Quantum key distribution protocols based on equiangular spherical codes are introduced and their behavior under the intercept/resend attack investigated. Such protocols offer a greater range of secure noise tolerance and speed options than protocols based on their cousins, the mutually-unbiased bases, while also enabling the determination of the channel noise rate without the need to sacrifice key bits. For fixed number of signal states in a given dimension, the spherical code protocols offer Alice and Bob more noise tolerance at the price of slower key generation rates.

Recall the general setting of quantum key distribution. Two parties, Alice and Bob, wish to make use of an authenticated public classical broadcast channel and an insecure quantum channel controlled by an adversary Eve to establish a secret key for the purposes of encrypting and sharing other data. They start by using the classical channel to fix a signal ensemble and a measurement for the quantum channel. Alice sends states drawn from the signal ensemble through the quantum channel to Bob, who performs the chosen measurement (in the case of signaling states drawn from mutually unbiased bases, the several measurement bases Bob chooses from for his measurement are here amalgamated into a single generalized measurement). Eve is free to exploit knowledge of the protocol and her control of the quantum channel to mount an attack on their protocol; she can in principle subject the signal states to any physical interaction that she wishes. Effectively, this process produces a sequence of samples from a certain tripartite probability distribution shared between the three parties. Alice and Bob then proceed to “distill” the key by communicating information based on their individual sequences over the classical channel. Their goal is to exploit the quantum nature of the channel to make Eve’s eavesdropping ineffective.

The relevant probability distribution is the joint probability $p(a_i,b_j,c_k)$ of Alice’s signal, Bob’s measurement result, and the result of whatever measurement Eve performs in the course of eavesdropping. Repeated use of the protocol yields a sequence of samples drawn from this distribution. Alice and Bob, however, must establish which distribution they are sampling from, as it depends on Eve’s attack. We imagine Eve has some physical setup which can give rise to many different distributions as she changes the strength of her interference with the channel. Given an assumption of the type of attack, Alice and Bob determine the extent of Eve’s interference by making public and comparing a fraction of the Alice’s signals and Bob’s measurement results. In this way they estimate the error rate of the channel, and together with
an assumption of the attack, determine the distribution \( p \). From the remaining samples, which are supposed to be an asymptotically large number \( M \), say, they can dis-  

\[ I_E \leq R \leq I(A:B|E) \tag{1} \]

where \( I(X:Y) = H(X) + H(Y) - H(XY) \) is the mutual information of \( X \) and \( Y \), \( H(\cdot) \) being the Shannon entropy, and \( I_E = I(A:B) - \min \{I(A:E), I(B:E)\} \). The lower bound obtains when the key is distilled using one-way communication \( 3 \); to progress beyond this requires a technique called advantage distillation, though this is of limited efficiency \( 10 \). These bounds provide a method of investigating the cryptographic usefulness of a protocol. Given a signal ensemble, Bob’s measurement, and an assumption about the nature of Eve’s attack, the probability distribution can be calculated, and the key rate bounds determined. In this way the security of the protocol against this attack is established. To say that a protocol is unconditionally secure is to demonstrate its security against all possible attacks.

The focus now turns to Alice’s signal ensemble and Bob’s measurement. An intuitively appealing ensemble is a spherical code, a complex-vector-space version of points on a sphere whose minimal pairwise distance is maximal. The complex version, called the Grassmann packing problem, asks for a set of unit vectors in \( \mathbb{C}^d \) with maximal pairwise overlap is minimal \( 12 \). When all these pairwise overlaps are equal, this equiangular spherical code is called a Grassmann frame; i.e., a set \( \mathcal{C} = \{ |\phi_k\rangle \in \mathbb{C}^d \}_{k=1}^n \) for \( n \geq d \) is a Grassmann frame if

\[ |\langle \phi_j | \phi_k \rangle|^2 = \frac{n-d}{d(n-1)} \quad \forall \ j \neq k \ . \tag{2} \]

Grassmann frames also arise as the solution to the “minimum energy problem.” For a set of unit vectors \( \mathcal{C} \), call \( V_1(\mathcal{C}) = \sum_{j<k} |\langle \phi_j | \phi_k \rangle|^2 \) the \( t \)-th order “potential energy” of the set of the vectors \( 13 \). The minimum energy problem is to find \( \mathcal{C} \) having \( n \geq d \) elements such that \( V_1 = n^2/d \) and \( V_2 \) is minimized. Note that \( n^2/d \) is the global minimum of \( V_1 \). This follows from considering the (at most) \( d \) nonzero (real) eigenvalues \( \gamma_j \) of the Gram matrix \( G_{jk} = \langle \phi_j | \phi_k \rangle \). Clearly \( \sum_k \gamma_k = n \) and \( \sum_k \gamma_k^2 = V_1(\mathcal{C}) \). These being the equations for a plane and a sphere, the minimum of \( V_1 \) occurs if and only if all the \( \gamma_k \) are equal to \( n/d \), whence \( V_1 \) is bounded below by \( n^2/d^2 \). Thus what is sought is the set of vectors with the minimum \( V_2 \) energy, given minimum \( V_1 \) energy.

To find a lower bound for the minimum of \( V_2 \), let \( \lambda_{jk} = |\langle \phi_j | \phi_k \rangle|^2 \), and employ the same method again. We have immediately that \( \sum_{j\neq k} \lambda_{jk} = V_1 - n = n(n-d)/d \) and \( \sum_{j\neq k} \lambda_{jk}^2 = V_2 - n \), whence the minimum of \( V_2 \) over all sets minimizing \( V_1 \) is bounded below by making all the \( \lambda_{jk} \) the same and given by Eq. \( 2 \). When this lower bound is achieved, i.e \( V_2 = n^2(n - 2d + d^2)/(n - 1) \), the result is a Grassman frame.

The existence of Grassmann frames isn’t established for arbitrary \( n \) and \( d \), though some general statements can be made \( 14 \). They always exist for \( n = d + 1 \) (a regular simplex), but never when \( n > d^2 \). For \( n \leq d^2 \), when a Grassman frame exists, it is a spherical code, but for \( n > d^2 \), spherical codes aren’t equiangular.

By minimizing \( V_1 \), Grassmann frames automatically form measurement POVMs, which can be used by Bob to detect Alice’s signal. This is true because \( S = \sum_k |\phi_k\rangle \langle \phi_k| = (n/d)I \), so that a POVM can be constructed from the subnormalized projectors \( (d/n)|\phi_k\rangle \langle \phi_k| \). To see this, fix an orthonormal basis \( \{ |e_k\rangle \} \) and consider the matrix \( T_{jk} = \langle e_j | \phi_k \rangle \). The Gram matrix can be written as \( G_{jk} = (T^T)_{jk} \) while \( S_{jk} = (TT^T)_{jk} \), so both have the same eigenvalues. When \( V_1 \) is minimized, these \( d \) eigenvalues are all \( n/d \), implying that the vectors form a resolution of the identity.

By using the same ensemble as Alice, Bob’s measurement confirms the signal she sent with probability \( d/n \). Bob may also choose a measurement which attempts to repudiate some of Alice’s signals, in a manner entirely similar to unambiguous state discrimination \( 15 \). Because the number of states is larger than the dimension of the space on which they are supported, it is impossible to unambiguously determine the signal Alice sent with nonzero probability. However, for any ensemble, a par-  

tial determination may be made, a scheme which works as follows. First, partition the signal ensemble into the set of all subsets of size \( b \). Then, for each subset, find the projector orthogonal to the span of the vectors in the subset. This procedure yields a measurement operator for each of the subsets. In order to find an orthogonal projector to the span, \( b \) must be restricted such that none of the subsets spans more than \( d - 1 \) dimensions.

For general ensembles, the operators constructed by this procedure do not quite form a measurement; some additional “failure” outcome is required to make the entire set form a resolution of identity. Interestingly, this additional outcome appears not to be required when the signal ensemble is an equiangular spherical code. Numerical constructions starting with ESCs in modest dimensions always yields a proper POVM in which no “failure” outcome is necessary. Though both types of measurements deserve further study, we shall specialize to the case of confirmation protocols; repudiation protocols are developed for qubits in \( 16 \).

Confirmation protocols are appealing because the ESC ensembles are the sets that are “least classical” in the following sense \( 17 \). Consider using these quantum states as signals on a classical channel as follows. Instead of sending the quantum state, Alice performs the associated measurement and communicates the result to Bob using a classical channel. Bob then prepares the asso-
associated quantum state at his end. The fidelity of Bob’s reconstruction with the input state, averaged over inputs and measurement results, measures how well the classical channel can be used to transmit quantum information. This fidelity is $dV_2/n^2$, so among all ensembles which themselves form POVMs, Grassmann frames are hardest to transmit “cheaply” in this way. Eavesdropping on the communication between Alice and Bob makes the channel more classical—Eve is essentially trying to copy the signal—so one might expect that Grassmann frames are useful in foiling the eavesdropper.

Before examining their resistance to eavesdropping, we remark on how Alice and Bob can concretely use the equiangular spherical codes to accomplish secure key distribution. Though a protocol satisfying the lower bound of equation 1 is guaranteed asymptotically, it may not be feasible in practice. Consider first the case of a noiseless quantum channel, i.e. no eavesdropping. In a length-$N$ string of samples Alice and Bob will agree with probability $d/n$. When labelling the states 0 to $n − 1$, Bob’s string $b$ is simply Alice’s string $a$ plus a string $δ$ having a fraction $(n − d)/n$ of non-zero elements. Alice can select a classical error-correcting code $C$ which can correct these errors, choose a codeword $c$ randomly, and send $a + c$ to Bob. He then simply subtracts this from his string to obtain $c + δ$, from which he uses the error-correcting property to determine the codeword $c$. From the Shannon noisy-channel coding theorem, there are roughly $N/2$ codewords, in accordance with the lower bound.

When Eve has no information about the quantum signals, the communicated string $a + c$ tells her nothing about $c$, since it’s effectively encrypted by $a$. Should Eve have some information about $a$, gleaned from her tampering with the quantum channel, Alice and Bob may proceed as before to establish $c$, and then use a privacy amplification procedure to shorten this string and remove any information Eve has about it.

As formulated, this protocol is not particularly robust; Alice and Bob can do better by first announcing some of the signals not received, a procedure analogous to sifting in the case of mutually unbiased bases. Upon receipt of each signal, Bob publicly broadcasts $m$ outcomes he did not obtain. If Alice’s signal is among these, they throw it away and proceed to the next. For the signals which pass the test Alice and Bob relabel the remaining states in order from 0 to $n − m − 1$ and follow the above procedure. This occurs with probability $m/(n − 1)$ as Bob could send any of $(n−1)$ outcomes and $(m−1)$ of these contain Alice’s signal. The protocol itself succeeds with probability

$$p_{\text{sift}} = \frac{n(n-1) - m(n-d)}{n(n-1)} = \frac{s}{n(n-1)},$$

where we have implicitly defined the constant $s$. Meanwhile, the key rate when using an $n$-word equiangular spherical code in $d$ dimensions excluding $m$ outcomes is given by

$$R = \log(n-m) + \frac{d(n-1)}{s} \log[d(n-1)]$$

$$+ \left(1 - \frac{d(n-1)}{s}\right) \log[n-d] - \log s.$$  \hspace{1cm} (4)

The real question is how well the protocol tolerates attempted eavesdropping. To begin formulating an answer it is useful to consider the intercept/resend attack. Because it is straightforward to analyze, it offers immediate insight into the usefulness of these protocols. A full treatment of the problem could involve Alice and Bob using quantum error-correcting codes to turn a noisy quantum channel into a smaller, noiseless channel. However, this is far too complicated for the first step in this analysis.

In the intercept/resend attack we assume that Eve fixes a fraction $q$ of signals to intercept. She measures those so chosen using the same equiangular spherical code as does Bob, resending him the output of this process. Eve simply guesses Alice’s signal to be her outcome, unless it is excluded by Bob’s announcement. In this case she may still guess, but retains the information that she was forced to do so.

By delineating the various cases, it is uncomplicated to arrive at the relevant quantities. First, the sift rate of the protocol depends on $q$ as in the following expression. Letting $t = s(n - 1) - qm(n - d)(d - 1)$, the sift rate is now

$$p_{\text{sift}} = t/n(n-1)^2.$$  \hspace{1cm} (5)

Alice and Bob’s joint probability distribution is determined by the agreement probability

$$p_{\text{ab}} = (n - 1)(d(n - 1) - q(n - d)(d - 1))/t.$$  \hspace{1cm} (6)

Since Alice and Bob use the same ESC ensemble, Eve’s joint probability with Alice is the same as with Bob. In order to account for the cases in which Eve measures the signal but this outcome is later excluded by the protocol, we may append an event to her probability distribution, denoted by ?. Now she has $n − m + 1$ total outcomes, and the ? outcome functions as a guess as to the key letter in the cases it occurs. The joint probability of such an exclusion and the particular signal $j$ is plainly the same for all $j$, and together with the probability of agreement between Alice and Eve, these quantities fully describe the overall distribution:

$$p_{\text{ae}} = qd(n-1)s/nt,$$  \hspace{1cm} (7)

$$p_{\gamma} = 1 - qs^2/nt.$$  \hspace{1cm} (8)

These quantities enable us to compute the lower bound on the optimal key rate in equation 1. Then we may determine the value of $q$ such that $R = 0$, and from this the maximum tolerable error rate for a given $n, m, d$ combination. Finally, we can convert this error rate into the
Eve’s mutual information is error is \( \frac{q}{k} \) makes Eve’s information correct basis. Letting \( k \) full information log Eve gains information only if she measures in the correct basis. For the spherical codes the depolarizing rate \( r \) is related to the error probability \( p_e = 1 - p_{adv} \) via the expression

\[
r = \frac{s}{m(d-1)} - \frac{n(n-1)(n-m-1)}{m(d-1)(n-1 + m(p_e - 1))}
\]

(9)

For protocols using unbiased bases these expressions are much simpler; in fact it is easier to work with the mutual information expressions themselves. For a given probability of error between Alice and Bob, their shared information is simply the full amount less the corruption caused by the error:

\[
I(A:B) = \log d + p_e \log p_e + (1 - p_e) \log(1 - p_e)/(d - 1)
\]

(10)

Eve gains information only if she measures in the correct basis, and the basis announcement step prevents her from incorrectly guessing Alice’s signal. Thus she gains the full information \( \log d \) if she manages to measure in the correct basis. Letting \( k \) be the number of bases, this makes Eve’s information \( q/k \log d \). The probability of error is \( (k-1)(d-1)/kd \), so in terms of error probability, Eve’s mutual information is

\[
I(A:E) = \frac{d}{(d-1)(k-1)} p_e \log d
\]

(11)

The sift rate of the protocol is always the probability 1\( k \) for Bob to measure in the same basis Alice prepared the state, regardless of Eve’s interference. Finally, the equivalent depolarizing rate \( r \) for a given error rate \( p_e \) is simply \( r = p_e d/(d-1) \).

Now we can compare the various aspects of both protocols. The first difference comes from the sift rate. When using unbiased bases, the protocol will fail with probability \( (k-1)/k \) no matter how noisy the channel. To determine the error rate, Alice and Bob must announce some of their created key letters so as to compare how often they agree. Obviously these letters cannot be used in the key so they are sacrificed. However, for equiangular spherical codes the sift rate depends on the intercept rate. Successful execution of the protocol relies on the correlation between Bob’s outcome and Alice’s signal, thus the probability of successful sifting decreases as the noise increases. Further, Eve can’t replace the signals with any others to increase the sift rate and attempt to mask her intervention on other signals, for she would need to know Alice’s signal. Thus, there is no need to sacrifice key letters in order to estimate Eve’s information.

The second immediate difference is the increased number of possible protocols using the equiangular spherical codes. With unbiased bases Alice and Bob have the choice of using anywhere from 2 to (possibly) \( d+1 \) bases. Spherical codes offer more possibilities, as \( n \) may range from \( d+1 \) to \( d^2 \) and \( m \) from 0 to \( n-2 \). This offers two advantages. The first is speed. In particular there are protocols involving fewer than \( 2d \) signals, the minimum for the unbiased bases. This translates into higher absolute key generation rates, the key generation rate times the sift rate. Suppose Alice and Bob use \( n = \alpha d \) spherical code states. Then the maximum key generation rate becomes \( \log(d)/\alpha + (\alpha-1)\alpha \log((\alpha-1)/\alpha) \) for large \( d \), in contrast to at best \( \log(d)/2 \) for unbiased bases. By choosing \( \alpha < 2 \) Alice and Bob can find spherical codes which are faster than two unbiased bases. For example, consider 35 spherical code states in 25 dimensions. The key rate is roughly 3.4 bits per signal as compared to 2.3 for two unbiased bases.

The second advantage is security. Increasing \( m \) is similar to increasing the number of unbiased bases in that both lead to decreased key rate and increased maximum tolerable noise. For maximum security Bob may elect to announce all but two of the outcomes he didn’t obtain, whereas for maximum speed Bob would choose \( m = 0 \). Figure 1 shows the maximum tolerable noise rate for two values of \( m \), as a function of number of signal states in ten dimensions. The corresponding maximum error rate when using various numbers of unbiased bases is included for comparison. The case of \( m = n-2 \) always yields an improved secure noise threshold over unbiased bases.

These two advantages hold when considering all possible ESC protocols in a fixed dimension and so are appropriate if only the dimension is constrained by the particular hardware Alice and Bob wish to use. However, a direct comparison of the two protocols involves a tradeoff between speed and security for a fixed number of states in a given dimension. When using spherical code protocols, Alice and Bob are free to choose \( m \) to match the perceived error rate in the channel, so for each style of signal ensemble, fixing \( n \) and \( d \) specifies a concrete physical setup with similar resources. For each protocol we may consider the pair consisting of maximum key generation rate and the maximum tolerable error rate. Plotting the pairs for protocols having \( n \) signals in various dimensions we can determine the tradeoff for the two protocols. Figure 2 reveals that for \( n = 2d \) signals, the spherical code protocols offer more security (larger vertical values) but at the price of slower key generation (smaller horizontal values).

Use of spherical codes has been applied here to a specific model of quantum key distribution. However, they are immediately applicable to two variants of the “prepare & measure” protocol discussed here. First is the co-
Potential processing tools to bear on the problem of security.

Equiangular spherical codes fit nicely into this framework, as they can always be realized from maximally entangled states. Thus they start on the same footing as unbiased bases, for which this is also true. To demonstrate this, consider a spherical code $C = \{|\phi_k\rangle\}$ and a “conjugate” code $C^* = \{|\phi^*_k\rangle\}$ formed by complex conjugating each code state in the standard basis. Then it is a simple matter to show that $|\Psi\rangle = (\sqrt{d}/n) \sum_k |\phi_k\rangle|\phi^*_k\rangle$ is maximally entangled. Thus if Alice prepares this state and sends the second half to Bob, they can realize the “prepare-and-measure” scheme by measurement.

Second, the full array of $d^2$ equiangular states is a tomographically-complete ensemble, like the full complement of $d+1$ unbiased bases $\{e_k\}$. Such sets are useful in a modified protocol in which Eve, instead of Alice, creates bipartite states and distributes one half to each of the two parties $\{e_k\}$, and as a test to ensure secret key generation is possible $\{e_k\}$. Now Alice and Bob need to perform quantum state tomography on their pieces to ensure the security of the protocol. Using the $d^2$ equiangular states again offers more security in this case.

Investigating the intercept/resend attack yields an insight into how the spherical codes ultimately perform in comparison with unbiased bases, when the strongest eavesdropping attacks are considered. The spherical codes’ flexibility in $n$ and $m$ leads to advantages in both speed and security, though not together in any one instance, all the while rendering unnecessary the procedure of sacrificing key letters to determine the error rate. For fixed resources, i.e. fixed number of states in a given dimension, spherical codes offer higher noise security thresholds, but slower key generation rates.

This hindrance may be possible to overcome, or at least ameliorate, for Bob’s use of Alice’s spherical code is almost certainly not his best choice. That is, assuming that the secrecy capacity will increase with increasing classical capacity of the quantum channel, Bob’s optimal measurement is likely not the spherical code used by Alice. This is evident in two dimensions for the two spherical codes, which we may think of in the Bloch-sphere representation as a regular tetrahedron and three equally-spaced coplanar vectors. By inverting each vector in the ensemble the resulting combination of spherical code encoder and inverse spherical code decoder achieves a higher classical capacity. In the case of the trine this is known to be the optimal measurement $\{e_k\}$ and conjectured to be for the tetrahedron $\{e_k\}$. These are in fact the two qubit protocols based on the repudiation measurement alluded to earlier. Much the same is true in three dimensions, at least for the case of six equiangular states: an unenlightening numerical maximization produced the result that a unitarily-transformed version of the encoding spherical code yields a capacity of 0.638, a roughly 50% improvement over the nominal capacity of 0.424 when using the same ensemble for encoding and decoding. An even greater classical capacity can be achieved by using a repudiation measurement, each outcome of which excludes two possible signal states. In this case the capacity jumps to 0.734. Neither of these strategies surpass the capacity...
generated by using two unbiased bases, \( \log[3]/2 \approx 0.792 \), but they narrow the gap.

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[1] C. H. Bennett and G. Brassard, in Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing (IEEE, New York, 1984), p. 175.
[2] D. Mayers, J. Assoc. Comput. Mach. 48, 351 (2001).
[3] E. Biham et al., in Proc. 32nd Annual ACM Symposium on the Theory of Computing (STOC), ACM Press, New York (2000), p. 715.
[4] H.-K. Lo and H. F. Chau, Science 283, 2050 (1999).
[5] P. W. Shor, J. Preskill, Phys. Rev. Lett. 85, 441 (2000).
[6] D. Bruß, Phys. Rev. Lett. 81, 3018 (1998).
[7] W. K. Wootters and B. D. Fields, Ann. Phys. (N.Y.) 191, 363 (1989).
[8] N. J. Cerf, M. Bourennane, A. Karlsson, and N. Gisin, Phys. Rev. Lett. 88, 127902 (2002).
[9] I. Csizsár and J. Körner, IEEE Trans. Inf. Theory, IT-24, 339 (1978).
[10] U. M. Maurer, IEEE Trans. Inf. Th. 39, 733 (1993).
[11] N. Gisin and S. Wolf, Phys. Rev. Lett. 83, 4200 (1999).
[12] T. Strohmer and R. Heath, Appl. Comp. Harm. Anal. 14, 257 (2003).
[13] J. J. Benedetto and M. Fickus, Adv. Comput. Math. 18, 357 (2003).
[14] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, J. Math. Phys. 45, 2171 (2004).
[15] A. Chefles, Phys. Lett. A 239, 339 (1998).
[16] J. M. Renes, quant-ph/0402135 Submitted to Phys. Rev. A.
[17] C. A. Fuchs and M. Sasaki, Quant. Info. Comp. 3, 377 (2003).
[18] Dagmar Bruß, et al., Phys. Rev. Lett. 91, 097901 (2003)
[19] M. Curty, M. Lewenstein, and N. Lütkenhaus, quant-ph/0307151.
[20] M. Sasaki, et al., Phys. Rev. A 59, 3325 (1999).
[21] E. B. Davies, IEEE Trans. Inf. Theory 24, 596 (1978).