Worst portfolios for dynamic monetary utility processes

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\textbf{ABSTRACT}

We study the worst portfolios for a class of law invariant dynamic monetary utility functions with domain in a class of stochastic processes. The concept of comonotonicity is introduced for these processes in order to prove the existence of worst portfolios. Using robust representations of monetary utility function processes in discrete time, a relation between the worst portfolios at different periods of time is presented. Finally, we study conditions to achieve the maximum in the representation theorems for concave monetary utility functions that are continuous for bounded decreasing sequences.

\textbf{1. Introduction}

In this paper, we present a definition of worst case portfolios for insurance versions of dynamic monetary utility functions, borrowing the notion of insurance version from Ruschendorf \cite{Ruschendorf}. This definition extends the ideas of worst case portfolios presented in \cite{Ruschendorf} to the dynamic framework of monetary utility functions studied in \cite{Cheridito}. Within the discrete time framework, given an agent interested in measuring the maximum risk, we study conditions under which a portfolio preserves the property of worst case portfolio along the time.

Monetary utility functions can be built through risk measures, and vice versa. When the aim is to look at worst case scenarios, risk measures are used to quantify the risk, while when the best possible performances are desired utility functions are employed. The axiomatic notion of monetary risk measure was first introduced by Artzner et al. \cite{Artzner1, Artzner2}, and has been extensively studied since then in different directions. The general framework defined by Follmer and Schied \cite{Follmer} in terms of convex functionals has also been analyzed in a dynamic setting, providing a systematic axiomatic approach to time-consistent monetary risk measures; see, for instance, Cheridito et al. \cite{Cheridito}, Cheridito and Kupper \cite{Cheridito2}, and the recent work of Vioglio et al. \cite{Vioglio}. The research work developed on this area has led to the general definition of conditional monetary risk measure \cite{Cheridito}, where the concept of monetary utility process is also defined. Under a different perspective, conditional monetary functions have also been studied by Filipovic et al. \cite{Filipovic}. On the other hand, from the regulators point of view, they are only interested in the amount and the...
intensity of the risk, and not in its operational nature, which has motivated the study of
law invariant risk measures [7,8,13]. The analysis of these monetary risk measures that are
continuous from below as well as their representation can be found in [11], for one period
of time.

The problem of identifying the worst case dependence structure of a d-dimensional
portfolio has been analyzed by Ruschendorf [14]. In this paper we relate this concept with
conditional monetary risk measures. Understanding the structure of worst case portfolios
associated to an specific monetary risk measure is important in order to calculate the
aggregated risk of a given portfolio [3,16]. As it is explained by Embrechts et al. [9], typically
only the marginal distributions are available and new techniques have to be developed to
calculate the most conservative values of the associated risk; for an interesting connection
of this problem with the mass transportation problem we refer the reader to [12].

In this work we are interested in measuring the average risk of an n-dimensional vector
of financial positions evolving in time; each financial position is modelled as a stochastic
process. One of our goals is to study properties of portfolios that are worst in the sense of
having maximum risk. From a practical point of view these type of portfolios are important,
since allow us to quantify the riskiest situation for aggregated positions. Our aim is to study
optimal portfolios in aggregated sense, with respect to fixed marginals of the different
stochastic process involved. More precisely, the initial aim of this paper is to describe the
worst case portfolios of insurance versions corresponding to law invariant conditional
monetary utility functions that are continuous for bounded decreasing sequences. In that
direction, we present a new definition of comonotonicity, which is fundamental in the
proof of the main results presented in this paper; this concept was also essential in the
work of Ekeland et al. [8]. The first result that we present, Theorem 3.1, establishes that
in order to study worst case portfolios, we should understand the comonotones portfolios
associated with the worst scenarios of the average risk function. This is a generalization
of Theorem 3.2 in [14]. Then, we make a transition to study the worst case portfolios of
insurance versions of discrete-time conditional monetary utility function processes. Before
proving our main result an invariance property of being worst case portfolio in three-time
steps is established, for certain class of insurance versions associated with time-consistent
monetary utility function processes. The main result of this work, stated as Theorem 3.3,
provides conditions on the insurance version of a monetary utility function process in
order to guarantee that the property of being a worst case portfolio does not change over
time. In the final part of this paper, we are interested in establishing conditions for reaching
the maximum in the dual Fenchel representation formulas for the insurance version of the
monetary utility function; see, for instance, Cheridito et al. [5]. This is used latter within
our study of worst portfolios.

The structure of the paper is the following. Section 2 is dedicated to introduce the
notation and spaces needed throughout the paper, while in Section 3 we begin defining a
new concept of comonotonicity which is based on conditional expectation. Moreover, we
provide a study of some aspects of worst case portfolios of insurance versions of monetary
utility function processes. Although Theorem 3.2 is interesting by itself, the main result
of that section, and of this paper, is Theorem 3.3. Finally, some results related with dual
representations are presented in Section 4, providing conditions to attain the maximum in
such representations.
2. Preliminaries

Throughout the remainder \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{N}}, P)\) is a filtered probability space, with \(\mathcal{F}_0 = \{\emptyset, \Omega\}\). All equalities and inequalities between random variables or stochastic processes are understood in the \(P\) – almost sure sense even without explicit mention. Also, \(T\) is a fixed deterministic finite time horizon in \(\mathbb{N}\). Denoting by \(E\) the expectation operator with respect to \(P\) we introduce the following spaces and operators, which are important to give the precise definitions.

- The space of \(\{\mathcal{F}_t\}\) – adapted stochastic processes is denoted by \(\mathcal{R}^0\), and define the subclass

\[
\mathcal{R}^\infty := \{X \in \mathcal{R}^0 : \|X\|_{\mathcal{R}^\infty} < \infty\},
\]

with \(\|X\|_{\mathcal{R}^\infty} := \inf \{m \in \mathbb{R} : \sup_{t \in \mathbb{N}} |X_t| \leq m\}\).

- \(A^1 := \{a \in \mathcal{R}^0 : \|a\|_{A^1} < \infty\}\), where \(a_{-1} = 0\), \(\Delta a_t := a_t - a_{t-1}\), \(t \in \mathbb{N}\), and \(\|a\|_{A^1} := E \left(\sum_{t \in \mathbb{N}} |\Delta a_t|\right)\).

- \(A^1_+ := \{a \in A^1 : \Delta a_t \geq 0\} \text{ for all } t \in \mathbb{N}\}, \text{ and the bilinear form } \langle \cdot, \cdot \rangle \text{ defined on } \mathcal{R}^\infty \times A^1 \text{ is given by}

\[
\langle X, a \rangle := E \left(\sum_{t \in \mathbb{N}} X_t \Delta a_t\right).
\]

- The space \(\mathcal{R}^\infty\) is endowed with the weak topology \(\sigma (\mathcal{R}^\infty, A^1)\), that makes all the linear functionals \(X \to \langle X, a \rangle\), \(a \in A^1\), continuous, and analogously, \(\sigma (A^1, \mathcal{R}^\infty)\) denotes the weak topology on \(A^1\).

- Given the \(\{\mathcal{F}_t\}\) – stopping times \(\tau\) and \(\theta\) such that \(0 \leq \tau < \infty\) and \(\tau \leq \theta \leq \infty\), the projection \(\pi_{\tau, \theta} : \mathcal{R}^0 \longrightarrow \mathcal{R}^0\) is given by \(\pi_{\tau, \theta} (X)_t := 1_{\{t \leq \tau\}} X_{t \wedge \theta}, t \in \mathbb{N}\). Define the vector space \(\mathcal{R}_{\tau, \theta}^\infty := \pi_{\tau, \theta} (\mathcal{R}^\infty)\).

For \(X \in \mathcal{R}^\infty\) and \(a \in A^1\), let

\[
\langle X, a \rangle_{\tau, \theta} := E \left(\sum_{t \in [\tau, \theta] \cap \mathbb{N}} X_t \Delta a_t | \mathcal{F}_t\right).
\]

- Define the following subsets of \(A^1\):

\[
\pi_{\tau, \theta} A^1, \quad (A^1_{\tau, \theta})_+ := \pi_{\tau, \theta} A^1_+, \quad D_{\tau, \theta} := \{a \in (A^1_{\tau, \theta})_+ : \langle a, 1 \rangle_{\tau, \theta} = 1\}.
\]

Finally, let \(D^c_{\tau, \theta} := \{a \in D_{\tau, \theta} : P \left(\sum_{j \geq \tau \wedge \theta} \Delta a_j > 0\right) = 1 \text{ for all } t \in \mathbb{N}\}\).

Now let us recall the definition of a monetary utility function in the static framework.

**Definition 2.1:** A mapping \(\phi : \mathcal{R}_{\tau, \theta}^\infty \longrightarrow L^\infty (\mathcal{F}_\tau)\) is a monetary utility function on \(\mathcal{R}_{\tau, \theta}^\infty\) if the following three properties hold:

1. \(\phi (1_A X) = 1_A \phi (X)\) for all \(X \in \mathcal{R}_{\tau, \theta}^\infty\) and \(A \in \mathcal{F}_\tau\).
2. \(\phi (X) \leq \phi (Y)\) for all \(X, Y \in \mathcal{R}_{\tau, \theta}^\infty\) such that \(X \leq Y\).
3. \(\phi (X + m 1_{[\tau, \infty)}) = \phi (X) + m\) for all \(X \in \mathcal{R}_{\tau, \theta}^\infty\) and \(m \in L^\infty (\mathcal{F}_\tau)\).

Such a mapping is said to be:

4. Concave if \(\phi (\lambda X + (1 - \lambda) Y) \geq \lambda \phi (X) + (1 - \lambda) \phi (Y)\) for all \(X, Y \in \mathcal{R}_{\tau, \theta}^\infty\) and \(\lambda \in L^\infty (\mathcal{F}_\tau)\) such that \(0 \leq \lambda \leq 1\).
Then the notion of relevance introduced by Follmer and Schied [11] captures the intuitive fact that if $X$ is a non-positive random variable with positive probability of being negative, then the risk of $X$ is higher than that of the position identically zero. The analogous version of this concept for $\mathcal{F}_t$ conditional monetary utility functions was given by Cheridito et al. [5].

Definition 2.2: A mapping $\gamma$ from $D_{t,T}$ to the space of measurable functions $f: (\Omega, \mathcal{F}_t) \rightarrow [-\infty, 0]$ is said to be a penalty function if

$$\text{ess sup}_{a \in D_{t,T}} \gamma(a) = 0.$$ 

Such a function is called local if $\gamma(1_A a + 1_{A^c} b) = 1_A \gamma(a) + 1_{A^c} \gamma(b)$ for all $a, b \in D_{t,T}$ and $A \in \mathcal{F}_t$.

Remark 2.1: If $\phi$ is a monetary utility function, we naturally define the insurance version by $\Psi(\cdot) = -\phi(-\cdot)$. It turns out that some times it is more convenient to work with the insurance version, and we often work with such functions instead of monetary utility functions. On the other hand, considering the negative of a monetary utility function, the result is a mapping that generalizes the original definition of a monetary risk measure [11], namely, the negative of a monetary utility function $\rho(\cdot) = -\phi(\cdot)$ defines a monetary risk measure.

Definition 2.3:

(a) Given $S \in \mathbb{N}$, with $S \leq T$, $(\phi_t)_{t \in [S,T]}$ is a monetary utility process if for each $t \in [S, T] \cap \mathbb{N}$, $\phi_t$ is a monetary utility function on $\mathcal{R}_{\tau,T}$. When the properties of concavity, coherence, decreasing monotonicity or relevancy are satisfied for $\phi_t$, for each $t \in [S, T] \cap \mathbb{N}$, we say that the utility function process $(\phi_t)_{t \in [S,T]}$ is concave, coherent, monotonically decreasing or relevant, respectively. If $\tau$ is an $(\mathcal{F}_t)$ stopping time, with $S \leq \tau \leq T$, we define the mapping $\phi_{\tau,T} : \mathcal{R}_{\tau,T} \rightarrow L^\infty(\mathcal{F}_t)$ by

$$\phi_{\tau,T}(X) := \sum_{t \in [S,T] \cap \mathbb{N}} \phi_t(1_{\{t=\tau\}}X).$$

(b) Such a utility function process $(\phi_t)_{t \in [S,T]}$ is time-consistent if

$$\phi_t(X) = \phi_t(X1_{\{t=\tau\}} + \phi_{\theta,T}(X1_{\{\theta,\infty\}}))$$

for each $t \in [S, T] \cap \mathbb{N}$, every finite $(\mathcal{F}_t)$ stopping time $\theta$ such that $t \leq \theta \leq T$ and all process $X \in \mathcal{R}_{\tau,T}$. 

(5) Coherent if $\phi(\lambda X) = \lambda \phi(X)$ for all $X \in \mathcal{R}_{\tau,T}$ and $\lambda \in L^\infty(\mathcal{F}_t)$.

(6) Continuous for bounded decreasing sequences if $\lim_{n \rightarrow \infty} \phi(X^n) = \phi(X)$ for every decreasing sequence $\{X^n\}_{n \in \mathbb{N}}$ in $\mathcal{R}_{\tau,T}$ and $X \in \mathcal{R}_{\tau,T}$, such that $X^n_T \rightarrow X_t$ for all $t \in \mathbb{N}$.

(7) $\theta$-relevant if $A \subset \{\phi(-\epsilon 1_{(t,\infty]}(\theta)) < 0\}$ for all $\epsilon > 0$, $t \in \mathbb{N}$ and $A \in \mathcal{F}_{t\wedge\theta}$.

The acceptance set of a monetary utility function $\phi$ is defined as

$$\mathcal{C}_\phi = \{X \in \mathcal{R}_{\tau,T} | \phi(X) \geq 0\}.$$
Example 2.1: Given $\alpha > 0$, let $u$ be the exponential utility function

$$u(x) = 1 - \exp(-\alpha x), \quad x \in \mathbb{R}.$$ 

The certainty equivalent of a probability measures of $\mathbb{R}$ or ‘lottery’ $\mu$, is defined as the number $c(\mu)$ for which the identity

$$u(c(\mu)) = U(\mu) := \int u(y)\mu(dy),$$ 

is satisfied.

Let $t \in [0, T] \cap \mathbb{N}$ and define $U_t : L^\infty(\mathcal{F}_T) \rightarrow L^\infty(\mathcal{F}_t)$ as

$$U_t(Y) = E\left(u(Y) \mid \mathcal{F}_t\right).$$

The function $C_t(Y) \in L^\infty(\mathcal{F}_t)$ is named the certainty equivalent of $Y$ at time $t$ if

$$u(C_t(Y)) = U_t(Y).$$

It can be verified that

$$C_t(Y) = -\frac{1}{\alpha} \log E\left[\exp\{-\alpha Y\} \mid \mathcal{F}_t\right].$$

Moreover, the utility function process $(\phi_{t,T})_{t \in [0,T] \cap \mathbb{N}}$ with $\phi_{t,T}(X) := C_t(X_T), X \in \mathcal{R}_{i,T}^\infty$, is time-consistent [5].

3. Worst portfolios of conditional monetary utility functions

In this section our aim is to analyze the relationship between the notions of worst portfolio and conditional utility functions, extending to the dynamic case the results established by Ruschendorf [14] for the static case between risk measures and worst portfolios. Given a risk measure, the study of the joint distribution of vector-valued portfolios evolving in time is crucial within the theory of Financial Mathematics, while in practice, agents are enforced by regulators to quantify the underlying risk associated with their positions, in order to take appropriate decisions. Also, from a different perspective, risk measures are useful to understand the endogenous effect of liquidity risk in economical crisis; see, for instance, the discussion presented by Danielson et al. [7]. Taking decisions using risk measures derives in some cases in the analysis of worst portfolios [13,14]; in order to describe worst d-dimensional portfolios we employ max correlation of conditional monetary utility functionals, using conditional expectation. Recall that there is a bijective relation between utility functions and risk measures (see Remark 2.1), and we shall formulate our results in terms of worst portfolios, which are obtained by quantifying the highest risk that the investor is facing; however, this quantification is restricted to a pre-specified set of marginal risks that are averaged to get the overall risk.

Let $X \in \mathcal{R}_{i,T}^\infty$, and express it as a random vector by $(X_t, \ldots, X_T)$; we write $X \sim X'$ for $X' \in \mathcal{R}_{i,T}^\infty$, when $X$ and $X'$ have the same distribution as random vectors. Moreover, we also define the function $\phi_{i,T}^x$ on $A_{i,T}^1$ given as
\[ \phi_{t,T}^g (a) := \text{ess inf } \inf_{X \in \mathcal{C}_{t,T}} < X, a >_{t,T}, \]

with \( \mathcal{C}_{t,T} \) defined as in (1).

We now introduce the definition of comonotonicity for the dynamic case in an analogous manner to [14] for the static case. Starting with a finite set of stochastic processes \( X^1, \ldots, X^n \in \mathcal{R}_{t,T}^\infty \) which represent the evolution in time of financial values, the approach presented below aims to quantify the worst case performance when conditional maximal correlations between these financial positions evolving in time and conditional densities in the space \( \Omega \times \mathbb{N} \) are computed. Throughout, \( X^1, \ldots, X^n \) remain fixed.

Below we shall show how the problem of calculating worst portfolios in terms of risk measures can be analyzed using comonotonicity.

**Definition 3.1:**

(i) Let \( a \in D_{t,T} \) be fixed, and define the mapping \( \psi_a : \mathcal{R}_{t,T}^\infty \rightarrow L^\infty (\mathcal{F}_t) \) as

\[ \psi_a (\hat{X}) = \text{ess sup } \hat{X}, a >_{t,T}, \hat{X} \in \mathcal{R}_{t,T}^\infty. \]

(ii) The average risk function \( F_{t,T} : D_{t,T} \rightarrow L^\infty (\mathcal{F}_t) \) is defined by

\[ F_{t,T} (a) = \frac{1}{n} \sum_{i=1}^n \psi_a (X^i) + \phi_{t,T}^g (a). \]

(iii) We call \( a^0 \in D_{t,T} \) the worst scenario of \( F_{t,T} \), if

\[ F_{t,T} (a^0) = \text{ess sup } a \in D_{t,T} F_{t,T} (a). \]

(iv) If \( a^0 \in D_{t,T} \) and \( \tilde{X}^i \sim X^i, \tilde{X}^i \in \mathcal{R}_{t,T}^\infty, i = 1, \ldots, n \), we call \( \tilde{X}^1, \ldots, \tilde{X}^n a^0 - \text{comonotone} \) when

\[ \psi_{a^0} (\tilde{X}^i) = < \tilde{X}^i, a^0 >_{t,T}, \]

and

\[ \psi_{a^0} \left( \sum_{i=1}^n \tilde{X}^i \right) = < \sum_{i=1}^n \tilde{X}^i, a^0 >_{t,T}. \]

Note that each function \( \psi_a \) can be interpreted as the maximum correlation between a financial position, constrained to have fixed marginals, and a conditional density \( a \in \Omega \times \mathbb{N} \). Moreover, Part (ii) in Definition 3.1 constructs an average between a maximum correlation function \( \psi_a \) and different positions plus a penalty on the density \( a \). On the other hand, maximizing \( F_{t,T} \) over the set of densities \( D_{t,T} \) resembles the problem of finding a density with maximum correlation with the average of the financial positions. The following remark illustrates the deep connection between the definition of comonotonicity and densities of maximal correlation, with given financial positions.

**Remark 3.1:** Let us illustrate the notion of comonotonicity with a simple example when \( t = 1 \) and \( T = 2 \). Given \( a \in D_{1,2} \), with \( \bar{a} = (\Delta a_1, \Delta a_2) \), choose \( X^0 \in \mathcal{R}_{1,2}^\infty \) such that \( X^0 \)
and $\tilde{a}$ are square integrable random vectors and $E\left[\Delta a_i\right] = \frac{1}{2}$, for $i = 1, 2$. If

$$\Psi_a\left(X^0\right) = \langle X^0, a \rangle > 1, 2,$$

taking expectation and multiplying by two it follows that

$$E\left[X^0 \cdot (2\tilde{a})\right] = \sup_{\tilde{X} \sim X^0} E\left[\tilde{X} \cdot (2\tilde{a})\right].$$

Therefore, from [13],

$$E\left[X^0 \cdot (2\tilde{a})\right] = \sup_{\tilde{X} \sim X^0, U \sim 2\tilde{a}} E\left[\tilde{X} \cdot U\right],$$

and hence

$$E\left[X^0 \cdot \tilde{a}\right] = \sup_{\tilde{X} \sim X^0, U \sim \tilde{a}} E\left[\tilde{X} \cdot U\right].$$

By Theorem 1 in [15], we conclude that $X^0 \in \partial f (\tilde{a})$, for some convex lower semi-continuous function $f$. Hence, if $\tilde{X}^1, \ldots, \tilde{X}^n$ are $a -$ comonotone, then $\sum_{i=1}^n \tilde{X}^i \in \partial f (\tilde{a})$ and $\tilde{X}^i \in \partial f (\tilde{a})$ for $i = 1, \ldots, n$.

Let us recall briefly the main objective of this paper: The aim is to propose natural conditions under which portfolios with maximum risk preserve such property over time. Ekeland, Galichon and Henry [8] considered law invariant, coherent and lower semi-continuous risk measures, and introduced strongly coherent measures. The idea behind these concepts was somehow to prevent unnecessary premium payments to conglomerates as well as to avoid imposing over-conservative rules to the banks. We follow these ideas, extending also the results of Ruschendorf [14], defining worst portfolios as those which maximize the aggregated risk over the set of all possible portfolios with the same marginals. The condition imposed, fixing the marginals, is a natural way of formalizing the notion that we are concerned only with the aggregate risk and not with its nature. With this purpose in mind we present now the definition of worst portfolios.

**Definition 3.2:** Given a monetary utility function $\phi_{t, T}$ and $\tilde{X}^i \in \mathcal{R}^{\infty}_{t, T}$ with $\tilde{X}^i \sim X^i$, $i = 1, \ldots, n$, we say that $(\tilde{X}^1, \ldots, \tilde{X}^n)$ is a worst case portfolio for the associated insurance version $\Psi_{t, T}$ if

$$\text{ess sup}_{\tilde{X}^i \sim X^i} \Psi_{t, T}\left(\frac{1}{n} \sum_{i=1}^n \tilde{X}^i\right) = \Psi_{t, T}\left(\frac{1}{n} \sum_{i=1}^n \tilde{X}^i\right).$$

When $n = 1$ the function $\Psi_{t, T}$ consists in evaluating a single financial position and the statement becomes trivial for law invariant measures, see the definition below. This is the case of the widely used value at risk measure. For $n > 1$, the definition of worst portfolios can be explained as selecting portfolios with maximum risk, recalling that the function $\Psi_{t, T}(- \cdot)$ is a risk measure. Thus, defining $\rho_{t, T} : \mathcal{R}^{\infty}_{t, T} \rightarrow L^{\infty}(\mathcal{F}_t)$ as

$$\rho_{t, T}(X) = \Psi_{t, T}( - X), \text{ for each } X \in \mathcal{R}^{\infty}_{t, T},$$
we have that \((\tilde{X}^1, \ldots, \tilde{X}^n)\) is a worst case portfolio for \(\Psi_{t,T}\) if and only if

\[
\text{ess sup}_{\tilde{X}^i \sim X^i} \rho_{t,T} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i \right) = \rho_{t,T} \left( \frac{1}{n} \sum_{i=1}^{n} -\tilde{X}^i \right).
\]

This means that, among all the portfolios with marginals specified by \(-X^1, \ldots, -X^n\), the portfolio \((-\tilde{X}^1, \ldots, -\tilde{X}^n)\) has the highest risk with respect to the dynamic monetary risk measure \(\rho_{t,T}\).

As it has been already mentioned, it is usual that investors deal with monetary utility functions, or equivalently, monetary risk measures, that depend only on the distribution of the portfolio process. We now present such definition in our context.

**Definition 3.3:** We say that a monetary utility function \(\phi_{t,T}\) is law invariant if for all \(X \sim X'\) for \(X, X' \in \mathcal{R}_{t,T}^{\infty}\) the following holds

\[
\phi_{t,T}(X) = \phi_{t,T}(X').
\]

We are now ready to state one of the main results of this section. This result connects the two fundamental concepts of comonotonicity and worst case portfolios introduced before. In the first part, it is shown that the value of the worst case portfolios is the same as the value of the average risk function, introduced in Definition 3.1, at its worst scenario. As a consequence of this result, the second part states sufficient conditions for being a worst portfolio, in terms of the comonotonicity property.

**Theorem 3.1:** Let \(\phi_{t,T}\) be a law invariant concave monetary utility process such that it is continuous for bounded decreasing sequences, and denote by \(\Psi_{t,T}\) its insurance version. Then, the following properties hold:

(i) \[
\text{ess sup}_{\tilde{X}^i \sim X^i} \Psi_{t,T} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i \right) = \text{ess sup}_{a \in D_{t,T}} F_{t,T}(a).
\]

(ii) If \(\tilde{X}^1, \ldots, \tilde{X}^n\) are \(a^0\) - comonotone, with \(\tilde{X}^i \sim X^i\), and \(a^0\) is a worst scenario of \(F_{t,T}\), then \((\tilde{X}^1, \ldots, \tilde{X}^n)\) is a worst case portfolio of \(\Psi_{t,T}\).

**Proof:** From Theorem 3.16 in [5] the representation

\[
\Psi_{t,T}(X) = \text{ess sup}_{a \in D_{t,T}} \{ < X, a >_{t,T} + \phi_{t,T}^\#(a) \}
\]

holds. This yields

\[
\text{ess sup}_{\tilde{X}^i \sim X^i} \Psi_{t,T} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i \right)
= \text{ess sup}_{\tilde{X}^i \sim X^i} \text{ess sup}_{a \in D_{t,T}} \left\{ < \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i, a >_{t,T} + \phi_{t,T}^\#(a) \right\}
= \text{ess sup}_{a \in D_{t,T}} \text{ess sup}_{\tilde{X}^i \sim X^i} \left\{ < \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i, a >_{t,T} + \phi_{t,T}^\#(a) \right\}
\]
This completes the proof of part (i). From the above argument, we get that

\[
\text{ess sup}_{\tilde{X}^i \sim X^i} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i \right) = F_{t,T}(a^0) = \frac{1}{n} \sum_{i=1}^{n} \text{ess sup}_{\tilde{X}^i \sim X^i} \tilde{X}^i, a^0 > t, T + \phi_{t,T}(a^0).
\]

Since \(\phi_{t,T}\) is law invariant, if \(X \in \mathcal{R}_{t,T}^{\infty}\), then

\[
\Psi_{t,T}(X) = \text{ess sup}_{\tilde{X} \sim X} \Psi_{t,T}(\tilde{X}) = \text{ess sup}_{\tilde{X} \sim X} \text{ess sup}_{a \in D_{t,T}} \left\{ < \tilde{X}, a > t, T + \phi_{t,T}(a) \right\} = \text{ess sup}_{a \in D_{t,T}} \left\{ \Psi_a(X) + \phi_{t,T}(a) \right\}.
\]

The last equation and comonotonicity imply that

\[
\Psi_{t,T} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i \right) \geq \Psi_{a^0} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i \right) + \phi_{t,T}(a^0) = \frac{1}{n} \sum_{i=1}^{n} \Psi_a(\tilde{X}^i) + \phi_{t,T}(a^0) = F_{t,T}(a^0),
\]

which combined with the previous equations yields (ii).

The above theorem transforms the problem of studying the worst case portfolios into one where comonotone portfolios of the worst scenarios are analyzed. Moreover, this result links the problem of studying the worst portfolios in a dynamic context to the theory of optimal coupling, as stated in Remark 3.1.

Let us now give an example of worst case portfolios in a dynamic setting, that is based on an example provided by Ruschendorf and Uckelmann [16]; similar extensions can be done for other examples presented in that paper.

**Example 3.1:** Define \(\phi_{0,T}\) for \(X \in \mathcal{R}_{0,T}^{\infty}\) as

\[
\phi_{0,T}(X) = E \left( 1_{C_1} \sum_{t=0}^{T} X_t \Delta a_t \right).
\]

for some \(a \in A^1\) and \(C_1 \in \mathcal{F}\). Then, we define \(X^i \in \mathcal{R}_{t,T}^{\infty}\) as

\[
X^i = 1_{C_1} B_i \Delta a + a_i.
\]
where $B_i$ is a $T \times T$ positive semidefinite matrix and $a_i \in \mathbb{R}^T$. Here the processes $X^i$ and $\Delta a$ are written as column random vectors in $\mathbb{R}^T$. Then, from [16], we obtain that $(X^1, \ldots, X^n)$ is a worst case portfolio of $\Psi_{0,T}$. Next we can go one step further, considering $\mathcal{F}_1 = \sigma \left( C_1^1, \ldots, C_{m_1}^1 \right)$ where $C_1^1, \ldots, C_{m_1}^1$ is a partition of $\Omega$. We define $\phi_{1,T}$ as

$$
\phi_{1,T} (X) = 1_{C_1} E \left( \sum_{t=1}^{T} X_t \Delta a_t | \mathcal{F}_1 \right),
$$

with $C_1 = C_1^1$. It is not difficult to see [16] that defining $X_{s,i}^{i,1} \in \mathcal{R}_{1,t}^\infty$ as $X_{s,i}^{i,1} := X_{s,i}$ for $s = 1, \ldots, T$ and $i = 1, \ldots, n$, we obtain that $(X^{1,1}, X^{2,1}, \ldots, X^{n,1})$ is a worst case portfolio of $\Psi_{1,t}$.

This construction can continue, with $\mathcal{F}_2 = \sigma \left( C_1^2, \ldots, C_{m_2}^2 \right)$ such that $\{C_1^1, \ldots, C_{m_2}^2\}$ is a partition of $\Omega$, and

$$
\phi_{2,T} (X) = 1_{A_2} E \left( \sum_{t=2}^{T} X_t \Delta a_t | \mathcal{F}_2 \right),
$$

with $C_2 = C_1$. And just as in the previous period of time $t = 1$, one can verify that $(X^{1,2}, X^{2,2}, \ldots, X^{n,2})$ is a worst case portfolio of $\Psi_{2,t}$ where $X_{s,i}^{i,1} := X_{s,i}$, for $s = 2, \ldots, T$ and $i = 1, \ldots, n$. Finally, proceeding by induction, we obtain that the restriction of $(X^{1}, \ldots, X^{n})$ to $\mathcal{R}_{t,T}^\infty$ is a worst case portfolio of $\Psi_{t,T}$ for all $t = 0, \ldots, T$.

Before presenting the main result of this section, we illustrate at smaller scale how the property of time consistency and the notion of worst case portfolios are related. In particular, Theorem 3.2 provides sufficient conditions to ensure that the worst case portfolios at time zero will also be worst case portfolios at time one. This is an important fact from the point of view of a practitioner, since at time zero it would allow him/her to know the worst scenario at the next time step.

**Theorem 3.2:** Let $(\phi_{t,T})_{t \in [0,2] \cap \mathbb{N}}$ be a time-consistent monetary utility function process such that the insurance version at the initial time satisfies

$$
\Psi_{0,2} (X) = \max_{Q \in \mathcal{Q}} \left\{ E_Q [X] - \alpha (Q) \right\},
$$

for all $X \in \mathcal{R}_{0,2}^\infty$ with $E_Q [X] := \sum_{t=0}^{2} E_{Q_t} [X_t]$ for all $Q \in \mathcal{Q} \subset \{(Q_1, Q_2, Q_3) \mid Q_1 \in \mathbb{R}^+, Q_2, Q_3 \in \mathcal{M}_p \}$, where $\mathcal{M}_p$ is the set of finitely additive measures equivalent to $P$. If $(\hat{X}^1, \ldots, \hat{X}^n)$ is a worst portfolio of $\Psi_{0,2}$ and $(\hat{\hat{X}}^1, \ldots, \hat{\hat{X}}^n)$ is a worst portfolio of $\Psi_{1,2}$ such that

$$
\hat{X}^i \sim \hat{\hat{X}}^i := \left( \hat{X}_0^i, \hat{X}_1^i, \hat{X}_2^i \right), \quad i = 1, \ldots, n,
$$

where $\sim$ denotes equality in distribution of vectors, then, $(\hat{X}^1, \ldots, \hat{X}^n)$ is a worst portfolio of $\Psi_{1,2}$.

**Proof:** Let $A \in \mathcal{F}_1$ be the event where

$$
\Psi_{1,2} \left( \frac{1}{n} \sum_{i=1}^{n} \hat{X}^i \right) < \Psi_{1,2} \left( \frac{1}{n} \sum_{i=1}^{n} \hat{\hat{X}}^i \right),
$$

and
and let us assume that $P(A) > 0$. Define the processes $\tilde{Y}$ and $\hat{Y}$ by

$$
\tilde{Y} = \left(\frac{1}{n}\sum_{i=1}^{n} \bar{X}^i\right) 1_{\{0\}} + \Psi_{1,2} \left(\frac{1}{n}\sum_{i=1}^{n} \bar{X}^i\right) 1_{[1,\infty)},
$$

$$
\hat{Y} = \left(\frac{1}{n}\sum_{i=1}^{n} \hat{X}^i\right) 1_{\{0\}} + \Psi_{1,2} \left(\frac{1}{n}\sum_{i=1}^{n} \hat{X}^i\right) 1_{[1,\infty)}.
$$

Clearly $\tilde{Y} \leq \hat{Y}$, and for $t \geq 1$

$$
\tilde{Y}_t < \hat{Y}_t, \quad \text{in } A.
$$

By hypothesis we can choose $Q^0 \in Q$ such that

$$
\Psi_{0,2} (\tilde{Y}) = E_{Q^0} [\tilde{Y}] - \alpha (Q^0).
$$

Since $P(A) > 0$,

$$
\Psi_{0,2} (\tilde{Y}) < E_{Q^0} [\tilde{Y}] - \alpha (Q^0) \leq \Psi_{0,2} (\hat{Y}).
$$

Finally, time-consistency and the last inequality imply that

$$
\Psi_{0,2} \left(\frac{1}{n}\sum_{i=1}^{n} \bar{X}^i\right) < \Psi_{0,2} \left(\frac{1}{n}\sum_{i=1}^{n} \hat{X}^i\right),
$$

which is a contradiction.

\[\square\]

**Remark 3.2:** Theorem 3.2 provides insight about the requirements in order that the property of being a worst portfolio is preserved in two stages. In fact, the key assumption is that the insurance version is a penalized expectation over a discrete set of lotteries. However, the idea of using a dynamic programming type of argument seems to be a natural venue to provide a more general result. In order to accomplish this aim, we shall modify accordingly the hypothesis on the discrete set of lotteries, which appear on the representation of the monetary utility function.

Now we move our analysis to temporal relations for a larger time horizon. With this objective in mind, we introduce before some relevant concepts.

**Definition 3.4:** Let $(\phi_{t,T})_{t \in [0,T]}$ be a monetary utility function process. We call $\{(X^{t,1}, \ldots, X^{t,n})\}_{t \in [0,T]}$ an adapted worst portfolio process for the insurance version $\Psi_{t,T}$ if $(X^{t,1}, \ldots, X^{t,n})$ is a worst portfolio of $\Psi_{t,T}$ for each $t = 0, \ldots, T$, and

$$
X^{t,i} \sim (X^{t,i}_t, X^{t+1,i}_{t+1}, \ldots, X^{t+1,i}_T), \quad i = 1, \ldots, n,
$$

for $t = 0, \ldots, T - 1$. Notice that, abusing of the notation, we are denoting by $X^{t,i}$ the process starting at time $t$ given by $(X^{t,i}_t, \ldots, X^{t,i}_T)$. 
Definition 3.4 can be explained as follows. An adapted worst portfolio process consists of worst case portfolios at each period of time such that, when any of the process at a given period of time \( t \) is constrained to the subsequent period \( t + 1 \), then it has the same marginals as the worst case portfolio process at time \( t \).

Next, we illustrate with an example the connection between adapted worst case portfolio process and time consistency. This will be put in a more broad perspective in the rest of this section.

**Example 3.2:** Let \( \left( \phi_{t, T} \right)_{t \in [0, T] \cap \mathbb{N}} \) be the utility function process described in Example 2.1. Given \( \{ (X_t^{s, 1}, \ldots, X_t^{s, n}) \}_{t \in [0, T] \cap \mathbb{N}} \) an adapted worst portfolio process of the respective insurance version process \( \left( \Psi_{t, T} \right)_{t \in [0, T] \cap \mathbb{N}} \), we claim that for each \( t_0 \in [0, T] \cap \mathbb{N} \) fixed, \( (X_t^{t_0, 1}, \ldots, X_t^{t_0, n}) \) is the worst portfolio of \( \Psi_{t, T} \), for \( t = t_0, \ldots, T \). To verify this, we just have to prove that \( (X_t^{t_0, 1}, \ldots, X_t^{t_0, n}) \) is the worst portfolio of \( \Psi_{t_0 + 1, T} \). By hypothesis, we know that

\[
\Psi_{t_0 + 1, T} \left( \frac{1}{n} \sum_{i=1}^{n} X_{t_0}^{i,i} \right) \leq \Psi_{t_0 + 1, T} \left( \frac{1}{n} \sum_{i=1}^{n} X_{t_0 + 1, i} \right),
\]

where \( X_{t_0 + 1, i} \in \mathcal{R}_{t_0, T}^{\infty}, i = 1, \ldots, n \), with \( X_{t_0 + 1, i} = X_{t_0}^{i,i} \), and \( X_{t_0 + 1, i} = X_{t_0 + 1, i} \), for \( s = t_0 + 1, \ldots, T \). Moreover, time-consistency and the definition of an adapted worst portfolio process yields

\[
\frac{1}{\alpha} \log E \left( \exp \left( \alpha \Psi_{t_0 + 1, T} \left( \frac{1}{n} \sum_{i=1}^{n} X_{t_0}^{i,i} \right) \right) \mid \mathcal{F}_{t_0} \right) = \Psi_{t_0, T} \left( \frac{1}{n} \sum_{i=1}^{n} X_{t_0}^{i,i} \right)
\]

From the last two inequalities we conclude that \( (X_t^{t_0, 1}, \ldots, X_t^{t_0, n}) \) is a worst portfolio of \( \Psi_{t_0 + 1, T} \) and the result follows.

We are now ready to present the main result of this paper. It puts on firm ground the intuitive idea that a worst case portfolio can be preserved over time, under suitable conditions. This implies that an agent will face the maximum aggregated risk across time given by the same portfolios. This has the following important financial implication: In order to find a worst case portfolio, it is sufficient to do it at the very beginning. Moreover, it is proved that there is strong connection between worst case portfolios across time and the temporal notion of time consistency, which can be interpreted as a dynamic programming condition [5]. Notice that this connection between dynamic programming and time consistency of dynamic risk measures has been explored already in different contexts, and next we show some other useful implications of this relationship.

**Theorem 3.3:** For each \( s \in [0, T] \cap \mathbb{N} \), let \( Q_s \subset D_{s,T}^\infty \) be a convex set with \( \Delta a_k \leq \Delta b_k \), \( \varepsilon_s \leq \sum_{j=1}^{T} \Delta a_j \), for all \( a \in Q_s \), \( k \in \mathbb{N} \), with \( b \in \mathcal{A}^1 \) and \( \varepsilon_s \in L^\infty_s (\mathcal{F}_s) \) such that \( P ( \varepsilon_s > 0 ) = 1 \). Now, we define \( \Psi_{s, T} ( \cdot ) = -\phi_{s, T} ( -\cdot ) \) and assume that \( (\phi_{s, T})_{s \in [0, T] \cap \mathbb{N}} \) is a time-consistent monetary utility function process, such that

\[
\phi_{s, T} (X) = \text{ess inf}_{a \in Q_s} \{ -X, a \succ_s T - \phi_{s, T}^s (a) \}, \quad X \in \mathcal{R}_{s,T}^{\infty},
\]

(2)
for all $s = 0, 1, \ldots, T$. If \( \{ (X^{s, 1}, \ldots, X^{s, n}) \}_{s \in [0, T]} \) is an adapted worst portfolio process of \( \{ \Psi_s, T \}_{s \in [0, T]} \), then \( (X^{0, 1}, \ldots, X^{0, n}) \) is a worst case portfolio of \( \Psi_{t, T} \), for all \( 0 \leq t \leq T \); see Definition 3.2.

**Proof:** It is enough to prove that the worst case portfolio at the time $t$ is also a worst case portfolio at time $t + 1$. Let $X \in \mathcal{R}^\infty_{0, T}$ be fixed. Choose $a, b \in \mathcal{Q}_t$ arbitrary and define $d = a1_U + b1_{U^c} \in \mathcal{Q}_t$, with $U$ defined by

\[
U := \{ < X, a >_{t, T} + \phi^a_{t, T} (a) > < X, b >_{t, T} + \phi^b_{t, T} (b) \} \in \mathcal{F}_t.
\]

Therefore,

\[
\max \{ < X, a >_{t, T} + \phi^a_{t, T} (a), < X, b >_{t, T} + \phi^b_{t, T} (b) \} = < X, d > + \phi^d_{t, T} (d),
\]

which implies that \( \{ < X, a >_{t, T} + \phi^a_{t, T} (a) \mid a \in \mathcal{Q}_t \} \) is directed upwards, and hence there exists a sequence \( \{ a^k \}_{k \in \mathbb{N}} \subset \mathcal{Q}_t \) such that

\[
\lim_{k \to \infty} \{ < X, a^k >_{t, T} + \phi^{a^k}_{t, T} (a^k) \} \uparrow \Psi_{t, T} (X).
\]

Now, define the event $A = \{ \Psi_{t+1, T} \left( \frac{1}{n} \sum_{i=1}^{n} X^{t+1, i} \right) < \Psi_{t+1, T} \left( \frac{1}{n} \sum_{i=1}^{n} X^{t+1, i} \right) \}$, and we shall assume that $P[A] > 0$. Define the processes $\bar{X}^t, \bar{X}^{t+1} \in \mathcal{R}^\infty_{t, T}$ as

\[
\bar{X}^t_s = \frac{1}{n} \sum_{i=1}^{n} X^{t, i}_s, \quad t \leq s \leq T
\]

\[
\bar{X}^{t+1}_s = \frac{1}{n} \sum_{i=1}^{n} X^{t+1, i}_s, \quad t + 1 \leq s \leq T
\]

\[
\bar{X}^{t+1}_t = \frac{1}{n} \sum_{i=1}^{n} X^{t, i}_t.
\]

Time-consistency and the previous arguments imply that there is a sequence \( \{ a^k \}_{k \in \mathbb{N}} \subset \mathcal{Q}_t \) such that

\[
\Psi_{t, T} (\bar{X}^t) = \Psi_{t, T} \left( (\bar{X}^t) 1_{\{t\}} + \Psi_{t+1, T} (\bar{X}^t) 1_{\{t+1, \infty\}} \right)
\]

\[
= \lim_{k \to \infty} \left\{ < (\bar{X}^t) 1_{\{t\}} + \Psi_{t+1, T} (\bar{X}^t) 1_{\{t+1, \infty\}}, a^k >_{t, T} + \phi^{a^k}_{t, T} (a^k) \right\}.
\]

Defining $Y = (\bar{X}^t) 1_{\{t\}} + \Psi_{t+1, T} (\bar{X}^t) 1_{\{t+1, \infty\}}$, for $k \in \mathbb{N}$ it follows that

\[
< Y, a^k >_{t, T} = \mathbb{E} \left( \bar{X}^t_t \Delta a^k_t + \Psi_{t+1, T} (\bar{X}^t) \left( \sum_{j=t+1}^{T} \Delta a^k_j \right) \mid \mathcal{F}_t \right).
\]
Clearly,
\[ Y^k_t := \tilde{X}^t \triangle a^k_t + \Psi_{t+1,T} (\tilde{X}^t) \left( \sum_{j=t+1}^{T} \Delta a^k_j \right) \]
\[ \leq \tilde{X}^{t+1} \triangle a^k_t + \Psi_{t+1,T} (\tilde{X}^{t+1}) \left( \sum_{j=t+1}^{T} \Delta a^k_j \right) =: Y^k_{t+1}. \]

For each \( k \in \mathbb{N} \) denote by \( C_k \) the set \( \{ \sum_{j=t+1}^{T} \Delta a^k_j > 0 \} \). Observe that, since \( a^k \in D^c_{0,T} \), we have that \( P(C_k) = 1 \), and hence the set \( C = \bigcap_{k \in \mathbb{N}} C_k \) is such that \( P(C) = 1 \). This implies that \( P(C \cap A) = P(A) > 0 \). By the definitions given above,
\[ \{ Y^k_t < Y^k_{t+1} \} = C_k \cap A, \quad \text{for all } k \in \mathbb{N}. \]

Let \( D := \bigcap_{k \in \mathbb{N}} B^c_k \), with \( B_k := E[Y^k_t | \mathcal{F}_t] = E[Y^k_{t+1} | \mathcal{F}_t] \) \( \in \mathcal{F}_t \), and observe that
\[ 0 = \int_{B_k} \left( E(Y^k_{t+1} | \mathcal{F}_t) - E(Y^k_t | \mathcal{F}_t) \right) dP \]
\[ = \int_{B_k} \left( Y^k_{t+1} - Y^k_t \right) dP \]
\[ = \int_{B_k \cap \{ Y^k_{t+1} > Y^k_t \}} \left( Y^k_{t+1} - Y^k_t \right) dP. \]

Then,
\[ P(B_k \cap C \cap A) = P(B_k \cap A) = P(B_k \cap C_k \cap A) = 0, \]
and,
\[ P(D^c \cap A) = P(D^c \cap C \cap A) = 0. \]

On the other hand, by time-consistency and hypothesis, we deduce that
\[ \Psi_{t,T} (\tilde{X}^t) = \lim_{k \to \infty} \{ E[Y^k_t | \mathcal{F}_t] + \phi^*_{t,T} (a^k) \} \]
\[ = \lim_{k \to \infty} \{ E[Y^k_{t+1} | \mathcal{F}_t] + \phi^*_{t,T} (a^k) \} \]
\[ = \Psi_{t,T} (\tilde{X}^{t+1}), \]
and hence
\[ \lim_{k \to \infty} | \left( E[Y^k_t | \mathcal{F}_t] + \phi^*_{t,T} (a^k) \right) - \left( E[Y^k_{t+1} | \mathcal{F}_t] + \phi^*_{t,T} (a^k) \right) | = 0. \]

Therefore,
\[ \lim_{k \to \infty} 1_D E \left[ \left( \sum_{j=t+1}^{T} \Delta a^k_j \right) (\Psi_{t+1,T} (\tilde{X}^{t+1}) - \Psi_{t+1,T} (\tilde{X}^t)) | \mathcal{F}_t \right] = 0, \]
which implies
\[
\lim_{k \to \infty} E \left[ 1_D \left( \sum_{j=t+1}^{T} \Delta a_j^k \right) \left( \Psi_{t+1,T} \left( \bar{X}_{t+1} \right) - \Psi_{t+1,T} \left( \bar{X}_t \right) \right) \right] = 0,
\]

since \( \left( \sum_{j=t+1}^{T} \Delta a_j^k \right) \leq \left( \sum_{j=t+1}^{T} \Delta b_j^k \right) \) for each \( k \in \mathbb{N} \). Thus,
\[
\lim_{k \to \infty} E \left[ 1_D \cap A \cap C \left( \sum_{j=t+1}^{T} \Delta a_j^k \right) \left( \Psi_{t+1,T} \left( \bar{X}_{t+1} \right) - \Psi_{t+1,T} \left( \bar{X}_t \right) \right) \right] = 0.
\]

Taking a subsequence if necessary, we conclude that
\[
\varepsilon_1 A \cap D \cap C \leq \lim_{k \to \infty} 1_D \cap A \cap C \sum_{j=t+1}^{T} \Delta a_j^k = 0,
\]
which is a contradiction, since \( P \left( A \cap D \cap C \cap \{ \varepsilon_s > 0 \} \right) = P \left( A \cap D \right) = P \left( A \right) > 0 \).

**Remark 3.3:**

1. The main hypothesis in Theorem 3.3 is concerned with the constraint on the set \( Q_s \) appearing in the dual representation (2), which should be bounded both from above and below. Observe that the first of this conditions, concerning the existence of \( b \in A^1 \) satisfying \( \Delta a_k \leq \Delta b_k \) for all \( a \in Q_s \), is not so restrictive, since the set \( Q_s \) is a subset of the class of positive conditional densities \( D_{s,T}^e \), while the set \( A^1 \) encompasses all the possible normalized conditional measures to the space \( \mathcal{R}^\infty \). Moreover, recall that when \( a \in D_{s,T}^e \), by our definition in the previous section, we have
\[
P \left( \sum_{j=s}^{T} \Delta a_j > 0 \right) = 1.
\]

Hence, we are asking that the increments of the elements of \( Q_s \) are uniformly away from zero, ensuring that there is a positive effect at each time step, along the evolution of the process.

2. It is worth noting the connection of Theorem 3.3 with the necessary conditions for time consistency presented in [5, Theorem 4.19]. Following the proof of such result, we obtain that for any \( t \) and \( s \) with \( 0 \leq t \leq s \leq T \), the penalty function \( \phi_{t,s}^a \) satisfies
\[
\phi_{t,s}^a (a) = \text{ess sup}_{b \in Q_s} \phi_{t,s}^a (a \oplus b) + E \left[ \phi_{s,T}^a (a) | \mathcal{F}_t \right], \quad \forall a \in Q_t.
\]

Here the expression \( a \oplus b \) refers to the concatenation of processes, presented below as Definition 4.1 in the next section. The above display can be thought as the time consistency property for the underlying process corresponding to the penalties.

3. The above theorems generalized the comonotonicity results from [14] to the dynamic framework. As a general conclusion, we can say that the property of comonotonicity seems to be the next step to characterize worst case portfolios.
Summarizing, we have shown in Theorem 3.1 that the problem of worst case portfolios can be translated into a problem of comonotonicity, while Theorem 3.3 presents a setting under which worst case portfolios can be time invariant.

While the conditions in Theorem 3.3 are sufficient to ensure the preservation of worst case portfolios as time evolves, it will be shown below that such conditions are not necessary. This is illustrated by considering a particular class of dynamic monetary utility functions with their corresponding insurance versions. Before such example is presented we introduce a technical definition.

**Definition 3.5:** Let \( D^{rel}_T \) be the class

\[
D^{rel}_T = \{ h \in L^1 (\mathcal{F}_T) \mid h > 0, E(h) = 1 \}.
\]

For \( f, g \in D^{rel}_T \), \( s \in [0, T] \cap \mathbb{N} \) and \( A \in \mathcal{F}_s \), we define the pasting \( f \otimes^s_A g \) by

\[
f \otimes^s_A g := \begin{cases} f & \text{on } A^c \cup \{ E(g \mid \mathcal{F}_s) = 0 \} \\ E[f \mid \mathcal{F}_s] \frac{g}{E[g \mid \mathcal{F}_s]} & \text{on } A \cap \{ E(g \mid \mathcal{F}_s) > 0 \} \end{cases}.
\]

A subset \( \mathcal{P} \) of \( D^{rel}_T \) is m1-stable if it contains \( f \otimes^s_A g \) for all \( f, g \in \mathcal{P} \), every \( s \in [0, T] \cap \mathbb{N} \) and \( A \in \mathcal{F}_s \).

We conclude this section with an example that leaves as an open question the problem of finding necessary and sufficient conditions on an insurance version process to have the same worst case portfolios across time. Thus, although the conditions in Theorem 3.3 ensure this property, we proceed to illustrate that the same conclusions can also be achieved with another set of assumptions. Within a more general framework, we shall see that using the concept of stability under concatenation within subsets of the density processes \( D^{rel}_{0,T} \) we can achieve this property for a certain class of utility function processes; see Theorem 4.2.

**Example 3.3:** Let \( \mathcal{P} \) be a non-empty m1-stable subset of \( D^{rel}_T \) and \( \alpha > 0 \). For \( t = 0, \ldots, T \) and \( X \in \mathcal{R}^\infty_{t,T} \), define

\[
\phi_{t,T}(X) := \text{ess inf}_{f \in \mathcal{P}} \left\{ \frac{-1}{\alpha} \log \frac{E(f \exp(-\alpha X_T) \mid \mathcal{F}_t)}{E(f \mid \mathcal{F}_t)} \right\}, \quad X \in \mathcal{R}^\infty_{0,T}.
\]

This is a time-consistent utility function process [5], which is a robust version of the utility function process given in Example 2.1. We consider the insurance version process \( \Psi_{t,T}(\cdot) = -\phi_{t,T}(\cdot) \). With this example we intend to illustrate that, even without the representation in the hypothesis of Theorem 3.3, it is possible to obtain the same conclusion. Namely, if \( \{(X^{t,1}, \ldots, X^{t,n}) \}_{t \in [0,T] \cap \mathbb{N}} \) is an adapted worst case portfolio process of \( (\Psi_{t,T})_{t \in [0,T] \cap \mathbb{N}} \), then \( (X^{0,1}, \ldots, X^{0,n}) \) is a worst case portfolio of \( \Psi_{t,T} \) for all \( 0 \leq t \leq T \). To this end for \( r \in \mathbb{N} \) we introduce the notation

\[
f_r = \frac{f}{E(f \mid \mathcal{F}_r)}
\]
and fix \( t_0 \in \{0, 1, \ldots, T\} \). Given \( X \in \mathcal{R}_{0, T}^{\infty} \), there exists a sequence, see Section 5.6 in [5], \( \{f^k\}_{k \in \mathbb{N}} \subset D_{T}^{rel} \) such that

\[
\left\{ \frac{1}{\alpha} \log E \left( f^k_{t_0} \exp (\alpha X_T) \mid F_{t_0} \right) \right\} \not\rightarrow \Psi_{t_0, T}(X), \quad \text{as } k \longrightarrow \infty,
\]

which implies

\[
\left\{ E \left( f^k_{t_0} \exp (\alpha X_T) \mid F_{t_0} \right) \right\} \not\rightarrow \exp \left( \alpha \Psi_{t_0, T}(X) \right), \quad \text{as } k \longrightarrow \infty.
\]

Now we follow the same lines as in the proof of Theorem 3.3. First, define the event

\[
A = \left\{ \Psi_{t_0+1, T} \left( \frac{1}{n} \sum_{i=1}^{n} X^{t_0,i} \right) < \Psi_{t_0+1, T} \left( \frac{1}{n} \sum_{i=1}^{n} X^{t_0+1,i} \right) \right\},
\]

and assume that \( P[A] > 0 \). The processes \( \tilde{X}^{t_0}, \tilde{X}^{t_0+1} \in \mathcal{R}_{t_0, T}^{\infty} \) are defined so as \( \tilde{X}^t, \tilde{X}^{t+1} \) in the proof of Theorem 3.3. Time-consistency and the previous arguments imply that there is a sequence \( \{a^k\}_{k \in \mathbb{N}} \subset D_{T}^{rel} \) such that

\[
\exp \left( \alpha \Psi_{t_0, T} \left( \tilde{X}^{t_0} \right) \right) \not\leq \lim_{k \to \infty} \left\{ E \left( a^k_{t_0} \exp \left( \alpha \Psi_{t_0+1, T} \left( \tilde{X}^{t_0+1} \right) \right) \mid F_{t_0} \right) \right\}
\]

\[
= \lim_{k \to \infty} \left\{ E \left( a^k_{t_0} \exp \left( \alpha \Psi_{t_0+1, T} \left( \tilde{X}^{t_0+1} \right) \right) \mid F_{t_0} \right) \right\}
\]

\[
= \exp \left( \alpha \Psi_{t_0, T} \left( \tilde{X}^{t_0+1} \right) \right).
\]

Therefore,

\[
\lim_{k \to \infty} \left\{ E \left( a^k_{t_0} \exp \left( \alpha \Psi_{t_0+1, T} \left( \tilde{X}^{t_0+1} \right) \right) \mid F_{t_0} \right) - E \left( a^k_{t_0} \exp \left( \alpha \Psi_{t_0+1, T} \left( \tilde{X}^{t_0} \right) \right) \mid F_{t_0} \right) \right\} = 0.
\]

Since for all \( k \in \mathbb{N} \),

\[
E \left( a^1_{t_0} \exp \left( \alpha \Psi_{t_0+1, T} \left( \tilde{X}^{t_0} \right) \right) \mid F_{t_0} \right) \leq E \left( a^k_{t_0} \exp \left( \alpha \Psi_{t_0+1, T} \left( \tilde{X}^{t_0} \right) \right) \mid F_{t_0} \right)
\]

\[
\leq E \left( a^k_{t_0} \exp \left( \alpha \Psi_{t_0+1, T} \left( \tilde{X}^{t_0+1} \right) \right) \mid F_{t_0} \right)
\]

\[
\leq \exp \left( \alpha \Psi_{t_0, T} \left( \tilde{X}^{t_0+1} \right) \right),
\]

we have that

\[
\lim_{k \to \infty} \left\{ E \left( a^k_{t_0} \exp \left( \alpha \Psi_{t_0+1, T} \left( \tilde{X}^{t_0+1} \right) \right) \mid F_{t_0} \right) - a^k_{t_0} \exp \left( \alpha \Psi_{t_0+1, T} \left( \tilde{X}^{t_0} \right) \right) \right\} = 0,
\]

and

\[
\lim_{k \to \infty} \left\{ E \left( \exp \left( \alpha \Psi_{t_0, T} \left( \tilde{X}^{t_0} \right) \right) \mid F_{t_0} \right) - a^k_{t_0} \exp \left( \alpha \Psi_{t_0+1, T} \left( \tilde{X}^{t_0} \right) \right) \right\} = 0 \right\}.\]
By the last two identities we conclude that there is a subsequence \( \{a^k_i\}_{i \in \mathbb{N}} \subset \{a^k\}_{k \in \mathbb{N}} \) for which

\[
\lim_{i \to \infty} a^k_i = 0 \quad \text{on} \quad A, \quad \text{and} \quad \lim_{i \to \infty} a^k_i \exp \left( \alpha \Psi_{t_0+1,T} \left( \bar{X}^{t_0} \right) \right) = \exp \left( \alpha \Psi_{t_0,T} \left( \bar{X}^{t_0} \right) \right).
\]

We conclude that \( P \left( \exp \left( \alpha \Psi_{t_0,T} \left( \bar{X}^{t_0} \right) \right) = 0 \right) > 0 \), but this is a contradiction.

4. Optimization in representation of insurance version functions

The results presented in the previous section are based on representation of conditional utility functions in terms of penalty functions, which can be thought as Fenchel conjugate representations; see (2). Now we revisit this type of representations and analyze conditions under which the maximum is attained. Using this representations, we will show a different venue, to Theorem 3.3, in order to attain the property of stability of worst case portfolios of time consistent insurance versions of utility processes. The main difference with Theorem 3.3 is that in this section we are concerned with insurance versions associated with coherent, relevant and time-consistent monetary utility function process. Such additional constraints will allow us to relax some of the assumptions of Theorem 3.3. Furthermore, we will also obtain as byproduct some properties in financial risk that can be deduced when the maximum in the Fenchel conjugate representations of conditional utility functions is attained.

We begin by defining the concatenation of density processes from a given period of time to another. This condition was introduced originally by Cheridito et al. [5]; see for instance their Theorem 3.16.

**Definition 4.1:** Let \( a, b \in A^1_+, \theta \) a finite \((\mathcal{F}_t)\) – stopping time and \( A \in \mathcal{F}_\theta \). Then the concatenation \( a \oplus^\theta_A b \) is defined by

\[
(a \oplus^\theta_A b)_t := 1_C a_t + 1_{C^c} \left( a_{\theta-1} + \min \{ 1, a_{\theta,0} \leq b_t - b_{\theta-1} \} \right),
\]

with \( C = \{ t < \theta \} \cup A^c \cup \{ < 1, b > \theta, \infty = 0 \} \). A subset \( \mathcal{M} \) of \( A^1_+ \) is said to be stable under concatenation if \( a \oplus^\theta_A b \in \mathcal{M} \) for all \( a, b \in \mathcal{M} \), each \((\mathcal{F}_t)\) – stopping time \( \theta \), and any \( A \in \mathcal{F}_\theta \).

The next theorem provides topological conditions on the set involved in the representation theorem to achieve the maximum. The main assumption is concerned with the compactness of the representation set with respect to the norm \( \| \cdot \|_{A^1} \). Although it is a restrictive assumption, we impose it for technical reasons which we were not able to overcome.

**Theorem 4.1:** For each \( X \in \mathcal{R}^{s,T} \), let \( \phi \left( X \right) = \text{ess inf}_{a \in \mathcal{M} \subset D_{s,T}} \left\{ < X, a > s,T - \phi \left( a \right) \right\} \) be a monetary utility function with \( s \in \mathbb{N}, s \leq T \). If \( \mathcal{M} \) is \( \| \cdot \|_{A^1} \)-compact, stable under concatenation, with \( \Delta a_t \leq \Delta b_t \) for all \( a \in \mathcal{M}, t \in \mathbb{N}, \) for some \( b \in A^1_+ \), then, the insurance version \( \Psi \left( \cdot \right) := -\phi \left( -\left( \cdot \right) \right) \) satisfies

\[
\Psi \left( X \right) = < X, a^X > s,T + \phi \left( a^X \right),
\]

where \( a^X \in \mathcal{M} \).
with \( a^X \in \mathcal{M} \) depending of \( X \).

**Proof:** Given \( X \in \mathcal{R}^\infty_{s,T} \), let \( \{a^k\} \) be a sequence in \( \mathcal{M} \) such that \( a^k \to \| \cdot \|_{A^1} a^* \), with \( a^* \in \mathcal{M} \). Passing to some subsequence if necessary, we have that

\[
< X, a^k >_{s,T} \to < X, a^* >_{s,T}.
\]

Since the last identity holds for all \( X \in \mathcal{R}^\infty_{s,T} \), it follows that

\[
\lim_{k \to \infty} \phi^\#(a^k) \leq \phi^\#(a^*),
\]

and hence

\[
\lim_{k \to \infty} (< X, a^k >_{s,T} + \phi^\#(a^k)) \leq < X, a^* >_{s,T} + \phi^\#(a^*).
\]

Now we prove that \( \{ < X, a >_{s,T} + \phi^\#(a) \mid a \in \mathcal{M} \} \) is directed upwards, which together with the previous arguments yield the result. Given \( b, c \in \mathcal{M} \), from the stability under concatenation of \( \mathcal{M} \), \( d = b \oplus_A c \in \mathcal{M} \), with \( A \) given by

\[
A = \left\{ < X, c >_{s,T} + \phi^\#(c) > < X, b >_{s,T} + \phi^\#(b) \right\},
\]

and

\[
d_r = b_r 1_{A \cap [r \leq s]} + (b_{s-1} + (c_r - c_{s-1})) 1_{A \cap [r \geq s]}.
\]

This implies that

\[
< X, d >_{s,T} + \phi^\#(d) = 1_{A^c} < X, b >_{s,T} + 1_A < X, c >_{s,T} + \phi^\#(1_{A^c} b + 1_A c) \\
= 1_{A^c} (< X, b >_{s,T} + \phi^\#(b)) + 1_A (< X, c >_{s,T} + \phi^\#(c)) \\
= \max \left\{ < X, b >_{s,T} + \phi^\#(b), < X, c >_{s,T} + \phi^\#(c) \right\},
\]

and the theorem follows. \( \square \)

Our next result is an application of Theorem 4.1, and shows that worst case portfolios do not necessary consists in imposing constrains in the marginal distributions. The aim is to illustrate that the general mathematical properties of worst case portfolios can be analyzed using Theorem 4.1. Later we shall go back to the definition of worst case portfolios given in Definition 3.2.

**Corollary 4.1:** Let \( \Psi \) be an insurance version satisfying the conditions of Theorem 4.1, with \( s = 1, T = 2 \), and \( \phi^\#(a) > -\infty \), for all \( a \in \mathcal{M} \). Let us assume that \( \mathcal{C} \subset \text{Mat}_{2 \times 2} (\mathcal{F}_1) \), \( \{(A_{11}, A_{12}, A_{21}, A_{22}) \mid A \in \mathcal{C} \} \) is a compact set in \( (L^\infty(\mathcal{F}_1))^4 \), and

\[
K_X = \{ < AX, a >_{1,2} + \phi^\#(a) \mid A \in \mathcal{C}, a \in \mathcal{M} \}
\]

is directed upwards for all \( X \in \mathcal{R}^\infty_{1,2} \). If \( X \in \mathcal{R}^\infty_{1,2} \), then

\[
es \sup_{A \in \mathcal{C}} \Psi(AX) = \Psi(A^0X),
\]
for some $A^0 \in C$.

**Proof:** Let us fix $X \in \mathcal{R}^{\infty}_{1,2}$. If $A, B \in C$, Theorem 4.1 implies that

$$
\Psi(A X) = \langle A X, a \rangle_{1,2} + \phi^*(a),
$$

$$
\Psi(B X) = \langle B X, b \rangle_{1,2} + \phi^*(b),
$$

for some $a, b \in \mathcal{M}$. Since $K_X$ is directed upwards, there are $C \in C$ and $c \in \mathcal{M}$ such that

$$
\langle C X, c \rangle_{1,2} + \phi^*(c) \geq \max\{ \langle A X, a \rangle_{1,2} + \phi^*(a), \langle B X, b \rangle_{1,2} + \phi^*(b) \},
$$

and by Theorem 4.1,

$$
\langle C X, d \rangle_{1,2} + \phi^*(d) = \Psi(C X) = \text{ess sup}_{e \in \mathcal{M}} \{ \langle C X, e \rangle_{1,2} + \phi^*(e) \}
$$

$$
\geq \langle C X, c \rangle_{1,2} + \phi^*(c),
$$

for some $d \in \mathcal{M}$. Consequently, $\{\Psi(A X) \mid A \in C\}$ is directed upwards, and hence there is a sequence $\{A^k\}_{k \in \mathbb{N}} \subset C$ such that

as $k \to \infty$, $\Psi(A^k X) \nearrow \text{ess sup}_{A \in C} \Psi(A X)$.

By compactness of $\{(A_{11}, A_{12}, A_{21}, A_{22}) \mid A \in C\}$, we can find $A^0 \in C$ such that

$$
A^k_{ij} \longrightarrow_{k \to \infty} A^0_{ij}, \quad \text{a.s for all } i, j \in \{1, 2\},
$$

passing through a subsequence if necessary. Given $a \in \mathcal{M}$, we have that

$$
\left| \langle A^k X, a \rangle_{1,2} + \phi^*(a) - \left( \langle A^0 X, a \rangle_{1,2} + \phi^*(a) \right) \right| = \left| \langle (A^k - A^0) X, a \rangle_{1,2} \right| \leq 2 \| X \|_{\infty} \max \left| A^k_{ij} - A^0_{ij} \right| \longrightarrow 0
$$

Therefore,

$$
\left| \text{ess sup}_{a \in \mathcal{M}} \{ \langle A^k X, a \rangle_{1,2} + \phi^*(a) \} - \text{ess sup}_{a \in \mathcal{M}} \{ \langle A^0 X, a \rangle_{1,2} + \phi^*(a) \} \right| \longrightarrow_{k \to \infty} 0,
$$

and the claim follows. \qed

The following theorem links some implications of attaining the maximum in the robust representation (2) with the notion of worst case portfolios, as presented in Section 3. Conclusions of this theorem are close to those of Theorem 3.3. Namely, we prove that for a certain class of dynamic utility functions process, the worst case portfolios of the insurance version process are preserved over time. Despite the similarity of our next result with Theorem 3.3, we now impose slightly different assumptions. The requirement of
boundedness from below in Theorem 3.3 is now replaced by coherency and stability under concatenation. Interested readers in the relationship between relevance, coherency, time-consistency and the representation given in Corollary 4.1 are referred to Corollary 4.16 from [5].

**Theorem 4.2:** Let \((\phi_s, T)_{s \in [0, T] \cap \mathbb{N}}\) be a coherent, relevant and time-consistent monetary utility function process such that

\[
\phi_{s, T}(X) = \text{ess inf}_{a \in \mathcal{M}} \frac{< X, a >_{s, T}}{< 1, a >_{s, T}}, \quad X \in \mathcal{R}_{s, T}^\infty,
\]

for some compact, convex set \(\mathcal{M} \subset D_{0, T}^s\) stable under concatenation, with \(\Delta a_k \leq \Delta b_k\) for all \(a \in \mathcal{M}, k \in \mathbb{N}\), with \(b \in A^1\). If \(\{(X^{s,1}, \ldots, X^{s,n})\}_{s \in [0, T] \cap \mathbb{N}}\) is an adapted worst portfolio process of the respective insurance version \((\Psi_s, T)_{s \in [0, T] \cap \mathbb{N}}\), then \((X^{0,1}, \ldots, X^{0,n})\) is a worst portfolio of \(\Psi_t, T\) for all \(0 \leq t \leq T\).

**Proof:** The first step is just to verify that for each \(X \in \mathcal{R}_{0, T}^\infty\), there is \(a \in \mathcal{M}\), such that

\[
\phi_{s, T}(X) = \frac{< X, a >_{s, T}}{< 1, a >_{s, T}}, \quad s \in [0, T] \cap \mathbb{N}.
\]

This can be done following similar arguments as in the proof of Theorem 4.1.

Hence, it is enough to prove that, for each \(t \in [0, T - 1]\), \((X^{t,1}, \ldots, X^{t,n})\) is a worst case portfolio of \(\Psi_{t+1, T}\). Define the event

\[
\mathcal{A} = \left\{ \Psi_{t+1, T} \left( \frac{1}{n} \sum_{i=1}^{n} X^{t,i} \right) < \Psi_{t+1, T} \left( \frac{1}{n} \sum_{i=1}^{n} X^{t+1,i} \right) \right\},
\]

and assume that \(P[\mathcal{A}] > 0\). Let \(\tilde{X}^t, \tilde{X}^{t+1} \in \mathcal{R}_{t, T}^\infty\) be the processes defined as

\[
\tilde{X}_s^t = \frac{1}{n} \sum_{i=1}^{n} X_s^{t,i}, \quad t \leq s \leq T,
\]

\[
\tilde{X}_{s+1}^{t+1} = \frac{1}{n} \sum_{i=1}^{n} X_{s+1}^{t+1,i}, \quad t + 1 \leq s \leq T,
\]

\[
\tilde{X}_{t+1}^{t+1} = \frac{1}{n} \sum_{i=1}^{n} X_{t+1}^{t,i}.
\]

Time-consistency and Theorem 4.1 imply that there is \(a^0 \in \mathcal{M}\) such that

\[
\Psi_t, T \left( \tilde{X}^t \right) = \Psi_{t, T} \left( \left( \tilde{X}^t \right)_{1|t} + \Psi_{t+1, T} \left( \tilde{X}^t \right)_{1(t+1, \infty)} \right) = \frac{< (\tilde{X}^t)_{1|t} + \Psi_{t+1, T} \left( \tilde{X}^t \right)_{1(t+1, \infty)}, a^0 >_{t,T}}{< 1, a^0 >_{t,T}}.
\]
Letting $Y := (\bar{X}^t)_{1[t]} + \Psi_{t+1,T} (\bar{X}^t)_{1[t+1,\infty]}$, we have that

$$< Y, a^0 > = E \left( \bar{X}_t^t \Delta a^0_t + \Psi_{t+1,T} (\bar{X}^t) \left( \sum_{j=t+1}^T \Delta a^0_j \right) \mid \mathcal{F}_t \right).$$

In addition,

$$Y_t := \bar{X}_t^t \Delta a^0_t + \Psi_{t+1,T} (\bar{X}^t) \left( \sum_{j=t+1}^T \Delta a^0_j \right)$$

$$\leq \bar{X}_t^{t+1} \Delta a^0_t + \Psi_{t+1,T} (\bar{X}^{t+1}) \left( \sum_{j=t+1}^T \Delta a^0_j \right)$$

$$=: Y_{t+1}.$$

Let $C := \left\{ \sum_{j=t+1}^T \Delta a^0_j > 0 \right\}$, and note that $P(C) = 1$, since $a^0 \in D^e_{0,T}$. Therefore,

$$P(A \cap C) = P(A) > 0$$

and

$$Y_t < Y_{t+1} \quad \text{in} \quad A \cap C.$$

Then, there exists an event $B \in \mathcal{F}_t$, with $P(B) > 0$, such that

$$E(Y_t \mid \mathcal{F}_t) < E(Y_{t+1} \mid \mathcal{F}_t) \quad \text{in} \quad B.$$

Finally, by time-consistency and last inequality, the following display holds in $B$

$$\Psi_{t,T} (\bar{X}^t) = E \left( Y_t \mid \mathcal{F}_t \right) \frac{< Y_t, a^0 >_{t,T}}{< 1, a^0 >_{t,T}}$$

$$\leq \text{ess sup}_{a \in \mathcal{M}} \frac{< (\bar{X}^{t+1})_{1[t]} + \Psi_{t+1,T} (\bar{X}^{t+1})_{1[t+1,\infty]}, a >_{t,T}}{< 1, a >_{t,T}}$$

$$= \Psi_{t,T} (\bar{X}^{t+1}),$$

which is a contradiction.

Notice that the proof of the previous theorem follows the same lines as that of Theorem 3.3. However, the key difference consist in avoiding one of the main assumptions in this theorem, concerning the boundedness from below in the set $\mathcal{Q}_s$. This is achieved noting that coherency and stability under concatenation, together with Theorem 4.1, allow us to write the monetary utility function process in a simple way. Thus, for each $X \in \mathcal{R}^\infty_{0,T}$, there is $a \in \mathcal{M}$, such that

$$\phi_{s,T} (X) = \frac{< X, a >_{s,T}}{< 1, a >_{s,T}}, \quad s \in [0, T] \cap \mathbb{N}.$$
Hence, it is not necessary to have a global property such as boundedness from below in $Q_s$. The result follows from the properties of the set $D^0_{t,T}$ and the time consistency assumption, illustrating once again the fact that the preservation of worst case portfolios of insurance versions is naturally linked with the property of time consistency.

We conclude this paper presenting a result that allow us to determine when the risk of modifying two given processes is comparable, even though the processes are not.

**Proposition 4.1:** Let $\phi$ be a concave monetary utility function that is continuous for bounded decreasing sequences in $R^\infty_{t,T}$. Assume that there exists a matrix $A \in Mat(T-t+1) \times (T-t+1)$ ($F_1$) with the following properties:

(i) $(1, \ldots, 1) \in \mathbb{R}^{T-t+1}$ is an eigenvector of $A$ with eigenvalue 1.

(ii) $(A_{ij}) \geq 0$, for all $i, j \in \{1, \ldots, T-t+1\}$.

(iii) $X \in R^\infty_{t,T}$ and $AX \in C_\phi$ imply $X \in C_\phi$.

(iv) $A \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i \right) \leq \sum_{i=1}^{n} \tilde{X}^i$,

for $\tilde{X}^i, \tilde{X}^i \in R^\infty_{t,T}, i = 1, \ldots, n$.

Then,

$$\Psi \left( A \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i \right) \right) \leq \Psi \left( A \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i \right) \right),$$

where $\Psi (X) := -\phi (-X)$ for all $X \in R^\infty_{t,T}$.

**Proof:** Define $\tilde{\phi} : R^\infty_{t,T} \rightarrow L^\infty (F_1)$ as

$$\tilde{\phi} (X) = \phi (AX), \quad X \in R^\infty_{t,T}.$$ 

It is not difficult to see that $\tilde{\phi}$ is a concave monetary utility function that is continuous for bounded decreasing sequences. Let us denote by $\tilde{\Psi}$ the functional given as $\tilde{\Psi} (X) := -\tilde{\phi} (-X)$ for all $X \in R^\infty_{t,T}$. By Theorem 3.16 from [5], we have that

$$\Psi (X) = \text{ess sup}_{a \in D_{t,T}} \{ < X, a >_{t,T} + \phi^\# (a) \},$$

$$\tilde{\Psi} (X) = \text{ess sup}_{a \in D_{t,T}} \{ < X, a >_{t,T} + \tilde{\phi}^\# (a) \}. $$

Since $\tilde{\Psi} (X) = \Psi (AX)$, then

$$\tilde{\Psi} (X) = \text{ess sup}_{a \in D_{t,T}} \{ < AX, a >_{t,T} + \phi^\# (a) \}.$$

By hypothesis, it is clear that

$$\phi^\# (a) = \text{ess inf}_{X \in C_\phi} < X, a >_{t,T} \leq \text{ess inf}_{X \in C_{\tilde{\phi}}} < X, a >_{t,T} = \tilde{\phi}^\# (a).$$
Therefore, for all $a \in D_{t,T}$ the following inequality holds
\[
\langle A \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}^i \right), a \rangle_{t,T} + \phi^a(a) \leq \langle \frac{1}{n} \sum_{i=1}^{n} X^i, a \rangle_{t,T} + \tilde{\phi}^a(a),
\]
and the conclusion follows.

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