Generalizing Two Structure Theorems of Lie Algebras to the Fuzzy Lie Algebras

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Abstract

In this paper we generalize two structure theorems of the class of Lie algebras to the class of fuzzy Lie algebras, namely the structure theorem of semisimple Lie algebras and the Levi’s decomposition theorem. Some open questions are also given.

Keywords: Semisimple fuzzy lie algebras; Levi's fuzzy decomposition; Fuzzy lie algebras

Introduction

Lie algebras were proposed by Sophus Lie [1] and there are many applications of them in several branches of physics [2]. The notion of fuzzy sets was introduced by Zadeh [3] and many mathematicians have been involved in extending the concepts and results of abstract Lie algebra to fuzzy theory. This paper is the continuation of the results obtained in [4], where we presented conditions to generalize the concepts of solvable and nilpotent radicals of Lie algebras (called of solvable and nilpotent fuzzy radicals, respectively) to a class of fuzzy Lie algebras. In this article we use the solvable fuzzy radical to generalize the structure theorem of semisimple Lie algebras and the Levi’s decomposition theorem to a class of the fuzzy Lie algebras. The results presented in this paper are still strongly connected with results proved in [5–10].

Fuzzy Sets, Fuzzy Lie Algebras and Fuzzy Lie Ideals

In this section we present the basic concepts of fuzzy sets, fuzzy Lie algebras, fuzzy ideals among others which will be used throughout this paper. More details referring to these concepts and its properties can be found in [4].

A mapping of a non-empty set \( X \) into the closed unit interval \([0, 1]\) is called a fuzzy set of \( X \) and the set \( \{ (x, \mu(x)) \mid x \in X \} \) is called the image of denoted by \( \mu(X) \). For all real \( \epsilon \in [0, 1] \) the subset \( \mu^{-1}([\epsilon, 1]) \) is called a \( \epsilon \)-level set of \( \mu \) and the set \( \{ x \mid x \in X, \mu(x) > 0 \} \) is called the support of \( \mu \) denoted by \( \mu^* \).

A set \( S \subseteq [0, 1] \) is said to be an upper well ordered set if for all non-empty subsets \( C \subseteq S \), then \( \sup \in C \). One defines the set

\[
F(X, S) = \{ \nu \mid \nu \text{ is an fuzzy set of } X \text{ such that } \nu(X) \subseteq S \}.
\]

Let \( L \) be a Lie algebra over a field \( F \). A fuzzy set \( \mu \) of \( L \) is called a fuzzy Lie algebra of \( L \) if satisfies the following conditions: (i) \( \mu(ax + by) \geq \mu(x) \wedge \mu(y) \), (ii) \( \mu(xy) \geq \mu(x) \wedge \mu(y) \) and (iii) \( \mu(0) = 1 \), for all \( a, b \in F \) and \( x, y \in L \). A fuzzy set \( \nu \) of \( L \) is called a fuzzy subalgebra of \( L \) if \( \nu \) is a fuzzy Lie algebra of \( L \) satisfying \( \nu(x) \leq \mu(x) \) for all \( x \in L \). One has that \( \mu \) is a fuzzy Lie algebra of \( L \) if, and only if, the \( \epsilon \)-level sets \( \mu^{-1}([\epsilon, 1]) \) are subalgebras of \( L \), for all \( \epsilon \in [0, 1] \). Also, \( \nu \) is a fuzzy subalgebra of \( \mu \) if, and only if, the \( \epsilon \)-level sets \( \nu^{-1}([\epsilon, 1]) \) are subalgebras of \( \mu^{-1}([\epsilon, 1]) \), for all \( \epsilon \in [0, 1] \). Moreover, if \( \mu \) is a fuzzy Lie algebra of \( L \) then \( \mu^{-1}(0) \) is a subalgebra of \( L \).

A fuzzy set \( \nu \) of \( L \) is called a fuzzy ideal of \( L \) if satisfies the following conditions: (i) \( \nu(ax + by) \geq \nu(x) \wedge \nu(y) \), (ii) \( \nu(xy) \geq \nu(x) \wedge \nu(y) \) and (iii) \( \nu(0) = 1 \), for all \( a, b \in F \) and \( x, y \in L \). A fuzzy set \( \nu \) of \( L \) is called a fuzzy ideal of \( L \) satisfying \( \nu(x) \leq \mu(x) \) for all \( x \in L \). One has that \( \nu \) is a fuzzy ideal of \( L \) if, and only if, the \( \epsilon \)-level sets \( \nu^{-1}([\epsilon, 1]) \) are ideals of \( L \), for all \( \epsilon \in [0, 1] \). Moreover, any fuzzy ideal of \( L \) is a fuzzy subalgebra of \( L \), any fuzzy ideal of \( \mu \) is a fuzzy subalgebra of \( \mu \) and if \( \nu \) is a fuzzy ideal of \( L \) then \( \nu \) is an ideal of \( L \).

We define the null fuzzy algebra of \( \mu \) as the fuzzy set of \( L \):

\[
\nu(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases}
\]

and as a consequence of this definition we assume that our upper set \( S \) has the real numbers 0 and 1.

A fuzzy Lie algebra \( \mu \) of \( L \) is called abelian if \( \mu^2 = 0 \) and non-abelian otherwise.

If \( \nu_1, \nu_2, \ldots, \nu_m \) are fuzzy sets of \( L \), one defines: (i) The fuzzy set \( \sum_{i=1}^{m} \nu_i \) (sum) as \( \sum_{i=1}^{m} \nu_i \) satisfying the condition \( \nu_i \geq 0 \) for all \( i \); (ii) The fuzzy set \( \bigoplus_{i=1}^{m} \nu_i \) (direct sum) as the fuzzy set \( \sum_{i=1}^{m} \nu_i \) satisfying the condition \( \nu_i \geq 0 \) for all \( i \); and (iii) The fuzzy set \( \nu_1 \nu_2 \) of \( L \) (product) as \( \nu_1 \nu_2 \).

To understand the main result of [4] which will be used at the end of this section and in the remainder of this article, we present the following definition.

For any fuzzy subalgebra \( \nu \) of a fuzzy algebra \( \mu \) one defines inductively the derived series of \( \nu \) as the descending chain of fuzzy subalgebras of \( \mu \) satisfying \( \nu^{(n+1)} = 0 \) for all \( n \geq 1 \) and the lower central series of \( \nu \) as the descending chain of fuzzy subalgebras of \( \mu \) satisfying \( \nu^{(n+1)} = 0 \) for all \( n \geq 1 \). The fuzzy subalgebra \( \nu \) is said solvable (resp., nilpotent) if there exists an integer \( k = k(\nu) \geq 1 \) such that \( \nu^{(k)} = 0 \) (resp., \( \nu^{(k)} = 0 \)).

Theorem 2.1. [4, Theorem 24] Let \( L \) be a finite dimensional Lie algebra.
algebra over a field \( F \) and \( S \) an upper well ordered set. Then every solvable (resp., nilpotent) fuzzy ideal \( \nu \) of \( \mu \) in \( F(L,S) \) is contained in a unique maximal solvable (resp., nilpotent) fuzzy ideal of \( \mu \) in \( F(L,S) \), called solvable (resp., nilpotent) fuzzy radical of \( \mu \) in \( F(L,S) \) and denoted by \( R(\mu; S) \) (resp., \( N(\mu; S) \)).

Let \( \mu \) be a fuzzy Lie ideal of \( L \). One says that \( \mu \) is a simple fuzzy ideal if: (i) \( \mu \) is a non-abelian fuzzy ideal and (ii) for all fuzzy ideals \( \nu \) of \( \mu \), one has either \( [\nu] = [\mu] \) or \( [\nu] = [0] \) for all \( t \in [0,1] \).

To conclude this section, we extend the notion of a semisimple fuzzy ideal [4, Definition 26] for a semisimple fuzzy algebra.

Let \( L \) be a finite dimensional Lie algebra over a field \( F \), \( S \) an upper well ordered set and \( \mu \) a fuzzy algebra of \( L \) in \( F(L,S) \). One says that \( \mu \) is a semisimple fuzzy algebra in \( F(L,S) \) if: (i) \( \mu \) is a non-abelian fuzzy algebra and (ii) its solvable fuzzy radical in \( F(L,S) \) is \( 0 \), that is, \( R(\mu; S) = 0 \).

### Semisimple Fuzzy Ideals

In this section, we generalize the theorem of decomposition of a semisimple Lie algebra as a direct sum of simple Lie ideals for the case of a semisimple fuzzy ideal, similarly to the crisp case. For this, we begin with the following definition.

**Definition 3.1.** Let \( L \) be a Lie algebra over a field \( F \), \( S \) an upper well ordered set and \( \mu \) a fuzzy ideal of \( L \). One says that a fuzzy set \( \pi \) of \( L \) is a fuzzy ideal of \( \mu \) relative to \( \pi' \) if the following conditions are satisfied: (i) \( \pi \preceq \mu \) (and hence \( [\pi] \subseteq [\mu] \), for all \( t \in [0,1] \)); (ii) \( [\pi] \) is an ideal of \( \mu \) for all \( t \in [0,1] \).

In this case, \( \pi' \) is also an ideal of \( \mu \).

If \( \pi \) is a fuzzy ideal of \( \mu \) relative to \( \mu' \), then one says that a fuzzy set \( \sigma \) of \( L \) is a fuzzy ideal of \( \nu \) relative to \( \pi' \) if the following conditions are satisfied: (iii) \( \sigma \preceq \pi \) (and hence \( [\sigma] \subseteq [\pi] \), for all \( t \in [0,1] \)); (iv) \( [\sigma] \) is an ideal of \( \pi \) for all \( t \in [0,1] \).

In this case, \( \sigma' \) is also an ideal of \( \pi' \).

One says that a fuzzy ideal \( \pi \) of \( \mu \) relative to \( \mu' \) is a simple fuzzy ideal of \( \mu \) relative to \( \pi' \) if the following conditions are satisfied: (v) \( \pi = \{0\} \) or \( \pi = \{\mu\} \) for all \( t \in [0,1] \).

Moreover:

**Theorem 3.3.** Let \( L \) be a finite dimensional Lie algebra over a field \( F \), \( \mu \) a fuzzy Lie ideal of \( L \) and \( \pi \) a non-abelian (in \( \mu \)) fuzzy ideal of \( \mu \) relative to \( \mu' \). If \( \pi \) is a simple fuzzy ideal of \( \mu \) relative to \( \mu' \), then \( \pi' \) is not a solvable ideal of \( \mu' \).

Moreover, \( \pi \) is a simple fuzzy ideal of \( \mu \) relative to \( \mu' \) if, and only if, \( \pi' \) is a simple ideal of \( \mu' \).

**Proof.** First, let us observe that \( (\pi')^t \) is not a zero ideal of \( \mu' \) since \( \pi' \) is non-null in \( \mu' \). Since \( L \) is finite dimensional, there is a finite set of real ideals \( \{\nu_1 = 0 < \nu_2 < \ldots < \nu_k < 1 = \nu_r\} \) such that \( [\pi'] = [\pi] \) for all \( t \in [0,1] \). This implies that \( [\pi'] = [\pi] \). In this case, \( \pi' \) is a solvable ideal of \( \mu' \), and \( \pi' \) is a simple ideal of \( \mu' \).

Now let us consider \( J \), an ideal of \( \pi' \). Again, by Nogota-Ralescu representation [7, Theorem 2.10], let us consider the fuzzy set \( S \) of \( L \) defined by the level sets: \( [\sigma] = \{[\sigma] \in [\pi] \} \) for all \( t \in [0,1] \). It follows that \( J = \{\nu \in \pi' \} \). So \( \pi' \) is a simple ideal of \( \mu' \). Reciprocally, let us consider a fuzzy ideal \( \pi' \) of \( \mu' \). Then \( [\sigma] \subseteq [\pi] \) and \( [\pi'] = [\pi] \). In this case, \( \pi' \) is a simple ideal of \( \mu' \).

**Theorem 3.4.** Let \( L \) be a finite dimensional Lie algebra over a field \( F \) of the characteristic 0, \( S \) an upper well ordered set and \( \mu \) a non-abelian fuzzy ideal of \( L \) in \( F(L,S) \). Then \( \mu \) is semisimple in \( F(L,S) \) if, and only if, \( \mu' \) is a semisimple ideal of \( L \).

**Proof.** Firstly, let us observe that \( R(\mu') = \mu' \cap R(L) \), where \( R(\mu') \) and \( R(L) \) are the ideals of \( \mu' \) and \( L \), respectively, by [12, Theorem 3.7]. Since \( R(\mu') \) is a solvable ideal of \( L \) contained in \( \mu' \), then \( R(\mu') \) is solvable in \( F(L,S) \), by [4, Theorem 3.6]. Thus \( \mu' \) is a semisimple ideal of \( L \). Reciprocally, it is immediate that \( R(\mu') = 0 \) by [12, Theorem 3.7].

**Theorem 3.5.** Let \( L \) be a finite dimensional Lie algebra over a field \( F \) of the characteristic 0, \( S \) an upper well ordered set and \( \mu \) a non-abelian fuzzy ideal of \( L \) in \( F(L,S) \). If \( \mu \) is semisimple in \( F(L,S) \), then there are simple fuzzy ideals \( \nu_1, \ldots, \nu_n \) of \( \mu \) relative to \( \mu' \) in \( F(L,S) \) such that \( \mu = \bigoplus_1^n \nu_i \).
Proof. From the hypothesis of the Theorem, we have that \( \mu' \) is an ideal of \( L \) which is a semisimple algebra, by Theorem 3.4. Hence, \( \mu = \mu_1 \oplus \mu_2 \oplus \ldots \mu_n \) for simple ideals of \( \mu' \), where every simple ideal of \( \mu' \) coincides with one of the \( \mu_i (i = 1, 2, \ldots, n) \) and each ideal of \( \mu' \) is a sum of certain simple ideals of \( \mu_i \), by [1, Theorem and Corollary, 5.2, pp. 23]. This implies that

\[
[\mu_i] = ([\mu_i] \cap [\mu]) \oplus ([\mu_i] \cap [\mu]) \oplus \ldots \oplus ([\mu_i] \cap [\mu]),
\]

where \( [\mu_i] \cap [\mu] = \{0\} \) or \( [\mu_i] \), for all \( i \in \{0,1\} \), since \( [\mu_i] \) is an ideal of \( \mu' \). By Negoita-Ralescu representation [7, Theorem 2.10], there are unique fuzzy sets \( \nu_i \), \( \ldots, \nu_n \) of \( L \) such that the \( t \)-level sets \( [v_{ij}] = \mu_i \cap [\mu] \) (\( i = 1, \ldots, n \)) for all \( \mu \in [0,1] \), which implies that the fuzzy sets \( \nu_i \) are fuzzy ideals of \( \mu' \) relative to \( \mu \). As \( L \) is finite dimensional, there is a finite set of real numbers \( 0 \leq r_1 < \ldots < r_n < 1 = r_n \leq S \) such that \( \mu(\Sigma_{i=1}^n r_i) \) is a fuzzy ideal in \( L \), where \( \mu(\Sigma_{i=1}^n r_i) \) is the fuzzy ideal generated by \( [\mu_i] \cap [\mu] \) for all \( i \). It follows that \( [v_{ij}] = \mu_i \cap [\mu] = [\mu_i]_r \), for all \( i \in \{1, \ldots, n\} \) and all \( r \in [0,1] \), and \( [\mu] = \mu \), for all \( r \in [0,1] \), then writing \( \mu(\Sigma_{i=1}^n r_i) = [\mu_i]_r \), for all \( i \in \{1, \ldots, n\} \) for all \( r \in [0,1] \), which results in \( \nu_i(x) = \mu_i(x) \) for all \( x \in L \). Thus, \( \nu_i \) is a fuzzy ideal in \( L \). Hence, for all fuzzy ideal \( \sigma_i \) of \( \mu' \) relative to \( \mu \), we have \( [\sigma_i] \subset [\mu_i] \) and \( [\sigma_i] \) is an ideal of \( \nu_i \) for all \( i \in \{0,1\} \). Hence, one has either \( [\sigma_i] = [\mu_i] \) or \( [\sigma_i] = \{0\} \) for all \( i \in \{0,1\} \), which implies that \( \nu_i \) is simple.

Now, let \( \pi \) be a simple fuzzy ideal of \( \mu \) relative to \( \mu' \). Then \( \pi \) is a sum of certain simple ideals of \( \mu' \), by [1, Theorem and Corollary, 5.2, pp. 23]. It follows that there is a unique ideal \( \mu_i (1 \leq i \leq n) \) such that \( \pi = \mu_i \), by [1, Theorem, and Corollary, 5.2, pp. 23] again. Finally, let \( \pi \) be a fuzzy ideal of \( \mu \) relative to \( \mu' \). Then we have an ideal of \( \mu' \) and therefore a sum of certain simple ideals of \( \mu' \), by [1, Theorem and Corollary, 5.2, pp. 23]. Hence there are fuzzy ideals \( v_{ij}, \ldots, v_n \) (\( 1 \leq i \leq \ldots \leq n \)) such that \( \pi = \bigoplus_{i} v_{ij} \). Levi Fuzzy Decomposition

In this section we generalize the theorem of Levi decomposition of a Lie algebra for the case of a class of fuzzy ideal, similarly to the crisp case.

Definition 4.1. Let \( L \) be a finite dimensional Lie algebra over a field \( F \), \( S \) an upper well ordered set and \( \mu \) a fuzzy ideal of \( L \) in \( F(L,S) \). One says that \( \mu \) has Levi’s hereditary if there is a semisimple subalgebra \( \nu \) of \( L \) such that:

(i) \( \nu = \nu \oplus R(\mu, S) \);

(ii) \( \nu = \nu \cap [\mu, R(\mu, S)] \), for all \( (e) \in [0,1] \).

In the following example we show that the conditions of the Definition 4.1 are not artificial.

Example 4.2. Let \( L \) be a finite dimensional Lie algebra over a field \( F \) of the characteristic 0, \( S = \{0, \frac{1}{2}, 1\} \) and \( \mu \) a fuzzy set of \( L \) defined by its \( t \)-level sets as: \( [\mu] = [\mu]_t \) for all \( t \in [0,1] \) and \( [\mu] = [\mu]_1 \) for all \( t \in [\frac{1}{2}, 1] \). It is easy to check that \( S \) is an upper well ordered set and \( \mu \) a fuzzy ideal of \( L \) in \( F(L,S) \) such that \( \mu(\mu) = L \). Also, \( [R(\mu, S)] = L \), \( R(\mu, S) = R(\mu, S) \), for all \( t \in [0,1] \) and \( [\mu, R(\mu, S)] = [\mu] \) for all \( t \in [\frac{1}{2}, 1] \), resulting in \( R(\mu, S') = R(L) \), where \( R(L) \) is the radical of \( L \). Moreover, there is a semisimple subalgebra \( \nu \) of \( L \) such that \( \nu = \nu \oplus R(\mu', S') \), by [1, Levi’s theorem], which implies \( \nu \cap [\mu] = \nu \) for all \( t \in [0,1] \).

Thus, has Levi’s hereditary.

Theorem 4.3. Let \( L \) be a finite dimensional Lie algebra over a field \( F \), \( S \) an upper well ordered set and \( \mu \) a fuzzy ideal of \( L \) in \( F(L,S) \). If \( \mu \) has Levi’s hereditary, then there exists a semisimple fuzzy subalgebra \( \nu \) of \( \mu \) in \( F(L,S) \) such that

\[ \mu = \nu \oplus R(\mu, S). \]

Proof. From the hypothesis of the Theorem, we have a unique fuzzy set \( \nu \) of \( S \) such that \( [\nu] = [\nu] \) for all \( [\nu] \in [0,1] \), by Negoita-Ralescu representation [4, Theorem 2.10]. The \( t \)-level sets \( [\nu] \) are fuzzy subalgebras of \( L \) which implies that the fuzzy set \( \nu \) is a fuzzy subalgebra of \( \mu \) satisfying \( \nu = \bigcup_{\nu \in [0,1]} [\nu] \). Now, using an argument quite similar to the one used in the Theorem 3.5, we can also conclude that \( \nu \) is in \( F(L,S) \). Furthermore, since \( \nu \) is a semisimple subalgebra of \( L \), then \( R(\nu, S) = [\nu] \) for all \( t \in [0,1] \). This implies that \( R(\nu, S) \) is a subalgebra of \( \nu \). For an arbitrary \( x \in \nu \), let us take \( \mu(x) = x \). If \( \mu(x) \), then \( \exists x \in \nu \) and \( \nu \in [0,1] \) and so \( \nu = \nu \). Let us show that \( \mu = \nu \oplus R(\mu, S) \). For an arbitrary \( x \in \nu \), let us take \( \mu(x) = x \). If \( \mu(x) \), then \( \exists x \in \nu \) and \( \nu = \nu \). Hence, \( \mu \) is a fuzzy subalgebra of \( \nu \) and \( \mu \) is a fuzzy ideal of \( \mu \) in \( F(L,S) \). Thus, has Levi’s hereditary.

Example 4.4. From the Theorem 4.3 we can conclude that in the Example 4.2 there exists a semisimple fuzzy subalgebra \( \nu \) of \( \mu \) in \( F(L,S) \) such that \( \mu = \nu \oplus R(\mu, S) \).

Open Questions

The history of the class of fuzzy Lie algebras proposed in [4] and in this paper is far from over. In fact, there are many unanswered questions that remain to be addressed.
questions and we list some of them as follows:

**Question 5.1** Find a fuzzy version of Malcev-Harish-Chandra’s Theorem.

**Question 5.2** Is it possible a fuzzy representation theory for the structure theory proposed?

**Question 5.3** Perspectives for applications in Physics. Reviewing the history of applications of Lie algebras in Physics since its origin, is it possible to determine methods based on the fuzzy Lie algebras presented in this work, to be applied in the Particle Physics?

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