Dissipation of energy in the zero temperature limit is a central question in the field of quantum turbulence. In helium superfluid turbulence is related to motion of quantized vortices that generate the superfluid velocity field. At scales smaller than the intervortex distance energy is expected to be cascaded to smaller scales via Kelvin waves (KWs) until it can be transformed into thermal excitations, phonons in case of $^4$He.$^{1,2}$ Therefore, the proper identification of Kelvin waves is important if we want to verify the theoretically predicted Kelvin spectrum.$^{3-5}$ Experimentally this is probably not possible, but numerically the identification is conceivable. Actually, several numerical simulations have tried to verify the correct spectrum using the vortex filament model$^{6-9}$. In our opinion, none of those is fully convincing. The cascade towards smaller scales is evident in simulations but the correct value for the slope of the spectrum is dominated by a numerical noise or distorted by the method used to determine the Kelvin amplitude. The simulations by Kozik et al.$^{7,8}$, using the Hamiltonian formulation of the filament model, correctly identify the Kelvin modes, but their spectrum may depend on the absolute amplitude of the initial spectrum.$^{5,10}$

In BEC’s the Kelvin spectrum has been determined by Yepez et al.$^{11}$ using the Gross-Pitaevskii equation. However, the interpretation of these results was strongly criticized.$^{12,13}$ In high-$k$ region the $k^{-3}$ spectrum (originally interpreted as Kelvin spectrum) most likely results from the velocity and density profile inside the vortex cores.$^{13}$ The low-$k$ region was originally interpreted it as Kolmogorov K41 law, even if the spectrum is likely visible at scales smaller than the intervortex distance (authors did not evaluate the mean vortex separation). One option is that this spectrum comes from Kelvin waves, but since the fluid is highly compressible the explanations presented in the Comment$^{12}$ are more likely.

In this article we list several ways that can be used to identify Kelvin waves. We work in the framework of the vortex filament model$^{14}$, where the vortices are considered to be thin and the superfluid velocity can be calculated simply from the vortex configuration $s(\xi)$, where $\xi$ is the length along the vortex, using the Biot-Savart integral. In other words, we consider only length scales much larger than the vortex core size, $a$. We start with two simple cases where the definition of the Kelvin waves is obvious: straight vortex with Kelvin waves and a vortex ring occupied by KWs. In both cases the Kelvin spectrum can be determined using a simple Fourier transformation. We use these two sample cases to test other methods that have been used previously in the literature. Such methods are, for example, curvature, energy spectrum, and fractal dimension.

II. KELVIN WAVES

Kelvin waves are helical distortions on a vortex. They can be generated by applying a counterflow $v_a - v_s$ along a vortex.$^{15}$ Vortex reconnection, and more generally, interaction with other vortices, with vortex itself, or with boundaries also induce Kelvin waves. In the small amplitude and long wave length limit the dispersion relation for a Kelvin wave with wave vector $k$ is given by

$$\omega_k = \frac{\kappa k^2}{4\pi} \left[ \ln(2/ka) - \gamma \right].$$

Here $\kappa = \hbar/m_4$ (for $^4$He) or $\hbar/2m_3$ (for $^3$He-B) is the circulation quantum, $a$ the vortex core size and $\gamma = 0.57721 \ldots$ is the Euler constant.

On a straight vortex (taken to be along $z$-axis) the definition and identification of the Kelvin waves is simple. Assume that vortex can be represented as coordinates $x(z)$ and $y(z)$ and define $w(z) = x(z) + iy(z)$. Now the vortex configuration can be presented as a sum of different Kelvin modes:

$$w(z) = \sum_{k} w_k \exp(ikz).$$

In case of periodic boundary conditions along the $z$-direction (typical for simulations), the $k$-vector is discrete and given by $k = 2\pi m/L_z$, where $m$ (mode) is integer and $L_z$ is the period. For a particular vortex configuration a Kelvin spectrum can be defined as $n_k = |w_k|^2 + |w_{-k}|^2$ with $k > 0$. Theoretical predictions, which apply for a statistical average, predict that the steady-state Kelvin spectrum takes the form$^{3-5}$

$$n_k = |w_k|^2 + |w_{-k}|^2 = N_0 k^{-2\eta}, \quad (k > 0)$$

with $\eta \approx 1.5 \ldots 2$. This spectrum results from interaction of different scale Kelvin waves and provides a constant ($k$-independent) energy flux, $\theta$, through different scales. If $\epsilon_m$
(with $m > 0$) determines the fraction of positive Kelvin modes, then we can parametrize the amplitudes using the modes:

$$w_m = \sqrt{\epsilon_m A m^{-\eta}},$$

$$|w_{-m}| = \sqrt{1 - \epsilon_m A m^{-\eta}}. \tag{4}$$

Numerically, a great care should be used when identifying the spectrum. If the vortex configuration is such that the local curvature approaches the numerical resolution, $k_{res}$, then even the interpolation used to obtain the vortex configuration at equidistant points (needed for FFT) can result in errors that distorts the spectrum for $k$-values $k \gtrsim k_{res}/5$. One way to avoid these errors is to directly solve the Fourier coefficients. But then one loses the speedup obtained by FFT. An alternative is to keep the vortex points equally $z$-distributed by removing the motion along the $z$-direction. By introducing a vector $\mathbf{r}(z,t) = (x(z,t), y(z,t))$, which only determines the point of intersection of the vortex line with the plane, one obtains that $^{1,16}$

$$\frac{\partial \mathbf{r}}{\partial t} = \mathbf{s} \cdot \frac{\partial \mathbf{s}}{\partial t} \left( \hat{z} + \frac{\partial \mathbf{r}}{\partial z} \right). \tag{5}$$

Here $\hat{z}$ is the unit vector along the $z$-axis. This formula, together with the Biot-Savart equation can be used to derive the Hamiltonian equation for the vortex motion. $^{1,16}$ This projection should be used with care, since it limits the real vortex motion. It, for example, prohibits a vortex to reconnect with itself.

Cascade due to Kelvin waves is typically very weak. The energy flux, $\theta$, can be increased by increasing the amplitudes of the Kelvin waves. Simulations always contain some dissipation (or noise) that typically is strongest at the smallest of the Kelvin waves. Simulations always contain some dissipated, this flow out from the numerical points. At zero temperature both the energy and momentum should be conserved. However, even if the energy is well conserved, this outflow from the numerical $k$-region can become smaller than the energy flux due to Kelvin-wave cascade. This results in a numerical bottleneck in which the small scales are removed of Kelvin waves near the resolution limit and generally resulting excessive fractality and noise. In simulations this can often be seen as a failure of the Fourier presentation for the vortex. Therefore, any numerical method that tries to determine the steady-state Kelvin spectrum should contain enough dissipation such that this bottleneck due to finite resolution is avoided. How this is implemented in numerics, especially with the vortex filament model, is still somewhat open question. One option is to work at fine scales and have high enough resolution such that all the energy can be dissipated by mutual friction. Important is, that a proper dissipation is arranged dynamically. Simply damping the highest modes might not be enough.

One should also note that near the resolution limit the dispersion relation, Eq. (1), becomes distorted. Simple estimation states that $\omega$ becomes flat near $k_{res}$. Because the dynamics is not accurately modeled at the smallest scales, these scales can result in different spectrum and should therefore be omitted from the fits.

### III. Kelvin Waves on a Vortex Ring

For a vortex ring with Kelvin waves one can also make a Fourier presentation. Consider a vortex ring of radius $R_0$ in the $z = z_0$ plane located symmetrically around $z$-axis. If we occupy the ring with Kelvin waves, in a similar fashion than a straight vortex, we may parametrize it as follows:

$$x = \sum_m R_m \cos(m \phi + \varphi_m) \cos \varphi \tag{6}$$

$$y = \sum_m R_m \cos(m \phi + \varphi_m) \sin \varphi \tag{6}$$

$$z = z_0 - \sum_m R_m \sin(m \phi + \varphi_m),$$

where $\phi = \arctan(y/x)$ is the azimuthal angle and $\varphi_m$ is the phase of the mode $m$. Amplitudes $R_m$ (which correspond to $|w_m|$, in case of a straight vortex) can be calculated using a FFT in the following way: First the points along the vortex must be interpolated at equidistant $\phi$ points. This can be realized using e.g. cubic interpolation. Now the Fourier transformation of

$$Z(x, y) = x \cos \phi + y \sin \phi - iz \tag{7}$$

directly determines the $R_m$‘s. The zero frequency term from the Fourier transformation gives both the ring radius $R_0$ (real part) and also the location of the ring, $z_0$ (imaginary part).

If we want to occupy the vortex with a particular spectrum, we can take

$$R_m = \sqrt{\epsilon_m A m^{-\eta}},$$

$$R_{-m} = \sqrt{1 - \epsilon_m A m^{-\eta}}. \tag{8}$$

Here $\epsilon_m$ (with $m > 0$) determines the fraction of positive and negative Kelvin modes.

At zero temperature, or when the normal fluid is at rest, kinetic energy and momentum are only due to superfluid component and completely determined by the vortex configuration (we ignore externally applied velocities). Additionally, if vortices form loops and the vorticity disappears at the infinity, one may evaluate kinetic energy, $E$, and momentum, $P$, in numerically convenient way by line integrals over the vortex configuration:

$$E = \rho_s \kappa \oint \mathbf{v} \cdot \mathbf{s} \times ds \tag{9}$$

$$P = \frac{1}{2} \rho_s \kappa \oint \mathbf{s} \times ds \tag{10}$$

These formulas make it possible to evaluate the accuracy of the numerical scheme used. At zero temperature both the energy and momentum should be conserved. However, even if the energy is well conserved, e.g. by accuracy of 0.1 percent, this fluctuation is still very large if we compare it with the energy related to smallest scale Kelvin waves. Therefore, the errors coming from the time iteration scheme can be large at the smallest length scales. In the following we are not concentrating on these errors but pay attention only to methods that try to identify the Kelvin waves on a tangle. In other words we assume that the vortex configuration is correct at the numerical points $s_i, i = 1, \ldots, N$. 
At least our test with spectrum given by Eq. (12) indicates that this kind of fit is poor (exponential fit is often better) and the obtained $\alpha$ is very sensitive to the fit region. More importantly, the location of the maximum and also the shape of the histogram at high curvature regions depends on the resolution for spectra $\eta \lesssim 2.5$, as seen in Fig. 1.

In case of straight vortex with small amplitude Kelvin waves one can relate the curvature spectrum with the Kelvin spectrum in the following way:

$$\int |c(\xi)|^2 d\xi \approx \int \langle |w''(z)|^2 \rangle dz = \sum_k k^4 |w_k|^2. \quad (11)$$

Therefore one might expect that the curvature spectrum could reveal the Kelvin spectrum, also in more complicated tangles. We observed that the curvature spectrum follows this law only when the amplitude is small and $\eta > 2$, i.e. only when the curvature spectrum $\propto k^2$ is such that $y < 0$. For any smaller $\eta$ the curvature spectrum becomes flat and nothing can be said about $\eta$. Improving the numerical scheme for determining the local curvature does not resolve this problem. If the Kelvin amplitude is not small, the above relation, Eq. (11), is satisfied even less accurately.

V. ENERGY SPECTRUM

Energy (or velocity) spectrum $E(k)$ describes the distribution of the kinetic energy at various length scales. At absolute zero temperature the total kinetic energy is given by superfluid component $E_k = \int \rho \partial_s |v|^2 r = \int E(k) dk$. Here $\rho$ is the three-dimensional vector in the momentum space and $k = |k|$. (Note that in case of Kelvin waves the $k$-vector was 1D object.) Assuming a quantized vortex with singular distribution of vorticity and that the vorticity disappears at the infinity one may derive following formula for the kinetic energy caused by the vortices:

$$E_{k,s} = \rho_s \frac{\kappa^2}{2(2\pi)^3} \int d^3 k \int d\xi_1 \int d\xi_2 e^{ik \cdot (s_1 - s_2)} (\hat{s}_1 \cdot \hat{s}_2). \quad (12)$$

Here the double integration is along vortices, described by coordinates $\xi_1$, and $\xi_2$. Notations $s_i = s(\xi_i)$ and $\hat{s}_i = ds(\xi_i)/d\xi_i$ are the location of the vortex core and tangent at $\xi_i$, respectively. Additionally one should emphasize that this formulation is exact only if the vortices form closed loops and that the vorticity disappears far away. Otherwise, at least the low $k$-values are calculated incorrectly, which is not emphasized in Refs. [17–19].

Fortunately, the integration over different $k$-directions in Eq. (12) can be done analytically, resulting that

$$E(k) = \rho \frac{\kappa^2}{(2\pi)^2} \int d\xi_1 \int d\xi_2 \frac{\sin(k|s_1 - s_2|)}{|s_1 - s_2|} (\hat{s}_1 \cdot \hat{s}_2). \quad (13)$$

For anisotropic situations, the dependence on the absolute value of the wave number, $k$, should be understood as an angle average.
In our numerical scheme vortices are described by sequence of points and we assume that between these points vortex is straight. This is a standard assumption used also by several other authors. The numerical integration of the above double integral is quite straightforward but, due to the oscillating part of the integrand, some care should be taken when \( k \) is large. However, we found that it is important to check this limit, because the energy spectrum at \( k \)-values that approach and exceed the numerical resolution limit should converge towards \( k^{-1} \) spectrum. This is not the case in some of the previous vortex filament calculations presented so far \cite{17,20,23} and therefore some caution should be kept in mind when reading those papers. Additionally, one numerical check for the spectrum is given by the absolute amplitude of the spectrum: in the high-\( k \) region, near the resolution, the amplitude should be given by the vortex length.

For a single vortex ring with radius \( R \), the line integration formula, Eq. (13), can be simplified further. Simply putting \( s_i = R(\cos \phi_i \hat{x} + \sin \phi_i \hat{y}) \) when \( s_i' \cdot s_i'' = \cos(\phi_1 - \phi_2) \) one obtains, after few standard integration techniques, that

\[
E(k) = \frac{\rho_0 k^2 R^2}{\pi} \int_0^\pi dx \frac{\sin(2kR \sin x)}{2kR \sin x} \cos(2x). \tag{14}
\]

This can be presented using generalized hypergeometric functions, but we found that it is much more convenient to do the integration numerically. The spectrum for vortex ring is presented in Fig. 2. At large scales the energy spectrum is given by \( E(k) = \rho_0 k^2 R^2 / 6 \) which can be obtained by doing a series expansion at small \( k \) for Eq. (14). At small scales the vortex ring looks like a straight vortex (of length \( 2\pi R \)) and the energy spectrum is given by \( E(k) = \rho_0 k^2 R^2 / (2k) \). The spectrum for single ring is very similar to the one produced by four rings in Ref. [21], but our better resolution makes it possible to observe small amplitude oscillations. We found that these oscillations, with period \( \pi / R \), around this limiting curve scale like \( k^{-3/2} \), which is illustrated in Fig. 2. To resolve these oscillations the discretization step for different \( k \)-values should be smaller than this oscillation period, which makes the calculations of high \( k \)-values time consuming.

An advantage of the line integration formula over the standard Fourier transformation used \textit{e.g.} by Kivotides \textit{et al.} \cite{21} is that one avoids the numerical problems that is caused by the divergent \( 1/r \) velocity field generated by vortices. A disadvantage of this method is that in the presence of the walls or with periodic boundaries some care should be taken since not all of the assumptions in deriving Eq. (12) are fulfilled. Typically the boundaries result in extra surface terms that make the correct calculation more cumbersome. For a single plane boundary the energy spectrum can be calculated by extending the second line integral in Eq. (13) to include also the image vortices. Nevertheless, the high \( k \)-values are correctly taken into account even with this simple formulation.

Even if the 3D energy spectrum \( E(k) \) is suitable for determining the large scale velocity structures appearing in vortex tangle, it is not sensitive for small scale Kelvin waves that have characteristic scale smaller than the intervortex separation. This is illustrated in Fig. 3 where we have plotted the 3D energy spectrum for a straight vortex with and without Kelvin waves. The \( 1/k \) contribution coming from the straight vortex is so dominant that very little can be said about the Kelvin spectrum, at least for spectra with \( \eta > 1.5 \). The increased length only raises the spectrum up without affecting the slope in the high-\( k \) region. In case of vortex ring the identification...
of Kelvin waves is even more complicated due to the oscillations that are already present for a ring without Kelvin waves. This just emphasizes that in order to identify the Kelvin spectrum the direct measurement of the vortex position becomes more important than determination of the 3D energy spectrum at high $k$-values.

VI. FRACTAL DIMENSION

At low temperature vortices typically looks quite wiggly, almost fractal like. This is due to Kelvin waves. Therefore, one way to characterize the tangle is to determine the fractal dimension for vortex lines.\(^9\)

Upper-bound estimate for the fractal dimension can be calculated by determining the correlation dimension $D$ introduced by Grassberger and Procaccia\(^22\). In case of $N$ discretization points, $D$ can be determined by calculating the number of point pairs, $n$, whose separation is less than $\varepsilon$. In the limit of $\varepsilon \to 0$ the correlation integral $K(\varepsilon) = n/N^2$ takes the form $K(\varepsilon) \propto \varepsilon^D$.

How are different Kelvin spectra and $D$ related? In order to answer this we have determined $D$ in case of vortex ring that is occupied by Kelvin waves with different spectra and with different amplitudes. Figure 4 summarizes our results. As expected, $D$ is clearly bigger than unity only when $\eta \lesssim 1.5$ and when the amplitude is large enough. This limits the usability of the fractal dimension to spectra that are less steep than predicted theoretically. The dependence on the amplitude makes it difficult to determine $\eta$ accurately. However, if one observes a correlation dimension that is clearly larger than unity, then one can almost certainly state that $\eta < 1.5$. Also, even if we used a simple vortex ring to determine the fractal dimension, the results should be valid more generally at scales smaller than the average intervortex separation, i.e. in the region where the Kelvin cascade is important.

The difficulty in determining the fractal dimension is that numerical noise near the resolution limit (appearing because there are too few pairs) causes that $D \to 3$. We observed that this happens especially when the amplitude is large and $\eta$ is small (near unity). Therefore, the fit region should be limited to scales larger than the numerical resolution. The incorrectly chosen fit region might explain why the calculated correlation dimensions (of order 1.5) in Ref. [9] are much bigger than the presented configuration (Fig. 20 in that article) would suggest. For example, Kelvin spectrum with $\eta = 1.20$ and with amplitude $A = 0.25R$, presented in the inset of Fig. 4, results in a fractal dimension of $D \approx 1.20$ only.

VII. SMOOTHED VORTEX

For a vortex tangle the identification of Kelvin waves can be done by using the concept of “smoothed vortex”\(^1,9\). From the original discretized vortex filaments, determined by points $s_i$ ($i = 1, \ldots, N$), one can construct a smoothed vortex line $s_{\text{smooth}}$ by using every $n$th points as nodes for a cubic-spline interpolation. One can then define the Kelvin-wave amplitude $a(\xi)$ as the distance between the smoothed and the original filament: $a(\xi) = |s - s_{\text{smooth}}|$. The amplitude spectrum $A(k)$ can be defined as

$$\frac{1}{2} \int a^2(\xi) d\xi = \int_0^\infty A(k) dk. \quad (15)$$

We have applied this method for a straight vortex with Kelvin waves, where the spectrum takes some known form. Figure 5 illustrates that the above scheme has a tendency to follow the correct spectrum. For type $A(k) \propto k^{-4}$ spectrum, $\hat{\beta}$ obtained from the fit is typically too big for spectra with $\beta \lesssim 4$, and too small for spectra with $\beta \gtrsim 4$. In some cases the method can result in a spectrum that is totally incorrect. This is also shown in Fig. 5. Better identification of Kelvin waves could perhaps be obtained if the smoothed vortex was allowed to avoid the original data points. This removes the discontinuities of the derivatives $a'(\xi)$ appearing when $a(\xi) = 0$, which is one reason why the spectrum tends towards $\beta = 4$. We have tried this type of interpolation but so far the improvements are only minor.

VIII. CONCLUSIONS

Here we have presented several methods that can be used to identify Kelvin waves. In case of a straight vortex or a vortex ring the identification can be done most reliable. However, in case of vortex tangle a great care should be used. The biggest problem is the lack of proper definition of a Kelvin wave on a curved vortex. In order to properly define a Kelvin wave there must exist a scale separation between the characteristic size of the underlying vortex and its disturbances.
Generally the Kelvin cascade to smaller scales can be qualitatively identified using, e.g. average curvature. An accurate determination of the Kelvin amplitude is still very demanding and can result in large errors, depending on the numerical method used. It is alarming that some numerical methods can result in a spectrum that is totally incorrect but still very close to the spectrum predicted theoretically. Therefore, all methods should be properly tested using a simple configuration where the spectrum is known in advance.

The highly dominating geometrical $1/k$ scaling in the 3D energy spectrum at high-$k$ illustrates that at small scales the determination of the vortex location becomes much more essential than the knowledge of the energy spectrum. Only after determining the vortex location, we have some hope for identifying the Kelvin spectrum. Identifying the vortex location is not a problem with the vortex filament model and is quite straightforward with the Gross-Pitaevskii equation. Experimentally the identification is challenging but recently the visualization of quantized vortices has become possible\textsuperscript{23}.

Furthermore, the accuracy of the numerical schemes typically limits the proper determination of the Kelvin spectrum to scales $k \lesssim k_{\text{res}}/5$, or even less. Similar requirement results from the dissipation, which must be present in order to avoid numerical bottlenecks that would otherwise generate in a non-physical fractalization of the vortex at the smallest scales. Therefore, the identification of Kelvin waves on a tangle is a great challenge and requires resolution that is much higher than the average vortex separation. Here faster computers and especially new numerical algorithms, like the tree method\textsuperscript{24}, are of great importance.

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1. B.V. Svistunov, *Phys. Rev. B* **52**, 3647 (1995).
2. W.F. Vinen and J.J. Niemela, *J. Low Temp. Phys.* **128**, 167 (2002).
3. E. Kozik and B. Svistunov, *Phys. Rev. Lett.* **92**, 035301 (2004).
4. V.S. L’vov and S. Nazarenko, JETP Lett. **91**, 428 (2010).
5. E. Sonin, *Phys. Rev. B* **85**, 104516 (2012).
6. W.F. Vinen M. Tsubota, A. Mitani, *Phys. Rev. Lett.* **91**, 135301 (2003).
7. E. Kozik and B. Svistunov, *Phys. Rev. Lett.* **94**, 025301 (2005).
8. E. Kozik and B. Svistunov, arXiv:1007.4927v1 (2010).
9. A.W. Baggaley, and C.F. Barenghi, *Phys. Rev. B* **83**, 134509 (2011).
10. R. Hänninen, arXiv:1104.4926v2 (2012).
11. J. Yepez, G. Vahala, L. Vahala, and M. Soe, *Phys. Rev. Lett.* **103**, 084501 (2009).
12. V. L’vov and S. Nazarenko, *Phys. Rev. Lett.* **104**, 219401 (2010).
13. G. Krstulovic, and M. Brachet *Phys. Rev. Lett.* **105**, 129401 (2010).
14. K.W. Schwarz, *Phys. Rev. B* **31**, 5782 (1985).
15. W.I. Glaberson, W.W. Johnson, and R.M. Ostermeier, *Phys. Rev. Lett.* **33**, 1197 (1974).
16. E.V. Kozik, and B.V. Svistunov, *J. Low Temp. Phys.* **156**, 215 (2009).
17. T. Araki, M. Tsubota, and S.K. Nemirovskii, *Phys. Rev. Lett.* **89**, 145301 (2002).
18. S.K. Nemirovskii, M. Tsubota, and T. Araki, *J. Low Temp. Phys.* **126**, 1535 (2002).
19. T. Araki, M. Tsubota, and S.K. Nemirovskii, *J. Low Temp. Phys.* **126**, 303 (2002).
20. L. Kondaurova, and S.K. Nemirovskii, *J. Low Temp. Phys.* **138**, 555 (2005).
21. D. Kivotides, J.C. Vassilicos, D.C. Samuels, and C.F. Barenghi, *Phys. Rev. Lett.* **86**, 3080 (2001).
22. P. Grassberger, and I. Procaccia, *Physica D* **9**, 189 (1983).
23. G.P. Bewley, D.P. Lathrop, and K.R. Sreenivasan, *Nature* **441**, 588 (2006).
24. A.W. Baggaley, and C.F. Barenghi, *J. Low Temp. Phys.* **166**, 3 (2012).