^2} r^j. \quad (27)$$

The mathematical results derived from (27) will then apply to the physical situation at hand once $\epsilon = \Omega^2$.

Finally, the order of equation (27) needs to be reduced in order to avoid a singular perturbation problem. To this end, in taking derivatives of $\hat{D}_{ij}$ substitutions for $\ddot{r}^i$ are made with the equation of motion (27), and terms to desired order are kept. In this case, we reduce (27) to a second-order equation that is linear in $\epsilon$ and $\delta$. See Appendix B for a complete derivation. What results is

$$\frac{d^2 r^i}{dt^2} = -\frac{r^i}{r^3} - \epsilon K_{ij} r^j - \delta R^i. \quad (28)$$

Here $R^i$ is the reduced radiation reaction term in Cartesian coordinates and is given by

$$R^i = \left( -\frac{24}{r} - 180v^2 + 72v^2 \right) \frac{v^i}{r^3} + \left( -\frac{8}{r} + 300v^2 - 216v^2 \right) \frac{rr^i}{r^4}. \quad (29)$$

where as before $r = |r|$, $v^i = \dot{r}^i$ and $v = |v|$.

4 NUMERICAL RESULTS

What results is a nonlinear two-dimensional second-order ODE that describes the relative motion between the two members of the binary. Because of the nonlinearity, a numerical solution is sought at this stage to elucidate the behavior of the system. The numerical integrator used in this study was furnished by MATHEMATICA. To produce a trajectory of the system, one needs values for the parameters $\epsilon, \delta$, and $\Omega$ plus the necessary initial conditions. For the purposes of generating numerical solutions for inspection, it becomes advantageous to generalize the system as a dynamical system at the expense of adherence to a strictly physical model. This relaxation of the parameter space allows one to more freely search for resonances in the generalized system.

Because near-Keplerian motion is under investigation, a natural analytical convention to use is the osculating ellipse. The otherwise constant elements of the ellipse for unperturbed motion become time-dependent osculating elements with the perturbations. Each instantaneous position in the motion is a point of an ellipse that is characterized by the osculating elements at that time. If at that time the perturbation were supressed, the motion would follow the path of the osculating ellipse uniquely described by the instantaneous position and velocity. One can, of course, transform between coordinate systems. The scheme for
numerical work, for example, was done in polar coordinates. When one takes equation (28), applies the transformation to polar coordinates

\[ x = r \sin \theta, \quad y = r \cos \theta, \quad (30) \]

and converts the second-order ODEs into equivalent first-order ODEs one obtains the following system of equations:

\[ \dot{r} = P_r, \quad (31) \]
\[ \dot{\theta} = \frac{P_\theta}{r^2}, \quad (32) \]
\[ \dot{P}_r = \frac{P_\theta^2}{r^3} - \frac{1}{r^2} + \frac{1}{2} cr^1 + 3 \cos (\Omega't - 2\theta) + \delta \frac{P_r}{r^3} (\frac{32}{r} + 24P_r^2 + 144P_\theta^2), \quad (33) \]
\[ \dot{P}_\theta = \frac{3}{2} cr^2 \sin (\Omega't - 2\theta) + \delta \frac{P_r}{r^3} (\frac{24}{r} + 108P_r^2 - 72\frac{P_\theta^2}{r^2}). \quad (34) \]

Here \( \Omega' = 2\Omega \); furthermore, this polar scheme with the usual variables \( r \) and \( \theta \) gives rise to the variables \( P_r \) and \( P_\theta \) defined above in (31) and (32) in this first order form. \( P_\theta \) represents the angular momentum of the two-body system per reduced mass.

One can generate a series of numerical integrations in search of resonances of various orders. Once the parameters that represent the amplitudes of the tidal perturbation due to the presence of the third mass and gravitational radiation damping are chosen, one can numerically generate an orbit starting from initial conditions. In anticipation of future analytical work to be done in Delaunay variables (Wardell, to be published), results of the numerical integration are graphed in terms of Delaunay variables. In figure 1, the orbit of the Delaunay action \( L \) is plotted versus time. The variable \( L \) is directly related to the semimajor axis by the relation \( L = a^{1/2} \), where \( a \) is the semimajor axis of the relative orbit. The left side of the orbit makes a sharp descent which accords with the expectation of semimajor axis decay due to the influence of gravitational radiation reaction. However, the orbit commences to undergo oscillations about an average value. A resonance occurs here. The amplitude of the oscillations increase until the orbit falls out of resonance. This is a nonlinear feature of the orbit. The orbit eventually falls out of the resonance and gives way to a descent whose slope is more gradual than that of the initial sharp descent due to the fact that the orbital eccentricity changes during resonance as a consequence of the variation of orbital angular momentum. The phenomena associated with this resonance are discussed in the next section.

5 RESONANCE

Numerical experiments show that a \((1:1)\) resonance exists for the system with the appropriate initial conditions. Checks on the numerical integration have been performed to rule out the possibility that the effects seen are numerical artifacts.
The gravitational environment of the binary comes from two contributors: the tidal influence of the third mass and the emission of gravitational waves. These combined effects enter as the perturbations in the dynamical system. The interplay of tidal energy input from the orbit around the third mass and the energy loss caused by the emission of gravitational waves comes to a place of balance during a resonance. A resonance is characterized by the resonance condition $m\omega = 2n\Omega$, where $\omega$ is the angular frequency of the perturbed Kepler orbit and $\Omega$ is the angular frequency of the binary’s orbit around the third mass. Because the radiation damping is a dissipative effect, decay of the semimajor axis might be expected as the dominant perturbative effect. In fact, the example given by the Hulse-Taylor binary pulsar where this decay is predominantly observed shows this effect quite well. However, in the presence of a third mass resonance, as seen in this system, fixes the average net flux of energy and gives rise to an interesting nonlinear dynamical system whose perturbations of tidal interaction and gravitational radiation damping offset each other on the average. Changes in the orbital angular momentum, though, can accompany this fixed average energy that follows from the resonance condition. The binary’s orbit around the third mass can result in an increase in its internal orbital angular momentum due to the tidal torque of the external mass. On the other hand, gravitational waves emitted from the binary carry with them angular momentum that decreases the orbital angular momentum in the binary.

5.1 CAPTURE INTO RESONANCE

Approaching from the left in Figure 1, the semimajor axis of the orbit steadily shrinks until it reaches the resonance manifold. The resonance manifold is defined by $L = L_*$, where $L_* = \omega_*^{-1/3}$ and the resonance condition for a $(m : n)$ resonance fixes $\omega_*$ such that $\omega_* = 2(n/m)\Omega$. This in turn fixes $L_*$. Resonance capture is noteworthy because the orbit of the system passes through many resonances unimpeded. The integers $m$ and $n$ must be relatively prime, making $n/m$ a rational number; since the rational numbers are dense with respect to the real numbers, the orbit is always close to a resonance without being captured, except under special circumstances. Numerical experiments show that capture more readily occurs for low-lying resonances such as $(1 : 1)$ as illustrated in Figures 1-3. Moreover, higher order resonances give rise to more chaotic structures in their corresponding figures for $L$, $P_\theta$, and $e$ (Chicone, Mashhoon, and Retzloff, 1997b). Here $P_\theta$ is the specific orbital angular momentum of the osculating ellipse and $e$ is its eccentricity defined by $(1 - P_\theta^2/L^2)^{1/2}$.

5.2 PASSAGE THROUGH RESONANCE

When an orbit is captured into a resonance it enters the resonance manifold. That is, the average $L$ value is fixed at $L_*$. The orbit then oscillates about this $L_*$ value with increasing amplitude. Eventually, it falls out of the resonance. While in resonance, the other action variable $P_\theta$ can on average change significantly. This is shown in Figure 2, which indicates that through the resonance $L$
oscillates about an average value $L_*$, $P_0$ increases on average during this interval. Thus in the course of resonance, the osculating ellipse associated with the orbit undergoes an average change in eccentricity, as demonstrated in Figure 3.

5.3 EXIT FROM RESONANCE

The energy loss due to the emission of gravitational waves depends on the eccentricity as $(1-e^2)^{-7/2}$ (Landau and Lifshitz, 1971). Therefore, the decrease in this orbital eccentricity accounts for the apparent lower rate of energy loss once the orbit leaves the resonance as in Figure 1. The anti-damping seen through the resonance eventually disrupts the resonance condition and leads to the orbit falling out of resonance. Where the orbit falls out of resonance the eccentricity is $e \simeq 0.73$ as compared to $e \simeq 0.85$ when the orbit was first captured into resonance.

An approximate analytic description of the phenomenon associated with resonance is the subject of a future publication (Wardell, to be published).

6 CONCLUSION

The main thrust of the analysis in this paper has been to introduce and develop a model of a binary system that is under the influence of gravitational radiation reaction and perturbed by a third body. The classical case of three gravitating bodies is considered here when the damping force due to the emission of gravitational waves by the system is included in the Newtonian equations of motion. After the pertinent equations of motion were derived, numerical analysis was done to explore important dynamics in the system – namely, resonance between the orbits of the relative and third body motions. This would optimistically be an astronomically observable effect.

An analytical approach that would resolve details about this nonlinear system would add to the confidence of the numerical results. What remains, to augment the results of this work, is to pursue an analytical solution. The previous work of Chicone, Mashhoon, and Retzloff (1996,1997) outlines a novel averaging approach that looks into orbits near resonance. It was applied to the case of a binary system perturbed by the emission and absorption of gravitational waves. The model presented in this paper along with numerical indication of the existence of resonances leads naturally to this analytical approach for the model developed here (Wardell, to be published).

ACKNOWLEDGEMENTS

I would like to thank B. Mashhoon for his indispensable help and guidance in this project. I would also like to thank B. DeFacio for generously offering his computer facilities.
Figure 1: This is a (1:1) resonance. The parameters are \( \epsilon = 0.00003 \), \( \delta/\epsilon = 10^{-3} \), and \( \Omega = 0.14286 \). The initial conditions at \( t = 0 \) are \((r, \theta, P_r, P_\theta) = (1, 0.57, 0.75246, 1)\).
Figure 2: Plot of $P_\theta$ versus time for the (1:1) resonance.
Figure 3: Plot of eccentricity versus time for the (1:1) resonance.
APPENDIX A: THE SECOND SCENARIO

Imagine a relativistic binary system consisting of masses $m_1$ and $m_2$ and a distant third mass $m_3$ (with $m_3 << m_1$, and $m_3 << m_2$) in a nearly circular orbit about the binary system such that $r << |x_3|$. Starting from (1)-(4), the equation of relative motion is

$$\frac{d^2 r^i}{dt^2} + \frac{GM r^i}{r^3} = Gm_3 \left( \frac{x^3_3 - x^3_2}{|x_3 - x_2|^3} - \frac{x^1_3 - x^1_2}{|x_3 - x_1|^3} \right) - \frac{2G}{15c^5} \frac{d^5 D^i_{ij}}{dt^5} x^j, \quad (A1)$$

Writing $x^1_i = X^i - \gamma_2 r^i$ and $x^2_i = X^i + \gamma_1 r^i$ as before, and recalling that $X^i = -m_3 x^i_3/M$, we find

$$\frac{x^3_3 - x^3_2}{|x_3 - x_2|^3} = \frac{(1 + \frac{m_3}{M})x^3_3 - \gamma_1 r^i}{|1 + \frac{m_3}{M}|x_3 - \gamma_1 r|^3}, \quad (A2)$$

$$\frac{x^3_3 - x^1_3}{|x_3 - x_1|^3} = \frac{(1 + \frac{m_3}{M})x^3_3 + \gamma_2 r^i}{|1 + \frac{m_3}{M}|x_3 + \gamma_2 r|^3}. \quad (A3)$$

Using $m_3 << M$ and neglecting $m_3/M$ in comparison to unity meanwhile expanding (A2) and (A3) as in equations (18) and (19) based on the fact that $r << |x_3|$, one finds that to lowest order the result is

$$\frac{d^2 r^i}{dt^2} = -\frac{GM r^i}{r^3} - \frac{Gm_3}{|x_3|^3} K_{ij} r^j - \frac{2G}{15c^5} \frac{d^5 D^i_{ij}}{dt^5} x^j, \quad (A4)$$
where $D_{ij} \simeq \mu (3r^i r^j - \delta_{ij} r^2)$ once the condition

$$\frac{m_3}{\mu} \left( \frac{r}{|x_3|} \right)^{11/2} < 1$$

(A5)

is imposed. In equation (A4), $K_{ij}$ is given by

$$K_{ij} = \delta_{ij} - 3 x_i^3 x_j^3 |x_3|^2.$$  

(A6)

It is clear that this scenario, suitably interpreted, gives essentially the same equation of relative motion that is given in equation (27).

**APPENDIX B: ITERATIVE REDUCTION**

To obtain the desired approximation of the radiation reaction force, one makes successive differentiations of the quadrupole moment tensor and substitutes the Keplerian equation of motion whenever the second time derivative of position appears. The reduced radiation reaction is:

$$R^i = \frac{d^5 \tilde{D}_{ij}}{dt^5} r^j.$$  

(B1)

The scaled Kepler equation of motion is:

$$\frac{d^2 r^i}{dt^2} = -\frac{r^i}{r^3}.$$  

(B2)

For brevity, $\dot{r}$ will be represented by $v$. One starts with the quadrupole moment tensor

$$\tilde{D}_{ij} = 3 r^i r^j - \delta_{ij} r^2$$

(B3)

and proceeds to take time derivatives:

$$\frac{d\tilde{D}_{ij}}{dt} = 3 (r^i v^j + v^i r^j) - \delta_{ij} (2r \cdot v),$$

(B4)

$$\frac{d^2 \tilde{D}_{ij}}{dt^2} = 3 \left( \dot{r}^i \dot{v}^j + \dot{v}^i \dot{r}^j + 2 v^i \dot{v}^j \right) - 2 \delta_{ij} \left( v^2 + r \cdot \dot{v} \right).$$

(B5)

Where $\dot{v}^i$ occurs, one substitutes the unperturbed Kepler equation of motion to arrive at the following

$$\frac{d^2 \tilde{D}_{ij}}{dt^2} = 6 \left( -\frac{r^i r^j}{r^3} + v^i v^j \right) - 2 \delta_{ij} \left( v^2 - \frac{1}{r} \right).$$

(B6)

Subsequent differentiations and substitutions yield the following expressions:

$$\frac{d^3 \tilde{D}_{ij}}{dt^3} = -\frac{12}{r^3} \left( r^i r^j + r^i v^j \right) + 18 \frac{r^i r^j}{r^4} \dot{r} - \delta_{ij} \left( -2 \frac{\dot{r}}{r^2} \right),$$

(B7)
\[
\frac{d^4\tilde{D}_{ij}}{dt^4} = -\frac{24v^i v^j}{r^3} + \frac{54(v^i r^j + r^i v^j)}{r^4} \dot{r} + \left(-90\dot{r} + 18v^2 + \frac{6}{r}\right) \frac{r^i r^j}{r^5} \tag{B8}
\]

\[
-\delta_{ij} \left(\frac{6}{r^3} \dot{r}^2 - \frac{2v^2}{r^3} + \frac{2}{r^4}\right),
\]

\[
\frac{d^5\tilde{D}_{ij}}{dt^5} = \frac{180v^i v^j}{r^4} \ddot{r} + \left(-\frac{24}{r} - 360\dot{r}^2 + 72v^2\right) \left(\frac{v^i r^j + r^i v^j}{r^5}\right) \tag{B9}
\]

\[
+ (630\dot{r}^2 - 270v^2) \frac{r^i r^j}{r^6} \dddot{r} - \delta_{ij} \left(-\frac{30\dot{r}^3}{r^4} + \frac{18v^2 \ddot{r}}{r^4} - \frac{16\dot{r}}{r^5}\right).
\]

Finally, when one contracts expression (B9) with \(r^j\) to compute the desired reduced radiation reaction expression as prescribed by (B1), one finds equation (29).