Energy-level structure of ion cloud and crystal in a linear Paul trap

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Abstract
Ion cloud and crystal are recognized as two distinct states in ion traps, but there is no theory that can describe the energy-level structures of these two states from the perspective of quantum theory. In this paper, we construct a model describing ions in a linear Paul trap to investigate the energy-level structure with the linear response function, which is often used in the field of condensed matter to compute the energy-level of electrons or plasmas. We support a method to calculate the energy-level of both the crystal and cloudy states of trapped ions in a linear Paul trap. Furthermore, we present the energy-level diagram of two trapped ions and give a quantum interpretation for the fluorescence hysteresis loops that were observed in our laboratory. This is a fundamental theory that will help us to understand the crystallization and cooling processes of ions in a linear Paul trap.

Keywords: ion trap, energy-level structure, quantum phase transition, linear response function

1. Introduction
Over the past several decades, experiments to control the level of trapped ions have been so widely developed that the experimental techniques for the confinement, cooling and coherent
manipulation of ions have been extensively exploited for quantum information processing purposes. During the laser cooling process, the ions undergo a series of phase transitions [1, 2]. These phase transitions are due to the competition between the repulsive Coulomb interaction and the kinetic energy of each ion, and they can be controlled by varying the detuning frequencies of a cooling laser. In the ion trap, the phenomena concerning the ‘ordered’ state, such as the phase transition of the ‘ordered’ state (zigzag transition) and the energy-level of the ‘ordered’ state (crystal state) [3–7], are mentioned more frequently than the ‘disordered’ (cloudy-crystal transition and cloudy state) [8–11].

However, the ‘disordered’ state has proven to be as important as the ‘ordered’ state, especially in the phase transition process. The ‘disordered’ state always seems to be an ion cloud. In early times, descriptions of the ion ‘cloud’ in the Penning trap and Paul trap were reported by Bollinger and Flory respectively [12, 13]. They showed that the temperature of the ion cloud is associated with its size. A further interpretation was made by G Z Li, who studied the density of ion cloud by using the Thomas–Fermi theory and gave a theoretic interpretation [14]. However, the description of the ion ‘cloud’ in a linear Paul trap still has no final conclusion.

In some situations, the theory of molecular dynamics can be used to simulate the actions of an ion cloud. However, it is still necessary to find a quantum theoretical description for two reasons. The first is because the classical theory will fail in the descriptions of phase transition parameters and the energy-levels of the ions, especially under conditions such that the kinetic energy of the ions is comparable to $\hbar\omega$, where $\omega$ is the secular frequency. In the case of this low kinetic energy, the quantum effects become significant and the classical theory description leads to erroneous results. In fact, Wineland explained that the energy of ions in a cloudy state is very low and can be comparable to $\hbar\omega$ [12], which forces the adopting of the quantum theoretical description. The other reason to employ a quantum theoretical description is that it is thought to be more exact than the classical theoretical description for trapped ions. It can reveal the real values for the state and energy of trapped ions in different states, and can also be used to predict new phenomena.

This paper explores a proper model that enables us to deal with the manipulation of both the crystal and cloudy states, using the original Hamiltonian. To eliminate the difficulty brought about by using a singular point of Coulomb potential, a proper field operator is chosen and the Coulomb potential is transferred to the second quantization formulation. Based on the calculations, the Hamiltonian is reduced further and to show this reduction can keep both the long-range and short-range actions of ions in the linear Paul trap; therefore, it can describe the ions’ behaviour no matter how near or how far they are. This is different from phonon models of ions, which can be used only in crystal states of low kinetic energy, or when the oscillation amplitude of ions in equilibrium is very small. If the kinetic energy of trapped ions is very large, the ions may be very close together and the phonon model will be invalid. An advantage of the model presented here, is that it can be used to describe the competition between the Coulomb interaction and the kinetic energy per ion, which leads to the phase transition between the cloudy state and crystal state according to this formulation.

Our model can show the energy variation during the process of phase transition and as well as the ion temperature in the Paul trap. By using the linear response function, it will be shown that the crystal state is stable, which corresponds to collective excitation, while the cloudy state is unstable, which corresponds to individual excitation. By using this model, we explain the hysteresis-loop-like behaviour formed by fluorescence intensity of trapped ions, which was proposed by Diedrich et al [15]. Furthermore, we show that the results arising from our formulation are in accordance with our experiments.
This paper falls into three parts. In the first part, we will introduce our ion-trap experiment and show the phenomena that we encounter in our experiments. In the next, we will construct a Hamiltonian model and show the way to calculate the energy-level of trapped ions with the linear response function. In this part, we will focus on the rationality of this model. In the last part, we will exhibit the comparison between our theory and the experiment.

2. The outline of experiment

In this experiment, a Paul trap was used to capture $^{40}\text{Ca}^+$. The set-up can be seen in figure 1. RF voltage was applied to the two pairs of blades and DC voltage was applied to the caps. The cooling laser used is a 397 nm laser, and the re-pumping laser is an 866 nm laser. The 397 nm laser was used for the initial cooling down of the ions in the trap. The trapped ions were then in the crystal state and formed a line along the axis. The frequency of the repumping laser was kept constant, but the 397 nm laser was detuned to heat the ions in the trap. Two directions are typically used for detuning, namely red-detuned and blue-detuned lasers. Red-detuning is usually used to cool down the ions, but when the detuning frequency difference is large, it may also heat the ions because of the broad kinetic energy range of the ions and the frequency of the cooling laser.

In the heating process, the ions became increasingly warmer. This was mainly reflected in their increasingly larger fluorescent images in EMCCD. This won’t last long if the cooling laser is detuned along one direction. It will experience a phase transition when the detuning frequency of the cooling laser, which is 397 nm in wavelength, is over 20 MHz for blue-detuning or 200 MHz for red-detuning in our laboratory. In the process of phase transition, the fluorescent image of the ions altered significantly to a large cloud. Sometimes, it is hard to cool the ion cloud to be a crystal because cooling and warming are not inverse processes in ion traps. To cool a big ion cloud to be a Coulomb crystal, two methods are commonly adopted. Either the 397 nm laser is red-detuned (in this process, the 866 nm laser should be continuously sweeping over one or more hours to assist the cooling when the compensation for micromotion is good) or the ion cloud is moved by reducing the voltage intensity of the caps (in this process, some ions in the trap escape, reducing the ion cloud, and then the phase transition happens).

Figure 1. The linear Paul trap set-up in our laboratory. (left) The sketch map and (right) one is the real set-up photo in our laboratory. The cooling laser is 397 nm laser and the repumping is 866 nm laser.
In fact, there is not an easy way to control the ion cloud in our experiment; therefore, to clearly define what the ion cloud is and how to control the process of phase transition is very important for us.

One puzzling phenomenon we observed in the laboratory is the hysteresis-loop-like behaviour of ion fluorescence, which was first detailed by Diedrich et al [15]. Diedrich et al found that during the process of a phase transition between the ‘crystalline’ and ‘gaseous’ state, the fluorescence intensity changes with a hysteresis-loop-like as a function of the cooling laser power, which had been never found before. However, they still did not give an explanation of the hysteresis-loop-like in theory. We observed this phenomenon in our laboratory, and it is shown in the left of figure 2. To interpret this phenomenon, physicists historically proposed some theories [16–23]. However, most of these theories are based on Newtonian mechanics and can not reveal ions’ energy varying in the process of phase transition. As we know, there are still no perfect ‘quantum’ theoretical explanations of the hysteresis loop mentioned above and present an explanation from the quantum perspective. Moreover, further hysteresis-loop-like behaviour that has not previously been reported is shown in the right of figure 2. In that process, the ions in the trap seemed to attract each other despite the Coulomb repulsion between them. This paper will interpret this interaction.

**Figure 2.** The phase transition between the cloudy state and crystal state in our laboratory. The fluorescent light is collected by EMCCD and imaging in a computer through special software processing. We observe two kinds of phase transition in the laboratory, the normal phase transition, shown in picture I, and abnormal phase transition, shown in II. The normal phase transition is only driven by cooling laser, but for the abnormal phase transition, we should use micro-amplification RF drive voltage under the compensating pole to assist the transition.
3. The model and method of calculation for energy levels of ions

3.1. The Hamiltonian model

To construct a proper Hamiltonian, we first consider the behaviour of the trapped ions. In the linear Paul trap, there are two main kinds of potential: the Coulomb potential, which comes from each pair of ions, and the trapped potential, which comes from both the axial DC voltage and radial RF voltage. The RF voltage can cause the micromotion of the trapped ions, and the amplitude of micromotion is proportional to the distance of the ion from the trap centre [1, 2]. Usually, this micromotion cannot be overlooked, but in low temperatures, if the number of ions is fewer than $10$ (this number is the best condition to observe the phase transition between the cloudy state and crystal state in our laboratory), the ions are crowded in the centre of the trap and their micromotion is close to zero and can be ignored. Under these conditions, the potential from the RF voltage is the pseudopotential of RF voltage, which is treated as a constant potential. Furthermore, the inner states of ions, spin states, are not considered in this paper. Therefore, the total Hamiltonian is

$$\hat{H} = \sum_{i=1}^{2m_p} \mathbf{\hat{p}}^2_i + \frac{1}{2} \sum_{i=1}^{2m_p} \sum_{\alpha=x,y,z} m_i \omega_\alpha^2 r_{\alpha, i}^2 + \frac{e^2}{4\pi\varepsilon_0} \sum_{i>j} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|},$$

(1)

where, $\omega_x$ and $\omega_y$ are the secular frequencies in the axial and radial directions respectively, and $m_p$ is the mass of the ions. $\hat{H}$ contains three parts, the kinetic part, the harmonic potential and the repulsive Coulomb potential. The Coulomb potential is always difficult to calculate for its singularities points at $\mathbf{r}_i = \mathbf{r}_j$, therefore, it is transferred to the second quantization to eliminate these singularities. We define the field operator as

$$\Psi^\dagger = \sum_{m,m_d} \psi_m(x) \phi_{m_d}^\dagger(\mathbf{r}) c_m^\dagger c_{m_d},$$

(2)

The subscript 'rd' means 'radial', and $c^\dagger$ is the creator operator. $\psi_m(x)$, $\phi_{m_d}(\mathbf{r})$ here are the eigenstates of Hamiltonians $\hat{H}_0^x = \hat{p}_x^2/(2m_p) + 1/2m_p\omega_x^2 x^2$ and $\hat{H}_0^{rd} = \hat{p}_r^2/(2m_p) + 1/2m_p\omega_{rd}^2 r^2$ respectively. After the second quantization, the Hamiltonian is described as

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1;$$

$$\mathbf{H}_0 = \sum_{m,\alpha} \left( m_\alpha + \frac{1}{2} \right) \hbar \omega_\alpha c_m^\dagger c_m + \sum_{m} \sum_{n} \kappa \left( \left| v^m_n \cdot v^m_n, \alpha \right| c_m^\dagger c_n^\dagger c_n,$$

$$\mathbf{H}_1 = \sum_{m_1} \sum_{m_2} \sum_{n_1} \sum_{n_2} \kappa \left( \left| v^{m_1}_{n_1} \cdot v^{m_2}_{n_2}, \alpha \right| c_{m_1}^\dagger c_{m_2}^\dagger c_{n_1} c_{n_2},$$

(3)

where, $\alpha = \sqrt{\omega_x/\omega_y}$. $\mathbf{H}$ in equation (3) contains two parts $\mathbf{H}_0$ and $\mathbf{H}_1$. $\mathbf{H}_0$ is the Hamiltonian with no transition-term, and $\mathbf{H}_1$ is the transition-terms Hamiltonian. For strong trapped potential in axial-direction, $\langle \mathbf{H}_0 \rangle \gg \langle \mathbf{H}_1 \rangle$, meaning $\mathbf{H} \approx \mathbf{H}_0$.

In this paper, we use the bold-face and lowercase letters to denote vectors and capital letters to denote matrices for simplification. In equation (3), the Hamiltonian after the second quantization is written as the bold-face $\mathbf{H}$ to be distinguished from $\hat{H}$ before the second quantization. The other undefined expressions in equation (3) are described in the following...
\[ \mathbf{v}_{m1} \cdot \mathbf{v}_{m2} = (v_{m1}, v_{m2}, v_{m3}, v_{m4}, v_{m5}), \quad (X, R, S); \]
\[ \mathbf{v}_{n1} \cdot \mathbf{v}_{n2} = (m1, n1, m2, n2); \]
\[ \mathbf{v}_{n} = (m, n)^T; \quad m = (m_x, m_y, m_z) = (m_x, m_{rd}). \] (4)

The functions \( \gamma \) and \( \kappa \) can be calculated as
\[ \gamma(\mathbf{v}_{n}^m \cdot |\mathbf{v}_{n}^m, \alpha) = \frac{e^2 p^r}{16 \pi c_0 \hbar} \left( \Theta(\mathbf{v}_{n}^m \cdot |\mathbf{v}_{n}^m, \alpha) \pm \Theta(\mathbf{v}_{n}^m \cdot |\mathbf{v}_{n}^m, \alpha) \right); \]
\[ \kappa(\mathbf{v}_{n1} \cdot \mathbf{v}_{n2}^m, \alpha) = \frac{e^2 p^r}{16 \pi c_0 \hbar} \left( \Theta(\mathbf{v}_{n1} \cdot \mathbf{v}_{n2}^m, \alpha) \pm \Theta(\mathbf{v}_{n1} \cdot \mathbf{v}_{n2}^m, \alpha) \right), \] (5)

where, ‘+’ corresponds to bosons and ‘−’ to fermions. \( \Theta \) here is a dimensionless function. Its concrete expressions can be seen in the appendix.

3.1.1. The reduction for \( \mathbf{H} \). The value \( \alpha \) describes the potential component in the ion trap. Let us look at two extreme conditions: \( \alpha \to 0 \), and \( \alpha \to \infty \). These extreme conditions help us to understand the ion phase transition and sort the Hamiltonian \( \mathbf{H} \). If \( \omega_x \to \infty \), corresponding to strong axial confinement, \( \alpha \to 0 \). In this condition, the terms that associate with conditions \( m1_x \neq m2_x \) or \( n1_x \neq n2_x \) in equation (A.9) equal zero. It means that in very strong axial potential, there is no transition term in the axial direction. It is because the energy gap along this direction is so big that the ions are not easily excited by Coulomb repulsion. In strong axial confinement, the ions need a considerable amount of energy to jump to the others energy level. This energy cannot be gotten from the Coulomb potential scattering because this scattering is weak.

For the other extreme, namely \( \omega_x \to 0 \), corresponding to weak axial confinement, \( \alpha \to \infty \), then, \( \Theta(\mathbf{v}_{n1} \cdot \mathbf{v}_{n2}^m, \alpha) \) would be zero if \( m1_{rd} \neq m2_{rd} \) or \( n1_{rd} \neq n2_{rd} \). It means that there is no coupling in the radial direction, consequently no excitation in the radial direction. This is because in the weak axial confinement, the ions repel each other, making the distance of each pair of ion so long that the overlapping of wave function in radial direction can be overlooked, and therefore, the ions cannot feel the force from the others in radial direction.

For different cases, we divide the transition-term \( \mathbf{H}_r \) into the axial transition \( \mathbf{H}_a \), the radial transition \( \mathbf{H}_r^{rd} \) and the coupling term \( \mathbf{H}_r^{rd} \)
\[ \mathbf{H}_r = \sum_{m1_x \neq m2_x \atop n1_x \neq n2_x} \kappa(\mathbf{v}_{n1} \cdot \mathbf{v}_{n2}^m, \alpha) c_{m1}^\dagger c_{m2} c_{n1}^\dagger c_{n2}; \]
\[ \mathbf{H}_r^{rd} = \sum_{m1_{rd} \neq m2_{rd} \atop n1_{rd} \neq n2_{rd}} \kappa(\mathbf{D}_{m} \cdot \mathbf{v}_{n1}^m \mathbf{v}_{n2}^m, \alpha) c_{m1}^\dagger c_{m2} c_{n1}^\dagger c_{n2}; \]
\[ \mathbf{H}_r^{rd} = x = \mathbf{H} - \mathbf{H}_a - \mathbf{H}_r^{rd}, \] (6)

where, \( D_m = \mathbf{v}_{m}^m \mathbf{v}_{m}^m \). In linear Paul trap, the secular frequency \( \omega_x \) is usually much less than \( \omega_{rd} \), namely \( \alpha \gg 1 \). In this condition, the terms about the excitation in the axial direction are much
bigger than those in the radial direction, namely, $\langle H_1^{rd} \rangle \ll \langle H_1^x \rangle$. Additionally, the coupling term is also much less than $H_1^x$. $H_1^x$ can be further reduced. As can be seen in figure 3, $\kappa \left( v_{m_1}^{n_1} \pm d_1 | v_{n_1}^{m_1}, D^{n_1}_{m_1}, \alpha \right) \approx 0$ if $d_1 \neq d_2$. Therefore

$$H_1^x = \sum_{m,n,d} \kappa \left( v_{n_1}^{m_1} \pm d | v_{n_2}^{m_1}, D^{n_2}_{m_1}, \alpha \right) c_{m_1} \pm d c_{n_1} \pm d c_{n_2}. \quad (7)$$

Let $\hat{Q}_{md} = c_{m} \pm d c_{m}$, which means the ion jumps to $m + d$ th level from $m$th level. Then, $H_1$ is written into two parts: $H_1^{IT}$ and $H_1^{IR}$.

$$H_1^{IT} = \sum_{m,n,d>0} \kappa \left( v_{n_1}^{m_1} + d | v_{n_2}^{m_1}, D^{n_2}_{m_1}, \alpha \right) \hat{Q}_{md} \hat{Q}_{nd} + H_c; \quad (8)$$

$$H_1^{IR} = \sum_{m,n,d>0} \kappa \left( v_{n_1}^{m_1} + d | v_{n_2}^{m_1}, D^{n_2}_{m_1}, \alpha \right) \hat{Q}_{md} \hat{Q}_{nd} + H_c.$$

The effect of $H_1^{IT}$ is to transfer any two levels $m$th and $n$th the that the ions occupy into the other two levels $m + d$th and $n + d$th. This is like the translation motion of energy levels $m$th and $n$th, and therefore, we use ‘T’ in $H_1^{IT}$ to indicate this effect. Contrarily, the effect of $H_1^{IR}$ is like the opposite movement of energy levels and we use ‘R’ in $H_1^{IR}$.

If we do not consider the zigzag phase transition, $H_1^{rd}$ can be overlooked. Therefore, Hamiltonian can be simply written as
The relations between the terms in equation (9) are\(\langle H_0 \rangle \gg \langle H_{1T} \rangle \approx \langle H_{1R} \rangle\). When the quantum numbers\(|m|, |n|\) are small, it follows that\(\gamma \gg \omega\). When quantum numbers\(|m|, |n|\) approach infinity, it follows that\(\gamma \ll \omega\).

3.1.2. Analysis of \(H_0\). We have successfully reduced \(H\) in the last section. Further, in this section, we will focus on the main Hamiltonian \(H_0\).

The main Hamiltonian \(H_0\) is very useful to analyze the energy of ions in the trap. The eigenstates of \(H_0\) can be expressed as \(\prod_l c^{\dagger}_{i_0} |0\rangle\) for \(l\) ions and the corresponding eigen energy is

\[
E_n^{(0)} = \sum_{l} \sum_{i=\alpha,x,y,z} \left( n_{l,\alpha} + \frac{1}{2} \right) \hbar \omega_{\alpha} + \sum_{l} \sum_{j>l} \gamma(D^m_n, \alpha).
\]  

The eigen energy of the system can be written as\(E_n = E_n^{(0)} + E_n^{(1)}\). \(E_n^{(0)} \gg E_n^{(1)}\) for a few ions in traps. \(H_{1T}\) and \(H_{1R}\) can be treated as the perturbation. They scatter any two states \(|m_\alpha\rangle\) and \(|n_\alpha\rangle\) into other two \(|m_\alpha\', \rangle\) and \(|n_\alpha\',\rangle\), but keep the parities of the differences of these two pairs of quantum numbers constant. Namely \(\text{Mod}\ (m_\alpha - n_\alpha, 2) = \text{Mod}\ (m_\alpha' - n_\alpha', 2)\).

In low temperature, the ions will tend to stay in low energy level. Therefore, it is useful for us to find out the minimum value of \(\gamma\). We find, if we let \(m\) and \(n\) keep constant but \(d\) vary in \(\gamma(D^m_{m - d}, \alpha)\) (or \(\gamma(D^m_{m - d}, \alpha)\)), \(\gamma\) will reach its minimum at \(d_{rd} = 0, d_{x} = 0\) or \(d_{rd} = 0, d_{x} = m_{x}\). We show the curve about the function \(\gamma(D^m_{m - d}, \alpha)\) in figure 4. In this figure, we can see the value of \(\gamma\) varying with the \(d_{x}\).

According to the characteristics of transition terms \(H_{1R}\) and \(H_{1T}\), the green line in the picture is the transition path. The ion tends to stay in low energy level at low temperature, which is corresponding to point \(A\) or \(C\) in figure 4. According to figure 4, the ions in point \(A\) are more stable than those in point \(C\), because the ions here need much more energy to jump to the next point \(A'\) than ions in point \(C\) to the next point \(C'\). On the contrary, point \(B\) is unstable, because \(B\)
lays in band of quasi continuous energy. Therefore, ions in $B$ will be quickly excited by thermal fluctuation. In fact, we will see, $B$ forms the energy band of cloudy state and $A$ constructs the energy level of crystal state.

### 3.2. Method: linear response function

To exactly calculate the energy of ions, we take the linear response function, which is usually applied in condense matter field for electrons or plasmas, to calculate the energy-level of ions [24]. The linear response function is defined on Green function,

$$G_{A,B}^+(t) = -\frac{i}{\hbar} \theta (\pm t) \langle [A(t), B] \rangle.$$  

(11)

The $\theta (t)$ is Heaviside step function here. It equals 1 when $t \geq 0$ and 0 when $t < 0$, if the Hamiltonian of system does not explicitly depend on time. In equation (11), $G_{A,B}^+$ is called retarded response, and $G_{A,B}^-$ is the advanced response. They are the same when we calculate energy levels in a system. Here, we choose $G_{A,B}^+$ to calculate the ions’ energy.

The Fourier transformation of $G_{A,B}^+$ is

$$\langle \langle A|B \rangle \rangle_{\omega + i\eta} = \int G_{A,B}^+(t)e^{i(\omega + i\eta)t}dt,$$

(12)

where, $\eta \to 0$. For a system with Hamiltonian $H$, the value of $\langle \langle A|B \rangle \rangle_{\omega}$ is

$$\omega \langle \langle A|B \rangle \rangle_{\omega} = \langle [A, B] \rangle + \langle \langle A_1|B \rangle \rangle_{\omega}, A_1 \equiv [A, H];$$

$$\omega \langle \langle A_1|B \rangle \rangle_{\omega} = \langle [A_1, B] \rangle + \langle \langle A_2|B \rangle \rangle_{\omega}, A_2 \equiv [A_2, H];$$  

(13)

To calculate $\langle \langle A|B \rangle \rangle_{\omega}$, we should get $\langle \langle A_1|B \rangle \rangle_{\omega}$, which is based on the value $\langle \langle A_2|B \rangle \rangle_{\omega}$, and so on. It is a multilevel computation. However, if $[A, H_0] = E_m^{(0)} A$, equation (13) can be simply expressed as

$$\left( \omega - E_m^{(0)} - \Sigma_m(\omega) \right) \langle \langle A|H_0 \rangle \rangle_{\omega} = \langle A \rangle.$$  

(14)

$\langle \langle A|H_0 \rangle \rangle_{\omega}$ has a singularity at $\omega_0 = E_m^{(0)} + \Sigma_m(\omega_0)$. This singularity $\omega_0$ is the eigen frequency in this system.

The more explicated introduction for linear response function can be seen in the references [25, 26]

### 3.3. Energy level of ions

Let us return to the expressions of $H_{IT}^+$ and $H_{TB}^+$. The task for us is to find out the appropriate operator that satisfies equation (14). We show the appropriate operator is a linear combination of $\hat{Q}_{md}$. The commutation relation about $[H_{IT}^+, \hat{Q}_{md}]$ can be calculated out.
\[
\begin{align*}
[\hat{Q}_{md}, \mathbf{H}^x_{1T}] &= 2 \sum_n \kappa \left( \frac{\hat{v}_n^{m+d}}{\hat{v}_n^m}, D_{n_{ad}}^{m_{ad}}, \alpha \right) \left( \hat{N}_{m+d} - \hat{N}_m \right) \hat{Q}_{nd}^+ \\
&+ 2 \sum_{d'} \left( \sum_{n \neq m-d} \kappa \left( \frac{\hat{v}_n^{m-d} + d'}{\hat{v}_n^{m-d}} \right), D_{n_{ad}}^{m_{ad}}, \alpha \right) \hat{Q}_{n,d'+d'}
\end{align*}
\]

Further, the other two commutations are
\[
\begin{align*}
[\hat{Q}_{md}, \mathbf{H}^x_{1R}] &= 2 \sum_n \kappa \left( \frac{\hat{v}_n^{m+d}}{\hat{v}_n^m}, D_{n_{ad}}^{m_{ad}}, \alpha \right) \left( \hat{N}_{m+d} - \hat{N}_m \right) \hat{Q}_{nd}^+ \\
[\hat{Q}_{md}, \mathbf{H}^0] &= d\hbar \omega \hat{Q}_{md} + 2 \sum_n \Gamma_{m,n,d} \hat{N}_n \hat{Q}_{nd}^+ \\
&= E^{(0)}_{m,d} \hat{Q}_{md},
\end{align*}
\]

The last four summation terms are heterogeneous transitions. They can be overlooked because their contribution to the final result is very limited. Similarly, we can get the commutation
\[
\begin{align*}
[\hat{Q}_{md}, \mathbf{H}_{1R}^x] &= 2 \sum_n \kappa \left( \frac{\hat{v}_n^{m+d}}{\hat{v}_n^m}, D_{n_{ad}}^{m_{ad}}, \alpha \right) \left( \hat{N}_{m+d} - \hat{N}_m \right) \hat{Q}_{nd}^+ \\
&+ \sum_{n \neq m} \kappa \left( \frac{\hat{v}_n^{m-d} + d'}{\hat{v}_n^{m-d}} \right), D_{n_{ad}}^{m_{ad}}, \alpha \right) \hat{Q}_{n,d'+d'}
\end{align*}
\]

Moreover,
\[
\begin{align*}
\hat{\beta}_{md} &= \hat{Q}_{md} + \hat{Q}_{md}, \text{ and } \hat{\beta} = \hat{Q}_{md} - \hat{Q}_{md}. \text{ Consequently,}
[\hat{\beta}_{md}, \mathbf{H}_{1R}^x] &= 2 \sum_n \Omega_{m,n,d} (\alpha) \left( \hat{N}_{m+d} - \hat{N}_m \right) \hat{\beta}_{nd}
\end{align*}
\]

The coefficients in the above equation are
\[
\begin{align*}
\Xi_{m,n,d} (\alpha) &= \kappa \left( \frac{\hat{v}_n^{m+d}}{\hat{v}_n^m}, D_{n_{ad}}^{m_{ad}}, \alpha \right) - \kappa \left( \frac{\hat{v}_n^{m+d}}{\hat{v}_n^m}, D_{n_{ad}}^{m_{ad}}, \alpha \right); \\
\Omega_{m,n,d} (\alpha) &= \kappa \left( \frac{\hat{v}_n^{m+d}}{\hat{v}_n^m}, D_{n_{ad}}^{m_{ad}}, \alpha \right) + \kappa \left( \frac{\hat{v}_n^{m+d}}{\hat{v}_n^m}, D_{n_{ad}}^{m_{ad}}, \alpha \right). \quad (20)
\end{align*}
\]

Based on the conclusions above, we show the operator is analogous to \( A \) in equation (14) and it can be written as
\(\hat{\omega}\left\langle \hat{\rho}_{md}|H_0\rangle \right\rangle + i\eta \left\langle \hat{\theta}_{md}|H_0\rangle \right\rangle = \left\langle E_{m,d}^{(0)}\hat{\rho}_{md}\right\rangle + \left\langle \left[\theta_{md}, H_0\right]|H_0\rangle \right\rangle; \)
\(\hat{\omega}\left\langle \hat{\theta}_{md}|H_0\rangle \right\rangle + i\eta \left\langle \hat{\rho}_{md}|H_0\rangle \right\rangle = \left\langle E_{m,d}^{(0)}\hat{\theta}_{md}\right\rangle + \left\langle \left[\rho_{md}, H_1\right]|H_0\rangle \right\rangle. \tag{21} \)

\(\left\langle \hat{\rho}_{md}|H_0\rangle \right\rangle\) can be calculated out according to equation (21). The singularities of its real part \(\text{Re} \left\langle \hat{\rho}_{md}|H_0\rangle \right\rangle\) are the eigen frequencies. To get the singularity, we let \(\eta \to 0\),
\(\hat{\omega}\left\langle \hat{\rho}_{md}|H_0\rangle \right\rangle = \sum_n U_{m,n}(d, \alpha) \left\langle \hat{\rho}_{nd}|H_0\rangle \right\rangle, \tag{22} \)
where,
\[U_{m,n}(d, \alpha) = 2\Omega_{m,n,d}(N_m - N_{m+d}) + E_{m,d}^{(0)}\delta_{m,n}. \tag{23}\]

\(U(d, \alpha)\) is the matrix. The eigen vector is denoted by \(g\) and the corresponding eigen value is \(\epsilon_d\). Therefore, equation (22) can be simplified into
\[(\hat{\omega} - \epsilon_d)\sum_m g_m \left\langle \hat{\rho}_{md}|H_0\rangle \right\rangle = \sum_m g_m E_{m,d}^{(0)} \left\langle \hat{\rho}_{md}\right\rangle. \tag{24}\]

Namely,
\[\left\langle \{X_d\}_d \right\rangle = \frac{C_d}{\hat{\omega} - \epsilon_d}. \tag{25}\]

The eigen energy of \(H\) is \(\hat{\omega}_H = \epsilon_d\). The number of \(\omega_H\) equals the dimensionality of matrix \(U(d, \alpha)\). For \(l\) ions in a trap, \(U(d, \alpha)\) is \(l \times l\) and then \(\omega_H\) has \(l\) values, namely, \(l\) modes in this system.

It should be noted that every \(\omega_H\) is the singularity of both \(\left\langle \hat{\rho}_{md}|H_0\rangle \right\rangle\) and \(\left\langle \hat{\theta}_{md}|H_0\rangle \right\rangle\), and it is also the singularity of \(\left\langle \hat{Q}_{md}|H_0\rangle \right\rangle\). The imaginary part of \(\left\langle \hat{Q}_{md}|H_0\rangle \right\rangle\) determines the lifetime of element excitation of ions in the trap. According to the formulas of equation (21), if \(\omega_H\) is close to \(E_{m,d}^{(0)}\), the imaginary part of \(\left\langle \hat{Q}_{md}|H_0\rangle \right\rangle\) must be nonzero. This is because, if \(\text{Im} \left\langle \hat{Q}_{md}|H_0\rangle \right\rangle = 0, \left\langle \hat{\rho}_{md}|H_0\rangle \right\rangle\) and \(\left\langle \hat{Q}_{md}|H_0\rangle \right\rangle\) are both real. This will contradict equation (21).

4. Comparison with experiments

We consider a situation in which there are only two ions in the Paul trap. This scenario was chosen in part because we observed the evidence of phase transition between the ion cloud and ion crystal in our laboratory. The other reason for using the two ions as the comparison example is that the possibility that the ion cloud is the electron gas or an order state can be ruled out if the ion cloud is formed by two ions.

Based on equation (22), the energy levels in figure 5 was determined. The energy of the cloudy state formed a quasi continuous area, which corresponds to the filled area in figure 5. The value of the cloudy state is \(E_{m,d}^{(0)}\), which corresponds to individual excitation, and the state has infinite lifetime. The line hanging on the bottom of the cloudy state area corresponds to
collective excitation, which represents the energy of the crystal state. Its elementary excitations are the phonons, which have a long lifetime and are hard to decay.

To prove the rationality of our theory, the crystal energy that arises from our theory is first observed. As can be seen in figure 5, the minimum energy of the ion trap system appears in the crystal state. Its energy is calculated to be $E = 6.30957 \times 10^{-24} J$ at $\alpha = \sqrt{\omega_{rd}/\omega_x} = 10$ and $\omega_{rd} = 20 \text{ MHz}$. The exact solution calculated from the force equilibrium equation is $E_0 = m\omega^2 l^2 + e^2/(8\pi\varepsilon l) = 6.21768 \times 10^{-24} J$, where $l$ is half of the distance between the equilibrium positions of these two ions. The agreement between the two values is evidence that the model in this paper is effective in calculating for the crystal state energy.

However, for the cloudy state, we have no other theoretical comparison; the results of the model can only be compared with the experimental results. According to our theory, in the cloudy state, the transition index $d_{eq}$ is proportional to the axial width of the movement areas of the ions in the trap. Therefore, we can evaluate the axial width of the fluorescence image of the cloudy state from EMCCD when the phase transition between the cloudy state and crystal states happens. The axial width is approximately $l_x \approx 76.3 \sim 98.6 \mu m$ at $\omega_x = 0.2 \text{ MHz}$ in theory. The limit superior of $l_x$ is the limit inferior of the valid area of the phonon model (introduced in the following text), and the limit inferior of $l_x$ is the point at which the energy of the crystal state equals that of the cloudy state. The experimental data of fluorescence images from EMCCD are shown in figure 6. To create strong radial confinement, the power of $RF$ should be increased, but the increased power of $RF$ causes the increased micromotion of ions in our laboratory. The radial RF power applied is 5 W, which corresponds to 1710 V in our laboratory. Under this high voltage, the axial width of fluorescence image became unstable and varied in every measurement. However, the width of the cloudy state measured in the experiments fell into the predicted area.

After the phase transition, the kinetic energy of the ions became smaller; hence, the distance between these two ions in trap was smaller than the width of the cloudy state. The distance $l_g \approx 59.83 \mu m$, which has a deviation of 7% in comparison to the exact result $l_0 = 55.60 \mu m$. The measurement that we get from the experiments is $l_x = 57.24 \pm 2.12 \mu m$.

Let us return to the phonon model. The phonon model is an approximated when used to calculate the ions’ energy in the trap. The approximation is to assume that the ions form a...
crystal, and their movements are treated as micro-oscillations around their equilibrium positions [27–30]. The question is how the phonon model can remain valid in calculating ion energy. We define the valid area for the phonon model as the 10% deviation between the phonon model and our model. We show the valid area varying with \( \omega_x \) in the right graph of figure 6. The valid areas for the breathing mode and the centre-of-mass mode are the same in theory. According to our theory, in weak axial confinement, the phonon model is exact in crystal energy level calculation at low temperature. However, at high temperature or strong axial trap, the phonon model will be invalid and the model in our paper provides a good substitution for the phonon model in these situations.

In the following discussion, two phenomena observed in the experiments, shown in figures 2 and 7 are described. We will use the model proposed in this paper to give their interpretations.

The transition process shown in picture I of figure 2, is described in the route map on the right side of the energy-levels in figure 8. The fluorescent images A, B, C,... in figure 8 are drawn according to the characteristics of the quantum states that they correspond.

The assembled ions are initially in state A, which has close to the lowest energy the ions are able to reach (in fact, the ions cannot reach this low energy solely through the Doppler cooling process). The cooling laser is then red-detuned with a large frequency difference. The ions are heated by RF, and the state transfers to B. The fluorescent image of the ions is much bigger at this stage. The size reflects the increased movement area that the ion can reach, which
means the ions have a higher kinetic energy. In stage $B$, the kinetic energy of the ions almost overcomes the Coulomb repulsion potential, and their fluorescence images almost overlap. At this stage, the ions are very unstable. If their kinetic energy increases slightly, the Coulomb repulsion decreases, the ions overcome the Coulomb repulsion to move together and the cloudy state $C$ forms. Points $B$ and $C$ have the same energy. In figure 8, these two points overlap. Point $B$ is on the crystal state line and $C$ is on the white line. The $B$-to-$C$ process is the phase transition between cloudy state and crystal state.

**Figure 7.** Graphs of the fluorescence intensity varying with the detuning frequencies, as observed in the laboratory. For the normal hysteresis loop, the fluorescence intensity of the ion cloud is larger than the ion crystal, which is in contrast to the abnormal hysteresis loop. The points $A$, $B$, $C$... in these graphs correspond to images $A$, $B$, $C$... in figure 2.

**Figure 8.** The explanation for phase transition between cloudy state and crystal state.
transition. In this process, the Coulomb potential converts to the kinetic energy but the total energy is unchanged. Because of the large amount of kinetic energy that ions get in state $C$, the temperature of the ion system increases.

From the cloudy state $C$, if we want the ions to be crystal again, we slowly detune back. In the back-scan process, the ions slowly cool down but they do not transfer back to $B$ and then to $A$. They transfer to stage $D$ first during the back-scan process. In the $C$-to-$D$ process, the kinetic energy of the ions in the radial direction is transferred to the axial direction to overcome "attractive" potential, which is essentially the Coulomb potential. The energy of these ions in state $D$ is almost lowest in cloudy states, depending on the balance of the cooling laser and heating RF. The ions’ fluorescence image in $D$ becomes very long and narrow in shape. The ions then become unstable and quickly decay into state $E$, the crystal state. In this process, the kinetic energy is used to overcome the ‘attractive’ potential and the temperature of the ion system drops. The state $E$ can be further cooled into $A$. Then, the normal hysteresis-loop-like forms.

The abnormal hysteresis-loop-like shown in II of figure 2 is the other evidence for our theory. 5 V RF is added under the compensating pole to excite the ions in the trap. When the RF frequency is near the radial secular frequency, the state $B'$ will be created. The cooling laser is then blue-detuned. The ions in the trap will quickly transfer to $E'$ through the stage $C'$. The ions in $C'$ are very unstable, rapidly decaying into $D$, then further transferring to $A$. In the process $B' \rightarrow C' \rightarrow D$, the ions exhibit a strong attractive force, which is contrary to our understanding because the Coulomb force among ions is a repulsive force. Why could the Coulomb potential support the ‘attractive’ force here? To answer this question, let us review a classical mode: the electrical potential of two electrified rings with the same radius $R$ in a plane. The electric potential of this system might reach a maximum when the distance between these two rings equals $2R$, and the electric potential will drop as the distance decreases. The ‘attractive’ force between two ions in a trap is similar to that of these two rings. The movement areas of the ions are the size of rings. If one ion, $P_1$, enters the movement area of the other, $P_2$, ion $P_1$ will not feel the force of $P_2$ because the average force of $P_2$ on $P_1$ in a very short time approximately equals zero. Then, the electric potential will quickly drop when the movement areas of the ions overlap.

5. Conclusion

This paper exhibited the drawbacks of previous work on the description of the cloudy state and proposed a new model that can calculate the energy-levels of both the cloudy state and crystal state. In accordance with the linear response function, we presented a method to calculate the energy-level of any number of trapped ions in the Paul trap and exhibited the energy level diagram of two trapped ions as an example. We showed the imaginary part of the function $\langle \hat{\rho}_{md} | H_0 \rangle_{\omega}$ is nonzero for the cloudy state, meaning the ions in cloudy state have finite lifetime. In contradiction to this, the crystal state is much more stable and the corresponding $\text{Im} (\langle \hat{\rho}_{md} | H_0 \rangle_{\omega})$ equals zero. We use this to interpret the phenomena that we encountered in our laboratory. We show that these quantum interpretations for the fluorescent hysteresis loops that are formed by two trapped ions in a Paul trap conform to the phenomena that we observed in the laboratory. This theory is fundamental and it can be used in the areas of ion cooling, ion state manipulation and ion transport in further studies.
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Appendix A. The derivation of Hamiltonian H

To calculate the energy-level, we should give the concrete expressions about $\gamma$ and $\kappa$, and reduce equation (3).

Before the derivation, we should first introduce some notes,

$$D_n^m = v_n^m \cdot |v_n^m|,
K_i = \begin{pmatrix} K_{1,C(T)} & K_{2,C(T)} \\ K_{1,C(T)} & K_{2,C(T)} \end{pmatrix},
C(R) = 1; C(S) = 2; C(X) = 3.$$ (A.1)

The second quantization for $\hat{H}$ can be written as

$$\hat{H} = \sum_m \sum_n \langle m | \hat{H}_0 | n \rangle c_m \dagger c_n + \frac{e^2}{8\pi^2 \varepsilon_0} \sum_{m_1} \sum_{m_2} \sum_{n_1} \sum_{n_2} \langle m_1, n_1 | \hat{H}_1 | m_2, n_2 \rangle c_{m_1} \dagger c_{m_2} c_{n_1} \dagger c_{n_2}.$$ (A.2)

The state $|m\rangle$ here is the eigenstate of $\hat{H}_0$ and $|m,m\rangle = |\Psi_m^\prime (r_1), \Psi_n^\prime (r_2)\rangle$, then the first term of equation (A.2) equals $(m_\alpha + 1/2)\hbar \omega_\alpha \delta_{m_\alpha, n_\alpha}$. The calculation of the second term of equation (A.2) is much more complicated because $\hat{H}_1$ has a singularity at $r_i = r_j$. To compute this formula, we write the coulomb potential into the following

$$\frac{1}{|r_i - r_j|} = \frac{1}{2} \int \int \int \frac{1}{q^2} \exp (i q \cdot (r_i - r_j)) dq^3.$$ (A.3)

The inner product $\langle m_1, n_1 | \exp (i q \cdot (r_1 - r_2)) | m_2, n_2 \rangle$ can be exactly calculated out by using the quantum algebra. Let $\Psi_n^\prime(k) = \langle k|m\rangle$, where $k$ is the momentum, then we get

$$\langle m_1 | \langle n_1 | \exp (i q \cdot (r_1 - r_2)) | m_2 \rangle | n_2 \rangle = \langle \Psi_{m_1}^\prime (k_1 + q) | \langle \Psi_{n_1}^\prime (k_2 - q) | \Psi_{m_2}^\prime (k_1) \rangle | \Psi_{n_2}^\prime (k_2) \rangle.$$ (A.4)
It has three dimensions. For the axial direction,

\[
\langle \psi_m(k_x + q_x) | \psi_n(k_x) \rangle = f(m, n, \alpha_x) \]

\[
= \sum_{l=0}^{\infty} \text{cof}(m, n, l)( - \alpha_x)^{m+n-2l} \exp \left( - |\alpha_x|^2 / 2 \right); \]

\[
\text{cof}(m, n, l) = \frac{(-1)^{m-l}}{(m - l)!(n - l)!}; \tag{A.5}
\]

where, \( \alpha_x = \hbar q_x / (\sqrt{2} p_0^x) \). To get this expression, the character of the coherent state should be used:

\[
\langle \psi_m(k_x + q_x) \rangle = \left( \hat{a}_x + \alpha_x \right)^m / \sqrt{m!} \langle \psi_0(k_x + q_x) \rangle; \]

\[
\langle \psi_0(k_x + q_x) \rangle = \exp \left( - \frac{\alpha_x^2}{2} \right) \sum_{n=0}^{\infty} \frac{(\alpha_x \hat{a}_x)^n}{n!} \langle \psi_0(k_x) \rangle. \tag{A.6}
\]

Quantum algebra is a simple way to calculate exactly inner products, especially in the condition that we cannot give a concrete expression of a wave function in coordinate or momentum representation. For the radial direction here, the formulas of wave functions are more complicated than the axial direction. It is hard to get the normalizing factors of them. In this condition, we must employ quantum algebra to calculate the inner products in radial direction. Considering the symmetry in the radial direction, we should choose the polar coordinate plane expression to calculate inner products. Under the polar coordinate, the basis can be written as

\[
\langle \phi_n(r, \theta) \rangle = |n_r, n_s\rangle; \]

\[
\hat{a}_r = \frac{1}{\sqrt{2}} \left( \hat{a}_y - i \hat{a}_z \right); \]

\[
\hat{a}_s = \frac{1}{\sqrt{2}} \left( \hat{a}_y + i \hat{a}_z \right). \tag{A.7}
\]

Based on the calculations in axial direction, the inner product can be expressed as

\[
\langle q_{n_r}(k_r + q_r) \rangle \langle q_{n_s}(k_r) \rangle = f(m_r, n_r, \alpha_r)f(m_s, n_s, \alpha_s); \]

\[
\alpha_r = - \frac{q_r e^{-i\phi}}{2p_0^r}; \quad \alpha_s = - \frac{q_r e^{i\phi}}{2p_0^r}; \quad p_0^r = \sqrt{m\hbar \omega_r}. \tag{A.8}
\]

Finally, the second quantization formula for \( \hat{H}_1 \) can be simply written into

\[
\langle m1 | \langle n1 | \hat{H}_1 | m2 \rangle | n2 \rangle
\]
\[ g_{r,s}(R, S, q_x) \] is the axial-radial coupling function. The closer the ions are, the bigger \( g_{r,s} \) is.

Let us look at the value of \( g_{r,s}(R, S, q_x) \) when \( q_x \to 0 \). For the given matrices \( R \) and \( S \), \( g_{r,s}(R, S, q_x) \) might be divergent or convergent at the point \( q_x \to 0 \). These matrices that make \( g_{r,s}(R, S, q_x) \) divergent are first and foremost considered in equation (A.9), because these corresponding \( g_{r,s}(R, S, q_x) \) can make much bigger integrate values than the convergent formula \( g_{r,s}(R, S, q_x) \). For example, \( g_{r,s}(0, 0, q_x) \) is divergent at the point \( q_x \to 0 \), and \( \Theta(q_x) = 3.35 \). However, \( g_{r,s}(0, 0, q_x) \) is convergent at the point (the value about it is \( -\pi/2 \)), and then \( \Theta(q_x) = 0.27 \). To simplify the expression \( H \), we discard small ones and keep big ones. Then, only the divergent \( g_{r,s}(R, S, \alpha) \) are kept in summation in equation (3). (In following text we can see, \( H^{rd} \) is kept and \( H^{el} \) is overlooked.)

The matrices \( R, S \) that correspond the divergent \( g_{r,s}(R, S, \alpha) \) are satisfied the formulas \( R = D_{n_r}^{m_r}, \) and \( S = D_{n_s}^{m_s} \).

Next, we will give a concrete expression of \( \Theta(v_{n_1}^{m_1} : v_{n_2}^{m_2}, \alpha) \). \( \Theta \) is a dimensionless quantity and only depends on structure parameter \( \alpha = \sqrt{\omega_{rd}/\omega_{el}} \).
integration \( \int q_x^m e^{-q_x^2} g_{r,s} dq_x \) is the key to get the expression of \( \Theta \).

\[
C_{RX}(X, R, S, \alpha) = \int_{-\infty}^{\infty} (aq_x)^{\Delta x^+} \exp \left( -\left( aq_x \right)^2 \right) g_{r,s}(R, S, q_x) dq_x
\]

\[
= \frac{1 + (-1)^{\Delta x^+} \alpha^{\Delta x^+}}{2^{(\Delta x^+,+2)/2} \left( 1 + \Delta_R^+ + \Delta_S^+ + \Delta_X^+ \right)} \Gamma \left( 1 + \frac{\Delta_R^+ + \Delta_S^+}{2} \right) \Gamma \left( 1 + \frac{\Delta_X^+}{2} \right)
\]

\[
2F_1 \left( \frac{1 + \Delta_X^+}{2}, \frac{1 + \Delta_R^+ + \Delta_S^+ + \Delta_X^+}{2}, \frac{3 + \Delta_R^+ + \Delta_S^+ + \Delta_X^+}{2}, 1 - \alpha^2 \right)
\]  \( \text{(A.12)} \)

Finally, we write the concrete expression about \( \Theta \) as the following.

\[
\Theta(X, R, S, \alpha) = \prod_{n=1,2} \left( \min(T_{1,n}, T_{2,n}) \right) \sum_{K_{n,T}=0} \text{cof}(T_{1,n}, T_{2,n}, K_{n,T}) \delta_{K_{n,T},0} C_{RX}(X - K_x, R - K_r, S - K_s, \alpha).
\]  \( \text{(A.13)} \)

References

[1] Häffner H, Roos C F and Blatt R 2008 Phys. Rep. 469 155
[2] Leibfried D, Blatt R, Monroe C and Wineland D 2003 Rev. Mod. Phys. 75 281–324
[3] Partoens B, Peeters F M and Apolinario S W S 2003 New J. Phys. 75 281–324
[4] Ludwig P, Kosse S and Bonitz M 2005 Phys. Rev. E 71 046403
[5] Fishman S, de Chiara G, Calarco T and Morigi G 2008 Phys. Rev. B 77 064111
[6] Dubin D H E 1993 Phys. Rev. Lett. 71 2753–6
[7] Enzer D G et al 2000 Phys. Rev. Lett. 85 2466–9
[8] Walther H 1995 *Phys. Scr.* **T59** 360–8
[9] Retzker A, Thompson R C, Segal D M and Plenio M B 2008 *Phys. Rev. Lett.* **101** 260504
[10] Shimshoni E, Morigi G and Fishman S 2011 *Phys. Rev. Lett.* **106** 010401
[11] Block M, Drakoudis A, Leuthner H, Seibert P, Werth G, Block M, Drakoudis A, Leuthner H and Seibert P 2000 *J. Phys.* B **33** 375
[12] Bollinger J J, Wineland D J and Itano W M 1985 *Proc. of the Int. Conf.* 727–30
[13] Cutler L S, Flory C A, Giffard R P and McGuire M D 1986 *Appl. Phys.* B **39** 251
[14] Li G Z 1989 Z. *Phys.* D **14** 1–8
[15] Diedrich F, Peik E, Chen J M, Quint W and Walther H 1987 *Phys. Rev. Lett.* **59** 2931–4
[16] Hoffnagle J, DeVoe R G, Reyna L and Brewer R G 1988 *Phys. Rev. Lett.* **61** 3
[17] Blümel R, Peik E, Quint W and Walther H 1990 *Acta Phys. Pol.* A **78** 3
[18] Dubin D H E and O’Neil T M 1988 *Phys. Rev. Lett.* **60** 511–4
[19] Thomas H M and Morfill G E 1996 *Nature* **379** 806–9
[20] Drewsen M, Brodersen C, Horneck L and Hanqst J S 1998 *Phys. Rev. Lett.* **81** 2878–81
[21] Nordlund K 1995 *Comput. Math. Sci.* **3** 448–56
[22] Schiller S and Lämmertzahi C 2003 *Phys. Rev.* A **68** 053406
[23] Prestage J D, Williams A, Maleki L, Djomehri M J and Harabetian E 1991 *Phys. Rev. Lett.* **66** 2964–7
[24] Altland A and Simons B D 2010 *Condensed Matter Field Theory* (New York: Cambridge University Press) p 372
[25] Nakajima S 1980 *The Physics of Elementary Excitation* (New York: Springer) chapter 3
[26] Kubo R 1957 *J. Phys. Soc. Japan* **12** 570
[27] Raizen M G, Gilligan J M, Bergquist J C, Itano W M and Wineland D J 1992 *J. Mod. Opt.* **39** 233
[28] Bermudez A and Plenio M B 2012 *Phys. Rev. Lett.* **109** 010501
[29] Cormick C and Morigi G 2012 *Phys. Rev. Lett.* **109** 053003
[30] James D F V 1998 *Appl. Phys.* B **66** 181–90