Weak and Semi-Contraction Theory with Application to Network Systems
Saber Jafarpour, Member, IEEE, Pedro Cisneros-Velarde, Student Member, IEEE, and Francesco Bullo, Fellow, IEEE

Abstract—We develop two generalizations of contraction theory: semi-contraction and weak-contraction theory. First, using the notion of semi-norm, we propose a geometric framework for semi-contraction theory. We introduce matrix semi-measures, characterize their properties, and provide techniques to compute them. We also propose two optimization problems for the spectral abscissa of a given matrix using its semi-measures. For dynamical systems, we use the semi-measure of their Jacobian to study convergence of their trajectories to invariant subspaces. Second, we provide a comprehensive treatment of weakly-contracting systems; we prove a dichotomy for asymptotic behavior of their trajectories and show that, for their equilibrium points, local asymptotic stability implies global asymptotic stability. Third, we introduce the class of doubly-contracting systems and show that every trajectory of a doubly-contracting system converges to an equilibrium point. Finally, we apply our results to various important network systems including affine averaging and affine flow systems, continuous-time distributed primal-dual algorithms, and networks of diffusively-coupled oscillators.

Index Terms—contraction theory, stability analysis, synchronization

I. INTRODUCTION

Problem description: Strict contractivity is a useful and classical property of dynamical systems, which ensures global exponential stability of a unique equilibrium. However, numerous example applications fail to satisfy this property and exhibit rich dynamic properties. In this paper, motivated by applications in network systems, we study systems that satisfy relaxed versions of the standard contractivity conditions. We characterize the implications of these relaxed conditions on the asymptotic behavior of the dynamical system. We aim to develop a generalized contractivity theory that can explain the rich behavior of some classic example systems, including affine averaging and flow systems, distributed primal-dual dynamics, and networks of diffusively-coupled oscillators.

Literature review: Studying contractivity of dynamical systems using matrix measures has a long history that can be traced back to Lewis [20] and Demidović [12]. In the control community, contraction theory was introduced by Lohmiller and Slotine [22]. We refer to [31] for a historical review and to the surveys [1], [13] for recent developments and applications to consensus and synchronization in complex networks.

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Several generalizations of contraction theory have been proposed in the literature. In [37], the notion of partial contraction is introduced to study convergence of system trajectories to a specific behavior or to a manifold. The idea is to impose contractivity only on a part of the states of the system. Partial contraction with respect to the $\ell_2$-norm has been further developed in [41], [32] to study synchronization in complex networks. A similar notion of partial contraction is studied by [13] in the context of convergence to invariant subspaces. Generalizations of contraction theory based on matrix measures have also been studied in the literature. In [38], it is shown that if the sum of the two largest eigenvalues of the symmetric part of the Jacobian of the system is negative, then every bounded trajectory of the system converges to an equilibrium point. In [19], a relaxed contraction property is defined based on the sign of the truncated trace of the symmetric part of the Jacobian.

Extensions of contraction theory to non-Euclidean norms have been explored in the context of monotone dynamical systems. For compartmental systems, contractivity with respect to the $\ell_1$-norm has been used to study convergence to the equilibrium point [23]. The connection between monotonicity and contractivity of dynamical systems has been studied in detail in [9], [16]. Other extensions of contraction theory include contraction theory on Finsler manifolds [15], contraction theory on Riemannian manifolds [36], transverse contraction for convergence to limit cycles [26], and contraction after transient [27].

Synchronization of diffusively-coupled oscillators has been extensively studied using contraction theory, e.g., see [42], [41], [11], [2]. Indeed the notion of partial contraction was developed in [41] precisely to study this class of dynamical systems. The early work [42] introduces what is now known as the QUAD condition for studying synchronization of coupled oscillators. In [24], local and global synchronization of linearly-coupled oscillators over directed graphs have been analyzed using the QUAD condition. While explaining the relationship between QUAD condition and contractivity of vector fields, [11] also studies diffusively-coupled identical nonlinear oscillators on undirected graphs. [3] proposes an $\ell_2$-norm condition for synchronization of diffusively-coupled oscillators with time-invariant interconnections. For diffusively-coupled networks with time-varying interconnections, synchronization has been studied using non-Euclidean matrix measures in [6], [2]. In [34], the synchronization of complex networks is studied using a contraction-based hierarchical approach via mixed norms.
Primal-dual algorithms for optimization problems have been widely studied and adopted in several applications; we refer to the survey [44] for a comprehensive review. In the last decade, there has been a renewed interest in convergence analysis [14] focusing on the convergence rate in centralized optimizations [33] and convergence guarantees in distributed optimizations [40]. Recently, contraction theory has been used to study convergence and robustness of continuous-time primal-dual algorithms [30], [8].

Contribution: In this paper, we develop two generalizations of contraction theory that broaden its range of applicability and allow for richer and more complex dynamic behaviors.

The first generalization, called semi-contractivity, allows the distance between trajectories to increase in certain directions, thus requiring the dynamical system to be contractive only on a certain subspace. Using the notion of semi-norm, we present a geometric framework for studying semi-contracting systems. We introduce the semi-measure of a matrix, which is associated with a semi-norm, and we study the linear algebra of semi-measures. We provide two optimization problems that establish a connection between semi-measures of a matrix and its spectral abscissa on certain subspaces. These optimization problems can be considered as extensions of [21, Proposition 2.3] to semi-norms and semi-measures, and they play a crucial role in studying the convergence rate of trajectories in our framework. Finally, we study the solutions of dynamical systems using the semi-measure of their Jacobian. We prove a generalization of the well-known Coppel’s inequality [10], which provides upper and lower bounds on the semi-norm of flows of linear time-varying systems. For a given semi-norm, we introduce the notion of semi-contracting dynamical systems with respect to the semi-norm. We prove that if a time-varying system is semi-contracting with respect to a semi-norm, then every trajectory converges to the kernel of the semi-norm.

The notion of semi-contractivity is closely related to the notion of partial contraction, as originally proposed in [37], elaborated in [41], [32], and surveyed in [13]. The framework proposed in this paper provides a geometric approach to contraction theory by adopting semi-norms – indeed the notion of semi-contracting system is independent under linear coordinate changes. Moreover, our approach is more general than that proposed by [41], whereby partial contraction is restricted to the $\ell_2$-norm, and that by [13], whereby partial contraction is characterized using an orthonormal set of basis.

The second generalization, called weak-contractivity, allows the distance between trajectories to be non-increasing. Our first result is a dichotomy for qualitative behavior of weakly-contracting systems; a trajectory of a weakly-contracting system is bounded if and only if the system has an equilibrium point. One interesting feature of this dichotomy is the connection between the topological properties of the dynamical system and the algebraic properties of its vector field. Moreover, we show that if a weakly-contracting system has a locally asymptotically stable equilibrium point, then this equilibrium point is globally asymptotically stable. This result can be considered as the generalization of [25, Theorem 2 and Theorem 3] and [23, Lemma 6] to weakly-contracting systems. Finally, for systems that are weakly-contracting with respect to a weighted $\ell_1$-norm or a weighted $\ell_\infty$-norm, we show that the dichotomy can be more elaborate; every trajectory converges to an equilibrium point if and only if the dynamical system has an equilibrium point. We conclude by providing an example which shows that, for weakly-contracting systems with respect to an arbitrary $\ell_p$-norm, a bounded trajectory does not necessarily converge to an equilibrium point.

In contrast to contractive systems whose trajectories converge to their globally stable equilibrium points, trajectories of semi-contracting or weakly-contracting system do not necessarily exhibit a unique asymptotic behavior. We illustrate this fact using examples of semi- and weakly-contracting systems. We introduce the class of doubly-contracting systems, i.e., systems which are both semi-contracting and weakly-contracting (possibly with respect to two different semi-norms). We show that, for a doubly-contracting system whose equilibrium points form a subspace of $\mathbb{R}^n$, every trajectory converges exponentially to an equilibrium point. Moreover, we provide a convergence rate that is independent of the norm and the semi-norm and depends only on the limiting equilibrium point.

We provide several applications of our results to network systems: affine averaging and flow systems, continuous-time primal-dual dynamics, and networks of coupled oscillators. First we show that affine averaging (affine flow) systems are weakly-contracting with respect to the $\ell_\infty$-norm ($\ell_1$-norm). Using our dichotomy result for weakly-contracting systems, we show that trajectories of both affine averaging and affine flow systems converge to an equilibrium point if and only if the system has an equilibrium point. We also show that both affine averaging and affine flow systems are doubly-contracting, thus recovering the exact rate of convergence of trajectories to the equilibrium points.

Second, we study a distributed implementation of the continuous-time primal-dual algorithm for optimizing a function over a connected network. We show that the resulting dynamical system is weakly-contracting and every trajectory of the system converges to an equilibrium point. As recently stated in [8], compared to the analysis in [33], [30], we show that the distributed primal-dual algorithm is weakly-contracting with respect to the $\ell_2$-norm and prove its global convergence to the solution of the optimization problem when the cost function is weakly convex.

Finally, for networks of diffusively-coupled identical oscillators, we use our semi-contraction framework to propose novel sufficient conditions for global synchronization. A key step in obtaining these synchronization conditions is the introduction of a new class of norms called $(2,p)$-tensor norms, with various useful properties. Compared to the contraction-based approaches using the $\ell_2$-norm [42], [41], [11], our synchronization conditions (i) are based on the weighted $\ell_p$-matrix measure of the Jacobian of oscillator dynamics for every $p \in [1, \infty]$, (ii) provide an explicit rate of convergence to the synchronized trajectories, and (iii) can recover the exact threshold of synchronization for diffusively-coupled linear oscillators. Compared to the results in [6], [2], our synchronization conditions (i) are applicable to arbitrary undirected
network topology, (ii) allow for a arbitrary class of weighted p-norms, for every p ∈ [1, ∞], and (iii) demarcate the roles of internal dynamics and the network connectivity.

**Paper Organization:** Section II introduces the notation. Section III introduces the matrix semi-measures and use them to study the semi-contracting systems. Sections IV and V study weakly-contracting and doubly-contracting systems, respectively. Finally, Section VI analyzes three applications of our semi-and weak-contraction theory to network systems.

II. NOTATION

For a set $S \subseteq \mathbb{R}^n$, its interior, closure, and diameter are denoted by $\text{int}(S)$, $\text{cl}(S)$, and $\text{diam}(S)$, respectively. The $n \times n$ identity matrix is $I_n$ and the all-ones and all-zeros column vectors of length $n$ are $\mathbf{1}_n$ and $\mathbf{0}_n$, respectively. For $A \in \mathbb{C}^{n \times m}$, the conjugate transpose of $A$ is $A^H$, the real part of $A$ is $\Re(A)$, the range of $A$ is $\text{Im}(A)$ and the kernel of $A$ is $\text{Ker}(A)$. The Moore–Penrose inverse of $A$ is the unique matrix $A^+ \in \mathbb{C}^{m \times n}$ such that $AA^+A = A$, $A^+A A^+ = A^+$ and $AA^+$ and $A^+A$ are Hermitian matrices. It can be shown that $AA^+$ is the orthogonal projection onto $\text{Im}(A)$ and $A^+A$ is the orthogonal projection onto $\text{Im}(A^H)$. Let $\lambda_1(A), \ldots, \lambda_n(A)$ and $\text{spec}(A)$ denote the eigenvalues and the spectrum of $A \in \mathbb{C}^{n \times n}$. Given a vector subspace $S \subseteq \mathbb{C}^n$, its orthogonal complement is $S^\perp$ and we define $AS := \{Au \mid u \in S\}$. The vector subspace $S \subseteq \mathbb{C}^n$ is invariant under $A \in \mathbb{C}^{n \times n}$ if $AS \subseteq S$. Given $A \in \mathbb{C}^{n \times n}$ and a vector subspace $S \subseteq \mathbb{C}^n$, $S$ is invariant under $A$ if and only if $S^\perp$ is invariant under $A^H$. Given $A \in \mathbb{C}^{n \times n}$ and a vector subspace $S \subseteq \mathbb{C}^n$ invariant under $A$, define

$$\text{spec}_S(A) := \{\lambda \in \text{spec}(A) \mid \exists v \in S \text{ s.t. } Av = \lambda v\}.$$ 

Note $\text{spec}(A) = \text{spec}_S(A) \cup \text{spec}_{S^\perp}(A^H)$. Define the spectral abscissa of $A$ restricted to $S$ by:

$$\alpha_S(A) := \max\{\Re(\lambda) \mid \lambda \in \text{spec}_S(A)\}.$$ 

We assume that $(t, x) \mapsto f(t, x)$ is twice-differentiable in $x$ and essentially bounded in $t$. Denote the flow of $(1)$ starting from $x_0$ by $t \mapsto \phi(t, x_0)$. A set $S \subseteq \mathbb{R}^n$ is invariant with respect to $(1)$ if $\phi(t, S) \subseteq S$ for every $t \in \mathbb{R}_{\geq 0}$. A vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ is piecewise real analytic if there exist closed sets $\{\Sigma_i\}_{i=1}^m$ which partition $\mathbb{R}^n$ and $X$ is real analytic on $\text{int}(\Sigma_i)$, for every $i \in \{1, \ldots, m\}$.

III. SEMI-CONTRACTING SYSTEMS

In this section we generalize the contracting theory by providing a geometric framework for semi-contracting systems via semi-norms.

A. Linear algebra and matrix semi-measures

We start with semi-norms and associated semi-measures.

**Definition 1** (Semi-norms). A function $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ is a semi-norm on $\mathbb{R}^n$, if it satisfies the following properties:

(i) $\|cv\| = |c|\|v\|$, for every $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$;

(ii) $\|v + w\| \leq \|v\| + \|w\|$, for every $v, w \in \mathbb{R}^n$.

For a semi-norm $\|\cdot\|$, its kernel is defined by

$$\text{Ker}(\|\cdot\|) = \{v \in \mathbb{R}^n \mid \|v\| = 0\}.$$ 

It is easy to see that $\text{Ker}(\|\cdot\|)$ is a subspace of $\mathbb{R}^n$ and $\|\cdot\|$ is a norm on the vector space $\text{Ker}(\|\cdot\|)^\perp$. Semi-norms can naturally arise from norms. Let $k \leq n$, and $\|\cdot\| : \mathbb{R}^k \to [0, \infty)$ be a norm on $\mathbb{R}^k$ and $R \in \mathbb{R}^{k \times n}$. Then the $R$-weighted semi-norm on $\mathbb{R}^n$ associated with the norm $\|\cdot\|_R$ on $\mathbb{R}^k$ is defined by

$$\|v\|_R = \|Rv\|,$$ 

for all $v \in \mathbb{R}^n$. (2)

It is easy to see that the $R$-weighted semi-norm $\|\cdot\|_R$ is a norm if and only if $k = n$ and $R$ is invertible.

A semi-norm $\|\cdot\|$ on $\mathbb{R}^n$ naturally induces a semi-norm on the space of real-valued matrices $\mathbb{R}^{n \times n}$.

**Definition 2** (Induced semi-norm). Let $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ be a semi-norm on $\mathbb{R}^n$, the induced semi-norm on $\mathbb{R}^{n \times n}$ (which without any confusion, we denote again by $\|\cdot\|$) is defined by

$$\|A\| = \sup\{\|Av\| \mid v \in \mathbb{R}^n, \|v\| = 1\}, \text{ for all } v \in \mathbb{R}^n.$$ 

The following properties of induced semi-norms are known [17] and we omit the proof in the interest of brevity.

**Proposition 3** (Properties of induced semi-norms). Let $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ be a semi-norm on $\mathbb{R}^n$ and assume that we denote the induced semi-norm on $\mathbb{R}^{n \times n}$ again by $\|\cdot\|$. Then, for every $A, B \in \mathbb{R}^{n \times n}$ and $c \in \mathbb{R}$,

(i) $\|I_n\| = 1$, $\|A\| \geq 0$, and $\|cA\| = |c|\|A\|$;

(ii) $\|A + B\| \leq \|A\| + \|B\|$;

(iii) $\|Av\| \leq \|A\|_R \|v\|$, for every $v \in \mathbb{R}^n$.

**Definition 4** (Matrix semi-measures). Let $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ be a semi-norm on $\mathbb{R}^n$ and we denote the induced semi-norm on $\mathbb{R}^{n \times n}$ again by $\|\cdot\|$. Then the matrix semi-measure associated with $\|\cdot\|$ is defined by

$$\mu_{\|\cdot\|}(A) = \lim_{h \to 0^+} \frac{\|I_n + hA\| - 1}{h}.$$ 

(3)
Theorem 5 (Properties of matrix semi-measures). Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a semi-norm on $\mathbb{R}^n$ and let $\mu_{\|\cdot\|}$ be the associated matrix semi-measure. Then, for every $A, B \in \mathbb{R}^{n \times n}$, the followings hold:

(i) $\mu_{\|\cdot\|}(A)$ is well-defined;
(ii) $\mu_{\|\cdot\|}(A + B) \leq \mu_{\|\cdot\|}(A) + \mu_{\|\cdot\|}(B)$;
(iii) $|\mu_{\|\cdot\|}(A) - \mu_{\|\cdot\|}(B)| \leq \|A - B\|$.

Moreover, if $\text{Ker } \|\cdot\|$ is invariant under $\Lambda$, then

(iv) $\alpha_{\text{Ker } \|\cdot\|}((A^T)^T) \leq \mu_{\|\cdot\|}(A)$.

Proof. Regarding part (i), define the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by $f(h) := \frac{\|f + hA\|_1}{h}$. It suffices to show that the limit

$$\lim_{h \to 0^+} f(h) \text{ exists.}$$

First note that for every $h \in \mathbb{R}_{\geq 0}$, we have $\|I_n + hA\|_1 \geq \|I_n - hA\|_1 = 1 - h\|A\|_1$. This implies that $f(h) \geq -h\|A\|_1$ for every $h \in \mathbb{R}_{\geq 0}$. Thus $f$ is bounded below. Moreover, for every $0 < h < h_2$, we have

$$\|(1/h_1)I_n + A\|_1 - \|(1/h_2)I_n + A\|_1 \leq \|(1/h_1 - 1/h_2)I_n\|_1 = 1/h_1 - 1/h_2.$$

As a result, we get

$$f(h_1) = \|(1/h_1)I_n + A\|_1 - (1/h_1) \leq \|(1/h_2)I_n + A\|_1 - (1/h_2) = f(h_2).$$

Therefore $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a strictly increasing function which is bounded below. Thus, $\lim_{h \to 0^+} f(h)$ exists. Regarding part (ii), the result is straightforward using the triangle inequality (Theorem 3(iii)) for the induced semi-norm. Regarding part (iii), the proof is straightforward using the triangle inequality proved in part (ii). Finally, regarding part (iv), suppose that $\|\cdot\|_{\Lambda}$ is the complexification of $\|\cdot\|$ and $\Lambda \in \mathbb{C}$ is an eigenvalue of $A^T$ with the right eigenvector $v \in \text{Ker } \|\cdot\|_{\Lambda} \subset \mathbb{C}^n$ normalized such that $\|\cdot\|_{\Lambda} = 1$. Then we have

$$\mathbb{R}(\lambda) = \lim_{h \to 0^+} \frac{\|v + h\Lambda v\|_{\Lambda} - 1}{h} = \lim_{h \to 0^+} \frac{\|v + hA^T v\|_{\Lambda} - 1}{h} \leq \lim_{h \to 0^+} \frac{\|I_n + hA^T\|_{\Lambda} - 1}{h} = \mu_{\|\cdot\|}(A^T).$$

Given the $R$-weighted semi-norm $\|\cdot\|_R$ as defined in (2), we denote its induced semi-norm on $\mathbb{R}^{n \times n}$ by $\|\cdot\|_R$ and its matrix semi-measure by $\mu_R$. Specifically, for the $\ell_p$-norm on $\mathbb{R}^k$, we let $\|\cdot\|_{p,R}$ and $\mu_{p,R}$ denote the associated $R$-weighted semi-norm on $\mathbb{R}^n$ and matrix semi-measure, respectively.

Lemma 6 (Computation of semi-measures). Let $\|\cdot\|$ be a norm with the associated matrix measure $\mu$ and let $k \leq n$ and $R \in \mathbb{R}^{k \times n}$ be full rank and $\xi \in \mathbb{R}^n_{\geq 0}$ then, for every $A \in \mathbb{R}^{n \times n}$,

(i) $\|A\|_R = \|RAR^T\|_R$,
(ii) $\mu_R(A) = \mu(RAR^T)$,
(iii) $\mu_{\text{diag}(\xi)}(A) = \max_{j: \xi_j \neq 0} \left\{a_{jj} + \xi_j \sum_{i: \xi_i \neq 0} \frac{|a_{ij}|}{\xi_i} \right\}$,
(iv) $\mu_{\infty,\text{diag}(\xi)}(A) = \max_{i: \xi_i \neq 0} \left\{a_{ii} + \xi_i \sum_{j: \xi_j \neq 0} \frac{|a_{ij}|}{\xi_j} \right\}$.

Moreover, if $\text{Ker } R$ is invariant under $\Lambda$, then

(iv) $\mu_{2,2}(A) = \frac{1}{2} \alpha_{\text{Ker } R^T} (A + P^T A^T P)$, with $P = R^T R$.

Proof. First note that $\text{Ker } \|\cdot\|_R = \{v \in \mathbb{R}^n \mid \|v\|_R = 0\}$. Moreover, we have $\|v\|_R = \|Rv\|$ and since $\|\cdot\|$ is a norm, we get $\|v\|_R = 0$ if and only if $Rv = 0_k$. This implies that $\text{Ker } \|\cdot\|_R = \text{Ker } R$. Regarding part (i), by definition of the induced-norm we have

$$\|A\|_R = \{\|RAR^T v\|_1 \mid \|v\|_1 = 1, v \perp \text{Ker } R\}.$$

Note that $R^T R$ is an orthogonal projection and $\text{Ker } (R^T R) = \text{Ker } (R)$. Thus, for every $v \perp \text{Ker } (R)$, there exists $x \in \mathbb{R}^n$ such that $v = R^T x$. As a result,

$$\|A\|_R = \{\|RAR^T R x\|_1 \mid \|RAR^T R x\|_1 = 1\} = \{\|RAR^T x\|_1 \mid \|x\|_1 = 1\} = \{\|RAR^T\|_R \mid \|x\|_1 = 1\},$$

where for the second equality, we used the fact that $RR^T R = R$. Since $R$ is full rank, for every $y \in \mathbb{R}^k$, there exists $x \in \mathbb{R}^n$ such that $y = Rx$. This means that

$$\|A\|_R = \sup\{\|RAR^T x\|_1 \mid \|x\|_1 = 1\} = \sup\{\|RAR^T y\|_1 \mid \|y\|_1 = 1\} = \|RAR^T\|_R.$$

Regarding part (ii), note that

$$\mu_R(A) = \lim_{h \to 0^+} \frac{\|I_n + hA\|_R - 1}{h} = \lim_{h \to 0^+} \frac{\|I_n + hRAR^T\|_R - 1}{h} = \mu(RAR^T),$$

where in the third equality, we used the fact that $R$ has full row rank and therefore $RR^T = I_k$. Regarding part (iii), define $\xi \in \mathbb{R}^{n \geq 0}$ by $\xi_i = \xi_i^{-1}$ if $\xi_i > 0$ and $\xi_i = 0$ if $\xi_i = 0$. Assume that $\xi$ has $r \leq n$ non-zero entries and define $Q_\xi \in \mathbb{R}^{x \times n}$ ($Q_\xi \in \mathbb{R}^{x \times n}$) as the matrix whose ith row is the ith non-zero row of $\text{diag}(\xi)$ (diag($\xi$)). Then it is easy to see that $Q_\xi$ is full rank and $Q_\xi^T = Q_\xi$. Note that $\|\cdot\|_{1,\text{diag}(\xi)} = \|\text{diag}((\xi)v)\|_1 = \|Q_\xi v\|_1$, for every $v \in \mathbb{R}^n$. Therefore, by part (ii),

$$\mu_{1,\text{diag}(\xi)}(A) = \mu_1(Q_\xi AQ_\xi^T) = \mu_1(Q_\xi AQ_\xi^T).$$

The result follows using the formula for $\ell_1$-norm matrix measure. The proof of part (iv) is similar to (iii) and we omit it. Regarding part (iv), note that by part (ii), we have $\mu_{2,2}(A) = \mu_2(RAR^T)$. Moreover, using the formula for the $\ell_2$-norm matrix measure,

$$\mu_2(RAR^T) = \frac{1}{2} \\text{max}\{\lambda \mid \lambda \in \text{spec}(RAR^T + (R^T)^T A^T R)\}.$$ 

Note that $\text{Ker } R$ is invariant under $A$. Therefore, using Lemma 30, we get $\text{spec}_{\text{Ker } R^T}(A^T) = \text{spec}(RAR^T + (R^T)^T A^T R)$. Noting the fact that $R$ is full rank and $RR^T = I_k$ and $R^T R = (R^T)^T = (R^T)^T$ and $PR^T = R^T$, we get

$$\mu_2(RAR^T) = \frac{1}{2} \\text{max}\{\lambda \mid \lambda \in \text{spec}_{\text{Ker } R^T}(A + P^T A^T P)\}.$$

B. Spectral abscessa as an optimal matrix measure

Theorem 5(iii) shows that the semi-measures of a matrix is lower bounded by its spectral abscessa. In the next theorem we study this gap and we show that on the space of all semi-measures this lower bound is tight. In this part, we use the generalization of the results in Section III-A to $\mathbb{C}^n$. Specifically, we use Lemma 6(i) and (ii) and and Theorem 5(iv) for matrices, norms, and matrix measures defined on $\mathbb{C}^n$. 
Theorem 7 (Optimal matrix measures and spectral abscissa). Let $A \in \mathbb{C}^{n \times n}$ and let $S \subseteq \mathbb{C}^n$ be a $(n-k)$-dimensional subspace which is invariant under $A$. Then

(i) $\alpha_{S^\perp}(A^H) = \inf\{\mu_{\|\cdot\|}(A) \mid \|\cdot\| \text{ a semi-norm with its associated matrix measure } \mu\}$

(ii) let $\|\cdot\|$ be an absolute norm with its associated matrix measure $\mu$, then $\alpha_{S^\perp}(A^H) = \inf\{\mu_R(A) \mid R \in \mathbb{C}^{k \times n}, \operatorname{Ker}(R) = S\}$.

Proof. First note that, by Theorem 5(iv), for every semi-norm $\|\cdot\|$ with $\operatorname{Ker}(\|\cdot\|) = S$, we have $\alpha_{S^\perp}(A^H) \leq \mu_{\|\cdot\|}(A)$. This implies that

$$\alpha_{S^\perp}(A^H) \leq \inf\{\mu_{\|\cdot\|}(A) \mid |||\cdot||| \text{ a semi-norm} \}, \operatorname{Ker}(|||\cdot|||) = S\}

\leq \inf\{\mu_R(A) \mid R \in \mathbb{C}^{k \times n}, \operatorname{Ker}(R) = S\}. \quad (4)$$

Therefore, in order to prove statements (i) and (ii), we need to show that

$$\alpha_{S^\perp}(A^H) \geq \inf\{\mu_R(A) \mid R \in \mathbb{C}^{k \times n}, \operatorname{Ker}(R) = S\}

\geq \inf\{\mu_{\|\cdot\|}(A) \mid |||\cdot||| \text{ a semi-norm} \}, \operatorname{Ker}(|||\cdot|||) = S\}. \quad (5)$$

We start by proving statement (ii). Consider the case that $A$ is diagonalizable. Suppose that $\operatorname{spec}_{S^\perp}(A^H) = \{\lambda_1, \ldots, \lambda_k\}$. Since $A$ is diagonalizable, there exists a set of linearly independent vectors $\{v_1, \ldots, v_k\} \subseteq S^\perp$ such that $A^H v_i = \lambda_i v_i$. Define the matrix $R \in \mathbb{C}^{k \times n}$, where $R_i$ (the $i$th row of matrix $R$) is equal to $v_i^H$, for every $i \in \{1, \ldots, k\}$. Note that $\{v_1, \ldots, v_k\}$ is linearly independent and therefore $R$ is full rank. Moreover, it is easy to see that $\operatorname{Ker} R = S$. On the other hand, we have $RA = \Lambda R$, where $\Lambda = \operatorname{diag}(\lambda_1^H, \ldots, \lambda_k^H) \in \mathbb{C}^{k \times k}$. This implies that $RAR^T = \Lambda$. As a result, we have $\mu_R(A) = \mu(RAR^T) = \mu(\Lambda)$ and therefore

$$\mu_R(A) = \mu(\Lambda) = \lim_{h \to 0^+} \frac{||K_h + h\Lambda|| - 1}{h} = \max_{i \in \{1, \ldots, k\}} |\Re(\lambda_i)| = \alpha_{S^\perp}(A^H),$$

where the third equality holds because $\|\cdot\|$ is absolute and so $||I_n + h\Lambda|| = \max_{i \in \{1, \ldots, k\}} (1 + |1 + \lambda_i|)$. Thus $\alpha_{S^\perp}(A^H) \geq \inf\{\mu_R(A) \mid R \in \mathbb{C}^{k \times n}, \operatorname{Ker}(R) = S\}$. Now, consider the case when $A$ is not diagonalizable. Note that, in the complex field $\mathbb{C}$, the set of diagonalizable matrices are dense in $\mathbb{C}^{n \times n}$. Therefore, for every $\varepsilon > 0$, there exists a diagonalizable $A_\varepsilon$ such that $\|A - A_\varepsilon\| \leq \varepsilon$. Theorem 5(iii) implies $|\mu_R(A) - \mu_R(A_\varepsilon)| \leq \varepsilon\|R\|\|R^T\|$. As a result

$$\mu_R(A) \leq \mu_R(A_\varepsilon) + \varepsilon\|R\|\|R^T\|.$$ 

By taking the infimum over the set of $R \in \mathbb{C}^{k \times n}$ with $\operatorname{Ker}(R) = S$ and noting the fact that $\sup\{\|R\|\|R^T\| \mid R \in \mathbb{C}^{k \times n}, \operatorname{Ker}(R) = S\} \leq M$, for some $M \in \mathbb{R}_{\geq 0}$, we get

$$\inf\{\mu_R(A) \mid R \in \mathbb{C}^{k \times n}, \operatorname{Ker}(R) = S\} \leq \inf\{\mu_R(A_\varepsilon) \mid R \in \mathbb{C}^{k \times n}, \operatorname{Ker}(R) = S\} + M\varepsilon = \alpha_{S^\perp}(A^H) + M\varepsilon,$$

where the last equality holds because $A_\varepsilon$ is diagonalizable. By continuity of eigenvalues, we get $\lim_{\varepsilon \to 0^+} \alpha_{S^\perp}(A^H) = \alpha_{S^\perp}(A^H)$. This implies that

$$\inf\{\mu_R(A) \mid R \in \mathbb{C}^{k \times n}, \operatorname{Ker}(R) = S\} \leq \alpha_{S^\perp}(A^H).$$

This completes the proof of statement (ii). Statement (i) is a consequence of (4) and statement (ii).

Remark 8. (i) Statement (i) is a generalization to semi-norms and semi-measures of [21, Proposition 2.3], which states $\alpha(A) = \inf\{\mu_{\|\cdot\|}(A) \mid \|\cdot\| \text{ a norm}\}$.

(ii) Statement (ii) is a generalization to absolute norms of [39], which essentially proves the result for the $\ell_2$ case. For the special case of $\ell_p$-norm on $\mathbb{R}^n$, Theorem 7 shows that the infimum of the weighted $\ell_p$-measure of a matrix with respect to the weights recovers the spectral abscissa of the matrix, that is, for any $p \in [1, \infty]$,

$$\alpha(A) = \inf_{R \text{ invertible}} \mu_{p,R}(A). \quad (5)$$

Next we present an application to algebraic graph theory.

Lemma 9 (Semi-measures of Laplacian matrices). Let $G$ be a weighted digraph with a globally reachable vertex and with Laplacian matrix $L$ (with eigenvalue $\lambda_1(L) = 0$). Let $\|\cdot\|$ be a norm with associated matrix measure $\mu$. Then

$$\inf\{\mu_R(-L) \mid R \in \mathbb{C}^{(n-1) \times n}, \operatorname{Ker}(R) = \operatorname{span}(1_n)\} = \alpha_{\mathbb{R}^+}(-L^T) = \alpha_{\text{ess}}(-L) < 0. \quad (6)$$

Moreover, if $G$ is undirected, then

$$\min\{\mu_R(-L) \mid R \in \mathbb{C}^{(n-1) \times n}, \operatorname{Ker}(R) = \operatorname{span}(1_n)\} \leq \mu_{R_V}(-L) = \alpha_{\mathbb{R}^+}(-L)\} = -\lambda_2(L) < 0, \quad (7)$$

where $V = \{v_2, \ldots, v_n\}$ are orthonormal eigenvectors of the Laplacian $L$ and

$$R_V = [v_2 \ldots v_n]^T \in \mathbb{R}^{(n-1) \times n}. \quad (8)$$

Proof. Theorem 7(ii) with $A = -L$ and $S = \operatorname{span}(1_n)$ gives

$$\inf_{R \in \mathbb{C}^{(n-1) \times n}} \mu_R(-L) = \alpha_{\mathbb{R}^+}(-L). \quad (9)$$

Since $G$ has a globally reachable node, $\lambda_1 = 0$ is a simple eigenvalue of $L$ with associated right eigenvector $1_n$ and all the other eigenvalues have positive real parts. This means that $\alpha_{\mathbb{R}^+}(-L^T) = \alpha_{\text{ess}}(-L) < 0$.

If $G$ is undirected, then $L$ is symmetric and all its eigenvalues are real. Suppose that $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$ are eigenvalues of $L$. Then it is clear that $\alpha_{\mathbb{R}^+}(-L^T) = \alpha_{\text{ess}}(-L) = -\lambda_2 < 0$. Moreover, we have $R_Y L R_Y^T = \operatorname{diag}(\lambda_2, \ldots, \lambda_n)$. Additionally, we know that $B_Y^T = R_Y^T (R_Y R_Y^T)^{-1} = R_Y^T$ [28, E4.5.20]. As a result,

$$\mu_{R_Y}(-L) = \mu(-R_Y L R_Y^T) = \mu(R_Y L R_Y^T) = -\lambda_2. \quad (\square)$$

Remark 10. Equation (6) implies that, for any $\varepsilon > 0$, there exists a matrix $R_\varepsilon \in \mathbb{C}^{(n-1) \times n}$ with $\operatorname{Ker}(R_\varepsilon) = \operatorname{span}(1_n)$ satisfying $\mu_{R_\varepsilon}(-L) \leq \alpha_{\text{ess}}(-L) + \varepsilon$. 

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C. Solution estimates for dynamical systems

Suppose that $t \mapsto A(t) \in \mathbb{R}^{n \times n}$ is a continuous map and consider the dynamical system

$$
\dot{x}(t) = A(t)x(t), \quad \text{for } t \geq t_0 \in \mathbb{R}
$$

(9)

with the initial condition $x(t_0) = x_0$.

**Theorem 11** (Generalized Coppel’s inequality). Let $\|\cdot\|$ be a semi-norm on $\mathbb{R}^n$ and let $\mu_{\|\cdot\|}$ be the associated matrix semi-measure. Assume that, for every $t \geq t_0$, $\text{Ker } \|\cdot\|$ is invariant under $A(t)$. Then, for every $t \geq t_0$, we have

$$
\exp \left( \int_{t_0}^{t} \mu_{\|\cdot\|}(A(\tau)) d\tau \right) \|x(0)\| \leq \|x(t)\| \leq \exp \left( \int_{t_0}^{t} \mu_{\|\cdot\|}(A(\tau)) d\tau \right) \|x(0)\|.
$$

Moreover, if $A$ is time-invariant, we have

$$
\exp(\mu_{\|\cdot\|}(A))\|x(0)\| \leq \|x(t)\| \leq \exp(\mu_{\|\cdot\|}(A))\|x(0)\|.
$$

**Proof.** Note that $t \mapsto A(t)$ is continuous. Therefore the solutions $t \mapsto x(t)$ of the time-varying dynamical system (9) are differentiable. Thus, for small enough $h$, we can write $x(t+h) = x(t) + hA(t)x(t) + o(h)$. Let $P$ be the orthogonal projection onto $\text{Ker } \|\cdot\|$. Then we have

$$
P x(t+h) = P x(t) + hPA(t)x(t) + o(h).
$$

where the last equality holds because $A(t) \text{Ker } \|\cdot\| \subseteq \text{Ker } \|\cdot\|$. Therefore, $\|P x(t+h)\| \leq \|x(t+h)\|$ and Theorem 3 together imply

$$
\|x(t+h)\| - \|x(t)\| \leq \|I_n + hA(t)\| \leq \|x(t)\| + o(h).
$$

Taking the limit as $h \to 0^+$, we obtain

$$
\frac{d}{dt} \|x(t)\| \leq \mu_{\|\cdot\|}(A)\|x(t)\|.
$$

The result then follows from Grönwall–Bellman inequality.

**Definition 12** (Semi-contracting systems). Let $C \subseteq \mathbb{R}^n$ be a convex set, $c > 0$ be a positive number, $\|\cdot\|$ be a semi-norm on $\mathbb{R}^n$, and $\mu_{\|\cdot\|}$ be its associated matrix semi-measure. The time-varying dynamical system (1) is semi-contracting on $C$ with respect to $\|\cdot\|$ with rate $c$, if the following conditions hold:

1. $\text{Ker } \|\cdot\|$ is an invariant subspace for (1);
2. for all $t \in \mathbb{R}_{\geq 0}, x \in C$,

$$
\mu_{\|\cdot\|}(Df(t, x)) \leq -c.
$$

Next, we study the asymptotic behavior of trajectories of the dynamical system (1) using semi-contracting theory.

**Theorem 13** (Trajectories of semi-contracting systems). Consider the time-varying dynamical system (1) with a convex invariant set $C$. Let $\|\cdot\|$ be a semi-norm on $\mathbb{R}^n$, $c$ be a positive number. If the system (1) is semi-contracting with respect to the semi-norm $\|\cdot\|$ with rate $c$, then

1. for every $x_0, y_0 \in C$ and every $t \in \mathbb{R}_{\geq 0}$, we have

$$
\|\phi(t, x_0) - \phi(t, y_0)\| \leq e^{-ct}\|x_0 - y_0\|;
$$

Moreover, if the dynamical system (1) is time-invariant, then
2. for every $x_0 \in C$ and every $t \in \mathbb{R}_{\geq 0}$, we have

$$
\|f(\phi(t, x_0))\| \leq e^{-ct}\|f(x_0)\|.
$$

Proving. Regarding part (i), for $\alpha \in [0, 1]$, define $\psi(t, \alpha) = \phi(t, \alpha x_0 + (1-\alpha)y_0)$ and note that we have $\psi(0, \alpha) = \alpha x_0 + (1-\alpha)y_0$ and $\frac{d}{d\alpha} \psi(0, \alpha) = x_0 - y_0$. Because $C$ is invariant, $\psi(t, \alpha) \in C$ for all $t \geq 0$ and $\alpha$. Then the time derivative of $\psi(t, \alpha)$ is given as follows:

$$
\frac{d}{d\alpha} \psi(t, \alpha) = \frac{\partial}{\partial \alpha} f(\phi(t, \psi(t, \alpha))) \frac{d}{d\alpha} \psi(t, \alpha).
$$

Therefore, $\frac{d}{d\alpha} \psi(t, \alpha)$ satisfies the linear time-varying differential equation $\frac{d}{d\alpha} = Df(t, \psi(t, \alpha)) \frac{d}{d\alpha}$. Now, using Theorem 11,

$$
\left\| \frac{d}{d\alpha} \psi(t, \alpha) \right\| \leq \left\| \frac{d}{d\alpha} \psi(0, \alpha) \right\| \exp \left( \int_0^t \mu(Df(t, \psi(t, \alpha))) \right) \leq e^{-ct}\|x_0 - y_0\|.
$$

(11)

where we used $\mu_{\|\cdot\|}(Df(t, x)) \leq -c$, for every $t \in \mathbb{R}_{\geq 0}$ and every $x \in C$. As a result, using inequality (11), we have

$$
\|\phi(t, x_0) - \phi(t, y_0)\| = \|\psi(t, 1) - \psi(t, 0)\|
$$

$$
= \left\| \int_0^1 \frac{d}{d\alpha} \psi(t, \alpha) \right\| d\alpha = \left\| \int_0^1 \frac{d}{d\alpha} \psi(t, \alpha) \right\| d\alpha \leq e^{-ct}\|x_0 - y_0\|.
$$

This completes the proof of part (i). Regarding part (ii), let $y_0 \in \text{Ker } \|\cdot\| \cap C$. Since both $\|\cdot\|$ and $C$ are invariant sets for (1), we get that $\phi(t, y_0) \in \text{Ker } \|\cdot\| \cap C$, for every $t \geq 0$. Suppose that $P$ is the orthogonal projection onto $\text{Ker } \|\cdot\|$. Then, for every $x_0 \in C$, we have

$$
\|P x(t)\| = \|P \phi(t, x_0) - \phi(t, y_0)\|
$$

$$
= \|\phi(t, x_0) - \phi(t, y_0)\| \leq e^{-ct}\|x_0 - y_0\|
$$

$$
\leq e^{-ct}\|x_0\|.
$$

This implies that $\lim_{t \to \infty} \|P x(t)\| = 0$. Since $\|\cdot\|$ is a norm on the vector space $\text{Ker } \|\cdot\|$, we get $\lim_{t \to \infty} \phi(t, x_0) = 0$ with rate $c$. Therefore, every trajectory of the system (1) converges to $\text{Ker } \|\cdot\|$ exponentially with rate $c$. Regarding part (iii), note that the map $t \mapsto f(\phi(t, x_0))$ satisfies $\frac{d}{dt} f(\phi(t, x_0)) = Df(\phi(t, x_0)) f(\phi(t, x_0))$. Now, we can use Theorem 11, to deduce that

$$
\|f(\phi(t, x_0))\| \leq \exp \left( \int_0^t \mu_{\|\cdot\|}(Df(\phi(\tau, x_0))) d\tau \right) \|f(\phi(0, x_0))\|
$$

$$
\leq e^{-ct}\|f(x_0)\|,
$$

where we used $\mu_{\|\cdot\|}(Df(\tau, x)) \leq -c$, for every $t \in \mathbb{R}_{\geq 0}$ and every $x \in C$.

**Remark 14.** Our semi-contraction theory is closely related to the notion of horizontal contraction introduced in [15]. We note that the horizontal contraction setting in [15] is developed
for Finsler manifolds and thus is more general than our semi-
contraction framework; see also [43]. However, our framework
provides a more comprehensive treatment of semi-measures,
characterizing their theoretical and computational properties.

Theorem 13 proves that every trajectory of a semi-
contracting system converges to the invariant subspace Ker ||·||. This theorem does not state that the asymptotic
behavior of the system is the same as the asymptotic behavior of the trajectories in the invariant subspace Ker ||·||. The following example elaborates on this aspect.

Example 15. The dynamical system on \( \mathbb{R}^2 \)

\[
\begin{align*}
\dot{x}_1 &= -x_1, \\
\dot{x}_2 &= x_1 x_2^2 
\end{align*}
\]  

(12)
is semi-contracting with respect to the semi-norm \( \|\cdot\| \) defined by \( \| (x_1, x_2)^T \| = |x_1| \), for every \( (x_1, x_2)^T \in \mathbb{R}^2 \). Therefore, by Theorem 13(ii), every trajectory of the system (12) converges to the invariant set \( S = \{(0, x_2) \mid x_2 \in \mathbb{R}\} \). Moreover, \( S \) is the set of equilibrium points for (12). On the other hand, the curve \( t \to (x_1(t), x_2(t)) := (e^{-t}, e^t) \) is a trajectory of (12) with the property that \( \lim_{t \to \infty} x_2(t) = \infty \). Therefore, this trajectory does not converge to any equilibrium point. In fact, one can show that, other than the equilibrium points, no system trajectory converges to an equilibrium point.

IV. WEAKLY-CONTRACTING SYSTEMS

We here generalize contraction theory by providing a com-
prehensive treatment to weakly-contracting systems.

Definition 16 (Weakly-contracting systems). Let \( C \subseteq \mathbb{R}^n \) be a convex set, \( \|\cdot\| \) be a norm on \( \mathbb{R}^n \), and \( \mu \) be its associated matrix
measure. The time-varying dynamical system (1) is weakly-
contracting on \( C \) with respect to \( \|\cdot\| \), if

\[
\mu(Df(t), x)) \leq 0, \quad \text{for all } t \in \mathbb{R}_{\geq 0}, x \in C. \tag{13}
\]

We aim to study the qualitative behaviors of weakly-contracting systems. For the rest of this section, we restrict
ourselves to the time-invariant dynamical systems:

\[
\dot{x} = f(x), \tag{14}
\]
on convex invariant sets and we assume that \( x \mapsto f(x) \) is twice-differentiable. We start with a useful and novel lemma which shows a weakly contracting system with a bounded trajectory has a compact convex invariant set.

Lemma 17 (Weak contraction and bounded trajectory imply invariant set). Consider the time-invariant dynamical system (14) with a closed convex invariant set \( C \). Suppose that \( \|\cdot\| \) is a norm on \( \mathbb{R}^n \), \( f \) is continuously differentiable and weakly
contracting with respect to the norm \( \|\cdot\| \), and \( t \to x(t) \) is a bounded trajectory of the system in \( C \). Then the system (14) has a compact convex invariant set \( W \subseteq C \).

Our first result is a dichotomy for qualitative behavior of trajectories of weakly contracting systems.

Theorem 18 (Dichotomy for weakly contracting systems). Consider the time-invariant dynamical system (14) with a convex invariant set \( C \subseteq \mathbb{R}^n \). Let \( \|\cdot\| \) be a norm on \( \mathbb{R}^n \) with the associated matrix measure \( \mu \). Suppose that the system (14) is weakly contracting on \( C \) with respect to \( \|\cdot\| \). Then either of the following exclusive conditions hold:

(i) the dynamical system (14) has at least one stable equilib-
rium point \( x^* \in C \) and every trajectory of (14) starting in \( C \) is bounded; or
(ii) the dynamical system (14) has no equilibrium point in \( C \) and every trajectory of (14) starting in \( C \) is unbounded.

Moreover, the following statements hold:

(iii) if \( x^* \) is locally asymptotically stable, then \( x^* \) is globally asymptotically stable in \( C \),
(iv) if additionally \( Df(x^*) \) is Hurwitz then \( x^* \) is the unique globally asymptotically stable equilibrium point in \( C \),
(v) if additionally \( \mu(Df(x^*)) \leq 0 \) then \( x^* \) is the unique globally asymptotically stable equilibrium point in \( C \) and \( x \mapsto \| x - x^* \| \) is a global Lyapunov function and \( x \mapsto \| f(x) \| \) is a local Lyapunov function.

Proof. Suppose that the equation \( f(x) = 0 \), at least one
solution \( x^* \) in \( C \). Let \( t \to x(t) \) be a trajectory of the system. Since the system (14) is weakly-contracting, we have

\[
\| x(t) - x^* \| \leq \| x(0) - x^* \|, \quad \forall t \geq 0.
\]

By setting \( M = \| x(0) - x^* \| \) and using triangle inequality, we see that \( \| x(t) \| \leq \| x(0) \| + M \), for every \( t \geq 0 \). This implies that \( t \to x(t) \) is bounded and thus statement (i) holds.

Now suppose that the algebraic equation \( f(x) = 0 \), does not have any solution in \( C \). We now need to show that every trajectory of (14) is unbounded. By contradiction, assume \( t \to x(t) \) is a bounded trajectory of (14). In Lemma 17 we establish that there exists a compact convex invariant set \( W \subseteq C \) for the dynamical system (14). Therefore, by Yorke Theorem [18, Lemma 4.1], the system (14) has an equilibrium point inside \( W \subseteq C \). This is in contradiction with the assumption that \( f(x) = 0 \), has no solution inside \( C \). Therefore, every trajectory of (14) is unbounded. This completes the proof of statement (ii). Regarding part (iii), since \( x^* \) is a locally exponentially stable equilibrium point for the dynamical system, there exists \( \epsilon, T > 0 \) such that \( \| x(T) \| \leq \| x^* \| + M \), for every \( z \in \mathbb{R}^n \). Therefore, after time \( T \), \( t \to x(t) \) is a trajectory of the dynamical system. Assume that \( \| y \| \leq \| x^* \| + \epsilon/2 \), and there is no point on the straight line connecting \( x(0) \) to the unique equilibrium point \( x^* \). Then we have

\[
\| x(T) - x^* \| \leq \| x(T) - \phi(T,y) \| + \| \phi(T,y) - x^* \| \leq \| x(0) - y \| + \epsilon/2 = \| x(0) - x^* \| - \epsilon/2,
\]

where the last equality holds because \( x^* \), \( y \), and \( x(0) \) are on the same straight line. Therefore, after time \( T \), \( t \to \| x(t) - x^* \| \) decreases by \( \epsilon/2 \). As a result, there exists a finite time \( T_{inf} \) such that, for every \( t \geq T_{inf} \), we have \( x(t) \in \mathbb{B}_{\|\cdot\|}(x^*,\epsilon) \). Since \( \mathbb{B}_{\|\cdot\|}(x^*,\epsilon) \) is in the region of attraction of \( x^* \) the trajectory \( t \to x(t) \) converges to \( x^* \). Regarding part (iv), \( Df(x^*) \) being Hurwitz implies that \( x^* \) is a locally exponentially stable equilibrium point for (14) and therefore it is a globally
exponentially stable equilibrium point by part (iii). Regarding part (v), \( \mu(Df(x^*)) < 0 \) implies that \( Df(x^*) \) is Hurwitz and the globally exponentially stability of the equilibrium follows from part (iv). Now we show that \( x \mapsto \|x - x^*\| \) is a global Lyapunov function. It is positive definite and radially unbounded. Since \( \mu(Df(x^*)) < 0 \) and \( f \) is continuously differentiable, there exists a closed ball \( \overline{B}_{\|\cdot\|}(x^*, r) \) and \( c > 0 \) such that

\[
\mu(Df(x)) \leq -c, \quad \forall x \in \overline{B}_{\|\cdot\|}(x^*, r).
\]

Suppose that \( t \mapsto x(t) \) is a trajectory of the system and, for every \( t \in \mathbb{R}_{\geq 0} \), assume that \( y(t) \) is the point on the intersection of the line connecting \( x(t) \) to \( x^* \) and \( \partial \overline{B}_{\|\cdot\|}(x^*, r) \). Then, for every \( s > t \), we have

\[
\|x(s) - x^*\| \leq \|x(s) - \phi(s-t, y)\| + \|\phi(s-t, y) - x^*\| \\
\leq \|x(t) - y\| + e^{-c(s-t)}\|y - x^*\| \\
< \|x(t) - x^*\|.
\]

This implies that \( t \mapsto \|x(t) - x^*\| \) is strictly decreasing. Since \( x \mapsto \|x - x^*\| \) is radially unbounded, we can deduce that \( x \mapsto \|x - x^*\| \) is a global Lyapunov function for the system. On the other hand, for every \( t \in \mathbb{R}_{\geq 0} \),

\[
\frac{d}{dt} f(x(t)) = Df(x(t)) f(x(t)).
\]

Note that the dynamical system (14) is weakly contracting with respect to the norm \( \|\cdot\| \). Therefore, for every \( x_0 \in \overline{B}_{\|\cdot\|}(x^*, r) \), the trajectory \( t \mapsto x(t) \) starting at \( x_0 \) remains inside \( \overline{B}_{\|\cdot\|}(x^*, r) \). This implies that, for every \( t \in \mathbb{R}_{\geq 0} \), we have \( \mu(Df(x(t))) \leq -c \). Using Theorem 11(iii), we get

\[
\|f(x(t))\| \leq e^{-ct}\|f(x(0))\|, \quad \text{for all } t \in \mathbb{R}_{\geq 0}.
\]

As a result \( x \mapsto \|f(x)\| \) is a local Lyapunov function for the dynamical system (14).

The dichotomy in Theorem 18 completely characterizes the qualitative behavior of the dynamical system when system has no equilibrium point. However, for weakly contracting systems with at least one equilibrium point, this theorem can only guarantee that the trajectories are bounded. The following theorem shows that, for some classes of norms, weak contractivity implies convergence of every trajectory to an equilibrium point.

Theorem 19 (Weak contraction and convergence to equilibria). Consider the time-invariant dynamical system (14) with a convex invariant set \( C \subseteq \mathbb{R}^n \). Suppose that the vector field \( f \) is differentiable and piecewise real analytic with an equilibrium point \( x^* \in C \). Suppose that there exists \( p \in \{1, \infty\} \) and an invertible matrix \( Q \in \mathbb{R}^{n \times n} \) such that \( \mu_{p, Q}(Df(x)) \leq 0 \), for every \( x \in C \), i.e., the system is weakly contracting with respect to \( \|\cdot\|_{p, Q} \) on \( C \). Then every trajectory of (14) starting in \( C \) converges to an equilibrium point.

Proof. We prove the theorem for \( p = 1 \). The proof for \( p = \infty \) is similar and we omit it. Suppose that the vector field \( f \) is piecewise real analytic and the system (14) is weakly contracting with respect to \( Q \)-weighted \( \ell_1 \)-norm. By Theorem 18(i) every trajectory of the system (14) is bounded. Now we use the LaSalle Invariance Principle [5, Theorem 15.7] for the function \( V(x) = \|f(x)\|_{1, Q} \) on the invariant convex set \( C \). We first show that, for every \( c > 0 \), \( V^{-1}(c) \) is a closed invariant set for the dynamical system (14). Since \( V \) is continuous, \( V^{-1}(c) \) is closed. Now we show that \( V^{-1}(c) \) is invariant. First note that, for every trajectory \( t \mapsto x(t) \) starting inside \( V^{-1}(c) \), we have \( \frac{d}{dt} f(x(t)) = Df(x(t)) f(x(t)) \). Using Theorem 11,

\[
\|f(x(t))\|_{1, Q} \leq \exp \left( \int_0^t \mu_{1, Q}(Df(x(\tau))) d\tau \right) \|f(x(0))\|_{1, Q} \leq \|f(x(0))\|_{1, Q} \leq c,
\]

where the second inequality holds because system (14) is weakly contracting with respect to \( Q \)-weighted \( \ell_1 \)-norm and thus \( \mu_{1, Q}(Df(x)) \leq 0 \), for every \( x \in C \) and the last inequality holds because \( x(0) \in V^{-1}(c) \). Thus, \( V^{-1}(c) \) is an invariant set for (14). Therefore, by the LaSalle Invariance Principle, the largest invariant set \( M \) inside the set

\[
\{ x \in C \mid \mathcal{L}_f V(x) = 0 \} \cap V^{-1}(c)
\]

is nonempty and every trajectory of (14) converges to \( M \). Suppose that \( t \mapsto y(t) \) is a trajectory in the set \( M \) (and therefore in the set \( C \)). Our goal is to show that \( t \mapsto y(t) \) is an equilibrium point. Since the vector field \( f \) is piecewise real analytic, there exists a partition \( \{\Sigma_j\}_{j=1}^m \) of \( C \) such that \( f \) is real analytic on \( \text{int}(\Sigma_j) \), for every \( j \in \{1, \ldots, m\} \). Now consider the trajectory \( t \mapsto y(t) \). It is clear that, there exists \( k \in \{1, \ldots, m\} \) and \( T > 0 \) such that \( y(t) \in \text{cl}(\Sigma_k) \), for every \( t \in [0, T] \). Since \( f \) is real analytic on \( \text{int}(\Sigma_k) \), there exists a real analytic vector field \( g : C \to \mathbb{R}^n \) such that \( g(x) = f(x) \), for every \( x \in \text{cl}(\Sigma_k) \). This implies that, for every \( t \in [0, T] \), the curve \( t \mapsto y(t) \) is a solution of the dynamical system

\[
y(t) = f(y(t)) = g(y(t)) \quad \text{on } [0, T].
\]

Therefore, the curve \( t \mapsto y(t) \) is real analytic for every \( t \in [0, T] \). Note that, for every \( t \in [0, T] \),

\[
0 = \frac{d}{dt} V(y(t)) = \sum_{i=1}^n \text{sign}\left( (Qf)_i(y(t)) \right) (Q\dot{f})_i(y(t))
\]

\[
= \sum_{i=1}^n \text{sign}\left( (Qg)_i(y(t)) \right) (Q\dot{g})_i(y(t)).
\]

Since \( g \) is real analytic and \( t \mapsto y(t) \) is real analytic on \([0, T] \), then \( t \mapsto (Qg)_i(y(t)) \) is real analytic, for \( i \in \{1, \ldots, n\} \). Therefore, either of the following conditions hold:

(i) \( (Qg)_i(y(t)) \neq 0 \), for every \( t \in [0, T] \), or
(ii) \( (Qg)_i(y(t)) = 0 \), for every \( t \in [0, T] \).

Since \( t \mapsto (Qg)_i(y(t)) \) is continuous, for every \( i \in \{1, \ldots, n\} \), this implies that \( \text{sign}((Qg)_i(y(t))) = \text{sign}((Qg)_i(y(0))) \), for every \( t \in [0, T] \) and every \( i \in \{1, \ldots, n\} \). Define \( w \in \mathbb{R}^n \) by

\[
w = \text{sign}(Qg(y(0))).
\]

Thus, we have

\[
0 = \mathcal{L}_f V(y(t)) = w^T Q\dot{g}(y(t)), \quad \forall t \in [0, T].
\]

By integrating the above condition, we get

\[
w^T Qg(y(t)) = \beta, \quad \forall t \in [0, T],
\]
for some constant $\beta \geq 0$. We first show that $\eta = 0$. Note that $g(y(t)) = f(y(t)) = \hat{y}(t)$. Therefore, we have $w^T Q \hat{y}(t) = \beta$, for every $t \in [t_1, t_2]$. Integrating with respect to time, we get

$$w^T Q \hat{y}(t) = \beta t + \eta, \quad \forall t \in [0, T].$$

for some constant $\eta \in \mathbb{R}$. Since every trajectory of (14) starting in $C$ is bounded, we have $\beta = 0$. Now, note that, for every $t \in [0, T]$

$$\frac{\|f(y(t))\|_1}{Q} = \sum_{i=1}^{n} \text{sign}(Qf_i(y(t))) \|Qf_i(y(t))\| = w^T Qg(y(t)) = 0.$$

This implies that $f(y(t)) = 0_n$, for every $t \in [0, T]$. Since $t \mapsto y(t)$ is a trajectory of (14) and it is continuous, we have $f(y(t)) = 0_n$, for every $t \geq 0$. This implies that every trajectory inside $M$ is an equilibrium point. Therefore, every trajectory of (14) starting in $C$ converges to the set of equilibrium points.

Remark 20. (i) Theorem 19(ii) is a generalization of [23, Lemma 6] to weakly contracting systems. Theorem 18(v) is due to [9, Corollary 8].

(ii) We believe that Theorem 19 can be generalized to polyhedral norms; we omit this generalization in the interest of brevity.

(iii) Theorem 19 does not hold in general for systems that are weakly contracting with respect to $\ell_p$-norms for $p \notin \{1, \infty\}$, as the following example illustrates.

Example 21. The dynamical system on $\mathbb{R}^2$

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1,
\end{align*}$$

(15)

has a unique equilibrium point $0_2$ and its vector field $f$ is piecewise real analytic. Let $\mu_{2}$ denote the matrix measure with respect to the $\ell_2$-norm on $\mathbb{R}^2$. Then

$$\mu_2(Df(x)) = \lambda_{\max} \left( \frac{Df(x) + Df^T(x)}{2} \right) = \lambda_{\max}(0_{2 \times 2}) = 0,$$

so that system (15) is weakly contracting with respect to $\|\cdot\|_2$. However, $t \mapsto (\sin(t), \cos(t))$ is a periodic trajectory of (15) which does not converge to the equilibrium point.

$\triangle$

V. DOUBLE CONTRACTING SYSTEMS

Examples 15 and 21 illustrate that semi-contractivity or weak-contractivity alone do not guarantee the convergence of trajectories to equilibrium points. Here we show that, for time-invariant systems, a combination of these two properties can ensure that every trajectory converges to an equilibrium point.

Theorem 22 (Double contraction and convergence to equilibrium). Consider the time-invariant dynamical system (14) with a convex invariant set $C \subseteq \mathbb{R}^n$. Suppose that $S \subseteq \mathbb{R}^n$ is a vector subspace consists of equilibrium points of (14). Assume that there exist

(A1) a norm $\|\cdot\|$ such that (14) is weakly-contracting with respect to $\|\cdot\|$ on $C$; and

(A2) a semi-norm $\|\cdot\|$ with Ker $\|\cdot\| = S$ such that (14) is semi-contracting with respect to $\|\cdot\|$ on $C$.

Then, for every trajectory $t \mapsto x(t)$ of (14) starting in $C$, there exists $x^* \in S$ such that $\lim_{t \to \infty} x(t) = x^*$ with exponential convergence rate $|\alpha_{S^\perp}(Df(x^*))|.$

We refer to systems satisfying (A1) and (A2) as doubly-contracting in the sense that their trajectories satisfy a weak contractivity and a semi-contractivity property:

$$\|\phi(t, x_0) - \phi(t, y_0)\| \leq \|x_0 - y_0\|,$$

$$\|\phi(t, x_0) - \phi(t, y_0)\| \leq e^{-c} \|x_0 - y_0\|,$$

for every $x_0, y_0 \in C$, every $t \in \mathbb{R}_{\geq 0}$, and some $c \in \mathbb{R}_{\geq 0}$.

Proof. Suppose that $t \mapsto x(t)$ is a trajectory of (14) starting from $x(0) \in C$. Let $P$ be the orthogonal projection onto the subspace $S^\perp = \text{Ker} \|\cdot\|$.$^n$. Note that, for every $t \in \mathbb{R}_{\geq 0}$, the orthogonal projection of $x(t)$ onto $S = \text{Ker} \|\cdot\|$ is given by $(I_n - P)x(t)$ and it is an equilibrium points for (14). Moreover, by Assumption (A1), system (14) is weakly-contracting with respect to the norm $\|\cdot\|$. This implies that, for every $t \in \mathbb{R}_{\geq 0}$ and every $s \geq t$, the point $x(s)$ remains inside the closed ball $B_{\|\cdot\|}[(I_n - P)x(t), \|Px(t)\|]$. Therefore, for every $t \geq 0$, the point $x(t)$ is inside the set $D_t$ defined by

$$D_t = \text{cl}(\bigcap_{\tau \in [0,t]} B_{\|\cdot\|}[(I_n - P)x(\tau), \|Px(\tau)\|]).$$

It is easy to see that, for $s \geq t$, we have $D_s \subseteq D_t$. This implies that the family $\{D_t\}_{t \in [0,\infty)}$ is a nested family of closed subsets of $\mathbb{R}^n$ such that $\text{diam}(D_t) \leq \|Px(t)\|$, for every $t \in [0, \infty)$. On the other hand, for every $t \in [0, \infty)$, the set $D_t$ is a closed convex invariant set for the system (14). Thus, using Assumption (A2) and Theorem 13(ii), the trajectory $t \mapsto x(t)$ converges to the subspace $S$. This means that $\lim_{t \to \infty} \text{diam}(D_t) = \lim_{t \to \infty} \|Px(t)\| = 0$. Thus, by the Cantor Intersection Theorem [29, Lemma 48.3], there exists $x^* \in C$ such that $\bigcap_{t \in [0,\infty)} D_t = \{x^*\}$. We show that $\lim_{t \to \infty} x(t) = x^*$. Note that $x^*, x(t) \in D_t$, for every $t \in \mathbb{R}_{\geq 0}$. This implies that $\|x(t) - x^*\| \leq \text{diam}(D_t).$ In turn means that $\lim_{t \to \infty} \|x(t) - x^*\| = 0$ and $t \mapsto x(t)$ converges to $x^*$. On the other hand, the trajectory $t \mapsto x(t)$ converges to the subspace $S$. Therefore, $x^* \in S$ and $x^*$ is an equilibrium point. Regarding the convergence rate, by Theorem 7, we have

$$\alpha_{S^\perp}(Df(x^*)) = \inf\{\mu_{\|\cdot\|}(Df(x^*)) \mid \|\cdot\| \text{ a semi-norm, Ker} \|\cdot\| = S\}.$$

First note that system (14) is semi-contracting with respect to $\|\cdot\|$ on $C$. This implies that $\alpha_{S^\perp}(Df(x^*)) = \mu_{\|\cdot\|}(Df(x^*)) < 0$. Let $\epsilon > 0$ be such that $\epsilon \leq \frac{1}{2} \mu_{\|\cdot\|}(Df(x^*))$. There exists a semi-norm $\|\cdot\|$, such that $\mu_{\|\cdot\|}(Df(x^*)) < \alpha_{S^\perp}(Df(x^*)) + \epsilon < 0$. Consider the family of closed convex invariant sets $\{D_t\}_{t \geq 0}$. Since $\lim_{t \to \infty} \text{diam}(D_t) = 0$, and $f$ is twice differentiable, there exists $t_\epsilon \in \mathbb{R}_{\geq 0}$ such that $\mu_{\|\cdot\|}(Df(x^*)) \leq \alpha_{S^\perp}(Df(x^*)) + 2\epsilon < 0$, for every $x \in D_{t_\epsilon}$. Therefore, using Theorem 13(ii), the trajectory $t \mapsto x(t)$ converges to the subspace $S$ with the convergence rate $|\alpha_{S^\perp}(Df(x^*))| + 2\epsilon$. 
Since $\epsilon$ can be chosen arbitrarily small, the trajectory $t \mapsto x(t)$ converges to the subspace $S$ with the rate $|\alpha_{2r}(Df(x^*))|$. \hfill \Box

Remark 23. Theorem 22 provides a convergence rate for trajectories of the system which depends only on the converging point and is independent of both the norm $\|\cdot\|$ in Assumption (A1) and the semi-norm $\|\|\|$ in Assumption (A2). This theorem also generalizes the classical results in contraction theory where the convergence rate depends on the norm.

VI. APPLICATION TO NETWORK SYSTEMS

In this section, we apply our results to example systems. We show that (i) affine averaging systems and affine flow systems are doubly-contracting, (ii) distributed primal-dual dynamics is weakly-contracting, and (iii) networks of diffusively-coupled oscillators with strong coupling are semi-contracting.

A. Affine averaging and flow systems are doubly-contracting

Theorem 24 (Affine averaging system). Let $L$ be the Laplacian matrix of a weighted digraph with a globally reachable node. Let $v$ denote the dominant left eigenvector of $L$ satisfying $\sum_i v_i = 1$. Let $b \in \mathbb{R}^n$. Then the affine averaging system

$$\dot{x} = -Lx + b$$  \hfill (16)

(i) is weakly-contracting with respect to $\|\cdot\|_\infty$ and semi-contracting with respect to $\|\cdot\|.R$, where $R_e$ is defined in Remark 10 as a function of a sufficiently small $\epsilon$,

(ii) if $v^Tb \neq 0$, then every trajectory is unbounded,

(iii) if $v^Tb = 0$, then the trajectory starting from $x(0) = x_0$ converges to the equilibrium point $L^\dagger b + (v^T x_0) \mathbb{1}_n$ with exponential rate $-\alpha_{\text{ess}}(-L)$.

Proof. Regarding part (i), for $f(x) := -Lx + b$, it is easy to see that $\mu_{\infty}(Df(x)) = \mu_{\infty}(-L) = 0$. Therefore, the dynamical system (16) is weakly-contracting with respect to $\ell_\infty$-norm. Regarding part (ii), if $v^Tb \neq 0$, then there does not exist $x \in \mathbb{R}^n$ such that $-Lx + b = 0$. Thus, Theorem 18(i) implies every trajectory of (16) is unbounded. Regarding part (iii), since $G$ has a reachable node, 0 is a simple eigenvalue of $L$ and the other eigenvalues of $L$ have positive real parts. Thus, if $v^Tb = 0$, then the algebraic equation $-Lx + b = 0$ has solutions $x = L^\dagger b + \beta x_0$, for $\beta \in \mathbb{R}$. Therefore, by Theorem 19, every trajectory of (16) converges to an equilibrium point in $\text{span}(\mathbb{1}_n)$. Let $t \mapsto x(t)$ be a trajectory of (16). Since $v^T L = 0$, we have $v^T x(t) = v^T x_0$. Thus, the trajectory $t \mapsto x(t)$ converges to the equilibrium point $L^\dagger b + (v^T x_0) \mathbb{1}_n$. In order to find the rate of convergence, we use the change of variable $z = x + L^\dagger b$ to get

$$\dot{z} = -Lz.$$  \hfill (17)

This implies that there exists $R \in \mathbb{C}^{(n-1)\times n}$ such that $\text{Ker}(R) = \text{span}(\mathbb{1}_n)$ and $\mu_{\infty,R}(-L) \leq -c$ for some positive $c > 0$. Thus the assumptions of Theorem 22 hold for the norm $\|\cdot\|_\infty$ and the semi-norm $\|\|_\infty$. This implies that every trajectory of the system (17) converges to $\text{span}(\mathbb{1}_n)$ with convergence rate $-\alpha_{\text{ess}}(-L) > 0$. \hfill \Box

Next, we state an analogous theorem for affine flow systems, whose proof is omitted in the interest of brevity.

Theorem 25 (Affine flow system). Under the same assumptions on $L$, $v$, and $b$ as in Theorem 24, the affine flow system

$$\dot{x} = -L^T x + b$$  \hfill (18)

(i) is weakly-contracting with respect to $\|\cdot\|_1$ and semi-contracting with respect to $\|\cdot\|_{1,R}$, where $R_e$ is defined in Remark 10 as a function of a sufficiently small $\epsilon$,

(ii) if $\sum_i b_i \neq 0$, then every trajectory is unbounded,

(iii) if $\sum_i b_i = 0$, then the trajectory starting from $x(0) = x_0$ converges to the equilibrium point $(L^T)^\dagger b + (v^T x_0) v$ with exponential rate $-\alpha_{\text{ess}}(-L)$.

B. Distributed primal-dual dynamics is weakly-contracting

In the second application, we study a well-known distributed implementation of unconstrained optimization problems. Suppose that we have $n$ agents connected though an undirected weighted connected graph with Laplacian $L$. We want to minimize an objective function $f : \mathbb{R}^k \to \mathbb{R}$ which can be represented as $f(x) = \sum_{i=1}^n f_i(x)$. This optimization problem can be written as

$$\min_{i=1}^n f_i(x_i),$$  \hfill (19)

When agent $i$ has access to only the objective function $f_i$, the optimization problem (19) can be implemented in a distributed fashion as:

$$\min_{i=1}^n f_i(x_i),$$  \hfill (20)

$$x_i = x_j, \quad \text{for every edge } i,j.$$  

The primal-dual algorithm associated with the distributed optimization problem (20) is given by

$$\dot{x}_i = -\nabla f_i(x_i) - \sum_{i=1}^n a_{ij}(\nu_i - \nu_j),$$  \hfill (21)

$$\dot{\nu}_i = \sum_{j=1}^n a_{ij}(x_i - x_j).$$

The following theorem characterizes the global convergence of the dynamics (21); for a more detailed treatment we refer to [8].

Theorem 26 (Distributed primal-dual algorithm). Consider the distributed optimization problem (20) and the primal-dual dynamics (21) over a connected undirected weighted graph with Laplacian matrix $L$. Assume that the optimization problem (19) has a solution $x^* \in \mathbb{R}^k$ and that, for each $i \in \{1, \ldots, n\}$, $f_i$ is convex and $\nabla^2 f_i(x^*) > 0$. Then
(i) system (21) is weakly contracting with respect to \( \| \cdot \|_2 \).
(ii) each trajectory \((x(t), \nu(t))\) of (21) converges exponentially to \((I_n \otimes x^*, I_n \otimes \nu^*)\), where \(\nu^* = \sum_{i=1}^{n} \nu_i(0)\).

Proof. We set \( x = (x_1, \ldots, x_n)^T \) and \( \nu = (\nu_1, \ldots, \nu_n)^T \), and define \( h(x) = \sum_{i=1}^{n} f_i(x_i) \). Then algorithm (21) can be written as

\[
\begin{align*}
\dot{x} &= -\nabla h(x) - (L \otimes I_k) \nu, \\
\dot{\nu} &= (L \otimes I_k) x.
\end{align*}
\]  

(22)

Regarding (i), let \((\dot{x}, \dot{\nu}) = F_{PD}(x, \nu)\). Then \(DF_{PD}(x, \nu) = \begin{bmatrix} -\nabla^2 h(x) & -L \otimes I_k \\ L \otimes I_k & 0 \end{bmatrix} \) and \(\mu_2(DF_{PD}(x, \nu)) = 0\). Therefore, the system (21) is weakly-contracting with respect to the \(\ell_2\)-norm. Regarding part (ii), let \( t \mapsto (x(t), \nu(t)) \) be a trajectory of the system (21). Note that, for every \( u \in \mathbb{R}^k \), we have \( (I_n \otimes u)^T \dot{\nu}(t) = (I_n \otimes u)^T (L \otimes I_k) x(t) = 0 \). This implies that \( \sum_{i=1}^{n} \nu_i(t) = \sum_{i=1}^{n} \nu_i(0) \) and the subspace \( V \subseteq \mathbb{R}^{nk} \times \mathbb{R}^{nk} \) defined by

\[
V = \{ (x, \nu) \in \mathbb{R}^{2nk} \mid \sum_{i=1}^{n} \nu_i = 0 \}
\]

is an invariant subspace of the system (21). For the Laplacian matrix \( L \), define \( R_L \) by equation (8). Define new coordinates \((\tilde{x}, \tilde{\nu})\) on \( V \) by \( \tilde{x} = x \) and \( \tilde{\nu} = (R_L \otimes I_k) \nu \). Thus, the dynamical system (21) restricted to \( V \) can be written in the new coordinate \((\tilde{x}, \tilde{\nu})\) as

\[
\begin{align*}
\dot{\tilde{x}} &= -\nabla h(\tilde{x}) - (LR_L^T \otimes I_k) \tilde{\nu}, \\
\dot{\tilde{\nu}} &= (R_L V \otimes I_k) \tilde{x}.
\end{align*}
\]  

(23)

Let \((\tilde{x}, \tilde{\nu}) := \tilde{F}_{PD}(\tilde{x}, \tilde{\nu})\) Note that \((\tilde{x}, \tilde{\nu}) = (I_n \otimes x^*, 0_{(n-1)k})\) is an equilibrium point of the dynamical system (23) and

\[
DF_{PD}(\tilde{x}, \tilde{\nu}) = \begin{bmatrix} -\nabla^2 h(\tilde{x}) & -LR_L^T \otimes I_k \\ R_l V \otimes I_k & 0 \end{bmatrix}.
\]

Again we note that \(\mu_2(DF_{PD}(\tilde{x}, \tilde{\nu})) = 0\), for every \((\tilde{x}, \tilde{\nu}) \in V\), and therefore, the system (23) is weakly-contracting with respect to the \(\ell_2\)-norm. Moreover, \(-\nabla^2 h(I_n \otimes x^*) < 0\) and \(\text{Ker}(LR_L^T \otimes I_k) = \emptyset\). Thus, by [7, Lemma 5.3], the matrix \(DF_{PD}(I_n \otimes x^*, 0)\) is Hurwitz and the equilibrium point \((I_n \otimes x^*, 0_{(n-1)k})\) is locally asymptotically stable for the dynamical system (23). Theorem 18(iii) applied to the weakly-contracting system (23) implies that \((I_n \otimes x^*, 0_{(n-1)k})\) is a globally attractively stable equilibrium point, i.e., \(\lim_{t \to \infty} (\tilde{x}(t), \tilde{\nu}(t)) = (I_n \otimes x^*, 0_{(n-1)k})\). Therefore, we get \(\lim_{t \to \infty} x(t) = I_n \otimes x^*\) and

\[
0_{nk} = (R_L^T \otimes I_k) \lim_{t \to \infty} \tilde{\nu} = (R_L^T \otimes I_k) \nu(t)
\]

\[
= \lim_{t \to \infty} \left( (I_n - \frac{1}{n} I_n I_n^T) \otimes I_k \right) \nu(t)
\]

\[
= \lim_{t \to \infty} \left( \nu(t) - \frac{1}{n} \sum_{i=1}^{n} \nu_i(t) \right),
\]

where the last equality holds because \(\frac{1}{n} \sum_{i=1}^{n} \nu_i(t) = \nu^*\). As a result, we get \(\lim_{t \to \infty} \nu(t) = I_n \otimes \nu^*\).

C. Strongly coupled oscillators are semi-contracting

As third application, we study synchronization phenomena in networks of identical diffusively-coupled oscillators. Consider \(n\) agents connected through a weighted undirected graph with Laplacian matrix \( L \). Suppose the agents have identical internal dynamics described by the time-varying vector field \( f : \mathbb{R}_{\geq 0} \times \mathbb{R}^k \rightarrow \mathbb{R}^k \). Then the overall network dynamics is

\[
\dot{x}_i = f(t, x_i) - \sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1, \ldots, n\}.
\]  

(24)

Next, we introduce a novel useful norm. For \( p \in [1, \infty] \), define the \((2, p)\)-tensor norm \( \| \cdot \|_{(2,p)} \) on \( \mathbb{R}^{nk} \approx \mathbb{R}^n \otimes \mathbb{R}^k \) by:

\[
\|u\|_{(2,p)} = \inf \left\{ \left( \sum_{i=1}^{n} \| v^i \|_2^p \| w^i \|_p^2 \right)^{\frac{1}{p}} \mid u = \sum_{i=1}^{n} v^i \otimes w^i \right\}.
\]  

(25)

The \((2, p)\)-tensor norm is closely related to, but different from, the well-known projective tensor product norm (see [35, Chapter 2] for definition and properties of this well-known norm). The well-definiteness and properties of the \((2, p)\)-tensor norm are studied in Lemma 31 in the Appendix. For the symmetric Laplacian \( L \), let \( R_L \) be defined by equation (8).

**Theorem 27** (Networks of diffusively-coupled oscillators). Consider the network of diffusively-coupled identical oscillators (24) over a connected weighted undirected graph with Laplacian matrix \( L \). Let \( Q \in \mathbb{R}^{k \times k} \) be an invertible matrix and \( p \in [1, \infty] \). Suppose there exists a positive \( c \) such that, for every \((t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^k \),

\[
\mu_{p,Q}(DF(t, x)) \leq \lambda_2(L) - c.
\]  

(26)

Then,

(i) system (24) is semi-contracting with exponential rate \( c \) with respect to \( \| \cdot \|_{(2,p)} \),

(ii) system (24) achieves global exponential synchronization with rate \( c \), i.e., for each trajectory \( (x_1(t), \ldots, x_n(t)) \) and for each pair \( i, j \), the distance \( \|x_i(t) - x_j(t)\|_2 \) vanishes exponentially fast with rate \( c \).

**Proof.** Set \( x = (x_1^T, \ldots, x_n^T) \in \mathbb{R}^{nk} \) and define \( x_{\text{ave}} := \frac{1}{n} \sum_{i=1}^{n} x_i, \quad z := x - x_{\text{ave}} \), and \( f_{\text{ave}}(t, x) := \frac{1}{n} \sum_{i=1}^{n} f(t, x_i) \). Using these variables, dynamics (24) can be written as:

\[
\dot{z} = F(t, z) - (L \otimes I_k) z,
\]  

(27)

where \( F(t, z) = \left( f(t, x_1) - f_{\text{ave}}(t, x), \ldots, f(t, x_n) - f_{\text{ave}}(t, x) \right)^T \). It is easy to see that \( S = \text{span}\{I_n \otimes u \mid u \in \mathbb{R}^k\} \) is an invariant subspace for (27) and consists only of equilibrium points of (27). Now, define \( P = I_n - \frac{1}{n} I_n 1_n^T \) and note that \( P = P^T \) is the orthogonal projection onto \( I_n \) and \( \text{Ker} P = \text{span}\{1_n\} \). A straightforward calculation shows that

\[
DF(t, z) = (P \otimes I_k) \Sigma (P^T \otimes I_k),
\]  

(28)

where

\[
\Sigma = \begin{bmatrix} Df(t, x_1) & \cdots & 0_{k \times k} \\
\vdots & \ddots & \vdots \\
0_{k \times k} & \cdots & Df(t, x_n) \end{bmatrix} \in \mathbb{R}^{nk \times nk}.
\]
Next, we show that \(\mu_{(2,p),R_{\Psi}\otimes Q}(DF(t,z) - L \otimes I_k) \leq -c\). First, note that

\[
\mu_{(2,p),R_{\Psi}\otimes Q}(DF(t,z) - L \otimes I_k) \\
\leq \mu_{(2,p),R_{\Psi}\otimes Q}(DF(t,z)) + \mu_{(2,p),R_{\Psi}\otimes Q}(-L \otimes I_k).
\]

From Lemma 9, recall that \(R_{\Psi} = R_{\Psi}(R_{\Psi}R_{\Psi}^T)^{-1} = R_{\Psi}^T\) and \(R_{\Psi}L_{\Psi} = \Lambda = \text{diag}(\lambda_2(L), \ldots, \lambda_n(L))\). Using Lemma 6(ii), this implies that

\[
\mu_{(2,p),R_{\Psi}\otimes Q}(L \otimes I_k) = \mu_{(2,p),R_{\Psi}\otimes Q}(-L \otimes I_k) \leq -\lambda_2(L).
\]

Thus, combining (29) and (30), we get

\[
\mu_{(2,p),R_{\Psi}\otimes Q}(L \otimes I_k) = \mu_{(2,p),R_{\Psi}\otimes Q}(-L \otimes I_k) \leq -\lambda_2(L).
\]

As a result, \(\lim_{t \to \infty} \|z(t)\|_2 = 0\) with rate \(c\). This implies that, we have \(\lim_{t \to \infty} \|x(t) - \frac{1}{n} \otimes x_{av}(t)\|_2 = 0\) with rate \(c\), which in turn means that \(\lim_{t \to \infty} \|x_i(t) - x_j(t)\|_2 = 0\) with rate \(c\), for every \(i, j \in \{1, \ldots, n\}\). \(\square\)

**Remark 28** (Comparison with literature). For networks of diffusively-coupled identical oscillators, condition (26) has two unique features: (i) it is based on matrix measures induced by weighted \(\ell_p\)-norm, for \(p \in [1, \infty]\), thus extending the QUAD-based global synchronization conditions in [24], [11] and the \(\ell_2\)-norm-based synchronization condition in [3]; and (ii) it is applicable to networks with arbitrary undirected topology, thus generalizing the synchronization conditions developed for specific network topologies in [2].

**Remark 29** (Diffusively-coupled linear systems). Given \(A \in \mathbb{R}^{k \times k}\) and a weighted undirected graph, consider the diffusively-coupled linear identical systems:

\[
\dot{x}_i = Ax_i - \frac{1}{n} \sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1, \ldots, n\}.
\]

It is known that the system (31) achieves synchronization if and only if \(A - \lambda_2(L)I_k\) is Hurwitz, e.g., see [5, Theorem 8.3(ii)]. For a fixed \(p \in [1, \infty]\), condition (26) reads

\[
\mu_{p,Q}(A) \leq -\lambda_2(L).
\]

Using Lemma 7(ii), we get that \(\inf_{Q \text{ invertible}} \mu_{p,Q}(A) = \lambda(A)\). Thus, per Theorem 27(ii), if \(\lambda(A) < \lambda_2(L)\), then the diffusively-coupled linear identical systems (31) achieve synchronization with exponential rate \(\lambda_2(L) - \mu_{p,Q}(A)\).

In other words, condition (26) recovers the exact threshold of synchronization for diffusively-coupled linear systems. We note that the contraction-based synchronization conditions for diffusively-coupled identical oscillators proposed in [3], [11], [2] do not recover this threshold for the linear systems.

**VII. CONCLUSION**

In this paper we have provided multiple analytic extensions of the basic ideas in contraction theory, including advanced results on semi-contracting, weakly-contracting and doubly-contracting systems. We have also illustrated how to apply our results to various network systems. Possible directions for future work include extensions to discrete-time and hybrid systems, to differential-geometric treatments, and to infinite-dimensional systems.

**APPENDIX A**

**USEFUL LEMMATA**

**Lemma 30.** Let \(A \in \mathbb{C}^{n \times n}\) and \(R \in \mathbb{C}^{k \times n}\) is a full rank matrix such that \(\text{Ker}(R)\) is invariant under \(A\). Then

\[
\text{spec}_{\text{Ker}(R^+)}(A^H) = \text{spec}(R^HA^H).
\]

**Proof.** Suppose that \(\lambda \in \text{spec}_{\text{Ker}(R^+)}(A^H)\). This means that there exist \(v \perp \text{Ker}(R)\) and \(\lambda \in \mathbb{C}\) such that \(A^Hv = \lambda v\).
Since $R^t R$ is an orthogonal projection and $v \perp \text{Ker}(R)$, there exists $x \in C^n$ such that $v = R^t Rx$. Thus, $A^H R^t R = \lambda R^t R x$. Multiplying both side by $R$, we get $RA^H R^t R x = \lambda RR^t R x = Rx$. This means that $\lambda \in \text{spec}(RA^H R^t)$. Thus, we have $\text{spec}_{\text{Ker}} R^t (A^H) \subseteq \text{spec}(RA^H R^t)$. On the other hand, since $A \ker(R) \subseteq \ker(R)$ and $R$ is a full-rank matrix, we have $|\text{spec}_{\text{Ker}} R^t (A^H)| = k$. It is easy to see that since $RA^H R^t \in C_{k \times k}$, we have $|\text{spec}(RA^H R^t)| = |\text{spec}_{\text{Ker}} R^t (A^H)| = k$. As a result, $\text{spec}_{\text{Ker}} R^t (A^H) = \text{spec}(RA^H R^t)$.

**Lemma 31 (Properties of (2, p)-tensor norms).** Consider the identification $R^{\otimes k} \simeq R^n \otimes R^k$ and the (2, $p$)-tensor norm defined in (25). Let $A \in R^{n \times k \times k}$, $\Lambda = \text{blkdiag}(\Lambda_1, \ldots, \Lambda_r)$ such that $\Lambda_i \in R^{k \times k}$, for every $i \in \{1, \ldots, n\}$ and $U \in R^{n \times n \times n}$. Then we have $\text{supp}(A) = \text{supp}(A)$.

(i) The (2, $p$)-tensor norm is well-defined:

(ii) $\|U \otimes I_k\|_{(2,p)} = \|U^T \otimes I_k\|_{(2,p)} = 1$.

(iii) $\|A\|_{(2,p)} \leq \max_i |\Lambda_i|$.

(iv) $\mu_{(2,p)}(U \otimes I_k) \leq \mu_{(2,p)}(A)$.

(v) $\mu_{(2,p)}(A) \leq \max_i |\Lambda_i|$.

**Proof.** Regarding part (i), we follow the argument in [35, Proposition 2.1] for projective tensor product norm to show that (2, $p$)-tensor norm is a norm. We first show that, for every $c \in R$, we have $\|cu\|_{(2,p)} = |c|\|u\|_{(2,p)}$. The proof is trivial for $c = 0$. Let $c \neq 0$ and $u = \sum_{i=1}^r v_i \otimes w_i$ be a representation of $u$. Then we have $cu = \sum_{i=1}^r cv_i \otimes w_i$. Using definition (25), this implies that $\|cu\|_{(2,p)} \leq |c|\|u\|_{(2,p)}$. Since $c \neq 0$, we can deduce that $\|u\|_{(2,p)} \leq (1/|c|)\|cu\|_{(2,p)}$. Thus, we get $\|u\|_{(2,p)} = |c|\|u\|_{(2,p)}$.

Now we show that $\|\cdot\|_{(2,p)}$ satisfies triangle inequality. Let $u, v, w \in R^n \otimes R^k$ and let $\epsilon > 0$. Then by definition (25), there exist representations $u = \sum_{i=1}^r v_i \otimes w_i$ and $z = \sum_{i=1}^s x_i \otimes y_i$ such that

$$\sum_{i=1}^r \|v_i\|^2_2 \|w_i\|^2_2 \leq \|u\|_{(2,p)} + \epsilon \frac{2}{2},$$

$$\sum_{i=1}^s \|x_i\|^2_2 \|y_i\|^2_2 \leq \|z\|_{(2,p)} + \epsilon \frac{2}{2}.$$ 

Note that $\sum_{i=1}^s x_i \otimes y_i + \sum_{i=1}^r v_i \otimes w_i$ is a representation of $u + z$. As a result, we have

$$\|u + z\|_{(2,p)} \leq \sum_{i=1}^s \|x_i\|^2_2 \|y_i\|^2_2 + \sum_{i=1}^r \|v_i\|^2_2 \|w_i\|^2_2 \leq \|u\|_{(2,p)} + \epsilon \frac{2}{2}.$$ 

Since $\epsilon$ can be chosen arbitrarily small, then we have $\|u + z\|_{(2,p)} \leq \|u\|_{(2,p)} + \|z\|_{(2,p)}$.

Finally, we show that if $\|u\|_{(2,p)} = 0$, then we have $u = 0_{nk}$. Since $\|u\|_{(2,p)} = 0$, there exists a representation $u = \sum_{i=1}^r v_i \otimes w_i$ such that $\sum_{i=1}^r \|v_i\|^2_2 \|w_i\|^2_2 \leq 1$. Let $\phi : R^n \rightarrow R$ and $\psi : R^k \rightarrow R$ be two linear functionals. Then we have $\sum_{i=1}^r (\phi (v_i))^2 (\psi (w_i))^2 \leq \epsilon \|\phi\|^2 \|\psi\|^2$. Since this inequality holds for every $\epsilon > 0$, then we have $\sum_{i=1}^r (\phi (v_i))^2 (\psi (w_i))^2 = 0$. This implies that, for every $i \in \{1, \ldots, r\}$, either $\phi (v_i) = 0$ or $\psi (w_i) = 0$. Thus, $\sum_{i=1}^r \phi (v_i) \psi (w_i) = 0$. Now using [35, Proposition 1.2], we get $u = 0_{nk}$.

Regarding part (ii), note that, for $u \in R^n \otimes R^k$ with a representation $u = \sum_{i=1}^r v_i \otimes w_i$,

$$\|(U \otimes I_k) u\|_{(2,p)}^2 = \sum_{i=1}^r \|U v_i \otimes w_i\|_{(2,p)}^2 \leq \sum_{i=1}^r \|v_i\|_2^2 \|w_i\|_p^2,$$

where the first inequality holds by definition of (2, $p$)-

Thus, we have $\|\Lambda u\|_{(2,p)}^2 = \sum_{i=1}^r \|v_i e_j \otimes \Lambda_j w_i\|_{(2,p)}^2 \leq \sum_{i=1}^r \|v_i\|_2^2 \|w_i\|_p^2 \leq \sum_{i=1}^r \|v_i\|_2^2 \|w_i\|_p^2$. Since the above inequality holds for every representation of $u$, then $\|(U \otimes I_k) u\|_{(2,p)} \leq \|u\|_{(2,p)}$, for every $u \in R^n \otimes R^k$. This implies that $\|U^T \otimes I_k\|_{(2,p)} \leq 1$. Similarly, one can show that $||U^T \otimes I_k\|_{(2,p)} \leq 1$. However, $(U \otimes I_k)(U^T \otimes I_k) = I_{nk}$, this implies that $\|(U \otimes I_k)(U^T \otimes I_k)\|_{(2,p)} \leq 1$.

Regarding part (iii), for $u \in R^n \otimes R^k$ with a representation $u = \sum_{i=1}^r v_i \otimes w_i$, we have $\Lambda u = \sum_{i=1}^r \sum_{j=1}^r v_i^j e_j \otimes \Lambda_j w_i$, where $e_k$ is the $k$th standard basis in $R^n$, for every $k \in \{1, \ldots, n\}$. This implies that

$$\|\Lambda u\|_{(2,p)}^2 = \sum_{i=1}^r \sum_{j=1}^r \|v_i^j e_j \otimes \Lambda_j w_i\|_{(2,p)}^2 \leq \sum_{i=1}^r \sum_{j=1}^r \|v_i^j\|_2^2 \|\Lambda_j w_i\|_{p}^2 \leq \sum_{i=1}^r \|\Lambda_i\|^2 \sum_{j=1}^r \|v_i^j\|_2^2 \|w_i\|_p^2 \leq \max_i \|\Lambda_i\|^2 \sum_{i=1}^r \|v_i\|_2^2 \|w_i\|_p^2.$$ 

where the first inequality holds by the definition of (2, $p$)-

Thus, we have $\|\Lambda u\|_{(2,p)} \leq \|\Lambda_j\|_{(p)} \|w_i\|_p$, for every $i \in \{1, \ldots, r\}$ and every $j \in \{1, \ldots, n\}$. For the third inequality, we use the fact that $\|\Lambda_j\|_{(p)} \leq \max_i \|\Lambda_i\|_{(p)}$. Finally, for the last equality, we used the fact that $\sum_{j=1}^r \|v_i^j\|_2^2 = \|v_i\|_2^2$. Since the above inequality holds for every representation of $u$, then we have $\|\Lambda u\|_{(2,p)} \leq \max_i \|\Lambda_i\| \|w_i\|_{(2,p)}$, for every $u \in R^n \otimes R^k$. This implies that $\|\Lambda u\|_{(2,p)} \leq \max_i \|\Lambda_i\|_{(p)}$.

Regarding part (iv), note that $U^T = U^t$. As a result, using Lemma 6(ii), we have

$$\mu_{(2,p)}(U \otimes I_k)(A) = \mu_{(2,p)}(U \otimes I_k)(A) = \lim_{h \rightarrow 0^+} \frac{I(n-1) - h(U \otimes I_k)(A)(U \otimes I_k)}{h}(2,p) - 1.$$
Note that, for every $h \in \mathbb{R}_{\geq 0}$,
\[
\|I_{(n-1)k} - h(U \otimes I)A(U^T \otimes I_k)\|_{(2,p)}
\leq \|U \otimes I_k\|_{(2,p)}\|I_{nk} - hA\|_{(2,p)}\|U^T \otimes I_k\|_{(2,p)},
\]
where for the first equality, we used the fact that $UU^T = I_n$, and the last equality holds by part (ii). As a result, we get
\[
\mu_{(2,p),U \otimes I_k}(A) \leq \mu_{(2,p)}(A).
\]
Regrading part (v), note that
\[
\mu_{(2,p)}(A) = \lim_{h \to 0^+} \frac{\|I_{nk} - hA\|_{(2,p)} - 1}{h}.
\]
On the other hand, we have $I_{nk} - hA = \text{blkdiag}(I_k - h\Lambda_1, \ldots, I_k - h\Lambda_n)$. Thus, by part (iii), we have
\[
\mu_{(2,p)}(A) = \lim_{h \to 0^+} \frac{\|I_{nk} - hA\|_{(2,p)} - 1}{h} \leq \lim_{h \to 0^+} \max_i \frac{\|I_k - h\Lambda_i\|_{(2,p)} - 1}{h} = \max_i \mu_p(\Lambda_i).
\]

**Appendix B**

**Proof of Lemma 17**

Since the trajectory $t \mapsto x(t)$ is bounded in $\mathbb{R}^n$, there exists $r > 0$ such that the trajectory $t \mapsto x(t)$ is contained in $\overline{B}_r(x(0)) \cap C$. For every $s \geq 0$, we define the set $U_s = \bigcap_{t \geq s} (\overline{B}_r(x(t)) \cap C)$. It is easy to show that the family of sets $\{U_s\}_{s \geq 0}$ satisfies the following monotonicity property: $U_{s_1} \subseteq U_{s_2}$, for $s_1 < s_2$. Additionally, we define the sets $U$ and $V$ by
\[
U = \bigcup_{s \geq 0} U_s, \quad V = \bigcup_{t \geq 0} (\overline{B}_r(x(t)) \cap C).
\]
Since the trajectory $t \mapsto x(t)$ is bounded, it is easy to see that the set $V$ is bounded and therefore $c(V)$ is a compact in $\mathbb{R}^n$. Moreover, it is easy to check that $U \subseteq c(V)$. We set $W = c(U)$ and we show that $W$ is a non-empty compact convex set with the property $\phi(t,W) \subseteq W$, for every $t \geq 0$.

**Step 1: $W$ is non-empty.** Consider $0 = t_0 \leq t_1 < \ldots < t_n$ and define the set $S = \bigcap_{i=0}^n (\overline{B}_r(x(t_i)) \cap C)$. Let $t \geq t_k$. For every $i \in \{1, \ldots, k\}$, we have
\[
\|x(t) - x(t_i)\| \leq \|x(t - t_i) - x(0)\| \leq r,
\]
where the first inequality holds because the system (14) is weakly contracting and the last inequality holds because $\overline{B}_r(x(t)) \cap C$ contains the whole trajectory $t \mapsto x(t)$. Therefore, the inequalities in (32) implies that $x(t) \in S$, for every $t \geq 0$. Thus $S$ is non-empty. Now consider the family of sets $\{\overline{B}_r(x(t), r) \cap C\}_{t \geq 0}$. By the above argument, it is clear that this family is inside the compact set $c(V)$ and, for every finite set $J \subseteq \mathbb{R}_{\geq 0}$, the intersection $\bigcap_{t \in J} (\overline{B}_r(x(t), r) \cap C)$ is non-empty. Therefore, by [29, Theorem 26.9], for every $s \geq 0$, the set $U_s$ is non-empty and, as a result, $W$ is non-empty.

**Step 2:** $W$ is convex and compact. We start by showing that $U$ is convex. Note that intersection of any collection of convex sets is convex [4]. Therefore, $U_s$ is convex, for every $s \geq 0$. Suppose that $x_1, x_2 \in U$ and $\alpha \in [0,1]$, we show that $\alpha x_1 + (1-\alpha)x_2 \in U$. By definition of $U$, there exists $s_1, s_2 \geq 0$ such that $x_1 \in U_{s_1}$ and $x_2 \in U_{s_2}$. Using the monotonicity of the family $\{U_s\}_{s \geq 0}$, we have that $x_1, x_2 \in U_{s_1+s_2}$. Since $U_{s_1+s_2}$ is convex, we have $\alpha x_1 + (1-\alpha)x_2 \in U_{s_1+s_2} \subseteq U$. This means that $U$ is convex. Moreover, the closure of any convex set is again convex, thus $W = \text{cl}(U)$ is convex. Since $U \subseteq V$ and $V$ is compact, we have $W = \text{cl}(U) \subseteq V$. Thus, $W$ is a closed subset of a compact set and, therefore, it is compact.

**Step 3:** $W$ is invariant under (14). Let $y \in U$. By definition, there exists $s \geq 0$ such that $y \in U_s$. For every $\eta > 0$ and every $s \geq 0$, we have
\[
\|\phi(\eta, y) - x(\eta + t)\| \leq \|y - x(t)\| \leq r,
\]
where the first inequality holds because the system (14) is weakly contracting and the last inequality holds because $y \in U$. This implies that, for every $\eta \geq 0$, we have $\phi(\eta, y) \in U_{s_1+s_2}$. This implies that $\phi(\eta, y) \in U$. So we have $\phi(\eta, U) \subseteq U$, for every $\eta > 0$. By a simple continuity argument and using the fact that $W$ is closed, we get $\phi(\eta, W) \subseteq W$.

**Appendix C**

**The Lotka–Volterra model**

In this appendix, we study the Lotka–Volterra model with mutualistic interactions and prove that it is weakly contracting. The Lotka–Volterra model is given by
\[
x = \text{diag}(x)(Ax + r),
\]
where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n_{>0}$ is the vector of populations, $A \in \mathbb{R}^{n \times n}$ is the mutual interaction matrix, and $r > 0_n$ is the intrinsic growth rate. We assume $A$ is Metzler, i.e., the interactions between any two species are mutualistic.

**Theorem 32** (Global asymptotic stability and Lyapunov functions). Consider the Lotka–Volterra model (33) with a Metzler and Hurwitz interaction matrix $A$. Let the positive vector $v \in \mathbb{R}^n_{>0}$ satisfy $v^TA \prec 0_n$. Then

(i) the open positive orthant $\mathbb{R}^n_{>0}$ is invariant, (ii) $x^* = -A^{-1}r > 0_n$, is the unique globally asymptotically stable equilibrium point of (33) restricted to $\mathbb{R}^n_{>0}$, (iii) the following distance between any two trajectories $x(t)$ and $z(t)$ is decreasing:
\[
d_{LV}(x(t), z(t)) = \sum_{i=1}^n v_i |\ln(x_i(t)/z_i(t))|,
\]
where the following functions are global Lyapunov functions:
\[
x \mapsto \sum_{i=1}^n v_i |\ln(x_i/z_i)|, \quad x \mapsto \sum_{i=1}^n v_i (Ax + r)_i.
\]

**Proof.** We omit the proof of statement (i) in the interest of brevity. For $x \in \mathbb{R}^n_{>0}$, let $y_i = \ln(x_i) \in \mathbb{R}$, $i \in \{1, \ldots, n\}$ and write the Lotka–Volterra model (33) as
\[
y = A\exp(y) + r := f_{LV}(y),
\]
where $y$ and its entry-wise exponential $\exp(y)$ are vectors in $\mathbb{R}^n$. Note that $Df_{\text{Lyap}}(y) = A \text{diag}(\exp(y))$ is Metzler since $\exp(y) > 0_n$. Since $v \in \mathbb{R}_{>0}$ satisfies $v^T A < 0_n$, there exists $c > 0$ such that $v^T c \preceq v^T c$. Therefore, for every $y \in \mathbb{R}^n$, $v^T Df_{\text{Lyap}}(y) = v^T A \text{diag}(\exp(y)) < -c v^T \text{diag}(\exp(y)) \preceq 0$.

We now recall [9] that, for a Metzler matrix $M$, a positive vector $v$ and a scalar $b$, $v^T M \leq b v^T \iff \mu_1(\text{diag}(v)^{-1}(M)) \leq b$.

This equivalence implies that $f_{\text{Lyap}}$ is weakly contracting in its domain. After a change of coordinates, this establishes statement (iii). At the equilibrium point $x^* = -A^{-1} r > 0_n$, that is, at $\exp(y^*) = -A^{-1} r \in \mathbb{R}_{>0}$, we know $v^T Df_{\text{Lyap}}(y^*) \leq -c v^T \text{diag}(\exp(y^*)) = -c \left( \min_{i \in \{1, \ldots, n\}} (-A^{-1} r)_i \right) v^T$, $\iff \mu_1(\text{diag}(v)) Df_{\text{Lyap}}(y^*) \leq -c \min_{i \in \{1, \ldots, n\}} (-A^{-1} r)_i$.

We now invoke Theorem 18(v): $f_{\text{Lyap}}$ is weakly contracting over the entire $\mathbb{R}^n$ and has strictly negative matrix measure at the equilibrium point $y^*$. Therefore, $y^*$ is the unique globally exponentially stable equilibrium with global Lyapunov functions $y \mapsto \|y - y^*\|_{1, \text{diag}(v)}$, and $y \mapsto \|f_{\text{Lyap}}(y)\|_{1, \text{diag}(v)}$.

Facts (ii) and (iv) follow from a change of coordinates.

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