Effects of a cloud of strings on the extended phase space of Einstein-Gauss-Bonnet AdS black holes

Hossein Ghaffarnejad∗ and Emad Yaraie†

Faculty of Physics, Semnan University, P.C. 35131-19111, Semnan, Iran

Abstract
In this paper we study the thermodynamics of Einstein-Gauss-Bonnet (EGB)-AdS black holes minimally coupled to a cloud of strings in an extended phase space where the cosmological constant is treated as pressure of the black holes and its conjugate variable is the thermodynamical volume of the black holes. To investigate the analogy between EGB black holes surrounded by a cloud of strings and liquid-gas system we derive the analytical solutions of the critical points and probe the effects of a cloud of strings on $P - V$ criticality. There is obtained resemblance between "small black hole/large black hole" (SBH/LBH) phase transition and the liquid-gas phase transition. We see that impact of a cloud of strings can bring Van der Waals-like behavior, in absence of the Gauss-Bonnet (GB) counterpart. In the other words, in the EGB black hole with $\alpha \to 0$ and when it is surrounded by a cloud of strings the Hawking-Page phase transition would be disappeared and SBH/LBH phase transition recovers. Also there is not happened Joule-Thomson effect.

1 Introduction
Black holes are one of the fascinating predictions of the Einstein’s theory of general relativity. The study of black hole as a thermodynamical system have been an attractive subject in theoretical physics for many years. In the recent years, the study of the thermodynamic properties of black holes have revealed many aspects of them. One of these aspects is thermodynamic phase transition in AdS black holes [1]. First works about the AdS black holes thermodynamics obtained from higher derivative gravity was studied in ref. [2] and for EGB type one can follow [3]. The study of the AdS black hole phase transition has been generalized to the extended phase space where the

∗E-mail: hghafarnejad@semnan.ac.ir
†E-mail: eyaraie@semnan.ac.ir
cosmological constant has been treated as the pressure of the black hole [4,5].
As we can see in [6] Van der Waals-like behavior of Reissner-Nordstrom black hole could be observed. For more study one can see [7-22]. The authors in [23] constructed a novel 5-dimensional black hole solution in the EGB gravity and in a cloud of strings background, then studied its aspects in a non-extended thermodynamics. Because the universe can be described by one-dimensional objects called strings, this solution has a great physical significance: A cloud of strings is introduced as a configuration of one-dimensional strings from the string theory as the most promising theory of quantum gravity which could be effective on gravitational fields such as black holes. This extension has some advantages like the ability of studying any higher dimensions and resemblance energetic field such as the global monopole in four dimension. Since strings are supposed to be fundamental objects in nature which are supported by observations, so it encourages us to study its effects on various gravitational theory. So the importance of studying the gravitational effects of the matter which is in the form of a cloud of strings arises. At first Strominger and Vafa showed a connection between counting string states and the entropy of the black hole in [24] , later in [25] it was considered a model for a cloud of strings that would be the equivalent of a perfect fluids. Implications of that for a broad range of black holes have been studied by many authors [23-32], as we can see some earlier works performed these effects on gravitational theories like Schwarzschild solution in [50], higher derivative theories like Lovelock theory in [30, 31, 32] and modified gravity in [51], and also by considering the effect of quintessence dark energy in [52, 53]. Gauss Bonnet terms in higher derivative theories could be seen in the low-energy effective action of superstring theories. They could be viewed as the corrections of large N expansion of boundary in the context of AdS/CFT duality in the strong coupling limit. Since such corrections have interesting effects, so it would be natural to study this black hole solution surrounded by a spherically symmetric string clouds as a thermodynamic system and seek criticality behavior and the effect of coupling constant of the model like the case for \( \alpha_{GB} \rightarrow 0 \) which leads to Schwarzschild black hole. We can also seek how the effect of the background string clouds can alter phase transition of black hole. In fact, this work is an extension of the previous work [10] to a cloud of strings background where we study thermodynamic aspects of the black hole with extended phase space and investigate the effects of a cloud of strings on \( P - V \) criticality. We also calculate critical exponents of the system.
Layout of the paper is as follows. In section 2 we define higher EGB higher derivative gravity in presence of a cloud of strings effects. We calculate equation of first law EGB black hole thermodynamics. In section 3 we use extended phase space thermodynamics to obtain its critical points at P-V hypersurface. Critical exponents are calculated in section 4 and finally we seek possibility of creation of the Joule Thomson effect. Section 5 denotes to conclusion and outlook of this work.

2 EGB black holes with cloud of strings

Let us we start with action of EGB gravity surrounded by a cloud of strings which in 5- dimension is given by [23]

$$S = \frac{1}{2} \int \sqrt{-|g|} \left( R - 2\Lambda + \alpha \mathcal{L}_{GB} \right) d^5x + I_{NG},$$

where $\alpha > 0$ is the GB coupling coefficient, $|g|$ is absolute value of determinant of the metric field $g_{\mu\nu}$ and $\mathcal{L}_{GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + R^2 - 4R_{\mu\nu}R^{\mu\nu}$, is the GB counterpart of the lagrangian which originates from quantum fields renormalization. Dynamics of a classical relativistic string is described by the Nambu-Goto action $I_{NG} = \int_\Sigma p \sqrt{-|\gamma|} d\lambda^0 d\lambda^1$, where $(\lambda^0, \lambda^1)$ are local coordinates of the string which makes parameterized the worldsheet. $|\gamma|$ is absolute value of determinant of an induced metric $\gamma_{ab}$ on the strings worldsheet for which $\gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b}$. On the other hand the bivector related to the strings worldsheet can be written as $\Sigma^{\mu\nu} = \epsilon^{ab} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b}$, where $\epsilon^{ab}$ is Levi-Civita tensor. So the energy momentum tensor for a cloud of strings is given by $T^{\mu\nu} = (-\gamma)^{-\frac{1}{2}} \rho \Sigma^{\mu\sigma} \Sigma^{\nu}_\sigma$, in which the proper density of a string cloud is described by $\rho$. Applying (2.1) and the static spherically symmetric metric $ds^2 = f(r) dt^2 - f(r)^{-1} dr^2 - r^2 d\Omega^2$ one can obtain metric field equation as follows.

$$\frac{1}{2r^3} \left( 2r^3 \Lambda + 6rf(r) - 6r + 3r^2 f(r)' \right) + \frac{6}{r^3} \alpha f(r)' (1 - f(r)) = -\frac{a}{r^3},$$

(2.2)

$$-1 + r^2 \left( \Lambda + \frac{f(r)''}{2} \right) + f(r) + 2rf(r)' - 2\alpha f(r)^2 + 2\alpha f(r)^2 (1 - f(r)) = 0,$$

(2.3)

with solution

$$f(r) = 1 + \frac{r^2}{4\alpha} \left( 1 - \sqrt{1 + \frac{32\alpha M}{r^4} - \frac{8\alpha}{\ell^2} + \frac{16\alpha\alpha}{3r^3}} \right),$$

(2.4)
where $a$ is real positive constant and $\ell$ corresponds to the AdS radius. It is related to the pressure of the black hole through $P = -\frac{1}{8\pi} = \frac{3}{4\pi\ell^2}$ in 5 dimension. Location of the event horizon is obtained by solving $f(r_+) = 0$, so the ADM mass $M$, entropy $S$ and Hawking temperature of the black hole can be derived respectively as follows.

\begin{equation}
M = \frac{1}{3}P\pi r_+^4 - \frac{1}{6}ar_+ + \frac{1}{4}r_+^2 + \frac{\alpha}{2}, \tag{2.5}
\end{equation}

\begin{equation}
S = \int_{0}^{r_+} \frac{1}{T} \left( \frac{\partial M}{\partial r_+} \right) dr_+ = \pi \left( \frac{r_+^3}{3} + 4ar_+ \right), \tag{2.6}
\end{equation}

\begin{equation}
T = \frac{f'(r_+)}{4\pi} = \frac{1}{6\pi} \frac{8P\pi r_+^3 - a + 3r_+}{r_+^4 + 4\alpha}. \tag{2.7}
\end{equation}

It is interesting to note that the entropy of a cloud of strings does not affect on the black hole entropy. The black hole mass $M$ in the extended thermodynamics is treated as enthalpy, so with respect to the above thermodynamics definitions the first law of the black hole thermodynamics in an extended phase space reads

\begin{equation}
dM = TdS + VdP + Ada + Bd\alpha, \tag{2.8}
\end{equation}

where $V = \left( \frac{\partial M}{\partial P} \right)_{S,a,\alpha} = \frac{1}{3}\pi r_+^4$ is the thermodynamic volume, $A = \left( \frac{\partial M}{\partial a} \right)_{S,P,\alpha} = \frac{1}{2}$, and $B = \left( \frac{\partial M}{\partial \alpha} \right)_{S,P,a} = -\frac{1}{2}r_+$, stand for the physical quantities conjugated to the parameters $a$ and $\alpha$ respectively. Physical meanings of the conjugated potentials $A$ and $B$ needs further investigation. Using the scaling argument we can obtain the generalized Smarr relation for the GB AdS black hole in the presence of cloud of strings as follows.

\begin{equation}
2M = 3TS - 2VP + 2A\alpha + B\alpha. \tag{2.9}
\end{equation}

Finally we would like to discuss some about the constraints and singularities: by attention to ansatz (2.4) in order to have a well defined (without any imaginary or naked singularity [45]) vacuum solution ($M = 0$ and $a = 0$), GB coefficient $\alpha_{GB}$ must be restricted as $0 \leq \alpha_{GB} \leq \ell^2/8$. Also for holographic purposes $\alpha_{GB}$ is also constrained as $-7/72 \leq \alpha_{GB} \leq 9/200$ due to causality and positive definiteness of the boundary energy density [46, 47]. The non-vacuum solutions $f(r)$ must be also a real-valued function, so in general we encounter by two types of singularities: Causal singularity at $r = 0$ and
branch singularity for \( r > r_{br} \) in which \( r_{br} \) is defined when the square root in (2.4) is non-negative leads to minimum non-negative real root of the following polynomial equation

\[
 r_{br}^4 \left( 1 - \frac{8\alpha}{\ell^2} \right) + 16\alpha \left( \frac{a r_{br}}{3} + 2M \right) = 0. \tag{2.10}
\]

Therefore the allowed domain of radius from \( 0 < r < \infty \) reduces to \( 0 < r < r_{br} \). On the other side if \( r \to 0 \) then \( f(r) \) approaches \( 1 - \sqrt{\frac{m}{\alpha}} \) which demonstrates that usual singularity at \( r = 0 \) removed by GB term.

It is interesting to discuss some about the causality constraints in our model which comes from the requirement that the equation of motions of perturbation must be hyperbolic [48,49], so the perturbations propagate in a causal way. The hyperbolicity condition of the equation of motion of Lovelock theories as necessary condition for causality is equivalent to Lorentzian effective metric in field space. So if the effective metric be non-Lorentzian, it implies causality violation. The effective metric components for 5-dimensional GB theory regarding [49] are:

\[
 [G^{\alpha\alpha}]_{ij} = g^{\alpha\alpha} \left( 1 - 2\alpha \frac{f'(r)}{r} \right), \quad [G^{kk}]_{ij} = g^{kk} \left( 1 - 2\alpha f''(r) \right), \tag{2.11}
\]

in which \( g^{\alpha\alpha} \) and \( g^{kk} \) are Lorentzian, so the signature of effective metric depends on factors. Large values of \( r \) lead to a Schwarzschild behavior of the metric and so the effective metric would be Lorentzian. But the case for small \( r \) is different: it can be seen that for all values of \( M, \alpha_{GB}, \ell \) and \( a \) the factor in \( [G^{kk}]_{ij} \) is negative, \( 1 - 2\alpha f''(r) < 0 \), but factor in \( [G^{\alpha\alpha}]_{ij} \) can be positive or negative. Actually by attention to (2.4) it is simple to find \( r_* \approx 3.36 \left( \frac{a\alpha}{\pi \ell^2} - 1 \right) \), that when \( r > r_* \) factor \( [G^{\alpha\alpha}]_{ij} \) would be negative and therefore the effective metric takes Lorentzian signature, however for \( r < r_* \) causality violations happens.

### 3 Thermodynamics behavior

Applying (2.7) and \( V = \left( \frac{\partial M}{\partial P} \right)_{s,a,\alpha} = \frac{4}{3} \pi r_+^4 \) one can obtain equation of state of this black hole as follows.

\[
P = \frac{T(v^2 + \frac{64}{9}\alpha) + \frac{8}{27}a - \frac{2}{3}v}{\frac{\pi v^3}{T}} \tag{3.1}
\]
where \( v = \frac{4}{3}r_+ \) is called as the specific volume [4]. Solving \( \frac{\partial P}{\partial v} \bigg|_{T=T_c} = 0 \) and \( \frac{\partial^2 P}{\partial v^2} \bigg|_{T=T_c} = 0 \), one can obtain the critical points as follows.

\[
v_c = \frac{2(a + \sigma)}{3}, \quad T_c = \frac{\sigma}{\pi(a^2 + a\sigma + 48\alpha)}, \quad P_c = \frac{a^3 + a^2\sigma + 48\alpha a + 24\sigma a}{\pi(a^2 + a\sigma + 48\alpha)(a + \sigma)^3}.
\]

where \( \sigma = \sqrt{a^2 + 48\alpha} \) the subscript \( c \) denotes to the critical point. By comparing our results with [10] for uncharged case it would be interesting to note that the effect of a cloud of strings reflected in all critical points which by putting \( a = 0 \) these critical points reduced to the result of an uncharged GB black hole solution. Note that universal Van der Waals ratio in this case depends on the values of GB and string cloud factors. We plot the \( P - v \) isotherm graph with different values of string cloud parameter \( a = \{0.1, 1\} \) and for weak GB counterpart \( \alpha = 0.001 \) in figure 1-a. The solid lines correspond to the ideal gas phase transition when temperature is above the critical value which shows a monotonically decreasing behavior. Upper than the critical temperature the dashed lines can be divided to three different branches. They indicate to large, small and medium black hole branches. The medium black hole branch is unstable while the large black hole and the small black hole branches are stable which mimic the Van der Waals liquid/gas phase transition. In figure 1-b we plotted \( P - v \) diagrams for temperature below the critical temperature in which \( \alpha \) alters. We can see the diagram takes a shifting and pressure goes to be negative by increasing GB coefficient. Also there exists a particular temperature \( T_0 \) similar to the one which is obtained for Van der Waals fluid. It is achieved by solving \( \frac{\partial P}{\partial v} = P = 0 \), which below this temperature the pressure is not be positive for some horizon radius, and of course could be remedied by Maxwell construction. However \( \frac{\partial P}{\partial v} = P = 0 \), reads \( v_0 = \frac{4}{9}(a + \xi) \) and \( T_0 = \frac{3}{4\pi(a^2 + a\xi + 72\alpha)} \).

where \( \xi = \sqrt{a^2 + 36\alpha} \) and is depicted for \( a = 1 \) and \( \alpha = 0.01 \) in figure 1-c. It is of vital importance to study the behavior of Gibbs free energy to analysis global stability of the system. In the extended phase space the Gibbs free energy is given by \( G = M - TS \) and for the black hole under consideration takes the form

\[
G = -\frac{1}{36} \frac{\left(144P\pi r^4 + 4ar^3 - 3r^4 + 18a^2r^2 - 72\alpha a^2\right)}{r^2 + 4\alpha}.
\]

By analysing the Gibbs free energy it can be revealed that at pressure lower
Figure 1: $P - V$ curves. (a): Solid lines correspond to $a = 0.1, \alpha = 0.01$ with $T_c = 0.397887$ and dash lines correspond to $a = 1, \alpha = 0.01$ with $T_c = 0.143605$, (b): diagrams are plotted for $a = 1, T < T_c$ with arbitrary value for critical temperature $T_c$. Solid red line for $\alpha = 0.00001$ ($T = 0.8T_c$), dash blue line for $\alpha = 0.01$ ($T = 0.8T_c$), dash green line for $\alpha = 0.1$ ($T = 0.8T_c$), dash dot orange line for $\alpha = 0.5$ ($T = 0.6T_c$) and dot black line for $\alpha = 1$ ($T = 0.4T_c$). (c): Diagrams are plotted for $a = 1, \alpha = 0.01$. Dot lines for upper temperature $T = 0.183605$, dash lines for critical temperature $T_c = 0.143605$ and solid line for $T_0 = 0.110208$. 
than the critical value the Gibbs function displays a characteristic swallow-tail behavior which implies that at this point the small black hole jumps into a large black hole via a first order phase transition. It must also be noted that an important behavior for $\alpha \rightarrow 0$ and $a \rightarrow 0$ at which branch of small black hole overlaps with $T$ axis, so that small black hole evaporates to AdS space. This means that the LBH/SBH phase transition reduces to "BH/AdS" phase transition which is well known as Hawking-Page phase transition. It is very interesting to note that in the limit $\alpha \rightarrow 0$ but surrounding by a cloud of strings, the Hawking-Page phase transition disappears and the LBH/SBH phase transition recovers. We can interpret figure 2 in such a way that in the absence of the GB term namely the Schwarzschild black hole, the impact of a cloud of strings can bring the SBH/LBH phase transition while this phase transition does not occur in Schwarzschild type. In figure 2 we can see the effect of GB coefficient on $G - T$ diagram which just move location of the phase transition on the diagram. The coexistence line in $P - T$ diagram in

![Figure 2](image-url)

Figure 2: $G - T$ curves. (a): for $a = 1, \alpha = 0.5$. (b): for $a = 0.1, \alpha = 0.001$. (c): for $a = 1$ and various $\alpha$ with $P < P_c$ as indicated by solid red line for $\alpha = 0.00001$ ($T = 0.4T_c$), dash blue line for $\alpha = 0.01$ ($T = 0.4T_c$), dash green line for $\alpha = 0.1$ ($T = 0.4T_c$), dash dot orange line for $\alpha = 0.5$ ($T = 0.5T_c$) and dot black line for $\alpha = 1$ ($T = 0.4T_c$) and arbitrary value on $T_c$.

Figure 3 can displays phase transition in a different view and happens where two surfaces of Gibbs free energy cross each other. Since the black hole undergoes a first order phase transition so the both phases have the same Gibbs free energy. To study the behavior of system along the coexistence
line it just enough to solve the following equalities between two phases at any arbitrary point on this line:

\[ G_1 = G_2, \quad T_1 = T_2, \quad 2T = T_1 + T_2, \]  

which are defined by regarding to \( \nu_1 \) and \( \nu_2 \) for two different phases. The two last equalities denote to isothermal transition. Solving these three equations we obtain equation of pressure with respect to temperature which is plotted in figure 3-a and 3-b for some values of \( a \) parameter and displays the effect of string clouds on \( P - T \) diagram. We can also study isenthalpic curves in

![Figure 3: The coexistence lines in \( p - T \) diagram which are ended at critical points. (a): for \( a = 1 \) and \( \alpha_{GB} = \{1, 0.5, 0.1\} \) up to down, respectively. (b): for \( \alpha_{GB} = 1 \) and \( a = \{1, 0.1, 0.001\} \) up to down, respectively.](image)

\[ T - P \] plan and see the behavior of our thermodynamic system in an expansion process so called Joule-Thomson expansion [35, 36]. Originally in classical thermodynamics this expansion describes the temperature change of a gas or liquid through a porous plug. In this process temperature of the system changes with respect to pressure in a constant enthalpy and it can show heating and cooling phases. Actually sign of the Joule-Thomson coefficient as \( \mu_{JT} = \left( \frac{\partial T}{\partial P} \right)_H \) indicates the phase of the gas. During the expansion \( \mu_{JT} > 0 \) denotes the cooling process in which pressure decreases and \( \mu_{JT} < 0 \) corresponds heating in which pressure increases. To find out the process in our model we can substitute \( r_+ \) from (2.5) into the Hawking
temperature (2.7) and plot $T - P$ curves for different constant mass $M$. We do it in figure 4. The process clearly has just a cooling phase and none of the model parameters could force it to enter a heating phase as we can see in [37] in which chemical potentials drive the process into a cooling-heating expansion. So in contrary with AdS black holes investigated in [37-44] our case doesn’t follow a heating-cooling process in a Joule-Thomson expansion and there is not any inversion temperature. At last, we apply the Ehrenfest’s equations to study an analytic verification of the nature of the phase transition in the critical points. As we know from classical thermodynamics the first and second equations of Ehrenfest are as follows [33].

$$\frac{\partial P}{\partial T} \bigg|_S = \frac{C_{P2} - C_{P1}}{TV(\zeta_2 - \zeta_1)} = \frac{\Delta C_P}{TV\Delta \zeta},$$

(3.5)

$$\frac{\partial P}{\partial T} \bigg|_V = \frac{\zeta_2 - \zeta_1}{\kappa_{T2} - \kappa_{T1}} = \frac{\Delta \zeta}{\Delta \kappa_T},$$

(3.6)

where $\zeta$ and $\kappa_T$ are coefficients of the volume expansion and isothermal compressibility of the system respectively. The volume expansion coefficient can be written as

$$V \zeta = \frac{\partial V}{\partial T} \bigg|_P = \frac{\partial V}{\partial S} \bigg|_P \times \frac{C_P}{T},$$

(3.7)
so the first Ehrenfest’s equation takes the form

\[ \frac{\partial P}{\partial T} \bigg|_S = \frac{\partial S}{\partial V} \bigg|_P. \]  

(3.8)

By using (3.4), the equation of (3.8) can be calculated as \( \frac{3(v^2 + 4\alpha)}{4r^3} = \frac{3(v^2 + 4\alpha)}{4r^3} \)

which shows that the first equation of Ehrenfest is satisfied at the critical point.

In order to verify the second Ehrenfest’s equation, we use the thermodynamic identity \( \frac{\partial V}{\partial P} \bigg|_T \times \frac{\partial P}{\partial T} \bigg|_V \times \frac{\partial T}{\partial V} \bigg|_P = -1 \) to obtain isothermal compressibility coefficient as

\[ \kappa_T = \frac{\partial T}{\partial P} \bigg|_V \zeta, \]  

(3.9)

So the second Ehrenfest’s equation would be automatically satisfied, as well. On the other hand, the ratio of Prigogine-Defay is calculated as follows

\[ \Pi = \frac{\Delta C_p \Delta \kappa_T}{T_v (\Delta \zeta)^2} = 1. \]  

(3.10)

So we can conclude that the second-order phase transition occurs in the black hole which is compatible with the transition of the liquid-gas phase in the classical thermodynamics. Comparing our results with the GB black holes [10], it seems quite interesting to note that a cloud of strings does not effect on the validation of Ehrenfest’s equations.

### 4 Critical exponents

To study the behavior of physical quantities near the critical point it would be useful to use rescaled quantities (3.4) for which (3.1) reaches to a re-scaled form as follows.

\[ p = \left( \frac{T_c}{v_c P_c} \right)^\tau + \left( \frac{64\alpha T_c}{9v_c^2 P_c} \right)^{\frac{\tau}{\nu^3}} - \left( \frac{2}{3\pi v_c^2 P_c} \right)^{\frac{1}{\nu^2}} + \left( \frac{8\alpha}{27\pi v_c^2 P_c} \right)^{\frac{1}{\nu^3}}. \]  

(4.1)

The above equation of state is called as “law of corresponding state”. Defining new expansion parameters \( t, \omega \) such that

\[ \tau = 1 + t, \quad \nu = 1 + \omega \]
which these are expanded around 1, we can seek thermodynamical behavior of the system near the critical points. Therefore the law of corresponding state (4.1) would be approximated as

\[ p = 1 + At + Bt\omega + C\omega^3 + \mathcal{O}(t\omega^2, \omega^4), \]  

(4.2)

where \( C = -1 \) and

\[ A = \frac{3\sigma(a\sigma + a^2 + 32\alpha)}{a^3 + a^2\sigma + 48a\alpha + 24\alpha\sigma}, \quad B = -\frac{3\sigma(a\sigma + a^2 + 48\alpha)}{a^3 + a^2\sigma + 48a\alpha + 24\alpha\sigma}. \]  

(4.3)

In the other hand critical exponents for \( T < T_c \) (or \( t < 0 \)) are defined by

\[ C_\upsilon = T_\frac{\partial S}{\partial T} \bigg|_{\upsilon} \propto |t|^{-\alpha}, \quad \eta = \upsilon_l - \upsilon_s \propto |t|^{\beta}, \quad \kappa_T = -\frac{1}{\upsilon} \frac{\partial \upsilon}{\partial P} \bigg|_T \propto |t|^{-\gamma}, \]  

and

\[ |P - P_c| \propto |\upsilon - \upsilon_c|^{\delta} \]  

in which \( \alpha \) shows the behavior of the specific heat at constant volume, \( \beta \) shows the isotherm behavior of the order parameter \( \eta \), \( \gamma \) describes the behavior of the isothermal compressibility and the last exponent \( \delta \) determines the behavior of pressure in an isothermal process corresponding to \( T = T_c \). The subscripts \( l \) and \( s \) denote the large black hole and small black hole respectively in the process of phase transition.

As we can see from (2.6) the entropy is independent of \( T \), so the specific heat vanishes \((C_\upsilon = 0)\) and so \( \alpha = 0 \). To obtain the second exponent we have to evaluate \( \upsilon_l \) and \( \upsilon_s \) to obtain the order parameter. The approximated pressure (4.2) does not changed during the isotherm phase transition, which means \( p_l = p_s \) for which

\[ Bt(\omega_l - \omega_s) + C(\omega_l^3 - \omega_s^3) = 0. \]  

(4.4)

Applying the Maxwells equal area law we lead to another relationship between \( \omega_l \) and \( \omega_s \) such that

\[ \int_{\omega_l}^{\omega_s} \omega \frac{dp}{d\omega} d\omega = 0 \rightarrow Bt(\omega_l^2 - \omega_s^2) + \frac{3}{2} C(\omega_l^3 - \omega_s^3) = 0. \]  

(4.5)

From two the above equations one can obtain \( \omega_l = -\omega_s = \sqrt{-Bt/3C} \), so \( \eta = \upsilon_l - \upsilon_s \propto \sqrt{-t} \) and therefore we can derive \( \beta = \frac{1}{2} \). The isothermal compressibility could be calculated for \( \upsilon = \upsilon_c(1 + \omega) \) as follows.

\[ \kappa_T = -\frac{1}{\upsilon_c(1 + \omega)} \frac{\partial \upsilon}{\partial \omega} \bigg|_T = -\frac{1}{P_c(1 + \omega)} \left( \frac{1}{Bt} + \mathcal{O}(\omega^2) \right), \]  

(4.6)
so the third exponent is achieved as $\gamma = 1$. The final exponent is obtained in an isotherm process at critical temperature $T = T_c$ or $t = 0$, so from (4.2) we have $p - 1 = C\omega^3$ that leads to $\delta = 3$. From the results in [8] we can see that the critical exponent are the same as the GB AdS black hole exponents, so the effect of string cloud would not change them and both models (with and without string cloud effects) have the same scaling laws.

5 Conclusion

In this paper, we extend the previous research [10] to the extended phase space of thermodynamics and studied the critical phenomena of the EGB black hole solution in a cloud of strings background. Here we have considered the cosmological constant as a thermodynamical pressure. Thermodynamical quantities such as temperature, entropy, pressure and the Gibbs free energy are studied to probe the thermodynamic stability of the black hole. We have investigated the analogy between the EGB black holes surrounded by a cloud of strings and Van der Waals fluid. We successfully derived the critical points of the system and studied $P - V$ critically in details. It is shown that the critical quantities and Van der Waals universal ratio is affected by string cloud. It is also shown that in the absence of the GB term (Schwarzschild black hole), the impact of a cloud of strings can almost bring SBH/LBH phase transition. In the other words, in the EGB black hole with $\alpha \rightarrow 0$ limit and surrounding by a cloud of strings, the Hawking-Page phase transition disappears and SBH/LBH phase transition recovers. By studying $T - P$ diagram we found this system never enter to a heating phase and in an Joule-Thomson expansion it cools forever. At last we investigated the behavior of our black hole solution near critical point by studying critical exponents and concluded that string could not be effective and does not change them. As a future work it also would be interesting to study thermodynamics of the Lovelock black hole solution surrounded by a cloud of strings in the extended phase space.

Acknowledgments

The authors are grateful to the editor and anonymous referees for their valuable comments and suggestions to improve the paper.

References

1. S. W. Hawking, D.N. Page, *Thermodynamics of black holes in anti-de
2. S. Nojiri and S. D. Odintsov ”Anti de Sitter black hole thermodynamics in higher derivative gravity and new confining deconfining phases in dual CFT”, Phys. Lett. B 521, 87 (2001); hep-th/0109122; Erratum: Phys. Lett. B 542, 301 (2002).

3. M. Cvetic, S. Nojiri and S. D. Odintsov, ”Black hole thermodynamics and negative entropy in de Sitter and anti de Sitter Einstein- Gauss-Bonnet gravity ”, Nucl. Phys. B628, 295 (2022).

4. D. Kastor, S. Ray, & J. Traschen, Enthalpy and mechanics of AdS black holes, Class. Quant. Grav. 26, 195011, (2009).

5. B. P. Dolan, The cosmological constant and black hole thermodynamic potential, Class. Quant. Grav. 28, 125020, (2011).

6. D. Kubiznak & R. B. Mann, P-V criticality of charged AdS black holes, JHEP 1207, 033, (2012).

7. S. Dutta, A. Jain and R. Soni, Dyonic Black Hole and Holography, JHEP 2013, 60, (2013); hep-th/1310.1748.

8. Gu-Qiang Li, Effects of dark energy on P-V criticality of charged AdS black holes, Physics Letters B 735, 256, (2014).

9. M. Zhang & W Liu,l, Coexistent physics of massive black holes in the phase transitions, gr-qc/1610.03648.

10. R Cai, L. Cao, & R. Yang,, P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space, JHEP1309, 005, (2013).

11. R. Cai, Y. Hu, Q. Pan, & Y. Zhang, Thermodynamics of black holes in massive gravity, Phys. Rev. D 91, 024032, (2015) hep-th/1409.2369.

12. R. Zhao, H. H. Zhao, M. S. Ma and L. C. Zhang, On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes, Eur. Phys. J. C 73, 2645, (2013).

13. J. Mo, G. Li, & X. Xu, Combined effects of f(R) gravity and conformally invariant Maxwell field on the extended phase space thermodynamics of higher-dimensional black holes, Eur. Phys. Jour. C 76, 545, (2016).
14. J. X. Mo, X. X. Zeng, G. Q. Li, X. Jiang, W. B. Liu, A unified phase transition picture of the charged topological black hole in Hořava-Lifshitz gravity,, JHEP 1310, 056, (2013).

15. N. Altamirano, D. Kubizňák, R. Mann and Z. Sherkatghanad, Kerr-AdS analogue of critical point and solid/liquid/gas phase transition, Class, Quantum, Grav. 31, 4, 042001, (2013), hep-th/1308.2672.

16. J. X. Mo and W. B. Liu, Ehrenfest scheme for $P-V$ criticality in the extended phase space of black holes, Phys. Lett. B 727, 336, (2013).

17. N. Altamirano, D. Kubizňák and R. Mann, Reentrant phase transitions in rotating AdS black holes, Phys. Rev. D 88, 101502, (2013).

18. D. C. Zou, S. J. Zhang and B. Wang, Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics, Phys. Rev. D 89, 044002, (2014).

19. R. Zhao, H. H. Zhao, M. S. Ma and L. C. Zhang, On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes, Eur. Phys. J. C 73, 2645, (2013).

20. H. Ghaffarnejad, E. Yaraie and M. Farsam, Quintessence Reissner Nordstrm anti de Sitter black holes and Joule Thomson effect., Int. J. Theor. Phys, 57, 1671, (2018), gr-qc/1802.08749.

21. H. Liu and X.-h. Meng, $PV$ criticality in the extended phase space of charged accelerating AdS black holes, Mod. Phys. Lett. A 31, 1650199, (2016).

22. D. Hansen, D. Kubiznak and R. B. Mann, Universality of $P-V$ Criticality in Horizon Thermodynamics, JHEP 01, 047 (2017), gr-qc/1603.05689.

23. H. Estanislao and M. G. Richarte, Black holes in Einstein-Gauss-Bonnet gravity with a string cloud background, Phys. Lett. B 689, 192 (2010).

24. A. Strominger, C. Vafa, Microscopic origin of the Bekenstein-Hawking entropy, Phys. Lett. B 379, 99, (1996).

25. P. S. Letelier, Clouds Of Strings In General Relativity, Phys. Rev. D 20, 1294, (1979).
26. M. G. Richarte and C. Simeone, *Traversable wormholes in a string cloud*, Int. J. Mod. Phys. D 17, 1179, (2008).

27. A. K. Yadav, V. K. Yadav and L. Yadav, *Cylindrically symmetric inhomogeneous universes with a cloud of strings*, Int. J. Theor. Phys. 48, 568, (2009).

28. A. Ganguly, S. G. Ghosh and S. D. Maharaj, *Accretion onto a black hole in a string cloud background*, Phys. Rev. D 90, 064037, (2014).

29. K. A. Bronnikov, S. W. Kim and M. V. Skvortsova, *The Birkhoff theorem and string clouds*, Class. Quantum Grav. 33, 195006, (2016).

30. S. G. Ghosh and S. D. Maharaj, *Cloud of strings for radiating black holes in Lovelock gravity*, Phys. Rev. D 89, 084027, (2014).

31. S. G. Ghosh, U. Papnoi and S. D. Maharaj, *Cloud of strings in third order Lovelock gravity*, Phys. Rev. D 90, 044068, (2014).

32. T. H. Lee, D. Baboolal and S. G. Ghosh, *Lovelock black holes in a string cloud background*, Eur. Phys. J. C 75, 297, (2015).

33. M. W. Zemansky and R.H. Dittman, *Heat and thermodynamics: an introduction*, McGraw-Hill (1997).

34. N. Goldenfeld, "Lectures on phase transitions and the renormalization group" CRC Press, (2018).

35. D. Winterbone and A. Turan. Advanced Thermodynamics for Engineers. Butterworth-Heinemann, (1996).

36. D. C. Johnston, ”Thermodynamic Properties of the van der Waals Fluid.” Cond-mat.soft/1402.1205 (2014).

37. J. X. Mo and G. Q. Li, ”Effects of Lovelock gravity on the Joule-Thomson expansion” gr-qc/1805.04327 (2018).

38. O. Ozgur and E. Aydner. ”JouleThomson expansion of the charged AdS black holes”, Eur. Phys. J. C 77, 24 (2017).

39. O. Ozgur and E. Aydner. ”JouleThomson expansion of KerrAdS black holes,” Eur. Phys. J. C 78, 123 (2018).
40. M. Chabab, H. E. Mounni, S. Iraoui, K. Masmar, and S. Zhizeh. "Joule-Thomson Expansion of RN-AdS Black Holes in $f(R)$ gravity" gr-qc/1804.10042 (2018).

41. J. X. Mo, G. Q. Li, S. Q. Lan, and X. B. Xu. "Joule-Thomson expansion of $d$-dimensional charged AdS black holes" gr-qc/1804.02650 (2018).

42. S. Q. Lan, "Joule-Thomson expansion of charged Gauss-Bonnet black holes in AdS space." gr-qc/1805.05817 (2018).

43. C. L. A. Rizwan, A. N. Kumara, D. Vaid, and K. M. Ajith. "Joule-Thomson expansion in AdS black hole with a global monopole." gr-qc/1805.11053 (2018).

44. Z. W. Zhao, Y. H. Xiu and N. Li. "On the throttling process of the Kerr–Newman–anti-de Sitter black holes in the extended phase space.” gr-qc/1805.04861 (2018).

45. D. L. Wiltshire, "Spherically symmetric solutions of Einstein-Maxwell theory with a Gauss-Bonnet term." Phys. Lett. B 169, 36 (1986).

46. M. Brigante, H. Liu, R. C. Myers, S. Shenker, and S. Yaida. "Viscosity bound violation in higher derivative gravity.” Phys. Rev. D 77, 126006 (2008).

47. A. Buchel and R. C. Myers. "Causality of holographic hydrodynamics.” JHEP 2009, 016 (2009).

48. H. S. Reall, N. Tanahashi and B. Way. "Causality and hyperbolicity of Lovelock theories." Class. Quantum Grav. 31, 205005 (2014).

49. R. Brustein and Y. Sherf. "Causality violations in Lovelock theories.” Phys. Rev. D 97, 084019 (2018); hep-th/1711.05140.

50. T. K. Dey, "Thermodynamics of AdS Schwarzschild black hole in the presence of external string cloud.” hep-th/1711.07008 (2017).

51. S. H. Mazharimousavi and M. Halilsoy. "Cloud of strings as source in 2 + 1-dimensional $f(R) = R^n$ gravity.” Eur. Phys. J. C 76, 95 (2016).

52. S. G. Ghosh, S. D. Maharaj, D. Baboolal and T. H. Lee. "Lovelock black holes surrounded by quintessence.” Eur. Phys. J. C 78, 90 (2018).
53. de M. T. Jefferson and V. B. Bezerra. "Black holes with cloud of strings and quintessence in Lovelock gravity." Eur. Phys. J. C 78, 534 (2018).

54. A. Anabalon, M. Appels, R. Gregory, D. Kubiznak, R. B. Mann, and A. Övgün. "Holographic Thermodynamics of Accelerating Black Holes." hep-th/1805.02687 (2018).

55. A. Övgün, $P - V$ criticality of a specific black hole in $f(R)$ gravity coupled with Yang-Mills field, Adv. High Energy Phys. 8153721 (2018); gr-qc/1710.06795.

56. R. Zhao, H. H. Zhao, M. S. Ma, and L. C. Zhang. "On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes." Eur. Phys. J. C 73, 2645 (2013); gr-qc/1305.3725.

57. M. S. Ma, F. Liu, and R. Zhao. "Continuous phase transition and critical behaviors of 3D black hole with torsion." Class. Quantum Grav. 31, 095001 (2014); gr-qc/1403.0449.

58. I. Sakalli, and A. A. Övgün. "Black hole radiation of massive spin-2 particles in (3+1) dimensions." Eur. Phys. J. Plus 131, 184 (2016); gr-qc/1605.02689.

59. I. Sakalli and A. Övgün. "Tunnelling of vector particles from Lorentzian wormholes in 3+ 1 dimensions." Eur. Phys. J. Plus 130, 110, (2015).

60. M. S. Ma and R. Zhao. "Phase transition and entropy spectrum of the BTZ black hole with torsion." Phys. Rev. D 89, 044005 (2014); gr-qc/1310.1491.

61. I Sakalli and A. Övgün. "Uninformed Hawking radiation." Eur. Phys. Let. 110, 10008 (2015); gr-qc/1409.5539.

62. I. Sakalli and A. Övgün, " Quantum tunneling of massive spin-1 particles from non-stationary metrics", Gen. Rel. Gravit. 48, 1, 1, (2016); gr-qc/1507.01753.

63. H. H. Zhao, L. C. Zhang, M. S. Ma, and R. Zhao, "P-V criticality of higher dimensional charged topological dilaton de Sitter black holes" Phys. Rev. D90, 064018 (2014).