Nonsingular vortices-skyrmions for odd Landau level fillings in 2d system

S.V.Iordanskii, S.G.Plyasunov
L.D.Landau Institute for Theoretical Physics Russian Ac.Sci.
142432 Chernogolovka Moscow district Russia

Abstract

Using gradient expansion method the particle number, the energy and the action were calculated at the formation of a vortex-skyrmion. Contrary to other papers on this subject no approximation of single Landau level was used. Taking into account the nearest Landau levels essentially changes the expression for the vortex energy and gives some simple physical interpretation of the main results. It is shown that the formation of a vortex skyrmion gives the gain in the thermodynamic energy and therefore they must be spontaneously created near the odd fillings. The Hopf term in the skyrmion action is calculated and corresponds to a fermionic picture.

A large number of theoretical works is devoted to the consideration of macroscopical skyrmioniclike excitations near odd Landau level fillings in 2d systems under Quantum Hall Effect conditions. In the work [1] the phenomenological approach of the Chern-Simons theory was used. It was shown the existence of such excitations and their energy was calculated. In [2] the energy was calculated using Hartree-Fock approximation for single Landau level projected wave functions. The gradient expansion method was used in [3] for the same model assuming the large size of a skyrmion. The energy and the particle number were found analitically. These results were confirmed in [4] and the general technique was developed for the calculation of arbitrary terms in gradient expansion. In the model of projected functions the skyrmion action was calculated [5] with the topological term (see the proper discussion [6]). The experiments give some evidence of skyrmion formation [7] but some experimentalists cast some doubts concerning these results [8].

Single Landau level projected functions approximation is usually justified by the large value of the cyclotron energy \( \hbar \omega_c \) compare to Coulomb interaction \( e^2/l_H \) where the magnetic length \( l_H = \hbar e/\omega_c \) and \( H \) is the external magnetic field. The appropriate calculations are rather elaborate and contain some additional assumptions. The final expressions for the electron energy and the density are obtained by cumbrsome calculations without any direct physical interpretation. In the present work we show that the approximation of the projected wave functions is not sufficient for the adequate skyrmion description. Some preliminary results were published in [9] and the present paper is more complete with the justification of the results by gradient expansion method.

Skyrmions correspond to the rotation of the second quantized operators-spinors using nonuniform rotation matrix \( U(\vec{r}) \). In that case the initial spinors in the laborator frame \( \psi \) are expressed in terms of the new spinors \( \chi \) with spin up in the local frame according to the relation \( \psi(\vec{r}) = U(\vec{r})\chi(\vec{r}) \). The matrix \( U(\vec{r}) \) can be parametrized by three Euler angles

\[
U(\vec{r}) = U_z(\gamma(\vec{r}))U_y(\beta(\vec{r}))U_z(\alpha(\vec{r}))
\]

\[
U_z(\alpha) = \cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2} \sigma_z
\]

\[
U_y(\beta) = \cos \frac{\beta}{2} + i \sin \frac{\beta}{2} \sigma_y,
\]

where \( \sigma_x, \sigma_y, \sigma_z \) are Pauli matrices. At large distances from the core for finite value of g-factor the average spin must be oriented along the external magnetic field to have a finite value of the total energy. Therefore the angle \( \beta \), counted from the magnetic field direction must
rapidly vanish for $r \to \infty$ (it can be shown exponentially) We assume that the matrix $U(\vec{r})$ has no singularities at any $\vec{r}$. That corresponds to a nonsingular behaviour of the matrices

$$A_k = -iU^+ \frac{\partial U}{\partial \vec{r}_k} = \Omega^i_k(\vec{r}) \sigma_i$$

where $k = x, y$ and $l = x, y, z$, Pauli matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The expression for the functions $\Omega^i_k$ are easily obtained by the direct differentiation of $U$

$$\Omega^x_k = \frac{1}{2}(\partial_k \alpha + \cos \beta \partial_k \gamma)$$

$$\Omega^y_k = \frac{1}{2}(\sin \beta \cos \alpha \partial_k \gamma - \sin \alpha \partial_k \beta)$$

$$\Omega^z_k = \frac{1}{2}(\cos \alpha \partial_k \beta + \sin \beta \sin \alpha \partial_k \gamma)$$

The non trivial topology defined by the matrix $U(\vec{r})$, is connected with the properties of the mappings $\alpha(\vec{r}), \gamma(\vec{r})$, where $\vec{r}$ is varied along a circumference of large radii. The degree of the mapping of 2d plain $x, y$ on the sphere parametrized by the angles $\gamma$ and $\beta$ coincides with the degree of the mapping of the circumference on the circumference i.e. the winding number for the vortex singularity of $\gamma(\vec{r})$. For a nonsingular $\Omega^l_k(\vec{r})$ it is necessary to have coinciding singularities in $\gamma(\vec{r})$ and $\alpha(\vec{r})$ at the the point where $\cos \beta = -1$. Therefore in the matrix $U$ must be represented all three Euler angles and the corresponding spinor $\psi(\vec{r})$ has vortex singularity of it’s phase at large distances with the integer winding number and is simple in 2d plain. For that reason it is more appropriate to talk about a non singular vortex with skyrmionic core by the analogy with $He_3$. The winding numbers are here any integers contrary to the case of $He_3$ where they are even [10]. The integral

$$\frac{1}{2\pi} \int rot \Omega^z d^2 r = Q$$

is a topological invariant and is expressed directly through the change of the phase for the spinor $\psi$ along a large circumference.

The pair interaction Lagrangian for electrons :

$$L = \int \left[ i\psi^+ \frac{\partial \psi}{\partial t} - \frac{1}{2m} \psi^+ (-i \partial_k - A_{0k})^2 \psi \right] d^2 r dt +\frac{1}{2} \int V(\vec{r} - \vec{r}') \psi^+(\vec{r}) \psi^+(\vec{r}') \psi(\vec{r}) \psi(\vec{r}') d^2 r' d^2 r dt - gH \int \psi^+ \sigma_z \psi d^2 r dt$$

transforms by the matrix $U$ into the Lagrangian in terms of spinors $\chi$

$$L' = \int i\chi^+ \left[ \frac{\partial \chi}{\partial t} - \Omega^i_k \sigma_i \chi - \frac{1}{2m} (i \partial_k - A_{0k} + \Omega^i_k) \sigma_i \chi \right] d^2 r dt$$

$$- \frac{1}{2} \int V(\vec{r} - \vec{r}') \chi^+(\vec{r}) \chi^+(\vec{r}') \chi(\vec{r}) \chi(\vec{r}') d^2 r' d^2 r dt + gH \int U^+ \sigma_z U \chi d^2 r dt$$

without any approximation. We assume that matrix $U$ depends on the time also. The quantity $\Omega^i_k$ corresponds to (11) but with the time derivative. $A_0$ is the vector potential of the external uniform magnetic field. The Lagrangian (11) can be easily obtained by direct differentiation taking into account some identity obtained by the differentiation of the of the identity $U^+ U = 1$. 

\[\text{Question: How can we interpret the results in the context of non-singular vortices with skyrmionic core?}\]
The size of the vortex-skyrmion core is defined by Coulomb interaction which tend to increase the domain with large change of the density and by Zeeman energy which tend to diminish this domain due to the positive energy of the spins non aligned to the external magnetic field. At large distances Coulomb is small and only Zeeman energy and the energy connected with spin direction gradients are essential. That gives the exponential decay of non aligned spins. For a small value of the g-factor the size of the core will be large and the derivatives of the matrix $U$ become small that justify the use of the gradient expansion for the calculations. We shall treat this case interesting mostly in topologically invariant terms not depending on deformations of $U$. That allow us to consider no Zeeman energy or taking it into account only in the first order of the perturbation theory.

The effective action depending on the matrix $U$ is defined by the integration out the fermions in the correspondent expression for the statistical sum and has the form \[ S = iTr \ln G, \] where $G$ is electron Green function. The Trace is over whole set of variables including spin and space-time. This action will be calculated by the gradient expansion in $U$ using Hartree-Fock approximation which we justify later.

**1 Green function in Hartree-Fock approximation**

We use Hartree-Fock approximation for the calculation of Green function which is valid for fully filled Landau level with the accuracy of the order $V_{\text{int}}/\hbar \omega_c$ assumed to be small. Such an approach allows to write the electronic Hamiltonian as

\[
H = \int \chi^{+} \left[ \Omega_{t}^{k} \sigma_{I} + \frac{1}{2m} (-i\partial_{k} - A_{0k} + \Omega_{k}^{I} \sigma_{I})^{2} \right] \chi d^{2}r + \int V(r - r') < \chi^{+}(\vec{r}) \chi(\vec{r}') > \chi^{+}(\vec{r}) \chi(\vec{r}) d^{2}r d^{2}r' - \int V(r - r') < \chi^{+}(\vec{r}') \sigma_{I} \chi(\vec{r}) > \chi^{+}(\vec{r}) \sigma_{I} \chi(\vec{r}') d^{2}r d^{2}r'
\]

The interaction term becomes the sum of exchange and direct interaction and small Zeeman energy is omitted. In the direct interaction we must omit zero Fourier component for Coulomb case due to compensating uniform charge (see e.g.\[11\]).

We assume that all the average quantities are close to their uniform values for fully filled Landau level. We restrict our consideration by the model of the local exchange neglecting the magnetic length. Probably the transition to the local exchange is quite unessential and topological terms does not depend on it but the use of such an approximation greatly simplifies the calculations. The final results are easily generalized on the case of the nonlocal exchange. Therefore we shall deal with the hamiltonian

\[
H = \int \left[ \chi^{+} \Omega_{t}^{k} \sigma_{I} \chi^{+} + \frac{1}{2m} \chi^{+} (-i\partial_{k} - A_{0k} + \Omega_{k}^{I} \sigma_{I})^{2} \chi - \gamma \rho \chi^{+} \sigma_{I} m_{l} \chi^{+} + V_{0} \rho \chi^{+} \chi \right] d^{2}r
\]

where $\gamma$ is the exchange constant and $V_{0}$ is the constant of direct interaction which we shall omit as in the case of Coulomb interaction, $\rho$ is the average density. We suppose that $\chi$ spinors are oriented in z-direction coinciding with the average local spin direction in the laboratory frame. Assuming a small value of the angle velocities $\Omega_{t}^{k}$ and their derivatives $\partial \Omega \sim \Omega^{2}$ we shall calculate Green function by the perturbation theory separating the hamiltonian $H = H_{0} + H_{1} + H_{2}$ into three parts

\[
H_{0} = \int \chi^{+} \left[ \frac{1}{2m} (-i\partial_{k} - A_{0k})^{2} - \gamma \rho \sigma_{I} m_{l} - \mu \right] \chi d^{2}r
\]

\[
H_{1} = \frac{1}{2} \int \chi^{+} \left[ \Omega_{t}^{k} \sigma_{I} (-i\partial_{k} - A_{0k}) + \Omega_{k}^{I} \sigma_{I} \right] \chi d^{2}r
\]
\[ H_2 = \frac{1}{2m} \int \chi^+ \left[ (\Omega_k^l \sigma_l)^2 - i \partial \Omega_k^l / \partial r_k \sigma_l \right] \chi^2 \]  

(8)

We use grand canonical ensemble introducing chemical potential \( \mu \).

Green function for the unperturbed Hamiltonian is Green function for noninteracting electrons in uniform magnetic field. We assume that at the last filled Landau level only the lowest spin sublevel is occupied. For simplicity we consider here only the case when Landau index \( s = 0 \) when unperturbed Green function acquires the form

\[ G_0(\vec{r}, \vec{r}', t, t') = -i < T \chi(\vec{r}, t) \chi^+(\vec{r}', t') > = \sum_{p,s} \int g_s(\omega) e^{i\omega(t' - t)} \Phi_{sp}(\vec{r}) \Phi_{sp}^*(\vec{r}') \frac{d\omega}{2\pi} \]  

(9)

Here \( T \) is a symbol of time product for fermions, the sum is over all \( s \) and \( p \), spin matrices \( g_s(\omega) \) corresponds to fully occupied Landau level \( s = 0 \) with spin up in the local frame. Other states are empty. All quantities are measured in units where \( H_0 = 1, l_H = 1 \) and \( \hbar = 1 \). The normalized wave functions \( \Phi_{sp}(\vec{r}) \) corresponds to eigenfunctions in Landau gauge.

It is easy to find matrices \( g_s(\omega) \) for the hamiltonian (8)

\[ g_0(\omega) = \frac{1}{\omega + (\gamma \rho - i\delta)\sigma_z + \mu} \]  

(10)

\[ g_s(\omega) = \frac{1}{\omega + \gamma \rho \sigma_z - \frac{\omega}{m} + \mu + i\delta} \]  

(11)

where \( \delta \to (+0) \), the energy of zero Landau level \( 1/2m \) is included in chemical potential.

Using the expression (8) we can develop the perturbation theory expansion for Green function of the total hamiltonian \( G = G_0 + G_1 + G_2 + ... \). The corresponding diagrams are shown in fig 1 up to the third order.

It is useful for further calculations to represent the operator entering (8) in the form

\[ \frac{1}{m} \Omega_k^l \sigma_l (-i \partial_k - A_{0k}) = \frac{1}{m} (\Omega_+^l \pi^+ + \Omega_-^l \pi^+ ) \sigma_l \]  

(12)

The operator \( \pi^+ \Phi_{sp} = \sqrt{2(s + 1)} \Phi_{(s+1)p} \) increases Landau index, and the operator \( \pi^- \Phi_{sp} = \sqrt{2s} \Phi_{(s-1)p} \) decreases Landau index what follows directly from the properties of oscillator wave functions and

\[ \Omega_+^l = -\frac{i\Omega_x^l + \Omega_y^l}{2}, \Omega_-^l = \frac{i\Omega_x^l - \Omega_y^l}{2} \]  

(13)

The correction of the first order in \( \Omega \) to Green function is

\[ G_1(\vec{r}, \vec{r}', t, t') = \int e^{-i\omega'(t - t')} g_s(\omega) \Phi_{sp}(\vec{r}) \Phi_{sp}^*(\vec{r}') \]

\[ \times \left[ \frac{1}{m} (\Omega_+^l(\vec{r}_1) \pi^+ + \Omega_-^l(\vec{r}_1) \pi^+ ) + \Omega_+^l(\vec{r}_1) \sigma_l g_s(\omega') \right] \]

\[ \times e^{-i\omega'(t_1 - t')} \Phi_{sp}'(\vec{r}_1') \Phi_{sp}'(\vec{r}_1') \frac{d\omega_1 d\omega_1}{2\pi 2\pi} d^2r_1 dt_1 \]  

(14)

We are interested in Green function \( G(r, r; t, t + \delta) \) defining the density. In that case the only nonvanishing terms are with \( s = 0, s' = 1 \), \( s = 1, s' = 0 \) and \( s = s' = 0 \). The others vanish because of analytical properties of \( g_s \) for \( s > 0 \) or have some additional derivatives. It is possible to neglect the difference \( t_1 - t \) for the same reason. We do not consider at first the term with \( \Omega_0^l \) which can be easily calculated. Integrand in (14) is vanishing rapidly when \( |\vec{r} - \vec{r}_1| \) is increased above magnetic length. Therefore it is possible to expand \( \Omega(\vec{r}_1) \) in terms of \( \vec{r} - \vec{r}_1 \) up to linear terms. Performing the summation in \( p \) we get the following expression

\[ G_1(\vec{r}, \vec{r}', t, t + \delta) = \int \frac{g_{0s} g_{0s'} e^{i\omega_0} \partial \Omega_0^l}{V R_{00}(\vec{r}, \vec{r}_1) R_{01}(\vec{r}, \vec{r}_1)(r_{1k} - r_k) d^2r d\omega} \]

\[ \frac{1}{2} \frac{d^2r_1 dt_1}{\omega_0} \]
\[
\frac{\sqrt{2}}{m} \int g_0(\omega)\sigma_l g_1 e^{i\omega\delta} \frac{d\omega}{2\pi} \int R_{00}(\vec{r}, \vec{r}) R_{01}(\vec{r}, \vec{r})(r_1 - r_k) \frac{\partial \Omega_{l}}{\partial r_k} d^2r_k + \\
\frac{\sqrt{2}}{m} \int g_1(\omega)\sigma_l g_0(\omega) e^{i\omega\delta} \frac{d\omega}{2\pi} \int R_{11}(\vec{r}, \vec{r}) R_{10}(\vec{r}, \vec{r})(r_1 - r_k) \frac{\partial \Omega_{l}}{\partial r_k} d^2r_k
\]

(15)

Here we introduce the functions

\[
R_{00}(\vec{r}, \vec{r}) = \frac{1}{2\pi} \exp(-\frac{(\vec{r} - \vec{r})^2}{4})
\]

(16)

\[
R_{11}(\vec{r}, \vec{r}) = \frac{1}{2\pi} \left[1 - \frac{(\vec{r} - \vec{r})^2}{2}\right] \exp(-\frac{(\vec{r} - \vec{r})^2}{4})
\]

(17)

\[
R_{10}(\vec{r}, \vec{r}) = \frac{1}{2\pi} \frac{x_1 - x - i(y_1 - y)}{\sqrt{2}} \exp(-\frac{(\vec{r} - \vec{r})^2}{4})
\]

(18)

\[
R_{01}(\vec{r}, \vec{r}) = \frac{1}{2\pi} \frac{x - x_1 - i(y_1 - y)}{\sqrt{2}} \exp(-\frac{(\vec{r} - \vec{r})^2}{4})
\]

(19)

After the integration in \(\vec{r}_1\) the term without the derivatives of \(\Omega\) vanish because the integrand is an odd function. Doing all integrations in (15) we get using (13), (11), (10), (9)

Finally the first order correction to Green function acquires the form

\[
G_1(\vec{r}, \vec{r}, t, t + \delta) = \frac{1}{4m\gamma} \sigma_l - \sigma_z \sigma_l \sigma_z \frac{1}{2}(\text{div}\vec{\Omega} + i\text{rot}\vec{\Omega})
\]

\[
\frac{1}{2\pi} \left[\frac{\sigma_z \sigma_l - \sigma_l \sigma_z}{4} \text{div}\vec{\Omega} + i \frac{1 + \sigma_z}{2} \sigma_l(1 + \sigma_z) \text{rot}\vec{\Omega}\right]
\]

here we consider only the main spin independent contribution to \(g_1 \sim m\). In the term with \(\Omega_{l}\) we must consider only the case \(s = s' = 0\) others are of the order of \(m\) and can be neglected. Only the terms with the first order poles in \(\omega\) are essential that results in

\[
G_1' = -i\Omega_{l} \frac{\sigma_l - \sigma_z \sigma_l \sigma_z}{8\pi\gamma \rho}
\]

Finally the first order correction to Green function acquires the form

\[
G_1(\vec{r}, \vec{r}, t, t + \delta) = \frac{1}{8\pi} \left[\sigma_z \sigma_l - \sigma_l \sigma_z \text{div}\vec{\Omega} + i((1 + \sigma_z)\sigma_l + \sigma_l(1 + \sigma_z)) \text{rot}\vec{\Omega}\right] + \\
\frac{1}{4m\gamma} \frac{\sigma_l - \sigma_z \sigma_l \sigma_z}{2} \text{div}\vec{\Omega} + i \Omega_{l} \frac{\sigma_z \sigma_l - \sigma_l \sigma_z}{8\pi\gamma}
\]

(20)

For the density \(\rho = -iSpG(\vec{r}, \vec{r}, t, t + \delta)\) we obtain

\[
\rho(\vec{r}, t) = \frac{1}{2\pi} (1 - \text{rot}\vec{\Omega})
\]

(21)

This result was firstly obtained in the approximation of the projected wave functions [3]. In the terms of unprojected wave functions this result has a simple physical meaning: the lowest Landau level in the local effective field \(H_{eff} = 1 - \text{rot}\vec{\Omega}\) is fully filled and the density coincides with the local density of states \(\rho = 1/2\pi l^2(H_{eff})\).

The results for the first order term in \(G\) contain the derivative of \(\Omega\) that corresponds to second order in \(\Omega\). Therefore it is necessary to calculate the formally second order term in \(G\) which can be written down symbolically as

\[
G_2(\vec{r}, \vec{r}, t, t + \delta) = \frac{1}{m^2} G_0(\Omega_{l+}^\pi - \Omega_{l-}^\pi) \sigma_l G_0(\Omega_{l+}^\pi - \Omega_{l-}^\pi) \sigma_l G_0 + G_0 H_2 G_0
\]

In the second order we can neglect the additional derivatives of \(\Omega\). Because the last term is formally of the second order we must take in it the states with \(s = 0\) with the second order pole in \(\omega\) which vanish after the integration. In the first term there are two types of
contributions with \( g_0 g_1 g_0 \) and \( g_1 g_0 g_1 \), where \( g \) corresponds to one of the three Green functions in this expression. The integration over space variables and \( p \) are easily performed and we get

\[
G_2 = \frac{1}{m^2 2\pi} \int [g_0 \sigma_l g_1 \sigma_f g_0 \Omega_+^{l'} + g_1 \sigma_l g_0 \sigma_f g_1 \Omega_+^{l'}] e^{i\omega \phi} d\omega d^2r dt
\]

To calculate the second order correction to the density we need trace of this expression. Transposing the matrices under the trace we get in the first case the square of diagonal matrix \( g_0^2(\omega) \) having the second order pole in \( \omega \). Using the standard expression for the proper integral we obtain

\[
SpG_2 = \frac{1}{2\pi m^2} \int \left[ -2g_1^2 \Omega_+^{l'} + 2g_1^2 \Omega_+^{l'} \right] d^2r dt = 0
\]

Therefore the density is given by the expression (21) up to the second order.

2 The calculation of the action, the energy and the particle number

For the calculation of the action \( S = iSp \ln G \) we use the perturbation theory for Green function and expand the logarithm

\[
S = iSp[H_1 G_0 + \frac{1}{2} H_1 G_0 H_1 G_0 + H_2 G_0 + \frac{1}{2}(H_2 G_0 H_1 G_0 + H_1 G_0 H_2 G_0) + \frac{1}{3} H_1 G_0 H_1 G_0 H_1 G_0 + ...]
\]

Here we omit nonessential part without the rotation matrix and only terms up to the third order are retained. In this section we limit our consideration by the second order terms. The action of the first order is given by

\[
S_1 = iSp \int \Omega_+^{l} \sigma_l g_0(\omega) e^{i\omega \phi} \frac{d^2r dt}{2\pi} \frac{d\omega}{2\pi} + iSp \int \sigma_l g_0(\omega) e^{i\omega \phi} \frac{d^2r dt}{2\pi} \frac{d\omega}{2\pi} (\Omega_+^{l} \pi^+ + \Omega_+^{l} \pi^-) \Phi_{sp}(r) \Phi_{sp}^*(r) d^2r dt
\]

The second term can be calculated using integration by parts and we get

\[
S_1 = -\frac{1}{2\pi} \int \Omega_+^{l} \pi^+ \frac{d^2r dt}{2m} \frac{d\omega}{2m} \int (i div \vec{\Omega} - rot \vec{\Omega}^z) d^2r dt \quad (22)
\]

The action of the second order consists of two terms one of which contains two \( H_1 H_1 \):

\[
S_2' = \frac{i}{2m^2} Sp \int (\Omega_+^{l'} \pi^+ + \Omega_+^{l'} \pi^-) \sigma_l g_s(\omega) \Phi_{sp}(\vec{r}) \Phi_{sp}^*(\vec{r}') (\Omega_+^{l'} \pi^+ + \Omega_+^{l'} \pi^-) \sigma_l g_s' (\omega) \Phi_{sp'}(\vec{r}') \Phi_{sp'}^*(\vec{r}') e^{i\omega \phi} \frac{d^2r' \pi^+ \pi^-}{2\pi}
\]

In this expression we make no difference between \( \Omega(\vec{r}, t) \) and \( \Omega(\vec{r}', t') \) neglecting derivatives of \( \Omega \). The only nonvanishing terms correspond to \( s = 0, s' = 1 \) and \( s = 1, s' = 0 \). Using the properties of the operators \( \pi^+, \pi^- \) and performing simple integration it is easy to get the second order action including the second term with \( H_2 \)

\[
S_2 = \frac{1}{2\pi} \int Sp \frac{\sigma_f \sigma_l (1 + \sigma_z) + \sigma_f \sigma_l (1 + \sigma_z)}{2} \Omega_+^{l} \Omega_+^{l'} d^2r dt - \frac{1}{2\pi} \int \frac{(\vec{\Omega})^2}{2m} - \frac{i}{2m} \frac{\partial \Omega_{k}^z}{\partial r_k} d^2r dt
\]

The symmetric part with \( l = l' \) of the first term is compensated by the proper part of the second term. The antisymmetric part contains only \( l, l' = x, y \) and can be rewritten using the identity \( rot \vec{\Omega}^z = 2(\Omega_x^x \Omega_y - \Omega_y^x \Omega_y^y) \) which is obtained by differentiation using the identity \( U^+ U = 1 \). Finally we get the second order action

\[
S_2 = \frac{1}{2\pi} \int rot \vec{\Omega}^z \frac{d^2r dt}{2\pi} + i \int div \vec{\Omega}^z \frac{d^2r dt}{2\pi} \quad (23)
\]
The total action up to the second order including (22) is

\[ S = \int \Omega_i^2 \frac{d^2 r dt}{2\pi} + \frac{1}{m} \int \text{rot}\Omega^2 \frac{d^2 r dt}{2\pi} - \mu \int \text{rot}\Omega^2 \frac{d^2 r dt}{2\pi} \]  

(24)

Here we add the term with chemical potential.

These results are in full agreement with the authors preliminary publication [9] and have simple physical interpretation. The electrons locally fill the lowest spin sublevel in the local “effective” magnetic field \( H_{\text{eff}} = 1 - \text{rot}\Omega^2 \). They fill it fully so the electron density coincides with the local density of states. This circumstance justifies the use of Hartree-Fock approximation in the same manner as it does for the fully filled Landau level. The “effective” magnetic field is less than the external one for vortices with positive \( Q \). It must be stressed that the gap between spin sublevels is defined by the exchange. The electrons with the spin direction opposite to average spin direction will see the “effective” field \( H_{\text{eff}} = 1 + \text{rot}\Omega^2 \).

Let us consider more detailed expression for the energy in a stationary case taking into account the change in the interaction terms due to the variation of the density. For the Coulomb interaction one must assume the existence of the compensating positive background requiring to vanish the zero Fourier of the interaction. Because of that there is no linear term in the density variation for direct interaction. Its’ expansion begins from the second order

\[ E_{\text{pot}} = \frac{1}{2} \int \frac{e^2}{|\vec{r} - \vec{r}'|} \text{rot}\Omega^2(\vec{r}) \text{rot}\Omega^2(\vec{r}') \frac{d^2 r d^2 r'}{(2\pi)^2} \]

The exchange energy begins with a linear term in the variation of the density. For it’s calculation one must express the density of the exchange energy as a function of the magnetic field for fully filled Landau level \(-\frac{e^2}{2\pi l_H^2} \sqrt{\frac{\pi}{2}} \) differentiate it in magnetic field and multiply by the variation in effective magnetic field due to vortex formation

\[ E_{\text{ex}} = \int \frac{3e^2}{4\sqrt{2\pi l_H}} \text{rot}\Omega^2 d^2 r \]

Also it is necessary to take into account an energy correction due to nonuniform direction of the average spin direction \( J \int \frac{\partial n}{\partial r_k} d^2 r \) where the quantity \( J = \frac{1}{\hbar v_{2\pi l_H^2}} \), (see e.g. [3,4], and add Zeeman energy. Finally we get the change of the thermodynamic energy due to vortex-skyrmion formation

\[ F = \delta <H - \mu N> = \frac{\hbar \omega c}{2} Q + \frac{3e^2}{2l_H} \sqrt{\frac{\pi}{2}} Q + \frac{e^2}{2} \int \frac{\text{rot}\Omega^2(\vec{r}) \text{rot}\Omega^2(\vec{r}')}{(2\pi)^2|\vec{r} - \vec{r}'|} d^2 r d^2 r' + \left[ \frac{1}{2} J \left( \frac{\partial n}{\partial r_k} \right)^2 + gH\vec{n} \cdot \frac{1}{2\pi l_H^2} \right] d^2 r \]  

(25)

We assume here the value of chemical potential \( \mu = \frac{\hbar \omega c}{2} \) corresponding to the fully filled lowest Landau spinsublevel at large distances from the core. By assumption the cyclotron is large compare to the interaction energy and consequently the quantity \( F \) is negative for positive topological number \( Q \) defining the change of the total number of electrons \( \delta N = -Q \) at vortex formation. Therefore the formation of nonsingular vortices-skyrmions give the gain in the thermodynamic energy and they must be spontaneously created.

It is possible to calculate also the energy of one particle excitations. This can be done by the investigation of the poles for \( G(\vec{r}, \vec{r}', \omega) \). However it is more simple to use the fact that these energies corresponds to to the variational derivatives of the total energy over the local density of the electrons with the different spin direction. It is possible to write down the quantity \( F \) as a functional depending on the densities \( \rho_+, \rho_- \) where signs denote the direction along and opposite to the average local spin

\[ F = \int \left[ \frac{\hbar \omega c (H_{\text{eff}}^+ \rho_+ + \hbar \omega c (H_{\text{eff}}^- \rho_-)}{2\pi l_H^2} - \mu (\rho_+ + \rho_-) - \frac{1}{2} (\rho_+ - \rho_-)^2 (\gamma' - J' \left( \frac{\partial n}{\partial r_k} \right)^2 ) \right] d^2 r + \int \left[ \frac{1}{2} \frac{\partial n}{\partial r_k} \right] d^2 r \]
\[ \int \frac{e^2}{2|\vec{r} - \vec{r}'|}(\delta \rho_+ (\vec{r} + \delta \rho_-(\vec{r})) (\delta \rho_+ (\vec{r}') + \delta \rho_-(\vec{r}')) d^2r d^2r' + \int g\vec{H} \vec{n}(\rho_+ - \rho_-) d^2r \]  

(26)

In this expression \( \gamma'_{\rho^2} = \frac{e^2}{2\pi l_H^2} \sqrt{\frac{\pi}{2}} - \frac{3e^2}{4\pi l_H^2} \sqrt{\frac{\pi}{2}} \text{rot} \vec{\Omega}^z \) and \( J'_{\rho^2} = J \). Variation in densities \( \rho_+, \rho_- \) gives us the energy of the hole in the filled states

\[ \epsilon_h = -\frac{\delta F}{\delta \rho_+} = \frac{2e^2}{l_H} \sqrt{\frac{\pi}{2}} (1 - \frac{3}{2} l_H^2 \text{rot} \vec{\Omega}^z) + \]

(27)

and the energy of the electron with the locally reversed spin

\[ \epsilon_e = \frac{2e^2}{l_H} \sqrt{\frac{\pi}{2}} (1 - \frac{3}{2} l_H^2 \text{rot} \vec{\Omega}^z) + \frac{\hbar \omega_c}{2} \text{rot} \vec{\Omega}^z + \int \frac{e^2}{|\vec{r} - \vec{r}'|} \text{rot} \vec{\Omega}^z(\vec{r}') \frac{d^2r'}{2\pi} - J4\pi l_H^2 \frac{\partial n^l}{\partial r_k}^2 - g\vec{H} \vec{n} \]

(28)

All previous results are obtained by using the perturbation theory in \( \Omega \). That means that one particle energy corrections due to \( \Omega \) must be small compared to the initial gap separating the lowest spin sublevel from the higher one. Therefore the largest additional term in one particle energy \( \hbar \omega_c/2l_H^2 \text{rot} \vec{\Omega}^z \sim \hbar \omega_c/2l_H^2L_c^2 \) where \( L_c \) is the core size must be small compare to the exchange energy \( \sim e^2/l_H \). That gives the inequality \( L_c \gg l_H \sqrt{\hbar \omega_c/e^2} \). The core size is defined by the competition of Coulomb energy \( e^2/L_c \) and Zeeman energy \( gH\rho L_c^3 \) that gives \( L_c^3 \sim (e^2 l_H^2)/gH \gg l_H^3 [(\hbar \omega_c l_H^2)/e^2]^{3/2} \). Therefore for the validity of the obtained results g-factor must be small enough. The addition of the electron with reversed spin increase the thermodynamic energy due to exchange and decrease it by Coulomb interaction for positive \( Q \) other terms are essentially less due to small value of \( \Omega \). Coulomb energy increases with \( Q \) and inverse size of core whereas exchange energy is local and does not depend on the core structure. Therefore it is possible to have metastable bound states and even the states below chemical potential. In that case the total amount of the electrons with “wrong” orientation in the external magnetic field will be diminished by one compare to the vortex without bound electron. The thermodynamic gain by the creation of the vortex will be conserved. For real experimental conditions the cyclotron energy is not very large compare to Coulomb energy and g-factor is also not very small and therefore these statements need computer check.

### 3 Hopf invariant in the action

Besides the mapping degree \( Q \) there is also topological Hopf invariant corresponding to the linking number of the lines with constant values of \( \vec{n}(\vec{r}, t) \) for nonstationary matrix \( U \) (see e.g.[13]). This invariant must be contained in the action with the coefficient determining the statistics of vortices or more precise the phase change when they are transposed [13]. This coefficient was calculated in [5] in the model of projected wave functions. However this calculation contains some additional assumptions and was questioned in the discussion [6]. The authors of [6] reference to their previously made semiclassical calculation which use the frequency and the momentum as a good quantum numbers for Green function calculation and cite the proper result. In this section we make a calculation of this coefficient directly in the limit of large magnetic field without additional assumptions. For that one needs to find the action in the third order of \( \Omega \) including only the terms containing two space and one time component of \( \Omega^t \).

Third order perturbation terms in the action are

\[ S = iTr[\frac{1}{2} H_1G_0 H_1G_0 + (H_1G_0 H_2G_0) + \frac{1}{3} H_1G_0 H_1G_0 H_1G_0] \]

(29)

The terms up to the second order were calculated in the previous section. However time derivatives were not included in these terms what is essential in the third order calculations. We shall calculate the various third order terms one by one.
We begin with the second order in $H_1$ term containing $\Omega_i^l$ omitted in the previous section assuming $\Omega_i \sim \bar{\Omega}^2$. After an easy calculation we get

$$S_2^1 = \frac{i}{m} \int Sp_{\sigma l g_0(\omega)\sigma_l g_0(\omega)}e^{i\omega t} \Omega_t^l(r, t) \Phi_{sp}(\vec{r}) \Phi_{sp}^*(\vec{r}_1) \times$$

$$(\Omega^l_i(\vec{r}_1, t) \tau + \Omega^l_i(\vec{r}_1, t) \tau^+) \Phi_{sp}(\vec{r}) \Phi_{sp}^*(\vec{r}_1) d^2r \frac{d\omega}{2\pi}$$

With the needed accuracy there are one term with $s = s' = 0$ and two terms with $s = 1$, $s' = 0$; $s = 0$, $s' = 1$. Introducing the new integration variables $\vec{R} = (\vec{r} + \vec{r}_1)/2$, $\vec{r} = \vec{r}_1 - \vec{r}$ and expanding $\Omega_t(\vec{r}), \bar{\Omega}(\vec{r}_1)$ in $\vec{r}$ up to the first order we obtain for the first term

$$S_{200}^1 = \frac{i\sqrt{\gamma}}{2m} \left[ \int Sp_{\sigma l g_0(\omega)\sigma l g_0(\omega)}e^{i\omega t} \Omega_t^l(\vec{R}) \frac{\partial \Omega^l_i}{\partial R_{k}} - \Omega^l_i(\vec{R}) \frac{\partial \Omega_t^l}{\partial R_{k}} \right] R_{000}(-\bar{\rho}) R_{10}(\bar{\rho}) \rho_k d^2\rho d^2R dt d\omega$$

Using expressions (10), (11) and performing the proper integrals in $\omega$ and $\bar{\rho}$ we have

$$S_{200}^1 = \frac{1}{4m\gamma} \int \left[ i\Omega_t^l \text{div} \bar{\Omega}^l_i - i(\bar{\Omega}^l_i \nabla) \Omega_t^l - \Omega_t^l \nabla \times \bar{\Omega}^l_i + \bar{\Omega}^l_i \times \nabla \Omega_t^l \right] \frac{d^2r dt}{2\pi}$$

The terms with $s = 0, s' = 1$; $s = 1, s' = 0$ are calculated in the same way

$$S_{201}^1 = \frac{1}{2} \left[ \Omega_t^l \nabla \times \Omega_t^l + \bar{\Omega}^l_i \times \nabla \Omega_t^l \right] \frac{d^2r dt}{2\pi}$$

There is also the term with two $\bar{\Omega}^l_i$ considered in the previous section not taking into account time derivatives. We can proceed quite similar introducing the new integration variables $T = (t + t')/2, \tau = t' - t$ and expanding in $\tau$. With the required accuracy it is sufficient to consider only the terms with $s = 1, s' = 0$ and $s = 0, s' = 1$ others contain extra derivatives. Omiting the second order term without time derivatives we get

$$S_2^2 = \frac{i}{m^2} \int Sp_{\sigma l g_0(\omega)\sigma l g_1(\omega')}e^{i\omega t} \exp i(\omega - \omega')\tau [\Omega_t^l \frac{\partial \Omega_t^l}{\partial T} - \Omega_t^l \frac{\partial \Omega_t^l}{\partial T} (2\pi)^2 d^2T d^2r dt \frac{d\omega}{2\pi}]$$

Representig $\tau$ as the derivative of the proper exponent in the integrand and integrating by parts we have

$$S_2^2 = \frac{1}{2} \left[ \bar{\Omega}^l_i \times \frac{\partial \bar{\Omega}^l_i}{\partial T} \frac{d^2r dt}{2\pi} - \frac{i}{4} \int Sp_{\sigma l \sigma l} \left[ \bar{\Omega}_t \frac{\partial \bar{\Omega}^l_i}{\partial T} - \bar{\Omega}^l_i \frac{\partial \bar{\Omega}_t}{\partial T} \right] d^2r dt \frac{d\omega}{2\pi}$$

In order to compare various terms in action they must be expressed in some standard way through $\Omega^l_i$ and it’s derivatives. Using the identity $\partial \eta_i \Omega^l_i - \partial_i \Omega^l_i = 2e^{ijm} \Omega^l_i \Omega^m_k$ where $e^{ijm}$ is antismetrical tensor it is easy to transform the previous expression to the following form

$$S_2^2 = \frac{1}{2} \left[ \left( \bar{\Omega}^l_i \times \nabla \Omega^l_i + e^{ijm} \bar{\Omega}^l_i \times \bar{\Omega}^m_j \Omega^l_i \right) \frac{d^2r dt}{2\pi} \right]$$

$$+ \frac{i}{2} \int Sp_{\sigma l \sigma l} \left[ \bar{\Omega}_t \nabla \Omega^l_i + 2e^{ijm} \bar{\Omega}^l_i \bar{\Omega}^m_j \Omega^l_i \right] \frac{d^2r dt}{2\pi}$$

Among the third order terms the simplest is containing $H_2$ and $H_1$

$$S_{21} = \frac{i}{2m} \int [(\bar{\Omega}^l_i)^2 - idiv\bar{\Omega} \sigma_l g_0(\omega) \sigma_l g_0(\omega) e^{i\omega t} \frac{d\omega}{2\pi}]$$

$$\frac{d^2r dt}{2\pi}$$
The other terms contain extra derivative. Only the term with the divergence gives nonvanishing contribution

\[ S_{21} = -\frac{i}{2m\gamma} \sum_{l \neq z} \int \Omega_l^i \text{div} \vec{\Omega}^l \frac{d^2rdt}{2\pi} \] (35)

The last remaining third order terms contain only \( H_1 \) in the combination with two space and one time \( \Omega^l \)

\[ S_3 = \frac{i}{m^2} \int \Omega^1 \Phi_{sp}(\vec{r})\Phi_{sp}^*(\vec{r}) (\Omega^1_{+}\pi^- + \Omega^1_{-}\pi^+) \Phi_{s1p1}(\vec{r}_1)\Phi_{s1p1}^*(\vec{r}_2) \]

\[ \times (\Omega^1_{+}\pi^- + \Omega^1_{-}\pi^+) \Phi_{s2p2}(\vec{r}_2)\Phi_{s2p2}^*(\vec{r}) d^2rd^2r_1d^2r_2 \int S\sigma_i g_s \sigma_i g_s \sigma_i g_s e^{i\omega \delta} \frac{d\omega}{2\pi} \]

Only the terms with \( s = s_2 = 0; s_1 = 1 \) and \( s = s_2 = 1; s_1 = 0 \) gives the needed third order and no derivatives of \( \Omega \) must be taken into account.

The calculations in the first case are very similar to the calculation of (31), (35) for double pole in the production of \( g_0 \). Using the standard form in terms of \( \Omega^l \) and \( \Omega^l \) we get

\[ S_3^0 = -\frac{i}{m\gamma} \sum_{l \neq z} S\sigma_i \sigma_i \int \Omega^1 \vec{\Omega}^l \times \vec{\Omega}^l \frac{d^2rdt}{2\pi} + \]

\[ \frac{i}{4} S\sigma_i \sigma_i \int \Omega^z \vec{\Omega}^l \times \vec{\Omega}^l \frac{d^2rdt}{2\pi} + \frac{1}{2} \sum_{l \neq z} \int \Omega^z \vec{\Omega}^l \times \vec{\Omega}^l \frac{d^2rdt}{2\pi} \] (36)

The calculation of the case \( s = s_2 = 1; s_1 = 0 \) is standard and gives

\[ S_3^1 = -\frac{1}{2} \int \Omega^1 \vec{\Omega}^z \frac{d^2rdt}{2\pi} + \frac{1}{2} \sum_{l \neq z} \int [\Omega^1 \vec{\Omega}^z - \Omega^z \vec{\Omega}^l] \frac{d^2rdt}{2\pi} + \frac{i}{4} S\sigma_i \sigma_i \int \Omega^1 \vec{\Omega}^l \times \vec{\Omega}^l \frac{d^2rdt}{2\pi} \] (37)

The final expression for the part of the action containing the Hopf invariant is obtained by the summation of all found terms (35), (36), (37). Some points in this summation must be explained. The terms with the factor \( 1/\gamma \) in \( S_3^0, S_3^1, S_{21} \) combine to the total derivative

\[ \frac{1}{4m\gamma} \sum_{l \neq z} \int [-i\text{div}(\Omega^1 \vec{\Omega}^l) - \nabla \times (\vec{\Omega}^l \vec{\Omega}^l)] \frac{d^2rdt}{2\pi} = 0 \]

because \( \Omega^l \) for \( l \neq z \) tends exponentially to zero at large distances. In the same way the term containing the other total derivative

\[ \frac{-i}{4} S\sigma_i \sigma_i \sigma_i \sigma_z \int [\Omega^1 \text{div} \vec{\Omega}^m + (\vec{\Omega}^m \nabla) \Omega^l_0] \frac{d^2rdt}{2\pi} = 0 \]

vanish.

The terms with \( \Omega^l(\vec{\Omega}^l)^2 \) and \( \Omega^1(\vec{\Omega}^1 \vec{\Omega}^1) \) combine also to zero after some algebraic manipulations with \( S_2^0 \). Finally only antisymmetric terms in \( S_3 \) are retained

\[ S_3 = \frac{i}{4} S\sigma_i \sigma_i \sigma_i \int \Omega^1 \vec{\Omega}^1 \times \vec{\Omega}^1 \frac{d^2rdt}{2\pi} + \frac{1}{2} \int \Omega^1 \nabla \times \vec{\Omega}^l \frac{d^2rdt}{2\pi} - \int e^{ijm} \Omega^1 \vec{\Omega}^l \times \vec{\Omega}^m \frac{d^2rdt}{2\pi} \]

Calculating the trace and using the identity for \( \nabla \times \vec{\Omega}^l \) we get

\[ S_3 = e^{ijm} \int \Omega^1 \vec{\Omega}^l \times \vec{\Omega}^m \frac{d^2rdt}{2\pi} \] (38)

According to [13], [12] the Hopf invariant can be expressed in terms of \( \Omega \)

\[ H = \frac{1}{2} \int e^{ijm} \Omega^1 \vec{\Omega}^l \times \vec{\Omega}^m \frac{d^2rdt}{2\pi} \]
Therefore the Hopfian term in vortex action is

$$S^H = \pi H$$  \hspace{1cm} (39)$$

This result coincide with those in [6]. According to the accepted point of view that means [13] the fermionic nature of vortices-skyrmions.

We must remain that the Hopf invariant $H$ is integer for the mapping of the sphere $S_3$ on the sphere $S_2$ that requires in our case vanishing of $\Omega$ at large $\vec{r}, t$ i.e. $Q = 0$. If the vortices are present at any time and $Q \neq 0$ then the topological invariant corresponding to the linking number changes a bit and is contained in some part of $H$ [15],[16]. However the found relation between $S$ and $H$ holds on as well as the statement about fermionic nature of vortices.

4 Conclusions

In this work the theory of nonsingular vortices-skyrmions was considered for 2d electron systems in a strong magnetic field. It was assumed that $g$-factor is sufficiently small to justify the perturbation theory in the derivatives of the rotation matrix. No other assumptions beside the large value of cyclotron energy $\hbar \omega_c$ compared to Coulomb energy $e^2/l_H$ was used. It was shown that currently used approximation of the single Landau level projected wave functions is not sufficient for the adequate description. Taking into account the nearest Landau levels is essential and changes the thermodynamic energy for vortex-skyrmion formation diminishing its value by $-\hbar \omega_c/2$ large by assumption. This fact must lead to the spontaneous creation of vortices with positive degree of mapping near odd Landau level fillings. Simple physical picture emerges for small enough $g$-factor providing validity of the perturbation theory: inside the vortex core exists an additional effective magnetic field with the total flux containing the number of the flux quanta equal to the degree of the mapping. The lowest Landau spin sublevel in the total (external plus effective)local field is fully filled and is separated by the exchange gap from the higher energy states. The total number of electrons changes because of the change of their density coinciding with the local density of states. As a consequence the electrical charge of the core emerges which equals $|q|$ that of electrons expelled from the core in the number of flux quanta for a positive degree of mapping.

There is a possibility to have bound electron states in the positive charge of the core. The position of the proper energy is defined by negative Coulomb energy and positive exchange energy not depending on the core. Under made assumptions it is above chemical potential. In real experimental situation this statement must be checked numerically.

The topological Hopf term in the vortex-skyrmion action is found. The calculated coefficient before the Hopf invariant coincides with semiclassical result [6]. Therefore the vortices are fermionic-like. The properties of vortices-skyrmions are close to “composite” fermions introduced phenomenologically in [14].

Authors express their gratitude to G.E.Volovik for valuable discussions.

The research described in this publication was made in part due to award RP1-273 US Civilian Research and Development Foundation for Independent States of Former Soviet Union. This work supported partially also by RFFI 95-02-05883 and the program “Statistical Physics” of the Russian Science Ministry.

References

[1] S.Sondhi, A.Kahlrede, S.Kivelson, E.Rezai Phys. Rev. B47, 16418 (1993)

[2] H.Fertig, L.Brey, R.Cote, A.Mac Donald Phys. Rev. B50, 11018 (1994)
[3] K.Moon, H.Mori, Kun Yang, S.Girvin, A.Mac Donald, Phys. Rev. B51, 5138 (1995)
[4] Yu.Bychkov, T.Maniv, I.Vagner Phys. Rev. B53, 10148 (1995)
[5] W.Apel, Yu.Bychkov, Phys. Rev. Lett. 78, 2188 (1997)
[6] G.Volovik, V.Yakovenko, cond-mat/9703228, 26 March (1997)
[7] S.Barret, G.Dabbagh, L.Pfeiffer, K.West, R.Tycko Phys. Rev. Lett. 74, 5112 (1995)
[8] I.Kukushkin, K.v Klitzing, K.Eberl Phys. Rev. B55,(16) (1997)
[9] ee.ejjanalss+?+, ee.A.5 sspilijo, A?ss .25a ayenAEoe 65, 248 (1997)
[10] M.Salomaa, G.Volovik, Rev. Mod. Phys. 59, 533 (1987)
[11] A.Abricosov, L.Gorkov, I.Dzyaloshinskii, Quantum field theory methods in statistical physics GIML, Moscow 1962
[12] G.Volovik, V.Yakovenko, J. Phys. Cond. Matter 1, 5263 (1989)
[13] F.Wilczek, A.Zee Phys. Rev. Lett. 51, 2250 (1983)
[14] K.Jain, Phys. Rev. Lett., 63, 199 (1989)
[15] V.Ruutu, U.Parts, J.Koivuniemi, M.Krusius, E.Thunenberg, G.Volovik Pisma ZhETF v 60 p 659 (1994)
[16] Yu.Makhlin, T.Misirpashaev, Pisma ZhETF v 61 p 48 (1995)