An Upper Bound on Burning Number of Graphs

Max Land ∗ Linyuan Lu †

June 27, 2016

Abstract

The burning number \( b(G) \) of a graph \( G \) was introduced by Bonato, Janssen, and Roshanbin [Lecture Notes in Computer Science 8882 (2014)] for measuring the speed of the spread of contagion in a graph. They proved for any connected graph \( G \) of order \( n \), \( b(G) \leq 2\lceil \sqrt{n} \rceil - 1 \), and conjectured that \( b(G) \leq \lceil \sqrt{n} \rceil \). In this paper, we proved \( b(G) \leq \lceil \frac{-3 + \sqrt{24n + 33}}{4} \rceil \), which is roughly \( \sqrt{\frac{2}{3} \sqrt{n}} \). We also settled the following conjecture of Bonato-Janssen-Roshanbin: \( b(G)b(\overline{G}) \leq n + 4 \) provided both \( G \) and \( \overline{G} \) are connected.

1 Introduction

The burning number of a graph was introduced by Bonato-Janssen-Roshanbin [3, 4, 10]. It is related to the contact processes on graphs such as the Firefighter problem [6, 8, 9]. In the paper [3, 4], Bonato-Janssen-Roshanbin considered a graph process which they called burning. At the beginning of the process, all vertices are unburned. During each round, one may choose an unburned vertex and change its status to burned. At the same time, each of the vertices that are already burned, will remain burned and spread to all of its neighbors and change their status to burned. A graph is called \( k \)-burnable if it can be burned in at most \( k \) steps. The burning number of a graph \( G \), denoted by \( b(G) \), is the minimum number of rounds necessary to burn all vertices of the graph. For example, \( b(K_n) = 2 \), \( b(P_3) = 2 \), and \( b(C_5) = 3 \). In the paper [4], they proved \( b(P_n) = \lceil n^{1/2} \rceil \). Based on this result, Bonato-Janssen-Roshanbin [4] made the following conjecture.

**Conjecture 1:** for any connected graph \( G \) of order \( n \), \( b(G) \leq \lceil n^{1/2} \rceil \).

Bonato-Janssen-Roshanbin [3, 4] proved \( b(G) \leq 2\lceil n^{1/2} \rceil - 1 \). The previous best known bound is due to Bonato et al. [7]:

\[
b(G) \leq \left( \sqrt{\frac{32}{19}} + o(1) \right) \sqrt{n}.
\]

∗Dutch Fork High School, Irmo, SC 29063, (max.ruikang.land@gmail.com).
†University of South Carolina, Columbia, SC 29208, (lu@math.sc.edu). This author was supported in part by NSF grant DMS 1300547.
In this paper, we improved the upper bound of $b(G)$ as follows.

**Theorem 1.** If $G$ is a connected graph of order $n$, then

$$b(G) \leq \left\lceil \frac{-3 + \sqrt{24n + 33}}{4} \right\rceil.$$ 

In the paper [4], Bonato, Janssen, and Roshanbin also considered Nordhaus-Gaddum Type problem on the burning number. Let $\bar{G}$ be the complement graph of the graph $G$. In [4], they proved $b(G) + b(\bar{G}) \leq n + 2$ and $b(G)b(\bar{G}) \leq 2n$. Both bounds are tight and are achieved by the complete graph and its complement. When both graphs $G$ and $\bar{G}$ are connected, they proved $b(G) + b(\bar{G}) \leq 3\lceil n^{1/2} \rceil - 1$ and $b(G)b(\bar{G}) \leq n + 6$ for all graph $G_n$ of order $n \geq 6$. The following conjecture has been made in [4]:

**Conjecture 2:** If both $G$ and $\bar{G}$ are connected graphs of order $n$, then $b(G)b(\bar{G}) \leq n + 4$.

Using Theorem 1, we settled this conjecture positively.

**Theorem 2.** If both $G$ and $\bar{G}$ are connected graphs of order $n$, then

$$b(G)b(\bar{G}) \leq n + 4.$$ 

The equality holds if and only if $G = C_5$.

### 2 Notations and Lemmas

For each positive integer $k$, let $[k]$ denote the set $\{1, 2, \ldots, k\}$. A graph $G = (V, E)$ consists of a set of vertices $V$ and edges $E$. The **order** of $G$, denoted by $|G|$, is the number of vertices in $G$. A graph $G$ is called **connected** if for any two vertices there is a path connecting them. In this paper, we always assume that $G$ is a connected graph. The **distance** between two vertices $u$ and $v$, denoted by $d(u, v)$, is the length of the shortest path from $u$ to $v$ in graph $G$. The **eccentricity** of a vertex $v$ is the maximum distance between $v$ and any other vertex in $G$. The maximum eccentricity is the **diameter** $D(G)$ while the minimum eccentricity is the **radius** $r(G)$. The **center** of $G$ is the set of vertices of eccentricity equal to the radius.

For any nonnegative integer $k$ and a vertex $u$, the **$k$-th closed neighborhood** of $u$ is the set of vertices whose distance from $u$ is at most $k$; denoted by $N_k[u]$. From the definition, a graph $G$ is $k$-burnable if there is a burning sequence $v_1, \ldots, v_k$ of vertices such that

$$V \subset \bigcup_{i=1}^{k} N_{k-i}[v_i] \quad (1)$$

$$\forall i, j \in [k]: d(x_i, x_j) \geq j - i. \quad (2)$$

The burning number $b(G)$ is the smallest integer $k$ such that $G$ is $k$-burnable. It has been shown that Condition (2) is redundant for the definition of burning.
number \( b(G) \) (see Lemma 1 of [7]). It is often convenient to rewrite Condition (1) by relabeling the vertices in the burning sequence as follows:

\[
V \subset \bigcup_{i=1}^{k} N_{r_1}[v_i].
\]  

(3)

This leads the following generalization, which is very useful for the purpose of induction. For a set (or multiset) \( A \) of \( k \) positive integers \( a_1, a_2, \ldots, a_k \) (not necessarily all distinct), we say a graph \( G \) is \( A \)-burnable, if there exist \( k \) vertices \( v_1, v_2, \ldots, v_k \) such that \( G \subset \bigcup_{i=1}^{k} N_{a_i}[v_i] \). Under this terminology, the burning number \( b(G) \) is the least \( k \) so that \( G \) is \([k]\)-burnable.

A tree is an acyclic connected graph. For any tree \( T \), it is well-known that the center of \( T \) consists of either one vertex or two vertices. If the center of \( T \) consists of one vertex, then \( D(T) = 2r(T) \); otherwise, \( D(T) = 2r(T) - 1 \). (See [2].)

A rooted tree is a tree with one vertex \( r \) designated as the root. The height of a rooted tree is the eccentricity of the root. In a rooted tree, the parent of a vertex is the vertex connected to it on the path to the root. A child of a vertex \( v \) is a vertex of which \( v \) is the parent. A descendant of any vertex \( v \) is any vertex which is either the child of \( v \) or is (recursively) the descendents of any of the children of \( v \). A leaf vertex is a vertex with degree 1 but not equal to the root. The subtree rooted at \( v \) is the induced subgraph on the set of \( v \) and its all descendents. The important observation is that if a subtree rooted at \( v \) is pruned from the whole tree, the remaining part (if non-empty) is still a tree. This observation is very useful for induction.

A spanning tree of graph \( G \) is a subtree of \( G \) that covers all vertices of \( G \). In the papers [3, 4], Bonato, Janssen, and Roshanbin proved

\[
b(G) = \min \{ b(T) : T \text{ is a spanning subtree of } G \}.
\]  

(4)

Thus, it is sufficient to only consider the burning number \( b(T) \) for tree \( T \).

First we prove a simple lemma, which illustrates the idea of the induction.

**Lemma 1.** Let \( A = \{a_1, a_2, \ldots, a_k\} \) be a set of \( k \) nonnegative integers. If a tree \( T \) has order at most \( \sum_{i=1}^{k} a_i + \max\{a_i : 1 \leq i \leq k\} - 1 \), then \( T \) is \( A \)-burnable.

**Proof.** With loss of generality, we can assume that \( a_1 \geq a_2 \geq \cdots \geq a_k \). We will use induction on \( k \). Initial case: \( k = 1 \), \( A = \{a_1\} \). We need to prove that if a tree \( T \) has at most \( 2a_1 - 1 \) vertices, then \( T \) is \( A \)-burnable. Note that

\[
 r(T) \leq \frac{D(T) + 1}{2} \leq \frac{n}{2} \leq a_1 - \frac{1}{2}.
\]

Since the radius \( r(T) \) is an integer, we must have \( r(T) \leq a_1 - 1 \). Thus \( T \) is \( \{a_1\} \)-burnable.

Now we assume the statement holds for any set of \( k - 1 \) integers. For any \( A \) of \( k \) integers \( a_1 \geq a_2 \geq \cdots \geq a_k > 0 \) and any tree \( T \) with at most \( 2a_1 + a_2 + \cdots + a_k - 1 \), we will prove that \( T \) is \( A \)-burnable. Pick an arbitrary vertex \( r \) as the root of \( T \). Let \( h \) be the height of this rooted tree. If \( h \leq a_1 - 1 \), then \( V(T) \subset N_{a_1-1}(r) \). I.e., \( T \) is \( \{a_1\} \)-burnable. Thus \( T \) is \( A \)-burnable.
Now we assume \( h \geq a_1 \). Select a leaf vertex \( u \) such that \( d(r, u) = h \). Let \( v_k \) be the vertex on the \( ru \)-path such that the distance \( d(u, v_k) = a_k - 1 \). (This is possible since \( h \geq a_1 > a_k - 1 \). Let \( T_1 \) be the subtree rooted at \( v_k \), and \( T_2 := T \setminus T_1 \) be the remaining subtree. Notice that \( |T_1| \geq a_k \). Thus,

\[
|T_2| = |T| - |T_1| \\
\leq 2a_1 + a_2 + \cdots + a_k - 1 - a_k \\
= 2a_1 + a_2 + \cdots + a_{k-1} - 1.
\]

By inductive hypothesis, \( T_2 \) is \( \{a_1, a_2, \ldots, a_{k-1}\} \)-burnable. Thus, there exists \( k - 1 \) vertices \( v_1, v_2, \ldots, v_{k-1} \) such that \( T_2 \subseteq \bigcup_{i=1}^{k-1} N_{a_i-1}[v_i] \). Also, notice \( T_1 \subseteq N_{a_k-1}[v_k] \). Therefore, \( T \subseteq \bigcup_{i=1}^{k-1} N_{a_i-1}[v_i] \). The proof of the lemma is finished. \( \square \)

**Remark 1.** The bound in Lemma 1 is tight.

**Proof.** Consider the following example: for any positive integer \( a \), let \( a_1 = a_2 = \cdots = a_k = a \), i.e. \( A \) is a multiset consisting of \( k \) \( a \)'s. Now we will construct a tree \( T \) as following. First construct \( k + 1 \) disjoint paths \( P_0, P_1, \ldots, P_k \) with each of order \( a \). Create tree \( T \) by connecting one endpoint of \( P_1, P_2, \ldots, P_k \) to the same endpoint of \( P_0 \) (see figure below).

![Tree Diagram]

The tree \( T \) has order \( (k + 1)a \), which is just one more than the amount of vertices in Lemma 1. Now we show \( T \) is not \( A \)-burnable. Otherwise, there exists \( v_1, v_2, \ldots, v_k \) such that \( T \) is covered by \( \bigcup_{i=1}^{k} N_{a_i-1}[v_i] \). By Pigeon-hole principle, one of the paths \( P_0, P_1, \ldots, P_k \) will not contain \( v_1, v_2, \ldots, v_k \), and the leaf vertex on this path is in any \( N_{a_i-1}[v_i] \). Thus, \( T \) is not \( A \)-burnable. \( \square \)

The following corollary is a slight improvement of Theorem 7 of [7].

**Corollary 1.** For any connected graph \( G \), \( b(G) \leq \frac{-3+\sqrt{8n+17}}{2} \approx \sqrt{2n} - \frac{3}{2} \).

**Proof.** Let \( A = \{k, k-1, \ldots, 1\} \). By Lemma 1 any Tree of order \( n \leq (\sum_{i=1}^{k} i) + k - 1 = \frac{k^2 + 3k - 2}{2} \) is \( A \)-burnable. Solving \( k \) we get \( k \leq \frac{-3+\sqrt{8n+17}}{2} \). Thus, \( b(T) \leq \frac{-3+\sqrt{8n+17}}{2} \). By Equation 1, the same bound holds true for \( b(G) \). \( \square \)
3 Proof of Theorems 1 and 2

We have seen that Lemma 1 is sharp when all $a_i$’s are equal. The improvement can be made when $a_i$’s are distinct. Let $g(A)$ be a function of $A$ so that any tree $T$ with order at most $g(A)$ is $A$-burnable. In the proof of Lemma 1, we show that

$$g(A) \leq g(A \setminus \{a_k\}) + a_k.$$  

The idea is to show a recursive bound

$$g(A) \leq \max_{1 \leq i \leq k-1} \{g(A \setminus \{a_i\}) + a_i\} + \left\lfloor \frac{k-1}{3} \right\rfloor$$

where $k$ is the number of (distinct) elements in $A$. We first prove the following Lemma.

**Lemma 2.** For any $k-1$ distinct positive integers $a_1 < a_2 < \cdots < a_{k-1}$, there exists an $a_i$ such that

$$2\left\lfloor \frac{k-1}{3} \right\rfloor \leq a_i \leq a_{k-1} - \left\lfloor \frac{k-1}{3} \right\rfloor.$$  

**Proof.** Let $j = \left\lfloor \frac{k-1}{3} \right\rfloor$ and $A = \{a_1, a_2, \ldots, a_{k-1}\}$. Divide $[1, a_{k-1}]$ into 3 intervals:

$$[1, 2j-1] \cup [2j, a_{k-1}-j] \cup [a_{k-1}-j+1, a_{k-1}].$$

There are at most $2j-1$ elements of $A$ in the first interval. There are at most $j$ elements of $A$ in the last interval. Since $3j-1 < k-1$, there exists at least one element of $A$ in the middle interval. Call this element $a_i$. □

**Lemma 3.** For all integer $k \geq 1$,

$$\sum_{i=1}^{k} \left\lfloor \frac{i-1}{3} \right\rfloor = \left\lfloor \frac{k^2 - 3k + 2}{6} \right\rfloor.$$  

**Proof.** For $k = 3s$, we have

$$\sum_{i=1}^{k} \left\lfloor \frac{i-1}{3} \right\rfloor = 3 \sum_{j=1}^{s} (j-1) = \frac{3s(s-1)}{2} = \left\lfloor \frac{k^2 - 3k + 2}{6} \right\rfloor.$$  

For $k = 3s+1$, we have

$$\sum_{i=1}^{k} \left\lfloor \frac{i-1}{3} \right\rfloor = 3 \sum_{j=1}^{s} (j-1) + s = \frac{3s(s-1)}{2} + s = \left\lfloor \frac{k^2 - 3k + 2}{6} \right\rfloor.$$  

For $k = 3s+2$, we have

$$\sum_{i=1}^{k} \left\lfloor \frac{i-1}{3} \right\rfloor = 3 \sum_{j=1}^{s} (j-1) + 2s = \frac{3s(s-1)}{2} + 2s = \left\lfloor \frac{k^2 - 3k + 2}{6} \right\rfloor.$$  

□
Theorem 3. Let $A$ be a set of $k$ distinct positive integers $a_1 < a_2 < \cdots < a_k$. If a tree $T$ has order at most

$$
\left( \sum_{i=1}^{k} a_i \right) + a_k - 1 + \left\lfloor \frac{k^2 - 3k + 2}{6} \right\rfloor .
$$

then $T$ is $A$-burnable.

Proof. Let $f(k) := \left\lfloor \frac{k^2 - 3k + 2}{6} \right\rfloor$. By Lemma 3, we have $f(k) = f(k-1) + \left\lfloor \frac{k-1}{3} \right\rfloor$.

Now we use induction on $k$.

Initial case $k = 1$: $A = \{a_1\}$. By Lemma 1, if a tree $T$ has order at most $2a_1 - 1$, then $T$ is $\{a_1\}$-burnable. The statement holds true for $k = 1$ since $f(1) = 0$.

Now assume this statement holds true for any set of $k-1$ distinct positive integers. Consider the case $A = \{a_1, a_2, \ldots, a_k\}$. We need to prove that if a tree $T$ has order at most $a_1 + a_2 + \cdots + 2a_k - 1 + f(k)$ then $T$ is $A$-burnable.

Let $j = \lfloor \frac{k-1}{3} \rfloor$. By Lemma 1, there exists $a_i$ that satisfies $2j \leq a_i \leq a_{k-1} - j$.

Choose an arbitrary root $r$ and view $T$ as a rooted tree. Let $u$ be the leaf vertex which has the farthest distance away from the root $r$. If $d(r, u) \leq a_k - 1$, then $V(T) \subseteq N_{a_k-1}(r)$; thus $T$ is $A$-burnable. So, we can assume $d(r, u) \geq a_k$. We will name three vertices $v_i, t, v_k$ on the $ru$-path such that $d(u, v_i) = a_i - 1$, $d(u, t) = a_i - 1 + j$, and $d(u, v_k) = a_{k-1}$. Let $T_1$ be the subtree rooted at $t$.

There are two cases:

Case 1: $T_1 \subseteq N_{a_i-1}[v_i]$. Let $T_2 = T \setminus T_1$. Notice $|T_1| \geq a_i + j$. Then,

$$
|T_2| \leq |T| - |T_1| \leq a_1 + a_2 + \cdots + 2a_k - 1 + f(k) - (a_i + j) = a_1 + a_2 + \cdots + a_i + \cdots + 2a_k - 1 + f(k - 1).
$$

By inductive hypothesis, $T_2$ is $(A \setminus \{a_i\})$-burnable. Thus, $T$ is $A$-burnable.

Case 2: $T_1 \nsubseteq N_{a_i-1}[v_i]$. Then there is a vertex $z \in T_1$ such that $d(v_i, z) \geq a_i$. Let $w$ be the closest vertex on the path $rt$ to $z$. Observe that $w$ is not in the subtree rooted at $v_i$. Thus, $w$ is between $v_i$ and $t$. We have

$$
d(w, z) = d(v_i, z) - d(v_i, w) \geq a_i - d(w, v_i) \geq a_i - d(v_i, t) \geq a_i - j \geq j.
$$
The last inequality uses Lemma 2 for the choice of $a_i$.

Let $v_k$ be a vertex on the path from $u$ to the root with distance $d(u, v_k)$. Let $T_3$ be the subtree rooted at $v_k$ and let $T_4 := T \setminus T_3$ be the remaining subtree. We have that $|T_3| \geq a_{k-1} + d(w, z) \geq a_{k-1} + j$.

$|T_4| \leq |T| - |T_3| \leq a_1 + a_2 + \cdots + a_k - 1 + f(k) - (a_{k-1} + j) = a_1 + a_2 + \cdots + a_{k-2} + 2a_k - 1 + f(k - 1)$.

By inductive hypothesis, $T_4$ is $(A \setminus \{a_{k-1}\})$-burnable. Clearly, $T_3$ is $\{a_{k-1}\}$-burnable. Putting together, $T$ is $A$-burnable.

The inductive proof is finished.

Proof. Proof of Theorem 1 Let $A = (1, 2, \ldots, k)$. Applying Theorem 3, any tree of $n$ vertices is $[k]$-burnable if

$$n \leq 1 + 2 + \cdots + k + k - 1 + \left\lfloor \frac{(k^2 - 3k + 2)}{6} \right\rfloor = \left\lfloor \frac{2k^2 + 3k - 2}{3} \right\rfloor.$$  

Note that $\left\lfloor \frac{2k^2 + 3k - 2}{3} \right\rfloor$ equals to $\frac{2k^2 + 3k - 2}{3}$ if $k$ is divisible by 3; equals to $\frac{2k^2 + 3k - 2}{3}$ otherwise. In either case, $G$ is $[k]$-burnable if $n \leq \frac{2k^2 + 3k - 2}{3}$. Solving for $k$, we have $k \geq \frac{-3 + \sqrt{24n + 33}}{4}$. Since $k$ is an integer, we can take ceiling on the bound of $k$. Thus for any tree $T$ of $n$ vertices,

$$b(T) \leq \left\lceil \frac{-3 + \sqrt{24n + 33}}{4} \right\rceil.$$  

By equation (4), the same bound holds for all connected graphs $G$.

Lemma 4. If $G$ is connected and the radius satisfies $r(G) \geq 3$, then the complement $\bar{G}$ is also connected and $r(\bar{G}) \leq 2$.

Proof. Since $r(G) \geq 3$, there exists a pair of vertex $(u, v)$ with distance at least 3. Let $S$ be the set of all neighbors of $v$ in the graph $G$. For any vertex not in $S \cup \{v\}$, it is directly connected to $v$ in the complement graph $\bar{G}$. For any vertex $x$ in $S$, both $xu$ and $uv$ are edges of $G$. Thus, the complement graph $\bar{G}$ has radius at most 2.

Proof of Theorem 2 By Lemma 4 either $r(G)$ or $r(\bar{G})$ is at most 2. Without loss of generality, we can assume $r(G) \leq 2$, which implies $b(\bar{G}) \leq 3$. We have the following cases.

Case 1 $n \leq 4$. Since both $G$ and $\bar{G}$ are connected, the only graph $G$ that can exist is the path $P_4$. In this case $G = \bar{G} = P_4$. Note, $b(P_4) = 2$. This satisfies $b(G) \cdot b(\bar{G}) = 4 < n + 4$. 


case 2 $n \geq 5$. By Theorem 1, $b(G_n) \leq \left\lceil -3 + \sqrt{24n + 33} \right\rceil / 4$.

$$b(G) \cdot b(\overline{G}) \leq 3 \cdot \left\lceil -3 + \sqrt{24n + 33} \right\rceil / 4.$$ 

Now we show this bound is at most $n + 4$. When $n = 5, 6, 7, 8$, $\left\lceil -3 + \sqrt{24n + 33} \right\rceil = 3$, so $3 \cdot 3 = 9 \leq n + 4$. It holds for $n = 5, 6, 7$.

Now we assume $n \geq 9$, we use $\left\lceil -3 + \sqrt{24n + 33} \right\rceil \leq -3 + \sqrt{24n + 33} + 1$. It is sufficient to show

$$-3 + \sqrt{24n + 33} + 1 < n + 4.$$ 

A simple calculation yields $0 < n^2 - 7n - 8$. This true is for all $n \geq 9$.

From above argument, the equality holds only when $n = 5$ and $b(G) = b(\overline{G}) = 3$. Now assume $n = 5$. If $G$ contains a vertex $v$ of degree 3 or 4, then $b(G) \leq 2$ since we $N[v]$ can covers at least 4 vertices. Thus all degrees of $G$ are at most 2. For the same reason, all degrees of $G$ are at most 2. This implies that all degrees in $G$ and in $\overline{G}$ are exactly 2. Since both $G$ and $\overline{G}$ are connected and $n = 5$, the only possible case is $G = \overline{G} = C_5$. □

References

[1] S. Bessy, D. Rautenbach, Bounds, Approximation, and Hardness for the Burning Number, [arXiv:1511.06023](https://arxiv.org/abs/1511.06023).

[2] N. L. Biggs, E. K. Lloyd, and R. J. Wilson, *Graph Theory 1736-1936*, Oxford, England: Oxford University Press, p. 49, 1976.

[3] A. Bonato, J. Janssen, E. Roshanbin, Burning a Graph as a Model of Social Contagion, *Lecture Notes in Computer Science* 8882 (2014) 13-22.

[4] A. Bonato, J. Janssen, E. Roshanbin, How to burn a graph, *Internet Mathematics* 1-2 (2016), 85-100.

[5] A. Bonato, J. Janssen, E. Roshanbin, Burning a Graph is Hard, [arXiv:1511.06774](https://arxiv.org/abs/1511.06774).

[6] S. Banerjee, A. Das, A. Gopalan, S. Shakkottai, Epidemic spreading with external agents, *IEEE Transactions on Information Theory* July 2014.

[7] S. Bessy, A. Bonato, J. Janssen, D. Rautenbach, and E. Roshanbin, Bounds on the Burning Number, personal communication.

[8] S. Finbow, A. King, G. MacGillivray, R. Rizzi, The firefighter problem for graphs of maximum degree three, *Discrete Mathematics* 307 (2007) 2094-2105.
[9] S. Finbow, G. MacGillivray, The Firefighter problem: a survey of results, directions and questions, *Australasian Journal of Combinatorics* 43 (2009) 5777.

[10] E. Roshanbin, Burning a graph as a model of social contagion, PhD Thesis, Dalhousie University, 2016.