Bitopological Approximation Space with Application to Data Reduction in Multi-valued Information Systems

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**Abstract.** We presented bitopological approximation space as a generalization of classical approximation space. This generalization is based on a topological space that have a subbases generated by a family of binary relations defined on the universe of discourse. We studied some properties of rough sets on bitopological approximation spaces. Many new membership functions and inclusion functions are defined and are used for redefining the rough approximations. Finally, some real life application examples are given to illustrate the benefit of our approach.

1. Introduction

For a long time we dreamed that the general topological spaces can apply in life sciences. Since 2004, I have been working on topology with rough sets and with information systems and tried to reduce the information system by topology by using general binary relations instead of equivalence relations. In 2010, I solved the problem of finding the missing attribute values by using the topological base relation that defined in [8]. More generalizations of the topological approach of information systems can do using near open sets such as pre-open sets, e-open sets and so on [5].

Many approaches generalized rough sets and fuzzy sets using topological spaces [9, 11, 12, 17, 18, 19, 20]. Liu in [6] has studied a comparison of two types of rough sets induced by coverings and he in [7] has introduced the axiomatic systems for rough sets and fuzzy rough sets. The main purpose from these generalizations is to add new objects in the positive region of decision categories. Deleting some objects from negative region is equivalent target for the addition, objects in the positive region. A topological generalizations achievement many aspects such as reduction of large data sets and generate decision rules that help in data mining. Also, attribute reduction in ordering information systems based on evidence theory is one of the directions of study on this topic [10].

Rough sets and topology with their generalizations have developed the similarity measure at the base of granular computing methodology. Knowledge discovery, according to granular computing models based on rough sets and topological spaces is a different technique for data pre-processing, reduction, and data mining [21, 22, 23, 24].

In [1] Abu-Donia introduced multi knowledge bases using rough approximations and topology, they used these knowledge bases in applications. He also in [2], introduced a comparison between different...
kinds of approximations using a family of binary relations. He with Salama in [3, 4] have generalized the
classical rough approximation spaces using topological near open sets called δβ-open sets.

δβ-open sets have opened the door about using other topological approaches for information systems.
In our approach we used any finite number of binary relations defined on attributes of an information
system to generate two topological spaces and used them in generalizations. We organized our work at the
following:

In Section 2 we defined the notion of bitopological approximation space with some important accuracy
measures on it. We introduced in Section 3 some important properties of bitopological spaces and we
defined the concept of rough set on it. In Section 4, we introduced an application approach for data
reduction in multi-valued information systems. The conclusion of our work is given in Section 5.

2. Bitopological Approximation Space

The approximation space \( A = (X, R) \) is the core of rough set theory, where \( X \) is the universe and \( R \) is an
equivalence relation [8, 14]. The equivalence class \( [x]_R \) is the basic tool for defining rough approximations,
lower \( R(A) = \{ x \in X : [x]_R \subseteq A \} \) and upper \( \bar{R}(A) = \{ x \in X : [x]_R \cap A \neq \emptyset \} \). In application this theory have a
wide range when we use positive \( \text{POS}_R(A) = R(A) \), negative \( \text{NEG}_R(A) = X - \bar{R}(A) \) and \( \text{BN}_R(A) = \bar{R}(A) - R(A) \)
regions of some categories \( X \) of the universe.

To measure the quality of applications Pawlak define the accuracy measure as follows:

\[
a_R(A) = \frac{|R(A)|}{|\bar{R}(A)|} \quad \text{where} \quad A \neq \emptyset.
\]

By accuracy measure we can capture the large of the boundary region of a given category. From the
advantage of this theory is the ability to handle a category only using the given data.

In the case of equivalence relations the essential rough set approximations lower and upper coincide
with the topological interior and the closure operation respectively. This open the door for using the theory
of topological spaces for more generalizations of rough sets. A topological space \( (X, \tau) \) consisting of a set \( X \)
and family \( \tau \) of subsets of \( X \) that satisfies the conditions:

1. The empty set and the universe are basic members in \( \tau \).
2. The arbitrary unions of members in \( \tau \) are again a member in \( \tau \).
3. The finite intersections of members in \( \tau \) are again a member in \( \tau \).

The topological interior and the topological closure of \( A \subset X \) are defined as follows:

\[
\begin{align*}
\text{int}(A) &= \bigcup \{ G \subseteq X : G \in \tau, G \subseteq A \}, \\
\text{cl}(A) &= \bigcap \{ F \subseteq X : X - F \in \tau, A \subseteq F \}.
\end{align*}
\]

We can generate a lower and upper approximation operator based on a topological bases generated by
binary relations.

Suppose \( R_i, i = 1, 2, 3, ..., n \) be any binary relations defined on a universe \( U \), if \( x, y \in U \) then the set
\( R_{i-R}(x) = \{ y : xR_iy, (x, y) \in R_i, \forall i \} \) is called the right blocks of \( x \). The family of all right blocks \( S_R = \{ R_{i-R}(x) : x \in U \} \) is a subbase of a topology \( \tau_R \) on the universe. In the same way the left blocks \( R_{i-L}(x) = \{ y : yRx, (x, y) \in R_i, \forall i \} \) define a subbase \( S_L = \{ R_{i-L}(x) : x \in U \} \) of another topology \( \tau_L \) on the universe. The
approximation space \( \text{BiT} = (U, R, \tau_R, \tau_L) \) is a generalization of the classical approximation space of Pawlak
and we named it biotopological approximation space.

The subbase \( S_R \) defines lower and upper approximations of \( A \subset U \) as follows:

\[
\begin{align*}
\text{S}_R(A) &= \bigcup \{ R_{i-R}(x) \in S_R : R_{i-R}(x) \subseteq A \}, \\
\bar{S}_R(A) &= \bigcap \{ R_{i-R}(x) \in S_R : R_{i-R}(x) \cap A \neq \emptyset \}.
\end{align*}
\]

Also, the subbase \( S_L \) defines lower and upper approximations of \( A \subset U \) as follows:
1. \( S_L(A) = \bigcup \{ R_{c-l}(x) \in S_L : R_{c-l}(x) \subseteq A \} \),
2. \( \overline{S_L}(A) = \bigcap \{ R_{c-l}(x) \in S_L : R_{c-l}(x) \cap A \neq \emptyset \} \).

The accuracy measures of a subset \( A \subseteq U \) of the above approximations defined as follows:

\[
\alpha(A) = \frac{|A \cap \overline{S_L}(A)|}{|A \cup \overline{S_L}(A)|},
\alpha_L(A) = \frac{|A \cap S_L(A)|}{|A \cup S_L(A)|}.
\]

A subset \( A \subseteq U \) of a bitopological space \( BiT = (U, R_o, \tau_R, \tau_L) \) is called \( \beta \)-open if \( A \subseteq cl_R(int_R(cl_L(A))) \) and \( \beta \)-closed if \( A \subseteq cl_L(int_L(cl_R(A))) \). The subset \( A \subseteq U \) is called \( \beta \)-open if \( A \subseteq cl_R(int_R(cl_L(A))) \cap cl_L(int_L(cl_R(A))) \).

A subset \( A \subseteq U \) that discernible the objects using the decision attribute in the bitopological space \( BiT = (U, R_o, \tau_R, \tau_L) \) is called \( \delta \)-open if \( A = \overline{S_R}(A) \), where \( \overline{S_R}(A) = \bigcap \{ G \in \tau_R : A \subseteq G \} \) and it called \( \delta \)-open if \( A = \overline{S_L}(A) \), where \( \overline{S_L}(A) = \bigcap \{ G \in \tau_L : A \subseteq G \} \). The subset \( A \subseteq U \) is called \( \delta \)-open if \( A = \overline{S_R}(A) \cap \overline{S_L}(A) \).

A subset \( A \subseteq U \) that discernible the objects using the decision attribute in the bitopological space \( BiT = (U, R_o, \tau_R, \tau_L) \) is called \( \delta \)-open if \( A \subseteq cl_R(int_R(cl_L(A))) \) and it is called \( \delta \)-open if \( A \subseteq cl_L(int_L(cl_R(A))) \). The subset \( A \subseteq U \) is called \( \delta \)-open if \( A \subseteq cl_R(int_R(cl_L(A))) \cap cl_L(int_L(cl_R(A))) \).

The family of all \( \delta \)-open (resp. \( \delta \)-closed) set is (resp. \( \delta \)-closed) set. We denote the set of all \( \delta \)-open (resp. \( \delta \)-closed) set by \( \delta \)-open (resp. \( \delta \)-closed) set.

Let \( BiT = (U, R_o, \tau_R, \tau_L) \) be a bitopological space. Lower approximation and Rupper approximation of the subset \( A \subseteq U \) is defined as follows:

1. \( \overline{BiT}_L(A) = \bigcup \{ G \in RO(U) : G \subseteq A \} \),
2. \( \overline{BiT}_R(A) = \bigcap \{ F \in RC(U) : F \supseteq A \} \).

Let \( BiT = (U, R_o, \tau_R, \tau_L) \) be a bitopological space. Lower approximation and Rupper approximation of the subset \( A \subseteq U \) is defined as follows:

1. \( \overline{BiT}_L(A) = \bigcup \{ G \in LO(U) : G \subseteq A \} \),
2. \( \overline{BiT}_R(A) = \bigcap \{ F \in LC(U) : F \supseteq A \} \).

The accuracy measures of a subset \( A \subseteq U \) of the above approximations defined as follows:

\[
\alpha(A) = \frac{|A \cap \overline{BiT}_L(A)|}{|A \cup \overline{BiT}_L(A)|},
\alpha_L(A) = \frac{|A \cap \overline{BiT}_R(A)|}{|A \cup \overline{BiT}_R(A)|}.
\]

**Example 2.1.** Let \( U = \{a, b, c, d, e\} \) be a universe we define three relations on \( U \) by:

\[
R_1 = \{(a, a), (a, c), (b, c), (b, d), (c, e), (d, a), (d, e), (e, c)\},
R_2 = \{(a, c), (a, a), (b, c), (b, d), (c, e), (d, a), (d, e), (e, c), (b, e)\},
R_3 = \{(c, c), (a, a), (a, c), (b, c), (b, d), (c, e), (d, a), (d, e), (e, c), (b, e)\}.
\]

Then we have:

\[ S_R = \{a, c, d, e\}, \]
\[ S_L = \{\{a, d\}, \{b\}, \{a, c, d, e\}\}. \]

Then the topologies associated with these relations are:

\[ \tau_R = \{U, \varphi, \{e\}, \{c, d\}, \{a, e\}, \{c, d, e\}, \{a, c, d, e\}\}, \]
\[ \tau_L = \{U, \varphi, \{b\}, \{a, d\}, \{a, b, d\}, \{a, c, d, e\}\}. \]

Table 1 below showing the degree of accuracy measure \( \alpha_R(A), \alpha_L(A) \) and \( \alpha_{\delta\beta}(A) \) for some subsets of the universe.

| Subset       | \( \alpha_R(A)\)% | \( \alpha_L(A)\)% | \( \alpha_{\delta\beta}(A)\)% |
|-------------|---------------------|---------------------|---------------------|
| \{a, c\}    | 100                 | 100                 | 100                 |
| \{b, e\}    | 33                  | 33                  | 100                 |
| \{a, b, e\} | 66                  | 100                 | 100                 |
| \{a, b, c\} | 50                  | 66                  | 100                 |
| \{a, c, d\} | 100                 | 100                 | 100                 |
| \{b, c\}    | 20                  | 75                  | 100                 |
| \{c, d\}    | 60                  | 75                  | 100                 |
| \{a, c, d, e\} | 80               | 80                  | 80                  |
| \{a, b, d\} | 40                  | 100                 | 100                 |
| \{b, c, d\} | 60                  | 80                  | 100                 |

Table 1: Accuracy measure \( \alpha_R(A), \alpha_L(A) \) and \( \alpha_{\delta\beta}(A) \) of some subsets

We see that the degree of exactness of the set \( \{c, d, e\} \) by using \( R \) accuracy measure equal to 60\%, by using \( L \) accuracy measure equal to 75\% and by using \( (\delta\beta) \) accuracy measure equal to 100\%. Consequently, \( (\delta\beta) \) accuracy measure is the best accuracy measure.

According to the above approximations any subset \( A \subseteq U \) has the following regions:

1. The Rinternal edg of \( A \), \( \underline{Edg}_R (A) = A - BiT_R (A) \).
2. The Linternal edg of \( A \), \( \underline{Edg}_L (A) = A - BiT_L (A) \).
3. The \( (\delta\beta) \)internal edg of \( A \), \( \underline{Edg}_{\delta\beta} (A) = A - BiT_{\delta\beta} (A) \).
4. The Rexternal edg of \( A \), \( \overline{Edg}_R (A) = BiT_R (A) - A \).
5. The Lexternal edg of \( A \), \( \overline{Edg}_L (A) = BiT_L (A) - A \).
6. The \( (\delta\beta) \) external edg of \( A \), \( \overline{Edg}_{\delta\beta} (A) = BiT_{\delta\beta} (A) - A \).
7. The Rboundary of \( A \), \( BON_R (A) = BiT_R (A) - BiT_L (A) \).
8. The Lboundary of \( A \), \( BON_L (A) = BiT_L (A) - BiT_{\delta\beta} (A) \).
9. The \( (\delta\beta) \) boundary of \( A \), \( BON_{\delta\beta} (A) = BiT_{\delta\beta} (A) - BiT_{\delta\beta} (A) \).
10. The R negative of \( A \), \( NEG_R (A) = U - BiT_R (A) \).
11. The L negative of \( A \), \( NEG_L (A) = U - BiT_L (A) \).
12. The \( (\delta\beta) \) negative of \( A \), \( NEG_{\delta\beta} (A) = U - BiT_{\delta\beta} (A) \).

Proposition 2.1. For any bitopological approximation space \( BiT = (U, R, \tau_R, \tau_L) \), and for any \( A \subseteq X \) we have:

1. \( BON_R (A) = \underline{Edg}_R (A) \cup \overline{Edg}_R (A) \)
2. \( BON_L (A) = \underline{Edg}_L (A) \cup \overline{Edg}_L (A) \)
3. \( BON_{\delta\beta} (A) = \underline{Edg}_{\delta\beta} (A) \cup \overline{Edg}_{\delta\beta} (A) \)
Proof. all three parts can proved as follows:

\[ \text{BON}_R(A) = \overline{\text{BiT}_I}(A) - \text{BiT}_I(A) = (\overline{\text{BiT}_I}(A) - A) \cup (A - \text{BiT}_I(A)) \]
\[ \text{Edg}_R(A) = \overline{\text{BiT}_I}(A) - A \text{ and } \text{Edg}_L(A) = A - \text{BiT}_I(A), \]
then we have \( \text{BON}_R(A) = \text{Edg}_L(A) \cup \text{Edg}_R(A) \).

The next two propositions give the connection between the classical lower and upper approximations of rough sets. □

**Proposition 2.2.** For any bitopological approximation space \( \text{BiT} = (U, R, \tau_R, \tau_L) \), and for any \( A \subseteq X \) we have:

1. \( \overline{R}(A) - \text{BiT}_I(A) = \overline{\text{Edg}}_L(A) \cup \text{Edg}_R(A) \)
2. \( \overline{R}(A) - \text{BiT}_I(A) = \overline{\text{Edg}}_L(A) \cup \text{Edg}_R(A) \)
3. \( \overline{R}(A) - \text{BiT}_{\{0\}}(A) = \overline{\text{Edg}}_L(A) \cup \text{Edg}_R(A) \)
4. \( \text{BiT}_I(A) - \overline{R}(A) = \overline{\text{Edg}}_L(A) \cup \text{Edg}_R(A) \)
5. \( \text{BiT}_I(A) - \overline{R}(A) = \overline{\text{Edg}}_L(A) \cup \text{Edg}_R(A) \)
6. \( \text{BiT}_{\{0\}}(A) - \overline{R}(A) = \overline{\text{Edg}}_L(A) \cup \text{Edg}_R(A) \)

Proof. Obvious. □

**Proposition 2.3.** For any bitopological approximation space \( \text{BiT} = (U, R, \tau_R, \tau_L) \), and for any \( A \subseteq X \) we have:

1. \( \text{Edg}(A) = \text{Edg}_L(A) \cup (\text{BiT}_I(A) - \overline{R}(A)) \)
2. \( \overline{\text{Edg}}_L(A) = \overline{\text{Edg}}_L(A) \cup (\text{BiT}_I(A) - \overline{R}(A)) \)
3. \( \overline{\text{Edg}}_L(A) = \overline{\text{Edg}}_L(A) \cup (\text{BiT}_{\{0\}}(A) - \overline{R}(A)) \)
4. \( \text{Edg}_L(A) = \text{Edg}_L(A) \cup (\overline{R}(A) - \text{BiT}_I(A)) \)
5. \( \overline{\text{Edg}}_L(A) = \overline{\text{Edg}}_L(A) \cup (\overline{R}(A) - \text{BiT}_I(A)) \)
6. \( \text{Edg}_L(A) = \text{Edg}_L(A) \cup (\overline{R}(A) - \text{BiT}_{\{0\}}(A)) \)

Proof. Obvious. □

Let \( \text{BiT} = (U, R, \tau_R, \tau_L) \) be a bitopological approximation space and \( A \subseteq U \). Then we define the following membership functions:

1. \( x \in A \text{ iff } x \in \overline{\text{BiT}}_I(A) \), that called \( R \)-strong membership function.
2. \( x \in A \text{ iff } x \in \text{BiT}_{\{0\}}(A) \), that called \( L \)-strong membership function.
3. \( x \in A \text{ iff } x \in \text{BiT}_{\{0\}}(A) \), that called \( \delta \)\( \beta \)-strong membership function.
4. \( x \in A \text{ iff } x \in \text{BiT}_{\{0\}}(A) \), that called \( R \)-weak membership function.
5. \( x \in A \text{ iff } x \in \text{BiT}_{\{0\}}(A) \), that called \( L \)-weak membership function.
6. \( x \in A \text{ iff } x \in \text{BiT}_{\{0\}}(A) \), that called \( \delta \)\( \beta \)-weak membership function.

According to the definitions of membership functions above we can redefine the lower and upper approximations of a set \( A \subseteq X \) as follows:

1. \( \text{BiT}_I(A) = \{ x \in A : x \in A \} \),
2. \( \text{BiT}_{\{0\}}(A) = \{ x \in A : x \in A \} \),
3. \( \text{BiT}_{\{0\}}(A) = \{ x \in A : x \in A \} \),
4. \( \text{BiT}_I(A) = \{ x \in A : x \in A \} \),
5. \( \text{BiT}_{\{0\}}(A) = \{ x \in A : x \in A \} \),
6. \( \text{BiT}_{\{0\}}(A) = \{ x \in A : x \in A \} \)

Let \( \text{BiT} = (U, R, \tau_R, \tau_L) \) be a bitopological approximation space and \( A, B \subseteq U \). Then we have the following cases with respect to \( A \) and \( B \):
1. R-roughly bottom equal \((A \equiv_R B)\) if \(\overline{B \uparrow T_s(A)} = B \uparrow T_s(B)\),
2. L-roughly bottom equal \((A \equiv_L B)\) if \(\overline{L \uparrow T_s(A)} = L \uparrow T_s(B)\),
3. \((\delta \beta)\)-roughly bottom equal \((A \equiv_{\delta \beta} B)\) if \(\overline{\delta \beta \uparrow T_s(A)} = \overline{\delta \beta \uparrow T_s(B)}\),
4. R-roughly top equal \((A \equiv_R B)\) if \(\overline{R \downarrow T_i(A)} = R \downarrow T_i(B)\),
5. L-roughly top equal \((A \equiv_L B)\) if \(\overline{L \downarrow T_i(A)} = L \downarrow T_i(B)\),
6. \((\delta \beta)\)-roughly top equal \((A \equiv_{\delta \beta} B)\) if \(\overline{\delta \beta \downarrow T_i(A)} = \overline{\delta \beta \downarrow T_i(B)}\),
7. R-roughly equal \((A \approx_R B)\) if \((A \equiv_R B)\) and \((A \approx_R B)\),
8. L-roughly equal \((A \approx_L B)\) if \((A \equiv_L B)\) and \((A \equiv_L B)\),
9. \((\delta \beta)\)-roughly equal \((A \approx_{\delta \beta} B)\) if \((A \equiv_{\delta \beta} B)\) and \((A \equiv_{\delta \beta} B)\).

Let \(BT = (U, R_i, \tau_R, \tau_L)\) be a bitopological approximation space and \(A, B \subseteq U\). Then we have the following cases with respect to \(A\) and \(B\):

1. \(A\) is R roughly bottom included in \(B\) \((A \subset^R B)\) if \(\overline{B \uparrow T_s(A)} \subseteq \overline{B \uparrow T_s(B)}\)
2. \(A\) is L roughly bottom included in \(B\) \((A \subset^L B)\) if \(\overline{L \uparrow T_s(A)} \subseteq \overline{L \uparrow T_s(B)}\)
3. \(A\) is \((\delta \beta)\) roughly bottom included in \(B\) \((A \subset^{\delta \beta} B)\) if \(\overline{\delta \beta \uparrow T_s(A)} \subseteq \overline{\delta \beta \uparrow T_s(B)}\)
4. \(A\) is R roughly top included in \(B\) \((A \subset^R B)\) if \(\overline{R \downarrow T_i(A)} \subseteq \overline{R \downarrow T_i(B)}\)
5. \(A\) is L roughly top included in \(B\) \((A \subset^L B)\) if \(\overline{L \downarrow T_i(A)} \subseteq \overline{L \downarrow T_i(B)}\)
6. \(A\) is \((\delta \beta)\) roughly top included in \(B\) \((A \subset^{\delta \beta} B)\) if \(\overline{\delta \beta \downarrow T_i(A)} \subseteq \overline{\delta \beta \downarrow T_i(B)}\)
7. \(A\) is R roughly included in \(B\) \((A \subset^R B)\) if \((A \subset^R B)\) and \((A \subset^R B)\)
8. \(A\) is L roughly included in \(B\) \((A \subset^L B)\) if \((A \subset^L B)\) and \((A \subset^L B)\)
9. \(A\) is \((\delta \beta)\) roughly included in \(B\) \((A \subset^{\delta \beta} B)\) if \((A \subset^{\delta \beta} B)\) and \((A \subset^{\delta \beta} B)\)

3. Properties of bitopological rough approximations

In this section, we introduced some important properties of bitopological spaces and we will define the concept of rough set in \(BT = (U, R_i, \tau_R, \tau_L)\). For any bitopological approximation space \(BT = (U, R_i, \tau_R, \tau_L)\) a subset \(A\) of \(U\) is called:

1. R definable set if \(\overline{B \uparrow T_s(A)} = \overline{B \uparrow T_s(A)}\) or \(\overline{B \uparrow T_s(A)} = \overline{B \uparrow T_s(A)}\)
2. L definable set if \(\overline{L \uparrow T_i(A)} = \overline{L \uparrow T_i(A)}\) or \(\overline{L \uparrow T_i(A)} = \overline{L \uparrow T_i(A)}\)
3. \((\delta \beta)\) definable set if \(\overline{\delta \beta \uparrow T_i(A)} = \overline{\delta \beta \uparrow T_i(A)}\) or \(\overline{\delta \beta \uparrow T_i(A)} = \overline{\delta \beta \uparrow T_i(A)}\)
4. R rough if \(\overline{B \uparrow T_i(A)} \neq \overline{B \uparrow T_s(A)}\) or \(\overline{B \uparrow T_i(A)} \neq \overline{B \uparrow T_s(A)}\)
5. L rough set if \(\overline{B \downarrow T_i(A)} \neq \overline{B \downarrow T_s(A)}\) or \(\overline{B \downarrow T_i(A)} \neq \overline{B \downarrow T_s(A)}\)
6. \((\delta \beta)\) rough set if \(\overline{\delta \beta \uparrow T_i(A)} \neq \overline{\delta \beta \uparrow T_s(A)}\) or \(\overline{\delta \beta \uparrow T_i(A)} \neq \overline{\delta \beta \uparrow T_s(A)}\)

For any bitopological approximation space \(BT = (U, R_i, \tau_R, \tau_L)\) a subset \(A\) of \(U\) is called:

1. Roughly R definable, if \(\overline{B \uparrow T_s(A)} \neq \overline{B \uparrow T_s(A)}\) and \(\overline{B \uparrow T_s(A)} \neq \overline{B \uparrow T_s(A)}\)
2. Roughly L definable, if \(\overline{L \uparrow T_i(A)} \neq \overline{L \uparrow T_i(A)}\) and \(\overline{L \uparrow T_i(A)} \neq \overline{L \uparrow T_i(A)}\)
3. Roughly \((\delta \beta)\) definable, if \(\overline{\delta \beta \uparrow T_i(A)} \neq \overline{\delta \beta \uparrow T_i(A)}\) and \(\overline{\delta \beta \uparrow T_i(A)} \neq \overline{\delta \beta \uparrow T_i(A)}\)
4. Internally R undefinable, if \(\overline{B \uparrow T_s(A)} = \overline{B \uparrow T_s(A)}\) and \(\overline{B \uparrow T_s(A)} = \overline{B \uparrow T_s(A)}\)
5. Internally L undefinable, if \(\overline{B \downarrow T_i(A)} = \overline{B \downarrow T_i(A)}\) and \(\overline{B \downarrow T_i(A)} = \overline{B \downarrow T_i(A)}\)
6. Internally \((\delta \beta)\) undefinable, if \(\overline{\delta \beta \uparrow T_i(A)} = \overline{\delta \beta \uparrow T_i(A)}\) and \(\overline{\delta \beta \uparrow T_i(A)} = \overline{\delta \beta \uparrow T_i(A)}\).
7. Externally \( R \) undefinable, if \( \text{BiT}_x(A) \neq \varnothing \) and \( \overline{\text{BiT}}_x(A) = U \).
8. Externally \( L \) undefinable, if \( \text{BiT}_x(A) \neq \varnothing \) and \( \overline{\text{BiT}}_x(A) = U \).
9. Externally \( (\delta \beta) \) undefinable, if \( \text{BiT}_x(A) \neq \varnothing \) and \( \overline{\text{BiT}}_x(A) = U \).
10. Totally \( R \) undefinable, if \( \text{BiT}_x(A) = \varnothing \) and \( \overline{\text{BiT}}_x(A) = U \).
11. Totally \( L \) undefinable, if \( \overline{\text{BiT}}_x(A) = \varnothing \) and \( \text{BiT}_x(A) = U \).
12. Totally \( (\delta \beta) \) undefinable, if \( \text{BiT}_x(A) = \varnothing \) and \( \overline{\text{BiT}}_x(A) = U \).

Proposition 3.1. For any bitopological approximation space \( \text{BiT} = (U, R, L) \) and for all \( x, y \in U \), we have:

1. If \( x \in \text{BiT}_x(y) \) and \( y \in \text{BiT}_x(x) \) then \( \text{BiT}_x(y) = \text{BiT}_x(x) \).
2. If \( x \in \text{BiT}_x(y) \) and \( y \in \text{BiT}_x(x) \) then \( \text{BiT}_x(y) = \text{BiT}_x(x) \).
3. If \( x \in \overline{\text{BiT}}_x(y) \) and \( y \in \overline{\text{BiT}}_x(x) \) then \( \overline{\text{BiT}}_x(y) = \overline{\text{BiT}}_x(x) \).

Proof. (1) By definition of \( R \) upper approximation of a set is the \( \tau_K \) closure of this set, and since \( cl_{\tau_K}(y) \) is \( R \) closed set containing \( x \) while \( \text{BiT}_x(x) \) is the smallest \( R \) closed set containing \( x \), thus \( \text{BiT}_x(x) \subset \text{BiT}_x(y) \). The opposite inclusion follows by symmetry \( \text{BiT}_x(y) \subset \text{BiT}_x(x) \). Hence \( \text{BiT}_x(x) = \text{BiT}_x(y) \). The proof of Parts (2) and (3) are by the same way. \( \square \)

4. Data deduction in information systems using bitopological spaces

Decision tables (information systems) are widely used in applications and it have many different types. Some of them, its rows are represented objects, while its columns are labeled by attributes. In this system, independent attributes named condition attributes and dependent attributes called decision attributes. For the decision table \( T = (U, C, D) \), \( U \) is the set of objects, \( C \) is the set of independent attributes and \( D \) is the dependent attribute. On any decision table we define an information function \( f \) that maps the direct product of \( U \) and \( C \) into the set of all attribute values \( V_C \). A decision table is called incomplete when some values of specified attributes are missing.

Reduction of attributes and derivation of decision rules from decision tables are important for applications and this will done using hybrid approach between rough set theory and topology. By reduction of the information table we mean smaller subsets \( B \subseteq C \) attributes that preserve the quality of approximations.

With respect to our approach the subset \( B' \subseteq B \subseteq C \) is a reduct of subset \( B \) with respect to the bitopological approximation space \( \text{BiT} = (U, C, \tau_R, \tau_L) \) if it is a minimal subset of \( B \) which keeps the quality of classification unchanged. Another important topic in rough set theory on decision table is called decision rules. A decision rule \( r \) generated by a reduct \( B \subseteq C \) is represented in the following form:

\[
r = \bigwedge\left\{ f(a(x), v) \mid (D(x), w) \right\} \rightarrow (D(x), w) \text{ where } v \in V_C \text{ and } w \in V_D.
\]

We denote by \( s = (a(x), v) \) and by \( t = (D(x), w) \) to the condition and decision parts of a decision rule, respectively. Let \( r \) and \( r_i \) be the set of objects satisfying condition and decision parts of the decision rule \( r \). Objects satisfying both condition and decision parts of the decision rule are called the support of this rule. Decision rules are certain if \( r_i \subseteq r \).

Another important issue in information systems is the discover of dependency among attributes. The set of attributes \( A \subseteq C \) depends totally on the set of attributes \( B \subseteq C \), denoted \( B \Rightarrow A \) if the set of all values of attributes from \( A \) are contained in the values of attributes from \( B \). In other words, \( A \) depends totally on \( B \), if there exists a functional dependency between values of \( A \) and \( B \).

When we need to measure the dependency of attributes we define three important regions of each approximation type of a subset \( A \subseteq C \) as follows:

1. The \( R \) positive region \( \text{POS}_R(A) = \text{BiT}_R(A) \).
2. The \( L \) positive region \( \text{POS}_L(A) = \overline{\text{BiT}}_L(A) \).
3. The \( (\delta \beta) \) positive region \( \text{POS}_{(\delta \beta)}(A) = \text{BiT}_{(\delta \beta)}(A) \).
4. The $R$ negative region $\text{NEG}_R(A) = U - \overline{\text{BIT}}_R(A)$
5. The $L$ negative region $\text{NEG}_L(A) = U - \overline{\text{BIT}}_L(A)$
6. The $(\delta \beta)$ negative region $\text{NEG}_{\delta \beta}(A) = U - \overline{\text{BIT}}_{\delta \beta}(A)$
7. The $R$ boundary region $\text{BON}_R(A) = \overline{\text{BIT}}_R(A) - \text{BIT}_R(A)$
8. The $L$ boundary region $\text{BON}_L(A) = \overline{\text{BIT}}_L(A) - \text{BIT}_L(A)$
9. The $(\delta \beta)$ boundary region $\text{BON}_{\delta \beta}(A) = \overline{\text{BIT}}_{\delta \beta}(A) - \text{BIT}_{\delta \beta}(A)$

Formally dependency can be defined in the following way. Let $A$ and $B$ be subsets of $C$. We will say that $A$ depends on $B$ with respect to $\tau$,$L$, denoted $B \Rightarrow_L A$, if $\gamma_L(A, B) = \frac{|\text{POS}_L(A)|}{|U|}$, and with respect to $\tau$,$R$, denoted $B \Rightarrow_R A$, if $\gamma_R(A, B) = \frac{|\text{POS}_R(A)|}{|U|}$.

**Example 4.1.** Consider the multi-valued information system given in Table 2 below:

In this table the set $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ is a set of objects and $C = \{A_1, A_2, A_3\}$ is the set of condition attributes and Decision = $\{D\}$ is the decision attribute. In this example we need to determine the condition attributes that support the decision attribute.

| $D$  | $A_3$ | $A_2$ | $A_1$ | $U$ |
|-----|-------|-------|-------|-----|
| Yes | [3]   | [1,2,3] | [0]   | $u_1$ |
| No  | [3,4] | [1,2]  | [0,1] | $u_2$ |
| Yes | [3]   | [1,3]  | [2]   | $u_3$ |
| No  | [4]   | [1,2,4] | [7]   | $u_4$ |
| Maybe | [3,4] | [4]    | [1]   | $u_5$ |
| Yes | [3]   | [1,2]  | [1,2] | $u_6$ |
| Maybe | [3,4] | [1,2,3] | [0,2] | $u_7$ |

Table 2: Multi-valued Information System

The power set of condition attributes is given as follows:

$P(C) = \{C, \emptyset, \{A_1\}, \{A_2\}, \{A_3\}, \{A_1, A_2\}, \{A_1, A_3\}, \{A_2, A_3\}\}$

Now we define the following binary relations on $U$ as follows:

$R_{B \subseteq C} = \{(x, y) : f_{B \subseteq C}(x) \subseteq f_{B \subseteq C}(y), \forall B \subseteq C, B \neq \emptyset, \forall x, y \in U\}$

For this done, we construct the following relations on condition attributes:

$R_{\{A_1\}} = \{(u_1, u_1), (u_1, u_2), (u_1, u_7), (u_2, u_2), (u_3, u_3), (u_3, u_7), (u_4, u_4), (u_4, u_2), (u_4, u_5), (u_4, u_6), (u_5, u_5), (u_5, u_2), (u_5, u_4), (u_5, u_6), (u_6, u_6), (u_7, u_7)\}$

$R_{\{A_2\}} = \{(u_1, u_1), (u_1, u_7), (u_2, u_2), (u_2, u_1), (u_2, u_4), (u_2, u_6), (u_2, u_7), (u_3, u_1), (u_3, u_3), (u_3, u_7), (u_4, u_4), (u_4, u_5), (u_5, u_4), (u_5, u_5), (u_6, u_6), (u_6, u_1), (u_6, u_2), (u_6, u_4), (u_6, u_7), (u_7, u_1), (u_7, u_7)\}$

$R_{\{A_3\}} = \{(u_1, u_1), (u_1, u_2), (u_1, u_3), (u_1, u_5), (u_1, u_6), (u_1, u_7), (u_2, u_2), (u_2, u_5), (u_2, u_7), (u_4, u_2), (u_4, u_5), (u_4, u_7), (4,4), (u_3, u_1), (u_3, u_2), (u_3, u_3), (u_3, u_5), (u_3, u_6), (u_3, u_7), (u_6, u_1), (u_6, u_2), (u_6, u_3), (u_6, u_5), (u_6, u_6), (u_6, u_7), (u_5, u_2), (u_5, u_5), (u_5, u_7), (u_7, u_2), (u_7, u_5), (u_7, u_7)\}$
\[ R_{[A_1,A_2]} = \{(u_1, u_1), (u_2, u_2), (u_3, u_3), (u_3, u_7), (u_4, u_4), (u_5, u_4), (u_5, u_5), (u_6, u_6), (u_7, u_7)\} \]

\[ R_{[A_1,A_2]} = \{(u_1, u_1), (u_1, u_7), (u_2, u_2), (u_3, u_3), (u_3, u_7), (u_4, u_4), (u_4, u_8), (u_5, u_5), (u_5, u_9), (u_6, u_6), (u_7, u_7)\} \]

\[ R_{[A_2,A_1]} = \{(u_1, u_1), (u_1, u_7), (u_2, u_2), (u_3, u_3), (u_3, u_1), (u_3, u_7), (u_4, u_4), (u_5, u_5), (u_6, u_6), (u_7, u_7)\} \]

\[ R_{C} = \{(u_1, u_1), (u_1, u_7), (u_2, u_2), (u_3, u_3), (u_3, u_7), (u_4, u_4), (u_5, u_5), (u_4, u_6), (u_6, u_6), (u_7, u_7)\} \]

The right and left blocks of \(R_{[A_1]}\) are a subbase as follows:
\[ S_{[A_1]}-\text{R} = \{(u_1, u_2, u_7), [u_1], [u_3, u_7], [u_2, u_4, u_5, u_6], [u_6], [u_7]\} \]

\[ S_{[A_1]}-\text{L} = \{(u_1), [u_1], [u_2], [u_3], [u_4, u_5], [u_4, u_5, u_6], [u_1, u_3, u_7]\} \]

The right and left blocks of \(R_{[A_2]}\) are a subbase as follows:
\[ S_{[A_2]}-\text{R} = \{(u_1, u_7), [u_1, u_2, u_4, u_6, u_7], [u_1, u_3, u_7], [u_4, u_5]\} \]

\[ S_{[A_2]}-\text{L} = \{(u_1, u_2, u_3, u_6, u_7), [u_2, u_4, u_5, u_6], [u_3, u_7], [u_2, u_4, u_5, u_6], [u_5, u_7]\} \]

The right and left blocks of \(R_{[A_3]}\) are a subbase as follows:
\[ S_{[A_3]}-\text{R} = \{(u_1, u_2, u_3, u_5, u_6, u_7), [u_2, u_3, u_7], [u_2, u_4, u_5, u_7]\} \]

\[ S_{[A_3]}-\text{L} = \{(u_1, u_3, u_4), [u_1, u_6]\} \]

The right and left blocks of \(R_{[A_1,A_2]}\) are a subbase as follows:
\[ S_{[A_1,A_2]}-\text{R} = \{(u_1, u_2), [u_3, u_7], [u_4, u_5], [u_5, u_6], [u_3, u_7]\} \]

\[ S_{[A_1,A_2]}-\text{L} = \{(u_1), [u_1], [u_2], [u_3], [u_4, u_5], [u_4, u_5, u_6], [u_7]\} \]

The right and left blocks of \(R_{[A_1,A_3]}\) are a subbase as follows:
\[ R_{[A_1,A_3]}-\text{R} = \{(u_1, u_2, u_7), [u_2, u_4, u_5, u_6, u_7], [u_2, u_4, u_5, u_6], [u_2, u_5, u_6], [u_7]\} \]

\[ R_{[A_1,A_3]}-\text{L} = \{(u_1, u_2, u_4, u_5, [u_3], [u_4], [u_3, u_6], [u_1, u_3, u_7]\} \]

The right and left blocks of \(R_{[A_2,A_3]}\) are a subbase as follows:
\[ S_{[A_2,A_3]}-\text{R} = \{(u_1, u_7), [u_2, u_4, u_5, u_6, u_7], [u_1, u_2, u_4, u_5, u_6, u_7]\} \]

\[ S_{[A_2,A_3]}-\text{L} = \{(u_1, u_3, u_6), [u_2, u_6], [u_3, u_4], [u_4, u_5], [u_6], [u_1, u_2, u_3, u_6, u_7]\} \]

The right and left blocks of \(R_{C}\) are a subbase as follows:
\[ S_{C-\text{R}} = \{(u_1, u_7), [u_2, u_3, u_7], [u_4, u_5], [u_6], [u_7]\} \]

\[ S_{C-\text{L}} = \{(u_1), [u_2], [u_3], [u_4], [u_5], [u_6], [u_1, u_3, u_7]\} \]

The two topologies we need in the reduction process is given by:
\[ t_\pi = \{[u, v], [u_2], [u_4], [u_5], [u_6], [u_7], [u_1, u_7], [u_3, u_7], [u_2, u_4], [u_4, u_6], [u_4, u_7], [u_2, u_5], [u_2, u_6], [u_2, u_7], [u_4, u_5], [u_5, u_6], [u_5, u_7], [u_6, u_7], [u_3, u_4, u_7], [u_1, u_4, u_7], [u_1, u_2, u_7], [u_3, u_5, u_7], [u_1, u_5, u_7], [u_3, u_6, u_7], [u_1, u_6, u_7], [u_1, u_3, u_7], [u_2, u_3, u_5, u_7], [u_2, u_4, u_5, u_6], [u_2, u_4, u_7], [u_4, u_5, u_6], [u_4, u_5, u_7], [u_2, u_3, u_4, u_7], [u_1, u_2, u_4, u_7], [u_3, u_4, u_5, u_7], [u_1, u_4, u_5, u_7], [u_2, u_4, u_5, u_6, u_7], [u_2, u_3, u_4, u_5, u_6, u_7], [u_1, u_2, u_4, u_5, u_6, u_7]\} \]
Now these topologies are considered as the basic knowledge base for our system that we can generate decision rules from Table 2.
The discernible subsets of the decision attribute are:

\[ D_1 = \text{Decision(Yes)} = \{u1, u3, u6\}, \]
\[ D_2 = \text{Decision(No)} = \{u2, u4\}, \]
\[ D_3 = \text{Decision(Maybe)} = \{u5, u7\}. \]

Now we need to calculate \( \mathbb{S}_{\varphi}(D_i), i = 1, 2, 3 \) and \( \overline{\mathbb{S}}_{\varphi}(D_i), i = 1, 2, 3 \), and then we know if the above categories about decision are \( \delta_{\text{open}}(D_i = \mathbb{S}_{\varphi}(D_i)) \) or \( \delta_{\text{open}}(D_i = \overline{\mathbb{S}}_{\varphi}(D_i)). \)

\[ \delta_{\text{open}}(D_1) = \mathbb{S}_{\varphi}(D_1) = \cap \{G \in \tau_{\varphi} : D_1 \subseteq G\} = \{u1, u3, u6, u7\}, \text{ then } D_1 \text{ is not } \delta_{\text{open}} \text{ category.} \]
\[ \delta_{\text{open}}(D_1) = \mathbb{S}_{\varphi}(D_1) = \cap \{G \in \tau_{\varphi} : D_1 \subseteq G\} = \{u1, u3, u6\}, \text{ then } D_1 \text{ is } \delta_{\text{open}} \text{ category.} \]
\[ \delta_{\text{open}}(D_2) = \mathbb{S}_{\varphi}(D_2) = \cap \{G \in \tau_{\varphi} : D_2 \subseteq G\} = \{u2, u4\}, \text{ then } D_2 \text{ is } \delta_{\text{open}} \text{ category.} \]
\[ \delta_{\text{open}}(D_3) = \mathbb{S}_{\varphi}(D_3) = \cap \{G \in \tau_{\varphi} : D_3 \subseteq G\} = \{u2, u4\}, \text{ then } D_3 \text{ is } \delta_{\text{open}} \text{ category.} \]
\[ \delta_{\text{open}}(D_1) = \mathbb{S}_{\varphi}(D_1) = \cap \{G \in \tau_{\varphi} : D_1 \subseteq G\} = \{u1, u3, u6, u7\}, \text{ then } D_1 \text{ is } \delta_{\text{open}} \text{ category.} \]
\[ \delta_{\text{open}}(D_2) = \mathbb{S}_{\varphi}(D_2) = \cap \{G \in \tau_{\varphi} : D_2 \subseteq G\} = \{u2, u4\}, \text{ then } D_2 \text{ is } \delta_{\text{open}} \text{ category.} \]
\[ \delta_{\text{open}}(D_3) = \mathbb{S}_{\varphi}(D_3) = \cap \{G \in \tau_{\varphi} : D_3 \subseteq G\} = \{u2, u4\}, \text{ then } D_3 \text{ is } \delta_{\text{open}} \text{ category.} \]

We notice that \( \delta_{\text{open}}(D_1) \cap \delta_{\text{open}}(D_2) = D_1 \cap \delta_{\text{open}}(D_2) = \delta_{\text{open}}(D_3) \cap \delta_{\text{open}}(D_3) = \delta_{\text{open}}(D_3) \), hence all decision categories are \( \delta_{\text{open}} \) open.

If we tried to separate among these decision categories using \( \delta_{\text{open}} \) approach, then we need to calculate \( \delta_{\text{open}}(\text{int}_{\varphi}(\delta_{\text{open}}(D_i))), i = 1, 2, 3. \)

\[ \delta_{\text{open}}(D_1) = \delta_{\text{open}}(\text{int}_{\varphi}(\delta_{\text{open}}(D_1))) = \cup \{\delta_{\text{open}}(D_1) : \delta_{\text{open}}(D_1) \subseteq \text{int}_{\varphi}(\delta_{\text{open}}(D_1))\} = \{u1, u3, u5, u6, u7\}, \]
\[ \delta_{\text{open}}(D_2) = \delta_{\text{open}}(\text{int}_{\varphi}(\delta_{\text{open}}(D_2))) = \cup \{\delta_{\text{open}}(D_2) : \delta_{\text{open}}(D_2) \subseteq \text{int}_{\varphi}(\delta_{\text{open}}(D_2))\} = \{u2, u4\}, \]
\[ \delta_{\text{open}}(D_3) = \delta_{\text{open}}(\text{int}_{\varphi}(\delta_{\text{open}}(D_3))) = \cup \{\delta_{\text{open}}(D_3) : \delta_{\text{open}}(D_3) \subseteq \text{int}_{\varphi}(\delta_{\text{open}}(D_3))\} = \{u3, u5, u6, u7\}. \]

Now we notice that \( \delta_{\text{open}}(D_1) \cap \delta_{\text{open}}(D_2) = D_1 \cap \delta_{\text{open}}(D_2) = \delta_{\text{open}}(D_3) \cap \delta_{\text{open}}(D_3) = \delta_{\text{open}}(D_3) \), \( \forall i = 1, 2, 3 \), then \( D_i, \forall i = 1, 2, 3 \) are \( \delta_{\text{open}} \) open. After many calculations using the topologies \( \tau_{\varphi} \text{ and } \tau_{\delta_{\text{open}}} \), the Lower approximation and Rupper approximation of the subset \( D_i, i = 1, 2, 3 \) are given by:

\[ \text{BiT}_{\varphi}(D_1) = \{u1, u6\}, \quad \text{BiT}_{\delta_{\text{open}}}(D_1) = \{u1, u3, u6, u7\}, \text{ then } \text{BON}_{\delta_{\text{open}}}(D_1) = \{u3, u7\}. \]
\[ \text{BiT}_{\varphi}(D_2) = \{u2, u4\}, \quad \text{BiT}_{\delta_{\text{open}}}(D_2) = \{u2, u4\}, \text{ then } \text{BON}_{\delta_{\text{open}}}(D_2) = \emptyset. \]
\[ \text{BiT}_{\varphi}(D_3) = \{u5, u7\}, \quad \text{BiT}_{\delta_{\text{open}}}(D_3) = \{u1, u3, u5, u6, u7\}, \text{ then } \text{BON}_{\delta_{\text{open}}}(D_3) = \{u1, u3, u6\}. \]

Also, the Lower approximation and Upper approximation of the subset \( D_i, i = 1, 2, 3 \) are given by:

\[ \text{BiT}_{\varphi}(D_1) = \{u1, u3, u6\}, \quad \text{BiT}_{\delta_{\text{open}}}(D_1) = \{u1, u3, u6, u7\}, \text{ then } \text{BON}_{\delta_{\text{open}}}(D_1) = \{u7\}. \]
\[ \text{BiT}_{\varphi}(D_2) = \{u2, u4\}, \quad \text{BiT}_{\delta_{\text{open}}}(D_2) = \{u2, u4\}, \text{ then } \text{BON}_{\delta_{\text{open}}}(D_2) = \emptyset. \]
The conclusions about the accurate approach (exactly(δβ) approach) with respect to the information given in Table 2 are given as the following:

1. The decision value “Yes” is not exactly such that {u7} is in the boundary region.
2. The decision value “No” is exactly 100% such that its boundary region is empty.
3. The decision value “Maybe” is not exactly such that {u7} is in its boundary region.

5. Conclusion

We generalized Pawlak approximation space by family of binary relations to bitopological approximation space. Using Rblocks and Lblocks of these relations we generated two topological spaces and using them we defined the lower and upper approximations of any subset in the universe. We generalized the rough approximations of any subset to three regions namely, R region, L region and (δβ)-region and used these regions for reduction of information systems.

The same category in a decision system can distinguishable by three accuracy measures \( a_\delta(D), a_L(D) \) and \( a_{(\delta\beta)}(D) \). The applications are proved that the measure \( a_{(\delta\beta)}(D) \) is the best than other measures.

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