Degenerations of Jordan Superalgebras

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Abstract. We describe degenerations of three-dimensional Jordan superalgebras over $\mathbb{C}$. In particular, we describe all irreducible components in the corresponding varieties.

Keywords: Jordan superalgebra, orbit closure, degeneration, rigid superalgebra

1. Introduction

Contractions of Lie algebras are limiting processes between Lie algebras, which have been studied first in physics [20,34]. For example, classical mechanics is a limiting case of quantum mechanics as $\hbar \to 0$, described by a contraction of the Heisenberg-Weyl Lie algebras to the Abelian Lie algebra of the same dimension. Description of contractions of low dimensional Lie algebras was given in [32]. The study of contractions and graded contractions of binary algebras has a very big history (see, for example, [13,18,36]). The study of graded contractions of Jordan algebras and Jordan superalgebras was initiated in [23]. The first attempt of the study of contractions of $n$-ary algebras stays in the variety of Filippov algebras [5].

In mathematics, often a more general definition of contraction is used, the so-called degeneration. Degenerations of algebras is an interesting subject, which was studied in various papers (see, for example, [6,8,10,11,16,17,29,33]). In particular, there are many results concerning degenerations of algebras of low dimensions in a variety defined by a set of identities. One of the important problems in this direction is the description of the so-called rigid algebras [22]. These algebras are of big interest, since the closures of their orbits under the action of generalized linear group form irreducible components of a variety under consideration (with respect to the Zariski topology). There are fewer works in which the full information about degenerations was found for some variety of algebras. This problem was solved for two-dimensional pre-Lie algebras in [6], for two-dimensional Jordan algebras in [2], for three-dimensional Novikov algebras in [7], for three-dimensional Jordan algebras [15], for four-dimensional Lie algebras in [8], for nilpotent four-dimensional Jordan algebras [3], for nilpotent four-dimensional Leibniz and Zinbiel algebras in [27], for nilpotent five- and six-dimensional Lie algebras in [16,33], for nilpotent five- and six-dimensional Malcev algebras in [26], and for all 2-dimensional algebras [28]. On the same time, the study of degenerations of superalgebras and graded algebras was initiated in [4] for associative case and in [11] for Lie superalgebras.
2. Definitions and notation

2.1. Jordan superalgebras. Jordan algebras appeared as a tool for studies in quantum mechanic in the paper of Jordan, von Neumann and Wigner [19]. A commutative algebra is called a Jordan algebra if it satisfies the identity

\[(x^2y)x = x^2(yx).\]

The study of the structure theory and other properties of Jordan algebras was initiated by Albert. Jordan algebras are related with some questions in differential equations [35], superstring theory [12], analysis, operator theory, geometry, mathematical biology, mathematical statistics and physics (see, the survey of Iordanescu [21]).

Let \( G \) be the Grassmann algebra over \( \mathbb{F} \) given by the generators \( 1, \xi_1, \ldots, \xi_n, \ldots \) and the defining relations \( \xi_i^2 = 0 \) and \( \xi_i \xi_j = -\xi_j \xi_i \). The elements \( 1, \xi_1, \xi_2 \ldots, \xi_k, \) \( i_1 < i_2 < \ldots < i_k \), form a basis of the algebra \( G \) over \( \mathbb{F} \). Denote by \( G_0 \) and \( G_1 \) the subspaces spanned by the products of even and odd lengths, respectively; then \( G \) can be represented as the direct sum of these subspaces, \( G = G_0 \oplus G_1 \). Here the relations \( G_i G_j \subseteq G_{i+j(mod \ 2)}, i, j = 0, 1 \), hold. In other words, \( G \) is a \( \mathbb{Z}_2 \)-graded algebra (or a superalgebra) over \( \mathbb{F} \). Suppose now that \( A = A_0 \oplus A_1 \) is an arbitrary superalgebra over \( \mathbb{F} \). Consider the tensor product \( G \otimes A \) of \( \mathbb{F} \)-algebras. The subalgebra

\[ G(A) = G_0 \otimes A_0 + G_1 \otimes A_1 \]

of \( G \otimes A \) is referred to as the Grassmann envelope of the superalgebra \( A \). Let \( \Omega \) be a variety of algebras over \( \mathbb{F} \). A superalgebra \( A = A_0 \oplus A_1 \) is referred to as an \( \Omega \)-superalgebra if its Grassmann envelope \( G(A) \) is an algebra in \( \Omega \). In particular, \( A = A_0 \oplus A_1 \) is referred to as a Jordan superalgebra if its Grassmann envelope \( G(A) \) is a Jordan algebra. The study of Jordan superalgebras has very big history (for example, see [9,14,24,25,30,31]).

2.2. Degenerations. Given an \((m, n)\)-dimensional vector superspace \( V = V_0 \oplus V_1 \), the set

\[ \text{Hom}(V \otimes V, V) = (\text{Hom}(V \otimes V, V))_0 \oplus (\text{Hom}(V \otimes V, V))_1 \]

is a vector superspace of dimension \( m^3 + 3mn^2 \). This space has a structure of the affine variety \( \mathbb{C}^{m^3+3mn^2} \). If we fix a basis \( \{e_1, \ldots, e_m, f_1, \ldots, f_n\} \) of \( V \), then any \( \mu \in \text{Hom}(V \otimes V, V) \) is determined by \( m^3 + 3mn^2 \) structure constants \( \alpha^k_{i,j}, \beta^q_{i,j}, \gamma^q_{p,i,j}, \delta^k_{p,q} \in \mathbb{C} \) such that

\[ \mu(e_i \otimes e_j) = \sum_{k=1}^m \alpha^k_{i,j} e_k, \quad \mu(e_i \otimes f_p) = \sum_{q=1}^n \beta^q_{i,p} f_q, \quad \mu(f_p \otimes e_i) = \sum_{q=1}^n \gamma^q_{p,i} f_q, \quad \mu(f_p \otimes f_q) = \sum_{k=1}^m \delta^k_{p,q} e_k. \]

A subset \( \mathbb{L}(T) \) of \( \text{Hom}(V \otimes V, V) \) is Zariski-closed if it can be defined by a set of polynomial equations \( T \) in the variables \( \alpha^k_{i,j}, \beta^q_{i,p}, \gamma^q_{p,i}, \delta^k_{p,q} \) \( (1 \leq i, j, k \leq m, 1 \leq p, q \leq n) \).

Let \( S_{m,n} \) be the set of all superalgebras of dimension \((m, n)\) defined by the family of polynomial superidentities \( T \), understood as a subset \( \mathbb{L}(T) \) of an affine variety \( \text{Hom}(V \otimes V, V) \). Then one can see that \( S_{m,n} \) is a Zariski-closed subset of the variety \( \text{Hom}(V \otimes V, V) \). The group \( G = (\text{Aut} V)_0 \simeq \text{GL}(V_0) \oplus \text{GL}(V_1) \) acts on \( S_{m,n} \) by conjugations:

\[ (g \ast \mu)(x \otimes y) = g \mu(g^{-1} x \otimes g^{-1} y) \]

for \( x, y \in V, \mu \in \mathbb{L}(T) \) and \( g \in G \).

Thus, \( S_{m,n} \) is decomposed into \( G \)-orbits that correspond to the isomorphism classes of superalgebras. Let \( O(\mu) \) denote the orbit of \( \mu \in \mathbb{L}(T) \) under the action of \( G \) and \( \overline{O(\mu)} \) denote the Zariski closure of \( O(\mu) \).

Let \( J, J' \in S_{m,n} \) and \( \lambda, \mu \in \mathbb{L}(T) \) represent \( J \) and \( J' \) respectively. We say that \( \lambda \) degenerates to \( \mu \) if \( \mu \in O(\lambda) \). Note that in this case we have \( O(\mu) \subset O(\lambda) \). Hence, the definition of a degeneration does not depend on the choice of \( \mu \) and \( \lambda \), and we will right indistinctly \( J \rightarrow J' \) instead of \( \lambda \rightarrow \mu \) and \( O(J) \) instead of \( O(\lambda) \). If \( J \nsubseteq J' \), then the assertion \( J \rightarrow J' \) is called a proper degeneration. We write \( J \nrightarrow J' \) if \( J' \notin O(J) \).

Let \( J \) be represented by \( \lambda \in \mathbb{L}(T) \). Then \( J \) is rigid in \( \mathbb{L}(T) \) if \( O(\lambda) \) is an open subset of \( \mathbb{L}(T) \). Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called irreducible component. In particular, \( J \)
is rigid in $\Sigma^{m,n}$ iff $\overline{O(\lambda)}$ is an irreducible component of $\mathcal{L}(T)$. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. We denote by $\text{Rig}(\Sigma^{m,n})$ the set of rigid superalgebras in $\Sigma^{m,n}$.

2.3. Principal notations. Let $\mathcal{J}^{m,n}$ be the set of all Jordan superalgebras of dimension $(m, n)$. Let $J$ be a Jordan superalgebra with fixed basis $\{e_1, \ldots, e_m, f_1, \ldots, f_n\}$, defined by

$$e_i e_j = \sum_{k=1}^{m} \alpha_{ij}^k e_k, \quad e_i f_j = \sum_{k=1}^{n} \beta_{ij}^k f_k, \quad f_i f_j = \sum_{k=1}^{m} \gamma_{ij}^k e_k.$$

We will use the following notation:

1. $\alpha(J)$ is the Jordan superalgebra with the same underlying vector superspace than $J$, and defined by $f_i f_j = \sum_{k=1}^{n} \gamma_{ij}^k e_k$.
2. $J^1 = J$, $J^r = J^{r-1} J + J^{r-2} J^2 + \cdots + J J^{r-1}$, and in every case $J^r = \langle J^r \rangle_0 \oplus \langle J^r \rangle_1$.
3. $c_{i,j} = \frac{\text{tr}(L(x)^i) \cdot \text{tr}(L(y)^j)}{\text{tr}(L(x)^i \cdot L(y)^j)}$ is the Burde invariant, where $L(x)$ is the left multiplication. This invariant $c_{i,j}$ is defined as a quotient of two polynomials in the structure constants of $J$, for all $x, y \in J$ such that both polynomials are not zero and $c_{i,j}$ is independent of the choice of $x, y$.

3. Methods

First of all, if $J \rightarrow J'$ and $J \not\equiv J'$, then $\dim \text{Aut}(J) < \dim \text{Aut}(J')$, where $\text{Aut}(J)$ is the space of automorphisms of $J$. Secondly, if $J \rightarrow J'$ and $J' \rightarrow J''$ then $J \rightarrow J''$. If there is no $J'$ such that $J \rightarrow J'$ and $J' \rightarrow J''$ are proper degenerations, then the assertion $J \rightarrow J''$ is called a primary degeneration. If $\dim \text{Aut}(J) < \dim \text{Aut}(J'')$ and there are no $J'$ and $J''$ such that $J' \rightarrow J$, $J'' \rightarrow J''$, $J' \not\rightarrow J''$ and one of the assertions $J' \rightarrow J$ and $J'' \rightarrow J''$ is a proper degeneration, then the assertion $J \not\rightarrow J''$ is called a primary non-degeneration. It suffices to prove only primary degenerations and non-degnerations to describe degenerations in the variety under consideration. It is easy to see that any superalgebra degenerates to the superalgebra with zero multiplication. From now on we will use this fact without mentioning it.

Let us describe the methods for proving primary non-degnerations. The main tool for this is the following lemma.

**Lemma 1.** If $J \rightarrow J'$ then the following hold:

1. $\dim(J^r)_i \geq \dim(J'^r)_i$, for $i \in \mathbb{Z}_2$;
2. $(J)_0 \rightarrow (J')_0$;
3. $\alpha(J) \rightarrow \alpha(J')$;
4. If the Burde invariant exist for $J$ and $J'$, then both superalgebras have the same Burde invariant;
5. If $J$ is associative then $J'$ must be associative. In fact, if $J$ satisfies a P.I. then $J'$ must satisfy the same P.I.

In the cases where all of these criteria can’t be applied to prove $J \not\rightarrow J'$, we will define $\mathcal{R}$ by a set of polynomial equations and will give a basis of $V$, in which the structure constants of $\lambda$ give a solution to all these equations. We will omit everywhere the verification of the fact that $\mathcal{R}$ is stable under the action of the subgroup of upper triangular matrices and of the fact that $\mu \not\in \mathcal{R}$ for any choice of a basis of $V$. These verifications can be done by direct calculations.

**Degenerations of Graded algebras.** Let $G$ be an abelian group and let $\mathcal{V}(\mathcal{F})$ be a variety of algebras defined by a family of polynomial identities $\mathcal{I}$. It is important to notice that degeneration on the $G$-graded variety $G\mathcal{V}(\mathcal{F})$ is a more restrictive notion than degeneration on the variety $\mathcal{V}(\mathcal{F})$. In fact, consider $A, A' \in G\mathcal{V}(\mathcal{F})$ such that $A, A' \in \mathcal{V}(\mathcal{F})$, a degeneration between the algebras $A$ and $A'$ may not give rise to a degeneration between the $G$-graded algebras $A$ and $A'$, since the matrices describing the basis changes in $G\mathcal{V}(\mathcal{F})$ must preserve the $G$-gradation. Hence, we have the following result.

**Lemma 2.** Let $A, A' \in G\mathcal{V}(\mathcal{F}) \cap \mathcal{V}(\mathcal{F})$. If $A \not\rightarrow A'$ as algebras, then $A \not\rightarrow A'$ as $G$-graded algebras.
4. Main result

In this section we describe all degenerations and non-degenerations of 3-dimensional Jordan superalgebras. Note that, there is only one (trivial) Jordan superalgebra of the type $(0, 3)$; the variety of 3-dimensional Jordan algebras (Jordan superalgebras of the type $(3, 0)$) has 19 non isomorphic algebras with non zero multiplication, in particularly, 5 algebras are rigid. The full description of all degenerations and non-degenerations of 3-dimensional Jordan algebras was given in [15]. The rest of the section is dedicated to study of degenerations of Jordan superalgebras of types $(1, 2)$ and $(2, 1)$.

4.1. Jordan Superalgebras of type $(1, 2)$.

4.1.1. Algebraic classification. As was noticed the algebraic classification of Jordan superalgebras of the type $(1, 2)$ was received in [30]. In the next table we give this classification with some additional useful information about these superalgebras.

| $J$ | multiplication tables | dim Aut($J$) | $c_{i,j}$ | type |
|-----|------------------------|-------------|-----------|------|
| $U^2_3$ | $e_1 e_1 = e_1$ | 4 | 1 | associative |
| $S^2_4$ | $e_1 e_1 = e_1, e_1 f_1 = \frac{1}{2} f_1$ | 2 | 2 | non-associative |
| $S^1_2$ | $e_1 f_1 = f_2, f_1 f_2 = e_1$ | 2 | \(\not\exists\) | non-associative |
| $S^1_2$ | $e_1 e_1 = e_1, e_1 f_1 = f_1$ | 2 | \(\not\exists\) | associative |
| $S^2_4$ | $e_1 f_2 = e_1$ | 4 | \(\not\exists\) | associative |
| $S^3_4$ | $e_1 e_1 = e_1, e_1 f_1 = f_1, e_1 f_2 = \frac{1}{2} f_2$ | 2 | \(\frac{2+\left(\frac{1}{2}\right)^i}{2+\left(\frac{1}{2}\right)^j}\) | non-associative |
| $S^3_4$ | $e_1 e_1 = e_1, e_1 f_1 = \frac{1}{2} f_1, e_1 f_2 = \frac{1}{2} f_2$ | 4 | \(\frac{1+2\left(\frac{1}{2}\right)^i}{1+2\left(\frac{1}{2}\right)^j}\) | non-associative |
| $S^3_4$ | $e_1 e_1 = e_1, e_1 f_1 = f_1, e_1 f_2 = f_2$ | 4 | 3 | associative |
| $S^3_4$ | $e_1 e_1 = e_1, e_1 f_1 = \frac{1}{2} f_1, e_1 f_2 = \frac{1}{2} f_2, f_1 f_2 = e_1$ | 3 | \(\frac{1+2\left(\frac{1}{2}\right)^i}{1+2\left(\frac{1}{2}\right)^j}\) | non-associative |
| $S^3_4$ | $e_1 e_1 = e_1, e_1 f_1 = f_1, e_1 f_2 = f_2, f_1 f_2 = e_1$ | 3 | 3 | non-associative |

4.1.2. Degenerations.

Theorem 3. The graph of primary degenerations for Jordan superalgebras of dimension $(1, 2)$ has the following form:

![Graph of primary degenerations for Jordan superalgebras of dimension $(1, 2)$]

Proof. We prove all required primary degenerations in Table 2 below. Recall that an associative superalgebra can only degenerate to an associative superalgebra. Let us consider the first degeneration $S^2_1 \rightarrow S^3_2$ to clarify this table. Write nonzero products in $S^2_1$ in the basis $E^t_1, F^t_1, F^t_2$:

$$E^t_1 E^t_1 = t E^t_1, \quad E^t_1 F^t_1 = t e_1 (f_1 - 2 t^{-1} f_2) = \frac{t}{2} (f_1 - 2 t^{-1} f_2) + f_2 = \frac{t}{2} F^t_1 + F^t_2.$$
It is easy to see that for \( t = 0 \) we obtain the multiplication table of \( S_3^3 \). The remaining degenerations can be considered in the same way.

Table 2. Primary degenerations of Jordan superalgebras of dimension \((1, 2)\).

| degenerations | parametrized bases |
|---------------|--------------------|
| \( S_1^1 \to S_2^1 \) | \( E_1 = te_1, \quad F_1 = f_1 - 2t^{-1}f_2, \quad F_2 = f_2 \) |
| \( S_2^1 \to S_2^1 \) | \( E_1 = te_1, \quad F_1 = f_1 - t^{-1}f_2, \quad F_2 = f_2 \) |
| \( S_1^1 \to S_3^1 \) | \( E_1 = e_1, \quad F_1 = tf_1, \quad F_2 = f_2 \) |
| \( S_3^1 \to S_2^1 \) | \( E_1 = e_1, \quad F_1 = tf_1, \quad F_2 = f_2 \) |
| \( S_1^1 \to S_2^1 \) | \( E_1 = te_1, \quad F_1 = f_1 - 2t^{-1}f_2, \quad F_2 = f_2 \) |
| \( S_2^1 \to S_2^1 \) | \( E_1 = te_1, \quad F_1 = f_1 - t^{-1}f_2, \quad F_2 = f_2 \) |
| \( S_1^1 \to S_1^1 \) | \( E_1 = e_1, \quad F_1 = tf_1, \quad F_2 = f_2 \) |
| \( S_3^1 \to S_2^1 \) | \( E_1 = te_1, \quad F_1 = f_1 - 2t^{-1}f_2, \quad F_2 = f_2 \) |
| \( S_2^1 \to S_2^1 \) | \( E_1 = te_1, \quad F_1 = f_1 - t^{-1}f_2, \quad F_2 = f_2 \) |
| \( S_1^1 \to S_2^1 \) | \( E_1 = e_1, \quad F_1 = tf_1, \quad F_2 = f_2 \) |
| \( S_3^1 \to S_2^1 \) | \( E_1 = e_1, \quad F_1 = tf_1, \quad F_2 = f_2 \) |
| \( S_2^1 \to S_3^1 \) | \( E_1 = e_1, \quad F_1 = tf_1, \quad F_2 = f_2 \) |
| \( S_3^1 \to S_3^1 \) | \( E_1 = e_1, \quad F_1 = tf_1, \quad F_2 = f_2 \) |

The primary non-degnerations are proved in Table 3.

Table 3. Primary non-degnerations of Jordan superalgebras of dimension \((1, 2)\).

| non-degnerations | reason |
|------------------|--------|
| \( S_1^1 \not\to S_2^1 \) | \( \dim(J^2)_0 < \dim(J^2)_0 \) |
| \( S_1^1 \not\to U_1^1, S_2^1; \quad S_1^1 \not\to S_2^1, S_3^1; \quad U_1^1, S_2^1, S_3^1 \) | \( J_0 \not\to J_0 \) |
| \( S_2^1 \not\to U_1^1, S_3^1 \) | \( \text{ci,j} \) |
| \( S_1^1 \not\to S_3^1, S_5^1, U_1^1, S_2^1, S_3^1, S_5^1, S_6^1 \) | \( a(J) \not\to a(J') \) |

\[ \square \]

4.1.3. Irreducible components and rigid algebras. Using Theorem 3 we describe the irreducible components and the rigid algebras in \( J^{S^{1,2}} \).

Corollary 4. The irreducible components of \( J^{S^{1,2}} \) are:

\[
\begin{align*}
\mathcal{C}_1 &= \overline{O(U_1^1)} = \{U_1^s, C^{1,2}\}; \\
\mathcal{C}_2 &= \overline{O(S_1^1)} = \{S_1^1, S_3^1, C^{1,2}\}; \\
\mathcal{C}_3 &= \overline{O(S_3^1)} = \{S_3^1, S_3^1, C^{1,2}\}; \\
\mathcal{C}_4 &= \overline{O(S_2^1)} = \{S_2^1, S_3^1, S_2^1, C^{1,2}\}; \\
\mathcal{C}_5 &= \overline{O(S_4^1)} = \{S_4^1, S_3^3, C^{1,2}\}; \\
\mathcal{C}_6 &= \overline{O(S_6^1)} = \{S_6^3, S_2^1, C^{1,2}\}; \\
\mathcal{C}_7 &= \overline{O(S_8^1)} = \{S_8^3, S_3^3, C^{1,2}\}; \\
\end{align*}
\]

In particular, \( \text{Rig}(J^{S^{1,2}}) = \{U_1^s, S_1^1, S_3^1, S_2^1, S_4^1, S_7^1, S_8^1\} \).

4.2. Superalgebras of type \((2, 1)\). In this section we describe all possible primary degenerations between Jordan superalgebras of dimension \((2, 1)\). First of all, notice that every Jordan superalgebra of dimension \((2, 1)\) has trivial odd product, so it can be considered as a Jordan algebra of dimension 3. However, the graph of primary degenerations of Jordan superalgebras of dimension \((2, 1)\) is not a subgraph of the primary degenerations of de Jordan algebras of dimension 3. In fact, there exist Jordan superalgebras, without degenerations between them, such that they degenerate as Jordan algebras. Take for example \( S_2^2 \oplus U_1^s \) and \( B_1^s \) (see Table 5). Taking the basis change: \( E_1 = e_1, E_2 = f_1 \) and \( F_1 = te_2 \), it follows that \( S_2^2 \oplus U_1^s \to B_1^s \) as Jordan algebras. Notice that this basis change does not preserve the \( \mathbb{Z}_2 \)-graduation. Moreover, we shall prove that \( S_2^2 \oplus U_1^s \not\to B_1^s \) as Jordan superalgebras.
4.2.1. **Algebraic classification.** In the next table we provide the classification of $(2, 1)$-dimensional Jordan superalgebras with some additional useful information about these superalgebras.

| $J$ | Multiplication tables | dim Aut($J$) | Type       |
|-----|------------------------|--------------|------------|
| $2U_1^s$ | $e_1e_1 = e_1, \ e_2e_2 = e_2$ | 1            | associative |
| $U_1$ | $e_1e_1 = e_1$          | 2            | associative |
| $B_1^s$ | $e_1e_1 = e_1, \ e_1e_2 = e_2$ | 2            | associative |
| $B_2^s$ | $e_1e_1 = e_1, \ e_1e_2 = \frac{1}{2}e_2$ | 3            | non-associative |
| $B_3^s$ | $e_1e_1 = e_2$          | 3            | associative |
| $S_2^2 \oplus U_1^s$ | $e_1e_1 = e_1, \ e_2e_2 = e_2, \ e_1f_1 = \frac{1}{2}f_1$ | 1            | non-associative |
| $S_2^2$ | $e_1e_1 = e_1, \ e_1f_1 = \frac{1}{2}f_1$ | 2            | non-associative |
| $S_3^2 \oplus U_1^s$ | $e_1e_1 = e_1, \ e_2e_2 = e_2, \ e_1f_1 = f_1$ | 1            | associative |
| $S_2^3$ | $e_1e_1 = e_1, \ e_1f_1 = f_1$ | 2            | associative |
| $S_9^3$ | $e_1e_1 = e_1, \ e_1e_2 = e_2, \ e_1f_1 = \frac{1}{2}f_1$ | 2            | non-associative |
| $S_{10}^3$ | $e_1e_1 = e_1, \ e_1e_2 = e_2, \ e_1f_1 = f_1$ | 2            | associative |
| $S_{11}^3$ | $e_1e_1 = e_1, \ e_1e_2 = \frac{1}{2}e_2, \ e_1f_1 = \frac{1}{2}f_1$ | 3            | non-associative |
| $S_{12}^3$ | $e_1e_1 = e_1, \ e_1e_2 = \frac{1}{2}e_2, \ e_1f_1 = f_1$ | 3            | non-associative |
| $S_{13}^3$ | $e_1e_1 = e_1, \ e_2e_2 = e_2, \ e_1f_1 = \frac{1}{2}f_1, \ e_2f_1 = \frac{1}{2}f_1$ | 1            | non-associative |

4.2.2. **Degenerations.**

**Theorem 5.** The graph of primary degenerations for Jordan superalgebras of dimension $(2, 1)$ has the following form:

![Graph of primary degenerations](image)

**Proof.** We prove all required primary degenerations in Table 5 below. For clarifying this table was considered an example in the proof of the theorem.
Primary non-degenerations between 3-dimensional Jordan algebras are given in [15], we use this result and Lemma 2 to prove some primary non-degenerations, (see table 6).

From [15] it follows that $2U^s_1 \rightarrow S^2_2, S^1_1 \rightarrow B^s_2, S^3_0 \rightarrow S^3_{12},$ and $S^2_2 \oplus U^s_1 \rightarrow B^s_1$ as Jordan algebras; we shall prove that they do not degenerate as Jordan superalgebras.

First, suppose that $S^2_2 \oplus U^s_1 \rightarrow B^s_1$ as Jordan superalgebras, then there exists a parameterized basis

$$E^t_1 = a(t)e_1 + b(t)e_2, \quad E^t_2 = c(t)e_1 + d(t)e_2, \quad F^t_1 = x(t)f_1$$

for $S^2_2 \oplus U^s_1$, such that for $t = 0$ we obtain $B^s_1$. Since $E^t_1 F^t_1 = a(t)F^t_1$ and $E^t_2 F^t_1 = c(t)F^t_1$, it follows that $a(0) = c(0) = 0$. Now, since $E^t_1 E^t_1 = E^t_1$ it follows that $a(t) = 0$ and $b(t) = 1$ for all $t$. Finally, since $E^t_2 F^t_1 = 0$ at $t = 0$, it follows that $d(0) = 0$, showing that $E^t_1 E^t_2 = 0$, for all $t$. This proves that $S^2_2 \oplus U^s_1 \not\rightarrow B^s_1$. For the remaining cases see table 6.

**Table 6. Primary non-degenerations of Jordan superalgebras of dimension (2, 1).**

| non-degnerations | reason |
|------------------|--------|
| $2U^s_1 \not\rightarrow S^2_1, S^2_2, S^3_0, S^3_{10}, S^3_1, S^3_{12}$ | dim($J'$)$_J < \dim((J')_J)$; |
| $S^2_2 \not\rightarrow B^s_2, S^3_0 \not\rightarrow S^3_{12}, S^3_1 \oplus U^s_1 \not\rightarrow B^s_2, S^3_1, S^3_{12}$; | $J \not\rightarrow J'$ as Jordan algebras |
| $2U^s_1 \not\rightarrow B^s_2, S^1_1, S^1_2, S^3_0 \not\rightarrow B^s_2, S^3_1, S^3_{12}$; | |
| $S^2_2 \not\rightarrow S^3_2, S^3_2 \not\rightarrow S^3_{10}, S^3_{13} \not\rightarrow U^s_1, B^s_1, S^3_2, S^3_3$ | |

4.2.3. **Irreducible components and rigid algebras.** Using Theorem 5 we describe the irreducible components and the rigid superalgebras in $\mathcal{S}^{2,1}$.

**Corollary 6.** The irreducible components of $\mathcal{S}^{2,1}$ are:

$$\mathcal{C}_1 = \overline{O(2U^s_1)} = \{2U^s_1, U^s_1, B^s_1, B^s_3, \mathbb{C}^{2,1}\};$$

$$\mathcal{C}_2 = \overline{O(B^s_2)} = \{B^s_2, \mathbb{C}^{2,1}\};$$

$$\mathcal{C}_3 = \overline{O(S^2_1 \oplus U^s_1)} = \{U^s_1, B^s_3, S^2_1 \oplus U^s_1, S^3_1, \mathbb{C}^{2,1}\};$$

$$\mathcal{C}_4 = \overline{O(S^2_2 \oplus U^s_1)} = \{U^s_1, B^s_3, S^2_2 \oplus U^s_1, S^2_2, \mathbb{C}^{2,1}\};$$

$$\mathcal{C}_5 = \overline{O(S^3_1)} = \{S^3_{11}, \mathbb{C}^{2,1}\};$$

$$\mathcal{C}_6 = \overline{O(S^3_{12})} = \{S^3_{12}, \mathbb{C}^{2,1}\};$$

$$\mathcal{C}_7 = \overline{O(S^3_{13})} = \{B^s_3, S^2_1, S^3_{10}, S^3_{13}, \mathbb{C}^{2,1}\}.$$
In particular, $\text{Rig}(\mathfrak{g}^{2,1}) = \{2U_1^*, B_2^*, S_1^2 \oplus U_1^*, S_2^2 \oplus U_1^*, S_3^0, S_{11}^3, S_{12}^3, S_{13}^3\}$.

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