Concerning pressure and entropy of shock-accelerated heliosheath electrons

Hans J. Fahr\textsuperscript{1}*, Robindro Dutta-Roy\textsuperscript{1}

\textsuperscript{1}Argelander-Institut für Astronomie der Universität Bonn, Auf dem Hügel 71, 53121 Bonn, Germany

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ABSTRACT
We study the behaviour of shocked wind-electrons leaving wind-driving stars after undergoing the outer wind termination shock. As an example, we describe the evolution of the keV-energetic electron distribution function downstream of the heliospheric termination shock. We start from a kinetic transport equation in the bulk frame of the heliosheath plasma flow taking into account shock-induced electron injection, convective changes, cooling processes, and whistler wave-induced energy diffusion. From this equation we proceed to an associated pressure moment of the electron distribution function arriving at a corresponding pressure transport equation which describes the evolution of the electron pressure in the bulk frame of the plasma along the plasma flow lines. We assume that the local distribution function, in view of the prevailing non-LTE conditions, is represented by a local kappa function with local kappa parameters that vary with the streamline coordinate $s$ downstream of the solar wind termination shock. We obtain the solution for the electron pressure as a function of the streamline coordinate $s$ from the pressure transport equation and demonstrate that, connected with this pressure, one obtains an expression for the entropy of the electron fluid which can also be derived as a streamline function. We show that the heliosheath electron fluid can essentially be characterized as an isobaric and isentropic flow. These results allow to generally conclude that astrotail plasma flows are characterized as such flows.

Key words: Shock waves – Solar wind – Magnetohydrodynamics (MHD) – Plasmas – Sun: heliosphere – Acceleration of particles

1 INTRODUCTION
It is well known that downstream of the solar wind termination shock energetic electrons in the range between 40 and 70 keV were observed by Voyager-2 (Krimigis 2015; Decker et al. 2015). These low- and middle-energetic electrons in these distant space plasma regions up to now play the role of unclassified and unpredicted particle species. Concerning theoretical studies, it is usually assumed at shocks like the solar wind termination shock that these light species, i.e. the electrons, at the transition from the upstream to the downstream side of the shock, strictly follow by velocity the Rankine-Hugoniot relations valid for the main momentum carrier, i.e. the ions. This means, it is generally assumed that the electrons react like the ions concerning their moment properties, i.e. attaining downstream of the shock densities, bulk velocities, and temperatures identical to those of the ions. In fact, electrons and ions at very localized shock structures cannot be taken as strongly coupled to each other, one species kinetically strongly bound to the other, bound to each other like electron and proton in the form of a neutral H-atom. Actually, the locally very strong, shock-induced electric fields lead to the phenomenon called "spontaneous demagnetization" (see Lembege et al. 2003), and due to this fact electrons in first order only react to the strong shock-induced electric fields essentially not recognizing Lorentz forces. This pinpoints the special role of electrons at shock passages which had been emphasized already in many papers of the recent past like e.g. by Chalov & Fahr (2013), Fahr et al. (2014), and Fahr et al. (2015). The background of this phenomenon has been clearly pointed out in a recent paper by Fahr & Verscharen (2016) showing that, as a consequence of a piece-wise "de-magnetization", electrons passing over the electric shock structure gain high overshoot velocities with energies of the order of several keV. In the following part of this paper, we start from these keV-energetic, shock-induced electrons and follow their kinetic fate in configuration and velocity space at their convection downstream from the termination shock.

\* E-mail: hfahr@astro.uni-bonn.de

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We first come back to the kinetic phase-space transport equation which we have already developed and used in a very similar application, namely to ions leaving the shock in Fahr et al. (2016). We analogously here assume that the locally prevailing electron distribution function $f_e$, shown to be initially given by a kappa function with parameters $\Theta = \Theta_0$ and $\kappa = \kappa_0$ (see e.g. Fahr & Verscharen 2016; Fahr et al. 2017), also further downstream of the termination shock can at least be expected as a non-equilibrium kappa-like function $f^\kappa_e(\kappa, \Theta, \Theta_0)$, characterized, however, by locally varying kappa parameters $\kappa = \kappa(s)$ and $\Theta = \Theta(s)$ varying with the streamline coordinate $s$ along the streamlines given by:

$$f^\kappa_e = f^\kappa_e(\kappa(s), \Theta(s), \nu)$$

with $n_e = \frac{n_0}{(\pi \kappa(s) \Theta^2(s))^3/2}$.

The function $\Gamma(s)$ is the well known mathematical Gamma function.

In fact, we are referring here to electrons that originate from upstream thermal solar wind electrons after they have undergone the differential acceleration due to the action of the shock-electric field. The basic theoretical description of that process has been given in our papers (Fahr et al. 2015; Fahr & Verscharen 2016). Here we show within a multifluid concept that electrons react to the shock-electric field in a very specific way leading to downstream superthermal, non-equilibrium electrons. The function $\Gamma(s)$ is the well known mathematical Gamma function.

The resulting distribution function is determined by the following kinetic transport equation

$$\frac{df^\kappa_e}{dt} = -U \frac{df^\kappa_e}{ds} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( v^2 D_{\nu \nu} \frac{\partial f^\kappa_e}{\partial \nu} \right)$$

where $U$ is the plasma bulk velocity. For more explanations of the explicit form of these terms the reader should also look into the paper by Fahr et al. (2016).

The processes taken into account in Eq. (2) are convective changes of the electron fluid moving with the bulk speed of the plasma flow, magnetic cooling, and energy diffusion by interaction with whistler wave turbulences. ‘Magnetic cooling’ we call the effect connected with the tendency of the electrons to keep their magnetic moment constant at the convection downstream. As we could show in many papers in the past (Fahr & Siewert 2010, 2013; Fahr 2007) this process operates on all time scales, even if other counteracting processes are operating. It does not mean that the magnetic moment in fact is conserved, but it means that the tendency to conserve the magnetic moment $(KT/mB)$ has to be accordingly respected in the transport equation.

Looking next only for stationary solutions of the problem (i.e. $\partial f/\partial t = 0$), multiplying Eq. (2) by $4\pi/mv^2$ and integrating over velocity space with $v^2 dv$ following the procedure practised by Fahr et al. (2016) (and futheron skipping the subindex "c", since only electrons are addressed in this paper), one obtains the following pressure-moment transport equation:

$$\frac{dp^\kappa_e}{ds} = 3 \pi m \int v^2 dv \left( \frac{\partial f^\kappa_e}{\partial \nu} \right)_{mag} + \frac{1}{v^2} \frac{\partial}{\partial v} \left( v^2 D_{\nu \nu} \frac{\partial f^\kappa_e}{\partial \nu} \right)$$

$$\left(3\right)$$

Eq. (3) contains two undeveloped integral terms which can be written in the form

$$T_1 = \int v^2 dv \frac{\partial f^\kappa_e}{\partial \nu}$$

$$\left(4\right)$$

and

$$T_2 = \int v^2 dv \frac{\partial}{\partial v} \left( v^2 D_{\nu \nu} \frac{\partial f^\kappa_e}{\partial \nu} \right)$$

$$\left(5\right)$$

The term $T_1$, describing magnetic cooling due to the conservation of the particle’s magnetic moment (see Fahr & Siewert 2013), has the following detailed representation:

$$T_1 = \int v^2 dv \left( \frac{\partial f^\kappa_e}{\partial \nu} \right)_{mag} = \int v^2 dv \left( \frac{\partial f^\kappa_e}{\partial \nu} \right)_{mag}$$

$$\left(6\right)$$

where the magnetic velocity drift $\nu_m$ due to magnetic moment conservation is given by (see Fahr 2007; Fahr & Fichtner 2011; Fahr & Siewert 2013)

$$\nu_m = U \frac{\partial \nabla}{\partial s} = \frac{2}{3} \int B \frac{\partial B}{\partial s}$$

$$\left(7\right)$$

with $B$ being the magnetic field magnitude.

Eq. (7) describes this magnetic moment conservation tendency by expressing the temporal change of the velocity perpendicular to the magnetic field due to the bulk motion in a direction into which the magnetic field magnitude $B$ changes. Hereby $\nu_m$ is the pitch-angle averaged velocity perpendicular to $B$ (see Fahr 2007). This leads to the following term $T_1$ in the pressure transport equation.

$$T_1 = \frac{2}{3} \int B \frac{\partial B}{\partial s} = \frac{4}{3} \int B \frac{\partial B}{\partial s}$$

$$\left(8\right)$$

where the streamline gradient of the magnetic field magnitude, i.e. $\partial B/\partial s$, can perhaps best be obtained from a paper by Suess & Nerney (1990) presenting analytic solutions for the frozen-in heliosheath magnetic field. $m$ is the electron mass.

Here in this calculation we use the assumption of a kinematically frozen-in magnetic field, in order to be able to use the available analytic solutions for $B = B(\vec{r})$. In fact the self-consistency in the field description is only relevant in the stagnation region of the heliosheath (close to the heliopause, where magnetic reconnection effects come into the game), but in the main region of the heliosheath which we are describing here the kinematical approach towards the frozen-in field $B$ given by Suess & Nerney (1990) is viable.
In Eq. (8), the local kappa pressure $P^\kappa(s)$ connected with the local parameters $\kappa(s)$ and $\Theta(s)$ is defined by

$$P^\kappa(s) = \frac{4\pi}{3} \int s f^\kappa(s) \, ds + \frac{1}{2} \frac{mn\Theta^2(s)}{\kappa(s) - 3/2}$$

(9)

Finally, the second term $T2$ has extensively been analysed in Fahr et al. (2014, 2016) and depends on the specific form of the velocity diffusion coefficient. For a velocity-
dependence of this diffusion coefficient in the form $D_{vv} = D_0 \cdot \nu^s$, it leads to the following expression:

$$T2 = 2D_0(3 + \alpha) \int \nu^{2s} f^s \, dv$$

(10)

The velocity diffusion process for electrons is based on electron interactions with selfconsistent whistler wave turbulence. The actual diffusion coefficient for this process is velocity-dependent and in our calculations further down from Eq. (10) we take it to depend on $\nu^s$ (i.e. $\alpha = 2$). The diffusion coefficient $D_0$ is a reference value for the velocity $\nu_0 = \sqrt{2} \text{keV}/mc$ and in our calculations is taken to be $D_0 = 10^{-9} \text{s}^{-1}$.

Adopting for the velocity-dependence of the diffusion coefficient (for the case of electrons) as a reasonable value $\alpha = 2$ (see Kallenbach et al. 2005), one is led to

$$T2 = 10D_0 \int \nu^{3s} \, dv = \frac{3}{4\pi m} 10D_0P^\kappa(s)$$

(11)

The moment transport equation derived above now attains the following explicit form:

$$\frac{dP^K}{ds} = \frac{1}{U} \left[ \frac{4\pi}{3} (T1 + T2) \right]$$

$$= \frac{1}{3} \frac{4\pi m}{3} \left[ \frac{1}{B} \frac{\partial B}{ds} + \frac{3}{4\pi m} P^\kappa(s) + \frac{3}{4\pi m} 10D_0 P^\kappa(s) \right]$$

(12)

$$= \frac{4}{3} \frac{\partial B}{ds} P^\kappa(s) + \frac{1}{U} 10D_0 P^\kappa(s)$$

$$= \frac{4}{3} \frac{\partial B}{ds} P^\kappa(s) + \frac{1}{U} 10D_0 P^\kappa(s)$$

From this transport equation (Eq. 12), one consequently derives the following more condensed form for the relevant pressure transport equation

$$\frac{dP^\kappa}{ds} = \frac{4}{3} \frac{\partial B}{ds} + \frac{10D_0}{U}$$

(13)

yielding $P^\kappa(s)$ as a function of the line element $s$, interestingly enough with knowledge of $\kappa(s)$ and $\Theta(s)$, in the following explicit form:

$$P^\kappa(s) = P^\kappa(s_0) \cdot \exp\left[ \int_{s_0}^{s} \left( \frac{4}{3} \frac{\partial B}{ds} + \frac{1}{U} 10D_0 \right) ds \right]$$

(14)

where $P^\kappa(s_0)$ denotes the initial pressure at the entrance coordinate $s_0$ into the heliosheath, the place where the plasma flow enters the heliosheath (i.e. at the termination shock with initial streamline coordinate $s = s_0$). This initial pressure $P^\kappa(s_0)$ near the nose (upwind direction) can be calculated in absolute values based on the following quantities taken from Fahr et al. (2017): $n_e(s_0) = 1.5 \cdot 10^{-3} \text{cm}^{-3}$, $m\Theta^2(s_0) = E_0 = 1 \text{keV}$, $s_0 = 7$, and allows to calculate the absolute electron pressure along heliosheath streamlines.

Astonishingly, it is evident from Eq. (14) that the actual pressure $P^\kappa(s)$ is influenced only by:

A: the change of the magnetic field magnitude along the streamline,

and by

B: the diffusivity parameter $d_{vv} = D_0 \int_{s_0}^{s} \frac{ds}{U}$ with $D_0 = 10^{-9} \text{s}^{-1}$.

As one can easily check, this latter factor in the upwind heliosheath evaluates to the order of $d_{vv} \leq 10^{-2}$ (see Chalov et al. 2007), meaning that by this expression pressure changes of about a factor of exp(1.0) are indicated. Also, the remaining first factor in the upper relation in front of the exponential function is given by the ratio of the field magnitudes at the streamline points $s$ and $s_0$, respectively. It is fairly evident that also this factor does not describe more essential pressure changes along the streamlines. Thus, it can be concluded that the heliosheath electron fluid behaves essentially as an "isobaric" medium. This means that the heliosheath streamlines (see Sylla & Fichtner 2015) essentially represent electron pressure isobars. Connected with earlier arguments given elsewhere (Fahr & Siewert 2015) which showed that due to the very low sonic Mach numbers the plasma behaves incompressible, this also means that $T(s) = P(s)/(\kappa(s)K)$ is fairly constant along streamlines and hence streamlines in the heliosheath characterize electron isentropals.

At a first glance, it is perhaps interesting to see that the resulting electron pressure $P^\kappa(s)$, as evident from Eq. (14), depends directly neither on $\kappa(s)$ nor on $\Theta(s)$ as could have been expected in principle from the fact that this pressure is locally defined on the basis of these two parameters through the expression (see Heerikhuizen et al. 2008; Scherer et al. 2018)

$$P^\kappa(s) = \frac{1}{2} mn[\Theta^2(s) \frac{\kappa(s)}{\kappa(s) - 3/2}]$$

(15)

with parameters $\kappa = \kappa(s)$ and $\Theta = \Theta(s)$ varying along the streamline.

However, it must first be kept in mind that any combination of parameters $\kappa$ and $\Theta$, solely fulfilling the following relation

$$\Theta^2(s) = \frac{\kappa(s)}{\kappa(s) - 3/2}$$

(16)

is a permitted solution and that in this respect $P^\kappa(s)$, as evident, depends directly neither on $\kappa$ nor on $\Theta$.

And second, it has to be kept in mind that the physically deeper-rooted explanation of the above mentioned independence of the pressure $P(s)$ on the local parameters $\kappa(s)$ and $\Theta(s)$ may, however, in truth rather be that the above pressure transport equation could have been derived without presupposing at all the pressure $P = P(s)$ as a kappa pressure $P^\kappa(s)$. One namely would arrive at an identical pressure transport equation (Eq. 14) without prescribing apriori anything about the underlying distribution function.

### 3 The Electron Entropy and a Thermodynamic Streamline Constant for Isentropic Flows

In the ongoing consideration we study the thermodynamical behaviour of the electron population being convected with the plasma bulk flow downstream from the termination shock. We start out from the well known thermodynamic relation connecting changes of the internal energy $de$
of a plasma volume \( dV \) and the pressure with changes of the entropy \( S \) in the well known classic thermodynamical form

\[
T \, dS = dP + \rho \, dV
\]

(17)

where \( T \) and \( P \) denote temperature and pressure of the electron fluid, \( dP \) is the internal energy of the volume \( dV \), and \( S = S_0 \) denotes the electron fluid entropy. Let us then first look here onto isentropic flows, i.e. those flows for which in the upper relation the change of the entropy \( S \) with the streamline coordinate \( s \) vanishes, i.e. \( dS/ds = 0 \). This request means that along streamlines the electron fluid behaves isentropic and purely adiabatic.

For such isentropic flows, the above relation then simply states that the work done by the pressure at the expansion of the co-moving plasma volume \( \Delta V \) is the only reason for a reduction or an increase of the inner electron energy \( d\epsilon \) of this volume. For that latter case, one hence obtains the following relation between pressure and energy density \( \epsilon \) for a co-moving plasma volume \( \Delta V \) (i.e. one which is flanked by adjacent streamlines) of the electron fluid at an increment \( ds \) of the streamline coordinate \( s \):

\[
-p \frac{d}{ds} \Delta V = \frac{d}{ds} (\epsilon \cdot \Delta V)
\]

(18)

where the relation between pressure and energy density \( \epsilon \) of the electrons via moment-definitions is given by \( p = \frac{m_e}{2} \epsilon \).

This furtheron leads to the following relation, if again here we assume to be based on kappa functions and the corresponding kappa pressure \( P_\kappa (s) \) (see Eq. (14)):

\[
-\frac{1}{2} m_e \Theta \langle \theta (s) \rangle^2 \frac{\kappa(s)}{\kappa(s) - 3/2} \frac{d}{ds} \Delta V = \frac{3}{4 \pi} \frac{d}{ds} \left( \Theta \langle \theta (s) \rangle^2 \frac{\kappa(s)}{\kappa(s) - 3/2} \right) \Delta V
\]

(19)

which furthermore, in view of the fact that the low-Mach-number heliosheath plasma flow with Mach numbers \( M_s \leq 0.1 \) can be considered as incompressible (i.e. \( \eta \) is approximated, see argumentation given e.g. in Fahr & Siewert (2015)), then leads to

\[
-\Theta^2 \langle \theta (s) \rangle^2 \frac{\kappa(s)}{\kappa(s) - 3/2} \frac{d}{ds} \Delta V = \frac{3}{4 \pi} \frac{d}{ds} \left( \Theta \langle \theta (s) \rangle^2 \frac{\kappa(s)}{\kappa(s) - 3/2} \right) \Delta V
\]

(20)

or yielding

\[
-\frac{1}{\Delta V} \frac{d}{ds} \Delta V = -\frac{1}{\Delta V} \frac{d}{ds} \left( \frac{d}{ds} \left( \Theta \langle \theta (s) \rangle^2 \frac{\kappa(s)}{\kappa(s) - 3/2} \right) \right) = \frac{3}{4 \pi} \frac{d}{ds} \left( \Theta \langle \theta (s) \rangle^2 \frac{\kappa(s)}{\kappa(s) - 3/2} \right)
\]

(21)

The relation above (Eq. (21)) can then be put into the following form:

\[
\frac{d}{ds} \left[ \ln \Theta \langle \theta (s) \rangle^2 \frac{\kappa(s)}{\kappa(s) - 3/2} \right] = \frac{d}{ds} \eta = 0
\]

(22)

where we have introduced in the formula above normalizations of the velocity by the reference value \( \Theta_0 \) and of the volume by the reference value \( V_0 \), respectively. Eq. (22) shows that the quantity \( \eta \) for isentropic flows represents a constant on streamlines, meaning that for any selected streamline parameter \( \Lambda \) (for its definition see Sylla & Fichtner 2015) it is thus given by

\[
\eta(\Lambda) = \left. \ln \Delta V + \frac{3}{3 + 4 \pi} \ln \Theta \langle \theta (s) \rangle^2 \frac{\kappa(s)}{\kappa(s) - 3/2} \right|_\Lambda
\]

(23)

and, when taking everything to the "e"-th power, leading to the requirement

\[
\exp[\eta] = \Delta V_1 \cdot \Theta \langle \theta (s_1) \rangle^2 \frac{\kappa(s_1)}{\kappa(s_1) - 3/2}
\]

\[
= \Delta V_2 \cdot \Theta \langle \theta (s_2) \rangle^2 \frac{\kappa(s_2)}{\kappa(s_2) - 3/2}
\]

(24)

Hereby the co-moving fluid volume \( \Delta V \) is connected with the streamline geometry of the plasma flow and can be defined by means of the particle conservation requirement for the co-moving fluid volume for \( x_0 < s_1 < s_2 \) in the following form:

\[
\frac{n_{e0} \Delta V_{x_0} U_{x_0}}{n_s \Delta V_{x_1} U_{x_1}} = n_{e2} \Delta V_{x_2} U_{x_2}
\]

(25)

At least in that region of the heliosheath where the kinematic approximation of the frozen-in \( B \)-field (Suess & Nerney 1990) can be used and sonic Mach numbers of the plasma are very low, the plasma (also the electrons) can be treated as incompressible. This may not be a viable approach near the heliopause where the plasma flow leads to a pile-up of magnetic field lines (see e.g. Drake et al. 2015). Here our transport equations would need corresponding changes.

In view of the electron incompressibility \( n_{e1} = n_{e2} \) this leads to the following simplified relation

\[
\frac{\Delta V_{x_1}}{\Delta V_{x_2}} = \frac{U_{x_2}}{U_{x_1}}
\]

(26)

and consequently leads to the following requirement for the upper streamline constant \( \eta \):

\[
\exp[\eta] = \frac{U \Theta^2 \kappa}{\kappa - 3/2} = \frac{U_0 \Theta_0^2 \kappa_0}{\kappa_0 - 3/2}
\]

(27)

and hence expresses the fact that - in case of an isentropic electron flow - there exists a concerted or intertwined change of the quantities \( U, \kappa, \) and \( \Theta \) along each streamline in order to fulfill the thermodynamical relation given in Eq. (27).

With the above streamline constant \( \exp[\eta] \) one obtains a new NLTE-behaviour of the electron fluid, namely a concerted variation with the streamline coordinate \( s \) of both kappa function parameters \( \Theta \) and \( \kappa \). This, however, has the consequence that the temperatures derived from the underlying kappa functions are reacting in an unexpected way on changing values of \( \kappa \) when it is taken into account that with a change in \( \kappa \) also a change in \( \Theta \) has to occur according to the following relation

\[
\frac{U \Theta^2 \kappa}{\kappa - 3/2} = \frac{U_0 \Theta_0^2 \kappa_0}{\kappa_0 - 3/2}
\]

(28)

in other words meaning that in an isentropic electron flow \( \Theta \) has to vary with \( \kappa \) according to the following relation

\[
\Theta^2 = \frac{\kappa - 3/2}{\kappa} \Theta_0^2 \frac{\kappa_0}{\kappa_0 - 3/2} \frac{U_0}{U}
\]

(29)

From the recent paper by Fahr et al. (2017), one can obtain as a starting value just after passage of the electrons over the termination shock, with \( [\Theta_0^2/U_0^2] = 0.09 \) \( \frac{m_e}{M} \) (\( M \) denoting the proton mass, \( m \) the electron mass), \( \kappa_0 = 4 \) and \( U_0 = 100 \) \( \text{km s}^{-1} \):

\[
[\Theta_0^2/U_0^2] \frac{\kappa_0}{\kappa_0 - 3/2} = 265
\]

(30)

These values are derived in the paper by Fahr et al.
(2018) with the restriction that the kappa distributed, shocked electrons represent the right spectral electron fluxes measured close to the termination shock by Voyager-1 in the energy range between 40 and 70 keV, namely measured close to the termination shock by Voyager-1.

This together with Eq. (29) shows the following variation \( \Theta = \Theta(\kappa) \) of the kappa-parameter \( \Theta \) with the kappa-parameter \( \kappa \) (Fig. 1):

\[
\Theta/U_0 = \sqrt{265 \kappa - 3/2 \frac{U_0}{U}}
\]

(31)

In this relation, one may recognize that whenever \( \kappa \) approaches the value \( (3/2) \) (i.e. extreme suprathermal case), the normalized value of the associated value \( \Theta(\kappa)/U_0 \) degenerates to 0 (i.e. collapse of the Maxwellian core). For large values of \( \kappa \) (i.e. Maxwellian case) the associated Maxwellian core temperature will be higher, the higher is the value \( (U/U_0) \). It is furthermore interesting to recognize that different, however permitted combinations of parameters \( \kappa \) and the associated \( \Theta(\kappa) \), as they can be extracted from Fig. 1, belong to different kappa functions which all have the same entropy, stating the astonishing fact that a complete power law distribution and the associated Maxwellian with a temperature spread \( \Theta = U_0\sqrt{265 \kappa} \) have identical entropies.

It is interesting to compare results of kappa-temperatures obtained on the basis of a constant value of \( \Theta \) at varying values of \( \kappa \) given by e.g. Heerikhuisen et al. (2008) in the form

\[
T_\Theta^\kappa(v) = \frac{m\Theta^2}{3k} \frac{\kappa}{\kappa - 3/2} = \frac{P^k(s)}{nK}(s)
\]

(32)

with corresponding moment-expressions obtained from integrations of the two-parameter kappa distribution function over the velocity space in the form:

\[
T_\Theta^\kappa(v) = \frac{m}{3K} \int_0^\infty v^4 f^\kappa(v, \Theta, \kappa)dv
\]

(33)

We have shown in Fig. 2 the distribution function \( f^\kappa(v, \eta(\Theta, \kappa)) \) (see Eq. 1) as a function of the velocity \( v \). In Fig. 3, we have displayed the differential temperature moment \( dT^\kappa/ds \) (i.e. the integrand of Eq. 33). Finally, in Fig. 4 we show the temperature moment itself as given in Eq. 33.

4 NON-ISENTROPIC ELECTRON FLOWS AND THE ELECTRON ENTROPY

In case that the electron flow along the streamlines, contrary to the assumption in the section before, develops non-isentropically due to the presence of energy sources or sinks, i.e. if the entropy of the electron fluid changes with the flow line element \( s \), then one has to consider the following more complicate relation along streamlines

\[
\frac{dS}{ds} = \frac{d}{ds} \left( \frac{Q}{T} \right) = \frac{d}{ds} \left( \frac{Q}{U} \frac{1}{T} \frac{\dot{T}}{T^2} \frac{1}{U} \right) = \frac{Q}{T} \left( \frac{\dot{Q}}{Q} \right) \left( \frac{\dot{T}}{T} \right) \left( \frac{1}{U} \right)
\]

(34)

where \( dQ \) according to thermodynamical convention describes energy gains and losses per unit of the streamline element \( s \) in the co-moving volume \( \Delta V \), and \( T = T^\kappa(s) \) is the local kappa temperature. Looking at Eq. 34, one can already qualitatively say that practically no change of the entropy has to be faced, if the temporal energy change \( \dot{Q} \)
compared to the given energy content $Q$ is negligible, i.e. the latter condition for $\frac{Q}{Q} = \frac{\varepsilon}{\varepsilon} \ll 1$ would lead to a quasi-isentropic behaviour.

In order to check on this condition, we may now look into the pressure transport equation (Eq. 14) where one can identify two processes contributing to the change of the energy density $Q(s) = \varepsilon(s)$, namely:

1) connected with the process of magnetic cooling (see Section 2, Eq. 8) which is given by

$$\dot{\varepsilon}_1 = m \cdot T1 = \frac{2}{3} \pi n U \frac{1}{B} \frac{\partial B}{\partial s} \int dv v \frac{\partial}{\partial v} (v^3 f^k) = \frac{4}{3} U_0 \frac{1}{B_0} \frac{\partial B}{\partial s} 3 \pi \kappa^s(s)$$

and

2) the process of velocity-space diffusion (see Section 2, Eq. 11) leading to the following term

$$\dot{\varepsilon}_2 = m \cdot T2 = 10 m D_0 \int v^4 f^k dv = \frac{3}{4 \pi} 10 D_0 P^\kappa(s)$$

Hence one can express the energy change per time by

$$\dot{Q} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2.$$

Since we have already derived the solution for the local pressure $P^\kappa(s)$, one thus, based on that, can also obtain the entropy evolution with the following equation (see Section 2, Eq. 14)

$$\frac{dS^\kappa}{ds} = \frac{d}{ds} \left( \frac{Q}{T^\kappa} \right) = \left[ \frac{\frac{1}{B} \frac{\partial B}{\partial s} U P^\kappa(s) + \frac{15}{2 \pi} D_0 P^\kappa(s)}{U \cdot T^\kappa(s)} \right]$$

where the actual kappa temperature is given by (see Heerikhuisen et al. 2008; Scherer et al. 2018)

$$T^\kappa(s) = \frac{m \Theta^3}{3K} \frac{\kappa}{\kappa - 3/2} \frac{P^\kappa(s)}{nK}$$

The kappa-entropy $S^\kappa(s)$ can then be calculated with the above differential equation, using the equation for $P^\kappa(s)$ (Eq. 14) derived further above in this paper in the form:

$$P^\kappa(s) = P^\kappa(s_0) \cdot \exp \left( \frac{4}{3} \frac{B(s) - B(s_0)}{U} + 10 D_0 \int_{s_0}^s \frac{ds}{U} \right)$$

and hence now obtaining the following differential equation

$$\frac{dS^\kappa}{ds} = \left[ \frac{\frac{1}{B} \frac{\partial B}{\partial s} U P^\kappa(s) + \frac{15}{2 \pi} D_0 P^\kappa(s)}{U \cdot T^\kappa(s)} \right]$$

which can be integrated to yield

$$S^\kappa(s) = S^\kappa(s_0) + \frac{nK}{\pi} \int_{s_0}^s \left( \frac{\frac{1}{B} \frac{\partial B}{\partial s} + \frac{15}{2 \pi} D_0}{} \right) ds$$

Now it is again interesting to see that also the entropy $S^\kappa(s)$, as the pressure $P^\kappa(s)$, depends directly obviously neither on $\kappa$ nor on $\Theta$, since it turns out accounting to the above relation that $\partial S^\kappa(s)/\partial \kappa = \partial S^\kappa(s)/\partial \Theta = 0$.

On the one hand we have found that isentropic electron flows require the combined change of the local kappa parameters $\kappa$ and $\Theta$, on the other hand the entropy itself, controlling the isentropic behaviour, does not directly depend on the kappa parameters themselves. Looking finally on the amount of entropy changes that are to be expected according to the formula above, it turns out, in view of the quantities entering Eq. (41), that the resulting entropy change, at least on the frontal side of the heliosheath (upwind side), will be fairly small.

5 CONCLUSIONS

In this paper, we have derived a kinetic transport equation describing the evolution of the heliosheath electron distribution along the flowlines of the plasma bulk flow downstream of the solar wind termination shock. From this kinetic transport equation one can proceed to moment transport equations. Here, we especially develop and mathematically treat the pressure transport equation which describes the evolution of the electron pressure along the plasma flow lines. It turns out that on the basis of the processes taken into account which influence electrons at their passage along the flowlines, namely convective phase-space changes, magnetic cooling processes and wave-particle diffusion, an analytic solution for the electron pressure as function of the streamline coordinate can be obtained. Introducing most probable parameter values for the relevant processes, it turns out that, at least on the upwind side of the heliosheath, the electron pressure is essentially constant along the flow lines, only varying from streamline to streamline (i.e. perpendicular to the streamlines) due to different boundary conditions at the flowline origin at the termination shock where, due to different injection conditions, the injected electrons lead to different initial pressures. This means, nevertheless, that the flowlines serve more or less as electron isolars.

It is then studied by us, how isentropic electron flows with $dS^\kappa/\pi ds = 0$ should behave, and it is shown, that, under isentropic electron flow conditions, the two parameters of the electron kappa function $\kappa(s)$ and $\Theta(s)$ should undergo concerted changes with the streamline coordinate $s$, i.e. in the form $\Theta(s) = \Theta(\kappa(s))$. We show that a streamline parameter $\eta$ exists which describes these concerted changes along streamlines in the form $d\eta/\pi ds = d[\eta(s, \kappa, \Theta)]/\pi ds = 0$. Thus, for isentropic electron flows, the two parameters $\kappa(s)$ and $\Theta(s)$
of the electron kappa-function along the streamline undergo concerted variations as described by Eq. (31).

This means that the entropy of a kappa function $S(\kappa, \Theta)$ for different values $\kappa$ and $\Theta$ leads to the same value $S(S(\kappa, \Theta(\eta)))$, if the two parameters are intertwined according to Eq. (31). This also allows to conclude that for $1 \ll \kappa$ one finds entropies identical to the associated Boltzmannian or Maxwellian quasi-LTE entropies $S(T_{\text{max}})$ connected with a temperature of

$$T_{\text{max}} = \frac{3}{2} m_\Theta^2 (\kappa \to \infty) = \frac{3}{2} m \cdot (265U_0/U).$$

(42)

On the other hand, we have studied non-isentropic electron flows, including energy sources and sinks operating on the convected electrons, like magnetic cooling or energy-diffusion processes. This allows to derive an entropy transport equation of the form given by Eq. (41) and to obtain an analytic solution for the electron entropy $S(s)$ as function of the streamline coordinate. Under the conditions prevailing in the heliosheath, i.e. subsonic flow, incompressible behaviour, etc., one finds that the variation of the electron entropy along flowlines is only weakly pronounced. The electron flows along streamlines in the heliosheath can thus be essentially characterized as incompressible, isobaric, isothermal and isentropic flows.

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