On a lower bound of the number of integers in Littlewood’s conjecture

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Littlewood’s conjecture

Littlewood’s conjecture (c.1930).

For every \((\alpha, \beta) \in \mathbb{R}^2\),

\[
\liminf_{n \to \infty} n\langle n\alpha \rangle \langle n\beta \rangle = 0,
\]

where \(\langle x \rangle = \min_{k \in \mathbb{Z}} |x - k|\).

- [Cassles, Swinnerton-Dyer, 1955].
  If \(\alpha\) and \(\beta\) are in the same cubic number field, then
  \[
  \liminf_{n \to \infty} n\langle n\alpha \rangle \langle n\beta \rangle = 0.
  \]

- [Pollington, Velani, 2000].
  For \(\forall \alpha \in \text{Bad} := \{\alpha \in \mathbb{R} | \liminf_{n \to \infty} n\langle n\alpha \rangle > 0\}\),
  \(\exists G(\alpha) \subset \text{Bad}\) s.t. \(\dim_H G(\alpha) = 1\) and, for \(\forall \beta \in G(\alpha)\),
  \[
  n\langle n\alpha \rangle \langle n\beta \rangle \leq \frac{1}{\log n}\]
  for infinitely many \(n\).
The set of exceptions for Littlewood’s conjecture has Hausdorff dimension zero.

Theorem [Einsiedler, Katok, Lindenstrauss, 2006].

\[
\dim_H \left\{ (\alpha, \beta) \in \mathbb{R}^2 \left| \liminf_{n \to \infty} n \langle n\alpha \rangle \langle n\beta \rangle > 0 \right. \right\} = 0.
\]

Furthermore, this set is an at most countable union of compact sets of box dimension zero.

This Theorem is obtained as a corollary of some property of the diagonal action on \( SL(3, \mathbb{R})/SL(3, \mathbb{Z}) \).
Littlewood’s conjecture says that, for every $(\alpha, \beta) \in \mathbb{R}^2$ and any $0 < \varepsilon < 1$, $n\langle n\alpha \rangle \langle n\beta \rangle < \varepsilon$ for infinitely many $n$.

Problem (Quantitative version of Littlewood’s conjecture).

For $(\alpha, \beta) \in \mathbb{R}^2$, $0 < \varepsilon < 1$ and sufficiently large $N \in \mathbb{N}$, how many integers $n \in [1, N]$ are there s.t.

$$n\langle n\alpha \rangle \langle n\beta \rangle < \varepsilon?$$
Result on quantitative LC / Main Theorem

- [Pollington, Velani, Zafeiropoulos, Zorin, 2022] (quantitative version of [Pollington, Velani, 2000]).
  For $\forall \alpha \in \text{Bad}, \forall \gamma \in [0, 1], \exists \mathbf{G}(\alpha, \gamma) \subset \text{Bad}$ s.t. $\dim_H \mathbf{G}(\alpha, \gamma) = 1$ and, for $\forall \beta \in \mathbf{G}(\alpha, \gamma)$, we have

$$\left| \left\{ n \in [1, N] \mid n\langle n\alpha \rangle \langle n\beta - \gamma \rangle \leq \frac{1}{\log n} \right\} \right| \gg \log \log N, \quad N \in \mathbb{N}.$$
Main Theorem [U., 2022+, 2024+].

For \(0 < \forall \gamma < 1/72\), there exists an “exceptional set” \(Z(\gamma) \subset \mathbb{R}^2\) with \(\dim_H Z(\gamma) \leq 90\sqrt{2\gamma}\) s.t., for \(\forall (\alpha, \beta) \in \mathbb{R}^2 \setminus Z(\gamma)\) and \(0 < \forall \varepsilon < 4^{-1}e^{-2}\),

\[
\liminf_{N \to \infty} \frac{(\log \log N)^2}{(\log N)^2} |\{n \in [1, N] | n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\}| \geq \gamma.
\]

Corollary.

There exists an “exceptional set” \(Z \subset \mathbb{R}^2\) with \(\dim_H Z = 0\) s.t., for \(\forall (\alpha, \beta) \in \mathbb{R}^2 \setminus Z\) and \(0 < \forall \varepsilon < 4^{-1}e^{-2}\),

\[
\liminf_{N \to \infty} \frac{(\log \log N)^2}{(\log N)^2} |\{n \in [1, N] | n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\}| \geq C_{\alpha,\beta},
\]

where \(C_{\alpha,\beta} > 0\) is a constant depending only on \((\alpha, \beta)\).
1. Littlewood’s conjecture, its quantitative version and Main Theorem

2. The diagonal action on $SL(3, \mathbb{R})/SL(3, \mathbb{Z})$ and its relation to Littlewood’s conjecture

3. About the proof of Main Theorem
The diagonal action on $\text{SL}(3, \mathbb{R})/\text{SL}(3, \mathbb{Z})$

We write

$$G := \text{SL}(3, \mathbb{R}), \quad \Gamma := \text{SL}(3, \mathbb{Z}), \quad X := G/\Gamma.$$ 

- By the one to one correspondence
  $$X = G/\Gamma \ni g\Gamma \mapsto g \cdot \mathbb{Z}^3 \in \{ \Lambda \subset \mathbb{R}^3 : \text{lattice of covolume 1} \},$$
  we can identify $X$ as the space of lattices in $\mathbb{R}^3$ of covolume 1.

- $X = G/\Gamma$ admits a unique $G$-invariant Borel probability measure $m_X$ on $X$, called the **Haar measure**. However, $X$ is not compact.

**Proposition (Mahler’s criterion).**

For a subset $B \subset X$, $B$ is unbounded in $X$ iff

$$0 < \forall \varepsilon < 1, \exists v \in \bigcup_{\Lambda \in B} \Lambda \setminus \{0\} \text{ s.t. } \|v\| < \varepsilon.$$
Let
\[ A := \left\{ \begin{pmatrix} e^{t_1} & \cdot & \cdot \\ \cdot & e^{t_2} & \cdot \\ \cdot & \cdot & e^{t_3} \end{pmatrix} \mid t_1, t_2, t_3 \in \mathbb{R}, \ t_1 + t_2 + t_3 = 0 \right\} < G. \]

The left action of \( A \)
\[ A \times X \ni (a, x) \mapsto ax \in X \]
is called the (higher rank) diagonal action on \( X \).

For the application to Littlewood’s conjecture, we consider the action of the positive cone \( A^+ \) of \( A \):
\[ A^+ := \left\{ a_{s,t} := \begin{pmatrix} e^{-s-t} & \cdot \\ \cdot & e^s \\ \cdot & \cdot & e^t \end{pmatrix} \mid s, t \geq 0 \right\}. \]
The relation between the diagonal action and LC

Let $U < G$ be the **unstable subgroup** for conjugation by $A^+$:

$$U := \left\{ u \in G \mid a^{-n}ua^n \xrightarrow[n \to \infty]{} e, \forall a \in A^+ \setminus \{e\} \right\}$$

$$= \left\{ u_{\alpha, \beta} := \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\} \cong \mathbb{R}^2.$$  

For $(\alpha, \beta) \in \mathbb{R}^2$, we write

$$\tau_{\alpha, \beta} = u_{\alpha, \beta} \Gamma \in X = G/\Gamma.$$  

By Mahler’s criterion for $B = A^+ \tau_{\alpha, \beta} \subset X$, we have the following:

**Proposition.**

For $(\alpha, \beta) \in \mathbb{R}^2$, $\liminf_{n \to \infty} n \langle n\alpha \rangle \langle n\beta \rangle = 0$ iff the $A^+$ orbit of $\tau_{\alpha, \beta}$ is unbounded in $X$.  

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Measure rigidity under positive entropy condition

For an $A$-invariant probability measure $\mu$ and $a \in A$, we write $h_\mu(a)$ for the entropy of the map $X \ni x \mapsto ax \in X$ w.r.t. $\mu$.

Theorem [Einsiedler, Katok, Lindenstrauss, 2006].

If $\mu$ is an $A$-invariant and ergodic Borel probability measure on $X = \text{SL}(3, \mathbb{R})/\text{SL}(3, \mathbb{Z})$ s.t. $h_\mu(a) > 0$ for $\exists a \in A$, then $\mu$ is the Haar measure $m_X$ on $X$.

As a corollary of this Theorem, we obtain that

$$\dim_H \{ u \in U \mid A^+ u \Gamma \subset X \text{ is bounded} \} = 0$$

and, by Proposition, this is equivalent to

$$\dim_H \left\{ (\alpha, \beta) \in \mathbb{R}^2 \mid \liminf_{n \to \infty} n \langle n\alpha \rangle \langle n\beta \rangle > 0 \right\} = 0.$$
Remarks on measure rigidity for the diagonal action

- Measure rigidity does not hold if $X = \text{SL}(2, \mathbb{R})/\text{SL}(2, \mathbb{Z})$.
- Theorem is the similar to the following measure rigidity for the $\times 2, \times 3$ action on $\mathbb{R}/\mathbb{Z}$ (by Rudolph, Johnson): if $a, b \in \mathbb{Z}_{\geq 2}$ are multiplicatively independent and $\mu$ is a $\times a, \times b$-invariant and ergodic Borel probability measure on $\mathbb{R}/\mathbb{Z}$ s.t. $\dim_H \mu > 0$, then $\mu$ is the Lebesgue measure.
- The positive entropy condition is believed to be dropped.

**Full measure rigidity conjecture [Margulis].**

For $n \geq 3$, every $A$-invariant and ergodic Borel probability measure on $\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$ is homogeneous.

It is known that if Full measure rigidity conjecture is true, then Littlewood’s conjecture follows from it.
Littlewood’s conjecture, it’s quantitative version and Main Theorem

The diagonal action on $\text{SL}(3, \mathbb{R})/\text{SL}(3, \mathbb{Z})$ and its relation to Littlewood’s conjecture

About the proof of Main Theorem
Main Theorem [U., 2022+, 2024+].

For $0 < \forall \gamma < 1/72$, there exists an “exceptional set” $Z(\gamma) \subset \mathbb{R}^2$ with $\dim_H Z(\gamma) \leq 90\sqrt{2}\gamma$ s.t., for $\forall (\alpha, \beta) \in \mathbb{R}^2 \setminus Z(\gamma)$ and $0 < \forall \varepsilon < 4^{-1}e^{-2}$,

$$\liminf_{N \to \infty} \frac{(\log \log N)^2}{(\log N)^2} |\{ n \in [1, N] | n\langle n\alpha \rangle \langle n\beta \rangle < \varepsilon \}| \geq \gamma.$$

To prove this, we want to define $Z(\gamma)$ properly and show that

- on the outside of $Z(\gamma)$, we can obtain the quantitative result, and
- $Z(\gamma)$ has small Hausdorff dimension.
Empirical measures w.r.t. the diagonal action

For \( x \in X = \text{SL}(3, \mathbb{R})/\text{SL}(3, \mathbb{Z}) \) and \( T > 0 \), we define the \( T \)-empirical measure of \( x \) w.r.t. \( A^+ \) by

\[
\delta_{A^+,x}^T := \frac{1}{T^2} \int_{[0,T]^2} \delta_{a_s,tx} \, dsdt.
\]

We are interested in the behavior of \( \delta_{A^+,x}^T \) as \( T \to \infty \). If \( \delta_{A^+,x}^T \) (\( T > 0 \)) accumulate to a measure \( \mu \) on \( X \), (w.r.t. the weak*-topology), then \( \mu \) is \( A \)-invariant but it may be that \( \mu(X) < 1 \) (since \( X \) is not compact).

**Definition (escape of mass).**

For \( x \in X \), a sequence \( (T_k)_{k=1}^\infty \) in \( \mathbb{R}_{>0} \) s.t. \( T_k \to \infty \) and \( 0 < \gamma < 1 \), we say that \( \delta_{A^+,x}^{T_k} \) (\( k = 1, 2, \ldots \)) exhibit \( \gamma \)-escape of mass if

\[
\limsup_{k \to \infty} \delta_{A^+,x}^{T_k}(K) \leq 1 - \gamma
\]

for any compact subset \( K \subset X \).
Outline of the proof

We take \((\alpha, \beta) \in \mathbb{R}^2\) and \((T_k)_{k=1}^\infty \subset \mathbb{R}_{>0}\) s.t. \(T_k \to \infty\). We consider the sequence of the empirical measures \((\delta^{T_k}_{A^+,\tau_{\alpha,\beta}})_{k=1}^\infty\) of \(\tau_{\alpha,\beta} \in X\).

Case 1 (large entropy case): For \(0 < \gamma < 1\), assume that \((\delta^{T_k}_{A^+,\tau_{\alpha,\beta}})_{k=1}^\infty\) converges to a Borel measure \(\mu\) on \(X\) s.t.

\[
1 - \gamma < \mu(X) \leq 1 \quad \text{and} \quad h_{\hat{\mu}}(a_1) > \gamma,
\]

where \(\hat{\mu} = \mu(X)^{-1}\mu\) and \(a_1 = \text{diag}(e^{-1}, e, 1) \in A^+ \setminus \{e\}\).

If we write \(\hat{\mu} = \int_{E_A(X)} \nu \ d\sigma(\nu)\) for the \(A\)-ergodic decomposition of \(\hat{\mu}\), then, by the measure rigidity,

\[
h_{\hat{\mu}}(a_1) = \int_{E_A(X)} h_\nu(a_1) \ d\sigma(\nu) = \sigma(\{m_X\}) h_{m_X}(a_1) = 4\sigma(\{m_X\}),
\]

and hence,
\[
\lim_{k \to \infty} \delta_{A^+, \tau, \alpha, \beta}^{T_k} = \mu \geq \mu(X) \cdot \sigma\{m_X\} m_X \geq 4^{-1}(1 - \gamma)\gamma \cdot m_X. \quad (1)
\]

(The following “tessellation” idea is from [Björklund, Fregoli, Gorodnik, 2022+].)

For \(T > 1\) and \(0 < \varepsilon < 4^{-1}e^{-1}\), we define

\[
\Omega_{T, \varepsilon} = \{(y, x_1, x_2) \in \mathbb{R}^3 \mid 0 < y < e^{2T}, \ 0 < |x_1|, |x_2| < 2^{-1}, \ y|x_1||x_2| < \varepsilon\} \subset \mathbb{R}^3.
\]

Then

\[
\left|\left\{n < e^{2T} \mid n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\right\}\right| = \left|\Omega_{T, \varepsilon} \cap (u_{\alpha, \beta} \cdot \mathbb{Z}^3)\right|.
\]

If we define

\[
\Delta_{\varepsilon} = \{(y, x_1, x_2) \in \mathbb{R}^3 \mid 0 < y < 1, \ (2e)^{-1} < |x_1|, |x_2| < 2^{-1}, \ y|x_1||x_2| < \varepsilon\} \subset \mathbb{R}^3,
\]

we have the following partial tessellation of \(\Omega_{T, \varepsilon}\):

\[
\Omega_{T, \varepsilon} \supset \bigcup_{m, n \in \mathbb{Z}_{\geq 0}, 0 \leq m, n \leq T} a_{m, n}^{-1} \Delta_{\varepsilon},
\]
and hence,

\[
\frac{1}{T^2} \left| \left\{ n < e^{2T} \mid n \langle n\alpha \rangle \langle n\beta \rangle < \varepsilon \right\} \right| = \frac{1}{T^2} \left| \Omega_{T,\varepsilon} \cap (u_{\alpha,\beta} \cdot \mathbb{Z}^3) \right| \\
\geq \frac{1}{T^2} \sum_{0 \leq m, n \leq T} \left| \Delta_{\varepsilon} \cap (a_{m,n}u_{\alpha,\beta} \cdot \mathbb{Z}^3) \right|.
\]

(2)

By using (1) and (2) and Siegel integral formula, we obtain:

**Theorem (large entropy case).**

Under our assumption of Case 1 (large entropy case), for

\[0 < \forall \varepsilon < 4^{-1}e^{-2},\]

we have

\[
\liminf_{k \to \infty} \frac{1}{T_k^2} \left| \left\{ n < e^{2T_k} \mid n \langle n\alpha \rangle \langle n\beta \rangle < \varepsilon \right\} \right| \geq \gamma(1 - \gamma)\varepsilon.
\]
Case 2 (escape of mass): For $0 < \gamma < 1$, assume that $(\delta_{A^+, \tau_{\alpha, \beta}}^{T_k})_{k=1}^{\infty}$ exhibits $\gamma$-escape of mass.

For $0 < \varepsilon < 1/2$, we define $X_{\varepsilon} = \{x \in X \mid \exists v \in x \setminus \{0\} \text{ s.t. } \|v\|_{\infty} \leq \varepsilon\}$. $X_{\varepsilon}$ is the complement of a bounded subset in $X$ (Mahler’s criterion), and hence,

$$
\liminf_{k \to \infty} \frac{m_{\mathbb{R}^2}}{T_k} \left( \left\{ (s, t) \in [0, T_k]^2 \mid a_s, t \tau_{\alpha, \beta} \in X_{\varepsilon} \right\} \right)
$$

$$
= \liminf_{k \to \infty} \frac{1}{T_k^2} m_{\mathbb{R}^2} \left( \bigcup_{n \in \mathbb{Z}^3 \setminus \{0\}} d_{\varepsilon, n} \cap [0, T_k]^2 \right)
$$

$$
\geq \gamma,
$$

where, for $n = ^t(n, m_1, m_2) \in \mathbb{Z}^3 \setminus \{0\}$ ($n > 0$),

$$
d_{\varepsilon, n} = \{(s, t) \in \mathbb{R}^2 \mid \|a_s, t u_{\alpha, \beta} n\|_{\infty} \leq \varepsilon\}
$$

$$
= \left\{ (s, t) \in \mathbb{R}^2 \mid s \leq \log \frac{\varepsilon}{|n\alpha + m_1|}, t \leq \log \frac{\varepsilon}{|n\beta + m_2|}, s + t \geq \log \frac{|n|}{\varepsilon} \right\}.
$$
For \( n = t(n, m_1, m_2) \in \mathbb{Z}^3 \setminus \{0\} \), if \( d_{\varepsilon, n} \cap [0, T_k]^2 \neq \emptyset \), we have

\[
n \leq \varepsilon e^{2T_k} < e^{2T_k} \quad \text{and} \quad n\langle n\alpha \rangle\langle n\beta \rangle \leq n|n\alpha + m_1||n\beta + m_2| \leq \varepsilon^3 < \varepsilon.
\]
Furthermore, if $d_{\varepsilon, n} \cap [0, T_k]^2$ is large, we can see that

$$kn < e^{2T_k} \quad \text{and} \quad kn\langle kn\alpha \rangle \langle kn\beta \rangle < \varepsilon \quad \text{for many} \quad k \in \mathbb{N}.$$ 

Using (3) and a counting method, we can obtain the following:

**Theorem (escape of mass).**

Under our assumption of Case 2 (escape of mass), for $0 < \forall \varepsilon < 1/2$, we have

$$\lim \inf_{k \to \infty} \frac{(\log T_k)^2}{T_k^2} \left| \left\{ n < e^{2T_k} \mid n\langle n\alpha \rangle \langle n\beta \rangle < \varepsilon \right\} \right| \geq \frac{\gamma}{18}.$$

If the sequence of the empirical measures $(\delta_{A^+,\tau_{\alpha,\beta}})_{k=1}^{\infty}$ of $\tau_{\alpha,\beta}$ converges to a measure with large entropy or exhibits escape of mass, then we can obtain a quantitative result on LC for $(\alpha, \beta)$. 
The exceptional case

To prove Main Theorem, we must consider the exceptional case, that is, \((\alpha, \beta) \in \mathbb{R}^2\) s.t. some subsequence \((\delta^{T_k}_{A^+,\tau_{\alpha,\beta}})_{k=1}^\infty\) of the empirical measures of \(\tau_{\alpha,\beta}\) converges to a measure \(\mu\) on \(X\) s.t.

\[
1 - \gamma < \mu(X) \leq 1 \quad \text{and} \quad h_{\hat{\mu}}(a_1) \leq \gamma.
\]

Actually, we can show that the set of such \((\alpha, \beta)\) has small Hausdorff dimension.

**Theorem (Hausdorff dimension of the exceptional set).**

Let \(x_0 \in X\) and \(0 < \gamma < 1\). We write \(Z_{x_0}(\gamma)\) for the set of \(u \in \overline{B_1^U} = \{u_{\alpha,\beta} \in U \mid |\alpha|, |\beta| \leq 1\}\) s.t. \(\delta^{T}_{A^+,ux_0} (T > 0)\) accumulate to some \(A\)-invariant measure \(\mu\) on \(X\) s.t. \(1 - \gamma < \mu(X) \leq 1\) and \(h_{\hat{\mu}}(a_1) \leq \gamma\). Then we have

\[
\dim_{H} Z_{x_0}(\gamma) \leq 15 \sqrt{\gamma}.
\]
This Theorem is based on the following result by R. Bowen.

**Proposition [Bowen, 1973].**

Let $T : X \rightarrow X$ be a continuous map on a compact metric space $X$. For $\gamma > 0$, we write $QR(\gamma)$ for the set of $x \in X$ s.t. the empirical measures $N^{-1} \sum_{n=0}^{N-1} \delta_{T^n x} (N > 0)$ accumulate to some $T$-invariant probability measure $\mu$ s.t. $h_\mu(T) \leq \gamma$. Then we have

$$h(T, QR(\gamma)) \leq \gamma,$$

where $h(T, A)$ for $A \subseteq X$ is Bowen’s topological entropy (for an arbitrary subset).

If $T$ is $\times a$ map on $\mathbb{R}/\mathbb{Z}$ ($a \geq 2$), $h(T, A) = \log a \cdot \dim_H A$. In Bowen’s argument and ours, the following combinatorial lemma is important:

**Lemma [Bowen, 1973].**

For $k \in \mathbb{N}$ and $\gamma > 0$,

$$\limsup_{N \to \infty} \frac{1}{N} \log \left| \left\{ c \in \{1, \ldots, k\}^N \mid H(\text{dist}(c)) \leq \gamma \right\} \right| \leq \gamma.$$
In our setting, empirical measures are of two-parameter action, but the entropy is of one-parameter subaction. In addition, the space \( X = \text{SL}(3, \mathbb{R})/\text{SL}(3, \mathbb{Z}) \) is not compact. Theorem is based on the following result on the \( \times a, \times b \) action.

**Theorem [U., 2023].**

Let \( a, b \in \mathbb{Z}_{\geq 2} \). We take \( 0 < \gamma < \min\{\log b, (\log a)^2/\log b\} \) and write \( K(\gamma) \) for the set of \( x \in \mathbb{R}/\mathbb{Z} \) such that the empirical measures \( N^{-2} \sum_{m,n=0}^{N-1} \delta_{am^nb^n}x \ (N \in \mathbb{N}) \) accumulate to a probability measure \( \mu \) s.t. \( h_{\mu}(\times a) \leq \gamma \). Then we have

\[
\text{dim}_H K(\gamma) \leq \frac{2\sqrt{\log b} \sqrt{\gamma}}{\log a + \sqrt{\log b} \sqrt{\gamma}}.
\]

The \( A^+ \)-action on each \( U \)-orbit is expanding. But, we need more argument because \( X = \text{SL}(3, \mathbb{R})/\text{SL}(3, \mathbb{Z}) \) is not compact. This can be done by the assumption that the accumulation measure \( \mu \) satisfies \( \mu(X) > 1 - \gamma \).