Maschke functors, semisimple functors and separable functors of the second kind. Applications

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Abstract

We introduce separable functors of the second kind (or $H$-separable functors) and $H$-Maschke functors. $H$-separable functors are generalizations of separable functors. Various necessary and sufficient conditions for a functor to be $H$-separable or $H$-Maschke, in terms of generalized (co)Casimir elements (integrals, in the case of Hopf algebras), are given. An $H$-separable functor is always $H$-Maschke, but the converse holds in particular situations. A special role will be played by Frobenius functors and their relations to $H$-separability. Our concepts are applied to modules, comodules, entwined modules, quantum Yetter-Drinfeld modules, relative Hopf modules.

0 Introduction

One of the fundamental results in classical representation theory is Maschke’s Theorem, stating that a finite group algebra $kG$ over a field $k$ is semisimple if and only if the characteristic of $k$ does not divide the order of $G$. Several generalizations of this result have appeared in the literature. To illustrate that there is a subtle difference, let us look more carefully at one of the earliest generalization, where the ground field $k$ is replaced by a commutative ring $k$. Algebras over a commutative ring are rarely semisimple, and one arrives at the following result: a finite group algebra $kG$ over a field $k$ is separable if and only if the characteristic of $k$ does not divide the order of $G$. The interesting thing is that, over a field $k$, a separable finite dimensional algebra is semisimple, but not conversely: it suffices to look at a purely inseparable field extension. A consequence of the two versions of Maschke’s Theorem is then that, for a finite group algebra (and, more generally, for a finite dimensional Hopf algebra) over a field, separability and semisimplicity are equivalent.

An elegant categorical definition of separability has been proposed by Năstăsescu et al. in [19]. A functor $F$ is called separable if and only if the natural transformation $\mathcal{F}$ induced by $F$ is split by a natural transformation $\mathcal{P}$. It is a proper generalization of the notion of separable algebra, in the sense that a $k$-algebra $A$ is separable if and only if the restriction of scalars functor $F : \mathcal{M}_A \to \mathcal{M}_k$ is separable [19, Prop. 1.3]. Moreover, a separable functor $F$ between two abelian categories satisfies the following version of Maschke’s Theorem: an exact sequence, that becomes split after applying $F$ is itself split. If we apply this property to the restriction of scalars functor in the case of an algebra $A$ over a field $k$, then we easily deduce that this algebra is semisimple. We also point out that many of the recent Maschke-type Theorems (see e.g. [8], [5], [4])... come down to proving that a certain functor is separable.

*Research supported by the bilateral project “Hopf Algebras in Algebra, Topology, Geometry and Physics” of the Flemish and Romanian governments.
Now consider a separable algebra $A$ over a field. What are its properties that distinguish it from a semisimple algebra? The answer is the following: $P$ allows to deform a $k$-linear splitting map $f$ between two $A$-modules in such a way that it becomes $A$-linear; this can also be done in the semisimple case, but in the separable case the deformation is natural in $f$!

In Section 3, we will propose categorical properties of functors that, when applied to the restriction of scalars functor in the case of an algebra over a field $k$, are equivalent to semisimplicity of the algebra. The starting point is the following: an algebra $A$ over a field is semisimple if and only if every $A$-module is projective, if and only if every $A$-module is injective. We will say that a functor $F : C \to D$ is a Maschke (resp. dual Maschke) functor if every object in $C$ is relative injective (resp. projective). A functor $F$ between abelian categories is called semisimple if and only if an exact sequence that becomes split after applying $F$ is itself split. The three notions are equivalent for a functor reflecting monics and epics (see Proposition 3.7).

In Section 2, we introduce another generalization of separable functors; consider an exact sequence of graded modules over a $G$-graded $k$-algebra $A$. Suppose that the sequence is split after we forget the $A$-action and the $G$-grading; separability of the functor forgetting action and grading would imply that the sequence is split as a sequence of graded $A$-modules. When can we conclude that the sequence is at least split as a sequence of $A$-modules? Or consider the following: an exact sequence of $A$-modules, with $A$ a $k$-algebra, which is split as a sequence of $k$-modules. Is it split as a sequence of $B$-modules, where $B$ is a given subalgebra of $A$.

This leads us to the following: let $F : C \to D$ and $H : C \to E$ be functors. We take an $H$-separable functor, or separable functor of the second kind, if the natural transformation $H$ induced by $H$ factorizes as a natural transformation trough $F$ induced by $F$:

$$H = P \circ F,$$

for a natural transformation $P$. If $H$ is the identity functor, then we recover the separable functors of $[19]$. Most properties of separable functors (Maschke’s Theorem, Rafael’s Theorem, the Frobenius-Rafael Theorem) can be generalized to $H$-separable functors, we discuss this in Section 3. Also the notion of Maschke functor, dual Maschke functor, and semisimple functor can be generalized in the same spirit; in fact, we decided to present at once the general theory of (dual) $H$-Maschke functors in Section 3.

In Section 4, we present some examples and applications, we look at the categories of modules, comodules, entwined modules, Hopf modules and relative Hopf modules. We present a structure Theorem for injective objects in the category of entwined modules, that arose from noncommutative geometry $[3]$. A separable functor is always Maschke (and dual Maschke), and in some particular cases we have the converse property. As a first example, we have modules over a group algebra or a Hopf algebra. The fact that Maschke implies separability comes from the fact that the separability of a Hopf algebra can be described in terms of integrals in the Hopf algebra. A similar phenomenon appears if we look at relative Hopf modules: if the functor forgetting action and coaction is $H$-Maschke ($H$ is the functor forgetting $A$-action), then it is also $H$-separable. Both conditions (Theorem 4.20) are equivalent to the fact that there exists a total integral in the sense of Doi $[13]$. This gives another motivation for introducing the $H$-separability concept.

1 Preliminaries

Let $k$ be a commutative ring. For a $k$-coalgebra $C$, we use the Sweedler-Heyneman notation for the comultiplication $\Delta_C$:

$$\Delta_C(c) = c_{(1)} \otimes c_{(2)}$$
following compatibility relation holds between the action and coaction:

$$\rho_M(m) = m_{[0]} \otimes m_{[1]} \in M \otimes C$$

$\mathcal{M}^C$ will be the category of right $C$-comodules and $C$-colinear maps. A right $C$-comodule $M$ is called relative injective if for any $k$-split monomorphism $i : U \rightarrow V$ in $\mathcal{M}^C$ and for any $C$-colinear map $f : U \rightarrow M$, there exists a $C$-colinear map $g : V \rightarrow M$ such that $g \circ i = f$. This is equivalent to the fact that $\rho_M : M \rightarrow M \otimes C$ splits in $\mathcal{M}^C$, i.e. there exists a $C$-colinear map $\lambda_M : M \otimes C \rightarrow M$ such that $\lambda_M \circ \rho_M = \mathrm{id}_M$. Of course, if $k$ is a field, $M$ is relative injective if and only if it is an injective object in $\mathcal{M}^C$. Relative projective modules over a $k$-algebra $A$ are defined dually.

A (right-right) entwining structure ([3]) is a triple $(A, C, \psi)$, where $A$ is a $k$-algebra, $C$ is a $k$-coalgebra, and $\psi : C \otimes A \rightarrow A \otimes C$ is a $k$-linear map satisfying the conditions

$$(ab) \otimes c^\psi = a \psi b \otimes c^{\psi \psi} \quad (1)$$
$$(1_A) \otimes c^\psi = 1_A \otimes c \quad (2)$$
$$a \psi \otimes \Delta_C(c^\psi) = a \psi \otimes c^\psi \otimes c_{(1)}^\psi \otimes c_{(2)}^\psi \quad (3)$$
$$\varepsilon_C(c^\psi) a \psi = \varepsilon_C(c) a \quad (4)$$

Here we used the sigma notation

$$\psi(c \otimes a) = a \psi \otimes c^\psi = a \psi \otimes c^\psi$$

Entwining structures where introduced with a motivation coming from noncommutative geometry: one can generalise the notion of principal bundles to a very general setting in which the role of coordinate functions on the base is played by a general noncommutative algebra $A$, and the fibre of the principal bundle by a coalgebra $C$, where $A$ and $C$ are related by a map $\psi : A \otimes C \rightarrow C \otimes A$, called the entwining map.

An entwining module $M$ is at the same time a right $A$-module and a right $C$-comodule such that the following compatibility relation holds between the action and coaction:

$$\rho_M(ma) = m_{[0]} a \psi \otimes m_{[0]}^\psi$$

$\mathcal{M}(\psi)^C_A$ is the category of entwined modules and $A$-linear $C$-colinear maps. The forgetful functors

$$F : \mathcal{M}(\psi)^C_A \rightarrow \mathcal{M}_A \text{ and } H : \mathcal{M}(\psi)^C_A \rightarrow \mathcal{M}^C$$

have respectively a right and a left adjoint (6)

$$G = \bullet \otimes C \text{ and } K = \bullet \otimes A.$$ 

A Doi-Koppinen datum $(H, A, C)$ consists of a bialgebra $H$, a right $H$-comodule algebra $A$ and a right $H$-module coalgebra $C$. Associated to it is an entwined structure $(A, C, \psi)$, with

$$\psi(c \otimes a) = a_{[0]} \otimes ca_{[1]}$$

The following special cases will be of interest to us:

1) $C = H$, where $H$ is a Hopf algebra. In this case $\mathcal{M}(\psi)^C_A = \mathcal{M}^H_A$, the category of relative Hopf modules.
2) $A = H$, where again $H$ is a Hopf algebra. Now $\mathcal{M}(\psi)^C_A = \mathcal{M}_H^C$, the category of Doi’s $[H, C]$-modules.
2 $H$-separable functors

Let $F : C \to D$ and $H : C \to E$ be covariant functors. We then have functors

$$\text{Hom}_C(\bullet, \bullet), \text{Hom}_D(F, F), \text{Hom}_E(H, H) : C^{\text{op}} \times C \to \text{Sets}$$

and natural transformations

$$\mathcal{F} : \text{Hom}_C(\bullet, \bullet) \to \text{Hom}_D(F, F) ; \quad \mathcal{H} : \text{Hom}_C(\bullet, \bullet) \to \text{Hom}_E(H, H)$$

given by

$$\mathcal{F}_{C, C'}(f) = F(f) ; \quad \mathcal{H}_{C, C'}(f) = H(f)$$

for $f : C \to C'$ in $C$.

**Definition 2.1** The functor $F$ is called $H$-separable if there exists a natural transformation

$$\mathcal{P} : \text{Hom}_D(F, F) \to \text{Hom}_E(H, H)$$

such that

$$\mathcal{P} \circ \mathcal{F} = \mathcal{H} \quad (5)$$

that is, $\mathcal{H}$ factors through $\mathcal{F}$ as a natural transformation, and we have a commutative diagram

$$\begin{array}{ccc} 
\text{Hom}_C(\bullet, \bullet) & \xrightarrow{\mathcal{F}} & \text{Hom}_D(F, F) \\
\downarrow{\mathcal{H}} & & \downarrow{\mathcal{P}} \\
\text{Hom}_E(H, H) & & 
\end{array}$$

**Remarks 2.2**

1) $F$ is $1_C$-separable if and only if $F$ is separable in the sense of [19]. Indeed, the functor $F$ is separable if and only if there exists a natural transformation $\mathcal{P}$ such that $\mathcal{P} \circ \mathcal{F} = 1_C$ (see [4]). We refer to [4] for a detailed study of separable functors. A finite extension of commutative fields $k \subset K$ is separable in the classical sense if and only if the forgetful functor $F : M_K \to M_k$ is separable. For the reader convenience we show how the above natural transformation $\mathcal{P}$ is constructed: let $K/k$ be a finite separable extension, $\alpha \in K$ be a primitive element (i.e. $K = k(\alpha)$) and $p \in K[X]$, $p(X) = X^n - \sum_{i=0}^{n-1} c_iX^i$ be the minimal polynomial of $\alpha$. Then the natural transformation $\mathcal{P}$ is constructed as follows: for $M, N$ two $K$-vector space we define

$$\mathcal{P}_{M,N} : \text{Hom}_k(M, N) \to \text{Hom}_K(M, N), \quad \mathcal{P}_{M,N}(f)(m) := p'(\alpha)^{-1} \sum_{i=0}^{n-1} \alpha^{-i-1}(\sum_{j=0}^{i} c_j\alpha^j)f(\alpha^im)$$

for any $f \in \text{Hom}_k(M, N)$ and $m \in M$. Then $\mathcal{P}$ is a natural transformation that splits $\mathcal{F}$. This is a one of remarkable property of classical separable fields extension $K/k$: any $k$-linear map $f$ between two $K$-vector spaces can be deformed, using the above formula, until it becomes a $K$-linear map.

As we will see below, most properties of separable functors can be generalized to $H$-separable functors.

2) The fact that $\mathcal{P}$ is natural means the following condition: for

$$u : X \to Y, v : Z \to T \text{ in } C \quad \text{and} \quad h : F(Y) \to F(Z) \text{ in } D,$$
we have
\[ P_{X,T}(F(v) \circ h \circ F(u)) = H(v) \circ P_{Y,Z}(h) \circ H(u) \]  
(6)

(5) can be rewritten as
\[ P_{C,C'}(F(f)) = H(f) \]  
(7)

for any \( f : C \to C' \) in \( \mathcal{C} \).

**Proposition 2.3** Consider functors

\[ \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{F_1} \mathcal{D}_1 \quad \text{and} \quad \mathcal{C} \xrightarrow{H} \mathcal{E} \]

1) If \( F_1 \circ F \) is H-separable, then \( F \) is H-separable.
2) If \( F \) is H-separable, and \( F_1 \) is separable, then \( F_1 \circ F \) is H-separable.

**Proof.** Obvious. \( \square \)

**Proposition 2.4** Let \( F \) be an H-separable functor. If \( f : C \to C' \) in \( \mathcal{C} \) is such that \( F(f) \) has a left, right, or two-sided inverse in \( \mathcal{D} \), then \( H(f) \) has a left, right, or two-sided inverse in \( \mathcal{E} \).

**Proof.** Let \( g \) be a left inverse of \( F(f) \). Using (6) and (7), we find
\[ P_{C,C'}(g) \circ H(f) = P_{C,C'}(g \circ F(f)) = P_{C,C'}(I_{F(C)}) \]
\[ = P_{C,C'}(F(I_C)) = H(I_C) = I_{H(C)} \]

The proof for right and two-sided inverses is similar. \( \square \)

**Corollary 2.5** (Maschke’s Theorem for H-separable functors) Let \( \mathcal{C}, \mathcal{D} \) and \( \mathcal{E} \) be abelian categories, and assume that \( F : \mathcal{C} \to \mathcal{D} \) is H-separable. An exact sequence in \( \mathcal{C} \) that becomes split after we apply the functor \( F \), also becomes split after we apply the functor \( H \).

Recall that Rafael’s Theorem (see [20]) gives an easy criterion for the separability a functor that has a left or right adjoint. We will now generalize Rafael’s Theorem to H-separable functors. First, we recall the following well-known result from category theory. For completeness sake, we include a brief sketch of proof, based on the well-known property that \((F,G)\) a pair of adjoint functors between the categories \( \mathcal{C} \) and \( \mathcal{D} \) if and only if there exist two natural transformations \( \eta : 1_{\mathcal{C}} \to GF \) and \( \varepsilon : FG \to 1_{\mathcal{D}} \), called the unit and counit of the adjunction, such that
\[ G(\varepsilon_D) \circ \eta_{G(D)} = I_{G(D)} \quad \text{and} \quad \varepsilon_{F(C)} \circ F(\eta_C) = I_{F(C)} \]  
(8)

for all \( C \in \mathcal{C} \) and \( D \in \mathcal{D} \).

**Proposition 2.6** Let \( G : \mathcal{D} \to \mathcal{C} \) be a right adjoint of \( F : \mathcal{C} \to \mathcal{D} \), and consider functors \( H : \mathcal{C} \to \mathcal{E} \) and \( K : \mathcal{D} \to \mathcal{E} \). Then we have isomorphisms

\[ \text{Nat}(HGF,H) \cong \text{Nat}\left(\text{Hom}_{\mathcal{D}}(F,F),\text{Hom}_{\mathcal{E}}(H,H)\right) \]

\[ \text{Nat}(K,KFG) \cong \text{Nat}\left(\text{Hom}_{\mathcal{C}}(G,G),\text{Hom}_{\mathcal{E}}(K,K)\right) \]
Proof. For \( \nu : HGF \to H \), we define
\[
\mathcal{P} = \alpha(\nu) : \text{Hom}_D(F, F) \to \text{Hom}_E(H, H)
\]
as follows: for \( g : F(C) \to F(C') \) in \( D \), we put
\[
\mathcal{P}_{C,C'}(g) = \nu_{C'} \circ HG(g) \circ H(\eta_C)
\]
Conversely, given \( \mathcal{P} : \text{Hom}_D(F, F) \to \text{Hom}_E(H, H) \), we define \( \alpha^{-1}(\mathcal{P}) : HGF \to H \) by
\[
\nu_C = \mathcal{P}_{GF(C), C}(\epsilon_{F(C)})
\]
for any \( C \in C \). \( \square \)

**Theorem 2.7 (Rafael’s Theorem for \( H \)-separability)** Let \( G : D \to C \) be a right adjoint of \( F : C \to D \), and consider functors \( H : C \to E \) and \( K : D \to E \). Then:
1) \( F \) is \( H \)-separable if and only if there exists a natural transformation \( \nu : HGF \to H \) such that
\[
\nu_C \circ H(\eta_C) = I_{H(C)} \tag{9}
\]
for any \( C \in C \).
2) \( G \) is \( K \)-separable if and only if there exists a natural transformation \( \zeta : K \to KFG \) such that
\[
K(\epsilon_D) \circ \zeta_D = I_{K(D)} \tag{10}
\]
for any \( D \in D \).

**Proof.** We only prove the first statement; the proof of the second one is similar. We use the notation introduced in the proof of Proposition 2.6. Assume that \( F \) is \( H \)-separable, and put \( \nu = \alpha^{-1}(\mathcal{P}) \). Then we compute
\[
\nu_C \circ H(\eta_C) = \mathcal{P}_{GF(C), C}(\epsilon_{F(C)}) \circ H(\eta_C) \tag{8}
\]
Conversely, assume that \( \nu \) satisfies (8), and take \( \mathcal{P} = \alpha(\nu) \). Using (8), we find
\[
\mathcal{P}(F(f)) = \nu_C \circ HGF(f) \circ H(\eta_C) = H(f) \circ \nu_C \circ H(\eta_C) = H(f)
\]
as needed. \( \square \)

Recall [7] that a functor \( F \) is called Frobenius if \( F \) has a right adjoint \( G \) that is also a left adjoint. \( (F, G) \) is then called a Frobenius pair. In [6], a Rafael-type criterion for the separability of a Frobenius functor is given. We will now generalize this to \( H \)-separability. First, we need the following standard fact from category theory.

**Proposition 2.8** Let \( G \) be a left adjoint of the functor \( F : C \to D \), and \( H : C \to E \) a functor. Then we have an isomorphism
\[
\text{Nat}(HGF, H) \cong \text{Nat}(HG, HG)
\]
Proof. (sketch) Let
\[ \mu : GF \to 1_C \text{ and } \chi : 1_D \to FG \]
be the counit and unit of the adjunction \((G, F)\). For \(v : HGF \to H\), we define \(\beta = \beta_v : HG \to HG\) by
\[ \beta_D = v_G(D) \circ HG(\chi_D) : HG(D) \to HG(D) \]
for every \(D \in D\). Conversely, given \(\beta : HG \to HG\), we define \(v = v_\beta : HGF \to H\) as follows:
\[ v_C = H(\mu_C) \circ \beta_F(C) : HFG(C) \to H(C) \]
for every \(C \in C\). \(\square\)

For a Frobenius pair of functors \((F, G)\), we will write
\[ \chi : 1_D \to FG \text{ and } \mu : GF \to 1_C \]
be the unit and counit of the adjunction \((G, F)\) and
\[ \eta : 1_C \to GF \text{ and } \varepsilon : FG \to 1_D \]
the unit and counit of the adjunction \((F, G)\).

**Proposition 2.9** Let \((F, G)\) be a Frobenius pair and \(H : C \to \mathcal{E}\) a functor. Then \(F\) is \(H\)-separable if and only if there exists a natural transformation \(\beta : HG \to HG\) such that
\[ H(\mu_C) \circ \beta_F(C) \circ H(\eta_C) = I_{H(C)} \]  
for all \(C \in C\).

**Proof.** First we will apply Theorem 2.7 to the adjunction \((F, G)\): we obtain that \(F\) is \(H\)-separable if and only if there exists a natural transformation \(\nu : HGF \to H\) such that (9) holds.

Now, we apply Proposition 2.8 to the adjunction \((G, F)\) to obtain the corresponding natural transformation \(\beta = \beta_\nu\). Furthermore, (9) holds for \(\nu\) if and only if (11) holds for \(\beta = \beta_\nu\). \(\square\)

### 3 Relative injectivity and Maschke functors

**Definition 3.1** Let \(F : C \to \mathcal{D}\) and \(H : C \to \mathcal{E}\) be covariant functors. An object \(M \in C\) is called \(F\)-relative \(H\)-injective if the following condition is satisfied: for any \(i : C \to C'\) in \(C\) with \(F(i) : F(C) \to F(C')\) a split monic in \(\mathcal{D}\), and for every \(f : C \to M\) in \(C\), there exists \(g : H(C') \to H(M)\) in \(\mathcal{E}\) such that \(H(f) = g \circ H(i)\), that is, the following diagram commutes in \(\mathcal{E}\):

\[
\begin{array}{ccc}
H(C) & \xrightarrow{H(i)} & H(C') \\
\downarrow H(f) & & \downarrow g \\
H(M) & & \\
\end{array}
\]

(12)

\(F\) is called an \(H\)-Maschke functor if any object of \(C\) is \(F\)-relative \(H\)-injective.

An \(F\)-relative \(1_C\)-injective object is also called an \(F\)-relative injective object. A \(1_C\)-Maschke functor is also called a Maschke functor.

\(P \in C\) is called \(F\)-relative \(H\)-projective if \(P\) is \(F^{\text{op}}\)-relative \(H^{\text{op}}\)-injective, where \(F^{\text{op}} : C^{\text{op}} \to \mathcal{D}^{\text{op}}\) is the functor opposite to \(F\).

\(F\) is called a dual \(H\)-Maschke functor if any object of \(C\) is \(F\)-relative \(H\)-projective.
Examples 3.2 1) Every object of $C$ is $1_C$-relative injective.
2) Let $A$ be an algebra over a field $k$, $D$ the category of $k$-vector spaces, and $C = M_k$ the category of right $A$-modules (or representations of $A$). The restrictions of scalars functor $F : M_k \rightarrow M_k$ is exact, and every monic (resp. epic) in $M_k$ splits (resp. cosplits), and therefore an $A$-module $M$ is $F$-relative injective or projective if and only if it is injective or projective as an $A$-module. Thus $F$ is Maschke if and only if every $A$-module is injective, and $F$ is dual Maschke if and only if every $A$-module is projective. It is well-known that both conditions are equivalent to $A$ being semisimple, see e.g. [9, Th. 5.3.7]. The classical Maschke Theorem can therefore be restated as follows in our terminology: for a finite group $G$, the restriction of scalars functor $F : M_kG \rightarrow M_k$ is a Maschke functor if and only if the order of $G$ does not divide the characteristic of $k$. We will come back to this in Proposition 4.2.

Proposition 3.3 Any $H$-separable functor $F : C \rightarrow D$ is at the same time an $H$-Maschke and a dual $H$-Maschke functor.

Proof. We will prove first that $F$ is a $H$-Maschke functor, and leave the proof of the second statement to the reader. Take an object $M \in C$, and let $i$ and $f$ be as in Definition 3.1. Then define
\[
g = H(f) \circ P_{C,C}(p)
\]
where $p$ is a left inverse of $F(i) : F(C) \rightarrow F(C')$ in $D$. Using (8), we obtain
\[
g \circ H(i) = H(f) \circ P_{C,C}(p) \circ H(i) = P_{C,M}(F(f) \circ p \circ F(i)) = P_{C,M}(F(f)) = H(f)
\]
as needed. □

Our next result is a Rafael-type Theorem for Maschke and dual Maschke functors.

Theorem 3.4 Assume that the functor $F : C \rightarrow D$ has a right adjoint $G : D \rightarrow C$.
1) $M \in C$ is $F$-relative $H$-injective if and only if $H(\eta_M) : H(M) \rightarrow HGF(M)$ has a left inverse in $E$. In particular, $F$ is an $H$-Maschke functor if and only if every $H(\eta_M)$ splits in $E$.
2) $P \in D$ is $G$-relative $H$-projective if and only if $H(\varepsilon_P) : HFG(P) \rightarrow H(P)$ has a right inverse. In particular, $G$ is a dual $H$-Maschke functor if and only if every $H(\varepsilon_P)$ cosplits in $E$.

Proof. 1) Assume first that $M$ is $F$-relative $H$-injective. Consider the unit map $\eta_M : M \rightarrow GF(M)$ in $C$. From (8), we know that $F(\eta_M)$ has a left inverse in $D$, so there exists a map $\nu_M : HGF(M) \rightarrow H(M)$ in $\mathcal{E}$ making the diagram
\[
\begin{array}{ccc}
H(M) & \xrightarrow{H(\eta_M)} & HGF(M) \\
I_{H(M)} & & \downarrow \nu_M \\
H(M) & \downarrow \\
\end{array}
\]
commutative. This means that $\nu_M$ is a left inverse of $H(\eta_M)$.
Conversely, assume that $H(\eta_M)$ has a left inverse $\nu_M$, and consider $i : C \rightarrow C'$, $f : C \rightarrow M$, with $p : F(C') \rightarrow F(C)$ a left inverse of $F(i)$. Then take
\[
g = \nu_M \circ HGF(f) \circ HG(p) \circ H(\eta_C) : H(C') \rightarrow H(M)
\]
\( \eta \) is a natural transformation, hence the diagrams

\[
\begin{array}{ccc}
C & \xrightarrow{i} & C' \\
\downarrow{\eta_C} & & \downarrow{\eta_{C'}} \\
GF(C) & \xrightarrow{GF(i)} & GF(C')
\end{array}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ effic
4 Examples and applications

Extension and restriction of scalars

We consider ring morphisms \( Q \to R \to S \) and \( T \to S \). Associated to these morphisms are the restriction of scalars functors

\[
\mathcal{M}_S \xrightarrow{G} \mathcal{M}_R \quad \mathcal{M}_S \xrightarrow{G_1} \mathcal{M}_T \quad \mathcal{M}_R \xrightarrow{G_2} \mathcal{M}_Q
\]

and their left adjoints, the induction functors

\[
S\mathcal{M} \xrightarrow{G'} \mathcal{M}_R \quad S\mathcal{M} \xrightarrow{G_1'} \mathcal{T}\mathcal{M} \quad R\mathcal{M} \xrightarrow{G_2'} \mathcal{Q}\mathcal{M}
\]

Proposition 4.1 The following assertions are equivalent:

- \( G : \mathcal{M}_S \to \mathcal{M}_R \) is \( G_1 \)-separable;
- \( G' : S\mathcal{M} \to \mathcal{M}_R \) is \( G'_1 \)-separable;
- there exists an element \( e = \sum e^1 \otimes_R e^2 \in S \otimes_R S \) such that

\[
\sum t e^1 \otimes_R e^2 = \sum e^1 \otimes_R e^2 t, \quad \text{for all } t \in T \tag{13}
\]

\[
\sum e^1 e^2 = 1 \tag{14}
\]

Proof. Basically, this follows from the fact that \( \text{Nat}(G_1, G_1 FG) \) is in bijective correspondence with the set of \( e \) satisfying (13): for a natural transformation \( \zeta : G_1 \to G_1 FG \), the map \( \zeta_S : S \to S \otimes_R S \) is right \( T \)-linear. For any \( a \in S \), consider \( f_a : S \to S \), \( f_a(s) = as \). Then \( f_a \in \mathcal{M}_S \), and the naturality of \( \zeta \) implies that

\[
(f_a \otimes I_S) (\zeta_S(s)) = \zeta_S(f_a(s))
\]

Let \( s = 1 \) and \( \zeta_S(1) = \sum e^1 \otimes_R e^2 \). Then (13) follows. Conversely, given \( e \) satisfying (13), we construct a natural transformation \( \zeta \) as follows:

\[
\zeta_M : M \to M \otimes_R S; \quad \zeta_M(m) = \sum me^1 \otimes_R e^2
\]

It follows from (13) that \( \zeta_M \) is right \( T \)-linear, and we leave it to the reader to show that \( \zeta \) is natural. If \( e \) satisfies (14), then for all \( M \in \mathcal{M}_R \) and \( m \in M \):

\[
(G_1(\varepsilon_M) \circ \zeta_M)(m) = \varepsilon_M(\sum me^1 \otimes_R e^2) = \sum me^1 e^2 = m
\]

and it follows from Theorem 2.7 that \( G \) is \( G_1 \)-separable. The converse, and the equivalence between the second and third assertion is done in a similar way. \( \square \)

Let us explain what this means. It is well-known [19], and actually a special case of Proposition 4.1, that \( \mathcal{M}_S \to \mathcal{M}_R \) is a separable functor if and only if \( R \to S \) is separable in the sense of [11], which means that there exists \( e \in S \otimes_R S \) satisfying (14), and (13) also, but for all \( t \in S \). In this situation, an exact sequence in \( \mathcal{M}_S \) that splits in \( \mathcal{M}_R \) also splits in \( \mathcal{M}_S \). In Proposition 4.1, we have \( e \in S \otimes_R S \) satisfying (14), and (13), but only for \( t \) in a subring \( T \) of \( S \). We then have the weaker conclusion that an exact sequence in \( \mathcal{M}_S \) that splits in \( \mathcal{M}_R \) also splits in \( \mathcal{M}_T \).

On the other hand, a nice ring-theoretical problem arises from the concept of (dual) \( H \)-Maschke functor: \( \textbf{1} \)

Let \( T \to S \) be a ring morphism. When is any right \( S \)-module is projective (injective) as a right \( T \)-module? Using Proposition 4.1 we obtain a sufficient condition is obtain:
Corollary 4.2 Let $T \rightarrow S$ be a morphism of $k$-algebras over a field $k$ such that $S$ is projective as a right $T$-module. Assume that there exists $e = \sum e^1 \otimes e^2 \in (S \otimes S)^T$ such that $\sum e^1 e^2 = 1_S$. Then any right $S$-module is projective as a right $T$-module.

Proof. We take $R = k$ is the Proposition 4.1. If such an $e$ exists, then the forgetful functor $G : M_k \rightarrow M_k$ is $G_1$-separable, where $G_1 : M_k \rightarrow M_T$ is the restriction of scalars functor. Hence, $G$ is a dual $G_1$-Maschke functor. Let $M$ be a right $S$-module; as $k$ is a field, the right $S$-module structure on $M$, $\nu_M : M \otimes S \rightarrow M$, has a section in $M_k$. Thus, there exists $f : M \rightarrow M \otimes S \cong S^{(M)}$ a right $T$-module map such that $\nu_M \circ f = \text{Id}_M$, i.e. $M$ is a direct summand of $S^{(M)}$ as a right $T$-submodule. As $S$ is projective in $M_T$, we obtain that $M$ is projective as a right $T$-module. \hfill \Box

We have a similar result for split extensions:

Proposition 4.3 The induction functor $F = - \otimes_R S$ is $G_2$-separable if and only if the ring morphism $R \rightarrow S$ is split as a map of $(R, Q)$-bimodules.

Proof. Assume that $F$ is $G_2$-separable. According to Theorem 2.7, there exists a natural transformation $\nu : G_2 GF \rightarrow G_2$ such that

$$\nu_M \circ G_2(\eta_M) = I_{G_2(M)}$$

for all $M \in M_R$. This means that $\nu_R : S \rightarrow R$ splits $R \rightarrow S$ as a map of right $Q$-modules. From the naturality of $\nu$, we can deduce that $\nu_R$ is also left $R$-linear, and it follows that $R \rightarrow S$ is split as a map of $(R, Q)$-bimodules. The converse is left to the reader. \hfill \Box

We now assume that $(F, G)$ is a Frobenius pair of functors; this means that the ring extension $R \rightarrow S$ is Frobenius, and it is equivalent to the existence of a Frobenius system (cf. e.g. [2] or [14]). A Frobenius system consists of a pair $(\overline{\eta}, f)$, where $\overline{\eta} : S \rightarrow R$ is an $R$-bimodule map, $f = \sum f^1 \otimes_R f^2 \in S \otimes_R S$ is a Casimir element i.e.

$$\sum s f^1 \otimes_R f^2 = \sum f^1 \otimes_R f^2 s$$

for all $s \in S$ and

$$\sum \overline{\eta}(f^1) f^2 = \sum f^1 \overline{\eta}(f^2) = 1$$

(15)

Our next two results can be deduced from Proposition 2.9, but it is easier to give a direct proof.

Proposition 4.4 We keep the notation from above, assuming that the ring extension $R \rightarrow S$ is Frobenius, with Frobenius system $(\overline{\eta}, f)$. Then $G$ is $G_1$-separable if and only if there exists an $(R, T)$-bimodule map $\alpha : S \rightarrow S$ such that $\sum f^1 \alpha(f^2) = 1$.

Proof. Assume that $G$ is $G_1$-separable, and take $e = \sum e^1 \otimes_R e^2 \in S \otimes_R S$ as in Proposition 4.1. We define $\alpha : S \rightarrow S$ by

$$\alpha(s) = \sum \overline{\eta}(se^1) e^2 = \sum \overline{\eta}(e^1) e^2 s$$

Using the fact that $\overline{\eta}$ is left $R$-linear, we easily prove that $\alpha$ is left $R$-linear. For all $s \in S$ and $t \in T$, we have

$$\alpha(st) = \sum \overline{\eta}(ste^1) e^2 = \sum \overline{\eta}(se^1) e^2 t = \alpha(s)t$$

so $\alpha$ is right $T$-linear. Finally

$$\sum f^1 \alpha(f^2) = \sum f^1 \overline{\eta}(f^2 e^1) e^2 = \sum e^1 f^1 \overline{\eta}(f^2 e^2) = \sum e^1 e^2 = 1$$
Conversely, suppose that we have an \((R, T)\)-bimodule map \(\alpha : S \to S\) such that \(\sum f^1 \alpha(f^2) = 1\). We then take
\[
e = \sum e^1 \otimes_R e^2 = \sum f^1 \otimes_R \alpha(f^2) \in S \otimes_R S
\]
and compute that \(\sum e^1 e^2 = 1\) and
\[
\sum t f^1 \otimes_R \alpha(f^2) = \sum f^1 \otimes_R \alpha(f^2 t) = \sum f^1 \otimes_R \alpha(f^2) t
\]
and it follows from Proposition 4.1 that \(G\) is \(G_1\)-separable. \(\square\)

**Proposition 4.5** We keep the notation from above, assuming that the ring extension \(R \to S\) is Frobenius, with Frobenius system \((\overline{\mu}, f)\). Then the following assertions are equivalent:

- \(F = - \otimes_R S\) is \(G_2\)-separable;
- \(F = - \otimes_R S\) is \(G'_2\)-separable;
- there exists \(x \in C_Q(S)\) such that \(\overline{\mu}(x) = 1\).

**Proof.** First assume that \(F\) is \(G_2\)-separable. From Proposition 4.3, we know that there exists an \((R, Q)\)-bimodule map \(\nabla : S \to R\) such that \(\nabla(1_S) = 1_R\). Take \(x = \sum f^1 \nabla(f^2)\). Then for all \(q \in Q\), we have
\[
qx = \sum q f^1 \nabla(f^2) = \sum f^1 \nabla(f^2 q) = \sum f^1 \nabla(f^2) q = xq
\]
and
\[
\overline{\mu}(x) = \sum \overline{\mu}(f^1 \nabla(f^2)) = \sum \overline{\mu}(f^1) \nabla(f^2) = \sum \nabla(\overline{\mu}(f^1) f^2) = \nabla(1_S) = 1_R
\]
Conversely, given \(x \in C_Q(S)\) such that \(\overline{\mu}(x) = 1\), we define \(\nabla : S \to R\) by \(\nabla(s) = \overline{\mu}(sx)\). Then \(\nabla(1) = \overline{\mu}(x) = 1\), and, for all \(r \in R, s \in S\) and \(q \in Q\), we have
\[
\nabla(rs) = \overline{\mu}(rsx) = r\overline{\mu}(sx) = r\nabla(s)
\]
\[
\nabla(sq) = \overline{\mu}(sqx) = \overline{\mu}(sxq) = \overline{\mu}(sx)q = \nabla(s)q
\]
and \(\nabla\) is an \((R, Q)\)-bimodule map, as needed.

The equivalence between the second and the third assertion can be shown in a similar way. \(\square\)

**Remark 4.6** It follows from Proposition 4.1 that the \(G_1\)-separability of the restriction of scalars functor \(G\) is left-right symmetric. It is remarkable that a similar property does not hold for the \(G_2\)-separability of the induction functor (see Proposition 4.3), unless we know that \(S/R\) is Frobenius (see Proposition 4.5).

**Hopf algebras**

A separable functor is always Maschke and dual Maschke, but the converse is in general not true, see Example 3.6. However, there are some particular situations where the converse property holds.

A classical result of Sweedler ([21]) states that a Hopf algebra over a field is semisimple if and only if there exists a (left or right) integral \(t \in H\) such that \(\varepsilon(t) = 1\). The generalization to Hopf algebras over a commutative ring \(k\) is the following: a Hopf algebra is separable if and only if there exists an integral \(t\)
with $\varepsilon(t) = 1$. Here the remarkable thing is that, over a field $k$, a separable algebra is semisimple, but not conversely: it suffices to look at a purely inseparable field extension. We can now explain this apparent contradiction. First observe the following.

An algebra $A$ over a field $k$ is semisimple if and only if the restriction of scalars functor $M_A \to M_k$ is a Maschke functor, if and only if it is a dual Maschke functor. This is a restatement of the classical result \[8, \text{Th. 5.3.7}\].

An algebra $A$ over a commutative ring $k$ is separable if and only if $M_A \to M_k$ is a separable functor (see Proposition \[4.1\] with $T = S$ or \[19\]).

With these observations in mind, we restate and prove Sweedler’s results in the following fashion.

**Proposition 4.7** Let $H$ be a Hopf algebra over a commutative ring $k$, and $G : M_H \to M_k$ the restriction of scalars functor. Then the following assertions are equivalent:

1) $G$ is a dual Maschke functor;
2) $G$ is a Maschke functor;
3) $G$ is a semisimple functor;
4) there exists a right integral $t \in H$ with $\varepsilon(t) = 1$;
5) $G$ is a separable functor.

*Proof.* The equivalence of 1), 2) and 3) follows immediately from Proposition \[3.7\], since $G$ reflects monomorphisms and epimorphisms.

1) $\implies$ 4). $k \in M_H$, with the trivial action: $x \cdot h = \varepsilon(h)x$. Then $\varepsilon : H \to k$ in $M_H$ is such that $G(\varepsilon)$ is a cosplit epimorphism. So we have a map $\tau \in M_H$ making the following diagram commutative in $M_H$:

\[
\begin{array}{ccc}
H & \xrightarrow{\varepsilon} & k \\
\downarrow I_H & & \downarrow \\
H & \xleftarrow{\tau} &
\end{array}
\]

$t = \tau(1)$ is then the required integral.

4) $\implies$ 5). Let $t$ be a right integral, with $\varepsilon(t) = 1$. $S(t(1)) \otimes t(2)$ is the required separability idempotent.

5) $\implies$ 1) follows from Proposition \[3.3\]. $\square$

The dual version of this result is the following; we leave the proof to the reader.

**Proposition 4.8** Let $H$ be a flat Hopf algebra over a commutative ring $k$, and $F : M^H \to M_k$ the forgetful functor. Then the following assertions are equivalent:

1) $F$ is a Maschke functor;
2) $F$ is a dual Maschke functor;
3) $F$ is a semisimple functor;
4) there exists a right integral $\varphi \in H^*$ with $\varphi(1) = 1$;
5) $F$ is a separable functor.

**The structure of injective objects in the category of entwined modules**

As an application of Theorem \[3.4\], we give the structure of injective objects in the category of entwined modules. For other results about injective objects in the category of graded modules and the category of modules graded by a $G$-set (which are special cases of entwined modules), we refer to \[7\, Th. 2.1, Cor.\]
Let \((A,C,\psi)\) be an entwining structure of a commutative ring \(k\). We will assume that \(C\) is flat as a \(k\)-module, to ensure that the category of \(C\)-comodules and the category of entwined modules is abelian. We will use the following notation for functors forgetting actions and coactions:

\[
\begin{array}{ccc}
\mathcal{M}(\psi)^C_A & \xrightarrow{F} & \mathcal{M}_A \\
\downarrow H & & \downarrow H_1 \\
\mathcal{M}^C & \xrightarrow{F_1} & \mathcal{M}_k
\end{array}
\] (16)

\(F\) has a right adjoint \(G = \bullet \otimes C\), and \(H_1\) has a right adjoint \(K_1\) given by

\[K_1(V) = \text{Hom}(A,V), \quad \text{with } (f \cdot a)(b) = f(ab)\]

for any \(f : A \to V\). Thus we have also an adjoint pair \((H_1F, GK_1)\) and the unit and counit of this adjoint pair are

\[\eta_M : M \to \text{Hom}(A,M) \otimes C, \quad \eta_M(m) = m[0] \bullet \otimes m[1]\]

\[\varepsilon_V : \text{Hom}(A,V) \otimes C \to V, \quad \varepsilon_V(f \otimes c) = f(1_A)\varepsilon_C(c)\]

For any \(m \in M \in \mathcal{M}_A\), we write \(m \bullet\) for the map \(A \to M\), sending \(a\) to \(ma\).

**Corollary 4.9** Let \((A,C,\psi)\) be an entwining structure over a field \(k\). \(Q\) is an injective object in \(\mathcal{M}(\psi)^C_A\) if and only if there exists a vector space \(V\) such that \(Q\) is isomorphic to a direct summand of \(\text{Hom}(A,V) \otimes C\).

**Proof.** As \(k\) is a field, then the category \(\mathcal{M}(\psi)^C_A\) is Grothendieck (see \([4]\)). The forgetful functor \(H_1F\) is exact and \(\mathcal{M}_k\) has enough injectives, so the right adjoint \(GK_1\) preserves injectives. Thus \(\text{Hom}(A,V) \otimes C\) is an injective object of \(\mathcal{M}(\psi)^C_A\), and so are its direct summands.

Conversely, assume that \(Q\) is an injective object of \(\mathcal{M}(\psi)^C_A\). As \(k\) is a field, \(Q\) is \(F\)-relative injective, and it follows from Theorem \([4,4]\) that the unit \(\eta_Q : Q \to \text{Hom}(A,Q) \otimes C\) has a retraction in the Grothendieck category \(\mathcal{M}(H)^C_A\), and this means that \(Q\) is isomorphic to a direct summand of \(\text{Hom}(A,Q) \otimes C\). \(\square\)

Let us present some examples, where the entwining structure comes from a Doi-Koppinen datum \((H,A,C)\).

**Examples 4.10** 1. Let \((H,A,C) = (k,A,k)\); then \(\mathcal{M}(k)^C_A = \mathcal{M}_A\), the category of right \(A\)-modules. From Corollary \([4,3]\), we recover the well-known result stating that a right \(A\)-module \(Q\) is injective if and only if there exists a vector space \(V\) such that \(Q\) is a direct summand of the right \(A\)-module \(\text{Hom}(A,V)\).

2. Now let \((H,A,C) = (k,k,C)\); then \(\mathcal{M}(k)^C = \mathcal{M}^C\), the category of right \(C\)-comodules, Corollary \([4,9]\) tells that the injective right \(C\)-comodule are the direct summands of \(C^{(i)}\), with \(i\) an index set.

3. Corollary \([4,9]\) can be used to describe injective modules graded by \(G\)-sets: let \(G\) be a group, \(X\) is a right \(G\)-set, \(A\) a \(G\)-graded \(k\)-algebra, and consider the Doi-Koppinen datum \((H,A,C) = (kG,A,kX)\). The corresponding Doi-Koppinen Hopf modules are then exactly the \(A\)-modules graded by \(X\), as introduced in \([18]\), and it follows that the injective objects in the category of \(A\)-modules graded by \(X\) are the direct summands of \(A\)-modules graded by \(X\) of the form \(\text{Hom}(A,V)^X = \oplus_{x \in X} \text{Hom}(A,V)_x\), with \(\text{Hom}(A,V)_x = \text{Hom}(A,V)\) for all \(x \in X\).
**Remark 4.11** If the forgetful functor $F : \mathcal{M}(\psi)^C_A \rightarrow \mathcal{M}_k$ has a left adjoint, then we can also describe the projective objects of $\mathcal{M}(\psi)^C_A$. Unfortunately, in general, $F$ has not a left adjoint: for instance, the forgetful functor $F : \mathcal{M}^C \rightarrow \mathcal{M}_k$ has a left adjoint if and only if $C$ is finite dimensional over the field $k$ (this result is a special case of [22, Proposition 1.10]); in this case $\mathcal{M}^C \cong c\cdot \mathcal{M}$, the category of modules over $C^*$.

**Entwined modules and separability**

We keep the notation [16]. Let us examine when $F$ is $H$-separable. In order to apply Theorem 2.7, we need to examine $\text{Nat}(HGF, H)$. In [2, Proposition 4.1], $\text{Nat}(GF, 1)$ has been computed, and an adaption of the arguments leads to a description of $\text{Nat}(HGF, H)$. We present a brief sketch: consider a natural transformation $\nu : HGF \rightarrow H$. $A \otimes C = G(A) \in \mathcal{M}(\psi)^C_A$, so we can consider the map

$$\nu_{A \otimes C} : HGF(A \otimes C) = A \otimes C \otimes C \rightarrow H(A \otimes C) = A \otimes C$$

in $\mathcal{M}^C$. Now we define $\theta : C \otimes C \rightarrow A$ by

$$\theta(c \otimes d) = (I_A \otimes \epsilon)(\nu_{A \otimes C}(1 \otimes c \otimes d))$$

Using the naturality of $\nu$, we can prove that $\theta$ satisfies the relation

$$\theta(c \otimes d_{(1)}) \otimes d_{(2)} = \theta(c_{(2)} \otimes d) \nu_{c_{(1)}}$$

(17) for all $c, d \in C$. Conversely, given a map $\theta : C \otimes C \rightarrow A$ satisfying (17), we can define a natural transformation $\nu : HGF \rightarrow H$ as follows: let

$$\nu_M : M \otimes C \rightarrow M : \nu_M(m \otimes c) = m\theta(c \otimes c)$$

for all $M \in \mathcal{M}(\psi)^C_A$. It is clear that $\nu_M \circ H(\eta_M) = I_H(M)$, for all $M \in \mathcal{M}(\psi)^C_A$ if and only if $\theta(\Delta_C(c)) = \epsilon_C(c) I_A$, for all $c \in C$. If such a map $\theta$ exists, then $\nu_M$ is a retraction in $\mathcal{M}^C$ of the $C$-coaction $\eta_M = \rho_M : M \rightarrow M \otimes C$. Thus any $M \in \mathcal{M}(\psi)^C_A$ is relative injective as a right $C$-comodule. We summarize our result in the following Proposition which is an equivalent version for entwining modules of [16, Theorem 2.6].

**Proposition 4.12** Let $(A, C, \psi)$ be an entwining structure, and consider the forgetful functors $F : \mathcal{M}(\psi)^C_A \rightarrow \mathcal{M}_A$ and $H : \mathcal{M}(\psi)^C_A \rightarrow \mathcal{M}^C$. Then $F$ is $H$-separable if and only if there exists a map $\theta : C \otimes C \rightarrow A$ such that

$$\theta(c \otimes d_{(1)}) \otimes d_{(2)} = \theta(c_{(2)} \otimes d) \nu_{c_{(1)}}$$

and $\theta \circ \Delta_C = \eta_A \circ \epsilon_C$ (18) for all $c, d \in C$. In this case any $M \in \mathcal{M}(\psi)^C_A$ is relative injective as a right $C$-comodule.

In a similar way, we can investigate when the functor $H$ is $F$-separable. We then obtain the following:

**Proposition 4.13** Let $(A, C, \psi)$ be an entwining structure, and consider the forgetful functors $F : \mathcal{M}(\psi)^C_A \rightarrow \mathcal{M}_A$ and $H : \mathcal{M}(\psi)^C_A \rightarrow \mathcal{M}^C$. Then $H$ is $F$-separable if and only if there exists a map $e : C \rightarrow A \otimes A, \ e(c) = \sum e^1(c) \otimes e^2(c)$ such that

$$\sum e^1(c) \otimes e^2(c)a = \sum a_{\psi}e^1(c^\psi) \otimes e^2(c^\psi)$$

and $m_A \circ e = \eta_A \circ \epsilon_C$ (19) for all $c \in C, a \in A$. In this case any $M \in \mathcal{M}(\psi)^C_A$ is relative projective as a right $A$-module.
Recall from [2, Theorem 3.4] that the pair \((F, G)\) is Frobenius if and only if there exists a \(k\)-linear map \(\theta : C \otimes C \to A\) and \(z = \sum_t a_t \otimes c_t \in A \otimes C\) such that

\[
\begin{align*}
\theta(c \otimes d)a &= a_{\psi\theta}(c^v \otimes d^v) \\
\theta(c \otimes d_1) \otimes d_2 &= \theta(c_2 \otimes d)\psi c_1^v + c^v \\
az &= za \\
\eta_A(\varepsilon_C(d)) &= \sum_t a_t \theta(c_t \otimes d) = \sum_t a_{\psi\theta}(d^v \otimes c_t)
\end{align*}
\]

for all \(a \in A\) and \(c \in C\). We will call \((\theta, z)\) a Frobenius system for the adjunction \((F, G)\). We give the unit and counit of the adjunctions \((F, G)\) and \((G, F)\):

\[
\begin{align*}
\eta : 1 &\to GF & \eta_M : M &\to M \otimes C & \eta_M(m) &= m_0 \otimes m_1 \\
\varepsilon : FG &\to 1 & \varepsilon_N : N \otimes C &\to N & \varepsilon_N(n \otimes c) &= \varepsilon_C(c)n \\
\nu : GF &\to 1 & \nu_M : M \otimes C &\to M & \nu_M(m \otimes c) &= m_0 \theta(m_1 \otimes c) \\
\zeta : 1 &\to FG & \zeta_N(n) &= \sum_t na_t \otimes c_t
\end{align*}
\]

Now assume that \((F, G)\) is a Frobenius pair, and that we know a Frobenius system \((\theta, z)\). Using Proposition \[2,9\], we can decide when \(F\) is separable or \(H\)-separable.

**Lemma 4.14** With notation as above,

\[
\Nat(HG, HG) \cong \Hom(C, A)
\]

and

\[
\Nat(G, G) \cong \{ \beta \in \Hom(C, A) \mid \beta(c)a = a_{\psi\beta}(c^v) \text{ for all } a \in A, c \in C \}
\]

**Proof.** Consider a natural transformation \(\alpha : HG \to HG\). Then the map \(\alpha_A : A \otimes C \to A \otimes C\) is right \(C\)-colinear, and, using the naturality of \(\alpha\), we find that \(\alpha_A\) is also left \(A\)-linear. Now consider the map \(\beta : C \otimes A\) defined by

\[
\beta(c) = (I_A \otimes \varepsilon_C)(\alpha_A(1_A \otimes c))
\]

Conversely given \(\beta : C \to A\), we define a natural transformation \(\alpha : HG \to HG\) by putting

\[
\alpha_N : N \otimes C \to N \otimes C, \quad \alpha_N(n \otimes c) = n\beta(c_1) \otimes c_2
\]

for every \(N \in \mathcal{M}_A\). It is obvious that \(\alpha_N\) is right \(C\)-colinear; let us check that \(\alpha\) is natural. For all \(f : N \to N'\) in \(\mathcal{M}_A\), we have

\[
\alpha_{N'}(f(n) \otimes c) = f(n)\beta(c_1) \otimes c_2 = f(n\beta(c_1)) \otimes c_2 = (f \otimes I_C)(\alpha_N(n \otimes c))
\]

If \(\alpha : G \to G\) is a natural transformation, then the map \(\alpha_A\) is also right \(A\)-linear, and it follows easily that \(\beta\) defined as above satisfies the centralizing condition

\[
\beta(c)a = a_{\psi\beta}(c^v)
\]  

(20)
If $\beta : C \rightarrow A$ satisfies (20), then we define $\alpha : G \rightarrow G$ by the same formula as above, and the second statement of the Lemma follows if we can prove that $\alpha_N$ is right $A$-linear. This goes as follows:

$$\alpha_N((n \otimes c)a) = \alpha_N(na_\psi \otimes c_\psi)$$

$$= na_\psi(\beta((c_\psi)_1) \otimes e_\psi)_2$$

$$= na_\psi(\beta(c_1)) \otimes c_2)$$

$$= n\beta(c_1) a_\psi \otimes c_2)$$

$$= (n\beta(c_1) \otimes c_2)a = \alpha_N(n \otimes c)a$$

Theorem 4.15 Consider the forgetful functors $F : \mathcal{M}(\psi)^C_A \otimes \mathcal{M}_A$ and $H : \mathcal{M}(\psi)_A \otimes \mathcal{M}_C^C$, and assume that the functor $F$ and its adjoint form a Frobenius pair, with Frobenius system $(\theta, z)$. Then $F$ is $H$-separable if and only if there exists a map $\beta : C \otimes A$ such that

$$\beta(c_3)_1 \psi \psi_2 \theta(c_2) \otimes c_4)_4) \psi_3 \otimes c_1 = 1 \otimes c$$

for all $c \in C$. $F$ is separable if and only if there exists a $\beta : C \otimes A$ satisfying (21) and (20).

Proof. This follows immediately from Proposition 2.9: if we apply (11) to $A$ then we find (21). Conversely, if $\beta$ satisfies (21), then we easily compute that the corresponding natural transformation $\alpha$ satisfies (11).

Let us now discuss the dual version of Theorem 4.15. Let $K$ be the left adjoint of the forgetful functor $H : \mathcal{M}(\psi)_A \rightarrow \mathcal{M}(\psi)_A$. In [4], Proposition 4.4, it is shown that $(H, K)$ is a Frobenius pair if and only if there exists a Frobenius system $(\vartheta, e)$, consisting of maps $\vartheta \in (C \otimes A)$ and $e : C \rightarrow A \otimes A$ such that

$$\vartheta(c_1) a \otimes c_2 = \vartheta(c_2 \otimes a)c_1$$

$$e_1(c_1) \otimes e_2(c_1) \otimes c_2) = e_1(c_1) \otimes e_2(c_1) \otimes c_2$$

$$e_1(c) \otimes e_2(c) a = a e_1(c) \otimes e^2(c)$$

$$e(c) 1 = \vartheta(c_1) a \otimes e_2(c_2) = \vartheta(c_1) \otimes e_2(c_2)$$

for all $c \in C$ and $a \in A$. We use the notation

$$e(c) = e_1(c) \otimes e_2(c)$$

with summation implicitly understood. The unit and counit of the adjunction $(H, K)$ is then given by

$$\zeta : 1 \rightarrow KH \quad \zeta_M : M \otimes M \otimes \mathcal{M}_A \quad \zeta_M(m) = m_{[0]} e_1(m_{[1]}) \otimes e_1(m_{[1]})$$

$$\psi : HK \rightarrow 1 \quad \psi_N : N \otimes \mathcal{M}_A \rightarrow N \quad \psi_N(n \otimes a) = \vartheta(n_{[1]} \otimes a)n_{[0]}$$

Lemma 4.16 We have isomorphisms

$$\text{Nat}(FK, FK) = \text{Hom}(C, A)$$

and

$$\text{Nat}(K, K) = \{ \beta \in \text{Hom}(C, A) \mid \beta(c_1) \otimes c_2 = \beta(c_2) \otimes c_1 \text{ for all } c \in C \}$$
Theorem 4.17 Consider the forgetful functors \( F : \mathcal{M}(\psi)^{\otimes}_{\Lambda} \otimes \mathcal{M}_{\Lambda} \) and \( H : \mathcal{M}(\psi)^{\otimes}_{\Lambda} \otimes \mathcal{M}_{\Lambda} \), and assume that the functor \( H \) and its adjoint form a Frobenius pair, with Frobenius system \((\theta, e)\). Then \( H \) is \( F \)-separable if and only if there exists a map \( \beta : C \otimes A \) such that

\[
\varepsilon(c) a = a_{\psi \varphi} \varphi e^\psi (c_{(2)}) \varphi \beta(c_{(1)}) \varphi e^\psi (c_{(2)})
\]

for all \( c \in C \) and \( a \in A \). \( H \) is separable if and only if there exists a \( \beta : C \otimes A \) satisfying (22) and

\[
\beta(c_{(1)}) \otimes c_{(2)} = \beta(c_{(2)}) \otimes c_{(1)}
\]

for all \( c \in C \).

**Yetter-Drinfeld modules and quantum integrals**

Proposition 4.12 and Proposition 4.13 can be applied in many situations: Doi-Koppinen modules, Yetter-Drinfeld modules, relative Hopf modules, graded modules, etc. are all special cases of the category \( \mathcal{M}(\psi)^{\otimes}_{\Lambda} \). In this subsection we shall apply the above results to the category \( \mathcal{Y} \mathcal{D}_H \) of Yetter-Drinfeld modules [23].

Let \((A, C, \psi) = (L, L, \psi)\), where \( L \) is a Hopf algebra and

\[
\psi : L \otimes L \to L \otimes L, \quad \psi(g \otimes h) = h_{(2)} \otimes S(h_{(1)}) g h_{(3)}
\]

for all \( g, h \in L \). The resulting category of entwined modules is just \( \mathcal{M}(\psi)^{\otimes}_L = \mathcal{Y} \mathcal{D}_L \), the category of Yetter-Drinfeld modules over \( L \).

Corollary 4.18 Let \( L \) be a Hopf algebra over a commutative ring \( k \) and consider the forgetful functors \( F : \mathcal{Y} \mathcal{D}_L \to \mathcal{M}_L \) and \( H : \mathcal{Y} \mathcal{D}_L \to \mathcal{M}_L \). 

1) The following statements are equivalent:
- \( F \) is \( H \)-separable;
- there exists a \( k \)-linear map \( \theta : L \otimes L \to L \) such that

\[
\theta(g \otimes h_{(1)})(h_{(2)}) = \theta(g_{(2)} \otimes h_{(1)} h_{(2)}) S(\theta(g_{(2)} \otimes h_{(1)}) g_{(1)} \theta(g_{(2)} \otimes h, h_{(3)}), \quad \theta(h_{(1)} \otimes h_{(2)}) = \varepsilon(h) 1_H
\]

for all \( g, h \in L \);
- there exists a \( k \)-linear map \( \gamma : L \to \text{End}(L) \) such that

\[
\gamma(h_{(1)})(g) \otimes h_{(2)} = \gamma(h)(g_{(2)})(h_{(2)}) \otimes S(\gamma(h)(g_{(2)})(h_{(1)}) g_{(1)} \gamma(h)(g_{(2)})(h_{(3)}), \quad \gamma(h_{(2)})(h_{(1)}) = \varepsilon(h) 1_H
\]

for all \( g, h \in L \). In this case any \( M \in \mathcal{Y} \mathcal{D}_L \) is relative injective as a right \( L \)-comodule.

2) The following statements are equivalent:
- \( H \) is \( F \)-separable;
- there exists a \( k \)-linear map \( e : L \to L \otimes L \) such that

\[
\sum e^1(g) \otimes e^2(h) = \sum h_{(2)} e^1(S(h_{(1)}) h_{(3)}) \otimes e^2(S(h_{(1)}) h_{(3)}) , \quad \sum e^1(g) e^2(h) = \varepsilon(g) 1_H
\]

for all \( g, h \in L \). In this case any \( M \in \mathcal{Y} \mathcal{D}_L \) is relative projective as a right \( L \)-module.

Furthermore, if \( L \) is finitely generated and projective over \( k \), these conditions are also equivalent to

- There exists an element \( \sum f_i \otimes h_i \in \text{End}(L) \otimes L \) such that

\[
\sum f_i(g) \otimes h = \sum h_{(2)} f_i(S(h_{(1)}) h_{(3)}) \otimes h_i, \quad \sum f_i(g) h_i = \varepsilon(g) 1_H
\]

for all \( g, h \in L \).
Proof. This follows from Proposition 4.12 and Proposition 4.13 applied to the above entwining structure. The equivalence between the maps $\theta : L \otimes L \to L$ and the maps $\gamma : L \to \text{End}(L)$ is given by the $k$-linear isomorphism given by the adjunction

$$\text{Hom}(L \otimes L, L) \cong \text{Hom}(L, \text{End}(L))$$

Hence, for any $\theta : L \otimes L \to L$ there exists a unique $\gamma = \gamma_\theta : L \to \text{End}(L)$ such that $\theta(g \otimes h) = \gamma(h)(g)$, for any $g, h \in L$.

In the case that $L$ is finitely generated and projective over $k$ we use the “Hom-tensor relations”

$$\text{End}(L) \otimes L \cong \text{Hom}(L, L \otimes L)$$

i.e. for any $e : L \to L \otimes L$ there exists a unique element $\sum_{i=1}^n f_i \otimes h_i \in \text{End}(L) \otimes L$ such that $e(g) = \sum_{i=1}^n f_i(g) \otimes h_i$, for any $g \in G$. □

Remarks 4.19

1. In [16], a map $\gamma : L \to \text{End}(L)$ satisfying the conditions of Corollary 4.18 has been called a total quantum integral.

2. Assume now that $L$ is a finite dimensional Hopf algebra over a field $k$ and let $\sum_{i=1}^n f_i \otimes h_i \in \text{End}(L) \otimes L$ be an element as in Corollary 4.18. Let $D(L)$ be the Drinfeld double of $L$. Then it is well known that there exists an equivalence of categories $D(L) \cong \text{Mod}(L)$ and the above functor $F$ is just the restriction of scalars. From ring theoretical point of view the ring extension $D(L)/L$ has a remarkable property: any right $D(L)$-module is projective as a right $L$-module.

Relative Hopf modules and total integrals

Let $L$ be a Hopf algebra over a commutative ring and $A$ a $L$-comodule algebra. Associated to this is an entwining structure $(A, L, \psi)$, with

$$\psi(h \otimes a) = a_0 \otimes h \psi = a_{[0]} \otimes h a_{[1]}$$

The resulting category of entwined modules is denoted

$$\mathcal{M}(\psi)_A^L = \mathcal{M}^L_A$$

and is usually called the category of relative Hopf modules. Now recall [13] that an $L$-colinear map $\phi : L \to A$ is called an integral. $\phi$ is called a total integral if $\phi(1_L) = 1_A$. We keep the notation introduced in (16), i.e.

$$\begin{array}{ccc}
\mathcal{M}^L_A & \xrightarrow{F} & \mathcal{M}_A \\
\downarrow H & & \downarrow H_1 \\
\mathcal{M}^L & \xrightarrow{F_1} & \mathcal{M}_k
\end{array}$$

(23)

Theorem 4.20 With notation as above, the following assertions are equivalent:

1) $F$ is $H$-separable;
2) $H_1 \circ F$ is $H$-separable;
3) $H_1 \circ F$ is $H$-Maschke;
4) $H_1 \circ F$ is dual $H$-Maschke;
5) $H_1 \circ F$ is $H$-semisimple;
6) there exists a map $\theta : L \otimes L \to A$ such that $\theta \circ \Delta_L = \eta_A \circ \varepsilon_L$ and
   \[ \theta(h \otimes k_{(1)}) \otimes k_{(2)} = \theta(h_{(2)} \otimes k)_{[0]} \otimes h \theta(h_{(2)} \otimes k)_{[1]} \]  
   (24)

for all $h, k \in L$;
7) there exists a total integral $\phi : L \to A$.

**Proof.** 2) $\Rightarrow$ 1): from Proposition 2.3.
2) $\Rightarrow$ 3): from Proposition 3.3.
3) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5): from Proposition 3.7.
1) $\Leftrightarrow$ 6): from Proposition 4.12.
6) $\Rightarrow$ 7): Define $\phi : L \to A$ by $\phi(h) = \theta(1 \otimes h)$ for all $h \in L$. A straightforward computation shows that $\phi$ is a total integral.
7) $\Rightarrow$ 6): Define $\theta : L \otimes L \to A$ by $\theta(h \otimes k) = \phi(S(h)k)$. It is easy to compute that $\theta$ satisfies (24) and that $\theta \circ \Delta_L = \eta_A \circ \varepsilon_L$.
3) $\Rightarrow$ 7): if $H_1F$ is $H$-Maschke, then
   \[ H(\eta_A) : H(A) = A \to H\text{GK}_1H_1F(A) = \text{Hom}(A, A) \otimes L \]
   has a left inverse $\nu$ in $\mathcal{M}^L$, by Theorem 3.4. Now let $\phi(h) = \nu(I_A \otimes h)$, for all $h \in L$. Then $\phi$ is an integral, since $\nu \in \mathcal{M}^L$, and
   \[ \phi(1) = \nu(I_A \otimes 1) = \nu(\eta_A(1_A)) = 1_A \]
7) $\Rightarrow$ 2): let $\phi$ be a total integral. We define a natural transformation $\nu : H\text{GK}_1H_1 \to H$ as follows:
   \[ \nu_M : \text{Hom}(A, M) \otimes L \to M ; \nu_M(f \otimes h) = f(1_A)_{[0]} \phi(S(f(1_A)_{[1]} h) \]

We leave it to the reader to verify that $f$ is natural. Finally
   \[ \nu_M(H(\eta_M)(m)) = \nu_M(m_{[0]} \bullet \otimes m_{[1]}) = m_{[0]} \phi(S(m_{[1]} m_{[2]}) = m \phi(1_H) = m \]

\[ \square \]

**Doi’s $[L, C]$-modules and augmented cointegrals**

Let us now discuss the dual situation. Let $L$ be a Hopf algebra, and $C$ a right $L$-module coalgebra. We then have an entwining structure $(L, C, \alpha)$, with
   \[ \alpha(c \otimes h) = h_{(1)} \otimes c = h_{(1)} \otimes ch_{(2)} \]

The associated entwining modules are called $[L, C]$-modules, and our diagram of forgetful functors now takes the form:

\[ \mathcal{M}_L^C \xrightarrow{F} \mathcal{M}_L \]
\[ \mathcal{M}_C^L \xrightarrow{F_1} \mathcal{M}_C \]
\[ H \quad H_1 \]
\[ \mathcal{M}_C^L \xrightarrow{F_1} \mathcal{M}_C \]  
(25)
$H_1 \circ F$ has a right adjoint, and this has been used in the proof of Theorem 4.20. $H$ has a left adjoint, but, in general, $F_1$ has no left adjoint, and this is the reason why the proof of Theorem 4.21 is different from the one of Theorem 4.20. We recall [12] that a right $L$-linear map $\psi : C \to L$ is called a cointegral. Furthermore, $\psi$ is called augmented if $\varepsilon_L \circ \psi = \varepsilon_C$.

**Theorem 4.21** With notation as above, the following assertions are equivalent:

1) $H$ is $F$-separable;
2) $F_1 \circ H$ is $F$-separable;
3) $F_1 \circ H$ is $F$-Maschke;
4) $F_1 \circ H$ is dual $F$-Maschke;
5) $F_1 \circ H$ is $F$-semisimple;
6) there exists a map $e : C \to L \otimes L$ such that $m_L \circ e = \eta_L \circ \varepsilon_C$ and
   \[ \sum e^1(c) \otimes e^2(c) h = h_1 e^1(ch_2) \otimes e^2(ch_2) \quad (26) \]
   for all $h \in L$ and $c \in C$;
7) there exists an augmented cointegral $\psi : C \to L$.

**Proof.** 2) $\Rightarrow$ 1): from Proposition 2.3;
2) $\Rightarrow$ 3): from Proposition 3.3;
3) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5): from Proposition 3.7;
1) $\Leftrightarrow$ 6): from Proposition 4.13;
6) $\Rightarrow$ 7): Let $e : C \to L \otimes L$ be as in 6); then $\psi : C \to L$, $\psi(c) = \varepsilon_L(e^1(c))e^2(c)$ is an augmented cointegral.
7) $\Rightarrow$ 2): Let $\psi : C \to L$ be an augmented cointegral. We define a natural transformation $P : \text{Hom}(F_1H, F_1H) \to \text{Hom}_L(F,F)$ as follows: for a $k$-linear map $f : M \to N$, with $M, N \in M_C^L$, we define $P_{M,N}(f)$ by
   \[ P_{M,N}(f)(m) = f(m_0 S(\psi(m_1))(1)) \psi(m_1)(2) \]
   It is straightforward to verify that $f_1 \circ H = P \circ f$.
4) $\Rightarrow$ 7): Assume that $F_1 \circ H$ is dual $F$-Maschke. The map
   \[ f : C \otimes L \to C, \quad f(c \otimes h) = ch \]
   is $L$-linear and $C$-colinear. As a $k$-linear map, $f$ is cosplit by the map $c \to c \otimes 1_L$, so there exists an $L$-linear map $g : C \to C \otimes L$, such that the following diagram commutes:

\[
\begin{array}{ccc}
C \otimes L & \xrightarrow{f} & C \\
\downarrow{g} & & \downarrow{1_C} \\
C & \xrightarrow{lc} & C \\
\end{array}
\]

Now consider $\psi = (\varepsilon_C \otimes 1_L) \circ g : C \to L$. $\psi$ is $L$-linear. For a fixed $c \in C$, write $g(c) = \sum_i c_i \otimes h_i$. Then $c = g(f(c)) = \sum_i c_i h_i$, and
   \[
   \varepsilon_L(\psi(c)) = (\varepsilon_C \otimes \varepsilon_H)(g(c)) = \sum \varepsilon_C(c_i) \varepsilon_H(h_i) = \varepsilon_C(c)
   \]
   and $\psi$ is an augmented cointegral. \qed
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